String Propagator: a Loop Space Representation

S. Ansoldi
Dipartimento di Fisica Teorica dell’Università,
Strada Costiera 11, 34014-Trieste, Italy

A. Aurilia
Department of Physics, California State Polytechnic University
Pomona, CA 91768

E. Spallucci
Dipartimento di Fisica Teorica dell’Università,
Istituto Nazionale di Fisica Nucleare, Sezione di Trieste,
Strada Costiera 11, 34014-Trieste, Italy

Abstract

The string quantum kernel is normally written as a functional sum over the string coordinates and the world-sheet metrics. As an alternative to this quantum field-inspired approach, we study the closed bosonic string propagation amplitude in the functional space of loop configurations. This functional theory is based entirely on the Jacobi variational formulation of quantum mechanics, without the use of a lattice approximation. The corresponding Feynman path integral is weighed by a string action which is a reparametrization invariant version of the Schild action. We show that this path integral formulation is equivalent to a functional “Schrodinger” equation defined in loop-space. Finally, for a free string, we show that the path integral and the functional wave equation are exactly solvable.

*E-mail address: ansoldi@vstst0.ts.infn.it
†E-mail address: aaurilia@csupomona.edu
‡E-mail address: spallucci@vstst0.ts.infn.it
I. INTRODUCTION

There are at least two approaches to the quantum theory of relativistic strings. One way is to look at a string model as a field theory in two spacetime dimensions. In this case, the string coordinates $x^\mu(\tau, \sigma)$ are reinterpreted as a multiplet of scalar fields defined over the string manifold parametrized by a Lorentzian coordinate mesh $(\tau, \sigma)$. The non-linearity of the Nambu-Goto action can be “softened” by assigning an auxiliary metric field $\gamma_{ab}(\tau, \sigma)$ over the string manifold, and then writing the action in the Howe-Tucker form \[1\]. After this reshuffling of variables, the original string model is converted into a local field theory and is quantized through canonical, or path integral methods \[2\]. Quantum fluctuations around a classical solution eventually give rise to a spectrum of elementary particles, and the string itself acquires the status of fundamental building block of everything in the universe.

On the other hand, one may regard a string as an elementary physical system by itself, and focus on the geometric and topological properties of the string manifold. Vortices in a superconducting medium \[3\] and cosmic strings \[4\] are two noteworthy examples of this geometrical approach. Quantum fluctuations are now interpreted as transitions between different string configurations. In particular, the quantum propagation kernel acquires the meaning of probability amplitude for the string shape to evolve from an initial configuration, represented by the non self-intersecting spatial loop $C^0: 0 \leq s \leq 1 \rightarrow x^\mu_0 = x^\mu(s)$, $x^\mu_0(0) = x^\mu_0(1)$, to a final, non self-intersecting configuration $C: x^\mu = x^\mu(s)$. Thus, in this functional approach, spatial deformations of the string shape are mapped into “translations” in the space of all possible loop configurations, and our major concern is to develop a “Hamiltonian” theory for the quantum mechanics of strings in loop space. As a matter of fact, the main purposes of this paper can be stated as follows:

i) to give a path integral definition for the string kernel;

ii) to prove that the path integral definition is equivalent to a functional wave equation defined over a “space of loops”, and to show that the same kernel can be derived from both formulations;

iii) to determine the exact form of the free string propagation kernel.

Accordingly, the paper is organized as follows.

In Sect.2, we obtain the Jacobi functional equation for a closed string. This equation is the essential link between the classical and quantum theory because it provides the WKB approximation to the whole functional “Schrodinger” equation.

In Sect.3, we define the Feynman path integral for the quantum propagation kernel. We find that the reparametrization invariant sum over world-sheets, weighed by the exponential of the Nambu-Goto action, is a Jacobi path integral, i.e. a sum over string histories at “fixed energy” $E = 1/4\pi\alpha'$.

In Sect.4, we present a path integral derivation of the string kernel wave equation. No discretization procedure is involved. Instead, we use Jacobi’s variational principle to derive the functional Schrodinger equation for the string in a manifestly reparametrization invariant form.

In Sect.5, we compute the quantum kernel for a free string in two different ways. In Subsect.5.1, we solve the kernel “Schrodinger” equation by exploiting the role of the Jacobi equation as the classical limit of the full quantum equation. In Subsect.5.2, the string quantum kernel is obtained from the path integral using a “trick” which bypasses the use of
Sect.6 is devoted to a brief summary of the results and to the discussion of their possible generalization.

II. FUNCTIONAL JACOBI EQUATION

As a starting point for the study of string dynamics one can choose either the Nambu–Goto action, or the Schild action: both functionals lead to the same classical dynamics [5]. It is not clear, however, whether or not such an equivalence persists unrestricted at the quantum level. This is because, in a quantum theory, the string propagation kernel reflects the different weight assigned to the string trajectories in the two classical frameworks. Another potential source of inequivalence stems from the fact that the Nambu–Goto action is reparametrization invariant but non linear with respect to the generalized velocity, whereas the Schild action is linear but at the expense of reparametrization invariance. Apart from all this, the standard procedure to construct the path integral in quantum mechanics applies to quadratic actions, which is not the case for relativistic systems. One way to deal with the problem would be to follow the Dirac quantization procedure for constrained systems. But then, a canonical evolution of the system does not make sense because of the vanishing of the Hamiltonian. To preserve a Hamiltonian–type evolution, it is necessary to start with a non–reparametrization invariant theory. Even in this case, the resulting dynamics for an extended object is non–canonical.

