1. Introduction

In this paper, we deal with divergence measures estimation using both wavelet and classical probability density functions. Let $\mathcal{P}$ be a class of two probability measures on $\mathbb{R}^d$, a divergence measure on $\mathcal{P}$ is an application,

$$D : \mathcal{P}^2 \rightarrow \mathbb{R} \quad (Q, L) \mapsto D(Q, L)$$

such that $D(Q, Q) = 0$ for any $Q \in \mathcal{P}$.

A divergence measure then is not necessarily symmetrical and it does neither have to be a metric. To better explain our concern, let us introduce some of the most celebrated divergence measures. Most of them are based on probability density functions. So let us suppose that all $Q \in \mathcal{P}$ have p.d.f. $f_Q$ with respect to a $\sigma$-finite measure $\nu$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ that is usually the Lebesgue measure. We have the $L_2$-divergence measure

$$D_{L_2}(Q, L) = \int_{\mathbb{R}^d} (f_Q(x) - f_L(x))^2 d\nu(x);$$

the family of Renyi divergence measures indexed by $\alpha \neq 1$, more known under the name of Renyi-$\alpha$,

$$D_{R,\alpha}(Q, L) = \frac{1}{\alpha - 1} \log \left( \int_{\mathbb{R}^d} f_Q^\alpha(x) f_L^{1-\alpha}(x) d\nu(x) \right);$$

the family of Tsallis divergence measures indexed by $\alpha \neq 1$, also known under the name of Tsallis-$\alpha$

$$D_{T,\alpha}(Q, L) = \frac{1}{\alpha - 1} \left( \int_{\mathbb{R}^d} f_Q^\alpha(x) f_L^{1-\alpha}(x) - 1 \right) d\nu(x);$$

and finally the Kulback-Leibler divergence measure

$$D_{KL}(Q, L) = \int_{\mathbb{R}^d} f_Q(x) \log(f_Q(x)/f_L(x)) d\nu(x).$$

The latter, the Kullback-Leibler measure, may be interpreted as a limit case of both the Renyi’s family and the Tsallis’ one by letting $\alpha \rightarrow 1$. As well, for $\alpha$ near 1, the Tsallis family may be seen as derived from a first order expansion $D_{R,\alpha}(Q, L)$ based on the first order expansion of the logarithm function in the neighborhood of the unity.

Although we are focusing on the aforementioned divergence measures, we have to attract the attention of the reader that there exist quite a few number of them. Let us cite for example the ones denamed as : Ali-Silvey distance or $f$-divergence [1], Cauchy-Schwarz, Jeffrey’s divergence (see [2]), Chernoff, Jensen-Shannon divergence etc. According to [3], there is more than a dozen of different divergence measures that one can find in the literature.
Before coming back to our divergence measures of interest, we want to highlight some important applications of them. Indeed, divergence has proven to be useful in applications. Let us cite a few of them:

(a) It may be as a similarity measure in image registration or multimedia classification (see [4]). It is also applicable as a loss function in evaluating and optimizing the performance of density estimation methods (see [6]).

(b) The estimation of divergence between the samples drawn from unknown distributions gauges the distance between those distributions. Divergence estimates can then be used in clustering and in particular for deciding whether the samples come from the same distribution by comparing the estimate to a threshold.

(c) Divergence estimates can also be used to determine sample sizes required to achieve given performance levels in hypothesis testing.

(d) Divergence gauges how differently two random variables are distributed and it provides a useful measure of discrepancy between distributions. In the frame of information theory, the key role of divergence is well known.

(e) There has been a growing interest in applying divergence to various fields of science and engineering for the purpose of estimation, classification, etc [7], [9].

(f) Divergence also plays a central role in the frame of large deviations results including the asymptotic rate of decrease of error probability in binary hypothesis testing problems.

The reader may find more applications descriptions in the following papers: [10], [11], [12], [14], [15], [16], [17], [18].

We may see two kinds of problems we encounter when dealing with these objects. First, the divergence measures may not be finite on the whole support of the distributions.

These two remarks apply to too many divergence measures. Both these problems are avoided with some boundedness assumption as in Singh et al. [25] and in Krishnamurthy et al. [23]. In the case where all \( Q \in \mathcal{P} \) have p.d.f. \( f_Q \) with respect to a \( \sigma \)-finite measure \( \nu \) on \( (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \), these authors suppose that there exist two finite numbers \( 0 < \kappa_1 < \kappa_2 < +\infty \) such that

\[
\kappa_1 \leq f_Q, f_L \leq \kappa_2.
\]

so that the quantities \( I_{\alpha}(\mathbb{P}_X, \mathbb{P}_Y, \alpha) \), for example, are finite in the expressions of Renyi-\( \alpha \) and Tsallis-\( \alpha \) measures and that the Kullback-Leibler is also finite. We will follow these authors by adopting the Assumption 1.1 throughout this paper.

1.1. Divergence measures as goodness-of-fit tests. The divergence measures may be applied to two statistical problems among others.

First, it may be used as a goodness-of-fit problem like that: let \( X_1, X_2, \ldots \) a sample from \( X \) with an unknown probability distribution \( \mathbb{P}_X \) and we want to test the hypothesis that \( \mathbb{P}_X \) is equal to a known and fixed probability \( \mathbb{P}_0 \).

For example Jager et al. (in [5]) proposed \( \mathbb{P}_0 \) to be the uniform probability distribution on \([0,1]\).
Theoretically, if we want to test the null hypothesis $H_0 : F = F_0$ versus $H_1 : F \neq F_0$ we have to use any of
general phi-divergence test statistic $\phi(f_n(x), f_0(x))$. Then our test statistic is of the form
$$\mathcal{D}(f_n, f_0) = \int \phi(f_n(x), f_0(x)) dx$$

Then we can answer this question by estimating a divergence measure $\mathcal{D}(\mathbb{P}_X, \mathbb{P}_0)$ by the plug-in estimator $\mathcal{D}(\mathbb{P}_X^{(n)}, \mathbb{P}_0)$ based on the sequences of empirical probabilities
$$\mathbb{P}_X^{(n)} = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}.$$  

From there establishing an asymptotic theory of $\Delta_n = \mathcal{D}(\mathbb{P}_X^{(n)}, \mathbb{P}_0) - \mathcal{D}(\mathbb{P}_X, \mathbb{P}_0)$ is necessary to conclude.

### 1.2. Divergence measures as a comparison tool problem.

As a comparison tool for two distributions, we may have two samples and wonder whether they come from the same probability measure. Here, we also may two different cases. In the first, we have two independent samples $X_1, X_2, ..., Y_1, Y_2, ...$ respectively from a random variable $X$ and $Y$. Here the empirical divergence $\mathcal{D}(\mathbb{P}_X^{(n)}, \mathbb{P}_Y^{(n)})$ is the natural estimator of $\mathcal{D}(\mathbb{P}_X, \mathbb{P}_Y)$ on which depends the statistical test of $\mathbb{P}_X = \mathbb{P}_Y$.

But the data may aslo be paired $(X,Y), (X_1,Y_1), (X_2,Y_2), ...$ that is $X_i$ and $Y_i$ are measurements of the same case $i = 1, 2, ...$ In that case, testing the equality of the margins $\mathbb{P}_X = \mathbb{P}_Y$ should be based on the empirical probabilities from the couple $(X,Y)$, that is
$$\mathbb{P}_{(X,Y)}^{(n)} = \frac{1}{n} \sum_{i=1}^{n} \delta_{(X_i,Y_i)}.$$  

### 1.3. Related work:

Krisnamurthy et al. (23), Singh and Poczos (25) studied mainly the independent case of the two distributions comparison. They both used divergence measures based on probability density functions and concentrated of Renyi-$\alpha$, Tsallis-$\alpha$ and Kullback-Leibler. Reiny:

- Singh and Poczos (25) proposed divergence estimators that achieve the parametric convergence of rate $n^{-N(s)}$ where $0 < N(s) \leq 1/2$ depends on the smoothness $s$ of the densities $f$ and $g$ both in a Holder class of smoothness $s$.

