HAUSDORFF DIMENSION AND KLEINIAN GROUPS

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ABSTRACT. Let $G$ be a non-elementary, finitely generated Kleinian group, $\Lambda(G)$ its limit set and $\Omega(G) = \mathbb{C} \setminus \Lambda(G)$ its set of discontinuity. Let $\delta(G)$ be the critical exponent for the Poincaré series and let $\Lambda_c$ be the conical limit set of $G$. We prove that

1. $\delta(G) = \dim(\Lambda_c)$.
2. A simply connected component $\Omega$ is either a disk or $\dim(\partial \Omega) > 1$.
3. $\Lambda(G)$ is either totally disconnected, a circle or has dimension $> 1$.
4. $G$ is geometrically infinite iff $\dim(\Lambda) = 2$.
5. If $G_n \to G$ algebraically then $\dim(\Lambda) \leq \lim \inf \dim(\Lambda_n)$.
6. The Minkowski dimension of $\Lambda$ equals the Hausdorff dimension.
7. If $\text{area}(\Lambda) = 0$ then $\delta(G) = \dim(\Lambda(G))$.

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1. Statement of results

Consider a group $G$ of Möbius transformations acting on the two sphere $S^2$. Such a group $G$ also acts on the hyperbolic 3-ball $B$ (with its boundary $S^2$ identified with the Riemann sphere). The limit set, $\Lambda(G)$, is the accumulation set (on $S^2$) of the orbit of the origin. We say the group is discrete if is discrete as a subgroup of $\text{PSL}(2, \mathbb{C})$, (i.e., if the identity element is isolated). The ordinary set of $G$, $\Omega(G)$, is the subset of $S^2$ where $G$ acts discontinuously, e.g., $\Omega(G)$ is the set of points $z$ such that there exists a disk around $z$ which hits itself only finitely often under the action of $G$. If $G$ is discrete, then $\Omega(G) = S^2 \setminus \Lambda(G)$. $G$ is called a Kleinian if it is discrete and $\Omega(G)$ is non-empty. The limit set $\Lambda(G)$ has either 0, 1, 2 or infinitely many points and $G$ is called elementary if $\Lambda(G)$ is finite.

If we form the Poincaré series for a discrete $G$ on $\mathbb{B}^3$,

$$\sum_{G} \exp(-s\rho(0, g(0))),$$

($\rho$ is the hyperbolic metric in $\mathbb{B}^3$) the series converges for all $s > s_0$ and the minimal such $s_0$ is called the Poincaré exponent of the group and is denoted $\delta(G)$. A point $x \in \Lambda(G)$ is called a conical limit point if there is a sequence of orbit points which converges to $x$ inside a (Euclidean) non-tangential cone with vertex at $x$ (such points are also called radial limit points or points of approximation). The set of such points is denoted $\Lambda_c(G)$. $G$ is called geometrically finite if there is a finite sided fundamental polyhedron for the $G$ action on $\mathbb{B}$. However, the only property we shall use is a result of Beardon and Maskit. 

[7]
that $G$ is geometrically finite iff $\Lambda(G)$ is the union of $\Lambda_c(G)$, the rank 2 parabolic cusps and doubly cusped rank 1 parabolic fixed points of $G$. This makes it clear that $\dim(\Lambda_c) = \dim(\Lambda)$ and $\text{area}(\Lambda) = 0$ in the geometrically finite case. (For our purposes we could take the Beardon-Maskit characterization to be the definition of geometrically finite.) By Selberg’s lemma any finitely generated discrete group contains a finite index subgroup without torsion. This subgroup must have the same limit set as the original group, so for our purposes it will always be sufficient to assume $G$ has no torsion.

In this paper “circle” will always refer to Euclidean circles or lines (e.g., circles on the sphere) as opposed to topological circles which we will always call closed Jordan curves. Similarly the terms “disk” or “ball” will always denote Euclidean balls.

The principle objective of this paper is to establish the following results.

**Theorem 1.1.** If $G$ is a non-elementary, discrete Möbius group on $\mathbb{B}$ then $\delta(G) = \dim(\Lambda_c(G))$.

**Theorem 1.2.** Suppose $G$ is a finitely generated Kleinian group and $\Omega$ is a simply connected, invariant component of $\Omega(G)$. Then $\dim(\partial \Omega) = 1$ iff $\partial \Omega$ is a circle.

**Theorem 1.3.** If $G$ is a finitely generated Kleinian group then its limit set is either totally disconnected, a circle or has Hausdorff dimension $> 1$. 
Theorem 1.4. If $G$ is a finitely generated Kleinian group which is geometrically infinite then $\dim(\Lambda(G)) = 2$.

Theorem 1.5. If $\{G_n\}$ is a sequence of $N$-generated Kleinian groups which converges algebraically to $G$ then $\dim(\Lambda(G)) \leq \lim \inf_n \dim(\Lambda(G_n))$.

Theorem 1.6. If $G$ is a finitely generated Kleinian group then the Minkowski dimension of $\Lambda$ exists and equals the Hausdorff dimension.

Theorem 1.7. Suppose that $G$ is a non-elementary, finitely generated Kleinian group and that $\text{area}(\Lambda(G)) = 0$. Then $\delta(G) = \dim(\Lambda)$.

These results are arranged roughly in order of dependence and the amount of required prerequisites from the theory of Kleinian groups and hyperbolic manifolds. Theorem 1.1 uses nothing but the definitions and a few simple properties of Möbius transformations. We have only been able to locate this result in the literature under the additional assumptions that $G$ is geometrically finite or Fuchsian (e.g., see [53], [56]). The direction $\dim(\Lambda_c(G)) \leq \delta(G)$ is easy and well known and the opposite inequality is fairly simply (and known) when the Poincaré series diverges at the critical exponent. We should also note that Theorem 1.1 holds for a discrete group of Möbius transformations acting on the hyperbolic ball in any dimension as well as the rank 1 case in general.

It is known that the conical limit set corresponds to the geodesics starting at $0 \in \mathbb{B}$ which return to some compact subset of $M$ infinitely
often. Our proof shows that for any \( \epsilon > 0 \) there is a subset of \( \Lambda_c(G) \) of dimension \( \geq \delta(G) - \epsilon \) which corresponds to geodesics which never leave some compact subset of \( M \). Thus our proof shows that if \( M \) is a hyperbolic manifold with finitely generated fundamental group then for any \( x \in M \), the set of directions corresponding to geodesics rays starting at \( x \) which have compact closure has dimension \( \delta(G) \).

Theorem 1.2 uses nothing about Kleinian groups except the definitions but does make use of some nontrivial results from the theory of conformal mappings and rectifiable sets. We shall actually give a long list of conditions which are equivalent to \( \Omega \) not being a disk (see the end of Section 3). Among them:

1. \( \delta(G) > 1 \).
2. \( \partial\Omega \) has tangents almost nowhere (with respect to harmonic measure).
3. \( \partial\Omega \) fails to have a tangent somewhere.

The elementary groups have to be excluded in Theorem 1.2 and Theorem 1.7 because a cyclic group consisting of parabolics has a one point limit set, but \( \delta(G) = 1/2 \).

Theorem 1.3 is essentially a corollary of Theorem 1.2 but requires the Ahlfors finiteness theorem and the Klein-Maskit combination theorems to reduce to the case considered in Theorem 1.2. The result was first formulated by Bowen in [19] in the case of quasi-Fuchsian groups with no parabolics. The convex, co-compact Kleinian case is proven in [53] and [20]. See also [55]. The general geometrically finite case is proven
in [26]. In this paper we complete the discussion by including the geometrically infinite groups.

We should also note that Theorem 1.2 and Theorem 1.3 could be deduced from Theorem 1.4 and the known results for geometrically finite groups. However, our proofs of these results seem new even in the geometrically finite case, so we have included them in this paper.

Theorem 1.4 is the first place we have to invoke some nontrivial results about 3-manifolds. We need a result of Sullivan [57] that \( \lambda_0 = \delta(G)(2 - \delta(G)) \) where \( \lambda_0 \) is the lowest eigenvalue for the Laplacian on \( M = \mathbb{B}/G \). We also need exponential decay (in time) for the heat kernel of a manifold with first eigenvalue bounded away from zero.

Theorem 1.4 was previously known in special cases. Examples of groups with \( \dim(\Lambda(G)) = 2 \) were constructed by Sullivan in [54], and Canary [25] proved Theorem 1.4 holds if \( M = \mathbb{B}/G \) is a “topologically tame” manifold and such that the thin parts have bounded type (in particular, if its injectivity radius is bounded away from zero). Our result shows these hypotheses are unnecessary.

Sullivan [56] and Tukia [60] independently showed that if \( G \) is a geometrically finite group then \( \dim(\Lambda(G)) < 2 \). Thus Theorem 1.4 implies that a finitely generated group is geometrically finite iff \( \dim(\Lambda(G)) < 2 \).

If \( M = \mathbb{B}/G \), the convex core of \( M \) is defined to be \( C(M) = C(\Lambda)/G \), where \( C(\Lambda) \) is the convex hull of the limit set in \( \mathbb{B} \). Burger and Canary prove in [22] that if \( G \) is a geometrically finite, \( M = \mathbb{B}/G \) has infinite
volume and $\delta(G) > 1$, then
\[
2 - \frac{2 \text{area}(\partial C(M))}{\text{vol}(C(M))} \leq \dim(\Lambda(G)) \leq 2 - \frac{K}{\text{vol}(C_1(M))},
\]
where $C_1(M)$ is a unit neighborhood of $C(M)$. Theorem 1.4 implies their result still holds in the geometrically infinite case (finitely generated groups are geometrically infinite iff $\text{vol}(C(M)) = \infty$; see Section 11).

To prove Theorem 1.5 we use our earlier results and an estimate of Canary which bounds the first eigenvalue of $M$ in terms of the volume of the convex core $C(M)$. We also need the use the Margulis lemma. These results are only needed in the case when $\Lambda(G)$ has positive area; otherwise Theorem 1.5 follows immediately from earlier results. We will also see that the assumption that $G$ and the $\{G_n\}$ are Kleinian is unnecessary; the result holds for any finitely generated Möbius groups.

Theorem 1.6 and Theorem 1.7 are both corollaries of a result relating $\delta(G)$ to Minkowski (or “box counting”) dimension which we will state in Section 15. This result uses only Theorem 1.1, the Ahlfors finiteness theorem and some geometric arguments. Theorem 1.6 was proven in the geometrically finite case by Stratmann and Urbanski 52, and our proof is quite simple in this case. The geometrically infinite case follows immediately from Theorem 1.4, although there is also a proof which does not require Theorem 1.4.

The Ahlfors conjecture states that the limit set of a finitely generated discrete group of Möbius transformations is either the whole sphere or has zero area. We do not address the Ahlfors conjecture in this paper,
but if it is true several of the arguments given here would simplify. As part of the proof of Theorem 1.4 we show that a finitely generated, geometrically infinite group with $\delta(G) < 2$ must have a limit set with positive area. Thus the Ahlfors conjecture implies $\delta(G) = 2$ for any geometrically infinite group. We do not know an argument for the converse direction, but both results are currently known to be true for topologically tame groups, [25].

In most of our results the case when $G$ has no rank 1 parabolics is easier because then the action of $G$ on $\Omega(G)$ has a compact fundamental polygon. When the parabolics introduce extra difficulties we usually give the proof first in the compact case and then go on to the general case. We will introduce the notion of “good” and “bad” horoballs. The good horoballs can be treated very similarly to the compact case but the bad horoballs require a little extra work (and can only occur in the geometrically infinite case). The rank 2 parabolics cause no difficulties since they do not correspond to cusps on $\Omega(G)/G$.

It might be worth noting that although the theory of hyperbolic 3-manifolds is intricate and highly developed, we we only need a few simple facts which we prove here. The only results we really use without proof are the Margulis lemma, Davies’ Gaussian upper bound on the heat kernel and Sullivan’s theorem relating the lowest eigenvalue for the Laplacian on $M = \mathbb{H}/G$ to the Poincaré exponent $\delta(G)$.

The remaining sections of this paper are organized as follows:
Section 2: We prove Theorem 1.1 and deduce from the proof that $\delta(G)$ is lower semi-continuous with respect to algebraic convergence.

Section 3: This is first of several sections devoted to Theorem 1.2 and Theorem 1.3. We define the "$\beta$"'s which measure the distance of a set to a line and we prove that a uniformly wiggley set (i.e., a set where the $\beta$'s are bounded away from zero) has dimension larger than one.

Section 4: We recall that large Schwarzian for a conformal map implies large $\beta$'s for the image domain. This implies that if $G$ has no parabolics then $\Lambda(G)$ is either a circle or is uniformly wiggley. Thus we obtain the compact case of Theorem 1.2 as a corollary.

Section 5: We prove that if a simply connected invariant component is not a disk then the boundary has tangents almost nowhere. Here we need some known results about conformal mappings and Schwarzian derivatives. We also prove a result on the length of level lines that is needed in the next section.

Section 6: We prove Theorem 1.2 with parabolics.

Section 7: We show that if $\Omega(G)$ is a union of infinitely many disks, then $\dim(\Lambda(G)) > 1$. This is easy and previously known.

Section 8: We show that the limit sets of degenerate groups (groups where $\Omega(G)$ is connected and simply connected) are uniformly wiggley. This gives an alternate proof of Theorem 1.2 in this case.
Section 9: We prove Theorem 1.3 and deduce a variety of corollaries.

Section 10: We prove $\Lambda(G)$ is uniformly perfect and introduce the idea of “good” and “bad” horoballs and prove some simple facts about them.

Section 11: Here we gather together the facts about the convex hull of a hyperbolic 3-manifold that we will need in later sections.

Section 12: We prove Theorem 1.4.

Section 13: We prove Theorem 1.5.

Section 14: We introduce Minkowski and upper Minkowski dimension and relate it to the Besicovitch-Taylor index.

Section 15: We prove Theorem 1.6 and Theorem 1.7 by proving the following result: if $G$ is finitely generated and $\Lambda(G)$ has zero area then $\delta(G)$ equals the upper Minkowski dimension of $\Lambda(G)$.

Section 16: We deduce some corollaries of our results in the special case when the groups belong to $\overline{T(S)}$, the closure of the Teichmüller space of a finite type surface $S$. For example, $\dim(\Lambda(G))$ is a lower semi-continuous function on $\overline{T(S)}$ and is continuous everywhere except at the geometrically finite cusps, where it is discontinuous.

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and for suggesting the problem of proving the Hausdorff dimension
of degenerate limit sets is strictly greater than 1. It was by considering
this problem that we were led to the other results in this paper.

2. The Conical Limit Set

First we recall the definition of Hausdorff dimension. Given an in-
creasing function $\varphi$ on $[0, \infty)$, we define

$$H^\delta_\varphi(E) = \inf \{ \sum \varphi(r_j) : E \subset \cup_j D(x_j, r_j), r_j \leq \delta \},$$

and

$$H_\varphi(E) = \lim_{\delta \to 0} H^\delta_\varphi(E).$$

This is the Hausdorff measure associated to $\varphi$. $H^\infty_\varphi$ is called the Haus-
dorff content. It is not a measure, but has the same null sets as $H^\alpha$.

When $\varphi(t) = t^\alpha$ we denote the measure $H_\varphi$ by $H_\alpha$ and we define

$$\dim(E) = \inf \{ \alpha : H_\alpha(E) = 0 \}.$$

For $\alpha = 1$ we sometimes denote $H_1$ by $\ell$ (for “length”). An upper bound
for $\dim(E)$ can usually be produced by finding appropriate coverings
of the set. We will be more interested in finding lower bounds. The
usual idea is to construct a positive measure $\mu$ on $E$ which satisfies
$\mu(D(x, r)) \leq Cr^\alpha$. This implies $\dim(E) \geq \alpha$ since for any covering of
$E$ we have

$$\sum_j r_j^\alpha \geq C^{-1} \mu(D(x_j, r_j)) \geq C^{-1} \mu(E) > 0.$$
Next we recall the definition of the critical exponent of Poincaré series of $G$,
\[
\delta(G) = \inf\{\alpha : \sum_{g \in G} (1 - |g(0)|)^\alpha < \infty\}.
\]

It is easy and well known that
\[
\dim(\Lambda_c) \leq \delta(G),
\]
so we will only prove the other direction. If the Poincaré series diverges at the critical exponent $\delta(G)$, then the Patterson measure ([17], [18], [19], [50]) lives on the conical limit set $\Lambda_c(G)$ and satisfies the estimate
\[
\mu(B(x, r)) \leq C r^{\delta(G)}.
\]

So the theorem is known in this case and we must only treat the case where the Poincaré series converges at the critical exponent. In this case the Patterson measure is supported on $\Lambda(G) \setminus \Lambda_c(G)$. Our proof does not use Patterson measure or any property of divergence or convergence at the critical index. It does construct a measure on the conical limit set which mimics the behavior of the Patterson measure.

In this section we will prove,

**Theorem 2.1.** Suppose $G$ is a group of Möbius transformations with more than two limit points. Then
\[
\delta(G) \leq \dim(\Lambda_c).
\]

**Proof.** Let $\{z_n\}$ denote the orbit of 0 under $G$ in the hyperbolic 3-ball, $\mathbb{B}$. Let $\delta = \delta(G)$ be the critical exponent for the Poincaré series of $G$
and let $\epsilon > 0$. Choose a point $x \in \overline{\mathcal{C}} = \partial \mathbb{B}$ so that
\[ \sum_{j : |z_j - x| < r} (1 - |z_j|)^{\delta - \epsilon} = \infty, \]
for every $r > 0$ (here $|z - x|$ denotes the spherical metric). We can do this by a simple compactness argument. Since $G$ is non-elementary, $x$ is not fixed by every element of $G$. Therefore we can choose elements \{g_1, \ldots, g_4\} \in G$ so that $x_i = g_i(x)$ are all distinct. Fix $r > 0$ to be so small that the balls $B_i$ on $\overline{\mathcal{C}}$ (in the spherical metric) of radius $2r$ around the points \{x_1, \ldots x_4\} are pairwise disjoint.

Suppose $M, N$ are large numbers (to be chosen below depending only on $G$ and $r$). Let $A_n = \{z \in \mathbb{B} : 2^{-n-1} \leq 1 - |z| < 2^{-n}\}$. If it were true that
\[ \sum_{j : z_j \in B_i \cap A_n} (1 - |z_j|)^{\delta - 2\epsilon} \leq M, \]
for all large enough $n$, then
\[ \sum_{j : z_j \in B_i} (1 - |z_j|)^{\delta - \epsilon} \leq C \sum_n 2^{-n\epsilon} \sum_{j : z_j \in B_i \cap A_n} (1 - |z_j|)^{\delta - 2\epsilon} \]
\[ \leq C M \sum_n 2^{-n\epsilon} \]
\[ < \infty. \]
This is a contradiction, so we must have
\[ \sum_{j : z_j \in B_i \cap A_n} (1 - |z_j|)^{\delta - 2\epsilon} \geq M, \]
for infinitely many values of $n$. For each $i = 1, 2, 3, 4$, fix a value of $n_i$ (depending on $M$ and hence of $r$) for which this inequality holds.

Since the $z_j$’s make up the orbit of a single point, they are uniformly separated in the hyperbolic metric of $\mathbb{B}$. Thus for any $A < \infty$ we may split the sequence into a finite number $B$ of sequences (depending on
A) each of which is separated by at least \( A \) in the hyperbolic metric. Therefore, to each point \( x_i \) we may associate a collection of points \( \mathcal{G}_i(t) \subset \{ x_i \} \) such that

\[
\mathcal{G}_i(t) \subset B_{i} \cap A_{i},
\]

where

\[
z, w \in \mathcal{G}_i(t) \implies |z - w| \geq \exists \mathcal{N} \in \mathbb{R},
\]

and

\[
\sum_{j: z_j \in \mathcal{G}_i(t)} (1 - |z_j|)^{\delta - 2\epsilon} \geq M/B.
\]

For each \( z \in \mathcal{G}_i(t) \) let \( z^* \) denote its radial projection onto the sphere \( \mathbb{C} = \partial \mathbb{B} \). For \( z \in \mathbb{B} \), let

\[
B(z_j) = B(z_j^*, N(1 - |z|)).
\]

By hypothesis, the balls \( B(z_j) \) are disjoint for all \( z_j \in \bigcup_{i=1}^{4} \mathcal{G}_i(t) \).

