A HODGE THEORETIC PROJECTIVE STRUCTURE ON RIEmann SURFACES

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Abstract. Given any compact Riemann surface $C$, there is a symmetric bidifferential $\tilde{\eta}$ on $C \times C$, with a pole of order two on the diagonal $\Delta \subset C \times C$, which is uniquely determined by the following two properties:

- the restriction of $\tilde{\eta}$ to $\Delta$ coincides with the constant function 1 on $\Delta$, and
- the cohomology class in $H^2(C \times C, \mathbb{C})/\langle [\Delta] \rangle$ corresponding to $\tilde{\eta}$ is of pure type $(1,1)$.

The restriction of $\tilde{\eta}$ to the nonreduced diagonal $3\Delta$ defines a projective structure on $C$. Since this projective structure on $C$ is completely intrinsic, it is natural to ask whether it coincides with the one given by the uniformization of $C$. Showing that the answer to it to be negative, we actually identify $\partial s$, where $s$ is this section of the moduli of projective structures over the moduli space of curves, to be the pullback of the Siegel form by the Torelli map.

1. introduction

Let $M_g$ denote the moduli space of smooth complex projective curves of genus $g$ and $A_g$ the moduli space of principally polarized complex abelian varieties of dimension $g$. Let

$$\tau : M_g \longrightarrow A_g$$

be the Torelli map that sends a curve to its Jacobian equipped with the theta-polarization; this $\tau$ is an orbifold immersion outside the hyperelliptic locus $[OS]$. The variety $A_g$ is equipped with an orbifold locally symmetric metric $\omega_S$ (the Siegel metric). The second fundamental form $\rho$ of the Torelli map with respect to $\omega_S$ was studied in [CPT], [CF], [CFG] (see also [GPT], [FP]).

In [CFG], an intrinsic holomorphic bidifferential

$$\tilde{\eta} \in H^0(C \times C, p^*K_C \otimes q^*K_C \otimes O_{C \times C}(2\Delta))$$

was constructed for every compact Riemann surface $C$, where $\Delta$ denotes the reduced diagonal in $S := C \times C$ and $K_C$ is the canonical line bundle of $C$, while $p$ and $q$ are the projections of $C \times C$ to the first and second factors respectively. This section $\tilde{\eta}$ is symmetric under the involution of $S$, and its restriction to the diagonal $\Delta$ is $1 \in H^0(\Delta, O_{\Delta}) = H^0(\Delta, K_{C \times C}(2\Delta)|_{\Delta})$. This bidifferential is also constructed in an unpublished book of Gunning [Gu2].

At the point $\tau(C) \in A_g$ corresponding to a non-hyperelliptic curve $C$, the second fundamental form $\rho$ of the Torelli map for the metric $\omega_S$ is the multiplication by the form $\tilde{\eta}$ [CFG].

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The above form $\hat{\eta}$ corresponds, via the projection formula, to a holomorphic homomorphism

$$\eta : TC \rightarrow p_*(q^*(K_C)(2\Delta)).$$

For any point $x \in C$, consider the natural homomorphism

$$j_x : H^0(C, K_C(2x)) \hookrightarrow H^1(C \setminus \{x\}, \mathbb{C}) = H^1(C, \mathbb{C}).$$

The restriction $\eta_x : T_x C \rightarrow H^0(C, K_C(2x))$ of $\eta$ is uniquely determined by the following two properties:

- the cohomology class $j_x(\eta_x(v)) \in H^1(C, \mathbb{C})$ is of type $(0, 1)$ for any $v \in T_x C$, and
- the composition of $\eta_x$ with the evaluation homomorphism $H^0(C, K_C(2x)) \rightarrow (K_C(2x))_x = T_x C$ is the identity map.

We prove the following (see (Theorem 6.1)):

The section $\hat{\eta}$ is the unique element of $H^0(S, K_S(2\Delta))$ of pure type $(1, 1)$ whose restriction to $\Delta$ is 1.

We give another interpretation of $\hat{\eta}$. It is shown here that the composition

$$\eta_x \circ \eta_x : T_x C \rightarrow H^1(C, \mathbb{C}),$$

where $\eta_x$ is the map in (1.2), actually coincides with the composition of the differential $T_x C \hookrightarrow H^1(C, \mathcal{O}_C)$, of the Abel-Jacobi map $C \rightarrow J^1(C)$, with the map $H^1(C, \mathcal{O}_C) \rightarrow H^1(C, \mathbb{C})$ given by the Hodge decomposition. More generally, $\eta$ corresponds to the Hodge decomposition (see Proposition 2.5 for the precise statement).

A projective structure on a Riemann surface $C$ is given by a holomorphic coordinate atlas on $C$ such that all the transition functions are Möbius transformations (not just holomorphic functions). Every Riemann surface admits a projective structure; the projective structures on $C$ form an affine space for the vector space $H^0(C, K_C^2)$. Note that $C$ has a natural projective structure given by the uniformization of $C$. (See [Gu1] for more on projective structures.)

The line bundle $(p^*K_C \otimes q^*K_C \otimes \mathcal{O}_{C \times C}(2\Delta))|_{2\Delta}$ over $2\Delta$ has a natural trivialization (details are in Section 3). Consider the space of all holomorphic sections $s$ of $p^*K_C \otimes q^*K_C \otimes \mathcal{O}_{C \times C}(2\Delta)$ over $3\Delta$ such that the restriction $s|_{2\Delta}$ coincides with the natural trivialization of $(p^*K_C \otimes q^*K_C \otimes \mathcal{O}_{C \times C}(2\Delta))|_{2\Delta}$. This space is evidently an affine space for $H^0(C, K_C^2)$. This affine space for $H^0(C, K_C^2)$ is canonically identified with the space of projective structures on $C$ (as mentioned above, it is an affine space for $H^0(C, K_C^2)$); the details are in Section 3. In particular, the restriction of the above bidifferential $\hat{\eta}$ to $3\Delta$ produces a projective structure on $C$. To clarify, this projective structure is completely intrinsic and defined by Hodge theory because $\hat{\eta}$ is so.

Although the general guess has been that this intrinsic projective structure on $C$ given by $\hat{\eta}$ should be the one given by the uniformization of $C$ (this is the only completely intrinsic one among the standard projective structures), we prove that they are different in general; see Theorem 4.2. Lemma 4.1 and Remark 4.5.

More precisely, we identify the projective structure on $C$ given by $\hat{\eta}$ up to a global automorphism of the moduli space of projective structures over $M_g$. To explain this, let $P_g$ be the moduli space of compact Riemann surfaces of genus $g$ equipped with a projective
structure. Let
\[ \Psi : P_g \rightarrow M_g \]
be the forgetful map that forgets the projective structure. For any \( C^\infty \) section \( f : M_g \rightarrow P_g \) of the above projection \( \Psi \), the section \( \overline{\partial}(f) \) is a \( C^\infty \) \((1,1)\)-form on \( M_g \), because \( P_g \) is a holomorphic affine bundle (torsor) over \( M_g \) for the holomorphic cotangent bundle \( \Omega^{1,0}_{M_g} \).

For the section \( \beta_u \) of \( \Psi \) given by the uniformization theorem, the \((1,1)\)-form \( \partial(\beta_u) \) coincides with the Weil–Petersson form on \( M_g \).

We prove the following (see Proposition 4.3 and Theorem 4.4):

For the section \( s : M_g \rightarrow P_g \) of \( \Psi \) given by the intrinsic bidifferential \( \hat{\eta} \), the \((1,1)\)-form \( \partial(s) \) on \( M_g \) coincides with a nonzero constant scalar multiple of \( \tau^*\omega_S \), where \( \omega_S \) is the Siegel \((1,1)\)-form on \( A_g \) and \( \tau \) is the Torelli map in (1.1).

The form \( \tau^*\omega_S \) is not a constant scalar multiple of the Weil–Petersson form on \( M_g \) for any \( g \geq 2 \).

2. Second fundamental form and the intrinsic bidifferential

We first recall some results from [CPT] and [CFG] on the second fundamental form of the Torelli locus. As before, \( M_g \) denotes the moduli space of smooth complex projective curves of genus \( g \) while \( A_g \) denotes the moduli space of principally polarized complex abelian varieties of dimension \( g \). Both \( M_g \) and \( A_g \) are complex orbifolds. We recall that \( A_g \) is the quotient of the Siegel space \( H_g \) by the action of the symplectic group \( \text{Sp}(2g, \mathbb{Z}) \), and hence the Hermitian symmetric metric on \( H_g \) descends to an orbifold locally symmetric metric (the Siegel metric) on \( A_g \); the corresponding Kähler form on \( A_g \) will be denoted by \( \omega_S \). Let \( \tau : M_g \rightarrow A_g \) be the Torelli map in (1.1).

Take a non-hyperelliptic curve \( C \in M_g \) with genus \( g \geq 2 \). Consider the short exact sequence
\[ 0 \rightarrow T_C \rightarrow \Omega \rightarrow N \rightarrow 0, \quad (2.1) \]
where \( d\tau \) is the differential of the map \( \tau \). Take the dual of the exact sequence in (2.1):
\[ 0 \rightarrow I_2(K_C) \rightarrow \text{Sym}^2(H^0(C, K_C)) \rightarrow H^0(C, 2K_C) \rightarrow 0, \quad (2.2) \]
where \( K_C \) is, as before, the canonical bundle of \( C \). Let \( II : \text{Sym}^2(T_C \rightarrow N_{JC, \Theta}) \) be the second fundamental form for the Torelli map, and let
\[ \rho := II^\vee : I_2(K_C) \rightarrow \text{Sym}^2(H^0(C, 2K_C)) \cong \text{Sym}^2(H^1(C, TC)) \]
be its dual.

Now take a point \( x \in C \). Let
\[ j_x : H^0(C, K_C(2x)) \hookrightarrow H^1(C \setminus \{x\}, \mathbb{C}) = H^1(C, \mathbb{C}) \]
be the injective homomorphism that associates to a meromorphic 1–form, with at most a double pole at \( x \), its de Rham cohomology class. Since
\[ h^0(C, K_C(2x)) = g + 1 \quad \text{and} \quad H^{1,0}(C) \subset j_x(H^0(C, K_C(2x))), \]
where $j_x$ is the homomorphism in [1,2], the inverse image $j_x^{-1}(H^{0,1}(C))$ has dimension 1; here $H^{0,1}(C)$ is considered as a subspace of $H^1(C, \mathbb{C})$ using the Hodge decomposition. If we fix a local holomorphic coordinate function $z$ on a neighborhood of $x$ with $z(x) = 0$, there exists a unique element $\mu$ in this line $j_x^{-1}(H^{0,1}(C))$ whose expression on $U$ is
\[
\mu|_U := \left(\frac{1}{z^2} + h(z)\right)dz,
\] where $h$ is a holomorphic function. So we have a map
\[
\eta_x : T_xC \rightarrow H^0(C, K_C(2x))
\]
that sends $\lambda \frac{\partial}{\partial z}(x)$ to $\lambda \mu$; this map is evidently independent of the choice of the coordinate function $z$.

