Cluster Percolation and Critical Behaviour in Spin Models and SU(N) Gauge Theories

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The critical behaviour of several spin models can be simply described as percolation of some suitably defined clusters, or droplets: the onset of the geometrical transition coincides with the critical point and the percolation exponents are equal to the thermal exponents. It is still unknown whether, given a model, one can define at all the droplets. In the cases where this is possible, the droplet definition depends in general on the specific model at study and can be quite involved. We propose here a simple general definition for the droplets: they are clusters obtained by joining nearest-neighbour spins of the same sign with some bond probability \( p_B \), which is the minimal probability that still allows the existence of a percolating cluster at the critical temperature \( T_c \). By means of lattice Monte Carlo simulations we find that this definition indeed satisfies the conditions required for the droplets, for many classical spin models, discrete and continuous, both in two and in three dimensions. In particular, our prescription allows to describe exactly the confinement-deconfinement transition of SU(\( N \)) gauge theories as Polyakov loop percolation.

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I. INTRODUCTION

The study of critical phenomena is one of the most fascinating topics in physics: phase transitions are processes that physicists continuously encounter in their investigations of nature, and they can occur in an incredible variety of systems. A phase transition is basically a change in the order of a system: by varying some parameter (usually the temperature), the microscopic constituents of the system, that we here call ‘spins’, choose a different way of staying together. If we go from high to low temperatures, we see that one passes from a situation in which the behaviour of each spin is totally independent of the others, to a situation in which several spins are correlated with each other and form ordered structures in the body of the system. These structures are local realizations of the new phase. If we further lower the temperature, the size of the ordered regions increases, until most of them stick to each other and form a domain which spans the whole system. When this happens, the system is in a new state of order, i.e. in a new phase.

This mechanism led already at the end of the 40’s to the idea that the physics of the phase transition is basically governed by the ordered domains built by the spin-spin correlations, and not by the individual spins. If the degrees of freedom relevant for the phase change are the ones of sets of particles, and not of single particles, it is likely that they do not depend on the details of the microscopic interaction, but only on its gross features (e.g. symmetries). That could explain why whole classes of systems, ruled by dynamics which appear very different from each other, happen to have the same behaviour at the phase transition. In particular, for second-order phase transitions, this simple picture could allow to explain the universality of the critical indices.

Percolation theory is the ideal framework for a geometrical description of phase transitions. The percolation phenomenon takes place when geometrical clusters, formed by elementary objects of some system, stick to each other giving rise to an infinite network, that spans the whole system. The analogy with the phase transition mechanism described above is evident, but there is much more than that. The real amazing thing is the fact that, in spite of the apparently different nature of percolation, which is a purely geometrical phenomenon, and second-order thermal phase transitions, these two types of processes are formally identical, as they are characterized by the same basic features, which are:

- Power law behaviour of the variables near the critical point, with relative exponents;
- Scaling relations between the exponents;
- Universality relations between the critical indices.

The temptation to identify a continuous thermal phase transition with a simple percolation transition is then very strong. This identification is possible provided one can establish a one-to-one correspondence between thermal and percolation variables. The main percolation variables are:

- the percolation strength \( P \), i.e. the probability that a site chosen at random belongs to a percolating cluster;
- the average cluster size \( S \),

\[
S = \frac{\sum n_s s^2}{\sum n_s s},
\]  

(1)

where \( n_s \) is the number of clusters with \( s \) sites and the sums exclude eventual percolating clusters.

Suppose we have defined how to group the spins of the configurations of a given model in clusters. Such clusters are the ‘physical’ clusters or droplets of the model if the following conditions are satisfied:
• the percolation point coincides with the thermal critical point;
• the connectedness length (average cluster radius) diverges as the thermal correlation length (same exponent);
• the percolation strength $P$ near the threshold varies like the order parameter $m$ of the model (same exponent);
• the average cluster size $S$ diverges as the physical susceptibility $\chi$ (same exponent).

The first studies in this direction concerned the Ising model. The simplest clusters one can think of are just the magnetic domains, i.e. the clusters built by joining nearest-neighbour spins of the same sign. In two dimensions these clusters indeed percolate at the critical temperature of the Ising model $T_p$; however, the critical percolation exponents differ from the thermal ones. So, the magnetic domains are not the Ising droplets. It was soon realized that the problem of these clusters is that they are too big due to purely geometrical effects. The average cluster size would be non-negligible also at extremely high temperatures, just because spins of the same sign may happen to lie close to each other, even in the absence of a physical correlation between them. In order to eliminate these artificial geometrical correlations, one can introduce a bond probability $p_B$ and join nearest-neighbouring spins with this probability, which automatically reduces the size of the clusters. Since the correlation changes with the temperature $T$, $p_B$ must be as well a function of $T$. If one chooses the expression $p_B = 1 - \exp(-2J/kT)$ ($J$ is the Ising coupling), the corresponding site-bond clusters are indeed the critical droplets of the Ising model, in any dimension.

