The emergent nature of time and the complex numbers in quantum cosmology

G W Gibbons
D.A.M.T.P.,
Cambridge University,
Wilberforce Road,
Cambridge CB3 0WA,
U.K.

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Abstract

The nature of time in quantum mechanics is closely related to the use of a complex, rather than say real, Hilbert space. This becomes particularly clear when considering quantum field theory in time dependent backgrounds, such as in cosmology, when the notion of positive frequency ceases to be well defined. In spacetimes lacking time orientation, i.e without the possibility of defining an arrow of time, one is forced to abandon complex quantum mechanics. One also has to face this problem in quantum cosmology. I use this to argue that this suggests that, at a fundamental level, quantum mechanics may be really real with not one, but a multitude of complex structures. I relate these ideas to other suggestions that in quantum gravity time evolution may not be unitary, possibly implemented by a super-scattering matrix, and the status of CPT.

1 Introduction

The topic of this conference is *The Arrow of Time*, but before asking that we should ask *What is the nature of time?*

Both Quantum Mechanics and General Relativity have something to say about this.

But what they say is not quite compatible.

For example, in quantum mechanics, there may be observables or operators corresponding to spatial positions but time is not an observable, i.e. it is not an operator. More precisely, by an argument going back to Pauli, commutation relations like

\[ [\hat{x}_\mu, \hat{P}_\nu] = i\delta_{\mu}^{\nu} \]  

\[ (1) \]

\[ ^1 \text{see Pullin's contribution in this volume.} \]
are incompatible with the spectrum of $\hat{p}^i$ lying in the future lightcone.

In General Relativity on the other hand, space and time are usually held to be on the same footing.

Because the nature of time in Quantum Mechanics is less familiar and less frequently discussed, than it is in General Relativity I shall begin by recalling [1, 8, 9, 11, 12, 16, 60] how time is intimately connected with the complex (Hilbert Space) structure of quantum mechanics.

In other words, the use of complex numbers and hence of complex amplitudes in Quantum Mechanics is intimately bound up with how Quantum States evolve in time.

$$i\frac{d\Psi}{dt} = H\Psi.$$  \hspace{1cm} (2)

In particular there can be no evolution if $\Psi$ is real.\footnote{Conversely, as shown by Dyson \cite{3} in his three-fold way, if $H$ is time-reversal invariant one may pass to a real (boson) or quaternionic (fermion) basis.}

To proceed it is helpful to contemplate more deeply than is usual in cosmology.

2 The Structure of Quantum Mechanics

If one analyzes the Logical Structure of Quantum Mechanics one discovers that it consists of two different types of statements:

- I Timeless\footnote{3Of course in splitting the discussion into two parts, in Part I we take the view that Quantum Logic like its classical Aristotelian special case is timeless. This avoids appealing to Temporal Logic to resolving such paradoxes as that of “the sea fight tomorrow” \cite{72, 73, 74} and puts the burden of its resolution firmly where it belongs, in Part II.} statements about states, propositions, the Principle of Superposition, probabilities, observables etc
- II Statements about how states and observables change, Schrödinger’s equation and Unitarity etc.

The upshot of an analysis of Part I (so called Quantum Logic) \cite{1, 7} is that pure states are points in a Projective Space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

$$\Psi \equiv \lambda \Psi, \quad \lambda \in \mathbb{R}, \mathbb{C} \text{ or } \mathbb{H}. \hspace{1cm} (3)$$

Now any vector space over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ is a vector space $V$ over $\mathbb{R}$ with some additional structure(cf. \cite{11, 7}) , so let’s use real notation. Observables are symmetric bilinear forms:

$$\langle \Psi O \Psi \rangle = \Psi^a O_{ab} \Psi^b, \quad O_{ab} = O_{ba}. \hspace{1cm} (4)$$

\footnote{4By the principle of binary coding, Classical Boolean Logic may, for finite sets at least, be thought of as projective geometry over the Galois field of two elements. We shall also ignore the exceptional case of the octonions}
\( a = 1, 2, \ldots, n = \dim \mathbb{R} V \). Mixed states \( \rho \) are positive definite observables dual to the observables

\[
\langle O \rho \rangle = \rho^{ab} O_{ab} = \text{Tr}(\rho O), \quad \rho^{ab} = \rho^{ba}.
\] (5)

There is a privileged density matrix \( \text{the completely ignorant density matrix} \) which we may think of as a metric \( g_{ab} \) on \( V \) and use it to normalize our states

\[
\langle \Psi | \Psi \rangle = g_{ab} \Psi^a \Psi^b, \quad g_{ab} = g_{ba}, \quad \text{Tr} \rho = g^{ab} \rho_{ab}.
\] (6)

The upshot of a conventional analysis of II (Dirac called it Transformation Theory) is that states change by by means of linear maps which preserves the metric (i.e. preserves complete ignorance)

\[
\Psi^a \rightarrow S^a_b \Psi^b, \quad g_{ab} S^a_c S^b_d = g_{cd}.
\] (7)

Thus \( S \in SO(n, \mathbb{R}), \ n = \dim \mathbb{R} V \). Infinitesimally

\[
S^a_b = \delta^a_b + T^a_b + \ldots,
\] (8)

where the endomorphism or Operator \( T^a_b \) gives a two-form when the index is lowered

\[
g_{ab} T^b_c := T^b_{ac} = -T^b_{ca}.
\] (9)

But Dirac taught us that, just as in Hamiltonian mechanics, to every (Hermitian) Operator there is an Observable and vice versa. How can this be? Our vector space \( V \) over \( \mathbb{R} \) needs some extra structure, in fact a complex structure \( J^a_b \) or privileged operator which also preserves the metric (i.e. preserves complete ignorance).

\[
g_{ab} J^a_c J^b_d = g_{cd}.
\] (10)

Then

\[
J^a_b J^b_c = -\delta^a_c \implies \omega_{ab} = -\omega_{ba},
\] (11)

where the symplectic two-form \( \omega_{ab} = g_{ac} J^c_b \) may be used to lower indices and obtain a symmetric tensor for every (Hermitian) observable (i.e. one that generates a transformation preserves the symplectic form)

\[
\omega_{ab} T^b_c := T_{bac} = +T_{bca}.
\] (12)

We can think of this more group theoretically. In regular Quantum Mechanics \( V \) is a Hermitian vector space its transformations should be unitary, but

\[
U\left(\frac{n}{2}, \mathbb{C}\right) = SO(n, \mathbb{R}) \cap GL\left(\frac{n}{2}, \mathbb{C}\right),
\] (13)
where $GL(n, \mathbb{C}) \subset GL(n, \mathbb{R})$ is the subgroup preserving $J$, and $SO(n, \mathbb{R}) \subset GL(n, \mathbb{R})$ is the subgroup preserving the metric $g$. One also has

$$U\left(\frac{n}{2}, \mathbb{C}\right) = SO(n, \mathbb{R}) \cap Sp(n, \mathbb{R}),$$

(14)

where $Sp(n, \mathbb{R}) \subset GL(n, \mathbb{R})$ is the subgroup preserving the symplectic form $\omega$, and of course

$$U\left(\frac{n}{2}, \mathbb{C}\right) = Sp(n, \mathbb{R}) \cap GL\left(\frac{n}{2}, \mathbb{C}\right).$$

(15)

### 2.1 A precautionary principle

Now the main message of this review is that given a vector space $V$ over $\mathbb{R}$ it may have no complex structure ($n$ must obviously be even!) or if it is does, the complex structure may not be unique (they are typically members of infinite families)

Thus on four dimensional Euclidean space $\mathbb{R}^4$ they belong (modulo a choice of orientation) to a two-sphere $S^2 = SO(4)/U(2)$.

More generally, every quaternion vector space has such a 2-sphere’s worth of complex structures, i.e. a 2-sphere’s worth of of times!

To bring out the fact that in physics we use many different complex structures for many different reasons it is occasionally helpful to indicate explicitly by the symbol $i_{qm}$ the very particular complex structure on the Hilbert space $\mathcal{H}_{qm}$ of the standard model and so that Schrödinger’s equation really reads

$$i_{qm} \frac{d\Psi}{dt} = H\Psi.$$  

(16)

At a more mundane level, the use of the notation $i_{qm}$ brings out how dangerous and misleading, certainly to the beginner, it can be to use complex notation too sloppily. Suppose one has a theory, with an $SO(2)$ symmetry (gauged or un-gauged). It is tempting to collect the fields, e.g. scalars $\phi_1, \phi_2$ in pairs

$$\phi = \phi_1 + i\phi_2.$$  

(17)

Now the $i$, which generates the $SO(2)$ action is (17) is not the same as $i_{qm}$.

This is clear from the fact that charge conjugation

$$C : \phi_1 + i\phi_2 \rightarrow \phi_1 - i\phi_2$$  

(18)

is anti-linear, i.e anti commutes with $i$ but is nevertheless represented on $\mathcal{H}_{qm}$ as a linear operator, i.e. one which commutes with $i_{qm}$. Note that there would be no temptation to indulge in such notational confusion if there were three scalar fields $\phi_1, \phi_2, \phi_3$ and the symmetry $SO(3)$.

Just how confusing the sloppy use of the somewhat ambiguous complex notations currently can be is nicely illustrated in [34] in the context of quantum field theory, an example which will be of relevance later.

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7 cf Hyper-Kähler manifolds such as K3
At a purely practical level, the avoidance of an excessive use of complex notation also helps in formulating action principles in an intelligible fashion. One is often instructed that in varying an action with, for example complex scalars, that one should vary the action regarding $\phi$ and its complex conjugate $\bar{\phi}$ as independent'. On the face of it this sounds ridiculous. What is actually meant is that one varies regarding the real $\phi_1$ and imaginary $\phi_2$ parts of $\phi$ as independent. It is easy to check that, as long as the action is real, then this cook book recipe will give the correct result, essentially because varying with respect to $\bar{\phi}$ gives the complex conjugate of the equation obtained by varying with respect to $\phi$. However as a general principle the cook book recipe cannot be of general validity. It fails, and is inconsistent, if, for example, one varies a complex valued function of a complex variable and its complex conjugate. If it only works in special cases and can lead to incorrect results, it seems best to avoid both the cook book recipe and the misleading notation that gives rise to it.

