SINGULAR DIFFUSION
WITH NEUMANN BOUNDARY CONDITIONS

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Abstract. In this paper we develop an existence theory for the nonlinear initial-boundary value problem with singular diffusion
\[ \partial_t u = \text{div}(k(x)\nabla G(u)), \]
\[ u|_{t=0} = u_0 \] with Neumann boundary conditions
\[ k(x)\nabla G(u) \cdot \nu = 0. \]
Here \( x \in B \subset \mathbb{R}^d \), a bounded open set with locally Lipchitz boundary, and with \( \nu \) as the unit outer normal. The function \( G \) is Lipschitz continuous and nondecreasing, while \( k(x) \) is a smooth diagonal matrix with positive entries \( k_i > 0 \). The fact that \( G' \) may vanish, allows for singular diffusion, and the factor \( k \) permits for spatially dependent diffusion. Due to the singular diffusion, this equation has weak solutions.

Our goal is to prove well-posedness of this equation in appropriate spaces. Working with weak solutions, we need to impose an appropriate entropy condition, and we here demand, see Definition 2.2, that
\[ \int_0^T \int_B \left[ \eta(u)\partial_t \varphi - \eta'(u)(k(x)\nabla G(u))\nabla \varphi \right] dxdt - \int_B \eta(u)\varphi dx \]
\[ = \left[ \eta(u_0)\varphi(x) - \eta(u(T))\varphi(x) \right]_0^T. \]

1. Introduction

We here study the nonlinear boundary value problem with singular diffusion
\[ \partial_t u = \text{div}(k(x)\nabla G(u)), \quad u|_{t=0} = u_0 \]
with Neumann boundary conditions
\[ k(x)\nabla G(u) \cdot \nu = 0. \]
Here \( x \in B \subset \mathbb{R}^d \), a bounded open set with locally Lipchitz boundary, and with \( \nu \) as the unit outer normal at the boundary of \( B \). The function \( G \) is Lipschitz continuous and nondecreasing, while \( k(x) = \text{diag}(k_1(x_1), \ldots, k_d(x_d)) \) is a smooth, diagonal matrix with positive entries \( k_i > 0 \). The fact that \( G' \) may vanish, allows for singular diffusion, and the factor \( k \) permits for spatially dependent diffusion. Due to the singular diffusion, this equation has weak solutions.

Our goal is to prove well-posedness of this equation in appropriate spaces. Working with weak solutions, we need to impose an appropriate entropy condition, and we here demand, see Definition 2.2, that
\[ \int_0^T \int_B \left[ \eta(u)\partial_t \varphi - \eta'(u)(k(x)\nabla G(u))\nabla \varphi \right] dxdt - \int_B \eta(u)\varphi dx \]
\[ = \left[ \eta(u_0)\varphi(x) - \eta(u(T))\varphi(x) \right]_0^T. \]

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\[ \eta''(u) |h(x)\nabla g(u)|^2 \varphi \, dx \, dt \]

holds for every smooth convex entropy \( \eta \) and for all nonnegative test functions \( \varphi \). Here \( \Omega_T = [0,T] \times B \). One of our main theorem reads, see Theorem 2.3, as follows:

For two weak entropy solutions \( u \) and \( v \) with initial data \( u_0 \) and \( v_0 \), respectively, we have

\[ \|u(t) - v(t)\|_{L^1(B)} \leq \|u_0 - v_0\|_{L^1(B)} e^{Ct}, \]

for almost every \( t \geq 0 \). The constant \( C \) depends on \( k \), \( G \), and \( B \). Here we only need to assume that \( k \) is a diagonal matrix.

As is common, existence and stability are proven by two independent arguments. Assuming existence of two weak entropy solutions \( u \) and \( v \), we proceed using the ingenious doubling of variables due to Kružkov, see [10]. In this approach, one starts with entropy conditions for \( u \) (in variables \( (t,x) \)) and \( v \) (in variables \( (s,y) \)), and considers a test function \( \varphi \) that depends on all four variables \( (t,x,s,y) \). Integrating over all variables, and adding the entropy expressions for \( u \) and \( v \), one finds the inequality (2.9). The next step is the choice of test function. Here we follow in the steps of Kružkov and choose an approximate Dirac delta function in the time variable \( t-s \) and space variables \( x-y \) in addition to a smooth cut-off function near the boundary. See equation (2.15). Next comes the delicate limits. We start by taking the \( s \to t \) limit. The fact that we deal with a bounded domain goes beyond the standard Kružkov theory. We find that we need to remove the spatial singularity and the cut-off at the boundary simultaneously, and, importantly, at a fixed ratio where the spatial singularity has to vanish at a faster rate than the boundary cut-off, see (2.23). This proves equation (1.1).

As for existence of weak entropy solutions, our starting point is the regularized equation where we add non-degenerate diffusion. Thus we consider for positive \( \mu \)

\[ \partial_t u^\mu = \text{div} \left( k(x) \nabla G(u^\mu) \right) + \mu \Delta u^\mu, \quad u^\mu|_{t=0} = u_0 \]

with boundary condition \( \left( k(x) \nabla G(u^\mu(t,x)) + \mu \nabla u^\mu(t,x) \right) \cdot \nu = 0 \). Existence of solutions for this problem follows from [17, Theorem 1.7.8].

In our case we have only been able to prove existence of a solution under the restriction that \( k_i = k_i(x_i) \). Our proof is based on showing compactness of the sequence \( \{u^\mu\}_{\mu>0} \), we were not able to deduce this without the simplifying assumption that \( k_i \) only depends on \( x_i \). Our main existence result, Theorem 3.1, reads as follows:

Assume that \( k_i = k_i(x_i) > 0 \). Then there exists a weak entropy solution \( u \) the initial-boundary value problem. In particular, we have, as \( \mu \to 0 \), that

\[ u^\mu \to u \quad \text{a.e. and in } L^p(\Omega_T) \text{ for every } T > 0 \text{ and } 1 \leq p < \infty. \]

Thus the problem is well-posed in the sense of Hadamard.

The proof starts with the entropy formulation for smooth solutions, which yields (cf. (3.5))

\[ \frac{d}{dt} \int_B \eta(u^\mu) \, dx + \int_B \eta''(u^\mu)G'(u^\mu)|h(x)\nabla u^\mu|^2 \, dx + \mu \int_B \eta''(u^\mu)|\nabla u^\mu|^2 \, dx = 0 \]

for convex entropies \( \eta \). By choosing different functions for the entropy, we can show a wide range of properties of the approximate solution \( u^\mu \), see Lemma 3.2. In particular, we show that \( \|\nabla G(u^\mu)\|_{L^2(\Omega_T)} \) is bounded independently of \( \mu \). Furthermore,
we verify that
\[ \| \partial_{x_j} u^\mu(t) \|_{L^1(B_\sigma)} \leq \| \partial_{x_j} u_0 \|_{L^1(B)} e^{C_\sigma t}, \]
where \( B_\sigma \) is the subset of \( B \) with distance at least \( \sigma \) from \( \partial B \), holds. From these estimates, we can prove, as \( \mu \to 0 \), that \( u^\mu \to u \) in \( L^p(\Omega_T) \) for every \( T > 0 \) and \( 1 \leq p < \infty \).

The problem of analyzing parabolic equations with singular diffusion has of course been studied by several researchers, and the literature is too comprehensive to be discussed in detail here. Our paper relies on the seminal paper by Carrillo [3] where the Kružkov doubling of variables, see, e.g., [10], is applied to study multi-dimensional degenerate parabolic problems, and with an entropy condition due to Vol’pert–Hudjaev [18]. Carrillo studied the case of Dirichlet boundary conditions. An early result can be found in [1]. Further generalizations of the results by Carrillo can be found in [6, 5] by Chen and Karlsen, extending the analysis to the whole space and allowing for spatial and temporal dependence the various terms, as well as nonlinear transport. We have relied on the work by Karlsen and Oehlberger [14], in particular their result regarding the weak chain rule, see equation (2.7). Different boundary conditions in one dimension are studied in [2]. Karlsen and Risebro [15] studied uniqueness and stability of nonlinear degenerate parabolic equations with rough coefficients. See also [4].

In Section 4 we study a convergent difference scheme for this equation in one dimension with \( B = (0, 1) \). Let \( u^n_j \) be an approximation to \( u(t_n, x_j) \) with \( t_n = n \Delta t \) and \( x_j = (j + 1/2) \Delta x \) for small positive numbers \( \Delta t, \Delta x \). The implicit first-order difference scheme is given by
\[ u^{n+1}_j - \mu \Delta_+(k_{j-1/2} \Delta_- G(u_j^{n+1})) = u^n_j, \quad j = 0, \ldots, N, \]
with boundary conditions that \( \Delta_- u^n_0 = \Delta_+ u^n_N = 0 \). Here \( \Delta_+ (\Delta_-) \) is the forward (backward) spatial difference, \( k_{j-1/2} = k(x_{j-1/2}) \), and \( \mu = \Delta t / \Delta x^2 \) (assumed to be bounded from below). We define the function \( u_{\Delta t}(t, x) \) on \([0, \infty) \times B\) by making it equal to \( u^n_j \) on the rectangle \([t_n, t_{n+1}) \times [x_{j-1/2}, x_{j+1/2})\). See [16] for related results on numerical methods.

Our interest in this equation stems from the modeling of multilane dense traffic. Each lane is modeled by the traditional Lighthill–Whitham–Richards model, which gives a scalar hyperbolic conservation law where the unknown function describes the density of vehicles. Our aim was to model unidirectional multilane traffic. We discovered, see [11, 12, 13], that the model we studied, allowed for an infinite number of lanes limit, resulting in an equation resembling the one of this paper.

