A MONOIDAL MODEL FOR GOODWILLIE DERIVATIVES

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Abstract. Using the category of finite sets and injections, we construct a lax monoidal model for Goodwillie’s derivatives of a functor between spaces or spectra. Along the way, we show that the cross effects of a monad form a functor-operad. We also recover a chain rule for endofunctors of spaces, expressing the derivatives of the composite $F \circ G$ as a derived composition product of the derivatives of $F$ and $G$.

In an effort to understand homotopy types, Goodwillie developed a theory of calculus for homotopy invariant functors from pointed topological spaces $T$ to the categories of spaces $T$ or spectra $S$ in a series of papers [Goo90, Goo92, Goo03].

Goodwillie defines the $n$-excisive (“polynomial”) approximation $P_n F$ of a homotopy invariant functor $F : C \to D$ with natural map $p_n F : F \to P_n F$ and shows that these functors and maps fit into a tower of fibrations, called the Taylor tower, analogous to a Taylor series expanded at the zero object.

$$
\begin{align*}
F(X) & \twoheadrightarrow P_1 F(X) \twoheadrightarrow P_0 F(X) \\
& \cdots \twoheadrightarrow P_n F(X) \twoheadrightarrow P_{n-1} F(X) \twoheadrightarrow \cdots \twoheadrightarrow P_1 F(X) \twoheadrightarrow P_0 F(X) \simeq F(*)
\end{align*}
$$

As in function calculus, one wishes to study the functor $F$ by studying its Taylor tower, and this is a good approximation when $F$ is analytic, a connectivity hypothesis which implies $F(X) \simeq \holim_n P_n F(X)$ for sufficiently connected $X$. Many functors are analytic; for example, the identity functor of spaces is analytic. Our work will mainly focus on analytic functors because of their nice stability properties.

The $n$-excisive approximations of $F$ are difficult to compute in general, so attention shifts to the homotopy fibers $D_n F = \hofib[P_n F \to P_{n-1} F]$, or layers, of the Taylor tower, with the hopes that the polynomial parts can be reconstructed once the layers are known.

One can consider the fiber $D_n F$ as a difference between the $n$th polynomial approximation and the $n - 1$st. We call $D_1 F$ the linearization of $F$. Indeed, this analogy is justified by Goodwillie’s classification of the layers, which look like the $n$-homogeneous pieces of the Taylor series, $\frac{f^{(n)}(0)}{n!}$.

Theorem 0.1. [Goo03]

$$
D_n F(X) \simeq \Omega^\infty (\partial_n F \wedge X^{\Sigma_n})_{h \Sigma_n}
$$

where $\partial_n F$ is a spectrum with $\Sigma_n$-action called the $n$th derivative of $F$.

The symmetric group $\Sigma_n$ acts on the smash product by permuting the factors and $(-)_{h \Sigma_n}$ denotes the homotopy orbits. Letting $n$ vary, the derivatives form a symmetric sequence in the category of spectra, which is just a sliver of the interesting structure they have been shown to possess [Joh95, AM99, Chi05]. Utilizing operadic duality, Arone and Ching have a significant body of work [AC11, AC15, AC16] developing which properties permit the derivatives to reconstruct the Taylor tower of a functor.
Goodwillie went further to identify the homotopy type of these derivatives, and we use $\partial^G F$ to denote his model for the derivatives.

**Theorem 0.2.** [Goo03] The $n$th derivative of $F$ is equivalent to the multilinearization of the $n$th cross effect.

$$(\Omega^\infty)^n \partial^G F \simeq \hocolim_{k_1, \ldots, k_n \to \infty} \Omega^{k_1} \cdots \Omega^{k_n} \text{cr}_n F(\Sigma^{k_1} S^0, \ldots, \Sigma^{k_n} S^0)$$

The $\Sigma_n$-action is induced by permuting the variables of $\text{cr}_n F$; in the multilinearization, this also permutes the loops. The $n$th cross effect is a functor of $n$ variables which can be thought of as a measurement of the failure of $F$ to be degree $n - 1$ (in an additive sense). For example, $\text{cr}_1 F(X) = \text{hofib}[F(X) \to F(*)]$, so if $F$ is degree 0 (or constant), $\text{cr}_1 F$ is trivial.

If we consider all the derivatives of a functor together, we see a symmetric sequence in spectra. Thus we may think of the derivatives as a functor

$$\partial_* : \text{Fun}(\mathcal{T}, \mathcal{T}) \to \text{SymmSeq}(\mathcal{S}p)$$

This paper investigates some of the extra structure that this map possesses; in particular, we consider a question posed by Arone and Ching in the introduction of [AC11].

**Question 1.** Is there a model for $\partial_*$ that is endowed with natural maps

$$\mu : \partial_* F \circ \partial_* G \to \partial_*(F \circ G)$$

and

$$\eta : 1 \to \partial_* \text{Id}$$

such that $\mu$ is associative and $\eta$ is unital. That is, is there a model for $\partial_*$ which is lax monoidal with respect to the composition of functors and the circle product of symmetric sequences?

It is easy to construct maps which are associative and unital up to homotopy, but strict associativity requires a different model for the derivatives. This problem is evident at the level of the cross effects that make up the derivatives as well as on the linearizations.

In this paper, we give a positive answer to this question for analytic endofunctors of spaces and spectra, proving the following theorem.

**Theorem 0.3.** The multilinearization of the cross effects over the category $I$ of finite sets and injections defines a lax monoidal model of $\partial_*$ on reduced endofunctors of spaces and spectra.

Some immediate consequences of this theorem are that if $F$ is a monad and $G$ is a module over $F$, then $\partial_* F$ is naturally an operad and $\partial_* G$ is a $\partial_* F$-module. For the monad $F = \text{Id}_\mathcal{T}$, these consequences have been proven. The first is due to work of Johnson [Job95], Arone-Mahowald [AM99], and Ching [Chi05], and the latter is work of Arone-Ching [AC11]. Arone and Ching went on to show that the derivatives satisfy a chain rule.

**Theorem 0.4.** [AC11] For reduced, finitary functors $F \circ G : \mathcal{C} \to \mathcal{D} \to \mathcal{E}$, where $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are either $\mathcal{T}$ or $\mathcal{S}p$,

$$\partial_* F \circ_{\partial_* \text{Id}_\mathcal{D}} \partial_* G \simeq \partial_*(F \circ G).$$

Taking the derived composition product over $\partial_* \text{Id}$ is a two-sided bar construction. The equivalence is given as a zigzag of equivalences, and there is no direct map for the chain rule. In this paper, we recover a chain rule as well, in which the equivalence is induced by the monoid map $\mu$. 
Theorem 0.5. For reduced, finitary, analytic functors $F \circ G : \mathcal{C} \to \mathcal{T} \to \mathcal{E}$, where $\mathcal{C}$ and $\mathcal{E}$ are either $\mathcal{T}$ or $\mathcal{Sp}$, the natural map is an equivalence

$$\partial_* F \circ_{\partial_* \text{id}_T} \partial_* G \to \partial_* (F \circ G).$$

The outline of this paper is as follows. In the first section, we review the definitions and conventions necessary for our models and for calculus. In section 2, we define our models for the cross effects of a functor and show that the cross effects of a monad form a functor-operad. In section 3, we define the new model for the derivatives and show that it is monoidal. Here is where the category $\mathbb{I}$ makes its debut, in order to give a strictly associative combination of homotopy colimits. We also employ the sphere operad of [AK14] to define the higher levels of the composition product map. Such structure sidesteps the technical aspects of Arone and Ching’s work and automatically produces a natural operad structure on the derivatives of the identity functor of spaces, a result which required a foray into operadic Koszul duality in [Chi05]. The derivatives of a functor also necessarily inherit the structure of a module over the derivatives of the identity, a hard-earned theorem in [ACT1]. Finally, in section 4, we prove a chain rule in this setting and indicate ways in which to extend this to functors of other categories.

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1. Background and Conventions

This paper investigates the calculus of functors between the categories of based topological spaces $\mathcal{T}$ and spectra $\mathcal{Sp}$. We will start with basic definitions necessary for Goodwillie’s calculus of functors and the use of the category $\mathbb{I}$ of finite sets and injective maps.

For technical reasons, we will work with pointed simplicial sets and EKMM’s category of S-modules [EKMM97]. Any good category of spectra will suffice for most of the results here; we only need the particularly useful properties of EKMM spectra to extend the models to functors of spectra.

We note that these assumptions are also made in Arone and Ching’s [ACT1]. The current paper extends and streamlines some topics there, so many technicalities and assumptions in this paper coincide with their basics.

Definition 1.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor where $\mathcal{C}$ and $\mathcal{D}$ are each either $\mathcal{Sp}$ or $\mathcal{T}$. Then we say

- $F$ is a homotopy functor if it preserves weak equivalences.
- $F$ is reduced if $F(*) \simeq *$, and pointed if $F(*) = *$.
- $F$ is continuous if the natural map $\mathcal{C}(X,Y) \to \mathcal{D}(F(X),F(Y))$ is a continuous homomorphism.
- $F$ is finitary if it preserves filtered homotopy colimits, i.e., for any filtered category $\mathcal{I}$ and diagram $X : \mathcal{I} \to \mathcal{C}$, $\text{hocolim}_{\mathcal{I}} F(X) \xrightarrow{\simeq} F(\text{hocolim}_{\mathcal{I}} X)$. We will also say that such functors satisfy the colimit axiom.