All of the above arguments converge to the focal question: if one insists on a Hamiltonian, albeit non canonical formulation of string dynamics, is there an evolution parameter which plays the role of “time variable”, and is this choice consistent with reparametrization invariance?

Previous attempts to deal with those questions lead to seemingly conflicting conclusions. For instance, the string propagator obtained in ref. [6], has been criticized in ref. [7]. From our vantage point, the critical issue is that of reparametrization invariance. Both authors are led to a string diffusion equation which is manifestly dependent on the string parameter $s$, leaving us with the impression that the lack of reparametrization invariance of the classical action manifests itself even at the quantum level. However, in ref. [4], the physical Green function is obtained by averaging over all the possible values of the proper evolution parameter. In ref. [7], instead, it is claimed that the parametric dependence of the propagation kernel is only apparent because the action is insensitive to the location of the area increment along the world–sheet boundary. The resulting wave equation is local, in the sense that it is defined at a single, representative, point on the string loop, and does not apply to the string as a whole. Evidently, in this approach the string is treated as a collection of constituent points, and this may well be a viable interpretation. However, inspired by our previous work on the classical dynamics of p–branes [8], [9], [10], we believe that the dynamics of each individual point on the string does not give a consistent account of the dynamics of the whole string.

The alternative point of view is that “the whole string is more than the sum of its parts”, and in this paper we wish to suggest a different approach which, in our view, addresses directly the question of the choice of dynamical variables and the related issue of reparametrization
invariance in the classical theory as well as in the quantum theory. The stipulation is also made that the classical theory must emerge as a well defined limiting case of the quantum theory. In order to fulfil this condition, we invoke a single dynamical principle encompassing both areas of string–dynamics, namely the Jacobi variational principle suitably adapted to the case in which the physical system is a relativistic extended object. Thus, the dynamical variables are restricted to vary within the family of string trajectories which are solutions of the classical equation of motion. In other words, the variational procedure applies only to the final configuration of the string, rather than to its spacetime history.

Against this conceptual backdrop, the formalism developed in this paper, largely inspired by the work of Nambu [11], [12] and Migdal [13], fully reflects our emphasis on the global structure of the string: our action functional is a reparametrized form of the Schild action, manifestly invariant under general coordinate transformation in the string parameter space, while preserving the polynomial structure in the dynamical variables; the natural candidate for the role of time variable is the proper area of the string world–sheet (equation 2.8), i.e., the invariant measure of the model manifold representing the evolution of the string. The final outcome is a manifestly reparametrization invariant Schrodinger equation which has the same form of the corresponding equation obtained from the Nambu–Goto action using a lattice approximation, and admits gaussian type wave packets as solutions. Our starting point is the Schild string action in Hamiltonian form

$$S[x(\xi), p(\xi)] = \frac{1}{2} \int_{X(\xi)} p_{\mu \nu} dx^\mu \wedge dx^\nu - \int_\Sigma d^2 \xi H(p)$$

(2.1)

$$H(p) \equiv \frac{1}{4m^2} p_{\mu \nu} p^{\mu \nu},$$

(2.2)

where $m^2 = 1/2\pi \alpha'$ is the string tension, $\Sigma$ represents the model manifold of the string in parameter space, and $X(\xi)$ represents its image in Minkowski space. Then, $p_{\mu \nu}$ stands for the linear momentum canonically conjugated to the world–sheet tangent element $\dot{x}^{\mu \nu} \equiv \epsilon^{ab} \partial_a x^\mu \partial_b x^\nu$. Both variables were originally introduced by Nambu [11], [12]. More recently, the same variables were used to formulate a gauge theory for the dynamics of strings and higher dimensional extended objects [8], [9], [10].

In order to cast the action (2.2) in a reparametrization invariant form, we introduce a new pair of world–sheet coordinates $(\sigma^0, \sigma^1)$ through the boundary preserving transformation $\xi^a \rightarrow \sigma^a = \sigma^a(\xi)$, and promote the original pair $(\xi^0, \xi^1)$ to the role of dynamical variables. Then, $S[x(\xi), p(\xi)]$ transforms into

$$S[x(\sigma), p(\sigma), \xi(\sigma)] = \frac{1}{2} \int_{X(\sigma)} p_{\mu \nu} dx^\mu \wedge dx^\nu - \frac{1}{2} \epsilon_{ab} \int_{\Sigma(\sigma)} d\xi^a \wedge d\xi^b H(p)$$

(2.3)

The new action (2.3) is numerically equivalent to (2.2) and leads to the same equation of motion for $x^\mu$, $p_{\mu \nu}$. Furthermore, variation with respect to the new fields $\xi^a(\sigma)$ leads to the energy–balance equation

$$\epsilon_{ab} \epsilon^{mn} \partial_m \xi^a \partial_n H = 0, \implies H_{\text{cl.}} = \text{const.} \equiv E$$

(2.4)

which, in our case, correctly shows that the Hamiltonian is constant along a classical solution. The action (2.3) is linear with respect to the “velocities” $\dot{x}^{\mu \nu}$ and $\dot{\xi}^{ab} \equiv \epsilon^{mn} \partial_m \xi^a \partial_n \xi^b$. Hence,
if one interprets \( \pi_{ab} \equiv \epsilon_{ab} H \) as the momentum canonically conjugated to \( \xi^a(\sigma) \), then \((2.3)\) acquires the form of a reparametrization invariant theory in six dimensions \([11]\).

