Quantum versus classical scattering of Dirac particles by a solenoidal magnetic field and the correspondence principle

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Abstract.

We present a detailed analysis of the scattering of charged particles by the magnetic field of a long solenoid of constant magnetic flux and finite radius. We study the relativistic and non-relativistic quantum and classical scenarios. The classical limit of the perturbative quantum expressions, understood as the Planck’s limit (making $\hbar$ going to zero) is analyzed and compared with the classical result. The classical cross section shows a general non-symmetric behavior with respect to the scattering angle in contradistinction to the quantum calculations performed so far. The various regimes analyzed show that the quantum cross sections do not satisfy the correspondence principle: they do not reduce to the classical result in any considered limit, an argument in favor of the interpretation of the process as a purely quantum phenomenon. We conclude that in order to restore the classical correspondence of the phenomenon, a complete non-perturbative quantum calculation for a finite solenoid radius is required.

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1. Introduction

The interactions of charged particles with magnetic fields have been widely studied. Important developments, both theoretical and experimental have appeared continuously, as those in Solid State Physics [1], and Particle Physics and Cosmology [2, 3, 4]. One of the most important effects with magnetic fields is perhaps the Aharonov-Bohm (AB) effect [5]. Since it was proposed, several experiments and studies about it have been developed [6, 7]. A lot of work has been done about this particular effect [8, 9, 10, 11], as well as in those that concern to the general properties of magnetic fields and its interaction with matter [12] and a few of them deal with the classical aspect of the AB-effect [13]. It is frequently mentioned that the AB-effect is a pure quantum effect and that it represents one of the hardest tests that quantum mechanics has successfully approved. In the zero radius limit of the solenoid the effect is purely quantum, because in \( \hbar \to 0 \) limit, the AB differential cross section (DCS) cancels exactly. However, one may wonder if there is some relation between the classical and quantum regimes in the case of finite solenoid radius. We pose to question if there is any particular limit in which the quantum result reduces to the classical one, otherwise we have an example in which the correspondence principle fails in its \( \hbar \to 0 \) version. In this paper we deal with one of this particular problems, the elastic scattering of charged spin-1/2 particles by an external solenoidal magnetic field. We carefully analyze both, the classical and quantum relativistic and non-relativistic scenarios. For the field theoretic calculation we argue that renormalization effects do not modify our conclusion.

The classical correspondence of the quantum scattering of particles by magnetic fields seems to be a puzzle still to be solved. Is this another pure quantum phenomena as was suggested before in Landau and Lifshitz book [14] in the zero-radius limit? Since the quantum field theory success in the Coulomb scattering, in which the Rutherford classical cross section is recovered, the following question arises: Can the classical limit of the scattering by a solenoid be taken for granted? AB-effect is a clear example that we should be prepared for surprises.

It is well known that the Planck’s limit, \( \hbar \to 0 \), is delicate [15, 16, 17]. The particular scattering problem analyzed in this paper is an example of this because the classical expressions are not obtained from the quantum ones. The search of a general prescription to obtain classical limits is still an open problem.

The paper is organized as follows. In section 2 we calculate the classical DCS of the scattering problem, although direct, this result is not readily found in the literature. There, we discuss two important limiting cases, which correspond to a uniform and constant magnetic field, and a thin solenoid of fixed flux (AB-case). In section 3 we present the calculations in both, relativistic [18] and non-relativistic [5, 14] quantum regimes. There we stress the consistently and completely symmetric behavior of the quantum DCS in the scattering angle. In section 4 we compare the classical and quantum results and we extend the discussion of the effect produced in the \( \hbar \) dependence of the relativistic perturbative result to all orders in \( \beta = e\Phi/2\pi c \). Finally, in section 5 we
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present our conclusions.

2. Classical Cross Section

The classical problem of the scattering of a charged particle by the magnetic field of a long solenoid of radius $R$ and constant magnetic flux is analyzed first. Besides we will show in section 3 that our results are relativistically correct and due to the delicate nature of the classical-quantum comparison, it is of basic importance to have a correct result for the classical differential cross section of charged particles by magnetic fields.

2.1. The scattering angle

In the figure, the scattering of a negatively charged particle coming from the left of the solenoid with velocity along the $x$-axis is schematically shown. This is a general situation due to the axial symmetry of the problem. The uniform and constant magnetic field $B$ differs from zero only inside of the solenoid and is defined to point outwards of the paper plane. The charged particle enters the magnetic field region with an impact parameter $b \in [-R, R]$, and the points in which it enters and leaves the solenoid are denoted by the position vectors $r_i$ and $r_f$ respectively. It is assumed that the particle does not radiate while it is interacting with the magnetic field, so its trajectory inside the solenoid will be an arc of a circumference of radius $r_L$ centered at $C$ respect to the axis of the solenoid. The particle leaves the solenoid at point $r_f$ corresponding to a scattering angle $\theta$ measured respect to the horizontal (incident direction). The non radiation assumption implies that $\theta \in [0, 2\pi)$.