The following is proved in [CFG].

Lemma 2.1 ([CFG, Lemma 3.5]). Identify $H^{0,1}(C)$ with $H^0(C, K_C)^*$ using Serre duality. Then the line $j_x(H^0(C, K_C(2x))) \cap H^{0,1}(C) \subset H^{0,1}(C)$ corresponds to the hyperplane in $H^0(C, K_C)$ defined by all 1-forms vanishing at $x$.

Set $S := C \times C$, and denote by $\Delta$ the reduced diagonal divisor $\{(x, x) \mid x \in C\} \subset S$. As before,
\[
p, q : S = C \times C \rightarrow C
\]
are the projections to the first and second factors respectively. Then $K_S = p^*K_C \otimes q^*K_C$. Define the line bundle
\[
L := (p^*K_C) \otimes (q^*K_C) \otimes O_S(2\Delta) = p^*K_C \otimes q^*K_C(2\Delta)
\]
on $S$. Set
\[
V := p_*(q^*K_C) \otimes O_S(2\Delta) \quad \text{and} \quad E := p_*L.
\]
The projection formula, [Ha, p. 426, A4], says that $E = K_C \otimes V$. Since $(q^*K_C \otimes O_S(2\Delta))|_{\{x\} \times C} = K_C(2x)$, we have
\[
H^0(p^{-1}(x), (q^*K_C \otimes O_S(2\Delta))|_{p^{-1}(x)}) = H^0(C, K_C(2x)).
\]
The fiber of the holomorphic vector bundle $V \rightarrow C$ over $x \in C$ is $H^0(C, K_C(2x))$, and the map $x \mapsto \eta_x$ (constructed in (2.5)) is a $C^\infty$ section of $K_C \otimes V = E$. This smooth section of $E$ will be denoted by $\eta$.

The following is proved in [CFG].

Proposition 2.2 ([CFG, Proposition 3.4]). The above $C^\infty$ section $\eta$ of $E$ is in fact holomorphic.

Since $E = p_*L$, there is an isomorphism $H^0(C, E) \cong H^0(S, L)$ that associates to any $\alpha \in H^0(C, E)$ the section $\tilde{\alpha} \in H^0(S, L)$ such that
\[
\alpha_x = \tilde{\alpha}|_{\{x\} \times C} \in T_x^*C \otimes H^0(C, K_C(2x)) = E_x.
\]
Thus there is a holomorphic section
\[
\tilde{\eta} \in H^0(S, L)
\]
corresponding to $\eta$.

The following is proved in [CFG].
Proposition 2.3 ([CFG, Lemma 3.5]). The section $\hat{\eta}$ in (2.7) is invariant under the tautological lift to $L$ of the involution of $C \times C$ defined by $(x, y) \mapsto (y, x)$.

Locally, we have

$$\hat{\eta} = \frac{d(z \circ p) \wedge d(z \circ q)}{(z \circ p - z \circ q)^2} + f(z \circ p, z \circ q)d(z \circ p) \wedge d(z \circ q)$$

around the diagonal, for any holomorphic coordinate function $z$ on $C$, where $f$ is a holomorphic function with $f(x, y) = f(y, x)$ (see Proposition 2.3). The form $\hat{\eta}$ also appears in an unpublished book of Gunning; he calls it an “intrinsic double differential of the second kind” [Gu2].

Using the natural identification of $H^0(C, K_C) \otimes H^0(C, K_C)$ with $H^0(S, p^*K_C \otimes q^*K_C)$, the kernel $I_2(K_C)$ in (2.2) is realized as a subspace of $H^0(S, p^*K_C \otimes q^*K_C(-\Delta))$. Since the elements of $I_2(K_C)$ are symmetric, they are in fact contained in $H^0(S, p^*K_C \otimes q^*K_C(-2\Delta))$. So for any $Q \in I_2(K_C)$, we have

$$Q \cdot \hat{\eta} \in H^0(S, (p^*K_C \otimes q^*K_C)^{\otimes 2}) = H^0(C, 2K_C) \otimes H^0(C, 2K_C).$$

The following Theorem 2.4 was proved in [CFG] using results of the earlier work [CPT].

We shall identify $(p^*K_C) \otimes (q^*K_C)$ with the canonical line bundle $K_S$ of $S = C \times C$ in the natural way. Note that this identification takes an invariant section of $(p^*K_C) \otimes (q^*K_C)$ under the involution of $C \times C$ to an anti-invariant section of $K_S$.

Theorem 2.4 ([CFG, Theorem 3.7]). Let $C$ be a non-hyperelliptic curve of genus $g \geq 4$. Then the homomorphism $\rho : I_2(K_C) \longrightarrow \text{Sym}^2(H^0(C, 2K_C))$ in (2.3) is the restriction to $I_2(K_C)$ of the multiplication map

$$H^0(S, K_S(-2\Delta)) \longrightarrow H^0(S, 2K_S), \; Q \longmapsto Q \cdot \hat{\eta}.$$ 

Consider the differential

$$\psi : TC \longrightarrow H^1(C, \mathcal{O}_C) \otimes \mathcal{O}_C$$

of the Abel-Jacobi map

$$C \longmapsto \text{Pic}^1(C), \; x \longmapsto \mathcal{O}_C(x).$$

It is the dual of the evaluation map $ev : H^0(C, K_C) \otimes \mathcal{O}_C \longrightarrow K_C$, after we identify $H^1(C, \mathcal{O}_C)^* \otimes H^0(C, K_C)$ using Serre duality. It can be shown that the following diagram is commutative

$$\begin{array}{ccc}
T_xC & \xrightarrow{\psi_x} & H^1(C, \mathcal{O}_C) \\
\downarrow \eta_x & & \downarrow i_C \\
H^0(C, K_C(2x)) = V_x & \xrightarrow{j_x} & H^1(C, \mathbb{C})
\end{array}$$

where $i_C$ is the natural identification of $H^1(C, \mathcal{O}_C)$ with $H^{0,1}(C)$ combined with the Hodge decomposition, while $j_x, \eta_x$ and $\psi_x$ are the homomorphisms in (1.2), (2.5) and (2.8) respectively. Indeed, (2.9) is exactly (3.1) in [CFG, p. 9], because $\psi_x(u), u \in T_xC$, is the element of $H^0(C, K_C)^* = H^1(C, \mathcal{O}_C)$ defined by $\omega \longmapsto \omega_x(u)$, where $\omega \in H^0(C, K_C)$. 
We shall first investigate the map \( j_x \) in (2.9). Let
\[
d : \mathcal{O}_S(\Delta) \longrightarrow \Omega^1_S(2\Delta) = p^*K_C(2\Delta) \oplus q^*K_C(2\Delta)
\]
be the de Rham differential. Let
\[
T : \mathcal{O}_S(\Delta) \longrightarrow q^*K_C(2\Delta)
\]
be the composition of this homomorphism \( d \) with the projection of \( p^*K_C(2\Delta) \oplus q^*K_C(2\Delta) \) to the second factor. We shall show that the kernel of \( T \) is the sheaf \( p^{-1}\mathcal{O}_C \). It is enough to prove this in terms of local holomorphic coordinates on the curve. Let \( z \) be a locally defined holomorphic coordinate function on \( C \); denote \( z \circ p \) and \( z \circ q \) by \( z_1 \) and \( z_2 \) respectively.

Set \( x = z_2 - z_1 \), \( y = z_1 + z_2 \), and take a local section \( f = \frac{a(y)}{x} + b(x, y) \) of \( \mathcal{O}_S(\Delta) \), where \( b \) is holomorphic. We have
\[
T(f) = \frac{\partial f}{\partial z_2} dz_2 = (-\frac{a(y)}{x^2} + \frac{a'(y)}{x}) dz_2 + \frac{\partial b}{\partial z_2} dz_2,
\]
ardence \( T(f) = 0 \) if and only if the following two hold:
\[
\begin{align*}
& \bullet a(y) \equiv 0, \text{ and } \\
& \bullet \frac{\partial b}{\partial z_2} = 0.
\end{align*}
\]

Therefore, \( T(f) = 0 \) if and only if \( f = b(z_1) \). Consequently, we have the exact sequence:
\[
0 \longrightarrow p^{-1}\mathcal{O}_C \longrightarrow \mathcal{O}_S(\Delta) \xrightarrow{T} T(\mathcal{O}_S(\Delta)) =: \mathcal{E} \longrightarrow 0. \tag{2.10}
\]

The sheaf \( \mathcal{E} \) in (2.10) is the kernel of the homomorphism
\[
r : q^*K_C(2\Delta) \longrightarrow p^{-1}\mathcal{O}_C
\]
constructed as follows: Let \( U \subset S \) be an open set and \( \omega \in q^*K_C(2\Delta)(U) \). If \( U \cap \Delta = \emptyset \), then set \( r(\omega) = 0 \). If \( (z_1, z_1) \in \Delta \cap U \), set
\[
r(\omega)(z_1) := \int_{\gamma_{z_1}} \omega,
\]
where \( \gamma_{z_1} \) is a small oriented circle around \( z_1 \) in the fiber \( \{z_1\} \times C \) \( \cap U \). In local coordinates, assuming \( D_{2\epsilon} \times D_{2\epsilon} \subset U \), we have \( r(\omega)(z_1) = \int_{|z_2-z_1|=\epsilon} \omega \). If \( \omega = \left( \frac{a(y)}{x^2} + \frac{b(y)}{x} + c(x, y) \right) dz_2 \), we have
\[
r(\omega)(z_1) = \int_{|z_2-z_1|=\epsilon} \left( \frac{a(z_1 + z_2)}{(z_2 - z_1)^2} + \frac{b(z_1 + z_2)}{z_2 - z_1} + c \right) dz_2
\]
\[
= \int_{|z_2-z_1|=\epsilon} \left( \frac{a(2z_1) + (z_2 - z_1)(a'(2z_1) + b(2z_1))}{(z_2 - z_1)^2} + \mathcal{H} \right) dz_2 = 2\pi \sqrt{-1} (a'(2z_1) + b(2z_1)),
\]
where \( \mathcal{H} \) is holomorphic. So \( r(\omega) \in \mathcal{O}(D_{2\epsilon}) \) and kernel(\( r \)) = \( \mathcal{E} \).

