Naturally reductive pseudo-Riemannian Lie groups in low dimensions

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Abstract. This work concerns the non-flat metrics on the Heisenberg Lie group of dimension three $H_3(\mathbb{R})$ and the bi-invariant metrics on the solvable Lie groups of dimension four. On $H_3(\mathbb{R})$ we prove that the property of the metric being naturally reductive is equivalent to the property of the center being non-degenerate. These metrics are Lorentzian algebraic Ricci solitons. We start with the indecomposable Lie groups of dimension four admitting bi-invariant metrics and which act on $H_3(\mathbb{R})$ by isometries and we finally study some geometrical features on these spaces.

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1. Introduction

Homogeneous manifolds constitute the goal of several modern research in pseudo-Riemannian geometry, for instance Lorentzian spaces for which all null geodesics are homogeneous became relevant in physics [13, 18]. This fact motivated several studies on g.o. spaces in the last years, see for instance [7, 8, 9, 11] and its references. In particular a three-dimensional connected, simply connected, complete homogeneous Lorentzian manifold is symmetric, or it is isometric to a three-dimensional Lie group equipped with a left-invariant Lorentzian metric [7].

In the case of the Heisenberg Lie group of dimension three $H_3(\mathbb{R})$ it was proved in [24] that there are three classes of left-invariant Lorentzian
metrics, and only one of them is flat (see also [19]), which is characterized by the property of the center being degenerate.

In this work we concentrate the attention to the other two non-flat metrics on $H_3(\mathbb{R})$ and their isometry groups. According to [22] any left-invariant metric on a Heisenberg Lie group, for which the center is non-degenerate is naturally reductive, so these spaces are geodesically complete and non-flat. Here we prove a partial converse to that result: Any naturally reductive Lorentzian metric on $H_3(\mathbb{R})$ admitting an action by isometric isomorphisms of a one-dimensional group, restricts to a metric on the center.

Thus for any left-invariant Lorentzian metric on $H_3(\mathbb{R})$ the following statements are equivalent:

- non-flat metric,
- non-degenerate center,
- naturally reductive metric.

The first equivalences follow from Theorem 1 in [15]. The statement above does not hold in higher dimensions: a flat left-invariant Lorentzian metric on $\mathbb{R} \times H_3(\mathbb{R})$ is proved to be naturally reductive in [23]. Properties of flat or Ricci-flat Lorentzian metrics were investigated for instance in [1] [2] [15] and references therein. Here we also compute the corresponding isometry groups following results on naturally reductive metrics in [22] (comparing with [6]) and we see that the non-flat metrics are algebraic Ricci solitons (see [5]).

The study of these naturally reductive non-flat metrics on $H_3(\mathbb{R})$ is motivated by the results on [21], which state that a naturally reductive pseudo-Riemannian space admits a transitive action by isometries of a Lie group equipped with a bi-invariant metric. Hence we start with the classification of all Lie algebras up to dimension four admitting an ad-invariant metric. It is important to remark that the method used here is constructive an independent of the classification of low dimensional Lie algebras.

So a naturally reductive Lorentzian metric on $H_3(\mathbb{R})$ admits an action by isometries of a Lie group $G$ with a bi-invariant metric. If $G$ has dimension four, it corresponds to one of the Lie algebras obtained before. This is a key point in the proof of the equivalence stated above.

Finally we complete the work by investigating the geometry of the bi-invariant metrics of the solvable Lie groups $G_0$ and $G_1$, which are associated to the non-flat metrics on $H_3(\mathbb{R})$. We compute the isometry groups $\mathfrak{l}(G_0)$ and $\mathfrak{l}(G_1)$ in the aim of establishing a relationship between them and $G_0$ and $G_1$ as isometry groups of $H_3(\mathbb{R})$. Also geodesics are described.

2. Lie algebras with ad-invariant metrics up to dimension four

In this section we revisit the Lie algebras of dimension $d \leq 4$ that can be furnished with an ad-invariant metric. The proofs given here are constructive and they do not make use of the double extension procedure [4] [12] [17].
Let $\mathfrak{g}$ be a real Lie algebra. A symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ is called \textit{ad-invariant} if the following condition holds:

$$\langle \text{ad}_X Y, Z \rangle + \langle Y, \text{ad}_X Z \rangle = 0 \quad \text{for all } X, Y, Z \in \mathfrak{g}.$$ 

Whenever $\langle \cdot, \cdot \rangle$ is non-degenerate the symmetric bilinear form is just called a \textit{metric}.

\textbf{Example 2.1.} The Killing form is an ad-invariant symmetric bilinear form on any Lie algebra $\mathfrak{g}$, which is non-degenerate if $\mathfrak{g}$ is semisimple. Moreover if $\mathfrak{g}$ is simple any ad-invariant metric on $\mathfrak{g}$ is a non-zero multiple of the Killing form.

Recall that the central descending series $\{C^r(\mathfrak{g})\}$ and central ascending series $\{C_r(\mathfrak{g})\}$ of a Lie algebra $\mathfrak{g}$, are for $r \geq 0$ respectively given by the ideals

\begin{align*}
C^0(\mathfrak{g}) &= \mathfrak{g} \quad C_0(\mathfrak{g}) = 0 \\
C^r(\mathfrak{g}) &= [\mathfrak{g}, C^{r-1}(\mathfrak{g})] \quad C_r(\mathfrak{g}) = \{X \in \mathfrak{g} : [X, \mathfrak{g}] \subseteq C_{r-1}(\mathfrak{g})\}.
\end{align*}

Fixing a subspace $\mathfrak{m}$ of $\mathfrak{g}$, its orthogonal subspace is defined as usual by $\mathfrak{m}^\perp = \{X \in \mathfrak{g} : \langle X, Y \rangle = 0, \forall Y \in \mathfrak{m}\}$.

The next result follows by applying the definitions above and an inductive procedure.

\textbf{Lemma 2.2.} Let $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ denote a Lie algebra endowed with an ad-invariant metric.

1. If $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ then $\mathfrak{h}^\perp$ is also an ideal of $\mathfrak{g}$.
2. $C^r(\mathfrak{g}) = (C_r(\mathfrak{g}))^\perp$ for all $r \geq 0$.

Notice that if the metric is indefinite, for any subspace $\mathfrak{m}$ the decomposition $\mathfrak{m} + \mathfrak{m}^\perp$ is not necessarily a direct sum. Nevertheless, the next formula holds

$$\dim \mathfrak{g} = \dim C^r(\mathfrak{g}) + \dim C_r(\mathfrak{g}) \quad \forall r \geq 0 \quad (1)$$

and in particular

$$\dim \mathfrak{g} = \dim C^1(\mathfrak{g}) + \dim J(\mathfrak{g}) \quad (2)$$

where $J(\mathfrak{g})$ denotes the center of $\mathfrak{g}$. Moreover

- if $\mathfrak{m} \subseteq C^1(\mathfrak{g})$ is a vector subspace such that $C^1(\mathfrak{g}) = (J(\mathfrak{g}) \cap C^1(\mathfrak{g})) \oplus \mathfrak{m}$, then $\mathfrak{m}$ is non-degenerate;

- if $\mathfrak{m}' \subseteq J(\mathfrak{g})$ is a vector subspace such that $J(\mathfrak{g}) = (J(\mathfrak{g}) \cap C^1(\mathfrak{g})) \oplus \mathfrak{m}'$, then $\mathfrak{m}'$ is non-degenerate.

