NAKAYAMA FUNCTOR FOR MONADS ON FINITE ABELIAN CATEGORIES

KENICHI SHIMIZU

Abstract. If $\mathcal{M}$ is a finite abelian category and $\mathbf{T}$ is a linear right exact monad on $\mathcal{M}$, then the category $\mathbf{T}$-mod of $\mathbf{T}$-modules is a finite abelian category. We give an explicit formula of the Nakayama functor of $\mathbf{T}$-mod under the assumption that the underlying functor of the monad $\mathbf{T}$ has a double left adjoint and a double right adjoint. As applications, we deduce formulas of the Nakayama functor of the center of a finite bimodule category and the dual of a finite tensor category. Some examples from the Hopf algebra theory are also discussed.

Contents

1. Introduction 2
1.1. Organization of this paper 4
1.2. Acknowledgment 6
2. Preliminaries 6
2.1. Basic notations 6
2.2. Finite abelian categories 6
2.3. Finite tensor categories and their modules 7
2.4. Nakayama functor 8
2.5. Radford isomorphism 8
3. Nakayama functors and double adjoints 9
3.1. Construction of the canonical isomorphism 9
3.2. Nakayama functor of the opposite category 11
3.3. Nakayama functor of the Deligne tensor product 12
3.4. Nakayama functor of the category of right exact functors 14
4. Nakayama functor for monads 15
4.1. Tensor products over monads 15
4.2. Adjoints of (bi)modules over monads 16
4.3. Adjunctions for monads 17
4.4. Nakayama functor for monads 18
5. Algebras with Frobenius traces 20
5.1. Modules in module categories 20
5.2. Algebras with a Frobenius trace 21
5.3. A formula of the Nakayama functor 22
5.4. Braided Hopf algebras 24
6. The center of a finite bimodule category 26
6.1. The canonical algebra and its variants 26
6.2. The canonical algebra and the twisted center 27
6.3. The canonical algebra and the Radford isomorphism 28
INTRODUCTION

Given a vector space $X$ over a field $k$, we denote its dual space by $X^*$. If $A$ is a finite-dimensional algebra over $k$, then the dual space $A^*$ has a natural structure of an $A$-bimodule. Hence we have an endofunctor

$$N_A : A\text{-mod} \to A\text{-mod}, \quad M \mapsto A^* \otimes_A M$$

on the category $A\text{-mod}$ of finite-dimensional left $A$-modules. This functor, called the Nakayama functor, plays an important role in the representation theory of finite-dimensional algebras; see, e.g., [SY11].

Recently, the Nakayama functor is attracted attention from a different viewpoint, that is, the theory of finite tensor categories and their modules. We recall that a finite abelian category [EGNO15] is a linear category that is equivalent to $A\text{-mod}$ for some finite-dimensional algebra $A$. Fuchs, Schaumann and Schweigert [FSS20] remarked that the Nakayama functor (1.1) is expressed as the coend

$$N_A(M) = \int^{X \in A\text{-mod}} \text{Hom}_A(M, X)^* \otimes_k X \quad (M \in A\text{-mod}).$$

Following this observation, one can define the Nakayama functor $N_M : M \to M$ for a finite abelian category $M$ by

$$N_M(M) := \int^{X \in M} \text{Hom}_M(M, X)^* \otimes X \quad (M \in M),$$

where $\otimes$ is the copower (see §2.2).

The point is that the functor $N_M$ is defined without referencing an algebra $A$ such that $M \cong A\text{-mod}$. In studying finite tensor categories and their modules, we often consider finite abelian categories for which no finite-dimensional algebras realizing them are given explicitly. The above abstractly redefined Nakayama functor is often useful in such a situation. A key feature of the Nakayama functor is the following relation to adjoint functors: Given a functor $F$, we denote by $F^{\text{la}}$ and $F^{\text{ra}}$ a left and a right adjoint of $F$, respectively, if it exists. Let $G : M \to N$ be a linear functor between finite abelian categories such that a double right adjoint $G^{\text{rra}} := (G^{\text{ra}})^{\text{ra}}$ exists and is right exact. By the universal property of Nakayama functors as coends, we have an isomorphism

$$\phi_G : G^{\text{rra}} \circ N_M \to N_N \circ G$$

of functors that is ‘natural’ and ‘coherent’ in a certain sense [FSS20].
As pointed out in [FSS20], some fundamental results on finite tensor categories and their modules are obtained by the isomorphism (1.2). For example, let \( C \) be a finite tensor category. Then the isomorphism (1.2) yields the formula

\[
\mathcal{N}_C(X) \cong \mathcal{N}_C(1) \otimes X^{\vee \vee} \quad (X \in C)
\]

of the Nakayama functor of \( C \), where \( \otimes \) is the tensor product, \( 1 \) is the unit object and \((-)^{\vee} \) is the left duality functor of \( C \). By this formula, we see that \( C \cong A\text{-}\text{mod} \) for some symmetric Frobenius algebra \( A \) if and only if \( C \) is unimodular (i.e., \( \mathcal{N}_C(1) \cong 1 \)) and the double dual functor of \( C \) is isomorphic to \( \text{id}_C \) [FSS20]. The above formula and its variant also give a categorical analogue of Radford’s \( S^2 \)-formula [ENO04] and a Frobenius type property of tensor functors between finite tensor categories [Shi17c].

For the above reasons, we think of the Nakayama functor as a fundamental tool in the theory of finite tensor categories and their modules. It is thus natural to ask how the Nakayama functor is obtained by the isomorphism (1.2). For example, let \( \mathcal{T} \) be an algebra in a monoidal category. Namely, let \( \mathcal{T} \) be an algebra in a monoidal category. Then the isomorphism (1.2) yields the formula (1.3)

\[
\mathcal{T}^{la} \otimes_{\mathcal{T}} \mathcal{M} := \text{coequalizer} \left( \begin{array}{c}
\mathcal{T}^{la}(M) \\
\mathcal{T}^{la}(a, M)
\end{array} \right) \in \mathcal{M}
\]

for \( \mathcal{M} = (\mathcal{M}, a_M : \mathcal{T}(M) \to M) \in \mathcal{T}\)-mod.

The assignment \( \mathcal{M} \mapsto \mathcal{T}^{la} \otimes_{\mathcal{T}} \mathcal{M} \) is not an endofunctor on \( \mathcal{T}\)-mod since, unlike the case of (1.1), the functor \( \mathcal{T}^{la} \) is not a ‘bimodule’ over \( \mathcal{T} \). Such asymmetry has already been noticed in the study of algebras and modules in a monoidal category. Namely, let \( A \) be an algebra in \( C \). Then, according to [DSPS20] §2.4.5, the left dual object of an \( A \)-bimodule in \( C \) is not an \( A \)-bimodule in \( C \) in general, but in fact an \( A^{\vee}\text{-}A \)-bimodule in \( C \). We now suppose that a double left adjoint \( \mathcal{T}^{dla} := (\mathcal{T}^{la})^{la} \) exists. Then \( \mathcal{T}^{dla} \) is naturally a monad on \( \mathcal{M} \), which we denote by \( \mathcal{T}^{dla} \). In the same way as [DSPS20] §2.4.5, we see that \( \mathcal{T}^{la} \) is a ‘\( \mathcal{T}^{dla} \)-\( \mathcal{T} \)-bimodule.’ Furthermore, the assignment \( \mathcal{M} \mapsto \mathcal{T}^{la} \otimes_{\mathcal{T}} \mathcal{M} \) gives rise to a linear functor

\[
(1.3) \quad \mathcal{T}^{la} \otimes_{\mathcal{T}} (-) : \mathcal{T}\text{-mod} \to \mathcal{T}^{dla}\text{-mod}.
\]

The Nakayama functor of \( \mathcal{M} \) lifts to the functor

\[
(1.4) \quad \mathcal{T}^{dla}\text{-mod} \to \mathcal{T}\text{-mod}, \quad (\mathcal{M}, a_M : \mathcal{T}^{la}(M) \to M) \mapsto (\mathcal{N}_{\mathcal{M}}(M), \tilde{a}_M),
\]

where \( \tilde{a}_M \) is the composition

\[
T(\mathcal{N}_{\mathcal{M}}(M)) \xrightarrow{(1.3) \text{ with } G = T^{la}} \mathcal{N}_{\mathcal{M}}(\mathcal{T}^{la}(M)) \xrightarrow{\mathcal{N}_{\mathcal{M}}(a_M)} \mathcal{N}_{\mathcal{M}}(M).
\]

Now the main result of this paper is stated as follows:
Theorem 1.1 (= Theorem 4.5). Let $\mathcal{M}$ be a finite abelian category, and let $T$ be a linear monad on $\mathcal{M}$ whose underlying functor admits a double left adjoint and a double right adjoint. Then the Nakayama functor of $T$-mod is given by the composition of the functors (1.3) and (1.4).

To define the functors (1.3) and (1.4), it suffices to assume that both $T^{\text{lla}}$ and $T^{\text{rra}}$ exist. Let $U : T$-mod $\to \mathcal{M}$ denote the forgetful functor. In the proof of the above theorem, we consider the isomorphism

\[(1.5) \quad N_{T}\text{-mod} \circ U^{\text{lla}} \cong U^{\text{rra}} \circ N_{\mathcal{M}}\]

obtained by letting $G = U^{\text{lla}}$ in (1.2). The existence of $T^{\text{rra}}$ is required to ensure that a double right adjoint of $U^{\text{lla}}$ is right exact. The assumption of the above theorem might be weakened by detouring the use of (1.5).

In the theory of finite tensor categories and their modules, there are many examples of monads to which Theorem 1.1 can be applied. Let $C$ be a finite tensor category, let $A$ be an algebra in $C$, and let $\mathcal{M}$ be a finite left $C$-module category with action $\triangleright : C \times \mathcal{M} \to \mathcal{M}$ (see §2.3). Then the monad $T := A \triangleright (-)$ on $\mathcal{M}$ induced by the algebra $A$ fulfills the assumption of Theorem 1.1. The theorem implies that the Nakayama functor of the category of modules over this monad is written as

\[(1.6) \quad M \mapsto N_{\mathcal{M}}(A^\vee \otimes_{A} M)\]

(see §5.1 for the precise meaning). This formula is used to, for example, compute the Nakayama functor of the category of modules over a Hopf algebra in a braided finite tensor category (see §5.4). The formula is also applied to compute the Nakayama functor of the center of a finite bimodule category.

1.1. Organization of this paper. We explain the organization of this paper as well as results yet to be described in the above.

In Section 2, we introduce some notations and recall basic results on finite tensor categories and their modules.

In Section 3, we review necessary results on the Nakayama functor. We explain the construction of the canonical isomorphism (1.2) and discuss how it looks like in various settings. Most of the contents of this section are found in [FSS20]. The aim of this section is, rather, to write down those results explicitly for the use of later sections.

In Section 4, we state and prove the main result of this paper. For this purpose, we first introduce the notion of bimodules over monads and the tensor product over a monad. Let $\mathcal{M}$ be a finite abelian category, and let $T$ be a monad on $\mathcal{M}$ satisfying the assumption of Theorem 1.1. The forgetful functor $U : T$-mod $\to \mathcal{M}$ has a left adjoint, called the free $T$-module functor. An easy but important observation is that the functor $T$-mod $\to \mathcal{M}$ given by $M \mapsto T^{\text{lla}} \otimes_{T} M$ is a left adjoint of $U^{\text{lla}}$ (Lemma 4.3). Thus, by (1.2), we have

\[\text{UN}_{T}\text{-mod}(M) \cong N_{\mathcal{M}}U^{\text{lla}}(M) \cong N_{\mathcal{M}}(T^{\text{lla}} \otimes_{T} M)\]

for $M \in T$-mod. This means that the underlying object of $N_{T}\text{-mod}(M)$ is given as stated in Theorem 1.1. To verify that $N_{T}\text{-mod}(M)$ is given as stated as a $T$-module, we need a bit technical discussion using (1.5), which we omit in Introduction.

In Section 5, we give applications of our results to the category of modules over an algebra in a finite tensor category. Let $C$ be a finite multi-tensor category, and let $\mathcal{M}$ be a finite left $C$-module category. We denote by $A\mathcal{M}$ the category of modules
over the monad induced by an algebra $A$ in $C$. By applying the main theorem, we see that the Nakayama functor of $\mathcal{A}\mathcal{M}$ is given by (1.6).

The case where $A$ is Frobenius may be of particular interest. In Section 5, we consider a more general setting that there is an invertible object $I \in C$ and $A$ is an algebra in $C$ equipped with a ‘non-degenerate’ morphism $\lambda: A \to I$ (such a morphism $\lambda$ is called an $I$-valued Frobenius trace in this paper). In this case, the Nakayama functor of $\mathcal{A}\mathcal{M}$ is given by $M \mapsto \mathcal{N}_{\mathcal{M}}(I \triangleright M)$ on the level of objects of $\mathcal{M}$, but the action of $A$ is twisted by an analogue of the Nakayama automorphism of $A$ as in the case of ordinary Frobenius algebras (Theorem 5.7). This result can be applied to Hopf algebras in a braided finite multi-tensor category (§5.4). Indeed, a cointegral and the ‘object of integrals’ play the role of $\lambda$ and $I$ of the above, respectively.

In Section 6, we explain how our results are applied to some categories appearing in the Hopf algebra theory. Let $H$ be a finite-dimensional Hopf algebra. In this section, we work in the category $C$ of finite-dimensional left $H$-comodules. Given two $A$ and $B$ in $C$, the category $\mathcal{A}\mathcal{C}_{B}$ of $A$-$B$-bimodules in $C$ is defined (an object of this category is called a relative Hopf bimodule). Since $\mathcal{A}\mathcal{C}_{B}$ is the category of modules over the monad $A \otimes (-) \otimes B$, its Nakayama functor can be computed by our result. For practical applications, we consider algebras in $C$ admitting non-degenerate grouplike-cointegral in the sense of [Kas18]. When $A$ and $B$ are such algebras, then the Nakayama functor of $\mathcal{A}\mathcal{C}_{B}$ takes a simple form (Theorem 6.8). By slightly extending the notion of unimodularity of a finite tensor category, we say that $\mathcal{A}\mathcal{C}_{A}$ is unimodular if the Nakayama functor of $\mathcal{A}\mathcal{C}_{A}$ fixes $A \in \mathcal{A}\mathcal{C}_{A}$ up to isomorphism. Theorem 6.8 yields handy criteria for $\mathcal{A}\mathcal{C}_{A}$ to be unimodular (Corollaries 6.10 and 6.11).

Some concrete examples are given in §6.4. For an algebra $A \in C$, the category $A\text{-mod}$ is in fact a left module category over the finite tensor category $D := H\text{-mod}$. According to [AM07], the category $\mathcal{A}\mathcal{C}_{A}$ is monoidally equivalent to the dual of $D$ with respect to $\mathcal{M}$. Thus our computation can also be thought of as examples of determining the unimodularity of the dual tensor category.
Our criteria for unimodularity (Corollaries 7.10 and 7.11) are applicable only for algebras in \( \mathcal{C} \) admitting non-degenerate grouplike-cointegrals. The presence of such a cointegral reduces the computation drastically, however, we shall note that it is sometimes absent. We give examples of coideal subalgebras without non-zero grouplike-cointegrals in \( \S 7.5 \). Such cases should also be treated, but it is beyond the scope of this paper.

1.2. **Acknowledgment.** The author thanks Taiki Shibata for discussion. The author is supported by JSPS KAKENHI Grant Number JP20K03520.

2. **Preliminaries**

2.1. **Basic notations.** Our basic reference on category theory is Mac Lane [ML98]. Given a category \( \mathcal{C} \), we denote its opposite category by \( \mathcal{C}^{\text{op}} \). An object \( X \in \mathcal{C} \) is written as \( X^{\text{op}} \) when it is viewed as an object of \( \mathcal{C}^{\text{op}} \). A similar notation will be used for functors.

Given a functor \( F \), we denote by \( F^{\text{la}} \) and \( F^{\text{ra}} \) a left and a right adjoint of \( F \), respectively, provided that they exist. We say that \( F \) admits a double left adjoint if a left adjoint of \( F \) exists and admits a left adjoint. The meaning of a ‘triple’ left adjoint should be clear. A similar phrase will be used for right adjoints.

Throughout this paper, we work over a field \( k \) (a technical assumption will be imposed on \( k \) in Section 6). Given a vector space \( X \), we denote by \( X^* \) the dual space of \( X \). By an algebra, we mean an associative and unital algebra over \( k \). Given two algebras \( A \) and \( B \), we denote by \( \text{A-mod, mod-}B \) and \( \text{A-mod-}B \) the category of finite-dimensional left \( A \)-modules, right \( B \)-modules and \( A-B \)-modules, respectively. We write \( k \)-mod as \( \text{Vec} \).

2.2. **Finite abelian categories.** A finite abelian category is a linear category that is equivalent to \( \mathcal{A} \)-mod for some finite-dimensional algebra \( \mathcal{A} \). We note that a linear category \( \mathcal{M} \) is a finite abelian category if and only if \( \mathcal{M}^{\text{op}} \) is, since the opposite category of \( \mathcal{A} \)-mod is equivalent to \( \mathcal{A}^{\text{op}} \)-mod by the duality.

Given two finite abelian categories \( \mathcal{M} \) and \( \mathcal{N} \), we denote by \( \text{Rex}(\mathcal{M}, \mathcal{N}) \) the category of linear right exact functors from \( \mathcal{M} \) to \( \mathcal{N} \). Let \( \mathcal{A} \) and \( \mathcal{B} \) be finite-dimensional algebras. The Eilenberg-Watts theorem states that the functor

\[
(2.1) \quad \text{B-mod-}A \to \text{Rex}(\text{A-mod, B-mod}), \quad M \mapsto M \otimes_A (-)
\]

is an equivalence of linear categories. By this equivalence, we see that \( \text{Rex}(\mathcal{M}, \mathcal{N}) \) is a finite abelian category.

The equivalence \( 2.1 \) also shows that a linear functor \( F : \mathcal{M} \to \mathcal{N} \) has a right adjoint if and only if \( F \) is right exact. Applying the same argument to \( F^{\text{op}} : \mathcal{M}^{\text{op}} \to \mathcal{N}^{\text{op}} \), we see that \( F \) has a left adjoint if and only if \( F \) is left exact.

Let \( \mathcal{M} \) be a finite abelian category, and let \( M \in \mathcal{M} \) be an object. Since the linear functor \( \text{Hom}_\mathcal{M}(M, -) : \mathcal{M} \to \text{Vec} \) is left exact, it has a left adjoint. We denote it by \( (-) \otimes M \). Thus, by definition, there is a natural isomorphism

\[
(2.2) \quad \text{Hom}_\mathcal{M}(X \otimes M, N) \cong \text{Hom}_k(X, \text{Hom}_\mathcal{M}(M, N))
\]

for \( X \in \text{Vec} \) and \( N \in \mathcal{M} \). The assignment \( (X, M) \mapsto X \otimes M \) extends to a bilinear functor from \( \text{Vec} \times \mathcal{M} \) to \( \mathcal{M} \), which we call the *copower*, in such a way that \( 2.2 \) is also natural in \( M \). Using the terminology to be introduced in \( \S 2.3 \) the category \( \mathcal{M} \) is a finite module category over \( \text{Vec} \).
2.3. Finite tensor categories and their modules. For basics on monoidal categories, we refer the reader to [ML98] and [EGNO15]. We assume that all monoidal categories are strict in view of Mac Lane’s strictness theorem. Given a monoidal category $C$, we usually denote by $\otimes$ and $I$ the monoidal product and the unit object of $C$, respectively. We denote by $C^{rev}$ the monoidal category obtained from $C$ by reversing the order of the monoidal product.

We follow [EGNO15, §2.10] for terminology for dual objects in a monoidal category. Let $C$ be a rigid monoidal category, that is, a monoidal category of which every object has a left and a right dual object. We usually denote a left dual object of $X \in C$ by $(X^\vee, ev_X : X^\vee \otimes X \to I, coev_X : I \to X \otimes X^\vee)$. The assignment $X \mapsto X^\vee$ gives rise to a contravariant monoidal equivalence $C \to C^{rev}$, which we call the left duality functor. A quasi-inverse of $(-)^\vee$, which we denote by $\vee(-)$, is given by taking a right dual object. By replacing $C$ with an equivalent one and choosing dual objects in an appropriate way, we may assume that $(-)^\vee$ and $\vee(-)$ are mutually inverse strict monoidal functors.