In conclusion therefore, it seems wise to adopt a course of action, particularly at the classical level before quantization, in which one proceeds as far as possible by considering all physical quantities and their related mathematical structures to be real until one is forced to introduce complex notation and $i_{qm}$ at the point where one introduces quantum mechanics.

In other words, in what follows, I plan to follow, in so far as is possible, Hamilton’s course of action \[27\]

The author acknowledges with pleasure that he agrees with M. CAUCHY, in considering every (so-called) Imaginary Equation as a symbolic representation of two separate Real Equations: but he differs from that excellent mathematician in his method generally, and especially in not introducing the sign $\sqrt{-1}$ until he has provided for it, by his Theory of Couples, a possible and real meaning, as a symbol of the couple $(0, 1)$.

### 2.2 Dyson’s Three-fold way

In this language, Dyson’s observation \[3\] is that in standard quantum mechanics an anti-linear involution $\Theta$ acting on rays may be normalized to satisfy

$$\Theta^2 = \pm 1, \quad (19)$$

where the plus sign corresponds to an even spin state and the odd sign to an odd spin state. To say that $\Theta$ is anti-linear is to say that it anti-commutes with the standard complex structure $i_{qm}$, $i_{qm}^2 = -1$

$$\Theta i_{qm} + i_{qm} \Theta = 0. \quad (20)$$

Now for the plus sign $\Theta$, is a projection operator and we get what is called a real structure on the original complex Hilbert space and if the Hamiltonian is time-reversal invariant, then we may use the projection operator to project onto the subspace of real states. On the other hand for the minus sign we construct
\[ K = \Theta i_{\text{qm}} \]  

and find that \( \Theta, i_{\text{qm}}, K \) satisfy the algebra of the quaternions.

## 2.3 Relation to Jordan Algebras

There is an interesting tie in here with the theory of Jordan algebras \[54, 55\] which were originally introduced by Jordan as a possible avenue for generalizing quantum mechanics but in the end led to the same three basic possibilities.

In all three varieties of quantum mechanics the states, i.e. the space of positive semi-definite Hermitian matrices in \( \text{Herm}_n(\mathbb{K}) \), \( \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H} \) form a homogeneous convex self-dual cone. Moreover as observed by Jordan, they satisfy an abelian, but non-associative, algebra whose multiplication law is

\[
(O_1, O_2) \rightarrow \frac{1}{2} (O_1 O_2 + O_2 O_1). \tag{22}
\]

The algebra \( J_n^\mathbb{K} \) thus obtained is real and power law associative and thus belongs to the class of what are now known as Jordan Algebras.

In fact the list of finite dimensional irreducible homogeneous self-dual cones is quite small and coincides with the list of finite dimensional irreducible Jordan algebras. The list is:

| Cone \( \mathbb{K} \) | Algebra \( J_n^\mathbb{K} \) | Reduced Structure Group | Automorphism Group |
|-------------------|----------------------|-------------------------|-------------------|
| \( C(\mathbb{E}^{n-1,1}) \) | \( \Gamma(n-1) \) | \( SO(n-1,1) \) | \( SO(n-1) \) |
| \( C_n(\mathbb{R}) \) | \( J_n^\mathbb{R} \) | \( PSL(k;\mathbb{R}) \) | \( SO(n) \) |
| \( C_n(\mathbb{C}) \) | \( J_n^\mathbb{C} \) | \( PSL(n;\mathbb{C}) \) | \( SU(n) \) |
| \( C_n(\mathbb{H}) \) | \( J_n^\mathbb{H} \) | \( SU^*(2n) \) | \( Sp(n) \) |
| \( C_3(\mathbb{O}) \) | \( J_3^\mathbb{O} \) | \( E_{6(-26)} \) | \( F_4 \) |

- \( C(\mathbb{E}^{n-1,1}) \subset \Gamma(n-1) \) is the usual Minkowski cone, in \( \mathbb{E}^{n-1,1} \) based on a the sphere \( S^{n-2} \). The automorphism group is the Lorentz group \( SO(n-1,1) \).
- \( C_k(\mathbb{R}) \subset J_n^\mathbb{R} \): the set of positive semi-definite \( n \times n \) real symmetric matrices. The reduced structure groups is \( PSL(n,\mathbb{R}) \) and the automorphism group is \( SO(n-1) \).
- \( C_n(\mathbb{C}) \subset J_n^\mathbb{C} \): the set of positive semi-definite \( n \times n \) hermitian matrices. The reduced structure group is \( PSL(k,\mathbb{C}) \) and the automorphism group is \( SO(n) \).
- \( C_n(\mathbb{H}) \subset J_n^\mathbb{H} \): \( n \times n \) positive definite quaternionic hermitian matrices. The reduced structure group is \( SU^*(2k) \) and the automorphism group is \( Sp(k) \).
- \( C_3(\mathbb{O}) \subset J_3^\mathbb{O} \): the set of positive semi-definite \( 3 \times 3 \) octonionic hermitian matrices. The reduced structure group is \( E_{6(-26)} \) and the automorphism group is \( F_4 \).
In all cases the automorphism group $\text{Aut}(J)$ of the Jordan algebra $J$ is the stability group of the unit element in the algebra, which may be taken as a unit matrix. The reduced structure group of the algebra $\text{St}_0(J) = \text{PLSG}$ is the subgroup of the structure group $G = \text{Str}(J)$, leaving the norm of the Jordan algebra invariant.

Note that these results subsume the foundational Alexandrow-Zeeman [56, 57] theorem which states that the automorphism group of the causal structure of Minkowski spacetime (defined by the cone $C(E^n_{-1,1})$) consists of dilations and Lorentz transformations [89].

In the case of $\Gamma(n-1)$ one may think of $v$ as an element of the Clifford algebra $\text{Cliff}(n-1,1;\mathbb{R})$, on sets $v = v^\mu \gamma_\mu$. However the Jordan algebra $\Gamma(n-1)$ is generated by $\gamma_i$ and the identity matrix. Then in all cases the commutative but not associative Jordan product is given by one half the anti-commutator, $u \circ v = \frac{1}{2}(uv + vu)$. The cone $C(J)$ is then obtained by taking the exponential $\exp(v)$ of elements $v \in J$. This is well defined because of the power associativity property $v \circ v^r = v^{r+1}$ of the algebra.

### 2.4 Special Cases: low order isomorphisms

It is a striking fact that in the case of $2 \times 2$ matrices over $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ the Jordan algebras coincide with the Clifford algebras, fact perhaps more familiar in the form

\begin{align*}
\text{Spin}(2,1) & \equiv \text{SL}(2,\mathbb{R}), \\
\text{Spin}(3,1) & \equiv \text{SL}(2,\mathbb{C}), \\
\text{Spin}(5,1) & \equiv \text{SL}(2,\mathbb{H}).
\end{align*}

There is also a closely related statement over the octonions for $\text{Spin}(9,1)$ which crops up in string theory.

The middle isomorphism in (25) has lead Penrose to attempt, in his Twistor theory, to connect the use of the complex numbers in spinor analysis with that in quantum mechanics. A connection which moreover seems to give a privileged position to four spacetime dimensions. I think one can take a very different view [59] but to appreciate it we need to make an excursion into

### 3 Spacetime Signature and the Real Numbers

The basic point being made here is that in $4+1$, and indeed $9+1$ and $10+1$, spacetime dimensions, it is possible, by choosing the spacetime signature appropriately, to develop spinor analysis at the classical level entirely over the reals. That is, to consistently use Majorana spinors whose components really are real. In four spacetime dimensions this requires the mainly plus signature convention (the opposite to that which Penrose uses). The complex numbers need only enter when one quantizes.

To see this in more detail we need some facts about
3.1 Clifford Algebras

Given a vector space $V$ with metric $g$, of signature $(s, t)$ where $s$ counts the positive and $t$ the negative signs, Clifford algebra $\text{Cliff}(s, t; \mathbb{R})$ is by definition the associative algebra over the reals generated by the relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}, \quad (26)$$

where $\gamma$ is a basis for $V$. As a real algebra, the signature does make a difference. For example

$$\text{Cliff}(0, 1; \mathbb{R}) \equiv \mathbb{C}, \quad (27)$$

while

$$\text{Cliff}(0, 1; \mathbb{R}) \equiv \mathbb{R} \oplus \mathbb{R}. \quad (28)$$

In fact $\text{Cliff}(0, 1; \mathbb{R})$ is identical with what are often called ‘double numbers’ or ‘hyperbolic numbers’, i.e numbers of the form.

$$a + eb, \quad a, b \in \mathbb{R}, \quad e^2 = 1. \quad (29)$$

As an algebra, $\text{Cliff}(0, 1; \mathbb{R})$ is not simple, $P_\pm = \frac{1}{2}(1 \pm e)$ are projectors onto two commuting sub-algebras.

In a matrix representation

$$i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (30)$$

However if we pass to the complex Clifford over $\mathbb{C}$ we lose the distinction since

$$\text{Cliff}(0, 1; \mathbb{C}) \equiv \text{Cliff}(0, 1; \mathbb{C}) \equiv M_2(\mathbb{C}), \quad (31)$$

where $M_2(\mathbb{C})$ is the algebra of all complex valued two by two matrices.

It is precisely at this point that the precautionary principle comes in. We should not rush into adopting

$$\text{Cliff}(3, 1; \mathbb{C}) \equiv \text{Cliff}(1, 3; \mathbb{C}) \equiv M_4(\mathbb{C}), \quad (32)$$

but rather enquire what are the possible differences between the two signatures.