Remark 1.2. As noted in Remark 1.25 of [ACT1], a reduced functor $F : \mathcal{C} \to \mathcal{D}$ is itself weakly equivalent to a pointed functor. This is also true for a multifunctor which is reduced in each variable. For a multi-reduced functor $F : \mathcal{C} \times \mathcal{C} \to \mathcal{D}$, we define the multi-pointed
replacement of $F$ for objects $X, Y$ as the total cofiber of the square

$$
\begin{array}{ccc}
F(\ast, \ast) & \longrightarrow & F(X, \ast) \\
\downarrow & & \downarrow \\
F(\ast, Y) & \longrightarrow & F(X, Y)
\end{array}
$$

Assembly maps are incredibly useful to the point of view offered here, so we review their construction.

**Lemma 1.3.** If $F$ is a continuous functor, then $F$ has assembly, a binatural transformation which is also natural in $F$ given by

$$
\alpha_F : Z \land F(X) \longrightarrow F(Z \land X).
$$

**Proof.** The assembly map is given by pushing the identity through the following maps

$$
\begin{align*}
\text{Hom}(Z \land X, Z \land X) & \xrightarrow{\cong} \text{Hom}(Z, \text{Hom}(X, Z \land X)) \\
F \circ \text{Hom}(Z, \text{Hom}(F(X), F(Z \land X))) & \xrightarrow{\cong} \text{Hom}(Z \land X, F(Z \land X))
\end{align*}
$$

Note that this requires $\text{Hom}(X, Y) \to \text{Hom}(F(X), F(Y))$ to be a pointed map, and so $X \to \ast \to Y$ must be sent to the basepoint of $\text{Hom}(F(X), F(Y))$, thus a functor $F$ must be pointed in order to be continuous. \qed

**Definition 1.4.** A functor $F : \mathcal{C} \to \mathcal{C}$ is a *monad* if it is a monoid in the category of endofunctors on $\mathcal{C}$. More explicitly, a monad $F$ is a functor equipped with natural transformations $\eta : Id \to F$ and $\gamma : F \circ F \to F$ satisfying associativity and unitality diagrams.

In [Goo03], Goodwillie constructs the Taylor tower of a homotopy functor from topological spaces to spaces or spectra, and Kuhn shows that Goodwillie’s work extends to functors between more general model categories [Kuh07]. We concentrate on the derivatives that show up in the layers of the Taylor tower, but some of our proofs rely on Goodwillie’s construction of the levels of the tower, so we review these definitions as well. In particular, we will need to consider the intermediate functors in the definition of linearization of a functor.

**Definition 1.5.** A *cubical diagram* is a functor $\mathcal{X} : \mathcal{P}(S) \to \mathcal{C}$, where $S$ is a finite set and $\mathcal{P}(S)$ is the poset of all subsets of $S$. An $n$-cube will be a functor $\mathcal{X}$ where the cardinality of $S$ is $n$, so 0-cubes are objects of $\mathcal{C}$, 1-cubes are morphisms of $\mathcal{C}$, 2-cubes are commutative squares, etc.

**Definition 1.6.** Let $\mathcal{P}_0(n)$ denote the poset of all nonempty subsets of $n = \{1, \ldots, n\}$. An $n$-cube is called *homotopy cartesian* if the map $a(\mathcal{X}) : \mathcal{X}(\emptyset) \to \text{holim}_{\mathcal{P}_0(n)} \mathcal{X}$ is a weak equivalence (note that $a(\mathcal{X})$ factors through the strict $\lim_{\mathcal{P}_0(n)} \mathcal{X}$). An $n$-cube is (homotopy) $k$-cartesian if $a(\mathcal{X})$ is $k$-connected. Similarly, an $n$-cube is *homotopy cocartesian* if the map $b(\mathcal{X}) : \text{hocolim}_{\mathcal{P}_0(n)} \mathcal{X} \to \mathcal{X}(S)$ is a weak equivalence. It is $k$-cocartesian if $b(\mathcal{X})$ is a $k$ connected map. An $n$-cube $\mathcal{X}$ is called *strongly homotopy (co)cartesian* if each face of dimension $\geq 2$ is (co)cartesian. If every two-dimensional face of $\mathcal{X}$ is (co)cartesian, then $\mathcal{X}$ is strongly (co)cartesian. The *total homotopy fiber* of a cube is the homotopy fiber of the map $a(\mathcal{X}) : \mathcal{X}(\emptyset) \to \text{holim}_{\mathcal{P}_0(n)} \mathcal{X}$.

We will often omit the word “homotopy” from our pushouts, pullbacks, fibers, and (co)cartesian cubes, but it is always intended, unless noted otherwise.
While the definition of $n$-excision is not necessary to the results of this paper, a variant (stable $n$-excision) does come up.

**Definition 1.7.** A homotopy functor $F: \mathcal{C} \to \mathcal{D}$ is $n$-excissive if for every strongly cocartesian $(n+1)$-cubical diagram $\mathcal{X}: \mathcal{P}(S) \to \mathcal{C}$, the diagram $F(\mathcal{X}): \mathcal{P}(S) \to \mathcal{D}$ is cartesian.

Goodwillie defines the $n$-excissive approximation $P_nF$ of a homotopy functor $F$ as the homotopy colimit of an infinite iteration of intermediate functors $T_nF$. For the purposes of this paper, we only need $T_1F$ for reduced functors $F$, so we substitute Goodwillie’s definition with the following equivalent one.

**Definition 1.8.** If $F$ is reduced, the functor $T_1F$ is equivalent to $\Omega \circ F \circ \Sigma$.

We see that $T_1: \text{Fun}(\mathcal{C}, \mathcal{D}) \to \text{Fun}(\mathcal{C}, \mathcal{D})$ can be iterated and thus for $F$ reduced, $T_1F = \Omega^1 \circ F \circ \Sigma$. We also note that there is a natural transformation $t_1: F \to T_1F$.

Goodwillie defines $P_nF(X)$ as the homotopy colimit of the diagram

$$F(X) \xrightarrow{T_nF(X)} T_{n+1}F(X) \xrightarrow{\vdots} \cdots$$

When $F$ is reduced and $n = 1$, $P_1F$ agrees with the *linearization* of the functor $F$, and we use the model $\text{hofib}(\Omega^i\mathcal{D})$ for linearization.

**Definition 1.9.** The layers of the Taylor tower of $F: \mathcal{C} \to \mathcal{D}$ are the functors $D_nF: \mathcal{C} \to \mathcal{D}$ for $n \geq 1$ given by

$$D_nF = \text{hofib}[P_nF \to P_{n-1}F]$$

The functor $D_nF$ is $n$-homogeneous, that is, both $n$-excissive ($P_nD_nF \simeq D_nF$) and $n$-reduced ($P_{n-1}(D_nF) = \ast$).

**Definition 1.10.** Let $F: \mathcal{C} \to \mathcal{D}$ be a homotopy functor. $F$ is stably $n$-excissive or satisfies stable $n$th order excision, if the following condition holds for some numbers $c$ and $\kappa$:

$E_n(c, \kappa)$: If $\mathcal{X}: \mathcal{P}(S) \to \mathcal{C}$ is any strongly co-cartesian $(n + 1)$-cube such that for all $s \in S$, the map $\mathcal{X}(s) \to \mathcal{X}(\{s\})$ is $k_s$-connected and $k_s \geq \kappa$, then the diagram $F(\mathcal{X})$ is $(-c + \Sigma k_s)$-cartesian.

**Definition 1.11.** The functor $F$ is $\rho$-analytic if there is some number $q$ such that $F$ satisfies $E_n(n\rho - q, \rho + 1)$ for all $n \geq 1$. (Note that it is the same $q$ for all $n$.)

**Example 1.12.** [Goo92] 4.3, 4.5] An analytic functor is one whose deviation from being $n$-excissive is bounded in a certain way for all $n$. The identity functor of spaces is $1$-analytic by the higher Blakers-Massey theorem, and the functor $\text{Hom}(\mathcal{K}, -)$ is $k$-analytic, where $k = \text{dim}(K)$.

**Theorem 1.13.** [Goo03] 1.13] If $F$ is $\rho$-analytic and $X$ is (at least) $\rho$-connected, then the connectivity of the map $F(X) \to P_\rho F(X)$ tends to infinity with $n$, so that $F(X)$ is equivalent to the homotopy limit $P_\rho F(X)$ of the tower. Thus, the number $\rho$ gives a sort of radius of convergence for the Taylor tower.

The following definition will make a cameo in Lemma 3.2. The notation $O$ stands for ‘osculating,’ according to Goodwillie.

