The Hamilton-Waterloo Problem with $C_4$ and $C_m$ Factors

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Abstract

The Hamilton-Waterloo problem with uniform cycle sizes asks for a $2-$ factorization of the complete graph $K_v$ (for odd $v$) or $K_v$ minus a $1-$factor (for even $v$) where $r$ of the factors consist of $n-$cycles and $s$ of the factors consist of $m-$cycles with $r + s = \left\lfloor \frac{v-1}{2} \right\rfloor$. In this paper, the Hamilton-Waterloo Problem with $4-$cycle and $m-$cycle factors for odd $m \geq 3$ is studied and all possible solutions with a few possible exceptions are determined.

Keywords: 2-factorizations, Hamilton-Waterloo Problem, Oberwolfach Problem, Resolvable decompositions, Cycle decompositions

1. Introduction

A decomposition of a graph $G$ is a set $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$ of edge-disjoint subgraphs of $G$ such that $\bigcup_{i=1}^{k} E(H_i) = E(G)$. A H-decomposition is a decomposition of $G$ such that $H_i \cong H$ for all $H_i \in \mathcal{H}$. If each $H_i$ is a cycle (or union of cycles), then $\mathcal{H}$ is called a cycle decomposition. A $\{F_1^{k_1}, F_2^{k_2}, \ldots, F_l^{k_l}\}-factorization$ of a graph $G$ is a decomposition which consists precisely of $k_i$ factors isomorphic to $F_i$.

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The case \( H \cong K_2 \) is known as 1-factorization. Another important case is 2-factorization where every vertex in the graph \( H \) has degree 2. Whether there exists a 2-factorization of \( K_v \) with prescribed 2-factors is a long standing important problem in combinatorial design theory.

One of the 2-factorization problems is the Oberwolfach Problem which was first formulated by Ringel at an Oberwolfach meeting in 1967 and is related to the possible seating arrangements at the conference. The problem was inspired by a question whether \( v \) mathematicians could be seated in such a way that each mathematician sits next to each other mathematician exactly once over \( \lfloor \frac{v-1}{2} \rfloor \) days, where there are \( k_i \) round tables with \( m_i \) seats for \( 1 \leq i \leq t \) satisfying \( \sum_{i=1}^{t} k_i m_i = v \). In graph theory language, the problem asks for a 2-factorization of the complete graph \( K_v \) (or for even \( v \), 2-factorization of \( K_v-I \)) into 2-factors each of which is isomorphic to a given 2-factor \( H \). If \( H \) consists of \( k_i \) \( m_i \)-cycles, \( 1 \leq i \leq t \), then the corresponding Oberwolfach problem is denoted by \( OP(m_1^{k_1}, m_2^{k_2}, \ldots, m_t^{k_t}) \).

It is known that the solutions to the cases \( OP(3^2), OP(3^4), OP(4,5) \) and \( OP(3^2,5) \) do not exist [2, 13, 17]. The Oberwolfach Problem for a single cycle size \( OP(m^k) \) for all \( m \geq 3 \) has been solved in two separate cases: odd cycles in [2] and the even cycle case in [13].

A generalization of the Oberwolfach Problem is the Hamilton-Waterloo Problem where the conference takes places in two venues; Hamilton and Waterloo, the first of which has \( k \) round tables, each seating \( n_i \) people for \( i = 1, \ldots, k \), the second of which has \( l \) round tables each seating \( m_i \) people for \( i = 1, \ldots, l \) (necessarily \( \sum_{i=1}^{k} n_i = \sum_{j=1}^{l} m_i = v \)).

If we let \( n = n_1 = n_2 = \ldots = n_k \) and \( m = m_1 = m_2 = \ldots = m_l \), then each 2-factor is composed of either \( n \)-cycles or \( m \)-cycles. This version of the Hamilton-Waterloo Problem, with uniform cycle sizes, has attracted most of the attention and we use the notation to denote the problem with \( r \) factors of \( n \)-cycles and \( s \) factors of \( m \)-cycles by \((n,m)\)-HWP\((v;r,s)\). The obvious necessary conditions for the existence of a solution to \((n,m)\)-HWP\((v;r,s)\) are
Lemma 1. Let $v$, $n$, $m$, $r$ and $s$ be non-negative integers with $n, m \geq 3$. If there exists a solution to $(n, m)\text{-HWP}(v; r, s)$, then

1) if $r > 0$, $v \equiv 0 \mod n$,

2) if $s > 0$, $v \equiv 0 \mod m$,

3) $r + s = \left\lfloor \frac{v - 1}{2} \right\rfloor$.