Nobody can yet say whether the phase transition of every model can be geometrically described as a percolation transition, i.e. whether one can always define the droplets. The Ising result, which is valid more in general for the $g$-state Potts model, can easily be extended to several spin systems, both discrete and continuous. In general, one finds that each interaction between a pair of spins corresponds to a bond in the percolation picture with an analogous bond probability as in Ising. In models with several spin-spin interactions of the same type (all ferro- or antiferromagnetic), one can still define a percolation picture by putting bonds between any pair of interacting spins with some probability, but the picture becomes quite involved: if two interacting spins are far from each other, the geometrical bond between them looks virtual, as the two spins are geometrically disconnected.

Recent investigations aimed at recovering the importance of the role of geometrical connectivity in the mapping between percolation and thermal critical behaviour. In bidimensional models, one can define simple site-bond clusters which show all features the droplets should have. For the models where a rigorous mapping between percolation and critical behaviour is possible, such site-bond clusters are in general different from the "exact" droplets (see), which are in general more complex, even if their behaviour at criticality is identical. Moreover, the result remains valid as well for models with competitive interactions (e.g. ferromagnetic + antiferromagnetic), for which an exact definition of the droplets is, at present, missing. In one examined theories with center symmetry $Z(2)$ and $Z(3)$, such that their critical behaviour is in the universality class of the model obtained by removing all interactions except the nearest-neighbour one (Ising for $Z(2)$, 3-state Potts for $Z(3)$). Therefore, for these models the nearest-neighbour spin-spin coupling is the fundamental interaction which determines the behaviour at the phase transition. This is probably the reason why, if one just considers geometrical connections between nearest-neighbours, weighted by some suitable bond probability, the corresponding clusters are at least a good approximation of the critical droplets of the model. The previous argument is of course independent of the number $d$ of space dimensions of the system. For this reason we believe that the result of must be valid in general, i.e. for $d > 2$ as well. In this paper we review the 2-dimensional results presented in and provide numerical evidence, based on Monte Carlo simulations, that our droplet definition holds true also in three dimensions.

We stress that the original target of our investigations was to provide a geometrical description in terms of percolation of the confinement-deconfinement transition in $SU(N)$ gauge theories. Early attempts focused on $SU(2)$ pure gauge theory, whose deconfining transition is second-order and in the universality class of the Ising model. The strategy we followed at that stage was to approximate the gauge model by means of simpler Polyakov loop effective theories for which an exact droplet definition exists. In this way one finds only an approximate solution of the problem, which strongly depends on the specific lattice regularization one chooses. This seemed to us unsatisfactory: the droplet prescription we propose here solves the problem in a simple and general way.

The paper is divided as follows: in Section II we discuss more in detail our droplet definition; in Section III we present the results of our simulations, distinguishing between spin models and $SU(2)$ pure gauge theory; finally the conclusions of our work are exposed.

II. THE DROPLET DEFINITION

Our droplet candidates are clusters built by joining nearest-neighbour spins of the same sign with a bond probability $p_B$; they are then uniquely defined once we specify $p_B$. In, the following criterion was proposed: $p_B$ is the minimal probability that still makes percolation
possible at the critical temperature $T_c$. This special minimal probability, that we will call $p_{CK}$, depends on the model at study. We recall that $p_{CK}$ is in general a function of the temperature, like in the Ising model (where $p_{CK}(T) = 1 - \exp(-2J/kT)$). We are only interested in the behaviour at the transition, i.e. near $T_c$. Therefore what matters is basically only the value $p_{CK}(T_c)$ of the bond weight at $T_c$. This is why, referring to $p_B$ ($p_{CK}$), we use the term “value” instead of “expression”.

In two dimensions, for any $p_B < p_{CK}$, the percolation temperature $T_p < T_c$ and the exponents are in the universality class of 2D pure random percolation. On the other hand, for any $p_B > p_{CK}$ (including the pure-site case $p_B = 1$), $T_p = T_c$, but the exponents do not coincide either with the thermal or with the random percolation ones. For $p_B = p_{CK}$ and only in this case, the site-bond clusters satisfy all conditions required for the droplets, i.e. both the critical temperature and the exponents of the geometrical transition are equal to the thermal counterparts. We see that there is a whole range of $p_B$ values, i.e. $p_{CK} \leq p_B \leq 1$, for which the relative site-bond clusters begin to percolate exactly at the onset of the thermal transition. Such feature is specific of bidimensional lattices. In three dimensions we shall see that there is just a single value of the bond probability $p_B$ for which the percolation threshold is exactly at $T_c$. If $p_B$ is greater than this value, the clusters begin to percolate at some $T_p > T_c$. However, this asymmetry between the 2D and the 3D cases does not represent a serious problem. In the Ising model, by using the Fortuin-Kasteleyn bond weight $p_{CK} = 1 - \exp(-2J/kT)$, one obtains the correct droplets in any dimension [22, 23]. In particular, for the 3-dimensional Ising model, $p_{CK}(T_c)$ is necessarily the unique $p_B$ value for which the two thresholds coincide [23]. This special $p_B$ value is again the minimal probability one needs in order to have a percolating cluster at $T_c$, so that the “criterion of the minimal bond probability” adopted in [23] would lead to the correct droplet definition in three dimensions too, at least in the Ising case. We then assume that such criterion is valid more in general, and we shall verify its validity by computer simulations of two models in three dimensions, the $O(2)$ spin model and $SU(2)$ lattice gauge theory.

