Energy Loss of Ultrahigh Energy Protons in Strong Magnetic Fields

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Abstract

Ultrahigh energy protons in magnetic fields produce pions and thus lose energy. The mean free path of such a process is worked out for Gaussian random fields. Two cases are considered: an isotropic and a cylindrically symmetric distribution. The energy loss is proportional to \( E^3 \langle B^2 \rangle \); it becomes significant for protons of energies \( \gtrsim 10^{19} \text{eV} \) and magnetic fields \( B \gtrsim 10^9 \text{Gauss} \). For energies and magnetic fields of this magnitude, a proton injected into the magnetic field loses a substantial fraction of its initial energy due to pion production.

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1 Introduction

It is well known that the propagation of ultra high energy (UHE) protons in the Universe is limited by the Greisen-Zatsepin-Kuzmin (GZK) mechanism, [1, 2]. The propagating protons undergo inelastic scattering on the photons of the cosmic microwave background radiation (CMBR) and produce pions. On the average, the initial energy is shared (roughly) equally by the nucleon

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and pion in the final state. The energy at which the GZK mechanism becomes important can be crudely estimated by saturating the inelastic cross section by the $\Delta$ resonance. Given the fact that the average energy of the CMBR photons is around $3 \times 10^{-4}$eV, one gets that pion production becomes significant around proton energies of the order of $10^{19}$eV. (In fact, a similar simple estimate was used in Greisen’s original paper.)

Here we discuss a mechanism for energy loss of UHE protons, hitherto apparently ignored, viz. by inelastic scattering on virtual photons. This mechanism plays no significant role in limiting the propagation of UHE protons in intergalactic space. However, it becomes significant when the propagation is considered in an environment where strong magnetic fields exist, e.g. in jets emerging from gamma ray bursters (GRB) or jets in active galactic nuclei (AGN).

The “average” energetics of pion production on virtual photons (typically, on an external magnetic field) is very different from that of the GZK mechanism. In fact, if the size of the external magnetic field is characterized by a length $L$, then the typical momentum of the virtual photon is of the order of $1/L$. Consequently, the average invariant center of mass (CMS) energy available for pion production is of the order,

$$s \approx m^2 + 2E/L,$$

where $m$ stands for the nucleon mass and $E$ is the energy of the incident proton in the rest frame of the local universe. Obviously, this energy is much less than the analogous quantity in the GZK process for any macroscopic $L$. Nevertheless, for a sufficiently strong and spatially confined magnetic field, one obtains an appreciable production rate, due to the fact that the Fourier spectrum of a confined field is rather slowly decreasing with the wave number (typically, as a power). As a consequence, Fourier components with $|k| \gg 1/L$ can play a significant role.

This paper is organized as follows. In the next section, we obtain a general expression of the cross section for the process $p + B \rightarrow X$, where $B$ stands for an external magnetic field. We also discuss the approximations one can make in order to simplify the calculation. The subsequent section, contains an evaluation of the interaction rate for random magnetic fields: we believe that this serves as a first model for energy loss in the chaotic fields present in typical astrophysical environments. Two situations are considered in detail: an isotropic and a cylindrically symmetric probability distribution of the random field.
While an isotropic environment largely serves to illustrate the physical features of the process in a simple context, it is also potentially applicable to a situation in which the size of the magnetic field is substantially larger than the interaction mfp. The calculation of the interaction rate for a cylindrically symmetric environment is relevant for jets, for instance, emerging from an AGN or GRB. The results are discussed in sec. 4.

2 General expression of the cross section in an external magnetic field.

The calculation described here is an elementary application of the optical theorem. The amplitude of a proton interacting with an external field and producing a final state $|X\rangle$ is given by

$$T(p + B \rightarrow X) = \int d^4x \ A_\mu(x) \langle X | j^\mu(x)| p \rangle$$  \hspace{1cm} (1)

Here, $A_\mu(x)$ stands for the vector potential of the external field and $j_\mu(x)$ is the density of the electromagnetic current. Squaring (1) and summing over the final states $|X\rangle$, one expresses the cross section in terms of the current correlation function. This is textbook material, see for instance [3]. We also assume that the external magnetic field is static. This assumption simplifies the calculation. From a physical point of view, it is justifiable even if astrophysical objects of bulk Lorentz factors of the order of a few hundred are considered: the protons we are interested in have Lorentz factors which are ten or eleven orders of magnitude larger.