With the geometry shown before,

$$r_i = (-\sqrt{R^2 - b^2}, b).$$

$r_f$ belongs to both circumferences, the one that describes the solenoid of radius $R$, and the one of Larmor radius $r_L$ that describes the trajectory of the particle inside the solenoid. Then

$$r_f = \left( \frac{\sqrt{R^2 - b^2} (r_L^2 - R^2)}{R^2 + 2br_L + r_L^2}, b - \frac{2(b - R)(b + R)r_L}{R^2 + 2br_L + r_L^2} \right).$$

The vector $P = r_f - C$, gives us information about the scattering angle $\theta$ as a function of the impact parameter $b$ accordingly to

$$\theta(b) = \arctan \left( \frac{|P_x|}{|P_y|} \right)$$

$$= 2 \arctan \left( \frac{\sqrt{R^2 - b^2}}{b + r_L} \right), \quad (1)$$

because

$$C = (-\sqrt{R^2 - b^2}, b + r_L).$$
In fact there are two solutions for $\theta(b)$ corresponding to the values of the two possible curvature concavities of the trajectories of the particles depending on the sign of $eB$. We determine the sign of $eB$ in such a way that there is only one physical solution for $\theta(b)$ which corresponds to that of equation (1).

Making use of the Newton’s Second Law combined with the Lorentz’s force,

$$\frac{dp}{dt} = -e\frac{p}{mc} \times B,$$

and using the fact that the momentum $p$ of the particle is at any given time perpendicular to the magnetic field $B$, one can show that

$$r_L = \frac{pc}{eB}, \quad (2)$$

which is relativistically correct [19]. In these expressions, $e$ and $m$ stand for the charge and mass of the scattered particle, and as usual, $c$ denotes the speed of light.

2.2. The differential cross section

To obtain the differential cross section (DCS) of the classical scattering problem, the impact parameter $b$ has to be expressed as a function of the scattering angle $\theta$, i.e.
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Table 1. Different limiting cases for the parameter $\rho_L$.

| $\rho_L \rightarrow 0$ | $R \rightarrow 0$ with $e, \Phi$ fixed |
|----------------------|----------------------------------------|
| $\rho_L \rightarrow \infty$ | $R \rightarrow \infty$ with $e, \Phi$ fixed |
|                       | $R \rightarrow 0$ with $e, B$ fixed |
| $e \rightarrow 0$ with $R, B$ fixed | $B \rightarrow 0$ with $e, R$ fixed |

Equation (1) has to be inverted. Then, the general solution for the differential cross section will be

$$
\frac{d\sigma(\theta)}{d\theta} = \sum_i \frac{db_i(\theta)}{d\theta},
$$

where $b_i(\theta)$ are the different branches of the solution.

Defining $\beta = e\Phi/2\pi c$ and introducing the dimensionless parameters

- $\rho_b = b/R$,
- $\rho_L = r_L/R = pR/2\beta$,

that express the impact parameter and the Larmor radius in units of the solenoid radius respectively ($\rho_b \in [-1, 1], \rho_L \in [0, \infty)$), once the value of $eB$ has been fixed and therefore the sign of equation (1) determined, the solutions for $\theta(\rho_b)$ are:

$$
\rho_b^\pm(\theta, \rho_L) = -\rho_L \sin^2(\theta/2) \pm \cos(\theta/2) \sqrt{1 - \rho_L^2 \sin^2(\theta/2)}. \tag{3}
$$

As indicated, the solutions $\rho_b^\pm(\theta, \rho_L)$ depend on the incident energy of the scattered particles. This is taken into account by the parameter $\rho_L$, the relevant one in the classical DCS. It is related to the typical magnetic coupling $eB$ as

$$
\rho_L = \frac{1}{R} \frac{pc}{eB}.
$$

There are several physical limiting cases of the classical DCS with respect to this parameter, corresponding to those shown in table 1.

For low energy, the Larmor radius is smaller than the solenoid one and the trajectories of the particles can turn inside the solenoid. Therefore the scattering angle $\theta$ will be found in the four quadrants: $\theta \in [0, 2\pi]$. In this case, the relation between $\theta$ and $b$ (equation (1)) is biunivocal, hence the inverse has a unique physical solution given by $\rho_b^+$. Thus the solution for $\rho_L < 1$ is

$$
\rho_b(\theta, \rho_L < 1) = \rho_b^+(\theta, \rho_L) \text{ for } \theta \in [0, 2\pi]. \tag{4}
$$

On the other hand, for high energy the particles will be scattered in the upper half plane only because their Larmor radius is larger than the solenoid one. Also notice that because $\rho_b$ is restricted to be real and to have values between $-1$ and $1$, for this $\rho_L \geq 1$
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In both cases, $\theta < \pi$ and $\rho_L \geq 1$, the expected asymmetry in the scattering predicted by the Newton’s Second Law for the Lorentz’s force will turn out.

