BIJECTIVE PROOFS OF SOME COINVOLUTION IDENTITIES RELATED TO
MACDONALD POLYNOMIALS

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Abstract. This paper gives bijective proofs of some novel coinversion identities first discovered by Ayyer, Mandelshtam, and Martin as part of their proof of a new combinatorial formula for the modified Macdonald polynomials $\tilde{H}_\mu$. Those authors used intricate algebraic manipulations of $q$-binomial coefficients to prove these identities, which imply the existence of certain bijections needed in their proof that their formula satisfies the axioms characterizing $\tilde{H}_\mu$. They posed the open problem of constructing such bijections explicitly. We resolve that problem here.

1. Introduction

We begin by reviewing the coinversion statistic and its relation to $q$-binomial coefficients and $q$-multinomial coefficients. Given a formal variable $q$ and a positive integer $n$, define the $q$-integer $\left[ n \right]_q = 1 + q + q^2 + \cdots + q^{n-1}$ and the $q$-factorial $\left[ n \right]_q! = \prod_{j=1}^n [j]_q$. We also set $[0]_q = 0$ and $[0]_q! = 1$.

For integers $k, n$ with $0 \leq k \leq n$, define the $q$-binomial coefficient $\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}$. We also define $\left[ \begin{array}{c} n \\ k \end{array} \right] = 0$ when $k < 0$ or $k > n$.

For nonnegative integers $k_1, k_2, \ldots, k_s$ with $k_1 + k_2 + \cdots + k_s = n$, define the $q$-multinomial coefficient $\left[ \begin{array}{c} n \\ k_1, k_2, \ldots, k_s \end{array} \right]_q = \frac{[n]_q!}{[k_1]_q! [k_2]_q! \cdots [k_s]_q!}$. If any $k_i$ is negative, the $q$-multinomial coefficient is defined to be 0.

For a word $w = w_1 w_2 \cdots w_n$ where each $w_i$ is an integer, the coinversion statistic $\text{coinv}(w)$ is the number of pairs $(i, j)$ with $i < j$ and $w_i < w_j$. For example, $\text{coinv}(231132) = 6$.

The following combinatorial formulas are well-known:

\begin{align*}
\left[ \begin{array}{c} n \\ k_1, k_2, \ldots, k_s \end{array} \right]_q &= \sum_{w \in \mathcal{R}(k_1^1 k_2^2 \cdots s^s)} q^{\text{coinv}(w)}, \\
\left[ \begin{array}{c} n \\ k \end{array} \right]_q &= \sum_{w \in \mathcal{R}(1^2 2^{n-2})} q^{\text{coinv}(w)}.
\end{align*}

These formulas are often stated with $\text{coinv}(w)$ replaced by the inversion count $\text{inv}(w)$, which is the number of $i < j$ with $w_i > w_j$. But the standard proofs using $\text{inv}(w)$ (see, for example, [4, Chpt. 8]) extend at once to $\text{coinv}(w)$ by reversing the natural ordering on $\mathbb{Z}$.

The purpose of this paper is to give bijective proofs of some novel identities involving the coinversion statistic. These identities were recently discovered and proved (algebraically) by Ayyer, Mandelshtam, and Martin as part of their study of the combinatorics of the modified Macdonald polynomials $\tilde{H}_\mu$. Macdonald polynomials are not needed in this paper, but the reader may consult references such as [2, 3, 5, 6] for more information.

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To proceed, we must recall some definitions and results from [1, Sec. 9]. Fix integers \( n \geq 3 \), \( L > 0 \), \( a_2, \ldots, a_{n-1} \geq 0 \), and define \( N = L + 2 a_2 + \cdots + a_{n-1} \). For \( 0 \leq k \leq L \), define \( W_k = R(1-L-k2^{N-L-k}) \) and \( W = \bigcup_{k=0}^{L} W_k \). For any word \( w \in W \), let \( p_n(w) \) be the position of the leftmost \( n \) in \( w \), counting from the left; let \( p_n(w) = \infty \) if no \( n \) occurs in \( w \). Let \( p_1(w) \) be the position of the rightmost 1 in \( w \), counting from the right end of \( w \) and ignoring occurrences of \( n \); let \( p_1(w) = \infty \) if no 1 occurs in \( w \). For example, when \( n = 4 \) and \( w = \) 3142241324243, we have \( p_4(w) = 3 \) and \( p_1(w) = 5 \). Note that the possible finite values of \( p_n(w) \) are \( 1, 2, \ldots, N - L + 1 \) since \( w \) has \( N - k \) symbols (ignoring all copies of \( n \)) and \( L - k + 1 \) copies of 1 must appear to the left of the rightmost 1 in \( w \). Similarly, the possible finite values of \( p_n(w) \) are \( 1, 2, \ldots, N - k + 1 \).