They showed that
$$E \left[ D_{T,\alpha}(\mathbb{P}_X^{(n)}, \mathbb{P}_Y^{(n)}) - D_{T,\alpha}(\mathbb{P}_X, \mathbb{P}_Y) \right] = O(\epsilon_n)$$

and
$$E \left[ D_{R,\alpha}(\mathbb{P}_X^{(n)}, \mathbb{P}_Y^{(n)}) - D_{R,\alpha}(\mathbb{P}_X, \mathbb{P}_Y) \right] = O(\epsilon_n)$$

where $\epsilon_n = \Omega(n^{-\gamma})$ and $\gamma = \min \{ \frac{4s}{4s+d}, 1/2 \}$.

Singh and Poczos (25), and Krishnamurthy et al (23) each proposed divergence estimators that achieve the parametric convergence rate ($O(\frac{1}{n})$) under weaker conditions than those given in [39].
Krishnamurthy et al. [23] proposed three estimators for the divergence measures $\mathcal{D}_{L_2}(P_X, P_Y)$, $\mathcal{D}_{R,\alpha}(P_X, P_Y)$, and for $\mathcal{D}_{T,\alpha}(P_X, P_Y)$: the plugging (pl), linear (lin), and the quadratic (qd) one. They showed that

$$\mathbb{E} \left( \mathcal{D}^{pl}_{T,\alpha}(P_X^{(n)}, P_Y^{(n)}) - \mathcal{D}_{T,\alpha}(P_X, P_Y) \right) = \mathcal{O} \left( n^{-\frac{1}{d} + \frac{1}{2d}} \right)$$

and

$$\mathbb{E} \left( \mathcal{D}^{lin}_{T,\alpha}(P_X^{(n)}, P_Y^{(n)}) - \mathcal{D}_{T,\alpha}(P_X, P_Y) \right) \leq c \left( n^{-\frac{1}{d} + \frac{1}{2d}} \right)$$

with the quadratic estimator

$$\mathbb{E} \left( \mathcal{D}^{qd}_{T,\alpha}(P_X^{(n)}, P_Y^{(n)}) - \mathcal{D}_{T,\alpha}(P_X, P_Y) \right) \leq c \left( n^{-\frac{1}{d} + \frac{1}{2d}} \right)$$

Poczos and Jeff ([26]) considered two samples not necessarily with the same size and used the $k$-Nearest Neighbour (kNN) based density estimators. They showed that, if $|\alpha - 1| < k$, then Reyni estimator est asymptotically unbiased that is

$$\lim_{n,m \to \infty} \mathbb{E} \left( \mathcal{D}_{R,\alpha}(P_X^{(n)}, P_Y^{(m)}) - \mathcal{D}_{R,\alpha}(P_X, P_Y) \right) = \mathcal{D}_{R,\alpha}(P_X, P_Y)$$

and it is consistent for $L_2$ norm that is

$$\lim_{n,m \to \infty} \mathbb{E} \left( \mathcal{D}_{R,\alpha}(P_X^{(n)}, P_Y^{(m)}) - \mathcal{D}_{R,\alpha}(P_X, P_Y) \right)^2 = 0.$$

All this is under conditions on the densities $f_0$ and $f_L$.

In Liu et al. [27] and worked with densities in Holder classes, whereas our work applies for densities in the Bessov class

In any case, the asymptotic distributions of the estimators in [30, 25, 23] are currently unknown.

But, in our view this case should rely on the available data so that using the same sample size may lead to a reduction (??). To apply their method, one should take the minimum of the two sizes and then loose information. We suggest to come back to a general case and then study the asymptotics of $\mathcal{D}(P_X^{(n)}, P_Y^{(m)})$ based on samples $X_1, X_2, ..., X_n$, and $Y_1, Y_2, ..., Y_m$.

As for the fitting approach, we may cite Hamza et al. ([28]) who used modern techniques of Mason and co-authors ([31], [32], [33]) on consistency bounds for p.d.f’s kernel estimators. But, these authors, hamza and al., in the current version of their work, did not address the existence problem of the divergence measures. We will seize the opportunity of these papers to correct this.

Also, for the fitting case, and when using Renyi-$\alpha$, Tsallis-$\alpha$ and Kulback-Leibler measures, we do not have symmetry. So we have to deal with the estimation of both $\mathcal{D}(P_X, P_0)$ by $\mathcal{D}(P_X^{(n)}, P_0)$ and that of $\mathcal{D}(P_0, P_X)$ by $\mathcal{D}(P_0, P_X^{(n)})$ and decided which of these cases is better.
As to the paired case, we are not aware of works on this. Yet, this approach is very important and should be addressed.

This paper will be devoted to a general study the estimation the Renyi-\(\alpha\), Tsallis-\(\alpha\), Kulback-Leibler, and \(L_2\) measures in the three level : fitting, independent comparison and paired comparison.

We will use empirical estimations of the density functions both by the Parzen estimator and the wavelet ones. The main novelty here resides in the wavelet approach. When using the Parzen statistics, our main tool will be modern techniques of Mason and co-authors ([31],[32],[33]) on consistency bounds for p.d.f’s kernel. For the wavelet approach, we will mainly back on the Giné and Nickl paper ([43]).

Since the tools we are using do not have the level of developpement, our results for the Parzen scheme will use \(k\)-dimensional distributions, while those pertaining to the wavelet frame are set for univariate distributions.

But, we will have to give a precise account of wavelet theory and its applications to statistical estimation, using Hardle et al.([37]), .....

The paper will be organized as follows. In section 3, we will describe how to use the density estimations both for Parzen and wavelets as well as the statements of the main hypothesis. As for wavelets, a broader account will be given in Appendix .. In Section 3, we deal with the fitting questions. Section 4 is devoted to independent distribution comparison. Finally in Section 5, we deal with margins distribution comparison. In all Sections 3, 4 and 5, we will establish strong efficiency and central limit theorems.

Under standards assumptions on the densities \(f_Q(x), f_L(x)\), on the scale function \(\varphi\) and on the wavelet kernel \(K\) (formalized in the sequel) we establish the following properties.

a) We define the linear wavelet density estimators and establish the consistency of these density estimators
b) We establish the asymptotic consistency showing that . . . (Theorem ??)

b) When .... we prove that the estimator is asymptotically normal (Theorem ??).

c) We derive . . .

d) We also prove
e) Lastly, we prove

Organization of the paper (plan)

2. RESULTS

We are going to establish general results both for consistency and asymptotic normality. Next results for particular divergences measures will follow as corollaries.
2.1. **General conditions.** Let $J(f,g)$ be a functional of two densities functions $f$ and $g$ satisfying **Assumption** 4 below of the form

$$J(f,g) = \int_D \phi(f(x), g(x))dx$$

where $\phi(s,t)$ is a function of $(s,t) \in \mathbb{R}^2$ of class $C^2$. We adopt the following notations with respect to the partial derivatives :

$$\phi_1^{(1)}(s,t) = \frac{\partial \phi}{\partial s}(s,t), \quad \phi_2^{(1)}(s,t) = \frac{\partial \phi}{\partial t}(s,t)$$

and

$$\phi_1^{(2)}(s,t) = \frac{\partial^2 \phi}{\partial s^2}(s,t), \quad \phi_2^{(2)}(s,t) = \frac{\partial^2 \phi}{\partial t^2}(s,t), \quad \phi_1^{(2)}(s,t) = \phi_2^{(2)}(s,t) = \frac{\partial^2 \phi}{\partial s \partial t}(s,t).$$

We require the following general conditions

**C1-φ :** The following integrals are finite

$$\int \left\{|\phi_1^{(1)}(f(x), g(x))| + |\phi_2^{(1)}(f(x), g(x))|\right\} dx < +\infty.$$  

**C2-φ :** For any measurable sequences of functions $\delta_n^{(1)}(x)$, $\delta_n^{(2)}(x)$, $\rho_n^{(1)}(x)$, and $\rho_n^{(2)}(x)$ of $x \in D$, uniformly converging to zero, that is

$$\max_{i=1,2} \sup_{j=1,2} \left\{|\delta_n^{(i)}(x)| + |\rho_n^{(j)}(x)|\right\} < +\infty,$$

then

(2.1) $$\int \phi_1^{(2)} \left(f(x) + \delta_n^{(1)}(x), g(x)\right) dx \rightarrow \int \phi_1^{(2)}(f(x), g(x))dx,$$

(2.2) $$\int \phi_2^{(2)} \left(f(x), g(x) + \delta_n^{(2)}(x)\right) dx \rightarrow \int \phi_2^{(2)}(f(x), g(x))dx$$

and

(2.3) $$\int \phi_1^{(2)} \left(f(x) + \rho_n^{(1)}(x), g(x) + \rho_n^{(2)}(x)\right) dx \rightarrow \int \phi_1^{(2)}(f(x), g(x))dx.$$

**Remark 1.** These results may result from the Dominated Convergence Theorem or the monotone Convergence Theorem or from other limit theorems. We may either express conditions under which these results hold true on the general function $\phi$. But we choose here to state the final results and next, to check them for particular cases, on which reside our real interests.

