Bound entangled states with extremal properties

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Following recent work of Beigi and Shor, we investigate PPT states that are “heavily entangled.” We first exploit volumetric methods to show that in a randomly chosen direction, there are PPT states whose distance in trace norm from separable states is (asymptotically) at least 1/4. We then provide explicit examples of PPT states which are nearly as far from separable ones as possible. To obtain a distance of $2 - \epsilon$ from the separable states, we need a dimension of $2^{\text{poly}(\log(\epsilon^{-1}))}$, as opposed to $2^{\text{poly}(1/\epsilon)}$ given by the construction of Beigi and Shor \cite{BeigiShor}. We do so by exploiting the so called private states, introduced earlier in the context of quantum cryptography. We also provide a lower bound for the distance between private states and PPT states and investigate the distance between pure states and the set of PPT states.

The set of PPT states (i.e. states with positive partial transpose) plays an important role in quantum information theory. While the PPT criterion perfectly discovers entanglement in pure states and for $2 \otimes 2$ and $2 \otimes 3$ systems, it is not always conclusive \cite{Horodecki:09} in higher dimensions. The entangled states that have the PPT property are known to be bound entangled: no pure entanglement can be distilled from them. It is a longstanding open problem whether this last property is equivalent to PPT (see \cite{Marshall:13} and references therein). On the other hand, it is possible to obtain cryptographic key from some PPT states \cite{Beigi:06}. In view of such operational characteristics – or conjectured characteristics – of the set of PPT states, it was often used as a first approximation of the set of separable states.

The geometric properties of sets of PPT states ($\text{PPT}$) and of separable states ($\text{SEP}$) were investigated starting with \cite{Pirandola:09}. Recently there has been interest in quantifying how different $\text{PPT}$ and $\text{SEP}$ are. It was shown in \cite{Marshall:13} that the ratio of the volumes of $\text{PPT}$ and $\text{SEP}$ grows super-exponentially in the dimension of the sets. The distance between a PPT state and $\text{SEP}$ was investigated in \cite{Henderson:01}, where it was proved that there exist PPT states that lie as far from separable states as it is possible, namely $2 - \epsilon$ in trace norm distance,\footnote{Here and further in this paper by trace norm distance we mean $\|\rho - \sigma\|_1$ were $\|\cdot\|_1$ is the trace norm. In \cite{Henderson:01} the distance with an additional factor 1/2 was used.} for any positive $\epsilon$, provided the dimension is large enough. Thresholds for the PPT property and for separability for random induced states were compared in \cite{Bennett:99} (see also \cite{Bennett:99,14}) and shown to be dramatically different.

In this paper we will revisit the phenomena studied in \cite{Henderson:01}. First, we will show how similar results can be deduced by well-known methods from the values of various geometric invariants of $\text{PPT}$ and $\text{SEP}$ calculated in \cite{Marshall:13}. A sample result states that a “generic witness” can detect a PPT state whose separability violation is about 1/4. Next, we provide an alternate (explicit) construction of a family of states that recovers the $2 - \epsilon$ bound from \cite{Henderson:01} and show how their dimensions scale depending on $\epsilon$.

With regards to the construction, our argument is based on private states, which were introduced in order to investigate the relationship between quantum security and entanglement \cite{Cleve:00}. They have been already used in the context of cryptography \cite{Dziembowski:01,12}, as well as in channel theory \cite{Devetak:03,15}. Here we use this class to investigate the geometry of the set $\text{PPT}$. The general idea is that every private state $\gamma$ is “rather far” from any separable state \cite{17,18}. If we can show that some PPT state $\rho$ is not “too far” from $\gamma$, we obtain easily a lower bound on the distance between $\rho$ and the set $\text{SEP}$ of separable states. Similarly as in \cite{Henderson:01}, our construction involves taking tensor power of some chosen state (here it is the one constructed in \cite{Beigi:06}). However, we do not use tools such as de Finetti theorem or quantum tomography, but instead rely on simple permanence properties of the sets in question. Our construction is essentially self-contained; it vastly improves the scaling of the dimension needed to obtain distance $2 - \epsilon$, which in our case is $2^{\mathcal{O}(\log(\epsilon^{-1}))}$, with $\mathcal{O} < 6$. (Here and in what follows all logarithms are to the base of 2.) No explicit formula is given in \cite{Henderson:01}, but an examination of the argument presented there shows that it requires the dimension to be of order $2^{(\kappa/\epsilon)^2}$, where $\kappa$ is at least 2 (and probably larger).