Since the balls \( \{ B_1, \ldots, B_4 \} \) have disjoint doubles, any sufficiently small disk (depending on \( r \)) can intersect at most one of the balls. For any point \( z = g(0) \) in the orbit of 0 consider the four balls \( \{ g(B_1), \ldots, g(B_4) \} \). The preceding statement implies that if \( N \) is sufficiently large (depending only on \( r \)) then at most one of these balls can intersect \( \mathbb{B} \setminus B(z) \).

This determines our choice of \( N \). Therefore, at least three of the balls are contained in \( B(z) \). Without loss of generality, assume they are \( g(B_1), g(B_2) \) and \( g(B_3) \).

The Möbius transformation \( g \) has bounded distortion as a map from \( S^2 \) to itself except possibly at one point. More precisely,

**Lemma 2.2.** Suppose \( r > 0 \). There is a \( C < \infty \) (depending only on \( r \)) such that given any Möbius transformation \( g \) of \( S^2 = \mathbb{C} \) to itself we
have

\[ C^{-1}(1 - |g(0)|) \leq |g'| \leq C(1 - |g(0)|), \]

except possibly on a disk \( D \) of radius \( r \) (both the derivatives and the disk are taken with respect to the spherical metric).

**Proof.** We may assume \( g(0) \neq 0 \) since otherwise the lemma is easy. Let \( z \) be the radial projection of \( g(0) \) onto \( S^2 \) and choose \( R \) so big (depending only on \( r \)) so that

\[ \omega(g(0), S^2 \setminus B(z, R(1 - |g(0)|)), B) \leq r^2. \]

Then \( D = g^{-1}(B(z, R(1 - |g(0)|))) \) is a disk of radius less than \( r \) and \( |g'| \) is comparable to \( 1 - |g(0)| \) on its complement. \( \square \)

So for any \( g \in G \), at least two of the disks (say \( B_1 \) and \( B_2 \)) are bounded away from this point so we get

\[ C^{-1}(1 - |g(0)|) \leq |g'| \leq C(1 - |g(0)|), \]

on \( B_1 \) and \( B_2 \) with constants depending only on \( r \).

Note that if we choose \( n_i \) large enough (depending only on \( N \)) we may assume

\[ \frac{1 - |z|}{2N} \geq 1 - |z_j| \geq \frac{1 - |z|}{CN}, \]

for some uniform \( C \) depending only on \( G \) and \( r \).
Now for the orbit point \( z = g(0) \) define \( \mathcal{G}(, ) = \{ \mathcal{G}_\infty(, ) \}. \) Thus

\[
\sum_{z \in \mathcal{G}(, )} (1 - |z|)^{\delta - 2\epsilon} \geq C^{-\delta + 2\epsilon}(1 - |z|)^{\delta - 2\epsilon} \sum_{z_k \in \mathcal{V}_\infty(, )} (1 - |z_k|)^{\delta - 2\epsilon}
\]

\[
\geq C^{-\delta + 2\epsilon}(1 - |z|)^{\delta - 2\epsilon} \frac{M}{B}
\]

\[
\geq C^{-\frac{2M}{B}}(1 - |z|)^{\delta - 2\epsilon}
\]

\[
\geq (1 - |z|)^{\delta - 2\epsilon},
\]

where the last line holds if \( M \) is large enough. Since \( C_1 \) depends only on \( r \) and \( B \) depends only on group \( G \) (more precisely it only depends on the injectivity radius of \( G \) at 0), this determines our choice of \( M \).

We have now constructed a set of points \( z_j = \mathcal{G}(, ) \) which satisfy the following conditions:

\[
z_j \in B(z^{*}, N(1 - |z|)),
\]

\[
B(z_j^{*}, 2N(1 - |z_j|)) \cap B(z_k^{*}, 2N(1 - |z_k|)) = \emptyset \text{ for } j \neq k,
\]

\[
\sum_j (1 - |z_j|)^{\delta - 2\epsilon} \geq (1 - |z|)^{\delta - 2\epsilon},
\]

\[
\frac{1 - |z|}{2N} \geq 1 - |z_j| \geq \frac{1 - |z|}{CN},
\]

for some uniform \( C \) depending only on \( G \) and \( r \) (because the points in \( G_1(0) \) do and the map \( g \) has uniformly bounded distortion on \( G_1(0) \)). It is now a standard argument to show that \( \dim(\Lambda_c) \geq \delta - 2\epsilon \). Briefly it goes as follows.

Construct generations of points starting with \( \mathcal{G}_0 = \{ \} \), and for each \( z \in \mathcal{G}_0 \), define points \( \{ z_j \} \) in \( \mathcal{G}_1(0) \) as above. To each point \( z \in \mathcal{G} = \cup \mathcal{G}_1(0) \), associate the disk \( B_z = B(z^{*}, 2N(1 - |z|)) \). Then let

\[
E_n = \cup_{z \in \mathcal{G}_1(0)} B_z,
\]
It is easy to see that \( E \subset \Lambda_c(G) \).

Define a probability measure \( \mu \) on \( E \) by setting \( \mu(E_0) = 1 \), and for \( z \in G\setminus \text{ with "parent" } z' \in G_{\infty} \), set

\[
\mu(B_z) = \frac{(1 - |z|)^{\delta-2\epsilon}}{\sum_{w \in G(z')}(1 - |w|)^{\delta-2\epsilon}} \mu(B_{z'}). 
\]

It is easy to see by induction that

\[
\mu(B_z) \leq (1 - |z|)^{\delta-2\epsilon} \leq C \text{diam}(B_z)^{\delta-2\epsilon},
\]

for each \( z \in G \). We want to show this inequality is true for any disk \( D \). Let \( D \) be any disk and let \( D_0 = B_z \) be the lowest generation disk in our construction so that \( D_0 \cap D \neq \emptyset \) but \( D_0 \subset 2D \). Let \( D_1 \) be the parent of \( D_0 \). By the maximality of \( D_0 \) we have \( D \subset 2D_1 \). Since \( 2D_1 \) is disjoint from any other balls of the same generation,

\[
\mu(D) \leq \mu(D_1) \leq C \text{diam}(D_1)^{\delta-2\epsilon} \leq C(NC)^{\delta-2\epsilon} \text{diam}(D)^{\delta-2\epsilon} 
\]

This is the desired inequality (the constant in front is larger, but is uniform over all disks; the power is the same).

Now we simply note that if \( \{D_j\} \) is any covering of \( E \) by disks, then

\[
0 < \mu(E) \geq \sum_j \mu(D_j) \leq C \sum_j \text{diam}(D_j)^{\delta-2\epsilon}.
\]

By the definition of Hausdorff dimension, this implies

\[
\dim(\Lambda_c(G)) \geq \dim(E) \geq \delta - 2\epsilon.
\]
Since $\epsilon$ was arbitrary, we get Theorem 2.1.

The proof of Theorem 2.1 has the following corollaries.

**Corollary 2.3.** Suppose $r > 0$ and let $C = C(r)$ and $N = N(r)$ be the constants in Lemma 2.2 and the proof above. Suppose $G$ is a group of Möbius transformations on $\mathbb{B}$ and suppose there are three disjoint balls $B_1, B_2, B_3$ of (spherical) radius $r$ and a collection of points $F \subset G(0) \cap A_n$ which satisfy

$$z, w \in F \text{ implies } |z - w| \geq N2^{-n},$$

$$\sum_{z \in F \cap B_i} (1 - |z|)^\alpha \geq C^{-2}.$$  

Then

$$\delta(G) \geq \alpha.$$ 

Once we have the conditions in the hypothesis, the proof of Theorem 2.1 proves the lemma. If $G$ is a group then it satisfies these conditions for every $\alpha < \delta(G)$. If $\tilde{G}$ is another group which is very close to $G$ (say $G$ and $\tilde{G}$ have generators which are very close in PSL(2, $\mathbb{C}$)) then $\tilde{G}$ will also satisfy these conditions (since they only involve a finite number of elements in the group). Thus

**Corollary 2.4.** Suppose $G$ is a Möbius group generated by $\{g_1, \ldots, g_n\}$. Given any $\delta_0 > 0$ there is an $\epsilon_0 > 0$ (depending only on $\delta$ and $G$) such that if $\tilde{G}$ is a group containing elements $\{\tilde{g}_1, \ldots, \tilde{g}_n\}$ with $\|g_i - \tilde{g}_i\| < \epsilon$ (as elements of PSL(2, $\mathbb{C}$)) then

$$\delta(G') \geq \delta(G) - \delta_0.$$
Suppose \( \{G_n\} \) is a sequence of \( m \)-generated M"obius groups each with a specific listing of its generators \( G_n = \{g_{1n}, \ldots, g_{mn}\} \). We say that \( G_n \) converges \emph{algebraically} to a Kleinian group \( G \) with generators \( \{g_1, \ldots, g_m\} \) if \( g_{jn} \to g_j \) for each \( 1 \leq j \leq m \), as elements of \( \text{PSL}(2, \mathbb{C}) \). See [38]. If we identify groups with points in \( \text{PSL}(2, \mathbb{C})^m \), this is just convergence in the product topology.

**Corollary 2.5.** If \( \{G_n\} \) is a sequence of discrete M"obius groups converging algebraically to \( G \), then

\[
\liminf_{n \to \infty} \delta(G_n) \geq \delta(G).
\]

This says that \( \delta(G) \) is lower semi-continuous with respect to algebraic convergence. Strict inequality is possible even for sequences of Kleinian groups (e.g., one can choose a sequence \( \{G_n\} \) of geometrically finite groups in \( T(S) \) converging to a a geometrically finite cusp group \( G \) so that \( \delta(G_n) \to 2 \) but \( \delta(G) < 2 \)).

We should also note that Corollary 2.5 is still true even if the groups involved are not discrete (assuming that we define \( \delta(G) \) appropriately). If \( G \) is a finitely generated M"obius group which is not discrete, then its closure is a closed subgroup of \( \text{PSL}(2, \mathbb{C}) \) and hence a Lie subgroup. The only possible limit sets (i.e., accumulation sets on \( S^2 \) for the orbit of 0) are: zero, one or two points, a circle or the whole sphere. Let us set \( \delta(G) \) to be 0, 1 or 2 in these three cases respectively. In each case, it is easy to check that Corollary 2.5 still holds if we allow either the \( \{G_n\} \) or \( G \) to be non-discrete groups.
3. Big $\beta$’s imply big dimension

Although Kleinian groups are closely associated with 3-manifolds, our approach to Theorem 1.2 is completely two dimensional, exploiting connections between conformal mappings, the Schwarzian derivative and certain geometric square functions which first arose in the study of the Cauchy integral. A dyadic square $Q$ in the plane is one of the form $Q = [2^{-n}j, 2^{-n}(j + 1)] \times [2^{-n}k, 2^{-n}(k + 1)]$. The side length of a square will be denoted $\ell(Q)$. For a positive number $\lambda > 0$, we let $\lambda Q$ denote the square concentric with $Q$ but with side length $\lambda \ell(Q)$, e.g., $2Q$ is the “double” of $Q$.

Given a set $E$ in the plane and a square $Q$ we define $\beta(Q)$ and $\delta(Q)$ as

$$\beta(Q) = t^{-1} \inf_{L \in \mathcal{L}} \sup_{z \in E \cap 3Q} \text{dist}(z, L),$$

where $\mathcal{L}$ is the set of all lines $L$ intersecting $Q$ and

$$\delta(Q) = \ell(Q)^{-1} \inf_{L \in \mathcal{L}} H(E \cap 3Q, L \cap 3Q),$$

where $H(E, F)$ denotes the Hausdorff distance between the sets,

$$H(E, F) = \max_{x \in E} \text{dist}(x, F) + \max_{y \in F} \text{dist}(y, E).$$

The second author proved in [36] that a planar set $E$ lies on a rectifiable curve iff

$$\sum_{Q} \beta(Q)^2 \ell(Q) < \infty,$$

where the sum is over all dyadic squares in the plane. We shall show
Theorem 3.1. Suppose $E$ is a closed, connected set in the plane and $\beta_E(Q) \geq \beta_0 > 0$ for every square $Q$ with $Q \cap E \neq \emptyset$ and $\ell(Q) \leq \text{diam}(E)$. Then $\text{dim}(E) \geq 1 + C\beta_0^2$, where $C$ is an absolute constant.

This is sharp (except for the choice of $C$) as can be seen by considering the standard iterative construction of a snowflake curve where each segment of length $r$ is divided into four equal segments and the middle two are replaced by segments of length $r(\frac{1}{4} + \beta^2)$.

We will see later that if $G$ has no parabolics then $\Lambda(G)$ has this property (i.e., is uniformly wiggley’’). However, a general limit set need not have $\beta$’s bounded away from zero everywhere (this fails near doubly cusped parabolic points), but they are large often enough to push the dimension above 1.

Proof of Theorem 3.1. Suppose $E_0$ is a connected compact set with large $\beta$’s, $Q$ is a square of side length $\ell(Q) = r = 2^{-N}$ with $\frac{1}{3}Q \cap E \neq \emptyset$ (here and later $\lambda Q$ denotes the square concentric with $Q$ of side length $\lambda \ell(Q)$). Our first objective is to show that for small $\epsilon > 0$ we can find more than $1000\epsilon^{-1}$ subsquares of $Q$ of sidelength $\epsilon \ell(Q)$ with disjoint doubles and such that each contains a point of $E_0$ in its middle third. We will then apply the same argument to each subsquare and use the resulting nested collection to squares to build a Frostman measure on $E_0$.

Define $E = (E_0 \cap Q) \cup \partial Q$. Note that $E$ is connected. Fix some integer $n$ so that $2^{-n} < r$ (possibly much smaller). Because $E$ is connected it is easy to check that there are more than $\frac{1}{3}r2^n$ dyadic subsquares
{Q_j} of Q which lie in \( \frac{2}{3} Q \) and such that \( Q_j \cap K \neq \emptyset \) (e.g., consider concentric “annuli” between \( \frac{1}{3} Q \) and \( \frac{2}{3} Q \) made of squares of size \( 2^{-n}r \); there are \( \frac{1}{3} r 2^n \) such and each must intersect \( E \)).

From this we deduce

\[
\sum_{Q: \ell(Q) = 2^{-n}} \beta^2_E(Q) \ell(Q) \geq \frac{1}{3} \beta^2_0 r.
\]

Thus for any integer \( k \geq 1 \) (recall \( r = 2^{-N} \)),

\[
\sum_{n=N+1}^{N+k} \sum_{Q: \ell(Q) = 2^{-n}} \beta^2_E(Q) \ell(Q) \geq \frac{1}{3} k \beta^2_0 r.
\]

Suppose \( \Gamma_k \) is the shortest curve in the plane with the property that for each \( z \in E \) we have \( \text{dist}(z, \Gamma_k) \leq 2^{-N-k} \). It is fairly easy to check that for squares with \( \ell(Q) \geq 10 \cdot 2^{-N-k} \beta^{-1}_0 \), we have

\[
\beta_{\Gamma_k}(Q) \geq \frac{1}{2} \beta_E(Q) \geq \frac{1}{2} \beta_0.
\]

Therefore

\[
\sum_{n=N+1}^{N+k} \sum_{Q: \ell(Q) = 2^{-n}} \beta^2_{\Gamma_k}(Q) \ell(Q) \geq (k - 10^{-1}) \beta^2_0 r.
\]

Choose \( k > 20 \beta^{-1}_0 \). Then the term on the right is \( \geq \frac{1}{6} k \beta^2_0 r \).

By the second author’s characterization of rectifiable curves in [36], the length of \( \Gamma_k \) is at least

\[
\ell(\Gamma_k) \geq C_0 \beta^2_0 kr,
\]

for some absolute constant \( C_0 \). We claim that this implies that there are more than \( (C_0 \beta^2_0 kr - 4r) C 2^{N+k} \) boxes \( \{Q_j\} \) of side length \( 2^{-N-k} \) such that \( \frac{1}{3} Q_j \cap E \neq \emptyset \).

To prove this claim let \( \{z_j\} \) be a collection of points on \( \Gamma_k \) so that \( j \neq k \) implies \( |z_j - z_k| \geq 2^{-N-k} \), but so that \( \bigcup_j B(z_j, 2^{-N-k+2}) \) covers
Let \( C \) be the collection \( \{ \partial Q_k \} \) of all dyadic squares of size \( 2^{-N-k} \) contained in \( Q \) which contain some \( z_j \). Let \( \Gamma = \bigcup_{Q_j \in C} \partial Q_j \cup \bigcup_j S_j \) be the union of the boundaries of these cubes, together with segments \( S_j \) which connect \( \partial Q_j \) with the point \( z_j \). Then obviously
\[
\ell(\Gamma) \leq 6 \cdot 2^{-N-k} |C|.
\]

Since \( \Gamma \) has the property that it passes within \( 2^{-N-k} \) of every point of \( E \) and since \( \Gamma_k \) was defined to be the shortest such curve, we must have
\[
6 \cdot 2^{-N-k} |C| \geq \ell(\Gamma) \geq \ell(\Gamma) \geq C_\beta \|e\|^{-N}.
\]
and hence
\[
|C| \geq \frac{\infty}{C_\beta \|e\|^{-N}}.
\]

So if \( k \) is large enough (e.g., \( k > 60000\beta_0^{-2}C_0^{-1} \)) Now set \( \epsilon = 2^{-k} \). Then \( |C| \geq \infty \epsilon^{-\infty} \), and consist of disjoint dyadic squares of size \( \epsilon \ell(Q) \) each of which contains a point of \( E \). By throwing away 9/10’s of the squares we can assume the remaining ones have disjoint triples, and this is what we wished to prove.

To finish the proof of the theorem, we build nested generations of squares using the construction above. The initial square \( Q_0 \) forms the first generation. The squares of size \( \epsilon \ell(Q_0) \) constructed above form the first generation. In general, given a \( n \)th generation square containing a point of \( E_0 \) in its middle third, we construct 1000\( \epsilon^{-1} \) subsquares as above (with disjoint triples and containing a point of \( E_0 \) in their middle thirds), and put these into the \((n+1)\)st generation.
We then define a measure $\mu$ by assigning each $n$th generation square equal mass (namely $(\epsilon/1000)^n$). Since a $n$th generation square has size $\epsilon^n$, this measure satisfies

$$\mu(Q) \leq C\ell(Q)^\alpha,$$

where

$$\alpha = \frac{\log \epsilon - \log 1000}{\log \epsilon} = 1 + \frac{\log 1000}{\log \frac{1}{\epsilon}} > 1,$$

and $Q$ is an $n$th generation square. Since $\epsilon = 2^{-k}$ and $k \sim \beta_0^{-2}$ we get $\log \epsilon^{-1} \sim \beta_0^{-2}$. This gives the estimate in the theorem. It only remains to check that this inequality holds for all squares in the plane, but this is a standard argument. Thus

$$\sum \text{diam}(D_j)^\alpha \geq \sum \mu(D_j) \geq \mu(E_0) > 0$$

for any covering $\{D_j\}$ of $E_0$, and therefore $\dim(E_0) \geq \alpha > 1$. \qed

4. Large Schwarzian implies large $\beta$’s

The Schwarzian derivative of a locally univalent function $F$ is defined by

$$S(F)(z) = \left[\frac{F''(z)}{F'(z)}\right]' - \frac{1}{2}\left[\frac{F''(z)}{F'(z)}\right]^2$$

$$= \left[\frac{F'''(z)}{F'(z)}\right] - \frac{3}{2}\left[\frac{F''(z)}{F'(z)}\right]^2.$$

If we write $F' = e^\varphi$ then it can be rewritten as

$$S(F)(z) = \varphi'' - \frac{1}{2}(\varphi')^2.$$

Recall that $S(F) \equiv 0$ iff $F$ is a Möbius transformation and that $S$ satisfies the composition law

$$S(F \circ G) = S(F)(G')^2 + S(G).$$
In particular, if $G$ is Möbius then
\[
S(F \circ G) = S(F)(G')^2
\]
\[
S(G \circ F) = S(F).
\]
In addition, given an $\epsilon > 0$, hyperbolic disk $D$ and a compact neighborhood $K$ of $D$, there is a $\delta > 0$ so that $|S(F)| \leq \delta$ on $D$ implies $F$ uniformly approximates a Möbius transformation on $K$ to within $\epsilon$.

