Inequalities for a Unified Integral Operator via 
\((\alpha, m)\)-Convex Functions

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Recently, a unified integral operator has been introduced by Farid, 2020, which produces several kinds of known fractional and conformable integral operators defined in recent decades (Kwun, 2019, Remarks 6 and 7). The aim of this paper is to establish bounds of this unified integral operator by means of \((\alpha, m)\)-convex functions. The resulting inequalities provide the bounds of all associated fractional and conformable integral operators in a compact form. Also, the results of this paper hold for different kinds of convex functions connected with \((\alpha, m)\)-convex functions.

1. Introduction

To prove the mathematical inequalities, fractional integral operators play an important role in the field of different branches of mathematics and engineering. Many mathematicians have used fractional integrals and conformable fractional integrals to develop integral inequalities [1–14]. We start from definitions of fractional integral operators which are direct consequences of unified integral operators given in (8) and (9).

Definition 1 (see [15]). Let \( \eta_1: [a, b] \to \mathbb{R} \) be an integrable function. Also, let \( \eta_2 \) be an increasing and positive function on \([a, b]\), having a continuous derivative \( \eta'_2 \) on \((a, b)\). The left-sided and right-sided fractional integrals of a function \( \eta_1 \) with respect to another function \( \eta_2 \) on \([a, b]\) of order \( \mu \), where \( \mathbb{R}(\mu) > 0 \), are defined by

\[
\frac{\mu}{\eta_2} \int_a^x \eta_1(t) dt = \frac{1}{\Gamma(\mu)} \int_a^x (\eta_2(x) - \eta_2(t))^{(\mu-1)} \eta'_2(t) \eta_1(t) dt, \quad x > a,
\]

\[
\frac{\mu}{\eta_2} \int_b^x \eta_1(t) dt = \frac{1}{\Gamma(\mu)} \int_b^x (\eta_2(t) - \eta_2(x))^{(\mu-1)} \eta'_2(t) \eta_1(t) dt, \quad x < b,
\]

where \( \Gamma(\cdot) \) is the gamma function.

A \( k \)-analogue of the above definition is defined as follows.

Definition 2 (see [16]). Let \( \eta_1: [a, b] \to \mathbb{R} \) be an integrable function. Also, let \( \eta_2 \) be an increasing and positive function on \([a, b]\), having a continuous derivative \( \eta'_2 \) on \((a, b)\). The left-sided and right-sided fractional integrals of a function \( \eta_1 \) with respect to another function \( \eta_2 \) on \([a, b]\) of order \( \mu \); \( \mathbb{R}(\mu) \), \( k > 0 \) are defined by

\[
\frac{\mu}{\eta_2} \int_a^x \eta_1(t) dt = \frac{1}{k \Gamma(\mu)} \int_a^x (\eta_2(x) - \eta_2(t))^{(k\mu-1)} \eta''_2(t) \eta_1(t) dt, \quad x > a,
\]

\[
\frac{\mu}{\eta_2} \int_b^x \eta_1(t) dt = \frac{1}{k \Gamma(\mu)} \int_b^x (\eta_2(t) - \eta_2(x))^{(k\mu-1)} \eta''_2(t) \eta_1(t) dt, \quad x < b,
\]
where $\Gamma_k(\cdot)$ is defined by [17]

$$
\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-(t+k)}dt, \quad \Re(x) > 0. \quad (4)
$$

A well-known function named Mittag-Leffler function is defined by [18]

$$
E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (5)
$$

where $a, z \in \mathbb{C}$ and $\Re(\alpha) > 0$.

One can see [19–22] to study the Mittag-Leffler function and its generalizations. Also, (2) and (3) produce many types of fractional integral operators (see [10], Remark 6).

A generalized fractional integral operator containing an extended generalized Mittag-Leffler function is defined as follows.

**Definition 3** (see [1]). Let $\omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, and $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$, and $0 < \kappa \leq \delta + \Re(\mu)$. Let $\eta_1 \in L_1[a,b]$ and $x \in [a, b]$. Then, the generalized fractional integral operators $E_{\mu,\alpha,\omega,c,\eta_1}$ defined by

$$
E_{\mu,\alpha,\omega,c,\eta_1}(x; p) = \int_a^x (x-t)^{\alpha-1}E_{\mu,\alpha,\omega,c,\eta_1}(\omega(x-t)^\mu; p)\eta_1(t)dt,
$$

$$
E_{\mu,\alpha,\omega,c,\eta_1}(b; p) = \sum_{n=0}^{\infty} \frac{\beta^p(y + nk, a - \gamma)}{\Gamma(\mu n + \gamma)} \frac{(c)_{nk}}{(l)_{nk}} \frac{t^m}{\Gamma(\mu n + \gamma)}.
$$

is the extended generalized Mittag-Leffler function.

