Inhomogeneous Perturbations and Stability Analysis of the Einstein Static Universe in $f(R,T)$ Gravity

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Abstract

The purpose of this paper is to analyze the existence of static stable Einstein universe using inhomogeneous linear perturbations in the context of $f(R,T)$ gravity ($R$ and $T$ denote the scalar curvature and trace of the stress-energy tensor, respectively). The static and perturbed field equations are constructed for perfect fluid parameterized by linear equation of state parameter. We obtain solutions manifesting the Einstein static state by considering peculiar $f(R,T)$ forms for vanishing and non-vanishing conservation of the stress-energy tensor. It is observed that stable static Einstein regions exist for both closed as well as open FLRW universe models for an appropriate choice of parameters. We conclude that this theory is efficient for presenting such cosmological solutions leading to emergent universe scenario.

Keywords: Einstein universe; Stability analysis; $f(R,T)$ gravity.
PACS: 04.25.Nx; 04.40.Dg; 04.50.Kd.

1 Introduction

Modified gravitational theories have attracted many researchers to discuss the accelerated expanding universe. The direct modification of general relativity (GR) is the $f(R)$ theory which is derived by introducing a general
function $f(R)$ in place of scalar curvature ($R$) in Einstein theory (Capozziello 2002; Nojiri and Odintsov 2003). One of the most stimulating characteristics of extended theories is the inclusion of coupling between gravitational and matter entities that has instigated several researchers to unveil the hidden mysteries of dark components. Harko et al. (2011) established such type of interaction in $f(R, T)$ theory. This modified theory can be regarded as an extended form of $f(R)$ theory. The motivation of introducing trace of the energy-momentum tensor (EMT) may originate from the results of some unknown gravitational interactions or the effects of some exotic fluid. It is predicted that such coupling provides the non-vanishing conservation of EMT. Consequently, an additional force arises due to which massive test particles follow the non-geodesic path while dust particles chase the geodesic lines. The $f(R, T)$ theory has extensively been studied for different tasks such as thermodynamics (Jamil et al. 2012; Sharif and Zubair 2012, 2013a), energy conditions (Alvarenga et al. 2013; Sharif et al. 2013; Sharif and Zubair 2013b), cosmological solutions (Shabani and Farhoudi 2013; Sharif and Zubair 2014a, b; Moraes 2015), dynamical instability (Noureen and Zubair 2015; Sharif and Waseem 2018a) and astrophysical scenarios (Sharif and Siddiqa 2018; Sharif and Waseem 2018b, 2019a, b; Deb et al. 2019; Maurya et al. 2019).

The big-bang singularity is another well-known issue in modern cosmology. In order to resolve this singularity issue, various speculations have been suggested to construct non-singular or past eternal cosmological models. The emergent universe conjecture has been developed in the background of GR (Gasperini and Veneziano 2003; Khoury et al. 2004) which says that the universe remains in an Einstein static state and then emerges into inflationary phase of cosmos (Ellis and Maartens 2004; Ellis et al. 2004). The successful emergent universe conjecture also demands the presence of stable Einstein universe (EU) with respect to all types of perturbations. Einstein universe is characterized by Friedmann-Lemaître-Robertson-Walker (FLRW) space-time with perfect fluid. Initially, this model was favorite to interpret the static universe but later, it was observed that for homogeneous and isotropic perturbations, the EU exhibits unstable regions around equilibrium state (Eddington 1930). It was also determined that EU always remains neutrally stable for inhomogeneous vector/tensor perturbations as long as the speed of sound fulfills the inequality $c^2_s > \frac{1}{3}$ and unstable otherwise (Harrison 1967; Barrow et al. 2003). Moreover, Barrow and Yamamoto (2012) analyzed the stability of EU with homogeneous perturbations for different kinds of matter.
fields and found unstable solutions.

Despite the fact that the basic component for developing emergent scenario is the Einstein static solution, the initial model does not prove successful to resolve the singularity issue, since scalar homogenous perturbations break the stability of primal cosmic static state in GR. Therefore, it was suggested to study Einstein static cosmos beyond the Einstein gravity. In this respect, the stability of EU has been investigated in several cosmological perspectives such as brane-world gravity (Gergely and Maartens 2002), loop quantum cosmology (Mulryne et al. 2005), Einstein Cartan scenario (Atazadeh 2014) and scalar fluid theories (Böhmer et al. 2015). The gravitational modified theories have become significant approaches to derive stable Einstein solutions. Böhmer et al. (2007) demonstrated the existence of stable EU solutions for particular $f(R)$ functions by adopting homogeneous scalar perturbations. They obtained stable EU by adding cosmological constant and described the comparison with GR.

