Finite-time asynchronous sliding mode control for Markov jump systems with actuator saturation

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ABSTRACT
This paper deals with the finite-time asynchronous sliding mode control for Markov jump systems subject to actuator saturation. The hidden Markov model is adopted to describe the modal information exchange between the system and the designed controller. A sufficient condition is established by employing the Lyapunov stability theorem, which guarantees the stochastic finite-time boundedness of sliding mode. In order to mitigate the effects of actuator saturation and chattering, an adaptive sliding mode controller is designed based on the auxiliary saturation compensation system. Subsequently, the reachability of sliding mode motion is analysed. The controller gain matrix is obtained by solving the minimization problem. Finally, a simulation example is used to demonstrate that the proposed control method can effectively restrain the disturbance and compensate saturation.

KEYWORDS
Sliding mode control; Markov jump systems; actuator saturation; hidden Markov model; stochastic finite-time boundedness

1. Introduction

Sliding mode control is a kind of control method with discontinuous control behaviour. Its basic principle is to design a sliding mode controller, such that the trajectory of system starting from any initial state can be driven to the predetermined sliding surface in finite-time and keep moving on it without being affected by other external factors. Different from the continuous sliding mode control, the discrete sliding mode control is quasi sliding mode motion, i.e. the system states are only forced to enter the small bounded neighbourhood near the sliding surface instead of staying on the sliding surface (Janardhanan & Bandyopadhyay, 2006). The sliding mode can be automatically switched according to the control requirements, and it is insensitive to matching uncertainty and external disturbances. Therefore, sliding mode control has the advantages of good transient performance, fast response speed, strong robustness and simple design ideas, which has been widely concerned (Ding et al., 2014; Du et al., 2019; Hu et al., 2012).

In industrial applications, the systems will always switch between different states when it faces machine failure, sudden change of environment (Ma & Liu, 2020). Hence, the single and deterministic model can not solve this type of control problem. But the Markov jump systems have great advantages in dealing with the problem (Liu et al., 2018). The Markov jump systems are a kind of sensitive stochastic systems, which can reflect the change of system structure and parameters through the jump of Markov chain. And the Markov jump systems are widely used in power systems, networked control systems and aircraft flight systems. In recent years, the researches on the stability, stabilization, filter and fault diagnosis of the Markov jump systems have achieved fruitful results (Song, H. et al., 2017; Wu & Mu, 2019; Zha et al., 2017; Zhan et al., 2010). For example, the event-triggered $H_\infty$ control problem has been studied for the networked Markov jump systems in Zha et al. (2017), where different modes correspond to different trigger thresholds. The event-triggered control problem has been investigated for a class of semi-Markov jump systems in Wu and Mu (2019), and the stochastic stability has been analysed for the closed-loop stochastic semi-Markov time delay systems with the help of stochastic system theory. In Song, H. et al. (2017), the sliding mode control problem has been discussed for the discrete systems with Markov jump systems in Zha et al. (2017), where different modes correspond to different trigger thresholds. The event-triggered control problem has been investigated for a class of semi-Markov jump systems in Wu and Mu (2019), and the stochastic stability has been analysed for the closed-loop stochastic semi-Markov time delay systems with the help of stochastic system theory. In Song, H. et al. (2017), the sliding mode control problem has been discussed for the discrete systems with Markov packet dropout. The influence of packet dropout on stability has been solved by using Markov jump systems. In Zhan et al. (2010), the static output feedback control has been studied for linear Markov jump systems, and an augmentation method has been proposed which overcomes the problem that the rank of the system matrix often be constrained when dealing with the output feedback control.
In the networked control systems, it is very popular to describe asynchronous phenomenon by introducing the hidden Markov model. The problem of asynchronous stochastic stabilization has been explored for Markov jump systems in Guan et al. (2019), and the necessary and sufficient conditions of asynchronous stochastic stability have been obtained. The hidden Markov model is a statistical model used in many real-world applications. It is a doubly stochastic finite model which calculates probability distribution over an infinite number of possible sequences. It is used for studying the observed items from a discrete-time series. States have assigned transition probabilities, and every state emits symbol according to the emission probability of the state. By adopting the hidden Markov model with partially acceptable modal detection probability to characterize the asynchronous phenomenon, the asynchronous sliding mode control has been studied for a class of uncertain Markov jump systems with time-delay and disturbance in Song, Niu et al. (2018), where the asynchronous sliding mode control law has been designed. In Song, J. et al. (2017), the asynchronous control has been discussed for time-varying Markov jump systems, after utilizing hidden Markov model to deal with the asynchronous phenomenon between the system mode and the controller mode, the sufficient condition of finite-time stochastic boundedness has been obtained which satisfies $H_\infty$ performance. In the research of networked systems, in order to save network resources and reduce data conflicts, the network communication protocol and the signal quantization technology are usually applied (Liu et al., 2014). Many research results show that the protocols and the quantification have played effective roles in coordinating network resources and reducing network burdens, and they have also played an important role in sliding mode control research. By constructing the Lyapunov function based on scheduling signals, the sliding mode control has been analysed for uncertain control systems with Round-Robin protocol in Song, Wang et al. (2018), where a novel output feedback sliding mode controller has been designed. A new event-triggered sliding mode control method has been proposed for networked switching systems in Shang and Zong (2020), besides, a sliding mode controller has been designed which relies on scheduling signals under Round-Robin protocol. The quantitative feedback sliding mode control has been proposed for linear uncertain systems in Zheng et al. (2014), and a static adjustment strategy has been designed for quantitative parameters to eliminate the influence of uncertainty. On the basis of Zheng et al. (2014), by introducing a dynamic uniform quantizer and combing the quantization error with the system output, the $H_\infty$ sliding mode controller has been constructed for the time-delay Markov jump systems in Zhang and Shen (2019).

In addition, the nonlinear constraints and the external disturbances are unavoidable in networked control systems (Han et al., 2020; Jenabzadeh & Safarianjad, 2018). There are challenging subjects to study the influence of different nonlinear constraints and various external disturbances on the system stability by using the sliding mode control method. Recently, the design problems of output feedback sliding mode controller and dynamic compensator have been studied for the systems with sector bounded actuator amplitude and rate saturation nonlinear constraint in Kapila and Hadj (2000), which has ensured the asymptotic stability of the closed-loop system. The asynchronous control problem has been researched for the Markov jump systems in Yang and Lin (2019), and the closed-loop system satisfies the performance index in a given finite-horizon. In order to compensate actuator saturation, the finite-time auxiliary systems have been proposed in Jia and Shan (2020), where the influence of actuator saturation on the auxiliary systems could be adjusted by parameters. Subsequently, a new auxiliary saturation compensation system has been proposed to eliminate the negative effects of asymmetric nonlinear actuator saturation in Guo et al. (2020). Assuming that the upper bound of the external disturbances is unknown, by employing the adaptive law to estimate the bound value of the disturbances, the robust adaptive sliding mode control has been researched in Xia et al. (2010) for discrete time-delay systems with external disturbances. When the upper bound of the sliding mode band is unknown, the reachability of the sliding mode surface has been guaranteed for the closed-loop systems by synthesizing an adaptive sliding mode controller in Yao et al. (2018) and Zhang and Xia (2010). The concept of finite-time stability has been extended to finite-time boundedness for the time-varying continuous systems affected by external disturbances, besides, related research has been presented for discrete systems in Amato and Ariola (2005). In recent years, the researches on the finite-time stability and finite-time boundedness have aroused the extensive interest of scholars.

