INDIVIDUAL AND AVERAGE BEHAVIOR OF PARTICLES IN A PROTOPLANETARY NEBULA

P. Garaud
Institute of Astronomy, University of Cambridge, Madingley Road, Cambridge CB3 0HA, UK

L. Barrière-Fouchet
Centre de Recherche Astronomique de Lyon, CNRS-UMR 5574, ENS Lyon, 46 allée d’Italie, F-69364 Lyon Cedex 07, France

AND

D. N. C. Lin
University of California Observatories, Lick Observatory, University of California, 1156 High Street, Santa Cruz, CA 95064

Received 2003 July 9; accepted 2003 November 14

ABSTRACT

We study the interaction between gas and particles in a protostellar disk, using both analytical and numerical approaches. We first present analytical expressions for the trajectories of individual particles undergoing gas drag in the disk, in the asymptotic cases of very small particles (Epstein regime) and very large particles (Stokes regime). Using a Boltzmann averaging method, we obtain an analytic expression for the evolution of the average density, velocity, and dispersion of the particles as a function of distance above the midplane of the disk. Using successive moments of the Boltzmann equation, we derive the equivalent fluid equations for the average motion of the particles; these are intrinsically different in the Epstein and Stokes regimes. A simple closure of the moment equations is proposed in both regimes. These fluid equations provide much better prospects for the study of more complex problems related to protoplanetary accretion disks, since for general initial size and phase-space distributions the evolution of the average behavior of the particles can be evaluated numerically with much less computational time than that required for the numerical integration of the orbits of all individual particles. In a companion paper, for instance, we use them for the analysis of a shearing instability induced by the sedimentation of the particles. In the present work we test the adequacy of the fluid formulation against a set of idealized numerical experiments. In the Epstein regime, we study an idealized uniform initial distribution of small particles. We obtain a set of analytic solutions for the fluid equations, which are found to be in good agreement with those obtained from numerical integration of the orbits of many particles. We also verify that any initial velocity dispersion is quickly damped out by the surrounding gas on the short stopping timescale, which provides closure and justifies the description of the particles as a fluid with a linear drag force and negligible pressure. In the Stokes regime, as the large particles oscillate across the midplane with declining amplitude, their velocity dispersion remains comparable to their average speed. Their sedimentation is analogous to the cooling of a pressure-supported fluid. We propose an empirical closure scheme for the moment equations of the Stokes particles fluid and test it against idealized numerical experiments. In both cases, this method can eventually be applied to study the evolution of particle distributions in protostellar disks after additional effects such as collision, sublimation, and condensation are included.

Subject headings: methods: analytical — methods: numerical — planetary systems: protoplanetary disks

1. INTRODUCTION

In an attempt to account for the coplanar nature of the orbits of all known solar system planets, Laplace (1796) postulated that they were formed in a common disk around the proto-Sun. Today, the detection of protostellar disks around most young T Tauri stars (Prosser et al. 1994) provides strong evidence that the Laplace nebula hypothesis is universally applicable. The recent discovery of planets around at least 10% of nearby solar-type stars (Marcy, Cochran, & Mayor 2000) suggests that their formation may be a robust process.

Conventional cosmogonical scenarios are based on the assumption that heavy elements in gas phase condensed to form grains, which then coagulated into planetesimals and grew into protoplanetary cores that can accrete, at least in some regions of the disk, massive gaseous envelopes around themselves (Pollack et al. 1996). The coexistence of gas and solid ice has been detected in some protostellar disks (Thi et al. 2002). In fact, protostellar disks are most conspicuous in their continuum radiation associated with the reprocessing of stellar light by the grains (Adams, Lada, & Shu 1987). The apparent wavelength dependence in the observed thickness of the disk dust layer has been interpreted as evidence of grain growth (Throop et al. 2001; D’Alessio, Calvet, & Hartmann 2001; Clarke, Gendrin, & Sotomayor 2001; McCabe, Duchene, & Ghez 2003) and settling (Shuping et al. 2003).

The micron-to-centimeter continuum radiation signatures of the dust are observed to fade on the timescale of a few megayears (Beckwith 1999; Haisch, Lada, & Lada 2001), signaling the depletion of grains in this size range. This evolution suggests that heavy elements initially contained in this size range are either evaporated, ejected to large distance, accreted onto the host stars, or have coagulated into larger particles. The first possibility is constrained by the concurrent decline in the CO gas (Zuckerman, Forveille, & Kastner 2001), whereas the last possibility is directly relevant to the process of planet formation.

Theoretical analysis suggests a very strong constraint on the growth of micron-size grains into kilometer-size planetesimals. Indeed, the orbital evolution of the particles is...
determined by both the gravity of the central star and the
drag of the disk gas. In the absence of turbulence, the disk
gas attains a dynamical equilibrium between gravity, pressure,
and centrifugal forces with zero velocity in both radial and
normal-to-the-disk directions and a slightly sub-Keplerian
velocity in the azimuthal direction. Particles in the disk un-
dergo both sedimentation toward the midplane and inward
drift in the radial direction (Whipple 1972; Weidenschilling
1977). In a minimum mass nebula (Hayashi, Nagazawa, &
Nagagawa 1985), the resulting orbital decay timescale at 1 AU
(for instance) is smallest for meter-size bodies (Adachi,
Nagagawa 1985), and is then less than about
10^2 yr. Unless the growth of planetesimals across this “most
vulnerable size” can occur faster than their orbital decay,
there would be no residual planetesimals left to provide the
building blocks of planets.

One possible channel of rapid grain growth is through
sedimentation into a sufficiently thin, gravitationally unstable
disk (Goldreich & Ward 1973). The critical thickness for
gravitational instability of such disks is less than ~10^{-5} of
their radii, and the characteristic size of the resulting fragment
is roughly a few kilometers. However, even a modest amount
of turbulence can provide adequate stirring to prevent the
sedimentation of grains into such a thin unstable layer
(Weidenschilling 1984; Supulver & Lin 2000). Although
turbulence is likely to occur in a magnetized disk (Balbus &
Hawley 1991) through magnetorotational instability, this
mechanism could well fail in regions of the disk where the
ionization fraction is too small. In these regions only, the
following alternative mechanism for turbulence has been
proposed.

Sedimentation toward the midplane of the disk leads to a
local concentration of massive particles; these particles en-
train the gas to a near-Keplerian velocity through drag,
thereby introducing a shear layer between the dust-dominated
midplane and the rest of the disk gas (Weidenschilling &
Cuzzi 1993). Such a flow pattern in the disk has the potential
to cause the onset of a shearing instability, a phenomenon
that was studied by Sekiya (1998) and Youdin & Shu (2002).
However, the stability analysis used by these authors is based
on a single-fluid approximation in which the particles are
assumed to be well coupled to the gas. Since the concen-
tration of the dust particles causes not only the shear but
also a stabilizing density stratification, the flow of dust and
gas should be treated separately. In a companion paper
(Garaud & Lin 2004) we carry out a two-component stability
analysis of the midplane dust layer. Such study is greatly
simplified by the treatment of the dust as a separate fluid
rather than a collection of particles. It is with this goal in
mind that we now present a system of averaged equations for
the evolution of a collection of dust particles in the form of
moments of the Boltzmann equation. This prescription could
also in principle be applied to studies of the evolution of
particles due to coagulation, sublimation, and condensation
(Supulver & Lin 2000) and under their interaction with
embedded planets (Wyatt et al. 1999) and stellar radiation
(Takeuchi & Artymowicz 2001; Klahr & Lin 2001; Takeuchi &
Lin 2002).

For the present calculation, we assume that the particles are
collisionless and indestructible spheres in a gaseous laminar
disk with no embedded planets and negligible stellar radiation.
In this paper we also neglect the feedback of the drag caused
by the particles on the motion of the gas. This issue is ad-
dressed in a following paper (Garaud & Lin 2004). In § 2 we
recall the general gas drag laws and their effects on particle
trajectories in a protoplanetary accretion disk. In §§ 3 and 4 we
solve for the orbits of individual particles and derive the
complementary set of dynamical equations for a collection
of particles in the form of low-order moments of the Boltzmann
equation (Boltzmann 1872). In § 3 we focus only on small
particles, which have sizes smaller than the mean free path of
the gas molecules, whereas in § 4 we develop analogous
equations for particles with sizes larger than the mean free
path of the gas. We obtain analytic solutions for the evolution
of the various moments of the distribution of particles under
the assumption of separation of the radial and normal-to-the-
disk variables, as well as a constant background gas density.
For more general initial and boundary conditions, the evolu-
tion of the average density, velocity, and velocity dispersion
distributions can be obtained numerically.

In § 5 we check the validity of our analytic descriptions
with a complete numerical calculation of the orbits of
thousands of particles in a simple-to-use model for the gas
nebula. We compare the results for both individual orbits and
the moments obtained from the Boltzmann averaging pro-
duce. This enables us to propose a simple closure model for
the moment equations that is valid for any set of physically
meaningful initial conditions, in both the Epstein and Stokes
regimes. We illustrate the use of this mathematical framework
by considering the spatial and temporal evolution of a given
size distribution of small particles in § 6 and show that par-
ticles with a narrowly selected size range sediment most
rapidly toward the midplane. Finally, in § 7 we summarize our
results and discuss their implications.

2. GAS DRAG LAWS AND PARTICLE MOTION

In this section we briefly recall the effect of gas drag and
gravitational forces on individual particle trajectories.

