Online bin packing with cardinality constraints revisited

György Dósa∗ Leah Epstein†

Abstract

Bin packing with cardinality constraints is a bin packing problem where an upper bound $k \geq 2$ on the number of items packed into each bin is given, in addition to the standard constraint on the total size of items packed into a bin. We study the online scenario where items are presented one by one. We analyze it with respect to the absolute competitive ratio and prove tight bounds of 2 for any $k \geq 4$. We show that First Fit also has an absolute competitive ratio of 2 for $k = 4$, but not for larger values of $k$, and we present a complete analysis of its asymptotic competitive ratio for all values of $k \geq 5$. Additionally, we study the case of small $k$ with respect to the asymptotic competitive ratio and the absolute competitive ratio.

1 Introduction

Bin packing with cardinality constraints (BPCC) is a variant of bin packing. The input consists of items, denoted by $1, 2, \ldots, n$, where item $i$ has a size $s_i > 0$ associated with it, and a global parameter $k \geq 2$, called the cardinality constraint. The goal is to partition the input items into subsets called bins, such that the total size of items of one bin does not exceed 1, and the number of items does not exceed $k$. In many applications of bin packing the assumption that a bin can contain any number of items is not realistic, and bounding the number of items as well as their total size provides a more accurate modeling of the problem. BPCC was studied both in the offline and online environments [16, 17, 15, 5, 1, 7, 9, 10].

In this paper we study online algorithms that receive and pack input items one by one, without any information on further input items. A fixed optimal offline algorithm that receives the complete list of items before packing it is denoted by OPT. For an input $L$ and algorithm $A$, we let $A(L)$ denote the number of bins that $A$ uses to pack $L$. We also use $OPT(L)$ to denote also the number of bins that $OPT$ uses for a given input $L$. The absolute approximation ratio of an algorithm $A$ is the supremum ratio over all inputs $L$ between the number of bins $A(L)$ that it uses and the number of bins $OPT(L)$ that $OPT$ uses. The asymptotic approximation ratio is the limit of absolute competitive ratios $R_K$ when $K$ tends to infinity and $R_K$ takes into account only inputs for which $OPT$ uses at least $K$ bins. Note that (by definition), for a given algorithm (for some online bin packing problem), its asymptotic approximation ratio never exceeds its absolute approximation ratio. If the algorithm is online, the term competitive ratio replaces the term approximation ratio. For an algorithm whose approximation ratio (or competitive ratio) does not exceed $R$, we say that it is an $R$-approximation (or $R$-competitive). We see a bin as a set of items, and for a bin $B$, we let $s(B) = \sum_{i \in B} s_i$ be its level.

∗Department of Mathematics, University of Pannonia, Veszprem, Hungary, dosagy@almos.vein.hu.
†Department of Mathematics, University of Haifa, Haifa, Israel. lea@math.haifa.ac.il.
Bin packing problems are often studied with respect to the asymptotic measures. Approximation algorithms were designed for the offline version (that is strongly NP-hard for \( k \geq 3 \)) [16, 15, 5, 9], and the problem has an asymptotic fully polynomial approximation scheme (AFPTAS) [5, 9]. Using elementary bounds, it was shown by Krause, Shen, and Schwetman [16] that the cardinality constrained variant of First Fit (FF), that packs an item \( i \) into a minimum index bin where it fits both with respect to size and cardinality (i.e., \( i \) has at most \( m > 1 \) items and level at most \( 1 - s_i \)), has an asymptotic competitive ratio of at most \( 2.7 - \frac{4}{k} \). For \( k \to \infty \), the competitive ratio is 2.7, since this is a special case of vector bin packing (with two dimensions) [11]. The case \( k = 2 \) is solvable using matching techniques in the offline scenario, but it is not completely resolved in the online scenario. Liang [19] showed a lower bound of \( \frac{3}{2} \) on the asymptotic competitive ratio for this case, Babel et al. [1] improved the lower bound to \( \sqrt{2} \approx 1.41421 \), and designed an algorithm whose asymptotic competitive ratio is at most \( 1 + \frac{1}{\sqrt{2}} \approx 1.44721 \) (improving over FF). Recently, Fujiwara and Kobayashi [10] improved the lower bound to 1.42764. For larger \( k \), there is a 2-competitive algorithm [1], and improved algorithms are known for \( k = 3, 4, 5, 6 \) (whose competitive ratios are at most 1.75, 1.86842, 1.93719, and 1.99306, respectively) [7]. Note that the upper bounds of [16] for FF and \( k = 3 \) is 1.9, and an algorithm whose competitive ratio is at most 1.8 was proposed by [1].

A full analysis of the cardinality constrained variant of the Harmonic algorithm [15] (that partitions items into \( k \) classes and packs each class independently, such that the classes are \( I_\ell = \left\{ \frac{1}{\ell + 1}, \frac{1}{\ell + 1} \right\} \) for \( 1 \leq \ell \leq k - 1 \) and \( I_k = \left\{ 0, \frac{1}{k} \right\} \), and for any \( 1 \leq \ell \leq k \), each bin of \( I_\ell \) possibly except for the last such bin, receives exactly \( \ell \) items) is given in [7], and its competitive ratio for \( k = 2, 3 \) is 1.5 and \( \frac{3}{2} \), respectively (its competitive ratio is in \([2, 12.69103]\) for \( k \geq 4 \), see TableI for some additional values). As for lower bounds, until recently, except for the case \( k = 3 \) for which a lower bound of 1.5 on the competitive ratio was proved in [1], most of the known lower bounds followed from the analysis of lower bounds for standard bin packing [27, 25, 2]. New lower bounds for many values of \( k \) were given in [10], and in particular, they proved lower bounds of 1.5 and \( \frac{25}{17} \approx 1.47058 \) for \( k = 4 \) and \( k = 5 \), respectively. For \( 6 \leq k \leq 9 \), the current best lower bound remained 1.5, that was implied by the lower bound of Yao [27], and for \( k = 10 \) and \( k = 11 \), lower bounds of \( \frac{80}{53} \approx 1.50943 \) and \( \frac{44}{25} \approx 1.51724 \), respectively, were proved in [10] (see [10] for the lower bounds of other values of \( k \)). In this paper we provide a complete analysis of FF with respect to the asymptotic competitive ratio. We find that its competitive ratio is at most \( 2.5 - \frac{2}{k} \) for \( k = 2, 3, 4 \), \( \frac{8(k-1)}{3k} = \frac{8}{3} - \frac{8}{3}k \) for \( 4 \leq k \leq 10 \), and \( 2.7 - \frac{3}{k} \) for \( k \geq 10 \) (we included each of the values \( k = 4 \) and \( k = 10 \) in two cases as \( 2.5 - 2/k = 8/3 - 8/(3k) \) for \( k = 4 \), and \( 8/3 - 8/(3k) = 2.7 - 3/k \) for \( k = 10 \)). Additionally, we provide improved lower bounds on the asymptotic competitive ratio of arbitrary online algorithms for \( k = 5, 7, 8, 9, 10, 11 \). The values of these lower bounds are 1.5 for \( k = 5 \), and approximately 1.51748, 1.5238, 1.5242, 1.526, 1.5255, for \( k = 7, 8, 9, 10, 11 \), respectively.

There are few known results for the absolute measures. The asymptotic \((1 + \varepsilon)\)-approximation algorithm of Caprara, Kellerer, and Pferschy [5] uses \((1 + \varepsilon)OPT + 1\) bins to pack the items, and thus, choosing \( \varepsilon > 0 \) to be small (for example \( \varepsilon = \frac{1}{100} \)) results in a polynomial time absolute \( \frac{3}{2} \)-approximation algorithm. This is the best possible unless \( P=NP \). In the online environment, it is not difficult to see that given the absolute upper bound of 1.7 on the competitive ratio of FF for standard bin packing [3], the upper bound of \( 2.7 - 2.4/k \) becomes an absolute one (we provide the proof here for completeness). In this paper we analyze the absolute competitive ratio, and show a tight bound of 2 on the absolute competitive ratio for any \( k \geq 4 \). The upper bound for \( k = 4 \) is proved for FF. An upper bound for \( k = 5 \) is proved using an algorithm that performs FF except for one case. We show that a variant of the algorithm of [1] has an absolute competitive ratio 2 for
any $k \geq 2$. In the case $k = 3$, we provide a lower bound of $\frac{7}{4} = 1.75$ on the absolute competitive ratio of any algorithm, and show that the absolute competitive ratio of FF is $\frac{7}{4} \approx 1.8333$. For $k = 2$, tight bounds of 1.5 on the best possible competitive ratio follow from previous work [3] (the upper bound is the absolute competitive ratio of FF).

For standard bin packing [24, 12, 13, 14, 18, 27, 20], it is known that the asymptotic competitive ratio is in $[1.5403, 1.58889]$ [2, 21], and the absolute competitive ratio is in $[\frac{5}{3}, 1.7]$ (see [28, 6]), where the upper bound is the competitive ratio of FF (without cardinality constraints). Interestingly, introducing cardinality constraints (with sufficiently large values of $k$) results in an increase of many competitive ratios by 1 [16, 14, 18, 7]. Another related problem is called class constrained bin packing [8, 22, 23, 26]. In that problem each item has a color, and a bin cannot contain items of more than $k$ colors (for a fixed parameter $k$). BPCC is the special case of that problem where all items have distinct colors.

We start with lower bounds in Section 2, where both the absolute competitive ratio and the asymptotic competitive ratio are studied. We consider algorithms afterwards, in Section 3, where we analyze FF, and in Section 4, where we consider algorithms whose absolute competitive ratio is at most 2.

| Value of $k$ | prev. LB | new LB | FF  | prev. UB for FF | Harmonic | best known |
|--------------|----------|--------|-----|-----------------|----------|------------|
| 2            | 1.42764  | 1.5    | 1.5 | 1.5             | 1.44721  | [1]        |
| 3            | 1.5      | 1.8333 | 1.9 | 1.8333         | 1.75     | [7]        |
| 4            | 1.5      | 2.1    | 2.1 | 2.1             | 1.86842  | [7]        |
| 5            | 1.47058  | 2.1333 | 2.22| 2.1             | 1.93719  | [7]        |
| 6            | 1.5      | 2.2222 | 2.3 | 2.3             | 2.16667  | 1.99306    |
| 7            | 1.5      | 2.2857 | 2.35714 | 2.2381   | 2        | [1]        |
| 8            | 1.5      | 2.3333 | 2.4 | 2.29167         | 2        | [1]        |
| 9            | 1.5      | 2.3704 | 2.4333 | 2.3333   | 2        | [1]        |
| 10           | 1.50943  | 2.4    | 2.46 | 2.36666        | 2        | [1]        |
| 11           | 1.51724  | 2.4273 | 2.481818 | 2.39394  | 2        | [1]        |
| 12           | 1.53488  | 2.45   | 2.5  | 2.41667        | 2        | [1]        |

Table 1: Bounds for $2 \leq k \leq 12$. The first column contains the previously known lower bounds on the asymptotic competitive ratio. The second column contains our improved lower bounds. The third column contains the tight asymptotic competitive ratio of FF (for $k = 2, 3, 4$ it is also the absolute competitive ratio), the fourth column contains the previous upper bound on FF’s asymptotic competitive ratio [16], the fifth column contains the tight asymptotic competitive ratio of Harmonic [7], and the last column contains the asymptotic competitive ratio of the current best algorithm.

2 Lower bounds

In this section we present lower bounds for the two measures.
2.1 Lower bounds on the absolute competitive ratio

We show that the absolute competitive ratio is at least 2 for \( k \geq 4 \). Together with the analysis of Section 4.1, we will find that this is the best possible competitive ratio.

**Proposition 1** The absolute competitive ratio of any algorithm for \( k \geq 4 \) is at least 2.

**Proof.** Let \( 0 < \epsilon < \frac{1}{3k} \). The input starts with \( k \) items, each of size \( \epsilon \), called tiny items. Since an optimal solution packs them into one bin, if an online algorithm uses two bins, then we are done. Otherwise the algorithm packs them into one bin, and no further items can be combined into this bin, since it already has \( k \) items. The next two items have sizes of \( \frac{1}{3} + \epsilon \). If the algorithm packs them into two new bins, then the next item has size \( \frac{2}{3} \) and it requires a new bin. An optimal solution packs the last item with \( k - 1 \) tiny items, and the remaining three items into another bin, while the algorithm uses four bins. Otherwise, the algorithm packs the two items of sizes \( \frac{1}{3} + \epsilon \) into one bin. In this case the last two items have sizes of \( \frac{1}{2} + \epsilon \). The algorithm now has four bins, where an optimal solution has two bins, each with an item of size \( \frac{1}{2} + \epsilon \), an item of size \( \frac{1}{3} + \epsilon \), and \( \lceil \frac{k}{2} \rceil \) or \( \lceil \frac{k}{2} \rceil \) items of size \( \epsilon \) each (which is possible since at most \( k - 2 \) items of size \( \epsilon \) are added; for \( k = 4 \) \( \frac{k}{2} = k - 2 = 2 \) holds, and \( \lceil \frac{k}{2} \rceil \leq \frac{k+1}{2} \leq k - 2 \) holds for \( k \geq 5 \)).

In the case \( k = 2 \), a lower bound of \( \frac{7}{4} \) on the absolute competitive ratio follows from an input that consists of two items, each of size \( \epsilon \), possibly followed by two items, each of size \( 1 - \epsilon \) (for \( 0 < \epsilon < \frac{1}{3} \)).

Next, we present a lower bound of \( \frac{7}{4} \) on the absolute competitive ratio of any algorithm for \( k = 3 \). Recall that the best asymptotic competitive ratio for \( k = 3 \) is in \([\frac{3}{2}, \frac{7}{4}]\). The upper bound of \( \frac{16}{11} \) for the asymptotic competitive ratio of FF is \( 2.7 - 2.4/3 = 1.9 \), and we will show a tight bound of \( \frac{11}{6} \) on the absolute and asymptotic competitive ratios of FF.

**Proposition 2** The absolute competitive ratio of any algorithm for \( k = 3 \) is at least \( \frac{7}{4} = 1.75 \).

**Proof.** Let \( 0 < \epsilon < \frac{1}{27} \). The input starts with three tiny jobs of size \( \epsilon \) each. Since an optimal solution can pack them into one bin, to avoid a competitive ratio of at least 2, the algorithm must do the same. Note that the bin containing these items cannot receive any additional items. Next, two items of sizes \( \frac{1}{3} + \epsilon \) arrive. If the last two items are packed into separate bins, an item of size \( \frac{2}{3} \) is presented. An optimal solution can pack the items into two bins; the last item is combined with two tiny items, and the remaining three items are packed into a second bin, which is possible given the value of \( \epsilon \).

Otherwise, the algorithm has two bins, where the second one can still receive one item of size at most \( \frac{1}{3} - 2\epsilon \). The remaining items will be larger, and thus they will be packed into new bins. Now, two items of sizes \( \frac{1}{3} + 3\epsilon \) arrive. If the algorithm uses two new bins to pack them, then two items whose sizes are equal to \( \frac{2}{3} - 2\epsilon \) arrive, and the algorithm is forced to use two new bins for them, for a total of six bins. An optimal solution uses three bins; two bins contain (each) an item of size \( \frac{2}{3} - 2\epsilon \), an item of size \( \frac{1}{3} + \epsilon \), and a tiny item. The remaining three items have total size \( \frac{2}{3} + 7\epsilon \) and can be packed into a third bin by an optimal solution. Thus, the competitive ratio is 2 in this case.

Otherwise, the algorithm has three bins, where the second and third bins can still receive one item each, but no item of size at least \( \frac{1}{3} \) can be packed there. The remaining items will be larger than \( \frac{1}{3} \), and thus they will be packed into new bins. Specifically, there are four items whose sizes are equal to \( \frac{2}{3} - 4\epsilon \) for \( \epsilon < \frac{1}{27} \). Each such item must be packed into a new dedicated bin, for a total of
seven bins. An optimal solution can combine each such item with an item of size $\frac{1}{3} + \varepsilon$ or an item of size $\frac{1}{3} + 3\varepsilon$, and three such bins also receive a tiny item.

2.2 Lower bounds on the asymptotic competitive ratio

In this section, we present improved lower bounds for ratio for $k = 5$ and $7 \leq k \leq 11$. To prove these lower bounds, we will consider inputs that consist of at most four batches. Let $0 < \delta < \frac{1}{2000}$, and let $N$ be a large integer divisible by $6k$. For $7 \leq k \leq 11$, the first batch contains $\frac{k \cdot \delta}{6} \cdot N$ items, and for $k = 5$, the first batch contains $\frac{N}{2}$ items, each of size $\frac{1}{12} - 3\delta > 0$. We use $\phi_k = \frac{k \cdot \delta}{6k}$ for $7 \leq k \leq 11$, and $\phi_5 = \frac{1}{12}$. Thus, the first batch consists of $k \phi_k \cdot N$ items. The second batch contains $N$ items, each of size $\frac{1}{12} + \delta$, the third batch contains $N$ items, each of size $\frac{1}{2} + \delta$, and the fourth batch contains $N$ items, each of size $\frac{1}{3} + \delta$. Note that the first batch has a smaller number of items than later batches for all values of $k$. The input may stop after any batch, and thus there are four possible inputs, denoted by $L_1$, $L_2$, $L_3$, and $L_4$. For $i = 1, 2, 3, 4$, an optimal solution for $L_i$ has bins packed almost identically. For $k = 5$, clearly, $OPT(L_1) \geq \frac{N}{10}$ and $OPT(L_2) \geq \frac{3N}{10}$, as no bin can have more than five items, $OPT(L_3) \geq \frac{N}{7}$, as no bin can have more than two items of the third batch, and $OPT(L_4) \geq N$, as no bin can have more than one item of the fourth batch. For $7 \leq k \leq 11$, clearly, $OPT(L_1) \geq \frac{N \cdot (k - 6)}{6k}$, as no bin can have more than $k$ items, $OPT(L_2) \geq \frac{N}{6}$, as no bin can have more than six items of the second batch, $OPT(L_3) \geq \frac{N}{7}$, as no bin can have more than two items of the third batch, and $OPT(L_4) \geq N$, as no bin can have more than one item of the fourth batch. Next, we define solutions that achieve those numbers and thus they are optimal. For $k = 5$, optimal solutions for $L_1$ and for $L_2$ have five items in each bin, which is possible since $5 \cdot \left(\frac{1}{12} + \delta\right) = \frac{5}{12} + 5\delta < 1$. Thus, $OPT(L_1) = \frac{N}{10}$ and $OPT(L_2) = \frac{3N}{10}$. An optimal solution for $L_3$ has two items of the third batch, two items of the second batch, and one item of the first batch in each bin, $2(\frac{1}{12} + \delta) + 2(\frac{1}{2} + \delta) + (\frac{1}{3} - 3\delta) < 1$, all items are packed, and $OPT(L_3) = \frac{N}{7}$. Finally, an optimal solution for $L_4$ has at most one item of the first batch and exactly one item of any other batch packed into each bin. All items are packed, $OPT(L_4) = N$, and $(\frac{1}{7} + \delta) + (\frac{1}{2} + \delta) + (\frac{1}{3} + \delta) + (\frac{1}{12} - 3\delta) = 1$. For $7 \leq k \leq 11$, an optimal solution for $L_1$ has $k$ items in each bin, which is possible since $k \cdot (\frac{1}{12} - 3\delta) < \frac{1}{12} < 1$.