On writing for the Fourier transform of the vector potential

$$A_\mu(q) = \delta(q_0) a_\mu(q)$$  \hspace{1cm} (2)

and using a gauge in which $A_0 = 0$, one gets:

$$\sigma = \frac{4\pi^2\alpha m}{E} \int d^3q \ a_i^*(q) a_k(q) W_{ik}$$  \hspace{1cm} (3)

In equation (3) $m$ and $E$ stand for the mass and energy of the incident proton, respectively and $W_{ik}$ is the spatial part of the standard polarization tensor:

$$W_{\mu\nu} = \frac{F_1}{m} \left( -g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) + \frac{F_2}{\nu} \left( p_\mu - q_\mu \frac{pq}{q^2} \right) \left( p_\nu - q_\nu \frac{pq}{q^2} \right)$$  \hspace{1cm} (4)
The notation is standard, $p$ and $q$ are the four momenta of the incident proton and virtual photon, respectively, $\nu = (pq)/m$.

A further simplification is possible due to the fact that the protons we are interested in are extreme relativistic and the average value of the momentum of the virtual photon is of the order of $1/L$. In order to motivate this simplification, we Lorentz transform to the rest frame of the proton. In that frame the components of the four momentum $q$ and the field quantities are distinguished by a prime. Components perpendicular to the direction of motion are denoted by capital letters; longitudinal components by a subscript $l$. Since we have $v \sim 1$, the transformation formulae are:

\begin{align}
q_0' \sim & \frac{1}{2} \exp(y) q_l, \quad q_l' \sim \frac{1}{2} \exp(y) q_l, \quad q_A' \sim q_A. \\
B'_l = & \quad B_l, \quad E'_l = E_l = 0, \\
B'_A \sim & \frac{1}{2} \exp(y) B_A, \quad E'_A \sim \frac{1}{2} \exp(y) \epsilon_{AB} B_B.
\end{align}

(5)

In the last two equations, $y$ stands for the rapidity.

As a consequence, apart from corrections of $O(\exp(-y))$,

\begin{equation}
q^2 \sim 0, \quad B \cdot E \sim 0, \quad B^2 - E^2 \sim 0.
\end{equation}

(7)

In a reference frame comoving with the proton, the magnetic field appears as a stream of (almost real) photons: consequently, the contribution of the structure function $F_2$ to the cross section is negligibly small.

One can then express the cross section on the external magnetic field in terms of the photoproduction cross section, $\sigma_\gamma$, viz.

\begin{equation}
\sigma \sim \frac{1}{E} \int d^3q \quad a_i(q)a_j(q)^* \frac{(p \cdot q)}{q^2} \sigma_\gamma \left( \delta_{ij} q^2 - q_i q_j \right)
\end{equation}

(8)

One readily recognizes that eq. (8) is equivalent to a Weizsäcker-Williams approximation to the cross section. Because of the presence of a transverse projector, that expression is a manifestly gauge invariant one. It is worth noticing that in the Weizsäcker-Williams approximation the expression of the cross section is independent of the mass of the projectile. Hence the same expression can be used to describe e.g. photon induced reactions in a magnetic field.
Finally, one considers the evaluation of $\sigma_\gamma$. Due to the fact that the photoabsorption cross section is to be evaluated near the pion production threshold, to a good approximation one can saturate it by the contribution of the $\Delta$ resonance. A narrow resonance approximation is sufficiently accurate. Hence we put
\begin{equation}
\sigma_\gamma \approx \sigma_0 \delta \left( s - m_\Delta^2 \right),
\end{equation}
where the dimensionless quantity, $\sigma_0$ is the integral of the pion photoproduction cross section across the resonance,
\begin{equation}
\sigma_0 = \int_{(res)} \sigma(s) ds.
\end{equation}
Using a standard invariant Breit-Wigner fit and the data available, one gets $\sigma_0 \approx 0.3$.

