HEIGHTS OF IDEALS OF MINORS

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Abstract. We prove new height inequalities for determinantal ideals in a regular local ring, or more generally in a local ring of given embedding codimension. Our theorems extend and sharpen results of Faltings [F] and Bruns [B1].

Introduction

Let $\varphi$ be a map of vector bundles on a variety $X$. A well-known theorem of Eagon and Northcott [EN] gives an upper bound for the codimension of the locus where $\varphi$ has rank $\leq s$ for any integer $s$.

Bruns [B1] improved this result by taking into account the generic rank $r$ of $\varphi$. We shall see below that unlike the Eagon-Northcott estimate, Bruns’ Theorem is sharp only when $X$ is singular. The first goal of this paper is to give stronger results when $X$ is nonsingular, and a little more generally.

Strengthening the Eagon-Northcott estimate in a different way from Bruns, Faltings [F1] gave an improved bound for the case $s = r - 1$ under the additional assumption that $X$ is nonsingular and the cokernel of $\varphi$ is torsion free. We also improve Faltings theorem to a result valid for all $s$.

Our results are actually local. Let $R$ be a local ring, and let $\varphi : R^m \to R^n$ be a matrix of rank $r$. We write $I_i = I_i(\varphi)$ for the ideal generated by $i \times i$ minors of $\varphi$, and we assume that $I_i \neq R$. Bruns’ Theorem says that

$$\text{height } (I_i) \leq (r - i + 1)(m + n - r - i + 1).$$

This formula is sharp for every $m, n, r, i$: take $\varphi$ to be the image of the generic $n \times m$ matrix

$$\Phi = (x_{ij}) \quad 1 \leq i \leq n, \quad 1 \leq j \leq m$$

over the ring $R = k[\{x_{ij}\}]/I_{r+1}(\Phi)$. Note that this ring is singular for $r > 0$.

Henceforth in this introduction we shall assume that $R$ is a regular local ring. Under this hypothesis we can improve Bruns’ bound as follows:

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**Theorem A.** \( \text{height}(I_i) \leq (r - i + 1)(\max(m, n) - i + 1) + i - 1. \)

Theorem A is a weak form of Corollary 3.6.1 below.

One should compare this result with the “trivial” case where the matrix \( \varphi \) contains only \( r \) nonzero rows (if \( m \geq n \)) or \( r \) nonzero columns (if \( n \geq m \)). In this case the codimension of the ideal of \( i \times i \) minors is given by the “Eagon-Northcott” formula

\[
\text{height}(I_i) \leq (r - i + 1)(\max(m, n) - i + 1),
\]

which is an equality if the nonzero rows (columns) of \( \varphi \) are generic. This formula coincides with ours when \( i = 1 \). Theorem A is also sharp for generic alternating \( 3 \times 3 \) matrices when \( i = 2 \).

A particularly interesting situation is that where the cokernel of \( \varphi \) is torsion free (or even a vector bundle on the punctured spectrum). In this torsion free case Faltings improved Bruns’ bound (for the \( r \times r \) minors only) and showed

\[
\text{height}(I_r) \leq n.
\]

Generalizing this to arbitrary size minors, and allowing the cokernel to be any module which is not the direct sum of a free module and a nonzero torsion module (that is, excluding the “trivial” case described above) we show that:

**Theorem B.** \( \text{height}(I_i) \leq n + (r - i)(\max(m, n) - i + 1). \)

Theorem B is a weak form of Corollary 3.6.2 below.

Theorem B is sharp in the case where \( n = 3, m \geq 3, r = i = 2 \) and \( \varphi \) is the generic alternating \( 3 \times 3 \) matrix followed by a \( 3 \times (m - 3) \) matrix of zeros. If on the other hand \( \varphi \) has one generic column, \( r - 1 \) generic rows, and the rest of its entries 0, then

\[
\text{height}(I_i) = (r - i)(m - i + 1) + \min\{m - i + 1, n - r + 1\}.
\]

This actual value is close to the bound given by Theorems A and B. Some less degenerate examples are given in section 4.

We can also ask for a bound on the height of one ideal of minors modulo the ideal of minors of the next larger size. By symmetry we may assume that \( m \geq n \). We prove

**Theorem C.** \( \text{height}(I_i/I_{i+1}) \leq \max(m - i + 1, n) + r - i. \)

Theorem C is a weak form of Corollary 3.9 below.

This result is comparable to Theorems A and B (or their sharpenings) in the case \( i = r \); but it does not follow from these results in general because \( R/I_{i+1} \) is not regular. However, if we have good information about the higher order minors of \( \varphi \), as in the case where the cokernel of \( \varphi \) is an ideal, then Theorem C gives results on the height of \( I_i(\varphi) \) that are better than those coming from Theorems A and B. In this way we reprove a theorem of Huneke [Hu] and extend it as follows:
Corollary D. Let $I$ be an ideal of $R$ of height $g$ that is minimally generated by $n$ elements.

(a) (Huneke) If $I$ is not a complete intersection, that is $n > g$, then the locus of primes $P$ such that $I_P$ is not a complete intersection has codimension $\leq n + 2g - 1$.

(b) If $R/I$ is a Cohen-Macaulay domain and $n > g + 1$ then the locus of primes $P$ such that $I_P$ cannot be generated by $g + 1$ elements has codimension $\leq n + 2g - 1$.

Corollary D is a weak form of Example 3.11 below. Huneke's result (which is sharp, for example, in case $I$ is the ideal of $2 \times 2$ minors of a $2 \times 3$ matrix) improves a formula of Faltings [F] by 1. One should compare this to a famous conjecture of Hartshorne [Ha1] saying that if $I$ is the homogeneous ideal of a smooth projective variety $X$ which is not a complete intersection, then the singular locus of $X$ has codimension $\leq 3g + 1$ in the ambient projective space.