The first result on the Hamilton-Waterloo Problem is settled by Adams et al. [1] in 2002. They solved the cases $(n, m) \in \{(4, 6), (4, 8), (4, 16), (8, 16), (3, 5), (3, 15), (5, 15)\}$ and settled the problem for all $v \leq 16$. With a few possible exceptions when $m = 24$ and 48, Danziger et al. [7] solved the problem for the case $(n, m) = (3, 4)$.

The case $n = 3$ and $m = v$, i.e. triangle-factors and Hamilton cycles, has attracted much attention and remarkable progresses are obtained by Horak et al. [10], Dinitz and Ling [8, 9]. In [4], Bryant et al. have settled the Hamilton-Waterloo Problem for bipartite $2$-factors, and in [6] Buratti and Rinaldi studied regular 2-factorizations leading to some cyclic solutions to Oberwolfach and Hamilton-Waterloo Problems, and also in [5], an infinite class of cyclic solutions to the Hamilton-Waterloo Problem is given.

Fu and Huang [11] solved the case of $4$-cycles and $m$-cycles for even $m$, and also settled all cases where $m = 2n$ and $n$ is even in 2008. Two years later Keranen and Özkan [14] solved the case of $4$-cycles and a single factor of $m$-cycles for odd $m$.

Most of the results involve the cases of even cycles or the cycles of same parity. Solving the Hamilton-Waterloo Problem for cycles with different parity is a more difficult problem and is not studied much.

Here we consider $4$-cycle and odd cycle factors, and complete the remaining cases in [14]. Our result also complements the results of Fu and Huang [11] and shows that the necessary conditions are sufficient also for odd $m$ with a few exceptions. Here is our main result.
Theorem 2. For all positive integers \( r, s \) and odd \( m \geq 3 \), a solution to \((4,m)\)-HWP\((v;r,s)\) exists if and only if \( 4|v, m|v \) and \( r + s = \frac{v-2}{2} \) except possibly when \( r = 2 \) and \( v = 8m \) or \( v = 24, 48 \) when \( m = 3 \) and \( r = 6 \).

2. Preliminary Results

If \( G_1 \) and \( G_2 \) are two edge disjoint graphs on the same vertex set, then \( G_1 \oplus G_2 \) will denote the graph on the same vertex set with \( E(G_1 \oplus G_2) = E(G_1) \cup E(G_2) \). Also \( \alpha G \) will denote the vertex disjoint union of the \( \alpha \) copies of \( G \).

We will denote a complete equipartite graph of \( b \) parts of size \( a \) each by \( K_{a:b} \). \( K_{a:b} \) is called complete bipartite graph and denoted by \( K_{a,a} \) as well.

Liu [16] gave a complete solution to the Oberwolfach Problem for complete equipartite graphs where all cycles have the same length and we will use this result in our main construction.

Theorem 3. [16] The complete equipartite graph \( K_{a:b} \) has a \( C_l \)-factorization for \( l \geq 3 \) and \( a \geq 2 \) if and only if \( l|ab, a(b-1) \) is even, \( l \) is even if \( b = 2 \) and \((a,b,l) \neq (2,3,3),(6,3,3),(2,6,3),(6,2,6)\).

Let \( H \) be a finite additive group and let \( S \) be a subset of \( H - \{0\} \) such that the inverse of every element of \( S \) also belongs to \( S \). The Cayley graph over \( H \) with connection set \( S \), denoted by \( Cay(H,S) \), is the graph with vertex set \( H \) and edge set \( E(Cay(H,S)) = \{(a,b)|a,b \in H, a-b \in S\} \). Note that, since \( S = S^{-1} \), here \( Cay(H,S) \) is not directed.

Let \( G \) be graph and \( G_0, G_1, G_2, \ldots, G_{k-1} \) be vertex disjoint copies of \( G \) with \( v_i \in V(G_i) \) for each \( v \in V(G) \). Then the graph \( G[k] \) is a graph with vertex set \( V(G[k]) = V(G_0) \cup V(G_1) \cup V(G_2) \cup \ldots \cup V(G_{k-1}) \) and edge set \( E(G[k]) = \{u_i v_j : uv \in E(G) \text{ and } 0 \leq i, j \leq k-1\} \). For example \( K_{m}[2] \cong K_{2m} - I \) and \( K_{2}[m] \cong K_{m,m} \) where \( I \) is a \( 1 \)-factor of \( K_{2m} \).

