Uplink Interference Reduction in Large Scale Antenna Systems

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Abstract

A massive MIMO system entails a large number (tens or hundreds) of base station antennas serving a much smaller number of terminals. These systems demonstrate large gains in spectral and energy efficiency compared with conventional MIMO technology. As the number of antennas grows, the performance of a massive MIMO system gets limited by the interference caused by pilot contamination [1]. In [5], [6] A. Ashikhmin and T. Marzetta proposed (under the name of Pilot Contamination Precoding) Large Scale Fading Precoding (LSFP) and Decoding (LSFD) based on limited cooperation between base stations. They showed that Zero-Forcing LSFP and LSFD eliminate pilot contamination entirely and lead to an infinite throughput as the number of antennas grows.

In this paper, we focus on the uplink and show that even in the case of a finite number of base station antennas, LSFD yields a very large performance gain. In particular, one of our algorithms gives a more than 140 fold increase in the 5% outage data transmission rate! We show that the performance can be improved further by optimizing the transmission powers of the users. Finally, we present decentralized LSFD that requires limited cooperation only between neighboring cells.

I. INTRODUCTION

In recent years, massive MIMO systems have become quite promising in terms of meeting the increasing demand to enable high data rates in cellular systems, see [7] and references within. In a massive MIMO system, the base station (BS) is equipped with a very large number of antennas that significantly exceeds the number of users. It was shown in [1] that when the number of

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antennas tends to infinity, the main limiting factor in performance is pilot contamination, which arises due to the fact that the users in different cells unavoidably use nonorthogonal pilot signals during estimation of the channel. A number of works in the literature have been devoted to the use of efficient schemes in order to mitigate the pilot contamination effect, see for example [2]-[6] and references therein. The works in [2]-[4] assume the channel to be low rank due to the presence of a smaller number of multipath components when compared with the number of antennas. Due to this reason, the channel matrices of the users span a low rank subspace. Exploting this idea, efficient precoders have been designed to make the resultant channel between the users orthogonal thereby mitigating the pilot contamination effect. The works [5], [6], however, do not make any assumption on the low rank property of the channel and alleviate the pilot contamination problem through the use of Large Scale Fading Precoding/Decoding schemes, referred to as LSFP and LSFD respectively. LSFP and LSFD assume two stage precoding/decoding. In particular, in LSFD, at the first stage, each base station equipped with $M$ antennas conducts $M$-dimensional MIMO decoding of received signals in order to get estimates of transmitted uplink signals. For instance, base station can use $M$-dimensional matched filtering, zero-forcing, or MMSE MIMO decoders. This stage is conducted completely locally and does not require any cooperation between base stations. At the second stage, each base station forwards the obtained uplink signal estimates to a network controller. The network controller uses this information for conducting an $L$-dimensional large scale fading coefficients decoding, where $L$ is the number of cells in the network. This decoding involves only large scale fading coefficients.

It is important to note that large-scale fading coefficients do not depend on antenna and OFDM frequency subcarrier index. Thus, between any base station and user, there is only one such coefficient. Therefore LSFD requires only small bandwidth on backhaul link between base stations and the network controller, and this bandwidth does not grow with $M$. Further, in the radius of 10 wavelengths the large-scale fading coefficients are approximately constant (see [8] and references there), while small-scale fading coefficients significantly change as soon as a
user moves by a quarter of the wavelength. Thus, large-scale fading coefficients change about 40 times slower and, for this reason, LSFD is robust to user mobility.

It is shown in [5] that when the number of antennas grows to infinity and the number of cells $L$ stays constant, Zero-Forcing LSFD (ZF LSFD) allows one to completely cancel interference and provides each user with SINR that grows linearly with the number of antennas. In real life scenarios, however, when the number of antennas $M$ is finite, other sources of interferences, beyond the one caused by pilot contamination, are significant. As a result, ZF LSFD begins providing performance gain only at very large number of antennas, like $M > 10^5$. In contrast, at a smaller number of antennas, ZF LSFD results in system performance degradation compared with the case when no cooperation between base stations is used.

A natural question therefore is to ask whether one can design LSFD so that LSFD would improve system performance for relatively small values of $M$, or LSFD is only a theoretical tool useful for analysis of asymptotic regimes. In this work, we design LSFDs that take into account all sources of interference and show that such LSFDs provide performance gain in the case of any finite $M$ (we are specifically interested in scenarios when $M$ is around 100).

As performance criteria, we use the minimum rate among all users and the 5%-outage rate, which is the smallest rate among 95% of the best users. In future generations of wireless systems, all or almost all users will have to be served with large rates. Therefore, we believe that these criteria are more meaningful than the often used sum throughput. For optimizing the above criteria, we consider max-min optimization problems. Though max-min optimization is strictly speaking not optimal for 5%-outage rate criterion, it gives very good results, and therefore can be considered as an engineering tool for optimizing the 5%-outage rate.

**Notation :** We use boldface capital and small letters for matrices and vectors respectively. $X^T$, $X^H$ and $X^{-1}$ denote the transpose, Hermitian transpose and inverse of $X$, $||x||$ denotes the vector 2-norm of $x$, and the identity matrix is denoted by $I$. 


II. System Model

We consider a multicell system comprised of $L$ cells with each cell having a BS equipped with $M$ antennas and serving $K$ single antenna users with random locations in the corresponding cell. We assume that the network uses frequency reuse factor 1 and consider a flat fading channel model for each OFDM subcarrier. In what follows, we omit the subcarrier index and focus on a single subcarrier. For a given subcarrier, the $M \times 1$ channel vector between the $k$th user in the $l$th cell to the BS in the $j$th cell is denoted by

$$g_{jkl} = \sqrt{\beta_{jkl}} h_{jkl}$$

where $\beta_{jkl}$ denotes the large scale fading coefficient that depends on the user location and the propagation environment between the user and the BS, and $h_{jkl} = (h_{jkl1}, \ldots, h_{jklM})^T$ denotes the small scale fading vectors whose entries $h_{jklm}, m = 1, M$, are small scale fading coefficients.

We assume that $h_{jkl} \sim CN(0, I_M)$. The coefficients $\beta_{jkl}$ are modeled, according to [15], as

$$10 \log_{10}(\beta_{jkl}) = -127.8 - 35 \log_{10}(d_{jkl}) + X_{jkl}$$

where $d_{jkl}$ denotes the distance (in km) between the user and base station and $X_{jkl} \sim CN(0, \sigma_{\text{shad}}^2)$, where the variance $\sigma_{\text{shad}}^2$ represents the shadowing.

We assume a time block fading model. Thus, small scale fading vectors $h_{jkl}$ stay constant during the coherence interval. It is convenient to measure the length $T$ of the coherence interval in terms of the number of OFDM symbols that can be transmitted within that interval. Similarly large scale fading coefficients $\beta_{jkl}$ stay constant during large scale coherence interval of $T_\beta$ OFDM symbols. A usual assumption is that $T_\beta$ is about 40 times larger than $T$. The vectors $h_{jkl}$ and coefficients $\beta_{jkl}$ are assumed to be independent in different coherence intervals and large scale coherence intervals respectively.

Finally, we assume reciprocity between uplink and downlink channels, i.e., $\beta_{jkl}$ and $h_{jkl}$ are the same for these channels.
It is important to note that small scale fading coefficients $h_{jklm}$ depend on antenna index and on OFDM subcarrier index. If $\Delta$ is the number of OFDM tones in the coherence bandwidth and $N$ is the total number of OFDM tones, then between a BS and a user, there are $MN/\Delta$ small scale fading coefficients and only one large scale fading coefficient.

III. TIME DIVISION PROTOCOL

We assume that in all cells, the same set of orthonormal training sequences $\phi_1, \ldots, \phi_K \in \mathbb{C}^{1 \times K}$, $\phi_i \phi_j^H = \delta_{ij}$, are used. We assume that in each cell, the users are enumerated and that the $k^{\text{th}}$ user uses the training sequence $\phi_k$. We will use the notation $\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_K \end{pmatrix}$.

Remark 1. One can also consider a system in which different sets of pilots are used in different cells. We leave this interesting topic for future work.

Let $s_{kl}$ be the uplink signal transmitted by the $k^{\text{th}}$ user in the $l^{\text{th}}$ cell. The TDD communication protocol consists of the following two steps as shown in Fig. [I]

TDD Protocol

1) all users synchronously transmit their training sequences $\phi_k$, $k = 1, K$;
2) The $l^{\text{th}}$ BS uses the received training sequences to get MMSE estimates $\hat{g}_{lkl}$ of the vectors $g_{lkl}$, $k = 1, \ldots, K$;
3) all users synchronously transmit their uplink signals $s_{kl}$, $k = 1, K$, $l = 1, L$;
4) The $l^{\text{th}}$ BS conducts an $M$-dimensional MIMO decoding of the received uplink signals and gets estimates $\tilde{s}_{kl}$ of $s_{kl}$, $k = 1, \ldots, K$.
5) BS $l$ transmits the estimates $\tilde{s}_{kl}$ to the network controller via a backhaul link.

The end.
At the first step of the TDD protocol, the $l^{th}$ BS receives the signal

$$T_l = \sum_{n=1}^{L} G_{ln} P_n^\frac{1}{2} \Phi + Z_l$$  \hspace{1cm} (3)$$

where $G_{ln} = [g_{l1n} g_{l2n} \cdots g_{lKn}]$ is the concatenation of the user channel vectors in the $n^{th}$ cell to the BS in the $l^{th}$ cell, $P_n = \text{diag}(p_{1n}, \ldots, p_{Kn})$ is the diagonal channel matrix of the training powers $p_{kn}$ used during the uplink training phase in the $n^{th}$ cell and $Z_l$ is AWGN with entries that are i.i.d. $CN(0,1)$ random variables.

