Uniqueness of ground states for combined power-type nonlinear scalar field equations involving the Sobolev critical exponent and a large frequency parameter in three and four dimensions

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Abstract

We prove the uniqueness of ground states for combined power-type nonlinear scalar field equations involving the Sobolev critical exponent and a large frequency parameter. This study is motivated by the paper [2] and aims to remove the restriction on dimension imposed there. In this paper, we employ the fixed-point argument developed in [7] to prove the uniqueness. Hence, the linearization around the Aubin-Talenti function plays a key role. Furthermore, we need some estimates for the associated perturbed resolvents (see Proposition 3.1).

Mathematics Subject Classification 35J20, 35B09, 35Q55

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1 Introduction

In this paper, we consider the following equation:

\[- \Delta u + \omega u - |u|^{p-1}u - \frac{|u|^{\frac{4}{d-2}}}{d-2}u = 0 \quad \text{in} \quad \mathbb{R}^d, \tag{1.1}\]

where \(d \geq 3\), \(1 < p < \frac{d+2}{d-2}\) and \(\omega > 0\).

This study is motivated by the paper [2] and aims to remove the restriction on dimension imposed there. More precisely, our aim is to prove the uniqueness of ground states to (1.1) when \(d = 3, 4\) and \(\omega\) is large (see Theorem 1.1). We remark that when \(\omega\) is small, the uniqueness had been proved in [3] for all \(d \geq 3\). We also remark that the properties of ground states to (1.1) are important in the study of the following nonlinear Schrödinger and Klein-Gordon equations (see [3, 4, 13]):

\[i \frac{\partial u}{\partial t} + \Delta u + |u|^{p-1}u + \frac{|u|^{\frac{4}{d-2}}}{d-2}u = 0, \tag{1.2}\]
\[\frac{\partial^2 u}{\partial t^2} - \Delta u + u - |u|^{p-1}u - \frac{|u|^{\frac{4}{d-2}}}{d-2}u = 0. \tag{1.3}\]

In order to describe our main result, we introduce a few symbols:

**Notation 1.1.**

1. For each \(d \geq 3\), we use \(2^*\) to denote the Sobolev critical exponent, namely

\[2^* := \frac{2d}{d-2}. \tag{1.4}\]

2. For \(d \geq 3\), \(1 < p < \frac{d+2}{d-2}\) and \(\omega > 0\), we define functionals \(S_\omega\) and \(N_\omega\) on \(H^1(\mathbb{R}^d)\) by

\[S_\omega(u) := \frac{\omega}{2} \|u\|^2_{L^2} + \frac{1}{2} \|\nabla u\|^2_{L^2} - \frac{1}{p+1} \|u\|^{p+1}_{L^{p+1}} - \frac{1}{2^*} \|u\|^{2^*}_{L^{2^*}}, \tag{1.5}\]
\[N_\omega(u) := \omega \|u\|^2_{L^2} + \|\nabla u\|^2_{L^2} - \|u\|^{p+1}_{L^{p+1}} - \|u\|^{2^*}_{L^{2^*}}. \tag{1.6}\]

Next, we clarify terminology:

**Definition 1.1.**

1. Let \(d \geq 3\), \(1 < p < \frac{d+2}{d-2}\) and \(\omega > 0\). Then, by a ground state to (1.1), we mean a minimizer for the following variational problem:

\[
\inf \{S_\omega(u) : u \in H^1(\mathbb{R}^d) \setminus \{0\}, \ N_\omega(u) = 0\}. \tag{1.7}
\]

2. We use the term “radial” to mean radially symmetric about \(0 \in \mathbb{R}^d\).

We refer to a known result for the existence of a ground state to (1.1) (see, e.g., [2]):

**Proposition 1.1.** Assume either \(d = 3\) and \(3 < p < 5\), or else \(d \geq 4\) and \(1 < p < \frac{d+2}{d-2}\). Then, for any \(\omega > 0\), there exists a ground state to (1.1). Furthermore, for any \(\omega > 0\), the following hold:

1. A ground state to (1.1) becomes a solution to (1.1) and is in \(C^2(\mathbb{R}^d)\).

2. For any ground state \(Q_\omega\) to (1.1), there exist \(y \in \mathbb{R}^d, \ \theta \in \mathbb{R}\) and a positive radial function \(\Phi_\omega\) such that \(Q_\omega(x) = e^{i\theta} \Phi_\omega(x - y)\); In particular, \(\Phi_\omega\) is a positive radial ground state to (1.1).
3. Let $\Phi_{\omega}$ be a positive radial ground state to (1.1). Then, $\Phi_{\omega} = \Phi_{\omega}(x)$ is strictly decreasing as a function of $|x|$. In particular, $\|\Phi_{\omega}\|_{L^\infty} = \Phi_{\omega}(0)$.

4. For any positive radial ground state $\Phi_{\omega}$ to (1.1), there exist $\nu > 0$ and $C > 0$ such that for any $x \in \mathbb{R}^d$,

$$|\Phi_{\omega}(x)| + |\nabla\Phi_{\omega}(x)| + |\Delta\Phi_{\omega}(x)| \leq Ce^{-\nu|x|}. \quad (1.8)$$

In particular, $\Phi_{\omega} \in H^2(\mathbb{R}^d)$.

Remark 1.1. In [5], it is proved that if $d = 3$ and $1 < p \leq 3$, then there exists $\omega_c > 0$ such that for any $\omega > \omega_c$, there is no ground state to (1.1).

By Proposition 1.1 we may assume that a ground state to (1.1) is positive and radial.

Now, we state the main result of this paper:

Theorem 1.1. Assume $d = 3, 4$ and $\frac{4}{d-2} - 1 < p < \frac{d+2}{d-2}$; namely either $d = 3$ and $3 < p < 5$, or $d = 4$ and $1 < p < 3$. Then, there exists $\omega_* > 0$ such that for any $\omega > \omega_*$, a positive radial ground state to (1.1) is unique.

Remark 1.2. 1. By the result of Pucci and Serrin [17] (see also Appendix C of [2]), we find that if $3 \leq d \leq 6$ and $\frac{4}{d-2} \leq p < \frac{d+2}{d-2}$, then for any $\omega > 0$, a positive radial solution to (1.1) is unique; in particular, a positive radial ground state is unique.

2. In [2], it had been proved that if $d \geq 5$ and $1 < p < \frac{d+2}{d-2}$, then there exists $\omega_* > 0$ such that for any $\omega \geq \omega_*$, a positive radial ground state to (1.1) is unique.

We give a proof of Theorem 1.1 in Section 4.

Acknowledgements

The authors would like to thank Professor Hiroaki Kikuchi for helpful discussion. This work was supported by JSPS KAKENHI Grant Number 20K03697.

2 Preliminaries

We collect symbols used in this paper, with auxiliary results:

1. For $d \geq 3$, $1 < p < \frac{d+2}{d-2}$ and $\omega > 0$, the symbol $\mathcal{G}_\omega$ denotes the set of positive radial ground states to (1.1).

2. $L^2_{\text{real}}(\mathbb{R}^d)$ denotes the real Hilbert space of complex-valued $L^2$-functions equipped with the inner product

$$\langle f, g \rangle := \Re \int_{\mathbb{R}^d} f(x)\overline{g(x)} \, dx.$$

3. For $u, v \in L^2(\mathbb{R}^d)$, we define

$$(u, v) := \int_{\mathbb{R}^d} u(x)\overline{v(x)} \, dx. \quad (2.1)$$
4. We define the Fourier and inverse Fourier transformations to be that for \( u \in L^1(\mathbb{R}^d) \),
\[
\mathcal{F}[u](\xi) := \int_{\mathbb{R}^d} e^{-i\xi \cdot x} u(x) \, dx, \quad \mathcal{F}^{-1}[u](\xi) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\xi \cdot x} u(x) \, dx.
\] (2.2)

Then, the relationship between Fourier transformation and convolution (see Theorem 0.1.8 of \cite{IS}), and Parseval’s identity (see Theorem 0.1.11 of \cite{IS}) are stated as
\[
\mathcal{F}\left[ \int_{\mathbb{R}^d} u(\cdot - y)v(y) \, dy \right] = \mathcal{F}[u] \mathcal{F}[v],
\] (2.3)
\[
(u, v) = \frac{1}{(2\pi)^d} \langle \mathcal{F}[u], \mathcal{F}[v] \rangle.
\] (2.4)

Furthermore, it is known (see, e.g., Theorem 2.4.6 of \cite{10}, or Theorem 5.9 of \cite{16}∗) that for any \( 0 < r < d \),
\[
\mathcal{F}[|x|^{-r}](\xi) = C_r |\xi|^{-(d-r)}, \quad \text{with} \quad C_r := \frac{2^{d-r} \pi^{\frac{d}{2}} \Gamma\left(\frac{d-r}{2}\right)}{\Gamma\left(\frac{r}{2}\right)}. \] (2.5)

5. We use \( W \) to denote the Aubin-Talenti function with \( W(0) = 1 \), namely
\[
W(x) := \left(1 + \frac{|x|^2}{d(d-2)}\right)^{-\frac{d-2}{2}}. \] (2.6)

Note that
\[
W(x) \sim (1 + |x|)^{-\frac{(d-2)}{2}}, \quad \|W\|_{L^d_{\text{weak}}} < \infty ,
\] (2.7)
where the implicit constant depends only on \( d \): in particular, if \( d = 3, 4 \), then \( W \notin L^2(\mathbb{R}^d) \). It is known that for \( d \geq 3 \), \( W \) is a solution to the following equation:
\[
\Delta u + |u|^\frac{4}{d-2} u = 0.
\] (2.8)

6. For \( d \geq 3 \), we define a potential function \( V \) as
\[
V := -\frac{d + 2}{d - 2} W^{\frac{1}{d-2}}.
\] (2.9)

Note that \( V \) is negative. Observe from (2.7) that for any \( d \geq 3 \),
\[
|V(x)| \lesssim (1 + |x|)^{-4},
\] (2.10)
where the implicit constant depends only on \( d \).

7. Let \( \lambda > 0 \). Then, we define the \( H^1 \)-scaling operator \( T_\lambda \) by
\[
T_\lambda[v](x) := \lambda^{-1} v(\lambda^{-\frac{2}{d-2}} x).
\] (2.11)

∗Notice that the definition of Fourier transformation in \cite{10} and \cite{16} is different from ours.
8. We use \( \Lambda W \) to denote the function defined as
\[
\Lambda W := \frac{d - 2}{2} W + x \cdot \nabla W = \left( \frac{d - 2}{2} - \frac{|x|^2}{2d} \right) \left( 1 + \frac{|x|^2}{d(d-2)} \right)^{-\frac{d}{2}}.
\] (2.12)
Observe that
\[
(-\Delta + V)\Lambda W = 0,
\] (2.13)
\[
|\Lambda W(x)| \lesssim (1 + |x|)^{-(d-2)},
\] (2.14)
where the implicit constant depends only on \( d \). In particular, (2.14) shows that if \( d = 3, 4 \), then \( \Lambda W \not\in L^2(\mathbb{R}^d) \). A computation involving integration by parts shows that for any \( \lambda > 0 \),
\[
\frac{d}{d\lambda} T_\lambda[W] = -\lambda^{-1} T_\lambda[W] - \frac{2}{d-2} T_\lambda[x \cdot \nabla W] = -\frac{2}{d-2} \lambda^{-1} T_\lambda[\Lambda W],
\] (2.16)
so that \( \Lambda W = -\frac{2}{d-2} \frac{d}{d\lambda} T_\lambda[W] \big|_{\lambda=1} \).
Observe from the fundamental theorem of calculus and (2.16) that for any \( \mu > 0 \),
\[
T_\mu[W] - W = \int_1^\mu \frac{d}{d\lambda} T_\lambda[W] \, d\lambda = -\frac{2}{d-2} \int_1^\mu \lambda^{-1} T_\lambda[\Lambda W] \, d\lambda.
\] (2.17)
Furthermore, by (2.17) and the fundamental theorem of calculus, we see that for any \( \mu > 0 \),
\[
T_\mu[W] - W + \frac{2}{d-2}(\mu - 1)\Lambda W
\]
\[
= -\frac{2}{d-2} \int_1^\mu \int_1^{\nu} \frac{d}{d\lambda} \left( \lambda^{-1} T_\lambda[\Lambda W] \right) \, d\lambda \, d\nu
\] (2.18)
\[
= \frac{2}{d-2} \int_1^\mu \int_1^{\nu} \lambda^{-2} T_\lambda \left[ 2\Lambda W + \frac{2}{d-2} x \cdot \nabla \Lambda W \right] \, d\lambda \, d\nu.
\]
9. We use \( \Gamma \) to denote the gamma function. Recall that \( \Gamma\left(\frac{d}{2}\right) = \frac{\sqrt{\pi}}{2} \) and \( \Gamma(3) = 2 \).
10. For \( d \geq 3 \), we use \( (-\Delta)^{-1} \) to denote the operator defined to be that for any function \( f \) satisfying \((1 + |x|)^{-(d-2)} f \in L^1(\mathbb{R}^d)\),
\[
(-\Delta)^{-1} f(x) ga := \int_{\mathbb{R}^d} \frac{\Gamma\left(\frac{d}{2}\right)}{(d-2)2^{d/2}\pi^{\frac{d}{2}}} |x - y|^{-(d-2)} f(y) \, dy.
\] (2.19)
Note that H"older’s inequality shows that if \( 1 \leq r < \frac{d}{2} \) and \( f \in L^r(\mathbb{R}^d) \), then \((1 + |x|)^{-(d-2)} f \in L^1(\mathbb{R}^d) \). The Hardy-Littlewood-Sobolev inequality shows that for any \( 1 < r < \frac{d}{2} \ (r \neq 1) \) and \( f \in L^r(\mathbb{R}^d) \),
\[
\|(-\Delta)^{-1} f\|_{L^{\frac{d}{d-2r}}} \lesssim \|f\|_{L^r},
\] (2.20)
where the implicit constant depends only on \( d \) and \( r \). Furthermore, the standard theory for Poisson’s equation (see, e.g., Theorem 6.21 of [16]) shows that for any \( 1 \leq r < \frac{d}{2} \) and \( f \in L^r(\mathbb{R}^d) \),
\[
(-\Delta)(-\Delta)^{-1}f = f \text{ in the distribution sense.} \tag{2.21}
\]
By (2.21) and \((-\Delta + V)\Lambda W = 0\) (see (2.13)), we see that
\[
(-\Delta)(-\Delta)^{-1}V\Lambda W = V\Lambda W = \Delta \Lambda W \text{ in the distribution sense,} \tag{2.22}
\]
which implies that there exists a harmonic function \( h \) on \( \mathbb{R}^d \) such that
\[
(-\Delta)^{-1}V\Lambda W = -\Lambda W + h. \tag{2.23}
\]
Note here that (2.20) shows that
\[
\|(-\Delta)^{-1}V\Lambda W\|_{L^{2*}} \lesssim \|V\Lambda W\|_{L^{\frac{2d}{d-2}}} \lesssim 1, \tag{2.24}
\]
whereas the maximum principle shows that a non-trivial harmonic function does not belong to \( L^{2*}(\mathbb{R}^d) \). Thus, \( h \equiv 0 \) in (2.23) and therefore
\[
(-\Delta)^{-1}V\Lambda W = -\Lambda W. \tag{2.25}
\]
11. For \( 1 \leq q \leq \infty \), we use \( L^q_{rad}(\mathbb{R}^d) \) to denote the set of radially symmetric functions about \( 0 \in \mathbb{R}^d \).
12. For \( 1 \leq q \leq \infty \), we define a set \( X_q \) as
\[
X_q := \{ f \in L^q_{rad}(\mathbb{R}^d) : \langle f, V\Lambda W \rangle = 0 \}. \tag{2.26}
\]
We can verify that \( X_q \) is a closed subspace of \( L^q(\mathbb{R}^d) \). Hence, \( X_q \) is a Banach space.
13. We define a function \( \delta \) on \((0, \infty)\) by
\[
\delta(s) := \begin{cases} 
  s^\frac{1}{2} & \text{if } d = 3, \\
  \frac{1}{\log(1 + s^{-1})} & \text{if } d = 4.
\end{cases} \tag{2.27}
\]
A computation involving the spherical coordinates shows that for any \( s > 0 \),
\[
\int_{|\xi| \leq 1} \frac{1}{(|\xi|^2 + s)|\xi|^2} d\xi = \frac{d\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2} + 1)} \begin{cases} 
  s^{-\frac{d}{2}} \arctan(s^{-\frac{1}{2}}) & \text{if } d = 3, \\
  \frac{1}{2} \log(1 + s^{-1}) & \text{if } d = 4.
\end{cases} \tag{2.28}
\]
Note here that \( \frac{d\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2} + 1)} \) is the surface area of unit sphere in \( \mathbb{R}^d \). Furthermore, we see that for any \( s > 0 \),
\[
\left| \frac{\pi}{2} - \arctan(s^{-\frac{1}{2}}) \right| \leq \int_{s^{-\frac{1}{2}}}^{\infty} \frac{1}{1 + t^2} dt \leq \int_{s^{-\frac{1}{2}}}^{\infty} t^{-2} dt \leq s^\frac{1}{2}. \tag{2.29}
\]
Observe from (2.28) and (2.29) that for \( d = 3, 4 \) and any \( s > 0 \),
\[
\left| \int_{|\xi| \leq 1} \frac{1}{(|\xi|^2 + s)|\xi|^2} d\xi - (5 - d)\pi^2 \delta(s)^{-1} \right| \lesssim 1, \tag{2.30}
\]
where the implicit constant depends only on \( d \).
14. We define a function $\beta$ on $(0, \infty)$ by

$$\beta(s) := \delta(s)^{-1}s, \quad (2.31)$$

where $\delta$ is the function given in (2.27).

Note that $\beta$ is strictly increasing on $(0, \infty)$, so that the inverse exists.

15. We use $\alpha$ to denote the inverse function of $\beta$.

When $d = 3$, $\beta(s) = s^{\frac{1}{3}}$; Hence, the domain of $\alpha$ is $(0, \infty)$.

When $d = 4$, the image of $(0, \infty)$ by $\beta$ is $(0, 1)$ (see Lemma C.1); Hence, the domain of $\alpha$ is $(0, 1)$.

Note that for any $t$ in the domain of $\alpha$,

$$t = \beta(\alpha(t)) = \delta(\alpha(t))^{-1}\alpha(t); \quad \text{or equivalently,} \quad \delta(\alpha(t)) = t^{-1}\alpha(t). \quad (2.32)$$

Since $\beta$ is strictly increasing and $\lim_{s \to 0} \beta(s) = 0$, we see that $\alpha$ is strictly increasing and

$$\lim_{t \to 0} \alpha(t) = 0, \quad \lim_{t \to 0} \delta(\alpha(t)) = 0. \quad (2.33)$$

Furthermore, the following holds (see Lemma C.1):

$$\alpha(t) = \begin{cases} 
    t^2 & \text{for } d = 3 \text{ and } 0 < t < \infty, \\
    \sim \delta(t)t & \text{for } d = 4 \text{ and } 0 < t \leq T_0, 
\end{cases} \quad (2.34)$$

where $T_0 > 0$ is some constant.

16. Let $d \geq 3$ and $p > 1$. Furthermore, let $t \geq 0$, $s \geq 0$, and let $\eta$ be a function on $\mathbb{R}^d$.

Then, we define $N(\eta; t)$ and $F(\eta; s, t)$ as

$$N(\eta; t) := |W + \eta|^\frac{d-2}{p-2}(W + \eta) - W^\frac{d+2}{p-2} - \frac{d+2}{d-2}W^\frac{4}{p-2}\eta, \quad (2.35)$$

$$ incentives$$

$$F(\eta; s, t) := -sW + tW^p + N(\eta; t). \quad (2.36)$$

17. For $d \geq 3$ and functions $\eta_1, \eta_2$ on $\mathbb{R}^d$, we define $D(\eta_1, \eta_2)$ as

$$D(\eta_1, \eta_2) := |W + \eta_1|^\frac{4}{d-2}(W + \eta_1) - |W + \eta_2|^\frac{4}{d-2}(W + \eta_2) - \frac{d+2}{d-2}W^\frac{4}{p-2}(\eta_1 - \eta_2). \quad (2.37)$$

18. For $d \geq 3$, $p > 1$ and functions $\eta_1, \eta_2$ on $\mathbb{R}^d$, we define $E(\eta_1, \eta_2)$ as

$$E(\eta_1, \eta_2) := |W + \eta_1|^{p-1}(W + \eta_1) - |W + \eta_2|^{p-1}(W + \eta_2). \quad (2.38)$$

Next, we make some remarks about $N(\eta; t)$, $D(\eta_1, \eta_2)$ and $E(\eta_1, \eta_2)$ (see (2.35), (2.37) and (2.38)). Let $d \geq 3$, $p > 1$, $t_1, t_2 \geq 0$, and let $\eta_1, \eta_2$ be functions on $\mathbb{R}^d$. Then, the following hold:
1. \[ N(\eta_1; t_1) - N(\eta_2; t_2) = D(\eta_1, \eta_2) + t_2 E(\eta_1, \eta_2) + (t_1 - t_2) E(\eta_1, 0). \] (2.39)

Note that \( D(\eta, \eta) = 0 \) and \( E(\eta, \eta) = 0 \).

Observe from \( N(0; 0) = 0 \) that
\[ N(\eta_1; t_1) = D(\eta_1, 0) + t_1 E(\eta_1, 0). \] (2.40)

Furthermore, observe from the fundamental theorem of calculus that
\[ |D(\eta_1, \eta_2)| \lesssim (W + |\eta_1| + |\eta_2|)^{\frac{2}{d-2}} (|\eta_1| + |\eta_2|) |\eta_1 - \eta_2|, \] (2.41)
\[ |E(\eta_1, \eta_2)| \lesssim (W + |\eta_1| + |\eta_2|)^{p-1} |\eta_1 - \eta_2|, \] (2.42)

where the first implicit constant depends only on \( d \), and the second one does only on \( d \) and \( p \).

We record well-known results for the resolvent of the Laplace operator. The standard theory for Yukawa’s equation (see, e.g., Theorem 6.23 and Lemma 9.11 of [16]) shows the following:

1. For any \( d \geq 1, s > 0, 1 \leq q \leq \infty \) and \( f \in L^q(\mathbb{R}^d) \),
\[ (-\Delta + s)^{-1} f(x) = \int_{\mathbb{R}^d} \int_0^\infty (4\pi t)^{-\frac{d}{2}} e^{-\frac{|y|^2}{4t}} e^{-st} dt f(y) dy. \] (2.43)

Observe from (2.43) that if \( |f| \leq |g| \), then
\[ |(-\Delta + s)^{-1} f| \leq (-\Delta + s)^{-1} |g|. \] (2.44)

Using integration by substitution, we see that for each \( x \in \mathbb{R}^d \setminus \{0\} \),
\[ \int_0^\infty (4\pi t)^{-\frac{d}{2}} e^{-\frac{|y|^2}{4t}} e^{-st} dt \leq \int_0^\infty (4\pi t)^{-\frac{d}{2}} e^{-\frac{|y|^2}{4t}} dt \lesssim |x|^{-(d-2)}, \] (2.45)

where the implicit constant depends only on \( d \).

2. For any \( d \geq 1, s > 0, 1 \leq q \leq \infty \) and \( f \in L^q(\mathbb{R}^d) \),
\[ (-\Delta + s)^{-1} f \in L^q(\mathbb{R}^d), \] (2.46)
\[ (-\Delta + s)(-\Delta + s)^{-1} f = f \quad \text{in the distribution sense}. \] (2.47)

Observe from (2.46) and (2.47) that \((-\Delta + s)^{-1} f \in W^{2,q}(\mathbb{R}^d)\).

3. For any \( d \geq 1, s > 0, 1 \leq q \leq \infty \) and \( f \in W^{2,q}(\mathbb{R}^d) \),
\[ (-\Delta + s)^{-1}(-\Delta + s) f = f. \] (2.48)

4. For any \( s > 0 \) and \( \xi \in \mathbb{R}^d \),
\[ \mathcal{F} \left[ \int_0^\infty (4\pi t)^{-\frac{d}{2}} e^{-\frac{|y|^2}{4t}} e^{-st} dt \right] (\xi) = (|\xi|^2 + s)^{-1}. \] (2.49)
At the end of this section, we collect known results about ground states:

**Lemma 2.1** (Lemma 2.3 of [2]). Assume either \( d = 3 \) and \( 3 < p < 5 \), or else \( d \geq 4 \) and \( 1 < p < \frac{d+2}{d-2} \). Then, we have

\[
\lim_{\omega \to \infty} \inf_{\Phi_\omega \in G_\omega} \Phi_\omega(0) = \infty, \tag{2.50}
\]

\[
\lim_{\omega \to \infty} \sup_{\Phi_\omega \in G_\omega} \omega \Phi_\omega(0)^{-2} = \lim_{\omega \to \infty} \sup_{\Phi_\omega \in G_\omega} \Phi_\omega(0)^{-2(p+1)} = 0. \tag{2.51}
\]

**Lemma 2.2** (Proposition 2.1 and Corollary 3.1 of [2]). Assume either \( d = 3 \) and \( 3 < p < 5 \), or else \( d \geq 4 \) and \( 1 < p < \frac{d+2}{d-2} \). Then, the following hold:

\[
\lim_{\omega \to \infty} \sup_{\Phi_\omega \in G_\omega} \| T_{\Phi(0)}[\Phi_\omega] - W \|_{H^1} = 0, \tag{2.52}
\]

\[
\lim_{\omega \to \infty} \sup_{\Phi_\omega \in G_\omega} \| T_{\Phi(0)}[\Phi_\omega] - W \|_{L^r} = 0 \quad \text{for all} \quad \frac{d}{d-2} < r < \infty. \tag{2.53}
\]

**Lemma 2.3** (Proposition 3.1 of [2]). Assume either \( d = 3 \) and \( 3 < p < 5 \), or else \( d \geq 4 \). Let \( 1 < p < \frac{d+2}{d-2} \). Then, for any sufficiently large \( \omega \), and any \( x \in \mathbb{R}^d \),

\[
\sup_{\Phi_\omega \in G_\omega} T_{\Phi(0)}[\Phi_\omega](x) \lesssim (1 + |x|)^{-(d-2)}, \tag{2.54}
\]

where the implicit constant depends only on \( d \) and \( p \).

### 3 Uniform estimates for perturbed resolvents

Our aim in this section is to prove the following proposition which plays a vital role for the proof of Theorem 1.1:

**Proposition 3.1.** Assume \( d = 3, 4 \), and let \( \frac{d}{d-2} < q < \infty \). If \( s > 0 \) is sufficiently small dependently on \( d \) and \( q \), then the inverse of the operator \( 1 + (-\Delta + s)^{-1} V : L^{q}_{\text{rad}}(\mathbb{R}^d) \to L^{q}_{\text{rad}}(\mathbb{R}^d) \) exists; and the following estimates hold:

1. If \( f \in L^{q}_{\text{rad}}(\mathbb{R}^d) \), then

\[
\| (1 + (-\Delta + s)^{-1} V)^{-1} f \|_{L^q} \lesssim \delta(s) s^{-1} \| f \|_{L^q}, \tag{3.1}
\]

where the implicit constant depends only on \( d \) and \( q \).

2. If \( f \in X_q \), then

\[
\| (1 + (-\Delta + s)^{-1} V)^{-1} f \|_{L^q} \lesssim \| f \|_{L^q}, \tag{3.2}
\]

where the implicit constant depends only on \( d \) and \( q \).

**Remark 3.1.**

1. When \( d = 3 \), the case \( q = \infty \) can be included in Proposition 3.1 (see Lemma 2.4 of [2]). On the other hand, from the point of view of Remark 3.5 below, when \( d = 4 \), it seems difficult to include the case \( q = \infty \).

2. By the substitution of variables \( \tau = \beta(s) := \delta(s)^{-1} s \), we may write (3.1) as

\[
\| (1 + (-\Delta + \alpha(\tau))^{-1} V)^{-1} f \|_{L^q} \lesssim \tau^{-1} \| f \|_{L^q}. \tag{3.3}
\]
3. Observe from $V = -\frac{d+2}{d-2}W^\frac{1}{d-2}$, $W^{\frac{d+2}{d-2}} = (-\Delta)W$ and $\Delta W = V AW$ that

$$\langle W, V AW \rangle = \frac{d+2}{d-2} \langle W, V W \rangle = \frac{d+2}{d-2} \langle W, V W \rangle = \frac{d+2}{d-2} \langle W, V W \rangle,$$

which implies $\langle W, V AW \rangle = 0$. Thus, we see that $W \in X_q$ for all $\frac{d}{d-2} < q \leq \infty$.

We will prove Proposition 3.1 in Section 3.2.

3.1 Preliminaries

In this subsection, we give basic estimates for free and perturbed resolvents.

Let us begin by recalling the following fact (see, e.g., Section 5.2 of [10], and Proposition 2.2 of [11]):

Lemma 3.1. Assume $d \geq 3$. Then, there exists $e_0 > 0$ such that $-e_0$ is the only one negative eigenvalue of the operator $-\Delta + V : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$. Furthermore, the essential spectrum of $-\Delta + V$ equals $[0, \infty)$ (hence $\sigma(-\Delta + V) = \{e_0\} \cup [0, \infty)$), and

$$\{u \in \dot{H}^1(\mathbb{R}^d) : (-\Delta + V)u = 0 \text{ in the distribution sense} \} = \text{span} \{AW, \partial_1 W, \ldots, \partial_d W \},$$

where $\partial_1 W, \ldots, \partial_d W$ denote the partial derivatives of $W$.

Lemma 3.2. Assume $d = 3, 4$. Let $\frac{d}{d-2} < r \leq \infty$. Then,

$$\{u \in L^r_{\text{rad}}(\mathbb{R}^d) : (-\Delta + V)u = 0 \text{ in the distribution sense} \} = \text{span} \{AW\}. \quad (3.6)$$

Proof of Lemma 3.2. Let $u \in L^r(\mathbb{R}^d)$ be a function such that $\Delta u = Vu$ in the distribution sense. Notice that $(1 + |x|)^{-2} Vu \in L^1(\mathbb{R}^d)$. Hence, the formula (2.21) is available. Furthermore, we see from (2.21), (2.19), the Hardy-Littlewood-Sobolev inequality and Hölder’s inequality that

$$\|\nabla u\|_{L^2} = \|(-\Delta)^{-1} Vu\|_{L^2} \leq \|\int_{\mathbb{R}^d} \frac{x - y}{|x - y|^d} V(y) u(y) dy\|_{L^2} \leq \|\int_{\mathbb{R}^d} |x - y|^{-(d-1)} V(y)|u(y)| dy\|_{L^2} \quad (3.7)$$

$$\leq \|Vu\|_{L^q} \lesssim \|u\|_{L^r},$$

so that $u \in \dot{H}^1(\mathbb{R}^d)$. Thus, we find from Lemma 3.1 that the claim (3.6) is true.

We record well-known inequalities for radial functions (see [11]):

Lemma 3.3. Assume $d \geq 3$, and let $g$ be a radial function on $\mathbb{R}^d$. Then, we have

$$|x|^{\frac{d-2}{2}} |g(x)| \lesssim \|\nabla g\|_{L^2}, \quad (3.8)$$

$$|x|^{\frac{d-1}{2}} |g(x)| \lesssim \|g\|_{L^2} \|\nabla g\|_{L^2}^{\frac{1}{2}}. \quad (3.9)$$

Remark 3.2. We see from Lemma 3.3 that for $d = 3, 4$,

$$|x||g(x)| \lesssim \|g\|_{L^2}^{\frac{4-d}{2}} \|\nabla g\|_{L^2}^{\frac{d-2}{2}}. \quad (3.10)$$
Lemma 3.4. Assume \( d \geq 1 \).