For \( 0 \leq k \leq L \), define \( W_k^> = \{ w \in W_k : p_n(w) > p_1(w) \} \) and \( W_k^< = \{ w \in W_k : p_n(w) \leq p_1(w) \} \). Since \( L > 0 \), we have the boundary cases \( W_0^> = W_0, W_0^< = \emptyset, W_L^< = W_L, \) and \( W_L^> = \emptyset \). Define \( W^> = \bigcup_{k=0}^{L} W_k^> \) and \( W^< = \bigcup_{k=0}^{L} W_k^< \). Our main goal is to construct explicit bijective proofs of the following identities, which are formulas (9.1) through (9.4) of [1].

**Theorem 1.** (a) For \( 0 \leq k \leq L \),

\[
\sum_{w \in W_k^>} q^{\text{coinv}(w)} = \left[ \frac{N}{L, a_2, \ldots, a_{n-1}} \right] q^k \left[ \frac{L-1}{k} \right]_q.
\]

(b) For \( 0 \leq k \leq L \),

\[
\sum_{w \in W_k^<} q^{\text{coinv}(w)} = \left[ \frac{N}{L, a_2, \ldots, a_{n-1}} \right] q^k \left[ \frac{L-1}{k-1} \right]_q.
\]

(c) For \( 0 \leq k \leq L \) and \( 1 \leq i \leq N - L + 1 \),

\[
\sum_{\substack{w \in W_k^> : \cr p_1(w) = i}} q^{\text{coinv}(w)} = q^{k+i-1} \left[ \frac{N-L}{a_2, \ldots, a_{n-1}} \right] q^k \left[ \frac{L-i}{L-k-1, N-L-i+1, k} \right]_q.
\]

(d) For \( 0 \leq k \leq L \) and \( 1 \leq j \leq N - k + 1 \),

\[
\sum_{\substack{w \in W_k^< : \cr p_n(w) = j}} q^{\text{coinv}(w)} = q^{(j-1)L} \left[ \frac{N-L}{a_2, \ldots, a_{n-1}} \right] q^{k} \left[ \frac{N-j}{L-k, N-L-j+1, k-1} \right]_q.
\]

Bijective proofs of parts (a) and (b) of the theorem combine to give bijections proving

\[
\sum_{w \in W_k^>} q^{\text{coinv}(w)} = q^k \sum_{w \in W_k^<} q^{\text{coinv}(w)},
\]

for \( 0 \leq k < L \). These are the crucial bijections the authors of [1] needed to complete their analysis of the quinv statistic in their novel combinatorial formula for modified Macdonald polynomials.

As was already noted in [1], the general formulas in Theorem 1 follow easily from the special case where \( n = 3 \). (This reduction can be done bijectively, as we see later.) Given \( 0 < L \leq N \) and \( 0 \leq k \leq L \), let \( X_k^> = R(1-L-k2^{N-L-k}) \), \( X_k^> = \{ w \in X_k : p_3(w) > p_1(w) \} \), and \( X_k^< = \{ w \in X_k : p_3(w) \leq p_1(w) \} \). For this three-letter case, we are reduced to proving the following formulas:

\[
\sum_{w \in X_k^>} q^{\text{coinv}(w)} = q^k \left[ \frac{N}{L, N-L} \right] q^k \left[ \frac{L-1}{L-1-k, k} \right]_q.
\]
weighted sets

$S$ is a weighted set (using the same weights), and $GF(S)$ is a Cartesian product $A\times B$. If $c$ is an integer, the following notation and facts about weighted sets. A weighted set indexing this sum, the leftmost 3 in $w$ occurs to the left of the rightmost 1 in $w$ if and only if $i + j + k \leq N + 1$.

(a) If $0 < k < L$, then

$$\sum_{w\in R(1-L-kN-L3): p_1(w)=i \text{ and } p_3(w)=j} q^{\text{coinv}(w)} = \left[ \frac{N-j}{k-1} \right] \left[ \frac{N-i-k}{L-k-1} \right] q^{(j-1)k+(L-k)(i-1)}.$$  

For the words $w$ indexing this sum, the leftmost 3 in $w$ occurs to the left of the rightmost 1 in $w$ if and only if $i + j + k \leq N + 1$.

(b) If $k = 0$, then

$$\sum_{w\in R(1-L2N-L): p_1(w)=i \text{ and } p_3(w)=\infty} q^{\text{coinv}(w)} = \left[ \frac{N-i}{L-1} \right] q^{L(i-1)}.$$  

(c) If $k = L$, then

$$\sum_{w\in R(2N3L): p_3(w)=j \text{ and } p_1(w)=\infty} q^{\text{coinv}(w)} = \left[ \frac{N-j}{L-1} \right] q^{(j-1)L}.$$  

This paper is organized as follows. Section 2 gives bijective proofs of some well-known identities for $q$-binomial coefficients and $q$-multinomial coefficients. Section 3 gives a bijective proof of Theorem 2. Section 4 gives a bijective proof of Theorem 3. Section 5 works out a detailed example illustrating all the bijections, which leads to a simplified description of the bijective proof of (3).