Multiplying $T_l$ by $\Phi^H$ and extracting the $k^{th}$ column of $T_l \Phi^H$, the $l^{th}$ BS gets

$$r_{kl} = \sum_{n=1}^{L} g_{lkn} \sqrt{p_{kn}} + \tilde{z}_l$$  \hspace{1cm} (4)$$

where $\tilde{z}_l \sim CN(0, I_M)$.

The MMSE estimate $\hat{g}_{lkl}$ of the channel vector $g_{lkl}$ is given by

$$\hat{g}_{lkl} = \mathbb{E}[g_{lkl} r_{kl}^H] \mathbb{E}[r_{kl} r_{kl}^H]^{-1} r_{kl} = \frac{\beta_{lkl} \sqrt{p_{kl}}}{1 + \sum_{n=1}^{L} \beta_{lkn} p_{kn}} r_{kl}$$  \hspace{1cm} (5)$$

Denote by $e_{lkl}$ the estimation error. Then, $g_{lkl} = \hat{g}_{lkl} + e_{lkl}$. It is well known that $e_{lkl}$ is independent and uncorrelated with $\hat{g}_{lkl}$ and

$$\hat{g}_{lkl} \sim CN\left(0, \frac{\beta_{lkl}^2 p_{kl}}{1 + \sum_{n=1}^{L} \beta_{lkn} p_{kn}} I_M \right), \quad e_{lkl} \sim CN\left(0, \left(\beta_{lkl} - \frac{\beta_{lkl}^2 p_{kl}}{1 + \sum_{n=1}^{L} \beta_{lkn} p_{kn}}\right) I_M \right)$$  \hspace{1cm} (6)$$

Invoking the MMSE decomposition, we can write $g_{lkm} = \hat{g}_{lkm} + e_{lkm}$, where, using (5), we have

$$\hat{g}_{lkm} = \mathbb{E}[g_{lkm} r_{kl}^H] \mathbb{E}[r_{kl} r_{kl}^H]^{-1} r_{kl} = \frac{\beta_{lkm} \sqrt{p_{km}}}{\beta_{lkl} \sqrt{p_{kl}}} \frac{\beta_{lkl} \sqrt{p_{kl}}}{1 + \sum_{n=1}^{L} \beta_{lkn} p_{kn}} r_{kl} = \frac{\beta_{lkm} \sqrt{p_{km}}}{\beta_{lkl} \sqrt{p_{kl}}} \hat{g}_{lkl},$$  \hspace{1cm} (7)$$
and
\[
\mathbf{e}_{lkm} \sim CN \left( \mathbf{0}, \left( \beta_{lkm} - \frac{\beta_{lkm}^2 P_{km}}{1 + \sum_{n=1}^{L} \beta_{lkn} P_{kn}} \right) \mathbf{I}_M \right)
\]

According to the TDD protocol, the signal received by the \( l \)-th BS at the third step is
\[
y_l = \sum_{n=1}^{L} \sum_{m=1}^{K} \mathbf{g}_{lmn} \sqrt{q_{mn}} s_{mn} + z_l
\]
where \( q_{mn} \) is the transmit power of the \( m \)-th user in the \( n \)-th cell and \( s_{mn} \) is its data symbol.

We can use several possible \( M \)-dimensional decoding algorithms for getting estimates of \( s_{kl} \) from \( y_l \). In particular, we can use matched filtering, zero-forcing or MMSE decoding.

The matched filtering operation has the smallest computational complexity among these three decoding algorithms. In addition, matched filtering does not require any cooperation between BS antennas and thus, significantly simplifies base station hardware. If the \( l \)-th BS uses matched filtering, then it gets for the \( k \)-th user of \( l \)-th cell the estimate
\[
\tilde{s}_{kl} = \mathbf{g}^H_{ilk} y_l = \sum_{n=1}^{L} \mathbf{g}^H_{ilk} \mathbf{g}^H_{lkn} \sqrt{q_{kn}} s_{kn} + \sum_{n=1}^{L} \sum_{m=1, m \neq k}^{K} \mathbf{g}^H_{ilk} \mathbf{g}^H_{lmn} \sqrt{q_{mn}} s_{mn} + \mathbf{g}^H_{ilk} z_l
\]
\[
= \underbrace{\mathbb{E}[\mathbf{g}^H_{ilk} \mathbf{g}_{kl}]}_{\text{Useful Term}} \sqrt{q_{kl}} s_{kl} + \underbrace{\sum_{n=1, n \neq l}^{L} \mathbb{E}[\mathbf{g}^H_{ilk} \mathbf{g}_{lkn}]}_{\text{Pilot Contamination Term}} \sqrt{q_{kn}} s_{kn}
\]
\[
+ \underbrace{\sum_{n=1}^{L} \left( \mathbf{g}^H_{ilk} \mathbf{g}^H_{lkn} - \mathbb{E}[\mathbf{g}^H_{ilk} \mathbf{g}_{lkn}] \right)}_{\text{Interference + Noise Terms}} \sqrt{q_{kn}} s_{kn} + \sum_{n=1}^{L} \sum_{m=1, m \neq k}^{K} \mathbf{g}^H_{ilk} \mathbf{g}^H_{lmn} \sqrt{q_{mn}} s_{mn} + \mathbf{g}^H_{ilk} z_l
\]

It is not difficult to see that in (10), the power of the useful term is proportional to \( \left| \mathbb{E}[\mathbf{g}^H_{ilk} \mathbf{g}_{kl}] \right|^2 \) and therefore is proportional to \( M^2 \). The powers of the pilot contamination terms are also proportional to \( M^2 \). At the same time, the powers of all other terms are proportional only to \( M \). These observations, after some additional analysis, lead to the following result obtained in [1] (see also [2], [10]).
Theorem 1. Assuming $p_{kl} = q_{kl}$ we have $\lim_{M \to \infty} \text{SINR}_{kj} = \frac{q_{kj} \beta_{klj}^2}{\sum_{l \neq i} q_{kl} \beta_{jkl}^2}$.

IV. LARGE SCALE FADING DECODING

Several techniques, such as power allocation algorithms, frequency reuse schemes, and others, have been proposed to mitigate the effect of pilot contamination, see [7], [9], [10]. These techniques allow one to mitigate the pilot contamination interference, but neither of them completely eliminates it. As a result, similar to Theorem I, the SINRs stay finite even in the asymptotic regime as $M$ tends to infinity.

For obtaining a system in which SINRs grow along with $M$, one may try to use a network MIMO scheme (see for example [12], [13], [14]). In such a system, the $j^{th}$ base station estimates the coefficients $\beta_{jkl}$ and $h_{jkl}$ for $k = 1, K$, $l = 1, L$, and $m = 1, M$, and sends them to the network controller (or other base stations). This allows all base stations to behave as one super large antenna array. This approach, however, seems to be infeasible for the following reason.

It can be seen that the number of small scale fading coefficients $h_{jklm}$ is proportional to $M$. Thus, in the asymptotic regime, as $M$ tends to infinity, the needed backhaul bandwidth grows infinitely, and the network MIMO scheme becomes infeasible. Even in the case of finite $M$, the needed backhaul bandwidth is tremendously large. For instance, assuming $M = 100$, the coherence bandwidth $\Delta = 14$ and the number of OFDM tones $N = 1400$, we obtain that the $j^{th}$ base station needs to transmit to the network controller $N M / \Delta \cdot K (L - 1) = 10000 K (L - 1)$ small scale fading coefficients. Note also that typically coherence interval is short, i.e., $T$ is small, since the small scale fading coefficients substantially change as soon as a mobile moves a quarter of the wavelength. Thus, those $10000 K (L - 1)$ coefficients will be sent quite frequently. All of these make the needed backhaul bandwidth hardly feasible.

A breakthrough was achieved in [5], [6] where it was proposed to organize cooperation between BSs on the level of large scale fading coefficients in order to cancel the pilot contamination terms in (10). In [5], this approach was called Pilot Contamination Postcoding. Since
this approach allows mitigation of all sources of interference, as we show below, we believe that a more appropriate name for it is Large Scale Fading Decoding (LSFD). In Section VII, we compare LSFD with a Network MIMO scheme in which the pilot contamination is taken into account. Our simulation results show that LSFD has virtually the same performance, while its communication and computation complexities are significantly lower. The formal description of LSFD is given below.

Large-Scale Fading Decoding

1) The $l^{th}$ BS estimates $\beta_{lkn}, k = 1, K, n = 1, L$, and sends them to a controller.

2) For each $k = 1, K$, the controller computes the $L \times L$ decoding matrix $A_k = (a_{k1}a_{k2} \ldots a_{kL}), k = 1, K$, as functions of $\beta_{lkn}, l, n = 1, L$.

3) The $l^{th}$ BS computes the MMSE estimates $\hat{g}_{lkl}$ according to (5).

4) The $l^{th}$ BS receives the vector $y_l$ defined in (8) and computes signals $\tilde{s}_{kl}, k = 1, K$, using an $M$-dimensional decoding, e.g., matched filtering (10), zero-forcing (20), or MMSE. It further sends $\tilde{s}_{kl}$ to the network controller.

5) The controller forms the vector $\tilde{s}_k = [\tilde{s}_{k1}, \ldots, \tilde{s}_{kL}]^T$ and computes the estimates $\hat{s}_{kl} = a_{kl}^H \tilde{s}_k, k = 1, K, l = 1, L$.

The end.

The network architecture for this protocol is shown in Fig. 2.

We would like to emphasize the following points.

- The large scale fading coefficients $\beta_{jkl}$ are easy to estimate since they are constant over the $M$ antennas, OFDM subcarriers, and over $T_\beta$ OFDM symbols.

- Steps 1 and 2 are conducted only once for every large scale coherence interval, i.e., every $T_\beta$ OFDM symbols.

- The estimate $\hat{g}_{lkl}$ in Step 3 is computed once for each coherence interval, i.e., every $T$ OFDM symbols.