Our general results concern the estimations of $J(f,g)$ in a one sample (see Theorem 2) and two samples problems (see Theorem 3). In both case, we use the linear wavelet estimators of $f$ and $g$, denoted $f_n$ and $g_n$, and defined in [37]. From there we mainly use results for Giné and Nickl [43].

Under their conditions, we define

(2.4) $$a_n = \|f_n - f\|_\infty, \quad b_n = \|g_n - g\|_\infty, \quad c_n = a_n \lor b_n,$$

where $\|h\|_\infty$ stands for $\sup_{x \in D} |h(x)|.$
2.2. Wavelet setting. The wavelet setting involves two functions \( \varphi \) and \( \psi \) in \( L_2(\mathbb{R}) \) such that
\[
\left\{ 2^{j/2} \varphi(2^j x - k), 2^{j/2} \psi(2^j x - k), (j,k) \in \mathbb{Z}^2 \right\}
\]
be a orthonormal basis of \( L_2(\mathbb{R}) \). The associated kernel function of the wavelets \( \varphi \) and \( \psi \) is defined by
\[
K_j(x,y) = 2^j K(2^j x, 2^j y), \ j \in \mathbb{N}
\]
where \( K(x,y) = \sum_{k \in \mathbb{Z}} \varphi(x-k)(y-k), \ x,y \in \mathbb{R} \).

For a measurable function we define \( K_j(h)(x) = \int K_j(x,y) h(y) dy \).

Assuming the following:

**Assumption 1.** \( (S) \), \( \varphi \) and \( \psi \) are bounded and have compact support and either (i) the father wavelet \( \varphi \) has weak derivatives up to order \( S \) in \( L_p(\mathbb{R}) \) or (ii) \( \psi \) has \( S \) vanishing moments, i.e \( \int x^m \varphi(x) dx = 0 \) for all \( m = 0, \ldots, S-1 \).

**Assumption 2.** \( \varphi \) is of bounded \( p \)-variation for some \( 1 \leq p < \infty \) and vanishes on \( (B_1, B_2) \) for some \( -\infty < B_1 < B_2 < \infty \).

**Assumption 3.** The resolution level \( j := j_n \) is such that \( 2^j \approx n^{1/4} \).

With this assumption, one has \( j_n \searrow \infty \) and
\[
(2.5) \quad \frac{j_n 2^j}{n} + 2^{-t/j_n} = \sqrt{\frac{1}{4 \log 2} \log n} + n^{-t/4} \to 0 \text{ as } n \to \infty, \ \forall t > 0
\]

\[
\frac{j_n}{ \log \log n} \to \infty \text{ as } n \to \infty, \ \text{and} \ \sup_{n \geq j_0} (j_2j - j_n) \leq \tau.
\]

These conditions allow the use of results of Giné [33].

**Definition 1.** Given two independent samples with size \( n \) \( X_1, \ldots, X_n \sim f \) and \( Y_1, \ldots, Y_n \sim g \) respectively from a random variable \( X \) and \( Y \) and absolute continuous law \( \mathbb{P}_X \) and \( \mathbb{P}_Y \) on \( \mathbb{R} \), straightforward wavelets estimators of \( f \) and \( g \) are defined by
\[
(2.6) \quad f_n(x) = \mathbb{P}_{n,X}(K_{j_n}(x,.)) = \frac{1}{n} \sum_{i=1}^{n} K_{j_n}(x,X_i)
\]
and
\[
(2.7) \quad g_n(x) = \mathbb{P}_{n,Y}(K_{j_n}(x,.)) = \frac{1}{n} \sum_{i=1}^{n} K_{j_n}(x,Y_i)
\]

In the sequel we suppose the densities \( f \) and \( g \) belong to the Besov space \( B^{s}_{\infty,\infty}(\mathbb{R}) \) (see [37]),
\[
\left\{ h \in L_\infty(\mathbb{R}) : \|h\|_{s,\infty,\infty} := \sup_{k \in \mathbb{Z}} |a_k(h)| + \sup_{l \geq 0} \sup_{k \in \mathbb{Z}} |2^{(l+1/2)} \beta_{lk}(h)| < \infty \right\}
\]
where \( a_k(h) = \int h(x) \varphi(x-k) dx \) and \( \beta_{lk}(h) = \int 2^{l/2} \varphi(x-k) \psi(2^l x - k) dx \) are the wavelet coefficients of the function \( h \).

The spaces \( B^{s}_{\infty,\infty}(\mathbb{R}) \) are the Holder-Zygmund spaces, which contain the classical Holder-Lipschitz spaces.

Given these definitions, we now describe how we will use the wavelet approach.
It is remarkable from Theorem 3 (in [43]) that, if the densities \( f \) and \( g \) belong to \( B^{t,∞}(\mathbb{R}) \), \( ϕ \) satisfies Assumption 2, and \( ϕ,ψ \) satisfy Assumption 1 \( (T) \) then \( a_n, b_n \) and \( c_n \) are all of them \( O\left(\sqrt{\frac{\log n}{n^{3/4}}} + n^{-t/4}\right) \) almost surely and converge all to zero at this rate (with \( 0 < t < T \)).

In order to establish the asymptotic normality of the divergences estimators, we need to recall some facts about kernels wavelets.

For \( h \in B^{t,∞}(\mathbb{R}) \), the Theorem 1 below provides the asymptotic normality of

\[
\sqrt{n} \int (f_n(x) - f(x))h(x)dx
\]

necessary for setting the asymptotic normality of divergence measure, provided the finiteness of \( \mathbb{P}_X(K_{jn}(h)(X))^2 \).

**Theorem 1.** Under Assumption 4 and 2 and if \( h \in B^{t,∞}(\mathbb{R}) \), then we have

\[
\sqrt{n} \int (f_n(x) - f(x))h(x)dx \rightsquigarrow N(0,σ^2_h) \quad \text{as} \quad n \to \infty,
\]

where

\[
(2.8) \quad σ^2_h = \mathbb{P}_X(K_{jn}(h)(X))^2 - (\mathbb{P}_X(K_{jn}(h)(X))^2)
\]

The symbol \( \rightsquigarrow \) denotes the convergence in law.

\( \mathbb{P}_X(h) = \int h(x)f(x)dx \) denotes the expection of the measurable function \( h \).

The proof of this theorem is postponed to Subsection 4.

2.3. Main results. In the sequel \( J(f,g) \) is a functional of two densities functions \( f \) and \( g \) satisfying Assumption 4 and defined by

\[
J(f,g) = \int_D \phi(f(x),g(x))dx
\]

where \( \phi(s,t) \) is a function of \( (s,t) \in \mathbb{R}^2 \) of class \( C^2 \).

Define the functions \( h_1 \) and \( h_2 \) by

\[
h_1(x) = \phi^{(1)}_1(f(x),g(x)) \quad \text{and} \quad h_2(x) = \phi^{(1)}_2(f(x),g(x))
\]

and the constants \( A_1 \) and \( A_2 \) by

\[
A_1 = \int |h_1(x)| dx \quad \text{and} \quad A_2 = \int |h_2(x)| dx
\]

Suppose that \( A_1 \) and \( A_2 \) are both finites.
2.3.1. One side estimation. Suppose that either we have a sample $X_1, \ldots, X_n$ with unknown p.d.f $f$ and a known p.d.f $g$ and we want to study the limit behavior of $J(f_n, g)$ or we have a sample $Y_1, \ldots, Y_n$ with unknown p.d.f $g$ and a known p.d.f. $f$ and we want to study the limit behavior of $J(f, g_n)$.

$f_n$ or $g_n$ are as in (2.6) or in (2.7).