Both Theorems A and C are direct consequences of our other main theorem, which gives the bound on the codimension of the ideals of minors of a matrix $\varphi$ over a ring $\bar{R} = R/J$ obtained by reducing $\varphi$ modulo $J$. We may assume that $m \geq n$. We write $\bar{r}$ for the rank of $\bar{\varphi}$ and set $\delta = r - \bar{r}$.

Theorem E. $\text{height}(I_i(\bar{\varphi})) \leq (\bar{r} - i)(m - i + \delta + 1) + \max(m - t + 1, n) + \delta$.

Theorem E is a weak form of Theorem 3.1.1 below. As with Faltings' work, we do not need $R$ to be regular, but can give bounds in terms of certain embedding codimensions.

We now describe the key ideas of our proofs. To establish height bounds for ideals of minors it is helpful to identify as “many” row ideals of $\varphi$ as possible that have “small” height. As it turns out, the behavior of $\varphi$ in this respect is determined by the analytic spread $\ell$ of $M = \text{Coker}(\varphi)$ (see Section 1 for the definition of analytic spread). If $\ell$ has the maximal possible value $n$ then all row ideals of $\varphi$ have height at most $r$, and (under weak conditions) the converse holds as well. Thus, whenever $\ell < n$ there have to exist row ideals whose height exceeds $r$. On the other hand we prove in this case that after a flat local base change, at least $\ell$ row ideals have height $\leq r - n + \ell < r$. To paraphrase, if the analytic spread of $M$ is not maximal, then the behavior of the row ideals is more unbalanced, but not necessarily worse for our purposes. This is the content of Theorem 2.2, the main technical result of the paper. A complicated induction then completes the proofs of our formulas in Section 3.

We finish this introduction with a list of open problems specifically suggested by the results of this paper. Of course the biggest open problem is the conjecture of Hartshorne mentioned above.

Problem 1: Let $\varphi$ be a symmetric $n \times n$ matrix of rank $r$, and suppose that 2 is invertible in $R$ (but not necessarily that $R$ is regular). We conjecture that for $i \leq r$,

$$\text{height}(I_i) \leq \binom{n - i + 2}{2} - \binom{n - r + 1}{2}.$$  

In Section 5 we prove this conjecture for the cases $i = 1$ and $i = n - 1$ if $R$ is regular.
If the conjecture is true, it is sharp, for example for the generic matrix, taken modulo the ideal of \((r + 1) \times (r + 1)\) minors. This formula is the analogue of Bruns’ bound for general matrices; it is computed as the difference between the heights of the ideals of \(i \times i\) and the \((r + 1) \times (r + 1)\) minors of a generic symmetric matrix. Notice that the conjecture fails in characteristic 2, as can be seen by taking \(\varphi\) to be a generic alternating \(3 \times 3\) matrix and \(i = 2\).

**Problem 2:** Are there better bounds than the ones of Theorems A and B if we assume that \(\varphi\) has no (generalized) rows or columns of zeros?

**Problem 3:** Are there better bounds if \(\varphi\) is a matrix of linear forms?

**Problem 4:** Find sharp bounds assuming the ranks are small. For example, what about \(I_2\) for a \(4 \times 4\) matrix of rank 2? Is the height bounded by 3?

1. Basic Results

In this section we fix our notation and review some basic facts, mainly about Rees algebras of modules, that will be used throughout.

Let \(R\) be a Noetherian ring and \(I\) an ideal of \(R\). We write \(\text{ht}(I)\) for the height of \(I\) and \(\text{bight}(I)\) for its big height, which is the maximum of the heights of minimal primes of \(I\). Let \(M\) a finitely generated \(R\)-module and \(\varphi\) an \(n \times m\) matrix with entries in \(R\). By the \(i\)th row ideal of \(\varphi\) we mean the ideal generated by the entries of the \(i\)th row of \(\varphi\), and the rank of \(\varphi\) is the integer \(r = \max\{i | I_i(\varphi) \neq 0\}\). We say that \(M\) has a rank and write \(\text{rank}(M) = e\) if \(M \otimes_R K\) is a free \(K\)-module of rank \(e\), with \(K\) denoting the total ring of quotients of \(R\). Notice that if \(\varphi\) presents \(M\) and \(M\) has a rank, then \(r + e = n\).

Let \(\varphi\) be a matrix presenting \(M\) and \(T = T_1, \ldots, T_n\) a row of variables. The row ideals of \(\varphi\) are related to the symmetric algebra \(\text{Sym}(M)\) of \(M\) via the homogeneous presentation \(\text{Sym}(M) \cong R[T_1, \ldots, T_n]/I_1(T \cdot \varphi)\) (see also [EHU1], where this fact has been exploited systematically). Since the symmetric algebra fails to be equidimensional in general, we are lead to consider the Rees algebra \(\mathcal{R}(M)\) of \(M\) instead. The general notion of Rees algebra has been introduced in [EHU2, 0.1]. In the present paper however we will restrict ourselves to considering modules that have a rank. In this case \(\mathcal{R}(M)\) is equal to \(\text{Sym}(M)\) modulo \(R\)-torsion. We say that \(M\) is of linear type if the natural map from \(\text{Sym}(M)\) to \(\mathcal{R}(M)\) is an isomorphism. If \(R\) has dimension \(d\) and \(E\) has a rank \(e\), then \(\dim \mathcal{R}(M) = d + e\) (see, e.g., [SUV, 2.2]). Suppose in addition that \(R\) is equidimensional, universally catenary and local. Under this assumption \(\mathcal{R}(M)\) is equidimensional. Thus we may write \(\mathcal{R}(M) \cong R[T_1, \ldots, T_n]/b\) with \(\text{bight}(b) = \text{ht}(b)\). In fact \(\text{ht}(b) = r\), the rank of any matrix with \(n\) rows that presents \(M\).

