CLASS INVARIANTS FOR QUARTIC CM FIELDS

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Abstract. One can define class invariants for a quartic primitive CM field $K$ as special values of certain Siegel (or Hilbert) modular functions at CM points corresponding to $K$. Such constructions were given in [DSG] and [Lau]. We provide explicit bounds on the primes appearing in the denominators of these algebraic numbers. This allows us, in particular, to construct $S$-units in certain abelian extensions of $K$, where $S$ is effectively determined by $K$.

1. Introduction

One of the main problems of algebraic number theory is the explicit description of ray class fields of a number field $K$. Besides the case of the field of rational numbers, the theory is most advanced in the case where $K$ is a complex multiplication (CM) field. Effective constructions are available using modular functions generalizing the elliptic modular function $j$; one constructs modular functions as quotients of two modular forms on a Siegel upper half space and evaluates at CM points corresponding to $K$. The values lie in an explicitly determined extension of the reflex field $K^*$ of $K$, that depends on the field over which the Fourier coefficients of the modular function are defined, on the level of the modular function, and on the conductor of the order of $K$ corresponding to the CM point. We loosely call magnitudes constructed this way “class invariants” of $K$. The terminology is proposed because when the Fourier coefficients are rational and the level is 1 the values of the modular function at CM points lie in ray class fields of $K^*$.

An outstanding problem is the effective construction of units in abelian extensions of number fields, even in the case of complex multiplication. A solution of this problem is expected to have significant impact on obtaining additional cases of Stark’s conjecture. The case of cyclotomic units and elliptic units is well developed, but in higher dimensional cases little was known. The essential problem is that divisors of modular functions cannot be supported at the boundary of the moduli space. The purpose of this paper is to provide explicit bounds on the primes appearing in the denominators of class invariants of a primitive quartic CM field $K$. This yields, in particular, an explicit bound on the primes dividing the invariants $u(a,b)$ constructed in [DSG], thus yielding $S$-units lying in a specific abelian extension of $K^*$ for an explicit finite set of primes $S$. It also yields class polynomials for primitive quartic CM fields whose coefficients are $S$-integers as conjectured in [Lau].

2. Elements of small norm in a definite quaternion algebra

2.1. Simultaneous embeddings. Let $B = B_{p,\infty}$ be “the” quaternion algebra over $\mathbb{Q}$ ramified at $\{p, \infty\}$. Concrete models for $B$ can be found in e.g. [Vig, p. 98]. Let $\operatorname{Tr}$ and $\operatorname{N}$ be the (reduced) trace and norm on $B$ and $x \mapsto \pi = \operatorname{Tr}(x) - x$ its canonical involution. Let $R$ be a maximal order of $B$. The discriminant of $R$ is $p^2$; if we choose a $\mathbb{Z}$-basis $v_1, \ldots, v_4$ for $R$ then $\det(\operatorname{Tr}(v_i \overline{v_j})) = p^2$; cf. [Piz, Prop. 1.1]. Further, using this basis we may identify $B \otimes \mathbb{R}$ with $\mathbb{R}^4$. The bilinear form $\langle \alpha, \beta \rangle = \operatorname{Tr}(\alpha \overline{\beta})$ is represented with respect to this basis by an integral symmetric $4 \times 4$ matrix.
matrix $M$ with even diagonal entries, which is positive definite and satisfies $\det(M) = p^2$. It defines an inner product on $\mathbb{R}^4$. We let $\|r\| = \sqrt{(r,r)} = \sqrt{2N(r)}$. Note that the co-volume of $R$ (the absolute value of the volume of a fundamental parallelepiped) is $p$.

Let $K_i = \mathbb{Q}(\sqrt{D_i}) \subset \mathcal{B}$ be quadratic imaginary fields, $D_i$ a square free integer, and let $\mathcal{O}_i$ be orders of $K_i$ of conductor $m_i$ contained in $R$. We assume that one of the following equivalent conditions holds: (i) $K_1 \neq K_2$; (ii) $K_1$ does not commute with $K_2$; (iii) $K_1 \cap K_2 = \mathbb{Q}$. Let $k_i$ be an element of $\mathcal{O}_i$ such that $\{1, k_i\}$ is a basis of $\mathcal{O}_i$ and $k_i$ has minimal possible norm.

Consider the sublattice $L$ of $R$ spanned by $1, k_1, k_2, k_1k_2$. It is a full-rank sublattice. Hence, $\text{co-vol}(L) \geq \text{co-vol}(R)$, while on the other hand

\begin{equation}
\text{co-vol}(L) \leq \|1\| \cdot \|k_1\| \cdot \|k_2\| \cdot \|k_1k_2\| = 4 \cdot N(k_1) \cdot N(k_2).
\end{equation}

We therefore get

**Lemma 2.1.1.** Let $K_i, i = 1, 2$, be quadratic imaginary fields of discriminant $d_{K_i}$ contained in $\mathcal{B}$ and let $\mathcal{O}_i$ be the order of conductor $m_i$ of $K_i$. Assume that both $\mathcal{O}_1, \mathcal{O}_2$ are contained in $R$, a maximal order of $\mathcal{B}$, and that $K_1 \neq K_2$. Then

\begin{equation}
p \leq \frac{(m_1^2d_{K_1} - 1)(m_2^2d_{K_2} - 1)}{4}.
\end{equation}

**Proof.** One verifies that $k_i = \pm m_i \sqrt{D_i}$ if $D_i \equiv 2, 3 \pmod{4}$, $(\pm 1 \pm m_i \sqrt{D_i})/2$ if $D_i \equiv 1 \pmod{4}$ and $m_i$ is odd, and $\pm m_i \sqrt{D_i}/2$ if $D_i \equiv 1 \pmod{4}$ and $m_i$ is even. The norms are, respectively, $m_1^2|D_i|/(1 + m_1^2|D_i|)/4$ and $m_2^2|D_i|/4$.

2.2. **Corollaries.** We draw some corollaries from Lemma 2.1.1. While some of the corollaries are weaker than what can be drawn from the work of Gross-Zagier [GZ] and Dorman [Dor], the techniques are much easier and generalize to higher dimensional situations. Let $R$ be a maximal order of $\mathcal{B}$, the quaternion algebra over $\mathbb{Q}$ ramified at $p$ and $\infty$.

**Corollary 2.2.1.** If $x, y \in R$ and $N(x), N(y) < \sqrt{p}/8 \ (< \sqrt{p}/2$ if $\text{Tr}(x) = \text{Tr}(y) = 0$) then $xy = yx$.

**Proof.** One reduces to the case of trace zero elements by replacing $x$ and $y$ by $2x - \text{Tr}(x)$ and $2y - \text{Tr}(y)$. If $x$ and $y$ have trace zero we get an embedding of the imaginary quadratic orders $\mathbb{Z}[\sqrt{-N(x)}], \mathbb{Z}[\sqrt{-N(y)}]$ into $R$. The Corollary follows from the same argument as above, see Equation (2.1), taking $k_1 = x, k_2 = y$.

**Corollary 2.2.2.** There is an order $\mathcal{O}_1$ of a quadratic imaginary field $K_1, \mathcal{O}_1 \subset R$ such that all elements of $R$ of norm less than $\sqrt{p}/8$ belong to $\mathcal{O}_1$. In particular, for every constant $A < \sqrt{p}/8$ the number of elements of $R$ of norm $A$ is at most the number of elements in $\mathcal{O}_{K_1}$ of norm $A$, which is at most $4\sqrt{A} + 2$.