2. Space dependent singular diffusion

We are interested in the boundary value problem
\[
\begin{align*}
\partial_t u &= \text{div} (k(x) \nabla G(u)), \quad (t, x) \in (0, \infty) \times B, \\
(k(x) \nabla G(u)) \cdot \nu &= 0, \quad (t, x) \in (0, \infty) \times \partial B, \\
u(0, x) &= u_0(x), \quad x \in B,
\end{align*}
\]
where we shall assume that
\[ (H.1) \quad B \subset \mathbb{R}^d \text{ is a bounded open set with locally Lipchitz boundary and } \nu \text{ the unit outer normal to its boundary}; \]
Theorem 2.3 \( \phi \) and for every test function \( \eta \) if it is a weak solution of \( \phi \) for almost every \( \tau \), \( \eta \) is a weak entropy solution in the sense of Definition 2.2 then

\[
\int_{\Omega_T} \left[ \eta(u) \partial_t \varphi - \eta'(u)(k(x) \nabla G(u)) \cdot \nabla \varphi \right] \, dx \, dt + \int_{B} u_0(x) \varphi(0, x) \, dx = 0.
\]

Definition 2.1. A function \( u \in C([0, \infty); L^1(B)) \) is a weak solution of \( (2.1) \) if

\[
(\text{2.2}) \quad u \in L^\infty(\Omega),
\]

\[
(\text{2.3}) \quad \nabla G(u) \in L^2(\Omega_T; \mathbb{R}^d),
\]

\[
(\text{2.4}) \quad (k(x) \nabla G(u)) \cdot \nu = 0 \text{ in the sense of traces on } \partial B \text{ for a.e. } t,
\]

and for every test function \( \varphi \in C_0^\infty(\Omega) \)

\[
(\text{2.5}) \quad \int_{\Omega} \left[ u \partial_t \varphi - (k(x) \nabla G(u)) \cdot \nabla \varphi \right] \, dt \, dx + \int_{B} u_0(x) \varphi(0, x) \, dx = 0.
\]

Definition 2.2. A function \( u \in C([0, \infty); L^1(B)) \) is an entropy solution of \( (2.1) \) if it is a weak solution of \( (2.1) \) in the sense of Definition 2.1 and for every convex entropy \( \eta \in C^2(\mathbb{R}) \) and for all nonnegative test functions \( \varphi \in C_0^\infty(\Omega) \)

\[
\int_{\Omega_T} \left[ \eta(u) \partial_t \varphi - \eta'(u)(k(x) \nabla G(u)) \cdot \nabla \varphi \right] \, dx \, dt + \int_{B} u_0(x) \varphi(0, x) \, dx
\]

\[
- \int_{B} \eta(u(T, x)) \varphi(T, x) \, dx + \int_{B} \eta(u_0(x)) \varphi(0, x) \, dx 
\]

\[
\geq \int_{\Omega_T} \eta''(u) |h(x) \nabla g(u)|^2 \, \varphi \, dx \, dt.
\]

Recall that the following weak chain rule holds, see [14]. Let \( A \) be a continuous function on \([0, \infty)\) with \( A(0) = 0 \) and \( b \) a continuous function. If \( u \) is an entropy solution in the sense of Definition 2.2 then

\[
(\text{2.7}) \quad b(u) \nabla \left( \int_{u}^{u} A \left( \sqrt{G'(\xi)} \right) \, d\xi \right) = \nabla \left( \int_{u}^{u} b(\xi) A \left( \sqrt{G'(\xi)} \right) \, d\xi \right),
\]

weakly in \( \Omega \). Then the following theorem holds.

Theorem 2.3 (Uniqueness and Stability). Assume that \( (H.1), (H.2), (H.3), \) and \( (H.4) \) hold. Let \( u \) and \( v \) be two weak entropy solutions of the initial-boundary value problem \( (2.1) \). Then

\[
(\text{2.8}) \quad \|u(t, \cdot) - v(t, \cdot)\|_{L^1(B)} \leq \|u(0, \cdot) - v(0, \cdot)\|_{L^1(B)} e^{Ct},
\]

for almost every \( t \geq 0 \). Here \( C \) is a finite positive constant depending on \( k, G, \) and \( B \).
Proof. We define the signum function as
\[
\text{sign}(\xi) = \begin{cases} 
1 & \xi > 0, \\
0 & \xi = 0, \\
-1 & \xi < 0,
\end{cases}
\]
and its regularized version
\[
\text{sign}_\epsilon(\xi) = \begin{cases} 
1 & \xi > \epsilon, \\
\sin\left(\frac{\xi}{\epsilon}\right) & |\xi| \leq \epsilon, \\
-1 & \xi < -\epsilon.
\end{cases}
\]
Now set
\[
|\sigma|_\epsilon = \int_0^\sigma \text{sign}_\epsilon(\xi) \, d\xi, \quad \eta_\epsilon(u,v) = |u - v|_\epsilon.
\]
Let now \( u = u(t,x) \) and \( v = v(s,y) \) be two entropy solutions with initial data \( u_0 \) and \( \nu_0 \), respectively. The maps \( u \mapsto \eta_\epsilon(u,v) \) and \( v \mapsto \eta_\epsilon(u,v) \) are admissible entropies. Let \( \varphi = \varphi(t,x,s,y) \) be an admissible test function both in \( (t,x) \) and in \( (s,y) \). By adding the entropy condition for \( u \) and \( v \) we get
\[
\int_{\Omega_T^2} \left[ |u - v|_\epsilon (\partial_t + \partial_s) \varphi - \text{sign}_\epsilon(u - v)(k(x)\nabla_x G(u)) \cdot \nabla_x \varphi 
+ \text{sign}_\epsilon(u - v)(k(y)\nabla_y G(v)) \cdot \nabla_y \varphi \right] \, dX
- \int_{B \times \Omega_T} |u - v|_\epsilon \varphi \, dx dy ds \bigg|_{t=T}^{t=0} - \int_{\Omega_T \times B} |u - v|_\epsilon \varphi \, dx dt dy \bigg|_{s=T}^{s=0}
\geq \int_{\Omega_T^2} \text{sign}'_\epsilon(u - v) \left[ |h(x)\nabla_x g(u)|^2 + |h(y)\nabla_y g(v)|^2 \right] \varphi \, dX,
\]
where
\[
dX = dt dx dy,
\]
with \( dx = dx_1 \ldots dx_d \) and similarly for \( dy = dy_1 \ldots dy_d \). Using the basic inequality \( a^2 + b^2 \geq 2ab \), this can be rewritten
\[
\tau_\epsilon(\varphi) + i_\epsilon(\varphi) - f_\epsilon(\varphi) - B_\epsilon(\varphi)
+ \int_{\Omega_T^2} \text{sign}_\epsilon(u - v) \left[ (k(x)\nabla_x G(u)) \cdot \nabla_x \varphi - (k(y)\nabla_y G(v)) \cdot \nabla_x \varphi \right] \, dX
\geq \pm 2 \int_{\Omega_T^2} \text{sign}'_\epsilon(u - v)(h(x)\nabla_x g(u)) \cdot (h(y)\nabla_y g(v)) \varphi \, dX,
\]
where
\[
\tau_\epsilon(\varphi) = \int_{\Omega_T^2} |u - v|_\epsilon (\partial_t + \partial_s) \varphi \, dX,
\]
\[
i_\epsilon(\varphi) = \int_{B \times \Omega_T} |u(0,x) - v(s,y)|_\epsilon \varphi(0,x,s,y) \, dx dy ds
+ \int_{\Omega_T \times B} |u(t,x) - v(0,y)|_\epsilon \varphi(t,x,0,y) \, dt dx dy,
\]
\[
f_\epsilon(\varphi) = \int_{B \times \Omega_T} |u(T,x) - v(s,y)|_\epsilon \varphi(T,x,s,y) \, dx dy ds.
\]
By the weak chain rule (2.7)

\[ \text{sign}_x(u - v) \nabla_x G(u) = \frac{\partial g}{\partial u}(u) \nabla_x \gamma^2(\xi) \frac{d\xi}{d\xi} \]

\[ = \nabla_x \left( \int_v^u \text{sign}_x(\xi - v) \gamma^2(\xi) d\xi \right), \]

\[ \text{sign}_x(u - v) \nabla_y G(v) = - \text{sign}_x(u - v) \nabla_y \left( \int_v^u \gamma^2(\xi) d\xi \right) \]

\[ = - \nabla_y \left( \int_v^u \text{sign}_x(u - \xi) \gamma^2(\xi) d\xi \right), \]

where \( \gamma(u) = g'(u) = \sqrt{G'(u)}. \) Therefore, using the diagonal structure of \( k \) and the fact that \( \varphi = \varphi(t,x,s,y) \) vanishes for \( x \in \partial B \) or \( y \in \partial B \), we have

\[ \int_{\Omega_t^x} \text{sign}_x(u - v) \left[ (k(x) \nabla_x G(u) \cdot \nabla_y \varphi - (k(y) \nabla_y G(v) \cdot \nabla_x \varphi) \right] dX \]

\[ = \int_{\Omega_t^x} \left[ (k(x) \nabla_y \varphi) \cdot (\text{sign}_x(u - v) \nabla_x G(u)) \right. \]

\[ - (k(y) \nabla_x \varphi) \cdot (\text{sign}_x(u - v) \nabla_y G(v)) \] \[ dX \]

\[ = \int_{\Omega_t^x} \left[ (k(x) \nabla_y \varphi) \cdot \nabla_x \left( \int_v^u \text{sign}_x(\xi - v) \gamma^2(\xi) d\xi \right) \right. \]

\[ + (k(y) \nabla_x \varphi) \cdot \nabla_y \left( \int_v^u \text{sign}_x(u - \xi) \gamma^2(\xi) d\xi \right) \] \[ dX \]

\[ = - \int_{\Omega_t^x} \left[ \nabla_{xy}^2 (k(x) \varphi) \left( \int_v^u \text{sign}_x(\xi - v) \gamma^2(\xi) d\xi \right) \right. \]

\[ + \nabla_{xy}^2 (k(y) \varphi) \left( \int_v^u \text{sign}_x(u - \xi) \gamma^2(\xi) d\xi \right) \] \[ dX, \]

where

\[ \text{div}_x (k(x)) = \left( \begin{array}{c} \partial_{x_1} k_1(x) \\ \vdots \\ \partial_{x_d} k_d(x) \end{array} \right), \]

\[ \nabla_{xy}^2 (k(x) \varphi) = \sum_{i=1}^d \partial_{x_i} \partial_{y_i} (k_i(x) \varphi). \]

Similarly,

\[ \text{sign}^\prime_x(u - v) \nabla_x g(u) = \text{sign}^\prime_x(u - v) \nabla_x \left( \int_v^u \gamma(\xi) d\xi \right) \]

\[ = \nabla_x \left( \int_v^u \text{sign}^\prime_x(\xi - v) \gamma(\xi) d\xi \right), \]
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\[ \text{sign}_e'(\xi - v) \nabla_y g(v) = \text{sign}_e'(\xi - v) \nabla_y \left( \int_{\xi}^v \gamma(\sigma) \ d\sigma \right) = \nabla_y \left( \int_{\xi}^v \text{sign}_e'(\xi - \sigma) \gamma(\sigma) \ d\sigma \right), \]

so that

\[ \text{sign}_e'(u - v) (\nabla_x g(u) \cdot \nabla_y g(v)) = \nabla^2_{xy} \left( \int_{\xi}^u \int_{\xi}^v \text{sign}_e'(\xi - \sigma) \gamma(\xi) \gamma(\sigma) \ d\sigma \ d\xi \right). \]

Using the diagonal structure of \( h \) and the fact that \( \phi = \phi(t, x, s, y) \) vanishes for \( x \in \partial B \) or \( y \in \partial B \), we have

\[
\int_{\Omega_T^2} \text{sign}_e'(u - v) (h(x) \nabla_x g(u)) \cdot (h(y) \nabla_y g(v)) \varphi \ dX \\
= \int_{\Omega_T^2} \text{sign}_e'(u - v) (h(x) h(y)) (\nabla_x g(u) \cdot \nabla_y g(v)) \varphi \ dX \\
= \int_{\Omega_T^2} \left( \int_{\xi}^u \int_{\xi}^v \text{sign}_e'(\xi - \sigma) \gamma(\xi) \gamma(\sigma) \ d\sigma \ d\xi \right) \nabla^2_{xy} (h(x) h(y) \varphi) \ dX.
\]

