Vector meson masses in 2+1 flavor Wilson Chiral Perturbation Theory

S. Aoki
Graduate School of Pure and Applied Sciences,
University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan and
Riken BNL Research Center, Brookhaven National Laboratory, Upton, New York 11973, USA

O. Bär and S. Takeda*
Graduate School of Pure and Applied Sciences,
University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan
(Dated: November 30, 2021)

Abstract

We calculate the vector meson masses in $N_f = 2 + 1$ Wilson chiral perturbation theory at next-to-leading order. Generalizing the framework of heavy vector meson chiral perturbation theory, the quark mass and the lattice cutoff dependence of the vector meson masses is derived. Our chiral order counting assumes that the lattice cut-off artifacts are of the order of the typical pion momenta, $p \sim a \Lambda_{QCD}^2$. This counting scheme is consistent with the one in the pseudo scalar meson sector where the $O(a^2)$ terms are included in the leading order chiral Lagrangian.

PACS numbers: 11.30.Hv, 11.30.Rd, 12.39.Fe, 12.38.Gc

*Present address: Institut für Physik, Humboldt Universität zu Berlin, Newtonstr. 15, 12489 Berlin, Germany.
I. INTRODUCTION

This is the second in a series of papers where we compute a variety of mesonic quantities in 2+1 flavor Wilson Chiral Perturbation Theory (Wilson $\chi$PT). After having computed the pseudo scalar mesons masses $^1$, we present here the results for the vector meson masses. The calculation of the pseudo scalar decay constants and the axial vector Ward identity quark mass is in progress $^2$. The main goal of this series of papers is to provide the necessary chiral fit forms for unquenched 2+1 flavor Lattice QCD simulations with improved Wilson fermions, as they have been currently performed by the CP-PACS/JLQCD collaboration $^3$.

The quark masses in the CP-PACS/JLQCD simulations are heavier than their physical values. The ratio of the pseudo scalar to vector meson mass is in the range $m_{PS}/m_V \simeq 0.62 - 0.78$. A chiral extrapolation in the light up and down quark masses is thus required. In order to perform the chiral extrapolation before taking the continuum limit we formulate $\chi$PT at non-zero lattice spacing, as originally proposed in Refs. $^4,^5$. A variety of pseudo scalar quantities has been already computed, mainly for 2 flavor Wilson $\chi$PT (see Ref. $^6$ and references therein). Although the vector meson masses were calculated recently in 2 flavor partially quenched Wilson $\chi$PT $^7$, the result for 2+1 flavors was missing.

In this paper we follow the heavy vector meson formalism first introduced by Jenkins et. al. in Ref. $^8$. The generalization to Lattice QCD at non-zero lattice spacing $a$ is straightforward and mirrors the strategy spelled out in Refs. $^4,^5$. However, since the lattice spacing is an additional expansion parameter the power counting requires some care. Here we adopt a power counting which assumes that lattice cut-off artifacts are of the order of the typical pion momenta, $a\Lambda_{QCD}^2 \sim p$, as it was assumed in Ref. $^9$. Previous results in unquenched 2-flavor simulations $^{10}$ seem to indicate that this is the appropriate power counting for describing the lattice results of the CP-PACS/JLQCD collaboration.

There is a price to pay if one wants to perform the chiral extrapolation before taking the continuum limit. The chiral fit forms contain terms proportional to powers of the lattice spacing accompanied by additional unknown low-energy constants. These constants are essentially unconstrained and serve as additional fit parameters. Obviously, the presence of too many of these additional parameters would limit or even spoil the chiral extrapolation. Our one loop results for the $\rho$ and $K^*$ meson masses contain seven unknown fit parameters compared to three in the corresponding continuum $\chi$PT result of Ref. $^8$. This number is still small enough for the results to be useful for the chiral extrapolation of the CP-PACS/JLQCD collaboration data.
This paper is organized as follows. In Sec. II we briefly review the heavy meson formalism, as it was introduced in Ref. [8]. In Sec. III we first summarize the meson and spurion fields which are necessary in our calculation. After discussing the power counting we derive the chiral effective Lagrangian and compute the vector meson masses to one loop. Concluding remarks are given in section IV.

II. HEAVY VECTOR MESON EFFECTIVE THEORY

We adopt the heavy vector meson effective theory to derive the quark mass and lattice spacing dependence of the vector meson masses.\(^1\) This effective theory can deal with vector number conserving decay processes like \(V \rightarrow V'X\), where \(V\) and \(V'\) are vector mesons, and \(X\) represents a state with one or more low momentum pseudo scalar mesons. On the other hand, it cannot be applied to the process \(\rho \rightarrow \pi\pi\), for example, since this decay includes hard pions in the final state.

Some physical quantities have been calculated within the continuum formulation of the heavy vector meson effective theory. The first result for the vector meson masses up to \(O(p^3)\) can be found in Ref. [8]. The effects of isospin breaking and electromagnetic corrections are included in Ref. [12]. The calculation of the vector meson masses to \(O(p^4)\) was done in Ref. [13], while the results for the decay constants are given in Ref. [14]. For results in quenched and partially quenched \(\chi\)PT, see Ref. [15] and Ref. [16], respectively.

Let us briefly review the main idea behind the heavy vector meson formalism. The following argument can be found in Ref. [14]. The vector meson sector introduces a typical energy scale, the vector meson mass \(m_V\). This mass does not vanish in the chiral limit, and it is of about the same size as the chiral symmetry breaking scale, \(4\pi f_\pi \sim 1\) GeV. The vector mesons are described by heavy matter fields. In order to obtain the Lagrangian of the effective theory one expands the relativistic Lagrangian in powers of \(1/m_V\). The starting point is the free relativistic vector meson Lagrangian,

\[ \mathcal{L}_R = -\frac{1}{4} V^{\mu\nu} V_{\mu\nu} + \frac{1}{2} m_V^2 V\mu V\mu, \] 

with \(V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu\). The relativistic field \(V_\mu\) is decomposed into two parts, a parallel and a perpendicular component with respect to the 4-velocity \(v_\mu (v^2 = 1)\) of the vector meson,

\[ V_\mu = P_{\mu\nu} V^\nu + v_\mu (v \cdot V) = V_{\perp\mu} + v_\mu V_\parallel, \] 

\(^1\) This formalism is very similar to heavy baryon chiral perturbation theory [11].
where

\[ P_{\mu\nu} = g_{\mu\nu} - v_\mu v_\nu, \]

(3)
is the projector which selects the perpendicular component of the vector field \( P^2 = P, \) \( v^\mu P_{\mu\nu} = 0 \).

\( V_\perp \) and \( V_\parallel \) can further be written as

\[ V_\perp = \frac{1}{\sqrt{2m_V}}[e^{-imVv\cdot x}W_\mu + e^{imVv\cdot x}W_\mu^\dagger], \]

(4)

\[ V_\parallel = \frac{1}{\sqrt{2m_V}}[e^{-imVv\cdot x}W_\parallel + e^{imVv\cdot x}W_\parallel^\dagger], \]

(5)

where \( W_\mu \) and \( W_\mu^\dagger \) are effective vector meson fields, and \( P_\mu W_\nu = W_\mu \) and \( P_\mu W_\nu^\dagger = W_\mu^\dagger \) are understood. By neglecting terms proportional to \( \exp(\pm 2imVv\cdot x) \), which oscillate rapidly if one takes \( m_V \) to infinity, the Lagrangian \( \mathcal{L}_R \) in terms of the \( W \) fields is given by

\[
\mathcal{L}_R \rightarrow \frac{1}{2m_V} \left[ -\partial_\mu W_\parallel^\dagger \partial^\mu W_\parallel + (v\cdot \partial W_\parallel)(v\cdot \partial W_\parallel) \right] + \frac{m_V}{2} W_\parallel^\dagger W_\parallel \\
+ \frac{1}{2} \left[ -i\partial_\mu W_\parallel^\dagger W_\mu + iW_\mu^\dagger \partial^\mu W_\parallel \right] \\
+ \frac{1}{2m_V} \left[ \partial_\mu W_\parallel^\dagger (v\cdot \partial W_\mu) + (v\cdot \partial W_\mu^\dagger) \partial^\mu W_\parallel \right] \\
- iW_\mu^\dagger (v\cdot \partial)W_\mu.
\]

(6)