The Jacobi equation for the string is obtained by varying \( S[ x(\sigma), p(\sigma), \xi(\sigma) ] \) within the family of world–sheets which solve the string equations of motion. We emphasize that this type of variation corresponds to a deformation of the only free boundary of the world–sheet, i.e. \( C \), and corresponds to the more familiar variation of the world–line end–point in the case of a particle. Then, the steps leading to the Jacobi equation are as follows. First, the contribution from the variation of the world–sheet itself vanishes by definition, and we obtain:

\[
\delta S_{\text{cl}}[\partial X; A] = \int_{\partial X} p_\mu \delta x^\mu - H_{\text{cl}} \delta A. \tag{2.5}
\]

Next, we note that in view of the constancy of the Hamiltonian over a classical trajectory, we can vary the area of the \( \Sigma \) domain without reference to the specific point along the boundary \( \partial \Sigma \) where the infinitesimal variation takes place. In other words, we can move “\( \delta \)” in front of the area integral and then trade the functional variation \( \delta A \) for an ordinary differential variation \( dA \), and define

\[
p_\mu(s) \equiv \frac{\delta S_{\text{cl}}}{\delta x^\mu(s)} = p_{\mu\nu} x'^\nu\tag{2.6}
\]

as the boundary momentum density. Similarly, the area–energy density \( E \) can be written as the partial derivative of the classical action with respect to the invariant measure of the \( \Sigma \) domain in parameter space:

\[
E = -\frac{\partial S_{\text{cl}}}{\partial A} \tag{2.7}
\]

\[
A \equiv \frac{1}{2} \epsilon_{ab} \int_\Sigma d\xi^a \wedge d\xi^b. \tag{2.8}
\]

Hence, the Jacobi variational principle in the form of equation \((2.5)\) shows that \( p_\mu(s) \) is conjugated to the spacetime world–sheet boundary variation, while \( H_{\text{cl}} \) describes the response of the classical action to an arbitrary area variation in parameter space. Thus, if we consider string dynamics from the loop space point of view \([13]\), then \( A \) and \( x^\mu(s) \) can be interpreted as the “time” and “space” positions of the final string \( C \) with respect to the initial one \( C_0 \), which we assume to be fixed at the outset. In this perspective, \( H_{\text{cl}} \) is the area–Hamiltonian, or generator of the classical evolution from the “initial time” \( T = 0 \) to the final time \( T = A \). Accordingly, \( p_\mu(s) \) is the generator of infinitesimal “translations” in loop space, which are perceived as infinitesimal deformations \( x^\mu(s) \to x^\mu(s) + \delta x^\mu(s) \) of the string shape in Minkowski space.

Finally, note that in this formulation, \( E \) represents the energy per unit area associated with an extremal world–sheet of the action \((2.3)\), while \( p_\mu(s) \) is the momentum per unit length of the string loop \( C \). Therefore, the energy–momentum dispersion relation can be written either as an equation between densities

\[
\frac{1}{2m^2} p_\mu p^\mu = \frac{1}{4m^2} p_{\mu\nu} p^{\mu\nu} x'^2(s) = E x'^2(s) \tag{2.9}
\]

5
or, as an integrated relation

\[
\frac{1}{2m^2} \int_0^1 ds \frac{ds}{\sqrt{x'^2}} p{\mu}p^\mu = E \int_0^1 ds \sqrt{x'^2}.
\]  

(2.10)

The above equation, once written in terms of \(S_{cl}\), turns into the promised functional Jacobi equation for the string:

\[
\left( \int_0^1 ds \sqrt{x'^2} \right)^{-1} \int_0^1 ds \frac{\delta S_{cl}}{\sqrt{x'^2}} \delta x{\mu}(s) = -2m^2 \partial x{\mu}(s) \partial A.
\]  

(2.11)

Looking in more detail at this equation, we observe that the covariant integration over \(s\) takes into account all the possible locations of the point, along the contour \(C\), where the variation can be applied. But, in this way, every point of \(C\) is overcounted a “number of times” equal to the string proper length. The first factor, in round parenthesis, is just the string proper length and removes such overcounting. In other words, we sum over all the possible ways in which one can deform the string loop, and then divide by the total number of them. The net result is that the l.h.s. of equation (2.11) is insensitive to the choice of the point where the final string \(C\) is deformed. Therefore the r.h.s. is a genuine reparametrization scalar which describes the system’s response to the extent of area variation, irrespective of the way in which the deformation is implemented. With hindsight, the wave equation proposed in \([6]\), \([7]\) appears to be more restrictive than equation (2.11), in the sense that it requires the second variation of the line functional to be proportional to \(x'^2(s)\) at any point on the string loop, in contrast to equation (2.11) which represents an integrated constraint on the string as a whole.

Equation (2.11) is the starting point in the first quantization program via the Correspondence Principle: one introduces the quantum operators

\[
\hat{p}{\mu}(s) \equiv i\hbar \frac{1}{\sqrt{x'^2}} \frac{\delta}{\delta x{\mu}(s)}, \quad \hat{H} \equiv -i\hbar \frac{\partial}{\partial A},
\]  

(2.12)

and imposes the operatorial form of the dispersion relation (2.10) on the string wave functional \(\psi[C; A]\). Alternatively, one can focus directly on the string propagation kernel \(K[x(s), x_0(s); A]\), in which case we turn to Feynman’s “sum over histories” method since this is probably the most natural and effective way to define \(K[x(s), x_0(s); A]\) in quantum string–dynamics.