We also analyze limitations of the approach via private states due to the fact that, in finite dimension, there is always a nonzero gap between private states and PPT states. We obtain a lower bound on this gap in the case of $\mathbb{C}^{2d} \otimes \mathbb{C}^{2d}$ states (private bits), extending results of \cite{Muller-Ammann:07}. We also find that the “distance” of pure states from PPT states in terms of fidelity is the same as that from separable states. This shows that our construction could not work with the set of pure states instead of the set of private states as a starting point.
In this work we use the following notation. For a state \( \rho_{AB} \) on a composite system \( A \otimes B \) we denote the partial transpose on system \( B \) as \( \rho_{AB}^{T_B} := (I_T \otimes T)(\rho_{AB}) \), where \( T \) is the transpose map (while the result does depend on what system we perform the partial transpose, its positivity does not). We denote the trace norm by \( \|X\|_1 := \text{Tr}\sqrt{XX^\dagger} \) and, more generally, the \( p \)-Schatten norm by \( \|X\|_p := (\text{Tr}(XX^\dagger)^{p/2})^{1/p} \). When talking about the distance of a state \( \rho \) from the set of separable states we will always mean the quantity
\[
\text{dist}(\rho, \mathcal{SEP}) := \min_{\sigma \in \mathcal{SEP}} \|\rho - \sigma\|_1.
\]
However, analogous expressions for other norms, and for properties different from PPT and separability, may be of interest and can also be studied by some of the methods we employ below.

I. BOUNDS BASED ON GLOBAL GEOMETRIC INVARANTS

One of the results of [8] (Theorem 1) asserts that for \( \mathbb{C}^d \otimes \mathbb{C}^d \), the ratio of the volumes of \( \mathcal{PPT} \) and \( \mathcal{SEP} \) is at least \((cd)^{m/2} \), where \( m := d^4 - 1 \) is the dimension of these sets and \( c > 0 \) is a universal (explicit and not too small) constant. This implies immediately that there is a PPT state \( \rho \) whose robustness \([21]\) is at least of order \( d^{1/2} \); if \( \varepsilon > c^{-1}d^{-3/2} \), then the mixture \( \rho + (1 - \varepsilon)I/d^2 \) is entangled. (The same assertion holds with the maximally mixed state \( I/d^2 \) replaced by any other separable state \( \sigma \), which is called in [21] robustness relative to \( \sigma \).)

The geometric invariant that played more fundamental role than volume in the arguments of [8] was the mean width, which is defined as follows. If \( K \) is a subset of a (real) Euclidean space we define its mean width (actually mean half-width), denoted \( w(K) \), as
\[
w(K) := \int_{S} \max_{x \in K} \langle x, u \rangle \, du,
\]
where \( S \) is the unit sphere of the space in question and the integration is performed with respect to the normalized invariant measure on \( S \). For a given unit vector \( u \in S \), the expression
\[
h_K(u) := \max_{x \in K} \langle x, u \rangle
\]
is usually called the width of \( K \) in the direction of \( u \) (“the extent of \( K \) in the direction of \( u \)” would be perhaps more appropriate). See Fig. 1 for graphical interpretation of the quantity.

The mean width is related to the volume by the classical inequality of Urysohn \( \text{vrad}(K) \leq w(K) \), where \( \text{vrad}(K) \) (the volume radius of \( K \)) is the radius of a Euclidean ball whose volume is equal to that of \( K \).

The asymptotic order of the mean widths of \( \mathcal{PPT} \) and \( \mathcal{SEP} \) – with respect to the Euclidean structure induced by the Hilbert-Schmidt (or Frobenius) norm and as the dimension goes to infinity in various regimes – was determined in [8]. For the bipartite systems \( \mathbb{C}^d \otimes \mathbb{C}^d \) we have the inequalities (valid for all \( d \))
\[
\frac{1}{6}d^{-3/2} \leq \text{vrad}(\mathcal{SEP}) \leq w(\mathcal{SEP}) \leq 4d^{-3/2},
\]
\[
\frac{1}{4}d^{-1} \leq \text{vrad}(\mathcal{PPT}) \leq w(\mathcal{PPT}) \leq 2d^{-1},
\]
and the limit relation
\[
\lim inf d \ w(\mathcal{PPT}) \geq \lim inf d \ \text{vrad}(\mathcal{PPT}) \geq \frac{1}{2}.
\]

The details of some of the calculations that lead to the specific numerical values of the multiplicative constants that appear above are contained in [22–24]. In fact, the expectation is that the limits \( \lim d \ w(\mathcal{PPT}) \) and \( \lim d^{3/2} \ w(\mathcal{SEP}) \) exist (and, \textit{a posteriori}, belong to the intervals \([1/2, 2]\) and \([1/6, 4]\) respectively), but we do not know of a rigorous argument to that effect. By comparison, the precise asymptotic order of the mean width of the set of all states on \( \mathbb{C}^n \) is known to be \( 2n^{-1/2} \) (i.e., \( 2d^{-1} \) in our setting; that’s where the upper estimate in (5) comes from). However, even this fact far from being obvious: the reason for the factor 2 is the “radius 2” in Wigner’s Semicircle Law [25, 26]; cf. Lemma 2 below and the comments following it.

