Patchwork quilts:
From positional information
to self-affine surfaces

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Finite automata working bitwise on the local coordinates of points in the plane are constructed and shown to lead to self-affine surfaces (‘patchwork quilts’) under general circumstances. We prove that these models give rise to a roughness exponent that shapes the resulting spatial patterns: Larger values of the exponent lead to coarser surfaces. We suggest that finite automata provide the mathematical link between the concept of positional information of modern theoretical biology and the emergence of fractal self-affine surfaces ubiquitously found in nature.
I. INTRODUCTION

Fractals [1,2] are ubiquitously found in physical systems at all scales [3–5] as well as in biological systems. Protein surfaces, for example, are known to have fractal self-affine features [7,8]. A surface $F(x,y)$ is self-affine if it satisfies [7,9]

$$F(x,y) \sim b^{-H}F(bx,by)$$  
(1)

where $H \in \mathbb{R}$ is the roughness exponent characterizing the self-affine scaling [1] [10].

Mathematically, fractals arise from iterated function systems [11] [12], strange attractors of chaotic dynamical systems [13], critical phenomena [14], cellular automata [15] [16] [17], arithmetic sequences [18], and any problems where some hierarchical structure is present [3] [19]. Because of these connections, fractals are also important in some recent approaches to nonequilibrium statistical mechanics of steady states [20] [21] [22] [23].

In this article we constructively prove the rather general fact that finite automata locally operating on each point in the plane $\mathbb{R}^2$ lead to fractal self-affine surfaces by providing explicit mathematical expressions for such surfaces. Our approach is inspired by the concept of positional information in biology. The latter proposes that cells in an organism have detailed information of their position coordinates and that they work out this information giving rise to spatial patterns [24]. In this article we do not discuss the physical/biological basis (e.g. morphogen gradients) that leads to establish this positional information. Rather, we take the existence of positional information for granted and we show how, by means of a local finite automaton at each point ($u,v$) in the plane (representing the hypothetical operations of a cell) fractal self-affine surfaces always emerge. The finite automaton works out bitwise on the digits of the position coordinates. We explicitly construct the generalized bitwise operators involved in this process by making use of the framework of digital calculus, that we have very recently introduced [25] [26].

In Section II we introduce the main concepts of digital calculus, the $B$-function and the digit function, as well as the discrete operators that we shall need. Then, in Section III with the aid of these discrete operators, we construct local finite automata models acting on the plane and we rigorously prove that all give rise to self-affine surfaces satisfying Eq. (1). We then also prove that the roughness exponent behaves in the normal, expected way, larger values of it leading to coarser surfaces [10] [27] [28].

II. THE $B$-FUNCTION, THE DIGIT FUNCTION AND DISCRETE OPERATORS

Digital calculus is based on the use of the $B$-function [29] and the digit function [25] and we define these first. The $B$-function for any real numbers $x$, $y$ is given by

$$B(x,y) \equiv \frac{1}{2} \left( \frac{x+y}{|x+y|} - \frac{x-y}{|x-y|} \right) = \frac{1}{2} \left( \text{sign}(x+y) - \text{sign}(x-y) \right)$$  
(2)

with $\text{sign}(x) \equiv \frac{x}{|x|}$ being the sign function. Since $\text{sign}(0) \equiv 0$, at $x = \pm y$ (singular borders), $B(\pm y, y) = \frac{y}{2|y|} = \text{sign}(y)/2$ and at the origin $B(0,0) \equiv 0$.

If, $x$ is integer valued and $y = 1/2$ as it shall be the case here, then the $B$-function behaves as a Kronecker delta: Let $a$ and $b$ be two integers; then $B(a-b) = \delta_{ab}$ i.e. equal to one if $a = b$ and zero otherwise.
The $B$-function allows a universal map for cellular automata to be formulated \[29\]. In this article we shall denote by $S$ the set of non-negative integers $\in \[0, p - 1\]$ where $p > 1$ is a natural number. Let $t$, a non-negative integer denote 'time' and $i \in \mathbb{Z}$ denote a 'position' on a 1D ring. Let $x_i^t \in S$ denote the dynamical state of a cell $i$ at time $t$. In \[29\] we found the following universal map for cellular automata \[29\]

$$x_{i+1}^t = \sum_{n=0}^{p^{r+l+1}-1} a_n B \left( n - \sum_{k=-r}^{l} p^{k+r} x_i^{i+k}, \frac{1}{2} \right)$$