**Corollary 2.2.3.** Let $j_i, i = 1, 2$, be two singular $j$-invariants, $j_i$ corresponding to an elliptic curve $E_i$ with complex multiplication by an order $\mathcal{O}_i$ of conductor $m_i$ in a quadratic imaginary field $K_i \subset \mathbb{C}$. Suppose that $K_1 \neq K_2$. Let $p$ be a prime of $\mathbb{Q}$ dividing $p$. If $(j_1 - j_2) \in p$ then $p \leq \frac{(m_1^2d_{K_1} - 1)(m_2^2d_{K_2} - 1)}{4}$.

**Proof.** If $(j_1 - j_2) \in p$ then $E_1 \cong E_2 \pmod{p}$. Let $E = E_1 \pmod{p}$. Since $K_1 \neq K_2$, $E$ is supersingular and after fixing an isomorphism $\text{End}(E) \otimes \mathbb{Q} \cong B$, $\text{End}(E)$ is a maximal order of $\mathcal{B}$ containing $\mathcal{O}_1, \mathcal{O}_2$. \(\square\)
2.3. A remark on successive minima. Let $E, E'$ be two supersingular elliptic curves over $\mathbb{F}_p$. Then $\text{Hom}(E, E')$ is a free abelian group of rank 4 equipped with a quadratic form $f \mapsto \deg(f)$. The associated bilinear form is $\langle f, g \rangle \mapsto \deg(f + g) - \deg(f) - \deg(g) = f \circ g' + g \circ f'$. Let $M$ be a matrix representing $\langle \cdot, \cdot \rangle$. It is known to be an integral symmetric positive definite matrix whose diagonal entries are even and whose determinant is $p^2$; cf. [Piz, Prop. 1.1]. We are interested in studying the successive minima $\mu_1, \ldots, \mu_4$ of the homogenous function of weight 1 ("gauge function"). $f(x) = (\frac{1}{2} x M x)^{1/2}$. Geometry of numbers, see [Sie, III §§3-4, X §3], gives $2^4(4V)^{-1} \leq \mu_1 \cdot \mu_2 \cdot \mu_3 \cdot \mu_4 \leq 2^4V^{-1}$, where $V$ is the volume of the unit ball with respect to $f$. Since $V = \frac{2\pi^2}{p}$ we find that

$$\frac{1}{3\pi^2} \cdot \frac{1}{p} \leq \mu_1 \cdot \mu_2 \cdot \mu_3 \cdot \mu_4 \leq \frac{8}{\pi^2} \cdot \frac{1}{p}. \quad (2.3)$$

**Proposition 2.3.1.** Assume that $E = E'$. Let $x$ be an element for which $\mu_2$ is obtained, $K_1 = \mathbb{Q}(x)$. Then $\mathbb{Z}[x]$ is an order of $K_1$, optimally embedded in $\text{End}(E)$. We have $\mu^2_2 \leq \frac{4}{\pi^2} \cdot \frac{1}{p^{2/3}}$ and

$$\frac{1}{2} \cdot \frac{1}{p^{1/2}} \leq \mu_2 \mu_3 \leq \frac{4}{\pi^3} \cdot \frac{1}{p^{2/3}}, \quad \max \left\{ \frac{1}{3^{1/3}} \cdot \frac{1}{\pi^{2/3}}, \frac{1}{\sqrt{2}} \cdot \frac{1}{p^{1/4}} \right\} \leq \mu_4 \leq \frac{2\sqrt{2}}{\pi} \cdot \frac{1}{p^{1/2}}.$$

**Proof.** If $E = E'$, then $\mu_1 = 1$. The embedding is optimal because the element of minimal norm independent of 1 in order determines the order, cf. proof of Lemma 2.1.1. Let $y$ be an element for which the third successive minimum is obtained and $K_2 = \mathbb{Q}(y)$. By definition $y \not\in K_1$ and we are in the situation of §2.1. We get using Equation (2.1), $\sqrt{p} \leq 2\sqrt{N(x)}\sqrt{N(y)} = 2\mu_2\mu_3$; since $xy$ is independent of $\{1, x, y\}$ we deduce that $\mu_4 \leq \sqrt{N(xy)} = \mu_2\mu_3$. We also have $\mu^2_4 \geq \mu_2\mu_3 \geq 1$. Analysis of the inequalities gives the result. \hfill $\square$

By Equation (2.3) $\mu^2_1 \leq \frac{2\sqrt{2}}{\pi} \sqrt{p}$; applying that to any pair of elliptic curves, we get:

**Proposition 2.3.2.** Fix a supersingular elliptic curve $E$ over $\mathbb{F}_p$. Let $N$ be the minimal integer such that for any supersingular $E'$ over $\mathbb{F}_p$ there is an isogeny of degree less or equal to $N$ between $E$ and $E'$. Then

$$N \leq \frac{2\sqrt{2}}{\pi} \sqrt{p} \approx 0.9004 \sqrt{p}.$$

**Remark 2.3.3.** By estimating the number of subgroups of $E$ of order $\leq N$, one finds that $N \geq 0.3\sqrt{p}$. Numerical evidence shows that $N \geq 0.7\sqrt{p}$. See §6.3. Such results can be interpreted in terms of coefficients of theta series.

3. An embedding problem

Let $K$ be a primitive quartic CM field and $K^+$ its totally real subfield. Write $K^+ = \mathbb{Q}(\sqrt{d})$, for $d > 0$ a square free integer. Write $K = K^+ (\sqrt{r})$ with $r \in \mathbb{Z}[\sqrt{d}]$ a totally negative element. Every quartic CM field can be written this way and there is much known on the index of $\mathbb{Z}[\sqrt{d}, \sqrt{r}]$ in $\mathcal{O}_K$. The following are equivalent for CM fields of degree 4: (i) $K$ is primitive, i.e., does not contain a quadratic imaginary field; (ii) $K$ is either non-Galois, or a cyclic Galois extension; (iii) $N_{\mathbb{Q}(\sqrt{d})/\mathbb{Q}}(r)$ is not a square in $\mathbb{Q}$.

Let $E_1, E_2$ be supersingular elliptic curves over $\mathbb{F}_p$. Let $a = \text{Hom}(E_2, E_1), a' := \text{Hom}(E_1, E_2), \quad R_i = \text{End}(E_i)$. Then $\text{End}(E_1 \times E_2) = \left( \begin{array}{cc} R_1 & a' \\ a & R_2 \end{array} \right)$. The product polarization induced by the divisor $E_1 \times \{0\} + \{0\} \times E_2$ on $E_1 \times E_2$ induces a Rosati involution denoted by $\vee$. This involution
is given by \((\frac{a}{b} \ \frac{c}{d}) \mapsto (\frac{a}{b} \ \frac{c}{d})^\vee = (\frac{a^\vee}{b^\vee} \ \frac{c^\vee}{d^\vee})\), where \(a^\vee, b^\vee\) etc. denotes the dual isogeny. Note that \(a^\vee = \overline{a}, d^\vee = \overline{d}\). The Rosati involution is a positive involution.

**The embedding problem:** To find a ring embedding \(\iota : \mathcal{O}_K \hookrightarrow \text{End}(E_1 \times E_2)\) such that the Rosati involution coming from the product polarization induces complex conjugation on \(\mathcal{O}_K\).

As we shall see below, the problem is intimately related with bounding primes in the denominators of class invariants.