2. Preliminary Bijections

This section describes some preliminary bijections needed to prove the main results. We use the following notation and facts about weighted sets. A weighted set is a set $S$ and a weight function $wt: S \rightarrow \mathbb{Z}_{\geq 0}$. The generating function for this weighted set is $GF(S) = \sum_{s\in S} q^{wt(s)}$. For any integer $c$ and weighted set $S$, the symbol $q^cS$ refers to the set $S$ with shifted weight function $wt'(s) = wt(s) + c$. Note that $GF(q^cS) = q^cGF(S)$. If $A$ and $B$ are weighted sets, then the Cartesian product $A \times B = \{(a, b) : a \in A, b \in B\}$ is a weighted set with $wt((a, b)) = wt(a) + wt(b)$, and $GF(A \times B) = GF(A)GF(B)$. If $S_1, \ldots, S_n$ are pairwise disjoint weighted sets, then $S_1 \cup \cdots \cup S_n$ is a weighted set (using the same weights), and $GF(S_1 \cup \cdots \cup S_n) = GF(S_1) + \cdots + GF(S_n)$. For weighted sets $S$ and $T$, we write $S \equiv T$ to mean there is a weight-preserving bijection between $S$
Proposition 4. Factorization Identities for $A$, $B$, and $C$. NICHOLAS A. LOEHR

Let $q$ be the following well-known identity for summing certain $q$-binomial coefficients.

Proposition 3. Fix integers $A, B$ with $1 \leq B \leq A + 1$. There are weight-preserving bijections

$$F : \mathcal{R}(0^{A+1-B}3B) \rightarrow \bigcup_{s=0}^{A+1-B} q^{sB}\mathcal{R}(0^{A+1-B-s}3B^{-1}),$$

$$G : \mathcal{R}(1B2A+1-B) \rightarrow \bigcup_{s=0}^{A+1-B} q^{sB}\mathcal{R}(1B-12A+1-B-s),$$

and therefore

$$\sum_{s=0}^{A-B+1} \left[\frac{A-s}{B-1} q^s\right] = \begin{bmatrix} A+1 \\ B \end{bmatrix}_q.$$

Proof. Given a word $w \in \mathcal{R}(0^{A+1-B}3B)$, write $w = 0^s3w'$ where the displayed 3 is the leftmost 3 in $w$. Define $F(w) = w'$, which belongs to $\mathcal{R}(0^{A+1-B-s}3B^{-1})$ for some $s$ between 0 and $A + 1 - B$. Each of the $s$ copies of 0 at the start of $w$ causes $B$ coinversions with the 3s later in $w$. These coinversions are not present in $w'$, but all other coinversions in $w$ and $w'$ are the same. Thus, $\text{coinv}(w) = sB + \text{coinv}(w')$, so that $F$ preserves weights. $F$ is a bijection with inverse $F^{-1}(w') = 0^s3w'$. When computing the inverse, we can deduce $s$ from $w'$ by counting the 0s in $w'$ (since $A$ and $B$ are fixed and known). In detail, writing $n_0(w')$ for the number of 0s in $w'$, we have $s = A + 1 - B - n_0(w')$.

The bijection $G$ is defined and analyzed similarly: given $w \in \mathcal{R}(1B2A+1-B)$, write $w = w'12s$ where the displayed 1 is the rightmost 1 in $w$, and let $G(w) = w'$. Each 2 at the end of $w$ causes coinversions with all $B$ copies of 1 appearing earlier, so passing from $w$ to $w'$ reduces $\text{coinv}(w)$ by $sB$. So $G$ is a weight-preserving bijection. \hfill $\square$

2.2. Factorization Identities for $q$-Multinomial Coefficients.

Proposition 4. Fix integers $A, B, C \geq 0$. There is a weight-preserving bijection

$$H : \mathcal{R}(1^{A}2B3C) \rightarrow \mathcal{R}(0^{A+B}3C) \times \mathcal{R}(1^{A}2B),$$

and therefore

$$\begin{bmatrix} A+B+C \\ A,B,C \end{bmatrix}_q = \begin{bmatrix} A+B+C \\ A+B,C \end{bmatrix}_q \begin{bmatrix} A+B \\ A,B \end{bmatrix}_q.$$

Proof. Let $H$ map $v \in \mathcal{R}(1^{A}2B3C)$ to $(y, z) \in \mathcal{R}(0^{A+B}3C) \times \mathcal{R}(1^{A}2B)$, where $y$ is obtained from $v$ by replacing each occurrence of 1 or 2 by 0, and $z$ is obtained from $v$ by erasing all 3s. For example, $H(231132) = (030030, 2112)$. It is routine to check that $\text{coinv}(v) = \text{coinv}(y) + \text{coinv}(z) = \text{wt}((y, z))$, so $H$ is weight-preserving. We invert $H$ by using the 1s and 2s in $z$ (reading left to right) to replace the $A + B$ copies of 0 in $y$. For example, $H^{-1}(303000, 1212) = 313212$. \hfill $\square$