(3)

where the coefficients $a_n \in S$ specify the cellular automata rule, together with the number of dynamical states $p$, the number of neighbors to the left $l$ and to the right $r$ and the Wolfram code $R$ given by

$$R \equiv \sum_{n=0}^{p^{r+l+1}-1} a_n p^n.$$  (4)

Thus, one can speak of the cellular automata operator $I R_p^r$ with $R$ given by \[4\] and $l$, $r$ and $p$ being parameters that specify the rule. This operator acts on the dynamical value at position $i$ giving rise to the cellular automata map $x_{i+1}^t = I R_p^r(x_i^{i+l}, x_i^{i+l}, ..., x_i^{i-r})$, which we abbreviate as $x_{i+1}^t = I R_p^r(x_i^t)$.

If we take $N = l + r + 1$, we can use these cellular automata operators on $N$ different variables taking values over the finite set $S$ without regarding them as associated to a specific topology. Thus, we define the discrete operator $N R_p(x_0, ..., x_{N-1})$ as

$$y = \sum_{n=0}^{p^{N-1}-1} a_n B \left( n - \sum_{k=0}^{N-1} p^k x_k, \frac{1}{2} \right) \equiv N R_p(x_0, ..., x_{N-1})$$

(5)

where $R$ is given by Eq. \[4\] as above. Such an operator introduces thus the mapping $S^N \rightarrow S$. In the case of 2-variable operators $N = 2$, Eq. \[5\] reduces to

$$y = 2 R_p(x_0, x_1) = \sum_{n=0}^{p^2-1} a_n B \left( n - p x_1 - x_0, \frac{1}{2} \right)$$

(6)

Such discrete 2-variable operators are termed laws of composition or magmas, i.e. the mappings $S \times S \rightarrow S$ \[30\] \[31\]. Note that closure is automatically warranted since $y$ is equal to any of the $a_n$'s which are all in $S$. Eq. \[6\] contains all laws of composition that can be established on $S$. These include all quasigroups, loops, semigroups, monoids, groups and abelian groups \[31\]. (If the restriction that all $a_n$'s must be in $S$ is removed, then one can also describe categories, groupoids and semicategories as well. These structures shall not occupy us here.)

Since $S$ is finite, we can tabulate all values that result from the operation $2 R_p(x_0, x_1)$. Thus, if the rows denote $x_1$ and the columns $x_0$, Eq. \[6\] provides the Cayley table of the magma as

| $2 R_p$ | 0 | 1 | $\ldots$ | $k$ | $\ldots$ | $p - 2$ | $p - 1$ |
|---------|---|---|---------|-----|---------|--------|--------|
| 0       | $a_0$ | $a_1$ | $\ldots$ | $a_k$ | $\ldots$ | $a_{p-2}$ | $a_{p-1}$ |
| 1       | $a_p$ | $a_{p+1}$ | $\ldots$ | $a_{p+k}$ | $\ldots$ | $a_{2p-2}$ | $a_{2p-1}$ |
| 2       | $a_{2p}$ | $a_{2p+1}$ | $\ldots$ | $a_{2p+k}$ | $\ldots$ | $a_{3p-2}$ | $a_{3p-1}$ |
| $\ldots$ | $a_{kp}$ | $a_{kp+1}$ | $\ldots$ | $a_{kp+k}$ | $\ldots$ | $a_{(k+1)p-2}$ | $a_{(k+1)p-1}$ |
| $p - 2$ | $a_{p(p-2)}$ | $a_{p(p-2)+1}$ | $\ldots$ | $a_{p(p-2)+k}$ | $\ldots$ | $a_{p(p-1)-2}$ | $a_{p(p-1)-1}$ |
| $p - 1$ | $a_{p(p-1)}$ | $a_{p(p-1)+1}$ | $\ldots$ | $a_{p(p-1)+k}$ | $\ldots$ | $a_{p^2-2}$ | $a_{p^2-1}$ |
Very recently, we have constructed such Cayley tables for some infinite families of finite groups by means of a digit function, which is closely related to the B-function above. We introduce now this digit function indicating its relationship to the B-function. The digit function \( d_p(k, x) \), for \( p \in \mathbb{N} \), \( k \in \mathbb{Z} \) and \( x \in \mathbb{R} \) is a surjective mapping \( \mathbb{R} \to S \) defined as