1. Let \( 1 \leq q_1 \leq q_2 \leq \infty \) and \( d\left(\frac{1}{q_1} - \frac{1}{q_2}\right) < 2 \). Then, the following holds for all \( s > 0 \):

\[
\|(-\Delta + s)^{-1}\|_{L^{q_1}(\mathbb{R}^d) \to L^{q_2}(\mathbb{R}^d)} \lesssim s^{\frac{d}{2}\left(\frac{1}{q_1} - \frac{1}{q_2}\right)-1},
\]

(3.11)

where the implicit constant depends only on \( d, q_1 \) and \( q_2 \).

2. Let \( 1 < q_1 \leq q_2 < \infty \) and \( d\left(\frac{1}{q_1} - \frac{1}{q_2}\right) < 2 \). Then, the following holds for all \( s > 0 \):

\[
\|(-\Delta + s)^{-1}\|_{L^{q_1}_{\text{weak}}(\mathbb{R}^d) \to L^{q_2}(\mathbb{R}^d)} \lesssim s^{\frac{d}{2}\left(\frac{1}{q_1} - \frac{1}{q_2}\right)-1},
\]

(3.12)

where the implicit constant depends only on \( d, q_1 \) and \( q_2 \).

Proof of Lemma 3.4. First, we prove (3.11). Let \( 1 \leq q_1 \leq q_2 \leq \infty \) and \( f \in L^{q_1}(\mathbb{R}^d) \). Then, it follows from Young’s inequality and elementary computations that

\[
\|(-\Delta + s)^{-1}f\|_{L^{q_2}} \lesssim \int_0^\infty (4\pi t)^{-\frac{d}{2}} e^{-\left|\frac{|x|^2}{4t}\right|} e^{-st} dt \|f\|_{L^{q_1}}
\]

(3.13)

\[
\lesssim \int_0^\infty t^{-\frac{d}{2} + \frac{d}{2}\left(\frac{1}{q_1} - \frac{1}{q_2}\right)} e^{-st} dt \|f\|_{L^{q_1}}
\]

\[
= s^{\frac{d}{2}\left(\frac{1}{q_1} - \frac{1}{q_2}\right)-1} \int_0^\infty r^{-\frac{d}{2}\left(\frac{1}{q_1} - \frac{1}{q_2}\right)} e^{-r} dr \|f\|_{L^{q_1}},
\]

where the implicit constants depend only on \( d, q_1 \) and \( q_2 \). The above right-hand side is finite if and only if \( \frac{d}{2}\left(\frac{1}{q_1} - \frac{1}{q_2}\right) < 1 \). Hence, we see that (3.11) holds. Similarly, by the weak Young inequality (see, e.g., (9) in Section 4.3 of [16]), we see that if \( 1 < q_1 < q_2 < \infty \) and \( \frac{d}{2}\left(\frac{1}{q_1} - \frac{1}{q_2}\right) < 1 \), then (3.12) holds.

Note that Lemma 3.4 does not deal with the case \( d\left(\frac{1}{q_1} - \frac{1}{q_2}\right) = 2 \). The following lemma compensates for Lemma 3.4.

Lemma 3.5. Assume \( d \geq 3 \), and let \( s_0 \geq 0 \). If \( \frac{d}{d+2q} < q < \infty \), then

\[
\|(-\Delta + s_0)^{-1}\|_{L^{\frac{dq}{d+2q}}(\mathbb{R}^d) \to L^{q}(\mathbb{R}^d)} \lesssim 1,
\]

(3.14)

where the implicit constant depends only on \( d \) and \( q \).

Remark 3.3. Put \( r := \frac{dq}{d+2q} \). Then, \( q = \frac{dr}{d-2r} \); and the condition \( \frac{d}{d-2r} < q < \infty \) is equivalent to \( 1 < r < \frac{d}{2} \).

Proof of Lemma 3.5. When \( s_0 = 0 \), the claim is identical to (3.14). Assume \( s_0 > 0 \). Then, by (2.21) and Hardy-Littlewood-Sobolev’s inequality (see (2.20)), we see that

\[
\|(-\Delta + s_0)^{-1}f\|_{L^{q}} \lesssim \|\|x|^{-(d-2)} * |f||\|_{L^{\frac{dq}{d+2q}}},
\]

(3.15)

where the implicit constants depend only on \( d \) and \( q \). Thus, we have proved the lemma.
Lemma 3.6. Assume \( d \geq 3 \). Let \( s_0 \geq 0 \) and \( 0 < \varepsilon \leq d - 2 \). Then, the following inequality holds:

\[
\|(-\Delta + s_0)^{-1}\|_{L^{\frac{d+\varepsilon}{d}} \to L^{\infty}} \lesssim 1, \tag{3.16}
\]

where the implicit constant depends only on \( d \) and \( \varepsilon \).

**Proof of Lemma 3.6.** Let \( f \in L^{\frac{d+\varepsilon}{d}}(\mathbb{R}^d) \cap L^{\frac{d+\varepsilon}{d}}(\mathbb{R}^d) \). Then, by (2.45) and Hölder’s inequality, we see that

\[
\|(-\Delta + s_0)^{-1}f\|_{L^{\infty}} \lesssim \|x|^{-(d-2)}f\|_{L^{\infty}}
\]

\[
\leq \sup_{x \in \mathbb{R}^d} \int_{|y| \leq 1} |y|^{-(d-2)}|f(x-y)| \, dy + \sup_{x \in \mathbb{R}^d} \int_{1 \leq |y|} |y|^{-(d-2)}|f(x-y)| \, dy \tag{3.17}
\]

\[
\lesssim \|f\|_{L^{\frac{d+\varepsilon}{d}}} + \|f\|_{L^{\frac{d+\varepsilon}{d}}} \lesssim \|f\|_{L^{\frac{d+\varepsilon}{d}}},
\]

where the implicit constant depends only on \( d \) and \( \varepsilon \). Thus, the claim is true. \( \square \)

Let us recall the resolvent equation:

\[
(-\Delta + s)^{-1} - (-\Delta + s_0)^{-1} = -(s-s_0)(-\Delta + s)^{-1}(-\Delta + s_0)^{-1}. \tag{3.18}
\]

This identity holds in the following sense:

**Lemma 3.7.** Assume \( d \geq 3 \). Let \( s > 0 \) and \( s_0 \geq 0 \).

1. Let \( \frac{d}{d-2} < q < \infty \). Then, the resolvent identity (3.18) holds as an operator from \( L^{\frac{d+\varepsilon}{d}}(\mathbb{R}^d) \) to \( L^q(\mathbb{R}^d) \).

2. Let \( 0 < \varepsilon \leq d - 2 \). Then, the resolvent identity (3.18) holds as an operator from \( L^{\frac{d+\varepsilon}{d}}(\mathbb{R}^d) \cap L^{\frac{d+\varepsilon}{d}}(\mathbb{R}^d) \) to \( L^\infty(\mathbb{R}^d) \).

**Remark 3.4.** Let \( \frac{d}{d-2} < q \leq \infty \). Then, Lemma 3.7 together with \(|V(x)| \lesssim (1 + |x|)^{-4}\) shows that the following identity holds as an operator from \( L^q(\mathbb{R}^d) \) to \( L^q(\mathbb{R}^d) \):

\[
1 + (-\Delta + s)^{-1}V = 1 + (-\Delta)^{-1}V - s(-\Delta + s)^{-1}(-\Delta)^{-1}V. \tag{3.19}
\]

**Proof of Lemma 3.7.** By (2.21), (2.47) and (2.48), we see that

\[
(-\Delta + s)^{-1} - (-\Delta + s_0)^{-1}
\]

\[
= \{(-\Delta + s)^{-1}(-\Delta + s_0) - 1\}(-\Delta + s_0)^{-1} \tag{3.20}
\]

\[
= \{(-\Delta + s)^{-1}(-\Delta + s) - (s-s_0)(-\Delta + s)^{-1} - 1\}(-\Delta + s_0)^{-1}
\]

\[
= -(s-s_0)(-\Delta + s)^{-1}(-\Delta + s_0)^{-1}.
\]

Note that each identity in (3.20) can be justified by Lemma 3.4, Lemma 3.5 and Lemma 3.6. Thus, the claims of the lemma are true. \( \square \)

**Lemma 3.8.** Assume \( 3 \leq d \leq 7 \). Let \( 2 \leq q \leq \infty \) and \( s > 0 \). Then, the following inequality holds:

\[
\|(-\Delta)(-\Delta + s)^{-1}V\|_{L^q \to L^q} \lesssim 1 + |1-s|s^{-1}, \tag{3.21}
\]

where the implicit constant depends only on \( d \) and \( q \). In particular, if \( f \in L^q(\mathbb{R}^d) \), then

\[
(-\Delta + s)^{-1}Vf \in H^2(\mathbb{R}^d). \tag{3.22}
\]
Proof of Lemma 3.8. Let \( f \in L^q(\mathbb{R}^d) \). Observe from \(|V(x)| \lesssim (1 + |x|)^{-4}\) (see (2.10)) that if \(3 \leq d \leq 7\), then
\[
\|Vf\|_{L^2 \cap L^q} \lesssim \|f\|_{L^q}, \tag{3.23}
\]
where the implicit constant depends only on \(d\) and \(q\). Then, by (2.47), Lemma 3.1 and (3.23), we see that
\[
\|(1 - \Delta)(-\Delta + s)^{-1}Vf\|_{L^2 \cap L^q} = \|\{(\Delta + s) + (1 - s)\}(\Delta + s)^{-1}Vf\|_{L^2 \cap L^q}
\leq \|Vf\|_{L^2 \cap L^q} + |1 - s|\|\Delta + s\|^{-1}Vf\|_{L^2 \cap L^q}
\lesssim \|Vf\|_{L^2 \cap L^q} + |1 - s|s^{-1}\|Vf\|_{L^2 \cap L^q} \lesssim (1 + |1 - s|s^{-1})\|f\|_{L^q},
\tag{3.24}
\]
where the implicit constants depend only on \(d\) and \(q\). Thus, we have obtained the desired estimate.

Lemma 3.9. Assume \(3 \leq d \leq 5\), and let \(\frac{d}{d-2} < q \leq \infty\). Then, \((-\Delta)^{-1}V\) is a compact operator from \(X_q\) to itself. In particular, \((-\Delta)^{-1}V : X_q \to X_q\) is bounded.

Proof of Lemma 3.9. We prove the lemma in two steps.

**Step 1.** We shall show that for any \(f \in L^q(\mathbb{R}^d)\),
\[
\|(-\Delta)^{-1}Vf\|_{L^q} \lesssim \|f\|_{L^q}, \tag{3.25}
\]
\[
\|\nabla(-\Delta)^{-1}Vf\|_{L^q} \lesssim \|f\|_{L^q}, \tag{3.26}
\]
where the implicit constants depend only on \(d\) and \(q\).

Assume \(\frac{d}{d-2} < q < \infty\), and let \(f \in L^q(\mathbb{R}^d)\). Then, applying (2.20) as \(r = \frac{dq}{2q + d}\), we see that that
\[
\|(-\Delta)^{-1}Vf\|_{L^q} \lesssim \|Vf\|_{L^{\frac{dq}{2q + d}}} \lesssim \|f\|_{L^q}, \tag{3.27}
\]
where the implicit constants depend only on \(d\) and \(q\). Furthermore, by (2.19) and the Hardy-Littlewood-Sobolev inequality, we see that
\[
\|\nabla(-\Delta)^{-1}Vf\|_{L^q} \lesssim \int_{\mathbb{R}^d} |x - y|^{-(d-1)}V(y)f(y)\,dy\|_{L^q} \lesssim \|Vf\|_{L^{\frac{dq}{2q + d}}} \lesssim \|f\|_{L^q}, \tag{3.28}
\]
where the implicit constants depend only on \(d\) and \(q\).

Next, assume \(q = \infty\), and let \(f \in L^\infty(\mathbb{R}^d)\). Then, Lemma 3.6 shows that
\[
\|(-\Delta)^{-1}Vf\|_{L^\infty} \lesssim \|Vf\|_{L^\infty} \lesssim \|Vf\|_{L^\frac{d}{d-2}} \lesssim \|f\|_{L^\frac{d}{d-2}} \lesssim \|f\|_{L^\infty}, \tag{3.29}
\]
where the implicit constants depend only on \(d\). Furthermore, by (2.19) and Hölder’s inequality, we see that
\[
\|\nabla(-\Delta)^{-1}Vf\|_{L^\infty} \lesssim \int_{\mathbb{R}^d} |y|^{-(d-1)}V(x - y)\,dy\|_{L^\infty} \lesssim \|f\|_{L^\infty}
\lesssim \left\{ \int_{|y| \leq 1} |y|^{-(d-1)}\,dy + \left( \int_{1 \leq |y|} |y|^{\frac{(d-1)}{d-2}}\,dy \right)^\frac{d-2}{d} \|Vf\|_{L^\frac{d}{2}} \right\} \|f\|_{L^\infty}, \tag{3.30}
\]
where the implicit constants depend only on \(d\). Thus, we have proved (3.25) and (3.26).

**Step 2.** We shall finish the proof of the lemma.
We will use Lemma A.1. Let \( \{f_n\} \) be a bounded sequence in \( X_q \).

Observe from (3.25) that
\[
\sup_{n \geq 1} \|(-\Delta)^{-1} V f_n\|_{L^q} \leq \sup_{n \geq 1} \|f_n\|_{L^q} < \infty. \tag{3.31}
\]

Furthermore, observe from the fundamental theorem of calculus and (3.26) that
\[
\limsup_{y \to 0, n \geq 1} \|(-\Delta)^{-1} V f_n - (-\Delta)^{-1} V f_n(\cdot + y)\|_{L^q}
= \limsup_{y \to 0, n \geq 1} \left\| \int_0^1 \frac{d}{d\theta} (-\Delta)^{-1} V f_n(\cdot + \theta y) \, d\theta \right\|_{L^q}
= \limsup_{y \to 0, n \geq 1} \left\| \int_0^1 y \cdot \nabla (-\Delta)^{-1} V f_n(\cdot + \theta y) \, d\theta \right\|_{L^q}
\leq \limsup_{y \to 0, n \geq 1} \|y\| \|\nabla (-\Delta)^{-1} V f_n\|_{L^q} \lesssim \lim_{y \to 0} \|y\| \sup_{n \geq 1} \|f_n\|_{L^q} = 0.
\tag{3.32}
\]

Note that \((-\Delta)^{-1} V f_n\) is radial, as so is \(f_n\). Let \(\varepsilon_0 := \min\left\{\frac{1}{2} (q - \frac{d}{q-2}), 1\right\}\). When \(q \neq \infty\), Lemma 3.3 shows that
\[
\limsup_{R \to \infty, n \geq 1} \|(-\Delta)^{-1} V f_n(x)\|_{L^q(|x| \geq R)}
\leq \limsup_{R \to \infty, n \geq 1} R^{-\frac{d-2}{\frac{d}{q}+\varepsilon_0}} \|x\|^{-\frac{2d}{\frac{d}{q}+\varepsilon_0}} (-\Delta)^{-1} V f_n\|_{L^\infty}
\lesssim \limsup_{R \to \infty, n \geq 1} R^{-\frac{(d-2)\varepsilon_0}{\frac{d}{q}+\varepsilon_0}} \|\nabla (-\Delta)^{-1} V f_n\|_{L^2} (-\Delta)^{-1} V f_n\|_{L^{\frac{d}{q}+\varepsilon_0}}, \tag{3.33}
\]
where the implicit constant depends only on \(d\) and \(q\). Similarly, when \(q = \infty\), we see that
\[
\limsup_{R \to \infty, n \geq 1} \|(-\Delta)^{-1} V f_n(x)\|_{L^\infty(|x| \geq R)}
\leq \limsup_{R \to \infty, n \geq 1} R^{-\frac{d-2}{\frac{d}{q}} \|x\|^{\frac{2d}{\frac{d}{q}}} (-\Delta)^{-1} V f_n\|_{L^\infty}
\lesssim \limsup_{R \to \infty, n \geq 1} R^{-\frac{d-2}{\frac{d}{q}}} \|\nabla (-\Delta)^{-1} V f_n\|_{L^2}, \tag{3.34}
\]
where the implicit constant depends only on \(d\).

Observe from 3 \(d \leq 5\) and \(\frac{d}{d-2} < q \leq \infty\) that \(\frac{2d}{d+2} < q\) and \(\frac{2dq}{d+2q-2d} > \frac{d}{4}\) \((\frac{2d}{d+2} > \frac{d}{4}\) if \(q = \infty\). Hence, by (2.19), the Hardy-Littlewood-Sobolev inequality, Hölder’s one, and \(V(x) \sim (1 + |x|)^{-4}\), we see that
\[
\|\nabla (-\Delta)^{-1} V f_n\|_{L^2} \lesssim \| |x|^{-(d-1)} * V f_n\|_{L^2} \lesssim \|V f_n\|_{L^{\frac{2d}{d-2}}}
\leq \|V\|_{L^{\frac{2dq}{d+2q-2d}}} \|f_n\|_{L^q} \lesssim \|f_n\|_{L^q}, \tag{3.35}
\]
where the implicit constants depend only on \(d\) and \(q\). Similarly, if \(q \neq \infty\), then
\[
\|(-\Delta)^{-1} V f_n\|_{L^{\frac{d}{q}+\varepsilon_0}} \lesssim \|V f_n\|_{L^{\frac{d}{q}+\varepsilon_0}} \leq \|V\|_{L^{\frac{2dq}{d+2q}+\varepsilon_0}} \|f_n\|_{L^q} \lesssim \|f_n\|_{L^q}, \tag{3.36}
\]

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Lemma 3.2 together with (3.40) implies that we can verify that for any $\phi$ it follows from

$$\langle -\Delta \rangle^{-1} V f_n(x) \rangle_{L^q(|x| \geq R)} = 0. \quad (3.37)$$

Thus, it follows from Lemma A.1 that $\{(-\Delta)^{-1} V f_n\}$ contains a convergent subsequence in $L^q(\mathbb{R}^d)$. We denote the convergent subsequence by the same symbol as the original one, and the limit by $g$. Note that $g \in L^q_{\text{rad}}(\mathbb{R}^d)$.

It remains to prove that the limit $g$ satisfies the orthogonality condition $\langle g, VAW \rangle = 0$. Since $V\Lambda W = \Delta \Lambda W$ and $\langle f_n, V\Lambda W \rangle = 0$, we can verify that for any $n \geq 1$,

$$\langle (-\Delta)^{-1} V f_n, V\Lambda W \rangle = \langle (-\Delta)^{-1} V f_n, -(-\Delta)\Lambda W \rangle = -\langle V f_n, \Lambda W \rangle = 0, \quad (3.38)$$

which together with $V\Lambda W \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ yields

$$\langle g, V\Lambda W \rangle = \lim_{n \to \infty} \langle (-\Delta)^{-1} V f_n, V\Lambda W \rangle = 0. \quad (3.39)$$

Thus, we have completed the proof. \hfill \Box

**Lemma 3.10.** Assume $3 \leq d \leq 5$ and $\frac{d}{d-2} < q < \infty$. Then, the inverse of $1 + (-\Delta)^{-1} V$ exists as a bounded operator from $X_q$ to itself.

**Proof of Lemma 3.10.** We see from Lemma 3.9 that $(-\Delta)^{-1} V$ is a compact operator from $X_q$ to itself. Furthermore, by a standard theory of compact operators, if $z \in \mathbb{C}$ is neither zero nor an eigenvalue of $(-\Delta)^{-1} V$, then $z$ belongs to the resolvent. In particular, if $-1$ is not an eigenvalue of $(-\Delta)^{-1} V$, then the inverse of $1 + (-\Delta)^{-1} V$ exists as a bounded operator from $X_q$ to itself. Hence, what we need to prove is that $-1$ is not an eigenvalue. Suppose for contradiction that there exists $f \in X_q$ such that $(-\Delta)^{-1} V f = -f$. Then, we can verify that for any $\phi \in C_c^\infty(\mathbb{R}^d)$,

$$\langle (-\Delta + V) f, \phi \rangle = \langle (-\Delta) \{f + (-\Delta)^{-1} V f\}, \phi \rangle = \langle f + (-\Delta)^{-1} V f, -\Delta \phi \rangle = 0. \quad (3.40)$$

Lemma 3.2 together with (3.40) implies that $f = \kappa \Lambda W$ for some $\kappa \in \mathbb{R}$. Furthermore, it follows from $\langle f, V\Lambda W \rangle = 0$ and $\langle \Lambda W, V\Lambda W \rangle \neq 0$ that $\kappa = 0$. Thus, $f$ is trivial and therefore $-1$ is not an eigenvalue of $(-\Delta)^{-1} V$. \hfill \Box

**Lemma 3.11.** Assume $d = 3, 4$. Then, the following holds for all $\frac{d}{d-2} < q < \infty$, $f \in L^q_{\text{rad}}(\mathbb{R}^d)$ and $s > 0$:

$$|\langle (-\Delta + s)^{-1} V f, |x|^{-(d-2)} \rangle - \frac{(5 - d)}{2d-2 \Gamma(d-2)} \mathcal{F}[V f](0) \delta(s)^{-1} | \lesssim \|f\|_{L^q}, \quad (3.41)$$

where the implicit constant depends only on $d$ and $q$.

**Remark 3.5.** When $d = 4$, it is essential that $q < \infty$ in Lemma 3.11. More precisely, since $V \not\in L^1(\mathbb{R}^4)$, $\mathcal{F}[V f](0)$ makes no sense for $f \in L^\infty_{\text{rad}}(\mathbb{R}^d)$. 

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Proof of Lemma 3.11: Let \( \frac{d}{2} < q < \infty \), \( f \in L^q(d\mathbb{R}^d) \) and \( s > 0 \). Then, by (2.21), (2.35), (2.3), (2.41) and (2.49), we see that

\[
\langle (-\Delta + s)^{-1}Vf, |x|^{-(d-2)} \rangle = \frac{1}{(2\pi)^d} \frac{4\pi^d}{\Gamma((\frac{d}{2} - 2))} \langle (|\xi|^2 + s)^{-1} \mathcal{F}[Vf], |\xi|^{-2} \rangle
\]

\[
= C_d \mathcal{R} \int_{|\xi| \leq 1} \frac{\mathcal{F}[Vf](0)}{(|\xi|^2 + s)|\xi|^2} d\xi + C_d \mathcal{R} \int_{|\xi| \leq 1} \frac{\mathcal{F}[Vf](\xi) - \mathcal{F}[Vf](0)}{(|\xi|^2 + s)|\xi|^2} d\xi
\]

\[
+ C_d \mathcal{R} \int_{1 \leq |\xi|} \frac{\mathcal{F}[Vf](\xi)}{(|\xi|^2 + s)|\xi|^2} d\xi,
\]

where \( C_d := \{2^{d-2} \pi^d \Gamma((\frac{d}{2} - 2))\}^{-1} \).

Consider the first term on the right-hand side of (3.42). By (2.31) and \( |V(x)| \lesssim (1 + |x|)^{-d} \) (see (2.10)), we see that

\[
\left| C_d \mathcal{R} \int_{|\xi| \leq 1} \frac{\mathcal{F}[Vf](0)}{(|\xi|^2 + s)|\xi|^2} d\xi \right| \leq C_d (5 - d)\pi^d \mathcal{R} \mathcal{F}[f](0) \delta(s)^{-1}
\]

\[
\lesssim |\mathcal{F}[Vf](0)| \leq \|Vf\|_{L^1} \lesssim \|f\|_{L^q},
\]

where the implicit constants depend only on \( d \) and \( q \). Next, consider the second term on the right-hand side of (3.42). Observe that

\[
\int_{|\xi| \leq 1} \frac{\xi}{(|\xi|^2 + s)|\xi|^2} d\xi = 0.
\]

Furthermore, observe that for any \( x \in \mathbb{R}^d \) and \( \xi \in \mathbb{R}^d \),

\[
|e^{-ix \cdot \xi} - 1 + ix \cdot \xi| \leq 2 \min\{||\xi|, |x|^2|\xi|^2\}.
\]

Fix a number \( \varepsilon_0 \) such that \( 4 - d < \varepsilon_0 < 4 - \frac{d(\alpha - 1)}{q} \); The dependence of a constant on \( \varepsilon_0 \) is absorbed into than on \( d \) and \( q \). Observe from \( d = 3, 4 \) and \( \frac{d}{2} < q < \infty \) that \( 0 < \varepsilon_0 < 2 \). Then, by (3.41), \( |V(x)| \lesssim (1 + |x|)^{-d} \), Hölder’s inequality, and \( \frac{d(4 - \varepsilon_0)}{q - 1} > d \), we see that

\[
|\mathcal{F}[Vf](\xi) - \mathcal{F}[Vf](0) - \xi \cdot \nabla \mathcal{F}[Vf](0)|
\]

\[
= \left| \int_{\mathbb{R}^d} (e^{-ix \cdot \xi} - 1 + ix \cdot \xi) V(x)f(x) dx \right|
\]

\[
\lesssim |\xi|^\varepsilon_0 \int_{\mathbb{R}^d} |x|^\varepsilon_0 (1 + |x|)^{-4} |f(x)| dx
\]

\[
\lesssim |\xi|^\varepsilon_0 \|1 + |x|\|^{-4 - \varepsilon_0} \|f\|_{L^q} \lesssim |\xi|^\varepsilon_0 \|f\|_{L^q},
\]

where the implicit constants depend only on \( d \) and \( q \). Hence, by (3.44), (3.46) and \( d - \varepsilon_0 < d \), we see that

\[
\left| \int_{|\xi| \leq 1} \frac{\mathcal{F}[Vf](\xi) - \mathcal{F}[Vf](0)}{(|\xi|^2 + s)|\xi|^2} d\xi \right|
\]

\[
= \left| \int_{|\xi| \leq 1} \frac{\mathcal{F}[Vf](\xi) - \mathcal{F}[Vf](0) - \xi \cdot \nabla \mathcal{F}[Vf](0)}{(|\xi|^2 + s)|\xi|^2} d\xi \right|
\]

\[
\lesssim \|f\|_{L^q} \int_{|\xi| \leq 1} \frac{|\xi|^\varepsilon_0}{(|\xi|^2 + s)|\xi|^2} d\xi \leq \|f\|_{L^q} \int_{|\xi| \leq 1} |\xi|^{-(4 - \varepsilon_0)} d\xi \lesssim \|f\|_{L^q},
\]

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where the implicit constants depend only on \(d\) and \(q\).

Consider the last term on the right-hand side of (3.42). By (3.10), Plancherel’s theorem, \(|V(x)| \lesssim (1 + |x|)^{-d}\) and \(q > 2\), we see that

\[
\|\xi|\mathcal{F}[V f]|\|_{L^\infty} \lesssim \|\mathcal{F}[V f]\|_{L^2}^{\frac{4-d}{2}} \|\nabla \mathcal{F}[V f]\|_{L^2}^{\frac{4-d}{2}} = \|\mathcal{F}[V f]\|_{L^2}^{\frac{4-d}{2}} \|\mathcal{F}[x V f]\|_{L^2}^{\frac{4-d}{2}}
\]  
(3.48)

where the implicit constants depend only on \(d\) and \(q\). Using (3.48), we see that

\[
\int_{1 \leq |\xi| \leq \langle \xi \rangle} \frac{\mathcal{F}[V f](\xi)}{|\langle \xi \rangle^2 + s|^2} |\xi|^2 d\xi \lesssim \|f\|_{L^q} \int_{1 \leq |\xi|} \frac{1}{|\xi|^2} |\xi|^2 d\xi \lesssim \|f\|_{L^q},
\]  
(3.49)

where the implicit constants depend only on \(d\) and \(q\).

Now, combining (3.42), (3.43), (3.47) and (3.49), we find that (3.41) holds.

\[\square\]

**Lemma 3.12.** Assume \(d = 3, 4\). Then, the following holds for all \(s > 0\):

\[
\|(-\Delta + s)^{-1} \Lambda W\|_{L^\infty} \lesssim 1 + \delta(s)^{-1},
\]  
(3.50)

where the implicit constant depends only on \(d\).

**Proof of Lemma 3.12.** Observe from (2.49) and \(-\Lambda W = (-\Delta)^{-1} V AW\) (see (2.25)) that

\[
\|(-\Delta + s)^{-1} \Lambda W\|_{L^\infty} \sim \|e^{ix\cdot \xi} \mathcal{F}[V AW](\xi) \|_{L^\infty} \|\frac{1}{(\langle \xi \rangle^2 + s)}\|_{L^\infty}^{\frac{1}{\delta}}
\]  
(3.51)

where the implicit constant depends only on \(d\).

Consider the first term on the right-hand side of (3.51). By (2.30), we see that

\[
\int_{|\xi| \leq 1} \frac{\mathcal{F}[V AW](\xi)}{|(\langle \xi \rangle^2 + s)|^2} |\xi|^2 d\xi \lesssim \{\delta(s)^{-1} + 1\} \|V AW\|_{L^2} \lesssim \delta(s)^{-1} + 1,
\]  
(3.52)

where the implicit constants depend only on \(d\). Move on to the second term on the right-hand side of (3.51). By (3.10) and Plancherel’s theorem, we see that

\[
\|\xi|\mathcal{F}[V AW]\|_{L^\infty} \lesssim \|\mathcal{F}[V AW]\|_{L^2}^{\frac{4-d}{4}} \|\nabla \mathcal{F}[V AW]\|_{L^2}^{\frac{4-d}{4}}
\]  
(3.53)

where the implicit constants depend only on \(d\). Hence, we find from (3.53) and \(d = 3, 4\) that

\[
\int_{1 \leq |\xi|} \frac{|\mathcal{F}[V AW](\xi)|}{(\langle \xi \rangle^2 + s)|^2} |\xi|^2 d\xi \lesssim \int_{1 \leq |\xi|} \frac{1}{|\xi|^2} d\xi \lesssim 1,
\]  
(3.54)

where the implicit constants depend only on \(d\). Plugging (3.52) and (3.54) into (3.51), we obtain the desired estimate (3.50). \[\square\]
Lemma 3.13. Assume $d = 3, 4$. Then, there exists a constant $C > 0$ depending only on $d$ such that the following holds for all $\frac{d}{\pi^2} < q < \infty$, $g \in L^q_{rad}(\mathbb{R}^d)$ and $s > 0$:

$$
\left| \langle (\Delta + s)^{-1} Vg, \Lambda W \rangle + \mathcal{RF}[Vg](0)\delta(s)^{-1} \right| \lesssim \|g\|_{L^q},
$$

where the implicit constant depends only on $d$ and $q$.

