Lattice Anderson model
Consider $H_\omega = -\Delta + V_\omega$ on $\ell^2(\mathbb{Z}^d)$: $-\Delta \psi(x) = \sum_{\|x-y\|=1} \psi(y)$ and $V_\omega \psi(x) = \omega_x \psi(x)$

$\omega = (\omega_x)_{x \in \mathbb{Z}^d}$ collection of i.i.d. random variables.
Consider restrictions of $H_\omega$ to regions $\Omega \subset \mathbb{Z}^d$ i.e. $(H_\omega)_\Omega = 1^T\Omega H_\omega 1_\Omega$.

Localization holds in $I \subset \mathbb{R}$ if

$\exists \mu > 0, \exists q > 0$ s.t. for $L$ large, with $\mathbb{P} \geq 1 - L^{-p}$, any e.v. $E \in I$ of $(H_\omega)|_\Lambda_L$ assoc. to norm. eigenfct $\varphi_E$ s.t. $\exists x_E \in \Lambda_L$, one has

$$\max_{|x| \leq L} |\varphi_E(x)| e^{\mu|x-x_E|} \lesssim \begin{cases} L^q & \text{in mathematical papers (M)} \\ 1 & \text{in physics papers (P)} \end{cases}$$

Known:
- the bound (P) cannot hold for all eigenfunctions with good probability (Lifshits tails states).
- the bound (M) is optimal (again Lifshits tails states).

Questions: in the localized region, where does the truth lie between (M) and (P)?
More precisely,
- how many states satisfy (P)?
- how many states satisfy no estimate “better than (M)”?
- how many states satisfy an estimate “in between (P) and (M)”?
Fix region $\Omega \subset \mathbb{Z}^d$, let $\mathcal{E}( (H_\omega)_{\Omega}) := \{ \text{the set of eigenvalues of } (H_\omega)_{\Omega} \}$.

Known:
(A1) $\exists \Sigma_p \subset \Sigma \subset \mathbb{R}$ s.t. $\sigma(H_\omega) = \Sigma$ and $\sigma_p(H_\omega) = \Sigma_p$ a.s.
(A2) $\forall E \in \mathbb{R}, \omega$-a.s., the integrated density of states exists and is indep. of $\omega$ a.s. i.e.
\[
N(E) := \lim_{\Omega \uparrow \mathbb{Z}^d, \Omega \text{ finite}} \frac{\# \mathcal{E}( (H_\omega)_{\Omega}) \cap (-\infty, E]}{\# \Omega}
\]  
(1)

To $\varphi \in \ell^2(\Omega)$, with $\Omega \subset \mathbb{Z}^d$, we associate the set of localization centers:
\[
\mathcal{C}(\varphi) := \{ x \in \Omega : |\varphi(x)| = ||\varphi||_\infty \}.
\]  
(2)

(A3) $\exists I_{AL} \subseteq \Sigma$, a union of finitely many open intervals, s.t. for any region $\Omega$, with probability one, $(H_\omega)_{\Omega}$ has pure point spectrum on $I_{AL}$. Furthermore, there are constants $A_{AL} < \infty$, $\mu > 0$ s.t. if $\Omega \subset \mathbb{Z}^d$ is a region, $S \subset \Omega$ is a finite set, and $0 < \varepsilon < 1$, then, with probability larger than $1 - \varepsilon$, any $\ell^2$-normalized eigenfunction $\varphi_E$ of $(H_\omega)_{\Omega}$ with eigenvalue $E \in I_{AL} \cap \mathcal{E}( (H_\omega)_{\Omega})$ and $\mathcal{C}(\varphi_E) \cap S \neq \emptyset$ satisfies
\[
\max_{y \in \mathcal{C}(\varphi_E) \cap S} \left( \sum_{x \in \Omega} e^{2\mu|x-y|} |\varphi_E(x)|^2 \right)^{\frac{1}{2}} \leq A_{AL} \left( \frac{\# S}{\varepsilon} \right)^{\frac{1}{2}}.
\]  
(3)

Given $\mu > 0$, define a decreasing family of weighted $\ell^2$-norms by
\[
M_\ell^\mu(\varphi; y) := \left( \sum_{x \in \Omega} e^{2\mu|x-y|+\ell} |\varphi(x)|^2 \right)^{\frac{1}{2}} \quad \text{where } \ell = 0, 1, 2, \ldots
\]  
(4)

If $M_\ell^\mu(\varphi; y)$ is finite, then
\[
\lim_{\ell \to \infty} M_\ell^\mu(\varphi; y) = ||\varphi||_{\ell^2}.
\]

Onset length: $\ell_\mu(\varphi; y) := \min\{ \ell : M_\ell^\mu(\varphi; y) \leq 2 ||\varphi||_{\ell^2} \}$.

