EXTREMAL CURVES IN 2 + 1-DIMENSIONAL YANG-MILLS THEORY

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Abstract

We examine the structure of the potential energy of 2+1-dimensional Yang-Mills theory on a torus with gauge group SU(2). We use a standard definition of distance on the space of gauge orbits. A curve of extremal potential energy in orbit space defines connections satisfying a certain partial differential equation. We argue that the energy spectrum is gapped because the extremal curves are of finite length. Though classical gluon waves satisfy our differential equation, they are not extremal curves. We construct examples of extremal curves and find how the length of these curves depends on the dimensions of the torus. The intersections with the Gribov horizon are determined explicitly. The results are discussed in the context of Feynman’s ideas about the origin of the mass gap.

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1 Introduction

To understand how the non-Abelian structure of gauge theories can lead to confinement and a dynamical mass gap is a classic problem. In this paper, we examine the geometry of the gauge-theoretic field-configuration space in two space and one time dimension and the potential-energy functional on this space. We examine the region of this infinite-dimensional space where the potential energy is small (i.e. of order one over the size of the system). A major question is whether this region is of finite extent. In this paper, we partially answer this question. We discuss how our results support the hypothesis that the shape of the potential-energy functional and the geometry of configuration space lead to the presence of a mass gap when the theory is quantized.

Early attempts to understand confinement and the gap in terms of the properties of the configuration space (or orbit space) were made by Gribov [1], Feynman [2] and Singer [3]. Gribov and Feynman attempted to use gauge transformations to minimize the Pythagorean distance between configurations. Feynman, in particular made certain conjectures concerning the mass spectrum and confinement in 2 + 1 dimensions. The first general use of this minimal distance in the literature may have been in reference [4].

Singer defined a metric of Riemannian form on orbit space, for the purpose of studying the spectrum of the Laplacian on this space in a gauge-invariant way [3]. Singer’s metric is the infinitesimal version of that discussed in references [4] and [2]. Further discussion of the metric can be found in [3] and in particular [6]. Other important observations were made by Zwanziger, van Baal and Cutkowsky [7]. A recent related development is the important work in 2 + 1 dimensions of Karabali and Nair who solved Gauss’ law and have demonstrated the existence of confinement and a mass gap at strong coupling with no cut-off [8] and developed strong-coupling expansions which seem quite reliable [9].

The degrees of freedom of the gauge theory are the set of gauge connections modulo gauge transformations. These degrees of freedom are called gauge orbits. The geometry of the space of gauge orbits (called orbit space) is known to be quite complicated [3, 6, 11, 12]. It was shown in reference [12] that the distance between configurations is a metric (in the real-analytic sense) and is just the length of a minimal geodesic with the local metric of Singer, as had been conjectured by Babelon and Viallet [6]. Using the lattice formulation of gauge theory, it was first shown in reference [13] and later in reference [14] that the heat kernel of the kinetic term of the Hamiltonian is proportional to the exponential of the square of this metric.

Our starting point on this problem is the intuitive picture of the dynamics of 2 + 1-dimensional gauge theory, proposed by Feynman [2]. Feynman worked in the Schrödinger representation and analyzed the structure of the potential energy functional. We interpret Feynman’s approach [2] as the study of the dependence of the

We refer only to Feynman’s published article [2] and not to the preprint on the subject written earlier by him.
potential energy on a coordinate called the radius. This coordinate is the distance
from the pure gauge orbit. Feynman argued that, unlike the case of Abelian gauge
theories, all non-Abelian orbits with minimal potential energy occur within a region
of finite radius (incidentally, this conjecture is false for Yang-Mills theories in three
space dimensions \[12, 13\] and the O(n) nonlinear sigma model in one space dimension
\[13, 15\]). Feynman gave some heuristic arguments to this effect. We construct extrema
of the potential energy on spheres of fixed distance from the pure gauge orbit in orbit
space for the SU(2) gauge theory. We find a differential equation which must be satis-
fied by these extrema. On a case-by-case basis, we determine which of the solutions of
this equation are truly extrema and call these solutions extremal curves \[15\]. We also
show that our extremal curves are actually minimal curves or river valleys \[13, 15\]. The
only extremal curves we have succeeded in finding have finite length in orbit space. If
all the extremal curves of the 2 + 1-dimensional SU(2) Yang-Mills theory are of finite
length, then we expect a gap in the energy spectrum directly above the ground state.

While we have not yet constructed all the extremal curves, we have found and
analyzed two interesting subclasses when space is a flat torus:

1. Flat families of orbits. These are nontrivial zero-curvature orbits.

2. A family of “special” non-Abelian orbits. These are proved to be the only
extremal-curve orbits with constant potential-energy density.

We prove that 1. and 2. have a fixed maximum radius and are river valleys.

We also consider some “Abelian” families of orbits. These contain representative
gauge fields in a U(1) subalgebra of the SU(2) theory that we consider. These are
standing color waves, i.e. the classical analogues of gluons. We show that these families
of orbits are not extremal curves. This result is important because it strongly indicates
that gluons are not physical excitations. In contrast, the river valleys of an Abelian
gauge theory are precisely such standing waves; this is why the physical excitations of
this theory are photons.

There is a subtlety in this analysis which has to do with the fact that gauge con-
nections lie in the adjoint representation of the Lie algebra. The true gauge group of
pure Yang-Mills theory (in the continuum) is SU(2)/\(Z_2 \cong SO(3)\), rather than SU(2).
In fact, we shall have a choice whether to use SO(3) or SU(2) as the gauge group.
The difference is important when the space has non-contractable loops and must be
considered when periodic boundary conditions are imposed.

In the next section we briefly review the motivation for the metric we use on orbit
space. In Section 3 we show how extremal curves are solutions of a nonlinear hyperbolic
or parabolic differential equation, and discuss how to distinguish between river valleys,
other extremal curves and unphysical solutions. In section 4, some general features of
the 2 + 1-dimensional SU(2) gauge theory are summarized. Embedded Abelian curves,
are discussed in Section 5; these turn out to be irrelevant. The simplest river valleys,
non-pure-gauge connections of zero curvature on the torus are investigated in Section
6. We show that the elements of these river valleys are within a finite distance of the pure gauge orbit. In Section 7, we discuss the solutions of our differential equation with constant potential energy density (many of the details are presented in the appendix). We show in Section 8 that these constant-potential-energy solutions are not extremal curves, except one, which happens to be a river valley (even this is not an extremal curve unless the volume of space is finite). The points of this river valley are again with a finite distance of the pure gauge orbit. In Section 9, we compare our results with the arguments made by Feynman [2] and briefly discuss why solutions of the elliptic case of our differential equation are of no physical significance. We summarize our basic results and discuss future directions of this research in Section 10.

2 The metric on Yang-Mills orbit space

In this section, the dimension of spacetime is $D+1$. We denote the space of connections on a flat $D$-dimensional manifold by $\mathcal{A}$. A connection is a Lie-algebra-valued field $A_i(x)$, $i = 1, \ldots, D$, which can also be written in terms of a real isovector $A^a_i(x)$ as $A_i(x) = A^a_i(x)t^a$, where $t^a$, $a = 1, \ldots, n$, are the generators of the Lie group $G$. The structure coefficients are $C^{abc}$, defined by $[t^a, t^b] = iC^{abc}t^c$ Denote the set of local gauge transformations $g(x)$ by $\mathcal{G}$. We will sometimes write connections leaving the index and space dependence implicit, e.g. $A_i$ for $A^a_i(x)$. The connection $A_i$ changes to $A^g$ under a local gauge transformation:

$$A^g_i(x) = g^{-1}(x)A_i(x)g(x) + ig^{-1}(x)\partial_i g(x).$$

(2.1)

The covariant derivative is $D_i = \partial_i - ig^{-1}\partial_i g(x)$.

Orbit space $\mathcal{O} \equiv \mathcal{A}/\mathcal{G}$ is a metric space $[12]$, provided that definitions are made carefully\(^2\). The metric is a function of two variables $\alpha$ and $\beta$ in $\mathcal{O}$ containing a representative connection $A$ and $B$ respectively. Its definition is

$$\rho[\alpha, \beta]^2 = \inf_{g \in \mathcal{G}} \frac{1}{2} \int d^Dx \, \text{Tr}[A^g_i(x) - B_i(x)]^2,$$

(2.2)

where the sum on the space index $i$ is implicit.

The potential energy is $U[\alpha]/\epsilon_0^2$, where $\epsilon_0$ is the coupling constant and the functional $U[\alpha]$ on an orbit $\alpha$ containing a representative $A$ is

$$U[\alpha] = \frac{1}{4} \int d^Dx \, \text{Tr} F_{ij}(x)F_{ij}(x),$$

where $F_{ij}(x) = \partial_i A_j(x) - \partial_j A_i(x)$.

\(^2\)Lebesgue measure is used, gauge fields $A$ are $L^2$ functions, the connections $ig^{-1}\partial_i g$ are also $L^2$ for allowed gauge transformations $g$ and gauge equivalence is defined through sequences of gauge transformations. To prove that gauge equivalence is indeed an equivalence relation and that the axioms of a metric space and the completeness property are satisfied takes considerable work. The proofs are much easier on the lattice $[13]$. 

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where \( F_{ij}(x) \) is the curvature, or field strength, defined in the usual way as
\[
F_{ij}(x) = i[D_i, D_j] = \partial_i A_j(x) - \partial_j A_i(x) - i[A_i(x), A_j(x)] .
\] (2.3)

The pure gauge orbit, whose curvature vanishes, will be denoted by \( \alpha_0 \).

The heat kernel of the kinetic term \( K \), for short time intervals \( \varepsilon \) behaves as \[ [13, 14]
\[
\langle \beta \left| e^{-K\varepsilon} \right| \alpha \rangle \sim \exp \left( -\frac{1}{2\varepsilon} \rho[\beta, \alpha]^2 \right) .
\]

This term therefore describes Brownian motion in orbit space with the metric \( \rho \).

3 Extremal curves in orbit space

We would like to study the region of orbit space \( \mathcal{O} \) where the potential energy (or magnetic energy) is “small”. To do this we will attempt to find the orbits \( \alpha \) in \( \mathcal{O} \) such that \( U[\alpha] \) is an extremum on the sphere of fixed \( \rho[\alpha, \alpha_0] = \rho_0 \). We will then view \( \rho_0 \) as a coordinate along which this minimum changes. The result is a hypersurface in \( \mathcal{O} \) which we call an extremal curve. If the extremum is a minimum, we call the extremal curve a river valley \[ [12, 13] \).