**Theorem 2.** Under Assumption [7] and [2] we have:

- **Consistency**:
  \[
  \limsup_{n \to \infty} \left| \frac{J(f_n, g) - J(f, g)}{a_n} \right| \leq A_1, \text{ a.s.}
  \]

  and

  \[
  \limsup_{n \to \infty} \left| \frac{J(f, g_n) - J(f, g)}{b_n} \right| \leq A_2, \text{ a.s.}
  \]

  where $a_n$ and $b_n$ are as in (2.4).

- **Asymptotic normality**:
  \[
  \sqrt{n} (J(f_n, g) - J(f, g)) \rightsquigarrow N(0, \sigma_1^2) \text{ as } n \to \infty
  \]

  and

  \[
  \sqrt{n} (J(f, g_n) - J(f, g)) \rightsquigarrow N(0, \sigma_2^2) \text{ as } n \to \infty
  \]

  where

  \[
  \sigma_1^2 = \mathbb{P}_X (K_{h_1}(X))^2 - (\mathbb{P}_X (K_{h_1}(h_1)(X)))^2
  \]

  and

  \[
  \sigma_2^2 = \mathbb{P}_Y (K_{h_2}(Y))^2 - (\mathbb{P}_Y (K_{h_2}(h_2)(Y)))^2.
  \]

2.3.2. Two sides estimation. Suppose that we have two samples $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ with respectively unknown p.d.f $f$ and $g$ and we want to study the limit behavior of $J(f_n, g_n)$.

**Theorem 3.** Under Assumption [7] and [2] we have

\[
\limsup_{n \to \infty} \left| \frac{J(f_n, g_n) - J(f, g)}{c_n} \right| \leq A_1 + A_2
\]

and

\[
\sqrt{n} (J(f_n, g_n) - J(f, g)) \rightsquigarrow N(0, \sigma_3^2) \text{ as } n \to \infty
\]

where $\sigma_3^2 = \sigma_1^2 + \sigma_2^2$.

$\sigma_1^2$ and $\sigma_2^2$ are as in (2.13) and (2.14).

The proofs are given in Section [4].

Right now, we are going to apply these results to particular divergence measures estimations. We will have to check the conditions (2.1), (2.2), and (2.3).
2.4. Particular cases. Results for Tsallis-α, Renyi-α, Kullback-Leibler and $L_2$ divergences measures will follow as corollaries since they are particular cases of $J(f, g)$. To ensure the general conditions $C1 – \phi$ and $C2 – \phi$ we begin by giving the main assumption on the densities $f$ and $g$.

Assumption 4. There exists a compact $K \subset \mathbb{R}$ containing the supports of the densities $f$ and $g$ and such that

$$\exists (\kappa_1, \kappa_2) \in \mathbb{R}^2, \text{such that}, 0 < \kappa_1 \leq f(x), g(x) \leq \kappa_2 < \infty.$$ 

Throughout this subsection, we will use the Assumption 4. The integrals are on $K$ and the constantes are integrables. We use the Dominate convergence theorem based on this remark. Meaning that with Assumption 4, then the conditions (2.1), (2.2), and (2.3) are satisfied.

In the following the divergence measures, the functions $h_1$ and $h_2$ should be updated in each cases, in the same way that $A_1, A_2, \sigma^2_1$, and $\sigma^2_2$ since they depend on the bessov functions $f$ and $g$ in $B^2_{1, \infty}(\mathbb{R})$ and on the randoms variables $X$ and $Y$.

2.4.1. Case 1: Hellinger integral of order $\alpha$. We start by the Hellinger integral of order $\alpha$ defined by

$$I(f, g) = \int_K f^\alpha(x) g^{1-\alpha}(x) \, dx$$

Here $\phi(s, t) = s^\alpha t^{1-\alpha}$ and one has

$$\phi_1^{(1)}(s, t) = \alpha s^{\alpha-1} t^{1-\alpha}, \phi_2^{(1)}(s, t) = (1-\alpha) s^\alpha t^{-\alpha}$$

and

$$\phi_1^{(2)}(s, t) = (\alpha - 1) s^{\alpha-2} t^{1-\alpha}, \phi_2^{(2)}(s, t) = -\alpha (1-\alpha) s^{\alpha-1} t^{-\alpha}, \phi_{1,2}^{(2)}(s, t) = \phi_{2,1}^{(2)}(s, t) = \alpha (1-\alpha) s^{\alpha-1} t^{-\alpha}.$$ 

Now let $h_1(x) = \alpha f^{\alpha-1}(x) g^{1-\alpha}(x)$ and $h_2(x) = (1-\alpha) f^{\alpha}(x) g^{-\alpha}(x)$

$A_1 = \int h_1(x) \, dx < \infty$ and $A_2 = \int |h_2(x)| \, dx < \infty$.

Corollary 1. (One sample estimation). We have

- **Consistency**
  $$\limsup_{n \to \infty} \frac{|I(f_n, g) - I(f, g)|}{a_n} \leq A_1, \ a.s$$
  $$\limsup_{n \to \infty} \frac{|I(f_n, g_n) - I(f, g)|}{b_n} \leq A_2, \ a.s$$

- **Asymptotic normality**
  $$\sqrt{n}(I(f_n, g) - I(f, g)) \rightsquigarrow \mathcal{N}(0, \sigma^2_{I,1}) \text{ as } n \to \infty$$
  $$\sqrt{n}(I(f_n, g_n) - I(f, g)) \rightsquigarrow \mathcal{N}(0, \sigma^2_{I,2}) \text{ as } n \to \infty$$

where

$$\sigma^2_{I,1} = \mathbb{P}_X(K_{j_n}(h_1)(X))^2 - (\mathbb{P}_X(K_{j_n}(h_1)(X))^2)^2$$

and

$$\sigma^2_{I,2} = \mathbb{P}_Y(K_{j_n}(h_2)(Y))^2 - (\mathbb{P}_Y(K_{j_n}(h_2)(Y))^2)^2.$$. 


with \( h_1(x) = \alpha f^{\alpha-1}(x)g^{1-\alpha}(x) \) and \( h_2(x) = (1-\alpha)f^\alpha(x)g^{-\alpha}(x) \).

**Corollary 2.** (Two side estimation) We have

- **Consistency**
  \[
  \limsup_{n \to \infty} \frac{|I(f_n, g_n) - I(f, g)|}{c_n} \leq A_1 + A_2, \text{ a.s.}
  \]

- **Asymptotic normality**
  \[
  \sqrt{n}(I(f_n, g_n) - I(f, g)) \to N(0, \sigma_3^2) \text{ as } n \to \infty
  \]

where \( \sigma_3^2 = \sigma_1^2 + \sigma_2^2 \).

In the following, handling \( I \) the Hellinger integral of order \( \alpha \), conditions (2.1), (2.2), and (2.3) are satisfied from **Assumption**[4].

**2.4.2. Case 2 : Tsallis Divergence measure.**

\[
D_{T,\alpha}(f, g) = \frac{1}{\alpha - 1} (I(f, g) - 1)
\]

**Corollary 3.** (One side estimation) We have

- **Consistency**
  \[
  \limsup_{n \to \infty} \frac{|D_{T,\alpha}(f_n, g_n) - D_{T,\alpha}(f, g)|}{a_n} \leq \frac{1}{|\alpha - 1|} A_1, \text{ a.s.}
  \]

- **Asymptotic normality**
  \[
  \sqrt{n}(D_{T,\alpha}(f_n, g_n) - D_{T,\alpha}(f, g)) \to N(0, \sigma_T^2) \text{ as } n \to \infty
  \]

where \( \sigma_T^2 = \sigma_{T,1}^2 + \sigma_{T,2}^2 \).

**Corollary 4.** (Two sides estimation)

Under conditions of theorem, we have

- **Consistency**
  \[
  \limsup_{n \to \infty} \frac{|D_{T,\alpha}(f_n, g_n) - D_{T,\alpha}(f, g)|}{c_n} \leq A_1 + A_2, \text{ a.s.}
  \]

- **Asymptotic normality**
  \[
  \sqrt{n}(D_{T,\alpha}(f_n, g_n) - D_{T,\alpha}(f, g)) \to N(0, \sigma_T^2)
  \]

where \( \sigma_T^2 = \sigma_{T,1}^2 + \sigma_{T,2}^2 \).
2.4.3. Case 3 : Reyni Divergence measure.

\[ D_{R,\alpha}(f,g) = \frac{1}{\alpha - 1} \log I(f,g) \]

Corollary 5. (One side estimation) We have

- Consistency

\[ D_{R,\alpha}(f_n,g) - D_{R,\alpha}(f,g) = O_{a.s}(a_n) \]

\[ D_{R,\alpha}(f,g_n) - D_{R,\alpha}(f,g) = O_{a.s}(b_n) \]

where \( a_n \) and \( b_n \) are as in (2.4).