As it turns out, much more information is available in addition to the bounds on the averages of the width functions of \( h_{\mathcal{PPT}} \) and \( h_{\mathcal{SEP}} \) given by (4–6): one has essentially the same pointwise estimates for \( h_{\mathcal{PPT}}(u) \) and \( h_{\mathcal{SEP}}(u) \) for all but a very small fraction of directions \( u \in S \). This is a consequence of the classical Levy’s concentration inequality.
Lemma 1. Let $m > 2$ and let $f$ be an $L$-Lipschitz function on the sphere $S$ in the $m$-dimensional Euclidean space. Then, for every $t > 0$,

$$P(|f - M| > t) \leq \exp\left(-\frac{mt^2}{2L^2}\right),$$

(7)

where $M$ is the median of $f$ and $P$ is the normalized invariant measure on $S$.

For functions of the form $f$, the Lipschitz constant $L$ equals the outradius of $K$. The outradius of the set of all states on $\mathbb{C}^n$ is $\sqrt{1 - \frac{1}{n}} < 1$ (provided the center of the circumscribed sphere is chosen to be at the maximally mixed state $1/d^2$, which is the natural choice) and so – for width functions of sets of states such as $f = h_{\text{PPT}}$ or $f = h_{\mathcal{SEP}}$ – the constant $L$ disappears from the estimate. Since the dimension of the space is then $m = d^4 - 1$, it follows that the probability in (7) is small if $t > d^{-2}$. Still another elementary consequence of (7) is that the mean in (8) is approximately $2d^{-1/2}$.

Theorem 1. Let $\epsilon > 0$. Then, for $d$ large enough (depending on $\epsilon$),

$$\max_{\rho \in \text{PPT}} \text{dist}(\rho, \mathcal{SEP}) \geq \frac{1}{4} - \epsilon. \quad (11)$$

Moreover, this distance is witnessed in most directions $u \in S$.

For the proof, consider any direction $u \in S$ for which $h_{\text{PPT}}(u) - h_{\mathcal{SEP}}(u) > (\frac{1}{2} - \frac{\epsilon}{4})d^{-1}$; by (9) and the comments following it this happens with probability close to 1 if $d$ is large. Assume also that $u$ does not belong to the (small) exceptional set given by the condition from Lemma 2, so that in particular $|u|_\infty < (2 + \epsilon)n^{-1/2}$.

\[ \langle \rho - \sigma, u \rangle = h_{\text{PPT}}(u) - h_{\mathcal{SEP}}(u) \geq \frac{1}{2} - \frac{\epsilon}{4}d^{-1}. \quad (12) \]

On the other hand,

\[ \langle \rho - \sigma, u \rangle \leq \|\rho - \sigma\|_1 |u|_\infty \leq (2 + \epsilon)d^{-1}\|\rho - \sigma\|_1. \quad (13) \]

Combining these inequalities leads to

\[ \|\rho - \sigma\|_1 \geq \frac{1}{2} - \frac{\epsilon}{4}/(2 + \epsilon) > \frac{1}{4} - \epsilon. \quad (14) \]

This means that the distance of $\rho$ to $\mathcal{SEP}$ in trace distance is at least $\frac{1}{4} - \epsilon$ and that such distance can be certified by nearly all witnesses $u \in S$ (for an appropriate $\rho \in \text{PPT}$, depending on $u$).

II. PPT STATES DISTANT FROM SEPARABLE STATES: A CONSTRUCTION BASED ON PRIVATE STATES

In the preceding section we showed that, in sufficiently large dimension, PPT states that are quite far
from the set of separable states are ubiquitous. However, our argument was of a probabilistic nature, hence non-constructive.

In the present section we will give an explicit construction of PPT states that are nearly as far from separable states as possible. The main result is stated in Theorem 2, which provides a bipartite PPT state whose distance from the set of separable states is larger than \(2 - \epsilon\), with the dimension of the system scaling like \(2^{O((\log^2(1/\epsilon)))}\), i.e., involving the number of qubits that is polylogarithmic in \(1/\epsilon\). We thus recover the main result of [1], with a much better dependence of \(\epsilon\) on the dimension. We also consider limitations of our approach, generalizing results of [20] in Proposition 2 which investigates distance between PPT states and the so-called “private states,” introduced originally in the context of quantum cryptography in [4].

1. Private, separable and PPT states

In our construction of PPT states which are far from separable states, we will employ “private states” [3]. Their precise definition will be given later in Appendix A but here we will only need their features listed below:

- any private state is far from separable states, the distance increasing with the dimension (Lemma 3),
- at the expense of dimension, there are private states arbitrary close to PPT states (Eq. (17)),
- tensor product of private states is again a private state.

