Deformed super brackets on forms of a manifold

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December 14, 2021

1 Introduction

Let $M$ be a differentiable manifold and $T(M)$ and $T^*(M)$ be the tangent and cotangent bundle of $M$ (or the spaces of the sections). In addition to the typical example of Lie superalgebra $\mathfrak{g} = \bigoplus_{p=1}^{\dim M} \Lambda^p T(M)$ with the Schouten bracket, the space of differential forms $\mathfrak{h} = \bigoplus_{q=0}^{\dim M} \Lambda^q T^*(M)$ with the bracket

\begin{equation}
\{\alpha, \beta\} = (-1)^a d(\alpha \wedge \beta) \quad \text{where} \ \alpha \in \Lambda^a T^*(M) \ \text{and} \ \beta \in \Lambda^b T^*(M)
\end{equation}

becomes a Lie superalgebra, where the grading of $\Lambda^a T^*(M)$ is $-a - 1$, and is often referred to as $a'$ (cf. [5]). The grading of $\Lambda^a T(M)$ is $a - 1$, and is also represented by $a'$.

There is a notion of deformation of the exterior differentiation $d$ by a 1-form $\phi$ defined by $d_t \alpha = d\alpha + t\phi \wedge \alpha$ where $t$ is a scalar parameter runs at least $[0, 1]$ interval, and it is well-known that $d_t \circ d_t = 0$ if $\phi$ is a 1-cocycle. It is natural to expect $d_\phi$ defines a Lie superalgebra structure, namely

$\{\alpha, \beta\}^t = (-1)^a d_t (\alpha \wedge \beta) = \{\alpha, \beta\} + \alpha \wedge (t\phi) \wedge \beta$

will be a super bracket for each $t$. Super symmetry holds good. About super Jacobi identity,

$\{\{\alpha, \beta\}^t, \gamma\}^t$

$= \{\alpha, \beta\}, \gamma\} + (-1)^{a+1} d(\alpha \wedge \beta \wedge t\phi \wedge \gamma) + (-1)^a d(\alpha \wedge \beta) \wedge t\phi \wedge \gamma$

$= \{\alpha, \beta\}, \gamma\} + (-1)^{b+c+1} \alpha \wedge \beta \wedge d\gamma \wedge t\phi + (-1)^{b+1} \alpha \wedge \beta \wedge \gamma \wedge d(t\phi)$

$= \{\alpha, \beta\}, \gamma\} + (-1)^{b+c+1} \alpha \wedge \beta \wedge d\gamma \wedge t\phi \quad \text{if} \ \phi \ \text{is closed}.$

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Claim 2: A deformation of the standard bracket: For a given closed 1-form $\omega$,

\[
\{\alpha, \beta\}^{t, \phi} = (-1)^{a} d(\alpha \wedge \beta) + (-1)^{b} t^2 \phi + \alpha \wedge (t \phi \wedge (t \phi \wedge \beta))
\]

\[
\{\alpha, \beta\}^{t, \phi} = \alpha \wedge (t \phi \wedge (t \phi \wedge \beta))
\]

where $\{\alpha, \beta\}^{t, \phi}$ is a super bracket on $\mathfrak{g}$ when $F$ is a symmetric function.

Claim 2: A deformation of the standard bracket: For a given closed 1-form $\phi$,

\[
\{\alpha, \beta\}^{t, \phi} = (\alpha \wedge \beta) + \frac{a+b}{2} \alpha \wedge (t \phi \wedge \beta)
\]

\[
\{\alpha, \beta\}^{t, \phi} = \alpha \wedge (t \phi \wedge (t \phi \wedge \beta))
\]

is a super bracket on $\mathfrak{g}$. 

Claim 3: An extension of the deformation of the standard bracket: For a given closed 1-form $\phi$, Claim 2 says $\mathfrak{g} = \{\mathfrak{g}, \{\cdot, \cdot\}^{t, \phi}\}$ is Lie superalgebra. Let $\mathfrak{g}' = \{X \in T(M) | L_X \phi = 0\}$, which is a subalgebra of $T(M)$. Then

\[
(\mathfrak{g}, \{\cdot, \cdot\}^{t, \phi}) \oplus \mathfrak{g}'
\]

becomes a Lie superalgebra naturally by the Lie derivative for each $t$. 

