LOGISTIC TYPE ATTRACTION-REPPULSION CHEMOTAXIS SYSTEMS WITH A FREE BOUNDARY OR UNBOUNDED BOUNDARY. I. ASYMPTOTIC DYNAMICS IN FIXED UNBOUNDED DOMAIN

LIANZHANG BAO*

School of Mathematical Science, Zhejiang University
Hangzhou 310027, China
and
School of Mathematics, Jilin University
Changchun, Jilin 130012, China
and
Department of Mathematics and Statistics, Auburn University
AL 36849, USA

WENXIAN SHEN

Department of Mathematics and Statistics, Auburn University
AL 36849, USA

(Communicated by Michael Winkler)

Abstract. The current series of research papers is to investigate the asymptotic dynamics in logistic type chemotaxis models in one space dimension with a free boundary or an unbounded boundary. Such a model with a free boundary describes the spreading of a new or invasive species subject to the influence of some chemical substances in an environment with a free boundary representing the spreading front. In this first part of the series, we investigate the dynamical behaviors of logistic type chemotaxis models on the half line $\mathbb{R}^+$, which are formally corresponding limit systems of the free boundary problems. In the second of the series, we will establish the spreading-vanishing dichotomy in chemoattraction-repulsion systems with a free boundary as well as with double free boundaries.

1. Introduction. The current series of research papers is to study spreading and vanishing dynamics of the following attraction-repulsion chemotaxis system with a free boundary and time and space dependent logistic source,

2010 Mathematics Subject Classification. Primary: 35R35, 35J65, 35K20; Secondary: 92B05.

Key words and phrases. Chemoattraction-repulsion system, nonlinear parabolic equations, free boundary problem, spreading-vanishing dichotomy, invasive population.

The first author is supported by China Postdoctoral Science Foundation (183816). The second author is supported by NSF grant DMS–1645673.

* Corresponding author: Lianzhang Bao.
and to study the asymptotic dynamics of

\[
\begin{aligned}
    u_t &= u_{xx} - \chi_1 (uv_{1,x})_x + \chi_2 (uv_{2,x})_x + u(a(t,x) - b(t,x)u), \quad 0 < x < h(t) \\
    0 &= \partial_{xx} v_1 - \lambda_1 v_1 + \mu_1 u, \quad 0 < x < h(t) \\
    0 &= \partial_{xx} v_2 - \lambda_2 v_2 + \mu_2 u, \quad 0 < x < h(t) \\
    h(t) &= -\nu u(x,h(t)) \\
    u_x(t,0) &= v_{1,x}(t,0) = v_{2,x}(t,0) = 0 \\
    u(t,h(t)) &= v_{1,x}(t,h(t)) = v_{2,x}(t,h(t)) = 0 \\
    h(0) &= h_0, \quad u(x,0) = u_0(x), \quad 0 \leq x \leq h_0,
\end{aligned}
\]

(1)

and

\[
\begin{aligned}
    u_t &= u_{xx} - \chi_1 (uv_{1,x})_x + \chi_2 (uv_{2,x})_x + u(a(t,x) - b(t,x)u), \quad x \in (0, \infty) \\
    0 &= v_{1,xx} - \lambda_1 v_1 + \mu_1 u, \quad x \in (0, \infty) \\
    0 &= v_{2,xx} - \lambda_2 v_2 + \mu_2 u, \quad x \in (0, \infty) \\
    u_x(t,0) &= v_{1,x}(t,0) = v_{2,x}(t,0) = 0,
\end{aligned}
\]

(2)

where \( \nu > 0 \) in (1) is a positive constant, and in both (1) and (2), \( \chi_i, \lambda_i, \) and \( \mu_i \) \( (i = 1, 2) \) are nonnegative constants, and \( a(t,x) \) and \( b(t,x) \) satisfy the following assumption,

(H0) \( a(t,x) \) and \( b(t,x) \) are bounded \( C^1 \) functions on \( \mathbb{R} \times [0, \infty) \), and

\[
\inf_{t \in \mathbb{R}, x \in [0, \infty)} a(t,x) > 0, \quad \inf_{t \in \mathbb{R}, x \in [0, \infty)} b(t,x) > 0.
\]

Chemotaxis is the influence of chemical substances in the environment on the movement of mobile species. This can lead to strictly oriented movement or to partially oriented and partially tumbling movement. The movement towards a higher concentration of the chemical substance is termed positive chemotaxis and the movement towards regions of lower chemical concentration is called negative chemotaxis. The substances that lead to positive chemotaxis are chemoattractants and those leading to negative chemotaxis are so-called repellents.

One of the first mathematical models of chemotaxis was introduced by Keller and Segel ([13], [14]) to describe the aggregation of certain type of bacteria. A simplified version of their model involves the distribution of the density of the slime mold \( Dicytostelum discoideum \) and the concentration \( v \) of a certain chemoattractant satisfying the following system of partial differential equations

\[
\begin{aligned}
    u_t &= \nabla \cdot (\nabla u - \chi u \nabla v) + G(u), \quad x \in \Omega \\
    \nu v_t &= d\Delta v + F(u,v), \quad x \in \Omega
\end{aligned}
\]

(3)

complemented with certain boundary condition on \( \partial \Omega \) if \( \Omega \) is bounded, where \( \Omega \subset \mathbb{R}^N \) is an open domain, \( \epsilon \geq 0 \) is a non-negative constant linked to the speed of diffusion of the chemical, \( \chi \) represents the sensitivity with respect to chemotaxis, and the functions \( G \) and \( F \) model the growth of the mobile species and the chemoattractant, respectively.

Since their publication, considerable progress has been made in the analysis of various particular cases of (3) on both bounded and unbounded fixed domains (see [1], [3], [4], [6], [10], [12], [18], [24], [25], [26], [29], [30], [31], [32], [33], [34], [36], and the references therein). Among the central problems are the existence of nonnegative solutions of (3) which are globally defined in time or blow up at a finite time and the asymptotic behavior of time global solutions. When \( \epsilon > 0 \)
(3) is referred to as the parabolic-parabolic Keller-Segel model and \( \epsilon = 0 \), which models the situation where the chemoattractant diffuses very quickly, is the case of parabolic-elliptic Keller-Segel model. The reader is referred to [7, 8] for some detailed introduction into the mathematics of KS models.

When the cells undergo random motion and chemotaxis towards attractant and away from repellent [17] on a fixed domain, we have a chemoattraction-repulsion process, which combined with proliferation and death of cells leads to the following parabolic-elliptic-elliptic differential equations,

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u - \chi_1 \nabla \cdot (u \nabla v_1) + \chi_2 \nabla \cdot (u \nabla v_2) + G(u), \quad x \in \Omega \\
\epsilon \frac{\partial v_1}{\partial t} &= d_1 \Delta v_1 + F(u, v_1), \quad x \in \Omega \\
\epsilon \frac{\partial v_2}{\partial t} &= d_2 \Delta v_2 + H(u, v_2), \quad x \in \Omega,
\end{align*}
\]

where \( \chi_1 \) and \( \chi_2 \) are nonnegative constants. Note that, when \( \chi_2 = 0 \), the first two equations in (4) are decoupled from the third equation and form system (3) with \( \chi = \chi_1 \) and \( v = v_1 \). Compared to the studies of (3), the global existence of classical solutions on bounded or unbounded domain, and the stability of equilibrium solutions of (4) are also studied in many papers (see [5, 9, 11, 15, 16, 17, 22, 27, 28, 35, 37] and the references therein).

System (1) describes the movement of a mobile species with population density \( u(t, x) \) in an environment with a free boundary subject to a chemoattractant with population density \( v_1(t, x) \), which diffuses very quickly, and a repellent with population density \( v_2(t, x) \), which also diffuses very quickly. Due to the lack of first principles for the ecological situation under consideration, a thorough justification of the free boundary condition is difficult to supply. As in [2], we present in the following a derivation of the free boundary condition in (1) based on the consideration of “population loss” at the front and the assumption that, near the propagating front, population density is close to zero. Then, in the process of population range expansion, on one hand, the individuals of the species are suffering from the Allee effect near the propagating front. On the other hand, as the front enters new unpopulated environment, the pioneering members at the front, with very low population density, are particularly vulnerable. Therefore it is plausible to assume that as the expanding front propagates, the population suffers a loss of \( \kappa \) units per unit volume at the front.

By Fick’s first law, for a small time increment \( \Delta t \), during the period from \( t \) to \( t + \Delta t \), the number of individuals of the population that enter the region (through diffusion, or random walk) bounded by the old front \( x = h(t) \) and new front \( x = h(t + \Delta t) \) is approximated by \( -d_u(x, h(t)) \Delta t \) (note that \( u_x(t, h(t)) \leq 0 \) for \( u(t, x) \geq 0 \) on \( [0, h(t)] \)), where \( d \) is some positive constant. The population loss in this region is approximated by

\[ \kappa \times (\text{volume of the region}) = \kappa \times [h(t + \Delta t) - h(t)]. \]

So the average density of the population in the region bounded by the two fronts is given by

\[ \frac{-d_u(x, h(t)) \Delta t}{h(t + \Delta t) - h(t)} - \kappa. \]

As \( \Delta t \to 0 \), the limit of this quantity is the population density at the front, namely \( u(t, h(t)) \), which by assumption is 0. This implies that

\[ h'(t) = -\nu u_x(t, h(t)) \]
with \( \nu = d/\kappa \), and the free boundary condition in (1) is then derived. Consider (1), it is interesting to know whether the species will spread into the whole region \([0, \infty)\) or will vanish eventually. Formally, (2) can be viewed as the limit system of (1) as \( h(t) \to \infty \). The study of the asymptotic dynamics of (2) plays an important role in the characterization of the spreading-vanishing dynamics of (1) and is also of independent interest. The objective of this series is to investigate the asymptotic dynamics of (2) and the spreading and vanishing scenario in (1).

In this first part of the series, we investigate the asymptotic dynamics of (2) as well as the asymptotic dynamics of the following chemotaxis system on the whole line,

\[
\begin{align*}
\begin{cases}
  u_t = u_{xx} - \chi_1 (uv_{1,x})_x + \chi_2 (uv_{2,x})_x + u(a(t, x) - b(t, x)u), & x \in \mathbb{R} \\
  0 = v_{1,xx} - \lambda_1 v_1 + \mu_1 u, & x \in \mathbb{R} \\
  0 = v_{2,xx} - \lambda_2 v_2 + \mu_2 u, & x \in \mathbb{R}.
\end{cases}
\end{align*}
\]

Formally, (5) can be viewed as the limit of the following free boundary problem with double free boundaries

\[
\begin{align*}
\begin{cases}
  u_t = u_{xx} - \chi_1 (uv_{1,x})_x + \chi_2 (uv_{2,x})_x + u(a(t, x) - b(t, x)u), & x \in (g(t), h(t)) \\
  0 = (\partial_{xx} - \lambda_1 I) v_1 + \mu_1 u, & x \in (g(t), h(t)) \\
  0 = (\partial_{xx} - \lambda_2 I) v_2 + \mu_2 u, & x \in (g(t), h(t)) \\
  g'(t) = -\nu u_x(g(t), t), h'(t) = -\nu u_x(h(t), t) \\
  u(g(t), t) = v_{1,x}(g(t), t) = v_{2,x}(g(t), t) = 0 \\
  u(h(t), t) = v_{1,x}(h(t), t) = v_{2,x}(h(t), t) = 0
\end{cases}
\end{align*}
\]

as \( g(t) \to -\infty \) and \( h(t) \to \infty \). The investigation of the asymptotic dynamics of (5) then plays a role in the characterization of the spreading-vanishing dynamics of (6) and is also of independent interest.

In the second of the series, we will establish spreading and vanishing dichotomy scenario in (1) and (6).