Recently, Farid defined a unified integral operator which unifies several kinds of fractional and conformable integrals in a compact formula which is defined as follows.

**Definition 4** (see [23]). Let $\eta_1, \eta_2: [a, b] \rightarrow \mathbb{R}, 0 < a < b$, be the functions such that $\eta_1$ be positive and $\eta_1 \in L_1[a,b]$ and $\eta_2$ be differentiable and strictly increasing. Also, let $\phi(x)$ be an increasing function on $[a, \infty)$ and $a, l, \gamma, c \in \mathbb{C}$, $\Re(\alpha), \Re(l) > 0$, $\Re(\gamma) > 0$, $\mu, \delta \geq 0$, and $0 < \kappa \leq \delta + \mu$. Then, for $x \in [a, b]$, the left and right integral operators are defined by

$$
K_{\mu,\alpha,\omega,c,\eta_1}(x; p) = \int_a^x \frac{\Gamma_k(x)}{\Gamma_k(\mu n + \gamma)} \frac{(c)_{nk}}{(l)_{nk}} \frac{t^m}{\Gamma(\mu n + \gamma)}
$$

$$
K_{\mu,\alpha,\omega,c,\eta_1}(b; p) = \sum_{n=0}^{\infty} \frac{\beta^p(y + nk, a - \gamma)}{\Gamma(\mu n + \gamma)} \frac{(c)_{nk}}{(l)_{nk}} \frac{t^m}{\Gamma(\mu n + \gamma)}.
$$

where

$$
K_{\mu,\alpha,\omega,c,\eta_1}(x; p) = \phi(\eta_2(x) - \eta_2(\gamma))/\eta_2(x) - \eta_2(\gamma)), \quad \eta_2(\gamma) = 0.
$$

**Theorem 1.** Let $\eta_1: [a, b] \rightarrow \mathbb{R}$ be a positive integrable $(a,m)$-convex function with $m \in (0, 1]$. Let $\eta_2: [a, b] \rightarrow \mathbb{R}$ be differentiable and strictly increasing function, and let $\phi(x)$ be an increasing function on $[a, b]$. If $a, l, \gamma, c \in \mathbb{C}$, $\Re(\alpha), \Re(l) > 0$, $\Re(\gamma) > 0$, $\mu, \delta \geq 0$, and $0 < \kappa \leq \delta + \mu$, then for $x \in (a, b)$, we have

For suitable settings of function $\phi, \eta_2$, and certain values of parameters included in Mittag-Leffler function (7), very interesting consequences are obtained which are comprised in Remarks 6 and 7 of [10].

The objective of this paper is to obtain bounds of unified integral operators explicitly which are directly linked with various fractional and conformable integrals. The $(a,m)$-convexity has been established for these bounds. The notion of $(a,m)$-convexity is defined by Mihesan in [24].

**Definition 5.** A function $\eta_1: [0, b] \rightarrow \mathbb{R}, b > 0$, is said to be $(a,m)$-convex, where $(a,m) \in [0,1]^2$, if

$$
\eta_1(tx + m(1 - t)y) \leq t^a \eta_1(x) + m(1 - t^a) \eta_1(y),
$$

holds for all $x, y \in [0, b]$ and $t \in [0, 1]$.

**Remark 1**

(i) If we put $(a,m) = (1,m)$, then (10) gives the definition of $m$-convex function

(ii) If we put $(a,m) = (1,1)$, then (10) gives the definition of convex function

(iii) If we put $(a, m) = (1,0)$, then (10) gives the definition of star-shaped function

For some recent citations and utilizations of $(a,m)$-convex functions, one can see [9, 25–28] and references therein. In the upcoming section, bounds of unified integral operators are established by using $(a,m)$-convexity. These bounds provide general formulas to obtain bounds of fractional and conformable integral operators described in Remarks 6 and 7 of [10]. Among the well-known inequalities which are related to the integral mean of a convex function, the Hadamard inequality is of great importance. Many mathematicians worked on new types of Hadamard inequalities using convex functions, see [8, 29–31]. We also established the general Hadamard-type inequality by applying Lemma 1 which further produces various inequalities of Hadamard type for fractional and conformable integrals. At the end, by using $(a,m)$-convexity of $|\eta_1|$, a modulus inequality is obtained.