Then, (2.10) can be written

\[
\tau_\epsilon(\varphi) + i_\epsilon(\varphi) - f_\epsilon(\varphi) - B_\epsilon(\varphi) \\
-\int_{\Omega_T^2} \left[ \left( \int_{\xi}^u \text{sign}_e'(\xi - v) \gamma^2(\xi) \ d\xi \right) \nabla^2_{xy} (k(x) \varphi) \\
+ \left( \int_{\xi}^u \text{sign}_e'(u - \xi) \gamma^2(\xi) \ d\xi \right) \nabla^2_{xy} ((k(y) \varphi) \right] \ dX \\
\geq \pm \int_{\Omega_T^2} \left( \int_{\xi}^u \int_{\xi}^v \text{sign}_e'(\xi - \sigma) \gamma(\xi) \gamma(\sigma) \ d\sigma \ d\xi \right) \nabla^2_{xy} (2h(x) h(y) \varphi) \ dX.
\]

Next, we observe that

\[
\lim_{\epsilon \to 0} \int_{\xi}^u \text{sign}_e'(\xi - \sigma) \gamma(\sigma) \ d\sigma = - \text{sign} (\xi - v) \gamma(\xi),
\]

and consequently

\[
\lim_{\epsilon \to 0} \int_{\xi}^u \text{sign}_e'(\xi - v) \gamma^2(\xi) \ d\xi = \Gamma(u, v),
\]

\[
\lim_{\epsilon \to 0} \int_{\xi}^u \text{sign}_e'(u - \xi) \gamma^2(\xi) \ d\xi = \Gamma(u, v),
\]

\[
\lim_{\epsilon \to 0} \int_{\xi}^u \int_{\xi}^v \text{sign}_e'(\xi - \sigma) \gamma(\sigma) \gamma(\xi) \ d\sigma \ d\xi = -\Gamma(u, v),
\]

where

\[ \Gamma(u, v) := \int_{\xi}^u \text{sign} (\xi - v) \gamma^2(\xi) \ d\xi. \]
Choosing the plus sign in the above inequality (2.11) and using the definition of $h$, we get

$$
\tau(\varphi) + i(\varphi) - f(\varphi) - B(\varphi) \\
\geq - \int_{\Omega_T^\sigma} \Gamma(u,v) \nabla_{xy}^2 ((k(x) - 2h(x)h(y) + k(y)) \varphi) \, dX \\
= - \int_{\Omega_T^\sigma} \Gamma(u,v) \nabla_{xy}^2 \left( (h(x) - h(y))^2 \varphi \right) \, dX =: -C(\varphi),
$$

with

$$
\begin{align*}
\tau(\varphi) &= \int_{\Omega_T^\sigma} |u-v| (\partial_t + \partial_s) \varphi \, dX, \\
i(\varphi) &= \int_{B \times \Omega_T} |u_0-v| \varphi(0,x,s,y) \, dxdsdy \\
&\quad + \int_{\Omega_T \times B} |u-v_0| \varphi(t,x,0,y) \, dt dx dy, \\
f(\varphi) &= \int_{B \times \Omega_T} |u(T,x) - v(s,y)| \varphi(T,x,s,y) \, dxdsdy \\
&\quad + \int_{\Omega_T \times B} |u(t,x) - v(T,y)| \varphi(t,x,T,y) \, dt dx dy, \\
B(\varphi) &= \int_{\Omega_T^\sigma} \text{sign}(u-v) [k(x)\nabla_x G(u) - k(y)\nabla_y G(v)] (\nabla_x + \nabla_y) \varphi \, dX.
\end{align*}
$$

The inequality (2.12) implies the bound (2.8) as we shall now demonstrate. We choose a suitable test function $\varphi$. To this end let $\omega_\sigma(\xi)$ be a standard\footnote{We let $\omega_\sigma(\xi) = \frac{1}{\sigma} \omega(\frac{\xi}{\sigma})$ where $\omega: \mathbb{R} \to [0, \infty)$, $\omega \in C^\infty$, $\text{supp}(\omega) = [-1,1]$, $\int_{\mathbb{R}} \omega(x) dx = 1$.} non-negative smooth mollifier with support inside $[-\sigma, \sigma]$. Recall that

$$
|\omega_\sigma^{(k)}(\xi)| = O\left(\sigma^{-(k+1)}\right) \quad \text{and} \quad \int_{\mathbb{R}} |\xi|^n |\omega_\sigma^{(k)}(\xi)| \, d\xi = O\left(\sigma^{n-k}\right),
$$

for $k \geq 0$ and $n \geq 0$. We then define

$$
B_\sigma = \{ x \in B \mid \text{dist}(x, \partial B) \geq \sigma \}.
$$

Let $\chi_\sigma$ is a $C^\infty$ function such that $\chi_\sigma(\xi) = 0$ for $\xi \notin B$, $\chi_\sigma(\xi) = 1$ for $\xi \in B_\sigma$, $0 \leq \chi_\sigma(\xi) \leq 1$. Due to the smoothness of $\partial B$, we have that $|\nabla \chi_\sigma(\xi)| \leq C/\sigma$. Furthermore, the smoothness of $\partial B$ implies that we can choose $\chi_\sigma$ such that for every smooth vector field $F$

$$
\lim_{\sigma \to 0} \int_B F \cdot \nabla \chi_\sigma \, dx = \int_{\partial B} F \cdot \nu \, ds.
$$

Then we choose $\varphi_\varepsilon$ as

$$
\varphi_\varepsilon(t,x,s,y) = \omega_{\varepsilon_0}(t-s)W_{\varepsilon_1}(x-y)\chi_{\varepsilon_2}(x)\chi_{\varepsilon_3}(y),
$$

where

$$
\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2), \quad W_{\varepsilon_1}(x-y) = \omega_{\varepsilon_1}(x_1-y_1) \cdots \omega_{\varepsilon_1}(x_d-y_d).
$$

With this choice

$$
(\partial_t + \partial_s) \varphi_\varepsilon = 0,
$$
\((\nabla_x + \nabla_y) \varphi_{\varepsilon} = \omega_{\varepsilon_0}(t-s)W_{\varepsilon_1}(x-y)(\nabla_x \chi_{\varepsilon_2}(x)\chi_{\varepsilon_2}(y) + \chi_{\varepsilon_2}(x)\nabla_y \chi_{\varepsilon_2}(y)),\)

it is straightforward to show that

\[
\lim_{\varepsilon \to 0} i(\varphi_{\varepsilon}) = \int_B |u_0(x) - v_0(x)| \, dx,
\]

and

\[
\lim_{\varepsilon \to 0} f(\varphi_{\varepsilon}) = \int_B |u(T, x) - v(T, x)| \, dx.
\]

Next we claim that

\[
\lim_{\varepsilon \to 0} \left( \lim_{\varepsilon_0 \to 0} |B(\varphi_{\varepsilon})| \right) = 0.
\]

To prove this claim first observe that

\[
B(\varphi_{\varepsilon}) = \int_{\Omega_t^2} \text{sign} \, (u - v) \, \left( k(x)\nabla_x G(u) - k(y)\nabla_y G(v) \right) \times \omega_{\varepsilon_0}(t-s)W_{\varepsilon_1}(x-y)(\nabla_x \chi_{\varepsilon_2}(x)\chi_{\varepsilon_2}(y) + \chi_{\varepsilon_2}(x)\nabla_y \chi_{\varepsilon_2}(y)) \, dX,
\]

and hence

\[
\lim_{(\varepsilon_0, \varepsilon_1) \to 0} |B(\varphi_{\varepsilon})| \leq 2 \int_{\Omega_T} |(k(x)\nabla_x G(u)) \cdot \nabla_x \chi_{\varepsilon_2}(x)| \, dxdt
+ 2 \int_{\Omega_T} |(k(y)\nabla_y G(v)) \cdot \nabla_y \chi_{\varepsilon_2}(y)| \, dyds.
\]

Since both \(u\) and \(v\) satisfy the Neumann boundary conditions (see (2.14))

\[
\lim_{\varepsilon_2 \to 0} \int_B |(k(x)\nabla_x G(u)) \cdot \nabla_x \chi_{\varepsilon_2}(x)| \, dx = 0 \text{ a.e. } t,
\]
equation (2.18) holds. Next we tackle the troublesome term, cf. (2.12),

\[
C(\varphi_{\varepsilon}) := \int_{\Omega_t^2} \Gamma(u, v)\nabla^2_{xy} \left( (h(x) - h(y))^2 \varphi_{\varepsilon} \right) \, dX.
\]

We start by computing

\[
\nabla^2_{xy} \left( (h(x) - h(y))^2 \varphi_{\varepsilon} \right) = \nabla^2_{xy} \begin{pmatrix}
(h_1(x) - h_1(y))^2 \varphi_{\varepsilon} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & (h_d(x) - h_d(y))^2 \varphi_{\varepsilon}
\end{pmatrix}
= \text{div}_y \begin{pmatrix}
\partial_{x_1}((h_1(x) - h_1(y))^2 \varphi_{\varepsilon}) \\
\vdots \\
\partial_{x_d}((h_d(x) - h_d(y))^2 \varphi_{\varepsilon})
\end{pmatrix}
= \sum_{i=1}^d \partial^2_{x_i y_i}((h_i(x) - h_i(y))^2 \varphi_{\varepsilon})
= -2 \sum_{i=1}^d \partial_{x_i} h_i(x) \partial_{y_i} h_i(y) \varphi_{\varepsilon}
+ 2 \sum_{i=1}^d (h_i(x) - h_i(y)) \partial_{x_i} h_i(x) \partial_{y_i} \varphi_{\varepsilon}
\]
Below we shall repeatedly use that since \( v \) is an entropy solution, \( v \) has a modulus of continuity, i.e., there is a continuous function \( \nu: [0, 1] \to [0, \infty) \) such that \( \nu(0) = 0 \) and
\[
\sup_{t \in [0,T]} \left( \sup_{|\sigma| \leq \sigma} \int_{|x| < \varepsilon_1} |v(t, x) - v(t, x + \sigma)| \, dx \right) \leq \nu(\sigma).
\]

With our choice of test function it readily follows that
\[
\lim_{\varepsilon_0 \to 0} |C_1(\varphi_\varepsilon)| \leq C \int_{\Omega_T} \int_B |u - v| W_{\varepsilon_1}(x - y) \, dy \, dx \, dt.
\]
\[
\leq C \int_{\Omega_T} |u(t, x) - v(t, x)| \int_{|y - x| < \varepsilon_1} W_{\varepsilon_1}(x - y) \, dy \, dx \, dt
\]
\[
+ C \int_{\Omega_T} \int_{|y - x| < \varepsilon_1} |v(t, x) - v(t, y)| W_{\varepsilon_1}(x - y) \, dy \, dx \, dt.
\]