So far, there is no affirmative statement in general setting. On the other hand, there is a result of deformation of the Schouten bracket by D. Iglesias and J. C. Marrero in [11]. They say for a 1-cocycle $\phi$,

\[
(P, Q)^{\phi} = (P, Q) + (-1)^{p} P(\phi) \wedge (q - 1)Q + (p - 1)P \wedge Q(\phi) \quad \text{where} \quad P(\phi) = t_{\phi}P
\]

satisfies the axioms of bracket of Lie superalgebra.

1. $[P, Q]^{\phi} = -(-1)^{(p-1)(q-1)}[Q, P]^{\phi}$

2. $[P, [Q, R]]^{\phi} = [[P, Q], R]^{\phi} + (-1)^{(p-1)(q-1)}[Q, [P, R]]^{\phi}$

Inspired by the work above, in this note for a given 1-form $\phi$, we fix properties of a function $F$ so that

\[
(-1)^{a} d(\alpha \wedge \beta) + F(a, b) \alpha \wedge (t \phi) \wedge \beta \quad (\alpha \in L^{a} T^{*}(M), \beta \in L^{b} T^{*}(M))
\]

becomes super bracket for $t$. The function $F$ should be defined on $\{(a, b) \in \mathbb{Z}^2 | a + b \leq \dim M - 1\}$.

Main results in this note is that there are deformations of two super brackets on the space $\mathfrak{g} = \sum_{q=0}^{\dim M} L^{q} T^{*}(M)$, and there is a natural extension to a subalgebra of $T(M)$.

Claim 1: A deformation of the trivial bracket: For a given 1-form $\phi$

\[
\{\alpha, \beta\}^{t, \phi} = F(a, b) \alpha \wedge (t \phi) \wedge \beta \quad (\alpha \in L^{a} T^{*}(M), \beta \in L^{b} T^{*}(M))
\]

is a super bracket on $\mathfrak{g}$ when $F$ is a symmetric function.

Claim 2: A deformation of the standard bracket: For a given closed 1-form $\phi$,

\[
\{\alpha, \beta\}^{t, \phi} = (-1)^{a} d(\alpha \wedge \beta) + \frac{a+b}{2} \alpha \wedge (t \phi \wedge \beta) \quad (\alpha \in L^{a} T^{*}(M), \beta \in L^{b} T^{*}(M))
\]

is a super bracket on $\mathfrak{g}$. 

Claim 3: An extension of the deformation of the standard bracket: For a given closed 1-form $\phi$, Claim 2 says $\mathfrak{g} = \{\mathfrak{g}, \{\cdot, \cdot\}^{t, \phi}\}$ is Lie superalgebra. Let $\mathfrak{g}' = \{X \in T(M) | L_X \phi = 0\}$, which is a subalgebra of $T(M)$. Then

\[
(\mathfrak{g}, \{\cdot, \cdot\}^{t, \phi}) \oplus \mathfrak{g}'
\]

becomes a Lie superalgebra naturally by the Lie derivative for each $t$. 

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Based on these results, there are many issues to be studied. We would like to develop homology theory of deformed superalgebra. When a Lie group $G$ acts on $M$, we get $\sum_{p=1}^{\dim M} \Lambda^p T_G(M)$ of $G$-invariant multivector fields, and $\sum_{q=0}^{\dim M} \Lambda^q T^*_G(M)$ of $G$-invariant differential forms. Since the action preserves the Jacobi-Lie bracket, the Schouten bracket is preserved by the action. Also the action commutes with the differentiation $d$, the Lie superalgebra bracket is preserved by the action. In short, $\sum_{p=1}^{\dim M} \Lambda^p T_G(M)$ and $\sum_{q=0}^{\dim M} \Lambda^q T^*_G(M)$ have Lie superalgebra structures.

The simplest case is a Lie group acts on itself. $\mathfrak{g} = \sum_{p=1}^n \Lambda^p \mathfrak{g}$ where $\mathfrak{g}$ = Lie algebra of $G$, has a Lie superalgebra structure by the Schouten bracket (cf. [4]). The differential $d$ gives a Lie superalgebra structure on $\mathfrak{h} = \sum_{p=0}^n \Lambda^p \mathfrak{h}$, where $\mathfrak{h} = \mathfrak{g}^*$ (cf. [5]). Concrete and fancy examples are presented from those superalgebras.

