Abstract. The programme of discretization of famous completely integrable systems and associated linear operators was launched in the 1990s. In particular, the properties of second-order difference operators on triangulated manifolds and equilateral triangular lattices have been studied by Novikov and Dynnikov since 1996. This study included Laplace transformations, new discretizations of complex analysis, and new discretizations of $GL_n$-connections on triangulated $n$-dimensional manifolds. A general theory of discrete $GL_n$-connections ‘of rank one’ has been developed (see the Introduction for definitions). The problem of distinguishing the subclass of $SL_n$-connections (and unimodular $SL_\pm^n$-connections, which satisfy $\det A = \pm 1$) has not been solved. In the present paper it is shown that these connections play an important role (which is similar to the role of magnetic fields in the continuous case) in the theory of self-adjoint Schrödinger difference operators on equilateral triangular lattices in $\mathbb{R}^2$. In Appendix 1 a complete characterization is given of unimodular $SL_\pm^1$-connections of rank 1 for all $n > 1$, thus correcting a mistake (it was wrongly claimed that they reduce to a canonical connection for $n > 2$). With the help of a communication from Korepanov, a complete clarification is provided of how the classical theory of electrical circuits and star-triangle transformations is connected with the discrete Laplace transformations on triangular lattices.\(^1\)

Bibliography: 29 titles.

Keywords: triangulated manifolds with black and white colouring, discrete connections, discrete complex structures, factorization of self-adjoint operators, Darboux and Laplace transformations, discrete integrable systems.

\(^1\)The papers of S.P. Novikov on this topic (partly with collaborators) can be found on his homepage http://www.mi.ras.ru/~snovikov, items 136–138, 140, 146, 148, 159, 163, 173–175. Click on Scientific Publications to pass to the list of papers.

AMS 2010 Mathematics Subject Classification. Primary 39A12, 39A70.
1. Introduction. Discrete $GL_n$-connections

The notion of a differential-geometric connection in bundles has two sources. First, there is holonomy, that is, parallel displacement of a point of the fibre along a path in the base. The result of the displacement belongs to a group $G$. It is clear how to discretize this notion: simply associate an element of $G$ with every oriented edge. Physicists separately decide the question of how to assign an ‘action functional’ to a connection (a Yang–Mills field). For this it is convenient to work with complexes whose 2-skeleton consists of squares. Physicists earlier worked with regular lattices, but later extended this class. In particle theory the group is compact, and so the definition of the action involves the Killing form.

The second source is the theory of overdetermined systems of first-order linear differential equations. This is what we are going to discretize. The resulting discretization arises from constructing for the Schrödinger equation an optimal discretization that preserves the fundamental property of ‘factorization’.

We briefly explain the main notions. Consider a system of equations

$$\frac{\partial \psi^i}{\partial x_l} = A^i_{kl}(x) \psi^k(x)$$

or, in vector notation,

$$\partial_l \Psi(x) = A_l(x) \Psi(x).$$

Here $i, k = 1, 2, \ldots, m$ and $l = 1, \ldots, n$. This system is generally overdetermined for $n > 1$. The obstruction to its solubility is a non-zero curvature, which is given by the commutator

$$R_{ij} = [\partial_i - A_i, \partial_j - A_j].$$

The matrix-valued form $A = A_i dx^i$ determines the connection. Solution of the equation along a path gives the parallel displacement. The curvature is given by

$$R = R_{ij} dx^i \wedge dx^j = -dA + A \wedge A$$

(wedge product of differentials). For the group $U_m \subset GL_m(\mathbb{C})$ we have $\bar{A}_l^i = -A_i$. The expressions $\text{Tr} R^s$ form a basis among all polynomials in the Chern classes.
(over the field of rational numbers). For

\[ SO_m \subset GL_m(\mathbb{R}), \quad s = 2p, \quad A_l^1 = -A_l \]

we obtain polynomials in the Pontryagin classes. Under a gauge transformation

\[ \Psi = B(x)\Phi(x) \]

we have

\[ A_l \rightarrow B^{-1}A_lB - B^{-1}\partial_l B, \quad R_{ij} \rightarrow B^{-1}R_{ij}B. \]

It is clear that the Chern and Pontryagin classes remain unchanged. They are
closed scalar-valued forms.

How can this approach to the notion of a connection be discretized? For this
purpose we define a class of first-order operators.

Consider a simplicial complex \( M^n \), in particular, a triangulated manifold. We
fix a family \( X \) of \( n \)-simplices. With every pair \((T, P)\), where \( T \) is an \( n \)-simplex
in \( X \) and \( P \) is any vertex of \( T \), we associate a non-singular \( r \times r \) matrix \( u_{T:P} \),
\( \det u_{T:P} \neq 0 \).

These data determine a first-order difference operator \( Q \) that maps vector-valued
functions on vertices to vector-valued functions on \( n \)-simplices in \( X \) by the formula

\[ Q\psi_T = \sum_P u_{T:P}\psi_P. \]

**Definition 1.** A discrete \( GL_{rn} \)-connection of rank \( r \) is given by the equation
\( Q\psi = 0 \), where the family \( X \) of simplices consists of all \( n \)-simplices.

The coefficients of such a connection are determined up to a gauge transforma-
tion, which is given by a pair of non-degenerate \( r \times r \) matrix-valued functions \( g_T, h_P \)
defined on \( n \)-simplices and vertices respectively:

\[ \psi \rightarrow h_P\psi_P, \quad u_{T:P} \rightarrow g_Tu_{T:P}h_P^{-1}, \quad Q \rightarrow g_TQh_P^{-1}. \]

The connection actually depends only on the ratios

\[ \mu_{T,P}\rightarrow u_{T:P}^{-1}u_{T:P'}, \quad \mu_{P',P} \rightarrow h_P\mu_{T,P'}h_P^{-1}. \]

The problem of recovering a connection from the holonomy data is discussed below
together with the very definition of holonomy. It is important to note here that
our discrete case exhibits the following two kinds of holonomy in contrast to the
continuous case (see the definitions below).

1. The **non-Abelian holonomy** in the group \( GL_{rn} \) is defined for all **thick
   paths** consisting of \( n \)-simplices \( T_0 \ldots T_m \) such that every intersection
   \( T_i \cap T_{i+1} = F_i \) is a common \((n - 1)\)-face of them (see Fig. 1).

2. The **framed holonomy**, which in the case of rank 1 is also referred to
   as Abelian, is defined for all **framed paths** \([P_0P_1 \ldots P_m; T_1 \ldots T_m]\), where the
   edges \( P_{i-1}P_i \) belong to the simplices \( T_i \) (see Fig. 2).

In our papers we studied connections of rank 1. All essential results
concern this case.
The simplest **parity connection** is the map sending each closed thick path $\gamma$ to its ‘parity’ $P(\gamma) = (-1)^{m(\gamma)}$, where $m(\gamma)$ is the number of $n$-simplices in the thick path $\gamma$.

Clearly, the parity connection is trivial if and only if the $n$-simplices admit a black and white colouring. In our previous works we called such colourings **discrete conformal structures**. In what follows we always work with $SL_n$-connections of rank 1 if the parity connection is trivial, and with $SL_n^\pm$-connections if it is non-trivial (the superscript $\pm$ will be omitted in some formulae).

The **canonical connection** corresponds to the case when all the coefficients $u_{T,P}$ are equal to 1.

We already mentioned that only the ratios of the coefficients of $Q$ are important for the connection $Q\psi = 0$. In the case of rank 1 we have

$$
\mu_{P,P'}^T = \frac{u_{T,P}}{u_{T,P'}}.
$$
A ‘framed path’ is a set \( \gamma_{fr} = [P_0P_1 \ldots P_m; T_1T_2 \ldots T_m] \), where \( P_{i-1}P_i \) is an edge of the \( n \)-simplex \( T_i \), \( i = 1, \ldots, m \).

Every connection of rank \( r \) induces a framed holonomy representation which associates an \( r \times r \) matrix with every framed path \( \gamma_{fr} \):

\[
\gamma_{fr} \rightarrow \prod_{i} \mu_{T_i}^{P_i} = \mu(\gamma_{fr}).
\]

This is a map from the semigroup of closed framed paths with a fixed initial point \( P_0 \) to the group \( GL_r(k) \), where \( k \) is our base field. In this paper we focus on the Abelian case \( r = 1 \) and assume that all \( u_{PP'} \) are strictly greater than 0, that is, \( u \in \mathbb{R}^+ \subset k^* = \mathbb{R}^* \):

\[
\mu : \Omega_{fr} \rightarrow \mathbb{R}^+.
\]

The interesting case \( k = \mathbb{C} \) will be studied elsewhere. The non-uniqueness of the square root leads to slight additional topological complications.