Proof of Lemma 3.13. Decompose $\Lambda W$ as

$$
\Lambda W(x) = -\frac{|x|^2}{2d} \left( \frac{|x|^2}{d(d-2)} \right)^{\frac{d}{2}} + \frac{d-2}{2} \left( 1 + \frac{|x|^2}{d(d-2)} \right)^{-\frac{d}{2}} + Z(x),
$$

where

$$
Z(x) := -\frac{|x|^2}{2d} \left( 1 + \frac{|x|^2}{d(d-2)} \right)^{-\frac{d}{2}} + \frac{|x|^2}{2d} \left( \frac{|x|^2}{d(d-2)} \right)^{-\frac{d}{2}}.
$$

Let $\frac{d}{\pi^2} < q < \infty$, $g \in L^q_{rad}(\mathbb{R}^d)$ and $s > 0$. Then, it follows from (3.58) that

$$
\langle (\Delta + s)^{-1} Vg, \Lambda W \rangle = -\frac{d(d-2)}{2d} \left( (\Delta + s)^{-1} Vg, |x|^{-(d-2)} \right) + \frac{d-2}{2} \left( (\Delta + s)^{-1} Vg, \left( 1 + \frac{|x|^2}{d(d-2)} \right)^{-\frac{d}{2}} \right) + \langle (\Delta + s)^{-1} Vg, Z \rangle.
$$

Consider the first term on the right-hand side of (3.58). Lemma 3.11 shows that there exists $A > 0$ depending only on $d$ such that

$$
\left| -\frac{d(d-2)}{2d} \left( (\Delta + s)^{-1} Vg, |x|^{-(d-2)} \right) \right| \lesssim \|g\|_{L^q},
$$

where the implicit constant depends only on $d$ and $q$. Consider the second term on the right-hand side of (3.58). Observe from $d = 3, 4$ that $\frac{4d}{2d+3} > \frac{d}{\pi^2}, \frac{4d}{2d+3} > 1$ and $\frac{4d}{2d+3} < \frac{d}{\pi^2} < q$. Then, by Hölder’s inequality and Lemma 3.13, we see that

$$
\left| \frac{d-2}{2} \left( (\Delta + s)^{-1} Vg, \left( 1 + \frac{|x|^2}{d(d-2)} \right)^{-\frac{d}{2}} \right) \right| \lesssim \| (\Delta + s)^{-1} Vg \|_{L^\frac{4d}{2d+3}} \| (1 + |x|)^{-d} \|_{L^\frac{4d}{2d+3}} \lesssim \| Vg \|_{L^\frac{4d}{2d+3}} \lesssim \| g \|_{L^q},
$$

where the implicit constants depend only on $d$ and $q$. It remains to estimate the last term on the right-hand side of (3.58). Observe from the fundamental theorem of calculus, $\frac{4d}{2d+3} < d$ and $\frac{4d}{2d+3} > 1$ that

$$
\| Z \|_{L^\frac{4d}{2d+3}} = \left\| \frac{|x|^2}{4} \int_0^1 \left( \theta + \frac{|x|^2}{d(d-2)} \right)^{-\frac{d}{2}-1} d\theta \right\|_{L^\frac{4d}{2d+3}} \lesssim \int_0^1 \theta^{-\frac{d}{2}} d\theta \| |x|^{-(d-1)} \|_{L^\frac{4d}{2d+3}(\{x:|x| \leq 1\})} + \| |x|^{-d} \|_{L^\frac{4d}{2d+3}(1 \leq |x|)} \lesssim 1,
$$

where the implicit constants depend only on $d$. Then, by (3.61) and the same computation as (3.60), we see that

$$
\left| \langle (\Delta + s)^{-1} Vg, Z \rangle \right| \lesssim \| (\Delta + s)^{-1} Vg \|_{L^\frac{4d}{2d+3}} \| Z \|_{L^\frac{4d}{2d+3}} \lesssim \| g \|_{L^q},
$$

where the implicit constants depend only on $d$ and $q$.

Putting (3.58), (3.59), (3.60) and (3.62) together, we find that (3.55) holds. \(\square\)
The following lemma can be proved in a way similar to Lemma 3.13.

**Lemma 3.14** (cf. Lemma 2.5 of [7]). Assume $d = 3, 4$. Then, there exists a constant $C > 0$ depending only on $d$ such that the following holds for all $\frac{d}{d-2} < q < \infty$, $g \in L^q(\mathbb{R}^d)$ and $s > 0$:
\[
\left| \langle (-\Delta + s)^{-1} V g, W \rangle - C \mathcal{R}\mathcal{F}[V g](0) \delta(s)^{-1} \right| \lesssim \|g\|_{L^q},
\] (3.63)
where the implicit constant depends only on $d$ and $q$.

**Proof of Lemma 3.14.** Rewrite $W$ as
\[
W(x) = \left( \frac{|x|^2}{d(d - 2)} \right)^{-\frac{(d-2)}{2}} + Z(x),
\] (3.64)
where
\[
Z(x) := \left( 1 + \frac{|x|^2}{d(d - 2)} \right)^{-\frac{(d-2)}{2}} - \left( \frac{|x|^2}{d(d - 2)} \right)^{-\frac{(d-2)}{2}}.
\] (3.65)
Then, it is easy to see that the same proof as Lemma 3.13 is applicable.

**Lemma 3.15.** Assume $d = 3, 4$. Then, there exists a constant $A_1 > 0$ depending only on $d$ such that the following holds for all $s > 0$:
\[
\left| \langle (-\Delta + s)^{-1} W, VAW \rangle - A_1 \delta(s)^{-1} \right| \lesssim 1,
\] (3.66)
where the implicit constant depends only on $d$.

**Proof of Lemma 3.15.** Observe from (2.13) that
\[
\mathcal{R}\mathcal{F}[VAW](0) = -\frac{d+2}{d-2} \left( W^{\frac{d+2}{2}}, \Lambda W \right) = \frac{d-2}{2} \|W\|_{L^\frac{d+2}{d-2}}^{d+2}.
\] (3.67)
Then, applying Lemma 3.14 as $g = AW$ and $q = 2^*$, we find that the claim of the lemma is true.

**Lemma 3.16** (cf. Lemma 2.5 of [7]). Assume $d = 3, 4$ and $\frac{2}{d-2} < p < \frac{d+2}{d-2}$. Then, the following holds for all $s > 0$ and $\max\{1, \frac{d}{(d-2)p}\} < r < \frac{d}{2}$:
\[
\left| \langle (-\Delta + s)^{-1} W^p, VAW \rangle - K_p \right| \lesssim s^{\frac{d}{p} - 1},
\] (3.68)
where the implicit constant depends only on $d$, $p$, and $r$, and
\[
K_p := \langle W^p, \Lambda W \rangle = \frac{4 - (d-2)(p-1)}{2(p+1)} \|W\|_{L^{p+1}}^{p+1}.
\] (3.69)

**Proof of Lemma 3.16.** Let $s > 0$. Observe from $VAW = \Delta W$ (see (2.13)) that
\[
\langle (-\Delta + s)^{-1} W^p, VAW \rangle = \langle W^p, (-\Delta + s)^{-1} \{(-\Delta + s) + s\} \Lambda W \rangle = \langle W^p, \Lambda W \rangle + s \langle W^p, (-\Delta + s)^{-1} \Lambda W \rangle.
\] (3.70)
Furthermore, by Hölder’s inequality and Lemma 3.4, we see that for any $\max\{1, \frac{d}{(d-2)p}\} < r < \frac{d}{2}$,
\[
s \langle W^p, (-\Delta + s)^{-1} \Lambda W \rangle \lesssim s \|W^p\|_{L^r} \|(-\Delta + s)^{-1} \Lambda W\|_{L^\frac{d}{d-r}} \lesssim s \frac{d-2 - \frac{(d-1)}{2r}}{2r} \|\Lambda W\|_{L^\frac{d}{d-r}} \lesssim s^{\frac{d}{p} - 1},
\] (3.71)
where the implicit constants depend only on $d$, $p$, and $r$. Putting (3.70) and (3.71) together, and using (2.13), we obtain the desired estimate (3.68).
3.2 Proof of Proposition 3.1

In this section, we give a proof of Proposition 3.1. Assume \( d = 3, 4 \), and let \( \frac{d}{d-2} < q \leq \infty \).

We introduce operators \( Q \) and \( \Pi \) from \( L^q(\mathbb{R}^d) \) to itself as

\[
Qf := \frac{\langle f, VAW \rangle}{\langle AW, VAW \rangle} AW, \quad \Pi f := \frac{\langle f, VAW \rangle}{\|VAW\|_{L^2}^2} VAW.
\] (3.72)

Observe that \( Q^2 = Q \), \( \Pi^2 = \Pi \), and

\[
\|Q\|_{L^q \rightarrow L^q} \lesssim 1,
\] (3.73)

where the implicit constant depends only on \( d \) and \( q \). Furthermore, it follows from \((-\Delta)AW = -VAW\) that

\[
\{1 + (-\Delta)^{-1}V\}Q = 0, \quad (3.74)
\]

\[
\Pi\{1 + (-\Delta)^{-1}V\} = 0. \quad (3.75)
\]

Lemma 3.17. Assume \( d = 3, 4 \), and let \( \frac{d}{d-2} < q \leq \infty \). Then, the following hold:

\[
X_q = (1 - Q)L^q_{rad}(\mathbb{R}^d) := \{(1 - Q)f : f \in L^q_{rad}(\mathbb{R}^d)\};
\]

\[
X_q = (1 - \Pi)L^q_{rad}(\mathbb{R}^d) := \{(1 - \Pi)f : f \in L^q_{rad}(\mathbb{R}^d)\}.
\]

Furthermore, \((1 - Q)|_{X_q}\) and \((1 - \Pi)|_{X_q}\) are the identity maps, namely

\[
(1 - Q)f = (1 - \Pi)f = f \quad \text{for all } f \in X_q. \quad (3.76)
\]

Proof of Lemma 3.17. Observe that for any \( f \in L^q(\mathbb{R}^d)\),

\[
\langle(1 - Q)f, VAW \rangle = 0. \quad (3.77)
\]

Hence, \((1 - Q)L^q_{rad}(\mathbb{R}^d) \subset X_q\). On the other hand, if \( f \in X_q \), then \( \langle f, VAW \rangle = 0 \) and therefore \( f = Qf + (1 - Q)f = (1 - Q)f \). Thus, \( X_q \subset (1 - Q)L^q_{rad}(\mathbb{R}^d) \). Similarly, we can prove \((1 - \Pi)L^q_{rad}(\mathbb{R}^d) = X_q\). \( \square \)

Next, for \( d \geq 3 \) and \( \frac{d}{d-2} < q \leq \infty \), we introduce linear subspaces \( X_q \) and \( Y_q \) of \( L^q(\mathbb{R}^d) \) equipped with the norms \( \| \cdot \|_{X_q} \) and \( \| \cdot \|_{Y_q} \) as

\[
X_q := X_q \times QL^q_{rad}(\mathbb{R}^d), \quad \|(f_1, f_2)\|_{X_q} := \|f_1\|_{L^q} + \|f_2\|_{L^q},
\] (3.78)

\[
Y_q := X_q \times \Pi L^q_{rad}(\mathbb{R}^d), \quad \|(f_1, f_2)\|_{Y_q} := \|f_1\|_{L^q} + \|f_2\|_{L^q}.
\] (3.79)

Furthermore, for \( \varepsilon > 0 \), we define operators \( B_\varepsilon : X_q \rightarrow L^q_{rad}(\mathbb{R}^d) \) and \( C : L^q_{rad}(\mathbb{R}^d) \rightarrow Y_q \) by

\[
B_\varepsilon(f_1, f_2) := \varepsilon f_1 + f_2, \quad \text{for all } (f_1, f_2) \in X_q,
\] (3.80)

\[
Cf := ((1 - Q)f, \Pi f) \quad \text{for all } f \in L^q_{rad}(\mathbb{R}^d). \quad (3.81)
\]

Lemma 3.18. Assume \( d = 3, 4 \). Let \( \frac{d}{d-2} < q \leq \infty \). Then, the following hold:
1. For any \( \varepsilon > 0 \), \( B_\varepsilon : \mathbb{X}_q \to L^q_{\text{rad}}(\mathbb{R}^d) \) is surjective.

2. \( C : L^q_{\text{rad}}(\mathbb{R}^d) \to \mathbb{Y}_q \) is injective.

Proof of Lemma 3.18. Let \( g \in L^q_{\text{rad}}(\mathbb{R}^d) \), and define \( f_1 := \varepsilon^{-1}(1 - Q)g \) and \( f_2 := Qg \). Then, Lemma 3.17 shows that \( (f_1, f_2) \in \mathbb{X}_q \). Furthermore, it is obvious that \( B_\varepsilon(f_1, f_2) = g \). Hence, \( B_\varepsilon \) is surjective.

Let \( f \in L^q_{\text{rad}}(\mathbb{R}^d) \). Suppose \( Cf = 0 \), so that \( (1 - Q)f = 0 \) and \( \Pi f = 0 \). In particular, we have \( f = Qf = c\Lambda W \) for some \( c \in \mathbb{R} \). Furthermore, this together with \( \Pi f = 0 \) implies \( c = 0 \). Thus, \( f = 0 \) and \( C \) is injective.

**Lemma 3.19.** Assume \( d = 3, 4 \). Then, all of the following hold:

1. For any \( \frac{d}{\pi - 2} < q < \infty \) and \( s > 0 \),
   \[
   s\|(-\Delta + s)^{-1}(-\Delta)^{-1}VQ\|_{L^q \to L^q} \lesssim s^{\frac{d - 2}{\pi}} \frac{d}{d - 2},
   \tag{3.82}
   \]
   where the implicit constant depends only on \( d \) and \( q \).

2. For any \( 0 < s \leq 1 \),
   \[
   s\|(-\Delta + s)^{-1}(-\Delta)^{-1}VQ\|_{L^\infty \to L^\infty} \lesssim \delta(s)^{-1}s,
   \tag{3.83}
   \]
   where the implicit constant depends only on \( d \).

3. For any \( \frac{d}{\pi - 2} < q \leq \infty, \frac{d}{\pi - 2} < r < \infty \) and \( 0 < s \leq 1 \),
   \[
   s\|\Pi(-\Delta + s)^{-1}(-\Delta)^{-1}V\|_{L^r_{\text{rad}} \to \Pi L^q_{\text{rad}}} \lesssim \delta(s)^{-1}s,
   \tag{3.84}
   \]
   where the implicit constant depends only on \( d, \) \( d \), and \( r \) (independent of \( s \)); If \( d = 3 \), then (3.84) still holds for \( r = \infty \).

Proof of Lemma 3.19. We shall prove (3.82) and (3.83). Let \( \frac{d}{\pi - 2} < q \leq \infty, f \in L^q(\mathbb{R}^d) \) and \( s > 0 \). Observe from \((-\Delta)^{-1}V \Lambda W = -\Lambda W \) (see (2.25)) and Hölder’s inequality that

\[
\begin{align*}
   s\|(-\Delta + s)^{-1}(-\Delta)^{-1}VQf\|_{L^q} & = s\frac{|\langle f, VQW \rangle|}{|\langle \Lambda W, VQW \rangle|} \|(-\Delta + s)^{-1}(-\Delta)^{-1}V\Lambda W\|_{L^q} \\
   & \lesssim s\|f\|_{L^r} \|(-\Delta + \alpha)^{-1}\Lambda W\|_{L^q},
\end{align*}
\]

where the implicit constant depends only on \( d \) and \( q \). Assume \( q \neq \infty \). Then, applying Lemma 3.12 to the right-hand side of (3.85), we see that

\[
\begin{align*}
   s\|(-\Delta + s)^{-1}(-\Delta)^{-1}VQf\|_{L^q} & \lesssim s^{\frac{d - 2}{\pi}} \frac{d}{d - 2} \|f\|_{L^q} \|\Lambda W\|_{L^q} \|f\|_{L^q} \lesssim s^{\frac{d - 2}{\pi}} \frac{d}{d - 2} \|f\|_{L^q},
\end{align*}
\]

where the implicit constants depend only on \( d \) and \( q \). Thus, (3.82) holds. When \( q = \infty \) and \( 0 < s \leq 1 \), applying Lemma 3.12 to the right-hand side of (3.85), we see that

\[
\begin{align*}
   s\|(-\Delta + s)^{-1}(-\Delta)^{-1}VQf\|_{L^\infty} & \lesssim s\delta(s)^{-1}\|f\|_{L^\infty},
\end{align*}
\]

where the implicit constant depends only on \( d \). Thus, (3.83) holds.
Next, we shall prove \((3.84)\). Let \(g \in L^r_{\text{rad}}(\mathbb{R}^d)\). Observe from \(|V(x)| \lesssim (1 + |x|)^{-4}\) and \(r < \infty\) that
\[
|\mathcal{F}[Vg](0)| \leq \|Vg\|_{L^1} \lesssim \|g\|_{L^r},
\] (3.88)
where the implicit constant depends only on \(d\) and \(r\); Note here that when \(d = 3\), (3.88) still holds for \(r = \infty\). Then, by \((-\Delta)^{-1}V\mathcal{W} = -\mathcal{W}\), Lemma 3.13 and (3.88), we see that
\[
s\|\Pi(-\Delta + s)^{-1}(-\Delta)^{-1}Vg\|_{L^q} = s\|\Pi(-\Delta)^{-1}(-\Delta + s)^{-1}Vg\|_{L^q}
\]
\[
= s\frac{|((-\Delta)^{-1}(-\Delta + s)^{-1}Vg, V\mathcal{W})|}{\|V\mathcal{W}\|^2_{L^2}}\|V\mathcal{W}\|_{L^q}
\]
(3.89)
\[
= s\frac{|((\Delta + s)^{-1}Vg, \mathcal{W})|}{\|V\mathcal{W}\|^2_{L^2}}\|V\mathcal{W}\|_{L^q} \lesssim s\delta(s)^{-1}\|g\|_{L^r},
\]
where the implicit constant depends only on \(d\), \(q\) and \(r\). Thus, (3.84) holds.

Now, we are in a position to prove Proposition 3.1.

**Proof of Proposition 3.1.** Assume \(d = 3, 4\). Let \(\frac{d}{d-2} < q < \infty\), and let \(0 < s < 1\) be a small number to be specified later. Furthermore, let \(\varepsilon > 0\) be a small constant to be chosen later dependently only on \(d\) and \(q\).

Observe from Lemma 3.17 that \(1 + (-\Delta + s)^{-1}V\) maps \(L^q_{\text{rad}}(\mathbb{R}^d)\) to itself. Recall from Lemma 3.18 that \(B_{\varepsilon}: X_q \to L^q_{\text{rad}}(\mathbb{R}^d)\) is surjective, and \(C: L^q_{\text{rad}}(\mathbb{R}^d) \to Y_q\) is injective. Then, we define an operator \(A_{\varepsilon}(s)\) from \(X_q\) to \(Y_q\) as
\[
A_{\varepsilon}(s) := C\{1 + (-\Delta + s)^{-1}V\}B_{\varepsilon}.
\] (3.90)

By Lemma 3.12 of [14] (see Lemma C.3 in Appendix C), we see that if the inverse of \(A_{\varepsilon}(s)\) exists, then the operator \(1 + (-\Delta + s)^{-1}V: L^q_{\text{rad}}(\mathbb{R}^d) \to L^q_{\text{rad}}(\mathbb{R}^d)\) has the inverse and
\[
\{1 + (-\Delta + s)^{-1}V\}^{-1} = B_{\varepsilon}\{A_{\varepsilon}(s)\}^{-1}C.
\] (3.91)
This fact is the main point of the proof.

Let \((f_1, f_2) \in X_q\). We may write \(A_{\varepsilon}(s)(f_1, f_2)\) as follows (we do not care about the distinction between column and row vectors):
\[
A_{\varepsilon}(s)(f_1, f_2) = (A_{11}f_1 + A_{12}f_2, A_{21}f_1 + A_{22}f_2) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},
\] (3.92)
where
\[
A_{11} := \varepsilon(1 - Q)\{1 + (-\Delta + s)^{-1}V\}|_{X_q},
\] (3.93)
\[
A_{12} := (1 - Q)\{1 + (-\Delta + s)^{-1}V\}Q,
\] (3.94)
\[
A_{21} := \varepsilon\Pi\{1 + (-\Delta + s)^{-1}V\}|_{X_q},
\] (3.95)
\[
A_{22} := \Pi\{1 + (-\Delta + s)^{-1}V\}Q.
\] (3.96)

Recall that \(1 + (-\Delta + s)^{-1}V: L^q_{\text{rad}}(\mathbb{R}^d) \to L^q_{\text{rad}}(\mathbb{R}^d)\). Then, observe from Lemma 3.17 and (3.66) that
\[
A_{11}: X_q \to X_q, \quad A_{12}: QL^q_{\text{rad}}(\mathbb{R}^d) \to X_q.
\] (3.97)
Furthermore, it is obvious that
\[ A_{21} : X_q \to \Pi L^q_{rad}(\mathbb{R}^d), \quad A_{22} : Q L^q_{rad}(\mathbb{R}^d) \to \Pi L^q_{rad}(\mathbb{R}^d). \] (3.98)

We divide the proof into several steps:

**Step 1.** We consider the operator \( A_{11} : X_q \to X_q \) (see (3.93)).
Observe from Lemma 3.10 that \( \{1 + (\Delta)^{-1}V\}|_{X_q} : X_q \to X_q \) has the inverse, say \( K_1 \), such that
\[ \|K_1\|_{X_q \to X_q} \lesssim 1, \] (3.99)
where the implicit constant depends only on \( d \) and \( q \). Furthermore, (3.70) in Lemma 3.17 shows that
\[ (1 - Q)\{1 + (\Delta)^{-1}V\}|_{X_q} = \{1 + (\Delta)^{-1}V\}|_{X_q}. \] (3.100)
Then, by (3.19) and (3.100) we may rewrite \( A_{11} \) as
\[ A_{11} = \varepsilon \{1 + (\Delta)^{-1}V\}|_{X_q} + \varepsilon S_{11} = \varepsilon \{1 + (\Delta)^{-1}V\}|_{X_q} \{1 + K_1 S_{11}\}, \] (3.101)
where
\[ S_{11} := -s(1 - Q)(\Delta + s)^{-1}(\Delta)^{-1}V|_{X_q}. \] (3.102)
Now, fix \( 1 < r_0 < \frac{d q}{d + 2q} \): The dependence of a constant on \( r_0 \) can be absorbed into that on \( d \) and \( q \). Note that \( q > \frac{d r_0}{d - 2r_0} \). Then, by (3.99), (3.73), Lemma 3.4 and (2.20), we see that for any \( f \in X_q \),
\[ \|K_1 S_{11} f\|_{L^q} \lesssim s\|V\|_{L^\infty} \lesssim s^{\frac{d - 2r_0}{d - 2r_0} - \frac{1}{q}} \|V f\|_{L^q} \lesssim s^{\frac{d - 2r_0}{d - 2r_0} - \frac{1}{q}} \|f\|_{L^q}, \] (3.103)
where the implicit constants depend only on \( d \) and \( p \). Observe from the computations in (3.103) and Lemma 3.17 that \( S_{11} : X_q \to X_q \). Hence, the Neumann series shows that if \( s \) is sufficiently small dependently only on \( d \) and \( q \), then the operator \( 1 + K_1 S_{11} : X_q \to X_q \) has the inverse, say \( K_2 \), which satisfies
\[ \|K_2\|_{X_q \to X_q} \leq (1 - \|K_1 S_{11}\|_{X_q \to X_q})^{-1} \lesssim 1, \] (3.104)
where the implicit constant depends only on \( d \) and \( q \). We summarize:
\[ A_{11}^{-1} = \varepsilon^{-1} K_2 K_1, \quad \|A_{11}^{-1}\|_{X_q \to X_q} \lesssim \varepsilon^{-1}, \] (3.105)
where \( K_1 \) and \( K_2 \) are the inverses of \( \{1 + (\Delta)^{-1}V\}|_{X_q} : X_q \to X_q \) and \( 1 + K_1 S_{11} : X_q \to X_q \), respectively, and the implicit constant depends only on \( d \) and \( q \).

**Step 2.** We consider the operator \( A_{12} : Q L^q_{rad}(\mathbb{R}^d) \to X_q \) (see (3.94)). Our aim is to show that
\[ \|A_{12}\|_{Q L^q_{rad} \to X_q} \lesssim s^{\frac{d - 2}{2} - \frac{d}{p}}, \] (3.106)
where the implicit constant depends only on \( d \) and \( q \).
Let \( f \in Q L^q_{rad}(\mathbb{R}^d) \). Then, by (3.19), (3.74) and (3.73), we see that
\[ \|A_{12} f\|_{L^q} = \|(1 - Q)\{1 + (\Delta)^{-1}V - s(\Delta + s)^{-1}(\Delta)^{-1}V\} Q f\|_{L^q} \]
\[ = \|(1 - Q)\{s(\Delta + s)^{-1}(\Delta)^{-1}V\} Q f\|_{L^q} \]
\[ \lesssim s\|(-\Delta + s)^{-1}(\Delta)^{-1}V Q f\|_{L^q}, \] (3.107)
where the implicit constant depends only on \( d \) and \( q \). Applying (3.82) in Lemma 3.19 to the right-hand side of (3.107), we obtain (3.106).

**Step 3.** We consider the operator \( A_{21} : X_q \to \Pi L^q_{\text{rad}}(\mathbb{R}^d) \) (see (3.85)). By (3.49), (3.75), and (3.84) in Lemma 3.19 with \( r = q \), we see that

\[
\|A_{21}\|_{L^q_{\text{rad}} \to \Pi L^q_{\text{rad}}} \lesssim \varepsilon \delta(s)^{-1}s, \tag{3.108}
\]

where the implicit constant depends only on \( d \) and \( q \).

**Step 4.** We consider the operator \( A_{22} : Q L^q_{\text{rad}}(\mathbb{R}^d) \to \Pi L^q_{\text{rad}}(\mathbb{R}^d) \) (see (3.96)).

Let \( f \in Q L^q_{\text{rad}}(\mathbb{R}^d) \). Observe from (3.19), (3.74) and \((\Delta)^{-1}VAW = -AW\) (see (2.25)) that

\[
A_{22}f = -s\Pi(\Delta + s)^{-1}(\Delta)^{-1}VQf
\]

\[
= -s\frac{\langle f, VAW \rangle}{\langle AW, VAW \rangle \|
\]

\[
\| VAW \| ^2_{L^2} - (\Delta + s)^{-1}(\Delta)^{-1}VAW, VAW \rangle VAW
\]

\[
= s\langle AW, (-\Delta + s)^{-1}VAW \rangle \Pi f.
\]

Furthermore, applying Lemma 3.13 to the right-hand side of (3.109) as \( g = AW \), we find that \( A_{22}f \) can be written as follows:

\[
A_{22}f = \mathcal{C}_0 \delta(s)^{-1}s\Pi f + \mathcal{C}_1 s\Pi f, \tag{3.110}
\]

where \( \mathcal{C}_0 \) and \( \mathcal{C}_1 \) are some constants depending only on \( d \). Define the operator \( G \) with domain \( Q L^q_{\text{rad}}(\mathbb{R}^d) \) as

\[
G := \mathcal{C}_0 \delta(s)^{-1}s\Pi|QL^q_{\text{rad}}(\mathbb{R}^d). \tag{3.111}
\]

Observe that \( G : Q L^q_{\text{rad}}(\mathbb{R}^d) \to \Pi L^q_{\text{rad}}(\mathbb{R}^d) \) is bijective, and the inverse \( G^{-1} : \Pi L^q_{\text{rad}}(\mathbb{R}^d) \to Q L^q_{\text{rad}}(\mathbb{R}^d) \) is given by

\[
G^{-1}g = \mathcal{C}_0^{-1}s^{-1}\delta(s)\frac{\langle g, AW \rangle}{\langle AW, VAW \rangle} AW. \tag{3.112}
\]

Hence, we may write \( A_{22} : Q L^q_{\text{rad}}(\mathbb{R}^d) \to \Pi L^q_{\text{rad}}(\mathbb{R}^d) \) as

\[
A_{22} = G(1 + s\mathcal{C}_1 G^{-1}\Pi). \tag{3.113}
\]

Observe that

\[
\|s\mathcal{C}_1 G^{-1}\Pi f\|_{L^q} \lesssim \delta(s)\|\Pi f, AW\| \|AW\|_{L^q} \lesssim \delta(s)\|\langle f, VAW \rangle \| \lesssim \delta(s)\|f\|_{L^q}, \tag{3.114}
\]

where the implicit constants depend only on \( d \) and \( q \). Hence, the Neumann series shows that if \( s \) is sufficiently small dependently only on \( d \) and \( q \), then the operator \( 1 + s\mathcal{C}_1 G^{-1}\Pi : Q L^q_{\text{rad}}(\mathbb{R}^d) \to Q L^q_{\text{rad}}(\mathbb{R}^d) \) has the inverse, say \( L \), such that

\[
\|L\|_{Q L^q_{\text{rad}} \to Q L^q_{\text{rad}}} \lesssim 1, \tag{3.115}
\]

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where the implicit constant depends only on $d$ and $q$. Then, we see that
\[ A^{-1}_{22} = LG^{-1}, \quad \|A_{22}^{-1}\|_{\Pi L^q_{\text{rad}} \to QL^q_{\text{rad}}} \lesssim \delta(s)s^{-1}, \] (3.116)
where the implicit constant depends only on $d$ and $q$.

**Step 5.** We shall finish the proof. To this end, we define
\[ \tilde{\mathbf{A}}_\varepsilon := \begin{pmatrix} 1 & A_{11}^{-1}A_{12} \\ A_{22}^{-1}A_{21} & 1 \end{pmatrix}, \quad \mathbf{I} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \] (3.117)

Note that $\tilde{\mathbf{A}}_\varepsilon : \mathbf{X}_q \to \mathbf{X}_q$. Observe from (3.92) that
\[
\begin{align*}
\|\mathbf{I} - \tilde{\mathbf{A}}_\varepsilon\|_{\mathbf{X}_q \to \mathbf{X}_q} &\leq \|A_{11}^{-1}A_{12}\|_{QL^q_{\text{rad}} \to QL^q_{\text{rad}}} + \|A_{22}^{-1}A_{21}\|_{\mathbf{X}_q \to QL^q_{\text{rad}}} \\
&\lesssim \varepsilon^{-1}A_{12}\|_{QL^q_{\text{rad}} \to QL^q_{\text{rad}}} + \|A_{21}\|_{\mathbf{X}_q \to QL^q_{\text{rad}}} \\
&\lesssim \varepsilon^{-1}s^{\frac{d-2}{2} + \frac{d}{2g} + \varepsilon},
\end{align*}
\] (3.119)
where the implicit constants depend only on $d$ and $q$. Thus, we find that there exist $s_0 > 0$ and $\varepsilon_0 > 0$, both depending only on $d$ and $q$, such that for any $0 < s < s_0$,
\[ \|\mathbf{I} - \tilde{\mathbf{A}}_{\varepsilon_0}\|_{\mathbf{X}_q \to \mathbf{X}_q} \leq \frac{1}{2}, \] (3.120)
and $\tilde{\mathbf{A}}_{\varepsilon_0}$ has the inverse $\mathbf{D} := \tilde{\mathbf{A}}_{\varepsilon_0}^{-1}$ satisfying
\[ \|\mathbf{D}\|_{\mathbf{X}_q \to \mathbf{X}_q} \leq 2. \] (3.121)

The dependence of a constant on $\varepsilon_0$ can be absorbed into that on $d$ and $q$. Furthermore, we may assume that $\varepsilon_0 \leq \delta(s_0)s_0^{-1}$. Then, for any $0 < s < s_0$, the inverse of $\mathbf{A}_{\varepsilon_0}(s)$ exists and is given by
\[ \mathbf{A}_{\varepsilon_0}(s)^{-1} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{pmatrix}, \] (3.122)
where $D_{jk}$ is the $(j,k)$-entry of $\mathbf{D}$. Furthermore, we see from (3.105), (3.116) and (3.108), we have
\[ \|\mathbf{A}_{\varepsilon_0}(s)^{-1}\|_{\mathbf{Y} \to \mathbf{X}} \lesssim \|A_{11}^{-1}\|_{\mathbf{X}_q \to \mathbf{X}_q} + \|A_{22}^{-1}\|_{\Pi L^q_{\text{rad}} \to QL^q_{\text{rad}}} \lesssim \delta(s)s^{-1}, \] (3.123)
where the implicit constants depend only on $d$ and $q$. Thus, Lemma 3.12 of [14] (see Lemma C.23 in Appendix C) shows that for any $0 < s < s_0$, the inverse of $1 + (-\Delta + s)^{-1}V$ exists as an operator from $L^q_{\text{rad}}(\mathbb{R}^d)$ to itself and
\[ \{1 + (-\Delta + s)^{-1}V\}^{-1} = B_{\varepsilon_0}\mathbf{A}_{\varepsilon_0}(s)^{-1}\mathbf{C}. \] (3.124)
Observe that
\[ \|B_{\varepsilon}\|_{X_{s} \rightarrow L_{rad}^{q}} \lesssim 1, \quad \|C\|_{L_{rad}^{q} \rightarrow Y} \lesssim 1. \]  
(3.125)
Hence, we see from (3.123) through (3.125) that for any \( 0 < s < s_0 \),
\[ \|\{1 + (-\Delta + s)^{-1}V\}^{-1}\|_{L_{rad}^{q} \rightarrow L_{rad}^{q}} \lesssim \delta(s)s^{-1}, \]  
(3.126)
which proves (3.1).