\textbf{Remark 1.} Suppose $\mathfrak{g}$ admits an ad-invariant metric and $J(\mathfrak{g}) \neq 0$. Then as said above any complementary space $\tilde{J}$ such that $J(\mathfrak{g}) = \tilde{J} \oplus (J(\mathfrak{g}) \cap C^1(\mathfrak{g}))$ is non-degenerate. It follows that $\mathfrak{g} = \tilde{J} \oplus \tilde{\mathfrak{g}}$ as a direct sum of non-degenerate ideals where $\tilde{\mathfrak{g}} = J^\perp$ each of them having ad-invariant metrics. In addition $J(\tilde{\mathfrak{g}}) = J(\mathfrak{g}) \cap C^1(\mathfrak{g})$.

Now suppose $\mathfrak{g}$ is solvable. Then by (2) it has non-trivial center. If moreover $\mathfrak{g}$ is nonabelian then both $\mathfrak{C}^1(\mathfrak{g})$ and $\mathfrak{J}(\mathfrak{g})$ are non-trivial and $\mathfrak{C}^1(\mathfrak{g}) \cap \mathfrak{J}(\mathfrak{g}) \neq 0$. In fact using the decomposition described above $\mathfrak{g} = \tilde{J} \oplus \tilde{\mathfrak{g}}$ where
\( \mathfrak{g} \) turns to be a solvable Lie algebra with an ad-invariant metric. Then its center \( \mathfrak{z}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}) \cap C^1(\mathfrak{g}) \) is not trivial.

**Proposition 2.3.** Let \( \mathfrak{g} \) denote a real Lie algebra of dimension two or three. If it can be endowed with an ad-invariant metric, then

- in dimension two \( \mathfrak{g} \) is abelian and
- in dimension three \( \mathfrak{g} \) is abelian or simple.

**Proof.** Assume first that \( \mathfrak{g} \) has dimension two. Then it is either abelian or isomorphic to the solvable Lie algebra spanned by the vectors \( X, Y \) in dimension three, there exist \( \langle \, , \, \rangle \).

Assume now that \( \mathfrak{g} \) has dimension 3. It is well known that it must be either solvable or simple. If it is abelian or simple, it admits an ad-invariant metric (see Example 2.1).

Suppose now \( \mathfrak{g} \) is a non-abelian solvable Lie algebra equipped with an ad-invariant bilinear form \( \langle \, , \, \rangle \). Since \( \mathfrak{z}(\mathfrak{g}) \cap C^1(\mathfrak{g}) \) is non-trivial (see Remark 1), there exist \( X, Y \in \mathfrak{g} \) such that \( [X, Y] = Z \in C^1(\mathfrak{g}) \cap \mathfrak{z}(\mathfrak{g}) \). It is not difficult to see that the vectors \( X, Y, Z \) form a basis of \( \mathfrak{g} \). Since \( Z \in C^1(\mathfrak{g}) \cap (C^1(\mathfrak{g}))^\perp \) then \( \langle Z, Z \rangle = 0 \). Furthermore,

\[
\langle Z, X \rangle = \langle [X, Y], X \rangle = -\langle Y, [X, X] \rangle = 0
\]

and in the same way one gets \( \langle Z, Y \rangle = 0 \). Thus any ad-invariant bilinear form on \( \mathfrak{g} \) must be degenerate. \( \square \)

A Lie algebra \( (\mathfrak{g}, \langle \, , \, \rangle) \) is called **indecomposable** if it has no non-degenerate ideals.

Observe that if a Lie algebra \( \mathfrak{g} \) with an ad-invariant metric admits a non-degenerate ideal \( \mathfrak{i} \), then \( \mathfrak{i}^\perp \) is also a non-degenerate ideal and so \( \mathfrak{g} = \mathfrak{i} \oplus \mathfrak{i}^\perp \).

**Remark 2.** By Remark 1 if \( (\mathfrak{g}, \langle \, , \, \rangle) \) is indecomposable and with non-trivial center, then the center is contained in the commutator \( \mathfrak{z}(\mathfrak{g}) \subseteq C^1(\mathfrak{g}) \).

**Lemma 2.4.** Let \( \mathfrak{g} \) denote a Lie algebra of dimension four furnished with an ad-invariant metric. If it is non-solvable then it is decomposable.

**Proof.** Let \( \mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s} \) be a Levi decomposition of \( \mathfrak{g} \), where \( \mathfrak{r} \) denotes the radical. Since \( \mathfrak{g} \) is not solvable \( \dim \mathfrak{r} < 4 \). Moreover since there are no simple Lie algebras of dimension one or two, it holds \( \dim \mathfrak{r} = 1 \) and \( \mathfrak{s} \) is either \( \mathfrak{sl}(2) \) or \( \mathfrak{so}(3) \). In every case the action \( \mathfrak{s} \to Der(\mathfrak{r}) \) is trivial. In fact let \( \mathfrak{r} = \mathbb{R}e_0 \) and \( \mathfrak{s} = Span\{e_1, e_2, e_3\} \).

Assume \( [e_i, e_0] = \lambda_i e_0 \). For all \( i, j = 1, 2, 3 \) there exist \( \xi_{ij} \in \mathbb{R} - \{0\} \) such that \( [e_i, e_j] = \xi_{ij} e_k \) for some \( k = 1, 2, 3 \) (see the Lie brackets in \( \mathfrak{sl}(2) \) or \( \mathfrak{so}(3) \)) and where \( \xi_{ij} \neq 0 \) for all \( i, j \). Since \( [\mathfrak{s}, \mathfrak{s}] = \mathfrak{s} \) from \( Ad([e_i, e_j]) e_0 = \xi_{ij} Ad(e_k) e_0 \) one gets \( \lambda_k = 0 \) for all \( k \).

Let \( \langle \, , \, \rangle \) denote an ad-invariant metric on \( \mathfrak{g} \) and denote \( \mu_k = \langle e_0, e_k \rangle \). So

\[
\xi_{ij} \mu_k = \xi_{ij} \langle e_0, e_k \rangle = \langle e_0, [e_i, e_j] \rangle = \langle [e_j, e_0], e_i \rangle = 0
\]
and since $\xi_{ij} \neq 0$ it must holds $\mu_k = 0$ for all $k$. Hence since $\langle , \rangle$ is non-degenerate, it follows $\langle e_0, e_0 \rangle \neq 0$, so that $\mathfrak{t}$ is a non-degenerate ideal and the proof is finished. \hfill $\square$

To complete the description of all the Lie algebras of dimension four admitting ad-invariant metrics we have the following result.