Goswami et al. (2008) discussed the stable EU modes in $f(R)$ background and observed the appearance of static Einstein solutions only for $c_s > \sqrt{0.21}$ which is very close to the value determined in GR. Böhmer and Lobo (2009) examined the same scenario using linear homogeneous perturbations in $f(G)$ framework and constructed stable static solutions corresponding to distinct values of equation of state (EoS) parameter. For generic $f(R)$ models, Seahra and Böhmer (2009) inspected the stability of EU with barotropic EoS and displayed that the unstable regions are obtained for inhomogeneous perturbations. Li et al. (2013) developed stable EU regions with respect to open and closed FLRW models by applying homogeneous perturbations in generalized teleparallel gravity. Huang et al. (2014) investigated the same scenario using all kinds of perturbations in Jordan Brans-Dicke theory. They also observed static EU by implementing perturbations on matter variables in $f(G)$ gravity for closed cosmic model (Huang et al. 2015).

In the context of curvature-matter coupled gravity, the conjecture of emergent universe has gathered the attention of many researchers. Shabani and Ziaie (2017) explored stable modes of EU in $f(R, T)$ background using perturbation technique as well as phase space analysis. They obtained EU solutions corresponding to three particular $f(R, T)$ models and analyzed their stability through graphical analysis. On the same ground, Sharif and his collaborators (2017; 2018; 2018c, d; 2019) obtained the stable EU regions by considering homogeneous, inhomogeneous and anisotropic perturbations in the framework of minimal as well as non-minimal coupled theories. They
also observed their solutions graphically and presented a detail comparison with the existing literature.

This paper demonstrates the stability analysis of Einstein static cosmos by employing scalar inhomogeneous perturbations in $f(R, T)$ framework. This study would be useful to analyze the role of inhomogeneous perturbations as well as curvature-matter coupling on the stable eras of EU. The next section manifests the formulation of $f(R, T)$ field equations with respect to Einstein static state. Section 3 provides the description about inhomogeneous linear perturbations while section 4 deals with the stability analysis of Einstein static solutions for conservation as well as non-conservation of EMT corresponding to some particular $f(R, T)$ models. The last section presents the summary of our work.

## 2 Einstein Static Universe in $f(R, T)$ Scenario

The $f(R, T)$ gravity with Lagrangian density of matter ($\mathcal{L}_m$) is characterized by the action (Harko et al. 2011)

$$\mathcal{A} = \int \left( \frac{f(R, T)}{2\kappa^2} + \mathcal{L}_m \right) \sqrt{-g} d^4x,$$  \hspace{1cm} (1)

where $\kappa^2 = 1$ stands for coupling constant and $g$ indicates determinant of the metric tensor ($g_{\gamma\eta}$). The EMT corresponding to $\mathcal{L}_m$ is expressed as (Landau and Lifshitz 1971)

$$T^{\gamma\eta} = \frac{2}{\sqrt{-g}} \frac{\delta(\mathcal{L}_m\sqrt{-g})}{\delta g_{\gamma\eta}} = g^{\gamma\eta} \mathcal{L}_m + \frac{2\delta \mathcal{L}_m}{\delta g_{\gamma\eta}}.$$  \hspace{1cm} (2)

The $f(R, T)$ field equations can be evaluated through varying the action (1) with respect to $g_{\gamma\eta}$ and are represented by

$$R_{\gamma\eta} f_R(R, T) = \frac{1}{2} g_{\gamma\eta} f(R, T) - (\nabla_\gamma \nabla_\eta - g_{\gamma\eta} \Box) f_R(R, T) = T_{\gamma\eta} - (\Theta_{\gamma\eta} + T_{\gamma\eta}) f_T(R, T),$$  \hspace{1cm} (3)

where $f_R(R, T)$ and $f_T(R, T)$ depict differentiation of generic function corresponding to $R$ and $T$, respectively, $\Box = g^{\gamma\eta} \nabla_\gamma \nabla_\eta$, $\nabla_\gamma$ acts as the covariant derivative and $\Theta_{\gamma\eta}$ is defined as

$$\Theta_{\gamma\eta} = g^{\mu\nu} \frac{\delta T_{\mu\nu}}{\delta g^{\gamma\eta}} = g_{\gamma\eta} \mathcal{L}_m - 2T_{\gamma\eta} - 2g^{\mu\nu} \frac{\partial^2 \mathcal{L}_m}{\partial g^{\gamma\eta} \partial g^{\mu\nu}}.$$  \hspace{1cm} (4)
The covariant divergence of Eq. (3) leads to

$$\nabla^\gamma T_{\gamma \eta} = \frac{f_T}{1 - f_T} \left[ (T_{\gamma \eta} + \Theta_{\gamma \eta}) \nabla^\gamma (\ln f_T) - \frac{g_{\gamma \eta}}{2} \nabla^\gamma T + \nabla^\gamma \Theta_{\gamma \eta} \right]. \quad (5)$$

We consider that the universe is comprised of perfect fluid given by

$$T_{\gamma \eta} = (\rho + p)U_{\gamma}U_{\eta} - pg_{\gamma \eta}, \quad (6)$$

where $\rho$ denotes the matter density, isotropic pressure is depicted by $p$ and $U_{\gamma}$ reveals the four velocity in comoving frame. For perfect fluid configuration, we assume $L_m = -p$ which displays that matter Lagrangian depends only on $g_{\gamma \eta}$ and not on its derivatives (Landau and Lifshitz 1971). Hence, $\Theta_{\gamma \eta} = -2T_{\gamma \eta} - pg_{\gamma \eta}$.

