Scattering from rough surfaces: A simple reflection phenomenon in fractional space

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Abstract

In this paper, scattering of incident plane waves from rough surfaces have been modeled in a fractional space. It is shown how wave scattering from a rough surface, could be a simple reflection problem in a fractional space. In the integer space, fluctuations of the surface result in wave scattering while in the fractional space these fluctuations are compensated by the geometry of the space. In the fractional space, reflection leads to the same results as the scattering in the integer space. To make it more clear, scattered wave function in the framework of Kirchhoff theory is considered in a fractional space and results are compared with those from a self-affine surfaces. Our results show that these two approaches are comparable.
I. INTRODUCTION

The problem of wave scattering from rough surfaces has been of substantial interest for more than a century. Extensive studies have been carried out at experimental and theoretical levels until now [1–3]. Indeed, this problem has gained considerable attention in diverse areas of science and engineering [4–7]. In reality, no surface is ideally smooth. In fact, wave scattering from a surface is affected by the morphology and roughness of this surface. Small perturbation method (SPM) [8] and Kirchhoff theory [9–11] are the two oldest and most employed approximate approaches to this problem.

When the surface under study is slightly rough, then SPM can be used for finding the solution to the scattering problem. In this case, the rough surface is assumed as a height perturbation to a smooth surface and the resulting changes due to the roughness is considered in the scattering coefficient. This approach requires that the height function is everywhere small compared to the wavelength of the incident wave and its gradient is also small in comparison to unity [9] but it is independent from the radius of curvature of the surface. When the points on the surface have a large radius of curvature relative to the wavelength of the incident field and the surface roughness may not be small compared to it, Kirchhoff theory can be applied. Kirchhoff theory, also known as the "tangent plane theory" is used in conjunction with an integral formula, to give an expression for the scattered field from the surface in terms of the approximated surface field. The physical basis for this theory is tangent plane approximation: any point on the surface is assumed to have the same optical behavior as if the surface was locally flat. In fact, the scattered field and its normal derivative at the boundary can be expressed through the incident field which help to reconstruct the scattered field in the total space [10]. Here, we focus on the Kirchhoff theory which is a local approximation method and could be used for surfaces much rougher than those considered by SPM [9]. We use the approach of the fractional calculus to deal with this scattering problem. In general, fractional calculus is the generalization of the classical calculus which deals with the integrals and derivatives of arbitrary (real or even complex) orders [12–14]. Remarkable attention and significance for this area have been achieved in recent decades because of its diverse application in varies fields of science and technology ranging from physics [15–18], plasma [19] and polymer [20] to engineering [21, 22], biology [23, 25], finance [26] and ... . Indeed, there are situations that fractional calculus are so useful. For instance, the concept
of nonlocality, memory and hereditary properties could be imported using the fractional operators.

Here, we consider the problem of scattering of a monochromatic plane wave from a rough surface and reformulate the problem in a fractional space. In the latter space, which is described by a constant non-integer fractal dimension, the surface is not rough anymore. In the integer space, fluctuations of the surface lead to wave scattering while in the fractional space these fluctuations are compensated by the topology of the space. Thus, the problem changes into the reflection of plane waves from a flat surface in a fractional space of order $\alpha$, $2 < \alpha < 3$. By considering the scattered wave function in the framework of Kirchhoff theory we show how reflection from a flat surface in the fractional space leads to the same results as the scattering in the integer space for the self-affine surfaces.

The paper is organized as follows. In Sec. II, we briefly review the Kirchhoff theory of wave scattering. In Sec. III, we discuss our method and our results are presented. Finally, Sec. IV summarizes our conclusion.