III. RESULTS

A. Numerical Analysis

Our aim is to investigate the percolation transition of special site-bond clusters, determining in particular the percolation temperature and the critical exponents.

To produce the equilibrium configurations we made use of standard Monte Carlo algorithms, like Metropolis or heat bath; for some models we adopted cluster updates, like the Wolff algorithm, which allows to reduce sensibly the correlation of the data and save a lot of CPU time. At each iteration, once the configuration to be examined is determined, all lattice sites are grouped in clusters by means of the algorithm devised by Hoshen and Kopelman [24]; for the cluster labeling we have always used free boundary conditions. After that we are left with a set of clusters of various sizes, and we can calculate the percolation variables. If a cluster connects the top with the bottom side (face in 3D) of the lattice, we say that it percolates [24]. Besides the percolation strength $P$ and the average cluster size $S$, we also calculate the size $S_M$ of the largest cluster of the configuration, since from it one can derive the fractal dimension $D$ of the spanning cluster at the threshold [25]. At each iteration the energy density $\epsilon$ and the lattice average $m$ of the order parameter of the thermal transition were also stored [26].

The first step is of course the determination of the percolation temperature. This can be effectively done by using a variable that can be extracted from the data sample of the percolation strength $P$. Suppose we have performed a number $N_I$ of iterations for one of our models at a given temperature and lattice size. Looking at the column of the $P$ data, say $N_P$, the number of configurations of our sample with (at least) a percolating cluster (for those configurations $P \neq 0$). The ratio $N_P/N_I$ is the percolation cumulant $\Pi$, which shares the same properties of the well-known Binder cumulant, namely:

1. if one plots $\Pi$ as a function of $T$, all curves corresponding to different lattice sizes cross at the same temperature $T_p$, which marks the threshold of the percolation transition;

2. the percolation cumulants for different values of the lattice size $L$ coincide, if considered as functions of $t_p L^{1/\nu_p}$ ($t_p = (T - T_p)/T_p$, $\nu_p$ is the exponent of the connectedness length);

3. the value of $\Pi$ at $T_p$ is a universal quantity, i.e. it labels a well defined set of critical indices.

The first property suggests that it is enough to make simulations on two different lattices to determine the critical point. The result is of course the more precise the larger the size of the lattices.

After evaluating the percolation temperature $T_p$, we extracted the values of the critical indices by means of standard finite-size scaling techniques at the critical point. If corrections to scaling do not play an important role, the finite-size scaling laws of the percolation variables at $T_p$ are given by the simple formulas

$$ P(T_p) \propto L^{-\beta_p/\nu_p} $$

(2)

$$ S(T_p) \propto L^{\gamma_p/\nu_p} $$

(3)

$$ S_M(T_p) \propto L^D $$

(4)

where $L$ is the lattice side and $\beta_p$, $\gamma_p$ are the exponents that rule the power law behaviour at criticality of $P$ and $S$, respectively. In order to improve the precision of the fits and to keep disturbing finite-size effects under
control, for each model four to six different lattices were used. In all our analyses we found that corrections to scaling do not perturb appreciably the leading behaviour expressed by Eqs. (2)-(4).

B. Spin Models

We start by reviewing the investigations of the bidimensional models presented in [13]. There, we analyzed two classes of systems: models with $Z(2)$ global symmetry and a magnetization transition with Ising exponents and models with $Z(3)$ global symmetry and a magnetization transition with exponents belonging to the 2-dimensional 3-state Potts model universality class. The spin systems belonging to the first group are:

1. the Ising model, $\mathcal{H} = -J \sum_{ij} s_i s_j$ ($J > 0$, $s_i = \pm 1$);
2. a model with nearest-neighbour (NN) ferromagnetic coupling and a weaker next-to-nearest (NTN) antiferromagnetic coupling: $\mathcal{H} = -J_1 \sum_{NN} s_i s_j - J_2 \sum_{NTN} s_i s_j$ ($J_1 > 0$, $J_2 < 0$, $|J_2/J_1| = 1/10$, $s_i = \pm 1$);
3. the continuous Ising model, $\mathcal{H} = -J \sum_{ij} S_i S_j$ ($J > 0$, $-1 \leq S_i \leq +1$).