Definition 2.1 ([EGNO15]). A finite multi-tensor category is a finite abelian category equipped with a structure of a rigid monoidal category whose monoidal product is bilinear. A finite tensor category is a finite multi-tensor category whose unit object is simple.

Given a monoidal category $C$, a left $C$-module category is a category $M$ endowed with a functor $\triangleright : C \times M \to M$, called the action of $C$ on $M$, and natural isomorphisms

\[(X \otimes Y) \triangleright M \cong X \triangleright (Y \triangleright M) \quad \text{and} \quad \mathbf{1} \triangleright M \cong M \quad (X, Y \in C, M \in M)\]

satisfying certain axioms similar to those for monoidal categories. A right module category and a bimodule category are defined analogously.

Let $\mathcal{M}$ and $\mathcal{N}$ be left $C$-module categories. A lax left $C$-module functor ([DSPS19] from $\mathcal{M}$ to $\mathcal{N}$ is a pair $(F, \xi)$ consisting of a functor $F : \mathcal{M} \to \mathcal{N}$ and a natural transformation $\xi_{X, M} : X \triangleright F(M) \to F(X \triangleright M) \quad (X \in C, M \in \mathcal{M})$ obeying certain axioms similar to those for monoidal functors. A lax left $C$-module functor $(F, \xi)$ is said to be strong if the structure morphism $s$ is invertible. When $C$ is rigid, every lax left $C$-module functor is strong ([DSPS19], Lemma 2.10) and thus the adjective ‘lax’ is usually omitted.

Left $C$-module categories, lax left $C$-module functors and their morphisms form a 2-category. An equivalence of left $C$-module categories is defined as an equivalence in this 2-category. A left $C$-module category is said to be strict if the natural isomorphisms (2.3) are the identities. One can prove an analogue of Mac Lane’s strictness theorem for module categories ([EGNO15], Remark 7.2.4]. Hence we may, and do, assume that all module categories are strict in this paper.

Definition 2.2. Let $C$ be a finite multi-tensor category. A finite left $C$-module category is a finite abelian category $\mathcal{M}$ equipped with a structure of a left $C$-module category such that the action of $C$ on $\mathcal{M}$ is linear and right exact in each variable. A finite right $C$-module category and a finite $C$-bimodule category are defined in an analogous way.

Despite that we only assume its right exactness, the action of $C$ on a finite left $C$-module category is exact in each variable ([DSPS19], Corollary 2.26].
2.4. Nakayama functor. Let $\mathcal{M}$ be a finite abelian category. In [FSS20], the Nakayama functor of $\mathcal{M}$ is defined to be the endofunctor $N_{\mathcal{M}}$ on $\mathcal{M}$ given by

$$N_{\mathcal{M}}(M) = \int^{X \in \mathcal{M}} \text{Hom}_{\mathcal{M}}(M, X)^* \otimes X$$

for $M \in \mathcal{M}$, where the integral means coends [ML98]. When the category $\mathcal{M}$ is clear from the context, we write $N_{\mathcal{M}}$ as $N$.

A finite abelian category $\mathcal{M}$ is said to be Frobenius if the class of projective objects of $\mathcal{M}$ coincides with the class of injective objects of $\mathcal{M}$. We say that $\mathcal{M}$ is symmetric Frobenius if $\mathcal{M} \approx A\text{-mod}$ for some symmetric Frobenius algebra $A$.

According to [FSS20, Proposition 3.24], $\mathcal{M}$ is Frobenius (respectively, symmetric Frobenius) if and only if the functor $N_{\mathcal{M}}$ is an equivalence (respectively, $N_{\mathcal{M}}$ is isomorphic to id$_{\mathcal{M}}$).

One of key features of the Nakayama functor is the following canonical isomorphism: Let $G : \mathcal{M} \to \mathcal{N}$ be a linear functor between finite abelian categories. If $G$ admits a triple right adjoint, then there is a natural isomorphism

$$\phi_G(M) : G^{rra}N_{\mathcal{M}}(M) \to N_{\mathcal{N}}G(M) \quad (M \in \mathcal{M})$$

that is ‘natural’ in $G$ and ‘coherent’ in a certain sense [FSS20, Theorem 3.18]. The construction of (2.4) will be recalled in Section 3.

2.5. Radford isomorphism. Let $\mathcal{C}$ be a finite multi-tensor category, and let $\mathcal{M}$ be a finite left $\mathcal{C}$-module category with action $\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$. Then the functor

$$\text{act}_{\mathcal{M}} : \mathcal{C} \to \text{Rex}(\mathcal{M}) := \text{Rex}(\mathcal{M}, \mathcal{M}), \quad \text{act}_{\mathcal{M}}(X)(M) = X \triangleright M$$

is a linear strong monoidal functor, where $\text{Rex}(\mathcal{M})$ is viewed as a monoidal category by the composition of functors. Since a strong monoidal functor preserves duals, there is an adjunction $\text{act}_{\mathcal{M}}(X) \dashv \text{act}_{\mathcal{M}}(\triangleright X)$ for each object $X \in \mathcal{C}$. The functor $N_{\mathcal{M}}$ is a ‘twisted’ left $\mathcal{C}$-module functor by the natural isomorphism

$$(2.5) \quad \triangleright X \triangleright X \triangleright N_{\mathcal{M}}(M) \cong N_{\mathcal{M}}(X \triangleright M) \quad (X \in \mathcal{C}, M \in \mathcal{M})$$

obtained by letting $G = \text{act}_{\mathcal{M}}(X)$ in (2.4).

An analogous result holds for finite right module categories: Let $\mathcal{C}$ be as above, and let $\mathcal{M}$ be a finite right $\mathcal{C}$-module category with action $\triangleleft : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$. Then $N_{\mathcal{M}}$ has a twisted right $\mathcal{C}$-module structure

$$(2.6) \quad N_{\mathcal{M}}(M) \triangleleft X \triangleright \cong N_{\mathcal{M}}(M \triangleleft X) \quad (X \in \mathcal{C}, M \in \mathcal{M}).$$

The category $\mathcal{C}$ itself is a finite $\mathcal{C}$-bimodule category by the tensor product of $\mathcal{C}$. Hence, as noted in [FSS20], there are natural isomorphisms

$$(2.7) \quad \triangleright X \otimes N_{\mathcal{C}}(1) \cong N_{\mathcal{C}}(X \otimes 1) = N_{\mathcal{C}}(X) = N_{\mathcal{C}}(1 \otimes X) \cong N_{\mathcal{C}}(1) \otimes X$$

for $X \in \mathcal{C}$. The object $N_{\mathcal{C}}(1)$ is of particular importance. If $\mathcal{C} = H\text{-mod}$ for some finite-dimensional (quasi-)Hopf algebra $H$, then $N_{\mathcal{C}}(1)$ is given by the modular function on $H$. Accordingly, we introduce the following terminology:

Definition 2.3. Let $\mathcal{C}$ be a finite abelian category equipped with a structure of a monoidal category. We call $\alpha_{\mathcal{C}} := N_{\mathcal{C}}(1)$ the modular object of $\mathcal{C}$. We say that $\mathcal{C}$ is unimodular if $N_{\mathcal{C}}(1)$ is isomorphic to the unit object.
Radford’s $S^4$-formula for finite-dimensional Hopf algebras has been extended to finite multi-tensor category as a formula of the quadruple dual [ENO04]. As pointed out in [FSS20], the $S^4$-formula of [ENO04] follows from a basic property of the Nakayama functor. To give a detail, we first introduce:

**Definition 2.4.** For a finite multi-tensor category $C$, we define

$$R_X := \left( \alpha_C \otimes X^{\vee\vee} \xrightarrow{\text{id} \otimes \alpha_C} \mathbb{N}_C(X) \xrightarrow{\text{id} \otimes R_X} \alpha_C \otimes X^{\vee\vee} \right) \quad (X \in C)$$

and call $R$ the *Radford isomorphism* of $C$.

Since a finite multi-tensor category is Frobenius [EGNO15], the functor $\mathbb{N}_C$ is an equivalence. By the formula (2.7) of the Nakayama functor, the modular object $\alpha_C$ is invertible. Hence the Radford isomorphism induces a natural isomorphism

$$X^{\vee\vee\vee} \xrightarrow{\text{id} \otimes R_X} \alpha_C \otimes X^{\vee\vee\vee} \xrightarrow{\text{id} \otimes R_X} \alpha_C \otimes X \otimes \alpha_C$$

for $X \in C$. As has been explained in [Shi17a, Section 4], this isomorphism coincides with the Radford $S^4$-formula given in [ENO04].

**Remark 2.5.** Let $C$ be a finite multi-tensor category. In view of (2.7), we may assume that $\mathbb{N}_C$ is given by $\mathbb{N}_C(X) = \vee^\vee X \otimes \alpha_C$ for $X \in C$. Under this choice of the Nakayama functor (and our assumption that the double dual functor of $C$ is strict monoidal), the twisted left $C$-module structure (2.5) of $\mathbb{N}_C$ is the identity morphism. The twisted right $C$-module structure (2.6) is given by

$$(\text{id} \otimes M) \otimes R_X : \mathbb{N}_C(M) \otimes X^{\vee\vee} \to \mathbb{N}_C(M \otimes X)$$

for $M, X \in C$.

### 3. Nakayama functors and double adjoints

#### 3.1. Construction of the canonical isomorphism

In this section, we collect useful formulas of Nakayama functors from [FSS20] and discuss how the canonical isomorphism (2.4) looks like in various settings. We first recall the following lemma for (co)ends:

**Lemma 3.1.** Let $T : B^{\text{op}} \times A \to V$ and $F : A \to B$ be functors, where $A$, $B$ and $V$ are categories. If $F$ has a left adjoint, then we have

$$\int_{X \in A} T(F(X), X) \cong \int_{Y \in B} T(Y, F^{\text{la}}(Y)),$$

meaning that the end of the left hand side exists if and only if so does the right hand side and, if they exist, they are canonically isomorphic. If $F$ has a right adjoint, then we have

$$\int_{X \in A} T(F(X), X) \cong \int_{Y \in B} T(Y, F^{\text{ra}}(Y))$$

with a similar meaning as the case of ends.

The isomorphism (3.2) is given in [BV12, Lemma 3.8], and the isomorphism (3.1) is obtained by the dual argument. For reader’s convenience, we include how the isomorphism (3.2) is established. Let $C$ and $D$ be the left and the right hand side of (3.2), respectively, and let

$$i_X : T(F(X), X) \to C \quad (X \in A) \quad \text{and} \quad j_Y : T(Y, F^{\text{ra}}(Y)) \to D \quad (Y \in B)$$


be the universal dinatural transformations for these coends. Then the isomorphism of the above lemma, which we denote by \( \phi : C \to D \), and its inverse are characterized as unique morphisms in \( V \) such that the equations

\[
\phi \circ i_X = j_{F(X)} \circ T(\text{id}_{F(X)}, \eta_X) \quad \text{and} \quad \phi^{-1} \circ j_Y = i_{F^{ra}(Y)} \circ T(\varepsilon_Y, \text{id}_{F^{ra}(Y)})
\]

hold for all objects \( X \in A \) and \( Y \in B \), where \( \eta \) and \( \varepsilon \) are the unit and the counit of the adjunction \( F \dashv F^{ra} \).

Let \( \mathcal{M} \) and \( \mathcal{N} \) be finite abelian categories, and let \( G : \mathcal{M} \to \mathcal{N} \) be a linear functor admitting a triple right adjoint. The canonical isomorphism mentioned in §2.4 is given by the composition

\[
G^{\text{ra}} N(M) = G^{\text{ra}} \left( \int_{X \in \mathcal{M}} \text{Hom}_{\mathcal{M}}(M, X)^* \otimes X \right) \\
\cong \int_{X \in \mathcal{M}} \text{Hom}_{\mathcal{M}}(M, X)^* \otimes G^{\text{ra}}(X) \\
\cong \int_{X \in \mathcal{N}} \text{Hom}_{\mathcal{M}}(M, G^{\text{ra}}(X))^* \otimes X \\
\cong \int_{X \in \mathcal{N}} \text{Hom}_{\mathcal{M}}(G(M), X)^* \otimes X = \mathbb{N}G(M),
\]

where the first isomorphism follows from that \( G^{\text{ra}} \) preserves copowers and colimits as it has a right adjoint, the second one is given by Lemma 3.1 and the last one is the adjunction isomorphism.

There are natural isomorphisms

\[
\text{Hom}_{\mathcal{M}}(N(M), M') \cong \int_{X \in \mathcal{M}} \text{Hom}_{\mathcal{M}}(\text{Hom}_{\mathcal{M}}(X, M'), \text{Hom}_{\mathcal{M}}(X, M))^* \otimes M, X) \\
\cong \int_{X \in \mathcal{M}} \text{Hom}_{\mathcal{M}}(\text{Hom}_{\mathcal{M}}(X, M'), \text{Hom}_{\mathcal{M}}(M, X)) \\
\cong \int_{X \in \mathcal{M}} \text{Hom}_{\mathcal{M}}(X, M') \otimes_{k} \text{Hom}_{\mathcal{M}}(M, X)
\]

for \( M, M' \in \mathcal{M} \). By the construction of the isomorphism (2.4), it is straightforward to verify the following lemma:

**Lemma 3.2.** For \( G \) as above, the diagram of Figure 1 commutes for all objects \( M \in \mathcal{M} \) and \( N \in \mathcal{N} \) (the Hom functor is denoted by \([-,-]\) to save spaces in the diagram).
3.2. Nakayama functor of the opposite category. We discuss the Nakayama functor of the opposite category of a finite abelian category. Let $\mathcal{M}$ be a finite abelian category. There is an endofunctor $N_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$, called the left exact analogue of the Nakayama functor in [FSS20, Definition 3.14]. The functor $N_{\mathcal{M}}$ is nothing but the Nakayama functor of $\mathcal{M}^{op}$ viewed as an endofunctor on $\mathcal{M}$. Namely, we have

$$N_{\mathcal{M}^{op}} = (N_{\mathcal{M}})^{op},$$

as noted in [FSS20, Proposition 3.20].

Since the functor $\text{Hom}_{\mathcal{M}}(M, -) : \mathcal{M} \to \text{Vec}$ preserves ends and copowers, there is a natural isomorphism

$$(3.5) \text{Hom}_{\mathcal{M}}(M, N_{\mathcal{M}}(M')) \cong \int_{X \in \mathcal{M}} \text{Hom}_{\mathcal{M}}(M, X) \otimes k \text{Hom}_{\mathcal{M}}(X, M')$$

for $M, M' \in \mathcal{M}$. Thus we obtain a natural isomorphism

$$(3.6) \text{Hom}_{\mathcal{M}}(N_{\mathcal{M}}(M), M') \cong \text{Hom}_{\mathcal{M}}(M, N_{\mathcal{M}}(M')) \quad (M, M' \in \mathcal{M})$$

by composing (3.3) and (3.5). This means that $N_{\mathcal{M}}$ is right adjoint to $N_{\mathcal{M}}$ [FSS20, Lemma 3.16]. In what follows, we choose $N_{\mathcal{M}}$ as a right adjoint of $N_{\mathcal{M}}^{op}$ together with the adjunction isomorphism (3.6).

Now let $F : \mathcal{N} \to \mathcal{M}$ be a linear functor between finite abelian categories $\mathcal{M}$ and $\mathcal{N}$. We assume that $F$ admits a triple left adjoint so that $F^{op} : \mathcal{N}^{op} \to \mathcal{M}^{op}$ admits a triple right adjoint. Then there is a canonical isomorphism

$$\phi_{F^{op}} : (F^{op})^{ra} \circ N_{\mathcal{N}^{op}} \to N_{\mathcal{M}^{op}} \circ F^{op}$$

by (2.14) with $G = F^{op}$. We note that $\phi_{F^{op}}$ is an isomorphism of functors from $\mathcal{N}^{op}$ to $\mathcal{M}^{op}$. Thus $\phi_{F^{op}}$ is actually a family

$$\phi_{F^{op}} = \{\phi_{F^{op}}(N) : N^{ra} F(N) \to F^{lla} N^{ra}(N)\}_{N \in \mathcal{N}}$$

of isomorphisms in $\mathcal{M}$.

Lemma 3.3. Let $F$ be as above, and let $G = F^{lla}$ be a triple left adjoint of $F$. Then, as a natural transformation between functors from $\mathcal{N}$ to $\mathcal{M}$, the canonical isomorphism $\phi_{F^{op}}$ is given by the composition

$$\phi_{F^{op}} = \left( N^{ra} F \xrightarrow{\cong} (G^{ra} N)^{ra} \xrightarrow{(\phi_G^{-1})^{ra}} (NG)^{ra} \xrightarrow{\cong} F^{lla} N^{ra} \right),$$

where the first and the third arrows represent the canonical isomorphism $(S \circ T)^{ra} \cong T^{ra} \circ S^{ra}$ for composable functors $S$ and $T$ admitting right adjoints.

Proof. We fix objects $M \in \mathcal{M}$ and $N \in \mathcal{N}$, and consider the diagram of Figure 2 (where the Hom functor is denoted by $[-,-]$ to save spaces). The left rectangle is the commutative diagram of Figure 1 with $G = F^{lla}$. The right rectangle is also commutative by Lemma 3.2 applied to $F^{op}$. Hence the diagram of Figure 2 is
commutative. It shrinks to the following commutative diagram:

\[
\begin{array}{cccc}
\text{Hom}_\mathcal{A}(\mathbb{N}G(M), N) & \xrightarrow{3.7} & \text{Hom}_\mathcal{A}(G(M), \mathbb{N}^{ra}(N)) \\
\downarrow \text{adjunction} & & \downarrow \text{adjunction} \\
\text{Hom}_\mathcal{A}(\mathbb{F}laN(M), N) & & \text{Hom}_\mathcal{M}(M, \mathbb{F}la\mathbb{N}^{ra}(N)) \\
\downarrow \text{adjunction} & & \downarrow \text{adjunction} \\
\text{Hom}_\mathcal{M}(M, \mathbb{N}^{ra}F(N)) & \xrightarrow{3.8} & \text{Hom}_\mathcal{M}(M, \mathbb{N}^{ra}F(N))
\end{array}
\]

By letting \( M = \mathbb{F}la\mathbb{N}^{ra}(N) \) and chasing \( \text{id}_M \in \text{Hom}_\mathcal{M}(M, \mathbb{F}la\mathbb{N}^{ra}(N)) \) in this diagram, we see that the inverse of \( \Phi_{\text{pop}} \) is given by

\[
\Phi_{\text{pop}}^{-1} = \left( \mathbb{F}la\mathbb{N}^{ra} \xrightarrow{\sim} \mathbb{N}^{ra} \xrightarrow{(\Phi_G)^{ra}} \mathbb{F}la\mathbb{N}^{ra} \xrightarrow{\sim} \mathbb{N}^{ra}F \right).
\]

Now the desired formula is obvious. The proof is done. \( \square \)

### 3.3. Nakayama functor of the Deligne tensor product

Here we give technical remarks on the Nakayama functor of the Deligne tensor product. Given finite abelian categories \( \mathcal{M} \) and \( \mathcal{N} \), we denote by \( \mathcal{M} \boxtimes \mathcal{N} \) their Deligne tensor product [EGNO15]. It was pointed out in [FSS20, Subsection 3.4] that a coend over \( \mathcal{M} \boxtimes \mathcal{N} \) can be computed as a double coend \( \int_{X \in \mathcal{M}, Y \in \mathcal{N}} \) under suitable assumptions on the integrand. As a consequence, we see that there is an isomorphism

\[
\theta : \mathbb{N}_\mathcal{M} \boxtimes \mathbb{N}_\mathcal{N} \to \mathbb{N}_\mathcal{M} \boxtimes \mathbb{N}_\mathcal{N}
\]

given as follows: Given a finite abelian category \( \mathcal{A} \), we denote by

\[
i_{M,X} : \text{Hom}_\mathcal{A}(M, X) \otimes X \to \mathbb{N}_\mathcal{A}(M) \quad (M, X \in \mathcal{A})
\]

the universal dinatural transformation for the coend \( \mathbb{N}_\mathcal{A}(M) \). We fix objects \( M \in \mathcal{M} \) and \( N \in \mathcal{N} \). By the Fubini theorem for coends and the exactness of \( \boxtimes \), the

Figure 2.
object \( \mathbb{N}_M(M) \boxtimes \mathbb{N}_N(N) \) is a coend

\[
\int^{X \in M, Y \in N} (\text{Hom}_M(X, M)^* \otimes X) \boxtimes (\text{Hom}_N(Y, N)^* \otimes Y)
\]

with universal dinatural transformation \( i_{M,X} \boxtimes i_{N,Y} \). For a \( \boxtimes \)-decomposable object \( T = M \boxtimes N \), the isomorphism \( \theta_T \) is defined to be the unique morphism in \( \mathcal{M} \boxtimes \mathcal{N} \) such that the diagram

\[
\begin{array}{ccc}
(\text{Hom}_M(M, X)^* \otimes X) \boxtimes (\text{Hom}_N(Y, N)^* \otimes Y) & \xrightarrow{i_{X,Y,M,N}} & \mathbb{N}_M(M) \boxtimes \mathbb{N}_N(N) \\
\downarrow \cong & & \downarrow \theta_T \\
\text{Hom}_M(M \boxtimes N, X \boxtimes Y)^* \otimes (X \boxtimes Y) & \xrightarrow{i_{X,Y,M,N}} & \mathbb{N}_{M \boxtimes N}(T)
\end{array}
\]

commutes for all \( X \in \mathcal{M} \) and \( Y \in \mathcal{N} \). We extend \( \theta_T \) to all objects \( T \in \mathcal{M} \boxtimes \mathcal{N} \) by the right exactness of Nakayama functors and a resolution of \( T \) by \( \boxtimes \)-decomposable objects.