**Definition 1.14.** A map $\alpha: F \to G$ between two functors from $\mathcal{C}$ to $\mathcal{D}$ satisfies $O_n(c, \kappa)$ if, for every $k \geq \kappa$, for every object $X$ of $\mathcal{C}$ such that $X \to *$ is $k$-connected, the map $\alpha_X: F(X) \to G(X)$ is $(-c + (n + 1)k)$-connected.

This paper exploits the properties of a particular indexing category $I$ used by Bökstedt to define topological Hochschild homology. The use of the category $I$ has found great success in the area of algebraic K-theory and the newer field of representation stability.
Definition 1.15. Let \( \mathbb{I} \) denote (the skeleton of) the category of finite sets and injective maps. Let \( \mathbb{N} \) denote the category of finite sets with only the standard inclusions (those induced by subset inclusion). Let \( \Sigma \) denote the category of finite sets with only bijections.

Bökstedt showed that under certain conditions on a functor \( G : \mathbb{I} \to \mathcal{T} \), \( \text{hocolim}_\mathbb{I} G \to \text{hocolim}_1 G \) is an equivalence. Essentially, the condition is that maps further in the diagram become more and more connected.

Lemma 1.16. \((\text{Bok85})\) Let \( G : \mathbb{I} \to \mathcal{T} \) be a functor, \( x \in \text{ob} \, \mathbb{I} \), and let \( x \downarrow \mathbb{I} \) be the full subcategory of \( \mathbb{I} \) of objects supporting maps from \( x \). If \( G \) sends maps in \( x \downarrow \mathbb{I} \) to \( n_{|x|} \)-connected maps and \( n_{|x|} \to \infty \) as \( |x| \to \infty \), then \( \text{hocolim}_\mathbb{I} G \to \text{hocolim}_1 G \) is an equivalence.

A published proof can be found in [DGM13, 2.2.2.2].

We only require the following definition and lemma for \( n = 1 \) (and \( F \) reduced), but we provide the more general statement.

Definition 1.17. Let \( \mathbb{P}_n F = \text{hocolim}_{k\in \mathbb{I}} T^k_n F \).

Lemma 1.18. When \( F \) is stably \( n \)-excisive, \( \mathbb{P}_n F \to \mathbb{P}_n F \) is an equivalence.

Proof. We will show that the functor \( \Theta : \mathbb{I} \to \text{Fun}(\mathcal{T}, \mathcal{T}) \) defined by \( \Theta(k) = T^k_n F \) satisfies the hypotheses of Bökstedt’s lemma \([1.16] \) when \( F \) satisfies \( E_n(c, \kappa) \). By \([\text{Good93} \, \text{Prop 1.4}] \), if \( F \) satisfies \( E_n(c, \kappa) \), then \( T^k_n F \) satisfies \( E_n(c-1, \kappa-1) \) and \( t_n F : F \to T^k_n F \) satisfies \( O_n(c, \kappa) \).

By induction on \( i \), \( T^k_n F \) satisfies \( E_n(c-i, \kappa-i) \), and \( T^k_n F \to T^{k+1}_n F \) satisfies \( O_n(c-i, \kappa-i) \).

By the definition of \( O_n \), all the maps \( T^k_n F(X) \to T^{k+1}_n F(X) \) are at least \((i-c+(n+1)\ell)\)-connected for \( \ell \geq \kappa \), where \( \ell \) is the connectivity of \( X \to * \). Since \((i-c+(n+1)\ell)\) increases as \( i \) increases, \( \Theta \) satisfies the condition of Bökstedt’s lemma. \( \square \)

The results of this paper are framed in terms of symmetric sequences, so we review the pertinent definitions now.

Definition 1.19. Let \( \mathcal{C} \) be a category. A symmetric sequence in \( \mathcal{C} \) is a functor \( A : \Sigma \to \mathcal{C} \). This is a sequence \( \{A(n)\} \) of objects of \( \mathcal{C} \) with a \( \Sigma_n \)-action on \( A(n) \) for each \( n \geq 1 \). A morphism of symmetric sequences \( f : A \to B \) is a natural transformation of functors or, explicitly, a sequence of \( \Sigma_n \)-equivariant morphisms \( f(n) : A(n) \to B(n) \). We denote the category of symmetric sequences in \( \mathcal{C} \) by SymmSeq(\( \mathcal{C} \)).

Definition 1.20. If \( \mathcal{C} \) is a cocomplete closed symmetric monoidal category with monoidal product denoted \( \wedge \) and if \( A, B \) are symmetric sequences in \( \mathcal{C} \), then the composition product or \( o \)-product of \( A \) and \( B \) is the symmetric sequence \( A \circ B \) defined by

\[
(A \circ B)(n) = \bigvee \text{unordered partitions of } \{1, \ldots, n\} \quad A(k) \wedge B(n_1) \wedge \cdots \wedge B(n_k).
\]

The composition product defines a monoidal product on the category of symmetric sequences in \( \mathcal{C} \). If the unit of \( \mathcal{C} \) is \( S \) and zero-object *, the unit object of SymmSeq(\( \mathcal{C} \)) is given by

\[
1(n) = \begin{cases} 
S & \text{if } n = 1 \\
* & \text{else}
\end{cases}
\]

Definition 1.21. An operad in \( \mathcal{C} \) is a monoid in SymmSeq(\( \mathcal{C} \)) under the composition product; that is, an operad is a symmetric sequence \( \mathcal{O} \) with a composition map \( \gamma : \mathcal{O} \circ \mathcal{O} \to \mathcal{O} \) and a unit map \( \eta : 1 \to \mathcal{O} \) satisfying associativity and unitality diagrams.
Definition 1.22. Let $O$ be an operad in $C$. A right $O$-module (respectively, left $O$-module) is a symmetric sequence $M$ with an action map $M \circ O \to M$ (respectively, $O \circ M \to M$) satisfying associativity and unitality diagrams.

Definition 1.23. Let $O$ be an operad in $C$ with right and left modules $R$ and $L$, respectively. The simplicial bar construction on $O$ with coefficients in $R$ and $L$ is a simplicial object $B_k(R, O, L)$ in the category of symmetric sequences in $C$. The $k$-simplices are given by

$$B_k(R, O, L) = R \circ O \circ \cdots \circ O \circ L$$

The face maps are given by operad composition and module action maps and the degeneracy maps are given by the unit map.

We take termwise geometric realization to obtain the derived composition product $R \circ L$.

Definition 1.24. A functor $F : C \to D$ between monoidal categories $(C, \otimes_C, 1_C)$ and $(D, \otimes_D, 1_D)$ is monoidal if there is a morphism $\epsilon : 1_D \to F(1_C)$ and a natural transformation $\mu_{X,Y} : F(X) \otimes_D F(Y) \to F(X \otimes_C Y)$ satisfying associativity and unitality diagrams.

2. Key properties of cross effects

In this section, we define our model for the cross effects of a functor and show that the cross effects of a monad form a functor-operad. The technical results here form the core of the main theorem for the derivatives.

We will start with the definition of the cross effects of an endofunctor $F$ of spaces. It is important to choose our model for the homotopy fiber carefully so that the desired maps exist.

Definition 2.1. [May99, 8.6] The homotopy fiber of a map $f : X \to Y$ is given by the strict limit

$$\text{hofib } f = \lim X \leftarrow \begin{array}{c} f \\ \downarrow \end{array} \begin{array}{c} Y' \xrightarrow{e_Y} Y \\ \downarrow \end{array}$$

where $Y'$ is the pointed path space of $Y$. This diagram is a fibrant replacement for the diagram with the terminal object $*$ in the place of $Y'$, so the homotopy limit agrees with the strict limit.

Definition 2.2. For a category $C$ with coproducts, let $\amalg_n : C^n \to C$ be defined by $\amalg_n(X_1, \ldots, X_n) = X_1 \vee \cdots \vee X_n$.

Definition 2.3. The 1st cross effect of $F$ on the space $X$ is given by

$$cr_1 F(X) = \text{hofib}[F(X) \to F(*)]$$

The $n$th cross effect of $F$ on the spaces $X_1, \ldots, X_n$ is given by

$$cr_n F(X_1, \ldots, X_n) = cr_1^{(n)} \cdots cr_1^{(1)} (F \circ \amalg_n)(X_1, \ldots, X_n)$$

where $cr_1^{(i)} G$ denotes the first cross effect applied to the $i$th variable of the multifunctor $G$.