Thus, $OPT(L_1) = \frac{N \cdot (k - 6) / 6}{k}$. An optimal solution for $L_2$ has six items of the second batch and $k - 6$ items of the first batch in each bin, and $OPT(L_2) = \frac{N}{6}$. The solution is valid as each bin has $k$ items, $6(\frac{1}{12} + \delta) + (k - 6)(\frac{1}{12} - 3\delta) = \frac{6}{12} + \frac{k \cdot \delta}{6} + (24 - 3k)\delta \leq \frac{6}{12} + \frac{5}{12} + 24\delta = \frac{41}{12} + 24\delta < 1$, as $\delta < \frac{1}{2000}$, and all the items of the first two batches are packed. An optimal solution for $L_3$ has two items of the third batch, two items of the second batch, and at most two items of the first batch in each bin, $2(\frac{1}{12} + \delta) + 2(\frac{1}{2} + \delta) + 2(\frac{1}{12} - 3\delta) = 1 - 2\delta < 1$, all items are packed, and $OPT(L_3) = \frac{N}{7}$. Finally, an optimal solution for $L_4$ has at most one item of the first batch and exactly one item of any other batch packed into each bin. All items are packed, $OPT(L_4) = N$, and $(\frac{1}{7} + \delta) + (\frac{1}{2} + \delta) + (\frac{1}{3} + \delta) + (\frac{1}{12} - 3\delta) = 1$. In all cases we have $OPT(L_i) = N \cdot \phi_k$. We use $\phi'_k = \frac{OPT(L_i)}{N}$, i.e., $\phi'_5 = \frac{3}{10}$, and $\phi'_6 = \frac{1}{6}$ for $7 \leq k \leq 11$.

Consider a deterministic or randomized algorithm $A$. Let $X_i$ be the number of bins (or expected number of bins) that the algorithm opens while packing the items of batch $i$. Assume that the competitive ratio is $R$. Let $f$ be a function where $f(n) = o(n)$ such that for any input $I$, $A(I) \leq R \cdot OPT(I) + f(OPT(I))$. We have $A(L_i) \leq R \cdot OPT(L_i) + f(OPT(L_i))$. Note that $A(L_i) = \sum_{j=1}^{4} X_j$. For any set of four parameters $\alpha_i \geq 0$ for $i = 1, 2, 3, 4$ (constants that are independent of $N$), we find $\sum_{i=1}^{4} \alpha_i A(L_i) \leq \sum_{i=1}^{4} \alpha_i (R \cdot OPT(L_i) + f(OPT(L_i)))$. Letting $\beta_i = \sum_{j=1}^{4} \alpha_i$ and rewriting it
we get \( \sum_{i=1}^{4} \beta_i X_i \leq RN \cdot (\alpha_1 \phi_k + \frac{\alpha_3}{3} + \frac{\alpha_4}{3} + \alpha_1) + \sum_{i=1}^{4} \alpha_i f(OPT(L_i)). \)

We define weights for the items. Let \( w_i \geq 0 \) be the weight of an item of batch \( i \). The total weight of items of \( L_4 \) is \( w_1 \cdot k \phi_k \cdot N + w_2 \cdot N + w_3 \cdot N + w_4 \cdot N \). Let \( W_i \) denote the maximum weight of any bin opened by the algorithm for batch \( i \) or a later batch (possibly used for additional items later). We have \( W_1 \geq W_2 \geq W_3 \geq W_4 \). The total weight of items is, denoted by \( W \) satisfies \( W \leq \sum_{i=1}^{4} W_i X_i \). Thus \( W = w_1 \cdot k \phi_k \cdot N + w_2 \cdot N + w_3 \cdot N + w_4 \cdot N = W \leq \sum_{i=1}^{4} W_i X_i \). Let \( W_5 = 0 \), and \( \beta_i = W_i \) (i.e., \( \alpha_i = W_i - W_{i+1} \) for \( i = 1, 2, 3, 4 \)).

**Lemma 3** For \( 7 \leq k \leq 11 \), we have \( R \geq \frac{W_1}{W_1/6-W_1/k+W_2/k+W_3/3+W_4/2}. \) For \( k = 5 \) we have \( R \geq \frac{w_1/2+w_2+w_3+w_4}{W_1/10+W_2/5+W_3/8+W_4/2}. \)

**Proof.** We have \( N(w_1 \cdot k \phi_k + w_2 + w_3 + w_4) \leq \sum_{i=1}^{4} \beta_i X_i \leq RN((W_1 - W_2)\phi_k + (W_2 - W_3)\phi_k' + \frac{W_3 - W_4}{2} + W_3) + \sum_{i=1}^{4}(W_i - W_{i+1})f(OPT(L_i)). \)

We find \( \sum_{i=1}^{4}(W_i - W_{i+1})f(OPT(L_i)) = o(N) \), since \( OPT(L_i) = \Theta(N) \) for \( i = 1, 2, 3, 4 \), and the values \( W_i \) are constants independent of \( N \).

Thus, \( w_1 \cdot k \phi_k + w_2 + w_3 + w_4 \leq R((W_1 - W_2)\phi_k + (W_2 - W_3)\phi_k' + \frac{W_3 - W_4}{2} + W_4) \), or alternatively, \( w_1 \cdot k \phi_k + w_2 + w_3 + w_4 \leq R(W_1\phi_k + W_2(\phi_k' - \phi_k) + W_3(\frac{\phi_j}{2} - \phi_k') + \frac{W_3}{2}). \)

For \( k = 5 \), let \( w_2 = 1 \) and \( w_3 = w_4 = 2 \). For \( k = 7, 8 \), let \( w_1 = 1, w_2 = 1, w_3 = w_4 = 2 \). For \( k = 9, 10, 11 \), let \( w_1 = 1, w_2 = 2, w_3 = w_4 = 4 \). For \( k = 10, 11 \), let \( w_1 = 1, w_2 = 3, w_3 = w_4 = 6 \).

**Lemma 4** For \( k = 5 \), we have \( W_1 \leq 10, W_2 \leq 6, W_3 \leq 4, \) and \( W_4 \leq 2 \).

For \( k = 7, 8 \), \( W_1 \leq k + 2, W_2 \leq 6, W_3 \leq 4, \) and \( W_4 \leq 2 \).

For \( k = 9, W_1 \leq 16, W_2 \leq 12, W_3 \leq 8, \) and \( W_4 \leq 4 \).

For \( k = 10, 11 \), \( W_1 \leq k + 12, W_2 \leq 18, W_3 \leq 12, \) and \( W_4 \leq 6 \).

**Proof.** The claim regarding \( W_4 \) holds since \( W_4 = w_4 \) must hold (as every bin opened for the last batch contains exactly one item). Moreover, any bin opened for the third batch will contain only items of sizes strictly above \( \frac{1}{3} \), so it can contain at most two items and since \( w_3 = w_4 \) in all cases, \( W_3 \leq 2w_3 \).

Let \( k = 5 \). Consider a bin that was opened for the second batch. If it has two items of later batches, it can have at most two items of the second batch, so the total weight is at most \( 2w_2 + 2w_3 = 6 \). If it has at most one item of later batches, then all its items have weight 1 except for at most one item whose weight is at most 2, for a total of 6, since there are at most five items in total. Finally, consider a bin that was opened for the first batch. Every item has weight of at most 2, and there are at most five items, thus the total weight is at most 10.

We are left with \( 7 \leq k \leq 11 \), and bounding \( W_1 \) and \( W_2 \). A bin that was opened for the second batch can contain at most six items of that batch. If it contains one larger item, then it can contain at most four items of the second batch, and if it contains two larger items, then it can contain at most two items of the second batch. It cannot contain more than two larger items, and since \( w_3 = w_4 \), we do not distinguish the items of the last two batches. Thus \( W_2 \leq \max\{6w_2, w_3 + 4w_2, 2w_3 + 2w_2\} = 6w_2 \). We find \( W_2 = 6 \) for \( k = 7, 8 \), and \( W_2 = 12 \) for \( k = 9 \), and \( W_2 = 18 \) for \( k = 10, 11 \).

Finally, consider bins opened for the first batch. We start with the case \( k = 7, 8 \). A bin contains at most \( k \) items, where items of the last two batches have weights of 2 and items of the first two batches have weights of 1. Thus, the total weight is at most \( k + \) the number of items of sizes above \( \frac{1}{3} \), which is at most 2. The total weight is therefore at most \( k + 2 \).
Consider the cases \( k = 9, 10, 11 \). Given a bin that contains at most three items of batches 2, 3, 4, it can contain at most two items of batches 3, 4, so the total weight is at most \( k + 2(w_3 - 1) + (w_2 - 1) \). This last value is equal to \( 2w_3 + w_2 + k - 3 = 5w_2 + k - 3 \), which is equal to 16 for \( k = 9 \) and to \( k + 12 \) for \( k = 10, 11 \). We are left with the case that there are at least four items of batches 2, 3, 4 packed into the bin. If all those items are of batch 2, then there are at most six such items, and the total weight is at most \( k + 6(w_2 - 1) = 6w_2 + k - 6 \leq 5w_2 + k - 3 \) as \( w_2 \leq 3 \). If there is one item of the last two batches, there are at most four items of batch 2. If there are at most three items of batch 2, then the weight is at most \( k + 3(w_2 - 1) + (w_3 - 1) = 3w_2 + w_3 + k - 4 = 5w_2 + k - 5 \). If there is one item of the last two batches and four items of the second batch, their total size is at least \( \frac{1}{2} + 6(\frac{1}{2} + \delta) = \frac{19}{21} + 5\delta \). The remaining space can only contain four items of the first batch as \( 5(\frac{1}{3} - 3\delta) + \frac{19}{21} + 5\delta = 1 + \frac{1}{72} - 10\delta > 1 \). The total weight is therefore at most \( 4w_1 + 4w_2 + w_3 \), which is equal to 16 for \( k = 9 \), and to \( 22 \leq k + 12 \) for \( k = 10, 11 \).

If there are two items of the last two batches, there are at most two items of batch 2. If there is at most one item of batch 2, then the weight is at most \( k + (w_2 - 1) + 2(w_3 - 1) = w_2 + 2w_3 + k - 3 = 5w_2 + k - 3 \). If there are two items of the last two batches and two items of the second batch, their total size is at least \( 2(\frac{1}{2} + \delta) + 2(\frac{1}{2} + \delta) = \frac{20}{21} + 4\delta \). The remaining space can only contain two items of the first batch as \( 3(\frac{1}{3} - 3\delta) + \frac{20}{21} + 4\delta = 1 + \frac{1}{72} - 10\delta > 1 \). The total weight is therefore at most \( 2w_1 + 2w_2 + 2w_3 \), which is equal to 14 for \( k = 9 \), and to \( 20 \leq k + 12 \) for \( k = 10, 11 \).

**Corollary 5** The following values are lower bounds on the competitive ratios.

- \( \frac{3}{2} = 1.5 \) for \( k = 5 \).
- \( \frac{k^2 + 24k}{k^2 + 10k + 24} \) for \( k = 7, 8 \). This value is equal to \( 217/143 \approx 1.5174825 \) for \( k = 7 \) and to \( \frac{32}{21} \approx 1.5238095 \) for \( k = 8 \).
- \( \frac{10.5}{62/9} = \frac{189}{121} \approx 1.52419355 \), for \( k = 9 \).
- \( \frac{k^2 + 84k}{k^2 + 48k + 36} \) for \( k = 10, 11 \). This value is equal to \( 235/154 \approx 1.525974 \) for \( k = 10 \) and to \( \frac{209}{137} \approx 1.525547 \) for \( k = 11 \).

Note that in the cases \( k = 6 \) and \( k = 12 \), our methods do not produce improved lower bounds, and they give exactly the known lower bound.

### 3 A complete analysis of First Fit

We provide a complete analysis of the asymptotic competitive ratio. For \( k = 2, 3, 4 \), the bounds that we find are the absolute competitive ratios as well. In the analysis, a bin of FF that has \( j \) items for \( j \leq k \) is called a \( j \)-bin, and a bin whose number of items is in \([j, k - 1]\) for some \( 1 \leq j < k \) is called a \( j^\uparrow \)-bin.

We find that the asymptotic competitive ratio of FF is \( 2.5 - \frac{2}{k} \) for \( k = 2, 3, 4 \), \( \frac{8(k - 1)}{3k} = \frac{8}{3} - \frac{8}{3k} \) for \( 4 \leq k \leq 10 \), and \( 2.7 - \frac{3}{k} \) for \( k \geq 10 \) (recall that the values \( k = 4 \) and \( k = 10 \) are included in two cases each). The values for \( k = 2, 3, \ldots, 12 \) are given in Tabletabtab. An interesting property is that for large values of \( k \) (\( k \) tending to infinity) both the competitive ratio of the cardinality constrained Harmonic algorithm and FF have competitive ratios that are larger by 1 than their competitive ratios for standard bin packing. Thus, Harmonic has a slightly smaller competitive
ratio of 1.69103. Moreover, it can be verified that the worst-case examples of Harmonic are valid (but not tight) for FF. For $k = 2, 3, 4$ they have the same competitive ratios, but not for $k \geq 5$, and in many cases the competitive ratio of Harmonic is much smaller (see examples in Table 1).

We start with examples showing that the asymptotic competitive ratios cannot be smaller. For $2 \leq k \leq 4$, let $\ell \geq 0$ be a large integer, and let $0 < \varepsilon < \frac{1}{10}$.

Consider an input consisting of $2k(k - 2)\ell$ items of size $\varepsilon$ each (smallest items), $2k\ell$ items of size $\frac{1}{2} - k\varepsilon > \frac{1}{3}$ each (medium size items), and $2k\ell$ items of size $\frac{1}{2} + \varepsilon$ each (largest items). The items are presented in this order. FF creates $2(k - 2)\ell$ bins containing $k$ smallest items each. Then, as further items are larger than $\frac{1}{3}$, FF creates $k\ell$ bins containing pairs of medium size items, and as the remaining items are larger than $\frac{1}{2}$, the largest items are packed into $2k\ell$ dedicated bins. For this input $L_\ell$, $OPT(L_\ell) = 2k\ell$, since it is possible to pack a largest item, a medium size item, and $k - 2$ smallest items into a bin as $\frac{1}{2} + \varepsilon + \frac{1}{2} - k\varepsilon + (k - 2)\varepsilon < 1$. While $FF(L_\ell) = 2(k - 2)\ell + k\ell + 2k\ell = 5k\ell - 4\ell$. This shows that the asymptotic competitive ratio of FF is at least $2.5 - \frac{\varepsilon}{k}$, that is, at least $\frac{11}{6}$ for $k = 3$ and at least 2 for $k = 4$. The example is valid for $k = 2$ too, giving the value 1.5 (in this case there are no smallest items).

In the case $5 \leq k \leq 10$, let $\ell$ be a positive integer divisible by $k$, let $0 < \varepsilon < \frac{1}{120}$ and $\delta < \frac{\varepsilon}{3^k}$ be small positive values, and consider the following input. There are $3\ell$ items of size $\frac{1}{2} + \delta$, $\ell$ items of size $\frac{1}{2} - 10\delta$, $\ell$ items of size $\frac{1}{2} + 20\delta$, $\ell$ items of size $\frac{1}{2} - 30\delta$, $(3k - 8)\ell$ items of size $\delta$, and for $1 \leq p \leq \ell$ there is a pair of items of sizes $\frac{1}{2} + \frac{\varepsilon}{2^p}$ and $\frac{1}{2} - \frac{\varepsilon}{2^p} - 10\delta$. Since $\delta < \varepsilon < \frac{1}{120}$, all sizes are strictly positive. An optimal solution has three types of bins. There are $\ell$ bins with an item of size $\frac{1}{2} + \delta$, an item of size $\frac{1}{2} - 10\delta$, and $k - 2 \leq 8$ items of size $\delta$ each. There are $\ell$ bins with an item of size $\frac{1}{2} + \delta$, an item of size $\frac{1}{2} + 20\delta$, an item of size $\frac{1}{2} - 30\delta$ and $k - 3 \leq 7$ items of size $\delta$ each. Finally, there are $\ell$ bins, where the $p$th bin has an item of size $\frac{1}{2} + \delta$, the pair of items of sizes $\frac{1}{2} + \frac{\varepsilon}{2^p}$ and $\frac{1}{2} - \frac{\varepsilon}{2^p} - 10\delta$, and $k - 3 \leq 7$ items of size $\delta$ each. Remove the items of sizes $\frac{1}{2} + \frac{\varepsilon}{2^p}$ and $\frac{1}{2} - \frac{\varepsilon}{2^p} - 10\delta$ from the input. Obviously, an optimal solution still requires at most $3\ell$ bins. For $1 \leq p \leq \ell - 1$, the items of sizes $\frac{1}{2} + \frac{\varepsilon}{2^p}$ and $\frac{1}{2} - 10\delta - \frac{\varepsilon}{3^{2p}}$ are called a modified pair of index $p$.