### III. FEYNMAN AND JACOBI PATH INTEGRALS

The path integral approach to string–dynamics is not only a useful check of the quantization procedure, but gives a new insight into the quantum theory itself. In particular, it gives a better insight into the meaning of the equivalence between the Nambu–Goto action and the Schild action. Furthermore, it provides a physically transparent relationship between the Feynman “sum over histories” integral and the Jacobi path integral. However, to see how this comes about, one must keep in mind that the momenta \(p{\mu}\) and \(\pi_{ab}\) cannot vary freely, because \(\pi_{ab}\) is merely a shorthand notation for the function \(\epsilon_{ab}H(p)\), and thus must satisfy the constraint equation

\[
\epsilon_{ab} \pi_{ab} = 0.
\]
\[ \pi_{ab} - \epsilon_{ab} H(p) = 0 . \]  

(3.1)

This constraint may be incorporated in the action \((2.3)\) by means of a Lagrange multiplier \(N^{ab}(\sigma)\)

\[
S[x(\sigma), p(\sigma), \xi(\sigma), N(\sigma); A] = \frac{1}{2} \int_{\Sigma} p_{\mu}\, dx^\mu \wedge dx^\nu + \frac{1}{2} \int_{\xi(\sigma)} \pi_{ab} d\xi^a \wedge d\xi^b - \frac{1}{2} \int \sigma N^{ab}(\sigma) \left[ \pi_{ab} - \epsilon_{ab} H(p) \right] ,
\]

(3.2)

and its physical meaning can be read out of the classical equation of motion obtained by varying \(\pi_{ab}\):

\[ N^{ab}(\sigma) = \epsilon^{mn} \partial_m \xi^a \partial_n \xi^b . \]

(3.3)

Thus, \(N^{ab}\) is the transformed \(\epsilon\)-tensor in the new coordinate system. Then, apart from an overall normalization constant, the amplitude for the initial string \(C_0\) to “evolve” into the final string \(C\) in a lapse of “time \(A\)”, can be represented by the path integral

\[
K[x(s), x_0(s); A] = \int_{x_0(s)}^{x(s)} \left[ D\mu(\sigma) \right] e^{iS[x,p,\xi,N;A]/\hbar} [D\mu(\sigma)] \equiv [Dx^\lambda(\sigma)][D\xi^a(\sigma)][Dp_{\mu\nu}(\sigma)][D\pi_{ab}(\sigma)][DN_{cd}(\sigma)].
\]

(3.5)

At the classical level, \(H\), or \(\pi_{ab}\), is independent of the coordinates \(\sigma^m\) because of the balance equation \((2.4)\). The same result is obtained at the quantum level by integrating out the \(\xi^a(\sigma)\) fields:

\[
\int_{\xi_0(s)}^{\xi(s)} \left[ D\xi^a(\sigma) \right] \exp \left\{ \frac{i}{2\hbar} \int_{\xi(\sigma)} d\xi^a \wedge d\xi^b \pi_{ab} \right\} = \delta \left[ \epsilon^{mn} \partial_m \pi_{an} \right] \exp \frac{i}{2\hbar} \int_{\xi(\sigma)} d \left( \pi_{ab} \xi^a d\xi^b \right) .
\]

(3.6)

(3.7)

The functional Dirac–delta requires \(\pi_{ab}\) to satisfy the classical equation of motion, i.e. \(\pi_{ab} = \epsilon_{ab} \times \text{const.} \equiv \epsilon_{ab} E\). Therefore,

\[
\int D[\pi_{ab}] \delta \left[ \epsilon^{mn} \partial_m \pi_{an} \right] \exp \left\{ \frac{i}{2\hbar} \left[ \pi_{ab} \int_{\xi(\sigma)} d \left( \xi^a d\xi^b \right) - \int \Sigma d^2 \sigma N^{ab}(\sigma) \pi_{ab} \right] \right\} = \int_0^\infty dE e^{iEA/\hbar} \exp \left\{ -\frac{iE}{2\hbar} \int_\Sigma d^2 \sigma N^{ab}(\sigma) \epsilon_{ab} \right\} .
\]

(3.8)

\[ 1 \text{The normalization constant will be fixed after all the functional integrations are carried out, by imposing the boundary condition}
\]

\[
\lim_{A \to 0} K[x(s), x_0(s); A] = \delta[x(s), x_0(s)] .
\]

(3.4)

Such a procedure effectively amounts to a renormalization of the field–dependent determinants produced by gaussian integration.
where we have assumed that the Hamiltonian is bounded from below and is normalized in such a way that \( E \geq 0 \). Then, the Feynman path integral can be written as follows

\[
K[x(s), x_0(s); A] = \int_0^\infty dE e^{iE A/\hbar} \int_{x_0(s)}^{x(s)} [Dx^\mu(\sigma)][DP_{\mu\nu}(\sigma)][DN_{\alpha\beta}(\sigma)] \times \\
\exp \left\{ \frac{i}{2\hbar} \int_{X(\sigma)} p_{\mu\nu} dx^\mu \wedge dx^\nu - \frac{i}{2\hbar} \epsilon_{ab} \int_\Sigma d^2\sigma N^{ab}(\sigma) [E - H(p)] \right\} \\
\equiv 2i\hbar m^2 \int_0^\infty dE e^{iE A/\hbar} G[C, C_0; E], \tag{3.9}
\]

where we have introduced the Jacobi path integral \( G[C, C_0; E] \) as the amplitude for a string to propagate from \( C_0 \) to \( C \), at fixed energy \( E \).

