Dynamical systems method (DSM) for nonlinear equations in Banach spaces

A.G. Ramm
Mathematics Department, Kansas State University, Manhattan, KS 66506-2602, USA
ramm@math.ksu.edu

Abstract

Let \( F : X \to X \) be a \( C^2_{loc} \) map in a Banach space \( X \), and \( A \) be its Fréchet derivative at the element \( w := w_\varepsilon \), which solves the problem \((*) \dot{w} = -A^{-1}_\varepsilon(F(w) + \varepsilon w), \ w(0) = w_0 \), where \( A_{\varepsilon} := A + \varepsilon I \). Assume that \( \|A^{-1}_\varepsilon\| \leq c\varepsilon^{-k}, \ 0 < k \leq 1, \ 0 < \varepsilon > \varepsilon_0 \). Then \((*)\) has a unique global solution, \( w(t) \), there exists \( w(\infty) \), and \((** ) F(w(\infty)) + \varepsilon w(\infty) = 0 \). Thus the DSM (Dynamical Systems Method) is justified for equation \((**)\). The limit of \( w_\varepsilon \) as \( \varepsilon \to 0 \) is studied.

1 Introduction

In [1] the DSM (Dynamical Systems Method) was developed for solving operator equations
\[ F(u) = 0 \] (1.1)
in a Hilbert space. In this note we generalize the DSM for Banach spaces and a more general class of nonlinear operators.

Let \( X \) be a Banach space, not necessarily reflexive, and \( F : X \to X \) be a \( C^2_{loc} \) map. This means that \( F \) is twice Fréchet differentiable and
\[ \sup_{u \in B(u_0,R)} \|F^{(j)}(u)\| \leq M_j(R), \quad j = 1, 2, \] (1.2)
where \( B(u_0, R) := \{u : \|u - u_0\| \leq R\} \). Let \( F'(u) := A := A(u), A_{\varepsilon} := A + \varepsilon I \), where \( I \) is the identity operator.

Math subject classification: 47J05, 47J06, 47J25
key words: dynamical systems method, operator equations, ill-posed problems, nonlinear problems
Assumption A.

\[ \|A_\varepsilon^{-1}\| \leq c_0 \varepsilon^{-k}, \quad \varepsilon \in (0, \varepsilon_0), \quad k = \text{const} > 0, \quad (1.3) \]

where \( c_0 = \text{const} > 0, \varepsilon_0 > 0 \) is an arbitrary small fixed number.

**Theorem 1.** If (1.2) and (1.3) hold, then equation

\[ F(u) + \varepsilon u = 0, \quad \varepsilon \in (0, \varepsilon_0), \quad (1.4) \]

has a solution.

In Section 2 this result is proved.

In Theorem 2 of Section 3 conditions for the convergence \( u := u_\varepsilon \to u_0 \) as \( \varepsilon \to \infty \) are given, where \( u_0 \) solves (1.1).

### 2 Proof of Theorem 1

Consider the equation

\[ \dot{w} = -A_\varepsilon^{-1}(w)[F(w) + \varepsilon w], \quad w(0) = w_0, \quad \varepsilon \in (0, \varepsilon_0), \quad (2.1) \]

where \( \dot{w} = \frac{dw}{dt} \) is the strong derivative, and \( w_0 \in X \) is arbitrary. Let \( h \in X^* \) be an arbitrary linear bounded functional on \( X \). Define \( g(t) := (F(w) + \varepsilon w, h) \), where \( (u, h) \) is the value of the functional \( h \) on the element \( u \in X \), and \( w = w(t) \) is the local solution to (2.1). From the assumptions (1.2) and (1.3) it follows that the right-hand side of (2.1) is locally Lipschitz, so (2.1) has a unique local solution \( w \). We wish to justify the DSM for solving equation (1.4).

The DSM consists of:

a) proving that \( w(t) \) exists globally, i.e., \( \forall t > 0 \),

b) the limit \( \lim_{t \to \infty} w(t) := w(\infty) \) exists, and

c) \( w(\infty) \) solves (1.4).

To prove a), b) and c), we start with the equation

\[ \dot{g}(t) = (A_\varepsilon(w)\dot{w}, h) = -g(t), \quad (2.2) \]

which implies:

\[ g(t) = g(0)e^{-t}, \quad \|g(t)\| = \|g(0)\|e^{-t}. \quad (2.3) \]

Thus

\[ \|F(w) + \varepsilon w\| = \sup_{\|h\| \leq 1} |g(t)| \leq \|F(w_0) + \varepsilon w_0\|e^{-t} : = F_0 e^{-t}, \quad (2.4) \]

and (2.1) implies:

\[ \|\dot{w}\| \leq c_0 \varepsilon^{-k} F_0 e^{-t}. \quad (2.5) \]

From (2.4) it follows that \( \|\dot{w}\| \in L^1(0, \infty) \). This and the Cauchy test for the existence of the limit \( w(\infty) := \lim_{t \to \infty} w(t) \) imply that \( w(\infty) \) exists,

\[ \|w(t) - w(\infty)\| \leq F_0 c_0 \varepsilon^{-k} e^{-t}, \quad (2.5) \]
and
\[ \|w(t) - w_0\| \leq F_0 c_0 \varepsilon^{-k}. \] (2.6)
From (2.4) and (2.1), passing to the limit \( t \to \infty \), one gets
\[ 0 = -A_\varepsilon^{-1}(w(\infty)) [F(w(\infty)) + \varepsilon w(\infty)]. \]
This implies that \( w(\infty) \) solves (1.4). The DSM is justified. Theorem 1 is proved. \( \square \)

**Remark 1.** The solution \( w(\infty) \) depends on the choice of \( w_0 \). Equation (1.4) may have many solutions under the assumptions of Theorem 1.