We introduce some notation from the second author’s paper [37].

For a domain $\Omega$ and a point $z_0 \in \Omega$ let $z_1 \in \partial \Omega$ be the closest point in the Euclidean metric and let $\theta_0 = \arg(z_0 - z)$. Let $\tilde{z}_1 = z_1 + \delta(z_0, z_1)$ were $0 < \delta < 1$ is fixed. Define
\[
L_{z_0}^\delta = \{ z_1 + t \exp(i(\theta_0 - \frac{\pi}{2})) : t \in \mathbb{R}, |t| \leq \delta^{-1}|z_1 - z_0| \},
\]
\[
S_{z_0}^\delta = \{ z : |z - z_1| \leq \delta^{-1}|z_1 - z_0|, |\theta_0 - \arg(z - \tilde{z}_1)| \leq \frac{\pi}{2} \}.
\]
Then $L_{z_0}^\delta$ is a sort of tangent line to $\partial \Omega$ (as seen from $z_0$) and $S_{z_0}^\delta$ is a half-disk which approximates $\Omega$ near $z_0$. We say that $\Omega$ satisfies condition $M(\delta)$ at $z_0$ if either there exists $z \in L_{z_0}^\delta$ such that
\[
\text{dist}(z, \partial \Omega) \geq \delta \text{dist}(z_0, \partial \Omega),
\]
or
\[
S_{z_0}^\delta \cap \partial \Omega \neq \emptyset.
\]
Thus if $M(\delta)$ is satisfied at $z_0$ then the boundary of $\Omega$ “wiggles” to order $\delta$ near $z_0$. The following is Theorem 2 of [37]

**Lemma 4.1.** Suppose $\Omega$ is simply connected and $\Phi : \mathbb{D} \to \Omega$ is a Riemann mapping. For every $\delta > 0$ there is a $\epsilon > 0$ so that if $s(z) = |S(\Phi)(z)|(1 - |z|)^2 \geq \epsilon$ then $\Omega$ satisfies $M(\delta)$ at $\Phi(z)$. Furthermore,
$M(\delta)$ is satisfied at every point of $\Omega$ iff there are $C < \infty$ and $\epsilon > 0$ so that $s(z) > \epsilon$ on a $C$-dense set of the disk (with respect to the hyperbolic metric).

The proof is a normal families argument. Since $\partial \Omega$ is connected it is easy to see that condition $M(\delta)$ implies that the square centered at $z_0$ of size $\delta^{-1}\text{dist}(z_0, \partial \Omega)$ has $\beta_{\partial \Omega}(Q) \geq \delta^2$. Thus large Schwarzian implies large $\beta$'s. Another version (more precise than we need in this paper) is given by

**Lemma 4.2.** Suppose $E$ is compact, $\Omega = \overline{C \setminus E}$ and $\Phi : \mathbb{D} \to \Omega$ the universal covering map. Suppose $\Phi$ is univalent on $D(w, \epsilon(1 - |w|))$. Then

$$|S(\Phi)(w)|(1 - |w|^2)^2 \leq C \epsilon^{-2} \sum_{n=0}^{\infty} \delta_E(2^n Q)2^{-\mu n}$$

where $r = \text{dist}(\Phi(w), \partial \Omega)$. The number $\mu$ satisfies $0 < \mu < 1$ but can be taken as close to 1 as we wish. The constant $C$ depends only on the choice of $\mu$.

Using either of these estimates we can prove the following

**Corollary 4.3.** Suppose $\Omega$ is simply connected and $\Phi : \mathbb{D} \to \Omega$ is a Riemann mappings. Suppose also that there is a $C < \infty$ and $\epsilon > 0$ so that every point of the disk is at most hyperbolic distance $C$ from a point $z$ where $|S(\Phi)(z)(1 - |z|^2)| \geq \epsilon$. Then there is a $\beta = \beta(C, \epsilon) > 0$ so that $\beta_{\partial \Omega}(Q) \geq \beta > 0$ for every square $Q$ such that $\ell(Q) \leq \text{diam}(Q)$ and $\frac{1}{3}Q \cap \partial \Omega \neq \emptyset$. In particular, $\text{dim}(\partial \Omega) > 1$ with estimates depending only on $C$ and $\epsilon$. 
Proof. We use the second lemma. Let $E = \partial \Omega$ and let $Q$ be a square such that $\ell(Q) \leq \text{diam}(Q)$ and $\frac{1}{3}Q \cap E \neq \emptyset$. Let $\beta_0 = e^{-10C}$. We may also suppose $\epsilon = 2^{-\mu N}$ for some integer $N$. If $\beta_E(Q) \geq \epsilon \beta_0$, there is nothing to do, so assume $\beta_E(Q) \leq \epsilon \beta_0$. By the Koebe 1/4 theorem, the hyperbolic metric on $\Omega$ is comparable to $\text{dist}(z, E)^{-1} ds$, so by our hypothesis we can find a point $z \in \frac{1}{3}Q \cap \Omega$ with $\epsilon \beta_0 \ell(Q) \leq \text{dist}(z, E) \leq \epsilon \ell(Q)$, and such that if $w = \Phi^{-1}(z)$ then $|S(\Phi)(w)(1 - |w|)^2| \geq \epsilon$.

Let $Q'$ be the square centered at $z$ with side length $2\text{dist}(z, E)$. By the estimate above we know that one of the squares $Q', 2Q', \ldots, 2^N Q'$ where $N \sim \log_2 \epsilon$ has $\delta_E \geq \epsilon$. Call this square $Q''$. Since $z \in \frac{1}{3}Q$ and $2^N \ell(Q) \leq \frac{1}{4} \ell(Q), Q''$ must be a subsquare of $Q$. Since $\ell(Q'') \geq \beta_0 \ell(Q)$, we deduce

$$\delta_E(Q) \geq \beta_0 \epsilon \delta_E(Q'') \geq \beta_0 \epsilon^2 \equiv \beta.$$ 

This is what we wished to prove. The final statement is simply an application of Theorem 3.1.

Let $\Phi : \mathbb{D} \to \Omega$ be a Riemann mapping and let $\hat{G} = \Phi \circ G \circ \Phi^{-1}$ denote the Fuchsian equivalent of $G$. We can now prove Theorem 1.2 in the case when $\hat{G}$ has no parabolics. In this case, the surface $\Omega/G$ is compact so that any point of $\mathbb{D}$ is within some bounded distance $C$ on any orbit of $\hat{G}$. If $\Omega$ is not a disk then $\Phi$ is not Möbius, so $S(\Phi)$ does not vanish identically. Choose a point $z$ so that $s(z) = |S(\Phi)(z)(1 - |z|^2)| = \epsilon \neq 0$. Since $S(z)$ is constant on orbits of $\hat{G}$, we can apply Corollary 4.3 to deduce $\dim(\partial \Omega) > 1$. 

5. Non-differentiability of the boundary

In this section we describe our “non-differentiability” result for invariant components. First we need some definitions which capture what we mean by “non-differentiable”. A point \( x \in \partial \Omega \) is called a (inner) tangent point of \( \Omega \) if for any \( \theta < \pi \), \( x \) is the vertex of a cone in \( \Omega \) with angle \( \theta \), and this is not true for any \( \theta > \pi \). The set of tangents is the same, up to a set of 1-dimensional measure zero, as the set \( E \subset \partial \Omega \) of points which are vertices of some cone in \( \Omega(G) \). If \( \Omega \) is simply connected and \( x \in \partial \Omega \), then \( x \) is called a twist point of \( \Omega \) if

\[
\limsup_{z \to x, z \in \Omega} \arg(x - z) = +\infty, \quad \liminf_{z \to x, z \in \Omega} \arg(x - z) = -\infty.
\]

The harmonic measure \( \omega \) for the domain \( \Omega \) is the push forward of Lebesgue measure on the circle under a Riemann mapping onto \( \Omega \) (the measure depends on the choice of Riemann mapping, but its null sets do not). A probability measure \( \mu \) is said to be singular with respect to a measure \( \nu \) if it gives full measure to a null set for \( \nu \).

**Theorem 5.1.** Suppose \( G \) is a finitely generated Kleinian group with simply connected invariant component \( \Omega \). Then the following are all equivalent.

1. \( \Omega \) is not a disk.

2. The set of inner tangent points of \( \partial \Omega \) has zero 1-dimensional measure.

3. Harmonic measure for \( \Omega \) is singular to 1-dimensional measure.
4. Almost every (with respect to harmonic measure) point of \( \partial \Omega \) is a twist point.

Conditions (2), (3) and (4) are equivalent for any simply connected domain and they each imply (1). Thus the main point is to show (1) implies any of (2), (3) or (4). Using known results on the behavior of harmonic measure (to be described below) this result implies,

**Corollary 5.2.** Suppose \( G \) is finitely generated and \( \Lambda(G) \) is not a circle. Then the harmonic measures for different components of \( \Omega(G) \) are mutually singular.

This last corollary is well known in many cases. For example, when \( G \) is a quasi-Fuchsian group it corresponds exactly to Mostow’s theorem: given two finitely generated, first kind Fuchsian groups on the unit disk and a homeomorphism \( \phi \) of the boundary which conjugates the actions on the circle, then either \( \phi \) is Möbius or \( \phi \) is singular (i.e., maps full measure to zero measure). A stronger version of Mostow’s result is proven in \( [10] \) where it is shown that such a \( \phi \) is either Möbius or maps a set of dimension \(< 1\) to the complement of a set of dimension \(< 1\).

If \( \Omega \) is a component of \( \Omega(G) \) then the stabilizer of \( \Omega \) in \( G \) is also finitely generated (this is a consequence of the Ahlfors finiteness theorem \( [3], [5] \)). Thus if \( \Omega \) is simply connected and \( \Phi: \mathbb{D} \to \Omega_\varphi \) is a Riemann mapping, \( \hat{G} = \Phi \circ G \circ \Phi^{-1} \) is a finitely generated Fuchsian group of the first kind (i.e., its limit set is the whole circle) called the Fuchsian equivalent of \( G \) on \( \Omega \). We can actually prove Theorem 5.1 using the weaker hypothesis that the Fuchsian equivalent of \( G \) on \( \Omega \) has
a non-tangentially dense orbit. This happens if it is divergence type, i.e.,

$$\sum_{\gamma \in \hat{G}} (1 - |\gamma(0)|) = \infty.$$  

This is the best one can expect since Astala and Zinsmeister [6] have shown that any convergence group (i.e., $\sum_{\gamma \in \hat{G}} (1 - |\gamma(0)|) < \infty$) has a quasiconformal deformation to a Kleinian group whose limit set is a rectifiable curve (not a circle).

Suppose $\Omega$ is simply connected and $\Phi : \mathbb{D} \to \Omega$ is a Riemann mapping. By Plessner’s theorem the unit circle $\mathbb{T}$ can be divided into two set $E_0, E_1$ such that almost every (Lebesgue measure) point of $E_0$, $\Phi'$ has a finite, non-zero non-tangential limit and almost everywhere on on $E_1$, $\Phi'$ is non-tangentially dense in the plane. Let $\omega$ denote the harmonic measure on $\Omega$, i.e., $\omega(E) = |\Phi^{-1}|/2\pi$. By McMillan’s theorem [14], almost every point of $\partial \Omega$ is either the vertex of a cone in $\Omega$ or is a twist point (see introduction) and these two sets correspond a.e. to $E_0$ and $E_1$ respectively via the map $\Phi$. The set of the cone points has $\sigma$-finite 1-dimensional measure and on this set $\omega$ is mutually absolutely continuous with 1-dimensional Hausdorff measure. Makarov [11] proved that there is a subset $F$ of the twist points which has zero 1-dimensional measure but the same harmonic measure as the full set of twist points. Thus harmonic measure on a simply connected domain always “lives” on a set of dimension 1, regardless of the Hausdorff dimension of the entire boundary. Pommerenke [50] proved that harmonic measure gives
full measure to a set of \(\sigma\)-finite 1-dimensional measure and Wolff \(\cite{62}\) has even extended this to arbitrary planar domains.

Given two disjoint simply connected domains \(\Omega_1, \Omega_2\) with overlapping boundary \(E = \partial \Omega_1 \cap \partial \Omega_2\), the two harmonic measures \(\omega_1, \omega_2\) are mutually absolutely continuous on a subset \(F \subset E\) iff almost every point (with respect to both measures) of \(F\) is the vertex of two cones, one in each of the two domains \(\cite{13}\). In this case, both harmonic measures are also mutually absolutely continuous with respect to 1-dimensional Hausdorff measure on \(F\). Thus the measures \(\omega_1\) and \(\omega_2\) are mutually singular iff the set of “double cone points” has zero 1-dimensional measure. In particular, if almost every point (with respect to \(\omega_1\)) of \(\partial \Omega_1\) is a twist point then \(\omega_1\) is singular to 1-dimensional measure and singular to harmonic measure on any other disjoint domain. The converse is not true; there is a closed Jordan curve so that the harmonic measures for the two complementary domains are mutually singular but each is absolutely continuous with respect to 1-dimensional measure \(\cite{11}\). This is not possible for quasicircles however. A criterion for singularity of harmonic measures of two general disjoint domains is given in \(\cite{12}\).

There is also a connection between rectifiability and the Schwarzian derivative.

**Lemma 5.3.** \(\cite{15}\) *There is a \(C < \infty\) such that if \(\Phi : \mathbb{D} \to \Omega\) is univalent and \(\partial \Omega\) is rectifiable (i.e., has finite 1-dimensional measure)*
then
\[ \int_{D} |\Phi'(z)| |S(\Phi)(z)|^2 (1 - |z|^2)^3 dx \, dy < C \ell(\partial \Omega). \]

The same holds if \( \Phi \) is only defined on a Lipschitz subdomain of \( \mathbb{D} \). If \( \Omega \) is a quasicircle and dist(\( \Phi(0), \partial \Omega \)) \( \sim \) diam(\( \Omega \)) then the two sides are comparable with a constant depending only on the quasicircle bound.

This is Lemma 3.7 of \cite{[15]} and is an easy computation involving Green’s theorem and basic estimates for univalent mappings.

From the lemma we now deduce Theorem \ref{thm:main}. Suppose \( \Omega \) is a simply connected component of \( \Omega(\hat{G}) \). If \( \Omega \) is not a circle, then \( S(\Phi) \) is non-zero somewhere in the disk and therefore the invariant quantity \( s(z) = |S(\Phi)(z)|(1 - |z|^2)^2 \) equals \( \epsilon > 0 \) on some orbit \( \{z_n\} \) of the Fuchsian equivalent \( \hat{G} \). Since \( \hat{G} \) is a finitely generated Fuchsian group of the first kind, the orbit of any point is non-tangentially dense, i.e., for almost every point \( x \in \mathbb{T} \) there is a sequence of points in the orbit approaching \( x \) through a Stolz cone with vertex \( x \). Alternatively, if we associate to each orbit point \( z_n \) the interval \( I_n \) of length \( 1 - |z_n| \) centered and \( z_n/|z_n| \), then almost every point if the circle is in infinitely many of the intervals \( \mathcal{F} = \{I_n\} \).

Now suppose \( \Phi' \) has non-tangential limits on a set \( E \subset \mathbb{T} \) of positive Lebesgue measure. Then by taking a union of small cones we could construct a “sawtooth” domain \( W \subset \mathbb{D} \) with \( |E \cap \partial W| \geq \frac{1}{2} |E| \) and such that \( M^{-1} \leq |\Phi'| \leq M \) on \( W \). Let \( F = E \cap \partial W \). Because of the non-tangential density of the orbit we can use the Vitali covering lemma (e.g. page 109 of \cite{[61]}) to obtain a disjoint covering of almost
every point of $F$ by intervals of the form $\{I_n\}$. In fact, by repeated applications of covering lemmas we can find infinitely many collections $F_{\parallel} = \{I_{\parallel}\} \subset F$ each of which is a disjoint covering of almost all of $F$, and so that no interval belongs to more than one collection.

For each orbit point $z_n$, let $D_n$ denote the disk centered at $z_n$ of radius $\frac{1}{2}(1 - |z_n|)$. By the Koebe 1/4 theorem

$$\iint_{D_n} |\Phi'(z)||S(\Phi)(z)|^2(1 - |z|^2)^3 dxdy \geq C \epsilon^2 \iint_{D_n} |\Phi'(z)|(1 - |z|)^{-1} dxdy \geq CM^{-1} \epsilon^2(1 - |z_n|) \geq CM^{-1} \epsilon^2 |I_n|.$$ 

Therefore

$$\iint_{W} |\Phi'(z)||S(\Phi)(z)|^2(1 - |z|^2)^3 dxdy \geq CM^{-1} \epsilon^2 \sum_{n: z_n \in W} |I_n| \geq CM^{-1} \epsilon^2 \sum_{k} \sum_{I_n \in F_{\parallel}} |I_n| = \infty.$$ 

Therefore $\Phi(W)$ cannot have a finite length boundary by Lemma 5.3. However, $|\Phi'|$ is bounded by $M$ on $W$, so $\Phi(W)$ must have a rectifiable boundary. This contradiction implies that $\Phi'$ cannot have non-tangential limits on any set of positive Lebesgue measure. Thus in terms of earlier remarks, almost every point of $\partial \Omega$ must be a twist point, the set of cone points has zero 1-dimensional measure and harmonic measure on $\Omega$ is singular to harmonic measure for any simply connected domain which is disjoint from $\Omega$. 
If we apply the proof not to the whole disk but to a “Carleson square” $Q$ corresponding to an interval $I \subset \partial \mathbb{D}$,

$$Q = \{re^{i\theta} : e^{i\theta} \in I, 1 - |I| < r < 1\},$$

then the proof shows that the length of $\Phi(rI)$ must tend to infinity. If we divide the circle into $N$ equal intervals, we can apply the proof to each interval and then by my taking the minimum growth rate, get an estimate which is valid for all $N$ intervals. Since any interval of length $\geq 4\pi/N$ contains at least one interval from our collection, the growth rate also holds for such an interval. Thus

**Lemma 5.4.** Suppose $G$ is finitely generated with invariant component $\Omega$ which is not a disk. If $\Phi$ is a Riemann mapping onto $\Omega$ and $I \subset \partial \mathbb{D}$ then $\ell(\Phi(rI)) \to \infty$ as $r \to 1$ with estimates that only depend on $G$ and the length of $I$.

We finish this section with a remark concerning Makarov’s law of the iterated logarithm from [41]. Not only did Makarov prove that the harmonic measure for a simply connected domain gives full mass to a set of dimension 1 (as discussed earlier), but also the following sharp converse. If we define

$$\varphi_C(t) = t \exp(-C \sqrt{\log \frac{1}{t} \log \log \log \frac{1}{t}}),$$

then there is an absolute constant $C_1$ so that harmonic measure for any simply connected domain is absolutely continuous with respect to the Hausdorff measure for the gauge function $\varphi_{C_1}$. Conversely, there is a $C_2$ and examples of simply connected domains for which harmonic
measure gives positive mass to a set of zero $H_{PC^2}$ measure. Such domains are called Makarov domains and have the worse possible behavior for harmonic measure. In [37] the second author proved that $\Omega$ is a Makarov domain if its boundary is “wiggley” on all scales (in a sense made precise in that paper). It is easy to check that $\Omega$ satisfies his condition if every point of the disk is within a bounded hyperbolic distance of a point where $s(z) = |S(\Phi)(z)(1 - |z|^2)^2$ is bounded away from zero. We have already observed that if $\Omega$ is a component of a finitely generated group $G$ such that the Fuchsian equivalent $\hat{G}$ contains only hyperbolics, then this condition is satisfied. Thus all such domains are Makarov domains. Presumably the same is true with parabolics, but we have not attempted to prove this.