The most important property of \( G[C, C_0; E] \) is reparametrization invariance. If we integrate out the area momentum \( p_{\mu\nu} \), we obtain

\[
G[C, C_0; E] = \int_{x_0(s)}^{x(s)} [Dx^\mu(\sigma)] [DN(\sigma)] \times \\
\exp \left\{ -\frac{i}{\hbar} \int_\Sigma d^2\sigma \left[ -\frac{m^2}{4N} \ddot{x}^{\mu\nu} \ddot{x}_{\mu\nu} + NE \right] \right\} \tag{3.10}
\]

where \( N(\sigma) \equiv \epsilon_{ab} N^{ab}(\sigma)/2 \). Equation (3.10) is manifestly invariant under reparametrization

\[
\ddot{x}^{\mu\nu}(\sigma) \rightarrow \text{det} \left( \frac{\partial u^a}{\partial \sigma^m} \right) \ddot{x}^{\mu\nu}(u), \quad N(u) \rightarrow \text{det} \left( \frac{\partial u^a}{\partial \sigma^m} \right) N(\sigma). \tag{3.11}
\]

Finally, if we estimate the path integral (3.10) around the saddle point \( \ddot{N}(\sigma) = (-m^2 \ddot{x}^{\mu\nu} \ddot{x}_{\mu\nu}/4E)^{1/2} \), we find

\[
G[C, C_0; E] = \int_{x_0(s)}^{x(s)} [Dx^\mu(\sigma)] \exp \left\{ -\frac{i}{\hbar} \sqrt{m^2} E \int_\Sigma d^2\sigma \sqrt{-\dddot{x}^{\mu\nu} \dddot{x}_{\mu\nu}} \right\}, \tag{3.12}
\]

which is the usual path integral weighed by the Nambu–Goto action, once we fix \( E = m^2/2 \).

The result (3.12) suggests the following concluding remarks for this section:

i) equation (3.12) could be assumed at the outset and taken as a starting point for string quantization by means of functional techniques. The advantage of our derivation is that it clarifies the physical meaning of such a reparametrization invariant path integral: it represents the string propagation amplitude at fixed “area–energy” \( E = 1/4\pi\alpha' \).

ii) We can invert the Fourier transform (3.9) and define the reparametrization invariant path integral in terms of the Feynman propagation amplitude at fixed “area–lapse” \( A \)

\[
G[C, C_0; m^2] \equiv \frac{1}{2i\hbar m^2} \int_0^\infty dA e^{-im^2A/2\hbar} K[x(s), x_0(s); A]. \tag{3.13}
\]

Then, reparametrization offers an alternative definition of the sum over histories: first, sum over all world–sheets of fixed area; then, integrate over all possible values of the world–sheet area.

iii) Equation (3.9) represents the quantum counterpart of the classical equivalence \[5\] between the Nambu–Goto action and the Schild action.
IV. THE STRING KERNEL WAVE EQUATION

The purpose of this section is to show how to derive the functional wave equation for \( K[x(s), x_0(s); A] \) from the corresponding path integral. This equation describes how the string responds to a variation of the final boundary \( \xi^a = \xi^a(s) \), just as the ordinary Schrödinger equation describes a particle reaction to a shift of the time interval end–point. As we have seen in section (II), the string “natural” evolution parameter is the area \( A \) of the string manifold, so that functional or area derivatives generate “translations” in loop space, or string deformations in Minkowski space. Thus, we expect the functional wave equation to be of order one in \( \partial / \partial A \), and of order two in \( \delta / \delta x^\mu(s) \), or \( \delta / \delta \sigma^{\mu\nu}(s)^2 \).

The standard procedure to arrive at the kernel wave equation goes through a recurrence relation satisfied by the discretized version of the Jacobi path integral \([14], [15]\). However, such a construction is well defined only when the action is a polynomial in the dynamical variables. For a non–linear action such as the Nambu–Goto area functional, a lattice definition of the path integral is much less obvious. Moreover, the continuum functional wave equation is recovered through the highly non–trivial limit of vanishing lattice step \([14], [15]\). In any case, the whole procedure seems disconnected from the classical approach to string dynamics, whereas we would like to see a logical continuity between quantum and classical dynamics. Against this background, it seems useful to offer an alternative path integral derivation of the string functional wave equation which is deeply rooted in the Hamiltonian formulation of string dynamics discussed in Sect.(II), and is basically derived from the same Jacobi variational principle which we have consistently adopted so far.