The items are presented to FF in the following order. First, all items of size $\delta$ are presented and packed into $(3k - 8)\ell$ bins that cannot be used again. Next, for $1 \leq p \leq \ell - 1$, the modified pair of items of index $p$ is presented, followed by an item of size $\frac{1}{2} - 30\delta$. The total size of these three items is $\frac{1}{2} + \frac{\varepsilon}{3^p} + \frac{1}{2} - 10\delta - \frac{\varepsilon}{3^{2p}}, \frac{1}{2} - 30\delta = \frac{3}{4} - 40\delta + \frac{2\varepsilon}{3^p} > \frac{3}{4} - \frac{40\delta}{3^p} + \frac{2\varepsilon}{3^p} \geq \frac{3}{4} - \frac{40\delta}{3^p} + \frac{2\varepsilon}{3^p} > \frac{3}{4} + \frac{\varepsilon}{2^{3p}}$, while further items have sizes of $\frac{1}{2} + \delta > \frac{1}{2} - 10\delta > \frac{1}{4} + 20\delta > \frac{1}{4} + \frac{\varepsilon}{2^p} > \frac{1}{4} + \frac{\varepsilon}{2^p} > \frac{1}{4} - 30\delta > \frac{1}{4} - 10\delta - \frac{20\varepsilon}{3^p} \geq \frac{1}{4} - 10\delta - \frac{\varepsilon}{3^{2p}}, \frac{1}{4} - 10\delta - \frac{\varepsilon}{3^{2p}},$ where $p' \geq p + 1$ (further modified pairs exist only if $p < \ell - 1$). We have $\frac{3}{4} + \frac{\varepsilon}{2^{3p}} + \frac{1}{2} - \frac{\varepsilon}{3^{2p}} - 10\delta \geq 1 + \frac{\varepsilon}{2^{3p}} - \frac{\varepsilon}{3^{2p}} - \frac{10\varepsilon}{3^{2p}} = 1 + \frac{(3^{2p}-3^{2p}-10\varepsilon)}{3^{2p}} > 1$. This proves that after a bin of a modified pair and an item of size $\frac{1}{2} - 30\delta$ is created, no further items can be packed into that bin. When no modified pairs remain, pairs of items of sizes $\frac{1}{2} - 10\delta$ and $\frac{1}{2} + 20\delta$ are presented (there are $\ell$ such pairs). Each bin receives such a pair, whose total size is $\frac{1}{2} + 10\delta$. Since all remaining items have sizes above $\frac{1}{2}$, each created bin will not be used for further items. Finally, all remaining items (of sizes $\frac{1}{2} + \delta$) are packed into dedicated bins. The total number of bins is $(3k - 8)\ell + \ell - 1 + \ell + 3\ell = \frac{8k - 8}{k}\ell - 1$. Since an optimal solution has at most $3\ell$ bins, we find that the asymptotic competitive ratio is at least $\frac{8(k - 1)}{4k}$.

In the case $k \geq 10$, we adapt the lower bound example of FF [1] by adding tiny items. The original construction was such that almost every bin of OPT (all bins except for a constant number of bins) had an item whose size was $\frac{1}{2} + \delta$. We replace those items with items of sizes $\frac{1}{2} + \frac{\varepsilon}{3}$, and add
claim that $k - 3$ tiny items of sizes $\frac{k}{3}$ to each such bin. All bins of OPT that had an item of size $\frac{1}{3} + \delta$ have at most three items each, so tiny items can be added to almost all bins of OPT, and this modification keeps those bins valid. The tiny items are presented to FF before other items, so they are packed into bins containing $k$ items each, that cannot be used for other items. The items of sizes $\frac{1}{3} + \frac{2}{3}$ are presented last and must be packed into dedicated bins, as any previous bin either has $k$ items or total size above $\frac{1}{2}$. Thus, the modified construction will give a lower bound of $1.7 + \frac{k-3}{k}$ on the asymptotic competitive ratio of FF. We describe the exact construction for completeness.

Let $\ell$ be a positive integer, such that $\ell - 1$ is divisible by $k$ and by $k - 3$, let $0 < \epsilon < \frac{1}{120}$ and $\delta < \frac{1}{3\ell}$ be small positive values, and consider the following input. The instance consists of $10(k - 3)(\ell - 1)$ tiny items of size $\frac{2}{3}$ and $30\ell$ larger items. We describe the items and the packing of FF, where the larger items are packed into $17\ell$ bins: $B_1, \ldots, B_{17\ell}$. The tiny items are presented first, and they are packed into $10(k - 3)\frac{\ell-1}{k}$ bins by FF.

The first $10\ell$ larger items are denoted by $a_{i,p}$ for $i = 1, \ldots, 10$ and $p = 1, \ldots, \ell$, and their sizes are defined as follows. The item $a_{i,p}$ has size $\frac{1}{3} + \frac{2}{3} - \delta$ for $1 \leq i \leq 3$, $\frac{1}{6} + \frac{2}{3} - 2\delta$ for $4 \leq i \leq 5$, $\frac{1}{6} - \frac{2}{3} - \delta$ for $6 \leq i \leq 7$, and $\frac{1}{6} - \frac{2}{3} - 2\delta$ for $8 \leq i \leq 10$. Note that all item sizes are in $(\frac{1}{3} - \epsilon, \frac{1}{3} + \epsilon)$ and the largest item has size $\frac{1}{6} + \frac{2}{3} - \delta < \frac{1}{3}$. Every ten items are packed into two bins as follows. The items $a_{1,p}, a_{2,p}, a_{3,p}, a_{6,p}, a_{7,p}$ are packed into one bin and $a_{4,p}, a_{5,p}, a_{8,p}, a_{9,p}, a_{10,p}$ are packed into another bin. We call these bins $B_{2p-1}$ and $B_{2p}$. The total size of items in $B_{2p-1}$ is $\frac{5}{6} + \frac{2}{3} - 10\delta > \frac{5}{6} + \frac{2}{3} - \frac{10\ell}{3\ell+1}$, since $10\delta < \frac{10\ell}{3\ell+1} < \frac{2\ell}{3\ell+1}$.

The next $10\ell$ items are denoted by $b_{i,p}$ for $i = 1, \ldots, 10$ and $p = 1, \ldots, \ell$. Their sizes are defined as follows. The size of $b_{i,p}$ for $1 \leq i \leq 5$ is $\frac{2}{3} + \frac{2}{3} - i\delta$ and for $6 \leq i \leq 10$ it is $\frac{1}{3} - \frac{2}{3} - (i - 5)\delta$.

There are $5\ell$ bins are created from these items, bins $B_{2p} + 5(p-1) + j$, where $1 \leq j \leq 5$ contains items $b_{j,p}$ and $b_{j+5,p}$. The total size of items in each such bin is $\frac{2}{3} + \frac{2}{3} - 2j\delta$. The least loaded bin has a load of $\frac{2}{3} + \frac{2}{3} - 10\delta > \frac{2}{3} + \frac{2}{3} - \frac{10\ell}{3\ell+1} > \frac{2}{3} + \frac{2}{3}$.

The last $10\ell$ items are denoted by $c_i$ for $i = 1, 2, \ldots, 10\ell$, each of these has size of $\frac{1}{3} + \frac{2}{3}$. These items are packed into the dedicated bins $B_{l+1}$ for $1 \leq j \leq 10\ell$.

Using the result of [14], the larger items can be packed into $10\ell + O(1)$ bins. We show that the tiny items can be combined into these bins, where every such bin receives $k - 3$ tiny items, as there will be three larger items packed into each bin. For $i = 1, \ldots, 5$ and $p = 1, \ldots, \ell$ there is a bin containing $\{a_{i,p}, b_{5i+1,p}, c_{5i(p-1)+1}\}$. For $i = 1, \ldots, 5$ and $p = 3, \ldots, \ell$, there is a bin containing $\{a_{3i+p-2}, b_{i,p}, c_{5(p-5)+3+p}\}$. This gives a total of $10\ell - 10$ bins, each having a level of at most $1 - \delta$, and thus $k - 3$ tiny items can be added to each bin. All tiny items are packed, and this leaves thirty unpacked items. The remaining items are packed into 12 additional bins: five bins containing $\{c_{10(l-1)+i}, b_{i,1}\}$ for $i = 1, \ldots, 5$, five bins containing $\{c_{10(l-1)+5+i}, b_{i,1}\}$ for $i = 1, \ldots, 5$, and two bins with five items each, a bin with $\{a_{6,\ell}, a_{7,\ell}, a_{8,\ell}, a_{9,\ell}, a_{10,\ell}\}$, and a bin with $\{a_{6,\ell-1}, a_{7,\ell-1}, a_{8,\ell-1}, a_{9,\ell-1}, a_{10,\ell-1}\}$.

The number of bins used by FF is $10(k - 3)(\ell - 1)/k + 17\ell$, while an optimal solution requires at most $10\ell + 2$ bins. Thus, the asymptotic competitive ratio is at least $2.7 - \frac{2}{3}$.

Next, we prove upper bounds. The next two lemmas will be used for all values of $k \geq 2$.

**Claim 6** Every bin of OPT has at most one item of a 1-bin of FF.

**Proof.** Assume by contradiction that this is not the case, and items $i, j$ of one bin of OPT are packed into 1-bins by FF. When $j$ arrives, since $s_i + s_j \leq 1$, FF does not open a new bin for $j$, as
there is at least one existing bin where it can be packed, a contradiction. ■

**Claim 7** Let $1 \leq j \leq k-1$. Every $j$-bin except for at most one bin has level above $\frac{j}{j+1}$. Moreover, every $j^+$-bin except for at most one bin has level above $\frac{j}{j+1}$.

**Proof.** Assume that there exists a $j$-bin (or $j^+$-bin) whose level is at most $\frac{j}{j+1}$. All further $j^+$-bins (that appear later in the ordering of FF) only have items of sizes above $\frac{j}{j+1}$, and each such bin has at least $j$ items, so their levels are above $\frac{j}{j+1}$. ■

We start with a simple proof that the upper bound $2.7 - 2.4/k$ on the competitive ratio of FF holds in the absolute sense, that is based on the proof of [16].

**Proposition 8** The absolute competitive ratio of FF is at most $2.7 - 2.4/k$.

**Proof.** Let $L$ be an input, and partition it into two subsequences, $L_1$ that consists of all items that are packed into bins eventually having $k$ items, and $L_2 = L \setminus L_1$. By the definition of FF, running it on $L_1$ results in the same bins for these items as in the run on $L$, and the same is true for $L_2$, even if FF is applied without taking the cardinality constraint into account. Let $M_1 = FF(L_1)$ and $M_2 = FF(L_2)$ be the resulting numbers of bins, where $FF(L) = M_1 + M_2$. We will use $OPT(L) \geq \frac{|L|}{k}$ and $|L_2| = |L_1| = kM_1$. Since the output for $L_2$ is valid without cardinality constraints, we have $FF(L_2) \leq 1.7OPT(L_2)$.

First, consider the case $M_2 \leq OPT(L)$. Since every bin of FF has at least one item, we have $|L_2| \geq M_2$, and therefore $M_1 + M_2 = M_1 + M_2/k + (1 - 1/k)M_2 \leq \frac{|L_1|}{k} + \frac{|L_2|}{k} + (1 - 1/k)M_2 = \frac{|L|}{k} + (1 - 1/k)M_2 \leq (2 - 1/k)OPT(L)$.

Otherwise, as the number of 1-bins is at most $OPT(L)$, and the remaining bins for $L_2$ have at least two items, thus, $|L_2| \geq M_2 + (M_2 - OPT(L))$, and we get $M_1 + M_2 \leq M_1 + 2M_2/k + (k - 2)M_2/k \leq \frac{|L_1|}{k} + |L_2|/k + OPT(L)/k + (k-2)M_2/k = |L|/k + OPT(L)/k + (k-2)/k \cdot 1.7OPT(L_2) \leq OPT(L) + OPT(L)/k + (1.7k - 3.4)OPT(L)/k = (2.7 - 2.4/k)OPT(L).$ ■

**Claim 9** For every input $\sigma$ for FF there exists an input $\sigma'$ for FF that contains the same items (possibly in a different order), $FF(\sigma) = FF(\sigma')$, $OPT(\sigma) = OPT(\sigma')$, and in the output of FF for $\sigma'$, all $k$-bins appear before all bins that are not $k$-bins, and all 1-bins appear after all $2^+$-bins.

**Proof.** Given $\sigma$, and the output for FF for it, we remove all items of $k$-bins and of 1-bins of FF from $\sigma$, we append all items of 1-bins at the end of the input in some order, and we insert all items of $k$ bins in the beginning of the input, in the same order as they appear in $\sigma$. This defines $\sigma'$. Obviously $OPT(\sigma') = OPT(\sigma)$. The items of $k$-bins are packed for $\sigma'$ exactly as they are packed for $\sigma$, as all the items of $k$-bins are presented in the same order. Afterwards, the items of $2^+$-bins are packed exactly as for $\sigma$, since no further item can be packed into a bin already containing $k$ items. Finally, since no two items of 1-bins can be packed into a bin together, and they cannot join $k$-bins, it remains to show that no such item can join a $2^+$-bin. Let $B$ be a $2^+$-bin, and let $i$ be an item of a 1-bin $B'$. If $B$ appears earlier than $B'$ in the ordering of FF (applied on $\sigma$), then when item $i$ is presented, it cannot be packed into $B$ (which at that time contains a subset of the items that $B$ receives). If $B$ appears later than $B'$ in the ordering of FF (applied on $\sigma$), then no item of $B$ can be packed with $i$ into a bin, and obviously $i$ cannot be added to $B$. ■

By Claim[9] in what follows we will only analyze inputs where the condition of the claim for $\sigma'$ holds.
3.1 Analysis of the absolute competitive ratio for the cases $k = 2, 3, 4$

We start with the simple case $k = 2$. A simple upper bound of \(\frac{3}{2}\) is achieved by a greedy matching algorithm, which is a generalization of FF. It is folklore that this algorithm matches at least half of the edges that an optimal solution can match and therefore it translates into a \(\frac{3}{2}\)-competitive algorithm for bin packing (where an edge between two items exists if they can be packed together into a bin). Moreover, for this case the upper bound \(2.7 - 2.4/k\) is equal to 1.5. For completeness, and as an introductory case for analysis using weights, we show how FF can be analyzed using weights for the case $k = 2$. The usage of weights is slightly different from their usage for proving lower bounds. We usually use a weight function $w$, that is applied on sizes of items. Thus, we define $w(a)$ for $a \in (0, 1]$, where the variable $a$ denotes the size of an item. For a set of items $A$ and a set of bins $\mathcal{A}$, let $w(A)$ and $w(\mathcal{A})$ denote the total weight of all items of $A$ or $\mathcal{A}$. Furthermore, let $W = w(I)$ be the total weight of all items of the input $I$. In this kind of analysis, the weights of bins of the algorithm and of OPT are compared, using the property that for a fixed input, the total weight of bins of equal for all algorithms. An item of an $i$-bin of FF is assigned a weight of \(\frac{1}{i}\) (for $i = 1, 2$). Obviously, any bin of FF has weight 1, and we analyze the total weight of bins of OPT. A bin of OPT cannot have two items of 1-bins, and therefore its weight cannot exceed \(\frac{3}{2}\).

We find that for any input $L$, the total weight satisfies $FF(L) = W \leq 1.5OPT(L)$.

3.1.1 The case $k = 3$

In this section we show that the absolute competitive ratio of FF for $k = 3$ is exactly \(\frac{11}{6} < 2\) (and that the asymptotic competitive ratio of FF is also equal to this value).

**Theorem 10** The absolute approximation ratio of FF for $k = 3$ is at most \(\frac{11}{6}\).

**Proof.** Next, let $I$ be an input sequence of items. Recall that it can be assumed without loss of generality that 3-bins are positioned in the beginning of the output, while 1-bins are positioned in the end of the output. Thus, the output is sorted by the numbers of items in the bins. Restricting our attention to the 2-bins and 1-bins we can see that these bins would have been created by running FF only on the subsequence of the items packed into them, even if the cardinality constraint is not taken into account. Thus, as in [4], it can be assumed that no 2-bin contains two items that are packed together in an optimal solution, since merging them into one item would result in the same packing (both for the application of FF on the original input and for the application of FF on the items of 2-bins and 1-bins). Moreover, if the number of 1-bins is $OPT(I)$, then no bin of the optimal solution contains two items that are packed into 2-bins (as in [4]). For completeness, we prove this property. Consider two 2-bins $B$ and $B'$ (where $B'$ appears later than $B$ in the ordering). Let $i_1$ and $i_2$ be the items of $B$, and let $i_3$ be the item of $B'$. Assume that $i_2$ and $i_3$ are packed into the same bin of $OPT(I)$. Let $i_4$ be the item of that bin of $OPT(I)$ that is packed into a 1-bin of $FF(I)$ and $i_5$ is the item of a 1-bin of $FF$ that is packed with $i_1$ in $OPT(I)$. We find $s_{i_3} + s_{i_1} + s_{i_2} > 1$ and $s_{i_4} + s_{i_5} > 1$, as $i_3$ was not packed into $B$, and $i_4, i_5$ are packed into 1-bins (the item out of $i_4$ and $i_5$ that arrives later was not packed with the other item out of these two items). On the other hand, $s_{i_3} + s_{i_2} + s_{i_4} \leq 1$ and $s_{i_1} + s_{i_5} \leq 1$. We have $2 < s_{i_1} + s_{i_2} + s_{i_3} + s_{i_4} + s_{i_5} \leq 2$, a contradiction.