In the following, we state the main results of this paper. Let

\[
C_{\text{unif}}(\mathbb{R}^+) = \{ u \in C(\mathbb{R}^+) \mid u(x) \text{ is uniformly continuous and bounded on } \mathbb{R}^+ \}
\]

with norm \( \| u \|_{\infty} = \sup_{x \in \mathbb{R}^+} |u(x)| \), and

\[
C_{\text{unif}}(\mathbb{R}) = \{ u \in C(\mathbb{R}) \mid u(x) \text{ is uniformly continuous and bounded on } \mathbb{R} \}
\]

with norm \( \| u \|_{\infty} = \sup_{x \in \mathbb{R}} |u(x)| \). Define

\[
M = \min \left\{ \frac{1}{\lambda_2} \left( (\chi_2 \mu_2 \lambda_2 - \chi_1 \mu_1 \lambda_1)_+ + \chi_1 \mu_1 (\lambda_1 - \lambda_2)_+ \right), \right\}
\]

and

\[
K = \min \left\{ \frac{1}{\lambda_1} \left( |\chi_1 \mu_1 \lambda_1 - \chi_2 \mu_2 \lambda_2| + \chi_1 \mu_1 |\lambda_1 - \lambda_2| \right), \right\}
\]

Let (H1)- (H3) be the following standing assumptions.

(H1) \( b_{\text{inf}} > \chi_1 \mu_1 - \chi_2 \mu_2 + M \).
(H2) $b_{\inf} > \left(1 + \frac{a_{\sup}}{a_{\inf}}\right)\chi_1\mu_1 - \chi_2 \mu_2 + M$.

(H3) $b_{\inf} > \chi_1 \mu_1 - \chi_2 \mu_2 + K$.

The main results of this first part are stated in the following theorems.

Theorem 1.1 (Global existence). Consider (2). If (H1) holds, then for any $t_0 \in \mathbb{R}$ and any nonnegative function $u_0 \in C^b_{\text{unif}}(\mathbb{R}^+)$, (2) has a unique solution $(u(t; x; t_0, u_0), v_1(t; x; t_0, u_0), v_2(t; x; t_0, u_0))$ with $u(t_0; x; t_0, u_0) = u_0(x)$ defined for $t \geq t_0$. Moreover,

$$0 \leq u(t, x; t_0, u_0) \leq C(u_0) \quad \forall \, t \in [t_0, \infty), \, x \in [0, \infty),$$

and

$$\limsup_{t \to \infty} ||u(t; \cdot; t_0, u_0)||_{\infty} \leq \frac{a_{\sup}}{b_{\inf} + \chi_2 \mu_2 - \chi_1 \mu_1 - M},$$

where

$$C(u_0) = \max\{||u_0||_{\infty}, \frac{a_{\sup}}{b_{\inf} + \chi_2 \mu_2 - \chi_1 \mu_1 - M}\}. \tag{9}$$

Theorem 1.2 (Persistence). Consider (2).

1. If (H1) holds, then for any $u_0 \in C^b_{\text{unif}}(\mathbb{R}^+)$ with $\inf_{x \in \mathbb{R}^+} u_0(x) > 0$, there is $m(u_0) > 0$ such that

$$m(u_0) \leq u(t, x; t_0, u_0) \leq C(u_0) \quad \forall \, t \geq t_0, \, x \in \mathbb{R}^+.$$

2. If (H2) holds, then there are $0 < m_0 < M_0$ such that for any $u_0 \in C^b_{\text{unif}}(\mathbb{R}^+)$ with $\inf_{x \in \mathbb{R}^+} u_0(x) > 0$, there is $T(u_0) > 0$ such that

$$m_0 \leq u(t, x; t_0, u_0) \leq M_0 \quad \forall \, t_0 \in \mathbb{R}, \, t \geq t_0 + T(u_0), \, x \in \mathbb{R}^+.$$

Theorem 1.3 (Positive entire solution). Consider (2).

1. (Existence of strictly positive entire solution) If (H1) holds, then (2) admits a strictly positive entire solution $(u^+(t, x), v_1^+(t, x), v_2^+(t, x))$. Moreover, if $a(t; x, t) \equiv a(t, x)$ and $b(t; x, t) \equiv b(t, x)$, then (2) admits a strictly positive $T$-periodic solution $(u^+(t, x), v_1^+(t, x), v_2^+(t, x)) = (u^+(t + T, x), v_1^+(t + T, x), v_2^+(t + T, x))$.

2. (Stability and uniqueness of strictly positive entire solution)

   (i) Assume (H3) and $a(t, x) \equiv a(t)$ and $b(t, x) \equiv b(t)$, then for any $u_0 \in C^b_{\text{unif}}(\mathbb{R}^+)$ with $\inf_{x \in \mathbb{R}} u_0(x) > 0$,

$$\lim_{t \to \infty} ||u(t + t_0, \cdot; t_0, u_0) - u^+(t + t_0, \cdot)||_{\infty} = 0, \forall \, t_0 \in \mathbb{R}. \tag{10}$$

   (ii) Assume (H3). There are $\chi_1^* > 0$ and $\chi_2^* > 0$ such that, if $0 \leq \chi_1 \leq \chi_1^*$ and $0 \leq \chi_2 \leq \chi_2^*$, then for any $u_0 \in C^b_{\text{unif}}(\mathbb{R}^+)$ with $\inf_{x \in \mathbb{R}} u_0(x) > 0$,

$$\lim_{t \to \infty} ||u(t; \cdot; t_0, u_0)||_{\infty} \leq \frac{a_{\sup}}{b_{\inf} + \chi_2 \mu_2 - \chi_1 \mu_1 - M}, \tag{10'}$$

Similar results to Theorems 1.1-1.3 hold for (5). More precisely, we have

Theorem 1.4. Consider (5). The following hold.

1. (Global existence) If (H1) holds, then for any $t_0 \in \mathbb{R}$ and any nonnegative function $u_0 \in C^b_{\text{unif}}(\mathbb{R})$, (5) has a unique solution $(u(t; x; t_0, u_0), v_1(t; x; t_0, u_0), v_2(t; x; t_0, u_0))$ with $u(t_0; x; t_0, u_0) = u_0(x)$ defined for $t \geq t_0$. Moreover,

$$0 \leq u(t, x; t_0, u_0) \leq C(u_0) \quad \forall \, t \in [t_0, \infty), \, x \in \mathbb{R},$$

and

$$\limsup_{t \to \infty} ||u(t; \cdot; t_0, u_0)||_{\infty} \leq \frac{a_{\sup}}{b_{\inf} + \chi_2 \mu_2 - \chi_1 \mu_1 - M},$$
Remark 1. (1) Note that $b_{\text{inf}} \geq \chi_1 \mu_1$ implies (H1), $b_{\text{inf}} \geq (1 + \frac{a_{\text{sup}}}{a_{\text{inf}}}) \chi_1 \mu_1$ implies (H2), and $b_{\text{inf}} > 2\chi_1 \mu_1$ implies (H3). In the case $\chi_2 = 0$, we can choose $\lambda_2 = \lambda_1$, and (H1) becomes $b_{\text{inf}} > \chi_1 \mu_1$, (H2) becomes $b_{\text{inf}} \geq (1 + \frac{a_{\text{sup}}}{a_{\text{inf}}}) \chi_1 \mu_1$, and (H3) becomes $b_{\text{inf}} > 2\chi_1 \mu_1$.

(2) In [22], an attraction-repulsion chemotaxis system with constant logistic source $u(a - bu)$ on the whole space is studied. Among others, it is proved in [22] that if (H1) holds, then (5) with $a(t, x) \equiv a$ and $b(t, x) \equiv b$ has a unique globally defined solution for any nonnegative, bounded, and uniformly continuous initial function (see [22, Theorem A]), and that if (H3) holds, then the constant solution $(\frac{a}{b}, \frac{a}{b}, \frac{a}{b}, \frac{a}{b})$ is globally stable with strictly positive perturbations (see [22, Theorem B]). Theorem 1.4 extends [22, Theorem A] and [22, Theorem B] for (5) with constant logistic source to time and space dependent logistic source. It should be mentioned that in [35], an attraction-repulsion chemotaxis system with constant logistic source $u(a - bu)$ on a bounded domain with Neumann boundary conditions is studied.

(3) (5) with $\chi_2 = 0$ is a special cases of the parabolic-elliptic chemotaxis model with space-time dependent logistic sources on $\mathbb{R}^N$ studied in [20] and [21].

Theorem 1.4 in the case $\chi_2 = 0$ is proved in [20] and [21] (see [20, Theorem 1.1], [21, Theorem 1.4], and [21, Theorem 1.5]). Theorem 1.4 also extends [20, Theorem 1.1], [21, Theorem 1.4], and [21, Theorem 1.5]) for the parabolic-elliptic chemotaxis model with space-time dependent logistic sources on the whole space to the parabolic-elliptic-elliptic chemotaxis model with space-time dependent logistic sources on the whole space.

where

$$C(u_0) = \max\{\|u_0\|_\infty, \frac{a_{\text{sup}}}{b_{\text{inf}} + \chi_2 \mu_2 - \chi_1 \mu_1 - M}\}.$$  

(2) (Persistence)

(i) If (H1) holds, then for any $u_0 \in C_{\text{unif}}(\mathbb{R})$ with $\inf_{x \in \mathbb{R}} u_0(x) > 0$, there is $m(u_0) > 0$ such that

$$m(u_0) \leq u(t, x; t_0, u_0) \leq C(u_0) \quad \forall t \geq t_0, \ x \in \mathbb{R}.$$

(ii) If (H2) holds, then there are $0 < m_0 < M_0$ such that for any $u_0 \in C_{\text{unif}}(\mathbb{R})$ with $\inf_{x \in \mathbb{R}} u_0(x) > 0$, there is $T(u_0) > 0$ such that

$$m_0 \leq u(t, x; t_0, u_0) \leq M_0 \quad \forall t \geq t_0 + T(u_0), \ x \in \mathbb{R}.$$

(3) (Existence of strictly positive entire solution) If (H1) holds, then (5) admits a strictly positive entire solution $(u^+(t, x), v_1^+(t, x), v_2^+(t, x))$. Moreover, if $a(t + T, x) \equiv a(t, x)$ and $b(t + T, x) \equiv b(t, x)$, then (5) admits a strictly positive $T-$ periodic solution $(u^+(t, x), v_1^+(t, x), v_2^+(t, x)) = (u^+(t + T, x), v_1^+(t + T, x), v_2^+(t + T, x)).$

(4) (Stability and uniqueness of strictly positive entire solution)

(i) Assume (H3) and $a(t, x) \equiv a(t)$ and $b(t, x) \equiv b(t)$, then for any $u_0 \in C_{\text{unif}}(\mathbb{R})$ with $\inf_{x \in \mathbb{R}} u_0(x) > 0$,

$$\lim_{t \to \infty} \|u(t + t_0, ; t_0, u_0) - u^+(t + t_0, \cdot)\|_\infty = 0, \forall t_0 \in \mathbb{R}. \quad (12)$$

(ii) Assume (H3). There are $\chi_1 > 0$ and $\chi_2 > 0$ such that, if $0 \leq \chi_1 \leq \chi_1^*$ and $0 \leq \chi_2 \leq \chi_2^*$, then for any $u_0 \in C_{\text{unif}}(\mathbb{R})$ with $\inf_{x \in \mathbb{R}} u_0(x) > 0,$ (12) holds.
(4) Logistic type attraction-repulsion chemotaxis systems on a half space are studied for the first time. The results stated in Theorems 1.1-1.3 are similar to those stated in Theorem 1.4 for logistic type attraction-repulsion chemotaxis systems on the whole space. Several existing techniques developed for the study of logistic type attraction-repulsion chemotaxis systems on a whole space are applied for the study of (2) with certain modifications. But, due to the presence of the boundary \( x = 0 \) as well as the unboundedness of the domain, such modifications are nontrivial and some other technical difficulties also arise in the study of (2).

The rest of this paper is organized in the following way. In section 2, we present some preliminary lemmas to be used in the proofs of the main results in later sections. We prove the main results of the paper in section 3. In this section, we present some lemmas to be used in the proof of the main results in later sections.