- Steps 4 and 5 are conducted for each OFDM symbol.
Taking into account the above points, we conclude that in LSFD, the backhaul traffic between base stations and the network controller grows marginally compared to the TDD protocol. Though LSFD requires some additional computations (Steps 2 and 5) at the network controller, these computations, especially in the case of the decentralized LSFD (see Section VI) when the size of matrices $A_k$ is small, are not overwhelming.

A. LSFD with Matched Filtering $M$-dimensional Receiver

First we assume that in Step 4 of LSFD, matched filtering (10) is used. Let $a_{klj}$ be the $j$th element of $a_{kl}$. It is useful to represent estimates $\hat{s}_{kl}$ as follows

$$
\hat{s}_{kl} = a_{kl}^H \tilde s_k = \sum_{j=1}^{L} a_{klj}^* \tilde s_{kj} = \sum_{j=1}^{L} a_{klj}^* \mathbb{E}[g_{jkj}^H g_{jkl}] \sqrt{q_{kl}} s_{kl} + \sum_{j=1}^{L} a_{klj}^* \sum_{n=1,n\neq l}^{L} \mathbb{E}[g_{jkj}^H g_{jkn}] \sqrt{q_{kn}} s_{kn}
$$

$$
+ \sum_{n=1}^{L} \sum_{j=1}^{L} a_{klj}^* \left( g_{jkj}^H g_{jkn} - \mathbb{E}[g_{jkj}^H g_{jkn}] \right) \sqrt{q_{kn}} s_{kn}
$$

$$
+ \sum_{n=1}^{L} \sum_{m=1,m\neq k}^{L} \sum_{j=1}^{L} a_{klj}^* g_{jmj}^H g_{jmn} \sqrt{q_{mn}} s_{mn} + \sum_{j=1}^{L} a_{klj}^* g_{jkj}^H \tilde z_j
$$

$$
\text{Useful Signal} \quad \text{Pilot Contamination} \quad \text{Interference plus Noise Terms}
$$
Taking into account that \( s_{kl} \) and \( s_{mn} \) are independent if \( (k, l) \neq (m, n) \), that \( h_{jkl} \) are independent from \( h_{nms} \) if \( (j, k, l) \neq (n, m, s) \), and that \( \hat{g}_{jkl} \) are uncorrelated with \( e_{jkl} \), it is easy to show that all terms in the above expression are uncorrelated. Thus, we can apply Theorem 1 from [16]. According to this theorem, the channel that minimizes \( I(\hat{s}_{kl}; s_{kl}|\hat{g}_{klt}) \) is the AWGN channel with noise variance equal to the sum of the variances of interferences and noise in the above expression. In other words we have

\[
I(\hat{s}_{kl}; s_{kl}|\hat{g}_{klt}) \geq \log(1 + \text{SINR}_{klt}),
\]

where

\[
\text{SINR}_{klt} = \frac{\mathbb{E}[[\text{Useful Signal}]^2]}{\text{Var}[\text{Pilot Cont. Interf.}] + \text{Var}[\text{Interf. plus Noise Terms}]}
\]

This leads to the following Theorem, which was proved in [6]. Slightly different notations are used in [6], hence, for the sake of self-completeness, we present the proof of this theorem in the Appendix. To shorten expressions, we use the notation

\[
\hat{a}_{klj} = \frac{\beta_{jkl} \sqrt{p_{kl}}}{1 + \sum_{i=1}^{L} \beta_{jki} p_{ki}} \cdot a_{klj}
\]

**Theorem 2.** If matched filtering decoding is used in Step 4 of LSFD, then the achievable SINR for the \( k \)th user in \( l \)th cell is given by

\[
\text{SINR}_{klt} = \frac{\left| \sum_{j=1}^{L} \hat{a}_{klj} \beta_{jkl} \right|^2 p_{kl}q_{kl}M}{\sum_{n \neq l}^{L} \left| \sum_{j=1}^{L} \hat{a}_{klj} \beta_{jkn} \right|^2 p_{kn}q_{kn}M + \sum_{j=1}^{L} \left| \hat{a}_{klj} \right|^2 (1 + \sum_{i=1}^{L} \beta_{jki} p_{ki}) \left( 1 + \sum_{n=1}^{K} \sum_{m=1}^{L} \beta_{jmn} q_{mn} \right)}
\]

**Proof:** See Appendix IX-A.

In [5], the following way of LSFD, called Zero-Forcing LSFD (ZF LSFD), was proposed:

\[
A_k = B_k^{-1}, \quad \text{and} \quad B_k = \begin{pmatrix}
\beta_{1k1} & \cdots & \beta_{1kL} \\
\vdots & & \vdots \\
\beta_{Lk1} & \cdots & \beta_{LkL}
\end{pmatrix}.
\]

It is not difficult to see that with this choice of \( A_k \), the numerator becomes equal to \( p_{ql}q_{kl}M \) and the first term in the denominator of (12) becomes equal to zero, while its two other terms do not depend on \( M \). Thus, we obtain that

\[
\lim_{M \to \infty} \text{SINR}_{klt} \overset{a.s.}{=} \infty,
\]
which is a drastic improvement over Theorem 1.

It happens, however, that in the case of \( M < 10^5 \), all terms in the denominator of (12) have comparable magnitudes with each other and for getting good performance, it is not enough to cancel only the first term which is caused by the pilot contamination interference. For this reason, ZF LSFD has very bad performance unless \( M \) is very large (see Fig.3). The natural question is whether LSFD can be designed so as to mitigate the most significant interference terms of (10) for a given \( M \). We answer positively to this question below. To keep notations short, we will use \( \hat{a}_{kl} = (\hat{a}_{kl1}, \ldots, \hat{a}_{klL})^T \).

**Theorem 3.** *If matched filtering decoding is used in Step 4 of LSFD, then the optimal LSFD coefficients \( \hat{a}_{kl} \) and the corresponding SINRs are*

\[
\hat{a}_{kl,\text{opt}} = \left( \sum_{n=1, n \neq l}^{L} \beta_{kn} \beta_{kn}^H P_{kn} q_{kn} M + \Lambda_k \right)^{-1} \beta_{kl},
\]

\[
\text{SINR}_{kl,\text{opt}} = \beta_{kl}^H \left( \sum_{n=1, n \neq l}^{L} \beta_{kn} \beta_{kn}^H P_{kn} q_{kn} M + \Lambda_k \right)^{-1} \beta_{kl} P_{kl} q_{kl} M,
\]

where \( \beta_{kn} = [\beta_{1kn} \ldots \beta_{Lkn}]^T \) and \( \Lambda_k = \text{diag}(\lambda_{1k}, \ldots, \lambda_{Lk}) \) with

\[
\lambda_{jk} = (1 + \sum_{i=1}^{L} \beta_{jki} P_{ki}) \left( 1 + \sum_{n=1}^{L} \sum_{m=1}^{K} \beta_{jmn} q_{mn} \right).
\]

**Proof:** Let \( C \) be a Hermitian matrix. According to the Rayleigh-Ritz theorem, see for example [11], the maximum of

\[
\frac{x^H uu^H x}{x^H C x}
\]

is achieved at

\[
x = C^{-1} u
\]

After some efforts, we transform (12) into the following form

\[
\text{SINR}_{kl} = \frac{\hat{a}_{kl}^H \beta_{kl} \beta_{kl}^H P_{kl} q_{kl} M \hat{a}_{kl}}{\hat{a}_{kl}^H \left( \sum_{n=1, n \neq l}^{L} \beta_{kn} \beta_{kn}^H P_{kn} q_{kn} M + \Lambda_k \right) \hat{a}_{kl}}
\]
It is easy to check that the matrix in the denominator is Hermitian. Hence we can apply (18). After simple computations, we obtain the assertions. ■

It is important to note that the vector \( \hat{\mathbf{a}}_{kl,\text{opt}} \) that maximizes the SINR of user \( k \) in cell \( l \) can be computed independently of the other vectors \( \mathbf{a}_{mn,\text{opt}} \).

B. LSFD with Zero-Forcing \( M \)-dimensional decoding

Now let us consider the scenario where zero forcing is used as an \( M \)-dimensional receiver in Step 4 of LSFD. The BS in cell \( l \) conducts linear zero forcing by taking the Moore-Penrose pseudoinverse of the estimated channel matrix as

\[
V_l = [v_{l1}, \ldots, v_{lK}] = \hat{G}_l^H (\hat{G}_l \hat{G}_l^H)^{-1}.
\]

and computing

\[
\tilde{s}_{kl} = \mathbf{v}_{lk}^H \mathbf{y}_l,
\]

where \( \mathbf{v}_{lk} \) denotes the \( k \)th column of \( V_l \). Therefore,

\[
\mathbf{v}_{lk}^H \hat{g}_{lml} = 0, \quad \forall \ m \neq k, \quad \text{and} \quad \mathbf{v}_{lk}^H \hat{g}_{lkl} = 1.
\]