The models belonging to the second group are:

1. the 3-state Potts model, $\mathcal{H} = -J \sum_{ij} \delta(s_i, s_j)$ ($J > 0$, $s_i = 1, 2, 3$);
2. a model obtained by adding to 1) a weaker next-to-nearest (NTN) antiferromagnetic coupling: $\mathcal{H} = -J_1 \sum_{NN} \delta(s_i, s_j) - J_2 \sum_{NTN} \delta(s_i, s_j)$ ($J_1 > 0$, $J_2 < 0$, $|J_2/J_1| = 1/10$, $s_i = 1, 2, 3$).

The strategy we followed in the numerical analysis was to tune by hand the value of the bond probability $p_B$ until, at the critical temperature $T_c$ of the model, the percolation cumulant $\Pi$ takes the same value for each of the lattices we used. The smallest $p_B$ value for which this is still possible is the minimal bond probability $p_{CK}$ we look for. The threshold value of $\Pi$ gives a strong indication on the universality class of the geometrical transition (property 3 of the percolation cumulant). To calculate the error on $p_{CK}$ we decreased $p_B$ until the $\Pi$ values of all lattices at $T_c$ were offset by more than one $\sigma$.

From a strictly numerical point of view one should take care to interpret the data of the simulations when $p_B$ is close to $p_{CK}$. In this case, in fact, the system finds itself in the neighbourhood of a discontinuity and if the lattice is not large enough, its behaviour is influenced by that. The simulations on small lattices would produce configurations which represent a sort of mixture of the two situations at $p_B = p_{CK}$ and $p_B \neq p_{CK}$. To recover the real behaviour of the system one should then go to very large lattices and disregard the small ones.

In two dimensions, as we said above, there is a whole range of $p_B$ values such that the onset of the percolation transition is exactly at the thermal critical point. The calculation of the exponents is then crucial to distinguish the various geometrical transitions.

We start by discussing the spin systems with $Z(2)$ symmetry. Fig. 1 shows the threshold value of the percolation cumulant $\Pi$ in the Ising model as a function of the bond probability $p_B$. For $p_B \approx p_{CK}$ we plotted the value of $\Pi$ for the largest lattice we took ($1000^2$), as $\Pi$ changes sensibly with the lattice size for the reason we explained above.

![FIG. 1: 2D Ising model: variation of $\Pi$ at the percolation temperature $T_p$ with the bond weight $p_B$.](image)

The vertical dashed line in the plot marks the minimal probability $p_{CK} = 1 - \exp(-2J/kT_c)$, which is $0.58578$. We see that the cumulant is quite stable to the right and to the left of $p_{CK}$, and that the plateau values correspond to two different universality classes. For $p_B < p_{CK}$ the site-bond clusters percolate at a temperature $T_p < T_c$, and the percolation exponents are in the random percolation universality class [25]. When $p_B > p_{CK}$ $T_p = T_c$, and the exponents belong to a special universality class (we call it 2D $Z(2)$ site percolation universality class because it includes the pure site percolation case, $p_B = 1$, and is the same for all $Z(2)$ models we studied). For $p_B = p_{CK}$ we recover the Fortuin-Kasteleyn mapping [1] and the site-bond clusters are the exact critical droplets of the system. We see that in this case the $\Pi$ threshold value does not lie on either of the plateaus, since the exponents are now in a different universality class, i.e. in the class of the 2D Ising model.

For the other $Z(2)$ spin systems the situation is analogous and we could show identical pictures as Fig. 1, except that the minimal probabilities $p_{CK}$ are different ($p_{CK} = 0.583(1)$ for Model 2 and $p_{CK} = 0.615(9)$ for Model 3). We remark that for the continuous Ising model we bound nearest-neighbour spins of the same sign, independently of their absolute values, although they play a key role in the definition of the "exact" droplets [1]. However, there is no proof of the existence of an exact
correspondence between the percolation transition of the site-bond clusters for \( p_B = p_{CK} \) and the magnetization transition. So it is essential to calculate precisely the critical exponents to show that the "minimal" clusters are indeed droplets for the system. The results are shown in Table 1, where we can see that the agreement with the critical indices of the 2D Ising droplets is very good.

|                | \( \alpha_p / \nu_p \) | \( \gamma_p / \nu_p \) | \( D \) | \( \Pi \) at \( T_p \) |
|----------------|-----------------|-----------------|---|----------------|
| 2D Ising       | 1/8=0.125       | 7/4=1.75        | 15/8=1.875 | 0.585(1)       |
| Model 2        | 0.131(10)       | 1.742(12)       | 1.862(20)  | 0.583(4)       |
| Model 3        | 0.121(9)        | 1.764(14)       | 1.870(11)  | 0.587(3)       |

**TABLE I:** Critical percolation indices for the site-bond clusters of the \( Z(2) \) models when \( p_B = p_{CK} \), compared with the values of the 2D Ising droplets.

