A note on optimization in $\mathbb{R}^n$

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Abstract In this article, we develop an algorithm suitable for constrained optimization in $\mathbb{R}^n$. The results are developed through standard tools of $n$-dimensional real analysis and basic concepts of optimization. Indeed, the well known Banach fixed point theorem has a fundamental role in the main result establishment.

Keywords Optimization · Inequality constraints · Convergence

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1 Introduction

In this short letter we develop a proximal algorithm for constrained optimization.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^2$ class function. Consider the problem of minimizing locally $f$ subject to $g(x) \leq 0$, where $g : \mathbb{R}^n \to \mathbb{R}$ is a given $C^2$ class function.

The lagrangian for this problem, denoted by $L : \mathbb{R}^{n+1} \to \mathbb{R}$ may be expressed by

$$L(x, \lambda) = f(x) + \lambda^2 g(x).$$

We define the proximal formulation for such a problem, denoted by $L_p$ by

$$L_p(x, \lambda, x_k) = f(x) + \lambda^2 g(x) + \frac{K}{2} |x - x_k|^2.$$
2 The main result

Linearizing $L_p$, we propose the following procedure for looking for a critical point of such a function:

Consider

$$L_p(x, \lambda, x_k) = f(x_k) + f'(x_k) \cdot (x - x_k) + \frac{1}{2} [f''(x_k)(x - x_k)] \cdot (x - x_k) + \lambda^2 (g(x_k) + g'(x_k) \cdot (x - x_k)) + \frac{K}{2} |x - x_k|^2.$$ 

Hence from

$$\frac{\partial L_p(x, \lambda, x_k)}{\partial x} = 0$$
we obtain,

$$f''(x_k)(x - x_k) + K(x - x_k) + f'(x_k) + \lambda^2 g'(x_k) = 0,$$
that is,

$$x - x_k = -(f''(x_k) + K I_d)^{-1}(f'(x_k) + \lambda^2 g'(x_k)),$$
and therefore

$$x(\lambda, x_k) = x_k - (f''(x_k) + K I_d)^{-1}(f'(x_k) + \lambda^2 g'(x_k)),$$
where $I_d$ denotes the $n \times n$ identity matrix.

We define $L_1(\lambda, x_k) = \tilde{L}_p(x(\lambda, x_k), x_k, \lambda)$ so that

$$L_1(\lambda, x_k) = -\frac{1}{2} [(f''(x_k) + K I_d)^{-1}(f'(x_k) + \lambda^2 g'(x_k))] \cdot (f'(x_k) + \lambda^2 g'(x_k)) + f(x_k) + \lambda^2 g(x_k)$$

From

$$\frac{\partial L_1(\lambda, x_k)}{\partial \lambda} = 0,$$
we get

$$[(f''(x_k) + K I_d)^{-1}(f'(x_k) + \lambda^2 g'(x_k))] \cdot g'(x_k) \lambda - \lambda g(x_k) = 0,$$
so that we have two solutions,

$$\lambda_1 = 0$$
and

$$\lambda_2^2(x_k) = -\left( \frac{[(f''(x_k) + K I_d)^{-1}f'(x_k)] \cdot g'(x_k) - g(x_k)}{[(f''(x_k) + K I_d)^{-1}g'(x_k)] \cdot g'(x_k)} \right).$$

Observe that if $(\lambda_2^2(x_k) < 0$ then $\lambda_2^2(x_k)$ is complex so that, from the condition $\lambda^2 \geq 0$, we obtain

$$\lambda^2(x_k) = \max\{0, (\lambda_2^2(x_k))\}.$$
Also, from the generalized inverse function theorem $\lambda^2(x)$ is locally Lipschtzian (see [3,11,12] for details). Hence, we may infer that for a given $x_0 \in \mathbb{R}^n$ there exists $r > 0$ and $\hat{K}_3 > 0$ such that

$$|\lambda^2(x) - \lambda^2(y)| \leq \hat{K}_3|x - y|,$$

$\forall x, y \in B_r(x_0)$. With such results in mind, for such an $x_0 \in \mathbb{R}^n$, define $\{x_k\}$ by

$$x_1 = x_0 - (f''(x_0) + KId)^{-1}(f'(x_0) + \lambda^2(x_0)g'(x_0)),$$

$$x_{k+1} = x_k - (f''(x_k) + KId)^{-1}(f'(x_k) + \lambda^2(x_k)g'(x_k)), \forall k \in \mathbb{N}.$$ 