After zero forcing for the \( k \)th user of \( l \)th cell, we get

\[
\tilde{s}_{kl} = \mathbf{v}_{lk}^H \mathbf{y}_l = \sum_{n=1}^{L} \mathbf{v}_{lk}^H \hat{g}_{lkn} \sqrt{q_{kn}s_{kn}} + \sum_{n=1}^{L} \mathbf{v}_{lk}^H \mathbf{e}_{lkn} \sqrt{q_{kn}s_{kn}} + \sum_{n=1}^{L} \sum_{m \neq k} \mathbf{v}_{lk}^H \hat{g}_{lmn} \sqrt{q_{mn}s_{mn}}
\]

\[
+ \sum_{n=1}^{L} \sum_{m \neq k} \mathbf{v}_{lk}^H \mathbf{e}_{lmn} \sqrt{q_{mn}s_{mn}} + \mathbf{v}_{lk}^H \mathbf{z}_l,
\]

\[
\overset{(a)}{=} \sqrt{q_{kl}s_{kl}} + \sum_{n=1, n \neq l}^{L} \beta_{ln} \sqrt{p_{ln}} \mathbf{v}_{lk}^H \hat{g}_{lkn} \sqrt{q_{kn}s_{kn}} + \sum_{n=1}^{L} \sum_{m=1}^{K} \mathbf{v}_{lk}^H \mathbf{e}_{lmn} \sqrt{q_{mn}s_{mn}} + \mathbf{v}_{lk}^H \mathbf{z}_l,
\]

where \( (a) \) follows from (20).
After LSFD, following the same steps as in Section IV, we get
\[
\hat{s}_{kl} = \frac{A_{kl}^H s_k}{|A_{kl}|^2} = \sum_{j=1}^{L} a_{klj}^* \hat{s}_{kj} = \sum_{j=1}^{L} a_{klj}^* \frac{\beta_{jkl} \sqrt{p_{kl}}}{\beta_{jkn} \sqrt{p_{kj}}} \sqrt{q_{kl}} \hat{s}_{kl} + \sum_{n=1, n \neq l}^{L} \sum_{j=1}^{L} a_{klj}^* \frac{\beta_{jkn} \sqrt{p_{kn}}}{\beta_{jkn} \sqrt{p_{kj}}} \sqrt{q_{kn}} \hat{s}_{kn} + \sum_{n=1, m=1}^{L} \sum_{j=1}^{L} a_{klj}^* \frac{\beta_{jln} \sqrt{p_{ln}}}{\beta_{jkn} \sqrt{p_{kn}}} \sqrt{q_{ln}} \hat{s}_{ln} + \sum_{j=1}^{L} a_{klj}^* \beta_{jln} \sqrt{p_{ln}} \sqrt{q_{ln}} \hat{s}_{ln}
\]

Useful Signal

\[
+ \sum_{n=1, m=1}^{L} \sum_{j=1}^{L} a_{klj}^* v_{jmn}^H \sqrt{q_{mn}} s_{mn} + \sum_{j=1}^{L} a_{klj}^* v_{jk}^H z_j
\]

Interference + Noise Terms

The variances of terms in the denominator can be found as follows. Since \( s_{kn} \) and \( s_{jl} \) are independent if \((k, n) \neq (j, l)\) we have
\[
\text{Var} \left[ |\text{Pilot Contamination}|^2 \right] = \sum_{n=1, n \neq l}^{L} \sum_{j=1}^{L} a_{klj}^* \frac{\beta_{jkn} \sqrt{p_{kn}}}{\beta_{jkn} \sqrt{p_{kj}}} \sqrt{q_{kn}} \right]^2 \mathbb{E}(|s_{kn}|^2)
\]
\[
= \sum_{n=1, n \neq l}^{L} \sum_{j=1}^{L} a_{klj}^* \frac{\beta_{jkn} \sqrt{p_{kn}}}{\beta_{jkn} \sqrt{p_{kj}}} \sqrt{q_{kn}}
\]

Since \( s_{kn} \) are independent from all \( z_j \), and \( z_i \) is independent from \( z_j \) if \( i \neq j \), we have
\[
\text{Var} \left[ |\text{Interference + Noise Terms}|^2 |\beta_{lkn}, l, n = 1, L \right]
\]
\[
= \sum_{n=1}^{L} \sum_{m=1}^{L} \mathbb{E} \left[ \left( \sum_{j=1}^{L} a_{klj}^* v_{jkn}^H e_{jmn} \right)^2 \right] q_{mn} \mathbb{E}(|s_{mn}|^2) + \mathbb{E} \left[ \left( \sum_{j=1}^{L} a_{klj}^* v_{jkn}^H z_j \right)^2 \right]
\]
\[
= \sum_{n=1}^{L} \sum_{m=1}^{L} \sum_{j=1}^{L} a_{klj}^* \frac{\beta_{jkn} \sqrt{p_{kn}}}{\beta_{jkn} \sqrt{p_{kj}}} \sqrt{q_{kn}} + \sum_{j=1}^{L} a_{klj}^* \frac{\beta_{jkn} \sqrt{p_{kn}}}{\beta_{jkn} \sqrt{p_{kj}}} \sqrt{q_{kn}}
\]

Using standard result from random matrix theory [17], we obtain
\[
\mathbb{E} v_{jkn}^H v_{jkn} = \frac{1 + \sum_{i=1}^{L} \beta_{jki} p_{ki} (M - K)}{\beta_{jkn}^2 p_{kn}}.
\]

All terms in (22) are uncorrelated. Thus, according to [16], the worst case channel is the AWGN channel with variance equal to the sum of the interferences and noise variances found above. Using \( \hat{a}_{klj} = \frac{a_{klj}}{\beta_{jkl} \sqrt{p_{kl}}} \), after some computations, we get the following result.
Theorem 4. If zero-forcing decoding is used in Step 4 of LSFD, then the achievable SINRs are

\[
\text{SINR}_{kl} = \frac{\left| \sum_{j=1}^{L} \hat{a}_{klj} \beta_{jkl} \right|^2 p_{kl} q_{kl}}{\sum_{n=1, n \neq l}^{L} \left| \sum_{j=1}^{L} \hat{a}_{klj} \beta_{jkn} \right|^2 p_{kn} q_{kn} + \sum_{j=1}^{L} \frac{|\hat{a}_{klj}|^2}{M-K} (1 + \sum_{i=1}^{L} \beta_{jki} p_{ki}) \left( 1 + \sum_{n=1}^{L} \sum_{m=1}^{K} \left[ \beta_{jmn} - \frac{\beta_{jmn}^2 p_{mn}}{1 + \sum_{i=1}^{L} \beta_{jmi} p_{mi}} \right] q_{mn} \right)}
\]

Let us define diagonal matrices \( \Lambda_k = \text{diag}(\lambda_{1k}, \lambda_{2k}, \ldots, \lambda_{Lk}) \) with

\[
\lambda_{jk} = \frac{M}{M-K} \left( 1 + \sum_{i=1}^{L} \beta_{jki} p_{ki} \right) \times \left( 1 + \sum_{n=1}^{L} \sum_{m=1}^{K} \left[ \beta_{jmn} - \frac{\beta_{jmn}^2 p_{mn}}{1 + \sum_{i=1}^{L} \beta_{jmi} p_{mi}} \right] q_{mn} \right)
\]

and let again \( \beta_{kn} = (\beta_{1kn} \beta_{2kn} \ldots \beta_{Lkn})^T \). With these notations, we have the following theorem.

Theorem 5. If zero-forcing decoding is used in Step 4 of LSFD, then the optimal \( \hat{a}_{kl,\text{opt}} \) and the corresponding achievable SINRs are defined by (14) and (15) with \( \lambda_{jk} \) defined in (23).

A proof of this result is similar to the proof of Theorem 3.

V. TRANSMIT POWER OPTIMIZATION

Using optimal LSFD coefficients obtained in Theorems 3 and 4 already give significant improvement compared to the case when LSFD is not used. However, even greater improvements can be obtained if we optimize the transmit powers \( p_{kl} \) and \( q_{kl} \). Denote

\[
p = (p_{kl} : k = 1, K, l = 1, L), \quad \text{and} \quad q = (q_{kl} : k = 1, K, l = 1, L).
\]

In this work, we will assume constant powers \( p = P_{\text{max}} \mathbf{1} \) during the training phase and focus on optimization of transmit powers \( q \) during the data transmission phase. As indicated in Section I, we shall optimize system performance with respect to the max-min criterion

\[
\max_{q} \min_{k,l} P_{\text{max}} M q_{kl} \cdot \beta_{kl}^H \left( \sum_{n=1, n \neq l}^{L} \beta_{kn} \beta_{kn}^H q_{kn} P_{\text{max}} M + \Lambda_k \right)^{-1} \beta_{kl}
\]

subject to \( 0 \leq q \leq Q_{\text{max}} \mathbf{1} \),

(24)
where \( \mathbf{1} \) is the \( KL \times 1 \) all ones vector, and \( \Lambda_k \) is defined in (16) with \( p_{ki} = P_{\text{max}}, \forall k, i \). Let

\[
\gamma = \min_{k,l} \beta_{kl}^H \left( \sum_{n=1,n\neq l}^L \beta_{kn} \beta_{kn}^H P_{\text{max}} q_{kn} M + \Lambda_k \right)^{-1} \beta_{kl} \times P_{\text{max}} q_{kl} M
\]

It is convenient to reformulate the optimization problem (24) in the following form

\[
\max \quad \gamma
\]

subject to \( 0 \leq q \leq Q_{\text{max}} \mathbf{1} \),

\[
P_{\text{max}} M q_{kl} \cdot \beta_{kl}^H \left( \sum_{n=1,n\neq l}^L \beta_{kn} \beta_{kn}^H P_{\text{max}} q_{kn} M + \Lambda_k \right)^{-1} \beta_{kl} \geq \gamma, \forall k, l.
\]

(25)

We solve the problem (25) in an iterative fashion. We start with an initial value of \( \gamma = \frac{\gamma_{\text{max}} + \gamma_{\text{min}}}{2} \) where \( \gamma_{\text{max}} \) and \( \gamma_{\text{min}} \) are chosen apriori and follow the bisection algorithm until the difference between \( \gamma_{\text{max}} \) and \( \gamma_{\text{min}} \) becomes small. The algorithm can be summarized as follows.

**Optimal Power Allocation**

**Input:** \( P_{\text{max}}, Q_{\text{max}}, \beta_{jkl}, j, l = 1, L; k = 1, K \).

**Output:** \( \gamma_{\text{opt}} = \max_{0 \leq q \leq Q_{\text{max}} \mathbf{1}} \min_{k,l} \text{SINR}_{kl} ; q_{kl}^{\text{opt}}, \forall k, l \).

1) **Step 1:** Set \( \gamma_{\text{max}} = \max_{k,l} ||\beta_{kl}||^2 P_{\text{max}} Q_{\text{max}} M \) and \( \gamma_{\text{min}} = 0 \).

2) **Step 2:** Assign \( \gamma = \frac{\gamma_{\text{max}} + \gamma_{\text{min}}}{2} \).