Thus
\[
(2.19) \quad \lim_{\varepsilon_0 \to 0} |C_1(\varphi_\varepsilon)| \leq C \left( \int_{\Omega_T} |u(t, x) - v(t, x)| \, dx \, dt + \nu(\varepsilon_1) \right).
\]

To estimate \( C_2(\varphi_\varepsilon) \) we observe that
\[
\partial_y \varphi_\varepsilon = \omega_{\varepsilon_0}(t - s) \left[ -\partial_y W_{\varepsilon_1}(x - y) \chi_{\varepsilon_2}(x) \chi_{\varepsilon_2}(y) + W_{\varepsilon_1}(x - y) \chi_{\varepsilon_2}(x) \partial_y \chi_{\varepsilon_2}(y) \right],
\]
and we split $C_2(\varphi_\varepsilon) = C_{2,1}(\varphi_\varepsilon) + C_{2,2}(\varphi_\varepsilon)$ accordingly. Then

$$\lim_{\varepsilon_0 \to 0} |C_{2,1}(\varphi_\varepsilon)|$$

$$\leq \sum_{i=1}^{d} \int_{\Omega_T} \int_{B} |u(t, x) - v(t, y)| |h_i(x) - h_i(y)|$$

$$\times |\partial_x h_i(x)| |\partial_v W_{\varepsilon_1}(x - y)\chi_{\varepsilon_2}(x)| dydxdt$$

$$\leq C \int_{\Omega_T} \int_{B} |u(t, x) - v(t, y)| |x - y| |\nabla_y W_{\varepsilon_1}(x - y)| dydxdt$$

$$\leq C \int_{\Omega_T} |u(t, x) - v(t, x)| \int_{B} |x - y| |\nabla_y W_{\varepsilon_1}(x - y)| dydxdt$$

$$+ C \int_{\Omega_T} \int_{|y - x| < \varepsilon_1} |v(t, x) - v(t, y)| |x - y| |\nabla_y W_{\varepsilon_1}(x - y)| dydxdt$$

$$\leq C \int_{\Omega_T} |u(t, x) - v(t, x)| dxdt + C\nu(\varepsilon_1).$$

Similarly we find that

$$\lim_{\varepsilon_0 \to 0} |C_{2,2}(\varphi_\varepsilon)|$$

$$\leq \sum_{i=1}^{d} \int_{\Omega_T} \int_{B} |u(t, x) - v(t, y)| |h_i(x) - h_i(y)|$$

$$\times |\partial_x h_i(x)| W_{\varepsilon_1}(x - y)|\partial_y \chi_{\varepsilon_2}(y)| dydxdt$$

$$\leq C \int_{\Omega_T} |u(t, x) - v(t, x)| \frac{1}{\varepsilon_2} \int_{|y - x| < \varepsilon_1} |x - y| W_{\varepsilon_1}(x - y) dydxdt$$

$$+ \frac{C}{\varepsilon_2} \int_{\Omega_T} \int_{|y - x| < \varepsilon_1} |v(t, x) - v(t, y)| |x - y| W_{\varepsilon_1}(x - y) dydxdt$$

$$\leq C \frac{\varepsilon_1}{\varepsilon_2} \left( \int_{\Omega_T} |u(t, x) - v(t, x)| dxdt + \nu(\varepsilon_1) \right).$$

Therefore we have that

$$\lim_{\varepsilon_0 \to 0} |C_2(\varphi_\varepsilon)| \leq C \left( 1 + \frac{\varepsilon_1}{\varepsilon_2} \right) \left( \int_{\Omega_T} |u(t, x) - v(t, x)| dxdt + \nu(\varepsilon_1) \right).$$

The term $C_3(\varphi_\varepsilon)$ can similarly be bounded as

$$\lim_{\varepsilon_0 \to 0} |C_3(\varphi_\varepsilon)| \leq C \left( 1 + \frac{\varepsilon_1}{\varepsilon_2} \right) \left( \int_{\Omega_T} |u(t, x) - v(t, x)| dxdt + \nu(\varepsilon_1) \right).$$

To estimate the final term $C_4(\varphi_\varepsilon)$ we first note that

$$\partial_{x,y}^2 \varphi_\varepsilon = \omega_{\varepsilon_0}(t - s) \left[ - \partial_{x,y}^2 W_{\varepsilon_1}(x - y)\chi_{\varepsilon_2}(x)\chi_{\varepsilon_2}(y) + \partial_x W_{\varepsilon_1}(x - y)\chi_{\varepsilon_2}(x)\partial_y \chi_{\varepsilon_2}(y) - \partial_y W_{\varepsilon_1}(x - y)\partial_x \chi_{\varepsilon_2}(x)\chi_{\varepsilon_2}(y) + W_{\varepsilon_1}(x - y)\partial_x \chi_{\varepsilon_2}(x)\partial_y \chi_{\varepsilon_2}(y) \right].$$
and we split \( C_4(\varphi_\varepsilon) = C_{4,1}(\varphi_\varepsilon) + C_{4,2}(\varphi_\varepsilon) + C_{4,3}(\varphi_\varepsilon) + C_{4,4}(\varphi_\varepsilon) \) accordingly. These terms are estimated as follows:

\[
\lim_{\varepsilon_0 \to 0} |C_{4,1}(\varphi_\varepsilon)| \\
\leq \sum_{i=1}^{d} \int_{\Omega_T} \int_{B} |u - v| (h_i(x) - h_i(y))^2 |\partial_{x, y}^2 W_{<1}(x - y)| \chi_{\varepsilon_2}(x) \chi_{\varepsilon_2}(y) \, dy \, dx \, dt \\
\leq C \sum_{i=1}^{d} \int_{\Omega_T} \int_{B} |u - v| |x - y|^2 |\partial_{x, y}^2 W_{<1}(x - y)| \, dy \, dx \, dt \\
\leq C \sum_{i=1}^{d} \int_{\Omega_T} |u(t, x) - v(t, x)| \int_{|y - x| < \varepsilon_1} |x - y|^2 |\partial_{x, y}^2 W_{<1}(x - y)| \, dy \, dx \, dt \\
+ \sum_{i=1}^{d} \int_{\Omega_T} \int_{|y - x| < \varepsilon_1} |v(t, x) - v(t, y)| |x - y|^2 |\partial_{x, y}^2 W_{<1}(x - y)| \, dy \, dx \, dt \\
\leq C \int_{\Omega_T} |u(t, x) - v(t, x)| \, dx \, dt + C \nu(\varepsilon_1),
\]

and

\[
\lim_{\varepsilon_0 \to 0} |C_{4,2}(\varphi_\varepsilon)| \\
\leq \sum_{i=1}^{d} \int_{\Omega_T} \int_{B} |u - v| (h_i(x) - h_i(y))^2 |\partial_{x} W_{<1}(x - y)| \chi_{\varepsilon_2}(x) \chi_{\varepsilon_2}(y) \, dy \, dx \, dt \\
\leq C \sum_{i=1}^{d} \int_{\Omega_T} |u(t, x) - v(t, x)| \int_{|y - x| < \varepsilon_1} |x - y|^2 |\partial_{x} W_{<1}(x - y)| \, dy \, dx \, dt \\
+ \sum_{i=1}^{d} \int_{\Omega_T} |v(t, x) - v(t, y)| \int_{|y - x| < \varepsilon_1} |x - y|^2 |\partial_{x} W_{<1}(x - y)| \, dy \, dx \, dt \\
\leq C \varepsilon_1 \left( \int_{\Omega_T} |u(t, x) - v(t, x)| \, dx \, dt + \nu(\varepsilon_1) \right).
\]

The term \( C_{4,3}(\varphi_\varepsilon) \) is similar,

\[
\lim_{\varepsilon_0 \to 0} |C_{4,3}(\varphi_\varepsilon)| \leq C \varepsilon_1 \varepsilon_2 \left( \int_{\Omega_T} |u(t, x) - v(t, x)| \, dx \, dt + \nu(\varepsilon_1) \right).
\]

It remains the term \( C_{4,4}(\varphi_\varepsilon) \),

\[
\lim_{\varepsilon_0 \to 0} |C_{4,4}(\varphi_\varepsilon)| \\
\leq \sum_{i=1}^{d} \int_{\Omega_T} \int_{B} |u - v| (h_i(x) - h_i(y))^2 W_{<1}(x - y) |\partial_{x, y} W_{<1}(x - y)| \, dy \, dx \, dt \\
\leq C \int_{\Omega_T} \int_{B} |u - v| |x - y|^2 W_{<1}(x - y) \frac{1}{\varepsilon_2} \, dy \, dx \, dt \\
\leq C \int_{\Omega_T} |u(t, x) - v(t, x)| \frac{1}{\varepsilon_2} \int_{|y - x| < \varepsilon_1} |x - y|^2 W_{<1}(x - y) \, dy \, dx \, dt.
\]
\[ + C \int_{\Omega_T} |v(t, x) - v(t, y)| \frac{1}{\varepsilon_2} \int_{|y-x|<\varepsilon_1} |x-y|^2 W_{\varepsilon_1}(x-y) \, dy \, dx \, dt \]

\[ \leq C \frac{\varepsilon_2^2}{\varepsilon_2} \left( \int_{\Omega_T} |u(t, x) - v(t, x)| \, dx \, dt + \nu(\varepsilon_1) \right). \]

Then we can conclude that

\[ (2.22) \quad \lim_{\varepsilon_0 \to 0} |C_4(\varphi_\varepsilon)| \leq C \left( 1 + \frac{\varepsilon_1}{\varepsilon_2} + \left( \frac{\varepsilon_1}{\varepsilon_2} \right)^2 \right) \left( \int_{\Omega_T} |u(t, x) - v(t, x)| \, dx \, dt + \nu(\varepsilon_1) \right). \]

Using (2.19)–(2.22) we find

\[ (2.23) \quad \lim_{\varepsilon_0 \to 0} |C(\varphi_\varepsilon)| \leq C \left( 1 + \frac{\varepsilon_1}{\varepsilon_2} + \left( \frac{\varepsilon_1}{\varepsilon_2} \right)^2 \right) \left( \int_{\Omega_T} |u(t, x) - v(t, x)| \, dx \, dt + \nu(\varepsilon_1) \right), \]

and hence we can send \( \varepsilon_1 \) and \( \varepsilon_2 \) to zero in such a manner that the quotient \( \varepsilon_1/\varepsilon_2 \) remains finite to conclude that

\[ (2.24) \quad \lim_{\varepsilon \to 0} |C(\varphi_\varepsilon)| \leq C \int_0^T \int_B |u(t, x) - v(t, x)| \, dx \, dt. \]