Assume

$$g(x_0) < 0$$

and there exists $\hat{K}_1$ such that $|f''(x)| \leq \hat{K}_1, \forall x \in B_r(x_0)$. Define

$$K_3 = \hat{K}_3 \left( \sup_{x \in B_r(x_0)} |g'(x)| \right),$$

$$\alpha_1 = \frac{2K_3}{|K - \hat{K}_1|}$$

and suppose

$$f''(x) + \lambda^2(x)g''(x) \geq \alpha_1(\hat{K}_1 + K)Id, \forall x, y \in B_r(x_0).$$

(5)

Suppose also $K$ is such that $K > \hat{K}_1$,

$$0 < \alpha_1 < 1,$$

$$\left(1 - \frac{\alpha_1}{4}\right)Id \leq ((f''(x) + KId)^{-1}(f''(y) + KId) \equiv H(x,y) \leq \left(1 + \frac{\alpha_1}{4}\right)Id,$$

$\forall x, y \in B_r(x_0)$ and

$$0 \leq \frac{f''(x) + \lambda(x)g''(x)}{K - \hat{K}_1} \leq \left(1 - \frac{\alpha_1}{2}\right)Id, \forall x, y \in B_r(x_0).$$

(7)

Observe that since $|f''(x)| \leq \hat{K}_1$, we have

$$0 \leq (K - \hat{K}_1)Id \leq f''(x) + KId,$$

so that

$$(f''(x) + KId)^{-1} \leq \frac{1}{K - \hat{K}_1}Id,$$

(8)

and

$$|(f''(x) + KId)^{-1}|K_3 \leq \frac{K_3}{|K - \hat{K}_1|} = \frac{\alpha_1}{2}, \forall x \in B_r(x_0).$$

(9)

Assume $K > 0$ is such that

$$x_1 \in B_{r(1-\alpha_0)}(x_0)$$
and suppose the induction hypotheses
\[ x_2, \ldots, x_{k+1} \in B_r(x_0). \]
where \( 0 < \alpha_0 < 1 \) is specified in the next lines.

Note that,
\[ x_{k+2} - x_{k+1} = -(f''(x_{k+1}) + KI_d)^{-1}(f'(x_{k+1}) + \lambda^2(x_{k+1})g'(x_{k+1})), \]
and
\[ x_{k+1} - x_k = -(f''(x_k) + KI_d)^{-1}(f'(x_k) + \lambda^2(x_k)g'(x_k)), \]
so that,
\[ (f''(x_{k+1}) + KI_d)(x_{k+2} - x_{k+1}) = -(f'(x_{k+1}) + \lambda^2(x_{k+1})g'(x_{k+1})), \]
and
\[ (f''(x_k) + KI_d)(x_{k+1} - x_k) = -(f'(x_k) + \lambda^2(x_k)g'(x_k)). \]

Therefore,
\[ (f''(x_{k+1}) + KI_d)(x_{k+2} - x_{k+1}) = (f''(x_k) + KI_d)(x_{k+1} - x_k) \]
\[ = (f''(x_k) + KI_d)(x_{k+1} - x_k) - (f'(x_{k+1}) + \lambda^2(x_{k+1})g'(x_{k+1}))(x_{k+1} - x_k) \]
\[ = (f''(x_k) + KI_d)(x_{k+1} - x_k) - (f'(x_{k+1}) + \lambda^2(x_{k+1})g'(x_{k+1}))(x_{k+1} - x_k) \]
where \( \tilde{x}_k \) is on the line connecting \( x_k \) and \( x_{k+1} \).

Thus,
\[ x_{k+2} - x_{k+1} = (f''(x_{k+1}) + KI_d)^{-1}[f''(x_k) + KI_d](x_{k+1} - x_k) \]
\[ = (f''(x_k) + KI_d)(x_{k+1} - x_k) - (f'(x_{k+1}) + \lambda^2(x_{k+1})g'(x_{k+1}))(x_{k+1} - x_k) \]
\[ = (f''(x_k) + KI_d)(x_{k+1} - x_k) - (f'(x_k) + \lambda^2(x_k)g'(x_k)), \]
so that
\[ |x_{k+2} - x_{k+1}| \leq |H(x_{k+1}, x_k) - (f''(x_{k+1}) + KI_d)^{-1}(f''(\tilde{x}_k) \]
\[ + \lambda^2(x_{k+1})g''(\tilde{x}_k))||x_{k+1} - x_k| \]
\[ + |(f''(x_{k+1}) + KI_d)^{-1}K_3||x_{k+1} - x_k|. \]