We split the analysis into cases.

**Case 1.** The number of 1-bins is $OPT(I)$. An optimal solution has at most one such item in each bin, and thus every bin of OPT contains such an item. Additionally it can contain at most
one item packed into a 2-bin by FF. We define a weight function based on the packing of FF. An item packed into an i-bin has weight $\frac{i}{2}$. We find that any bin of the optimal solution has weight of at most $1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$.

**Case 2.** The number of 1-bins is at most $OPT(I) - 1$. In this case there exists at least one bin of the optimal solution that does not contain an item packed into a 1-bin by FF. We define slightly different weights in this case. An item packed into an i-bin by FF, where $i \neq 2$ has weight $\frac{1}{i}$. For the 2-bins, we define weights as a function of the sizes of items. An item of size above $\frac{1}{2}$ (called a big item) has weight $\frac{2}{3}$, an item of size in ($\frac{1}{4}, \frac{1}{2}$) (called a medium item) has weight $\frac{1}{2}$, and an item of size in (0, $\frac{1}{4}$] (called a tiny item) has weight $\frac{1}{3}$. Note that there is at most one item packed into a 1-bin whose size does not exceed $\frac{1}{2}$. If such an item exists, then we call it the special item. If its size at most $\frac{1}{4}$, then we say that the special item is small, and otherwise it is large. Note that the weight of a bin of OPT that does not have an item of an 1-bin of FF has a total weight of at most $\frac{11}{6}$.

Obviously, the only bins whose total weights can be below 1 are 2-bins. We claim that there exists at most one 2-bin whose total weight is strictly below 1, and if there exists a special item and it is small, then no such bin can exist. Consider a 2-bin that contains a big item. The weight of the big item is $\frac{2}{3}$, and the weight of the other item is at least $\frac{1}{3}$. Consider a 2-bin that has level above $\frac{3}{4}$. If the bin contains a big item, then we are done. Otherwise, since both the items packed in it have sizes of at most $\frac{1}{2}$, none of them can be tiny, and each one of the items has weight $\frac{1}{2}$. Assume now that there are two 2-bins, each of total weight below 1. Each such bin contains no big items, and at most one medium item. Thus, each such bin contains a tiny item. This contradicts the action of FF as the tiny item that is packed into the bin that appears later in the ordering could be packed into the bin that appears earlier in the ordering. If there is a special item that is tiny, and there exists a 2-bin with level at most $\frac{3}{4}$, then the special item could be packed there by FF contradicting its action.

Let $W$ denote the total weight. We split the analysis further.

**Case 2.1** There is no special item. We calculate the total weight. There is at most one 2-bin of weight below 1, and this bin still has weight of at least $\frac{2}{3}$ (as it has two items). Thus, $W \geq (FF(I) - 1) + \frac{3}{2} = FF(I) - \frac{1}{3}$. Every bin of the optimal solution that has an item of a 1-bin has an item of weight 1 and size above $\frac{1}{2}$. As there is no special item, and it can have only one further item of weight above $\frac{1}{4}$, and if it exists, then this item must have size in ($\frac{1}{4}, \frac{1}{2}$] and weight $\frac{1}{2}$. Thus, the total weight of the bin is at most $\frac{11}{6}$. A bin that only has items of 3-bins and 2-bins cannot have an item of weight 1. It can have at most one item of weight $\frac{2}{3}$, in which case the total weight of the two additional items is at most $\frac{1}{2} + \frac{3}{2} = \frac{1}{2}$ (bins with less than three items can only have smaller total weights). If the bin does not have an item of weight $\frac{2}{3}$, then the total weight is at most $\frac{1}{2}$ as well, since the weight of each item is at most $\frac{1}{2}$. We find $W \leq \frac{11}{6}(OPT(I) - 1) + \frac{3}{2} = \frac{11}{6}OPT(I) - \frac{1}{3}$.

We found that $FF(I) \leq \frac{11}{6}OPT(I)$ holds in this case.

**Case 2.2** There is a small special item. In this case $W \geq FF(I)$. The total weights of bins of the optimal solution not containing the special item are as computed before. Recall that the special item is an item of a 1-bin, thus there exists at least one bin whose total weight is at most $\frac{3}{2}$. The weight of the bin containing the special item can be larger by $\frac{2}{3}$ compared to a bin containing an item of the same size that is not the special item, but not containing an item of a 1-bin. Therefore, $W \leq \frac{11}{6}(OPT(I) - 2) + 2 \cdot \frac{2}{3} + \frac{2}{3} = \frac{11}{6}OPT(I)$. We found that $FF(I) \leq \frac{11}{6}OPT(I)$ holds in this case as well.
Case 2.3 There is a large special item. In this case, if there exists a 2-bin of FF of level at most \(\frac{3}{4}\), still its level is above \(\frac{1}{2}\), as otherwise FF could pack the special item in this bin. At least one of its two items is medium, and the total weight of the bin is at least \(\frac{5}{6}\). Therefore, \(W \geq FF(I) - \frac{1}{6}\). The total weights of bins of the optimal solution not containing the special item are as computed before. As before, the special item is an item of a 1-bin, thus there exists at least one bin whose total weight is at least \(\frac{5}{6}\). Therefore, \(W \geq FF(I) - \frac{1}{6}\).

The total weights of bins of the optimal solution not containing the special item are as computed before. As before, the special item is an item of a 1-bin, thus there exists at least one bin whose total weight is at most \(\frac{3}{2}\). The weight of the bin containing the special item can be larger by \(\frac{1}{2}\) compared to a bin containing an item of the same size that is not the special item, but not containing an item of a 1-bin. Therefore, \(W \leq FF(I) - \frac{1}{6} + 2 \cdot \frac{3}{2} = \frac{11}{6}OPT(I) - \frac{1}{6}\). We found that \(FF(I) \leq \frac{11}{6}OPT(I)\) holds in this case as well.

3.1.2 The case \(k = 4\)

We prove that \(FF\) is 2-competitive in the absolute sense for \(k = 4\). We define weights as follows. A large item, i.e. any item whose size exceeds \(\frac{1}{2}\) has weight 1. A medium item, i.e., an item of size in \((\frac{1}{4}, \frac{1}{2}]\) has weight \(\frac{1}{2}\). A small item, i.e., an item of size at most \(\frac{1}{4}\) has weight \(\frac{1}{4}\). Recall that the total weight of the item is denoted by \(W\).

**Lemma 11** The weight of any bin of \(OPT\) is at most 2.

**Proof.** Consider a bin \(B\) of \(OPT\). Bin \(B\) can contain at most one large item. If \(B\) does not contain a large item, then the weight of any item is at most \(\frac{1}{2}\), and since \(|B| \leq 4\), the total weight is at most 2. Suppose now that \(B\) contains a large item. Out of the remaining (at most) three items, at most one item can be medium, and the total weight is at most \(1 + \frac{1}{2} + 2 \cdot \frac{1}{4} = 2\), and the claim follows. ■

**Claim 12** Every bin that has a large item has total weight of at least 1. Every 4-bin has total weight of at least 1. Every 2+-bin that does not have any small items has total weight of at least 1. Every bin that has items of total size above \(\frac{3}{4}\) has total weight of at least 1.

**Proof.** The first property holds as the weight of a large item is 1. The second property holds as the weight of any item is at least \(\frac{1}{4}\). The third claim holds as two medium items have total weight of 1. Finally, we prove the last claim. The total size of items of a 2-bin that does not have a large item but has a small item is at most \(\frac{3}{4}\). If a 3-bin only has small items, then the total size of its items is at most \(\frac{3}{4}\). A 3-bin that has at most two small items has an item of weight at least \(\frac{1}{2}\), so its total weight is at least 1. ■

**Claim 13** Given an input \(L\), the total weight of the input items is at least \(FF(L) - \frac{3}{4}\).

**Proof.** We calculate the total weight based on the packing of FF. Every dedicated bin, except for possibly one such bin, has a large item, thus there is at most one dedicated bin whose weight is below 1 (and it is at least \(\frac{1}{2}\)). If all 2+-bins have weights of at least 1, then we are done as all 2+-bins, all 4-bins, and all 1-bins except for possibly one bin (that has weight of at least \(\frac{1}{2}\)) have total weights of at least 1.

Otherwise, there must be a 2+-bin of level of at most \(\frac{3}{4}\), that does not have a large item. Consider the first such bin according to the ordering of FF, and call it \(B\). This bin must have a small item. All items that are packed into bins that appear later in the ordering of FF can only be medium or large. Thus, all further 2+-bins have total weights of at least 1, and if there is a
1-bin that does not have a large item, then it must have a medium item. Moreover, if $B$ has level of at most $\frac{1}{2}$, then all dedicated bins have large items. In this last case, all bins except for $B$ have weights of at least 1, while $B$ has weight of at least $\frac{1}{2}$. Otherwise, if the level of $B$ is in $\left(\frac{1}{2}, \frac{2}{3}\right]$, then we have the following cases. If $B$ is a 2-bin, then it has one medium item and one small item, and its weight is $\frac{2}{3}$. If $B$ is a 3-bin, then it has weight of $\frac{3}{4}$ as well (as its weight is below 1, and a 3-bin has weight of at least $\frac{3}{4}$). Thus, all bins except for $B$ and possibly one 1-bin with a medium item have total weights of at least 1, while these two bins have weights of at least $\frac{3}{4}$ and $\frac{1}{2}$, respectively, and the claim follows. ■

We have $W \leq 2 \cdot \text{OPT}(L)$ and $W \geq FF(L) - \frac{3}{4}$. Thus, $\text{FF}(L) - 2 \cdot \text{OPT}(L) \leq \frac{3}{4}$, which implies (by integrality) $\text{FF}(L) \leq \text{OPT}(L)$.

### 3.2 The case $k = 5$

We analyzed the cases $k = 2, 3, 4$, and it remains to analyze the asymptotic competitive ratio for the cases $k \geq 5$, which are more complicated. In particular, items that are packed in $k$-bins will be treated separately. These items are called $\alpha$-items, and the weight of every such item will be equal to $\frac{1}{k}$ in all remaining cases. Often these items will be analyzed together with very small items. Items that are not $k$-items will be called additional items. Thus, for $k = 5$, the weight of any $\alpha$-item is $\frac{1}{5}$, and we define weights and types for the additional items as follows. Recall that the variable $a$ denotes the size of an item.

$$w(a) = \begin{cases} 
1/5 & \text{if } a \leq 1/6, \text{ in this case the item is tiny} \\
4/15 & \text{if } 1/6 < a \leq 1/4, \text{ in this case the item is small} \\
7/15 & \text{if } 1/4 < a \leq 1/3, \text{ in this case the item is medium} \\
8/15 & \text{if } 1/3 < a \leq 1/2, \text{ in this case the item is big} \\
1 & \text{if } 1/2 < a \leq 1, \text{ in this case the item is huge}
\end{cases}$$

We will show that the weight of any bin of $\text{OPT}$ is at most $32/15$, while the weight of any bin of $\text{FF}$ is at least 1, except for a constant number of special bins.

**Lemma 14** For every bin $B$ of $\text{OPT}$, $w(B) \leq 32/15$ holds.

**Proof.** Bin $B$ can contain at most one huge item. Assume first that $B$ contains a huge item. If it also contains a big item, then every remaining item is either tiny or an $\alpha$-item, that is, an item of weight $\frac{1}{5}$. The total weight is therefore at most $1 + \frac{8}{15} + 3 \cdot \frac{1}{5} = \frac{32}{15}$. If $B$ does not contain a big item, then it can have at most two items that are medium or small, out of which at most one can be medium, and the remaining items have weights of $\frac{1}{5}$. In this case the total weight is at most $1 + \frac{7}{15} + \frac{7}{15} + 2 \cdot \frac{1}{5} = \frac{32}{15}$.

If $B$ does not contain a huge item, then it can contain at most three items of sizes above $\frac{1}{4}$, out of which at most two can have sizes above $\frac{1}{4}$, and the remaining items have weights of at most $\frac{4}{15}$. The total weight is at most $2 \cdot \frac{8}{15} + \frac{7}{15} + 2 \cdot \frac{1}{15} = \frac{31}{15}$. ■

**Lemma 15** The total weight of $j$-bins of $\text{FF}$ for a given input $L$ is at least $\text{FF}(L) - 4$.

**Proof.** As $k$-bins always have weight 1, it is left to consider $j$-bins for $1 \leq j \leq 4$, which contain additional items, and thus their weights are according to $w$. We remove all bins containing total
weight at least 1 from the considered set of bins, and we will prove that at most four bins are left. As FF acts in the same way on subsequences where the complete sets of items of a subset of bins is given, this will prove the claim. Any bin containing a huge item has weight of at least 1, and thus no such bins remain. Since all 1-bins except for at most one bin have huge items, at most one 1-bin remained, and if there is such a bin, then it must appear last. Assume by contradiction that at least five bins have remained, denote the $\ell$th bin by $B_\ell$. The first four bins are 2+-bins.

In the next three claims we consider the possible contents of 2+-bins that have total weights below 1 together with lower bounds on total sizes of items of such bins.

**Claim 16** A 2-bin $B$ such that $w(B) < 1$ has items of total size at most $\frac{3}{4}$.

**Proof.** Assume by contradiction that the total size of items is above $\frac{3}{4}$. At least one of the items of $B$ must be big, as otherwise the total size is at most $\frac{1}{2}$. The second item must be either medium of big, as otherwise the total size is at most $\frac{3}{4}$. Thus, the total weight is at least 1, a contradiction. □

**Claim 17** Bins $B_1$ and $B_2$ have total sizes of items above $\frac{3}{4}$.

**Proof.** Assume by contradiction that the claim does not hold. The further bins cannot have items of sizes in $(0, \frac{1}{4}]$ as such an item could be packed into one of $B_1$, $B_2$. A 3+-bin with medium and big items has a weight above 1, thus $B_3$ and $B_4$, that are 2+-bins must be 2-bins. None of them can have a big item, since the total weight of big item and a medium item is 1. Thus, $B_3$ and $B_4$ are two medium items each. However, the first medium item of $B_4$ could be packed into $B_3$, which is a contradiction.

We find that $B_1$ and $B_2$ are 3+-bins.

**Claim 18** The total size of the items of $B_1$ is at most $\frac{5}{6}$.

**Proof.** Assume by contradiction that the total size of items is above $\frac{5}{6}$. We analyze the contents of a 3+-bin with items of total size above $\frac{5}{6}$ and weight below 1.

If $B_1$ is a 4-bin, then it cannot contain an item of size above $\frac{1}{4}$, as in such a case the total weight is at least $\frac{7}{15} + 3 \cdot \frac{1}{3} > 1$. It cannot have at least two tiny items, since the total size of two tiny items and two small items is at most $\frac{5}{6}$. However, the total weight of three small items and another small or tiny item is at least 1. Thus $B_1$ is a 3-bin. The total size of three small items is at most $\frac{3}{4}$, thus $B_1$ has a medium or big item. It cannot have more than one such item, as in this case the total weight is at least $2 \cdot \frac{7}{15} + \frac{1}{3} > 1$. Since the total size of a medium item and two small items is at most $\frac{5}{6}$, we find that $B_1$ has a big item and two items that are small or tiny. Bin $B_1$ has at most one tiny item as the total size of a big item and two tiny items is at most $\frac{5}{6}$. The total weight of a big item, a small item, and another small or tiny item, is at least 1. We reached a contradiction.

We find that the further bins do not have tiny items (as a tiny item could be packed into $B_1$). Thus, $B_2$ is a 3-bin, as a 4-bin with no tiny items has a weight of at least $\frac{16}{15}$, while $w(B_2) < 1$.

**Claim 19** The total size of the items of $B_2$ is below $\frac{3}{4}$.

**Proof.** Assume by contradiction that the total size of items is above $\frac{3}{4}$. As $B_2$ is a 3-bin, it must have at least one item that is not small, while the total weight of a medium item and two small items is 1. □

We have reached a contradiction, and thus the lemma is proved. □

We found that $FF(L) \leq W \leq \frac{32}{15} OPT(L) + 4$ for any input $L$. 15
Theorem 20  The asymptotic approximation ratio of FF for $k = 5$ is at most $\frac{32}{13}$.

3.3 The cases $k = 6, 7, 8$

In this case the definitions of the different types of additional items remain the same, but the weights of such items are defined differently. The weight of any huge additional item is 1. Next, we consider the remaining items, i.e., the additional items with sizes at most $1/2$. The weight $w(a)$ of any additional item of size $a \leq \frac{1}{2}$ consists of three parts. The first part is the ground weight, the second part is the scaled size, and the third part is the bonus. Each part is non-negative. The ground weight of any item of size $a$, is $g(a) = 1/k$. This ensures that the weight of any item (no matter how small it is) is at least $1/k$. The scaled size of an additional item of size $a \leq 1/2$, is defined by $s(a) = \frac{2(2k-11)}{3k}a$.

