RICCI FLOW, COURANT ALGEBROIDS, AND
RENNORMALIZATION OF POISSON–LIE T-DUALITY

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ABSTRACT. We use a generalized Ricci tensor, defined for generalized metrics in Courant algebroids, to show that Poisson-Lie T-duality is compatible with the 1-loop renormalization group.

1. INTRODUCTION

If \((M, g)\) is a Riemannian manifold, the 1-loop renormalization group flow of the standard 2-dimensional \(\sigma\)-model with the action functional

\[ S(f) = \int_{\Sigma} g(\partial f, \bar{\partial} f) \quad (f : \Sigma \to M) \]

is (up to an inessential coefficient), as found by Friedan [4], the Ricci flow

\[ \frac{dg}{dt} = -2 \text{Ric}_g + L_X g, \]

where \(X\) is an arbitrary vector field on \(M\).

In a more general situation \(\text{Ric}_g\) gets replaced by \(\text{Ric}_{(g, H)}\) where \(H\) is a closed 3-form and \(\text{Ric}_{(g, H)}\) is the Ricci tensor of the \(g\)-preserving connection on \(M\) with the torsion \(T\) given by \(g(T(X, Y), Z) = -H(X, Y, Z)\). Namely, if \(b \in \Omega^2(M)\) is a 2-form and \(H_0 \in \Omega^3(M)\) a closed 3-form, the corresponding \(\sigma\)-model has the action functional

\[ S(f) = \int_{\Sigma} g(\partial f, \bar{\partial} f) + \int_{\Sigma} f^* b + \int_Y f^* H_0 = \int_{\Sigma} e(\partial f, \bar{\partial} f) + \int_Y f^* H_0 \]

where \(e = g + b \in \Gamma((T \oplus T^*) \otimes^2 M)\) and \(Y\) is an oriented 3-manifold bounded by \(\Sigma\). We set \(H = db + H_0\) and the 1-loop renormalization group flow is now [3]

\[ \frac{de}{dt} = -2 \text{Ric}_{(g, H)} + L_X e + i_X H_0 - da \]

where \(X\) and \(a\) are an arbitrary vector field and a 1-form respectively.

It is natural to interpret the generalized Ricci flow (1) in terms of Courant algebroids. The pair \((X, a)\) can be seen as a section of an exact Courant algebroid \((T \oplus T^*) M\) and its contribution to the flow as the Courant bracket \([[(X, a), \cdot], \cdot]\). The tensor field \(e\) can be replaced by its graph which is a subbundle of \((T \oplus T^*) M\) and (1) can be seen as an evolution of this subbundle. This point of view was suggested by Streets in [13].

The main problem in this approach is to find a version of the flow (1) for arbitrary Courant algebroids, not just for \((T \oplus T^*) M\) (i.e. exact ones). The motivation for this problem comes mainly from T-duality. T-duality [1], and its extension Poisson-Lie T-duality [8], is an equivalence of \(\sigma\)-models with different target spaces \(M\). It has a natural formulation in terms of Courant algebroids (see [2] for the ordinary and [12] for the Poisson-Lie case). A suitable definition of a generalized Ricci flow for an arbitrary Courant algebroid would immediately imply the compatibility of

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T-duality with 1-loop renormalization group flow. In this paper we give such a
definition and prove that Poisson-Lie T-duality is indeed compatible with the 1-
loop renormalization group flow. This result was known only in the case of no
spectators [16] and the new proof is much simpler.

We should mention that this is not the first attempt to define the Ricc i tensor of
a generalized metric. A previous, and more conceptual way, was to use Gualtieri’s
definition of a generalized connection and its generalized curvature [6], choose the
connection so that it preserves the generalized metric and a suitable part of its gen-
eralized torsion vanishes, and take a suitable trace of the curvature. This approach
was used by Garcia-Fernandez [5] to get a new interpretation of the equations of
motion of heterotic supergravity. Our approach is more utilitarian.

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Ricci tensor [5].

2. Courant algebroids

In this section we shall summarize basic definitions and results concern ing Courant
algebroids. Courant algebroids were introduced by Liu, Weinstein and Xu in [11].