The same relabeling idea gives bijective proofs of related identities for $q$-multinomial coefficients. For example, assuming $a_2 + \cdots + a_{n-1} = N - L$, we get

$$\mathcal{R}(1^{L-k}a_2 \cdots (n-1)^{a_{n-1}}n^k) \equiv \mathcal{R}(1^{L-k}2N-L3^k) \times \mathcal{R}(2^{a_2}3^{a_3} \cdots (n-1)^{a_{n-1}}).$$
by mapping $v$ to $(y, z)$, where $y$ is $v$ with the middle letters $2, \ldots, n - 1$ all relabeled as 2 and the biggest letter $n$ relabeled as 3, and $z$ is $v$ with all copies of 1 and $n$ erased. We also get
\begin{equation}
\mathcal{R}(0^L3^{N-L}) \times \mathcal{R}(2^{a_2}3^{a_3} \cdot \cdot \cdot (n-1)^{a_n-1}) \equiv \mathcal{R}(1^L2^{a_2}3^{a_3} \cdot \cdot \cdot (n-1)^{a_n-1})
\end{equation}
by mapping $(y, z)$ to $v$, where $v$ is $y$ with each 0 replaced by 1 and the subword $3^{N-L}$ replaced by the word $z$. It is routine to check that these maps are weight-preserving bijections.

2.3. Symmetry Identities for $q$-Multinomial Coefficients.

Proposition 5. For all integers $A, B, C \geq 0$, there is a weight-preserving bijection
\[ K : \mathcal{R}(1^A2^B3^C) \rightarrow \mathcal{R}(1^A2^B3^C). \]

Proof. Let $K$ act on $v \in \mathcal{R}(1^A2^B3^C)$ by replacing each 2 by 3 and each 3 by 2 in $v$, then reversing the subword of 2s and 3s in $v$. For example, $K(2311323331) = 2211232231$. We invert $K$ by performing the same actions on the output word. It is routine to check that $K$ preserves coinversions. \qed

More generally, if $b_1, \ldots, b_n$ is any permutation of $a_1, \ldots, a_n$, we have the algebraically obvious symmetry property
\begin{equation}
\left[ a_1 + \cdot \cdot \cdot + a_n \right]_{q^b_1,\ldots, b_n} = \left[ a_1 + \cdot \cdot \cdot + a_n \right]_{b_1,\ldots, b_n}.
\end{equation}
The idea in the preceding proof generalizes at once to give a bijection interchanging the frequencies of any two adjacent letters. Composing several bijections of this form, we can transform the initial frequencies $(a_i$ copies of $i$ for all $i)$ to the final frequencies $(b_i$ copies of $i$ for all $i)$. This gives a bijective proof of (16) by showing $\mathcal{R}(1^{a_1}2^{a_2} \cdot \cdot \cdot n^{a_n}) \equiv \mathcal{R}(1^{b_1}2^{b_2} \cdot \cdot \cdot n^{b_n})$.

3. BIJECTIVE PROOF OF THEOREM 2

To prove Theorem 2(a), let $L, N, i, j, k$ be integers with $0 < L \leq N$, $0 < k < L$, $1 \leq i \leq N - L + 1$, and $1 \leq j \leq N - k + 1$. It suffices to define a weight-preserving bijection $P = P_{i,j}$ mapping the domain
\begin{equation}
\mathcal{R}(0^{N-j-k+1}3^{k-1}) \times \mathcal{R}(1^{L-k+1}2^{N-L-i+1}) q^{(j-1)k+(L-k)(i-1)}
\end{equation}
one-to-one onto the codomain
\[ \{ w \in \mathcal{R}(1^{L-k}2^{N-L}3^k) : p_1(w) = i \text{ and } p_3(w) = j \}. \]
Given an input $(y, z)$ in the domain of $P$, we build $w = P(y, z)$ as follows. Start with $N$ empty slots for the $N$ symbols in $w$. Put a 3 in slot $j$; we refer to this 3 as L3 (the leftmost 3 in $w$). There are $N - j$ slots to the right of position $j$, and the remaining $k - 1$ copies of 3 must go in these slots to ensure that $p_3(w) = j$. Place the word $y$ in these $N - j$ slots, regarding a 0 in $y$ as a slot in $w$ that still remains empty for now. Next, visit the empty slots in $w$ from right to left, placing $i - 1$ copies of 2 followed by a 1, which ensures that $p_1(w) = i$. We refer to this copy of 1 as R1 (the rightmost 1 in $w$). Finally, fill the remaining empty slots in $w$ (to the left of the 1 just placed) with the remaining $L - k - 1$ copies of 1 and the remaining $N - L - i + 1$ copies of 2. Do this by reading $z$ (left to right) and filling the empty slots (left to right) using the symbols in $z$. It is routine to check that this procedure is invertible, so $P$ is a bijection.