We will show that \( p_*\mathcal{E} \cong p_* q^*K_C(2\Delta) \). For that, take an open subset \( U \subset C \) biholomorphic to the disk. To prove that \( p_*\mathcal{E}(U) = p_* q^*K_C(2\Delta)(U) \), it suffices to show that if \( \omega \in p_* q^*K_C(2\Delta)(U) = q^*K_C(2\Delta)(U \times C) \), then \( \omega \in \mathcal{E}(U \times C) \).

Now, if \( x \in U \) is fixed, then \( \omega(x, \cdot) \in H^0(C, K_C(2x)) \) does not have residue at \( x \), and hence \( r(\omega)(x) = 0 \). From this it follows that \( \omega \in \mathcal{E}(U \times C) \).
Note that \( p^{-1}\mathcal{O}_C = p^{-1}\mathcal{O}_C \otimes_C q^{-1}\mathcal{C}_C \), and by K"unneth’s formula, [De, p. 244], for an open set \( U \subset C \),
\[
H^1(U \times C, p^{-1}\mathcal{O}_C \otimes_C q^{-1}\mathcal{C}_C) \cong (H^0(U, \mathcal{O}_C) \otimes H^1(C, \mathbb{C})) \oplus (H^1(U, \mathcal{O}_C) \otimes H^0(C, \mathbb{C})) = \mathcal{O}_C(U) \otimes H^1(C, \mathbb{C}),
\]
because \( H^1(U, \mathcal{O}_C) = 0 \). Consequently,
\[
R^1 p_*(p^{-1}\mathcal{O}_C)(U) = \mathcal{O}_C(U) \otimes H^1(C, \mathbb{C}).
\]
Using the same method we get that
\[
R^2 p_* p^{-1}(\mathcal{O}_C) = \mathcal{O}_C \otimes H^2(C, \mathbb{C}) = \mathcal{O}_C.
\]
So, applying \( p_* \) to the exact sequence
\[
0 \to \mathcal{E} \to q^* K_C(2\Delta) \to p^{-1}\mathcal{O}_C \to 0
\]
we conclude that \( R^1 p_* \mathcal{E} \cong \mathcal{O}_C \).

Now we apply \( p_* \) to the exact sequence (2.10). Since \( p_* (p^{-1}(\mathcal{O}_C)) = p_* \mathcal{O}_S(\Delta) = \mathcal{O}_C \), the following exact sequence is obtained:
\[
0 \to p_* \mathcal{E} \cong p_* (q^* K_C(2\Delta)) \xrightarrow{j} R^1 p_* (p^{-1}(\mathcal{O}_C)) \to R^1 p_* (\mathcal{O}_S(\Delta)) \to 0. \tag{2.11}
\]
So (2.11) becomes
\[
0 \to V \xrightarrow{j} H^1(C, \mathbb{C}) \otimes \mathcal{O}_C \to R^1 p_* \mathcal{O}_S(\Delta) \to 0.
\]
Notice that the above homomorphism \( j \) at any point \( x \in C \) is the map \( j_x \) in (2.9).

The following commutative diagram is obtained from (2.9):
\[
\begin{array}{ccc}
H^1(C, \mathcal{O}_C) \otimes \mathcal{O}_C & \xrightarrow{\psi} & TC \\
\downarrow i & & \downarrow j \\
\mathcal{O}_C & \xleftarrow{-\eta} & H^1(C, \mathbb{C}) \otimes \mathcal{O}_C
\end{array}
\tag{2.12}
\]
where \( \eta : TC \to p_* (q^* K_C(2\Delta)) \) is the map in (2.5).

We will construct the extension of the diagram in (2.12) to families of curves. So consider a family of curves
\[
\pi : \mathcal{C} \to B. \tag{2.13}
\]
Define the fiber product
\[
\mathcal{S} := \mathcal{C} \times_B \mathcal{C},
\]
and let
\[
\Delta_B := \{(x, x) \mid x \in \mathcal{C}\} \subset \mathcal{S}
\]
be the relative reduced diagonal divisor. Let
\[
\tilde{\pi}, \tilde{\eta} : \mathcal{S} \to \mathcal{C} \tag{2.14}
\]
be the projections to the first and second factors respectively. Let
\[
\Pi = \pi \circ \tilde{\pi} = \pi \circ \tilde{\eta} : \mathcal{S} \to B
\]
be the projection.

Now consider the map

\[ 0 \to T_{C/B} \xrightarrow{\psi} R^1\tilde{\varphi}_*\mathcal{O}_S \]

which is the dual of the evaluation map \( \tilde{\varphi}_*\tilde{q}^*K_{C/B} \to K_{C/B} \). The restriction of this map to the fiber over any point of \( B \) coincides with \( \psi \) in (2.12) (this re-use of notation should not cause any confusion).

To construct the map \( j \) in the relative setting, notice that we have

\[ \Omega^1_{S/B} = \tilde{\varphi}^*K_{C/B} \oplus \tilde{q}^*K_{C/B}, \]

so, it can be proved as above that there is a short exact sequence

\[ 0 \to \tilde{\varphi}^{-1}\mathcal{O}_C \to \mathcal{O}_S(\Delta_B) \xrightarrow{\tilde{T}} \text{image}(\tilde{T}) =: \tilde{\mathcal{E}} \to 0, \]

where \( \tilde{T} \) is the composition of the de Rham differential with the projection

\[ \Omega^1_{S/B}(2\Delta_B) = \tilde{\varphi}^*K_{C/B}(2\Delta_B) \oplus \tilde{q}^*K_{C/B}(2\Delta_B) \to \tilde{q}^*K_{C/B}(2\Delta_B). \]

Applying \( \tilde{\varphi}_* \) to (2.16) we get that

\[ 0 \to \tilde{\varphi}_*\tilde{\mathcal{E}} \to R^1\tilde{\varphi}_*(\tilde{\varphi}^{-1}\mathcal{O}_C) \to R^1\tilde{\varphi}_*(\mathcal{O}_S(\Delta_B)) \to 0. \]

It can be shown likewise above that

\[ \tilde{\varphi}_*\tilde{\mathcal{E}} \cong \tilde{\varphi}_*(\tilde{q}^*K_{C/B}(2\Delta_B)), \]

and \( R^1\tilde{\varphi}_*(\tilde{\varphi}^{-1}\mathcal{O}_C) \cong R^1\tilde{\varphi}_*\mathcal{C}_S \otimes \mathcal{O}_C \), so we have

\[ j : \tilde{\varphi}_*\tilde{\mathcal{E}} \cong \tilde{\varphi}_*(\tilde{q}^*K_{C/B}(2\Delta_B)) \to R^1\tilde{\varphi}_*(\tilde{\varphi}^{-1}\mathcal{O}_C) \cong R^1\tilde{\varphi}_*\mathcal{C}_S \otimes \mathcal{O}_C. \]

We have the relative version

\[ \eta : T_{C/B} \to \tilde{\varphi}_*(\tilde{q}^*K_{C/B}(2\Delta_B)) \]

of the map \( \eta \) in (2.12), and also the composition

\[ T_{C/B} \xrightarrow{\eta} \tilde{\varphi}_*(\tilde{q}^*K_{C/B}(2\Delta_B)) \xrightarrow{j} R^1\tilde{\varphi}_*\mathcal{C}_S \otimes \mathcal{O}_C. \]

Consider the variation of Hodge structure for the family \( \tilde{\varphi} : S \to C \):

\[ 0 \to \tilde{\varphi}_*\tilde{q}^*K_{C/B} \to R^1\tilde{\varphi}_*\mathcal{C}_S \otimes \mathcal{O}_C \to R^1\tilde{\varphi}_*\mathcal{O}_S \to 0. \]

Let

\[ i : R^1\tilde{\varphi}_*\mathcal{O}_S \to R^1\tilde{\varphi}_*\mathcal{C}_S \otimes \mathcal{O}_C \]

be the \( C^\infty \) splitting of it given by the Hodge decomposition. The homomorphism in (2.19) is the relative version of the map \( i \) in (2.12).

Hence we have proved the following:
Proposition 2.5. The diagram

\[ \begin{array}{ccc}
T_{C/B} & \xrightarrow{\psi} & R^1\tilde{p}_*\mathcal{O}_S \\
\downarrow \eta & & \downarrow i \\
\tilde{p}_*(q^*K_{C/B}(2\Delta_B)) & \xrightarrow{\psi} & R^1p_*c_S \otimes \mathcal{O}_C \\
\end{array} \]

is commutative.

Let

\[ 0 \rightarrow \mathcal{F}^1 := \pi_*K_{C/B} \rightarrow R^1\pi_*\mathcal{C}_C \rightarrow R^1\pi_*\mathcal{O}_C \rightarrow 0 \]

be the variation of Hodge structure for the family \( \pi \) in (2.13). Note that its pullback

\[ 0 \rightarrow \pi^*\mathcal{F}^1 \rightarrow \pi^*(R^1\pi_*\mathcal{C}_C) \rightarrow \pi^*(R^1\pi_*\mathcal{O}_C) \rightarrow 0, \]

(2.20)

to \( \mathcal{C} \) coincides with (2.18). We have the following isomorphisms:

\[ R^1\tilde{p}_*\mathcal{O}_S \cong \pi^*(R^1\pi_*\mathcal{O}_C) \quad \text{and} \quad R^1\tilde{p}_*\mathcal{C}_S \otimes \mathcal{O}_C \cong \pi^*(R^1\pi_*\mathcal{C}_C \otimes \mathcal{O}_B). \]

Consider the pull-back of (2.21) via the map \( \psi \):

\[ \begin{array}{ccc}
0 & \xrightarrow{\pi^*\mathcal{F}^1} & R^1\tilde{p}_*\mathcal{C}_S \otimes \mathcal{O}_C \\
\uparrow i_{\psi} = -j_{\eta} & & \updownarrow \psi \\
0 & \xrightarrow{\pi^*\mathcal{F}^1} & \mathcal{H} \\
& \rightarrow & \rightarrow T_{C/B} \\
& \rightarrow & \rightarrow 0 \\
\end{array} \]

(2.22)

where \( i \) is the homomorphism in (2.19), and \( \mathcal{H} := j(\tilde{p}_*(q^*K_{C/B}(2\Delta_B))). \)

Corollary 2.6. The image of the above \( \mathcal{C}^\infty \) homomorphism

\[ -j \circ \eta = i \circ \psi : T_{C/B} \rightarrow R^1\tilde{p}_*\mathcal{C}_S \otimes \mathcal{O}_C \]

lies in \( \mathcal{H} \), and it gives a \( \mathcal{C}^\infty \) splitting of the bottom exact sequence in the diagram (2.22).