Definition 1. A Courant algebroid (CA) is a vector bundle $E \to M$ equipped with
a non-degenerate symmetric bilinear form $(\cdot, \cdot)$, with a vector bundle map
$\rho : E \to T M$ (the anchor map) and with a $\mathbb{R}$-bilinear map (Courant bracket)
$\{,\} : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$
satisfying

- $\{[s, t, u]\} = [[s, t], u] + [t, [s, u]]$ for any $s, t, u \in \Gamma(E)$
- $\rho([s, t]) = [\rho(s), \rho(t)]$ for any $s, t \in \Gamma(E)$
- $[s, ft] = f[s, t] + (\rho(s)) f t$ for any $s, t \in \Gamma(E)$, $f \in C^\infty(M)$
- $\rho(s)(t, u) = \langle [s, t], u \rangle + \langle t, [s, u] \rangle$
- $\{s, t\} + \{t, s\} = \rho^t (d\langle s, t \rangle)$, where

$$\rho^t : T^* M \to E^* \xrightarrow{(\cdot, \cdot)} E$$

is the transpose of $\rho$.

One can reformulate the first four properties as follows: if $s \in \Gamma(E)$ then the
map $\Gamma(E) \to \Gamma(E)$, $x \mapsto [s, x]$, is a derivation of the Courant algebroid $E$, i.e. it is
given by a vector field $Z_s$ on $E$ whose flow is an automorphism of the CA $E$, such
that $\rho_* Z_s = \rho(s)$. We shall call the map $x \mapsto [s, x]$ an inner derivation of $E$.

Example 1. If $M$ is a point then $E$ is a Lie algebra with invariant non-degenerate
quadratic form $\langle \cdot, \cdot \rangle$.

Example 2. A Courant algebroid $E \to M$ is called exact if

(2) $0 \to T^* M \xrightarrow{\rho^*} E \xrightarrow{(\cdot, \cdot)} TM \to 0$

is an exact sequence (it is a chain complex for any Courant algebroid). Exact CAs
over $M$ are classified by $H^3(M, \mathbb{R})$. Namely, if we choose a splitting of (2) by a
$\langle \cdot, \cdot \rangle$-isotropic subbundle of $E$, we get $E \cong (T \oplus T^*)M$ with

\begin{align}
(3a) \quad \langle [X, \alpha], (Y, \beta) \rangle &= \alpha(Y) + \beta(X), \\
(3b) \quad \rho(X, \alpha) &= X, \\
(3c) \quad [\langle X, \alpha \rangle, (Y, \beta)] &= \langle [X, Y], L_X \beta - i_Y d\alpha + H(X, Y, \cdot) \rangle
\end{align}

for some closed 3-form $H \in \Omega^3(M)$, and vice-versa, any closed $H$ makes $(T \oplus T^*)M$ in this way to an exact CA. A different choice of the splitting replaces $H$ by $H + dB$ for an appropriate $B \in \Omega^2(M)$.

It is convenient to write the Courant bracket $[,]$ on $E$ in terms of a connection on $E$:

**Proposition 1.** Let $E \to M$ be a CA and let $\nabla$ be a connection on the vector bundle $E$ preserving the pairing $\langle \cdot, \cdot \rangle$. Then there is a section $c_\nabla \in \Gamma(\wedge^3 E)$ such that for every $u, v \in \Gamma(E)$ we have

\begin{equation}
[u, v] = c_\nabla\langle u, v, \cdot \rangle + \nabla_{\rho(u)} v - \nabla_{\rho(v)} u + \rho^i \langle \nabla u, v \rangle
\end{equation}

where we identify $E^\ast$ with $E$ via the pairing $\langle \cdot, \cdot \rangle$.

**Proof.** Let us set

$$[u, v]_\nabla := [u, v] - \left( \nabla_{\rho(u)} v - \nabla_{\rho(v)} u + \rho^j \langle \nabla u, v \rangle \right).$$

One easily sees that $[u, v]_\nabla$ is $C^\infty(M)$-bilinear in $u$ and $v$, i.e. it is a vector bundle map $E \otimes E \to E$.