2. Main Results

Bounds of unified integral operators (8) and (9) using $(a,m)$-convexity are studied in the following result:

**Theorem 1.** Let $\eta_1: [a, b] \rightarrow \mathbb{R}$ be a positive integrable $(a,m)$-convex function with $m \in (0, 1]$. Let $\eta_2: [a, b] \rightarrow \mathbb{R}$ be differentiable and strictly increasing function, and let $\phi(x)$ be an increasing function on $[a, b]$. If $a, l, \gamma, c \in \mathbb{C}$, $\Re(\alpha), \Re(l) > 0$, $\Re(\gamma) > 0$, $\mu, \delta \geq 0$, and $0 < \kappa \leq \delta + \mu$, then for $x \in (a, b)$, we have

...
and hence,
\[
\begin{align*}
\left( \eta_2, \mathcal{E}_{\mu,a,l}^{\gamma,\delta,k,c} \eta_1 \right)(x, \omega; p) & \leq K_\alpha \left( \mathcal{E}_{\mu,a,l}^{\gamma,\delta,k,c} \eta_1, \eta_2, \phi \right) \left( m \eta_1 \left( \frac{x}{m} \right) \eta_2(x) - \eta_1(a) \eta_2(a) \right) \\
& \leq K_\alpha \left( \mathcal{E}_{\mu,a,l}^{\gamma,\delta,k,c} \eta_1, \eta_2, \phi \right) \left( m \eta_1 \left( \frac{x}{m} \right) \eta_2(x) - \eta_1(a) \eta_2(a) \right) \right) \\
& - \frac{\Gamma(a+1)}{(x-a)^a} \left( m \eta_1 \left( \frac{x}{m} \right) \eta_2(x) - \eta_1(a) \eta_2(a) \right) \eta_1(I_{\alpha}, \eta_2(x)) \\
& + K_\alpha \left( \mathcal{E}_{\mu,a,l}^{\gamma,\delta,k,c} \eta_1, \eta_2, \phi \right) \left( \eta_1(b) \eta_2(b) - m \eta_1 \left( \frac{x}{m} \right) \eta_2(x) \right) \\
& - \frac{\Gamma(a+1)}{(b-x)^a} \left( \eta_1(b) - m \eta_1 \left( \frac{x}{m} \right) \right) \eta_1(I_{\alpha}, \eta_2(x))
\end{align*}
\]

Proof. Under the assumptions of \( \phi \) and \( \eta_2 \), one can write the following inequality:
\[
\frac{\phi(\eta_2(x) - \eta_2(t))}{\eta_2(x) - \eta_2(t)} \leq \frac{\phi(\eta_2(x) - \eta_2(a))}{\eta_2(x) - \eta_2(a)}, \quad t \in [a,x], x \in (a,b).
\]

Multiplying with \( \mathcal{E}_{\mu,a,l}^{\gamma,\delta,k,c} \left( \omega(\eta_2(x) - \eta_2(t))^\mu; p \right) \eta_2'(t) \), we can obtain
\[
K_\alpha \left( \mathcal{E}_{\mu,a,l}^{\gamma,\delta,k,c} \eta_1, \eta_2, \phi \right) \eta_2'(t) \\
\leq K_\alpha \left( \mathcal{E}_{\mu,a,l}^{\gamma,\delta,k,c} \eta_1, \eta_2, \phi \right) \eta_2'(t)
\]

By using \( \mathcal{E}_{\mu,a,l}^{\gamma,\delta,k,c} \left( \omega(\eta_2(x) - \eta_2(t))^\mu; p \right) \leq \mathcal{E}_{\mu,a,l}^{\gamma,\delta,k,c} \left( \omega(\eta_2(x) - \eta_2(a))^\mu; p \right) \), the following inequality is obtained:
\[
K_\alpha \left( \mathcal{E}_{\mu,a,l}^{\gamma,\delta,k,c} \eta_1, \eta_2, \phi \right) \eta_2'(t) \leq K_\alpha \left( \mathcal{E}_{\mu,a,l}^{\gamma,\delta,k,c} \eta_1, \eta_2, \phi \right) \eta_2'(t).
\]

Using the definition of \( (a,m) \)-convexity for \( \eta_1 \), the following inequality is valid:
\[
\eta_1(t) \leq \eta_1(a) + m \left( 1 - \left( \frac{x-t}{x-a} \right)^a \right) \eta_1 \left( \frac{x}{m} \right).
\]