3.1 The Yang-Mills-Proca equation

To find extremal curves we vary the functional
\[
Q[\alpha] = U[\alpha] + \lambda(\rho[\alpha, \alpha_0]^2 - \rho_0^2) ,
\] (3.1.1)
where the number \( \lambda \) is a Lagrange multiplier. Once obtained, it can be checked whether this curve is a river valley.

At first glance extremizing (3.1.1) appears intractable. The constraint implemented by the Lagrange multiplier contains the radius \( \rho[\alpha, \alpha_0] \), which from (2.2) is
\[
\rho[\alpha, \alpha_0]^2 = \inf_{g \in \mathcal{G}} \frac{1}{2} \int d^Dx \text{Tr}(A^g_i)^2
\] (3.1.2)
which, for most connections, is impossible to evaluate (it is this expression that was first studied by Gribov \[ 1 \]). However, the situation is not hopeless. Any extremum of \( Q \) defined in (3.1.1) is also an extremum of the functional \( Q_1 \) of two fields \( A \) and \( g \)
\[
Q_1[A, g] = \int d^Dx \left[ \frac{1}{4} \text{Tr}F_{ij}^2 + \frac{1}{2} \lambda \text{Tr}(A^g_i)^2 \right] - \lambda \rho_0^2
\] (3.1.3)
(though the converse is not necessarily true - it only would be true if the value of \( g(x) \) at the extremum were actually the infimum in eqn. (3.1.2) ). Extremizing the functional \( Q_1 \) gives a nonlinear field equation for \( A \) and \( g \). The problem is still difficult, but a
substantial simplification over (3.1.1). Once an extremum \( \{A, g\} \) of \( Q_1 \) is found, the orbit \( \alpha \) containing the resulting connection \( A \) can be then be tested to see if it is an extremum of \( Q \). Suppose we have found an extremum \( \alpha \) of \( Q_1 \) defined in (3.1.3), with representative connection \( A \). Then there is some \( g \in G \) such that the radius is given by

\[
\rho[\alpha, \alpha_0]^2 = \frac{1}{2} \int d^D x \; \text{Tr}(A_i^g)^2 .
\]  

(3.1.4)

By virtue of the fact that \( g \) minimizes the quantity on the right-hand side of (3.1.4), the connection \( A^g \) satisfies the Coulomb gauge condition:

\[
\partial_i A_i^g = 0 .
\]  

(3.1.5)

Parenthetically we remark that we ultimately seek \( g \in G \) which is the absolute minimum of the right-hand side of (3.1.4). But this is a local extremum as well, so that (3.1.3) is, in fact, satisfied. The extremal condition for \( Q_1 \) generates the equations (3.1.5) as well as

\[
- [D_i, F_{ij}] + \lambda (A_j + i \partial_j g \ g^{-1}) = 0 .
\]  

(3.1.6)

Now redefine \( A \) by the gauge-transformed connection \( A^g \). This is simply a particular choice of representative connection of \( \alpha \). Then this choice of \( A \) satisfies

\[
\rho[\alpha, \alpha_0]^2 = \frac{1}{2} \int d^D x \; \text{Tr}(A_i)^2 .
\]  

(3.1.7)

We say that the connection \( A \) satisfying (3.1.7) lies in the fundamental region [1], [5], [6], [7]. Furthermore the equations (3.1.5) and (3.1.6) become

\[
\partial_i A_i = 0
\]  

(3.1.8)

and

\[
- [D_i, F_{ij}] + \lambda A_j = 0 ,
\]  

(3.1.9)

respectively.

A further simplification is possible. Clearly from the form of \( Q_1[A, g] \) in (3.1.3), \( Q_1[A, g] = Q_1[A^g, 1] = Q_2[A^g] \). Therefore, one can simply vary \( A^g \) instead of \( A \), in this new functional \( Q_2[A^g] \). We can now relabel \( A^g \) by \( A \). Therefore, all that has to be done is to find the extrema of

\[
Q_2[A] = \int d^D x \left[ \frac{1}{4} \text{Tr} F_{ij}^2 + \frac{1}{2} \lambda \text{Tr} A_i^2 \right] - \lambda \rho_0^2 .
\]

At the risk of belaboring the obvious, we will show that the gauge connection which extremizes \( Q_2[A] \) automatically obeys the Coulomb gauge condition (3.1.5), which is a
necessary condition for $\int Tr(A^0)^2 d^D x$ to be minimized at $g = 1$. In fact, this condition follows from the Proca-Yang-Mills equation (3.1.9) by virtue of the Jacobi identity. Taking the commutator of $D_j$ with both sides of (3.1.9) yields

$$i[D_j, [D_i, [D_i, D_j]]] = \lambda \partial_i A_i.$$  

(3.1.10)

From the Jacobi identity this becomes

$$-i[[D_i, D_j], [D_j, D_i]] - i[[D_i, D_j], D_j] = \lambda \partial_i A_i,$$

which simplifies to

$$-i[D_i, [D_j, [D_j, D_i]]] = \lambda \partial_i A_i.$$  

Notice that this is the same as (3.1.10) except the sign of the left-hand side has changed and (3.1.8) follows. Therefore all of the extremal curve configurations have a representative connection satisfying (3.1.9). We therefore proceed by finding solutions of the “massive” Yang-Mills or Proca equations (3.1.9), then analyzing them to see if they lie on river-valley gauge orbits. We put “massive” in quotes, for as we shall see in examples below, the “mass squared” $\lambda$ is negative, so the equation (3.1.9) is hyperbolic rather than elliptic (the quotes will be dropped henceforth).

A necessary condition for a family of solutions of the Yang-Mills-Proca equation to be an extremal curve is that (3.1.7) is true. This means that the Faddeev-Popov functional

$$F.P. = \int d^D x \, \frac{1}{2} \left[ (\partial_i h^a)^2 + C^{abc} A_i^a \partial_i h^b h^c \right],$$  

(3.1.11)

which is the second variation of the integral under a gauge transformation, is positive.

### 3.2 Potential-energy stability analysis

Suppose we wish to know whether a given gauge orbit with a representative connection in the fundamental region and satisfying the Proca equation is a local minimum of the potential energy. In other words, we seek information as to whether such a connection lies on a river valley or not. The question is similar to that asked in Hamilton-Jacobi theory, i.e. whether a local extremum of some variational principle is a local minimum. However, in Hamilton-Jacobi theory it is sufficient to study the spectrum of linear operators; a luxury which will be denied us.

Suppose that there is a solution of the non-Abelian Proca equation $A$ contained in a gauge orbit $\alpha$, such that $\rho[\alpha, \alpha_0] = \rho_0$. Consider a small variation $A \rightarrow A + \delta A = B$, where the connection $B$ is contained in the gauge orbit $\beta$. We note that it is nontrivial to prove that $\rho[\alpha, \beta]$ is then of order $(\delta A)^2$ on the sphere of radius $\rho_0$, i.e. $\rho[\beta, \alpha_0] = \rho_0$ (this is done in reference [12]). We wish to know whether the new potential energy
$U[\beta]$ is greater than $U[\alpha]$. If for any variation $\delta A$, we have that $U[\beta] > U[\alpha]$ then $\alpha$ lies on a river valley.

Let us try to formulate the statements in the last paragraph more precisely. If both $\alpha$ and $\beta$ lie on the sphere of radius $\rho_0$ about $\alpha_0$, then

$$\rho_0^2 = \inf_h \frac{1}{2} \int d^D x \ Tr[(A_i + \delta A_i)^h]^2 = \inf_h \frac{1}{2} \int d^D x \ Tr(A_i^g)^2 = \frac{1}{2} \int d^D x \ Tr(A_i)^2,$$

where in the last step, we have assumed that the connection $A_i$ is in the fundamental region, i.e. is the absolute minimum of $\frac{1}{2} \int d^D x \ Tr(A_i^g)^2$ with respect to $g$. In (3.2.1), the gauge transformation $h$ minimizing the second integral can be assumed to be of the form $e^{i\delta H}$, where the Lie-algebra valued field $\delta H$ is infinitesimal. Let us define a new variation of $A$, which we call for the moment $\delta A'$ given by

$$A_j + \delta A_j' = (A_j + \delta A_j)^h = A_j + \delta A_j - i[\delta H, A_j + \delta A_j] + i\partial_j \delta H + \ldots.$$

In general, the new gauge connection $A + \delta A'$ remains in the fundamental domain of orbit space (the exceptional situations occur when $A$ is at the boundary of the fundamental region). It is a particular gauge connection lying within the orbit $\beta$. In particular, $A + \delta A'$ satisfies Coulomb gauge and half the integral of its trace squared is equal to $\rho_0^2$:

$$\partial_j \delta A_j' = 0, \quad \rho_0^2 = \frac{1}{2} \int d^D x \ Tr A_j^2 = \frac{1}{2} \int d^D x \ Tr(A_j + \delta A_j')^2. \quad (3.2.2)$$

For convenience, we now relabel $\delta A'$ by $\delta A$.

We must impose the constraints (3.2.2) when investigating the variation of the potential energy $U$. This is because we wish to see how variations $\delta A$ subject to the condition that the gauge orbit remains on the sphere of radius $\rho_0$ change $U$. If such extremal variations produce only positive changes in $U$, then $\alpha$ lies on a river valley. Our problem has been reduced to the study of the functional

$$J[A; \delta A] \equiv U[\beta] - U[\alpha] = \frac{\lambda}{2} \int d^D x \ (\delta A_j^a)^2 + \frac{1}{2} \int d^D x \ \delta A_j^a \left[-(D^2)^{ab}\delta_{jk} + (D_j D_k)^{ab} + 2C^{acb} F_{ck}\right] \delta A_k^b, \quad (3.2.3)$$

where $\delta A$ is subject to the conditions (equivalent to (3.2.2))

$$\partial_j \delta A_j^a = 0, \quad \int d^D x \ [2(\delta A_j^a)^2 + (\delta A_j^a)^2] = 0 \quad (3.2.4)$$

We have used the fact that $A$ satisfies the Proca equation and then used the second of equations (3.2.2) to simplify the first term on the right-hand side of (3.2.3). The precise formulation of our question is the following: is $J[A; \delta A]$ positive for variations $\delta A$ satisfying (3.2.4)?
4 SU(2) in 2 + 1 dimensions

In the remainder of this paper, we will only consider the case of two space and one time dimension.