6. Proof of Theorem 1.2 with parabolics

We have already proven Theorem 1.2 when the Fuchsian equivalent $\hat{G}$ of $G$ on $\Omega$ has no parabolic elements. In this section we will prove it assuming $\hat{G}$ does contain parabolics. This will also give a different proof for the co-compact case.

This section is not strictly necessary for proof of Theorem 1.2. We shall later give two more proofs for the case of degenerate groups and Theorem 1.2 can be deduced from this and the known results for geometrically finite groups. However, this argument seems to be necessary to prove $\delta(G) > 1$. In fact, we will show $\delta(G) > 1$ and use the fact that if $\Omega$ is $G$-invariant, then $\partial\Omega = \Lambda(G)$. Thus by Theorem 1.1 we deduce Theorem 1.2.
First we want to show that it is enough to estimate the Poincaré series along an orbit in \( \Omega(G) \) (rather than along the orbit of \( 0 \in \mathbb{B} \)).

Recall that for \( z \in \Omega(G) \) we set

\[
d(z) = \text{dist}(z, \partial \Omega(G)).
\]

**Lemma 6.1.** If \( G \) is a Kleinian group, \( 0 \) is the center of \( \mathbb{B} \) and \( z \in \Omega(G) \subset \partial \mathbb{B} \), then

\[
d(g(z)) \sim 1 - |g(0)|, \quad \text{for all } g \in G
\]

with constants that depend on \( z \) and \( G \), but not on \( g \).

**Proof.** Choose a ball \( B \subset \Omega \) centered at \( z_0 \) so that \( \text{diam}(B) \leq \frac{1}{2} \text{dist}(z_0, \partial \Omega) \).

Let \( \omega_1 = \omega(0, B, \mathbb{B}) \) be the harmonic measure of this ball in \( \mathbb{B} \) with respect to the point zero and let \( \omega_2 \) be the harmonic measure of \( \frac{1}{2} B \) with respect to 0. Then by the conformal invariance of harmonic measure, for any \( g \in G \), \( g(0) \) is the unique point \( z \) so that

\[
\omega(z, g(B), \mathbb{B}) = \omega_1,
\]

\[
\omega(z, g(\frac{1}{2} B), \mathbb{B}) = \omega_2.
\]

By our choice of \( B \) and the Koebe 1/4 theorem, \( g(\frac{1}{2} B) \subset \lambda g(B) \) for some \( \lambda < 1 \) independent of \( g \). Therefore any \( z \in \mathbb{B} \) which satisfies the two equalities above must satisfy

\[
|g(z_0) - z| \leq C \text{diam}(g(B)),
\]

\[
1 - |z| \geq \frac{1}{C} \text{diam}(g(B)).
\]

This proves the lemma. \( \square \)
Note that this implies that the accumulation set of any point in $\Omega(G)$ is all of $\Lambda(G)$. In particular, if $\Omega$ is an invariant component of $\Omega(G)$ then $\partial \Omega = \Lambda(G)$.

Let $\hat{G}$ denote the Fuchsian equivalent of $G$ and suppose 0 is a point where the Schwarzian derivative of the Riemann mapping $D \to \Omega$ is large. Suppose $z_j = g(0) \in D$ is an orbit point of zero and let $I_j \subset \partial D$ be the interval on the boundary centered at $z_j/|z_j|$ with length $1 - |z_j|$. We let $S_j$ denote the Carleson square with base $I_j$,

$$S_j = \{ z : z/|z| \in I_j, 1 - |I_j| \leq |z| < 1 \}.$$

Let

$$d_j = d(\Phi(z_j)) = \text{dist}(\Phi(z_j), \partial \Omega) \sim (1 - |z_j|)|\Phi'(z_j)|.$$

We will show

**Lemma 6.2.** There is a $C < \infty$ (depending only on $G$) such that if $g \in \hat{G} \setminus \{Id\}$, and $z_0 = g(0)$ then there is a collection of orbit points $\{z_k\} = \{g_k(0)\} \subset T_0$ such that

1. $\sum_k d_k \geq 2d_0$,
2. $(1 - |z_k|) \geq (1 - |z_0|)/C$.
3. the intervals $\{I_k\}$ are disjoint.

Note that conditions (1) and (2) imply

$$\sum_k d_k^{1+\epsilon} \geq d_0^{1+\epsilon},$$
if $\epsilon$ is small enough (depending on $C$). Using condition (3), we can break the orbit of $0 \in \mathbb{D}$ into generations $\mathcal{G}_{\lambda}$ so that
\[
\sum_{z_k \in \mathcal{G}_{\lambda}} d_k^{1+\epsilon} \geq \sum_{z_k \in \mathcal{G}_{\lambda_{-\infty}}} d_k^{1+\epsilon} \geq 1,
\]
which implies
\[
\sum_{z_k \in \mathcal{G}(0)} d_k^{1+\epsilon} = \infty.
\]
By Lemma 6.1, this proves $\delta(G) > 1$.

Thus it suffices to prove Lemma 6.2 Let $\Phi : \mathbb{D} \to \Omega$ be a Riemann map, normalized so that $S(\Phi)(0) \neq 0$. Conjugate $g$ so that $\Phi(0) = \infty$ and $\text{diam}(\partial \Omega) = 1$. If $\Omega/G$ is a surface with punctures, then we can find a $G$ invariant collection of disjoint balls $\mathcal{B}_{\infty} = \{B_j^\infty\}$ in $\Omega$, each invariant under a parabolic element of $G$ and so that $\Omega/G$ is compact (i.e., we are taking a neighborhood of each puncture on $\Omega/G$ and lifting it to $\Omega$. Each $B_j^1$ is thus conjugate to one of a finite subcollection and each ball has a parabolic fixed point of $G$ on its boundary. To each ball in $\mathcal{B}_{\infty}$ we associate smaller invariant balls $B_j^2 \subset B_j^1$ so that the hyperbolic distances between $\partial B_j^1$ and $\partial B_j^2$ is 1. Also any point of $\Omega$ which is outside $\cup_j B_j^2$ is within a bounded hyperbolic distance of the orbit of $\infty$. Let $C_1$ denote this bound.

If $\Omega/G$ has no punctures, just replace the collections $\mathcal{B}_j$ by the empty set in the proof that follows. Note that in this case every point of $\Omega$ is within a bounded distance of the orbit of $\infty$.

For each $t > 0$ consider the level line of Green’s function in the disk
\[
\Gamma_t = \{z : |z| = 1 - e^{-t}\},
\]
and for each orbit point $z_j \in \hat{G}(0)$ let

$$\Gamma^j_t = \Gamma_t(1-|z_j|) \cap S_j.$$  

The first thing we want to see is that the image of $\Gamma^j_t$ is very long if $t$ is small enough (independent of $j$).

**Lemma 6.3.** For any $M > 0$ there is a $t_0$ such that if $t \leq t_0$ then

$$\ell(\Phi(\Gamma^j_y)) \geq Md_j = M \text{dist}(\Phi(z_j), \partial \Omega).$$

**Proof.** This is just Lemma 5.4 after rescaling $z_j$ to be the origin. \qed

We would like to take our orbits in Lemma 6.2 to be (in some sense) taken along the curve $\Gamma^j_t$. More precisely, we will break $\Gamma^j_t$ into unit hyperbolic segments $\{\gamma_k\}$ and to each segment associate the closest orbit point $z_k$. If there were no parabolic points then each orbit point would be associated to a bounded number of segments, say $N$ (depending on only $G$) and by the standard distortion theorems for conformal maps,

$$\ell(\Phi(\gamma_k)) \sim d(z_k).$$

Thus by throwing out repeats we would have a collection of points $z_k \in S_j$ with

$$\sum_k d_k \geq \frac{M}{N},$$

and

$$d_k \geq Cd_j,$$

(where $C$ depends on $t$ and constants in certain distortion theorems for conformal maps). This proves the Lemma 6.2 when there are no parabolics.
If the surface $\Omega/G$ has punctures, then there may be points of $\Gamma^j_t$ which are very far from the closest orbit of 0, and we need to replace such pieces by new curves which are closer to orbit points. The idea is that if $\Gamma^j_t$ passes though the “bottom half” of a horoball $B$ in the unit disk, then we can should replace $\Gamma^j_t \cap B$ by the arcs

$$\Gamma_B = \partial B \cap \{z : \frac{1}{2}t(1 - |z_j|) \leq 1 - |z| \leq t(1 - |z_j|)\}.$$ 

These arcs have similar hyperbolic length to the arc $\Gamma^j_t \cap B$, they are only slightly closer to the boundary of the disk, and they remain within a bounded hyperbolic distance of the orbit of 0. Most importantly, the Koebe 1/4-theorem implies that

$$\ell(\Phi(\Gamma_B)) \sim \ell(\Phi(\Gamma^j_t \cap B)).$$

Thus we have

**Lemma 6.4.** For each $z_j$ there is an arc $\tilde{\Gamma}^j_t$ (consisting of pieces of $\Gamma^j_t$ and arcs of horoballs such that

1. $\tilde{\Gamma}^j_t \subset S_j \cap \{|z| \leq 1 - \frac{1}{2}(1 - |z_j|)\}$.
2. $\ell(\Phi(\tilde{\Gamma}^j_t)) \geq Md_j$.
3. every component of $\tilde{\Gamma}^j_t$ has hyperbolic length at least 1.

We can now finish the proof of Lemma 6.2 just as in the case without parabolics described above. This finishes the lemma and hence completes the proof of Theorem 1.2.

Let us summarize the results from the last three sections. Suppose $G$ is a finitely generated Kleinian group with a simply connected invariant component $\Omega$. Recall that $G$ is called an extended Fuchsian group if it
is Fuchsian or has an index two Fuchsian subgroup. Then the following are equivalent:

1. $G$ is not an extended Fuchsian group.
2. $\partial \Omega$ is not a circle.
3. $\partial \Omega$ has infinite 1-dimensional measure.
4. $\dim(\partial \Omega) > 1$.
5. $\dim(\Lambda_{c}(G)) > 1$.
6. $\delta(G) > 1$.
7. $\partial \Omega$ fails to have a tangent somewhere.
8. $\partial \Omega$ fails to have an inner tangent and all points except possibly the rank 1 parabolic fixed points.
9. The set of inner tangent points of $\partial \Omega$ has zero 1-dimensional measure.
10. Harmonic measure for $\Omega$ is singular to 1-dimensional measure (i.e., there is a subset of $\partial \Omega$ of full harmonic measure and 1-dimensional measure zero).
11. Almost every (with respect to harmonic measure) point of $\partial \Omega$ is a twist point.
12. Harmonic measures for distinct components of $\Omega(G)$ are mutually singular.

7. Groups with round components

Suppose $G$ is a finitely generated Kleinian group and $\Omega(G)$ contains a component $\Omega$ which is a disk. Then either $\Omega(G)$ consists of exactly two components (both round disks) or has infinitely many components
which are disks. In the first case, $G$ is an extended Fuchsian group and the limit set is a circle. The second case is described by the following result.

**Theorem 7.1.** Suppose $G$ is a finitely generated Kleinian group and $\Omega(G)$ contains infinitely many components which are disks. Then $\dim(\Lambda(G)) > 1$.

This is slightly different than Larmen’s result mentioned in the introduction, because we do not insist that every component be a disk (it could happen, for example, that there is another component of $\Omega(G)$ with accidental parabolics). This is known result; it is a special case of Theorem 1 of [26] which Canary and Taylor prove using a result of Furusawa [31]. It is also contained in results of Sullivan and of Patterson. Roughly speaking, such a group $G$ must contain 2 Fuchsian subgroups $G_1, G_2$ whose limit circles are each contained in a fundamental domain for the other group. Furusawa’s theorem then says that $\delta(G_1 \ast G_2) > \delta(G_1) = 1$. Since these groups are geometrically finite the same inequality holds for the dimension of the limit sets. An alternate approach is to note that the sum of the diameters of the orbit of $\Lambda(G_1)$ under $G_2$ diverges because $G_2$ is divergence type. Thus $\Lambda(G_1 \ast G_2)$ has infinite length. Sullivan’s results then imply it has dimension $> 1$.

Some very interesting pictures of this type of limit set appear in [21]. Further results on such limit sets and the corresponding groups are given in [39].
We include a proof of Theorem 7.1 for completeness. It is a fairly standard computation involving Hausdorff measures. As usual, we pass to a finite index subgroup if necessary to remove any elliptic elements. Let \( D_1, D_2 \) be distinct round components (which are not just opposite sides of one circle) with stabilizers \( G_1, G_2 \). Since the orbit of \( D_1 \) under \( G_2 \) accumulates densely on \( \partial D_2 \) and vice versa, we may assume (by choosing new disks if necessary) that

\[
\text{dist}(D_1, D_2) \geq 1,
\]

\[
\text{diam}(D_1) \leq 1/1000,
\]

and the double of each disk is contained in a fundamental polygon of the other group (so the translates of the doubles are disjoint).

Fix values of \( \delta > 0 \) and \( N < \infty \). Suppose we construct a Cantor set \( E \) by an iterative construction in which a disk \( D \) is replaced by at most \( N \) disks \( \{D_j\} \) such that

1. \( D_j \subset 2D \), and \( \{2D_j\} \) are disjoint.
2. \( \delta \text{diam}(D) \leq \text{diam}(D_j) \leq \text{diam}(D)/100. \)
3. \( \sum_j \text{diam}(D_j) \geq 2\text{diam}(D). \)

Then it is easy to see that \( \text{dim}(E) \geq \alpha(\delta, N) > 1. \)

For \( D_1 \) we can choose \( \delta, N \) and such disks \( \{D^1_j\} \) by taking part of the orbit of \( D_2 \) under \( G_1 \) (using the fact that \( G_1 \) is divergence type). Similarly for \( D_2 \) and \( \{D^2_j\} \). At a general step in the construction suppose we have a disk \( D' \) which is a “child” of \( D'' \) (i.e., \( D' \subset 2D'' \) and \( \text{diam}(D') \leq \text{diam}(D'')/100 \)). Then \( D' \) corresponds to either \( D_1 \) or \( D_2 \).
under the action of $G_1 * G_2$, so assume it is $D_1$. Then there is an element $g \in G_1 * G_2$ so that $g(D_1) = D'$ and $g(D_2) = D$. It's easy to check that $g$ has bounded distortion on $2D_1$ (since it corresponds to the much smaller disk $D'$). Thus $\{g(D^j_1)\}$ satisfy the desired conditions with respect to $D'$ and the construction may be continued (the constants may be different, but we have uniform bounds).

8. Degenerate limit sets have large $\beta$'s

Suppose $G$ is a finitely generated degenerate group, i.e., $\Omega(G) = \Omega$ has a unique component and this component is simply connected. If the Fuchsian equivalent $\hat{G} = \Phi \circ G \circ \Phi^{-1}$ has no parabolics, then $\Lambda(G)$ has “large $\beta$’s” by the argument in Section 4. We shall show this is true even if $\hat{G}$ contains parabolics. The point is that a rank 1 parabolic in a degenerate group cannot be doubly cusped, and this implies big $\beta$’s.

**Theorem 8.1.** If $G$ is a degenerate group there is a $\beta_0 = \beta_0(G) > 0$ such that if $E = \Lambda(G)$, then $\beta_E(Q) > \beta > 0$ for every square $Q$ such that $\ell(Q) \leq \text{diam}(Q)$ and $\frac{1}{2}Q \cap \partial \Omega \neq \emptyset$. In particular, $\dim(\Lambda(G)) > 1$.

**Proof.** As usual, by Selberg’s lemma we may pass to a finite index subgroup without torsion (which clearly has the same limit set). Thus we assume $G$ has no elliptics and let $R = \Omega/G = \mathbb{D}/\hat{G}$ denote the Riemann surface represented by $G$ and $\hat{G}$. If $\hat{G}$ has no parabolics then Theorem 8.1 follows from Theorem 4.3 as noted in Section 4, so assume
$\hat{g} \in \hat{G}$ is parabolic and let $g = \Phi \circ \hat{g} \circ \Phi^{-1}$ be the corresponding element of $G$. Note that $g$ must also be parabolic and call its fixed point $z_0$.

Let $B$ denote a horoball in $\mathbb{D}$ corresponding to $\hat{g}$. Let $\hat{\gamma}$ denote its boundary and $h$ the hyperbolic translation length of $\hat{G}$ along $\hat{\gamma}$. Let $\hat{\gamma}_0$ denote a segment of hyperbolic length $h$ on $\hat{\gamma}$. If we conjugate the group $G$ so that the fixed point of $g$ becomes $\infty$ then $g$ becomes a translation. Then $\hat{\gamma}_0$ maps to a smooth arc and thus its easy to see that $\hat{\gamma}$ maps to a smooth quasicircle passing through infinity. In particular, its lies in the strip between two parallel lines. Thus before conjugating the fixed point to infinity, we see that $\gamma = \Phi(\hat{\gamma})$ is a quasicircle passing through $z_0$ and is contained between two circles tangent at $z_0$, say $B_1 \subset \Phi(B) \subset B_2$. Note that these two balls are invariant under $g$ since they correspond to half-planes when $z_0$ is conjugated to $\infty$.

By taking $B_1$ smaller, if necessary, we may assume that $B_3$, the reflection of $B_1$ through the fixed point $z_0$, does not intersect $B_2$. Then $B_3$ is also invariant under $g$. We wish to show that $B_3$ must contain points of $\Lambda(G)$ that are not too close to $z_0$. Change variables so that $z_0 = 0$, $B_3$ lies in the upper half plane and has diameter 1. Let $z = iy$ lie on the positive imaginary axis and note that $|z - g(z)| = O(|y|^2)$, as $y \to 0$. However, $\hat{g}$ moves every point of the disk outside $B$ by at least some hyperbolic distance $\delta$ (depending on $\hat{g}$ and $B$) and therefore since $z \in B_3$ lies outside $B_2 \supset \Phi(B)$, $g$ must move $z$ by more than $\delta$ in the hyperbolic metric on $\Omega$. Since $\Omega$ is simply connected the hyperbolic metric on $\Omega$ is comparable to $\text{dist}(z, \partial \Omega)^{-1} ds$. Thus for $z = iy$ small
we must have
\[ \text{dist}(z, \partial \Omega) \leq C \delta^{-1} |z - g(z)| \leq C \delta^{-1} y^2. \]

By considering the point \( z = iy \) with \( 0 < y < \delta/C \) we see that \( B_3 \cap \Lambda(G) \) must contain a point \( z_1 \) within \( |y|^2 \) of \( z \). The orbit of \( z_1 \) under powers of \( g \) is a \( y^2 \)-dense set on a circle of radius larger that \( y/2 \) tangent at the fixed point \( z_0 \). Since \( \Lambda(G) \) is invariant under \( g \) it contains this sequence of points.

In fact, we have shown that a point \( z \in B_3 \cap \Omega \) at distance \( r \) from \( z_0 \) can be at most distance \( \sim r^2 \) from \( \partial \Omega = \Lambda(G) \). In particular, if \( Q \) is any square so that \( Q \cap B_3 \) is large (say contains a disk or radius \( > \ell(Q)/100 \) then \( \beta_{\Lambda(G)}(Q) \geq \beta_0 > 0 \) (and \( \beta \geq 1/4 \) for all sufficiently small square containing 0).