**Theorem 3.0.4.** If the embedding problem has a positive solution then \(p \leq 16 \cdot d^2 (\text{Tr}(r))^2\).

**Proof.** Assume such an embedding \(\iota\) exists. Then \(\iota(\mathcal{O}_K^+)\) is fixed by the Rosati involution, thus \(\sqrt{d} \mapsto M = (\frac{a}{b} \ \frac{c}{d})\), for some \(a, e, b \in \mathbb{Z}, b \in \mathfrak{a}, b^\vee = c\). Moreover, \(M^2 = (\frac{d}{0} 0\))\). This gives the following conditions on the entries of \(M\):

\[
\begin{align*}
a^2 + bb^\vee &= d \\
b(a + e) &= 0 \\
b^\vee(a + e) &= 0 \\
b^\vee b + e^2 &= d.
\end{align*}
\]

If \(a \neq -e\) then \(b = 0\) and hence \(d\) is a square - a contradiction. Thus, \(a = -e\), and we can write the embedding as

\[
(3.1) \quad \sqrt{d} \mapsto M = \left(\frac{a}{b} \ \frac{b}{-a}\right), \quad a \in \mathbb{Z}, \ b \in \mathfrak{a}, \ a^2 + bb^\vee = d.
\]

Let us write

\[
r = \alpha + \beta \sqrt{d}, \quad \alpha < 0, |\alpha| > |\beta \sqrt{d}|.
\]

The condition of the Rosati involution inducing complex conjugation is equivalent to \(\iota(\sqrt{r})^\vee = -\iota(\sqrt{r})\). So, if \(\iota(\sqrt{r}) = (\frac{x}{y} \ \frac{z}{w})\) then \(\left(\frac{x^\vee}{y^\vee} \ \frac{z^\vee}{w^\vee}\right) = -\left(\frac{x}{y} \ \frac{z}{w}\right)\). This translates into the conditions \(x = -x^\vee, \ w = -w^\vee, \ y = -z^\vee\), implying in particular that \(x\) and \(w\) have trace zero, which we write as \(x \in R_1^0\) and \(w \in R_2^0\). It follows that

\[
(3.2) \quad \iota(\sqrt{r}) = \left(\frac{x}{-y^\vee} \ \frac{y}{w}\right), \quad x \in R_1^0, \ w \in R_2^0, \ y \in \mathfrak{a}.
\]

A further condition is obtained from \(\iota(\sqrt{r})^2 = r\), i.e., \(\left(\frac{x}{-y^\vee} \ \frac{y}{w}\right)^2 = \left(\frac{\alpha + \beta a}{\beta b^\vee} \ \frac{\beta b}{\alpha - \beta a}\right)\), that is,

\[
(3.3) \quad \left(\frac{x^2 - yy^\vee}{-y^\vee x - wy^\vee} \ \frac{xy + yw}{w^2 - y^\vee y}\right) = \left(\frac{\alpha + \beta a}{\beta b^\vee} \ \frac{\beta b}{\alpha - \beta a}\right).
\]

Since \(yy^\vee = y^\vee y \in \mathbb{Z}\), this leads to the following necessary conditions

\[
(*) \quad x^2 - yy^\vee = \alpha + \beta a \quad xy + yw = \beta b \\
w^2 - yy^\vee = \alpha - \beta a \quad a^2 + bb^\vee = d,
\]

where \(x \in R_1^0, \ w \in R_2^0, \ b, y \in \mathfrak{a}, \ \alpha, \beta, a \in \mathbb{Z}\).

Note that \(y = 0\) implies that either \(b = 0\) or \(\beta = 0\). The case \(b = 0\) gives that \(d\) is a square, hence is not possible; the case \(\beta = 0\) is possible, but leads to \(K\) a bi-quadratic field, contrary to our assumption.
We use the notation \( N(s) = ss^\vee, N(y) = yy^\vee \), etc.. Note that for \( s \in R \) this definition of the norm is the usual one and, in any case, under the interpretation of elements as endomorphisms \( N(s) = \deg(s) \) and so \( N(st) = N(s) \cdot N(t) \) when it makes sense. It follows from (**) that

\[
(\ast\ast) \quad N(x) + N(y) = -(\alpha + \beta a) \\
N(w) + N(y) = -(\alpha - \beta a).
\]

Let \( \varphi : E_1 \to E_2 \) be a non-zero isogeny of degree \( \delta \). For \( f \in \text{End}(E_1 \times E_2) \) the composition of rational isogenies

\[
E_1 \times E_1 \xrightarrow{(1, \varphi)} E_1 \times E_2 \xrightarrow{f} E_1 \times E_2 \xrightarrow{(1, \delta^{-1} \varphi^\vee)} E_1 \times E_1,
\]

gives a ring homomorphism \( \text{End}^0(E_1 \times E_2) \to \text{End}^0(E_1 \times E_1) \) that can be written in matrix form as

\[
f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \mapsto \begin{pmatrix} f_{11} & f_{12} \delta^{-1} \varphi^\vee f_{21} \\ f_{21} & f_{22} \delta^{-1} \varphi^\vee f_{21} \end{pmatrix}.
\]

Let \( \psi \) be the composition \( K \to \text{End}^0(E_1 \times E_2) \to \text{End}^0(E_1 \times E_1) \). Then \( \psi \) is an embedding of rings with the property

\[
(3.4) \quad \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \psi(O_K) \subset M_2(R_1).
\]

Choose \( \varphi = y^\vee \). Taking \( f = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \) (corresponding to \( \sqrt{d} \)), or \( \begin{pmatrix} x & y \\ -y & w \end{pmatrix} \) (corresponding to \( \sqrt{r} \)), the embedding \( \psi \) is determined by

\[
(3.5) \quad \psi(\sqrt{d}) = \begin{pmatrix} a & by^\vee \\ yb^\vee & -a \end{pmatrix}, \quad \psi(\sqrt{r}) = \begin{pmatrix} x & \delta \\ -1 & ywy^\vee / \delta \end{pmatrix}.
\]

We conclude that

\[
S = \{by^\vee, yb^\vee, x, ywy^\vee \} \subset R_1.
\]

Let

\[
\delta_1 = \min\{-(\alpha - \beta a), -(\alpha + \beta a)\} = |\alpha| - |\beta| \cdot |a|,
\]
\[
\delta_2 = \max\{-(\alpha - \beta a), -(\alpha + \beta a)\} = |\alpha| + |\beta| \cdot |a|.
\]

It follows from (3.1) and (**) \( N(by^\vee) = N(yb^\vee) \leq d\delta_1, \ N(x) \leq \delta_2. \)

Assume that \( p > 16 \cdot d^2(\text{Tr}(r))^2 \geq \max\{64d^2\delta_1^2, 64\delta_2^2\} \). Then \( N(by^\vee), N(yb^\vee) \) and \( N(x) \) are all smaller than \( \sqrt{p}/8 \). By Corollary 2.2.2, the elements \( x, y^\vee b, yb^\vee \) belong to some imaginary quadratic field \( K_1 \). The equation \( xy + yw = \beta b \) appearing in (**) gives the relation \( xyy^\vee + ywy^\vee = \beta by^\vee \), which shows that \( ywy^\vee \in K_1 \). We conclude that \( \psi \) is an embedding \( K \to M_2(K_1) \). This implies that \( K_1 \hookrightarrow K \) (else consider the commutative subalgebra generated by \( K \) and \( K_1 \) in \( M_2(K_1) \)), contrary to our assumption. It follows that if there is a solution to the embedding problem \( \iota \) then \( p \leq 16 \cdot d^2(\text{Tr}(r))^2 \). \( \square \)
4. Bad reduction of CM curves

In this section we discuss the connection between solutions to the embedding problem and bad reduction of curves of genus two whose Jacobian has complex multiplication. We shall assume CM by the full ring of integers, but the arguments can easily be adapted to CM by an order, at least if avoiding primes dividing the conductor of the order.

