TWO COUNTEREXAMPLES FOR POWER IDEALS OF HYPERPLANE ARRANGEMENTS.

FEDERICO ARDILA AND ALEXANDER POSTNIKOV

Abstract. We disprove Holtz and Ron’s conjecture that the power ideal \( C_{A,-2} \) of a hyperplane arrangement \( A \) (also called the internal zonotopal space) is generated by \( A \)-monomials. We also show that, in contrast with the case \( k \geq -2 \), the Hilbert series of \( C_{A,k} \) is not determined by the matroid of \( A \) for \( k \leq -6 \).

Remark. This note is a corrigendum to our article [1], and we follow the notation of that paper.

1. Introduction.

Let \( A = \{H_1, \ldots, H_n\} \) be a hyperplane arrangement in a vector space \( V \); say \( H_i = \{x \mid l_i(x) = 0\} \) for some linear functions \( l_i \in V^* \). Call a product of (possibly repeated) \( l_i \)'s an \( A \)-monomial in the symmetric algebra \( C[V^*] \). Let \( \text{Lines}(A) \) be the set of lines of intersection of the hyperplanes in \( A \). For each \( h \in V \) with \( h \neq 0 \), let \( \rho_A(h) \) be the number of hyperplanes in \( A \) not containing \( h \). Let \( \rho = \rho(A) = \min_{h \in V} (\rho_A(h)) \). For all integers \( k \geq -(\rho + 1) \), consider the power ideals:

\[
I_{A,k} := \left\langle h^{\rho_A(h)+k+1} \mid h \in V, h \neq 0 \right\rangle, \quad I'_{A,k} := \left\langle h^{\rho_A(h)+k+1} \mid h \in \text{Lines}(A) \right\rangle
\]

in the symmetric algebra \( C[V] \). It is convenient to regard the polynomials in \( I_{A,k} \) as differential operators, and to consider the space of solutions to the resulting system of differential equations:

\[
C_{A,k} = I_{A,k}^\perp := \left\{ f(x) \in C[V^*] \mid h \left( \frac{\partial}{\partial x} \right)^{\rho_A(h)+k+1} f(x) = 0 \text{ for all } h \neq 0 \right\}
\]

which is known as the inverse system of \( I_{A,k} \). Define \( C'_{A,k} \) similarly. These objects arise naturally in numerical analysis, algebra, geometry, and combinatorics. For references, see [1, 3].

One important question is to compute the Hilbert series of these spaces of polynomials, graded by degree, as a function of combinatorial invariants of \( A \). Frequently, the answer is expressed in terms of the Tutte polynomial of \( A \). This has been done successfully in many cases. One strategy used independently by different authors has been to prove the following:

\[\text{Supported in part by NSF Award DMS-0801075 and CAREER Award DMS-0956178.}\]
\[\text{Supported in part by NSF CAREER Award DMS-0504629.}\]
(i) There is a spanning set of \( A \)-monomials for \( C_{A,k} \).
(ii) There is an exact sequence \( 0 \to C_{A \setminus H,k}(-1) \to C_{A,k} \to C_{A/H,k} \to 0 \) of graded vector spaces.
(iii) Therefore, the Hilbert series of \( C_{A,k} \) is an evaluation of the Tutte polynomial of \( A \).

Here \( A \setminus H \) and \( A/H \) are the deletion and contraction of \( H \), respectively.

For \( k \geq -1 \), this method works very nicely. Dahmen and Michelli [2] were the first ones to do this for \( C'_{A,-1} \). Postnikov-Shapiro-Shapiro [5] did it for \( C_{A,0} \), while Holtz and Ron [3] did it for \( C'_{A,0} \). In [1] we did it for \( C_{A,k} \) for all \( k \geq -1 \), and showed that \( C'_{A,0} = C_{A,0} \) and \( C'_{A,-1} = C_{A,-1} \).

For \( k \leq -3 \) this approach does not work in full generality. In [1] we showed that (i) is false in general for \( C_{A,k} \), and left (ii) and (iii) open, suggesting the problem of measuring \( C_{A,k} \). For \( k \leq -6 \), (ii) and (iii) are false, as we will show in Propositions 4 and 5, respectively. In fact, we will see that the Hilbert series of \( C_{A,k} \) is not even determined by the matroid of \( A \).