The idea is now to exploit the first two features and the triangle inequality to obtain PPT states whose distance to any separable state is at least about 1 (see Fig. 2). We next consider tensor powers of that state, which of course remain PPT, and show – by combining the first and the third feature – that their distance to separable states can be boosted as closely to 2 as desired, at the expense of increasing the dimension.

Private states are states of four systems \(A, B, A',\) and \(B'\). The systems \(A\) and \(B\) constitute the key part, while \(A'\) and \(B'\) - the so-called shield part. The corresponding Hilbert spaces are \(\mathcal{H}_A = \mathcal{H}_B = C^{d_k}\) and \(\mathcal{H}_{A'} = \mathcal{H}_{B'} = C^{d_s}\). When \(d_k = 2\), we will call a private state a private bit. It is immediately seen from the definition that the tensor product of two private states is again a private state, with the key and shield dimensions of the product state being products of those of original states. The following lemma 17 (cf. 18) quantifies the distance of an arbitrary private state from the set of separable states.

**Lemma 3.** For any private state \(\gamma\) with the key part of dimension \(d_k \times d_k\) we have

\[
\text{dist}(\gamma, \mathcal{SEP}) \geq 2 - \frac{2}{d_k}, \tag{15}
\]

Now set \(d_k = 2\) and consider the following state constructed in [19]

\[
\rho = (1 - p)\gamma + p\gamma', \tag{16}
\]

where \(p = \frac{1}{\sqrt{d_s + 1}}\) and where \(\gamma, \gamma'\) are mutually orthogonal private states given by Eq. (A7) (Appendix A). The matrix form of \(\rho\) is also presented in Eq. (A8). The state \(\rho\) has the following properties: (i) it is PPT, since by construction it is invariant under the partial transpose; (ii) it is close to the private state \(\gamma\) since we have

\[
\|\rho - \gamma\|_1 = 2p = \frac{2}{\sqrt{d_s + 1}}. \tag{17}
\]

Consider now the closest, in trace norm, separable state to \(\rho\), call it \(\sigma\). Using Lemma 3 for \(d_k = 2\) and the triangle inequality we obtain

\[
\|\rho - \sigma\|_1 + \|\rho - \gamma\|_1 \geq \|\sigma - \gamma\|_1 \geq \text{dist}(\gamma, \mathcal{SEP}) \geq 1 \tag{18}
\]

Applying now (17) we obtain the following

**Proposition 1.** Let \(\rho\) be the state given by (16). Then its distance to the set of separable states satisfies

\[
\text{dist}(\rho, \mathcal{SEP}) \geq 1 - \frac{2}{\sqrt{d_s + 1}}. \tag{19}
\]

We see that this lower bound improves with a larger shield part, and is the worst for \(d_s = 2\) (then \(\rho\) is four-qubit state). In that case we have \(\text{dist}(\rho, \mathcal{SEP}) \geq 0.58579\).

It is known that the state \(\rho\) lies on the boundary of PPT states (see [19] Observation 2), so its choice is in a sense optimal. To see to what extent the estimate could be improved, we recall that if a PPT state
\[ \rho = \sum_{ijkl=0}^{1} |ij\rangle \langle kl| \otimes A_{ijkl} \text{ on } \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{d_l} \otimes \mathbb{C}^{d_s}, \]
with hermitian \( A_{0011} \), approximates a private state with shield of dimension (necessarily) \( d_s \times d_s \), then by \[ |\rangle \langle \rho - \gamma| \geq \frac{1}{2(d_s + 1)}. \] (20)

We will show here that the above bound holds in general for private bits, i.e., even if \( A_{0011} \) is not hermitian.

**Proposition 2.** Let \( \rho, \gamma \) be states on \( \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{d_l} \otimes \mathbb{C}^{d_s} \) such that \( \rho \) is PPT and \( \gamma \) is private. Then the bound \[ |\rangle \langle \rho - \gamma| \geq \frac{1}{2(d_s + 1)}. \] (20)

We prove this result in Appendix \[ \text{B} \] Thus – in general – the lower bound of \[ \text{[19]} \] could not be made larger than \( 1 - \frac{1}{2(d_s + 1)} \).

Finally, one can ask if this approach could be simplified by working with separable states instead of private states. In Appendix \[ \text{C} \] we will show that such approach could not work since the distance – in the appropriate sense – between a pure state and the set of PPT states is achieved on separable states. Consequently, one can not find a PPT state which is close to a pure state and far from separable states.

