Research Article

Yohannis Alemayehu Wakjira and Gemechis File Duressa*

Exponential spline method for singularly perturbed third-order boundary value problems

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Abstract: The exponential spline function is presented to find the numerical solution of third-order singularly perturbed boundary value problems. Convergence analysis of the method is briefly discussed, and it is shown to be sixth order convergence. To validate the applicability of the method, some model problems are solved for different values of the perturbation parameter, and the numerical results are presented both in tables and graphs. Furthermore, the present method gives more accurate solution than some methods existing in the literature.

Keywords: exponential spline, singular perturbation, boundary layers, convergence

MSC 2020: 65L10, 65L11

1 Introduction

Singly perturbed problems arise frequently in the mathematical modelling of real-life phenomena in science and engineering areas such as fluid mechanics, elasticity, quantum mechanics, chemical-reactor theory, aerodynamics, plasma dynamics, rarefied-gas dynamics, oceanography, meteorology, modelling of semiconductor devices, geophysics, optimal control theory, diffraction theory and reaction–diffusion processes [1,2]. Solutions of singularly perturbed boundary value problems manifest multi-scale character. Due to the presence of perturbation parameter, \( \varepsilon \), the solution varies quickly near thin transition layer and performs regularly and varies slowly away from the layer. Hence, the main concern with such problems is the swift growth or deterioration of their solutions in one or more narrow boundary layer region(s). As a result, not only determining analytical solutions to such problems is difficult but also the convergence analysis.

In recent years, a considerable number of numerical methods that deal with quartic, quintic and septic splines with polynomials and non-polynomials; combination of asymptotic expansion approximations; shooting method and finite difference methods; subdivision collocation methods and B-splines collocation methods have been developed for solving singularly perturbed boundary value problems using various splines [1–8]. Furthermore, associating quadratic spline method with other techniques was introduced for the time fractional sub-diffusion and the Helmholtz equation with the Sommerfeld boundary conditions; for details one can refer to [9,10] and references therein.

However, classical finite difference methods are not reliable to preserve the stability property as they require the introduction of very fine meshes inside the boundary layers, which requires more computational cost. Furthermore, they could not capture the solutions in the layer region of the domain as the solution...
profile depends on the perturbation parameter [3,11]. Thus, it is crucial to develop more accurate numerical method which works suitably for $\varepsilon \ll h$, where most of the numerical methods fail to give smooth solution.

Hence, the purpose of the study is to develop a convergent and more accurate spline method for solving third-order singularly perturbed boundary value problem and that works for the cases where other numerical methods fail to give good results. This method depends on exponential spline function which has exponential and polynomial parts.

We consider singularly perturbed reaction–diffusion boundary value problems of the form:

$$
\begin{align*}
Ly(x) & \equiv -\varepsilon y'''(x) + u(x)y = f(x), \\
y(0) & = \alpha_1, \quad y(1) = \beta_1, \quad y'(0) = y_1
\end{align*}
$$

(1)

or

$$
\begin{align*}
Ly(x) & \equiv -\varepsilon y'''(x) + u(x)y = f(x), \\
y(0) & = \alpha_1, \quad y(1) = \beta_1, \quad y'(0) = y_2,
\end{align*}
$$

(2)

where $\alpha_1, \beta_1, y_1$ and $y_2$ are constants; $\varepsilon$ is the small positive parameter; and $u(x)$ and $f(x)$ are sufficiently smooth functions. The spline function proposed in this paper has the form: $T_n = \text{span}\{1, x, x^2, x^3, e^{kx}, e^{-kx}\}$, where $k$ can be real or imaginary.

## 2 Formulation of the method

We consider a uniform mesh $\Delta$ with nodal points $x_i$ on interval $[a, b]$, such that:

$\Delta: a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b, \quad x_i = x_0 + ih, \quad i = 0, 1, \ldots, n, \quad h = \frac{b-a}{n}.$

An exponential spline function $S_0(x)$ of a class $C^4[a, b]$ which interpolates $y(x)$ at mesh points $x_i$, $i = 0, 1, 2, \ldots, n$, depends on a parameter $k$. If $k \to 0$, then it reduces to ordinary quintic spline in $[a, b]$.

For each subinterval $[x_i, x_{i+1}]$, $i = 1, 2, \ldots, n - 1$, we consider the exponential spline, $S_j(x)$, of the form:

$$
S_j(x) = a_i e^{k(x-x_i)} + b_i e^{-k(x-x_i)} + c_i(x-x_i)^3 + d_i(x-x_i)^2 + e_i(x-x_i) + f_i,
$$

(3)

where $i = 1, 2, \ldots, n-1$; $a_i$, $b_i$, $c_i$, $d_i$, $e_i$ and $f_i$ are unknown coefficients, and $k$ is a free parameter which will be used to raise the accuracy of the method.

Let $y(x)$ be the exact solution of Eqs. (1) and (2) and $S_j$ be an approximation to $y_j = y(x_i)$ obtained by the spline function $S_j(x)$ passing through the points $(x_i, S_i)$ and $(x_{i+1}, S_{i+1})$.

To develop the consistency relations between the value of spline and its derivatives at knots, let

$$
\begin{align*}
S_j(x_i) &= y_i, \quad S_j(x_{i+1}) = y_{i+1}, \\
S_j''(x_i) &= D_i, \quad S_j''(x_{i+1}) = D_{i+1}, \\
S_j'''(x_i) &= T_i, \quad S_j'''(x_{i+1}) = T_{i+1}, \\
S_j^{(4)}(x_i) &= F_i, \quad S_j^{(4)}(x_{i+1}) = F_{i+1},
\end{align*}
$$

for $i = 0, 1, \ldots, n$.