The intermediate cases are interesting and subtle, and deserve further study; notably the case \( k = -2 \), which Holtz and Ron call the \textit{internal zonotopal space}. In [3] they proved (ii) and (iii) and conjectured (i) for \( C'_{A,-2} \).

In [1, Proposition 4.5.3] – a restatement of Holtz and Ron’s Conjecture 6.1 in [3] – we put forward an incorrect proof of this conjecture; the last sentence of our argument is false. In fact their conjecture is false, as we will see in Proposition 2.

2. THE CASE \( k = -2 \): INTERNAL ZONOTOPAL SPACES.

Before showing why Holtz and Ron’s conjecture is false, let us point out that the remaining statements about \( C_{A,-2} \) that we made in [1] are true. The easiest way to derive them is to prove that \( C_{A,-2} = C'_{A,-2} \), and simply note that Holtz and Ron already proved those statements for \( C'_{A,-2} \):

\textbf{Lemma 1.} We have \( C_{A,k} = C'_{A,k} \) for any \( k \) with \(-\rho + 1 \leq k \leq 0\).

\textbf{Proof.} By [1, Theorem 4.17] we have \( I_{A,0} = I'_{A,0} \), so it suffices to show that \( I_{A,j} = I'_{A,j} \) implies that \( I_{A,j-1} = I'_{A,j-1} \) as long as these ideals are defined. If \( I_{A,j} = I'_{A,j} \), then for any \( h \in V \setminus \{0\} \) we have \( h^{p_A(h)+j+1} = \sum f_i h_i^{p_A(h_i)+j+1} \) for some polynomials \( f_i \), where the \( h_i \)'s are the lines of the arrangement. As long as the exponents are positive, taking partial derivatives in the direction of \( h \) gives \( h^{p_A(h)+j} = \sum g_i h_i^{p_A(h_i)+j} \) for some polynomials \( g_i \). \hfill \Box

The following result shows that (i) does not hold for \( C_{A,-2} \).

\textbf{Proposition 2.} [3, Conjecture 6.1] is false: The “internal zonotopal space” \( C_{A,-2} \) is not necessarily spanned by \( A \)-monomials.

\textbf{Proof.} Let \( \mathcal{H} \) be the hyperplane arrangement in \( \mathbb{C}^4 \) determined by the linear forms \( y_1, y_2, y_3, y_1 - y_4, y_2 - y_4, y_3 - y_4 \). We have

\( I'_{\mathcal{H},-2} = \langle x_1^1, x_2^1, x_3^1, (\epsilon_1 x_1 + \epsilon_2 x_2 + \epsilon_3 x_4 + x_4)^2 \rangle = \langle x_1, x_2, x_3, x_4^2 \rangle \)
as \( \epsilon_1, \epsilon_2, \epsilon_3 \) range over \( \{0,1\} \). The other generators of \( I_{H,-2} \) are of degree at least 3, and are therefore in \( I_{H,-2}' \) already, so

\[
I_{H,-2} = \langle x_1, x_2, x_3, x_4^2 \rangle, \quad C_{H,-2} = \text{span}(1, y_4).
\]

Therefore \( C_{H,-2} \) is not spanned by \( H \)-monomials. \( \Box \)

As Holtz and Ron pointed out, if [3, Conjecture 6.1] had been true, it would have implied [3, Conjecture 1.8], an interesting spline-theoretic interpretation of \( C_{A,-2} \) when \( A \) is unimodular. The arrangement above is unimodular, but it does not provide a counterexample to [3, Conjecture 1.8]. In fact, Matthias Lenz [4] has recently put forward a proof of this weaker conjecture.

3. The case \( k \leq -6 \)

In this section we show that when \( k \leq -6 \), the Hilbert series of \( C_{A,k} \) is not a function of the Tutte polynomial of \( A \). In fact, it is not even determined by the matroid of \( A \). Recall that \( \rho = \rho(A) := \min_{h \in V} (\rho_A(h)) \). Say \( h \in V \) is large if it is on the maximum number of hyperplanes, so \( \rho_A(h) = \rho \).