In remains to prove (3.2). Assume \( f \in X_{q} \). Then, \( \Pi f = 0 \) and therefore
\[ \{1 + (-\Delta + s)^{-1}V\}^{-1}f = B_{\varepsilon_0} \left( \begin{array}{cc} D_{11} & D_{12} \\ D_{21} & D_{22} \end{array} \right) \left( \begin{array}{cc} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{array} \right) C f \]
\[ = D_{11}A_{11}^{-1}(1 - Q)f + D_{21}A_{11}^{-1}(1 - Q)f. \]  
(3.127)
Then, (3.2) follows from (3.127), (3.105) with \( \varepsilon = \varepsilon_0 \), and (3.121). Thus, we have completed the proof.

4 Uniqueness

In this section, we give a proof of Theorem 1.1. Our proof is based on the scaling argument and the fixed-point argument developed in [7]. We will see that “the uniqueness provided by the fixed-point argument” is helpful in proving the uniqueness of ground states to (1.1).

First, observe that if \( \Phi \) is a solution to (1.1) and \( \lambda > 0 \), then \( u := T_{\lambda}[\Phi] \) satisfies
\[ -\Delta u + \omega \lambda^{-2^* + 2}u - \lambda^{-2^* + p + 1}|u|^{p-1}u - |u|^{\frac{4}{d-2}}u = 0. \]  
(4.1)
For \( s > 0 \) and \( t > 0 \), we consider a general form of (4.1):
\[ -\Delta u + su - t|u|^{p-1}u - |u|^{\frac{4}{d-2}}u = 0. \]  
(4.2)
When \( s = t = 0 \), (4.2) agrees with the equation (2.8). Hence, we may expect that (4.2) admits a ground state asymptotically looks like \( W \) as \( s,t \to 0 \).

We look for a radial solution to (4.2) of the form \( W + \eta \), which leads us to
\[ (-\Delta + V + s)\eta = -sW + tW^p + N(\eta; t), \]  
(4.3)
where \( N(\eta; t) \) is the function defined by (2.33). By the decomposition \( -\Delta + V + s = (-\Delta + s)\{1 + (-\Delta + s)^{-1}V\}^{-1} \), and the substitution \( s = \alpha(\tau) \), we can rewrite (4.3) as
\[ \eta = \{1 + (-\Delta + \alpha(\tau))^{-1}V\}^{-1}(-\Delta + \alpha(\tau))^{-1}F(\eta; \alpha(\tau), t), \]  
(4.4)
where \( F(\eta; \alpha(\tau), t) := -\alpha(\tau)W + tW^p + N(\eta; t) \) (see 2.36). The significance of the substitution \( s = \alpha(\tau) \) is that \( \alpha(\tau) \) is written as \( \tau \delta(\alpha'(\tau)) \) (see 2.32); We will use this fact later (see 4.4 below). Note that a solution to (4.2) which asymptotically looks like \( W \) corresponds to a solution to (4.4) which vanishes as \( \tau \to 0 \) (in \( L^{q}(\mathbb{R}^{d}) \) for all \( q \geq \frac{d}{d-2} \)). Furthermore, recall from Proposition 3.4 (see also 5.3) in Remark 4.1 that for any \( \frac{d}{d-2} < q < \infty \),
\[ \|\{1 + (-\Delta + \alpha(\tau))^{-1}V\}^{-1}\|_{L_{rad}^{q} \rightarrow L_{rad}^{q}} \lesssim \tau^{-1}, \]  
(4.5)
where the implicit constant depends only on \(d\) and \(q\). From these point of view, we need to eliminate the singular behavior of a solution to \((4.4)\) arising from \((4.5)\). To this end, we require the following condition, as well as \([7]\):

\[
((-\Delta + \alpha(\tau))^{-1} F(\eta; \alpha(\tau), t), VAW) = 0.
\]  (4.6)

As a result, we consider the system of \((4.4)\) and \((4.6)\). We will find a solution by using Banach fixed-point theorem, as in \([7]\). To this end, using \((2.36)\) and \((2.37)\), we rewrite \((4.6)\) as

\[
\tau = t((-\Delta + \alpha(\tau))^{-1} W^p, VAW) + ((-\Delta + \alpha(\tau))^{-1} N(\eta; t), VAW) .
\]  (4.7)

Here, we introduce several symbols:

**Notation 4.1.**

1. For \(t > 0, 0 < \tau < 1\), and \(\eta \in H^1(\mathbb{R}^d)\), we define \(\mathfrak{X}(\tau), W_p(\tau)\) and \(N(t, \tau, \eta)\) as

\[
\mathfrak{X}(\tau) := \delta(\alpha(\tau))((-\Delta + \alpha(\tau))^{-1} W, VAW),
\]  (4.8)

\[
W_p(\tau) := ((-\Delta + \alpha(\tau))^{-1} W^p, VAW),
\]  (4.9)

\[
N(t, \tau, \eta) := ((-\Delta + \alpha(\tau))^{-1} N(\eta; t), VAW).
\]  (4.10)

2. For \(t > 0, 0 < \tau < 1\) and \(\eta \in H^1(\mathbb{R}^d)\), we define \(s(t; \tau, \eta)\) as

\[
s(t; \tau, \eta) := \frac{t W_p(\tau) + N(t, \tau, \eta)}{\mathfrak{X}(\tau)}.
\]  (4.11)

3. For \(t > 0, 0 < \tau < 1\) and \(\eta \in H^1(\mathbb{R}^d)\), we define \(g(t; \tau, \eta)\) as

\[
g(t; \tau, \eta) := \{1 + (-\Delta + \alpha(\tau))^{-1} W\}^{-1}((-\Delta + \alpha(\tau))^{-1} F(\eta; \alpha(\tau), t).
\]  (4.12)

4. We use \(A_1\) and \(K_p\) to denote the constants given in Lemma 3.4.3 and Lemma 3.4.6, respectively.

5. For \(t > 0\), we define an interval \(I(t)\) as

\[
I(t) := \left[ \frac{K_p}{2A_1}, \frac{3K_p}{2A_1} \right].
\]  (4.13)

In stead of \(I(t)\), we may use any interval which contains \(\frac{K_p}{A_1} t\) and whose length is comparable to \(t\).

6. For \(\frac{d}{d-2} < q < \infty\) we define a number \(\Theta_q\) as

\[
\Theta_q := \frac{d-2}{2} - \frac{d}{2q}.
\]  (4.14)

7. For \(\frac{d}{d-2} < q < \infty\), \(R > 0\) and \(0 < t < 1\), we define \(Y_q(R, t)\) as

\[
Y_q(R, t) := \{ \eta \in L^q_{\text{rad}}(\mathbb{R}^d): \|\eta\|_{L^q} \leq R\alpha(t)^{\Theta_q} \}.
\]  (4.15)

Note here that when \(d = 4\), \(\alpha\) is defined only on \((0, 1)\) (see Lemma 6.7).
Remark 4.1. Observe from $s = \alpha(\beta(s))$ that
\[\delta(s)((-\Delta + s)^{-1}W, VAW) = \mathcal{X}(\beta(s)),\]
\[((-\Delta + s)^{-1}W^p, VAW) = \mathcal{W}_p(\beta(s)),\]
\[((-\Delta + s)^{-1}N(\eta; t), VAW) = N(t, \beta(s), \eta).\]
\[\tag{4.16} \text{ } \tag{4.17} \text{ } \tag{4.18}\]

Remark 4.2. 1. We can verify that for any $\frac{d+2}{d-2} < q < \infty$, $R > 0$ and $0 < t < 1$, the set $Y_q(R, t)$ is complete as a subspace of $L^q(\mathbb{R}^d)$ with the induced metric (see Section [3]).

2. Since $\alpha(t)$ is strictly increasing on $(0, 1)$ and $\lim_{t \to 0} \alpha(t) = 0$ (see Section [2]), we find that: for any $\theta > 0$ and $R > 0$, there exists $T(\theta, R) > 0$ depending only on $d$, $\theta$ and $R$ such that
\[ (1 + R)^{\frac{d+2}{d-2}} \alpha(t) \theta \leq 1 \text{ for all } 0 < t \leq T(\theta, R). \]
\[ \tag{4.19} \]

Now, we use the symbols $(4.11)$ and $(4.12)$ to rewrite the system of the equations $(4.4)$ and $(4.6)$ as follows:
\[ (\tau, \eta) = (\sigma(t; \tau, \eta), \rho(t; \tau, \eta)). \]
\[ \tag{4.20} \]

We can find a solution to $(4.20)$:

Proposition 4.1. Assume $d = 3, 4$ and $\frac{4}{d-2} - 1 < p < \frac{d+2}{d-2}$. Let $\max\{2^*, \frac{2^*}{p+1}\} < q < \infty$ and $R > 0$. Then, there exists $0 < T_1(q, R) < 1$ depending only on $d$, $p$ and $R$ with the following property: for any $0 < t < T_1(q, R)$, the product metric space $I(t) \times Y_q(R, t)$ admits one and only one solution $(\tau_t, \eta_t)$ to $(4.20)$. Furthermore, $\tau_t$ is continuous and strictly increasing with respect to $t$ on $(0, T_1(q, R))$.

Observe that if $(\tau_t, \eta_t)$ is a solution to $(4.20)$, then $W + \eta_t$ is one to the following equation:
\[ -\Delta u + \alpha(\tau_t)u - t|u|^{p-1}u - |u|^{\frac{4}{d-2}}u = 0. \]
\[ \tag{4.21} \]

We denote the action and Nehari functional associated with $(4.21)$ by
\[ \tilde{S}_t(u) := \frac{\alpha(\tau_t)}{2} \|u\|_{L^2}^2 + \frac{1}{2} \|
abla u\|_{L^2}^2 - \frac{t}{p+1} \|u\|_{L^{p+1}}^{p+1} - \frac{1}{2^*} \|u\|_{L^{2^*}}^{2^*}, \]
\[ \tag{4.22} \]
\[ \tilde{N}_t(u) := \alpha(\tau_t)\|u\|_{L^2}^2 + \|
abla u\|_{L^2}^2 - t\|u\|_{L^{p+1}}^{p+1} - \|u\|_{L^{2^*}}^{2^*}. \]
\[ \tag{4.23} \]

Furthermore, we define $\tilde{G}_t$ as the set of all positive radial ground states to $(4.21)$, namely, $\tilde{G}_t$ denotes the set of positive radial minimizers for the following variational problem:
\[ \inf \{ \tilde{S}_t(u) : u \in H^1(\mathbb{R}^d) \setminus \{0\}, \tilde{N}_t(u) = 0 \}. \]
\[ \tag{4.24} \]

The following proposition is a key to proving the uniqueness of ground states to $(4.1)$:

Proposition 4.2 (cf. Lemma 3.11 of [7]). Assume $d = 3, 4$ and $\frac{4}{d-2} - 1 < p < \frac{d+2}{d-2}$. Let $\max\{2^*, \frac{2^*}{p+3-2^*}\} < q < \infty$. Furthermore, for $R > 0$, let $T_1(q, R)$ denote the number given in Proposition [4.1]. Then, there exist $R_0 > 0$ and $0 < T_* < T_1(q, R_0)$, both depending only on $d$, $p$ and $q$, with the following property: Let $0 < t < T_*$, and let $(\tau_t, \eta_t)$ be a unique solution to $(4.20)$ in $I(t) \times Y_q(R_0, t)$ (see Proposition [4.1]). Then, $W + \eta_t$ is a unique positive radial ground state to $(4.21)$; in other words, $\tilde{G}_t = \{W + \eta_t\}$.
We give a proof of Proposition 4.2 in Section 4.3.

Now, we are in a position to prove Theorem 1.1:

**Proof of Theorem 1.1.** Let $d = 3, 4$ and \[ \frac{4}{d-2} < p < \frac{d+1}{d-2}, \]
so that the dependence of a constant on $q$ can be absorbed into that on $d$ and $p$. Then, by Proposition 4.1 and Proposition 4.2, we see that there exist $R_*>0$ and $0<T_1<T_*$ depending only on $d$ and $p$ such that for any $0<t\leq T_*$, $I(t) \times Y_q(R_*,t)$ admits a unique solution $(\tau_t, \eta_t)$ to (4.20), and $\tilde{G}_t = \{W + \eta_t\}$.

We shall show that there exists $\omega^* > 0$ with the following property: for any $\omega > \omega^*$, there exists $0 < t(\omega) < T_*$ such that

\[ \alpha(\tau_t(\omega)) = \omega t(\omega) \frac{2^*-2}{2*(p+1)}. \]  

(4.25)

Observe from $\beta(s) = \delta(s)^{-1}s$ (see (2.31)) that for any $0 < t < T_*$,

\[ \beta(\omega t^{\frac{2^*-2}{2*(p+1)}}) = \begin{cases} \sqrt{\omega t^{\frac{2}{p}}} & \text{if } d = 3, \\ \log (1 + \omega^{-1} t^{\frac{3}{p}}) \omega t^{\frac{2}{p}} & \text{if } d = 4. \end{cases} \]  

(4.26)

Then, by $\tau_t \in I(t) := [\frac{K_p}{2A_1}t, \frac{3K_p}{2A_1}t]$ and (4.26), we find (see Figure 1) that there exists $\omega_* > 0$ with the following property: for any $\omega > \omega_*$, there exists $0 < t(\omega) < T_*$ such that

\[ \tau_t(\omega) = \beta(\omega t(\omega) \frac{2^*-2}{2*(p+1)}). \]  

(4.27)

Since $\alpha$ is the inverse function of $\beta$, (4.27) implies (4.25).

Now, we shall finish the proof of Theorem 1.1. Let $\omega_*$ be a frequency found in the above, so that for any $\omega > \omega_*$, there exists $0 < t(\omega) < T_*$ such that (4.25) holds. Suppose for contradiction that there exists $\omega > \omega_*$ for which the equation (1.1) admits two distinct positive ground states, say $\Phi_\omega$ and $\Psi_\omega$. Then, we define $\lambda(\omega) > 0$ by

\[ \lambda(\omega)^{-1} (2^*-(p+1)) = t(\omega). \]  

(4.28)

Furthermore, we define $\tilde{\Phi}_\omega$ and $\tilde{\Psi}_\omega$ as

\[ \tilde{\Phi}_\omega := T_{\lambda(\omega)}[\Phi_\omega], \quad \tilde{\Psi}_\omega := T_{\lambda(\omega)}[\Psi_\omega], \]  

(4.29)

where $T_{\lambda(\omega)}$ is the scaling operator defined by (2.11). We shall derive a contradiction by showing that

\[ \tilde{\Phi}_\omega, \tilde{\Psi}_\omega \in \tilde{G}_t(\omega). \]  

(4.30)
Indeed, (4.30) contradicts Proposition 4.2. Thus, the claim of Theorem 1.1 is true.

Let us prove (4.30). Observe from $N_\omega(\Phi_\omega) = 0$, a computation involving the scaling, and (4.25) that

$$\tilde{N}_t(\omega)(\tilde{\Phi}_\omega) = 0,$$

(4.31)

where $N_\omega$ and $\tilde{N}_t(\omega)$ are the functionals defined by (1.6) and (4.23), respectively. Furthermore, (4.31) implies

$$\tilde{S}_t(\omega)(\tilde{\Phi}_\omega) \geq \inf \{ \tilde{S}_t(\omega)(u) : u \in H^1(\mathbb{R}^d) \setminus \{0\}, \tilde{N}_t(\omega)(u) = 0 \}. $$

(4.32)

If $\tilde{\Phi}_\omega$ were not in $\tilde{G}_t(\omega)$, then the equality failed in (4.32) and there existed a function $u \in H^1(\mathbb{R}^d) \setminus \{0\}$ such that

$$\tilde{S}_t(\omega)(u) < \tilde{S}_t(\omega)(\tilde{\Phi}_\omega),$$

(4.33)

$$\tilde{N}_t(\omega)(u) = 0.$$ 

(4.34)

Then, define $U_\omega$ by $U_\omega(x) := T_{\lambda(\omega)^{-1}}[u](x) = \lambda(\omega)u(\lambda(\omega)^{-2}x)$. By (4.25), (4.33), (4.34) and a computation involving the scaling, we see that

$$S_\omega(U_\omega) < S_\omega(\Phi_\omega), \quad N_\omega(U_\omega) = 0.$$ 

(4.35)

However, (4.35) contradicts that $\Phi$ is a ground state to (1.1). Thus, it must hold that $\tilde{\Phi}_\omega \in \tilde{G}_t(\omega)$. Similarly, we can verify that $\tilde{\Psi}_\omega \in \tilde{G}_t(\omega)$. Thus, we have proved (4.30), which completes the proof.

4.1 Basic estimates

In this section, we prepare several estimates for the proof of Proposition 4.1. In order to describe them, we use the following symbols:

**Notation 4.2.**

1. For $d = 3, 4$, $p > 1$ and $q > 1$, define a number $\nu_q$ as

$$\nu_q := \begin{cases} 
\frac{d}{2q} & \text{if } d = 3, 4 \text{ and } \frac{d}{d-2} \leq p < \frac{d+2}{d-2}, \\
1 - \frac{(d-2)(p-1)}{4} & \text{if } d = 3, 4 \text{ and } 1 < p < \frac{d}{d-2}.
\end{cases}$$

Note that

$$\frac{(d-2)p-2}{2} > \frac{d - 2}{2} - \nu_q.$$ 

(4.37)

Furthermore, if $p > \frac{4}{d-2} - 1$ and $q > \max\{2^*, \frac{2^*}{p-1}\}$, then

$$\Theta_q - \nu_q > 0, \quad \Theta_q + \frac{(d-2)(p-1)}{2} - 1 > 0.$$ 

(4.38)

2. For $d = 3, 4$, $p > \frac{4}{d-2} - 1$ and $q > \max\{2^*, \frac{2^*}{p-1}\}$, define $\theta_q > 0$ as

$$\theta_q := \frac{1}{2} \min\{\Theta_q - \nu_q, \Theta_q + \frac{(d-2)(p-1)}{2} - 1\}.$$ 

(4.39)
Remark 4.3. Assume \( d = 3, 4 \) and \( p > \frac{4}{d-2} - 1 \), and let \( q > \max\{2^*, \frac{2}{p-1}\} \). Then, we see from (2.34) that if \( t > 0 \) is sufficiently small depending only on \( d, p \) and \( q \), then

\[
 t \leq \alpha(t)^{\Theta_q},
\]

\[
 t^{-1} \alpha(t)^{2\Theta_q} \leq \alpha(t)^{\Theta_q - \nu_q}.
\]

Lemma 4.1. Assume \( d = 3, 4 \). Let \( 2^* < q < \infty \). Then, for any \( R > 0 \), \( s > 0 \), \( 0 < t_1 \leq t_2 < 1 \), \( \eta_1 \in Y_q(R,t_1) \) and \( \eta_2 \in Y_q(R,t_2) \), the following holds:

\[
 \|(-\Delta + s)^{-1} D(\eta_1, \eta_2)\|_{L^q} \lesssim \left\{ R\alpha(t_2)^{\Theta_q} + s^{2^* - 1} R^{\frac{4}{d-2}} \alpha(t_2)^{\frac{4}{d-2} \Theta_q} \right\} \|\eta_1 - \eta_2\|_{L^q},
\]

where the implicit constant depends only on \( d \) and \( q \).

Proof of Lemma 4.1. Note that

\[
 \frac{(6-d) dq}{(d-2)(2q-d)} > \frac{d}{d-2}.
\]

Hence, we see from (2.7) that

\[
 W \in L^{(6-d) dq/(d-2)(2q-d)}(\mathbb{R}^d).
\]

Furthermore, observe from \( 0 < t_1 \leq t_2 \) that

\[
 \max_{j=1,2} \|\eta_j\|_{L^q} \leq R\alpha(t_2)^{\Theta_q}.
\]

Then, by (2.44), (2.41), Lemma 3.5, Lemma 3.4, Hölder’s inequality, (4.31) and (4.44), we see that

\[
 \|(-\Delta + s)^{-1} D(\eta_1, \eta_2)\|_{L^q}
\]

\[
 \lesssim \|(-\Delta + s)^{-1}\left\{ W^{\frac{6-d}{2q}}(|\eta_1| + |\eta_2|) + (|\eta_1| + |\eta_2|)\right\} \|\eta_1 - \eta_2\|_{L^q}
\]

\[
 \lesssim \|W \left(\frac{6-d}{2q}\right) (|\eta_1| + |\eta_2|)\|_{L^\frac{dq}{6-d q}} + s^{\frac{2^* - 1}{q}} (|\eta_1| + |\eta_2|)\|\eta_1 - \eta_2\|_{L^q}
\]

\[
 \lesssim \|W \|_{L^{\frac{6-d}{2q}}(\mathbb{R}^d)} \|\eta_1 + \eta_2\|_{L^q} \|\eta_1 - \eta_2\|_{L^q} + s^{\frac{2^* - 1}{q} - 1} \|\eta_1 + \eta_2\|_{L^q} \|\eta_1 - \eta_2\|_{L^q}
\]

\[
 \lesssim \left\{ R\alpha(t_2)^{\Theta_q} + s^{\frac{2^* - 1}{q} - 1} R^{\frac{4}{d-2}} \alpha(t_2)^{\frac{4}{d-2} \Theta_q} \right\} \|\eta_1 - \eta_2\|_{L^q},
\]

where the implicit constants depend only on \( d \) and \( q \). This proves the lemma.

Lemma 4.2. Assume \( d = 3, 4 \) and \( 1 < p < \frac{d+2}{d-2} \). Let \( 2^* \leq q < \infty \). Then, for any \( R > 0 \), \( s > 0 \), \( 0 < t_1 \leq t_2 < 1 \), \( \eta_1 \in Y_q(R,t_1) \) and \( \eta_2 \in Y_q(R,t_2) \), the following holds:

\[
 \|(-\Delta + s)^{-1} E(\eta_1, \eta_2)\|_{L^q} \lesssim \left\{ s^{-\nu_q} + s^{\frac{d(p-1)}{2q} - 1} R^{p-1} \alpha(t_2)^{(p-1)\Theta_q} \right\} \|\eta_1 - \eta_2\|_{L^q},
\]

where the implicit constant depends only on \( d \), \( p \) and \( q \).

Proof of Lemma 4.2. First, assume that \( 1 < p < \frac{d}{d-2} \). Let \( 0 < \varepsilon_0 < \frac{(d-2)(p-1)}{2} \), and define \( q_0 \) as

\[
 q_0 := \frac{dq}{d + (d-2)(p-1)q - 2q\varepsilon_0}.
\]

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Observe that $1 < q_0 < q$ and $\frac{d}{d-2} < \frac{(p-1)q_0}{q-q_0}$. Furthermore, by (2.7), we see that

$$W \in L^{\frac{(p-1)q_0}{q-q_0}}(\mathbb{R}^d).$$

(4.48)

Then, by (2.44), (2.42), Lemma 3.1, Hölder’s inequality, (4.48) and $\alpha(t_1) \leq \alpha(t_2)$, we see that

$$\|(-\Delta + s)^{-1} E(\eta_1, \eta_2)\|_{L^q} \leq s^{\frac{d(\frac{1}{p} - \frac{1}{q})}{2q-d}} \|W\|_{L^{\frac{(p-1)q_0}{q-q_0}}} \|\eta_1 - \eta_2\|_{L^{\frac{p}{2q}}} + s^{\frac{d(p-1)}{2q}} \|\|\eta_1\| + |\eta_2|\|^{p-1}\|\eta_1 - \eta_2\|_{L^q}$$

(4.51)

$$\|(-\Delta + s)^{-1} E(\eta_1, \eta_2)\|_{L^q} \leq s^{\frac{d}{2q}} \|W\|_{L^{\frac{(p-1)q_0}{q-q_0}}} \|\eta_1 - \eta_2\|_{L^q} + s^{\frac{d(p-1)}{2q}} \|\|\eta_1\| + |\eta_2|\|^{p-1}\|\eta_1 - \eta_2\|_{L^q}$$

(4.49)

where the implicit constants depend only on $d$, $p$, $q$ and $\varepsilon_0$. Note that we may choose $\varepsilon_0 = \frac{(d-2)(p-1)}{4}$ in (4.49).

Next, assume that $\frac{d}{d-2} \leq p < \frac{d+2}{d-2}$. Observe from the assumptions about $p$ and $q$ that

$$\frac{(p-1)q}{2q-d} > \frac{d}{d-2} \geq \frac{d}{2}, \quad \frac{d(p-1)}{2q} < 1, \quad p < q.$$

(4.50)

Then, a computation similar to (4.49) shows that

$$\|(-\Delta + s)^{-1} E(\eta_1, \eta_2)\|_{L^q} \leq s^{\frac{d}{2q}} \|W\|_{L^{\frac{(p-1)q_0}{q-q_0}}} \|\eta_1 - \eta_2\|_{L^q} + s^{\frac{d(p-1)}{2q}} \|\|\eta_1\| + |\eta_2|\|^{p-1}\|\eta_1 - \eta_2\|_{L^q}$$

(4.52)

where the implicit constants depend only on $d$, $p$ and $q$.

Putting (4.49) with $\varepsilon_0 = \frac{(d-2)(p-1)}{4}$, and (4.51), we find that (4.52) holds.

The following lemma follows immediately from Lemma 4.1, Lemma 4.2, (2.40) and (2.39):

**Lemma 4.3.** Assume $d = 3, 4$ and $\frac{4}{d-2} - 1 < p < \frac{d+2}{d-2}$. Let $2^* < q < \infty$.

1. For any $R > 0$, $s > 0$, $t > 0$, $0 < t_0 < 1$ and $\eta \in Y_q(R, t_0)$, the following holds:

$$\|(-\Delta + s)^{-1} N(\eta; t)\|_{L^q} \leq R^2 \alpha(t_0)^{2\Theta_4} + s^{\frac{2^*}{q'-1}} R^d \alpha(t_0)^{\frac{d+2}{d-2} \Theta_4}$$

$$+ t \left\{ s^{-\Theta_4} R \alpha(t_0)^{\Theta_4} + s^{\frac{d(p-1)}{2q}} R^d \alpha(t_0)^{p\Theta_4} \right\},$$

where the implicit constant depends only on $d$, $p$ and $q$. 

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2. For any $R > 0$, $s > 0$, $t > 0$, $0 < t_1 \leq t_2 < 1$, $\eta_1 \in Y_q(R, t_1)$ and $\eta_2 \in Y_q(R, t_2)$, the following holds:

$$\|(-\Delta + s)^{-1} \{N(\eta_1; t) - N(\eta_2; t)\}\|_{L^q} \lesssim \{R\alpha(t_2)\Theta_q + s^{\frac{d}{2} - 1} R^{\frac{d}{p} - 1} \alpha(t_2) \Theta_q\} \|\eta_1 - \eta_2\|_{L^q}$$

(4.53)

where the implicit constant depends only on $d$, $p$ and $q$.

**Lemma 4.4.** Assume $d = 3, 4$. Then, there exists $0 < T < 1$ depending only on $d$ such that if $0 < t_1 \leq t_2 < T$, then

$$|\alpha(t_2) - \alpha(t_1)| \leq 2\delta(\alpha(t_2)|t_2 - t_1|,$$

(4.54)

$$|\delta(\alpha(t_2)) - \delta(\alpha(t_1))| \leq (1 + \alpha_1(1)) t_1^{-1} \delta(\alpha(t_1)) \delta(\alpha(t_2))^{d-3}|t_2 - t_1|.$$  

(4.55)

**Proof of Lemma 4.4.** We may assume that $d = 4$, as the case $d = 3$ ($\alpha(t) = t^2$) is almost trivial. Thus, $\delta(s) = \frac{1}{\log(1 + s^{-1})}$. Note that $\frac{d\delta}{ds}(s) = \{(s + s^2) \log^2(1 + s^{-1})\}^{-1}$ and $(s + s^2) \log^2(1 + s^{-1})$ is strictly increasing on $(0, e^{-2})$.

Let $0 < t_1 \leq t_2 < 1$. Then, by the fundamental theorem of calculus and (2.32), we see that if $t_2 \leq \beta(e^{-2})$ (hence $\alpha(t_2) \leq e^{-2}$), then

$$\delta(\alpha(t_2)) - \delta(\alpha(t_1)) = \int_{\alpha(t_1)}^{\alpha(t_2)} \frac{1}{(s + s^2) \log^2(1 + s^{-1})} ds$$

$$\leq \frac{\alpha(t_2) - \alpha(t_1)}{\alpha(t_1) + \alpha(t_1)^2 \log^2(1 + \alpha(t_1)^{-1})}$$

$$\leq \alpha(t_1)^{-1} \delta(\alpha(t_1))^2 \{\alpha(t_2) - \alpha(t_1)\} = t_1^{-1} \delta(\alpha(t_1)) \{\alpha(t_2) - \alpha(t_1)\}.$$  

(4.56)

Furthermore, by (2.32), we see that

$$\alpha(t_2) - \alpha(t_1) = \delta(\alpha(t_2))(t_2 - t_1) + \{\delta(\alpha(t_2)) - \delta(\alpha(t_1))\} t_1.$$  

(4.57)

Plugging (4.56) into (4.57), we see that

$$\alpha(t_2) - \alpha(t_1) \leq \delta(\alpha(t_2))(t_2 - t_1) + \delta(\alpha(t_1)) \{\alpha(t_2) - \alpha(t_1)\},$$

(4.58)

which implies that

$$\alpha(t_2) - \alpha(t_1) \leq (1 + \alpha_1(1)) \delta(\alpha(t_2))(t_2 - t_1).$$

(4.59)

If $t_1$ is sufficiently small depending only on $d$, then (4.59) yields (4.54). Furthermore, plugging (4.54) into (4.56), we obtain (4.55).  

Now, observe from Lemma 3.15, Lemma 3.16 and $\lim_{s \to 0} \delta(s) = 0$ if $s > 0$ is sufficiently small depending only on $d$ and $p$, then

$$|\delta(s)\langle(-\Delta + s)^{-1}W, V\Lambda W\rangle| \sim A_1 \sim 1,$$

(4.60)

$$|\langle(-\Delta + s)^{-1}W^p, V\Lambda W\rangle| \sim K_p \sim 1,$$

(4.61)

where the implicit constants depend only on $d$ and $p$.  

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Lemma 4.5. Assume $d = 3, 4$. Then, the following holds for all $0 < s_1 \leq s_2 < 1$:

$$
\left| \delta(s_1)((-\Delta + s_1)^{-1}W, V\Lambda W) - \delta(s_2)((-\Delta + s_2)^{-1}W, V\Lambda W) \right| \lesssim \log (1 + s_2^{-1})|s_1 - s_2| + |\delta(s_1) - \delta(s_2)|,
$$

(4.62)

where the implicit constant depends only on $d$.