It is easy to see that if a graph \( G \) has an \( H \)-decomposition, then there exists an \( H[k] \)-decomposition of \( G[k] \). Moreover if a graph \( G \) has an \( H \)-factorization, then there exists an \( H[k] \)-factorization of \( G[k] \).
In fact, this graph operation is a generalization of Häggkvist’s doubling construction and it coincides with a special case of a graph product called lexicographic product. Häggkvist \[12\] constructed 2-factorizations containing even cycles using \(G[2]\).

**Lemma 4.** \[12\] Let \(G\) be a path or a cycle with \(m\) edges and let \(H\) be a 2-regular graph on \(2m\) vertices where each component of \(H\) is a cycle of even length. Then \(G[2]\) has an \(H\)-decomposition.

Baranyai and Szasz \[3\] have shown that if \(G\) consists of \(x\) Hamilton cycles and if \(H\) has \(y\) vertices and consists of \(z\) Hamilton cycles then their lexicographic product is decomposable into \(xy + z\) Hamilton cycles. So, \(C_m[n]\) has a \(C_{mn}\)-factorization. Also Alspach et al. \[2\] have shown that for an odd integer \(m\) and a prime \(p\) with \(3 \leq m \leq p\), \(C_m[p]\) has a \(C_p\)-factorization.

By \[2\] and \[12\], solutions to \(OP(4^{v/4})\) and \(OP(m^{v/m})\) exist except \(m = 3\) and \(v = 6\) or \(v = 12\). That is a solution to \((4, m)\)-HWP\((v; r, s)\) exists for \(r = 0\) or \(s = 0\) with exceptions \((v, m, r) = (6, 3, 0)\) and \((v, m, r) = (12, 3, 0)\). So, we can assume that \(r \neq 0\) and \(s \neq 0\).

In our case \(4|v, m|v\) and \(m\) is odd. Then there exists a \(t \in \mathbb{Z}^+\) such that \(v = 4mt\).

Note that:

\[
K_{4mt} \cong K_{mt}[4] \oplus mtK_4
\]

or equivalently

\[
K_{4mt} - I \cong K_{mt}[4] \oplus mtC_4
\]

where \(V(K_{4mt}) = V(K_{mt}[4])\) and \(I\) is a 1-factor in \(K_{4mt}\). So \(K_{4mt}\) has a \(\{(C_m[4])^{(mt-1)/2}, K_4\}\)-factorization for odd \(t\) by \[3\] and, \(K_{mt}\) has a \(C_m\)-factorization for odd \(t\) by \[1\]. In short, for odd \(t\) we have

\[
K_{4mt} \cong tC_m[4] \oplus tC_m[4] \oplus \ldots \oplus tC_m[4] \oplus mtK_4.
\]

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Similarly, $K_{4mt}$ has a $\{ (C_m[4])^{(mt-2)/2}, K_{4,4}, K_4 \}$-factorization for even $t$ by \[1\] and, $K_{mt}$ has a $\{ C_m^{(mt-2)/2}, K_2 \}$-factorization for even $t$ by \[13\]. In short, for even $t$, we have

$$K_{4mt} \cong tC_m[4] \oplus tC_m[4] \oplus \ldots \oplus tC_m[4] \oplus \frac{mt}{2}K_{4,4} \oplus mtK_4.$$  

(4)

with exceptions $m = 3$ and $t = 2$ or $t = 4$.

In our proofs, we will use these decompositions with appropriate factorizations of $C_m[4]$’s.

It is obvious that a $2$–factorization of $C_m[4]$ has exactly four factors. The followings will be shown:

(i) $C_m[4]$ has a $C_4$–factorization (Lemma 5),

(ii) $C_m[4]$ has a $C_m$–factorization (Lemma 6),

(iii) $C_m[4]$ has a $\{ C_2^4, C_m^2 \}$–factorization (Lemma 7),

(iv) $C_m[4]$ has no $\{ C_4^1, C_m^3 \}$–factorization for odd $m$ (Lemma 8).

**Lemma 5.** For every integer $m \geq 3$, $C_m[4]$ has a $C_4$–factorization.

**Proof.** Note that $C_m[4] \cong C_m[2][2]$. By Lemma 4, $C_m[2]$ can be decomposed into $C_{2m}$–factors, and each $C_{2m}$ can be decomposed into two $1$–factors. So $C_m[2]$ has a $1$–factorization. If $F$ is a $1$–factor in $C_m[2]$, $F[2]$ is a $C_4$–factor in $C_m[4]$ since $K_2[2] \cong C_4$. Hence $C_m[4]$ has a $C_4$–factorization.