2. Preliminary lemmas. The first lemma is on the local existence of solutions of (2) and (5).

**Lemma 2.1.** (1) Consider (5). For any \( t_0 \in \mathbb{R} \) and any nonnegative function \( u_0 \in C^b_{\text{unif}}(\mathbb{R}) \), there is \( T_{\text{max}} > 0 \) such that (5) has a unique solution \((u(t, x; t_0, u_0), v_1(t, x; t_0, u_0), v_2(t, x; t_0, u_0))\) defined on \([t_0, t_0 + T_{\text{max}}]\) with \( u(t_0, x; t_0, u_0) = u_0(x) \). Moreover, if \( T_{\text{max}} < \infty \), then

\[
\limsup_{t \to T_{\text{max}}} \|u(t_0 + t, :, t_0, u_0)\|_\infty = \infty.
\]

(2) Consider (2). For any \( t_0 \in \mathbb{R} \) and any nonnegative function \( u_0 \in C^b_{\text{unif}}(\mathbb{R}^+) \), there is \( T_{\text{max}} > 0 \) such that (2) has a unique solution \((u(t, x; t_0, u_0), v_1(t, x; t_0, u_0), v_2(t, x; t_0, u_0))\) defined on \([t_0, t_0 + T_{\text{max}}]\) with \( u(t_0, x; t_0, u_0) = u_0(x) \). Moreover, if \( T_{\text{max}} < \infty \), then

\[
\limsup_{t \to T_{\text{max}}} \|u(t_0 + t, :, t_0, u_0)\|_\infty = \infty.
\]

**Proof.** (1) It follows from the similar arguments used in the proof of [23, Theorem 1.1]. For the reader’s convenience and for the proof of (2), we outline the proof in the following.

First, let \( T(t) \) be the semigroup generated by \( \partial_{xx} - I \) on \( C^b_{\text{unif}}(\mathbb{R}) \). Then for any \( u_0 \in C^b_{\text{unif}}(\mathbb{R}) \),

\[
(T(t)u_0)(x) = \frac{e^{-t}}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{t}} e^{-\frac{(x-y)^2}{4t}} u_0(y) dy
\]

for \( t > 0 \) and \( x \in \mathbb{R} \). Let \( u \in C^b_{\text{unif}}(\mathbb{R}) \) and set \( v = (\partial_{xx} - \lambda I)^{-1} u \). Then we have

\[
v(x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{\lambda s}{\sqrt{s}}} e^{-\frac{|x-y|^2}{4s}} u(z) dz ds.
\]

By [23, Lemma 3.2], \( T(t)\partial_x \) can be extended to \( C^b_{\text{unif}}(\mathbb{R}) \), and for any \( u \in C^b_{\text{unif}}(\mathbb{R}) \), there holds

\[
\|(T(t)\partial_x)u\|_\infty \leq \frac{1}{\sqrt{\pi}} t^{-\frac{1}{2}} \|u\|_\infty.
\]

By [23, Lemma 3.3], for any \( u \in C^b_{\text{unif}}(\mathbb{R}) \),

\[
\|\partial_x(\partial_{xx} - \lambda I)^{-1} u\|_\infty \leq \frac{1}{\sqrt{\lambda}} \|u\|_\infty.
\]
By the similar arguments as those in [23, Theorem 1.1], there is $\tau > 0$ such that (5) has a unique solution $u(t, x), v_1(t, x), v_2(t, x)) = (u(t, x; t_0, u_0), v_1(t, x; t_0, u_0), v_2(t, x; t_0, u_0))$ with $u(t_0, x; t_0, u_0) = u_0(x)$ defined on $[t_0, t_0 + \tau]$ and satisfying

$$u(t, \cdot) = T(t - t_0)u_0 + \frac{1}{\sqrt{t}} \int_{t_0}^{t} (T(t - s)\partial_x)(u(s, \cdot)\partial_x(\partial_{xx} - \lambda_1 I)^{-1}u(s))ds$$

$$- \frac{1}{\sqrt{t}} \int_{t_0}^{t} (T(t - s)\partial_x)(u(s, \cdot)\partial_x(\partial_{xx} - \lambda_2 I)^{-1}u(s))ds$$

$$+ \frac{1}{\sqrt{t}} \int_{t_0}^{t} T(t - s)(1 + a(s, \cdot))u(s, \cdot)ds - \int_{t_0}^{t} T(t - s)b(s, \cdot)u^2(s, \cdot)ds.$$

Now, by the standard extension arguments, there is $T_{\max} > 0$ such that (5) has a unique solution $(u(t, x; t_0, u_0), v_1(t, x; t_0, u_0), v_2(t, x; t_0, u_0))$ with $u(t_0, x; t_0, u_0) = u_0(x)$ defined on $[t_0, t_0 + T_{\max}]$, and if $T_{\max} < \infty$, then

$$\lim_{t \to t_0 + T_{\max}} \|u(t, \cdot; t_0, u_0)\|_{\infty} = \infty.$$

(2) It can be proved by the arguments in (1). To be more precise, first, let $\hat{T}(t)$ be the semigroup generated by $\partial_{xx} - I$ on $C^{\text{unif}}(\mathbb{R}^+)$ with Neumann boundary at 0. Then for any $u_0 \in C^{\text{unif}}(\mathbb{R}^+)$,

$$\hat{T}(t)u_0(x) = e^{-t} \int_{0}^{\infty} \sqrt{\pi} e^{-\frac{(\tau - y)^2}{4\tau}} u_0(y)dy$$

$$= e^{-t} \int_{-\infty}^{\infty} \sqrt{\pi} e^{-\frac{(\tau - y)^2}{4\tau}} \tilde{u}_0(y)dy$$

$$= T(t)\tilde{u}_0$$

for $t > 0$ and $x \in \mathbb{R}^+$, where $\tilde{u}_0(x) = u_0(|x|)$ for $x \in \mathbb{R}$. Let $u \in C^{\text{unif}}(\mathbb{R}^+)$ and set $v = (\partial_{xx} - \lambda I)^{-1}u$ on $[0, \infty)$. Then we have

$$v(x) = \frac{1}{2\sqrt{\pi}} \int_{0}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x + s}{2}} e^{-\frac{|z|^2}{4\tau}} \tilde{u}(z)dzds$$

(14)

for every $x \in \mathbb{R}^+$, where $\tilde{u}(z) = u(|z|)$. Hence by the arguments in (1), $\hat{T}(t)\partial_x$ can be extended to $C^{\text{unif}}(\mathbb{R}^+)$, and for any $u \in C^{\text{unif}}(\mathbb{R}^+)$, there holds

$$\|\hat{T}(t)\partial_x u\|_{\infty} \leq \frac{1}{\sqrt{\pi}} t^{-\frac{1}{2}}\|u\|_{\infty}.$$

Also, for any $u \in C^{\text{unif}}(\mathbb{R}^+)$,

$$\|\partial_x(\partial_{xx} - \lambda I)^{-1}u\|_{\infty} \leq \frac{1}{\sqrt{\lambda}} \|u\|_{\infty}.$$

We can then apply the arguments in [23, Theorem 1.1] to prove that there is $\tau > 0$ such that (2) has a unique solution $u(t, x), v_1(t, x), v_2(t, x)) = (u(t, x; t_0, u_0), v_1(t, x; t_0, u_0), v_2(t, x; t_0, u_0))$ with $u(t_0, x; t_0, u_0) = u_0(x)$ defined on $[t_0, t_0 + \tau]$ and satisfying

$$u(t, \cdot) = \hat{T}(t - t_0)u_0 + \frac{1}{\sqrt{t}} \int_{t_0}^{t} (\hat{T}(t - s)\partial_x)(u(s, \cdot)\partial_x(\partial_{xx} - \lambda_1 I)^{-1}u(s))ds$$

$$- \frac{1}{\sqrt{t}} \int_{t_0}^{t} (\hat{T}(t - s)\partial_x)(u(s, \cdot)\partial_x(\partial_{xx} - \lambda_2 I)^{-1}u(s))ds$$
Lemma 2.2. Assume $u$

For the completeness, we provide a proof for (2). It can be proved similarly for (5).

(1) It can be proved by some similar arguments as those in [22, Theorem A].

$\chi$

The second lemma is on the estimate of $\chi_2 \lambda_2 v_2 - \chi_1 \lambda_1 v_1$.

**Lemma 2.2.** Assume (H1) holds.

1. Suppose that $(u(t, x; t_0, u_0), v_1(t, x; t_0, u_0), v_2(t, x; t_0, u_0))$ is a solution of (2) (resp. (5)) on $[t_0, t_0 + T]$ with $u(t_0, \cdot; t_0, u_0) = u_0(\cdot)$, where $u_0 \in C^{b}_{\text{unif}}(\mathbb{R}^+)$ (resp. $u_0 \in C^{b}_{\text{unif}}(\mathbb{R})$). If $\|u(t, \cdot; t_0, u_0)\|_{\infty} \leq C_0 := C(u_0)$ for $t \in [t_0, t_0 + T]$, then

\[
(\chi_2 \lambda_2 v_2 - \chi_1 \lambda_1 v_1)(t, x; t_0, u_0) \leq M C_0 \quad \forall t \in [t_0, t_0 + T],
\]

where $M$ and $C_0(u_0)$ are as in (7) and (11), respectively.

2. Let $u(\cdot, \cdot) \in C^{b}_{\text{unif}}([t_0, \infty) \times \mathbb{R}^+)$ and $v_1(t, x; u)$ and $v_2(t, x; u)$ be the solutions of

\[
\begin{cases}
v_{1,xx} - \lambda_1 v_1 + \mu_1 u = 0, & x \in (0, \infty) \\
v_1(t, 0) = 0
\end{cases}
\]

and

\[
\begin{cases}
v_{2,xx} - \lambda_2 v_2 + \mu_2 u = 0, & x \in (0, \infty) \\
v_2(t, 0) = 0
\end{cases}
\]

respectively. Then

\[
|\chi_2 \lambda_2 v_2(t, \cdot; u) - \chi_1 \lambda_1 v_1(t, \cdot; u)|_{\infty} \leq K \|u(t, \cdot)|_{\infty} \quad \forall t \in [t_0, \infty),
\]

where $K$ is as in (8).

**Proof.** (1) It can be proved by some similar arguments as those in [22, Theorem A]. For the completeness, we provide a proof for (2). It can be proved similarly for (5).

Let $(u(t, x), v_1(t, x), v_2(t, x)) = (u(t, x; t_0, u_0), v_1(t, x; t_0, u_0), v_2(t, x; t_0, u_0))$. Note that $v_1(t, x)$ is the solution of

\[
\begin{cases}
v_{1,xx} - \lambda_1 v_1 + \mu_1 u(t, x) = 0, & 0 < x < \infty \\
v_1(t, 0) = 0
\end{cases}
\]

and $v_2(t, x)$ is the solution of

\[
\begin{cases}
v_{2,xx} - \lambda_2 v_2 + \mu_2 u(t, x) = 0, & 0 < x < \infty \\
v_2(t, 0) = 0
\end{cases}
\]

Let $T(s)$ be the semigroup generated by $\partial_{xx}$ on $C^{b}_{\text{unif}}(\mathbb{R}^+)$ with Neumann boundary at $x = 0$. Then

\[
v_i(t, \cdot) = \mu_i \int_0^{\infty} e^{-\lambda_i s} T(s) u(t, \cdot) ds, \quad i = 1, 2.
\]
We then have
\[(\chi_2 \lambda_2 v_2 - \chi_1 \lambda_1 v_1)(t, x) = \chi_2 \lambda_2 \mu_2 \int_0^\infty e^{-\lambda_2 s} \bar{\mathcal{T}}(s) u(t, \cdot) ds \]
\[- \chi_1 \lambda_1 \mu_1 \int_0^\infty e^{-\lambda_1 s} \bar{\mathcal{T}}(s) u(t, \cdot) ds \]
\[= (\chi_2 \lambda_2 - \chi_1 \lambda_1) \int_0^\infty e^{-\lambda_2 s} \bar{\mathcal{T}}(s) u(t, \cdot) ds \]
\[+ \chi_1 \lambda_1 \mu_1 \int_0^\infty (e^{-\lambda_2 s} - e^{-\lambda_1 s}) \bar{\mathcal{T}}(s) u(t, \cdot) ds \]
\[\leq (\chi_2 \lambda_2 - \chi_1 \lambda_1) + \int_0^\infty e^{-\lambda_2 s} \bar{\mathcal{T}}(s) u(t, \cdot) ds \]
\[+ \chi_1 \lambda_1 \mu_1 \int_0^\infty (e^{-\lambda_2 s} - e^{-\lambda_1 s}) \bar{\mathcal{T}}(s) u(t, \cdot) ds \]

Note that
\[\bar{\mathcal{T}}(s) u(t, \cdot) \leq \bar{\mathcal{T}}(s) C_0 = C_0.\]

Hence
\[\int_0^\infty e^{-\lambda_2 s} \bar{\mathcal{T}}(s) u(t, \cdot) ds \leq \int_0^\infty e^{-\lambda_2 s} C_0 ds \]
\[+ \int_0^\infty (\chi_1 \lambda_1 - \lambda_2) C_0 ds \]
\[= \frac{C_0}{\lambda_2} \left( \chi_2 \lambda_2 - \chi_1 \lambda_1 + \chi_1 \lambda_1 \lambda_1 - \lambda_2 \right).\]

Similarly, we can prove that
\[\int_0^\infty e^{-\lambda_2 s} \bar{\mathcal{T}}(s) u(t, \cdot) ds \leq \frac{C_0}{\lambda_1} \left( \chi_2 \lambda_2 - \chi_1 \lambda_1 + \chi_2 \lambda_2 - \chi_1 \lambda_1 \lambda_1 \right).\]

(1) then follows.