\section{Deformation from the trivial bracket}

In this section, we study deformed super bracket of the trivial bracket on $\mathfrak{h}$, namely

\begin{equation}
\{\alpha, \beta\}^{t, \phi} = F(a, b) \alpha \wedge (t\phi) \wedge \beta \quad (\alpha \in \Lambda^a T^*(M), \beta \in \Lambda^b T^*(M)).
\end{equation}

Then the symmetric property of $F$ implies the super symmetric property of $\{\cdot, \cdot\}^{t, \phi}$ because of

$$\{\alpha, \beta\}^{t, \phi} + (-1)^{(1+a)(1+b)} \{\beta, \alpha\}^{t, \phi} = (F(a, b) - F(b, a)) \alpha \wedge (t\phi) \wedge \beta.$$

The super Jacobi identity holds automatically because

$$\{\{\alpha, \beta\}^{t, \phi}, \gamma\}^{t, \phi} = \{F(a, b) \alpha \wedge (t\phi) \wedge \beta, \gamma\}^{t, \phi} = F(a, b) F(a + b + 1, c) \alpha \wedge (t\phi) \wedge \beta \wedge (t\phi) \wedge \gamma = 0.$$

\textbf{Proposition 2.1} The bracket (2.1) is a super bracket on $\mathfrak{h}$ when $F$ is a symmetric function on $\{(a, b) \in \mathbb{Z}^2 \mid a + b \leq \dim M - 1\}$.

\textbf{Remark 2.1} In the proposition above, $\phi$ is not necessarily closed. Like nilpotent subalgebras of a Lie algebra, $\{\{\mathfrak{h}, \mathfrak{h}\}^{t, \phi}, \mathfrak{h}\}^{t, \phi} = 0$ holds. In contrast, $(-1)^{a'b'} \{\{\alpha, \beta\}, \gamma\} = (-1)^{a'b'} \alpha \wedge d\beta \wedge \gamma - (-1)^{b'a'} \beta \wedge d\gamma \wedge d\alpha$ holds for $\{\alpha, \beta\} = (-1)^a d(\alpha \wedge \beta)$.

We already know that the superalgebra $\mathfrak{h} = \sum_{i=0}^{\dim M} \Lambda^i T^*(M)$ with the bracket $\{\alpha, \beta\} = (-1)^a d(\alpha \wedge \beta)$ has an extension by $\mathfrak{g}_0 = T(M)$ through Lie derivative in [5]. Here we study the deformed superalgebra given by the bracket (2.1) has an extension by a subalgebra of $T(M)$ through the Lie derivative $L_X = i_X \circ d + d \circ i_X$ with respect to $X$. 

\begin{equation}
\{\mathfrak{h}, \mathfrak{h}\}^{t, \phi} = 0
\end{equation}
Let $F(a, b)$ be a symmetric function on $\mathbb{Z}^2$ and $\phi$ be a 1-form on $M$ and

\[
(2.2) \quad \{\alpha, \beta\}^{t, \phi} = F(a, b)\alpha \wedge t\phi \wedge \beta \quad (\alpha \in \Lambda^aT^*(M), \beta \in \Lambda^bT^*(M)).
\]

For 1-vector field $X$, we define $\{X, \alpha\}^{t, \phi} = L_X\alpha$ and $\{\alpha, X\}^{t, \phi} = -L_X\alpha$. So the super symmetry holds good. About super Jacobi identity, we check two cases: one is $X, Y, \alpha$ and the other is $X, \alpha, \beta$. The first case the super Jacobi identity is just the formula $L_{[X,Y]} = L_XL_Y - L_YL_X$. So the super symmetry holds good. We treat the other case:

Since

\[
\{X, \{\alpha, \beta\}^{t, \phi}\}^{t, \phi} - \{\{X, \alpha\}^{t, \phi}, \beta\}^{t, \phi} - \{\alpha, \{X, \beta\}^{t, \phi}\}^{t, \phi}
\]

\[
= tF(a, b)L_X(\alpha \wedge \phi \wedge \beta) - tF(a, b)L_X\alpha \wedge \phi \wedge \beta - tF(a, b)\alpha \wedge \phi \wedge L_X\beta
\]

\[
= tF(a, b)\alpha \wedge (L_X\phi) \wedge \beta
\]

we see that $L_X\phi = 0$ is an efficient condition for super Jacobi identity.

**Proposition 2.2** Let $g'_0 = \{X \in g_0 \mid L_X\phi = 0\}$, which is a subalgebra of $g_0$. Then $(\mathfrak{h} \oplus g'_0, \{\cdot, \cdot\}^{t, \phi})$ is a deformed superalgebra.