- Asymptotic normality

\[ \sqrt{n}(D_{R,\alpha}(f_n,g) - D_{R,\alpha}(f,g)) \rightsquigarrow N(0,\sigma_{R,1}^2) \text{ as } n \to \infty \]

\[ \sqrt{n}(D_{R,\alpha}(f,g_n) - D_{R,\alpha}(f,g)) \rightsquigarrow N(0,\sigma_{R,2}^2) \text{ as } n \to \infty \]

where \( \sigma_{R,1}^2 = \frac{\sigma_{I,1}^2}{(\alpha - 1)I(f,g)} \) and \( \sigma_{R,2}^2 = \frac{\sigma_{I,2}^2}{(\alpha - 1)I(f,g)} \).

Corollary 6. (Two sides estimation) We have

- Consistency

\[ D_{R,\alpha}(f_n,g_n) - D_{R,\alpha}(f,g) = O_{a.s}(c_n) \]

where \( c_n \) is as in (2.4).

- Asymptotic normality

\[ \sqrt{n}(D_{R,\alpha}(f_n,g_n) - D_{R,\alpha}(f,g)) \rightsquigarrow N(0,\sigma_R^2) \text{ as } n \to \infty \]

where \( \sigma_R^2 = \sigma_{R,1}^2 + \sigma_{R,2}^2 \).

The proofs of Corollaries 5 and 6 are postponed to Section 4.

2.4.4. Case 4 : Kulback-Leib Divergence measure.

\[ D_{KL}(f,g) = \int_K f(x) \log \frac{f(x)}{g(x)} \, dx \]

In this case \( \phi(s,t) = s \log \frac{s}{t} \) and one has

\[ \phi_1^{(1)}(s,t) = 1 + \log \frac{t}{s}, \quad \phi_2^{(1)}(s,t) = -s \frac{t}{s} \]

and

\[ \phi_1^{(2)}(s,t) = \frac{1}{s}, \quad \phi_2^{(2)}(s,t) = s \frac{t}{s}, \quad \phi_{1,2}^{(2)}(s,t) = \phi_{2,1}^{(2)}(s,t) = -\frac{1}{t}, \]

Thus \( h_1(x) = 1 + \log \frac{f(x)}{g(x)}, \quad h_2(x) = \frac{f(x)}{g(x)}, \quad A_1 = \int_K \left( 1 + \log \frac{f(x)}{g(x)} \right) \, dx < \infty, \) and \( A_2 = \int_K \left| \frac{f(x)}{g(x)} \right| \, dx < \infty. \)

With the Assumption 4 the conditions (2.1), (2.2), and (2.3) are satisfied for any measurables sequences of functions \( \delta_{n}^{(1)}(x), \delta_{n}^{(2)}(x), \rho_{n}^{(1)}(x), \) and \( \rho_{n}^{(2)}(x) \) of \( x \in D, \) uniformly converging to zero.
Corollary 7. (One side estimation) We have

- **Consistency**

\[
\limsup_{n \to \infty} \frac{|\mathcal{D}_{KL}(f_n, g) - \mathcal{D}_{KL}(f, g)|}{a_n} \leq A_1, \text{ a.s}
\]

\[
\limsup_{n \to \infty} \frac{|\mathcal{D}_{KL}(f, g_n) - \mathcal{D}_{KL}(f, g)|}{b_n} \leq A_2, \text{ a.s}
\]

- **Asymptotic normality**

\[
\sqrt{n}(\mathcal{D}_{KL}(f_n, g) - \mathcal{D}_{KL}(f, g)) \rightarrow_{	ext{d}} N(0, \sigma_{K,1}^2)
\]

\[
\sqrt{n}(\mathcal{D}_{KL}(f, g_n) - \mathcal{D}_{KL}(f, g)) \rightarrow_{	ext{d}} N(0, \sigma_{K,2}^2)
\]

where

\[
\sigma_{K,1}^2 = \frac{1}{(\alpha - 1)^2} \left( \mathbb{P}(K_j(h_1)(X))^2 - \mathbb{P}(K_j(h_1)(X))^2 \right)
\]

and

\[
\sigma_{K,2}^2 = \frac{1}{(\alpha - 1)^2} \left( \mathbb{P}(K_j(h_2)(Y))^2 - \mathbb{P}(K_j(h_2)(Y))^2 \right)
\]

with \(h_1(x) = 1 + \log \frac{f(x)}{g(x)}\) and \(h_2(x) = \frac{f(x)}{g(x)}\).

Corollary 8. (Two sides estimation) We have

- **Consistency**

\[
\limsup_{n \to \infty} \frac{|\mathcal{D}_{KL}(f_n, g_n) - \mathcal{D}_{KL}(f, g)|}{c_n} \leq A_1 + A_2, \text{ a.s}
\]

- **Asymptotic normality**

\[
\sqrt{n}(\mathcal{D}_{KL}(f_n, g_n) - \mathcal{D}_{KL}(f, g)) \rightarrow_{	ext{d}} N(0, \sigma_{K}^2)
\]

where \(\sigma_{K}^2 = \sigma_{K,1}^2 + \sigma_{K,2}^2\)

2.4.5. Case 5 : \(L_2\) Divergence measure.

\[
\mathcal{D}_{L_2}(f, g) = \int_K (f(x) - g(x))^2 dx
\]

Here \(\phi(s, t) = |f(x) - g(x)|^2\), but we proceed by a different route. One has

\[
\mathcal{D}_{L_2}(f_n, g) - \mathcal{D}_{L_2}(f, g) = \int_K \left( (f_n(x) - g(x))^2 - (f(x) - g(x))^2 \right) dx
\]

\[
= \int_K (f_n(x) - f(x)) (f_n(x) + f(x) - 2g(x)) dx
\]

\[
= 2 \int_K (f_n(x) - f(x)) (f(x) - g(x)) dx + \int_K (f_n(x) - f(x))^2 dx
\]

and also

\[
\mathcal{D}_{L_2}(f, g_n) - \mathcal{D}_{L_2}(f, g) = -2 \int_K (g_n(x) - g(x)) (f(x) - g(x)) dx + \int_K (g_n(x) - g(x))^2 dx
\]
Let \( h_1(x) = 2(f(x) - g(x)) \) and \( h_2 = -h_1 \). Then we deduce

\[
\sqrt{n} \left( \mathcal{D}_{L_2}(f_n, g) - \mathcal{D}_{L_2}(f, g) \right) = \sqrt{n} \int (f_n(x) - f(x))h_1(x)dx + o_P(1)
\]

\[
\sqrt{n} \left( \mathcal{D}_{L_2}(f_n, g_n) - \mathcal{D}_{L_2}(f, g) \right) = \sqrt{n} \int (g_n(x) - g(x))h_2(x)dx + o_P(1)
\]

Let \( A_1 = A_2 = 2\int_K |f(x) - g(x)|dx \). Then we give

**Theorem 4. (One side estimation)**

- **Consistency**

\[
\limsup_{n \to \infty} \frac{|\mathcal{D}_{L_2}(f_n, g) - \mathcal{D}_{L_2}(f, g)|}{a_n} \leq A_1, \text{ a.s}
\]

\[
\limsup_{n \to \infty} \frac{|\mathcal{D}_{L_2}(f_n, g_n) - \mathcal{D}_{L_2}(f, g)|}{b_n} \leq A_2, \text{ a.s}
\]

- **Asymptotic normality**

\[
\sqrt{n} \left( \mathcal{D}_{L_2}(f_n, g) - \mathcal{D}_{L_2}(f, g) \right) \sim \mathcal{N}(0, \sigma_{L_2}^2) \text{ as } n \to \infty
\]

\[
\sqrt{n} \left( \mathcal{D}_{L_2}(f_n, g_n) - \mathcal{D}_{L_2}(f, g) \right) \sim \mathcal{N}(0, \sigma_{L_2}^2) \text{ as } n \to \infty
\]

where

\[
\sigma_{L_2,1}^2 = \mathbb{P}_X(K_{j_n}(h_1)(X))^2 - (\mathbb{P}_X(K_{j_n}(h_1)(X))^2)
\]

and

\[
\sigma_{L_2,2}^2 = \mathbb{P}_Y(K_{j_n}(h_2)(Y))^2 - (\mathbb{P}_Y(K_{j_n}(h_2)(Y))^2)
\]

with \( h_1(x) = 2(f(x) - g(x)) \) and \( h_2(x) = 2(g(x) - f(x)) \).