Suppose R1 is placed to the left of L3 in $w$. This forces R1 to be the leftmost symbol in the collection $C$ consisting of the $k - 1$ copies of 3 to the right of L3, the $i - 1$ copies of 2 to the right of R1, and R1 itself. Since there are $N - j$ available slots to the right of L3, we must have $(k-1)+(i-1)+1 > N-j$, so $i+j+k > N+1$. Conversely, if $i+j+k > N+1$, then the $N-j$
slots to the right of L3 cannot accommodate all symbols in C, which forces R1 to be placed to the left of L3 in w.

To see that P preserves weights, we show that \( \text{coinv}(w) = \text{coinv}(y) + \text{coinv}(z) + (j - 1)k + (L - k)(i - 1) \). Note that every 0 in y is placed in w to the right of L3 and eventually gets relabeled as a 1 or 2. Thus, coinv(y) counts all coinversions in w involving a 1 or 2 to the right of L3 followed by a 3 to the right of L3. Similarly, coinv(z) counts all coinversions in w involving a 1 to the left of R1 followed by a 2 to the left of R1. We finish counting the coinversions of w as follows. First, each of the \( j - 1 \) symbols to the left of L3 (which must be 1 or 2) causes a coinversion with each of the \( k \) copies of 3 in w, giving \( (j - 1)k \) coinversions. Second, each of the \( L - k \) copies of 1 in w causes a coinversion with each of the \( i - 1 \) copies of 2 to the right of R1, giving \( (L - k)(i - 1) \) coinversions. This explains the weight-shifting factor in the domain \( \{17\} \).

For example, let \( N = 13, L = 6, k = 4, i = 5, j = 3, y = 0030003030 \), and \( z = 2122 \). Then \( P(y, z) = w = 2132231223232 \) where \( p_1(w) = 5 \) and \( p_3(w) = 3 \). Note that coinv(y) = 13, coinv(z) = 2, and coinv(w) = 31 = 13 + 2 + 2.4 + 2.4.

We can prove parts (b) and (c) of Theorem 2 by using degenerate versions of the bijection used to prove (a). For the \( k = 0 \) case, we define a bijection

\[
\mathcal{R}(1^{L-1}2^{N-(L+i+1)})q^{j-1} \rightarrow \{ w \in \mathcal{R}(1^{L-1}2^{N-L}) : p_1(w) = i \}
\]

by mapping \( z \) in the domain to \( w = z12^{j-1} \) in the codomain. The extra \( q \)-power appears since each of the \( L \) copies of 1 in w causes a coinversion with each of the \( i - 1 \) copies of 2 added at the end. For the \( k = L \) case, we define a bijection

\[
\mathcal{R}(2^{N-L-j+1}3^{L-1})q^{j-1} \rightarrow \{ w \in \mathcal{R}(2^{N-L}3^{L}) : p_3(w) = j \}
\]

by mapping \( y \) in the domain to \( w = 2^j 13y \) in the codomain. The extra \( q \)-power appears since each of the \( j - 1 \) copies of 2 at the start of \( w \) causes a coinversion with all \( L \) copies of 3 in w.

4. Bijective Proof of Theorem 1

4.1. Proof of \( \{19\} \). Fix \( i \) with \( 1 \leq i \leq N - L + 1 \). To prove \( \{19\} \) when \( 0 < k < L \), we must define a weight-preserving bijection

\[
\{ w \in \mathcal{R}(1^{L-k}2^{N-L}3^k) : p_3(w) > p_1(w) = i \} \rightarrow \mathcal{R}(1^{L-k-1}2^{N-L-i+1}3^k)q^{k+(i-1)L}.
\]

Take the disjoint union of the bijections \( P_{i,j}^{-1} \) as \( j \) ranges over possible values larger than \( i \). This maps the domain in \( \{20\} \) to the disjoint union

\[
\bigcup_{j=i+1}^{N-k+1} \mathcal{R}(0^{N-j-k+1}3^{k-i}) \times \mathcal{R}(1^{L-k-1}2^{N-L-i+1})q^{j-1-i+k+L(i-1)+k}.
\]

Letting \( s = j - i - 1 \), we can write this union as

\[
\bigcup_{s=0}^{N-k-i} \mathcal{R}(0^{N-i-k-s}3^{k-i})q^{s} \times \mathcal{R}(1^{L-k-1}2^{N-L-i+1})q^{k+L(i-1)}.
\]

The parenthesized piece is the codomain of the bijection \( F \) in Proposition \( \{13\} \) taking \( B = k \) and \( A = N - i - 1 \). Applying \( F^{-1} \) to this piece, we get a bijection to

\[
\mathcal{R}(0^{N-i-k}3^k) \times \mathcal{R}(1^{L-k-1}2^{N-L-i+1})q^{k+L(i-1)}.
\]

Applying \( H^{-1} \) from Proposition \( \{3\} \) we reach

\[
\mathcal{R}(1^{L-k-1}2^{N-L-i+1}3^k)q^{k+L(i-1)},
\]
which is the codomain in (20). When \( k = 0 \), (20) reduces to
\[
\{ w \in \mathcal{R}(1^{L-2^N-L}) : p_1(w) = i \} \to \mathcal{R}(1^{L-12^N-L-1})q^{L(i-1)},
\]
which is the inverse of the bijection (18). When \( k = L \), both sides of (9) are 0.