A non-Abelian gauge theory with no matter in two space dimensions is simpler than a theory in three space dimensions. The 2 + 1-dimensional Yang-Mills theory is ultraviolet finite and therefore the bare dimensionful coupling constant can be fixed to some finite nonzero value. The metric and the potential energy are not renormalized by infinite constants. We take gauge group \( G = SU(2) \) and generators \( t^a = \sigma^a / 2 \), where \( \sigma^1, \sigma^2 \) and \( \sigma^3 \) denote the three Pauli matrices. The structure coefficients are therefore

\[
C^{abc} = \varepsilon^{abc}.
\]

We shall find it necessary to impose an infrared cutoff. To illustrate the reason for this, consider the problem of extremizing (3.1.1) when the space is the infinite plane. Consider an arbitrary connection \( A_i(x) \). Under the scale transformation, \( A_i(x) \rightarrow A_i'(x) = sA_i(sx) \), where \( s \) is a positive real number, the orbit \( \alpha \) containing \( A_i(x) \) is mapped to an orbit \( \alpha' \) (for further discussion, see Section 10 of reference [12]). Under this transformation of orbits, \( \rho_0 \) is unchanged but \( U(\alpha) \) changes by an overall factor \( U(\alpha') = s^2 U(\alpha) \). Therefore, we can transform the potential energy to a value as small as desired for a given value of \( \rho_0 \). For this reason, Yang-Mills theory on the infinite plane has no non-trivial extremal curves with finite \( \rho_0 \). Moreover, to lower its potential energy and preserve its value of \( \rho_0 \), a gauge field tends to spread out to infinite size. In order to cutoff this infrared behavior, we will put the system in a box with periodic boundary conditions, i.e. a torus with coordinates \( 0 \leq x^i < L_i \), for \( i = 1, 2 \) and all functions of \( x \) to be doubly-periodic with periods \( L_1 \) and \( L_2 \).

We do not consider twisted boundary conditions in this paper [16]. However, we should like to mention that the case of a twist can easily be incorporated by doubling one of our torus dimensions \( L_1 \) or \( L_2 \). Any SU(2) Yang-Mills-Proca solution on the twisted torus of dimensions \( L_1 \) by \( L_2 \) is automatically a Yang-Mills-Proca solution on the periodic torus of dimensions \( L_1 \) by \( 2L_2 \).

5 Embedded Abelian curves

The first set of solutions to the Proca equation (3.1.3) we will consider are those which lie entirely in an Abelian subalgebra, \( A_i(x) = A_i^1(x)t^1 \). The most general solutions of this kind on the torus are simple to find:

\[
A_i^1(x) = q_2 \left( \cos q_1 x^1, \sin q_1 x^1 \right) \left[ -\sin q_2 x^2, \cos q_2 x^2 \right],
\]

\[
A_i^2(x) = q_1 \left( \sin q_1 x^1, -\cos q_1 x^1 \right) \left[ \cos q_2 x^2, \sin q_2 x^2 \right],
\]

\[
\lambda = -q_1^2 - q_2^2,
\]

(5.1)
where \( q_i = \frac{2\pi l_i}{L_i} \), \( l_i \) are integers, the case \( q_1 = q_2 = 0 \) is excluded and \( M \) is an arbitrary real two-by-two matrix.

It is clear that (5.1) is precisely of the form of the river valleys of an Abelian gauge theory. Calculating \( \rho_0 \) and \( U \) is quite easy and reveal a harmonic oscillator potential for each choice of \( q_i \). The excitations are simply those of the oscillators; they are photons. The oscillator frequency is minimized by taking one of the \( q_i \) equal to zero and the other equal to its smallest allowed value, i.e. \( \frac{2\pi l_i}{L_i} \). The gap between the ground state and the first excited state vanishes in the thermodynamic limit. However, for the SU(2) theory, the situation is quite different.

For the solution in (5.1), the potential energy is

\[
U = \frac{1}{16} (q^2)^2 L_1 L_2 \text{Tr}(MM^T)
\]

and

\[
\mathcal{I}(A, 1) = \frac{1}{2} \int d^2 x \text{Tr}[A_i(x)]^2 = \frac{1}{16} L_1 L_2 q^2 \text{Tr}(MM^T) .
\]

In the region where \( \mathcal{I}(A, 1) \) can be identified with \( \rho_0^2 \), the potential energy behaves as

\[
U = q^2 \rho_0^2 ,
\]

which grows quadratically with \( \rho_0 \). This is always the case for the gauge group \( U(1) \); it is easy to see that any gauge copy of \( A \) under Abelian gauge transformations \( (A \rightarrow A + t^i \nabla \chi) \) has a larger value of the integral:

\[
\mathcal{I}(A, \exp(it_1 \chi)) \geq \mathcal{I}(A, 1) .
\]

However, in the SU(2) gauge theory, it is necessary to check whether non-Abelian gauge copies of \( A \) can have a smaller value of the integral, i.e. whether \( g(x) \) exists such that \( \mathcal{I}(A, g) \leq \mathcal{I}(A, 1) \). If this is the case, then \( \rho_0^2 \) is smaller than \( \mathcal{I}(A, 1) \) and the potential energy rises faster with increasing \( \rho_0 \) than the quadratic behavior in (5.2).

One approach to this problem is to check whether \( \mathcal{I}[A, g] \) is a local minimum at \( g(x) = 1 \). Indeed, since \( A_i(x) \) satisfies the Coulomb gauge condition, it is guaranteed to be an extremum. To determine whether this extremum is a local minimum, we must examine the spectrum of the quadratic form in the Faddeev-Popov functional (3.1.11). Indeed, Taylor expanding \( \mathcal{I}[A, g] \) in the Lie-algebra valued quantity \( h = h^a t^a \) where \( g = e^{ih} \) yields

\[
\mathcal{I}[A, g] = \mathcal{I}[A, 1] + \text{F.P.} = \mathcal{I}[A, 1] + \frac{1}{2} \int d^2 x h^a(x) (M)^{ab} h^b(x) + \ldots ,
\]

where \( M \) denotes the Faddeev-Popov operator

\[
(M)^{ab} = -\partial^2 \delta^{ab} - \epsilon^{cab} A_c^i(x) \partial_i .
\]
We will show that this operator always has a vanishing eigenvalue for the connection (5.1). This means that this connection lies on the so-called Gribov horizon.

We first diagonalize $\mathcal{M}$ in the color indices to obtain

$$
\mathcal{M} = \begin{pmatrix}
-\partial^2 & 0 & 0 \\
0 & -\partial^2 + iA^1(x) \cdot \partial & 0 \\
0 & 0 & -\partial^2 - iA^1(x) \cdot \partial
\end{pmatrix},
$$

(5.3)

The following straightforward variational argument shows that $-\partial^2 \pm iA^1(x) \cdot \partial$ will have a vanishing eigenvalue for any choice of the matrix $M$. In other words, the connections (5.1) all sit on the Gribov horizon. Consider the normalizable trial function

$$
\Psi = e^{iR_1 x^1 + iR_2 x^2} \left[ 1 + (\cos q_1 x^1, \sin q_1 x^1) P \begin{pmatrix} \cos q_2 x^2 \\ \sin q_2 x^2 \end{pmatrix} \right]
$$

(5.4)

where $P$ is a two-by-two complex matrix and $R_i = \frac{2\pi s_i}{L_i}$, $s_i$ an integer. Consider the matrix element of one of the components of the operator (5.3)

$$
G_+ \equiv \frac{1}{L_1 L_2} \int d^2 x \Psi^* (x) \left[ -\partial^2 \pm iA^1(x) \cdot \partial \right] \Psi (x) = R^2 + \frac{1}{2} (R^2 + q^2) \text{Tr} P^\dagger P \\
\pm \frac{1}{2} \text{Tr} \left[ R_2 q_1 \begin{pmatrix} -M_{21} & -M_{22} \\ M_{11} & M_{12} \end{pmatrix} \right] - R_1 q_2 \begin{pmatrix} -M_{12} & M_{11} \\ -M_{22} & M_{21} \end{pmatrix} \\
\pm \frac{1}{2} \text{Tr} \left[ R_2 q_1 \begin{pmatrix} -M_{21} & M_{11} \\ -M_{22} & M_{12} \end{pmatrix} \right] - R_1 q_2 \begin{pmatrix} -M_{12} & -M_{22} \\ M_{11} & M_{21} \end{pmatrix} \right] P.
$$

It is straightforward to minimize this expression with respect to $P$ by completing the square. The smallest value of $G_+$ is

$$
\min G_+ = R^2 + \frac{1}{8(R^2 + q^2)} \left[ (R_1^2 q_2^2 + R_2^2 q_1^2) \text{Tr} M^T M + 4R_1 R_2 q_1 q_2 \det M \right].
$$

(5.5)

Among our choices of $\Psi$, there is always one for which this expression is zero, namely that with $R_1 = R_2 = 0$. While the connection (5.1) is a solution of the Proca equation, it lies on the Gribov horizon for all $M$. Therefore, these connections do not lie in the fundamental region and do not constitute an extremal curve.

We have found a striking difference between the Abelian and non-Abelian gauge theory. In the Abelian theory, standing-wave connections lie in the orbits of true river valleys. Excitations in these river valleys are photons. In the Yang-Mills theory, such connections are not extremal curves. This result strongly suggests that the excitations are not perturbative gluons.

## 6 River valleys containing flat connections

An obvious strategy to find minima of the potential is to look for configurations of zero field strength. Not all such configurations are pure gauge [16]. The idea is to
find solutions of the field equations $F = 0$ which are at a particular distance from the trivial configuration $A = 0$,

$$\rho_0^2 = \inf_g \frac{1}{2} \int d^2x \text{Tr} [A^g(x)]^2,$$

where $A^g$ is a static gauge transform of $A$. The space of gauge orbits containing such connections is automatically a river valley, since the potential energy has saturated its lower bound, namely zero.

Consider solutions of the condition $F = 0$ on the torus. A connection is flat when it is written as

$$A_i(x) = ig^{-1}(x) \partial_i g(x), \quad (6.1)$$

where $g(x)$ is a unitary matrix. A gauge transformation

$$A_i(x) \rightarrow ih^{-1}(x) (\partial_i - iA_i(x)) h(x),$$

is equivalent to

$$g(x) \rightarrow g(x) h(x).$$

These are trivially solutions of the Proca equation (3.1.9).

The gauge field must satisfy the boundary conditions of the torus,

$$A_i(x^1 + L_1, x^2) = A_i(x^1, x^2),$$
$$A_i(x^1, x^2 + L_2) = A_i(x^1, x^2).$$

This will occur if $g(x)$ in (6.1) obeys the condition

$$g(x^1 + L_1, x^2) = u_1 g(x^1, x^2),$$
$$g(x^1, x^2 + L_2) = u_2 g(x^1, x^2),$$

where $u_1$ and $u_2$ are constant unitary matrices. The consistency condition

$$g(x^1 + L_1, x^2 + L_2) = u_1 u_2 g(x^1, x^2),$$
$$g(x^1 + L_1, x^2 + L_2) = u_2 u_1 g(x^1, x^2), \quad (6.2)$$

requires that $u_1$ and $u_2$ commute.