**Theorem 5. (Two sides estimation)**

- **Consistency**

\[
\limsup_{n \to \infty} \frac{\mathcal{D}_{L_2}(f_n, g_n) - \mathcal{D}_{L_2}(f, g)}{c_n} \leq A_1 + A_2 \text{ a.s}
\]

- **Normality**

\[
\sqrt{n} \left( \mathcal{D}_{L_2}(f_n, g_n) - \mathcal{D}_{L_2}(f, g) \right) \sim \mathcal{N}(0, \sigma_{L_2}^2) \text{ a.s } n \to \infty
\]

where \( \sigma_{L_2}^2 = \sigma_{L_2,1}^2 + \sigma_{L_2,2}^2 \)

### 3. Applications

3.1. **Statistics tests.** The divergence measures may be applied to two statistical problems among others. First, it may be used as a goodness-of-fit problem like that: let \( X_1, X_2, \ldots \) a sample from \( X \) with an unknown probability density function \( f \) and we want to test the hypothesis that \( f \) is equal to a known and fixed probability density function \( g \). We want to test

\[
H_0 : f = g \quad \text{versus} \quad H_1 : f \neq g,
\]

both functions \( f \) and \( g \) in Besov space \( B^r_{\infty, \infty}(\mathbb{R}) \).

For a fixed \( x \in D \), we can test the (pointwise) null hypothesis

\[
H_0 : f(x) = g(x) \quad \text{versus} \quad H_1 : f(x) \neq g(x)
\]

(3.1)
using particular divergences measure like $\alpha$-Tsallis, $\alpha$-Renyi, KB, or $L_2$ divergences.

Then our proposed test statistics are of the form

$$D^\phi(f_n, g) = \int \phi(f_n(x), g(x)) dx$$

As particular cases we consider

$$\phi_1(s, t) = s^{\alpha t^{1-\alpha}}$$

$$\phi_2(s, t) = s \log \frac{s}{t}$$

$$\phi_3(s, t) = |s - t|^2$$

3.1.1. Limit distribution under null hypothesis $H_0$. In testing the null hypothesis, we propose tests statistics using Tsallis, Renyi, Kulback and $L_2$ divergence measures. Suppose that the null hypothesis $H_0$ holds so that $g$ is a known p.d.f.

Then it follows, from the previous work that

$$\sqrt{n}(D_{T,\alpha}(f_n, g) - D_{T,\alpha}(f, g)) \rightsquigarrow \mathcal{N}(0, \sigma_{T,1}^2)$$

where

$$\sigma_{T,1}^2 = \frac{1}{(\alpha - 1)^2} \left( \mathbb{P}_X(K_{j_n}(h_T)(X))^2 - \mathbb{P}_X(K_{j_n}(h_T)(X))^2 \right)$$

with $h_T(x) = \alpha f^{\alpha-1}(x)g^{1-\alpha}(x)$ and $K_{j_n}(h_T)(X) = \int K_{j_n}(x, t)h_1(t)f(t)dt$

Reny divergence measure

$$\sqrt{n}(D_{R,\alpha}(f_n, g) - D_{R,\alpha}(f, g)) \rightsquigarrow \mathcal{N}(0, \sigma_{R,1}^2)$$

where

$$\sigma_{R,1}^2 = \frac{\sigma_{T,1}^2}{(\alpha - 1)^2 f(g)}$$

whith $h_R(x) = \alpha f^{\alpha-1}(x)g^{1-\alpha}(x)$

Kulback-Leib :

$$\sqrt{n}(D_{KL}(f_n, g) - D_{KL}(f, g)) \rightsquigarrow \mathcal{N}(0, \sigma_{K,1}^2)$$

where

$$\sigma_{K,1}^2 = \frac{1}{(\alpha - 1)^2} \left( \mathbb{P}_X(K_{j_n}(h_K)(X))^2 - \mathbb{P}_X(K_{j_n}(h_K)(X))^2 \right)$$

where $h_K(x) = 1 + \log \frac{f(x)}{g(x)}$

3.2. Confidence bands. We want to obtain

4. PROOFS

The rest of this section, proceeds as follows. In Subsection 4.1 we establish the proof of the Theorem 1. Subsection 4.2 is devoted to the proof of the Theorem 2. In Subsection 4.3 we present the proof of the theorem 3. The Subsection 4.4 is devoted to proofs of the corollaries 5 and 6.
4.1. Proof of the Theorem

Proof. Suppose assumptions [1] and [2] are satisfied and \( h \in B_{\infty,\infty}^2(\mathbb{R}) \).

We start by showing first that \( \sqrt{n} \int (f_n(x) - f(x))h(x)dx \) is a sum of an empirical process based on the sample \( X_i \) and applied on the function \( K_{j_n}(h) \) and a random variable.

We have, by definition \( K_{j_n}(h)(.) = \int K_{j_n}(x,.)h(x)dx \). Write

\[
\int (f_n(x) - f(x))h(x)dx = \int \left( \frac{1}{n} \sum_{i=1}^{n} K_{j_n}(x,X_i)h(x) - f(x)h(x) \right) dx
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \int K_{j_n}(x,X_i)h(x)dx - \int f(x)h(x)dx
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} K_{j_n}(h)(X_i) - \int f(x)h(x)dx
\]

\[
= \mathbb{P}_{n,X}(K_{j_n}(h)) - \mathbb{P}_{X}(h(X)) = (\mathbb{P}_{n,X} - \mathbb{P}_{X})(K_{j_n}(h)) + \mathbb{P}_{X}(\{K_{j_n}(h)(X) - h(X)\}).
\]

Therefore

\[
\sqrt{n} \int (f_n(x) - f(x))h(x)dx = \sqrt{n}(\mathbb{P}_{n,X} - \mathbb{P}_{X})(K_{j_n}(h)) + \sqrt{n}R_{1,n}
\]

where \( \sqrt{n}R_{1,n} = \sqrt{n}\mathbb{P}_{X}(\{K_{j_n}(h)(X) - h(X)\}) \).

One has

\[
\mathbb{P}_{X}(K_{j_n}(h)(X))^2 = \int (K_{j_n}(h)(x))^2 f(x)dx
\]

\[
= \int \left( \int K_{j_n}(x,t)h(t)dt \right)^2 f(x)dx
\]

\[
= 2^{j_n} \int \left( \int K(2^{j_n}x,2^{j_n}t)h(t)dt \right)^2 f(x)dx.
\]

Boundedness and support compactness of \( \varphi \) and \( \psi \) give \( K(2^{j_n}x,2^{j_n}t) = \sum_{k} \varphi(2^{j_n}x - k)\psi(2^{j_n}t - k) \leq C_1 \).

Now \( \left( \int K(2^{j_n}x,2^{j_n}t)h(t)dt \right)^2 \leq C_1^2 C_2 \) since \( \varphi \) vanishes on \( (B_1, B_2)^c \) and \( h \) is bounded. Finally

\[
\mathbb{P}_{X}(K_{j_n}(h)(X))^2 \leq \kappa_2 2^{2j_n} C_1^2 C_2 \text{ with } C_1, C_2 > 0.
\]

Now the usual C.L.T gives

\[
(4.1) \quad \sqrt{n}(\mathbb{P}_{n,X} - \mathbb{P}_{X})(K_{j_n}(h)) \sim N(0, \sigma_{\mathbb{P}}^2) \text{ as } n \to \infty
\]

where \( \sigma_{\mathbb{P}}^2 = \mathbb{P}_{X}(K_{j_n}(h)(X))^2 - \mathbb{P}_{X}(K_{j_n}(h)(X))^2 \).

Then the theorem will be proved if we show that \( \sqrt{n}R_{1,n} = o_p(1) \) and it is in this step that we use the fact that \( h \in B_{\infty,\infty}^2(\mathbb{R}) \).

