A FOOTNOTE TO A FOOTNOTE TO A PAPER OF B. SEGRE

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Dedicated to Ciro Ciliberto, for his 70th birthday.

_Si canimus silvas, silvae sint consule dignae._

_(Vergilius)_

1. Introduction

The paper is devoted to a detailed study of sextics in three variables having a decomposition as a sum of nine powers of linear forms. This is indeed the unique case of a Veronese image $X$ of the plane which, in the terminology introduced by Ciliberto and the first author in [12], is _weakly defective_, and non-identifiable: a general sextic of the 9-secant variety of $X$ has two minimal decompositions.

The title originates from a famous paper of 1981, [4], where Arbarello and Cornalba state and prove a result on plane curves with preassigned singularities, which is relevant to extend the studies of B. Segre on special linear series on curves. The result (Theorem 3.2) says that the linear system of sextics with 9 general nodes in $\mathbb{P}^2$ is the unique non-superabundant system of plane curves with general nodes whose (unique) member is non-reduced.

The result is of course relevant also in the theory of interpolation, and in the study of secant varieties to Veronese varieties, with consequences for the Waring decomposition of forms. The matter turns out to be strictly related to the uniqueness of a minimal Waring decomposition of a general form. From this point of view Theorem 3.2 of [4] and Theorem 2.9 of [13] imply that the unique case of general ternary forms of fixed (Waring) rank with a finite number greater than one of minimal decomposition holds for degree 6 and rank 9. Since the unique sextic with 9 general nodes is twice an elliptic curve, it turns out by proposition 5.2 of [13] that a general ternary sextic of rank 9 has exactly two minimal decompositions. A complete list of cases, in any number of variables, in which the previous phenomenon appears is contained in [14].

Remaining in the case of ternary sextics, already in [4] the authors observe that if the 9 nodes are in special position (e.g. when they are complete intersection of two cubics), then the linear system of nodal curves has also reduced members. Our analysis starts here, and aims to describe the decompositions of specific ternary sextics, with respect to the postulation of the corresponding sets of projective points in $\mathbb{P}^2$.

We consider a fixed sextic form $F$ in the polynomial ring $R = \mathbb{C}[x_0, x_1, x_2]$ in three variables. In order to effectively produce decompositions of $F$, the first natural step is to consider the _apolar_ ideal $F^\perp$ of $F$.

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In general, the apolar ideal $F^\perp$ of a form $F$ of degree $d$ is defined via the natural action of the dual ring $R^\vee$ on $R$. $F^\perp$ is the ideal of elements $D \in R^\vee$ that kill $F$. A classical theorem by Sylvester says that a set of points is a decompositions of $F$ if and only if the corresponding ideal in $R^\vee$ is contained in $F^\perp$. Indeed, in order to find properties of the decompositions of a specific form $F$, it is often sufficient to look at homogeneous pieces of the ideal $F^\perp$, which correspond to the kernels of the (catalecticant) maps $(R^\vee)_k \to R_{d-k}$ induced by $F$, and their projective versions $C^k_d$. In [20], as well as in [24], it is explained that if the image of $C^k_d$ has the expected dimension and cuts the corresponding Veronese variety in $\mathbb{P}(R_{d-k})$ in a finite set $A$, then the set $A$ determines a decomposition of $F$, and many minimal decompositions can be found in this way.

Consider, in particular, the case of a sextic $F$ of rank 8. In this situation, the image of the catalecticant map of order 3 determines a linear space of codimension 2 in $\mathbb{P}(R_3)$, which cuts the 3-Veronese of $\mathbb{P}^2$ in 9 points. The set $Z$ of 9 points is the image of a complete intersection of two cubics in $\mathbb{P}^2$, via the Veronese map. Then the 8 points of a minimal decomposition of $F$ are among the points of $Z$. It is quite easy to find the coefficients of $F$ with respect to the points of $Z$, and then determine the 8 points that give a minimal decomposition. The procedure shows that there are sextic forms which lie in between forms of rank 8 and forms of rank 9. These are forms of rank 9, with a minimal decomposition $Z$ coming from a complete intersection of two plane cubics. Forms of this sort fill a subvariety $W$ of the secant variety $S^9(X)$ ($X$ being the 6-Veronese image of $\mathbb{P}^2$) which contains $S^8(X)$.

We show that a general sextic in $W$ has a unique decomposition with 9 powers of linear forms. From a certain point of view, this is quite surprising. A general sextic $F'$ of rank 9 has two minimal decompositions of length 9, coming from the existence of an elliptic normal curve in $X$ which spans $F'$. When $F'$ sits in $W$, then there is a pencil of elliptic curves in $X$ which span $F'$, so one may expect infinitely many minimal decompositions, one for each elliptic curve.

The phenomenon can be investigated in terms of the map $s^9$ from the abstract secant variety $\text{AS}^9(X)$ to $S^9(X)$, which is generically 2:1. The component of the ramification locus $\mathcal{R}$ is strictly connected with the theory of Terracini loci, introduced in [6]. Roughly speaking, Terracini loci contain finite subsets of $X$ such that the tangent spaces to the points are not independent. A subset $A \subset X$ consisting of 9 points belongs to the Terracini locus $T_9(X)$ exactly when the linear span of $v_9(A)$ is contained in $\mathcal{R}$. Thus, when a sextic form $F$ has a decomposition $A$ complete intersection of two cubics, which obviously lie in the Terracini locus, then in a neighborhood of $F$ the secant map has degree 2, so it has degree 2 also in $F$, but the fiber is non-reduced.

The subvariety $W$ of $S^9(X)$ lies in between $S^9(X)$ and $S^8(X)$, and it is easy to detect because it is easy to compute the rank and the kernel of the catalecticant map $C^3_9$. There are other relevant varieties of $S^9(X)$, which can be studied in terms of the geometry of the decomposition. Given a general set $A$ of 9 points in $\mathbb{P}^2$, and a general sextic $F$ in the span of $v_9(A)$, a second decomposition of $F$ comes out by taking a geometry of 9 points $B$ linked to $A$ by a cubic and a sextic. In this case, as explained in [2], the sum $I_A + I_B$ of the homogeneous ideals of $A$ and $B$ determines a subspace of $R_9$, whose orthogonal direction gives coefficients for the (unique) form $F$ in the intersection $\langle v_9(A) \rangle \cap \langle v_9(B) \rangle$. In general, the cubic is uniquely determined
by $A$ but the sextic moves and then the set of forms defined by $(I_A + I_B)_6$ dominates $\langle v_6(A) \rangle$. On the contrary, when $A$ is complete intersection, then also $B$ is complete intersection. We obtain that starting from a complete intersection $A$ the linked sets $B$ determine only a subvariety of forms in $\langle v_6(A) \rangle$, each having a 2-dimensional set of decompositions. When $A$ varies in the Hilbert stratum of complete intersections of type $(3,3)$ in $\mathbb{P}^2$, we obtain a subvariety $W' \subset W$ of sextics in $S^6(X)$ with a 2-dimensional set of decompositions, which can be easily described because for such forms $F$ the kernel of $C^3_F$ has dimension 3, so that $W'$ does not contain $S^6(X)$, but properly contains $S^7(X)$.

Let us notice that the picture is completed by another subvariety $\mathcal{R}'$ of the ramification locus, given by forms $F$ with two different and linked decompositions $A, B$, both sets of nodes of irreducible sextics. The variety $\mathcal{R}'$ parametrizes forms with a 1-dimensional family of decompositions, and its intersection with $W$ contains $W'$. The variety $\mathcal{R}'$ is the most mysterious object in the picture, for it cannot be completely described in terms of the catalecticant map $C^3_F$ and its kernel. While the third section is devoted to the geometry (and, somehow, parametric) description of $S^6(X)$, in the last section we provide for most of them a set of equations. Equations for $S^6(X)$, $W$, $W'$ can be easily described generically by the vanishing of appropriate minors of the catalecticant map $C^3_F$.

Equations for $S^8(X)$ and $S^7(X)$ require the determinant of a $27 \times 27$ matrix $A_f$ which represents a flattening, introduced in [21]. It is remarkable that the invariant $H_{27} = \det A_f$, joint with the catalecticant matrix $C^3$, allows to describe the equations of all the secant varieties to $X = v_6(\mathbb{P}^3)$. The ramification locus turns out to be described also in terms of $\det(A_f)$ (see Theorem 4.4). We wonder if a description of equations for $\mathcal{R}'$ can be achieved in terms of minors of $A_f$.

2. Preliminaries

2.1. Notation. In this section, as in most of the paper, we write $\mathbb{P}^n$ for the projective space of linear forms in $n + 1$ variables, with complex coefficients.

In the sequel, by abuse, we will identify a form $F$ with the point in the projective space $\mathbb{P}(\text{Sym}^d(\mathbb{C}^{n+1}))$ associated to $F$, and also with the hypersurface of equation $F = 0$.

Fix $N = -1 + (n+d)$ and identify $\mathbb{P}^N$ with $\mathbb{P}(\text{Sym}^d(\mathbb{C}^{n+1}))$, the projective space of degree $d$ forms in $n + 1$ variables.

If we denote with $R$ the polynomial ring $R = \mathbb{C}[x_0, \ldots, x_n]$, then $\mathbb{P}^N$ is the projective space over the degree $d$ piece $R_d$, while $\mathbb{P}^n$ is the projective space over $R_1$.

We denote with $v_d : \mathbb{P}^n \to \mathbb{P}^N$ the $d$-th Veronese map of $\mathbb{P}^n$ which, in the previous notation, maps $L \in \mathbb{P}^n$ to $L^d$.

A (Waring) expression of length $r$ of $F$ is an equality

$$F = a_1 L_1^d + \cdots + a_r L_r^d$$

where the $L_i$'s are linear forms and the $a_i$'s are complex coefficients. Each $L_i$ represents a point in $\mathbb{P}^n$. The expression is non-redundant if the $L_i^d$ are linearly independent in $\text{Sym}^d(\mathbb{C}^{n+1})$, and all the coefficients are non-zero. Non-redundant means that one cannot find a proper subset $A' \subset A$ such that also $A'$ is a decomposition of $F$. 
We are aware that, since we are working over the algebraically closed field $\mathbb{C}$, we could get rid of the coefficients $a_i$’s in a Waring expression of $F$. It is convenient for us to maintain the coefficients, because we will compare, below, Waring expressions in which the linear forms $L_i$’s are fixed and the coefficients move.