It remains to check that

$$c_\nabla\langle u_1, u_2, u_3 \rangle = \langle [u_1, u_2]_\nabla, u_3 \rangle$$

is antisymmetric in its 3 arguments $u_1, u_2, u_3 \in \Gamma(E)$. To do it, we choose $u_i$’s such that $\nabla u_i$’s vanish at a point $P \in M$. In that case $\langle [u_1, u_2]_\nabla, u_3 \rangle(P) = \langle [u_1, u_2, u_3] \rangle(P)$ and $\langle [u_1, u_2, u_3] \rangle(P)$ is antisymmetric as $d(u_1, u_2)$’s vanish at $P$. \hfill $\square$

**Remark 1.** In [6], the section $c_\nabla$ is called the torsion of $\nabla$ and is defined for generalized connections on $E$.

### 3. Generalized metric

A **generalized metric** on a Courant algebroid $E \to M$, as defined in [7], is a vector subbundle $V_+ \subset E$ such that $\langle \cdot, \cdot \rangle$ is positive definite on $V_+$ and negative definite on $V_- := V_+^\perp$.

If $E \to M$ is an exact CA with a chosen splitting, i.e. if $E = (T \oplus T^*)M$ with the structure given by [8], then a generalized metric $V_+ \subset E$ is the graph of a bilinear form $e = g + b$ on $TM$ such that the symmetric part $g$ of $e$ is a Riemannian metric on $M$. The skew-symmetric part $b$ of $e$ depends on the splitting $(g$ does not), and for a given $V_+$ there is a unique splitting of $E$ such that $b = 0$ (see [7]); the closed 3-form corresponding to this splitting will be denoted $H$. An exact CA with a generalized metric is thus equivalent to a pair $(g, H)$.

Let now $\nabla^\pm$ be connections on the vector bundles $V_\pm$ preserving $\langle \cdot, \cdot \rangle$, and let $\nabla = \nabla^+ \oplus \nabla^-$ be the resulting connection on $E = V_+ \oplus V_-$. We shall call such a connection $\nabla$ **compatible** with the generalized metric $V_+ \subset E$. In the case of an exact CA there is a canonical connection $\nabla^-$ for which the “$-\$”-part of $c_\nabla \in \Gamma(\wedge^3 E)$ vanishes. This result can be found in [6], but we include a proof for completeness.
Proposition 2. If $E \to M$ is an exact CA and $V_+ \subset E$ a generalized metric, there is a unique $(\cdot,\cdot)$-preserving connection $\nabla_{\text{can}}$ on $V_-$ such that, for any $(\cdot,\cdot)$-preserving $\nabla^+$ on $V_+$, we have

\begin{equation}
\tag{5}
c_{\nabla^+ \oplus \nabla_{\text{can}}} (x_+, y_-, z_-) = 0
\end{equation}

for any $x_+ \in \Gamma(V_+)$ and $y_-, z_- \in \Gamma(V_-)$. When we identify $V_-$ with $TM$ via the anchor $\rho$ then $\nabla_{\text{can}}$ is the the $g$-preserving connection with the torsion $T$ given by $g(T(X,Y), Z) = -H(X,Y,Z)$.

Proof. Let $\nabla = \nabla^+ \oplus \nabla^-$ be a compatible connection on $E$. The relation (4) gives

\begin{equation}
\tag{6}
\langle [x_+, y_-], z_- \rangle = c_\nabla(x_+, y_-, z_-) + \langle \nabla_{\rho(x_+)} y_-, z_- \rangle.
\end{equation}

As a result, if we change $\nabla$ by a 1-form

\[ a = a_+ + a_- \quad a_\pm \in \Omega^1(M, \wedge^2 V_\pm), \]

we get

\begin{equation}
\tag{7}
(c_{\nabla^+ a} - c_{\nabla})(x_+, y_-, z_-) = -a_-(\rho(x_+))(y_-, z_-).
\end{equation}

For a given $\nabla = \nabla^+ \oplus \nabla^-$ we then define $a_- \in \Omega^1(M, \wedge^2 V_-)$ via

\[ a_-(\rho(x_+))(y_-, z_-) = c_{\nabla}(x_+, y_-, z_-) \]

(we use the fact that $\rho : V_+ \to TM$ is bijective) and see that the connection $\nabla_{\text{can}} = \nabla^- + a_-$ on $V_-$ is the unique solution of our problem.

