ON THE FIRST ORDER OPERATORS
IN BIMODULES

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Abstract

We analyse the structure of the first order operators in bimodules introduced by A. Connes. We apply this analysis to the theory of connections on bimodules generalizing thereby several proposals.

1 Introduction

It is well known that the notion of linear partial differential operator admits the following algebraic formulation. Let \( \mathcal{A} \) be a commutative associative algebra over \( \mathbb{C} \) and let \( \mathcal{M} \) and \( \mathcal{N} \) be two \( \mathcal{A} \)-modules. A linear mapping \( D \) of \( \mathcal{M} \) into \( \mathcal{N} \) is called an operator of order 0 of \( \mathcal{M} \) into \( \mathcal{N} \) if it is a \( \mathcal{A} \)-module homomorphism, i.e. if \([D,f] = 0, \forall f \in \mathcal{A}, (f \in \mathcal{A} \text{ being identified with the multiplication by } f \text{ in } \mathcal{M} \text{ and in } \mathcal{N})\). Then, one defines inductively the operators of order \( k \in \mathbb{N} \): A linear mapping \( D \) of \( \mathcal{M} \) into \( \mathcal{N} \) is an operator of order \( k+1 \) of \( \mathcal{M} \) into \( \mathcal{N} \) if \([D,f] \) is an operator of order \( k \) of \( \mathcal{M} \) into \( \mathcal{N} \), \( \forall f \in \mathcal{A} \). For bad choices of \( \mathcal{A} \), it may happen that this notion is not very appealing, however it makes sense and when \( \mathcal{A} \) is the algebra of smooth functions on a smooth manifold \( X \) and when \( \mathcal{M} \) and \( \mathcal{N} \) are the modules of smooth sections of two smooth complex vector bundles \( E \) and \( F \) over \( X \) then this notion is just the usual notion of linear partial differential operators of order \( k \in \mathbb{N} \) of \( E \) into \( F \). The troubles start if one tries to replace \( \mathcal{A} \) by a noncommutative associative algebra and \( \mathcal{M} \) and \( \mathcal{N} \) by left (or right) \( \mathcal{A} \)-module. Then the definition breaks down because multiplications by elements \( f \) and \( g \) of \( \mathcal{A} \) do not commute. However, it was noticed by A. Connes [3] that if one works with bimodules, there is a natural noncommutative generalization of first order operators since the multiplications on the right and on the left commute. Moreover, in [3] it was pointed out that the first order condition for the generalized Dirac operator plays an important role in the connection...
between the noncommutative Poincaré duality and cyclic cohomology. On the other hand, it was observed at several places, e.g. [6][7], that the natural generalization of the notion of module over a commutative algebra is not necessarily the notion of left (or right) \( A \)-module over a noncommutative algebra \( A \) but can be a notion of bimodule over \( A \). Furthermore it was claimed in [7] that the latter point of view is unavoidable if one wants to discuss reality conditions and if one takes the Jordan algebras of hermitian elements of complex \(*\)-algebras as the noncommutative analogue of algebras of real functions (e.g. as in quantum theory).

In section 2 we recall the definition of first order operators in a form which is convenient for our purposes.

In section 3 we give some basic general examples.

In section 4 we establish the general structure of first order operators and describe the appropriate notion of symbol.

In section 5 we apply our result to the theory of connections on bimodules and show the relation between the symbols and previously introduced twistings (generalized transpositions) in the case of noncommutative generalizations of linear connections [9].

## 2 First order operators in bimodules

In this paper, \( \mathcal{A} \) and \( \mathcal{B} \) are unital associative algebras over \( \mathbb{C} \), their units \( 1_{\mathcal{A}} \)