Proof. This follows from Proposition 2.5. \( \square \)

3. Projective structures on a Riemann surface

Let \( \mathbb{V} \) be a complex vector space of dimension two. Let \( \mathbb{P}(\mathbb{V}) \) be the projective space parametrizing the lines in \( \mathbb{V} \). The group \( \text{PGL}(\mathbb{V}) = \text{GL}(\mathbb{V})/\mathbb{C}^* \) acts faithfully on \( \mathbb{P}(\mathbb{V}) \). The holomorphic cotangent bundle of \( \mathbb{P}(\mathbb{V}) \) will be denoted by \( K_{\mathbb{P}(\mathbb{V})} \).

Let \( C \) be a compact connected Riemann surface. A holomorphic coordinate chart on \( C \) is a pair of the form \( (U, \phi) \), where \( U \subset C \) is an open subset and \( \phi : U \rightarrow \mathbb{P}(\mathbb{V}) \) is a holomorphic embedding. A holomorphic coordinate atlas on \( C \) is a collection of coordinate charts \( \{(U_i, \phi_i)\}_{i \in I} \) such that \( C = \bigcup_{i \in I} U_i \). A projective structure on \( C \) is given by a coordinate atlas \( \{(U_i, \phi_i)\}_{i \in I} \) such that for all pairs \( i, j \in I \times I \) for which \( U_i \cap U_j \neq \emptyset \), there is an element \( \tau_{j,i} \in \text{PGL}(\mathbb{V}) \) such that \( \phi_j \circ \phi_i^{-1} \) is the restriction of \( \tau_{j,i} \) to \( \phi_i(U_i \cap U_j) \) of the automorphism of \( \mathbb{P}(\mathbb{V}) \) given by \( \tau_{j,i} \). Two such collections of pairs
\{(U_i, \phi_i)\}_{i \in I} \text{ and } \{(U_i, \phi_i)\}_{i \in J} \text{ are called equivalent if their union } \{(U_i, \phi_i)\}_{i \in I \cup J} \text{ is again a part of a collection of pairs satisfying the above condition. A projective structure on } C \text{ is an equivalence class of collection of pairs satisfying the above condition; see [Gu1]. There are projective structures on } C, \text{ for example, the uniformization of } C \text{ produces a projective structure on } C. \text{ In fact, the space of all projective structures on } C \text{ is an affine space for the vector space } H^0(C, K_C^\otimes 2) \text{ [Gu1].}

For } i = 1, 2, \text{ let } p_i : \mathbb{P}(V) \times \mathbb{P}(V) \longrightarrow \mathbb{P}(V) \text{ be the projection to the } i\text{-th factor. Consider the holomorphic line bundle }

L_0 := (p_1^*K_{\mathbb{P}(V)}) \otimes (p_2^*K_{\mathbb{P}(V)}) \otimes \mathcal{O}_{\mathbb{P}(V) \times \mathbb{P}(V)}(2\Delta_0) \longrightarrow \mathbb{P}(V) \times \mathbb{P}(V),

where } \Delta_0 \subset \mathbb{P}(V) \times \mathbb{P}(V) \text{ is the reduced diagonal divisor. Since } \text{Pic}(\mathbb{P}(V) \times \mathbb{P}(V)) = \text{Pic}(\mathbb{P}(V)) \oplus \text{Pic}(\mathbb{P}(V)), \text{ the line bundle } L_0 \text{ is trivializable. Also, using the Poincaré adjunction formula it follows that } L_0|_{\Delta_0} \text{ is canonically trivialized. Hence combining these it follows that } L_0 \text{ is canonically trivialized. Let }

\sigma_0 \in H^0(\mathbb{P}(V) \times \mathbb{P}(V), L_0)

be the section giving the canonical trivialization. Note that the diagonal action of PGL(V) on } \mathbb{P}(V) \times \mathbb{P}(V) \text{ has a canonical lift to an action of PGL(V) on } L_0. \text{ The section } \sigma_0 \text{ in (3.1) is fixed by this action of PGL(V) on } L_0. \text{ The involution of } \mathbb{P}(V) \times \mathbb{P}(V) \text{ defined by } (x, y) \longmapsto (y, x) \text{ lifts canonically to an involution of } L_0, \text{ and the section } \sigma_0 \text{ is evidently preserved by this involution of } L_0.

Let } C \text{ be a compact connected Riemann surface. As before } p, q : C \times C \longrightarrow C \text{ are the projections to the first and second factors respectively. Let }

L = (p^*K_C) \otimes (q^*K_C) \otimes \mathcal{O}_{C \times C}(2\Delta) \longrightarrow C \times C

be the holomorphic line bundle in (2.6), where } \Delta \subset C \times C \text{ is the reduced diagonal divisor. }

As noted before, using the adjunction formula, the restriction } L|_{\Delta} \text{ is the trivial line bundle on } \Delta. \text{ In fact, the trivialization of } L|_{\Delta} \text{ extends to a canonical trivialization of } L|_{2\Delta} \text{ (see [BR1] p. 754, Theorem 2.1, [BR2] p. 688, Theorem 2.2)).}

Let } P = \{(U_i, \phi_i)\}_{i \in I} \text{ be a projective structure on } C. \text{ For any } i \in I, \text{ there is a natural isomorphism }

(\phi_i \times \phi_i)^*L_0 \sim \longrightarrow L|_{U \times U}

given by the differential of the map } \phi_i. \text{ Using this isomorphism, the section } \sigma_0 \text{ in (3.1) produces a section }

\sigma_{0,i} = (\phi_i \times \phi_i)^*\sigma_0 \in H^0(U \times U, L|_{U \times U}).

Since } \sigma_0 \text{ is fixed by the action of PGL(V) on } L_0, \text{ these sections } \sigma_{0,i} \text{ patch together compatibly to define a section }

\tilde{\sigma} \in H^0(U, L|_U), \tag{3.2}

where } U \subset C \times C \text{ is an analytic open subset containing } \Delta. \text{ Let }

\nu : C \times C \longrightarrow C \times C \tag{3.3}

be the involution defined by } (x, y) \longmapsto (y, x). \text{ This involution lifts canonically to an involution of } L. \text{ The section } \tilde{\sigma} \text{ in (3.2) has the following two properties:

(1) The above involution of } L \text{ lifting } \nu \text{ in (3.3) preserves } \tilde{\sigma}, \text{ and
(2) the restriction $\tilde{\sigma}|_{2\Delta}$ coincides with the canonical trivialization of $L|_{2\Delta}$; this follows from [BR1, p. 756, Proposition 2.10] (note that $\tilde{\sigma}|_{\Delta}$ coincides with the canonical trivialization of $L|_{\Delta}$).

Let

$$S(C) \subset H^0(3\Delta, L|_{3\Delta})$$

be the locus of all sections $s$ such that the restriction $s|_{2\Delta}$ coincides with the canonical trivialization of $L|_{2\Delta}$. This $S(C)$ is evidently an affine space for $H^0(C, K_C^\otimes 2)$. Let

$$\sigma \in H^0(3\Delta, L|_{3\Delta}) \quad (3.4)$$

be the restriction of the section $\tilde{\sigma}$ in (3.2) to the nonreduced divisor $3\Delta$. As noted above, we have $\sigma \in S(C)$.

Let $P(C)$ denote the space of all projective structures on $C$. We have a map

$$\Phi : P(C) \rightarrow S(C) \quad (3.5)$$

that sends any $P \in P(C)$ to $\sigma \in S(C)$ constructed in (3.4) using $P$. This map $\Phi$ is an isomorphism of affine spaces for $H^0(C, K_C^\otimes 2)$ (see [BR1, p. 757, Theorem 3.2] and [BR1, p. 758, Lemma 3.6]; see also [BR2, p. 688, Theorem 2.2]).

Let $\pi : C \rightarrow B$ be a smooth holomorphic family of irreducible complex projective curves of genus $g$. Let $\tilde{p}, \tilde{q} : C \times_B C \rightarrow C$ be the projections to the first and second factors respectively. The reduced relative diagonal divisor in $C \times_B C$ will be denoted by $\Delta_B$. Let $K_C \rightarrow C$ be the relative canonical bundle for the projection $\pi$. Consider the family of surfaces

$$\Pi : C \times_B C \rightarrow B. \quad (3.6)$$

Let

$$L := (\tilde{p}^*K) \otimes (\tilde{q}^*K) \otimes \mathcal{O}_{C \times_B C}(2\Delta_B) \quad (3.7)$$

be the line bundle on this family $C \times_B C$. Using the map $\Pi$ in (3.6), construct the direct images

$$V := \Pi_*\left(\mathcal{O}_{C \times_B C}(-3\Delta_B) \otimes L\right) \rightarrow B \quad (3.8)$$

and

$$V_2 := \Pi_*\left(\mathcal{O}_{C \times_B C}(-2\Delta_B) \otimes L\right) \rightarrow B$$

which are holomorphic vector bundles over $B$. There is a natural surjective homomorphism

$$\Psi : V \rightarrow V_2 \quad (3.9)$$

given by restriction of sections to $2\Delta_B$. The vector bundle $V_2$ has a tautological holomorphic section given by the earlier mentioned canonical trivialization of $L|_{2\Delta}$ for any curve $C$. This holomorphic section of $V_2$ will be denoted by $s_0$. Now define

$$\hat{V} := \Psi^{-1}(s_0) \subset V, \quad (3.10)$$

where $\Psi$ is the projection in (3.9). We note that $\hat{V}$ is an affine bundle over $B$ for the vector bundle $\pi_*K_C^\otimes 2$. Indeed, this follows immediately from the fact that

$$\ker(\Psi) = \pi_*K_C^\otimes 2. \quad (3.11)$$
The following lemma is an immediate consequence of the isomorphism $\Phi$ in (3.5).