To sum up, for the nonlinear networked Markov jump systems with time delay, there are few reports to handle the resource constraints by considering the communication protocols and the dynamic quantization. In the Markov jump systems with unknown bounded non-linearity, cumulative bounded disturbance and actuator saturation, the problems should be further discussed: the boundedness analysis of the sliding mode in finite-time and the design of an adaptive sliding mode control strategy based on auxiliary saturation compensation, etc. Therefore, the main purpose of this paper is to construct...
Consider control systems is shown in Figure 1.

The control performance of the actuator. The structure of the feedback sliding mode control strategy with auxiliary saturation compensation system.

Figure 1. Structure diagram of networked control system based on actuator saturation.

2. Problem formulation and sliding surface design

2.1. Model description

Consider the discrete nonlinear time delay Markov jump systems with actuator saturation

\[
\begin{align*}
    x_{k+1} &= A(r_k)x_k + A_d(r_k)x_k - d_k \\
    &+ B(r_k)(\text{sat}(u_k) + f(y_k, k)) \\
    &+ D(r_k)\omega_k \\
    y_k &= C_ix_k \\
    x_k &= \phi_k, \forall k \in [-d_M, 0]
\end{align*}
\]

where \( x_k \in \mathbb{R}^n \), \( u_k \in \mathbb{R}^m \) and \( y_k \in \mathbb{R}^p \) are the state vectors, the control input and the measurement output of the systems, respectively. \( f(y_k, k) \) is an unknown nonlinear function. \( d_k \) represents the varying time delay satisfying \( 0 \leq d_m \leq d_k \leq d_M \), here \( d_M \) and \( d_M \) are known upper and lower bounds of time delay. \( \phi_k \) denotes the initial state vector. \( \omega_k \in \mathbb{R}^q \) describes the cumulative bounded disturbance input, i.e.

\[
\sum_{k=1}^{N} \omega_k^T \omega_k \leq \delta
\]

here \( \delta \) is a given positive scalar, \( N \in \mathbb{N}^+ \) is a known positive integer.

\( \text{sat}(u_k) : \mathbb{R}^m \to \mathbb{R}^m \) represents the actuator saturation function, it is defined as:

\[
\text{sat}(u_k) = \left[ \text{sat}(u_{1,k}), \text{sat}(u_{2,k}), \ldots, \text{sat}(u_{m,k}) \right]^T
\]

where \( \text{sat}(u_{j,k}) = \text{sign}(u_{j,k}) \min\{u_{j,max} - |u_{j,k}|, 0\} \), \( j = 1, 2, \ldots, m \). The constant \( u_{j,max} \) denotes the saturation level of the jth input.

The stochastic processes \( \{r_k = i, k \geq 0\} \) are homogeneous Markov chains taking values in the finite set \( \mathcal{N}_1 = \{1, 2, \ldots, N_1\} \). The transition probability matrix is \( \Pi = (\pi_{ij}) \), \( \forall \{i, j \in \mathcal{N}_1\} \), and \( \sum_{j=1}^{N_1} \pi_{ij} = 1 \). For convenience, for any \( r_k = i \in \mathcal{N}_1 \), the system matrix \( A_i(r_k) \) is denoted as \( A_i \), other system matrices are similarly denoted as \( A_{di}, B_i, C_i \) and \( D_i \), respectively. It is assumed that \( A_i, B_i \) is controllable. The input matrix \( B_i \) and the output matrix \( C_i \) are column full rank and row full rank, respectively. It is assumed that \( \Pi \in \mathcal{N}_1, \text{rank}(C_iB_i) = m_r \).

The system (1) can be simplified as

\[
\begin{align*}
    x_{k+1} &= A_ix_k + A_{di}x_k - d_k \\
    &+ B_i(\text{sat}(u_k) + f(y_k, k)) \\
    &+ D_i\omega_k \\
    y_k &= C_ix_k \\
    x_k &= \phi_k, \forall k \in [-d_M, 0]
\end{align*}
\]

Assumption 2.1: The bounded matching nonlinearity \( f(y_k, k) \) satisfies

\[
\|f(y_k, k)\| \leq a + b\|y_k\|
\]

where \( a \) and \( b \) are unknown scalars.

Since rank \( (C_iB_i) = m_r \), it is obvious that there is a nonsingular matrix \( T_i \in \mathbb{R}^{n \times n} \), the system (3) can be transformed into the following form under the state transition \( z_k = T_ix_k \),

\[
\begin{align*}
    z_{k+1} &= \tilde{A}_iz_k + \tilde{A}_{di}z_k - d_k \\
    &+ \left[ O_{(n-m) \times m} \right] \left( B_{2i} \right) \text{sat}(u_k) + f(y_k, k)) \\
    &+ \tilde{D}_i\omega_k \\
    y_k &= \left[ O_{p \times (n-p)} \right] C_{2i}z_k
\end{align*}
\]

where \( B_{2i} \in \mathbb{R}^{m \times m} \) and \( C_{2i} \in \mathbb{R}^{p \times p} \) are nonsingular matrices, \( \tilde{A}_i = \left[ \tilde{A}_{11i}, \tilde{A}_{12i}, \tilde{A}_{21i}, \tilde{A}_{22i}, \tilde{A}_{d1i}, \tilde{A}_{d2i} \right] \), \( \tilde{A}_{11i} \in \mathbb{R}^{(n-m) \times (n-m)}, \tilde{A}_{12i} \in \mathbb{R}^{(n-m) \times m}, \tilde{A}_{21i} \in \mathbb{R}^{m \times (n-m)}, \tilde{A}_{22i} \in \mathbb{R}^{m \times m}, \) and \( \tilde{D}_i = \left[ \tilde{D}_{11i}, \tilde{D}_{12i}, \tilde{D}_{21i}, \tilde{D}_{22i} \right] \), \( \tilde{D}_{11i} \in \mathbb{R}^{(n-m) \times q}, \tilde{D}_{21i} \in \mathbb{R}^{m \times q} \).
With loss of generality assume $C = [0 \ I_p]$. Since the matrix $B_l$ is column full rank in system (1), so the matrix $B_l$ can be written as $B_l = \begin{bmatrix} \frac{B_l}{I_{m}} \end{bmatrix}$, where $B_l$ is an invertible matrix. Let $T_r = \begin{bmatrix} \frac{1}{0} \ -B_l \end{bmatrix}$, then $T_r \beta = \begin{bmatrix} 0 \ \beta \end{bmatrix}$, $C_r T_r^{-1} = \begin{bmatrix} 0 \ I_p \end{bmatrix}$.