If \(U\) is a submodule of \(M\), we say that \(U\) is a reduction of \(M\) or, equivalently, \(M\) is integral over \(U\) if the ring \(\mathcal{R}(M)\) is integral over its subalgebra \(R[U]\). In case \(R\) is local with residue
field \( k \), the analytic spread \( \ell(M) \) of \( M \) is defined to be the Krull dimension \( \dim R(M) \otimes_R k \). The two notions are related by the fact that \( \ell(M) = \min \{ \mu(U) | U \text{ a reduction of } M \} \) whenever \( k \) is infinite (here \( \mu(\cdot) \) denotes minimal number of generators). One always has \( \text{rank}(M) \leq \ell(M) \leq \mu(M) \) (see, e.g. [SUV, 2.3]), and the last inequality is an equality if and only if \( M \) has no proper reduction, at least in the case of an infinite residue field.

Before describing more refined estimates, we need to review the property \( G_s \), where \( s \) is an integer: A module \( M \) of rank \( e \) is said to satisfy \( G_s \) if \( \mu(M_P) \leq \dim R_P + e - 1 \) for every prime ideal \( P \) with \( 1 \leq \dim R_P \leq s - 1 \). What makes the concepts of integral dependence and analytic spread play a central role in this paper is their relation to the height of certain colon ideals:

**Theorem 1.1.** ([R, 2.5], [EHU3, 1.2]) Let \( R \) be an equidimensional universally catenary Noetherian local ring, let \( M \) be a finitely generated \( R \)-module having a rank \( e \), and let \( U \) be a submodule of \( M \) with \( \mu(U) \geq e \). If

\[
\text{ht}(U :_R M) > \mu(U) - e + 1
\]

then \( U \) is a reduction of \( M \).

This theorem yields the upper bound \( \ell(M) \leq \mu(U) \) when the hypothesis is satisfied. Conversely, one has:

**Proposition 1.2.** ([EHU3, 3.7bis]) Let \( R \) be a Noetherian local ring with infinite residue field, let \( M \) be a finitely generated torsion free \( R \)-module having a rank \( e \), and assume that \( M \) satisfies \( G_{s+1} \). If \( \text{ht}(U :_R M) \leq \mu(U) - e + 1 \) for every submodule \( U \) generated by \( e + s - 1 \) general linear combinations of generators of \( M \), then \( \ell(M) \geq e + s \).

In a more general setting one still has the following weaker bounds:

**Proposition 1.3.** ([SUV, 4.1]) Let \( R \) be a Noetherian local ring and let \( M \) be a finitely generated \( R \)-module having a rank \( e \). If \( M \) is not a direct sum of a torsion module and a free module, and \( M_P \) is free for every prime ideal \( P \) with depth \( R_P \leq 1 \), then \( \ell(M) \geq e + 1 \).

If \( R \to S \) is a homomorphism of rings, \( J^c \) will denote the contraction to \( R \) of an \( S \)-ideal \( J \), and \( -_S \) will stand for the functor \(- \otimes_R S \). We will denote \( \text{Hom}_R(\cdot, R) \) by \(-^* \). The embedding codimension ecodim(\( R \)) of a Noetherian local ring \( (R, m) \) is defined as the difference \( \mu(m) - \dim R \); equivalently, writing \( \hat{R} \cong S/J \) with \( (S, n) \) a regular local ring and \( J \) an \( S \)-ideal contained in \( n^2 \), one has ecodim(\( R \)) = \text{ht}(J).

### 2. Choosing Row Ideals of Small Height

Let \( R \) be an equidimensional universally catenary Noetherian local ring, and \( M \) a finitely generated \( R \)-module having a rank \( e \) with \( n = \mu(M), \ell = \ell(M) \). Theorem 1.1 shows that...
if \( \ell = n \), then every row ideal of any matrix minimally presenting \( M \) has height at most \( r = n - e \). According to Proposition 1.2, the converse holds in case \( M \) satisfies \( G_{r+1} \). Thus, whenever \( \ell < n \) there tend to exist row ideals of height strictly greater than \( r \). On the other hand, we will prove below that it is possible in this case to find “many” row ideals whose height is strictly less than \( r \). More precisely, over a flat local extension ring \( S \) of \( R \) there exists a matrix \( \phi \) minimally presenting \( M_S \) such that at least \( \ell \) row ideals of \( \phi \) have height at most \( r - n + \ell = \ell - e \). These row ideals are constructed as defining ideals of Rees algebras of certain modules. The local homomorphism \( R \to S \) has a complete intersection closed fiber, but regularity may fail to pass from \( R \) to \( S \). This will require some extra care since the height of ideals in \( S \) may no longer be subadditive.

We begin by recording a weaker version of the above estimate, which has the advantage that \( S \) can be chosen to be a localization of a polynomial ring over \( R \). This theorem was inspired by a result of Evans and Griffith saying that if \( R \) is a universally catenary domain with algebraically closed residue field and \( N \) is a finitely generated nonfree \( R \)-module of rank \( r \) then there exists a minimal generator \( x \in N \) with \( \text{ht}(N^*(x)) \leq r \) ([EG2, 2.12]).

**Theorem 2.1.** Let \( R \) be an equidimensional universally catenary Noetherian local ring and let \( M \) be a finitely generated \( R \)-module with rank \( e \). Write \( \ell = \ell(M) \) and \( r = \mu(M) - e \). Then there exists a local homomorphism \( R \to S \), with \( S \) a localization of a polynomial ring over \( R \), and a minimal presentation matrix of \( M_S \) over \( S \) that has \( \ell \) row ideals of height at most \( r \).