The map defined above is independent of the choice of the initial point for \( r = 1 \), since the framed holonomy is Abelian. All the expressions \( \mu(\gamma_{fr}) \) are gauge invariant. The simplest such invariant is

\[
\rho_{PP'}^{TT'} = \mu_{PP'}^{TT'} = \rho_{PP'}^{TT'} = \frac{\mu_{PP'}}{\mu_{PP'}},
\]

which corresponds to the framed path \( \gamma_{fr} = [PP'P; TT'] \). These invariants are also defined in the non-Abelian case, but they are not sufficient to solve the problem of recovering the connection for \( r > 1 \). For \( r = 1 \) this problem was completely solved in [1]. We describe the solution in the simplest case \( n = 2 \).

**Reconstruction of rank-1 connections in dimension \( n = 2 \).** To reconstruct a connection from its invariants, it suffices to know the functions \( \rho_{PP'}^{TT'} \) and the set of numbers \( \mu(\gamma_{fr}^{ij}) \) corresponding to the generators of the group \( \mathcal{H}_1(M, \mathbb{Z}) \). The corresponding procedure is described in [1]. It simplifies considerably in dimension \( n = 2 \), especially when the manifold is oriented. We now describe this case.

Consider an oriented 2-dimensional manifold \( M \).

We define a multiplicative 2-cochain \( \rho(T) \) by the formula

\[
\rho(T) = \prod_{R_q \in T} \rho_{R_q}^{TT_q},
\]

where \( T_q \cap T = R_q = ij, T = ijk \), and \( R_q \) is an edge of the triangle \( T \). It was proved in [1] that this cochain is cohomologous to zero. Indeed, this is obvious in the non-compact case. For compact oriented manifolds one easily verifies that

\[
\prod_{T \in M} \rho(T) = 1.
\]

By definition, \( \rho(T) = \prod_{R_i} \rho_{R_i}^{TT_i} \), where the \( R_i \) are all the oriented edges of \( T \) and \( T_i \cap T = R_i \). Since \( \rho_{ij}^{TT'} \rho_{ji}^{TT'} = 1 \), our product is equal to 1. This means that the multiplicative cochain is cohomologous to the trivial one.
Consider a 1-chain $\lambda$ such that $\delta \lambda = \rho^{-1/2} > 0$. We see that
\[
\lambda_{ij} \lambda_{jk} \lambda_{ki} = \rho^{-1/2}(T), \quad T = ijk, \quad \lambda_{ij} \lambda_{ji} = 1.
\]

Putting
\[
\mu_{ij}^T = \lambda_{ij} \cdot (\rho_{ij}^{TT})^{1/2},
\]
we arrive at the formula for the connection as found in [1].

Of course, the cochain $\lambda$ is not unique: one can multiply it by an arbitrary cocycle. However, this does not change the cohomology classes, since these transformations correspond precisely to the Abelian gauge changes. Thus, we obtain invariants of connections in the group $H^1(M)$ which are in addition to the ‘local’ invariants like $\rho_{ij}^{TT}$.

The non-Abelian holonomy representation is defined using ‘thick paths’, as already stated above (see Fig. 1).

By definition, a thick path $\gamma_{\text{thick}} = T_1 \ldots T_m$, where $T_m = T_0$, is a sequence of $n$-simplices such that the intersection $F_i = T_i \cap T_{i+1}$ of any two neighbours is a common $(n - 1)$-dimensional face and $F_i \neq F_{i+1}$. Every closed thick path determines a linear map
\[
K : \Omega_{\text{thick}}(M, T_0) \to GL_n(\mathbb{R})
\]
by the following rule. On every $n$-simplex $T_i$ the equation $Q\psi = 0$ induces a map from the space of $r$-vector-valued functions on the vertices of the face $F_i$ to the space of $r$-vector-valued functions on the vertices of $F_{i+1}$. We consider the composite of these maps starting at $T_0 = T_m$.

The non-Abelian curvature at a point $P$ is defined for $n = 2$ as the non-Abelian holonomy corresponding to the thick path $\gamma_P^{\text{thick}} = T_1 \ldots T_m$ formed by all the 2-simplices $T_i = PP_i P_{i+1}$ that surround the vertex $P$ (see Fig. 3).

Figure 3. The star of a vertex $P$.

This set is called the star of the vertex $P$: $\text{St}(P) = T_1 \cup T_2 \cup \cdots \cup T_m$. The non-Abelian curvature for the group $GL_2$ and for $r = 1$ is the upper triangular $2 \times 2$ matrix $K(P, T_1)$ with diagonal $(1, \pm \mu_P)$, where $\mu_P = \mu(\gamma_P^{\text{thick}})$ and the second row is $(\alpha(P_i, P), \pm \mu_P)$. Here $\gamma_P^{\text{thick}}$ is the closed thick path $\partial \text{St}(P)$ beginning and ending at any vertex $P_i \in T_i \in \text{St}(P)$, $P_i \neq P$. The following lemma is easily verifiable.
Lemma 1. For the closed path surrounding the vertex $P$,

$$\mu_P = \prod_{i=1,\ldots,m} T_i T_{i+1}$$

where the index $i$ takes all values modulo $m$.

Following [1], we now give expressions for the coefficients of the non-Abelian curvature for all $n$.

The non-Abelian curvature can be defined for all $n \geq 2$ by using the star $\text{St}(\sigma^{n-2})$ for every $(n-2)$-dimensional simplex $\sigma$ with vertices $0, \ldots, n-2$. Here, just as for $n=2$, we encounter upper-triangular matrices $A_p$ of transition from the face $\sigma, p$ to $\sigma, p + 1$. The vertices of the star are labelled by the numbers $p$ for the $(n-1)$-faces $\sigma, p$, where $p$ runs through the whole cycle $p = 1, 2, \ldots, m$, as for $n = 2$. The full cyclic product of the matrices $A_{p+j}$ over $j$ gives the curvature

$$K_{\sigma, p} = A_{p+m-1} \cdots A_p$$

around the simplex of codimension 2 with initial point $p$. It has diagonal entries of the form

$$1, \ldots, 1, \mu_{p}, \ldots, \mu_{s} = (\mu_{s+1})^{s=p+m-1}.$$ 

Here the $n$-simplices $T_s$ are of the form $\sigma, s - 1, s$, where $s$ is considered modulo $m$.

The last row is the only non-trivial one. It is given by

$$\alpha_{\sigma,0,p}, \ldots, \alpha_{\sigma,m-2,p}, \mu_{\sigma}.$$ 

The following formula expresses the coefficients in terms of the connection:

$$\alpha_{\sigma,q,p} = -\mu_{p+q, p} + \mu_{p-1, q, p} \mu_{q, p-1} - \cdots + (-1)^m \mu_{p-1, q, p} \mu_{p-2, q, p} \cdots \mu_{q, p-m+1}.$$ 

The quantities

$$\alpha^*_{\sigma,q,p} = \alpha_{\sigma,q,p} \mu_{q,p}$$

are gauge invariant and satisfy

$$\alpha^*_{\sigma,q,p+1} = -\frac{\alpha^*_{\sigma,q,p}}{\mu_{pq} \mu_{qp}} + 1 - (-1)^{m(\gamma)} \mu_{\sigma},$$

where $m(\gamma)$ is the parity connection. These quantities are expressed in terms of the framed holonomy $\rho_{ij}^{T,T'}$ by the formula

$$\alpha^*_{\sigma,q,p} = \sum_{k=0}^{m-1} (-1)^k \prod_{j=k}^{j=k} \rho_{p-j, q}^{T_{p-j+1}, T_{p-j}}, \quad \prod_p \rho_{qp}^{T_p} = (-1)^m \mu_{\sigma}.$$ 

We see from this formula that the expression improves considerably in the case of $SL_n$-connections corresponding to the trivial parity connection, since $\mu_{\sigma} = 1$.

In this paper we solve the problem of reduction of a connection to the group $SL^\pm_n$.

Our definition of a connection does not enable us to effectively distinguish the compact groups needed in particle physics. We work with linear operators and apply these ideas to the theory of discrete Schrödinger operators in electric and magnetic fields in dimension $n = 2$. 

Discrete $SL_n$-connections
2. The optimal discretization and integrable systems

In our approach, an optimal discretization of an operator must preserve the features connecting the operator with completely integrable systems of KdV type and with isospectral deformations of the type of Lax pairs for \( n = 1 \), or Manakov triples (studied by Dubrovin, Krichever, and Novikov since 1976; see [2]) for \( n = 2 \), where only one energy level is deformed. In both cases, the main property of a good discretization is the preservation of factorability of the operator. This gives rise to discrete quasi-isospectral transformations which are called either Darboux transformations at all energy levels (\( n = 1 \)), where one needs strong factorization, or Laplace transformations at one level for \( n = 2 \), where one uses weak factorization (see below).