and \( 1_{\mathcal{B}} \) will be simply denoted by \( 1 \) when no confusion arises. Let \( \mathcal{M} \) and \( \mathcal{N} \) be two \((\mathcal{A}, \mathcal{B})\)-bimodules, (i.e. \( \mathcal{A} \otimes \mathcal{B}^{\text{op}} \)-modules), and let \( \mathcal{L}(\mathcal{M}, \mathcal{N}) \) be the vector space of all linear mappings of \( \mathcal{M} \) into \( \mathcal{N} \). Among the elements \( \mathcal{L}(\mathcal{M}, \mathcal{N}) \), one can distinguish several natural subclasses. The most natural subspace of \( \mathcal{L}(\mathcal{M}, \mathcal{N}) \) is the space \( \text{Hom}_{\mathcal{A}}^\mathcal{B}(\mathcal{M}, \mathcal{N}) \) of all \((\mathcal{A}, \mathcal{B})\)-bimodule homomorphisms
of $\mathcal{M}$ into $\mathcal{N}$. Other natural subspaces are the space $\text{Hom}^B(\mathcal{M}, \mathcal{N})$ of all right $B$-module homomorphisms of $\mathcal{M}$ into $\mathcal{N}$ and the space $\text{Hom}_A(\mathcal{M}, \mathcal{N})$ of all left $A$-module homomorphisms of $\mathcal{M}$ into $\mathcal{N}$. However, from the point of view of the $(\mathcal{A}, \mathcal{B})$-bimodule structure, it was pointed out in [3] that there is a more symmetrical subspace of $\mathcal{L}(\mathcal{M}, \mathcal{N})$ which contains both $\text{Hom}^B(\mathcal{M}, \mathcal{N})$ and $\text{Hom}_A(\mathcal{M}, \mathcal{N})$ which we now describe. One has the following lemma.

**Lemma 1** The following conditions a) and b) are equivalent for an element $D$ of $\mathcal{L}(\mathcal{M}, \mathcal{N})$.

a) for any $f \in \mathcal{A}$, $m \mapsto D(fm) - fD(m)$ is a right $B$-module homomorphism.

b) for any $g \in \mathcal{B}$, $m \mapsto D(mg) - D(m)g$ is a left $A$-module homomorphism.

**Proof** [3]. Let $L_f$ be the left multiplication by $f \in \mathcal{A}$ and let $R_g$ be the right multiplication by $g \in \mathcal{B}$ in $\mathcal{M}$ and $\mathcal{N}$. The condition a) reads $[[D, L_f], R_g] = 0$, $\forall f \in \mathcal{A}$, $\forall g \in \mathcal{B}$, and the condition b) reads $[[D, R_g], L_f] = 0$, $\forall f \in \mathcal{A}$, $\forall g \in \mathcal{B}$. On the other hand one has $[L_f, R_g] = 0$ which implies that $[[D, L_f], R_g] = [[D, R_g], L_f]$. □

An element $D$ of $\mathcal{L}(\mathcal{M}, \mathcal{N})$ which satisfies the above equivalent conditions a) and b) is called [3] a first order operator or an operator of order 1 of $\mathcal{M}$ into $\mathcal{N}$. The set of all first order operators of $\mathcal{M}$ into $\mathcal{N}$ is a subspace of $\mathcal{L}(\mathcal{M}, \mathcal{N})$ which will be denoted by $\mathcal{L}_1(\mathcal{M}, \mathcal{N})$:

$\mathcal{L}_1(\mathcal{M}, \mathcal{N}) = \{ D \in \mathcal{L}(\mathcal{M}, \mathcal{N}) | [[D, L_f], R_g] = 0, \forall f \in \mathcal{A}, \forall g \in \mathcal{B} \}$.

This terminology is of course suggested by the fact that when $\mathcal{A}$ and $\mathcal{B}$ coincide with the algebra $C^\infty(X)$ of smooth functions on a smooth manifold $X$ and when $\mathcal{M}$ and $\mathcal{N}$ are the smooth sections of smooth vector bundles $E$ and $F$ over $X$, then $\mathcal{L}_1(\mathcal{M}, \mathcal{N})$ is just the space of ordinary first order
differential operators of \(E\) into \(F\).

**Remark.** One has \(D \in \text{Hom}_A(M,N) \iff [D, L_f] = 0 \forall f \in A\) and \(D \in \text{Hom}_B(M,N) \iff [D, R_g] = 0 \forall g \in B\). Therefore, \(\text{Hom}_A(M,N)\) and \(\text{Hom}_B(M,N)\) are subspaces of \(L_1(M,N)\). When \(f\) runs over \(A\), \([D, L_f]\) are the obstructions for \(D\) to be a left \(A\)-module homomorphism and thus, the condition a) of lemma 1 means that these obstructions are right \(B\)-module homomorphisms. The condition b) can be formulated similarly by exchange of left and right.