Let \( F : \mathcal{M}_1 \to \mathcal{M}_2 \) and \( G : \mathcal{N}_1 \to \mathcal{N}_2 \) be linear functors, where \( \mathcal{M}_1, \mathcal{M}_2, \mathcal{N}_1 \) and \( \mathcal{N}_2 \) are finite abelian categories. We assume that both \( F \) and \( G \) admit triple right adjoint and choose \( F^{\text{tra}} \boxtimes G^{\text{tra}} \) as a double right adjoint of \( F \boxtimes G \). In view of the above discussion, we may also choose \( \mathbb{N}_M \boxtimes \mathbb{N}_N \) \((i = 1, 2)\) as the Nakayama functor of \( \mathcal{M}_i \boxtimes \mathcal{N}_i \). Then, as one may expect, the canonical isomorphism (2.4) for \( F \boxtimes G \) is decomposed as follows:

**Lemma 3.4.** For \( F \) and \( G \) as above, we have

(3.8) \( \phi_{F \boxtimes G} = \phi_F \boxtimes \phi_G : F^{\text{tra}} \boxtimes G^{\text{tra}} \to \mathbb{N}F \boxtimes \mathbb{N}G. \)

**Proof.** Since the source and the target of (3.8) are linear right exact functors, it suffices to show that the equation

(3.9) \( \phi_{F \boxtimes G}(M \boxtimes N) = \phi_F(M) \boxtimes \phi_G(N) \)

holds for all objects \( M \in \mathcal{M}_1 \) and \( N \in \mathcal{N}_1 \). For simplicity of notation, we set \( \mathcal{L}_i = \mathcal{M}_i \boxtimes \mathcal{N}_i \) \((i = 1, 2)\) and introduce the functor

\( T : \mathcal{L}_1^{\text{op}} \times \mathcal{L}_1 \to \mathcal{L}_2 \), \( (K^{\text{op}}, L) \mapsto (F^{\text{tra}} \boxtimes G^{\text{tra}})(\text{Hom}_{\mathcal{L}_1}(M \boxtimes N, K)^* \otimes L) \).

The source of (3.9) is a coend of \( T \). We denote by \( j_L \) \((L \in \mathcal{L}_1)\) the universal dinatural transformation of this coend. For a \( \boxtimes \)-decomposable object \( L = X \boxtimes Y \) of \( \mathcal{L}_1 \), the morphism \( j_L \) is given by the composition of the isomorphism

\( T(L, L) \cong F^{\text{tra}}(\text{Hom}_{\mathcal{M}_1}(M, X)^* \otimes X) \boxtimes G^{\text{tra}}(\text{Hom}_{\mathcal{N}_1}(N, Y)^* \otimes Y) \)

and \( F^{\text{tra}}(i_{M,X}) \boxtimes G^{\text{tra}}(i_{N,Y}) \). By this observation and the definition of the canonical isomorphism (2.4), it is straightforward to verify that the equation

(3.10) \( \phi_{F \boxtimes G}(M \boxtimes N) \circ j_L = (\phi_F(M) \boxtimes \phi_G(N)) \circ j_L \)

holds for all \( \boxtimes \)-decomposable object \( L \in \mathcal{L}_1 \).

To complete the proof, we shall show that the equation (3.10) holds for all objects \( L \in \mathcal{L}_1 \). Let \( L \in \mathcal{L}_1 \) be an arbitrary object, and let \( q : X \boxtimes Y \to L \) be an epimorphism in \( \mathcal{L}_1 \) from a \( \boxtimes \)-decomposable object. Then we have

\[
(\phi_F(M) \boxtimes \phi_G(N)) \circ j_L \circ T(id_L, q) = (\phi_F(M) \boxtimes \phi_G(N)) \circ j_{X \boxtimes Y} \circ T(q, id_{X \boxtimes Y})
\]

\[
= \phi_{F \boxtimes G}(M \boxtimes N) \circ j_{X \boxtimes Y} \circ T(q, id_{X \boxtimes Y}) = \phi_{F \boxtimes G}(M \boxtimes N) \circ j_L \circ T(id_L, q)
\]
3.4. Nakayama functor of the category of right exact functors. For two finite abelian categories \( \mathcal{M} \) and \( \mathcal{N} \), there is an equivalence

\[
(\text{3.11}) \quad \text{EW}_{\mathcal{M}, \mathcal{N}} : \mathcal{M}^{\text{op}} \boxtimes \mathcal{N} \to \text{Rex}(\mathcal{M}, \mathcal{N}), \quad M^{\text{op}} \boxtimes N \mapsto \text{Hom}_{\mathcal{M}}(-, M)^* \otimes N
\]

of linear categories \( \text{Shi17c}, \text{Lemma 2.3} \), which may be thought of as a ‘Morita invariant’ version of (2.1). Given linear right exact functors \( F : \mathcal{M}_2 \to \mathcal{M}_1 \) and \( G : \mathcal{N}_1 \to \mathcal{N}_2 \) between finite abelian categories \( \mathcal{M}_1, \mathcal{M}_2, \mathcal{N}_1 \) and \( \mathcal{N}_2 \), we define

\[
(\text{3.12}) \quad \text{Rex}(F, G) : \text{Rex}(\mathcal{M}_1, \mathcal{N}_1) \to \text{Rex}(\mathcal{M}_2, \mathcal{N}_2), \quad X \mapsto G \circ X \circ F.
\]

The equivalence (3.11) is ‘natural’ in the sense that the diagram

\[
\begin{array}{ccc}
(M_1)^{\text{op}} \boxtimes N_1 & \xrightarrow{\text{EW}_{M_1, N_1}} & \text{Rex}(M_1, N_1) \\
\downarrow (F^{\text{ra}})^{\text{op}} \boxtimes G & & \downarrow \text{Rex}(F, G) \\
(M_2)^{\text{op}} \boxtimes N_2 & \xrightarrow{\text{EW}_{M_2, N_2}} & \text{Rex}(M_2, N_2)
\end{array}
\]

commutes up to isomorphism.

**Lemma 3.5.** For finite abelian categories \( \mathcal{M} \) and \( \mathcal{N} \), we have

\[
\text{N}_{\text{Rex}(\mathcal{M}, \mathcal{N})} = \text{Rex}(\text{N}_{\mathcal{M}}, \text{N}_{\mathcal{N}}).
\]

**Proof.** Despite that this formula has been given in \( \text{Shi17c}, \text{Lemma 3.21} \), we include a proof to explain how we view the functor \( \text{Rex}(\text{N}_{\mathcal{M}}, \text{N}_{\mathcal{N}}) \) as a Nakayama functor of \( \mathcal{F} := \text{Rex}(\mathcal{M}, \mathcal{N}) \). By the ‘naturality’ of the equivalence (3.11) mentioned in the above, we have isomorphisms

\[
\text{Rex}(\text{N}_{\mathcal{M}}, \text{N}_{\mathcal{N}}) \circ \text{EW} \cong \text{EW} \circ ((\text{N}^{\text{ra}}_{\mathcal{M}})^{\text{op}} \boxtimes \text{N}_{\mathcal{N}})
\]

(3.13) \quad \cong \text{EW} \circ (\text{N}^{\text{ra}}_{\mathcal{M}})^{\text{op}} \boxtimes \text{N}_{\mathcal{N}} \cong \text{N}_{\mathcal{F}} \circ \text{EW},

where \( \text{EW} = \text{EW}_{\mathcal{M}, \mathcal{N}} \). Hence \( \text{N}_{\mathcal{F}} \cong \text{Rex}(\text{N}_{\mathcal{M}}, \text{N}_{\mathcal{N}}) \). The proof is done. \( \square \)

Now let \( F \) and \( G \) be as in (3.12). We assume that \( F \) has a triple left adjoint and \( G \) has a triple right adjoint. Then the functor \( T := \text{Rex}(F, G) \) has a triple right adjoint. Indeed, we have a chain of adjunctions

\[
\text{Rex}(F, G) \dashv \text{Rex}(F^{\text{lla}}, G^{\text{rra}}) \dashv \text{Rex}(F^{\text{lla}}, G^{\text{rra}}) \dashv \text{Rex}(F^{\text{lla}}, G^{\text{rra}}).
\]

Thus there is a canonical isomorphism

\[
\phi_T : T^{\text{rra}} \circ \text{N}_{\mathcal{F}_1} \to \text{N}_{\mathcal{F}_2} \circ T,
\]

where \( \mathcal{F}_i = \text{Rex}(\mathcal{M}_i, \mathcal{N}_i) \) \( (i = 1, 2) \). We choose \( \text{Rex}(\text{N}_{\mathcal{M}_i}, \text{N}_{\mathcal{N}_i}) \) as a Nakayama functor of \( \mathcal{F}_i \). We also choose \( \text{Rex}(F^{\text{lla}}, G^{\text{rra}}) \) as a double right adjoint of \( T \). Under our choice of adjoints and Nakayama functors, \( \phi_T \) is a family

\[
\phi_T = \{ \phi_T(X) : G^{\text{rra}} \circ \text{N}_{\mathcal{N}_1} \circ X \circ \text{N}_{\mathcal{M}_1} \circ F^{\text{lla}} \to \text{N}_{\mathcal{N}_2} \circ G \circ X \circ F \circ \text{N}_{\mathcal{M}_2} \}_{X \in \mathcal{F}_1}
\]

of morphisms in \( \mathcal{F}_2 \). We give an explicit formula of \( \phi_T \) as follows:

**Lemma 3.6.** Notations are as above. For \( X \in \mathcal{F}_1 \), we have

\[
\phi_T(X) = \phi_G \circ \text{id}_X \circ (\phi_{F^{\text{lla}}})^{-1},
\]

where \( \circ \) means the horizontal composition of natural transformations.
Proof. We choose \((\mathcal{N}_{\mathcal{T}_r}^{\mathcal{M}})_{\mathcal{M}}^{\mathcal{N}}\) as a Nakayama functor of \(\mathcal{T}_r := \mathcal{M}^{\mathcal{S}} \mathcal{N}^{\mathcal{S}}\) as in the previous subsection. By Lemmas 3.3 and 3.4 the canonical isomorphism
\[
\phi_{(F^{\mathcal{T}_r})_{\mathcal{M}}} : \mathcal{N}_{\mathcal{T}_r} \circ ((F^{\mathcal{T}_r})_{\mathcal{M}} \mathcal{N}) \to ((F^{\mathcal{T}_r})_{\mathcal{M}} \mathcal{N})^{\mathcal{T}_r} \circ \mathcal{N}_{\mathcal{T}_r}
\]
is given by \(\phi_{(F^{\mathcal{T}_r})_{\mathcal{M}}} = (\phi_{F^{\mathcal{T}_r}})_{\mathcal{M}} \mathcal{N}\), where \(\phi_{F^{\mathcal{T}_r}}\) is the composition
\[
F^{\mathcal{T}_r} \mathcal{N}_{\mathcal{M}}^{\mathcal{N}} \cong \to (\mathcal{N}_{\mathcal{M}} F)^{\mathcal{T}_r} \to (F^{\mathcal{T}_r} \mathcal{N}_{\mathcal{N}}^{\mathcal{N}}) \to \mathcal{N}_{\mathcal{N}}^{\mathcal{T}_r}(F^{\mathcal{T}_r})^{\mathcal{N}}.
\]
The proof is completed by translating this formula through the Eilenberg-Watts equivalence \(3.11\). \(\square\)

4. Nakayama functor for monads

4.1. Tensor products over monads. A monad \([ML98]\) on a category \(\mathcal{M}\) is a triple \(\mathcal{T} = (T, \mu, \eta)\) consisting of an endofunctor \(T\) on \(\mathcal{M}\) and natural transformations \(\mu : TT \to T\) and \(\eta : \text{id}_\mathcal{M} \to T\) satisfying the associative and unit laws. Let \(\mathcal{T} = (T, \mu, \eta)\) be a monad on \(\mathcal{M}\). A \(\mathcal{T}\)-module in \(\mathcal{M}\) \((=\ a \mathcal{T}\text{-algebra }[ML98])\) is a pair \(\mathcal{M} = (M, a)\) consisting of an object \(M \in \mathcal{M}\) and a morphism \(a : T(M) \to M\) in \(\mathcal{M}\), called the action, satisfying the associative and the unit laws. Modules over \(\mathcal{T}\) form a category, which we denote by \(\mathcal{T}\)-mod.

If \(\mathcal{T}\) is a linear right exact monad on a finite abelian category \(\mathcal{M}\), then \(\mathcal{T}\)-mod is a finite abelian category such that the forgetful functor \(U : \mathcal{T}\)-mod \to \mathcal{M}\) preserves and reflects exact sequences. The goal of this section is to prove Theorem 4.5 which provides a formula of \(\mathcal{N}_{\mathcal{T}\text{-mod}}\) under the assumption that
\[
T \text{ admits a double left adjoint and a double right adjoint.}
\]

To state and prove the theorem, we introduce the tensor product over a monad as a slight generalization of the tensor product over a ring. This idea comes from monads in a bicategory and related notions. We write down necessary definitions in the form specialized to the 2-category of categories. Given a functor \(F\) and an object \(M\) of the source of \(F\), we often, but not always, write \(F(M)\) as \(F \otimes M\) in this section. In response, the symbol \(\otimes\) will also be used to express the composition of functors and the horizontal composition of natural transformations. Let \(\mathcal{S} = (S, \mu^S, \eta^S)\) and \(\mathcal{T} = (T, \mu^T, \eta^T)\) be monads on categories \(\mathcal{L}\) and \(\mathcal{M}\), respectively. Then a left \(\mathcal{S}\text{-module}\) in \([\mathcal{M}, \mathcal{L}]\) is a pair \((F, a)\) consisting of a functor \(F : \mathcal{M} \to \mathcal{L}\) and a natural transformation \(a : S \otimes F \to F\) satisfying
\[
a \circ (\text{id}_S \otimes a) = a \circ (\mu^S \otimes \text{id}_F) \quad \text{and} \quad a \circ (\eta^S \otimes \text{id}_F) = \text{id}_F.
\]
A right \(\mathcal{T}\text{-module}\) in \([\mathcal{M}, \mathcal{L}]\) is defined analogously. An \(\mathcal{S} : \mathcal{T}\text{-bimodule}\) in \([\mathcal{M}, \mathcal{L}]\) is a triple \(\mathcal{F} = (F, a^\ell, a^r)\) consisting of a functor \(F : \mathcal{M} \to \mathcal{L}\) and natural transformations \(a^\ell : S \otimes F \to F\) and \(a^r : F \otimes T \to F\) such that \((F, a^r)\) is a left \(\mathcal{S}\text{-module}\) in \([\mathcal{M}, \mathcal{L}]\), \((F, a^\ell)\) is a right \(\mathcal{T}\text{-module}\) in \([\mathcal{M}, \mathcal{L}]\), and the following equation is satisfied:
\[
a^r \circ (a^\ell \otimes \text{id}_F) = a^\ell \circ (\text{id}_S \otimes a^r).
\]

Now we suppose that the category \(\mathcal{L}\) admits coequalizers. For a right \(\mathcal{T}\text{-module}\) \(\mathcal{F} = (F, a^r)\) in \([\mathcal{M}, \mathcal{L}]\) and a \(\mathcal{T}\text{-module}\) \(\mathcal{M} = (M, a_M)\) in \(\mathcal{M}\), we define
\[
\mathcal{F} \otimes \mathcal{T} \mathcal{M} = \text{coequalizer}\left( F \otimes T \otimes M \xrightarrow{\alpha^r \otimes \text{id}_M} F \otimes M \right)
\]
and call it the tensor product of $F$ and $M$ over $T$. If $F = (F, a^r_F, a^l_F)$ is an $S$-$T$-module in $\mathcal{L}$ and $S$ preserves coequalizers, then $F \otimes_T M$ is an $S$-module by the action induced by $a^r_F \otimes \text{id}_F \otimes \text{id}_M$ and this construction gives rise to a functor

$$F \otimes_T (-) : \text{T-mod} \to \text{S-mod}, \quad M \mapsto F \otimes_T M.$$  

If, furthermore, $F$ preserves coequalizers, then the functor $F \otimes_T (-)$ also preserves coequalizers.

4.2. Adjoints of (bi)modules over monads. As the second important ingredient to state the main theorem, we discuss a natural module structure of an adjoint of a (bi)module over monads. Throughout this subsection, $S = (S, \mu^S, \eta^S)$ and $T = (T, \mu^T, \eta^T)$ are monads on categories $\mathcal{L}$ and $\mathcal{M}$, respectively. We first discuss a left adjoint of a left $S$-module. Let $\mathcal{N}$ be a category. We note that if $F : \mathcal{M} \to \mathcal{L}$ is a functor admitting a left adjoint, then there is a canonical bijection

$$(4.2) \quad \text{Nat}(X, F \otimes Y) \cong \text{Nat}(F^\text{la} \otimes X, Y)$$

for functors $X : \mathcal{N} \to \mathcal{L}$ and $\mathcal{N} \to \mathcal{M}$. There is also a canonical bijection

$$(4.3) \quad \text{Nat}(X \otimes F, Y) \cong \text{Nat}(X, Y \otimes F^\text{la})$$

for functors $X : \mathcal{M} \to \mathcal{N}$ and $Y : \mathcal{L} \to \mathcal{N}$.

**Lemma 4.1.** Let $F = (F, a^l_F)$ be a left $S$-module in $[\mathcal{M}, \mathcal{L}]$ whose underlying functor $F$ admits a left adjoint. Then the left adjoint $F^\text{la}$ is a right $S$-module in $[\mathcal{L}, \mathcal{M}]$ by the natural transformation corresponding to $a^l_F$ via

$$\text{Nat}(S \otimes F, F) \xrightarrow{(4.1)} \text{Nat}(S, F \otimes F^\text{la}) \xrightarrow{(4.2)} \text{Nat}(F^\text{la} \otimes S, F^\text{la}).$$

We denote by $F^\text{la}$ the right $S$-module in $[\mathcal{M}, \mathcal{L}]$ of the above lemma. We suppose that the functor $T$ has a double left adjoint. Then the functor $T^\text{la}$ is naturally a monad on $\mathcal{M}$, which we denote by $T^\text{lla}$. As has been observed at least for algebras and modules in a monoidal category [DPS20 §2.4.5], a left adjoint of an $S$-$T$-module in $[\mathcal{M}, \mathcal{L}]$ has no natural structure of a $T$-$S$-bimodule in general, however, it becomes a $T^\text{lla}$-$S$-bimodule in the following way:

**Lemma 4.2.** Suppose that $T$ has a double left adjoint. Let $F = (F, a^r_F)$ be a right $T$-module in $[\mathcal{M}, \mathcal{L}]$ such that $F$ admits a left adjoint. Then $F^\text{la}$ is a left $T^\text{lla}$-module in $[\mathcal{L}, \mathcal{M}]$ by the natural transformation corresponding to $a^r_F$ via

$$\text{Nat}(F \otimes T, F) \xrightarrow{(-)^\text{la}} \text{Nat}(F^\text{la}, T^\text{la} \otimes F^\text{la}) \xrightarrow{(4.2)} \text{Nat}(T^\text{lla} \otimes F^\text{la}, F^\text{la}).$$

If $T$ is an $S$-$T$-bimodule, then $F^\text{la}$ is a $T^\text{lla}$-$S$-bimodule by this left action and the right action given by the previous lemma.