![Graph](https://example.com/graph.png)

**FIG. 2:** 2D 3-state Potts model: variation of \( \Pi \) at the percolation temperature \( T_p \) with the bond weight \( p_B \).

As far as the analysis of the \( Z(3) \) spin systems is concerned, we attain the same conclusions. Fig. 2 shows the dependence on \( p_B \) of \( \Pi \) at the percolation threshold for the 3-state Potts model. We notice that we obtain the same pattern we found for the \( Z(2) \) models (see Fig. 1). Since the magnetization transition is characterized by exponents which are different from the Ising ones, one expects to find another set of critical indices to the right of \( p_{CK} = 0.6339736 \). As we can see from the figure, the plateau for \( p_B > p_{CK} \) lies indeed at a different height compared to the one we have in the Ising plot (0.932 instead of 0.9832). Analogously as we did above, we call this new set of critical indices 2D \( Z(3) \) site percolation universality class. The values of these indices and of their \( Z(2) \) counterparts were predicted in [24, 27]; our numerical findings confirm such theoretical predictions, which are listed in Table 1.

For the \( Z(3) \) model with competitive interactions the results are the same, apart from the value of the minimal probability (\( p_{CK} = 0.61(1) \)). Again, in order to prove that the minimal clusters are droplets for the system we can only rely on the numerical evaluation of the exponents. We report our estimates in Table 11; the agreement with the indices of the 3-state Potts droplets is good.

|                | \( \alpha_p / \nu_p \) | \( \gamma_p / \nu_p \) | \( D \) | \( \Pi \) at \( T_p \) |
|----------------|-----------------|-----------------|---|----------------|
| 2D 3S Potts    | 2/15            | 26/15           | 28/15 | 0.649(9)       |
| Model 2        | 0.143(17)       | 1.725(21)       | 1.858(18) | 0.646(11) |

**TABLE III:** Critical percolation indices for the site-bond clusters of the \( Z(3) \) models when \( p_B = p_{CK} \).

In three dimensions, as we said in Section II, the situation looks quite different. We performed simulations of the 3D Ising model, using several values for the bond probability above and below \( p_{CK} = 1 - \exp(-2J/kT) = 0.35808 \). We studied the variation of \( \Pi \) with \( p_B \) as we did in two dimensions: the result is shown in Fig. 3. For \( p_B \neq p_{CK} \), the percolation temperature \( T_p \) is different from the magnetization temperature \( T_c \) and we recover the 3D random percolation exponents (see [29]). Only for \( p_B = p_{CK} \), \( T_p = T_c \) and we could eventually get the thermal critical indices (for the 3D Ising model this is exactly what happens).

![Graph](https://example.com/graph.png)

**FIG. 3:** 3D Ising model: variation of \( \Pi \) at the percolation temperature \( T_p \) with the bond weight \( p_B \).

We notice that, even if the 3D pattern is very different from the 2D one, the special (Fortuin-Kasteleyn) proba-
bility $p_{CK}$ is still the smallest probability for which the site-bond clusters can form a percolating cluster at $T_c$ (for $p_B < p_{CK}$, $T_B < T_c$ and all clusters at $T_c$ are finite).

We want to verify whether this is also valid for other 3D spin systems, and we analyze here the $O(2)$, or $XY$, model. For $O(n)$ models a rigorous mapping between percolation and critical behaviour was established in [1]. The droplets are built in two steps:

1. choose a random vector $r$ of $O(n)$;
2. bind together any pair of nearest-neighbouring spins $s_i$, $s_j$ with the probability

$$p(i, j) = 1 - \exp\{\min[0, -2\beta(s_i \cdot r)(s_j \cdot r)]\} \quad (\beta = J/kT).$$

Such droplets are just the clusters devised by Wolff in his famous algorithm [28] for $O(n)$ spin systems. We see that only pairs of spin vectors having both a positive/negative projection on the random vector $r$ can be joined to each other. The random vector $r$, therefore, divides the spin space in two hemispheres, separating the spins which have a positive projection onto it from the ones which have a negative projection. The droplets are made out of spins which all lie either in the one or in the other hemisphere. In this respect, we can again speak of ‘up’ and ‘down’ spins, like for the Ising model. In addition to that, the bond probability is local, since it explicitly depends on the spin vectors $s_i$ and $s_j$, and not only on the temperature like the Fortuin-Kasteleyn factor.

The situation is similar as in the 2D continuous Ising model we considered above, and we proceed in the same way, i.e. we reduced the $O(2)$ configurations to Ising configurations, according to the sign of the projection of the spins on $r$, so disregarding the length of the projection. The bond weight $p_B$ we introduced is the same for each pair of nearest-neighbouring sites.