This result is a special case of the next theorem. Before stating the theorem we remark on some notation and terminology. Let \( M \) be a finitely generated module over a local Noetherian ring \((R, m)\), and assume that \( n = \mu(M) \). When we speak of a *generic* generating set of \( M \) in a local ring \( R' \) obtained by \( R \) by a purely transcendental residue field extension we mean the following: Let \( X = (x_{ij}) \) be a generic \( n \) by \( n \) matrix over \( R \). Fix a generating set \( m_1, ..., m_n \) of \( M \), and let \( v_i = \sum_{i=1}^n x_{ij} m_j \) and set \( R' = R[x_{ij}]_{mR[x_{ij}]} \). Then \( M_{R'} := M \otimes_R R' \) is generated by \( v_1, ..., v_n \). Furthermore, if \( \ell = \ell(M) \), then any \( \ell \) of the \( v_j \) form a minimal reduction of \( M_{R'} \). (This can be shown by first proving that the correct number of generic elements always give Noether normalizations for finitely generated algebras over fields, and which is explicitly shown in [FUV, Thm. 7.3].)

**Theorem 2.2.** Let \((R, m)\) be an equidimensional universally catenary Noetherian local ring, let \( M \) be a finitely generated \( R \)-module with rank \( e \), and set \( n = \mu(M), \ell = \ell(M), r = n - e \). Let \( v_1, ..., v_n \in M_{R'} \) be a generic generating set defined over a local ring \( R' \) that is obtained from \( R \) by a purely transcendental residue field extension and let \( \psi \) be an \( n \times m \) matrix presenting \( M_{R'} \) with respect to \( v_1, ..., v_n \). Further let \( T \) be an \( n \) by \( n \) matrix of the form

\[
T = \begin{bmatrix}
T_1 \\
\vdots \\
T_n
\end{bmatrix} = \begin{bmatrix}
1_{n-\ell} & 0 \\
\downarrow & \downarrow \\
& T'
\end{bmatrix}
\]
with rows $T_i$, where $T'$ is a generic $\ell$ by $n$ matrix over $R'$. Set $\phi = T\psi$.

There exists a local ring $R'$ obtained from $R'$ by another purely transcendental residue field extension and a prime ideal $Q$ of $A = R''[T]$ with $\det(T) \notin Q$ and $m \subset Q$ having the following property: for $n - \ell + 1 \leq t \leq n$ there exist $A_Q$-regular sequences $a_t$ each of length $n - \ell$, so that the following holds: given an arbitrary (possibly empty) set $\Lambda = \{t_1, \ldots, t_d\}$ of integers $n - \ell + 1 \leq t_1 < \cdots < t_d \leq n$ and writing $B = R''[\{T_i|i \notin \Lambda\}]Q^c$, $a = a_{t_1}, \ldots, a_{t_d}$, and $S = A_Q/(a)$:

1. The homomorphism $R \to B$ is local (and regular), and the homomorphism $B \to S$ is local flat with a complete intersection closed fiber.
2. $\phi_S$ is a presentation matrix of $M_S$, and for $n - \ell + 1 \leq i \leq n$, the $i^{th}$ row ideal $J_i$ of $\phi_S$ has height at most $r - n + \ell = \ell - e$ if $i \notin \Lambda$ and at most $r$ otherwise.
3. A form a regular sequence on $A_Q/I_A$ for every proper ideal $I$ of $R$.
4. $\text{ecodim}(S_P) = \text{ecodim}(R_{P \cap R})$ for every prime ideal $P$ of $S$ with $P \notin V(J_{t_1} \cdots J_{t_d})$.

Proof. Write $U = R'/v_{n-\ell+1} + \cdots + R'v_n \subset M_{R'}$. Since $U$ is generated by $\ell$ generic elements of $M$ it follows that $U$ is a minimal reduction of $M_{R'}$. To simplify notation we write $R$ instead of $R'$ from now on.

For $n - \ell + 1 \leq i \leq n$ let $a_i$ be the ideal of $A_i = R[T_i]$ generated by the $i^{th}$ row ideal of $\phi$. We obtain isomorphisms

$$A_i/a_i \cong \text{Sym}(M)$$

sending the $(i, j)$ entry $T_{ij}$ of $T$ to $v_j$. Since $\text{Sym}(M)$ maps onto $\mathcal{R}(M)$, there are $A_i$-ideals $b_i$ containing $a_i$ such that $A_i/b_i \cong \mathcal{R}(M)$. Observe that $\text{bight}(b_i) = \text{bight}(b_iA) = r$ (see the remarks at the start of this section).

Let $(R'', m'')$ be the local ring obtained from $R = R'$ by a purely transcendental residue field extension of transcendence degree $(n - \ell)(\sum \mu(b_i))$, let $k'' = R''/m''$, $E = \otimes_{R}^\ell R(M) \otimes_{R} k''$. Note that $E$ is isomorphic to a polynomial ring over $k''$ in $\ell^2$ variables which are the images of $T_{ij}$ in $E$ for $n - \ell + 1 \leq i \leq n$ and $n - \ell + 1 \leq j \leq n$.

The above isomorphisms induce an isomorphism

$$A/(m, b_{n-\ell+1}, \ldots, b_n) \cong E.$$

Moreover, the natural map of $k''$-algebras $E \to F$ is module finite since $U$ is a reduction of $M$. Its image is generated by the images in $F$ of $T_{ij}$ for $n - \ell + 1 \leq i \leq n$ and $n - \ell + 1 \leq j \leq n$. Hence these elements of $F$ are algebraically independent over $k''$, because $\text{dim} F = \ell^2$. It follows that the image of $\Delta = \det(T)$ in $F$ is not nilpotent. Thus there exists a prime ideal $Q$ of $A$ with $\Delta \notin Q$ and $(m, b_{n-\ell+1}, \ldots, b_n) \subset Q$.