Proof of Lemma 4.5. Since $-\Delta W = W^{\frac{d+2}{d-2}}$, we may write the Fourier transform of $W$ as

$$
\mathcal{F}[W](\xi) = |\xi|^{-2}\mathcal{F}[W^{\frac{d+2}{d-2}}](\xi).
$$

(4.63)

Furthermore, using the spherical coordinate system, we can verify that

$$
\left| \int_{|\xi| \leq 1} \frac{\delta(s_1)}{(|\xi|^2 + s_1)|\xi|^2} d\xi - \int_{|\xi| \leq 1} \frac{\delta(s_2)}{(|\xi|^2 + s_2)|\xi|^2} d\xi \right| \lesssim |\delta(s_1) - \delta(s_2)| \quad \text{if } d = 3,
$$

(4.64)

$$
= 0 \quad \text{if } d = 4.
$$

Next, we define a function $g$ as

$$
g := \mathcal{F}[V\Lambda W]^{\frac{d}{d-2}} = \mathcal{F}[V\Lambda W \ast W^{\frac{d+2}{d-2}}].
$$

(4.65)

Observe from $|V\Lambda W(x)| \sim (1 + |x|)^{-(d+2)}$ and $|W^{\frac{d+2}{d-2}}(x)| \sim (1 + |x|)^{-(d+2)}$ that

$$
|V\Lambda W \ast W^{\frac{d+2}{d-2}}(x)| \lesssim (1 + |x|)^{-(d+2)}.
$$

(4.66)

Then, by (4.66) and (4.66), we see that

$$
|g(\xi) - g(0) - \xi \cdot \nabla g(0)| = \left| \int_{\mathbb{R}^d} (e^{-ix \cdot \xi} - 1 + i\xi \cdot x)\mathcal{F}^{-1}[g](x) \, dx \right|
$$

$$
\lesssim \int_{\mathbb{R}^d} \min\{|x||\xi|, |x|^2|\xi|^2\} |V\Lambda W \ast W^{\frac{d+2}{d-2}}(x)| \, dx
$$

(4.67)

$$
\lesssim \int_{\mathbb{R}^d} \min\{|x||\xi|, |x|^2|\xi|^2\}(1 + |x|)^{-(d+2)} \, dx.
$$

Now, we shall derive the desired estimate (4.62). We see from Parseval’s identity, $(-\Delta + s_j) = \mathcal{F}[(|\xi|^2 + s_j)]\mathcal{F}$, (4.63) and (4.64) that

$$
\left| \delta(s_1)\langle R(s_1)W, V\Lambda W \rangle - \delta(s_2)\langle R(s_2)W, V\Lambda W \rangle \right|
$$

$$
= \left| \left\langle \{\delta(s_1)R(s_1) - \delta(s_2)R(s_2)\} V\Lambda W, W \right\rangle \right|
$$

$$
\lesssim \left| \int_{\mathbb{R}^d} \left\{ \frac{\delta(s_1)}{(|\xi|^2 + s_1)|\xi|^2} - \frac{\delta(s_2)}{(|\xi|^2 + s_2)|\xi|^2} \right\} g(\xi) \, d\xi \right|
$$

$$
\lesssim \delta(s_1) \left| \int_{|\xi| \leq 1} \left\{ \frac{g(\xi) - g(0)}{(|\xi|^2 + s_1)|\xi|^2} - \frac{g(\xi) - g(0)}{(|\xi|^2 + s_2)|\xi|^2} \right\} \, d\xi \right| + |\delta(s_1) - \delta(s_2)||g(0)|
$$

(4.68)

$$
+ |\delta(s_1) - \delta(s_2)| \int_{|\xi| \leq 1} \frac{g(\xi) - g(0)}{(|\xi|^2 + s_2)|\xi|^2} \, d\xi
$$

$$
+ \int_{1 \leq |\xi|} \left| \frac{\delta(s_1)}{(|\xi|^2 + s_1)|\xi|^2} - \frac{\delta(s_2)}{(|\xi|^2 + s_2)|\xi|^2} \right| |g(\xi)| \, d\xi.
$$
We move on to the third term on the right-hand side of (4.68). We see from (3.44), (4.66), and Lemma C.2 in the appended section C that
\[
\delta(s_1) \left| \int_{|\xi| \leq 1} \left\{ \frac{g(\xi) - g(0)}{(|\xi|^2 + s_1)|\xi|^2} - \frac{g(\xi) - g(0)}{(|\xi|^2 + s_2)|\xi|^2} \right\} d\xi \right| \\
= \delta(s_1) |s_1 - s_2| \int_{|\xi| \leq 1} \frac{g(\xi) - g(0) - \xi \cdot \nabla g(0)}{(|\xi|^2 + s_1)(|\xi|^2 + s_2)|\xi|^2} d\xi \\
\lesssim \delta(s_1) |s_1 - s_2| \int_{\mathbb{R}^d} \int_{|\xi| \leq 1} \min\{ |x||\xi|, |x|^2|\xi|^2 \} \left( 1 + |x| \right)^{-(d+2)} d\xi dx \\
\lesssim |s_1 - s_2| \log (1 + s_2^{-2}).
\] (4.69)

Next, we consider the second term on the right-hand side of (4.68). We see from (4.66) that
\[
|\delta(s_1) - \delta(s_2)||g(0)| \leq |\delta(s_1) - \delta(s_2)| \int_{\mathbb{R}^d} |V A W * W_{\frac{d+2}{2}}^2(x)| dx \\
\lesssim |\delta(s_1) - \delta(s_2)| \int_{\mathbb{R}^d} (1 + |x|)^{-(d+2)} dx \lesssim |\delta(s_1) - \delta(s_2)|.
\] (4.70)

We move on to the third term on the right-hand side of (4.68). We see from (3.44) and (4.67) that
\[
|\delta(s_1) - \delta(s_2)| \left| \int_{|\xi| \leq 1} \frac{g(\xi) - g(0)}{(|\xi|^2 + s_1)|\xi|^2} d\xi \right| \\
= |\delta(s_1) - \delta(s_2)| \left| \int_{|\xi| \leq 1} \frac{g(\xi) - g(0) - \xi \cdot \nabla g(0)}{(|\xi|^2 + s_2)|\xi|^2} d\xi \right| \\
\lesssim |\delta(s_1) - \delta(s_2)| \int_{|\xi| \leq 1} \int_{\mathbb{R}^d} \left( 1 + |x| \right)^{-(d+2)} d\xi dx \lesssim |\delta(s_1) - \delta(s_2)|.
\] (4.71)

It remains to estimate the last term on the right-hand side of (4.68). Note that \( g \) is radial. Furthermore, it follows from Remark 3.2 and (4.66) that
\[
|\xi| \|g(\xi)\| \lesssim \|g\|_{L^2}^{\frac{d+2}{2}} \|\nabla g\|_{L^2}^{\frac{d-2}{2}} = \|V A W * W_{\frac{d+2}{2}}^2\|_{L^2}^{\frac{d+2}{2}} \|\|V A W * W_{\frac{d+2}{2}}^2\|_{L^2}^{\frac{d-2}{2}} \lesssim 1.
\] (4.72)

Hence, we see from (4.72) that
\[
\int_{1 \leq |\xi|} \left| \frac{\delta(s_1)}{|\xi|^2 + s_1)|\xi|^2} - \frac{\delta(s_2)}{|\xi|^2 + s_2)|\xi|^2} \right| |g(\xi)| d\xi \\
\leq \delta(s_1) \int_{1 \leq |\xi|} \left| \frac{1}{(|\xi|^2 + s_1)|\xi|^2} - \frac{1}{(|\xi|^2 + s_2)|\xi|^2} \right| |g(\xi)| d\xi \\
+ |\delta(s_1) - \delta(s_2)| \int_{1 \leq |\xi|} \left| \frac{|g(\xi)|}{(|\xi|^2 + s_2)|\xi|^2} \right| d\xi \\
\leq \delta(s_1) \log (1 + s_2^{-2}) + \delta(s_1) - \delta(s_2) \int_{1 \leq |\xi|} \left| \frac{1}{|\xi^6|} \right| d\xi \\
\lesssim \delta(s_1) \log (1 + s_2^{-2}) + \delta(s_1) - \delta(s_2) |s_1 - s_2|.
\] (4.73)
Putting the estimates (4.68), (4.69), (4.70), (4.71) and (4.73) together, we obtain the desired estimate (4.62). □

**Lemma 4.6.** Assume $d = 3,4$ and $\frac{2}{d-2} < p < \frac{d+2}{d-2}$. Let $\max\{\frac{d}{(d-2)p}, \frac{d}{2}\} < r < \frac{d}{2}$. Then, the following holds for all $s_1, s_2 > 0$:

\[
\mathcal{N}_{\mathcal{Q}} - \mathcal{N}_{s_1} - \mathcal{N}_{s_2} \lesssim \frac{1}{s_1 - s_2},
\]

where the implicit constant depends only on $d, p$ and $r$.

**Proof of Lemma 4.6.** Observe from (2.13) that for any $R > d$ where the implicit constant depends only on $d$,

\[
\mathcal{N}_{\mathcal{Q}} = -\mathcal{N}_{s_1} - \mathcal{N}_{s_2}
\]

Then, by (4.75), Lemma 3.7, Hölder’s inequality and Lemma 3.4 we see that

\[
\begin{align*}
&\mathcal{N}_{\mathcal{Q}} - \mathcal{N}_{s_1} - \mathcal{N}_{s_2} \\
&= s_1(\mathcal{Q}, \mathcal{N}_{s_1} - \mathcal{N}_{s_2}) - s_2(\mathcal{Q}, \mathcal{N}_{s_1} - \mathcal{N}_{s_2}) \\
&\leq s_1(\mathcal{Q}, \mathcal{N}_{s_1} - \mathcal{N}_{s_2}) \\
&+ |s_1 - s_2||\mathcal{Q}||\mathcal{N}_{s_1} - \mathcal{N}_{s_2}|
\end{align*}
\]

where the implicit constant depends only on $d, p$ and $r$. This proves the lemma. □

**Lemma 4.7.** Assume $d = 3,4$ and $\frac{4}{d-2} - 1 < p < \frac{d+2}{d-2}$. Let $2^\ast < q < \infty$. Then, for any $R > 0, s_1, s_2 > 0, t > 0, 0 < t_1 \leq t_2 < 1, \eta_1 \in \mathcal{Y}_q(R, t_1)$ and $\eta_2 \in \mathcal{Y}_q(R, t_2)$, the following holds:

\[
\begin{align*}
&\mathcal{N}_{\mathcal{Q}}(\eta_1, t) - \mathcal{N}_{\mathcal{Q}}(\eta_2, t) \\
&\lesssim s_1^{-\frac{d}{q-1}} \left\{ R^2 \alpha(t_1)^{2q} + s_2^{-\frac{d}{q-1}} \frac{\alpha(t_1)}{s_2^{\frac{4}{q-2}}} \right\} |s_1 - s_2| \\
&+ t s_1^{-\frac{d}{q-1}} \left\{ s_2^{-\frac{d}{q-1}} \alpha(t_1)^{p} + s_2^{-\frac{d}{q-1}} \frac{\alpha(t_1)}{s_2^{\frac{4}{q-2}}} \right\} |s_1 - s_2| \\
&+ \left\{ \frac{R^4}{d} \alpha(t_2)^{p} + s_2^{-\frac{d}{q-1}} \frac{\alpha(t_2)}{s_2^{\frac{4}{q-2}}} \right\} ||\eta_1 - \eta_2||_{L^p}
\end{align*}
\]

where the implicit constant depends only on $d, p$ and $q$. 

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Remark 4.4. Observe from an elementary computation and Lemma 3.7 that

\[
\langle (-\Delta + s_1)^{-1} N(\eta_1; t), V\Lambda W \rangle - \langle (-\Delta + s_2)^{-1} N(\eta_2; t), V\Lambda W \rangle
\]

\[
= -(s_1 - s_2) \langle (-\Delta + s_1)^{-1} (-\Delta + s_2)^{-1} N(\eta_1; t), V\Lambda W \rangle
\]

\[
+ \langle (-\Delta + s_2)^{-1} \{N(\eta_1; t) - N(\eta_2; t)\}, V\Lambda W \rangle.
\]

Proof of Lemma 4.8. Assume \(d = 3, 4\) and \(\frac{4}{d-2} - 1 < p < \frac{d+2}{d-2}\). Let \(\max\{2^*, \frac{2}{p-1}\} < q < \infty\) and \(R > 0\). Then, there exists \(0 < T(q, R) < 1\) depending only on \(d, p, q\) and \(R\) such that for any \(0 < t < T(q, R)\), \((\tau, \eta) \in I(t) \times Y_q(R, t)\) and \(\max\{1, \frac{d}{(d-2)p}\} < r < \frac{d}{2}\), the following estimate holds:

\[
\left| \varrho(t; \tau, \eta) - \frac{K_p}{A_1} t \right| \lesssim t \left\{ \alpha(t)^{\frac{d}{2r} - 1} + \delta(\alpha(t)) + \alpha(t)^{\theta q} \right\},
\]

where the implicit constant depends only on \(d, p, q\) and \(r\). Furthermore, applying Lemma 4.3 to the right-hand side of (4.79), we find that the claim of the lemma is true. 

**Lemma 4.8.** Assume \(d = 3, 4\) and \(\frac{4}{d-2} - 1 < p < \frac{d+2}{d-2}\). Let \(\max\{2^*, \frac{2}{p-1}\} < q < \infty\) and \(R > 0\). Then, there exists \(0 < T(q, R) < 1\) depending only on \(d, p, q\) and \(R\) such that for any \(0 < t < T(q, R)\), \((\tau, \eta) \in I(t) \times Y_q(R, t)\) and \(\max\{1, \frac{d}{(d-2)p}\} < r < \frac{d}{2}\), the following estimate holds:

\[
\left| \varrho(t; \tau, \eta) - \frac{K_p}{A_1} t \right| \lesssim t \left\{ \alpha(t)^{\frac{d}{2r} - 1} + \delta(\alpha(t)) + \alpha(t)^{\theta q} \right\},
\]

where the implicit constant depends only on \(d, p, q\) and \(r\).

**Remark 4.4.** Fix \(\max\{1, \frac{d}{(d-2)p}\} < r < \frac{d}{2}\), so that the dependence of a constant on \(r\) can be absorbed into that on \(d\) and \(p\). Then, Lemma 4.8 shows that if \(0 < t < 1\) is sufficiently small depending only on \(d, p, q\) and \(R\), then

\[
\varrho(t; \tau, \eta) \in I(t).
\]

Proof of Lemma 4.8. Let \(\max\{1, \frac{d}{(d-2)p}\} < r < \frac{d}{2}\). By (4.60), Hölder’s inequality, Lemma 3.16 and Lemma 3.15, we see that for any sufficiently small \(t > 0\) depending only on \(d\) and \(p\), and any \((\tau, \eta) \in I(t) \times Y_q(R, t)\),

\[
\left| \varrho(t; \tau, \eta) - \frac{K_p}{A_1} \right| = \left| \frac{tW_p(\tau) + N(t, \tau, \eta)}{\lambda(\tau)} - \frac{K_p}{A_1} \right|
\]

\[
\lesssim t \left| \frac{tA_1W_p(\tau) - K_p\lambda(\tau)}{A_1\lambda(\tau)} \right| + \left| \frac{N(t, \tau, \eta)}{\lambda(\tau)} \right|
\]

\[
\lesssim t \left| A_1 \{W_p(\tau) - K_p\} - K_p\lambda(\tau) - A_1 \right| + |N(t, \tau, \eta)|
\]

\[
\lesssim t |A_1 \{W_p(\tau) - K_p\} - K_p\lambda(\tau) - A_1| + \|(-\Delta + \alpha(\tau))^{-1} N(\eta; t)\|_{L^q} \|V\Lambda W\|_{L^{\frac{q}{q-r}}}
\]

\[
\lesssim t \alpha(\tau)^{\frac{d}{2r} - 1} + t\delta(\alpha(\tau)) + \|(-\Delta + \alpha(\tau))^{-1} N(\eta; t)\|_{L^q},
\]

(4.82)
where the implicit constants depend only on $d$, $p$, $q$ and $r$. Note here that $\tau \in I(t)$ implies $\alpha(\tau) \sim \alpha(t)$. Hence, applying Lemma 4.3 to the right-hand side of (4.82) as $s = \alpha(\tau)$, and using (4.11), $\Theta_q < 1$ (hence $\frac{d}{2} - 1 + \frac{d+2}{2} \Theta_q > 2 \Theta_q$), and (4.19) with $\theta = \Theta_q$, we see that if $t > 0$ is sufficiently small depending only on $d$, $p$, $q$ and $R$, then

$$|tW_p(\tau) + N(t, \tau, \eta)| = \frac{K_p}{A_1}$$

$$\lesssim t\alpha(\tau)^{\frac{d}{2}-1} + t\delta(\alpha(\tau)) + R^2 \alpha(t)^{2\Theta_q} + \alpha(\tau)^{\frac{d}{2}+1} R^{d-1} \alpha(t)^{\frac{d}{2}+1} \Theta_q$$

$$+ t\left\{\alpha(\tau)^{-\nu_q} R\alpha(t)^{\Theta_q} + \alpha(\tau)^{\frac{d}{2}+1} R^\nu \alpha(t)^{\nu\Theta_q}\right\}$$

$$\lesssim t\alpha(t)^{\frac{d}{2}-1} + t\delta(\alpha(t)) + t\alpha(t)^{\frac{1}{2}(\Theta_q - \nu_q)} + t\alpha(t)^{\frac{1}{2}(\Theta_q - \nu_q) + \frac{(d-2)(p-1)}{2} - 1},$$

where the implicit constants depend only on $d$, $p$, $q$ and $r$. Thus, we have proved the lemma.

**Lemma 4.9.** Assume $d = 3, 4$ and $\frac{d}{2} - 1 < p < \frac{d+2}{2}$. Let max$\{2^s, \frac{2^s}{p}\} < q < \infty$ and $R > 0$. Then, there exists $0 < T(q, R) < 1$ depending only on $d$, $p$, $q$ and $R$ with the following property: Let $0 < t_1 \leq t_2 < T(q, R)$, and assume that $t_2 \leq 2t_1$. Furthermore, let $(\tau_1, \eta_1) \in I(t_1) \times Y_q(R, t_1)$ and $(\tau_2, \eta_2) \in I(t_2) \times Y_q(R, t_2)$. Then, the following holds for all max$\{1, \frac{d}{d-2}p\} < r < \frac{d}{2}$:

$$\left|g(t_2; \tau_2, \eta_2) - g(t_1; \tau_1, \eta_1) - \frac{K_p}{A_1} (t_2 - t_1)\right|$$

$$\lesssim \left\{\alpha(t_1)^{\frac{d}{2}-1} + \delta(\alpha(t_1)) + \alpha(t_1)^{\Theta_q}\right\}|t_1 - t_2|$$

$$+ \left\{\delta(\alpha(t_1))^{\frac{d}{2}-2} + \alpha(t_1)^{\frac{d}{2}-1} + \alpha(t_1)^{\frac{d}{2}}\right\} |\tau_1 - \tau_2| + t_1 \alpha(t_1)^{-\Theta_q + \nu_q} \|\eta_1 - \eta_2\|_{L^r},$$

where the implicit constant depends only on $d$, $p$, $q$ and $r$.

**Remark 4.5.** Fix max$\{1, \frac{d}{d-2}p\} < r < \frac{d}{2}$, so that the dependence of a constant on $r$ can be absorbed into that on $d$ and $p$. Then, we may write (4.84) as

$$\left|g(t_2; \tau_2, \eta_2) - g(t_1; \tau_1, \eta_1) - \frac{K_p}{A_1} (t_2 - t_1)\right|$$

$$\lesssim \alpha(t_1) \left\{|t_1 - t_2| + |\tau_1 - \tau_2|\right\} + t_1 \alpha(t_1)^{-\Theta_q + \nu_q} \|\eta_1 - \eta_2\|_{L^r},$$

where the implicit constant depends only on $d$, $p$ and $q$.

**Proof of Lemma 4.9.** First, observe from the assumptions about $t_1, t_2, \tau_1, \tau_2$ that if $t_1$ and $t_2$ are sufficiently small depending only on $d$ and $p$, then

$$\tau_1 \sim t_1 \sim \tau_2 \sim t_2, \quad \alpha(\tau_1) \sim \alpha(t_1) \sim \alpha(\tau_2) \sim \alpha(t_2),$$

where the implicit constants depend only on $d$ and $p$. 

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Now, observe from (4.11) that
\[
\left| s(t_2; \tau_2, \eta_2) - s(t_1; \tau_1, \eta_1) - \frac{K_p}{A_1}(t_2 - t_1) \right|
\leq |t_2 - t_1| \left| \frac{W_p(\tau_2)}{X(\tau_2)} - \frac{K_p}{A_1} \right| + \left| \frac{N(t_2, \tau_2, \eta_2) - N(t_1, \tau_2, \eta_2)}{X(\tau_2)} \right|
\leq |t_2 - t_1| \left| \frac{W_p(\tau_2)}{X(\tau_2)} - \frac{K_p}{A_1} \right| + \left| \frac{N(t_1, \tau_2, \eta_2)}{X(\tau_2)} - \frac{N(t_1, \tau_1, \eta_1)}{X(\tau_1)} \right|. \tag{4.87}
\]

We shall estimate each term on the right-hand side of (4.87) separately:

**Estimate for 1st term.** We shall derive an estimate for the first term on the right-hand side of (4.87). Let \( \max\{1, \frac{d}{(d-2)p}\} < r < \frac{d}{2} \). Then, by (4.60), Lemma 3.15, Lemma 3.16 and (4.86), we see that for any
\[
\left| \frac{W_p(\tau_2)}{X(\tau_2)} - \frac{K_p}{A_1} \right| \lesssim |A_1 W_p(\tau_2) - K_p X(\tau_2)|
\leq A_1 |((-\Delta + \alpha(\tau_2))^{-1}W, VAW) - K_p|
\leq K_p \delta(\alpha(\tau_2)) |A_1 \delta(\alpha(\tau_2))^{-1} - ((-\Delta + \alpha(\tau_2))^{-1}W, VAW)|
\lesssim \alpha(t_1)^{\frac{d}{2r}-1} + \delta(\alpha(t_1)),
\]
where the implicit constants depend only on \( d, p \) and \( r \). Thus, we find that if \( t_1 \) and \( t_2 \) are sufficiently small depending only on \( d, p \) and \( q \), then
\[
|t_2 - t_1| \left| \frac{W_p(\tau_2)}{X(\tau_2)} - \frac{K_p}{A_1} \right| \lesssim \{ \alpha(t_1)^{\frac{d}{2r}-1} + \delta(\alpha(t_1)) \} |t_1 - t_2|, \tag{4.89}
\]
where the implicit constants depend only on \( d, p \) and \( r \).

**Estimate for 2nd term.** Consider the second term on the right-hand side of (4.87).
We see from (2.69) that
\[
N(t_1, \tau_2, \eta_2) - N(t_2, \tau_2, \eta_2) = (t_1 - t_2) ((-\Delta + \alpha(\tau_2))^{-1}E(\eta_2, 0), VAW). \tag{4.90}
\]
Then, by (4.60), (4.90), Hölder’s inequality, Lemma 4.2 with \( s = \alpha(\tau_2) \), (4.86) and (4.19) with \( \theta = \theta_q \), we see that if \( t_1 \) and \( t_2 \) are sufficiently small depending only on \( d, p, q \) and \( R \), then
\[
\left| \frac{N(t_2, \tau_2, \eta_2) - N(t_1, \tau_2, \eta_2)}{X(\tau_2)} \right| \lesssim |N(t_1, \tau_2, \eta_2) - N(t_2, \tau_2, \eta_2)|
\lesssim |t_1 - t_2| \left\{ \alpha(t_2)^{-\nu_\eta} + \alpha(t_2)^{\frac{d(p-1)}{2q}-1}R^{p-1} \alpha(t_2)^{(p-1)\Theta_q} \right\} R \alpha(t_2)^{\Theta_q}
\lesssim \{ \alpha(t_1)^{\frac{1}{2}(\Theta_q - \nu_\eta)} + \alpha(t_1)^{\frac{1}{2}(\Theta_q + \frac{(d-2)(p-1)}{2} - 1)} \} |t_1 - t_2|,
\]
where the implicit constants depend only on \( d, p \) and \( q \).
Lemma 4.3, (4.86), $\Theta$

where the implicit constants depend only on Lemma 4.4, we see that

Consider the third term on the right-hand side of (4.87). Let

Estimate for 3rd term. Consider the third term on the right-hand side of (4.87). Let

Observe from an elementary computation and (4.60) that

Consider the first term on the right-hand side of (4.93). By Lemma 4.5, (4.86), Lemma 4.6, (4.86), Lemma 4.7, we see that

where the implicit constant depends only on $d$, $p$ and $r$.

Estimate for the last term. We shall derive an estimate for the last term on the right-hand side of (4.87), and finish the proof of the lemma.

Observe from an elementary computation and (4.60) that

\[
|N(t_1, \tau_2, \eta_2) - N(t_1, \tau_1, \eta_1)| \leq \left| \frac{X(\tau_1) - X(\tau_2)}{X(\tau_1)} \right| |N(t_1, \tau_2, \eta_2)| + \left| \frac{X(\tau_2)}{X(\tau_1)} \right| |N(t_1, \tau_1, \eta_1)|
\]

(4.93)

where the implicit constant depends only on $d$ and $p$.

Consider the first term on the right-hand side of (4.93). By Lemma 4.5, (4.86), Lemma 4.7, we see that

where the implicit constants depend only on $d$ and $p$. Furthermore, by Hölder’s inequality, Lemma 4.2, (4.86), $\Theta_q \leq 1$, (4.41), and (4.19) with $\theta = \theta_q$, we see that

(4.94)

(4.95)
where the implicit constants depend only on $d$, $p$ and $q$. Then, we find from (4.87), (4.89), (4.91), (4.92) and (4.99) together, we find that (4.84) holds. Thus, we have completed the proof. \qed
4.2 Proof of Proposition 4.1

In this section, we give a proof of Proposition 4.1. The following lemma plays an essential role to prove Proposition 4.1:

Lemma 4.10. Assume \( d = 3, 4 \) and \( \frac{4}{d-2} - 1 < p < \frac{d+2}{d-2} \). Let \( \max\{2^*, \frac{2^*}{p-1}\} < q < \infty \) and \( R > 0 \). Then, there exists \( 0 < T(q, R) < 1 \) depending only on \( d, p, q \), and \( R \) such that the following estimates hold:

1. Let \( 0 < t < T(q, R) \) and \( \eta \in Y_q(R, t) \). Furthermore, let \( \tau \in I(t) \) with \( \tau = s(t; \tau, \eta) \). Then,
   \[
   \|g(t; \tau, \eta)\|_{L^q} \lesssim \alpha(t)\Theta_q, \tag{4.100}
   \]
   where the implicit constant depends only on \( d, p \) and \( q \).

2. Let \( 0 < t_1 \leq t_2 < T(q, R) \) with \( t_2 - t_1 \leq 2t_1 \), and let \( \eta_1 \in Y_q(R, t_1) \) and \( \eta_2 \in Y_q(R, t_2) \). Furthermore, let \( \tau_1 \in I(t_1) \) with \( \tau_1 = s(t_1; \tau_1, \eta_1) \), and let \( \tau_2 \in I(t_2) \) with \( \tau_2 = s(t_2; \tau_2, \eta_2) \). Then,
   \[
   \|g(t_2; \tau_2, \eta_2) - g(t_1; \tau_1, \eta_1)\|_{L^q} \lesssim o_t(1)t_1^{-1}\alpha(t_1)\Theta_q|t_2 - t_1| + |t_2 - t_1| + \alpha(t_1)\Theta_q\|\eta_2 - \eta_1\|_{L^q}, \tag{4.101}
   \]
   where the implicit constant depends only on \( d, p \) and \( q \).

Proof of Lemma 4.10. By (2.36), \( \alpha(s(t; \tau, \eta)) = s(t; \tau, \eta)\delta(\alpha(s(t; \tau, \eta))) \) (see (2.32)), and (4.8) through (4.10) that

\[
((\Delta + \alpha(s(t; \tau, \eta)))^{-1}F(\eta; \alpha(s(t; \tau, \eta)), t), V\Lambda W) = -s(t; \tau, \eta)\chi(s(t; \tau, \eta)) + t\mathcal{W}_p(s(t; \tau, \eta)) + N(s(t; \tau, \eta)), \tag{4.102}
\]

which together with (4.11) implies that

\[
((\Delta + \alpha(\tau))^{-1}F(\eta; \alpha(\tau), t), V\Lambda W) = 0, \quad \text{provided} \quad \tau = s(t; \tau, \eta). \tag{4.103}
\]

Now, we shall prove (4.100):

Proof of (4.100). Let \( 0 < t < 1 \) be a sufficiently small number to be specified in the middle of the proof, dependently on \( d, p, q \), and \( R \). Furthermore, let \( \eta \in Y_q(R, t) \), and let \( \tau \in I(t) \) with \( \tau = s(t; \tau, \eta) \). Then, (3.2) in Proposition 3.1 together with (4.103) shows that

\[
\|g(t; \tau, \eta)\|_{L^q} \lesssim \|((\Delta + \alpha(\tau))^{-1}F(\eta; \alpha(\tau), t))\|_{L^q}, \tag{4.104}
\]

where the implicit constant depends only on \( d, p \) and \( q \). Furthermore, by Lemma 3.3, Lemma 3.5, (4.52) in Lemma 4.3, \( \tau \in I(t) \) (hence \( \tau \sim t \)), and \( \Theta_q < 1 \) (hence \( \frac{2^*}{q} - 1 + \frac{2}{p-1} \)) together with (4.104)
$d+2\Theta_q > 2\Theta_q$), we see that
\[
\|(-\Delta + \alpha(\tau))^{-1} F(\eta; \alpha(\tau), t)\|_{L^q}
\leq \alpha(\tau)\|(-\Delta + \alpha(\tau))^{-1} W\|_{L^q} + t\|(-\Delta + \alpha(\tau))^{-1} W^p\|_{L^q}
+ ||(-\Delta + \alpha(\tau))^{-1} N(\eta; t)\|_{L^q}
\lesssim \alpha(\tau)\|W\|_{L^\text{weak}} + t\|W^p\|_{L^\frac{d}{p}}
+ R^2 \alpha(t)^{2\Theta_q} + \alpha(\tau) R^{\frac{d+2}{4}} \alpha(t) R^{\frac{d+2}{2}}\Theta_q
+ t\left\{ \alpha(\tau)^{-\nu_q} R\alpha(t)^{\Theta_q} + \alpha(\tau) R^{\frac{(d-2)(p-1)}{2}}\Theta_q \right\}
\lesssim \alpha(t)^{\Theta_q} + (R^2 + R^{\frac{d+4}{2}})\alpha(t)^{2\Theta_q} + tR\alpha(t)^{\Theta_q-\nu_q} + tR\alpha(t)^{\Theta_q} + (d-2)(p-1)^{-1},
\]
where the implicit constants depend only on $d$, $p$, and $q$. Plugging (4.105) into (4.104), and then using (4.19) with $\theta = \Theta_q$, and (4.40), we see that if $t$ is sufficiently small depending only on $d$, $p$, and $R$, then
\[
\|g(t; \tau, \eta)\|_{L^q} \lesssim \alpha(t)^{\Theta_q} + \alpha(t)^{2\Theta_q-\Theta_q} + t\alpha(t)^{\Theta_q} \lesssim \alpha(t)^{\Theta_q},
\]
where the implicit constants depend only on $d$, $p$, and $q$. Thus, we find that (4.100) holds.

Next, we shall prove (4.101):
**Proof of (4.101):** Let $0 < t_1 \leq t_2 < 1$ with $t_2 \leq 2t_1$, and let $\eta_1 \in Y_q(R, t_1)$ and $\eta_2 \in Y_q(R, t_2)$. We will specify $t_1$ and $t_2$ in the middle of the proof, dependently on $d$, $p$, and $q$. Furthermore, let $\tau_1 \in I(t_1)$ with $t_1 = s(t_1; \tau_1, \eta_1)$, and let $\tau_2 \in I(t_2)$ with $\tau_2 = s(t_2; \tau_2, \eta_2)$. Observe from $t_1 \leq t_2 \leq 2t_1$, $\tau_1 \in I(t_1)$ and $\tau_2 \in I(t_2)$ that
\[
t_1 \sim \tau_1 \sim t_2 \sim \tau_2, \quad \alpha(t_1) \sim \alpha(\tau_1) \sim \alpha(t_2) \sim \alpha(\tau_2),
\]
where the implicit constants depend only on $d$ and $p$. By (4.103), we see that
\[
\langle (-\Delta + \alpha(\tau_1))^{-1} F(\eta_1; \alpha(\tau_1), t_1) - (-\Delta + \alpha(\tau_2))^{-1} F(\eta_2; \alpha(\tau_2), t_2), V AW \rangle = 0. \tag{4.108}
\]
Furthermore, Lemma 4.3 (see also Remark 4.3) together with $\tau_j = s(t_j; \tau_j, \eta_j)$ shows that if $t_1$ and $t_2$ are sufficiently small depending only on $d$, $p$, and $R$, then
\[
|\tau_2 - \tau_1| \lesssim o_1(1)|t_2 - t_1| + t_1 \alpha(t_1)^{-\Theta_q} \eta_2 - \eta_1 \|_{L^q}, \tag{4.109}
\]
where the implicit constant depends only on $d$, $p$, and $q$.