3) **Step 3:** Check feasibility of the following problem

\[
\min \sum_{k=1}^K \sum_{l=1}^L q_{kl}
\]

subject to \( 0 \leq q \leq Q_{\text{max}} \mathbf{1} \),

\[
p_{kl} q_{kl} M \cdot \beta_{kl}^H \left( \sum_{n=1,n\neq l}^L \beta_{kn} \beta_{kn}^H P_{\text{max}} q_{kn} M + \Lambda_k \right)^{-1} \beta_{kl} \geq \gamma, \quad k = 1, K; l = 1, L.
\]

(26)

4) **Step 4:** If \( \gamma \) is feasible, set \( \gamma_{\text{min}} = \gamma \) and go to Step 5, else set \( \gamma_{\text{max}} = \gamma \).

5) **Step 5:** If \( \gamma_{\text{max}} - \gamma_{\text{min}} < \epsilon \), where \( \epsilon \) is a small number, stop and assign \( \gamma_{\text{opt}} = \gamma_{\text{min}} \) and \( q_{kl}^{\text{opt}} = q_{kl} \), where \( q_{kl} \) are solutions of (26) with \( \gamma = \gamma_{\text{min}} \), otherwise go to Step 2.

**The end**
Using the techniques of [18], [19], we show in Section VI that if the problem (26) is feasible, then it has a unique solution and that there are iterative algorithms that converge to it.

The optimality of the proposed algorithm can be proved by contradiction. Let the solution obtained by the algorithm is $\gamma_1$, and the optimal solution is $\gamma_2 > \gamma_1$. Then there exists $\gamma_2 > \gamma_3 > \gamma_1$ such that $\gamma_3$ is infeasible (else our algorithm would have returned $\gamma_3$ as the optimal solution.). Since $\gamma_2 > \gamma_3$, we can reduce one user’s power to make its SINR equal to $\gamma_3$. This results in a reduction of interference to all the other users, making the SINR of the other users $\geq \gamma_2 > \gamma_3$. This means $\gamma_3$ is also feasible, which is a contradiction.

The key step of the algorithm is Step 3. In Section VI we propose a nice decentralized algorithm for implementing it. Now we can formulate the following communication protocol.

**An LSFD with Transmit Power Optimization**

1) All $L$ base stations estimate their large scale fading coefficients (the $j$-th base station estimates the coefficients $\beta_{jkl}, k = 1, K; l = 1, L$) and send them to a controller.

2) The controller runs the **Optimal Power Allocation** algorithm.

3) The controller sends the optimal transmit powers $q_{kl}^{opt}$ to the corresponding users (perhaps via the corresponding Base Stations).

4) The users transmit data with $q_{kl}^{opt}$.

5) The controller runs an LSFD to get estimates $\hat{s}_{kl}$.

The end.

Simulation results in Section VII show that power optimization gives large performance gain.

**Remark 2.** Note that the problem (24) can also be formulated as a power optimization problem over the user powers during the training phase while keeping the powers in the data transmission phase fixed. This problem can be solved using the same techniques described in Section V.
VI. DECENTRALIZED LSFD

The assumption that the network controller coordinates all base stations across the entire network is reasonable only for small networks, like a network for a campus, stadium, small town or similar facility. In a large network, we have to use decentralized algorithms and protocols that require coordination of only a small number of BSs. In this section, we propose a decentralized version of LSFD. We assume that the \( l \)th BS has access only to its \( L' \) neighboring cells. Let

\[
\Omega^{(l)} = \{ l \cup \{ \text{indices of } L' \text{ neighboring cells of cell } l \} \}, \text{ and } \hat{L} = L' + 1 = |\Omega^{(l)}|.
\]

To make the description of the following protocol short, it will be convenient to assume that the \( l \)th BS plays the role of the network controller for the network formed by the cells from \( \Omega^{(l)} \) (a number of other possibilities for organizing a network controller or controllers exist). We assume for each \( l \), the elements of \( \Omega^{(l)} \) are ordered in a certain order and the order is fixed.

Decentralized Uplink LSFD

1) The \( l \)th BS estimates \( \beta_{lkn}, n = 1, L \), and computes

\[
\eta_{kl} = \frac{\beta_{lkl}\sqrt{P_{kl}}}{1 + \sum_{i=1}^{L} \beta_{lki}p_{ki}}, k = 1, K.
\]  
(27)  
(See notes at the end of this section on an empirical way of computation of \( \eta_{kl} \).)

2) The \( l \)th BS computes \( \bar{s}_{kl}, k = 1, K \), using an \( M \)-dimensional decoding procedure. In particular, it may apply matched filtering [10] or zero-forcing [19] decoding.

3) The \( l \)th BS collects from neighboring BSs symbols \( \bar{s}_{kj}, j \in \Omega^{(l)} \), and forms the vectors

\[
\bar{s}_{k}^{(l)} = [\bar{s}_{kj} : j \in \Omega^{(l)}]^T, k = 1, K.
\]  
(28)

4) The \( l \)th BS collects coefficients \( \eta_{kj}, j \in \Omega^{(l)} \), and computes \( \hat{L} \)-dimensional vectors \( a_{kl,dec} = (a_{kl,dec} : j \in \Omega^{(l)}) \) (see explanations below). Here ‘\( \text{dec} \)’ stands for ‘decentralized’.

5) The \( l \)th BS computes the LSFD estimates as \( \hat{s}_{kl,dec} = a_{kl,dec}^H \bar{s}_{k}^{(l)}, k = 1, K \).

The end.
A. Decentralized LSFD with Matched Filtering $M$-dimensional Receiver

If we use matched filtering in Step 2 of LSFD, we get

$$
\hat{s}_{kl,dec} = a_{kl,dec}^H s_k = \sum_{j \in \Omega^{(l)}} a_{klj,dec}^* \tilde{s}_j
$$

$$
= \sum_{j \in \Omega^{(l)}} a_{klj,dec}^* E[g_{jkl}^H g_{jkl}] \sqrt{q_{kl}} \tilde{s}_j + \sum_{j \in \Omega^{(l)}} a_{klj,dec}^* E[g_{jkl}^H g_{jkn}] \sqrt{q_{kn}} \tilde{s}_kn
$$

$$
+ \sum_{n=1}^L \sum_{j \in \Omega^{(l)}} a_{klj,dec}^* (g_{jkl}^H g_{jkn} - E[g_{jkl}^H g_{jkn}]) \sqrt{q_{kn}} \tilde{s}_kn
$$

$$
+ \sum_{n=1}^L \sum_{j \in \Omega^{(l)}} \sum_{m \neq k} a_{klj,dec}^* g_{jmn} \sqrt{q_{mn}} \tilde{s}_{mn} + \sum_{j \in \Omega^{(l)}} a_{klj,dec}^* \tilde{z}_j
$$

(29)

Let us, similar to (11), define $\hat{a}_{klj,dec} = \eta_{kj} a_{klj,dec}$. Conducting derivations similar to the ones used in Theorem 2, we obtain the following expression

$$
\text{SINR}_{kl,dec} = \frac{M p_{kl} q_{kl} |\sum_{j \in \Omega^{(l)}} \hat{a}_{klj,dec}^* \beta_{jkl}|^2}{M \cdot \sum_{n=1,n \neq l}^L p_{kn} q_{kn} |\sum_{j \in \Omega^{(l)}} \hat{a}_{klj,dec}^* \beta_{jkl}|^2 + \sum_{j \in \Omega^{(l)}} |\hat{a}_{klj,dec}^* \beta_{jkl}|^2 (1 + \sum_{i=1}^L \beta_{jki} p_{ki}) \left(1 + \sum_{n=1}^L \sum_{m=1}^K \beta_{jmn} q_{mn}\right)}
$$

(30)

Further, by defining vectors $\beta^{(l)}_{kn} = (\beta_{jkn} : j \in \Omega^{(l)})^T$ and using arguments similar to the ones used in Theorem 3, we conclude that $\text{SINR}_{kl,dec}$ is maximized at

$$
\hat{a}_{kl,\text{opt,dec}} = \left(\sum_{n=1,n \neq l}^L \beta^{(l)}_{kn} \beta^{(l)}_{kn} p_{kn} q_{kn} M + \Lambda^{(l)}_k\right)^{-1} \beta^{(l)}_{kl},
$$

(31)

where $\Lambda^{(l)}_k = \text{diag}(\lambda_{jk} : j \in \Omega^{(l)})$ and

$$
\lambda_{jk} = (1 + \sum_{i=1}^L \beta_{jki} p_{ki}) \left(1 + \sum_{n=1}^L \sum_{m=1}^K \beta_{jmn} q_{mn}\right)
$$

(32)

Let $D_{kl} = \text{diag}(\eta_{kj} : j \in \Omega^{(l)})$. Coefficients $\eta_{kj}, j \in \Omega^{(l)}$, are passed to the $l$th BS in Step 4. So, if the $l$th BS possesses $\hat{a}_{kl,\text{opt,dec}}$, it could compute $a_{kl,\text{opt,dec}} = D_{kl}^{-1} \hat{a}_{kl,\text{opt,dec}}$ and use it in
Step 5 of the algorithm. The problem is, however, that the \( l \)th BS does not know the powers \( p_{kn} \) and \( q_{kn} \) for \( n \not\in \Omega^{(l)} \). Thus it can not compute optimal \( \hat{a}_{kl,dec,opt} \) according to (31). Using in (31), for instance, maximum powers \( p_{kn} = P_{\text{max}} \) and \( q_{kn} = Q_{\text{max}} \) or minimum powers \( p_{kn} = 0 \) and \( q_{kn} = 0 \) for \( n \not\in \Omega^{(l)} \) results in significant performance degradation.