If $\phi$ is a 1-cocycle, then $(\mathfrak{h} \oplus g''_0, \{\cdot, \cdot\}^{t, \phi})$ is a deformed superalgebra, where $g''_0 = \{X \in g_0 \mid i_X\phi = 0\}$, which is a subalgebra of $g'_0$.

Some concrete example will appear in the tail of the next section.

### 3 Deformation from the standard bracket

It is known in [5] that $\{\alpha, \beta\} = (-1)^a d(\alpha \wedge \beta)$ defines a super bracket on $\sum_{i=0}^{\dim M} \Lambda^iT^*(M)$. Looking at the deformed Schouten bracket, the bracket we expect is of form $\{\alpha, \beta\}^{t, \phi} = \{\alpha, \beta\} + F(a, b)\alpha \wedge t\phi \wedge \beta$ for some function $F$ on $\{(a, b) \in \mathbb{Z}^2 \mid a + b \leq \dim M - 1\}$.

About super symmetric property, we have

\[
(3.1) \quad \{\alpha, \beta\}^{t, \phi} + (-1)^{ab'} \{\beta, \alpha\}^{t, \phi} = (F(a, b) - F(b, a))\alpha \wedge t\phi \wedge \beta
\]

About Jacobi identity, we see

\[
\sum_{a,b, \gamma} (-1)^{a'c} \left(\{\{\alpha, \beta\}^{t, \phi}, \gamma\}^{t, \phi} - \{\alpha, \gamma\}^{t, \phi}\right)
\]

\[
= (F(1 + b + c, a) - F(a, b) - F(b, c) - F(c, a) + F(1 + c + a, b)) (-1)^{ac + a + c} (\alpha \wedge t\phi \wedge \beta \wedge d(\gamma))
\]
\[-(-1)^{ac+ab+c} (F(1+a+b,c)+F(1+b+c,a)-F(a,b)-F(b,c) - F(c,a)) (\alpha \wedge d (\beta) \wedge t\phi \wedge \gamma) \]
\[+ (F(a,b) + F(b,c) + F(c,a)) (-1)^{ac+a+b+c} (\alpha \wedge d (t\phi) \wedge \beta \wedge \gamma) \]
\[-(-1)^{ac+c} (F(1+a+b,c) - F(a,b) - F(b,c) - F(c,a) + F(1+c+a,b)) (d (\alpha) \wedge \beta \wedge t\phi \wedge \gamma) \text{.} \]

Thus, if \(\phi\) is a 1-cocycle, then we get the following sufficient conditions for super Jacobi identity

\[(3.2) \quad 0 = F(1+b+c,a) + F(1+c+a,b) - F(a,b) - F(b,c) - F(c,a) \text{ ,} \]
\[(3.3) \quad 0 = F(1+a+b,c) + F(1+b+c,a) - F(a,b) - F(b,c) - F(c,a) \text{ ,} \]
\[(3.4) \quad 0 = F(1+c+a,b) + F(1+a+b,c) - F(a,b) - F(b,c) - F(c,a) \text{ .} \]

The difference \((3.2) - (3.3) = 0\) implies \(F(1+c+a,b) = F(1+a+b,c)\). Putting \(c = 0\), we have \(F(1+a,b) = F(1+a+b,0)\) and \(F(a,0) = F(0,0)(\frac{a}{2} + 1)\), we see that the symmetric function \(F\) satisfying the above 3 conditions is

\[(3.5) \quad F(a,b) = \kappa(a+b+2) \text{ where } \kappa \text{ is a constant.} \]

Assume 1-form \(\phi\) is not closed. Then we get the following sufficient conditions for super Jacobi identity

\[(3.6) \quad 0 = F(1+b+c,a) + F(1+c+a,b) \]
\[(3.7) \quad 0 = F(1+a+b,c) + F(1+b+c,a) \]
\[(3.8) \quad 0 = F(a,b) + F(b,c) + F(c,a) \]
\[(3.9) \quad 0 = F(1+c+a,b) + F(1+a+b,c) \]

We conclude a symmetric \(F\) satisfying the 4 above conditions is trivial as follows. Putting \(c = 0\) in \((3.8)\)

\[(3.10) \quad F(a,b) = -G(a) - G(b) \text{ where } G(a) = F(0,a) = F(a,0) \text{ .} \]

Applying this expression to \((3.8)\), we have

\[(3.11) \quad 0 = -2(G(a) + G(b) + G(c)) \text{ , and so } G(a) = 0 \text{ , } F(a,b) = 0 \text{ , i.e., trivial.} \]

We summarize above discussion.