### 3 Examples

#### 3.1 The structure of \(L_1(A,N)\) for a \((A,A)\)-bimodule \(N\)

Recall that a *derivation of \(A\) into a bimodule \(N\) over \(A\)* is a linear mapping \(\delta : A \to N\) satisfying \(\delta(fg) = \delta(f)g + f\delta(g), \forall f, g \in A\). The space of all derivations of \(A\) into \(N\) is denoted by \(\text{Der}(A,N)\). For each element \(n \in N\) one defines a derivation \(ad(n) \in \text{Der}(A,N)\) by \(ad(n)(f) = nf - fn, \forall f \in A\). The subspace \(ad(N)\) of \(\text{Der}(A,N)\) is denoted by \(\text{Int}(A,N)\) and its elements are called *inner (or interior) derivations of \(A\) into \(N\)*. By the very definition, a \(D \in L(A,N)\) is a first order operator, i.e. \(D \in L_1(A,N)\), if and only if one has:

\[
D(fg) = D(f)g + fD(g) - fD(1)g, \forall f, g \in A.
\]

It follows that the derivations of \(A\) into \(N\) are exactly the first order operators of \(A\) into \(N\) which vanish on the unit \(1\) of \(A\), i.e. one has:

\[
\text{Der}(A,N) = \{D \in L_1(A,N) | D(1) = 0\}.
\]
One defines two projections $p_R$ and $p_L$ of $L_1(A, \mathcal{N})$ onto $\text{Der}(A, \mathcal{N})$ by setting $p_R(D)(f) = D(f) - D(1)f$ and $p_L(D)(f) = D(f) - fD(1)$. Notice that $p_L(D) - p_R(D) = \text{ad}(D(1))$ so $\text{Im}(p_L - p_R) = \text{Int}(A, \mathcal{N})$. It is clear that $D \mapsto (p_L(D), D(1))$ and $D \mapsto (p_R(D), D(1))$ are both isomorphisms of $L_1(A, \mathcal{N})$ onto $\text{Der}(A, \mathcal{N}) \oplus \mathcal{N}$. Thus one has

$$L_1(A, \mathcal{N}) \simeq \text{Der}(A, \mathcal{N}) \oplus \mathcal{N}.$$  

In fact, one has $\ker(p_R) = \text{Hom}^A(A, \mathcal{N})$, $\ker(p_L) = \text{Hom}_A(A, \mathcal{N})$ and the mapping $D \mapsto D(1)$ of $L(A, \mathcal{N})$ into $\mathcal{N}$ induces isomorphisms

$$\alpha_R : \text{Hom}^A(A, \mathcal{N}) \cong \mathcal{N}$$

and

$$\alpha_L : \text{Hom}_A(A, \mathcal{N}) \cong \mathcal{N},$$

so the two above isomorphisms of $L_1(A, \mathcal{N})$ onto $\text{Der}(A, \mathcal{N}) \oplus \mathcal{N}$ correspond to the canonical splitting of the exact sequences of vector spaces

$$0 \to \text{Hom}^A(A, \mathcal{N}) \xrightarrow{\alpha_R} L_1(A, \mathcal{N}) \xrightarrow{\text{pr}} \text{Der}(A, \mathcal{N}) \to 0$$

and

$$0 \to \text{Hom}_A(A, \mathcal{N}) \xrightarrow{\alpha_L} L_1(A, \mathcal{N}) \xrightarrow{\text{pr}} \text{Der}(A, \mathcal{N}) \to 0$$

associated to the inclusion $\text{Der}(A, \mathcal{N}) \subset L_1(A, \mathcal{N})$.