**Proposition 2.5.** Let $\mathfrak{g}$ denote a real Lie algebra of dimension four which can be endowed with an ad-invariant metric. Then $\mathfrak{g} = \text{span}\{e_0, e_1, e_2, e_3\}$ is isomorphic to one of the following Lie algebras:

- $\mathbb{R}^4$
- $\mathbb{R} \oplus \mathfrak{sl}(2, \mathbb{R})$
- $\mathbb{R} \oplus \mathfrak{so}(3, \mathbb{R})$
- the oscillator Lie algebra $\mathfrak{g}_0 = \text{span}\{e_0, \cdots, e_3\}$ with the non-zero Lie brackets:
  \[
  [e_0, e_1] = e_2 \quad [e_0, e_2] = -e_1 \quad [e_1, e_2] = e_3 \tag{3}
  \]
- $\mathfrak{g}_1 = \text{span}\{e_0, \cdots, e_3\}$ with the non-zero Lie brackets:
  \[
  [e_0, e_1] = e_1 \quad [e_0, e_2] = -e_2 \quad [e_1, e_2] = e_3. \tag{4}
  \]

**Proof.** Let $\mathfrak{g}$ be a Lie algebra equipped with an ad-invariant metric $\langle , \rangle$. If $\mathfrak{g}$ is decomposable then $\mathfrak{g}$ corresponds to one of the following Lie algebras: $\mathbb{R}^4$, $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{so}(3, \mathbb{R})$ (by Proposition 2.3).

Assume now $\mathfrak{g}$ is indecomposable. From Lemma 2.4 the Lie algebra $\mathfrak{g}$ is solvable and hence $C^1(\mathfrak{g}) \neq \mathfrak{g}$. By Remark 2, $\mathfrak{z}(\mathfrak{g}) \subseteq C^1(\mathfrak{g})$ and $4 = \dim \mathfrak{z}(\mathfrak{g}) + \dim C^1(\mathfrak{g}) \leq 2 \dim C^1(\mathfrak{g})$. It follows that $\dim \mathfrak{z}(\mathfrak{g}) = 1$ or $\dim \mathfrak{z}(\mathfrak{g}) = 2$. But since we cannot have $\mathfrak{z}(\mathfrak{g}) = C^1(\mathfrak{g})$ (in dimension four), it should be $\dim \mathfrak{z}(\mathfrak{g}) = 1$ and $\dim C^1(\mathfrak{g}) = 3$.

Let $e_3$ be a generator of $\mathfrak{z}(\mathfrak{g})$ and let $e_0 \in \mathfrak{g} - C^1(\mathfrak{g})$ such that $\langle e_0, e_3 \rangle = 1$. Denote by $\mathfrak{m} = \text{span}\{e_0, e_3\}^\perp$. Then $\mathfrak{m} \subseteq \mathfrak{z}(\mathfrak{g})^\perp = C^1(\mathfrak{g})$, $\mathfrak{m}$ is non-degenerate and it is not difficult to see that $C^1(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{m}$. Then there exists a basis $\{e_1, e_2\}$ of $\mathfrak{m}$ such that the matrix of the metric in this basis takes one of the following forms

\[
B^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B^{1,1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad -B^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Thus $C^1(\mathfrak{g}) = \text{span}\{e_1, e_2, e_3\}$ and $e_0$ acts on $C^1(\mathfrak{g})$ by the adjoint action. Due to the ad-invariance property of $\langle , \rangle$ it follows that $\text{ad}(e_0) \mathfrak{m} \subseteq \mathfrak{m}$.

Assume that $\mathfrak{m}$ has the metric given by $B^0$, hence $\text{ad}(e_0) \in \mathfrak{so}(2)$ for $B^0$, implying that

\[
\text{ad}(e_0) = \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix} \tag{5}
\]

for some $\lambda \neq 0$. In the case that the metric is given by $-B^0$ the same matrix is obtained for $\text{ad}(e_0)$. Similarly $\text{ad}(e_0) \in \mathfrak{so}(1, 1)$ for $B^{1,1}$, implying that

\[
\text{ad}(e_0) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \tag{6}
\]
for some $\lambda \neq 0$.

In either case, since $\langle [e_0, e_1], e_2 \rangle = \langle e_0, [e_1, e_2] \rangle$ one gets that $[e_1, e_2] = \lambda e_3$.

In the basis $\{ \frac{1}{\lambda} e_0, e_1, e_2, \lambda e_3 \}$ the action of $\text{ad}(\frac{1}{\lambda} e_0)$ on $\mathfrak{m}$ is as in (5) taking $\lambda = 1$ while the metric obeys the rules

$$1 = \langle \frac{1}{\lambda} e_0, \lambda e_3 \rangle = \langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle \quad \langle e_0, e_0 \rangle = \mu \in \mathbb{R}$$

and this is for $\mathfrak{g}_0$. In fact, in this basis the relations of (3) are verified.

In the other case a similar reasoning gives the results of the statement, that is, one gets the basis $\{ e_1, e_2, e_3 \}$ for the action (6) and proceeding as above one gets the Lie algebra $\mathfrak{g}_1$ together with the ad-invariant metric given by:

$$1 = \langle \frac{1}{\lambda} e_0, \lambda e_3 \rangle = \langle e_1, e_2 \rangle \quad \langle e_0, e_0 \rangle = \mu \in \mathbb{R}.$$  \hspace{1cm} (8)

□

Remark 3. The ad-invariant metric on the Lie algebra $\mathfrak{g}_0$ (resp. $\mathfrak{g}_1$) can be taken with $\mu = 0$. In fact it suffices to change $e_0$ by $\sqrt{\frac{2}{\mu}} e_0 - e_3$ whenever $\mu > 0$ and by $\sqrt{\frac{2}{-\mu}} e_0 + e_3$ if $\mu < 0$. This gives the following matrices for the ad-invariant metrics

$$\mathfrak{g}_0: \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \mathfrak{g}_1: \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

which will be used from now on.

3. Naturally reductive metrics on the Heisenberg Lie group

Let $G$ denote a Lie group with Lie algebra $\mathfrak{g}$ and let $H < G$ be a closed Lie subgroup of $G$ whose Lie algebra is denoted by $\mathfrak{h}$. A homogeneous pseudo-Riemannian manifold $(M = G/H, \langle , \rangle)$ is said to be naturally reductive if it is reductive, i.e. there is a reductive decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \quad \text{with} \quad \text{Ad}(H)\mathfrak{m} \subseteq \mathfrak{m}$$

and

$$\langle [x, y]_\mathfrak{m}, z \rangle + \langle y, [x, z]_\mathfrak{m} \rangle = 0 \quad \text{for all} \quad x, y, z \in \mathfrak{m}.$$

We shall say that a metric on $M$ is naturally reductive if the conditions above are satisfied for some pair $(G, H)$. If $M$ is naturally reductive the geodesics passing through the point $o \in M$ are

$$\gamma(t) = \exp tx \cdot o \quad \text{for some} \quad x \in \mathfrak{m},$$

which implies that these spaces are geodesically complete. For the Heisenberg Lie group of dimension $2n+1$, $\text{H}_{2n+1}(\mathbb{R})$, one has the next result.
Theorem 3.1 follows from the next result and the previous lemma.