Multiplying (15) with (16) and integrating over \([a,x]\), one can obtain
\[
\int_a^x K_\alpha \left( \mathcal{E}_{\mu,a,l}^{\gamma,\delta,k,c} \eta_1, \eta_2, \phi \right) \eta_1(t) d(\eta_2(t)) \leq \eta_1(a) K_\alpha \left( \mathcal{E}_{\mu,a,l}^{\gamma,\delta,k,c} \eta_1, \eta_2, \phi \right) \int_a^x \frac{x-t}{x-a} \left( \frac{x-t}{x-a} \right)^a d(\eta_2(t)) \\
+ m \eta_1 \left( \frac{x}{m} \right) K_\alpha \left( \mathcal{E}_{\mu,a,l}^{\gamma,\delta,k,c} \eta_1, \eta_2, \phi \right) \int_a^x \left( 1 - \left( \frac{x-t}{x-a} \right)^a \right) d(\eta_2(t)).
\]
Now, on the other side, for \( t \in (x, b) \) and \( x \in (a, b) \), the following inequality holds:

\[
K_{\delta}^{\phi}(E_{\mu, \alpha, \delta^{k}, \kappa}(\eta_{1}, \phi))_t \leq K_{\delta}^{\phi}(E_{\mu, \alpha, \delta^{k}, \kappa}(\eta_{2}, \phi))_t.
\]  

(19)

Using \((a, m)\)-convexity of \( \eta_{1} \), we have

\[
\left( E_{\mu, \alpha, \delta^{k}, \kappa}(\eta_{1}) \right)(x, \omega; p) \leq K_{\delta}^{\phi}(E_{\mu, \alpha, \delta^{k}, \kappa}(\eta_{2}, \phi))_t \left( (\eta_{1}(b))_t(b) - m\eta_{1}( \frac{x}{m} )_t(x) \right) - \frac{\Gamma(a + 1)}{(b - x)^{a}} \left( \eta_{1}(b) - m\eta_{1}( \frac{x}{m} )_t(x) \right) I_{\nu, \alpha, \delta^{k}}(x).
\]  

(21)

By adding (18) and (21), (12) can be obtained. \( \square \)

**Remark 2**

(i) If we consider \((a, m) = (1, 1)\) in (12), Theorem 8 in [10] is obtained

(ii) If we consider \( \phi(t) = (\Gamma(\mu))^{\mu k}/K_{\delta}^{\phi}(\mu) \) for the left-hand integral and \( \phi(t) = (\Gamma(\nu))^{\nu k}/K_{\delta}^{\phi}(\nu) \) for the right-hand integral and \( p = \omega = 0 \in (12) \), then Theorem 1 in [9] can be obtained

(iii) If we consider \( \mu = \nu \) in the result of (ii), then Corollary 1 in [9] can be obtained

(iv) If we consider \( \phi(t) = (\Gamma(\mu))^{\mu k}/K_{\delta}^{\phi}(\mu) \) for the left-hand integral and \( \phi(t) = (\Gamma(\nu))^{\nu k}/K_{\delta}^{\phi}(\nu) \) for the right-hand integral, \((a, m) = (1, 1)\), then Theorem 1 in [6] is obtained

(v) If we consider \( \mu = \nu \) in the result of (iv), Corollary 1 in [6] is obtained

(vi) If we consider \( \phi(t) = (\Gamma(\mu))^{\mu k}/K_{\delta}^{\phi}(\mu) \) for the left-hand integral and \( \phi(t) = (\Gamma(\nu))^{\nu k}/K_{\delta}^{\phi}(\nu) \) for the right-hand integral, \((a, m) = (1, 1)\), then Theorem 1 in [3] can be obtained

(vii) If we consider \( \mu = \nu \) in the result of (vi), then Corollary 1 in [4] can be obtained

(viii) If we consider \( \phi(t) = (\Gamma(\mu))^{\mu k}/K_{\delta}^{\phi}(\mu) \) for the right-hand integral and \( \phi(t) = (\Gamma(\nu))^{\nu k}/K_{\delta}^{\phi}(\nu) \) for the left-hand integral, \((a, m) = (1, 1)\), then Theorem 1 in [5] can be obtained

(ix) By setting \( \mu = \nu = 1 \) and \( x = a = b = x/m \) in the result of (ix), Corollary 2 in [5] can be obtained

(x) By setting \( \mu = \nu = 1 \) and \( x = a = b/2 \) in the result of (ix), Corollary 3 in [5] can be obtained

To prove the next result, we need the following lemma [9].

