AN INTERIOR ESTIMATE FOR CONVEX SOLUTIONS AND A RIGIDITY THEOREM

MING LI, CHANGYU REN, AND ZHIZHANG WANG

Abstract. We establish an interior $C^2$ estimate for $k + 1$ convex solutions to Dirichlet problems of $k$-Hessian equations. We also use such estimate to obtain a rigidity theorem for $k + 1$ convex entire solutions of $k$-Hessian equations in Euclidean space.

1. Introduction

In this paper, we consider an interior $C^2$ estimate for the following Dirichlet problem for $k$-Hessian equations,

\begin{align*}
\sigma_k(D^2u) &= f(x,u,\nabla u), \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega.
\end{align*}

Here, $u$ is a function defined in some domain $\Omega$. $\nabla u$ is the gradient of $u$ and $D^2u$ is the Hessian of $u$. We also require $f > 0$ and smooth enough respect to every variables.

The interior $C^2$ estimates for Monge-Ampère equations were studied at first by A.V. Pogorelov [13, 9]. Then, K.S. Chou and X.-J. Wang extended Pogorelov’s estimates to the case of $k$-Hessian equations of [8, 16]. Explicitly, in their paper, for any function $f$ not depending $\nabla u$ in (1.1), they have proved that, for any small positive constant $\varepsilon$, the following estimates hold,

\begin{align*}
(-u)^{1+\varepsilon}\Delta u &\leq C.
\end{align*}

Here, constant $C$ depends on the domain $\Omega$, $k$, $f$ and $\sup_{\Omega} |\nabla u|$.

Maybe a natural question is whether these interior estimates are still valid for that $f$ does depend on the gradient term $\nabla u$, namely, interior estimates for (1.1). For the 2-Hessian equation, we can get this type of interior estimates.

Theorem 1. For 2-Hessian equations, i.e. $k = 2$ in (1.1), there is some constant $\beta > 0$, such that

\begin{align*}
(-u)^{\beta}\Delta u &\leq C.
\end{align*}

Here positive constants $\beta$ and $C$ depend on the domain $\Omega$, the function $f$, $\sup_{\Omega} |u|$ and $\sup_{\Omega} |\nabla u|$.

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By some reasons, the small constant \( \varepsilon \) should not be zero in Chou-Wang’s proof. On the other hand, for Monge-Ampère equation case, namely, \( k = n \) in (1.1), we can drop the small \( \varepsilon \). It reminds us that if the convexity is better, estimate (1.2) can be improved. Using techniques developing in [11], we can get the following theorem,

**Theorem 2.** Suppose that function \( u \) is a \( k + 1 \) convex solution for the Dirichlet problem of \( k \)-Hessian equations (1.1). Namely, function \( u \) is in \( k + 1 \) convex cone. We have,

\[
(-u)\Delta u \leq C.
\]

Here, positive constant \( C \) depends on \( \sup_{\Omega} |\nabla u|, \sup_{\Omega} |u| \), the function \( f \) and the domain \( \Omega \).

Here the definition of the \( k \)-convex cone is following Caffarelli-Nirenberg-Spruck [3],

**Definition 3.** For a domain \( \Omega \subset \mathbb{R}^n \), a function \( v \in C^2(\Omega) \) is called \( k \)-convex if the eigenvalues \( \kappa(x) = (\kappa_1(x), \cdots, \kappa_n(x)) \) of the hessian \( \nabla^2 v(x) \) is in \( \Gamma_k \) for all \( x \in \Omega \), where \( \Gamma_k \) is the Garding’s cone

\[
\Gamma_k = \{ \kappa \in \mathbb{R}^n \mid \sigma_m(\kappa) > 0, \ m = 1, \cdots, k \}.
\]

Note that the constant \( \beta \) is large in Theorem 1. We can not improve \( \beta \) to be 1 or \( 1 + \varepsilon \) as Chou-Wang’s paper [8] or Theorem 2.

An application of the interior estimates may to prove rigidity theorems for \( k \)-Hessian equations. Consider the entire solutions \( u \) in \( n \)-dimensional Euclidean spaces of the following equations,

\[
\sigma_k(D^2u) = 1.
\]

S.-Y. A. Chang and Y. Yuan in [6] proposed a problem that: Are the entire solutions of (1.5) with lower bound only quadratic polynomials ?

Let’s review known results related the above problem. For \( k = 1 \), (1.5) is a linear equation. It is a obvious result coming from the Liouville property of the harmonic functions. For \( k = n \), Monge-Ampère equation case, it is a well know theorem. For \( n = 2 \), K. Jörgens [12] proved that every entire strictly convex solution is a quadratic polynomial. Then, E. Calabi [4] obtained the same result for \( n = 3, 4, 5 \). At last, A.V. Pogorelov [13, 14] gave a proof for all dimensions. Then, S.Y. Cheng and S.T. Yau [7] gave another more geometry proof. In 2003, L. Caffarelli and Y. Li, [5] extended the theorem of Jörgens, Calabi and Pogorelov.

For \( k = 2 \), S.-Y. A. Chang and Y. Yuan [6] have proved that, if

\[
D^2u \geq \delta - \sqrt{\frac{2n}{n-1}},
\]

for any \( \delta > 0 \), then the entire solution of the equation (1.5) only have quadratic polynomials. For general \( k \), it is still open, but J. Bao, J.Y. Chen, B. Guan and M. Ji in [2] obtained that, strictly convex entire solutions of (1.5), satisfying a quadric
growth are quadratic polynomials. Here, the quadratic growth means that, there is some positive constant \(c, b\) and sufficiently large \(R\), such that,
\[
(1.6) \quad u(x) \geq c|x|^2 - b, \text{ for } |x| \geq R.
\]
Note that, our interior estimates Theorem 2 holds for \(k+1\) convex solutions. Hence, we can relax their restriction. In deed, we have proved,

**Theorem 4.** The entire solutions in \(k+1\) convex cone of the equations (1.5) defined in \(\mathbb{R}^n\) with quadratic growth are quadratic polynomials.

In our proof, we don’t need the assumption of strictly convexity. Hence, we do not use the estimates of L. Caffaralli. Now, we give the following two Lemmas, which will be needed in our proof.

**Lemma 5.** Set \(k > l\). For \(\alpha = \frac{1}{k-l}\), we have,
\[
(1.7) \quad -\frac{\sigma_{pp,qq}^k}{\sigma_k} u_{pph} u_{qqh} + \frac{\sigma_{pp,qq}^l}{\sigma_l} u_{pph} u_{qqh} \geq \left( \frac{(\sigma_k)_h}{\sigma_k} - \frac{(\sigma_l)_h}{\sigma_l} \right) \left( (\alpha - 1) \frac{(\sigma_k)_h}{\sigma_k} - (\alpha + 1) \frac{(\sigma_l)_h}{\sigma_l} \right).
\]

Further more, for sufficiently small \(\delta > 0\), we have,
\[
(1.8) \quad -\sigma_{pp,qq}^k u_{pph} u_{qqh} + (1 - \alpha + \frac{\alpha}{\delta}) \frac{\sigma_k^2_h}{\sigma_k} \geq \sigma_k(\alpha + 1 - \delta\alpha) \left[ \frac{(\sigma_l)_h}{\sigma_l} \right]^2 - \frac{\sigma_k}{\sigma_l} \sigma_{pp,qq}^l u_{pph} u_{qqh}.
\]

The another one is,

**Lemma 6.** Denote \(\text{Sym}(n)\) the set of all \(n \times n\) symmetric matrices. Let \(F\) be a \(C^2\) symmetric function defined in some open subset \(\Psi \subset \text{Sym}(n)\). At any diagonal matrix \(A \in \Psi\) with distinct eigenvalues, let \(\tilde{F}(B, B)\) be the second derivative of \(C^2\) symmetric function \(F\) in direction \(B \in \text{Sym}(n)\), then
\[
(1.9) \quad \tilde{F}(B, B) = \sum_{j,k=1}^n \tilde{f}^j k B_{jj} B_{kk} + 2 \sum_{j<k} \frac{j \cdot j - j \cdot k}{\kappa_j - \kappa_k} B_{jk}^2.
\]

The proof of the first Lemma can be found in [10] and [11]. The second Lemma can be found in [1] and [3].

The paper is organized by three sections. The first section gives the interior estimates of 2-Hessian case. The second section gives the interior estimates for \(k+1\) convex solutions. The last section proves the rigidity theorem.