Our aim here is to characterize the Lorentzian naturally reductive metrics on the Heisenberg Lie group of dimension three. We shall prove a converse of the result above.

Theorem 3.1. If $H_3(\mathbb{R})$ is endowed with a naturally reductive pseudo-Riemannian left-invariant metric with pair $(G, \mathbb{R})$ where $G$ has dimension four and $\mathbb{R} < G$ acts by isometric automorphisms on $H_3(\mathbb{R})$, then the center of $H_3(\mathbb{R})$ is non-degenerate.

Thus the property of the center being non-degenerate characterizes the naturally reductive metrics on $H_3(\mathbb{R})$ whenever the isometries fixing a point act by isometric isomorphisms.

As known there is a one-to-one correspondence between left-invariant pseudo-Riemannian metrics on $H_3(\mathbb{R})$ and metrics on the corresponding Lie algebra $\mathfrak{h}_3$, which is generated by $e_1, e_2, e_3$ obeying the non-trivial Lie bracket relation $[e_1, e_2] = e_3$. In order to prove the theorem above we start with the next result, which does not make use of any metric.

Lemma 3.2. Let $\mathfrak{g} = \mathbb{R}e_0 \oplus \mathfrak{h}_3$ where the commutator $C^1(\mathfrak{g}) \subseteq \mathfrak{h}_3$ and the restriction of $\text{ad}(e_0)$ to $\mathfrak{v} = \text{span}\{e_1, e_2\}$ is non-singular. If $\mathfrak{m} \subset \mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$ which is isomorphic to $\mathfrak{h}_3$ then $\mathfrak{m} = \mathfrak{h}_3 = \text{span}\{e_1, e_2, e_3\}$.

Proof. Let $\mathfrak{m}$ denote a subalgebra of $\mathfrak{g}$ such that $\mathfrak{m} = \text{span}\{v_1, v_2, v_3\}$ with $[v_1, v_2] = v_3$ and $[v_i, v_3] = 0$ for $i = 1, 2$. Take

$$v_1 = a_0 e_0 + w_1 + a_3 e_3 \quad v_2 = b_0 e_0 + w_2 + b_3 e_3 \quad v_3 = c_0 e_0 + w_3 + c_3 e_3$$

where $w_i \in \text{span}\{e_1, e_2\}$ for all $i = 1, 2$. Since $C^1(\mathfrak{g}) \subseteq \text{span}\{e_1, e_2, e_3\}$ it follows that $c_0 = 0$. Let $A$ denote the restriction of $\text{ad}(e_0)$ to $\mathfrak{v}$, thus we have the following equations

$$v_3 = [v_1, v_2] = A(a_0 w_2 - b_0 w_1) + \omega(w_1, w_3) e_3$$

$$0 = [v_1, v_3] = a_0 Aw_3 + \omega(w_1, w_3) e_3$$

$$0 = b_0 Aw_3 + \omega(w_2, w_3) e_3.$$ 

If $a_0$ or $b_0$ is different from zero, then $w_3 = 0$ and so $v_3 = c_3 e_3$. Therefore $a_0 w_2 - b_0 w_1 = 0$ and so we can write $w_2$ in terms of $w_1$ or $w_1$ in terms of $w_2$ depending on $a_0 \neq 0$ or $b_0 \neq 0$ respectively. It is not hard to see that putting these conditions in $v_1, v_2, v_3$ then one gets that the set $v_1, v_2, v_3$ is linearly dependent which is a contradiction. So $a_0 = b_0 = 0$ and $\mathfrak{m} = \text{span}\{e_1, e_2, e_3\}$.

Now if $G$ is a Lie group acting by isometries on $H_3(\mathbb{R})$ which is naturally reductive with pair $(G, H)$, then $G$ is a semidirect extension of $H_3(\mathbb{R})$ and it admits a bi-invariant metric (according to Theorem 2.2 in [21]). Hence the Lie algebra of $G$ should be a solvable Lie algebra of dimension four admitting an ad-invariant metric, therefore either $\mathfrak{g}_0$ or $\mathfrak{g}_1$ of the previous section. Thus Theorem 3.1 follows from the next result and the previous lemma.
Lemma 3.3. Let $h_3$ denote the Heisenberg Lie algebra of dimension three equipped with a naturally reductive metric with pair $(g_i, \mathbb{R})$ $i=0,1$ where $\mathbb{R} \simeq g_i/h_3$ acts by skew-adjoint derivations on $h_3$. Then the center of $h_3$ is non-degenerate.

Proof. Let $v \in g_i$ be an element which is not in $\text{span}\{e_1, e_2, e_3\}$. Thus $g_i = \mathbb{R}v \oplus h_3$ and we may assume $v = e_0 + \alpha e_1 + \beta e_2 + \gamma e_3$ and $[v, h_3] \subseteq h_3$.

For $g_0$ the action of $\text{ad}(v)$ is given by

$$\text{ad}(v)e_1 = e_2 - \beta e_3 \quad \text{ad}(v)e_2 = -e_1 + \alpha e_3 \quad \text{ad}(v)e_3 = 0.$$

Let $Q$ denote a metric on $h_3$ such that $b_{ij} = Q(e_i, e_j)$ and for which $\text{ad}(v)$ is skew-adjoint. The condition $Q(\text{ad}(v)x, y) = -Q(x, \text{ad}(v)y)$ for all $x, y \in h_3$ gives rise to a system of equations on the coefficients $b_{ij}$:

$$b_{12} - \beta b_{13} = 0 \quad b_{22} - \beta b_{13} = b_{11} - \alpha b_{13} \quad b_{23} - \beta b_{33} = 0 \quad b_{12} - \alpha b_{23} = 0 \quad b_{13} - \alpha b_{33} = 0.$$

It is not hard to see that if we write $B = (b_{ij})$ then $\det B \neq 0$ implies $b_{33} \neq 0$, that is $Q$ non-degenerate implies the center of $h_3$ non-degenerate.