The bonus of an item of size $a$, denoted by $b(a)$ is defined as follows.

$$
\begin{align*}
    b(a) &= \begin{cases}
        0 & \text{if } a \leq 1/6 \quad \text{(tiny)} \\
        \frac{2(2k-11)}{3k}(a - \frac{1}{4}) + \frac{10-k}{2k} & \text{if } 1/6 < a \leq 1/4 \quad \text{(small)} \\
        \frac{2(2k-11)}{3k}(a - \frac{1}{4}) + \frac{3}{2k} & \text{if } 1/4 < a \leq 1/3 \quad \text{(medium)} \\
        \frac{2}{k} & \text{if } 1/3 < a \leq 1/2 \quad \text{(big)}
    \end{cases}
\end{align*}
$$

Note that $b(a)$ (and therefore also $w(a)$) is a piecewise linear function. The value of the bonus is zero if $a \leq 1/6$, and the bonus is constant $(2/k)$ for $a \in (1/3, 1/2]$. It is monotonically non-decreasing for $a \in (0, 1/2]$. The weight of an additional item of size $a \leq 1/2$, is $w(a) = g(a) + s(a) + b(a)$. The weight function has the discontinuity points, $1/6, 1/4, 1/3$, and $1/2$ (this is not exactly the same set of discontinuity points are the weight function of FF and standard bin packing [14]).

The bonus of a medium item is at least $\frac{3}{2k}$ and at most $\frac{4(2k-11)}{3k} + \frac{10-k}{3k} = \frac{k+8}{9k} < \frac{2}{k}$, and the bonus of a small item is at least $\frac{10-k}{9k}$ and at most $\frac{4(2k-11)}{3k} + \frac{7-k}{3k} = \frac{1}{2k} < \frac{3}{2k}$.

3.3.1 Properties of the weighting and the asymptotic bound

Lemma 21  For every bin $B$ of OPT, $w(B) \leq \frac{8(k-1)}{3k}$ holds.

Proof.  We will assume that all items packed into $B$ are additional items, as an additional item has larger weight than an $\alpha$-item of the same size.

Case 1:  $B$ contains no huge item. The bin can contain at most $k$ items, thus the total ground weight is at most 1. Similarly, the total scaled size is at most $\frac{2(2k-11)}{3k}$. Thus it remains to bound $b(B)$, it suffices to show that the total bonus of the items in the bin is at most $b(A) \leq \frac{8k-8}{3k} - 1 - \frac{2(2k-11)}{3k} = \frac{k+14}{3k}$.

If there are two big items in the bin, there can be at most one further item with a positive bonus, and $b(A) \leq 3 \cdot \frac{2}{k} \leq \frac{k+14}{3k}$, for $k \geq 4$. If there is only one big item in the bin, there can be at most three further items having positive bonuses. Then $b(A) \leq \frac{3}{k} + 3 \cdot \frac{k+8}{9k} = \frac{k+14}{3k}$. Now suppose that any item of $B$ has size at most 1/3. There can be at most five items in the bin having positive bonuses, and there can be at most three medium items among them. Thus the total bonus is at most $b(A) \leq 3 \cdot \frac{k+8}{9k} + 2 \cdot \frac{1}{2k} = \frac{k+11}{3k}$.

Case 2:  $B$ contains a huge item. Recall that the weight of the huge item is 1, and its size is bigger than $\frac{1}{2}$. There can be at most $k-1$ further items in the bin, their total ground weight
is at most $\frac{k-1}{3k}$, and their total scaled size is at most $\frac{2(2k-11)}{3k} \cdot \frac{1}{2}$, thus it suffices to show that the total bonus of the further items in the bin is at most $\frac{8k-8}{3k} - 1 - \frac{k-1}{k} - \frac{2k-11}{3k} = \frac{2}{k}$. The total size of remaining items is below $\frac{1}{7}$, thus the bin can contain at most two items with positive bonuses. Moreover, if $B$ contains only one item with a positive bonus, then this bonus is at most $\frac{2}{7}$, and we are done. Otherwise, if it contains two items of positive bonuses, none of them can be big, and at least one of them is small. If both are small, then their total bonus is at most $\frac{1}{k}$. We are left with the case that $B$ contains items of sizes $a_1$ and $a_2$ where $\frac{1}{6} < a_1 \leq \frac{1}{4} < a_2 \leq \frac{1}{3}$. Then applying $a_1 + a_2 < \frac{1}{2}$, we get that the total bonus is

$$\frac{2(2k-11)}{3k}a_1 + \frac{7-k}{3k} + \frac{2(2k-11)}{3k}a_2 + \frac{10-k}{3k} \leq \frac{2(2k-11)}{3k} \cdot \frac{1}{2} + \frac{17-2k}{3k} = \frac{2}{3k}. \quad \blacksquare$$

Now, we find a lower bound on the total weight of the bins created by FF for an input $L$ and a given $k \in \{6, 7, 8\}$. The total weight of 1-bins is at least their number minus 1, as all 1-bins except possibly one bin have huge items. The total weight of $k$-bins is exactly their number. We will show that for each one of the four sets: 2-bins, 3-bins, 4-bins, and 5+-bins, the total weight of items packed into bins of this set is at least the number of such bins minus 2 (for 5+-bins it is at least their number minus 1). This will show that $W \geq FF(L) - 8$. Since the weight of every bin that contains a huge item is at least 1, we can restrict the analysis to bins that do not contain such items, and for $2 \leq j \leq k-1$ we will only consider $j$-bins that have no huge items.

Claim 22 Every 5+-bin of level above $\frac{5}{6}$ has weight of at least 1, and the total weight of 5+-bins is at least their number minus 1.

Proof. All 5+-bins, except for at most one bin, have levels above $\frac{5}{6}$. Consider a $j$-bin $A$ where $5 \leq j \leq k-1$. The ground weight of its items is $\frac{j}{k}$, and their scaled size is at least $\frac{5}{6} \cdot \frac{2(2k-11)}{3k}$. If $j = 5$, then at least one item has a positive bonus (otherwise the total size is at most $\frac{2}{3}$), and the weight of the bin is $w(A) = g(A) + s(A) + b(A) \geq \frac{5}{6} + \frac{5}{6} \cdot \frac{2(2k-11)}{3k} + \frac{10-k}{9k} = 1$, since the value of any positive bonus is at least $\frac{10-k}{9k}$. Otherwise, $k \geq 6$, so the ground weight is at least $\frac{2}{3}$, and we are done since $\frac{1}{k} \geq \frac{10-k}{9k}$. Since there is at most one 5+-bin whose level is at most $\frac{5}{6}$, and all 5+-bins with level above $\frac{5}{6}$ have weights of at least 1, we find that the total weight of 5+-bins is at least their number minus 1. ■

It remain to consider only the 2-bins, 3-bins, and 4-bins. For all of these cases we consider two subcases. We will show that if the level of a bin is sufficiently large (above $\frac{3}{4}$ for 2-bins, and above $\frac{5}{6}$ otherwise), then the total weight of the bin is at least 1. Then, we will consider $j$-bins of smaller levels for for $j = 2, 3, 4$.

Lemma 23 Consider a 2-bin of level above $\frac{3}{4}$, the weight of the bin is at least 1.

Proof. The bin must have a big item and another item that is either medium or big. The ground weight is $\frac{2}{3}$, and the scaled size is at least $\frac{2(2k-11)}{2k} \cdot \frac{3}{4}$. The total bonus is at least $\frac{2}{k} + \frac{3}{2k} = \frac{7}{2k}$. The total weight is therefore at least $\frac{2}{k} + \frac{2k-11}{2k} + \frac{7}{2k} = \frac{4+2k-11+7}{2k} = 1$. ■

Lemma 24 Let $j \in \{3, 4\}$. Consider a $j$-bin of level above $\frac{5}{6}$, the weight of the bin is at least 1.
Proof. The scaled size of the bin is at least \( \frac{2(2k-11)}{3k} \cdot \frac{5}{6} = \frac{10k-55}{9k} = 1 + \frac{k-55}{9k} \). For \( k = 3 \), the bin has ground weight \( \frac{2}{3} \), and for \( k = 4 \), the bin has ground weight \( \frac{4}{5} \). If the bin has at least two items of sizes above \( \frac{1}{3} \), then their combined bonuses are at least \( \frac{3}{k} \), and the total weight is at least \( 1 + \frac{k-55}{9k} + \frac{2}{3} + \frac{2}{k} = 1 + \frac{k-1}{9} > 1 \). Similarly, if a bin has a big item and at least one other item with a positive bonus, their combined bonuses are at least \( \frac{2}{k} + \frac{10-k}{9k} = \frac{28-k}{9k} \), and the total weight is at least \( 1 + \frac{k-55}{9k} + \frac{3}{k} + \frac{28-k}{9k} = 1 \). The remaining cases are considered separately for \( k = 3 \) and \( k = 4 \).

For \( k = 3 \), a bin that has a big item and two tiny items has level of at most \( \frac{5}{6} \), and a bin that has a medium item and two small items also has level of at most \( \frac{5}{6} \). Thus there are no additional cases and we are done. For \( k = 4 \), a bin that has a big item has weight of at least \( 1 + \frac{k-55}{9k} + \frac{4}{k} + \frac{2}{k} = 1 + \frac{k-1}{9} > 1 \). We are left with the case where \( k = 4 \), the bin has no big items, and it has at most one medium item. If the bin has one medium item, then (since the size of a medium item and three tiny items is at most \( \frac{5}{6} \)), it must have also one small item and their total bonus is at least \( \frac{3}{2k} + \frac{10-k}{9k} = \frac{47-2k}{18k} \), and the bin has weight of at least \( 1 + \frac{k-55}{9k} + \frac{4}{k} + \frac{47-2k}{18k} = 1 + \frac{1}{2k} > 1 \). Finally, if it has no medium items, then it must have at least three small items. These three items must have combined total size of at least \( \frac{2}{3} \), and their total bonus is at least \( \frac{2(2k-11)}{3k} + \frac{3(7-k)}{3k} = \frac{19-k}{9k} \). In this case the weight of the bin is at least \( 1 + \frac{k-55}{9k} + \frac{4}{k} + \frac{19-k}{9k} = 1 \). 

Lemma 25 The total weight of the 2-bins of levels in \( \left( \frac{2}{3}, \frac{3}{4} \right] \) is at least their number minus 1.

Proof. Consider two consecutive 2-bins of levels in \( \left( \frac{2}{3}, \frac{3}{4} \right] \), \( B_i \) and \( B_j \) (these bins become consecutive if we remove all other kinds of bins from the list of bins). We prove that \( g(B_i) + s(B_i) + b(B_j) \geq 1 \). Let the level of \( B_i \) be \( 2/3 + x \) with some \( 0 < x \leq 1/12 \). There are two items in \( B_j \), their sizes are above \( 1/3 - x \geq \frac{1}{k} \) (since they were not packed into \( B_i \)), and moreover one of them must be big (and the other one is either medium or big). Their total bonus is at least \( \frac{2}{k} + \frac{2(2k-11)}{3k} \cdot (1/3 - x) + \frac{10-k}{3k} = \frac{26+k-6x(2k-11)}{9k} \). We get

\[
\frac{g(B_i) + s(B_i) + b(B_j)}{\frac{2}{k} + \frac{2(2k-11)}{3k}} \cdot (1/3 - x) + \frac{10-k}{3k} = 1.
\]

The number of pairs \( i, j \) that are considered is the number of considered bins minus 1 and the claim follows. 

Since there is at most one 2-bin whose level is at most \( \frac{2}{3} \), and all 2-bins with level above \( \frac{3}{4} \) have weights of at least 1, we find that the total weight of 2-bins is at least their number minus 2.

Lemma 26 The total weight of the 3-bins of levels in \( \left( \frac{3}{4}, \frac{5}{6} \right] \) is at least their number minus 1.

Proof. Suppose that \( B_i \) and \( B_j \) are two consecutive 3-bins. We prove that \( g(B_i) + s(B_i) + b(B_j) \geq 1 \). Let the level of \( B_i \) be \( 3/4 + x \) with some \( 0 < x \leq 1/12 \). Then there are three items in \( B_j \), of sizes \( a_1 \geq a_2 \geq a_3 \), such that all are bigger than \( 1/4 - x \), and in particular, all are bigger than \( 1/6 \). At least one of them must be also bigger than \( 1/4 \), otherwise the level of the bin is at most \( 3/4 \).

We have \( g(B_i) + s(B_i) = \frac{3}{k} + \frac{2(2k-11)}{3k} \cdot \left( \frac{3}{4} + x \right) = \frac{2k-5}{2k} + \frac{2x(2k-11)}{3k} \). Thus, it is sufficient to show \( b(B_j) \geq \frac{5}{2k} - \frac{2x(2k-11)}{3k} \). This holds if \( a_2 > \frac{1}{3} \), as the bonus of a medium or big item is at least \( \frac{3}{2k} \). If none of the items is big, using \( a_1 + a_2 + a_3 \geq \frac{3}{4} \) we have \( b(B_j) \geq \frac{2(2k-11)}{3k} \cdot (a_1 + a_2 + a_3) + 2 \cdot \frac{7-k}{3k} + \frac{10-k}{3k} \geq \frac{2(2k-11)}{3k} \cdot \frac{3}{2} + \frac{24-3k}{3k} = \frac{5}{2k} \).

We are left with the case that \( a_1 > \frac{1}{3} \), and \( \frac{1}{4} - x < a_3 \leq \frac{1}{4} \). The bonus of each small item is at least \( \frac{2(1/4-x)(2k-11)}{3k} + \frac{7-k}{3k} \), and \( b(B_j) \geq \frac{2}{k} + 2 \cdot \frac{2(2k-11)}{3k} \left( \frac{1}{4} - x \right) + \frac{7-k}{3k} = \frac{3}{k} - \frac{2x(2k-11)}{3k} = \)
\[ \frac{5}{2k} - \frac{2x(2k-11)}{3k} + \frac{1}{2k} - \frac{2x(2k-11)}{3k}, \] where by using \( x \leq \frac{1}{2} \), we get \( \frac{1}{2k} - \frac{2x(2k-11)}{3k} \geq \frac{1}{2k} - \frac{2k-11}{18k} = \frac{20-2k}{18k} \geq 0. \]

Since there is at most one 3-bin whose level is at most \( \frac{3}{4} \), and all 3-bins with level above \( \frac{5}{6} \) have weights of at least 1, we find that the total weight of 3-bins is at least their number minus 2.

**Lemma 27** The total weight of the 4-bins of levels in \( \left( \frac{4}{5}, \frac{5}{6} \right) \) is at least their number minus 1.

**Proof.** Suppose that \( B_i \) and \( B_j \) are two consecutive such 4-bins. We prove that \( g(B_i) + s(B_i) + b(B_j) \geq 1 \). Let the size of \( B_i \) be \( 5/6 - x \) with some \( 0 \leq x \leq 1/30 \). Then there are four items in \( B_j \), of sizes \( a_1 \geq a_2 \geq a_3 \geq a_4 \), all are bigger than \( 1/6 + x \). If any of them is also bigger than 1/4, we have
\[
g(B_i) + s(B_i) + b(B_j) \geq \frac{4}{k} + \frac{4}{5} \cdot \frac{2(2k-11)}{3k} + \frac{3}{2k} = \frac{32k-11}{30k} \geq 1,
\]
since \( k \geq 6 \) and we are done. Otherwise all four items are small. Note that the three biggest items among them have total size at least \( 3/4 \cdot 4/5 = 3/5 \), so the total size of all four of them is at least \( \frac{3}{5} + \frac{1}{6} + x = \frac{23}{30} + x \) and the total bonus is at least \( \frac{2(2k-11)}{3k} \left( \frac{23}{30} + x \right) + \frac{4(7-k)}{3k} \). Thus we have
\[
g(B_i) + s(B_i) + b(B_j) \geq \frac{4}{k} + \frac{2(2k-11)}{3k} \left( \frac{5}{6} - x \right) + \frac{2(2k-11)}{3k} \left( \frac{23}{30} + x \right) + \frac{4(7-k)}{3k} = \frac{40-4k}{3k} + \frac{2(2k-11)}{3k} \left( \frac{5}{6} - x + \frac{23}{30} + x \right) = \frac{40-4k}{3k} + \frac{2(2k-11)}{3k} 8 \geq 1,
\]
since \( k \leq 8 \). Since there is at most one 4-bin whose level is at most \( \frac{4}{5} \), and all 4-bins with level above \( \frac{5}{6} \) have weights of at least 1, we find that the total weight of 4-bins is at least their number minus 2.

We proved \( FF(L) - 8 \leq W \leq (8/3 - 8/(3k))OPT(L) \).  

**Theorem 28** The asymptotic approximation ratio of FF for any \( 6 \leq k \leq 8 \) is at most \( 8/3 - 8/(3k) \).

### 3.4 The case \( k = 9 \)

We consider this case separately, as the proof for smaller and larger \( k \) fails in this case. We combine methods from all other proofs here. The weighting function is similar to that is used in the previous section, in the sense that it has discontinuity points at 1/6 and 1/3. It is also similar to the weighting used in the next section as the intervals for small and medium sizes are divided to two parts. In this section we introduce a new method that was not used in the previous cases. We distinguish the bins of \( OPT \) according to the number of additional items packed into them. Since \( \alpha \)-items always have weights of \( \frac{1}{k} \), bins that contain \( k \) such items will still have weights of exactly 1. Thus, in the analysis we can assume that there is at least one additional item in each bin of \( OPT \).

**Case a.** Consider bins of \( OPT \) containing one or two additional items (and the remaining items are \( \alpha \)-items). Such bins are called \( \gamma \)-bins, and the additional items packed into such bins (in \( OPT \)) are called \( \gamma \)-items. The largest \( \gamma \)-item of such a bin is called a \( \gamma_1 \)-item (breaking ties arbitrarily). Any \( \gamma_1 \)-item has weight 1. If the bin contains another \( \gamma \)-item, this item is called a \( \gamma_2 \)-item, and its weight is defined to be \( \frac{16}{27} \).