Observe that, from [3],
\[ f''(\tilde{x}_k) + \lambda^2(\tilde{x}_{k+1})g'(x_k) \geq \alpha_1(\tilde{K}_1 + K)I_d \geq \alpha_1(f''(x_{k+1}) + KI_d), \]
so that
\[ ((f''(x_{k+1}) + KI_d)^{-1}(f''(\tilde{x}_k) + \lambda^2(x_{k+1})g''(\tilde{x}_k)) \geq \alpha_1I_d. \]
Hence, from this, (6), (8) and (7), we obtain
\[
I_d \left(1 + \frac{\alpha_1}{4}\right) - \alpha_1 I_d
\geq H(x_{k+1}, x_k) - (f''(x_{k+1}) + K I_d)^{-1}(f''(\tilde{x}_k) + \lambda^2(x_{k+1})g''(\tilde{x}_k))
\geq I_d \left(1 - \frac{\alpha_1}{4}\right) - (K I_d - \hat{K}_1 I_d)^{-1}(f''(\tilde{x}_k) + \lambda^2(x_{k+1})g''(\tilde{x}_k))
\geq I_d \left(1 - \frac{\alpha_1}{4}\right) - I_d \left(1 - \frac{\alpha_1}{2}\right)
= \frac{\alpha_1}{4} I_d
\geq 0,
\]
and therefore,
\[
|H(x_{k+1}, x_k) - (f''(x_k) + K I_d)^{-1}(f''(\tilde{x}_k) + \lambda^2(x_{k+1})g''(\tilde{x}_k))| \leq 1 - \frac{3\alpha_1}{4}.
\]
On the other hand, from (9) we have,
\[
|(f''(x_k) + K I_d)^{-1}K_3| \leq \frac{\alpha_1}{2}.
\]
From (11) and these last two inequalities, we obtain
\[
|x_{k+2} - x_{k+1}| \leq \left(1 - \frac{3\alpha_1}{4} + \frac{\alpha_1}{2}\right)|x_{k+1} - x_k| = \left(1 - \frac{\alpha_1}{4}\right)|x_{k+1} - x_k|.
\]
Thus, denoting \(\alpha_0 = 1 - \alpha_1/4\), we have obtained,
\[
|x_{j+2} - x_{j+1}| \leq \alpha_0|x_{j+1} - x_j|, \forall j \in \{1, \cdots, k + 1\}
\]
so that
\[
|x_{j+2} - x_{j+1}| \leq \alpha_0|x_{j+1} - x_j|
\leq \alpha_0^2|x_{j} - x_{j-1}|
\leq \cdots
\leq \alpha_0^{j+1}|x_1 - x_0|, \forall j \in \{1, \cdots, k\}.
\]
Thus,
\[
|x_{k+2} - x_1|
= |x_{k+2} - x_{k+1} + x_{k+1} - x_k + x_k - x_{k-1} + \cdots + x_2 - x_1|
\leq |x_{k+2} - x_{k+1}| + |x_{k+1} - x_k| + \cdots + |x_2 - x_1|
\leq \sum_{j=1}^{k+1} \alpha_0^j |x_1 - x_0|
\leq \sum_{j=1}^{+\infty} \alpha_0^j |x_1 - x_0|
= \frac{\alpha_0}{1 - \alpha_0} |x_1 - x_0|,
\]
so that

\[
|x_{k+2} - x_0| \leq |x_{k+2} - x_1| + |x_1 - x_0| \\
\leq \frac{\alpha_0}{1 - \alpha_0} |x_1 - x_0| + |x_1 - x_0| \\
= \frac{1}{1 - \alpha_0} |x_1 - x_0| \\
< \frac{1}{1 - \alpha_0} r(1 - \alpha_0) \\
= r.
\]

(15)

Hence \(x_{k+2} \in B_r(x_0)\), and therefore the induction is complete, so that, \(x_k \in B_r(x_0), \forall k \in \mathbb{N}\).

Moreover, \(\{x_k\}\) is a Cauchy sequence, so that there exists \(\tilde{x}\), such that

\(x_k \to \tilde{x}, \text{ as } k \to \infty\).

Finally

\[
0 = \lim_{k \to \infty} (x_{k+1} - x_k) \\
= \lim_{k \to \infty} [-((f''(x_k) + KI_d)^{-1}(f'(x_k) + \tilde{\lambda}^2(x_k)g'(x_k)))] \\
= -(f''(\tilde{x}) + KI_d)^{-1}(f'(\tilde{x}) + \tilde{\lambda}^2g'(\tilde{x})).
\]

(16)

Hence, from this and

\[det(f''(\tilde{x}) + KI_d) \neq 0,\]

we obtain

\[f'(\tilde{x}) + \tilde{\lambda}^2g'(\tilde{x}) = 0\]

In such a case, from (22) letting \(k \to \infty\), we also obtain

\[\tilde{\lambda}^2 g(\tilde{x}) = 0.\]

Thus if \(\tilde{\lambda}^2 > 0\), then \(g(\tilde{x}) = 0\).

