On a comparison method for a parabolic–elliptic system of chemotaxis with density-suppressed motility and logistic growth

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Abstract

We consider a parabolic–elliptic system of partial differential equations with chemotaxis and logistic growth given by the system

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta (u \gamma(v)) &= \mu u (1 - u), \\
-\Delta v + v &= u,
\end{align*}
\]

under Neumann boundary conditions and appropriate initial data in a bounded and regular domain \(\Omega\) of \(\mathbb{R}^N\) (for \(N \geq 1\)), where \(\gamma \in C^3([0, \infty))\) and satisfies

- \(\gamma(s) > 0\), \(\gamma'(s) \leq 0\), \(\gamma''(s) \geq 0\), \(\gamma'''(s) \leq 0\) for any \(s \geq 0\)
- \(-2\gamma'(s) + \gamma''(s)s \leq \mu_0 < \mu\)
- \([\gamma'(s)]^2 \leq c\), for any \(s \in [0, \infty)\).

We obtain the global existence and uniqueness of bounded in time solutions and the following asymptotic behavior

\[\|u - 1\|_{L^\infty(\Omega)} + \|v - 1\|_{L^\infty(\Omega)} \to 0, \quad \text{when} \ t \to +\infty.\]

Keywords Comparison methods for parabolic–elliptic systems of PDEs · Chemotaxis · Asymptotic behaviour · Logistic growth term

Mathematics Subject Classification 35K55 · 35K57 · 35B35

Dedicated to Professor J. Ildefonso Díaz on the Occasion of his 70th Birthday

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1 Introduction

Chemotaxis has been studied from a mathematical point of view in the last decades, especially significant are the works of Keller and Segel [23, 24] modeling the phenomenon. The model presented in [23, 24] is a fully parabolic system of two equations involving a chemical stimuli and a biological species orientating its movement in response to the mentioned stimuli.

There exists an extensive mathematical literature studying chemotaxis systems of Partial Differential Equations, see for instance the surveys Horstmann [18, 19], Bellomo et al. [2], Hillen and Painter [17] and references therein for more details.

One of the main topics studied in the literature is under which assumptions the solution blows up or it remains bounded for any \( t < \infty \). Such a dichotomy is also present in chemotaxis systems with logistic growth terms. In the pioneering works of Díaz and Nagai [5], Díaz et al. [6] where symmetrization technics are used to get global existence of solutions and its asymptotic behavior.

By using comparison strategies, in bounded and regular domains of \( \mathbb{R}^N \), the solution to the parabolic elliptic system

\[
\begin{cases}
    u_t - \Delta u = -d i v(u \chi \nabla v) + \mu u (1 - u), \\
    -\Delta v + v = u,
\end{cases}
\]

under homogeneous Neumann boundary conditions, globally exists and it is bounded in time when

\[
\chi < \frac{N-2}{N} \mu, \quad \text{if } N > 2 \quad \text{and } \mu > 0, \quad \text{if } N = 2,
\]

see Tello and Winkler [35]. In Khan and Stevens [25], the global existence of solutions is obtained for the limit case \( \chi = \frac{N-2}{N} \mu \). The result is also valid if the domain is \( \mathbb{R}^N \) and \( \chi = \mu \), see Salako and Shen [32]. Galakhov et al. [15] study the system

\[
\begin{cases}
    u_t - \Delta u = -d i v(u \chi \nabla v) + \mu u (1 - u^k), \\
    -\Delta v + v = u^\theta,
\end{cases}
\]

in a bounded and regular domain with homogeneous Neumann boundary conditions. In [15], the authors obtain that, under any of the assumptions

\[
\theta > m + \gamma - 1, \\
\theta = m + \gamma - 1 \quad \text{and } \mu > \frac{N\theta - 2}{2(m-1)+N\theta} \chi,
\]

the solution exists globally in time for regular initial data. Moreover, if

\[
\theta \geq m + \gamma - 1 \quad \text{and } \mu > 2\chi,
\]

and the initial data satisfy \( 0 < u_0 \leq u_0 \leq \bar{u}_0 < \infty \), for some positive constants \( u_0 \) and \( \bar{u}_0 \) we have the following asymptotic behavior

\[
\| u - 1 \|_{L^\infty(\Omega)} + \| w - 1 \|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty.
\]

The blow up of solutions for the parabolic elliptic problem with logistic term in the form

\[
\begin{cases}
    u_t - \Delta u = -d i v(u \nabla v) + \lambda u - \mu u^k, \\
    -\Delta v = u - \frac{1}{|\Omega|} \int_{\Omega} u \, dx,
\end{cases}
\]

in a \( N \)-dimensional bounded domains have been studied in Fuest [10]. The author obtain finite time blow up of solutions for some initial data under assumptions

\[
k \in (0, 1), \quad \mu > 0, \quad N \geq 4; \\
k = 2, \quad \mu \in (0, (N-4)/N), \quad N \geq 5.
\]
see also Winkler [37, 38].

Recently, in Liu et al. [28], the authors propose a fully parabolic system of two parabolic equations to model the pattern formation of e-coli bacteria (see also Fu et al. [9]). In [28], \( u \) denotes the e-coli density and \( v \) the molecule acyl-homoserine lactone concentration (AHL) which is excreted by the e-coly cells. The system proposed in [28] reads as follows

\[
\begin{aligned}
  u_t - \Delta(\tilde{\gamma}(v)u) &= \mu u(1 - u/\rho_s), \\
  v_t - D_h \Delta v + \alpha v &= \beta u,
\end{aligned}
\]

where \( \tilde{\gamma} \) is given by a bounded function

\[
\tilde{\gamma}(v) = \frac{D_\rho + D_{\rho,0} v^m}{1 + \frac{v^m}{K_h^m}}
\]

for \( m = 20 \) and positive constants \( D_\rho, D_{\rho,0}, K_h, \mu, \rho_s, D_h, \alpha \) and \( \beta \). Notice that the parabolic equation satisfied by \( u \) in [28] can be written as a particular case of the classical Keller–Segel system

\[
\begin{aligned}
  u_t - d v(\tilde{\gamma}(v)\nabla u) &= d v(u\tilde{\gamma}'(v)\nabla v) + \mu u(1 - u/\rho_s),
\end{aligned}
\]

where \( \tilde{\gamma} \) represents the diffusion coefficient and \( \tilde{\gamma}'(v) \) is the chemotactant coefficient which are clearly linked.

In [28], the diffusion coefficient of \( v \) is taken \( D_h \sim 400 \mu m^2 s^{-1} \) and \( D_{\rho,0}/D_\rho \ll 1 \). Considering such a range of values for \( D_h \) and \( D_{\rho,0}/D_\rho \) and the range of data for \( v \), a natural simplification of the fully parabolic system proposed in [28] is to take

\[
\tilde{\gamma}(v) = \frac{D_\rho}{1 + \frac{v^m}{K_h^m}}
\]

for \( v \) satisfying the elliptic equation

\[
-D_h \Delta v + \alpha v = \beta u.
\]

Such simplification transforms the fully parabolic problem into a parabolic–elliptic system. Now we introduce the rescaled variables and parameters

\[
\tilde{u} = \frac{u}{\rho_s}, \quad \tilde{v} = \alpha v, \quad \tilde{x} = \frac{a^2}{D_h^2} x, \quad \tilde{\beta} = \rho_s \frac{\beta}{\alpha}
\]

and the system becomes

\[
\begin{aligned}
  u_t - \Delta(\gamma(\tilde{v})\tilde{u}) &= \mu \tilde{u}(1 - \tilde{u}), \\
  -\Delta \tilde{v} + \tilde{v} &= \beta \tilde{u}.
\end{aligned}
\]

For simplicity we drop the tilde, assume that \( \tilde{\beta} = 1 \) and complete the system with Neumann boundary conditions and appropriate initial data in a bounded and regular domain \( \Omega \)

\[
\begin{aligned}
  u_t - \Delta(\gamma(v)u) &= \mu u(1 - u), & x \in \Omega, & t > 0, \\
  -\Delta v + v &= u, & x \in \Omega, & t > 0, \\
  \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} &= 0, & x \in \partial \Omega, & t > 0, \\
  u(0, x) &= u_0(x), & x \in \Omega.
\end{aligned}
\]
The fully parabolic system for $\mu = 0$ have been considered from a mathematical point of view in Tao and Winkler [34]. In [34], the function $\gamma$ belongs to $C^3((0, \infty))$ and satisfies

$$k_0 \leq \gamma(s) \leq k_1, \quad |\gamma'| \leq k_2,$$

for some positive constants $k_i (i = 0 \cdots 2)$. The authors prove that the solution is uniformly bounded when $\Omega$ is a two dimensional bounded and regular domain. If the dimension is bigger than 2, there exists a global-in-time weak solution in the appropriate Sobolev space. Moreover, for a range of parameters, the solution becomes classical provided $t > t_0$ for some $t_0 < \infty$. After [34], Jin et al. [22] proved that the fully parabolic system with logistic term possesses global classical solutions with a uniform-in-time bounds when the limit

$$\lim_{v \to \infty} \frac{\gamma'(v)}{\gamma(v)}$$

exists and

$$\lim_{v \to \infty} \gamma(v) = 0$$

in a two dimensional bounded domain. Moreover, if $\mu$ is large enough the unique positive constant steady state $u = v = 1$, is asymptotically stable.