**Lemma 1.** Let \( \eta_{1}: [0, \infty) \rightarrow \mathbb{R} \) be an \((a, m)\)-convex function with \( m \in (0, 1) \). If \( \eta_{1}(x) = \eta_{1}(a + b - x/m) \), \( 0 < a < b \), then the following inequality holds:

\[
\eta_{1}(a + b/2) \leq \frac{1}{2^{a}} (1 + m(2^{a} - 1)) \eta_{1}(x),
\]  

(22)

for all \( x \in [a, b] \) and \( m \in (0, 1) \).

The following result provides upper and lower bounds of the sum of operators (8) and (9) in the form of a Hadamard inequality.

**Theorem 2.** With the assumptions of Theorem 1 in addition, if \( \eta_{1}(x) = \eta_{1}(a + b - x/m) \), then we have

\[
\frac{2^{a} \eta_{1}(a + b/2)}{1 + m(2^{a} - 1)} \left( E_{\mu, \alpha, \delta^{k}, \kappa}(\eta_{1}, \phi) \right)(b, \omega; p) \preceq \left( E_{\mu, \alpha, \delta^{k}, \kappa}(\eta_{1}, \phi) \right)(b, \omega; p)
\]

(23)

**Proof.** Under the assumptions of \( \phi \) and \( \eta_{2} \), we have

\[
\frac{\phi(\eta_{2}(x) - \eta_{2}(a))}{\eta_{2}(x) - \eta_{2}(a)} \leq \frac{\phi(\eta_{2}(b) - \eta_{2}(a))}{\eta_{2}(b) - \eta_{2}(a)}.
\]  

(24)

Multiplying with \( E_{\mu, \alpha, \delta^{k}, \kappa}(\omega(\eta_{2}(x) - \eta_{2}(a))^\alpha; p) \eta_{2}(x) \), we can obtain from (24) the following inequality:

\[
K_{\delta}^{\phi}(E_{\mu, \alpha, \delta^{k}, \kappa}(\eta_{1}, \phi))_t(\eta_{2}(x)) \preceq \frac{\phi(\eta_{2}(b) - \eta_{2}(a))}{\eta_{2}(b) - \eta_{2}(a)} E_{\mu, \alpha, \delta^{k}, \kappa}(\omega(\eta_{2}(x) - \eta_{2}(a))^\alpha; p) \eta_{2}(x).
\]  

(25)

By using \( E_{\mu, \alpha, \delta^{k}, \kappa}(\omega(\eta_{2}(x) - \eta_{2}(a))^\alpha; p) \preceq E_{\mu, \alpha, \delta^{k}, \kappa}(\omega(\eta_{2}(b) - \eta_{2}(a))^\alpha; p) \), the following inequality is obtained:

\[
K_{\delta}^{\phi}(E_{\mu, \alpha, \delta^{k}, \kappa}(\eta_{1}, \phi))_t(\eta_{2}(x)) \preceq \frac{\phi(\eta_{2}(b) - \eta_{2}(a))}{\eta_{2}(b) - \eta_{2}(a)} E_{\mu, \alpha, \delta^{k}, \kappa}(\eta_{2}(x)) \eta_{2}(x).
\]  

(26)

Using \((a, m)\)-convexity of \( \eta_{1} \) for \( x \in (a, b) \), we have
By adding (29) and (31), the following inequality can be obtained:

\[
\begin{align*}
\eta_1(x) & \leq \left( \frac{x-a}{b-a} \right)^{\alpha} \eta_1(b) + m \left( 1 - \left( \frac{x-a}{b-a} \right)^{\alpha} \right) \eta_1\left( \frac{a}{m} \right) \\
& \leq m \eta_1\left( \frac{a}{m} \right) K_a^\alpha \left( E^{\gamma,\delta,k,c}_{\mu,a,l}, \eta_2; \phi \right) \int_a^b \left( 1 - \left( \frac{x-a}{b-a} \right)^{\alpha} \right) d(\eta_2(x)) \\
& \leq \eta_1 \left( \frac{b}{a} \right) \phi \left( \eta_2(b) - \eta_2(a) \right) K_a^\alpha \left( E^{\gamma,\delta,k,c}_{\mu,a,l}, \eta_2; \phi \right) \int_a^b \left( \frac{x-a}{b-a} \right)^{\alpha} d(\eta_2(x)).
\end{align*}
\]