We note that there is a natural transformation $cr_1 F \to F$ and thus also a natural transformation $cr_n F \to (F \circ \amalg_n)$ for all $n$. Traditionally, $cr_n F$ is defined as the total fiber of a cube constructed from coproducts of the inputs [Goo03], and we will now demonstrate that the model given above is equivalent to the usual cubical model, $cr_n^\Box$. 
Definition 2.4. Goodwillie’s $n^{th}$ cross effect of $F$ on the spaces $X_1, \ldots, X_n$ is given by

$$cr_n^i F(X_1, \ldots, X_n) = \text{tothofib}_{U \in \mathcal{P}(\Omega)} F\left( \bigvee_{i=1}^n X_i \right) = \text{hofib} \left[ F \left( \bigvee_{i=1}^n X_i \right) \to \text{holim} \left[ F\left( \bigvee_{i \in U} X_i \right) \right] \right]$$

Thus Goodwillie’s second cross effect is defined as

$$cr_2^1 F(X, Y) = \text{tothofib} \left( F(X \vee Y) \to F(Y) \right)$$

$$cr_1^1 (F \circ \sqcup_2)(X, Y) = \text{hofib} \left[ F(X \vee Y) \to F(* \vee Y) \right]$$

$$cr_1^1 (F \circ \sqcup_2)(X, *) = \text{hofib} \left[ F(X \vee *) \to F(* \vee *) \right]$$

Then the vertical fiber $\text{hofib}[cr_1^1 (F \circ \sqcup_2)(X, Y) \to cr_1^1 (F \circ \sqcup_2)(X, *)]$ is equivalent to $cr_1^2 (F \circ \sqcup_2)(X, Y)$, so the iterated first cross effects is equivalent to the standard cubical model, $cr_2^1 F(X, Y) \simeq cr_2^1 F(X, Y)$. The generalization to higher cubes is straightforward.

Note that if $F$ is reduced, $cr_1 F$ is also reduced, and the choice of model for the homotopy fiber yields isomorphisms

$$cr_1^1 cr_1^2 (F \circ \sqcup_2)(X, Y) \simeq cr_1^2 cr_1^1 (F \circ \sqcup_2)(X, Y).$$

Lemma 2.5. The multifunctor $cr_n F$ has assembly maps in each variable.

Proof. Since $cr_n F$ is continuous and reduced in each variable, the assembly map is given by following the identity through the maps

$$\text{Hom}(Z \land X, Z \land X) \xrightarrow{z} \text{Hom}(Z, \text{Hom}(X, Z \land X))$$

$$\xrightarrow{cr_2^1 F(-, Y)} \text{Hom}(Z, \text{Hom}(cr_2^1 F(X, Y), cr_2^1 F(Z \land X, Y)))$$

$$\xrightarrow{z} \text{Hom}(Z \land cr_2^1 F(X, Y), cr_2^1 F(Z \land X, Y))$$

\[\square\]

Lemma 2.6. If $F$ is stably $n$-excisive, then $cr_k F$ is stably $n$-excisive in each variable.

Proof. If $F$ is stably $n$-excisive, then $F$ satisfies $E_n(c, \kappa)$ for some $c$ and $\kappa$. We will show that $cr_k^0 F(-, Y_2, \ldots, Y_k)$ is stably $n$-excisive. Let $\mathcal{X}$ be a strongly cocartesian $(n + 1)$-cube such that $\mathcal{X}(\emptyset) \to \mathcal{X}(\{s\})$ is $k_s$-connected with $k_s \geq \kappa$. Then $cr_k^0 F(\mathcal{X}, Y_2, \ldots, Y_k)$ is a cube of homotopy fibers (so is of the form to apply Proposition 1.18 of [Goo92])

$$V \mapsto cr_k^0 F(\mathcal{X}(V), Y_2, \ldots, Y_k)$$

$$= \text{hofib} \left[ F(\mathcal{X}(V) \vee Y_2 \vee \cdots \vee Y_k) \to \text{holim}_{U \in \mathcal{P}(\Omega)} F\left( \mathcal{X}(\emptyset) \vee \bigvee_{j \in U} Y_j \right) \right]$$

$$= \text{hofib} \left[ \mathcal{Y} \to \mathcal{Y}' \right]$$
where \( \delta_U = \begin{cases} 0 & \text{if } 1 \in U \\ 1 & \text{if } 1 \notin U \end{cases} \), i.e. \( \mathcal{X}(V) \) is in the last sum if \( 1 \notin U \). Since \( \mathcal{Y} \) is an \((n+1)\)-cube of homotopy limits of punctured \( k \)-cubes, by \cite[1.22]{Goo92}, \( \mathcal{Y} \) is \( \ell \)-cartesian where \( \ell = \min\{1 - |U| + \ell_U \} \), and \( \ell_U \) is the cartesianness of the \((n+1)\)-cube at \( U \). If \( 1 \notin U \), the \((n+1)\)-cube at \( U \) is given by \((F \circ \Delta_2)(\mathcal{X}, \vee_{j \in U} Y_j)\), which gives \( \ell_U = \Sigma k_s - c \). If \( 1 \in U \), the \((n+1)\)-cube at \( U \) is constant, so cartesian. Then the largest that \( U \) can be (where the cube at \( U \) is not cartesian) is \( k - 1 \), so \( \ell = \Sigma k_s - c - k + 2 \).

The \((n+1)\)-cube \( \mathcal{Y} = (F \circ \Delta_k)(\mathcal{X}, Y_2, \ldots, Y_k) \) is \((\Sigma k_s - c)\)-cartesian, so by \cite[1.6, 1.18]{Goo92}, the cube of fibers \( cr_k^n F(\mathcal{X}, Y_2, \ldots, Y_k) \) is \((\Sigma k_s - c - k + 1)\)-cartesian. Thus \( cr_k^n F \) satisfies \( E_\gamma(c + k - 1, \kappa) \), and is stably \( n \)-excisive. By equivalence, \( cr_k F \) is also stably \( n \)-excisive in each variable. \( \square \)

We now recall the definition of functor-operad given in \cite{MS04} and show that the cross effects of a monad provide an example.

**Definition 2.7.** For a simplicial category \( \mathcal{C} \), a *functor-operad* in \( \mathcal{C} \) is a sequence of continuous functors \( \mathcal{F}_k : \mathcal{C}^k \to \mathcal{C} \) together with natural isomorphisms \( \sigma : \mathcal{F}_k \to \mathcal{F}_k \circ \sigma^* \) for each \( \sigma \in \Sigma_k \), and natural transformations \( Id \to \mathcal{F}_1 \) and \( \gamma : \mathcal{F}_k(\mathcal{F}_{j_1}, \ldots, \mathcal{F}_{j_k}) \to \mathcal{F}_{j_1 + \ldots + j_k} \) which satisfy the usual associativity and equivariance diagrams.

The original definition requires that \( \mathcal{F}_1 \) is the identity functor, but we argue that this should have been called a *reduced* functor-operad, and we prefer to work with the more general notion.

**Theorem 2.8.** For a monad \( F : \mathcal{C} \to \mathcal{C} \), the cross effects of \( F \) form a functor-operad.

First, note that the \( \Sigma_k \)-action on \( cr_k F \) gives the natural isomorphisms \( cr_k F \to cr_k F \circ \sigma^* \) for \( \sigma \in \Sigma_k \). Since the identity functor is reduced, the monad map \( Id \to F \) factors through \( cr_1 F \). Finally, the maps \( \gamma \) are defined in the next proposition, which is proven by induction after a series of lemmas.

**Proposition 2.9.** Let \( F, G : \mathcal{C} \to \mathcal{C} \) be functors. For natural numbers \( k, j_1, \ldots, j_k \) and objects \((X_i, \ell_{i j})_{1 \leq i \leq k, 1 \leq j \leq k} \), there are natural associative maps

\[
\begin{align*}
\text{cr}_k F(\text{cr}_{j_1} G(X_{1,1}, \ldots, X_{1,j_1}), \ldots, \text{cr}_{j_k} G(X_{k,1}, \ldots, X_{k,j_k})) & \xrightarrow{\gamma_{cr_s}} \\
\text{cr}_{j_1 + \ldots + j_k} (F \circ G)(X_{1,1}, \ldots, X_{k,j_k}) & 
\end{align*}
\]

**Lemma 2.10.** There is a natural transformation \( F \circ \text{cr}_1 G \to \text{cr}_1 (F \circ G) \)

**Proof.** Since \( cr_1 G(X) \) is a strict limit, \( F \circ cr_1 G(X) \) fits into the following commuting diagram

\[
\begin{array}{ccc}
F \circ \text{cr}_1 G(X) & \longrightarrow & FG(X) \\
\downarrow & & \downarrow \\
F(G(*))^l & \longrightarrow & FG(*)
\end{array}
\]
There is a map from $F \circ cr_1G(X)$ to the strict limit of the rest of the diagram. A map from this limit to $cr_1(F \circ G)$ is induced by the map of diagrams

\[
\begin{array}{c}
\text{FG}(X) \\
\downarrow \downarrow \\
\text{FG}(*) \\
\end{array}
\begin{array}{ccc}
\text{F(G{*})} & \xrightarrow{\text{F(\text{ev}1)}} & \text{FG(\text{ev})} \\
\downarrow \alpha_{I} & & \downarrow \alpha_{G0} \\
\text{FG{(*)}I} & \xrightarrow{\text{ev}1} & \text{FG(\text{ev})} \\
\end{array}
\]

where $\alpha_Z$ is the natural transformation given by the adjoint of the composite

\[
Z \land F(\text{Hom}(Z,G(\text{*}))) \xrightarrow{\text{assembly}} F(Z \land \text{Hom}(Z,G(\text{*}))) \xrightarrow{F(\text{evaluation})} F(G(\text{*})).
\]

The map of diagrams commutes by viewing the evaluation at 1 map as $\text{Hom}(I,Y) \to \text{Hom}(S^0,Y) \cong Y$, and using naturality. That is, we use the commutativity of

\[
\begin{array}{c}
\text{F(\text{Hom}(I,G(\text{*})))} \\
\downarrow \alpha_{I} \\
\text{Hom}(I,F(G(\text{*})))
\end{array}
\begin{array}{ccc}
\xrightarrow{\text{ev}1} & & \\
\xrightarrow{\alpha_{G0}} & & \\
\text{Hom}(S^0,F(G(\text{*})))
\end{array}
\]

Thus, there is a map $F \circ cr_1G \to cr_1(F \circ G)$.