(4)

To define spline in terms of $y_j$, $y_{i+1}$, $D_i$, $D_{i+1}$, $T_i$, $T_{i+1}$, $F_i$ and $F_{i+1}$, the coefficients introduced in Eq. (3) are calculated as:

$$
\begin{align*}
a_i &= \frac{F_{i+1} - F_i}{2k \sinh(\theta)}, \\
b_i &= \frac{F_i e^{k\theta} - F_{i+1} e^{-k\theta}}{2k \sinh(\theta)}, \\
c_i &= \frac{D_{i+1} - D_i}{6h} - \frac{F_{i+1} - F_i}{6h \theta^2}, \\
d_i &= \frac{1}{2} \left( D_i + \frac{F_i}{k^2} \right), \\
e_i &= \frac{y_{i+1} - y_i}{h} - \frac{F_{i+1} - F_i}{hk^2} - \frac{h}{6}(D_{i+1} + 2D_i) - \frac{h}{6}(F_{i+1} + 2F_i), \\
f_i &= y_i - \frac{F_i}{k^2}.
\end{align*}
$$

(5)

where $\theta = kh$ and $i = 0, 1, 2, \ldots, n$. 

Using the continuity condition of the first derivatives at knots, we have:

\[ S'_{\Delta_1}(x_i) = S'_{\Delta_2}(x_i). \quad (6) \]

Then from Eq. (6), we do have:

\[ -a_{i-1} k \sinh(\theta) + b_{i-1} k \cosh(\theta) + 3c_{i-1} h^2 + 2d_{i-1} h + e_{i-1} = b_i k + e_i. \quad (7) \]

By reducing indices of Eq. (5) by one and replacing in Eq. (7), we obtain:

\[
\frac{F_{i-1} \sinh(\theta)}{k^3} + \frac{F_i \cos(\theta)}{k^3 \sinh(\theta)} - \frac{F_{i+1} \cos^2(\theta)}{k^3 \sin(\theta)} - \frac{3hD_i}{6} - \frac{3hD_{i-1}}{6} + \frac{3hF_i}{6k^2} - \frac{3hF_{i+1}}{6k^2} \\
+ hD_{i-1} + \frac{hF_{i-1}}{k^3} + \frac{F_i}{h k^2} - \frac{F_{i-1}}{h k^2} - \frac{2hD_{i-1}}{6} - \frac{hF_i}{6k^2} - \frac{2hF_{i-1}}{6k^2} \\
= F_{i+1} \left( \frac{1}{k^3 \sinh(\theta)} - \frac{h}{6k^2} - \frac{1}{h k^5} \right) - F_i \left( \frac{\coth(\theta)}{k^3} - \frac{2h}{6k^2} + \frac{1}{h k^5} \right) - \frac{2hD_i}{6} - \frac{hD_{i+1}}{6} + \frac{y_{i+1} - y_i}{h}. \quad (8) \]

On simplification, Eq. (8) yields:

\[ D_{i+1} + 4D_i + D_{i-1} = \frac{6}{h^2} (y_{i+1} - 2y_i + y_{i-1}) - 6h^2 (\lambda_1 F_{i+1} + 2p_1 F_i + \lambda_2 F_{i-1}), \quad (9) \]

where \( \lambda_1 = \frac{3h}{4k^2} - \frac{3h}{k^3} + \frac{1}{k^2 \sinh(\theta)}, \) \( \rho_1 = \frac{6 \coth(\theta)}{k^3}. \)

Again, using the continuity condition of the third derivatives at knots, we have:

\[ S^{(3)}_{\Delta_1}(x_i) = S^{(3)}_{\Delta_2}(x_i). \quad (10) \]

Then, from Eq. (10), we do have:

\[ a_{i-1} k^3 \sinh(\theta) - b_{i-1} k^3 \cosh(\theta) + 6c_{i-1} = -b_i k^3 + 6c_i. \quad (11) \]

By reducing indices of Eq. (5) by one and replacing in Eq. (11), we obtain:

\[
\frac{\sinh(\theta)}{k^3} - \frac{F_i}{k \sinh(\theta)} + \frac{\cos^2(\theta)}{k \sinh(\theta)} - \frac{D_i}{h} - \frac{D_{i-1}}{h} + \frac{F_i}{h k^2} - \frac{F_{i-1}}{h k^2} \\
- \frac{F_{i+1}}{k \sinh(\theta)} + \frac{F_i \cosh(\theta)}{k \sinh(\theta)} + \frac{D_{i+1}}{h} - \frac{D_i}{h} + \frac{F_{i+1}}{h k^2} - \frac{F_i}{h k^2}. \quad (12) \]

On simplifying Eq. (12), we obtain:

\[ D_{i+1} - 2D_i + D_{i-1} = \lambda_2 F_{i+1} + 2p_2 F_i + \lambda_2 F_{i-1}, \quad (13) \]

where \( \lambda_2 = \frac{h}{k^3} - \frac{1}{k^2} \) and \( \rho_2 = \frac{1}{k^2} - \frac{h \coth(\theta)}{k^3}. \)

Now, subtracting Eq. (13) from Eq. (9), we obtain:

\[ D_i = \frac{1}{h^2} (y_{i+1} - 2y_i + y_{i-1}) - h^2 \left( \lambda_1 + \frac{\lambda_2}{6} \right) F_{i+1} + 2 \left( \rho_1 + \frac{\rho_2}{6} \right) F_i + \left( \lambda_1 + \frac{\lambda_2}{6} \right) F_{i-1}. \quad (14) \]