Given $\rho_b(\theta, \rho_L)$, it is straightforward to write down the classical DCS for the scattering of a charged particle by the magnetic field of a solenoid:

$$\frac{1}{R} \frac{d\sigma}{d\theta} = \frac{\sin \theta}{2} \left( \rho_L + \frac{1 + \rho_L^2 \cos \theta}{2 \cos (\theta/2) \sqrt{1 - \rho_L^2 \sin^2 (\theta/2)}} \right) + \frac{\sin \theta}{2} \left( \rho_L - \frac{1 + \rho_L^2 \cos \theta}{2 \cos (\theta/2) \sqrt{1 - \rho_L^2 \sin^2 (\theta/2)}} \right) \Theta(|\rho_L| - 1)$$

where $\theta \in [0, 2\pi)$ if $\rho_L < 1$ and $\theta \in [0, \theta_{\text{max}}]$ if $\rho_L \geq 1$. $\Theta(x)$ is the usual Heaviside (unit step) function.

In general, this DCS presents an asymmetric behavior under the $\theta \rightarrow 2\pi - \theta$ transformation. According to figure 1, this behavior is related to the reflection with respect to the $x$–axis. Notice that when $e \rightarrow -e$ or $\rho_L \rightarrow -\rho_L$ the same asymmetry results in equation (7), that is, all these transformations are totally equivalent.

2.3. Zero radius limit (Low energy incident particles)

As expected, the Lorentz’s force produces in general a classical DCS that is not symmetric with respect to the scattering angle $\theta = 0$. Nevertheless, there is an interesting limit, $pR \rightarrow 0$ ($\rho_L \rightarrow 0$), for which the cross section behaves symmetrically with respect to the change $\theta \rightarrow 2\pi - \theta$. This case is equivalent to consider small energy incident particles, hence backscattering is greatly favored:

$$\left| \frac{d\sigma}{d\theta} \right|_{\rho_L \rightarrow 0} = \frac{R}{2} |\sin(\theta/2)| .$$

Figure 2 shows how the DCS behavior becomes symmetrical as the parameter $\rho_L$ approaches to zero, whereas for values of $\rho_L$ below unity, the DCS results totally asymmetric in the full range of $\theta \in [0, 2\pi)$. 

$\sin(\theta_{\text{max}}/2) = 1/\rho_L.$ (5)

and both solutions $\rho_b^\pm$ are physically acceptable. In this case, the scattering angle will be found in the first and second quadrants: $\theta \in [0, \theta_{\text{max}}]$. The $\theta - b$ relation is not biunivocal and both solutions $\rho_b^\pm$ must be taken into account:

$$\rho_b(\theta, \rho_L \geq 1) = \rho_b^\pm(\theta, \rho_L)$$ for $\theta \in [0, \theta_{\text{max}}]$. (6)

Given $\rho_b(\theta, \rho_L)$, it is straightforward to write down the classical DCS for the scattering of a charged particle by the magnetic field of a solenoid:
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Figure 2. Classical cross section for the scattering by a solenoidal magnetic field for several values of $\rho_L = pR/2\beta \leq 1$.

2.4. High energy incident particles

Consider the limit $pR \to \infty$ ($\rho_L \gg 1$) that, for fixed magnetic flux (fixed $\beta$), corresponds to very high energy incident particles. In this case, the DCS as given by equation (7) diverges in the direction of the maximum scattering angle $\theta_{\text{max}}$ (notice that the poles of the cross section correspond to the zeros of $|d\theta/d\rho_b|$). There is a manifestly asymmetric behavior of the cross section with respect to the scattering angle when it changes from $\theta$ to $2\pi - \theta$, and as long as the condition $\theta \leq \theta_{\text{max}}$ is fulfilled, it is possible to show from equation (7) that the cross section reduces to

$$
\left. \frac{d\sigma}{R \, d\theta} \right|_{\rho_L \gg 1} \approx R \theta \frac{1 + \rho_L^2}{\sqrt{4 - \rho_L^2 \theta^2}} \text{ for } \theta \in [0, \theta_{\text{max}}].
$$

Notice that taking the limit $e \to 0$ in equation (7) is equivalent to consider $\rho_L \to \infty$. And even more, for fixed solenoid radius $R$ this result is also obtained when $eB \to 0$.