## 3 Random magnetic fields

We model the chaotic magnetic fields present in the astrophysical environments of interest by means of a Gaussian random field of zero mean. The central object in the theory of random fields is the generating functional of the correlation functions. In the case of a Gaussian field, only the second cumulant is different from zero. We write the generating functional as follows:
\begin{equation}
Z[j] = \int \mathcal{D}a \exp \left[ -S + i \int d^3k j_r(k) a_r(k) \right] \tag{10}
\end{equation}
In eq. (10) $j$ stands for an external source, $a$ is the Fourier transform of the vector potential, cf. eq. (2). The functional $S$ is a generalized entropy; for a Gaussian field it is a (gauge invariant) quadratic functional of $a$. We write $S$ as follows.
\begin{equation}
S = \int d^3k \quad a^*_i(k) a_j(k) \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \frac{K^2}{4\pi L^3\langle B^2 \rangle} \left( 1 + L^2 k_r k_s u_{rs} \right)^2. \tag{11}
\end{equation}
In eq. (11), $L$ stands for the root mean square correlation length (the average is taken over directions). The tensor $u_{rs}$ characterizes the directional distribution of the probability density. For a general, arbitrarily anisotropic distribution, $u_{rs}$ has 6 independent components: e.g. the three, mutually
orthogonal, principal correlations and the three angles describing the orientation of the principal correlations. In what follows, however, we consider environments of high symmetry; consequently, fewer parameters are sufficient. The factor \((1 + L^2k_ru_{rs}k_s)^2\) ensures an exponential decrease of the correlation function with distance, cf. ref. [5].

In considering particle production in a random field, one has to replace factors such as \(a_i^*a_j\) in eq. (3) and subsequent ones by their expectation values in the ensemble defined by eqs. (10) and (11).

We now consider two special cases of the ensembles in order to calculate particle production cross sections.

### 3.1 Isotropic ensemble.

This ensemble is characterized by the tensor \(u_{ij}\) in eq. (11) being the unit tensor, \(u_{ij} = \delta_{ij}\). One finds:

\[
\langle a_i(q)a_j(q')^* \rangle = \delta^3(q-q') \left( \delta_{ij} - \frac{q_i q_j}{q^2} \right) \frac{4\pi \langle B^2 \rangle L^3}{q^2} \left( 1 + L^2 q^2 \right)^{-2} \tag{12}
\]

In the expression of the cross section, however, one finds a factor \(\langle a_i(q)a_j^*(q)\rangle\), which is infinite, see the last equation. This is due to the fact that we idealized a region of non vanishing magnetic field by one of infinite extent albeit of an exponentially decreasing correlation function. In order to correct for the inconsistency caused by this idealization, we replace the delta function of vanishing argument by a quantity proportional to the volume, \(vU\).

\[
\delta^3(0) \rightarrow \frac{1}{8\pi^3} V.
\]

(This is a consistent procedure provided the density of levels can be approximated by the Rayleigh-Jeans formula, as done here. In the problem under consideration, the conditions for the validity of that approximation are satisfied.) We take \(V\) to be the correlation volume; thus, in the isotropic case, \(V = 4\pi L^3/3\); clearly, different geometries give rise to different expressions of the correlation volume. The important fact is, however that the incident flux is \(\propto 1/V\). Thus the reciprocal mfp is independent of the choice of the volume.