2.1. Drag Force Description

In particle number density regimes where collisions are
negligible, the trajectory of a particle is given by

\[ \ddot{r} = -\nabla \Phi - \frac{1}{m_p} F_d, \]

where \( q' \) denote derivatives of \( q \) with respect to time, \( r \) is the
position vector of the particle, \( \Phi \) is the externally imposed
gravitational potential, \( m_p \) is the mass of the particle, and \( F_d \)
is the drag force between the particle and the gas. The amplitude
of the drag force depends on the radius of the particle \( s \)
compared to the mean free path of the gas \( \lambda \), and one typically
distinguishes two regimes, \( s \ll \lambda \) (Epstein regime) and \( s \gg \lambda \)
(Stokes regime).

In the small-particle limit (\( s \ll \lambda \)), i.e., the Epstein regime,
the drag is caused by the thermal agitation of the gas and is
proportional to the velocity of the particle relative to that of the
gas:

\[ F_d = \frac{m_p}{\rho_s} \frac{c}{\rho} \frac{\dot{r} - \mathbf{v}_g}{s} \quad \text{when } s \ll \lambda, \]

where \( \mathbf{v}_g \) is the gas velocity, \( \rho \) is the local density of the gas,
\( c \) is the local sound speed, and \( \rho_s \) is the solid (i.e., internal)
density of the particle. In a standard solar nebula, this for-
mula is typically valid up to decimeter-sized particles at 1 AU.
In the opposite limit \((s \gg \lambda)\), large particles see the gas as a fluid and experience a drag force through the laminar or turbulent wake that they create as they move through the gas. This is the Stokes regime. Whipple (1972) reported that the drag force on a sphere is

\[
F_d = m_p \frac{\rho}{\rho_s} \frac{C(\text{Re})}{s} |\Delta \mathbf{v}| (\mathbf{r} - \mathbf{v}_g) \quad \text{when } s \gg \lambda,
\]
where \(|\Delta \mathbf{v}|\) is the norm of \(\mathbf{r} - \mathbf{v}_g\) and \(\text{Re}\) is the particle Reynolds number of the flow \((\text{Re} = 2s|\Delta \mathbf{v}|/\nu, \text{where } \nu = \lambda \rho / 3\) is the molecular viscosity of the gas). Experimental results suggest that the drag coefficient \(C(\text{Re})\) varies as

\[
C = \begin{cases} 
9\text{Re}^{-1} & \text{for } \text{Re} \leq 1, \\
9\text{Re}^{-0.6} & \text{for } 1 \leq \text{Re} \leq 800, \\
0.165 & \text{for } 800 < \text{Re}.
\end{cases}
\]

Note that the definition of the aerodynamic drag coefficient given here differs by a factor of \(\frac{1}{2}\) from the one used by Whipple (1972), owing to a slightly different expression for the actual amplitude of the drag force. This prescription ensures a smooth transition between the Epstein regime and the Stokes regime for intermediate particle sizes \((s \approx \lambda)\). Note that in the Stokes regime \((\text{Re} \rightarrow \infty)\), the amplitude of the drag force depends on the total velocity of the particle with respect to the gas, and not simply its component parallel to the direction considered. This expresses the fact that a strong wake caused by the motion of the particle in one direction also affects even the slightest motion in another perpendicular direction.

### 2.2. Separation of Vertical and Radial Motions

As particles undergo gas drag, they lose angular momentum, which causes them to drift slowly inward. For particles strongly coupled to the gas (i.e., very small particles), this drift is slowed down by the indirect support provided by the gas pressure, via the drag exerted by the gas on the particles. For weakly coupled particles (i.e., very large particles), the angular momentum loss is very small and the drift is equally slow (see the next section). There exists an intermediate regime, however, where orbital decay can occur very rapidly. Weidenschilling (1977) carried out a first quantitative study of the effect of the gas drag on particles of various sizes and determined the inward drift velocity as a function of particle size. He found that the maximal drift occurs for particles for which the typical stopping timescale \(t_s = |r - v_g|/[F_d]\) is of the order of the orbital timescale \(\Omega_k^{-1}\) and decays very quickly for particles much larger or much smaller than that. As a result, unless the particle is of intermediate size, there will be a marked separation between the dynamical timescale and the orbital decay timescale.

Moreover, there is also a marked separation between radial and vertical length scales; indeed, numerical integration of the orbits of particles in a gaseous disk shows that the typical radial drift velocity is of the same order (in the case of small particles) or much smaller (in the case of large particles) than the vertical settling velocity (see § 5.2). Hence, in a thin disk the particles settle to the midplane with very little radial excursion and then undergo radial decay within a very thin dust layer. For this reason, we suggest that it is justified to assume (at least for the purpose of analytical work) that the vertical motion of particles is more or less independent of their radial motion (although not of their radial position) and that their radial motion occurs mostly at \(z = 0\). Since Weidenschilling (1977) has already derived an analytical description of the particle motion in the radial direction, we shall concentrate on the problem of vertical settling. A short derivation of the case of radial and azimuthal motion of the particles in a gas disk is presented in the Appendix for completeness. In the numerical calculations presented in § 5, we check the assumption of separability of the variables and timescales.

### 3. The Collective Evolution of Small Particles

#### 3.1. Vertical Motion of Small Particles in the Epstein Regime

From the previous sections we infer the vertical equation of motion of a small dust particle in the Epstein regime:

\[
z'' = -\frac{\partial \Phi}{\partial z} - \frac{\rho_g}{\rho_s} z',
\]
where \(z\) is the height above the disk. In a central potential, the gravitational force close to the midplane can be rewritten as \(-\Omega_k^{-2}(r)z\), where \(\Omega_k(r)\) is the Keplerian angular velocity at the radius considered. We renormalize the time variable to the orbital timescale \(t_o = \Omega_k^{-1}\) at the radial position considered and the distances to the orbital radius \(r\); we deduce the nondimensional equation

\[
\tilde{z} = -z - \mu \tilde{z},
\]
where \(\mu = \Omega_k^{-2}(r)\rho(r)/\rho_s [c(r)/s]\). In terms of this normalization, the magnitude of \(\mu\) in this regime is typically much larger than unity, which means that the stopping time \(1/\mu\) is much smaller than the orbital time. The solution to equation (6) is straightforward, and the trajectory of a particle initially positioned at height \(z_p\) with initial velocity \(w_i\) is

\[
z_p(\mu, t; z_i, w_i) = \frac{1}{5} \left[ \left( z_i + \frac{2w_i + \mu z_i}{\sqrt{\mu^2 - 4}} \right) \exp \left( -\mu + \frac{\sqrt{\mu^2 - 4}}{2} t \right) 
+ \left( z_i - \frac{2w_i + \mu z_i}{\sqrt{\mu^2 - 4}} \right) \exp \left( -\mu - \frac{\sqrt{\mu^2 - 4}}{2} t \right) \right].
\]

Note that for \(\mu \gg 1\) this expression simplifies to

\[
z_p(\mu, t; z_i, w_i) = \left( z_i + \frac{w_i}{\mu} \right) e^{-t/\mu} - \frac{w_i}{\mu} e^{-\mu t}.
\]

The second term of equation (8) describes the rapid deceleration of the particle by the gas drag, while the first represents a slow settling toward the midplane. Let us recast equation (8) into

\[
z_p(\mu, t; z_i, w_i) = \alpha(\mu, t) z_i + \beta(\mu, t) w_i,
\]
which defines the functions \(\alpha\) and \(\beta\) uniquely. The instantaneous vertical velocity of dust particles is then given by

\[
w_p(\mu, t; z_i, w_i) = \dot{z}_p = \dot{\alpha} z_i + \dot{\beta} w_i.
\]
Note that for \(\mu \approx 1\) (which describes the intermediate parameter range between the Epstein regime and the Stokes
regime) the motion of the particle contains an oscillatory component decaying exponentially on timescale $\mu/2$. The transition between the two trajectory regimes (i.e., the point at which critical damping occurs) has been studied by Weidenschilling (1980) and Nakagawa et al. (1986).

3.2. Boltzmann Description and Continuum Equations

Given a set of particles at initial positions $z_i$ and with initial velocities $w_i$, the Boltzmann distribution function of these particles is

$$f(z, w, t) = \sum_i m_i \delta(z - z_p(\mu, t; z_i, w_i)) \delta(w - w_p(\mu, t; z_i, w_i)).$$

(11)

One can also take the continuum limit of this description in the case of a very large number of particles:

$$f(z, w, t) = \int dz_i \int dw_i \rho_i(z_i) g(w_i, z_i) \delta(z - z_p(\mu, t; z_i, w_i)) \times \delta(w - w_p(\mu, t; z_i, w_i)),$$

(12)

where $\rho_i$ is the initial particle density distribution and $g$ is the initial velocity distribution of particles. The freedom in the relative choices of the initial distribution functions $\rho_i$ and $g$ is lifted by requiring that for all $z_i$

$$\int dw_i g(w_i, z_i) = 1.$$  

(13)

The mass density of the particles is obtained by integrating $f$ over all possible velocities:

$$\rho_p(z, t) = \int dw f(z, w, t)$$

$$= \int dz_i \int dw_i \rho_i(z_i) g(w_i, z_i) \delta(z - z_p(\mu, t; z_i, w_i)).$$

(14)

Substitution of the equations for the particle trajectories into equation (14) and integration over all possible initial positions yield

$$\rho_p(z, t) = \frac{1}{\alpha} \int dw_i \rho_i \left(\frac{z - \beta w_i}{\alpha}\right) g \left(w_i, \frac{z - \beta w_i}{\alpha}\right).$$

(15)

The first moment of the Boltzmann function is the average vertical velocity:

$$\rho_p(z, t) \bar{w} = \int dw \, w f(z, w, t)$$

$$= \int dz_i \int dw_i \rho_i(z_i) g(w_i, z_i)$$

$$\times w_p \delta(z - z_p(\mu, t; z_i, w_i)).$$

(16)

The same manipulations yield

$$\rho_p \bar{w} = \int \frac{dw_i}{\alpha} \rho_i \left(\frac{z - \beta w_i}{\alpha}\right)$$

$$\times g \left(w_i, \frac{z - \beta w_i}{\alpha}\right) \left(\frac{\alpha z - \beta w_i}{\alpha} + \beta w_i\right).$$

(17)

Let us assume for a short while that the distribution of the initial velocities is independent of height above the midplane. Then we can verify easily that the quantities determined in equations (15) and (17) satisfy the standard continuity equation

$$\frac{\partial \rho_p}{\partial t} + \frac{\partial}{\partial z} (\rho_p \bar{w}) = 0,$$

(18)

regardless of the form of the “trajectory functions” $\alpha$ and $\beta$.