Lemma 2.11. The composite $cr_1F \circ cr_1G \to F \circ cr_1G \to cr_1(F \circ G)$ is associative.

Proof. Consider the diagram

\[
\begin{array}{ccc}
\text{cr}_1F \circ \text{cr}_1G \circ \text{cr}_1H & \to & \text{cr}_1F \circ \text{G} \circ \text{cr}_1H \\
\downarrow & & \downarrow \\
\text{F} \circ \text{cr}_1G \circ \text{cr}_1H & \to & \text{F} \circ \text{G} \circ \text{cr}_1H \\
\downarrow & & \downarrow \\
\text{cr}_1(M \circ G) \circ \text{cr}_1H & \to & \text{F} \circ \text{G} \circ \text{cr}_1H \\
\downarrow & & \\
\text{cr}_1(F \circ G) \circ \text{cr}_1H & \to & \text{cr}_1(F \circ G) \circ \text{cr}_1H
\end{array}
\]

The top two squares commute by naturality of the map $cr_1F \to F$, and the bottom two commute by naturality of $\beta$, which is the following composite.

\[
F \left( \lim \left( \begin{array}{c} G(X) \\
\downarrow \downarrow \\
G(Y) \rightarrow G(Y) \end{array} \right) \right) \to \lim \left( \begin{array}{c} FG(X) \\
\downarrow \downarrow \\
F(G(Y)) \rightarrow F(G(Y)) \end{array} \right) \to \lim \left( \begin{array}{c} FG(X) \\
\downarrow \downarrow \\
FG(Y) \rightarrow FG(Y) \end{array} \right)
\]

Letting $(Y,Z) = (\text{*},I)$ or $(X,S^0)$ shows that the vertical maps of the bottom row are instances of $\beta$.

Now we will use induction to show the existence of the maps of Proposition 2.9.

Proof of Proposition 2.9. To keep notation at bay, we’ll prove the case $k = 2$ and note that the general case follows easily; that is, we will show existence of natural maps

\[
\text{cr}_2F(\text{cr}_{j_1}G(X_1,\ldots,X_{j_1}),\text{cr}_{j_2}G(Y_1,\ldots,Y_{j_2})) \to \text{cr}_{j_1+j_2}(F \circ G)(X_1,\ldots,X_{j_1},Y_1,\ldots,Y_{j_2}).
\]
We describe the map in parts. First note that by definition the domain can be written as the composite
\[ cr_2 F (cr_1^{(1)} \cdots cr_1^{(j-1)} -, cr_{j_2} G(Y_1, \ldots, Y_{j_2})) \circ cr_1^{(j_1)} (G \circ \bigcup_j_1) (X_1, \ldots, X_{j_1}). \]

By Lemma 2.10 there is a map to the following, where a single first cross effect has been pulled out front
\[ cr_1^{(j_1)} cr_2 F (cr_1^{(1)} \cdots cr_1^{(j-1)} (G \circ \bigcup_j_1) (X_1, \ldots, X_{j_1}), cr_{j_2} G(Y_1, \ldots, Y_{j_2})). \]

Iteration of this map lets one pull all the first cross effects to the front
\[ cr_1^{(j_1)} \cdots cr_1^{(1)} cr_2 F ((G \circ \bigcup_j_1) (X_1, \ldots, X_{j_1}), cr_{j_2} G(Y_1, \ldots, Y_{j_2})). \]

Repeating this process for \( cr_{j_2} G \) yields a map to the following, where all first cross effects from inside \( cr_2 F \) are out front
\[ cr_1^{(j_1) + j_2)} \cdots cr_1^{(1+j_2)} cr_1^{(j_1) \cdots cr_1^{(1)}} cr_2 F ((G \circ \bigcup_j_1) (X_1, \ldots, X_{j_1}), (G \circ \bigcup_j_2) (Y_1, \ldots, Y_{j_2})). \]

Finally, we use the natural map \( cr_2 F (Z_1, Z_2) \to (F \circ \bigcup_j) (Z_1, Z_2) \) and a natural transformation
\[ \bigcup (G \circ \bigcup_j_1, G \circ \bigcup_j_2) \to G \circ \bigcup_{j_1+j_2}. \]

Applying the first cross effects to these maps yields the codomain
\[ cr_1^{(j_1) + j_2)} \cdots cr_1^{(1+j_2)} cr_1^{(j_1) \cdots cr_1^{(1)}} (F \circ G) (X_1, \ldots, X_{j_1}, Y_1, \ldots, Y_{j_2}), \]
which is, by definition, \( cr_{j_1+j_2} (F \circ G)(X_1, \ldots, X_{j_1}, Y_1, \ldots, Y_{j_2}) \). \( \square \)

This monoidal structure on the cross effects endows the derivatives with a monoidal structure, although we must deal with a few roadblocks presented by the linearizations.

3. Main theorem

In this section, we give a model for the derivatives of an endofunctor of spaces \( \partial_* F \) and show that this model has a positive answer to the question of Arone-Ching. We first give a definition for the derivatives in terms of the underlying space, which we decorate with a \( \mathcal{T} \) for clarity. We will give the full spectrum in Definition 3.7.

**Definition 3.1.** Let \( F \) be a continuous homotopy functor \( F : \mathcal{T} \to \mathcal{T} \) and define
\[ \partial_n^\mathcal{T} F = \text{hocolim} \Omega^U U \circ cr_n F (\Sigma^U S^0, \ldots, \Sigma^U S^0). \]

The \( \Sigma_n \)-action is induced by permuting the \( n \) inputs of \( cr_n F \), which also permutes the loops in the multilinearization. For example, the \( \Sigma_2 \)-action on \( \partial_1^\mathcal{T} F \) is the conjugate action which block swaps the sphere coordinates of the loops and variables, given by sending \( f \in \Omega^U \Omega^V cr_2 F (S^U, S^V) \) to the composite
\[ S^V \wedge S^U \xrightarrow{\chi^U \wedge V} S^U \wedge S^V \xrightarrow{f} cr_2 F (S^U, S^V) \xrightarrow{cr_2 F (\tau)} cr_2 F (S^V, S^U). \]

**Lemma 3.2.** If \( F \) is stably 1-excisive, then the natural map \( \partial_n^\mathcal{T} F \to \partial_n^\mathcal{T} F \) is an equivalence.

**Proof.** By Lemma 2.6 if \( F \) satisfies \( E_k (c, \kappa) \), then \( cr_n F \) satisfies \( E_k (c + n - 1, \kappa) \) in each variable. By Proposition 1.4 of [Goo03], \( T_1 F \) satisfies \( E_1 (c - 1, \kappa - 1) \) and \( T_1^j F \to T_1^{j+1} F \) satisfies \( O_1 (c - j, \kappa - j) \). Thus the natural transformation \( T_1^j cr_n F \to T_1^{j+1} cr_n F \) satisfies
\[ \Omega^{U_i \cup U_j} \text{cr}_n F(S^{U_1}, \ldots, -, \ldots, S^{U_n})(S^0) \rightarrow \Omega^{U_i \cup U_j} T^{i+1} \text{cr}_n F(S^{U_1}, \ldots, -, \ldots, S^{U_n})(S^0) \]

are \((j - \Sigma \ell \neq |U_i| - c - n + 1)\)-connected. As \(j\) goes to infinity, so does the connectivity of this map. Thus for each \((U_1, \ldots, U_n)\), the map of homotopy colimits induced by the inclusion \(N \hookrightarrow I\) is an equivalence. For each \(1 \leq i \leq n\),

\[
\text{hocolim}_{U_i \in N} \Omega^{U_i \cup U_j} \text{cr}_n F(S^{U_1}, \ldots, S^{U_n}) \xrightarrow{\sim} \text{hocolim}_{U_i \in \mathbb{N}} \Omega^{U_i \cup U_j} \text{cr}_n F(S^{U_1}, \ldots, S^{U_n})
\]

One can iterate this for each of the \(n\) sets \(U_i\) to get the desired equivalence on the homotopy colimits indexed by \(N^n \hookrightarrow I^n\). Note that analytic functors are, in particular, stably 1-excisive, so this lemma holds for all analytic functors.

**Theorem 3.3.** The model for \(\partial^T : \text{Fun}^{\text{red}}(\mathcal{T}, \mathcal{T}) \rightarrow \text{SymmSeq}(\mathcal{T})\) given in Definition 3.1 is monoidal.