This also applies for $g_1$. One writes down the action of $\text{ad}(v)$ and from $Q(\text{ad}(v)x, y) = -Q(x, \text{ad}(v)y)$ the equations follow

$$b_{11} - \beta b_{13} = 0 \quad b_{12} - \beta b_{23} = b_{12} - \alpha b_{13} \quad b_{13} - \beta b_{33} = 0 \quad b_{22} - \alpha b_{23} = 0 \quad b_{23} - \alpha b_{33} = 0.$$

In this case also $b_{33} \neq 0$ says that the center of $h_3$ must be non-degenerate. □

The simply connected Lie group $H_3(\mathbb{R})$ with Lie algebra $h_3$ can be realized on the usual differentiable structure of $\mathbb{R}^3$ together with the next multiplication

$$(v, z) \cdot (v', z') = (v + v', z + z' + \frac{1}{2}v^T J v'),$$

where $v, v' \in \mathbb{R}^2$, $v^T$ denotes the transpose matrix of the $2 \times 1$ matrix $v$, and $J$ denotes the matrix given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

A basis of left-invariant vector fields at every point $(x, y, z) \in \mathbb{R}^3$ satisfying the non-trivial Lie bracket relation $[X_1, X_2] = X_3$ is given by

$$X_1 = \partial_x - \frac{y}{2} \partial_2$$
$$X_2 = \partial_y + \frac{z}{2} \partial_2$$
$$X_3 = \partial_z.$$

Two non-isometric Lorentzian metrics on $H_3(\mathbb{R})$ can be taken by defining

$$1 = \langle X_1, X_1 \rangle = \langle X_2, X_2 \rangle = -\langle X_3, X_3 \rangle \quad (10)$$
$$1 = \langle X_1, X_2 \rangle = \langle X_3, X_3 \rangle \quad (11)$$
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and the other relations are zero. Each of them is a naturally reductive pseudo-Riemannian metric on $H_3(\mathbb{R})$ with the following expression in the usual coordinates of $\mathbb{R}^3$:

$$h_1 = (1 - \frac{y^2}{4})dx^2 + (1 - \frac{x^2}{4})dy^2 - \frac{1}{4} x y dxdy - \frac{y}{2} dxdz + \frac{x}{2} dydz$$

$$h_2 = \frac{y^2}{4}dx^2 + \frac{x^2}{4}dy^2 + dz^2 + \frac{1}{4} x y dxdy + \frac{y}{2} dxdz - \frac{x}{2} dydz.$$ 

Making use of this information one can compute several geometrical features on $H_3(\mathbb{R})$ [22]. Recall that an algebraic Ricci soliton on $H_3(\mathbb{R})$ is a left-invariant pseudo-Riemannian metric such that its Ricci operator $Rc$ satisfies the equality

$$Rc(g) = c \text{id} + D$$

where $c \in \mathbb{R}$ and $D$ is a derivation of $\mathfrak{h}_3$,

that is $D : \mathfrak{h}_3 \to \mathfrak{h}_3$ is a linear map which satisfies $D[x, y] = [Dx, y] + [x, Dy]$ for all $x, y \in \mathfrak{h}_3$.

A pseudo-Riemannian manifold is called locally symmetric if $\nabla R \equiv 0$, where $\nabla$ denotes the covariant derivative with respect to the Levi-Civita connection and $R$ denotes the curvature tensor. The Ambrose-Hicks-Cartan theorem (see for example [20, Thm. 17, Ch. 8]) states that given a complete locally symmetric pseudo-Riemannian manifold $M$, a linear isomorphism $A : T_pM \to T_pM$ is the differential of some isometry of $M$ that fixes the point $p \in M$ if and only if it preserves the symmetric bilinear form that the metric induces into the tangent space and if for every $u, v, w \in T_pM$ the following equation holds:

$$R(Au, Av)Aw = AR(u, v)w.$$ (12)

In [10] it was proved that the isometry group corresponding to a pseudo-Riemannian left-invariant metric on a 2-step nilpotent Lie algebra is a semidirect product $I(N) = N \rtimes F(N)$, where $F(N)$ denotes the isotropy subgroup at the identity element. Thus $I(N)$ is essentially determined by $F(N)$. Moreover

- if $h_0$ is a flat metric on $H_3(\mathbb{R})$ then $(H_3(\mathbb{R}), h_0)$ is a locally symmetric space and therefore it applies the Ambrose-Hicks-Cartan theorem for the computation of $F(N)$.

- for the non-flat metrics the action of the isotropy subgroup (of the full isometry group) at the identity element is given by isometric automorphisms [10] so that $I(H_{2n+1}(\mathbb{R})) = H_{2n+1}(\mathbb{R}) \rtimes H$, where $H$ denotes the group of isometric automorphisms. In [22] this group is described.

**Proposition 3.4.** The isometry groups for the Lorentzian left-invariant metrics on $H_3(\mathbb{R})$ are given by

- $I(H_3(\mathbb{R}), h_0) = H_3(\mathbb{R}) \rtimes O(2, 1)$,

- $I(H_3(\mathbb{R}), h_1) = H_3(\mathbb{R}) \rtimes O(2)$,

- $I(H_3(\mathbb{R}), h_2) = H_3(\mathbb{R}) \rtimes O(1, 1)$.

Moreover both Lorentzian left-invariant non-flat metrics are algebraic Ricci solitons.
Proof. The description of the isometry group for a 2-step nilpotent Lie group equipped with a left-invariant metric obtained in [22] and the observations above give the proofs of the isometry groups. Notice that the connected component of the identity are $G_0$ and $G_1$ for $h_1$ and $h_2$ respectively (see the description of $G_0$ and $G_1$ in the next section).

By computing the Ricci tensor in the case of the naturally reductive metrics $h_1$ and $h_2$ one verifies that the corresponding Ricci operators satisfy

$$\text{Rc}(h_1) = \text{Rc}(h_2) = \frac{3}{2} \text{Id} - D$$  \hspace{1cm} (13)

where $D$ is the derivation of $h_3$ given by

$$D(X_1) = -X_1 \quad D(X_2) = -X_2 \quad D(X_3) = -2X_3,$$

showing that both $h_1$ and $h_2$ are algebraic Ricci solitons.  \hspace{1cm} $\square$

Remark 4. It can be verified that the Lie groups $G_0$ and $G_1$ act by isometries on $(H_3(\mathbb{R}), h_1)$ and $(H_3(\mathbb{R}), h_2)$ respectively. Compare with [6] for the isometry groups. For Ricci solitons see [5].

Remark 5. A left-invariant Lorentzian metric on $H_3(\mathbb{R})$ is flat if and only if the center is degenerate [15]. In [23] the flat Lorentzian metric on $\mathbb{R} \times H_3(\mathbb{R})$ given in [15] is proved to be naturally reductive and it admits an action by isometries of the free 3-step nilpotent Lie group in two generators.

Left-invariant pseudo-Riemannian metrics on 2-step nilpotent Lie groups are geodesically complete [14, 10].

4. Simply connected solvable Lie groups with a bi-invariant metric in dimension four

Our aim now is to describe geometrical features of the simply connected solvable Lie groups of dimension four provided with a bi-invariant metric. More precisely those corresponding to the Lie algebras $g_0$ and $g_1$ described in Proposition [2, 3].