The kernel variation under infinitesimal deformations of the field variables is

\[
\delta K[x(s), x_0(s); A] = \frac{i}{\hbar} \int_{x_0(s)}^{x(s)} \int_{\xi_0(s)}^{\xi(s)} [D\mu(\sigma)] \delta S \exp \left( iS / \hbar \right).
\] (4.2)

As usual, only boundary variations will contribute to equation (4.2) if we restrict the fields to vary within the family of classical solutions corresponding to a given initial string configuration. Then,

\[
\delta S_{cl}[C; A] = \oint_C p_{\mu\nu} \delta x^\mu(s) \, dx^\nu - E \, dA.
\] (4.3)

From equations (4.2) and (4.3), we obtain

\[
\frac{\partial}{\partial A} K[x(s), x_0(s); A] = -\frac{iE}{\hbar} K[x(s), x_0(s); A],
\] (4.4)

---

2Functional and area derivatives are related by \([13]\)

\[
\frac{\delta}{\delta x^\mu(s)} = x'^\nu \frac{\delta}{\delta \sigma^{\mu\nu}[C]}, \quad x'^\nu \equiv \frac{dx^\nu}{ds}.
\] (4.1)

Therefore, the functional wave equation can be written in terms of either type of derivative. Contrary to the statement in ref. \([7]\), area derivatives are regular even when ordinary functional derivatives are not \([13]\).
\[ \frac{\delta}{\delta x^\mu(s)} K[x(s), x_0(s); A] = i \hbar \int_{x_0(s)}^{x(s)} [D\mu(\sigma)] p_{\mu\nu} \exp (iS/\hbar) . \] (4.5)

Then, by comparison of (4.4), (4.5) and (2.11), one obtains immediately the kernel wave equation

\[ -\frac{\hbar^2}{2m^2} \left( \int_0^1 ds \sqrt{x'^2} \right)^{-1} \int_0^1 ds \frac{\delta^2}{\delta x^\mu(s) \delta x_\mu(s)} K[x(s), x_0(s); A] = \]

\[ i\hbar \frac{\partial}{\partial A} K[x(s), x_0(s); A] . \] (4.6)

Thus, \( K[x(s), x_0(s); A] \) can be determined either by solving the functional wave equation (4.6), or by evaluating the path integral (3.5).

Once equation (4.6) is given, it is straightforward to show that \( G[C, C_0; m^2] \) satisfies the following equation

\[ \left[ -\frac{\hbar^2}{2m^2} \left( \int_0^1 ds \sqrt{x'^2} \right)^{-1} \int_0^1 ds \frac{\delta^2}{\delta x^\mu(s) \delta x_\mu(s)} + m^4 \right] G[C, C_0; m^2] = -\delta[C - C_0] . \] (4.7)

Therefore, \( G[C, C_0; m^2] \) can be identified with the Green function for the string.

V. COMPUTING THE KERNEL

A. Integrating the functional wave equation

It is possible to compute \( K[x(s), x_0(s); A] \) exactly in the “free” case because the Lagrangian corresponding to the Hamiltonian (2.2) is quadratic with respect to the generalized velocities \( \dot{x}^{\mu\nu} \). Previous experience with this class of Lagrangians suggests the following ansatz for the string quantum kernel:

\[ K_0[x(s), x_0(s); A] = \mathcal{N} A^\alpha \exp (iI[x(s), x_0(s); A]/\hbar) , \] (5.1)

where \( \mathcal{N} \) is a normalization constant, and \( \alpha \) a real number. Substituting this ansatz into eq.(4.6) gives two independent equations for the amplitude and the phase respectively,

\[ \frac{2\alpha m^2}{A} = - \left( \int_0^1 ds \sqrt{x'^2} \right)^{-1} \int_0^1 ds \frac{\delta^2 I}{\delta x^\mu(s) \delta x_\mu(s)} , \] (5.2)

\[ 2m^2 \frac{\partial I}{\partial A} = - \left( \int_0^1 ds \sqrt{x'^2} \right)^{-1} \int_0^1 ds \frac{\delta I}{\delta x^\mu(s) \delta x_\mu(s)} . \] (5.3)

Comparing equations (5.3) and (2.11), we see that \( I = S_{cl}[x(s), x_0(s); A] \) and (5.3) is just the classical Jacobi equation. Therefore, the main problem is to determine the form of \( S_{cl} \) in the string case. We do so by analogy with the relativistic point–particle case, where \( S_{cl} \) is a functional of the world–line length element. Accordingly, we first introduce the oriented surface element as a functional of the surface boundary \( C \)
\[ \sigma^{\mu\nu}[C] \equiv \oint_C x^\mu \, dx^\nu = \int_0^1 du \, x^\mu(u) \frac{dx^\nu}{du}. \]  

(5.4)

Then, from the above definition we obtain

\[ \frac{\delta \sigma^{\mu\nu}[C]}{\delta x^\alpha(s)} = \delta^\alpha_\mu x^{\prime \nu}(s) - \delta^\alpha_\nu x^{\prime \mu}(s), \]  

(5.5)

\[ \frac{\delta^2 \sigma^{\mu\nu}[C]}{\delta x^\alpha(s) \delta x^\beta(u)} = (\delta^\alpha_\mu \delta^\nu_\beta - \delta^\alpha_\nu \delta^\mu_\beta) \frac{d}{ds} \delta(s - u). \]  

(5.6)

Next, we introduce the trial solution

\[ S_{\text{cl}}[x(s), x_0(s); A] = \frac{\beta}{4A} \left( \sigma^{\mu\nu}[C] - \sigma^{\mu\nu}[C_0] \right) \left( \sigma_{\mu\nu}[C] - \sigma_{\mu\nu}[C_0] \right), \]

\[ \equiv \frac{\beta}{4A} \Sigma^{\mu\nu}[C - C_0] \Sigma_{\mu\nu}[C - C_0] \]  

(5.7)

where \( \beta \) is a second parameter to be fixed by the equations (5.2), (5.3). By taking into account (5.5), (5.6), we find

\[ \frac{\delta S_{\text{cl}}}{\delta x^\mu(s)} = \frac{\beta}{A} \Sigma^{\mu\nu}[C - C_0] x^{\prime \nu}(s). \]  