2. Boosting the distance via tensoring

We will now take \( l \) copies of the state \( \rho \) of \[ \text{[16]} \] and consider the PPT state \( \rho^{\otimes l} \) and the private state \( \gamma^{\otimes l} \). By similar argument as in \[ \text{[18]} \] we obtain, for any separable state \( \sigma \),

\[ |\rangle \langle \| \rho^{\otimes l} - \sigma \|_1 \geq |\rangle \langle \| \sigma - \gamma^{\otimes l} \|_1 - \| \rho^{\otimes l} - \gamma^{\otimes l} \|_1 \geq \text{dist}(\gamma^{\otimes l}, \mathcal{SEP}) - \| \rho^{\otimes l} - \gamma^{\otimes l} \|_1 \geq 2 \sqrt{2l} - \| \rho^{\otimes l} - \gamma^{\otimes l} \|_1, \] (21)

where the last inequality follows from Lemma \[ \text{[3]} \] and the fact that the key-part of \( \gamma^{\otimes l} \) is \( 2^l \times 2^l \) dimensional. Next, using \[ |\rangle \langle \| \rho^{\otimes l} - \gamma^{\otimes l} \|_1 \leq l |\rangle \langle \| \rho - \gamma \|_1 \] (which follows by expressing \( \rho^{\otimes l} - \gamma^{\otimes l} = \sum_{i=1}^{l} \rho^{\otimes i} \otimes \gamma^{\otimes i-l} \) \[ \text{[22]} \] and by multiplicativity of \( |\rangle \langle \|_1 \) under tensoring), we are led to

\[ |\rangle \langle \| \rho^{\otimes l} - \sigma \|_1 \geq 2 \left( \frac{\sqrt{2}}{\sqrt{d_s + 1}} \right)^l \] (22)

It is now clear that by appropriately choosing \( l \) and \( d_s \), we can make the last two terms on the right as small as we wish. Indeed, fix \( \epsilon > 0 \) and let \( l \) be the smallest integer that satisfies \( \frac{2l}{\sqrt{d_s + 1}} \leq \frac{\epsilon}{2}, \) i.e., \( l := [\log \frac{\epsilon}{2} \frac{2}{\sqrt{d_s + 1}}] \). Next, let \( d_s \) be the smallest integer satisfying \( \frac{2l}{\sqrt{d_s + 1}} \leq \frac{\epsilon}{\sqrt{d_s}} \), i.e.,

\[ d_s = \left( \frac{4l}{\epsilon} - 1 \right)^2 \] (23)

With such choices, we will have \( |\rangle \langle \| \rho^{\otimes l} - \sigma \|_1 \geq 2 - \epsilon \) for any separable state \( \sigma \). Recall that \( \rho^{\otimes l} \) is, by construction, a PPT state on \( \mathbb{C}^d \otimes \mathbb{C}^d \), where

\[ d = 2^l d_s = 2^l \left( \frac{4l}{\epsilon} - 1 \right)^2 \frac{2}{\sqrt{d_s}}. \] (24)

Recalling that \( l = [\log \frac{\epsilon}{2}] \) and streamlining the formula for \( d \) we obtain

**Theorem 2.** For arbitrary \( \epsilon \) there exists a PPT state \( \rho' \) acting on the space \( \mathbb{C}^d \otimes \mathbb{C}^d \) with \( d \leq 2^C(\log \frac{\epsilon}{2})^2 \) and

\[ \text{dist}(\rho', \mathcal{SEP}) \geq 2 - \epsilon. \] (25)

Here \( C > 0 \) is absolute constant. The state \( \rho \) is given by \( \rho' = \rho^{\otimes l} \) with \( l = [\log \frac{\epsilon}{2}] \) and \( \rho \) given by Eq. \[ \text{[16]} \].

**Remark** It is straightforward to analytically upper-bound the constant \( C \) by 12; numerically we find \( C < 6 \).

One can obtain a slightly better estimate by appealing to equivalence of trace distance and fidelity \( F(\rho_1, \rho_2) := Tr \sqrt{\rho_1 \rho_2} \sqrt{\rho_2 \rho_1} \) \[ \text{[33]} \] and, more precisely, to the (second part of the) relation \[ \text{[34]} \]

\[ 2(1 - F(\rho_1, \rho_2)) \leq |\rangle \langle \| \rho_1 - \rho_2 \|_1 \leq 2 \sqrt{1 - F(\rho_1, \rho_2)^2} \]

specified to \( \rho_1 = \rho^{\otimes l}, \rho_2 = \gamma^{\otimes l} \). Since \( F(\rho^{\otimes l}, \gamma^{\otimes l}) = F(\rho, \gamma)^l \), we can focus on calculating the fidelity between \( \rho \) and \( \gamma \). This is easy since \( \rho = (1 - p) \gamma \oplus p \gamma' \) and so

\[ F(\rho, \gamma) = Tr \sqrt{\sqrt{\gamma} \rho \sqrt{\gamma}} = Tr \sqrt{(1 - p) \gamma^2} = \sqrt{1 - p} \] (27)