### 3 Limiting behavior of the solution.

Denote \( w(\infty) := v_\varepsilon := v \). We want to give conditions sufficient for the existence of the limit
\[ \lim_{\varepsilon \to 0} v_\varepsilon = y, \] (3.1)
where \( y \) solves equation (1.1).

First, note that equation (1.4) can be solvable for any \( \varepsilon \in (0, \varepsilon_0) \), but equation (1.1) may have no solution. For example, let \( F(u) = Au - f \), where \( A \geq 0 \) is a bounded selfadjoint operator in \( X = H \), where \( H \) is a Hilbert space and \( f \notin R(A) \), where \( R(A) \) is the range of \( A \). The equation \( A\varepsilon v_\varepsilon - f = 0 \) has a unique solution for any \( \varepsilon > 0 \), but the limiting equation \( Ay - f = 0 \) has no solution. That is why we assume that equation (1.1) is solvable: \( F(y) = 0 \). If \( F(y) = 0 \), then
\[ 0 = F(v) + \varepsilon v - F(y) = F'(y)(v - y) + R + \varepsilon(v - y) + \varepsilon y. \]
By Taylor’s formula one has \( F(v) - F(y) = F'(y)(v - y) + R \). Let \( z := v - y \). Then
\[ A_\varepsilon z + R = -\varepsilon y, \quad A := F'(y). \] (3.2)
Assume that
\[ y = A\psi, \quad \|\psi\| \ll 1, \] (3.3)
where \( \|\psi\| \ll 1 \) means that \( \|\psi\| \) is sufficiently small (see (3.8) below).

Then (3.2) is equivalent to
\[ z = -A_\varepsilon^{-1}R - \varepsilon A_\varepsilon^{-1}A\psi := T(z), \] (3.4)
where
\[ R := F(v) - F(y) - F'(y)z = \int_0^1 ds(1 - s)F''(y + sz)zz. \] (3.5)
Let us check that the map \( T \) maps a ball \( B(0, z) := B_r := \{u : \|z\| \leq r\}, z = v - y, \) into itself and is a contraction in \( B_r \) for a suitable \( r > 0 \). Indeed,
\[ \|T(z)\| \leq \frac{c_0}{\varepsilon^{k}} \frac{M_2}{2} r^2 + \varepsilon \|\psi\| \leq r, \] (3.6)
provided that
\[ r = \frac{\varepsilon^k}{c_0 M_2} \left( 1 - \sqrt{1 - 2c_0M_2\|\psi\|\varepsilon^{1-k}} \right), \] (3.7)
and
\[ \rho := 2c_0 M_2 \| \psi \| \varepsilon^{1-k} < 1. \] (3.8)
Condition (3.8) is satisfied if \( k < 1 \) and \( \varepsilon \) is sufficiently small, or if \( k = 1 \) and \( \| \psi \| \) is sufficiently small. If \( k > 0 \) then \( r = r(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \), and \( T \) maps \( B_{r(\varepsilon)} \) into itself.

Let us check the contraction mapping property. Let \( z, p \in B_r \). Then, using (3.5), one gets
\[ \| T(z) - T(p) \| \leq c_0 \frac{\varepsilon^k}{\varepsilon^k} \| R(z) - R(p) \| \leq c_0 \frac{\varepsilon^k}{\varepsilon^k} \int_0^1 ds(1-s) \left[ \| F''(y + sz) - F''(y + sp) \| r^2 + 2M_2r \| z - p \| \right] \leq c_0 \frac{\varepsilon^k}{\varepsilon^k} \int_0^1 ds(1-s) \left( sM_3 r^2 + 2M_2 r \right) \| z - p \|. \] (3.9)
Thus \( T \) is a contraction on \( B_r \) if
\[ \eta := c_0 \frac{\varepsilon^k}{\varepsilon^k} \left( \frac{M_3 r^2}{6} + M_2 r \right) \leq q < 1. \] (3.10)
If (3.7) and (3.8) hold, then
\[ \eta = O(\varepsilon^k) + 1 - \rho := q < 1 \] (3.11)
if \( \varepsilon \in (0, \varepsilon_0) \) is sufficiently small.

We have proved:

**Theorem 2.** Assume that: 1) equation (1.1) is solvable, \( F(y) = 0 \), 2) (1.2) holds for \( j \leq 3 \), 3) (1.3) holds, 4) (3.3) holds and 5) (3.8) holds. Then there exists and is unique a solution \( v_\varepsilon \) to equation (1.4) such that
\[ \| v_\varepsilon - y \| = O(\varepsilon^k). \] (3.12)

**Remark 2.** One may drop assumption (3.3) and consider in place of equation (1.4) the following one:
\[ F(p) + \varepsilon(p - q) = 0, \quad p := p_\varepsilon, \] (3.13)
where \( q \) is any element such that
\[ y - q = A\psi, \quad \| \psi \| \ll 1. \] (3.14)
Then (3.12) holds with \( p_\varepsilon \) in place of \( v_\varepsilon \) and the proof is essentially the same. Note that (3.14) holds, with \( A := F'(y) \) if \( AB_r \cap (B_{r_0}\{0\}) \neq \emptyset \) for any \( r \in (0, r_0) \), where \( r_0 > 0 \) is a fixed number.
References

[1] Ramm, A. G., Dynamical systems method for solving operator equations, Comm. in Nonlinear Sci. and Numer. Simulation, 9, N2, (2004), 383-402.