Notice finally that one has in $L_1(A, \mathcal{N})$

$$\text{Int}(A, \mathcal{N}) = \text{Der}(A, \mathcal{N}) \cap (\text{Hom}^A(A, \mathcal{N}) + \text{Hom}_A(A, \mathcal{N})).$$

### 3.2 First order operators of $A$ into itself

In the case where $\mathcal{N} = A$, the space $L_1(A, A)$, $\text{Der}(A, A)$ and $\text{Int}(A, A)$ will be simply denoted by $L_1(A)$, $\text{Der}(A)$ and $\text{Int}(A)$. If $D_1$ and $D_2$ are elements of $L_1(A)$, it is easy to see that $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ is again an element of $L_1(A)$. Thus $L_1(A)$ is a Lie algebra; it is also, in an obvious way, a module over the center $Z(A)$ of $A$. The subspace $\text{Der}(A)$ is a Lie subalgebra
and a $Z(\mathcal{A})$-submodule of $\mathcal{L}_1(\mathcal{A})$ and $\text{Int}(\mathcal{A})$ is a Lie ideal of $\text{Der}(\mathcal{A})$ and also a $Z(\mathcal{A})$-submodule. The quotient $\text{Out}(\mathcal{A}) = \text{Der}(\mathcal{A})/\text{Int}(\mathcal{A})$ is a Lie algebra and a $Z(\mathcal{A})$-module. The spaces $\text{Hom}^A(\mathcal{A}, \mathcal{A})$ and $\text{Hom}_A(\mathcal{A}, \mathcal{A})$ are both Lie ideals and $Z(\mathcal{A})$-submodules of $\mathcal{L}_1(\mathcal{A})$ and the exact sequences

$$0 \to \text{Hom}^A(\mathcal{A}, \mathcal{A}) \to \mathcal{L}_1(\mathcal{A}) \xrightarrow{pr} \text{Der}(\mathcal{A}) \to 0$$

and

$$0 \to \text{Hom}_A(\mathcal{A}, \mathcal{A}) \to \mathcal{L}_1(\mathcal{A}) \xrightarrow{pl} \text{Der}(\mathcal{A}) \to 0$$

are now exact sequences of Lie algebras and of $Z(\mathcal{A})$-modules, whereas the corresponding isomorphisms $\mathcal{L}_1(\mathcal{A}) \simeq \text{Der}(\mathcal{A}) \oplus \mathcal{A}$ are isomorphisms of Lie algebras and of $Z(\mathcal{A})$-modules. One has the isomorphism of Lie algebra and of $Z(\mathcal{A})$-module

$$\text{Out}(\mathcal{A}) \simeq \frac{\mathcal{L}_1(\mathcal{A})}{(\text{Hom}^A(\mathcal{A}, \mathcal{A}) + \text{Hom}_A(\mathcal{A}, \mathcal{A}))}.$$ 

### 3.3 First order operators associated with derivations

Let $\Omega^1_L$ be a bimodule over $\mathcal{A}$, let $\Omega^1_R$ be a bimodule over $\mathcal{B}$ and let $d_L$ be a derivation of $\mathcal{A}$ into $\Omega^1_L$ and $d_R$ be a derivation of $\mathcal{B}$ into $\Omega^1_R$. Let $\mathcal{M}$ and $\mathcal{N}$ be two $(\mathcal{A}, \mathcal{B})$-bimodules and let $D$ be a linear mapping of $\mathcal{M}$ into $\mathcal{N}$.

1. Assume that there is a $(\mathcal{A}, \mathcal{B})$-bimodule homomorphism $\sigma_L$ of $\Omega^1_L \otimes_A \mathcal{M}$ into $\mathcal{N}$ such that $D(fm) = fD(m) + \sigma_L(d_L(f) \otimes m)$, $\forall m \in \mathcal{M}$ and $\forall f \in \mathcal{A}$, then $D$ is a first order operator.

2. Assume that there is a $(\mathcal{A}, \mathcal{B})$-bimodule homomorphism $\sigma_R$ of $\mathcal{M} \otimes_B \Omega^1_R$ into $\mathcal{N}$ such that $D(mg) = D(m)g + \sigma_R(m \otimes d_R(g))$, $\forall m \in \mathcal{M}$ and $\forall g \in \mathcal{B}$, then $D$ is a first order operator.
Thus any of the two above conditions implies that $D$ is of first order. We shall now show that, conversely, if $D$ is a first order operator of $\mathcal{M}$ into $\mathcal{N}$, these conditions are satisfied with an appropriate choice of the $(d_L, \Omega_L)$ and $(d_R, \Omega_R)$.