Now we can deduce that \( \Lambda(G) \) has large \( \beta \)'s everywhere. Suppose we have a square \( Q \) such that \( \ell(Q) \leq \text{diam}(Q) \) and \( \frac{1}{3} Q \cap \partial \Omega \neq \emptyset \).

If \( \Omega \cap \frac{1}{10} Q \) contains no disk of size \( \ell(Q)/100 \), then \( \beta_{\Lambda(G)}(Q) \geq 1/20 \). So assume \( \Omega \cap Q \) does contain a large disk and let \( z_1 \in \frac{1}{10} Q \) be its center. If \( z_1 \) lies outside all the balls \( B_1 \) corresponding to parabolics in \( \hat{G} \) described above, then \( w_1 = \Phi^{-1}(z) \) lies only a bounded hyperbolic distance from an orbit where the Schwarzian derivative \( |S(\Phi)|(z)(1 - |z|^2)^2 \) is bounded away from zero. Therefore in this case \( \beta_{\Lambda(G)}(Q) \geq \beta_0 \) (where \( \beta_0 \) depends on \( G \), but not on \( Q \)).

Otherwise, \( z_1 \) must lie inside a horoball \( B_1 \). Let \( g \) be the corresponding element of \( G \) with fixed point \( z_0 \). By the choice of \( B_1, B_3 \) described above and since \( z_1 \in \frac{1}{10} Q, Q \cap B_3 \) must contain a disk of size \( \geq \frac{1}{100} \ell(Q) \).
So by our remarks above, $\beta_{\Lambda(G)}(Q) \geq \beta_0$ (in fact it is quite large). This completes the proof of Theorem 8.1.

One can give yet another proof of $\dim(\Lambda(G)) > 1$ for degenerate groups with parabolics using the observations above, but without using Theorem 3.1. One notes that the horoball $B_3$ can be packed by an infinite number of ball accumulating to the fixed point, each containing points of the limit set and so that the sum of the radii diverges. This, together with the observation that conjugates of $B_1$ take up a fixed fraction of the area of $\Omega(G)$ in each such ball, allows one to define a nested sequence of balls such that the resulting Cantor set lies in the limit set and has dimension larger than one. The packing in question is roughly the one corresponding (via a Möbius transformation) to a unit packing of the upper half plane. The dimension of Cantor sets arising from this kind of packing of the disk were studied in [32].

9. PROOF OF THEOREM 1.3 AND SOME COROLLARIES

Suppose $G$ has a connected limit set. Then any component $\Omega$ of $\Omega(G)$ is simply connected. The subgroup fixing any component $\Omega$ of $\Omega(G)$ is a finitely generated Kleinian group $G_\Omega$ and $\Omega$ is an invariant component of its ordinary set. Thus by Theorem 1.2 either $\dim(\Lambda(G)) > 1$ or every component of $\Omega(G)$ is a disk. If the latter case holds then either $\Omega(G)$ has two components or infinitely many. If it has two then $\Lambda(G)$ is a circle. Otherwise $\Lambda(G) = \mathbb{C} \setminus \bigcup_j D_j$ for some infinite collection of disjoint open disks. Thus $\dim(\Lambda(G)) > 1$ by Theorem 7.1. (Actually, Larman [40] proved that such a set has dimension $> 1$, regardless of whether
the disks are associated to any group.) It follows from the Klein-Maskit combination theorems that either $\Lambda(G)$ is totally disconnected or $\Lambda(G)$ contains a component which is itself the limit set of finitely generated subgroup (see e.g. [2], [13]). Thus we get

**Theorem 9.1.** If $G$ is a finitely generated Kleinian group then its limit set is either totally disconnected, a circle or has Hausdorff dimension $> 1$.

Our proof of Theorem 1.2 showed that if $\Omega$ is a simply connected component which is not a disk then $\delta(G) > 1$. Thus we obtain

**Corollary 9.2.** If $G$ is finitely generated and $\Omega$ is a simply connected invariant component of $\Omega(G)$ which is not a disk, then $\delta(G) = \dim(\Lambda_c(G)) > 1$.

It follows from the Klein-Maskit combination theorems that if $G$ is finitely generated and $\Lambda(G)$ is totally disconnected then $G$ is geometrically finite and hence $\delta(G) = \dim(\Lambda_c(G)) = \dim(\Lambda(G))$. Thus Theorem 9.1 implies

**Corollary 9.3.** If $G$ is a finitely generated Kleinian group then $\dim(\Lambda(G)) > 1$ iff $\delta(G) > 1$.

Recall that $G$ is called degenerate if $\Omega(G)$ is connected and simply connected. As a special case of Theorem 1.2 we get

**Corollary 9.4.** If $G$ is degenerate then $\dim(\Lambda(G)) > 1$ and $\delta(G) > 1$. 
Greenberg [34] showed that degenerate groups are geometrically infinite so Theorem 1.4 will imply that the limit set of a degenerate group has dimension 2, but we mention this weaker result here because it is a new result whose proof easy, whereas the stronger result requires much more. In [24] Canary shows that \( \delta(G) = 2 \) for any degenerate group.

A web group is one in which each component subgroup is quasi-Fuchsian (so the ordinary set is a union of quasidisks). By the decomposition theorem of Abikoff and Maskit [2] any finitely generated, geometrically infinite group contains either a degenerate group or a geometrically infinite web group (all component subgroups quasi-Fuchsian). In either case the limit set must have dimension > 1, so

**Corollary 9.5.** *If G is a finitely generated, geometrically infinite Kleinian group then \( \dim(\Lambda(G)) > 1 \) and \( \delta(G) > 1 \).*

Later we will show that \( \dim(\Lambda(G)) = 2 \) for geometrically infinite groups, and \( \delta(G) = 2 \) if \( \Lambda(G) \) has zero area, but this will be our best result for \( \delta(G) \) in general.

**10. Geometry of \( \Omega(G) \)**

In this section we will prove some lemmas about the geometry of \( \Lambda(G) \) and \( \Omega(G) \) that we will use later. The two main facts we need are

1. \( \Lambda(G) \) is uniformly perfect.
2. Show bad horoballs must be singly cusped and far apart.

For \( z \in \Omega(G) \) define

\[
d(z) = \text{dist}(z, \partial \Omega(G)) = \text{dist}(z, \Lambda),
\]

where
where “distance” means spherical distance. Suppose $\Omega$ is a component of $\Omega(G)$ and let $\rho$ denote the hyperbolic metric on $\Omega$. Then a standard estimate says that

$$|d\rho(z)| \leq 2\frac{|dz|}{d(z)}.$$

Since an orbit is uniformly separated in the hyperbolic metric this implies its is also uniformly separated in the “$|dz|/d(z)$” metric, and hence each Whitney box contains only a bounded number of orbit points. Except near parabolic horoballs it is also true that Whitney cubes are near at least one orbit point. To prove this we need to show $\Lambda(G)$ is uniformly perfect.

A compact set $K$ is called \textit{uniformly perfect} if there exists $\epsilon > 0$ such that for any $x \in K$ and $r < \text{diam}(K)$ there exists $y \in K$ such that

$$\epsilon r \leq |x - y| \leq r.$$

There are several well known equivalent formulations of this condition (e.g., [30], [33]). Suppose $K$ is compact and $\Omega$ is its complement. Then the following are known to be equivalent:

1. $K$ is uniformly perfect.
2. There is a positive lower bound for the length of the shortest closed hyperbolic geodesic in $\Omega$.
3. There is a constant $C < \infty$ so that

$$\frac{1}{C} \frac{|dz|}{d(z)} \leq |d\rho(z)| \leq 2\frac{|dz|}{d(z)}.$$

\textbf{Lemma 10.1.} If $G$ is a finitely generated, non-elementary Kleinian group then $\Lambda$ is uniformly perfect. In particular, if $\rho$ is the hyperbolic
metric on a component $\Omega$ of $\Omega(G)$, then

$$|d\rho(z)| \sim \frac{|dz|}{d(z)}.$$ 

Proof. From the discussion above it is enough to verify condition (2) for $\Omega$: there exists $\epsilon$ so that every closed geodesic for $\rho$ in $\Omega$ has hyperbolic length $\geq \epsilon$. By the Ahlfors finiteness theorem, $\Omega(G)/G = \bigcup_{i=1}^{N} R_i$ is a finite union of compact Riemann surfaces with a finite number of punctures, so there is a lower bound $\epsilon_0$ for the length of the shortest closed geodesic (although if there are punctures there is no lower bound for homotopically non-trivial loops). Suppose there was a homotopically non-trivial closed loop $\gamma$ in $\Omega(G)$ with length less than $\epsilon_0$. Then $\Gamma$ projects to a loop in a cusp region on $\Omega(G)/G$ which can be homotoped to a puncture. Thus $\gamma$ can be homotoped in $\Omega(G)$ to curve with arbitrarily short hyperbolic length, which implies $\partial\Omega(G)$ must have an isolated boundary point. This is impossible if $G$ is non-elementary. Thus $\Omega(G)$ has the desired property. \[\square\]

It is worth noting that the constants in the previous result depend on $G$ and cannot be taken to depend, say, only on the number of generators. It has also been pointed out to us that this lemma (with almost the same proof) was earlier proven by Canary in [23].

Next we define what we mean by “good” and “bad” horoballs. Since $G$ is finitely generated, the Ahlfors finiteness theorem [1] says that $\Omega(G)/G$ is a finite union of finite Riemann surfaces $R_1, \ldots, R_N$, i.e., each is a compact surface with at most a finite number of punctures. Let $\{p_1, \ldots, p_m\}$ be the punctures in $R = \bigcup_{i=1}^{N} r_i$, and for each $p_i$ let
$B_i^*$ be a neighborhood of $p_i$ which lifts to a Euclidean ball $B_i$ in $\Omega$ which is invariant under some parabolic element of $G$ (see Lemma 1 of [3]). Then $X = \bigcup_i R_j \setminus \bigcup_j B_j^*$ is compact, so as above, we can choose a finite set of points $E = \{z_1, \ldots, z_P\} \subset \Omega(G)$ which project to an $\epsilon$ dense subset of $X$. It will also be convenient to assume (as we may) that each of the chosen horoballs $B_j$, is contained in a horoball in $\Omega$ of twice the diameter (so $\partial B_j$ does not come too close to $\partial \Omega$ except near the parabolic fixed point.

Since there are only a finite number of surfaces in $\Omega(G)/G$, it is easy to see that there is an $\epsilon_0 > 0$ so that if $\gamma$ is homotopically non-trivial loop on one of the surfaces $R_i$ of length less than $\epsilon_0$ then $\gamma$ must be contained in one of the neighborhoods $B_i^*$. Thus any closed curve in $\Omega(G)$ of hyperbolic length $\leq \epsilon_0$ is either homotopically trivial or lies in an image of one of the balls $B_i$.

Suppose $B = g(B_i)$ for some $g \in G$. Suppose $B$ is fixed by a parabolic element $h \in G$ with fixed point $p \in \partial B$. We say that $B$ is doubly cusped it there is a another (disjoint) ball $B_1$ fixed by $h$.

Normalize, so that $\Lambda(G)$ has diameter 1. Given a $\eta > 0$ we say $B$ is a “$\eta$-bad” horoball if

$$\sup_{z \in \partial B} d(z) \leq \eta \text{diam}(B).$$

Otherwise we say $B$ is “$\eta$-good”. We will need the following simple facts about good and bad horoballs.

**Lemma 10.2.** Suppose $\text{diam}(\Lambda) = 1$. 

1. There is a $C_1$ (depending only on $\eta$) so that for any $\eta$-good horoball $B$, and any $w \in \partial B$, there is a point $z \in G(E)$ such that

$$C_1^{-1}d(w) \leq d(z) \leq C_1d(w),$$

$$C_1^{-1}d(w) \leq |z - w| \leq C_1d(w).$$

2. There is a $\eta_2$ (depending on $G$) so that if $B$ is doubly cusped then $B$ is $\eta_2$-good.

3. If $\eta$ is small enough (depending only on $G$), and if $B$ is an $\eta$-bad horoball and $D \subset \Omega \cap \eta^{-1/2}B$ is a disk, then $\text{diam}(D) \leq \eta^{1/6}\text{diam}(B)$. (the powers are not sharp).

4. For any $\delta > 0$ there is a $\eta_3 > 0$ (depending only on $\delta$) such that if $B$ is a $\eta_3$-bad horoball then there is a disk $D \subset 3B$ such that $\text{diam}(D) \geq \frac{1}{3}\text{diam}(B)$, and $D \setminus \Lambda$ contains no balls of radius $\geq \delta\text{diam}(D)$.

5. There is $\eta_4 > 0$ so that if $B_1, B_2$ are $\eta_4$-bad horoballs with $\text{diam}(B_1) \leq \text{diam}(B_2)$ then

$$\text{dist}(B_1, B_2) \geq 100\text{diam}(B_1).$$

6. If $B_1, B_2$ are horoballs with

$$\text{diam}(B_1) \leq \text{diam}(B_2) \leq 2\text{diam}(B_1),$$

and $\text{dist}(B_1, B_2) \leq A\text{diam}(B_1)$, then both $B_1$ and $B_2$ are $A^{-3}$-good (the power is not sharp).

**Proof.** The first claim is easy, since if $w \in \partial B$ then it is within hyperbolic distance $\epsilon$ of a point in $G(E)$, and these points lie in the same or adjacent Whitney squares.
To prove (2) suppose $B = g(B_i)$. Then $B$ is doubly cusped iff $B_i$ is. Suppose they are and let $\hat{B}_i$ be the other ball corresponding to $B_i$ and let $\hat{B} = g(\hat{B}_i)$. Choose $\eta$ so that $B_i$ is $\eta$-good. If $B$ is $\eta_2$-bad with $\eta_2 << \eta$, it’s easy to see that $g$ maps $\hat{B}_i$ to the exterior of a ball of diameter $\leq 2\text{diam}(B)$. But this means that $\text{diam}(\Lambda) \leq 2\text{diam}(B) < 1$, a contradiction. Thus $B$ must be $\eta_2$-good, as desired.

To prove (3), conjugate $B$ by a linear map so it becomes the ball of radius $1/2$ centered at $i/2$ and it is fixed under an element of the form

$$h(z) = \frac{z}{1 + az}.$$ 

If $B$ is $\eta$-bad then we must have $a \leq C\eta$. Therefore, a point $z \in \eta^{-1/2}B$, is displaced at most

$$|z - h(z)| = \left|\frac{az^2}{1 + az}\right| \leq C\eta|z|^2 \leq C\eta^{1/3} < \eta^{1/6}\epsilon_0/100,$$

if $\eta$ is small enough. Suppose $z \in 3B \setminus 2B$ is the center of a disk $D \subset \Omega$ of radius $\eta^{1/6}$. Then $D$ intersects $h(D)$ and connecting the centers gives a circular arc which projects to a loop on $\Omega(G)/G$ of length less than $\epsilon_0$. Thus iterating $D$ under powers of $h$ gives a circle fixed by $h$ and lying in $\Omega$. Moreover, this circle projects to a curve in $\Omega(G)/G$ of length less than $\epsilon_0$. Therefore it lies in a horoball. Hence side of this circle must lie in $\Omega$. It cannot be the side containing $B$ because this component contains points of $\Lambda$ (the ones that are within distance $\eta$ of $i \in \partial B$). Therefore the other side must be contained in $\Omega$. This implies $B$ is doubly cusped, hence $\eta_2$-good. This is a contradiction, so there are no such small disks.

Parts (4), (5), and (6) are all special cases of (3), so we are done. $\square$
11. The convex hull of the limit set

A discrete group of Möbius transformations is called geometrically finite if there is a finite sided fundamental polyhedron for its action on \( \mathbb{B} \). For our purposes, however, the following characterization due to Beardon and Maskit of geometrically finite groups is much more useful. Recall that a rank 1 parabolic fixed point \( p \) is called doubly cusped if there are two disjoint balls in \( \Omega(G) \), tangent at \( p \), and both invariant under the parabolic subgroup fixing \( p \).

**Proposition 11.1.** If \( G \) is a Kleinian group then \( G \) is geometrically finite iff \( \Lambda(G) \) is the union of \( \Lambda_c(G) \), the rank 2 parabolic fixed points and the doubly cusped rank 1 parabolic fixed points of \( G \).

Geometrical finiteness can also be characterized in terms of the convex hull of the limit set. If \( K \) is a compact set on \( S^2 = \partial \mathbb{B} \) we will let \( C(K) \subset \mathbb{B} \) denote its convex hull with respect to the hyperbolic metric on \( \mathbb{B} \). If (as usual in this paper) \( G \) is a Kleinian group without torsion, we let \( M = \mathbb{B}/G \) be the hyperbolic 3-manifold associated to \( G \). Then \( C(M) = C(\Lambda(G))/G \subset M \) is called the convex core of \( M \). Much of the interesting topology of \( M \) is associated to the topology of the convex core.

For \( r > 0 \) we define the radius \( r \) neighborhood of \( C(M) \) as

\[
C_r(M) = \{ z \in \mathbb{B} : \text{dist}(z, C(M)) < r \},
\]

where distance is measure in the hyperbolic metric. Although we don’t need it here, we should point out that Thurston has shown that \( G \) is
geometrically finite iff $C_r(G)$ has finite volume for some (all) $r > 0$. We cannot take $r = 0$ because if $G$ is any Fuchsian group, then $\Lambda(G)$ is contained in a circle, so the convex hull of $\Lambda(G)$ is contained in a hyperplane and hence has zero volume. Thus $C(M)$ has finite volume even for infinitely generated Fuchsian groups. However, for finitely generated groups this is not a problem.

**Proposition 11.2.** If $G$ is a finitely generated Kleinian and $C(M)$ has finite volume then $G$ is geometrically finite.

This is well known, but we will deduce it as a corollary of Lemma 11.3 later of this section. The equivalence of the many equivalent formulations of geometric finiteness is discussed in [18].

Our first goal is to see that $C(M)$ can be separated from $\Omega(G)$ by finite area surfaces. More precisely,

**Lemma 11.3.** Suppose $G$ is finitely generated Kleinian group (without torsion) and $M$ is the corresponding hyperbolic 3-manifold. Let $\{\Omega_j\}_1^N$ be conjugacy classes of components of $\Omega(G)$ (i.e., the geometrically finite ends of $M$). For each $\Omega_j$ there is a surface $\{S_j\}$ in $M$ so that the following holds.

1. $\text{dist}(S_j, C(M)) > 2$.
2. Each $S_j$ has finite area.
3. If $U_j = \{z \in M : \rho(z, S_j) < 1\}$ is a unit hyperbolic neighborhood of $S_j$ then $U_j$ has finite hyperbolic volume.
4. The function \( f(y) = (\text{vol}(B(y, 1)))^{-1/2} \) is integrable over \( U_j \), i.e.,

\[
\int_{U_j} (\text{vol}(B(y, 1)))^{-1/2} dy < \infty.
\]

5. The surfaces \( S_j \) separate \( C(M) \) from the geometrically finite ends of \( M \), i.e., there is a \( \epsilon_0 > 0 \) so that \( w(z) \leq 1 - \epsilon_0 \) on the component \( M_1 \) of \( M \setminus \cup_j S_j \) containing \( C(M) \).

Proof: This consists of known facts, as described, for example, in [29]. In fact, we do not even need the sophisticated machinery developed there. For each component \( \Omega_j \) we can simply take \( S_j \) to be the intersection of a fundamental polygon in \( B \) for \( G \) with the surface

\[
\{ z \in B : 1 - |z| = \epsilon_0 \text{dist}(z/|z|, \Lambda(G)) \}.
\]

By Lemma 10.1 (which says \( \Lambda(G) \) is uniformly perfect) hyperbolic area on this surface is boundedly equivalent to hyperbolic area on \( \Omega(G) \) under the radial projection if \( \epsilon_0 \) is small enough (depending only on the constants in Lemma 10.1; these depend on \( G \), but may be taken absolute if \( \Lambda(G) \) is connected). Thus (2) holds. Condition (3) follows because \( U_j \) is boundedly equivalent to \( S_j \times [0, 1] \). To prove (4) we note that the integral obvious converges on any compact piece of \( U_j \) and use the fact that the injectivity radius decreases exponentially to bound the integral in the cusps (also note that for \( y \in U \), \( \text{vol}(B(y, 1)) \sim \text{inj}(y) \)). Conditions (1) and (5) are easy to check if \( \epsilon_0 \) is small enough.