Next, (4) can be further decomposed into the sliding mode motion (6) and the approaching motion (7) as follows:

$$z_{1,k+1} = A_{11} z_{1,k} + A_{12} z_{2,k} + d_k + \bar{A}_{12} z_{2,k} + \bar{D}_1 (\omega_k),$$  

(6)

$$z_{2,k+1} = A_{21} z_{1,k} + A_{22} z_{2,k} + d_k + \bar{A}_{22} z_{2,k} + B_2 (\text{sat}(u_k) + f(y_k, k)) + \bar{D}_2 (\omega_k),$$  

(7)

where $z_{1,k+1} \in \mathbb{R}^{n-m}, z_{2,k+1} \in \mathbb{R}^m$.

### 2.2. Asynchronous sliding surface synthesis

It is difficult to measure the system mode $r_k$. Therefore, the detector can be used to obtain the estimated value of $r_k$ with a certain probability (Costa et al., 2015). This leads to the fact that the system modal information of real observation from the controller is often inaccurate, i.e. the signal $l_k$ from the detector to the controller may not be synchronized with the system modal $r_k$, but they are not completely independent, and the phenomenon can be characterized by the given conditional probability. Consequently, the hidden Markov model $(\tau, l_k, \mathcal{N}_1, \mathcal{N}_2)$ is constructed to describe this asynchronous phenomenon. The random variables $l_k$ take values in $\mathcal{N}_2 = \{1, 2, \ldots, N_2\}$. The description is as follows:

$$\Pr(l_k = \mu | r_k = i) = \chi_{\mu i}, \quad \forall i \in \mathcal{N}_1, \mu \in \mathcal{N}_2$$  

(8)

where, the conditional probability of modal detection $\chi_{\mu i} \in [0, 1]$ satisfies $\sum_{\mu=1}^{N_2} \chi_{\mu i} = 1$. The conditional probability matrix of modal detection is defined as $\Omega = [\chi_{\mu i}]$.

**Remark 2.1:** Note that $\chi_{\mu i}$ is the probability of the systems running in the $i$th mode but the controller in the $j$th mode. Therefore, our problems are more general. The above hidden Markov model (8) includes two cases. 1) For $\mathcal{N}_2 = \mathcal{N}_2$ and $\chi_{\mu i} = 1$ when $\mu = i$, it is called modal dependence. Meanwhile, the designed controller is synchronous controller. 2) For $\mathcal{N}_2 = 1$, it is called modal independence. The controller is changed into a single-mode controller. Thus, asynchronous controller includes synchronous controller and single-mode controller.

Based on the measurement output, the following asynchronous sliding surface function is constructed

$$s_k = [-K_\mu \ l_m] C^{-1}_2 y_k$$  

(9)

where $\mu \in \mathcal{N}_2$, $K_\mu$ is the parameter matrix to be designed. Substituting (5) into (9), we can get

$$s_k = [-K_\mu \ l_m] C^{-1}_2 \begin{bmatrix} 0 & C_2 \end{bmatrix} z_k$$  

$$= [-K_\mu \ l_m] \begin{bmatrix} 0(n-m) \times (n-m) & l_m \end{bmatrix} \begin{bmatrix} y_k \end{bmatrix}$$  

$$= [-K_\mu C_1 l_m \ z_k]$$  

$$= -K_\mu C_1 z_{1,k} + z_{2,k} = 0$$

where $C_1 = [0 \ I_{p-m} \times p-m \ l_{p-m}]$. It can be seen from $s_k = 0$ that $z_{2,k} = K_\mu C_1 z_{1,k}$. The sliding mode equation is obtained by substituting $z_{2,k}$ into the sliding mode motion Equation (6)

$$z_{1,k+1} = A_{11} z_{1,k} + A_{12} z_{2,k} + d_k + \bar{A}_{12} z_{2,k}$$  

(10)

where $A_{11} \triangleq A_{11} + \bar{A}_{12} K_\mu C_1$, $A_{12} \triangleq A_{12} + A_{12} K_\mu C_1$.

We define $\eta_k = z_{1,k+1} - z_{1,k}$ and assume that for any $k \in [-d_M, \ldots, 0]$ there is a positive number $b_1$ such that

$$\eta_k \in \Omega \subset [0, 1].$$  

**Definition 2.1 (Niu et al., 2015):** Consider $N > 0 (N \in \mathbb{N}^+)$ and non-zero disturbance $\omega_k$ satisfying (2). Let scalars $c_2 > c_1 > 0$, weighted matrix $R_i > 0$. If $E[Z_{1,k+1} R_i Z_{1,k}] \leq c_1 (\forall k \in [-d_M, 0])$, there are $\in [1, N]$. Then, the system (10) is called stochastic finite-time bounded with respect to $(c_1, c_2, N, R_i, \delta)$.

**Remark 2.2:** The finite-time boundedness is different from the finite-time stability in Zuo et al. (2013). When the disturbance input is $0$ ($\omega_k = 0$), the finite-time boundedness degenerates into finite-time stability. In addition, the delay systems considered in this paper are quite diverse from the systems without delay. Specifically, due to the existence of systems delays, the initial states $\phi_k = T \phi_k$, $\forall k \in [-d_M, 0]$ satisfy $E[Z_{1,k+1} R_i Z_{1,k}] \leq c_1 (\forall k \in [1, N])$. On the other hand, in order to improve the performance of Markov jump systems, it is meaningful to minimize the trajectory boundary $c_2$ and maximize the finite-time interval $N$.