4 General structure of first order operators

Recall that in the category of derivations of $\mathcal{A}$ into the bimodules over $\mathcal{A}$ there is an initial object, $d_u : \mathcal{A} \to \Omega^1_u(\mathcal{A})$, which is obtained by the following standard construction [2], [1]. The bimodule $\Omega^1_u(\mathcal{A})$ is the kernel of the multiplication $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, $(m(f \otimes g) = fg)$, and the derivation $d_u : \mathcal{A} \to \Omega^1_u(\mathcal{A})$ is defined by $d_u(f) = 1 \otimes f - f \otimes 1$. This derivation has the following universal property: For any derivation $\delta : \mathcal{A} \to \mathcal{M}$ of $\mathcal{A}$ into a bimodule $\mathcal{M}$ over $\mathcal{A}$, there is a unique bimodule homomorphism $i_\delta$ of $\Omega^1_u(\mathcal{A})$ into $\mathcal{M}$ such that $\delta = i_\delta \circ d_u$. As left (resp. right) $\mathcal{A}$-module $\Omega^1_u(\mathcal{A})$ is isomorphic to $\mathcal{A} \otimes d_u \mathcal{A}$ (resp. $d_u \mathcal{A} \otimes \mathcal{A}$) whereas the kernel of $d_u$ is $\mathbb{C} 1$, i.e. $d_u \mathcal{A} \simeq \mathcal{A}/\mathbb{C} 1$ as vector space. One has the following structure theorem for first order operators.

**Theorem 1** Let $\mathcal{M}$ and $\mathcal{N}$ be two $(\mathcal{A}, \mathcal{B})$-bimodules and let $D$ be a first order operator of $\mathcal{M}$ into $\mathcal{N}$. Then, there is a unique $(\mathcal{A}, \mathcal{B})$-bimodule homomorphism $\sigma_L(D)$ of $\Omega^1_u(\mathcal{A}) \otimes \mathcal{A} \mathcal{M}$ into $\mathcal{N}$ and there is a unique $(\mathcal{A}, \mathcal{B})$-bimodule homomorphism $\sigma_R(D)$ of $\mathcal{M} \otimes \mathcal{B} \Omega^1_u(\mathcal{B})$ into $\mathcal{N}$ such that one has:

$$D(fmg) = fD(m)g + \sigma_L(D)(d_u f \otimes m)g + f\sigma_R(D)(m \otimes d_u g)$$

for any $m \in \mathcal{M}$, $f \in \mathcal{A}$ and $g \in \mathcal{B}$.

**Proof.** By definition, one has

$$D(fmg) - D(fm)g - fD(mg) + fD(m)g = 0$$
which can be rewritten

\[ D(fm) = fD(m)g + (D(fm) - fD(m))g + f(D(mg) - D(m)g). \]

Now we know (lemma 1, a), etc. that \( m \mapsto D(fm) - fD(m) \) is a right \( \mathcal{B} \)-module homomorphism, furthermore it vanishes whenever \( f \in \mathbb{C} \mathbb{1} \), so one defines a \((\mathcal{A}, \mathcal{B})\)-bimodule homomorphism \( \tilde{\sigma}_L(D) \) of \( \Omega^1_u(\mathcal{A}) \otimes \mathcal{M} \) into \( \mathcal{N} \) by setting \( \tilde{\sigma}_L(D)(f_0d_u f_1 \otimes m) = f_0(D(f_1m) - f_1D(m)) \). Moreover, one has

\[
\begin{align*}
\tilde{\sigma}_L(D)(f_0d_u(f_1)h \otimes m) &= \tilde{\sigma}_L(D)(f_0d_u(f_1h) \otimes m) - \tilde{\sigma}_L(D)(f_0f_1d_u h \otimes m) \\
&= f_0(D(f_1hm) - f_1D(hm)) - f_0f_1(D(hm) - hD(m)) \\
&= \tilde{\sigma}_L(D)(f_0d_u f_1 \otimes hm).
\end{align*}
\]