Lemmas [(4.1)] and [(4.2)] are proved straightforwardly. Instead of giving the detailed proof of these lemmas, we provide diagrammatic expressions of the actions of monads on the functor $F^\text{la}$. Thanks to our convention of denoting the composition of functors by $\otimes$, we may divert the standard graphical calculus in a monoidal category to express natural transformations. We use the diagrams $\begin{tikzpicture}[baseline=-.5ex]
\node (a) at (0,0) {$(4.1)$};
\node (b) at (1.5,0) {$(4.2)$};
\end{tikzpicture}$ and $\begin{tikzpicture}[baseline=-.5ex]
\node (a) at (0,0) {$(4.3)$};
\node (b) at (1.5,0) {$(4.4)$};
\end{tikzpicture}$ to represent a left and a right action of a monad on a functor. We also use a cap and a cup to represent the unit and the counit for an adjunction, respectively. Then
the right action of $S$ and the left action of $T^{la}$ on $F^{la}$ given in the above lemmas are expressed as follows:

\[ F^{la} S := F^{la} F^{la} S \]
\[ T^{la} F^{la} := F^{la} F^{la} T^{la} \]

Similarly, if $F$ is a right $T$-module in $[M, L]$ whose underlying functor $F$ admits a right adjoint, then $F^{ra}$ has a natural structure of a left $T$-module in $[L, M]$. As in the case of left adjoints, we denote this right $T$-module by $F^{ra}$. If, furthermore, $F$ is an $S$-$T$-bimodule in $[M, L]$ and $S$ admits a double right adjoint, then $F^{ra}$ is a $T$-$S^{ra}$-bimodule in $[L, M]$. Graphically, the actions on $F^{ra}$ are expressed by the left-right inverted ones of the above diagrams expressing the actions on $F^{la}$.

4.3. Adjunctions for monads. Let $M$ be a category admitting coequalizers, and let $T = (T, \mu, \eta)$ be a monad on $M$. There is the forgetful functor $U : T\text{-mod} \to M$, $(M, a_M) \mapsto M$. As is well-known, $U$ has a left adjoint $F : M \to T\text{-mod}$, $X \mapsto (T^*(X), \mu_X)$ called the \textit{free} $T$-module functor. Below we describe a left adjoint of $F$ and a right adjoint of $U$ by using adjoints of bimodules.

Lemma 4.3. Suppose that $T$ admits a left adjoint. Then the functor

\[ F^{la} : T\text{-mod} \to M, \quad M \mapsto T^{la} \otimes T M \]

is left adjoint to $F$.

\textbf{Proof.} Let $M = (M, a_M) \in T\text{-mod}$ and $X \in M$ be objects. One can check that a morphism $f : M \to T \otimes X$ in $M$ satisfies the equation

\[ f \circ a_M = \mu_X \circ T(f) \]

if and only if the corresponding morphism $g : T^{la} \otimes M \to X$ in $M$ satisfies

\[ g \circ (a^\tau \otimes \text{id}_X) = g \circ (\text{id}_{T^{la}} \otimes a_M), \]

where $a^\tau : T^{la} \otimes T \to T^{la}$ is the right action of $T$ on $T^{la}$. Hence we have natural isomorphisms

\[ \text{Hom}_{T\text{-mod}}(M, F(X)) = \{ f \in \text{Hom}_M(M, T \otimes X) \mid f \text{ satisfies } (4.4) \} \]
\[ \cong \{ g \in \text{Hom}_M(T^{la} \otimes M, X) \mid g \text{ satisfies } (4.5) \} \]
\[ \cong \text{Hom}_M(T^{la} \otimes T M, X), \]

where the first isomorphism is the adjunction isomorphism and the second one follows from the universal property of the coequalizer defining the tensor product over $T$. The proof is done. \hfill \square

Given a left $T$-module $F = (F, a_F)$ in $[M, M]$ and $X \in M$, we define

\[ F \otimes X := (F \otimes X, a_F \otimes \text{id}_X) \in T\text{-mod}. \]
Lemma 4.4. Suppose that $T$ admits a right adjoint. Then the functor
$$U^{ra} : M \to T\text{-mod}, \quad X \mapsto T^{ra} \otimes X \quad (X \in M)$$
is right adjoint to $U$.

Proof. The functor $C := T^{ra}$ has a natural structure of a comonad on $M$, and the category of $T$-modules is identified with the category of $C$-comodules. Under this identification, a right adjoint of $U$ is given by the free $C$-comodule functor $F_C : M \to \text{(the category of } C\text{-comodules)}, X \mapsto (C \otimes X, \Delta \otimes \text{id}_X)$,

where $\Delta : C \to C \otimes C$ is the comultiplication of $C$. The $T$-module $T^{ra} \otimes X$ in concern is nothing but the $C$-comodule $F_C(X)$ viewed as a $T$-module. Hence a right adjoint of $U$ is given as stated. $\Box$

4.4. Nakayama functor for monads. Let $T = (T, \mu_T, \eta_T)$ be a linear monad on a finite abelian category $M$ satisfying (4.1). As in the previous subsection, we denote by $F$ and $U$ the free and the forgetful functor for the monad $T$. Since $T$ is right exact, the category $T\text{-mod}$ is a finite abelian category such that $U$ preserves and reflects exact sequences. By the coherence property of the canonical isomorphism (2.4), the Nakayama functor $N_M$ lifts to the functor
$$(4.6) \quad T^{lla}\text{-mod} \to T\text{-mod}, \quad (M, a_M) \mapsto (N_M(M), \tilde{a}_M),$$

where $\tilde{a}_M$ is the composition
$$T \otimes N_M(M) \xrightarrow{\phi_{T^{lla}}(M)} N_M(T^{lla} \otimes M) \xrightarrow{N_M(a_M)} N_M(M).$$

Now the main result of this section is stated as follows:

Theorem 4.5. The Nakayama functor of $T\text{-mod}$ is given by the composition
$$(4.7) \quad T\text{-mod} \xrightarrow{T^{ra} \otimes \text{(-)}} T^{lla}\text{-mod} \xrightarrow{\psi} T\text{-mod}.$$

We recall that $F$ has a left adjoint given by Lemma 4.3. Since $F$ is left adjoint to $U$, there is a natural isomorphism
$$\psi_X : F^{la}F(X) = F^{la}U^{la}(X) \xrightarrow{\simeq} (UF)^{la}(X) = T^{la} \otimes X \quad (X \in M)$$

by the uniqueness of adjoints. To prove the above theorem, we give the following technical remark:

Lemma 4.6. The isomorphism $\psi_X$ is a morphism
$$\psi_X : T^{la} \otimes_T (T \otimes X) \to T^{la} \otimes X$$
of $T^{lla}\text{-modules in } M$.

Proof. Let $a^l : T^{lla} \otimes T^{la} \to T^{lla}$ and $a^r : T^{lla} \otimes T \to T^{lla}$ be the left and the right action of $T^{lla}$ and $T$ on $T^{la}$, respectively. We note that the action $\pi^l$ of $T^{lla}$ on the source of $\psi_X$ is determined by the equation
$$\pi^l \circ (id_{T^{lla}} \otimes \pi_X) = \pi_X \circ (a^l \otimes \text{id}_X),$$
where $\pi_X : T^{la} \otimes T \otimes X \to T^{la} \otimes_T (T \otimes X)$ is the canonical epimorphism.
There are natural isomorphisms

\[(4.8) \quad \text{Hom}_M(T^{la} \otimes X, Y) \cong \text{Hom}_M(X, T \otimes Y)\]
\[(4.9) \quad \cong \text{Hom}_{T\text{-mod}}(F(X), F(Y))\]
\[(4.10) \quad \cong \text{Hom}_M(F^{la}F(X), Y)\]

for \(X, Y \in M\). The isomorphism \(\psi_X\) is the image of \(id_{T^{la} \otimes X}\) under the above chain of isomorphisms with \(Y = T^{la} \otimes X\). Let \(\eta\) and \(\varepsilon\) be the unit and the counit for the adjunction \(T^{la} \dashv T\). The element \(id_{T^{la} \otimes X}\) is sent as follows:

\[
\begin{align*}
\text{id}_{T^{la} \otimes X} & \Rightarrow \eta \otimes id_X \\
& \Rightarrow (\mu^T \otimes id_{T^{la}} \otimes id_X) \circ (id_T \otimes \eta \otimes id_X).
\end{align*}
\]

By the proof of Lemma 4.3, the isomorphism \((4.10)\) sends an element \(f\) to the unique morphism \(g\) in \(M\) such that the equation

\[
g \circ \pi_X = (\varepsilon \otimes id_T \otimes id_X) \circ (id_{T^{la}} \otimes f)
\]

holds. In conclusion, the isomorphism \(\psi_X\) is the unique morphism in \(M\) such that the equation \(\psi_X \circ \pi = a^r \otimes id_X\) holds. By this equation and the fact that \((T^{la}, a^r, a^l)\) is a bimodule, we have the equation

\[
(a^r \otimes id_X) \circ (id_{T^{la}} \otimes \psi_X \pi_X) = (\mu^r \otimes id_X) \circ (id_{T^{la}} \otimes \psi_X \pi_X),
\]

which implies that \(\psi_X\) is a morphism of left \(T^{la}\)-modules in \(M\).

**Proof of Theorem 4.3.** By Lemma 4.3, \(U\) has a right adjoint. Since \(U\) is a double right adjoint of \(F^{la}\), we have a canonical isomorphism

\[
\xi_M := \phi_{F^{la}}(M) : \mathcal{U}N_{T\text{-mod}}(M) \overset{(2.3)}{\longrightarrow} N_MF^{la}(M) \quad (M \in T\text{-mod}).
\]

By Lemma 4.3, the functor \(N_MF^{la}\) is the composition of \((4.7)\) and \(U\). Thus, to complete the proof, it suffices to show that \(\xi_M\) is in fact a morphism

\[(4.11) \quad \xi_M : N_{T\text{-mod}}(M) \to N_M(T^{la} \otimes_T M)\]

of \(T\)-modules in \(M\) for all \(M \in T\text{-mod}\).

We first verify that \((4.11)\) is a morphism of \(T\)-modules in the case where \(M\) is a free \(T\)-module. By Lemma 4.3 and the assumption that \(T\) has a double right adjoint, \(U^{la}\) is right exact. This implies that \(F\) has a triple right adjoint. Hence there is the following natural isomorphism of \(T\)-modules:

\[(4.12) \quad \phi_T(X)^{-1} : N_{T\text{-mod}}F(X) \overset{(2.3)}{\longrightarrow} T^{la} \otimes N_M(X) \quad (X \in M)\]

By the naturality and the coherence property, we see that

\[(4.13) \quad \phi_{T^{la}}(X) : T^{la} \otimes N_M(X) \overset{(2.3)}{\longrightarrow} N_M(T^{la} \otimes X) \quad (X \in M)\]

is an isomorphism of \(T\)-modules. By composing \((4.12)\), \((4.13)\) and \(N_M(\psi_X^{-1})\), we obtain the following natural isomorphism of \(T\)-modules:

\[
\zeta_X : N_{T\text{-mod}}F(X) \to N_M(T^{la} \otimes_T F(X)) \quad (X \in M).
\]

We recall that \(\psi_X\) is defined by the uniqueness of adjoints. By the naturality and the coherence property of the canonical isomorphism \((2.4)\), we see that the following
diagram is commutative:

\[
\begin{array}{cccc}
UU^T \mathcal{N}_M(X) & \xrightarrow{U(\phi_E(X))} & U \mathcal{N}_M F(X) & \xrightarrow{\phi_{T^E}(X)} & \mathcal{N}_M(\mathcal{F}^T \mathcal{F}(X)) \\
T^{ra} \otimes \mathcal{N}_M(X) & \xrightarrow{} & \mathcal{N}_M(\mathcal{F}^T \mathcal{F}(X)) & \xrightarrow{} & \mathcal{N}_M(T^T \otimes X).
\end{array}
\]

Hence \( \xi_{F(X)} = \zeta_X \) for all \( X \in \mathcal{M} \). By the construction, \( \zeta_X \) is an isomorphism of \( \mathcal{T} \)-modules. Thus \( \xi_{F(X)} \) is also an isomorphism of \( \mathcal{T} \)-modules.

Now let \( \mathcal{M} \) be an arbitrary \( \mathcal{T} \)-module in \( \mathcal{M} \). Then there is an exact sequence in \( \mathcal{T} \text{-mod} \) of the form \( \mathcal{F}(X) \rightarrow \mathcal{F}(Y) \rightarrow \mathcal{M} \rightarrow 0 \) for some \( X, Y \in \mathcal{M} \). We consider the following diagram:

\[
\begin{array}{cccc}
\mathcal{N}_{T \text{-mod}} \mathcal{F}(X) & \xrightarrow{\xi_{F(X)}} & \mathcal{N}_{T \text{-mod}} \mathcal{F}(Y) & \xrightarrow{\xi_{F(Y)}} & \mathcal{N}_{T \text{-mod}}(\mathcal{M}) & \rightarrow 0 \\
\mathcal{N}_\mathcal{M}(T^T \otimes \mathcal{F}(X)) & \xrightarrow{\xi_{F(x)}} & \mathcal{N}_\mathcal{M}(T^T \otimes \mathcal{F}(Y)) & \xrightarrow{\xi_{F(Y)}} & \mathcal{N}_\mathcal{M}(T^T \otimes \mathcal{M}) & \rightarrow 0
\end{array}
\]

Since \( \mathcal{N}_{T \text{-mod}} \) and \( \xi_{F(X)} \) are right exact functors, the rows are exact. Thus \( \xi_{\mathcal{M}} \) is characterized as a unique morphism in \( \mathcal{M} \) such that the above diagram commutes when the dashed arrow is filled with it. Since both \( \xi_{F(X)} \) and \( \xi_{F(Y)} \) are morphisms of \( \mathcal{T} \)-modules, so is \( \xi_{\mathcal{M}} \). The proof is done.

\[\square\]

5. **Algebras with Frobenius Traces**

5.1. **Modules in module categories.** Let \( \mathcal{C} \) be a finite multi-tensor category, and let \( \mathcal{A} \) and \( \mathcal{B} \) be algebras in \( \mathcal{C} \). If \( \mathcal{M} \) is a finite left \( \mathcal{C} \)-module category with action \( \triangleright : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M} \), then the endofunctor \( \mathcal{A} \triangleright ( - ) \) on \( \mathcal{M} \) has a natural structure of a monad on \( \mathcal{M} \). We define \( \mathcal{A} \mathcal{M} \) to be the category of modules over this monad and call an object of \( \mathcal{A} \mathcal{M} \) an left \( \mathcal{A} \)-module in \( \mathcal{M} \). In a similar way, the category \( \mathcal{M}^B \) of right \( \mathcal{B} \)-modules in \( \mathcal{M} \) and the category \( \mathcal{A} \mathcal{M}^B \) of \( \mathcal{A} \)-\( \mathcal{B} \)-bimodules in \( \mathcal{M} \) are defined when \( \mathcal{M} \) is a finite right \( \mathcal{C} \)-module category and a finite \( \mathcal{C} \)-bimodule category, respectively.

Since \( \mathcal{C} \) itself is a finite \( \mathcal{C} \)-bimodule category, the category \( \mathcal{A} \mathcal{C}_B \) of \( \mathcal{A} \)-\( \mathcal{B} \)-bimodules in \( \mathcal{C} \) is defined. Let \( \mathcal{M} \) be a finite left \( \mathcal{C} \)-module category. Given \( X \in \mathcal{A} \mathcal{C}_B \) and \( M \in \mathcal{B} \mathcal{M} \), their tensor product over \( \mathcal{B} \) is defined by

\[
X \otimes_\mathcal{B} M := \text{coequalizer} \left( \begin{array}{cc} X \triangleright \mathcal{B} \triangleright M & \xrightarrow{id_X \triangleright a^t} \xrightarrow{a^r \triangleright id_M} \mathcal{X} \triangleright M \end{array} \right) \in \mathcal{A} \mathcal{M},
\]

where \( a^t : \mathcal{B} \triangleright M \rightarrow M \) and \( a^r : X \otimes \mathcal{B} \rightarrow X \) are the left and the right action of \( \mathcal{B} \) on \( M \) and \( X \), respectively. This construction defines a functor \( \otimes_\mathcal{B} : \mathcal{A} \mathcal{C}_B \times \mathcal{B} \mathcal{M} \rightarrow \mathcal{A} \mathcal{M} \) that is linear and right exact in each variable. When \( \mathcal{M} \) is a finite right \( \mathcal{C} \)-module category, there is a functor \( \otimes_\mathcal{A} : \mathcal{M}_A \times \mathcal{A} \mathcal{C}_B \rightarrow \mathcal{M}_B \) defined in a similar way (these functors are a special case of the tensor product of modules over monads introduced in the previous section). Theorem 1.13 gives a formula for the Nakayama functor of categories of the form \( \mathcal{A} \mathcal{M}, \mathcal{M}^B \) or \( \mathcal{A} \mathcal{M}^B \) as follows:

**Theorem 5.1.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be algebras in a finite multi-tensor category \( \mathcal{C} \).

1. For a finite left \( \mathcal{C} \)-module category \( \mathcal{M} \), we have

\[
\mathcal{N}_{A \mathcal{M}}(M) = \mathcal{N}_{A \mathcal{M}}(A^\vee \otimes_A M) \quad (M \in \mathcal{A} \mathcal{M}).
\]
(2) For a finite right $C$-module category $\mathcal{M}$, we have
\[ N_{\mathcal{M}_B}(\mathcal{M}) = N_{\mathcal{M}}(\mathcal{M} \otimes_B \mathcal{M}) \quad (\mathcal{M} \in \mathcal{M}_B). \]

(3) For a finite $C$-bimodule category $\mathcal{M}$, we have
\[ N_{\mathcal{A} \mathcal{M}_B}(\mathcal{M}) = N_{\mathcal{M}}(\mathcal{A} \otimes_A \mathcal{A} \otimes_B \mathcal{M}) \quad (\mathcal{M} \in \mathcal{A} \mathcal{M}_B). \]

In this section, we further investigate the Nakayama functor of the category of modules in the case where the algebra is a kind of Frobenius algebra.

5.2. Algebras with a Frobenius trace. Let $\mathcal{C}$ be a rigid monoidal category. The notion of a Frobenius algebra in $\mathcal{C}$ is defined as a generalization of a Frobenius algebra (see, e.g., [FS08]). If $A$ is a Frobenius algebra in $\mathcal{C}$, then $A \cong \mathcal{M}_A$ and $A \cong A^\mathcal{M}$ as left and right $A$-modules in $\mathcal{C}$, respectively. Hence the Nakayama functor of the category of $A$-modules would look a simpler form than that given in Theorem 5.1.

In view of further applications, we consider a more general setting:

**Definition 5.2.** Given an invertible object $I \in \mathcal{C}$, an $I$-valued Frobenius trace on an algebra $A$ in $\mathcal{C}$ is a morphism $\lambda : A \to I$ in $\mathcal{C}$ such that the morphism $\phi := (\text{ev}_I \otimes \text{id}_A) \circ (\text{id}_I \otimes \lambda m \otimes \text{id}_A) \circ (\text{id}_I \otimes \text{coev}_A) : I \otimes_A A \to A$ is invertible, where $m : A \otimes A \to 1$ is the multiplication of $A$.