4.2. Proof of (10). Fix \( j \) with \( 1 \leq j \leq N-k+1 \). To prove (10) when \( 0 < k < L \), we must define a weight-preserving bijection
\[
(21) \quad \{ w \in \mathcal{R}(1^{L-k2^N-L-3^k}) : j = p_3(w) \leq p_1(w) \} \to \mathcal{R}(1^{L-k2^N-L-j+13^k-1})q^{L(j-1)}.
\]
Take the disjoint union of the bijections \( P_{i,j}^{-1} \) as \( i \) ranges over its possible values that are at least \( j \). This maps the domain in (21) to the disjoint union
\[
\bigcup_{i=j}^{N-L+1} \mathcal{R}(0^{N-j-k+13^k-1}) \times \mathcal{R}(1^{L-k-12^N-L-1})q^{L(j-1)k+(L-k)(i-1)}.
\]
Letting \( s = i - j \), we can write this union as
\[
q^{L(j-1)} \mathcal{R}(0^{N-j-k+13^k-1}) \times \left( \bigcup_{s=0}^{N-L-j+1} \mathcal{R}(1^{L-k-12^N-L-s-j+1})q^{(L-k)s} \right).
\]
The parenthesized piece is the codomain of the bijection \( G \) in Proposition 3, taking \( B = L - k \) and
\( A = N - k - j \). Applying \( G^{-1} \) to this piece, we get a bijection to
\[
(22) \quad q^{L(j-1)} \mathcal{R}(0^{N-j-k+13^k-1}) \times \mathcal{R}(1^{L-k2^N-L-j+1}).
\]
Applying \( H^{-1} \) from Proposition 4, we reach
\[
\mathcal{R}(1^{L-k2^N-L-j+13^k-1})q^{L(j-1)},
\]
which is the codomain in (21). When \( k = L \), (21) reduces to
\[
\{ w \in \mathcal{R}(1^{L-k2^N-L-3^k}) : p_3(w) = j \} \to \mathcal{R}(1^{L-k2^N-L-j+13^k-1})q^{L(j-1)},
\]
which is the inverse of the bijection (19). When \( k = 0 \), both sides of (10) are 0.

4.3. Proof of (7). To prove (7) for fixed \( k \), we build a weight-preserving bijection
\[
(23) \quad \{ w \in \mathcal{R}(1^{L-k2^N-L-3^k}) : p_3(w) > p_1(w) \} \to q^{k} \mathcal{R}(0^{L-3^N-L}) \times \mathcal{R}(1^{L-k-12^k}).
\]
Take the disjoint union of the bijections (20) over all possible \( i \). This maps the domain in (23) to the disjoint union
\[
\bigcup_{i=1}^{N-L+1} \mathcal{R}(1^{L-k-12^N-L-i+13^k})q^{L(i-1)+k}.
\]
Use the bijection \( K \) of Proposition 5 to interchange the frequencies of 2s and 3s, which yields
\[
\bigcup_{i=1}^{N-L+1} \mathcal{R}(1^{L-k-12^k3^N-L-i+1})q^{L(i-1)+k}.
\]
Next use the bijection \( H \) of Proposition 4 to reach
\[
\left( \bigcup_{i=1}^{N-L+1} \mathcal{R}(0^{L-13^N-L-i+1})q^{L(i-1)} \right) \times q^{k} \mathcal{R}(1^{L-k-12^k}).
\]
To finish, use \( G^{-1} \) from Proposition 4 (taking \( s = i - 1 \), \( A = N - 1 \), \( B = L \) and replacing each 1 by 0 and each 2 by 3) to reach
\[
q^{k} \mathcal{R}(0^{L-3^N-L}) \times \mathcal{R}(1^{L-k-12^k}).
\]
4.4. Proof of (8). To prove (8) for fixed \( k \), we build a weight-preserving bijection
\[
(24) \quad \{w \in \mathcal{R}(1^{L-k}2^{N-L-3}k) : p_3(w) \leq p_1(w)\} \rightarrow \mathcal{R}(0^L3^{N-L}) \times \mathcal{R}(1^{L-k}2^{k-1}).
\]
Take the disjoint union of the bijections (21) over all possible \( j \). (Since \( j \leq p_1(w) \) here, the upper limit for \( j \) is \( N - L + 1 \).) We thereby map the domain in (24) to the disjoint union
\[
\bigcup_{j=1}^{N-L+1} \mathcal{R}(1^{L-k}2^{N-L-j+1}3^{k-1})q^L(j-1).
\]
Use the bijection \( K \) of Proposition 5 to interchange the frequencies of 2s and 3s, which yields
\[
\bigcup_{j=1}^{N-L+1} \mathcal{R}(1^{L-k}2^{k-1}3^{N-L-j+1})q^L(j-1).
\]
Next use the bijection \( H \) of Proposition 4 to reach
\[
\left( \bigcup_{j=1}^{N-L+1} \mathcal{R}(0^L3^{N-L-j+1})q^L(j-1) \right) \times \mathcal{R}(1^{L-k}2^{k-1}).
\]
To finish, use \( G^{-1} \) from Proposition 3 (taking \( s = j - 1, A = N - 1, B = L \) and replacing each 1 by 0 and each 2 by 3) to reach
\[
\mathcal{R}(0^L3^{N-L}) \times \mathcal{R}(1^{L-k}2^{k-1}).
\]