**Theorem 3.1** The super symmetry of the bracket \((1.3)\)

\[-(-1)^ad(\alpha \wedge \beta) + F(a,b)\alpha \wedge (t\phi) \wedge \beta \]

yields \(F(a,b)\) is a symmetric function, i.e., \(F(a,b) = F(b,a)\).

The super Jacobi identity implies if \(\phi\) is not a cocycle, i.e., not exact then \(F(a,b) = 0\). If \(\phi\) is a cocycle, i.e., if \(\phi\)
We summarize the kernel dimensions and Betti numbers as follows, we assume
is exact then the super Jacobi identify yields \( F(a, b) = \kappa(a + b + 2) \).

**Corollary 3.1** Let \( \phi \) be an exact 1-form.

\[
\{\alpha, \beta\}^{\phi} = \{\alpha, \beta\} + \frac{a+b+2}{2}\alpha \wedge t\phi \wedge \beta
\]

where \( \{\alpha, \beta\} = (-1)^a d(\alpha \wedge \beta), \alpha \in \Lambda^a T^*(M) \) and \( \beta \in \Lambda^b T^*(M) \). This bracket satisfies super symmetry and super Jacobi identity, and the space \( \mathfrak{h} \) with this bracket is a Lie superalgebra.

**Remark 3.1** This bracket is a super bracket from Theorem above. If one proceed to reconfirm that this bracket is a super bracket, it is one page exercise. Symbol calculus, Maple has a package `diffforms` and it is helpful to study of properties of this bracket.

**Example 3.1** Take a 2-dimensional Lie algebra with the Lie bracket relations \([y_1, y_2] = y_1 \) and the \( z_1, z_2 \) is the
dual basis so that \( dz_1 = -z_1 \wedge z_2 \) and \( dz_2 = 0 \). The chain complex of weight \(-3\) is given by \( C_1[-3] = \mathbb{R}(z_1 \wedge z_2), C_2[-3] = \mathbb{R}(z_1 \wedge 1) + \mathbb{R}(z_2 \wedge 1), C_3[-3] = \mathbb{R}(1 \wedge 1 \wedge 1) = \mathbb{R}(\Delta^3 1) \). We refer to the appendix or [5] about odd notations.

Fix \( \phi = z_2 \). \( \partial(z_1 \wedge z_2) = 0 \) for 1-chain.

\[
\partial(z_j \wedge 1) = \{z_j, 1\}^{t, z_2} = \begin{cases} 
  c_1 \delta_j^1 t z_1 \wedge z_2 & \text{c_1 is constant, deform of trivial} \\
  \delta_j^1 (1 + \frac{3}{2} t) z_1 \wedge z_2 & \text{deform of standard}
\end{cases}
\]

\[
\partial(\Delta^3 1) = \left(\frac{3}{2}\right) \{1, 1\}^{t, z_2} \wedge 1 = \begin{cases} 
  c_0 t z_2 \wedge 1 & \text{c_0 is constant, deform of trivial} \\
  t z_2 \wedge 1 & \text{deform of standard}
\end{cases}
\]

We summarize the kernel dimensions and Betti numbers as follows, we assume \( c_0 c_1 \neq 0 \).

| trivial | 1 | 2 | 3 |
|---------|---|---|---|
| dim     | 1 | 2 | 1 |
| ker dim | 1 | 1 + \delta_0^0 | \delta_t^0 |
| Betti   | \delta_t^0 | 2\delta_t^0 | \delta_t^0 |

| standard | 1 | 2 | 3 |
|----------|---|---|---|
| dim      | 1 | 2 | 1 |
| ker dim  | 1 | 1 + \delta_{2+3t}^0 | \delta_t^0 |
| Betti    | \delta_{2+3t}^0 | \delta_{2+3t}^0 + \delta_t^0 | \delta_t^0 |