Figure 3 shows the behavior of the classical DCS for some values of the parameter $\rho_L$ when it is greater than unity. As can be observed, the DCS behaves each time more singular and the maximum scattering angle tends to zero as the parameter $\rho_L$ increases.

2.5. The asymmetry function

As stated before the probability to find scattered particles in the upper positive half-plane is in general different from the one to find scattered particles in the lower negative
Figure 3. Classical cross section for the scattering by a solenoidal magnetic field for several values of $\rho_L = pR/2\beta \geq 1$.

half-plane. To show this explicitly let us introduce the asymmetry of the cross section as a function of the parameter $\rho_L$,

$$A(\rho_L) = \frac{\sigma_+(\rho_L) - \sigma_-(\rho_L)}{\sigma(\rho_L)},$$

with $\sigma(\rho_L) = \sigma_+(\rho_L) + \sigma_-(\rho_L)$. $\sigma_\pm$ correspond to the total cross section for $0 \leq \theta < \pi$ and $\pi \leq \theta < 2\pi$, respectively. For all values of $\rho_L$, $\sigma(\rho_L) = 2R$. As it is evident from equation (7), for $\rho_L \geq 1$ the cross section is completely asymmetric. In this case

$$\sigma_-(\rho_L \geq 1) = 0,$$

$$\sigma_+(\rho_L \geq 1) = \int_{\theta_{\text{max}}}^{0} \frac{d\sigma_{\rho_L}(\theta)}{d\theta} d\theta = 2R,$$

and

$$A(\rho_L \geq 1) = 1;$$

whereas for $\rho_L < 1$,

$$\sigma_+(\rho_L < 1) = \int_{0}^{\pi} \frac{d\sigma_{\rho_L}(\theta)}{d\theta} d\theta = R (\rho_L + 1),$$

$$\sigma_-(\rho_L < 1) = \int_{\pi}^{2\pi} \frac{d\sigma_{\rho_L}(\theta)}{d\theta} d\theta = R (1 - \rho_L).$$

In this form,

$$A(\rho_L < 1) = \rho_L.$$

(9)
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It follows from equation (9) that only in the limit $\rho_L \to 0$ the cross section is completely symmetric, $A(\rho_L \to 0) \to 0$. In such a case, for fixed solenoid radius, the Larmor radius of the scattered particles is smaller than the solenoid one and backscattering is greatly favored.

Figure 4 depicts the general behavior of $A(\rho_L)$.

![Graph](image)

**Figure 4.** Asymmetry function of the classical cross section of the scattering by a solenoidal magnetic field.

### 3. Quantum mechanical results

Both the relativistic and non-relativistic quantum mechanical scenarios for the scattering of charged particles by a solenoidal magnetic field are presented below. The non-relativistic case can be exactly solved in the zero radius limit. We briefly recall the most important results obtained years ago by Aharonov and Bohm [5] and Landau and Lifshitz [14] in the non-relativistic regime. The more general case of finite non-zero radius of the solenoid is discussed in the relativistic regime section using first order perturbation theory.

#### 3.1. Non relativistic regime

A landmark result for the non relativistic scattering of electrons by solenoidal magnetic fields was presented by Aharonov and Bohm [5]. They obtained the exact solution for the scattering problem in the zero radius limit of the solenoid for a constant magnetic flux $\Phi$,

$$
\left. \frac{d\sigma}{d\theta} \right|_{AB} = \hbar \frac{\sin^2 (e\Phi/2hc)}{2\pi p \sin^2 (\theta/2)}.
$$

(10)
This result is manifestly symmetric in the scattering angle under the $\theta \rightarrow 2\pi - \theta$ transformation.

Landau and Lifshitz [14] studied the same scattering problem in the eikonal approximation. Including only the contribution of the vector potential from the exterior of the solenoid, they obtained precisely the same result as Aharonov and Bohm. These authors also studied the case of small scattering angles for a small magnetic flux, $e\Phi/2hc \ll 1$, where perturbation theory is applicable, the resulting cross section is again symmetric in the scattering angle with respect to the incident direction of the particles ($\theta = 0$):

$$\left. \frac{d\sigma}{d\theta} \right|_{LL, \theta \ll 1} = \hbar \frac{(e\Phi/\hbar c)^2}{2\pi p} \frac{1}{\theta^2}. \tag{11}$$

They mention that the singular behavior of the total cross section for $\theta$ going to zero is specifically a quantum effect, without any further comment.

In any case the limit $R = 0$, clearly separates the classical regime (in which $d\sigma/d\theta$ cancels) with respect to the quantum regime (equations (10) and (11)). Consequently any further comparison of the classical and quantum solutions requires to consider the finite radius situation.

### 3.2. Relativistic regime

The relativistic quantum mechanical problem of the scattering of electrons by the magnetic field of a long solenoid of radius $R$ with axial axis in the $x_3$ direction is considered here. The magnetic flux $\Phi = \pi R^2 B_0$ will be kept constant.