Let us now compute the torsion of $\nabla_{\text{can}}$ using (5), i.e. using (cf. (6))

\begin{equation}
\tag{8}
\langle [x_+, y_-], z_- \rangle = \langle (\nabla_{\text{can}})_{\rho(x_+)} y_-, z_- \rangle.
\end{equation}

We split $E$ to $(T \oplus T^*) M$ via the splitting for which $V_+$ is the graph of the Riemann metric $g$, and thus $V_-$ is the graph of $-g$. Notice that for $v_-, w_- \in \Gamma(V_-)$ we have

\begin{equation}
\tag{9}
\langle v_-, w_- \rangle = -2g(\rho(v), \rho(w)).
\end{equation}

Let $X$ and $Y$ be vector fields on $M$ such that $[X, Y] = 0$, and let

\[ x_\pm = (X, \pm g(X, \cdot)), \quad y_\pm = (Y, \pm g(Y, \cdot)) \in \Gamma(V_\pm) \]

be their lifts to $V_\pm$. Let $z_- = (Z, -g(Z, \cdot))$ be a section of $V_-$. Then (3c) gives us

\[ [x_+, y_-] - [y_+, x_-] = (0, 2H(X,Y, \cdot)) \]

and (3) and (9) give us

\[ -2g(T(X,Y), Z) = \langle [x_+, y_-] - [y_+, x_-], z_- \rangle = 2H(X,Y,Z) \]

as we wanted to show. \qed

Let us now introduce the following graphical notation. We will systematically identify $E^*$ with $E$ via the pairing $(\cdot, \cdot)$. The section $c_\nabla \in \Gamma(\wedge^3 E)$ will be represented by a trivalent vertex

\[ c_\nabla = \begin{tikzpicture}[baseline=-0.5ex]
\node (v1) at (0,0) {};
\node (v2) at (0,-0.5) {};
\node (v3) at (0,-1) {};
\draw (v1) -- (v2);
\draw (v2) -- (v3);
\end{tikzpicture} \]

with the counter-clockwise orientation, i.e. for any $u, v, w \in \Gamma(E)$

\[ c_\nabla(u,v,w) = u \begin{tikzpicture}[baseline=-0.5ex]
\node (v1) at (0,0) {};
\node (v2) at (0,-0.5) {};
\node (v3) at (0,-1) {};
\draw (v1) -- (v2);
\draw (v2) -- (v3);
\end{tikzpicture} \]

where $u$ is counterclockwise oriented.
If $V_+ \subset E$ is a generalized metric, let the orthogonal projections $E \to V_\pm$ be denoted by $\pi^+$ and $\pi^-$ respectively. In particular, if $E$ is exact and if $\nabla^- = \nabla^\text{can}$ then

$$\pi^+ \circ \pi^- = 0.$$  

Finally, let us also use the notation

$$\rho = \ldots \quad \text{and} \quad \nabla c = \ldots \text{.}$$

where dotted lines signify vector fields on $M$; more generally, $\ldots$ will stand for $\nabla$ (always applied to some section of $E^\otimes n$).

4. Generalized Ricci tensor

Let $V_+ \subset E$ be a generalized metric. An infinitesimal deformation of $V_+$, i.e. a tangent vector to $V_+$ in the space of generalized metrics in $E$, is given by a linear map $S : V_+ \to E/V_+ \cong V_-$, or equivalently by the corresponding bilinear form $C : V_+ \otimes V_- \to \mathbb{R}$, $C(u_+, v_-) = \langle Su_+, v_- \rangle$.

Deformations by inner derivations $[s, \cdot]$ of $E$ have the bilinear form $C(u_+, v_-) = \langle [s, u_+], v_- \rangle$.

These deformations are trivial in the sense that they don’t change the isomorphism class of the pair $V_+ \subset E$.

We can now define the main object of this paper.

**Definition 2.** The generalized Ricci tensor of a generalized metric $V_+$ in a Courant algebroid $E$ with a compatible connection $\nabla = \nabla^+ \oplus \nabla^-$ is the bilinear form $\text{GRic}^{(V)}_{V_+} : V_+ \otimes V_- \to \mathbb{R}$ given by, for $u_+ \in \Gamma(V_+)$ and $v_- \in \Gamma(V_-)$,

$$\text{GRic}^{(V)}_{V_+}(u_+, v_-) := \text{Tr}_{V_-} \left( x_- \mapsto R_{\nabla^-} \left( \rho(x_-), \rho(u_+) \right) v_- \right)$$

where $R_{\nabla^-}$ is the curvature of $\nabla^-$.  