**Proposition**

Let $\Omega \subset \mathbb{Z}^d$ be region and $S \subset \Omega$ a finite set. If $0 < \varepsilon < 1$, then, with probability larger than $1 - \varepsilon$, any $\ell^2$-normalized eigenfunction $\varphi_E$ of $(H_\omega)_{\Omega}$ with eigenvalue $E \in I_{AL} \cap \mathcal{E}( (H_\omega)_{\Omega})$ and $\mathcal{C}(\varphi_E) \cap S \neq \emptyset$ satisfies
\[
\left( \sum_{x \in \Omega} e^{2\mu|x-x_E|+\ell_E} |\varphi_E(x)|^2 \right)^{\frac{1}{2}} \leq 2, \quad \text{for any } x_E \in \mathcal{C}(\varphi_E) \cap S
\]  
(5)

with onset length $\ell_E = \ell_\mu(\varphi_E; x_E) < \frac{1}{\mu} \left( \log A_{AL} + \frac{1}{2} \log \#S - \frac{1}{2} \log 3\varepsilon \right) + 1$. 

---
Given two localization centers \(x_E, x'_E \in \mathcal{C}(\varphi_E)\) i.e.

**Proposition**

\[
|\ell_\mu(\varphi_E; x_E) - \ell_\mu(\varphi_E; x'_E)| \leq |x_E - x'_E|
\]  

(6)

The onset length \(\ell_\mu(\varphi_E; x_E)\) also gives an upper bound on the diameter of \(\mathcal{C}(\varphi_E)\), namely,

**Proposition**

Pick \(\kappa > 0\) such that \(8e^{-2\mu \kappa} = 1\). If \((x_E, x'_E) \in \mathcal{C}(\varphi_E)^2\) then

\[
|x_E - x'_E| \leq \ell_\mu(\varphi_E; x_E) + \frac{d}{2\mu} \log \left(2\ell_\mu(\varphi_E; x_E) + 2\kappa + 1\right) + \frac{3\log 2}{2\mu}
\]

(7)

This yields pointwise bound \(|\varphi_E(x)| \leq \|\varphi_E\|_{\infty} e^{-\mu(|x - x_E| - \tilde{\ell}_{\mu,E})}\) where

\[
\tilde{\ell}_{\mu,E} = \ell_\mu(\varphi_E; x_E) + \frac{d}{2} \log(2\ell_\mu(\varphi_E; x_E) + 2\kappa + 1)
\]

Assume

(A4) \(\exists A_M > 0\) s.t., for any finite region \(\Omega \subset \mathbb{Z}^d\), one has

\[
\sup_{E \in \mathcal{I}_{AL}} \mathbb{P}\left(\{\text{tr}(1_{[E-\varepsilon,E+\varepsilon]}((H_\omega)\Omega)) \geq 2\}\right) \leq A_M |\Omega|^2 \varepsilon^2
\]

for any \(\varepsilon > 0\).

Main results quantify the distribution of onset lengths for eigenfunctions with localization centers in a given bounded region.

Set \(\Lambda = \Lambda_L(x_0) = \left(x_0 + \left[-\frac{L}{2}, \frac{L}{2}\right]^d\right) \cap \mathbb{Z}^d\).

**Theorem**

Let \(\mu\) and \(I_{AL}\) be as in (A3) and let \([a, b] \subset I_{AL}\). Then, for any \(\nu < \mu\) and any \(p > 0\) there exist \(C_\nu > 0, \ell_0 > 0,\) and \(L_0 > 0\) such that if \(\Omega \subset \mathbb{Z}^d\) is a region with \(\Lambda \subset \Omega\) with \(L \geq L_0\), then with probability larger than \(1 - L^{-p}\), for all \(\ell \geq \ell_0\), one has

\[
\#\{E \in \mathcal{E}((H_\omega)\Omega) \cap [a, b] \text{ s.t. } \exists x_E \in \mathcal{C}(\varphi_E) \cap \Lambda \text{ and } \ell_\nu(\varphi_E; x_E) \geq \ell\} \leq L^d e^{-C_\nu \ell}.
\]

(9)
Theorem

Let $\mu$ and $I_{AL}$ be as in (A3) and let $[a, b] \subset I_{AL}$. Pick $\alpha > d$. Then, for any $\nu < \mu$, there exists $c > 0$ s.t. for any $\ell \geq 1$, one has

$$\sup_{x \in \Omega} \mathbb{E} \left[ \sum_{E \in \mathcal{E}(H_\omega) \cap [a, b]} \sum_{x_E \in \mathcal{C}(\varphi_E) : \ell_{\nu}(\varphi_E ; x_E) \geq \ell} \left( 1 + \frac{1}{\ell_{\nu}(\varphi_E ; x_E)} \right)^{\alpha} \right] \leq \frac{1}{c} e^{c\ell \ell^{d+1}}. \quad (10)$$

Thus, the localization centers corresponding to large onset length are more or less uniformly distributed and quite remote from one another.

Corollary

For $\nu < \mu$, $a < b$ real such that $[a, b] \subset I_{AL}$, and $\ell > 0$, the limit

$$N_{\nu}([a, b], \ell) := \lim_{L \to +\infty} \frac{\# \{ E \in \mathcal{E}(H_\omega) \cap [a, b] : \exists x_E \in \mathcal{C}(\varphi_E) \cap \Lambda \text{ s.t. } \ell_{\nu}(\varphi_E ; x_E) \geq \ell \} \} {N(I) \cdot L^d}$$

exists a.s. and is independent of $\omega$.