**3.1 An extension**

We mentioned before that the superalgebra \( \mathfrak{h} = \sum_{i=0}^{\dim M} \Lambda^i T^*(M) \) with the bracket \( \{\alpha, \beta\} = (-1)^a d(\alpha \wedge \beta) \) has an
extension by \( T(M) \) through Lie derivation in [5]. Here we study two deformed superalgebras have an extension
by a subalgebra of \( \mathfrak{g}_0 = T(M) \) through the Lie derivative \( L_X = t_X \circ d + d \circ t_X \) with respect to \( X \).
The superalgebra $\mathfrak{h}$ has the deformed bracket $(\alpha, \beta)^{t, \phi} = (-1)^a d(\alpha \wedge \beta) + \frac{a + b + c}{2} \alpha \wedge t \phi \wedge \beta$, where $\phi$ is a 1-cocycle. Let $\{\cdot, \cdot\}'$ be a candidate of superbracket on $\mathfrak{h} \oplus T(M)$, i.e., $(\alpha, \beta)' = (\alpha, \beta)^{t, \phi}$, $(X, Y)' = [X, Y]$, $(X, \beta)' = -\{\beta, X\}' = L_X \beta$ for forms $\alpha, \beta$ and 1-vectors $X, Y$. We have to check super Jacobi identity for two cases. Again we abbreviate $t \phi$ by $\phi$. One case is all right as following.

$\{\{X, Y\}', \alpha\} + \{\{Y, \alpha\}', X\} + \{\{\alpha, X\}', Y\}'$

$= \{X, [Y, \alpha]\} - L_X L_Y \alpha + L_Y L_X \alpha = (L_{[X,Y]} - L_X L_Y + L_Y L_X) \alpha = 0$.

We try the other.

$(X, \{\alpha, \beta\}') + (\{\alpha, \beta\}', X) + (-1)^{b' a'} \{\beta, (X, \alpha)\}'$

$= L_X (\alpha, \beta)' + (-1)^a d(\alpha \wedge \{\beta, X\}') + \frac{a' + b'}{2} \alpha \wedge \phi \wedge \{\beta, X\}'$

$+ (-1)^{1 + a' b'} ((-1)^{b' d}(\beta \wedge L_X \alpha) - \frac{a' + b'}{2} \beta \wedge \phi \wedge L_X \alpha)$

$= L_X ((-1)^a d(\alpha \wedge \beta) + \frac{a' + b'}{2} \alpha \wedge \phi \wedge \beta) + (-1)^{a + 1} d(\alpha \wedge L_X \beta)$

$- \frac{a' + b'}{2} \alpha \wedge \phi \wedge L_X \beta + (-1)^{1 + a' b'} ((-1)^{b' d}(\beta \wedge L_X \alpha) - \frac{a' + b'}{2} \beta \wedge \phi \wedge L_X \alpha)$

$= (-1)^a d L_X (\alpha \wedge \beta) + \frac{a' + b'}{2} L_X (\alpha \wedge \phi \wedge \beta)$

$+ (-1)^{a + 1} d(\alpha \wedge L_X \beta) - \frac{a' + b'}{2} \alpha \wedge \phi \wedge (L_X \beta)$

$+ (-1)^{1 + a' b'} (\beta \wedge L_X \alpha) + (-1)^{a' b'} \frac{a'' + b''}{2} \beta \wedge \phi \wedge (L_X \alpha + (X, 0) \alpha)$

$= + \frac{a' + b'}{2} L_X (\alpha \wedge \phi \wedge \beta) + (-1)^{a' b'} \frac{a'' + b''}{2} \beta \wedge \phi \wedge L_X \alpha - \frac{a' + b'}{2} \alpha \wedge \phi \wedge (L_X \beta)$

$+ (-1)^a d L_X (\alpha \wedge \beta) + (-1)^{a + 1} d(\alpha \wedge L_X \beta) + (-1)^{a + 1} d(L_X \alpha \wedge \beta)$

$= + \frac{a' + b'}{2} L_X (\alpha \wedge \phi \wedge \beta) + (-1)^{a' b'} \frac{a'' + b''}{2} (\beta \wedge \phi \wedge L_X \alpha) - \frac{a' + b'}{2} (\alpha \wedge \phi \wedge L_X \beta)$

$= + \frac{a' + b'}{2} (L_X (\alpha \wedge \phi \wedge \beta) - L_X \alpha \wedge \phi \wedge \beta - \alpha \wedge \phi \wedge L_X \beta)$

$= + \frac{a' + b'}{2} (\alpha \wedge L_X (\phi \wedge \beta)$

This vanishes if $L_X \phi = 0$. We see that $\{X \in T(M) \mid L_X \phi = 0\}$ forms a $\mathbb{R}$ subalgebra of $T(M)$. Thus we have the following result.