Given a Waring expression of $F$, we call (Waring) decomposition of $L$ the finite set $A = \{L_1, \ldots, L_r\} \subset \mathbb{P}^n$. The length of the decomposition $A$ is the cardinality of the finite set $A$, often denoted with $\ell(A)$.

From the geometric point of view, $A$ is a decomposition of $F$ if and only if $F$ belongs to the linear span $\langle v_d(A) \rangle$ of $v_d(A)$, and $A$ is non-redundant if one cannot find a proper subset $A' \subset A$ such that also $A'$ is a decomposition of $F$.

The rank of $F$ is the minimal $r$ for which $F$ has a Waring expression of length $r$. The border rank of $F$ is the minimal $r'$ such that $F$ is limit of forms of rank $r'$.

2.2. Catalecticant maps. The projective catalecticant map associated to a form $F$, as suggested in [20], is defined as follows.

Denote with $R' = \mathbb{C}[\partial_0, \ldots, \partial_n]$ the ring of linear operators. There is a natural contraction map $\mathcal{C} : R' \times R \to R$ defined by linearity and by

$$\mathcal{C}(\partial_i \cdots \partial_{i_k}, F) = \frac{\partial^k F}{\partial x_{i_1} \cdots \partial x_{i_k}}.$$  

Notice that $\mathcal{C}$ maps $(R')_k \times R_d$ bilinearly to $R_d$.

The $k$-polar space of $F$ the subspace $\text{Im}(\mathcal{C}_F^k)$ of $R_{d-k}$, and call $k$-polar projective space of $F$ its projectification $P_F^k = \mathbb{P}(\text{Im}(\mathcal{C}_F^k))$.

We will use in the sequel the correspondence between $R_d$ and $R_d'$ induced by the monomial basis.

Properties of catalecticant maps in connection with decompositions of a form $F$ are deeply studied in [20]. We recall some basic facts.

**Theorem 2.1.** If $A = \{L_1, \ldots, L_r\} \subset \mathbb{P}^n$ is a decomposition of $F$, then the projective polar space $P_F^k$ is contained in the linear span of $L_1^{d-k}, \ldots, L_r^{d-k}$.

Thus, $\dim(P_F^k) \leq \text{rank}(F)$.

Comparing dimensions, one obtains.

**Corollary 2.2.** In the setting of Theorem 2.1 if moreover $P_F^k$ has (projective) dimension $r-1$, then the points $L_1^{d-k}, \ldots, L_r^{d-k}$ are linearly independent, and

$$P_F^k = \langle L_1^{d-k}, \ldots, L_r^{d-k} \rangle.$$  

In particular, if $\dim(P_F^k) = \text{rank}(F)$, then any minimal decomposition $A$ of $F$ satisfies

$$v_{d-k}(A) \subset P_F^k \cap v_{d-k}(\mathbb{P}^n).$$

We notice that it is easy to construct the matrix of $\mathcal{C}_F^k$ with respect to the monomial basis, and then check the dimension of the image of $\mathcal{C}_F^k$.

The Apolarity Theorem (see [20] Section 4) determines a fundamental link between catalecticant maps and decompositions. We will use it in the form below.
Theorem 2.3. The direct sum of the kernels of the catalecticant maps \( C^k_R \) is an (artinian, homogeneous) ideal in \( R' \), called the apolar ideal \( F' \) of \( F \). If the homogeneous ideal \( I_A^\vee \) of a finite set \( A^\vee \), in the projective space \( (\mathbb{P}^n)^\vee \), associated to \( R' \), is contained in \( F' \), then the corresponding set \( A \subset \mathbb{P}^n \) determines a decomposition of \( F \).

Notice that when \( I_A^\vee \subset F' \), then the \( k \)-polar space of \( F \) is contained in the span of \( v_{d-k}(\mathbb{P}^n) \) for all \( k \). It is well known that the converse fails. Thus in general one cannot hope to find a decompositions of \( F \) by looking only to one catalecticant map. Yet, we will see that there are cases in which just one catalecticant map is sufficient.

Remark 2.4. Consider an element \( D \) of the kernel of \( C^k_R \), and an element \( D' \) of the kernel of \( C^k_{R'} \). We can consider \( D, D' \) as polynomials in \( R^\vee \), with degrees \( k, k' \) resp. Assume that \( D, D' \) have no common factors.

By the Apolarity Theorem 2.3 the complete intersection of \( D, D' \) determines a decomposition \( Z \) of \( F \), of length \( kk' \).

Since \( F' \) is strictly bigger than the ideal generated by \( D, D' \), for \( F' \) is artinian, one can have decompositions \( Z' \) of \( F \) which are different, even disjoint, from \( Z \).

2.3. Hilbert functions. Let \( Z \) be a finite, reduced subset of length \( r \) in \( \mathbb{P}^n \). Fix a set of representatives for the coordinates points of \( Z \). The evaluation of polynomials at the chose representatives provides for any degree \( d \) a linear map \( \rho_r : R_d \rightarrow \mathbb{C}^r \) whose image has dimension which depends only on \( Z \) and not on the choice of the representatives. The Hilbert function of \( Z \) is the map

\[
 h_Z : \mathbb{Z} \rightarrow \mathbb{Z} \quad h_Z(d) = \dim \text{Im}(\rho_d).
\]

We will often use the difference \( Dh_Z(d) = h_Z(d) - h_Z(d-1) \), and the h-vector of \( Z \), which is the \( t \)-uple of non-zero values of \( Dh_Z \).

Clearly \( h_Z(i) = Dh_Z(i) = 0 \) when \( i \) is negative. Also \( h_Z(0) = Dh_Z(0) = 1 \). Moreover \( h_Z(d) \) is non decreasing, and \( h_Z(d) = r \) for all large \( d \). Consequently \( Dh_Z \) is non-negative and \( Dh_Z(d) = 0 \) if \( d \) is sufficiently large.

Other standard properties of \( h_Z \) and \( Dh_Z \) are known in the literature, and are collected e.g. in [11]. We list the most relevant of them.

a1) for all \( d > 0 \), \( \sum_{i=0}^{d} Dh_Z(i) = h_Z(d) = r \);
a2) if \( Dh_Z(i) = 0 \) for some \( i > 0 \), then \( Dh_Z(j) = 0 \) for all \( j \geq i \);
a3) if \( Z' \subset Z \) then for all \( i \), \( h_{Z'}(i) \leq h_Z(i) \) and \( Dh_{Z'}(i) \leq Dh_Z(i) \).
a4) if \( Z \) is contained in a plane curve of degree \( q \), then \( Dh_Z(i) \leq q \) for all \( i \).

The application of Hilbert functions to tensor analysis is based on the trivial observation that \( h_Z(1) < \text{min}\{n+1, r\} \), i.e. the evaluation map in degree 1 is not of maximal rank, if and only if \( Z \) is linearly dependent and fails to span \( \mathbb{P}^n \). This implies that when \( \binom{n+d}{n} \geq r \), then \( h_Z(d) < r + 1 \) if and only if \( v_d(Z) \) is linearly dependent.

It follows that if \( A, B \) are two different, non redundant decompositions of the same form \( F \in R_d \), and we take \( Z \) to be the union \( Z = A \cup B \), then \( h_Z(d) < r \), so that

b0) \( Dh_Z(d+1) \geq 0 \).
The next properties of the union $Z = A \cup B$, which will be used in the sequel, follow from results in the theory of finite subsets in projective space, as the Cayley-Bacharach theorem, or the Macaulay Maximal Growth principle. For an account, we refer to [11] or [2].

**Proposition 2.5.** Let $A, B$ be two different, non-redundant decomposition of a form $F$ of degree $d$ in $n + 1$ variables. Put $Z = A \cup B$ and assume $A \cap B = \emptyset$.

b1) For all $j \leq d + 1$ one has
\[
\sum_{i=0}^{j} Dh_Z(i) \leq \sum_{i=d+1-j}^{d+1} Dh_Z(i).
\]

b2) The projective dimension of the intersection $\langle v_d(A) \rangle \cap \langle v_d(B) \rangle$ is equal to $(\sum_{i>d} Dh_Z(i)) - 1$. In particular, if $Dh_Z(d+1) = 1$ and $Dh_Z(d+2) = 0$, then the intersection contains only $F$.

b3) Assume $n = 2$, i.e. $F$ is a ternary form. If for some $0 < i < j$ one has $Dh_A(i) > Dh_Z(j) > 0$ then $Dh_Z(j+1) < Dh_Z(j)$.

2.4. **Ramification and the Terracini locus.** Following [6], Terracini loci can be defined as follows.

**Definition 2.6.** The Terracini locus $T_r(X)$ of a variety $X \subset \mathbb{P}^N$ is the closure of the locus in $X^{(r)}$ of subsets $A$ of cardinality $r$ of the regular part $X_{\text{reg}}$, such that the span of the tangent spaces to $X$ at the points of $A$ has dimension smaller than the expected value $r(n+1) - 1$.

The locus $T_r(X)$ is connected to the structure of the natural map (projection) from the abstract secant variety $AS^r(X)$ and $S^r(X)$,
\[
s^r : AS^r(X) \to S^r(X).
\]

Given a linearly independent finite set $Y \subset X_{\text{reg}}$, by Terracini’s Lemma and its proof the span of the tangent spaces to $X$ at the points of $Y$ correspond to the image of the differential of $s^r$. Since for a general $P \in \langle Y \rangle$, $AS^r(X)$ is smooth of dimension $r(n+1) - 1$ at $(P, Y) \in AS^r(X)$, then $Y$ belongs to $T_r(X)$ if and only if the linear span $\langle Y \rangle$ is contained in the ramification locus of the map $s^r$.

