An Infinite Product for $e^\gamma$ via Hypergeometric Formulas for Euler's Constant $\gamma$

Jonathan Sondow

INTRODUCTION. Euler's constant, $\gamma = 0.5772156649\ldots$, is defined as the limit

$$\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right).$$

(1)

In this note we prove the infinite product formula for $e^\gamma = 1.7810724179\ldots$

$$e^\gamma = \left(\frac{2}{1}\right)^{1/2} \left(\frac{2^2}{1 \cdot 3}\right)^{1/3} \left(\frac{2^3 \cdot 4}{1 \cdot 3^3}\right)^{1/4} \left(\frac{2^4 \cdot 4^4}{1 \cdot 3^6 \cdot 5}\right)^{1/5} \cdots,$$

(2)

where the $n$th factor is

$$a(n) = \prod_{k=0}^{n} \left(\frac{(-1)^{k+1} (n+k)}{(k+1)!}\right)^{1/(n+1)}.$$

Figure 1. The $n$th partial product $a(n)$ of the infinite product for $e^\gamma$

The path that led to (2) began with the proof [5] of the double integral formula for Euler's constant

$$\gamma = \iint_{[0,1]^2} -\frac{1-x}{(1-xy) \ln xy} \, dx \, dy,$$

(3)
an analog of double integrals \([4]\) for \(\zeta(2)\) and \(\zeta(3)\), where \(\zeta(s) = \sum_{n \geq 1} n^{-s}\). A series \(\sum a_n\) in the proof (essentially the first series in (4) below), with term ratio a quotient of polynomials in \(n\), gave the idea to write the series as a hypergeometric function (see [2, p. 208]). A hypergeometric transformation then paved the way to (2).

We give four proofs of (2). The first two are hypergeometric; the second uses the digamma function \(\psi(t) = \Gamma'(t)/\Gamma(t)\) and is shorter. The last two proofs, which avoid hypergeometrics, depend on integral formulas for \(\gamma\): a classical one in the third proof, and (3) in the fourth. Using the classical integral, we end the note with a quick proof of (3).

**PROOF 1.** We follow the latter part of the path described. Using \(\ln((n+1)/n) \to 0\) to replace \(\ln n\) by \(\ln(n+1)\) in (1), we convert the limit to the series

\[
\gamma = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{\ln n + 1}{n} \right) = \sum_{n=1}^{\infty} \int_1^{\infty} \frac{1}{(t+n-1)^2} - \frac{1}{(t+n-1)(t+n)} \ dt
\]

\[
= \sum_{n=1}^{\infty} \int_1^{\infty} \frac{1}{(t+n-1)^2(t+n)} \ dt.
\]

Interchanging summation and integration (justified by uniform convergence, the integrand being bounded by \(n^{-2}(n+1)^{-1}\)), we write the formula as

\[
\gamma = \int_1^{\infty} \sum_{n=1}^{\infty} \frac{1}{(t+n-1)^2(t+n)} \ dt = \int_1^{\infty} \frac{1}{t^2(t+1)} \sum_{n=1}^{\infty} \frac{t^2(t+1)}{(t+n-1)^2(t+n)} \ dt
\]

\[
= \int_1^{\infty} \frac{1}{t^2(t+1)} \left( 1 + \frac{t^2}{(t+1)(t+2)} + \frac{t(t+1)^2}{(t+1)(t+2)(t+3)} + \cdots \right) \ dt
\]

in order to replace the series by a hypergeometric function [2, p. 205], [3, p. 101]

\[
F\left(1, \frac{s}{u}, \frac{t}{v} \right) = _3F_2\left(1, \frac{s}{u}, \frac{t}{v} \left| 1 \right) = 1 + \frac{s \cdot t}{u \cdot v} + \frac{s(s+1) \cdot t(t+1)}{u(u+1) \cdot v(v+1)} + \cdots, \right.
\]

obtaining

\[
\gamma = \int_1^{\infty} \frac{1}{t^2(t+1)} \cdot F\left(1, \frac{t}{t+1}, \frac{t}{t+2} \right) dt.
\]
The transformation

\[
F\left(\frac{1, s, t}{s + 1, v}\right) = \frac{s}{v - t} \cdot F\left(\frac{1, v - t, v - s}{v - t + 1, v}\right)
\]  

(6)

(valid if \(s > 0\) and \(v - t > 0\), and proved by applying the identity \(F\left(\frac{r, s, t}{u, v}\right) = F\left(\frac{s, r, t}{u, v}\right)\), then Thomae's transformation [3, pp. 104-105, 111], then the identity) gives the simpler expression

\[
\gamma = \int_1^\infty \frac{1}{2t(t + 1)} \cdot F\left(\frac{1, 2, 2}{3, t + 2}\right) dt.
\]  

(7)

Expanding by (5), and simplifying terms, we get the key formula

\[
\gamma = \int_1^\infty \sum_{n=1}^\infty \frac{n!}{(n + 1)t(t + 1) \cdots (t + n)} dt.
\]  

(8)

Integrating termwise (the \(n\)th term being bounded by \((n + 1)^{-2}\)) by the method of partial fractions, we arrive at the series

\[
\gamma = \sum_{n=1}^\infty \frac{1}{n + 1} \sum_{k=0}^n (-1)^{k+1} \binom{n}{k} \ln(k + 1),
\]

and exponentiation yields the desired product (2).