(2) Note that
\[v_i(t, \cdot; u) = \mu_i \int_0^\infty e^{-\lambda_i s} \bar{\mathcal{T}}(s) u(t, \cdot) ds, \quad i = 1, 2.\]

Hence
\[\left| (\chi_2 \lambda_2 v_2 - \chi_1 \lambda_1 v_1)(t, x; u) \right| = |\chi_2 \lambda_2 \mu_2 | \int_0^\infty e^{-\lambda_2 s} \bar{\mathcal{T}}(s) u(t, \cdot) ds \]
\[- \chi_1 \lambda_1 \mu_1 \int_0^\infty e^{-\lambda_1 s} \bar{\mathcal{T}}(s) u(t, \cdot) ds| \]
\[\leq \left| (\chi_2 \lambda_2 - \chi_1 \lambda_1) \int_0^\infty e^{-\lambda_2 s} \bar{\mathcal{T}}(s) u(t, \cdot) ds \right| \]
\[+ \left| \chi_1 \lambda_1 \mu_1 \int_0^\infty (e^{-\lambda_2 s} - e^{-\lambda_1 s}) \bar{\mathcal{T}}(s) u(t, \cdot) ds \right| \]
\[\leq |\chi_2 \lambda_2 - \chi_1 \lambda_1| \int_0^\infty e^{-\lambda_2 s} \bar{\mathcal{T}}(s) \| u(t, \cdot) \|_\infty ds \]
\[+ \chi_1 \lambda_1 \mu_1 \int_0^\infty | e^{-\lambda_2 s} - e^{-\lambda_1 s} | \bar{\mathcal{T}}(s) \| u(t, \cdot) \|_\infty ds \]
for $t \in [t_0, \infty)$. Similarly, it can be proved that
\[
|(\chi_2 \lambda_2 v_2 - \chi_1 \lambda_1 v_1)(t, x; u)| \leq \left(\frac{|\chi_2 \lambda_2 v_2 - \chi_1 \lambda_1 v_1|}{\lambda_1} + \frac{|\chi_2 \lambda_2 - \chi_1 \lambda_1|}{\lambda_2}\right)\|u(t, \cdot)\|_\infty
\]
for $t \in [t_0, \infty)$. It then follows that
\[
|(\chi_2 \lambda_2 v_2 - \chi_1 \lambda_1 v_1)(t, x; u)| \leq K\|u(t, \cdot)\|_\infty
\]
for $t \in [t_0, \infty)$. Hence (2) holds. \hfill \Box

The next lemma is on the upper bound of $u(t, x; t_0, u_0)$.

**Lemma 2.3.** Consider (2) and assume (H1). For any given $t_0 \in \mathbb{R}$ and $u_0 \in C^0_{\text{unif}}(\mathbb{R}^+)$ with $u_0 \geq 0$ and $u_0 \neq 0$, if $u(t, x; t_0, u_0)$ exists on $[t_0, \infty)$ and
\[
\limsup_{t \to \infty} \|u(t, \cdot; t_0, u_0)\| < \infty,
\]
then
\[
\limsup_{t \to \infty} \|u(t, \cdot; t_0, u_0)\|_\infty \leq \frac{\alpha_{\text{sup}}}{b_{\text{inf}} + \chi_2 \lambda_2 - \chi_1 \lambda_1 - M}.
\]

**Proof.** (1) For given $t_0 \in \mathbb{R}$ and $u_0 \in C^0_{\text{unif}}(\mathbb{R}^+)$ with $u_0 \geq 0$ and $u_0 \neq 0$, assume that $u(t, x; t_0, u_0)$ exists on $[t_0, \infty)$ and $\limsup_{t \to \infty} \|u(t, \cdot; t_0, u_0)\| < \infty$. Let
\[
\bar{u} = \limsup_{t \to \infty} \sup_{x \in \mathbb{R}^N} u(x, t; t_0, u_0).
\]
By the assumption, $\bar{u} < \infty$. Then for every $\varepsilon > 0$, there is $T_\varepsilon > 0$ such that
\[
u(t + t_0, x; t_0, u_0) \leq \bar{u} + \varepsilon \quad \forall \; x \in \mathbb{R}^+, \; \forall \; t \geq T_\varepsilon.
\]
Hence, it follows from comparison principle for elliptic equations, that
\[
\lambda_i v_i(t + t_0, x; t_0, u_0) \leq \lambda_i (\bar{u} + \varepsilon), \forall \; x \in \mathbb{R}^+, \; \forall \; t \geq T_\varepsilon, \; i = 1, 2.
\]

By similar arguments as those in Lemma 2.2, we have
\[
(\chi_2 \lambda_2 v_2 - \chi_1 \lambda_1 v_1)(t, x; t_0, u_0) \leq \frac{\bar{u} + \varepsilon}{\lambda_2} \left((\chi_2 \lambda_2 v_2 - \chi_1 \lambda_1 v_1) + \chi_1 \lambda_1 (\lambda_1 - \lambda_2)\right)
\]
and
\[
(\chi_2 \lambda_2 v_2 - \chi_1 \lambda_1 v_1)(t, x; t_0, u_0) \leq \frac{\bar{u} + \varepsilon}{\lambda_1} \left((\chi_2 \lambda_2 - \chi_1 \lambda_1) + (\chi_2 \lambda_2 - \chi_1 \lambda_1)\right)
\]
for $t \geq t_0 + T_\varepsilon$. This implies that
\[
(\chi_2 \lambda_2 v_2 - \chi_1 \lambda_1 v_1)(t, x; t_0, u_0) \leq M(\bar{u} + \varepsilon)
\]
for $t \geq t_0 + T_\varepsilon$ and then
\[
u_t \leq \nu_{xx} \left((\chi_2 v_2 - \chi_1 v_1) \nu_x + (\alpha_{\text{sup}} + M(\bar{u} + \varepsilon)) \nu - (b_{\text{inf}} + \chi_2 \lambda_2 - \chi_1 \lambda_1) \nu\right)
\]
for $t \geq t_0 + T_\varepsilon$.

By (16) and comparison principle for parabolic equations,
\[
\nu(t, x; t_0, u_0) \leq U_\varepsilon(t), \quad \forall \; t \geq t_0 + T_\varepsilon, \; x \in \mathbb{R}^+,
\]
where $U_\varepsilon(t)$ is the solution of
\[
\begin{cases}
U' = (\alpha_{\text{sup}} + M(\bar{u} + \varepsilon))U - (b_{\text{inf}} + \chi_2 \lambda_2 - \chi_1 \lambda_1)U)U
\\U(t_0 + T_\varepsilon) = \|u(t_0 + T_\varepsilon, \cdot; t_0, u_0)\|_\infty.
\end{cases}
\]
Note that
\[
\lim_{t \to \infty} U_\varepsilon(t) = \frac{a_{\text{sup}} + M(\bar{u} + \varepsilon)}{b_{\text{inf}} + \chi_2 \mu_2 - \chi_1 \mu_1}.
\]
It then follows that
\[
\bar{u} = \limsup_{t \to \infty} \|u(t, \cdot; t_0, u_0)\|_\infty \leq \frac{a_{\text{sup}} + M(\bar{u} + \varepsilon)}{b_{\text{inf}} + \chi_2 \mu_2 - \chi_1 \mu_1}
\]
and then
\[
\bar{u} \leq \frac{a_{\text{sup}}}{b_{\text{inf}} + \chi_2 \mu_2 - \chi_1 \mu_1} - \varepsilon.
\]
The lemma is thus proved.

Before we state the next lemma, let \(a_0 = \frac{a_{\text{inf}}}{3}\) and \(L > 0\) be a given constant. Consider
\[
\begin{cases}
    u_t = u_{xx} + a_0 u, & x \in (-L, L) \\
    u(t, -L) = u(t, L) = 0,
\end{cases}
\]
and its associated eigenvalue problem
\[
\begin{cases}
    u_{xx} + a_0 u = \sigma u, & x \in (-L, L) \\
    u(t, -L) = u(t, L) = 0.
\end{cases}
\]
Let \(\sigma_L\) be the principal eigenvalue of (18) and \(\phi_L(x)\) be its principal eigenfunction with \(\phi_L(0) = 1\). Note that
\[
\sigma_L = -\frac{\pi^2}{4L^2} + a_0
\]
and
\[
\phi_L(x) = \cos\left(\frac{\pi}{2L} x\right) \quad \text{and} \quad 0 < \phi_L(x) \leq \phi_L(0), \quad \forall x \in (-L, L).
\]
Note also that \(u(t, x) = e^{\sigma_L t} \phi_L(x)\) is a solution of (17). Let \(u(t, x; u_0)\) be the solution of (17) with \(u_0 \in C([-L, L])\). Then
\[
u(t, x; \kappa \phi_L) = \kappa e^{\sigma_L t} \phi_L(x)
\]
for all \(\kappa \in \mathbb{R}\). Moreover, we have that \(\phi_L(x)\) satisfies
\[
\begin{cases}
    u_{xx} + a_0 u = \sigma_L u, & x \in (0, L) \\
    u_x(t, 0) = u(t, L) = 0,
\end{cases}
\]
and (19) also holds when \(u(t, x; \kappa \phi_L)\) is the solution of
\[
\begin{cases}
    u_t = u_{xx} + a_0 u, & x \in (0, L) \\
    u_x(t, 0) = u(t, L) = 0
\end{cases}
\]
with \(u(0, x; \kappa \phi_L) = \kappa \phi_L(x)\) for \(x \in [0, L]\).

In the following, fix \(T_0 > 0\) and let \(L_0 \gg 0\) be such that \(\sigma_{L_0} > 0\). Note that \(\sigma_L\) is increasing as \(L\) increases. Choose \(N_0 \in \mathbb{N}\) and \(\alpha_1 > 1\) such that
\[
\min\{e^{\sigma_{L_0} T_0}, e^{\sigma_{N_0} L_0 T_0} \cos\left(\frac{\pi}{2N_0}\right)\} \geq \alpha_1.
\]
Then
\[
\begin{cases}
    u(T_0, 0; \kappa \phi_L) = \kappa e^{\sigma_L T_0} \geq \alpha_1 \kappa \\
    u(T_0, x; \kappa \phi_{N_0 L}) = \kappa e^{\sigma_{N_0 L} T_0} \phi_{N_0 L}(x) \geq \alpha_1 \kappa \quad \text{for} \quad |x| \leq L
\end{cases}
\]
for any \(L \geq L_0\).
Lemma 2.4. Consider (2) and assume (H1). There is

\[ 0 < \delta_0^* < M^+ = \frac{a_{\sup}}{b_{\inf} + \chi_2 \mu_2 - \chi_1 \mu_1 - M} + 1 \]

such that for any \( 0 < \delta \leq \delta_0^* \) and for any \( u \in C^{b}_{\text{unif}}(\mathbb{R}^+) \) with \( \delta \leq u_0 \leq M^+ \),

\[ \delta \leq u(t_0 + T_0, x; t_0, u_0) \leq M^+ \quad \forall \ x \in \mathbb{R}^+, \forall \ t_0 \in \mathbb{R}. \quad (23) \]

Proof. It can be proved by applying properly modified arguments in [20, Lemma 3.5]. But the modification is not trivial. For the reader’s convenience, we provide some outline of the proof in the following.