For example, we can say that a Frobenius algebra in $\mathcal{C}$ is an algebra $A$ in $\mathcal{C}$ endowed with a $1$-valued Frobenius trace. A non-zero cointegral on a Hopf algebra in a braided finite multi-tensor category is an $I$-valued Frobenius trace with $I$ the ‘object of integrals’ of $H$ [Tak99, BKLT00].

Now we fix an invertible object $I \in \mathcal{C}$ and an algebra $A$ in $\mathcal{C}$ equipped with an $I$-valued Frobenius trace $\lambda$. We define $\phi$ as above, and set $\beta := \text{ev}_I \circ (\text{id}_I \otimes \lambda m) = \text{ev}_A \circ (\phi \otimes \text{id}_A)$. The following graphical expressions of $\beta$ and $\phi$ may be helpful:

\[ \beta = \begin{array}{c} I \otimes A \\ \lambda \end{array} \begin{array}{c} \phi = \begin{array}{c} I \otimes A \\ \lambda \\ A \end{array} \\ \begin{array}{c} m = \begin{array}{c} A \\ \lambda \end{array} \end{array} \end{array} \]

It is easy to see that $\phi$ is an isomorphism of right $A$-modules. The map $\lambda \mapsto \phi$ gives a bijection between the set of $I$-valued Frobenius traces on $A$ and the set of isomorphisms $I^\mathcal{M} \otimes A \to A$ of right $A$-modules in $\mathcal{C}$.

**Definition 5.3.** The *Nakayama isomorphism* of $A$ associated to $\lambda$ is
\[ \nu := (\phi^{-1} \otimes \text{id}_I) \circ \phi : A \otimes I \to I \otimes A \otimes I. \]

When $\mathcal{C}$ is pivotal and $I = 1$, the composition of $\nu$ and the pivotal structure of $\mathcal{C}$ is equal to the Nakayama automorphism of $A$ introduced in [FS08, Section 5] (since $\beta, \phi$ and $\nu$ correspond to $\kappa, \Phi_{\kappa, \ell}$ and $\Phi_{\kappa, \ell}$ of [FS08], respectively). Thus we may view $\nu$ as an analogue of the Nakayama automorphism of a Frobenius algebra, although it is no more an ‘automorphism’ of $A$. 




The Nakayama automorphism of a Frobenius algebra is defined by an equation like \( \lambda(ab) = \lambda(v(a)b) \). The following lemma can be thought of as an analogue of such an equation.

**Lemma 5.4.** With notation as above, we have

\[
\begin{array}{c}
A^\vee \downarrow \quad I^\vee \quad A \\
\beta \\
\end{array}
= 
\begin{array}{c}
A^\vee \downarrow \quad I^\vee \quad A \\
\nu \quad \nu \\
\beta
\end{array}
\]

(5.2)

**Proof.** The equation is obvious if one rewrites \( \beta \) and \( \nu \) by \( \phi \). \qed

For simplicity of notation, we write \( A^I = I^\vee \otimes A \otimes I^\vee \) and

\[
\rho = (\text{id}_{I^\vee} \otimes m) \circ (\text{id}_{I^\vee} \otimes \text{id}_A \otimes \text{ev}_{I^\vee} \otimes \text{id}_A) : A^I \otimes I^\vee \otimes A \to I^\vee \otimes A.
\]

The object \( A^I \) is an algebra in \( \mathcal{C} \) with multiplication \( \rho \otimes \text{id}_{I^\vee} \). By using the above lemma, one can show that \( \nu : A^\vee \to A' \) is an isomorphism of algebras in \( \mathcal{C} \) in a similar way as [FS08, Section 5]. The object \( I^\vee \otimes A \) is an \( A^I \)-\( A \)-bimodule in \( \mathcal{C} \) by the left action \( \rho \) and the right action \( \text{id}_{I^\vee} \otimes m \).

**Lemma 5.5.** The isomorphism \( \phi : I^\vee \otimes A \to A^\vee \) in \( \mathcal{C} \) is actually an isomorphism of \( A^\vee \)-\( A \)-bimodules in \( \mathcal{C} \) if we view the source of \( \phi \) as a left \( A^\vee \)-module through the Nakayama isomorphism \( \nu : A^I \to A^\vee \).

**Proof.** Let \( \rho \) be as above, and let \( m' : A^\vee \otimes A \to A^\vee \) be the left action of \( A^\vee \) on \( A^\vee \). This lemma claims that the following equation holds:

\[
\phi \circ \rho \circ (\nu \otimes \text{id}_{I^\vee} \otimes \text{id}_A) = m' \circ (\text{id}_{A^\vee} \otimes \phi).
\]

(5.3)

This is verified as in Figure 3 by the graphical calculus. Here, the first and the fourth equality follow from the right \( A \)-linearity of \( \phi \), the second and the fifth from the definition of the right action of \( A^\vee \) on \( A \), the third from the definition of \( \phi^\vee \), and the last from the definition of the left action of \( A^\vee \) on \( A^\vee \). \qed

**Remark 5.6.** Let \( m'' \) be the right-hand side of (5.3). Then we have

\[
\nu' = (\phi^{-1} \circ m'' \circ (\text{id}_{A^\vee} \otimes \text{id}_{I^\vee} \otimes u)) \circ (\text{id}_{A^\vee} \otimes \text{coev}_{I^\vee}) = \nu,
\]

where \( u : 1 \to A \) is the unit of \( A \). This means that the above lemma can also be used as the definition of the Nakayama isomorphism of \( A \).

### 5.3. A formula of the Nakayama functor.

Let \( \mathcal{C} \) be a finite multi-tensor category, and let \( \mathcal{M} \) be a finite left \( \mathcal{C} \)-module category. Given a left \( A \)-module \( M = (M, a_M) \) in \( \mathcal{M} \), one can make \( I^\vee \triangleright M \) into a left \( A^\vee \)-module in \( \mathcal{M} \) in a similar way as the left \( A^\vee \)-module \( I^\vee \otimes A \) in \( \mathcal{C} \) appeared in Lemma 5.5. This construction gives rise to a functor

\[
I^\vee \triangleright (-) : A\mathcal{M} \to A^\vee \mathcal{M}, \quad (M, a_M) \mapsto (I^\vee \triangleright M, \tilde{a}_M)
\]

(5.4)

\[
(\tilde{a}_M := (\text{id}_{I^\vee} \triangleright a_M) \circ (\text{id}_{I^\vee} \triangleright \text{id}_A \triangleright \text{ev}_{I^\vee} \triangleright \text{id}_M) \circ (\nu \triangleright \text{id}_M)).
\]

Now we express the Nakayama functor of \( A\mathcal{M} \) by using this functor as follows:
Theorem 5.7. Let \( A \) and \( \mathcal{M} \) be as above. Then the Nakayama functor of \( \mathcal{A}\mathcal{M} \) is given by the composition
\[
\mathcal{A}\mathcal{M} \xrightarrow{\phi} \mathcal{A}\mathcal{V}\mathcal{M} \xrightarrow{\text{id}} \mathcal{A}\mathcal{M}.
\]

Proof. By Lemma 5.5 we have natural isomorphisms
\[
\mathcal{A}\mathcal{V} \otimes \mathcal{A}\mathcal{M} \cong (I \otimes \mathcal{A}) \otimes \mathcal{A}\mathcal{M} \cong I \otimes \mathcal{M}
\]
for \( \mathcal{M} \in \mathcal{A}\mathcal{M} \). Now this theorem follows from Theorem 5.1.

We note some immediate consequences of this theorem. In the below, \( \mathcal{C} \) is a finite multi-tensor category, \( I \) is an invertible object of \( \mathcal{C} \), and \( \mathcal{M} \) is a finite left \( \mathcal{C} \)-module category.

Corollary 5.8. Suppose that the module category \( \mathcal{M} \) is Frobenius. Let \( A \) be an algebra in \( \mathcal{C} \). If \( A \) admits an \( I \)-valued Frobenius trace, then \( \mathcal{A}\mathcal{M} \) is Frobenius.

The converse of this corollary does not hold: For an algebra \( A \) in \( \mathcal{C} = \text{Vec} \), the category \( \mathcal{A}\mathcal{C} (= A\text{-mod}) \) is Frobenius if and only if \( A \) is self-injective, however, there are self-injective algebras which are not Frobenius [SY11].

Corollary 5.9. Suppose that \( \mathcal{C} \) is pivotal. Let \( A \) be a Frobenius algebra in \( \mathcal{C} \), and let \( \mathcal{U}_A : \mathcal{A} \to \mathcal{A} \) be the Nakayama automorphism of \( A \) defined in [FS08]. Then the Nakayama functor of \( \mathcal{A}\mathcal{C} \) is isomorphic to the functor
\[
\mathcal{A}\mathcal{C} \to \mathcal{A}\mathcal{C}, \quad (M, a_M) \mapsto (M \otimes \alpha_C, a_M \mathcal{U}_A \otimes \text{id}_a),
\]
where \( \alpha_C = \mathcal{N}_C(1) \). Thus, when \( \mathcal{C} \) is unimodular, then the Nakayama functor of \( \mathcal{A}\mathcal{C} \) is given by twisting the action of \( A \) by the Nakayama automorphism \( \mathcal{U}_A \), as in the case of ordinary Frobenius algebras.
Proof. We identify an object $X \in C$ with $X^{\vee \vee}$ by the pivotal structure of $C$. Then, as noted in the above, $\mathcal{O}$ is identified with the isomorphism $\nu$ defined by (1.1). By the formula (2.7) of the Nakayama functor of $C$, we have $\text{N}_C(X) = X \otimes \alpha_C$ for $X \in C$. Now the result follows from Theorem 5.7. □

If $C$ is pivotal and $A$ is a symmetric Frobenius algebra in the sense of [FS08], then we may assume that the Nakayama automorphism of $A$ is the identity. Hence, by the above corollary, we have:

**Corollary 5.10.** Suppose that $C$ is pivotal and unimodular. Let $A$ be a symmetric Frobenius algebra in $C$. Then $\mathcal{A}C$ is symmetric Frobenius.

The unimodularity assumption cannot be dropped from the above corollary. Indeed, the trivial algebra $A := 1$ is a symmetric Frobenius algebra in $C$, while $\mathcal{A}C(\sim = C)$ is symmetric Frobenius only if $C$ is unimodular.

5.4. Braided Hopf algebras. Let $B$ be a braided finite multi-tensor category with braiding $\sigma$, and let $H$ be a Hopf algebra in $B$ with comultiplication $\Delta$, counit $\varepsilon$ and antipode $S$. Then the category $\mathcal{H}B$ is a finite multi-tensor category. Here we determine the modular object of $\mathcal{H}B$ by using the results of this section.

We recall from [Tak99, BKLT00] basic notions and results on (co)integrals. There is an invertible object $I := \text{Int}(H) \in B$ called the object of integrals in [BKLT00]. A right integral in $H$ is a morphism $\Lambda : H \to I$ in $B$ satisfying $m \circ (\Lambda \otimes \text{id}_H) = \Lambda \otimes \varepsilon$, where $m$ is the multiplication of $H$. A right cointegral on $H$ is a morphism $\lambda : H \to I$ in $B$ satisfying $(\lambda \otimes \text{id}_H) \circ \Delta = \lambda \otimes u$, where $u$ is the unit of $H$. It is known that a non-zero right (co)integral exists. Furthermore, a non-zero right cointegral on $H$ is, in our terminology, an $I$-valued Frobenius trace on the algebra $H$.

We fix a non-zero right cointegral $\lambda : H \to I$ on $H$. Below we compute the Nakayama isomorphism $\nu$ associated to $\lambda$. For this purpose, we introduce some notations: For $X \in B$, we define the morphism $\psi_X$ in $B$ by

$$\psi_X = (\text{ev}_X \otimes \text{id}_{X^{\vee \vee}}) \circ (\sigma_X^{-1} \otimes \text{id}_X) \circ (\text{id}_X \otimes \text{coev}_X) : X \to X^{\vee \vee}.$$ 

This morphism and its inverse are graphically expressed as follows:

$$\psi_X = \begin{array}{c}
\xymatrix{
X \ar[rr]^-{\psi_X^{-1}} & & X^{\vee \vee} \ar[rr]^-{\psi_X} & & X
}
\end{array}$$

The notion of a left integral in $H$ is defined in a similar way as a right one. We fix a non-zero left integral $\Lambda^\ell : I \to H$ in $H$. The left modular function is the morphism $\alpha_H : H \to 1$ in $C$ determined by

$$\Lambda^\ell \otimes \alpha_H = m \circ (\Lambda^\ell \otimes \text{id}_H).$$

Given a morphism $f : H \to 1$ in $C$, we define

$$\tilde{f} := (f \otimes \text{id}_H) \circ \Delta : H \to H.$$

**Lemma 5.11.** The isomorphism $\nu : H^{\vee \vee} \to H^I$ is given by

$$\nu = (\sigma_{H,I^\vee} \otimes \text{id}_{1^{\vee \vee}}) \circ ((S^2 \circ \alpha_H \circ \psi_H^{-1}) \otimes \text{coev}_{1^\vee}).$$
Proof. The monodromy \([\text{BKLT00]}\) around an invertible object \(K \in B\) is the natural isomorphism \(\Omega(K) : \text{id}_B \rightarrow \text{id}_B\) uniquely determined by the equation

\[ \text{id}_K \otimes \Omega(K)_X = \sigma_{K,X} \circ \sigma_{K,X} \quad (X \in B). \]

For an invertible object \(K \in B\), one has

\[ \Omega(K^\vee)_X = \Omega(K)^{-1}_X, \quad \Omega(K)_X \otimes \text{id}_K = \sigma_{K,X} \circ \sigma_{X,K} \quad (X \in B). \]  

There is an isomorphism \(N : H \rightarrow H\) in \(B\) determined by

\[ \lambda \circ m \circ \sigma_{H,H} = \lambda \circ m \circ (\text{id}_H \otimes N). \]

An explicit description of \(N\) is given in \([\text{Shi17c}, \text{Lemma 5.7}]\) in terms of the antipode, the monodromy and the right modular function of \(H\). We note that, in \([\text{Shi17c}, \text{Lemma 5.7}]\), the symbol \(\alpha_H\) is used to express the right modular function of \(H\), which is the inverse of the left one with respect to the convolution product. Thus, in our notation, the inverse of \(N\) is given by

\[ N^{-1} = \Omega(I)^{-1}_H \circ S^2 \circ \alpha_H. \]

By \([\text{5.6}]\), we have \(\lambda \circ m = \lambda \circ m \circ \sigma_{H,H} \circ (\text{id}_H \otimes N^{-1})\). Thus the left hand side of \((5.2)\) is computed as follows:

\[ H^\vee \otimes I^\vee \otimes \lambda = H \otimes \alpha_H \otimes \text{id}_H \]

Equation \((5.2)\) can actually be used as the definition of \(\nu\). By comparing the above result with the right hand side \((5.2)\), we obtain

\[ \nu = (\sigma^{-1}_{I^\vee,H} \otimes \text{id}_{I^\vee}) \circ (N^{-1} \psi^{-1}_H \otimes \text{coev}_{I^\vee}). \]

The desired formula is obtained by rewriting the right hand side by \((5.5)\). □

By the above lemma, we have

\[ (\text{id}_{I^\vee} \otimes \varepsilon \otimes \text{id}_{I^\vee}) \circ \nu = \text{coev}_{I^\vee} \circ \alpha_H^\vee. \]

Thus, by Theorem \([\text{5.7}]\) we immediately have:

**Theorem 5.12.** The modular object of \(C := \mathcal{H}B\) is given by

\[ \alpha_C = (K, \alpha_H \otimes \text{id}_K : H \otimes K \rightarrow K) \quad (K := I^\vee \otimes \alpha_B). \]

This formula has been obtained in \([\text{Shi17c}]\) in a different method (though, strictly speaking, the category of right \(H\)-modules is considered in \([\text{Shi17c}]\)). Since \(\mathcal{H}B\) is a finite multi-tensor category, the above formula can also be read as a formula of the Nakayama functor of \(\mathcal{H}B\).
6. The center of a finite bimodule category

6.1. The canonical algebra and its variants. In this section, we assume that the base field $k$ is perfect. Thanks to this, the class of finite multi-tensor categories over $k$ is closed under the Deligne tensor product [Del90, EGO15].

Let $C$ be a finite multi-tensor category. A finite $C$-bimodule category $M$ can be regarded as a finite left module category over $C_{\text{env}} := C \boxtimes C_{\text{env}}$ by the action $\triangleright$ determined by $(X \boxtimes Y) \triangleright M = X \triangleright M \otimes Y$ for $X, Y \in C$ and $M \in M$. Let $\text{Hom} : C^{\text{op}} \times C \to C_{\text{env}}$ be the internal Hom functor for the left $C_{\text{env}}$-module category $C$ [EGNO15]. Then $A := \text{Hom}(1, 1)$ is an algebra in $C_{\text{env}}$ and the functor

$$(6.1) \quad K : C \to (C_{\text{env}})_A, \quad V \mapsto (V \boxtimes 1) \otimes A$$

is an equivalence of left $C_{\text{env}}$-module categories [ENO04].

The algebra $A$ is called the canonical algebra [EGNO15]. It is known that $A$ is the coend $A = \int_{X \in C} X \boxtimes Y X$ as an object of $C_{\text{env}}$. In terms of the universal dinatural transformation $i_X : X \boxtimes Y X \to A$ ($X \in C$), the multiplication $m : A \otimes A \to A$ and the unit $u : 1 \boxtimes 1 \to A$ of $A$ are given by $m \circ (i_X \otimes i_Y) = i_{X \otimes Y}$ and $u = i_1$ for $X, Y \in C$, respectively [Shi17a].

By Lemma 3.1 and the succeeding remark, there is a unique isomorphism

$$\tau_Y : (1 \boxtimes Y) \otimes A \to (Y \boxtimes 1) \otimes A$$

in $C_{\text{env}}$ such that the diagram

$$
\begin{array}{ccc}
(1 \boxtimes Y) \otimes A & \xleftarrow{\text{id} \otimes i_X} & (1 \boxtimes Y) \otimes (X \boxtimes Y) \\
\downarrow{\tau_Y} & & \downarrow{\text{id} \otimes (\text{coev}_Y \otimes i_X) \otimes \text{id}_{Y \boxtimes X}} \\
(Y \boxtimes 1) \otimes A & \xleftarrow{\text{id} \otimes (i_Y \otimes \text{id}_X \otimes \text{coev}_X \otimes \text{id}_{X \boxtimes Y})} & (Y \boxtimes 1) \otimes ((Y \boxtimes Y) \boxtimes Y \boxtimes X)
\end{array}
$$

commutes for all objects $X \in C$. According to [Shi17a, Appendix A], the structure morphism of the $C_{\text{env}}$-module functor $K$, which we denote by

$${\tau}_{M,Y} : M \otimes K(V) \to K(M \triangleright V) \quad (M \in C_{\text{env}}, V \in C),$$

is given by

$$(X \boxtimes Y) \otimes K(V) = ((X \boxtimes V) \boxtimes 1) \otimes (1 \boxtimes Y) \otimes A$$

$$\xleftarrow{\text{id} \otimes \tau_Y} ((X \boxtimes V) \boxtimes 1) \otimes (Y \boxtimes 1) \otimes A = K((X \boxtimes Y) \triangleright V)$$

if $M = X \boxtimes Y$ for some $X, Y \in C$. For later use, we note:

Lemma 6.1. The following equation holds:

$$m \circ (i_Y \otimes \text{id}_A) = ((\text{ev}_Y \boxtimes 1) \otimes \text{id}_A) \circ (\text{id}_{Y \boxtimes 1} \otimes \tau_Y) \quad (Y \in C).$$

Proof. By the definition of $m$ and $\tau_Y$, we have

$$(m \circ (i_Y \otimes \text{id}_A) \circ (\text{id}_{Y \boxtimes 1} \otimes \tau_Y) \circ (\text{id}_{Y \boxtimes 1} \otimes i_X)$$

$$= i_{Y \otimes X} = m \circ (i_Y \otimes \text{id}_A) \circ (\text{id}_{Y \boxtimes 1} \otimes i_X)$$

for $X, Y \in C$. The claim now follows from the universal property. \hfill \Box

In this section, the following variants of the canonical algebra are useful:
Definition 6.2. For integers $p$ and $q$, we define
\[
\mathbb{A}_{2p,2q} := \int_{X \in \mathcal{C}} S^{2p}(X) \boxtimes S^{2q-1}(X),
\]
where $S : \mathcal{C} \to \mathcal{C}$ is the left duality functor.