The emergent universe conjecture yields a viable alternative to the initial singularity only with spatially non-flat FLRW spacetime whose line element is expressed by (Huang et al. 2015)

$$ds^2 = a^2(\tau) \left[ d\tau^2 - \left( \frac{1}{1 - K\chi^2} d\chi^2 + \chi^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right) \right], \quad (7)$$

where $a(\tau)$ manifests the conformal scale factor which is determined by the conformal time ($\tau$) and expresses the connection as $a(\tau)d\tau = dt$ whereas $K$ denotes the parameter of spatial curvature which provides open, closed and flat cosmic models for $K = -1, 1$ and 0, respectively. The scalar curvature and trace of EMT become

$$R = -6 \left( \frac{aK + \ddot{a}}{a^3} \right), \quad T = \rho - 3p,$$

where dot reveals differentiation associated with conformal time. The corresponding field equations for the line element (7) give

$$3 \left[ \left( \frac{\dot{a}}{a} \right)^2 + K \right] = \frac{1}{f_R} \left[ \rho a^2 + \frac{a^2}{2} f(R, T) + a^2(p + \rho) f_T + 3 \left( K + \frac{\ddot{a}}{a} \right) \right] \times f_R - 3 \frac{\dot{a}}{a} \partial_t f_R], \quad (8)$$

$$\left( \frac{\dot{a}}{a} \right)^2 - 2\frac{\ddot{a}}{a} - K = \frac{1}{f_R} \left[ a^2 p - \frac{a^2}{2} f(R, T) - 3 \left( K + \frac{\ddot{a}}{a} \right) f_R + \frac{\ddot{a}}{a} \partial_t f_R \right. + \left. \partial_t f_R ]. \quad (9)$$
In the past decades, the question about the beginning and origin of cosmos has provided fascinating results depending on the observations of GR as well as modern cosmology. In accordance with the fundamental physical perceptions on cosmic matter configuration, GR equations signify that the current expanding cosmos must be anticipated by a singularity, where the physical parameters such as spacetime curvature and density diverge. To resolve this problem, enormous research has been accomplished to assemble different singularity free cosmological scenarios. In this respect, the emergent universe conjecture has gained substantial importance to solve the issue of primordial singularity (Gasperini and Veneziano 2003; Khoury et al. 2004). According to this conjecture, the universe initiates asymptotically from Einstein static state and then it moves into an expanding state that yields inflationary scenario. The emergent universe model has interesting characteristics like there is no primordial singularity, the universe is eternal and static cosmic behavior in infinite past \((t \rightarrow -\infty)\). Thus, the aim of this speculation is to analyze the presence of stable Einstein solutions. For static EU characterized by FLRW universe model, we consider \(a(\tau) = a_0 = \text{constant} \) and the associated forms of \(R \) and \(T \) provide

\[
R(a_0) = R_0 = -\frac{6K}{a_0^2}, \quad T_0 = \rho_0 - 3p_0, \quad (10)
\]

where \(\rho_0 \) and \(p_0 \) exhibit the static forms of matter density and pressure, respectively. Equations (8) and (9) turn into

\[
3K = \frac{1}{f_R} \left( a_0^2 \rho_0 + \frac{a_0^2}{2} f(R_0, T_0) + a_0^2 (p_0 + \rho_0) f_T + 3K f_R \right), \quad (11)
\]

\[
-K = \frac{1}{f_R} \left( a_0^2 p_0 - \frac{a_0^2}{2} f(R_0, T_0) - 3K f_R \right). \quad (12)
\]

It is worth mentioning here that EU presents a rotation, expansion as well as shear free cosmos. Using linear EoS \(p = \omega \rho \) with \(\omega \) being an EoS parameter, addition of Eqs. (11) and (12) leads to

\[
a_0^2 = \frac{2K f_R}{\left(1 + \omega\right) \left(1 + f_T\right)}. \quad (13)
\]