However, the original Boltzmann equation (Boltzmann 1872) describes the evolution of the distribution function $f$ in phase space through

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial z}(\rho_p \bar{w}) + \frac{\partial}{\partial w} \frac{\partial f}{\partial w} = \Gamma,$$

(19)

where $\Gamma$ is a collision term that reproduces the interaction of the particles with each other and with the surrounding medium. If we integrate the Boltzmann equation with respect to the velocity space, we get

$$\frac{\partial \rho_p}{\partial t} + \frac{\partial}{\partial z} (\rho_p \bar{w}) + \int dw \frac{\partial f}{\partial w} = \int \Gamma \, dw.$$

(20)

Substitution of the equation of motion given by equation (6) into the last term of the left-hand side of equation (20) provides an expression for the interaction term $\Gamma$ as a condition for the mass continuity equation to be satisfied:

$$\Gamma = \mu f.$$  

(21)

This result can be generalized in the case where $g$ depends on $z_i$.

Subtracting the mass continuity equation from the first moment of the Boltzmann equation reveals the importance of the particle velocity correlations $\bar{w}^2$:

$$\rho_p \frac{\partial \bar{w}}{\partial t} + \rho_p \bar{w} \frac{\partial \bar{w}}{\partial z} = -\rho_p \frac{\partial \Phi}{\partial z} - \mu \rho_p \bar{w} - \frac{\partial}{\partial z} (\rho_p \sigma^2),$$

(22)

where $\sigma^2 = \bar{w}^2 - \bar{w}^2$ represents the velocity dispersion function of the particles. This equation is very similar to that of a standard fluid, with the following caveats: (1) it contains an explicit drag term in the form of $-\mu \bar{w}$; and (2) the very last term, which usually represents the pressure term in a standard fluid, must be written explicitly as a function of known quantities. We expect this term to be null in the case of collisionless particles with negligible relative velocity with respect to the gas; in order to double-check this conjecture, we now evaluate $\sigma^2$ explicitly.

Let us first evaluate $\bar{w}^2$:

$$\rho_p \bar{w}^2 = \int \frac{dw_i}{\alpha} \rho_i \left(\frac{z - \beta w_i}{\alpha}\right)$$

$$\times g \left(w_i, \frac{z - \beta w_i}{\alpha}\right) \left(\frac{\alpha z - \beta w_i}{\alpha} + \beta w_i\right)^2.$$  

(23)

This term is related to thermal elastic collisions of the gas particles in a standard fluid and tends to depend only on the local density and temperature of the fluid.
The behavior of $\sigma^2$ at short times (i.e., for times shorter than the stopping time $t_s = 1/\mu$) is difficult to extract analytically while keeping the functions $\rho_s$ and $g$ unsupervised. However, for large $\mu$ and large times one can simplify the expressions for the functions $\alpha$ and $\beta$ to

$$
\alpha = \exp\left(-t/\mu\right) + O(\exp(-\mu t)),
$$
$$
\beta = \frac{1}{\mu}\left[\exp\left(-t/\mu\right) - \exp(-\mu t)\right].
$$

In that case, the expressions for the first and second velocity moments simplify largely to

$$
\bar{w} = -\frac{z}{\mu} + O(\exp(-\mu t)),
$$
$$
\bar{w}^2 = \frac{z^2}{\mu^2} + O(\exp(-\mu t)),
$$

and one can show that, provided $g$ is an even function of velocity,

$$
\sigma^2 = O(\exp(-2\mu t)).
$$

This result is expected: any initial velocity dispersion is quickly damped out by the surrounding gas on the short stopping timescale $t_s$. Hence, we deduce a simple closure for the moment equations as $\sigma^2 = 0$. Note that in the case where $\rho_s$ and $g$ are uniformly distributed, this expression is also valid for very short times. The long-term evolution of the collection of particles becomes a slow average settling, on a timescale of the order of unity or smaller.

$$
\sigma^2 = O(\exp(-2\mu t)).
$$

4. THE COLLECTIVE EVOLUTION OF LARGE PARTICLES

The main drag effect of gas on a large particle occurs through the formation of a wake behind the particle. If the velocity of the particle relative to the gas is small enough, the wake is laminar and the drag force is a linear function of the relative velocity. In that case, the particle trajectory is actually the same as that obtained in the case of the Epstein regime, and the results of the previous section apply. However, as the relative velocity between the particle and the gas increases, the wake becomes turbulent and the amplitude of the drag force becomes a power law of the Reynolds number of the particle.

The vertical component of the drag force is

$$
F_d \cdot \hat{e}_z = -m_p \frac{\rho(r) C(Re)}{\rho_s} \frac{\pi}{s} \sqrt{r^2 + z'^2 + (r \theta' - v_3)'^2} z',
$$

if one assumes that the only important component of the gas velocity is in the azimuthal direction. This formula can in principle be used only in conjunction with a complete evaluation of the orbit of the particle. However, one can assume that the largest typical contrasts in velocity between particle and gas occur in the vertical direction, where the particles oscillate across the midplane with an epicyclic frequency whereas the gas is more or less stationary [so that $z'^2 \gg (r \theta' - v_3)^2$ and $z'^2 \gg r'^2$]. In that case, the expression for the vertical component of the drag force reduces to

$$
F_d \cdot \hat{e}_z = -m_p \frac{\rho(r) C(Re)}{\rho_s} \frac{\pi}{s} |z'| z'.
$$

Note that since the typical velocity of the particles across the midplane is of the order of $\Delta \Omega_c$, where $\Delta$ is the typical height of particles above the disk, the asymptotic Stokes regime occurs for particles of size

$$
\frac{s}{z} = \frac{\text{Re}_c}{6} \frac{r \Omega_c}{c},
$$

where $\text{Re}_c = 800$. This condition corresponds, for example, to 10 m size bodies at $r = 1$ AU and $\Delta = 1/10$ of the disk scale height in a standard solar nebula.

4.1. Particles’ Trajectories

Using the same normalizations as in the Epstein regime, the vertical equation of motion can then be written as

$$
\ddot{z} = -z - \mu |\dot{z}| \dot{z},
$$

where $\mu = C(Re) (r/s) (\rho/\rho_s)$ and $r = 1$ AU. In the asymptotic Stokes regime, $C(Re)$ is simply a constant and $\mu$ is typically of the order of unity or smaller.

This equation can be solved exactly through the introduction of the quantity $w = \dot{z}$. On the positive branch (when $w > 0$) and negative branch (when $w < 0$) the solutions are, respectively,

$$
w^+_z = A \exp(-2\mu z) - \frac{z}{\mu} + \frac{1}{2\mu^2},
$$
$$
w^-_z = B \exp(2\mu z) + \frac{z}{\mu} + \frac{1}{2\mu^2}.
$$

As the particle oscillates about the midplane, it follows one branch or the other. The constants $A$ and $B$ are determined by matching the first branch to the boundary conditions and the successive alternative ones to each other at the turning point (i.e., when $w = 0$). For instance, the trajectory of a particle starting from rest at $z = z_0$ above the midplane satisfies the equation

$$
w^+_z = \dot{z}^2 = \left( -\frac{z_0}{\mu} - \frac{1}{2\mu^2} \right) \exp(2\mu z - 2\mu z_0) + \frac{z}{\mu} + \frac{1}{2\mu^2}.
$$

It accelerates toward the midplane and then decelerates on the other side until it reaches a turning point $z_1$, which satisfies

$$
0 = \left( -\frac{z_0}{\mu} - \frac{1}{2\mu^2} \right) \exp(2\mu z_1 - 2\mu z_0) + \frac{z_1}{\mu} + \frac{1}{2\mu^2}.
$$

These solutions indicate the existence of two regimes. When the initial height $z_0$ is much larger than the reference height $1/\mu$, the turning point $z_1$ is roughly equal to $1/2\mu$ regardless
of the height from which it originated. This pattern corresponds to a rapid stopping phase, when the particle quickly drops toward the midplane. When the amplitude of \( z_0 \) is reduced much below \( 1/\mu \), the turning point \( z_1 \) is roughly equal to \(-z_0\), which corresponds to the oscillatory phase, when the particle follows an epicyclic motion about the midplane with a slowly decaying amplitude.