Once the gauge has been fixed, the magnetic vector potential of the solenoid is

$$\mathbf{A} = A_\mu \gamma^\mu = \frac{\Phi}{2\pi} \epsilon_{ijk} x_i \gamma^j \begin{cases} \frac{1}{R^2} & \text{for } r < R \\ \frac{1}{x_1^2 + x_2^2} & \text{for } r > R, \end{cases} \tag{12}$$

with scalar potential $A^0 = 0$. $\epsilon_{ijk}$ is the Levi-Civita symbol in three indexes.

The first order matrix element in $e$, $S^{(1)}_{fi}$, that has to be computed is

$$S^{(1)}_{fi} = \delta_{fi} - ie \int \bar{\psi}_f(y) \mathbf{A}(y) \psi_i(y) d^4y,$$

with $\psi_i(y)$ and $\psi_f(y)$ free particle incident and final asymptotic states. A detailed calculation of the first order Born approximation for the DCS yields [18]

$$\frac{d\sigma}{d\theta} = \hbar \left( \frac{e\Phi}{Rc} \right)^2 \frac{|J_1(2\frac{p}{R}R|\sin (\theta/2)|)|^2}{8\pi p^3 \sin^4 (\theta/2)}, \tag{13}$$

where $J_1$ are the first order Bessel functions of first kind. The previous result has the same form whether or not the final polarization of the beam is actually measured. As can be observed, the lowest perturbative order in $\alpha = e^2/\hbar c$ of the relativistic quantum mechanical cross section is also symmetric in $\theta$. 

### References

[14] Landau, L. D., & Lifshitz, E. M. (1959). *Quantum Mechanics*. Pergamon Press.
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An analysis of the classical Planck’s limit (taking $\hbar \to 0$) of equations (10) and (13), with fixed $e, p, R, \Phi$ and $\theta$, yields a consistent zero limit, contrary to the asymmetric finite value of the classical DCS in equation (7). From equation (13) we have

$$\lim_{\hbar \to 0} \frac{d\sigma}{d\theta} = \lim_{\hbar \to 0} \hbar^2 \left( \frac{e\Phi}{2\pi c} \right)^2 \frac{\cos^2 \left(\frac{2p}{\hbar} R|\sin (\theta/2)| - \frac{3\pi}{4}\right)}{2R^3p^4 |\sin^5 (\theta/2)|} = 0. \quad (14)$$

Notice that the same limit is recovered by taking $pR \to \infty$ with $\hbar$ and $R$ fixed [21].

Equation (14) establishes that, to first order in $\alpha$, there is not a classical correspondence of the phenomena of scattering of charged fermions by the magnetic field of a solenoid with finite non-zero radius. It is instructive to compare this with the Coulomb scattering, in which both the relativistic and non-relativistic quantum results reproduce the well known Rutherford classical cross section in the classical limit $\hbar \to 0$. Notice that the classical-quantum correspondence for the Coulomb field takes place in a perturbative regime to first order in $\alpha$.

In this quantum relativistic regime, the asymmetry function is also equal to zero, because $d\sigma(2\pi - \theta) = d\sigma(\theta)$ and therefore $\sigma_- = \sigma_+$. Figure 5 shows the symmetric behavior of the relativistic quantum cross section of equation (13). Notice that the scattering is directed mainly in the forward direction.

![Figure 5. Typical behavior of the relativistic quantum cross section of the scattering by a solenoidal magnetic field.](image)

The essentially different behavior between the classical and quantum DCS becomes evident from the symmetric behavior of the quantum result, equation (13), as compared
to the asymmetric structure of the classical one, equation (7). Furthermore, notice that
the total cross section in both quantum regimes is infinite, in contrast to the finite value
of $2R$ obtained for the classical case.

In order to throw further understanding of these facts, we notice that the classical
result depends on two length-dimensional parameters: the solenoid radius $R$ and the
Larmor radius $r_L$. In the quantum regime we can add two more length-dimensional
parameters, the de Broglie length $\lambda = \hbar/p$ and the magnetic length $\ell_B = \sqrt{\hbar c/eB} = R\sqrt{\hbar c\pi/e\Phi}$. Not all these parameters are independent, in fact
$r_L = \ell_B^2/\lambda$. The quantum
dCS in equation (13) can be expressed in terms of three of these parameters. e.g. $R, r_L$
and $\lambda$. Unlike the Rutherford formula, the cross section in equation (13) can not be
written purely in terms of classical length parameters.