4.1. Bad reduction solves the embedding problem. Fix a quartic primitive CM field $K$. Write $K = \mathbb{Q}(\sqrt{d})(\sqrt{r})$, $r \in \mathcal{O}_K^+$, $d$ a positive integer. Let $\mathcal{C}$ be a smooth projective genus 2 curve over a number field $L$. We say that $\mathcal{C}$ has CM (by $\mathcal{O}_K$) if $\text{Jac}(\mathcal{C})$ has CM by $\mathcal{O}_K$. By passing to a finite extension of $L$ we may assume that $\mathcal{C}$ has a stable model over $\mathcal{O}_L$ and that all the endomorphisms of $\text{Jac}(\mathcal{C})$ are defined over $L$. Since $K$ is primitive, $\text{Jac}(\mathcal{C})$ is a simple abelian variety and so $\text{End}^0(\text{Jac}(\mathcal{C})) = K$. In particular, the natural polarization of $\text{Jac}(\mathcal{C})$, associated to the theta divisor $\mathcal{C} \subset \text{Jac}(\mathcal{C})$, preserves the field $K$ and acts on it by complex conjugation.

It is well known that $\text{Jac}(\mathcal{C})$ has everywhere good reduction. It follows that for every prime ideal $\mathfrak{p} \not| \mathcal{O}_L$ either $\mathcal{C}$ has good reduction modulo $\mathfrak{p}$ or is geometrically isomorphic to two elliptic curves $E_1, E_2$ crossing transversely at their origins. In the latter case we have an isomorphism of principally polarized abelian varieties over $k(\mathfrak{p}) = \mathcal{O}_L/\mathfrak{p}$, $\text{(Jac}(\mathcal{C}), \mathcal{C}) \cong (E_1 \times E_2, E_1 \times \{0\} + \{0\} \times E_2)$. Since $K \hookrightarrow \text{End}(E_1 \times E_2) \otimes \mathbb{Q}$ we see that $E_1$ must be isogenous to $E_2$. Moreover, $E_1$ cannot be ordinary; that implies that $K \hookrightarrow M_2(K_1)$ for some quadratic imaginary field $K_1$ and one concludes that $K_1 \hookrightarrow K$, contradicting the primitivity of $K$. We conclude

**Lemma 4.1.1.** Let $\mathcal{C}/L$ be a non-singular projective curve of genus 2 with CM by $\mathcal{O}_K$. Assume that $\mathcal{C}$ has a stable model over $\mathcal{O}_L$. If $\mathcal{C}$ has bad reduction modulo a prime $\mathfrak{p} \mid \mathcal{O}_L$ then the embedding problem has a positive solution for the prime $\mathfrak{p}$.

The following theorem now follows immediately using Theorem 3.0.4.

**Theorem 4.1.2.** Let $\mathcal{C}$ be a non-singular projective curve of genus 2 with CM by $\mathcal{O}_K$ and with a stable model over the ring of integers $\mathcal{O}_L$ of some number field $L$. Let $\mathfrak{p} \mid \mathcal{O}_L$ be a prime ideal of $\mathcal{O}_L$. Assume that $\mathfrak{p}$ is greater or equal to $16 \cdot d^2(\text{Tr}(r))^2$ then $\mathcal{C}$ has good reduction modulo $\mathfrak{p}$.

4.2. A solution to the embedding problem implies bad reduction.

**Theorem 4.2.1.** Assume that the embedding problem of §3 has a solution with respect to a primitive quartic CM field $K$. Then there is a smooth projective curve $\mathcal{C}$ of genus 2 over a number field $L$ with CM by $\mathcal{O}_K$, whose endomorphisms and stable model are defined over $\mathcal{O}_L$, and a prime $\mathfrak{p}$ of $\mathcal{O}_L$ such that $\mathcal{C}$ has bad reduction modulo $\mathfrak{p}$.

Our strategy for proving the theorem is the following. We consider a certain infinitesimal deformation functor $\mathbf{N}$ for abelian surfaces with CM by $\mathcal{O}_K$. We show that $\mathbf{N}$ is pro-representable by a $W(\mathbb{F}_p)$-algebra $R^a$, and that a solution to the embedding problem can be viewed as an $\mathbb{F}_p$-point $x$ of $\text{Spec}(R^a)$. We prove that $R^a$ is isomorphic to the completed local ring of a point on a suitable Grassmann variety and deduce that $R^a \otimes \mathbb{Q} \neq 0$. We conclude that $x$ can be lifted to characteristic zero and finish using classical results in the theory of complex multiplication. Before beginning the proof proper, we need some preliminaries about Grassmann varieties.

4.3. Grassmann schemes. The following applies to any number field $K$ with an involution $\ast$; we denote the fixed field of $\ast$ by $K^+$. Put $[K : \mathbb{Q}] = 2g$. 
4.3.1. Consider the module $M_0 := \mathcal{O}_K \otimes_{\mathbb{Z}} W$, $W = W(\mathbb{F}_p)$, equipped with an alternating perfect $W$-linear pairing $\langle \cdot, \cdot \rangle$ with values in $W$, such that for $s \in \mathcal{O}_K$ we have $\langle sr, r' \rangle = \langle r, s^*r' \rangle$. Note that this also holds for $s \in \mathcal{O}_K \otimes_{\mathbb{Z}} W$ if $*$ denotes the natural extension of the involution to this ring.

This defines a Grassmann problem: classify for $W$-algebras $W'$ the isotropic, locally free, locally direct summands $W'$-submodules of $M_0 \otimes_W W'$ of rank $g$ that are $\mathcal{O}_K$-invariant. This is representable by a projective scheme $\mathbf{G}' \to \text{Spec}(W)$ (a closed subscheme of the usual (projective) Grassmann scheme). We claim that $\mathbf{G}'$ is topologically flat: namely, that every $\mathbb{F}_p$-point of it lifts to characteristic zero. That means that for every submodule $N_1$ of $\mathcal{O}_K \otimes_{\mathbb{Z}} W$, satisfying the conditions above, there is a flat $W$-algebra $W'$ and such submodule $N_0$ of $\mathcal{O}_K \otimes_W W'$ that lifts $N_1$.

4.3.2. First note that for $k \supset W$ an algebraically closed field of characteristic zero, the $k$-points of $\mathbf{G}'$ are in bijection with “CM types”. Indeed, we are to classify the isotropic, rank $g$, $k$-vector spaces of $\mathcal{O}_K \otimes_{\mathbb{Z}} k = \oplus_{(\varphi : k \to k)} k(\varphi)$, where $k(\varphi)$ is $k$ on which $\mathcal{O}_K$ acts via $\varphi$. It is easy to see that the pairing decomposes as a direct sum of orthogonal pairings on the $g$ subspaces $k(\varphi) \oplus k(\varphi \circ *)$ (use that for $r \in k(\varphi), r' \in k(\varphi')$ we have $\varphi(s)r = \langle sr, r' \rangle = \langle r, s^*r' \rangle = (\varphi'^* \circ *)/r(s)(r, r')$). On $k(\varphi) \oplus k(\varphi \circ *)$ the pairing is non-degenerate so every maximal isotropic subspace is a line and vice-versa. The condition of being an $\mathcal{O}_K$-submodule leaves us with precisely two submodules of $k(\varphi) \oplus k(\varphi \circ *)$, viz. $k(\varphi), k(\varphi \circ *)$. Thus, the choice of an isotropic, $\mathcal{O}_K$-invariant $k$-subspace of dimension $g$ of $\mathcal{O}_K \otimes_{\mathbb{Z}} k$ corresponds to choosing an element from each of the $g$ pairs $\{\varphi, \varphi \circ *\}$.