For our simulations we applied the Wolff algorithm and used four lattices: $24^3$, $48^3$, $72^3$ and $96^3$. At each run, 40000 to 100000 measurements were taken. We tuned the bond probability so to make the percolation point coincide with the magnetization point. At the end we found the same scenario that we had seen for the Ising model (Fig. 4).

Our estimate of the minimal bond probability is $p_{CK} = 0.374(1)$. Finally we calculated the critical indices of the percolation transition when $p_B = p_{CK}$. Fig. 4 shows the corresponding finite size scaling plots of the percolation strenght $P$ (top) and the average cluster size $S$ (bottom) at the critical point. The $\chi^2$ of the fits improves considerably if the smallest lattice ($24^3$) is excluded, this is why we put only three points in the plots.

The critical indices we extracted agree with the thermal $O(2)$ values (Table IV).

C. SU(2) Pure Gauge Theory

The deconfining transition from hadronic matter to a plasma of quarks and gluons has been object of intensive investigations over the last two decades. Though the concrete goal is to try to produce the quark-gluon plasma by means of high energy heavy ion collisions in the lab, i.e. in tiny and in general non-equilibrated fireballs, it is crucial from a theoretical point of view to study...
Table IV: Critical percolation indices for the site-bond clusters of 3D O(2) when \( p_B = p_C K \); for comparison we also report the O(2) thermal exponents (from [29]).

| \( \beta_p/\nu_p \) | \( \gamma_p/\nu_p \) | D |
|---------------------|---------------------|-----|
| 3D O(2)            | 0.5189(3)           | 1.9619(5) | 2.4808(8) |
| Perc. Exponents    | 0.530(15)           | 1.971(13) | 2.484(7)  |

The ideal situation of an infinite system of strongly interacting matter in thermal equilibrium at a temperature \( T \). This could be effectively done after the discovery of the lattice approach [30], and indeed finite temperature Quantum Chromodynamics (QCD) has been extensively simulated on the lattice since then.

The group that rules the gauge invariance of QCD is \( SU(3) \), which is non-abelian. Because of that the gauge fields are self-interacting and it makes sense to study systems constituted only by gluons. This simpler situation is described by the so-called \( SU(3) \) pure gauge theory. Since any \( SU(N) \) group is non-abelian, the study of the relative pure gauge theories may be of interest also for \( N \neq 3 \).

Suppose we have a \( d \)-dimensional box containing gluons at a temperature \( T \). The discretization of space-time returns a \((d+1)\)-dimensional lattice, with \( N_x \) spacings in each space direction and \( N_t \) spacings in the imaginary time (or temperature) direction. The partition function of finite temperature \( SU(N) \) pure gauge theories on this lattice takes the form

\[
Z(N_x, N_T; g^2) = \int \prod_{\text{links}} \, dU_{ij} \exp[-S(U)],
\]

where \( S(U) \) is the Wilson action

\[
S(U) = \frac{2N}{g^2} \sum_{\text{plaq}} \left( 1 - \frac{1}{N} \text{Re} \text{Tr} \, UUUU \right). \tag{7}
\]

Here \( g \) is the (temperature-dependent) coupling and \( U_{ij} \) the so-called link variable, which is a function of the gauge fields set between a nearest-neighbouring sites \( i \) and \( j \). The product in Eq. (6) runs over all links of the lattice, the sum in Eq. (7) over all the smallest closed paths (plaquettes), which are formed by four links; \( UUUU \) is the product of the link variables corresponding to each side of a plaquette.

All \( SU(N) \) pure gauge theories undergo a transition from a phase in which the gluons are bound in glueballs to a phase of free gluons. Such deconfining transition is due to the spontaneous breaking of a global \( Z(N) \) symmetry which results from the periodicity of the gauge fields in the temperature direction [31]. The order parameter is the lattice average of the Polyakov loop, defined as

\[
L = |\langle L_{\vec{x}} \rangle| \tag{8}
\]

with

\[
L_{\vec{x}} = \frac{1}{N} \text{Tr} \prod_{t=1}^{N_r} U_{\vec{x},t,t+1}, \tag{9}
\]

The product in (9) runs over all the \( U \)'s in the temperature direction taken at a given spatial site \( \vec{x} \). In the confined phase \( L = 0 \), whereas at deconfinement \( L \neq 0 \). The main features of the deconfining transition are then all in the Polyakov loop configurations one obtains by projecting out the temperature direction of the lattice through the matrix product of Eq. (9).