The expression (6.1) is unchanged if we replace $g(x)$ by $vg(x)$ where $v$ is a constant unitary matrix. Under this replacement $u_1$ and $u_2$ are replaced by $vu_1v^{-1}$ and $vu_2v^{-1}$, respectively. Since $u_1$ and $u_2$ commute, they can be simultaneously diagonalized by a judicious choice of $v$. Thus, for SU(2) they can be taken to have the form

$$u_i = \left( \begin{array}{cc} e^{i\phi_i} & 0 \\ 0 & e^{-i\phi_i} \end{array} \right), \quad (6.3)$$

where the phases lie in the fundamental region

$$|\phi_i| \leq \pi. \quad (6.4)$$
A gauge potential which corresponds to a particular \( u_i \) is

\[
A_i(x) = g^{-1}(x) \left( i\partial_i - 2\phi_i t^3/L_i \right) g(x),
\]

where \( g(x) \) satisfies periodic boundary conditions, \( g(x^1 + L_1, x^2) = g(x^1, x^2 + L_2) = g(x^1, x^2) \). Furthermore,

\[
\rho_0^2 = \inf_h \frac{1}{2} \int d^2 x \text{Tr} \left[ (gh)^{-1} \left( i\partial_i - 2\phi_i t^3/L_i \right) (gh) \right]^2.
\]

Though \( g(x) \) is periodic on the torus, \( h(x) \) need not be. The transformation \( h(x) \) can be periodic up to an element of the center of SU(2). For example, we can choose \( h(x) \) so that

\[
g(x)h(x) = \exp \left( i \sum_{i=1}^2 2\pi n_i x^i t^3/L_i \right).
\]

This yields an upper bound on \( \rho_0^2 \),

\[
\rho_0^2 \leq \inf_{n_i} L_1 L_2 \sum_{i=1}^2 \left( \frac{n_i \pi + \phi_i}{L_i} \right)^2 \leq \frac{\pi^2}{4} \left( \frac{L_1}{L_2} + \frac{L_2}{L_1} \right).
\]

The Gribov horizon is located where the inequality is saturated. Note that, if we had instead required that the gauge transformations be strictly periodic on the torus, the above equation would read \( \rho_0^2 \leq \pi^2 \left( \frac{L_1}{L_2} + \frac{L_2}{L_1} \right) \).

7 Constant-magnitude curvature solutions I

In this section we will find solutions of the Yang-Mills-Proca equations with the supplementary condition that the field strength has constant magnitude. What is remarkable is that it is possible to find all such solutions. We answer the question of which of these are extremal curves in the next section.

We will start with the simplest case, where the field strength is simply constant. Later (in the next section) we shall see that this case is an example of the general solution. The equations (3.1.9) and (3.1.7) simplify considerably if \( F_{ij} \) is assumed to be a constant. It will turn out that these are the only constant-magnitude field strength solutions which are extremal curves.

We begin by looking for a solution for which

\[
F_{ij} = \epsilon_{ij} ft^3,
\]

with \( f \) constant. We have two reasons for treating this case first. The first is that it is considerably easier than the more general situation of curvature of constant magnitude. The second reason is that, as we show in Section 8, these configurations are the only
constant-magnitude curvature extremal curves, and therefore it seems worthwhile to derive them simply.

The equation (3.1.9) becomes

$$\epsilon^{ab} \epsilon_{ij} f A_i^b + \lambda A_j^a = 0 ,$$

(7.1)

from which $A_j^3 = 0$ immediately follows.

Define $a$ to be the real $2 \times 2$ matrix whose $b - i$ component is $A_i^b$. Then (7.1) becomes

$$\sigma^2 a \sigma^2 = \lambda f a .$$

(7.2)

There are solutions of (7.2) only if $\lambda = \pm f$. If $\lambda = f$, then

$$a = a_+ \equiv a_0 1 + i a_2 \sigma^2 ,$$

while if $\lambda = -F$, then

$$a = a_- \equiv a_1 \sigma^1 + a_3 \sigma^3 .$$

These results may be written in terms of the corresponding connections $A_{\pm i}$ as

$$A_{+1} = a_3 t^1 + a_1 t^2 , \quad A_{+2} = a_1 t^1 - a_3 t^2 ,$$

$$A_{-1} = a_0 t^1 - a_2 t^2 , \quad A_{-2} = a_2 t^1 + a_0 t^2 .$$

The curvature must given by

$$f = -\frac{1}{2} \text{Tr} a_\pm \sigma^2 a_\pm \sigma^2 = \mp \frac{1}{2} \text{Tr} a_\pm^2 ,$$

and the Abelian piece $\partial_1 A_2 - \partial_2 A_1$ must vanish. This latter condition means that for $\pm = +$, $a_\pm^2 \equiv a_1^2 + a_3^2$ is constant, while if $\pm = -$, $a_- \equiv a_1^2 + a_3^2$ is constant. The Coulomb gauge condition (3.1.8) then implies that the numbers $a_1, a_2, a_3$ and $a_4$ are everywhere harmonic functions, $\partial^2 a_q = 0, q = 1,2,3,4$. Since these functions are doubly-periodic, Liouville’s theorem implies that they are constants.

To summarize the results obtained in this section thus far, we have found that the solutions to the Proca equation with constant field strength are of the form

$$A_i^a = f^{1/2} \delta_i^a 1 , \quad A_2^a = f^{1/2} \delta_2^a ,$$

(7.3)

up to global gauge rotations. Such constant non-Abelian potentials were discussed long ago by Brown and Weisberger [17].

Recall that $\rho_0^2$ is the minimum of $\mathcal{I}[A, g] = \frac{1}{4} \int d^2 x \{(A_i^1)^2 + (A_2^1)^2\}$ with respect to gauge transformations $g$. Each of our constrained Proca solutions is a local extremum of $\mathcal{I}[A, g]$ at $g = 1$. However, further work needs to be done to check to see whether such a solution is an extremal curve configuration; in other words, that it is the absolute
minimum of the integral of the square of the gauge field, so that $\mathcal{I}[A, 1] = \rho_0^2$. Given $\mathcal{I}[A, 1]$ it is possible to determine $F$ and $\lambda$. We find that for both $f = \lambda$ and $f = -\lambda$,

$$\lambda = -\frac{2\mathcal{I}[A, 1]}{L_1L_2}.$$

An extensive analysis to determine whether $\mathcal{I}[A, 1]$ can be decreased by a gauge transformation will be done in Section 9. The gauge orbit of only one of the resulting solutions (up to translations and rotations) is found to lie on an extremal curve.

Next we turn to the more general case of solutions of (3.1.9) for which the curvature is not assumed constant, but the magnitude of the curvature is assumed constant. The key to finding these solutions is the analyticity of certain quadratic polynomials of the connection. Liouville’s theorem then guarantees that these quadratic polynomials are constants. This enables us to use a particular parametrization of the connections and the curvature. The constant-curvature solutions of the last section are a special case of those we find here.

The equations (3.1.9) may be written as

$$\partial_i F^a + \epsilon^{abc} A_i^b F^c = \lambda \epsilon_{ij} A_j^a, \quad F^a_{ij} \equiv \epsilon_{ij} F^a.$$  \hfill (7.4)

We supplement these equations with the condition

$$(F^a)^2 \equiv f^2.$$  \hfill (7.5)

We call (7.4) together with (7.5) the **constrained Proca equations**.

The general solutions are found in the appendix. These have spatial dependence on only $x^1$ or $x^2$. If we choose the dependence to be on $x^1$, these have the form

$$A_1^a = \frac{f}{|\lambda|^{3/2}} \gamma_0^a,$$

$$A_2^a = \frac{1}{|\lambda|^{1/2}} \beta_0^a \cos \left( \frac{f^2 - \lambda^2}{|\lambda|^{3/2}} (x^1 - x_0^1) \right) - \frac{1}{|\lambda|^{1/2}} F_0^a \sin \left( \frac{f^2 - \lambda^2}{|\lambda|^{3/2}} (x^1 - x_0^1) \right),$$

$$F^a = \beta_0^a \sin \left( \frac{f^2 - \lambda^2}{|\lambda|^{3/2}} (x^1 - x_0^1) \right) - F_0^a \cos \left( \frac{f^2 - \lambda^2}{|\lambda|^{3/2}} (x^1 - x_0^1) \right),$$  \hfill (7.6)

where $F_0^a = F^a(x^0)$, and the isovectors $\gamma_0^a$, $\beta_0^a$ satisfy $(\gamma_0^a)^2 = (\beta_0^a)^2 = f^2$, and $\beta_0^a = \frac{1}{f} \epsilon^{abc} F^b \gamma^c$. Notice that this solution (7.6) is consistent with periodic boundary conditions, provided

$$\frac{f^2 - \lambda^2}{|\lambda|^{3/2}} = \frac{2\pi n}{L_1},$$  \hfill (7.7)

where $n$ is an integer. This is a quartic equation for $|\lambda|$ in terms of the integer $n$ and has one real root. Thus the solutions on the torus are quantized for each $f$. 

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The field strength $F^a$ in (7.6) can always be made equal to $f \delta^{a3}$ by a gauge transformation, no matter what the choice of $|\lambda|$. However, the gauge orbit containing the gauge connection in (7.6) is different for each $\lambda$. Though the field strength for two such gauge fields, each with a different $\lambda$ is the same, it is in general impossible to gauge transform one gauge field to the other (this will be shown in the next section). This is an example of the Wu-Yang ambiguity [18].

We now have all the solutions to the Proca equations which have constant potential-energy density. We note that these include the constant-field-strength configurations of (7.3). These are simply (7.6) when $|\lambda| = f$, i.e. global gauge rotations of (7.3). We now need to see under which circumstances these are absolute minima of $I[A, g] = \frac{1}{4} \int d^2 x \{ [(A^q_1)^a]^2 + [(A^q_2)^a]^2 \}$ at $g = 1$.

8 Constant-magnitude curvature solutions II

Any river-valley gauge orbit of constant potential energy density must contain a gauge configuration satisfying the constrained Proca equations (7.4) and (7.5). We proved in the appendix that such a connections must always be of the form (7.6). In this section we determine which of the orbits containing (7.6) are in river valleys. We begin this analysis by working in the full plane $\mathbb{R}^2$, where it is somewhat easier than on the torus. In the plane we shall find that none of the solutions (7.6) are river valleys. This conclusion is not so surprising, for without a careful restriction on gauge orbits [12] the sphere of constant $\rho_0$ is not a compact space. We then show that on the torus only one of the gauge configurations $A^a_i$ given by (7.6) is a local minimum of $I[A, g]$ at $g = 1$. However for sufficiently large $f$, at the Gribov horizon, even this configuration is not a local minimum. The horizon is reached at a finite value of $\rho_0$.