**Lemma 3.1.** The $C^\infty$ (respectively, holomorphic) sections of the fiber bundle $\hat{V} \rightarrow B$ in (3.10) are in a natural bijective correspondence with the $C^\infty$ (respectively, holomorphic) families of projective structures for the family of curves $C$.

Let

$$\beta : B \rightarrow \hat{V}$$

be a $C^\infty$ section. Denote the Dolbeault operator for the holomorphic vector bundle $V$, defined in (3.8), by $\overline{\partial}V$. Since $\beta$ is also a section of $V$, we have

$$\overline{\partial}V(\beta) \in \Omega^{0,1}(B, V).$$

The projection $\Psi$ in (3.9) is evidently holomorphic, and recall that $\Psi(\beta)$ is a holomorphic section of $V_2$. These imply that for the section $\Psi \circ \overline{\partial}V(\beta) \in \Omega^{0,1}(B, V_2),$

$$\Psi \circ \overline{\partial}V(\beta) = \overline{\partial}V_2(\Psi(\beta)) = 0,$$

where $\overline{\partial}V_2$ is the Dolbeault operator for the holomorphic vector bundle $V_2$. Hence from (3.11) it follows immediately that

$$\overline{\partial}V(\beta) \in \Omega^{0,1}(B, \pi_*K^{\otimes 2}). \quad (3.12)$$

Let

$$\Gamma : B \rightarrow \mathbb{M}_g \quad (3.13)$$

be the holomorphic map to the moduli space of curves corresponding to the above family $C$ over $B$.

Let $\omega_{wp}$ be the Weil–Petersson Kähler form on $\mathbb{M}_g$; it is actually an orbifold form. The uniformization theorem gives a $C^\infty$ section of $\hat{V}$. Let

$$\beta^u : B \rightarrow \hat{V} \quad (3.14)$$

be the section given by the uniformization theorem. For this $\beta^u$ it was shown by Takhtajan and Zograf that

$$\overline{\partial}V(\beta^u) = \Gamma^*\omega_{wp} \in \Omega^{0,1}(B, \Gamma^*\Omega^{1,0}_{\mathbb{M}_g}) \quad (3.15)$$

[ZT, p. 310, Theorem 2], [ZT, p. 311, Remark 3]; see [IV, p. 214, Theorem 1.7] for an excellent exposition (see also [Mc, p. 355, Theorem 9.2]).

Let $\beta_1, \beta_2 : B \rightarrow \hat{V}$ be two $C^\infty$ sections of $\hat{V} \rightarrow B$ such that

$$\overline{\partial}V(\beta_1) = \overline{\partial}V(\beta_2).$$

Since $\Psi(\beta_1 - \beta_2) = 0$, where $\Psi$ is the projection in (3.9), from (3.11) we conclude that

$$\beta_1 - \beta_2 \in H^0(B, \Gamma^*\Omega^{1,0}_{\mathbb{M}_g}), \quad (3.16)$$

where $\Gamma$ is the map in (3.13).

**Lemma 3.2.** Let $\beta_1, \beta_2 : B \rightarrow \hat{V}$ be two $C^\infty$ sections of $\hat{V} \rightarrow B$ such that

$$\overline{\partial}V(\beta_1) = \overline{\partial}V(\beta_2).$$
Then $\beta_1 = T \circ \beta_2$, where $T$ is a holomorphic automorphism of the $\Gamma^*\Omega^{1,0}_{M_g}$-torsor $\hat{V}$ over $B$. Conversely, for any holomorphic automorphism $T$ of the $\Gamma^*\Omega^{1,0}_{M_g}$-torsor $\hat{V}$, and for any $C^\infty$ section $\beta : B \to \hat{V}$ of $\hat{V} \to B$, 
$$\frac{\partial}{\partial V} (\beta) = \frac{\partial}{\partial V} (T \circ \beta).$$

**Proof.** This follows from (3.16), because $H^0(B, \Gamma^*\Omega^{1,0}_{M_g})$ is in fact the group of holomorphic automorphisms of the $\Gamma^*\Omega^{1,0}_{M_g}$-torsor $\hat{V}$ over $B$. □

Take any $b \in B$. For the curve $C := \pi^{-1}(b)$, consider the section $\hat{\eta}$ constructed in (2.7). Its restriction to $3\Delta$ is a projective structure on $C$. Denote this projective structure on $C$ by $P_b$. So we have a $C^\infty$ section $\beta^\eta : B \to \hat{V}, b \mapsto P_b$.

(3.17)

### 4. Geometric structure

Consider the universal family of curves $\pi : C \to M_g$ over the moduli space of curves, which exists in the orbifold category. Let

$$\Pi : S := C \times_{M_g} C \to M_g$$

be the projection from the fiber product. Let

$$E := \Pi_* \mathbb{L}$$

be the direct image, where $\mathbb{L}$ is defined in (3.7). Let

$$\tilde{\eta} \in C^\infty(M_g, E)$$

be the relative version of the section in (2.7), so $\tilde{\eta}$ corresponds to a homomorphism $\eta$ constructed as in (2.17). This section $\tilde{\eta}$ should be considered in the orbifold category.

The section $\tilde{\eta}$ in (4.2) produces a $C^\infty$ section of the fiber bundle $\hat{V}$ in (3.10), simply by restricting a section of $\mathbb{L}|_{C \times C}$ over $C \times C$ to the nonreduced diagonal $3\Delta \subset C \times C$. The $C^\infty$ sections of $\hat{V}$ are in a bijective correspondence with the $C^\infty$ families of projective structures on $C$ (see Lemma 3.1).

**Lemma 4.1.** Assume that $g \geq 3$. The $C^\infty$ section $\beta^\eta$ (see (3.17)) of $\hat{V}$ produced by $\tilde{\eta}$ does not coincide with the section of $\hat{V}$ produced by the uniformization of Riemann surfaces (it is the section $\beta^u$ in (3.14)).

**Proof.** Let $K := K_{C/M_g} \to C$ be the relative canonical bundle for the projection $\pi$ to $M_g$. Define

$$F := \Pi_* ((\tilde{p}^* \mathcal{K}) \otimes (\tilde{q}^* \mathcal{K}))$$

and

$$F_1 := \Pi_* (\tilde{p}^* \mathcal{K} \otimes \tilde{q}^* \mathcal{K}(\Delta_{M_g})), $$

where $\tilde{p}$ and $\tilde{q}$ are the projections as in (2.13). We have $F = F_1$, because $H^0(C \times C, K_{C \times C}) = H^0(C \times C, K_{C \times C}(\Delta))$ for any compact Riemann surface $C$. Hence there is a short exact sequence

$$0 \to F \to E \to \mathcal{O}_{M_g} \to 0,$$

where $E$ is constructed in (4.1). The projection $E \to \mathcal{O}_{M_g}$ in (4.3) sends the smooth section $\tilde{\eta}$ of $E$ in (4.2) to the constant function $1$ on $M_g$. 

The involution of $S$ defined by $(x, y) \mapsto (y, x)$ lifts canonically to both $(\tilde{p}^* K) \otimes (\tilde{q}^* K)$ and $\tilde{p}^* K \otimes \tilde{q}^* K(\Delta M_g)$; these lifts of action produce decompositions

$$F = F^s \oplus F^a$$
$$E = E^s \oplus E^a$$

into the symmetric and anti-symmetric parts. Note that $F^s = \text{Sym}^2(\mathcal{F}^1)$, where $\mathcal{F}^1 \to M_g$ is the Hodge bundle defined as in (2.20). From (4.3) we have the short exact sequence

$$0 \to F^s = \text{Sym}^2(\mathcal{F}^1) \to E^s \to \mathcal{O}_{M_g} \to 0.$$  

(4.4)

The fiberwise multiplication map $H^0(C, K_C) \otimes H^0(C, K_C) \to H^0(C, K_C^2)$ produces a $\mathcal{O}_{M_g}$–linear homomorphism

$$m : \text{Sym}^2(\mathcal{F}^1) \to \pi_* K^\otimes 2 = \Omega_{M_g}^{1,0},$$

(4.5)

which is the dual of the differential of the Torelli map $\tau$ in (1.1). This map $m$ produces a homomorphism

$$m' : \Omega^{0,1}_{M_g}(\text{Sym}^2(\mathcal{F}^1)) \to \Omega^{1,1}_{M_g},$$

(4.6)

by tensoring it with $\text{Id}_{\Omega^{0,1}_{M_g}}$. From (4.3) we have the commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & \text{Sym}^2(\mathcal{F}^1) \\
\bigg\downarrow m & & \bigg\downarrow \approx \\
T'^* M_g & \longrightarrow & \mathcal{V}' \\
\bigg\downarrow r & & \\
0 & \longrightarrow & \mathcal{O}_{M_g} \\
\end{array}
$$

(4.7)

where $\mathcal{V}'$ is the subbundle of $\mathcal{V}$ (defined in (3.8)) generated by $\tilde{V}$ defined in (3.10), the map $r$ is the restriction of sections to $3\Delta M_g$ and $m$ is the homomorphism in (4.5).

From Proposition 2.3 it follows that

$$\tilde{\eta} \in C^\infty(M_g, E^s),$$

where $\tilde{\eta}$ is the section in (4.2). Note that

$$\beta^n \defeq r \circ \tilde{\eta} \in C^\infty(M_g, \mathcal{V}'),$$

where $r$ is the restriction homomorphism in (4.7), is the $C^\infty$ section $\beta^n$ in (3.17) for the family parametrized by $B = M_g$.

Since the surjective homomorphism in (4.4) sends $\tilde{\eta}$ to the function 1 on $M_g$, it follows that $\overline{\partial}\tilde{\eta}$ is a section of $\mathcal{A}^{0,1}\text{Sym}^2(\mathcal{F}^1)$. As the homomorphism $r$ in (4.7) is holomorphic, from the commutativity of (4.7) we conclude that

$$m'(\overline{\partial}\tilde{\eta}) = \overline{\partial} \beta^n,$$

(4.8)

where $m'$ is the homomorphism in (4.6).