To resolve this problem, instead of computing \( \hat{a}_{kl,\text{opt,dec}} \) according to (31), we propose that the \( l \)th BS empirically estimates the matrix \( \mathbb{E} \left[ \tilde{s}_k^{(l)} \tilde{s}_k^{(l)H} \right] \), and further computes
\[
a_{kl}^{\text{MMSE}} = \mathbb{E} \left[ \tilde{s}_k^{(l)} \tilde{s}_k^{(l)H} \right]^{-1} \mathbb{E} \left[ \tilde{s}_k^{(l)} s_{kl}^{*} \right]
\]

It is shown in the next theorem that vector \( \mathbb{E} \left[ \tilde{s}_k^{(l)} s_{kl}^{*} \right] \) can be computed by the \( l \)th BS directly. Estimation of matrix \( \mathbb{E} \left[ \tilde{s}_k^{(l)} \tilde{s}_k^{(l)H} \right] \) can be obtained, for instance, by collecting sufficiently many samples of \( \tilde{s}_k^{(l)} \) to get an estimate of the matrix, and by further updating the matrix using \( \tilde{s}_k^{(l)} \) obtained in Step 3. The following theorem shows that the vectors \( a_{kl}^{\text{MMSE}} \) are optimal.

**Theorem 6.** If matched filtering decoding is used in Step 4 of LSFD, then vectors
\[
a_{kl}^{\text{MMSE}} = \mathbb{E} \left[ \tilde{s}_k^{(l)} \tilde{s}_k^{(l)H} \right]^{-1} \mathbb{E} \left[ \tilde{s}_k^{(l)} s_{kl}^{*} \right] = D_k^{-1} \times \text{const} \times a_{kl,\text{opt,dec}},
\]
are optimal and lead to the optimal value
\[
\text{SINR}_{kl,\text{opt,dec}} = p_{kl} q_{kl} M \cdot \beta_{kl}^{(l)H} \left( \sum_{n=1,n \neq l}^{L} \beta_{kn}^{(l)H} \beta_{kn} p_{kn} q_{kn} M + \Lambda_k^{(l)} \right)^{-1} \beta_{kl}^{(l)}. \quad (33)
\]

**Proof:** To simplify notations, let us denote
\[
\mathbf{x} = (x_1, \ldots, x_L)^T = (\mathbf{g}_{jk}^{H} \mathbf{g}_{jkl} : j \in \Omega^{(l)})^T, \quad \mathbf{\hat{x}} = (\mathbf{\hat{x}}_1, \ldots, \mathbf{\hat{x}}_L)^T = (\mathbf{\hat{g}}_{jk}^{H} \mathbf{\hat{g}}_{jkl} : j \in \Omega^{(l)})^T \quad \text{and} \quad \mathbf{b} = (b_1, \ldots, b_L) = \left( \frac{\beta_{kl}\sqrt{p_{kl}}}{\beta_{jk}\sqrt{p_{jk}}} : j \in \Omega^{(l)} \right).
\]

Then we have
\[
\mathbb{E} \left[ \tilde{s}_k^{(l)} s_{kl}^{*} \right] = \mathbb{E} [\mathbf{x}] \sqrt{q_{kl}} = \begin{bmatrix} b_1 \mathbb{E} [\hat{x}_1] \\
\vdots \\
... \\
\sqrt{q_{kl}} = \left( \frac{\beta_{jkl}\sqrt{p_{kl}}\beta_{jk}\sqrt{p_{jk}}}{1 + \sum_{i=1}^{L} \beta_{jki} p_{ki}} : j \in \Omega^{(l)} \right)^T M \sqrt{q_{kl}} \\
b_L \mathbb{E} [\hat{x}_L] \end{bmatrix} = M D_k \beta_{kl}^{(l)} \sqrt{q_{kl} \sqrt{p_{kl}}}.
\]
\[
= M D_k \beta_{kl}^{(l)} \sqrt{q_{kl} \sqrt{p_{kl}}}. \quad (34)
\]
All components of this equation are available to the \( l \)th BS, so it can compute this vector directly.

It is not difficult to show that if \( (n, m, t) \neq (j, k, l) \), then

\[
E \left[ \left| g_{jkl}^H g_{jkl} \right|^2 \right] = \beta_{jkl}^2 (M^2 + M) \quad \text{and} \quad E \left[ \left| g_{jkl}^n g_{jkl}^m \right|^2 \right] = \beta_{jkl} \beta_{nml} M
\]  

(35)

Using (28) and (9), and further (35), after some efforts (we omit tedious computations), we get

\[
E \left[ s_k^{(l)} s_k^{(l)H} \right] = M^2 \sum_{n=1}^{L} D_k \beta_{kn}^l \beta_{kn}^l p_{kn} q_{kn} D_k + M D_k \Lambda_k^{(l)} D_k
\]

As a result, we have

\[
a_{kl}^{\text{MMSE}} = \left[ M^2 \sum_{n=1}^{L} D_k \beta_{kn}^l \beta_{kn}^l p_{kn} q_{kn} D_k + M D_k \Lambda_k^{(l)} D_k \right]^{-1} M D_k \beta_{kl}^{(l)} \sqrt{q_{kl}} \sqrt{p_{kl}}
\]

\[
= D_k^{-1} \left[ M \sum_{n=1}^{L} \beta_{kn}^l \beta_{kn}^l p_{kn} q_{kn} + \Lambda_k^{(l)} \right]^{-1} \beta_{kl}^{(l)} \sqrt{q_{kl}} \sqrt{p_{kl}}
\]

\[
\times D_k^{-1} \left[ M \sum_{n=1, n \neq l}^{L} \beta_{kn}^l \beta_{kn}^l p_{kn} q_{kn} + \Lambda_k^{(l)} \right]^{-1} \beta_{kl}^{(l)} = \text{const} \times D_k^{-1} \times \hat{a}_{kl, \text{opt, dec}},
\]

(36)

where \( (a) \) is due to the fact that

\[
(K + xx^H)^{-1} x = \left[ K^{-1} - \frac{K^{-1} xx^H K^{-1}}{1 + xx^H K^{-1} x} \right] x = \frac{1}{1 + xx^H K^{-1} x} K^{-1} x.
\]

From (30), it follows that vectors \( \text{const} \times \hat{a}_{kl} \) lead to the same \( \text{SINR}_{kl, \text{dec}} \) as vectors \( \hat{a}_{kl} \). Hence, vectors \( a_{kl}^{\text{MMSE}} \) are optimal and being used in Decentralized Uplink LSFD allow achieving \( \text{SINR}_{kl, \text{opt, dec}} \) defined in (33). \( \blacksquare \)

Now, let us return to the computation of coefficients \( \eta_{kj} \) defined in Step 1 of the algorithm. If all users use the same training powers, i.e., \( p_{kl} = p, \forall k, l \), and we assume that the \( j \)th BS knows all \( \beta_{jkl}, \forall k, l \), i.e., all the large scale fading coefficients between itself and all users across the entire network, then coefficients \( \eta_{kj} \) can be computed directly. If users use different training powers \( p_{kl} \), then \( \eta_{kj} \) can be computed empirically as follows. According to (5),

\[
\eta_{kj} = \frac{\mathbb{E}[g_{jkl}^H r_{kl}]}{\mathbb{E}[r_{kl}^H r_{kl}]}.
\]

January 19, 2017 DRAFT
The quantity $\mathbb{E}[\mathbf{g}_{kl}^{H} \mathbf{r}_{kl}] = M \beta_{jk} \sqrt{p_{kj}}$ can be computed directly, and $\mathbb{E}[\mathbf{r}_{kl}^{H} \mathbf{r}_{kl}]$ can be computed empirically over multiple realizations of $\mathbf{r}_{kl}$.

**B. Decentralized LSFD with Zero-Forcing $M$-dimensional decoding**

Let now $M$-dimensional zero-forcing decoding is used. Similar to the previous subsection, let

$\Lambda^{(l)}_{k} = \text{diag}(\lambda_{jk} : j \in \Omega(l))$ but with $\lambda_{jk}$ defined by (23). Using arguments similar to the ones used in Sections IV-B and VI-A, we obtain that the SINR value is defined in (37), where

$$\hat{a}_{klj,dec} = \frac{a_{klj,dec} \beta_{jkj} \sqrt{p_{kj}}}{\beta_{jkj} \sqrt{p_{kj}}}.$$

$$\text{SINR}_{kl,dec} = \frac{\sum_{j=1}^{L} |\hat{a}_{klj,dec} \beta_{jkl}|^2 p_{kl} q_{kl}}{\sum_{n=1, n \neq l}^{L} L_{j=1}^{L} |\hat{a}_{klj,dec} \beta_{jkn}|^2 p_{kn} q_{kn}}$$

Combining arguments of Sections IV-B and VI-A, we obtain that optimal $\hat{a}_{kl,opt,dec}$ and corresponding SINR$_{kl,opt,dec}$ are defined by (31) and (33) respectively with $\lambda_{jk}$ defined in (23). The $l$th BS can not directly compute $\hat{a}_{kl,opt,dec}$. Instead, it should empirically estimate the vector

$$\hat{a}_{kl}^{\text{MMSE}} = \mathbb{E} \left[\mathbf{s}_{k}^{(l)} \mathbf{s}_{k}^{(l)^{H}}\right]^{-1} \mathbb{E} \left[\mathbf{s}_{k}^{(l)} \mathbf{s}_{kl}^{*}\right],$$

where vectors $\bar{\mathbf{s}}^{(l)}$ are obtained in Step 3 with $M$-dimensional zero-forcing decoding.

Simulation results (see Section VII) show that Decentralized LSFD with $M$-dimensional zero-forcing decoding visibly outperforms the one with $M$-dimensional matched filtering decoding.