2. AN INTERIOR \(C^2\) ESTIMATE FOR \(\sigma_2\) EQUATIONS

In this section, we prove Theorem 1. We consider the following test function,
\[
M = \max_{|\xi|=1, x \in \Omega} (-u)^\beta \exp\left\{ \frac{\varepsilon}{2} |Du|^2 + \frac{a}{2} |x|^2 \right\} u_{\xi\xi},
\]

3
where $\beta, \varepsilon$ and $a$ are three constants which we will be determined later. Suppose that $M$ achieve its maximum value in $\Omega$ at some point $x_0$ along some direction $\xi$. We can assume that $\xi = (1, 0, \cdots, 0)$. By rotating the coordinate, we diagonal the matrix $(u_{ij})$, and we also can assume that $u_{11} \geq u_{22} \cdots \geq u_{nn}$.

Hence, at $x_0$, differentiating the test function twice, we have

\begin{equation}
(2.1) \quad \frac{\beta u_i}{u} + \frac{u_{11i}}{u_{11}} + \varepsilon u_i u_{11i} + ax_i = 0,
\end{equation}

and,

\begin{equation}
(2.2) \quad \frac{\beta u_{11}}{u} - \frac{\beta u_{11}^2}{u^2} + \frac{u_{111i}}{u_{11}} - \frac{u_{111i}^2}{u_{11}^2} + \sum_k \varepsilon u_{kk} u_{11i} + \varepsilon u_{11i}^2 + a \leq 0.
\end{equation}

In the above inequality, contracting with $\sigma_{ii}^2$, we have,

\begin{equation}
(2.3) \quad \sigma_{ii}^2 u_{11j} = f_j + u_{11j} + f_{pj} u_{jj},
\end{equation}

and,

\begin{equation}
(2.4) \quad \sigma_{ii}^2 u_{11j} + \sigma_{pq,rs}^2 u_{pqj} u_{rsj} \geq -C - C u_{11j}^2 + \sum_k f_{pk} u_{kk}.
\end{equation}

Inserting (2.1) into (2.2), we have,

\begin{equation}
(2.5) \quad 0 \geq \frac{2 \beta \sigma_2}{u} - \frac{\beta \sigma_{ii}^2 u_{11i}^2}{u^2} + \frac{1}{u_{11}} [-C - C u_{11j}^2 + \sum_k f_{pk} u_{kk} - K(\sigma_2)^2 + K(\sigma_2)^2 - \sigma_{pq,rs}^2 u_{pqj} u_{rsj}] - \frac{\sigma_{ii}^2 u_{11i}^2}{u_{11}^2} + \sum_k \varepsilon u_{kk} \sigma_{11i}^2 u_{11i} + \varepsilon \sigma_{11i}^2 u_{11i}^2 + (n - 1)a \sigma_1.
\end{equation}

Using (2.1) and (2.3), we have,

\begin{equation}
\frac{1}{u_{11}} \sum_k f_{pk} u_{kk} + \sum_k \varepsilon u_{kk} \sigma_{11i}^2 u_{11i} \geq -C - \sum_k \frac{\beta u_k f_{pk}}{u}.
\end{equation}

Note that

\begin{equation}
-\sigma_{pq,rs}^2 u_{pqj} u_{rsj} = -\sigma_{pq,qq}^2 u_{pqj} u_{qqj} + \sum_{p \neq q} u_{pqj}^2 \\
\geq -\sigma_{pq,qq}^2 u_{pqj} u_{qqj} + 2 \sum_{i \neq 1} u_{11i}^2.
\end{equation}

Using Lemma 5 there exists some sufficiently large constant $K$ depending on $f$, such that,

\begin{equation}
K(\sigma_2)^2 - \sigma_{pq,qq}^2 u_{pqj} u_{qqj} \geq 0.
\end{equation}
Using the above two formulas, inequality (2.5) becomes,

\[
\frac{-C}{u} \geq -\frac{\beta \sigma_{ii}^2 u_i^2}{u^2} + \frac{2}{u_{11}} \sum_{i \neq 1} u_{11i}^2 - \frac{\sigma_{ii}^2 u_{11i}^2}{u_{11}^2} + \varepsilon \sigma_{ii}^2 u_{ii}^2 + (n-1)a\sigma_1 - Cu_{11} - C.
\]

Take a sufficiently large \( a \) such that,

\[
(n-1)a\sigma_1 - Cu_{11} - C \geq a\sigma_1.
\]

Here, we always assume that \( u_{11} \) is sufficiently large. Now we should divide into two cases to deal with other third order derivatives.

(A) Suppose \( \sum_{i=2}^{n-1} \lambda_i \leq \lambda_1/3 \). In this case, using (2.1), we have,

\[
-\frac{\beta \sigma_{ii}^2 u_i^2}{u^2} \geq -\frac{2\sigma_{ii}^2 u_{11i}^2}{u_{11}^2} - \frac{2\sigma_{ii}^2 u_{11i}^2}{u_{11}^2} (\varepsilon u_{ii} + ax_i)^2.
\]

Using (2.6) and (2.7), we have,

\[
-\frac{C}{u} \geq -\frac{\beta \sigma_{ii}^2 u_i^2}{u^2} + \frac{2}{u_{11}} \sum_{i \neq 1} u_{11i}^2 - (1 + \frac{2}{\beta}) \sum_{i \neq 1} \frac{\sigma_{ii}^2 u_{11i}^2}{u_{11}^2}
\]

\[
- \frac{\sigma_{ii}^2 u_{11i}^2}{u_{11}^2} + \varepsilon \sigma_{ii}^2 u_{ii}^2 - \sum_{i \neq 1} \frac{2\sigma_{ii}^2 u_{11i}^2}{u_{11}^2} (\varepsilon u_{ii} + ax_i)^2 + a\sigma_1.
\]

Since, \( \sum_{i=2}^{n-1} \lambda_i \leq \frac{\lambda_1}{3} \), we have, for sufficiently large \( \beta \),

\[
2 \frac{\sigma_{ii}^2 u_{11i}^2}{u_{11}^2} - (1 + \frac{2}{\beta}) \sum_{i \neq 1} \frac{\sigma_{ii}^2 u_{11i}^2}{u_{11}^2} \geq 0.
\]

Again, using (2.7), we have,

\[
-\frac{\sigma_{ii}^2 u_{11i}^2}{u_{11}^2} \geq -2\sigma_{ii}^2 \left( \frac{\beta u_{11}}{u} \right) (\varepsilon u_{ii} + ax_i)^2.
\]

Then we obtain,

\[
\frac{-C}{u} + \frac{(\beta + 2\beta^2)\sigma_{ii}^2 u_i^2}{u^2}
\]

\[
\geq \varepsilon \sigma_{ii}^2 u_{ii}^2 - \sum_{i \neq 1} \frac{2\sigma_{ii}^2}{\beta} (\varepsilon u_{ii} + ax_i)^2 - 2\sigma_{ii}^2 (\varepsilon u_{ii} + ax_i)^2 + a\sigma_1
\]

\[
\geq \sum_{i} \varepsilon \sigma_{ii}^2 u_{ii}^2 - \sum_{i \neq 1} \frac{4\sigma_{ii}^2}{\beta} \varepsilon^2 u_{ii}^2 u_i^2 - \sum_{i \neq 1} \frac{4\sigma_{ii}^2}{\beta} a^2 x_i^2 - 4\sigma_{ii}^2 \varepsilon^2 u_{ii}^2 u_i^2
\]

\[
- 4\sigma_{ii}^2 a^2 x_i^2 + au_{11}.
\]
We choose \( \varepsilon \) and \( \beta \), such that
\[
\varepsilon > 8\varepsilon^2 \max_{\Omega} |Du|^2, \quad \text{and} \quad \beta > a^2.
\]

Hence, (2.9) becomes,
\[
(2.10) 
-C \frac{u}{u^2} + \frac{C\sigma_{11}}{u^2} \geq \varepsilon \frac{\sigma_{11}}{2} u_{11}^2 - 4\sigma_{11} a^2 x_1^2 + (a - C)u_{11}.
\]

Taking \( a \) and \( u_{11} \) sufficiently large, we obtain (1.3).

(B) If \( \sum_{i=2}^{n-1} \lambda_i \geq \frac{\lambda_1}{3} \), then we have \( \frac{\lambda_1}{3(n-2)} \leq \lambda_2 \leq \lambda_1 \). Using (2.7), (2.6) becomes,
\[
(2.11) 
-C \frac{u}{u^2} + \sum_i \frac{(\beta + 2\beta^2) \sigma_{ii}^2 u_{ii}^2}{u^2} 
\geq \sum_i \varepsilon \sigma_{ii}^2 u_{ii}^2 - 4 \sum_i \sigma_{ii} \varepsilon u_{ii}^2 u_{ii}^2 - 4 \sum_i \sigma_{ii} a^2 x_i^2 + a\sigma_1.
\]

We should divide this case into two subcases, (B1) \( \sigma_{22}^2 \geq 1 \) and (B2) \( \sigma_{22}^2 < 1 \). We also take a sufficiently small \( \varepsilon \), such that \( \varepsilon > 8\varepsilon^2 \max_{\Omega} |Du|^2 \). In both subcases, the right hand side of the above inequality always has high order term \( u_{11}^2 \) or \( u_{31}^2 \), then we have (1.3). See [11] for detail.