For every $t, n - \ell + 1 \leq t \leq n$, let $a_t \subset R''[T_t]$ be a sequence of $n - \ell$ generic elements for $b_t \subset R[T_t]$, defined using indeterminates over $R = R'$ as coefficients. Such sequences exist by the definition of $R''$. As $(m, b_t)A_t/mA_t$ is an ideal in a polynomial ring over a field of height $\dim(A_t/mA_t) - \ell(M) = n - \ell$, it follows that $a_t$ form a regular sequence on $A_t/mA_t \otimes_{k} k''$.

We are now ready to verify statements (1) – (4) in the theorem. Write $n$ for the maximal ideal of $B$. As $m \subset Q$ we have that $m \subset n$ and thus the map $R \to B$ is a (regular) local
homomorphism. Furthermore, \( a \subset Q \) and \( A_Q/n_A \) is flat over \( (A_{t_i}/(mA_{t_i})) \otimes_k \cdots \otimes_k (A_{t_d}/(mA_{t_d})) \otimes_k k'' \). Thus \( a \) form a regular sequence on \( A_Q/n_A \), the closed fiber of the map \( B \to A_Q \). Consequently, the (local) homomorphism \( B \to S = A_Q/(a) \) is flat with complete intersection closed fiber, and \( a \) form a regular sequence on \( A_Q/IA_Q \) for any \( R \)-ideal \( I \subset m \) ([Ma, p. 177]). This proves (1) and (3).

To show (2), observe that the image of \( \Delta \) is a unit in \( S \) since \( \Delta \notin Q \). Thus \( \phi_S \) is a presentation matrix of \( M_S \). Obviously \( J_i = a_i S \subset b_i S \). If \( i \notin \Lambda \) then \( S \) is flat over \( A_i \) and hence \( \text{ht}(J_i) \leq \text{ht}(b_i S) = \text{ht}(b_i(A_i)_{Q \cap A_i}) \) \( \leq \text{bht}(b_i) = r \). If on the other hand \( i \in \Delta \) then \( a_i \subset b_i \), which together with the \( A_Q \)-regularity of \( a \) gives \( \text{ht}(J_i) \leq \text{ht}(b_i S) \leq \dim(S) - \dim(S/b_i S) = \dim(A_Q) - d(n - \ell) - (\dim(A_Q) - (d - 1)(n - \ell) - \text{ht}(b_i A_Q)) = \text{ht}(b_i A_Q) - (n - \ell) \leq \text{bht}(b_i A) - n + \ell = r - n + \ell = \ell - e \). This proves (2).

Finally, to show (4) notice that if \( P \in \text{Spec}(S) \setminus V(J_{t_1} \cdots J_{t_d}) \), then \( P \notin V(b_t S) \) for every \( t \in \Lambda \). Thus by the generic choice of \( a_t \) in \( b_t \), the ring \( S_P \) is a localization of a polynomial ring over \( R_{P \cap R} \). \( \square \)

We will often apply Theorem 2.2 in conjunction with the following generalization of a theorem of Serre:

**Lemma 2.3.**

1. Let \( f : (A, m) \to (B, n) \) be a local homomorphism of equidimensional and universally catenary Noetherian local rings, with \( A \) regular. Let \( I \) be an ideal of \( A \) and \( J \) be an ideal of \( B \). Then

\[
\text{ht}(IB + J) \leq \text{ht}(I) + \text{ht}(J).
\]

2. Let \( B \to S \) be a local homomorphism of Noetherian local rings with \( S \) equidimensional and universally catenary, let \( K \) be an ideal of \( B \), and let \( J \) be an ideal of \( S \). Then \( \text{ht}(J + KS) \leq \text{ht}(J) + \text{ht}(K) + \text{ecodim}(B) \).

**Proof.** We first prove (1). Suppose first that \( f \) is onto, and write \( B = A/K \). Lift \( J \) to an ideal \( L \) in \( A \), so that \( J = L/K \). Since \( B \) is equidimensional and universally catenary, \( \text{ht}(IB + J) = \text{ht}((I + L)/K) = \text{ht}(I + L) - \text{ht}(K) \leq \text{ht}(I) + \text{ht}(L) - \text{ht}(K) = \text{ht}(I) + \text{ht}(J) \), where the middle inequality follows from the subadditivity of height in regular local rings [S, Chap. V, Thm. 3].

We now do the general case. Without loss of generality, we may assume both \( A \) and \( B \) are complete: our assumptions do not change (see [Ma, 31.7]), nor does the conclusion. We use a Cohen factorization of \( f \) as in [AFH, Thm 1.1]. There is a factorization of \( f \), \( A \xrightarrow{g} C \xrightarrow{h} B \) where \( C \) is local, \( g \) is flat and \( C/mC \) is regular, and \( h \) is surjective. Since \( g \) is flat with regular closed fiber and \( A \) is regular, it follows that \( C \) is also regular by [Ma, Thm. 23.7]. As \( C \) maps onto \( B \), to finish the proof it suffices to prove that \( \text{ht}(I) \geq \text{ht}(IC) \).