**C. Decentralized LSFD with Transmit Power Optimization**

The performance of Decentralized LSFD can be significantly enhanced by choosing optimal transmit powers. We formulate the following optimization problem

$$\max_{\mathbf{q}} \quad \gamma$$

subject to $\mathbf{0} \leq \mathbf{q} \leq Q_{\text{max}} \mathbf{1}$ and $\text{SINR}_{kl,dec} \geq \gamma, \ k = 1, K, \ l = 1, L.$
This optimization problem cannot be solved in a centralized manner, but we can solve the following optimization problem in a distributed manner for a given target SINR $\gamma$ \cite{13}, i.e.,

$$
\min_{q_{kl}} \sum_{k=1}^{K} \sum_{l=1}^{L} q_{kl}
$$

subject to $0 \leq q \leq Q_{\text{max}} \mathbf{1}$ and $\text{SINR}_{kl,\text{dec}} \geq \gamma$, $k = 1, K$, $l = 1, L$. \hspace{1cm} (39)

**Decentralized LSFD with Transmit Power Optimization**

1) Identical to Step 1 of **Decentralized LSFD**.

2) Assign $n = 1$ and repeat steps 3-9 until $|\text{SINR}_{kj} - \gamma| < \epsilon, k = 1, K; j = 1, L$.

3) The $l^{th}$ BS computes $\hat{s}_{kl}, k = 1, K$, with matched filtering (10) or zero-forcing (19).

4) The $l^{th}$ BS collects signals $\hat{s}_{kj}$ for $j \in \Omega^{(l)} \setminus l$, $k = 1, K$, forms the vectors $\hat{s}_k = [\hat{s}_{kl}, l \in \Omega^{(l)}]$, and estimates $\mathbb{E} \left[ \hat{s}_k \hat{s}_k^H \right]^{-1}$ (over multiple realizations of $\hat{s}_k$).

5) The $l^{th}$ BS computes $\hat{a}_{kl,\text{dec}} = a_{kl}^{\text{MMSE}} = \mathbb{E} \left[ \hat{s}_k \hat{s}_k^H \right]^{-1} \sqrt{q_{kl}} \sqrt{P_{kl}} M D_k \beta_{kl}^{(l)}$.

6) The $l^{th}$ BS computes the LSFD estimates $\hat{s}_{kl,\text{dec}} = a_{kl,\text{dec}}^H \hat{s}_k$, $k = 1, K$.

7) The $l^{th}$ BS estimates $\text{SINR}_{kl}, k = 1, K$, (over multiple realizations) and sends them to the corresponding users.

8) The $k^{th}$ user in the $l^{th}$ cell computes its new transmit power as

$$
q_{kl}^{(n)} = \begin{cases} 
q_{kl}^{(n-1)} \frac{\gamma}{\text{SINR}_{kl}^{(n-1)}}, & \text{if } q_{kl}^{(n-1)} \leq \frac{Q_{\text{max}}}{\gamma} \\
\frac{Q_{\text{max}}^2}{\gamma}, & \text{if } q_{kl}^{(n-1)} > \frac{Q_{\text{max}}}{\gamma}
\end{cases}
$$

9) Assign $n = n + 1$;

The end.

**Theorem 7.** The algorithm always converges and when (39) is feasible, it converges to the optimal powers $q_{kl}$.

In order to prove this theorem, we notice that (40) resembles the power update function of \cite{20}, Definition 1. To prove the convergence of the decentralized algorithm, we first show that
\[ I_{kl}(q) = \frac{q_{kl}}{\text{SINR}_{kl}} \quad \text{and} \quad \frac{1}{I_{kl}(q)} = \frac{\text{SINR}_{kl}}{q_{kl}} \] are two-sided scalable functions and then invoke the result from [20], Theorem 1, to complete the proof. The full proof can be found in Appendix IX-B.

VII. NUMERICAL RESULTS

We consider a network consisting of \( L = 19 \) cells of radius 1 km wrapped into a torus (see [14]). The wrapping allows us to mimic an infinite network of cells. We assume that \( K = 5 \) and \( M = 100 \). For decentralized LSFD, we set \( L' = 6 \). The maximum transmit power of each user during the pilot and data transmission phase is set to \( P_{\text{max}} = Q_{\text{max}} = 200 \) mW, and the large scale fading coefficients \( \beta_{jkl} \) are computed according to (2), with \( \sigma_{\text{shad}}^2 = 8 \) dB.

![CDF of user rates](image)

Figure 3 shows the CDF of the achievable rates for the various schemes considered in the paper. We mark the 5% outage rates by the dashed “black” horizontal line. In addition to the results presented in Theorems 3 and 5, we also derived SINR expressions for a Network MIMO
scheme where the BSs cooperate by sharing between themselves all the channel state information.

We considered two variants of the scheme, one where the BSs have access (magically) to the actual \( g_{lkj} \) (Network MIMO Perfect CSI) and the other where they only have \( \hat{g}_{lkj} \) defined in (5) (Network MIMO Imperfect CSI). The analysis of these results is omitted due to space limitations, but we use them in Fig. 3.

We observe that ZF-LSFD defined in (13) performs very poorly even in comparison with no LSFD case. ZF-LSFD starts visibly gain only at \( M > 10^6 \). At the same time we observe a 62.5 fold increase in the 5% outage rates when going from no LSFD (“dashed red”) to LSFD (“blue”) with matched filtering and transmit power optimization. When the BS uses LSFD (“black”) with
zero forcing and transmit power optimization, a 140 fold increase is observed, showing that the obtained gains are truly significant. It is also remarkable to see that LSFD with zero forcing performs close to full cooperation with imperfect CSI.

Figure 4 shows the fraction of users achieving a certain target SINR for varying target SINRs for global and decentralized LSFDs (with matched filtering) with and without power optimization. Again, by looking at the 5% outage rates, we observe a 16 dB provided by the transmit power optimization (“blue” curves). We observe only a minor 0.5 dB loss in going from global LSFD to decentralized LSFD, as for the case of power optimization as well as without it.

VIII. CONCLUSION

Large Scale Fading Decoding allows one to overcome the pilot contamination effect. LSFD assumes a two level structure. First, BSs locally (independent from each other) conduct $M$-dimensional linear decoding and obtain first level estimates of the transmitted uplink signals. Next, a network controller collects these estimates and conducts a second level linear decoding, which is based solely on the large scale fading coefficients between BSs and users.

In this paper, we considered LSFDs with two $M$-dimensional linear decodings: matched filtering and zero-forcing. We first derived SINR expressions as functions of an LSFD decoding matrix used by the network controller. We further derive optimal LSFD decoding matrices that maximize SINRs of all users simultaneously. Next, we proposed a decentralized version of LSFD in which only a small number of neighboring base stations participate in LSFD. The problem of finding optimal LSFD matrices is significantly more difficult in this case. One of the reasons for this is that transmit powers of users located outside of the neighboring cells is not known. We found a way around this problem and proposed a technique for empirical computation of optimal matrices for decentralized LSFD. Finally, we proposed a decentralized algorithm for uplink transmit power optimization, which provides additional system performance gain.

Simulation results show that LSFD with zero forcing $M$-dimensional decoding and power
optimization gives a 140-fold gain over MIMO systems without LSFD.

IX. APPENDIX

A. Proof of Theorem 2

We start by computing the power of useful signal:

\[
\mathbb{E}\left[|\text{Useful Signal}|^2\right] = \mathbb{E}\left[|s_{kl}|^2\right] q_{kl} \left| \sum_{j=1}^{L} a_{klj}^* \left( \mathbb{E}[\hat{g}_{jkl}\hat{g}_{jkl}] + \mathbb{E}[\hat{g}_{jkl}^H e_{jkl}] \right) \right|^2
\]

\[
= q_{kl} \left| \sum_{j=1}^{L} a_{klj}^* \frac{\beta_{jkl} \sqrt{p_{kl}}}{\sqrt{p_{kj}}} \mathbb{E}[\hat{g}_{jkl}^H \hat{g}_{jkl}] \right|^2 \tag{41}
\]

where (a) follows from the fact that \( \mathbb{E}[\hat{g}_{jkl}^H e_{jkl}] = 0 \) and (7), and (b) follows from (6). Next

\[
\text{Var}\left[\text{Pilot Contamination}\right] = \sum_{n=1,n\neq l}^{L} \left| \sum_{j=1}^{L} a_{klj}^* \left( \mathbb{E}[\hat{g}_{jkl}^H \hat{g}_{jkl}] + \mathbb{E}[\hat{g}_{jkl}^H e_{jkl}] \right) \right|^2 q_{kl} = \sum_{n=1,n\neq l}^{L} \left| \sum_{j=1}^{L} a_{klj}^* \frac{\beta_{jkl} \sqrt{p_{kl}}}{\sqrt{p_{kj}}} \mathbb{E}[\hat{g}_{jkl}^H \hat{g}_{jkl}] \right|^2 \frac{\beta_{jkl}^2 p_{kl}}{1 + \sum_{i=1}^{L} \beta_{jki} p_{ki}} M q_{kn} \tag{41}
\]