3. **Plane sextics**

In this section we will analyze in details the situation of forms of degree 6 in three variables, i.e. sextic plane curves. We will see how the study of the catalecticant map determines several loci in the secant varieties of the surface $X = v_6(\mathbb{P}^2)$.

**Remark 3.1.** Sextics in three variables are parameterized by $\mathbb{P}(\text{Sym}^6(\mathbb{C}^3)) = \mathbb{P}^{27}$. By [1], the (Waring) rank of a general ternary sextic is 10. Sextics of (border) rank 9 determine a hypersurface in the space of sextics: the 9-secant variety to the 6-Veronese of $\mathbb{P}^2$. Moreover, by [11] and by [13], a general form of rank 9 has exactly two minimal decompositions.

Sextics of rank 8 determine an irreducible subvariety of dimension 23 in $\mathbb{P}^{27}$, i.e. the 8-secant variety to $v_6(\mathbb{P}^2)$. By [13], a general sextic of rank 8 has a unique decomposition of length 8.
Following [24], let us look what happens to the catalecticant map of order three applied to a sextic $F$ of rank $<10$. We will focus mainly on the cases in which $F$ has rank either 8 or 9.

We start by recalling the following, elementary fact.

**Remark 3.2.** Let $C$ be a form in the kernel of $\mathcal{C}^r_F$. Then $C$ corresponds by duality to a form in $R$ (that we continue to denote as $C$), and its coefficients correspond to the orthogonal to a hyperplane in $\mathbb{P}(R^3)$ that contains $P^r_F$. Since the Veronese map and the duality are both given in terms of the monomial basis, then the intersection of $P^r_F$ with $v_k(\mathbb{P}^2)$ corresponds to a divisor that defines $v_k$, i.e. to a curve of degree $k$ in $\mathbb{P}^2$, which is exactly $C$. Thus $P^r_F \cap v_k(\mathbb{P}^2) = v_k(C)$.

**Proposition 3.3.** If $F$ has rank $r < 10$ then $\dim(P^r_F) \leq r - 1$. If $\dim(P^r_F) = 8$ and the kernel of $\mathcal{C}^3_F$ is generated by an irreducible form $C$, then $F$ has border rank at most 9, and rank $\geq 9$.

**Proof.** If we take a minimal decomposition $A$ of $F$, then the ideal of $A$ certainly contains $10 - r$ independent cubics. Thus from Theorem 2.1 we know that the kernel of $\mathcal{C}^3_F$ has dimension at least $10 - r$, so that $\dim(P^3_F) \leq r - 1$.

Now assume that the dimension of $P^3_F$ is exactly equal to 8, so that $F$ cannot have rank smaller than 9. $P^3_F$ is then a hyperplane in $\mathbb{P}(\text{Sym}^3(C^3)) = \mathbb{P}^9$ which intersects $v_3(\mathbb{P}^2)$ in $v_3(C)$. Since $F^\perp$ certainly contains some form $C'$ independent from $C$, and by the Apolarity Theorem $C \cap C'$ is a decomposition of $F$, hence $F$ lies in the span of $v_6(C)$, which is the Veronese image of an irreducible plane cubic of arithmetic rank 1. Since irreducible curves are never defective, then $F$ has border rank at most 9 with respect to $v_6(C)$, hence also with respect to $v_6(\mathbb{P}^2)$. \hfill \Box

Notice that not every general set of 9 points in $C$ determines a decomposition of $F$. So, it is not sufficient that the span of $v_3(A)$ contains $P^3_F$ for $A$ to be a decomposition of $F$.

**Example 3.4.** Let $F$ be a sextic of rank 8. Since 8 points always lie in a pencil of cubic curves, then $\dim(P^5_F) \leq 7$. Assume that $P^3_F$ has dimension exactly 7 and look at the intersection of $P^3_F$ with the 3-Veronese surface $S = v_3(\mathbb{P}^2)$. Since $\deg(S) = 9$, it is clear that $P^3_F$ intersects $S$ either in a curve $\Gamma$, or in a scheme of length 9.

The former case cannot happen. Namely the curve $\Gamma$ would be the image in $v_3$ of a plane curve whose ideal in degree 3 has linear dimension 1, but no such plane curves exist.

Thus $Z = P^3_F \cap S$ is a subscheme of length 9. It is easy to see ( [24], Section 4) that for a general choice of the form $F$ the scheme $Z$ is reduced, hence it consists of 9 distinct points.

It follows that any decomposition of length 8 of $F$ is contained in $Z$, for any set of length 8 sits in two independent cubics, and by the Apolarity Theorem these two cubics are in the kernel of $\mathcal{C}^3_F$.

When $Z$ is reduced, it consists of a minimal decomposition, plus one extra point.

**Proposition 3.5.** Let $F$ be a sextic ternary form such that $\dim(P^3_F) = 7$ and $Y = P^3_F \cap S$ is reduced. Assume that $F$ has rank 8. Then there exists only one decomposition of $F$ of length 8, i.e. $F$ is identifiable, in the sense of [13].
Lemma 3.7. (Intersection Lemma) Let $F \in R_d$ be a form with two non-redundant decompositions $A, B$, of length $\ell(A) \geq \ell(B)$. Assume $A \cap B \neq \emptyset$. Then there exists a form $F' \in R_d$ with two non-redundant decompositions $A' \subset A, B' \subset B$ such that $A' \cap B' = \emptyset$ and $\ell(A') \geq \ell(B')$, and moreover either $\ell(A') > \ell(B')$ or $A' \neq A$.

Proof. Let $A = \{L_1, \ldots, L_r\}$ and $B = \{L_1, \ldots, L_j, M_{j+1}, \ldots, M_r\}$ with $L_i \notin A$ for $i = j+1, \ldots, r'$ ($r' = \ell(A), r'' = \ell(B)$). Thus $A \cap B = \{L_1, \ldots, L_j\}$ and $j > 0$. Write

$$F = a_1 L_1^d + \cdots + a_j L_j^d + a_{j+1} L_{j+1}^d + \cdots + a_r L_r^d$$

and

$$F = b_1 L_1^d + \cdots + b_j L_j^d + b_{j+1} M_{j+1}^d + \cdots + b_{r'} M_{r'}^d.$$

In conclusion, we get

Corollary 3.6. Let $F$ be a sextic ternary form such that $\dim(P^3_F) = 7$ and $Y = P^3_F \cap S$ is reduced. Then $Z = v_3^{-1}(Y) \subset P^2$ is a complete intersection of two cubics, and either

- $F$ has rank 8 and a unique decomposition of length 8, contained in $Z$; or
- $F$ has rank 9, and $Z$ is a minimal decomposition of $F$, i.e. there exists a minimal decomposition of $F$ which is a complete intersection of 2 cubics.

An algorithm that can guarantee that a ternary form $F$ of degree 6 has rank 8, and find a decomposition, is the following (see [24]).

Compute $\dim(P^3_F)$. If $\dim(P^3_F) > 7$, then $F$ cannot have rank 8. Assume that $\dim(P^3_F) = 7$ (this will be true for general forms of rank 8).

Compute two generators $D, D'$ of the kernel of $C^3_F$ and compute the coordinates of the points of the complete intersection $Z$ of $D$ and $D'$ (this can require approximation). Assume that $Z$ has length 9 (this will be true for general forms of rank 8).

Then $Z = \{L_1, \ldots, L_9\}$. If we identify each $L_i$ with a linear form in $R$, then there exists a linear combination $F = a_1 L_1^d + \cdots + a_9 L_9^d$ and we know that the linear combination is unique, for $v_6(Z)$ is linearly independent. Compute the $a_i$'s.

By Proposition 3.5, $F$ has rank 8 if and only if one coefficient $a_i$ is 0. In this case, by dropping the summand $a_i L_i^d$, we also obtain the unique decomposition of length 8 of $F$.

It is known that since a general point in $\mathbb{P}^{17}$ has exactly two different decompositions with respect to an elliptic normal curve, then a general sextic of rank 9 has exactly two minimal general decompositions.

A geometric way to find the second decomposition, once one of them is known, can be described in terms of liaison.

We need first a series of preparatory results on the interaction between Hilbert functions and decompositions.
Let \( r \) be a sextic curve. If \( F' \) has \( 5 \) aligned points, a contradiction. Thus \( \mathbf{h} \)-vector of \( A \) is non-redundant since \( b_j \neq 0 \) and \( \mathbf{v}(B') \) is linearly independent. Let \( A' = A \setminus \{L_i : i \leq j \text{ and } a_i = b_i\} \). Then also \( A' \) is a decomposition of \( F' \), which is non-redundant since \( \mathbf{v}(A') \) is linearly independent and \( a_j = 1, \ldots, a_r \neq 0 \). \( A' \) and \( B' \) are clearly disjoint, and \( \ell(A') \geq r - j \geq r' - j = \ell(B') \). Finally, if \( A' = A \) then \( \ell(A') = r > r' - j \).

The next result tells us where we must look, geometrically, in order to find minimal decompositions of a sextic form.

We give a long proof in which we analyze one by one the several cases, just because we want to stress the use of the Hilbert function for tensor analysis. Indeed, some steps could be shortened by using [5] or Section 3 of [2]. See also [23] for an approach to the problem.

**Proposition 3.8.** Let \( F \) be a ternary sextic with two different non-redundant decompositions \( A, B \), with length \( \ell(A) \leq 9 \) and \( \ell(B) \leq \ell(A) \). Assume that no 4 points of \( A \) are aligned, and that \( A \) does not lie in a conic.

Then \( \ell(B) = \ell(A) = 9 \) and \( A \cap B = \emptyset \).

Furthermore \( A \cup B \) lies in a cubic curve \( C \), and it is complete intersection of \( C \) and a sextic curve.

**Proof.** We set \( r = \ell(A), r' = \ell(B) \) and we make induction on \( r, r' \). Put \( Z = A \cup B \) and notice that \( \mathbf{Dh}(7) > 0 \), by b0).

If \( r \leq 3 \), for any \( r' \) we have a contradiction with [9] Corollary 2.2.1.

First assume that \( A, B \) are disjoint.