There are conjectures suggesting that the deconfining transition of \( SU(N) \) pure gauge theories is intimately related to the magnetization transition of the \( N \)-state Potts model, with which they share the \( Z(N) \) symmetry [32]. In particular, it was predicted that if the deconfining transition is second-order, the critical indices are in the universality class of the corresponding Potts model: this prediction has been confirmed by computer simulations without exceptions. This is actually the reason why we investigated simple Potts-like spin systems. As our droplet prescription seems to work for these models, we tried to see whether it is correct for \( SU(N) \) gauge theories as well, at least in the cases in which the deconfining transition is continuous. In this way we would have for the confinement-deconfinement transition the same geometrical picture as for magnetization in the Potts model.

We start by exposing the results for \( 2 + 1 \) SU(2). We carried on our simulations on four different lattices,
64^2 \times 2, 96^2 \times 2, 128^2 \times 2 and 200^2 \times 2. For each run we collected from 10000 to 40000 measurements. The result of our analysis is identical as for the Z(2) spin systems (Fig. 2).

FIG. 6: 2 + 1 SU(2): variation of Π at the percolation temperature \( T_p \) with the bond weight \( p_B \).

If we compare Fig. 6 to Fig. 1 we see no differences, except in the value of the minimal bond probability (for SU(2), \( p_{CK} = 0.6275(5) \)).

This is truly remarkable and shows the substantial analogy of the two cases. The exponents we calculated for \( p_B = p_{CK} \) are in good accord with the 2D Ising ones, which shows that the minimal clusters are critical droplets for SU(2) as well (Table V).

| Bond Probability \( p_B \) | Threshold Value of the Percolation Cumulant | 2D Z(2) SP Univ. Class |
|-----------------------------|-------------------------------------------|------------------------|
| 0.3                         | 0.6                                      | 1                      |
| 0.4                         | 0.7                                      | 2D Ising Univ. Class   |
| 0.5                         | 0.8                                      | 2D RP Univ. Class      |

TABLE V: Critical percolation indices for the site-bond clusters of 2 + 1 SU(2) (\( N_c = 2 \)) when \( p_B = p_{CK} \), compared with the values of the 2D Ising droplets.

| Bond Probability \( p_B \) | Threshold Value of the Percolation Cumulant | 3D Ising Univ. Class |
|-----------------------------|-------------------------------------------|---------------------|
| 0.3                         | 0.4                                      | 1.875               |
| 0.4                         | 0.5                                      | 1.75                |
| 0.5                         | 0.6                                      | 1.6                 |

FIG. 7: 3 + 1 SU(2): variation of Π at the percolation temperature \( T_p \) with the bond weight \( p_B \).

TABLE VI: Critical percolation indices for the site-bond clusters of 3 + 1 SU(2) (\( N_c = 2 \)) when \( p_B = p_{CK} \), compared with the values of the 3D Ising droplets (from [33]).

| Bond Probability \( p_B \) | Threshold Value of the Percolation Cumulant | 3D Ising Univ. Class |
|-----------------------------|-------------------------------------------|---------------------|
| 0.3                         | 0.4                                      | 1.875               |
| 0.4                         | 0.5                                      | 1.75                |
| 0.5                         | 0.6                                      | 1.6                 |

IV. CONCLUSIONS

We have shown that the phase transition of many systems can be interpreted as a geometrical percolation transition of simple site-bond clusters, like in the Ising model. We have found a general criterion to define the “correct” bond probability \( p_{CK} \) of geometrical connection between nearest-neighbouring sites carrying spins of the same sign: \( p_{CK} \) is the minimal probability for which it is still possible to have a percolating cluster at the critical temperature \( T_c \) of the system. We remark that this criterion looks a bit artificial, since it imposes by hand the coincidence of the percolation with the thermal threshold. So, if one studies a new model, it would be impossible to define the droplets until one finds the critical temperature of the system. However, this is not relevant for us, as our aim was just to show that the droplets exist. We investigated a wide variety of models, from spin systems (discrete and continuous) to SU(2) lattice gauge theory, both in two and in three dimensions. To the extent of these models our recipe provides indeed a solution of the problem, valid even for systems for which “exact” droplets could not so far be identified (models with competitive interactions, SU(2)). The generality of the solution is clearly shown by its validity for a complex theory like SU(2): the effective theory of the Polyakov loop for SU(3) gauge theory consists of a mixture of many different interactions, short- and long-ranged, ferromag-
netic and antiferromagnetic, including couplings between more than two spins (like plaquette-interactions, six-spin couplings, etc.) and self-interactions. Our analysis is entirely numerical, but the result is most likely exact, as it is for Ising.

The droplet definition we propose puts in evidence the key role of geometrical connectivity in the mechanism of the phase transition. This is easy to understand for the models we have considered, where the nearest-neighbour coupling is by far the most important compared to eventual longer-ranged interactions and determines the critical behaviour. However this may not be valid when the strength of other couplings is comparable to the nearest-neighbour one: in this case the phase transition could be influenced as well by the other interactions (e.g. the critical indices might change) and a droplet definition based only on nearest-neighbour connections is probably inadequate.