The coend $\mathbb{A}_{2p,2q}$ is an algebra in $\mathcal{C}_{\text{env}}$ as the image of the canonical algebra $\mathbb{A}$ under the tensor autoequivalence $S^{2p} \boxtimes S^{2q}$ of $\mathcal{C}_{\text{env}}$. Since $S^2 \boxtimes S^{-2}$ is the double left dual functor of $\mathcal{C}_{\text{env}}$, we have $\mathbb{A}_{2p,2q}$ as algebras in $\mathcal{C}_{\text{env}}$.

6.2. The canonical algebra and the twisted center. Let $\mathcal{C}$ be a finite multitensor category, and let $\mathcal{M}$ be a finite $\mathcal{C}$-bimodule category. The center of $\mathcal{M}$, denoted by $Z(\mathcal{M})$, is the category defined as follows: An object of this category is a pair $(M, \sigma)$ consisting of an object $M \in \mathcal{M}$ and a natural transformation $\sigma(X) : X \triangleright M \to M \triangleright X$ ($X \in \mathcal{C}$) satisfying
\[
\sigma(1) = \text{id}_M \quad \text{and} \quad \sigma(X \otimes Y) = (\sigma(X) \circ \text{id}_Y) \circ (\text{id}_X \triangleright \sigma(Y))
\]
for all objects $X, Y \in \mathcal{C}$. Given two objects $M = (M, \sigma)$ and $N = (N, \tau)$ of $Z(\mathcal{M})$, a morphism from $M$ to $N$ is a morphism $f : M \to N$ in $\mathcal{M}$ satisfying
\[
\tau(X) \circ (\text{id}_X \triangleright f) = (f \circ \text{id}_X) \circ \sigma(X)
\]
for all objects $X \in \mathcal{C}$. Lemma 6.3 below is well-known for $\mathcal{M} = \mathcal{C}$. The general case can be proved in the same way.

Lemma 6.3. Let $(M, \sigma)$ be an object of $Z(\mathcal{M})$. Then, for all objects $X \in \mathcal{C}$, the morphism $\sigma(X)$ is an isomorphism with the inverse
\[
\sigma(X)^{-1} = (\text{id}_X \triangleright \text{id}_M \circ \text{id}_X) \circ (\text{id}_X \triangleright \sigma(X^\vee) \circ \text{id}_X) \circ (\text{coev}_X \triangleright \text{id}_M \circ \text{id}_X).
\]

Given a tensor autoequivalence $F$ of $\mathcal{C}$ and a left $\mathcal{C}$-module category $\mathcal{N}$, we denote by $(F)_!$ the left $\mathcal{C}$-module category obtained from $\mathcal{N}$ by twisting the action of $\mathcal{C}$ by $F$. A similar notation will be used for right module categories and bimodule categories. For two integers $p$ and $q$, we define the twisted center of $\mathcal{M}$ by
\[
Z_{2p,2q}(\mathcal{M}) := Z((\mathcal{C}^{p})!\mathcal{M}(\mathcal{C}^{q}),
\]
where $S : \mathcal{C} \to \mathcal{C}$ is the left duality functor $X \mapsto X^\vee$. Now we fix two integers $p$ and $q$ and let $\mathbb{A}_{2p,2q} \in \mathcal{C}_{\text{env}}$ be the algebra of Definition 6.2. According to [Shi17a, Section 3],

Lemma 6.4. The category of $\mathbb{A}_{2p,2q}$-modules in $\mathcal{M}$ is isomorphic to $Z_{2p,2q}(\mathcal{M})$.

For reader’s convenience, we include how the category isomorphism of this lemma is established. We first note that the action of $\mathcal{C}_{\text{env}}$ on $\mathcal{M}$ preserves coends as it is right exact. Thus we have isomorphisms
\[
\text{Hom}_{\mathcal{C}_{\text{env}}}(\mathbb{A}_{2p,2q} \triangleright M, M) \cong \text{Hom}_{\mathcal{C}_{\text{env}}}(\int_{X \in \mathcal{C}} S^{2p}(X) \triangleright M \bowtie S^{2q-1}(X), M)
\]
\[
\cong \int_{X \in \mathcal{C}} \text{Hom}_{\mathcal{C}_{\text{env}}}(S^{2p}(X) \triangleright M \bowtie S^{2q-1}(X), M)
\]
\[
\cong \int_{X \in \mathcal{C}} \text{Hom}_{\mathcal{C}_{\text{env}}}(S^{2p}(X) \triangleright M, M \bowtie S^{2q}(X))
\]
\[
\cong \text{Nat}(S^{2p}(\cdot) \triangleright M, M \bowtie S^{2q}(\cdot))
\]
for $M \in \mathcal{M}$. The category isomorphism is obtained by showing that a morphism $\mathbb{A}_{2p,2q} \triangleright M \to M$ in $\mathcal{M}$ makes $M$ an $\mathbb{A}_{2p,2q}$-module in $\mathcal{M}$ if and only if the corresponding natural transformation makes $M$ an object of $Z_{2p,2q}(\mathcal{M})$. 

6.3. The canonical algebra and the Radford isomorphism. Let $\mathcal{C}$ be a finite multi-tensor category, and let $\alpha := \mathbb{N}_C(\mathbb{1})$. Since the equivalence $K : \mathcal{C} \rightarrow (\text{C}^{\text{env}})_\mathbb{A}$ of (6.1) is a left $\text{C}^{\text{env}}$-module functor, it induces an equivalence
\[ K : \mathbb{A}^\vee \mathcal{C} \rightarrow \mathbb{A}^\vee (\text{C}^{\text{env}})_\mathbb{A} \]
between the categories of left $\mathbb{A}^\vee$-modules. The source of (6.2) can be identified with $\mathcal{Z}_{2,-2}(\mathcal{C})$ by Lemma 6.4 and the succeeding remark. We note that $\mathbb{A}^\vee$ belongs to the target of (6.2).

**Definition 6.5.** The object $D_\mathcal{C} \in \mathcal{Z}_{2,-2}(\mathcal{C})$ is defined as an object corresponding to the $\mathbb{A}^\vee\mathbb{A}$-bimodule $\mathbb{A}^\vee$ via the equivalence (6.2).

We write $D_\mathcal{C} = (D, R')$. The distinguished invertible object of $\mathcal{C}$ [ENO04] is defined as an object of $\mathcal{C}$ corresponding to the right $\mathbb{A}$-module $\mathbb{A}^\vee$ via (6.1). Thus $D$ in the above is the distinguished invertible object of $\mathcal{C}$. It is explained in [Shi17a, Section 4] that $D = \overline{\mathbb{N}}_\mathcal{C}(\mathbb{1})$, where $\overline{\mathbb{N}}_\mathcal{C}$ is the right adjoint of $\mathbb{N}_\mathcal{C}$ given in §3.2. Furthermore, the natural isomorphism $R'$ is given by
\[
R_X = (X^\vee \otimes D) \xrightarrow{\cong} \int_{Y \in \mathcal{C}} \text{Hom}_{\mathcal{C}}(Y, \mathbb{1}) \otimes (X^\vee \otimes Y) \\
\xrightarrow{\cong} \int_{Y \in \mathcal{C}} \text{Hom}_{\mathcal{C}}(X^\vee \otimes Y, \mathbb{1}) \otimes Y \\
\xrightarrow{\cong} \int_{Y \in \mathcal{C}} \text{Hom}_{\mathcal{C}}(Y, X) \otimes Y \\
\xrightarrow{\cong} \int_{Y \in \mathcal{C}} \text{Hom}_{\mathcal{C}}(Y \otimes Y, \mathbb{1}) \otimes Y \\
\xrightarrow{\cong} \int_{Y \in \mathcal{C}} \text{Hom}_{\mathcal{C}}(Y, \mathbb{1}) \otimes (Y \otimes Y) \xrightarrow{\cong} D \otimes Y
\]
for $X \in \mathcal{C}$, where Lemma 3.1 is used at the second and the fourth isomorphisms.

We have introduced the Radford isomorphism in (6.5) in terms of the Nakayama functor. There is the following relation between the isomorphisms $R$ and $R'$. The left dual of $\alpha$ is also an end of the same form as $\overline{\mathbb{N}}_\mathcal{C}(\mathbb{1})$. Hence we can identify $\alpha^\vee$ with $D$ by the universal property. Under this identification, it is straightforward to see that the equation
\[ R_X = (R_{\alpha X})^\vee : X^{\vee \vee} \otimes \alpha^\vee \rightarrow \alpha^\vee \otimes \vee^\vee X \quad (X \in \mathcal{C}) \]
holds by the construction of the Radford isomorphism.

6.4. Nakayama isomorphism of the canonical algebra. Let $\mathbb{A} \in \text{C}^{\text{env}}$ be the canonical algebra. By the definition of $D_\mathcal{C} \in \mathcal{Z}_{2,-2}(\mathcal{C})$, there is an isomorphism $\phi : K(D_\mathcal{C}) \rightarrow \mathbb{A}^\vee$ of $\mathbb{A}^{\vee \vee} \mathbb{A}$-modules in $\text{C}^{\text{env}}$. As a right $\mathbb{A}$-module, the source of $\phi$ is $(\alpha \boxtimes \mathbb{1})^\vee \otimes \mathbb{A}$. Thus there is an $(\alpha \boxtimes \mathbb{1})$-valued Frobenius trace $\lambda$ on $\mathbb{A}$ inducing the isomorphism $\phi$. The Nakayama isomorphism
\[
\nu = (\phi^{-1} \otimes (\alpha^\vee \boxtimes \mathbb{1}))) \circ \phi^\vee : \mathbb{A}^\vee \rightarrow \mathbb{A}^\alpha := (\alpha^\vee \boxtimes \mathbb{1}) \otimes \mathbb{A} \otimes (\alpha^{\vee \vee} \boxtimes \mathbb{1})
\]
associated to $\lambda$ is given as follows:

**Lemma 6.6.** For all objects $X \in \mathcal{C}$, we have
\[
\nu \circ i^\vee_X = (\text{id}_{\alpha^\vee \boxtimes \mathbb{1}} \otimes i^{\vee X} \otimes \text{id}_{\alpha^{\vee \vee} \boxtimes \mathbb{1}}) \\
\circ ((R'_{\mathbb{A}} \otimes \text{id}_{\alpha^\vee}) \circ (\text{id}_{X^{\vee \vee} \otimes \text{coev}_{\alpha^\vee}}) \boxtimes \text{id}_{\alpha^{\vee \vee} X}),
\]
where $R_X$ is given by (6.3).
We first introduce some morphisms. Let \( \rho : A^{\vee} \otimes \alpha \to \alpha^{\vee} \) denote the left action of \( A^{\vee} \) on \( \alpha^{\vee} \) corresponding to \( R' \). Thus we have
\[
\rho \circ (i^{\vee} \otimes \alpha) = (id_{A^{\vee}} \otimes ev_{\vee YX}) \circ (R'_X \otimes id_{\alpha^{\vee}}) \quad (X \in C).
\]
The left action of \( A^{\vee} \) on \( K(\alpha) \) is \( \rho_1 := K(\rho) \circ \tilde{\tau}_{A^{\vee}} \), where \( \tilde{\tau} \) is the left \( C^{\text{env}} \)-module structure of \( K \). The object \( K(\alpha) \) is also a left \( A^{\alpha} \)-module by the action
\[
\rho_2 := id_{\alpha^{\vee}} \otimes (m \circ (id_{A^{\vee}} \otimes ev_{\vee YX} \otimes id_{A})) : A^{\alpha} \otimes K(\alpha) \to K(\alpha).
\]
For \( X \in C \), we set \( \nu_X = (R'_X \otimes id_{\alpha^{\vee}}) \circ (id_{X^{\vee}} \otimes coev_{\alpha^{\vee}}) \). By the universal property, there is a unique morphism \( \nu' : A^{\vee} \to A^{\alpha} \) such that the equation
\[
\nu' \circ i^{\vee}_X = (id_{\alpha^{\vee}} \otimes i^{\vee} X \otimes id_{\alpha^{\vee}}) \circ (\nu'_X \otimes id_{\alpha^{\vee}}).
\]
This lemma claims that the equation \( \nu' = \nu' \) holds. Thus, in view of Lemma [5.5] and the succeeding remark, it suffices to show that the following equation holds:
\[
\rho_1 = \rho_2 \circ (\nu' \otimes id_{K(\alpha)})
\]
For \( X \in C \), we compute \( \rho_1 \circ (i^{\vee} \otimes id_{K(\alpha)}) \)
\[
= K(\rho) \circ \tilde{\tau}_{A^{\vee}} \circ (i_X \otimes id_{K(\alpha)})
= K(\rho) \circ (i_X \otimes \alpha) \circ \tilde{\tau}_{X^{\vee}} \otimes ev_{\vee YX, \alpha}
= K((id_{\alpha^{\vee}} \otimes ev_{\vee X}) \circ (R'_X \otimes id_{\vee X})) \circ \tilde{\tau}_{X^{\vee} \otimes ev_{\vee X, \alpha}}
= K((id_{\alpha^{\vee}} \otimes ev_{\vee X}) \circ (R'_X \otimes id_{\vee X})) \circ (id_{X^{\vee} \otimes \alpha}) \otimes \nu'_X \otimes id_{\alpha^{\vee}}.
\]
where the second equality follows from the naturality of \( \tilde{\tau} \). We also have
\[
\rho_2 \circ (\nu' \otimes id_{K(\alpha)}) \circ (i^{\vee} \otimes id_{K(\alpha)})
= \rho_2 \circ (id_{\alpha^{\vee}} \otimes i^{\vee} X \otimes id_{\alpha^{\vee}} \otimes id_{K(\alpha)}) \circ (\nu'_X \otimes id_{\alpha^{\vee}}) \circ (id_{\alpha^{\vee}} \otimes id_{K(\alpha)})
= (id_{\alpha^{\vee}} \otimes (m \circ (i^{\vee} X \otimes ev_{\vee X} \otimes id_{A})) \circ ((R'_X \otimes id_{\alpha^{\vee}}) \otimes id_{A}))
= (id_{\alpha^{\vee}} \otimes i^{\vee} X \otimes id_{A}) \circ ((R'_X \otimes id_{\alpha^{\vee}}) \otimes id_{A})
= ((id_{\alpha^{\vee}} \otimes ev_{\vee X}) \otimes id_{A}) \circ (id_{\alpha^{\vee}} \otimes i^{\vee} X) \otimes (id_{\alpha^{\vee}} \otimes coev_{\alpha^{\vee}}) \otimes \nu'_X \otimes id_{\alpha^{\vee}}.
\]
The proof is done. \( \square \)

6.5. **Nakayama functor of the center.** Let \( C \) be a finite multi-tensor category, and let \( M \) be a finite \( C \)-bimodule category. We fix integers \( p \) and \( q \). Given objects \( V = (V, \sigma_V) \in Z_{2p,2q}(C) \) and \( M = (M, \sigma_M) \in Z(M) \), we define
\[
V \triangleright M := (V \triangleright M, \sigma_{V \triangleright M}) \in Z(M),
\]
where \( \sigma_{V \triangleright M}(X) \) for \( X \in C \) is given by the composition
\[
S^{2p}(X) \triangleright (V \triangleright M) \xrightarrow{\sigma_V(X) \triangleright id_M} V \triangleright S^{2q}(X) \triangleright M
\]
\[
\xrightarrow{id_V \triangleright \sigma_M(S^{2q}(X))} V \triangleright M \triangleleft S^{2q}(X).
\]
This construction gives rise to the functor
\[
\triangleright : Z_{2p,2q}(C) \times Z(M) \to Z_{2p,2q}(M).
\]
In particular, the object \( D_C \) of Definition [6.5] yields the functor
\[
(6.4) \quad D_C \triangleright (-) : Z(M) \to Z_{2,-2}(M).
\]
The Nakayama functor $N_M : M \to M$ becomes a $C$-bimodule functor from $M$ to $(S^{-1}, M((S^2)$ together with the natural isomorphisms $2.4$ and $2.6$ (as noted in [FSS20] as the bimodule Radford $S^2$-formula). Hence we have a functor

$Z_{2p,2q}(M) \to Z_{2p-2,2q+2}(M), \quad (M, \sigma) \mapsto (N_M(M), \tilde{\sigma})$

for integers $p$ and $q$, where $\tilde{\sigma}$ is given by

\[
\tilde{\sigma}(X) = (S^{2p-2}(X) \triangleright N_M(M) \overset{2.5}{\longrightarrow} N_M(S^{2p}(X) \triangleright M) \overset{N_M(\sigma(X))}{\longrightarrow} N_M(M \triangleleft S^{2q}(X)) \overset{2.5}{\longrightarrow} N_M(M \triangleleft S^{2q+2}(X))
\]

for $X \in C$. Now we state and prove the following main result of this section:

**Theorem 6.7.** With notation as above, the Nakayama functor of $Z(M)$ is given by the composition

$Z(M) \overset{D_{Z\triangleright Y,(-)} \circ (6.3)}{\longrightarrow} Z_{2,2}(M) \overset{2.5}{\longrightarrow} Z_M(M)$.

**Proof.** Let $\nu$ be the Nakayama isomorphism of $A$ associated to the $(\alpha \triangleright 1)$-valued Frobenius trace mentioned in the previous subsection. If we identify $Z_{2p,2q}(M)$ with the category of $A_{2p,2q}$-modules in $M$ by Lemma 6.4, then the Nakayama functor of $Z(M)$ is given by the composition of (6.5) and the functor

$A_{1, M} \to A_{v_{1,1}, M}, \quad (M, a_M) \mapsto (\nu \triangleright M, \tilde{a}_M)$

$(\tilde{a}_M := (id_{A \triangleright 1} \triangleright a_M) \circ (id_{A \triangleright 1} \triangleright id_A \triangleright ev_{A \triangleright 1} \triangleright id_M) \circ (\nu \triangleright id_M))$.

By Lemma 6.4, this functor is identical to (6.4). Thus the Nakayama functor of $Z(M)$ is given as stated. The proof is done. $\square$

**6.6. Applications.** Here we exhibit some corollaries of Theorem 6.7 and give some related remarks. The theorem has some applications as many basic constructions in the theory of finite tensor categories are viewed as the center of a particular bimodule category.

**6.6.1. Frobenius property of the center.** Let $C$ be a finite multi-tensor category, and let $M$ be a finite bimodule category over $C$. By the description of $N_Z(M)$ given by Theorem 6.7, we have:

**Corollary 6.8.** If $M$ is Frobenius, then so is its center $Z(M)$.

We note that the center of a bimodule category is not always Frobenius: Take a finite-dimensional algebra $A$ such that $M := A$-mod is not Frobenius and regard $M$ as a finite bimodule category over $C := Vec$. Then $Z(M)$ is identified with $M$, which is not Frobenius.

**6.6.2. The centralizer of a tensor functor.** Let $C$ and $D$ be finite multi-tensor categories, and let $F : C \to D$ be a tensor functor. Then $D$ is a finite $C$-bimodule category by the action given by $X \triangleright M \triangleleft Y = F(Y) \otimes M \otimes F(Y)$ for $X, Y \in C$ and $M \in D$. The centralizer of $F$, which we denote by $Z(F)$, is defined as the center of the finite $C$-bimodule category $D$. The category $Z(F)$ is a finite multi-tensor category by the monoidal structure inherited from $D$.