Let \( r = 4, 5 \). Since four points of \( A \) are not aligned, in both cases \( \mathbf{Dh}(3) = 0 \), which is non-redundant, and \( \mathbf{Dh}(7) > 0 \), by b2).

Assume \( r = 6 \), so that \( \ell(Z) \leq 12 \). Since \( A \) is not aligned, then \( \mathbf{Dh}(1) = 2 \). If \( \mathbf{Dh}(2) \leq 1 \), then \( A \) sits in two independent conics, which, by Bezout, is possible only if \( A \) has 5 aligned points, a contradiction. Thus \( \mathbf{Dh}(2) \geq 2 \), so that the \( \mathbf{h} \)-vector of \( A \) is either \( (1, 2, 3) \), or \( (1, 2, 2, 1) \). In the former case, arguing as in cases \( r = 4, 5 \), we get \( \mathbf{Dh}(3) = 0 \), a contradiction. In the latter case, we have \( \sum_{i=0}^{3} \mathbf{Dh}(i) \geq 6 \), thus by b1) also \( \sum_{i=4}^{7} \mathbf{Dh}(i) \geq 6 \). Since \( \ell(Z) \leq 12 \), we must have \( \sum_{i=0}^{3} \mathbf{Dh}(i) = 6 \) which, by a2), implies \( \mathbf{Dh}(i) = \mathbf{Dh}(A) \) for \( i = 0, \ldots, 3 \). Then \( \mathbf{Dh}(3) = 1 \), so that, by b3), \( \mathbf{Dh}(4) = 0 \), which is not consistent with b0).

Assume \( r = 7, 8 \). \( A \) cannot be contained in conics, so that \( \mathbf{Dh}(2) = 3 \). If \( \mathbf{Dh}(3) \leq 2 \), then by applying b3) for \( i = 4, 5 \) we see that \( \mathbf{Dh}(5) = 0 \), which contradicts b0).

Thus \( \mathbf{Dh}(3) \geq 3 \), so that \( \sum_{i=0}^{3} \mathbf{Dh}(i) \geq 9 \). Then by b1) also \( \sum_{i=4}^{7} \mathbf{Dh}(i) \geq 9 \), which is not consistent with \( \ell(Z) \leq 2r \).

So, we are left with the case \( r = 9 \). As above \( \mathbf{Dh}(2) = 3 \), and \( \mathbf{Dh}(3) \leq 2 \) yields a contradiction. Thus \( \mathbf{h} \)-vector of \( Z \) is not consistent with \( (1, 2, 3, \ldots, q) \) with \( q = \mathbf{Dh}(3) \geq 3 \). Then \( \sum_{i=0}^{3} \mathbf{Dh}(i) = 6 + q \), so that by b1) \( \sum_{i=0}^{7} \mathbf{Dh}(i) \geq 6 + q \).
Since \( \ell(Z) \leq 18 \), it follows \( q = 3 \). This means that \( \ell(Z) = 18 \), i.e. \( \ell(B) = 9 \), and moreover \( h_2(3) = 9 \), so \( Z \) lies in a cubic curve.

By Proposition 5.6 one computes that the unique possibility for the h-vector of \( Z \) is \((1, 2, 3, 3, 3, 2, 1)\). By \cite{3} Lemma 5.3 we know that \( Z \) has the Cayley-Bacharach property. Since the h-vector of \( Z \) is the same than the h-vector of a complete intersection of type \((3, 6)\), by \cite{10} \( Z \) is complete intersection of a cubic and a sextic curve. Then \( A \) and \( B \) are linked by a cubic and a sextic curve. Finally, the case where \( A \cap B \neq \emptyset \) implies, by Lemma 3.7 the existence of a sextic \( F' \) with non-redundant decompositions \( A' \subset A \) and \( B' \), such that \( \ell(B') \leq \ell(A') \), and either \( \ell(A') \leq 8 \) or \( \ell(B') \leq 8 \). The existence of \( F' \) is excluded by the previous argument. \( \square \)

Given a general sextic \( F \) of rank 9, and a decomposition \( A \) of \( F \), we can find a second decomposition \( B \) of \( F \) with a procedure introduced e.g. in \cite{2}, Section 4.

**Remark 3.9.** Let \( F \) be a sextic of rank 9, with a decomposition \( A \) of length 9.
From the kernel of \( C_F^3 \), which has dimension 1 is \( F \) is general, we find a cubic \( C \) containing \( A \). A resolution of the homogeneous ideal of \( A \) is given by

\[
0 \to R(-5)^3 \to R(-4)^3 \oplus R(-3) \to I_A \to 0.
\]

\( M \) is the Hilbert-Burch matrix of \( A \), whose minors provide generators for \( I_A \). The degrees of \( M \) are

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
2 & 2 & 2
\end{pmatrix}.
\]

We know that \( A \cup B \) is complete intersection of \( C \) with a sextic. The homogeneous ideal of the residue \( B \) of \( A \) in the intersection is obtained by erasing the bottom row of \( M \) and adding a column of three quadrics. The maximal minors of the matrix \( M' \) that we obtain generate the homogeneous ideal of a scheme \( B \) linked to \( A \) in a complete intersection \((3, 6)\).

The previous procedure produces a set \( B \) which decomposes some form \( F' \in v_0(A) \).
In other words, the intersection \( \langle v_6(A) \rangle \cap \langle v_6(B) \rangle \) is a form \( F' \) in \( v_6(A) \) which depends on the choice of the three quadrics in \( M' \).

One can find \( F' \) by taking the sum \( I_A + I_B \), which determines in \( R_6 \) a linear subspace \( H \) of codimension 1. The coefficients of \( F' \) form a vector orthogonal to \( H \).

Thus, in order to find the second decomposition of the given \( F \), one needs to choose the three quadrics \( Q_1, Q_2, Q_3 \) in \( M' \) so that the orthogonal to \( H \) matches with the coefficient of \( F \). In practice, one needs to solve a linear system whose matrix has entries linear in the coefficients of the \( Q_i \)'s. A convenient way to write this linear system into \( M_2 \) \cite{15} is to ask that the minors of \( M' \) are apolar to \( F \).

With a procedure similar to the proof of Proposition 3.8, we can prove the following property of sextics of rank 9 with a decomposition \( A \) complete intersection of two cubics.

**Proposition 3.10.** Let \( F \) be a sextic ternary form such that \( \dim(P_F^3) = 7 \) and \( Y = P_F^3 \cap S \) is reduced. Assume that \( F \) has rank 9. Then \( A = v_5^{-1}(Y) \) is the unique decomposition of \( F \) of length 9, i.e. \( F \) is identifiable, in the sense of \cite{13}.

**Proof.** From Proposition 3.8 we know that \( A \) is a complete intersection of two cubics. In particular, no 4 points of \( A \) are aligned and \( A \) does not lie in a conic.
The h-vector of \( A \) is \( (1, 2, 3, 2, 1) \). Assume that there exists another decomposition \( B \) of length 9. Then by Proposition 3.10, we know that \( A \cap B = \emptyset \) and \( Z = A \cup B \) lies in a cubic, so that the h-vector of \( Z \) starts with \((1, 2, 3, 3, \ldots)\). Since \( Z \) lies in a plane curve of degree 3, by \( a4) \) \( Dh_Z(i) \leq 3 \) for all \( i \).

Next, we claim that \( Dh_Z(4) = 3 \). Indeed if \( Dh_Z(4) \leq 2 \) then by \( b3) \) \( Dh_Z(5) \leq 1 \) and \( Dh_Z(6) = 0 \), which contradicts \( b0) \). If \( Dh_Z(4) \geq 4 \), then \( \sum_{i=5}^{7} Dh_Z(i) < 6 \), which contradicts \( b1) \). The same argument proves that \( Dh_Z(5) = 3 \), and \( Dh_Z(6) \leq 2 \). Then \( Dh_Z(6) + Dh_Z(7) = 3 \). Since by \( b3) \) \( Dh_Z(6) > Dh_Z(7) \geq 1 \), then the h-vector of \( Z \) is \((1, 2, 3, 3, 3, 2, 1)\).

By Lemma 5.3 we know that \( Z \) has the Cayley-Bacharach property \( \text{CB}(6) \). Since moreover the h-vector of \( Z \) is the same than the h-vector of a complete intersection of type \((3, 6)\), by \([16]\) \( Z \) is complete intersection of a cubic and a sextic curve. Then \( A \) and \( B \) are linked by a cubic and a sextic curve. This implies, by \([22]\) that the h-vector of \( B \) is \((1, 2, 3, 2, 1)\). Thus \( B \) lies in the base locus of a pencil of cubic curves. Since \( B \) is a decomposition of \( F \), and \( \dim(F^9) = 7 \), there is only one pencil of cubic curves that contain \( B \), and the base locus of the pencil is \( A \). Comparing the degrees, we get \( A = B \). \( \square \)

from a certain point of view, Proposition 3.10 is rather surprising. Namely, as explained in [4] Theorem 3.2, the 6-Veronese surface \( v_6(\mathbb{P}^2) \) is 9-weakly defective, in the sense of [12]: a general hyperplane which is tangent to \( v_6(\mathbb{P}^2) \) at 9 points is tangent along an elliptic normal curve \( C \). Thus, by [14] Theorem 2.9 and Proposition 5.2, \( F \) has two minimal decompositions of length 9, both lying in \( C \). Notice that, in general, none of the two decompositions lies in two independent cubic curves. If the form \( F \) has a decomposition \( Z \) of length 9 which is a complete intersections of two cubics, then there are infinitely many elliptic normal curves in \( v_6(\mathbb{P}^2) \) containing \( v_6(Z) \), so one may expect the existence of infinitely many decompositions of length 9 of \( F \), for any elliptic normal curves \( v_6(Z) \) could provide a second decomposition \( Z' \neq Z \) of \( F \). On the contrary, in this case the construction collapses and any elliptic normal curves provides only one decomposition for \( F \), namely \( A \) alone.

A motivations for the peculiar behavior of sextics with a decomposition complete intersection of two cubics can be explained in terms of the ramification of the map \( s^9 \) from the abstract secant variety \( \text{AS}^9(v_6(\mathbb{P}^2)) \) to \( S^9(v_6(\mathbb{P}^2)) \), and the Terracini locus of \( v_6(\mathbb{P}^2) \).

