Random runners are very lonely

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Abstract. Suppose that \(k\) runners having different constant speeds run laps on a circular track of unit length. The Lonely Runner Conjecture states that, sooner or later, any given runner will be at distance at least \(1/k\) from all the other runners. We prove that, with probability tending to one, a much stronger statement holds for random sets in which the bound \(1/k\) is replaced by \(1/2 - \varepsilon\). The proof uses Fourier analytic methods. We also point out some consequences of our result for colouring of random integer distance graphs.

1. Introduction

Suppose that \(k\) runners run laps on a unit-length circular track. They all start together from the same point and run in the same direction with pairwise different constant speeds \(d_1, d_2, \ldots, d_k\). At a given time \(t\), a runner is said to be lonely if no other runner is within a distance of \(1/k\), both in front and rear. The Lonely Runner Conjecture states that for every runner there is a time at which he is lonely. For instance if \(k = 2\), one can imagine easily that at some time or other, the two runners will find themselves on antipodal points of the circle, both becoming lonely at that moment.

To give a precise statement, let \(T = [0, 1)\) denote the circle (the one-dimensional torus). For a real number \(x\), let \(\{x\}\) be the fractional part of \(x\) (the position of \(x\) on the circle), and let \(\|x\|\) denote the distance of \(x\) to the nearest integer (the circular distance from \(\{x\}\) to zero). Notice that \(\|x - y\|\) is just the length of the shortest circular arc determined by the points \(\{x\}\) and \(\{y\}\) on the circle. It is not difficult to see that the following statement is equivalent to the Lonely Runner Conjecture.

Conjecture 1. For every integer \(k \geq 1\) and for every set of positive integers \(\{d_1, d_2, \ldots, d_k\}\) there exists a real number \(t\) such that

\[
\|td_i\| \geq \frac{1}{k + 1}
\]

for all \(i = 1, 2, \ldots, k\).

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The above bound is sharp as is seen for the sets \{1, 2, \ldots, k\}. The paper of Goddyn and Wong \[11\] contains items of interesting exemplars of such extremal sets. The problem was posed for the first time by Wills \[16\] in connection to Diophantine approximation. Cusick \[8\] raised the same question independently, as a view obstruction problem in Discrete Geometry (cf. \[5\]). Together with Pomerance \[9\], he confirmed the validity of the conjecture for \(k \leq 4\). Bienia et al. \[3\] gave a simpler proof for \(k = 4\) and found interesting application to flows in graphs and matroids. Next the conjecture was proved for \(k = 5\) by Bohman et al. \[4\]. A simpler proof for that case was provided by Renault \[13\]. Recently the case \(k = 6\) was established by Barajas and Serra \[2\], using a new promising idea.

Let \(D = \{d_1, d_2, \ldots, d_k\}\) be a set of \(k\) positive integers. Consider the quantity

\[\kappa(D) = \sup_{x \in \mathbb{T}} \min_{d_i \in D} \|xd_i\|\]

and the related function \(\kappa(k) = \inf \kappa(D)\), where the infimum is taken over all \(k\)-element sets of positive integers. So, the Lonely Runner Conjecture states that \(\kappa(k) \geq \frac{1}{k+1}\). The trivial bound is \(\kappa(k) \geq \frac{1}{2k}\), as the sets \(\{x \in \mathbb{T} : \|xd_i\| < \frac{1}{2k}\}\) simply cannot cover the whole circle (since each of them is a union of \(d_i\) open arcs of length \(\frac{1}{kd_i}\) each). Surprisingly, nothing much better was proved so far. Currently the best general bound is

\[\kappa(k) \geq \frac{1}{2k - 1 + \frac{1}{2k-3}}\]

for every \(k \geq 5\) \[6\]. A slightly improved inequality \(\kappa(k) \geq \frac{1}{2k-3}\) holds when \(k \geq 4\) and \(2k - 3\) is prime \[7\]. Using the probabilistic argument we proved in \[10\] that every set \(D\) contains an element \(d\) such that

\[\kappa(D \setminus \{d\}) \geq \frac{1}{k}\]

In this paper we prove another general result supporting the Lonely Runner Conjecture.

**Theorem 1.** Let \(k\) be a fixed positive integer and let \(\varepsilon > 0\) be fixed real number. Let \(D \subseteq \{1, 2, \ldots, n\}\) be a \(k\)-element subset chosen uniformly at random. Then the probability that \(\kappa(D) \geq \frac{1}{2} - \varepsilon\) tends to 1 with \(n \to \infty\).