**Case b.** Consider the other bins of \( OPT \) (containing at least three additional items). Each such bin has at most six \( \alpha \)-items, we call it a \( \phi \)-bin, and its items that are not \( \alpha \)-items are called \( \phi \)-items.
The weighting function of the $\phi$-items is more complicated. The weight of any huge $\phi$-item (i.e. a $\phi$-item with size strictly above $1/2$) is exactly 1. The weight of a $\phi$-item of size $a \leq 1/2$ is $w(a) = s(a) + b(a)$, where $s(a) = \frac{32}{27}a$ is called the scaled size, and $b(a)$ is the bonus of the item. Note that there is no ground weight in this case. Below we give the bonus function of the $\phi$-items of sizes no larger than $1/2$. The functions $b(a)$ and $w(a)$ are piecewise linear, and breakpoints where it is continuous are $1/5$ and $3/10$.

$$b(a) = \begin{cases} 
0 & \text{if } a \leq 1/6 \quad \text{(tiny)} \\
\frac{32}{27}(a - \frac{1}{6}) + \frac{1}{81} = \frac{32}{27}a - \frac{5}{27} & \text{if } 1/6 < a \leq 1/5 \quad \text{(very small)} \\
-\frac{28}{27}(a - \frac{1}{5}) + \frac{7}{135} = -\frac{28}{27}a + \frac{7}{27} & \text{if } 1/5 < a \leq 1/4 \quad \text{(larger small)} \\
-\frac{28}{27}(a - \frac{1}{4}) + \frac{7}{9} = -\frac{28}{27}a + \frac{10}{27} & \text{if } 1/4 < a \leq 3/10 \quad \text{(smaller medium)} \\
\frac{32}{27}(a - \frac{3}{10}) + \frac{8}{135} = \frac{32}{27}a - \frac{8}{27} & \text{if } 3/10 < a \leq 1/3 \quad \text{(larger medium)} \\
\frac{1}{5} & \text{if } 1/3 < a \leq 1/2 \quad \text{(big)} 
\end{cases}$$

The value of the bonus is zero if $a \leq 1/6$ and it is constant ($\frac{1}{7}$) between $1/3$ and $1/2$. The bonus function is not continuous at the points $1/6$, $1/4$, and $1/3$. The bonus function is monotonically increasing in $(1/6, 1/5)$ and in $(3/10, 1/3)$. It is monotonically decreasing in $(1/5, 1/4)$ and in $(1/4, 3/10)$ (which is less typical for weight functions). Nevertheless, the weight function remains monotonically increasing for the whole interval $0 < a \leq 1/2$, and the value of the bonus is nonnegative for the whole interval.

Next, we state several additional properties of the bonus function. For small items (very small and larger small items, i.e., items of sizes in $(1/6, 1/4]$), the maximum value of the bonus for very small items is given for $a = 1/5$, and the bonus at this point is $7/135$. The bonus of very small items is at least $\frac{1}{27}$, and the smallest bonus of larger small items is zero. For smaller medium items, the bonus decreases from $\frac{1}{27}$ to $\frac{5}{135}$, and for larger medium items, the bonus increases from $\frac{8}{135}$ to $\frac{8}{27}$. The weight of a big $\phi$-item is at least $41/81$, the weight of a $\phi$-item with size more than $1/4$ is at least $11/27$, and for any $\phi$-item with size $0 < a \leq 1$, and for any $\gamma_2$-item, the next inequality holds: $w(a) \geq \frac{32}{27}a > \frac{7}{6}a$. This is true since bonuses of $\phi$-items are non-negative, and since the size of any $\gamma_2$-item is at most $\frac{1}{2}$ (as the bin of $OPT$ that contains it has another item of at least the size of the $\gamma_2$-item) while its weight is $\frac{16}{27}$.

### 3.4.1 Properties of the weighting and the asymptotic bound

**Lemma 29** For every bin $B$ of $OPT$, $w(B) \leq 8/3 - 8/27 = 2 + \frac{10}{27} = \frac{64}{27}$ holds.

**Proof.** Consider the case that $B$ is a $\gamma$-bin. In this case $B$ has one item of weight 1, possibly an item of weight $\frac{16}{27}$, and each remaining item is an $\alpha$-item and has weight $\frac{1}{6}$. Thus, the total weight is at most $1 + \frac{16}{27} + \frac{1}{6} = \frac{64}{27}$. It remains to consider the case that $B$ is a $\phi$-bin. It contains at most six $\alpha$-items, of total weight at most $6/9$. Thus it suffices to show that the total weight of the $\phi$-items is at most $\frac{64}{27} - \frac{6}{9} = \frac{46}{27}$.

First, assume that $B$ contains no huge item. The total scaled size of the $\phi$-items is at most $32/27$. It suffices to show that the total bonus of $\phi$-items the bin is at most $46/27 - 32/27 = 14/27$. Since the bonus is zero if the size of the item is at most $1/6$, it follows that at most five items can have positive bonuses. Moreover at most three items can have size above $1/4$, and the bonus of
each such item is at most $\frac{1}{7}$, and the bonus of any other item is at most $7/135$. Thus the total bonus of the bin is at most $3 \cdot \frac{1}{7} + 2 \cdot \frac{7}{135} = 59/135 < 14/27$.

Next, assume that $B$ contains a huge item. The weight of a huge item is exactly $1$. We will show that if there are six $\alpha$-items, then the total weight of the further additional items of $B$ is at most $46/27 - 1 = 19/27$, and consider also the case that the number of $\alpha$ items is smaller. Since the total size of remaining additional items is below $\frac{1}{27}$, their scaled size is at most $\frac{10}{27}$, and it suffices to show that their total bonus is at most $19/27 - 16/27 = 1/9$. Since only items of size above $\frac{1}{9}$ have positive bonuses, there can be at most two further items in the bin having positive bonuses. If there is only one further item having positive bonus, we are done, since no bonus is above $\frac{1}{9}$. If there are two items with bonuses, but there are at most five $\alpha$-items, then the total weight of $\alpha$-items is at most $\frac{5}{9}$, and we are done as well. Thus, it is left to consider the case where there are two further $\phi$-items in the bin both having positive bonuses, and there are no other $\phi$-items packed into $B$ except for the huge item and these two items. Let their sizes be denoted as $a_1$ and $a_2$, where $1/6 < a_1 \leq a_2$, and thus $a_2 < 1/3$ as $a_1 + a_2 < 1/2$. The claim holds if $a_2 \leq 1/4$, since then the total bonus is at most $2 \cdot 7/135 = \frac{14}{135} < \frac{1}{9}$. Thus the only remaining case is where the item of size $a_1$ is small and the item of size $a_2$ is a medium item. We will show $w(a_1) + w(a_2) \leq \frac{19}{27}$. There are three cases, as $a_1 > \frac{1}{3}$ and $a_2 > 0.3$ cannot hold simultaneously. In all cases $s(a_1) + s(a_2) = \frac{32}{27} (a_1 + a_2)$ and $a_1 + a_2 < \frac{1}{2}$.

If $a_1 \leq \frac{1}{3}$ and $a_2 \leq \frac{5}{18}$, $w(a_1) + w(a_2) = \frac{32}{27}(a_1 + a_2) + \frac{32}{27}a_1 + \frac{28}{27}a_2 + \frac{20}{27} = \frac{64}{27}a_1 + \frac{4}{27}a_2 + \frac{20}{27} \leq \frac{64}{27} \cdot \frac{5}{18} + \frac{4}{27} \cdot \frac{5}{18} + \frac{20}{27} \leq \frac{10}{27}$. If $a_1 > \frac{1}{3}$ and $a_2 \leq \frac{5}{18}$, $w(a_1) + w(a_2) = \frac{32}{27} (a_1 + a_2) - \frac{28}{27} (a_1 + a_2) + \frac{20}{27} = \frac{4}{27} (a_1 + a_2) + \frac{20}{27} \leq \frac{10}{27}$. If $a_1 \leq \frac{1}{3}$ and $a_2 > \frac{5}{18}$, $w(a_1) + w(a_2) = \frac{32}{27} (a_1 + a_2) + \frac{32}{27} (a_1 + a_2) - \frac{13}{27} = \frac{64}{27} (a_1 + a_2) - \frac{13}{27} \leq \frac{10}{27}$.

Now, we will analyze the total weight of the bins of FF. Once again, we split the analysis according to the number of items in these bins. The 9-bins that have weight 1. Moreover, every item of size above $\frac{1}{9}$ packed into any bin of FF that is not a 9-bin is either a huge $\phi$-item, or it is a $\gamma$-item, in which case this must be the largest item of its bin of OPT, i.e., it is a $\gamma_1$-item, and its weight is 1. Thus, we neglect all bins containing items of size above $\frac{1}{9}$ from the analysis. At most one 1-bin is left, and we neglect that bin (if it exists) as well. In what follows we analyze 2+-bins of FF that contain items of sizes in $(0, \frac{1}{9}]$. These bins only contain $\phi$-items, and $\gamma_2$-items.

**Lemma 30** The weight of any bin with level above $\frac{6}{7}$ is at least 1. There is at most one 6+-bin whose weight is below 1.

**Proof.** For any $\phi$-item or for any $\gamma_2$-item, the weight of the item is at least $\frac{32}{27}$ times the size of the item. Since except for at most one bin, the level of 6+-bin is at most $\frac{6}{7}$, the weights of these bins, except for at most one bin, are at least 1.

In the following we concentrate on the 2-bins, 3-bins, 4-bins and 5-bins. We start with analyzing bins containing a $\gamma_2$-item. Note that a bin that has two $\gamma_2$-items has weight above 1, and thus we consider bins that contain one $\gamma_2$-item and the remaining items are $\phi$-items.

**Lemma 31** The weight of any 2-bin that has a $\gamma_2$-item is at least 1, except for at most one such bin. The weight of any 3-bin that has a $\gamma_2$-item is at least 1, except for at most two such bins. The weight of any bin of 4 or 5-bin that has a $\gamma_2$-item is at least 1, except for at most one such bin.

**Proof.** Any bin that has a $\gamma_2$-item and a $\phi$-item of size above $\frac{1}{9}$ has total weight of at least 1, since any $\phi$-item of size above $\frac{1}{9}$ has weight at least $\frac{11}{27}$, and each $\gamma_2$-item has weight $\frac{16}{27}$. Moreover,
We get level of at most \( \frac{32}{27} \) times its size, if the level of the bin is at least \( \frac{27}{32} \), then the total weight is at least 1. In each one of the cases we assume that there exist two bins, called \( B_i \) and \( B_j \), were \( B_i \) appears earlier than \( B_j \) in the ordering of FF, each having a \( \gamma_2 \)-item, and each one of these two bins has weight below 1 (and thus a level below \( \frac{27}{32} \)), all its \( \phi \)-items have sizes of at most \( \frac{1}{4} \), and the sizes of items of \( B_j \) are above \( \frac{1}{3} \) (since none of them could not be packed into \( B_j \)).

Bins \( B_i, B_j \) cannot be 2-bins; in such a case \( B_i \) has one \( \gamma_2 \)-item (of size at most \( \frac{1}{2} \)) and one \( \phi \)-item (of size at most \( \frac{1}{4} \), so its level is at most \( \frac{1}{1} \), and in this case \( B_j \) cannot have an item of size at most \( \frac{1}{4} \). Thus, there cannot be a pair \( i, j \) that are both 2-bins.

For 4-bins and 5-bins, \( B_j \) has at least three \( \phi \)-items of sizes at least \( \frac{5}{32} \), whose total weight is at least \( 3 \cdot \frac{32}{27} \cdot \frac{5}{32} = \frac{5}{9} \), so the weight of \( B_j \) is above 1, a contradiction. Thus, there cannot be a pair \( i, j \), each of which containing four or five items.

For 3-bins, we assume that there is a third 3-bin \( B_r \) that appears after \( B_j \) in the ordering, where \( B_r \) has a \( \gamma_2 \)-item and weight below 1 as well. We split the analysis to the cases where \( B_j \) has level above \( \frac{5}{6} \), and the case that it does not. If it has level above \( \frac{5}{6} \), then the total size of its \( \phi \)-items is above \( \frac{1}{3} \) (as the \( \gamma_2 \)-item has size of at most \( \frac{1}{2} \)), and at least one of them has size above \( \frac{1}{4} \). If at least one of the \( \phi \)-items has size above \( \frac{1}{3} \), then their total size is above \( \frac{5}{32} + \frac{1}{3} = \frac{57}{160} \), and the weight of all items is at least \( \frac{32}{27} \cdot \frac{57}{160} > 1 \). Otherwise, there is an item that has a bonus of at least \( \frac{1}{81} \), and the total weight is at least \( \frac{32}{27} + \frac{27}{9} \cdot \frac{1}{3} + \frac{1}{4} = 1 \). Therefore, we find that \( B_j \) has a level of at most \( \frac{5}{6} \). However, in this case the two \( \phi \)-items of \( B_r \) have sizes above \( \frac{1}{4} \). The weight of such an item is at least \( \frac{32}{27} \cdot \frac{1}{6} + \frac{1}{81} = \frac{17}{81} \), and the total weight of \( B_r \) is at least \( \frac{16}{27} + 2 \cdot \frac{17}{81} > 1 \), a contradiction.

We are left with bins containing only \( \phi \)-items of sizes at most \( \frac{1}{4} \), that are 2-bins, 3-bins, 4-bins, and 5-bins. Before analyzing their total weights we discuss some properties of \( \phi \)-items.

**Lemma 32** Consider two \( \phi \)-items of sizes \( a_1 \leq a_2 \leq 1/2 \). If \( 1 \geq a_1 + a_2 > 1 - a_1 \) holds, then the total weight of the two items is at least 1.

**Proof.** We have \( a_1 > \frac{1-a_2}{2} > \frac{1}{4} \). If \( a_1 > \frac{1}{4} \), then both items are big, and since the weight of any big item is at least \( \frac{41}{81} > 1/2 \), the claim holds in this case. Next, we consider the case \( 1/4 < a_1 \leq 1/3 \), where \( a_2 > 1 - 2a_1 \geq 1/3 \), thus the item of size \( a_2 \) is big.

If \( a_1 \) is smaller medium, then let \( a_1 = 1/4 + x \) for some \( 0 < x \leq 1/20 \), and \( a_1 + a_2 > 1 - a_1 = \frac{3}{4} - x \).

The total weight of the items is

\[
\frac{32}{27} (a_1 + a_2) + b(a_1) + b(a_2) \geq \frac{32}{27} (3/4 - x) - \frac{28}{27} x + \frac{1}{9} + \frac{1}{9} = \frac{10}{9} - \frac{20}{9} x \geq 1.
\]

If \( a_1 \) is larger medium, then let \( a_1 = 3/10 + x \) for some \( 0 < x \leq 1/30 \), and \( a_1 + a_2 > 1 - a_1 = \frac{7}{10} - x \).

We get

\[
\frac{32}{27} (a_1 + a_2) + b(a_1) + b(a_2) \geq \frac{32}{27} (7/10 - x) + \frac{32}{27} x + \frac{8}{135} + \frac{1}{9} = 1.
\]

**Lemma 33** Consider three \( \phi \)-items of sizes \( a_1 \leq a_2 \leq a_3 \leq 1/2 \). If \( 1 \geq a_1 + a_2 + a_3 > 1 - a_1 \) holds, then the total weight of the three items is at least 1.

**Proof.** We have \( \frac{3}{7} (a_1 + a_2) \geq 2a_1 + a_2 > 1 - a_3 \geq \frac{1}{7} \), thus \( a_1 + a_2 > \frac{1}{3} \).
If \( a_1 > \frac{1}{4} \), then the claim holds since the weight of an item with size above 1/4 is at least 11/27 (so the total weight is at least \( \frac{11}{27} \)). In what follows we assume that \( a_1 \leq 1/4 \), and thus \( a_1 + a_2 + a_3 > 1 - a_1 \geq \frac{3}{4} \). If the largest item is big, then its bonus is \( \frac{1}{9} \), and the total weight of the three items is at least \( \frac{32}{27} (a_1 + a_2 + a_3) + \frac{1}{9} \geq \frac{32}{27} \cdot \frac{3}{4} + \frac{1}{9} = 1 \). If the largest item is not big, i.e., \( a_3 \leq \frac{1}{9} \), then using \( \frac{2}{9} \geq 2a_3 \geq a_2 + a_3 > 1 - 2a_1 \geq \frac{1}{4} \) we get \( a_1 > \frac{1}{6} \) and \( a_3 > \frac{1}{9} \), thus the smallest item is small, and the largest item is medium. If there are two medium items, then the bonus of each one of them is at least \( \frac{8}{135} \), and the total weight is at least \( \frac{32}{27} \cdot \frac{3}{4} + \frac{16}{135} = \frac{136}{135} > 1 \). We are left with the case where the smallest two items are small and the largest item is medium. We find that \( 2a_1 > 1 - a_2 - a_3 \geq 1 - \frac{4}{9} - \frac{3}{9} = \frac{2}{9} \), thus the smallest item is larger small, and so is the item of size \( a_2 \).

If the largest item is smaller medium, we have a total weight of \( \frac{32}{27} (a_1 + a_2 + a_3) - \frac{28}{27} (a_1 + a_2 + a_3) + \frac{24}{27} \geq \frac{2}{9} \cdot \frac{3}{4} + \frac{24}{27} = 1 \). If the largest item is larger medium, we have \( a_1 + a_2 > \frac{2}{9} (1 - a_3) \). The total weight is at least \( \frac{32}{27} (a_1 + a_2 + a_3) - \frac{28}{27} (a_1 + a_2) + \frac{14}{27} + \frac{32}{27} a_3 - \frac{8}{27} = \frac{1}{27} (a_1 + a_2) + \frac{64}{27} a_3 + \frac{8}{27} \geq \frac{4}{27} \cdot \frac{2}{3} (1 - a_3) + \frac{64}{27} a_3 + \frac{8}{9} = \frac{154}{27} a_3 + \frac{26}{81} \geq \frac{154}{27} \cdot \frac{3}{10} + \frac{26}{81} = \frac{406}{406} > 1 \).

**Lemma 34** Consider four \( \phi \)-items of sizes \( a_1 \leq a_2 \leq a_3 \leq a_4 \leq 1/2 \). If \( 1 \geq a_1 + a_2 + a_3 + a_4 > 1 - a_1 \) holds, then the total weight of the four items is at least 1.