**Definition 2.2 (Park et al., 2011):** Let $f_1, f_2, \ldots, f_N : \mathbb{R}^m \mapsto \mathbb{R}^n$ be positive definite functions on the open subset $D$ of $\mathbb{R}^m$. The interactive convex combination of the defined function on $D$ can be expressed as

$$\frac{1}{\alpha_1} f_1 + \frac{1}{\alpha_2} f_2 + \cdots + \frac{1}{\alpha_N} f_N : D \mapsto \mathbb{R}^n$$

where $\alpha_i > 0$ and $\sum_{i=1}^{N} \alpha_i = 1$.
Lemma 2.1 (Park et al., 2011): If \( f_1, f_2, \ldots, f_N : \mathbb{R}^m \mapsto \mathbb{R}^n \) is a set of positive definite functions on the open subset \( D \) of \( \mathbb{R}^m \), then the interactive convex combination of \( f_i \) over \( D \) has the following properties:

\[
\min \sum_{i=1}^{N} \frac{1}{\alpha_i} f_i(t) = \sum_{i=1}^{N} f_i(t) + \max_{i \neq j} g_{ij}(t)
\]

where \( g_{ij}(t) : \mathbb{R}^m \mapsto \mathbb{R}^n, g_{ij}(t) = g_{j}(t), \left[ g_{ij}(t) g_{ij}(t) \right] \geq 0, \alpha_i > 0 \) and \( \sum_{i=1}^{N} \alpha_i = 1 \).

Lemma 2.2 (Mathiyalagan et al., 2012): For any symmetric positive definite matrix \( E \in \mathbb{R}^{n \times n} \), the scalars \( \tau_m \) and \( \tau_M \) satisfy \( \tau_m < \tau_M \), and the vector \( \eta_k = x_{k+1} - x_k (k \in \mathbb{Z}^+) \) as well as the following inequalities hold

\[
-k^{-\tau_m-1} - \sum_{i=k-\tau_M}^{k-\tau_m} \eta_i^T E \eta_i \\
\leq -\frac{1}{\tau_M - \tau_m} \sum_{i=k-\tau_M}^{k-\tau_m} \eta_i^T E \sum_{i=k-\tau_m}^{k-\tau_m} \eta_i \\
\leq -\tau_m^{-2} \sum_{j=-\tau_m}^{k-\tau_m} \sum_{i=k+j}^{k-1} \eta_i^T E \eta_i \\
\leq -\kappa \sum_{j=-\tau_m}^{k-\tau_m} \sum_{i=k+j}^{k-1} \eta_i \sum_{j=-\tau_m}^{k-\tau_m} \sum_{i=k+j}^{k-1} \eta_i
\]

where \( \kappa = \frac{2}{(\tau_m-2)(\tau_m+\tau_M+1)} \).

3. Stochastic finite-time boundedness analysis of sliding modes

The objectives of this section are to construct the Lyapunov–Krasovskii functional and give the sufficient condition for the stochastic finite-time boundedness of the sliding mode (10) with respect to \( (c_1, c_2, N, R_i, \delta) \).

Theorem 3.1: For each mode \( i \in \mathcal{N}_1 \) and \( \mu \in \mathcal{N}_2 \), the scalars \( \beta > 1 \) and \( b_1 > 0 \) are given. Suppose that the state of the system can reach the sliding surface in finite time. If there exist positive definite matrices \( P_i, Q_1, Q_2, S_1, S_2, S_3, W_i \), real matrix \( G_i \) and positive scalars \( \lambda_{0i}, \lambda_{1i}, \lambda_{2i}, \lambda_{3i}, \lambda_{4i}, \lambda_{5i} \) and \( \lambda_{6i} \), such that the following inequalities hold

\[
\begin{bmatrix}
S_{2i} & * \\
G & S_{2i}
\end{bmatrix} > 0 \quad \ldots \quad (12)
\]

\[
[\Theta_{1i} \quad \Theta_{2i} \quad \Theta_{3i}] < 0 \quad \ldots \quad (13)
\]

\[
\begin{bmatrix}
\lambda_{0i} I \leq \hat{P}_i \leq \lambda_{1i} I, & 0 < \hat{Q}_1 \leq \lambda_{2i} I \\
0 < \hat{Q}_2 \leq \lambda_{3i} I, & 0 < \hat{S}_1 \leq \lambda_{4i} I \\
0 < \hat{S}_2 \leq \lambda_{5i} I, & W_i \leq \lambda_{6i} I
\end{bmatrix}
\]

\[
\begin{bmatrix}
\Theta_{1i} & \Theta_{2i} & \Theta_{3i}
\end{bmatrix} < 0 \quad \ldots \quad (13)
\]

\[
\begin{bmatrix}
\lambda_{0i} I \leq \hat{P}_i \leq \lambda_{1i} I, & 0 < \hat{Q}_1 \leq \lambda_{2i} I \\
0 < \hat{Q}_2 \leq \lambda_{3i} I, & 0 < \hat{S}_1 \leq \lambda_{4i} I \\
0 < \hat{S}_2 \leq \lambda_{5i} I, & W_i \leq \lambda_{6i} I
\end{bmatrix}
\]

\[
\begin{bmatrix}
\Theta_{1i} & \Theta_{2i} & \Theta_{3i}
\end{bmatrix} < 0 \quad \ldots \quad (13)
\]

\[
\begin{bmatrix}
\Theta_{1i} & \Theta_{2i} & \Theta_{3i}
\end{bmatrix} < 0 \quad \ldots \quad (13)
\]
Proof: Construct the Lyapunov–Krasovskii functional for the sliding mode (10)

\[ V(z_{1,k}, i) = \sum_{h=1}^{4} V_h(z_{1,k}, i) \]

where

\[ V_1(z_{1,k}, i) = z_{1,k}^T P_i z_{1,k} \]
\[ V_2(z_{1,k}, i) = \sum_{s=k-d_m}^{k-1} \beta^{k-1-s} z_{1,s}^T Q_1 z_{1,s} \]
\[ + \sum_{s=k-d_m}^{k-1} \beta^{k-1-s} z_{1,s}^T Q_2 z_{1,s} \]
\[ V_3(z_{1,k}, i) = d_m \sum_{t=-d_m}^{t=k-t} \sum_{s=k-t}^{k-1} \beta^{k-1-s} s_T S_1 s \]
\[ V_4(z_{1,k}, i) = (d_M - d_m) \sum_{t=-d_m}^{t=k-t} \sum_{s=k-t}^{k-1} \beta^{k-1-s} s_T S_2 s. \]

Define \( E(\Delta V_k) \triangleq E[V(z_{1,k+1}, r_{k+1}) | z_{1,k}, r_k = i] - V(z_k, r_k = i). \) And calculate the \( \Delta V_1, \Delta V_2, \Delta V_3 \) and \( \Delta V_4, \) respectively. We have

\[ E(\Delta V_1) = E[ V_1(k + 1, z_{1,k+1}, r_{k+1}) | k, z_{1,k}, r_k = i ] \]
\[ = E \left( z_{1,k+1}^T \left( \sum_{j=1}^{N_1} \pi_j P_j \right) z_{1,k+1} \right) \]
\[ - V_1(k, z_{1,k}, i) \]
\[ = \sum_{j=1}^{N_1} \pi_j \sum_{\mu=1}^{N_2} x_{j\mu} \left( (A_{1j} z_{1,k} + \hat{A}_{d1j} z_{1,k-d_k}) + \hat{D}_{1i} z_{1,k} + \hat{A}_{d1} z_{1,k-d_k} + \hat{D}_{1i} \right) \]
\[ - \beta V_1(k, z_{1,k}, i) \]
\[ = \xi_k^T \left( \hat{P}_{1i} \eta_{1} \xi_k - \beta \xi_k^T \hat{P}_1 \xi_k \right) \]
\[ + (\beta - 1) V_1(k, z_{1,k}, i). \]  