3. An interior \( C^2 \) estimate for \( k + 1 \) convex solutions

In this section, we consider the interior estimates for \( k \) Hessian equations (1.1). We will prove Theorem 2. Before we start our proof, we need the following fact.

**Lemma 7.** Suppose \( u \) is a \( k + 1 \) convex solution for equation (1.1). Then, there is some constant \( K_0 > 0 \) depending on the diameter of the domain \( \Omega \), \( \sup_{\Omega} |u| \) and \( \sup_{\Omega} |\nabla u| \), such that,
\[
D^2 u + K_0 I \geq 0.
\]
Here \( \geq 0 \) means the matrix is semi positive definite.

**Proof.** We choose \( K_0 \) satisfying
\[
\left( \frac{K_0}{n} \right)^k \geq \sup_{\Omega} f(x, u, \nabla u).
\]

Suppose \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) is the eigenvalues of the Hessian \( D^2 u \). Then, we have, using \( u \in \Gamma_{k+1} \),
\[
\sigma_k = \sigma_{k-1} \left( \lambda_1^k \right) \lambda_1 + \sigma_k \left( \lambda_1^1 \right) \lambda_1 \geq \sigma_{k-1} \left( \lambda_1^1 \right) \lambda_1 \\
= \sigma_{k-2} \left( \lambda_1^{12} \right) \lambda_1 \lambda_2 + \lambda_1 \sigma_{k-1} \left( \lambda_1^{12} \right) \lambda_1 \lambda_2 \\
= \cdots \geq \cdots \\
\geq \lambda_1 \lambda_2 \cdots \lambda_k \geq \lambda_k^k.
\]

Hence, \( \lambda_k \leq K_0/n \). Since, \( u \in \Gamma_k \), we have,
\[
\sum_{i=k}^{n} \lambda_i > 0,
\]
which implies that $\lambda_n + K_0 \geq 0$. We obtain the Lemma. □

We use the $m$-polynomials. Here, $m$ should be sufficiently large to give more convexity, since we have more negative terms. Let’s consider the following test function,

$$
\varphi = m \log(-u) + \log P_m + \frac{mN}{2} |Du|^2,
$$

where \( P_m = \sum_j \kappa_j^m \), and \( \kappa_j = \lambda_j + K_0 \),

and \( N \) is some undetermined constant. The \( \lambda_1, \lambda_2, \cdots, \lambda_n \) are eigenvalues of the Hessian \( D^2u \). By Lemma 7, \( \kappa_1, \kappa_2, \cdots, \kappa_n \) are non negative. Suppose that function \( \varphi \) achieves its maximum value in \( \Omega \) at some point \( x_0 \). Rotating the coordinates, we assume that \( (u_{ij}) \) is diagonal matrix at \( x_0 \), and \( \kappa_1 \geq \kappa_2 \cdots \geq \kappa_n \).

Differentiating our test function twice and using Lemma 6, at \( x_0 \), we have,

$$
\sum_j \frac{\kappa_j^{m-1} u_{jj}i}{P_m} + Nu_{ii} + \frac{u_i}{u} = 0,
$$

and,

$$
0 \geq \frac{1}{P_m} \left[ \sum_j \kappa_j^{m-1} u_{jj}i + (m - 1) \sum_j \kappa_j^{m-2} u_{jj}i + \sum_{p \neq q} \frac{\kappa_p^{m-1} - \kappa_q^{m-1}}{\kappa_p - \kappa_q} u_{pqi}^2 \right]
$$

- \( \frac{m}{P_m^2} \left( \sum_j \kappa_j^{m-1} u_{jj}i \right)^2 \) + \( \sum_s N u_{si} u_{si} + N u_{ii}^2 + \frac{u_i}{u} - \frac{u_i^2}{u^2} \).

At \( x_0 \), differentiating the equation (1.1) twice, we have,

$$
\sigma_{ii}^k u_{ij} = \psi_p u_{jj} + \psi_u u_{jj} + \psi_j,
$$

and

$$
\sigma_{ii}^k u_{ij} + \sigma_{ii}^{pq,rs} u_{pqi} u_{rsj} \geq -C - C u_{11}^2 + \sum_s \psi_p, u_{kkj}.
$$

Here, \( C \) is a constant depending on \( f \), the diameter of the domain \( \Omega \), \( \sup_{\Omega} |u| \) and \( \sup_{\Omega} |\nabla u| \). Contacting \( \sigma_{ii}^k \) in both side of (3.3), and using (3.4) (3.5), we get,

$$
0 \geq \frac{1}{P_m} \left[ \sum_l \kappa_l^{m-1} (-C - C u_{11}^2 + \sum_s \psi_p, u_{ssl} - K(\sigma_k)_l^2 + K(\sigma_k)_l^2 - \sigma_{pq,rs}^k u_{pqi} u_{rsj}) + (m - 1) \sigma_{ii}^k \sum_j \kappa_j^{m-2} u_{jj}i + \sigma_{ii}^k \sum_{p \neq q} \frac{\kappa_p^{m-1} - \kappa_q^{m-1}}{\kappa_p - \kappa_q} u_{pqi}^2 \right]
$$

- \( \frac{m}{P_m^2} \left( \sum_j \kappa_j^{m-1} u_{jj}i \right)^2 \) + \( \sum_s N u_{si} u_{si} \sigma_{ii}^k + N u_{ii}^2 \sigma_{ii}^k + \frac{k \sigma_k}{u} - \frac{\sigma_{ii}^k u_i^2}{u^2} \).
Using (3.2) and (3.4), we have,

\[
\frac{1}{P_m} \sum_l \sum_s \kappa_l^{m-1} \psi_p u_{sll} + \sum_s N u_s \sigma_k^{ii} u_{ssi} \geq - \sum_s \psi_p \frac{u_s}{u} - C.
\]

On the other hand, we have,

\[
-\sigma_k^{pq,rs} u_{pql} u_{rsl} = -\sigma_k^{qq,pp} u_{qpl} u_{qql} + \sigma_k^{pp,qq} u_{pql}^2.
\]

Then, using the previous two formulas, (2.8) becomes,

\[
(3.7)
0 \geq \frac{1}{P_m} \left[ \sum_l \kappa_l^{m-1} (-C - Cu^2_1 - K \psi_p^2 u_{ll}^2 + K (\sigma_k)_l^2 - \sigma_k^{pp,qq} u_{pql} u_{qql} + \sigma_k^{pp,qq} u_{pql}^2) \right]
+ (m - 1) \sigma_k^{ii} \sum_j \kappa_j^{m-2} u_{jj}^2 + \sigma_k^{ii} \sum_{p \neq q} \kappa_p^{m-1} - \kappa_q^{m-1} \frac{u_{pq}^2}{\kappa_p - \kappa_q}
- \frac{m \sigma_k^{ii}}{P_m^2} \left( \sum_j \kappa_j^{m-1} u_{jj}^2 \right)^2 + Nu_{ii}^2 \sigma_k^{ii} + \frac{k \sigma_k^{ii}}{u} - \frac{\sigma_k^{ii} u^2_i}{u^2} - \sum_s \psi_p \frac{u_s}{u}.
\]

Let’s deal with the third order derivatives. Denote,

\[
A_i = \frac{\kappa_i^{m-1}}{P_m} (K (\sigma_k)_l^2 - \sum_{p,q} \sigma_k^{pp,qq} u_{ppl} u_{qql}), B_i = \frac{2 \kappa_i^{m-1}}{P_m} \sum_j \sigma_k^{jj,ii} u_{jjji}^2,
\]

\[
C_i = \frac{m - 1}{P_m} \sigma_k^{ii} \sum_j \kappa_j^{m-2} u_{jjj}^2, D_i = \frac{2 \sigma_k^{jj}}{P_m} \sum_{j \neq i} \kappa_j^{m-1} - \kappa_i^{m-1} \frac{u_{jjj}^2}{\kappa_j - \kappa_i},
\]

\[
E_i = \frac{m \sigma_k^{ii}}{P_m^2} \left( \sum_j \kappa_j^{m-1} u_{jjj}^2 \right)^2.
\]

We divide two cases to deal with the third order derivatives, \(i \neq 1\) and \(i = 1\).

**Lemma 8.** For any \(i \neq 1\), we have

\[
A_i + B_i + C_i + D_i - (1 + \frac{1}{m}) E_i \geq 0,
\]

for sufficiently large \(m\).