### 3 Inhomogeneous Scalar Perturbations

The perturbations play a vital role to convert a difficult mathematical problem into a simpler one. There are different forms of perturbations such as
isotropic, anisotropic, homogeneous/inhomogeneous scalar, vector and tensor perturbations. For successful realization of emergent universe, the EU must display stable regions against all kinds of perturbations. Several researchers have adopted these perturbations to examine the stable state of EU. It is observed that the inhomogeneous perturbations lead to unstable solutions in $f(R)$ framework (Seahra and Böhmer 2009). What will happen in $f(R, T)$ scenario? Will the stable EU solution exist under the influence of inhomogeneous perturbations? To answer these queries, here we inspect the stable modes of EU by implementing inhomogeneous linear perturbations. We consider Newtonian/longitudinal gauge whose perturbed line element is expressed by (Huang et al. 2015)

$$
ds^2 = (1 - 2\vartheta) a_0^2 d\tau^2 - (1 - 2\varphi) a_0^2 \left( \frac{1}{1 - K\chi^2} d\chi^2 + \chi^2 d\theta^2 + \chi^2 \sin^2 \theta d\phi^2 \right),$$

where $\vartheta$ exhibits the Bardeen potential and $\varphi$ represents the perturbation to spatial curvature. The linear perturbations in matter components yield

$$p = p_0(1 + \delta p), \quad \rho = \rho_0(1 + \delta \rho),$$

where $\delta \rho$ and $\delta p$ depict the perturbed matter density and pressure, respectively. The harmonic decomposition of inhomogeneous linear perturbations are (Seahra and Böhmer 2009)

$$\begin{align*}
\delta p &= \delta p_l(\tau) \Omega_l(\mu^i), \\
\delta \rho &= \delta \rho_l(\tau) \Omega_l(\mu^i), \\
\vartheta &= \vartheta_l(\tau) \Omega_l(\mu^i), \\
\varphi &= \varphi_l(\tau) \Omega_l(\mu^i).
\end{align*}$$

Here, $\mu^i$ demonstrates the spatial entities $(\chi, \theta, \phi)$, when summation on $l$ is considered. The harmonic function $\Omega_l(\mu^i) \equiv \Omega_l$ for different cosmic models describes the following relations

$$\Delta \Omega_l \equiv -h^2 \Omega_l = \begin{cases}
-(l^2 + 1) \Omega_l, & l^2 \geq 0, \quad \mathcal{K} = -1, \\
-l^2 \Omega_l, & l^2 \geq 0, \quad \mathcal{K} = 0, \\
-l(l + 2) \Omega_l, & l = 0, 1, 2, ..., \quad \mathcal{K} = 1,
\end{cases}$$

where $\Delta$ acts as the three-dimensional Laplacian operator.

These inhomogeneous perturbations yield discrete spectrum for open geometry of cosmos while a continuous spectrum is produced for closed and flat cosmic models (Huang et al. 2015). It is noted that for $l = 0$, one can
recover the homogeneous scalar perturbations. Implementing Taylor series on $f(R,T)$ function and applying inhomogeneous perturbations, we obtain $\delta R$ and $\delta T$

$$\delta R = -\frac{2}{a_0^2} \left(3\dot{\varphi} - 6K\varphi - 2a_0^2\dot{\varphi} + a_0^2\dot{\varphi} + a_0^2\dot{\varphi} \right), \quad \delta T = (1 - 3\omega)\rho_0\delta \rho.$$ \hspace{1cm} (15)

Substituting inhomogeneous perturbations and Eq. (15) in (3), the linearized $\tau\tau$ and diagonal entities associated with perturbed line element (14) provide

$$\begin{align*}
(6K + 2a_0^2h^2)\varphi f_R(R_0, T_0) + a_0^2\rho_0 \left[1 - \frac{(\omega - 3)}{2} f_T(R_0, T_0) + (1 - 3\omega)\rho_0 \right] \\
\times (1 + \omega) f_{TT}(R_0, T_0) \delta \rho + a_0^2h^2 f_{RR}(R_0, T_0) \delta R = 0, \quad (16)
\end{align*}$$

$$\begin{align*}
2 \left[6K\varphi - a_0^2h^2(\varphi - 2\dot{\varphi}) - 3\dot{\varphi} \right] f_R(R_0, T_0) + a_0^2\rho_0 \left[1 - 3\omega + (3 - 5\omega) \right] f_T(R_0, T_0) + (1 - 3\omega) \\
\times f_{TT}(R_0, T_0) \delta \rho - 3a_0^2\rho_0(1 - 3\omega) f_{RT}(R_0, T_0) \delta \rho - 3a_0^2f_{RR}(R_0, T_0) \delta \rho \\
+ 3a_0^2 \left(\frac{f_R(R_0, T_0)}{2} + \left(\frac{2K}{a_0^2} + \hbar^2 \right) f_{RR}(R_0, T_0) \right) \delta R = 0.
\end{align*} \hspace{1cm} (17)$$

For perfect matter configuration, the non-diagonal constituents yield the following connection

$$\vartheta(\tau) = \varphi(\tau), \hspace{1cm} (18)$$

while anisotropic matter distribution does not satisfy this relation.

In order to observe $f(R,T)$ theory as a feasible gravitational theory, one must consider an effective and viable expression of $f(R,T)$ function. The models of this gravity are displayed in the following ways (Harko et al. 2011: Harko and Lobo 2019).