Substituting \( p = \frac{l}{\sqrt{d_s + 1}} \) and arguing as earlier we obtain

\[ |\rangle \langle \| \rho^{\otimes l} - \sigma \|_1 \geq 2 - \frac{2}{2l} - 2 \sqrt{1 - \left( \frac{\sqrt{d_s}}{\sqrt{d_s + 1}} \right)^l}. \] (28)

By comparing \[ \text{[22]} \] and \[ \text{[28]} \], and then expanding in powers of \( \alpha = pl \), one finds that the above bound is better than \[ \text{[22]} \]. As previously, we can deduce from \[ \text{[28]} \] how the dimension \( d \) will scale with \( \epsilon \). However, the scaling is pretty much the same, possibly with a better constant.

**Appendix A: Private states**

We present here some basic properties of private states and of the PPT state \( \rho \) of Eq. \[ \text{[16]} \]. These properties can be found, e.g., in \[ \text{[18]} \text{[19]} \], while the construction itself was provided in \[ \text{[19]} \]. We start with the definition of private states.

**Definition 1.** A state \( \rho_{ABA'B'} \) is called a private state if it is of the form

\[ \rho_{ABA'B'} = \sum_{i,j=1}^{d_k} \frac{1}{d_k} |e_i\rangle \langle f_i| |e_j\rangle \langle f_j| \otimes U_i \sigma_{A'B'} U_j^\dagger, \] (A1)

where \( \{ |e_i\rangle \} \) and \( \{ |f_i\rangle \} \) are bases in \( \mathcal{H}_A \) and \( \mathcal{H}_B \) respectively, \( U_i \)'s are unitary transformations acting on the system \( A'B' \), and \( \sigma_{A'B'} \) is a state of that system.
Any private state with $d_k = 2$ can be written (up to change of basis in the key part) in the form

$$\gamma_{ABA'B'} = \gamma(X) = \frac{1}{2} \begin{bmatrix} \sqrt{XX^\dagger} & 0 & 0 & X \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ X^\dagger & 0 & 0 & \sqrt{XX^\dagger} \end{bmatrix}, \quad \text{(A2)}$$

where $X$ is some operator with trace norm one. Note that $X$ completely describes any private state with $d_k = 2$ (again, up to change of basis in the key part).

We next describe the state of Eq. (16) constructed in Appendix B: Distance between PPT states and private states in finite dimension. Consider two matrices of unit trace norm:

$$X = \frac{1}{d_s\sqrt{d_s}} \sum_{i,j=1}^{d_s} u_{ij} |ij\rangle \langle ji|, \quad \text{(A3)}$$

and

$$Y = \sqrt{d_s}X^\Gamma = \frac{1}{d_s} \sum_{i,j=1}^{d_s} u_{ij} |ii\rangle \langle jj|, \quad \text{(A4)}$$

where $u_{ij}$ are matrix elements of some (arbitrary) unitary matrix $U$ acting on $C^{d_s}$ with $|u_{ij}| = 1/\sqrt{d_s}$ for all $i,j$. For definiteness, we may set $U$ to be quantum Fourier transform

$$U|k\rangle = \sum_{j=1}^{d_s} \frac{1}{d_s} e^{2\pi ijk/d_s} |j\rangle. \quad \text{(A5)}$$

The state $\rho$ is then given by

$$\rho = (1-p)\gamma + p\gamma', \quad \text{(A6)}$$

where

$$\gamma = \gamma(X), \quad \gamma' = \sigma_A^x \otimes I_{B'A'B'} \gamma(Y) \sigma_A^x \otimes I_{B'A'B'}, \quad \text{with} \quad p = \frac{1}{1+\sqrt{d_s}}, \quad \sigma_x \text{ being a Pauli matrix. More explicitly} \rho \text{ equals}$$

$$\frac{1}{2} \begin{bmatrix} (1-p)\sqrt{XX^\dagger} & 0 & 0 & (1-p)X \\ 0 & p\sqrt{YY^\dagger} & pY & 0 \\ 0 & pY^\dagger & p\sqrt{YY^\dagger} & 0 \\ (1-p)X^\dagger & 0 & 0 & (1-p)\sqrt{XX^\dagger} \end{bmatrix}. \quad \text{(A7)}$$

**Appendix B: Distance between PPT states and private states in finite dimension**

For the proof of Proposition 2 we need the following simple (and presumably well-known) lemma.

**Lemma 4.** For any operator $A$ in $C^d \otimes C^d$ we have

$$\|A\|_1 \leq d\|A^\Gamma\|_1 \quad \text{and} \quad \|A^\Gamma\|_1 \leq d\|A\|_1. \quad \text{(B1)}$$

The proof of the first inequality uses the following chain of (in)equalities:

$$\|A\|_1 \leq d\|A^\Gamma\|_2 = d\|A^\Gamma\|_2 \leq d\|A^\Gamma\|_1,$$

where the equality follows from the fact that $\Gamma$ only permutes the elements of a matrix, and the inequalities from the bounds $\|A\|_2 \leq \|A\|_1 \leq n^{1/2}\|A\|_2$ valid for any $n \times n$ matrix. The second inequality in (B1) follows then from $\Gamma$ being an involution.