In [1], the existence of global in time bounded solutions is proved for the parabolic–elliptic system (for $\mu = 0$) when $\gamma(v) = v^{-\kappa}$ for any $\kappa > 0$ if $\Omega$ is a one or two-dimensional bounded domain. If $\Omega$ is a $N$-dimensional bounded domain for $N > 2$ and $\kappa < \frac{2}{N-2}$, the global in time bounded solutions is also obtained. The linear stability is also presented in both cases.

The system (1.1) has been already studied in Fujie and Jiang [11], the authors consider the parabolic–elliptic system with logistic growth in a 2-dimensional bounded domain $\Omega$. In that case, the solution is uniformly bounded provided

$$\frac{|\gamma'(s)|^2}{\gamma(s)} \leq k_0 < \infty \quad \text{for any } s \geq 0.$$

Moreover, the global existence is also obtained in a 2-dimensional bounded domain when $\mu = 0$ and $\gamma$ satisfies

$$\gamma' \leq 0, \quad \lim_{s \to +\infty} s^k \gamma(s) = +\infty$$

or $\gamma(s) = e^{-s}$ and the initial mass is small enough. In the last case, the asymptotic stability of the solution converging to the average of the initial data is also given. A complementary case, large initial mass and the same function $\gamma$, i.e. $\gamma(s) = e^{-s}$ in the unit ball, produces blow up at $t = +\infty$. See also Fujie and Senba [13, 14].

In Fujie and Jiang [12] the parabolic–elliptic system is also considered for $\mu = 0$. The authors obtain the existence of a unique global classical solution which is uniformly-in-time bounded under the assumption

$$\lim_{s \to \infty} e^{\alpha s} \gamma(s) = +\infty, \quad \text{for any } \alpha > 0.$$ 

If the previous assumption is only satisfied for large $\alpha$, the solution also exists in time provided the initial data is small enough. The proofs are given in bounded domains in arbitrary dimension $N$.

The parabolic elliptic case is also studied in Jiang [20] for $\mu = 0$ and $\gamma$ satisfying

$$\frac{N+2}{4} |\gamma'(s)|^2 \leq \gamma(s)\gamma''(s), \quad \text{for } s \geq 0.$$
In [20], the author proved that the unique solution is global in time and converges to the average of the initial data in $L^\infty(\Omega)$. Moreover, if $\gamma = v^{-k}$ for $k$ satisfying
\[
  k \in (0, 1), \quad \text{if } N = 4, 5 \\
  k \in (0, \frac{4}{N-2}), \quad \text{if } N \geq 6
\]
the same result is also obtained. The steady states of the system with logistic term has been considered in Wang and Xu [39], where bifurcation of such steady states is obtained.

In this article we study the solutions of Eq. (1.1) in the following sense.

**Definition 1.1** We say that $(u, v)$ is a weak solution to (1.1) if 
\[
u \in L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega)) \cap C(0, T; L^2(\Omega)),
\]
\[
v \in C(0, T; H^1(\Omega))
\]
and for any $\phi \in L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega)) \cap C(0, T; L^2(\Omega))$ such that 
\[
\frac{\partial \phi}{\partial n} = 0, \quad \text{in } x \in \partial \Omega
\]
we have that
\[
- \int_0^T \int_\Omega u \phi_t \, dx \, dt - \int_0^T \int_\Omega u \gamma(v) \Delta \phi \, dx \, dt = \mu \int_0^T \int_\Omega u(1-u) \phi \, dx \, dt
\]
\[
+ \int_\Omega [u_0 \phi_0 - u(T) \phi(T)] \, dx
\]
and
\[
\int_\Omega \nabla v \nabla \phi \, dx + \int_\Omega v \phi \, dx = \int_\Omega u \phi \, dx.
\]

The problem is studied under the assumptions
\[
\gamma \in C^3([0, \infty)), \quad \gamma(s) > 0,
\]
\[
\gamma'(s) \leq 0, \quad \gamma''(s) \geq 0, \quad \gamma'''(s) \leq 0, \quad \text{for any } s \geq 0,
\]
\[
-2\gamma'(s) + \gamma''(s)s \leq \mu_0 < \mu.
\]
\[
\left| \frac{\gamma'(s)}{\gamma(s)} \right|^2 \leq c_\gamma < \infty, \quad \text{for any } s \in [0, \infty),
\]
where the initial datum $u_0$ satisfies 
\[
u_0 \in C^{2,\alpha}(\overline{\Omega}), \quad \frac{\partial u_0}{\partial n} = 0 \text{ in } \partial \Omega.
\]

There exists positive constants $u_0, \overline{u}_0$ such that
\[
0 < u_0 \leq u \leq \overline{u}_0 < \infty.
\]

Notice that assumptions (1.2)–(1.5) are satisfied for instance by 
\[
\gamma_1(s) := e^{-\alpha_1 s}, \quad \text{for any } \alpha_1 > 0, \quad \gamma_2(s) := [\epsilon_2 + s]^{-\alpha_2} \quad \text{for any } \epsilon_2, \alpha_2 > 0
\]
and $\mu$ large enough.

The main result of the article is enclosed in the following theorem.
Theorem 1.1 Under assumption (1.2)–(1.7) there exists a unique solution \((u, v)\) in the sense of Definition 1.1 in the time interval \((0, \infty)\) satisfying

\[
\lim_{t \to \infty} \|u - 1\|_{L^\infty(\Omega)} + \|v - 1\|_{L^\infty(\Omega)} = 0.
\]

The article is organized as follows. In Sect. 2, we present the local in time existence of solutions. The proof is obtained by standard arguments using a priori estimates in the appropriate Sobolev spaces. In Sect. 3, an auxiliary system of ODE is introduced, the solutions of such system are used in Sect. 4 as sub and super solutions of the equation. In Sect. 5, the global existence of solutions is given. Also in Sect. 5, the proof of the asymptotic behavior of the solution is presented as a consequence of the asymptotic behavior of the sub and super solutions.

The comparison method used in the article is known as rectangle method, similar comparison methods have been already applied to reaction-diffusion systems by different authors, see for instance Pao [29], Conway and Smoller [4], Fife and Tang [7] and also Negreanu and Tello [30]. Parabolic–elliptic chemotaxis problems are among the systems where the rectangle method have been successfully applied, in this case, the sub- and super-solutions are defined as the solutions of a coupled nonlinear ODE’s systems, see for instance [8, 31]. The method have been also applied to parabolic–elliptic chemotaxis systems of two species, see for instance Tello and Winkler [36] and Stinner et al. [33]. The method can be applied to reaction–diffusion system with general non-linearities and non-local terms if the crossed terms are of first order and the non-linearities are given by \(C^0\) functions. Nevertheless, the rectangle method have been resulted unaplicable to proof blow up of solutions for chemotaxis system because the sub-solution remains below the average mass, which is uniformly bounded.

Throughout the article we consider the constant \(c_{\Omega}\) defined in the following definition.

**Definition 1.2** Let \(\Omega\) be a regular bounded domain of \(\mathbb{R}^N\), we define \(c_{\Omega}\) and \(c_p\) (for \(p > N/2\)) as follows

\[
c_{\Omega} := \sup_{f \in L^\infty(\Omega)} \left\{ \frac{\|\nabla \phi\|_{L^\infty(\Omega)}}{\|f\|_{L^\infty(\Omega)}} \right\}
\]

and

\[
c_p := \sup_{f \in L^p(\Omega)} \left\{ \frac{\|\phi\|_{L^\infty(\Omega)}}{\|f\|_{L^p(\Omega)}} \right\}
\]

where \(\phi\) is the solution to the problem

\[
\begin{cases}
-\Delta \phi + \phi = f, & x \in \Omega, \\
\frac{\partial \phi}{\partial n} = 0, & x \in \partial \Omega
\end{cases}
\]

for \(f \not= 0\), \(f \in L^\infty(\Omega)\) or \(f \in L^p(\Omega)\), respectively.