The proof uses elementary Fourier analytic technique for subsets of \(\mathbb{Z}_p\). We give it in the next section. In the last section we point to a striking consequence of our result for colouring of integer distance graphs.

2. Proof of the main result

Let \(k\) be a fixed positive integer and let \(p \geq k\) be a prime number. For \(a \in \mathbb{Z}_p\), let \(\|a\|_p = \min\{a, p - a\}\) be the circular distance from \(a\) to zero in \(\mathbb{Z}_p\). We will need the following notion introduced by \[14\]. Let \(L\) be a fixed
positive integer. A set $D = \{d_1, \ldots, d_k\} \subseteq \mathbb{Z}_p$ is called $L$-independent in $\mathbb{Z}_p$ if equation
\[ d_1 x_1 + d_2 x_2 + \ldots + d_k x_k = 0 \]
has no solutions satisfying
\[ 0 < \sum_{i=1}^{k} \|x_i\|_p \leq L. \]

We will show that for appropriately chosen $L$, any $L$-independent set can be pushed away arbitrarily far from zero. Then we will demonstrate that for such $L$, almost every set in $\mathbb{Z}_p$ is $L$-independent.

Let $f : \mathbb{Z}_p \to \mathbb{C}$ be any function and let $\hat{f} : \mathbb{Z}_p \to \mathbb{C}$ denote its Fourier transform, that is
\[ \hat{f}(r) = \sum_{x \in \mathbb{Z}_p} f(x) \omega^{rx}, \]
where $\omega = e^{\frac{2\pi i}{p}}$. For a set $A \subseteq \mathbb{Z}_p$, by $A(x)$ we denote its characteristic function. We will make use of the following basic properties of the Fourier transform:

(F1): $|\hat{f}(r)| = |\hat{f}(-r)|$ for every $r \in \mathbb{Z}_p$.
(F2): $f(x) = \frac{1}{p} \sum_{r \in \mathbb{Z}_p} \hat{f}(r) \omega^{-rx}$ for every $x \in \mathbb{Z}_p$.
(F3): $\hat{A}(0) = |A|$ for every subset of $\mathbb{Z}_p$.

In the lemma below we give a bound for the Fourier coefficient $\hat{A}(r)$ for the sets of the form
\[ (*) \quad A = \{s, s+1, \ldots, l\}, \]
where $l$ and $s$ are elements of $\mathbb{Z}_p$, such that $s < l$. This bound does not depend on $l$ and $s$. The following lemma can be easily proved, as for instance in [12](p. 39). We proved this for a reader convenience.

**Lemma 1.** If $0 < r < \frac{p}{2}$, then
\[ |\hat{A}(r)| \leq \frac{p}{2r}. \]

**Proof.** By simple calculations we have
\[ |\hat{A}(r)| = \left| \sum_{x=s}^{l} \omega^{rx} \right| = \left| \frac{\omega^{r(l+1)} - \omega^{rs}}{\omega^r - 1} \right| = \left| \frac{\omega^{\frac{r(l+1)}{2}} \cdot \omega^{\frac{r(l+1-s)}{2}} - \omega^{\frac{r(l+1-s)}{2}}}{\omega^{\frac{r}{2}} - \omega^{\frac{-r}{2}}} \right| = \left| \frac{\sin(\frac{2x}{p})}{\sin(\frac{2r}{p})} \right|. \]

Using inequality $\sin(x) \geq \frac{2x}{\pi}$ for $x < \frac{\pi}{2}$, we get
Now, we state and prove the aforementioned property of $L$-independent sets.

**Theorem 2.** Let $0 < \varepsilon < \frac{1}{2}$ be a fixed real number. Let $D$ be a $k$-element, $L$-independent set in $\mathbb{Z}_p$, where

$$L > \sqrt{\frac{k^33k-1}{2k+1\varepsilon^2k}}.$$  

Then

$$\kappa(D) \geq 1/2 - \varepsilon.$$

**Proof.** Let

$$C = \{x \in \mathbb{Z}_p : \left(\frac{1}{4} - \frac{\varepsilon}{2}\right)p < x < \left(\frac{1}{4} + \frac{\varepsilon}{2}\right)p\}$$

and let $C(x)$ be the characteristic function of the set $C$. Define convolution of two functions $f$ and $g$ by

$$(f * h)(x) = \sum_{y \in \mathbb{Z}_p} f(y) \cdot g(x - y).$$

Denote by $B(x) = (C * C)(x)$ convolution of function $C$ with itself. It is easy to see that $\hat{B}(r) = \hat{C}(r) \cdot \hat{C}(r)$ for all $r \in \mathbb{Z}_p$.