First, choose \( \alpha_2 \) and \( \alpha_3 \) such that \( \alpha_1 > \alpha_2 > \alpha_3 > 1 \). Consider

\[
\begin{cases}
  u_t = u_{xx} + b_1(x, t)u_x + a_0u, & x \in (-L, L) \\
  u(t, -L) = u(t, L) = 0
\end{cases}
\]

and

\[
\begin{cases}
  u_t = u_{xx} + b_1(t, x)u_x + a_0u, & x \in (0, N_0L) \\
  u_x(t, 0) = u(t, N_0L) = 0,
\end{cases}
\]

where \( |b_1(x, t)| < \epsilon \) and \( t_0 \leq t \leq t_0 + T_0 \). Let \( u_{b_1}(t, x; t_0, u_0) \) be the solution of (24) (resp. (25)) with \( u_{b_1}(t_0, x; t_0, u_0) = u_0(x) \). By the similar arguments as those in Step 1 of [20, Lemma 3.5], it can be proved that there is \( \epsilon_0 > 0 \) such that for any \( L \geq L_0, k > 0 \), and \( 0 \leq \epsilon \leq \epsilon_0 \),

\[
u_{b_1,L}(t_0 + T_0, 0; t_0, \kappa \phi_L) \geq \alpha_2 \kappa
\]

(26)

(resp. \( u_{b_1}(t_0 + T_0, x; t_0, \kappa \phi_{N_0L}) \geq \alpha_2 \kappa \), for \( 0 \leq x \leq L \) provided that \( |b_1(t, x)| < \epsilon \) for \( x \in [-L, L] \) (resp. \( x \in [0, N_0L] \)); and for any \( L \geq L_0 \) and \( 0 \leq \epsilon \leq \epsilon_0 \),

\[
\begin{cases}
  0 \leq u_{b_1}(t + t_0, x; t_0, \kappa \phi_L) \leq e^{\alpha_2 \kappa} \quad \forall \ 0 \leq t \leq T_0, \ x \in [-L, L] \\
  (\text{resp.} \ 0 \leq u(t + t_0, x; t_0, \kappa \phi_{N_0L}) \leq e^{\alpha_2 \kappa} \quad \forall \ 0 \leq t \leq T_0, \ x \in [0, N_0L]).
\end{cases}
\]

(27)

Second, consider

\[
\begin{cases}
  u_t = u_{xx} + b_1(t, x)u_x + u(2a_0 - c(t, x)u), & x \in (-L, L) \\
  u(t, -L) = u(t, L) = 0
\end{cases}
\]

and

\[
\begin{cases}
  u_t = u_{xx} + b_1(t, x)u_x + u(2a_0 - c(t, x)u), & x \in (0, N_0L) \\
  u_x(t, 0) = u(t, N_0L) = 0,
\end{cases}
\]

where \( 0 \leq c(t, x) \leq b_{\sup} + \chi_2 \mu_2 \). Let \( u(t, x; t_0, u_0) \) be the solution of (28) (resp. (29)) with \( u(t_0, x; t_0, u_0) = u_0(x) \). Assume \( L \geq L_0 \) and \( 0 \leq \epsilon \leq \epsilon_0 \). By the similar arguments as those in Step 2 of [20, Lemma 3.5], it can be proved that there is \( \kappa_0 > 0 \) such that

\[
u_{\kappa_0}(t_0 + T_0, 0; t_0, \kappa \phi_L) \geq \alpha_3 \kappa \quad \text{(resp.} \ u_{\kappa_0}(t_0 + T_0, 0; t_0, \kappa \phi_{N_0L}) \geq \alpha_3 \kappa \quad \text{for} \ 0 \leq x \leq L)
\]

(30)

for all \( 0 < \kappa \leq \kappa_0 \).

Third, assume that \( (u(t, x; t_0, u_0), v_1(t, x; t_0, u_0), v_2(t, x; t_0, u_0)) \) is the solution of (2) on \( [t_0, t_0 + T_0] \) with \( u(t_0, x; t_0, u_0) = u_0(\cdot) \in C^{b}_{\text{unif}}(\mathbb{R}^+) \). By the similar arguments as those in Step 3 of [20, Lemma 3.5], it can be proved that there is \( 0 < \delta_0 \leq \kappa_0 \).
such that for any $u_0 \in C^b_{\text{unif}}(\mathbb{R}^+)$ and $x_0 \in \mathbb{R}^+$ with $0 \leq u_0 \leq M^+$ and $u_0(x) < \delta_0$ for $x \in \mathbb{R}^+$, $|x - x_0| \leq 3N_0L$, there holds

$$
\begin{align*}
0 & \leq \lambda_1 v_1(t, x; t_0, u_0) \leq \frac{\alpha_0}{4x_1} \\
0 & \leq \lambda_2 v_2(t, x; t_0, u_0) \leq \frac{\alpha_0}{4x_2} \\
|\nabla v_1(t, x; t_0, u_0)| & < \frac{\alpha_0}{4x_1} \\
|\nabla v_2(t, x; t_0, u_0)| & < \frac{\alpha_0}{4x_2}
\end{align*}
$$

(31)

for $t_0 \leq t \leq t_0 + T_0$, $x \in \mathbb{R}^+$ with $|x - x_0| \leq N_0L$ provided that $L \gg 1$.

Fourth, we claim that there is $0 < \delta_0^* < \min\{\delta_0, M^+\}$ such that for any $0 < \delta \leq \delta_0^*$ and for any $u_0 \in C^b_{\text{unif}}(\mathbb{R}^+)$ with $\delta \leq u_0 \leq M^+$,

$$
\delta \leq u(t_0 + T_0, x; t_0, u_0) \leq M^+ \quad \forall \ x \in \mathbb{R}^+.
$$

(32)

Assume that the above claim does not hold. Then there are $\delta_n \to 0$, $t_{0n} \in \mathbb{R}$, $u_{0n} \in C^b_{\text{unif}}(\mathbb{R}^+)$ with $\delta_n \leq u_{0n} \leq M^+$, and $x_n \in \mathbb{R}^+$ such that

$$
u(t_{0n} + T_0, x_n; t_{0n}, u_{0n}) < \delta_n.
$$

(33)

For fixed $L \gg 1$, let

$$
D_{0n} = \{x \in \mathbb{R}^+ \mid |x - x_n| < 3N_0L, \ u_{0n}(x) > \frac{\delta_0}{2}\}.
$$

Without loss of generality, we may assume that $\lim_{n \to \infty} |D_{0n}|$ exists, where $|D_{0n}|$ is the Lebesgue measure of $D_{0n}$.

Assume that $\lim_{n \to \infty} |D_{0n}| = 0$. Let \{\tilde{u}_{0n}\}_{n \geq 1} be a sequence of elements of $C^b_{\text{unif}}(\mathbb{R}^+)$ satisfying

$$
\begin{align*}
\delta_n & \leq \tilde{u}_{0n}(x) \leq \frac{\delta_0}{2}, \ x \in \mathbb{R}^+, |x - x_n| \leq 3N_0L \quad \text{and} \\
\|\tilde{u}_{0n}(\cdot) - u_{0n}(\cdot)\|_{L^p(\mathbb{R}^+)} & \to 0, \ \forall p > 1.
\end{align*}
$$

Let $w_n(x, t) := u(t + t_{0n}, x; t_{0n}, u_{0n}(\cdot)) - u(t + t_{0n}, x; t_{0n}, \tilde{u}_{0n}(\cdot))$ and $v_{i,n}(x, t) := v_i(t + t_{0n}, x; t_{0n}, u_{0n}(\cdot)) - v_i(t + t_{0n}, x; t_{0n}, \tilde{u}_{0n}(\cdot))$, $i = 1, 2$. By the similar arguments as those in Step 4 of [20, Lemma 3.5], it can be proved that

$$
\lim_{n \to \infty} \sup_{t_{0n} \leq t \leq t_{0n} + T_0} \|w_n(t, \cdot)\|_{L^p(\mathbb{R}^+)} = 0
$$

(34)

and

$$
\lim_{n \to \infty} \sup_{t_{0n} \leq t \leq t_{0n} + T_0} \|v_{i,n}(t, \cdot)\|_{C^b_{\text{unif}}(\mathbb{R}^+)} = 0, \ i = 1, 2.
$$

(35)

By (31), for every $n \geq 1$,

$$
\begin{align*}
0 & \leq \chi_1 v_1(t + t_{0n}, x; t_{0n}, \tilde{u}_{0n}) \leq \frac{\alpha_0}{4x_1} \\
0 & \leq \chi_2 v_2(t + t_{0n}, x; t_{0n}, \tilde{u}_{0n}) \leq \frac{\alpha_0}{4x_2} \\
|\chi_1 v_1(t + t_{0n}, x; t_{0n}, \tilde{u}_{0n})| & \leq \frac{\alpha_0}{4} \\
|\chi_2 v_2(t + t_{0n}, x; t_{0n}, \tilde{u}_{0n})| & \leq \frac{\alpha_0}{4}
\end{align*}
$$

for all $0 \leq t \leq T_0$ and $x \in \mathbb{R}^+$ with $|x - x_n| \leq N_0L$. This together with (35) implies that, for $n \gg 1$, there holds

$$
\begin{align*}
0 & \leq \chi_1 \lambda_1 v_1(t + t_{0n}, x; t_{0n}, u_{0n}) \leq \frac{\alpha_0}{2} \\
0 & \leq \chi_2 \lambda_2 v_2(t + t_{0n}, x; t_{0n}, u_{0n}) \leq \frac{\alpha_0}{2} \\
|\chi_1 v_1(t + t_{0n}, x; t_{0n}, u_{0n}(\cdot))| & \leq \frac{\alpha_0}{2} \\
|\chi_2 v_2(t + t_{0n}, x; t_{0n}, u_{0n}(\cdot))| & \leq \frac{\alpha_0}{2}
\end{align*}
$$
for all $0 \leq t \leq T_0$ and $x \in \mathbb{R}^+$ with $|x - x_n| \leq N_0 L$. Hence

$$
\begin{cases}
|\chi_1 v_{1x}(t + t_0, x; t_0, u_{0n}) - \chi_2 v_{2x}(t + t_0, x; t_0, u_{0n})| \leq \varepsilon_0 \\
|\chi_1 \lambda v_1(t + t_0, x; t_0, u_{0n}) - \chi_2 \lambda v_2(t + t_0, x; t_0, u_{0n})| \leq a_0
\end{cases}
$$

for all $0 \leq t \leq T_0$ and $x \in \mathbb{R}^+$ with $|x - x_n| \leq N_0 L$.

Let

$$
b_n(t, x) = -\chi_1 v_{1x}(t + t_0, x; t_0, u_{0n}) + \chi_2 v_{2x}(t + t_0, x; t_0, u_{0n})
$$

and

$$
u_n(t, x) = u(t + t_0, x + x_n; t_0, u_{0n}).
$$

In the case $x_n > L$, $u_n(t, x)$ satisfies

$$
\begin{cases}
u_t \geq u_{xx} + b_n(t, x)u_x + u(2a_0 - (b_{sup} + \chi_2 \mu_2)u), & -L < x < L \\
u(t, -L) > 0, \quad u(t, L) > 0 \\
u(t_0, x) \geq \delta_n, \quad x \in [-L, L].
\end{cases}
$$

In the case $x_n \leq L$, $u_n(t, x)$ satisfies

$$
\begin{cases}
u_t \geq u_{xx} + b_n(t, x)u_x + u(2a_0 - (b_{sup} + \chi_2 \mu_2)u), & -x_n < x < N_0 L \\
u_x(t, -x_n) = 0, \quad u(t, L) > 0 \\
u(t_0, x) \geq \delta_n, \quad x \in [-x_n, N_0 L].
\end{cases}
$$

In either case, it follows from the arguments of (30) that

$$u(T + t_0, x_n; t_0, u_{0n}) = u_n(T_0, 0) > \delta_n,$$

which is a contradiction. Hence $\lim_{n \to \infty} |D_{0n}| \neq 0$.