This means that \( \tilde{\sigma}_L(D)(\alpha h \otimes m) - \tilde{\sigma}_L(D)(\alpha \otimes hm) = 0 \), for any \( \alpha \in \Omega^1_u(\mathcal{A}) \), \( m \in \mathcal{M} \) and \( h \in \mathcal{A} \), i.e. that \( \tilde{\sigma}_L(D) \) passes to the quotient and defines a \((\mathcal{A}, \mathcal{B})\)-bimodule homomorphism \( \sigma_L(D) \) of \( \Omega^1_u(\mathcal{A}) \otimes \mathcal{M} \) into \( \mathcal{N} \). The uniqueness of \( \sigma_L(D) \) is obvious by setting \( g = \mathbb{1} \) in the statement. One proceeds similarly on the other side for \( \sigma_R(D) \). □

In view of 3.3, the converse is also true: any element \( D \) of \( \mathcal{L}(\mathcal{M}, \mathcal{N}) \) for which there are \( \sigma_L(D) \) and \( \sigma_R(D) \) as above is a first order operator.

It is clear that \( \sigma_L(D) \) and \( \sigma_R(D) \) are the appropriate generalization of the notion of symbol in this setting. We shall refer to them as the left and the right universal symbol of \( D \) respectively.

In order to make contact with this notion, let us investigate the examples of the last section. So let \( D \) be a first order operator of \( \mathcal{A} \) into a bimodule \( \mathcal{N} \) and let \( p_R(D) \) and \( p_L(D) \) be the corresponding derivations of \( \mathcal{A} \) into \( \mathcal{N} \) as in 3.1,
then $\sigma_D(D) = i_{pR(D)}$ and $\sigma_L(D) = i_{pL(D)}$ are the canonical homomorphisms of bimodule over $\mathcal{A}$ of $\Omega^1_u(\mathcal{A})$ into $\mathcal{N}$ associated with the derivations $p_R(D)$ and $p_L(D)$, (i.e. such that $p_R(D) = i_{pR(D)} \circ d_u$ and $p_L(D) = i_{pL(D)} \circ d_u$); in particular, if $\delta$ is a derivation of $\mathcal{A}$ into $\mathcal{N}$, one has $\sigma_R(\delta) = \sigma_L(\delta) = i_\delta$. In the cases of the examples of 3.3, for the case 1, one has $\sigma_L(D) = \sigma_L \circ (id_L \otimes id_M)$ and the case 2, one has $\sigma_R(D) = \sigma_R \circ (id_M \otimes id_R)$, where $id_L$ is the canonical homomorphism of bimodule over $\mathcal{A}$ of $\Omega^1_u(\mathcal{A})$ into $\Omega^1_L$ associated with the derivation $d_L$, $id_R$ is the canonical homomorphism of bimodule over $\mathcal{B}$ of $\Omega^1_u(\mathcal{B})$ into $\Omega^1_R$ associated with the derivation $d_R$ and where $id_M$ is the identity mapping of $M$ onto itself.

**Remark**

In the case where the right and the left module structures are related, the $\sigma_R(D)$ and the $\sigma_L(D)$ are also related. For instance if $\mathcal{A} = \mathcal{B}$ is a commutative algebra and if $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{A}$-modules then $\sigma_R(D) = \sigma_L(D) \circ T$ where $T$ is the transposition of $\mathcal{M} \otimes_A \Omega^1_u(\mathcal{A})$ onto $\Omega^1_u(\mathcal{A}) \otimes_A \mathcal{M}$ defined by $T(m \otimes \alpha) = \alpha \otimes m$, $\forall m \in \mathcal{M}$ and $\forall \alpha \in \Omega^1_u(\mathcal{A})$. In particular, this applies if $\mathcal{A} = \mathcal{B} = C^\infty(X)$ and if $\mathcal{M}$ and $\mathcal{N}$ are the smooth sections of smooth vector bundles $E$ and $F$ over the smooth manifold $X$; in this case one has $\sigma_R(D) = \sigma(D) \circ (id \otimes id_M)$ where $\sigma(D)$ is the usual symbol of the first order linear partial differential operator $D$ of $E$ into $F$ and $id$ is the canonical homomorphism of $\Omega^1_u(C^\infty(X))$ into $\Omega^1(X)$ (= the ordinary 1-forms on $X$) associated with the ordinary differential $d : C^\infty(X) \to \Omega^1(X)$. 