However, as \( A \) and \( C \) are regular and \( g \) is flat,

\[
\text{ht}(I) = \dim(A) - \dim(A/I) = \dim(C) - \dim(C/mC) - (\dim(C/IC) - \dim(C/mC)) = \text{ht}(IC).
\]
We prove (2). We can pass to the completions of $B$ and $S$ and assume both rings are complete. Write $B = A/I$, where $A$ is a regular local ring, and lift $K$ to an ideal $L$ in $B$, so that $K = L/I$. Note that $LS = KS$, so to prove (2), it is enough to prove that $\text{ht}(K) + \text{ecodim}(B) = \text{ht}(L)$ and then apply (1). But this equality is immediate. □

Next we give a short proof of a modified version of Theorem 2.2. It requires the following definition:

**Definition 2.4.** Let $R$ be a Noetherian local ring with residue field $k$ (or a positively graded $k$-algebra), let $M$ be a finitely generated (graded) $R$-module having a rank, and write $\mathcal{R} = \mathcal{R}(M)$. We set

$$s(M) = \dim_k[(\mathcal{R} \otimes_R k)/\sqrt{0}]_1.$$  

**Remark 2.5.** Observe that in general $\ell(M) \leq s(M) \leq \mu(M)$. If $M$ is graded and generated by forms of the same degree, then $\mathcal{R} \otimes_R k$ embeds into a polynomial ring over $R$ and therefore $s(M) = \mu(M)$ as long as $R$ is reduced and $M$ is torsionfree.

**Theorem 2.6.** Let $R$ be an equidimensional universally catenary Noetherian local ring with algebraically closed residue field $k$, let $M$ be a finitely generated $R$-module with rank $e$, and write $r = \mu(M) - e$, $s = s(M)$. There exists a minimal presentation matrix of $M$ that has $s$ row ideals of height at most $r$.

**Proof.** Write $\mathcal{R}$ for the Rees algebra of $M$ and set $V = [(\mathcal{R} \otimes_R k)/\sqrt{0}]_1$, which we identify with affine space of dimension $s$. Consider the closed subset $X$ of $V$ whose coordinate ring is the homogeneous $k$-algebra $(\mathcal{R} \otimes_R k)/\sqrt{0}$. Since $k$ is algebraically closed there exists a basis $v_1, \ldots, v_s$ of $V$ contained in $X$, and then the lines $kv_1, \ldots, kv_s$ all lie on $X$.

Let $z_1, \ldots, z_n$ be a minimal generating set of $M$ chosen so that $z_i$ maps to $v_i$ for $1 \leq i \leq s$, and let $\phi$ be a presentation matrix with respect to $z_1, \ldots, z_n$. Set $J_i$ equal to the ideal generated by the $i^{\text{th}}$ row of $\phi$. We claim $\text{ht}(J_i) \leq r$ for $1 \leq i \leq s$.

Let $A = R[T_1, \ldots, T_n]$ be a polynomial ring, let $\mathfrak{m}$ denote the maximal ideal of $R$, and for $1 \leq i \leq s$ consider the prime ideals $Q_i = (\mathfrak{m}, T_1, \ldots, \hat{T}_i, \ldots, T_n)$ of $A$. Mapping $T_j$ to $z_j$ for $1 \leq j \leq n$, we obtain presentations $\text{Sym}(M) \cong A/\mathfrak{a}$ and $\mathcal{R} \cong A/\mathfrak{b}$, where $\mathfrak{a} \subset \mathfrak{b}$ are $A$-ideals. As $X$ contains the line $kv_i$, we have $b \subset Q_i$ for $1 \leq i \leq s$. Thus $\text{ht}(a_{Q_i}) \leq \text{ht}(b_{Q_i}) \leq \text{bight}(b) = r$. Let $\pi_i : A_{Q_i} \to R[T_i]$ be the $R[T_i]$-epimorphism whose kernel is generated by the $A_{Q_i}$-regular sequence $T_1, \ldots, \hat{T}_i, \ldots, T_n$. Since $\text{ht}(\pi_i(a_{Q_i})) + n - 1 = \text{ht}(\pi_i(a_{Q_i}T_1, \ldots, \hat{T}_i, \ldots, T_n)) = \text{ht}(a_{Q_i}T_1, \ldots, \hat{T}_i, \ldots, T_n) \leq \text{ht}(a_{Q_i}) + n - 1$, it follows that $\text{ht}(\pi_i(a_{Q_i})) \leq \text{ht}(a_{Q_i})$. But $\pi_i(a_{Q_i}) = J_iR(T_i)$, which gives $\text{ht}(J_i) \leq r$. □

We finish the section with two immediate consequences of Theorem 2.6. Both are first height estimates for ideals of minors of matrices, stated more conveniently in terms of Fitting ideals of modules.
Corollary 2.7. Let $R$ be a regular local ring with perfect residue field $k$ and let $M$ be a finitely generated $R$-module of rank $e$, and write $r = \mu(M) - e$, $s = s(M)$. For every $1 \leq i \leq s$,

$$\text{ht}(\text{Fitt}_{i-1}(M)) \leq ir.$$  

Proof. There exists a flat local homomorphism $R \to S$ where $S$ is a regular local ring with algebraically closed residue field $K$ [G,(10.3)]. Since $S$ is flat over $R$ and $k$ is perfect, one has that $(R(M) \otimes_S K)/\sqrt{0} \cong (R(M) \otimes_R k)/\sqrt{0} \otimes_k K$ and therefore $s(M) = s(M_S)$. We replace $R$ and $M$ by $S$ and $M_S$, and assume that $k$ is algebraically closed.

By Theorem 2.6 there exists a minimal presentation matrix of $M$ that has $i$ row ideals $J_1, \ldots, J_i$ of height at most $r$. As $\text{Fitt}_{i-1}(M) \subset J_1 + \cdots + J_i$ and $R$ is a regular local ring, we conclude that $\text{ht}(\text{Fitt}_{i-1}(M)) \leq ir$. \hfill \Box

Corollary 2.8. Let $R$ be a polynomial ring over a field, let $M$ be a torsion-free graded $R$-module of rank $e$ minimally generated by $n$ homogeneous elements of the same degree, and write $r = n - e$. For every $1 \leq i \leq n$, $\text{ht}(\text{Fitt}_{i-1}(M)) \leq ir$. In particular, for every submodule $U$ of $M$ generated by $t < n$ elements, $\text{ht}(U :_R M) \leq (t + 1)r$.