So, if we find $t \in \mathbb{Z}_p$ such that $tD \subseteq \text{supp} B$, where $\text{supp} B = \{x \in \mathbb{Z}_p : B(x) \neq 0\}$, then at the same time we push the set $D$ away into the small arc $(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)$ on the torus $\mathbb{T}$.

Then the expression

$$I = \sum_{t \in \mathbb{Z}_p} B(td_1)B(td_2) \cdots B(td_k)$$

counts those numbers $t$ which push the set $D$ away to a distance $\frac{1}{2} - \varepsilon$ from zero. We will show that $I \neq 0$. From properties of the Fourier transform results that

$$I = \sum_{t \in \mathbb{Z}_p} \left(\frac{1}{p} \sum_{r_1 \in \mathbb{Z}_p} \hat{B}(r_1)\omega^{-td_1r_1}\right) \cdots \left(\frac{1}{p} \sum_{r_k \in \mathbb{Z}_p} \hat{B}(r_k)\omega^{-td_kr_k}\right).$$

Denoting $\vec{r} = (r_1, r_2, \cdots, r_k)$, we get

$$p^k I = \sum_{\vec{r} \in \mathbb{Z}_p^k} B(r_1) \cdots B(r_k) \sum_{t \in \mathbb{Z}_p} \omega^{-t(d_1r_1 + \cdots + d_kr_k)}.$$  

The expression $\sum_t \omega^{-t(d_1r_1 + \cdots + d_kr_k)}$ is equal to $p$ when

$$d_1r_1 + \cdots + d_kr_k \equiv 0 \pmod{p},$$

which implies $I \neq 0$. 

and is equal to zero in the contrary case. As a consequence we may write
\[ p^{k-1} 1 = \sum_{\vec{r} \in \mathbb{Z}_p^k} \hat{B}(r_1) \cdots \hat{B}(r_k) R(\vec{r}), \]
where \( R(\vec{r}) = 1 \) for \( r_1, \ldots, r_k \) satisfying the equation (**), and \( R(\vec{r}) = 0 \) in the opposite situation. Since \( D \) is \( L \)-independent, the identity \( R(\vec{r}) = 1 \) holds only for those \( r_1, \ldots, r_k \) satisfying condition \( \sum_{i=1}^{k} \|r_i\|_p > L \), or \( r_1 = r_2 = \ldots = r_k = 0 \). Hence,
\[ p^{k-1} 1 - |C|^{2k} = \sum_{\vec{r} \in \mathbb{Z}_p^k, \|r\|_p \leq L} \hat{B}(r_1) \cdots \hat{B}(r_k), \]
as for \( r_i = 0 \) the Fourier coefficient \( \hat{B}(r_i) \) is equal to square of the size of \( C \). So, by showing that
\[ |C|^{2k} > \sum_{\|r\|_p \leq L} \left| \hat{B}(r_1) \cdots \hat{B}(r_k) \right| R(\vec{r}) \]
we will confirm that \( I \neq 0 \).

The property of \( L \)-independence of the set \( D \) implies that in any non-trivial solution of (**) there is some \( r_i \) satisfying \( \|r_i\|_p > \frac{L}{k} \). The estimates for those \( r_i \)
\[ \left| \hat{B}(r_i) \right| = \left| \hat{C}(r_i) \right|^2 \leq \left( \frac{p}{2r_i} \right)^2 \leq \left( \frac{kp}{2L} \right)^2 \]
results from Lemma [1]

Denote by \( \vec{r}_j = (r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_k) \), the vector \( \vec{r} \) with \( j \) th coordinate missing. Substituting this to the previous sum we obtain
\[ \sum_{\|r\|_p \leq L} \left| \hat{B}(r_1) \cdots \hat{B}(r_k) \right| R(r_1, \ldots, r_k) \]
\[ \leq \left( \frac{kp}{2L} \right)^2 \sum_{j=1}^{k} \sum_{\vec{r}_j \in \mathbb{Z}_p^{k-1}} \left| \hat{B}(r_1) \cdots \hat{B}(r_{j-1}) \right| \left| \hat{B}(r_{j+1}) \cdots \hat{B}(r_k) \right| \]
\[ \leq k \left( \frac{kp}{2L} \right)^2 \sum_{\vec{r}_k \in \mathbb{Z}_p^{k-1}} \left| \hat{B}(r_1) \cdots \hat{B}(r_{k-1}) \right|. \]