**Proof.** At first, by Lemma 5 for sufficiently large \(K\), we have,

\[
(3.8) \quad K (\sigma_k)_l^2 - \sigma_k^{pp,qq} u_{pql} u_{qql} \geq \sigma_k (1 + \frac{\alpha}{2}) \frac{(\sigma_1)_l^2}{\sigma_1} \geq 0.
\]

Hence, \(A_i \geq 0\).
Then, we also have,

\begin{equation}
(3.9) \quad P_m^2[B_i + C_i + D_i - (1 + \frac{1}{m})E_i] \\
= \sum_{j \neq i} P_m [2\kappa_j^{m-1}\sigma_{k,j}^{m-1} + (m - 1)\kappa_j^{m-2}\sigma_{k,j}^{m-2} + 2\sigma_k^{j_j} \sum_{l=0}^{m-2} \kappa_i^{m-2-l}\kappa_j^{l}u_{jji}^2] \\
+ P_m (m - 1)\sigma_{k,j}^{m-2}u_{jji}^2 \\
- (m + 1)\sigma_k^{j_j}(\sum_{j \neq i} \kappa_j^{m-2}u_{jji}^2 + \sum_{p \neq q} \kappa_p^{m-1}\kappa_q^{m-1}u_{ppi}u_{qqi}).
\end{equation}

Note that

\begin{equation}
(3.10) \quad \kappa_j^{j_j}\sigma_k^{j_j} + \sigma_k^{j_j} = (\lambda_j + K_0)\sigma_{k,j}^{j_j} + \sigma_k^{j_j} \\
= K_0\sigma_{k,j}^{j_j} + \sigma_k^{j_j} - \sigma_{k-1}(\lambda|i) + \lambda_i\sigma_{k-2}(\lambda|i) + \sigma_{k-1}(\lambda|i) \\
= (K_0 + \lambda_i)\sigma_{k,j}^{j_j} + \sigma_k^{j_j} \\
\geq \sigma_k^{j_j}.
\end{equation}

For any index $j \neq i$, using the above inequality, we have,

\begin{equation}
(3.11) \quad P_m [2\kappa_j^{m-1}\sigma_{k,j}^{m-1} + (m - 1)\kappa_j^{m-2}\sigma_{k,j}^{m-2} + 2\sigma_k^{j_j} \sum_{l=0}^{m-2} \kappa_i^{m-2-l}\kappa_j^{l}u_{jji}^2] \\
- (m + 1)\sigma_k^{j_j}\kappa_j^{m-2}u_{jji}^2 \\
\geq P_m (m - 1)\sigma_{k,j}^{j_j}\kappa_j^{m-2}u_{jji}^2 - (m + 1)\sigma_k^{j_j}\kappa_j^{m-2}u_{jji}^2 \\
+ 2P_m\sigma_k^{j_j}(\sum_{l=0}^{m-3} \kappa_i^{m-2-l}\kappa_j^{l}u_{jji}^2) \\
\geq (m + 1)(P_m - \kappa_j^m)\sigma_{k,j}^{j_j}\kappa_j^{m-2}u_{jji}^2 + 2P_m\sigma_k^{j_j}(\sum_{l=0}^{m-3} \kappa_i^{m-2-l}\kappa_j^{l}u_{jji}^2)
\end{equation}

Using Cauchy-Schwarz inequalities, we have,

\begin{equation}
(3.12) \quad 2 \sum_{j \neq i} \sum_{p \neq i,j} \kappa_j^{m-2}\kappa_p^{m}u_{jji}^2 \\
= \sum_{p \neq i} \sum_{q \neq i,p} \kappa_p^{m-2}\kappa_q^{m}u_{ppi}^2 + \sum_{q \neq i} \sum_{p \neq i,q} \kappa_q^{m-2}\kappa_p^{m}u_{qqi}^2 \\
\geq 2 \sum_{p \neq q;p,q \neq i} \kappa_p^{m-1}\kappa_q^{m-1}u_{ppi}u_{qqi}.
\end{equation}
Hence, by \(3.9\), \(3.11\) and \(3.12\), we obtain,

\[
(3.13) \quad \sum_{j \neq i} (m + 1) \kappa_i^m \kappa_j^{m-2} \sigma_k^{i \mid j} u_{jj}^i + ((m - 1)(P_m - \kappa_i^m) - 2\kappa_i^m) \kappa_i^{m-2} \sigma_k^{i \mid j} u_{ii}^i
\]

\[-2(m + 1)\sigma_k^{i \mid j} \kappa_i^{m-1} u_{ii}^j \sum_{j \neq i} \kappa_j^{m-1} u_{jj}^i + 2P_m \sum_{j \neq i} \sigma_j^{i \mid j} \sum_{l=0}^{m-3} \kappa_i^{m-2-l} \kappa_j^l u_{jj}^i \]

\[
\geq \sum_{j \neq i} [(m + 1)\kappa_i^m \kappa_j^{m-2} \sigma_k^{i \mid j} + 2\kappa_i^m \kappa_j^{m-2-l} \kappa_j^l u_{jj}^i + ((m - 1)(P_m - \kappa_i^m) - 2\kappa_i^m) \kappa_i^{m-2} \sigma_k^{i \mid j} u_{ii}^j - 2(m + 1)\sigma_k^{i \mid j} \kappa_i^{m-1} u_{ii}^j \sum_{j \neq i} \kappa_j^{m-1} u_{jj}^i.
\]

We divide two cases to discuss.
Case (A) For \(\lambda_j \geq \lambda_i\), we divide into two sub cases to discuss. If \(\lambda_i \geq K_0\), for \(1 \leq l \leq m - 3\), we have,

\[
(3.14) \quad 2\kappa_i^m \sigma_k^{i \mid j} \kappa_i^{m-2-l} \kappa_j^l = 2\kappa_1^m (\lambda_i \sigma_k^{i \mid j} + \sigma_{k-1}(\lambda_i | j)) \kappa_i^{m-2-l} \kappa_j^l
\]

\[
\geq \kappa_1^m (\kappa_i \sigma_k^{i \mid j} + \sigma_{k-1}(\lambda_i | j)) \kappa_i^{m-2-l} \kappa_j^l
\]

\[
\geq \kappa_1^m (\kappa_j \sigma_k^{i \mid j} + \sigma_{k-1}(\lambda_i | j)) \kappa_i^{m-2-l} \kappa_j^l
\]

\[
\geq \kappa_1^m (\lambda_i \sigma_k^{i \mid j} + \sigma_{k-1}(\lambda_i | j)) \kappa_i^{m-2-l} \kappa_j^l
\]

\[
= \kappa_1^m \kappa_i^{m-2-l} \kappa_j^l \sigma_k^{i \mid j}.
\]

Here, we have used \(\sigma_{k-1}(\lambda_i | j) > 0\) since \(u\) is a \(k + 1\) convex solution.
If \(\lambda_i < K_0\), for all \(k \leq l \leq k + 8\), we have,

\[
\kappa_1^{l+1} \sigma_k^j \geq \kappa_1^l \lambda_i \sigma_k^j \geq c_0 \sigma_{k-1} \kappa_i^l \sigma_k^j \geq \sigma_k^j
\]

when \(\lambda_i\) is sufficiently large. Here, we have used \(\lambda_i \sigma_k^j \geq c_0 \sigma_{k-1}\). Hence, we have,

\[
(3.15) \quad \kappa_1^m \sigma_k^{i \mid j} \kappa_i^{m-2-l} \kappa_j^l \geq \kappa_1^{l+1} \sigma_k^j \kappa_i^{m-2-l} \kappa_j^l \kappa_i^{m-2-l} \kappa_1 \geq \sigma_k^j \kappa_i^{m-2-l} \kappa_i^m \frac{\kappa_1}{\kappa_i^{l+2}}
\]

\[
\geq \sigma_k^j \kappa_i^{m-2-l} \kappa_i^m.
\]

Since \(\lambda_i < K_0\), we have used \(\kappa_i \geq \kappa_i^l+2\) for sufficiently large \(\lambda_i\).
Case (B) For \(\lambda_j < \lambda_i\), obviously, we have,

\[
2\kappa_1^m \sigma_k^{i \mid j} \kappa_i^{m-2-l} \kappa_j^l \geq 2\kappa_1^m \kappa_i^{m-2-l} \kappa_j^l \sigma_k^j.
\]