\( n = 1 \) (strong factorability):

\[
L = -\partial_x^2 + u(x) = Q^+Q + \text{const}, \quad Q = \partial_x + a(x), \quad a_x + a^2 = u + C.
\]

Discretization:

\[
L = c_{n-1}T^{-1} + c_nT + u_n = (Q')^+Q' + C' = Q^+Q + C,
\]

where \( Q = aT + b \), \( Q' = aT^{-1} + b \), and \( T(n) = n + 1 \).

The Darboux transformations (right and left) are given by

\[
\psi \to Q\psi = \psi', \quad L \to L' = QQ^+
\]

and, respectively,

\[
\psi \to Q'\psi = \psi''', \quad L \to L''' = Q'(Q')^+
\]

for all solutions of the equation \( L\psi = \lambda\psi \).

\( n = 2 \) (weak factorability, modulo diagonal terms).

A. Hyperbolic case:

\[
L\psi = 0, \quad L = Q_1Q_2 + W(x) = Q_2Q_1 + V(x), \quad Q_i = \partial_i + A_i.
\]

The discretization of the operator (see Fig. 4) corresponds to a square lattice with basic shifts \( T_1, T_2 \):

\[
L = a + bT_1 + cT_2 + dT_1T_2.
\]

The gauge group of the equation \( L\psi = 0 \) is given by

\[
L \to fLg^{-1}, \quad \psi \to g\psi.
\]

These transformations have two invariants, which look like analogues of curvature. The Laplace transformation (see below) can be expressed in terms of these invariants.
Discrete $SL_n$-connections

Figure 4. The basic shifts on a square lattice.

The operator admits right and left factorizations:

$$L = f[(1 + uT_1)(1 + vT_2) + w],$$
$$L = f'[(1 + v'T_2)(1 + u'T_1) + w'].$$

B. Elliptic case:

$$L = Q^+Q + W = QQ^+ + V,$$
$$Q = \partial + A(z, \bar{z}),$$

where $\partial = \partial_x - i\partial_y$, $z = x + iy$, $i^2 = -1$.

The discretization of the operator corresponds to an equilateral triangular lattice with basic shifts $T_1$, $T_2$, $T_1^{-1}T_2$:

$$L = a + bT_1 + cT_2 + dT_1^{-1}T_2 + T_1^{-1}b + T_2^{-1}c + T_2^{-1}T_1d,$$

where the coefficients are real. We have a black and white colouring (see Fig. 5) and factorizations

$$L = Q^+Q + F = P^+P + G, \quad Q = u + vT_1 + wT_2, \quad P = u' + v'T_1^{-1} + w'T_2^{-1}.$$

The Laplace transformations (right and left) are given by

$$\psi \rightarrow Q\psi$$

in the continuous hyperbolic case, and accordingly in the discrete case in terms of the factorization.

In the elliptic case they are given by

$$\psi \rightarrow Q\psi \quad \text{(right)}, \quad \psi \rightarrow P\psi \quad \text{(left)}.$$

**Definition.** The direct sum of operators $Q \oplus P$ is called the connection determined by the operator (1). This connection has rank 1 by definition.

Below we shall prove that this is an $SL_2$-connection. This construction extends to all triangulated surfaces and, most naturally, to surfaces with a discrete conformal structure, that is, a black and white colouring (see Figs. 5 and 6). In this case we have the following three families of $n$-simplices.
1. All simplices $Q$. This is the connection.
2. The black simplices $Q^b$. In the case of a canonical connection, this operator is taken as the discretization of $\bar{\partial}$ when we construct a discrete complex analysis.
3. The white simplices $Q^w$. This operator is taken as the discretization of $\partial$.

We accordingly encounter the operators that play the role of covariant derivatives in our theory:

$$Q^b, \quad Q^w, \quad Q = Q^b \oplus Q^w.$$

3. The reduction problem

**Problem.** How can we effectively construct all discrete $SL_2^\pm$-connections on a triangulated manifold $M^2$?

**Construction.** We use the following procedure. Assign a number $A(R) > 0$ to each edge $R \in M$. Each triangle $T \in M$ has precisely 3 vertices $P_1, P_2, P_3$ and 3 edges $R^T_1, R^T_2, R^T_3$. Here $R_i = P_iP_{i+1}$ and $i$ is taken modulo 3. Consider the following connection operator $Q$:

$$Q \psi_T = \sum_i u_{T:P_i} \psi_{P_i}, \quad (2)$$

where the $u_{T:P_i}$ are defined as solutions of the system of equations (see Fig. 7)

$$u_{T:P_i}u_{T:P_{i+1}} = A(P_iP_{i+1}).$$
**Theorem 1.** 1. The discrete connections determined by this operator $Q$ belong to $SL_2^{\pm}$ locally and globally.
2. All $SL_2^{\pm}$-connections are obtained by this construction.

**Proof of the first part.** Consider a thick path $\gamma = T_1 \ldots T_m$, $T_m = T_0$, where $F_i = T_i \cap T_{i+1}$. The determinant of the matrix of transition from the space connected with the edge $F_{i-1}$ to the space connected with $F_i$ is equal to

$$\det M_i = -\frac{u_{T_i;P'}}{u_{T_i;P''}} = -\frac{u_{T_i;P}u_{T_i;P'}}{u_{T_i;P}u_{T_i;P''}} = -\frac{A(F_{i-1})}{A(F_i)},$$

where $P' \in F_{i-1}$, $P' \notin F_i$, $P'' \notin F_{i-1}$, $P'' \in F_i$, and $P = F_{i-1} \cap F_i$. For a closed thick path, each factor occurs exactly once in the numerator and exactly once in the denominator. This means that the monodromy matrix is unimodular. □

**Proof of the second part.** Every edge $F$ of our triangulation belongs to exactly two triangles: $F = T' \cap T''$. We claim that the connection can be normalized (by a gauge transformation $Q \rightarrow fQ$, where $f$ is some positive function on the triangles) in such a way that for all edges $F = PP'$ we have

$$A(T' : PP') = A(T'' : PP'),$$

where $A(T' : PP') = u_{T' ; P}u_{T ; P'}$ and $A(T'' : PP') = u_{T'' ; P}u_{T ; P'}$.

Indeed, fix a triangle $T_0$ and let $f(T_0) = 1$. Consider an arbitrary thick path beginning at $T_0$ and ending at $T$. The condition $A(T' : PP') = A(T'' : PP')$ uniquely fixes the normalization on all the triangles of this path. Consider a closed thick path beginning and ending at $T_0$. The monodromy matrix is unimodular precisely when the circuit along such a path yields again $f(T_0) = 1$. Hence the procedure is consistent. □

**Remark.** Every connection that locally belongs to $SL_2^{\pm}$ may be regarded as an $SL_2^{\pm}$-connection on the universal covering space of $M$, where the operator $Q$ is determined by a set of numbers $A(E)$ on the edges, and its coefficients $u_{T ; P}$ satisfy for every $g \in \pi_1(M)$ the equality

$$u_{gT : gP} = \lambda_g u_{T ; P}$$

or, equivalently, $g^* Q = \lambda_g Q$ for some representation $\pi_1(M) \rightarrow \mathbb{R}^+$. 

---

**Figure 7.** Constructing a connection from functions on edges.
4. $SL_2$-connections and self-adjoint difference operators

We consider an arbitrary triangulated 2-manifold $M^2$ endowed with a ‘discrete conformal structure’, that is, a black and white colouring of 2-simplices (triangles) of the triangulation, as in [3]–[6], and we take a scalar real self-adjoint operator on such a manifold:

$$L \psi_P = \sum_{P'} b_{P,P'} \psi_{P'} = \sum_{P \neq P'} b_{P,P'} \psi_{P'} + W(P) \psi_P,$$

where either $P = P'$, or the points $P$ and $P'$ are connected by an edge. The self-adjointness means that $b_{P,P'} = b_{P',P}$ for all pairs $P, P'$. The coefficients $b_{P,P'}$ are regarded as a potential: $W(P) = b_{P,P}$. We also assume that $b_{P,P'} > 0$ for all $P, P'$.

We observed in [7]–[9] that every such operator can be written in a factorized form: there is a unique pair of operators $Q^b$ and $Q^w$ acting on the black and white triangles respectively such that

$$L = Q^b + Q^w = Q^w + Q^b + W.$$

Here $Q^b \psi_T = \sum_{P \in T} u_{T,P} \psi_P$, where $T$ is any black triangle, and $Q^w_T = \sum_{P \in T} u_{T,P} \psi_P$, where $T$ is any white triangle (see Figs. 5 and 6).

Consider the combined operator $Q_L = \{Q^b, Q^w\}$ which coincides with $Q^b$ on black triangles and with $Q^w$ on white triangles. It determines on $M^2$ a discrete $GL_2$-connection associated with the real operator $L$.