4.2. The Stopping Phase

The stopping phase corresponds to the limit where \( z \) is much larger than the reference height \( 1/\mu \). Note that this situation may not always occur since in the Stokes regime \( \mu \leq 1 \). This condition corresponds to particles that start at a considerable distance above or below the disk itself. Nonetheless, in this case the solutions simplify to

\[
w_\pm \approx \frac{z}{\mu} = \dot{z} \approx -\frac{z_{1/2}}{\mu^{1/2}}.
\]  

(for the downward branch), and this differential equation can easily be solved with

\[
z_p(t) = \left( z_0^{1/2} - \frac{t}{2\mu^{1/2}} \right)^2.
\]  

Note that the particle velocity in that case varies linearly with time:

\[
w_p(t) = -2\left( \frac{z_0}{\mu} \right)^{1/2} + \frac{t}{2\mu}.
\]  

so that the typical duration of the stopping phase is \( t_s = 2\mu^{1/2}z_0^{1/2} \). Finally, we also note that for a particle starting from \( z_0 \) above the disk, for example, the lower turning point is more or less independent of \( z_0 \) and is roughly equal to \( z_1 = -1/2\mu \).

4.3. The Slowly Decaying Oscillating Phase

The oscillating phase corresponds to the limit \( z \ll 1/\mu \). In that case, the equation for the trajectory of the particle (in the downward branch, for example) simplifies to

\[
w^2 = z^2 = 2z_0(z_0 - z) - (1 + 2\mu z_0)(z_0 - z)^2.
\]  

This first-order differential equation can be solved with conventional substitutions to yield

\[
z_p(t) = z_0 - \frac{z_0}{1 + 2\mu z_0} \left[ 1 - \cos \left( 1 + 2\mu z_0 \right)^{1/2} t \right].
\]  

The lower turning point is then simply

\[
z_1 = z_0 - \frac{2z_0}{1 + 2\mu z_0}.
\]  

Starting from that point, the upward branch is then obtained through a similar analysis and becomes

\[
z_p(t) = z_1 - \frac{z_1}{1 - 2\mu z_1} \left[ 1 - \cos \left( 1 - 2\mu z_1 \right)^{1/2} t \right].
\]  

The following upper turning point is \( z_2 \) such that

\[
z_2 = z_1 - \frac{2z_1}{1 - 2\mu z_1}.
\]  

These equations show that the amplitude \( \gamma(t) \) of the oscillation slowly decays with time and can be approximated by

\[
\gamma(t) \simeq \frac{3\pi}{4\mu + K}.
\]  

where \( K \) is a constant that is determined in such a way as to satisfy the initial conditions. If the particle is released from rest from \( z = z_0 \) at \( t = 0 \) (with \( z_0 \ll 1/\mu \)), \( K = 3\pi/z_0 \). Moreover, the frequency of the oscillation also varies with time and converges to the epicyclic frequency.

To recapitulate briefly, we find that after a brief stopping phase on a timescale proportional to \( \mu^{1/2} \), the particle oscillates around the midplane with frequency that is close to the epicyclic frequency and with an amplitude that decays slowly in time roughly as given by equation (42). These solutions are illustrated in Figure 1. The analytic expression in equation (42) is also a solution of the differential equation (4.16) deduced by Adachi et al. (1976) for the sedimentation of large particles in the limit of negligible eccentricity for the particles and Keplerian azimuthal speed for the gas.

4.4. Continuum Equations in the Stokes Regime

The true particle trajectory in the asymptotic Stokes regime cannot be expressed analytically in any simple way. As in the Epstein case, we only look at the long-term behavior. In that case, we can approximate the trajectory of a particle by

\[
z_p(t, \mu; K_i, \varphi_i) = \gamma(\mu, t; K) \cos (t + \varphi),
\]  

![Figure 1](image-url)
where $K_i$ and $\varphi_i$ are complicated functions of the initial position and velocity of the particle. For simplicity, we consider only the asymptotic limit for times $4\mu t \gg K_i$. In this limit, equation (43) with $\gamma = 3\pi/4\mu$ is a good approximation, since the amplitude of oscillation of the particle becomes independent of the point of release (see Fig. 1). Let us also assume for the purpose of this work that the phase simply has a given distribution function $g(\varphi)$ so that the Boltzmann function for the Stokes regime is

$$f(z, w, t) = \int d\varphi \rho_s(z_0) \int_0^{2\pi} d\varphi g(\varphi) \delta(z - z_p(\mu, t; \varphi)) \quad \equiv \Sigma_p \int d\varphi g(\varphi) \delta(z - z_p(\mu, t; \varphi)), \quad (44)$$

where $\Sigma_p$ is the column density of particles and the normalization of $g$ is chosen such that $\int_{-\pi}^{\pi} d\varphi g(\varphi) = 1$. The function $g$ is periodic with period $2\pi$, but the initial distribution of particles can be described with $0 \leq \varphi \leq 2\pi$. Using the well-known relation

$$\delta(h(x)) = \delta(x - x_i) \left| \frac{dh}{dx}(x = x_i) \right|^{-1} \quad \text{where} \quad h(x_i) = 0, \quad (45)$$

we can determine the particle mass density profile,

$$\rho_p(z, t) = \frac{\Sigma_p}{\gamma} \mathcal{H}(\gamma - |z|) \left( 1 - \frac{z^2}{\gamma^2} \right)^{-1/2} \times \left[ g\left( \cos^{-1}\left( \frac{z}{\gamma} \right) - t \right) + g\left( -\cos^{-1}\left( \frac{z}{\gamma} \right) - t \right) \right], \quad (46)$$

where $\mathcal{H}$ is a Heaviside function and the function $\cos^{-1}$ takes values in the interval $[0, \pi]$.

As in the Epstein case, we compare the fluid equation for mass conservation to the first moment of the Boltzmann equation and obtain an expression for the collision term in the Stokes regime:

$$\Gamma = 2\mu|w|f. \quad (47)$$

Similar operations with the second moment of the Boltzmann equation yield the equivalent of the fluid equation of motion,

$$\rho_p \frac{\partial \bar{w}}{\partial t} + \rho_p \bar{w} \frac{\partial \bar{w}}{\partial z} = -\rho_p \frac{\partial \Phi}{\partial z} - \int \mu f|w|w dw - \frac{\partial}{\partial z} (\rho_p \sigma^2). \quad (48)$$

It is extremely tempting to write the drag term as

$$\int \mu f|w|w dw = \mu \rho_p |\bar{w}| \bar{w}. \quad (49)$$

However, a proper evaluation of the left-hand side integral shows that this cannot be done formally, and the complete expression for the drag force term should be kept. From equations (43) and (44),

$$\int f|w|w dw = \frac{\Sigma_p}{\gamma} \left( 1 - \frac{z^2}{\gamma^2} \right)^{-1/2} \mathcal{H}(\gamma - |z|) \times \left[ g\left( \cos^{-1}\left( \frac{z}{\gamma} \right) - t \right) \left( \frac{z^2}{\gamma} - \gamma \sqrt{1 - \frac{z^2}{\gamma^2}} \right) \right. \times \left. \left( \frac{z^2}{\gamma} - \gamma \sqrt{1 - \frac{z^2}{\gamma^2}} \right) \right] = \left[ \frac{\gamma^2}{\gamma^2} - \gamma \sqrt{1 - \frac{z^2}{\gamma^2}} \right] \right]. \quad (50)$$

In the case where $g$ is uniformly distributed, this expression reduces to

$$\int f|w|w dw = \frac{\rho_p}{2} \left[ \left( \frac{z^2}{\gamma^2} - \gamma \sqrt{1 - \frac{z^2}{\gamma^2}} \right) \left( \frac{z^2}{\gamma^2} - \gamma \sqrt{1 - \frac{z^2}{\gamma^2}} \right) + \left( \frac{z^2}{\gamma^2} - \gamma \sqrt{1 - \frac{z^2}{\gamma^2}} \right) \right], \quad (51)$$

whereas the simpler expression $\rho_p |\bar{w}| \bar{w}$ in equation (49), which is rewritten $\rho_p (\bar{w}^2/\gamma) |\bar{w}| \gamma$ in that case, would be wrongly applied.

As before, we are interested in evaluating the velocity dispersion $\sigma^2$. We need to calculate

$$\frac{\rho_p \bar{w}}{\Sigma_p} = \int d\varphi g(\varphi) \delta(z - \gamma \cos(\varphi + t)) \times \left[ \gamma \cos(t + \varphi) - \gamma \sin(t + \varphi) \right]. \quad (52)$$

In the asymptotic limit

$$\frac{\rho_p \bar{w}}{\Sigma_p} = \frac{\mathcal{H}(\gamma - |z|)}{\gamma} \left[ \left( \gamma \gamma \left( 1 - \frac{z^2}{\gamma^2} \right)^{-1/2} - \gamma \right) g\left( \cos^{-1}\left( \frac{z}{\gamma} \right) - t \right) \right. \times \left. \left[ \gamma \gamma \left( 1 - \frac{z^2}{\gamma^2} \right)^{-1/2} + \gamma \right] g\left( -\cos^{-1}\left( \frac{z}{\gamma} \right) - t \right) \right] \right]. \quad (53)$$

Similarly, one can show that

$$\frac{\rho_p \sigma^2}{\Sigma_p} = \Sigma_p \int_0^{2\pi} d\varphi g(\varphi) \delta(z - \gamma \cos(\varphi + t)) \times \left[ \gamma \cos(t + \varphi) - \gamma \sin(t + \varphi) \right]^2, \quad (54)$$
so that in the asymptotic limit

\[
\rho \frac{\partial w}{\partial t} = \rho \left[ \frac{z^2 \gamma^2 + (\gamma^2 - z^2)}{\gamma^2} \right] - 2\Sigma_p G(\gamma - |z|) \gamma z \\
\times \left[ g \left( \cos^{-1} \left( \frac{z}{\gamma} \right) - t \right) - g \left( - \cos^{-1} \left( \frac{z}{\gamma} \right) - t \right) \right].
\] (55)

In the case where the initial distribution function \( g \) is uniform, we can simplify these expressions greatly and show that for \( |z| \leq \gamma \) (which corresponds to the thickness of the dust layer)

\[
\sigma^2 = (\gamma^2 - z^2) H(\gamma - |z|).
\] (56)

This expression means that within the dust disk, the particle velocity dispersion remains important at all times. The velocity dispersion mimics the effect of a pressure term that effectively slows down the settling. In this simplified case where \( g(\varphi) \) was taken to be a uniform distribution, it is actually possible to relate \( \sigma \) to intrinsic large-scale properties of the system. However, in the more general case where \( g(\varphi) \) is not a uniform distribution there exists no simple relationship between \( \sigma^2 \) and the local density of the gas as there would normally be in a standard fluid. Instead, this mock pressure term would depend in a complicated manner on the initial configurations of the particles in phase space. This prevents any further progress, at this stage, in the use of a fluid description for these large particles. In §5, however, we shall infer from numerical experiments a simple empirical approximation to the evolution of \( \sigma^2 \) for more general initial conditions.