Taking the limit \( \varepsilon \to 0 \) in (2.12), using (2.16), (2.17), (2.18) and (2.24) we get

\[ \int_B |u(T, x) - v(T, x)| \, dx \leq \int_B |u_0(x) - v_0(x)| \, dx + C \int_0^T \int_B |u(t, x) - v(t, x)| \, dx \, dt. \]

The inequality (2.8) then follows by Gronwall’s lemma. \( \square \)

**Remark 2.4.** If \( u \) is an entropy solution, then \( u(t, x) \geq y \) a.e. \( (t, x) \) for some finite constant \( y \). Since \( \nabla G(u) = \nabla (G(u) - G(y)) \), there is no loss of generality in assuming \( G(y) = 0 \). Then define

\[ \Psi(u) = \int_{\mathbb{R}} u \, G(\sigma) \, d\sigma. \]

The \( \Psi(u(t, x)) \geq 0 \) for almost all \( (t, x) \), and

\[ \frac{d}{dt} \int_B \Psi(u(t, x)) \, dx = - \int_B (k(x) \nabla G(u(t, x))) \cdot \nabla G(u(t, x)) \, dx \leq 0. \]

Since \( k_i \geq k > 0 \) this means that

\[ \int_{\Omega_T} |\nabla G(u(t, x))|^2 \, dx \, dt \leq \frac{1}{k} \int_B \Psi(u_0(x)) - \Psi(u(T, x)) \, dx \]

\[ \leq \frac{1}{k} \int_B \Psi(u_0(x)) \, dx < \infty. \]

We then claim that an entropy solution satisfies the following regularity estimate

\[ (2.25) \quad \int_0^T |G(u(t, x)) - G(u(t, y))| \, dt \leq CT \sqrt{|x - y|}, \]

for some constant \( C \) which depends only on \( u_0 \) and \( T \), and for almost all \( x \) and \( y \) in \( B \). To substantiate this claim we calculate

\[ \int_0^T |G(u(t, x)) - G(u(t, y))| \, dt \]

\[ \leq \int_0^T \left| \int_0^1 \nabla G(u(t, \theta y + (1 - \theta)x)) \cdot (y - x) \, d\theta \right| \, dt. \]
Proof.

If we multiply with (3.3c) \( B \) Integrating over (3.3e) \( d \) (3.5) \( 14 \) G. M. COCLITE, H. HOLDEN, AND N. H. RISEBRO

Lemma 3.2. We have that

\[ \frac{d}{dt} \int_B u^\mu(t,x) \, dx = 0. \]

3. Existence of a solution

Given \( \mu > 0 \), let \( u^\mu \) be the unique classical solution of the initial-boundary value problem [17, Theorem 1.7.8]

\[
\begin{cases}
\partial_t u^\mu = \text{div} (k(x) \nabla G(u^\nu)) + \mu \Delta u^\mu, & t > 0, \ x \in B, \\
(k(x) \nabla G(u^\mu(t,x)) + \mu \nabla u^\mu(t,x)) \cdot \nu = 0, & t > 0, \ x \in \partial B, \\
u^\mu(0,x) = u_0(x), & x \in B.
\end{cases}
\]

The main result of this section is the following.

Theorem 3.1 (Existence). Assume that (H.1), (H.2), (H.3), (H.4) hold and that every \( k_i \) depends only on \( x_i \). Then there exists an entropy solution \( u \) in the sense of Definition 2.2 to the initial-boundary value problem (2.1). In particular, we have, as \( \mu \to 0 \), that

\[ u^\mu \to u \quad \text{a.e. and in } L^p(\Omega_T) \text{ for every } T > 0 \text{ and } 1 \leq p < \infty. \]

First observe that by the Neumann boundary conditions, the solution operator is conservative, i.e.,

\[ \frac{d}{dt} \int_B u^\mu(t,x) \, dx = 0. \]

Lemma 3.2. We have that

\[ \| u^\mu(T, \cdot) \|^2_{L^2(B)} + \int_{\Omega_T} 2G(u^\mu)|\nabla u^\mu|^2 \, dx \, dt \]

(3.3a) \[ + 2\mu \int_{\Omega_T} |\nabla u^\mu|^2 \, dx \, dt = \| u_0 \|^2_{L^2(B)}, \]

(3.3b) \[ \inf_{x \in B} u_0(\bar{x}) \leq u^\mu(t,x) \leq \sup_{x \in B} u_0(\bar{x}), \quad (t,x) \in \Omega \]

(3.3c) \[ \| u^\mu(t, \cdot) \|_{L^1(B)} \leq \| u_0 \|_{L^1(B)}, \]

(3.3d) \[ \| \nabla G(u^\mu) \|_{L^2(\Omega_T)} \leq \frac{\| G(u_0) \|_{L^\infty(B)} \| u_0 \|_{L^1(B)}}{\sqrt{\min_{i,x \in B} k_i(x)}}. \]

(3.3e) \[ \| \partial_k u^\mu(t, \cdot) \|_{L^1(B)} \leq C \| u_0 \|_{W^{2,1}(B)}, \]

(3.3f) \[ \| \text{div} (k \nabla G(u^\mu)) (t, \cdot) \|_{L^1(B)} \leq C \| u_0 \|_{W^{2,1}(B)}, \]

for every \( \mu > 0 \) and \( t \in [0, \infty) \).

Proof. If we multiply with \( \eta'(u^\mu) \) where \( \eta \) is a smooth convex function (entropy), using that \( u^\mu \) is a classical smooth solution, we get

\[
\partial_t \eta(u^\mu) = \eta'(u^\mu) \text{div} (k(x) \nabla G(u^\mu)) + \mu \eta'(u^\mu) \Delta u^\mu
\]

(3.4)

\[ = \eta''(u^\mu) (k(x) \nabla G(u^\mu) + \mu \nabla u^\mu) - \eta''(u^\mu) G'(u^\mu) \nabla u^\mu |\nabla u^\mu|^2 \]

\[ - \mu \eta''(u^\mu) |\nabla u^\mu|^2. \]

Integrating over \( B \) and using the boundary conditions we get

\[ \frac{d}{dt} \int_B \eta(u^\mu) \, dx + \int_B \eta''(u^\mu) G'(u^\mu) |\nabla u^\mu|^2 \, dx + \mu \int_B \eta''(u^\mu) |\nabla u^\mu|^2 \, dx = 0. \]
Choosing $\eta(u) = \frac{1}{2} u^2$ we get (3.3a). By an approximation argument, we can choose $\eta(u) = (u - c)\pm$ where $c$ is a constant, to get
$$\int_B (u^\mu(t,x) - c)\pm \, dx \leq \int_B (u_0 - c)\pm \, dx.$$ 
This implies the bound (3.3b). Setting $c = 0$ and adding the $(u)^+$ and $(u)^-$ inequalities we get (3.3c).

Without loss of generality, we can assume that $G(0) = 0$. If necessary by an approximation argument, we can use the entropy $\eta(u) = \int u_0 G(v) \, dv$.

Note that $\eta(u) \geq 0$. Since $\eta'(u) = G(u)$ we get the bound
$$\frac{d}{dt} \int_B \eta(u^\mu) \, dx + \int_B |h(x) \nabla G(u^\mu)|^2 \, dx \leq 0.$$ 
This means that $\nabla G(u^\mu)$ is uniformly bounded in $L^2(\Omega_T)$ and (3.3d) holds.

Since, from (3.1),
$$(k(x) \nabla \partial_t G(u^\mu(t,x)) + \mu \nabla \partial_t u^\mu(t,x)) \cdot \nu = 0, \quad t > 0, \ x \in \partial B,$$
differentiating the equation in (3.1) we gain
$$\frac{d}{dt} \int_B |\partial_t u^\mu| \, dx \leq \int_B \partial_t^2 u^\mu \text{sign} (\partial_t u^\mu) \, dx$$
$$= \int_B \text{div} ((k(x) \nabla \partial_t G(u^\mu)) + \mu \nabla \partial_t u^\mu) \text{sign} (\partial_t u^\mu) \, dx$$
$$= - \int_B \left( G'(u^\mu)(k(x) \nabla \partial_t u^\mu) \cdot \nabla \partial_t u^\mu + \mu |\nabla \partial_t u^\mu|^2 \right) \text{sign} (\partial_t u^\mu) \, dx$$
$$\leq 0$$
$$- \int_B G''(u^\mu) \partial_t u^\mu (k(x) \nabla u^\mu) \cdot \nabla \partial_t u^\mu \text{sign} (\partial_t u^\mu) \, dx$$
$$= 0$$
$$+ \int_{\partial B} (k(x) \nabla \partial_t G(u^\mu) + \mu \nabla \partial_t u^\mu) \cdot \nu \text{sign} (\partial_t u^\mu) \, ds.$$ 
The middle term disappears as the integrand contains a term of the form $\psi \text{sign}'(\psi)$ (with $\psi = \partial_t u^\mu$). Integrating over $(0,t)$ we get (3.3e) and from the equation (3.3f).

**Lemma 3.3.** For every $\sigma > 0$ there exists a constant $C^* > 0$ independent of $\mu$ such that
$$\|\partial_x u^\mu(t, \cdot)\|_{L^1(B_\sigma)} \leq \|\partial_x u_0\|_{L^1(B)} e^{C^*t},$$
for every $\mu > 0$, $t \geq 0$, and $j \in \{1, \ldots, d\}$, where $B_\sigma$ is defined in (2.13).