We will also need the Margulis lemma (e.g., [3]). This says that there is an \( \epsilon > 0 \) so that if

\[
M_{\text{thin(}\epsilon)} = \{ x \in M : \text{inj}(x) \leq \epsilon \},
\]
then every component of $M_{\text{thin}(\epsilon)}$ is one of three kinds:

1. a torus cusp, i.e., a horoball in $\mathbb{B}$ modulo a rank 2 parabolic subgroup.
2. a rank one cusp, i.e., a horoball in $\mathbb{B}$ modulo a rank 1 parabolic subgroup.
3. a solid torus neighborhood of a closed geodesic, i.e., a $r$-neighborhood of a geodesic in $\mathbb{B}$ modulo powers of a loxodromic element fixing the geodesic.

Moreover, the components of $M_{\text{thin}(\epsilon)}$ (i.e., the horoballs in (1) and (2) and the neighborhoods in (3)) may be taken to be pairwise disjoint. Note that this implies that if $g \in G$ and $B$ is a “thin component” (one of the three types of regions described above), then $g(B)$ hits $B$ iff $g(B) = B$ and $g$ is in the subgroup associated to $B$ above. Actually we will not need the full strength of the Margulis lemma. All we will use is that there is an $\epsilon > 0$ as described above, but this number may be allowed to depend on $G$.

We will need the following lemma in the proof of Theorem 1.5.

**Lemma 11.4.** If $G$ is a Kleinian group (not necessarily finitely generated) and $\Lambda(G)$ has positive area then $C(M)$ has infinite volume.

Note that we need the hypothesis that $G$ is Kleinian, since it is possible for $G$ to be a discrete group with $\Lambda(G) = S^2$, but $C(M) = M$ to have finite volume (e.g., if $G$ is co-compact).
Proof. We will prove this by showing there is a sequence \( \{x_n\} \in C(M) \) with

\[
\text{dist}(x_n, \partial C(M)) \to \infty,
\]

and \( \text{inj}(x_n) > \epsilon \) for all \( n \).

For \( z \in \mathbb{B} \) define

\[
w(z) = \max_{D \subset \Omega(G)} \omega(z, D, \mathbb{B}),
\]

where the max is over all round disks in \( \Omega(G) \). Then \( C(\Lambda) = \{ z : w(z) \leq 1/2 \} \). Since \( w \) is a max of harmonic functions it is sub-harmonic but we will not need this. Instead of \( w \) we could simply deal with the distance function \( \text{dist}(z, \partial C(M)) \), which is essentially \( -\log w \). It is also \( G \) invariant, so defines a function on \( M \). It is easy to see that for any \( R > 0 \) there is an \( \epsilon \) so that \( w(x) < \epsilon \) implies \( \text{dist}(x, \partial C(M)) > R \).

Since \( \Lambda(G) \) has positive area the Lebesgue density theorem gives us a point of density \( z_0 \in \Lambda(G) \). Let \( \gamma \) be the hyperbolic geodesic connecting the origin to \( z_0 \) (i.e., a radius of \( \mathbb{B} \)) and consider points \( x \in \gamma \) converging to \( z_0 \). Clearly \( w(x) \to 0 \), so \( \text{dist}(x, \partial C(M)) \to \infty \) as \( x \to z_0 \). Therefore we only have to show that the injectivity radius of \( x \) is \( \geq \epsilon \) along some subsequence converging to \( z_0 \).

But if this is false then eventually \( \gamma \) must be in one of the three types of thin regions for all points close enough to the boundary. This implies \( z_0 \) is fixed by some element of \( G \). Since such points form a countable subset of the limit set we may certainly assume \( z_0 \) is not one of them and we are done.
We will use the following result in the proof of Theorem 1.4 (but it is not essential; we will also sketch a proof which does not require it). We will use part of the proof in the proof of Theorem 1.6 and Theorem 1.7.

**Lemma 11.5.** Suppose $G$ is finitely generated and geometrically infinite. Then there is a sequence $\{x_n\} \in C(M)$ with $\text{dist}(x_n, \partial C(M)) \to \infty$ and $\text{inj}(x_n) > \epsilon$ for all $n$.

*Proof.* The proof is similar to the last lemma but with a few extra technicalities because of the possibility of “bad” horoballs. We will carry out the proof in two cases

1. Every rank 1 parabolic is doubly cusped.
2. There is a rank 1 parabolic which in not doubly cusped.

Note that case (1) contains the case when $G$ has no parabolic elements.

*Proof for case (1) of Lemma 11.5.* In this case, there is an $\eta > 0$ so that every horoball (if any exist) is $\eta$-good by Lemma 10.2. Let $z_0 \in \Lambda(G)$ be a point which is neither a parabolic fixed point nor in $\Lambda_c(G)$ (such a point exists by Proposition 11.1). Let $\gamma$ be the hyperbolic geodesic from the origin in $\mathbb{B}$ to $z_0$. We will first show

$$\lim_{x \to z_0, x \in \gamma} w(x) = 0.$$  

Suppose this is false, i.e., suppose there is an $\epsilon_0$ such that $w(x) > \epsilon_0$ for all $x \in \gamma$. We will prove that $z_0$ is either in $\Lambda_c(G)$ or is a parabolic fixed point.

Let $\{\Omega_j\}_{j=1}^N$ be a maximal collection of non-conjugate components of $\Omega(G)$. In each we can find a finite sided fundamental polygon $P_j$ (in
\( \Omega_j \)'s hyperbolic geometry). Cover \( P_j \) by a finite number of Euclidean disks, including a horoball for each parabolic cusp. Let \( \mathcal{D} = \{ D_j \} \) denote the (finite) collection of disks obtained in this way. Note that there is a constant \( C_0 \) (depending only on our choice of \( \mathcal{D} \)) such that for any disk \( D \) in \( \Omega(G) \) there is a \( D_k \) and a \( g \in G \) so that \( D \cap D_k \neq \emptyset \) and \( \text{diam}(g(D_k)) \geq C \text{diam}(D) \). Thus there is a \( C_1 > 0 \) so that for any \( x \in \mathbb{B} \) there is a \( D_k \) and \( g \in G \) so that

\[ \omega(x, g(D_k), \mathbb{B}) \geq C_1 w(x). \]  

(11.1)

Therefore we may suppose we have \( z_0 \in \Lambda(G) \) and there is an \( \epsilon_0 \) such that for every \( x \in \gamma \) there exists \( D_k \in \mathcal{D} \) and \( g \in G \) such that

\[ \omega(x, g(D_k), \mathbb{B}) \geq \epsilon_0. \]

Let \( D(x) = g(D_k) \) be a choice of disk for each \( x \in \gamma \).

If \( z_0 \) is not a parabolic fixed point then \( D(x) \) must change infinitely often as \( x \to z_0 \). Since \( D(x) \) is either compactly contained in \( \Omega \) or a \( \eta \)-good horoball, the proof of Lemma \( 6.1 \) and Lemma \( 10.2 \) shows that there is a constant \( C_3 > 0 \) (depending only on our choice of \( \mathcal{D} \) and \( \epsilon_0 \)) so that if

\[ \omega(x, g(D_k), \mathbb{B}) \geq \epsilon_0, \]

for some \( D_k \in \mathcal{C} \), then the orbit point \( g(0) \) is contained in the cone with angle \( \pi - C_3 \) with vertex \( z_0 \). Thus in this case, there is a cone with vertex \( z_0 \) which contains infinitely many images of 0, i.e., \( z_0 \in \Lambda_c(G) \).

Thus we have shown that \( w(x) \to 0 \) as \( x \to z_0 \), or equivalently \( \text{dist}(x, \partial C(M)) \to \infty \). The proof now finishes just as in the previous lemma (if \( z_0 \) is not a fixed point of some element \( g \in G \) then there is
a sequence \( \{ x_n \} \to z_0 \) in \( \gamma \) which lies outside the thin regions). This finishes the proof for case (1).

**Proof for case (2) of Lemma 11.3.** Now we consider the second case when there is a rank 1 cusp in \( \Omega(G) \) which is not doubly cusped. Normalize so that this parabolic is \( z \to z+1 \). Since \( \infty \) is not doubly cusped, we may also assume that \( \Omega(G) \cap \mathbb{H} \) contains no invariant horoball for this parabolic. The corresponding thin part in \( \mathbb{B} \) will be

\[
B_0 = \{ (x, y, z) : z > C \},
\]

for some \( C \). If \( \infty \) is singly cusped, normalize so the lower half plane is the largest invariant half plane in \( \Omega \). Consider the points \( y^* = (0, y, C) \). Since these are on the boundary of the thin part the injectivity radii are all \( \epsilon \). Therefore we need only show \( w(y^*) \to 0 \) as \( y \to \infty \).

Suppose not, say \( w(y^*) \geq \nu > 0 \) and let \( D(y) \) be the largest disk seen from \( y^* \). Suppose \( D(y) \) is not the lower half plane. Then \( \text{diam}(D_y) \leq 1 \) if \( y \) is large enough, because otherwise \( D(y) \) would not be disjoint from its translate under the parabolic element \( z \to z+1 \). This would give a line in \( \Omega \) and everything below this line would be in \( \Omega \), contradicting that the lower half plane is the largest half plane in \( \Omega \).

If \( D(y) \) is constant for all large enough \( y \) then clearly the harmonic measure of \( D(y) \) from the point \( y^* \) tends to zero, which is what we want. Otherwise, \( D(y) \) must change infinitely often. Assume this is the case.

If there is a sequence \( y_n \to \infty \) such that \( D(y_n) \) is \( \eta_n \)-bad with \( \eta_n \to 0 \), then we must have \( \text{diam}(D(y_n)) \to 0 \). This holds because the estimates
in the proof of Lemma 10.2 show that for bad horoballs the associated thin part in $B$ has (Euclidean) diameter which is much greater than the diameter of the horoball in $\Omega(G)$. Since the thin regions in $B$ are disjoint the thin region corresponding to $D(y_n)$ must have diameter $\leq C$ (by the normalization above). Thus $\text{diam}(D(y_n)) \to 0$. Therefore we are done in this case.

Next suppose that there are no $\eta$-bad horoballs for some $\eta > 0$. Then we get a subsequence of disks $D(y_n)$ which are all images of the same disk in $D$ and so that the diameters converge to some $\nu_0$, $0 < \nu \geq \nu_0 \geq 1$. For two disks in this sequence consider the group element $g_{nm}$ which maps $D(y_n)$ to $D(y_m)$. Also consider a linear conformal map $h_{nm}$ from $D(y_m)$ to $D(y_n)$. By compactness we can choose $n, m$ and $h_{nm}$ so that $g_{nm} \circ h_{nm}$ is arbitrarily close to the identity in $\text{PSL}(2, \mathbb{C})$. Since $h_{nm}$ preserves $B_0$, this contradicts the fact (the Margulis lemma) that $g_{nm}(B_0)$ can’t intersect $B_0$. Thus the diameters of $D(y^*)$ must tend to zero. This completes the proof of Lemma 11.5.

Richard Canary has pointed out to us that Lemma 11.5 also follows easily from known results. His argument goes as follows: If $C(N)$ is non-compact, then (see [17]), there exists a sequence $\{\gamma_i\}$ of closed geodesics leaving every compact set. Pick $\epsilon$ less than the Margulis constant. Either, $\text{length}(\gamma_i) < \epsilon$ or some point on $\gamma_i$ lies in the $\epsilon$-thick part. In the first case, let $C_i$ denote the component of the thin part containing $\gamma_i$, then $\partial C_i$ (the boundary of $C_i$) intersects $C(N)$ (otherwise $C(N) \subset \partial C_i$ which implies that $N$ is elementary), so pick $x_i$ in the the
intersection of $\partial C_i$ and $C(N)$. In the second case pick $x_i \in \gamma_i \cap N_{thick}$. Then, \{x_i\} leaves every compact set and every point in the sequence has injectivity radius at least $\epsilon$.

Note that we obtain Proposition 11.2 as a corollary of Lemma 11.3 since the lemma gives us infinitely many disjoint balls which all have volume bounded uniformly from below. The proof also shows the following known fact (e.g., [43], it also follows from Proposition 11.1):

**Corollary 11.6.** *If $G$ is a finitely generated, geometrically finite group, then every parabolic cusp of $\Omega(G)$ is doubly cusped (hence $\eta$-good for some $\eta$ depending on $G$).*

We will use this in the proof of Theorem 1.6 and Theorem 1.7.
12. Geometrically infinite groups

In this section we will prove,

**Theorem 12.1.** *If $G$ is a finitely generated and geometrically infinite then*

$$\dim(\Lambda(G)) = 2.$$  

If $\delta(G) = 2$ then this follows from Theorem 1.1. Therefore we may assume $\delta(G) < 2$. In this case Theorem 12.1 follows from

**Theorem 12.2.** *If $G$ is a finitely generated, geometrically infinite group and $\delta(G) < 2$ then $\Lambda(G)$ has positive area.*

Note that the Ahlfors conjecture and Theorem 12.2 imply that $\delta(G) = 2$ for any finitely generated, geometrically infinite group $G$.

**Proof of Theorem 12.2:** Let $\Lambda = \Lambda(G)$ be the limit set of $G$. As usual, we assume $G$ has no torsion. Let $\delta = \delta(G)$ be the critical index for the Poincaré series and $\lambda_0$ the base eigenvalue for the Laplacian on $M = \mathbb{B}/G$. By Corollary 9.3 we know that if $G$ is geometrically infinite then $\delta(G) > 1$. Therefore by a result of Sullivan (Theorem 2.18 [57])

$$\lambda_0 = \delta(G)(2 - \delta(G)).$$

Let $C(\Lambda)$ be the convex hull of the limit set in $\mathbb{B}$ and $C(M) = C(\Lambda)/G \subset M$ be the convex core of $M$ (see previous section). Let $S = \cup_j S_j$ and $U = \cup_j U_j$ be the surfaces and their unit neighborhoods given by Lemma 11.3. Let $M_1$ be the component of $M \setminus \cup_j S_j$. We will prove that Brownian motion is transient in $M_1$. More precisely, we will show that
Lemma 12.3. Suppose $M, S, M_1$ are as above. Then there is a point $x_0 \in M_1$ such that a Brownian motion started at $x_0$ has a positive probability of never hitting $S$.

First let us see why this implies $\Lambda$ has positive area. Let $\tilde{x}$ be a point in the hyperbolic ball $\mathbb{B}$ which corresponds to the point $x_0$ in the previous paragraph and let $\tilde{M}_1$ be the lift of $M_1$ to the ball. Then a Brownian motion started at $\tilde{x}_0$ has a positive probability of hitting $S^2 = \partial \mathbb{B}$ without leaving $\tilde{M}_1$. Since $w < 1 - \epsilon_0$, such a path must first hit $S^2$ at a point of $\Lambda$. Since harmonic measure for $\mathbb{B}$ is mutually absolutely continuous with respect to area measure on $S^2$ this means that $\Lambda$ has positive area. Therefore we need only prove Lemma 12.3.

Proof of Lemma 12.3: The lemma follows from known estimates on the heat kernel $K(x, y, t)$ for a Riemannian manifold.

Proposition 12.4 (Davies, Theorem 17 [27]). If $\delta > 0$ then there is a constant $C = C(\delta)$ such that

$$0 \leq K(x, y, t) \leq C \text{vol}(B(x, 1))^{-1/2} \text{vol}(B(y, 1))^{-1/2} e^{(\delta - \lambda_0)t} e^{-\rho(x, y)^2(4+\delta)^{-1}t^{-1}},$$

where $\rho(x, y)$ denotes the hyperbolic distance between $x$ and $y$ in $M$.

See also [28] and [35] and their references. To apply Davies’ result we let $\delta = \lambda_0/2$. Choose $R > 0$ very large. By Lemma 11.5 we can choose a point $x \in C(M)$ where the injectivity radius $\text{inj}(x) > \epsilon > 0$ is uniformly bounded below and

$$\text{dist}(x, U) \geq \text{dist}(x, \partial C(M)) \geq R.$$
The expected time a Brownian motion started at $x$ spends in $U$ is
\[
\int_0^\infty \left[ \int_U K(x, y, t) dy \right] dt.
\]

By Davies’ estimate this is bounded by
\[
\text{vol}(B(x, 1))^{-1/2} \int_0^\infty e^{-\lambda_0 t/2} e^{-R^2/5t} dt \cdot \int_U \text{vol}(B(y, 1))^{-1/2} dy.
\]

Using the fact that inj($x$) is bounded below and part (4) of Lemma 11.3 this is bounded by
\[
C \int_0^\infty e^{-\lambda_0 t/2} e^{-R^2/5t} dt.
\]

This can be estimated by breaking the integral at $t = R$ to get
\[
C \int_0^R e^{-\lambda_0 t/2} e^{-R^5/5} dt + C \int_R^\infty e^{-\lambda_0 t/2} dt \leq \frac{C}{\lambda_0} (e^{-\frac{3}{2}R} + e^{-\frac{3}{4}R}).
\]

Thus by choosing $R$ sufficiently large we can make the expected time a Brownian motion started at $x$ spends in $U$ is a small as we wish.

From this we wish to deduce that a Brownian motion started at $x_0$ never hits $S$ with some positive probability. To do this we simply note that the expected time it takes a Brownian motion started at a point $y$ of $S$ to first leave $U$ (i.e., to travel unit distance from $S$) is bounded away from zero independent of the starting point $y$. This is because the expected time to travel distance 1 in $M$ is greater than or equal to the expected time to travel this distance in the covering space $B$, and this is bounded away from zero. Let $t_0 > 0$ be a lower bound for the expected time to travel distance 1.

Thus the expected time a Brownian motion started at $x$ spends in $U$ is at least the probability that it every hits $S$ times the bound $t_0$. Now choose $x_1$ so that the expected time spent in $U$ is less than $t_0/2$. 
Then the probability that a Brownian motion starting at \( x_1 \) ever hits \( S \) is less than \( 1/2 \). This completes the proof of Theorem 12.1.

We do not actually need the full strength of Lemma 11.5. All we need is that there is a sequence \( \{x_n\} \) with \( \text{inj}(x_n) \geq \epsilon \) and \( x_n \) leaving every compact set (rather than \( \text{dist}(x_n, \partial C(M)) \to \infty \)). This is because we can cut \( U \) up into a compact piece and a finite number of small tips of cusps. If the volume of the small cusp pieces are small enough, then any uniform bound on the heat kernel shows that the expected time in these pieces is small. The time in the remaining compact part is small as long as the distance of \( x_n \) form the compact part is large. This is true as long as \( x_n \) tends to infinity in \( M \).

**Sketch of second proof of Lemma 12.3:** It actually suffices to use a weaker estimate on the heat kernel and less information on the convex hull. All we need to know is that Brownian motion is transient on a complete, connected, infinite volume Riemannian manifold with lowest eigenvalue bounded away from zero. For example, the estimate

\[
K(x, y, t) \leq C_1 e^{-C_2 t},
\]

for some fixed \( x, y \) in some ball and \( t > T_0 \) would be sufficient for this.