**Remark 3.11.** For \( X = v_6(\mathbb{P}^2) \), by Definition 2.3 a linearly independent set \( v_6(A) \) of length 9 belongs to \( \mathcal{T}_9(X) \) if the 9 tangent planes at the points of \( v_6(A) \) span a subspace of dimension \( \leq 25 \). When \( A \) is complete intersection of two cubics \( \Gamma_1, \Gamma_2 \), then certainly \( v_6(A) \) belongs to the Terracini locus. Indeed, there is a 2-dimensional family of sextic curves in \( \mathbb{P}^2 \) which are singular at the points of \( A \): the sextics formed by the union of two cubics in the pencil generated by \( \Gamma_1, \Gamma_2 \). Any sextic like that corresponds to the intersection of \( X \) with a hyperplane which contains the tangent planes to \( X \) at the points of \( v_6(A) \). Thus, the span of the tangent planes at the points of \( v_6(A) \) has codimension at least 2 in \( \mathbb{P}^N = \mathbb{P}^{27} \).

By Remark 3.1 the map \( s^9 : \text{AS}^9(X) \to \mathbb{P}^{27} \) is generically 2 : 1, when \( A \) is a general set of 9 points in the plane. By Proposition 3.10 we see that general points \( (F, v_6(A)) \) such that \( A \) is complete intersection of two cubics are in the ramification.
locus of $s^9$.

The last observation is consistent with the fact that the differential of $s^9$ degenerates at points $(F, v_6(A))$ such that $A$ is complete intersection of two cubics.

**Definition 3.12.** We denote with $\mathcal{R}$ the closure of the set of sextics with a decomposition formed by a set $A$ of nine points which are nodes of an irreducible sextic. By [19], $\mathcal{R}$ is irreducible.

We denote by $W$ the closure of the set of sextics with a decomposition formed by a set $A$ of nine points, complete intersection of two cubics.

**Remark 3.13.** The previous discussion proves that, in the new notation, $W$ properly contains $S^9(X)$, and both $W$ and $\mathcal{R}$ are contained in the Terracini locus $T_9(X)$, which is the closure of the image of the locus of smooth points in $AS^9(X)$ in which the differential of the map $s^9$ drops rank.

Indeed, we will show in the next section that $\mathcal{R}$ is the unique component of $T_9(X)$ which intersect the smooth locus of $S^9(X)$, while $W$, whose codimension is bigger than 1, sits into the singular locus of $S^9(X)$.

We will also prove in the next section that $W$ is not contained in $\mathcal{R}$, which means that the Terracini locus has at least two components.

**Remark 3.14.** For the 6-Veronese variety $X$ of $\mathbb{P}^2$, the Terracini locus $T_9(X)$ corresponds to sets $A$ of 9 points which are singular in a pencil of sextics. General sets in $T_9(X)$ are easy to describe (see Example 5.1 in [9]): just take $C$ to be a reduced elliptic plane sextic whose singularities are nodes. $C$ has a set $A$ of 9 nodes which also sit in a cubic curve $D$. All sextics in the pencil generated by $C$ and $D^2$ are singular at $A$. It follows that the 9 tangent planes to $X$ at the points of $v_6(A)$ span a linear space of dimension 25 in $\mathbb{P}^{27}$, the linear space spanned by $X$.

Notice that a $T_9(X)$ does not contain a general set of 9 points in the plane, and a general $A \in T_9(X)$ is not complete intersection of two cubics.

General forms contained in the variety $\mathcal{R}$ have only one decomposition with 9 summands. On the other hand, both $\mathcal{R}$ and $W$ contain subvarieties whose elements have many decompositions of length 9.

Namely, if $A$ lies in the Terracini locus $T_9(X)$, then the liaison procedure introduced in Remark 3.15 determine a second decomposition for some forms in $\langle v_6(A) \rangle$.

**Lemma 3.15.** Let $C$ be a general (smooth) cubic plane curve and let $A$ be a reduced divisor of degree 9 on $C$ such that $2A \notin |O_C(6)|$, but $A \notin |O_C(3)|$. Then there exists a sextic plane curve $G$ which is singular at the points of $A$ and irreducible.

**Proof.** Call $P_1, \ldots, P_9$ the points of $A$. For each $i$ call $\epsilon_i$ the double structure on $P_i$ contained in $C$. For each $P_i$ choose a scheme $\mu_i$ of length 2 in $\mathbb{P}^2$, supported at $P_i$ such that $\mu_i \notin C$. By assumption there exists a sextic $G'$ such that $G'$ does not contain $C$ and $\epsilon_i \in G'$ for all $i$. Thus, the linear system $\mathcal{M}$ of sextics in $\mathbb{P}^2$ which contain $\{\epsilon_1, \ldots, \epsilon_9\}$ has (projective) dimension 10. Note that $\mathcal{M}$ contains the double curve $2C$. Since $\epsilon_i \cup \mu_i$ has length 3, then every $\mu_i$ imposes at most one condition to curves in $\mathcal{M}$. Then there exists a sextic $G''$ which contains $\epsilon_i \cup \mu_i$ for all $i$ and $G''$ is different from $2C$. Since $G''$ contains $\epsilon_i \cup \mu_i$, then $G''$ is singular at the points of $A$. $G''$ cannot contain $C$, because $A \notin |O_C(3)|$. Thus a general curve $G$ in the pencil generated by $2C$ and $G''$ is irreducible, by Bertini. □

**Proposition 3.16.** Let $A$ be a set of 9 points in $\mathbb{P}^2$, which are nodes of a general reduced elliptic sextic curve $G$. The generality of $G$ implies that there exists a unique
cubic curve $C$ containing $A$, and moreover $C$ is smooth. Let $B$ be the residue of $A$ in the intersection of $C$ with a general sextic $S$ that passes through $A$. Then also $B$ is the set of nodes of a reduced sextic $C'$.  

**Proof.** The first claim is classical. Just to see an argument, observe that for a general set $D$ of 8 points in a general cubic curve $C$, the divisor $2D$ on $C$ lies in some divisor $D'$ of the linear series $|\mathcal{O}_C(6)|$. $D' - D$ consists of 2 points, that determine a linear series $\mathcal{L} = \mathcal{L}_2$. Choose a Weierstrass point $P_9$ of $\mathcal{L}$. Since $\mathcal{L}$ cannot have 8 Weierstrass points, then by monodromy for a general choice of $D$ we can assume that $P_9 \notin D$. Then $2D + 2P_9 \in |\mathcal{O}_C(6)|$ and then $D \cup \{P_9\}$ is contained in the singular locus of a sextic curve, by Lemma 3.15. Since the variety that parametrizes sextic plane curves with 9 singular points is irreducible, we see that on a general cubic curve we can find a set of 9 nodes of an irreducible sextic.

For the second claim, observe that on $C$ the divisor $2A$ belongs to $|\mathcal{O}_C(6)|$, and also the divisor $A + B$ belongs to $|\mathcal{O}_C(6)|$. Thus $2A + 2B$ belongs to $|\mathcal{O}_C(12)|$. Then also $2B \in |\mathcal{O}_C(6)|$, and the claim follows from the generality of $G, C, S$ and from Lemma 3.15.

In other words, the previous proposition says that some forms $F$ of rank 9 with a decomposition $A$ which lies in the Terracini locus $T_9(X)$, have a second decomposition $B$ of length 9 which also sits in the Terracini locus $T_9(X)$. Thus the subscheme of the abstract 9-secant variety which maps to $F$ is supported at two points, but it has length 4.

The closure $\mathcal{R}'$ of the locus of forms $F$ as above provides a proper subvariety of $\mathcal{R}$, whose geometry has non-trivial aspects.

**Proposition 3.17.** A general sextic in $\mathcal{R}'$ has infinitely many decompositions of length 9. Thus $\mathcal{R}'$ is the locus in $\mathcal{R}$ in which the dimension of fibers of the map $s^9 : \mathcal{A}^9(X) \to \mathcal{S}^9(X)$ jumps.

**Proof.** Fix a general cubic curve $C$ and a general set $A \subset C$ of length 9, such that there is an irreducible sextic $S$ singular at the points of $A$. For a general sextic $S'$ through $A$ the intersection $S' \cap C$ is a set $A \cup B$ of 18 points, with $B$ disjoint from $A$. As explained in Proposition 3.13, the linear spans $\langle \nu_6(A) \rangle$ and $\langle \nu_6(B) \rangle$ meet in a point $F \in \mathbb{P}^{27}$ which represents a point in $\mathcal{R}'$. Since any element of $\mathcal{R}' \cap \langle \nu_6(A) \rangle$ arises in this way, we get a surjective map form the projective space $\mathbb{P}$ over $(IA)_6$ mod multiples of $C$, to $\mathcal{R}' \cap \langle \nu_6(A) \rangle$. Since $A$ is separated by cubics, then $\mathbb{P}$ has dimension $27 - 19 = 8$. Since $\langle \nu_6(A) \rangle$ also has dimension 8, and $\mathcal{R}' \cap \langle \nu_6(A) \rangle$ is a proper subvariety, because $\mathcal{R} \neq \mathcal{R}'$, the claim follows.

The same procedure of liaison produces a special subvariety of $W$.

**Example 3.18.** Let $A$ be a set of 9 points, complete intersection of two cubics $C_1, C_2$ Fix a general cubic curve $C_3$ and consider the set $B = C_1 \cap C_3$. Then $A \cap B = \emptyset$, and $Z = A \cup B$ is a complete intersection of a cubic and a sextic curve. Thus the h-vector of $Z$ is $(1, 2, 3, 3, 3, 3, 2, 1)$. By Proposition 2.19, it follows that the spans $\langle \nu_6(A) \rangle$ and $\langle \nu_6(B) \rangle$ meet in exactly one point, corresponding to a sextic form $F$.