**Remark 2.** In this definition one can replace $\nabla$ with a generalized connection in the sense of [6]. Moreover, the result is probably equal, up to inner derivations, to the generalized Ricci tensor of a torsion-free generalized connection [5]. For our purposes ordinary connections are sufficient.

The infinitesimal deformation of $V_+$ given by $\text{GRic}^{(V)}_{V_+}$ is independent of $\nabla$ modulo inner derivations (and is actually fully independent of $\nabla^+$):

**Theorem 1.** Let $V_+ \subset E$ be a generalized metric in a Courant algebroid $E$ and let $\nabla = \nabla^+ \oplus \nabla^-$ be a compatible connection. If $\nabla^+ + a, a = a_+ + a_-, a_\pm \in \Omega^1(M, \Lambda^\pm V_\pm)$, is another compatible connection then

$$\text{GRic}^{(V_+ + a)}(u_+, v_-) - \text{GRic}^{(V)}_{V_+}(u_+, v_-) = \langle [s_-, u_+], v_- \rangle$$
where $s_- \in \Gamma(V_-)$ is

\begin{equation}
(11) \quad s_- = a_-
\end{equation}

Here we use the graphical notation

$$a_\pm(X)(x,y) = X \cdots a_\pm$$

for a vector field $X$ and sections $x,y \in \Gamma(E)$.

The proof of Theorem 1 is a straightforward calculation and can be found in Appendix A.

Let us recall that a generalized metric in an exact CA is equivalent to a Riemannian metric $g$ and a closed 3-form $H$. Let $\text{Ric}_{(g,H)}$ be the Ricci tensor of the $g$-preserving connection with the torsion given by $g(T(X,Y),Z) = -H(X,Y,Z)$ (cf. Proposition 2).

**Theorem 2.** If $V_+ \subset E$ is a generalized metric in an exact CA $E$ and $\nabla = \nabla^+ \oplus \nabla^-$ a compatible connection then

\begin{equation}
(12) \quad \text{GRic}_{V_+}^{(\nabla)}(u_+,v_-) = \text{Ric}_{(g,H)}(\rho(u_+),\rho(v_-)) + \langle [s_-,u_+],v_- \rangle
\end{equation}

where $s_- \in \Gamma(V_-)$ is given by (11) with $a_- = \nabla^- - \nabla^-_{\text{can}}$.

**Proof.** If $\nabla^- = \nabla^-_{\text{can}}$ then

$$s_- = 0$$

and so the second and third term of (11) vanish and we have

$$\text{GRic}_{V_+}^{(\nabla)}(u_+,v_-) = \text{Tr}_{V_+^-} \left( x_- \mapsto R_{\nabla^-_{\text{can}}}(\rho(x_-),\rho(u_+))v_- \right) = \text{Ric}_{(g,H)}(\rho(u_+),\rho(v_-)).$$

For any compatible $\nabla$ we therefore have

$$\text{GRic}_{V_+}^{(\nabla-a_-)}(u_+,v_-) = \text{Ric}_{(g,H)}(\rho(u_+),\rho(v_-))$$

where $a_- = \nabla^- - \nabla^-_{\text{can}}$. Equation (12) now follows from Theorem 1. \qed

Theorem 2 has the following meaning. If $E = (T \oplus T^*)M$ is exact given by a closed 3-form $H_0$ and $V_+ \subset E$ is the graph of $e = g + b$, then an infinitesimal deformation of $V_+$ can be given either by a bilinear form $C : V_+ \otimes V_- \to \mathbb{R}$, or by a deformation $\frac{de}{dt}$ of $e$; they are linked via

$$\frac{de}{dt}(\rho(u_+),\rho(v_-)) = C(u_+,v_-).$$

The deformation of $V_+$ via $-2\text{GRic}_{V_+}^{(\nabla)}$ is thus precisely the generalized Ricci flow (11) with $(X,\alpha) = -2s_-$ (notice that $\alpha = -e(\cdot,X)$ as $s_- \in \Gamma(V_-)$).