In our analysis of the models with $Z(2)$ symmetry we have seen that the essential spin feature for the droplet definition is the $Z(2)$ variable, i.e. the sign of the spin. In fact, we built the clusters in the same way, no matter if the spin is discrete like in Ising or continuous like in $SU(2)$. This shows once more the crucial role played by the $Z(2)$ symmetry, whose spontaneous breaking is indeed responsible for the phase transition. Since the relationship between $SU(N)$ pure gauge theories and $Z(N)$ spin models holds for any $N$, our result should be valid for all $SU(N)$ pure gauge theories with a continuous deconfining transition, i.e. also for $SU(3)$ and $SU(4)$ in $2 + 1$ dimensions. The special lattice regularization of the gauge theory does not play a role, as the critical behaviour has the same features for any $N$. So, we found that the confinement-deconfinement phase transition of $SU(N)$ pure gauge theory, if second-order, is equivalent to a percolation transition of special site-bond clusters of like-signed Polyakov loops: this is the most important result of our work. Very recent studies show that percolation can help to describe as well the chiral transition of special fermion lattice models.

We focused only on systems with a continuous second-order transition because the percolation phenomenon is typically smooth. This does not mean that we cannot use percolation to describe discontinuous phase changes. The Fortuin-Kasteleyn mapping is valid for any $q$-state Potts model in any dimension, and the equivalence between the magnetization and the percolation order parameter holds for first-order phase transitions as well. In this case at the critical point there is a coexistence of a paramagnetic and a ferromagnetic phase, which correspond to two different “geometrical phases”, characterized by small and large clusters, respectively. Therefore, the percolation order parameter jumps at the threshold exactly as the magnetization (from zero to a non-zero value). Because of that we believe that the criterion of the minimal bond probability we adopted here can be directly extended to systems undergoing a discontinuous phase transition. Now it is however more difficult to support the idea of droplets because there are no critical indices to reproduce. One could nevertheless try to see whether the percolation variables $(P, S)$ vary near $T_c$ like their thermal counterparts $(m, \chi)$. In this way one could study in particular the “physical” confinement-deconfinement transition of $SU(3)$ pure gauge theory in $3 + 1$ dimensions (quenched QCD).

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[17] The studies reported in [5, 6] concerned lattice regularizations of $2 + 1$ and $3 + 1$ $SU(2)$ gauge theory which correspond to the so-called strong coupling limit: the number $N_c$ of lattice spacings in the temperature direction is then constrained to be small, e.g. $N_c = 2, 3, 4$. 
The proposed percolation picture fails for \( N_{\tau} \gg 4 \).

By approximating the bond weight \( p_{CK}(T) \) with its value \( p_{CK}(T_c) \), the power law behaviour of the percolation variables at criticality would not be appreciably affected and the critical exponents would remain the same. Besides, we calculated the exponents through finite size scaling fits of the percolation variables at \( T_c \), so that the above-mentioned approximation does not affect at all our numerical results.

We refer here explicitly to three dimensions because we could prove our conjecture numerically for models in 3D. However, at least for the Ising model, whatever we said about the 3D case is valid in general for any dimension higher than two.

Below the upper critical dimension \( (d = 4 \text{ for Ising-like systems}) \) and in the infinite volume limit there can be just a single percolating cluster at the critical point (see A. Coniglio, Physica A 281, 129 (2000)). Nevertheless, since we are forced to simulate our systems on finite lattices, it may happen that one finds more than a spanning cluster in the same configuration, although this is quite rare. In such cases we consider all spanning clusters as "pieces" of one larger percolating cluster, whose size is then given by the sum of the individual sizes.

Between the size \( s \) and the radius \( R \) of a cluster there is a relation of this type: \( s \propto R^D \). The exponent \( D \) is the fractal dimension of the cluster.

This is necessary to determine the thermal critical temperature of some of the models we have studied, for which it is unknown.

Very near the critical probability \( p_{CK} \), the infinite volume limit behaviour cannot be reproduced even on the largest lattice we have used. Therefore, the value of \( \Pi \) at the threshold on the largest lattice still differs from the asymptotic universal value by some amount and the corresponding point in plots like Fig. 3 does not sit on the relative plateau.

If the percolation transition occurs at \( T_p \neq T_c \), the physical correlation length \( \xi(T_p) \) is finite; therefore, clusters separated by a distance \( d > \xi(T_p) \) are not influenced by each other, and the long-range behaviour is the same as in random percolation.

\[ Z(N) \text{ is the center of the SU}(N) \text{ group, i.e. the subset of those elements which commute with each element of SU}(N). \text{ Such elements are the matrices obtained by multiplying the unit matrix by the complex } N\text{-roots of 1}. \]

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