**Theorem 3.2** The superalgebra $(\mathfrak{h}, \{\cdot, \cdot\}^{t, \phi})$ allows an extension by $\mathfrak{g}_0' = \{X \in T(M) \mid L_X \phi = 0\}$, namely, $\mathfrak{h} \oplus \mathfrak{g}_0'$ becomes a Lie superalgebra extension of $\mathfrak{h}$. 

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Example 3.2 We extend \( \mathfrak{h} \) in Example 3.1 by \( \mathfrak{g}_0 \) and show the \((-3)\)-weighted chain complex. We denote \( y_1 \triangle y_2 \) by \( U \), and \( z_1 \wedge z_2 \) by \( V \).

| \( C_m^{[3]} \) | 1 | 2 | 3 | 4 | 5 | if |
|-------------|---|---|---|---|---|----|
| dim         | 1 | 4 | 6 | 4 | 1 |    |
| basis       | \( V \) | \( z_j \triangle 1 \) | \( y_i \triangle 1 \) | \( U \triangle 1 \) | \( U \triangle 1 \) |    |
| ker dim     | 1 | 3 | 4 | 2 | 0 | \( t(1 + 3t/2) = 0 \) |
|             | 3 | 1 |   |   |   | \( t(1 + 3t/2) \neq 0 \) |
| Betti       | 0 | 1 | 2 | 1 | 0 | \( t(1 + 3t/2) = 0 \) |
|             | 0 | 0 | 0 |   |   | \( t(1 + 3t/2) \neq 0 \) |

By the direct computation, we see that the boundary image is spanned as follows.

\[
\partial C_1^{[-3]} = \{0\}, \quad \partial C_2^{[-3]} = \{(1 + 3t/2)V, V\}
\]

\[
\partial C_3^{[-3]} = \{3tz_2 \triangle 1, z_2 \triangle 1 + (1 + 3t/2)y_1 \triangle 1, z_1 \triangle 1 - (1 + 3t/2)y_2 \triangle V\},
\]

\[
\partial C_4^{[-3]} = \{ty_1 \triangle z_2 \triangle 1, ty_2 \triangle z_2 \triangle 1, -y_2 \triangle z_2 \triangle 1 + (1 + 3t/2)y_1 \triangle y_2 \triangle V, y_1 \triangle z_2 \triangle 1\},
\]

\[
\partial C_5^{[-3]} = \{y_1 \triangle^2 1 + 3ty_1 \triangle y_2 \triangle z_2 \triangle 1\}.
\]

Thus, the kernel dimensions of \( \partial \) for \( m=1,3,5 \) are 1,3,0 and those of \( m = 3,4 \) are 4,2 if \( t(1 + 3t/2) = 0 \) else 3,1. Finally the Betti numbers are 0,1,2,1, 0 if \( t(1 + 3t/2) = 0 \) else 0,0,0,0,0.

To get the weighted homology groups of Lie algebras, even of 2-dimensional, we need hard work. We prepare reporting the general weighted homology groups of 2-dimensional Lie algebra.

In [2] and [3], we introduced double weight for the algebra of homogeneous polynomial coefficient multi-vector fields on \( \mathbb{R}^n \). By the similar way, we get examples of double weighted super algebras of homogeneous polynomial coefficient forms and 1-vector fields on \( \mathbb{R}^n \) in [5]. It may be interesting to study those deformed double weighted superalgebras and their homology groups.

Appendix: A Quick review of the homology groups of Lie superalgebra

Let \( \mathfrak{g} = \sum_{i \in \mathbb{Z}} \mathfrak{g}_i \) be a Lie superalgebra. From super symmetry \([X,Y] + (-1)^{xy}[Y,X] = 0\) for \( X \in \mathfrak{g}_x, Y \in \mathfrak{g}_y \), \( m \)-th chain space is given by \( C_m = \otimes^m \mathfrak{g} / \text{Ideal of}(X \otimes Y + (-1)^{xy} Y \otimes X) \). We denote the class of \( A_1 \otimes \cdots \otimes A_p \)
by $A_1 \triangle \cdots \triangle A_p$. Let $\widetilde{A} = A_1 \triangle \cdots \triangle A_p$ and $\widetilde{B} = B_1 \triangle \cdots \triangle B_q$. The boundary operator $\partial : C_m \to C_{m-1}$ called (boundary homomorphism) is defined by

(Appendix: A.1) $\partial(Y_1 \triangle \cdots \triangle Y_m) = \sum_{i < j} (-1)^{i+j}(\sum_{s<i} y_s)Y_1 \triangle \cdots \triangle \widetilde{Y}_i \cdots \Delta [Y_i, Y_j] \triangle \cdots \triangle Y_m$

for a decomposable element, where $y_i$ is the degree of homogeneous element $Y_i$, i.e., $Y_i \in g_{y_i}$. It is clear that $\partial \circ \partial = 0$ and we have the homology groups $H_m(g, \mathbb{R}) = \ker(\partial : C_m \to C_{m-1})/\partial(C_{m+1})$.