**Remark 3.** Theorems 1 and 2 explain why $\text{GRic}_{V_+}^{(\nabla)}$ is useful, but its definition is somewhat ad hoc. A natural guess is that $\text{GRic}_{V_+}^{(\nabla)}$ comes from one-loop renormalization of the Courant $\sigma$-model given by $E$ with the boundary condition given by $V_+$, introduced in [13].
Remark 4. We defined GRic$_{V_+}^{(\nabla)}$ as an infinitesimal deformation of $V_+$. An obvious question is whether it defines, say in the case of a compact $M$, a well-defined flow for some finite time. In other words, whether one can find $t_0 > 0$, a family $V_+(t)$ parametrized by $t \in [0, t_0)$ such that $V_+(0) = V_+$, and a family of connections $\nabla(t)$ compatible with $V_+(t)$, and possibly a family of sections $z_-(t) \in \Gamma(V_-(t))$, such that
\[
\frac{dV_+(t)}{dt} = -2 \text{GRic}_{V_+}^{(\nabla(t))} + \langle [z_-(t), \cdot], \cdot \rangle
\]
(where, as before, we identify tangent vectors to $V_+$ in the space of generalized metrics in $E$ with bilinear forms $V_+ \otimes V_- \to \mathbb{R}$). The section $z_-(t) \in \Gamma(V_-(t))$ is added because of the possibly arbitrary family of connections $\nabla(t)$ (cf. Theorem 1). We leave this question open.

Remark 5. Another question is, supposing the flow exists, how to describe its outcome in a way that would be independent of $\nabla(t)$ and $z_-(t)$, i.e. how to deal with the fact that GRic$_{V_+}^{(\nabla)}$ depends on the auxiliary connection $\nabla$, but only by inner derivations by sections of $V_-$. Here the answer is simple. The outcome should be a submersion $p : \tilde{M} \to [0, t_0)$, a Courant algebroid $\tilde{E} \to \tilde{M}$ such that $dp \circ \rho$ is surjective at all points of $\tilde{M}$, and a vector subbundle $\tilde{V}_+ \subset \tilde{E}$ such that $\langle \cdot, \cdot \rangle|_{\tilde{V}_+}$ is positive definite and $dp \circ \rho|_{\tilde{V}_+} = 0$.

If we set $M = p^{-1}(0)$, we can find a CA $E \to M$, a (at least local, and global if $p$ is proper) diffeomorphism $\tilde{M} \cong M \times [0, t_0)$ compatible with $p$, and an isomorphism of CAs $\tilde{E} \cong E \times (T \otimes T^*)[0, t_0)$ such that $\tilde{V}_+$ becomes a family $V_+(t)$ of subbundles of $\tilde{E}$ (these isomorphisms can be obtained by choosing a section $x \in \Gamma(\tilde{V}_+)$ such that $p_*(\rho(x)) = \partial/\partial t$ and $(x, x) = 0$ and using the flow in $E$ generated by the inner derivation $[x, \cdot]$). The resulting family $V_+(t)$ depends on the choices, and it changes by the flow of a time-dependent inner derivation generated by a section of $V_-(t)$ if we make a different choice.

5. Poisson-Lie T-duality is compatible with the 1-loop renormalization group flow

Theorem 3. Let $E \to M$ be a CA, $\phi : M' \to M$ a smooth map, and let $\phi^*E$ be endowed with a CA structure satisfying the condition
\[
[\phi^*u, \phi^*v] = \phi^*[u, v] \quad \text{and} \quad \phi_*(\rho(\phi^*u)) = \rho(u) \quad \forall u, v \in \Gamma(E).
\]

Let $V_+ \subset E$ be a generalized metric and $\nabla = \nabla^+ \oplus \nabla^-$ a compatible connection on $E$. Then the generalized Ricci tensors of $V_+ \subset E$ and of $\phi^*V_+ \subset \phi^*E$ satisfy
\[
\text{GRic}_{\phi^*V_+}^{(\phi^*\nabla)} = \phi^* \text{GRic}_{V_+}^{(\nabla)}.
\]

Proof. If $u, v, w \in \Gamma(E)$ then $\langle [u, v], w \rangle = \langle [\phi^*u, \phi^*v], \phi^*w \rangle$. When we express both sides of this equality using Proposition 1, we get $\phi_0^*\nabla = \phi^*c\nabla$. From this and from the definition of GRic the statement follows readily. \hfill $\square$