Proof. We may assume that the ground field is perfect. Writing $m$ for the irrelevant maximal ideal of $R$ we observe that $s(M_m) = s(M) = n$ and $(U :_R M)_m \subset \text{Fitt}_t(M_m)$. The assertions now follow from Corollary 2.7. \hfill \Box

3. Heights of Determinantal Ideals

The classical theorem of Bruns ([B1, Cor. 1]) states that in a Noetherian ring $R$, the height of the ideal of $i$ by $i$ minors of an $n$ by $m$ matrix of rank $r$ cannot exceed the “generic” value $N(i, r, m, n)$ defined as follows: let $X$ be a generic $n$ by $m$ matrix and set $N(i, r, m, n) := \text{ht}(I_i(X)) - \text{ht}(I_{r+1}(X)) = (r - i + 1)(m + n - r - i + 1)$. This is exactly the height of the ideal of $i$ by $i$ minors of the image of $X$ in the ring $R[X]/I_{r+1}(X)$ (note the image of $X$ has rank $r$ in this ring). However, if we also insist that the base ring $R$ be regular, then it is by no means clear that this maximum is ever attained. The main results known for the regular case are due to Bruns and Faltings ([B1, Thm. 3], [F]), and their results apply only to the case $i = r$. In Corollary 3.6.1 below we establish a bound for the height of the ideal of $i$ by $i$ minors of an $n$ by $m$ matrix of rank $r$ over a regular ring that is roughly $(r - i)(\max\{m, n\} - i + 1) + \max\{m - i + 1, n\}$.

A second, related problem is to estimate the height of the ideal of $i$ by $i$ minors modulo the ideal of $i + 1$ by $i + 1$ minors. Again, the best general bound is $N(i, i, m, n) = m + n - 2i + 1$, but one may expect better results if $R$ is regular and the rank $r$ of the matrix is not maximal. We address this issue in Corollary 3.9.1, where the bound $\max\{m - i + 1, n\} + r - i$ is established.

Both problems are special cases of the following, more general question: How can one estimate the height of the ideal of $i$ by $i$ minors of a matrix of rank $\bar{r}$ that can be “lifted”
to a matrix of rank $r$ over a ring $R$. Theorem 3.1, the main result of this section, gives such a bound involving the difference $r - \bar{r}$ of the ranks and the embedding codimension of $R$. The proof of this result relies on the work of Section 2 about row ideals of small height. The theorem gives particularly strong estimates if the matrix can be lifted in such a way that the increase in the rank is compensated by a decrease in the embedding codimension of the ambient ring.

**Theorem 3.1.** Let $R$ be an equidimensional universally catenary Noetherian local ring, let $\varphi$ be an $n$ by $m$ matrix of rank $r$ with entries in $R$, and let $I$ be an $R$-ideal. Assume that $M = \text{Coker} (\varphi)$ has a rank, and write $\ell = \ell (M)$, $\bar{R} = R/I$, $\bar{\varphi} = \varphi_{\bar{R}}$, $\bar{r} = \text{rank}(\bar{\varphi})$. Let $i \leq \bar{r}$ be an integer so that $I_i (\bar{\varphi}) \neq \bar{R}$. Set $\delta = r - \bar{r}$ and $\epsilon = \max_P \{\text{ecodim}(R_P)\}$, where the maximum is taken over all prime ideals $P$ of $R$ not containing $I_i (\varphi)$.

\begin{align*}
(1) \quad \text{ht}(I_i (\bar{\varphi})) & \leq \max\{\min\{n - \ell, \bar{r}\} - i + 1)(m - i + 1 + \max\{0, n - \ell - \bar{r}\}) ,
\end{align*}

\begin{align*}
& (\bar{r} - i)(\max\{m, n + \epsilon\} - i + \delta + 1) + \ell + \delta + \text{ecodim}(R) \\
& \leq (\bar{r} - i)(\max\{m, n + \epsilon\} - i + \delta + 1) + \max\{m - i + 1, \ell + \text{ecodim}(R)\} + \delta.
\end{align*}

(2) If the $\bar{R}$-module $\bar{M} = M_{\bar{R}}$ is not a direct sum of a torsion module and a free module, $\bar{M}_P$ is free for every prime $P$ of $R$ with depth($R_P$) \leq 1 and $M_P$ is of linear type for every associated prime $P$ of $I$, then

\begin{align*}
\text{ht}(I_i (\bar{\varphi})) & \leq (\bar{r} - i)(\max\{m, n + \epsilon\} - i + \delta + 1) + \ell + \delta + \text{ecodim}(R).
\end{align*}

Before proving the theorem we wish to make several comments. First notice that $\epsilon = 0$ in case $R$ is locally regular on the punctured spectrum. If the $\bar{R}$-module $\bar{M}$ is a direct sum of a torsion module and a free module then trivially $\text{ht}(I_i (\bar{\varphi})) \leq (\bar{r} - i + 1)(m - i + 1)$. It is also obvious that one can replace the bound of part (1) by the better formula of (2) whenever $i \geq n - \ell + 1$. Finally, the estimates of Theorem 3.1 are sharp for $\varphi$ a generic matrix with entries in the localization of a polynomial ring over a regular ring and $I = I_{\bar{r}+1}(\varphi)$, if $n \leq m$ or $i = 1$.