Using continuity of third derivative and Eq. (14), we obtain the relation:

\[
T_i = \frac{1}{h^2} (y_{i+1} - 3y_{i+1} + 3y_{i-1} - y_{i-1}) - h(pF_{i+2} + (p_0 - p + \alpha) F_{i+1} + (p - p_0 + \beta) F_i - pF_{i-1}), \quad (15) \]

where \( p = \lambda_1 + \frac{\lambda_2}{6}, \) \( p_0 = 2 \left( \rho_1 + \frac{\rho_2}{6} \right), \) \( \beta = \frac{1}{\alpha} (1 - \theta \coth(\theta)) \) and \( \alpha = \frac{1}{\alpha} (\theta \csch(\theta) - 1). \)

Defining the operator \( L \) by \( LT_i \equiv p(T_{i+2} + T_{i-2}) + sT_i + q(T_{i+1} + T_{i-1}), \) for any function \( T \) evaluated at the mesh points [12], we have:

\[ LT \equiv p(T_{i+2} + T_{i-2}) + sT_i + q(T_{i+1} + T_{i-1}). \quad (16) \]

Using Eqs. (15) and (16), we obtain the relation:

\[
\left( \frac{\alpha + \beta}{h^3} \right) (y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}) = p(T_{i+2} + T_{i-2}) + sT_i + q(T_{i+1} + T_{i-1}), \quad (17) \]

for \( i = 2, 3, \ldots, n - 2. \)
At nodal point $x_i$, the proposed singularly perturbed Eq. (1) can be evaluated as:

$$- ey''(x_i) + u(x_i) y(x_i) = f(x_i).$$

(18)

From Eq. (18) we obtain:

$$y''(x_i) = \frac{u(x_i) y(x_i) - f(x_i)}{\varepsilon},$$

where $f_i = f(x_i)$ and $u_i = u(x_i)$.

Using spline’s third derivative, we have:

$$T_i = \frac{u_i y_i - f_i}{\varepsilon}, \quad T_{i-1} = \frac{u_{i-1} y_{i-1} - f_{i-1}}{\varepsilon}, \quad T_{i-2} = \frac{u_{i-2} y_{i-2} - f_{i-2}}{\varepsilon}, \quad T_{i+1} = \frac{u_{i+1} y_{i+1} - f_{i+1}}{\varepsilon}, \quad T_{i+2} = \frac{u_{i+2} y_{i+2} - f_{i+2}}{\varepsilon}.$$  

(19)

Substituting Eq. (19) into Eq. (17) and simplifying, we obtain:

$$((\alpha + \beta)\varepsilon + u_{i-2} p h^3) y_{i-2} - ((\alpha + \beta)\varepsilon - qu_{i-1} h^2) y_{i-1} + su_i h^3 y_i + ((\alpha + \beta)\varepsilon + u_{i+1} q h^3) y_{i+1} - ((\alpha + \beta)\varepsilon + u_{i+2} p h^3) y_{i+2} = -h^3 (p f_{i+2} + f_{i-2}) + s f_i + q (f_{i-1} + f_{i+1}) + t_i,$$

for $i = 0, \ldots, n - 2$ and $t_i$ is the local truncation error with:

$$t_i = 12 h (2 \alpha - \beta) y_i' + \frac{1}{3} h^3 (6 p + 6 q + 3 s - 10 \alpha - 4 \beta) y_i^{(2)} + \frac{1}{30} h^5 (-30 (4 p + q) + 17 \alpha + 14 \beta) y_i^{(5)} + \frac{17}{6048} h^9 (\alpha + \beta) - \frac{8 p}{45} - \frac{q}{360} y_i^9 + O(h^{11}).$$

(21)

Again, by truncating terms in Eq. (20) that contains $h^9$ and above, for arbitrary $\alpha$ and $\beta$ provided that $\alpha + \beta = \frac{1}{2}$, and evaluated for free parameters $p$, $q$ and $s$, we obtain:

$$2 p + 2 q + s = 1, \quad 120 p + 30 q = 7.5, \quad 16080 p + 105 q = 31.5.$$

On solving we obtain: $(\alpha, \beta, p, q, s) = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{260}, \frac{7}{30}, \frac{21}{40}\right)$, and the local truncation error for Eq. (20) is as follows:

$$t_i = -\frac{1}{6048} e h^9 y_i^9 + O(h^{11}), \quad i = 2, 3, \ldots, n - 2.$$

### 3 End conditions

The relation given in Eq. (20) has $(n - 3)$ linear algebraic equations in the $(n - 1)$ unknown $y_i$, for $i = 1, 2, \ldots, n - 1$. So we need two more equations at each end. Following the procedure given in [13], the required end condition of Eq. (1) can be written as:

$$\sum_{i=0}^{3} b_i y_i + c_i h y_i' + h^3 \sum_{i=0}^{3} d_i y_i''' + t_i = 0, \quad i = 0,$$

(22)

$$\sum_{i=-n-3}^{n} m_i y_i + h^3 \sum_{i=-n-4}^{n} k_i y_i''' + t_{n-1} = 0, \quad i = n - 1,$$

(23)
where \( b_i, c_i, d_i, m_i \) and \( k_i, l = 0, 1, 2, 3, n - 4, n - 3, n - 2, n - 1, n \) are arbitrary parameters which can be calculated using the method of undetermined coefficients.