II. KIRCHHOFF THEORY OF WAVE SCATTERING

In Kirchhoff theory of wave scattering, valid in the far field region of the rough surface, we consider a incident plane wave, $\psi^{inc}(r) = \exp(-ik_{inc}.r)$, $k$ and $r$ stand for wave number and position, respectively. The geometry used to study the scattering phenomena from a rough surface, is shown in Fig. 1. By making the assumption that there is no point on the scatterer surface with infinite gradient, we work under the Dirichlet boundary condition, i.e., the surface reflectance, $R_0 = -1$. Therefore, the total scattered field, $\psi^{sc}(r)$, over the mean reference plane $A_M$, is given by

$$\psi^{sc}(r) = \frac{ie^{ikr}}{4\pi r} \int_{A_M} \left( \frac{\partial h}{\partial x_0} + b \frac{\partial h}{\partial y_0} - c \right)$$

$$\times \exp (ik[Ax_0 + By_0 + Ch(x_0, y_0)]) \, dx_0 dy_0,$$

(1)
where

\[ A = \sin \theta_1 - \sin \theta_2 \cos \theta_3, \]
\[ B = -\sin \theta_2 \sin \theta_3, \]
\[ C = -(\cos \theta_1 + \cos \theta_2), \]
\[ a = \sin \theta_1(1 - R_0) + \sin \theta_2 \cos \theta_3(1 + R_0), \]
\[ b = \sin \theta_2 \sin \theta_3(1 + R_0), \]
\[ c = \cos \theta_2(1 + R_0) - \cos \theta_1(1 - R_0). \]  

(2)

Here, \( h(x_0, y_0) \) is the height of the surface at position \((x_0, y_0)\) from the reference surface. The total scattered field intensity, the experimentally measurable quantity, \( I_{tot} = I_{coh} + I_d = \langle \psi^{sc}(r)\psi^{sce}(r) \rangle \), is consisted of two parts; the coherent intensity (\( I_{coh} \)) and the diffuse one (\( I_d \)). The major contribution of the coherent part is seen in the specular direction while for the diffuse part it is in other directions. Root mean square, \( \sigma \), is a scale for the surface roughness that affects the specular part; besides \( \sigma \), correlation function is another parameter which has only impacts on the diffuse part. In previous works on Kirchhoff theory, these two parts of intensity (coherent, \( I_{coh} \), and diffuse, \( I_d \), parts) were separately studied, but here, we consider the total scattered intensity, \( I_{tot} \), itself.
III. EQUIVALENCE OF 3D SCATTERING AND REFLECTION IN THE FRACTIONAL SPACE

In our new perspective, to solve this integral and get the scattered field, instead of dealing with roughness, we take the scatterer surface as a completely smooth one and compensate the outcomes by considering a fractal dimension for our space. As a consequence, the dimension of the space is no longer the integer dimension of the Euclidian embedding space. In other words, we have a fractional integration of a plane wave over a flat surface. Subsequently, the concept of fractional calculus will appear and in lieu of scattering from a rough surface we have just reflection from a flat surface in fractional space. The coherent intensity corresponds to reflection in specular direction. Also there are reflections in other directions than specular one, which are equivalent to diffuse scattering from rough surfaces.

Since there is no height fluctuation, the mean height from the reference surface is zero everywhere, \( h(x_0, y_0) = 0 \). Finally, the scattered field will be of the following form,

\[
\psi_{sc}(r) = \frac{i(\vec{c})ke^{ikr}}{4\pi r} \int_{s_0} \exp(ikAs) d^\alpha s,
\]

where the integral is on the light spot size, \( s_0 \). Here \( \alpha \) is the order of fractional integration which shows the fractal dimension of the surface, in this case \( 2 < \alpha < 3 \) and \( s \) is the flat surface in fractional space. Fractal dimension or equivalently Hurst exponent, \( H = d - \alpha \) (where \( d = 3 \) is the dimension of the embedded space so \( 0 < H < 1 \)) is a measure for the roughness of the fractal surface. For integer spaces with fluctuations, small \( H \) values represent high irregularity, while for \( H \) close to 1 the surface is more regular; hence, for fractional dimensional spaces, small \( H \) values correspond to surfaces with higher dimensionality while large \( H \) values correspond to smaller dimensions.