**Theorem 2.** The connection $Q_L$ is an $SL_2$-connection.

**Proof.** Every edge of the triangulation is the intersection of a black triangle and a white triangle: $R = T \cap T'$. Since $L = Q^b + Q^b = Q^w + Q^w$ up to diagonal terms, we see that the off-diagonal coefficients of $L$ can be represented as products of pairs of coefficients of the operators $Q^b$ or $Q^w$, respectively (see Fig. 7):

$$u_{T,P} u_{T',P'} = u_{T,P} u_{T',P'} = A(PP').$$

We proved in Theorem 2 that prescribing a set of numbers $A(R)$ on the edges $R = PP'$ determines an $SL_2 \pm$-connection. Since every thick path on a manifold with discrete conformal structure contains an even number of simplices, we have an $SL_2$-connection. □

5. The Laplace–Darboux transformations, finite-zone operators, and two-dimensional Toda lattice

The **Darboux transformations** for the one-dimensional problem $L \psi = \lambda \psi$:

$$L = Q^+ Q \rightarrow Q Q^+ = \tilde{L}, \quad \psi \rightarrow \tilde{\psi} = Q \psi,$$

and the **Laplace transformations** for the 2-dimensional equation $L \psi = 0$ (see § 2) are deeply connected with completely integrable systems of Korteweg–de Vries type, Kadomtsev–Petviashvili type, and other types.
In the continuous case $L = -\partial_x^2 + u$, cyclic chains of Darboux transformations of odd length give rise to finite-zone operators and interesting generalizations of them (see the papers [10]–[12] of Weiss, Veselov, and Shabat). Their difference analogue is still incomplete (see [13]).

In the 2-dimensional continuous hyperbolic case

$$L \psi = 0, \quad L = \partial_x \partial_y + a \partial_x + b \partial_y + c$$

there is a connection between Laplace transformations and three-dimensional geometry. Their chains (including cyclic ones) were studied by Darboux’s school in the 19th century. It is convenient to express the Laplace transformation in terms of gauge invariants.

The 2-dimensional continuous elliptic case was studied by Novikov and Veselov [9]. In this case the operator

$$L = -\Delta + A \partial + B \bar{\partial} + c = L^+ = Q^+ Q + V,$$

$$Q^+ = -\partial + A, \quad Q = \bar{\partial} + \bar{A}$$

(with charge 1) is self-adjoint and periodic. The invariants are the magnetic field $2H = (\bar{\partial}A - \partial B)$, $B = \bar{A}$, and the potential $V = e^f$, which are required to be smooth and periodic:

$$H' = H + \frac{1}{2} \Delta f, \quad e^{f'} = e^f + H'.$$

An operator is said to be topologically trivial if the integral over an elementary cell of the lattice is equal to zero: $\bar{H} = 0$ or, equivalently, the coefficients $A, B, V$ of the operator are periodic. Consider a chain of Laplace transformations

$$\cdots \to L_{-n} \to L_{-n+1} \to \cdots \to L_{-1} \to L_0 \to L_1 \to \cdots.$$  

Expressing the magnetic fields in terms of the potentials: $H' = e^{f'} - e^f$ for all $n$, we get that

$$\frac{1}{2} \Delta f_n = H_{n+1} - H_n = (e^{f_{n+1}} - e^{f_n}) - (e^{f_n} - e^{f_{n-1}}).$$

The substitution $f_n = g_n - g_{n-1}$ yields the 2-dimensional Toda lattice

$$\frac{1}{2} \Delta g_n = e^{g_{n+1}} - g_n - e^{g_n} - g_{n-1}$$

in the elliptic case. The same argument enables one to deduce the Toda lattice in the hyperbolic case.

The difference is in the global properties. By the theorem of Novikov and Veselov [9], every cyclic Laplace chain of smooth periodic operators $L_0, L_1, \ldots, L_n = L_0$ in the elliptic case consists of algebro-geometric operators (at the zero level). All of them are topologically trivial. There is no similar result in the hyperbolic case: this is a result of global analysis based on the fact that the spaces of solutions of elliptic non-linear equations on a compact manifold (here a torus) are finite dimensional.
We have also studied the chains
\[ L_0 \rightarrow L_1 \rightarrow \cdots \rightarrow L_n \]
with both operators \( L_0 = Q_0^+ Q_0 + C \) and \( L_n = Q_n^+ Q_n \) strongly factorable. These operators possess remarkable spectral properties: they are always topologically non-trivial and have two infinitely degenerate levels, one of which is the ground level. For chains of length \( n = 1 \) there is only one such operator: the Landau operator in a homogeneous magnetic field and zero electric field. For \( n = 2 \), such operators form finite-dimensional families generated by arbitrary doubly periodic smooth solutions of the equation
\[ \frac{1}{2} \Delta f = e^f + C, \]
where \( C \) is a non-zero topological charge of multiplicity \( n \) (see [9]). We conjecture that this phenomenon occurs only for small chain lengths \((n = 2, 3, \ldots)\).

We now pass to discrete systems. The following proposition is easily proved.

**Proposition.** Let \( M^2 \) be a triangulated surface. Every real self-adjoint second-order difference operator
\[ L \psi_P = \sum_{P'} b_{P,P'} \psi_{P'} \]
with \( b_{P',P} = b_{P,P'} > 0 \) for \( P \neq P' \) can be written in the form
\[ L = Q^+ Q + W. \]
Here \( Q \) is an \( SL_2^\pm \)-connection operator and \( W \) is a potential. If the triangulation has a black and white colouring, then the operator can be written uniquely in the form
\[ L = (Q^b)^+ Q^b + U = (Q^w)^+ Q^w + V = \frac{1}{2} Q^+ Q + W, \]
where \( Q = Q^b \oplus Q^w \).

The proof follows easily from the description of \( SL_2^\pm \)-connections in Theorem 2. The coefficient \( b_{P',P} \) factors uniquely as a product of the coefficients of the connection operator \( Q \) along any triangle adjacent to the edge \( PP' \) (see Fig. 7). This uniquely determines the connection \( Q \).

When is the Laplace transformation well defined? All possible Laplace transformations for coloured triangulations were found by Novikov and Dynnikov in [8] (1997). Appendix 2 on electrical circuits contains an interesting physical realization of this result in the special gauge where \( L(\text{const}) = 0 \).

The situation is as follows. Suppose that \( L \psi = 0 \). We define the transformation as in [8]:
\[ L = Q^+ Q + W \rightarrow Q W^{-1} Q^+ + 1 = \tilde{L}, \]
\[ \psi \rightarrow \tilde{\psi} = Q \psi. \]

The operator \( Q \) can be replaced by \( Q^b \) or \( Q^w \). This Laplace transformation is well defined and gauge invariant under the transformations \( L \rightarrow U^{1/2} L U^{1/2} \). However, the square of such a Laplace transformation is the identity.
In the 2-dimensional case we have found only the following classes on which the Laplace transformation is always well defined and can be non-trivially iterated infinitely many times, forming a chain.

I. Square lattices. Consider a square lattice with basic shifts $T_1, T_2$. The operators $L$ act on the space of functions of its vertices and are hyperbolic (see [7], [8]):

$$L = a(m, n) + b(m + 1, n)T_1 + c(m, n + 1)T_2 + d(m + 1, n + 1)T_1T_2.$$ 

At the zero level of the hyperbolic problem $L\psi = 0$ we have an action of the gauge group

$$L \rightarrow fLg, \quad \psi \rightarrow g^{-1}\psi,$$

where both functions $f, g$ are assumed to have no zeros. There are two factorizations ('left' and 'right'):

$$L = f[(1 + uT_1)(1 + vT_2) + w] = g[(1 + vT_2)(1 + uT_1) + w_L].$$

The Laplace transformation is defined in the standard way:

$$\tilde{L} = f'[1 + vT_2]w^{-1}(1 + uT_1) + 1].$$

To apply the Laplace transformation, we use the following representative $L$ of the gauge equivalence class of operators at the zero energy level: $L = (1 + uT_1)(1 + vT_2) + w$. Here $u = u(m + 1, n), v = v(m, n + 1)$, and $w = w(m, n)$. The coefficients of the operator $\tilde{L}$ resulting from the Laplace transformation are denoted by $u', v'$, and $w'$, respectively. They are taken at the same points.