Combing the above two cases, we get, for \(k \leq l \leq k + 8\),

\[
(3.16) \quad 2\kappa_1^m \sigma_k^{i \mid j} \kappa_i^{m-2-l} \kappa_j^l \geq \kappa_1^m \kappa_i^{m-2-l} \sigma_k^j.
\]
Thus, (3.13) becomes,

\[(3.17) \quad P_m^2(B_1 + C_1 + D_1 - (1 + \frac{1}{m})E_i)\]
\[\geq \sum_{j \neq i} (m + 8)\kappa_i^m \kappa_j^m \sigma_k^2 u_{jj}^2 + ((m - 1)(P_m - \kappa_i^m) - 2\kappa_i^m)\kappa_i^m \sigma_k^2 u_{ii}^2\]
\[-2(m + 1)\sigma_k^2 \kappa_i^m \kappa_j^m u_{ii} u_{jj} + \sum_{j \neq i} \kappa_j^m \kappa_i^m \kappa_j^m u_{ii} u_{jj}\]
\[\geq (m + 8)\kappa_i^m \kappa_j^m \sigma_k^2 u_{jj}^2 + ((m - 1)(P_m - \kappa_i^m) - 2\kappa_i^m)\kappa_i^m \sigma_k^2 u_{ii}^2\]
\[-2(m + 1)\sigma_k^2 \kappa_i^m \kappa_j^m u_{ii} u_{jj} + \sum_{j \neq i} \kappa_j^m \kappa_i^m \kappa_j^m u_{ii} u_{jj}\]
\[\geq 0.\]

Here, we have used, for \(m \geq 10,\)
\[(m + 8)(m - 3) \geq (m + 1)^2.\]

So, we take
\[m = \max\{10, k + 11\},\]
which is sufficiently large. \(\square\)

The left case is \(i = 1.\) Let's begin with the following Lemma which is modified from [11].

**Lemma 9.** For \(\mu = 1, \ldots, k - 1,\) if there exists some positive constant \(\delta \leq 1,\)
such that \(\lambda_\mu/\lambda_1 \geq \delta.\) Then there exists two sufficiently small positive constants \(\eta, \delta'\)
depending on \(\delta,\) such that, if \(\lambda_{\mu+1}/\lambda_1 \leq \delta',\) we have,
\[A_1 + B_1 + C_1 + D_1 - (1 + \frac{\eta}{m})E_1 \geq 0.\]

**Proof.** At first, we have,
\[(3.18) \quad P_m^2(B_1 + C_1 + D_1 - (1 + \frac{\eta}{m})E_1)\]
\[\geq \sum_{j \neq 1} ((1 - \eta)P_m + (m + \eta)\kappa_j^m)\kappa_j^m \sigma_k^2 u_{jj}^2\]
\[+ ((m - 1)(P_m - \kappa_j^m) - (1 + \eta)\kappa_j^m)\kappa_j^m \sigma_k^2 u_{11}^2\]
\[-2(m + \eta)\sigma_k^2 \kappa_j^m \kappa_j^m \kappa_j^m u_{ii} u_{jj} + 2P_m \sum_{j \neq 1} \sigma_k^j \left( \sum_{l=0}^{m-3} \kappa_j^{m-2-l} \kappa_j^l \right) u_{jj}^2.\]

Since \(\sigma_k^j \geq \sigma_k^{11}\) for any \(j \neq 1,\) for \(m \geq 5,\) it is obvious,
\[2P_m \sum_{j \neq 1} \sigma_k^j \left( \sum_{l=0}^{m-3} \kappa_j^{m-2-l} \kappa_j^l \right) u_{jj}^2 \geq 3 \sum_{j \neq 1} \kappa_j^m \sigma_k^2 u_{jj}^2 + 2P_m \sum_{j \neq 1} \kappa_j^m \sigma_k^2 u_{jj}^2.\]
Hence, by (3.18), we obtain,

\[
\begin{align*}
P_m^2 (B_1 + C_1 + D_1 - (1 + \frac{\eta}{m})E_1) \\
\geq \sum_{j \neq 1} (m + 4)\kappa_1^{m-1}k_j \kappa_1^{m-2}\sigma_k^{11}u_{jj1} + (m - 1) \sum_{j \neq 1} \kappa_j^{m-1}\kappa_1^{m-2}\sigma_k^{11}u_{111} \\
- 2(m + \eta)\sigma_k^{11}\kappa_1^{m-1}u_{111} \sum_{j \neq 1} \kappa_j^{m-1}u_{jj1} \\
- (1 + \eta)\kappa_1^{m-2}\sigma_k^{11}u_{111} + 2P_m\kappa_1^{m-2} \sum_{j \neq 1} \sigma_k^{jj}u_{jj1} \\
\geq - (1 + \eta)\kappa_1^{m-2}\sigma_k^{11}u_{111} + 2P_m\kappa_1^{m-2} \sum_{j \neq 1} \sigma_k^{jj}u_{jj1}.
\end{align*}
\]

Here, we have used

\[(m + 4)(m - 1) \geq (m + 1)^2,\]

for \(m \geq 5\). By Lemma 5, we have,

\[
\begin{align*}
A_1 & \geq \frac{\kappa_1^{m-1}}{P_m} \left[ \sigma_k (1 + \frac{\alpha}{2}) (\frac{\sigma_\mu}{\sigma_\mu})_1^2 - \frac{\sigma_k^{pp,qq}u_{pp1}u_{qq1}}{\sigma_\mu} \right] \\
& \geq \frac{\kappa_1^{m-1}}{P_m \sigma_\mu} \left[ (1 + \frac{\alpha}{2}) \sum_a (\sigma_\mu^{aa}u_{aa1})^2 + \frac{\alpha}{2} \sum_{a \neq b} \sigma_\mu^{aa} \sigma_\mu^{bb}u_{aa1}u_{bb1} \\
& \quad + \sum_{a \neq b} (\sigma_\mu^{aa} - \sigma_\mu^{bb})u_{aa1}u_{bb1} \right].
\end{align*}
\]

For \(\mu = 1\), notice that \(\sigma_1^{aa} = 1\) and \(\sigma_1^{aa,bb} = 0\). Then, we have,

\[
(1 + \frac{\alpha}{2}) \sum_{a, b} u_{aa1}u_{bb1} \geq 2(1 + \frac{\alpha}{2}) \sum_{a \neq 1} u_{aa1}u_{111} + (1 + \frac{\alpha}{2})u_{111}^2 \\
\geq (1 + \frac{\alpha}{4})u_{111}^2 - C_\alpha \sum_{a \neq 1} u_{aa1}^2.
\]

Then, we get,

\[
\begin{align*}
P_m^2 A_1 & \geq \frac{P_m\kappa_1^{m-1}\sigma_k (1 + \frac{\alpha}{4})u_{111}^2 - \kappa_1^{m-1}P_mC^\alpha \sum_{a \neq 1} u_{aa1}^2}{\sigma^2_1} \\
& \geq \frac{P_m\kappa_1^{m-2}\sigma_k^{11}}{(1 + \sum_{j \neq 1} \lambda_j/\lambda_1)^2 (1 + \frac{\alpha}{4})u_{111}^2 - C_\alpha \sum_{a \neq 1} u_{aa1}^2}{\sigma^2_1} \\
& \geq (1 + \eta)P_m\kappa_1^{m-2}\sigma_k^{11}u_{111}^2 - C_\alpha P_m\kappa_1^{m-1}u_{aa1}^2.
\end{align*}
\]

The last two inequalities come from,

\[
\sigma_k \geq \lambda_1 \sigma_k^{11}.
\]
for sufficiently large $\lambda_1$, and
\begin{equation}
1 + \frac{\alpha}{4} \geq (1 + \eta)(1 + (n - 1)\delta')^2.
\end{equation}

For $\mu \geq 2$, obviously, for $a \neq b$, we have,
\begin{equation}
\sigma_{\mu}^{\alpha a} - \sigma_{\mu}^{\alpha b} = (\lambda_b \sigma_{\mu - 2}(\lambda|ab) + \mu_1 - 1(\lambda|ab)) (\lambda_a \sigma_{\mu - 2}(\lambda|ab) + \mu_1 - 1(\lambda|ab))
\end{equation}
\begin{equation}
- (\lambda_a \lambda_b \sigma_{\mu - 2}(\lambda|ab) + \mu_1 - 1(\lambda|ab) + \mu_1 - 1(\lambda|ab) + \sigma_{\mu}(\lambda|ab)) \sigma_{\mu - 2}(\lambda|ab)
\end{equation}
\begin{equation}
\geq 0.
\end{equation}

The last inequality comes from Newton inequality. Since $u \in \Gamma_{k+1} \subset \Gamma_{n+2}$, we have, for any $a \leq \mu$,
\begin{equation}
\sigma_{\mu}^{aa} \geq \frac{\lambda_1 \cdots \lambda_\mu}{\mu_a}.
\end{equation}