8.1 The effects of gauge transformations in the plane

We first examine (7.6) in the plane $\mathbb{R}^2$. We shall find that for any of these connections there always exists a $g(x) \in \text{SU}(2)$, $g(x) \neq 1$ such that $[(A^q_1)^a]^2 + [(A^q_2)^a]^2 < (A^q_1)^2 + (A^q_2)^2$. The expression on each side of this inequality is constant, so this is similar to saying that $I[A, g] < I[A, 1]$, though in truth, neither integral exists. A careful definition of gauge orbits excludes gauge fields which are not square-integrable (clearly (7.6) is not square-integrable in the plane). Nonetheless, it is useful to examine $[(A^q_1)^a]^2 + [(A^q_2)^a]^2$ for (7.6) in $\mathbb{R}^2$, as it will shed light on the situation on the torus. Since this quantity happens to be independent of $x$ in this subsection, it is analogous to the integral $I[A, g]$ in the general case.

We should like to mention that the $f > |\lambda|$ connections are gauge equivalent to the $f < |\lambda|$ connections, with the space-coordinate axes interchanged. To show this, we make a rigid (i.e. independent of $x$) gauge transformation of (7.6) so that $q_0^a \rightarrow f \delta^{a1}$, $\beta_0^a \rightarrow f \delta^{a2}$ and $F_0^a \rightarrow f \delta^{a3}$ and to make a translation so that $x_0^1 = 0$. We then find for
the two-by-two Hermitian matrices $A_i$ and $F$

\[
A_1(x) = \frac{f^2}{2|\lambda|^{3/2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_2(x) = \frac{f}{2|\lambda|^{1/2}} \begin{pmatrix} -\sin kx^1 & -i \cos kx^1 \\ i \cos kx^1 & \sin kx^1 \end{pmatrix}, \quad F(x) = \frac{f}{2} \begin{pmatrix} \cos kx^1 & -i \sin kx^1 \\ i \sin kx^1 & -\cos kx^1 \end{pmatrix},
\]

where $k = (f^2 - \lambda^2)/|\lambda|^{3/2}$. If we perform the gauge transformation (2.1)

\[
g(x) = \begin{pmatrix} \cos kx^1/2 & i \sin kx^1/2 \\ i \sin kx^1/2 & \cos kx^1/2 \end{pmatrix},
\]

we can bring $A_i$ to a constant gauge field $(A^g)_i$, for which the isovectors $(A^g)_a^i$ are

\[
(A^g)_1^i = |\lambda|^{1/2} \delta^{a1}, \quad (A^g)_2^i = \frac{f}{|\lambda|^{1/2}} \delta^{a2}, \quad (F^g)_a^i = f \delta^{a3}.
\]

Notice that under $\lambda \to f^2/\lambda$, the 1-component and 2-component of the gauge connection (8.1.2) are interchanged. Thus an $f > |\lambda|$ solution and an $f < |\lambda|$ solution are equivalent under a gauge transformation.

Under a gauge transformation (8.1.1) it is possible to lower the value of $[(A^g)_1^i]^2 + [(A^g)_2^i]^2$. We will show this by a nonrigorous argument. Considering formally the second variation of the integral of this expression (i.e. $\mathcal{I}[A,g]$) with respect to a small gauge transformation $g \approx 1 - ih - h^2/2$. The result is again the Faddeev-Popov functional (3.1.11)

\[
\text{F.P.} = \int d^2x \frac{1}{2} \left[ (\partial_i h^a)^2 + \epsilon^{abc} A^a_i \partial_i h^b h^c \right].
\]

Even though $\mathcal{I}[A,g]$ is not well-defined, the functional F.P. exists for appropriately defined gauge transformations. This quantity can be negative for some choice of $h^a$. For by Fourier transforming and replacing $\partial_i$ by $ip_i$, the eigenvalues of the quadratic form (8.1.3) are $(p_i)^2$, $(p_i)^2 \pm 2f^{1/2} \sqrt{(p_i)^2}$. One of these eigenvalues is negative for any $p_i$ such that $(p_i)^2 < 4f$. Since the values of $p_i$ are continuous, a negative eigenvalue is always present.

It is actually quite easy to show that the gauge connections (7.6) are different for different $|\lambda| < f$ (we have already showed that each $|\lambda| < f$ solution is gauge equivalent to a unique $|\lambda| > f$ solution). By making the further gauge transformation $g'(x)$,

\[
g'(x) = \exp(i|\lambda|^{1/2}x^1),
\]

on (8.1.2), the transformed component $A^g_1$ is brought to zero, while

\[
A^g_2 = \frac{f}{|\lambda|^{1/2}} \left[ \cos \left( \frac{|\lambda|^{1/2}}{2} x^1 \right) \delta^{a2} + \sin \left( \frac{|\lambda|^{1/2}}{2} x^1 \right) \delta^{a3} \right].
\]
Though these axial-gauge-fixed connections are all distinct, there do exist gauge transformations such that each has the same field strength $F^a$, explicitly illustrating the Wu-Yang ambiguity [18].

8.2 The effects of gauge transformations on the torus

We show below that only the $k = 2\pi n/L_1 = 0$ solution of (7.6) can possibly lie in the fundamental region, when the inequality

$$
\sqrt{5} > \frac{L_1}{L_2} > \frac{1}{\sqrt{5}}
$$

is satisfied. It may be that only the $k = 0$ solution lies within the Gribov horizon for other choices of $L_1/L_2$ as well, but we have not yet proved this.

The Faddeev-Popov operator $M$ for (7.6) is

$$
M^{bc}(x, y) = [-\delta^{bc}\partial^2 - \epsilon^{1bc}\frac{2f^2}{|\lambda|^{3/2}}\partial_1 \hspace{1cm} - \epsilon^{2bc}\frac{2f}{|\lambda|^{1/2}}\cos(kx^1)\partial_2 + \epsilon^{3bc}\frac{2f}{|\lambda|^{1/2}}\sin(kx^2)\partial_3] \delta^2(x - y)
$$

We wish to determine the eigenvalues of $M$. Define the unitary matrix $S^{ab}$, acting only on color indices:

$$
S = \begin{pmatrix}
    e^{ikx^1} & 0 & 0 \\
    0 & 1/\sqrt{2} & i/\sqrt{2} \\
    0 & i/\sqrt{2} & 1/\sqrt{2}
\end{pmatrix}.
$$

Then $S^\dagger MS$ has the same eigenvalues as $M$. This new operator has the form

$$
(S^\dagger MS)(x, y) = \begin{pmatrix}
    -\partial^2 - 2ik\partial_1 + k^2 & -i\frac{\sqrt{2}f}{|\lambda|}\partial_2 & -\frac{\sqrt{2}f}{|\lambda|^{1/2}}\partial_2 \\
    -i\frac{\sqrt{2}f}{|\lambda|}\partial_2 & -\partial^2 - 2i\frac{f^2}{|\lambda|^{3/2}}\partial_1 + k^2 & 0 \\
    \frac{\sqrt{2}f}{|\lambda|^{1/2}}\partial_2 & 0 & -\partial^2 + 2i\frac{2f^2}{|\lambda|^{3/2}}\partial_1 + k^2
\end{pmatrix} \delta^2(x - y),
$$

which contains no functions of the coordinates $x$. Thus this operator has the same eigenvalues as that with $\partial_1$ and $\partial_2$ replaced with $p_1 = \frac{2\pi j_1}{L_1}$ and $p_2 = \frac{2\pi j_2}{L_2}$, respectively. The secular determinant of the resulting matrix vanishes at the eigenvalues of $M$. For each choice of $j_1$ and $j_2$ there are three eigenvalues $w(j_1, j_2; 1)$, $w(j_1, j_2; 2)$ and $w(j_1, j_2; 3)$ The eigenvalues $w(j_1, j_2; q)$ are determined by

$$
w(j_1, j_2; q) = p^2 + W(j_1, j_2; q),
$$
\[(W - 2kp_1 - k^2)\left(W^2 - \frac{4f^4}{|\lambda|^3 p_1^2}\right) - \frac{4f^2}{|\lambda|} p_2^2 W = 0. \quad (8.2.2)\]

We will show that if (8.2.1) holds, then there are negative \(w(j_1, j_2; q)\) for some \(j_1, j_2\) and \(q\), unless \(n = 0\).

First consider the solutions of (8.2.2) with \(j_2 = 0\). These are
\[
w(j_1, 0; 1) = (p_1 + k)^2, \quad w(j_1, 0; 2) = |p_1| \left(|p_1| + \frac{2f^2}{|\lambda|^{3/2}}\right),
\]
\[
w(j_1, 0; 3) = |p_1| \left(|p_1| - \frac{2f^2}{|\lambda|^{3/2}}\right).
\]
In order for both \(w(j_1, 0; 2)\) and \(w(j_1, 0; 3)\) to be nonnegative we must have
\[
\left|\frac{2\pi p_1}{L_1}\right| = |p_1| \geq \frac{2f^2}{|\lambda|^{3/2}} \geq 0,
\]
where the second inequality follows from the fact that the right hand side is nonnegative. Using (7.7) to eliminate \(f\) this becomes
\[
|j_1| \geq 2n + \frac{L_1|\lambda|^{1/2}}{2\pi} \geq 0.
\]
If \(n\) is any positive integer, this inequality must be violated for some \(j_1\). Therefore \(n \leq 0\), and there is a negative eigenvalue unless
\[
1 \geq 2n + \frac{L_1|\lambda|^{1/2}}{2\pi} \geq 0.
\]
Writing \(-n\) as \(|n|\) and adding \(2|n|\) to all three sides gives the necessary conditions for eigenvalues of the Faddeev-Popov operator to be nonnegative :
\[
2|n| + 1 \geq \frac{L_1|\lambda|^{1/2}}{2\pi} \geq 2|n|, \quad n \leq 0. \quad (8.2.3)
\]