Note that we must choose the function $f'$ appropriately in order to preserve the gauge form after the Laplace transformation: $f' = (1 + w')w/(1 + w)$. We get that

$$\tilde{L} = f'[(1 + vT_2)w^{-1}(1 + uT_1) + 1] = (1 + u'T_1)(1 + v'T_2) + w'.$$

Here we use the following notation for functions with shifted arguments: $u_i = T_i^*(u), v'_i = T_i^*(v'), \ldots$

A direct substitution yields

$$u' = \frac{f'u}{w}, \quad v' = \frac{f'v}{w_2}, \quad u'v'_1 = \frac{f'vu_2}{w_2}.$$ 

We still have gauge transformations that preserve the form of the operator: $L \rightarrow f^{-1}Lf, \psi \rightarrow f\psi$. The invariants of these transformations are the potential $w$ in the factorized form and the curvature $H = vu_2w^{-1}v_1^{-1}$, which is analogous to the magnetic field in the continuous case. Applying the Laplace transformation, we find that

$$H' = \frac{1 + w'}{1 + w}, \quad 1 + w' = (1 + w)w_{m-1,n}w_{m,n+1}w^{-1}w_{m-1,n+1}H_{m-1,n}$$

or, after a shift by $T_1$,

$$1 + w'_1 = (1 + w_1)ww_{1,2}^{-1}H.$$
Thus, following the scheme of [8], we can express the invariants $H', w'$ in terms of the invariants $H, w$ by analogy with the continuous case. Then we eliminate $H, H'$ from the equations and express the whole chain in terms of $w$ only (see [9]).

As a result, following the classical scheme of Darboux and his school and using chains of Laplace transformations (see the references in [9]), we obtain a discretization of the 2-dimensional Toda lattice. The chain of Laplace transformations

$$\cdots \rightarrow L_{k-1} \rightarrow L_k \rightarrow L_{k+1} \rightarrow \cdots$$

is described by the equation

$$\frac{w''_1 + 1}{w'_1 + 1} \frac{w_2 + 1}{w'_2 + 1} = \frac{w'w'_{1,2}}{w'w_2},$$

where $f'$ (respectively, $f''$) stands for the function obtained from $f$ by applying the Laplace transformation once (respectively, twice). In particular, $w, w', w''$ correspond to the discrete times $k - 1, k, k + 1$ in our lattice: $w^{k-1}(m,n) = w, w^k(m,n) = w', w^{k+1}(m,n) = w''$ at the point $(m,n)$, and $w_i$ means the shift of this function in the direction $T_i, i = 1, 2$, that is, $w_{m+1,n}$ and $w_{m,n+1}$ respectively. Hence, the equation for $w^k(m,n)$ takes the form

$$\frac{w^{k+1}_1 + 1}{w^k_1 + 1} \frac{w^{k-1}_2 + 1}{w^k_2 + 1} = \frac{w^kw_{1,2}^k}{w^kw_2^k}.$$ 

A similar investigation was carried out by Doliwa [14].

**Example.** Following [8], we consider the problem of a cyclic chain of period 2 whose continuous analogue leads to the sine-Gordon (or sinh-Gordon) equation. Put $a = w^{2k}, b = w^{2k+1}$ for all $k$. We obtain a pair of equations for the functions $a(m,n)$ and $b(m,n)$. Introducing a new function $G = ab$, we have

$$\frac{GG_{12}}{G_1G_2} = 1.$$ 

We consider particular solutions with $G = C$, where $C$ is a constant. This results in the system

$$\frac{C + b_1}{1 + b_1} \frac{C + b_2}{1 + b_2} = bb_{12}.$$ 

As mentioned in [8] (1997), this system may be regarded as a discrete analogue of the sinh-Gordon system. For $C = 1$ it degenerates into the trivial form $bb_{12} = 1$.

The equivalence between this discrete analogue of the 2-dimensional Toda lattice and the known totally discrete 3-dimensional Hirota system was established by Doliwa several years later (see [15]).

The family of Hirota systems can be written in the form

$$\gamma F(k + 1, m + 1, n)F(k - 1, m, n + 1) + \alpha F(k, m, n)F(k, m + 1, n + 1) + \beta F(k, m + 1, n)F(k, m, n + 1) = 0,$$

assuming that $\alpha + \beta + \gamma = 0$. 


II. Trivalent trees. We consider a real self-adjoint fourth-order operator $L$ acting on functions of the vertices of a trivalent tree (see [16]). It is given by

$$L \psi_P = \sum_{P''} a_{P,P''} \psi_{P''} + \sum_{P'} b_{P,P'} \psi_{P'} + W_P \psi_P,$$

where $PP'P''$ is a short path of length 2, $|PP'| = 1$, and $|P'P''| = 1$ (see Fig. 8).

![Figure 8. A trivalent tree.](image)

The operator $L$ admits a factorization $L = Q^+ Q + u_P$, where

$$Q \psi_P = \sum_{P'} d_{P,P'} \psi_{P'} + v_P \psi_P,$$

$$a_{P,P''} = d_{P,P} d_{P,P''}, \quad b_{P,P'} = d_{P,P'} v_{P'} + d_{P,P} v_P,$$

$$W_P = v_P^2 + \sum_{P'} d_{P,P'}^2 + u_P.$$

The Laplace transformation is defined by the standard formulae

$$\tilde{L} = Qu^{-1}Q^+ + 1, \quad \tilde{\psi} = Q \psi$$

in the equivalence class of self-adjoint operators $L \rightarrow fLf$, $\psi \rightarrow f^{-1}\psi$. Iterating the Laplace transformation, we obtain Laplace chains

$$\cdots \rightarrow L_n \rightarrow L_{n+1} \rightarrow \cdots,$$

where $\tilde{L}_n = L_{n+1}$.

In this situation the choice of factorization depends on a continuous parameter, since the equation

$$b_{P,P'} = d_{P,P'} v_P + d_{P,P'} v_{P'},$$

has a one-parameter family of solutions $v$ depending on the initial value $v_{P_0}$ at a given point $P_0 \in \Gamma$. We can perform a series of Laplace transformations, each with its own value of the parameter. This easily yields non-trivial and long Laplace chains. They are similar to Darboux chains for the continuous 1-dimensional Schrödinger operators studied by Weiss, Shabat, and Veselov in [10]–[12], where the factorization also depends on a parameter that labels solutions of the Riccati
equation. We plan to study the corresponding analogues of discrete Toda lattices elsewhere using sequences of Laplace–Darboux chains. These systems seem to be much simpler than those discussed above.

**III. Equilateral triangular lattices.** Consider a real self-adjoint operator $L$ acting on functions of the vertices of an equilateral triangular lattice with basic shifts $T_1^\pm, T_2^\pm, (T_1^{-1}T_2)^\pm$ of equal length:

$$L = a(m,n) + \{b(m+1,n)T_1 + c(m,n+1)T_2 + d(m-1,n+1)T_1^{-1}T_2 + (\text{adjoint})\},$$

where $T_i^+ = T_i^{-1}$. We have right and left factorizations:

$$L = Q^+Q + W, \quad Q = (u + vT_1 + wT_2),$$

and

$$L = Q'^+Q' + W', \quad Q' = u' + v'T_1^{-1} + w'T_2^{-1}.$$  

This lattice naturally determines a black and white coloured triangulation of the plane $\mathbb{R}^2$, where $Q = Q^b$ and $Q' = Q^w$ (see Fig. 5).

The coefficients $b, c, d$ are associated with the edges $R = PP', PP'', PP''$, where $P = (m,n), P' = (m+1,n), P'' = (m-1,n+1)$.

This lattice is a particular case of triangulated 2-manifolds with black and white colouring of triangles (see § 1). Here we have a natural isomorphism between both the set of black triangles and the set of white triangles, and the set of vertices $(m,n)$. We can use it to write $Q^+, Q$ and $Q'^+, Q'$ as operators from the space of functions of the vertices into itself. Therefore, the Laplace transformation can be applied infinitely many times. Infinite chains of Laplace transformations are necessary in the construction of discretizations of the 2-dimensional Toda system.

In our case there is no gauge freedom, since the transformation $L \rightarrow f^{-1}Lf$, $\psi \rightarrow f^{-1}\psi$ leads to non-self-adjoint operators while the transformations $L \rightarrow fLf$ destroy the potentials $W$ and $W'$, where $L = Q^+Q + W = Q'^+Q' + W'$. To construct a discretization of the 2-dimensional Toda lattice, we must consider the Laplace transformations associated with all the factorizations $L = Q_j^+Q_j + W_j$ corresponding to the basic shifts $T_{j,1}, T_{j,2}, j = 0, 1, 2, 3, 4, 5$, where $T_{0,1} = T_1, T_{0,2} = T_2$, and the others are obtained from these by a rotation through the angle $2j\pi/6$. The basic shifts with numbers $j$ and $j+3 \pmod{6}$ are inverse to each other:

$$L = Q_j^+Q_j + \text{const} \rightarrow \tilde{L}_j = Q_jQ_j^+ + \text{const} = \tilde{\psi}_j,$$

$$\psi \rightarrow Q_j\psi = \tilde{\psi}_j.$$

One can also discretize the 2-dimensional Toda lattice using another gauge at the zero energy level $L\psi = 0$:

$$L \rightarrow fLf, \quad \psi \rightarrow f^{-1}\psi.$$  

This gauge can be fixed by the condition $W = \text{const}$, where const $= 0$ or const $= 1$. The Laplace transformation takes the following form in this gauge:

$$L = Q_j^+Q_j + \text{const} \rightarrow \tilde{L}_j = Q_jQ_j^+ + \text{const},$$

$$\psi \rightarrow Q_j\psi = \tilde{\psi}_j.$$
At each step we perform a new factorization and then make a gauge transformation to reduce the potential again to a constant. The whole collection of Laplace transformations in this case can be optimally organized using the relations between them. In any case, we arrive at some discrete Toda-type system on the 3-dimensional lattice in $\mathbb{R}^3$. Without presenting the calculations, we note that a similar system has been known since the beginning of 1980s (see [17]). We call it the ‘4-term Hirota–Miwa system’. It is plausible that nothing else can be obtained here. Thus, discrete 2-dimensional Toda-type completely integrable systems were already found in the early 1980s during a period of intensive search for new completely integrable systems by many research groups.