**Lemma 6.** For every integer $m \geq 3$, $C_m[4]$ has a $C_m$–factorization.

**Proof.** We can represent $C_m[4]$ as the Cayley graph over $V_4 \times Z_m$ with connection set $V_4 \times \{ 1, -1 \}$ where $V_4$ is the additive group of $F_4 = \{ 0, 1, x, x^2 \}$. Let $C = (v_0, v_1, \ldots, v_{m-1})$ be an $m$–cycle of $C_m[4]$ where $v_i = (x^i, i)$ for $0 \leq i \leq m - 1$. In the case of $m \equiv 1 \pmod{3}$ replace $v_{m-1}$ with $(x, m-1)$. It can be checked that

$$F = C \cup (x, 1) \cdot C \cup (x^2, 1) \cdot C \cup (0, 1) \cdot C$$
is a $2-$ factor of $C_m[4]$. It is also easy to check that
\[ F = \{ F, F + (1, 0), F + (x, 0), F + (x^2, 0) \} \]
is a $2-$ factorization of $C_m[4]$.

It is evident that the addition by $(1, 0)$ and multiplication by $(x, 1)$ are automorphisms of the above factorization $F$. These automorphisms clearly generate $AGL(1, 4)$ (the $1-$dimensional affine general linear group over $F_4$).

**Lemma 7.** For every integer $m \geq 3$, $C_m[4]$ has a $\{ C^2_4, C^2_m \}-factorization.

**Proof.** We can represent $C_m[4]$ as the Cayley graph $\Gamma$ over $\mathbb{Z}_4 \times \mathbb{Z}_m$ with connection set $\mathbb{Z}_4 \times \{ 1, -1 \}$.

When $m$ is even, let $C = (v_0, v_1, \ldots, v_{m-1})$ and $C' = (v'_0, v'_1, \ldots, v'_{m-1})$ be the $m-$cycles of $\Gamma$ where $v_i = (2i, i)$ and $v'_i = (0, i)$ for $0 \leq i \leq m - 1$. Then
\[
F_1 = C \cup (C + (1, 0)) \cup (C + (2, 0)) \cup (C + (3, 0)) \quad \text{and} \quad F'_1 = C' \cup (C' + (1, 0)) \cup (C' + (2, 0)) \cup (C' + (3, 0))
\]
are two edge-disjoint $m$-cycle factors of $\Gamma$.

Also let $C_* = ((0, 1), (1, 0), (2, 1), (3, 0))$ be a $4-$cycle of $\Gamma$. Then
\[
F_2 = \bigcup_{i=0}^{m-1} (C_* + (0, i))
\]
is a 4-cycle factor of $\Gamma$. It can be checked that
\[ F = \{ F_1, F'_1, F_2, F_2 + (1, 0) \} \]
is a $2-$ factorization of $\Gamma$.

When $m$ is odd, let $C, C'$ and $C_*$ be defined as above with $v_{m-1} = (1, m-1)$.

Also let $C'_* = ((0, 0), (2, m-1), (1, m-2), (3, m-1))$ be a $4-$cycle of $\Gamma$. Then
\[
F_1 = C \cup (C + (1, 0)) \cup (C + (2, 0)) \cup (C + (3, 0)) \quad \text{and} \quad F'_1 = C' \cup (C' + (1, 0)) \cup (C' + (2, 0)) \cup (C' + (3, 0)) \quad \text{and} \quad F_2 = \bigcup_{i=0}^{m-3} (C_* + (0, i)) \cup C'_* \cup (C'_* + (2, 0))
\]
are $2-$factors of $\Gamma$. It can be checked that
\[ F = \{ F_1, F'_1, F_2, F_2 + (1, 0) \} \]
Lemma 8. For any odd integer $m \geq 3$, $C_m[4]$ has no $\{C_4^1, C_m^3\}$-factorization; that is, $C_m[4] \not\cong mC_4 \oplus 4C_m \oplus 4C_m \oplus 4C_m$.

Proof. Consider $C_m[4]$ as the Cayley graph $\Gamma$ over $\mathbb{Z}_4 \times \mathbb{Z}_m$ with connection set $\mathbb{Z}_4 \times \{1, -1\}$ as before.

We prove the Lemma by contradiction. So assume that $\Gamma$ can be decomposed into three $C_m$-factors and a single $C_4$-factor.

Since $m$ is odd, each $m$-cycle in $\Gamma$ contains one and only one vertex $(a, i)$ of $\Gamma$ for each $i \in \mathbb{Z}_m$. When we remove the three $C_m$-factors, we are left with a $2$-regular graph where each vertex $(a, i)$ is adjacent to only one vertex $(b, i - 1)$ and only one vertex $(c, i + 1)$ for some $b, c \in \mathbb{Z}_4$. So, this $2$-regular graph can not contain any $4$-cycles.