We pay for the less precise estimate by a more involved construction on the manifold. Let \( S \) be the surfaces described by Lemma 11.3 which separate the convex core from the geometrically finite ends. Cut \( M \) along \( S \) and let \( M_1 \) be the component containing \( C(M) \). Glue two copies of \( M_1 \) along \( S \). We claim that the resulting manifold \( N \) (the double of \( M_1 \)) has lowest eigenvalue bounded away from zero. If so
then the heat kernel estimates apply to the new manifold $N$ and we deduce that the expected time a Brownian motion spends in $U$ (the unit radius neighborhood of the $S$) is finite. By the Borel-Cantelli lemma this says that there are points $x$ in $M$ from which the probability of ever hitting $S$ is strictly less than 1 (in fact is as small as we wish). Thus there is a point $x \in N \setminus U$ from which the chance of ever hitting $U$ is less than $1/2$. But the two components of $N \setminus U$ are both exactly $M_1$. Thus Brownian motion in $M_1$ must have a positive probability of tending to infinity without ever hitting $S$.

This proves the lemma, except for verifying that $N$ has first eigenvalue bounded away from 0. We will not verify this in detail, but simply note that since $M$ has constant negative curvature and lowest eigenvalue $> 0$, Buser’s inequality (e.g. [24]) implies the Cheeger constant for $M$ is bounded away from zero. From this one proves that the Cheeger constant for the manifold with boundary $M_1$ is non-zero, and from this that the Cheeger constant for the doubled manifold $N$ is non-zero. Then Cheeger’s estimate say the first eigenvalue for $N$ is non-zero, as desired. \qed

13. Lower semi-continuity of Hausdorff dimension

In this section we will prove,

**Theorem 13.1.** If $G$ is a finitely generated Möbius group and $\{G_n\}$ is a sequence of Möbius groups converging algebraically to $G$ then

$$\liminf_n \dim(\Lambda(G_n)) \geq \dim(\Lambda(G)).$$
In particular, if $G$ is geometrically infinite then this result and Theorem 1.4 imply that \( \lim_n \dim(\Lambda(G_n)) = 2 \).

If $G$ is not discrete then Theorem 13.1 follows from Theorem 1.1 and the remarks following Corollary 2.5. If all but finitely many of the \( \{G_n\} \) are non-discrete then by passing to a subsequence we may assume \( \Lambda(G_n) \) is always all of \( S^2 \), are all circles or are always two or less points. We may also assume that the limit sets \( \Lambda_n = \Lambda(G_n) \) converge in the Hausdorff metric to a set \( \Lambda_\infty \). Moreover, because of the special form of the sets we must have

\[
\dim(\Lambda_\infty) \leq \lim \inf \dim(\Lambda_n).
\]

It is easy to check that the limit set of $G$ is a subset of the limit of the sets $\Lambda_n$.

Therefore we may assume all the groups are discrete. If all but finitely many of the \( \{G_n\} \) satisfy \( \Lambda(G_n) = S^2 \), there is nothing to do, so we may as well assume (by passing to a subsequence) that the \( \{G_n\} \) are all Kleinian groups.

If \( \Lambda(G) \) has zero area or \( \delta(G) = 2 \) then \( \delta(G) = \dim(\Lambda(G)) \) and this follows from Corollary 2.3. Therefore we need only consider the case when \( \delta(G) < 2 \) and \( \Lambda(G) \) has positive area (possibly the whole sphere). If $G$ where geometrically finite then either \( \Lambda(G) = 0 \) (4) or \( \delta(G) = 2 \). In either case we are finished, so we may assume that $G$ is geometrically finite.

By passing to a subsequence we may assume

\[
\lim_n \dim(\Lambda(G_n)) = 2.
\]
exists and equals the liminf of the original sequence. If all but finitely many of the groups \( \{G_n\} \) are geometrically infinite then
\[
\liminf_n \dim(\Lambda(G_n)) = 2,
\]
by Theorem 12.1 so there is nothing to do. Otherwise, by passing to another subsequence, we may assume all the \( G_n \)'s are geometrically finite. We may also assume \( \lim_n \delta(G_n) = \alpha \) exists and is strictly less than 2 (otherwise the result follows from the inequality \( \dim(\Lambda(G_n)) \geq \delta(G_n) \)). To finish the proof we need

**Theorem 13.2.** If \( \{G_n\} \) is a sequence of geometrically finite Kleinian groups which converges algebraically to a finitely generated, geometrically infinite discrete group \( G \) then \( \delta(G_n) \to 2 \).

This result follows from two known results

**Proposition 13.3** (Canary). If \( G \) is a \( n \)-generated, geometrically finite group then
\[
\lambda_0 \leq \frac{A_n}{\text{vol}(C(M))},
\]
where \( A_n \) is a constant that only depends on the number of generators of \( G \).

**Proof:** This is essentially Theorem A of [24] except that there Canary proves
\[
\lambda_0 \leq A \frac{\chi(\partial C(M))}{\text{vol}(C(M))},
\]
where \( A \) is an absolute constant and where \( \chi \) denotes the Euler characteristic. However, the Euler characteristic of \( \partial C(M) \) is the same as
that of \( \Omega(G)/G \), because there is always a homeomorphism between the two (e.g., see Epstein and Marden’s paper \cite{EpsteinMarden}). By the Bers inequality (a quantitative version of the Ahlfors finiteness theorem) the area, and hence the Euler characteristic, of \( \Omega(G)/G \) can be bounded in terms of \( n \), the number of generators of the group \( G \). Thus Canary’s results says

\[
\lambda_0 \leq \frac{A_n}{\text{vol}(C(M))},
\]

where \( A_n \) depends only on the number of generators. \( \square \)

**Proposition 13.4.** Suppose \( \{G_n\} \) is a sequence of \( n \)-generated, geometrically finite Kleinian groups such that

\[
\sup_n \text{vol}(C(M)) \leq M < \infty.
\]

If the sequence \( \{G_n\} \) converges algebraically to a finitely generated, geometrically infinite, discrete group \( G \) then \( \Lambda(G) \) has zero area.

*Proof:* This is an easy case of a result obtained by E. Taylor in \cite{Taylor} and is probably well known. Here we will sketch a proof which follows an argument given by Jørgensen and Marden in \cite{JorgensenMarden}.

Suppose \( \Lambda(G) \) has positive area. We will derive a contradiction. Suppose \( \{G_n\} \) is a sequence of \( N \)-generated Kleinian groups converging algebraically to a Kleinian group \( G \). The Borel selection theorem says that the set of compact subsets of a compact metric space is itself compact with the Hausdorff metric

\[
d(E, F) = \max_{z \in E} \text{dist}(z, F) + \max_{w \in F} \text{dist}(w, E),
\]
so by passing to a subsequence (which we also denote \( \{G_n\} \)) we may assume the sets \( \Lambda_n = \Lambda(G_n) \) converge in the Hausdorff metric to a compact set \( \Lambda_\infty \).

We say a sequence \( \{G_n\} \) converges polyhedrally to a group \( H \) if \( H \) is discrete and for some \( x_0 \in \mathbb{B} \), the fundamental polyhedra (the Dirichlet polyhedron)
\[
P(G_n) = \{ z \in \mathbb{B} : \rho(z, x_0) \leq \rho(z, g(x_0)) \text{ for all } g \in G_n \},
\]
converge to \( P(H) \) uniformly on compact subsets of \( \mathbb{B} \). By Proposition 3.8 of [38] any algebraically converging subsequence has a polyhedrally convergent subsequence and the polyhedral limit contains the algebraic limit (but they need not be equal).

A third notion of convergence of groups is geometric convergence. Given a sequence of groups \( \{G_n\} \) we define
\[
\text{Env}\{G_n\} = \{ g \in \text{PSL}(2, \mathbb{C}) : g = \lim_n g_n, g_n \in G_n \},
\]
and we say \( G_n \to H = \text{Env}\{G_n\} \) geometrically if for every subsequence \( \{G_{n_j}\} \), \( \text{Env}\{G_{n_j}\} = \text{Env}\{G_n\} \). Proposition 3.10 of [38] says that \( G_n \) converges geometrically to \( H \) if and only if it converges polyhedrally to \( H \).

Thus we may assume that we have groups \( G \subset H \) such that

1. \( G_n \to G \) algebraically,
2. \( G_n \to H \) polyhedrally and geometrically,
3. \( \Lambda_n \to \Lambda_\infty \) in the Hausdorff metric.

We first claim that \( \Lambda(H) \subset \Lambda_\infty \). If \( \Lambda_\infty = S^2 \) there is nothing to do, so we may assume \( \Lambda_\infty \) is not the whole sphere. In this case we follow
the proof of Proposition 4.2 of [38]. Let \( \Omega = S^2 \setminus \Lambda_\infty \) and suppose \( K, K' \) are compact sets such that

\[
K \subset \text{int}(K') \subset K' \subset \Omega.
\]

Suppose \( h \in H \). Because \( \{G_n\} \) converges geometrically to \( H \) we can write \( h = \lim_n g_n \) with \( g_n \in G_n \).

We claim that \( h(K) \subset \Omega \). If not then \( h(K) \) intersects \( \Lambda_\infty \), so \( \text{int}(h(K')) \) also hits \( \Lambda_\infty \). This implies that \( \text{int}(h(K')) \) intersects \( \Lambda_n \) for all large enough \( n \), say \( n \geq N_1 \). Therefore \( \text{int}(g_n(K')) \) hits \( \Lambda_m \) for all \( m \geq N_1 \), for all sufficiently large \( n \), say \( n \geq N_2 \). So if \( N_3 = \max(N_1, N_2) \), then \( n \geq N_3 \) implies

\[
\text{int}(g_n(K')) \cap \Lambda_n \neq \emptyset.
\]

Therefore

\[
\text{int}(K') \cap g_n^{-1}(\Lambda_n) = \text{int}(K') \cap \Lambda_n \neq \emptyset.
\]

This is a contradiction, so we must have \( h(K) \subset \Omega \). This implies \( h(\Omega) \subset \Omega \). Since the same argument applies to \( h^{-1} \), we see that \( h(\Omega) = \Omega \), or equivalently, \( h(\Lambda_\infty) = \Lambda_\infty \). Since \( \Lambda_\infty \) is a closed set which is invariant under the group \( H \) we must have \( \Lambda(H) \subset \Lambda_\infty \) as desired (recall that the limit set is the smallest closed \( H \)-invariant set if \( H \) is non-elementary).

Since \( \Lambda(H) \subset \Lambda_\infty \) the convex hull \( C(\Lambda(H)) \) of \( \Lambda(H) \) in \( \mathbb{B} \) is contained in the convex hull \( C(\Lambda_\infty) \) of \( \Lambda_\infty \). The convex hulls \( C(\Lambda_n) \) of the sets \( \Lambda_n \) converge, uniformly on compacta, to \( C(\Lambda_\infty) \). Thus for any \( R < \infty \),

\[
\text{vol}(C(\Lambda(H)) \cap P(H) \cap B(x_0, R)) \leq \lim \inf_n \text{vol}(C(\Lambda_n) \cap P(G_n) \cap B(x_0, R)) \leq M.
\]
Thus $C(\Lambda(H)) \cap P(H)$ has finite volume. Here we are using the convexity of the sets $\Lambda_n$; in general, sets of zero volume can converge in the Hausdorff metric to sets of infinite volume.

The volume of the quotient manifold is easily seen to be continuous under geometric convergence, so we deduce that $\mathbb{B}/H$ has infinite volume. If $\Lambda(H) = S^2$ then $\text{vol}(M) = \text{vol}(C(M))$ is finite, which is a contradiction. Thus $\Lambda(H) \neq S^2$.

If $\Lambda(H)$ is a proper subset of $S^2$ it still has positive area since it contains the set $\Lambda(G)$. Thus by Lemma 11.4 the convex core of $H$ has infinite volume. This is another contradiction and so completes the proof of Proposition 13.4. □

Now that we have the two propositions, we can can finish the proof of Theorem 13.2. Suppose $G$ is a finitely generated, geometrically infinite discrete group and $\{G_n\}$ are geometrically finite groups converging to $G$ algebraically. If $\Lambda(G)$ has zero area then $\delta(G) = \dim(\Lambda(G)) = 2$ and $\delta(G_n) \to 2$ by Corollary 2.5. Thus we may assume $\Lambda(G)$ has positive area. By Proposition 13.4 we must have

$$\text{vol}(C(G_n)) \to \infty,$$

so by Proposition 13.3 we get

$$\lambda_0(G_n) \to 0.$$

Since $G$ is geometrically infinite we have $\delta(G) > 1$ by Corollary 9.3 and so by Corollary 2.3 we have

$$\lim \inf_n \delta(G_n) \geq \delta(G) > 1.$$
Therefore we may as well assume $\delta(G_n) \geq 1$ for all $n$. By Sullivan’s result \cite{57} $\lambda_0 = \delta(2 - \delta)$ if $\delta \geq 1$ so,

$$\delta(G_n) \to 2,$$

as $n \to \infty$. This completes the proof of Theorem \ref{13}. \hfill \Box

14. Upper Minkowski dimension

In this section we introduce upper Minkowski dimension and prove a simple lemma which we will need in the next section.

Suppose $K$ is a bounded set in $\mathbb{R}^d$ (or any metric space for that matter) and let $N(K, \epsilon)$ be the minimal number of $\epsilon$ balls needed to cover $K$. We define the upper Minkowski dimension as

$$\overline{\dim}(K) = \limsup_{\epsilon \to 0} \frac{\log N(K, \epsilon)}{\log 1/\epsilon},$$

and the lower Minkowski dimension

$$\underline{\dim}(K) = \liminf_{\epsilon \to 0} \frac{\log N(K, \epsilon)}{\log 1/\epsilon}.$$

If the two values agree, the common value is simply called the Minkowski dimension of $K$ and is denoted $\dim(K)$. The Minkowski dimension says the the number of balls of size $\epsilon$ needed to cover $K$ grows like $\epsilon^{-\dim(K)}$. It is clear that we always have

$$\dim(K) \leq \overline{\dim}(K) \leq \underline{\dim}(K),$$

and strict inequality is possible in both places. A simple example where the Minkowski dimension exists, but is strictly larger than the Hausdorff dimension, is given by the countable set $K = \{0\} \cup \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. 
Here it is not too difficult to prove

\[ N(K, \epsilon) \sim \frac{1}{\sqrt{\epsilon}}, \]

so that Mdim\( (K) = 1/2 \). However, since the set is countable, dim\( (K) = 0 \). A much more sophisticated class of sets where the two dimensions disagree is given in [43]. Given a compact set \( K \) in the plane let \( \Omega = \mathbb{R}^2 \setminus K \) be its complement. A Whitney decomposition of \( \Omega \) is a collection of squares \( \{Q_j\} \) which are disjoint, except along their boundaries, and such that

\[ \frac{1}{10} \text{dist}(Q_j, \partial \Omega) \leq \ell(Q_j) \leq 10 \text{dist}(Q_j, \partial \Omega). \]

The existence of Whitney decomposition for any open set is a standard fact in real analysis (e.g., [51]). The squares \( \{Q_j\} \) may also be taken to be dyadic.

For any compact set \( K \) we can define an exponent of convergence similar to the exponent of convergence of a Poincaré series,

\[ \kappa = \kappa(K) = \inf \left\{ \alpha : \sum_j \ell(Q_j)^\alpha < \infty \right\}, \]

where the sum is taken over all squares in Whitney decomposition of \( \Omega = K^c \) which are within distance 1 of \( K \) (we have to drop the “far away” squares or the series might not converge). It is easy to check that this does not depend on the particular choice of Whitney decomposition. This number has been rediscovered many times in the literature, but seem to have been first used by Besicovitch and Taylor in [10]. See also Tricot’s paper [59] where he shows that it agrees with
the upper Minkowski dimension. We include a proof of this fact since it is central to our argument.

**Lemma 14.1.** For any compact set $K$, $\kappa \leq \overline{Mdim}(K)$. If $K$ also has zero area then $\kappa = \overline{Mdim}(K)$.

**Proof.** We start with the easy direction, $\kappa \leq \overline{Mdim}(K)$. Let $D = \overline{Mdim}(K)$ and choose $\epsilon > 0$. Then by the definition of $\overline{Mdim}(K)$ we have that

$$N(K, 2^{-n}) \leq C2^{D+\epsilon}.$$  

Let $\mathcal{S}\setminus$ be a covering of $K$ by fewer than $C2^{D+\epsilon}$ squares of size $2^{-n}$. If $Q$ is a Whitney square with $2^{-n-1} \leq \ell(Q) < 2^{-n}$, then choose a point $x \in K$ with $\text{dist}(x, Q) \leq \ell(Q)$. Let $S(Q) \in \mathcal{S}\setminus$ be the square containing $Q$. Since $S(Q)$ and $Q$ have comparable sizes and there distance apart is at most $\ell(Q)$, we easily see that each $S \in \mathcal{S}\setminus$ can only be associated to a uniformly bounded number of $Q$’s in the Whitney decomposition, say $A$.

Let $W_n$ be the number of Whitney squares with size $2^{-n-1} \leq \ell(Q) < 2^{-n}$. Then

$$\sum_{j} \ell(Q_j)^{D+2\epsilon} \leq \sum_{n=0}^{\infty} W_n 2^{-n(D+2\epsilon)} \leq \sum_{n=0}^{\infty} C N(K, 2^{-n}) 2^{-n(D+2\epsilon)} \leq C \sum_{n=0}^{\infty} 2^{n(D+\epsilon)} 2^{-n(D+2\epsilon)} \leq C \sum_{n=0}^{\infty} 2^{-n\epsilon} < \infty,$$
which proves \( \kappa \leq D + 2\epsilon \). Taking \( \epsilon \to 0 \) gives \( \kappa \leq D = \overline{\text{Mdim}}(K) \).

Now we assume \( K \) has zero area and will prove \( \kappa \geq D = \overline{\text{Mdim}}(K) \). As above, let \( \epsilon > 0 \) and suppose \( \{Q_j\} \) is a Whitney decomposition of \( \Omega = K^c \). By the definition of \( \overline{\text{Mdim}}(K) \) we have

\[
N(K, 2^{-n}) \geq 2^{n(D-\epsilon)},
\]

for infinitely many \( n \). Suppose \( n_0 \) is a value where this occurs and let \( S = \{S_\parallel\} \) be a covering of \( K \) with dyadic squares of size \( 2^{-n_0} \). Let for each \( S_k \in S, \ C_\parallel = \{Q_\parallel\} \) be the collection of Whitney squares which intersect \( S_k \). If we assume the \( Q_j \) are dyadic, then every square hitting \( S_k \) is contained in \( S_k \). Since the area of \( K \) is zero, this gives

\[
2^{-2n_0} = \text{area}(S_k) = \text{area}(S_k \setminus K) = \text{area}(S_k \cap \Omega) = \sum_{C_\parallel} \ell(Q_{jk})^2.
\]

Therefore,

\[
\sum_{C_\parallel} \ell(Q_{jk})^{D-2\epsilon} = \sum_{C_\parallel} \ell(Q_{jk})^2 \ell(Q_{jk})^{-2+D-2\epsilon} \\
\geq \sum_{C_\parallel} \ell(Q_{jk})^2 \ell(S_k)^{-2+D-2\epsilon} \\
= \ell(S_k)^{-2+D-2\epsilon} \sum_{C_\parallel} \ell(Q_{jk})^2 \\
= \ell(S_k)^{-2+D-2\epsilon} \ell(S_k)^2 \\
= \ell(S_k)^{D-2\epsilon} \\
= 2^{-n_0(D-2\epsilon)}.
\]
Hence,
\[
\sum_j \ell(Q_j)^{D-2\epsilon} \geq \sum_k \sum_{c_{jk}} \ell(Q_{jk})^{D-2\epsilon} \\
\geq \sum_k \ell(S_k)^{D-2\epsilon} \\
\geq N(K, 2^{-n_0}) 2^{-n_0(D-2\epsilon)} \\
\geq 2^{n_0(D-\epsilon)} 2^{-n_0(D-2\epsilon)} \\
= 2^{n_0\epsilon}.
\]

Taking \(n_0 \to \infty\), we get \(\sum_j \ell(Q_j)^{D-2\epsilon} = \infty\), and hence \(\kappa \geq D - 2\epsilon\).