**Lemma 3.** The degree 1 component of \( C_{A,-\rho} \) is

\[
(C_{A,-\rho})_1 = (\text{span}\{h \in V : h \text{ is large}\})^\perp
\]

in \( V^* \).

**Proof.** An element \( f \) of \( C_{A,-\rho} \) needs to satisfy the differential equation

\[
h (\partial/\partial x)^{\rho_A(h) - \rho + 1} f(x) = 0 \quad \text{for all non-zero } h \in V.
\]

If \( f \) is linear, this condition is trivial unless \( h \) is large; and in that case it says that \( f \perp h \). \( \Box \)

**Proposition 4.** For \( k \leq -6 \), the Hilbert series of \( C_{A,k} \) is not determined by the matroid of \( A \).

**Proof.** First assume \( k = -2m \). Let \( L_1, L_2, L_3 \) be three lines through 0 in \( \mathbb{C}^3 \) and consider an arrangement \( A \) of 3m (hyper)planes consisting of \( m \) generically chosen planes \( H_{i1}, \ldots, H_{im} \) passing through \( L_i \) for \( i = 1, 2, 3 \). Then \( \rho = 2m \) and the only large lines are \( L_1, L_2, \) and \( L_3 \). Therefore \( \dim(C_{A,-2m})_1 = 1 \) if \( L_1, L_2, L_3 \) are coplanar, and 0 otherwise. However, the matroid of \( A \) does not know whether \( L_1, L_2, L_3 \) are coplanar.

More precisely, consider two versions \( A_1 \) and \( A_2 \) of the above construction; in \( A_1 \) the lines \( L_1, L_2, L_3 \) are coplanar, and in \( A_2 \) they are not. Then \( A_1 \) and \( A_2 \) have the same matroid but \( \dim(C_{A_1,-2m})_1 \neq \dim(C_{A_2,-2m})_1 \).

The case \( k = -2m - 1 \) is similar. It suffices to add a generic plane to the previous arrangements. \( \Box \)

**Proposition 5.** For \( k \leq -6 \), the sequence of graded vector spaces

\[
0 \to C_{A\backslash H,k}(-1) \to C_{A,k} \to C_{A/H,k} \to 0
\]

of [1, Proposition 4.4.1] is not necessarily exact, even if \( H \) is neither a loop nor a coloop.
Proof. We will not need to recall the maps that define this sequence; we will simply show an example where right exactness is impossible because \( \dim(C_{A,k})_1 = 0 \) and \( \dim(C_{A/H,k})_1 = 1 \). We do this in the case \( k = -2m \); the other one is similar.

Consider the arrangement \( A = A_2 \) of the proof of Proposition 4 and the plane \( H = H_{11} \). We have \( \dim(C_{A,-2m})_1 = 0 \). In the contraction \( A/H \), the planes \( H_{12}, \ldots , H_{1m} \) become the same line \( L_1 \) in \( H \), while the other \( 2m \) planes of \( A \) become generic lines in \( H \). Therefore \( \rho(A \setminus H) = 2m \) and \( (C_{A/H,-2m})_1 = L_1^\perp \) in \( H^* \), which is one-dimensional. □

Acknowledgments. We are very thankful to Matthias Lenz for pointing out the error in [1], and to Andrew Berget and Amos Ron for their comments on a preliminary version of this note.

References

1. F. Ardila and A. Postnikov. Combinatorics and geometry of power ideals. Transactions of the American Mathematical Society 362 (2010), 4357-4384.
2. W. Dahmen and C. Micchelli. On the local linear independence of translates of a box spline. Studia Math. 82(3) (1985) 243-263.
3. O. Holtz and A. Ron. Zonotopal algebra. Adv. Math. 227 (2011) 847-894.
4. M. Lenz. Interpolation, box splines, and lattice points in zonotopes. Preprint, 2012.
5. A. Postnikov, B. Shapiro, M. Shapiro. Algebras of curvature forms on homogeneous manifolds. Differential Topology, Infinite-Dimensional Lie Algebras, and Applications: D. B. Fuchs 60th Anniv. Collection, AMS Translations, Ser. 2 194 (1999) 227–235.