Take a hyperelliptic curve $C \in M_g$, and take any nonzero $v \in T_C M_g$ that is sent to $-v$ by the hyperelliptic involution $\xi$ of $C$; since $g \geq 3$, such a $v$ exists. It can be shown that

$$(\overline{\partial}\tilde{\eta}(C))(\overline{v}) = 0.$$  

Indeed, $\overline{\partial}\tilde{\eta} \in C^\infty(M_g, \Omega_{M_g}^{0,1}\text{Sym}^2(\mathcal{F}^1))$ defines a $C^\infty$ homomorphism

$$\overline{\partial}\tilde{\eta}(C) : T^{0,1} M_g \to \text{Sym}^2(\mathcal{F}^1).$$
A HODGE THEORETIC PROJECTIVE STRUCTURE

(15)

Since the hyperelliptic involution \( \xi \) acts trivially on the fiber \((\text{Sym}^2(F))_C\) and the homomorphism \( \tilde{\eta}(C) \) is \( \xi \)-invariant, it follows that \((\tilde{\eta}(C))C\) = 0. Therefore, from (4.3) it follows that \((\tilde{\beta}(\eta))(\tau) = 0\). Hence the \((1,1)\)-form \( \tilde{\beta} \) fails to be positive at \( C \). This implies that \( \tilde{\beta} \) is not a nonzero scalar multiple of the Weil–Petersson \( \omega_{wp} \) form on \( M_g \), because \( \omega_{wp} \) is Kähler. Consequently, from (3.13) we conclude that the section \( \beta \eta \) of \( \hat{V} \) produced by \( \tilde{\eta} \) does not coincide with the section \( \beta u \) in (3.14) produced by the uniformization of Riemann surfaces.

**Theorem 4.2.** Consider the projective structure given by the uniformization of Riemann surfaces. Let \( \beta u \in C^\infty(M_g, V') \) be the corresponding section (as in (3.14)). There is no \( C^\infty \) section \( \gamma : M_g \longrightarrow E^a \) such that \( r(\gamma) = \beta u \), where \( r \) is the restriction map in (4.7).

**Proof.** This follows from the proof of Lemma 4.1 in a straightforward way. We omit the details.

Let

\[ \varphi : U \longrightarrow A_g \]

be the universal family of principally polarized abelian varieties. Let

\[ 0 \longrightarrow \mathcal{F} \longrightarrow R^1\varphi_*\mathcal{C}_U \otimes \mathcal{O}_{A_g} \longrightarrow R^1\pi_*\mathcal{O}_U \cong (\mathcal{F})^\vee \longrightarrow 0 \quad (4.9) \]

be the exact sequence for the Hodge filtration. The notation \( \mathcal{F} \) is re-used; note that \( \mathcal{F} \) in (2.2) is the pullback of \( \mathcal{F} \) in (4.9) by the map \( \tau \) in (1.1). Let

\[ i : R^1\pi_*\mathcal{O}_U = (\mathcal{F})^\vee \longrightarrow R^1\varphi_*\mathcal{C}_U \otimes \mathcal{O}_{A_g} \quad (4.10) \]

be the \( C^\infty \) splitting of (4.9) given by the Hodge decomposition.

Tensoring (4.9) with \( \mathcal{F} \), we get

\[ 0 \longrightarrow \mathcal{F} \otimes \mathcal{F} \longrightarrow (R^1\varphi_*\mathcal{C}_U) \otimes \mathcal{F} \overset{\chi}{\longrightarrow} (\mathcal{F})^\vee \otimes \mathcal{F} \longrightarrow 0 \quad (4.11) \]

Let

\[ i' = i \otimes \text{Id}_{\mathcal{F}} : (\mathcal{F})^\vee \otimes \mathcal{F} \longrightarrow R^1\varphi_*\mathcal{C}_U \otimes \mathcal{F} \]

be the \( C^\infty \) splitting. Define the homomorphism

\[ s : \mathcal{O}_{A_g} \longrightarrow (\mathcal{F})^\vee \otimes \mathcal{F} = \text{End}(\mathcal{F}) \neq c \longmapsto c \cdot \text{Id} \, . \]

Define

\[ \mathcal{G} := \chi^{-1}(s(\mathcal{O}_{A_g})) \subset (R^1\varphi_*\mathcal{C}_U) \otimes \mathcal{F} \, , \]

where \( \chi \) is the projection in (4.11). Now we have the commutative diagram

\[ 0 \longrightarrow \mathcal{F} \otimes \mathcal{F} \longrightarrow R^1\varphi_*\mathcal{C}_U \otimes \mathcal{F} \longrightarrow (\mathcal{F})^\vee \otimes \mathcal{F} \longrightarrow 0 \]

\[ 0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_{A_g} \longrightarrow 0 \quad (4.12) \]

The image of \( i' \circ s \) clearly lies in \( \mathcal{G} \), and the \( C^\infty \) homomorphism \( i' \circ s \) : \( \mathcal{O}_{A_g} \longrightarrow \mathcal{G} \) is a \( C^\infty \) splitting of the bottom exact sequence in (4.12). Taking the quotient by \( \Lambda^2 \mathcal{F} \) of the bottom exact sequence in (4.12) yields the exact sequence

\[ 0 \longrightarrow \text{Sym}^2(\mathcal{F}) \longrightarrow \mathcal{G}^+ \overset{f}{\longrightarrow} \mathcal{O}_{A_g} \longrightarrow 0 \quad (4.13) \]
it has a $C^\infty$ splitting

\[ \sigma := q^1 \circ i' \circ s : \mathcal{O}_{A_g} \longrightarrow \mathcal{G}^+, \]

where $q^1 : \mathcal{G} \longrightarrow \mathcal{G}^+$ is the projection.

Since the homomorphism $f$ in (4.13) is holomorphic, for the section $h := \sigma(1)$ of $\mathcal{G}^+$,

\[ f(\overline{\partial}h) = \overline{\partial}(f \circ h) = \overline{\partial}(1) = 0, \]

and consequently from (4.13) it follows that

\[ \omega := \overline{\partial}h \]

is a $(0,1)$-form on $A_g$ with values in $\text{Sym}^2(F^1) = T^*A_g$. In other words, $\omega$ is a $(1,1)$-form on $A_g$.

**Proposition 4.3.** The $(1,1)$-form $\omega$ in (4.14) is a nonzero constant scalar multiple of the Kähler form $\omega_S$ for the Siegel metric on $A_g$.

**Proof.** From the construction of $\omega$ it follows that the pullback of $\omega$ to the Siegel space $H_g$ is preserved by the action of $\text{Sp}(2g,\mathbb{R})$ on $H_g$. This implies that $\omega$ is a constant scalar multiple of the Kähler form $\omega_S$ on $A_g$. This scalar factor is nonzero because the Hodge decomposition $i$ in (4.10) is not holomorphic. \[ \square \]

Consider $\tilde{\mathcal{N}}$ in (3.10) for the universal family of curves $\pi : \mathcal{C} \longrightarrow M_g$. Let

\[ \beta^n \in C^\infty(M_g, \tilde{\mathcal{N}}) \]

be the section in (3.17) given by $\tilde{\eta}$ in (4.2).

**Theorem 4.4.** For the section $\beta^n$ in (4.15),

\[ \overline{\partial}(\beta^n) = \tau^*\omega, \]

where $\omega$ and $\tau$ are constructed in (4.14) and (1.1) respectively.

**Proof.** First tensoring the diagram in (2.22) with the relative canonical bundle $K := K_{C/M_g}$, we get

\[
\begin{array}{cccccc}
0 & \longrightarrow & \pi^*F^1 \otimes K & \longrightarrow & R^1\tilde{p}_*\mathcal{C}_S \otimes K & \longrightarrow & R^1\tilde{p}_*\mathcal{O}_S \otimes K & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \psi \otimes \text{Id}_K & & \\
0 & \longrightarrow & \pi^*F^1 \otimes K & \longrightarrow & \mathcal{H} \otimes K & \longrightarrow & \mathcal{O}_C & \longrightarrow & 0
\end{array}
\]

together with a $C^\infty$ splitting $\mathcal{O}_C \longrightarrow \mathcal{H} \otimes K$ given by Corollary 2.6; as before, $S = C \times_{M_g} C$. Now applying $\pi_*$ yields

\[
\begin{array}{cccccc}
0 & \longrightarrow & F^1 \otimes F^1 & \longrightarrow & R^1\pi_*\mathcal{C}_C \otimes F^1 & \longrightarrow & (F^1)\nu \otimes F^1 & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F^1 \otimes F^1 & \longrightarrow & \pi_*(\mathcal{H} \otimes K) \cong \Pi_*(\tilde{p}^*K \otimes \tilde{q}^*K(2\Delta_{M_g})) & \longrightarrow & \mathcal{O}_{M_g} & \longrightarrow & 0
\end{array}
\]
where the right vertical map is \( c \mapsto c \cdot \text{Id} \); the maps \( \tilde{p} \) and \( \tilde{q} \) are as in (2.14). Taking quotient of the bottom exact sequence by the second exterior power, produces the exact sequence
\[
0 \to \text{Sym}^2(F^1) \to (\pi_* (\tilde{p}^* K \otimes \tilde{q}^* K(2\Delta_{M_g})))^+ \to \mathcal{O}_{M_g} \to 0 \tag{4.16}
\]
which has a \( C^\infty \) splitting \( \mathcal{O}_{M_g} \to (\pi_* (\tilde{p}^* K \otimes \tilde{q}^* K(2\Delta_{M_g})))^+ \) given by \( \tilde{\eta} \) in (4.2). Composing with \( (d\tau)^* : \text{Sym}^2(F^1) \to \Omega^1_{M_g} \), where \( d\tau \) is the differential of the Torelli map in (1.1), we obtain the diagram
\[
\begin{array}{cccccc}
0 & & \to & & \text{Sym}^2(F^1) & & \to & & (\pi_* (\tilde{p}^* K \otimes \tilde{q}^* K(2\Delta_{M_g})))^+ & & \to & & \mathcal{O}_{M_g} & & \to & & 0 \\
\downarrow & & & & \downarrow & & & & \downarrow & & & & \downarrow & & & & \downarrow & & & & \\
0 & & \to & & T^*_{M_g} & & \to & & \mathcal{V}' & & \to & & \mathcal{O}_{M_g} & & \to & & 0 \\
\end{array} \tag{4.17}
\]
where \( \mathcal{V}' \subset \mathcal{V} \) is the subbundle in (4.7) generated by \( \hat{\mathcal{V}} \). We note that diagram in (4.17) coincides with the one in (4.7). The bottom exact sequence in (4.17) admits a \( C^\infty \) splitting \( \mathcal{O}_{M_g} \to \mathcal{V}' \) given by \( \beta \eta \) in (4.15). The \((1,1)\)–form \( \partial(\beta \eta) \) over \( M_g \) coincides, by construction, with \( \tau^* \omega \), where \( \omega \) and \( \tau \) are constructed in (4.14) and (1.1) respectively.

\[ \square \]

**Remark 4.5.** Using Theorem 4.4 and Proposition 4.3 it can be deduced that Lemma 4.1 remains valid for \( g = 2 \). Indeed, in view of Theorem 4.4 and Proposition 4.3, it suffices to prove that the \((1,1)\)–form \( \tau^* \omega_S \), where \( \omega_S \) is the Siegel \((1,1)\)–form on \( A_2 \) and \( \tau \) is the map in (1.1), is not a constant scalar multiple of the Weil–Petersson Kähler form \( \omega_{wp} \) on \( M_2 \). To prove that \( \tau^* \omega_S \) is not a constant scalar multiple of \( \omega_{wp} \), note that \( \tau^* \omega_S \) extends smoothly when a one-parameter family of smooth curves of genus 2 degenerates to a reducible stable curve (the limit is two elliptic curves touching at a point). On the other hand, a theorem of Masur says that the Weil–Petersson blows up in such a situation (see [Ma, p. 624, Theorem 1]). Therefore, \( \tau^* \omega_S \) is not a constant scalar multiple of \( \omega_{wp} \).