Remark 6. If $E \to M$ is a CA and $\phi : M' \to M$ a smooth map, the CA structures on $\phi^*E$ satisfying the condition (13) were characterized by Li-Bland and Meinrenken [10] as follows: if $\rho : \phi^*E \to TM'$ is a vector bundle map then such a CA structure on $\phi^*E$ with the anchor $\rho$ exists (and moreover is unique) iff
- $\phi_*(\rho(\phi^*u)) = \rho(u) \quad \forall u \in \Gamma(E)$
- $[\rho(\phi^*u), \rho(\phi^*v)] = \rho([\phi^*[u, v]]) \quad \forall u, v \in \Gamma(E)$
- for any $p \in M'$ the kernel of $\rho$ at $p$ is a coisotropic subspace of $E_{\phi(p)}$. 

In particular, if \( M \) is a point and hence \( E = g \) is a Lie algebra, a CA structure on \( g \times M' \), such that the Courant bracket of constant sections is the Lie bracket in \( g \), is equivalent to an action \( \rho \) of \( g \) on \( M' \) with coisotropic stabilizers. This CA is exact iff the action is transitive with Lagrangian stabilizers, i.e. if \( M' \) is (locally diffeomorphic to) \( G/H \) where \( h \subset g \) satisfies \( h^\perp = h \).

Theorem 3 implies that Poisson-Lie T-duality is compatible with the 1-loop renormalization group flow. Let us summarize the needed definitions. Suppose that \( E \to M \) is a CA, \( \phi_i : M_i \to M \), \( i = 1, 2 \), are surjective submersions, and that we have exact CA structures on \( \phi_i^* E \) satisfying (13)\footnote{If \( G \) is a connected Lie group and \( \langle \_ \_ \_ \_ \_ \rangle \) an invariant inner product on \( g \), then any principal \( G \)-bundle \( P \to M \) with vanishing 1st Pontryagin class \( ([F, F]) \in H^4(M, \mathbb{R}) \) gives a transitive CA \( E \to M \) (depending on a choice of \( \omega \in \Omega^3(M)/d\Omega^2(M) \) such that \( d\omega = \langle F, F \rangle \)) and for any Lie subgroup \( H \subset G \) with \( h^\perp = h \) we have a compatible exact CA structure on \( \phi^* E \to P/H \) where \( \phi : P/H \to M \) is the projection. This is the main source of examples. (In the case of \( M = \) point we have \( E = g \) and so \( \phi^* E = g \times G/H. \) See [12] for details.}. If \( V_i \subset E \) is a generalized metric then \( \phi_i^* V_i \subset \phi_i^* E \) are generalized metrics in the exact CAs \( \phi_i^* E \) and these generalized metrics on \( M_1 \) and \( M_2 \) are said to be Poisson-Lie T-dual to each other\footnote{It implies that the 2-dimensional \( \sigma \)-models with the target spaces \( M_1 \) and \( M_2 \) are (after a suitable reduction) isomorphic as Hamiltonian systems. See [13] for details.}

If we choose a compatible connection \( \nabla \) on \( E \) and deform \( V_+ \) by \(-2 \mathrm{GRic}^{(\nabla)}_{V_+} \) then by Theorem 3 the subbundle \( \phi_i^* V_i \subset \phi_i^* E \) gets deformed by \(-2 \mathrm{GRic}^{(\phi_i^* \nabla)}_{\phi_i^* V_i} \), i.e. by the 1-loop renormalization group flow (1). The deformed \( \phi_i^* V_i \)'s stay Poisson-Lie T-dual to each other, and so Poisson-Lie T-duality is indeed compatible with the 1-loop renormalization group flow.

\[ \mathrm{GRic}_{V_+} (u_+, v_-) = - u_+ \longrightarrow v_- \]

This expression was discovered in [14] as the 1-loop renormalization of a duality-invariant Hamiltonian version of the corresponding \( \sigma \)-models from [9]. Proving that this expression really gives the generalized Ricci flow (1) and extending it to Poisson-Lie T-duality with spectators (i.e. with non-trivial \( M \)) was the original motivation of this paper.