**Proof.** Recall from Definition 1.24 that we must define a morphism \(\epsilon : 1 \rightarrow \partial^T \text{Id}\) and a natural transformation \(\mu_{F,G} : \partial^T F \circ \partial^T G \rightarrow \partial^T (F \circ G)\). First, we define the morphism \(\epsilon\). Since the unit of \(\mathcal{T}^S\) is the symmetric sequence with \(S^0\) in level 1 and the trivial space elsewhere, \(\epsilon\) is determined by the map \(S^0 \rightarrow \partial^T \text{Id}\), given by the inclusion of the first object in the homotopy colimit \(S^0 \rightarrow \text{hocolim}_{k \in I^0} \Omega^k \text{cr}_1 \text{Id}(S^k)\).

The natural transformation \(\mu\) is a map of symmetric sequences, thus a levelwise equivariant map. On level \(j\), this is

\[
\bigvee_{\text{partitions of } \{1, \ldots, j\}} \partial^T F \wedge \partial^T G \wedge \cdots \wedge \partial^T G \longrightarrow \partial^T (F \circ G)
\]

which boils down to defining maps

\[
\partial^T F \wedge \partial^T G \wedge \cdots \wedge \partial^T G \longrightarrow \partial^T (F \circ G) \quad \text{for all } j = j_1 + \cdots + j_k.
\]

We start by defining the map for the first level, \(\partial^T F \wedge \partial^T G \rightarrow \partial^T (F \circ G)\). Recall that the homotopy colimit and loops functors are both continuous, so by Lemma 2.10 they have assembly maps \(\alpha\). The first step is to assemble the homotopy colimits and loops out of the smash product of cross effects. Next, we use assembly for the cross effects to compose them and use the map defined in Lemma 2.10 to get a single cross effect. Finally, we use the map induced by \(\amalg : I \times I \rightarrow I\) to reindex the homotopy colimit.
\[
\begin{align*}
&\text{hocolim}_{U \in I} \Omega^U cr_1 F(S^U) \land \text{hocolim}_{V \in I} \Omega^V cr_1 G(S^V) \\
&\xrightarrow{\alpha_{\text{hocolim}_U \otimes \alpha_{\text{hocolim}_V}}}
&\text{hocolim}_{U \in I} \text{hocolim}_{V \in I} \Omega^U \Omega^V cr_1 F(S^U) \land cr_1 G(S^V) \\
&\xrightarrow{\alpha_{cr_1}}
&\text{hocolim}_{U \in I} \text{hocolim}_{V \in I} \Omega^U \Omega^V cr_1 F(cr_1 G(S^U \land S^V)) \\
&\xrightarrow{\text{Remark 2.10}}
&\text{hocolim}_{(U,V) \in I \times I} \Omega^U \Omega^V cr_1 (F \circ G)(S^U \cupdot S^V) \\
&\xrightarrow{\cupdot}
&\text{hocolim}_{W \in I} \Omega^W cr_1 (F \circ G)(S^W)
\end{align*}
\]

**Remark 3.4.** The last step is the key reason for using $I$; if the homotopy colimit is defined over $\mathbb{N}$, the map $\mu$ can be defined, but it will not be strictly associative on homotopy colimits. This is similar to the reason naive spectra do not have a good smash product, but symmetric spectra have enough extra structure to encode the smash product in an associative way.

To define the composition map in general, we will first define a map

\[
\otimes^U F(S^U) \to \Omega^U F(S^U).
\]

To define \( \otimes \) in an equivariant and associative way, we make use of the sphere operad defined in \[AK14\]. We recall its definition and salient properties here.

The sphere operad $S$ is the one-point compactification of a nonunital simplex operad, whose $n$th space is the open \((n-1)\)-dimensional simplex, so the $n$th space of $S$ is homeomorphic to $S^{n-1}$. The operad composition maps are homeomorphisms

\[
S^{k-1} \land S^{j_{i-1}} \land \cdots \land S^{j_{k-1}} \to S^{j_{i} + \cdots + j_{k} - 1}.
\]

There is a map of operads $S \to \text{Coend}(S^1)$ such that for each $n \geq 1$ the map $S_n \cong S^{n-1} \to \Omega S^n$ is adjoint to a homeomorphism $S^{n-1} \land S^1 \to S^n$. Since the $\Sigma_n$-action on the coendomorphism operad of $S^1$ permutes the $n$ coordinates of $S^n$, this defines a $\Sigma_n$-equivariant map $S^1 \land S_n \cong S^n$. Finally, there is a map of operads $\text{Com} \to S$ such that the composite $\text{Com} \to S \to \text{Coend}(S^1)$ is levelwise the canonical map adjoint to the diagonal map $S^1 \to S^n$.

We define a related operad $S_U^U$ whose $n$th space is the smash product of $U$ copies of $S_n$. This has the diagonal $\Sigma_n$-action induced by that on $S_n$, and composition maps require a shuffling of coordinates before applying the composition maps of $S$. The desired map is defined by smashing with the $j$th space of $S_U^U$ then assembling the sphere into $F$. That is,

\[
s_U^U \boxtimes F(S_U^U) \maps S_U^U \land S_U^U \xrightarrow{\alpha_p} S_U^U \land F(S_U^U) \xrightarrow{\alpha_p} F(S_U^U \land S_U^U) \cong F(S_U^U).
\]
Remark 3.5. This has the necessary equivariance and associativity because the sphere operad $S^U$ has these properties. For example, associativity can be seen by considering the maps $\Omega S^1 \to \Omega^2 S^2 \to \Omega^3 S^3$ where $f \in \Omega S^1$ is sent to $S_2 \wedge S_2 \wedge S_1 \wedge f$, which is equivariantly homeomorphic to $S_3 \wedge f$, the image of $f$ under $\Omega S^1 \to \Omega^3 S^3$.

We will introduce new notation to save some ink in the definition of the general $\mu_{F,G}$. If $U, V_1, \ldots, V_k$ are finite sets, let $S^{\mu_{U, V_1, \ldots, V_k}}$ denote the $k$-tuple of spheres $(S^{V_1}, \ldots, S^{V_k})$ and let $S^U \sqcup S^V = (S^U \sqcup V_1, \ldots, S^U \sqcup V_k)$. As in Proposition 2.9, we will restrict to the case $\partial_T^j F \wedge \partial_T^j G \to \partial_T^{j+1} (F \circ G)$, and note that the general case follows easily. Note also that the superscripts are indices and do not indicate powers.

The map $\mu$ is defined as a long composition, with most maps the same as in the level 1 case. As before, we assemble the homotopy colimits and loops out of the smash product of cross effects first, then we apply the map constructed in 2.9. Finally, as before, we use the map $\gamma$ from Proposition 2.9 to combine the cross effects, and use the amazing properties of the $\gamma$ to reindex the homotopy colimit. That is, $\mu$ is defined by the following composite.

\[ hocolim \Omega^{U_1} \Omega^{U_2} \cdots \Omega^{U_k} F(S^{U_1}, S^{U_2}) \wedge hocolim \Omega^{V_1} \cdots \Omega^{V_k} G(S^{V_1}) \wedge hocolim \Omega^{V_1} \cdots \Omega^{V_k} F(S^{V_1}, \ldots, S^{V_k}) \]

\[ \downarrow \alpha_{hocolim} \alpha_{\Omega} \]

\[ hocolim \Omega^{U_1} \Omega^{U_2} \cdots \Omega^{U_k} \Omega^{V_1} \cdots \Omega^{V_k} \cdots F(S^{U_1}, \ldots, S^{U_k}, S^{V_1}, \ldots, S^{V_k}) \wedge \cdots \wedge G(S^{V_1}, \ldots, S^{V_k}) \]

\[ \downarrow \alpha_{cr} \alpha_{F} \]

\[ hocolim \Omega^{U_1} \Omega^{U_2} \cdots \Omega^{U_k} \Omega^{V_1} \cdots \Omega^{V_k} \cdots F(S^{U_1}, \ldots, S^{U_k}, S^{V_1}, \ldots, S^{V_k}) \wedge \cdots \wedge G(S^{V_1}, \ldots, S^{V_k}) \]

\[ \downarrow \alpha_{cr} \gamma \]

\[ hocolim \Omega^{U_1} \Omega^{U_2} \cdots \Omega^{U_k} \Omega^{V_1} \cdots \Omega^{V_k} \cdots F(S^{U_1}, \ldots, S^{U_k}, S^{V_1}, \ldots, S^{V_k}) \wedge \cdots \wedge G(S^{V_1}, \ldots, S^{V_k}) \]

Note that the assembly maps are equivariant and associative, as is the map described in 2.9, and the cross effect map from Proposition 2.9 is also equivariant with respect to permuting the variables, so the composition above is equivariant and associative.

Remark 3.6. It is easy to generalize the construction above to say that if $\mathcal{F}$ is a functor-operad with assembly in each variable, then the multilinearizations evaluated at the unit, $hocolim_{U_1, \ldots, U_n \in I} \Omega^{U_1} \mathcal{F}(S^{U_1}, \ldots, S^{U_n})$, form an operad.

The rest of this section is dedicated to defining a spectrum level description of the derivatives which agrees with Goodwillie’s definition and maintains monoidicity.