- $f(R,T) = f_1(R) + f_2(T)$. This choice of $f(R,T)$ function corresponds to the minimal interaction and can be regarded as a linear extension to $f(R)$ theory. By adopting any linear combination of $f_2$, various models can be obtained for different choices of $f_1(R)$ function. If we consider $f_1(R) = R$ and $f_2(T) = 2h(T)$, then the outcomes of this model show consistency with ΛCDM cosmological model.

- $f(R,T) = f_1(R) + f_2(T)f_3(R)$. This form describes the non-minimal coupling between matter and geometry. The results obtained from this choice may be different from the minimal coupled models.
To evaluate the stable modes of EU, we adopt the first form of \( f(R, T) \) gravity. The field equations (16) and (17) corresponding to minimally coupled \( f(R, T) \) function yield

\[
(6\mathcal{K} + 2a_0^2\hbar^2)f_1'(R_0) + a_0^2\rho_0 \left[ 1 - \frac{\omega - 3}{2} f_2'(T_0) + (1 - 3\omega)\rho_0 \right]
\times (1 + \omega)f_2''(T_0) \] \delta \rho + a_0^2f_1''(R_0)\delta R = 0, \tag{19}
\]

\[
2 \left[ 6\mathcal{K}\phi - a_0^2\hbar^2(\vartheta - 2\varphi) - 3\varphi \right] f_1'(R_0) + a_0^2\rho_0 \left[ (1 - 3\omega) + (3 - 5\omega) \right]
\times f_2'(T_0) + (\omega + 1)(1 - 3\omega)\rho_0 f_2''(T_0) \] \delta \rho - 3a_0^2f_1''(R_0)\delta \tilde{R} + 3a_0^2
\times \left( \frac{f_1'(R_0)}{2} + \left( \frac{2\mathcal{K}}{a_0^2} + \hbar^2 \right)f_1''(R_0) \right) \delta R = 0, \tag{20}
\]

where \( f_1'(R) = df_1(R)/dR \) and \( f_2'(T) = df_2(T)/dT \). Inserting Eq. (15) in the elimination of \( \vartheta \) and \( \delta \rho \) from Eqs. (19) and (20), it follows that

\[
18 \left[ 1 - \frac{\omega - 3}{2} f_2'(T_0) + \rho_0(\omega + 1)(1 - 3\omega)f_2''(T_0) \right] f_1''(R_0)\phi^{(iv)} + \left[ f_1''(R_0) \right]
\times \left\{ - 6\left( \hbar^2(2 + 3\omega + a_0^2) + 6\mathcal{K}\left( 1 + \frac{1}{a_0^2} \right) \right) - 3\left( \hbar^2(3 + 7\omega - (\omega - 3)a_0^2) \right) \right. \\
- 6\mathcal{K}(\omega - 3)\left( 1 + \frac{1}{a_0^2} \right)f_2'(T_0) - 6\rho_0(1 - 3\omega)(1 + \omega)\left( 6\mathcal{K}\left( 1 + \frac{1}{a_0^2} \right) + \hbar^2 \right) \\
\times \left. \left( 2 + a_0^2 \right) f_2''(T_0) \right\}\varphi + \left[ f_1''(R_0) \right] \left\{ 12\mathcal{K}\hbar^2(5 - 3\omega) + \frac{72\mathcal{K}^2}{a_0^2} + 2a_0^2\hbar^2(4 - 3\omega) \right. \\
+ f_2'(T_0) \left( 12\mathcal{K}\hbar^2(9 - 7\omega) + a_0^2\hbar^4(15 - 13\omega) - \frac{36\mathcal{K}^2(\omega - 3)}{a_0^2} \right) + f_2''(T_0) \right. \\
\times \left. \left( 4\hbar^2(15\mathcal{K} + 2a_0^2\hbar^2) + \frac{72\mathcal{K}^2}{a_0^2} \right) \right\} + f_1'(R_0) \left\{ 18\mathcal{K}(2 - \omega) + a_0^2\hbar^2(7 - 6\omega) \right. \\
+ f_2'(T_0) \left( 36\mathcal{K}(1 - \omega) + 9(3 - \omega) + \frac{a_0^2\hbar^2}{2}(27 - 25\omega) \right) + 7a_0^2\hbar^2\rho_0(1 - 3\omega) \\
\times \left. (1 + \omega)f_2''(T_0) \right\}\varphi = 0. \tag{21}
\]