\[
d_p(k, x) = \left \lfloor x \cdot p^k \right \rfloor - p \left \lfloor x \cdot p^{k+1} \right \rfloor (7)
\]

and gives the \( k \)-th digit of the real number \( x \) (when it is non-negative) in a positional numeral system in radix \( p > 1 \). If \( p = 1 \) the digit function satisfies \( d_1(k, x) = d_1(0, x) = 0 \) and it does not relate to a positional numeral system. In Eq. (7) \( \lfloor \ldots \rfloor \) denotes the floor function (lower closest integer) of the quantity between the brackets.

With the digit function we can express any real number \( x \) as

\[
x = \text{sign}(x) \sum_{k=\infty}^{\lfloor \log_p |x| \rfloor} p^k d_p(k, |x|) (8)
\]

The integer part of \( x \) is given by

\[
\text{sign}(x) |x| = \text{sign}(x) \sum_{k=0}^{\lfloor \log_p |x| \rfloor} p^k d_p(k, |x|) (9)
\]

Example: In the decimal radix \( p = 10 \), the number \( \pi \equiv 3.1415 \ldots \) has digits \( d_{10}(0, \pi) = 3, d_{10}(-1, \pi) = 1, d_{10}(-2, \pi) = 4, d_{10}(-3, \pi) = 1, d_{10}(-4, \pi) = 5 \), etc.

The digit function possess an important scaling property that we shall use below. We have

\[
d_p(k, p^m x) = \left \lfloor \frac{p^m x}{p^k} \right \rfloor - p \left \lfloor \frac{p^m x}{p^{k+1}} \right \rfloor = \left \lfloor x \cdot \frac{1}{p^{k-m}} \right \rfloor - p \left \lfloor x \cdot \frac{1}{p^{k-m+1}} \right \rfloor = d_p(k-m, x) (10)
\]

An alternative expression for Eq. (5) is provided by the digit function as

\[
y = d_p \left( \sum_{k=0}^{N-1} p^k x_k, \sum_{n=0}^{N-1} p^n a_n \right) = d_p \left( \sum_{k=0}^{N-1} p^k x_k, R \right) (11)
\]

This expression most concisely defines the operator \( NR_p \).

III. MAIN RESULTS: PATCHWORK QUILTS, SELF-AFFINE SURFACES AND THE ROUGHNESS EXPONENT

We now draw inspiration from the concept of positional information in biology as follows. We assume that at each point with coordinates \((u, v)\) in the plane \( \mathbb{R}^2 \) there is a local finite automata so that:

- The operations that can be made are performed on the coordinates \( u, v \) numerically given by strings of digits, also called \textit{words}, corresponding to the numerical real value that the coordinates have (the digits giving its radix \( p \) representation). Although these words are infinite, they can always be truncated in practice to a specific number of significant digits: In nature there is never an infinitesimally tiny length scale.
• Such operations can be interpreted in terms of bitwise arithmetic carried on the digits to locally produce the digits of another word (the resulting local output). This leads to a spatial distribution of the words according to the local operations performed by each cell on its position coordinates.

• If one in turn interprets the resulting words as real scalars, the digits giving a radix $q$ representation (with $q$ not necessarily equal to $p$) the spatial distribution of this scalar field is a discontinuous function on the plane $\mathbb{R}^2$.