Definition 3.7. Let $\partial_n F$ be the spectrum defined in level $\ell$ by

\[ (\partial_n F)_\ell = hocolim_{U_1, \ldots, U_n \in I} \Omega^{U_1} \cdots \Omega^{U_n} F(S^{U_1}, \ldots, S^{U_n}, \ldots, S^0). \]
Lemma 3.8. If \( F \) is analytic, \( \Omega^\infty \partial_k F \simeq \partial_k^G F \).

Proof. If \( F \) is analytic, \( \Omega^\infty \partial_k F \simeq \hocolim_{\mathbb{N}^k} \Omega^t \hocolim_{\mathbb{N}^k} \Omega^{\Sigma_1^{v_1} \cdots \Sigma_1^{v_k}} \partial_k F(S^V) \), where \( S^V = (S^{v_1}, \ldots, S^{v_k}) \), and the loops commute with the directed homotopy colimit over \( \mathbb{N} \). Thus it suffices to show that the following map is an equivalence.

\[
\partial_k^G F \simeq \hocolim_{\mathbb{N}^k} \Omega^{\Sigma_1^{v_1} \cdots \Sigma_1^{v_k}} \partial_k F(S^V) \rightarrow \hocolim_{\mathbb{N}^{k+1}} \Omega^{\Sigma_1^{v_1} \cdots \Sigma_1^{v_k}} \partial_k F(S^V) \simeq \Omega^\infty \partial_k F
\]

Suppose \( F \) is analytic and satisfies \( E_{k-1}(c, \kappa) \). The \( k \)-cube \( \mathcal{X} : U \rightarrow \bigvee_{j} S^V \) is strongly cocartesian with \( v_k \)-connected maps \( \mathcal{X}(\emptyset) \rightarrow \mathcal{X}(\{s\}) \), so \( F(\mathcal{X}) \) is \((\Sigma v_k - c)\)-cartesian, and the total homotopy fiber of \( F(\mathcal{X}) \) is \((\Sigma v_k - c - 1)\)-connected.

By the Blakers-Massey theorem, the map \( \partial_k F(S^V) \rightarrow \Omega \Sigma \partial_k F(S^V) \) is \((2(\Sigma v_k - c - 1) - 1)\)-connected, so the map \( \partial_k F(S^V) \rightarrow \Omega^t \Sigma^t \partial_k F(S^V) \) is also \((2(\Sigma v_k - c - 1) - 1)\)-connected.

Thus the map \( \Omega^{\Sigma_1^{v_1} \cdots \Sigma_1^{v_k}} \partial_k F(S^V) \rightarrow \Omega^{\Sigma_1^{v_1} \cdots \Sigma_1^{v_k}} \partial_k F(S^V) \) is \((\Sigma v_k + 2(c - 1) - 1)\)-connected, and as \( v_k \rightarrow \infty \), the map on homotopy colimits becomes an equivalence. \( \Box \)

Lemma 3.9. If \( F \) is analytic, \( \partial_k^G F \simeq \partial_k F \).

Proof. Recall from [Good3] that Goodwillie defines the \( t \)-th component of \( \partial_k^G F \) to be

\[
(\partial_k^G F)_t = \Omega^t \hocolim_{\mathbb{N}^k} \partial_k F(S^t, \ldots, S^t)
\]

where \( \mathcal{V}_k \) is the reduced standard representation of \( \Sigma_k \), so has dimension \( k-1 \). The equivalent associated \( \Omega \)-spectrum is given in level \( t \) by

\[
\hocolim_{\mathbb{N}^k} \Omega^u \Omega^{(k-u)(t+u)} \partial_k F(S^{t+u}, \ldots, S^{t+u}) \simeq \hocolim_{\mathbb{N}^k} \Omega^{k(t+u)} \Omega^t \partial_k F(S^{t+u}, \ldots, S^{t+u})
\]

Reindexing by \( t = \ell + u \) and using the Blakers-Massey argument of Lemma 3.8 this is equivalent to

\[
\hocolim_{\mathbb{N}^k} \Omega^{k(t+u)} \Omega^t \partial_k F(S^{t}, \ldots, S^t).
\]

Finally, the diagonal map and Lemma 5.2 show that this is equivalent to \((\partial_k F)_t\). \( \Box \)

An easy modification of the proof of Theorem 5.3 yields the following result.

Theorem 3.10. The model for \( \partial_* : \text{Fun}^{red}(T, T) \rightarrow \text{SymmSeq}(\delta \mathbb{P}) \) given in Definition 5.4 is monoidal.

Using the adjunction \( \Sigma^\infty : T \longrightarrow \delta \mathbb{P} : \Omega^\infty \), we can extend this result to reduced functors \( F : C \rightarrow D \) where \( C \) and \( D \) are either spaces or spectra. In order to make this adjunction strict, and for the functors to be homotopy invariant, we need to use EKMM spectra for \( \delta \mathbb{P} \) and use pointed simplicial sets for \( T \). Since the sphere spectrum is not cofibrant in EKMM spectra, we use the cofibrant replacement \( S^c \) to define \( \Sigma^\infty \).

The derivatives of a functor as defined by Goodwillie are equivalent as a symmetric sequence to the derivatives of the associated endofunctor of spaces. That is, for \( G : \delta \mathbb{P} \rightarrow T \), \( \partial_k^G G \simeq \partial_* (G \Sigma^\infty) \) as symmetric sequences. Similarly, for functors \( G : T \rightarrow \delta \mathbb{P} \), the derivatives \( \partial_k^G G \simeq \partial_* (G \Omega^\infty) \) are equivalent as symmetric sequences. Thus we can define the derivatives of a functor \( C \rightarrow D \), for \( C, D \) either \( T \) or \( \delta \mathbb{P} \), as the derivatives of the associated endofunctor of spaces.

Corollary 3.11. For composable functors \( F \circ G : C \rightarrow D \rightarrow E \), with \( C, D, E \) each either \( T \) or \( \delta \mathbb{P} \), there are natural maps \( \partial_* F \circ \partial_* G \rightarrow \partial_* (F \circ G) \) and \( 1 \rightarrow \partial_* \text{Id} \) that satisfy associativity and unitality diagrams.
Proof. When $F$ and $G$ are endofunctors of spectra, this map is a composite where the first map is the one described by Theorem 3.10

$$\partial_* F \circ \partial_* G = \partial_*(\Omega^\infty F \Sigma^\infty) \circ \partial_*(\Omega^\infty G \Sigma^\infty) \xrightarrow{\mu} \partial_*(\Omega^\infty F \Sigma^\infty \Omega^\infty G \Sigma^\infty)$$

Because of the adjunction, $\Sigma^\infty \Omega^\infty$ is a comonad, so there is a natural transformation $\Sigma^\infty \Omega^\infty \rightarrow Id$, and thus a map

$$\partial_*(\Omega^\infty F \Sigma^\infty \Omega^\infty G \Sigma^\infty) \rightarrow \partial_*(\Omega^\infty F \circ G \Sigma^\infty) = \partial_*(F \circ G).$$

With these definitions, we also get a map $1 \rightarrow \partial_* Id_{Sp}$ since $\partial_* Id_{Sp} = \partial_*(\Omega^\infty \Sigma^\infty) \simeq 1$.

The other cases are similar. □

We take a moment now to reiterate some consequences of this theorem, compare it with existing work in the area, and discuss its limitations. The derivatives of a reduced monad, for example the identity functor, have the structure of an operad and derivatives of other functors (respectively, modules over monads) inherit a module structure over the derivatives of the identity (respectively, the monad). In particular, there is an operad structure on the derivatives of the monad associated to any operad. The operad structure is explicit; this differs from the proof that $\partial^C_* Id_T$ forms an operad in the literature ([Joh95, AM99, Chi05]), which utilizes a cooperad structure on dual spectra.

It is of note that this paper presents a simple solution for endofunctors of spaces, while the easier case in [AC11] was endofunctors of spectra, and the results for spaces were achieved using adjunctions. We expect that Theorem 3.10 could transfer as-is to continuous endofunctors of spectra, but the only finitary spectral functors with assembly are linear, so the result is not as interesting for finitary functors. This is why we utilize adjunctions to extend to functors of spectra.

We expect that any simplicial category $\mathcal{C}$ with fibrant objects which has an adjunction with spaces $c \wedge - : \mathcal{T} \mathcal{C} : \text{Hom}(c, -)$ for $c \in \mathcal{C}$ will have a monoidal model for derivatives of its endofunctors. This generalization of Theorem 3.10 to functors between other categories would give natural operad and module structures in new settings. For example, the derivatives of the identity functor in the category of algebras over an operad are conjectured to be equivalent to the operad itself, but this has not been shown explicitly (see [Per13] for the proof of equivalence as symmetric sequences). We expect to return to this question in future work.

4. Chain rule

In this section, we prove a chain rule for reduced, finitary, analytic endofunctors of spaces, and give a slight extension. This has advantages to the chain rule of [AC11], in that the monoid map $\partial_* F \circ \partial_* G \rightarrow \partial_*(F \circ G)$ defines a spectrum level map on the derived composition product $\partial_* F \circ \partial_* G \rightarrow \partial_*(F \circ G)$ in contrast with a map existing solely in the homotopy category. Another difference is that the following chain rule is for analytic functors, while Arone and Ching do not make this assumption.