In fact

$$\text{Cliff}(3, 1; \mathbb{R}) \equiv M_4(\mathbb{R}), \quad \text{Cliff}(1, 3; \mathbb{R}) \equiv M_2(\mathbb{H}), \quad (33)$$

where

$$\mathbb{H} \equiv \text{Cliff}(0, 2; \mathbb{R}) \quad (34)$$

are the quaternions. Despite the differences the spin groups are identical

$$\text{Spin}(3, 1) \equiv \text{Spin}(1, 3) \equiv SL(2, \mathbb{C}), \quad (35)$$

$^8$V is not $\mathbb{H}^\text{sep}$ thought of as real! A good reference for the properties of Clifford algebras used here is [112], see also [111].

$^9$A similar point has been made recently by Schucking [62] but he plumps for the quaternions.
but if discrete symmetries are taken into account they differ:

\[ \text{Pin}(3,1) \neq \text{Pin}(1,3). \quad (36) \]

This has important consequences in spacetimes which are time, space or space-time non-orientable \[35, 32, 13, 36, 38\].

### 3.2 Chiral rotations

Independently of signature

\[ \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3, \quad \gamma_5^2 = -1. \quad (37) \]

Moreover if \(\gamma_\mu\) generate a Clifford algebra, then so do

\[ e^{\alpha\gamma_5}\gamma_\mu e^{-\alpha\gamma_5} = \cos 2\alpha\gamma_\mu + \sin 2\alpha\gamma_5\gamma_\mu, \quad \alpha \in \mathbb{R}. \quad (38) \]

Thus by choosing \(\alpha = \pi\) we can reverse the sign of the \(\gamma_\mu\) and so we expect that no physical consequences should follow from the choice of sign.

The chiral rotations maintain the reality properties of the gamma matrices. Multiplication by \(i\) of course reverses the signature.

### 3.3 Majorana Spinors

It is a striking and, I believe, a possibly rather significant fact that the signature (3, 1) leads directly to a Majorana representation, in which all \(\gamma\) matrices are real. Certainly if one holds that \(N = 1\) supersymmetry and \(N = 1\) supergravity are important, this fact renders the mainly positive signature rather attractive.

The precautionary principle would lead one to adopt the signature (3, 1) and use a real notation for as long as one can, certainly at the classical level where one need never introduce complex numbers. Thus the basic entities are Majorana spinors \(\psi\) belonging to a four dimensional real vector space \(\mathbb{M}\) with real, or real Grassmann number components \(\psi^a, a = 1, 2, 3, 4\).

Note that if \(\psi\) is a Majorana spinor then so is its chiral rotation \(e^{\alpha\gamma_5}\psi\).

The charge conjugation matrix \(C = -C^t\) satisfies

\[ C\gamma_\mu C^{-1} = -\gamma_\mu^t, \quad C\gamma_5 C^{-1} = -\gamma_5^t. \quad (39) \]

It serves as a Lorentz-invariant symplectic form on \(\mathbb{M}\). Thus \(\text{Spin}(3, 1) \subset \text{Sp}(4; \mathbb{R}) \equiv \text{Spin}(3, 2)\).

### 3.4 Dirac Spinors

To incorporate Dirac spinors, one considers pairs of Majorana spinors \(\psi^i, i = 1, 2\) which are elements of \(\mathbb{R}^4 \oplus \mathbb{R}^4 \equiv \mathbb{R}^4 \otimes \mathbb{C}^2 \equiv \mathbb{R}^8\) if \(\delta_{ij}\) is the metric and \(\epsilon_{ij} = \delta_{ik}J^k_{\ j}\), the symplectic and \(J^k_{\ j}\) the complex structure which rotates the two summands into each other, we can endow \(\mathbb{D} \equiv \mathbb{R}^8\) with a symplectic form
\( \omega \) and a pseudo-riemannian metric \( g \), and hence a pseudo-hermitean structure. In components, for commuting spinors,

\[
g(X, Y) = X^{ia} C_{ab} \epsilon^{ij} Y^{ja} = g(Y, X) \tag{40}
\]

\[
\omega(X, Y) = X^{ia} C_{ab} \delta^{ij} Y^{ja} = -\omega(Y, X) \tag{41}
\]

so that

\[
\omega(X, Y) = g(JX, Y) . \tag{42}
\]

The signature of the metric \( g \) is \((4, 4)\) and of the hermitian form, which is usually written

\[
\bar{\psi} \psi, \tag{43}
\]

where the Dirac conjugate

\[
\bar{\psi} = \psi\dagger \beta \tag{44}
\]

is \((2, 2)\). The ‘light cone’on which \( \bar{\psi} \psi \) consists of Majorana Spinors

Not that electromagnetic rotations and chiral rotations commute with one another.

Alternatively we can think of the Dirac spinors as elements of a four dimensional complex vector space \( D = M_C \equiv \mathbb{C}^4 \), the complexification of the real space of of Majorana spinors \( M \).

### 3.5 Weyl Spinors

To see where Weyl spinors fit in we observe that \( \gamma_5 \) acts as a complex structure converting \( M \equiv \mathbb{R}^4 \) to \( W \equiv \mathbb{C}^2 \). In other words, we write

\[
M \otimes_{\mathbb{R}} \mathbb{C} = D = W \oplus \overline{W} , \tag{45}
\]

Elements of \( W^2 \) are chiral spinors for which

\[
\gamma_5 \psi_R = i \psi_R , \tag{46}
\]

Elements of \( \overline{W} \) are anti-chiral spinors for which

\[
\gamma_5 \psi_L = -i \psi_L , \tag{47}
\]

The projectors \( \frac{1}{2} (1 - i \gamma_5) \) and \( \frac{1}{2} (1 + i \gamma_5) \) project onto chiral and anti-chiral Weyl spinors respectively.

It is of course possible to treat Weyl spinors without the explicit introduction of complex numbers at the expense of introducing pairs of Majorana spinors \( \psi_1, \psi_2 \) subject to the constraint that

\[
\gamma_5 \psi_1 = -\psi_2 , \quad \gamma_5 \psi_2 = \psi_1 . \tag{48}
\]

One then has

\[
\psi_R = \psi_1 + i \psi_2 , \psi_L = \psi_1 - i \psi_2 . \tag{49}
\]
3.6 Signature reversal non-invariance

Many people would argue that after all, a choice of signature is only a convention. That is true, but as we have seen above, this choice of convention comes with consequences. Moreover reversal of signature is not a symmetry of the basic equations of physics. as has emerged very clearly recently in work aimed at understanding why the observed cosmological constant is so small in comparison with its expected value. There have been a number of suggestions [30, 31, 39] that this might be due to a symmetry, analogous to chiral symmetry which is used to account for the smallness of of the electron mass. One candidate for such a symmetry, which may be expressed in a manifestly generally covariant, and simple fashion is the symmetry under change of spacetime signature

\[ g_{\mu\nu} \rightarrow -g_{\mu\nu}. \] (50)

For flat spacetime this is equivalent to the transformation [30]

\[ x^\mu \rightarrow ix^\mu, \] (51)

taking West Coast to East Coast,

\[ E^{3,1} \rightarrow E^{1,3}, \] (52)

but complexifying or analytic continuation of coordinates are not without problems in curved spacetime and so I prefer (50) which does the job just as well. I have a similar prejudice against formulations in terms of non-generally covariant concepts such as energy [31].

Under (50) one has

\[ R_{\mu\nu} \rightarrow R_{\mu\nu} \] (53)

and so, if \( \Lambda \neq 0 \), (50) is definitely not a symmetry of the equations

\[ R_{\mu\nu} = \Lambda g_{\mu\nu}. \] (54)

and hence is violated by a non-vanishing cosmological constant.

If scalar fields are present, then (50) is violated by mass or potential terms since under (50) the Christoffel symbols and hence the connection are unchanged

\[ \{\mu^\nu\sigma\} \rightarrow \{\mu^\nu\sigma\}, \quad \nabla_\mu \rightarrow \nabla_\mu, \] (55)

but the equation

\[ g^{\mu\nu}\nabla_\mu \nabla_\nu \phi = V'(\phi) \] (56)

is not invariant and neither is the Einstein equation

\[ \frac{1}{8\pi G}R_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu}V(\phi), \] (57)
On the other hand, the source free Maxwell equations are invariant, but the Einstein equation.

\[ \frac{1}{8\pi G} R_{\mu\nu} = g^{\sigma\tau} F_{\mu\sigma} F_{\nu\tau} - \frac{1}{4} g_{\mu\nu} g^{\alpha\beta} g^{\sigma\tau} F_{\alpha\sigma} F_{\beta\tau} \]  

(58)

is not.

This is part of a more general pattern, for a massless \( p \)-form field strength in \( n \) spacetime dimensions (so that \( p = 1 \) corresponds to a scalar and in four dimensions, \( p = 3 \) to a pseudoscalar or axion) then while the equation of motion is invariant, (50) induces

\[ T_{\mu\nu} \rightarrow (-1)^{p+1} T_{\mu\nu}. \]  

(59)

From this it is clear that the Maxwell equations coupled to a complex scalar field, that is the Abelian Higgs or Landau Ginzburg model are not invariant. This can be seen from

\[ \nabla_{\nu} F^{\mu\nu} = J^{\mu}. \]  

(60)

Under (50)

\[ F^{\mu\nu} \rightarrow F^{\mu\nu}, \]  

(61)

but

\[ J^{\mu} \rightarrow -J^{\mu}. \]  

(62)

Similarly the Lorentz equation

\[ \frac{d^2 x^\mu}{d\tau^2} + \{_{\sigma} \} \frac{dx^\sigma d^\tau}{d\tau} = e g^{\mu\alpha} F_{\alpha\beta} \frac{dx^{\beta}}{d\tau}, \]  

(63)

is not invariant under (50).