Substituting the expression of \( a_0^2 \) from Eq. (13) in (21), the resulting fourth-ordered perturbed equation in terms of \( \varphi \) takes the form

\[
36\mathcal{K}\rho_0(1 + \omega)(1 + f_2'(T_0)) \left[ (\omega + 1)(1 - 3\omega)\rho_0 f_2''(T_0) - \frac{\omega - 3}{2} f_2'(T_0) + 1 \right]
\]
\[ \times f_1'(R_0)f_2''(R_0)\varphi''(\nu) + \left[f_2''(R_0) \left\{ -6\left(2K\rho f_1'(R_0)(\rho_0(1 + \omega)(2 + 3\omega)(1 + f_2'(T_0)) + 6K\rho_0(1 + \omega)(1 + f_2'(T_0))(2Kf_1'(R_0) + \rho_0(1 + \omega) \times \left(1 + f_2'(T_0)\right) \right) - 3f_2'(T_0)\right) 2K\rho f_1'(R_0)(\rho_0(3 + 7\omega)(1 + \omega)(1 + f_2'(T_0)) - 2K(\omega - 3)f_1'(R_0) - 6K\rho_0(1 + \omega)(1 + f_2'(T_0))(2Kf_1'(R_0) + \rho_0(1 + \omega) \times \left(1 + f_2'(T_0)\right) \right) - 12K\rho_0(\omega + 1)(1 - 3\omega)f_2''(R_0)\left(2h^2 f_1'(R_0)(Kf_1'(R_0) + \rho_0(1 + \omega) \times \left(1 + f_2'(T_0)\right) \right) - 36K\rho_0(1 + \omega)(1 + f_2'(T_0))f_1'(R_0)^2 \left\{ \frac{\omega - 3}{2}f_2'(T_0) - 1 - \rho_0(1 - 3\omega)(1 + \omega)f_2''(T_0) \right\} \right] \varphi + \left[f_1''(R_0) \left\{ 24K^2\rho_0(1 + \omega)(1 + f_2'(T_0))(h^2 \times (5 - 3\omega)f_1'(R_0) + 3\rho_0(\omega + 1)(1 + f_2'(T_0))) + 8K^2h^2(4 - 3\omega)f_1'(R_0)^2 + f_2'(T_0)\left(24K^2\rho_0(\omega + 1)(1 + f_2'(T_0))(h^2(9 - 7\omega)f_1'(R_0) - 3\rho_0(1 + \omega)(1 + f_2'(T_0)))) + 12K^2h^2(5 - \omega)f_1'(R_0)^2 \right) + 8K^2f_2''(T_0)\left(h^2 f_1'(R_0)(15 \times \rho_0(1 + \omega)(1 + f_2'(T_0)) + 4h^2 f_1'(R_0)) + 9\rho_0^2(1 + \omega)^2(1 + f_2'(T_0))^2 \right) \right\} \right] \varphi = 0. \]

In the following section, we analyze the existence as well as stability of Einstein static solutions corresponding to the specific choices of \(f_1(R)\) and \(f_2(T)\) functions.

### 4 Stability Analysis of Einstein Universe

In this section, we evaluate three classes of solutions that can be regarded as EU models with respect to the conserved and non-conserved forms of EMT for \(f_1(R) = R\). First, we consider the conserved case to obtain the particular form of \(f_2(T)\) and investigate the stability of Einstein solution through graphical analysis. Second, we assume the non-conserved case in
which two different forms of \( f_2(T) \) are used to examine the stable modes of EU.

**Case I: Conserved EMT**

The modified theories comprising curvature-matter coupling do not satisfy the conservation law. The continuity equation in the context of generic FLRW universe model is demonstrated as

\[
\dot{\rho} + \frac{3(1 + \omega)\dot{a}}{a}\rho = \frac{-1}{1 + f_T(R, T)} \left[ \rho(1 + \omega)f_T(R, T) + \frac{1 - \omega}{2}\dot{f}_T(R, T) \right].
\]

(23)

Here, we consider that the conservation law holds in \( f(R, T) \) gravity and consequently, the differential equation for minimally coupled \( f(R, T) \) model leads to

\[(1 - \omega)f'_2(T) + 2(1 + \omega)Tf''_2(T) = 0,\]

whose solution yields a unique expression of \( f_2(T) \) for which EMT remains conserved and it is given by

\[f_2(T) = \left(\frac{1 + \omega}{1 + 3\omega}\right)T^{\frac{1 + 3\omega}{2(1 + \omega)}}c_1 + c_2 = 0,\]

(24)

with \( c_1 \) and \( c_2 \) as integration constants. Implementing this solution in Eq. (22) with \( f_1(R) = R \), we obtain the inhomogeneous perturbed differential equation of the form

\[A_1\varphi - A_2\ddot{\varphi} = 0,\]

(25)

where \( A_1 \) and \( A_2 \) are

\[
A_1 = \mathcal{K}\left[18\mathcal{K}\rho_0^2(1 + 3\omega)(1 + \omega)^2\left\{1 + \frac{c_1}{2}\left((1 - 3\omega)\rho_0\right)^\frac{1 + \omega}{2 + 3\omega}\right\} + 2\mathcal{K}h^2(7 - 6\omega) + \frac{c_1}{2}\left((1 - 3\omega)\rho_0\right)^\frac{1 + \omega}{2 + 3\omega}\left[9\rho_0(1 + \omega)(4\mathcal{K}(1 - \omega) + 3 - \omega)\left(1 + \frac{c_1}{2}\right) + \mathcal{K}h^2(27 - 25\omega)\right] + \frac{7\mathcal{K}h^2(\omega + 1)(7 - 6\omega)c_1}{2}\times\left((1 - 3\omega)\rho_0\right)^\frac{1 + \omega}{2 + 3\omega}\right] + \mathcal{K}h^2,\]