From Theorem 9.3 in [7] one has

\[
|\mathbb{P}_{X}(K_{j_n}(h)) - h| \leq \int |K_{j_n}(h)(x) - h(x)| f(x)dx
\]

\[
\leq C_3 \|K_{j_n}(h) - h\|_{\infty} \|f\|_{\infty}
\]

\[
\leq \kappa_2 C_3 2^{-j_n t}.
\]
Therefore
\[ \sqrt{n}R_{1,n}(h) \leq \kappa_2 C_3 \sqrt{n} 2^{-j-t} = \kappa_2 C_3 n^{(1-2t)/8} = o_p(1), \]
for any $1/2 < t < T$.

Note the moment condition in theorem quoted above is equivalent to Assumption 11 (S) (see [37] page 85). This justify its use in our context.

Finally we conclude by
\[ \sqrt{n} \int (f_n(x) - f(x))h(x)dx \sim N(0, \sigma_n^2) \text{ as } n \to \infty \]
where $\sigma_n^2$ is defined above. \(\square\)

4.2. Proof of the Theorem 2

Proof. In the following development we are going to use systematically the Mean Value Theorem (M.V.T) in a bivariate dimensional and with real functions $\theta_i(i = 1, 2, \ldots, 6)$ depending on $x \in K$ but always satisfying $|\theta_i(x)| \leq 1$.

For ease of notation, we introduce the two following notations used in the sequel
\[ \Delta_n f(x) = f_n(x) - f(x) \quad \text{and} \quad \Delta_n g(x) = g_n(x) - g(x) \]
such that
\[ a_n = \|\Delta_n f\|_{\infty} \quad \text{and} \quad b_n = \|\Delta_n g\|_{\infty}. \]

Let $c_n = \max(a_n, b_n)$. Recall $a_n$, $b_n$ and $c_n$ are all $o_p(1)$.

We start by the one side asymptotic estimation.

One has
\[ \phi(f_n(x), g(x)) = \phi(f(x) + \Delta_n f(x), g(x)). \]

By an application of the M.V.T to the function $u_1(x) \mapsto \phi(u_1(x), g(x))$ one has that there exists $\theta_1(x) \in (0, 1)$ such that
\[ \phi(f_n(x), g(x)) = \phi(f(x), g(x)) + \Delta_n f(x)\phi^{(1)}_1(f(x) + \theta_1(x)\Delta_n f(x), g(x)) \]
where
\[ \Delta_n f(x)\phi^{(1)}_1(f(x) + \theta_1(x)\Delta_n f(x), g(x)) = \Delta_n f(x) \phi^{(1)}_1(f(x), g(x)) + \theta_1(x)(\Delta_n f(x))^2 \phi^{(2)}_1(f(x) + \theta_2(x)\Delta_n f(x), g(x)) \]
by an application of the M.V.T to the function $u_2(x) \mapsto \phi^{(1)}_1(u_2(x), g(x))$ and with $\theta_2(x) \in (0, 1)$.

We can write (4.2) as
\[ \phi(f_n(x), g(x)) = \phi(f(x), g(x)) + \Delta_n f(x)\phi^{(1)}_1(f(x), g(x)) + \theta_1(x)(\Delta_n f(x))^2 \phi^{(2)}_1(f(x) + \theta_2(x)\Delta_n f(x), g(x)) \]
Now we has
\[ J(f_n, g) - J(f, g) = \int \Delta_n f(x) \phi^{(1)}_1(f(x), g(x))dx + \int \theta_1(x)(\Delta_n f(x))^2 \phi^{(2)}_1(f(x) + \theta_2(x)\Delta_n f(x), g(x)) dx, \]
hence
\[ |J(f_n, g) - J(f, g)| \leq a_n \int |\phi^{(1)}_1(f(x), g(x))| dx + a_n^2 \int |\phi^{(2)}_1(f(x) + \theta_2(x)\Delta_n f(x), g(x))| dx. \]
Therefore
\[
\limsup_{n \to \infty} \frac{|J(f_n, g) - J(f, g)|}{a_n} \leq A_1 + a_n \int \phi_1^{(2)}(f(x) + \theta_2(x)\Delta_n f(x), g(x)) \, dx,
\]
where 
\[A_1 = \int \phi_1^{(1)}(f(x), g(x)) \, dx.\]

This with \(2.1\) yield and prove \(2.9\).

Now let prove \(2.10\).

By swapping the roles of \(f\) and \(g\) one obtains
\[
J(f, g_n) - J(f, g) = \int \Delta_n g(x) \phi_2^{(1)}(f(x), g(x)) \, dx + \int \theta_3(x)(\Delta_n g(x))^2 \phi_2^{(2)}(f(x), g(x) + \theta_4(x)\Delta_n g(x)) \, dx
\]
then
\[
|J(f, g_n) - J(f, g)| \leq b_n \int \phi_2^{(1)}(f(x), g(x)) \, dx + b_n^2 \int \phi_2^{(2)}(f(x), g(x) + \theta_4(x)\Delta_n g(x)) \, dx
\]
one obtains
\[
|J(f, g_n) - J(f, g)| \leq A_2 + b_n \int \phi_2^{(2)}(f(x), g(x) + \theta_4(x)\Delta_n g(x)) \, dx
\]
where 
\[A_2 = \int \phi_1^{(1)}(f(x), g(x)) \, dx.\]

This and \(2.2\) give and prove \(2.11\).

We focus now on the asymptotic normality for one sample estimation.

Going back to \(4.3\), we have
\[
\sqrt{n}(J(f_n, g) - J(f, g)) = \int \phi_1^{(1)}(f(x), g(x)) \, \alpha_n(x) \, dx + \int \theta_1(x)\sqrt{n}(\Delta_n f(x))^2 \phi_1^{(2)}(f(x) + \theta_2(x)\Delta_n f(x), g(x)) \, dx.
\]
\[= \sqrt{n} \left( n \int (f_n(x) - f(x))h_1(x) \, dx + \sqrt{n}R_{2,n} \right)
\]
where \(h_1(x) = \phi_1^{(1)}(f(x), g(x))\).

Now by theorem \(4.3\), \(\sqrt{n} \int (f_n(x) - f(x))h_1(x) \, dx \sim \mathcal{N}(0, \sigma_1^2)\) as \(n \to \infty\) where
\[
\sigma_1^2 = \mathbb{P}_X(K_{j_n}(h_1)(X))^2 - (\mathbb{P}_X(K_{j_n}(h_1)(X)))^2
\]
and provided that \(h_1 \in B_{\infty, \infty}(\mathbb{R})\).

Thus, \(2.11\) will be proved if we show that \(\sqrt{n}R_{2,n} \to o_p(1)\). One has
\[
|\sqrt{n}R_{2,n}| \leq \sqrt{\sigma_n^2} \int \phi_1^{(1)}(f(x) + \theta_2(x)\Delta_n f(x), g(x)) \, dx
\]
Let show that \(\sqrt{\sigma_n^2} = o_p(1)\).

By Chebyshev’s inequality, one has for any \(\epsilon > 0\)
\[
P \left( \sqrt{\sigma_n^2} > \epsilon \right) = P \left( a_n > \frac{\sqrt{\epsilon}}{\sqrt{n}^{1/4}} \right) \leq \frac{n^{1/4}}{\sqrt{\epsilon}} \mathbb{E} \left[ a_n^2 \right].
\]
From theorem 3 in Gine [43], one has

\[(E\sigma_n^2)^{1/2} = O\left(\sqrt{\frac{j_n^2}{n} + 2^{-t_j_n}}\right)\]

\[= O\left(\sqrt{\frac{1}{4 \log 2} \frac{\log n}{n^{3/4}} + n^{-t/4}}\right)\]

where we use the fact that $2^{j_n} \approx n^{1/4}$. Thus

\[(P \left(\sqrt{n\sigma_n^2} > \epsilon\right))^2 = O\left(\sqrt{\frac{1}{4 \log 2} \frac{\log n}{n^{1/2}} + n^{(1-2t)/8}}\right)\]

Finally $\sqrt{n\sigma_n^2} = o_P(1)$ since

\[\sqrt{\frac{1}{4 \log 2} \frac{\log n}{n^{1/2}} + n^{(1-2t)/8}} \to 0 \text{ as } n \to +\infty\]

for any $1/2 < t < T$.

Finally from (4.6) and using (2.1), one has $\sqrt{\pi R_{2,n}} \to 0$ as $n \to +\infty$.