**Proof.** First, consider the case \( a_1 \leq \frac{1}{6} \). In this case \( a_1 + a_2 + a_3 + a_4 > 1 - a_1 \geq \frac{5}{6} \), and if at least one item is very small or medium, there is at least one item with a bonus of at least \( \frac{1}{8} \), and the total weight is at least \( \frac{32}{27} \cdot \frac{5}{6} + \frac{1}{8} = 1 \). Otherwise, all items are larger small and tiny. At least three items must be larger small as \( 2a_1 + a_2 + a_3 + a_4 > 1 \), which is impossible in the case where \( a_2 \leq \frac{1}{10} \) and \( a_4 \leq \frac{1}{4} \). The total weight if all four items are larger small is at least \( \frac{4}{27} \cdot (a_1 + a_2 + a_3 + a_4) + \frac{28}{27} > 1 \). Otherwise, the total weight is at least \( \frac{32}{27} a_1 + \frac{14}{27} (a_2 + a_3 + a_4) + \frac{21}{27} > \frac{32}{27} a_1 + \frac{14}{27} (1 - 2a_1) + \frac{21}{27} = \frac{1}{27} a_1 + \frac{25}{27} \). Since the three largest items are larger small, \( a_2 + a_3 + a_4 \leq \frac{4}{3} \), and \( 2a_1 > 1 - (a_2 + a_3 + a_4) \geq \frac{1}{4} \), so \( a_1 > \frac{1}{8} \). We find that the total weight is at least \( \frac{28}{27} > 1 \).

Next, consider the case \( a_1 > 1/6 \). It follows that the total weight of the three smallest items is at least \( \frac{32}{27} \cdot \frac{7}{6} = \frac{16}{9} \). If the biggest item is bigger than 1/4, the total weight is at least \( \frac{16}{9} + \frac{11}{27} = 1 \). Then it is left to consider only the case where \( 1/6 < a_1 \leq a_2 \leq a_3 \leq a_4 \leq 1/4 \), i.e., all items are small.

Note that the weight of a larger small item is at least \( \frac{32}{27} \cdot \frac{1}{6} + \frac{7}{135} = \frac{13}{35} \). If all four items are larger small, then their total weight is above 1. Otherwise, the smallest item is very small. The total size of the items is above \( 1 - a_1 \), and the bonus of the smallest item is at least \( \frac{8}{15} a_1 - \frac{5}{27} \). The total weight of the four items is at least \( \frac{32}{27} (1 - a_1) + \frac{32}{27} a_1 - \frac{5}{27} = 1 \).

**Lemma 35** Consider five \( \phi \)-items of sizes \( a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq 1/2 \). If \( 1 \geq a_1 + a_2 + a_3 + a_4 + a_5 > 1 - a_1 \) holds, then the total weight of the five items is at least 1.

**Proof.** If \( a_1 \leq \frac{1}{6} \), then \( a_1 + a_2 + a_3 + a_4 + a_5 > 1 - a_1 \geq \frac{5}{6} \) holds. Otherwise, \( a_1 + a_2 + a_3 + a_4 + a_5 \geq 5a_1 \geq \frac{5}{6} \) holds too. If at least one item is not larger small, then its bonus is at least \( \frac{1}{8} \) and the total weight is at least \( \frac{28}{27} \cdot \frac{7}{6} + \frac{11}{27} = 1 \). If all items are larger small, then their total size is above 1, contradicting the assumption.

**Lemma 36** The total weight of the 2-bins, 3-bins, 4-bins, and 5-bins, containing \( \phi \)-bins is at least their number minus 1.
Proof. Consider bins of FF whose numbers of items is in $[2, 5]$, that contain only $\phi$-items, and their weights are below 1. Obviously these bins have no huge items. If the level of a given bin is bigger than 1 minus the size of the smallest item in the bin, then the weight of the bin is at least 1 by the previous lemmas. Thus, we only consider bins that do not satisfy this property. If there is at most one bin to consider, then we are done. Otherwise, in the list of remaining bins, consider two consecutive bins $B_i$ and $B_j$ (such that $B_j$ appears after $B_i$ in the ordering). Let $i_1$ denote the smallest item of $B_i$ and $j_1$ the smallest item of $B_j$ (breaking ties in favor of items of smaller indices). Let $S = s(B_i)$. We will show that the total weight of the items of $B_i$ excluding $i_1$ together with the weight of $j_1$ is at least 1. Applying this property to every such consecutive pair of bins will show that the total weight is at least the number of bins in the list of remaining bins minus 1. If $S - s_{i_1} + s_{j_1} > 1$, then their total weight is above $\frac{32}{27} > 1$. Otherwise, we have the following properties. First, $s_{j_1} > 1 - S$ since $j_1$ was not packed into $B_i$. Additionally, by assumption, $S \leq 1 - s_{i_1}$. Therefore $s_{j_1} > s_{i_1}$. Finally, $(S - s_{i_1}) + 2s_{j_1} > S + s_{j_1} > 1$. Thus, the set of items of $B_i$ together with $j_1$ and excluding $i_1$ satisfies the condition of one of Lemmas $32, 33, 34, 35$ (the one where the considered number of items is equal to that of this set (which is equal to the number of items of $B_i$ and therefore it is in $\{2, 3, 4, 5\}$), and the total weight of this set is at least 1.

We proved $FF(L) - 7 \leq W \leq (64/27)OPT(L)$.

Theorem 37 The asymptotic approximation ratio of FF for $k = 9$ is at most 64/27.

3.5 The case $k \geq 10$

The case $k \geq 10$ is studied similarly to previous cases. In this case we also distinguish the definitions of weights based on the bins of $OPT$ according to the number of additional items packed into these bins. The weight of an $\alpha$-item remains $\frac{1}{4}$.

Case a. Consider bins of $OPT$ containing one or two additional items (and the remaining items are $\alpha$-items). Such bins are called $\gamma$-bins again, and the additional items in the bin are called $\gamma$-items. The largest $\gamma$-item of a bin is called a $\gamma_1$-item (breaking ties arbitrarily). Any $\gamma_1$-item has weight 1. If the bin contains another $\gamma$-item, this item is called a $\gamma_2$-item. If $10 \leq k \leq 19$, then the weight of the $\gamma_2$-item is $\frac{7}{10} - \frac{1}{k}$, and otherwise (if $k \geq 20$), then its weight is $\frac{13}{20} = 0.65$. The smallest weight of a $\gamma_2$-item is 0.6, and its size is at most $\frac{1}{2}$, therefore the ratio between the weight of such an item and its size is at least 1.2.

Case b. Consider the remaining bins of $OPT$ (containing at least three additional items). Each such bin has at most $k - 3$ $\alpha$-items, and we call it a $\phi$-bin. The items packed into $\phi$-bins that are not $\alpha$-items are called $\phi$-items. The weighting function of the $\phi$-items is again more complicated. The weight of any huge $\phi$-item (i.e., a $\phi$-item with size strictly above 1/2) is exactly 1. The weight of a $\phi$-item of size $a \leq 1/2$ is $w(a) = s(a) + b(a)$, where $s(a) = \frac{4}{a}$ is the scaled size, and $b(a)$ is the bonus of the item. Below we give the bonus function of the $\phi$-items of sizes no larger than 1/2.

For $k \geq 20$, the classical weighting function of FF [14] is appropriate, in this case the bonus function is defined as follows.

$$b(a) = \begin{cases} 0 & \text{if } a \leq 1/6 \quad \text{(tiny)} \\ \frac{2}{3}(a - \frac{1}{6}) = 0.6a - 0.1 & \text{if } 1/6 < a \leq 1/3 \quad \text{(small or medium)} \\ 0.1 & \text{if } 1/3 < a \leq 1/2 \quad \text{(big)} \end{cases}$$
The weight function in this case is continuous in the interval \((0, \frac{1}{7})\). The bonus is piecewise linear (and so is the weight function). In the interval \((\frac{1}{6}, \frac{1}{3})\), the bonus increases from 0 to 0.1.

For \(10 \leq k \leq 19\), we use additional modifications to the classic weight function. In some of the cases the bonus function is still equal to the one in the classic analysis. More specifically, these are the cases where the size is in \((1/6, 1/5)\) and \((3/10, 1/3)\). In these intervals the slope of the weight function is 1.8, i.e., the slope of the bonus function is 0.6. The bonus function and the weight function are piecewise linear, and continuous in \((0, \frac{1}{7})\) and \((\frac{1}{7}, \frac{1}{5})\). The partition into types is as in the case \(k = 9\).

\[
b(a) = \begin{cases} 
0 & \text{if } a = 0 \\
\frac{3}{5}(a - \frac{1}{6}) = 0.6a - 0.1 & \text{if } 0 < a \leq \frac{1}{6} \\
(1.6 - 20/k)(a - \frac{1}{6}) + \frac{1}{50} = (1.6 - 20/k)a - 0.3 + \frac{4}{k} & \text{if } \frac{1}{6} < a \leq \frac{1}{5} \\
(1.6 - 20/k)(a - \frac{1}{5}) + \frac{1}{10} = (1.6 - 20/k)a - 0.4 + \frac{6}{k} & \text{if } \frac{1}{5} < a \leq \frac{1}{4} \\
\frac{3}{5}(a - \frac{3}{10}) + \frac{8}{100} = 0.6a - 0.1 & \text{if } \frac{1}{4} < a \leq 3/10 \\
0.1 & \text{if } 3/10 < a \leq 1/3 \\
& \text{if } 1/3 < a \leq 1/2 
\end{cases}
\]

This bonus function is monotonically non-decreasing for \(k \geq 13\), but not in the cases \(k = 10, 11, 12\), whereas the resulting weight function is monotonically increasing for \(10 \leq k \leq 19\). The value of the bonus is zero for \(a \leq 1/6\) and it is 0.1 between 1/3 and 1/2. We have \(b(\frac{1}{5}) = 0.02\) (and \(w(\frac{1}{5}) = 0.26\)), \(b(\frac{1}{4}) = 0.1 - \frac{1}{k}\), thus for \(a \in (\frac{1}{5}, \frac{1}{4})\) the bonus is in \([0.1 - \frac{1}{k}, 0.02]\) for \(k = 10, 11, 12\) and in \((0.02, 0.1 - \frac{1}{k})\) for \(13 \leq k \leq 19\). For \(a \in (\frac{1}{4}, 0.3)\) the bonus is in \([0.08, \frac{1}{k})\) for \(k = 10, 11, 12\) and in \((\frac{1}{k}, 0.08)\) for \(13 \leq k \leq 19\) (we have \(w(0.3) = 0.44\)).

For \(k \geq 10\), since the bonus function is non-negative, for any \(\phi\)-item of size \(0 \leq a \leq \frac{1}{2}\), \(w(a) \geq \frac{6}{5}a\) holds. The bonus of every item of size in \((0, \frac{1}{2})\) is in \([0, 0.1]\). The weight of a big item is at least 0.3. The weight of a medium item is at least \(0.3 + \frac{1}{k} > 0.35\) for \(k \leq 19\) and at least 0.35 for \(k \geq 20\).

### 3.5.1 Properties of the weighting and the asymptotic bound

**Lemma 38** For every bin \(B\) of OPT, \(w(B) \leq 2.7 - 3/k\) holds.

**Proof.** The claim holds for bins having only additional items. For a \(\gamma\)-bin, if there is just one \(\gamma\)-item, then the total weight is at most \(\frac{k-1}{k} + 1 < 2\). Otherwise, if \(k \leq 19\), then the total weight is at most \(\frac{k-2}{k} + 1 + 0.7 - \frac{1}{k} = 2.7 - \frac{3}{k}\), and if \(k \geq 20\), then the total weight is at most \(\frac{k-2}{k} + 1 + 0.65 = 2.65 - \frac{2}{k} \leq 2.7 - \frac{3}{k}\).

Next, consider \(\phi\)-bins. For \(k \geq 20\), the proof follows from the standard analysis [14], and we include it for completeness. There are at most \(k - 3\) \(\alpha\)-items, and their total weight never exceeds \(\frac{k-3}{k}\). If a bin does not contain a huge \(\phi\)-item, then it has at most five \(\phi\)-items of positive bonuses (each bonus is at most 0.1), and their scaled size is at most 1.2. This gives a total weight of at most \(1 - \frac{3}{k} + 1.2 + 0.5 = 2.7 - \frac{3}{k}\). Note that this total weight cannot be achieved as both situations where there are \(k - 3\) \(\alpha\)-items and five \(\phi\)-items cannot occur simultaneously. If a bin contains a huge item, then there are at most two (other) \(\phi\)-items with positive bonuses. The scaled size of all \(\phi\)-items except for the huge item is at most 0.6, and the total weight excluding the bonuses of \(\phi\)-items is at most \(2.6 - \frac{2}{k}\). If there is only one \(\phi\)-item with a positive bonus, then the total weight is at most \(2.7 - \frac{3}{k}\). Assume that there are two items with positive bonuses. None of these items can be larger than \(\frac{1}{k}\), as their total size is below \(\frac{1}{2}\). If both items have bonuses of 0.6 times their sizes
minus 0.1, then their total bonus is at most \(0.6 \cdot \frac{1}{3} - 0.2 = 0.1\). In the case \(k \geq 20\), this is the only remaining option (as each of these items is small or medium), and we are left with the case \(k \leq 19\), and moreover, in the remaining case there are two items with positive bonuses, and these bonuses are not both equal to the sizes times 0.6 minus 0.1. Let \(a_1 \leq a_2\) be the sizes of the items. We have \(a_2 \in (0.2, 0.3]\) (otherwise either both items are very small, or the larger item is larger medium and the smaller one is very small, and both items have bonuses of the form 0.6 times the size minus 0.1, a case that was analyzed earlier). Thus, the larger item of the two is either larger small or smaller medium. We will bound the total weight of the two items and show that it does not exceed 0.7. Since the weight function is monotonically non-decreasing, we analyze \(w(a_2) + w(\frac{1}{2} - a_2)\). If \(\frac{1}{5} < a_2 \leq \frac{1}{3}\), then \(w(a_2) + w(\frac{1}{2} - a_2) = (2.8 - 20/k)\frac{1}{2} - 0.7 + \frac{10}{k} = 1.4 - 10/k - 0.7 + 10/k = 0.7\). The case \(\frac{1}{5} < a_2 \leq 0.3\) is symmetric. 

Now, we bound the total weight of the bins of FF. Once again we split the analysis into several cases according to the number of items packed into the bins. In this case we can also neglect \(k\)-bins and 1-bins, as the total weight of a \(k\) bin is 1, and all items of size above \(\frac{1}{2}\) are either huge \(\phi\)-items, or \(\gamma_1\)-items. Moreover, any bin that contains a huge \(\phi\)-item or a \(\gamma_1\)-item can be removed from the analysis. Thus, we are left with 2\(^+\)-bins that do not contain such items. Additionally, the weight of any bin with level at least \(5/6\) is at least 1, as the weight of any \(\phi\)-item and of a \(\gamma_2\)-item is at least \(6/5\times\) times the size of the item. Since there can be at most one \(5^+\)-bin whose level is below \(5/6\), the weight of any \(5^+\)-bin (except for at most one bin) is at least 1. In the following we concentrate on the 2-bins, 3-bins and 4-bins.

**Lemma 39** The weight of any 2-bin containing a \(\gamma_2\)-item is at least 1, except for at most one bin. The weight of any 3-bin or 4-bin, containing a \(\gamma_2\)-item, is at least 1, except for at most one bin.

**Proof.** Assume that at least two bins have \(\gamma_2\)-items, and each one has weight below 1. Denote them by \(B_i\) and \(B_j\) such that \(B_j\) appears after \(B_i\) in the ordering of FF. Each of them can have at most one \(\gamma_2\)-item, as the total weight of two \(\gamma_2\) items is above 1. None of them has a level of at least \(\frac{5}{6}\), as in such a case the weight is at least 1.

Assume that both these bins are 2-bins. The total weight of a \(\gamma_2\)-item and a \(\phi\)-item of size above \(\frac{1}{4}\) is at least \(0.35 + 0.65 = 1\) for \(k \geq 20\), and at least \(0.3 + \frac{1}{4} + 0.7 - \frac{1}{k} = 1\) for \(k \leq 19\). Thus, each of these 2-bins has a \(\phi\)-item of size at most \(\frac{1}{4}\) (as there is only one \(\gamma_2\)-item packed into each of the two bins). We find \(s(B_i) \leq \frac{3}{4}\), as the size of the \(\gamma_2\)-item is at most \(\frac{1}{2}\), and therefore \(B_j\) cannot have any item of size at most \(\frac{1}{4}\), a contradiction.

Next, assume that \(B_i\) and \(B_j\) contain 3 or 4 items each and have weights below 1, such that each of them contains one \(\gamma_2\)-item, and the other items are \(\phi\)-items. Similarly to the proof for 2-bins, none of them has a \(\phi\)-item of size above \(\frac{1}{4}\). If all the \(\phi\)-items of \(B_j\) have sizes of at least 1/6, then their total size is at least \(\frac{1}{6}\), and their total weight is at least \(6/5 \cdot \frac{1}{3} = 4/10\), and we reach a contradiction, since the \(\gamma_2\)-item of that bin has weight of at least 0.6. Otherwise, since \(B_j\) has an item of size below \(\frac{1}{6}\), the level of \(B_i\) is above 5/6, a contradiction. 

We are left with bins containing only \(\phi\)-items that are not huge.

**Lemma 40** Consider two \(\phi\)-items of sizes \(a_1 \leq a_2 \leq 1/2\). If \(1 \geq a_1 + a_2 > 1 - a_1\) holds, then the total weight of the two items is at least 1.