Hence, $\Gamma$ has no $\{C_4^1, C_m^3\}$-factorization.

3. When $r$ is odd

Now, we can prove that for odd $m \geq 3$, a solution to $(4, m)-\text{HWP}(v; r, s)$ exists for all odd $r$ (or even $s$) satisfying the necessary conditions.

Theorem 9. For all positive odd integers $r$ and $m \geq 3$, a solution to $(4, m)-\text{HWP}(v; r, s)$ exists if and only if $4|v$, $m|v$ and $r + s = \frac{v - 2}{2}$ except possibly $v = 24, 48$ when $m = 3$.

Proof. If a solution to $(4, m)-\text{HWP}(v; r, s)$ exists, then by Lemma 1, $m|v$, $4|v$ and $r + s = \frac{v - 2}{2}$ since $v$ is even.

For the sufficiency part, assume $m \geq 3$ is odd, $m|v$ and $4|v$. Then, since $\gcd(4, m) = 1$, $4m|v$. Thus, there exists an integer $t$ such that $v = 4mt$.

We will prove the theorem in two cases; $t$ is odd or even.

Case 1: Assume $t$ is odd.

By Lemma 3, $K_{4mt} - I$ has a $\{(C_m[4])^{(mt-1)/2}, C_4\}$-factorization. Now, let $r_1, s_1$ and $x$ be non-negative integers with $r_1 + s_1 + x = \frac{mt-1}{2}$. Placing a $C_4$-factorization on $r_1$ of the $C_m[4]$-factors by Lemma 3, a $C_m$-factorization on $s_1$.
of the $C_m[4]$–factors by Lemma 6 and a $\{C_1^2, C_m^2\}$–factorization on the remaining $x$ $C_m[4]$–factors by Lemma 7 gives us a $\{C_4^{4r_1+2x+1}, C_m^{4s_1+2x}\}$–factorization of the $K_{4mt} - I$. That is, a solution to $(4, m)$–HWP($4mt; r, s$) exists for $r = 4r_1 + 2x + 1$ (any positive odd integer can be written in this form for non-negative $r_1$ and $x$) and $s = 4s_1 + 2x$. It is not difficult to see that $1 \leq r$ is odd and $0 \leq s$ is even with $r + s = 4r_1 + 2x + 1 + 4s_1 + 2x = 2mt - 1 = \frac{v-2}{2}$.

Therefore, a solution to $(4, m)$–HWP($4mt; r, s$) exists for all odd integers $r$ and $t$ with $r + s = 2mt - 1$.

**Case 2**: Now assume $t$ is even, except $t \neq 2, 4$ when $m = 3$.

By (4), $K_{4mt} - I$ has a $\{(C_m[4])^{(mt-2)/2}, C_4^3\}$–factorization.

Similarly, placing a $C_4$–factorization on $r_1$ of the $C_m[4]$–factors by Lemma 5, a $C_m$–factorization on $s_1$ of the $C_m[4]$–factors by Lemma 6, and a $\{C_4^2, C_m^2\}$–factorization on the remaining $x$ $C_m[4]$–factors by Lemma 7 gives us a $\{C_4^{4r_1+2x+3}, C_m^{4s_1+2x}\}$–factorization of the $K_{4mt} - I$.

Since any odd integer $r \geq 3$ can be written as $r = 4r_1 + 2x + 3$ for non-negative integers $r_1$ and $x$, we obtain that for even $t$, a solution to $(4, m)$–HWP($4mt; r, s$) exists for all odd integers $3 \leq r$ (or even $0 \leq s$) with $r + s = \frac{v-2}{2}$ and $m \geq 3$, except $t \neq 2, 4$ and $m = 3$.

For $r = 1$, by the equivalence (2), and $K_{mt}[4] \cong K_{4:mt}$, we can write $K_{4mt} - I \cong K_{4:mt} \oplus mtC_4$. From (1), $K_{4:mt}$ has a $C_m$–factorization. So, placing a $C_m$–factorization on the $K_{4:mt}$–factor yields a $\{C_4^1, C_m^s\}$–factorization of $K_{4mt} - I$ with $s = 2mt - 2$.