\section*{Appendix A. Proof of Theorem 1}

Let us compute
\[ \delta_a \mathrm{GRic}^{(\nabla)}_{V_+} := \frac{d}{dt} \mathrm{GRic}^{(\nabla + ta)}_{V_+} \big|_{t=0} . \]

The three terms of \( \mathrm{GRic}^{(\nabla)}_{V_+} \) contribute
\[ \delta_a \mathrm{Tr}_{V_-} (x_+ \mapsto R_{V_-} (\rho(x_+), \rho(u_+)) v_-) = \]

\[ u_+ \to u_+ \quad v_- \to v_- \quad u_+ \to u_+ \quad v_- \to v_- \quad u_+ \to u_+ \quad v_- \to v_- \]

\[ = \]

\[ = \]
\[ -\delta_a u_+ v_- = u_+ a_+ v_- + u_+ a_- v_- \]

\[ \delta_a u_+ v_- = - u_+ a_+ v_- + a_- u_+ v_- \]

\[ + u_+ a_- v_- - a_+ u_+ v_- \]

Since

\[ \langle [s_-, u_+], v_- \rangle = - \langle [u_+, s_-], v_- \rangle = \]

\[ \delta_a \text{GRic}(\nabla)(u_+, v_-) = \langle [s_-, u_+], v_- \rangle \]

we get

\[ \text{GRic}(\nabla)(u_+, v_-) = \langle [s_-, u_+], v_- \rangle \]

as

\[ - u_+ a_+ v_- + u_+ a_- v_- + a_- u_+ v_- = 0 \]

and the remaining terms cancel in pairs.

Finally, since \( \langle [s_-, u_+], v_- \rangle \) is independent of \( \nabla \), we get

\[ \text{GRic}(\nabla)(u_+, v_-) - \text{GRic}(\nabla)(u_+, v_-) = \langle [s_-, u_+], v_- \rangle \]

as we wanted to show.

References

[1] T. Buscher: Path-integral derivation of quantum duality in nonlinear sigma-models, Phys. Lett. B 201(4), 1988, 466–472
[2] G. Cavalcanti, M. Gualtieri: Generalized complex geometry and T-duality. In: A Celebration of the Mathematical Legacy of Raoul Bott (CRM Proceedings & Lecture Notes), American Mathematical Society, 2010, pp. 341–366.
[3] B.E. Fridling, A.E.M. van de Ven: Renormalization of non-linear \( \sigma \)-models, Nuclear Physics B268 (1986) 719–736
[4] D. Friedan: Non-linear models in 2 + \( \epsilon \)-dimensions, Phys.Rev.Lett. 45 (1980), 1057–1060.
[5] M. Garcia-Fernandez: Torsion-Free Generalized Connections and Heterotic Supergravity, Commun.Math.Phys. 332 (2014) no. 1, 89–115
[6] M. Gualtieri: Branes on Poisson varieties. In: The Many Facets of Geometry: A Tribute to Nigel Hitchin, Oxford, 2010.
[7] M. Gualtieri: Generalized Kähler geometry, Commun.Math.Phys. 331 (2014) no.1, 297–331
[8] C. Klimčík, P. Ševera: Dual non-Abelian T-duality and the Drinfeld double. Phys.Lett. B 351 (1995), 455–462.
[9] C. Klimčík, P. Ševera: Poisson-Lie T-duality and Loop Groups of Drinfeld Doubles, Phys.Lett. B372 (1996) 65-71
[10] D. Li-Bland, E. Meinrenken: Courant algebroids and Poisson geometry, Int Math Res Notices (2009) 2009 (11): 2106–2145.
[11] Zh.-J. Liu, A. Weinstein, P. Xu: Manin triples for Lie bialgebroids. J. Differential Geom. 45 (1997), no. 3, 547–574
[12] P. Ševera: Poisson-Lie T-duality and Courant algebroids, Lett.Math.Phys. 105 (2015) no.12, 1689–1701
[13] P. Ševera: Poisson-Lie T-duality as a boundary phenomenon of Chern-Simons theory, JHEP 1605 (2016) 044
[14] K. Sfetsos, K. Siampos, D.C. Thompson: Renormalization of Lorentz non-invariant actions and manifest T-duality, Nucl. Phys. B827 (2010) 545–564
[15] J. Streets: Generalized geometry, T-duality, and renormalization group flow, arXiv:1310.5121
[16] G. Valent, C. Klimčík, R. Squellari: One loop renormalizability of the Poisson-Lie sigma models, Phys.Lett. B678 (2009) 143-148

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