As it has been noticed throughout this paper, $\rho_L = r_L/R$ results to be the significant
parameter in the classical regime. In the quantum case, we can identify two action-
dimension parameters: $pR$ and $e\Phi/c$. A comparison of these parameters respect to $\hbar$
helps to have a better understanding of the structure of the quantum DCS. In terms of
the dimensionless parameters, $s_p = pR/\hbar$ and $s_\Phi = e\Phi/\hbar c$, the DCS of Aharonov
and Bohm (AB-DCS), equation (10), looks like

$$\frac{d\sigma}{d\theta} \bigg|_{AB} = R \frac{\sin^2 (s_\Phi/2)}{2\pi s_p \sin^2 (\theta/2)},$$

while the perturbative relativistic DCS of equation (13) can be expressed as

$$\frac{d\sigma}{d\theta} = \frac{R \ s_\Phi^2}{8\pi \ s_p^3} \left| J_1(2s_p|\sin (\theta/2)|) \right|^2 \sin^2 (\theta/2).$$

(15)

The former equation results from a perturbative expansion in $s_\Phi$; it is therefore obtained
for arbitrary values of $s_p$, but small $s_\Phi$. On the other hand, the range of validity of the
AB-DCS is arbitrary $s_\Phi$ but small $s_p$ (in fact, it was obtained in the special case of
$s_p = 0$).

In terms of the same parameters $s_p$ and $s_\Phi$, $\ell_B = R\sqrt{\pi/s_\Phi}$, $\lambda = R/s_p$ and
$\rho_L = \pi s_p/s_\Phi$, and the classical DCS can be expressed as a function of the ratio $s_p/s_\Phi$
and $\theta$ only.

Thus the results in (10) and (13) do not reduce to the classical result in the
$\hbar \to 0$ limit. One may wonder if there is still a situation in which it is possible
to establish a correspondence between the quantum and classical results. It becomes
relevant to observe, that according to Berry and Mount [16] and Gutzwiller [17], the
implementation of the classical limit requires to look at the situation in which the action
quantities that appear in the corresponding classical problem are considered very large
as compared to $\hbar$ [22]. In the problem at hand we can select $s_p$ and $s_\Phi$ as the relevant
parameters. Consequently, the classical limit of the relativistic DCS vanishes, because
for $s_p$ arbitrary large, equation (15) behaves like $s_\Phi^2/s_p^4$. A similar situation happens for
the non-relativistic case.
4. Quantum vs Classical results

Here we extend the discussion of the $\hbar$ dependence of the relativistic DCS to all orders in $\beta = e\Phi/2\pi c$ and $\alpha = e^2/\hbar c$, a consideration that becomes crucial for the inspection of the classical limit of the scattering process.

The consistency between the non-relativistic and the relativistic quantum results has been well established before [18, 23, 24, 25] in the $R \to 0$ limit. However, we want to stress that there is still an unsolved problem with the classical Planck’s limit that shows itself clearly in the lack of asymmetry of the quantum calculations.

Higher order processes can be calculated using the Feynman rules for the electron-solenoid scattering that are presented in the appendix. Here, we are interested in counting the $\hbar$ power contributions to higher order diagrams as the ones depicted in figure 6. Assuming free particle asymptotic states, we notice that any extra insertion of the external magnetic field (i.e. in $\beta = e\Phi/2\pi c$) to the scattering matrix contributes with a magnetic interaction and a free-fermion-propagator (see figure 6). Each magnetic interaction contributes with a $n = +1$ power of $\hbar$, while for each free fermion propagator there is a $n = -1$ power in addition to the global $\hbar$ factor of the scattering matrix in momentum space (see the appendix). This means that higher orders in $\beta$ do not modify the leading $\hbar$ power contribution to the scattering matrix.

![Figure 6. Feynman diagram and $\hbar$ power counting for an arbitrary order in $\beta = e\Phi/2\pi c$ of the scattering matrix for a solenoidal magnetic field. The wiggled lines represent the interaction with the external magnetic field while the straight lines represent the free-fermion propagators.](image-url)

We recall that as is well known, usual radiative corrections (higher powers in $\alpha = e^2/\hbar c$) will in general contribute with positive $\hbar$ powers to the matrix elements, hence they are not expected to be relevant in the classical limit.

Consequently, for all orders in both $\alpha$ and $\beta$ in the perturbative expansion, the
classical Planck’s limit of this process is proportional to $\hbar$ and the classical result can not be recovered with a perturbative calculation in $\beta$ and $\alpha$. As discussed in the last section, the same zero classical limit is found analyzing the DCS behavior in terms of the $s_\Phi$ and $s_p$ variables.