There is a natural isomorphism $\xi_X : F(X^\vee) \to F(X)^\vee$ ($X \in C$) defined by the uniqueness of a left dual object (the duality transformation [NS07 Section 1]). For
for $X \in \mathcal{C}$, we set $\zeta_X = (\xi_X^{-1} \circ \xi_X^*$. By the construction of the isomorphism (2.6), we see that the diagram

$$
\begin{array}{ccc}
\mathbb{N}_D(M \triangleleft X) & \xrightarrow{\xi} & \mathbb{N}_D(M) \triangleleft X^{\vee\vee} \\
\quad & \downarrow & \quad \\
\mathbb{N}_D(M \otimes F(X)) & \xrightarrow{\xi} & \mathbb{N}_D(M) \otimes F(X)^{\vee\vee}
\end{array}
$$

commutes for all $M \in \mathcal{D}$ and $X \in \mathcal{C}$. There is an analogous commutative diagram for the left action of $\mathcal{C}$ on $\mathcal{D}$. Theorem 6.7 now yields the following corollary:

**Corollary 6.9.** For $F$ as above, the modular object of $\mathcal{Z}(F)$ is given by

$$
\alpha_{\mathcal{Z}(F)} = (\alpha_C^\vee \otimes \alpha_D, \gamma),
$$

where $\gamma$ is the natural isomorphism given by the composition

$$
\begin{align*}
F(X) \otimes F(\alpha_C^\vee) & \otimes \alpha_D \\
\cong F(\alpha_C^\vee) \otimes F(X^{\vee\vee}) \otimes \alpha_D & \text{(by the Radford isomorphism in } \mathcal{C}) \\
\cong F(\alpha_C^\vee) \otimes (F(X)^{\vee\vee} \otimes \alpha_D) & \text{(by the duality transformation)} \\
\cong F(\alpha_C^\vee) \otimes \alpha_D \otimes F(X) & \text{(by the Radford isomorphism in } \mathcal{D})
\end{align*}
$$

for $X \in \mathcal{C}$.

Since the Drinfeld center of $\mathcal{C}$ is the centralizer of $\text{id}_{\mathcal{C}}$, we obtain:

**Corollary 6.10** (cf. [ENO04]). The Drinfeld center of a finite multi-tensor category is unimodular.

6.6.3. The category of module functors. Let $\mathcal{C}$ be a finite multi-tensor category, and let $\mathcal{M}$ and $\mathcal{N}$ be finite left $\mathcal{C}$-module categories.

**Definition 6.11.** We denote by $\text{Rex}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ the category of linear right exact left $\mathcal{C}$-module functors from $\mathcal{M}$ to $\mathcal{N}$.

Let $\text{act}_\mathcal{M} : \mathcal{C} \to \text{Rex}(\mathcal{M})$ and $\text{act}_\mathcal{N} : \mathcal{C} \to \text{Rex}(\mathcal{N})$ be the functors induced by the actions of $\mathcal{C}$ on $\mathcal{M}$ and $\mathcal{N}$, respectively. The category $\mathcal{E} := \text{Rex}(\mathcal{M}, \mathcal{N})$ is a finite $\mathcal{C}$-bimodule category by the action given by

$$
X \triangleleft F \triangleleft Y = \text{act}_\mathcal{N}(X) \circ F \circ \text{act}_\mathcal{M}(Y) \quad (X, Y \in \mathcal{C}, F \in \mathcal{E}),
$$

and the category $\mathcal{F} := \text{Rex}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ is precisely the center of $\mathcal{E}$. Thus, by Lemma 3.26 and Theorem 6.7, the Nakayama functor of $\mathcal{F}$ is given as follows:

**Corollary 6.12.** For an object $F = (F, \xi)$ of the above $\mathcal{F}$, we have

$$
\mathbb{N}_\mathcal{F}(F) = (\mathbb{N}_\mathcal{N} \circ \text{act}_\mathcal{N}(\alpha_C^\vee) \circ F \circ \mathbb{N}_\mathcal{M}, \bar{\xi}),
$$

where the natural isomorphism $\bar{\xi}$ is given by the composition

$$
\begin{align*}
\text{act}_\mathcal{N}(X) \circ \mathbb{N}_\mathcal{N} \circ \text{act}_\mathcal{N}(\alpha_C^\vee) \circ F \circ \mathbb{N}_\mathcal{M} & \cong \mathbb{N}_\mathcal{N} \circ \text{act}_\mathcal{N}(X^{\vee\vee}) \circ \text{act}_\mathcal{N}(\alpha_C^\vee) \circ F \circ \mathbb{N}_\mathcal{M} \quad \text{(use (2.6))} \\
\cong \mathbb{N}_\mathcal{N} \circ \text{act}_\mathcal{N}(\alpha_C^\vee) \circ \text{act}_\mathcal{N}(\gamma) \circ F \circ \mathbb{N}_\mathcal{M} \quad \text{(use (6.3))} \\
\cong \mathbb{N}_\mathcal{N} \circ \text{act}_\mathcal{N}(\alpha_C^\vee) \circ F \circ \text{act}_\mathcal{M}(\gamma) \circ \mathbb{N}_\mathcal{M} \quad \text{(use the structure morphism } \xi) \\
\cong \mathbb{N}_\mathcal{N} \circ \text{act}_\mathcal{N}(\alpha_C^\vee) \circ F \circ \mathbb{N}_\mathcal{M} \circ \text{act}_\mathcal{M}(X) \quad \text{(use (2.7))}
\end{align*}
$$

for $X \in \mathcal{C}$. 

Given a finite left \( C \)-module category \( \mathcal{M} \), we set \( C^*_\mathcal{M} := \text{Rex}_C(\mathcal{M}, \mathcal{M}) \) and view it as a linear monoidal category by the composition of module functors (the order of the tensor product is reversed in \([EGNO15]\), but this will not matter since we only mention its unimodularity). By the above corollary, we have:

**Corollary 6.13.** \( \alpha_{C^*_\mathcal{M}} = N_\mathcal{M} \circ \text{act}_\mathcal{M}(\alpha_C^\vee) \circ N_\mathcal{M} \).

Suppose that \( \mathcal{M} \) is indecomposable and exact in the sense of \([EGNO15]\). Then the linear monoidal category \( C^*_\mathcal{M} \) is in fact a finite tensor category called the *dual* of \( \mathcal{M} \) with respect to \( \mathcal{M} \). The (dual of) the modular object of \( C^*_\mathcal{M} \) has been obtained in \([FGJS22]\) with the use of relative Serre functors. The formula of the above corollary coincides with that of \([FGJS22]\) Proposition 4.13] if we rewrite \( N_\mathcal{M} \) by the relative Serre functor of \( \mathcal{M} \).

### 7. Examples from Hopf Algebra Theory

#### 7.1. Radford isomorphism

In this section, we explain how our results are applied to some categories appearing in the Hopf algebra theory. Unless otherwise noted, the base field \( k \) is arbitrary in this section. The unadorned tensor symbol \( \otimes \) means the tensor product over \( k \). Given a Hopf algebra \( H \), we denote by \( \Delta, \varepsilon \) and \( S \) the comultiplication, the counit and the antipode of \( H \), respectively. We use the Sweedler notation, such as \( \Delta(h) = h_{(1)} \otimes h_{(2)} \), to express the comultiplication. Accordingly, the left coaction of \( H \) is written as \( m \mapsto m_{(-1)} \otimes m_{(0)} \) for an element \( m \) of a left \( H \)-comodule.

Let \( H \) be a finite-dimensional Hopf algebra, and let \( \mathcal{C} := H\text{-comod} \) be the category of finite-dimensional left \( H \)-comodules. As is well-known, \( \mathcal{C} \) is a finite tensor category. Given an object \( X \in \mathcal{C} \) and a coalgebra automorphism \( f \) of \( H \), we denote by \( X^{(f)} \) the left \( H \)-comodule obtained from \( X \) by twisting the coaction by \( f \). We note that a left dual object of \( X \in \mathcal{C} \) is the vector space \( X^\vee \) equipped with the left coaction \( f \mapsto f_{(-1)} \otimes f_{(0)} \) determined by

\[
\langle f_{(-1)}, x \rangle f_{(0)} = \langle f, x_{(0)} \rangle S^{-1}(x_{(1)}) \quad (x \in X, f \in X^\vee),
\]

and thus \( X^{\vee\vee} \) is identified with \( X^{(S^{-2})} \) via the canonical isomorphism of vector spaces. Similarly, the double right dual \( X^{\vee\vee} \) is identified with \( X^{(S^2)} \).

We give a description of the Radford isomorphism of \( \mathcal{C} \) (see \([SS21]\) Section 6) for the case of right comodules. We denote by \( G(H) \) the set of grouplike elements of \( H \). We fix a non-zero right cointegral \( \lambda_H : H \rightarrow k \). Then there is a unique element \( g_H \in G(H) \), called the distinguished grouplike element, such that

\[
\lambda_H(h_{(1)}) \lambda_H(h_{(2)}) = \lambda_H(h) g_H \quad (h \in H).
\]

(7.1)

Since \( \lambda_H \) is non-degenerate, one can define \( \nu_H : H \rightarrow H \) to be the unique linear map such that \( \lambda_H(ba) = \lambda_H(\nu_H(a)b) \) for all \( a, b \in H \) (namely, \( \nu_H \) is the Nakayama automorphism of \( H \) with respect to the Frobenius trace \( \lambda_H \)). We fix a non-zero left integral \( \Lambda^\ell \) in \( H \) and define the left modular function \( \alpha_H : H \rightarrow k \) in the same way as \([SS21]\). By definition, it is characterized by the equation

\[
\Lambda^\ell \cdot h = \alpha_H(h) \Lambda^\ell \quad (h \in H).
\]

(7.2)

Lemma \([5.11]\) reduces to the well-known formula

\[
\nu_H(h) = S^2(h \mapsto \alpha_H) \quad (h \in H),
\]
where $h \leftarrow f = (f, h_{(1)})h_{(2)}$ for $f \in H^*$ and $h \in H$. In particular, we have

$$\alpha_H = \varepsilon \circ \nu_H.$$  

(7.3)

A right cointegral on $H$ is the same thing as a left cointegral on the Hopf algebra $H^{\text{cop}}$ obtained from $H$ by reversing the order of the comultiplication. By equations (7.4), we find that our $\alpha_H$ and $g_H$ are $\alpha_{H^{\text{cop}}}$ and $g_{H^{\text{cop}}}$, respectively, in the notation of [SS21 Section 6]. By the discussion of [SS21 Section 6] applied to $H^{\text{cop}}$, we see that the modular object of $C$ is $kg$ and, under an appropriate identification of double duals, the Radford isomorphism of $C$ is viewed as the following map:

$$R_X : kg \otimes X(S^{-2}) \rightarrow X(S^2) \otimes kg,$$

(7.4)

$$gH \otimes x \mapsto (x \leftarrow \alpha_H) \otimes gH \quad (x \in X).$$

**Notation 7.1.** The symbols $H, C, gH$ and $\alpha_H$ introduced in the above will be used till the end of §7.3. We set $\tau_H = \alpha_H \circ S$ (this is the inverse of $\alpha_H$ with respect to the convolution product).

### 7.2. Cointegrals on comodule algebras.

An algebra in $C$ is the same thing as a finite-dimensional left $H$-comodule algebra. According to [Kas18], we introduce the following terminology:

**Definition 7.2.** Let $A$ be an algebra in $C$, and let $g \in G(H)$. A $g$-cointegral on $A$ is a linear map $\lambda : A \rightarrow k$ satisfying the equation $a_{(-1)}(\lambda a_{(0)}) = \lambda(a)g$ for all $a \in A$. We say that a $g$-cointegral on $A$ is non-degenerate if it is a Frobenius trace on $A$. A (non-degenerate) grouplike-cointegral is a linear map $A \rightarrow k$ being a (non-degenerate) $g$-cointegral for some $g \in G(H)$.

Some existence criteria for (non-degenerate) grouplike-cointegrals are given in [Kas18] and [Shi19 §4.10]. Given an element $g \in G(H)$, we denote by $k_g \in C$ the vector space $k$ equipped with the left $H$-coaction given by $1_k \mapsto g \otimes 1_k$. With this notation, a $g$-cointegral is the same thing as a morphism $A \rightarrow k_g$ in $C$. Thus a non-degenerate $g$-cointegral is nothing but a $k_g$-valued Frobenius trace on $A$ in the sense of Definition 5.2.

Now let $A$ be an algebra in $C$ and suppose that there are an element $g \in G(H)$ and a non-degenerate $g$-cointegral $\lambda : A \rightarrow k$. We define the morphism

$$\nu : A^{\text{cop}} \rightarrow (k_g)^{\text{cop}} \otimes A \otimes (k_g)^{\text{cop}}$$

in $C$ by (5.1). Given an object $X \in C$, we denote by $X^g$ the vector space $X$ equipped with the left $H$-coaction $x \mapsto g^{-1}x_{(-1)}g \otimes x_{(0)}$ ($x \in X$). The source and the target of $\nu$ are identified with $A(S^{-2})$ and $A^g$, respectively, and hence $\nu$ induces a morphism from $A(S^{-2})$ and $A^g$, which, by abuse of notation, we denote by the same symbol $\nu : A(S^{-2}) \rightarrow A^g$.

**Lemma 5.3** implies the equation $\lambda(ba) = \lambda(b)a$ ($a, b \in A$). Namely, $\nu$ is the Nakayama automorphism of $A$ with respect to the Frobenius trace $\lambda$. A trivial but important point is that $\nu$ is a morphism in $C$.

**Lemma 7.3.** The Nakayama automorphism $\nu$ of $A$ with respect to the non-degenerate $g$-cointegral $\lambda$ satisfies the equations

$$\nu(a)_{(-1)} \otimes \nu(a)_{(0)} = gS^{-2}(a_{(-1)})g^{-1} \otimes \nu(a_{(0)}),$$

(7.5)

$$\nu^{-1}(a)_{(-1)} \otimes \nu^{-1}(a)_{(0)} = g^{-1}S^2(a_{(-1)})g \otimes a_{(0)}$$

(7.6)

for all $a \in A$. 

Proof. Since $\nu : A^{(S^{-2})} \to A^g$ is a morphism in $\mathcal{C}$, we have
\[ \nu(a)(-1) \otimes g^{-1}\nu(a)(0)g = S^{-2}(a(-1)) \otimes \nu(a(0)) \]
for all $a \in A$. The first equation follows from this. The second one is obtained by replacing $a$ with $\nu^{-1}(a)$ in the first one. \qed

A left coideal subalgebra of $H$ is a subalgebra of $H$ that is also a left coideal of $H$, that is, $\Delta(A) \subseteq H \otimes A$ holds. A left coideal subalgebra of $H$ is an algebra in $\mathcal{C}$ with respect to the restriction of the comultiplication of $H$. The above proposition yields the following formula:

**Theorem 7.4.** Let $A$ be a left coideal subalgebra of $H$ and suppose that there is an element $g \in G(H)$ and a non-zero $g$-cointegral $\lambda : A \to k$. Then $\lambda$ is non-degenerate, and thus the Nakayama automorphism $\nu_A$ of $A$ with respect to $\lambda$ is defined. For every integer $k$, we have
\[ \nu_A^k(a) = g^kS^{-2k}(\varepsilon\nu_A^k \to a)g^{-k} \quad (a \in A), \]
where $f \mapsto a = a(-1)(f, a(0))$ for $f \in A^*$ and $a \in A$. The order of $\nu_A$ is finite.

The element $f \mapsto a$ in the above is well-defined because $\Delta(A) \subseteq H \otimes A$, but we should be careful that it may be no more an element of $A$.

**Proof.** The non-degeneracy of $\lambda$ follows from [Kas18 Theorem 4.11] (see also [Shi19 Proposition 4.22]). Equation (7.7) is trivial when $k = 0$. By (7.5), we have
\[ \beta \nu_A(a) = g_A S^{-2}(a(-1))g_A^{-1}(\beta, \nu_A(a(0))) = g_A S^{-2}(\beta \nu_A \to a)g_A^{-1} \]
for all $a \in A$ and $\beta \in A^*$. By this equation and induction on $k$, we show that (7.7) holds for all positive integer $k$. For $k < 0$, use (7.6) instead of (7.5).

The finiteness of the order of $\nu_A$ is proved as follows: Since the set of algebra maps $A \to k$ is at most finite, there is a positive integer $k$ such that $\varepsilon\nu_A^k = \varepsilon$. For this $k$ and a positive integer $\ell$, we have $\nu_A^k(a) = g^{k\ell}S^{-2k\ell}(a)g^{-k\ell}$. Since both $g$ and $S$ are of finite order, we conclude that so is $\nu_A$. \qed

**Remark 7.5.** Let $A$ be as above. As in the case of Hopf algebras, a left integral in $A$ is defined to be an element $\Lambda \in A$ such that $a\Lambda = \varepsilon(a)\Lambda$ for all $a \in A$. It is known that a coideal subalgebra of a finite-dimensional Hopf algebra is Frobenius [Skr08]. By the general argument for Frobenius algebras noted in [HN99 Section 4], the space of left integrals in $A$ is one-dimensional. We fix a non-zero left integral $\Lambda \in A$. Then the algebra map $\alpha_A := \varepsilon \circ \nu_A$ is characterized by the property that the equation $\Delta a = \alpha_A(a)\Lambda$ holds for all $a \in A$.

**Remark 7.6.** The Hopf algebra $H$ itself is a left coideal subalgebra of $H$ and a right cointegral $\lambda : H \to k$ is a $g_H$-cointegral in the sense of Definition 7.2. Thus we have
\[ \nu_H(h) = g_H S^{-2}(\alpha_H \to h)g_H^{-1} \quad (h \in H) \]
by the above proposition. Radford’s $S^4$-formula
\[ S^4(h) = g_H(\alpha_H \to h \to \bar{\alpha}_H)g_H^{-1} \quad (h \in H) \]
is obtained by comparing two descriptions (7.2) and (7.8) of the Nakayama automorphism of $H$. This formula implies that $g_H$ is central in the group $G(H)$.
7.3. **Relative Hopf bimodules.** Let \( A \) and \( B \) be algebras in \( C \). An object of the category \( {_A C_B} \) is often called a relative Hopf bimodule. The Nakayama functor of \( {_A C_B} \) is obtained by Theorem 7.1. Here we examine the case where both \( A \) and \( B \) admit non-degenerate grouplike-cointegrals.

**Remark 7.7.** Before we go into the detail, we note an interpretation of the category \( {_A C_B} \) from the viewpoint of the theory of finite tensor categories. According to [AM07], the category \( R\text{-mod} \) for an algebra \( R \in C \) has a natural structure of a finite left module category over \( D := H\text{-mod} \). There is an equivalence

\[
_A C_B \rightarrow \text{Rex}_D(B\text{-mod}, A\text{-mod}), \quad M \mapsto M \otimes_B (-)
\]

of categories. Furthermore, when \( A = B \), this equivalence is monoidal (where the monoidal product of \( {_A C_A} \) is given by the tensor product over \( A \)). Thus \( {_A C_A} \) is equivalent to the dual \( D_A\text{-mod} \) as a linear monoidal category. Below we compute the Nakayama functor of \( {_A C_B} \) and the modular object of \( {_A C_A} \) by viewing \( {_A C_B} \) as the category of modules over the monad \( A \otimes (-) \otimes B \) on \( C \). The same results can of course be obtained from Corollary 6.12.

Now we assume that for each \( X \in \{A, B\} \), there are an element \( g_X \in G(H) \) and a non-degenerate \( g_X \)-cointegral \( \lambda_X \) on \( X \). For \( X \in \{A, B\} \), we denote by \( \nu_X \) the Nakayama automorphism of \( X \) associated to \( \lambda_X \). We introduce the endofunctor \( \mathcal{N} \) on \( {_A C_B} \) as follows: As a vector space, \( \mathcal{N}(M) = M \) for \( M \in {_A C_B} \). The left coaction of \( H \) on \( \mathcal{N}(M) \) is defined by

\[
m \mapsto g_A^{-1}S^2(m(-1))g_B^{-1}g_H \otimes m(0),
\]

where \( m \mapsto m(-1) \otimes m(0) \) is the original coaction of \( H \) on \( M \). The left action \( \triangleright \) of \( A \) and the right action \( \triangleleft \) of \( B \) on \( \mathcal{N}(M) \) are defined by

\[
a \triangleright m \triangleleft b = \nu_A(a) \cdot m \cdot \nu_B^{-1}(b \leftarrow \alpha_H) \quad (a \in A, b \in B, m \in M),
\]

where ‘‘\( \cdot \)’’ means the original actions of \( A \) and \( B \) on \( M \).