4.5. Proof of (4) and (5). To prove (4), we need a weight-preserving bijection
\[
(25) \quad \{w \in W_k : p_1(w) = i\} \rightarrow q^{k+(i-1)L} \mathcal{R}(1^{L-k-1}2^{N-L-i+1}3^k) \times \mathcal{R}(2^{a_2} \cdots (n-1)^{a_{n-1}}).
\]
Restricting the bijection (14) to the domain of (25), we get a bijection mapping that domain to
\[
\{w' \in X_k : p_1(w') = i\} \times \mathcal{R}(2^{a_2} \cdots (n-1)^{a_{n-1}}).
\]
Now, apply (20) to the first factor to reach the codomain of (25). We prove (5) in the same way, using (21).

4.6. Proof of (2) and (3). To prove (2), we need a weight-preserving bijection
\[
(26) \quad W_k^r \rightarrow q^k \mathcal{R}(1^{L-2^{a_2}} \cdots (n-1)^{a_{n-1}}) \times \mathcal{R}(1^{L-1-k}2^k).
\]
Restricting the bijection (14) to the domain of (26), we get a bijection mapping that domain to
\[
X_k^r \times \mathcal{R}(2^{a_2} \cdots (n-1)^{a_{n-1}}).
\]
Apply (23) to the first factor to reach
\[
q^k \mathcal{R}(0^L3^{N-L}) \times \mathcal{R}(1^{L-k-1}2^k) \times \mathcal{R}(2^{a_2} \cdots (n-1)^{a_{n-1}}).
\]
Finally, apply bijection (15) to the first and third factors in this Cartesian product to obtain
\[
q^k \mathcal{R}(1^{L-2^{a_2}} \cdots (n-1)^{a_{n-1}}) \times \mathcal{R}(1^{L-k-1}2^k).
\]
We prove (3) in the same way, using (24).
5. A Detailed Example and the Proof of (3).

In this section, we start with a specific \( w \in W_k^\geq \) and trace through all the bijections in the proof of (2) to find the image of \( w \) in the codomain of (26). We continue by dropping the weight-shift factor \( q^k \), replacing \( k \) by \( k + 1 \), and tracing the proof of (3) backwards from this codomain to get \( w' \in W_{k+1}^\leq \) such that \( \text{coinv}(w) = \text{coinv}(w') + k \). Some intermediate bijections cancel out in this two-step process, leading us to a simpler bijective proof of (3).

5.1. Mapping \( w \in W_k^\geq \) to the Intermediate Object. Let \( w = 3112443214243 \in \mathcal{R}(1^32^33^44^4) \), so \( n = 4, k = 4, L = 7, a_2 = 3, a_3 = 3, N = 13, \text{coinv}(w) = 39, p_4(w) = 5, p_1(w) = 3, \) and \( w \in W_4^\geq \). We follow the proof of (2) to send \( w \) to an intermediate object in the codomain of the map (26).

- **Step 1.** Apply bijection (14) to convert \( w \) to the pair \((y, z)\), where \( y = 21123322132323 \) and \( z = 323223 \). Note \( y \in X_4^\geq \) with \( p_3(y) = 5 > 3 = p_1(y) \), \( \text{coinv}(y) = 35 \), and \( \text{coinv}(z) = 4 \). The next six steps apply to \( y \) alone.
- **Step 2.** Apply bijection \( P_{1, j}^{-1} \) to \( y \), where \( i = p_1(y) = 3 \) and \( j = p_3(y) = 5 \). We get the pair \((30003030 \cdot q^4, 211222 \cdot q^{18})\), where the extra \( q \)-powers indicate weight-shifting amounts for each component word.
- **Step 3.** Apply bijection \( F^{-1} \) (Proposition 3) with \( A = 9, B = 4, s = j - i - 1 = 1 \) to the first component, producing \((0330003030, 211222 \cdot q^4)\).
- **Step 4.** Apply bijection \( H^{-1} \) (Proposition 4) to change this pair to \( 2331123232 \cdot q^{18} \).
- **Step 5.** Apply bijection \( K \) (Proposition 5) to reach \( 3231123223 \cdot q^{18} \).
- **Step 6.** Apply bijection \( H \) to get the pair \((3030003003 \cdot q^{14}, 211222 \cdot q^4)\).
- **Step 7.** Apply bijection \( G^{-1} \) (with \( i = 3, s = i - 1 = 2, A = 12, B = 7 \)) to get \((3030003003033, 211222 \cdot q^4)\).
- **Step 8.** Apply bijection (15) to combine the first component here with \( z \). We thereby reach the intermediate object

\[ q^4(31211113112123, 211222) \in q^k \mathcal{R}(1^72^33^3) \times \mathcal{R}(1^22^4) \]

This object has weight \( 4 + 29 + 6 = 39 = \text{coinv}(w) \).