\[
A_2 = 36\mathcal{K}\rho_0(\omega + 1)\left[1 + \frac{c_1}{2}\left(\rho_0(1 - 3\omega)\right)^\frac{1 + \omega}{2 + 3\omega}\right]^2.
\]
The solution of Eq. (25) is

\[ \varphi(\tau) = a_1 e^{\varpi \tau} + a_2 e^{-\varpi \tau}, \]

where \(a_1\) and \(a_2\) are constants of integration. The parameter \(\varpi\) manifests the frequency of perturbation represented by

\[ \varpi_1^2 = A_1 / A_2. \] (26)

The existence of unstable/stable eras of EU is based only on the exponential growth of perturbations. The inequality \(\varpi_1^2 > 0\) leads to the unstable solutions while the stable ones exist for \(\varpi_1^2 < 0\). When \(l = 0\), the frequency corresponding to the homogenous perturbations is obtained and in general relativistic limit, i.e., \(c_1 = 0\), this frequency reduces to

\[ \varpi_1^2 = \frac{\mathcal{K}}{2} \rho_0 (1 + \omega)(1 + 3\omega), \]

which yields the stable solutions for \(-1 < \omega < -\frac{1}{3}\) (Böhmer and Lobo 2009). To inspect the stable eras of EU graphically, we choose current value of \(\rho_0\) as \(\rho_0 = 0.3\) (Ade et al. 2016). Figure 1 demonstrates the existence of stable EU modes for both closed and open universe models with respect to two different values of \(l\). It is observed that the stability of EU increases towards positive values of \(\omega\) with the increasing value of \(l\) in the framework of closed cosmic model while it slightly reduces for \(\mathcal{K} = -1\) as \(l^2\) enhances. It is also found that more stable modes exist for positive and negative values of \(c_1\) with respect to \(\mathcal{K} = 1\) and \(\mathcal{K} = -1\), respectively. However, in both plots, the stable EU regions appear for \(\omega > -1\) which is consistent with GR.

**Case II: Non-conserved EMT**

Here, we inspect the stable Einstein static solution when covariant divergence of EMT is not zero. For this purpose, we assume two particular choices of \(f(R, T)\) function that describe a direct relation between \(R\) and \(T\). First, we take

\[ f(R, T) = R + m \sqrt{T}, \] (27)

where \(m\) is a coupling constant. Shabani and Ziaie (2017) have used this model to examine the solution for homogeneous perturbations and determined that stable modes of EU are obtained only for positive values of \(m\).
Figure 1: Stability of EU in $(\omega, c_1)$ space with $l = 2$ (blue), $l = 15$ (orange) for $\mathcal{K} = 1$ (left) and $l^2 = 2$ (blue), $l^2 = 15$ (orange) for $\mathcal{K} = -1$ (right).

Inserting this model in Eq. (22), the equation in $\varphi$ is acquired whose solution provides the frequency as follows

$$\omega_2^2 = A_3/A_4,$$  \hspace{1cm} \text{(28)}

where $A_i$’s ($i = 3, 4$) are represented by

$$A_3 = \mathcal{K} \left[ 18\mathcal{K}\rho_0^2(1 + 3\omega)(1 + \omega)^2 \left(1 + \frac{m}{2\sqrt{\rho_0(1 - 3\omega)}}\right) + 2\mathcal{K}h^2(7 - 6\omega) \right.$$  
\left. + \frac{m}{2\sqrt{\rho_0(1 - 3\omega)}} \left\{9\rho_0(1 + \omega)(4\mathcal{K}(1 - \omega) + 3 - \omega) \left(1 + \frac{m}{2\sqrt{\rho_0(1 - 3\omega)}}\right) \right. \right.$$  
\left. + \mathcal{K}h^2(27 - 25\omega) \right\} - \frac{7\mathcal{K}h^2(\omega + 1)(7 - 6\omega)m}{2\sqrt{\rho_0(1 - 3\omega)}} \right].$$

$$A_4 = -36\mathcal{K}\rho_0(1 + \omega) \left(1 + \frac{m}{2\sqrt{\rho_0(1 - 3\omega)}}\right) \left(\frac{m(\omega - 1)}{2\sqrt{\rho_0(1 - 3\omega)}} - 1\right).$$

The graphical interpretation of stable modes in non-conserved state for different values of $l$ is exhibited in Figure 2. For closed universe model, it is observed that stability increases as $l$ increases for positive values of $m$ whereas it almost remains the same in case of open cosmic model. However, for $\mathcal{K} = -1$, more stable eras exist in comparison with $\mathcal{K} = 1$. For $m = 0 = l$, the
frequency can retrieve the results of GR as displayed in the case of conserved EMT.