We say a non-zero $m$-th decomposable element $Y_1 \triangle \cdots \triangle Y_m$ has the weight $\sum_{i=1}^{m} y_i$ where $Y_i \in g_{y_i}$. The weight is preserved by $\partial$, i.e., $\partial(C_m[w]) \subset C_{m-1}[w]$ where $C_m[w]$ is the subspace of $w$-weighted $m$-th chains and we have the weighted homology groups.

If all $y_i$ are even in (Appendix: A.1), then

(Appendix: A.2) $\partial(Y_1 \triangle \cdots \triangle Y_m) = -\sum_{i < j} (-1)^{i+j} [Y_i, Y_j] \triangle Y_1 \triangle \cdots \triangle \widetilde{Y}_i \cdots \Delta Y_m$.

If all $y_i$ are odd in (Appendix: A.1), then

(Appendix: A.3) $\partial(Y_1 \triangle \cdots \triangle Y_m) = \sum_{i < j} [Y_i, Y_j] \triangle Y_1 \triangle \cdots \triangle \widetilde{Y}_i \cdots \Delta Y_m$.

**Definition 1** Let $A = A_1 \triangle \cdots \triangle A_{\bar{a}}$ ($A_i \in g_{a_i}$) and $B = B_1 \triangle \cdots \triangle B_{\bar{b}}$ ($B_j \in g_{b_j}$). Define

(Appendix: A.4) $[A, B]_{res} = \partial(A \triangle B) - (\partial A) \triangle B - (-1)^{\bar{a}} A \triangle \partial B$.

It satisfies

(Appendix: A.5) $[A, B]_{res} = \sum_{i,j} (-1)^{i+a_i} \sum_{s=1}^{i+j} a_s + j + b_j \sum_{s=1}^{j} b_s A_1 \triangle \cdots \triangle \tilde{A}_i \cdots \triangle A_{\bar{a}} \triangle [A_i, B_j] \triangle B_1 \triangle \cdots \triangle B_{\bar{b}}$.

If all $a_i$ are even and all $b_j$ are odd in (Appendix: A.5), then

(Appendix: A.6) $[A, B]_{res} = \sum_{i,j} (-1)^{i+j} A_1 \triangle \cdots \triangle \tilde{A}_i \cdots \triangle A_{\bar{a}} \triangle [A_i, B_{\bar{j}}] \triangle B_1 \triangle \cdots \triangle B_{\bar{b}}$.

Let $C = C_1 \triangle \cdots \triangle C_{\bar{c}}$ ($C_k \in g_{c_k}$) with $c_k$ are all even. Then

(Appendix: A.7) $[A, C \triangle B]_{res} = \sum_{i,k} (-1)^{i+k} A_1 \triangle \cdots \triangle \tilde{A}_i \cdots \triangle A_{\bar{a}} \triangle [A_i, C_k] \triangle C_1 \triangle \cdots \triangle \tilde{C}_i \cdots \triangle C_{\bar{c}} \triangle B$ + $\sum_{i,j} (-1)^{i+j} A_1 \triangle \cdots \triangle \tilde{A}_i \cdots \triangle A_{\bar{a}} \triangle C \triangle [A_i, B_j] \triangle B_1 \triangle \cdots \triangle B_{\bar{b}}$.

(Appendix: A.8) $= \sum_{i,k} (-1)^{i+k} A_1 \triangle \cdots \triangle \tilde{A}_i \cdots \triangle A_{\bar{a}} \triangle C_1 \triangle \cdots \triangle [A_i, C_{\bar{k}]} \triangle \cdots \triangle C_{\bar{c}} \triangle B$. 

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\[ + \sum_{i,j} (-1)^{i+1} A_1 \Delta \cdots \A_i \cdots A_{\a} \Delta C \Delta [A_i, B_j] \Delta B_1 \Delta \cdots \B_j \cdots \Delta B_{\b} \].

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