4.3.3. We now prove topological flatness for $\mathbf{G}'$. We first make a series of reductions. Let $p = \prod p_i^{n_i}$ in $\mathcal{O}_{K^+}$. We have $\mathcal{O}_{K^+} \otimes_{\mathbb{Z}} W = \oplus_{p \mid p} \mathcal{O}_{K^+_p} \otimes_{\mathbb{Z}} \mathbb{F}_p$ (with corresponding idempotents $\{e_p\}$) and $\mathcal{O}_{K^+} \otimes_{\mathbb{Z}} \mathbb{F}_p = \oplus_{p \mid p} \mathcal{O}_{K^+_p} \oplus \mathbb{F}_p$. The modules $M_0 = \mathcal{O}_K \otimes_{\mathbb{Z}} W$, $M_1 := \mathcal{O}_{K^+} \otimes_{\mathbb{Z}} \mathbb{F}_p$, which are, respectively, free $\mathcal{O}_{K^+} \otimes_{\mathbb{Z}} W$ and $\mathcal{O}_{K^+} \otimes_{\mathbb{Z}} \mathbb{F}_p$ modules of rank $2$, decompose accordingly as $\oplus_{p \mid p} M_0(p), \oplus_{p \mid p} M_1(p)$. We claim that the submodules $\{M_0(p) : p|p\}$ (resp. $\{M_1(p) : p|p\}$) are orthogonal. Indeed, this follows from the fact that for the idempotents $\{e_p\}$ we have $\langle e_p r, e_p r' \rangle = \delta_{p, p'} \langle e_p r, e_p r' \rangle = \delta_{p, p'} \langle e_p r, e_p r' \rangle$. We may thus assume without loss of generality that $p = p^e$ with residue degree $f$ in $\mathcal{O}_{K^+}$ (note that the global nature of the rings $\mathcal{O}_{K^+}$ plays no role). Let $W^+ = W(\mathbb{F}_p)$, considered as the maximal unramified sub-extension of $\mathcal{O}_{K^+_p}$. A further reduction is possible: Since $\mathcal{O}_{K^+_p} \otimes_{\mathbb{Z}} W = \oplus_{p^f = W \to W} \mathcal{O}_{K^+_p} \otimes_{W} W$, the same arguments as above (using idempotents etc.) allow us to assume with out loss of generality that $f = 1$. Thus, the problem reduces to the following:

4.3.4. One is given a $p$-adic ring of integers $A$, finite of rank $e$ over $W$, and a free $A$-algebra $B$ of rank $2$ with an involution $*$ whose fixed points are $A$. Also given is a perfect alternating pairing $\langle \cdot, \cdot \rangle : B \times B \to W$ such that for $s \in B$ we have $\langle sr, r' \rangle = \langle r, s^*r' \rangle$. One needs to show that every maximal isotropic $B \otimes_W \mathbb{F}_p$ submodule of $B \otimes_W \mathbb{F}_p$ lifts to characteristic zero in the sense previously described.

Note that $B$ is either an integral domain that is a ramified extension of $A$ or isomorphic as an $A$-algebra to $A \oplus A$ with the involution being the permutation of coordinates. The first case is immediate: We have $B \otimes_W \mathbb{F}_p \cong \mathbb{F}_p[t]/(t^{2e})$ and it has a unique submodule of rank $e$ over $\mathbb{F}_p$, viz. $(t^e)$. Since the Grassmann scheme $\mathbf{G}'$ always has characteristic zero geometric points and is projective, a lift is provided by (any) characteristic zero point of $\mathbf{G}'$. 
In the second case we have $B \otimes_{W} \overline{F}_p \cong \overline{F}_p[t]/(t^e) \oplus \overline{F}_p[t]/(t^e)$. Every submodule of $B \otimes_{W} \overline{F}_p$ of rank $e$ over $\overline{F}_p$ is a direct sum $(t^i) \oplus (t^{e-i})$. Such submodules are automatically isotropic. We claim that the submodule $(t^i)$ of $A[t]/(t^e)$ can be lifted to characteristic zero, that such a lifting corresponds to a choice of $e-i$ embeddings $A \to \overline{O}_p$ over $W$ and that each lifting is isotropic when considered as a submodule of $B \otimes_{W} W' = A \otimes_{W} W' \oplus A \otimes_{W} W'$, where $W'$ is a “big enough” extension of $W$. Indeed, every geometric point of the appropriate Grassmann scheme, being proper over Spec($W$), extends to an integral point (defined over a finite integral extension $W'/W$). Such a geometric point corresponds to a choice of $(\xi_i)$ embeddings $A \to \overline{\Omega}_p$ over $W$ and is isotropic (cf. §4.3.2 – when we view $A \otimes W$ as a $B$-submodule of $B \otimes W$ via the first (or second) component, it is isotropic). Moreover, since the submodule $(t^i)$ is uniquely determined by its rank, every such integral point indeed provides a lift of $(t^i)$. It now easily follows that $(t^i) \oplus (t^{e-i})$ can be lifted in $A[S]$ ways.

4.4. Proof of Theorem 4.2.1. By an abelian scheme with CM we mean in this section a triple $(A/S, \lambda, \iota)$, consisting of a principally polarized abelian scheme over $S$ with an embedding of rings $\iota: \mathcal{O}_K \to \text{End}_S(A)$ such that the Rosati involution defined by $\lambda$ induces complex conjugation on $\mathcal{O}_K$. We denote complex conjugation on $K$ by $*$ and let $K^+$ be the totally real subfield of $K$.

As before, $W = W(\overline{F}_p)$. The following lifting lemma, that holds for any CM field $K$ and whose proof is given in §§4.4.1 - 4.4.4, is the key point.

Lemma 4.4.1. Let $(A, \lambda, \iota)$ be an abelian variety with CM over $\overline{F}_p$ then $(A, \lambda, \iota)$ can be lifted to characteristic zero.

4.4.1. Let $S$ be a local artinian ring with residue field $\overline{F}_p$. Let $(A', \lambda', \iota')$ be an abelian scheme over $S$ with CM. We claim that $H^1_{dR}(A'/S)$ is a free $\mathcal{O}_K \otimes_{\mathbb{Z}} S$-module of rank 1. Since $H^1_{dR}(A'/S)$ is a free $S$-module of rank 2, to verify it is a free $\mathcal{O}_K \otimes_{\mathbb{Z}} S$-module it is enough to prove that modulo the maximal ideal of $S$ (cf. [DP, Rmq. 2.8]), namely, that $H^1_{dR}(A' \otimes_S \overline{F}_p/\overline{F}_p)$ is a free $\mathcal{O}_K \otimes_{\mathbb{Z}} \overline{F}_p$-module. This is [Rap, Lem. 1.3]. In fact, loc. cit. gives that $H^1_{crys}(A' \otimes_S \overline{F}_p/W)$ is a free $\mathcal{O}_K \otimes_{\mathbb{Z}} W$-module.

4.4.2. The polarization $\lambda$ induces a perfect alternating pairing $\langle \cdot, \cdot \rangle$ on the free $\mathcal{O}_K \otimes_{\mathbb{Z}} W$-module $H^1_{crys}(A/W)$, which we identify with $M_0 := \mathcal{O}_K \otimes_{\mathbb{Z}} W$. This pairing induces complex conjugation on $\mathcal{O}_K$ and reduces modulo $p$ to the pairing induced by $\lambda$ on $H^1_{dR}(A/F_p)$. Moreover, there exists a finite extension $\Lambda$ of $W$ such the Hodge filtration $0 \to H^0(A, \Omega^1_{A/F_p}) \to H^1_{dR}(A/F_p)$ can be lifted to $M_0 \otimes_W \Lambda$. This follows from the discussion in §4.3. In fact, the results of that section show that such a lift is uniquely determined by its generic point, a subspace of $K \otimes_{\mathbb{Q}} \overline{Q}_p = \bigoplus_{\phi: K \to \overline{Q}_p} \overline{Q}_p(\phi)$, consisting of a choice of one subspace out of each pair $\overline{Q}_p(\phi) \oplus \overline{Q}_p(\phi \circ \iota)$.