Thus, the end condition of Eq. (1) can be calculated as:

\[
(b_0, b_1, b_2, b_3, c_0, d_0, d_1, d_2, d_3) = \left( \frac{800}{9}, -1600, 2720, -\frac{9280}{9}, -\frac{2240}{3}, \frac{448}{9}, \frac{2200}{3}, \frac{1472}{3}, \frac{56}{9} \right).
\]

\[
(m_{n-3}, m_{n-2}, m_{n-1}, m_n, k_{n-4}, k_{n-3}, k_{n-2}, k_{n-1}, k_n) = (480, -1440, 1440, -480, 2, -8, 252, 232, 2).
\]

By using Eqs. (18) and (24), we obtain the first boundary equation as:

\[
-1600 \varepsilon + \frac{2200}{3} u_1 h^3 + 2720 \varepsilon + \frac{1472}{3} u_2 h^3 + \left( -\frac{9280}{9} + \frac{56}{9} u_3 h^3 \right) y_1 = -\frac{9280}{9} + \frac{56}{9} u_3 h^3.
\]

Again using Eqs (18) and (25) we obtain the other end condition as:

\[
2u_{n-4} h^3 y_{n-4} + (480 \varepsilon + 8u_{n-3} h^3)y_{n-3} + (-1440 \varepsilon + 252u_{n-2} h^3)y_{n-2} + (1440 \varepsilon + 232u_{n-1} h^3)y_{n-1} = (-480 \varepsilon + 2u_{n} h^3) \beta_1 + h^3 (2f_{n-4} - 8f_{n-3} + 252f_{n-2} + 232f_{n-1} + 2f_{n}) + t_n - t_{n-1}.
\]

Similarly, besides Eq. (23), it requires additional equation to determine the end condition of Eq. (2) which can be written as:

\[
\sum_{i=0}^{3} b_i y_i + c_i h y_i'' + h^3 \sum_{i=0}^{3} d_i y_i'''' + t_1 = 0, \quad i = 1.
\]

After solving coefficients of Eq. (28) using Eq. (18), we obtain end condition of Eq. (2) as:

\[
\left[ -\frac{9360}{11} \varepsilon + \frac{248}{11} u_1 h^3 \right] y_1 + \left[ \frac{10080}{11} \varepsilon + \frac{2346}{11} u_2 h^3 \right] y_2 + \left[ -\frac{3600}{11} \varepsilon + \frac{1704}{11} u_3 h^3 \right] y_3 = \frac{248}{11} f_0 + \frac{2346}{11} f_1 + \frac{1704}{11} f_2 + 2 f_3 + t_1.
\]

Using values given on the Eqs. (24) and (25), we get the local truncation error for Eqs. (26) and (27) as:

\[
t_i = \begin{cases} 
\frac{123}{105} \varepsilon y_i^9 + O(h^{11}), & i = 1, \\
\frac{251}{126} \varepsilon y_i^9 + O(h^{11}), & i = n - 1.
\end{cases}
\]

4 Convergence analysis

The main purpose here is to drive a bound on \( \|E\|_{\infty} \). Considering Eqs. (20), (26) and (27), we obtain linear system of order \((n - 1) \times (n - 1)\) and that can be rewritten in the matrix form:

\[
AY = G + T,
\]

where \( A = N + h^3BU, U = \text{diag} (u_i) \),

\[
N = \begin{pmatrix} -1600 \varepsilon & 2720 \varepsilon & -\frac{9280}{9} \varepsilon & \hdots \\
-\varepsilon & 0 & \varepsilon & -0.5 \varepsilon & \hdots \\
0.5 \varepsilon & -\varepsilon & 0 & \varepsilon & -0.5 \varepsilon & \hdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & 0.5 \varepsilon & -\varepsilon & 0 & \varepsilon & -0.5 \varepsilon & \hdots \\
\vdots & \vdots & \vdots & \vdots & 0.5 \varepsilon & -\varepsilon & 0 & \varepsilon & \hdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & 0 & 480 \varepsilon & -1440 \varepsilon & 1440 \varepsilon \end{pmatrix}.
\]
\[
B = \begin{pmatrix}
(d_1 & d_2 & d_3 & \ldots \\
q & s & q & p & \ldots \\
p & q & s & q & p & \ldots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
p & q & s & q & p \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
k_{n-4} & k_{n-3} & k_{n-2} & k_{n-1} & k_n & \ldots \\
\end{pmatrix},
\]

where

\[
B = \begin{pmatrix}
g_1, g_2, \ldots, g_{n-2}, g_{n-1} \end{pmatrix}^T,
\]

and

\[
Y = (y_1, y_2, \ldots, y_{n-2}, y_{n-1})^T.
\]

Assuming that \( Y = (y(x_i), y(x_i), \ldots, y(x_{n-1}))^T \) is the exact solution of Eq. (1) at the points: \( x_i \), for \( i = 1, 2, \ldots, n - 1 \), we have:

\[
A Y = G
\]

with truncation error: \( T = (t_1, t_2, \ldots, t_{n-1})^T \) and

\[
g_i = \begin{cases}
\frac{1}{2} \varepsilon h y_i^9 - \left( \frac{800}{9} \varepsilon - \frac{448}{9} u_0 h^3 \right) a_1 + h^3 \sum_{j=0}^{3} d_j f_j, & i = 1, \\
(h^3 (p(f_{i+2} + f_{i-2}) + sf_i + q(f_{i+1} + f_{i+1})), & i = 2, \\
(0.5 \varepsilon - u_0 p h^2) a_1 + h^3 (p(f_{i+2} + f_{i-2}) + sf_i + q(f_{i+1} + f_{i+1})), & i = 3, 4, \ldots, n - 3, \\
(-480 \varepsilon + 2 u_0 h^2) \beta_1 + h^3 \sum_{j=0}^{n} k_n f_{n-j}, & i = n - 1.
\end{cases}
\]

Now, subtracting Eq. (30) from Eq. (31), we obtain:

\[
A(Y - Y) = AE = T,
\]

where \( E \) is the discretization error, and \( E = Y - Y = (e_1, e_2, \ldots, e_{n-1})^T \).