Since $F$ has a decomposition given by $C_1 \cap C_2$, then $F$ corresponds to a point in the 9-secant variety of $X$ spanned by a set in the Terracini locus. Moreover, from the Apolarity Theorem, $C_1, C_2$ lie in the apolar ideal of $F$. For the same reason, looking at the decomposition $C_1 \cap C_3$ of $F$, one obtains that also $C_3$ lies in $F^\perp$. 

Thus we have an example of a form $F$ whose apolar ideal contains 3 independent cubics $C_1, C_2, C_3$. Notice that, by the generality of $C_3$, the intersection $C_1 \cap C_2 \cap C_3$ is empty. Thus the image of $C^3_3$, which is a $\mathbb{P}^6$, will not cut the 3-Veronese surface in $\mathbb{P}^9$. In particular, $F$ has not rank 7.

By the Apolarity Theorem, a decomposition of $F$ can be found by taking any general pair of cubics in the linear system spanned by $C_1, C_2, C_3$. Thus $F$ has a 2-dimensional family of decompositions.

**Definition 3.19.** We denote with $W'$ the closure of the locus of sextic forms $F$ such that the catalecticant map $C^3_3$ has a kernel of dimension 3.

A general choice of three cubics $C_1, C_2, C_3$ determines $F \in W'$, by taking the intersection of the spans of $v_0(C_1 \cap C_2)$ and $v_0(C_1 \cap C_3)$ (any general choice of two pairs will be suitable and determine the same $F$). Thus $W'$ has the dimension of the Grassmannian of planes in $\mathbb{P}^9$, i.e. 21.

We stress that there is no way to determine if $F$ belongs to $W'$ only by looking at the 9 projective points of a decomposition of $F$. Once one knows that $F$ has a decomposition $A$ which is complete intersection, and one fixes representatives $\{L_1, \ldots, L_9\}$ for the points of $A$, then the membership of $F$ in $W'$ depends on the coefficients of a linear combination of powers $L_1^p, \ldots, L_9^p$ that determine $F$ in $\langle v_0(A) \rangle$.

Observe that the locus $S^7(X)$ lies in the intersection of $S^8(X)$ with $W'$. We will see in the next section that, at least set-theoretically, $S^7(X) = S^8(X) \cap W'$.

**Example 3.20.** Many forms $F$ in the subvariety $W'$ defined above can be computed following the procedure introduced in Section 4 of [2], as explained in Remark 3.9. Just to see an example of such a form, consider

$$F = (x_0x_1x_2)^2.$$ 

One easily sees that $F^\perp$ contains three independent cubics, which are $x_0^3, x_1^3, x_2^3$.

We have the explicit decomposition of rank 9 [8]

$$810(x_0x_1x_2)^2 = \sum_{p,q=0}^2 e^{2\pi i (p+q)/3} (x_0 + e^{2\pi ip/3}x_1 + e^{2\pi i q/3}x_2)^6.$$ 

In the next section we will compute equations for the 8-th secant variety $S^8(X)$. By applying these equations to $F$, one realizes that $F$ does not belong to $S^8(X)$. Thus $F$ has rank (and border rank) 9. This proves that $W'$ is not contained in $S^8(X)$.

The relations among the subvarieties $\mathcal{R}, \mathcal{R}', W, W'$ of $S^9(X)$ reflect the rich geometry of secant varieties, as soon as the genus approaches the generic value. The most complicated object to describe remains $\mathcal{R}'$, for which we do not have a set of equations. We hope that the geometric structure of $\mathcal{R}'$ will be clarified, in a future footnote, maybe.

4. **Equations for loci in $S^9$.**

We introduce a Young flattening to get a more refined study of plane sextics. Let $T$ be the tangent bundle of $\mathbb{P}^2$, we consider the rank three bundle $E = \text{Sym}^2 T$. ...
The space of sections $H^0(\Sym^2 T)$ is 27-dimensional and can be identified with the $SL(3)$-module $\Gamma_{4,2}^2 \C^3$, corresponding to the Young diagram

\[ \begin{array}{ccc} & & \\
& & \\
& & \\
& & \\
\end{array} \]

A presentation of $\Sym^2 T$ can be described as follows. First we recall that $T$ is presented by the following exact sequence

$$0 \to O \to \C^3 \otimes O(1) \to T \to 0$$

which dualizes to

$$0 \to T^\vee \to \C^3 \otimes O(-1) \to O \to 0$$

and since $T^\vee = T(-3)$ we get

$$0 \to T \to \C^3 \otimes O(2) \to O(3) \to 0$$

The first and third sequence fit together into the presentation of $T$

$$\begin{array}{ccc}
\C^3 \otimes O(1) & \xrightarrow{f} & \C^3 \otimes O(2) \\
\downarrow & & \downarrow \\
T & & T
\end{array}$$

where the horizontal skew-symmetric map $f(v) = v \wedge x$ is given by the matrix

$$
\begin{pmatrix}
0 & x_2 & -x_1 \\
-x_2 & 0 & x_0 \\
x_1 & -x_0 & 0
\end{pmatrix}
$$

The second symmetric power $\Sym^2 T = E$ appears in the exact sequence

$$0 \to \C^3 \otimes O(1) \to \Sym^2 \C^3 \otimes O(2) \to \Sym^2 T \to 0$$

and repeating the above argument, thanks to the identification $\Sym^2 T^\vee = \Sym^2 T(-6)$ we get the presentation

$$\begin{array}{ccc}
\Sym^2 \C^3 \otimes O(2) & \xrightarrow{\Sym^2 f} & \Sym^2 \C^3 \otimes O(4) \\
\downarrow & & \downarrow \\
\Sym^2 T & & \Sym^2 T
\end{array}$$

where the horizontal symmetric map $\Sym^2 f(v^2) = (v \wedge x)^2$ is given by the matrix

\begin{equation}
B = 
\begin{pmatrix}
0 & 0 & 0 & z_2^2 & -2z_1z_2 & z_1^2 \\
0 & -2z_2^2 & 2z_1z_2 & 0 & 2z_0z_2 & -2z_0z_1 \\
0 & 2z_1z_2 & -2z_1^2 & -2z_0z_2 & 2z_0z_1 & 0 \\
z_2^2 & 0 & -2z_0z_2 & 0 & 0 & z_0^2 \\
-2z_1z_2 & 2z_0z_2 & 2z_0z_1 & 0 & -2z_0^2 & 0 \\
z_1^2 & -2z_0z_1 & 0 & z_0^2 & 0 & 0
\end{pmatrix}
\end{equation}

Note that for $L = O(6)$ we have $E = E^\vee \otimes L$ hence, as in [21] we have a contraction map

\begin{equation}
\begin{array}{ccc}
End(\Sym^2 \C^3) & \xrightarrow{P_1} & End(\Sym^2 \C^3) \\
\downarrow & & \downarrow \\
H^0(E) & \xrightarrow{A_1} & H^0(E^\vee \otimes L)^\vee
\end{array}
\end{equation}
where the horizontal map $P_f$ for $f = v^6 \in v_6(P^2)$ is defined as

$$P_{v^6}(M^2)(w^2) = (M(v) \wedge v \wedge w)^2v^2 \quad \forall M \in \text{End}(\mathbb{C}^3), w^2 \in \text{Sym}^2\mathbb{C}^3$$

where $M^2 \in \text{End}(\text{Sym}^2\mathbb{C}^3)$ is defined by $M^2(v^2) = (M(v))^2$ and then extended by linearity to any $N \in \text{End}(\text{Sym}^2\mathbb{C}^3)$ and to any plane sextic $f$. Comparing with \cite{25, §2} we see that the invariant $\det A_f$ of degree 27 of plane sextics has a construction similar to the Aronhold invariant of degree 4 of plane cubics.

The coordinate description of $P_f$ is the following. Differentiate the $6 \times 6$ catalecticant $C(f)$ (given by differentiating on rows and columns by monomials of degree 2) by $B$ in (1). The output is a $36 \times 36$ matrix obtained by replacing any entry of $B$ by a $6 \times 6$ catalecticant block.

The M2 commands to get $P_f$ are the following, after the matrix $B$ has been defined as in (1)

\begin{verbatim}
R=QQ[z_0..z_2]
f=z_0^3+z_1^3+z_2^3+5*z_0*z_1*z_2---any cubic polynomial
P=diff(transpose basis(2,R),diff(basis(2,R),diff(B,f)))
\end{verbatim}

The map $A_f$ is symmetric and could be obtained by a convenient submatrix of $P_f$. An alternative way to get a coordinate description is to employ the M2 package "PieriMaps" by Steven Sam with the M2 commands \cite{18}

\begin{verbatim}
loadPackage "PieriMaps"
pieri({6,4,2},{1,1,2,2,3,3},QQ[x_0..x_2]);
\end{verbatim}

although the symmetry is not transparent from the coordinates chosen by the system.

The contraction $A_f$ in (2) can be pictorially described (compare with \cite{25}) by

\begin{align*}
\begin{array}{|c|c|c|c|c|c|}
\hline
& & & & & \\
\hline
& & & & & \\
\hline
& & & & & \\
\hline
\end{array}
\otimes
\begin{array}{|c|c|c|c|c|c|}
\hline
& & & & & \\
\hline
& & & & & \\
\hline
& & & & & \\
\hline
\end{array}
\rightarrow
\begin{array}{|c|c|c|c|c|c|}
\hline
& & * & & & \\
\hline
& & * & & & \\
\hline
& & * & & & \\
\hline
\end{array}
\simeq
\begin{array}{|c|c|c|c|c|c|}
\hline
& & & & & \\
\hline
& & & & & \\
\hline
& & & & & \\
\hline
\end{array}
\end{align*}

We have the decomposition

$$\text{End}(\text{Sym}^2\mathbb{C}^3) = \Gamma^{4,2}\mathbb{C}^3 \oplus \Gamma^{2,1}V \oplus \wedge^3\mathbb{C}^3$$

corresponding to

\begin{align*}
\begin{array}{|c|c|c|c|c|c|}
\hline
& & & & & \\
\hline
& & & & & \\
\hline
& & & & & \\
\hline
\end{array}
\otimes
\begin{array}{|c|c|c|c|c|c|}
\hline
& & & & & \\
\hline
& & & & & \\
\hline
& & & & & \\
\hline
\end{array}
= \begin{array}{|c|c|c|c|}
\hline
& & & \\
\hline
& & & \\
\hline
& & & \\
\hline
\end{array}
\oplus
\begin{array}{|c|c|c|c|}
\hline
& & & \\
\hline
& & & \\
\hline
& & & \\
\hline
\end{array}
\oplus
\begin{array}{|c|c|c|c|}
\hline
& & & \\
\hline
& & & \\
\hline
& & & \\
\hline
\end{array}
\end{align*}

Note that the last two summands correspond to $\text{End}(\mathbb{C}^3)$.