(20)

where \( \xi_k = [z_{1,k}^T, z_{1,k-d_m}^T, z_{1,k-d_k}^T, s_{k-1}^T]^T. \)

\[ E(\Delta V_2) = E \left( \sum_{s=k-d_m}^{k-1} \beta^{k-s} z_{1,s}^T \left( \sum_{j=1}^{N_1} \pi_j Q_{1j} \right) z_{1,s} \right) \]
\[ - \beta^{k-1} z_{1,k}^T Q_1 z_{1,k} \]
\[ + (\beta - 1) V_2(k, z_{1,k}, i). \]  

(21)

When the condition (19) is satisfied, according to (21), we have

\[ E(\Delta V_2) = z_{1,k}^T Q_{11} z_{1,k} - \beta^{d_1} z_{1,k-d_m}^T Q_{11} z_{1,k-d_m} \]
\[ + (\beta - 1) V_2(k, z_{1,k}, i). \]  

(22)

Similarly, it is not hard to get

\[ E(\Delta V_3) = d_m \sum_{t=-d_m}^{t=k-t} \sum_{s=k-t}^{k-1} \beta^{k-s} s_T S_1 s \]
\[ - d_m \sum_{t=-d_m}^{t=k-t} \sum_{s=k-t}^{k-1} \beta^{k-s} s_T S_1 s \]
\[ + (\beta - 1) V_3(k, z_{1,k}). \]  

(23)

According to Lemma 2.2, one has

\[ - d_m \sum_{s=k-d_m}^{k-1} \beta^{k-s} s_T S_1 s \]
where

\[
\begin{align*}
\text{Substituting (27) into (26), we get}
\end{align*}
\]

\[
\begin{align*}
E\{\Delta V_k\} & \leq -\beta^{d_m+1}_k \left\{ \zeta_k^T \bar{Z}_{1,k} \mathcal{H}_i \bar{Z}_{1,k}^T \right\} + d_m \sum_{s=-d_m}^{-1} \beta^{-\lambda_{max}(\mathcal{Q}_i)}(\zeta_k^T S_{2j} \beta_3 \delta_k) \\
& \quad + (\beta - 1) V_k(k, z_{1,k}).
\end{align*}
\]

where

\[
\begin{align*}
\bar{Z}_{1,k} &= \begin{bmatrix} z_{1,k}^T & z_{1,k-d_m}^T & z_{1,k-d_m}^T \end{bmatrix}^T \\
\mathcal{H}_i &= \begin{bmatrix} S_{2j} & * & * \\
S_{2j} & * & * \\
-S_{2i} & -S_{2i} & S_{2j} \end{bmatrix}
\end{align*}
\]

Next, construct an auxiliary function

\[
J_k = E\{\Delta V_k - (\beta - 1) V_k - \omega_k^T \Omega \omega_k\}.
\]

Combining with (20)–(28), \(\Omega = \tilde{\mathcal{O}}_1 + \tilde{\mathcal{O}}_2 \tilde{P}_1 \tilde{P}_1 + \tilde{\mathcal{O}}_2 S_1 \tilde{P}_2 + \tilde{\mathcal{O}}_3 S_2 \tilde{P}_3 < 0\) can ensure \(J_k < 0\). Further, we are not hard to get \(E[V_{k+1}] - V_k \leq (\beta - 1) V_k + \omega_k^T \Omega \omega_k\), that is,

\[
E[V_{k+1}] \leq \beta V_k + \lambda_{max}(\mathcal{W}_k) E[\omega_k^T \Omega \omega_k].
\]

Summing \(E[V_{k+1}]\) from 0 to \(k-1\) yields

\[
E[V_k] \leq \beta^k E[V_0] + \lambda_{max}(\mathcal{W}_k) \sum_{j=0}^{k-1} \beta^{k-1-j} \omega_k^T \omega_k \\
\leq \beta^k E[V_0] + \lambda_{max}(\mathcal{W}_k) \beta^k \delta.
\]

On the other hand, for \(x_k = \psi_k (k \in [-d_m, 0])\), we have

\[
E[V_0] = z_{1,0}^T P_1 z_{1,0} + \sum_{s=-d_m}^{-1} \beta^{-\lambda_{max}(\mathcal{Q}_i)}(\zeta_k^T S_{1,s} Q_{i,s} z_{1,s}) \\
+ \sum_{s=-d_m}^{-1} \beta^{-\lambda_{max}(\mathcal{Q}_i)}(\zeta_k^T S_{1,s} Q_{i,s} z_{1,s}) \\
+ d_m \sum_{s=-d_m}^{-1} \beta^{-\lambda_{max}(\mathcal{Q}_i)}(\zeta_k^T S_{1,s} Q_{i,s} z_{1,s}) \\
+ (d_m - d_m) \sum_{s=-d_m}^{-1} \beta^{-\lambda_{max}(\mathcal{Q}_i)}(\zeta_k^T S_{2j} \beta_3 \delta_k) \\
\leq \lambda_{max}(P_1) z_{1,0}^T R_1 z_{1,0} \\
+ \beta^d \lambda_{max}(\mathcal{Q}_1) \sum_{s=-d_m}^{-1} z_{1,s}^T R_1 z_{1,s} \\
+ \beta^d \lambda_{max}(\mathcal{Q}_2) \sum_{s=-d_m}^{-1} z_{1,s}^T R_1 z_{1,s} \\
\leq \lambda_{max}(P_1) z_{1,0}^T R_1 z_{1,0}.$
In addition, The proof is complete.

4. Sliding mode controller design and reachability analysis

4.1. Design of adaptive sliding mode controller with saturation compensation

In this section, an adaptive sliding mode controller is constructed and an auxiliary saturation compensation system is designed to mitigate the negative effects of actuator saturation.

Similar to Yao et al. (2018) and Zhang and Xia (2010), define a small neighbourhood $S$ near the sliding surface before designing the controller

$$S = \{z_k \in \mathbb{R}^n : \|z_k\| \leq \sigma\}, \quad k \in [-d_2, N] \quad (35)$$

where $\sigma > 0$ is an unknown parameter.