In the quantum regime the perturbative cross section has a symmetric behavior with respect to the scattering angle when it changes form $\theta$ to $2\pi - \theta$, whereas in the classical scenario, the symmetry in the scattering angle only occurs in the limit $pR \to 0$ ($\rho_L \to 0$). Higher orders in $\beta$ are expected to give rise to up-down asymmetries with respect to the scattering angle in the quantum DCS for the finite solenoid radius case, but notice that as we stated before, the renormalized series in $\beta$ of the scattering matrix leads to a null classical limit. Therefore these asymmetries in the perturbative quantum calculations clearly are not the classical ones.

In this sense, the renormalized perturbative calculation of scattering of Dirac particles by a solenoidal magnetic field has to be understood as a pure quantum one. Only an exact non-perturbative calculation at finite radius $R$ could possibly produce a consistent result from which the classical limit could be possible recovered.

5. Conclusions

In this paper we present an analysis for the scattering of charged particles by a solenoidal magnetic field of finite radius $R$ and constant magnetic flux $\Phi$ in both, classical and quantum relativistic and non-relativistic regimes. We focus on the classical Planck’s limit, taking $\hbar \to 0$, of the quantum results in the framework of a perturbation theory in $\alpha = e^2/\hbar c$ and $\beta = e\Phi/2\pi c$.

In order to have a clear reference to compare with, and as we did not find it reported in the literature, we explicitly calculated the differential cross section (DCS) for the classical scenario. There, we found a general asymmetric behavior of the DCS with respect to the scattering angle $\theta$. This result differs with respect to the symmetric DCS obtained in the quantum regime. Also notice that in the quantum regime the total cross section of the scattering problem becomes infinite (for both cases, $R \neq 0$ and $R = 0$), while the classical total cross section is finite and equal to $2R$.

As we have showed, the quantum results for the DCS are proportional to $\hbar$. In this paper we studied the contribution to the power counting of $\hbar$ of the perturbative expansion to all orders in both, $\alpha$ and $\beta$, showing that in the classical limit, in the Planck’s sense, the leading term will be proportional to $\hbar$, so it vanishes. This means that the perturbative evaluation of the scattering matrix can not converge to the classical solution in the $\hbar \to 0$ limit, this one understood as the limit in which all the classical action variables are very large as compared to the Planck’s constant $\hbar$. Our conclusion is that only an exact non-perturbative calculation for a finite solenoid radius, offers the possibility to obtain a consistent result for the DCS, from which the classical limit could be possible recovered.
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Appendix. Feynman rules for the solenoidal case

The general Feynman rules for the scattering amplitude of electrons by a solenoidal magnetic field of finite non-zero radius $R$ are presented here.

Notice first that in standard perturbation theory, using free particle asymptotic states, the lowest order contribution in $e$ to the scattering matrix is given by the Fourier transformation of the four-vector potential $A^\mu(x)$:

$$S^{(1)}_{fi} = \delta_{fi} - i \int \bar{\psi}_f(x) \frac{eA^\mu(x)}{\hbar c} \psi_i(x) d^4x,$$

with

$$\psi(x) = \sqrt{mc^2} u(p, s) e^{-ip \cdot x/\hbar}.$$

For the solenoidal potential given by equation (12), the first order matrix element results in

$$S^{(1)}_{fi} = \sqrt{\frac{mc^2}{Ev} u(p, s) e^{-ip \cdot x/\hbar}} \bar{u}_f \left[ -2i \frac{\hbar^2}{R} J_1(q \cdot R/\hbar) \epsilon_{ij3} \frac{q_i}{q_\perp} \gamma_j \right] u_i (2\pi)^2 \delta^2(q || / \hbar),$$

where $q = p_f - p_i$ is the transferred momentum. We have defined $a^\mu = a^\mu_\perp + a^\mu_\parallel$, with $a^\mu_\perp = (0, a_1, a_2, 0)$ and $a^\mu_\parallel = (a_0, 0, 0, a_3)$, where the subscripts $\perp$ and $\parallel$ refer to the components of the four-vector that are perpendicular or parallel to the direction of the solenoidal magnetic field.

$S^{(1)}_{fi}$ has the structure of the product: final state spinor $\times$ magnetic-interaction vertex $\times$ magnetic propagator $\times$ initial state spinor $\times$ energy-momentum conservation. Precisely this structure determines the Feynman rules of the invariant amplitude $\mathcal{M}$ of the problem:

- Each external incoming (anti)fermionic line contributes with a factor $u(p, s)$ ($v(p, s)$).
- Each external out-coming (anti)fermionic line contributes with a factor $\bar{u}(p, s)$ ($\bar{v}(p, s)$).
- For each magnetic-fermionic vertex a dimensionless factor appears:

$$\frac{e \Phi}{\hbar c}.$$

- Each magnetic propagator contributes to $\mathcal{M}$ with

$$-2i \frac{\hbar^2}{R} J_1(q \cdot R/\hbar) \epsilon_{ij3} \frac{q_i}{q_\perp} \gamma_j,$$
which has the dimensions of length. This propagator will contribute with a factor of $\hbar^2$ to the amplitude. Notice that this is a general result for any electromagnetic interaction in a bi-dimensional problem.