If \(\lambda = 0\), then \(f'(\tilde{x}) = 0\) and

\[(\lambda_1^2)(\tilde{x}) \leq 0\]

so that from (23), since \((f''(\tilde{x}) + KI_d)^{-1}\) is positive definite, letting \(k \to \infty\), we get

\[g(\tilde{x}) = (\lambda_2^2)(\tilde{x})[(f''(\tilde{x}) + KI_d)^{-1}g'(\tilde{x})] \cdot g'(\tilde{x}) \leq 0.\]

That is, in any case,

\[g(\tilde{x}) \leq 0.\]
Remark 1 For the more general case with $m_1$ equality scalar constraints
\[ h_j(x) = 0, \forall j \in \{1, \ldots, m_1\} \]
and $m_2$ inequality scalar constraints
\[ g_l(x) \leq 0, \forall l \in \{1, \ldots, m_2\}, \]
where $h_j, g_l : \mathbb{R}^n \to \mathbb{R}$ are $C^2$ class functions, $\forall j \in \{1, \ldots, m_1\}$ and $\forall l \in \{1, \ldots, m_2\}$, we assume $m_1 + m_2 < n$ and define the Lagrangian $L_p$ by
\[
L_p(x, \lambda, x_k) = f(x) + \sum_{j=1}^{m_1} (\lambda_h) h_j(x) + \sum_{l=1}^{m_2} (\lambda_g) g_l(x) + \frac{K}{2} |x - x_k|^2.
\]
Linearizing $L_p$, we propose the following procedure for looking for a critical point of such a function:
Consider
\[
\tilde{L}_p(x, \lambda, x_k) = f(x_k) + f'(x_k) \cdot (x - x_k) + \frac{1}{2} [f''(x_k) (x - x_k)] \cdot (x - x_k)
\]
\[ + \sum_{j=1}^{m_1} (\lambda_h) h_j(x_k) + h_j'(x_k) \cdot (x - x_k)
\]
\[ + \sum_{l=1}^{m_2} (\lambda_g) g_l(x_k) + g_l'(x_k) \cdot (x - x_k) + \frac{K}{2} |x - x_k|^2.
\]
Hence from
\[
\frac{\partial \tilde{L}_p(x, \lambda, x_k)}{\partial x} = 0,
\]
we obtain,
\[
f''(x_k)(x - x_k) + K(x - x_k) + f'(x_k) + \sum_{j=1}^{m_1} (\lambda_h) h_j'(x_k) + \sum_{l=1}^{m_2} (\lambda_g) g_l'(x_k) = 0,
\]
that is,
\[
x - x_k = -(f''(x_k) + K I_d)^{-1} \left( f'(x_k) + \sum_{j=1}^{m_1} (\lambda_h) h_j'(x_k) + \sum_{l=1}^{m_2} (\lambda_g) g_l'(x_k) \right),
\]
and therefore
\[
x(\lambda, x_k) = x_k - (f''(x_k) + K I_d)^{-1} \left( f'(x_k) + \sum_{j=1}^{m_1} (\lambda_h) h_j'(x_k) + \sum_{l=1}^{m_2} (\lambda_g) g_l'(x_k) \right),
\]
where $I_d$ denotes the $n \times n$ identity matrix.
We define $L_1(\lambda, x_k) = \tilde{L}_p(x(\lambda, x_k), x_k, \lambda)$, so that
\[ L_1(\lambda, x_k) = -\frac{1}{2} \left( f''(x_k) + KI_d \right)^{-1} \left( f'(x_k) + \sum_{j=1}^{m_1} (\lambda_h)_j h'_j(x_k) + \sum_{l=1}^{m_2} (\lambda_g)_l^2 g'_l(x_k) \right) \cdot \left( f'(x_k) + \sum_{j=1}^{m_1} (\lambda_h)_j h'_j(x_k) + \sum_{l=1}^{m_2} (\lambda_g)_l^2 g'_l(x_k) \right) + f(x_k) + \sum_{j=1}^{m_1} (\lambda_h)_j h_j(x_k) + \sum_{l=1}^{m_2} (\lambda_g)_l^2 g_l(x_k). \] (18)

From \[ \frac{\partial L_1(\lambda, x_k)}{\partial (\lambda g)_l} = 0, \] we get
\[ \left( f''(x_k) + KI_d \right)^{-1} \left( f'(x_k) + \sum_{j=1}^{m_1} (\lambda_h)_j h'_j(x_k) + \sum_{l=1}^{m_2} (\lambda_g)_l^2 g'_l(x_k) \right) \cdot g'_l(x_k) (\lambda g)_l - (\lambda g)_l g_l(x_k) = 0, \] (19)