We thus seek a function $b_q: \mathbb{R}^2 \rightarrow \mathbb{R}$ of a local discrete operator $2R_p$ given by Eq. (6) (or Eq. (11) with $N = 2$) and of the coordinates $u$ and $v$. The parameter $q \in \mathbb{N}$, $q \geq p$ is the radix of the output number. Since the operator $2R_p$ acts only on $S^2 = S \times S$ where $S$ is the set of the integers $\in [0, p-1]$, a map $\mathbb{R}^2 \rightarrow S^2$ is necessary first to send the coordinates $u$ and $v$ to elements of $S$. This map is provided by the digit function above, by extracting the digits of the coordinates. Since the function $b_q$ is real-valued we then find, in consistency with Eq. (8) and with the above remarks that the function sought has necessarily the following structure:

$$b_q(2R_p; u,v) \equiv \sum_{k=-\infty}^{k_{max}} 2R_p(d_p(k,u), d_p(k,v)) q^k$$

$$= \sum_{k=-\infty}^{k_{max}} q^k \sum_{n=0}^{p^2-1} a_n \mathcal{B}\left(n - d_p(k,u) - p d_p(k,v), \frac{1}{2}\right)$$

$$= \sum_{k=-\infty}^{k_{max}} q^k d_p\left(d_p(k,u) + p d_p(k,v), R\right)$$

(12)

where $k_{max} = \max\{\lfloor \log_p u \rfloor, \lfloor \log_p v \rfloor\}$.

Example: Let us find what is the numerical value of $b_2(2142; 5, 11)$. We have that 5 and 11 are given respectively in base $p = 2$ as '0101' and '1011'. Now the operator 142 has vector $(a_0, a_1, a_2, a_3) = (0, 1, 1, 1)$ which means that it returns one when the digit $k$ of $u$ or $v$ or both is one and zero otherwise. Then, the resulting string is (1 1 1 1), a number in base $q = 2$ which corresponds to 15 in the decimal base. We thus have $b_2(142; 5, 11) = 15$.

In Figs. 1 and 2 $b_q$ is plotted as a function of $(u, v)$ and the operator indicated in each case. Patchwork quilts are obtained in every case when projecting the 3D function in the plane (the figures just only show a finite portion of the plane). Such representations on the plane are useful to get a glimpse of the behavior of the operator $2R_p$ within $b_q$. For example that $262$ is commutative can be clearly seen from the reflection axis on the southwest-northeast diagonal exhibited by $b_2(262; u, v)$ (see Fig. 2 top left). The different patches give information on how $b_q$ organizes in the plane.
It is now easy to prove that the function $b_q$ defined by Eq. (12) is self-affine for every possible choice of the local operator $2R_p$ (but the trivial one $R = 0$). We have

$$b_q (2R_p; pu, pv) = \sum_{k=\infty}^{k_{\max}} 2R_p \left( d_p(k, pu), d_p(k, pv) \right) q^k \quad (13)$$

$$= \sum_{k=\infty}^{k_{\max}} 2R_p \left( d_p(k - 1, u), d_p(k - 1, v) \right) q^k$$

$$= \sum_{k'=\infty}^{k'_{\max}} 2R_p \left( d_p(k', u), d_p(k', v) \right) q^{k'+1}$$

$$= p^{\log_p q} \sum_{k'=\infty}^{k'_{\max}} 2R_p \left( d_p(k', u), d_p(k', v) \right) q^{k'} = p^{\log_p q} b_q (2R_p; u, v)$$

where we have used Eq. (10). We then, finally, obtain

$$b_q (2R_p; u, v) = p^{-\log_p q} b_q (2R_p; pu, pv) \quad (14)$$

which proves that $\log_p q = H$ is the roughness exponent of the self-affine surface by simple comparing this latter expression with Eq. (1).
FIG. 2: From top to bottom and left to right, the functions $b_2(262; u, v)$, $b_3(274173; u, v)$, $b_3(294073; u, v)$ and $b_5(2134274173; u, v)$, with $u \in [0, 100]$ and $v \in [0, 100]$.