Thus the classical equations of motion of the bosonic sector of the standard model are certainly not invariant under (50). To make them so would entail adding additional fields whose energy momentum tensor is opposite in sign to the standard case. These fields would antigravitate rather than gravitate. Various schemes of this sort have been discussed in the literature (e.g. [41, 42, 43]).

If, therefore, the signature reversal is not a symmetry of our world, then it seems reasonable to me to suppose that one signature is preferred over the other, and that is the view being advocated here.

Of course one could follow Duff and Kalkkinen that we have simply mistaken the dimension we are in, [64, 65] or conclude that the signature of spacetime may vary from place to place, some regions having signature \(- + + +\) and some signature \(+ - + +\). Perhaps one should say that spacetime signature is an emergent property.

## 4 More than one time: Signature Change

We have been arguing that time, or at least a universal complex structure on the quantum mechanical Hilbert space single may be an emergent, or historical phenomenon.
This seems clearest in certain, instanton based, approximate, treatments of the birth of the universe based on what Hartle and I have called Tunnelling Metrics [15], in which a Riemannian manifold $M_R$ and a Lorentzian manifold $M_L$ are joined on across a surface $\Sigma$ of time symmetry which may be regarded as the origin of time surface. There is no time in $M_R$ where the metric signature is $++++$. The metric signature flips to $-+++$ across $\Sigma$. If that can happen why can’t it flip to $-+--$ across some other surface, as suggested by Eddington long ago [55]?

Signature flip also arises in brane-world scenarios in which the brane bends over in time while remaining a smooth sub-manifold of the Lorentzian bulk spacetime, ceases to be timelike, but rather spacelike with positive definite (i.e. Riemannian) induced metric [86, 87, 88]. However unless the bulk as more than one time the transition can only be from Lorentzian to Riemannian. In the model studied in [86] time certainly emerges after the collision of two branes.

The question therefore arises, could two, or possibly more than two times have emerged? There has been a fairly large amount of work on the possibility of two or more times, i.e on spacetimes of signature $-+,+,+,+$ or $-,+,+,+,+$ or $-,+,+,+,+$. A very early example is hinted at by Halsted [113] A later, and for me difficult to understand example is [93], where the extra temporal coordinate is called ‘anti-time ‘or ‘eternity ‘).

As far as a can see, little attempt to relate them to the algebraic structure of quantum mechanics, although a theory of Kostant comes quite close.

In fact, the standard reason for rejecting such theories is the existence of the instabilities and causality violations that result as a consequence of the fact that the interior of the light cone is no longer convex. This is clearly shown by Dorling’s argument [45] that the lowest mass particle is such a spacetime could decay into particles of heavier mass. In Kaluza-Klein theories, timelike extra dimensions lead to negative energies for vector fields on dimensional reduction [46] and provides limits on their size [90].

One way to say this is that necessarily such spacetimes cannot admit a time orientation and hence, in accordance with our general outlook, cannot admit standard complex quantum mechanics.

Among multi-time theories, a particularly intriguing case from the mathematical point of view is that of six-dimensional manifolds with neutral or Kleinian signature $(++|--)$. In other words where there is a complete symmetry between space and time. This has been energetically pursued by Cole over many years [105, 104, 103, 102, 101, 100, 99, 98, 97, 96, 95] in an attempt to make physical sense of it. I am skeptical but believe it may ultimately play a role in string theory.

One has the isomorphisms

$$SO(3, 3) \equiv SL(4, \mathbb{R})/\mathbb{Z}_2, \quad \text{Cliff}(3, 3; \mathbb{R}) \equiv M_8(\mathbb{R}).$$

The first isomorphism links us to real three-dimensional projective geometry, via a a real form of Twistor theory [59]. One may think of $\mathbb{E}^{3,3}$ as the space of
bi-vectors in $L^{\mu \nu} = -L^{\nu \mu}$ in $\mathbb{R}^4$ endowed with the metric
\[ \frac{1}{4} \epsilon_{\mu \nu \sigma \tau} L^{\mu \nu} L^{\sigma \tau}. \] (65)

By the well-known Plücker correspondence, lines in $\mathbb{R}P^3$ correspond to simple bi-vectors in and hence to null 6-vectors in $\mathbb{E}^{3,5}$. It is also possible to regard $\mathbb{R}^4$ or its projectivization $\mathbb{R}P^3$ as the space of Majorana spinors in four spacetime dimensions. Conformally $SO(3,3)$ is the conformal group of $\mathbb{E}^{2,2}$. In Penrose’s Twistor Theory one complexifies and another real form is $SO(4,2)$ the conformal group of ordinary Minkowski spacetime $\mathbb{E}^{3,1}$.

Kostant [106, 107] has made the imaginative proposal that our spacetime (with signature $(3,1)$) is a 3-brane embedded in a six-dimensional bulk spacetime with a metric of signature $(3,3)$. The restriction of the ambient metric to the normal bundle has signature $(0,2)$ and the associated $SO(2)$ symmetry allows him to think of the normal bundle as a complex line bundle over spacetime. This is the origin of electromagnetism in his theory.

To obtain examples, Kostant noted that the conformal group of $\mathbb{E}^{3,3}$ is $SO(4,4)$. More accurately $SO(4,4)$ acts globally on the conformal compactification of $\mathbb{E}^{3,3}$, which may be regarded as the space of null rays in $\mathbb{E}^{4,4}$. To get compactified Minkowski spacetime $(S^1 \times S^3/\mathbb{Z}_2$ one intersects the null cone of the origin of $\mathbb{E}^{4,4}$ with a 6-plane through the origin of signature $(4,2)$. An interesting aspect is that since we are dealing with $SO(4,4)$ there is a triality which acts. Mathematically, Kostant’s model has many intriguing features (see [108, 109]) but so far clearly fails to make much contact with the real world.

Another, purely technical use of three times is to study integrability. Because $\mathbb{E}^{3,3}$ admits a para-hermitean structure, in other words it admits an isometric involution $J$ on the tangent space which $J^2 = 1$, (i.e. para-complex) and such that $g(JX,JY) = g(X,Y)$, this may be used to obtain the KP equations via a self-duality condition on $\mathfrak{sl}(2,\mathbb{R})$ gauge fields [110].

5 Examples

The general algebraic considerations may seem rather abstract, but they have already arisen in the application of quantum mechanics to cosmology.

In what follows, I shall give some examples. Before doing so I note that

The much discussed question of whether black hole evaporation is unitary is meaningless if there is no complex structure, and ill-posed if there is more than one.

5.1 Quantum Field Theory in Curved Spacetime

In Quantum Field Theory in Curved Spacetime the main problem is that there is no unique definition of “positive frequency”. In the free theory, $V = \mathcal{H}_{\text{one particle}}$
is the space of real-valued solutions of wave equations. $V$ is naturally (and covariantly) a symplectic (boson), or orthogonal (fermion) vector space.

$$\omega(f, g) = \int (\dot{f}g - \dot{g}f)\,dx = -\omega(g, f)$$ (66)

$$g(\psi, \chi) = \int (\psi^*\chi)\,dx = g(\chi, \psi)$$ (67)

To quantize we complexify and decompose

$$V_C = \mathbb{C} \otimes V = V^+ \oplus V^-$$ (68)

This decomposition (which defines a complex structure) is not unique.

This non-uniqueness corresponds physically to the possibility of particle production and is an essential part of our current understanding of black hole evaporation and inflationary perturbations.

At this point it may be instructive to recall why commutation relations of the form

$$[\hat{x}^\mu, \hat{P}^\nu] = i\delta^\mu_\nu$$ (69)

don’t apply in quantum field theory in Minkowski spacetime. If they did, then they would have, up to natural equivalence, to be represented in the standard Stone-Von-Neumann fashion on $L^2(\mathbb{R}^{3,1})$. But then the energy $\hat{P}^0$, could not be bounded below. Thus $L^2(\mathbb{R}^{3,1})$ is not the quantum mechanical Hilbert space. Rather, as stated above, it is the space of positive frequency solutions of the Klein-Gordon or Dirac equations. These are much more subtle objects and certainly not uniquely defined in a curved spacetime manifold $\{M, g\}$, unlike $L^2(M, \sqrt{-gd^4x})$, the obvious generalization of $L^2(\mathbb{R}^{3,1})$, which is unambiguous even in a curved spacetime.

5.2 The Wave Function of the Universe

In Hartle and Hawking’s Wave Function for The Universe

$$\Psi(h_{ij}, \Sigma) = \int d[g]e^{-\frac{1}{2}mc(g)}, \quad h_{ij} = g_{ij}\big|_{\Sigma=\partial M}$$ (70)

Is real valued. To get a notion of time one typically passes to a Lorentzian WKB approximation $S_c$

$$\Psi = Ae^{iS_c} + \bar{A}e^{-iS_c}$$ (71)

but this is only a semi-classical approximation, in other words

Time, the complex numbers and the complex structure $i_{qm}$ of quantum mechanics emerge only as an approximation at late times

\footnote{We use real (Majorana) commuting spinors for convenience: there use is not essential}
5.3 Euclidean Quantum Field Theory

In fact in Euclidean Quantum Field Theory it is not sufficient just to compute correlators.

In order to recover Quantum Mechanics, rather than merely to indulge in an unphysical case of Statistical Mechanics, the correlators must exhibit Reflection Positivity \[20, 21\]. This guarantees the possibility of analytically continuing to real time.

This can be done for Riemannian backgrounds if they admit a suitable reflection map, for example static or time-symmetric metrics such as Real tunneling geometries \[15, 9, 16, 17, 18, 24, 22, 23\].

However most Riemannian metrics do not admit such a reflection map.

Thus generically in such approaches one would not recover standard complex quantum mechanics. Only for very special classical saddle points of the functional integral would a well defined complex structure emerge.