In the infinite mass limit, \( m_V \rightarrow \infty \), the \( W_\parallel \) field decouples because of the presence of the mass term \( m_V W_\parallel^\dagger W_\parallel/2 \). In order to remove the parallel component, one can impose the constraint \( v\cdot V = 0 \) or \( v\cdot W = 0 \). Alternatively, we can remove \( W_\parallel \) by making use of the equation of motion,

\[ W_\parallel = \frac{-i}{m_V} \partial_\mu W_\mu + O(1/m_V^2), \]

(7)

which also shows that \( W_\parallel \) is suppressed by powers of \( 1/m_V \) relative to the perpendicular component \( W_\mu \). Having removed the parallel component the resulting effective Lagrangian simplifies to

\[ \mathcal{L}_R = -iW_\mu^\dagger (v\cdot \partial)W_\mu + O(1/m_V). \]

(8)

The first term is the kinetic term for the vector meson in the heavy effective theory. Vector meson fields which appear in the following are the effective fields \( W_\mu \) and \( W_\mu^\dagger \). The partial derivative \( i\partial_\mu \) acting on these effective fields produces a small residual momentum \( r_\mu \), defined as

\[ k_\mu = m_V v_\mu + r_\mu, \]

(9)

where \( k_\mu \) is the usual four-momentum of the vector meson. We assume here that the residual momentum is of the size of the low momentum of the pseudo scalar meson. This assumption holds in interaction processes with soft pions.
III. WILSON CHIRAL PERTURBATION THEORY FOR VECTOR MESONS

The chiral effective Lagrangian for heavy vector mesons is expanded in powers of the small pseudo scalar momenta and masses, and the residual momentum $r_\mu$ of the vector mesons. The symmetries of QCD, in particular chiral symmetry, dictate the form of the terms in the chiral Lagrangian. Even though the quark masses explicitly break chiral symmetry, their effect is properly taken into account by a spurion analysis, where the quark mass matrix is assumed to transform non-trivially under chiral transformations in an intermediate step.

The same principles apply to Wilson $\chi$PT, the low energy effective theory for Lattice QCD with Wilson fermions. The main difference is the presence of an additional expansion parameter, the lattice spacing $a$. The non-zero lattice spacing is also taken into account by the spurion analysis, and the transformation behaviour of the corresponding spurion field is exactly the same as for the quark masses. Constructing the terms in the chiral Lagrangian of Wilson $\chi$PT for vector mesons is therefore straightforward. Apart from the presence of two instead of one spurion field there is essentially no difference to Ref. [8]. The order counting, however, is not entirely obvious, since both the quark masses and the lattice spacing are expansion parameters and their relative size is important for a consistent power counting, as we will discuss in section III B.

A. Matter and spurion fields

The pseudo scalar meson field is introduced as usual as an SU(3) unitary matrix

$$\Sigma = \exp \left( \frac{2i\Pi}{f} \right),$$

where

$$\Pi = \sqrt{2}\pi^a T^a = \begin{bmatrix}
\pi^0 + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\
\pi^- & -\pi^0 + \frac{\eta}{\sqrt{6}} & K^0 \\
K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}}
\end{bmatrix}.$$ (11)

The SU(3) generators $T^a$ ($a = 1, \ldots, 8$) are normalized such that

$$\text{tr}[T^a T^b] = \frac{1}{2} \delta^{ab},$$

and $f$ is the leading order pseudo scalar decay constant in the chiral limit. The field $\Sigma$ transforms according to $\Sigma \to L\Sigma R^{-1}$ under chiral rotations with $L \in SU(3)_L$ and $R \in SU(3)_R$. The square

---

$^2$ Our normalization corresponds to $f \approx 132$ MeV.
The root of $\Sigma$,

$$\xi = \exp \left( \frac{i\Pi}{f} \right) = \sqrt{\Sigma}, \quad (13)$$

is needed to describe the interaction of the pseudo scalars with the vector mesons. It transforms under chiral transformations according to

$$\xi \rightarrow L\xi U^\dagger = U\xi R^{-1}, \quad (14)$$

with an SU(3) matrix $U$. In fact, eq. (14) defines $U$, which is a function of $L, R, \Pi$. For vector transformations with $L = R$ one finds $U = L = R$.

The vector meson fields are introduced as an octet

$$O_\mu = \sqrt{2}\rho_\mu^a T^a = \begin{bmatrix} \rho^0 \sqrt{2}/2 + \phi_\mu^{(8)}/\sqrt{6} & \rho_\mu^+ & K_\mu^{*+} \\ \rho_\mu^- & -\rho^0 \sqrt{2}/2 + \phi_\mu^{(8)}/\sqrt{6} & K_\mu^{*0} \\ K_\mu^{*-} & \tilde{K}_\mu^{*0} & -2\phi_\mu^{(8)}/\sqrt{6} \end{bmatrix}, \quad (15)$$

and a singlet $S_\mu = \phi^0_\mu$. These fields are required to satisfy the constraints

$$v \cdot S = v \cdot O = 0, \quad (16)$$

in order to describe spin 1 particles with three polarization states. As stated in the previous section, this constraint can be enforced by applying the projector $P_{\mu\nu}$ on the vector meson fields. In the following we assume that this projector has been applied, i.e. we implicitly assume $S_\mu = P_{\mu\nu}S^\nu$ and $O_\mu = P_{\mu\nu}O^\nu$.

For the construction of the chiral Lagrangian the quark masses and the lattice spacing are treated as spurion fields. For the quark mass matrix we use

$$\tilde{M}_q = \text{diag}(\tilde{m}, \tilde{m}, \tilde{m}_s) = \tilde{M}_0 I + \tilde{M}_8 T^8, \quad (17)$$

where $\tilde{M}_0$ and $\tilde{M}_8$ are expressed in terms of $\tilde{m}$ and $\tilde{m}_s$

$$\tilde{M}_0 = \frac{2\tilde{m} + \tilde{m}_s}{3}, \quad \tilde{M}_8 = \frac{2(\tilde{m} - \tilde{m}_s)}{\sqrt{3}}. \quad (18)$$

We use the tilde in order to highlight that the quark masses $\tilde{m}$ and $\tilde{m}_s$ denote shifted quark masses, which are defined such that the tree level pseudo scalar meson masses become zero if $\tilde{m} = \tilde{m}_s = 0$. See Ref. [1] and also appendix A for details.

We introduce a spurion field $A$ to include the effect of a non-zero lattice spacing $a$. This field transforms under chiral transformations just as the quark mass field $M$ ($M = \tilde{M}_q$ after the spurion analysis), i.e.

$$A \rightarrow LAR^{-1}. \quad (19)$$
TABLE I: Transformation properties under the group $G = SU(3)_L \times SU(3)_R$, $(L \in SU(3)_L$ and $R \in SU(3)_R)$, charge conjugation $C$ and parity $P$. In the parity transformed expressions it is understood that the argument is $(-x,t)$.

Once the chiral Lagrangian is derived, the spurion is set to $aI$ where $I$ denotes the unit matrix in flavor space.

For the construction of the chiral Lagrangian it will also be useful to introduce the following quantities:

\[ M_{\pm} = \frac{1}{2} (\xi M^\dagger \xi \pm \xi^\dagger M \xi), \]
\[ W_{\pm} = \frac{1}{2} (\xi A^\dagger \xi \pm \xi^\dagger A \xi), \]
\[ V_\mu = \frac{1}{2} (\xi \partial_\mu \xi^\dagger + \xi^\dagger \partial_\mu \xi), \]
\[ A_\mu = \frac{i}{2} (\xi \partial_\mu \xi^\dagger - \xi^\dagger \partial_\mu \xi), \]
\[ D_\nu O_\mu = \partial_\nu O_\mu + [V_\nu, O_\mu]. \]

The transformation behavior of these quantities as well as of the meson and spurion fields under chiral rotations in $G = SU(3)_L \times SU(3)_R$, charge conjugation $C$ and parity $P$ are summarized in Table I.
B. Power counting

Having defined the meson and spurion fields it is straightforward to write down the most general chiral Lagrangian which is compatible with chiral symmetry, charge conjugation and parity. However, since we have two sources for explicit chiral symmetry breaking, the quark masses and the lattice spacing, their relative size matters for an appropriate power counting. Moreover, the power counting one wants to employ for the vector meson chiral Lagrangian should be consistent with the one adopted for the pure pseudo scalar chiral Lagrangian.