All we have to do now is to find the fractional integral of order \( \alpha \) of a plane wave. In fractional calculus, the \( \alpha \) order integral of an exponential function is given as \[12\],

\[
_0D_x^{-\alpha}e^{mx} = x^\alpha e^{mx} \gamma*(\alpha, mx) = E_x(\alpha, m),
\]

where \( _0D_x^{-\alpha} \) is the left Riemann-Liouville integral of order \( \alpha \), \( m \) is an arbitrary constant that is complex here, \( x \) is the integral variable. In this formula, \( \gamma^* \) is the incomplete Gamma function which is defined as,

\[
\gamma^*(\alpha, mx) = \frac{1}{(mx)^\alpha \Gamma(\alpha)} \times \int_0^{mx} \xi^{\alpha-1}e^{-\xi}d\xi,
\]
FIG. 2: (a) Dependence of the total scattered field intensity, $I_{\text{tot}}$, on scattering angle, $\theta_2$, for different values of Hurst exponent, vertical incident, $\theta_1 = 0$, and for monochromatic light $\lambda = 500$ nm and $\theta_3 = 0$.

and $E_x(\alpha, m)$ is the Miller-Ross function which is related to the Mittag-Leffler functions as follows,

$$E_x(\alpha, m) = x^\alpha E_{1,\alpha+1}(mx).$$  \hspace{1cm} (6)

Fig. 2 shows the dependence of the total normalized scattered field intensity, $I_{\text{tot}}$, on the scattering angle, $\theta_2$, for different values of Hurst exponent, $H$. By decreasing $H$, fractional dimension of the surface will increase which is equivalent to increasing the roughness of the surface in the Euclidian space. As a result, the amount of reflected intensity in specular direction will decrease and contribution of reflection in other directions which appear in this approach and corresponds to diffuse component from rough surface will increase, which is in good agreement with the results obtained by Kirchhoff theory for rough surfaces \[27\].

Reflection in specular angle, $\theta_1 = \theta_2$ is equivalent to coherent part of scattering, and in other directions is its diffuse part. So, intensity in angles far away from specular directions, gives us the diffuse intensity.

Wavelength is the observation scale of the surface. When wavelength of the incident beam is shorter than the correlation length of the surface, the more the wavelength is decreased, the more the roughness of the surface is being sensed by the incident beam. For a flat surface in the fractional space, the correlation length is infinite so it is larger than the incident wavelength and roughness could be sensed more. Thus, the diffuse reflected intensity should
FIG. 3: Dependence of the diffuse scattered field intensity $I_d$ on the angle of scattering $\theta_2$, for $H = 0.7$, $\theta_3 = 0$, vertical incident and for different values of wavelength ($\lambda = 500, 1000$ and 1500 nm).

increase by decreasing the wavelength which could be seen in Fig. 3. The total intensity for angles far from zero (non-specular angles) shows the diffuse intensity and it decreases by decreasing the wavelength which is in good agreement with the results obtained in [27].

Two characteristic scales of the problem are roughness and wavelength and in fact scattering could be modeled by $k\sigma$. When $H$ is constant for a surface, by increasing $\sigma$ (or increasing $\xi$), diffuse intensity increases for the specular angle. In other words, increasing $\sigma$ is equivalent to decreasing the wavelength and because the wavelength is smaller than the distance between two peaks or correlation length of the surface, reflection or scattering is larger for the specular angle. But for larger wavelengths (which are equivalent to smaller $\sigma$ or $\xi$), the wavelength is larger than (or of the same order as) the distance between two peaks and the intensity is of the same order for all angles.

IV. CONCLUSION

In this paper, we proposed a new perspective to the problem of scattering from a rough surface in an integer space. We have shown how this scattering problem is equivalent to reflection phenomenon in fractional space. In the integer space, fluctuations of the surface are the main reasons for scattering while in the fractional space these fluctuations are compensated by the topology of the problem. To clarify our perspective, we have considered the
Kirchhoff theory of scattering both in the integer dimensional and fractional spaces and we have shown that the two approaches are comparable.