Recall that if $G$ is a connected real Lie group, its Lie algebra $\mathfrak{g}$ is identified with the Lie algebra of left-invariant vector fields on $G$. Assume $G$ is endowed with a left-invariant pseudo-Riemannian metric $\langle \cdot, \cdot \rangle$. Then the following statements are equivalent (see [20, Ch. 11]):

1. $\langle \cdot, \cdot \rangle$ is right-invariant, hence bi-invariant;
2. $\langle \cdot, \cdot \rangle$ is $\text{Ad}(G)$-invariant;
3. the inversion map $g \rightarrow g^{-1}$ is an isometry of $G$;
4. $\langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0$ for all $X, Y, Z \in \mathfrak{g}$;
5. $\nabla_X Y = \frac{1}{7}[X, Y]$ for all $X, Y \in \mathfrak{g}$, where $\nabla$ denotes the Levi Civita connection;
6. the geodesics of $G$ starting at the identity element $e$ are the one parameter subgroups of $G$.  

By (3) the pair \((G, \langle , \rangle)\) is a pseudo-Riemannian symmetric space. Furthermore by computing the curvature tensor one has
\[
R(X, Y) = -\frac{1}{4} \text{ad}([X, Y]) \quad \text{for } X, Y \in \mathfrak{g}.
\]

\section{4.1. Structure of the Lie groups}

The action of \(e_0\) on \(h_3\) on both Lie algebras \(\mathfrak{g}_0\) and \(\mathfrak{g}_1\), lifts to a Lie group homomorphism \(\rho : \mathbb{R} \to \text{Aut}(H_3(\mathbb{R}))\) which on \((v, z) \in \mathbb{R}^2 \oplus \mathbb{R}\) has a matrix of the form
\[
\rho(t) = \begin{pmatrix} R_i(t) & 0 \\ 0 & 1 \end{pmatrix} \quad i = 0, 1
\]
where
\[
R_0(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \quad \text{for } \mathfrak{g}_0,
\]
\[
R_1(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad \text{for } \mathfrak{g}_1.
\]

Let \(G_0\) and \(G_1\) denote the simply connected Lie groups with respective Lie algebras \(\mathfrak{g}_0\) and \(\mathfrak{g}_1\). Then \(G_0\) and \(G_1\) are modeled on the smooth manifold \(\mathbb{R}^4\), where the algebraic structure is the resulting from the semidirect product of \(\mathbb{R}\) and \(H_3(\mathbb{R})\), via \(\rho\). Thus on \(G_i\) for \(i = 0, 1\), the multiplication is given by
\[
(t, v, z) \cdot (t', v', z') = (t + t', v + R_i(t)v', z + z' + \frac{1}{2}v^T J R_i(t)v').
\]

This information is useful in order to find a basis of the left-invariant vector fields. For \(G_0\) such a basis at every point \((t, x, y, z) \in \mathbb{R}^4\) is given by the following vector fields, each of them evaluated at \((t, x, y, z)\):
\[
X_0 = \partial_t
\]
\[
X_1 = \cos t \partial_x + \sin t \partial_y + \frac{1}{2}(x \sin t - y \cos t) \partial_z
\]
\[
X_2 = -\sin t \partial_x + \cos t \partial_y + \frac{1}{2}(x \cos t + y \sin t) \partial_z
\]
\[
X_3 = \partial_z
\]
and for \(G_1\) it is given by
\[
X_0 = \partial_t
\]
\[
X_1 = e^t \partial_x - \frac{1}{2}y e^t \partial_z
\]
\[
X_2 = e^{-t} \partial_y + \frac{1}{2}x e^{-t} \partial_z
\]
\[
X_3 = \partial_z.
\]

These vector fields verify the relations given in (3) and (4) respectively.

For every \(i = 0, 1\) the bi-invariant metric on \(G_i\) induced by the ad-invariant metric on \(\mathfrak{g}_i\) described in (9) induces on \(\mathbb{R}^4\) the next pseudo-Riemannian metric (in the usual coordinates):
\[
g_0 = dz \, dt + dx^2 + dy^2 + \frac{1}{2}(ydx \, dt - xdy \, dt) \quad \text{for } G_0
\]
\[
g_1 = dz \, dt + dx \, dy + \frac{1}{2}(ydx \, dt - xdy \, dt) \quad \text{for } G_1.
\]
4.2. Geodesics

Computing the Christoffel symbols of the Levi-Civita connection for the metrics $g_0, g_1$ (cf. [20]), a curve $\alpha(s) = (t(s), x(s), y(s), z(s))$ is a geodesic in $G_i$ if its components satisfy the second order system of differential equations:

- for $G_0$
  \[
  \begin{aligned}
  t''(s) &= 0, \\
  x''(s) &= -t'(s)y'(s), \\
  y''(s) &= t'(s)x'(s), \\
  z''(s) &= \frac{1}{2} t'(s)(x(s)x'(s) + y(s)y'(s)).
  \end{aligned}
  \]

- for $G_1$
  \[
  \begin{aligned}
  t''(s) &= 0, \\
  x''(s) &= t'(s)x'(s), \\
  y''(s) &= -t'(s)y'(s), \\
  z''(s) &= -\frac{1}{2} t'(s)(x(s)y'(s) + y(s)x'(s)).
  \end{aligned}
  \]

On the other hand, if $X_e = \sum_{i=0}^{3} a_i X_i(e) \in T_e G_0$, then the geodesic $\alpha$ through $e$ with initial condition $\alpha'(0) = X_e$ is the integral curve of the left-invariant vector field $X = \sum_{i=0}^{3} a_i X_i$. Suppose $\alpha(s) = (t(s), x(s), y(s), z(s))$ is the curve satisfying $\alpha'(s) = X_{\alpha(s)}$, then its coordinates are as below.

On $G_0$, for $a_0 \neq 0$:
\[
\begin{align*}
t(s) &= a_0 s, \\
x(s) &= \frac{a_1}{a_0} \sin a_0 s + \frac{a_2}{a_0} \cos a_0 s - \frac{a_2}{a_0}, \\
y(s) &= -\frac{a_1}{a_0} \cos a_0 s + \frac{a_2}{a_0} \sin a_0 s + \frac{a_1}{a_0}, \\
z(s) &= \frac{1}{2} \left( \left( \frac{a_2}{a_0} + \frac{a_3}{a_0} \right) - \left( \frac{a_3}{a_0} \right) \right) \sin a_0 s.
\end{align*}
\]

If $a_0 = 0$, it is easy to see that $\alpha(s) = (0, a_1 s, a_2 s, a_3 s)$ is the corresponding geodesic.

On $G_1$ for $a_0 \neq 0$:
\[
\begin{align*}
t(s) &= a_0 s, \\
x(s) &= \frac{a_1}{a_0} e^{a_0 s} - \frac{a_1}{a_0}, \\
y(s) &= -\frac{a_2}{a_0} e^{-a_0 s} + \frac{a_2}{a_0}, \\
z(s) &= \left( \frac{a_1 a_2}{a_0} + a_3 \right) s - \frac{a_1 a_2}{a_0^2} \sinh(a_0 s).
\end{align*}
\]

If $a_0 = 0$ again $\alpha(s) = (0, a_1 s, a_2 s, a_3 s)$ is the corresponding geodesic.