5.2. Mapping the Intermediate Object to \( w' \in W_{k+1}^\leq \). We continue operating on the intermediate object, dropping the \( q^4 \) shift and working backwards through the proof of (3), taking \( k = 5 \) now. We discover that the first five steps undo the last five steps of the previous algorithm:

- **Step 8'.** Apply the inverse of bijection (15) to produce the triple \((3030003003033, 211222, z = 323223)\). We save \( z \) for later and keep acting on the first two components.
- **Step 7'.** Apply \( G \) to the first component to get \((3030003003 \cdot q^{14}, 211222)\).
- **Step 6'.** Apply \( H^{-1} \) to this pair to get \( 3231123223 \cdot q^{14} \).
- **Step 5'.** Apply \( K^{-1} \) to get \( 2331123232 \cdot q^{14} \).
- **Step 4'.** Apply \( H \) to get \((0330003030, 211222) \cdot q^{14} \). Comparing to (22), we see \( j = 3 \) now.
- **Step 3'.** Apply \( G \) (with \( j = 3, s = 3, A = 5, B = 2, i = s + j = 6 \)) to get \((0330003030 \cdot q^{14}, 21 \cdot q^6)\).
- **Step 2'.** Apply \( P_{6, 3} \) to get \( 21313322223232 \).
- **Step 1'.** Apply the inverse of (14) to the pair \((21313322223232, z = 323223)\) to get the final output \( w' = 3141442324243 \). Note \( w' \in W_5^\leq \) has \( p_4(w') = 3 \leq 6 = p_1(w') \), and \( \text{coinv}(w') = 35 = \text{coinv}(w) - 4 \).
5.3. Simplified Bijection Proving $W_k^> \equiv q^kW_{k+1}^\leq$. To prove (6) bijectively, we need a weight-preserving bijection from $W_k^>$ to $q^kW_{k+1}^\leq$ for $0 \leq k < L$. The cancellation of the actions in Steps 3 through 8 of the example holds in general. Thus we arrive at the following simplified description of how the required bijection acts on $w \in W_k^>$:

(a) Use bijection (14) to convert $w$ to $(y, z)$ where $y \in X_k^>$.
(b) Apply $P_{i,j}^{-1}$ to $y$, where $i = p_1(y)$ and $j = p_3(y)$, so that $(y, z)$ becomes $(u, v, z)$.
(c) Apply $F^{-1}$ to $u$ by prepending $0^k3$, where $s = N - i - k - n_0(u)$, to get $(u', v, z)$.
(d) Apply $G$ to $v$, which removes a suffix $12^{s'}$ and leaves us with $(u', v', z)$. Hereafter, we use new variables $k' = k + 1$, $j' = N - k' + 1 - n_0(u') = i$, and $l' = s' + j' \geq j'$.
(e) Apply $P_{i',j'}$ to $(u', v', z)$ to change $(u', v', z)$ to $(y', z)$, where $y' \in X_{k'}^\leq = X_{k+1}^\leq$.
(f) Use the inverse of bijection (14) to map $(y', z)$ to the final output word $w' \in W_{k'}^\leq = W_{k+1}^\leq$.

In the boundary cases where $k = 0$ or $k' = L$, we modify these steps as follows. When $k = 0$, replace $P_{i,j}^{-1}$ in step (b) by the inverse of bijection (15), which transforms $y$ to $v$; and replace step (c) by setting $u' = 0^N-i$. For any $k$ in the range $0 \leq k < L$, steps (b) and (c) send a word $y \in X_k^>$ with $p_1(y) = i$ to a pair $(u', v')$ in

\[ (27) \quad \mathcal{R}(0^{N-i-k}3^k) \times \mathcal{R}(1^{L-k-1}2^{N-L-i+1})q^{L+i-1}. \]

When $k' = L$, we must have $v = 2^{N-L-j'+1}$. Here, we modify step (d) by discarding $v$ from $(u', v, z)$ and replacing all $0$s in $u'$ by $2$s. In step (e), we replace $P_{i',j'}$ by the bijection (19), which transforms the modified $u'$ to a word $y' \in X_{k'}^\leq$. For any $k'$ in the range $0 < k' \leq L$, doing the inverse of step (e) followed by the inverse of step (d) sends a word $y' \in X_{k'}^\leq$ with $p_3(y') = j'$ to a pair $(u', v)$ in

\[ (28) \quad \mathcal{R}(0^{N-j'-k'+1}3^{k'-1}) \times \mathcal{R}(1^{L-k'}2^{N-L-j'+1})q^{L-j'+1}. \]

Since $k' = k + 1$ and $j' = i$, the intermediate collections (27) and (28) match after shifting the latter by $q^k$, as needed.

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