5. NUMERICAL CALCULATIONS

Most studies of the evolution of dust particles in astrophysics require the determination of the evolution of their phase-space moments (local average density, velocity), as well as the evolution of their size distribution. Although numerical integration of the orbital evolution of all individual particles can provide that information, such an approach is too time consuming to be used for a wide range of initial conditions. In contrast, the moments of the Boltzmann equation (as given by eqs. [20] and [22]) provide much simpler description of the evolution of the particles for any set of initial conditions.

With the simplified analytical approach presented in §§3 and 4, three principal approximations were performed:

1. We neglected the radial motion of the particles and therefore implicitly assumed that it is possible to perform a separation of the variables and of the equations of motion into individual components.
2. We neglected the vertical variation of \( \mu \) and used a simplified expression for the gravity.
3. We neglected the contribution of the azimuthal motion of the particles through the gas in the calculation of the gas drag in the Stokes regime. This approximation effectively underestimates the gas drag.

In order to check the validity of these approximations, we begin by integrating the orbits of particles in a realistic gaseous protoplanetary nebula, with the complete expression for the gas drag force. We compare them to the corresponding analytical expressions. We then release various single-size sets of 10,000 particles in the disk from a small region and follow their evolution in time. At a given time \( t \) we then perform the Boltzmann averaging procedure and compare the results with those obtained in §§3 and 4.

5.1. Numerical Integration: Method and Model Parameters

The equation of motion of a particle in a gaseous accretion disk is given by equation (1). Expanding this equation into its components in a cylindrical coordinate system \((r, \theta, z)\) yields

\[
\begin{align*}
\dot{r} &= r \dot{\theta}^2 - \frac{GM_r}{(r^2 + z^2)^{3/2}} - \frac{F_d}{m_p} \hat{e}_r, \\
\dot{\theta} &= -2 \frac{\dot{r}}{r} - \frac{F_d}{m_p} \hat{e}_\theta, \\
\dot{z} &= -\frac{GM_z}{(r^2 + z^2)^{3/2}} - \frac{F_d}{m_p} \hat{e}_z,
\end{align*}
\] (57)

where for small particles we use the Epstein drag force expression

\[
\frac{F_d}{m_p} = \frac{\rho}{\rho_s} \frac{C(Re)}{s} \left[ r \dot{e}_r + (r \dot{\theta}' - v_g) \dot{e}_\theta + z' \dot{e}_z \right]
\] (58)

and for very large particles we use the Stokes drag force

\[
\frac{F_d}{m_p} = \frac{\rho}{\rho_s} \frac{C(Re)}{s} \sqrt{r^2 + (r \dot{\theta}' - v_g)^2 + z^2} \times \left[ r \dot{e}_r + (r \dot{\theta}' - v_g) \dot{e}_\theta + z' \dot{e}_z \right].
\] (59)

We normalize these expressions using \( R = 1 \text{ AU} = 1.5 \times 10^{13} \text{ cm} \) as a unit distance and \( \Omega_k^{-1}(R) = 2.0 \times 10^{-7} \text{ s} \) as a unit time. This yields

\[
\begin{align*}
\dot{r} &= r \dot{\theta}^2 - \frac{r}{(r^2 + z^2)^{3/2}} - \mu(r,z) \dot{r}, \\
\dot{\theta} &= -2 \frac{\dot{r}}{r} - \mu(r,z) \left[ \dot{\theta} - \frac{v_g(r,z)}{\Omega_k(R)} \right], \\
\dot{z} &= -\frac{z}{(r^2 + z^2)^{3/2}} - \mu(r,z) \dot{z},
\end{align*}
\] (60)

where \( v_g(R) \) is the linear azimuthal Keplerian velocity at \( R = 1 \text{ AU} \), with

\[
\mu_E(r,z) = \frac{\rho(r,z) c(r,z)}{\rho_s s \Omega_k(R)}
\] (61)

in the Epstein regime, and

\[
\mu_S(r,z) = \frac{\rho(r,z) C(Re) R}{\rho_s s}
\] (62)

in the Stokes regime. The numerical integration of the trajectories of the particles in this model is performed using a fourth-order Range-Kutta integrator.

The quantities related to the disk structure (as the gas density \( \rho \), the gas velocity \( v_g \) and the sound speed \( c \)) are derived from the protoplanetary nebula model of Supulver & Lin (2000). In this model, the gas follows a polytropic equation of state with

\[
p = K \rho^2, \quad c = \sqrt{\gamma K \rho},
\] (63)
where \( p \) is the gas pressure, with \( K = 6.9 \times 10^{20} \text{ dyn cm}^4 \text{ g}^{-2} \), 
\( \gamma = 1.4 \) for an ideal diatomic gas (composed of pure molecular hydrogen, for example), and

\[
\rho(r, z) = 8.5 \times 10^{-12} \left( \frac{r}{6} \right)^{-3/4} \left[ 1 - \frac{z^2}{H^2(r)} \right] \text{ g cm}^{-3},
\]

where \( r \) is in AU and \( H^2(r) = 34 \times 10^{-12}K(r/6)^{-3/4} \)
\( \Omega_k(r)^{-2}R^{-2} \) is the square of the disk scale height (in the dimensionless units). Finally, the dimensionless gas velocity is

\[
r_v(r) = \frac{v_K(r)}{v_K(R)} = \frac{1}{2\rho \frac{\partial}{\partial r} v_K(R) \Omega_k(r)} \frac{1}{\frac{8.5 \times 10^{-12}KR^3}{GM} \frac{3}{4} r^{-1/4}} = r^{-1/2} - 2\Omega_k(r)^{-2} r^{-1/4},
\]

if we ignore the variation of the gas density scale height with radius. This expression defines \( \epsilon \) uniquely; with the numerical values of \( K, G \) given above and using \( M = 2 \times 10^{33} \text{ g} \), we have \( \epsilon = 1.8 \times 10^{-3} \). This expression describes how gas pressure reduces the effective gravity on the gas; this results in the slightly sub-Keplerian character of the gas velocity as found in equation (66).

### 5.2. Numerical Integration: Particle Trajectories

#### 5.2.1. Small Particles

Figure 2 shows the trajectories of small particles (0.1 mm to 1 m) released at 1 AU from height 0.01 AU, as calculated numerically using the procedure described in § 5.1. It compares the results to those obtained analytically assuming separation of variables and constant \( \mu \), which are summarized in equation (7). The analytical fit is indistinguishable from the complete numerical solution, despite these approximations.

#### 5.2.2. Large Particles

Figure 3 shows the numerically integrated orbit of a 10 m–size body in a protoplanetary nebula, when released from rest at radius 1 AU and height 0.01 above the midplane. It compares the numerical results to the theoretical predictions given by equation (43), and here again, despite the approximations, the analytical fit is excellent. Note that integration for much longer times (about 1000 orbits) reveals a small but growing discrepancy in the amplitude of the oscillation. This is due to the misrepresentation of the analytical expression for the amplitude of the drag force compared to its true value (see eqs. [27] and [28]). This discrepancy is discussed in more detail later.

### 5.3. Numerical Integration: Boltzmann Averaging and Collective Behavior

In this section we now follow the evolution of a collection of 10,000 uniformly sized particles, in the Epstein regime (of size 1 mm) and in the Stokes regime (with size 10 m). We release all particles from a small interval in radius, for \( 0.9 < r < 1.1 \). The initial radial positions are uniformly distributed in that interval. Other initial conditions depend on the regime studied, in order to be best able to perform the adequate comparisons between theory and numerical experiment. After a time \( t \), the positions and velocities of all particles are gathered and binned into regular intervals in radius and in height. The total number of particles, the average velocity, and the average second velocity moment are then calculated for each bin according to the formulae

\[
\rho_p(r, z) = m_p N(r, z),
\]

\[
\overline{w}(r, z) = \sum_p w_p / N(r, z),
\]

\[
\overline{w^2}(r, z) = \sum_p w_p^2 / N(r, z),
\]

where the sums are carried out over all particles found within the individual bin centered on the position \( (r, z) \).

When comparing the numerical data to the analytical solutions of the previous sections, we are careful to apply the correct normalization to the analytical formulae: indeed the normalization of the time variable depends on radial position in the analytical solutions, whereas in the numerical computations it is normalized to the orbital frequency at 1 AU.