Next let us examine the case \(j_1 = 0\). We shall see that this implies another set of conditions, which together with (8.2.1) and (8.2.3) implies \(n = 0\). The eigenvalues of the Faddeev-Popov operator \(\mathcal{M}\) are now
\[
w(0, j_2; 1) = p_2^2, \quad w(0, j_2; 2) = p_2^2 + \frac{k^2}{2} + \sqrt{\frac{k^2}{4} + \frac{4f^2}{|\lambda|} p_2^2},
\]
\[
w(0, j_2; 3) = p_2^2 + \frac{k^2}{2} - \sqrt{\frac{k^2}{4} + \frac{4f^2}{|\lambda|} p_2^2}.
\]
If there are no negative eigenvalues \( w(0, j_2; q) \), then for any \( j_2 \) we must have

\[
\left( p_2^2 + \frac{k^2}{2} \right) \geq \frac{k^4}{4} + \left| \frac{f^2 p_2}{q} \right|.
\]

Eliminating \( f \) with (7.7) and simplifying gives

\[
4 \left( \frac{|\lambda|^{1/2} L_1}{2\pi} \right)^2 - 4|n| \frac{|\lambda|^{1/2} L_1}{2\pi} - n^2 - \frac{L_1^2}{L_2^2} j_2^2 \leq 0,
\]

which is an inequality for a quadratic polynomial in \( |\lambda|^{1/2} L_1/(2\pi) \) with positive coefficient in the highest order (quadratic) term. Therefore, the inequality is satisfied if \( |\lambda|^{1/2} L_1/(2\pi) \) is between the roots of the polynomial, i.e.

\[
\frac{|n|}{2} + \sqrt{\frac{n^2}{2} + \frac{L_1^2}{4L_2^2}} \geq |\lambda|^{1/2} \frac{L_1}{2\pi} \geq \frac{|n|}{2} - \sqrt{\frac{n^2}{2} + \frac{L_1^2}{4L_2^2}}.
\]

To satisfy both (8.2.3) and (8.2.4), the left-hand side of (8.2.4) must be greater or equal to the right-hand side of (8.2.3), i.e.

\[
\frac{3}{2} |n| - \sqrt{\frac{n^2}{2} + \frac{L_1^2}{4L_2^2}} \geq 0,
\]

which impossible for \( n \neq 0 \) if \( L_1^2/L_2^2 < 5 \). Thus if (8.2.1) is satisfied, \( \mathcal{M} \) has a negative eigenvalue for a gauge connection (7.6), unless the connection is constant. Hence only the \( n = 0 \) solution can be an extremal curve. Furthermore none of the connections (7.6) with \( L_1 \) and \( x^1 \) interchanged with \( L_2 \) and \( x^2 \), respectively can be extremal curves if \( L_2^2/L_1^2 < 5 \) unless the connection is constant.

We have proved that the only constant-magnitude-curvature solutions of the Yang-Mills-Proca equation which lie within the Gribov horizon are constant gauge connections, provided (8.2.1) is satisfied. We conjecture that examining other choices of \( j_1 \) and \( j_2 \) will eliminate the non-constant connections, even without imposing (8.2.1).

Incidentally, it is quite simple to establish the Wu-Yang ambiguity on the torus, i.e. to prove that many of the connections (7.6) are not equivalent under gauge transformations, despite their having the same field strength after gauge transformations. Consider the Wilson loop on a closed line of length \( L_1 \), parallel to the \( x^1 \)-axis:

\[
\text{Tr } \mathcal{P} \exp i \int_0^{L_1} A_1(x^1, x^2) \, dx^1 = 2 \cos \frac{f^2 L_1}{2|\lambda|^{3/2}}, \tag{8.2.5}
\]

and the Wilson loop on a closed line of length \( L_2 \), parallel to the \( x^2 \)-axis:

\[
\text{Tr } \mathcal{P} \exp i \int_0^{L_2} A_2(x^1, x^2) \, dx^2 = 2 \cos \frac{f L_2}{2|\lambda|^{1/2}}. \tag{8.2.6}
\]
These gauge-invariant quantities clearly depend on $\lambda$. This does not prove that all the configurations (7.6) are gauge-inequivalent, because of the periodicity of the right-hand sides of (8.2.5) and (8.2.6). However, it is clear that most of the connections (7.6) really are gauge-inequivalent, because the values of $\lambda$ are quantized according to (7.7).

### 8.3 A river valley of the gauge theory on the torus

We have shown that the only possible river-valley gauge connections on the torus with constant $(F^a)^2$ are given by with $|\lambda| = f$. These have have, of course, constant gauge connections $A_i^q$. We will show that these are indeed river-valley configurations for $f$ sufficiently small.

Let us first check whether $I[A, 1]$, in (8.1.11), is a minimum of $I[A, g]$ for this configuration. If this is the case, then the distance of the gauge orbit containing $A_i^q$ from the pure-gauge orbit is given by $\rho_0 = I[A, 1]$. Let us begin by repeating the arguments in Subsection 8.1. The second variation of this integral with respect to small gauge transformations $g \approx 1 - ih - h^2/2$ is again the Faddeev-Popov functional (3.1.11). However, the integral is now performed over the torus instead of the plane. As done for (8.1.3), Fourier transforming and replacing $\partial_i$ by $ip_i$, in (3.1.11), the eigenvalues of the quadratic form (8.1.3) are $(p_i)^2$, $(p_i)^2 \pm 2f^{1/2}/(p_i)^2$. The new feature is that $p_i$ is quantized, and may only take the values $p_i = 2\pi j_i/L_i$, for some integers $j_1$ and $j_2$. With the exception of the zero modes of global gauge transformations when $p_1 = p_2 = 0$, the eigenvalues are all positive, if and only if

$$f < \pi^2 \min \left\{ \frac{1}{L_1^2}, \frac{1}{L_2^2} \right\}. \tag{8.3.1}$$

This inequality implies that the length of this curve is bounded in the thermodynamic limit:

$$\rho_0^2 \leq I[A, 1] = \frac{fL_1L_2}{2} < \frac{\pi^2}{2} \min \left\{ \frac{L_2}{L_1}, \frac{L_1}{L_2} \right\}. \tag{8.3.2}$$

We will argue, using the methods of Subsection 3.2, that the family of gauge orbits (7.9) with $|\lambda| = f$ is indeed a river valley. Let us examine of the functional $J[A; \delta A]$ defined in (3.2.3). As discussed in Subsection 3.2, we wish to determine whether this functional is positive, subject to the conditions (3.2.4). Since the gauge connection $A$ is translation invariant, we must actually uncover choices of $\delta A$ for which $J[A; \delta A]$ vanishes. However, under the variations $\delta A$ orthogonal to these translations, the functional $J[A; \delta A]$ is positive within the Gribov horizon. This establishes that the family of gauge orbits parametrized by $f$ is indeed a river valley.

The gauge connections we are considering are of the form $A_i^q = f^{1/2} \delta_i^q$. We can solve the first of (3.2.4) by writing the variation of gauge connection as a Fourier series

$$\delta A_i^q(x) = \frac{i\epsilon_{lm}}{L_1L_2} \sum_{p_i=2\pi j_i/L_i} p_m C^a(p) e^{ip\cdot x}, \tag{8.3.3}$$
with the choice of wavenumbers \( p_1 = p_2 = 0 \) automatically excluded. Substituting (8.3.3) into (3.2.3) yields

\[
J[A; \delta A] = \frac{1}{2L_1 L_2} \sum_{\nu} (\mathcal{E}^1(p), \mathcal{E}^2(p), \mathcal{E}^3(p)) \times \left( \begin{array}{ccc}
(p^2)^2 & -fp_1p_2 & -2if^{1/2}p_2p^2 \\
-fp_1p_2 & (p^2)^2 & 2if^{1/2}p_1p^2 \\
2if^{1/2}p_2p^2 & 2if^{1/2}p_1p^2 & (p^2)^2
\end{array} \right) \left( \begin{array}{c}
\mathcal{E}^1(p) \\
\mathcal{E}^2(p) \\
\mathcal{E}^3(p)
\end{array} \right) .
\] (8.3.4)

The matrix in (8.3.4) has eigenvalues

\[
\omega_0 = (p^2)^2, \quad \omega_{\pm} = (p^2)^2 \pm \sqrt{4f(p^2)^3 + f^2p_1^2p_2^2},
\]

of which only \( \omega_- \) can possibly be zero or negative. The eigenvalue \( \omega_- \) is positive provided (8.3.1) holds.

We have proved that the connections (7.6) with \( |\lambda| = f \) lie within the Gribov horizon and are stable under those variations which do not change \( I[A, 1] \). What remains to show is that they lie within the fundamental region. To prove this rigorously we would have to establish that there is no choice of \( g \) such that \( I[A, g] < I[A, 1] \). Though we have not done this, we have not succeeded in reducing \( I \) by a gauge transformation and have convinced ourselves that this is not possible. Consequently, we assert that these gauge connections lie in gauge orbits \( \alpha \), the set of which is a river valley. From (8.3.2) we conclude that the \( \rho_0^2 \) has values on this river valley over the interval from zero to \( \frac{\pi^2}{2} \min \{L_2/L_1, L_1/L_2\} \).

9 Feynman’s arguments and vortex solutions

We now attempt to compare the results of this paper to that of Feynman [2]. In our language, Feynman argued that a slowly-varying connection \( A \) in an orbit \( \alpha \) has a value of \( \rho_0 \) which is considerably smaller than \( I[A, 1] \). He thereby concluded that the lowest-lying excitations were not gluons. This conclusion is strongly supported by our results. Feynman also argued that in reducing \( I[A, 1] \) to \( I[A, g] \), slowly-varying connections \( A \) would be gauge transformed to connections \( A^g \) with a periodic structure, of period \( \approx 1/\sqrt{F} \). He went on to argue that the genuine excitations are described by small oscillating domains of magnetic flux whose size is equal to this period. However, we find no evidence of this. We believe the reason Feynman came to this conclusion is that his method of lowering the integral \( I[A, 1] \) to \( I[A, g] \) was heuristic and did not provide a lower bound. He carried out this reduction in the thermodynamic limit, starting with connections of constant field strength of the form \( A_1 = 0, A_2 = f\delta^{a3}x^1 \). His gauge-transformed gauge potential is not included among the general solutions we have found (in a finite volume, Feynman’s connection is inequivalent to any we consider) [17, 18].
We have found the smallest $I[A, g]$ for configurations of constant field strength in the previous two sections and do not see evidence of a periodic structure in $A^\rho$.

An excitation of the form suggested by Feynman would occur if there existed a river valley whose elements were described by elliptic solutions of (3.1.9), i.e. solutions with $\lambda > 0$. In fact there are such elliptic solutions; they are non-Abelian vortices [19]. This is seen by introducing a chiral field $g$ and writing (3.1.9) and (3.1.8) as

$$-[D_i^g, F_{ij}^g] + \lambda (A^\rho)_j = 0,$$

$$\partial_j(A^\rho)_j = 0,$$

where $A^\rho$, is defined by (2.1) as before, $D_i^g = g^{-1}D_i g$ and $F_{ij}^g = g^{-1}F_{ij} g$ (We are reinterpreting our Proca equation as the unitary gauge formulation of the gauged chiral sigma model). However these vortex solutions have logarithmically-divergent potential energy. This energy can be regularized with a cut-off, but in 2+1 dimensions, this field theory requires only finite renormalization, as the cut-off is removed. Upon removal of the cut-off, vortices are suppressed.