We will need the following lemma in the proof of the chain rule; a proof for Goodwillie’s derivatives can be found in [AC11 2.13], and thus it will hold for our model of derivative for analytic functors.

Lemma 4.1. $\partial_k$ preserves fiber sequences of analytic functors.

Note that if $\partial_k$ preserves fiber sequences of functors, then it also preserves finite products of functors, using the fiber sequence $F \rightarrow F \times G \rightarrow G$. 
**Theorem 4.2 (Chain rule).** Let $F, G : T \to T$ be reduced, analytic, homotopy functors and let $F$ be finitary. The natural map $\partial_* F \circ \partial_{i,1d} \partial_* G \to \partial_* (F \circ G)$ is an equivalence.

**Proof.** First, consider the case when $X$ is a finite CW complex and $F = \text{Hom}(X,-)$. We will use induction up the skeletal filtration for $X$. We filter $X$ by its skeleton $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$, which yields cofiber sequences $X_i \to X_{i+1} \to \bigvee S^{i+1}$, and these produce fiber sequences

$$\text{Hom}(\bigvee S^{i+1}, -) \to \text{Hom}(X_{i+1}, -) \to \text{Hom}(X_i, -).$$

Since $\partial_*$ preserves fiber sequences (by Lemma 4.1), we have fiber sequences in spectra:

$$\partial_k \text{Hom}(\bigvee S^{i+1}, -) \to \partial_k \text{Hom}(X_{i+1}, -) \to \partial_k \text{Hom}(X_i, -)$$

Taken together (for all $k$), the derivatives form a fiber sequence of symmetric sequences in spectra. Since the derivatives land in spectra, each level is also a cofiber sequence, so together they form a cofiber sequence in $\text{SymmSeq}(\mathbb{S}p)$. The bar construction $B(-, \partial_* \text{Id}, \partial_* G)$ preserves cofiber sequences, and thus the following is a fiber sequence in $\text{SymmSeq}(\mathbb{S}p)$.

$$\partial_* \text{Hom}(\bigvee S^{i+1}, -) \circ \partial_{i,1d} \partial_* G \to \partial_* \text{Hom}(X_{i+1}, -) \circ \partial_{i,1d} \partial_* G \to \partial_* \text{Hom}(X_i, -) \circ \partial_{i,1d} \partial_* G$$

Similarly, since $\text{Hom}(\bigvee S^{i+1}, -) \circ G \to \text{Hom}(X_{i+1}, -) \circ G \to \text{Hom}(X_i, -) \circ G$ is a fiber sequence, there is another fiber sequence in $\text{SymmSeq}(\mathbb{S}p)$ and the maps $\mu_{F,G}$ described in Theorem 4.3 yield a map of fiber sequences:

$$\partial_* \text{Hom}(\bigvee S^{i+1}, -) \circ \partial_{i,1d} \partial_* G \to \partial_* \text{Hom}(X_{i+1}, -) \circ \partial_{i,1d} \partial_* G \to \partial_* \text{Hom}(X_i, -) \circ \partial_{i,1d} \partial_* G$$

We will proceed by induction, referring to this diagram as $(\exists)$. First, we will show that the left vertical map of $(\exists)$ is an equivalence. Recall that $\bigvee S^i \to \ast \to \bigvee S^{i+1}$ is a cofiber sequence of spaces, so again there is a map of fiber sequences

$$\partial_* \text{Hom}(\bigvee S^{i+1}, -) \circ \partial_{i,1d} \partial_* G \to \partial_* \text{Hom}(\ast, -) \circ \partial_{i,1d} \partial_* G \to \partial_* \text{Hom}(\bigvee S^i, -) \circ \partial_{i,1d} \partial_* G$$

We use induction, noting that the middle terms are trivial, so the middle vertical map is an equivalence.

In the base case, $i = 0$, the right vertical map is an equivalence. That is, $\text{Hom}(S^0, \ast) = \text{Id}$, and there is a contracting simplicial homotopy $\partial_* G \simeq \partial_* \text{Id} \circ \partial_{i,1d} \partial_* G$. For $\beta > 1$, $\text{Hom}(\bigvee S^0, \ast) \cong \prod \text{Id}$, and it suffices to show an equivalence

$$\partial_* (\prod \text{Id}) \circ \partial_{i,1d} \partial_* G \to \partial_* (\prod (\text{Id} \circ G)) = \partial_* (\prod G).$$

The identity functor is analytic, as are products of the identity, so $\partial_* G (\prod \text{Id}) \simeq \partial_* (\prod \text{Id})$. The cross effect functors and $\Omega$ commute with products, and filtered homotopy colimits commute with finite limits, so $\partial_* G (\prod \text{Id}) \cong \prod \partial_* G (\text{Id})$. Similarly, when $G$ is analytic, $\prod G$ is also analytic, so $\partial_* (\prod G) \cong \partial_* G (\prod G) \cong \prod \partial_* G \cong \prod \partial_* G.$

It is not hard to show that the bar construction of symmetric sequences in spectra commutes with finite products,

$$\prod_\ast \text{Id} \circ \partial_{i,1d} \partial_* G \to \prod \left( \partial_* \text{Id} \circ \partial_{i,1d} \partial_* G \right).$$
Finally, we use the equivalence \( \partial_*Id \circ \partial_{*,Id} \partial_* G \simeq \partial_* G \) given by contracting homotopy on each factor. We have shown

\[
\partial_*(\prod Id) \circ \partial_{*,Id} \partial_* G \simeq (\prod \partial_* Id) \circ \partial_{*,Id} \partial_* G \simeq \prod (\partial_* Id \circ \partial_{*,Id} \partial_* G) \simeq \prod (\partial_* G) \simeq \partial_*(\prod G).
\]

Thus the base case \( i = 0 \) for the right vertical map is an equivalence. A map of fiber sequences in spectra in which two maps are equivalences yields an equivalence on the third map. Thus induction shows that \( \mu_{\text{Hom}(\nu S^i,-)G} \) is an equivalence for all \( i \).

In the diagram of fiber sequences (\( \triangledown \)), the left map is an equivalence and by induction the right map is an equivalence (the base case is taken care of by the base case for spheres) and so induction yields the chain rule for representable functors

\[
\partial_* \text{Hom}(X_{i+1},-) \circ \partial_{*,Id} \partial_* G \xrightarrow{\simeq} \partial_*(\text{Hom}(X_{i+1},-) \circ G).
\]

Arone and Ching show in [AC15] that a cofibrant model for the derivatives of representable functors can be extended to a model for all functors.

By [Kel05 Prop 4.23], any cofibrant functor is equivalent to its left Kan extension along the identity functor, so we may rewrite \( F(-) \simeq \text{Hom}(X,-) \wedge_{X \in \mathcal{T}} F(X) \).

Then

\[
\partial_* F \circ \partial_{*,Id} \partial_* G \simeq (\partial_* \text{Hom}(X,-) \circ \partial_{*,Id} \partial_* G) \wedge_{X \in \mathcal{T}} F(X)
\]

\[
\simeq \partial_*(\text{Hom}(X,G(-)) \wedge_{X \in \mathcal{T}} F(X))
\]

\[
\simeq \partial_*(F \circ G).
\]

Thus the chain rule extends to all (analytic, finitary) functors built out of representable functors.

\[\square\]

**Corollary 4.3.** Let \( F : \mathcal{T} \to \mathcal{E} \) and \( G : \mathcal{C} \to \mathcal{T} \) be reduced, analytic, homotopy functors with \( \mathcal{C} \) and \( \mathcal{E} \) each equal to either \( \mathcal{T} \) or \( \mathcal{Sp} \), and suppose \( F \) is finitary. Then the natural map induced by \( \mu \) is an equivalence

\[
\partial_* F \circ \partial_{*,Id} \partial_* G \to \partial_*(F \circ G)
\]

We expect this chain rule to extend to functors \( F \circ G : \mathcal{C} \to \mathcal{D} \to \mathcal{E} \) with \( \mathcal{D} \) either \( \mathcal{T} \) or \( \mathcal{Sp} \), and will return to this in future work. While the module structures allow the derived composition product to be defined over \( \partial_* Id_{\mathcal{Sp}} \), the map induced by \( \mu \) is

\[
\partial_*(F \Sigma^\infty) \circ \partial_{*,(\Omega^\infty)^\infty} \partial_*(\Omega^\infty G) \to \partial_*(F \Sigma^\infty \Omega^\infty G).
\]

This clearly maps to \( \partial_*(F \circ G) \) by the counit of the comonad \( \Sigma^\infty \Omega^\infty \), but neither of these are a priori equivalences. The additional effort could be anticipated, as it is similar to the extra work in [AC11] to obtain a chain rule where the intermediate category is \( \mathcal{T} \). We also expect this chain rule to extend to functors on simplicial categories with an appropriate adjunction with \( \mathcal{T} \), using the same techniques as outlined at the end of section 3.

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