For $a, b \leq \mu$, we claim,
\begin{equation}
\sigma_{\mu - 1}(\lambda|ab) \leq C \frac{\lambda_1 \cdots \lambda_{\mu + 1}}{\lambda_a \lambda_b}, \quad \sigma_{\mu}(\lambda|ab) \leq C \frac{\lambda_1 \cdots \lambda_{\mu + 2}}{\lambda_a \lambda_b}
\end{equation}
\begin{equation}
\sigma_{\mu - 2}(\lambda|ab) \leq C \frac{\lambda_1 \cdots \lambda_{\mu}}{\lambda_a \lambda_b}.
\end{equation}

The proof of the above three inequalities are same. We only give more detail for the first one. Since, $u \in \Gamma_{n+2}$, then, for any index $i \geq \mu + 1$, there is some constant $C$ such that,
\begin{equation}
|\lambda_i| \leq C \mu_{\mu + 1}.
\end{equation}

We write down the expression of $\sigma_{\mu}$ and replace any $\lambda_i$ for $i \geq \mu + 1$ by $\mu_{\mu + 1}$, then we obtain the first inequality. Using (3.26) and (3.25), we get, for $a, b \leq \mu$,
\begin{equation}
\sigma_{\mu - 1}(\lambda|ab) - \sigma_{\mu}(\lambda|ab) \sigma_{\mu - 2}(\lambda|ab) \leq C_1 \frac{\lambda_{\mu + 1}}{\lambda_b} (\sigma_{\mu}^{\alpha a})^2.
\end{equation}

Then, by (3.27), we have, for any undetermined positive constant $\epsilon$,
\begin{equation}
\sum_{a \neq b, a \in \mu} (\sigma_{\mu}^{aa} - \sigma_{\mu}^{ab})(u_{aa1} u_{bb1}) \geq - \sum_{a \neq b, a \in \mu} (\sigma_{\mu - 1}(\lambda|ab) - \sigma_{\mu}(\lambda|ab) \sigma_{\mu - 2}(\lambda|ab)) u_{aa1}^2
\end{equation}
\begin{equation}
\geq - \sum_{a \neq b, a \in \mu} C_1 \frac{\lambda_{\mu + 1}}{\lambda_b} (\sigma_{\mu}^{aa} u_{aa1})^2
\end{equation}
\begin{equation}
\geq - C_2 \frac{\lambda_{\mu + 1}}{\lambda_1} (\sigma_{\mu}^{aa} u_{aa1})^2 \geq - \epsilon \sum_{a \leq \mu} (\sigma_{\mu}^{aa} u_{aa1})^2.
\end{equation}

Here, we choose a sufficiently small $\delta'$, such that,
\begin{equation}
\delta' \leq \delta \sqrt{\epsilon/C_2}.
\end{equation}
By (3.27), we also have,

\[(3.30)\]
\[
\begin{align*}
2 & \sum_{a \leq \mu; b > \mu} \left( \sigma_{\mu}^{aa} \sigma_{\mu}^{bb} - \sigma_{\mu}^{aa,bb} \right) u_{aa1} u_{bb1} \\
\geq & -2 \sum_{a \leq \mu; b > \mu} \sigma_{\mu}^{aa} \sigma_{\mu}^{bb} \left| u_{aa1} u_{bb1} \right| \\
\geq & -\epsilon \sum_{a \leq \mu; b > \mu} \left( \sigma_{\mu}^{aa} u_{aa1} \right)^2 - \frac{1}{\epsilon} \sum_{a \leq \mu; b > \mu} \left( \sigma_{\mu}^{bb} u_{bb1} \right)^2.
\end{align*}
\]

Again by (3.27), we have,

\[(3.31)\]
\[
\begin{align*}
\sum_{a \neq k; a, b > \mu} \left( \sigma_{\mu}^{aa} \sigma_{\mu}^{bb} - \sigma_{\mu}^{aa,bb} \right) u_{aa1} u_{bb1} & \geq - \sum_{a \neq k; a, b > \mu} \sigma_{\mu}^{aa} \sigma_{\mu}^{bb} \left| u_{aa1} u_{bb1} \right| \\
& \geq - \sum_{a \neq k; a, b > \mu} \left( \sigma_{\mu}^{aa} u_{aa1} \right)^2.
\end{align*}
\]

Hence, combing (3.30), (3.28), (3.30) and (3.31), then taking \(\alpha = 0\) in (3.20), we get,

\[(3.32)\]
\[
A_1 \geq \frac{\kappa_1^{m-1}}{P_m \sigma_{\mu}^2} \sigma_k^{11} \left| (1 - 2\epsilon) \sum_{a \leq \mu} \left( \sigma_{\mu}^{aa} u_{aa1} \right)^2 - C_\epsilon \sum_{a > \mu} \left( \sigma_{\mu}^{aa} u_{aa1} \right)^2 \right|.
\]

For \(a > \mu\), we have,

\[
\sigma_{\mu}^{aa} \leq C \lambda_1 \cdots \lambda_{\mu - 1}, \quad \text{and} \quad \sigma_{\mu} \geq \lambda_1 \cdots \lambda_\mu.
\]

For \(a \leq \mu\), we have,

\[
\sigma_{\mu}(\lambda | a) \leq C \frac{\lambda_1 \cdots \lambda_{\mu + 1}}{\lambda_a}.
\]

Then, we have, for \(\lambda_1 \geq K_0\),

\[(3.33)\]
\[
\begin{align*}
P_m^2 A_1 & \geq \frac{P_m \kappa_1^{m-1}}{\sigma_{\mu}^2} \lambda_1 \sigma_k^{11} (1 - 2\epsilon) \sum_{a \leq \mu} \left( \sigma_{\mu}^{aa} u_{aa1} \right)^2 - \frac{P_m \kappa_1^{m-1}}{\sigma_{\mu}^2} \sigma_k C_\epsilon \sum_{a > \mu} \left( \sigma_{\mu}^{aa} u_{aa1} \right)^2 \\
& \geq \frac{P_m \kappa_1^{m-1}}{\lambda_1} \sigma_k^{11} (1 - 2\epsilon) \sum_{a \leq \mu} \left( \frac{\lambda_a \sigma_{\mu}^{aa}}{\sigma_{\mu}} \right)^2 u_{aa1}^2 - \frac{P_m \kappa_1^{m-3}}{\sigma_{\mu}^2} \sigma_k^{11} \frac{\lambda_1 C_\epsilon}{\delta^2} \sum_{a > \mu} \left( \sigma_{\mu}^{aa} u_{aa1} \right)^2 \\
& \geq \kappa_1^{m-2} \sigma_k^{11} (1 - 2\epsilon)(1 + \delta^m) \sum_{a \leq \mu} \left( \frac{C_3 \lambda_{\mu + 1}}{\lambda_a} \right)^2 u_{aa1}^2 - \frac{P_m \kappa_1^{m-3}}{\sigma_{\mu}^2} \sigma_k^{11} \frac{\lambda_1 C_\epsilon}{\delta^2} \sum_{a > \mu} u_{aa1}^2 \\
& \geq \kappa_1^{m-2} \sigma_k^{11} (1 - 2\epsilon)(1 + \delta^m)(1 - \frac{C_3 \lambda_{\mu + 1}}{\delta^2} \sum_{a \leq \mu} u_{aa1}^2 - \frac{P_m \kappa_1^{m-3} C_\epsilon}{\delta^2} \sum_{a > \mu} u_{aa1}^2 \\
& \geq (1 + \eta) \kappa_1^{m-2} \sigma_k^{11} \sum_{a \leq \mu} u_{aa1}^2 - \frac{P_m \kappa_1^{m-3}}{\delta^2} C_\epsilon \sum_{a > \mu} u_{aa1}^2.
\end{align*}
\]
Here, the last inequality comes from that we choose \( \delta', \eta \) and \( \epsilon \) satisfying
\[
(3.34) \quad \delta' C_3 \leq 2 \epsilon \delta, \quad (1 - 2 \epsilon)^2 (1 + \delta^m) \geq 1 + \eta.
\]

Using (3.19) and (3.22) or (3.33), we have,
\[
(3.35) \quad P_m(A_1 + B_1 + C_1 + D_1 - (1 + \eta/m) E_1) \geq 2 P_m^m (\sum_{j \neq 1} \sigma_j^2 - C \epsilon P_m^m - \sum_{j > \mu} \sigma_j^2).
\]

Now, for \( k \geq j > \mu \), we have,
\[
(3.36) \quad \lambda_{k-1} \sigma_k = \frac{\lambda_1 \cdots \lambda_{k-1} \cdot \sigma_k}{\lambda_k} \geq \frac{\sigma_k}{C \delta'}.
\]

For both cases, chose \( \delta' \) small enough such that,
\[
(3.37) \quad \delta' < \frac{\sigma_k \delta'}{C \epsilon},
\]

then (3.35) is nonnegative. We complete the proof. \( \square \)

Hence, a directly corollary of Lemma 8 and Lemma 9 is the following.