With this and using eq. (12) the cross section can be evaluated in terms of elementary functions. Quoting directly the inverse of the absorption mfp
which is the relevant quantity for the applications, one finds:

\[
\frac{1}{\lambda_a} = \frac{L}{\pi} \frac{\sigma_0}{m_\Delta^2 - m^2} f(w). \tag{13}
\]

Here \( f \) is a function of the dimensionless variable, \( w = (m_\Delta^2 - m^2) L / 2E \). Its explicit form is:

\[
f(w) = w^2 \left[ \ln \left( \frac{1 + w^2}{w^2} \right) + \frac{1}{1 + w^2} \right]. \tag{14}\]

However, it was pointed out in the Introduction that \( 1/L \) is a small momentum. As a consequence, we only need eq. (14) for large values of \( w \). In that case, eq. (14) simplifies to:

\[
f(w) \sim \frac{1}{2w^2} \quad (w \gg 1).
\]

Hence, the expression of \( \lambda_a \) becomes:

\[
\frac{1}{\lambda_a} \sim \frac{2\sigma_0}{L\pi} \frac{E^2}{(m_\Delta^2 - m^2)^3} \tag{15}\]

### 3.2 Cylindrically symmetric ensemble

This geometry is a more realistic one. In particular, an astrophysical jet as emerging, for instance, from a GRB or an AGN can be approximated by a cylindrical geometry at the early stages of expansion. (At early stages, the lateral expansion is negligibly small compared to the longitudinal one.) Approximating the jet by one of cylindrical geometry means that the lateral expansion is neglected altogether. It has been known for a long time that this is an acceptable approximation in the initial stages of expansion of a relativistic fluid [3].

In this case, the tensor \( u_{ij} \) in eq. (11) effectively depends on one parameter only. It is convenient to introduce the longitudinal and transverse correlation lengths with respect to the axis of the cylinder and an anisotropy parameter, \( \alpha \), such that

\[ L_T^2 = \alpha L^2, \quad L_L^2 = L^2(1 - \alpha). \]

In practice, \( \alpha \ll 1 \), say \( \alpha \approx 0.1 \) or so\footnote{E. Vishniac, private communication.} Using this parametrization, we have:

\[
(1 + L^2 q_i u_{ij} q_j) = \left( 1 + (\alpha q_T^2 + (1 - \alpha) q_L^2) \right). \tag{16}\]
In eq. (16), $q_T$ and $q_l$ stand for the momentum components perpendicular and parallel to the axis of the cylinder, respectively.

In the case of such a geometry, the integral occurring in eq. (8) cannot be calculated in a closed form. In essence, this is due to the fact that the expression of the absorption cross section now contains two directions: that of the incident proton and the axis of the cylinder. However, instead of resorting to a numerical evaluation, we observe that the variable $w$ is large and the absorption can only be significant if the angle between the incident proton and the axis of the cylinder is not too large: efficient absorption requires a coherent magnetic field.

These simplifications allow a calculation of the absorption mfp in a closed form. A somewhat tedious, but elementary calculation leads to the result:

$$\frac{1}{\lambda_a} \sim \frac{4 \sigma_0}{\pi L} F(\Theta) \frac{E^2 \langle B^2 \rangle}{(m^2 - m^2)\alpha}$$

(17)

The factor $F(\Theta)$ is given by the expression:

$$F(\Theta) \approx (\cos \Theta)^3 \ln \left(\frac{(\cos \Theta)^2}{\alpha} - 1\right)$$

(18)

In eq. (18) $\Theta$ stands for the angle between the incident proton and the axis of the cylinder. Obviously, this expression holds only if the angle $\Theta$ is small. From the physical point of view, however, this is not a serious limitation: due to the presence of the factor $\propto (\cos \Theta)^3$, the mean free path becomes very large unless the angle of incidence with respect to the axis of the cylinder is small.