**Theorem 3.** The black and white operators $Q^b = Q_j = u_j + v_jT_{j1} + wT_{j2}$ and $Q^w = P_j = u_j' + v_j'T_{j1}^{-1} + w_j'T_{j2}^{-1}$ constructed above determine an $SL_2$-discrete connection $\{Q^w, Q^b\}$ which is independent of $j$.

**Proof.** We easily see that this connection coincides exactly with the connection obtained by factorization of real self-adjoint operators in § 1 and written in terms of the three different pairs of shift operators.

By Theorem 2, this connection is always of class $SL_2$ because the product relations between the coefficients of the black and white operators follow immediately from the fact that we have two different factorizations of the same operator, as pointed out above. □

**Conclusion.** The canonical $SL_2$-connection associated with a real self-adjoint operator on the equilateral triangular lattice is analogous to the magnetic part of the Schrödinger operator in quantum mechanics.

**Appendix 1. Discrete $SL_n$-connections**

We introduce the following notation. Consider an $(n-1)$-dimensional face $F$ of an $n$-dimensional simplex $T$ with vertices $P_0, P_1, \ldots, P_n, P_n \notin F$. Write $A(T : F)$ for the product of all the coefficients of a connection at the vertices of $F$:

$$A(T : F) = \prod_{0 \leq i < n} u_{T : P_i}.$$ 

**Theorem 4.** A discrete connection $Q$ belongs to the class $SL_n\pm$ if and only if there is a gauge transformation $Q \rightarrow fQ$, where $f$ is a positive function on $n$-simplices, such that

$$A(T' : F) = A(T'' : F) = A(F)$$

for any pair $T', T''$ of $n$-simplices intersecting in a face $F$ of maximal dimension: $F = T' \cap T''$ with $\dim F = n - 1$.

**Proof.** This repeats almost verbatim the proof of Theorem 1.

Indeed, suppose that (3) holds for all hyperfaces. Consider a thick path $\gamma = T_1 \ldots T_m, T_m = T_0$, where $F_i = T_i \cap T_{i+1}$. The determinant of the matrix of transition from the space connected with the face $F_{i-1}$ to the space connected with $F_i$ is equal to

$$\det M_i = -\frac{u_{T_{i} : P'}}{u_{T_{i} : P''}} = -\frac{\prod_{P \in F_{i-1}} u_{T_{i} : P}}{\prod_{P \in F_{i}} u_{T_{i} : P}} = -\frac{A(F_{i-1})}{A(F_i)},$$
where \( P' \in F_{i-1} \), \( P' \notin F_i \), \( P'' \notin F_{i-1} \), and \( P'' \in F_i \). For a closed thick path, each factor \( A(F_i) \) occurs once in the numerator and once in the denominator. This means that the monodromy matrix is unimodular.

Now suppose that the monodromy matrix along every thick path is unimodular. Fix an \( n \)-simplex and denote it by \( T_0 \). Put \( f(T_0) = 1 \). Consider an arbitrary thick path beginning at \( T_0 \) and ending at some simplex \( T \). The condition \( A(T' : F) = A(T'' : F) \) uniquely fixes a normalization on all the \( n \)-simplices of this path. Consider a closed thick path beginning and ending at \( T_0 \). The monodromy matrix is unimodular precisely when the circuit along such a path yields again \( f(T_0) = 1 \). Hence the procedure is consistent. \[ \square \]

**Corollary.** The complete classification of \( SL_n^\pm \)-connections is as follows. A connection is determined by an arbitrary positive real-valued function \( A(\Delta) \in \mathbb{R}^+ \) on the \((n - 1)\)-dimensional faces, up to the gauge equivalence

\[
A(\Delta) \rightarrow \prod_{P_i \in \Delta} f(P_i)A(\Delta)
\]

for any function of the vertices.

**Example.** For the simplest triangulations of \( n \)-spheres as faces of the simplices \( S^n = \partial \Delta^{n+1} \) we have \( s_k = (n + 2)!/((k + 1)! (n - k + 1)! \) simplices of dimension \( k \). The space of gauge equivalence classes of \( SL_n^\pm \)-connections has dimension

\[
s_{n-1} - s_0 = \frac{1}{2} (n + 1)(n + 2) - (n + 2) = \frac{1}{2} (n + 2)(n - 1).
\]

This number is equal to 2, 5, 9, 14 for \( n = 2, 3, 4, 5 \) and then grows as a quadratic function of \( n \).

We have already used canonical connections to construct discretizations of complex analysis for \( n = 2 \) (see [3], [6]). For all \( n \geq 2 \) we know that the curvature of a canonical connection is vanishing (that is, trivial) at every point if and only if the simplicial star \( St(\sigma^{n-2}) \) of every \((n - 2)\)-simplex contains an even number of vertices. A black and white colouring of \( n \)-simplices exists if and only if every closed thick path contains an even number of \( n \)-simplices. The holonomy group \( G \) is always a subgroup of the permutation group: \( G \subset S_{n+1} \).

To construct a discrete analogue of complex analysis, we must have a globally flat canonical connection. In this case we have a black and white colouring of \( n \)-simplices and an \( n \)-dimensional family of covariant constants \( Q\psi = 0 \), \( \psi \in \mathbb{R}^n \). Hence, the black and white operators \( Q^b \) and \( Q^w \) are well defined and the connection operator is their direct sum \( Q = Q^b \oplus Q^w \). A covariant constant is determined by an \((n + 1)\)-tuple of real numbers \( \psi_j \) with \( \sum_j \psi_j = 0 \) on any \( n \)-simplex. Analogues of this theory for \( n > 2 \) have not yet been constructed.

**Appendix 2. Electrical circuits and Laplace transformations**

We consider a graph \( \Gamma \) (that is, a one-dimensional simplicial complex). Suppose that \( \Gamma \) is the 1-dimensional skeleton of a 2-dimensional complex \( K \) with the following properties.
1) Each edge of $\Gamma$ lies on the boundary of precisely one 2-dimensional triangle.

2) Each vertex belongs to at least three triangles.

Regarding this graph as an electrical circuit, we assign a positive number $c(I) > 0$ called the conductance to every edge $I$. The resistance of an edge is the reciprocal of the conductance: $r(I) = 1/c(I)$. Given any voltage function $U(P)$ of vertices of the graph, one can find the current through every oriented edge $\hat{I} = P_0P_1$ by the formula

$$J(P_0P_1) = c(I) \cdot (U(P_1) - U(P_0)) = (C\partial^* U)(P_0P_1), \quad U = U(P).$$

The current $J$ may be naturally regarded as a one-dimensional circuit on the graph. A vertex of the graph is said to be free if it does not occur in the boundary $\partial J$ of this circuit (or, more precisely, occurs with coefficient 0). The voltage at a free vertex $P$ satisfies

$$U(P) = \sum_i U(P_i)c(PP_i) / \sum_i c(PP_i),$$

where the $P_i$ are all the neighbours of $P$. If all the vertices of the graph are free, then the function $U(P)$ satisfies a linear second-order difference equation $LU = 0$. In general, the image $LU(P)$ is the total current through the vertex $P$.

We now describe the star-triangle transformation\(^2\) for one triangle $P_1, P_2, P_3$ with conductances $c_3$ for $P_2P_1$, $c_1$ for $P_3P_2$, and $c_2$ for $P_1P_3$ (see Fig. 9). We add a new vertex $T$ at the centre of the triangle $T$ (we regard it as a ‘black triangle’) and connect it to the vertices $P_1, P_2, P_3$ by new edges with conductances $c'_1$ for $P_1T$, $c'_2$ for $P_2T$, and $c'_3$ for $P_3T$, where

$$c'_1(T) = \frac{c_1c_2 + c_1c_3 + c_2c_3}{c_1}, \quad c'_2(T) = \frac{c_1c_2 + c_1c_3 + c_2c_3}{c_2},$$

$$c'_3(T) = \frac{c_1c_2 + c_1c_3 + c_2c_3}{c_3}.$$

We remove all three edges $P_1P_2$, $P_2P_3$, $P_3P_1$ from the resulting graph and define a new voltage function $U'$ on the new graph by the following conditions:

1) $U' = U$ at all vertices except for the new vertex $T$;
2) the total current through $T$ is equal to 0.