For notational convenience, we put
\[
A_j := 1 + (-\Delta + \alpha(\tau_j))^{-1} V \quad \text{for} \quad j = 1, 2. \tag{4.110}
\]
Observe from $A_2 - A_1 = \{(-\Delta + \alpha(\tau_2))^{-1} - (-\Delta + \alpha(\tau_1))^{-1}\}V$ that
\[
\|g(t_2; \tau_2, \eta_2) - g(t_1; \tau_1, \eta_1)\|_{L^q}
\leq \|A_1^{-1} \{(-\Delta + \alpha(\tau_1))^{-1} F(\eta_1; \alpha(\tau_1), t_1) - (-\Delta + \alpha(\tau_2))^{-1} F(\eta_2; \alpha(\tau_2), t_2)\}\|_{L^q}
+ \|A_1^{-1} \{(-\Delta + \alpha(\tau_2))^{-1} - (-\Delta + \alpha(\tau_1))^{-1}\}V g(t_2; \tau_2, \eta_2)\|_{L^q}. \tag{4.111}
\]
Then, the desired estimate \( (4.101) \) follows from the following estimates:

\[
\| A^{-1}_1 \{ (\Delta + \alpha(t_1))^{-1} F(\eta_1; \alpha(t_1), t_1) - (\Delta + \alpha(t_2))^{-1} F(\eta_2; \alpha(t_2), t_2) \} \|_{L^q} \\
\lesssim o_t(1) t_1^{-1} \alpha(t_1) \Theta(t_1) |t_1 - t_2| + |t_1 - t_2| + \alpha(t_1)^\gamma \| \eta_1 - \eta_2 \|_{L^q},
\]

(4.112)

where the implicit constant depends only on \( d, p \) and \( q \); and

\[
\| A^{-1}_1 \{ (\Delta + \alpha(t_1))^{-1} - (\Delta + \alpha(t_2))^{-1} \} V(g(t_2; \tau_2, \eta_2)) \|_{L^q} \\
\lesssim o_t(1) t_1^{-1} \alpha(t_1) \Theta(t_1) |t_2 - t_1| + \alpha(t_1)^\gamma \| \eta_2 - \eta_1 \|_{L^q},
\]

(4.113)

where the implicit constant depends only on \( d, p \) and \( q \).

It remains to prove (4.112) and (4.113).

**Proof of (4.112).** By (3.2) in Proposition 3.1 and (4.108), we see that

\[
\| A^{-1}_1 \{ (\Delta + \alpha(\tau_1))^{-1} F(\eta_1; \alpha(\tau_1), t_1) - (\Delta + \alpha(\tau_2))^{-1} F(\eta_2; \alpha(\tau_2), t_2) \} \|_{L^q} \\
\lesssim \| (\Delta + \alpha(\tau_1))^{-1} F(\eta_1; \alpha(\tau_1), t_1) - (\Delta + \alpha(\tau_2))^{-1} F(\eta_2; \alpha(\tau_2), t_2) \|_{L^q},
\]

(4.114)

where the implicit constant depends only on \( d, p \) and \( q \). Furthermore, we estimate the right-hand side of (4.114) as follows:

\[
\| (\Delta + \alpha(\tau_1))^{-1} F(\eta_1; \alpha(\tau_1), t_1) - (\Delta + \alpha(\tau_2))^{-1} F(\eta_2; \alpha(\tau_2), t_2) \|_{L^q} \\
\leq \| (\Delta + \alpha(\tau_1))^{-1} \left\{ F(\eta_1; \alpha(\tau_1), t_1) - F(\eta_2; \alpha(\tau_2), t_2) \right\} \|_{L^q} \\
+ \| (\Delta + \alpha(\tau_1))^{-1} - (\Delta + \alpha(\tau_2))^{-1} \| F(\eta_2; \alpha(\tau_2), t_2) \|_{L^q}.
\]

Consider the first term on the right-hand side of (4.115). Observe that \( \frac{p q}{d+2q} > \frac{d}{d+2} \) for \( p > \frac{4}{d-2} - 1 \) and \( q > \frac{2}{p-1} \). Hence, we see that

\[
\| W_p \|_{L^\frac{d}{d+2q}} \lesssim 1,
\]

(4.116)

where the implicit constants depend only on \( d, p \) and \( q \). Furthermore, observe from (4.10) that

\[
N(\eta_1; t_1) - N(\eta_2; t_2) = N(\eta_1; t_1) - N(\eta_2; t_1) + N(\eta_2; t_1) - N(\eta_2; t_2) \\
= N(\eta_1; t_1) - N(\eta_2; t_1) + (t_1 - t_2) E(\eta_2, 0).
\]

(4.117)

Then, by (2.30), (4.54) in Lemma 4.4 Lemma 3.4 \( \tau_1 \sim \tau_2 \sim t_1 \) (see (4.107)), Lemma 3.5 (4.110) and (4.111), we see that

\[
\| (\Delta + \alpha(\tau_1))^{-1} \left\{ F(\eta_1; \alpha(\tau_1), t_1) - F(\eta_2; \alpha(\tau_2), t_2) \right\} \|_{L^q} \\
\leq |\alpha(t_1) - \alpha(t_2)| \| (\Delta + \alpha(\tau_1))^{-1} W \|_{L^q} + |t_1 - t_2| \| (\Delta + \alpha(\tau_1))^{-1} W^p \|_{L^q} \\
+ \| (\Delta + \alpha(\tau_1))^{-1} \left\{ N(\eta_1; t_1) - N(\eta_2; t_2) \right\} \|_{L^q} \\
\lesssim \delta(\alpha(t_1)) |\tau_1 - \tau_2| |\alpha(t_1) \Theta(t_1)|^{-1} \| W \|_{L^\frac{d}{d+2q}} + |t_1 - t_2| \\
+ \| (\Delta + \alpha(\tau_1))^{-1} \left\{ N(\eta_1; t_1) - N(\eta_2; t_1) \right\} \|_{L^q} \\
+ |t_1 - t_2| \| (\Delta + \alpha(\tau_1))^{-1} E(\eta_2, 0) \|_{L^q},
\]

(4.118)
where the implicit constants depend only on $d$, $p$ and $q$. Recall from (2.36) that
\begin{equation}
\delta(\alpha(t_1)) = t_1^{-1} \alpha(t_1).
\tag{4.119}
\end{equation}

Then, by (4.109) and (4.110), we see that the first term on the right-hand side of (4.118) is estimated as follows:
\begin{equation}
\delta(\alpha(t_1))||_{t_1 - t_2|\alpha(t_1)^{\Theta_{\eta^{-1}}}W||_{L^p_{\text{weak}}} = o_1(1)\delta(\alpha(t_1))||_{t_1 - t_2|\alpha(t_1)^{\Theta_{\eta^{-1}}} + \delta(\alpha(t_1))t_1 \alpha(t_1)^{\Theta_{\eta^{-1}}}||_{\eta_2 - \eta_1||_{L^p}}
\tag{4.120}
\end{equation}
where the implicit constants depend only on $d$, $p$ and $q$. Applying Lemma 4.3 to the third term on the right-hand side of (4.118), and using (4.107), (4.40), (4.19) with $\delta = \delta_q$, and the definition of $\delta_q$ (see (4.39)), we see that
\begin{equation}
\|(-\Delta + \alpha(t_1))^{-1}\{N(\eta_1; t_1) - N(\eta_2; t_1)\}||_{L^p} = o_1(1)\delta(\alpha(t_1))||_{t_1 - t_2|\alpha(t_1)^{\Theta_{\eta^{-1}}}W||_{L^p_{\text{weak}}} = o_1(1)\delta(\alpha(t_1))||_{t_1 - t_2|\alpha(t_1)^{\Theta_{\eta^{-1}}} + \delta(\alpha(t_1))t_1 \alpha(t_1)^{\Theta_{\eta^{-1}}}||_{\eta_2 - \eta_1||_{L^p}}
\tag{4.121}
\end{equation}
where the implicit constants depend only on $d$, $p$ and $q$. Furthermore, using Lemma 4.2 (4.107), (4.40), (4.19) with $\delta = \delta_q$, and the definition of $\delta_q$ (see (4.39)), we see that the last term on the right-hand side of (4.118) is estimated as follows:
\begin{equation}
||t_1 - t_2||(-\Delta + \alpha(t_1))^{-1} E(\eta_2, 0)||_{L^p} = o_1(1)\delta(\alpha(t_1))||_{t_1 - t_2|\alpha(t_1)^{\Theta_{\eta^{-1}}}W||_{L^p_{\text{weak}}} = o_1(1)\delta(\alpha(t_1))||_{t_1 - t_2|\alpha(t_1)^{\Theta_{\eta^{-1}}} + \delta(\alpha(t_1))t_1 \alpha(t_1)^{\Theta_{\eta^{-1}}}||_{\eta_2 - \eta_1||_{L^p}}
\tag{4.122}
\end{equation}
where the implicit constants depend only on $d$, $p$ and $q$. Plugging (4.120), (4.121) and (4.122) into (4.118), we find that
\begin{equation}
||(-\Delta + \alpha(t_1))^{-1}\{F(\eta_1; \alpha(t_1), t_1) - F(\eta_2; \alpha(t_2), t_2)\}||_{L^p} = o_1(1)\delta(\alpha(t_1))||_{t_1 - t_2|\alpha(t_1)^{\Theta_{\eta^{-1}}}W||_{L^p_{\text{weak}}} = o_1(1)\delta(\alpha(t_1))||_{t_1 - t_2|\alpha(t_1)^{\Theta_{\eta^{-1}}} + \delta(\alpha(t_1))t_1 \alpha(t_1)^{\Theta_{\eta^{-1}}}||_{\eta_2 - \eta_1||_{L^p}}
\tag{4.123}
\end{equation}
where the implicit constants depend only on $d$, $p$ and $q$. Plugging (4.120), (4.121) and (4.122) into (4.118), we find that
\begin{equation}
||(-\Delta + \alpha(t_1))^{-1}\{F(\eta_1; \alpha(t_1), t_1) - F(\eta_2; \alpha(t_2), t_2)\}||_{L^p} = o_1(1)\delta(\alpha(t_1))||_{t_1 - t_2|\alpha(t_1)^{\Theta_{\eta^{-1}}}W||_{L^p_{\text{weak}}} = o_1(1)\delta(\alpha(t_1))||_{t_1 - t_2|\alpha(t_1)^{\Theta_{\eta^{-1}}} + \delta(\alpha(t_1))t_1 \alpha(t_1)^{\Theta_{\eta^{-1}}}||_{\eta_2 - \eta_1||_{L^p}}
\tag{4.124}
\end{equation}
where the implicit constants depend only on $d$, $p$ and $q$.
Move on to the second term on the right-hand side of (4.112). By Lemma 3.4, Lemma 3.5, Lemma 4.3, (4.116), (4.107), and \( \delta(t_1) = t_1^{-1} \alpha(t_1) \) (see (2.32), we see that

\[
\| \{ (-\Delta + \alpha(t_1))^{-1} - (-\Delta + \alpha(t_2))^{-1} \} F(\eta_2; \alpha(t_2), t_2) \|_{L^q} \\
\leq \| \alpha(t_1) - \alpha(t_2) \|_{L^\infty} \| (-\Delta + \alpha(t_1))^{-1} (-\Delta + \alpha(t_2))^{-1} F(\eta_2; \alpha(t_2), t_2) \|_{L^q} \\
\lesssim \delta(t_1) \| \tau_1 - \tau_2 \| \| (-\Delta + \alpha(t_1))^{-1} F(\eta_2; \alpha(t_2), t_2) \|_{L^q},
\]

where the implicit constants depend only on \( d, p, q \). Furthermore, by (2.33), Lemma 3.4, Lemma 3.5, Lemma 4.3, (4.116), (4.107), and (4.19) with \( \theta = \theta_q \), we see that

\[
\| (-\Delta + \alpha(t_1))^{-1} F(\eta_2; \alpha(t_2), t_2) \|_{L^q} \\
\lesssim \alpha(t_2) \| (-\Delta + \alpha(t_1))^{-1} W \|_{L^\infty} + t_2 \| (-\Delta + \alpha(t_1))^{-1} W \|_{L^q} \\
+ \| (-\Delta + \alpha(t_1))^{-1} N(\eta_2; t_2) \|_{L^q} \\
\lesssim \alpha(t_2) \alpha(t_1)^{\Theta_q-1} \| W \|_{L^{\frac{1}{\theta_q}}} + t_2 \| W \|_{L^{\frac{1}{\theta}}} \\
+ R^2 \alpha(t_2)^{2\theta_q} + \alpha(t_1)^{\frac{2^*}{q}-1} R^{\frac{d}{2}-2} \alpha(t_2)^{\frac{d}{2}-2} \Theta_q \\
+ t_2 \{ \alpha(t_1)^{-\nu} R \alpha(t_2)^{\Theta_q} + \alpha(t_1)^{\frac{d(p-1)}{2q}-1} R^n \alpha(t_2)^{\theta} \}
\]

(4.125)

where the implicit constants depend only on \( d, p, q \). Plugging (4.109) and (4.125) into (4.131), and using \( t_1 \leq \alpha(t_1)^{\Theta_q} \) (see (4.40)), we see that

\[
\| \{ (-\Delta + \alpha(t_1))^{-1} - (-\Delta + \alpha(t_2))^{-1} \} F(\eta_2; \alpha(t_2), t_2) \|_{L^q} \\
\lesssim t_1^{-1} \{ \alpha(t_1) \| t_2 - t_1 \| + t_1 \alpha(t_1)^{-\Theta_q+\theta_q} \| \eta_2 - \eta_1 \|_{L^q} \} \{ \alpha(t_1)^{\Theta_q} + t_1 + t_1 \alpha(t_1)^{\theta_q} \}
\]

(4.126)

where the implicit constants depend only on \( d, p, q \).

Now, putting (4.112), (4.113), (4.123) and (4.126) together, we find that (4.112) holds.

**Proof of (4.113).** For notational convenience, we put

\[
g_2 := g(t_2; \tau_2, \eta_2).
\]

(4.127)
In order to treat the singularity coming from $A_1^{-1}$, we will use the operator $\Pi$ introduced in Section 3.2. Then, by Lemma 3.7, (1.51) in Lemma 4.4 and (4.107), we see that

$$
\|A_1^{-1}\{(-\Delta + \alpha(t_2))^{-1} - (-\Delta + \alpha(t_1))^{-1}\} V g(t_2; \tau_2, \eta_2)\|_{L^q}
$$

$$
= |\alpha(t_1) - \alpha(t_2)| \|A_1^{-1}(-\Delta + \alpha(t_2))^{-1}(-\Delta + \alpha(t_1))^{-1} V g_2\|_{L^q}
$$

$$
\lesssim \delta(\alpha(t_1)) |\tau_1 - \tau_2| \|A_1^{-1}(-\Delta + \alpha(t_2))^{-1}(-\Delta + \alpha(t_1))^{-1} V g_2\|_{L^q},
(4.128)
$$

$$
\lesssim \delta(\alpha(t_1)) |\tau_1 - \tau_2| \|A_1^{-1} \Pi(-\Delta + \alpha(t_2))^{-1}(-\Delta + \alpha(t_1))^{-1} V g_2\|_{L^q}
$$

$$
+ \delta(\alpha(t_1)) |\tau_1 - \tau_2| \|A_1^{-1}(1 - \Pi)(-\Delta + \alpha(t_2))^{-1}(-\Delta + \alpha(t_1))^{-1} V g_2\|_{L^q},
$$

where the implicit constants depend only on $d$ and $q$. 

Consider the first term on the right-hand side of (4.128). By (3.1) in Proposition 3.1 and (4.107), we see that

$$
\delta(\alpha(t_1)) |\tau_1 - \tau_2| \|A_1^{-1} \Pi(-\Delta + \alpha(t_2))^{-1}(-\Delta + \alpha(t_1))^{-1} V g_2\|_{L^q}
$$

$$
\lesssim \delta(\alpha(t_1))^2 |\tau_1 - \tau_2| \|\Pi(-\Delta + \alpha(t_2))^{-1}(-\Delta + \alpha(t_1))^{-1} V g_2\|_{L^q},
(4.129)
$$

where the implicit constants depend only on $d$ and $q$. Observe from the definition of $\Pi$ (see (3.72)) and $V \Lambda W = \Delta \Lambda W = (-\Delta + \alpha(t_2))\Lambda W + \alpha(t_2)\Lambda W$ that

$$
\|\Pi(-\Delta + \alpha(t_2))^{-1}(-\Delta + \alpha(t_1))^{-1} V g_2\|_{L^q}
$$

$$
\lesssim \left|\langle(-\Delta + \alpha(t_2))^{-1}(-\Delta + \alpha(t_1))^{-1} V g_2, V \Lambda W\rangle\right|
$$

$$
\leq \left|\langle(-\Delta + \alpha(t_1))^{-1} V g_2, \Lambda W\rangle\right|
$$

$$
+ \left|\langle(-\Delta + \alpha(t_1))^{-1} V g_2, \alpha(t_2)(-\Delta + \alpha(t_2))^{-1} \Lambda W\rangle\right|,
(4.130)
$$

where the implicit constants depend only on $d$ and $q$. Applying Lemma 3.13 to the first term on the right-hand side of (4.130), and using $V(x) \lesssim (1 + |x|)^{-4}$ (see (2.10) and (4.107)), we see that

$$
\left|\langle(-\Delta + \alpha(t_1))^{-1} V g_2, V \Lambda W\rangle\right| \lesssim \delta(\alpha(t_1))^{-1} \|V g_2\|_{L^1} + \|g_2\|_{L^q} \lesssim \delta(\alpha(t_1))^{-1} \|g_2\|_{L^q},
(4.131)
$$

where the implicit constants depend only on $d$ and $q$. On the other hand, by Hölder’s inequality, Lemma 5.4, (4.107) and $V(x) \lesssim (1 + |x|)^{-4}$, the second term on the right-hand side of (4.130) is estimated as

$$
\left|\langle(-\Delta + \alpha(t_1))^{-1} V g_2, \alpha(t_2)(-\Delta + \alpha(t_2))^{-1} \Lambda W\rangle\right|
$$

$$
\lesssim \delta(\alpha(t_1))^{-1} V g_2 \|_{L^{\frac{d}{d-2}}} \|\alpha(t_2)\|_{L^{\frac{d}{d-2}}} \|\Lambda W\|_{L^{\frac{d}{d-2}}}
$$

$$
\lesssim \alpha(t_1)^{\frac{d-2}{d}} \frac{\frac{d}{d-2}-1}{\frac{d}{d-2}} \|V g_2\|_{L^1} \|\alpha(t_2)\|_{L^{\frac{d}{d-2}}} \|\Lambda W\|_{L^{\frac{d}{d-2}}}
$$

$$
\lesssim \alpha(t_1)^{\frac{d-2}{d}} \|V g_2\|_{L^1} \|\alpha(t_2)\|_{L^{\frac{d}{d-2}}} \|\Lambda W\|_{L^{\frac{d}{d-2}}}
$$

$$
\lesssim \alpha(t_1)^{\frac{d-2}{d}} \|V g_2\|_{L^1} \|\alpha(t_2)\|_{L^{\frac{d}{d-2}}} \|\Lambda W\|_{L^{\frac{d}{d-2}}}
$$

$$
\lesssim \alpha(t_1)^{\frac{d-2}{d}} \|V g_2\|_{L^1} \|\alpha(t_2)\|_{L^{\frac{d}{d-2}}} \|\Lambda W\|_{L^{\frac{d}{d-2}}}
$$

(4.132)

$$
\lesssim \alpha(t_1)^{\frac{d-2}{d}} \|V g_2\|_{L^1} \|\alpha(t_2)\|_{L^{\frac{d}{d-2}}} \|\Lambda W\|_{L^{\frac{d}{d-2}}}
$$

$$
\lesssim \alpha(t_1)^{\frac{d-2}{d}} \|V g_2\|_{L^1} \|\alpha(t_2)\|_{L^{\frac{d}{d-2}}} \|\Lambda W\|_{L^{\frac{d}{d-2}}}
$$

$$
\lesssim \alpha(t_1)^{\frac{d-2}{d}} \|V g_2\|_{L^1} \|\alpha(t_2)\|_{L^{\frac{d}{d-2}}} \|\Lambda W\|_{L^{\frac{d}{d-2}}}
$$

$$
\lesssim \alpha(t_1)^{\frac{d-2}{d}} \|V g_2\|_{L^1} \|\alpha(t_2)\|_{L^{\frac{d}{d-2}}} \|\Lambda W\|_{L^{\frac{d}{d-2}}}
$$

$$
\lesssim \alpha(t_1)^{\frac{d-2}{d}} \|V g_2\|_{L^1} \|\alpha(t_2)\|_{L^{\frac{d}{d-2}}} \|\Lambda W\|_{L^{\frac{d}{d-2}}}
$$

(4.133)

$$
= |\tau_1 - \tau_2| \|t_1^{-1} + \delta(\alpha(t_1)) t_1^{-1} \alpha(t_1)^{\frac{d-2}{d}}\|_{L^q} \lesssim |\tau_1 - \tau_2| \|t_1^{-1} \alpha(t_1)^{\frac{d-2}{d}}\|_{L^q},
$$

47
where the implicit constants depend only on \(d\) and \(q\).

Move on to the second term on the right-hand side of (4.128). Observe from the definition of \(\Pi\) (see (3.72)) that
\[
\|1 - \Pi\|_{L^q \to L^p} \lesssim 1, \tag{4.134}
\]
where the implicit constant depends only on \(d\) and \(q\). Note that the functions appear in what follows are radial, and recall from Lemma 3.17 that \(X_q = (1 - \Pi)L^q_{rad}(\mathbb{R}^d)\). Then, by (3.2) in Proposition 3.1, (4.134), Lemma 3.5, Lemma 3.4, (4.107) and \(\delta(t_1) = t_1^{-1}\alpha(t_1)\) (see (2.32)), we see that
\[
\delta(\alpha(t_1))|\tau_1 - \tau_2||A_1^{-1}(1 - \Pi)(-\Delta + \alpha(\tau_2))^{-1}(-\Delta + \alpha(\tau_1))^{-1}V\mathfrak{g}_2|_{L^q}
\lesssim \|1 - \Pi\|_{L^q \to L^p} \lesssim 1, \tag{4.135}
\]
the implicit constants depend only on \(d\) and \(q\).

Plugging (4.133) and (4.135) into (4.128), and using (4.109) and (4.106), we see that
\[
\|A_1^{-1}\{-\Delta + \alpha(\tau_2)\} - \{\alpha(t_1)^{-1}(-\Delta + \alpha(\tau_1))\}V\mathfrak{g}(t_2, \tau_2, \eta_2)|_{L^q}
\lesssim |\tau_1 - \tau_2||t_1^{-1}\alpha(t_1)^{-1}V\mathfrak{g}_2|_{L^q} \lesssim |\tau_1 - \tau_2|t_1^{-1}\|\mathfrak{g}_2|_{L^q}
\lesssim \{o_1(1)|t_2 - t_1| + t_1\alpha(t_1)^{-\delta q}||\eta_2 - \eta_1|_{L^q}\}t_1^{-1}\|\mathfrak{g}_2|_{L^q}
\lesssim o_1(1)|t_1^{-1}\alpha(t_1)^{-\delta q}|t_2 - t_1| + \alpha(t_1)^{\delta q}||\eta_2 - \eta_1|_{L^q}, \tag{4.136}
\]
the implicit constant depends only on \(d, p\) and \(q\). Thus, we have proved (4.133) and completed the proof of the lemma.

Now, we are in a position to prove Proposition 4.4.

**Proof of Proposition 4.4.** First, we claim that if \(t > 0\) is sufficiently small depending only on \(d, p, q\) and \(R\), then for any \(\eta \in Y_q(R, t)\), the interval \(I(t)\) admits one and only one element \(\tau(t; \eta)\) such that
\[
\tau(t; \eta) = \mathfrak{s}(t; \tau(t; \eta), \eta). \tag{4.137}
\]
We prove this claim by applying the Banach fixed-point theorem to the mapping \(\tau \in I(t) \mapsto \mathfrak{s}(t; \tau, \eta), \) as in [8]. By Lemma 4.8 (see also Remark 4.4), we see that if \(t\) is sufficiently small depending only on \(d, p, q\) and \(R\), then for any \(\eta \in Y_q(R, t)\) and \(\tau \in I(t)\), \(\mathfrak{s}(t; \tau; \eta) \in I(t)\). Furthermore, applying Lemma 4.9 as \(t_1 = t_2 = t\) and \(\eta_1 = \eta_2 = \eta\) (see also Remark 4.6), we see that if \(t\) is sufficiently small depending only on \(d, p, q\) and \(R\), then for any \(\eta \in Y_q(R, t)\) and \(\tau_1, \tau_2 \in I(t)\),
\[
|\mathfrak{s}(t; \tau_2, \eta) - \mathfrak{s}(t; \tau_1, \eta)| \lesssim o_1(1)|\tau_1 - \tau_2|, \tag{4.138}
\]
where the implicit constant depends only on \(d, \ p\) and \(q\). Then, the Banach fixed-point theorem shows that the claim is true.

Next, we claim that if \(t > 0\) is sufficiently small depending only on \(d, \ p\) and \(R\), then \(Y_q(R, t)\) admits one and only one element \(\eta_t\) such that

\[
\eta_t = g(t; \tau(t; \eta_t), \eta_t),
\]

\((4.139)\)

where \(\tau(t; \eta)\) denotes the unique number in \(I(t)\) satisfying \((4.137)\). Indeed, applying Lemma 4.10 (use \((4.101)\) as \(t_j = t\) and \(\tau_j = \tau(t; \eta_j)\)), we see that the mapping \(\eta \in Y_q(R, t) \rightarrow g(t; \tau(t; \eta), \eta)\) is contraction in the metric induced from \(L^p(\mathbb{R}^d)\). Hence, the claim follows from the Banach fixed-point theorem.

By the above claims, we conclude that if \(t > 0\) is sufficiently small depending only on \(d, \ p, \ q\) and \(R\), then \(I(t) \times Y_q(R, t)\) admits at least one solution to \((4.20)\). Furthermore, the uniqueness of such a solution follows from Lemma 4.9 and Lemma 4.10; indeed, by these lemmas and \((4.40)\), we see that if \((\tau_1, \eta_1)\) and \((\tau_2, \eta_2)\) are solutions to \((4.20)\) in \(I(t) \times Y_q(R, t)\), then

\[
|\tau_t - \tau_0| + \|\eta_t - \eta_0\|_{L^p}
\]

\((4.140)\)

where the implicit constants depend only on \(d, \ p, \ q\). This estimate implies that if \(t > 0\) is sufficiently small depending only on \(d, \ p, \ q\) and \(R\), then the uniqueness holds.

Now, for a sufficiently small \(t > 0\), we use \((\tau_t, \eta_t)\) to denote the unique solution to \((4.20)\) in \(I(t) \times Y_q(R, t)\).

We shall show that \(\tau_t\) and \(\eta_t\) are continuous with respect to \(t\). Let \(t_0, t > 0\) be sufficiently small numbers with \(\frac{1}{2}t \leq t_0 \leq 2t\). Then, by Lemma 4.9 (see also Remark 4.5, 4.40) and Lemma 4.10, we see that

\[
|\tau_t - \tau_{t_0}| + \|\eta_t - \eta_{t_0}\|_{L^p}
\]

\((4.141)\)

where the implicit constant depends only on \(d, \ p, \ q\). Note that if \(t_0\) is sufficiently small, then the first and last terms on the right-hand side of \((4.141)\) can be absorbed into the left-hand side. Thus, \((4.141)\) implies the continuity of \(\tau_t\) and \(\eta_t\).

It remains to prove that \(\tau_t\) is strictly increasing with respect to \(t\). Let \(0 < t_1 < t_2\) be sufficiently small numbers to be specified later, depending on \(d, \ p, \ q\) and \(R\). By the continuity of \(\tau_t\) with respect to \(t\), we may assume that \(t_1\) and \(t_2\) are sufficiently close; in particular, we may assume \(t_2 \leq 2t_1\). Then, by \(\tau_{t_j} = s(t_j; \tau_{t_j}, \eta_{t_j})\), Lemma 4.9 (see also Remark 4.5, 4.40), we see that

\[
\tau_{t_2} - \tau_{t_1} \geq \frac{K_{s\theta}}{A_1} (t_2 - t_1) - a_1(1) |t_2 - t_1| - a_1(1) |\tau_{t_2} - \tau_{t_1}|
\]

\((4.142)\)

\[
- C_0 t_1 \Theta_{-\theta_\eta} \|\eta_{t_2} - \eta_{t_1}\|_{L^p},
\]
where \( C_0 > 0 \) is some constant depending only on \( d, p \) and \( q \). Furthermore, we find from Lemma 4.10 and \( \eta_j = g(t_j; \tau_j, \eta_j) \) that
\[
\| \eta_{t_2} - \eta_{t_1} \|_{L^q} \lesssim o_h(1) t_1^{-1} \alpha(t_1) |t_2 - t_1| + |t_2 - t_1|, \tag{4.143}
\]
where the implicit constant depends only on \( d, p \) and \( q \). Plugging (4.143) into (4.142), and using (4.40), we see that
\[
\tau_{t_2} - \tau_{t_1} \geq \frac{K_p}{A_1} (t_2 - t_1) - o_h(1) |t_2 - t_1| - o_h(1) \tau_{t_2} - \tau_{t_1} - C_0 o_h(1) \alpha(t_1) |t_2 - t_1| + C_0 \alpha(t_1) |t_2 - t_1|, \tag{4.144}
\]
where the implicit constant depends only on \( d, p \) and \( q \). This implies that \( \tau_t \) is strictly increasing with respect to \( t \). Thus, we have completed the proof of the proposition. \( \square \)

### 4.3 Proof of Proposition 4.2

In this section, we prove Proposition 4.2.

Recall that \( \tilde{S}_t \) and \( \tilde{N}_t \) denote the action and Nehari functional associated with (4.24), respectively (see (4.1)); precisely, we have:
\[
\tilde{S}_t(u) = \{ \Phi \} \quad \text{and} \quad \tilde{N}_t(u) = \{ \Phi \}.
\]
Furthermore, recall that \( \tilde{G}_t \) denotes the set of positive radial minimizers for (4.24).