Without loss of generality, we may then assume that $\inf_{n \geq 1} |D_{0n}| > 0$ and there is $L > 0$ such that

$$\inf_{n \geq 1} \{|x \in \mathbb{R}^+ | x \in D_{0n} \cap [x_n - 3N_0 L, x_n + 3N_0 L]|\} > 0.$$

By the similar arguments as those in Step 4 of [20, Lemma 3.5], it can be proved that there is $0 < \tilde{T}_0 < T_0$ such that

$$\inf_{n \geq 1} \|u(t_0 + \tilde{T}_0, \cdot; t_0, u_{0n})\|_{C([x_n - 3N_0 L, x_n + 3N_0 L] \cap [0, x_n + 3N_0 L])} > 0.$$

Moreover, we might suppose that $x_n \to x^* \in [0, \infty]$ and $u(T_0 + t_0, x_n + \cdot; t_0, u_{0n}(\cdot + x_n)) \to u^*(\cdot)$ locally uniformly on $(-x^*, \infty)$ and $\|u^*_0\|_{C([x_n - 3N_0 L, x_n + 3N_0 L] \cap [-x^*, 3N_0 L])} > 0$. Also, we might assume that $(u(t + t_0, x_n + \cdot; t_0, u_{0n}), v_1(t + t_0, x_n + \cdot; t_0, u_{0n}), v_2(t + t_0, x_n + \cdot; t_0, u_{0n})) \to (u^*(t, x), v_1^*(t, x), v_2^*(t, x))$ locally uniformly on $[T_0, \infty) \times (-x^*, \infty)$, $a(t, x + x_n) \to a^*(t, x)$, and $b(t, x + x_n) \to b^*(t, x)$, where $(u^*, v_1^*, v_2^*)$ satisfies

$$
\begin{cases}
u_{xx}^* - \chi_1 u^* v_{1x}^* + \chi_2 u^* v_{2x}^* + (a^*(t, x) - b^*(t, x)u^*)u^*, & -\infty < x < \infty \\
0 = v_{1xx}^* - \lambda_1 v_{1x}^* + \mu_1 u^*, & -\infty < x < \infty \\
0 = v_{2xx}^* - \lambda_2 v_{2x}^* + \mu_2 u^*, & -\infty < x < \infty \\
u^*(T_0, \cdot) = u_0^*
\end{cases}
$$
Proofs of the main results. In this section, we prove Theorems 1.1-1.3 and Theorem 1.4. We mainly provide the proof for Theorems 1.1-1.3. Theorem 1.4 can be proved by the similar arguments of Theorems 1.1-1.3.

3.1. Global existence. In this subsection, we prove Theorem 1.1 for the global existence of solutions of (2) with nonnegative initial functions.

Proof of Theorem 1.1. By Lemma 2.1, for any $t_0 \in \mathbb{R}$ and any nonnegative function $u_0 \in C^b_{\text{unif}}(\mathbb{R}^+)$, (2) has a unique solution $(u(t, x; t_0, u_0), v_1(t, x; t_0, u_0), v_2(t, x; t_0, u_0))$ with $u(t_0, x; t_0, u_0) = u_0(x)$ defined on $[t_0, t_0 + T_{\text{max}}]$. Moreover, if $T_{\text{max}} < \infty$, then

$$
\limsup_{t \to T_{\text{max}}} \|u(t_0 + t, \cdot; t_0, u_0)\|_\infty = \infty.
$$

Let $C_0 = C_0(u_0)$ be as in Lemma 2.2. For any give $0 < T < T_{\text{max}}$, let

$$
\mathcal{E}^T = C^b_{\text{unif}}([0, T] \times \mathbb{R}^+)
$$

endowed with the norm

$$
\|u\|_{\mathcal{E}^T} := \sup_{k=1}^{\infty} \frac{1}{2^k} \|u\|_{L^\infty([0, T] \times [0, k])}.
$$

(36)

Consider the subset $\mathcal{E}$ of $\mathcal{E}^T$ defined by

$$
\mathcal{E} := \{u \in C^b_{\text{unif}}([0, T] \times \mathbb{R}^+) \mid u(0, \cdot) = u_0, 0 \leq u(x, t) \leq C_0, x \in \mathbb{R}^+, 0 \leq t \leq T\}.
$$

It is clear that

$$
\|u\|_{\mathcal{E}^T} \leq C_0, \quad \forall u \in \mathcal{E}.
$$

(37)

Moreover, $\mathcal{E}$ is a closed bounded and convex subset of $\mathcal{E}^T$. We shall show that $u(t_0 + \cdot, \cdot; t_0, u_0) \in \mathcal{E}$.

To this end, for any given $u \in \mathcal{E}$, let $v_i(t, x; u)$ be the solution of

$$
\begin{cases}
0 = v_{i,xx} - \lambda_i v + \mu_i u(t, x), \quad x \in \mathbb{R}^+ \\
\frac{\partial v_i}{\partial n}(t, 0) = 0.
\end{cases}
$$

Let $U(t, x; u)$ be the solution of the initial value problem

in the case $x^* = \infty$, and satisfies

$$
\begin{aligned}
u^*_{xx} &= u^* - \chi_1(u^*v^*_1)_x + \chi_2(u^*v^*_2)_x + (a^*(t, x) - b^*(t, x)u^*)_x, \quad -x^* < x < \infty \\
0 &= v^*_1_{xx} - \lambda_1 v^*_1 + \mu_1 u^*, \quad -x^* < x < \infty \\
0 &= v^*_2_{xx} - \lambda_2 v^*_2 + \mu_2 u^*, \quad -x^* < x < \infty \\
u^*_x(t, -x^*) &= v^*_1(t, -x^*) = v^*_2(t, -x^*) = 0 \\
u^*(T_0, \cdot) &= v_0
\end{aligned}
$$

in the case $x^* < \infty$. Since $\|u^*_0\|_\infty > 0$ and $u^*(t, x) \geq 0$, it follows from comparison principle for parabolic equations that $u^*(t, x) > 0$ for every $x \in (-x^*, \infty)$ and $t \in (T_0, \infty)$. In particular $u^*(T_0, 0) > 0$. Note by (33) that we must have $u^*(T_0, 0) = 0$, which is a contradiction. Hence the claim (32) holds. The lemma is thus proved. \qed
Lemma 2.4. For any $u$\footnote{We refer to [19, Section 4.3] for the details.} \begin{equation}
abla\chi_2v_2(t,x) - \chi_1v_1(t,x)U \leq MC_0 \quad \forall t \in [t_0,t_0 + T].
\end{equation}
Hence for $t \in (t_0,t_0 + T)$,
\begin{equation}
\begin{cases}
U_t - \Delta U + \nabla(\chi_2v_2(t,x) - \chi_1v_1(t,x)U) + U \left( a_{sup} + C_0M - (b_{inf} + \chi_2\mu_2 - \chi_1\mu_1)U \right) \\
\frac{\partial U}{\partial n}(t,0) = 0 \\
U(t_0,\cdot;u) = u_0(\cdot).
\end{cases}
\end{equation}
Observe that $U \equiv C_0$ is a super-solution of (39). Hence by comparison principle
for parabolic equations, we have
\begin{equation}
U(t,\cdot;u) \leq C_0 \quad t \in [t_0,t_0 + T], \ x \in \mathbb{R}^+.
\end{equation}
Therefore, $U(\cdot,\cdot;u) \in \mathcal{E}$.

By the similar arguments as those in [19, Lemma 4.3], the mapping $E \ni u \mapsto U(\cdot,\cdot;u) \in \mathcal{E}$ is continuous and compact, and then by Schauder’s fixed theorem, it has a fixed point $u^*$. Clearly $(u^*(\cdot,\cdot),v_1(\cdot,\cdot;u^*),v_2(\cdot,\cdot;u^*))$ is a classical solution
of (2). Thus, by Lemma 2.1, we have
\begin{equation}
\lim_{t \rightarrow \infty} \sup_{R} \|u(t,\cdot;u_0)\|_{\infty} \leq \frac{a_{sup}}{b_{inf} + \chi_2\mu_2 - \chi_1\mu_1 - M}.
\end{equation}

Theorem 1.1 then follows. \hfill \Box

3.2. Persistence. In this subsection, we prove Theorem 1.2 on the persistence of solutions of (2) with strictly positive initial functions.

**Proof of Theorem 1.2.** (1) Assume that (H1) holds. Fix $T_0 > 0$. Let $\delta_{0}^+$ and $M^+$ be as in Lemma 2.4. For any $u_0 \in C^{b}_{uni}(\mathbb{R}^+)$ with $\lim_{x \rightarrow \infty} u_0(x) > 0$, by Theorem 1.1,
\begin{equation}
\begin{aligned}
\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^+} u(t_0,\cdot;u_0) &\leq C(u_0) \quad \forall t \geq t_0, \ x \in \mathbb{R}^+.
\end{aligned}
\end{equation}

By Lemma 2.3, there is $T_1 > 0$ such that
\begin{equation}
\begin{aligned}
u(t,\cdot;u_0) &\leq M^+ \quad \forall t \geq t_0 + T_1, \ x \in \mathbb{R}^+.
\end{aligned}
\end{equation}

Observe that
\begin{equation}
\begin{aligned}
\inf_{x \in \mathbb{R}^+} u(t_0 + T_1,\cdot;u_0) > 0.
\end{aligned}
\end{equation}
Then there is $0 < \delta \leq \delta_{0}^+$ such that
\begin{equation}
\begin{aligned}
\delta \leq u(t_0 + T_1,\cdot;u_0) \leq M^+ \quad \forall x \in \mathbb{R}^+.
\end{aligned}
\end{equation}

\begin{equation}
\begin{cases}
U_t = \Delta U + \nabla(\chi_2v_2(t,x;u) - \chi_1v_1(t,x;u))U \\
\frac{\partial U}{\partial n}(t,0) = 0 \\
U(t_0,\cdot;u) = u_0(\cdot).
\end{cases}
\end{equation}
By Lemma 2.4, 
\[ \delta \leq u(t_0 + T_1 + nT_0, x; t_0, u_0) \leq M^+ \quad \forall \ n \in \mathbb{N}, \ x \in \mathbb{R}^+. \]
This implies that there is \( m(u_0) > 0 \) such that
\[ m(u_0) \leq u(t, x; t_0, u_0) \leq M^+ \quad \forall \ t \geq t_0, \ x \in \mathbb{R}^+. \]

(2) Assume that \((H2)\) holds. Let
\[ M_0 = \frac{a_{\sup}}{b_{\inf} - \chi_1 \mu_1 + \chi_2 \mu_2 - M} \]
and
\[ m_0 = \frac{a_{\inf}(b_{\inf} - (1 + \frac{a_{\sup}}{a_{\inf}})\chi_1 \mu_1 + \chi_2 \mu_2 - M)}{(b_{\inf} - \chi_1 \mu_1 + \chi_2 \mu_2 - M)(b_{\sup} - \chi_1 \mu_1 + \chi_2 \mu_2)}. \]
By \((H2)\), \( m_0 > 0 \). For given \( u_0 \in C^b_{\text{unif}}(\mathbb{R}^+) \) with \( \inf_{x \in \mathbb{R}^+} u_0(x) > 0 \), define
\[ \underline{v} := \liminf_{t \to \infty} \inf_{x \in \mathbb{R}^+} u(x, t + t_0; t_0, u_0) \quad \text{and} \quad \overline{v} := \limsup_{t \to \infty} \sup_{x \in \mathbb{R}^+} u(x, t + t_0; t_0, u_0). \]
If suffices to prove that
\[ m_0 \leq \underline{v} \leq \overline{u} \leq M_0. \]