It is also straightforward to prove that the roughness exponent that we have obtained above has the nice properties reported in experimental studies on fractal self-affine surfaces [10] [27] [28]: A larger value of the roughness exponent corresponds to a coarser (i.e. ‘smoother’) surface. To see this note that when $q$ is increasingly larger, so that $q \gg p$, Eq. (12) scales as

$$b_q (2R_p; u, v) \sim q^{k_{max}} d_p (d_p(k_{max}, u) + p d_p(k_{max}, v), R)$$

(15)
FIG. 3: Gradual spatial coarsening of the bitwise operation function $b_q$ as the parameter $q$ is increased. Shown is the function $b_q(139033; u, v)$ for all integer values of $q \in [3, 11]$ as indicated besides each panel ($u \in [0, 100]$ and $v \in [0, 100]$).

since $0 \leq d_p(d_p(k_{\text{max}}, u) + p d_p(k_{\text{max}}, v), R) \leq p - 1$, $\forall u, v$. Note furthermore that this latter quantity, which is bounded everywhere, is only sensitive to the most significant digit $k_{\text{max}}$ of the largest of the numbers $u$ and $v$. Thus, changing the coordinates $u$ and $v$ the latter quantity varies slowly for $q$ large, because the most significant digits of the coordinates change on a longer length scale. This, in turn means that $b_q$ changes on more coarser structures for larger $q$ (if all other parameters are kept constant).

Since all 'fine details' of the surface are contained in the less significant digits of $b_q$, the latter are lost as $q$ is increased because they become negligible, due to the polynomial form of $b_q$ and to the fact that the coefficients of the polynomial are bounded functions of the coordinates (sent by the digit function to the finite set of nonnegative integers $S$). This observation is illustrated in Fig. [3] for the function $b_q(139033; u, v)$ and the values of $q \in [3, 11]$ ($p = 3$): For $q$ increasingly large, details are gradually lost and only the coarser structures survive. In the Figure it is observed that for $q = 10$ the asymptotic limit is already being approached. Remarkably, for $q$ prime details sometimes reemerge (but in a coarser scale): This rather subtle phenomenon poses an interesting number theoretical
problem for which we have not yet a solution, and shall be addressed elsewhere.

IV. CONCLUSIONS

In this article we have provided a new general method to mathematically design fractal self-affine surfaces. The method is inspired by the concept of positional information in biology which proposes that cells in an organism have detailed information of their local position coordinates and that they work out this information giving rise to spatial patterns \[24\]. In view that many entities of interest in biology, as proteins, have fractal properties \[7\] \[35\] \[36\] \[8\] \[37\], we have investigated the mathematical connection between positional information and the emergence of such hierarchical structures.

We have rigorously proved that finite automata acting on real scalar fields on each point in the plane \(\mathbb{R}^2\) lead to self-affine surfaces. We have obtained the roughness exponent from these rather abstract models and shown that it has the characteristic, 'nice' properties expected for such an exponent as widely reported in experimental systems: the larger this exponent, the coarser the resulting surface \[10\] \[27\] \[28\].

We believe that the result here reported is important because of its wide generality and because it shows the connection between algebraic operators defined on finite sets, the breaking of the continuum and the emergence of fractal structures. These operators are quite general (magmas) and only closure has been demanded. In a previous work \[33\] we have shown that if, furthermore, the relevant finite operators involved have Cayley tables with the latin square property, finite families of fractals related by symmetry can be generated out of any function introducing a nontrivial partition of the numerical value of that function. These fractals are, however, of a very different and more special kind to the ones constructed here.

The surfaces here obtained come under the general name 'patchwork quilt' because we are following the acknowledgment made by Donald Knuth in \[38\] to one pattern designed by D. Sleator in 1976 (unpublished result, displayed on p. 136 of \[38\]) resembling the ones obtained here. No relationship of such pattern to fractals and self-affinity seems to have been ever attempted and no systematic construction of such patterns is found in the literature. The study of generalized bitwise arithmetic, first introduced in \[38\] (Chapter 7.1.3) in the context of computer programming, is at its very infancy. Here, Eq. (12) extends the binary-radix generalized bitwise arithmetic of \[38\] to any radices \(p\) and \(q\) and this article constitutes a systematic investigation of this concept, suggesting the importance that it may find in physics. Our approach has also the advantage of considerably simplifying the notation employed by Knuth, besides generalizing it while making such computations considerably straightforward. This is achieved thanks to digital calculus, a mathematical formalism that we have very recently introduced in a formulation of quantum mechanics \[25\] and whose building blocks we have just briefly described here.

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