The existence of a minimizer for (4.24) follows from that for (1.1) via the \( H^1 \)-scaling (see (4.1)); precisely, we have:

**Lemma 4.11.** Assume \( d = 3, 4 \) and \( \frac{1}{d-2} - 1 < p < \frac{d+2}{d-1} \). Let max \( \{ \frac{2^*}{p+2}, \frac{2^*}{p-1} \} < q < \infty \) and \( R > 0 \), and let \( T_1(q, R) \) be the number given in Proposition 4.7, hence for any \( 0 < t < T_1(q, R), I(t) \times Y(q, R, t) \) admits a unique solution to (4.20), say \( (\tau_t, \eta_t) \). Furthermore, let \( 0 < t < T_1(q, R) \), and define \( \lambda(t) \) and \( \omega(t) \) by
\[
\lambda(t)^{-2^*+p+1} = t, \quad \omega(t) = \alpha(\tau_t) \lambda(t)^{2^* - 2} = \alpha(\tau_t) t^{\frac{2^* - 2}{2^* - (p+1)}}. \tag{4.145}
\]
Then, the following hold:

1. Let \( \Phi \in \tilde{G}_{\omega(t)}; \) Note that by Proposition 4.1, \( \tilde{G}_{\omega(t)} \neq \emptyset \). Then, \( T_{\lambda(t)}[\Phi] \in \tilde{G}_t; \) Hence, \( \tilde{G}_t \neq \emptyset \).
2. Let \( u \in \tilde{G}_t \). Then, \( T_{\lambda(t)}^{-1}[u] \in \tilde{G}_{\omega(t)} \), and
\[
\tilde{S}_t(u) = \frac{1}{d} \| \nabla u \|_{L^2}^2 \leq \frac{1}{d} \| \nabla W \|_{L^2}^2. \tag{4.146}
\]

**Proof of Lemma 4.11.** We shall prove the first claim. Observe from a computation involving the scaling that
\[
\tilde{N}_t(T_{\lambda(t)}[\Phi]) = N_{\omega(t)}(\Phi) = 0, \tag{4.147}
\]
which implies that
\[
\tilde{S}_t(T_{\lambda(t)}[\Phi]) \geq \inf \{ \tilde{S}_t(u); u \in H^1(\mathbb{R}^d) \setminus \{0\}, \tilde{N}_t(u) = 0 \}. \tag{4.148}
\]
If the equality failed in (4.148), then we could take a function \( u_0 \in H^1(\mathbb{R}^d) \setminus \{0\} \) such that
\[
\tilde{S}_t(u_0) < \tilde{S}_t(T_{\lambda(t)}[\Phi]), \quad \tilde{N}_t(u_0) = 0. \tag{4.149}
\]
Furthermore, (4.149) together with computations involving the scaling shows that
\[ S_{\omega(t)}(T_{\lambda(t)}\{u_0\}) = \tilde{S}_t(u_0) \leq \tilde{S}_t(T_{\lambda(t)}[\Phi]) = S_{\omega(t)}(\Phi), \] (4.150)
\[ N_{\omega(t)}(T_{\lambda(t)}\{u_0\}) = \tilde{N}_t(u_0) = 0. \] (4.151)

However, (4.150) together with (4.151) contradicts \( \Phi \in \mathcal{G}_{\omega(t)} \). Thus, the equality must hold in (4.148), which together with (4.147) shows that \( T_{\lambda(t)}[\Phi]\in \mathcal{G}_t \).

Move on to the second claim. As well as the first claim, we can verify that \( T_{\lambda(t)}\{u\} \in \mathcal{G}_{\omega(t)} \). It remains to prove (4.154). Observe from computations involving the scaling that
\[ \tilde{S}_t(u) = S_{\omega(t)}(T_{\lambda(t)}\{u\}), \quad \|\nabla u\|_{L^2} = \|\nabla T_{\lambda(t)}\{u\}\|_{L^2}. \] (4.152)

Then, by Lemma 2.2 of \[2\], and (4.152), we find that (4.156) holds. \( \square \)

The following fact follows from the second claim in Lemma 4.11 (cf. Proposition 1.1):

**Corollary 4.1.** Assume \( d = 3,4 \) and \( \frac{4}{d-1} - 1 < p \leq \frac{d+2}{2} \). Let \( \max\{2^*, \frac{2^*}{p^*}\} < q < \infty \), \( R > 0 \) and let \( T_{1}(q,R) \) be the number given in Proposition 4.1; hence for any \( 0 < t < T_{1}(q,R) \), \( I(t) \times Y_{q}(R,t) \) admits a unique solution to (4.20), say \((\tau_t, \eta_t)\). Furthermore, let \( 0 < t < T_{1}(q,R) \) and \( u_t \in \mathcal{G}_t \). Then, \( u_t \) is a positive radial solution to (4.21). Furthermore, \( u_t \in C^2(\mathbb{R}^d) \cap H^2(\mathbb{R}^d) \), and \( u_t(x) \) is strictly decreasing as a function of \( |x| \).

Next, we give properties of ground states to (4.21) (Lemma 4.12 and Lemma 4.13):

**Lemma 4.12.** Assume \( d = 3,4 \) and \( \frac{4}{d-1} - 1 < p \leq \frac{d+2}{2} \). Let \( \max\{2^*, \frac{2^*}{p^*}\} < q < \infty \), \( R > 0 \) and let \( T_{1}(q,R) \) be the number given in Proposition 4.1; hence for any \( 0 < t < T_{1}(q,R) \), \( I(t) \times Y_{q}(R,t) \) admits a unique solution to (4.20), say \((\tau_t, \eta_t)\). Then, the following hold:
\[ \lim_{t \to 0} \alpha(\tau_t) = 0, \] (4.153)
\[ \lim_{t \to 0} \sup_{u_t \in \mathcal{G}_t} t\|u_t\|_{L^{p+1}} = \sup_{t \to 0} \alpha(\tau_t)\|u_t\|_{L^2}^2 = 0. \] (4.154)

**Proof of Lemma 4.12.** By (2.34), the assumption about \( p \) and \( \tau_t \in I(t) \) (hence \( \tau_t \sim t \)), we see that (4.153) holds.

We shall prove (4.154). Let \( 0 < t < T_{1}(q,R) \) be a number to be taken \( t \to 0 \), and let \( u_t \in \mathcal{G}_t \). Note that \( \mathcal{N}_t(u_t) = 0 \). Then, by (4.146) in Lemma 4.11 we see that
\[ \frac{1}{d}\|\nabla W\|^2_{L^2} \geq \tilde{S}_t(u_t) - \frac{1}{p+1}\tilde{N}_t(u_t) \]
\[ = \frac{p-1}{2(p+1)}\alpha(\tau_t)\|u_t\|_{L^2}^2 + \frac{p-1}{2(p+1)}\|\nabla u_t\|_{L^2}^2 + \frac{2^* - (p+1)}{2^*(p+1)}\|u_t\|_{L^{2^*}}^2. \] (4.155)

Furthermore, by Hölder’s inequality, we see that
\[ t\|u_t\|_{L^{p+1}} \leq \|u_t\|_{L^2}^{2(p^* - (p+1)) - 2} \|u_t\|_{L^{2^*}}^{2^* - (p+1)} \|u_t\|_{L^2}^{2^* - (p+1)} \|u_t\|_{L^{2^*}}^{2^* - (p+1)}. \] (4.156)

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Then, we find from (4.153), (4.155) and (4.156) that
\[
\lim_{t \to 0} \sup_{u \in \tilde{G}_t} t \|u\|_{L^{p+1}}^{p+1} = 0. \tag{4.157}
\]
Furthermore, a standard argument (see [6]) shows that the following “Pohozaev identity” holds for all \( u \in \tilde{G}_t \):
\[
\frac{1}{d^2} \alpha(\tau_t) \|u\|_{L^2}^2 = \frac{2^* - (p + 1)}{2^*(p + 1)} t \|u\|_{L^{p+1}}^{p+1}. \tag{4.158}
\]
Putting (4.157) and (4.158) together, we find that (4.154) is true.

**Lemma 4.13.** Assume \( d = 3, 4 \) and \( \frac{d}{d-2} - 1 < p < \frac{d+2}{d-2} \). Let \( \max\{2^*, \frac{2^*}{p-1}\} < q < \infty \), \( R > 0 \) and let \( T_1(q, R) \) be the number given in Proposition 4.1; hence for any \( 0 < t < T_1(q, R) \), \( I(t) \times Y_q(R, t) \) admits a unique solution to (4.20), say \((\tau_t, \eta_t)\). Then, all of the following hold:

1. \( \lim_{t \to 0} \sup_{u \in \tilde{G}_t} \|T_{u(0)}[u] - W\|_{H^1} = 0. \tag{4.159} \)

2. If \( 0 < t < T_1(q, R) \) is sufficiently small depending only on \( d, p, q \) and \( R \), then
\[
\sup_{u \in \tilde{G}_t} |T_{u(0)}[u](x)| \lesssim (1 + |x|)^{-(d-2)}, \tag{4.160}
\]
where the implicit constant depends only on \( d \) and \( p \); in particular, for any \( u \in \tilde{G}_t \) and \( \frac{d}{d-2} < r \leq \infty \),
\[
\|T_{u(0)}[u]\|_{L^r} \lesssim 1, \tag{4.161}
\]
where the implicit constant depends only on \( d, p \) and \( r \).

3. \( \lim_{t \to 0} \sup_{u \in \tilde{G}_t} u(0)^{-2^* + 2} \alpha(\tau_t) = 0. \tag{4.162} \)

**Remark 4.6.** By (4.159) and (4.161), we see that
\[
\lim_{t \to 0} \sup_{u \in \tilde{G}_t} \|T_{u(0)}[u] - W\|_{L^r} = 0 \quad \text{for all } \frac{d}{d-2} < r < \infty. \tag{4.163}
\]

**Proof of Lemma 4.13.** Let \( 0 < t < T_1(q, R) \) be a number to be taken \( t \to 0 \), and let \( u \in \tilde{G}_t \). Define \( \lambda(t) \) and \( \omega(t) \) as in (4.145). Furthermore, put \( \Psi_{\omega(t)} := T_{\lambda(t)-1}[u] \). Then, Lemma 4.11 shows that \( \Psi_{\omega(t)} \in \tilde{G}_{\omega(t)} \). Observe that
\[
T_{\Psi_{\omega(t)}[0]}[\Psi_{\omega(t)}] = T_{u(0)}[u]. \tag{4.164}
\]
Furthermore, observe from (4.153) in Lemma 4.12 that
\[
\lim_{t \to 0} \omega(t) = \lim_{t \to 0} \alpha(\tau_t) \frac{-(2^* - (p+1))}{2^* - 2} \left\{ \frac{(2^* - 2)}{2^* - (p+1)} \right\} \frac{(2^* - 2)}{2^* - (p+1)} = \infty \quad \text{as } d \to 0. \tag{4.165}
\]
Then, (4.158) follows from \( \Psi_{\omega(t)} \in \tilde{G}_{\omega(t)} \), (4.161) and Lemma 2.2. Furthermore, (4.160) follows from Lemma 2.3.
It remains to prove the last claim (4.162). Suppose for contradiction that the claim is false. Then, we can take a sequence \( \{t_n\} \) with the following property: \( \lim_{n \to \infty} t_n = 0 \), and for any \( n \geq 1 \) there exists \( u_n \in \mathcal{G}_I \) such that

\[
\lim_{n \to \infty} u_n(0)^{-2\alpha(t_n)} > 0. \tag{4.166}
\]

Observe from a computation involving the scaling, and (4.164) in Lemma 4.12 that

\[
\lim_{n \to \infty} u_n(0)^{-2\alpha(t_n)} \|T_{u_n(0)}[u_n]\|_{L^2}^2 = \lim_{n \to \infty} \alpha(t_n) \|u_n\|_{L^2}^2 = 0. \tag{4.167}
\]

By (4.166) and (4.167), we see that

\[
\lim_{n \to \infty} \|T_{u_n(0)}[u_n]\|_{L^2} = 0. \tag{4.168}
\]

Furthermore, by (4.159), (4.161) and (4.168), we see that

\[
\|W\|_{L^{2^*}} = \lim_{n \to \infty} \|T_{u_n(0)}[u_n]\|_{L^{2^*}} \leq \lim_{n \to \infty} \|T_{u_n(0)}[u_n]\|_{L^2}^{\frac{d-2}{2}} \|T_{u_n(0)}[u_n]\|_{L^\infty}^{\frac{2}{2}} \leq \lim_{n \to \infty} \|T_{u_n(0)}[u_n]\|_{L^2}^{\frac{d-2}{2}} = 0.
\]

This is a contradiction. Thus, the claim (4.162) is true.

We will prove Proposition 4.2 by showing that a minimizer for (4.24) coincides with the unique solution to (4.20) obtained in Proposition 4.1. The first step to accomplish this plan is the following lemma (cf. Lemma 3.8 in [7]):

**Lemma 4.14.** Assume \( d = 3, 4 \) and \( \frac{4}{d-2} - 1 < p < \frac{4}{d-2} + 1 \). Let \( \max\{2^*, \frac{2^*}{p+3-2^*}\} < q < \infty \). Furthermore, for \( R > 0 \), let \( T_1(q, R) \) denote the number given in Proposition 4.1. Then, there exist \( R_* > 0 \) and \( 0 < T_* < T_1(q, R_*) \), both depending only on \( d, p \) and \( q \), with the following property: For any \( 0 < t < T_* \) and \( u_t \in \mathcal{G}_I \), there exists \( \mu(t) = 1 + o_1(1) \) such that, denoting a unique solution to (4.20) in \( I(t) \times Y_q(R_*, t) \) by \( (\tau_1, \eta_1) \) (see Proposition 4.1), and defining \( \mu^t(t), t^t, \alpha^t(t) \) and \( \eta^t \) as

\[
\mu^t(t) := \mu(t) u_t(0), \tag{4.170}
\]

\[
t^t := \mu^t(t)^{-(2^* - (p+1))} t, \quad \alpha^t := \mu^t(t)^{-(2^* - 2)} \alpha(t), \tag{4.171}
\]

\[
\eta^t := T_{\mu^t(t)}[u_t] - W, \tag{4.172}
\]

we have the following:

1. \[
\langle (-\Delta + \alpha^t(t)\rangle^{-1} F(\eta^t, \alpha^t(t), t^t), VAW \rangle = 0, \tag{4.173}
\]

\[
\lim_{t \to 0} \sup_{u_t \in \mathcal{G}_I} \alpha^t(t) = 0, \quad \lim_{t \to 0} \sup_{u_t \in \mathcal{G}_I} t^t = 0, \tag{4.174}
\]

\[
\lim_{t \to 0} \sup_{u_t \in \mathcal{G}_I} \alpha^t(t) \frac{(2^* - (p+1))}{2^* - 2} t^t = 0, \tag{4.175}
\]

\[
\lim_{t \to 0} \|\eta^t\|_{H^1} = 0, \quad \lim_{t \to 0} \|\eta^t\|_{L^r} = 0 \quad \text{for all} \quad \frac{d}{d-2} < r < \infty. \tag{4.176}
\]
2. \[ (\beta(\alpha^1(t)), \eta^1_t) \in I(t^1) \times Y_q(R_s, t^1). \] (4.177)

3. If \( 0 < t < T_s \), then \( 0 < t^1 < T_1(q, R_s) \). Furthermore, let \( 0 < t < T_s \), and let \((\tau_t^1, \eta^1_t)\) denote a unique solution to (4.120) with \( t = t^1 \) in \( I(t^1) \times Y_q(R_s, t^1) \). Then, \[ (\beta(\alpha^1(t)), \eta^1_t) = (\tau_t^1, \eta_t^1). \] (4.178)

**Proof of Lemma 4.14.** Since the proof is long, we divide it into three steps:

**Step 1.** Let \( R > 0 \). Then, we shall show that there exists \( 0 < T_2(q, R) < T_1(q, R) \), depending only on \( d, q, p, q, R \), with the following property: For any \( 0 < t < T_2(q, R) \) and \( u_t \in \mathcal{G} \), there exists \( \mu(t) = 1 + \alpha_t(1) \) such that defining \( \mu^1(t), t^1, \alpha^1(t) \) and \( \eta_t^1 \) in the same manner as (4.14) through (4.17), we have (4.170) through (4.174).

First, observe from (4.162) in Lemma 4.13 that

\[ \lim_{t \to 0} \sup_{u_t \in \mathcal{G}} \{ \mu u_t(0) \}^{-2^{n-2}} \alpha(\tau_t) = 0. \] (4.179)

Furthermore, observe from (4.153) in Lemma 4.12 that

\[ \lim_{t \to 0} \sup_{u_t \in \mathcal{G}} \{ \mu u_t(0) \}^{-2^{n-2}} \alpha(\tau_t) = 0. \] (4.180)

Next, let \( 0 < T_2 < T_1(q, R) \) be a sufficiently small number to be specified in the middle of the proof, dependently on \( d, p, q, R \) (we will see that the dependence of \( T_2 \) on \( q, R \) from \( T_2 < T_1(q, R) \) only). Furthermore, let \( 0 < t < T_2 \), \( u_t \in \mathcal{G} \) and \( \frac{1}{2} < \mu < \frac{3}{2} \). By Corollary 4.1 and a computation involving the scaling, we see that

\[ -\Delta T_{\mu u_t(0)}[u_t] + \{ \mu u_t(0) \}^{-2^{n-2}} \alpha(\tau_t) T_{\mu u_t(0)}[u_t] \]
\[- \{ \mu u_t(0) \}^{-2^{n-2}} \alpha(\tau_t) T_{\mu u_t(0)}[u_t] = 0. \] (4.181)

Define \( t(\mu), \alpha_t(\mu) \) and \( \eta_t(\mu) \) as

\[ t(\mu) := \{ \mu u_t(0) \}^{-2^{n-2}} \alpha(\tau_t), \quad \alpha_t(\mu) := \{ \mu u_t(0) \}^{-2^{n-2}} \alpha(\tau_t), \] (4.182)

\[ \eta_t(\mu) := T_{\mu u_t(0)}[u_t] - W. \] (4.183)

By (4.179), we may assume that \( \alpha_t(\mu) < 1 \), \( \delta(\alpha_t(\mu)) < 1 \). (4.184)

Observe from (4.181) and (4.1) that

\[ (-\Delta + \alpha_t(\mu) + V) \eta_t(\mu) = F(\eta_t(\mu); \alpha_t(\mu), t(\mu)). \] (4.185)

Furthermore, observe from \((-\Delta + V) AW = 0 \) (see (2.13)) that

\[ V AW = \alpha_t(\mu) AW - (-\Delta + \alpha_t(\mu)) AW. \] (4.186)
By (4.185) and (4.186), we see that
\[ \langle (-\Delta + \alpha_\ell(\mu) )^{-1} F(\eta(\mu); \alpha_\ell(\mu), t(\mu)), V \Lambda W \rangle \]
\[ = \langle (-\Delta + \alpha_\ell(\mu) )^{-1}(-\Delta + \alpha_\ell(\mu) + V) \eta(\mu), V \Lambda W \rangle \]
\[ = \langle \eta(\mu), V \Lambda W \rangle + \langle (-\Delta + \alpha_\ell(\mu) )^{-1} V \eta(\mu), V \Lambda W \rangle \]
\[ = \langle (-\Delta + \alpha_\ell(\mu) )^{-1} V \eta(\mu), \alpha_\ell(\mu) \Lambda W \rangle = \langle \eta(\mu), \alpha_\ell(\mu) V (-\Delta + \alpha_\ell(\mu) )^{-1} \Lambda W \rangle. \]  
(4.187)

Note that \( \eta(\mu) \) can be written as
\[ \eta(\mu) = T_\mu [ T_{u(0)} [ u_t ] - W ] + T_\mu [ W ] - W. \]  
(4.188)

Plugging (4.188) into (4.187), we see that for any \( \mu > 0 \),
\[ \langle (-\Delta + \alpha_\ell(\mu) )^{-1} F(\eta(\mu); \alpha_\ell(\mu), t(\mu)), V \Lambda W \rangle \]
\[ = \langle T_\mu [ T_{u(0)} [ u_t ] - W ], \alpha_\ell(\mu) V (-\Delta + \alpha_\ell(\mu) )^{-1} \Lambda W \rangle \]
\[ + \langle T_\mu [ W ] - W + \frac{2}{d - 2} (\mu - 1) \Lambda W, \alpha_\ell(\mu) V (-\Delta + \alpha_\ell(\mu) )^{-1} \Lambda W \rangle \]
\[ - \frac{2}{d - 2} (\mu - 1) \langle \Lambda W, \alpha_\ell(\mu) V (-\Delta + \alpha_\ell(\mu) )^{-1} \Lambda W \rangle. \]  
(4.189)

Furthermore, by (2.18), we may rewrite (4.189) as
\[ \langle (-\Delta + \alpha_\ell(\mu) )^{-1} F(\eta(\mu); \alpha_\ell(\mu), t(\mu)), V \Lambda W \rangle \]
\[ = \langle T_\mu [ T_{u(0)} [ u ] - W ], \alpha_\ell(\mu) V (-\Delta + \alpha_\ell(\mu) )^{-1} \Lambda W \rangle \]
\[ + \frac{2}{d - 2} \int_1^\mu \int_1^\nu \lambda^{-2} T_\lambda [ 2 \Lambda W + \frac{2}{d - 2} x \cdot \nabla \Lambda W ] d\lambda d\nu, \alpha_\ell(\mu) V (-\Delta + \alpha_\ell(\mu) )^{-1} \Lambda W \rangle \]
\[ - \frac{2}{d - 2} (\mu - 1) \langle \Lambda W, \alpha_\ell(\mu) V (-\Delta + \alpha_\ell(\mu) )^{-1} \Lambda W \rangle. \]  
(4.190)

Observe from the duality, Lemma 3.13, (4.184) and \( \beta(s) := \delta(s)^{-1}s \) that
\[ \| \alpha_\ell(\mu) V (-\Delta + \alpha_\ell(\mu) )^{-1} \Lambda W \|_{L^{2\mu}_{x\sigma}} = \sup_{g \in L^{2\mu}_{x\sigma}} \| \langle \alpha_\ell(\mu) V (-\Delta + \alpha_\ell(\mu) )^{-1} \Lambda W, g \rangle \| \]
\[ = \alpha_\ell(\mu) \sup_{g \in L^{2\mu}_{x\sigma}} \| \langle (-\Delta + \alpha_\ell(\mu) )^{-1} V g, \Lambda W \rangle \| \]
\[ \lesssim \alpha_\ell(\mu) \sup_{g \in L^{2\mu}_{x\sigma}} \| \delta(\alpha_\ell(\mu) )^{-1} \|_{L^{2\mu}_{x\sigma}} \| \| g \|_{L^{2\mu}_{x\sigma}} + \| g \|_{L^{2\mu}_{x\sigma}} \| \lesssim \beta(\alpha_\ell(\mu)), \]  
(4.191)

where the implicit constants depend only on \( d \). Furthermore, by H"older's inequality, a computation involving the scaling and (4.191), the first term on the right-hand side of (4.190) is estimated as follows:
\[ \| \langle T_\mu [ T_{u(0)} [ u_t ] - W ], \alpha_\ell(\mu) V (-\Delta + \alpha_\ell(\mu) )^{-1} \Lambda W \rangle \|_{L^{2\mu}_{x\sigma}} \]
\[ \lesssim \| T_{u(0)} [ u_t ] - W \|_{L^{2\mu}_{x\sigma}} \beta(\alpha_\ell(\mu)), \]  
(4.192)
where the implicit constant depends only on $d$. Similarly, the second term on the right-hand side of (4.190) is estimated as follows:

$$
\left| \int_1^{\mu} \int_1^{\nu} \lambda^{-2} T_{\lambda} \left[ 2A W + \frac{2}{d-2} x \cdot \nabla A W \right] d\lambda d\nu, \alpha_t(\mu)V(-\Delta + \alpha_t(\mu))^{-1} A W \right|
$$

$$
\lesssim \beta(\alpha_t(\mu)) \left[ \int_1^{\mu} \int_1^{\nu} \lambda^{-2} T_{\lambda} \left[ 2A W + \frac{2}{d-2} x \cdot \nabla A W \right] d\lambda d\nu \right]_{L^2^*}
$$

$$
\leq \beta(\alpha_t(\mu)) \int_1^{\mu} \int_1^{\nu} \lambda^{-2} \|2A W + \frac{2}{d-2} x \cdot \nabla A W\|_{L^2^*} d\lambda d\nu
$$

$$
\lesssim \beta(\alpha_t(\mu)) \int_1^{\mu} \int_1^{\nu} \lambda^{-2} d\lambda d\nu \lesssim \beta(\alpha_t(\mu))|\mu - 1|^2,
$$

where the implicit constants depend only on $d$. Consider the last term on the right-hand side of (4.190). Note that

$$
\langle A W, \alpha_t(\mu)V(-\Delta + \alpha_t(\mu))^{-1} A W \rangle = \alpha_t(\mu)\langle(-\Delta + \alpha_t(\mu))^{-1} V A W, A W \rangle.
$$

(4.194)

Furthermore, observe from (2.15) that $\Re F[V A W](0) = -\frac{d+4}{d-2} \langle W^{\frac{d-2}{2}}, A W \rangle > 0$. Then, applying Lemma 3.13 to (4.194) as $q = A W$ and $q = 2^*$, and using $\delta(s) := \delta(s)^{-1}s$, we see that there exists $C_0 > 0$ depending only on $d$ such that

$$
\left| -\frac{2}{d-2} (\mu - 1) \langle A W, \alpha_t(\mu)V(-\Delta + \alpha_t(\mu))^{-1} A W \rangle - C_0 \beta(\alpha_t(\mu))|\mu - 1| \right|
$$

$$
\lesssim \delta(\alpha_t(\mu))\beta(\alpha_t(\mu))|\mu - 1|,
$$

(4.195)

where the implicit constant depends only on $d$.

Plugging (4.192), (4.193) and (4.195) into (4.190), we see that

$$
\left| \langle(-\Delta + \alpha_t(\mu))^{-1} F(\eta_t(\mu); \alpha_t(\mu),\mu), V A W \rangle - C_0 \beta(\alpha_t(\mu))\{\mu - 1\} \right|
$$

$$
\leq C_1 \beta(\alpha_t(\mu)) \left\{ \sup_{u_t \in \tilde{G}_t} \|T_{\lambda}[u_t - W]\|_{L^2^*} + |\mu - 1|^2 + \delta(\alpha_t(\mu))|\mu - 1| \right\},
$$

(4.196)

where $C_1 > 0$ is some constant depending only on $d$. Furthermore, by (4.196), (4.197) in Lemma 4.13 and $\lim_{t \to 0} \delta(\alpha_t(\mu)) = 0$ (see (4.179)), we see that there exists $T_2 > 0$ depending only on $d$ and $p$ such that for any $0 < t < T_2$, $u_t \in \tilde{G}_t$ and $\frac{1}{2} < \mu < \frac{3}{2}$,

$$
\left| \langle(-\Delta + \alpha_t(\mu))^{-1} F(\eta_t(\mu); \alpha_t(\mu),\mu), V A W \rangle - C_0 \beta(\alpha_t(\mu))\{\mu - 1\} \right|
$$

$$
\leq \beta(\alpha_t(\mu)) \left\{ C_1 + C_1 |\mu - 1|^2 + \frac{C_0}{4} |\mu - 1| \right\},
$$

(4.197)

which together with the intermediate value theorem implies that for any $0 < t < T_2$ and $u_t \in \tilde{G}_t$, there exists $\mu(t) > 0$ with $|\mu(t) - 1| \leq \min(\frac{\mu(0)}{4(1 + C_1)}, \mu(t) - 1)$ such that

$$
\langle(-\Delta + \alpha_t(\mu(t)))^{-1} F(\eta_t(\mu(t)); \alpha_t(\mu(t)), \mu(t)), V A W \rangle = 0.
$$

(4.198)

Furthermore, when $\mu = \mu(t)$, (4.196) together with (4.198) implies that

$$
\lim_{t \to 0} |\mu(t) - 1| = 0.
$$

(4.199)
Thus, we have proved that for any $R > 0$, there exists $0 < T_2(q, R) < T_1(q, R)$ such that for any $0 < t < T_2(q, R)$ and $u_t \in \tilde{G}_t$, there exists $\mu(t) = 1 + o_t(1)$ for which (4.173) holds. Define $\mu^1(t)$, $t^1$, $\alpha^1(t)$ and $\eta^1_t$ as in (4.170) through (4.172). Then, the claims (4.171) and (4.175) follow from (4.179) and (4.180): We remark that $\lim_{t \to 0} \sup_{u_t \in \tilde{G}_t} \alpha^1(t) = 0$ and (4.175).

It remains to prove (4.170). By the fundamental theorem of calculus (see (2.17)), a computation involving the scaling, (4.159) in Lemma 4.1.3, and (4.159) in Lemma 4.1.13 and $\mu(t) = 1 + o_t(1)$, we see that

$$
\|\eta^1_t\|_{H^1} \leq \|T_{\mu(t)[u_t]} - W\|_{H^1} + \|T_{\mu(t)}[W] - W\|_{H^1}
$$

$$
\lesssim \|T_{u_t[0]}[u_t] - W\|_{H^1} + \int_1^\mu(\| \lambda^{-1} \| \Lambda W \|_{H^1}) \, d\lambda \to 0 \quad \text{as} \quad t \to 0.
$$

Similarly, using (4.163) instead of (4.159), we see that for any $\frac{d}{\sqrt{q} - 2} < r < \infty$,

$$
\|\eta^1_t\|_{L^r} \lesssim \mu(t) \|t^{\frac{2}{2r}} \|T_{u_t[0]}[u_t] - W\|_{L^r} + \int_1^\mu(\| \lambda^{-1} + \| \Lambda W \|_{L^r}) \, d\lambda \to 0 \quad \text{as} \quad t \to 0.
$$

Thus, we have proved (4.176).

**Step 2.** Let $R > 0$, and let $T_2(q, R)$ denote the number given in the previous step. Furthermore, let $0 < t < T_2(q, R)$, $u_t \in \tilde{G}_t$, and let $\mu(t) = 1 + o_t(1)$ be a number such that (4.173) through (4.176) hold. Define $\mu^1(t)$, $t^1$, $\alpha^1(t)$ and $\eta^1_t$ as in (4.170) through (4.172). Then, our aim in this step is to show that, taking $T_2(q, R)$ even smaller dependently on $d$, $p$ and $q$, we have

$$
\|\eta^1_t\|_{L^p} \lesssim \alpha^1(t)^{\theta_q} + t^1,
$$

where the implicit constant depends only on $d$, $p$ and $q$.

Note that $\eta^1_t$ satisfies

$$
\eta^1_t = \{1 + (-\Delta + \alpha^1(t))^{-1}V\}^{-1}(-\Delta + \alpha^1(t))^{-1}F(\eta^1_t; \alpha^1(t), t^1).
$$

By (2.36), (3.12) in Lemma 3.4 with $q_1 = \frac{d}{\sqrt{q} - 2}$, Lemma 3.5, $\frac{d_{\alpha^1}}{d_{\alpha^1}} > \frac{d}{\sqrt{q} - 2}$ and (2.40), we see that

$$
\|(-\Delta + \alpha^1(t))^{-1}F(\eta^1_t; \alpha^1(t), t^1)\|_{L^q}
$$

$$
\leq \alpha^1(t)\|(-\Delta + \alpha^1(t))^{-1}W\|_{L^q} + t^1\|(-\Delta + \alpha^1(t))^{-1}Wp\|_{L^q}
$$

$$
+ \|(-\Delta + \alpha^1(t))^{-1}D(\eta^1_t, 0)\|_{L^q}
$$

$$
\leq \alpha^1(t)^{\theta_q} + t^1
$$

$$
+ \|(-\Delta + \alpha^1(t))^{-1}D(\eta^1_t, 0)\|_{L^q} + t^1\|(-\Delta + \alpha^1(t))^{-1}E(\eta^1_t, 0)\|_{L^q},
$$

where the implicit constant depends only on $d$, $p$ and $q$. Consider the third term on the right-hand side of (4.201). By (2.43), (2.41), Lemma 3.3, Hölder’s inequality, (4.176),

$$
\frac{(6^{-d}dq)}{(d-2)(2q-d)} > \frac{d}{d-2}
$$

and $q > \frac{d}{d-2}$, we see that

$$
\|(-\Delta + \alpha^1(t))^{-1}D(\eta^1_t, 0)\|_{L^q} \lesssim \|W + |\eta^1_t|^{\frac{2d}{2d-2}}|\eta^1_t|^2\|_{L^{\frac{d}{d-2}}}
$$

$$
\lesssim \|W + |\eta^1_t|^{\frac{2d}{2d-2}}\|_{L^{\frac{d}{d-2}}}||\eta^1_t||_{L^q} \lesssim ||\eta^1_t||_{L^q}^2 = o_t(1)||\eta^1_t||_{L^q},
$$

(4.205)
where the implicit constants depend only on $d$ and $q$. Consider the last term on the right-hand side of (4.204). Introduce an exponent $q_1$ as

$$q_1 := \frac{2^* q}{(p-1)q + 2^*} = \frac{2^*}{p-1 + \frac{2^*}{q}}.$$ (4.206)

Observe from $q > \max\{2^*, \frac{2^*}{p-1}\}$ that

$$1 < q_1 < \frac{q}{2}, \quad \frac{d}{2} \left( \frac{1}{q_1} - \frac{1}{q} \right) - 1 = -\frac{(2^* - (p + 1))}{2^* - 2}, \quad \frac{d}{d-2} < \frac{(p-1) q q_1}{q-q_1}. $$ (4.207)

Furthermore, observe from the last condition in (4.207) and (4.176) that

$$\text{Proposition 3.1 and (4.210), we see that}$$

$$\text{where the implicit constant depends only on} \ d.$$ (4.209)

Introduce an exponent $q_1$, then by (2.41), (2.42), Lemma 3.4 Hölder’s inequality, the second condition in (4.207), (4.209) and (4.176), we see that

$$t^\|\langle -\Delta + \alpha^\dagger(t) \rangle^{-1} E(\eta_{\dagger}^I, 0) \|_L^q \sum_1, \quad \| \eta_{\dagger}^I \|_{L^\frac{(p-1)q_1}{q_1}} = \mathcal{O}(1),$$

(4.209)

where the implicit constant depends only on $d, p$ and $q$. Then, by (4.208), (4.209), (4.209) into (4.204), we see that

$$\|(-\Delta + \alpha^\dagger(t))^{-1} F(\eta_{\dagger}^I; \alpha^\dagger(t), t^\dagger) \|_{L^q} \lesssim \alpha^\dagger(t) \Theta_q + t^\dagger + \mathcal{O}(1) \| \eta_{\dagger}^I \|_{L^q},$$

(4.210)

where the implicit constant depends only on $d, p$ and $q$. Furthermore, by (4.203), (4.173), Proposition 3.1 and (4.210), we see that

$$\| \eta_{\dagger}^I \|_{L^q} \lesssim \|(-\Delta + \alpha^\dagger(t))^{-1} F(\eta_{\dagger}^I; \alpha^\dagger(t), t^\dagger) \|_{L^q} \lesssim \alpha^\dagger(t) \Theta_q + t^\dagger + \mathcal{O}(1) \| \eta_{\dagger}^I \|_{L^q},$$

(4.211)

where the implicit constant depends only on $d, p$ and $q$. Since the last term on the right-hand side of (4.211) can be absorbed into the left-hand side, (4.211) implies (4.202).