PROOF 2. Using values of the \(\psi\) function and a series for \(\psi'\) (see [1, pp. 15, 16, 22, equations (4), (8), (9), resp.], we shorten the first proof. From (1) and the formulas \(\psi(1) = -\gamma\) and \(\psi(n + 1) = 1 + 1/2 + \cdots + 1/n - \gamma\), we deduce that

\[\int_1^\infty \left(\psi'(t) - \frac{1}{t}\right) dt = \gamma.\]

Employing (5) and (6), we transform the series \(\psi'(t) = \sum_{n \geq 0} (t + n)^{-2}\) into the function

\[\psi'(t) = \frac{1}{t} \sum_{n=0}^\infty \frac{t^2}{(t + n)^2} = \frac{1}{t^2} \cdot F\left(\frac{1, t, t}{t + 1, t + 1}\right) = \frac{1}{t} \cdot F\left(\frac{1, 1, 1}{2, t + 1}\right).
\]

Using (5) (but not (6)), we verify the identity

\[\frac{1}{t} \left[F\left(\frac{1, 1, 1}{2, t + 1}\right) - 1\right] = \frac{1}{2t(t + 1)} \cdot F\left(\frac{1, 2, 2}{3, t + 2}\right).
\]
and (7) follows. The rest of the proof is the same.

Remark. One can accelerate the convergence of such hypergeometric representations by computing some terms in (1). For example,

\[
\gamma = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \ln n + \int_{n}^{\infty} \frac{1}{t} \left[ F\left(1, 1, 1; \frac{1}{2}, \frac{1}{t+1}\right) - 1 \right] dt.
\]

PROOF 3. We reduce the key formula (8) to the classical formula for Euler’s constant

\[
\gamma = \int_{0}^{1} \frac{1}{\ln u} + \frac{1}{1-u} \, du. \tag{9}
\]

(One way to derive (9) is by inserting the value \(\psi(1) = -\gamma\) into Gauss’s integral for the \(\psi\) function [6, p. 246, Example 1].) Denoting integral (8) by \(I\), we substitute Euler’s beta integral

\[
\int_{0}^{1} \frac{(1-u)^{n}}{n+1} \, du = \frac{n!}{(n+1)(t+1)\cdots(t+n)}
\]

(evaluated by \(n\)-fold integration by parts), then replace \(t\) by \(t+1\), obtaining

\[
I = \int_{0}^{\infty} \sum_{n=1}^{\infty} \int_{0}^{1} \frac{u'(1-u)^{n}}{n+1} \, du \, dt.
\]

For fixed \(t > 0\), a little calculus shows that

\[
\max_{0 \leq u \leq 1} u'(1-u)^{n} \frac{n+1}{n+1} \cdot \frac{n^{n} t^{n}}{(n+t)^{n+t}} < \frac{t^{n}}{n^{1+n}},
\]

guaranteeing uniform convergence of the series

\[
\sum_{n=1}^{\infty} \frac{u'(1-u)^{n}}{n+1} = \frac{u'}{1-u} \sum_{n=1}^{\infty} \frac{(1-u)^{n+1}}{n+1} = u'\left(\frac{-\ln u}{1-u} - 1\right)
\]

on the interval \(0 \leq u \leq 1\). This permits interchanging summation and integration with respect to \(u\), which gives

\[
I = \int_{0}^{\infty} \int_{0}^{1} u'\left(\frac{-\ln u}{1-u} - 1\right) \, du \, dt. \tag{10}
\]

Reversing the order of integration (the integrand being non-negative), we integrate with respect to \(t\), and arrive at integral (9). This proves (8), and (2) follows as before.
**Proof 4.** We show that integrals (8) and (3) are equal. Repeating the previous proof through (10), we substitute the integral

\[
\frac{1}{1-u} \int_0^{1-u} \frac{v}{1-v} dv = -\ln u - 1,
\]

reverse the order of integration from \(dv du dt\) to \(dt dv du\) (non-negativity again), and integrate with respect to \(t\), obtaining

\[
I = \int_0^1 \int_0^{1-u} \frac{v}{(1-v)(1-u)\ln u} dv du .
\]  

(11)

Since the change of variables \(v = 1 - x, u = xy\) in integral (3) transforms it into integral (11), we infer (8), and (2) follows as usual.

**Remark.** Integration with respect to \(v\) in integral (11) produces integral (9) for \(\gamma\). Combined with the equality of integrals (3) and (11) by change of variables, this gives a short proof of (3). (The proof in [5] is longer, but uses only the definition of Euler's constant.)

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**References**

1. A. Erdélyi et al., *Higher Transcendental Functions*, vol. 1, McGraw-Hill, New York, 1953.
2. R. Graham, D. Knuth, O. Patashnik, *Concrete Mathematics*, 2nd edition, Addison-Wesley, Boston, 1994.
3. G. H. Hardy, *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work*, Amer. Math. Soc., Providence, 1999.
4. D. Huylebrouck, Similarities in irrationality proofs for \(\pi, \ln 2, \zeta(2)\) and \(\zeta(3)\), *Amer. Math. Monthly* 108 (2001) 222-231.
5. J. Sondow, A double integral for Euler's constant, submitted to *Amer. Math. Monthly*.
6. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th edition, Cambridge University Press, Cambridge, 1988.