Now, we consider power-law model of $f(R, T)$ gravity presented by (Shabani and Ziaie 2017)

$$f(R, T) = R + \alpha T^\beta,$$

(29)

with arbitrary constants displayed by $\alpha$ and $\beta$. For this model, the solution of differential equation (22) yields the following form of frequency

$$\omega_3^2 = \mathcal{A}_5/\mathcal{A}_6,$$

(30)

where $\mathcal{A}_5$ and $\mathcal{A}_6$ are

$$\mathcal{A}_5 = \mathcal{K}\left[18\mathcal{K}_0^2(1 + \omega)^2(1 + 3\omega)(1 + \alpha\beta((1 - 3\omega)\rho_0)^{\beta-1}) + 2\mathcal{K}_0^2(7 - 6\omega)
+ \alpha\beta((1 - 3\omega)\rho_0)^{\beta-1}\left\{9\rho_0(1 + \omega)(4\mathcal{K}(1 - \omega) + 3 - \omega)(1 + \alpha\beta(\rho_0(1
- 3\omega))^{\beta-1}) + \mathcal{K}_0^2(27 - 25\omega)\right\} + 14\mathcal{K}_0^2(1 + \omega)(7 - 6\omega)(\beta - 1)
\times (\rho_0(1 - 3\omega))^{\beta-1}\right],$$

$$\mathcal{A}_6 = -36\mathcal{K}_0(1 + \omega)\left(1 + \alpha\beta(\rho_0(1 - 3\omega))^{\beta-1}\right)\alpha\beta\left(\frac{\omega - 3}{2} - (1 + \omega)(\beta - 1)\right).$$
$\omega(1 - 3\omega)^{\beta - 1} - 1$.

For $\alpha = 0$, this frequency reduces to GR.

Figures 3 and 4 manifest the stable eras of EU for distinct values of $l$ and $\alpha$ with respect to closed and open geometries of cosmos, respectively. From these Figures, we observe that the stability slightly enhances and reduces with increasing values of $l$ in case of closed cosmic model for positive and negative value of $\alpha$, respectively. For $K = -1$, both values of $\alpha$ correspond to increasing stable modes of EU as $l$ increases. It is also found that more stable regions appear for positive and negative values of $\beta$ in the background of $K = 1$ and $K = -1$, respectively. From these graphical analysis, we can conclude that the considered $f(R, T)$ models provide more stable regions of EU with inhomogeneous perturbations as compared to homogeneous perturbations (Shabani and Ziaie 2017).

5 Concluding Remarks

The conjecture of emergent universe has been identified as a feasible alternative to the big-bang singularity and modified theories have been proved successful tool to discuss this conjecture. In this paper, we have analyzed
the existence of stable EU in the domain of different $f(R, T)$ models. The static EU solutions have been examined by employing scalar inhomogeneous perturbations characterized by linear EoS. We have obtained the second order perturbed differential equations for three specific $f(R, T)$ functions with respect to conservation and non-conservation of EMT.

For the conserved EMT, we have evaluated a peculiar expression of $f(R)$ for which the continuity equation satisfies in $f(R, T)$ framework. We have examined the EU solutions against integration constant $c_1$ for different values of $l$. We have observed that more stable eras of EU appear for positive and negative values of $c_1$ with respect to closed and open FLRW models, respectively. In case of non-conserved EMT, we have considered the power-law forms of $f(R, T)$ gravity and derived the stable EU solutions for appropriate choices of model parameters. It is worthy to mention that our solutions can be transformed to homogeneous perturbations for $l = 0$ and to GR when the model parameters become zero.

Shabani and Ziaie (2017) investigated the presence of stable EU regions in the same gravity using dynamical system approach and homogeneous linear perturbations. They found that in comparison with $f(R)$ gravity in which generally unstable solutions appear (Seahra and Böhmer 2009), some particular $f(R, T)$ models lead to stable EU regions. From our graphical analysis, we conclude that using scalar inhomogeneous perturbations, more

Figure 4: Stability of EU in $(\omega, \beta)$ space for $\mathcal{K} = -1$ with $\alpha = 1$ (left), $\alpha = -1$ (right), $l^2 = 2$ (blue) and $l^2 = 15$ (orange).
stable regions exist in $f(R, T)$ gravity as compared to homogeneous linear perturbations (Shabani and Ziaie 2017). It is noticed that our all stable modes of EU lie in the interval $-1 < \omega < 0.35$ which is consistent with GR. Hence, $f(R, T)$ gravity can provide such environment in which EU is associated with asymptotic emergent universe conjecture. It would be worthwhile to discuss this scenario on the ground of anisotropic perturbations in the same gravity.

Acknowledgment

One (AW) of us would like to thank the Higher Education Commission, Islamabad, Pakistan for its financial support through the Indigenous Ph.D. 5000 Fellowship Program Phase-II, Batch-III.

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