#### 5.3.1. Epstein Regime

Using the normalizations adequate to the numerical calculations, the theoretical solutions for the average particle velocity and dispersion become

\[
\overline{w} = -\frac{z^{3/2}}{\mu(r)} + O(\exp(-\mu(r)^{3/2}t)),
\]

\[
\sigma^2 \propto O(\exp(-2\mu(r)^{3/2}t)).
\]
distribution in heights in the interval \([-0.01, 0.01]\) AU and a uniform distribution in velocities with \(-10^{-3} < w_i < 10^{-3}\) (in units of the Keplerian velocity at \(r = 1\)). For this figure, the binning in the averaging process is fairly coarse (10 bins in each direction) in order to generate adequate statistics and to visualize the results more clearly. The velocity dispersion is indeed found to decay exponentially in accordance with equation (67).

The trajectory of these particles is then followed up to time \(t = 100\) (i.e., about 16 orbits), and their average velocity is computed with a finer binning (50 bins in each direction). The results are presented in Figure 5 and compared to the analytical solutions in equation (67). Once again in this regime, the analytical solutions are found to match the numerical results very precisely.

Following on the adequate verification of the analytic predictions for the evolution of the moments in the case of a uniform initial distribution, we can safely assume that the numerical evaluations of equations (15), (17), and (23) for more general initial particle distributions will also yield the correct evolution laws. Note that in a very general sense, the average evolution of particles in the Epstein regime is not very sensitive to their initial velocity distribution because any initial velocity dispersion is quickly damped out by the surrounding gas on the short stopping timescale.

5.3.2. Asymptotic Stokes Regime

At 1 AU in a laminar minimum mass nebula, small particles grow through coagulation to less than 1 cm before they settle to the midplane of the disk (Safronov 1969; Goldreich & Ward 1973). Nevertheless, particles can grow up to sizes typical of the Stokes regime along the path of their radial drift (Supulver & Lin 2000). These particles may also be scattered into large inclination through encounters with embedded planetesimals (Aarseth, Lin, & Palmer 1993; Kokubu & Ida 2002) and thereby attain a spatial distribution with a large scale height.

In any case, analytical predictions using the assumptions described in the beginning of this section suggest that the velocity dispersion of the particles should remain important at all times and is directly related to the thickness of the particle layer. We now wish to test these predictions against the various moments of the particle distribution in the case where the orbits are integrated numerically in a realistic description of the gas disk and with a correct description of the Stokes drag law.

In order to simplify the comparison between the analytical predictions given by equations (46), (53), and (55) in § 4 and the full numerical results, we begin by releasing 10,000 10 m size particles in the disk with similar initial conditions as the ones that were used in the analytical work (i.e., uniform distribution in height and uniform distribution of the phase of the oscillation). Although these initial conditions may seem perhaps artificial, they are the only ones that can actually provide an analytical solution. Since the purpose of this section is only to verify the applicability and consequences of the assumptions used to calculate analytically these integral expressions (i.e., the separation of spatial variables, the constant uniform background density, and the reduced form of the drag term), we choose here a set of idealized initial density distributions for which analytical solutions exist and can be checked simply. Once the validity of the assumptions is checked, the integrals in equations (46), (53), and (55) can be evaluated numerically for more general initial density and velocity distributions of particles.

Fig. 3.—Top: Trajectory of a 10 m size body in the \((r, z)\)-plane, released from radius 1 AU and height \(z = 0.01\) AU above the midplane (to avoid crowding in the figure). The markers mimic a Poincaré map, i.e., are positioned at the points where the particle crosses the \(\theta = 0\) plane. Bottom: Trajectory of the same particle in the \((z, t)\)-plane; again, the symbols represent positions of intersection with the \(\theta = 0\) plane. The analytical trajectory as given by eq. (43), using simply an average value of \(\mu\) in that region (\(\mu_0 = 0.08\)), is shown by dotted lines: they are virtually indistinguishable from the numerically integrated trajectories. Note that one must take into account the fact that the normalization of the time variable varies with radius in the simple analytical model. The dashed line represents the envelope of the oscillation as given by eq. (42).

Fig. 4.—Left: Evolution of the velocity dispersion profile for a collection of millimeter-size particles. The style of the symbols used corresponds to those in the right panel. Right: Evolution of the velocity dispersion of the particles located around \(r = 1\). The symbols correspond to the results of the numerical experiment, and the solid line is the analytical prediction of eq. (67), using a value of \(\mu\) that is an average value for this quantity around \(r = 1\).
observe the settling and (negligible) radial drift in this diagram, as well as the
particles were released with a uniform distribution of heights in the interval
[0.01, 0.01] AU and of radii in the interval [0.9, 1.1] AU. One can easily observe the settling and (negligible) radial drift in this diagram, as well as the
excellent matching between theory and this numerical experiment.

For illustration purposes, we specify the initial values
\[ z_i = \frac{3\pi}{K} \cos(\varphi_i), \]
\[ w_i = -4\mu(r_i, z_i) \frac{3\pi}{K^2} \cos(\varphi_i) - \frac{3\pi}{K} \sin(\varphi_i), \] (68)
where \( \varphi_i \) is uniformly distributed in the interval \([0, 2\pi]\). The azimuthal velocity of the particles is the Keplerian velocity at the point of release.

Using the same normalization we adopted in our numerical simulations, we find the theoretical solutions for the velocity dispersion to be
\[ \sigma^2 = \left(\frac{9\pi^2}{4\mu(r, z) t + K^2} - z^2\right)r^{-3}. \] (69)

Figure 6 compares the results of the numerical calculations to the analytical solutions given by equation (69). For some time after the onset of the calculation, the numerically obtained dispersion is indeed extremely well approximated by the analytical formula. However, over much longer times, a systematic shift emerges, in which the numerical values are slightly lower than the theoretical values. This slight overestimate of the velocity dispersion comes from the fact that the analytical solutions underestimate the total drag force by neglecting the contributions from the motion of the particles in the radial and azimuthal direction. In order to verify this conjecture, another numerical experiment is carried out in which the drag force is artificially set to
\[ \Delta = \frac{3\pi}{4\mu(r, z) t + 3\pi/\Delta_0}, \]
\[ \sigma^2 = \frac{\Delta^2(t)}{2} \exp\left(-\frac{2z^2}{\Delta^2(t)}\right), \] (71)
where \( \Delta_0 = 0.01 \). This formula was obtained from the following heuristic arguments: According to equation (43), the typical vertical velocity of particles near the midplane is of

In that case, the contribution of the drag force in the vertical direction is exactly that used in analytical calculation. As expected, the analytical expression is now able to reproduce the experimental results much more accurately.

Having gained some experience with the Boltzmann averaging methods, and having moreover established the adequacy of the numerical simulations to the level where the initial conditions are set randomly (rather than in such a way as to allow easy analytical calculations). In the follow-up numerical computation, 10,000 10 m size particles are released with a uniform initial height and vertical velocity distribution chosen in the intervals \([-0.01, 0.01]\) AU and \([-0.01, 0.01]\) AU. The azimuthal velocity of the particles is set to be the Keplerian velocity at the point of release. With these initial conditions, the initial velocity dispersion should be uniform within the dust layer.

Figure 7 shows the evolution with time of this dispersion profile at radius \( r = 1 \) AU, when the particles are binned in 20 intervals in radius and heights. After a brief adjustment phase (not shown in the figure), the dispersion profile seems to be well approximated by a Gaussian distribution. We attempt to match such a Gaussian profile to the numerical data points and find that the best fit is given by
\[ \Delta = \frac{3\pi}{4\mu(r, z) t + 3\pi/\Delta_0}, \]
\[ \sigma^2 = \frac{\Delta^2(t)}{2} \exp\left(-\frac{2z^2}{\Delta^2(t)}\right), \] (71)
where \( \Delta_0 = 0.01 \). This formula was obtained from the following heuristic arguments: According to equation (43), the typical vertical velocity of particles near the midplane is of

In that case, the contribution of the drag force in the vertical direction is exactly that used in analytical calculation. As expected, the analytical expression is now able to reproduce the experimental results much more accurately.

Having gained some experience with the Boltzmann averaging methods, and having moreover established the adequacy of the numerical simulations to the trajectories of particles in the Stokes regime, we now attempt to use the results of the numerical simulations to find the evolution of the dispersion in the case where the initial conditions are chosen randomly.

Figure 6 compares the results of the numerical calculations to the analytical solutions given by equation (69). For some time after the onset of the calculation, the numerically obtained dispersion is indeed extremely well approximated by the analytical formula. However, over much longer times, a systematic shift emerges, in which the numerical values are slightly lower than the theoretical values. This slight overestimate of the velocity dispersion comes from the fact that the analytical solutions underestimate the total drag force by neglecting the contributions from the motion of the particles in the radial and azimuthal direction. In order to verify this conjecture, another numerical experiment is carried out in which the drag force is artificially set to
\[ \Delta = \frac{3\pi}{4\mu(r, z) t + 3\pi/\Delta_0}, \]
\[ \sigma^2 = \frac{\Delta^2(t)}{2} \exp\left(-\frac{2z^2}{\Delta^2(t)}\right), \] (71)
where \( \Delta_0 = 0.01 \). This formula was obtained from the following heuristic arguments: According to equation (43), the typical vertical velocity of particles near the midplane is of

In that case, the contribution of the drag force in the vertical direction is exactly that used in analytical calculation. As expected, the analytical expression is now able to reproduce the experimental results much more accurately.

Having gained some experience with the Boltzmann averaging methods, and having moreover established the adequacy of the numerical simulations to the trajectories of particles in the Stokes regime, we now attempt to use the results of the numerical simulations to find the evolution of the dispersion in the case where the initial conditions are chosen randomly (rather than in such a way as to allow easy analytical calculations). In the follow-up numerical computation, 10,000 10 m size particles are released with a uniform initial height and vertical velocity distribution chosen in the intervals \([-0.01, 0.01]\) AU and \([-0.01, 0.01]\) AU. The azimuthal velocity of the particles is set to be the Keplerian velocity at the point of release. With these initial conditions, the initial velocity dispersion should be uniform within the dust layer.