5.4 Lorentzian Creation ex nihilo

The next example involves a Lorentzian Born From Nothing Scenario \[13, 14, 116\]. Essentially, one considers de-Sitter spacetime modded out by the antipodal map \(dS/\mathbb{Z}_2\) (so-called elliptic interpretation).

\[- (X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = \frac{3}{\Lambda} \mathbb{Z}_2 : X^A \equiv -X^A. \tag{72}\]

Now the antipodal map preserves space orientation but reverse time orientation. But in quantum mechanics a time reversing transformation is represented by an anti-unitary operator \(\Theta\) and if all states are invariant up to a factor

\[\Theta \Psi = \lambda \Psi \tag{73}\]

then only real linear combinations are allowed.

Thus Quantum Mechanics in \(dS/\mathbb{Z}_2\) is Real Quantum Mechanics.

This jibes with the fact that under the action of the antipodal map is anti-symplectic on the bosonic space of solutions \(V\)

\[\omega(\cdot, \cdot) \rightarrow -\omega(\cdot, \cdot). \tag{74}\]

This renders imposing the CCR’s impossible \[11\]

Compare regular time reversal

\[(p_i, q^i) \rightarrow \omega = (-p_i, q^i) \implies dp_i \wedge dq^i \rightarrow -dp_i \wedge dq^i = -\omega \tag{75}\]

If there is no symplectic form then the Heisenberg commutation relations make no sense, one cannot geometrically quantize.

\[11\] Bernard Kay has implemented this argument more rigourously within an algebraic framework \[120\].
This pathology arises quite generally for spacetimes which do not admit a
\textit{Time Orientation}, i.e. a smooth choice of future lightcone. Such spacetimes
always have a double cover which is time orientable and so may be regarded as
the quotient of a time orientable spacetime by a generalized, time orientation
reversing, antipodal map. The double cover thus realizes various speculative
ideas of the past and not so recent past \cite{47, 48, 49, 50, 51} of spacetimes in
which the arrow of time runs one way in one part and the other way in the
other.

In other words quantum field theory is not defined unless one may define an
\textit{Arrow of Time}.\footnote{Amusingly CTC’s seem to be quiet innocuous from this point of view. It seems that they can be compatible with quantum mechanics.}

Amusingly CTC’s seem to be quiet innocuous from this point of view. It
seems that they can be compatible with quantum mechanics, but not necessarily
locality.

\section{Topology, Time Reversal and the Arrow of Time}

An interesting question, discussed by Chamblin and myself \cite{26}, is whether this
arrow is intrinsically defined, or whether both possibilities are on the same
footing.

In other words, do there exist time-orientable spacetimes which have an
\textit{intrinsic direction of time}?\footnote{see Hartle and Witt \cite{114}}

The analogy here is with a quartz crystal which is either left-handed or right
handed. This is because the point group contains no reflections or inversions.

For a spatial manifold $\Sigma$ one asks: does $\Sigma$ there exist an orientation reversing
diffeomorphism. In other words is there a diffeomorphism taking $\Sigma$ with one
orientation to $\Sigma$ with the opposite orientation?. For such manifolds a \textit{Parity
Map} cannot be defined. Such “handed ”manifolds are quite common, certain
Lens Spaces and $\mathbb{C}P^2$ being examples.\footnote{For a spatial manifold $\Sigma$ one asks: does $\Sigma$ there exist an orientation reversing
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Map} cannot be defined. Such “handed ”manifolds are quite common, certain
Lens Spaces and $\mathbb{C}P^2$ being examples.}

For spacetimes the analogous question is whether there exist a time reversing
diffeo $\Theta$?

We found some rather exotic examples, based on higher dimensional Taub-
NUT spacetimes for which no such diffeo $\Theta$ exists.

The question can be formulated in \textit{Hamiltonian Mechanics}. Does there exist
a symplectic manifold $\{M, \omega\}$ admitting no anti-symplecto-morphism, i.e. a
time reversal map $\Theta$ such that

$$\Theta^* \omega = -\omega. \quad (76)$$

The answer to this topological question, which should not be confused with
asking whether any particular Hamiltonian, on a symplectic manifold which
does admit a time reversal map, is time-reversal invariant, i.e. whether
\( \Theta^* H = H \),

(77)

does not seem to be known.

\section{Unification and Spin(10).}

If the viewpoint advocated here is on the right track, one might expect that should be signs in what little information we have about possible unification schemes. A very popular one is based on the group \( SO(10) \) and it is perhaps gratifying that it seems to fit with the philosophy espoused here.

In the standard electro-weak model, the neutrinos are purely left-handed and a description of the fundamental degrees of freedom in terms of Weyl spinors is often felt to be appropriate. One may then argue that this more more convenient with the mainly minus signature. However nothing prevents one describing it using Majorana notation and the mainly plus signature. Moreover the discovery of the non-zero neutrino masses and the so-called see-saw mechanism make it plausible that there is a right handed partner for the neutrinos and the fact that then each family would fit into a chiral (i.e. \( 16 \)) representation of Spin(10) makes it perhaps more attractive to describe the fundamental fields in Majorana notation. This would tend to favour the use of the mainly plus signature.

To see this in more detail recall

\begin{equation}
\text{Cliff}(10, 0; \mathbb{R}) \equiv M_{32}(\mathbb{R}).
\end{equation}

(78)

Let \( \Gamma_a, a = 1, 2, \ldots, 10 \) be a representation of the generators by real \( 32 \times 32 \) matrices and

\begin{equation}
\Gamma_{11} = \Gamma_1 \Gamma_2 \ldots \Gamma_{10},
\end{equation}

so that

\begin{equation}
\Gamma_{11}^2 = -1.
\end{equation}

(80)

It is customary to describe the Spin(10) model in terms of 16 left handed spacetime Weyl fermions which are then placed in a single complex chiral \( 16 \), \( \Psi \) of Spin(10)

\begin{equation}
\Gamma_{11} \Psi = i \Psi,
\end{equation}

(81)

but this is completely equivalent ,and notationally simpler to regard the 16 spacetime Weyl fermions as 32 spacetime Majorana fermions and then to regard \( \Psi \) as a 32 dimensional Majorana spinor of Spin(10) subject to the constraint

\begin{equation}
\Gamma_{11} \Psi = \gamma_5 \Psi.
\end{equation}

\begin{footnotesize}
\begin{enumerate}
\item[14]This is clear from the periodicity modulo eight of Clifford algebras \( \text{Cliff}(s + 8, t) \equiv \text{Cliff}(s, t) \otimes M_{16}(\mathbb{R}) \) and the easily verified fact that the that \( \text{Cliff}(2, 0; \mathbb{R}) \equiv M_2(\mathbb{R}) \).
\item[15]The matrices \( \Gamma_a, \Gamma_{11} \) generate the M-theory Clifford algebra \( \text{Cliff}(10, 1; \mathbb{R}) \equiv M_{32}(\mathbb{R}) \oplus M_{32}(\mathbb{R}) \).
\end{enumerate}
\end{footnotesize}
In more detail, we start with the 15 observed left handed Weyl fermions of the electro-weak theory with their weak hypercharges $Y = Q - t_3$, where $Q$ is the electric charge and $t_3$ the third component of weak isospin

$$
\begin{pmatrix}
u_L \\
e_L
\end{pmatrix}, \ Y = \frac{1}{2} \quad \begin{pmatrix} u_L \\ d_L \\
v_L \\ e_L \end{pmatrix}, \ Y = -\frac{1}{2}
$$

(83)

$$
u^c_L, \ Y = -\frac{2}{3} \quad d^c_L, \ Y = \frac{1}{3} \quad e^c_L, \ Y = 1.
$$

(84)

The first row consists of 4 iso-doublets and the second row of 7 iso-singlets. The up and down quarks $u_L$ and $d_L$ are in a 3 of $SU(3)$ colour and their charge conjugates $u^c_L, d^c_L$ are in a 3 of $SU(3)$. In fact the, because effective group is $S(U(3) \times U(2)) \equiv SU(3) \times SU(2) \times U(1)/Z_3 \times Z_2$, where $Z_3$ and $Z_2$ are the centres of $SU(3)$ and $SU(2)$ respectively [33]. This is because the electric charge assignments are such that acting with $Z_3 \times Z_2 \equiv Z_6$ can always be compensated by a $U(1)$ rotation.

Now $S(U(3) \times U(2))$ is a subgroup of $SU(5)$ and is well known one may fit all 15 left handed Weyl spinors in a 5 and a 10. However it is more elegant to adjoin the charge conjugate of the right handed neutrino, $\nu^c_L$ to make up a complex 10 of Spin(10). In fact the multiplets may be organized into multiplets of the Spin(6) $\equiv$ SU(4) $\times$ SU(2) $\times$ SU(2) subgroup of Spin(10)

$$
\begin{pmatrix} u_L \\ d_L \\
\nu_L \\ e_L \\
u^c_L \\ d^c_L \\
e^c_L \\
e^c_L \end{pmatrix}
$$

(85)

In this formalism we have left-right symmetry with the first row consisting of 4 weak iso-doublets and the bottom row of 4 doublets of some other, as yet unobserved $SU(2)$. The quarks and leptons also form two Spin(6) $\equiv SU(4)$ quartets.

### 8 Gravitational CP violation?

To conclude I would like to illustrate once more the advantages of the reality viewpoint by addressing a question of some current interest which is relevant to the present proceedings. That is whether CP-violating Dirac and Majorana mass terms for spin half fermions can give rise to detectably different behaviour as the particles fall in a gravitational field of a rotating body, due to the Lense-Thirring effect [69, 68, 67].

If they could, then a violation of the Weak Equivalence Principle in the form of the Universality of Free Fall would be entailed, which seems rather unlikely. The calculations given in [66, 67] are rather complicated and in view of the great importance of the issue, it seems worth while examining the question in a more elementary fashion. There are also potential implications for the quantum theory of black holes.