Finally,

\[
\text{Var}\left[\text{Interference plus Noise Terms}\right] = \sum_{n=1}^{L} \mathbb{E}\left[\left| \sum_{j=1}^{L} a_{klj}^* \left( \hat{g}_{jkl}^H \hat{g}_{jkl} - \mathbb{E}[\hat{g}_{jkl}^H \hat{g}_{jkl}] \right) \right|^2 \right] q_{kn} \mathbb{E}[|s_{kn}|^2] \\
+ \sum_{n=1}^{L} \sum_{m \neq k} \mathbb{E}\left[\left| \sum_{j=1}^{L} a_{klj}^* \hat{g}_{jkl}^H \hat{g}_{jmn} \right|^2 \right] q_{mn} \mathbb{E}[|s_{mn}|^2] \tag{42}
\]

\[
+ \sum_{n=1}^{L} \sum_{j=1}^{L} \frac{\beta_{jkl}^2 p_{kl}}{\beta_{jkl}^2 p_{kj}} \mathbb{E}\left[\left| \hat{g}_{jkl}^H \hat{g}_{jkl} - \mathbb{E}[\hat{g}_{jkl}^H \hat{g}_{jkl}] \right|^2 \right] q_{kn} \\
+ \sum_{n=1}^{L} \sum_{j=1}^{L} \left| \sum_{j=1}^{L} a_{klj}^* \hat{g}_{jkl}^H e_{jkl} \right|^2 q_{kn} \tag{42}
\]
To compute the first term in this expression, we note that, using (6), we have
\[ \mathbb{E} \left[ \left| \left( \hat{g}_{j,k}^H \hat{g}_{j,k} \right) \right|^2 \right] = \left( \frac{\beta_{jk,l}^2 P_{kj}}{1 + \sum_{l=1}^{L} \beta_{j,k,l} P_{ki}} \right)^2 (M + M^2), \quad \mathbb{E} \left[ \hat{g}_{j,k}^H g_{j,k} \right] = \frac{\beta_{jk,l}^2 P_{kj}}{1 + \sum_{l=1}^{L} \beta_{j,k,l} P_{ki}} M \]
giving \[ \text{Var} \left[ \left( \hat{g}_{j,k}^H \hat{g}_{j,k} \right) \right] = \left( \frac{\beta_{jk,l}^2 P_{kj}}{1 + \sum_{l=1}^{L} \beta_{j,k,l} P_{ki}} \right)^2 M. \] Hence the first term in (42) is equal to
\[ \sum_{n=1}^{L} \sum_{j=1}^{L} |a_{kl}|^2 \frac{\beta_{j,k,l}^2 P_{kn}}{1 + \sum_{l=1}^{L} \beta_{j,k,l} P_{ki}} M \left( \sum_{n=1}^{L} \sum_{m=1}^{K} \beta_{j,m,n} q_{mn} + 1 \right) \]
We compute other terms in (42) in a similar way, and, after some calculations, obtain
\[ \text{Var} \left[ \text{Inter. plus Noise Terms} \right] = \sum_{j=1}^{L} |a_{kl}|^2 \frac{\beta_{j,k,l}^2 P_{kn}}{1 + \sum_{l=1}^{L} \beta_{j,k,l} P_{ki}} M \left( \sum_{n=1}^{L} \sum_{m=1}^{K} \beta_{j,m,n} q_{mn} + 1 \right) \]
Combining these expressions and using (11), we obtain the claim. \( \square \)

B. Proof of Theorem 7

We provide here a proof of the two sided scalability of the functions \( I_{kl}(q) = \frac{q_{kl}}{\text{SINR}_{kl}} \) and \( \frac{1}{I_{kl}(q)} = \frac{\text{SINR}_{kl}}{q_{kl}}. \) A function \( f(x) \) is a two-sided scalable function \([20]\) if it satisfies the property:
\[ \text{for all } \alpha > 1 \text{ and vectors } x_1, x_2, \frac{1}{\alpha} x_1 \leq x_2 \leq \alpha x_1 \text{ implies } \frac{1}{\alpha} f(x_1) < f(x_2) < \alpha f(x_1). \]

In order to prove that \( I_{kl}(q) \) and \( \frac{1}{I_{kl}(q)} \) are two-sided scalable, we first need to show that \( I_{kl}(q) \) is a standard interference function \([18]\), which means that it satisfies the following three properties:
1) \( I_{kl}(q) \geq 0 \) \( \forall q \geq 0 \),
2) \( I_{kl}(q_1) \geq I_{kl}(q_2) \),
3) for any \( \alpha > 1 \), \( I_{kl}(\alpha q) < \alpha I_{kl}(q) \).

Clearly, \( I_{kl}(q) \geq 0 \) since both \( q_{kl} \) and \( \text{SINR}_{kl} \) are positive quantities. Using (17), we obtain
\[
I_{kl}(q) = \frac{q_{kl}}{\text{SINR}_{kl}} = \frac{1}{\beta_{kl}^H \left( \sum_{n=1,n\neq l}^{L} \beta_{kn}^\prime \beta_{kn}^H P_{kn} q_{kn} M + \Lambda_k^\prime(q) \right)^{-1} \beta_{kl}^\prime P_{kl} M} = \left[ \max_u u^H \beta_{kl}^\prime \beta_{kl}^H u P_{kl} M \right]^{-1} \left( \sum_{n=1,n\neq l}^{L} \beta_{kn}^\prime \beta_{kn}^H P_{kn} q_{kn} M + \Lambda_k^\prime(q) \right) u \] (43)

If \( q_1 \geq q_2 \), then from (32), it follows that \( \Lambda_k^\prime(q_1) - \Lambda_k^\prime(q_2) \) is positive definite. Denote by \( q_{1,kl} \) and \( q_{2,kl} \) the corresponding entries of \( q_1 \) and \( q_2 \). The matrices \( \beta_{kl}^\prime \beta_{kl}^H P_{kn}(q_{1,kl} - q_{2,kl}) \) are also positive definite. Hence, we have for any vector \( u \):
\[
\left( \sum_{n=1,n\neq l}^{L} \beta_{kn}^\prime \beta_{kn}^H P_{kn} q_{1,kl} M + \Lambda_k^\prime(q_1) \right) u \geq \left( \sum_{n=1,n\neq l}^{L} \beta_{kn}^\prime \beta_{kn}^H P_{kn} q_{2,kl} M + \Lambda_k^\prime(q_2) \right) u.
\]
Let
\[ \hat{u} = \arg \max_u \frac{u^H \beta'_kl \beta'_l^H u_{kl} M}{u^H \left( \sum_{n=1, n \neq l}^L \beta'_kn \beta'^H_{kn} p_{kn} q_{1, kn} M + \Lambda'_k(q_1) \right) u} \]

Now, from (43), it follows that
\[
\max_u \frac{u^H \beta'_kl \beta'_l^H u_{kl} M}{u^H \left( \sum_{n=1, n \neq l}^L \beta'_kn \beta'^H_{kn} p_{kn} q_{1, kn} M + \Lambda'_k(q_1) \right) u} \leq \max_u \frac{u^H \beta'_kl \beta'_l^H u_{kl} M}{u^H \left( \sum_{n=1, n \neq l}^L \beta'_kn \beta'^H_{kn} p_{kn} q_{2, kn} M + \Lambda'_k(q_2) \right) u}
\]
From this, it follows that \( I_{kl}(q_1) \geq I_{kl}(q_2). \) Finally, for any \( \alpha > 1, u \) and \( j \), we have
\[
\alpha \left( 1 + \sum_{i=1}^L \beta_j p_{ki} \right) \left( 1 + \sum_{n=1}^L \sum_{m=1}^K \beta_{jmn} q_{mn} \right) \left( 1 + \sum_{i=1}^L \beta_j p_{ki} \right) \left( 1 + \sum_{n=1}^L \sum_{m=1}^K \beta_{jmn} \alpha q_{mn} \right)
\]
This implies \( \alpha u^H \Lambda_k'(q) u > u^H \Lambda_k'(\alpha q) \) and further
\[
\alpha u^H \left( \sum_{n=1, n \neq l}^L \beta'_kn \beta'^H_{kn} p_{kn} q_{kn} M + \Lambda'_k(q_1) \right) u > u^H \left( \sum_{n=1, n \neq l}^L \beta'_kn \beta'^H_{kn} p_{kn} \alpha q_{kn} M + \Lambda'_k(\alpha q) \right) u
\]
Hence,
\[
\max_u \frac{u^H \beta'_kl \beta'_l^H u_{kl} M}{u^H \left( \sum_{n=1, n \neq l}^L \beta'_kn \beta'^H_{kn} p_{kn} q_{kn} M + \Lambda'_k(q) \right) u} \leq \max_u \frac{u^H \beta'_kl \beta'_l^H u_{kl} M}{u^H \left( \sum_{n=1, n \neq l}^L \beta'_kn \beta'^H_{kn} p_{kn} q_{2, kn} M + \Lambda'_k(q) \right) u}
\]
From this, it follows that \( \alpha I_{kl}(q) > I_{kl}(\alpha q). \) Thus, we proved that \( I_{kl}(q) \) is a standard interference function. Replacing \( q \) by \( \frac{1}{\alpha} q \) in Property 3 of the standard interference function, we have
\[
I_{kl}(q) < \alpha I_{kl} \left( \frac{1}{\alpha} q \right) \implies \frac{1}{\alpha} I_{kl}(q) < I_{kl} \left( \frac{1}{\alpha} q \right) \tag{44}
\]
Now, for any \( q_1 \) and \( q_2 \) and all \( \alpha > 1 \) such that \( \frac{1}{\alpha} q_1 \leq q_2 \leq \alpha q_1 \), we have
\[
\frac{1}{\alpha} I_{kl}(q_1) \overset{(a)}{<} I_{kl} \left( \frac{1}{\alpha} q_1 \right) \overset{(b)}{\leq} I_{kl}(q_2) \overset{(c)}{\leq} \alpha I_{kl}(q_1) \overset{(d)}{\implies} \frac{1}{\alpha} I_{kl}(q_1) \overset{(a)}{<} I_{kl}(q_2) < \alpha I_{kl}(q_1)
\]
where (a) follows from (44), (b) and (c) follow from Property 2 of the standard interference function, and (d) follows from Property 3. Thus, \( I_{kl}(q) \) is a standard interference function. In a
similar fashion, we have

\[
\frac{1}{\alpha I_{kl}(\mathbf{q}_1)} > (a) \frac{1}{I_{kl}(\alpha \mathbf{q}_1)} \geq (b) \frac{1}{I_{kl}(\mathbf{q}_2)} \geq (c) \frac{1}{I_{kl}(\mathbf{q}_1)} > (d) \frac{1}{\alpha I_{kl}(\mathbf{q}_1)} \implies \frac{1}{\alpha I_{kl}(\mathbf{q}_1)} < \frac{1}{I_{kl}(\mathbf{q}_1)} < \frac{1}{I_{kl}(\mathbf{q}_1)} < \frac{1}{\alpha I_{kl}(\mathbf{q}_1)}
\]

where (a) follows from Property 3 of the standard interference function, (b) and (c) follow from Property 2, and (d) follows from (44). Thus, \( \frac{1}{I_{kl}(\mathbf{q})} \) is also a standard interference function.

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