As a consequence if $X = \sum_{i=0}^{3} a_i X_i(e)$, the exponential map is

- On $G_0$, if $a_0 \neq 0$,
  \[
  \exp(X) = \left( a_0, \frac{1}{a_0} (R_0(a_0)J - J)(a_1, a_2)^t, a_3 + \frac{1}{2} \left( \frac{a_1^2}{a_0} - \frac{a_2^2}{a_0} \right) \left( 1 - \frac{\sin a_0}{a_0} \right) \right)
  \]

- On $G_1$, if $a_0 \neq 0$,
if $a_0 = 0$, 
\[
\exp(X) = (0, a_1, a_2, a_3).
\]

- On $G_1$, if $a_0 \neq 0$
\[
\exp(X) = \left(a_0, \frac{a_1}{a_0}(e^{a_0} - 1), \frac{a_2}{a_0}(1 - e^{-a_0}), \frac{a_1 a_2}{a_0} + a_3 - \frac{a_1 a_2}{a_0^2} \sinh(a_0)\right)
\]

if $a_0 = 0$, 
\[
\exp(X) = (0, a_1, a_2, a_3).
\]

In both cases the geodesic passing through the point $g \in G_i, i = 0, 1$ and with derivative the left-invariant vector field $X$, is the translation on the left of the one-parameter group at $e$, that is $\gamma(s) = g \exp(sX)$ for $\exp(sX)$ given above.

### 4.3. Isometries

Let $G$ be a connected Lie group with a bi-invariant metric, and let $\mathfrak{l}(G)$ denote the isometry group of $G$. This is a Lie group when endowed with the compact-open topology. Let $\varphi$ be an isometry such that $\varphi(e) = x$, for $x \neq e$. Then $L_{x^{-1}} \circ \varphi$ is an isometry which fixes the element $e \in G$. Therefore $\varphi = L_x \circ f$ where $f$ is an isometry such that $f(e) = e$. Let $F(G)$ denote the isotropy subgroup of the identity $e$ of $G$ and let $L(G) := \{L_g : g \in G\}$, where $L_g$ is the translation on the left by $g \in G$. Then $F(G)$ is a closed subgroup of $G$ and the explanation above says
\[
\mathfrak{l}(G) = L(G)F(G) = \{L_g \circ f : f \in F(G), g \in G\}. \quad (18)
\]

Thus $\mathfrak{l}(G)$ is essentially determined by $F(G)$.

The following lemma is proved by applying the Ambrose-Hicks-Cartan Theorem \([12]\) to the Lie group $G$ equipped with a bi-invariant metric and whose curvature formula was given in \([14]\). In this way one gets a geometric proof of the next result (see \([16]\)).

**Lemma 4.1.** Let $G$ be a simply connected Lie group with a bi-invariant pseudo-Riemannian metric. Then a linear endomorphism $A : \mathfrak{g} \to \mathfrak{g}$ is the differential of some isometry in $F(G)$ if and only if for all $X, Y, Z \in \mathfrak{g}$, the linear map $A$ satisfies the following two conditions:

(i) $\langle AX, AY \rangle = \langle X, Y \rangle$;

(ii) $A[[X, Y], Z] = [[AX, AY], AZ]$.

Notice that if $G$ is simply connected, every local isometry of $G$ extends to a unique global one. Therefore the full group of isometries of $G$ fixing the identity is isomorphic to the group of linear isometries of $\mathfrak{g}$ that satisfy condition (ii) of Lemma 4.1. By applying this to our case, one gets the next result.

**Theorem 4.2.** Let $G$ be a non-abelian, simply connected solvable Lie group of dimension four endowed with a bi-invariant metric. Then the group of isometries fixing the identity element $F(G)$ is isomorphic to:

- $(\{1, -1\} \times O(2)) \ltimes \mathbb{R}^2$ for $G_0$, 

• \((\{1, -1\} \times O(1, 1)) \times \mathbb{R}^2\) for \(G_1\).

In particular the connected component of the identity of \(F(G)\) coincides with the group of inner automorphisms \(\{I_g : G_0 \to G_0, I_g(x) = g x g^{-1}\}_{g \in G}\).

**Proof.** We proceed with Lemma 4.1.

Let \(A : g_0 \to g_0\) be a linear isometry that satisfies the conditions of Lemma 4.1.

Since \(C^1(g_0)\) coincides with \(C^2(g_0)\) it follows that \(AC^1(g_0) \subseteq C^1(g_0)\). We also have \([C^1(g_0), C^1(g_0)] = \text{span}\{e_3\}\) and from the relation \(-A e_3 = [A e_1, [A e_1, A e_0]]\) one has \(A e_3 = a_{33} e_3\). Thus we may assume that in the basis \(\{e_0, e_1, e_2, e_3\}\) the map \(A\) has a matrix of the form

\[
\begin{pmatrix}
  a_{00} & 0 & 0 & 0 \\
  a_{10} & a_{11} & a_{12} & 0 \\
  a_{20} & a_{21} & a_{22} & 0 \\
  a_{30} & a_{31} & a_{32} & a_{33}
\end{pmatrix}.
\]

From \(\langle A e_0, A e_3 \rangle = 1\) it follows that

\[a_{00} a_{33} = 1. \tag{19}\]

From \(\langle A e_i, A e_j \rangle = \delta_{ij}\), for \(i, j = 1, 2\) one gets that

\[A : \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in O(2). \tag{20}\]

Now \(A[e_0, [e_1, e_0]] = [A e_0, [A e_1, A e_0]] = A e_0\) implies

\[a_{00}^2 a_{11} = a_{11}, \quad a_{00}^2 a_{21} = a_{21} \tag{21}\]

and

\[a_{31} = -a_{00} (a_{10} a_{11} + a_{20} a_{21}). \tag{22}\]

Equations (19), (20) and (21) assert

\[a_{00} = a_{33} = \pm 1. \tag{23}\]

Now from \(A[e_0, [e_2, e_0]] = [A e_0, [A e_2, A e_0]] = A e_2\) one has

\[a_{32} = -a_{00} (a_{10} a_{12} + a_{22} a_{20}). \tag{24}\]

Set \(w = (a_{10}, a_{20})^T\), from (22) and (24) it follows that \((a_{31}, a_{32}) = \mp w^T \tilde{A}\).