**Proof.** Define
$$v^\mu_i = \partial_x_i u^\mu.$$
The equation (3.1) reads
\[ \partial_t \mu = \sum_{i=1}^{d} \partial_{x_i} \left( k_i(x_i) G'(u^\mu) v_i^\mu \right) + \mu \Delta u^\mu. \]

Differentiating with respect to \( x_j \) we get
\[ \partial_{x_j} u^\mu = \sum_{i=1}^{d} \partial_{x_i} \left( k_i(x_i) G'(u^\mu) \partial_{x_j} v_i^\mu \right) + \sum_{i=1}^{d} \partial_{x_i} \left( k_i(x_i) G''(u^\mu) v_i^\mu v_j^\mu \right) + \mu \Delta e_j. \]

Let \( \chi \) be a cut-off function such that
\[ \chi \in C^\infty(\mathbb{R}^d), \quad 0 \leq \chi \leq 1, \quad \chi(x) = \begin{cases} 1, & \text{if } x \in B_\sigma, \\ 0, & \text{if } x \notin B_\sigma, \end{cases} \]
\[ \left| \partial^2_{x_i x_j} \chi \right| \leq c(\sigma) \chi, \quad \left| \partial_{x_i} \chi \right| \leq c(\sigma) \chi. \]

Using the facts
\[ \chi|_{\partial B} = \partial_{x_i} \chi|_{\partial B} = 0, \quad i \neq j \Rightarrow \partial_{x_i} k_i = 0, \]
we have that
\[ \frac{d}{dt} \int_{B} |v_j^\mu| \chi dx = \int_{B} \partial_t v_j^\mu \text{sign}(v_j^\mu) \chi dx = \sum_{i=1}^{d} \int_{B} \partial_{x_i} \left( k_i(x_i) G'(u^\mu) \partial_{x_j} v_i^\mu \right) \text{sign}(v_j^\mu) \chi dx \]
\[ + \sum_{i=1}^{d} \int_{B} \partial_{x_i} \left( k_i(x_i) G''(u^\mu) v_i^\mu v_j^\mu \right) \text{sign}(v_j^\mu) \chi dx \]
\[ + \int_{B} \partial_{x_j} \left( (\partial_{x_i} k_i(x)) G'(u^\mu) v_j^\mu \right) \text{sign}(v_j^\mu) \chi dx \]
\[ + \mu \int_{B} \Delta u^\mu \text{sign}(v_j^\mu) \chi dx \]
\[ = - \sum_{i=1}^{d} \int_{B} k_i(x_i) G'(u^\mu) \left( \partial_{x_i} v_i^\mu \right)^2 \text{sign}(v_j^\mu) \chi dx \]
\[ \leq 0 \]
\[ - \sum_{i=1}^{d} \int_{B} k_i(x_i) G'(u^\mu) \partial_{x_i} |v_j^\mu| \partial_{x_j} \chi dx \]
\[ - \sum_{i=1}^{d} \int_{B} \partial_{x_i} k_i(x_i) G''(u^\mu) v_i^\mu v_j^\mu \left( \partial_{x_i} v_j^\mu \right) \text{sign}(v_j^\mu) \chi dx \]
\[ = 0 \]
\[ - \sum_{i=1}^{d} \int_{B} k_i(x_i) G''(u^\mu) v_i^\mu |v_j^\mu| \partial_{x_i} \chi dx \]
The function $u$ can find a sequence $\{u_j\}$ and using a diagonal argument we can find a subsequence $\{u_j\}$. Consider the sequence $\{u_j\}$. In light of [4, Theorem 2.1], we have that (3.7) holds. The dominated convergence theorem, Lemmas 3.2, 3.3, (3.4), and (3.7) guarantee that (2.2), (2.3), (2.5), (2.6) hold. Regarding (2.4), we observe that thanks to (3.3e) and (2.4) holds.

$$\int_B (\partial_{x_i} k_j(x)) G'(u^\mu)v_j^\mu(\partial_{x_i} v_j^\mu)\text{sign}'(v_j^\mu)\chi dx$$

$$= 0$$

$$- \int_B (\partial_{x_i} k_j(x)) G'(u^\mu)v_j^\mu(\partial_{x_i} v_j^\mu)\chi dx$$

$$- \mu \int_B |\nabla u_j^\mu|^2\text{sign}'(v_j^\mu)\chi dx + \mu \int_B |v_j^\mu| \Delta \chi dx$$

$$\leq - \sum_{i=1}^d \int_B k_i(x_i) \partial_{x_i} \left( G'(u^\mu)|v_j^\mu| \right) \chi dx$$

$$- \int_B (\partial_{x_i} k_j(x)) G'(u^\mu)v_j^\mu(\partial_{x_i} v_j^\mu)\chi dx + \mu \int_B |v_j^\mu| \Delta \chi dx$$

$$= \sum_{i=1}^d \int_B (\partial_{x_i} k_i(x_i)) G'(u^\mu)|v_j^\mu| \partial_{x_i} \chi dx$$

$$+ \sum_{i=1}^d \int_B k_i(x_i) G'(u^\mu)|v_j^\mu| \partial_{x_i} \chi dx$$

$$- \int_B (\partial_{x_i} k_j(x)) G'(u^\mu)v_j^\mu(\partial_{x_i} v_j^\mu)\chi dx + \mu \int_B |v_j^\mu| \Delta \chi dx$$

$$\leq c(\sigma) \int_B |v_j^\mu| \chi dx.$$
Finally, we can improve the convergence along a subsequence in (3.7) to the one along all the family in (3.2) due to the uniqueness of the entropy solutions. □

4. CONVERGENCE OF A DIFFERENCE SCHEME IN ONE SPACE DIMENSION

We now consider the problem in one space dimension with $B = (0, 1)$. Thus

$$
\partial_t u = \left( k(x) \partial_x G(u) \right)_x, \quad u|_{t=0} = u_0,
$$

with Neumann boundary conditions $\left( k(x) \partial_x G(u) \right)|_{x=0,1} = 0$.

Let $\Delta x = 1/(N+1)$ for some positive integer $N$. We use the notation

$$
\Delta \pm a_j = \pm (a_{j+1} - a_j),
$$

and $x_j = (j + 1/2)\Delta x$, $t_n = n\Delta t$, and $k_{j+1/2} = k(x_{j+1/2})$ for (small) positive numbers $\Delta x$ and $\Delta t$. With the shift operator

$$
(S^\pm a)_j = a_{j\pm 1},
$$

we can write

$$
S^\pm \Delta x = \Delta \pm.
$$

We use the common convention that $u^n_j$ is an approximation to $u(t_n, x_j)$. Let $u^n_j$, $j = 0, \ldots, N$, $n \geq 0$ be the solution to the following system of equations

(4.1) \hspace{1em} u^{n+1}_j - \mu \Delta_+ \left( k_{j-1/2} \Delta_- G \left( u^{n+1}_j \right) \right) = u^n_j, \quad j = 0, \ldots, N,

with the boundary conditions that $\Delta_- u_0^{n+1} = \Delta_+ u_N^n = 0$. Here $\mu = \Delta t/\Delta x^2$ (which is only assumed to be bounded from below). As to the initial condition we define

(4.2) \hspace{1em} u^n_j = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x) \, dx, \quad j = 0, \ldots, N.

For later use we also define the function $u_{\Delta t} : [0, \infty) \times B \to \mathbb{R}$ as

(4.3) \hspace{1em} u_{\Delta t}(t, x) = u^n_j \text{ for } (t, x) \in [t_n, t_{n+1}) \times [x_{j-1/2}, x_{j+1/2}).

Regarding the solvability of (4.1), we have the following result.

**Lemma 4.1.** For any given $u^n = (u^n_0, \ldots, u^n_N) \in \mathbb{R}^{N+1}$, the equation (4.1) has a unique solution $u^{n+1} \in \mathbb{R}^{N+1}$.

**Proof.** Let $\alpha = (\alpha_0, \ldots, \alpha_N) \in \mathbb{R}^{N+1}$ be given. Consider the map $\mathbb{R}^{N+1} \ni u \mapsto F(u) \in \mathbb{R}^{N+1}$, where

$$
F(u)_j = u_j - \mu \Delta_+ \left( k_{j-1/2} \Delta_- G(u_j) \right) - \alpha_j.
$$

Then the statement of the lemma is equivalent to the existence of a unique $u$ such that $F(u) = 0$. We have

$$
(F(u), u) = \|u\|^2 - \mu \sum_{j=0}^{N+1} \Delta_+ \left( k_{j-1/2} \Delta_- G(u_j) \right) u_j - (\alpha, u)
$$

$$
= \|u\|^2 + \mu \sum_{j=0}^{N+1} k_{j-1/2} \Delta_- G(u_j) \Delta_- u_j - (\alpha, u)
$$

$$
= \|u\|^2 + \mu \sum_{j=0}^{N+1} k_{j-1/2} G'(\tilde{u}_j) |\Delta_- u_j|^2 - (\alpha, u)
$$
so that the scheme (4.1) is valid for $n$ the left vanishes, and we are left with

\[
\eta(j_n) \geq \left( \|u\| - \|\alpha\| \right) \|u\|.
\]

Here $\tilde{u}_j$ is a number between $u_j$ and $u_{j+1}$, and we have used the monotonicity of $G$. For a given $\alpha$ with norm $r = \|\alpha\|$, we have that $(F(u), u) \geq 0$ for all $u$ with $\|u\| \geq r$. Then [7, Thm. 9.9-3] gives the existence of an $\tilde{u}$ such that $F(\tilde{u}) = 0$. Uniqueness follows from (4.9). More precisely, we consider two solutions

\[
\eta(j_n+1) \geq \eta(j_n) \geq \left( \|u\| - \|\alpha\| \right) \|u\|.
\]

with the boundary conditions $\Delta_- u_0 = \Delta_+ u_N = 0$ and $\Delta_+ v_0 = \Delta_- v_N = 0$. After a detailed analysis of each term we conclude (cf. (4.9))

\[
\sum_{j=0}^N |u_j - v_j| \leq \sum_{j=0}^N |f_j - g_j|.
\]

It will also be useful to define the (artificial) value $u^{-1}$ by taking an explicit step in the negative $\xi$ direction,

\[
u^{-1} = u_0 - \mu \Delta_+ (k_{\xi^{-2}} \Delta_- G(u_0)), \quad j = 0, \ldots, N,
\]

so that the scheme (4.1) is valid for $n \geq 1$.

Due to the boundary condition, the scheme is conservative,

\[
\sum_{j=0}^N u_j = \sum_{j=0}^N u_j + \mu \sum_{j=0}^N \Delta_+ (k_{\xi^{-2}} \Delta_- G(u_j))
\]

\[
= \sum_{j=0}^N u_j + \mu (k_{\xi^{-2}} \Delta_- G(u_0) - k_{\xi^{-2}} \Delta_+ G(u_N))
\]

\[
= \sum_{j=0}^N u_j + \mu (k_{\xi^{-2}} \Delta_- G(u_0) - k_{\xi^{-2}} \Delta_+ G(u_N))
\]

If $\eta(u)$ is a twice continuously differentiable function,

\[
\eta'(u_j^{n+1}) (u_j^{n+1} - u_j^n) = \eta(u_j^{n+1}) - \eta(u_j^n) + \frac{1}{2} \eta''(u_j^{n+1}) (u_j^{n+1} - u_j^n)^2,
\]

where $u_j^{n+1}$ is some value between $u_j^{n+1}$ and $u_j^n$. We can then multiply the scheme (4.1) by $\eta'(u_j^{n+1})$ to find that

\[
\eta(u_j^{n+1}) - \eta(u_j^n) = \frac{1}{2} \eta''(u_j^{n+1}) (u_j^{n+1} - u_j^n)^2.
\]

Using the “Leibniz rule” $\Delta_+ (a_j b_j) = a_j \Delta_+ b_j + b_j \Delta_+ a_j$, this can be rewritten

\[
\eta(u_j^{n+1}) - \mu \eta' (k_{\xi^{-2}} \Delta_- G(u_j)) \Delta_+ (u_j^{n+1}) + \frac{1}{2} \eta''(u_j^{n+1}) (u_j^{n+1} - u_j^n)^2.
\]