In Ref. [1] the $O(a^2)$ terms are included in the leading order chiral Lagrangian. To be consistent we adopt here the following power counting scheme:

\[
\begin{align*}
\text{LO} & : O(p), O(a), \\
\text{NLO} & : O(m), O(a^2), O(ap), O(p^2), \\
\text{NNLO} & : O(mp), O(p^3), O(am), O(a^3), O(ap^2), O(a^2p).
\end{align*}
\]

Here $p$ represents both the residual momentum of the vector meson and the momentum of the pseudo scalar meson, and $m$ denotes the quark mass. This counting scheme assumes $p \sim a$. This implies $p^2 \sim a^2$, and therefore maintains consistency with the power counting of Ref. [1] for the pure pseudo scalar sector.

We have listed the NNLO contributions in eq. (25), because some NNLO terms enter already the one-loop calculation of the vector meson masses. To discuss this point, let us define a generic power counting scale $E$ of order $p \sim a$, so that the LO terms or of $O(E)$, the NLO terms are $O(E^2)$ and so on. Just from a dimensional analysis one finds that a one-loop integral involving a vertex from the LO Lagrangian gives an $O(E^3)$ contribution to the vector meson mass. Therefore, some of the coefficients in the NNLO Lagrangian are needed as counterterms for the cancellation of the one-loop divergences. The coefficients of the NLO Lagrangian, on the other hand, are not needed as counterterms, since there are no one loop corrections with dimension $E^2$. This situation is different from the pure pseudo scalar sector where the divergences in the one-loop contribution from the LO Lagrangian are canceled by the tree level terms coming from the NLO Lagrangian.
The structure of the correction to the vector meson masses can be summarized as follows:

\[ m_V = \mu_c + O(E) \]

\[ + \text{[tree level contribution from the NLO Lagrangian]} \]

\[ + O(E^2) \]

\[ + \text{[one-loop contribution from the LO Lagrangian]} \]

\[ + O(E^3) \]

\[ + \text{[tree level contribution from the NNLO Lagrangian]} + O(E^4), \quad (26) \]

where \( \mu_c \) is the vector meson mass in the chiral limit. In fact, the second term on the right hand side of eq. (26) comes from the \( O(a) \) Lagrangian only. As has been shown in Ref. [8], the \( O(p) \) term does not contribute at tree level and the chiral correction to the vector meson mass starts at \( O(E^2) \).

In this paper we are only interested in deriving the vector meson masses through \( O(E^3) \). In that case we neither need the NLO terms of \( O(ap, p^2) \) nor the NNLO terms of \( O(mp, p^3, ap^2, a^2p) \). The reason is that all these terms contain at least one momentum factor \( p \). By using the lowest order equation of motion, we can replace the residual vector meson momentum with the momentum \( p_\pi \) of the pseudo scalar and a contribution to proportional to \( a^3 \). The resulting term has a part with at least one \( p_\pi \) and a part of \( O(a^2, am, a^3) \). For the former, \( p_\pi \sim \partial_\mu \pi \) implies that this term always contains at least one pseudo scalar field together with a derivative \( \partial_\mu \). This kind of term results in a three-point vertex (vector-vector-pseudo scalar) as a non-vanishing leading term and does not give a tree level contribution to the vector meson mass. Hence the terms of \( O(ap, p^2, mp, p^3, ap^2, a^2p) \) are not needed for our calculation and we do not list these terms in the next subsection.

C. Effective Lagrangian

In this section, we show our results for the terms in the chiral Lagrangian which contribute to the vector meson mass through order \( E^3 \). By construction it involves the meson fields and is invariant under Lorentz and chiral transformations, charge conjugation and parity.\(^4\)

\[ O(p): \]

\(^3\) The leading order equation of motion, \(( -iv \cdot \partial + a\alpha_s )S^\mu \), is easily read off from the LO chiral Lagrangian, see eq. (A2) in appendix A.

\(^4\) Just for convenience we present our results in Minkowski space, which does not make a difference for the computation of the vector meson masses. The underlying lattice theory, however, is usually formulated for Euclidean space time.
\[ L_p = -iS^\dagger_\mu (v \cdot \partial) S^\mu \]
\[ -i(O^\dagger_\mu (v \cdot D) O^\mu) \]
\[ +ig_1 \left( S^\dagger_\mu (O_\nu A_\lambda) - S^\mu (O^\dagger_\nu A_\lambda) \right) v_\sigma \epsilon^{\mu \nu \lambda \sigma} \]
\[ +ig_2 \{O^\dagger_\mu, O_\nu \} A_\lambda v_\sigma \epsilon^{\mu \nu \lambda \sigma}, \]

with \( \langle X \rangle = \text{tr}(X) \) for flavor indices. The derivative \( i\partial_\mu \) acting on the vector mesons provides the residual momentum \( r_\mu = k_\mu - \mu_v v_\mu \). We therefore consider the kinetic terms for the vector mesons as \( O(p) \). The terms proportional to \( g_1 \) and \( g_2 \) involve \( A_\lambda \), defined in eq. (23). Expanding in terms of the pseudo scalar fields this quantity starts as \( A_\lambda = \partial_\lambda \Pi/f + O(\Pi^3) \). The presence of the derivative means that both the \( g_1 \) and \( g_2 \) term are of \( O(p) \).

As is well known, the vector mesons are not stable. This effect can be taken into account by including an anti-hermitian term in the Lagrangian whose coefficient is proportional to the decay width. However, the decay width is rather small and this contribution is usually ignored. We also neglect this term in this paper.

\( O(a) \):

\[ L_a = \alpha_1 \langle W_+ \rangle S^\dagger_\mu S^\mu \]
\[ +\alpha_2 \left( \langle O^\dagger_\mu W_+ \rangle S^\mu + \langle O_\mu W_+ \rangle S^\mu \dagger \right) \]
\[ +\alpha_3 \{O^\dagger_\mu, O_\mu \} W_+ \]
\[ +\alpha_4 \langle W_+ \rangle \langle O^\dagger_\mu O_\mu \rangle. \]

This \( O(a) \) Lagrangian is a new element of this study. Since the spurion field \( A \) associated with the lattice spacing transforms exactly like the quark mass spurion field (cf. eq. (19)), this part of the Lagrangian has the same structure as the following \( O(m) \) Lagrangian.

\( O(m) \):

\[ L_m = \lambda_s \langle M_+ \rangle S^\dagger_\mu S^\mu \]
\[ +\lambda_0 \left( \langle O^\dagger_\mu M_+ \rangle S^\mu + \langle O_\mu M_+ \rangle S^\mu \dagger \right) \]
\[ +\lambda_{01} \langle M_+ \rangle \langle O^\dagger_\mu O_\mu \rangle \]
\[ +\lambda_{02} \{O^\dagger_\mu, O_\mu \} M_+ \]
\[ +\Delta \mu S^\dagger_\mu S^\mu. \]

Note that the leading order mass terms \( \mu_s S^\dagger_\mu S^\mu \) and \( \mu_o \langle O^\dagger_\mu O_\mu \rangle \), where \( \mu_s \) and \( \mu_o \) are the masses for the singlet and octet vector meson in the chiral limit, respectively, are absent in the \( O(p) \) Lagrangian. Their effect is already included through the phase factor \( \exp(-im \cdot v \cdot x) \) in the definition of the heavy meson fields. The mass difference

\[ \Delta \mu \equiv \mu_s - \mu_o \]
between the singlet and octet vector meson mass, however, needs to be introduced explicitly. This term, which is the last term in eq. (29), is considered to be of $O(\mu)$, since $\Delta \mu < 200$ MeV $\sim m_s$.

$O(a^2)$:

\[
\mathcal{L}_{a^2} = \beta_1 (W_+ W_+) S^\dagger_\mu S^\mu \\
+ \beta_2 (W_+)^2 S^\dagger_\mu S^\mu \\
+ \beta_3 (\langle O^\dagger_\mu W_+ W_+ \rangle S^\mu + \langle O_\mu W_+ W_+ \rangle S^\mu) \\
+ \beta_4 (W_+) (\langle O^\dagger_\mu W_+ \rangle S^\mu + \langle O_\mu W_+ \rangle S^\mu) \\
+ \beta_5 \{\{O^\dagger_\mu, O_\mu\}\} W_+ W_+ \\
+ \beta_6 \langle O^\dagger_\mu W_+ O_\mu W_+ \rangle \\
+ \beta_7 \langle W_+ \rangle \{\{O^\dagger_\mu, O_\mu\}\} W_+ \\
+ \beta_8 \langle W_+ W_+ \rangle \langle O^\dagger_\mu O_\mu \rangle \\
+ \beta_9 (W_+)^2 \langle O^\dagger_\mu O_\mu \rangle \\
+ \beta_{10} \langle O^\dagger_\mu W_+ \rangle \langle O_\mu W_+ \rangle \\
+ (W_+ \rightarrow W_-).
\]

In total we find twenty terms, however, the parts involving $W_-$ are not needed. By expanding in powers of the pseudo scalar field one finds $W_- = (2ia/f) \Pi$ at leading order. Hence, all these terms have at least two pseudo scalar fields and therefore do not contribute to the vector meson masses at tree level.

\[5\] We assumed the definition $m_V = \mu_o$. If we had chosen $m_V = \mu_s$ the last term in eq. (29) would be $-\Delta \mu \langle O^\dagger_\mu O_\mu \rangle$.