In order to derive the bound on \( \| E \|_\infty \) the following two lemmas are important.

**Lemma 1.** If \( H \) is the square matrix of order \( n \) and \( \| H \|_\infty < 1 \), then \( (I + H)^{-1} \) exists and \( \| (I + H)^{-1} \|_\infty < \frac{1}{1 - \| H \|_\infty} \).

**Proof.** For the details of the proof, one can refer to [14, 15].

**Lemma 2.** The matrix \( A \) is non-singular, if \( \| \mathbf{u} \| < \frac{162}{w} \), where \( w = 480(b - a)^3 \left( 2 + \frac{3h^3}{(b - a)^2} \right) \).

**Proof.** According to [16, 17], the matrix \( N \) is non-singular and its inverse satisfies the inequality:

\[
\| N^{-1} \|_\infty \leq \frac{2}{81} (b - a)^3 \left( 1 + \frac{3h^3}{2(b - a)^2} \right).
\]

Since \( N \) is the non-singular matrix, we have:

\[
A = N + h^3 B U = (I + N^{-1} h^3 B U) N.
\]

So, to prove the non-singularity of \( A \), it is sufficient to show that \( I + N^{-1} h^3 B U \) is non-singular. Moreover, \( \| U \|_\infty \leq \| \mathbf{u} \| = \max_{a \leq x \leq b} | \mathbf{u}(x) | \) [18].
By Cauchy–Schwarz and triangle inequalities [19], we get:

\[ \|N^{-1}h^3BU\|_{\infty} \leq \|N^{-1}\|_{\infty} h^3\|U\|_{\infty} \leq \|N^{-1}\|_{\infty} h^3\|B\|_{\infty} \|U\|_{\infty} \leq h^3\|N^{-1}\|_{\infty} \|B\|_{\infty} \|U\|_{\infty}, \]  

(34)

where \( \|B\|_{\infty} = k_{n-1} + k_{n-2} + k_{n-3} + k_{n-4} = 480. \)

Therefore, substituting \( \|N\|_{\infty}, \|B\|_{\infty} \) and \( \|U\| \) in Eq. (34), we get the following relation:

\[ \|N^{-1}h^3BU\|_{\infty} \leq 1, \]  

(35)

Using Lemma 1 and Eq. (35), we deduce that the matrix \( A \) is non-singular.

Since \( A \) is the non-singular matrix, Eq. (33) can be written as:

\[ E = A^{-1}T = (N + h^3BU)^{-1}T. \]

Since \( N \) is non-singular, we can re-write the matrix \( E \) in the form:

\[ E = (I + N^{-1}h^3BU)^{-1}N^{-1}T, \]

and using the Cauchy–Schwarz inequality we obtain:

\[ \|E\|_{\infty} \leq \|(I + N^{-1}h^3BU)^{-1}\|_{\infty}\|N^{-1}\|_{\infty}\|T\|_{\infty}. \]  

(36)

Hence, from Eq. (36) and Lemma 1, it follows that:

\[ \|E\|_{\infty} \leq \frac{\|N^{-1}\|_{\infty}\|T\|_{\infty}}{1 - \|N^{-1}h^3BU\|_{\infty}}. \]

Furthermore, from Eq. (32), we have:

\[ \|T\|_{\infty} = \frac{1}{60480}eh^6M_0, \]

where \( M_0 = \max_{a \leq x \leq b} |y'(x_i)| \) and hence

\[ \|E\|_{\infty} \leq \frac{\|N^{-1}\|_{\infty}\|T\|_{\infty}}{1 - h^3\|N^{-1}\|_{\infty}\|B\|_{\infty} \|U\|} \equiv O(h^6), \]  

(37)

\[ \|E\|_{\infty} \leq Mh^6, \]

where \( M \) is a constant independent of \( h. \)

Therefore, from Eq. (37) it follows that \( \|E\|_{\infty} \leq O(h^6). \)

5 Numerical examples and results

To demonstrate the applicability of the method, four singularly perturbed model problems were considered. These examples were chosen because they have been widely discussed in the literature, and their exact solutions were available for comparison.

**Example 1.** Consider the following singularly perturbed problem:

\[ -\varepsilon y'''(x) + y(x) = f(x), \quad 0 \leq x \leq 1, \]

\[ y(0) = 0, \quad y(1) = 0, \quad y'(0) = 9\varepsilon, \]

where \( f(x) = 6\varepsilon(1-x)^5x^3 - 6\varepsilon^2(6(1-x)^5 - 90(1-x)x + 180(1-x)^3x^2 - 60(1-x)^2x^3) \) and the analytic solution is

\[ y(x) = 6\varepsilon^2(1-x)^5. \]