**Remark 1.1** Notice that for any \(p > N + 1\)

\[
\|f\|_{L^{N+1}(\Omega)} \leq |\Omega|^{\frac{p-N-1}{p(N+1)}} \|f\|_{L^p(\Omega)}
\]

we have that

\[
c_p = \sup_{f \in L^p(\Omega)} \left\{ \frac{\|\phi\|_{L^\infty(\Omega)}}{\|f\|_{L^p(\Omega)}} \right\} \leq \sup_{f \in L^{N+1}(\Omega)} \left\{ |\Omega|^{\frac{p-N-1}{p(N+1)}} \frac{\|\phi\|_{L^\infty(\Omega)}}{\|f\|_{L^{N+1}(\Omega)}} \right\} = |\Omega|^{\frac{p-N-1}{p(N+1)}} c_{N+1}
\]
and taking limits when \( p \to \infty \), it results
\[
c_{\Omega} \leq |\Omega|^{\frac{1}{p+1}}c_{N+1}.
\]

## 2 Local existence of solutions

We first consider the approximated problem

\[
\begin{aligned}
    &u_{n,t} - \Delta (\gamma(v_n)u_n) = \mu u_n (1 - (u_n)_+) , & & \; x \in \Omega , \; \; t > 0, \\
    &-\Delta v_n + v_n = \frac{u_n}{1 + (\frac{u_n}{p})^+} , & & \; x \in \Omega , \; \; t > 0, \\
    &\frac{\partial u_n}{\partial n} = \frac{\partial v_n}{\partial n} = 0 , & & \; x \in \partial \Omega , \; \; t > 0, \\
    &u_n(0,x) = u_{n0}(x) , & & \; x \in \Omega ,
\end{aligned}
\]

\[(2.1)\]

where \(( \cdot )_+\) indicates the positive part function. We work with weak solutions of the approximated problem, which are given in the following definition.

**Definition 2.1** We say that \((u_n, v_n)\) is a weak solution to \((2.1)\) if

\[
\begin{aligned}
    u_n &\in L^2(0, T : H^2(\Omega)) \cap H^1(0, T : L^2(\Omega)) \cap C(0, T : L^2(\Omega)), \\
v_n &\in C(0, T : H^1(\Omega)) \text{ and for any } \phi \in L^2(0, T : H^2(\Omega)) \cap H^1(0, T : L^2(\Omega)) \cap C(0, T : L^2(\Omega)) \text{ such that}
\end{aligned}
\]

\[
\frac{\partial \phi}{\partial n} = 0 , \quad \text{in } x \in \partial \Omega
\]

we have that

\[
\begin{aligned}
    - \int_0^T \int_{\Omega} u_n \phi_t dx dt - \int_0^T \int_{\Omega} u_n \gamma(v_n) \Delta \phi dx dt &= \mu \int_0^T \int_{\Omega} u_n (1 - (u_n)_+) \phi dx dt \\
    &+ \int_{\Omega} [u_n\phi_0 - u(T)\phi(T)] dx
\end{aligned}
\]

and for any \( t \in (0, T) \)

\[
\begin{aligned}
    \int_{\Omega} \nabla v_n \nabla \phi dx + \int_{\Omega} v_n \phi dx &= \int_{\Omega} \frac{u_n}{1 + (\frac{u_n}{p})^+} \phi dx.
\end{aligned}
\]

Now we introduce the following ODE problem for \( p \geq \max\{4, N + 1\} \)

\[
\frac{1}{p} y' = \mu |\Omega| + c_\Omega c_\gamma y^{\frac{p+2}{p}}.
\]

\[(2.2)\]

**Lemma 2.1** Let \( p > \max\{4, N\} \) and \( y_0 \geq 0 \), then, there exists \( T_p > 0 \) and a unique solution to \((2.2)\) satisfying \( y(0) = y_0 \) in \((0, T_p)\).

**Proof** Existence of solutions is a consequence of Peano’s Theorem. Uniqueness is deduced from the fact that the right-hand side term is locally Lipschitz.

In the following lemmas we obtain some a priori estimates to finally prove the local existence of solutions.
Lemma 2.2 Let \( u_n \) be a weak solution to (2.1) in the sense of Definition 2.1, then, we have that
\[
u_n \geq 0.\]

**Proof** Since
\[
\frac{u_n}{1 + (u_n)_+} \in L^\infty(\Omega),
\]
we have that \( v_n \in W^{1,\infty}(\Omega) \).

We multiply by \( \phi(u_n) := -(u_n)_+ \) and after integration by parts we obtain
\[
\frac{d}{dt} \int_\Omega \left[ \phi(u_n) \right]^2 dx + \int_\Omega \gamma(v) \nabla u_n \nabla \phi(u_n) dx + \int_\Omega \gamma'(v_n) u_n \nabla v_n \nabla \phi(u_n) dx = \mu \int_\Omega u_n \phi(u_n) (1 - (u_n)_+) dx.
\]
(2.3)

Since
\[
\int_\Omega \gamma(v) \nabla u_n \nabla \phi(u_n) dx = \int_\Omega \gamma(v) |\nabla \phi(u_n)|^2 dx;
\]
\[
\int_\Omega \gamma'(v) u \nabla v \nabla \phi(u_n) dx \leq \epsilon \int_\Omega \gamma(v) |\nabla \phi(u_n)|^2 dx + c(\epsilon) \int_\Omega \frac{|\gamma'(v_n)|^2}{\gamma(v_n)} |\nabla v_n|^2 |\phi(u_n)|^2 dx;
\]
\[
\mu \int_\Omega u_n \phi(u_n) (1 - (u_n)_+) dx \leq \mu \int_\Omega \phi(u_n)^2 dx,
\]
thanks to assumption (1.5) and in view of \( v_n \in L^\infty(\Omega) \), (2.3) becomes
\[
\frac{d}{dt} \frac{1}{2} \int_\Omega [\phi(u_n)]^2 dx + (1 - \epsilon) \int_\Omega \gamma'(v) |\nabla \phi(u_n)|^2 dx \leq c \int_\Omega |\phi(u_n)|^2 dx.
\]
Standard comparison’s results of O.D.E.s ends the proof.

Lemma 2.3 Let \( u_n \) be a weak solution to (2.1) in the sense of Definition 2.1, then, we have that
\[
\int_\Omega |u_n| dx \leq c.
\]

**Proof** We integrate over \( \Omega \) to obtain
\[
\frac{d}{dt} \int_\Omega u_n dx = \mu \int_\Omega u_n dx - \mu \int_\Omega u_n^2 dx
\]
Thanks to Cauchy-Schwarz inequality we get
\[
\int_\Omega |u_n| dx \leq |\Omega|^\frac{1}{2} \left[ \int_\Omega u_n^2 dx \right]^\frac{1}{2}
\]
which implies
\[
\frac{d}{dt} \int_\Omega u_n dx + \frac{\mu}{|\Omega|^\frac{1}{2}} \left[ \int_\Omega u_n dx \right]^\frac{1}{2} \leq \mu \int_\Omega u_n dx.
\]
Growall’s Lemma ends the proof.
Lemma 2.4 Let \( p \geq \max\{4, N + 1\} \), then, there exists \( T_p > 0 \) independent of \( n \) such that
\[
\int_{\Omega} |u_n|^p \, dx \leq c \quad \text{for} \ t < T_p.
\]

Proof Let \( p \geq \max\{4, N + 1\} \) be a given finite number. We now take \( u^{p-1} \) as test function in (2.1) to obtain
\[
\frac{d}{dt} \frac{1}{p} \int_{\Omega} |u_n|^p \, dx + (p - 1) \int_{\Omega} \gamma(v) u_n^{p-2} |\nabla u_n|^2 \, dx \\
+ (p - 1) \int_{\Omega} \gamma'(v_n) u_n^{p-1} \nabla v_n \nabla u_n \, dx = \mu \int_{\Omega} u_n^p (1 - u_n) \, dx.
\]

Notice that
\[
(p - 1) \int_{\Omega} \gamma(v) u_n^{p-2} |\nabla u_n|^2 \, dx = \frac{4(p - 1)}{p^2} \int_{\Omega} \gamma(v) |\nabla u_n^p|^2 \, dx,
\]
\[
\mu \int_{\Omega} u_n^p (1 - u_n) \, dx \leq \mu |\Omega|,
\]
\[
(p - 1) \int_{\Omega} \gamma'(v_n) u_n^{p-1} \nabla v_n \nabla u_n \, dx \leq \epsilon \int_{\Omega} \gamma(v) |\nabla u_n^p|^2 \, dx \\
+ c(\epsilon) \int_{\Omega} \frac{|\gamma'(v_n)|^2}{\gamma(v_n)} |\nabla v_n|^2 u_n^p \, dx,
\]
\[
\int_{\Omega} \frac{|\gamma'(v_n)|^2}{\gamma(v_n)} |\nabla v_n|^2 u_n^p \, dx \leq ||\gamma'(v_n)||_{L^\infty(\Omega)}^2 ||\nabla v_n||_{L^\infty(\Omega)}^2 \left[ \int_{\Omega} u_n^p \, dx \right]^\frac{p+2}{p},
\]
where \( c_p \) has been defined in (1.2). Then, (2.4) becomes
\[
\frac{d}{dt} \frac{1}{p} \int_{\Omega} |u_n|^p \, dx \leq \mu |\Omega| + c_p^2 ||\gamma'(v_n)||_{L^\infty(\Omega)}^2 \left[ \int_{\Omega} u_n^p \, dx \right]^\frac{p+2}{p}.
\]