4 Discussion

Our approach has the advantage that it does not depend on the details of the production process, since it is based on the use of the optical theorem. However, its limitation is that the absorption cross section is obtained to lowest order in the fine structure constant. This poses no problem as long as $\sqrt{\langle B^2 \rangle} \lesssim m^2/e = B_{\text{crit}}$, where $m$ is the mass of a charged particle involved in the process. For electrons and light quarks (u,d), the value of $B_{\text{crit}}$ is around $10^{14}$Gauss. In magnetic fields of this order of magnitude, radiative corrections and pair production become important. To our knowledge, no results are
available for such field strengths. Existing calculations, such as Erber’s, assume a homogeneous magnetic field. Calculations of this type can be used to estimate energy losses as long as the magnetic fields are approximately homogeneous on the scale of the Larmor radius of the propagating charged particle. For realistic circumstances, however, this is hardly the case. Thus, the question about the energy loss of charged particles in astrophysically important magnetic field approaching the critical value of the field, is still an open one. Our formulae, however, are expected to give at least a qualitative insight into the question of absorption even for near-critical fields.

Our results show that the circumstances needed for the applicability of eq. (15) are hardly met: one needs magnetic fields with a coherence length substantially in excess of the Larmor radius at high energies. Nevertheless, that equation is an instructive one: due to its simplicity, the general features of the absorption cross section is easily understood.

From the physical point of view, eq. (17) is more interesting. In order to assess the importance of the process discussed it is worth converting eq. (17) into a form permitting numerical estimates. The value of \( \sigma_0 \) has been quoted before; the rest of the numbers is also taken from ref. [4]. One obtains:

\[
\frac{1}{\lambda_a} \approx \frac{5.5}{L} F(\Theta) \left( \frac{E}{10^{20}\text{eV}} \right) \left( \frac{\langle B^2 \rangle}{(10^9\text{Gauss})^2} \right) (19)
\]

(We used the usual conversion factor between the natural and conventional units of the magnetic field, viz. \( B/1(\text{MeV})^2 = 1.9 \times 10^{14} B/1\text{Gauss}. \))

We find that a paraxially propagating proton of energy \( \approx 10^{20}\text{eV} \) traversing a – relatively modest – magnetic field of \( 10^9\text{Gauss} \) has an absorption mfp about \((1/5)^{th}\) the size of the magnetic field.

In a collision at the relevant energies, on the average the nucleon and the produced pion in the final state share the incident energy equally. As a consequence, using the continuous energy loss approximation, the energy loss per unit path length is:

\[
\frac{dE}{dx} = -\frac{\epsilon E}{\lambda} (20)
\]

In the last equation, \( \epsilon \) stands for the fractional energy loss of a nucleon (in the laboratory system) due to pion production. Assuming as we do throughout this paper that the pion production cross section is dominated by the \( \Delta \) resonance, one gets,

\[
\epsilon \approx \frac{m^2}{2m_\Delta m}
\]
Due to the fact that $1/\lambda \propto E^2$, the energy loss per unit path length grows as $E^3$.

In fact, by inserting the expression for the mfp given by eq. (19) into eq. (20), the equation for the energy loss is readily integrated. We exhibit the result for the energy loss of paraxial protons ($F(\Theta) \approx 1$) over one correlation length, $L$. We get:

$$\frac{E(x)}{E(0)} \approx \left[ 1 + 0.1 \left( \frac{E}{10^{20}\text{GeV}} \right)^2 \left( \frac{\langle B^2 \rangle}{(10^6\text{Gauss})^2} \right) \frac{x}{L} \right]^{-1/2}$$

(21)

We used the mass values listed in ref. [4] in order to arrive at eq. (21).

We conclude that the mechanism described in this paper appears to be a major obstacle to accelerating protons up to energies of the order of $10^{19}$eV or more by a conventional Fermi acceleration mechanism. One notices for instance that a proton of $E = 10^{20}$eV injected into a field of $\sqrt{\langle B^2 \rangle} = 10^{10}$Gauss loses about 70% of its initial energy over a correlation length.

This adds to the puzzle of the highest energy cosmic rays: it is known that particles of energy about $10^{20}$eV arrive to the Earth and they give rise to extensive air showers. At the same time, it appears to be increasingly difficult to find an efficient mechanism for producing them at the usually suspected sites, for instance in active galactic nuclei or gamma ray bursters.

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