There are two aspects of the problem:
• The emission and detection of the fermions by ordinary matter
• Their propagation from source to detector through an intervening gravitational field.

It is the latter which I will be discuss here. If the fermions are assumed to be electrically neutral and with vanishing electric and magnetic dipole moments, this is a well defined problem in general relativity. Clearly if the fermions are moving in an electromagnetic field and they possess electric charges and/or magnetic and electric dipole moments the conclusions might be modified, but then the question is no longer one of pure gravity.

With our conventions, A system of \( k \) Majorana fermions \( \psi \) has Lagrangian

\[
L = \frac{1}{2} \psi^t C \partial \psi - \frac{1}{2} \psi^t C (m_1 + m_2 \gamma^5) \psi .
\]

(87)

where \( m_1 \) and \( m_2 \) are real symmetric \( k \times k \) matrices.

The kinetic term, but not the mass term, is invariant under \( SO(k) \) transformations

\[
\psi \to O \psi , \quad O^t O = 1 .
\]

(88)

Note that one may write

\[
O = \exp \omega_{ij} , \quad \omega_{ij} = -\omega_{jk} .
\]

(89)

The kinetic term, but not the mass term is also invariant under chiral rotations

\[
\psi \to P \psi ,
\]

\[
P = \exp \nu_{ij} \gamma^5 , \quad \nu_{ij} = \nu_{ji} .
\]

(90)

(91)

Combining these two sets of transformations we see that the kinetic term, but not the mass term is in fact invariant under the action of \( U(k) \), i.e. under

\[
\psi \to S \psi ,
\]

\[
S = \exp (\omega_{ij} + i\nu_{ij} \gamma^5) .
\]

(92)

(93)

The \( U(k) \) invariance is perhaps more obvious in a Weyl basis. Since

\[
(\gamma^5)^2 = -1 ,
\]

(94)

one may regard \( \gamma^5 \) as providing a complex structure on the space of \( 4k \) real dimensional Majorana spinors, converting it to the \( 2k \) complex dimensional space of positive chirality Weyl spinors for which

\[
\gamma^5 = i .
\]

(95)

Clearly \( S \) then becomes the exponential of the \( k \times k \) anti-hermitian matrix

\[
\omega_{ij} + i\nu_{ij} .
\]

(96)
Thus  

\[ SS^t = 1. \]  

(97)

The mass matrix is then a complex symmetric matrix

\[ m = m_1 + im_2, \]  

(98)

and under a \( U(k) \) transformation

\[ m \rightarrow S^t m S. \]  

(99)

At this point we invoke the result of Zumino \[70\] that \( S \) may chosen to render the matrix \( m \) diagonal with real non-negative entries.

This implies that \( k \) free massive Majorana (or equivalently Weyl) fermions \( \psi^i \) moving in a gravitational field will satisfy

\[ D\psi^k - \mu_k \psi^k = 0, \]

with no sum over \( k \) and where the masses \( \mu_k \) may be taken to be real and non-negative. There are no exotic non-trivial effects moving past a spinning object due to the Lense-Thirring effect. In particular there are no CP violating effects and gravity alone cannot distinguish ‘Majorana’ from ‘Dirac’ masses.

8.1 Behaviour in a Gravitational field

From now on, we assume that the mass matrix \( m \) is real and diagonal. If one iterates the Dirac equation and uses the cyclic Bianchi identity in a curved space one gets

\[ - \nabla^2 \psi + \frac{1}{4} R \psi + m^2 \psi = 0. \]  

(100)

As is well known, there is no ‘gyro-magnetic’ coupling between the spin and the Ricci or Riemann tensors \[71\]. To proceed, one may pass to a Liouville-Green-Wentzel-Kramers-Brilouin approximation of the form

\[ \psi = \chi e^{iS}. \]  

(101)

One obtains

\[ (i\gamma^\mu \partial_\mu S + m) \chi = 0, \]  

(102)

and

\[ \partial_\mu S \nabla^\mu \chi = 0. \]  

(103)

The analogue of the Hamilton Jacobi equation is

\[ \left( g^{\mu \nu} \partial_\mu S \partial_\nu S + m^2 \right) \chi = 0. \]  

(104)

Now since \( m \) is diagonal with diagonal entries \( \mu_i \), say, then each eigenspinor \( \chi_i \) propagates independently along timelike geodesics via

\[ \mu_i \frac{dx^\mu}{d\tau} = g^{\mu \nu} \partial_\nu S. \]  

(105)
The spinor amplitude $\chi_i$ is parallelly transported along these geodesics. Of course the geodesics are independent of the mass eigenvalue $\mu_i$ and the polarization state given by $\chi_i$. Indeed if the fermion starts off in a given polarization state with (with the associated mass), it remains in it. In other words, at the L-G-W-K-B level, the Weak Equivalence Principle, in the form of the Universality of Free Fall holds.

9 Pure States $\rightarrow$ Mixed States?

The completely thermal character of Hawking radiation (at the semi-classical level) and the apparent violation of Global Symmetries if black hole decay leaves no remnants led Hawking [80] to suggest that while the standard propositional structure of quantum mechanics, and its complex structure, should remain in a full quantum theory of gravity, the evolution law should change. In particular the evolution law should allow pure states to evolve to mixed states. In what follows I shall review the formalism suggested (and now abandoned) by Hawking and then comment on its relation to the suggestion I am making about the complex structure of quantum mechanics. I shall also relate this discussion to issues of reversibility and the arrow of time.

9.1 Density Matrices

Are positive semi-definite Hermitean operators acting on a quantum mechanical Hilbert space $\mathcal{H}$ with unit trace

$$\rho = \rho^\dagger, \quad \text{Tr} \rho = 1, \quad \langle \psi | \rho \psi \rangle \geq 0, \quad \forall |\psi\rangle.$$ (106)

If one diagonalizes

$$\rho = \sum_n P_n |n\rangle \langle n|$$ (107)

where $P_n \geq 0$ is the probability one is in the (normalized) state $|n\rangle$, and

$$\sum_n P_n = 1.$$ (108)

A pure state is one for which

$$\text{Tr} \rho = 1,$$ (109)

in which case, all but one of the $P_n$ vanishes and one is in that state with certainty. In a general orthonormal basis with one writes

$$\rho = \rho_{mn} |m\rangle \otimes \langle n|$$ (110)

with

$$\rho_{mm} = 1, \quad \rho_{mn} = \bar{\rho}_{nm}$$ (111)

There is a distinguished density matrix $\iota$ associated with complete ignorance for which $P_n = \frac{1}{N}$ where $N = \dim \mathcal{H}$.  

22
9.2 Gibbs Entropy

Normalized density matrices form a convex cone in the space of all Hermitian operators and the Gibbs entropy

\[ S = -\text{Tr}\rho \ln \rho = -\sum_n P_n \ln P_n \]  

(112)

is a convex function on the cone which is largest at the completely ignorant density matrix \( \iota \) and vanishes for any pure state.

If \( N = 2 \) we may set

\[ \rho = \frac{1}{2} \left( x^0 I_2 + x^i \sigma_i \right) \]  

(113)

where \( \sigma_i, i = 1, 2, 3 \) are Pauli matrices and the cone corresponds to the future light cone of four dimensional Minkowski spacetime

\[ x^0 \geq \sqrt{x^i x^i} = r. \]  

(114)

The unit trace condition implies that \( x^0 = 1 \) and thus \( r - \sqrt{x^i x^i} \leq 1 \) One finds that

\[ S = -\ln \left[ \left( \frac{1 + r}{2} \right)^{\frac{1}{2}} \left( \frac{1 - r}{2} \right)^{\frac{1}{2}} \right]. \]  

(115)

The entropy is maximum at the origin and goes to zero on the boundary of unit ball .

9.3 Evolution by an S-matrix

In general we might be interested in situations where there is an in and an out Hilbert space, \( \mathcal{H}^{\text{in}} \) and \( \mathcal{H}^{\text{out}} \) respectively. Conventionally one thinks of \( \mathcal{H}^{\text{in}} \) and \( \mathcal{H}^{\text{out}} \) as being isomorphic , except possibly described in a different basis but one could envisage more general situations. One has an associated set of states or density matrices for \( \mathcal{H}^{\text{in}} \) and \( \mathcal{H}^{\text{out}} \). The set of such (unormalized ) mixed states we call \( \mathcal{N}^{\text{in}} \) or \( \mathcal{N}^{\text{out}} \) respectively.

Conventionally one postulates there is a unitary map \( S : \mathcal{H}^{\text{in}} \to \mathcal{H}^{\text{out}} \) called an S-matrix such that

\[ |\text{out}\rangle = S|\text{in}\rangle \]  

(116)

which acts by conjugation on mixed states or density matrices

\[ \rho^{\text{out}} = S\rho^{\text{in}}S^{\dagger}. \]  

(117)

9.4 Tracing out

A situation which often arises is when the out Hilbert space \( \mathcal{H}^{\text{out}} \) is a tensor product

\[ \mathcal{H}^{\text{out}} = \mathcal{H}^{\text{out}1} \otimes \mathcal{H}^{\text{out}2} \]  

(118)
An initial state \( |\text{in}\rangle \) which remains pure will have an expansion
\[
|\text{in}\rangle = c_{mM} |m\rangle \otimes |M\rangle
\]
where \( |m\rangle \) is a basis for \( \mathcal{H}^{\text{out}1} \) and \( |M\rangle \) a basis for \( \mathcal{H}^{\text{out}2} \). An observable \( O_1 \) which acts as the identity on \( \mathcal{H}^{\text{out}2} \) will have an expectation value
\[
\langle \text{in}|O_1|\text{in}\rangle = \rho_{mn} |m\rangle \otimes \langle n|,\]
where
\[
\rho_{mn} = \bar{c}_{mM} c_{nM},
\]
where we use the fact that
\[
\langle n|O_1|m\rangle = \operatorname{Tr}(0_1 |m\rangle \otimes \langle n|).
\]

In other words observations made only in \( \mathcal{H}^{\text{out}1} \) can tell us nothing about \( \mathcal{H}^{\text{out}2} \) and hence will in general behave as if the final state were mixed.