Multiplying (26) and (27) and integrating the resulting inequality over \([a,b]\), one can obtain

\[
\eta_1(x) \leq \left( \frac{x-a}{b-a} \right)^{\alpha} \eta_1(b) + m \left( 1 - \left( \frac{x-a}{b-a} \right)^{\alpha} \right) \eta_1\left( \frac{a}{m} \right).
\]

By using Definition 4 and integrating by parts, the following inequality is obtained:

\[
\begin{align*}
\left( \eta \cdot E^{\gamma,\delta,k,c}_{\mu,a,l}, \eta_1 \right)_{\eta_2} \left( x \right) & \leq K_b^\alpha \left( E^{\gamma,\delta,k,c}_{\mu,a,l}, \eta_2; \phi \right) \left( \eta_1(b) \eta_2(b) - m \eta_1\left( \frac{a}{m} \right) \eta_2(a) \right) \\
& \leq \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left( \eta_1(b) - m \eta_1\left( \frac{a}{m} \right) \right)^{\alpha} \bar{I}_{\eta_2} (a).
\end{align*}
\]

On the other hand, for \( x \in (a,b) \), the following inequality holds true:

\[
\begin{align*}
\left( \eta \cdot E^{\gamma,\delta,k,c}_{\mu,a,l,b^r}, \eta_1 \right)_{\eta_2} \left( x \right) & \leq K_b^\alpha \left( E^{\gamma,\delta,k,c}_{\mu,a,l}, \eta_2; \phi \right) \left( \eta_1(b) \eta_2(b) - m \eta_1\left( \frac{a}{m} \right) \eta_2(a) \right) \\
& \leq \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left( \eta_1(b) - m \eta_1\left( \frac{a}{m} \right) \right)^{\alpha} \bar{I}_{\eta_2} (a).
\end{align*}
\]

Adopting the same pattern of simplification as we did for (26) and (27), the following inequality can be observed from (27) and (30):

\[
\begin{align*}
\eta_1 \left( \frac{a+b}{2} \right) & \leq \frac{1}{2^{\alpha}} \left( 1 + tmn \left( 2^a - 1 \right) \right) \times \int_a^b K_a^\alpha \left( E^{\gamma,\delta,k,c}_{\mu,a,l,b}, \eta_2; \phi \right) d(\eta_2(x)).
\end{align*}
\]

From Definition 4, the following inequality is obtained:

\[
\begin{align*}
\eta_1 \left( \frac{a+b}{2} \right) & \leq \frac{2^{\alpha}}{(1 + m(2^a - 1))} \left( \eta_1 \cdot E^{\gamma,\delta,k,c}_{\mu,a,l,b} \right)_{\eta_2} (a; p) \\
& \leq \left( \eta \cdot E^{\gamma,\delta,k,c}_{\mu,a,l,b}, \eta_1 \right)_{\eta_2} (a; p).
\end{align*}
\]
Similarly, multiplying both sides of (22) by \(K_b^x(E_{\mu,1}^{\beta,k,c},\eta_1;\phi)d(\eta_2(x))\) and integrating over \([a,b]\), we have
\[
\eta_1\left( \frac{a+b}{2} \right) \frac{2^a}{(1 + m(2^a - 1))} \left( \eta_1 F_{\mu,1}^{\beta,k,c} + 1 \right) (b; p) \\
\leq \left( \eta_1 F_{\mu,1}^{\beta,k,c} \right) (b; p).
\]  
(35)

By adding (34) and (35), the following inequality is obtained:
\[
\eta_1\left( \frac{a+b}{2} \right) \frac{2^a}{(1 + m(2^a - 1))} \left( \eta_1 F_{\mu,1}^{\beta,k,c} + 1 \right) (a; p) \]
\[
\leq \left( \eta_1 F_{\mu,1}^{\beta,k,c} \right) (a; p) + \left( \eta_1 F_{\mu,1}^{\beta,k,c} \right) (b; p).
\]  
(36)

Using (32) and (36), inequality (23) can be achieved. \(\square\)

**Remark 3**

(i) If we consider \((a,m) = (1,1)\) in (23), Theorem 22 in [10] is obtained

(ii) If we consider \(\phi(t) = \Gamma(\mu)t^{\gamma+1}\) for the left-hand integral and \(\phi(t) = \Gamma(\nu)t^{\beta+1}\) and \(p = \omega = 0\) in (23), then Corollary 3 in [9] can be obtained

(iii) If we consider \(\mu = 0\) in the result of (ii), then Corollary 3 in [9] can be obtained