Our final results for the vector meson masses are, of course, independent of this particular choice.
\[ \mathcal{L}_{am} = \gamma_1 \langle W_+ M_+ \rangle S_\mu^\dagger S^\mu + \gamma_2 \langle W_+ \rangle \langle M_+ \rangle S_\mu^\dagger S^\mu + \gamma_3 \langle M_+ \rangle (\langle O_\mu^\dagger W_+ \rangle S^\mu + \langle O_\mu W_+ \rangle S^{\mu\dagger}) + \gamma_4 \langle S_\mu^\dagger O^\mu \{ M_+, W_+ \} \rangle + \gamma_5 \langle W_+ \rangle (\langle O_\mu^\dagger M_+ \rangle S^\mu + \langle O_\mu M_+ \rangle S^{\mu\dagger}) + \gamma_6 \langle M_+ \rangle (\{ O_\mu^\dagger, O^\mu \} W_+) + \gamma_7 \langle O_\mu^\dagger O^\mu \rangle \langle M_+ W_+ \rangle + \gamma_8 \langle W_+ \rangle \langle M_+ \rangle \langle O_\mu^\dagger O^\mu \rangle + \gamma_9 \{ \langle O_\mu^\dagger, O^\mu \} \{ M_+, W_+ \} \rangle + \gamma_{10} (\langle O_\mu^\dagger M_+ O^\mu W_+ \rangle + \langle O_\mu^\dagger W_+ O^\mu M_+ \rangle) + \gamma_{11} \langle W_+ \rangle \{ \langle O_\mu^\dagger, O^\mu \} M_+ \rangle + \gamma_{12} (\langle O_\mu^\dagger M_+ \rangle \langle O^\mu W_+ \rangle + \langle O_\mu^\dagger W_+ \rangle \langle O^\mu M_+ \rangle) + (M_+, W_+ \to M_-, W_-). \] (32)

One can easily check that \( M_- \sim \Pi \) at leading order. Therefore, the terms involving \( M_- \) and \( W_- \) do not contribute to the vector meson mass at the order we are working.

For our purposes it is not necessary to list the full \( O(a^3) \) Lagrangian, which is quite cumbersome. Since we will not encounter any divergences proportional to \( a^3 \) in our calculation, we do not need the coefficients in the \( O(a^3) \) Lagrangian as counterterms. The tree level contribution of the \( O(a^3) \) Lagrangian merely gives an analytic \( a^3 \) term to the singlet and octet vector meson mass, which we can simply add to our final one loop result. Note, however, that the \( O(a^3) \) Lagrangian does not give an off diagonal contribution to the singlet-octet two point function at \( O(E^3) \). All possible ways to take the trace in terms like

\[ S_\mu^\dagger O^\mu W_+ W_+ W_+ + h.c. \quad \text{or} \quad S_\mu^\dagger O^\mu W_+ W_- W_- + h.c. \] (33)

yield at least two pseudo scalar fields in the non-vanishing leading term when one expands \( W_\pm \) in powers of the \( \Pi \) field. By noting \( \langle O^\mu \rangle = 0, \ W_+ = 1 + O(\pi) \) and \( W_- = O(\pi) \), one can easily check this statement.

Having determined the chiral effective Lagrangian we can set the spurion fields to their constant values, i.e. we take \( A \to aI \) in eqn. (27) – (29) and eqn. (31) – (32). This replacement simplifies the fields \( W_\pm \) to

\[ W_\pm \to \frac{a}{2} (\Sigma \pm \Sigma^\dagger). \] (34)
The Lagrangian in terms of the component fields $\pi^a$, $\rho^a_\mu$ for the pseudo scalar and vector mesons is presented in appendix A together with the Feynman rules relevant for the calculation of the vector meson masses.

D. Vector meson masses

The full propagator for the $\rho$ and the $K^*$ is given by

$$i\Delta^{ab}_{\mu\nu}(k) = \frac{-iP_{\mu\nu}}{v \cdot r - a \alpha_{o1}} \delta^{ab} + \frac{-iP_{\mu\alpha}}{v \cdot r - a \alpha_{o1}} \left[-iP^{\alpha\beta}\Sigma^{ab}_O(k)\right] \frac{-iP_{\beta\nu}}{v \cdot r - a \alpha_{o1}} + \cdots$$

$$= -iP_{\mu\nu}[v \cdot k - \mu_o - a\alpha_o + \Sigma_{O}(k)]^{-1},$$

where $a, b$ runs from 1 to 7. Recall that the external four-momentum for the octet vector meson is written as $k_\nu = \mu_o v_\nu + r_\nu$ where $\mu_o$ is the octet mass in the chiral limit. Using the Feynman rules summarized in appendix A we obtain for the octet-to-octet self energy through $O(E^3)$

$$\Sigma^{ab}_O(k) = -[(N M_0 \lambda_{o1} + a^2 \beta_o + N a \tilde{M}_0 \gamma_{o1})\delta^{ab} + (\lambda_{o2} + a \gamma_{o2})(2\tilde{M}_0 \delta_{aa} + \tilde{M}_8 d^{a8})]$$

$$+ \frac{f^2}{2} \delta^{ab} 2C_{21}(w_s, \tilde{m}_a) + \frac{2\alpha_1^2}{f^2} \sum_{c,e} d^{ace} d^{bce} \tilde{C}_{21}(w_o, \tilde{m}_c)$$

$$+ a \frac{2\alpha_{o1}}{N f^2} \delta^{ab} \sum_c I_0(\tilde{m}_c) + a \frac{2\alpha_{o2}}{N f^2} \sum_{c,e} d^{abc} d^{ece} I_0(\tilde{m}_c),$$

where

$$w_{o,s} = v \cdot r - a\alpha_{o1,s},$$

and $d^{abc}$ is the totally symmetric $d$–symbol of $\text{su}(N)$. The coefficients $\alpha_{oi}$ are particular combinations of the low energy constants $\alpha_1, \ldots, \alpha_4$ which enter eq. (28). Similarly, the coefficients $\beta_o$ and $\gamma_{oi}$ are combinations of the low energy constants $\beta_1, \ldots, \beta_{10}$ and $\gamma_1, \ldots, \gamma_{12}$, respectively. The full expressions for these combinations can be found in appendix A even though it will not be necessary to keep track of the original constants in the following. The functions $C_{21}(\omega, M)$ and

---

6 We keep the number of flavors $N$ undetermined at intermediate stages since $N$ provides some useful checks for our equations. In the final results we will set $N$ equal to 3. Note, however, that intermediate results like eq. (36) do not hold for arbitrary $N$ unless all masses are degenerate.
$I_0(M)$ stem from the loop integral and are defined as

$$C_{21}(w, M) = \frac{1}{3} (M^2 - w^2) J(w, M) + w I_0(M) - \frac{w}{12\pi^2} \left( \frac{M^2}{2} - \frac{w^2}{3} \right),$$  

(38)

$$J(w, M) = \frac{w}{8\pi^2} \left[ R + \ln \frac{M^2}{\mu^2} - 1 \right]$$

$$+ \frac{1}{8\pi^2} \left[ 2\sqrt{M^2 - w^2} \arccos \left( \frac{w}{M} \right) \right], \quad \text{for} \quad w^2 < M^2;$$

(39)

$$I_0(M) = \frac{M^2}{16\pi^2} \left[ R + \ln \frac{M^2}{\mu^2} \right],$$

(40)

$$R = \frac{2}{n-4} - \left[ \ln(4\pi) + \Gamma'(1) + 1 \right],$$

(41)

with the dimension $n$. The one-loop integrals encountered here are essentially the same as those in baryon chiral perturbation theory, and a detailed derivation of the formulae (38) - (41) can be found in Ref. [18].