### 9.5 Evolution by an \$ matrix

Taking \( \mathcal{H}^{\text{out}1} \) to be states at infinity and \( \mathcal{H}^{\text{out}2} \) horizon states shows that in general outgoing radiation from a black holes with a permanent horizon will be in a mixed state.

However back reaction means that the horizon is not permanent and the issue arises whether taking back reaction into account would give a pure or a mixed state.

More generally, one may try to construct a generalization of standard quantum mechanics in which in general pure states evolve to mixed states. One postulates that there is a linear map \( \$ : \mathcal{N}^{\text{in}} \rightarrow \mathcal{N}^{\text{in}} \) such that
\[
\rho^{\text{out}} = \$ \rho^{\text{in}}.
\]

One further postulates that \( \$ \) is hermiticity, and trace-preserving
\[
\begin{align*}
(a) \quad (\$ \rho)^\dagger &= \rho^\dagger, \\
(b) \quad \operatorname{Tr}\$ \rho &= \operatorname{Tr} \rho, \\
(c) \quad \$ \iota &= \iota.
\end{align*}
\]

One also demands that \( \$ \) takes positive semi-definite operators to positive semi-definite operators.

Some comments are in order.

- The assumption of linearity is a form of locality assumption since it amounts to assuming ‘non-interference of probabilities’. It should be possible to lump together results of two independent experiments and obtain the same probabilities.

Thus if in one ensemble consisting of 100 states with 30 in state 1 and 70 in state 2 these go to states 3 and 4 in 45 and 55 times respectively, and in
a second run of the same experiment 30 in state 1 and 70 in state 2 go to 72 and 28 in states 3 and 4 respectively than it should be the case, if the usual idea of probabilities is to make sense, that run in which 85=30+55 are in state 1 and 115=70+45 in state 2, then 117=45+72 should land up in state 3 and 83=55+28 should land up in state 4.

Of course strictly speaking, this argument only works for commuting density matrices but, by continuity it seems reasonable to assume linearity for all density matrices.

- The assumption that the completely ignorant density matrix $\mathbb{1}$ is preserved in time would seem to be necessary for any type of thermodynamics to be possible, not least because the completely ignorant density matrix $\mathbb{1}$ has the largest Gibbs entropy.

### 9.6 Invertibility and Factorisability

Standard S-matrix evolution is such that

$$\rho = S\rho S^\dagger.$$

Such $\mathcal{S}$-matrices are said to be factorisable and factorisable density matrices clearly take pure states to pure states, but in a general $\mathcal{S}$ matrix will take pure states to mixed states. In fact, in general, a $\mathcal{S}$ matrix acts as a contraction on the convex cone of positive definite Hermitian operators. Thus in general it is not invertible \cite{81,82}. Indeed there is a

**Theorem** A super-scattering matrix $\mathcal{S}$ is invertible iff it is factorisable

**Proof** Assume the contrary. Then there exits a mixed out-state $\rho_{\text{out}}$ which is mapped to a pure state $\rho_{\text{in}} = |\text{in}\rangle \langle \text{in}|$. Thus

$$\mathcal{S} \sum_n P_n |n,\text{out}\rangle \langle n,\text{out}| = |\text{in}\rangle \langle \text{in}|.$$  \hspace{1cm} (128)

Let $|\psi^{\text{in}}\rangle$ be any in state orthogonal to $|\text{in}\rangle$. One has

$$\sum_n P_n (\psi^{\text{rmin}}) (\mathcal{S} |n,\text{out}\rangle \langle n,\text{out}| |\psi^{\text{in}}\rangle = 0.$$ \hspace{1cm} (129)

But $\mathcal{S} |n,\text{out}\rangle \langle n,\text{out}|$ is a density matrix and so positive semi-definite. Thus a if $|n,\text{out}\rangle$ has $P_n \neq 0$, then it must be orthogonal to every pure state $|\psi^{\text{in}}\rangle$ orthogonal to $|\text{in}\rangle$ and hence

$$\mathcal{S} |n,\text{out}\rangle = |\text{in}\rangle \langle \text{in}|, \quad \forall \{P_n |P_n \neq 0\}.$$ \hspace{1cm} (130)

But if $\mathcal{S}$ takes all such states $|n,\text{out}\rangle$ to the same state $|\text{in}\rangle$ it cannot be invertible.
9.7 Irreversibility and CPT

Thus, as one might have expected, evolution by a superscattering matrix would irreversible. How does this square with our prejudices about \( CPT \)? This is usually taken to be an anti-unitary invertible (since \( \theta^2 = 1 \) \( \theta : N^{\text{out}} \rightarrow N^{\text{out}} \) which takes pure states to pure states, and preserves traces and preserves ignorance. In fact one usually has

Let us call its restriction to pure states

9.8 Strong CPT

assumes an invertible map \( \Theta \) from in states to out states

\[
\Theta = \$ \Theta^{-1} \$ .
\] (131)

Thus

\[
\$^1 = \Theta^{-1} \$ \Theta^{-1} .
\] (132)

In other words Strong CPT implies that the evolution is invertible. Note that this rather strong result does not assume that either \( \Theta \) or \$ is a linear map. However if \$ satisfies the requirements for a superscattering matrix and strong CPT, then it must be invertible and hence factorisable.

9.9 Weak CTP

Faced with the result above, one could argue that only \textit{probabilities} are related by CPT. this

\[
\text{Prob}(|\psi\rangle \rightarrow |\phi\rangle) = \text{Prob}(\Theta^{-1}|\phi\rangle \rightarrow \Theta|\psi\rangle) .
\] (133)

That is

\[
\langle \phi|\$ \langle |\psi\rangle |\psi\rangle \phi \rangle = \langle \Theta \phi|\$ \langle \Theta^{-1} \phi \rangle \langle \Theta^{-1} \phi \rangle \Theta \psi \rangle ,
\] (134)

that is

\[
\$^\dagger = \Theta^{-1} \$ ^\dagger \Theta^{-1} .
\] (135)

Of course for a factorisable \$ matrix \ref{135} holds by unitarity of the \( S \) matrix. Moreover \( \ref{135} \) implies that the superscattering operator is ignorance preserving

\[
\$ i = i .
\] (136)

An interesting set of questions is

- Is \( \ref{135} \) equivalent to \textit{detailed balance}?
- Does \( \ref{135} \) imply the H theorem?
- Does \( \ref{135} \) imply that only the microcanonical ensemble, i.e. the perfectly ignorant density matrix \( i \) is left-invariant by \$?

A full answer to these questions appears not be known but what is well known is the situation when all density matrices are assumed diagonal.
9.10  Pauli Master Equation

This is essentially the case when the density matrix remains diagonal. One sets

$$\dot{P}_r = \sum_{s \neq r} U_{rs} P_s - P_r \sum_{s \neq r} U_{sr}$$  \hspace{1cm} (137)

where $U_{rs} \geq 0$ may be interpreted as the transition probability per unit time if a transition from state $|s\rangle\langle s|$ to state $|r\rangle\langle r|$. In perturbation theory

$$U_{rs} = |\langle r|H_{\text{pert}}|s\rangle|^2 = \langle r|H_{\text{pert}}|s\rangle^* \langle r|H_{\text{pert}}|s\rangle$$  \hspace{1cm} (138)

and hence from the Hermiticity of the Hamiltonian

$$\langle r|H_{\text{pert}}|s\rangle^* = \langle s|H_{\text{pert}}|r\rangle$$  \hspace{1cm} (139)

we have detailed balance or microscopic reversibility

$$U_{rs} = U_{sr}$$  \hspace{1cm} (140)

Under this assumption and that all transitions take place, i.e $U_{rs} > 0 \; \forall r, s$ we have the two following [75, 76, 77, 78, 79]

Theorem A (Existence Uniqueness of Equilibrium) there is a unique equilibrium state $i$ of total ignorance such that $P_r = P_s, \; \forall r, s$ and

Theorem B (‘H-Theorem’) The entropy $S = -\sum_r P_r \ln P_r$ is monotonic increasing $\dot{S} \geq 0$.

Proof of A under these assumptions

$$\dot{P}_r = \sum_{s, s \neq r} U_{rs} (P_s - P_r).$$  \hspace{1cm} (141)

If we order the $P_s$ in numerical order the r.h.s is non-negative and vanishes iff $P = P_s \forall r, s$

Proof of B under these assumptions it is also true that

$$-\dot{S} = \sum_{r, s, r \neq s} U_{rs} (P_s - P_r) \ln P_r.$$  \hspace{1cm} (142)

$$= -\frac{1}{2} \sum_{r, s, r \neq s} U_{rs} (P_s - P_r) (\ln P_s - \ln P_r)$$  \hspace{1cm} (143)

But

$$(x - y)(\ln x - \ln y) \geq 0.$$  \hspace{1cm} (144)

The problem is that in general $U_{rs} \neq U_{sr}$. In fact

$$U_{rs} = |\langle r|T|s\rangle|^2,$$  \hspace{1cm} (145)
where the S-matrix is given by

$$S = 1 + iT.$$  \hfill (146)

*Unitarity* then implies that

$$\sum_s U_{rs} = \sum_r U_{rs}. \hfill (147)$$

### 9.11 Consequence of Symmetries

It is well known that in standard S-matrix quantum mechanics that that symmetries and conservation laws are closely related. In the case of $S$-matrix quantum mechanics the connection is much less close.