**Corollary 10.** There exists two finite sequence of positive numbers \( \{\delta_i\}_{i=1}^k \) and \( \{\epsilon_i\}_{i=1}^k \), such that, if the following inequality holds for some index \( 1 \leq r \leq k - 1 \),
\[
(3.38) \quad \frac{\lambda_r}{\lambda_1} \geq \delta_r, \quad \text{and} \quad \frac{\lambda_{r+1}}{\lambda_1} \leq \delta_{r+1},
\]

then, for sufficiently large \( K \), we have,
\[
(3.39) \quad A_1 + B_1 + C_1 + D_1 - (1 + \rho/m) E_1 \geq 0.
\]

**Proof.** We use induction to find the sequence \( \{\delta_i\}_{i=1}^k \) and \( \{\epsilon_i\}_{i=1}^k \). Let \( \delta_1 = 1/2 \). Then \( \lambda_1/\lambda_1 = 1 > \delta_1 \). Assume that we have determined \( \delta_r \) for \( 1 \leq r \leq k - 1 \). We want to search for \( \delta_{r+1} \). In Lemma 9 we may choose \( \mu = r \) and \( \delta = \delta_r \). Then there is some \( \delta_{r+1} \) and \( \epsilon_r \) such that, if \( \lambda_{r+1} \leq \delta_{r+1} \lambda_1 \), we have (3.36). We have \( \delta_{r+1} \) and \( \epsilon_r \). \( \square \)

Now, we continue to prove Theorem 2.

By Corollary 10 there exists some sequence \( \{\delta_i\}_{i=1}^k \). We divide two cases to deal with.

Case(A): \( \lambda_k \geq \delta_k \lambda_1 \). Then, obviously we have,
\[
(3.40) \quad f = \sigma_k > \lambda_1 \cdots \lambda_k \geq \delta_k^{k-1} \lambda_1^k,
\]

which implies \( \lambda_1 \leq C \). Hence, we have proved Theorem 2.
Case (B): There exists some index \(1 \leq r \leq k - 1\) such that,

\[ \lambda_r \geq \delta_r \lambda_1 \text{ and } \lambda_{r+1} \leq \delta_{r+1} \lambda_1. \]

By Corollary 10 and Lemma 8, we have,

\[
\sum_{i=1}^{n} \left( A_i + B_i + C_i + D_i \right) - E_1 - \left( 1 + \frac{1}{m} \right) \sum_{i=2}^{n} E_i \geq 0.
\]

Using the definitions of \( A_i, B_i, C_i, D_i, E_i \) and (3.7), we have,

\[
(3.37) \quad 0 \geq \frac{1}{P_m} \sum_{i=2}^{n} \kappa_i^{m-1} ( -C - C u_{11}^2 - K \psi_{m}^2 u_{li}^2 ) + \sum_{i=2}^{n} \frac{\sigma_k^{ii}}{P_m} ( \sum_{j} \kappa_j^{m-1} u_{jj} )^2
\]

\[
+ Nu_{ii}^2 \sigma_k^{ii} + \frac{k \sigma_k}{u} - \frac{\sigma_k^{ii} u_{ii}^2}{u^2} - \sum_{s} \psi_{ps} u_{s}.\]

By (3.2), we have, for any fixed \( i \geq 2 \),

\[
- \frac{\sigma_k^{ii} u_{ii}^2}{u^2} = - \frac{\sigma_k^{ii} ( \sum_{j} \kappa_j^{m-1} u_{jj} )^2}{u^2} + \sigma_k^{ii} N^2 u_{ii}^2 + 2N \sigma_k^{ii} u_{ii}^2 u_{ii}.
\]

Hence, (3.37) becomes,

\[
(3.38) \quad 0 \geq -C(K) \lambda_1 + \sum_{i=2}^{n} \left( \sigma_k^{ii} N^2 u_{ii}^2 + 2N \sigma_k^{ii} u_{ii}^2 u_{ii} \right)
\]

\[
+ Nu_{ii}^2 \sigma_k^{ii} + \frac{k \sigma_k}{u} - \frac{\sigma_k^{ii} u_{ii}^2}{u^2} - \sum_{s} \psi_{ps} u_{s}.\]

Since, there is some positive constant \( c_0 \) such that,

\[ u_{11} \sigma_k^{11} \geq c_0 > 0, \]

then we have,

\[ 0 \geq \left( \frac{c_0 N}{2} - C(K) \right) \lambda_1 + \sum_{i=2}^{n} \left( \sigma_k^{ii} N^2 u_{ii}^2 + 2N \sigma_k^{ii} u_{ii}^2 u_{ii} \right)
\]

\[
+ Nu_{ii}^2 \sigma_k^{ii} + \frac{k \sigma_k}{u} - \frac{\sigma_k^{ii} u_{ii}^2}{u^2} - \sum_{s} \psi_{ps} u_{s}.\]

Here, we have used

\[ \sigma_k = \lambda_i \sigma_k^{ii} + \sigma_k (\lambda | i) \geq \lambda_i \sigma_k^{ii}. \]

Hence, we obtain, for \( N \geq \frac{4C(K)}{c_0} \),

\[
- \frac{C}{u} + \frac{C \sigma_k^{11}}{u^2} \geq \frac{N}{4} \lambda_1 + \frac{N}{2} \sigma_k^{11}\lambda_1^2.\]

If at maximum value point \( p \), \(-u \geq \sigma_k^{11}\), the above inequality becomes,

\[
\frac{2C}{-u} \geq \frac{N}{4} \lambda_1,
\]

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which implies our result. If $-u \leq \sigma_{k}^{11}$, the inequality becomes,

$$\frac{2C\sigma_{k}^{11}}{(-u)^2} \geq \frac{N}{2}\sigma_{k}^{11} \lambda_1^2,$$

which also implies our result. We complete the proof of Theorem 2.

4. A RIGIDITY THEOREM FOR $k + 1$ CONVEX SOLUTIONS

In this section, we prove Theorem 4. At first we have the following Lemma.

**Lemma 11.** We consider the Dirichlet problem of the $k$-Hessian equations,

$$\begin{aligned}
\sigma_k(D^2u) &= f(x) \text{ in } \Omega \\
u &= 0 \text{ on } \partial \Omega.
\end{aligned}$$

(4.1)

Here, $f$ is a smooth function defined in $\Omega$. For $k + 1$ convex solutions, we have the following type of interior estimates,

$$(-u)^{\beta} \Delta u \leq C.$$  

(4.2)

for sufficiently large $\beta > 0$. Here constant $C$ and $\beta$ only depends on the diameters of the domains $\Omega$ and $k$.

**Proof.** Obviously, for sufficiently large $a$ and $b$, the function $w = \frac{a}{2}|x|^2 - b$ can control $u$ by comparison principal (see [3] for detail), namely,

$$w \leq u \leq 0.$$  

Here $a, b$ depends on the diameter of the domain $\Omega$. Hence, in the following proof, the constant $\beta, C$ in (4.2) can contains $\sup_{\Omega}|u|$.

Since $u$ is a $k + 1$ convex solution, by Lemma 7 there is some constant $K_0 > 0$, such that $D^2u + K_0 I \geq 0$. We consider the following test functions,

$$\varphi = m\beta \log(-u) + \log P_m + \frac{m}{2}|x|^2.$$  

where $P_m = \sum_{j} \kappa_{j}^{m}$, $\kappa_{i} = \lambda_i + K_0 > 0$. Suppose $\varphi$ achieves its maximal value at $x_0 \in \Omega$. We may assume $(u_{ij})$ is diagonal by rotating the coordinate and $u_{11} \geq u_{22} \cdots \geq u_{nn}$. We always denote $u_{ii} = \lambda_i$.