In order to cancel the divergence in $R$, defined in eq. (41), the $O(am)$ coefficients $\gamma_i (i = o1, o2)$ need to be properly renormalized,

$$\gamma_i = \gamma_i^r(\mu) + \frac{2BC_i}{8\pi^2f^2} R,$$

(42)

where the coefficient $B$ is a low-energy parameter in the LO chiral Lagrangian for the pseudo scalars (see Ref. [1] and appendix A). The coefficients $\gamma_i^r(\mu)$ denote renormalized parameters, $\mu$ is the renormalization scale$^7$, and the coefficients $C_i$ are given by

$$C_{o1} = \alpha_{o1} \frac{N^2 - 1}{N} - \alpha_{o2} \frac{5}{3N},$$

(43)

$$C_{o2} = \alpha_{o2} \frac{5}{6N}.$$  

(44)

The on-shell condition for the $\rho$ and $K^*$ meson ($a = 1, \cdots, 7$) is given by (no sum over $a$)

$$0 = v \cdot k - \mu_o - a\alpha_{o1} + \Sigma_{O}^{aa}(k),$$

(45)

$$k_\mu = m_a v_\mu, \quad k^2 = m_a^2,$$

where $m_a$ is the physical mass. So far we have chosen to express the self energy as a function of the four-momentum $k_\mu$. $\Sigma_{O}^{aa}(k)$ may also be parametrized by the scalar variables $w_o$ and $w_s$, defined in eq. (57), and we can write $\Sigma_{O}^{aa}(k) = \Sigma_{O}^{aa}(w_o, w_s)$. It is easy to see that the definition of $w_o$ together with the on-shell condition (45) yields,

$$w_o = m_a - \mu_o - a\alpha_{o1} = -\Sigma_{O}^{aa}(k = m_a v).$$

(46)

$^7$ In the following we will simply write $\gamma_i^r$ for the renormalized parameters and suppress the dependence on $\mu$. 

14
$w_o$ is considered to be of $O(E^2)$ since the self energy $\Sigma_{O}^{aa}$ is of $O(E^2)$. For $w_s$ we find the similar result

$$w_s = m_a - \mu_o - a\alpha_s = -\Sigma_{O}^{aa}(k = m_a v) - a(\alpha_s - \alpha_o). \quad (47)$$

Here we assume that the difference between the leading lattice artifacts for the octet and the singlet mass is of order $E^2$ and therefore small\(^8\),

$$a(\alpha_s - \alpha_o) = a\Delta\alpha = O(E^2). \quad (48)$$

This is supported by a strong coupling analysis performed in Ref. \[19\]. Expanding the effective potential through second order in terms of the vector meson fields, it has been shown that there exists no difference between the singlet and octet vector meson to all orders in the strong coupling expansion. Whether $w_s$ and $w_o$ is small away from the strong coupling limit can and should be checked in the actual lattice simulation.

Provided the assumption \[18\], both $w_s$ and $w_o$ are of $O(E^2)$ and we can set $w_{o,s} = 0$ in $\Sigma_{O}^{aa}$, since the difference is beyond the order we are working to. We therefore conclude that

$$m_a = \mu_o + a\alpha_o - \Sigma_{O}^{aa}(w_o = 0, w_s = 0). \quad (49)$$

Note that the function $C_{21}$ becomes much simpler when the first argument is set to zero,

$$C_{21}(w = 0, M) = \frac{1}{24\pi} M^3. \quad (50)$$

Let us now consider the sector of the $\rho^8$ and the $\phi^0$ mesons. Due to mixing, the propagator for this sector is written as a $2\times2$ matrix,

$$i\Delta_{\mu\nu}(k) = -iP_{\mu\nu} \begin{bmatrix} v \cdot r - a\alpha_o + \Sigma_O(k) & \Sigma_{OS}(k) \\ \Sigma_{OS}(k) & v \cdot r - a\alpha_s + \Sigma_S(k) \end{bmatrix}^{-1}, \quad (51)$$

where the self energies for octet-to-octet ($a = 8$), singlet-to-singlet and octet-to-singlet are given

\(^8\text{If this assumption is not valid, then } w_s \text{ is of } O(a) \sim O(E). \text{ In this case, the function } C_{21}(w, \tilde{m}_a) \text{ in eq.}(50) \text{ contains divergences whose coefficients are of order } anm \text{ and } a^3. \text{ Such divergences would alter the renormalization conditions for } \gamma_{o1,o2} \text{ in eq. (12). Also the } O(a^2) \text{ counterterms may receive a divergent renormalization in order to cancel additional divergences. Furthermore, if } w_s \text{ is of } O(a), \text{ the condition } w_s^2 < M^2 \text{ in eq. (39) may not be valid. Hence the functional form of } J(w, M) \text{ might be changed according to the relation between } w \text{ and } M \text{ as in Ref.}\[15\] eq.(C.27).
by
\[ \Sigma_O(k) = \Sigma_O^{a=8,b=8}(k), \] (52)
\[ \Sigma_S(k) = -(N\tilde{M}_0\lambda_s + a^2\beta_s + Na\tilde{M}_0\gamma_s + \Delta\mu)
+ \sum_c \frac{g_1^2}{f^2} 2C_{21}(w_o, \tilde{m}_c) + \frac{2\alpha_s}{N f^2} \sum_c I_0(\tilde{m}_c), \] (53)
\[ \Sigma_{OS}(k) = -\frac{\tilde{M}_s}{\sqrt{2}}(\lambda_{os} + a\gamma_{os})
+ \frac{\sqrt{2}g_1g_2}{f^2} \sum_c d^{cess}2C_{21}(w_o, \tilde{m}_c) + \frac{\alpha_{os}\sqrt{2}}{N f^2} \sum_c d^{cess} I_0(\tilde{m}_c). \] (54)

For the cancellation of \( R \), in a similar way to the case of \( \rho-K^* \) sector, \( O(am) \) coefficients \( \gamma_i \) \((i = s, os)\) should be renormalized as in eq.\((42)\) with
\[ C_s = \alpha_s \frac{N^2 - 1}{N^2}, \] (55)
\[ C_{os} = \alpha_{os} \frac{5}{6N}. \] (56)

The physical masses \( m_i, i = 1, 2 \), for this sector are determined by the on-shell condition
\[ \det \begin{bmatrix} v \cdot r - a\alpha_{o1} + \Sigma_O(k) & \Sigma_{OS}(k) \\ \Sigma_{OS}(k) & v \cdot r - a\alpha_s + \Sigma_S(k) \end{bmatrix} = 0, \] (57)
at \( k_\mu = m_i v_\mu \). The task is to solve the equation
\[ \det \begin{bmatrix} m - \mu_o - a\alpha_{o1} + \Sigma_O(k = mv) & \Sigma_{OS}(k = mv) \\ \Sigma_{OS}(k = mv) & m - \mu_s - a\alpha_s + \Sigma_S(k = mv) \end{bmatrix} = 0, \] (58)
with respect to the mass \( m_i \), where we have defined \( \Sigma'_S \) as \( \Sigma_S = \Sigma'_S - \Delta\mu \). The solutions of the equation are given by
\[ m_\pm = \frac{1}{2} \left[ \mu_s + \mu_o + a(\alpha_s + \alpha_{o1}) - (\Sigma'_S + \Sigma_O) \right]
\pm \sqrt{ \left\{ \Delta\mu + a\Delta\alpha - (\Sigma'_S - \Sigma_O) \right\}^2 + 4(\Sigma_{OS})^2}. \] (59)

Assuming that the SU(3) breaking \((\tilde{m} - \tilde{m}_s)/3 \) is smaller than the average \((2\tilde{m} + \tilde{m}_s)/3 \) of the quark masses\(^9\) together with \( \{\Delta\mu + a\Delta\alpha - (\Sigma'_S - \Sigma_O)\} > 0 \), the physical masses are given by
\[ m_+ = \mu_s + a\alpha_s - \Sigma'_S + \frac{(\Sigma_{OS})^2}{\Delta\mu + a\Delta\alpha - (\Sigma'_S - \Sigma_O)}, \] (60)
\[ m_- = \mu_o + a\alpha_{o1} - \Sigma_O - \frac{(\Sigma_{OS})^2}{\Delta\mu + a\Delta\alpha - (\Sigma'_S - \Sigma_O)}. \] (61)