**Proof.** We have \(a_1 > (1 - a_2)/2 \geq \frac{1}{4}\). If both items have sizes at least 1/3, since \(w(1/3) = 1/2\), the claim holds, since \(w\) is monotonically non-decreasing. Thus it suffices to consider the case \(1/4 < a_1 \leq 1/3\). In this case \(a_2 > 1 - 2a_1 \geq \frac{1}{3}\). If \(k \geq 20\), then the total weight of the two items
is $1.2(a_1 + a_2) + (0.6a_1 - 0.1) + 0.1 = 1.8a_1 + 1.2a_2 = 0.9(2a_1 + a_2) + 0.3a_2 > 0.9 \cdot 1 + 0.3 \cdot \frac{1}{4} = 1$.

If $k \leq 19$, we consider two cases. If $a_1 > 0.3$, then the calculation is the same as for $k \geq 20$. Otherwise, the total weight of the two items is $1.2(a_1 + a_2) + (1.6 - 20/k)a_1 - 0.4 + \frac{2}{3}$, and since the total weight of the items is $1.2(a_1 + a_2) + 0.3 + 0.6/k = 1.2 > 1$, since $0.4 - 20/k < 0$ and $a_1 \geq 0.3$. □

Lemma 41 Consider three φ-items of sizes $a_1 \leq a_2 \leq a_3 \leq 1/2$. If $1 \geq a_1 + a_2 + a_3 > 1 - a_1$ holds, then the total weight of the three items is at least 1.

Proof. We have $4a_3 \geq 2a_1 + a_2 + a_3 > 1$, so $a_3 > \frac{1}{4}$. If $a_1 > \frac{1}{4}$, then the result holds since the total weight of an item with size above $1/4$ is at least 0.35. If $a_1 \leq \frac{1}{6}$, then $a_1 + a_2 + a_3 > \frac{5}{6}$, and the total weight is at least 1. Thus, we are left with the case $\frac{1}{6} < a_1 \leq \frac{1}{4}$, and thus $a_1 + a_2 + a_3 > \frac{3}{4}$.

If $a_3 > \frac{1}{3}$, then its bonus is 0.1, and the total weight of the three items is at least $\frac{2}{3} \cdot \frac{3}{4} + 0.1 = 1$. We are left with the case $\frac{1}{6} < a_1 \leq a_2 \leq a_3 \leq \frac{1}{3}$. We find that in the case $k \geq 20$, as all three items have sizes in $(\frac{1}{6}, \frac{1}{3})$, the total weight of the items is $1.8(a_1 + a_2 + a_3) - 0.3 > 1.8 \cdot \frac{2}{3} - 0.3 = 1.05 > 1$.

We are left with the case $k \leq 19$. If $a_2 > \frac{1}{4}$, then since the bonus of any item of size above $\frac{1}{4}$ is above $1/20$, the total weight of the items is at least $1.2 \cdot \frac{2}{3} + 2 \cdot 0.05 = 1$. If $a_1 \leq \frac{1}{6}$, then $a_1 + a_2 + a_3 > \frac{1}{4}$, and since the bonus of the largest item is above $1/20$, we get a total weight of at least $1.2 \cdot \frac{1}{4} + 0.05 = 1.01 > 1$. We are therefore left with the case $\frac{1}{6} < a_1 \leq a_2 \leq \frac{1}{4}$. If $a_3 \leq 0.3$, then the total weight is at least $2(2.8 - 20/k)\frac{3}{4} + 2(-0.3 + 4/k) + (-0.4 + 6/k) = 1.1 - 1/k \geq 1$.

If $a_3 > 0.3$, then the total weight is $(2.8 - 20/k)(a_1 + a_2) + 1.8a_3 + 2(-0.3 + 4/k) - 0.1 > (2.8 - 20/k)a_1 + 1.8a_3 + 0.7 + 8/k + (2.8 - 20/k)(1 - 2a_1 - a_3) = (20/k - 2.8)a_1 + (20/k - 1)a_3 + 2.1 - 12/k \geq (20/k - 2.8)/4 + (20/k - 1) \cdot 0.3 + 2.1 - 12/k = 1.1 - 1/k \geq 1$ since $k \geq 10$, $a_1 \leq \frac{1}{6}$, and $a_3 \geq 0.3$. □

Lemma 42 Consider four φ-items of sizes $a_1 \leq a_2 \leq a_3 \leq a_4 \leq 1$. If $1 \geq a_1 + a_2 + a_3 + a_4 > 1 - a_1$ holds, then the total weight of the four items is at least 1.

Proof. We have $5a_4 \geq 3a_1 + a_2 + a_3 + a_4 > 1$, so $a_4 > \frac{1}{5}$. If $a_1 \leq \frac{1}{5}$, then $a_1 + a_2 + a_3 + a_4 > \frac{5}{5}$, and the total weight is at least 1. Otherwise, we find $a_1 + a_2 + a_3 + a_4 \geq \max\{1 - a_1, 4a_1\} \geq \frac{4}{5}$.

If $a_4 > \frac{1}{5}$, then its bonus is above $1/20$, and the total weight is at least $1.2 \cdot \frac{4}{5} + 0.05 \geq 1$. Thus, $\frac{1}{5} < a_4 \leq \frac{1}{4}$. If $k \geq 20$, as the sizes of all items are in $(\frac{4}{5}, \frac{1}{2})$, the total weight of all four items is $1.8(a_1 + a_2 + a_3 + a_4) - 0.4 \geq 1.04 \geq 1$. We are left with the case $k \leq 19$. If $a_1 > \frac{1}{6}$, then the total weight of all four items is $(2.8 - 20/k)(a_1 + a_2 + a_3 + a_4) - 1.2 + 16/k \geq (2.8 - 20/k) \cdot 0.8 - 1.2 + 16/k = 1.04 > 1$. Otherwise, $\frac{1}{6} < a_1 \leq \frac{1}{5}$, $a_4 \leq \frac{1}{5}$, and the total weight is $1.8a_1 - 0.1 + (2.8 - 20/k)(a_2 + a_3 + a_4) - 0.9 + 8/k > 1.8a_1 + (2.8 - 20/k)(1 - 2a_1 - 12/k = a_1(40/k - 3.8) + 0.8 - 0.8/k$. If $40/k - 3.8$ is non-negative, then using $8/k \leq 0.8$ we find a total weight of at least 1. Otherwise, using $a_1 \leq \frac{1}{6}$, we find a total weight of at least $40/k - 3.8 \cdot \frac{1}{5} + 0.8 - 0.8/k = 1.04 > 1$. If $a_2 \leq \frac{1}{4}$, then the scaled size is 1.2$(a_1 + a_2 + a_3 + a_4) > 1.2(1 - a_1)$. Thus bonus of the two smallest items is $0.6(a_1 + a_2) - 0.2 \geq 0.2a_1 - 0.2$. Thus, the total weight is at least 1. □

Lemma 43 The total weight of the 2-bins, 3-bins and 4-boxes of FF that contain only φ-items is at least their number minus 1.

Proof. The proof is the same as for Lemma 36 (the only difference is that 5-bins are not considered).

□

We proved $FF(L) - 5 \leq W \leq (2.7 - 3/k)OPT(L)$.

Theorem 44 The asymptotic approximation ratio of FF for any $k \geq 10$ is at most $2.7 - 3/k$.
4 Other algorithms

4.1 A 2-competitive algorithm for all $k \geq 2$

Kotov et al. \cite{1} designed an algorithm that is 2-competitive in the asymptotic sense. We present a simplified version of that algorithm and prove that it is 2-competitive in the absolute sense. Our algorithm Thin and Fat (TF) has three kinds of bins.

1. Paired bins. Those are bins partitioned into pairs such that the total size of items for each pair is strictly above 1, and the total number of items packed into the two bins is at least $k$. Those bins will not be used for packing further items.
2. Fat bins. Those are bins containing exactly $k - 1$ items.
3. Thin bins. Those are non-empty bins containing at most $k - 2$ items.

After we define TF, we will prove that if it has at least one fat bin, then it has at most one thin bin. The algorithm acts as follows. Initially all three sets of bins are empty. Let $i \geq 1$ be a new item. The following steps are processed for $i$ until it is packed.

1. If there is a fat bin $B$ such that $s(B) + s_i > 1$, pack $i$ into a new bin, match $B$ and the new bin, these bins become paired.
2. If there are no thin bins, pack $i$ into a new bin.
3. If there exists a thin bin $B$ such that $s(B) + s_i \leq 1$, then pack $i$ into $B$. If $B$ becomes fat and there is a thin bin $B' \neq B$, match $B$ and $B'$, these bins become paired.
4. If there are no fat bins, pack $i$ into a new bin.
5. Pack $i$ into a fat bin $B$, match $B$ with a thin bin $B'$, these bins become paired.

Lemma 45 i. In all cases, the actions described above can be performed.

ii. Every two thin bins have total sizes above 1.

iii. Every two bins that are matched have sufficient total sizes and sufficient numbers of items.

iv. If there is at least one fat bin, then there is at most one thin bin.

Proof. We start with proving part i, i.e., we show that any item $i$ can be packed into the bin that it is assigned to. In steps 1, 2, and 4, the item is packed into a new bin. Step 3 is applied provided that $B$ exists. Such a bin has at most $k - 2$ items, and has sufficient space. Assume that step 5 is reached. Since $i$ is not packed in step 1, every fat bin can receive $i$ since it has $k - 1$ items and sufficient space. Since step 4 was not applied, a fat bin must exist. The only other action that is performed unconditionally is matching bins in step 5. The thin bin must exist as $i$ was not packed in step 2.

Next we consider part ii. Note that a thin bin can become fat, but a fat bin cannot become thin. Thus, a new thin bin is created only by packing an item into a new bin. The bin remains thin as long as it is not paired, and it has at most $k - 2$ items. Obviously, the total size packed into a bin cannot decrease over time. Thus, to prove this part, it is sufficient to prove that a pair of a thin bin and a thin bin that was just created and was not paired immediately have a total size of items above 1. A new bin $B'$ that it not paired immediately can be created in steps 2 and 4. In step 2 it becomes the only thin bin. In step 4, it is created since the item was not packed in step 3, thus for any existing thin bin $B$, $s(B) + s(B') = s(B) + s_i > 1$.

Consider part (iii). Bins are matched only in steps 1, 3, 5. In step 1, the two bins will have $k$ items and total size above 1. In step 3, the pair is created only if a thin bin $B$ becomes fat, i.e., $B$ now has $k - 1$ items and $B'$ has at least one item. Moreover, since $B$ and $B'$ were thin, their total...
Lemma 47. The current level of conditions (if no such bin exists, then it is assigned into an empty bin).

If $B$ currently contains four items, then after assigning $i$, the third condition must hold, and therefore the first condition does not hold, and the total size of items packed into the two bins exceeds 1. In step 5, since step 3 was not applied, we have $s(B') + s_i > 1$, and the total number of items is at least $k + 1$.

To verify the last property (part iv), consider the cases where $i$ is packed into a bin that is not paired immediately. In step 2, there will be a unique thin bin. In step 3, if $B$ becomes fat and it is not paired, then no thin bins remain. Otherwise, there is no change in the numbers of fat and thin bins. In step 4, there will be no fat bins after $i$ is packed. ■

**Theorem 46** For any input $L$, $TF(L) \leq 2 \cdot OPT(L)$.

**Proof.** Let $p$, $f$, and $t$ be the numbers of paired bins, fat bins, and thin bins when the algorithm terminates. Let $W$ be the total size of items, and $n$ their number.

Assume first that $f = 0$. If $t \leq 1$, and $p = 0$, then $TF(L) = t = OPT(L)$. Otherwise, if $p > 0$ (and $t \leq 1$), we find that $W > \frac{k}{2}$. Therefore $OPT(L) \geq W > \frac{k}{2}$ and $OPT(L) \geq \frac{k+1}{2}$. Thus, $p + f + t \leq p + 1 \leq 2OPT(L)$. If $t \geq 2$, since the total size of items of every pair of thin bins is above 1, we have $OPT(L) \geq W > \frac{k}{2} + \frac{2}{2}$, while $TF(L) = p + t$.

Otherwise, $f \geq 1$. In this case $t \leq 1$. We have $n \geq \frac{k}{2} \cdot k + (k - 1)f + t$. If $t = 0$, then we get $n \geq \frac{k}{2}(p + f) = \frac{k}{2}TF(L)$, since $k - 1 \geq \frac{1}{2}$. Otherwise, $n \geq \frac{k}{2} \cdot k + (k - 1)f + 1 = \frac{k}{2}(p + f + 1) + (\frac{k}{2} - 1)f - \frac{k}{2} + 1 = \frac{k}{2}(p + f + 1) + (\frac{k}{2} - 1)(f - 1) \geq \frac{k}{2}(p + f + 1) = \frac{k}{2}TF(L)$. Since $OPT(L) \geq \frac{k}{2}$, and we get $TF(L) \leq 2OPT(L)$. ■

**4.2 A simple algorithm with an absolute competitive ratio 2 for $k = 5$**

We present a different algorithm that is based on an adaptation of FF. The algorithm $ALG$ acts as follows. A new item $i$ is assigned into a minimum index bin $B$ that satisfies all the following conditions (if no such bin exists, then it is assigned into an empty bin).

1. The current level of $B$ is at most $1 - s_i$.
2. The current number of items of $B$ is at most 4.
3. If $B$ currently contains four items, then after assigning $i$, its level will be at least $\frac{1}{2}$.

A regular bin is a bin of $ALG$ that is a 2-bin or a 3-bin. We treat 1-bins (also called dedicated bins), 4-bins, and 5-bins separately.

**Lemma 47** The level of any 5-bin is at least $\frac{1}{2}$. The level of any regular bin, except for at most one bin is at least $\frac{3}{2}$. For a pair of bins $B$, $B'$ (where $B'$ appears after $B$ in the ordering), the total size of items packed into $B$ and $B'$ is above 1 in the following two cases.

1. None of the bins contains more than three items. 2. One of the bins is a large dedicated bin, and the other contains four items.

**Proof.** By the third condition, $ALG$ never creates a 5-bin whose level is below $\frac{1}{2}$. For regular bins, $ALG$ simply applies FF (without cardinality constraints) on the subsequence of items of these bins, and thus the claim follows from Claim 7.

If $B$ contains at most three items, then when the first item of $B'$ is packed, the third condition is irrelevant, the second condition holds, and thus the first condition does not hold, and the total size of items packed into the two bins exceeds 1.

We are left with the case that $B$ contains four items when the unique item $i$ of $B'$ arrives, and $s_i > \frac{1}{2}$ (the case that $B'$ has four items but $B$ has one item was already considered). Thus, if $i$ is packed into $B$, the third condition must hold, and therefore the first condition does not hold, and the total size of items packed into the two bins exceeds 1. ■
Consider an input $L$. For the output of ALG applied on $L$, let $f$ denote the number of 4-bins, let $d_1$ be the number of dedicated bins whose items have sizes above $\frac{1}{2}$ (such bins and items are called large dedicated bins and large dedicated items), and let $d_0$ be the number of dedicated bins whose items have sizes no larger than $\frac{1}{2}$ (such bins and items are called small dedicated bins and small dedicated items). By Lemma 47, $d_0 \leq 1$ must hold.

**Theorem 48** The absolute competitive ratio of ALG is at most 2.

**Proof.** We distinguish two cases as follows.

**Case 1.** $f < d_1$. We match 4-bins and large dedicated bins into pairs arbitrarily, leaving at least one large dedicated bin unmatched. The remaining bins that are not $k$-bins (regular bins, and a small dedicated bin, if it exists) are also matched into pairs, and if the number of these bins is odd, one of them is matched to an unmatched large dedicated bin. The total size of items of any matched pair is above than 1, the level of every remaining large dedicated bin is above $\frac{1}{2}$, and the level of any $k$-bin is at least $\frac{1}{2}$, by Lemma 47. We find that the total size of items $S$ satisfies $S \geq \frac{5}{2}$, thus $\text{OPT}(L) \geq S \geq \frac{5}{2}$.

**Case 2.** $f \geq d_1$. For an item of size $x$, we define its weight to be $w(x) = 1 + 3x$. Let $W$ denote the total weight of all items of $L$. For any bin, the total weight of its items is at most 8, as it has at most five items of a total size of at most 1. Match every large dedicated bin to a 4-bin. For each such pair, the total size is above 1, and the number of items is 5, thus the total weight of the items of every such pair of bins is at least 8. Every remaining 4-bin has four items, and their total weight is at least 4. Similarly, every 5-bin has a total weight above 5. Every regular bin, except for at most one such bin, has a total size of items of at least $\frac{2}{3}$, and at least two items, so its weight is at least $2 + \frac{2}{3} \cdot 3 = 4$. Thus, on average, all bins have weights of at least 4, except for possibly a small dedicated bin, if it exists, and a regular bin of level below $\frac{2}{3}$, if it exists. If none of those exists, we find $8 \cdot \text{OPT}(L) \geq W \geq 4 \cdot \text{ALG}(L)$, and we are done. If both such bins exist, then the total size of items in those bins is above 1 by Lemma 47 and there are at least three items, so the total weight is at least 6. We find $W \geq 4 \cdot (\text{ALG}(L) - 2) + 6$. If $d_0 = 1$ but no regular bin of level below $\frac{2}{3}$ exists, then the weight of the small dedicated bin is at least 1, and $W \geq 4 \cdot (\text{ALG}(L) - 1) + 1$. If $d_0 = 0$, but there is a regular bin of level below $\frac{2}{3}$, then the weight of this regular bin is at least 2, and $W \geq 4 \cdot (\text{ALG}(L) - 1) + 2$. In all three last cases, $8 \text{OPT}(L) \geq W \geq 4 \cdot \text{ALG}(L) - 3$, or alternatively, $\text{ALG}(L) \leq 2 \cdot \text{OPT}(L) + 3/4$. By integrality, we get $\text{ALG}(L) \leq 2 \cdot \text{OPT}(L)$. 

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