Recall that a CM type $\Phi$ of $K$ is a subset of $\text{Hom}(K, \mathbb{C})$ (or of $\text{Hom}(K, \overline{Q}_p)$) that is disjoint from its complex conjugate, equivalently, a subset that induces $\text{Hom}(K^+, \mathbb{C})$ (or $\text{Hom}(K^+, \overline{Q}_p)$). A choice of lift of the Hodge filtration provides us with CM type $\Phi$. Let $K^*$ be the reflex field defined by $\Phi$. We see that, in fact, a lift of the Hodge filtration is defined over $\Lambda$, where $\Lambda$ is the compositum of $W$ with the valuation ring of the $p$-adic reflex field associated to $\Phi$.

4.4.3. Let $V = \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{C}$ - a complex vector space on which $\mathcal{O}_K$ acts. Choose a $\mathbb{Z}$-basis $e_1, \ldots, e_{2g}$ for $\mathcal{O}_K$ and consider $f_{\Phi}(\overline{z}) \det(\sum e_i x_i \text{Lie}(A))$. This a polynomial in $x_1, \ldots, x_{2g}$ with coefficients in $\mathcal{O}_K^*$ that depends only on $\Phi$ and determines it.

Let $\mathbf{M}: \text{Sch}_{\mathcal{O}_K^*} \to \text{Sets}$ be the functor from the category of schemes over $\mathcal{O}_K^*$ to the category of sets such that $\mathbf{M}(S)$ is the isomorphism classes of triples $(A/S, \lambda, \iota: \mathcal{O}_K \to \text{End}_S(A))$,
where $A/S$ is an abelian scheme with CM and $\det(\sum e_i x_i, \text{Lie}(A)) = f_\Phi(x)$. That is, the triple $(A/S, \lambda, \iota: \mathcal{O}_K \hookrightarrow \text{End}_S(A))$ satisfies the Kottwitz condition [Kot, §5] uniquely determined by $\Phi$.

For the given point $x = (A, \lambda, \iota) \in \text{M}({\mathbb{F}}_p)$ we consider the local deformation problem induced by $\text{M}$. This is the functor $\text{N}$ from the category $\text{C}_\Lambda$ of local artinian $\Lambda$-algebras with residue field $F_p$ to the category $\text{Sets}$ associating to a ring $R$ in $\text{C}_\Lambda$ those elements of $\text{M}(R)$ specializing to $x$. We remark that the Kottwitz condition is closed under specialization. It is thus fairly standard that $\text{N}$ is pro-represented by a complete noetherian $\Lambda$-algebra $R^a$; cf. [Oor, §2] and [Dok, §4].

4.4.4. Let $G \to \text{Spec}(\Lambda)$ be the Grassmann variety parameterizing for a scheme $S \to \text{Spec}(\Lambda)$ the set of $\mathcal{O}_K$-invariant, isotropic, locally free, locally direct summands $\mathcal{O}_S$-submodules of rank $g$ of $M_0 \otimes \mathcal{O}_S$ (with the pairing coming from $x$ as above) and satisfying Kottwitz condition $f_\Phi$ for a CM type $\Phi$. (In fact, one can deduce that $G \cong \text{Spec}(\Lambda)$ but we don't need it here.) Let $x$ be the point of $G$ corresponding to $H^0(A, \Omega^1_{A/{\mathbb{F}}_p}) \to \mathbb{H}^1_{\text{dR}}(A/{\mathbb{F}}_p)$.

Given the results of §4.4.1, the theory of local models furnishes an isomorphism $\mathcal{O}^\wedge_{G,x} \cong R^a$; cf. [DP, §3], [Dok, Thm. 4.4.1] – the arguments easily extend to allow a Kottwitz condition. We conclude therefore that there is a triple $(A, \lambda, \iota)$ lifting $x$ defined over the $p$-adic field $K_1 = \Lambda \otimes \mathbb{Q}$. This concludes the proof of the lemma.

4.4.5. Let $K$ be a primitive quartic CM field. A solution of the embedding problem for $p$ provides us with a triple $(A/{\mathbb{F}}_p, \lambda, \iota) = (E_1 \times E_2/{\mathbb{F}}_p, \lambda = \lambda_1 \times \lambda_2, \iota: \mathcal{O}_K \to \text{End}_{{\mathbb{F}}_p}(E_1 \times E_2))$. By Lemma 4.4.1, we may lift $(A/{\mathbb{F}}_p, \lambda, \iota)$ to a triple $(A_0/{\mathbb{F}}_p, \lambda_0, \iota_0)$ defined over some $p$-adic field $K_1$ and so, by Lefschetz principle, defined over $\mathbb{C}$. By the theory of complex multiplication $(A_0, \lambda_0, \iota_0)$ is defined over some number field $K_2$. Since the CM field $K$ is primitive, $A_0$ is simple and principally polarized. By a theorem of Weil [Wei] the polarization is defined by a non singular projective genus 2 curve $\mathcal{C}$ and it follows that $A_0 \cong \text{Jac}(\mathcal{C})$ as polarized abelian varieties. Furthermore, $\mathcal{C}$ is defined over a number field $K_3$ (that is at most a quadratic extension of $K_2$). By passing to a finite extension $L$ of $K_3$, we get a stable model.

5. Applications

5.1. A general principle. The following lemma is folklore and easy to prove:

**Lemma 5.1.1.** Let $\pi: S \to R$ be a proper scheme over a Dedekind domain $R$ with quotient field $H$. Let $\mathcal{L} \to S$ be a line bundle on $S$ and $f, g: S \to \mathcal{L}$ sections. Let $x \in S(H')$ be a point, where $H'$ is a finite field extension of $H$. Let $u = (f/g)(x) \in H'$. Let $p$ be a prime of $R'$, the integral closure of $R$ in $H'$. Let $\bar{x}$ be the $R'$-point corresponding to $x$. Then $\text{val}_p(u) < 0$ implies that $\bar{x}$ intersects the divisor of $g$ in the fiber of $S$ over $p$.

**Corollary 5.1.2.** Let $\mathcal{A}_2 \to \text{Spec}(\mathbb{Z})$ be the moduli space of principally polarized abelian surfaces and let

$$
\Theta(\tau) = \frac{1}{2^{12}} \prod_{(c', c)} \left( \Theta \left[ \frac{c}{c'} \right](0, \tau) \right)^2. 
$$

(5.1)

even char.

Let $f$ be a Siegel modular form with $q$-expansion $\sum a(\nu) q^{2\nu i \text{Tr}(\nu \tau)}$, where $\nu$ runs over $g \times g$ semi-integral, semi-definite symmetric matrices. Assume that all the Fourier coefficients $a(\nu) \in \mathcal{O}_L$,
the ring of integers of a number field $L$, and that the weight of $f$ is of the form $10k$, $k$ a positive integer.

Let $\tau$ be a point on $\text{Sp}_4(\mathbb{Z})/\mathcal{H}_2$ corresponding to a smooth genus 2 curve $\mathcal{C}$ with CM by the full ring of integers of a primitive CM field $K$. Then $(f/\Theta^k)(\tau)$ is an algebraic number lying in the compositum $H_K\cdot L$ of the Hilbert class field of $K^* \cdot L$. If a prime $p$ divides the denominator of $(f/\Theta^k)(\tau)$ then $\mathcal{C}$ has bad reduction modulo $p$.