**Example 2.** Consider the following singularly perturbed problem:

\[ -\varepsilon y'''(x) + y(x) = f(x), \quad 0 \leq x \leq 1, \]

\[ y(0) = 0, \quad y(1) = 0, \quad y''(0) = 0, \]

where \( f(x) = 6\varepsilon(1-x)^5x^3 - 6\varepsilon^2(6(1-x)^5 - 90(1-x)x + 180(1-x)^3x^2 - 60(1-x)^2x^3) \) and the analytic solution is

\[ y(x) = 6\varepsilon^2(1-x)^5. \]
Example 3. Consider the following singularly perturbed problem:
\[-\varepsilon y'''(x) + y(x) = f(x), \quad 0 \leq x \leq 1,
\]
\[y(0) = 0, \quad y(1) = 3\varepsilon \sin(3), \quad y'(0) = 9\varepsilon,
\]
where \(f(x) = 81\varepsilon^2 \cos 3x + 3\varepsilon \sin 3x\), and the analytic solution is
\[y(x) = 3\varepsilon \sin(3x).
\]

Example 4. Consider the following singularly perturbed problem:
\[-\varepsilon y'''(x) + y(x) = f(x), \quad 0 \leq x \leq 1,
\]
\[y(0) = 0, \quad y(1) = 3\varepsilon \sin(3), \quad y''(0) = 0,
\]
where \(f(x) = 81\varepsilon^2 \cos 3x + 3\varepsilon \sin 3x\), and the analytic solution is
\[y(x) = 3\varepsilon \sin(3x).
\]

Table 1: Maximum absolute errors and numerical rate of convergence for Example 1

\[\begin{array}{c|c|c|c}
\epsilon & N = 10 & N = 20 & N = 40 \\
\hline
\text{New method} & & & \\
1/16 & 1.0028 \times 10^{-6} & 7.7262 \times 10^{-9} & 5.9445 \times 10^{-11} \\
r\rightarrow & 1.3824 & 3.2636 & 3.2736 \\
1/32 & 4.2762 \times 10^{-7} & 3.2959 \times 10^{-9} & 2.5316 \times 10^{-11} \\
r\rightarrow & 3.2610 & 3.2660 & 3.2743 \\
1/64 & 1.7927 \times 10^{-7} & 1.3211 \times 10^{-9} & 1.0236 \times 10^{-11} \\
r\rightarrow & 3.3258 & 3.2535 & 3.2756 \\
\text{Reference [4]} & & & \\
1/16 & 6.8572 \times 10^{-6} & 1.1698 \times 10^{-7} & 1.8578 \times 10^{-9} \\
1/32 & 2.9156 \times 10^{-6} & 4.9916 \times 10^{-8} & 7.9252 \times 10^{-10} \\
1/64 & 1.2223 \times 10^{-6} & 2.0000 \times 10^{-8} & 3.2111 \times 10^{-10} \\
\text{Reference [5]} & & & \\
1/16 & 4.8700 \times 10^{-4} & 1.8600 \times 10^{-5} & 1.9500 \times 10^{-5} \\
1/32 & 1.9500 \times 10^{-4} & 8.7600 \times 10^{-6} & 8.6300 \times 10^{-6} \\
1/64 & 7.9700 \times 10^{-5} & 4.0000 \times 10^{-6} & 3.6100 \times 10^{-6} \\
\end{array}
\]

Table 2: Maximum absolute errors and numerical rate of convergence for Example 2

\[\begin{array}{c|c|c|c}
\epsilon & N = 10 & N = 20 & N = 40 \\
\hline
\text{New method} & & & \\
1/16 & 1.5925 \times 10^{-5} & 2.5004 \times 10^{-7} & 3.9129 \times 10^{-9} \\
r\rightarrow & 2.2345 & 2.2393 & 2.2416 \\
1/32 & 5.6471 \times 10^{-6} & 8.8838 \times 10^{-8} & 1.3917 \times 10^{-9} \\
r\rightarrow & 2.2317 & 2.2470 & 2.2517 \\
1/64 & 1.8565 \times 10^{-6} & 2.8912 \times 10^{-8} & 4.5267 \times 10^{-10} \\
r\rightarrow & 2.2463 & 2.2386 & 2.3977 \\
\text{Reference [2]} & & & \\
1/16 & 2.8930 \times 10^{-4} & 5.3006 \times 10^{-6} & 2.6033 \times 10^{-8} \\
1/32 & 1.0962 \times 10^{-4} & 1.9394 \times 10^{-6} & 1.3221 \times 10^{-8} \\
1/64 & 3.8007 \times 10^{-5} & 6.8026 \times 10^{-7} & 6.2298 \times 10^{-9} \\
\text{Reference [3]} & & & \\
1/16 & 6.2854 \times 10^{-3} & & \\
1/32 & 1.9707 \times 10^{-3} & & \\
1/64 & 3.9065 \times 10^{-4} & & \\
\end{array}
\]
Table 3: Maximum absolute errors and numerical rate of convergence for Example 3