Thanks to assumption (1.5), it becomes
\[
\frac{d}{dt} \frac{1}{p} \int_{\Omega} |u_n|^p \, dx \leq \mu |\Omega| + c_p^2 c_{\gamma} \left[ \int_{\Omega} u_n^p \, dx \right]^\frac{p+2}{p}.
\]

In view of Lemma 2.1, standard comparison methods prove the existence of a positive \( T_p > 0 \) such that \( \int_{\Omega} |u_n|^p \, dx \) is bounded for any \( t < T_p \). \( \square \)

Lemma 2.5 Let \( T_* := \frac{1}{2} T_{\max\{4, N+1\}} \) for \( T_{\max\{4, N+1\}} \) defined in Lemma 2.4, then, for any \( t < T_* \) there exists \( c_v \) such that
\[
\| \nabla v_n \|_{L^\infty(\Omega)} + \| v_n \|_{L^\infty(\Omega)} \leq c_v
\]
for any \( t < T_* \).

Proof Thanks to Lemma 2.3 and Theorem 8.31 in Gilbard and Trudinger [16], we have that
\[
v_n \in W^{2,q}(\Omega)
\]
for any \( q < \infty \), the Sobolev embedding \( W^{2,q}(\Omega) \hookrightarrow W^{1,\infty}(\Omega) \), for \( q > N \) proves the result. \( \square \)
Lemma 2.6 Let $T_n$ be a positive constant defined in Lemma 2.5,

$$
\lambda := \sup_{t \in [0, T_n]} \{ \| \gamma''(v_n) \|_{L^\infty(\Omega)} \| \nabla v_n \|_{L^\infty(\Omega)}^2 + \gamma' v_n + (\| u_0 \|_{L^\infty(\Omega)} + 2) | \gamma'| e + \mu \}
$$

and $T_{\min} \leq \min\{ \frac{1}{\lambda}, T_n \}$. Then, there exists a positive constant $c$ such that

$$
\| u_n \|_{L^\infty(\Omega)} \leq c.
$$

Proof Let $c_0$ be positive constants defined by

$$
c_0 := \| u_0 \|_{L^\infty(\Omega)} + 1.
$$

We consider the change of unknown $u_n = e^{\lambda t} w_n$, then, $w_n$ satisfies

$$
w_{nt} - div(\gamma(v_n) \nabla w_n) + \lambda w_n = \gamma' \nabla w_n \nabla v_n + w_n [\gamma''|\nabla v_n|^2 + \gamma'(v_n - e^{\lambda t} w_n) + \mu (1 - e^{\lambda t} w_n)]. \tag{2.5}
$$

We introduce the truncated problem

$$
\tilde{w}_{nt} - div(\gamma(v_n) \nabla \tilde{w}_n) + \tilde{\lambda} \tilde{w}_n
\quad
= \gamma' \nabla \tilde{w}_n \nabla v_n + \tilde{w}_n [\gamma''|\nabla v_n|^2 + \gamma' v_n + T_k(\tilde{w}_n) |\gamma'| e^{\lambda t} + \mu (1 - e^{\lambda t} \tilde{w}_n)] \tag{2.6}
$$

where $T_k(\tilde{w}_n) = \tilde{w}_n$ if $\tilde{w}_n < k$ and $T_k(\tilde{w}_n) = k$ otherwise, for

$$
k = c_0 + 1.
$$

Now, we multiply (2.6) by $(\tilde{w}_n - c_0)_+$, after some computations, in view of

$$
\gamma''|\nabla v_n|^2 + |\gamma'| v_n + k |\gamma'| e^{\lambda t} + \mu \leq \lambda,
$$

for $t \leq \frac{1}{\lambda}$, we get

$$
\frac{d}{dt} \frac{1}{2} \int_{\Omega} (\tilde{w}_n - c_0)^2_+ dx \leq c \int_{\Omega} (\tilde{w}_n - c_0)^2_+ dx
$$

which implies, thanks to Gronwall’s Lemma that $\tilde{w}_n < c_0$. Since $T_k(\tilde{w}_n) = \tilde{w}_n$ for $k = c_0 + 1$ we have that $\tilde{w}_n = w_n$ and it satisfies (2.5) for any $t \leq T_{\min}$, therefore $u_n \leq c_0 e$ for $t < T_{\min}$.

□

Lemma 2.7 Let $T_{\min}$ be defined in Lemma 2.6 then, we have that

$$
\int_{\Omega} u_n^2(T_{\min}) dx + \int_0^{T_{\min}} |\nabla u_n|^2 dx dt \leq c(T_{\min})
$$

and

$$
\int_0^{T_{\min}} \int_{\Omega} |u_{nt}|^2 dx dt \leq c(T_{\min}).
$$

Proof Let $T \leq T_{\min}$, we take $u_n$ as test function in the weak formulation of (2.1) to obtain

$$
\int_{\Omega} u_n^2(T) dx + c_0 \int_0^T \int_{\Omega} |\nabla u_n|^2 dx dt \leq \int_0^T \int_{\Omega} \gamma'(v_n) u_n \nabla u_n \nabla v_n dx dt + \mu \int_0^T \int_{\Omega} u_n^2(1 - u_n) dx dt + c.
$$
Since
\[
\int_0^T \int_\Omega \gamma'(v_n)u_n \nabla u_n \cdot \nabla v_n \, dx \, dt \leq \epsilon \int_0^T \int_\Omega |\nabla u_n|^2 \, dx \, dt \\
+ \int_0^T \int_\Omega |\gamma'(v_n)|^2 u_n^2 |\nabla v_n|^2 \, dx \, dt
\]
we have that
\[
\int_\Omega u_n^2 \, dx + \int_0^T \int_\Omega |\nabla u_n|^2 \, dx \, dt \leq c(T),
\]
which proves the first part of the theorem. To prove the second part, we first notice that
\[
-\Delta (\gamma(v_n)u_n) = -\gamma(v_n)\Delta u_n - 2\gamma'(v_n)\nabla v_n \cdot \nabla u_n - \gamma''(v_n) |\nabla v_n|^2 u_n - \gamma'(v_n)u_n \Delta v_n.
\]
Now, we take $\Delta u_n$ in the weak formulation to get
\[
\frac{1}{2} \int_\Omega |\nabla u_n|^2 \, dx \bigg|_0^T + \int_0^T \int_\Omega \gamma'(v_n) |\Delta u_n|^2 \, dx \, dt \leq \int_0^T \int_\Omega u_n |\gamma'(v_n) |\Delta v_n| |\Delta u_n| \, dx \, dt \\
+ \int_0^T \int_\Omega \left[ |\gamma'(v_n)||\nabla v_n||\nabla u_n| + |\gamma''(v_n)||\nabla v_n|^2 u_n \right] |\Delta u_n| \, dx \, dt \\
+ \mu \int_0^T \int_\Omega (u_n + u_n^2) |\Delta u_n| \, dx \, dt.
\]
Since
\[
\int_\Omega |\Delta v_n|^p \, dx \leq c(T), \\
\gamma(v_n) \geq c > 0, \quad |\gamma'| + |\gamma''| < c, \quad \text{in } (0, T_{min}), \\
\|\nabla v\|_{L^\infty(\Omega)}, \quad \|v\|_{L^\infty(\Omega)}, \quad \|u\|_{L^\infty(\Omega)} \leq c
\]
and
\[
|\nabla u||\Delta u| \leq \epsilon |\Delta u|^2 + \frac{1}{4\epsilon} |\nabla u|^2,
\]
we get
\[
\int_\Omega |\nabla u_n|^2 \, dx \bigg|_0^T + \int_0^T \int_\Omega |\Delta u_n|^2 \, dx \, dt \leq c(T).
\]
Notice that in view of assumption (1.2), (1.3) and Lemma 2.5, we have that
\[
\gamma(v_n) \geq \gamma(\|v_n\|_{L^\infty(\Omega)}) > 0, \quad \text{for any } T < T_{min}
\]
and therefore
\[
\int_\Omega \gamma(v_n) |\Delta u_n|^2 \, dx \geq \gamma(\|v_n\|_{L^\infty(\Omega)}) \int_\Omega |\Delta u_n|^2 \, dx.
\]
We multiply now by $u_{mt}$ and, in view of
\[
\int_\Omega u_t \Delta \gamma(v) \, dx \leq \epsilon \int_\Omega |u_{nt}|^2 \, dx \\
+ c \int_\Omega |\Delta u_{nt}|^2 \, dx + c \int_\Omega |\Delta v_{nt}|^2 \, dx + c \int_\Omega |\nabla u_{nt}|^2 |\nabla v_{nt}|^2 \, dx
\]
and previous lemmas we get that
\[ \int_0^T \int_{\Omega} |u_n|^2 \, dx \leq c \]
which ends the proof. \( \square \)