If $f \in v_6(P^2)$ then $rk(A_f) = rk(E) = 3$, so that if $f \in S^9(v_3(P^2))$ then $rk(A_f) \leq 3 \cdot 8 = 24$ and in particular $\det A_f = 0$. This was observed already in \cite[Theorem 4.2.9]{21}.

**Proposition 4.1.** (1) The variety $W$ of sextics in $S^9(v_3(P^2))$ such that their Waring decomposition comes from a complete intersection of two cubics is cut by the 9-minors of $\mathbb{C}^3$. It has codimension 3 and degree 165 and it coincides with the singular locus of $S^9(v_3(P^2))$.\]
(2) The secant variety $S^8(v_3(\mathbb{P}^2))$ coincides with the reduced structure on $W \cap V(\det A_f)$, it has codimension 4 and degree $1485 = 165 \cdot 9$.

Proof. First recall that the variety of symmetric $10 \times 10$ matrices of rank $\leq 8$ is irreducible of codimension 3 and degree 165 by Segre formula \[27\]. Since the variety $C_9$ given by the 9-minors of $C^3$ has again codimension 3, being a linear section of the previous one, it has the same degree 165. This variety $C_9$ corresponds to the row with $s = 8$ of Table 3.1 in \[20\], indeed with the notations of \[20\], it is the union of irreducible strata $\text{Gor}(T)$ for several Hilbert functions $T$, and one may check using Conca-Valla formula (\[15\] or \[20\] Theorem 4.26) that only one strata

is irreducible and contained in the variety given by the 9-minors of $C^3$. We know that $W$ is irreducible and contained in the variety given by the 9-minors of $C^3$, since the two varieties have the same dimension, the equality follows, proving i). The claim about the singular locus follows because by direct computation on a random linear subspace the singular locus of $S^9(v_3(\mathbb{P}^2))$, which obviously contains $W$, has codimension 3 and degree 165.

In order to prove ii), recall that if $f \in S^8(v_3(\mathbb{P}^2))$ then $\det(A_f) = 0$. Pick 9 general points $p_i$ for $i = 0, \ldots, 8$ and consider the linear span $f = \sum_{i=0}^{8} k_i p_i^6$ with coefficients $k_i$, in other words we restrict $f$ to a general 9-secant space. The expression $\det A_f$ is symmetric in $k_i$, has total degree 27 and vanishes when $\prod_{i=0}^{8} k_i$ vanishes. An explicit computation with the coordinate expression found by the package “PieriMaps” as described above shows that $\det(A_f)$ coincides, up to scalar multiples, with $\left(\prod_{i=0}^{8} k_i\right)^3$. We get that the condition that $\det(A_f)$ vanishes on a general 9-secant space is equivalent to $f$ being of rank 8. This expression is unchanged if $f \in W$, namely if $f$ belongs to a 9-secant space corresponding to a complete intersection of two general cubics. This computation shows that $V(\det A_f)$ meets $W$ with multiplicity 3 at a general point of the intersection, so that the reduced structure on $W \cap V(\det A_f)$ has degree $1485 = 165 \cdot 9$. Note the rank of $A_f$ drops by 3 on $W$ the degree of $S^8(v_3(\mathbb{P}^2))$ can be computed with Numerical Algebraic Geometry. The M2 package NumericalImplicitization by Cho and Kileel shows indeed that its degree is 1485, which can be certified by the trace test. Since $S^8(v_3(\mathbb{P}^2)) \subseteq (W \cap V(\det A_f))_{\text{red}}$, this proves ii) \[\square\]

Remark 4.2. ii) of Prop. 4.1 strengthens \[21\] Theorem 4.2.9] where the 25-minors of $A_f$ were considered.

We wish now to find the ramification locus of the $2:1$ map $s^0 : AS^9(v_6(\mathbb{P}^2)) \to S^9(v_6(\mathbb{P}^2))$. Let $p_1, \ldots, p_{10}$ be points in $\mathbb{P}^2$. Let $C(p_1, \ldots, p_{10})$ be a multihomogeneous polynomial of degree 3 in the coordinates of the points $p_1, \ldots, p_{10}$, skew-symmetric with respect to them and that vanishes if and only if $p_1, \ldots, p_{10}$ are on a cubic. A coordinate expression is given by the $10 \times 10$ determinant such that its $i$-th row is the evaluation on $F_i$ of the monomial basis of $\text{Sym}^3 C^3$.

We construct now the Terracini matrix of the points $p_1, \ldots, p_9$ which defines the tangent space at $S^3(v_6(\mathbb{P}^2))$ at any point in $(p_1^6, \ldots, p_9^6)$. Consider the $3 \times 28$ Jacobian matrix $J$ of the monomial basis of $\text{Sym}^6 C^3$. Let $J(p_i)$ be the evaluation of $J$ at $p_i$ and let $T(p_1, \ldots, p_9)$ be the $27 \times 28$ Terracini matrix obtained by stacking $J(p_1), \ldots, J(p_9)$. Let $R(p_1, \ldots, p_9; p_{10}) = \det \left( \frac{T(p_1, \ldots, p_9)}{(p_{10})^3} \right)$.
be the determinant of the $28 \times 28$ matrix obtained by stacking $T(p_1, \ldots, p_9)$ and
the monomial basis of $\text{Sym}^6 \mathbb{C}^3$ evaluated at $p_{10}$.

$R(p_1, \ldots, p_9; p_{10})$ is a multihomogeneous polynomial of degree 15 in the coordi-
nates of the points $p_1, \ldots, p_9$, of degree 6 in the coordinates of $p_{10}$, skew-symmetric
with respect to $p_1, \ldots, p_9$ and that vanishes if and only if $p_{10}$ lies on a sextic singular
at $p_1, \ldots, p_9$.

Since $C(p_1, \ldots, p_{10})^2$ is a (non reduced) sextic singular at $p_1, \ldots, p_{10}$ we have the
factorization

$$R(p_1, \ldots, p_9; p_{10}) = C(p_1, \ldots, p_{10})^2 N(p_1, \ldots, p_9)$$

where $N(p_1, \ldots, p_9)$ is a multihomogeneous polynomial of degree 9 in the coordi-
nates of the points $p_1, \ldots, p_9$, symmetric with respect to $p_1, \ldots, p_9$ and that van-
ishes if and only if there is a reduced sextic singular at $p_1, \ldots, p_9$. Note that the
result in [4] that the unique sextic singular at $p_1, \ldots, p_9$ is the non-reduced curve
$C(p_1, \ldots, p_9, p)^2$ in $p$ is equivalent to the fact that $T$ has maximal rank for a general
choice of $p_1, \ldots, p_9$, which can be checked by a random choice. This proves that
$N(p_1, \ldots, p_9)$ is a nonzero polynomial, which defines the codimension 1 condition
that the space of sextics singular at $p_1, \ldots, p_9$ has dimension $\geq 2$. It can be checked
that at general $p_1, \ldots, p_9$ on the hypersurface $N(p_1, \ldots, p_9) = 0$ then $T$ has corank
1, while if $p_1, \ldots, p_9$ are distinct points defined by a general complete intersection
then both $R(p_1, \ldots, p_9; p_{10})$ and $C(p_1, \ldots, p_{10})$ vanish $\forall P_{10}$ and
$N(p_1, \ldots, p_9) \neq 0$.

We note also that the same argument given in the proof of [26, Theor. 45] shows that $N(p_1, \ldots, p_9)$ is irreducible, since there are no nonzero symmetric cubic
equations of 9 points in $\mathbb{P}^2$.

**Remark 4.3.** For general $p_1, \ldots, p_9$, the locus of ninth point $p_{10}$ such that there is
a reduced sextic singular at $p_1, \ldots, p_9, p_{10}$ has two irreducible components, namely
the nonic $N(p_1, \ldots, p_9, p) = 0$ and the point $p_{10}$ in the base locus of the linear system
of cubics through $p_1, \ldots, p_8$. The general sextic arising from the first (resp. second)
choice of $p_1, \ldots, p_9$ has maximal rank for a general
component is irreducible (resp. reducible) and lies in $R$ (resp. in $W$), see Definition
3.12. More precisely, $R \subset S^9(v_6(\mathbb{P}^2))$ is the closure of $\cup \{ p_i^{6}, \ldots, p_9^{6}\}$, where the union
is taken for distinct $p_1, \ldots, p_9$ satisfying $N(p_1, \ldots, p_9) = 0$. As noted in Remark
3.13 $R$ and $W$ are two irreducible components of the ramification locus of the $2 : 1$
map $AS^9(v_6(\mathbb{P}^2)) \rightarrow S^9(v_6(\mathbb{P}^2))$. They are distinct since $H_{27}$ vanishes on $R$ and
not on $W$. When $N(p_1, \ldots, p_9) = 0$ then $Z = \{ p_1, \ldots, p_9 \}$ is self-linked on the
cubic $C(p_1, \ldots, p_9, p)$ with respect to any irreducible sextic singular at $p_1, \ldots, p_9$, which
meets the cubic in $2p_1 + \ldots + 2p_9$.

**Theorem 4.4.** Let $f = \sum_{i=1}^{9} p_i^6$ be a rank 9 sextic, in affine notation. Then

$$\det A_f = \lambda N(p_1, \ldots, p_9)^2$$

for a nonzero scalar $\lambda \in \mathbb{C}^*$.