| $\varepsilon$ | $N = 10$            | $N = 20$            | $N = 40$            |
|--------------|---------------------|---------------------|---------------------|
| New method   |                     |                     |                     |
| 1/16         | 4.4336 $\times 10^{-8}$ | 2.0866 $\times 10^{-10}$ | 1.0750 $\times 10^{-12}$ |
| r$\rightarrow$ | 3.9727             | 3.8422              | 3.9422              |
| 1/32         | 1.8916 $\times 10^{-8}$ | 8.8814 $\times 10^{-11}$ | 4.5211 $\times 10^{-13}$ |
| r$\rightarrow$ | 2.0969             | 3.8595              | 4.9501              |
| 1/64         | 7.9396 $\times 10^{-9}$ | 3.5668 $\times 10^{-11}$ | 1.8448 $\times 10^{-13}$ |
| r$\rightarrow$ | 4.0398             | 3.8366              | 4.9522              |
| Reference [4] |                    |                     |                     |
| 1/16         | 3.1247 $\times 10^{-7}$ | 4.9269 $\times 10^{-9}$ | 7.4543 $\times 10^{-11}$ |
| 1/32         | 1.3421 $\times 10^{-7}$ | 2.1095 $\times 10^{-9}$ | 3.1741 $\times 10^{-11}$ |
| 1/64         | 5.6587 $\times 10^{-8}$ | 8.4937 $\times 10^{-10}$ | 1.2904 $\times 10^{-11}$ |
| Reference [5] |                    |                     |                     |
| 1/16         | 2.3200 $\times 10^{-4}$ | 6.1200 $\times 10^{-5}$ | 1.5200 $\times 10^{-6}$ |
| 1/32         | 9.7700 $\times 10^{-5}$ | 2.5900 $\times 10^{-5}$ | 6.4500 $\times 10^{-6}$ |
| 1/64         | 3.7800 $\times 10^{-5}$ | 1.0000 $\times 10^{-6}$ | 2.5000 $\times 10^{-6}$ |

Table 4: Maximum absolute errors and numerical rate of convergence for Example 4

| $\varepsilon$ | $N = 10$            | $N = 20$            | $N = 40$            |
|--------------|---------------------|---------------------|---------------------|
| New method   |                     |                     |                     |
| 1/16         | 7.1881 $\times 10^{-7}$ | 6.8962 $\times 10^{-9}$ | 6.9667 $\times 10^{-11}$ |
| r$\rightarrow$ | 2.9452             | 2.8707              | 3.4707              |
| 1/32         | 2.5476 $\times 10^{-7}$ | 2.4500 $\times 10^{-9}$ | 2.4726 $\times 10^{-11}$ |
| r$\rightarrow$ | 2.9417             | 2.8722              | 3.5260              |
| 1/64         | 8.3850 $\times 10^{-8}$ | 7.9666 $\times 10^{-10}$ | 8.0634 $\times 10^{-12}$ |
| r$\rightarrow$ | 2.9592             | 2.8680              | 3.8707              |
| Reference [2] |                    |                     |                     |
| 1/16         | 9.4405 $\times 10^{-6}$ | 5.4886 $\times 10^{-7}$ | 2.5658 $\times 10^{-8}$ |
| 1/32         | 3.1645 $\times 10^{-6}$ | 1.9215 $\times 10^{-7}$ | 9.1282 $\times 10^{-9}$ |
| 1/64         | 9.9920 $\times 10^{-7}$ | 6.1969 $\times 10^{-8}$ | 2.9364 $\times 10^{-9}$ |

Figure 1: The graph of exact and numerical solutions of Example 1 for $N = 40$ and $\varepsilon = \frac{1}{64}$. 
Figure 2: The graph of exact and numerical solutions of Example 2 for $N = 40$ and $\varepsilon = \frac{1}{32}$.

Figure 3: The graph of exact and numerical solutions of Example 3 for $N = 20$ and $\varepsilon = \frac{1}{32}$.

Figure 4: The graph of exact and numerical solutions of Example 4 for $N = 20$ and $\varepsilon = \frac{1}{32}$.
The exponential spline method is developed to approximate solution of a third-order singularly perturbed two point boundary value problems. The convergence analysis is investigated and revealed that the present method is of sixth order convergence. Moreover, the study analysed by taking different mesh size $h$ and sufficiently small perturbation parameter $\varepsilon$. From the results in Tables 1–4, one can see that maximum absolute error decreases as both the perturbation parameter and mesh size decrease, which in turn shows