**Lemma 2.8** Let \( T < T_{\text{min}} \) small enough, then, there exists a unique weak solution \((u_n, v_n)\) to (2.1) in the sense of definition 2.1 such that
\[ u_n \in L^2(0, T : H^2(\Omega)) \cap L^\infty(0, T : H^1(\Omega)), \quad v \in C^0(0, T : C^1(\overline{\Omega})). \]

**Proof** We consider a fixed point argument, for a given \( \tilde{u}_n \in C_{t,x}^{1+\beta}([0, T] \times \overline{\Omega}), \tilde{u}_n \geq 0 \), we consider the solution \( v_n \) to the problem
\[ -\Delta v_n + v_n = \frac{\tilde{u}_n}{1 + \tilde{u}_n}, \quad (2.7) \]
with Neumann boundary condition. Then, we consider \( u_n \) the solution to the parabolic problem
\[ \begin{cases} u_{nt} - div(\gamma(v_n)\nabla u_n) = -div(u_n\gamma'(v_n)\nabla v_n) + \mu u_n(1 - \tilde{u}_n), \\ u(0, x) = u_0(x), \end{cases} \quad (2.8) \]
with Neumann boundary conditions. Lax–Milgram theorem proves the existence of a unique solution \( v_n \) to (2.7), thanks to Gilbart and Trudinger [16] Theorem 8.34 we have that \( v_n \in C_{t,x}^{1+\beta}([0, T] \times \overline{\Omega}). \) We replace \( v_n \) into (2.8) to obtain thanks to Lieberman [26], Theorem 4.30, page 79, that \( u_n \in C_{t,x}^{1+\beta}([0, T] \times \overline{\Omega}). \) Schauder fixed point Theorem proves the existence of solutions for \( T \) small enough. Since, for any \( n \), the term \( \frac{\tilde{u}_n}{1 + \tilde{u}_n} \) is bounded in \( L^\infty([0, T_{\text{min}}] \times \Omega) \) we extend the solution up to \( T = T_{\text{min}} \).

Uniqueness of solutions follows from standard arguments, see for instance Theorem 2.1 in Tello and Winkler [35]. For readers convenience we detail the proof. We assume the existence of, at least, two different solutions \((u_{n1}, v_{n1})\) and \((u_{n2}, v_{n2})\) and proceed by contradiction. We denote by \( w \) the difference of both solutions, i.e. \( w = u_{n1} - u_{n2} \) which satisfies
\[ w_t - div(\gamma(v_{n1})\nabla w) - div(\gamma(v_{n1}) - \gamma(v_{n2}))\nabla u_{n1} = g(u_{n1}, v_{n1}, \nabla u_{n1}, \nabla v_{n1}) - g(u_{n2}, v_{n2}, \nabla u_{n2}, \nabla v_{n2}) \]
where
\[ g(u, v, \nabla u, \nabla v) := 2\gamma' (v_n) \nabla v_n \nabla u_n + u_n \gamma''(v_n) |\nabla v_n|^2 + u_n \gamma'(v_n) \left( v_n - \frac{u_n}{1 + \frac{u_n}{\tilde{u}_n}} \right) + \mu u_n(1 - u_n). \]
Notice that \( v_{n1} \) is uniformly bounded in \((0, T_{\text{min}})\) and \( g \) is a Lipschitz function, we multiply by \( w \) and integrate by parts to get, after some computations
\[ \frac{d}{dt} \int_{\Omega} w^2 \, dx + \int_{\Omega} \gamma(v_{n1})|\nabla w|^2 \, dx + \int_{\Omega} |\gamma(v_{n1}) - \gamma(v_{n2})| \nabla u_{n1} \nabla w \, dx \leq c \int_{\Omega} w^2 \, dx + c \int_{\Omega} [v_{n1} - v_{n2}]^2 \, dx + c \int_{\Omega} |\nabla w| w \, dx. \]
Since
\[
\int_\Omega [\nabla (v_{n1} - v_{n2})]^2 dx + \int_\Omega [v_{n1} - v_{n2}]^2 dx \leq c \int_\Omega w^2 dx,
\]
\[
\int_\Omega [\gamma(v_{n1}) - \gamma(v_{n2})] \nabla u_{n1} \nabla w dx \leq c(\epsilon) \| \gamma' \|^2_{L^\infty} \int_\Omega [v_{n1} - v_{n2}]^2 |\nabla u_{n1}|^2 dx + \epsilon \int_\Omega |\nabla w|^2 dx,
\]
\[
\int_\Omega [v_{n1} - v_{n2}]^2 |\nabla u_{n1}|^2 dx \leq \| \nabla u_{n1} \|^2_{L^\infty(\Omega)} \int_\Omega [v_{n1} - v_{n2}]^2 dx
\]
and \( u_n \in C^{1+\frac{\beta}{2},2+\beta}([0,T] \times \Omega) \) we conclude, after some computations
\[
\frac{d}{dt} \int_\Omega w^2 dx \leq c \int_\Omega w^2 dx.
\]
Gronwall’s Lemma ends the proof in view of \( w(0,x) = 0 \). \( \square \)

**Theorem 2.1** Let \( T < T_{\text{min}} \), then, there exists a unique weak solution \((u,v)\) to (1.1) in the sense of definition 1.1 such that
\[
u \in L^2(0,T:H^2(\Omega)) \cap L^\infty(0,T:H^1(\Omega)), \quad v \in C^0(0,T:C^1(\overline{\Omega})).
\]

**Proof** We have that for any \( T < T_{\text{min}} \)
\[
\int_0^T \int_\Omega |u_{nt}|^2 dx \leq c(T),
\]
\[
\int_0^T \int_\Omega |\Delta u_n|^2 dx \leq c(T),
\]
and
\[
\| \nabla u_n \|_{[L^2(\Omega)]^N} \leq c(T).
\]
Then, \( v_n \) is bounded in \( L^\infty(0,T:W^{2,p}(\Omega)) \). Moreover if we derivate respect to \( t \) the equation of \( v_n \) it results
\[
-\Delta v_{nt} + v_{nt} = \frac{u_{nt}}{(1 + u_{nt})^2}
\]
we take squares in the previous inequality to get, after integration
\[
\int_0^T \int_\Omega [|\Delta v_{nt}|^2 + |u_{nt}|^2 + |\nabla u_{nt}|^2] dx dt \leq c(T).
\]
Thanks to Aubin–Lions Lemma, and the compact embeddings \( H^2(\Omega) \hookrightarrow H^1(\Omega) \hookrightarrow L^2(\Omega) \), there exists \( u \in C(0,T: L^2(\Omega)) \) such that
\[
u_n \to u \quad \text{strong in } L^2(0,T:H^1(\Omega))
\]
\[
u_n \rightharpoonup v \quad \text{weak in } L^2(0,T:H^2(\Omega))
\]
and since \( W^{2,p}(\Omega) \hookrightarrow C^1(\overline{\Omega}) \hookrightarrow L^2(\Omega) \), there exists \( v \in C(0,T:C^1(\overline{\Omega})) \) satisfying
\[
v_n \to v \quad \text{strong in } C^0(0,T:C^1(\overline{\Omega})).
\]
Then
\[ \gamma(v_n) \to \gamma(v), \quad \text{strong in } C^0(0, T : C^1(\overline{\Omega})) \]
and
\[ \gamma'(v_n) \to \gamma'(v), \quad \text{strong in } C^0(0, T : C^1(\overline{\Omega})) \]
and therefore we get the following convergence of the integrals
\[
\begin{align*}
\int_0^T \int_\Omega u_n \phi_t dx dt & \to \int_0^T \int_\Omega u \phi_t dx dt, \\
\int_0^T \int_\Omega \gamma(v_n) \nabla u_n \nabla \phi dx dt & \to \int_0^T \int_\Omega \gamma(v) \nabla u \nabla \phi dx dt, \\
\mu \int_0^T \int_\Omega u_n (1 - (u_n)_+) \phi dx dt & \to \mu \int_0^T \int_\Omega u (1 - (u)_+) \phi dx dt, \\
\int_\Omega u_n(T) \phi(T) dx & \to \int_\Omega u(T) \phi(T) dx, \\
\int_\Omega \nabla v_n \nabla \phi dx & \to \int_\Omega \nabla v \nabla \phi dx, \\
\int_\Omega v_n \phi dx & \to \int_\Omega v \phi dx, \\
\int_\Omega \frac{u_n}{1 + (u_n)_+} \phi dx & \to \int_\Omega u \phi dx.
\end{align*}
\]

We take limits in (2.1) to obtain that \((u, v)\) is a weak solution of (1.1) in the sense of definition 1.1. Since \(\nabla v\) is bounded in \([0, T] \times \Omega\) for any \(T < T_\ast\) we have that standard parabolic regularity shows \(u \in L^q(0, T : W^{2,q}(\Omega)) \cap W^{1,q}(0, T : L^q(\Omega))\) for any \(q < \infty\).