Finally, the relation \(\langle A e_0, A e_0 \rangle = 0\) implies \(a_{30} = \pm \frac{1}{2} ||w||^2\). Therefore

\[A = \begin{pmatrix}
  \pm 1 & 0 & 0 & 0 \\
  w & 0 & 0 & 0 \\
  \mp \frac{1}{2} ||w||^2 & \mp w^T \tilde{A} & \mp \frac{1}{2} ||w||^2 & a_{33} \\
 0 & \mp w \mp \frac{1}{2} ||w||^2 & \mp w^T \tilde{A} & \mp \frac{1}{2} ||w||^2
\end{pmatrix}. \tag{25}\]

where \(w \in \mathbb{R}^2\) and \(\tilde{A} \in O(2)\). Moreover any matrix of the form (25) verifies (i) and (ii) of Lemma 4.1 This gives a group isomorphic to \((\{1, -1\} \times O(2)) \times \mathbb{R}^2\) for which the identity component corresponds to those matrices of the form (25) with \(a_{00} = a_{33} = 1\) and \(\tilde{A} \in SO(2) = \{R_0(t) : t \in \mathbb{R}\}\).
Naturally reductive pseudo-Riemannian Lie groups in low dimensions

On the other hand, the set of isometric automorphisms of \( g_0 \) coincides with the set \( \text{Ad}(G_0) \), that is, the matrices of the form

\[
\text{Ad}(t, v) = \begin{pmatrix}
1 & 0 & 0 \\
Jv & R_0(t) & 0 \\
-\frac{1}{2}||v||^2 & -(Jv)^T R_0(t) & 1
\end{pmatrix}, \quad v \in \mathbb{R}^2.
\]

being \( A(t, v) = \text{Ad}(t, v, z) \) for \( v = (x, y) \). By dimension and since \( \text{Ad}(G_0) \) is connected, it must coincide with the identity component.

The procedure for \( g_1 \) is the same. In this case we obtain that in the basis \( \{e_0, \ldots, e_3\} \), the matrix of a linear isometry of \( g_1 \) that satisfies the conditions of Lemma 4.1 is of the form

\[
A = \begin{pmatrix}
\pm 1 & 0 & 0 \\
w & \tilde{A} & 0 \\
\mp \frac{1}{2}||w||^2 & \mp w^T \tilde{J} \tilde{A} & \pm 1
\end{pmatrix}, \quad (26)
\]

with \( w = (x, y)^T \in \mathbb{R}^2, ||w||^2 = 2xy, \tilde{A} \in O(1, 1) \) and \( \tilde{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

The matrix \( A(t, v) \) of \( \text{Ad}(t, v, z) \) with \( v = (x, y) \) is of the form (26) with \( a_{00} = 1, w = (-x, y) \) and \( \tilde{A} = R_1(t) \).

\( \square \)

Remark 6. For \( G_0 \) compare with [3].

References

[1] M. Aitbenhaddou, M. Boucetta, H. Lebzioui, Left-invariant Lorentzian flat metrics on Lie groups, J. Lie Theory 22 (1) (2012), 269–289. (arXiv:1103.0650v1 (2011)).

[2] M. Boucetta, Ricci flat left invariant Lorentzian metrics on 2-step nilpotent Lie groups, arXiv:0910.2563 (2009).

[3] F. Bourseau, Die Isometrien der Oszillatorgruppe und einige Ergebnisse über Prämorphismen Liescher Algebren, Diplomarbeit, Fak. der Math., Univ. Bielefeld (1989).

[4] H. Baum, I. Kath, Doubly extended Lie groups – curvature, holonomy and parallel spinors, Differ. Geom. Appl. 19 (3) (2003), 253–280.

[5] W. Batat and K. Onda, Algebraic Ricci Solitons of three-dimensional Lorentzian Lie groups, arxiv 1112.2455v2 (2012).

[6] W. Batat, and S. Rahmani, Isometries, Geodesics and Jacobi Fields of Lorentzian Heisenberg Group, Mediterr. J. Math. 8 (2011), 411-430.

[7] G. Calvaruso, Homogeneous structures on three dimensional Lorentzian Lie manifolds, J. Geom. Phys. 57 (2007), 1279–1291.

[8] G. Calvaruso, R. A. Marinosci, Homogeneous geodesics of three dimensional unimodular Lorentzian Lie groups, Mediterr. J. Math. 3 (2006), 467–481.

[9] G. Calvaruso, R. A. Marinosci, Homogeneous geodesics of non unimodular Lorentzian Lie groups and naturally Lorentzian spaces in dimension three, Adv. Geom. 8 (2008), 473–489.
[10] L. Cordero, P. Parker, Isometry groups of pseudoriemannian 2-step nilpotent Lie groups, Houston J. Math. 35 (1) (2009), 49 - 72.
[11] Z. Dusek, Survey on homogeneous geodesics, Note Mat. 1 (suppl. no. 1) (2008), 147–168.
[12] G. Favre, L. Santharoubane, Symmetric, invariant, non-degenerate bilinear form on a Lie algebra, J. of Algebra, 105 (1987), 451–464.
[13] J. Figueroa O’Farrill, P. Meessen, S. Philip, Supersymmetry and homogeneity of M-theory backgrounds, Class. Quant. Grav. 22 (1) (2005), 207–226.
[14] M. Guediri, Sur la complétude des pseudo-métriques invariantes à gauche sur les groupes de Lie nilpotents, Rend. Sem. Mat. Univ. Pol. Torino 52 (1994), 371–376.
[15] M. Guediri, On the noneexistence of closed timelike geodesics in flat Lorentz 2-step nilmanifolds, Trans. AMS 355 (2) (2003), 775–786.
[16] D. Müller, Isometries of bi-invariant pseudo-Riemannian metrics on Lie groups. Geom. Dedicata 29 (1) (1989), 65–96.
[17] A. Medina, P. Revoy, Algèbres de Lie et produit scalaire invariant (French) [Lie algebras and invariant scalar products], Ann. Sci. École Norm. Sup. (4) 18 (3) (1985), 553–561.
[18] P. Meessen, Homogeneous Lorentzian spaces whose null-geodesics are canonically homogeneous, Lett. Math. Phys. 75 (2006), 209–212.
[19] K. Nomizu, Left-invariant Lorentz metrics on Lie groups, Osaka J. Math 16 (1) (1979), 143–150.
[20] B. O’Neill, Semi-Riemannian geometry with applications to relativity, Academic Press (1983).
[21] G. Ovando, Naturally reductive pseudo-Riemannian spaces, J. Geom. Phys. 61 (2011), 157–171.
[22] G. Ovando, Naturally reductive pseudo Riemannian 2-step nilpotent Lie groups, to appear in Houston J. Math., (see arXiv:0911.4067).
[23] G. Ovando, Examples of naturally reductive pseudo-Riemannian Lie groups, AIP Conference Proc. 1360 (2011), 157–163.
[24] S. Rahmani, Métriques de Lorentz sur les groupes de Lie unimodulaires de dimension 3, J. Geom. Phys. 9 (1992), 295–302.
[25] N. Rahmani, S. Rahmani, Lorentzian Geometry of the Heisenberg Group, Geom. Dedicata 118 (2006), 133–140.

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