Optimality of the upper bound on the counting function: a lower bound

One has the deterministic lower bound

Theorem

Let $\mu$ and $I_{AL}$ be as in (A3). Let $E_-$ be the infimum of $\Sigma$ the almost sure spectrum of $H_\omega$ and assume $E_- > -\infty$. Then, $\exists c > 0$, $\ell_0 > 0$ and $L_0 > 0$ s.t. for any $\nu < \mu$, for $\Lambda = \Lambda_L$ with $L \geq L_0$, and all $\ell \geq \ell_0$, with probability 1, one has

$$\# \{ E \in \mathcal{E}(H_\omega) : \exists E \in \mathcal{C}(\varphi_E) \cap \Lambda \neq \emptyset \text{ s.t. } \ell_{\nu}(\varphi_E ; x_E) \geq \ell \} \geq \# \{ E \in \mathcal{E}(H_\omega) \cap [E_-, E_- + c\ell^{d-1}] : \mathcal{C}(\varphi_E) \cap \Lambda \neq \emptyset \}. \quad (11)$$

Corollary

Let $E_-$ be the infimum of $\Sigma$ the almost sure spectrum of $H_\omega$ and assume $E_- > -\infty$. Then there exists $\ell_0 > 0$ such that, for any $\nu < \mu$ and $\ell \geq \ell_0$, one has

$$N_{\nu}(\Sigma, \ell) \geq N(E_- + c\ell^{d-1}) \quad (12)$$

Known (Lifshits tails): $N(E_- + \lambda) \geq e^{-f(\lambda)\lambda^{-d/2}}$. 
Figure: Three eigenfunctions for the 1D Anderson model on $\Lambda = [1, 2000]$ with $\lambda = 1$.

(a) Lyapunov exponent and density of states versus energy
(b) Normalized eigenfunction correlators $C_2(\mu, I, \Lambda)$ vs. exponent $\mu$ for the energy intervals shown. The Lyapunov exponent $L(I)$ for each interval is indicated as a vertical line. Here

$$C_2(\mu, I, \Lambda) = \frac{1}{\# \Lambda} \sum_{x, y \in \Lambda} e^{2\mu|x-y|} \left( \sum_{E \in I \cap \sigma((H_\omega)_\Lambda)} |\phi_E(x)|^2 |\phi_E(y)|^2 \right).$$
Figure: Exponents (a) and onset length (b) for eigenfunctions of the 1D Anderson model on an interval of 2000 with disorder $\lambda = 1$.

(a) On each energy interval with observed eigenvalues, the exponent $\mu$ at which $C_2(\mu, I, \Lambda) / C_2(0, I, \Lambda) = 2$ is shown in blue and the ratio $\mu / L(I)$ is plotted in red, where $L(I)$ is the minimal Lyapunov exponent on $I$.

(b) Onset length $\ell_E$ versus energy $E$ for the eigenfunctions from 240 samples (480,000 eigenfunctions in total). Only 6,313 (1.3% of the total) eigenfunctions have $\ell_E > 0$.

Figure: Cumulative distributions

(a) Cumulative distribution of onset lengths for eigenfunctions of the 1D Anderson model.
(b) Cumulative distribution of onset lengths for 240 samples with $\lambda = 1$ on an interval of length 2000.
Numerics show onset lengths as large as 262.

**Question:** consistency with the estimation that onset lengths “of order \( \log \# \Lambda = \log 2000 \approx 7.6 \)?

Correlator bound \( C_2(\mu, I, \Lambda) \leq 2 \cdot C_2(0, I, \lambda) \) provides an \textit{a priori} bound on localization lengths consistent with this observation.

Indeed, from the Markov inequality, with proba \( \geq 1 - \varepsilon \),

\[
\frac{1}{\# \Lambda} \sum_{x,y \in \Lambda} e^{2\mu |x-y|} \sum_{E \in I \cap \mathcal{E}(H_\omega)} |\varphi_E(x)|^2 |\varphi_E(y)|^2 \leq \frac{2C_2(0, I, \lambda)}{\varepsilon}.
\]

Taking \( y = x_E \) (\( E \) fixed), each eigenfunction satisfies

\[
\sum_x e^{2\mu |x-x_E|} |\varphi_E(x)|^2 \leq \frac{2C_2(0, I, \Lambda) \# \Lambda}{\varepsilon \|\varphi_E\|_\infty^2}.
\]

First proposition implies that

\[
\ell_E \leq \frac{1}{2\mu} (\log 2C_2(0, I, \Lambda) + \log \# \Lambda - \log 3\varepsilon - 2\log \|\varphi_E\|_\infty) + 1. \tag{14}
\]

Numerical context: \( \varepsilon = 1/240 \) (as 240 samples).

Key point: onset lengths to be no larger than \( \frac{1}{2\mu} (\log \# \Lambda - \log 3\varepsilon) \).

\( \# \Lambda = 2000, \varepsilon = 1/240, \) and \( \mu \approx 0.01 \implies \text{rough bound of order 600.} \)