(iv) If we consider \(\phi(t) = \Gamma(\mu)t^{\gamma+1}\) for the left-hand integral and \(\phi(t) = \Gamma(\nu)t^{\beta+1}\) for the right-hand integral in (23), \(p = 0\), and \((a,m) = (1,1)\) in (23), Theorem 3 in [6] is obtained

(v) If we consider \(\mu = 0\) in the result of (iv), Corollary 3 in [6] is obtained

(vi) If we consider \(\phi(t) = \Gamma(\mu)t^{\gamma+1}\) for the left-hand integral and \(\phi(t) = \Gamma(\nu)t^{\beta+1}\) for the right-hand integral, \((a,m) = (1,1)\), \(\eta_1(x) = x\), and \(p = \omega = 0\) in (23), then Theorem 3 in [4] can be obtained

(vii) If we consider \(\mu = 0\) in the result of (vi), then Corollary 6 in [4] can be obtained

(viii) By setting \(\phi(t) = \Gamma(\nu)t^{\beta+1}\) for the left-hand integral and \(\phi(t) = \Gamma(\gamma)t^{\gamma+1}\) for the right-hand integral, \(p = 0\), \((a,m) = 1\), and \(g(t) = t\) in (23), Theorem 3 in [5] can be obtained

(ix) By setting \(\mu = 0\) in the result of (viii), Corollary 6 in [5] can be obtained

**Theorem 3.** Let \(\eta_1\colon [a,b] \rightarrow \mathbb{R}\) be a differentiable function. \(\eta_1^\prime\) is \((a,m)\)-convex with \(m \in (0,1)\), and let \(\eta_2\colon [a,b] \rightarrow \mathbb{R}\) be differentiable and strictly increasing function; also, let \(\phi(x)\) be an increasing function on \([a,b]\). If \(a,\beta,\gamma,\mu,\nu \in \mathbb{C}\), \(\mathfrak{R}(a),\mathfrak{R}(\beta) > 0\), \(\mathfrak{R}(\gamma) > 0\), \(p,\mu,\delta \geq 0\), and \(0 < k \leq \delta + \mu\), then for \(x \in (a,b)\), we have
\[
\int_a^x K_b^x(E_{\mu,1}^{\beta,k,c},\eta_1;\phi)d(\eta_2(t)) \\
\leq \left( \eta_1 F_{\mu,1}^{\beta,k,c} \right) (a; p) + \left( \eta_1 F_{\mu,1}^{\beta,k,c} \right) (b; p) \\
\leq \left( \eta_1 F_{\mu,1}^{\beta,k,c} \right) (a; p) + \left( \eta_1 F_{\mu,1}^{\beta,k,c} \right) (b; p).
\]  
(37)

where
\[
\left( \eta_1 F_{\mu,1}^{\beta,k,c} \right) (x; p) = \int_a^x K_b^x(E_{\mu,1}^{\beta,k,c},\eta_1;\phi)d(\eta_2(t)), \\
\left( \eta_1 F_{\mu,1}^{\beta,k,c} \right) (x; p) = \int_a^x K_b^x(E_{\mu,1}^{\beta,k,c},\eta_1;\phi)d(\eta_2(t)).
\]  
(38)

**Proof.** Let \(x \in (a,b)\) and \(t \in [a,x]\). Then, using \((a,m)\)-convexity of \(\eta_1^\prime\), we have
\[
|\eta_1^\prime(t)| \leq \left( \frac{x-t}{x-a} \right)^a |\eta_1^\prime(a)| + m \left( 1 - \left( \frac{x-t}{x-a} \right)^a \right) |\eta_1^\prime(x)|. 
\]  
(39)

Inequality (39) can be written as follows:
\[
- \left( \frac{x-t}{x-a} \right)^a |\eta_1^\prime(a)| + m \left( 1 - \left( \frac{x-t}{x-a} \right)^a \right) |\eta_1^\prime(x)| \\
\leq \eta_1^\prime(t) \leq \left( \frac{x-t}{x-a} \right)^a |\eta_1^\prime(a)| + m \left( 1 - \left( \frac{x-t}{x-a} \right)^a \right) |\eta_1^\prime(x)|.
\]  
(40)

Let us consider the second inequality of (40):
\[
|\eta_1^\prime(t)| \leq \left( \frac{x-t}{x-a} \right)^a |\eta_1^\prime(a)| + m \left( 1 - \left( \frac{x-t}{x-a} \right)^a \right) |\eta_1^\prime(x)|.
\]  
(41)