### 4. When $r$ is even

Since $C_m[4]$ has no $\{C_4^1, C_m^3\}$–factorization, we can not obtain a solution to $(4, m)$–HWP($4mt; r, s$) for even $r$ (or equivalently odd $s$) using the construction in the proof of Theorem 9. However, we will use a similar construction via switching the edges of a $1$–factor from $K_4$’s with some edges of $C_m[4]$ in (3) and (4), then we will get a $\{C_4^2, C_m^3\}$–factorization of the new graph. In short, if we let $C_m^* [4] \cong C_m[4] \oplus mK_4$ and $I$ is a $1$–factor of $C_m[4]$, then we will show
that
\[ C_m'[4] - I \cong mC_4 \oplus mC_4 \oplus 4C_m \oplus 4C_m \oplus 4C_m, \]
that is, \( C_m'[4] - I \) has a \( \{C_4^2, C_m^3\} \)-factorization.

**Lemma 10.** \((C_m[4] - I) \oplus mK_4\) has a \( \{C_4^2, C_m^3\} \)-factorization for some \( 1 \)-factor \( I \) in \( C_m[4] \) where each \( K_4 \) consists of four copies of the vertex \( v_i \) for any \( v_i \in C_m \).

**Proof.** Consider \( C_m[4] \) as the Cayley graph \( \Gamma \) over \( \mathbb{Z}_4 \times \mathbb{Z}_m \) with connection set \( \mathbb{Z}_4 \times \{1, -1\} \), so each \( K_4 \) consists of the vertices \((0, i), (1, i), (2, i), (3, i)\) for \( i \in \mathbb{Z}_m \). And let \( C_{(1)} = (u_0, u_1, \ldots, u_{m-1}) \), \( C_{(2)} = (v_0, v_1, \ldots, v_{m-1}) \) and \( C_{(3)} = (y_0, y_1, \ldots, y_{m-1}) \) be \( m \)-cycles of \( \Gamma \) defined by the vertices \( u_i = (0, i), v_i = (i^2, i) \) and \( y_i = (-i^2, i) \) for \( 0 \leq i \leq m - 2 \) and \( u_{m-1} = (3, m - 1), v_{m-1} = (1, m - 1), y_{m-1} = (0, m - 1) \). Then
\[
F_1 = C_{(1)} \cup (C_{(1)} + (1, 0)) \cup (C_{(1)} + (2, 0)) \cup (C_{(1)} + (3, 0))
\]
\[
F_2 = C_{(2)} \cup (C_{(2)} + (1, 0)) \cup (C_{(2)} + (2, 0)) \cup (C_{(2)} + (3, 0))
\]
\[
F_3 = C_{(3)} \cup (C_{(3)} + (1, 0)) \cup (C_{(3)} + (2, 0)) \cup (C_{(3)} + (3, 0))
\]
are \( m \)-cycle factors of \( \Gamma \).

Let \( C_{(4)} = ((1, 0), (2, 0), (0, 1), (3, 1)) \) and \( C_{(5)} = ((0, 0), (1, 0), (3, 0), (2, 0)) \) be \( 4 \)-cycles of \( \Gamma \). Then
\[
F_4 = \bigcup_{i=0}^{m-1} (C_{(4)} + (0, i))
\]
\[
F_5 = \bigcup_{i=0}^{m-1} (C_{(5)} + (0, i))
\]
are 4-cycle factors of \( \Gamma \). Then
\[
\mathcal{F} = \{F_1, F_2, F_3, F_4, F_5\}
\]
is a 2-factorization of \((\Gamma - I) \oplus mK_4\) where \( I \) is a 1-factor of \( \Gamma \) with the edges \( \{(0, i)(2, i + 1)\} \) and \( \{(3, i)(1, i + 1)\} \) for each \( i \in \mathbb{Z}_m \).

Now, we give solutions to the Hamilton-Waterloo Problem for some small cases and improve the results given in [7].

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Theorem 11. For all positive integer $r$, a solution to $(4,3)\text{-HWP}(24;r,s)$ exists if and only if $r + s = 11$ except possibly when $r = 2$ and $r = 6$.

Proof. All the cases are covered by [7] with possible exceptions when $r = 2, 4, 6$.

Let the vertex set of $K_{24}$ be $Z_{24}$. Then, let

$F_1 = (0, 1, 10, 9) \cup (2, 3, 17, 16) \cup (4, 5, 19, 18) \cup (6, 7, 8, 15) \cup (11, 12, 21, 20) \cup (13, 14, 23, 22),$

$F_2 = (0, 2, 4, 6) \cup (1, 3, 5, 7) \cup (10, 12, 14, 8) \cup (16, 18, 20, 22) \cup (17, 19, 21, 23) \cup (9, 11, 13, 15),$

$F_3 = (0, 3, 4, 7) \cup (1, 2, 5, 6) \cup (10, 11, 14, 15) \cup (16, 19, 20, 23) \cup (17, 18, 21, 22) \cup (9, 12, 13, 8),$