If we sum this over $j$, then, due to the boundary conditions, the second term on the left vanishes, and we are left with
If \( \eta \) we can also choose

\[
\text{Setting } \eta(u) = \frac{1}{2} u^2, \text{ we get the } L^2 \text{ bound}
\]

\[
\frac{1}{2} \sum_{j=0}^N (u_j^{n+1})^2 + \mu \sum_{j=0}^{M-1} \sum_{j=0}^N k_{j+1/2} (\Delta_+ u_j^n) (\Delta_+ G(u_j^n)) + \frac{1}{2} \mu^2 \sum_{j=0}^N (\Delta_+ (k_{j+1/2} \Delta_- G(u_j^n)))^2 = \frac{1}{2} \sum_{j=0}^N (u_j^n)^2.
\]

Summing this over \( n = 0, \ldots, M - 1 \) we find that

\[
\sum_{j=0}^N (u_j^M)^2 + 2\mu \sum_{n=0}^{M-1} \sum_{j=0}^N k_{j+1/2} (\Delta_+ u_j^n) (\Delta_+ G(u_j^n)) + \mu^2 \sum_{n=0}^{M-1} \sum_{j=0}^N (\Delta_+ (k_{j+1/2} \Delta_- G(u_j^n)))^2 = \sum_{j=0}^N (u_j^n)^2.
\]

In particular, this implies the uniform \( L^2 \) bound on \( u_{\Delta t} \),

\[
\|u_{\Delta t}(t, \cdot)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)}.
\]

We can also choose \( \eta(u) = |u|_\varepsilon \) in (4.6), and then let \( \varepsilon \to 0 \) to conclude that

\[
\|u_{\Delta t}(t, \cdot)\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}.
\]

If we choose \( \eta(u) = \int_0^u G(v) \, dv \) in equation (4.6) we find

\[
\sum_{j=0}^N \eta(u_j^{n+1}) + \mu \sum_{j=0}^N k_{j+1/2} (\Delta_+ G(u_j^{n+1})) \leq \sum_{j=0}^N \eta(u_j^n).
\]

If \( \Delta x \sum_j \eta(u_j^n) \) is uniformly bounded, then

\[
\Delta t \Delta x \sum_{j=0}^M \sum_{n=0}^N k_{j+1/2} \left( \frac{\Delta_+ G(u_j^{n+1})}{\Delta x} \right)^2 \leq C.
\]

Next we show stability of solutions to (4.1) with respect to the initial data. To keep the notation simple, let \( u \) and \( v \) solve

\[
\begin{align*}
&u_j - \mu \Delta_+ (k_{j-1/2} \Delta_- G(u_j)) = f_j \quad \text{for } j = 0, \ldots, N, \\
v_j - \mu \Delta_+ (k_{j-1/2} \Delta_- G(v_j)) = g_j \quad \text{with the boundary conditions } \\
&\Delta_- u_0 = \Delta_+ u_N = 0, \\
&\Delta_- v_0 = \Delta_+ v_N = 0.
\end{align*}
\]

Subtracting these equations

\[
(u_j - v_j) - \mu \Delta_+ (k_{j-1/2} \Delta_- (G(u_j) - G(v_j))) = f_j - g_j.
\]

Multiplying with \( \text{sign}_\varepsilon(u_j - v_j) \)

\[
\text{sign}_\varepsilon(u_j - v_j) (u_j - v_j) - \mu \Delta_+ \left[ k_{j-1/2} \text{sign}_\varepsilon(u_j - v_j) \Delta_- (G(u_j) - G(v_j)) \right]
\]
Consider the second sum on the left, each summand reads
\[ \sum_j \sum_{j=0}^N |u_j - v_j| + \mu \sum_j k_{j+1/2} \left[ \Delta_+ \text{sign}(u_j - v_j) \right] \left[ \Delta_+ (G(u_j) - G(v_j)) \right] = \text{sign}(u_j - v_j) (f_j - g_j). \]

Summing over \( j = 1, \ldots, N \) and sending \( \varepsilon \to 0 \),
\[ \sum_{j=0}^N |u_j - v_j| + \mu \sum_{j=0}^N k_{j+1/2} \left[ \Delta_+ \text{sign}(u_j - v_j) \right] \left[ \Delta_+ (G(u_j) - G(v_j)) \right] \leq \sum_{j=0}^N |f_j - g_j|. \]

Consider the second sum on the left, each summand reads
\[ k_{j+1/2} \left[ \text{sign}(u_{j+1} - v_{j+1}) - \text{sign}(u_j - v_j) \right] \left[ G(u_{j+1}) - G(v_{j+1}) - (G(u_j) - G(v_j)) \right]. \]

We have that
\[ \text{sign}(u_{j+1} - v_{j+1}) - \text{sign}(u_j - v_j) = \begin{cases} 2 & \text{if } u_{j+1} > v_{j+1} \text{ and } u_j < v_j, \\ 1 & \text{if } u_{j+1} > v_{j+1} \text{ and } u_j = v_j, \\ 1 & \text{if } u_{j+1} = v_{j+1} \text{ and } u_j < v_j, \\ -1 & \text{if } u_{j+1} = v_{j+1} \text{ and } u_j > v_j, \\ -1 & \text{if } u_{j+1} < v_{j+1} \text{ and } u_j = v_j, \\ -2 & \text{if } u_{j+1} < v_{j+1} \text{ and } u_j > v_j, \\ 0 & \text{otherwise}. \end{cases} \]

Similarly we find that
\[ G(u_{j+1}) - G(v_{j+1}) - (G(u_j) - G(v_j)) \begin{cases} \geq 0 & \text{if } u_{j+1} > v_{j+1} \text{ and } u_j < v_j, \\ \geq 0 & \text{if } u_{j+1} > v_{j+1} \text{ and } u_j = v_j, \\ \geq 0 & \text{if } u_{j+1} = v_{j+1} \text{ and } u_j < v_j, \\ \leq 0 & \text{if } u_{j+1} = v_{j+1} \text{ and } u_j > v_j, \\ \leq 0 & \text{if } u_{j+1} < v_{j+1} \text{ and } u_j = v_j, \\ \leq 0 & \text{if } u_{j+1} < v_{j+1} \text{ and } u_j > v_j, \end{cases} \]

Thus each summand in the second sum over \( j \) above is nonnegative and we find that
\[ \sum_{j=0}^N |u_j - v_j| \leq \sum_{j=0}^N |f_j - g_j|. \]

This completes the argument used to prove (4.4).

If \( v^n_j \) is another solution of (4.1), with initial data \( v^0_j \), then we have that
\[ \sum_{j=0}^N |u^n_j - v^n_j| \leq \sum_{j=0}^N |u^0_j - v^0_j|. \]

Since the update \( u^n \to u^{n+1} \) is conservative, by the Crandall–Tartar lemma [10, Lemma 2.13], it is also monotone, i.e., if \( u^n \leq v^n \), then \( u^{n+1} \leq v^{n+1} \). If we set \( v^n_j = v^{n+1}_j \) we get the estimate
\[ \sum_{j=0}^N |u^{n+1}_j - u^n_j| \leq \sum_{j=0}^N |u^0_j - u^{-1}_j| = \mu \sum_{j=0}^N |\Delta_+ (k_{j-1/2} \Delta_- G(u^0_j))|. \]
Now we assume that $u_0$ is such that $k \partial_x G(u_0) \in BV(B)$. This then gives the estimates
\begin{equation}
\|u_{\Delta t}(t + \Delta t, \cdot) - u_{\Delta t}(t, \cdot)\|_{L^1(B)} \leq C \Delta t \int_B |\partial_x (k \partial_x G(u_0(x)))| \, dx
= C \Delta t |k \partial_x G(u_0)|_{BV(B)},
\end{equation}
and
\begin{equation}
\Delta x \sum_{j=0}^N \frac{\Delta_+ \left( k_{j-1/2} \Delta_- G(u_j^n) \right)}{\Delta x^2} \leq C |k \partial_x G(u_0)|_{BV(B)}.
\end{equation}

As for the viscous regularization (3.1), we now establish a $BV$ bound on $u_{\Delta x}$.

Define $v_{n+1}^{j-1/2} = \Delta_- u_j^n$. Then $v_{-1/2}^n = v_{N+1/2}^n = 0$, and $v_{j-1/2}^n$ solves the equation
\begin{equation}
v_{j-1/2}^{n+1} - \mu \Delta_+ \left( k_{j-1/2} \Gamma_{n+1}^{j-1/2} v_{j-1/2}^{n+1} \right) = v_{j-1/2}^n, \quad j = 1, \ldots, N,
\end{equation}
where
\[ \Gamma_{n+1}^{j-1/2} = \frac{\Delta_- G(u_j^{n+1})}{\Delta_- u_j^{n+1}}. \]

Set $\eta(v) = |v|_\varepsilon$, multiply with $\eta'(v_{j-1/2}^{n+1})$ to get
\[ \eta(v_{j-1/2}^{n+1}) - \mu \Delta_+ \left[ \eta'(v_{j-1/2}^{n+1}) \Delta_- \left( k_{j-1/2} \Gamma_{n+1}^{j-1/2} v_{j-1/2}^{n+1} \right) \right]
- \mu k_{j+1/2} \left[ \Delta_+ \eta'(v_{j-1/2}^{n+1}) \Delta_+ \left( k_{j-1/2} \Gamma_{n+1}^{j-1/2} v_{j-1/2}^{n+1} \right) \right] \leq \eta(v_{j-1/2}^n).
\]

Here we have used that
\[ \eta(v_{j-1/2}^n) - \eta(v_{j-1/2}^{n+1}) \geq (v_{j-1/2}^n - v_{j-1/2}^{n+1}) \eta'(v_{j-1/2}^{n+1}) \]

due to the convexity of $\eta$, and
\[ \eta'(v_{j-1/2}^{n+1}) \Delta_- \Delta_+ \left( k_{j-1/2} \Gamma_{n+1}^{j-1/2} v_{j-1/2}^{n+1} \right)
= \Delta_+ \left[ \eta'(v_{j-1/2}^{n+1}) \Delta_- \left( k_{j-1/2} \Gamma_{n+1}^{j-1/2} v_{j-1/2}^{n+1} \right) \right]
- (\Delta_+ \eta'(v_{j-1/2}^{n+1})) S^+ \left( \Delta_- \left( k_{j-1/2} \Gamma_{n+1}^{j-1/2} v_{j-1/2}^{n+1} \right) \right)
= \Delta_+ \left[ \eta'(v_{j-1/2}^{n+1}) \Delta_- \left( k_{j-1/2} \Gamma_{n+1}^{j-1/2} v_{j-1/2}^{n+1} \right) \right]
- (\Delta_+ \eta'(v_{j-1/2}^{n+1})) \Delta_+ \left( k_{j-1/2} \Gamma_{n+1}^{j-1/2} v_{j-1/2}^{n+1} \right).
\]