**Step 3.** We shall finish the proof of the lemma. Let $R > 0$, and let $T_2(q, R)$ denote the same number as in the previous step. Then, for any $0 < t < T_2(q, R)$ and $u_t \in \mathcal{G}_t$, we can take $\mu(t) = 1 + o_t(1)$ for which (4.173) through (4.176) and (4.202) hold.

Note that since $\alpha$ is the inverse function of $\beta$, we have

$$\alpha^\dagger(t) = \alpha(\beta(\alpha^\dagger(t))).$$

(4.212)

Observe from (4.173) that

$$0 = \langle (-\Delta + \alpha^\dagger(t))^{-1} F(\eta_{\dagger}^I; \alpha^\dagger(t), t^\dagger), VAW \rangle$$

$$= -\alpha^\dagger(t) \langle (-\Delta + \alpha^\dagger(t))^{-1} W, VAW \rangle + t^\dagger \langle (-\Delta + \alpha^\dagger(t))^{-1} W^p, VAW \rangle$$

(4.213)

$$+ \langle (-\Delta + \alpha^\dagger(t))^{-1} D(\eta_{\dagger}^I, 0), VAW \rangle + t^\dagger \langle (-\Delta + \alpha^\dagger(t))^{-1} E(\eta_{\dagger}^I, 0), VAW \rangle.$$
Recall that \( \beta(s) = \delta(s)^{-1}s \). Then, by Lemma 3.15 and Lemma 3.16, we see that
\[
\alpha(t)((-\Delta + \alpha(t))^{-1}W, VAW) = A_1 t \beta(\alpha(t)) + O(\alpha(t)), \tag{4.214}
\]
\[
t^\dagger((-\Delta + \alpha(t))^{-1}W^p, V\Lambda W) = K_p t^\dagger + o_t(1)t^\dagger. \tag{4.215}
\]
By (4.208) and (4.202), we see that
\[
|((-\Delta + \alpha(t))^{-1} D(\eta_t^1, 0), VAW)| \leq \|(\Delta + \alpha(t))^{-1} E(\eta_t^1, 0), VA^2W\|_{L^2} \lesssim \|\eta_t^1\|_{L^2} \lesssim \alpha(t)^{2\Theta_q} + o_t(1)t^\dagger,
\]
where the implicit constants depend only on \( d \), \( p \) and \( q \). Observe from \( q > \max\{2^\ast, \frac{2^\ast}{p+3-2^\ast}\} \) and \( p > \frac{1}{d-2} - 1 \) that
\[
\Theta_q - \frac{2^\ast - (p + 1)}{2-2} > 0, \quad 2\Theta_q > \frac{d - 2}{2}. \tag{4.217}
\]
By (4.209), (4.202), (1.214) and (1.173), we see
\[
\begin{align*}
t^\dagger|((-\Delta + \alpha(t))^{-1} E(\eta_t^1, 0), VAW)| \lesssim & \, t^\dagger|((-\Delta + \alpha(t))^{-1} E(\eta_t^1, 0)|_{L^2} \\= & \, t^\dagger \alpha(t)^{-\frac{(2^\ast - (p + 1))}{2-2}} \|\eta_t^1\|_{L^2} \lesssim \, t^\dagger \alpha(t)^{-\frac{(2^\ast - (p + 1))}{2-2}} \{\alpha(t)^{2\Theta_q} + t^\dagger\} = o_t(1)t^\dagger,
\end{align*}
\]
where the implicit constants depend only on \( d \), \( p \) and \( q \). Plugging (4.214), (4.215), (4.216) and (1.213) into (4.213), and using (4.212), (4.217) and (2.34), we see that
\[
\left|\beta(\alpha(t)) - \frac{K_p}{A_1} t^\dagger\right| \leq \omega_t(1)t^\dagger + o_t(1)t^\dagger + \alpha(t)^{2\Theta_q} \leq \omega_t(1)t^\dagger + o_t(1)\beta(\alpha(t)). \tag{4.219}
\]
This implies that \( \beta(\alpha(t)) \in I(t^\dagger) \). Furthermore, by (4.208), (4.212), (4.40) and \( \beta(\alpha(t)) \in I(t^\dagger) \) (hence \( \beta(\alpha(t)) \sim t^\dagger \)), we see that
\[
\|\eta_t^1\|_{L^q} \lesssim \alpha(t)^{2\Theta_q} + t^\dagger \lesssim \alpha(\beta(\alpha(t)))^{\Theta_q} + \alpha(t)^{\Theta_q} \lesssim \alpha(t)^{\Theta_q}, \tag{4.220}
\]
where the implicit constants depend only on \( d \), \( p \) and \( q \), so that there exists \( R_* > 0 \) depending only on \( d \), \( p \) and \( q \) such that
\[
\eta_t^1 \in Y_q(R_*, t^\dagger). \tag{4.221}
\]
Since \( R_* \) depends only on \( d \), \( p \) and \( q \), we may take \( R = R_* \) from the beginning of the proof; Put \( T_* := T_2(q, R_*) \). Thus, we have proved the claims (1.173) through (1.177).

It remains to prove (1.178). By (4.212), (4.7) and (1.11), we may rewrite (1.173) as
\[
\beta(\alpha(t)) = s(t^\dagger; \beta(\alpha(t)), \eta_t^1). \tag{4.222}
\]
Furthermore, by (1.212), we may write (4.203) as
\[
\eta_t^1 = g(t^\dagger; \beta(\alpha(t)), \eta_t^1). \tag{4.223}
\]
By (1.174), choosing \( T_* \) even smaller dependently only on \( d \), \( p \) and \( q \), we may assume that \( t^\dagger < T_1(q, R_*) \) for all \( 0 < t < T_* \). Then, the claim (1.178) follows from the uniqueness of solutions to (4.20) in \( I(t^\dagger) \times Y_q(R_*, t^\dagger) \) (see Proposition 4.4 and 4.177).

Thus, we have completed the proof of the lemma. \( \Box \)
Now, we are in a position to prove Proposition 4.2.

**Proof of Proposition 4.2.** We employ the idea from [7] (see the proof of Lemma 3.10 in [7]). For $R > 0$, let $T_1(q, R)$ denote the number given by Proposition 4.1. Furthermore, for $0 < t < T_1(q, R)$ and $0 < \lambda < T_1(q, R)$, let $(\tau_{\lambda t}, \eta_{\lambda t})$ denote a unique solution to (4.20) with $t = \lambda t$ in $I(\lambda t) \times Y_y(R, \lambda t)$ (see Proposition 4.1). Then, for $0 < t < T_1(q, R)$, we define a function $\Omega_t : (0, \frac{T_1(q, R)}{t}) \to (0, \infty)$ by

$$\Omega_t(\lambda) := \lambda^{\frac{(2^* - 2)}{2^* - (p + 1)}} \alpha(\tau_{\lambda t}). \quad (4.224)$$

Observe from the continuity of $\tau_t$ with respect to $t$ (see Proposition 4.1) that $\Omega_t(\lambda)$ is continuous with respect to $\lambda$.

We claim:

**Claim.** Let $R > 0$, and let $0 < t < T_1(q, R)$; We may choose $T_1(q, R)$ even smaller, depending on $d$, $p$, $q$ and $R$, if necessary. Then, $\Omega_t(\lambda)$ is injective on $(0, \frac{T_1(q, R)}{t})$.

To prove this claim, it suffices to show that $\Omega_t(\lambda)$ is strictly decreasing with respect to $\lambda$ on $(0, \frac{T_1(q, R)}{t})$. In addition, by the continuity of $\Omega_t(\lambda)$, it suffices to prove the following: There exists $0 < \varepsilon < 1$ depending only on $d$, $p$ and $q$ such that if $0 < \lambda_1 < \lambda_2 < \frac{T_1(q, R)}{t}$ and $\lambda_2 < (1 + \varepsilon)\lambda_1$, then

$$\Omega_t(\lambda_2) < \Omega_t(\lambda_1). \quad (4.225)$$

Let $0 < \varepsilon < 1$ be a constant to be specified later, depending on $d$, $p$ and $q$. Furthermore, let $0 < \lambda_1 < \lambda_2 < \frac{T_1(q, R)}{t}$, and assume that $\lambda_2 < (1 + \varepsilon)\lambda_1$. Observe that

$$\Omega_t(\lambda_2) - \Omega_t(\lambda_1) = \left\{ \lambda_2^{\frac{(2^* - 2)}{2^* - (p + 1)}} - \lambda_1^{\frac{(2^* - 2)}{2^* - (p + 1)}} \right\} \alpha(\tau_{\lambda_1 t}) + \lambda_2^{\frac{(2^* - 2)}{2^* - (p + 1)}} \left\{ \alpha(\tau_{\lambda_2 t}) - \alpha(\tau_{\lambda_1 t}) \right\}. \quad (4.226)$$

Note that for each $j = 1, 2$, $(\tau_{\lambda_j t}, \eta_{\lambda_j t})$ is a unique solution to (4.20) with $t = \lambda_j t$ in $I(\lambda_j t) \times Y_y(R, \lambda_j t)$; in particular, we have

$$\tau_{\lambda_j t} = \gamma(\lambda_j t; \tau_{\lambda_j t}, \eta_{\lambda_j t}), \quad \eta_{\lambda_j t} = \gamma(\lambda_j t; \tau_{\lambda_j t}, \eta_{\lambda_j t}). \quad (4.227)$$

Furthermore, observe from $\tau_{\lambda_j t} \in I(\lambda_j t)$ and $\lambda_1 \leq \lambda_2 \leq 2\lambda_1$ ($0 < \varepsilon < 1$) that

$$\tau_{\lambda_2 t} \sim \lambda_1 t \sim \lambda_2 t \sim \tau_{\lambda_2 t}, \quad (4.228)$$

$$\alpha(\tau_{\lambda_1 t}) \sim \alpha(\lambda_1 t) \sim \alpha(\lambda_2 t) \sim \alpha(\tau_{\lambda_2 t}), \quad (4.229)$$

where the implicit constants depend only on $d$ and $p$.

We consider the first term on the right-hand side of (4.226). By an elementary computation, we see that

$$\lambda_2^{\frac{(2^* - 2)}{2^* - (p + 1)}} - \lambda_1^{\frac{(2^* - 2)}{2^* - (p + 1)}} = -\lambda_2^{\frac{(2^* - 2)}{2^* - (p + 1)}} \left\{ \lambda_2^{\frac{2^*-2}{2^*-(p+1)}} - 1 \right\} \quad (4.230)$$

$$= -\lambda_2^{\frac{(2^* - 2)}{2^* - (p + 1)}} \frac{2^* - 2}{2^* - (p + 1)} \lambda_1 \left( \lambda_2 - \lambda_1 \right) + \lambda_2^{\frac{(2^* - 2)}{2^* - (p + 1)}} \alpha(\lambda_1^{-1} |\lambda_2 - \lambda_1|).$$
Furthermore, by \( \alpha(t) = \delta(\alpha(t))t \) (see (4.222) and Lemma 4.8), we see that
\[
\alpha(\tau_{\lambda t}) = \delta(\alpha(\tau_{\lambda t}))\tau_{\lambda t} = \delta(\alpha(\tau_{\lambda t}))\left\{ \frac{K_p}{A_1}\lambda t + o(\lambda t) \right\}.
\] (4.231)

Putting (4.230) and (4.231) together, we find that
\[
\begin{align*}
\left\{ \lambda_2^{-(2^*-2)} - \lambda_1^{-(2^*-2)} \right\} \alpha(\tau_{\lambda t}) \\
= -\delta(\alpha(\tau_{\lambda t}))\lambda_2^{-(2^*-2)} \frac{2^* - 2}{2^* - (p + 1)} \left\{ \frac{K_p}{A_1}\lambda t + o(\lambda t) \right\} \lambda_1^{-1}(\lambda_2 - \lambda_1) \\
+ \delta(\alpha(\tau_{\lambda t}))\lambda_2^{-(2^*-2)} \left\{ \frac{K_p}{A_1}\lambda t + o(\lambda t) \right\} o(\lambda t^{-1}|\lambda_2 - \lambda_1|).
\end{align*}
\] (4.232)

Next, we consider the second term on the right-hand side of (4.226). Since \( \tau_1 \) is strictly increasing with respect to \( t \), we have \( \tau_{\lambda t} - \tau_{\lambda_1 t} = |\tau_{\lambda_2 t} - \tau_{\lambda_1 t}| \). Hence, by \( \alpha(t) = \delta(\alpha(t))t \) (see (4.232)), (4.55) in Lemma 4.4, we see that
\[
\alpha(\tau_{\lambda_2 t}) - \alpha(\tau_{\lambda_1 t}) = \{ \delta(\alpha(\tau_{\lambda_2 t})) - \delta(\alpha(\tau_{\lambda_1 t})) \} \tau_{\lambda_2 t} + \delta(\alpha(\tau_{\lambda_1 t}))\{ \tau_{\lambda_2 t} - \tau_{\lambda_1 t} \}.
\] (4.233)

Here, observe from \( \tau_{\lambda t} = g(\lambda t; \tau_{\lambda t}, \eta_{\lambda t}) \) (see (4.222)) and Lemma 4.9 (see Remark 4.5) that
\[
|\tau_{\lambda_2 t} - \tau_{\lambda_1 t}| \lesssim |\lambda_2 t - \lambda_1 t| + \lambda_1 t a(\lambda_1 t)^{-\Theta_q + \Theta_q}||\eta_{\lambda t} - \eta_{\lambda_2 t}||_{L^q},
\] (4.234)
where the implicit constant depends only on \( d, p \) and \( q \). Furthermore, Lemma 4.9 (see Remark 4.5) together with (4.231) shows that
\[
\left| \tau_{\lambda_2 t} - \tau_{\lambda_1 t} - \frac{K_p}{A_1}\{ \lambda_2 t - \lambda_1 t \} \right| \lesssim o_{\lambda_1 t}(1)|\lambda_2 t - \lambda_1 t| + \lambda_1 t a(\lambda_1 t)^{-\Theta_q + \Theta_q}||\eta_{\lambda t} - \eta_{\lambda_2 t}||_{L^q},
\] (4.235)
where the implicit constant depends only on \( d, p \) and \( q \). Observe from \( \eta_{\lambda t} = g(\lambda t; \tau_{\lambda t}, \eta_{\lambda t}) \) (see (4.222)), and (4.101) in Lemma 4.10 that
\[
||\eta_{\lambda t} - \eta_{\lambda_2 t}||_{L^q} \lesssim \{ o_{\lambda_1 t}(1)(\lambda_1 t)^{-1} a(\lambda_1 t)^{-\Theta_q} + 1 \} |\lambda_1 t - \lambda_2 t|,
\] (4.236)
where the implicit constant depends only on \( d, p \) and \( q \). Plugging (4.236) into (4.235), and using (4.40), we see that
\[
\tau_{\lambda_2 t} - \tau_{\lambda_1 t} = \frac{K_p}{A_1}(\lambda_2 - \lambda_1) t + o_{\lambda_1 t}(1)(\lambda_2 - \lambda_1) t.
\] (4.237)

Furthermore, dividing both sides of (4.237) by \( \tau_{\lambda_1 t} \), and using (4.228) and \( \lambda_2 \leq (1 + \varepsilon)\lambda_1 \), we see that
\[
\frac{\tau_{\lambda_2 t}}{\tau_{\lambda_1 t}} - 1 \lesssim \frac{\lambda_2 - \lambda_1}{\lambda_1} \leq \varepsilon,
\] (4.238)
where the implicit constant depends only on $d$ and $p$. We may write (4.233) as

$$\frac{\tau_{\lambda_2 t}}{\tau_{\lambda_1 t}} = 1 + O(\varepsilon).$$

(4.239)

Putting (4.233), (4.237) and (4.238) together, we find that

$$\leq \lambda_2^{-(2^* - p - 2)} \left\{ (1 + o(\lambda_1 t(1) + O(\varepsilon)) \delta(\alpha(\tau_{\lambda_2 t}))d^3 + 1 \right\} \delta(\alpha(\tau_{\lambda_1 t}))
\times \left\{ \frac{K_p}{\lambda_1} + o(\alpha(\lambda_1 t(1)) \right\} (\lambda_2 - \lambda_1)t.

(4.240)

Now, plugging (4.232) and (4.240) into (4.226), and making a simple computation, we see that

$$\Omega_t(\lambda_2) - \Omega_t(\lambda_1)
\leq -\delta(\alpha(\tau_{\lambda_1 t})) \lambda_2^{-(2^* - p - 2)} \frac{2^* - 2}{2^* - (p + 1)} \frac{K_p}{\lambda_1} (\lambda_2 - \lambda_1)t
+ \delta(\alpha(\tau_{\lambda_1 t})) \lambda_2^{-(2^* - p - 2)} o(\lambda_2 - \lambda_1 t)
+ \delta(\alpha(\tau_{\lambda_1 t})) \lambda_2^{-(2^* - p - 2)} \left\{ \delta(\alpha(\tau_{\lambda_2 t}))d^3 + 1 \right\} \frac{K_p}{\lambda_1} (\lambda_2 - \lambda_1)t
+ \delta(\alpha(\tau_{\lambda_1 t})) \lambda_2^{-(2^* - p - 2)} \left\{ o(\lambda_1 t(1) + O(\varepsilon)) \delta(\alpha(\tau_{\lambda_2 t}))d^3 \right\} |\lambda_2 - \lambda_1| t.

(4.241)

We find from (4.241) that if $t$ and $\varepsilon$ are sufficiently small depending only on $d$, $p$ and $q$, then $\Omega_t(\lambda_2) - \Omega_t(\lambda_1) < 0$. Thus, we have proved that $\Omega_t(\lambda)$ is strictly decreasing with respect to $\lambda$, which implies that $\Omega_t$ is injective.

Now, we shall finish the proof of the proposition:

**End of the proof.** Let $R_*$ and $T_*$ be the numbers given in Lemma 4.14. Furthermore, let $0 < t < T_*$, and let $(\tau_t, \eta_t)$ be a unique solution to (4.20) in $I(t) \times Y_q(R_*, t)$ (see Proposition 1.1). Then, what we need to prove is that if $u_t \in \tilde{G}_t$, then $u_t = W + \eta_t$.

Let $u_t \in \tilde{G}_t$. Then, Lemma 4.14 shows that there exists $\mu(t) = 1 + o(1)$ such that, defining $t^\dagger$, $\alpha^\dagger(t)$ and $\eta^\dagger_t$ as

$$t^\dagger := \{ \mu(t)u_t(0) \}^{-2^* + p + 1}t, \quad \alpha^\dagger(t) := \{ \mu(t)u_t(0) \}^{-2^* + 2} \alpha(\tau_t),$$

$$\eta^\dagger_t := T_{\mu(t)u_t(0)}[u_t] - W,$$

(4.242)

(4.243)

we have the following:

$$\lim_{t \to 0} \alpha^\dagger(t) = 0, \quad \lim_{t \to 0} t^\dagger = 0,$$

(4.244)

$$\tau_{\alpha^\dagger(t)}(t) = (\beta(\alpha^\dagger(t)), \eta^\dagger_t) \in I(t^\dagger) \times Y_q(R_*, t^\dagger),$$

(4.245)

where $(\tau_{t^\dagger}, \eta_{t^\dagger})$ denotes a unique solution to (4.20) with $t = t^\dagger$ in $I(t^\dagger) \times Y_q(R_*, t^\dagger)$. Since $\alpha$ is the inverse function of $\beta$, (4.245) shows that

$$\alpha(\tau_{t^\dagger}) = \alpha(\beta(\alpha^\dagger(t))) = \alpha^\dagger(t).$$

(4.246)
Note that
\[ t^\dagger = \{ \mu(t)u_t(0) \}^{-2^{(r+p+1)}.} \tag{4.247} \]
Furthermore, observe from (4.247) and (4.246) that
\[ \Omega_t \left( t^\dagger \right) = \{ \mu(t)u_t(0) \}^{2^r - 2}\alpha(t) = \alpha(t) = \Omega_t(1). \tag{4.248} \]
Then, the injectivity of \( \Omega_t(\lambda) \) with respect to \( \lambda \), together with (4.248) and (4.247), implies that \( \mu(t)u_t(0) = 1 \). Hence, \( t^\dagger = t \). Furthermore, by \( t^\dagger = t \), \( \mu(t)u_t(0) = 1 \), (4.243) and (4.245), we see that \( \eta = \eta_t = u_t - W \). Thus, we have completed the proof. \( \square \)

A Compactness in Lebesgue spaces

We record a compactness theorem in Lebesgue spaces (see Proposition A.1 of [15]):

**Lemma A.1.** Assume \( d \geq 1 \), and let \( 1 \leq q \leq \infty \). A sequence \( \{ f_n \} \) in \( L^q(\mathbb{R}^d) \) has a convergent subsequence if and only if it satisfies the following properties:

1. \( \sup_{n \geq 1} \| f_n \|_{L^q} < \infty. \) \hspace{1cm} (A.1)
2. \( \lim_{y \to 0} \sup_{n \geq 1} \| f_n(\cdot) - f_n(\cdot + y) \|_{L^q} = 0. \) \hspace{1cm} (A.2)
3. \( \lim_{R \to \infty} \sup_{n \geq 1} \| f_n \|_{L^q(\|x\| \geq R)} = 0. \) \hspace{1cm} (A.3)

**Remark A.1.** Although Proposition A.1 of [15] does not refer to the case \( q = \infty \), we may include this case in the statement. Note that when \( q = \infty \), (A.2) implies that the sequence \( \{ f_n \} \) is continuous; hence the limit along the subsequence is also continuous.

B Completeness of \( Y_q(R, t) \)

Let \( \frac{d}{2} < q < \infty \), \( R > 0 \), \( 0 < t < 1 \), and let \( Y_q(R, t) \) be a set defined by (4.15). Then, we shall show that \( Y_q(R, t) \) is complete with the metric induced from \( L^q(\mathbb{R}^d) \).

Let \( \{ \eta_n \} \) be a Cauchy sequence in \( Y_q(R, t) \). Since \( L^q_{rad}(\mathbb{R}^d) \) is complete, there exists a radial function \( \eta_\infty \in L^q(\mathbb{R}^d) \) such that
\[ \lim_{n \to \infty} \eta_n = \eta_\infty \quad \text{strongly in } L^q(\mathbb{R}^d). \] \hspace{1cm} (B.1)

Furthermore, we see that
\[ \| \eta_\infty \|_{L^q} = \lim_{n \to \infty} \| \eta_n \|_{L^q} \leq R\alpha(t) \frac{d-2}{r} \frac{d}{r^q}, \] \hspace{1cm} (B.2)
so that \( \eta_\infty \in Y_q(R, t) \). Thus, we have proved the completeness.
C Elementary computations

Lemma C.1. Define a function $\beta$ on $(0, \infty)$ by $\beta(s) := s \log (1 + s^{-1})$. Then, $\beta$ is strictly increasing on $(0, \infty)$, and the image of $(0, \infty)$ by $\beta$ is $(0, 1)$; hence the inverse is defined on $(0, 1)$, say $\alpha : (0, 1) \to (0, \infty)$. Furthermore, there exists $0 < T_0 < 1$ such that if $0 < t \leq T_0$, then

$$\alpha(t) \sim \frac{t}{\log (1 + t^{-1})}. \quad (C.1)$$

Proof of Lemma C.1. We may write $\beta$ as

$$\beta(s) = s \int_0^1 \frac{d}{d\theta} \log (s + \theta) \, d\theta = 1 - \int_0^1 \frac{\theta}{s + \theta} \, d\theta. \quad (C.2)$$

This formula shows that $\beta$ is strictly increasing on $(0, \infty)$, $\lim_{s \to 0} \beta(s) = 0$, and $\lim_{s \to \infty} \beta(s) = 1$. In particular, the image of $(0, \infty)$ by $\beta$ is $(0, 1)$.

Next, let $\alpha : (0, 1) \to (0, \infty)$ be the inverse of $\beta$. Note that $\alpha$ is strictly increasing, continuous, and $\lim_{s \to 0} \alpha(s) = 0$. Hence, there exists $0 < T_0 < 1$ such that $\alpha(t) \leq e^{-1}$ for all $0 < t \leq T_0$. Let $0 < t \leq T_0$, and put $s := \alpha(t)$. Note that $0 < s \leq e^{-1}$ and $\beta(s) = t$.

Then, observe from $\beta(s) := s \log (1 + s^{-1})$ that $s \leq t \leq \frac{s^2}{2}$, which implies that

$$\frac{1}{2} \log (s^{-1}) \leq \log (t^{-1}) \leq \log (s^{-1}). \quad (C.3)$$

Furthermore, it follows from $0 < s, t \leq 1$, $(C.3)$ and $s = \alpha(t)$ that

$$t = \beta(s) \sim s \log (s^{-1}) \sim s \log (t^{-1}) \sim \alpha(t) \log (1 + t^{-1}). \quad (C.4)$$

Thus, we have proved the lemma.

Lemma C.2. Assume $d = 3, 4$. Let $0 < s_1 \leq s_2 < 1$. Then, we have

$$\int_{\mathbb{R}^d} \int_{|\xi| \leq 1} \frac{\min \{|x||\xi|, |x|^2|\xi|^2\}}{|\xi|^2(|\xi|^2 + s_1)(|\xi|^2 + s_2)} (1 + |x|)^{-(d+2)} \, d\xi dx \lesssim \log (1 + s_2^{-1}) \delta(s_2)^{-1}, \quad (C.5)$$

where the implicit constant depends only on $d$.

Proof of Lemma C.2. Observe that

$$\int_{\mathbb{R}^d} \int_{|\xi| \leq 1} \frac{\min \{|x||\xi|, |x|^2|\xi|^2\}}{|\xi|^2(|\xi|^2 + s_1)(|\xi|^2 + s_2)} (1 + |x|)^{-(d+2)} \, d\xi dx$$

\begin{align*}
&\lesssim \int_0^1 \int_0^1 \frac{r^{d+1}\lambda^{d-1}}{(1 + r)^{d+2}(\lambda^2 + s_1)(\lambda^2 + s_2)} \, d\lambda dr \\
&+ \int_1^\infty \int_0^1 \frac{r^{d+1}\lambda^{d-1}}{(1 + r)^{d+2}(\lambda^2 + s_1)(\lambda^2 + s_2)} \, d\lambda dr \\
&+ \int_1^\infty \int_0^1 \frac{r^d\lambda^{d-2}}{(1 + r)^{d+2}(\lambda^2 + s_1)(\lambda^2 + s_2)} \, d\lambda dr,
\end{align*} \quad (C.6)
where the implicit constant depends only on $d$. Consider the first term on the right-hand side of (C.6). By an elementary computation, and (2.29) (only for $B$), we see that

$$
\int_0^1 \int_0^1 \frac{r^{d+1}\lambda^{d-1}}{(1+r)^{d+2}(\lambda^2 + s_1)(\lambda^2 + s_2)} \, d\lambda dr \\
\lesssim \int_0^1 \frac{\lambda^{d-1}}{(\lambda^2 + s_1)(\lambda^2 + s_2)} \, d\lambda \lesssim \int_0^1 \frac{\lambda^{d-3}}{\lambda^2 + s_2} \, d\lambda \lesssim \delta(s_2)^{-1},
$$

where the implicit constant depends only on $d$. Next, consider the second term on the right-hand side of (C.6). By change of the order of integrals, and (2.29) (only for Lemma C.3), we see that

$$
\int_0^1 \int_0^1 \frac{r^{d+1}\lambda^{d-1}}{(1+r)^{d+2}(\lambda^2 + s_1)(\lambda^2 + s_2)} \, d\mu dr \\
\leq \int_0^1 \int_0^1 \frac{\frac{d-4}{s_2^2} \mu^{d-1}}{r(\mu^2 + s_2^2)(\mu^2 + 1)} \, d\mu dr \\
= \int_0^\infty \int_0^1 \frac{1}{\sqrt{s_2}} \frac{\frac{d-4}{s_2^2} \mu^{d-1}}{r(\mu^2 + s_2^2)(\mu^2 + 1)} \, d\mu dr \\
\leq \int_0^\infty \int_0^1 \frac{1}{\sqrt{s_2}} \frac{d-4}{s_2^2} \frac{s_2^2}{\lambda} \, d\lambda dr \\
\lesssim \delta(s_2)^{-1} \log (1 + s_2^{-1}),
$$

where the implicit constant depends only on $d$. Finally, we consider the last term on the right-hand side of (C.6). By change of the order of integrals, and (2.29) (only for $d = 3$), we see that

$$
\int_1^\infty \int_0^1 \frac{r^d\lambda^{d-2}}{(1+r)^{d+2}(\lambda^2 + s_1)(\lambda^2 + s_2)} \, d\lambda dr \\
\leq \int_1^\infty \int_0^1 \frac{\lambda^{d-2}}{r^2(\lambda^2 + s_1)(\lambda^2 + s_2)} \, d\lambda dr = \int_0^1 \frac{1}{\lambda^2 + s_1} \, d\lambda \int_0^\infty \frac{1}{r^2 \lambda^2} \, dr \\
= \int_0^1 \frac{\lambda^{d-1}}{(\lambda^2 + s_1)(\lambda^2 + s_2)} \, d\lambda \leq \int_0^1 \frac{\lambda^{d-3}}{\lambda^2 + s_2} \, d\lambda \lesssim \delta(s_2)^{-1},
$$

where the implicit constant depends only on $d$.

Putting (C.6) through (C.9) together, we find that (C.5) holds.

The following lemma is quoted from [11]:

**Lemma C.3** (Lemma 3.12 of [11]). Let $X$, $Y$, $X$, $Y$ be vector spaces. Let $A : X \to Y$, $B : X \to X$, $C : Y \to Y$ be linear operators. Define $A := CAB$. If $A^{-1}$ exists, then $A^{-1} = BA^{-1}C$ provided $B$ is surjective and $C$ is injective.
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