By (1), \( \underline{u} > 0 \). Using the definition of limsup and liminf, we have that for every \( 0 < \varepsilon < \underline{u} \), there is \( T_{\varepsilon} > 0 \) such that
\[ \underline{u} - \varepsilon \leq u(x, t; t_0, u_0) \leq \overline{u} + \varepsilon \quad \forall \ x \in \mathbb{R}^+, \ \forall \ t \geq t_0 + T_{\varepsilon}. \]
Hence, it follows from comparison principle for elliptic equations, that
\[ \mu_i(\underline{u} - \varepsilon) \leq \lambda_i v_i(x, t; t_0, u_0) \leq \mu_i(\overline{v} + \varepsilon), \ \forall \ x \in \mathbb{R}^+, \ \forall \ t \geq T_{\varepsilon}, \ i = 1, 2. \quad (40) \]
We then have
\[ u_t = u_{xx} + (\chi_2 v_2 - \chi_1 v_1) u_x + u(a(t, x) - \chi_1 \lambda_1 v_1 + \chi_1 \mu_1 v_2 - (b(t, x) - \chi_1 \mu_1 + \chi_2 \mu_2) u) \geq u_{xx} + (\chi_2 v_2 - \chi_1 v_1) u_x + u(a_{\inf} - \chi_1 \mu_1 \overline{u} + \varepsilon + \chi_2 \mu_2 (\underline{u} - \varepsilon) - (b_{\sup} - \chi_1 \mu_1 + \chi_2 \mu_2) u) \]
for \( t \geq t_0 + T_{\varepsilon} \). This together with comparison principle for parabolic equations implies that
\[ \underline{u} \geq \frac{a_{\inf} - \chi_1 \mu_1 (\overline{u} + \varepsilon) + \chi_2 \mu_2 (\underline{u} - \varepsilon)}{(b_{\sup} - \chi_1 \mu_1 + \chi_2 \mu_2)}. \]
Let \( \varepsilon \to 0 \), we have
\[ \underline{u} \geq \frac{a_{\inf} - \chi_1 \mu_1 \overline{u}}{(b_{\sup} - \chi_1 \mu_1 + \chi_2 \mu_2)}. \]
By Lemma 2.3,
\[ \overline{u} \leq M_0 = \frac{a_{\sup}}{b_{\inf} - \chi_1 \mu_1 + \chi_2 \mu_2 - M}. \]
It then follows that
\[ \underline{u} \geq m_0 = \frac{a_{\inf}(b_{\inf} - (1 + \frac{a_{\sup}}{a_{\inf}})\chi_1 \mu_1 + \chi_2 \mu_2 - M)}{(b_{\inf} - \chi_1 \mu_1 + \chi_2 \mu_2 - M)(b_{\sup} - \chi_1 \mu_1 + \chi_2 \mu_2)}. \]
\[ \square \]
3.3. **Positive entire solutions.** In this subsection, we prove Theorem 1.3 on the existence, uniqueness, and stability of strictly positive entire solutions of (2).

We first prove Theorem 1.3(1).

**Proof of Theorem 1.3(1).** It can be proved by applying properly modified arguments in [21, Theorem 1.4 (iii)]. For the reader’s convenience, we provide some outline of the proof.

First, Let

\[ M^+ = \frac{a_{\text{sup}}}{b_{\text{inf}} + \chi_2 \mu_2 - \chi_1 \mu_1} + 1. \]

By Lemma 2.4, there is \( 0 < \delta^*_0 < M^+ \) such that for any \( 0 < \delta \leq \delta^*_0 \) and for any \( u_0 \in C_{\text{unif}}^b(\mathbb{R}^+) \) with \( \delta \leq u_0 \leq M^+ \),

\[ \delta \leq u(t_0 + T_0, x; t_0, u_0) \leq M^+ \quad \forall \ x \in [0, \infty), \ t_0 \in \mathbb{R}. \quad (41) \]

Next, let

\[ u_n(t, x) = u(t - nT, x; -nT, \delta^*_0) \quad \forall \ t \geq -nT, \ x \in \mathbb{R}^+ \]

Then there is \( n_k \to \infty \) and \( u^*(t, x) \) such that

\[ \lim_{n \to \infty} u_n(t, x) = u^*(t, x) \]

locally uniformly on \( \mathbb{R} \times [0, \infty) \). It can then be verified that \( (u, v_1, v_2) = (u^*(t, x), v_1^*(t, x), v_2^*(t, x)) \) is a strictly positive entire solution of (2), where \( v_i^*(t, x) \) satisfies

\[
\begin{align*}
0 &= v_{i,xx} - \lambda_i v + \mu_i u^*(t, x), & 0 < x < \infty \\
v_i(t, 0) &= 0
\end{align*}
\]

for \( i = 1, 2 \).

Next, we show that, if \( a(t + T, x) = a(t, x) \) and \( b(t + T, x) = b(t, x) \), then (2) has a time \( T^- \) periodic positive solution. To this end, choose \( T_0 = T \) and let

\[ \mathcal{E} = \{ u \in C_{\text{unif}}^b(\mathbb{R}^+) | \delta^*_0 \leq u_{\text{inf}} \leq u_{\text{sup}} \leq M^+ \} \]

endowed with the open compact topology. For any \( u_0 \in \mathcal{E} \), define

\[ \mathcal{P} u_0 = u(T, \cdot) \]

By (41), \( \mathcal{P} u_0 \in \mathcal{E} \). By the similar arguments as those in [21, Theorem 1.4 (iii)], it can be proved that \( \mathcal{P} : \mathcal{E} \to \mathcal{E} \) is a continuous and compact map. Then Schauder’s fixed theorem implies that there is \( u^* \in \mathcal{E} \) such that \( u(T, \cdot; 0, u^*) = u^* \). Clearly \( (u(\cdot, \cdot; 0, u^*), v_1(\cdot, \cdot; 0, u^*), v_2(\cdot, \cdot; 0, u^*)) \) is a \( T^- \) periodic solution of (1) and hence is a positive entire solution. Theorem 1.3(1) is thus proved. \( \square \)

Next, we prove Theorem 1.3 (2).

**Proof of Theorem 1.3 (2).** (i) First, note that, by Lemmas 2.2 and 2.3, we only need to prove the statement for \( u_0 \in C_{\text{unif}}^b(\mathbb{R}^+) \) satisfying

\[ 0 < \inf_{x \in \mathbb{R}^+} u_0(x) \leq \sup_{x \in \mathbb{R}^+} u_0(x) < \frac{a_{\text{sup}} + 1}{b_{\text{inf}} + \chi_2 \mu_2 - \chi_1 \mu_1} - M. \quad (42) \]

Note also that \( (u(t, x), v_1(t, x), v_2(t, x)) = (u^*(t), v_1^*(t), v_2^*(t)) \) is a strictly positive periodic solution of (2), where \( u = u^*(t) \) is the unique positive \( T^- \) periodic solution of the ODE

\[ u' = (a(t) - b(t)u)u, \quad (43) \]
and $v_1^*(t) = \frac{\partial u^*}{\partial x} u^*(t)$, $v_2^*(t) = \frac{\partial v^*}{\partial x} u^*(t)$. It then suffices to prove that, for any given $t_0 \in \mathbb{R}$ and $u_0 \in C^b_{\text{unif}}(\mathbb{R}^+)$ satisfying (42),

$$\lim_{t \to \infty} \|u(t + t_0, \cdot; t_0, u_0) - u^*(t + t_0)\|_{\infty} = 0. \tag{44}$$

To prove (44), for given $t_0 \in \mathbb{R}$ and $u_0 \in C^b_{\text{unif}}(\mathbb{R}^+)$ with $\inf_{x \in \mathbb{R}^+} u_0(x) > 0$, define

$$U(t, x) = \frac{u(t, x; t_0, u_0)}{u^*(t)}, \quad V_1(t, x) = \frac{v_1(t, x; t_0, u_0)}{v_1^*(t)}, \quad V_2(t, x) = \frac{v_2(t, x; t_0, u_0)}{v_1^*(t)}.$$

It then suffices to prove that

$$\lim_{t \to \infty} \|U(t, \cdot) - 1\|_{\infty} = 0. \tag{45}$$

We claim that for any $\varepsilon > 0$, there are $T_{\varepsilon, n}$ ($n = 1, 2, \cdots$) such that for any $t \geq T_{\varepsilon, n}$,

$$\|U(t + t_0, \cdot) - 1\|_{\infty} \leq \left(\frac{K}{b_{\text{inf}} + \chi_2 \mu_2 - \chi_1 \mu_1}\right)^n \frac{a_{\text{sup}} + 1}{(b_{\text{inf}} + \chi_2 \mu_2 - \chi_1 \mu_1 - M) u_{\text{inf}}} + \varepsilon, \tag{46}$$

where $K$ is defined in (8). Note that (H3) implies that $\frac{K}{b_{\text{inf}} + \chi_2 \mu_2 - \chi_1 \mu_1} < 1$. Hence (45) follows from (46). In the following, we prove (46) by induction.

First, by direct calculation, we have

$$U_t = \Delta U - \chi_1 U x v_1 + \chi_2 U x v_2 + \chi_1 \mu_1 U(V_1 - 1) u^*(t) - \chi_2 \mu_2 U(V_2 - 1) u^*(t) + (b(t) - \chi_1 \mu_1 + \chi_2 \mu_2) U(1 - U) u^*(t). \tag{47}$$

Obverse that $V_1(t, x)$ satisfies

$$\begin{cases}
0 = (V_1 - 1)_{xx} - \lambda_1 (V_1 - 1) + \lambda_1 (U(t, x) - 1), & x \in (0, \infty) \\
(V_1 - 1)_x(t, 0) = 0
\end{cases}$$

and $V_2(t, x)$ satisfies

$$\begin{cases}
0 = (V_2 - 1)_{xx} - \lambda_2 (V_2 - 1) + \lambda_2 (U(t, x) - 1), & x \in (0, \infty) \\
(V_2 - 1)_x(t, 0) = 0
\end{cases}$$

for $t \geq t_0$. Then by Lemma 2.2(2),

$$\|\chi_2 \mu_2 (V_2(t, \cdot) - 1) - \chi_1 \mu_1 (V_1(t, \cdot) - 1)\|_{\infty} \leq K \|U(t, \cdot) - 1\|_{\infty} \quad \forall t \in [t_0, \infty),$$

where $K$ is as in (8). Observe also that

$$\limsup_{t \to \infty} \|U(t, \cdot) - 1\|_{\infty} \leq \frac{a_{\text{sup}}}{(b_{\text{inf}} + \chi_2 \mu_2 - \chi_1 \mu_1 - M) u_{\text{inf}}}.$$

Let

$$\tilde{M}_1 = \frac{a_{\text{sup}} + 1}{(b_{\text{inf}} + \chi_2 \mu_2 - \chi_1 \mu_1 - M) u_{\text{inf}}}.$$

Then there is $\tilde{T}_{1, \varepsilon} > 0$ such that

$$U_t \leq \Delta U - \chi_1 U x v_1 + \chi_2 U x v_2 + \left[K \tilde{M}_1 U + (b(t) - \chi_1 \mu_1 + \chi_2 \mu_2) U(1 - U)\right] u^*(t)$$

and

$$U_t \geq \Delta U - \chi_1 U x v_1 + \chi_2 U x v_2 - \left[K \tilde{M}_1 U + (b(t) - \chi_1 \mu_1 + \chi_2 \mu_2) U(1 - U)\right] u^*(t)$$

for $t \geq \tilde{T}_{1, \varepsilon}$.
Next, it is not difficult to see that
\[ \inf_{x \in \mathbb{R}^+} U(t, x) > 0 \quad \forall \ t \geq t_0. \]

Let \( \bar{U}(t) \) be the solution of
\[
\begin{align*}
U_t &= \left[ K \bar{M}_1 U + (b_{\inf} - \chi_1 \mu_1 + \chi_2 \mu_2) U(1 - U) \right] u^*(t) \\
U(\bar{T}_{1, \varepsilon}) &= \max\{\|U(\bar{T}_{1, \varepsilon}, \cdot)\|_\infty, 1 + \frac{K}{b_{\inf} + \chi_2 \mu_2 - \chi_1 \mu_1} \bar{M}_1 \}
\end{align*}
\]
and \( \bar{U}(t) \) be the solution of
\[
\begin{align*}
U_t &= \left[ -K \bar{M}_1 U + (b_{\inf} - \chi_1 \mu_1 + \chi_2 \mu_2) U(1 - U) \right] u^*(t) \\
U(\bar{T}_{1, \varepsilon}) &= \inf\{\inf_{x \in \mathbb{R}^+} U(\bar{T}_{1, \varepsilon}, x), 1 \}.
\end{align*}
\]
Then
\[
\lim_{t \to \infty} \bar{U}(t) = 1 + \frac{K}{b_{\inf} + \chi_2 \mu_2 - \chi_1 \mu_1} \bar{M}_1
\]
and
\[
\lim_{t \to \infty} \bar{U}(t) = 1 - \frac{K}{b_{\inf} + \chi_2 \mu_2 - \chi_1 \mu_1} \bar{M}_1.
\]
Moreover, it is not difficult to see that
\[ \bar{U}(t) \leq U(t, x) \leq \bar{U}(t) \quad \forall \ t \geq \bar{T}_{1, \varepsilon}. \]
It then follows that for any given \( \varepsilon > 0 \), there is \( T_{1, \varepsilon} \geq \bar{T}_{1, \varepsilon} \) such that (46) holds with \( n = 1 \).