Figure 7 shows the evolution with time of this dispersion profile at radius \( r = 1 \) AU, when the particles are binned in 20 intervals in radius and heights. After a brief adjustment phase (not shown in the figure), the dispersion profile seems to be well approximated by a Gaussian distribution. We attempt to match such a Gaussian profile to the numerical data points and find that the best fit is given by
\[ \Delta = \frac{3\pi}{4\mu(r, z) t + 3\pi/\Delta_0}, \]
\[ \sigma^2 = \frac{\Delta^2(t)}{2} \exp\left(-\frac{2z^2}{\Delta^2(t)}\right), \] (71)
where \( \Delta_0 = 0.01 \). This formula was obtained from the following heuristic arguments: According to equation (43), the typical vertical velocity of particles near the midplane is of
the order of $\Delta$, the maximum height of the particles above the midplane. Moreover, $\Delta$ decays with time as a result of gas drag according to equation (42), so that one can readily deduce the evolution of $\Delta$ with time for a set of particles released at $t = 0$ from typical heights ranging between 0 and $\Delta_0$. The numerical factors have been fitted to the data. The analytical fit is also shown in Figure 7 and is found to reproduce the evolution of $\sigma$ satisfactorily.

Finally, recalling that the analytical expression for the vertical motion of the particles slightly underestimates the true drag force, we compare the analytical fit given by equation (71) to an artificial simulation in which the drag force is given by equation (70). In principle, the fit should be much better in this case. However, we find that the evolution of the dispersion in that case is better fitted by

$$ \sigma^2 = \frac{2}{\pi} \Delta^2 \exp\left(-\frac{2z^2}{\Delta^2}\right), $$

while keeping the expression for $\Delta$ the same (see Fig. 7). Although this difference is small, it still suggests that any attempt to find an analytical fit to the collective motion of Stokes particles should be carefully calibrated before being used in further simulations.

To conclude this section, we found that the analytical evaluation of the orbit of single particles was extremely useful toward the determination of the average motion of a large number of particles and that the resulting averaged equations fit the numerical experiments extremely well. We have also been able to determine a simple heuristic way of closing the moment equations at low order in the case of particles in the Stokes regime, which emulates the effect of the particle dispersion by a mock pressure term that can be related to large-scale properties and initial conditions of the fluid.

6. EVOLUTION OF THE SIZE DISTRIBUTION OF PARTICLES IN A GASEOUS DISK

Having discussed the time evolution of a large number of particles of equal size, we now consider a population of different-size particles. Again, the Boltzmann approach proves to be powerful and simple to use. Let us assume for simplicity that for a given size and at a given radius the ratio of initial particles to gas mass is independent of $z$. Hence, the initial distribution of particles of a given size $s$ at time $t = 0$ is

$$ \rho_i(s, z) = \rho_0(s) \exp\left(-\frac{z^2}{2H^2}\right), $$

where $H$ is the typical gas disk height; for an isothermal disk, $H = c/r\Omega_K$. Let us also assume (again for simplicity) that the initial vertical velocity of particles is null. This idealized ad hoc setup could correspond to a condensation front where, at time $t = 0$, both hydrogen and heavy-elemental gas diffuses into a cool region of the disk. Indeed, the coexistence of small and medium-size particles well above the midplane requires runaway coagulation of initially steady, microscopic dust grains into grains up to a size $s$. This rapid in situ growth is possible in regions where the particles have large surface density and much greater than unity area filling factor. In those regions, the condensation and sublimation timescales are generally much shorter than the dynamical timescales (Supulver & Lin 2000) so that at least the small grains may form with a similar density distribution as the gas and without any initial in-falling motion. The particles then start settling from their initial positions and interact with the gas mostly with an Epstein drag law.

The particle density distribution at time $t$ is then simply

$$ \rho_p(s, z, t) = \frac{1}{\alpha} \rho_i\left(\frac{z}{\alpha}\right) = \rho_0(s) \exp\left(-\frac{z^2}{2\Delta^2}\right) \exp\left(\frac{t}{\mu(s)}\right), $$

where $\alpha$ is defined by equation (9) and $\Delta = H\alpha = H \exp (-t/\mu(s))$ is the evolving dust layer thickness. This equation was derived from equation (17) using the Ansatz given by equation (73) and $g(w_i, z) = \delta(w_i)$. We can deduce from this result the number distribution of particles of size $s$

$$ n(s, z, t) = n_0(s) \exp\left(-\frac{z^2}{2\Delta^2}\right) \exp\left(\frac{t}{\mu(s)}\right) $$

$$ = n_0(s) \exp\left(-\frac{z^2}{2H^2}\right) \exp\left(\frac{t\sigma^2}{H^2}\right), $$

where we have defined the constant $\kappa$ such that $\mu = \kappa/s$ in order to write explicitly this equation in terms of its dependence on the particle size. The initial particle size distribution function is taken to be that given by Hellyer (1970) and Mathis, Rumpl, & Nordsieck (1977),

$$ n_0(s) \propto s^{-3.5}. $$
At a given height above the disk, the distribution function $n(s)$ peaks for small particle sizes [which corresponds to the initial peak in $n_0(s)$] but has another maximum, which evolves with time, at approximately

$$s = \frac{2\kappa}{\tau} \ln \left( \frac{H}{z} \right),$$

(77)

regardless of the initial size distribution function of the particles. This result is illustrated in Figure 9.

The results shown in Figures 8 and 9 suggest the following comments. First, at a given height, successive fronts of particles with decreasing sizes pass through the gas as they settle, to leave the region completely depleted of the intermediate-sized particles. Second, at a given time, there is a definite stratification of the particles in terms of their respective sizes, with the larger particles concentrating tightly around the midplane. Note that this method is only valid for particles with sizes up to the order of the mean free path of the gas molecules. For the very large particles, the Stokes drag law must be applied. In that case, we have shown in § 4 that the large particles only settle algebraically toward the midplane. As a result, although their number density is small, the large particles remain important at all times and at all heights in the disk.

This work shows that after a few settling times (which are of the order of a few thousands of orbits for centimeter-sized particles), the bulk of the disk will be depleted of intermediate-sized particles, which have all settled down to the midplane.

7. SUMMARY AND DISCUSSION

In this paper we provide a set of analytic solutions to describe the sedimentation of dust particles due to gas drag in a protostellar accretion disk.

The main effect of the gas drag is to induce a loss of angular momentum and linear momentum for the particle, which in turn leads to orbital decay and vertical settling (Adachi et al. 1976; Weidenschilling 1977). Numerical integration of the trajectories of particles in a protoplanetary accretion disk is straightforward if we assume that the gas flow is laminar and occurs mostly in the azimuthal direction. However, analytical integration is also possible given a set of assumptions (which have been checked here to be reasonably well justified), including the separability of the radial and vertical motion and locally uniform density. Accordingly, we have extracted analytical formulae for the vertical motion of particles in a disk, both in the Epstein regime (for particles with size much smaller than the mean free path of the gas molecules) and in the Stokes regime (for particles much larger than the mean free path). Comparison of these analytical expressions with the complete numerically integrated orbits shows an excellent match in both cases.

By using the analytical expression for the orbits of particles, we were able to extract their average behavior using a standard Boltzmann averaging analysis (i.e., local averaging, or rather, coarse graining, of the positions and velocities of particles in phase space). The successive moments of the Boltzmann distribution function provide the evolution of the coarse-grained particle density, velocity, and velocity dispersion. The successive moments of the Boltzmann equation yield the equivalent of “fluid” equations for the collection of particles and, in particular, a mass conservation equation and a momentum equation.

Numerical experiments were performed in which 10,000 single-size particles were released in the disk from a localized region of phase space (typically uniformly distributed in space and with a uniform vertical velocity distribution). Using a similar coarse-graining method as the one used in the analytical analysis, we were able to extract the particle density profile, their average velocity, and velocity dispersion from the numerical integration. We compared them to the analytical formulae obtained previously. The main results are summarized here.

In the Epstein regime, there is an excellent agreement between the analytic formulae and the results of the numerical calculation, despite the approximations used. Individual particles have their velocity with respect to the gas quickly...
damped out on a very short stopping timescale. Correspondingly, their collective behavior is coherent: the typical velocity dispersion is found to decay on the particle stopping timescale, which is usually much smaller than the orbital timescale. For times longer than the stopping timescale, individual particles settle exponentially toward the midplane and the corresponding average momentum equation reduces to a simple standard fluid equation with a linear drag term and null pressure.

In the Stokes regime, there is a good agreement between the theory and the results of the numerical experiments, and there again despite the approximations used. Individual particles are found to oscillate across the midplane with an algebraically decaying amplitude. Although this behavior is very well reproduced by the analytical work, there exists a small discrepancy between analytical formulae and the results of numerical computation in the form of a slight systematic overestimate of the oscillation amplitude of the particle. This problem is due to an (unfortunately unavoidable) oversimplification of the amplitude of the drag force in the analytical work and results in similar discrepancies in the comparison of averaged quantities. Nonetheless, apart from this small identifiable and controllable error, the predictions of the analytical work match very well the numerical results. As expected, the velocity dispersion is found to remain large at all times within the dust layer, with a maximum near the midplane. This large dispersion is directly related to the continuous oscillatory motion of the particles across the midplane. We found it extremely difficult to obtain analytical predictions of the temporal evolution of the dispersion for anything but the simplest initial conditions, which were not necessarily always realistic. However, we were able to deduce from the numerical experiments a heuristic fit to the observed evolution of the velocity dispersion, for more general initial conditions. This result has interesting consequences with respect to the possibility of closing the moment equations at low order and describing accordingly the evolution of a collection of large particles with averaged fluid equations. We will test this theory in future work.