(5.8)

Note that the dependence on the parameter \( s \) is only through the factor \( x^{\prime \nu}(s) \). Then,

\[ \frac{\delta^2 S_{\text{cl}}}{\delta x^\mu(s) \delta x^\mu(s)} = \frac{3\beta}{A} x^{\prime 2}(s). \]  

(5.9)

Equations (5.3) and (5.2) now give

\[ \beta = -\frac{2\alpha}{3} m^2, \quad \alpha = -\frac{3}{2}. \]  

(5.10)

Finally, if we define the loop space Dirac delta function

\[ \delta[C - C_0] \equiv \lim_{\epsilon \to 0} \left( \frac{1}{\pi \epsilon} \right)^{3/2} \exp \left( -\frac{1}{2\epsilon} \Sigma^{\mu\nu}[C - C_0] \Sigma_{\mu\nu}[C - C_0] \right) \]  

(5.11)

then, the kernel normalization constant is fixed by the boundary condition (3.4), and we finally obtain the promised expression of the quantum kernel as an exact evaluation of the path integral

\[ K[x(s), x_0(s); A] = \left( \frac{m^2}{2i\pi \hbar A} \right)^{3/2} \exp \left( \frac{im^2}{4\hbar A} \Sigma^{\mu\nu}[C - C_0] \Sigma_{\mu\nu}[C - C_0] \right). \]  

(5.12)

The above equation, in turn, leads us to the following representation of the Nambu–Goto closed string propagator
\[\int_{x_0(s)}^{x(s)} [Dx^\mu(\sigma)] \exp \left\{ -\frac{im^2}{\hbar} \int_\Sigma d^2 \sigma \sqrt{-\frac{1}{2} \dot{x}^{\mu} \dot{x}_{\mu}} \right\} = \int_0^\infty dA e^{-im^2 A/2\hbar} \left( \frac{m^2}{2i\pi\hbar A} \right)^{3/2} \exp \left( \frac{i m^2}{4\hbar A} \Sigma_{\mu\nu} [C - C_0] \Sigma_{\mu\nu} [C - C_0] \right) \cdot \tag{5.13}\]

Note that, since no approximation was used to obtain equation (5.13), the above representation can also be interpreted as a new definition of the Nambu–Goto path integral. This definition is based on the classical Jacobi formulation of string dynamics rather than on the customary discretization procedure.

**B. Integrating the path integral**

As a consistency check on the above result, and in order to clarify some further properties of the path integral, it may be useful to offer an alternative derivation of equation (5.13) which is based entirely on the usual gaussian integration technique. As we have seen in the previous section, the Feynman amplitude can be written as follows

\[K[x(s), x_0(s); A] = \int_{x_0(s)}^{x(s)} [Dx^\mu(\sigma)] [Dp_{\mu\nu}(\sigma)] \times \exp \left\{ \frac{i}{2\hbar} \int_{X(\sigma)} p_{\mu\nu} dx^\mu \wedge dx^\nu - \frac{i}{2\hbar} \epsilon_{ab} \int_{\Sigma(\sigma)} d\xi^a \wedge d\xi^b H(p) \right\}. \tag{5.14}\]

In order to evaluate the functional integral (5.14), without discretization of the variables, we enlist the following equalities,

\[\int_{x_0(s)}^{x(s)} [Dx^\mu(\sigma)] \exp \left\{ \frac{i}{2\hbar} \int_{X(\sigma)} p_{\mu\nu} dx^\mu \wedge dx^\nu \right\} = \delta [d (p_{\mu\nu} dx^\nu)] \exp \left\{ \frac{i}{2\hbar} \int_{\partial X(s)} p_{\mu\nu} x^\mu dx^\nu \right\}. \tag{5.15}\]

The functional delta function has support on the classical, extremal trajectories of the string. Therefore, the momentum integration is restricted to the classical area–momenta and the residual integration variables are the components of the area–momentum along the world–sheet boundary \(p_{\mu\nu}(s)\). As a matter of fact, boundary conditions fix the initial and final string loops \(C_0\) and \(C\) but not the conjugate momenta. In analogy to the point particle case, the classical equations of motion on the final world–sheet boundary

\[d (p_{\mu\nu} dx^\nu) \big|_{x=x(s)} = 0 \tag{5.16}\]

require that the three normal components of \(p_{\mu\nu}\) be constant, i.e. \(p_{\mu\nu} x^{\nu}(s) = \text{const}\). Hence, the functional integral over the boundary momentum reduces to a three dimensional, generalized, Gaussian integral.
Moreover, the Hamiltonian is constant over a classical world-sheet and can be written in terms of the boundary $p_{\mu\nu}$. In such a way, the path integral is reduced to the Gaussian integral over the three components of $p_{\mu\nu}$ which are normal to the boundary

$$K[x(s), x_0(s); A] = \mathcal{N} \int [dp_{\mu\nu}] \exp \left\{ i \frac{\hbar}{2} p_{\mu\nu} \int_{\partial X} x^\mu(s) dx^\nu - \frac{A}{4m^2} p_{\mu\nu} p_{\mu\nu} \right\} =$$

$$\mathcal{N} \int [dp_{\mu\nu}] \exp \left\{ i \frac{\hbar}{2} p_{\mu\nu} \Sigma^{\mu\nu}[C - C_0] - \frac{A}{4m^2} p_{\mu\nu} p_{\mu\nu} \right\}. \quad (5.18)$$

The integral $(5.18)$ correctly reproduces the expression $(5.13)$.