**Proof.** Recall we denoted $E = \text{Sym}^2 T$ and denote $Z = \{ p_1, \ldots, p_9 \}$. Assume
that $\det A_f$ vanishes, then $\ker A_f$ is a nonzero subspace of $H^0(E)$. Consider the
restriction $H^0(E) \xrightarrow{\cdot j} H^0(E_{|Z})$, note both sides are 27-dimensional. Assume $j$
is injective. Then by [21, Lemma 5.4.1] we have that $H^0(I_Z \otimes E) = \ker A_f$ is nonzero,
which is a contradiction since a section vanishing on $Z$ is in the kernel of $j$. It
follows that $j$ is not injective and there is a section $s \in H^0(E)$ vanishing at $Z =
Again by \[21\] Lemma 5.4.1 the section \(s\) belongs to \(\ker A_f\). Then the contraction
\[
\text{Sym}^2 H^0(I_Z \otimes E) \to H^0(I_Z \otimes E) \otimes H^0(I_Z \otimes E^\vee \otimes \mathcal{O}(6)) \to H^0(I_Z^2 \otimes \mathcal{O}(6))
\]
as in \[21\] Theorem 5.4.3] takes \(s^2\) to a sextic singular at \(Z\). There are two cases, \(Z\) is a complete intersection or \(N(p_1, \ldots, p_9) = 0\). The first case has codimension 3 by 4.1 and can be excluded in proving a polynomial equality, like in our statement. It follows that \(\det A_f = 0\) implies generically \(N(p_1, \ldots, p_9) = 0\), hence a power of \(N(p_1, \ldots, p_9)\) is divided by \(\det A_f\). The multihomogeneous polynomial \(\det A_f\) is symmetric in \(p_1, \ldots, p_9\) and has total degree \(27 \cdot 6\), hence must have degree 18 in each \(p_i\). The result follows since \(N(p_1, \ldots, p_9)\) is irreducible. \(\square\)

**Corollary 4.5.** The variety \(\mathcal{R}\) is the complete intersection of the two hypersurfaces \(\det A_f\) and \(S^9(v_6(\mathbb{P}^2))\) and has degree \(10 \cdot 27 = 270\).

### Tables about the loci we have studied
In the next table we list some informations about the loci we have studied in \(\mathbb{P}^{27} = \mathbb{P}^{\text{Sym}^6 C^3}\). The main theme is that \((C)_9\) and the invariant \(H_{27} = \det A_f\) are enough to describe all \(k\)-secant varieties to \(v_6(\mathbb{P}^2)\).

| \(\text{dim}\) | \(\text{deg}\) | \(\text{equations}\) | \(\text{equations}\) |
|---|---|---|---|
| \(S^9\) | 26 | 10 | \(\det C^3\) | 9-secant |
| \(\mathcal{R}\) | 25 | 270 | \(\det C^3, H_{27}\) | comp. of ramif. locus of \(AS^9 \to S^9\) |
| \(W\) | 24 | 165 | \((C^3)_9, H_{27}\) | Sing \(S^9\), span of compl. inters. 9-ples |
| \(S^8\) | 23 | 1485 = \((165 \cdot 27)/3\) | \((C^3)_9, H_{27}\) | 8-secant |
| \(W'\) | 21 | 2640 | \((C^3)_8, H_{27}\) | sextics with three apolar cubics |
| \(S^7\) | 20 | 11880 = \((2640 \cdot 27)/6\) | \((C^3)_8, H_{27}\) | 7-secant |

The fact that \(\text{Sing } S^9 = W\) has been checked computationally, since the singular locus of \(S^9\) has codimension 3 and degree 165, and contains the variety \(W\) given by the 9-minors of \(C^3\), hence the equality follows. The following graph links the containments among the several loci.

\[
\begin{array}{c}
S^9 \\
\downarrow \ \\
\downarrow \\
W \ \\
\downarrow \ \\
W' \ \\
\downarrow \\
S^8 \\
\downarrow \ \\
\downarrow \\
S^7
\end{array}
\]

The right column is obtained by the left column after cut with the invariant hypersurface \(H_{27}\). The three diagonal links correspond to cut with the invariant \(H_{27}\). The matrix \(A_f\) drops rank by 1 in the first link, by 3 in the second one, by 6 in the third one. Hence a check on the degrees shows that the intersection with \(H_{27}\) does not contain other components in all the three cases.
The following table lists the properties related to Waring decomposition of general member of the loci (brk is the border rank)

| rk | C | brk | rk | #{ minimal Waring decompositions} |
|----|----|-----|----|----------------------------------|
| 9  | 9  | 9   | 2  |                                  |
| 9  | 9  | 9   | 1  |                                  |
| 8  | 9  | 9   | 1  | (Prop 3.10)                      |
| 8  | 8  | 8   | 1  |                                  |
| 7  | 9  | 9   | ∞  |                                  |
| 7  | 7  | 7   | 1  |                                  |

Tables about the dual loci

| dim | deg          |
|-----|--------------|
| (S^7)\vee | 20 | 34 435 125 [10] | irred. sextics with 7 singular pts |
| (W')\vee   | 26 | 83 200 [2]       | sums of three squares             |
| (S^8)\vee   | 19 | 58 444 767 [10] | irred. sextics with 8 singular pts |
| W\vee       | 18 | \frac{1}{2} (9^2) = 24 310 | reducible in 2 cubics, or sums of two squares |
| R\vee       | 18 | 57 435 240 [10][17] | irred. sextics with 9 singular pts |
| (S^9)\vee   | 9  | 2^9 = 512       | square of a cubic                 |

\[ (S^7)^\vee \]
\[ (S^8)^\vee \]
\[ (W')^\vee \]
\[ R^\vee \]
\[ W^\vee \]
\[ (S^9)^\vee \]

REFERENCES

1. J. Alexander and A. Hirschowitz, Polynomial interpolation in several variables, J. Algebraic Geom. 4 (1995), 201–222.
2. E. Angelini and L. Chiantini, On the identifiability of ternary forms, Lin. Alg. Applic. 599 (2020), 36–65.
3. E. Angelini, L. Chiantini, and A. Mazzon, Identifiability for a class of symmetric tensors, Mediterr. J. Math. 16 (2019), 97.
4. E. Arbarello and M. Cornalba, Footnotes to a paper of B. Segre, Math. Ann. 256 (1981), 341–362.
5. E. Ballico, An effective criterion for the additive decompositions of forms, Rend. Ist. Matem. Trieste 51 (2019), 1–12.
6. E. Ballico and L. Chiantini, On the terracini locus of projective varieties, available online arXiv:2011.13189, 2020.
7. G. Blekherman, J. Hauenstein, J. C. Ottem, K. Ranestad, and B. Sturmfels, Algebraic boundaries of Hilbert’s SOS cones, Compos. Math. 148 (2012), no. 6, 1717–1735. MR 2999301
8. W. Buczyński, J. Buczyński, and Z. Teitler, Waring decompositions of monomials, J. of Algebra 378 (2013), 45–57.
9. J. Buczyński, A. Ginensky, and J.M. Landsberg, Determinantal equations for secant varieties and the Eisenbud-Koh-Stillman conjecture, J. London Math. Soc. 88 (2013), 1–24.
10. L. Caporaso and J. Harris, *Counting plane curves of any genus*, Invent. Math. **131** (1998), no. 2, 345–392. MR 1608583
11. L. Chiantini, *Hilbert functions and tensor analysis*, Quantum Physics and Geometry, Lecture Notes of the Unione Matematica Italiana, vol. 25, Springer, Berlin, New York NY, 2019, pp. 125–151.
12. L. Chiantini and C. Ciliberto, *Weakly defective varieties*, Trans. Amer. Math. Soc. **354** (2002), 151–178.
13. L. Chiantini, *On the concept of k-secant order of a variety*, J. London Math. Soc. **73** (2006), 436–454.
14. L. Chiantini, G. Ottaviani, and N. Vannieuwenhoven, *On generic identifiability of symmetric tensors of subgeneric rank*, Trans. Amer. Math. Soc. **369** (2017), 4021–4042.
15. A. Conca and G. Valla, *Hilbert function of powers of ideals of low codimension*, Math. Zeit. **230** (1999), 753–784.
16. E. Davis, *Hilbert functions and complete intersections*, Rend. Seminario Mat. Univ. Politecnico Torino **42** (1984), 333–353.
17. E. Getzler, *Intersection theory on $\overline{M}_{1,4}$ and elliptic Gromov-Witten invariants*, J. Amer. Math. Soc. **10** (1997), no. 4, 973–998. MR 1451505
18. D. Grayson and M. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, Available at [http://www.math.uiuc.edu/Macaulay2/](http://www.math.uiuc.edu/Macaulay2/).
19. J. Harris, *On the Severi problem*, Invent. Math. **84** (1986), 445–461.
20. A. Iarrobino and V. Kanev, *Power sums, Gorenstein algebras, and determinantal loci*, Lecture Notes in Mathematics, vol. 1721, Springer, Berlin, New York NY, 1999.
21. J.M. Landsberg and G. Ottaviani, *Equations for secant varieties of Veronese and other varieties*, Ann. Mat. Pura Appl. **192** (2013), 549–606.
22. J. Migliore, *Introduction to liaison theory and deficiency modules*, Progress in Mathematics, vol. 165, Birkhäuser, Basel, Boston MA, 1998.
23. B. Mourrain and A. Oneto, *On minimal decompositions of low rank symmetric tensors*, Lin. Alg. Appl. **607** (2020), 347–377.
24. L. Oeding and G. Ottaviani, *Eigenvectors of tensors and algorithms for Waring decomposition*, J. Symbolic Comput. **54** (2013), 9–35.
25. G. Ottaviani, *An invariant regarding Waring’s problem for cubic polynomials*, Nagoya Math. J. **193** (2009), 95–110.
26. G. Ottaviani and E. Sernesi, *On the hypersurface of Lüroth quartics*, Michigan Math. J. **59** (2010), no. 2, 365–394.
27. C. Segre, *Gli ordini delle varietà che annullano i determinanti dei diversi gradi estratti da una data matrice*, Atti Accad. Lincei Clase Sci. **9** (1900), 253–260.

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