### 9.12 $S$-matrix case

Wigner’s theorem tells us that if $T$ be a norm preserving map acting on the pure states preserving probabilities, then $T$ must be unitary or anti unitary $T^{-1} = T^\dagger$. We also assume a similar map $T'$ acts on the out pure states $T'^{-1} = T'^\dagger$, then if the $S$ matrix is invariant

$$ST = T'S.$$  \hfill (148)

Thus

$$STS^{-1} = T'.$$  \hfill (149)

Now if

$$T = \exp i\epsilon G, \quad G = g^\dagger,$$  \hfill (150)

then

$$SGS^{-1} = G',$$  \hfill (151)

where $T' = \exp i\epsilon G'$.

Thus if $| \text{out} \rangle = \$| \text{in} \rangle$

$$\langle \text{in} | G | \text{in} \rangle = \langle \text{out} | G' | \text{out} \rangle.$$  \hfill (152)

In other words, $H$ is conserved. Moreover it also follows that any power $G^k$ of $G$ is conserved and that eigenstates of $H$ and are taken to eigenstates of $G'$.

### 9.13 $S$-matrix case

In the $S$-matrix case, a density matrix transforms under $T$ as

$$\rho \rightarrow \mathcal{T}\rho = T\rho T^\dagger.$$  \hfill (153)

with

$$\mathcal{T} = \mathcal{T}^\dagger.$$  \hfill (154)

The condition of symmetry is now

$$\$ = \mathcal{T}'^{-1}\$\mathcal{T} = \mathcal{T}'^\dagger\$\mathcal{T}.$$  \hfill (155)

It is easy to see with particular examples that, in general symmetries, do not imply conservation laws
10 Superscattering and the Reals

Hawking’s original proposal (now famously abandoned by him) assumed the standard complex structure of quantum mechanics. From the point of view of what I have been advocating it seems curiously conservative to maintain that while advocating a much more radical modification of what we mean by the laws of physics.

In fact the entire discussion above works just as well over the reals, that is when the density matrices are just real symmetric semi-definite.

The general theory of super-scattering matrices works over all three fields, $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ and interestingly the space of such matrices is itself a convex set. Now any convex set is, by a Theorem of Minkowski, the convex hull of its extreme points. In this case, the extreme points are unitary or anti-unitary purity preserving maps, i.e. $S$-matrices.

A simpler case to consider is restrict attention to the case of diagonal density matrices. In this case, $S$ matrices are the doubly stochastic matrices encountered in the theory of Markov processes. These are the convex hull of the permutation matrices which take pure states to pure states.

The general theory of $S$ matrices, at least in finite dimension, is nicely discussed in [84].

11 Conclusion

We have seen in this talk that

- Time and its arrow are intimately linked with the complex nature of quantum mechanics.
- It is not difficult to construct spacetimes for which no arrow of time exists and on which backgrounds only real quantum mechanics is possible
- Only Riemannian manifolds admitting a reflection map $\Theta$ allow the recovery of standard quantum mechanics
- Even if one can define an arrow of time it may not be possible to define an operator $\Theta$ which reverses it.

Why then do we have such a strong impression that time exists and that it has an arrow? When and how did the complex numbers get into quantum mechanics?

Like so many things in life: its all a matter of history. The universe “started” with very special initial conditions “when neither time nor quantum mechanics were present. Both are emergent phenomena. Both are consequences of the special state we find ourselves in.

Constructing and understanding that state, and its alternatives is the ongoing challenge of Quantum Cosmology.
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13 Appendix: Complex versus Real Vector spaces

In this appendix we review some mathematical facts about complex structures. The standard structure of quantum mechanics requires that (pure) states are rays in a Hilbert space $H_{q_m}$ which is a vector space over the complex numbers carrying a Hermitian positive definite inner product $h(U,V)$ such that

\[(i)\quad h(U,\lambda V) = \lambda h(U,V), \quad \forall \lambda \in \mathbb{C}. \quad (156)\]
\[(ii)\quad h(U,V) = \overline{h(V,U)}. \quad (157)\]
\[(iii)\quad h(U,U) > 0. \quad (158)\]

It follows that $h(U,V)$ is antilinear in the first slot

\[h(\lambda U,V) = \overline{\lambda} h(U,V), \quad \forall \lambda \in \mathbb{C}. \quad (159)\]
In Dirac's bra and ket notation elements of \( \mathcal{H}_{qm} \) are written as kets:

\[ V \leftrightarrow |V\rangle \quad (160) \]

and elements of the \( \mathbb{C} \)-dual space \( \mathcal{H}_{qm}^* \), the space of \( \mathbb{C} \)-linear maps \( \mathcal{H}_{qm} \to \mathbb{C} \) as bras: and there is an anti-linear map from \( \mathcal{H}_{qm} \) to \( \mathcal{H}_{qm}^* \) given by

\[ U \to \langle U | \quad (161) \]

such that

\[ h(U, V) = \langle U | V \rangle , \quad (162) \]

thus

\[ \langle U | = h(U, \cdot ) . \quad (163) \]

In components

\[ |V\rangle = V^i |i\rangle \quad (164) \]

and

\[ \langle U | = \langle j | \bar{U}^j \quad (165) \]

\[ \langle U | V \rangle = h(U, V) = h_{ij} \bar{U}^i V^j , \quad (166) \]

where

\[ h_{ij} = \langle j | i \rangle \quad (167) \]

and

\[ \bar{h}_{ij} = h_{ji} . \quad (168) \]

### 13.1 Complex Vector spaces as Real Vector spaces

A useful references for this material with a view to applications in physics are \[29, 2\].

For simplicity of exposition one may imagine that \( \mathcal{H}_{qm} \) as finite dimensional \( \dim_{\mathbb{C}} \mathcal{H}_{qm} = n < \infty \). Since a complex number is just a pair of real numbers \[27\], any Hermitian vector space may be regarded as a real vector space \( V \) of twice the dimension \( \dim_{\mathbb{R}} V = 2n \) with something added \[1\], a complex structure \( J \), i.e a real-linear map such that

\[ J^2 = -1 , \quad (169) \]

and a positive definite metric \( g \) such that \( J \) is an isometry, i.e.

\[ g(JX, JY) = g(X, Y) . \quad (170) \]

It follows that \( V \) is also a symplectic vector space, with symplectic form

\[ \omega(X, Y) = g(JX, Y) = -\omega(Y, X) , \quad (171) \]

and \( J \) acts canonically, i.e.

\[ \omega(JX, JY) = \omega(X, Y) . \quad (172) \]
Alternatively given $J$ and the symplectic form $\omega$ we obtain the metric $g$ via

$$g(X, Y) = \omega(X, JY). \quad (173)$$

The standard example is the complex plane $\mathbb{C} = \mathbb{R}^2$ where if

$$e_1 = (1, 0), \quad e_2 = (0, 1), \quad (174)$$

$J(e_1) = e_2, \quad J(e_2) = -e_1 \quad (175)$

or as a matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (176)$$

and thus

$$J(xe_1 + ye_2) = xe_2 - ye_1 \quad (177)$$

which is the same in the usual notation as

$$i(x + iy) = -y + ix, \quad (178)$$

where $1 \leftrightarrow (1, 0)$ and $i \leftrightarrow (0, 1)$.

A complex structure $J$ can be thought of as a rotation of ninety degrees in $n$ orthogonal two planes. To specify it therefore it suffices to specify the (unordered) set of planes and the sense of rotation in each 2-plane.

### 13.2 A Real vector space as a Complex Vector space

Given the original real vector space, how are the complex numbers actually introduced? We start with $V$ and pass to its *complexification*, the tensor product $V \otimes_{\mathbb{R}} \mathbb{C}$. \( (179) \)

Note that $\dim_{\mathbb{R}} V \otimes_{\mathbb{R}} \mathbb{C} = 4n = 2 \dim_{\mathbb{C}} V$. We now extend the action of $J$ to $V \otimes_{\mathbb{R}} \mathbb{C}$, so it commutes with $i \in \mathbb{C}$:

$$J\alpha X = \alpha JX, \quad \forall \quad \alpha \in \mathbb{C}, X \in \mathbb{C}. \quad (180)$$

We may now diagonalize $J$ over $\mathbb{C}$ and write

$$V_{\mathbb{C}} = W \oplus \overline{W} \quad (181)$$

where

$$JW = iW, \quad J\overline{W} = -i\overline{W}. \quad (182)$$

Clearly $\dim_{\mathbb{R}} W = 2n = 2 \dim_{\mathbb{C}} W = \dim_{\mathbb{R}} V$, and $W$ may be thought of as $V$ in complex notation.

Thus if $X \in V$, we have that

$$X = \frac{1}{2}(1 - iJ)X + \frac{1}{2}(1 + iJ)X, \quad (183)$$

with $\frac{1}{2}(1 - iJ)X \in W$ and $\frac{1}{2}(1 - iJ)X \in \overline{W}$. Vectors in $W$ are referred to as type $(1, 0)$ or holomorphic and vectors in $\overline{W}$ as type $(0, 1)$ or anti-holomorphic.
13.3 The metric on $V_\mathbb{C}$

If $V$ admits a metric for which $J$ acts by isometries, we may extend the metric $g$ to all of $V_\mathbb{C} = W \oplus \overline{W}$ by linearity over $\mathbb{C}$, we find that

(i) $g(\bar{U}, V) = \overline{g(U, V)}$ \hspace{1cm} (184)
(ii) $g(U, U) > 0$, \hspace{1cm} (185)
(iii) $g(U, V) = 0$, $\forall U, V \in W$, and $\forall U, V \in \overline{W}$, \hspace{1cm} (186)

13.4 Negative Probabilities?

The metric $g$ is usually assumed to positive definite because of the demand that probabilities be positive and lie in the interval $[0, 1]$. This requirement has been brought into question, notably by Feynman [28]. In the context of vacuum energy one should perhaps not be too quick in rejecting this possibility since the expectation value of the energy momentum tensor for negative probability states in such theories can of course have the opposite sign from the usual one. This could have applications the cosmological constant problem.