This yields and ends the proof of (2.11).

Going back to (4.4), one has

\[\sqrt{n}(J(f, g_n) - J(f, g)) = \int \phi_2^{(1)}(f(x), g(x)) \beta_n(x) dx + \sqrt{n} \int \theta_4(x)(\Delta_n g(x))^2 \phi_2^{(2)}(f(x), g(x) + \theta_4(x)\Delta_n g(x)) dx.\]

\[= \sqrt{n}\int (g_n(x) - g(x)) h_2(x) dx + \sqrt{n}R_{3,n}\]

where $h_2(x) = \phi_2^{(1)}(f(x), g(x))$.

Then by Theorem 1, one has $\sqrt{n} \int (g_n(x) - g(x)) h_2(x) dx \sim N(0, \sigma_2^2)$ where

\[\sigma_2^2 = P_Y(K_{j_n}(h_2)(Y))^2 - (P_Y(K_{j_n}(h_2)(Y)))^2\]

since $P_Y(h_2) < \infty$ and provided that $h_2 \in B_{\infty,\infty}(\mathbb{R})$.

Similarly,

\[\left|\sqrt{n}R_{3,n}\right| \leq \sqrt{n}b_n^2 \int \phi_2^{(2)}(f(x), g(x) + \theta_4(x)\Delta_n g(x)) dx,\]

while $\sqrt{n}b_n^2 = o_P(1)$ as previously. So this and (2.2) give $\sqrt{n}R_{3,n} = o_P(1)$.

Finally this shows that (2.12) holds and completes the proof of the Theorem. \[\square\]
4.3. Proof of Theorem 3

Proof. We proceed by the same techniques that led to the prove of (2.9).

We begin by breaking \( \phi(f_n(x), g_n(x)) - \phi(f(x), g(x)) \) into two terms we have already handled:

\[
\phi(f_n(x), g_n(x)) - \phi(f(x), g(x)) = \phi(f_n(x), g_n(x)) - \phi(f(x), g_n(x)) + \phi(f(x), g_n(x)) - \phi(f(x), g(x))
\]

By an application of the M.V.T to the function \( f_n(x) \mapsto \phi(f_n(x), g_n(x)) \), one has that there exists \( \theta_5(x) \in (0, 1) \) such that

\[
I_1 = \phi(f(x) + \Delta f_n(x), g_n(x)) - \phi(f(x), g_n(x)) = \Delta f_n(x) \phi_1^{(1)}(f(x) + \theta_5(x)\Delta_n f(x), g(x))
\]

By a second application of the M.V.T to the function \( f(x) + \theta_5(x)\Delta_n f(x) \mapsto \phi(f_n(x), g_n(x)) \), with \( \theta_5(x) \in (0, 1) \).

From (4.3), we get

\[
I_2 = \Delta_n g(x) \phi_2^{(1)}(f(x), g(x)) + \theta_3(x)(\Delta_n g(x))^2 \phi_2^{(2)}(f(x), g(x) + \theta_4(x)\Delta_n g(x))
\]

Therefore

\[
J(f_n, g_n) - J(f, g) = \int \Delta_n f(x) \phi_1^{(1)}(f(x), g(x)) dx + \int \Delta_n g(x) \phi_1^{(1)}(f(x), g(x)) dx + \int \theta_5(x)(\Delta_n f(x))^2 \phi_2^{(1)}(f(x) + \theta_6(x)\Delta_n f(x), g(x)) dx
\]

and

\[
|J(f_n, g_n) - J(f, g)| \leq c_n A_1 + c_n A_2 + c_n^2 \int \left| \phi_1^{(2)}(f(x) + \theta_6(x)\Delta_n f(x), g(x)) \right| dx + c_n^2 \int \left| \phi_2^{(2)}(f(x), g(x) + \theta_4(x)\Delta_n g(x)) \right| dx
\]

thus

\[
\limsup_{x \to \infty} \frac{J(f_n, g_n) - J(f, g)}{c_n} \leq A_1 + A_2 + c_n \int \left| \phi_1^{(2)}(f(x) + \theta_6(x)\Delta_n f(x), g(x)) \right| dx + c_n \int \left| \phi_2^{(2)}(f(x), g(x) + \theta_4(x)\Delta_n g(x)) \right| dx
\]

This proves the desired result.

It remains to prove (2.10)
4.4. Proofs of Corollaries 5 and 6

Proof. of Corollary 5

One has
\[ D_{R,\alpha}(f_n, g) - D_{R,\alpha}(f, g) = \frac{1}{\alpha - 1} (\log I(f_n, g) - \log I(f, g)) \]
but \( I(f_n, g) - I(f, g) = O_{a.s}(a_n) = o_P(1) \). Then, by using a Taylor expansion of \( \log(1 + y) \) it follows that almost surely,
\[ \log I(f_n, g) - \log I(f, g) = \log \left(1 + \frac{I(f_n, g) - I(f, g)}{I(f, g)}\right) = \frac{I(f_n, g) - I(f, g)}{I(f, g)} + O_{a.s}(a_n^2) = O_{a.s}(a_n). \]
That is
\[ D_{R,\alpha}(f_n, g) - D_{R,\alpha}(f, g) = O_{a.s}(a_n). \]
This proves the desired result.

The proof of (2.18) is similar to the previous proof.

To prove (2.19), recall
\[ \sqrt{n}(I(f_n, g) - I(f, g)) = \sqrt{n} \int (f_n(x) - f(x))h_1(x)dx + o_P(1) = O_P(1) \]
then
\[ \frac{I(f_n, g)}{I(f, g)} = 1 + \frac{\int (f_n(x) - f(x))h_1(x)dx}{I(f, g)} + o_P(1) \]
and by Taylor expansion of \( \log(1 + y) \) it follows that almost surely,
\[
\log I(f_n, g) - \log I(f, g) = \log \left( 1 + \frac{\int (f_n(x) - f(x)) h_1(x) dx}{I(f, g)} \right) = \frac{\int (f_n(x) - f(x)) h_1(x) dx}{I(f, g)} + O_P \left( \frac{1}{n} \right)
\]
therefore
\[
\sqrt{n}(\mathcal{D}_{R,\alpha}(f_n, g) - \mathcal{D}_{R,\alpha}(f, g)) = \frac{1}{\alpha - 1} \sqrt{n} \int (f_n(x) - f(x)) h_1(x) dx + O_P(1) \sim N(0, \sigma_1^R)
\]
as \( n \to \infty \).

where \( \sigma^2_{R,1} = \frac{\sigma^2_{I,1}}{(\alpha-1)^2 I(f,g)} \).

(2.20) is proved similarly.

Finally this ends the proof of the Corollary [5].

Proof of the Corollary [6]

Proof. We start by the consistency. From the previous work one gets
\[
\log I(f_n, g_n) - \log I(f, g) = \frac{I(f_n, g_n) - I(f, g)}{I(f, g)} + O_{a.s}(c_n^2) = O_{a.s}(c_n)
\]
hence \( (\mathcal{D}_{R,\alpha}(f_n, g_n) - \mathcal{D}_{R,\alpha}(f, g)) = O_{a.s}(c_n) \). That proves (2.21)

Let find the asymptotic normality. One gets
\[
\sqrt{n} (I(f_n, g_n) - I(f, g)) = \sqrt{n} \int (f_n(x) - f(x)) h_1(x) dx + \sqrt{n} \int (g_n(x) - g(x)) h_2(x) dx + o_P(1) = N_n
\]
where \( h_1(x) = \alpha f^{\alpha-1}(x) g^{1-\alpha}(x) \) and \( h_2(x) = (1-\alpha) f^{\alpha}(x) g^{-\alpha}(x) \).

Hence we obtain
\[
\log I(f_n, g_n) - \log I(f, g) = \frac{N_n}{\sqrt{n} I(f, g)} + O_P \left( \frac{1}{n} \right)
\]
Therefore
\[
\sqrt{n} (\mathcal{D}_{R,\alpha}(f_n, g_n) - \mathcal{D}_{R,\alpha}(f, g)) = \frac{1}{\alpha - 1} \frac{N_n}{I(f, g)} + o_P(1) \sim N(0, \sigma_2^R)
\]
as \( n \to \infty \).

where \( \sigma^2_R = \sigma^2_{R,1} + \sigma^2_{R,2} \).

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