### 5. Differentials of the Second Type on a Surface

In this section we recall some definitions and results on meromorphic differentials on surfaces that will be used in the next section to determine the cohomology class of the form \( \hat{\eta} \). Let \( S \) be a smooth complex projective surface.

Let \( D \subset S \) be a smooth curve. Let \( \mathcal{A}^{p,q} \) be the sheaf of smooth differential forms of type \( p, q \) on \( S \). Let
\[
\mathcal{A}^m = \bigoplus_{p+q=m} \mathcal{A}^{p,q}
\]
be the sheaf of the \( m \)–forms, with \( \mathcal{C}^\infty_S = \mathcal{A}^0 = \mathcal{A}^{0,0} \). Let \( \mathcal{A}^{p,q}(nD) \) be the sheaf having a pole of order at most \( n \) on \( D \); more precisely, if \( x = 0 \) is a local equation of \( D \) on \( U \cap D \) then
\[
\omega \in \mathcal{A}^{p,q}(nD)(U) \iff x^n \omega \in \mathcal{A}^{p,q}(U).
\]
Now define \( \mathcal{A}^m(D) = \bigoplus_{p+q=m} \mathcal{A}^{p,q}(D) \). We consider the complexes
\[
\mathcal{C}^\infty \xrightarrow{d} \mathcal{A}^{1,0}(D) \oplus \mathcal{A}^{0,1} \xrightarrow{d} \mathcal{A}^2(2D) \tag{5.1}
\]
\[
\mathcal{O}_S \xrightarrow{d} \Omega^1(\log D) \xrightarrow{d} \Omega^2(D) \tag{5.2}
\]
and
\[
\mathcal{O}_S \xrightarrow{d} \Omega^1(D) \xrightarrow{d} \Omega^2(2D). \tag{5.3}
\]

**Lemma 5.1.** The cohomology sheaves of all the above complexes are isomorphic to
\[ C_D = j_* \mathbb{C}, \]
where \( j : D \to S \) is the inclusion map.

**Proof.** We investigate the smooth case in (5.1). Define
\[ \mathcal{N} := \ker(d : \mathcal{A}^{1,0}(D) + \mathcal{A}^{0,1} \to \mathcal{A}^2(2D)). \]
Let \( U \subset S \) be an open subset with coordinate function \((x, y)\) such that \( x = 0 \) is the equation of \( D \cap U \). Take \( \omega \in \mathcal{N}(U) \). We have:
\[ \omega = \frac{f(x,y)dx + g(x,y)dy}{x} + \phi, \]
where \( \phi \) is a smooth 1–form. The terms with poles of \( \partial \omega \) in the coefficients of \( dx \wedge d\overline{x} \) and \( dx \wedge d\overline{y} \) are \(-\frac{\partial f}{\partial x}\) and \(-\frac{\partial f}{\partial y}\) respectively. Hence, if \( d\omega = 0 \), then \( f(x,y) \) is holomorphic.

Using the Taylor expansion of \( f \) with respect to \( x \) we can write
\[ \omega = \frac{h(y)dx + g(x,y)dy}{x} + \tilde{\phi}. \]

The polar terms of \( d\omega \) in \( dx \wedge dy \) is
\[ (-\frac{h'(y)}{x} + \frac{1}{x} \frac{\partial g}{\partial x} - \frac{g}{x^2})dx \wedge dy, \]
and hence if \( d\omega = 0 \), then \( g(x,y) = 0 \) and \( h'(y) = 0 \), in which case \( \omega = \frac{dx}{x} + \tilde{\phi} \), with \( \tilde{\phi} \) being closed and hence \( \omega \) is locally exact. The residue map
\[ \text{res} : \mathcal{N} \longrightarrow \mathbb{C}_D, \ \omega \longmapsto \lambda \]
is well defined, and its kernel is \( d\mathcal{C}^\infty \). \( \square \)

Consider (5.2) and (5.3). Define
\[ \mathcal{N}_h = \ker d : \Omega^1(D) \longrightarrow \Omega^2(2D). \]
Now the exact sequence in (5.3) yields
\[
\begin{align*}
0 & \longrightarrow \mathcal{N}_h \longrightarrow \Omega^1(D) \longrightarrow \Omega^2(2D) \longrightarrow 0 \tag{5.4} \\
0 & \longrightarrow \mathbb{C}_S \longrightarrow \mathcal{O}_S \xrightarrow{d} \mathcal{N}_h \longrightarrow \mathbb{C}_D \longrightarrow 0.
\end{align*}
\]
We also get
\[
\begin{align*}
0 & \longrightarrow L_h \longrightarrow \mathcal{N}_h \longrightarrow \mathbb{C}_D \longrightarrow 0 \tag{5.5} \\
0 & \longrightarrow \mathbb{C}_S \longrightarrow \mathcal{O}_S \longrightarrow L_h \longrightarrow 0, \tag{5.6}
\end{align*}
\]
where \( L_h = d\mathcal{O}_S \). Let us construct a homomorphism
\[ \Gamma : H^0(S, \Omega^2(2D)) \longrightarrow H^1(D, \mathbb{C}) \]
as follows: Take \( \zeta \in H^0(S, \Omega^2(2D)) \). Set \( \Gamma(\zeta) \in H^1(D, \mathbb{C}) \) to be the image of its coboundary \( \partial \zeta \in H^1(S, \mathcal{N}_h) \) (see (5.4)) under the homomorphism \( H^1(S, \mathcal{N}_h) \to H^1(\mathbb{C}_D) \cong H^1(D, \mathbb{C}) \) (see (5.5)).
We get that $\partial \beta = r(\beta)$, where $\beta \in H^1(S, L_h)$ and $r : H^1(S, L_h) \to H^1(S, \mathcal{N}_h)$ is the homomorphism of cohomologies given by the injective homomorphism of sheaves in (5.5). The coboundary homomorphism $\partial : H^1(S, L_h) \to H^2(S, \mathcal{C}_S)$ (see (5.6)) gives a class $\partial(\beta) = [\gamma] \in H^2(S, \mathcal{C}_S)$. By construction, we have $j^*([\gamma]) = [\zeta]$, where $j$ is the inclusion map in Definition 5.2, and $[\zeta] \mapsto 0 \in H^2(S, \mathcal{O}_S)$.

This means that the $(0, 2)$ part of $[\gamma]$ vanishes. Consequently, using the Hodge decomposition for $S$,

$$[\gamma] = \gamma^{2,0} + \gamma^{1,1},$$

(5.7)

where $\gamma^{2,0}$ is holomorphic and $\gamma^{1,1}$ is harmonic of type $(1, 1)$. Therefore

$$\zeta' := \zeta - \gamma^{2,0}$$

is a holomorphic differential of second type of pure type $(1, 1)$.

**Proposition 5.3.** Let $\zeta' \in H^0(S, \Omega^2_S(2D))$ be a holomorphic differential of the second type and of pure type $(1, 1)$. Then there is a class $\alpha \in H^0(S, \mathcal{A}_{1,0}(D))$ such that

$$\zeta' - \gamma^{1,1} = d\alpha,$$

that is $\partial \alpha = \zeta'$ and $\overline{\partial} \alpha = -\gamma^{1,1}$, where $\gamma^{1,1}$ is a harmonic $(1, 1)$ form on $S$.

**Proof.** From the sequence (5.1)

$$\mathcal{C}^\infty \xrightarrow{d} \mathcal{A}^{1,0}(D) \oplus \mathcal{A}^{0,1} \xrightarrow{d} \mathcal{A}^2(2D)$$

we get an exact sequence

$$0 \to \mathcal{C}_S \to \mathcal{C}^\infty \xrightarrow{d} \mathcal{N} \to \mathcal{C}_D \to 0.$$  

(5.8)

Let $\gamma^{1,1}$ be a $(1, 1)$ harmonic form on $S$ such that $j^*([\gamma]) = [\zeta']$. The form

$$\zeta' - \gamma^{1,1} \in H^0(S, d(A^{1,0}(D) \oplus A^{0,1}))$$

maps to zero in $H^1(D, \mathcal{C})$. We find then

$$\beta = \beta^{1,0} + \beta^{0,1} \in H^0(S, \mathcal{A}^{1,0}(D) \oplus \mathcal{A}^{0,1})$$

such that $d\beta = \zeta' - \gamma^{1,1}$. Since $\zeta'$ is of type $(2, 0)$,

$$\zeta' = \partial \beta^{1,0} \quad \text{and} \quad -\gamma^{1,1} = \overline{\partial} \beta^{1,0} + \partial \beta^{0,1}, \quad \overline{\partial} \beta^{0,1} = 0.$$  

Now $\partial \beta^{0,1}$ is a smooth form that is $\partial$ exact and $\overline{\partial}$ closed. By $\partial \overline{\partial}$ lemma, there is a smooth function $f$ such that $\partial \beta^{0,1} = \overline{\partial} f$. Set

$$\alpha = \beta^{1,0} - \partial f \in H^0(S, \mathcal{A}^{1,0}(D)).$$

We get that

$$\partial \alpha = \zeta', \quad \overline{\partial} \alpha = -\gamma^{1,1},$$

and the proof is complete. \qed
Let $C$ be a smooth complex projective curve of genus $g$. As before, $p$ and $q$ are the projections of $S := C \times C$ to the first and second factors respectively. Also, as before $\Delta \subset S$ is the reduced diagonal divisor. Using the natural identification of the canonical line bundle $\Omega^2_S$ with $p^*K_C \otimes q^*K_C(2\Delta)$ in (2.6), the line bundle $L$ in (2.6) will be identified with $\Omega^2_S(2\Delta)$.