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5 Application to the theory of connections

Let $\Omega$ be a graded differential algebra, with differential $d$, such that $\Omega^0 = A$. The restriction of the differential $d$ to $A$ is then a derivation of $A$ into the bimodule $\Omega^1$ over $A$ (the $\Omega^n$, $n \in \mathbb{N}$, are bimodules over $A$). Recall that a $\Omega$-connection on a left $A$-module $M$ [3], [4] is a linear mapping $\nabla$ of $M$ into $\Omega^1 \otimes_A M$ satisfying

$$
\nabla(fm) = f\nabla(m) + df \otimes m, \; \forall m \in M \text{ and } \forall f \in A.
$$

Suppose now that $M$ is not only a left $A$-module but is a bimodule over $A$. It follows then from 3.3 that a left $A$-module $\Omega$-connection $\nabla$ on $M$ as above is a first order operator of the bimodule $M$ into the bimodule $\Omega^1 \otimes_A M$ over $A$. Consequently, in view of the structure theorem of 4, there is a homomorphism of bimodule over $A$, $\sigma_R(\nabla)$, of $M \otimes_A \Omega^1_u(A)$ into $\Omega^1 \otimes_A M$ such that

$$
\nabla(mg) = \nabla(m)g + \sigma_R(\nabla)(m \otimes d_u g), \; \forall m \in M \text{ and } \forall g \in A.
$$

The homomorphism $\sigma_R(\nabla)$ is the right universal symbol of $\nabla$ whereas its left universal symbol is simply $\sigma_L(\nabla) = i_d \otimes id_M$ where $i_d$ is the canonical bimodule homomorphism of $\Omega^1_u(A)$ into $\Omega^1$ induced by $d$, (i.e. such that $d = i_d \circ d_u$), and $id_M$ is the identity mapping of $M$ onto itself.

In the case where $\sigma_R(\nabla)$ factorizes through a bimodule homomorphism $\sigma$ of $M \otimes_A \Omega^1$ into $\Omega^1 \otimes_A M$ as $\sigma_R(\nabla) = \sigma \circ (id_M \otimes i_d)$, we call $\nabla$ a (left) bimodule $\Omega$-connection on the bimodule $M$. (One defines similarly a (right) bimodule $\Omega$-connection by starting with a right $A$-module $\Omega$-connection on $M$).

In the case where $M$ is the bimodule $\Omega^1$ itself, a (left) bimodule $\Omega$-connection on $\Omega^1$ in the above sense is just what is the first part of the proposal of J.
Mourad [9] for the definition of linear connections in noncommutative geometry, (the second part which relates \( \sigma \) and the product \( \Omega^1 \otimes_A \Omega^1 \to \Omega^2 \) makes sense only in this case). This proposal was applied in some simple examples, e.g. [5], [8].

On the other hand, in [7] connections on central bimodules for the derivation-based differential calculus i.e. for \( \Omega = \Omega_{\text{Der}}(\mathcal{A}) \) were defined and it was pointed out in [9] that when the central bimodule is \( \Omega_{\text{Der}}^1(\mathcal{A}) \) itself the connections of [7] are linear connections in the sense of [9], (the choice of \( \sigma \) being fixed). By a similar argument one sees that the derivation-based connections on central bimodules of [7] are bimodules \( \Omega_{\text{Der}}^1(\mathcal{A}) \)-connections in the above sense. To be precise, this holds strictly in the finite dimensional cases, otherwise, one has to introduce completions of the \( \Omega_{\text{Der}}^1(\mathcal{A}) \otimes_A \mathcal{M} \), but this is merely a technicality.

Thus the above definition of (left) bimodule \( \Omega^1 \)-connections seems very general. However, it is worth noticing here that for the derivation-based connections on central bimodules, the curvatures of these connections are bimodule homomorphisms whereas this is generally not the case for the curvature of a (left) bimodule \( \Omega \)-connection, (it is of course always a left \( \mathcal{A} \)-module homomorphism), although differences of such connections for a fixed \( \sigma \) are bimodule homomorphisms, (see e.g. [5]).

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