\(^9\) This assumption means that the diagonal part is dominant compared to the off-diagonal element, \( \{\Delta\mu + a\Delta\alpha - (\Sigma'_S - \Sigma_O)\}^2 > 4(\Sigma_{OS})^2 \)
As in the case of the $\rho$ and $K^*$ sector, we consider the self energies $\Sigma_{O,OS}$ and $\Sigma'_S$ as functions of $w_{o,s}$ when we impose the on-shell condition. Setting the physical masses equal to $m_\pm$, the parameters $w_{o,s}$ are given as

\begin{align}
  w_o &= m - \mu_o - a\alpha_{o1} \\
  &= \begin{cases} 
    \Delta \mu + a\Delta \alpha - \Sigma'_S - \frac{(\Sigma_{OS})^2}{\Delta \mu + a\Delta \alpha - (\Sigma'_S - \Sigma_O)}, & \text{for } m = m_+, \\
    -\Sigma_O - \frac{(\Sigma_{OS})^2}{\Delta \mu + a\Delta \alpha - (\Sigma'_S - \Sigma_O)}, & \text{for } m = m_-,
  \end{cases} \\
  w_s &= m - \mu_o - a\alpha_s \\
  &= \begin{cases} 
    \Delta \mu - \Sigma'_S - \frac{(\Sigma_{OS})^2}{\Delta \mu + a\Delta \alpha - (\Sigma'_S - \Sigma_O)}, & \text{for } m = m_+, \\
    -a\Delta \alpha - \Sigma_O - \frac{(\Sigma_{OS})^2}{\Delta \mu + a\Delta \alpha - (\Sigma'_S - \Sigma_O)}, & \text{for } m = m_-.
  \end{cases}
\end{align}

Hence it turns out that both $w_o$ and $w_s$ are of $O(E^2)$, since $\Delta \mu, a\Delta \alpha, \Sigma_{O,S}$ and $\Sigma_{OS}$ are of $O(E^2)$. Therefore, we can set $w_{o,s} = 0$ in the self energies, as we did in the $\rho-K^*$ sector.

With these preparations our one-loop results for the vector meson masses are as follows:
\[
m_\rho = m_0(a) + \lambda_x(a)x + 2\lambda_y(a)y \\
- \frac{1}{12\pi f^2} \left[ (g_1^2 + \frac{2}{3} g_2^2)(x + 2y)^\frac{3}{2} + 2g_2^2(x - y)^\frac{3}{2} + \frac{2}{3} g_2^2(x - 2y)^\frac{3}{2} \right] \\
- \frac{2a}{3f^2} \left[ (3\alpha_{o1} + \alpha_{o2}) L_\pi + \left( 4\alpha_{o1} - \frac{2}{3} \alpha_{o2} \right) L_K + \left( \alpha_{o1} - \frac{1}{3} \alpha_{o2} \right) L_\eta \right],
\]

\[
m_{K^*} = m_0(a) + \lambda_x(a)x - \lambda_y(a)y \\
- \frac{1}{12\pi f^2} \left[ \frac{3}{2} g_2^2(x + 2y)^\frac{3}{2} + \left( g_1^2 + \frac{5}{3} g_2^2 \right) (x - y)^\frac{3}{2} + \frac{6}{1} g_2^2(x - 2y)^\frac{3}{2} \right] \\
- \frac{2a}{3f^2} \left[ (3\alpha_{o1} - \frac{1}{2} \alpha_{o2}) L_\pi + \left( 4\alpha_{o1} + \alpha_{o2} \right) L_K + \left( \alpha_{o1} + \frac{1}{6} \alpha_{o2} \right) L_\eta \right],
\]

\[
m_+ = m_{(00)} + \frac{m_{(08)}^2}{m_{(00)} - m_{(88)}},
\]

\[
m_- = m_{(88)} - \frac{m_{(08)}^2}{m_{(00)} - m_{(88)}},
\]

\[
m_{(88)} = m_0(a) + \lambda_x(a)x - 2\lambda_y(a)y \\
- \frac{1}{12\pi f^2} \left[ 2g_2^2(x + 2y)^\frac{3}{2} + \frac{2}{3} g_2^2(x - y)^\frac{3}{2} + \left( g_1^2 + \frac{2}{3} g_2^2 \right) (x - 2y)^\frac{3}{2} \right] \\
- \frac{2a}{3f^2} \left[ (3\alpha_{o1} - \alpha_{o2}) L_\pi + \left( 4\alpha_{o1} + \frac{2}{3} \alpha_{o2} \right) L_K + \left( \alpha_{o1} + \frac{1}{6} \alpha_{o2} \right) L_\eta \right],
\]

\[
m_{(00)} = m_S(a) + \sigma_x(a)x \\
- \frac{1}{12\pi f^2} g_1^2 \left[ 3(x + 2y)^\frac{3}{2} + 4(x - y)^\frac{3}{2} + (x - 2y)^\frac{3}{2} \right] \\
- \frac{2a}{3f^2} \alpha_s \left[ 3L_\pi + 4L_K + L_\eta \right],
\]

\[
m_{(08)} = \sigma_y(a)y \\
- \frac{1}{12\pi f^2} g_1 g_2 \sqrt{\frac{2}{3}} \left[ 3(x + 2y)^\frac{3}{2} - 2(x - y)^\frac{3}{2} - (x - 2y)^\frac{3}{2} \right] \\
- \frac{a}{3f^2} \alpha_{os} \sqrt{\frac{2}{3}} \left[ 3L_\pi - 2L_K - L_\eta \right],
\]

\[
\tan 2\theta = \frac{2m_{(08)}}{m_{(00)} - m_{(88)}},
\]

where \( \theta \) denotes the mixing angle in the \( \rho^8 - \phi^0 \) sector. We introduced the two parameters

\[
x = \frac{2B}{3}(2\bar{m} + \bar{m}_s),
\]

\[
y = \frac{B}{3}(\bar{m} - \bar{m}_s),
\]

as a short hand notation for convenient combinations of quark masses [1]. The leading order pseudo
scalar masses assume a very simple form in terms of $x$ and $y$,

$$
\begin{align*}
\tilde{m}_\pi^2 &= x + 2y, \\
\tilde{m}_K^2 &= x - y, \\
\tilde{m}_\eta^2 &= x - 2y.
\end{align*}
$$

For the chiral logarithms stemming from the loop integration we have introduced

$$
\begin{align*}
L_\pi &= \frac{\tilde{m}_\pi^2}{16\pi^2} \ln \left( \frac{\tilde{m}_\pi^2}{\mu^2} \right), \\
L_K &= \frac{\tilde{m}_K^2}{16\pi^2} \ln \left( \frac{\tilde{m}_K^2}{\mu^2} \right), \\
L_\eta &= \frac{\tilde{m}_\eta^2}{16\pi^2} \ln \left( \frac{\tilde{m}_\eta^2}{\mu^2} \right).
\end{align*}
$$

The functions

$$
\begin{align*}
m_O(a) &= V_0 + V_1a + V_2a^2 + V_3a^3, \\
m_S(a) &= S_0 + S_1a + S_2a^2 + S_3a^3, \\
\lambda_x(a) &= \lambda_x^{(0)} + \lambda_x^{(1)}a, \\
\lambda_y(a) &= \lambda_y^{(0)} + \lambda_y^{(1)}a, \\
\sigma_x(a) &= \sigma_x^{(0)} + \sigma_x^{(1)}a, \\
\sigma_y(a) &= \sigma_y^{(0)} + \sigma_y^{(1)}a,
\end{align*}
$$

parametrize the analytic lattice spacing dependence. We have, for convenience and for the sake of transparency, introduced new constants $V_i, S_i, i = 1, \ldots, 4$ and $\lambda_x^{(j)}, \lambda_y^{(j)}, j = 1, 2$. These new constants are combinations of the low energy constants in the chiral Lagrangian. They are explicitly given

$$
\begin{align*}
V_0 &= \mu_0, \quad V_1 = \alpha_o, \quad V_2 = \beta_0, \\
S_0 &= \mu_s, \quad S_1 = \alpha_s, \quad S_2 = \beta_s, \\
\lambda_x^{(0)} &= \frac{N\lambda_{o1} + 2\lambda_{o2}}{2B}, \quad \lambda_x^{(1)} = \frac{N\gamma_{o1}^r + 2\gamma_{o2}^r}{2B}, \\
\lambda_y^{(0)} &= \frac{\lambda_{o2}}{B}, \quad \lambda_y^{(1)} = \frac{\gamma_{o2}^r}{B}, \\
\sigma_x^{(0)} &= \frac{N\lambda_s}{2B}, \quad \sigma_x^{(1)} = \frac{N\gamma_s^r}{2B}, \\
\sigma_y^{(0)} &= \frac{\sqrt{6}\lambda_{os}}{B}, \quad \sigma_y^{(1)} = \frac{\sqrt{6}\gamma_{os}^r}{B}.
\end{align*}
$$

However, from a practical point of view there is no need to keep track of the original low energy constants. Note that we have added an $O(a^3)$ term for the octet and singlet vector meson mass, but not for the singlet-octet mixed contribution $m_{(08)}$, as discussed in sect. [HIC].
Note that there are chiral logarithms present in the results for the vector meson masses. These chiral corrections are a lattice artifact, as can be seen from the presence of the \( a \) in front of the logarithms. This chiral log contribution vanishes in the continuum limit and the mass formulae converge to the results given in Ref. [8]. However, as long as the lattice spacing is non-zero, there are two kinds of non-analytical quark mass dependence, proportional to \( m_q^{3/2} \) and \( m_q \log m_q \). It will be very interesting to study the competition of these two contributions in actual lattice simulations.