Using this identification, the intrinsic invariant bidifferential $\hat{\eta}$ constructed in (2.7) will be considered as a section of $\Omega^2_S(2\Delta)$. Since $\hat{\eta}$ is symmetric, the section of $\Omega^2_S(2\Delta)$ corresponding to it is anti-symmetric.

From the exact sequence of homology groups for the pair $(S, \Delta)$ and Poincaré and Lefschetz duality, it follows that the homomorphism $j^*: H^2(S, \mathbb{Z}) \rightarrow H^2(S \setminus \Delta, \mathbb{Z})$ is surjective with the kernel of $j^*$ being generated by the class of the diagonal. Consequently, every element of $H^0(\Omega^2_S(2\Delta))$ is of second type. In particular, the bidifferential $\hat{\eta}$ constructed in (2.7) is of second type.

**Theorem 6.1.** All the elements of $H^0(S, \Omega^2_S(2\Delta))$ of pure type $(1, 1)$ are contained in the line $C \cdot \hat{\eta}$.

**Proof.** Write $[\hat{\eta}] = j^*([\gamma])$ and $\gamma = \gamma^{2,0} + \gamma^{1,1}$ as in (5.7) and Definition 5.2. By Proposition 5.3 there is a form $\alpha = \alpha^{1,0} \in C^\infty(A^{1,0}(\Delta))$ such that $\hat{\eta} - \gamma^{2,0} = \partial \alpha$ and $\gamma^{1,1} = -\overline{\partial} \alpha$.

Since $\overline{\partial} \alpha$ is smooth it follows that the polar part of $\alpha$ is smooth, that is, in local coordinates around the diagonal,

$$\alpha = \frac{d(w + z)}{z - w} + h_1(z, w)dz + h_2(z, w)dw,$$

where $h_1$ and $h_2$ are smooth functions. We want to prove that $\gamma^{2,0} = 0$.

Since the decomposable forms generate the $(2, 0)$ cohomology, it suffices to prove that

$$\int_S \gamma \wedge \overline{\omega}_1 \wedge \overline{\beta}_2 = 0,$$  \hspace{1cm} (6.1)

where $\overline{\omega}_1 = p^*\overline{\omega}$ and $\overline{\beta}_2 = q^*\overline{\beta}$ with $\omega$ and $\beta$ being holomorphic 1–forms on $C$.

We write

$$\hat{\eta} = \gamma + d\alpha.$$

Let $U_r$ be a tubular neighborhood of the diagonal $\Delta$ and $\chi_r$ the characteristic function of the complement $S \setminus U_r$. We will show that

$$\lim_{r \to 0} \int_S \chi_r \hat{\eta} \wedge \overline{\omega}_1 \wedge \overline{\beta}_2 = 0$$  \hspace{1cm} (6.2)

and

$$\lim_{r \to 0} \int_S \chi_r d\alpha \wedge \overline{\omega}_1 \wedge \overline{\beta}_2 = 0.$$  \hspace{1cm} (6.3)

Note that (6.2) and (6.3) together imply (6.1).

Take $\{W, w\}$, where $W \subset C$ is an open subset and $w$ is a holomorphic coordinate function on $W$; define

$$V := q^{-1}(W) = C \times W.$$
On \( V \), we have \( \hat{\eta} = \mu \wedge dw \) where \( \mu \) is a \((1, 0)\) meromorphic form with pole on the diagonal of \( W \times W \). Using Fubini’s theorem,

\[
\int_V \chi_r \hat{\eta} \wedge \overline{\alpha}_1 \wedge \overline{\beta}_2 = \int_W \left( \int_{C \times \{t\}} \chi_r \mu_t \wedge \overline{\omega} \right) \wedge dw \wedge \overline{\beta}.
\]

The form \( \chi_r \mu_t \wedge \overline{\omega} \) is can be shown to be exact. Indeed, this follows from the fact that \( \mu_t \) is the form defined in \([2, 4]\), whose local expression in a coordinate \( z \) centered in \( t \) is

\[
\mu_t = \left( \frac{1}{t^2} + h(z) \right) dz = \partial \left( \frac{1}{t} + f(z) \right) \text{ with } f(z) \text{ holomorphic.}
\]

So,

\[
\int_{C \times \{t\}} \chi_r \mu_t \wedge \overline{\omega} = \int_{\partial \Delta_r} \left( \frac{1}{z} + f(z) \right) \overline{\omega}.
\]

Now writing \( \omega = g(\overline{z})d\overline{z} \),

\[
\int_{\partial \Delta_r} \left( \frac{1}{z} + f(z) \right) g(\overline{z})d\overline{z} = \frac{1}{\sqrt{-1}} \int_0^{2\pi} \left( \frac{-1}{re^{\sqrt{-1}\theta}} - e^{-\sqrt{-1}\theta} g(re^{-\sqrt{-1}\theta}) + f(re^{-\sqrt{-1}\theta}) g(re^{-\sqrt{-1}\theta})re^{-\sqrt{-1}\theta}d\theta \right).
\]

The integrand functions

\[-g(re^{-\sqrt{-1}\theta})e^{-2\sqrt{-1}\theta} + f(r \cdot e^{-\sqrt{-1}\theta})g(r \cdot e^{-\sqrt{-1}\theta})re^{-\sqrt{-1}\theta}\]

are integrable and bounded. The limit of the integral in \((6.4)\) for \( r \to 0 \) is 0 since \( g \) is anti-holomorphic and \( g(r \cdot e^{-\sqrt{-1}\theta}) = g(0) + r(\tilde{g}(r \cdot e^{-\sqrt{-1}\theta})) \). Then we can pass the limit under the integral sign:

\[
\lim_{r \to 0} \int_V \chi_r \hat{\eta} \wedge \overline{\alpha}_1 \wedge \overline{\beta}_2 = \lim_{r \to 0} \left( \int_W \int_{C \times \{t\}} \chi_r \mu_t \wedge \overline{\omega} \right) \wedge dw \wedge \overline{\beta} = \int_W \lim_{r \to 0} \left( \int_{C \times \{t\}} \chi_r \mu_t \wedge \overline{\omega} \right) \wedge dw \wedge \overline{\beta} = 0.
\]

This proves \((6.2)\).

To prove \((6.3)\),

\[
\lim_{r \to 0} \int_S \chi_r d\alpha \wedge \overline{\alpha}_1 \wedge \overline{\beta}_2 = \lim_{r \to 0} \int_{\partial U_r} \alpha \wedge \overline{\alpha}_1 \wedge \overline{\beta}_2.
\]

Cover the diagonal with a finite number of products of disks \( A_i \times B_i \), biholomorphic to \( \{(z, w) \in \mathbb{C}^2 \mid |z| < 1, |w| < 1\} \) (with compact closure). We take \( r \) small enough in a way that \( \partial U_r \subset \bigcup_i A_i \times B_i \) and we may assume that \( \partial U_r \cap (A_i \times B_i) \) corresponds to \( B_r = \{(z, w) \mid |z - w| = r\} \), hence \( w = z + re^{\sqrt{-1}\theta} \). We may also assume that \( \alpha \) is of the type

\[
\alpha = \frac{d(w + z)}{z - w} + h_1(z, w)dz + h_2(z, w)dw
\]

in local coordinates, where \( h_1 \) and \( h_2 \) are smooth functions. Write moreover \( \overline{\alpha}_1 = f_1(\overline{z})d\overline{z} \) and \( \overline{\alpha}_2 = f_2(\overline{w})d\overline{w} \) with the \( f_i \) being anti-holomorphic.

We have

\[
\int_{B_r} \frac{d(w + z)}{z - w} f_1(\overline{z})d\overline{z}f_2(\overline{w})d\overline{w} = 2\sqrt{-1} \int_{B_r} f_1(\overline{z})f_2(\overline{w})e^{-2\sqrt{-1}\theta}dzd\overline{z}d\theta
\]
\[ = 2\sqrt{-1} \left( \int_{|z|<1} f_1(z) f_2(z) e^{-2\sqrt{-1} \theta} dz d\theta + \int_{|z|<1} \int_0^{2\pi} r f_1(z) h(\theta, \bar{z}) e^{-2\sqrt{-1} \theta} dz d\theta \right). \]

Now we have \( \int_{|z|<1} f_1(z) f_2(z) dz d\theta = 0 \) and
\[
\int_{|z|<1} \int_0^{2\pi} |r f_1(z) h(\theta, \bar{z}) e^{-2\sqrt{-1} \theta} dz d\theta| = r \int_{|z|<1} \int_0^{2\pi} |f_1(\bar{z}) h(\theta, \bar{z})| d\theta dz d\bar{z} = r c,
\]
where \( c \) is a constant. Then the limit of the integral in (6.5) is zero as \( r \to 0 \).

It now follows that we have to evaluate only
\[
\int_{B_r} (h_1(z, w) dz + h_2(z, w) dw) f_1(z) f_2(w) d\bar{z} dw.
\]

We compute the two terms separately, let \( G = h_1(z, w) f_1(z) f_2(w) \):
\[
\int_{B_r} |G(z, w) dz d\bar{z} dw| = \int_{B_r} |(G(z, \theta) dz d\bar{z})(-\sqrt{-1} re^{-\sqrt{-1} \theta} d\theta)|
\]
\[
= r \int_{|z|<1} \int_0^{2\pi} |G(z, \theta)| d\theta dz d\bar{z} = rk,
\]
where \( k \) is a constant, and similarly for the second term. Then the limit as \( r \to 0 \) is zero and therefore summing
\[
\lim_{r \to 0} \int_S \chi_r d\alpha \wedge \overline{\omega}_1 \wedge \overline{\beta}_2 \leq \lim_{r \to 0} \sum_i \int_{A_i \times B_i} |\chi_r d\alpha \wedge \overline{\omega}_1 \wedge \overline{\beta}_2| = 0.
\]
This proves (6.3). \( \square \)

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