From \[ \frac{\partial L_1(\lambda, x_k)}{\partial (\lambda h)_j} = 0, \] we have
\[ \left( f''(x_k) + KI_d \right)^{-1} \left( f'(x_k) + \sum_{j=1}^{m_1} (\lambda_h)_j h'_j(x_k) + \sum_{l=1}^{m_2} (\lambda_g)_l^2 g'_l(x_k) \right) \cdot h'_j(x_k) - h_j(x_k) = 0, \] (20)

\[ \forall j \in \{1, \ldots, m_1\}. \] Solving the linear system which comprises these last \( m_1 \) equations and the \( m_2 \) equations
\[ \left( f''(x_k) + KI_d \right)^{-1} \left( f'(x_k) + \sum_{j=1}^{m_1} (\lambda_h)_j h'_j(x_k) + \sum_{l=1}^{m_2} (\lambda_g)_l^2 g'_l(x_k) \right) \cdot g'_l(x_k) - g_l(x_k) = 0, \] (21)

\[ \forall l \in \{1, \ldots, m_2\}, \] we may obtain a solution
\[ ((\lambda_h)_j(x_k), (\lambda_g)_l^2(x_k)). \]

Thus, to obtain a concerning critical point, we follow the following algorithm.
1. Choose $x_0 \in \mathbb{R}^n$, $K_{\text{max}} \in \mathbb{N}$ ($K_{\text{max}}$ is the maximum number of iterations), set $k = 0$ and $e_1 \approx 10^{-5}$.

2. Obtain a solution $((\lambda_h)_j(x_k), (\lambda_g)_l^2(x_k))$

   by solving the linear system (in $(\lambda_h)_j$ and $(\lambda_g)_l^2$) indicated in (20) and (21).

   Observe that if $(\lambda_g)_l^2 < 0$ then $(\lambda_h)_j(x_k)$ is complex.

   To up-date $\lambda_h$ and $\lambda_g$ proceed as follows:

3. For each $l \in \{1, \ldots, m_2\}$ if $(\lambda_g)_l^2(x_k) \leq 0$, then set $(\lambda_g)_l(x_k) = 0$.

4. Define $J = \{l \in \{1, \ldots, m_2\} \text{ such that } (\lambda_g)_l^2(x_k) > 0\}$.

5. Recalculate $(\lambda_h)_j(x_k)$ and the non-zero $(\lambda_g)_l^2(x_k)$ for $l \in J$ through the solution of the linear system (in $(\lambda_h)_j$ and $(\lambda_g)_l^2$)

   $$
   \left( (f''(x_k) + KI_d)^{-1} \left( f'(x_k) + \sum_{j=1}^{m_1} (\lambda_h)_j h_j'(x_k) 
   + \sum_{l \in J} (\lambda_g)_l^2 g_l(x_k) \right) \right) \cdot h_j'(x_k) - h_j(x_k) = 0,
   $$

   $\forall j \in \{1, \ldots, m_1\}$ and

   $$
   \left( (f''(x_k) + KI_d)^{-1} \left( f'(x_k) + \sum_{j=1}^{m_1} (\lambda_h)_j h_j'(x_k) 
   + \sum_{l \in J} (\lambda_g)_l^2 g_l(x_k) \right) \right) \cdot g_l(x_k) - g_l(x_k) = 0,
   $$

   $\forall l \in J$.

6. If $(\lambda_g)_l^2(x_k) \geq 0$, $\forall l \in \{1, \ldots, m_2\}$, then go to 7 otherwise go to item 8.

7. Up-date $x_k$ through the equation

   $$
   x_{k+1} = x_k - (f''(x_k) + KI_d)^{-1} \left( f'(x_k) + \sum_{j=1}^{m_1} (\lambda_h)_j(x_k) h_j'(x_k) 
   + \sum_{l=1}^{m_2} (\lambda_g)_l^2(x_k) g_l'(x_k) \right).
   $$

8. If $|x_{k+1} - x_k| < e_1$ or $k > K_{\text{max}}$, then stop, otherwise $k := k + 1$ and go to 2.

3 Conclusion

In this article we have developed an algorithm for constrained optimization in $\mathbb{R}^n$. We prove the main result only for the special case of a single scalar
inequality constraint. However, we highlight the proof of a more general result involving equality and inequality constraints may be developed in a similar fashion, as indicated in remark 1. We postpone the presentation of the formal details for such a more general case for a future work.

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