**Remark:** Note that the same bounds hold for the realignment [25], since it also preserves the Schatten 2-norm.

We now turn to the proof of Proposition 2. Let $\rho = \rho_{ABA'B'}$ be a PPT state and consider its block form

$$\rho_{ABA'B'} = \begin{bmatrix} A_{0000} & \times & A_{0011} \\ \times & A_{0101} & A_{0110} \times \\ \times & A_{1001} & A_{1010} \times \\ A_{1100} & \times & A_{1111} \end{bmatrix}, \quad \text{(B2)}$$

where $\times$ denotes unimportant (but not necessarily vanishing) matrix blocks. Our proof will be similar to that of [20]. We assume that $\rho_{ABA'B'}$ is $\epsilon$-close to some private state $\gamma$ in trace norm. To simplify notation, in the rest of the proof we will denote the trace norm $\|\cdot\|_1$ by $\|\cdot\|$.

We will use now the so-called privacy squeezing operation which turns the above state into a 2-qubit state of the form

$$\rho_{AB} = \begin{bmatrix} \|A_{0000}\| & \times & \|A_{0011}\| \\ \times & \|A_{0101}\| & \|A_{0110}\| \times \\ \times & \|A_{1001}\| & \|A_{1010}\| \times \\ \|A_{1100}\| & \times & \|A_{1111}\| \end{bmatrix}, \quad \text{(B3)}$$

where again $\times$ denotes unimportant but not necessarily zero matrix elements. The operation is given by applying first unitary transformation $15$ of the form

$$\sum_{i,j=0}^{1} |ij\rangle_{AB} \langle ij| \otimes U_{ij}^{A'B'} \quad \text{(B4)}$$

where $U_{00}$ and $U_{11}^\dagger$ come from the singular value decomposition (SVD) of $A_{0011}$, and $U_{01}$ and $U_{10}^\dagger$ from the SVD of $A_{0110}$, and then performing partial trace over the systems $A'B'$. Since the state $\rho$ is PPT, the operation applied both to the state itself, as well as to its partial transpose, produces again a state, in particular, a positive operator. Thus

$$\sqrt{\|A_{0000}\|\|A_{1111}\|} \geq \|A_{0011}\|, \quad \text{(B5)}$$

$$\sqrt{\|A_{0101}\|\|A_{1010}\|} \geq \|A_{0011}\|. \quad \text{(B6)}$$

Now, since $\|\rho - \gamma\| \leq \epsilon < 1$ by assumption, Proposition 3 of $15$ implies that

$$\|A_{0011}\| \geq \frac{1}{2} - \epsilon. \quad \text{(B7)}$$
Hence, by (B5), \( \sqrt{\|A_{0000}\| \cdot \|A_{1111}\|} \geq \frac{1}{2} - \epsilon \), and the arithmetic-geometric mean inequality shows then that \( \|A_{0000}\| + \|A_{1111}\| \geq 1 - 2\epsilon \). As a consequence, by the trace condition for \( \rho \), \( \text{Tr}A_{0101} + \text{Tr}A_{1010} \leq 2\epsilon \). Combining this with Eq. (B6) and appealing again to the arithmetic-geometric mean inequality (note that \( \Gamma \) preserves the trace, and \( A_{0101} \) and \( A_{1010} \) are non-negative), we obtain

\[
\|A_{f011}\| \leq \epsilon. \tag{B8}
\]

In this way, we have arrived at

\[
\frac{\|A_{0011}\|}{\|A_{f011}\|} \geq \frac{\frac{1}{2} - \epsilon}{\epsilon}. \tag{B9}
\]

We now use Lemma 4 as it provides a bound on the left hand side of the above inequality, namely

\[
d_s \geq \frac{\|A_{0011}\|}{\|A_{f011}\|}. \tag{B10}
\]

which combined with (B9) implies that the gap between PPT and PS states is

\[
\epsilon \geq \frac{1}{2(d_s + 1)}. \tag{B11}
\]

Thus we proved that the bound of [20] holds in general for private bits, as asserted in Proposition 2.

### Appendix C: Distance between pure states and PPT states

In this section we investigate the distance between a pure state and the set of PPT states. It turns out that the maximal fidelity between a given pure state and a (arbitrary) PPT state equals the maximal fidelity between that pure state and a separable state. Consequently, as we argue below, private states can not be replaced with pure states – in a scheme similar to ours – in order to construct a PPT state which is far from separable states.