$F_4 = (0, 4, 1, 5) \cup (2, 6, 3, 7) \cup (11, 15, 12, 8) \cup (16, 20, 17, 21) \cup (18, 22, 19, 23) \cup (9, 13, 10, 14),$

$F_5 = (0, 8, 16) \cup (1, 9, 17) \cup (2, 10, 18) \cup (3, 11, 19) \cup (4, 12, 20) \cup (5, 13, 21) \cup (6, 14, 22) \cup (7, 15, 23),$

$F_6 = (0, 13, 19) \cup (1, 14, 20) \cup (2, 15, 21) \cup (3, 8, 22) \cup (4, 9, 23) \cup (5, 10, 16) \cup (6, 11, 17) \cup (7, 12, 18),$

$F_7 = (0, 14, 18) \cup (1, 15, 19) \cup (2, 8, 20) \cup (3, 9, 21) \cup (4, 10, 22) \cup (5, 11, 23) \cup (6, 12, 16) \cup (7, 13, 17),$

$F_8 = (0, 15, 20) \cup (1, 8, 21) \cup (2, 9, 22) \cup (3, 10, 23) \cup (4, 11, 16) \cup (5, 12, 17) \cup (6, 13, 18) \cup (7, 14, 19),$

$F_9 = (0, 12, 23) \cup (1, 13, 16) \cup (2, 14, 17) \cup (3, 15, 18) \cup (4, 8, 19) \cup (5, 9, 20) \cup (6, 10, 21) \cup (7, 11, 22),$

$F_{10} = (0, 11, 21) \cup (1, 12, 22) \cup (2, 13, 23) \cup (3, 14, 16) \cup (4, 15, 17) \cup (5, 8, 18) \cup (6, 9, 19) \cup (7, 10, 20),$

$F_{11} = (0, 10, 17) \cup (1, 11, 18) \cup (2, 12, 19) \cup (3, 13, 20) \cup (4, 14, 21) \cup (5, 15, 22) \cup (6, 8, 23) \cup (7, 9, 16).$

It is easy to check that

$$
F = \{F_1, F_2, \ldots, F_{11}\}
$$

is a 2-factorization of $K_{24} - I$ with four $C_4$-factors where $I = \{(0, 22), (1, 23), (2, 11), (3, 12), (4, 13), (5, 14), (6, 20), (7, 21), (8, 17), (9, 18), (10, 19), (15, 16)\}$. 

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This completes the case $r = 4$.

**Theorem 12.** For all positive integers $r$, a solution to $(4,3)\text{-HWP}(48;r,s)$ exists if and only if $r + s = 23$ with a possible exception when $r = 6$.

**Proof.** It is known that $(4,3)\text{-HWP}(12;1,4)$ has a solution with the possible exceptions when $r = 6,8,10,14,16,18,20$. A solution to $(4,3)\text{-HWP}(12;1,4)$ has given in the Appendix of [1], and by (1), $K_{48} \cong K_{12}[4] \oplus 12K_4$. Hence a $(C_3[4])^4, C_4[4], K_4, K_4$—factorization of $K_{48}$ exists. Also by Lemma [4] $C_4[4]$ can be decomposed into four $C_4$—factors, by Lemma [10] $(C_3[4] - I) \oplus 3K_4$ has a $\{C_2^2, C_3^3\}$—factorization, and it is easy to see that $K_{4,4}$ can be decomposed into two $C_4$—factors. So, we now have $8 C_4$—factors and $3 C_3$—factors already. For the remaining three $C_3[4]$'s, decompose $r_1$ of them into $C_4$—factors, $s_1$ of them into $C_3$—factors and $x$ of them into two $C_4$—factors and two $C_3$—factors where $r_1 + s_1 + x = 3$ by Lemmas [9] and [7] respectively. Hence, we get $r = 4r_1 + 2x + 8$ and $s = 4s_1 + 2x + 3$ and this gives us a $\{C_4^r, C_3^s\}$—factorization of $K_{48} - I$ for $r = 8, 10, 12, 14, 16, 18, 20$.

We would like to note that, we get the information on S. Bonvicini and M. Buratti gave solutions to all nine remaining cases of [6] independently, using clear algebraic methods in their soon to be submitted paper "Sharply Vertex Transitive 2-Factorizations of Cayley Graphs". But for the sake of completeness of our paper, we presented our results on six cases we have solved and left 3 cases as exceptions in our theorems.