To obtain uniqueness we proceed as in Lemma 2.8 and assume the existence of two different solutions \((u_1, v_2)\) and \((u_2, v_2)\). We denote by \(w\) the difference of both solutions, i.e. \(w = u_1 - u_2\), to obtain, after some computations
\[
\frac{d}{dt} \int_\Omega w^2 dx + \epsilon \int_\Omega |\nabla w|^2 dx + \int_\Omega [\gamma(v_1) - \gamma(v_2)] \nabla u_1 \nabla w dx \leq c \int_\Omega w^2 dx.
\]
Since
\[
\int_\Omega [\gamma(v_1) - \gamma(v_2)] \nabla u_1 \nabla w dx \leq c(\epsilon) \|
\gamma'\|_L^\infty \int_\Omega [v_1 - v_2]^2 |\nabla u_1|^2 dx
\]
\[+ \epsilon \int_\Omega |\nabla w|^2 dx,\]
and
\[
\int_\Omega [v_1 - v_2]^2 |\nabla u_1|^2 dx \leq c \|
\nabla u_1\|_{L^q(\Omega)} \|v_1 - v_2\|_{L^{2+}(\Omega)}^2
\]
for \(q\) large enough and \(\epsilon\) small enough we get, in view of
\[
\|v_1 - v_2\|_{L^{2+}(\Omega)}^2 \leq c \|v_1 - v_2\|_{H^1(\Omega)}^2 \leq c \|w\|_{L^2(\Omega)}^2
\]
the following differential inequality
\[
\frac{d}{dt} \int_\Omega w^2 dx \leq c(t) \int_\Omega w^2 dx,
\]

\(\square\) Springer
where $c(t) \in L^1(0, T)$. Grownall’s Lemma ends the proof.

\[\square\]

**Remark 2.1** We may extend the solution up to $T_{\text{max}}$ where $T_{\text{max}}$ satisfies
\[
\lim \sup_{t \to T_{\text{max}}} \left[ \|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + t \right] = \infty.
\]

### 3 Auxiliary problem

We first notice that the term $\Delta[\gamma(v)u]$ in (1.1) is expressed as follows
\[
-\Delta(u\gamma(v)) = -\gamma(v)\Delta u - 2\gamma'(v)v\nabla u + u\gamma'(v)\Delta v - u\gamma''(v)|\nabla v|^2
\]
\[
= -\gamma(v)\Delta u - 2\gamma'(v)v\nabla u + u\gamma'(v)(u - v) - u\gamma''(v)|\nabla v|^2,
\]
we replace the previous expression into (1.1) to obtain
\[
u_t - \gamma(v)\Delta u = 2\gamma'(v)v\nabla u - u\gamma'(v)(u - v) + u\gamma''(v)|\nabla v|^2 + \mu u(1 - u).
\]
Let $a(t)$ defined by
\[
a(t) := \|\nabla u\|_{L^\infty(\Omega)}
\]
and consider the system of Ordinary Differential Equations in the time interval $[0, T]$ for some $T \in (0, T_{\text{min}})$,
\[
\begin{aligned}
\bar{u}_t &= \bar{u}[\gamma'(u)](\bar{u} - u) + \bar{u}\gamma''(u)a(t) + \mu(1 - \bar{u}), \\
u_t &= u[\gamma'(u)](u - \bar{u}) + \mu(1 - u), \\
\bar{u}(0) &= \bar{u}_0, \quad u(0) = u_0.
\end{aligned}
\]

To simplify the previous system, we divide by $\bar{u}$ the first equation and by $u$ the second one to have
\[
\begin{aligned}
\frac{d}{dt} \log(\bar{u}) &= [-\gamma'(u)](\bar{u} - u) + \gamma''(u)a(t) + \mu(1 - \bar{u}), \\
\frac{d}{dt} \log(u) &= [-\gamma'(u)](u - \bar{u}) + \mu(1 - u), \\
\bar{u}(0) &= \bar{u}_0, \quad u(0) = u_0.
\end{aligned}
\]

**Lemma 3.1** For any $T < T_{\text{max}}$, there exist $C^1(0, T)$ functions $\bar{u}, u$ such that $(\bar{u}, u)$ is the unique solution to (3.1) in $(0, T)$.

**Proof** Since $\gamma \in C^3$ and $a(t) \in C^0(0, T)$ (for $T < T_{\text{max}}$), then, thanks to Peano’s theorem there exists a local solution to (3.2). Moreover since the right-hand-side part of the system is locally Lipschitz in $\bar{u}$ and $u$ we deduce the uniqueness of solutions. We also may extend the solution to a maximal interval of existence given by maximum interval where $a(t)$ is bounded and continuous. Since $a(t)$ is a continuous function in $[0, T_{\text{max}})$ (for $T_{\text{max}}$ defined in Remark 2.1), we obtain the wished regularity and conclude the proof. \[\square\]

**Lemma 3.2** Let $\bar{u}$ and $u$ the solutions to (3.1) in $(0, T_{\text{max}})$, such that
\[
0 < u_0 < 1 < \bar{u}_0, \quad \text{for any } t < T_{\text{max}}
\]
then
\[
0 < u < 1 < \bar{u}.
\]

\[\square\]
\textbf{Proof} We argue by contradiction and assume that
\begin{equation}
\text{there exists } t_0 \in (0, T_{\text{max}}), \text{ such that (3.3) is satisfied for any } t < t_0
\end{equation}
and
(i) \( u(t_0) = 1 \) and \( \overline{u}(t_0) > 1 \);
(ii) \( u(t_0) < 1 \) and \( \overline{u}(t_0) = 1 \);
(iii) \( u(t_0) = 1 \) and \( \overline{u}(t_0) = 1 \);
or either
(iv) \( u(t_0) = 0 \) and \( \overline{u}(t_0) \geq 1 \).

In case (i), we have that, by substituting in (3.1),
\[ u'(t_0) < 0 \]
which contradicts (3.4) because at \( t = t_0 \), \( u \) gets its maximum of \((0, t_0] \). In the same fashion we see that (ii) is not possible. In case (iii), we substruct equations in (3.2) to get
\begin{align*}
\frac{d}{dt} \left( \log(\overline{u}) - \log(u) \right) &= 2 \left[ -\gamma'(u) (\overline{u} - u) + \gamma''(u) a(t) - \mu(\overline{u} - u) \right] \\
&= [-2\gamma'(u) - \mu](\overline{u} - u) + \gamma''(u) a(t) \\
&= [-2\gamma'(u) - \mu] \xi (\log(\overline{u}) - \ln(u)) + \gamma''(\overline{u}) a(t)
\end{align*}
for some \( \xi \in (u, \overline{u}) \). After integration over \((0, t)\) we have that
\[ e^{-\int_0^t [-2\gamma'(u) - \mu] \xi d\tau} \left( \log(\overline{u}) - \log(u) \right) = \log(\overline{u}_0) - \log(u_0) \\
+ \int_0^t e^{-\int_0^\tau [-2\gamma'(u) - \mu] \xi ds} \gamma''(u) a(\tau) d\tau.
\]
We take \( t = t_0 < T_{\text{max}} \) to obtain
\[ \log(\overline{u}(t_0)) - \log(u(t_0)) > 0 \]
which contradicts (iii). To see that (iv) is not possible, we just notice that \( u = 0 \) in \([0, t_0]\) is the backward solution to (3.1), thanks to uniqueness of solutions, we have that \( u(0) = 0 \) which contradicts that \( u_0 > 0 \) and the proof ends. \( \square \)

\section{4 Comparison principle ODEs system / PDEs system}

In this section we compare the solution of system (1.1) to \( u, \overline{u} \), the solution to system (3.1). We notice that to prove that \( u \) and \( v \) are bounded by \( \overline{u} \) is equivalent to obtain the non-positivity of the functions \( u - \overline{u} \) and \( v - \overline{v} \). In the same way we have to see that \( u - \overline{u} \) and \( v - \overline{v} \) are non-negative functions. To obtain such result we introduce the following functions
\begin{align*}
\overline{U} &= u - \overline{u}, & \underline{U} &= u - u, \\
\overline{V} &= v - \overline{v}, & \underline{V} &= v - v.
\end{align*}