**Proposition 3.** For a pure state \( |\psi\rangle \) with Schmidt decomposition \( |\psi\rangle = \sum_i a_i |e_i\rangle |f_i\rangle \) we have

\[
\sup_{\sigma \in \text{PPT}} F(|\psi\rangle \langle \psi|, \sigma) = \sup_{\sigma \in \text{SEP}} F(|\psi\rangle \langle \psi|, \sigma) = \max_i a_i =: M_a. \tag{C1}
\]

Before giving a proof of the proposition let us sketch a derivation of its consequences mentioned earlier: we can not find PPT states far from separable states by taking a pure state \( |\psi\rangle \) \( \equiv \sigma \) in place of private state \( \gamma \) from Eq. (16). Indeed, to obtain – for some PPT state \( \rho \) – a bound analogous to (22) via considerations going along the lines of (21), we would need (asymptotically, when dimension is large) both (i) \( \text{dist}(\sigma^{\otimes l}, \text{SEP}) \approx 2 \) and (ii) \( \|\rho - \sigma^{\otimes l}\|_1 \approx 0 \). The relation (26) between the trace distance and fidelity would then imply (i) \( \sup_{\rho \in \text{SEP}} F(\sigma^{\otimes l}, \sigma) \approx 0 \) and (ii) \( F(\rho, \sigma^{\otimes l}) \approx 1 \). However, by the Proposition, the conditions (i) and (ii) can not be simultaneously satisfied since, by (C1), the first implies \( F(\rho, \sigma^{\otimes l}) \approx 0 \), which contradicts the second.

Analogous argument shows that even the first step of the construction, Proposition 1, can not be implemented – at least via scheme similar to ours – with a pure state \( \tau \) as a starting point. Indeed, we can not simultaneously have \( \text{dist}(\tau, \text{SEP}) \geq c \) (where \( c > 0 \) is a universal constant) and \( \|\tau - \rho\|_1 \approx 0 \). This is not entirely surprising since – as is well known – the PPT criterion perfectly discovers entanglement in pure states, but having precise equality of the first two quantities in (C1) throughout their full range seems remarkable.

**Proof of Proposition 3.** To simplify the notation, assume that \( |e_i\rangle \) and \( |f_i\rangle \) are the computational bases (the argument carries over mutatis mutandis to the general case since the set of PPT states is invariant under unitary operations). Let \( \sigma = \sum_{rstv} b_{rstv} |rs\rangle \langle tv| \). We want to upper-bound \( \sup_{\sigma \in \text{PPT}} F(|\psi\rangle \langle \psi|, \sigma) \). We have

\[
F(|\psi\rangle \langle \psi|, \sigma) = \sqrt{\text{Tr}
(\langle \psi| \langle \psi| \sigma)}) = \sqrt{\text{Tr}
\left( \sum_{ij} a_i a_j |ii\rangle \langle jj| \sum_{rstv} b_{rstv} |rs\rangle \langle tv| \right)}
= \sqrt{\sum_{ij} a_i a_j b_{jjii}}. \tag{C2}
\]

Given that \( \sigma \) is PPT, \( \sigma^F \) is again a state, and so the inequality \( b_{jjii} b_{jjii} \leq b_{jjij} b_{jjij} \) holds for all \( i, j \). Since the elements \( b_{jjij}, b_{jjii} \) are diagonal, hence nonnegative, we can use the arithmetic-geometric mean inequality to obtain

\[
\frac{b_{jjij} + b_{jjii}}{2} \geq |b_{jjii}|. \tag{C3}
\]

Accordingly, (C2) can be upper-bounded using the following chain of relations

\[
\sum_{ij} a_i a_j b_{jjii} = \text{Re} \left( \sum_{ij} a_i a_j b_{jjii} \right)
\leq \left| \sum_{ij} a_i a_j b_{jjii} \right| \leq \sum_{ij} a_i a_j |b_{jjii}|
\leq \sum_{ij} a_i a_j \left( \frac{b_{jjij} + b_{jjii}}{2} \right), \tag{C4}
\]

where in the first equality we use the fact that fidelity is a real number, even though the \( b_{jjii} \)'s may be complex. Next, \( \max_{ij} a_i a_j \leq \max_i a_i^2 =: M_a^2 \) (note that \( a_i \geq 0 \)) and hence, by monotonicity of the square root function,

\[
\text{Tr}(|\psi\rangle \langle \psi| \sigma) \leq \sqrt{M_a^2 \sum_{ij} \left( \frac{b_{jjij} + b_{jjii}}{2} \right)}
= M_a \max_i a_i. \tag{C5}
\]
This bound is easily reached by separable (in fact product) states, and so PPT states are as close in fidelity to $|\psi\rangle\langle\psi|$ as are separable states, which we set out to prove.

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