These conditions uniquely determine the voltage at the new vertex $P$, and we have $\partial J' = \partial J$.

Successively applying the star-triangle transformation to all 2-dimensional (black) simplices $T$ of the complex $K$, we obtain a transformation on the whole of $K$. It induces a map from the space of voltage functions $U(P)$ of the old vertices to the space of voltage functions $U'(T)$ of the new vertices. Using the one-to-one correspondence between the black triangles and the new vertices, one can interpret this map as a map from the space of functions of vertices to the space of functions of black triangles.

\(^2\)It seems that this transformation was first introduced in 1899 by Kennelly [18] for classical electrical circuits.
The voltage function $U(P)$ on the original graph $\Gamma$ with vertices $P_i$ (which are all assumed to be free) satisfies the equation

$$LU = 0, \quad L = \partial C \partial^*, \quad C: I \mapsto c_I I,$$

for all edges $I$. Consider the **black triangle operator**

$$Q^b \psi(T) = \sum c'_i \psi(P_i), \quad i = 1, 2, 3,$$

which maps functions of vertices to functions of black triangles. It was introduced in papers of the authors and Dynnikov from 1997.

**Theorem 5.** The operator $L = \partial C \partial^*$ can be factorized in the black-triangular (Novikov–Dynnikov) form

$$L = Q^+(C')^{-1} Q - W,$$

for all black triangles $T$.

This theorem is easily proved by direct substitution. Novikov and Dynnikov (see [8], [19]) used the factorization $L = P^+ P - V$, where $P = (\sqrt{(C')})^{-1} Q$ is a black-triangular operator and $\psi' = P \psi$. In our case, $\psi' = (C')^{-1} Q \psi$. Here we work in the gauge $L \psi = 0$ for $\psi = \text{const}$. Using the gauge group $L \rightarrow f(P) L f(P)$, $\psi(P) \rightarrow f^{-1}(P) \psi(P)$, we can always write the transformation in a convenient special form.

**Theorem 6.** The operator $L'$ acting on functions of black triangles can be obtained from $L$ by the star-triangle transformation in the theory of electrical circuits. This operator is given by

$$L' = QW^{-1} Q^+ - C',$$

and $LU = 0$ implies that $L'U' = 0$, $U' = C' QU$. Hence, $L'$ is obtained from $L$ by a Novikov–Dynnikov Laplace-type transformation (written in a particular gauge) for any complex $K$ consisting of black triangles, as defined above.
This appendix appeared after Korepanov’s communication to the authors about the star-triangle transformations. Korepanov and Kashaev used it to construct solutions of Yang–Baxter-type equations (see [20], [21]). After a discussion with Korepanov, we concluded that this transformation is connected with Laplace-type transformations developed in our papers from the 1990s for discrete systems on triangulated structures.

Appendix 3. Symplectic properties of operators on graphs

In 1970 one of the authors (Novikov) published the papers [22], [23] devoted to an application of symplectic ideas (that is, ‘Hamiltonian formalism’) in differential topology and in the related area of algebraic $K$-theory. These ideas were not picked up at that time by the community at large: topologists restated them in an abstract language that forgets the general mathematical ideas. In the 1970s this area of topology slowly evolved towards isolation, becoming buried as if in deeply covered tombs.

However, Gelfand became acquainted with these ideas, and he liked them. He made an important observation and communicated it to Novikov in 1971. It turns out that von Neumann’s theory of extension of symmetric operators is, according to Gelfand, largely a chapter of symplectic algebra, where the principal role is played by Lagrangian subspaces in a symplectic vector space. This language was unknown in the 1930s, it came into being only in the 1960s. Everything was described only in the language of Hermitian operators.

Novikov found applications of this observation only after many years, starting with the paper [24] (1997). Here we present some of these ideas.

Consider a graph $\Gamma$, possibly infinite, but with finitely many edges incident to every vertex. We define a real self-adjoint operator of finite order

$$L\psi_P = \sum_{P'} b_{P,P'} \psi_{P'},$$

where the distance between $P$ and $P'$ is bounded: $d(P, P') \leq k$. The distance between two vertices is defined as the minimal number of edges in a path connecting them. The operator is self-adjoint (symmetric) if $b_{P,P'} = b_{P',P}$ or, in the real case, the coefficients themselves are equal. The order of the operator is by definition equal to $2k$.

We fix a shortest simple oriented path

$$\gamma(PP') = -\gamma(P'P)$$

connecting the vertices $P$, $P'$. For any pair of solutions

$$L\psi = \lambda \psi, \quad L\phi = \lambda \phi$$

we define their symplectic Wronskian

$$\langle \psi, \phi \rangle = \sum_{P,P'} b_{P,P'} [\gamma(PP')] (\psi_P \phi_{P'} - \phi_P \psi_{P'}).$$

This is a 1-chain. The following theorem holds.
Theorem 7. The symplectic Wronskian \( \langle \psi, \phi \rangle = -\langle \phi, \psi \rangle \) is a bilinear form on the space of solutions of the equation \( L\psi = \lambda \psi \) with values in the open 1-homology:

\[
0 = \partial (\psi, \phi) , \quad \langle \psi, \phi \rangle \in H^\text{open}_1(\Gamma, \mathbb{R}) .
\]

Before [24] (1997) this quantity did not appear in the literature and its symplectic properties were not discussed.

This condition basically means that the current \( J = \langle \Psi, \Psi^- \rangle / (2i) \) in a charged (that is, complex) quantum state \( \Psi = \psi + i\phi \) satisfies Kirchhoff’s rule on the graph without any redistribution of the current in other uncontrollable directions. In a pure state the quantum system described by the \( \psi \)-function is always closed and conservative. The current generated by a pure quantum state does not go through the vertices: the system is closed. Comparing with Appendix 2, we see that the state described there is principally a non-quantum state, except for the case \( L\psi = 0 \).

The vertices in a quantum system should be freely ‘hung in the air’.

The notion of scattering is defined for graphs with some ends going to infinity. If the operator has order 2 (that is, \( k = 1 \)) and the coefficients trivialize at infinity (that is, \( b_{P,P'} = 1, b_{P,P} = -2 \) for the vertices of the edge), then the behaviour at infinity for a given \( \lambda \) is determined by a Lagrangian subspace \( S \subset H_\infty \) of half the dimension in the symplectic space of all solutions defined near infinity. The subspace \( S \) consists of the solutions extensible to the whole graph. It turns out to be Lagrangian by the properties of the symplectic Wronskian.

The scattering zone in the quantum theory corresponds to those (real) values of the energy \( \lambda \) for which the basis of solutions at infinity is complex-conjugate and unimodular. The scattering matrix \( A \) turns out to be symmetric and unitary. From the point of view of symplectic algebra, this is just a physical way of parametrizing the Lagrangian subspaces. Indeed, we can write such a matrix in the form \( A = BB^t \), where \( B \) is unitary. For \( O \in O_n \) we have

\[
A = BB^t = BOO^t B^t
\]
since \( OO^t = 1 \). Therefore, the symmetric unitary matrices \( A \) are parametrized by the classes \( B \in U_n/O_n, B \in S \). This is the manifold of Lagrangian subspaces of half the dimension.

These ideas were developed in the papers [24]–[29] of Novikov, Schwarz, Schrader, and Kostrykin. In the continuous case, one takes Lagrangian subspaces at each vertex and imposes matching conditions. An analogue of the symplectic Wronskian for non-linear equations on graphs was defined by Novikov and Schwarz [27]. It is a closed 2-form on the manifold of solutions of a non-linear Euler–Lagrange equation on the graph with values in \( H^\text{open}_1(\Gamma, \mathbb{R}) \).

Bibliography

[1] С. П. Новиков, “Дискретные связности и разностные линейные уравнения”, Геометрическая топология и теория множеств, Сборник статей. К 100-летию со дня рождения профессора Людмилы Всеволодовны Келдыш, Тр. МИАН, 247, Наука, М. 2004, с. 186–201; English transl., S. P. Novikov, “Discrete connections and difference linear equations”, Proc. Steklov Inst. Math. 247 (2004), 168–183, arXiv: math-ph/0303035.
[2] Б. А. Дубровин, И. М. Кричевер, С. П. Новиков, “Уравнение Шрёдингера в периодическом поле и римановы поверхности”, Докл. АН СССР 229:1 (1976), 15–18; English transl., B. A. Dubrovin, I. M. Krichever, and S. P. Novikov, “The Schrödinger equation in a periodic field and Riemann surfaces”, Soviet Math. Dokl. 17 (1977), 947–951.