**Proof.** The argument is essentially that of [DSG, §4.4]: Igusa [Igu2] proved that $\Theta$ is a modular form on $\text{Sp}_4(\mathbb{Z})/\mathcal{H}_2$ (see [Igu2, Thm. 3]), $\Theta$ is denoted there $\chi_{10}$. It is well known to have weight 10 and a computation shows that its Fourier coefficients are integers and have g.c.d. 1. The $q$-expansion principle [FC, Ch. V, Prop. 1.8] shows that $f$ and $\Theta^k$ are sections of a suitable line bundle of the moduli scheme $\mathcal{A}_2 \otimes \mathcal{O}_L$. The value $(f/\Theta^k)(\tau)$ lies in $H_K\cdot L$ by the theory of complex multiplication.

It is classical that the divisor of $\Theta$ over $\mathbb{C}$, say $D_{\text{gen}}$, is the locus of the reducible polarized abelian surfaces — those that are a product of elliptic curves with the product polarization. The Zariski closure $D_{\text{gen}}$ of $D_{\text{gen}}$ in $\mathcal{A}_2$ is contained in the divisor $D_{\text{arith}}$ of $\Theta$, viewed as a section of a line bundle over $\mathcal{A}_2$, and therefore $D_{\text{gen}} = D_{\text{arith}}$, because by the $q$-expansion principle $D_{\text{arith}}$ has no “vertical components”. Since $D_{\text{gen}}$ also parameterizes reducible polarized abelian surfaces, it follows that $D_{\text{arith}}$ parameterizes reducible polarized abelian surfaces. (Furthermore, it is easy to see by lifting that every reducible polarized abelian surface is parameterized by $D_{\text{arith}}$.) The Corollary thus follows from Lemma 5.1.1. \hfill $\square$

**Corollary 5.1.3.** $(f/\Theta^k)(\tau)$ is an $S$-integer, where $S$ is the set of primes of lying over rational primes $p$ less than $16 \cdot d^2(\text{Tr}(r))^2$ and such that $p$ decomposes in a certain fashion in a normal closure of $K$ as imposed by superspecial reduction [Gor, Thms. 1, 2] (for example, if $K$ is a cyclic Galois extension then $p$ is either ramified or decomposes as $p_1p_2$ in $K$).

**5.2. Class invariants.** Igusa [Igu, p. 620] defined invariants $A(u), B(u), C(u), D(u)$ of a sextic $u_6X^6 + u_1X^5 + \cdots + u_6$, with roots $\alpha_1, \ldots, \alpha_6$, as certain symmetric functions of the roots. For example, $D(u) = u_0^{10} \prod_{i<j} (\alpha_i - \alpha_j)^2$ is the discriminant. Igusa also proved that if $k$ is a field of characteristic different from 2, the complement of $D = 0$ in $\text{Proj} k[A, B, C, D]$, where $A, B, C, D$ are of weights 2, 4, 6, 10 respectively, is the coarse moduli space for hyperelliptic curves of genus 2. Moreover, the ring of rational functions is generated by the “absolute invariants” $B/A^2, C/A^3, D/A^5$ (see [Igu1, p. 177], [Igu, p. 638]). One can choose other generators of course, and for our purposes it makes sense to choose generators with denominator a power of $D$. Choose then as in [Wam, p. 313] the generators

$$i_1 = A^5/D, \quad i_2 = A^3B/D, \quad i_3 = A^2C/D.$$  

One should note though that these invariants are not known a-priori to be valid in characteristic 2, since Weierstrass points “do not reduce well” modulo 2. The invariants $i_n$ can be expressed in terms of Siegel modular forms thus:

$$i_1 = 2 \cdot 3^5 \chi_{10}^{-6} \chi_{12}^{5}, \quad i_2 = 2^{-3} \cdot 3^3 \psi_4 \chi_{10}^{-4} \chi_{12}^{-3}, \quad i_3 = 2^{-5} \cdot 3^2 \psi_6 \chi_{10}^{-3} \chi_{12}^2 + 2^2 \cdot 3^3 \psi_4 \chi_{10}^{-4} \chi_{12}^3.$$  

See [Igu1, pp. 189, 195] for the definitions; $\psi_i$ are Eisenstein series of weight $i$, $-2^2 \chi_{10}$ is our $\Theta$.

Another interesting approach to the definition of invariants is the following: Let $I_2 = h_{12}/h_{10}$, $I_4 = h_4, I_6 = h_{16}/h_{10}, I_{10} = h_{10}$ be the modular forms of weight 2, 4, 6, 10, respectively, as in [Lau]. The appeal of this construction is that each $h_n$ is a simple polynomial expression in Riemann theta functions with integral even characteristics $[\zeta]$; for example, $h_4 = \sum_{10} (\Theta [\zeta](0, \tau))^4$,
$h_{10} = 2^{12} \Theta$. It is not hard to prove that the g.c.d. of the Fourier coefficients of $\Theta \left[ \frac{\tau}{\omega} \right] (0, \tau)$, for $\left[ \frac{\tau}{\omega} \right]$ an integral even characteristic, is 1 if $\epsilon \in \mathbb{Z}^2$ (that happens for 4 even characteristics) and 2 if $\epsilon \notin \mathbb{Z}^2$ (that happens for 6 even characteristics). Using that and writing $I_n = \varepsilon / \Theta$, one finds that the numerator of $I_n$ has an integral Fourier expansion. One then lets

$$j_1 := I_5^2 / 2^{-12} I_{10}, \quad j_2 := I_2^3 I_4 / 2^{-12} I_{10}, \quad j_3 := I_2^2 I_6 / 2^{-12} I_{10}.$$  

These are modular functions of the form $f / \Theta^k$, such that the numerator has integral Fourier coefficients. Slightly modifying the definition of $[\text{Lau}]$ (there one uses $a$ different from 5 the group generated by the

$$\frac{1}{\text{a}} = \frac{1}{\text{b}} = \frac{1}{\text{c}} = \frac{1}{\text{d}} = \frac{1}{\text{e}} = \frac{1}{\text{f}}$$

coefficients. Slightly modifying the definition of $[\text{Lau}]$ (there one uses $a$ different from 5 the group generated by the

$$\frac{1}{\text{a}} = \frac{1}{\text{b}} = \frac{1}{\text{c}} = \frac{1}{\text{d}} = \frac{1}{\text{e}} = \frac{1}{\text{f}}$$

functions in modular invariants, viz. the values of the functions $\Theta\left( \frac{a}{\omega} \right)$ associated to CM types $\Phi$. The construction essentially involves the evaluation of $\Theta$ at various CM points $\frac{a}{\omega}$. We deduce from the preceding results the following:

**Corollary 5.2.1.** The coefficients of the rational polynomials $H_i(X)$ are $S$-integers where $S$ is the set of primes smaller than $16 \cdot d^2 (\text{Tr}(r))^2$ and satisfying a certain decomposition property in a normal closure of $K$ as imposed by superspecial reduction.

We provide some numerical data in §§6.1 - 6.2.

**Remark 5.2.2.** Theorem 4.2.1 gives a partial converse to this corollary.

### 5.3. Units

Let $K$ be a primitive quartic CM field as before. In [DSG], De Shalit and the first named author constructed class invariants $u(\Phi; a), u(\Phi; a,b)$ associated to certain ideals of $K$ and a CM type $\Phi$. The construction essentially involves the evaluation of $\Theta$ at various CM points associated to $K$. Though the construction is general, we recall it only for the $u(\Phi; a)$ and under very special conditions. For the general case, refer to loc. cit.