The last sum may be estimated further. Let \( S_p = \{0, 1, \ldots, \frac{p-1}{2}\} \) and we get
\[ \sum_{\vec{r}_k \in \mathbb{Z}_p^{k-1}} \left| \hat{B}(r_1) \cdots \hat{B}(r_{k-1}) \right| \]
\[ \leq 2^{k-1} \sum_{\vec{r}_k \in S_p^{k-1}} \left| \hat{B}(r_1) \cdots \hat{B}(r_{k-1}) \right|. \]
Thus, applying Lemma 1 again we get
\[
\sum_{\|r_i\|_p > L} \left| \hat{B}(r_1) \cdots \hat{B}(r_k) \right| R(\vec{r}) \\
\leq k \left( \frac{kp}{2L} \right)^2 \cdot 2^{k-1} \cdot \left( \frac{p}{2^k-1} \right)^2 \cdot \left( 1 + \sum_{r \in S_p} \frac{1}{r^2} \right)^{k-1}
\]
since \(1 + \frac{\pi^2}{2} \leq 3\), we obtain
\[
\sum_{\|r_i\|_p > L} \left| \hat{B}(r_1) \cdots \hat{B}(r_k) \right| R(\vec{r}) \\
\leq k \left( \frac{kp}{2L} \right)^2 \cdot 2^{k-1} \cdot \left( \frac{p}{2^k-1} \right)^2 \cdot \left( 1 + \pi^2 \right)^{k-1}
\]
So, by the assumption on \(L\) we obtain
\[
\sum_{\|r_i\|_p > L} \left| \hat{B}(r_1) \cdots \hat{B}(r_k) \right| R(\vec{r}) < (\varepsilon p)^{2k} \leq \left| C \right|^{2k}.
\]
This completes the proof. \(\square\)

**Proof of Theorem 1.** Let \(L\) be a number satisfying inequalities
\[
\sqrt{\frac{k^33^{k-1}}{2^{k+1}2^k}} < L < \sqrt[3]{k^33^{k-1}}.
\]
Such numbers \(L\) exist provided that \(p\) is sufficiently large. By Theorem 2, \(\kappa(D) \geq \frac{1}{2} - \varepsilon\) for every \(L\)-independent set \(D\). We show that the second inequality implies that almost every set in \(\mathbb{Z}_p^*\) is \(L\)-independent. Indeed, the number of sets that are not \(L\)-independent is at most
\[
(2L + 1)^k \binom{p-1}{k-1}.
\]
So, the fraction of those sets in \(\mathbb{Z}_p^*\) is equal
\[
\frac{(2L + 1)^k \binom{p-1}{k-1}}{\binom{p-1}{k-1}} = \frac{(2L + 1)^k k}{p - k} < \frac{(2 \sqrt[3]{k^33^{k-1}} + 1)^k k}{p - k}.
\]
The last expression tends to zero with \(p\) tending to infinity. This completes the proof, as the ratios of two consecutive primes tend to one. \(\square\)

### 3. Integer distance graphs

We conclude the paper with a remark concerning integer distance graphs. For a given set \(D\), consider a graph \(G(D)\) whose vertices are positive integers, with two vertices \(a\) and \(b\) joined by an edge if and only if \(|a - b| \in D\). Let \(\chi(D)\) denote the chromatic number of this graph. It is not hard to see that \(\chi(D) \leq |D| + 1\).

To see a connection to parameter \(\kappa(D)\), put \(N = \lceil \kappa(D)^{-1} \rceil\) and split the circle into \(N\) intervals \(I_i = [(i - 1)/N, i/N), i = 1, 2, \ldots, N\) (cf. [15]). Let
t be a real number such that \( \min_{d \in D} \| dt \| = \kappa(D) \). Then define a colouring \( c : \mathbb{N} \to \{1, 2, \ldots, N\} \) by \( c(a) = i \) if and only if \( \{ ta \} \in I_i \). If \( c(a) = c(b) \) then \( \{ ta \} \) and \( \{ tb \} \) are in the same interval \( I_i \). Hence \( \| ta - tb \| < 1/N \leq \kappa(D) \), and therefore \( |a - b| \) is not in \( D \). This means that \( c \) is a proper colouring of a graph \( G(D) \). So, we have a relation

\[
\chi(D) \leq \left\lceil \frac{1}{\kappa(D)} \right\rceil.
\]

Now, by Theorem 1 we get that \( \chi(D) \leq 3 \) for almost every graph \( G(D) \).

A different proof of a stronger version of this result has been recently found by Alon [1]. He also extended the theorem for arbitrary Abelian groups, and posed many intriguing questions for general groups.

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