Now we sum this over $j = 0, \ldots, N$, the second term on the left vanishes due to the boundary condition on $v_{j-1/2}^{n+1}$ and the fact that $\eta'(0) = 0$. Then we can let $\varepsilon \to 0$ to conclude that
\[ \sum_{j=0}^{N-1} |v_{j+1/2}^{n+1}| + \mu \sum_{j=1}^{N} k_{j-1/2} \left[ \Delta_+ \text{sign} \left( v_{j-1/2}^{n+1} \right) \right] \left[ \Delta_+ \left( \Gamma_{j-1/2} v_{j-1/2}^{n+1} \right) \right] \leq \sum_{j=0}^{N-1} |v_{j+1/2}^n|. \]
Now

$$\text{sign} \left( v_{j+1/2}^{n+1} \right) - \text{sign} \left( v_{j-1/2}^{n+1} \right) = \begin{cases} 2 & v_{j-1/2}^{n+1} < 0 < v_{j+1/2}^{n+1}, \\ 1 & v_{j-1/2}^{n+1} < 0 = v_{j+1/2}^{n+1}, \\ 1 & v_{j-1/2}^{n+1} = 0 < v_{j+1/2}^{n+1}, \\ -1 & v_{j-1/2}^{n+1} = 0 > v_{j+1/2}^{n+1}, \\ -1 & v_{j-1/2}^{n+1} > 0 = v_{j+1/2}^{n+1}, \\ -2 & v_{j-1/2}^{n+1} > 0 > v_{j+1/2}^{n+1}, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\Gamma_{j+1/2}^{n+1} v_{j+1/2}^{n+1} - \Gamma_{j-1/2}^{n+1} v_{j-1/2}^{n+1} \geq \begin{cases} 0 & v_{j-1/2}^{n+1} < 0 < v_{j+1/2}^{n+1}, \\ 0 & v_{j-1/2}^{n+1} < 0 = v_{j+1/2}^{n+1}, \\ 0 & v_{j-1/2}^{n+1} = 0 < v_{j+1/2}^{n+1}, \\ 0 & v_{j-1/2}^{n+1} > 0 = v_{j+1/2}^{n+1}, \\ 0 & v_{j-1/2}^{n+1} > 0 > v_{j+1/2}^{n+1}, \\ 0 & \text{otherwise}. \end{cases}$$

Thus the second sum above is nonnegative, and we conclude that

$$\sum_{j=0}^{N-1} |u_{j+1}^n - u_j^n| \leq \sum_{j=0}^{N-1} |u_{j+1}^0 - u_j^0|, \quad n \geq 0. \tag{4.14}$$

We have established that \( \{u_{\Delta t}\}_{\Delta t > 0} \) is uniformly bounded in \( L^\infty(\Omega_T) \) and also \( C([0,T]; L^1(B)) \), and that we have a uniform \( (t, \Delta t) \) bound on \( |u_{\Delta t}(t, \cdot)|_{BV(B)} \).

By Kolmogorov–Riesz’s theorem (see [10, Thm. A11], [8, 9]), there exists a subsequence of \( \Delta t’s \) and a function \( u = u(t,x) \), such that \( u_{\Delta t} \to u \) in \( C([0,T]; L^1(B)) \). Since \( G \) is a Lipschitz continuous function, also \( G(u_{\Delta t}) \) converges to \( G(u) \). We shall have use for the notation

$$D_{\pm} = \frac{1}{\Delta x} \Delta_{\pm} \text{ and } D_{\pm} \eta \left( u_j^n \right) = \frac{\eta(u_{j+1}^n) - \eta(u_j^n)}{\Delta t}.$$ 

Define the continuous function \( kDG_{\Delta t}(t,x) \) by piecewise linear interpolation

$$kDG_{\Delta t}(t,x) = k_{j-1/2}D_- G(u_{\Delta t}(t,x)) + \frac{x-x_{j-1/2}}{\Delta x} D_+ \left( k_{j-1/2}D_- G(u_{\Delta t}(t,x)) \right),$$

for \( x \in (x_{j-1/2}, x_{j+1/2}] \). Then, for fixed \( t \), by the estimate (4.12), \( kDG_{\Delta t}(t, \cdot) \) is in \( BV(B) \) (with a uniformly bounded \( BV \) seminorm). Hence there is a (further) subsequence of \( \Delta t’s \) such that \( kDG_{\Delta t}(t, \cdot) \to H(t, \cdot) \) in \( L^1(B) \). Furthermore \( H = k\partial_x G(u) \) weakly in \( B \). This also implies that \( \partial_u G(u) \in L^\infty([0,T]; BV(B)) \).

The bound (4.8) implies that \( \partial_u G(u) \) is in \( L^2(\Omega_T) \). In order to prove that \( u \) is an entropy solution, we start with the discrete entropy inequality (4.5), which implies that

$$D_+ \eta \left( u_j^n \right) - D_+ \left( \eta(u_{j+1}^{n+1}) k_{j-1/2} D_- G(u_j^{n+1}) \right) + k_{j+1/2} \left( D_+ \eta(u_{j+1}^{n+1}) \right) (D_+ G(u_j^{n+1})) \leq 0.$$ 

Let \( \varphi \) be a suitable nonnegative test function and set

$$\varphi_j^n = \frac{1}{\Delta t \Delta x} \int_{t_n}^{t_{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \varphi(s,y) \, dy \, ds.$$
We multiply the above inequality with $\Delta t\Delta x \varphi_j^n$ and do summation by parts in $n$ and $j$ to get
\begin{align}
(4.15) \quad \Delta t\Delta x \sum_{n=0}^{M} \sum_{j=1}^{N} \eta^n_j D_{j} \varphi^n_j &= \Delta x \sum_{j=1}^{N} \eta^{n+1}_j \varphi^M_j + \Delta x \sum_{j=0}^{N} \eta^0_j \varphi^0_j \\
(4.16) \quad -\Delta t\Delta x \sum_{n=0}^{M} \sum_{j=0}^{N} (\eta')_j^{n+1} k_{j-1/2} D_- G_{j}^{n+1} D_- \varphi^n_j \\
(4.17) \quad -\Delta t\Delta x \sum_{n=0}^{M} \sum_{j=0}^{N} k_{j+1/2} \left( D_+ (\eta')_j^{n+1} \right) (D_+ G_{j}^{n+1}) \varphi^n_j \geq 0,
\end{align}
where $\eta^{n+1}_j = \eta(u_j^{n+1})$, etc. By using the compactness properties of $u_{\Delta t}$, in particular that $u_{\Delta t}(t, \cdot) \in BV(B)$ and that $\partial_x G(u_{\Delta t})$ is bounded, it follows by a standard Lax–Wendroff calculations, see [10, Thm. 3.4] that
\[\lim_{\Delta t \to 0} (4.15) = \int_{\Omega_T} \eta(u) \partial_x \varphi \, dx \, dt - \int_{B} \eta(u(T, x)) \varphi(T, x) \, dx + \int_{B} \eta(u_0(x)) \varphi(0, x) \, dx,\]
and
\[\lim_{\Delta t \to 0} (4.16) = -\int_{\Omega_T} \eta' \left( k \partial_x G(u) \partial_x \varphi \right) \, dx \, dt.\]
Regarding (4.17), by Jensen’s inequality
\[\left( \Delta_+ (\eta')_j^{n+1} \right) \left( \Delta_+ G_{j}^{n+1} \right) \varphi^n_j = \eta''(u_{j+1/2}^{n+1}/u_j^{n+1}) \Delta_+ u_j^{n+1} \Delta_+ G_{j}^{n+1}\]
\[= \eta''(u_{j+1/2}^{n+1}) \left( \Delta_+ u_j^{n+1} \right)^2 \left[ \frac{1}{\Delta_+ u_j^{n+1}} \int_{u_j^{n+1}}^{u_{j+1}^{n+1}} (g'(s))^2 \, ds \right]\]
\[\geq \eta''(u_{j+1/2}^{n+1}) \left( \Delta_+ u_j^{n+1} \right)^2 \left[ \frac{1}{\Delta_+ u_j^{n+1}} \int_{u_j^{n+1}}^{u_{j+1}^{n+1}} g'(s) \, ds \right]^2\]
\[= \eta''(u_{j+1/2}^{n+1}) \left( \Delta_+ g_{j}^{n+1} \right)^2,
\]
where $u_{j+1/2}^{n+1}$ is between $u_j^{n+1}$ and $u_{j+1}^{n+1}$ and $g(u) = \int u \sqrt{G'(s)} \, ds$. Since $\eta''$ is continuous, “$\eta''(u_{j+1/2}^{n+1}) \to \eta''(u)$” as $\Delta t \to 0$. We also have that $g(u_{\Delta t}) \to g(u)$ and that $\partial_x g(u_{\Delta t}) \to \partial_x g(u)$ weakly. Therefore
\[\lim_{\Delta t \to 0} \Delta t\Delta x \sum_{n=0}^{M} \sum_{j=0}^{N} k_{j+1/2} \left( D_+ (\eta')_j^{n+1} \right) (D_+ G_{j}^{n+1}) \varphi^n_j\]
\[\geq \lim_{\Delta t \to 0} \Delta t\Delta x \sum_{n=0}^{M} \sum_{j=0}^{N} k_{j+1/2} \eta''(u_{j+1/2}^{n+1}) \left( D_+ g_{j}^{n+1} \right)^2 \varphi^n_j\]
\[= \int_{\Omega_T} \eta''(u) k (\partial_x g(u))^2 \varphi \, dx \, dt.\]
We have now shown that the limit $u$ is an entropy solution, i.e., we have proved to following theorem:

**Theorem 4.2.** Assume that $G$ is a non-decreasing, Lipschitz continuous function, that $k \in C^1([0,1])$ is a strictly positive function, and that the initial data $u_0 \in
BV([0,1]) \cap L^\infty([0,1]). Then the sequence \( \{u_{\Delta t}\}_{\Delta t > 0} \) defined by (4.3), (4.2) and the scheme (4.1) converges to an entropy solution of (2.1) for \( B = (0, 1) \).

**Proof.** The compactness and convergence of a subsequence is already proved above, and we observe that since the entropy solution is unique, the whole sequence converges. \( \square \)

### 4.1. An example.

We present an example of how this method works in practice. Let

\begin{equation}
G(u) = \begin{cases}
0 & u < 0, \\
u(2 - u) & 0 \leq u \leq 1, \\
1 & 1 < u,
\end{cases}
\end{equation}

and

\begin{equation}
u_0(x) = 2 \sin(2\pi x).
\end{equation}

In Figure 1 we show the approximate solution for \( t = 0, t = 0.07, t = 0.13 \) and \( t = 0.2 \), computed with the implicit finite difference scheme (4.1) with \( N = 512, \Delta x = 1/513, \) and \( \Delta t = 0.01\Delta x^2 \). The nonlinear equation was solved numerically using Newton iteration which terminated when the error was less than 0.1\( \Delta x^2 \).

![Figure 1](image)

**Figure 1.** The approximate solution for various times.

Figure 2 we also show the same approximate solution as a function of \( (t, x) \).
Figure 2. The approximate solution for $0 \leq t \leq 0.2$.

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