Aiming at the nonlinear problem in the system (3), the following adaptive output feedback sliding mode controller with saturation compensation is designed based on the asynchronous sliding mode surface (9)

$$u_k = u_{k1} + u_{k2} \quad (36)$$

where $u_{k1} = -B_{2i}^{-1}y_1s_k + \|[-K_\mu C_{1i} \quad I_m](\hat{A}_i + \hat{A}_d)\|_{[s_k]^{1-\beta_2}}\hat{s}_k + \beta_3 u_{k2} + \beta_4 \hat{\psi}_k$.

The following adaptive laws are designed in the small neighbourhood defined by (35)

$$\Delta \hat{\psi}_k = q_0(-\epsilon_0 \hat{\psi}_k + \|B_{2i}^{-1}s_k\|)$$

$$\Delta \hat{\psi}_k = q_2[-\epsilon_2 \hat{\psi}_k + \|[-K_\mu C_{1i} \quad I_m](\hat{A}_i + \hat{A}_d)\|_{[s_k]^{1-\beta_2}} + \hat{A}_d)] \|\hat{s}_k\|| \quad (37)$$

where $K_\mu$ and $\hat{K}$ are symmetric positive definite matrices, $\hat{\psi}_1$, $\hat{\psi}_2$ are smaller positive scalars, $\gamma$, $\epsilon_0$, $\epsilon_1$, $\epsilon_2$, $q_0$, $q_1$ and $q_2$ are positive scalars, $\hat{\psi}_1$, $\hat{\psi}_2$ and $\hat{\psi}_3$ are the estimate values for unknown parameters $a$, $b$ and $\sigma$, respectively. Besides, $\Delta \hat{\psi}_k = \hat{\psi}_{k+1} - \hat{\psi}_k$, $\Delta \hat{\phi}_k = \hat{\phi}_{k+1} - \hat{\phi}_k$ and $\Delta \hat{\delta}_k = \hat{\delta}_{k+1} - \hat{\delta}_k$.

To compensate the effect of the actuator saturation, the variable $\psi_k$ in $u_2$ is obtained from the following auxiliary system

$$\Delta \psi_k = e_2 \Delta u_k - e_1 \psi_k - \|s_k^{1-\beta_2}\|_{[s_k]^{1-\beta_2}} - 0.5e_2 \Delta u_k^T \Delta u_k \psi_k$$

where $e_1$ and $e_2$ are positive constants.

**Remark 4.1:** In general, the symbolic function is used to design sliding mode controller when analysing the reachability of sliding surface. It usually exists in the form of $s_k\text{sign}(s_k) = \frac{s_k}{\|s_k\|}$, where $\frac{s_k}{\|s_k\|}$ is a discontinuous term. To reduce the chattering caused by the discontinuous term, $\frac{s_k}{\|s_k\|}$ is use to replace the $\frac{s_k}{\|s_k\|}$. Therefore, there is no symbolic function in the design of the controller. Specifically, the controller without symbolic function is designed by replacing discontinuous functions $\frac{s_k}{\|s_k\|}$ and $\frac{\hat{s}_k}{\|\hat{s}_k\|}$ with $\frac{s_k}{\|s_k\| + \epsilon_1}$ and $\frac{\hat{s}_k}{\|\hat{s}_k\| + \epsilon_2}$ in this paper, respectively. Design $u_2$ can compensate the negative effects of actuator saturation. By adjusting the parameter $e_2$ in the auxiliary system (38), the overcompensation can be avoided for the actuator saturation phenomenon.

4.2. Reachability analysis of the sliding mode motion

Suppose that the parameter $\sigma$ is unknown, the reachability of sliding mode surface is discussed under the adaptive sliding mode controller (36) based on the Lyapunov method.

According to (4) and (9), $s_k = [-K_\mu C_{1i} \quad I_m]z_k$, it is not difficult to get

$$s_{k+1} = [-K_\mu C_{1i} \quad I_m] \hat{A}_i z_k + \hat{A}_d z_k - \hat{\delta}_k$$

$$+ \left[ \begin{array}{c} 0 \\ B_{2i} \end{array} \right] \left( \text{sat}(u_k) + f(y_k, k) \right) + \hat{D}_d w_k \quad (39)$$
Theorem 4.1: Consider the discrete nonlinear Markov jump systems (4) with saturation and time-varying delay, suppose that the gain matrix $K_m$ has a feasible solution in the asynchronous sliding mode surface (9). For all $i \in \mathbb{N}_1$ and $\mu \in \mathbb{N}_2$, scalars $\gamma, e_1$, and $e_2$, design the sliding mode controller (36) based on the adaptive law (37) and the auxiliary system (38), if there exist positive definite matrix $K$ satisfying $-\frac{1}{2}K + e_1l_m - \frac{1}{2}e_2l_m > 0$, then the states of the system (4) would be driven onto near $z_k \in \mathbb{S}$.

Proof: Consider the Lyapunov function

$$V_k = \frac{1}{2} \left\{ s_k^T \Delta s_k + \frac{1}{q_0} \Delta \hat{a}_k \Delta \hat{a}_k + \frac{1}{q_1} \Delta \hat{b}_k \right\} + \frac{1}{q_2} \Delta \hat{s}_k \Delta \hat{s}_k + \psi_k^T \Delta \psi_k + \gamma.$$  (40)

where

$$\Delta s_k = s_{k+1} - s_k, \quad \Delta \hat{a}_k = \hat{a}_{k+1} - \hat{a}_k$$

$$\Delta \hat{b}_k = \hat{b}_{k+1} - \hat{b}_k, \quad \Delta \hat{s}_k = \hat{s}_{k+1} - \hat{s}_k$$

$$\gamma = \frac{1}{2} \left( \Delta \hat{a}_k^T \Delta \hat{a}_k + \frac{1}{q_0} \Delta \hat{a}_k^T \Delta \hat{a}_k + \frac{1}{q_1} \Delta \hat{b}_k^T \Delta \hat{b}_k + \frac{1}{q_2} \Delta \hat{s}_k^T \Delta \hat{s}_k + \psi_k^T \Delta \psi_k \right).$$

It is worth noting that $\Delta \hat{a}_k = -\Delta \hat{a}_k, \Delta \hat{b}_k = -\Delta \hat{b}_k$ and $\Delta \hat{s}_k = -\Delta \hat{s}_k$, then

$$E[\Delta \hat{V}_k] = E \left\{ s_k^T \Delta s_k - \frac{1}{q_0} \Delta \hat{a}_k \Delta \hat{a}_k - \frac{1}{q_1} \Delta \hat{b}_k \right\} + \frac{1}{q_2} \Delta \hat{s}_k \Delta \hat{s}_k + \psi_k^T \Delta \psi_k + \gamma.$$  (41)