10 Conclusions

In our studies of the structure of the potential-energy functional on orbit space we have found significant differences between the SU(2) and Abelian gauge theories. The fact that standing-wave Proca solutions of Section 5 lie on the Gribov horizon is evidence that the quantized particle excitations bear no resemblance to perturbative gluons. The families of gauge orbits of minimal potential energy lie in river valleys of finite length. If this is true for all such families of orbits, a mass gap must be present.

We are attempting to construct most of the river valleys for the 2 + 1-dimensional SU(2) theory, at least approximately. This may be sufficient to find the vacuum and the lowest excited states.

We are also studying the extremal-curve problem in 3 + 1 dimensions. There are nontrivial solutions of the Proca equation for the hyperbolic ($\lambda > 0$), parabolic ($\lambda = 0$) and elliptic ($\lambda < 0$) cases. Unlike the case of 2 + 1 dimensions, the elliptic solutions (which have ultraviolet-divergent potential energy) cannot be dismissed out of hand, since the coupling constant has an infinite renormalization. These elliptic solutions include vortices [19], as well as other configurations.

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Appendix: the general solution of the constrained Proca equation

A.1 Analyticity and parametrization of connections

By virtue of (7.4) and (7.5) we have that

$$A^a F_a = 0 .$$  \hspace{1cm} (A.1.1)

From the expression for $F$ in terms of $A$ we see that

$$A^a (\partial_1 A_2^a - \partial_2 A_1^a) = 0 .$$

Let us write these equations in complex coordinates, $z = x^1 + i x^2$ and $\bar{z} = x^1 - i x^2$; $\partial = \partial /\partial z = (\partial_1 - i \partial_2) /2$, $\bar{\partial} = \partial /\partial \bar{z} = (\partial_1 + i \partial_2) /2$. The Hermitian gauge connection becomes non-Hermitian in these coordinates: $A = (A_1 - i A_2) /2$, $\bar{A} = (A_1 + i A_2) /2$. Then

$$A^a (\partial \bar{A}^a - \bar{\partial} A^a) = \bar{A}^a (\partial \bar{A}^a - \bar{\partial} A^a) = 0 .$$  \hspace{1cm} (A.1.2)

The Coulomb gauge condition is

$$\partial \bar{A}^a + \bar{\partial} A^a = 0$$  \hspace{1cm} (A.1.3)

Equations (A.1.2) and (A.1.3) imply

$$\bar{\partial} (A^a)^2 = \partial (\bar{A}^a)^2 = 0 .$$  \hspace{1cm} (A.1.4)

The meaning of equations (A.1.4) is that $(A^a)^2$ is a complex-analytic function. If doubly-periodic boundary conditions are imposed, this function must be a constant by Liouville’s theorem. However, it is interesting to the general solution on the plane. We will see that even without doubly-periodic boundary conditions, $(A^a)^2$ is a constant.

We can derive the relations (A.1.4) another way. If we view our system as a Euclidean field theory in two dimensions, the two-dimensional energy-momentum tensor associated with (7.4) has components:

$$T_{11} = \frac{1}{2} (F^a)^2 + \frac{\lambda}{2} (A_1^a)^2 - \frac{\lambda}{2} (A_2^a)^2,$$

$$T_{22} = \frac{1}{2} (F^a)^2 - \frac{\lambda}{2} (A_1^a)^2 + \frac{\lambda}{2} (A_2^a)^2,$$

$$T_{12} = T_{21} = \lambda A_1^a A_2^a .$$

Since $(F^a)^2$ is constant, conservation of energy and momentum, $\partial_i T_{ij} = 0$ implies (A.1.4).
The analyticity conditions (A.1.4) imply that the Yang-Mills connection has the functional form

\[ A^a(z, \bar{z}) = Q(z)G_+^a(z, \bar{z}), \]

\[ \bar{A}^a(z, \bar{z}) = \bar{Q}(\bar{z})[\cosh \Phi(z, \bar{z})G_+^a(z, \bar{z}) + i \sinh \Phi(z, \bar{z})G_-^a(z, \bar{z})], \]  

(A.1.5)

where \( G_+^a \) is a complex-valued isovector satisfying

\[ (G_+^a)^2 = f^2, \quad G_+^a F^a = 0, \]  

(A.1.6)

and \( G_-^a \) is the complex-valued isovector defined by

\[ G_-^a = \frac{1}{f} \epsilon^{abc} F_b G^c. \]  

(A.1.7)

Notice that \( G_-^a \) automatically satisfies

\[ (G_-^a)^2 = f^2, \quad G_-^a F^a = 0. \]  

(A.1.8)

The isovectors \( G_+^a, G_-^a \) and \( F^a \) are not the basis of a right-handed coordinate system of \( \mathbb{R}^3 \), since \( G_+^a \) and \( G_-^a \) are not orthogonal (no complex conjugates are present in (A.1.6), (A.1.8)) or even real. In fact, the matrix \( \mathbf{G} \) defined by

\[ \mathbf{G} = \begin{pmatrix} G_+^T \\ G_-^T \\ F^T \end{pmatrix} = \begin{pmatrix} G_+^1 & G_+^2 & G_+^3 \\ G_-^1 & G_-^2 & G_-^3 \\ F^1 & F^2 & F^3 \end{pmatrix}, \]  

(A.1.9)

whose determinant equals unity by (A.1.7), is not in SO(3) but rather in the adjoint representation of SL(2, \( \mathbb{C} \)) (since \( F^a \) is real, \( \mathbf{G} \) cannot be an arbitrary element of adj[SL(2, \( \mathbb{C} \))]). In any case, the “vector products” in (A.1.6) and (A.1.8) are not inner products.

The function \( \Phi(z, \bar{z}) \) is not arbitrary, but is chosen so that

\[ \bar{G}_+^a(z, \bar{z}) = \cosh \Phi(z, \bar{z})G_+^a(z, \bar{z}) + i \sinh \Phi(z, \bar{z})G_-^a(z, \bar{z}). \]

Such a choice of \( \Phi \) can always be made. One may always write \( G_+^a \) as

\[ G_+^a(z, \bar{z}) = \gamma^a(z, \bar{z}) \cosh \frac{\Phi(z, \bar{z})}{2} - i \beta^a(z, \bar{z}) \sinh \frac{\Phi(z, \bar{z})}{2}, \]

\[ G_-^a(z, \bar{z}) = \beta^a(z, \bar{z}) \cosh \frac{\Phi(z, \bar{z})}{2} + i \gamma^a(z, \bar{z}) \sinh \frac{\Phi(z, \bar{z})}{2}, \]  

(A.1.10)

where the isovectors \( \beta^a, \gamma^a \) and \( F^a \) are real orthogonal and normalized, i.e. \( (\beta^a)^2 = (\gamma^a)^2 = f^2 \) and \( \beta^a = \epsilon^{abc} F^b \gamma^c \). We will find this expression (A.1.10) useful when we write down the explicit form of the gauge connection and the field strength.
A.2 Parallel transport

Before proceeding further, it is convenient to define the complex one-forms $U_a$, $V_a$ and $W_a$ which describe how the vectors $G^a_+$ and $F^a$ change under translations:

$$
\partial G^a_+ = U_a G^a_+ + V_a F^a_+ \quad \partial G^a_- = -U_a G^a_- + W_a F^a_- \quad \partial F^a = -V_a G^a_+ - W_a G^a_- \\
\bar{\partial} G^a_+ = \bar{U}_a G^a_+ + \bar{V}_a F^a_+ \quad \bar{\partial} G^a_- = -\bar{U}_a G^a_- + \bar{W}_a F^a_- \quad \bar{\partial} F^a = -\bar{V}_a G^a_+ - \bar{W}_a G^a_- .
$$

(A.2.1)

The form of (A.2.1) is dictated by (7.5), (A.1.6) and (A.1.8). These one-forms define a connection in the adjoint representation of $SL(2, \mathbb{C})$, namely

$$
B = dG G^{-1} = \begin{pmatrix} 0 & U & V \\
-U & 0 & W \\
-V & -W & 0 \end{pmatrix} , \quad \bar{B} = d\bar{G} \bar{G}^{-1} = \begin{pmatrix} 0 & \bar{U} & \bar{V} \\
-\bar{U} & 0 & \bar{W} \\
-\bar{V} & -\bar{W} & 0 \end{pmatrix} .
$$

This connection is obviously flat from the definition, i.e.

$$
[\partial - B, \bar{\partial} - \bar{B}] = 0 .
$$

(A.2.2)

For later convenience, we define the antisymmetric matrices $l_1$, $l_2$ and $l_3$ by

$$
l_1 = \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 \end{pmatrix} , \quad l_2 = \begin{pmatrix} 0 & 0 & -1 \\
0 & 0 & 0 \\
1 & 0 \end{pmatrix} , \quad l_3 = \begin{pmatrix} 0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 \end{pmatrix} ,
$$

so that $B = Wl_1 - Vl_2 + Ul_3$ and $\bar{B} = \bar{W}l_1 - \bar{V}l_2 + \bar{U}l_3$.

If we know $G$ at one point $z_0, \bar{z}_0$ and we also know the connection $B, \bar{B}$ everywhere in the plane, we can determine $F^a, A^a$ and $\bar{A}^a$ everywhere. This follows from the formula for parallel transport of the matrix $G$:

$$
G(z, \bar{z}) = \mathcal{P} e^{\int_{z_0}^{z} dz' B(z', \bar{z})} \mathcal{P} e^{\int_{\bar{z}_0}^{\bar{z}} d\bar{z}' \bar{B}(z_0, z')} G(z_0, \bar{z}_0) \quad \text{(A.2.3)}
$$

The path of the integration in (A.2.3) can be deformed to any path from $z_0, \bar{z}_0$ to $z, \bar{z}$ since $B, \bar{B}$ is flat.