\textbf{Lemma 4.1} \textbf{For any } \( t < T_{\text{max}} \text{ we have that} \)
\[ \underline{U} \leq 0, \quad \overline{U} \geq 0. \]

\textbf{Proof} We first consider the differential equations satisfied by \( \overline{U} \) and \( \underline{U} \). Since \( u \) fulfills
\[ u_t - d v(\gamma(v) \nabla u) = \gamma'(v) v \nabla v \nabla u + u v''(v) \nabla v^2 + u \left[ -\gamma'(v) \right] (u - v) + \mu u (1 - u), \]
\( \square \) Springer
\( U \) satisfies

\[
\begin{align*}
\frac{\partial}{\partial t} U(t) - \nabla (\gamma'(v) \nabla U) & = \gamma'(v) \nabla v \nabla U + (\bar{U} + \bar{u}) \gamma''(v) |\nabla v|^2 - \bar{u} \gamma''(u) a(t) \\
& + (\bar{U} + \bar{u}) [-\gamma'(v)](u - v) - \bar{u} [-\gamma'(u)](\bar{u} - u) + \mu(1 - u) - \mu\bar{u}(1 - \bar{u}) \\
& = \gamma'(v) \nabla v \nabla U + \bar{U} \gamma''(v) |\nabla v|^2 + \bar{u} \gamma''(v) |\nabla v|^2 - \gamma''(u) a(t) \\
& + \bar{U} [-\gamma'(v)](u - v) + \bar{u} \left\{ [-\gamma'(v)](u - v) - [-\gamma'(u)](\bar{u} - u) \right\} + \mu \bar{U} - \mu \bar{U} (u + \bar{u}) \\
& = \gamma'(v) \nabla v \nabla U + \bar{U} \gamma''(v) |\nabla v|^2 + \bar{u} \gamma''(v) (|\nabla v|^2 - a(t)) - [\gamma''(u) - \gamma''(v)] a(t) \\
& + \bar{U} [-\gamma'(v)](u - v) + \bar{u} \left\{ [-\gamma'(v)](u - v) - [-\gamma'(u)](\bar{u} - u) \right\} + \mu \bar{U} - \mu \bar{U} (u + \bar{u}).
\end{align*}
\]

In view of definition of \( a(t) \), we have that

\[ |\nabla v|^2 - a(t) \leq 0 \quad \text{a.e.} \]

and as a consequence of

\[ \bar{u} [-\gamma'(v)](u - v) = \bar{u} [-\gamma'(v)](\bar{U} - V) + \bar{u} [-\gamma'(v)](\bar{u} - u) \]

we get

\[
\begin{align*}
\frac{\partial}{\partial t} U(t) - \nabla (\gamma'(v) \nabla U) & \leq \gamma'(v) \nabla v \nabla U + \bar{U} \gamma''(v) |\nabla v|^2 - \bar{u} [\gamma''(u) - \gamma''(v)] a(t) \\
& + \bar{U} [-\gamma'(v)](u - v) \\
& + \bar{u} \left\{ [-\gamma'(v)](\bar{U} - V) - \gamma'(v)(\bar{u} - u) - [-\gamma'(u)](\bar{u} - u) \right\} \\
& + \mu \bar{U} - \mu \bar{U} (u + \bar{u}).
\end{align*}
\]

Since

\[
\begin{align*}
\gamma''(u) - \gamma''(v) & = -\gamma'''(\xi_1) V, \\
\gamma'(u) - \gamma'(v) & = -\gamma''(\xi_2) V
\end{align*}
\]

and

\[
0 \leq \bar{u} - u
\]

we have that

\[
\begin{align*}
\frac{\partial}{\partial t} U(t) - \nabla (\gamma'(v) \nabla U) & \leq \gamma'(v) \nabla v \nabla U + \bar{U} \gamma''(v) |\nabla v|^2 - \bar{u} [\gamma''(\xi_1)] a(t) V \\
& + \bar{U} [-\gamma'(v)](u - v) + \bar{u} \left\{ [-\gamma'(v)](\bar{U} - V) - \bar{u} \gamma''(\xi_2) V (\bar{u} - u) + \mu \bar{U} - \mu \bar{U} (u + \bar{u}).
\end{align*}
\]

We now multiply the previous equation by \( U_+ \) and integrate by parts over \( \Omega \) to obtain after some computations

\[
\frac{d}{dt} \int_\Omega \frac{1}{2} |U_+|^2 dx + e \int_\Omega |\nabla U_+|^2 dx \leq k \int_\Omega (1 + u) |U_+|^2 dx \\
- \int_\Omega \left\{ -\gamma'(v) + \gamma''(\xi_2)(\bar{u} - u) - \gamma'''(\xi_1) a(t) \right\} U_+ V dx
\]

thanks to assumption (1.3) we have that

\[
\int_\Omega \left\{ -\gamma'(v) + \gamma''(\xi_2)(\bar{u} - u) - \gamma'''(\xi_1) a(t) \right\} U_+ V dx \geq 0
\]

therefore

\[
- \int_\Omega \left\{ -\gamma'(v) + \gamma''(\xi_2)(\bar{u} - u) - \gamma'''(\xi_1) a(t) \right\} U_+ V dx \\
\leq \int_\Omega |U_+|^2 dx + k(t) \int_\Omega (-V_+)^2 dx.
\]
We notice that
\[
\int_{\Omega} u|\bar{U}_+|^2 dx \leq \|u\|_{L^p(\Omega)} \|\bar{U}_+\|_{L^{2p'}(\Omega)}^2
\]
for some \( p > N \) we have that \( 2p' = \frac{2p}{p-1} < \frac{2N}{N-2} \) therefore thanks to Gagliardo Nirenberg inequality we have that
\[
\|\bar{U}_+\|^2_{L^{2p'}(\Omega)} \leq c\|\bar{U}_+\|_{H^1(\Omega)}^{2a} \|\bar{U}_+\|_{L^2(\Omega)}^{2(1-a)} + c\|\bar{U}_+\|_{L^2(\Omega)}^2
\]
for \( a \) satisfying
\[
\frac{1}{2p'} = \left(\frac{1}{2} - \frac{1}{N}\right) a + \frac{(1-a)}{2}
\]
i.e.
\[
a = \frac{N(p'-1)}{2p'} = \frac{N}{2p} < \frac{1}{2}.
\]
Thanks to Young inequality and Lemma 2.4 we get
\[
\frac{d}{dt} \frac{1}{2} \int_{\Omega} |\bar{U}_+|^2 dx \leq k \int_{\Omega} |\bar{U}_+|^2 dx + k \int_{\Omega} |(-\nabla U)|^2 dx \tag{4.1}
\]
In the same fashion, in view of \( u\gamma''(v)|\nabla v|^2 \geq 0 \), the following inequality is satisfied by \( U \)
\[
U_t - div(\gamma'(v)\nabla U) \geq \gamma'(v)\nabla v \nabla U + U[-\gamma'(v)](u-v) + u \{[-\gamma'(v)](u-v)\} - u[\gamma'(v)](u-\mu) + \mu U - \mu U(u+\mu).
\]
Since
\[
u \{[-\gamma'(v)](u-v)\} = u \{[-\gamma'(v)](u-\mu)\} + u \{[-\gamma'(v)](u-v)\}
\]
we have that
\[
u \{[-\gamma'(v)](u-v)\} = -u[-\gamma'(v)](u-\mu)\]
\[
u \{[-\gamma'(v)](u-v)\} = u \{[-\gamma'(v)](u-\mu)\} - u(u-\mu) \{[-\gamma'(v)](u-\mu)\} + u \{[-\gamma'(v)](u-\mu)\}
\]
for some \( \xi \in (u, v) \) if \( u < v \) and \( \xi \in [v, u] \) otherwise. Then, it results
\[
U_t - div(\gamma'(v)\nabla U) \geq \gamma'(v)\nabla v \nabla U + U[-\gamma'(v)](u-v) + u \{[-\gamma'(v)](u-\mu)\} - u(u-\mu) \{\gamma''(\xi)\} + \mu U - \mu U(u+\mu).
\]
We take \(-[\bar{U}]_+\) as test function in the weak formulation to get, after some computations
\[
\frac{d}{dt} \frac{1}{2} \int_{\Omega} [-\bar{U}]_+^2 dx \leq k \int_{\Omega} [-\bar{U}]_+^2 dx + \int_{\Omega} u[-\gamma'(v)][-\bar{U}]_+ \nabla \bar{U} dx
\]
\[
+ \int_{\Omega} u(u-\mu) \gamma''(\xi) \nabla \bar{U}_+ dx
\]
where the term \( \int_{\Omega} u[-\bar{U}]_+^2 dx \) is treated as before, using Gagliardo Nirenberg inequality. Since
\[
\int_{\Omega} u[-\gamma'(v)][-\bar{U}]_+ \nabla \bar{U} dx \leq \int_{\Omega} u[-\gamma'(v)][-\bar{U}]_+ \nabla \bar{U} dx \leq c \int_{\Omega} [-\bar{U}]_+^2 dx + c \int_{\Omega} \nabla \bar{U}_+^2 dx
\]
and
\[ \int_{\Omega} u(u - \bar{u})\gamma''(\xi_3) V[-V]_+ dx = \int_{\Omega} u(\bar{u} - u)\gamma''(\xi_3)[-V]_+[-U]_+ dx \leq c \int_{\Omega} [-U]_+^2 dx + c \int_{\Omega} [-V]_+^2 dx \]
it results
\[ \frac{d}{dt} \frac{1}{2} \int_{\Omega} [-U]_+^2 dx \leq c \int_{\Omega} U_+^2 dx + c \int_{\Omega} V_+^2 dx + c \int_{\Omega} [-V]_+^2 dx. \] (4.2)
In the same way we have that
\[ -\Delta V + V = U \] (4.3)
we multiply by \([-V]_+\) and integrate by parts to get, after some computations and thanks to Young Inequality
\[ \int_{\Omega} [-V]_+^2 dx \leq \int_{\Omega} [-U]_+^2 dx. \] (4.4)
In the same fashion we obtain
\[ \int_{\Omega} V_+^2 dx \leq \int_{\Omega} U_+^2 dx. \] (4.5)
Thanks to (4.1)–(4.5) we have
\[ \frac{d}{dt} \left( \int_{\Omega} U_+^2 dx + \int_{\Omega} [-U]_+^2 dx \right) \leq c \left( \int_{\Omega} U_+^2 dx + \int_{\Omega} [-U]_+^2 dx \right). \]
Gronwall’s Lemma ends the proof.