VI. CONCLUSIONS

The main results of this paper can be summarized as follows. Starting with the canonical form of the Schild action for a closed, bosonic string, it is possible to formulate the Hamilton–Jacobi theory of string dynamics in loop space, with the proper area of the string manifold playing the role of evolution parameter. The conjugate dynamical quantity is an area–Hamiltonian which is quadratic in the corresponding momenta so that it is possible to extend to strings, indeed to any p-brane, many of the results which are applicable to a relativistic point particle. The Feynman path integral for quantum strings is then equivalent to the functional wave equation $(4.6)$ which was derived without recourse to a lattice approximation. For the kernel $(5.12)$, the path integral, or the corresponding wave equation can be solved exactly.

If, as it is normally done, one starts from a reparametrization invariant path integral over the string coordinates, the corresponding amplitude describes the propagation of strings with fixed “energy” $E = 1/2\pi\alpha'$. The relation between the two amplitudes is given by equation $(3.13)$.

Three generalizations of the above results are almost straightforward:

i) the wave equation for a closed string coupled to a Kalb–Ramond field can be obtained from $(4.6)$ through the replacement

$$\frac{\delta}{\delta x^\mu(s)} \longrightarrow \frac{\delta}{\delta x^\mu(s)} + i\kappa B_{\mu\nu}(x) x^\nu(s). \quad (6.1)$$

Unfortunately, there is no straightforward way to solve the wave equation, or to integrate the path integral, for an arbitrary gauge potential;

ii) some more work is required to extend the above formalism to the case of a closed p-brane imbedded into a $D$–dimensional spacetime. However, no essential difficulties arise in treating higher dimensional extended objects;

iii) once the quantum mechanical propagator is known, then one can second quantize the system. If we introduce the string wave function $\psi[x(s)]$ as an element of a functional space of string states, then we can write $(4.6)$ as
\[ i\hbar \frac{\partial}{\partial A} |\psi\rangle = H |\psi\rangle , \quad (6.2) \]

where \( H \) is the area–Hamiltonian operator. Then, the corresponding Green function can be written as follows

\[ G[x(s), x_0(s); m^2] = \frac{i}{\hbar} \int_0^\infty dA \langle x(s) | e^{iHA/\hbar} | x_0(s) \rangle = \langle x(s) | \frac{1}{H} | x_0(s) \rangle , \quad (6.3) \]

or

\[ G[x(s), x_0(s); m^2] = \mathcal{N} \int [D\psi^*][D\psi] \psi^*[x(s)]\psi[x_0(s)] \exp \left( iS[\psi^*, \psi]/\hbar \right) , \quad (6.4) \]

where, \( \psi^*[x(s)], \psi[x_0(s)] \) constitute a pair of complex functional fields, and

\[ S[\psi^*, \psi] = \int [Dx^\mu(s)] \psi^*[x(s)] \left[ -\hbar^2 \left( \int_0^1 ds \sqrt{x'^2} \right)^{-1} \int_0^1 ds \frac{\delta^2}{\sqrt{x'^2} \delta x^\mu(s) \delta x^\mu(s)} + m^4 \right] \psi[x(s)] \]

is the Marshall–Ramond \([16]\) action for the dual string model. Therefore, by extending the classical Hamilton–Jacobi formulation of string dynamics into the quantum domain, one arrives at a functional field theory in loop space.

As a final speculative remark, it seems worth observing that the functional approach to the quantum mechanics of strings is analogous, in several ways, to the functional approach to quantum cosmology. For instance, the “wave function of the universe” is defined in the functional space of all possible 3–geometries, and we suggest that spatial loop configurations in string theory play the same role as spatially closed 3–geometries in quantum cosmology. Likewise, the functional Schrödinger equation for strings, at fixed areal–time, plays the same role as the Wheeler–DeWitt equation in quantum cosmology. If this analogy is more than coincidental, then quantum string theory in loop space may shed some light on the many dark areas of quantum cosmology.
REFERENCES

[1] P.S. Howe and R.W. Tucker, J. Phys. A10, L155, (1977).
[2] A.M. Polyakov, “Gauge fields and strings”, Harwood Acad. Publ. 1987
[3] A.A. Abrikosov, Sovt. Phys. JEPT 5, 1174, (1957)
[4] T.W.B. Kibble, J. Phys. A9, 1387, (1976)
[5] A. Schild, Phys. Rev. D16, 1722, (1977)
[6] T. Eguchi, Phys. Rev. Lett. 44, 126, (1980)
[7] A.T. Ogielski, Phys. Rev. D22, 2407, (1980)
[8] A. Aurilia, A. Smailagic and E. Spallucci, Phys. Rev. D47, 2536, (1993)
[9] A. Aurilia, and E. Spallucci, Class. Quantum Grav. 10, 1217, (1993)
[10] A. Aurilia, E. Spallucci and I. Vanzetta, Phys. Rev. D50, 6490, (1994)
[11] Y. Nambu, Phys. Lett. 92B, 327, (1980)
[12] Y. Nambu, Phys. Lett. 102B, 149, (1981)
[13] A. Migdal, Phys. Rep. 102, 199, (1983)
[14] H. Kawai, Progr. Theor. Phys. 65, 351, (1981)
[15] K. Seo and A. Sugamoto, Phys. Rev. D24, 1630, (1981)
[16] C. Marshall and P. Ramond, Nucl. Phys. B85, 375, (1975)