[3] И. А. Дынников и С. П. Новиков, “Геометрия уравнения Шрёдингера на двуманifoldах”, Mosc. Math. J. 3:2 (2003), 419–438.

[4] С. П. Новиков, New discretization of complex analysis: the Euclidean and hyperbolic planes, arXiv:0809.2963.

[5] С. П. Новиков, “Four lectures on discrete systems”, Symmetries and integrability of difference equations (Montreal, QC, June 8–21, 2008), London Math. Soc. Lecture Note Ser., vol. 381, Cambridge Univ. Press, Cambridge 2011, pp. 191–206.

[6] П. Г. Гриневич, Р. Г. Новиков, “Ядро Коши для DN-дискретного комплексного анализа Новикова–Дынникова на треугольной решетке”, УМН 62:4 (376) (2007), 155–156; English transl., P. G. Grinevich and R. G. Novikov, “The Cauchy kernel for the Novikov–Dynnikov DN-discrete complex analysis in triangular lattices”, Russian Math. Surveys 62:4 (2007), 799–801.

[7] С. П. Новиков, “Алгебраические свойства двумерных разностных операторов”, УМН 52:1 (313) (1997), 225–226; English transl., С. П. Новиков, “Algebraic properties of two-dimensional difference operators”, Russian Math. Surveys 52:1 (1997), 226–227.

[8] С. П. Новиков, И. А. Дынников, “Дискретные спектральные симметрии маломерных дифференциальных операторов и разностных операторов на правильных решетках и двумерных многообразиях”, УМН 52:5 (317) (1997), 175–234; English transl., С. П. Новиков и И. А. Дынников, “Discrete spectral symmetries of low-dimensional differential operators and difference operators on regular lattices and two-dimensional manifolds”, Russian Math. Surveys 52:5 (1997), 1057–1116.

[9] С. П. Новиков и А. Веселов, “Exactly soluble two-dimensional Schrödinger operators and Laplace transformations”, Appendix 1 (by С. П. Новиков): Difference analogs of the Laplace transformations, Appendix 2 (by С. П. Новиков, И. А. Таиманов): Difference analogs of harmonic oscillator, Solitons, geometry, and topology: on the crossroad, Amer. Math. Soc. Transl. Ser. 2, vol. 179, Amer. Math. Soc., Providence, RI 1997, pp. 109–132.

[10] J. Weiss, “Periodic fixed points of Bäcklund transformations and the Korteweg–de Vries equation”, J. Math Phys. 27:11 (1986), 2647–2656.

[11] А. Б. Шабат, “К теории преобразований Лапласа–Дарбу”, ТМФ 103:1 (1995), 170–175; English transl., A. B. Shabat, “On the theory of Laplace–Darboux transformations”, Theoret. and Math. Phys. 103:1 (1995), 482–485.

[12] А. П. Веселов, А. Б. Шабат, “Одевающая цепочка и спектральная теория оператора Шрёдингера”, Функц. анализ и его прил. 27:2 (1993), 1–21; English transl., А. П. Веселов и А. Б. Шабат, “Dressing chains and the spectral theory of the Schrödinger operator”, Funct. Anal. Appl. 27:2 (1993), 81–96.

[13] V. Spiridonov, L. Vinet, and A. Zhedanov, “Difference Schrödinger operators with linear and exponential discrete spectra”, Lett. Math. Phys. 29:1 (1993), 63–73.

[14] A. Doliwa, “Geometric discretization of the Toda system”, Phys. Lett. A 234:3 (1997), 187–192.

[15] A. Doliwa, “Lattice geometry of the Hirota equation”, SIDE III – Symmetries and integrability of difference equations (Sabaudia, Italy, May 16–22, 1998), CRM Proc. Lecture Notes, vol. 25, Amer. Math. Soc., Providence, RI 2000, pp. 93–100.
[16] P. G. Grinevich and S. P. Novikov, “Trivalent graphs and solitons”, UMN 54:6(330) (1999), 149–150; English transl., I. M. Krichever and S. P. Novikov, “Trivalent graphs and solitons”, Russian Math. Surveys 54:6 (1999), 1248–1249.

[17] T. Miwa, “On Hirota’s difference equations”, Proc. Japan Acad. Ser. A Math. Sci. 58:1 (1982), 9–12.

[18] A. E. Kennelly, “The equivalence of triangles and three-pointed stars in conducting networks”, Electrical World and Engineer 34:12 (1899), 413–414.

[19] I. A. Dynnikov, S. P. Novikov, “Преобразования Лапласа и симплексы связи”, UMN 52:6(318) (1997), 157–158; English transl., I. A. Dynnikov and S. P. Novikov, “Laplace transforms and simplicial connections”, Russian Math. Surveys 52:6 (1997), 1294–1295.

[20] R. M. Kashaev, “On discrete three-dimensional equations associated with the local Yang–Baxter relation”, Lett. Math. Phys. 38:4 (1996), 389–397.

[21] I. G. Korepanov, “А dynamical system connected with an inhomogeneous 6-vertex model”, Дифференциальная геометрия, группы Ли и механика. 14, Зап. науч. сем. ПОМИ, 215, ПОМИ, СПб. 1994, с. 178–196; I. G. Korepanov, “A dynamical system connected with an inhomogeneous 6-vertex model”, J. Math. Sci. (New York) 85:1 (1997), 1671–1683.

[22] С. П. Новиков, “Алгебраическое построение и свойства эрмитовых аналогов $K$-теории над кольцами с инволюцией с точки зрения гамильтонова формализма. Некоторые применения к дифференциальной топологии и теории характеристических классов. I”, Изв. АН СССР. Сер. матем. 34:2 (1970), 253–288; English transl., S. P. Novikov, “Algebraic construction and properties of hermitian analogs of $K$-theory over rings with involutions from the viewpoint of hamiltonian formalism. Applications to differential topology and the theory of characteristic classes. I”, Math. USSR-Izv. 4:2 (1970), 257–292.

[23] С. П. Новиков, “Алгебраическое построение и свойства эрмитовых аналогов $K$-теории над кольцами с инволюцией с точки зрения гамильтонова формализма. Некоторые применения к дифференциальной топологии и теории характеристических классов. II”, Изв. АН СССР. Сер. матем. 34:3 (1970), 475–500; English transl., S. P. Novikov, “Algebraic construction and properties of hermitian analogs of $K$-theory over rings with involutions from the viewpoint of hamiltonian formalism. Applications to differential topology and the theory of characteristic classes. II”, Math. USSR-Izv. 4:3 (1970), 479–505.

[24] С. П. Новиков, “Оператор Шрёдингера на графах и топология”, UMN 52:6(318) (1997), 177–178; English transl., S. P. Novikov, “The Schrödinger operator on graphs and topology”, Russian Math. Surveys 52:6 (1997), 1320–1321.

[25] S. P. Novikov, “Discrete Schrödinger operators and topology”, Asian J. Math. 2:4 (1998), 921–933.

[26] S. P. Novikov, “Schrödinger operators on graphs and symplectic geometry”, The ArnoldFest. Proceedings of a conference in honour of V. I. Arnold for his 60th birthday (Toronto, ON, June 15–21, 1997), Fields Inst. Commun., vol. 24, Amer. Math. Soc., Providence, RI 1999, pp. 397–413.

[27] С. П. Новиков, А. С. Шварц, “Дискретные лагранжевы системы на графах. Симплекто-топологические свойства”, UMN 54:1(325) (1999), 257–258; English transl., S. P. Novikov and A. S. Shvarts (Schwarz), “Discrete Lagrangian systems on graphs. Symplectic-topological properties”, Russian Math. Surveys 54:1 (1999), 258–259.

[28] V. Kostrykin and R. Schrader, “Kirchhoff’s rule for quantum wires”, J. Phys. A 32:4 (1999), 595–630.
[29] V. Kostrykin and R. Schrader, “Kirchhoff’s rule for quantum wires. II. The inverse problem with possible applications to quantum computers”, *Fortschr. Phys.* **48**:8 (2000), 703–716.

**Petr G. Grinevich**  
Landau Institute for Theoretical Physics  
of the Russian Academy of Sciences  
*E-mail*: pgg@landau.ac.ru

**Sergei P. Novikov**  
Landau Institute for Theoretical Physics  
of the Russian Academy of Sciences;  
Institute for Physical Sciences and Technology,  
University of Maryland at College Park, USA  
*E-mail*: novikov@ipst.umd.edu

Received 04/SEP/13  
Translated by A. DOMRIN