Taking \(\epsilon \to 0\) gives the desired result. \(\square\)

15. **Minkowski dimension equals Hausdorff dimension**

In this section we prove,

**Theorem 15.1.** If \(G\) is a finitely generated Kleinian group then the Minkowski dimension of \(\Lambda\) exists and equals the Hausdorff dimension.

Finitely generated is necessary, because it is possible to construct infinitely generated Kleinian groups where the Minkowski dimension of \(\Lambda(G)\) fails to exist.

**Theorem 15.2.** If \(G\) is finitely generated Kleinian group and area(\(\Lambda(G)\)) = 0 then \(\delta(G) = \dim(\Lambda)\).

Thus if the Ahlfors conjecture holds, we would have \(\delta(G) = \dim(\Lambda(G))\) for all finitely generated groups. Theorem [15.1] and Theorem [15.2] follow from this more technical looking result.
Theorem 15.3. Suppose $G$ is a finitely generated Kleinian group. If $\text{area}(\Lambda(G)) = 0$ then $\delta(G) = \overline{M\text{dim}}(\Lambda(G))$.

Proof of Theorem 15.1 and Theorem 15.2: To see how Theorem 15.3 implies Theorem 15.1 we consider two cases. First, if $\dim(\Lambda) = 2$ then

$$2 = \dim(\Lambda) \leq \overline{M\text{dim}}(\Lambda) \leq \overline{M\text{dim}}(\Lambda) \leq 2,$$

so all are equal to 2. On the other hand, if $\dim(\Lambda) < 2$, then $\Lambda$ has zero area, so Theorem 15.3 applies and gives

$$\dim(\Lambda) \leq \overline{M\text{dim}}(\Lambda) \leq \overline{M\text{dim}}(\Lambda) = \delta(G) = \dim(\Lambda_c) \leq \dim(\Lambda),$$

so again, all these numbers are equal. Thus in both cases the Minkowski dimension exists and equals the Hausdorff dimension. The second case also proves Theorem 15.2.

Our proof of Theorem 15.3 breaks up into three cases:

1. $\Omega(G)/G$ is compact.
2. $\Omega(G)/G$ has punctures and $G$ is geometrically finite.
3. $\Omega(G)/G$ has punctures and $G$ is geometrically infinite (i.e., $\dim(\Lambda(G)) = 2$).

Note that the third case is not needed for Theorem 15.1, since the two dimensions must agree whenever the Hausdorff dimension is 2.

It is worth also noting that our proof uses Theorem 1.4, but this is not necessary. It is possible to prove Theorem 15.3 without using Theorem 1.4. This argument uses an intricate stopping time argument involving the “good” and “bad” horoballs and is completely “elementary” (e.g., two dimensional), but using Theorem 1.4 allows us to give a
simpler proof. It may be possible that the implication could be reversed and Theorem 15.3 used to give an alternate proof of Theorem 1.4 but we have not done this (it would suffice to show that if $G$ is geometrically infinite and $\delta(G) < 2$ then $\delta(G) < \overline{\operatorname{Mdim}}(\Lambda(G))$).

Since $\delta(G) = \dim(\Lambda_c) \leq \dim(\Lambda)$ and $\dim(\Lambda) \leq \overline{\operatorname{Mdim}}(\Lambda)$ we already know that

$$\delta(G) \leq \overline{\operatorname{Mdim}}(\Lambda),$$

but here is a simple proof of this fact which does not require Theorem 1.1. Chose a point $z_0 \in \Omega(G)$. By the Lemma 10.1 the critical exponent $\delta(G)$ for the Poincaré series of $G$ is the same as the critical exponent for the series

$$\sum_{g \in G} d(g(z))^{\alpha}.$$

Because each Whitney square contains only a bounded number of images of $z_0$,

$$\sum_{g \in G} d(g(z_0))^{\alpha} \leq C \sum_{j} \ell(Q_j)^{\alpha}.$$

Hence the sum on the left converges whenever the sum on the right does. Thus

$$\delta(G) \leq \kappa(\Lambda) \leq \overline{\operatorname{Mdim}}(\Lambda),$$

for any Kleinian group $G$.

We will now start the three cases of Theorem 15.3.

Proof of case (1) in Theorem 15.3: Now suppose $G$ is finitely generated, $\Omega(G)/G = R_1 \cup \cdots \cup R_N$ is a union of compact surfaces and $\Lambda(G)$ has zero area. Let \{Q_j\} be a Whitney decomposition of $\Omega(G)$. 
By the previous lemma there is an $\epsilon > 0$ so that every $Q_j$ contains a hyperbolic ball of radius $\epsilon$.

By the compactness of the surfaces $R_1, \ldots, R_N$, we can choose a finite number of points $E = \{z_1, \ldots, z_M\} \subset \Omega(G)$, so that $E$ projects to an $\epsilon$-dense set in $R_1 \cup \cdots \cup R_N$, i.e., every point of $\Omega$ is within distance $\epsilon$ of some point of $G(E) = \bigcup_{i=1}^{M} \bigcup_{g \in G} g(z_i)$. Thus each Whitney square $Q_j$ contains at least one point of $G(E)$. Therefore

$$\sum_j \ell(Q_j)^\alpha \leq C \sum_{z \in G(E)} d(z)^\alpha.$$  

By Lemma 6.1, the infinite series on the right hand side converges for $\alpha > \delta(G)$, hence so does the left hand side, i.e., $\kappa(\lambda) \leq \delta(G)$. Since we already proved the opposite inequality we have $\kappa(\Lambda) = \delta(G)$.

If $\Lambda(G)$ has zero area, then Lemma 14.1 implies

$$\underline{MDim}(\Lambda) = \kappa(\Lambda) = \delta(G),$$

as desired. □

Proof of case (2) in Theorem 15.3: In this case we assume that $G$ is geometrically finite, so by Corollary 11.6 every parabolic cusp of $\Omega(G)$ is double cusped. Thus by part (2) of Lemma 10.2 there is an $\eta > 0$ so that every horoball in $\Omega(G)$ is $\eta$-good.

Let $\epsilon > 0$ and choose $n_0$ so that

$$N(\Lambda, 2^{-n_0}) \geq 100 \cdot 2^{n_0(D-\epsilon/2)}.$$  

Let $r = 2^{-n_0}$. Let $S = \{S_k\}$ be a collection of $2^{n_0(D-\epsilon)}$ squares of size $r$ so that the triples $3S_k$ are pairwise disjoint and $\frac{1}{3}S_k \cap \Lambda \neq \emptyset$ for each $k$.  


For each \( \eta \)-good horoball with \( \text{diam}(B) \geq r/3 \), let \( G_B \) be the collection of squares in \( S \) which so that \( \frac{1}{3} S \) hits \( B \). Let \( G \) be the union of all the \( G_B \). Note that the number of squares in \( G_B \) is at most \( C \text{diam}(B)/r \). Therefore if \( R = \text{diam}(B) \),

\[
\sum_{S \in G_B} \ell(S)^{D-\epsilon} \leq C \frac{R}{r} r^{D-\epsilon} = CRr^{D-1-\epsilon} \leq CRR^{D-1-\epsilon} = CR^{D-\epsilon}
\]

By part (1) of Lemma \[10.2\] there is an orbit point \( z \in G(E) \cap B \) such that

\[
d(z) \sim \text{diam}(B).
\]

For this point,

\[
d(z)^{D-\epsilon} \geq C \sum_{S \in G_B} \ell(S)^{D-\epsilon}.
\]

If more than half the squares in \( S \) belong to \( G \) then this argument shows

\[
\sum_{z \in G(E)} d(z)^{D-\epsilon} \geq \frac{1}{2} C^2 n_0 \epsilon/2.
\]

If this happens for arbitrarily large \( n_0 \) then we have shown that \( \delta(G) \geq D - \epsilon \), as desired.

Thus we may assume that fewer than half the elements of \( S \) are in \( G \). In this case, part (1) of Lemma \[10.2\] implies that each horoball \( B \) is associated to an orbit point \( z \) so that

\[
diam(B) \sim d(z).
\]
Moreover, if we sum over all Whitney square hitting $B$ then

$$
\sum_{Q \cap B \neq \emptyset} \ell(Q)^{\alpha} \sim \text{diam}(B)^{\alpha} \sim d(z)^{\alpha}.
$$

Therefore the proof in this case is completed exactly as in the compact case.

Proof of case (3) of Theorem 15.3: We can consider the final case. Suppose that $G$ is geometrically infinite (so $\dim(\Lambda(G)) = 2$ by Theorem 1.4) and assume $\text{area}(\Lambda(G)) = 0$. By Lemma 14.1 $\kappa(\Lambda(G)) = M\dim(\Lambda(G)) = \dim(\Lambda(G)) = 2$. Let $\{B_j\}$ be a listing of the horoballs in $\Omega(G)$. Each horoball contains a Whitney square of comparable size so for any $\epsilon > 0$

$$
\sum_j \text{diam}(B_j)^{2-\epsilon/2} = \infty.
$$

Thus if

$$
\mathcal{B}_\lambda = \{B : \epsilon^{-\lambda} - \infty \leq \text{diam}(B) < \epsilon^{-\lambda}\},
$$

and $N_n = \#\mathcal{B}_\lambda$, we must have

$$
N_n \geq 2^{n(2-\epsilon)},
$$

for infinitely many values of $n$. Fix a value of $n_0$ where this holds and note that for at least half the balls $B$ in $\mathcal{B}_\lambda$ there is a second ball $B' \in \mathcal{B}_\lambda$ such that

$$
\text{dist}(B, B') \leq 2^{n(1-\epsilon)} \leq \text{diam}(B)2^{-n_\epsilon},
$$

(otherwise we would have so many disjoint balls that we contradict the assumption that $\Lambda(G)$ has diameter 1). By part (6) of Lemma 10.2, this implies that both $B$ and $B'$ are $2^{-3n_\epsilon}$-good horoballs. Let $\mathcal{G}_\eta \subset \mathcal{B}_\lambda$ be the subcollection of $2^{-3n_\epsilon}$-good horoballs. For any $\eta$-good horoball
Let $z$ be the point given in part (1) of Lemma 10.2 and let $h$ be the primitive parabolic element fixing $B_j$. Then an easy calculation shows

$$\sum_{k \in \mathbb{Z}} d(h^k(z))^\alpha \geq C \text{diam}(B)^\alpha \eta^{\alpha - 1},$$

for any $0 < \alpha < 2$ and some $C$ depending on $G$ and $\alpha$.

Thus if $z_j$ is the good point in $B_j$ given by (1) of Lemma 10.2,

$$\sum_{z \in G(E)} d(z)^\alpha \geq \sum_{B_j \in \mathcal{F}} \sum_{k \in \mathbb{Z}} d(h^k(z_j))^\alpha \geq C \sum_{B_j \in \mathcal{F}} \text{diam}(B)^\alpha 2^{-3n_0(\alpha - 1)} \geq C 2^{n_0(2-\epsilon)} 2^{-n_0\alpha} 2^{-3n_0(\alpha - 1)} \geq C 2^{n_0[(2-\epsilon)-\alpha - 3\epsilon(\alpha - 1)]} \geq C 2^{n_0[(2-\alpha)-\epsilon - 3\epsilon(\alpha - 1)]}$$

If we first choose any $\alpha < 2$ and then choose $\epsilon$ sufficiently small, the exponent is positive. Thus $\delta(G) > \alpha$ for any $\alpha < 2$, i.e., $\delta(G) = 2$. This completes the proof of the third and final case of Theorem 15.3.

\[\Box\]

This finishes the proofs of Theorem 15.1 and Theorem 15.3.
16. Teichmüller spaces

In this section we shall consider $\dim(\Lambda(G))$ as a function on the closure of Teichmüller space $T(S)$ of a finite type hyperbolic surface $S$.

Given a finite type surface $S$ (compact with a finite number of punctures (possible none)), the Teichmüller space $T(S)$ is the set of equivalence classes of quasiconformal mapping of $S$ to itself. Each such is represented by a Beltrami differential $\mu$ which may be lifted to a Beltrami differential $\mu$ on the the upper halfplane, $\mathbb{H}$. Let $\Gamma$ be a Fuchsian group acting on $\mathbb{H}$ such that $\mathbb{H}/\Gamma = S$. There is a quasiconformal mapping $F$ of the plane which fixes $0, 1, \infty$ and such that $\partial F/\partial F = \mu$ on $\mathbb{H}$ and so that $F$ is conformal on the lower half plane. On the lower half-plane the Schwarzian derivative $S(F)$ satisfies

$$\|S(F)\| = \sup_{z} |\text{Im}(z)|^2 |S(F)(z)| \geq 6 < \infty.$$ 

This realizes $T(S)$ as a bounded subset of a Banach space and gives a metric on $T(S)$. The closure of $T(S)$ with respect to this metric is denoted $\overline{T(S)}$ and the boundary by $\partial T(S)$. Points of $\overline{T(S)}$ may be identified with certain Kleinian groups which are isomorphic to $\Gamma$. Moreover, convergence in the Teichmüller metric above implies algebraic convergence of the groups. Recall that a group is called degenerate if $\Omega(G)$ has one component and this component is simply connected. Such groups must be geometrically infinite by a result of Greenberg [34]. $G \in \partial T(S)$ is called a cusp if there is a hyperbolic element in $\Gamma$ which becomes parabolic in $G$. Bers showed $\partial T(S)$ consists entirely of degenerate groups and cusps and that degenerate groups form a dense...
$G_\delta$ set in $\partial T(S)$ in [3] (in fact, the cusps lie on a countable union of real codimension 2 surfaces). McMullen [46] proved there is a dense set of geometrically finite cusps in $\partial T(S)$.

Recall that Theorem 1.5 says that if $\{g_n\}$ converges algebraically to $G$ then

$$\liminf_{n \to \infty} \dim(\Lambda(g_n)) \geq \dim(\Lambda(G)).$$

One special case where this holds is for $G \in \overline{T(S)}$, the closure of the Teichmüller space of a finite type hyperbolic Riemann surface $S$. Since $\Lambda(G)$ is at most 2 and is lower semi-continuous, it is continuous whenever it takes the value 2 (i.e., at the geometrically infinite groups). Since these points are dense on the boundary of Teichmüller space, this function must be discontinuous at the geometrically finite cusps on the boundary. Thus

**Corollary 16.1.** Suppose $S$ is a hyperbolic Riemann surface of finite type. Then $\dim(\Lambda(G))$ is lower semi-continuous on $\overline{T(S)}$, and is continuous everywhere except at the geometrically finite cusps in $\partial T(S)$ where it is discontinuous.

This also show that equality in Corollary 2.3 and Theorem 1.3 need not occur, because a geometrically finite cusp ($\delta(G) < 2$) can be approximated by by geometrically finite groups which are close to degenerate groups (so $\delta(G_n) \to 2$). The discontinuity at the geometrically finite cusped had been proved earlier by Taylor in [58]; he showed that for each geometrically finite cusp $G$ there is a sequence $G_n \to G$ algebraically, but $G_n \to H$ geometrically where $H$ is a geometrically finite
group containing $G$ and
\[ \dim(\Lambda(G_n)) \to \dim(\Lambda(H)) > \dim(\Lambda(G)). \]

If $f$ is lower semi-continuous then $\{ f \leq \alpha \}$ is closed. Thus,

**Corollary 16.2.** The set $E_\alpha = \{ G \in \overline{T(S)} : \dim(\Lambda(G)) \leq \alpha \}$ is closed in $\overline{T(S)}$. The set $F_\alpha = \{ G \in \partial T(S) : \dim(\Lambda(G)) \leq \alpha < 2 \}$ is a closed, nowhere dense subset of $\partial T(S)$.

Since a lower semi-continuous functions takes a minimum on a compact set, Theorem 1.2 implies

**Corollary 16.3.** $\dim(\Lambda(G))$ takes a minimum value on $\partial T(S)$ and this minimum is strictly larger than 1.

It is not clear where the minimum occurs. Canary has suggested it might occur at the cusp group corresponding to shrinking a minimum length geodesic on $S$ to a parabolic, since this requires the “least” deformation of the Fuchian group (in some sense). Since $\dim(\Lambda(G))$ takes a minimum on $\partial T(S)$ which is $> 1$, any group in $\overline{T(S)}$ with small enough dimension must be quasi-Fuchsian. Thus

**Corollary 16.4.** Suppose $\{ G_n \}$ is a sequence of quasiconformal deformations of a Fuchsian group $G$ (i.e., $\{ G_n \}$ is a sequence in $T(S)$, $S = \mathbb{D}/G$). If $\dim(\Lambda(G_n)) \to 1$, then $G_n \to G$.

**Proof.** This is immediate from Theorem 13.1 and the fact (from Theorem 1.2) that $G$ is the only point in $\overline{T(S)}$ where $\dim(\Lambda) = 1$. \qed
Corollary 16.5. Suppose $G$ is a finitely generated and has a simply connected invariant component $\Omega_0$ (possibly not unique). Let $\Omega_0/G = S$. Then for any $\epsilon > 0$ there is a $\delta$, depending only on $S$ and $\epsilon$, such that $\dim(\Lambda(G)) < 1 + \delta$ implies $G$ is a $\epsilon$-quasiconformal deformation of a Fuchsian group.

Proof. If $G$ is a quasi-Fuchsian group this follows from the previous result. Maskit [42] proved that a finitely generated Kleinian group with two invariant components is quasi-Fuchsian, so we may now assume $G$ is a $b$-group (i.e., has exactly one simply connected, invariant component). If $G$ is geometrically infinite then $\dim(\Lambda(G)) = 2$, which contradicts our assumption. Therefore, $G$ must be geometrically finite. Abikoff [1] proved that every geometrically finite $b$ group covering $S$ is on the boundary of the Teichmüller space $T(S)$, and so its dimension is bounded away from 1 by Corollary [16.3].

It is not true that the $\epsilon$ in Corollary [16.5] can be taken to depend only on the topological type of $S$ (e.g., the number of generators of $G$). For example, given a surface with punctures $S$, it is possible to use the combination theorems to construct a $b$-group $G$ with $\dim(\Lambda(G))$ as close to one as we wish and so that $\Omega_0/G$ is homeomorphic to (thought not conformally equivalent to) $S$.

Larman showed that there is an $\epsilon_0$, such that if $\{D_j\}$ is a collection of three or more disjoint open disks then the dimension of $\overline{\mathbb{C}} \setminus \cup_j D_j$ is larger than $1 + \epsilon_0$. A careful reading of Larman’s paper [40] shows that his proof gives
Proposition 16.6. There is an $\epsilon_0 > 0$ such that if $\{D_j\}$ is any collection of three or more disjoint open $\epsilon_0$-quasidisks, then $\dim(\mathbb{C} \setminus \bigcup_j D_j) > 1 + \epsilon_0$.

Recall that a web group is a finitely generated Kleinian groups each of whose component subgroups is quasi-Fuchsian. Suppose $G$ is web group. If $\Omega(G)$ has only two components then $G$ is quasi-Fuchsian [43]. So suppose $G$ has three or more components and let $\{G_1, \ldots, G_n\}$ representatives of each conjugacy class of component subgroups. By the last corollary either one of these has limit set with dimension $> 1 + \delta$ or all are $\epsilon_0$-quasicircle. In the latter case, Larman’s theorem implies that $\Lambda(G)$ has dimension bigger than $\epsilon_0$. In either case the dimension is bounded away from 1 by a number which only depends on the conformal structure of $\Omega(G)/G$. of generators for each component subgroup.

Corollary 16.7. Suppose $G$ is a finitely generated web group which is not quasi-Fuchsian. Then $\dim(\Lambda) > 1 + \epsilon$ where $\epsilon$ depends only on the conformal types of the components of $\Omega(G)/G$.

It is not true that the dimension of limit sets of proper web groups (i.e., not quasi-Fuchsian) is bounded uniformly away from 1. Canary, Minsky and Taylor have constructed examples of proper web groups (with a fixed number of generators) whose limits sets have dimension arbitrarily close to 1 (personal communication).

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