Finally, by combining the average evolution equations for single-size particles obtained in the Epstein regime to a plausible size distribution function for dust grains in the interstellar medium (Hellyer 1970; Mathis et al. 1977), we showed that it is possible to follow analytically the spatial and temporal evolution of such a size distribution as the particles settle toward the midplane. This analytical calculation was done for collisionless particles, and the extension of this work to include collisions, coagulation, and sublimation would probably require numerical analysis. Nonetheless, within this approximation we found that the sedimentation process results in a very strong segregation of particles according to their sizes, within a timescale of a few hundred years only. All intermediate-size particles quickly converge to the midplane, leaving behind only the very small particles, indirectly supported against the settling by gas pressure via the gas drag, and the very large particles, which keep oscillating across the midplane with a slowly decaying amplitude. This segregation process may well suggest that within the very thin dust layer at the midplane, one could approximate the population of dust particles by a single-size population.

In future work, we shall apply the results obtained here to two essential purposes:

1. The description of the particles as a continuum fluid will be used to study the stability of the dust layer against shearing (Garaud & Lin 2004), in order to test whether gravitational instability can indeed take place in the dust layer. More generally, we intend to implement this description into existing three-dimensional hydrodynamics codes for the computation of accretion disks, in order to obtain a fully self-consistent description of dust evolution and growth in these disks.

2. The temporal and spacial evolution of the size distribution of particles can be compared directly with observations of dust. We plan to construct models with which we can infer the extent of particle coagulation and the surface density distribution of the gas from high-resolution, multiwavelength (mostly in the submillimeter and millimeter range) maps of protostellar disks.

P. G. thanks New Hall (Cambridge) and PPARC for financial support toward the completion of this work. It was also supported in part by NASA through grant NAG5-10612 and the California Space Institute. The authors thank Neil Balmforth and Taku Takeuchi for many useful discussions.

APPENDIX

DERIVATION OF THE PROPERTIES OF RADIAL MOTION OF PARTICLES IN A GASEOUS DISK

Neglecting the vertical motion of the particles, the equations of motion in the radial and azimuthal directions in the Epstein regime are

\[
\dot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2} - \frac{c}{s} \frac{\rho}{\rho_s} \dot{r},
\]

(A1)

\[
r\ddot{\theta} + 2\dot{r}\dot{\theta} = -\frac{c}{s} \frac{\rho}{\rho_s} \left[ r\dot{\theta} - v_g(r) \right],
\]

(A2)

where \(v_g(r)\) is the slightly sub-Keplerian azimuthal velocity of the gas

\[
v_g(r) = \sqrt{\frac{GM}{r}} + \frac{1}{2} \sqrt{\frac{r^3}{GM\rho}} \frac{1}{\delta r} = \Omega(r) - \frac{\eta(r)}{\Omega(r)},
\]

(A3)
which defines \( \eta(r) = -(1/2\rho)\partial p/\partial r \). As in § 3.1, we use the normalizations \(|r| = R = 1 \) and \(|t| = \Omega_K^{-1}(R) \) to obtain

\[
\ddot{r} - r\dot{\theta}^2 = -\frac{1}{r^2} - \mu \dot{r},
\]

\[
r\ddot{\theta} + 2r \dot{r} \dot{\theta} = -\mu \left( r\dot{\theta} - \frac{1}{r^{1/2}} + \frac{\eta(r)}{\Omega_K^2(r)} \right).
\]

It is not possible to find simple analytical solutions of the full system. However, we can assume that deviations from Keplerian motions are small and that the particle velocity can be written as

\[
r = 1 + \epsilon, \quad \dot{\theta} = 1 + \dot{\phi}.
\]

The linearized system yields (assuming that the background quantities vary very little with \( r \))

\[
\ddot{\epsilon} + \mu \dot{\epsilon} - 3\epsilon = 2\dot{\phi},
\]

\[
\ddot{\phi} + \mu \dot{\phi} = -2\dot{\epsilon} - \frac{3}{2}\mu \epsilon - \mu \frac{\eta}{\Omega_K^2}.
\]

Substituting the first into the second yields

\[
\frac{d}{dt}\left[ \frac{1}{2} \dot{\epsilon} + \mu \dot{\epsilon} + \frac{1}{2} (1 + \mu^2)\epsilon \right] = -\mu \frac{\eta}{\Omega_K^2}.
\]

Hence, the local solution is

\[
\dot{\epsilon} = e^{-\mu t}(a \cos t + b \sin 2t) - 2 \frac{\mu}{1 + \mu^2} \frac{\eta}{\Omega_K^2}.
\]

This solution represents the local difference between the particle velocity and the Keplerian velocity, in units of the Keplerian velocity. As in the solution found in § 4, the first term represents a very quick stopping of the particle on timescale \( 1/\mu \), and the second term represents a constant inward drift. The variation of this drift velocity with particle size is given by the function \( \mu/(1 + \mu^2) \), which has a maximum for \( \mu = 1 \). This result reproduces the results found by Weidenschilling (1977).

REFERENCES

Aarseth, S. J., Lin, D. N. C., & Palmer, P. L. 1993, ApJ, 403, 351
Adachi, I., Hayashi, C., & Nakazawa, K. 1976, Prog. Theor. Phys., 56, 1756
Adams, F. C., Lada, C. J., & Shu, F. H. 1987, ApJ, 312, 788
Balbus, S. A., & Hawley, J. F. 1991, ApJ, 376, 214
Beckwith, S. V. W. 1999, in The Origin of Stars and Planetary System, ed. C. J. Lada & N. D. Kylafis (Dordrecht: Kluwer), 579
Bolzmann, L. 1872, Wien Berichten, 66, 275
Clarke, C. J., Gendrin, A., & Sotomayor, M. 2001, MNRAS, 328, 485
D’Alessio, P., Calvet, N., & Hartmann, L. 2001, ApJ, 553, 321
Garaud, P., & Lin, D. N. C. 2004, ApJ, submitted
Goldreich, P., & Ward, W. R. 1973, ApJ, 183, 1051
Haisch, K. E., Jr., Lada, E. A., & Lada, C. J. 2001, ApJ, 553, L153
Hayashi, C., Nagazawa, K., & Nagawaga, Y. 1985, in Protostars and Planets II, ed. D. Black & M. Mathews (Tucson: Univ. Arizona Press), 1100
Hellyer, B. 1970, MNRAS, 148, 383
Klahr, H. H., & Lin, D. N. C. 2001, ApJ, 554, 1095
Kokubu, E., & Ida, S. 2002, ApJ, 581, 666
Laplace, P. S. 1796, Exposition du Systeme du Monde (Courrier: Paris)
Marcy, G. W., Cochran, W. D., & Mayor, M. 2000, in Protostars and Planets IV, ed V. Mannings, A. P. Boss, & S. S. Russell (Tucson: Univ. Arizona Press), 1285
Mathis, J. S., Rumpl, W., & Nordsieck, K. H. 1977, ApJ, 217, 425
McCabe, C., Duchene, G., & Ghez, A. M. 2003, ApJ, 588, L113
Nakagawa, Y., Sekiya, M., & Hayashi, C. 1986, Icarus, 67, 375
Pollack, J. B., Hubickyj, O., Bodenheimer, P., Lissauer, J. J., Podolak, M., & Greenzweig, Y. 1996, Icarus, 124, 62
Prosser, C. F., Stauffer, J. R., Hartmann, L., Soderblom, D. R., Jones, B. F., Werner, M. W., & McCaughrean, M. J. 1994, ApJ, 421, 517
Safronov, V. S. 1969, Evolution of the Protoplanetary Cloud and Formation of the Earth and the Planets (Moscow: Nauka)
Sekiya, M. 1998, Icarus, 133, 298
Shuping, R. Y., Bally, J., Morris, M., & Throop, H. 2003, ApJ, 587, L109
Supulver, K. D., & Lin, D. N. C. 2000, Icarus, 146, 525
Takeuchi, T., & Artymowicz, P. 2001, ApJ, 557, 990
Takeuchi, T., & Lin, D. N. C. 2002, ApJ, 581, 1344
Thi, W. F., Pontoppidan, K. M., van Dishoeck, E. F., Dartois, E., & d’Hendecourt, L. 2002, A&A, 394, L27
Throop, H. B., Bally, J., Esposito, L. W., & McCaughrean, M. J. 2001, Science, 292, 1686
Weidenschilling, S. J. 1977, MNRAS, 180, 57
———. 1980, Icarus, 44, 172
———. 1984, Icarus, 60, 553
Weidenschilling, S. J., & Cuzzi, J. N. 1993, in Protostars & Planets III, ed. E. H. Levy & J. I. Lunine (Tucson: Univ. Arizona Press), 1031
Whipple, F. L. 1972, in From Plasma to Planet, ed. A. Elvius (New York: Wiley), 211
Wyatt, M. C., Dermott, S. F., Telesco, C. M., Fisher, R. S., Grogan, K., Holness, E. K., & Pina, R. K. 1999, ApJ, 527, 918
Youdin, A. N., & Shu, F. H. 2002, ApJ, 580, 494
Zuckerman, B., Forveille, T., & Kastner, J. H. 1995, Nature, 373, 494