- For each internal fermionic line, the free fermion propagator will appear:

$$-iS_F(q) = -i \frac{\hbar}{\slashed{q} - mc + i\epsilon}.$$  

Additionally there is an integral in $d^4q/\hbar^4$ and due to the cylindrical symmetry two of this variables are integrated out immediately using the momentum-energy conservation (see the next rule). Then, $S_F(q)$ will finally contribute with a $1/\hbar$ factor to the amplitude. This situation is equivalent to consider an internal loop in the $q_{\perp}$-space (see figure 1).

- For each magnetic-fermionic vertex, both the energy and the momentum along the direction of the magnetic field, are conserved. So, the additional term

$$(2\pi)^2\delta^2(q_{\parallel}/\hbar),$$

arises in the amplitude.

Additional terms have to be considered along with the phase space factors to construct the differential cross section. The normalization factors of the external fermionic lines have to be included in these extra-terms.

As an example, let us apply the Feynman rules just listed before to write down the second order matrix element of the scattering process, $\mathcal{M}^{(2)}$; figure 1 depicts the Feynman diagram that has to be evaluated. There, the $\hbar$ power contribution to $\mathcal{M}^{(2)}$ is shown explicitly for each part of the diagram.

**Figure 1.** Second order Feynman diagram of the scattering matrix for a solenoidal magnetic field.
According to the Feynman rules already established:

\[
\mathcal{M}^{(2)} = \left\{ \begin{array}{l}
\frac{1}{(2\pi)^4} \int \frac{d^4q}{\hbar^4} \left\{ \bar{u}(p_f, s_f) \\
\times \left( \frac{e\Phi}{\hbar c} \right)^2 \\
\times \left[ -2i\frac{\hbar^2}{R} J_1(|p_f - q| R / \hbar) \epsilon_{lm3} \frac{(p_f - q)_l}{|p_f - q|^3} \gamma^m \right] \\
\times \left[ -i \frac{\hbar}{\theta - mc + i\epsilon} \right] \\
\times \left[ -2i\frac{\hbar^2}{R} J_1(|q - p_i| R / \hbar) \epsilon_{ij3} \frac{(q - p_i)_j}{|q - p_i|^3} \gamma^i \right] \\
\times u(p_i, s_i) \\
\right. \\
\left. \times \left[ (2\pi)^2 \delta^2 ((p_f - q)_/|/\hbar) \right] \left[ (2\pi)^2 \delta^2 ((q - p_i)_/|/\hbar) \right] \right\}
\]

Out-coming external fermion

\[
\text{Vertexes } m \text{ and } j
\]

Magnetic propagator at \( m \)

Free fermion propagator

Magnetic propagator at \( j \)

Incoming external fermion

Energy-momentum conservation,

which with the aid of a pair of the Dirac-delta functions reduces to

\[
\mathcal{M}^{(2)} = (-i)^3 \left( \frac{2e\Phi}{\hbar c R} \right)^2 (2\pi)^2 \delta^2 ((p_f - p_i)_/|/\hbar) \epsilon_{lm3} \epsilon_{ij3}
\]

\[
\times \frac{1}{(2\pi)^2} \int \frac{d^2q_\perp}{\hbar^2} \bar{u}_f \gamma^m \frac{\hbar}{p_\parallel + q_\perp - mc + i\epsilon} \gamma^j u_i
\]

\[
\times \hbar^2 \frac{(p_f - q)_l}{|p_f - q|^3} J_1(|p_f - q| R / \hbar) \hbar^2 \frac{(q - p_i)_j}{|q - p_i|^3} J_1(|q - p_i| R / \hbar).
\]

In terms of the dimensionless action-related variables \( s_p = p R / \hbar \) and \( s_\Phi = e\Phi / \hbar c \), this last expression reads

\[
\mathcal{M}^{(2)} = (-i)^3 R (2s_\Phi)^2 (2\pi)^2 \delta^2 ((p_f - p_i)_/|/\hbar) \epsilon_{lm3} \epsilon_{ij3}
\]

\[
\times \frac{1}{(2\pi)^2} \int d^2s_\perp \bar{u}_f \gamma^m \frac{1}{\hat{s}_\parallel + s_\perp - mc + i\epsilon} \gamma^j u_i
\]

\[
\times \frac{(s_{p_f} - s_q)_l (s_q - s_{p_i})_j}{|s_{p_f} - s_q|^3 |s_q - s_{p_i}|^3} J_1(|s_{p_f} - s_q| \perp) J_1(|s_q - s_{p_i}| \perp).
\]

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