At the point $x_0$, we differentiate the test function twice and using Lemma 6. We have,

$$\sum_{j} \frac{\kappa_{j}^{m-1} u_{jjj}}{P_m} + x_i + \frac{\beta u_i}{u} = 0,$$

and,

$$0 \geq \frac{1}{P_m} \left[ \sum_{j} \kappa_{j}^{m-1} u_{jjj} + (m - 1) \sum_{j} \kappa_{j}^{m-2} u_{jjj}^2 + \sum_{p \neq q} \frac{\kappa_{p}^{m-1} - \kappa_{q}^{m-1}}{\kappa_{p} - \kappa_{q}} u_{pq}^2 \right] - \frac{m}{P_m^2} \left( \sum_{j} \kappa_{j}^{m-1} u_{jjj} \right)^2 + \frac{\beta u_{ii}}{u} - \frac{\beta u_{i}^2}{u^2} + 1.$$  

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Differentiating the equation (4.1) twice at $x_0$, we have,

\[ (\sigma_k)_j = \sigma^{ii}_k u_{iij} = f_j, \]

and

\[ \sigma^{ii}_k u_{iijj} + \sigma^{pq,rs}_k u_{pqj} u_{rsj} = f_{jj}, \]

Then, contracting $\sigma^{ii}_k$ in (4.4) and using the previous two equalities, we have,

\[ 0 \geq \frac{1}{P^m} \left[ \sum_l \kappa^{-1}_i (f_l - \sigma^{pq,rs}_k u_{pql} u_{qrl}) + (m - 1) \sigma^{ii}_k \sum_j \kappa^{-2}_j u_{jjj} \right. \]

\[ + \sigma^{ii}_k \sum_{p \neq q} \frac{\kappa^{-1}_p - \kappa^{-1}_q}{\kappa_p - \kappa_q} u_{pql}^2 ] - \frac{m \sigma^{ii}_k}{P^2 m} \left( \sum_j \kappa^{-1}_j u_{jjj} \right)^2 \]

\[ + \frac{\beta k}{u} - \frac{\beta \sigma^{ii}_k u_i^2}{u^2} + (n - k + 1) \sigma_{k-1}. \]

Using (4.3), we have,

\[ -\frac{\beta \sigma^{ii}_k u_i^2}{u^2} \geq -\frac{2 \sigma^{ii}_i (\sum_j \kappa^{-1}_j u_{jjj})^2}{\beta} - \frac{2 \sigma^{ii}_k x_i^2}{\beta} \]

Note that,

\[ -\sigma^{pq,rs}_k u_{pqj} u_{rsj} = -\sigma^{pp,qq}_k u_{pql} u_{qql} + \sigma^{pp,qq}_k u_{pql}^2. \]

For sequence $\{\varepsilon_i\}_{i=1}^k$ appears in Corollary 10. Let

\[ \varepsilon_\beta = \frac{2}{\beta} \min \left\{ \frac{1}{10}, \varepsilon_1, \ldots, \varepsilon_k \right\}, \]

then, (4.7) becomes

\[ 0 \geq \frac{1}{P^m} \left[ \sum_l \kappa^{-1}_i (f_l - \sigma^{pp,qq}_k u_{pql} u_{qql}) + 2 \sum_{j \neq i} \kappa^{-2}_j u_{jjj} \right. \]

\[ + (m - 1) \sigma^{ii}_k \sum_j \kappa^{-2}_j u_{jjj} + \sigma^{ii}_k \sum_{p \neq q} \frac{\kappa^{-1}_p - \kappa^{-1}_q}{\kappa_p - \kappa_q} u_{pql}^2 ] \]

\[ - \frac{(m + \varepsilon_\beta) \sigma^{ii}_k}{P^2 m} \left( \sum_j \kappa^{-1}_j u_{jjj} \right)^2 + \frac{\beta k}{u} - \frac{2 \sigma^{ii}_k x_i^2}{\beta} + (n - k + 1) \sigma_{k-1}. \]
Next we mainly deal with the third order derivative terms. We divide into two case: \( i \neq 1 \) and \( i = 1 \). By Lemma 8, we have, for sufficiently large \( K \),

\[
0 \leq \frac{1}{P_m} \left[ \sum_{l=2}^{n} \kappa_l^{m-1} (K(\sigma_k)^{l} - \sigma_k^{pp,qq} u_{ppl} u_{qql}) + 2 \sum_{i=2}^{n} \sum_{j \neq i} \kappa_j^{m-2} \sigma_k^{jjii} u_{jjii}^2 \right] + (m-1) \sum_{i=2}^{n} \sigma_k^{11} \sum_{j} \kappa_j^{m-2} u_{jjj}^2 + 2 \sum_{i=2}^{n} \sum_{j \neq i} \kappa_j^{m-1} \sigma_k^{jjii} (\kappa_j - \kappa_i - u_{jjj}^2) \right] \\
- \frac{m+1}{P_m^2} \sum_{i=2}^{n} \sigma_k^{ii} (\kappa_j^{m-1} - u_{jjj}^2)^2.
\]

Hence, (4.8) becomes,

\[
0 \geq \frac{1}{P_m} \left[ \kappa_1^{m-1} (C + K(\sigma_k)^{1}) - \sigma_k^{pp,qq} u_{pp1} u_{qq1} + 2 \sum_{j \neq 1} \kappa_j^{m-2} \sigma_k^{jj11} u_{jj1}^2 \right] + (m-1) \sigma_k^{11} \sum_{j} \kappa_j^{m-2} u_{jj1}^2 + 2 \sigma_k^{11} \sum_{j \neq 1} \kappa_j^{m-1} - \kappa_1^{m-1} u_{jjj}^2 \right] \\
- \frac{m+1}{P_m^2} \sigma_k^{11} (\sum_{j} \kappa_j^{m-1} - u_{jjj}^2)^2 + \frac{\beta_k}{u} - 2 \frac{\sigma_k^{ii} x_i^2}{\beta} + C_0 \sigma_{k-1}.
\]

Now, we divide two sub-cases to continue. By Corollary 10, there exists some sequence \( \{\delta_i\}_{i=1}^{k} \).

Case(A): \( \lambda_k \geq \delta_k \lambda_1 \). Then, obviously we have,

\[
f = \sigma_k > \lambda_1 \cdots \lambda_k \geq \delta_k^{k-1} \lambda_1,
\]

which implies \( \lambda_1 \leq C \). Hence, we have proved Lemma 11.

Case(B): There exists some index \( 1 \leq r \leq k-1 \) such that,

\[
\lambda_r \geq \delta_r \lambda_1 \text{ and } \lambda_{r+1} \leq \delta_{r+1} \lambda_1.
\]

By Corollary 10, (4.10) becomes,

\[
0 \geq \frac{\beta_k}{u} - 2 \frac{\sigma_k^{ii} x_i^2}{\beta} + C_0 \sigma_{k-1} - C.
\]

We take \( \beta \) sufficiently large, then, we have

\[
C \geq \frac{\beta_k}{u} + \frac{C_0}{2} \sigma_{k-1} \geq \frac{\beta_k}{u} + c_0 \sigma_k^{-\frac{1}{k-2}} \sigma_k^{-\frac{k-2}{k-1}},
\]

where we have used Newton-Maclaurin in the last inequality. Hence, we obtain Lemma 11.

**Proof of Theorem 4** The proof is classical [15]. Suppose \( u \) is an entrie solution of the equation \( 1.5 \). For arbitrary positive constant \( R > 1 \), we consider the set

\[
\Omega_R = \{ y \in \mathbb{R}^n; u(Ry) \leq R^2 \}.
\]
Let 
\[ v(y) = u(Ry) - \frac{R^2}{R^2}. \]

We consider the following Dirichlet problem,
\[
\begin{aligned}
\sigma_k [D^2 v] &= 1 \quad \text{in } \Omega_R, \\
v &= 0 \quad \text{on } \partial \Omega_R.
\end{aligned}
\]

(4.11)

Using Lemma 11, we have the following type estimates,
\[
(-v)^\beta \Delta v \leq C.
\]

(4.12)

Here \( \beta \) and \( C \) depend on \( k \), diameter of the \( \Omega_R \). Now using the quadratic growth condition appears in Theorem 4, we have
\[
c|Ry|^2 - b \leq u(Ry) \leq R^2,
\]

which implies
\[
|y|^2 \leq \frac{1 + b}{c}.
\]

Thus \( \Omega_R \) is bounded. Hence, the constant \( C, \beta \) become two absolutely constants. We now consider the domain
\[
\Omega'_R = \{ y; u(Ry) \leq R^2/2 \} \subset \Omega_R.
\]

In \( \Omega'_R \), we have,
\[
v(y) \leq -\frac{1}{2}.
\]

Hence, (4.12) implies that in \( \Omega'_R \), we have,
\[
\Delta v \leq 2^\beta C.
\]

Note that,
\[
\nabla_y^2 v = \nabla^2_x u.
\]

Thus, using previous two formulas, we have, in \( \Omega'_R = \{ x; u(x) \leq R^2/2 \} \),
\[
\Delta u \leq C,
\]

(4.13)

where \( C \) is a absolutely constant. Since \( R \) is arbitrary, we have the above inequality in whole \( \mathbb{R}^n \). Using Evans-Krylov theory [9], we have
\[
|D^2 u|_{C^\alpha(B_R)} \leq C \frac{|D^2 u|_{C^0(B_R)}}{R^\alpha} \leq \frac{C}{R^\alpha}.
\]

Hence, we obtain our theorem letting \( R \to +\infty \).

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Institute of Mathematics, Fudan University, Shanghai, China
E-mail address: leemingfudan@gmail.com

School of Mathematical Science, Jilin University, Changchun, China
E-mail address: renency@jlu.edu.cn

Institute of Mathematics, Fudan University, Shanghai, China
E-mail address: zzwang@fudan.edu.cn