5 Global existence of solutions and asymptotic behaviour

Thanks to Lemma 4.1, we have that
\[ u \leq u \leq \bar{u}. \] (5.1)

Lemma 5.1 Let \((u, v)\) be the weak solution of (1.1) and \((u, \bar{u})\) the unique solution to (2.1), then, for any \(t < T_{\text{max}}\) we have that
\[ \|\nabla v\|_{L^\infty(\Omega)} \leq c_\Omega |\bar{u} - u|. \]

Proof As a consequence of Lemma 4.1 we have that \(u \in L^\infty(0, T : L^\infty(\Omega))\), for any \(T < T_{\text{max}}\). Then, from Eq. (4.3) we deduce
\[ V \in W^{2,q}(\Omega) \]
for any \(t < T_{\text{max}}\) and \(q < \infty\). Therefore, the constant \(c_\Omega\) defined in Definition 1.2 provided us
\[ \|\nabla V\|_{L^\infty(\Omega)} \leq c_\Omega \|U\|_{L^\infty(\Omega)} \]
in view of Lemma 4.1 we have that
\[ \|U\|_{L^\infty(\Omega)} \leq \bar{u} - u \]
which ends the proof. \(\square\)
Lemma 5.2 Let \((u, \overline{u})\) the solution to (3.1), then, the solution remains uniformly bounded for any \(t < T_{\text{max}}\).

Proof We subtract in (3.2) the second equation to the first to obtain
\[
\frac{d}{dt} \left[ \log(\overline{u}) - \log(u) \right] = (2[-\gamma'(u)] - \mu)(\overline{u} - u) + \gamma''(u)a(t).
\]
Lemma 5.2 shows
\[
\frac{d}{dt} \left[ \log(\overline{u}) - \log(u) \right] \leq (2[-\gamma'(u)] - \mu + \gamma''(u))(\overline{u} - u),
\]
and assumption (1.4) implies
\[
\frac{d}{dt} \left[ \log(\overline{u}) - \log(u) \right] \leq -\mu_0(\overline{u} - u) \leq 0.
\]
After integration in the previous equation, we get
\[
\log(\overline{u}) - \log(u) \leq \log(\overline{u}_0) - \log(u_0) \leq c.
\]
Which shows, in view of
\[
0 < \overline{u} < 1 < u,
\]
the upper boundedness of \(u\) for any \(t \leq T_{\text{max}}\). \(\square\)

Lemma 5.3 Let \((u, v)\) the solution to (1.1) in \((0, T_{\text{max}})\), then \(T_{\text{max}} = \infty\).

Proof Remark 2.1 and Lemmata 4.1 and 5.2 prove the lemma. \(\square\)

Lemma 5.4 Let \((u, \overline{u})\) be the unique solution to (2.1), then, the solution satisfies
\[
|\overline{u} - u| \to 0 \quad \text{as} \quad t \to \infty.
\]

Proof In view of Lemma 5.1, we have that \(c_{\Omega}|u - \overline{u}|^2\) is an upper bound of \(a(t)\), therefore \(\overline{u}\) and \(u\) satisfy
\[
\begin{aligned}
\frac{d}{dt} \log(\overline{u}) &\leq [-\gamma'(u)](\overline{u} - u) + \gamma''(u)c_{\Omega}|\overline{u} - u|^2 + \mu(1 - \overline{u}), \quad t > 0 \\
\frac{d}{dt} \log(u) &\leq [-\gamma'(u)](u - \overline{u}) + \mu(1 - u), \quad t > 0 \\
\overline{u}(0) &= \overline{u}_0,
\end{aligned}
\]
we subtract both equations to get
\[
\frac{d}{dt} \left( \log(\overline{u}) - \log(u) \right) \leq 2[-\gamma'(u)](\overline{u} - u) + \gamma''(u)c_{\Omega}|\overline{u} - u|^2 - \mu(\overline{u} - u) \\
&= [-2\gamma'(u) - \mu](\overline{u} - u) + \gamma''(u)c_{\Omega}|\overline{u} - u|^2 \\
&\leq [-2\gamma'(u) - \mu + \gamma''(\overline{u})c_{\Omega}u](\overline{u} - u).
\]
Thanks to assumption (1.4) it results
\[
\frac{d}{dt} \left( \log(\overline{u}) - \log(u) \right) \leq (\mu_0 - \mu)(\overline{u} - u).
\]
After integration over \((0, t)\), in view of non-negativity of the righthand side term, it results
\[
\log(\overline{u}) - \log(u) \leq \log(\overline{u}_0) - \log(u_0).
\]
Thanks to assumption (1.6) we have
\[ \bar{u} \leq \frac{\bar{u}_0}{u_0} u \]
which implies
\[ 0 < \frac{u_0}{\bar{u}_0} \leq u. \] (5.2)

We apply Mean Value Theorem to get the inequality
\[ (\mu_0 - \mu)(\bar{u} - u) \leq (\mu_0 - \mu)\xi (\ln(\bar{u}) - \ln(u)) \]
where \( \xi \in (u, \bar{u}) \) i.e.
\[ (\mu_0 - \mu)(\bar{u} - u) \leq (\mu_0 - \mu)\frac{u_0}{\bar{u}_0} (\ln(\bar{u}) - \ln(u)) \]
which implies
\[ \frac{d}{dt} (\log(\bar{u}) - \log(u)) \leq (\mu_0 - \mu)\frac{u_0}{\bar{u}_0} (\ln(\bar{u}) - \ln(u)) \]

We solve the previous differential inequality and it results
\[ \log(\bar{u}) - \log(u) \leq [\log(\bar{u}_0) - \log(u_0)] e^{(\mu_0-\mu)\frac{T_0}{\bar{u}_0} t} \]
which implies, in view of (5.2) that \( u \) and \( \bar{u} \) are uniformly bounded in time and therefore \( T_0 = \infty \). We take exponentials in the previous inequality
\[ \frac{\bar{u}}{u} \leq \exp\left\{ e^{(\mu_0-\mu)\frac{T_0}{\bar{u}_0} t} \right\} \]
which proves
\[ \bar{u} - u \leq u(\exp\{e^{(\mu_0-\mu)\frac{T_0}{\bar{u}_0} t}\} - 1) \leq (\exp\{e^{(\mu_0-\mu)\frac{T_0}{\bar{u}_0} t}\} - 1) \]

Taking limits when \( t \to \infty \) we obtain the wished result. \( \square \)

**End of the proof of theorem 1.1**

Theorem 2.1 proves the local existence of solutions. Lemma 4.1 proves that the solution is bounded by the auxiliary functions \( \bar{u} \) and \( u \). Lemma 5.4 gives the existence of global in time solutions as a consequence of the global existence of the upper and lower functions. The asymptotic behavior is obtained in view of
\[ \|u - 1\|_{L^\infty(\Omega)} \leq |\bar{u} - 1| + |u - 1|, \]
\[ \|v - 1\|_{L^\infty(\Omega)} \leq |\bar{u} - 1| + |u - 1| \]

Lemmas 3.2 and 5.4.

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