One can verify that \( \mathcal{N}(M) \) is an object of \( {_A C_B} \) with the help of equations (7.5), (7.8) and (7.9). However, we omit the detail, since the reason why \( \mathcal{N}(M) \) belongs to \( {_A C_B} \) will be revealed in the proof of the following theorem:

**Theorem 7.8.** The functor \( \mathcal{N} \) is the Nakayama functor of \( {_A C_B} \).

**Proof.** We identify \( X^\vee \vee \) and \( \vee \vee X \) with \( X^{(S^2)} \) and \( X^{(S^2)} \), respectively, if \( X \) is an object of \( C \). In view of (2.7), we choose the functor

\[
\mathcal{N}_C : C \rightarrow C, \quad X \mapsto X^{(S^2)} \otimes k g_H \quad (X \in C)
\]

as the Nakayama functor of \( C \). As discussed in Subsection 5.1, there is an isomorphism \( A^\vee \cong k g_A^{-1} \otimes A \) of right \( A \)-modules in \( C \). The same argument applied to \( B^{op} \in C^{\text{rev}} \) yields an isomorphism \( \vee B \cong B \otimes k g_B^{-1} \) of left \( B \)-modules in \( C \). Hence, by Theorem 5.1 we have natural isomorphisms

\[
\mathcal{N}_{_A C_B}(M) \cong \mathcal{N}_C(A^\vee \otimes_A M \otimes_B \vee B) \cong \mathcal{N}_C(k g_A^{-1} \otimes M \otimes k g_B^{-1})
\]
\[
\cong k g_A^{-1} \otimes M^{(S^2)} \otimes k g_B^{-1} \otimes k g_H \cong \mathcal{N}(M) \quad (M \in {_A C_B})
\]

of right \( H \)-comodules.

Since the twisted left \( C \)-module structure of \( \mathcal{N}_C \) is the identity map, it is easy to see that the isomorphism \( \mathcal{N}_{_A C_B}(M) \cong \mathcal{N}(M) \) obtained in the above preserves left action of \( A \). The right action of \( B \) on \( \mathcal{N}_{_A C_B}(M) \) is defined via the twisted \( C \)-module
structure of $\mathbb{N}_C$ (cf. Remark 2.5). Thus, when an element $b \in B$ acts on $\mathbb{N}(M)$, it is first affected by the Radford isomorphism (7.4), and then acts on $M$ through the Nakayama automorphism of $B^{\text{op}} \in C^{\text{rev}}$, that is, $\nu_B^{-1}$. Therefore the action of $B$ on $\mathbb{N}(M)$ is given as stated. The proof is done. \hfill $\square$

Now we consider the case where $A = B$.

**Theorem 7.9.** Let $A$ be an algebra in $\mathcal{C}$ and suppose that there is an element $g_A \in G(H)$ and a non-degenerate $g_A$-cointegral $\lambda_A : A \to \mathbb{k}$. Then the modular object of $\mathcal{A}C_A$ is the vector space $\alpha := A$ equipped with the following structure morphisms: The left $H$-coaction on $\alpha$ is given by

$$x \mapsto g_A^{-2}g_Hx_{(-1)} \otimes x_{(0)} \quad (x \in \alpha),$$

where $a \mapsto a_{(-1)} \otimes a_{(0)}$ is the original left coaction of $H$ on $A$. The left and the right actions of $A$ on $\alpha$ are given by

$$a \triangleright x \triangleleft b = \nu_A(a \leftarrow \bar{\pi}_H) \cdot x \cdot b \quad (a, b \in A, x \in \alpha),$$

where $\nu_A$ is the Nakayama automorphism of $A$ with respect to $\lambda_A$.

**Proof.** Let, in general, $\beta : H \to \mathbb{k}$ be an algebra map. By (7.5), we have

$$\nu_A(a) \leftarrow \beta = (\beta, g_A S^{-2}(a_{(-1)})g_A^{-1}) \nu_A(a_{(0)}) = \nu_A(a \leftarrow \beta)$$

for all $a \in A$. Since the unit object of $\mathcal{A}C_A$ is $A$, the modular object of $\mathcal{A}C_A$ is $\mathbb{N}(A)$, where $N : \mathcal{A}C_A \to \mathcal{A}C_A$ is the functor introduced in the above. By (7.12), it is easy to see that the bijective linear map

$$\xi : \alpha \to \mathbb{N}(A), \quad x \mapsto \nu_A^{-1}(x \leftarrow \alpha_H) \quad (x \in \alpha)$$

preserves the left and the right actions of $A$. Let $\delta_{\mathbb{N}(A)}$ denote the left coaction of $H$ on $\mathbb{N}(A)$. The map $\xi$ also preserves the left coaction of $H$. Indeed, for $x \in \alpha$, we have

$$(\text{id}_H \otimes \xi^{-1})\delta_{\mathbb{N}(A)}(\xi(x)) = (\alpha_H, x_{(-1)}) (\text{id}_H \otimes \xi^{-1})\delta_{\mathbb{N}(A)}(\nu_A^{-1}(x_{(0)}))$$

$$= (\alpha_H, x_{(-1)}) g_A^{-1} S^2(\nu_A^{-1}(x_{(0)})(-1)) g_A^{-1} g_H \otimes \xi^{-1}(\nu_A^{-1}(x_{(0)}))$$

$$= (\alpha_H, x_{(-2)}) g_A^{-2} S^4(x_{(-1)}) g_H \otimes \xi^{-1}(\nu_A^{-1}(x_{(0)}))$$

$$= g_A^{-2} S^4(x_{(-1)} \leftarrow \alpha_H) g_H \otimes (x_{(0)} \leftarrow \bar{\pi}_H)$$

$$= g_A^{-2} g_H(x_{(-1)} \otimes x_{(0)} \leftarrow \bar{\pi}_H)$$

by (7.6) and (7.9). Thus $\alpha$ is an object of $\mathcal{A}C_A$ that is isomorphic to $\mathbb{N}(A)$. The proof is done. \hfill $\square$

**Corollary 7.10.** Notations are as above. Then $\mathcal{A}C_A$ is unimodular if and only if there is an invertible element $g \in A$ such that the equations

$$\begin{align*}
(7.13) & \quad ga = \nu_A^2(a \leftarrow \bar{\pi}_H) g, \\
(7.14) & \quad \delta_A(g) = g_H^{-1} g_A^2 \otimes g
\end{align*}$$

hold for all elements $a \in A$. 

Proof. Let $\alpha$ be the modular object of $AC_A$ given by the previous theorem. We shall discuss when $\alpha$ is isomorphic to $A$ as objects of $AC_A$. Given an element $g \in A$, we define $\phi_g : A \to A$ by $\phi_g(a) = ga (a \in A)$. The map $g \mapsto \phi_g$ gives a bijection between the set $A$ and the set of right $A$-linear maps $A \to A$. It is easy to see that $\phi_g$ is invertible if and only if $g$ is, $\phi_g$ preserves left $A$-action if and only if $(7.13)$ holds, and $\phi_g$ preserves the left $H$-coaction if and only if $(7.14)$ holds. The claim is proved by summarizing the discussion so far.

**Corollary 7.11.** Let $A$ be a left coideal subalgebra of $H$. Suppose that there are an element $g_A \in G(H)$ and a non-zero $g_A$-cointegral on $A$. Then $AC_A$ is unimodular if and only if $g_H^{-1}g_A^2 \in A$ and $\varepsilon \circ \nu_A^2 = \alpha_H|A$.

**Proof.** Suppose that there is an element $g \in A^\times$ satisfying $(7.13)$ and $(7.14)$. We may assume $\varepsilon(g) = 1$ by renormalizing $g$. By applying $id_H \otimes \varepsilon$ to the both sides of $(7.14)$, we obtain $g_H^{-1}g_A^2 = g \in A$. We also have

$$\varepsilon\nu_A^2 \to a \overset{\text{7.13}}{=} S^{-4}(g_A^{-2}\nu_A^2(a))g_A^2 \overset{\text{7.14}}{=} S^{-4}(g_A^{-2}g(a - \alpha_H)g^{-1}g_A^2)$$

$$= S^{-4}(g_H^{-1}(a - \alpha_H)g_H) \overset{\text{8.8}}{=} \alpha_H \to a$$

for all $a \in A$, and thus $\varepsilon\nu_A^2 = \alpha_H|A$. The ‘only if’ part is proved. The ‘if’ part is proved by showing that $g = g_H^{-1}g_A^2$ satisfies $(7.13)$ and $(7.14)$ straightforwardly. □

### 7.4. Examples of determining the unimodularity

From now on till the end of this paper, we assume that $k$ is an algebraically closed field of characteristic zero. We fix an odd integer $N > 1$ and a primitive $N$-th root $q \in k$ of unity, and consider the small quantum group $u_q(sl_2)$. We recall that, as an algebra, it is generated by $E$, $F$ and $K$ subject to the relations $K^N = 1$, $E^N = F^N = 0$, $KE = q^2FK$, $KF = q^{-2}FK$ and $EF - FE = (K - K^{-1})/(q - q^{-1})$. The Hopf algebra structure of $u_q(sl_2)$ is determined by

$$\Delta(K) = K \otimes K, \quad \Delta(E) = E \otimes 1 + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F. \quad \text{The set } \{E^rF^sK^t \mid r, s, t = 0, 1, \cdots, N - 1\} \text{ is a basis of } U := u_q(sl_2). \text{ A formula of the comultiplication of each of elements of this basis is written explicitly in, e.g., [Kas95] Chapter VII. One can check that the linear map}$$

$$\lambda_U : U \to k, \quad E^rF^sK^t \mapsto \delta_{r,N-1}\delta_{s,N-1}\delta_{t,1} \quad (r, s, t = 0, 1, \cdots, N - 1)$$

is a right cointegral on $U$. Furthermore, we have $g_U = K^2$. It is also known that $U$ is unimodular, i.e., $\alpha_U = \varepsilon$.

Below, for some algebras $A$ in $\mathcal{C} := U\text{-comod}$, we determine whether $AC_A$ is unimodular. In view of Remark 7.10, we may say that we will discuss whether the dual of $U\text{-mod}$ with respect to $A\text{-mod}$ is unimodular.

**Example 7.12.** We fix $\xi \in k$ and a divisor $d$ of $N$. We consider the algebra $A$ generated by $G$ and $Y$ subject to the relations $G^d = 1$, $GY = q^{-2m}YG$ and $Y^N = \xi 1_A$, where $m = N/d$. The algebra $A$ is a left $U$-comodule algebra by the left $U$-coaction determined by $G \mapsto K^m$ and $Y \mapsto F \otimes 1 + K^{-1} \otimes Y$. In a similar way as [Shi19] Section 5], we see that the linear map

$$\lambda_A : A \to k, \quad Y^rG^s \mapsto \delta_{r,N-1}\delta_{s,0} \quad (r = 0, \cdots, N - 1; s = 0, \cdots, d - 1)$$

is a non-degenerate $g_A$-cointegral on $A$ with $g_A = K$. The Nakayama automorphism $\nu_A$ of $A$ with respect to $\lambda_A$ is given by $\nu_A(G) = q^{2m}G$ and $\nu_A(Y) = Y$. 


Corollary 7.10 is effective to determine whether $AC_A$ is unimodular. Indeed, if $g$ is an invertible element of $A$ satisfying the condition (7.14) of the corollary, then $g$ must be 1, since $gU^{-1}g^{-1} = 1_U$. Thus $AC_A$ is unimodular if and only if $\nu_A = 1_{AC_A}$, or, equivalently, $d = 1$ (because the order of $q$ is odd).

Example 7.13. We choose a parameter $\xi \in k$ and consider the subalgebra $A$ of $U$ generated by $Y := F + \xi K^{-1}$. Since $\Delta(Y) = F \otimes 1 + K^{-1} \otimes Y$, the algebra $A$ is in fact a left coideal subalgebra of $U$. The linear map

$$\lambda_A : A \to k, \quad Y^r \mapsto \delta_{r, N-1} \quad (r = 0, 1, \cdots, N - 1)$$

is a non-degenerate $g_A$-cointegral on $A$ with $g_A = K$. By Corollary 7.11, we see that $AC_A$ is always unimodular (this is actually the case where $d = 1$ in the previous example).

Example 7.14. We choose a divisor $m$ of $N$ and consider the subalgebra $A$ of $U$ generated by $G := K^m$. The subalgebra $A$ is a left coideal subalgebra of $U$ (which is, in fact, a Hopf subalgebra of $U$). The linear map

$$\lambda_A : A \to k, \quad G^r \mapsto \delta_{r, 0} \quad (r = 0, 1, \cdots, N/m - 1)$$

is a non-degenerate $g_A$-cointegral on $A$ with $g_A = 1_A$. By Corollary 7.11, $AC_A$ is unimodular if and only if $m = 1$.

7.5. Coideal subalgebras without grouplike-cointegrals. Corollaries 7.10 and 7.11 are not applicable to comodule algebras without non-degenerate grouplike-cointegrals. We close this paper by giving an example of coideal subalgebras without non-zero grouplike-cointegrals. We fix an odd integer $N > 1$ and a primitive $N$-th root $q \in k$ of unity, and define $H$ to be the algebra generated by $a$, $b$, $c$ and $d$ subject to the following relations:

$$ba = qab, \quad ca = qac, \quad db = qbd, \quad dc = qcd, \quad bc = cb,$$

$$ad - q^{-1}bc = da - qbc = 1, \quad a^N = d^N = 1, \quad b^N = c^N = 0.$$ 

The algebra $H$ is a Hopf algebra with the comultiplication defined so that $a$, $b$, $c$ and $d$ form a matrix coalgebra. It is known that $H$ is dual to $U := u_q(\mathfrak{sl}_2)$. A detailed account is found in, e.g., [Kas99 VII.4] and [Chi19 Appendix A]. Specifically, there is an isomorphism $\phi : H \to U^*$ of Hopf algebras such that

$$\begin{pmatrix} \phi(a)(u) \\ \phi(c)(u) \\ \phi(d)(u) \end{pmatrix} = \rho(u) \quad (u \in U),$$

where $\rho : U \to \mathrm{Mat}_{2 \times 2}(\mathbb{C})$ is the algebra map determined by

$$E \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad K \mapsto \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}.$$ 

A right $H$-comodule is identified with a left $U$-comodule through the Hopf algebra isomorphism $\phi$. In particular, the action of $U$ on $H$ is computed by

$$E \cdot 1 = 0, \quad E \cdot a = 0, \quad E \cdot b = a, \quad E \cdot c = 0, \quad E \cdot d = c, \quad F \cdot 1 = 0, \quad F \cdot a = b, \quad F \cdot b = 0, \quad F \cdot c = d, \quad F \cdot d = 0,$$

$$K \cdot 1 = 1, \quad K \cdot a = qa, \quad K \cdot b = q^{-1}b, \quad K \cdot c = qc, \quad K \cdot d = q^{-1}d$$

and the rule $u \cdot (xy) = (u(1) \cdot x)(u(2) \cdot y)$ for $u \in U$ and $x, y \in H$.

For two integers $r$ and $s$, we put $\nu_rs = a^rb^{r-s}$ if $0 \leq s \leq r \leq N$ and $\nu_rs = 0$ otherwise. For every $r = 0, \cdots, N$, the subspace $V_r = \text{span}_k \{ \nu_rs \mid s = 0, \cdots, r \}$ of
'homogeneous polynomials' in a and b is an \((r + 1)\)-dimensional \(U\)-submodule of \(H\) (cf. [Kas95 IV.7]). The actions of the generators of \(U\) on \(V_r\) are given by
\[
E \cdot v_{rs} = [s]v_{r,s-1}, \quad F \cdot v_{rs} = [r-s]v_{r,s+1} \quad \text{and} \quad K \cdot v_{rs} = q^{r-2s}v_{rs},
\]
where \([m] = (q^m - q^{-m})/(q - q^{-1})\).

The \(U\)-submodule \(A := V_N\) of \(H\) is closed under the multiplication. Thus it is in fact a right coideal subalgebra of \(H\). The subspace of \(A\) spanned by \(1_A = v_{N,0}\) is a unique non-trivial \(U\)-submodule of \(A\). In particular, \(A\) has no quotient right \(H\)-comodule of dimension one. Thus \(A\), viewed as a left \(H^{\text{cop}}\)-comodule algebra, admits no non-zero grouplike-cointegral.

**Question 7.15.** Let \(H\) be a finite-dimensional Hopf algebra, and let \(A\) be an algebra in \(C := H\text{-comod}\), which may not admit a non-degenerate grouplike-cointegral. Is there an easy criterion for \(A C_A\) to be unimodular?

**References**

[AM07] N. Andruskiewitsch and M. Mombelli. On module categories over finite-dimensional Hopf algebras. *J. Algebra*, 314(1):383–418, 2007.

[BKLT00] Y. Bespalov, T. Kerler, V. Lyubashenko, and V. Turaev. Integrals for braided Hopf algebras. *J. Pure Appl. Algebra*, 148(2):113–164, 2000.

[BV12] A. Bruguières and A. Virelizier. Quantum double of Hopf monads and categorical centers. *Trans. Amer. Math. Soc.*, 364(3):1225–1279, 2012.

[Cli19] Z. Cline. On actions of Drinfel’d doubles on finite dimensional algebras. *J. Pure Appl. Algebra*, 223(8):3635–3664, 2019.

[Del90] P. Deligne. Catégories tannakiennes. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Progr. Math.*, pages 111–195. Birkhäuser Boston, Boston, MA, 1990.

[DSPS19] C. L. Douglas, C. Schommer-Pries, and N. Snyder. The balanced tensor product of module categories. *Kyoto J. Math.*, 59(1):167–179, 2019.

[DSPS20] C. L. Douglas, C. Schommer-Pries, and N. Snyder. Dualizable tensor categories. *Mem. Amer. Math. Soc.*, 268(1308):vii+88, 2020.

[EGNO15] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrom. *Tensor categories*, volume 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.

[ENO04] P. Etingof, D. Nikshych, and V. Ostrom. An analogue of Radford’s \(S^4\) formula for finite tensor categories. *Int. Math. Res. Not.* (54):2915–2933, 2004.

[FGJS22] J. Fuchs, C. Galindo, D. Jakobitz, and D. Schweigert. Spherical Morita contexts and relative Serre functors. arXiv:2207.07031.

[FS08] Jürgen Fuchs and Carl Stüger. On Frobenius algebras in rigid monoidal categories. *Arab. J. Sci. Eng. Sect. C Theme Issues*, 33(2):175–191, 2008.

[FSS20] J. Fuchs, G. Schaumann, and C. Schweigert. Eilenberg-Watts calculus for finite categories and a bimodule Radford \(S^4\) theorem. *Trans. Amer. Math. Soc.*, 373(1):1–40, 2020.

[HN99] F. Hausser and F. Nill. Integral Theory for Quasi-Hopf Algebras. 1999.

[Kas05] C. Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[Kas18] P. Kasprzak. Generalized (co)integrals on coideal subalgebras. arXiv:1810.07114.

[ML98] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[NS07] S.-H. Ng and P. Schauenburg. Higher Frobenius-Schur indicators for pivotal categories. In *Hopf algebras and generalizations*, volume 441 of *Contemp. Math.*, pages 63–90. Amer. Math. Soc., Providence, RI, 2007.

[Shi17a] K. Shimizu. Ribbon structures of the Drinfeld center. arXiv:1707.09651.

[Shi17b] K. Shimizu. On unimodular finite tensor categories. *Int. Math. Res. Not. IMRN*, 1(1):277–322, 2017.

[Shi17c] K. Shimizu. The relative modular object and Frobenius extensions of finite Hopf algebras. *J. Algebra*, 471:75–112, 2017.
[Shi19] K. Shimizu. Relative Serre functor for comodule algebras. arXiv:1904.00376.

[Skr08] S. Skryabin. Projectivity of Hopf algebras over subalgebras with semilocal central localizations. *J. K-Theory*, 2(1):1–40, 2008.

[SS21] T. Shibata and K. Shimizu. Nakayama functors for coalgebras and their applications to Frobenius tensor categories. arXiv:2110.08739.

[SY11] A. Skowroński and K. Yamagata. *Frobenius algebras*. I. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2011. Basic representation theory.

[Tak99] M. Takeuchi. Finite Hopf algebras in braided tensor categories. *J. Pure Appl. Algebra*, 138(1):59–82, 1999.

*Email address: kshimizu@shibaura-it.ac.jp*

Department of Mathematical Sciences, Shibaura Institute of Technology, 307 Fukasaku, Minuma-ku, Saitama-shi, Saitama 337-8570, Japan.