[1] T. A. Germer, Phys. Rev. Lett. 85, 349-352 (2000).
[2] T. M. Elfouhaily, C. A. Gurin, Waves in Random Media. 14, pp. 1-40 (2004).
[3] M. Salami, M. Zamani, S. M. Fazeli and G. R. Jafari, J. Stat. Mech. P08006 (2011).
[4] S. Schröder, A. Duparr, L. Coriand, A. Tnnermann, and D. H. Penalver, and J. E. Harvey, Optics Express. 19, 9820-9835 (2011).
[5] Q. Xua, H. Lin, X. Lid, J. Zuo, Q. Zhenge, W. G. Pichelf, and Y. Liug, International Journal of Remote Sensing. 31, 993-1008 (2010).
[6] A. Arnold-Bos, A. Khenchaf, and A. Martin, IEEE Transactions on Geoscience and Remote Sensing. 45, 3372-3383 (2007).
[7] S.K. Sinha, E.B. Sirotai, S. Garof and H.B. Stanley, Phys. Rev. B. 38, 2297-2311 (1988).
[8] J.T. Johnson, J. Opt. Soc. Am. A. 16, 2720 (1999).
[9] J.A. Ogilvy, Theory of WaveScattering from Random Rough Surfaces; Institute of Physics, Bristol (1991);
[10] A. G. Voronovich, Wave Scattering from Rough Surfaces, 2nd ed.; Springer, Heidelberg (1994).
[11] G.R. Jafari, P. Kaghazchi, R.S. Dariani, A. Iraji zad, S.M. Mahdavi, M. Reza Rahimi Tabar, and N. Taghavinia, J. Stat. Mech. (2005) P04013; G.R. Jafari, S.M. Mahdavi, A. Iraji zad and P. Kaghazchi, Surface and Interface Analysis. 37 (2005) 641-645.
[12] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations; Wiley-Interscience (1993).
[13] I. Podlubny, Fractional Differential Equations; Academic Press, New York (1998).
[14] R. Metzler and J. Klafter, Phys. Rep. 339, 1 (2000).
[15] D.A. Kessler and E. Barkai, Phys. Rev. Lett. 108, 230602 (2012).
[16] V. E. Tarasov, Chaos. 14, 123 (2004).
[17] F. Wilczek, Fractional Statistics and Anyon Superconductivity; World Scientific Pub Co Inc (1990).
[18] B.J. West, P. Grigolini, R.M. and T.F. Nonnenmacher, Phys. Rev. E. 55, 99-106 (1997).
[19] M. Vahabi, M.H. Allami, B. Shokri, Phys. Rev. E 84, 026401 (2011).
[20] N. Heymans, Nonlinear Dynamics. 38, 221-231 (2004).

[21] J. Sabatier, O. P. Agrawal, J. A. Tenreiro Machado, Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering; Springer (2007).

[22] A. Carpinteri, B. Chiaia, P. Cornetti, Engineering Fracture Mechanics. 70, 2321-2349 (2003).

[23] R. L. Magin, Computers and Mathematics with Applications. 59, 1586-1593 (2010).

[24] I. Bronstein, Y. Israel, E. Kepten, S. Mai, Y. Shav-Tal, E. Barkai and Y. Garini, Phys. Rev. Lett. 103, 018102 (2009).

[25] D. Brockmann, L. Hufnagel, and T. Geisel, The scaling laws of human travel, Nature. 439, 462-465 (2006).

[26] E. Scalas, R. Gorenflo, F. Mainardi, Physica A: Statistical Mechanics and its Applications. 284, 376-384 (2000).

[27] M. Zamani, M. Salami, S.M. Fazeli and G.R. Jafari, J. Opt. Mod. 59, 16 (2012).