Letting $\Delta u_k = \text{sat}(u_k) - u_k$, by substituting the adaptive laws (37) and (39) into (41), we have

$$E[\Delta \hat{V}_k] = s_k^T \left( -K_i C_i \quad I_m \right) \left( \hat{A}_i z_k + \hat{A}_{di} z_k - \delta_k \right) + s_k^T B_2 u_k + s_k^T B_2 \Delta u_k + s_k^T \tilde{B}_2 \left( \gamma y_k, k \right) + s_k^T \left( -K_i C_i \quad I_m \right) D_i \omega_k - s_k^T s_k - \frac{1}{q_0} \Delta \hat{a}_k q_0 \left( -\epsilon_0 \Delta \hat{a}_k + \|B_2^T s_k\|\right) - \frac{1}{q_1} \Delta \hat{b}_k q_1 \left( -\epsilon_1 \Delta \hat{b}_k + \|B_2^T s_k\| \right) \left( \gamma \right)^2.$$

Combining Assumption 2.1, the controller (36) with the auxiliary system (38), we can obtain

$$E[\Delta \hat{V}_k] \leq s_k^T \left( \gamma I_m + \frac{1}{2} \right) s_k + s_k^T \tilde{K} \psi_k + s_k^T \tilde{B}_2 \left( \gamma y_k, k \right) - \left\{ s_k^T \left( -K_i C_i \quad I_m \right) \left( \hat{A}_i + \hat{A}_{di} \right) \right\} \|s_k\| - \frac{1}{q_2} \Delta \hat{s}_k \left( -\epsilon_2 \Delta \hat{s}_k \right) + \|\left( -K_i C_i \quad I_m \right) \left( \hat{A}_i + \hat{A}_{di} \right) \|s_k\| + \psi_k \Delta \psi_k + \gamma.$$  (42)

According to the fundamental inequality and norm properties, there are

$$s_k^T B_2 \Delta u_k \leq s_k^T B_2 \left( \gamma y_k, k \right)$$

$$s_k^T \tilde{K} \psi_k \leq \frac{1}{2} s_k^T \tilde{K} s_k + \frac{1}{2} \psi_k^T \tilde{K} \psi_k$$  (43)

$$\psi_k \Delta u_k \leq \frac{1}{2} \psi_k^T \psi_k + \frac{1}{2} \Delta u_k^T \Delta u_k.$$  (44)

Then, (42) can be simplified as

$$E[\Delta \hat{V}_k] \leq s_k^T \left( \gamma I_m + \frac{1}{2} K \right) s_k - \epsilon_2 \left( \Delta \hat{a}_k - \frac{\sigma}{2} \right)^2 - \epsilon_0 \left( \Delta \hat{a}_k - \frac{\sigma}{2} \right)^2 - \epsilon_1 \left( \Delta \hat{b}_k - \frac{b}{2} \right)^2 - \psi_k \left( -\frac{1}{2} \Delta \hat{s}_k - e_1 \Delta \hat{s}_k \right) + \frac{1}{4} \epsilon_2 \sigma^2 + \frac{1}{4} \epsilon_2 \sigma^2 + \gamma.$$
Select the parameters $e_1$ and $e_2$, such that $e_1 l_m - \frac{1}{2} \mathbf{K} - \frac{1}{2} e_2 l_m > 0$, and then

$$
E(\Delta \hat{V}_k) \leq -s_k^T \left( \gamma l_m + l_m - \frac{1}{2} \mathbf{K} \right) s_k - e_2 \left( \hat{\sigma}_k - \frac{\sigma}{2} \right)^2 - e_0 \left( \hat{\sigma}_k - \frac{\sigma}{2} \right)^2 - \epsilon_1 \left( b_k - b \right)^2 + \frac{1}{4} \epsilon_1 b^2 + \frac{1}{4} e_2 \sigma^2 + \frac{1}{4} e_0 \sigma^2 + \gamma.
$$

(44)

The parameter $\gamma$ can be selected according to the practical situation. When the sliding function $s_k$ tends to the small bounded neighbourhood near the equilibrium point, $E(\Delta \hat{V}_k) < 0$ can be guaranteed by adjusting $\gamma$. Therefore, according to literatures (Wang & Liu, 2018; Yao et al., 2018), under the controller (36), the state trajectories of the system (4) can be driven to the small neighbourhood $\mathcal{B}$ near the sliding surface and kept on it all the time.

\section*{4.3. Solve the parameters}

Note that the conditions given in Theorem 3.1 are not linear matrix inequalities. Next, the nonlinear coupling terms in inequality (13) are dealt with.

\textbf{Theorem 4.2:} For given positive scalars $c_2 > c_1$, $\beta > 1$ and positive definite matrix $R_i$, each mode $i \in \mathcal{N}_1$ and $\mu \in \mathcal{N}_2$, assume that there exist positive definite matrices $P_i, Q_{ii}, Q_{2i}, S_{ii}, S_{2i}, \bar{W}_i$, real matrix $G_i$ and positive scalars $\lambda_{0i}, \lambda_{1i}, \lambda_{2i}, \lambda_{3i}, \lambda_{4i}, \lambda_{5i}$ and $\lambda_{6i}$, such that the following inequalities are solvable

$$
\begin{bmatrix}
S_{2i} & * \\
G_i & S_{2i}
\end{bmatrix} > 0
$$

(45)

$$
\Omega_i =
\begin{bmatrix}
\Theta_i & * & * & * \\
\theta_{1i} & \Omega_{22i} & * & * \\
\theta_{2i} & 0 & \Omega_{33i} & * \\
\theta_{3i} & 0 & 0 & \Omega_{44i}
\end{bmatrix} < 0
$$

(46)

$$
\lambda_0 R_i \leq P_i \leq \lambda_1 R_i, \quad 0 < Q_{ii} \leq \lambda_2 R_i
$$

(47)

$$
0 < Q_{2i} \leq \lambda_3 R_i, \quad 0 < S_{ii} \leq \lambda_4 R_i
$$

(48)

$$
0 < S_{2i} \leq \lambda_5 R_i, \quad W_i \leq \lambda_6 l_n
$$

(49)

$$
v_1c_1 + v_2b_1 + \lambda_2i \delta < \beta^{-N} c_2 \lambda_{0i}.
$$

(50)

where

$$\begin{align*}
\Omega_{22i} &= -2l_{n-m} + \bar{P}_i, \quad \Omega_{33i} = -2l_{n-m} + S_{ii} \\
\Omega_{44i} &= -2l_{n-m} + S_{2i}
\end{align*}
$$

Then, the sliding mode (11) on the sliding surface (9) is robust stochastic finite-time bounded with respect to $(c_1, c_2, N, R_i, \delta)$. Further, the output feedback gain matrix $K_\mu$ can be obtained by solving (45)–(50).