Next, assume that (46) holds for \( n = k \). For fixed \( \varepsilon > 0 \), let \( \bar{\varepsilon} = \varepsilon \frac{b_{\inf} + \chi_2 \mu_2 - \chi_1 \mu_1 - M}{K} \). Then, by the similar arguments as in the above, we have
\[
\|\chi_2 \mu_2(V_2(t, \cdot) - 1) - \chi_1 \mu_1(V_1(t, \cdot) - 1)\|_\infty
\]
\[ \leq K \left( \frac{K}{b_{\inf} + \chi_2 \mu_2 - \chi_1 \mu_1} \right)^k \frac{(a_{\sup} + 1)}{(b_{\inf} + \chi_2 \mu_2 - \chi_1 \mu_1 - M) u_{\inf}^*} + K \bar{\varepsilon} \]
for \( t \geq T_{\bar{\varepsilon}, k} \). Let
\[
\tilde{M}_k = \left( \frac{K}{b_{\inf} + \chi_2 \mu_2 - \chi_1 \mu_1} \right)^k \frac{(a_{\sup} + 1)}{(b_{\inf} + \chi_2 \mu_2 - \chi_1 \mu_1 - M) u_{\inf}^*} + \bar{\varepsilon}.
\]
Then for \( t \geq T_{\bar{\varepsilon}, k} \),
\[
U_t \leq \Delta U - \chi_1 U x v_{1, x} + \chi_2 U x v_{2, x} + \left[ K \tilde{M}_k U + (b(t) - \chi_1 \mu_1 + \chi_2 \mu_2) U(1 - U) \right] u^*(t)
\]
and
\[
U_t \geq \Delta U - \chi_1 U x v_{1, x} + \chi_2 U x v_{2, x} - \left[ K \tilde{M}_k U + (b(t) - \chi_1 \mu_1 + \chi_2 \mu_2) U(1 - U) \right] u^*(t).
\]
Moreover, we have
\[ \bar{U}(t) \leq U(t, x) \leq \bar{U}(t) \quad \forall \ t \geq T_{\bar{\varepsilon}, k}, \ x \in \mathbb{R}^+, \]
where \( \bar{U}(t) \) is the solution of
\[
\begin{align*}
U_t &= \left[ K \tilde{M}_k U + (b_{\inf} - \chi_1 \mu_1 + \chi_2 \mu_2) U(1 - U) \right] u^*(t) \\
U(T_{\bar{\varepsilon}, k}) &= \max\{\sup_{x \in \mathbb{R}^+} U(T_{\bar{\varepsilon}, k}, x), 1 + \frac{K}{b_{\inf} + \chi_2 \mu_2 - \chi_1 \mu_1} \tilde{M}_k \}.
\end{align*}
\]
and $U_k(t)$ be the solution of
\begin{align*}
\begin{cases}
U_t = \left[ -K\tilde{M}u + (b_{\inf} - \chi_1\mu_1 + \chi_2\mu_2)U(1-U) \right]u^*(t) \\
U(T_{\tilde{\epsilon},k}) = \inf\{ \inf_{x \in \mathbb{R}^+} U(T_{\tilde{\epsilon},k}, x), 1 \}.
\end{cases}
\end{align*}

Note that
\begin{align*}
\lim_{t \to \infty} \bar{U}_k(t) &= 1 + \frac{K}{b_{\inf} + \chi_2\mu_2 - \chi_1\mu_1 - M\tilde{M}} \\
\lim_{t \to \infty} U_k(t) &= 1 - \frac{K}{b_{\inf} + \chi_2\mu_2 - \chi_1\mu_1 - M\tilde{M}}.
\end{align*}

It then follows that there is $T_{\epsilon,k+1}$ such that
\[ \|U(t, \cdot) - 1\|_{\infty} \leq \left( \frac{K}{b_{\inf} + \chi_2\mu_2 - \chi_1\mu_1} \right)^{k+1} \left( a_{\sup} + 1 \right) \left( \frac{K}{b_{\inf} + \chi_2\mu_2 - \chi_1\mu_1 - M} \right) u^*_{\inf} + \epsilon. \]

By induction, (46) holds for any $n \geq 1$. Theorem 1.3(2)(i) is thus proved.

(ii) It can be proved by combing the arguments in (i) and properly modified arguments in [21, Theorem 1.5]. We do not provide the proof in this paper.

**Proof of Theorem 1.4.** (1) It follows from the similar arguments as those in Theorem 1.1.

(2) It follows from the similar arguments as those in Theorem 1.2.

(3) It follows from the similar arguments as those in Theorem 1.3(1).

(4) It follows from the similar arguments as those in Theorem 1.3(2).

**Acknowledgments.** The authors would like to thank Prof. Yihong Du for helpful comments and suggestions on the derivation of the free boundary condition in (1).

**REFERENCES**

[1] N. Bellomo, A. Bellouquid, Y. Tao and M. Winkler, Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues, *Math. Models Methods Appl. Sci.*, **25** (2015), 1663–1763.

[2] G. Bunting, Y.-H. Du and K. Krakowski, Spreading speed revisited: Analysis of a free boundary model, *Netw. Heterog. Media*, **7** (2012), 583–603.

[3] J. I. Diaz and T. Nagai, Symmetrization in a parabolic-elliptic system related to chemotaxis, *Advances in Mathematical Science and Applications*, **5** (1995), 659–680.

[4] J. I. Diaz, T. Nagai and J.-M. Rakotoson, Symmetrization techniques on unbounded domains: Application to a chemotaxis system on $\mathbb{R}^N$, *J. Differential Equations*, **145** (1998), 156–183.

[5] E. Espejo and T. Suzuki, Global existence and blow-up for a system describing the aggregation of microglia, *Applied Mathematics Letters*, **35** (2014), 29–34.

[6] E. Galakhov, O. Salieva and J. I. Tello, On a parabolic-elliptic system with chemotaxis and logistic type growth, *J. Differential Equations*, **261** (2016), 4631–4647.

[7] T. Hillen and K. J. Painter, A user’s guide to PDE models for chemotaxis, *J. Math. Biol.*, **58** (2009), 183–217.

[8] D. Horstmann, From 1970 until present: The keller-segel model in chemotaxis and its consequences, I, *Jahresber. Deutsch. Math.-Verein*, **105** (2003), 103–165.

[9] D. Horstmann, Generalizing the Keller–Segel model: Lyapunov functionals, steady state analysis, and blow-up results for multi-species chemotaxis models in the presence of attraction and repulsion between competitive interacting species, *Journal of Nonlinear Science*, **21** (2011), 231–270.

[10] D. Horstmann and M. Winkler, Boundedness vs. blow-up in a chemotaxis system, *J. Differential Equations*, **215** (2005), 52–107.

[11] H. Jin, Boundedness of the attraction-repulsion Keller-Segel system, *Journal of Mathematical Analysis and Applications*, **422** (2015), 1463–1478.
[12] K. Kanga and A. Steven, Blowup and global solutions in a chemotaxis-growth system, Nonlinear Analysis, 135 (2016), 57–72.
[13] E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theoret. Biol., 26 (1970), 399–415.
[14] E. F. Keller and L. A. Segel, A model for chemotaxis, J. Theoret. Biol., 30 (1971), 225–234.
[15] K. Lin, C. Mu and Y. Gao, Boundedness and blow up in the higher-dimensional attraction-repulsion chemotaxis system with nonlinear diffusion, Journal of Differential Equations, 261 (2016), 4524–4572.
[16] J. Liu and Z. Wang, Classical solutions and steady states of an attraction-repulsion chemotaxis in one dimension, Journal of Biological Dynamics, 6 (2012), 31–41.
[17] M. Luca, A. Chavez-Ross, L. Edelstein-Keshet and A. Mogilner, Chemotactic signaling, microglia, and Alzheimer’s disease senile plaques: Is there a connection?, Bulletin of Mathematical Biology, 65 (2003), 693–730.
[18] T. Nagai, T. Senba and K. Yoshida, Application of the trudinger-moser inequality to a parabolic system of chemotaxis, Funkcialaj Ekvacioj, 40 (1997), 411–433.
[19] R. B. Salako and W. Shen, Spreading speeds and Traveling waves of a parabolic-elliptic chemotaxis system with logistic source on $\mathbb{R}^N$, Discrete Contin. Dyn. Syst., 37 (2017), 6189–6225.
[20] R. B. Salako and W. Shen, Parabolic-elliptic chemotaxis model with space-time dependent logistic sources on $\mathbb{R}^N$. I. Persistence and asymptotic spreading, Mathematical Models and Methods in Applied Sciences, 28 (2018), 2237–2273.
[21] R. B. Salako and W. Shen, Parabolic-elliptic chemotaxis model with space-time dependent logistic sources on $\mathbb{R}^N$. II. Existence, uniqueness, and stability of strictly positive entire solutions, J. Math. Anal. Appl., 464 (2018), 883–910.
[22] R. B. Salako and W. Shen, Global classical solutions, stability of constant equilibria, and spreading speeds in attraction-repulsion chemotaxis systems with logistic source on $\mathbb{R}^N$, Journal of Dynamics and Differential Equations, 31 (2019), 1301–1325.
[23] R. B. Salako and W. Shen, Global existence and asymptotic behavior of classical solutions to a parabolic-elliptic chemotaxis system with logistic source on $\mathbb{R}^N$, J. Differential Equations, 262 (2017), 5635–5690.
[24] Y. Sugiyama, Global existence in sub-critical cases and finite time blow up in super critical cases to degenerate Keller-Segel systems, Differential Integral Equations, 19 (2006), 841–876.
[25] Y. Sugiyama and H. Kunii, Global Existence and decay properties for a degenerate keller-Segel model with a power factor in drift term, J. Differential Equations, 227 (2006), 333–364.
[26] L. Wang, C. Mu and P. Zheng, On a quasilinear parabolic-elliptic chemotaxis system with logistic source, J. Differential Equations, 256 (2014), 1847–1872.
[27] Y. Wang, Global bounded weak solutions to a degenerate quasilinear attraction-repulsion chemotaxis system with rotation, Computers and Mathematics with Applications, 72 (2016), 2226–2240.
[28] Y. Wang and Z.-Y. Xiang, Boundedness in a quasilinear 2D parabolic-parabolic attraction-repulsion chemotaxis system, Discrete and Continuous Dynamical Systems-Series B, 18 (2013), 1613–1626.
[29] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model, J. Differential Equations, 248 (2010), 2889–2905.
[30] M. Winkler, Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction, Journal of Mathematical Analysis and Applications, 384 (2011), 261–272.
[31] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system, J. Math. Pures Appl., 21 (2013), 300–384.
[32] M. Winkler, Global asymptotic stability of constant equilibria in a fully parabolic chemotaxis system with strong logistic dampening, J. Differential Equations, 257 (2014), 1056–1077.
[33] M. Winkler, How far can chemotactic cross-diffusion enforce exceeding carrying capacities? J. Nonlinear Sci., 24 (2014), 809–855.
[34] T. Yokota and N. Yoshino, Existence of solutions to chemotaxis dynamics with logistic source, Discrete Contin. Dyn. Syst. Dynamical Systems, Differential Equations and Applications, 10th AIMS Conference. Suppl., 2015, 1125–1133.
[35] Q. Zhang and Y. Li, An attraction-repulsion chemotaxis system with logistic source, Z. Angew. Math. Mech., 96 (2016), 570–584.
[36] P. Zheng, C. Mu, X. Hu and Y. Tian, Boundedness of solutions in a chemotaxis system with nonlinear sensitivity and logistic source, J. Math. Anal. Appl., 424 (2015), 509–522.
[37] P. Zheng, C. Mu, X. Hu and Y. Tian, Boundedness in the higher dimensional attraction-repulsion chemotaxis-growth system, *Computers and Mathematics with Applications*, 72 (2016), 2194–2202.

Received May 2019; revised August 2019.

E-mail address: lzbao@jlu.edu.cn

E-mail address: wenxish@auburn.edu