### Table 5: Maximum absolute errors for Example 2 when $\varepsilon < h$

| $\varepsilon$ | $N = 10$ | $N = 50$ | $N = 100$ | $N = 150$ | $N = 200$ | $N = 250$ |
|--------------|--------|--------|--------|--------|--------|--------|
| New method   |        |        |        |        |        |        |
| $10^{-1}$    | 3.0519 $\times 10^{-5}$ | 1.9136 $\times 10^{-9}$ | 3.0023 $\times 10^{-11}$ | 2.6799 $\times 10^{-13}$ | 4.3687 $\times 10^{-15}$ | 8.2174 $\times 10^{-17}$ |
| $10^{-2}$    | 8.4777 $\times 10^{-8}$ | 5.6414 $\times 10^{-11}$ | 8.8953 $\times 10^{-13}$ | 7.6186 $\times 10^{-15}$ | 1.0949 $\times 10^{-17}$ | 2.2856 $\times 10^{-19}$ |
| $10^{-3}$    | 1.7615 $\times 10^{-8}$ | 1.2017 $\times 10^{-12}$ | 1.8385 $\times 10^{-14}$ | 1.6631 $\times 10^{-15}$ | 2.7283 $\times 10^{-16}$ | 8.8663 $\times 10^{-17}$ |
| $10^{-4}$    | 1.7081 $\times 10^{-10}$ | 2.5046 $\times 10^{-14}$ | 4.0203 $\times 10^{-16}$ | 3.6209 $\times 10^{-17}$ | 7.5647 $\times 10^{-18}$ | 4.1056 $\times 10^{-18}$ |
| Reference [2] |        |        |        |        |        |        |
| $10^{-1}$    | 5.3363 $\times 10^{-4}$ | 5.2813 $\times 10^{-8}$ | 9.8418 $\times 10^{-9}$ | 2.1617 $\times 10^{-9}$ | 3.7808 $\times 10^{-10}$ | — |
| $10^{-2}$    | 1.8773 $\times 10^{-5}$ | 2.2337 $\times 10^{-9}$ | 2.5972 $\times 10^{-10}$ | 6.0042 $\times 10^{-11}$ | 2.0403 $\times 10^{-11}$ | — |
| $10^{-3}$    | 1.5441 $\times 10^{-6}$ | 5.6301 $\times 10^{-12}$ | 3.8697 $\times 10^{-13}$ | 8.5417 $\times 10^{-13}$ | 2.8902 $\times 10^{-13}$ | — |
| $10^{-4}$    | 1.8248 $\times 10^{-8}$ | 3.2291 $\times 10^{-12}$ | 5.6471 $\times 10^{-14}$ | 1.0858 $\times 10^{-14}$ | 3.3903 $\times 10^{-15}$ | — |
| Reference [3] |        |        |        |        |        |        |
| $10^{-1}$    | 1.6190 $\times 10^{-2}$ | 7.3371 $\times 10^{-4}$ | 6.4463 $\times 10^{-4}$ | 6.3671 $\times 10^{-4}$ | 6.3496 $\times 10^{-4}$ | 7.0694 $\times 10^{-4}$ |
| $10^{-2}$    | 5.4777 $\times 10^{-4}$ | 3.5302 $\times 10^{-5}$ | 3.2708 $\times 10^{-5}$ | 3.3005 $\times 10^{-5}$ | 3.3331 $\times 10^{-5}$ | 4.8167 $\times 10^{-5}$ |
| $10^{-3}$    | 4.3814 $\times 10^{-5}$ | 2.4150 $\times 10^{-6}$ | 1.3966 $\times 10^{-6}$ | 1.1544 $\times 10^{-6}$ | 1.2348 $\times 10^{-6}$ | 1.5754 $\times 10^{-6}$ |
| $10^{-4}$    | 7.5623 $\times 10^{-6}$ | 2.4329 $\times 10^{-7}$ | 1.1223 $\times 10^{-7}$ | 7.6323 $\times 10^{-8}$ | 6.1521 $\times 10^{-8}$ | 3.3448 $\times 10^{-8}$ |

### Table 6: Maximum absolute errors for Example 3 when $\varepsilon < h$

| $\varepsilon$ | $N = 10$ | $N = 50$ | $N = 100$ | $N = 150$ | $N = 200$ | $N = 250$ |
|--------------|--------|--------|--------|--------|--------|--------|
| New method   |        |        |        |        |        |        |
| $10^{-1}$    | 2.1007 $\times 10^{-10}$ | 8.4524 $\times 10^{-16}$ | 5.6292 $\times 10^{-16}$ | 1.7495 $\times 10^{-16}$ | 9.9739 $\times 10^{-17}$ | 2.5405 $\times 10^{-18}$ |
| $10^{-2}$    | 4.1057 $\times 10^{-12}$ | 4.0874 $\times 10^{-17}$ | 3.8625 $\times 10^{-18}$ | 1.4474 $\times 10^{-18}$ | 1.7093 $\times 10^{-18}$ | 5.4454 $\times 10^{-19}$ |
| $10^{-3}$    | 1.3838 $\times 10^{-13}$ | 1.9331 $\times 10^{-18}$ | 5.8445 $\times 10^{-20}$ | 1.5543 $\times 10^{-20}$ | 1.2955 $\times 10^{-20}$ | 4.8789 $\times 10^{-21}$ |
| Reference [3] |        |        |        |        |        |        |
| $10^{-1}$    | 7.4069 $\times 10^{-4}$ | 1.1109 $\times 10^{-4}$ | 4.4739 $\times 10^{-5}$ | 2.3062 $\times 10^{-5}$ | 1.2307 $\times 10^{-5}$ | 6.0161 $\times 10^{-6}$ |
| $10^{-2}$    | 3.0209 $\times 10^{-2}$ | 1.8618 $\times 10^{-5}$ | 9.0607 $\times 10^{-6}$ | 5.9279 $\times 10^{-6}$ | 4.3685 $\times 10^{-6}$ | 3.4355 $\times 10^{-6}$ |
| $10^{-3}$    | —      | 2.1405 $\times 10^{-6}$ | 1.0275 $\times 10^{-6}$ | 6.7970 $\times 10^{-7}$ | 5.0807 $\times 10^{-7}$ | 4.0537 $\times 10^{-7}$ |

The numerical solutions in terms of maximum absolute errors and comparison with other findings existing in the literature are given in Tables 1–4 along with its graph in Figures 1–4 for different values of the perturbation parameters $\varepsilon$ and mesh points $N$. The numerical rate of convergence for all the examples has been calculated by the formula:

$$r^N = \frac{\log(E_{N+1}) - \log(E_N)}{\log(2)},$$

where $r$ is the rate of convergence (Tables 5 and 6).

### 6 Conclusion

The exponential spline method is developed to approximate solution of a third-order singularly perturbed two point boundary value problems. The convergence analysis is investigated and revealed that the present method is of sixth order convergence. Moreover, the study analysed by taking different mesh size $h$ and sufficiently small perturbation parameter $\varepsilon$. From the results in Tables 1–4, one can see that maximum absolute error decreases as both the perturbation parameter and mesh size decrease, which in turn shows...
the convergence of the computed solution. Furthermore, the result of the present method is compared with the existing findings and observed that it gives more accurate solution than some existing numerical methods reported in the literature. The present method approximated the exact solution very well. Generally, the present method is convergent and more accurate for solving third-order singularly perturbed two point boundary value problems and the scheme can also work properly for the case of variable coefficients but it needs the linearization technique, a supporting technique to be applicable for non-linear problems.

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