Let us count the number of unknown parameters. In the case of the \( \rho \) and \( K^* \) meson, we find twelve unknown combinations of low-energy constants: \( V, \lambda(j), \lambda(j)_x, g_1, g_2, \alpha_{o1}, \alpha_{o2} \). However, as long as we are interested in performing fits to lattice data at a given and fixed lattice spacing \( a \), this number essentially reduces to seven independent constants: \( m_O, \lambda_x, \lambda_y, g_1, g_2, \alpha_{o1}, \alpha_{o2} \). For the \( m_{\pm} \) sector, we have additional ten low energy constants: \( S, \sigma(j), \sigma(j)_x, \alpha_{os}, \alpha_s \). When we consider at a fixed lattice spacing, there are essentially five independent constants: \( m_S, \sigma_x, \sigma_y, \alpha_{os}, \alpha_s \).

**IV. CONCLUDING REMARKS**

We have derived the one loop expressions for vector meson masses using an effective theory based on the heavy vector meson formalism of Ref. [8]. The effects due to a non-vanishing lattice spacing introduce a fair number of new unknown low-energy constants in the chiral Lagrangian. However, the actual number of fit parameters in the mass formulae seems small enough for the expressions to be useful for the chiral extrapolation of actual lattice data. In the case of the \( \rho \) and \( K^* \) meson mass, the number of unknown fit parameters is seven. The numerical simulations of the CP-PACS/JLQCD collaboration are carried out with five different (degenerate) up and down quark masses and two different values for the strange quark mass. The total number of independent data points is 18 without data for the \( \phi \) and \( \omega \) mesons. If one measures these masses including noisy disconnected contributions, the number of data points increases to 36. In both cases the number of data points is well above 7.

Here we have calculated the expressions for the vector meson masses only. In the calculation of the vector meson decay constants some additional low-energy constants enter, depending on the choice for the vector current. Apart from this the calculation is straightforward and currently under way [20].

Our main motivation for the calculation presented here was to derive fit forms for the chiral extrapolation of the vector meson masses. If our formulae describe the lattice QCD data well,
we obtain, as a byproduct, estimates for the low energy constants of the chiral effective theory for vector mesons. Particularly promising are the leading order parameters $g_1$ and $g_2$. These couplings are not corrected by lattice artifacts through $O(E^3)$. Provided that the corrections of $O(E^4)$ and higher are negligible, we have a good chance to determine $g_1$ and $g_2$ in a fit, even if lattice data is available for one lattice spacing only. The chiral quark model \[21\] predicts $g_2 = 0.75$ in the large $N_c$ limit where $g_1 = \frac{2}{\sqrt{3}}g_2$. On the other hand, the estimate $g_2 \sim 0.6$ is given in Ref. \[22\], where vector meson chPT is applied to $\tau$ decay processes together with a comparison of the results with experimental data. If we are able to extract $g_1$ and $g_2$ in a fit to lattice QCD data, we obtain results based on first principles QCD without relying on model dependent assumptions or the large $N_c$ limit.

The heavy vector meson formalism of Ref. \[8\] is not the only way to derive an effective theory for vector mesons. Another approach employs the so-called hidden local symmetry \[23\]. Also this approach has a definite counting scheme, and in principle one can formulate it for non-zero lattice spacing too. It would be interesting to perform the calculation of the vector meson masses based on this alternative approach. The comments in the previous paragraph about the extraction of low energy constants through a fit to lattice data applies to this effective theory as well.

**Acknowledgments**

This work is supported in part by the Grants-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology (Nos. 13135204, 15204015, 15540251, 16028201, 16-11968). O. B. is supported in part by the University of Tsukuba Research Project. S. T. is supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

**APPENDIX A: FEYNMAN RULES**

In this appendix, we present the Feynman rules needed in the calculation for the vector meson masses. The lattice spacing $a$ shown in the following equations stems from the spurion field $W_\pm$ once the replacement $A \rightarrow aI$ has been made.
1. Expanding the Lagrangian

In terms of the component fields, the LO Lagrangian which includes up to two pseudo scalar meson fields is expressed as

\[
\mathcal{L}_{\text{LO}} = \mathcal{L}_p + \mathcal{L}_a
\]

\[
= (\mathcal{L}_p^{0\pi} + \mathcal{L}_a^{0\pi}) + \mathcal{L}_p^{1\pi} + (\mathcal{L}_p^{2\pi} + \mathcal{L}_a^{2\pi}) + O(\pi^3), \tag{A1}
\]

\[
\mathcal{L}_p^{0\pi} + \mathcal{L}_a^{0\pi} = S_\mu^\dagger (iv \cdot \partial + a\alpha_s) S_\mu + \sum_a \rho_\mu^a (iv \cdot \partial + a\alpha_{o1}) \rho_\mu^a, \tag{A2}
\]

\[
\mathcal{L}_p^{1\pi} = \frac{ig_1}{f} \sum_a (S_\mu^\dagger \rho_\mu^a - S_\mu \rho_\mu^a) \partial_\lambda \pi_a v_\sigma \epsilon^{\mu\nu\lambda\sigma} + \frac{ig_2\sqrt{2}}{f} \sum_{a,b,c} d_{abc} \rho_\mu^{a\dagger} \rho_\nu^b \partial_\lambda \pi_c v_\sigma \epsilon^{\mu\nu\lambda\sigma}, \tag{A3}
\]

\[
\mathcal{L}_p^{2\pi} + \mathcal{L}_a^{2\pi} = \frac{i}{f^2} \sum_{a,b,c,d,e} f_{abc} f_{bce} \rho_\mu^{a\dagger} \rho_\nu^b \pi^c (v \cdot \partial \pi^d) \rho_\mu^d
\]

\[
- \frac{a\sqrt{2} \alpha_{os}}{N f^2} \sum_{a,b,c} d_{abc} (S_\mu^{a\dagger} \rho_\mu^{b} \pi^c + S_\mu^{a\dagger} \rho_\mu^{b} \pi^c)
\]

\[
- \frac{a}{N f^2} \sum_{a,b,c,d} (\alpha\alpha_1 \delta_\ab \delta_\cd + \alpha_{o2} \sum_e d_{abe} d_{cde} \rho_\mu^{a\dagger} \rho_\nu^{b\dagger} \pi^{c d})
\]

\[
- \frac{a}{N f^2} \sum_{a,b} \delta_\ab S_\mu^{a\dagger} S_\mu^b \pi^a \pi^b, \tag{A4}
\]

where the newly introduced coefficients relate to the ones defined in eq. \[28\] according to

\[
\alpha_s = N\alpha_1, \tag{A5}
\]

\[
\alpha_{os} = N\alpha_2, \tag{A6}
\]

\[
\alpha_{o1} = 2\alpha_3 + N\alpha_4, \tag{A7}
\]

\[
\alpha_{o2} = N\alpha_3. \tag{A8}
\]

The summation over roman indices is understood to range from 1 to 8. \[f_{abc}\] and \[d_{abc}\] are the structure constants and the totally symmetric \[d\]–symbol of \[\text{su}(N)\] respectively. Obviously, we are most interested in the case \[N = 3\].