**Theorem 13.** For all positive even $r$ and odd $m \geq 3$, a solution to

$(4,m)\text{-HWP}(v;r,s)$ exists if and only if $r + s = \frac{v-2}{2}$ except possibly $v = 8m$ when $r = 2$, and $v = 24, 48$ when $m = 3$.

**Proof.** We will consider two cases depending on the parity of $t$.

**Case 1:** Let $t$ be odd.

By [8], $K_{4mt}$ has a $\{(C_m[4])^{(mt-1)/2}, K_4\}$—factorization.
Let $I$ be a 1-factor of $K_{4m}$ as defined in Lemma 10 and, $r_1, s_1$ and $x$ be non-negative integers with $r_1 + s_1 + x = \frac{mt-3}{2}$. Placing a $C_4$-factorization on $r_1$ of the $C_m[4]$'s by Lemma 5 a $C_m$-factorization on $s_1$ of the $C_m[4]$'s by Lemma 6 a $\{C_2^4, C_2^m\}$-factorization on the $x$ of the $C_m[4]$'s by Lemma 7 and a $\{C_4^2, C_m^3\}$-factorization on the remaining $(C_m[4] - I) \oplus K_4$-factor by Lemma 10 gives us a $\{C_4^{r_1+2x+2}, C_m^{s_1+2x+3}\}$-factorization of the $K_{4mt} - tI$ where $tI$ gives a 1-factor in $K_{4mt}$.

Then, since any even integer $r \geq 2$ can be written as $r = 4r_1 + 2x + 2$ for non-negative integers $r_1$ and $x$, a solution to $(4, m)$-HWP$(4mt; r, s)$ exists for any even $r \geq 2$ and odd $t$ satisfying $r + s = 2mt - 1 = \frac{v-2}{2}$.

Case 2: Let $t$ be even.

By (1), $K_{4mt}$ has a $\{(C_m[4])^{(mt-2)/2}, K_{4, 4}, K_4\}$-factorization.

For $r_1 + s_1 + x = \frac{mt-2}{2}$, placing a $C_4$-factorization on $r_1$ of the $C_m[4]$'s, a $C_m$-factorization on $s_1$ of the $C_m[4]$'s, a $\{C_4^2, C_m^2\}$-factorization on the $x$ of the $C_m[4]$'s, two $C_4$-factors on the $K_{4, 4}$-factor and a $\{C_4^2, C_m^3\}$-factorization on the remaining $(C_m[4] - I) \oplus K_4$- factor yields a solution to $(4, m)$-HWP$(4mt; r, s)$ for all even $r \geq 4$ except $t = 2$ or $t = 4$ when $m = 3$.

Now, we consider the case $r = 2$ and $t$ is even. Partitioning the vertices of $K_{4mt}$ into $t$ sets of size $4m$ gives the equivalence: $K_{4mt} - I \cong t(K_{4m} - I') \oplus K_{4mt}$ where $I'$ is a 1-factor of $K_{4m}$. By Case 1, $K_{4m} - I'$ has a $\{C_4^2, C_m^{2m-3}\}$-factorization and also from Theorem 3 $K_{4m:t}$ has a $C_m$-factorization for $t \neq 2$. Thus, $K_{4mt} - I$ has a $\{C_4^2, C_m^{2mt-3}\}$-factorization.

5. Proof of Main Result and Conclusion

Combining the results of the previous section it is now possible to obtain the proof of the Theorem.

Proof of Theorem 2. Odd $r$ follows from Theorem 9 and even $r$ follows from Theorem 13 with possible exceptions when $r = 2$ and $v = 8m$, and $v = 24$ or $v = 48$ when $m = 3$. Theorem 11 and Theorem 12 cover some of these
exceptions for $m = 3$ and the remaining cases are $r = 2$ when $v = 8m$, and $v = 24, 48$ and $r = 6$ when $m = 3$.

Although our solution is for odd $m$, our results in Lemmas are valid for even $m$ as well and can be used in different constructions. Our result also complements the result of Fu and Huang [11]; altogether, existence of a solution to $(4, m) – \text{HWP}(v; r, s)$ is shown for all integers $m \geq 3$ with a few possible exceptions. Regarding the results of Bonvicini and Buratti, only exception would be $r = 2$ when $v = 8m$ for odd $m \geq 5$. We can combine these results as follows.

**Theorem 14.** For all positive integers $r$, $s$ and $m \geq 3$, a solution to $(4, m) – \text{HWP}(v; r, s)$ exists if and only if $4|v$, $m|v$ and $r + s = \frac{v + 2}{2}$ except possibly when $r = 2$ and $v = 8m$ for $m \geq 5$ odd.

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