A.3 Imposing the equations of motion

Simplifying the equations of motion (7.3) with (A.2.1) yields

$$
-V G^a_+ - W G^a_- + \epsilon^{abc} Q G^b_+ F^c = i\lambda Q G^a_+ ,
$$

$$
-\bar{V} G^a_+ - \bar{W} G^a_- + \epsilon^{abc} \bar{Q} (\cosh \Phi^b_+ + i \sinh \Phi^b_-) F^c = -i\lambda \bar{Q} (\cosh \phi G^a_+ + i \sinh \phi G^a_-) .
$$
From the linear independence of $G_a^\pm$, $F^a$, the equations of motion reduce to

\[ V = -i\lambda Q, \quad W = -fQ, \]

\[ \bar{V} = i\bar{Q}(f \sinh \Phi + \lambda \cosh \Phi), \quad \bar{W} = -\bar{Q}(\lambda \sinh \Phi + f \cosh \Phi). \]  \hspace{1cm} (A.3.1)

Notice that the equations of motion alone impose no restriction on $U$ or $\bar{U}$.

**A.4 Consistency of $F$ as curvature**

We next impose (2.3) and (A.1.1) on (A.1.5). Since $G_a^\pm$ and $F^a$ are linearly independent, (2.3) is equivalent to

\[ 0 = G_a^+(\bar{\partial A}^a - \partial \bar{A}^a + \epsilon^{abc} \bar{A}^b A^c), \]

\[ 0 = G_a^-(\bar{\partial A}^a - \partial \bar{A}^a + \epsilon^{abc} \bar{A}^b A^c), \]

\[ f^2 = 2i F^a(\bar{\partial A}^a - \partial \bar{A}^a + \epsilon^{abc} \bar{A}^b A^c). \]

Using (A.1.1) and (7.3), these equations simplify to

\[ 0 = \bar{Q} \sinh \Phi (i\partial \Phi + U), \] \hspace{1cm} (A.4.1)

\[ 0 = Q\bar{U} - QU \cosh \Phi (i\partial \Phi + U), \] \hspace{1cm} (A.4.2)

\[ -\frac{i}{2} = Q\bar{V} - \bar{Q}(\cosh \Phi V + i \sinh \Phi W) - iQ\bar{Q}f \sinh \Phi. \] \hspace{1cm} (A.4.3)

Substituting the equations of motion (A.3.1) into (A.4.1) and (A.4.2) means that either

- I. $\sinh \Phi = 0$, $Q\bar{U} = \bar{Q}U$, or
- II. $i\partial \Phi + U = 0$ and $\bar{U} = 0$.

We will see in the next subsection that I. can be eliminated. The third equation (A.4.3) upon substitution of (A.3.1) becomes

\[ \frac{1}{2QQ} = -2\lambda \cosh \Phi - f \sinh \Phi. \] \hspace{1cm} (A.4.4)

Notice that this equation implies that $\Phi$ is real.

By substituting the equations of motion into the expression for the curvature as we have done, the full content of the original constrained Proca equations (7.4), (7.3) is in (A.2.2) and (A.4.4) together with either I. or II.
A.5 Flatness and the solution for $B$ and $\bar{B}$

We next evaluate the commutator in (A.2.2) for I. and II.

If we assume I., then there must exist a function $c(z, \bar{z})$ such that $U(z, \bar{z}) = Q(z)c(z, \bar{z})$ and $\bar{U}(z, \bar{z}) = \bar{Q}(z)c(z, \bar{z})$. The flatness condition (A.2.2) is

$$(Q\partial c - \bar{Q}\partial c) l_3 + 2i\lambda Q\bar{Q}(f l_3 + c l_1) = 0.$$  

This equation implies that $c = 0$ and $2i\lambda f = 0$, and is therefore not viable.

We see that II. must be satisfied. Evaluating (A.2.2) yields

$$\partial\bar{\partial}\Phi + Q\bar{Q}[2\lambda f \cosh\Phi + (f^2 + \lambda^2) \sinh\Phi] = 0,$$

and is just the Lax pair of this equation. Defining

$$\xi = \sqrt{f^2 + \lambda^2} \int z Q(z)dz, \quad \bar{\xi} = \sqrt{f^2 + \lambda^2} \int \bar{z} Q(\bar{z})d\bar{z}$$

and the new field

$$\Psi = \Phi + \tanh^{-1} \frac{2\lambda f}{f^2 + \lambda^2},$$

reveals that (A.5.1) is the sinh-Gordon equation

$$\frac{\partial^2 \Psi(\xi, \bar{\xi})}{\partial\xi \partial\bar{\xi}} + \sinh\Psi(\xi, \bar{\xi}) = 0.$$  

(A.5.3)

However, this equation together with condition (A.4.4) actually implies that $\Psi$ is a constant.

The solution of (A.4.4) can be written in terms of $\Psi$ as

$$\Psi = \tanh^{-1} \frac{2\lambda f}{f^2 + \lambda^2} - \tanh^{-1} \frac{2\lambda}{f} - \sinh^{-1} \frac{1}{2Q\bar{Q}\sqrt{f^2 - 4\lambda^2}}.$$  

Substitution into (A.5.3) yields

$$\frac{\partial Q^3 \partial\bar{Q}^3}{\partial\xi \partial\bar{\xi}} = C_1[4(f^2 - 4\lambda^2)Q^2\bar{Q}^2 - 1]^{3/2} + C_2[4(f^2 - 4\lambda^2)Q^2\bar{Q}^2 - 1]^2,$$

where $C_1$ and $C_2$ are constants depending only on $f$ and $\lambda$. In order to have nonconstant solutions, the right-hand side of this equation would have to factorize into a function of $Q$ times a function of $\bar{Q}$. However such a factorization is impossible. Therefore $Q$, $\bar{Q}$ and $\Phi$ are all constants. Of course, we would also have found this result by simply imposing doubly-periodic boundary conditions on the analytic function $Q(z)$ and the antianalytic function $\bar{Q}(z)$ (by Liouville’s theorem). It is interesting that $Q$, $\bar{Q}$ and $\Phi$ are constant even without taking boundary conditions into account.
Our final expressions for $Q$, $\bar{Q}$ and $\Phi$ are

$$Q = \sqrt{\frac{f^2 - \lambda^2}{4\lambda^3}} e^{i\theta}, \quad \bar{Q} = \sqrt{\frac{f^2 - \lambda^2}{4\lambda^3}} e^{-i\theta},$$

$$\Phi = \tanh^{-1} \frac{2\lambda f}{f^2 + \lambda^2}.$$

From these expressions, we can evaluate the integral of $(A^a_i)^2$ to find

$$T[A, 1] = -L_1 L_2 \frac{f^4(f^2 + \lambda^2)}{4\lambda^3},$$

from which we see that $\lambda$ must be negative. The connections $B$ and $\bar{B}$ are

$$B = \sqrt{\frac{f^2 - \lambda^2}{4\lambda}} e^{i\theta} \begin{pmatrix} 0 & 0 & -i\lambda \\ 0 & 0 & -f \\ i\lambda & f & 0 \end{pmatrix},$$

$$\bar{B} = \sqrt{\frac{f^2 - \lambda^2}{4\lambda}} e^{-i\theta} \begin{pmatrix} 0 & 0 & -i\lambda \\ 0 & 0 & -f \\ i\lambda & f & 0 \end{pmatrix} \text{sgn}(f^2 - \lambda^2). \quad (A.5.4)$$

### A.6 Periodicity of the gauge fields

In order to determine the form of the gauge fields at all points on the plane, using (A.2.3), we must diagonalize the matrices (A.5.4). These constant matrices are simultaneously diagonalizable, as they must be in order for (A.2.2) to hold. The diagonalization is

$$B = S B_{\text{diag}} S^{-1}, \quad \bar{B} = S \bar{B}_{\text{diag}} S^{-1},$$

with

$$S = \begin{pmatrix} -f & -i\lambda & -i\lambda \\ i\lambda & -f & -f \\ 0 & i\sqrt{f^2 - \lambda^2} & -i\sqrt{f^2 - \lambda^2} \end{pmatrix}.$$

Using this diagonalization, (A.2.3) may be evaluated:

$$G(z, \bar{z}) = \begin{pmatrix} f^2 - \lambda^2 c & i|\lambda|(1 - c) & i|\lambda|\sqrt{f^2 - \lambda^2} s \\ i|\lambda|(1 - c) & \lambda^2 + f^2 c & -f\sqrt{f^2 - \lambda^2} s \\ -i|\lambda|\sqrt{f^2 - \lambda^2} s & f\sqrt{f^2 - \lambda^2} s & (f^2 - \lambda^2) c \end{pmatrix}$$

$$\times \frac{1}{f^2 - \lambda^2} G(z_0, \bar{z}_0), \quad (A.6.1)$$
where \( c \) and \( s \) are abbreviations for

\[
    c = \cos \left( \frac{f^2 - \lambda^2}{2|\lambda|^{3/2}} e^{i\theta} (z - z_0) + \frac{|f^2 - \lambda^2|}{2|\lambda|^{3/2}} e^{-i\theta} (\bar{z} - \bar{z}_0) \right),
\]

\[
    s = \sin \left( \frac{f^2 - \lambda^2}{2|\lambda|^{3/2}} e^{i\theta} (z - z_0) + \frac{|f^2 - \lambda^2|}{2|\lambda|^{3/2}} e^{-i\theta} (\bar{z} - \bar{z}_0) \right).
\]

While the expression (A.6.1) gives a real field strength \( F^a \) only if \( \lambda^2 \leq f^2 \), we can continue the solution to any \( \lambda < 0 \).

The expression (A.6.1) is simple to use to find the form of the gauge field and the field strength everywhere. If we choose \( \theta = 0 \), so that the spatial dependence is in the \( x^1 \)-direction, the result is

\[
    A^a_1 = \frac{f}{|\lambda|^{3/2}} \gamma^a_0,
\]

\[
    A^a_2 = \frac{1}{|\lambda|^{1/2}} \beta^a_0 \cos \left( \frac{f^2 - \lambda^2}{|\lambda|^{3/2}} (x^1 - x^1_0) \right) - \frac{1}{|\lambda|^{1/2}} F^a_0 \sin \left( \frac{f^2 - \lambda^2}{|\lambda|^{3/2}} (x^1 - x^1_0) \right),
\]

\[
    F^a = \beta^a_0 \sin \left( \frac{f^2 - \lambda^2}{|\lambda|^{3/2}} (x^1 - x^1_0) \right) - F^a_0 \cos \left( \frac{f^2 - \lambda^2}{|\lambda|^{3/2}} (x^1 - x^1_0) \right),
\]

where \( F^a_0, \gamma^a_0, \beta^a_0 \) are the field strength and the isovectors in (A.1.10), \( F^a(z, \bar{z}), \gamma^a(z, \bar{z}) \) and \( \beta^a(z, \bar{z}) \), respectively, evaluated at \( z_0 = x^1_0 + ix^2_0 \) and \( \bar{z}_0 = x^1_0 - ix^2_0 \).

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