The NLO and NNLO Lagrangian results in the following tree level contribution to the vector
meson masses:

\[ \mathcal{L}_{\text{NLO}} + \mathcal{L}_{\text{NNLO}} = (\mathcal{L}_m + \mathcal{L}_{a^2}) + \mathcal{L}_{am} \]
\[ = \mathcal{L}_{m}^{0\pi} + \mathcal{L}_{a^2}^{0\pi} + \mathcal{L}_{am}^{0\pi} + O(\pi), \]

\[ \mathcal{L}_m^{0\pi} + \mathcal{L}_{a^2}^{0\pi} + \mathcal{L}_{am}^{0\pi} = (NM_0\lambda_s + a^2\beta_s + N\tilde{M}_0a\gamma_s + \Delta \mu)S^\dagger_\mu S^\mu \]
\[ + (\lambda_{os} + a\gamma_{os})\frac{\bar{M}_8}{\sqrt{2}} \sum_a \delta_{as}(\rho^{a\dagger}_\mu S^\mu + \rho^a S^{\mu\dagger}) \]
\[ + (N\tilde{M}_0\lambda_{o1} + a^2\beta_{o} + N\tilde{M}_0a\gamma_{o1}) \sum_a \rho^{a\dagger}_\mu \rho^{\mu a} \]
\[ + (\lambda_{o2} + a\gamma_{o2}) \sum_{a,b}(2\tilde{M}_0\delta_{ab} + \tilde{M}_8d_{ab8})\rho^{a\dagger}_\mu \rho^{\mu b}, \]

where

\[ \beta_s = N\beta_1 + N^2\beta_2, \]
\[ \beta_o = 2\beta_5 + \beta_6 + 2N\beta_7 + N\beta_8 + N^2\beta_9, \]
\[ \gamma_s = \gamma_1 + N\gamma_2, \]
\[ \gamma_{os} = 2\gamma_4 + N\gamma_5, \]
\[ \gamma_{o1} = 2\gamma_6 + \gamma_7 + N\gamma_8, \]
\[ \gamma_{o2} = 2\gamma_9 + \gamma_{10} + N\gamma_{11}. \]

2. Propagators

The propagator for the singlet and octet vector meson is given by

\[ G^S_{\mu\nu}(r) = \frac{-iP_{\mu\nu}}{v \cdot r - aa_s}, \]
\[ G^{O,ab}_{\mu\nu}(r) = \frac{-iP_{\mu\nu}\delta^{ab}}{v \cdot r - aa_{o1}}, \]

where \( r_\mu \) is the residual momentum.

For the pseudo scalar meson, the propagator is

\[ G^{ab}_{PS}(p) = \frac{i\delta^{ab}}{p^2 - \tilde{m}_a^2}, \]

where the leading order pseudo scalar mass \( \tilde{m}_a^2 \) of Ref. [1] is used. The explicit form of the mass is

\[ \tilde{m}_a^2 = m_a^2(1 - Nc_3 - \tilde{c}_3) - m_{av}Nc_3 - Nc_2 - \tilde{c}_2, \]
with

\[ m^2_{av} = \frac{1}{N^2 - 1} \sum_a m^2_a, \]  

(A21)

where \( m_a^2 \) is the meson mass in the continuum limit (i.e. \( m_1^2 = m_2^2 = m_3^2 = m_\pi^2 \), \( m_4^2 = m_5^2 = m_\eta^2 \)). \( c_1, c_3, \tilde{c}_3 \) are \( O(a) \) terms [17], and \( c_2, \tilde{c}_2 \) are \( O(a^2) \) terms [9, 24]. They are low energy constants in the pure pseudo scalar meson sector. The mass \( \tilde{m}_a^2 \) is expressed in terms of the shifted quark masses,

\[
\tilde{m}_a^2 = \begin{cases} 
\tilde{m}_\pi^2 = 2B\tilde{m}, & a = 1, 2, 3, \\
\tilde{m}_K^2 = B(\tilde{m} + \tilde{m}_s), & a = 4, 5, 6, 7, \\
\tilde{m}_\eta^2 = \frac{2B}{3}(\tilde{m} + 2\tilde{m}_s), & a = 8.
\end{cases}
\]  

(A22)

3. Vertices

- NLO vector-vector 2-point vertex from \( \mathcal{L}_{\pi}^{0\pi} + \mathcal{L}_{a}^{0\pi} + \mathcal{L}_{am}^{0\pi} \)

\[
\nu, b \quad \begin{array}{c} \mu, a \\ \end{array} = iP_{\mu \nu} \delta_{ab} (N\tilde{M}_0 \lambda_{\alpha_1} + a^2 \beta_\alpha + Na\tilde{M}_0 \gamma_{\alpha_1}) \\
+ (2\tilde{M}_0 \delta_{ab} + \tilde{M}_8 \delta_{ab})(\lambda_{\alpha_2} + a\gamma_{\alpha_2}),
\]

\[
\nu, a \quad \begin{array}{c} \mu \\ \end{array} = iP_{\mu \nu} \delta_{ab} \tilde{M}_8 \sqrt{2}(\lambda_{\alpha_3} + a\gamma_{\alpha_3}),
\]

\[
\nu \quad \begin{array}{c} \mu \\ \end{array} = iP_{\mu \nu} (N\tilde{M}_0 \lambda_{\alpha} + a^2 \beta_\alpha + Na\tilde{M}_0 \gamma_{\alpha} + \Delta \mu).
\]

The oriented narrow line represents the singlet vector meson and the oriented wide line represents the octet vector meson.
• LO vector-vector-pseudo-scalar 3-point vertex from $\mathcal{L}^1_{\mu \pi}$

$$\frac{ig_1}{f} p_\lambda v_\sigma \epsilon^{\mu \nu \lambda \sigma} \delta_{ab},$$

The dotted line represents the incoming pseudo scalar meson.

• LO vector-vector-pseudo scalar-pseudo scalar 4-point vertex from $\mathcal{L}^2_{a \pi}$

$$\frac{ig_2 \sqrt{2}}{f} p_\lambda v_\sigma \epsilon^{\mu \nu \lambda \sigma} d_{abc},$$

$$\frac{ig_2 \sqrt{2}}{f} p_\lambda v_\sigma \epsilon^{\mu \nu \lambda \sigma} d_{abc},$$

$$\frac{ig_2 \sqrt{2}}{f} p_\lambda v_\sigma \epsilon^{\mu \nu \lambda \sigma} d_{abc}.$$
[1] S. Aoki, O. Bär, T. Ishikawa and S. Takeda, hep-lat/0509049.
[2] S. Aoki, O. Bär, T. Ishikawa and S. Takeda, work in progress.
[3] T. Ishikawa et al., Nucl. Phys. Proc. Suppl. 140 (2005) 225.
[4] S. R. Sharpe and J. Singleton, Robert, Phys. Rev. D58 (1998) 074501.
[5] W.-J. Lee and S. R. Sharpe, Phys. Rev. D60 (1999) 114503.
[6] O. Bär, Nucl. Phys. Proc. Suppl. 140 (2005) 106.
[7] H. R. Grigoryan and A. W. Thomas, hep-lat/0507028.
[8] E. Jenkins, A. V. Manohar and M. B. Wise, Phys. Rev. Lett. 75 (1995) 2272.
[9] S. Aoki, Phys. Rev. D68 (2003) 054508.
[10] Y. Namekawa et al., Phys. Rev. D70 (2004) 074503.
[11] E. Jenkins and A. V. Manohar, Phys. Lett. B255 (1991) 558.
[12] J. Bijnens and P. Gosdzinsky, Phys. Lett. B388 (1996) 203.
[13] J. Bijnens, P. Gosdzinsky and P. Talavera, Phys. Lett. B429 (1998) 111.
[14] J. Bijnens, P. Gosdzinsky and P. Talavera, Nucl. Phys. B501 (1997) 495.
[15] M. Booth, G. Chiladze and A. F. Falk, Phys. Rev. D55 (1997) 3092.
[16] C.-K. Chow and S.-J. Rey, Nucl. Phys. B528 (1998) 303.
[17] G. Rupak and N. Shoresh, Phys. Rev. D66 (2002) 054503.
[18] S. Scherer, Advances in Nuclear Physics. Vol. 27 (2002) 277.
[19] S. Aoki, Phys. Rev. D34 (1986) 3170.
[20] S. Aoki and S. Takeda, work in progress.
[21] A. Manohar and H. Georgi, Nucl. Phys. B234 (1984) 189.
[22] H. Davoudiasl and M. B. Wise, Phys. Rev. D53 (1996) 2523.
[23] M. Harada and K. Yamawaki, Phys. Rept. 381 (2003) 1.
[24] O. Bär, G. Rupak and N. Shoresh, Phys. Rev. D70 (2004) 034508.