\textbf{Proof:} With the help of $(l_{n-m} - \bar{P}_i)\bar{P}_i^{-1}(l_{n-m} - \bar{P}_i) \geq 0$, we can get

$$
-\bar{P}_i^{-1} \leq -2l_{n-m} + \bar{P}_i.
$$

Similarly, we have $-S_{ii}^{-1} \leq -2l_{n-m} + S_{ii}$ and $-S_{2i}^{-1} \leq -2l_{n-m} + S_{2i}$. Therefore, the matrix inequality (13) can be transformed into (46).

\textbf{Remark 4.2:} In Theorem 3.1, $c_2$ depends on the values of $c_1, \beta, N$. For given $\beta$ and $N$, if $c_1$ is fixed, then the minimum value of $c_2$ can be obtained by solving the following optimization problem

$$
\begin{align*}
& \min \quad c_2 \\
& \text{s.t.} \quad (45) - (50).
\end{align*}
$$

According to Theorem 4.2, we know that $c_2$ exist in the form of linear in the inequalities (45)–(50). Therefore, the optimization problem ‘min $c_2$, s.t. (45)–(50)’ is a convex optimization problem, which can be solved by using the convex optimization toolbox in Matlab to obtain the minimum value of $c_2$.

\section*{5. Numerical simulation and analysis}

In this section, a numerical example is given to illustrate the advantages of the designed controller based on an auxiliary saturation compensation system.

Consider the discrete Markov jump systems with two modes, and the controller also has two modes, that is $\mathcal{N}_1 = \{1, 2\}$ and $\mathcal{N}_2 = \{1, 2\}$, transition probability matrix $\Pi = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}$, modal detection conditional probability matrix $\Omega = \begin{bmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{bmatrix}$. The following system parameters in (4) are selected

$$
\begin{align*}
\bar{A}_1 &=
\begin{bmatrix}
0.695 & -0.05 & 0.0792 \\
-0.205 & 0.95 & 0.62 \\
0.01 & 0 & 0.98
\end{bmatrix} \\
\bar{A}_2 &=
\begin{bmatrix}
0.9 & 0.2 & 0.9 \\
-0.3 & 0.37 & 0.2 \\
0.8 & 0.1 & 0.82
\end{bmatrix} \\
\bar{A}_{d1} &=
\begin{bmatrix}
0.05 & 0.01 & 0.04 \\
0.03 & 0.01 & 0.01
\end{bmatrix} \\
\bar{A}_{d2} &=
\begin{bmatrix}
0.01 & -0.02 & 0.054 \\
0.034 & 0.01 & 0.034 \\
-0.04 & 0.01 & 0.001
\end{bmatrix} \\
B_2 &= [0.1, 0.2, 0.9], \quad C_1 = C_2 = l_2
\end{align*}
$$

\begin{align*}
\end{align*}
Figure 2. Mode evolution of system and controller.

Figure 3. The trajectories $z_k$ and $z^T_kR_1z_{1,k}$ of open-loop system.

\[
\tilde{D}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \tilde{D}_2 = \begin{bmatrix} 0 \\ 0.1 \\ 0.1 \end{bmatrix}
\]

and the nonlinear function \( f(y_k, k) = 0.3 \sin(3k)y_{1,k} \).

The time-varying delay is selected as \( \alpha_k \in [2, 5] \), the initial values of the system states are \( z_k = [1.9 \ 3 \ -0.9]^T (k \in [-5, 0]) \), the upper bound of the cumulative bounded disturbance is \( \delta = 1 \), \( N = 50 \) and the saturation level is \( \mu_{j,\text{max}} = 1.3 \).

Let \( c_1 = 1, \ c_2 = 15, \ \gamma = 1.01, \ e_1 = 1, e_2 = 0.1, \ \vartheta_1 = \vartheta_2 = 1, \ \beta = 1.001, \ b_1 = 0.01 \) and positive definite matrix \( R_1 = R_2 = \text{diag}(0.1, 0.07) \). Give the initial value of the adaptive laws \( \hat{a}_0 = 0.058, \ \hat{b}_0 = 0.0275, \ \hat{\sigma}_0 = 0.05 \). Obviously, the initial values satisfy \( z^T_kR_1z_{1,k} \leq c_1, \ \eta^T_k\eta_k \leq b_1 (k \in [-5, 0]) \).

Figure 2 shows the modal evolution of the systems and the controller under asynchronous control strategy. The trajectories of $z_k$ and sliding mode $z^T_kR_kz_k$ in the open-loop system (4) are shown in Figure 3. It is obvious that the systems are unstable without sliding mode control.
For given parameters $\epsilon_0 = \epsilon_1 = \epsilon_2 = 1$, $q_0 = q_1 = q_2 = 1$, by solving Theorem 4.2, we obtain the controller parameters: $K_1 = -0.1377$, $K_2 = 0.0136$ and $\tilde{K} = 0.9498$. According to whether the saturation phenomenon is compensated, the control performance is discussed in two cases for the adaptive output feedback sliding mode controller system (36).

Case 1: There is the phenomenon of actuator saturation but not compensated saturation. Under the controller $u_k = u_{1,k}$, the simulation results are shown in Figures 4–6.

Case 2: The auxiliary system (38) is designed to compensate the actuator saturation phenomenon. Under the adaptive sliding mode controller (36), the simulation results are shown in Figures 7–9.

It can be clearly seen that in Case 1, although the states can be driven to the origin in Figures 4–6, the unknown parameters can be estimated effectively, but compared with Figures 7–9 in Case 2, the actuator saturation phenomenon leads to poor control performance and slower convergence speed. Therefore, the designed control strategy based on the new auxiliary saturation
compensation system has the obvious control effect in this paper. The phenomenon of input saturation in Figures 4 and 7 roughly occurs in intervals [2 16] and [2 3], respectively. It can be seen that the design auxiliary system can reduce the impact of saturation through compensation. The simulation results verify the superiority and necessity of the design method.

6. Conclusions

For the actuator saturation phenomenon, based on the auxiliary saturation compensation system, the adaptive asynchronous sliding mode control problem has been studied for a class of networked discrete time-delay Markov jump systems with bounded noise and unknown disturbance. Firstly, since the system mode is not easy to
get accurately, by using the hidden Markov model, the asynchronous sliding surface has been designed based on the measured output, and the sliding equation has been obtained. Moreover, the Lyapunov–Krasovskii functional has been constructed, and the criterion of the sliding mode stochastic finite-time boundedness has been given. Secondly, in the case that the sliding mode region is bounded but the upper bound is unknown, in order to reduce the chattering and the negative impact caused by the actuator saturation, an adaptive output feedback sliding mode controller without the sign function has been designed via the auxiliary saturation compensation, which ensures that the system state trajectories can reach the sliding region and remain in the small neighbourhood. Finally, the effectiveness of the proposed control strategy has been verified by the numerical simulations.

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