**Example 5.3.1.** Assume that $K$ is a cyclic CM field with odd class number $h_K$, $h_K^+ = 1$. Let $\Phi$ be a CM type of $K$ and assume that the different ideal $D_{K/Q}(\delta)$ with $\delta = -\delta$ and $\text{Im}(\varphi(\delta)) > 0$ for $\varphi \in \Phi$. Let $a$ be a fractional ideal of $\mathcal{O}_K$ and choose $a \in K^+, a \geq 0$ such that $\mathfrak{a} \mathcal{O}_K = (a)$. The form $(f,g) = \text{Tr}_{K/Q}(fg/a\delta)$ induces a principal polarization on $\mathbb{C}^2/\Phi(a)$. Write the lattice $\mathfrak{a}(\Phi)$ as spanned by the symplectic basis formed by the columns of $(\omega_1 \omega_2)$ and consider $\Delta(\Phi(\mathfrak{a})) := \text{det}(\omega_2)^{-10} \Theta(\omega_2^{-1} \omega_1)$. It depends only on $\Phi, a$ and not on $a$. One then lets

$$u(\Phi; a) = \frac{\Delta(\Phi(\mathfrak{a}^{-1}))}{\Delta(\Phi(\mathcal{O}_K))}.$$  

See [DSG, §1.3] for remarkable properties of these invariants. In particular, if $h_K$ is a prime different from 5 the group generated by the $u(\Phi; a, b)$ in $H^+_K$ has rank $h_K - 1$. The following corollary holds in general.
Corollary 5.3.2. The invariants \(i(\Phi, a)\) are \(S\)-units for the set of primes of \(H_K\) that lie over rational primes \(p\) smaller than \(16 \cdot d^2(T_r(r))^2\) such that \(p\) decomposes in a certain fashion in a normal closure of \(K\) as imposed by superspecial reduction.

6. Appendix: Numerical data

6.1. Class invariants. Let \(K = \mathbb{Q}(x)/(x^2 + 50x^2 + 93)\) be the non-normal quartic CM field of class number 4 generated by \(i/25 + 2\sqrt{13}\) over its totally real subfield \(K_0 = \mathbb{Q}(\sqrt{13})\). The field discriminant of \(K\) is \(d = 3 \cdot 13^2\). The reflex field of \(K\) is the quartic CM field \(K^* = \mathbb{Q}(x)/(x^4 + 100x^2 + 2128)\) and has class number 4. The first class polynomial \(H_1(X)\) for \(K\) is:

\[
H_1(X) = X^8 + 104412714125638834470066767624487585591880751692147182604995283658326241284310965080758580599
\]

The other two class polynomials are not given here, as they have the same primes in their denominators. The polynomials were computed using PARI with 1000 digits of precision in about 8 hours each on an Intel Pentium 4, 2.4GHz, 512MB memory. The denominator factors as: 7 \(2^3 \cdot 3 \cdot 11\). Note that for the first invariant \(\Theta\) appears to the sixth power in the denominator, which agrees with the fact that all powers are a multiple of 6.

6.2. Curves with bad reduction. To illustrate the theory we give an example of a CM field \(K\) and two genus 2 curves over \(\mathbb{Q}\) with CM by \(K\). We list their invariants, and verify that they have bad reduction at the primes in the denominators of the invariants. In [Wam], van Wamelen gives a complete list of all isomorphism classes of genus 2 CM curves defined over the rationals with their Igusa invariants. For example, for the cyclic CM field \(K = \mathbb{Q}(\sqrt{13} - 3\sqrt{13})\) of class number 2, there are two non-isomorphic genus 2 curves defined over \(\mathbb{Q}\).

The curve with invariants equal to \(i_1 = \frac{21}{53} \cdot 6719^3 \cdot 70391^3 \cdot 2475892862351^3\), \(i_2 = \frac{2}{53} \cdot 6719^3 \cdot 7229^3 \cdot 2313^3\), \(i_3 = \frac{2}{13} \cdot 19 \cdot 53^2 \cdot 70391^3 \cdot 2475892862351^3\), and 

\[
y^2 = -7039944x^6 + 36128207x^5 + 262678342x^4 - 48855486x^3 - 112312588x^2 + 36312676x
\]

The reduction of a genus 2 curve at a prime can be calculated using [Liu, Thm 1, p. 204]. For these examples we actually calculated the reduction using the genus 2 reduction program written by Liu. The output of the program shows that at the primes \(p = 2, 3, 23, 131\), the curve
has potential stable reduction equal to the union of two supersingular elliptic curves $E_1$ and $E_2$ intersecting transversally at one point.

The second curve has invariants equal to $i_1 = \frac{2 \cdot 7^{10} \cdot 11^5 \cdot 21059^5}{3^2 \cdot 23^{12}}$, $i_2 = \frac{2 \cdot 5 \cdot 7^7 \cdot 11^3 \cdot 837 \cdot 21059^3}{3^3 \cdot 23^8}$, $i_3 = \frac{2 \cdot 7^6 \cdot 11^4 \cdot 21059^2 \cdot 71347 \cdot 739363}{3 \cdot 23^9}$, and has an affine model

$$y^2 = -243x^6 + 2223x^5 - 1566x^4 - 19012x^3 + 903x^2 + 19041x - 5882.$$ 

In this case, the output of the genus 2 reduction program shows that at the primes $p = 2, 3, 23$, the curve has potential stable reduction equal to the union of two supersingular elliptic curves $E_1$ and $E_2$ intersecting transversally at one point.

The reader may have noticed that $2$ does not appear in the denominator of the invariants. This is not due to the invariants $i_n$ being divisible by $2$. It is an artifact of cancellation between “values of the numerator and the denominator” and explains in which sense Theorem 4.2.1 may fail to provide a converse to Corollary 5.1.2. In fact, bad reduction of CM curves modulo primes over $2$ turns out to be prevalent. According to [IKO], there is no smooth superspecial curve in characteristic $2$. On the other hand, using complex multiplication, one can prove (e.g. for cyclic CM fields $K$ and primes decomposing as $(p) = p_1 p_2$ or $(p) = p_1^2$) superspecial reduction of principally polarized abelian surfaces with CM by $K$ (cf. [Gor]). This implies for $p = 2$ bad reduction of the corresponding curve.

### 6.3. Isogenies between two supersingular curves.

Let $p$ be a prime, $h$ the class number of $B_{p,\infty}$ and $N(p)$ the minimal integer for which there exists an isogeny of degree less or equal to $N(p)$ between any two supersingular elliptic curves over $\mathbb{F}_p$. Because of running time and memory restrictions we did only sample calculations. For $p = 10007$, the Total Computation Time was 22688.710 seconds, Total Memory Usage was 1213.97MB. The program ran on an Intel Pentium 4, 2.53 GHz, 1 GB memory.

| $p$  | $h$  | $\sqrt{p}$ | $N$  | $N/\sqrt{p}$ |
|------|------|-------------|------|--------------|
| 101  | 9    | 10          | 6    | 0.600        |
| 211  | 18   | 15          | 9    | 0.600        |
| 307  | 26   | 18          | 11   | 0.611        |
| 401  | 34   | 20          | 12   | 0.600        |
| 503  | 43   | 22          | 15   | 0.682        |
| 601  | 50   | 25          | 14   | 0.560        |
| 701  | 59   | 26          | 17   | 0.654        |
| 809  | 68   | 28          | 18   | 0.643        |
| 907  | 76   | 30          | 19   | 0.633        |
| 1009 | 84   | 32          | 20   | 0.625        |
| 2003 | 168  | 45          | 30   | 0.667        |
| 3001 | 250  | 55          | 34   | 0.618        |
| 4001 | 334  | 63          | 44   | 0.698        |
| 5003 | 418  | 71          | 46   | 0.648        |
| 6007 | 501  | 78          | 51   | 0.654        |
| 7001 | 584  | 84          | 56   | 0.667        |
| 8009 | 668  | 89          | 60   | 0.674        |
| 9001 | 750  | 95          | 59   | 0.621        |
| 10007| 835  | 100         | 70   | 0.700        |
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