Multiplying (15) and (41) and integrating over \([a,x]\), we can obtain
\[
\int_a^x K_b^x(E_{\mu,1}^{\beta,k,c},\eta_1;\phi)d( \eta_2(t)) \\
\leq \left( \eta_1 F_{\mu,1}^{\beta,k,c} \right) (a; p) + \left( \eta_1 F_{\mu,1}^{\beta,k,c} \right) (b; p) \\
+ m \left( \eta_1 F_{\mu,1}^{\beta,k,c} \right) (a; p) + \left( \eta_1 F_{\mu,1}^{\beta,k,c} \right) (b; p).
\]  
(42)

By using (8) of Definition 4 and integrating by parts, the following inequality is obtained:
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\( \left( \eta_l \right)_{\mu, \alpha, c}^\alpha \eta_2 \) (x, \omega; p)

\[ \leq K^2 \left( \eta_l \right)_{\mu, \alpha, c}^\alpha \eta_2 \phi \]

\[ \times \left( m \eta_1 \left( \frac{x}{m} \right) \eta_2 (x) - \eta_1 (a) \eta_2 (a) \right) \]

\[ - \frac{\Gamma (a + 1)}{(x - a)^a} \left( m \eta_1 \left( \frac{x}{m} \right) - \eta_1 (a) \right)^a I^a_x \eta_2 (x). \]

If we consider the left-hand side from inequality (40) and adopt the same pattern as we did for the right-hand side inequality, then

\[ \left( \eta_l \right)_{\mu, \alpha, c}^\alpha \eta_2 \phi \]

\[ \geq - K^2 \left( \eta_l \right)_{\mu, \alpha, c}^\alpha \eta_2 \phi \]

\[ \times \left( m \eta_1 \left( \frac{x}{m} \right) \eta_2 (x) - \eta_1 (a) \eta_2 (a) \right) \]

\[ - \frac{\Gamma (a + 1)}{(x - a)^a} \left( m \eta_1 \left( \frac{x}{m} \right) - \eta_1 (a) \right)^a I^a_x \eta_2 (x). \]

From (43) and (44), the following inequality is observed:

\[ \left( \eta_l \right)_{\mu, \alpha, c}^\alpha \eta_2 \phi \]

\[ \leq K^2 \left( \eta_l \right)_{\mu, \alpha, c}^\alpha \eta_2 \phi \times \left( m \eta_1 \left( \frac{x}{m} \right) \eta_2 (x) - \eta_1 (a) \eta_2 (a) \right) \]

\[ - \frac{\Gamma (a + 1)}{(x - a)^a} \left( m \eta_1 \left( \frac{x}{m} \right) - \eta_1 (a) \right)^a I^a_x \eta_2 (x). \]

Now, using (\(a, m\))-convexity of \(|\eta_1|\) on (\(x, b\]) for \(x \in (a, b)\), we have

\[ |\eta_1 (t)| \leq \left( \frac{t - x}{b - x} \right)^a |\eta_1 (b)| + m \left( 1 - \left( \frac{t - x}{b - x} \right)^a \right) |\eta_1 \left( \frac{x}{m} \right)|. \]

On the same procedure as we did for (15) and (39), one can obtain the following inequality from (19) and (46):

\[ \left( \eta_l \right)_{\mu, \alpha, c}^\alpha \eta_2 \phi \]

\[ \leq K^2 \left( \eta_l \right)_{\mu, \alpha, c}^\alpha \eta_2 \phi \]

\[ \times \left( \left( \eta_1 (b) \eta_2 (b) - m \eta_1 \left( \frac{x}{m} \right) \eta_2 (x) \right) \right) \]

\[ \times \left( \left( \eta_1 (b) \eta_2 (b) - m \eta_1 \left( \frac{x}{m} \right) \eta_2 (x) \right) \right) \]

\[ - \frac{\Gamma (a + 1)}{(b - x)^a} \left( \eta_1 (b) |m \eta_1 \left( \frac{x}{m} \right)| \right)^a I^a_\eta \eta_2 (x). \]

By adding (45) and (47), inequality (37) can be achieved.

\[ \square \]

3. Concluding Remarks

This research paper explores fractional and conformable fractional integral inequalities in a unified form, which provide the bounds of conformable fractional integral operators and fractional integral operators containing Mittag-Leffler functions in their kernels. The results of this paper hold for fractional and conformable integral operators and convex, \(m\)-convex, and star-shaped functions (see Remarks 6 and 7 of [10] and Remark 1) simultaneously.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors have declared that no conflicts of interest exist.

Authors’ Contributions

All authors have equal contribution to the formation of this manuscript.
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