**Proof of Theorem 3.1.** We first prove that the second inequality of (1) is true, namely that

\begin{align*}
\max\{\min\{n - \ell, \bar{r}\} - i + 1)(m - i + 1 + \max\{0, n - \ell - \bar{r}\}) ,
\end{align*}

\begin{align*}
& (\bar{r} - i)(\max\{m, n + \epsilon\} - i + \delta + 1) + \ell + \delta + \text{ecodim}(R) \\
& \leq (\bar{r} - i)(\max\{m, n + \epsilon\} - i + \delta + 1) + \max\{m - i + 1, \ell + \text{ecodim}(R)\} + \delta.
\end{align*}
We prove each term in the maximum on the left hand side of the inequality is at most the right hand side. This is clear for the second term. It remains to see why
\[
(\min\{n-\ell, \bar{r}\} - i + 1)(m - i + 1 + \max\{0, n - \ell - \bar{r}\}) \\
\leq (\bar{r} - i)(\max\{m, n + \epsilon\} - i + \delta + 1) + \max\{m - i + 1, \ell + \text{ecodim}(R)\} + \delta.
\]
By possibly lessening the right hand side and increasing the left-hand side, it is enough to prove that
\[
(\bar{r} - i + 1)(m - i + 1 + \max\{0, n - \ell - \bar{r}\}) \leq (\bar{r} - i)(m - i + \delta + 1) + m - i + 1 + \delta = (\bar{r} - i + 1)(m - i + \delta + 1),
\]
and for this it suffices to prove that \(\max\{0, n - \ell - \bar{r}\} \leq \delta = r - \bar{r}\). Clearly \(0 \leq \delta\). The inequality \(n - \ell - \bar{r} \leq r - \bar{r}\) is equivalent to the inequality \(n - r \leq \ell\), which is always true, since \(n - r = e = \text{rank}(M) \leq \ell\).

We use induction on \(n\) to prove the first inequality of Theorem 3.1. Suppose that \(n = 1\). In this case, \(M = R/J\), where \(J\) is an ideal with \(m\)-generators. By assumption, \(M\) has a rank, which is of necessity either 0 or 1. However the rank cannot be 1, since then \(J_P = 0\) for all associated primes of \(R\), and hence \(J = 0\) and \(M = R\) is free. Thus the rank of \(M\) is 0, and then \(J\) contains a non-zerodivisor. It follows that the analytic spread of \(M\) is 0, since we always mod out torsion to compute the analytic spread. Hence, \(\ell = 0, r = 1, n = 1\), and \(\bar{r}\) is either 0 or 1. If \(\bar{r} = 0\), then the theorem is vacuous. Hence we may assume that \(\bar{r} = 1\) also, and \(i = 1\). In this case the inequality reads:
\[
\text{ht}(I_1(\varphi)) \leq \max\{m, \text{ecodim}(R)\}.
\]
By the Krull height theorem, the height of \(I_1(\varphi)\) is at most its number of generators, which is bounded by \(m\), proving the case \(n = 1\).

We may assume that the entries of \(\varphi\) lie in the maximal ideal of \(R\). We claim that we may assume that \(I = P\) is a prime ideal. Let \(P\) be a minimal prime of \(I\) having maximal dimension. We write \(r_P\) for the rank of \(\phi_{R/P}\). There are three cases, depending on the relationship of \(r_P\) to \(i\) and \(\bar{r}\). Note that \(r_P \leq \bar{r}\).

Case 1. \(r_P = \bar{r}\). Since \(R\) is equidimensional and catenary, \(\text{ht}(I_1(\varphi)) = \text{ht}(I_1(\varphi_{R/P}))\). Hence the left-hand side of in the inequality of (1) doesn’t change, but neither does the right-hand side in this case.

Case 2. \(r_P < i\). Then \(I_1(\varphi) \subseteq P\), and \(\text{ht}(I_1(\varphi_{R/P})) = 0\). Since the right-hand side of (1) is nonnegative, the inequality holds.

Case 3. \(i \leq r_P\). In this case we prove that as a function of \(\bar{r}\), the right-hand side of (1) is nonincreasing as we decrease \(\bar{r}\) to \(i\). Since \(i \leq r_P \leq \bar{r}\) and since \(\text{ht}(I_1(\varphi)) = \text{ht}(I_1(\varphi_{R/P}))\), this will prove our claim. The right-hand side of (1) is a maximum of two terms. Decreasing \(\bar{R}\) by one changes the second term, \((\bar{r} - i)(\max\{m, n + \epsilon\} - i + \delta + 1) + \ell + \delta + \text{ecodim}(R)\), to \((\bar{r} - i - 1)(\max\{m, n + \epsilon\} - i + \delta + 2) + \ell + \delta + 1 + \text{ecodim}(R)\). Subtracting the first from the second gives the value \(\max\{m, n + \epsilon\} + r + 1 - 2\bar{r}\), which is always nonnegative.

The first term, \((\min\{n - \ell, \bar{r}\} - i + 1)(m - i + 1 + \max\{0, n - \ell - \bar{r}\})\) can only increase if \(n - \ell - \bar{r} \geq 0\). Then as \(\bar{r}\) decreases by 1, \(\max\{0, n - \ell - \bar{r}\}\) will increase by 1. However,
in this case \( \min \{ n - \ell, \bar{r} \} \) will be \( \bar{r} \) and will decrease by 1. Then the product has the form \((\bar{r} - i + 1)(m - i + 1 + (n - \ell - \bar{r}))\), and when we replace \( \bar{r} \) by \( \bar{r} - 1 \) we obtain \((\bar{r} - i)(m - i + 1 + (n - \ell - \bar{r} - 1))\). But \((\bar{r} - i + 1)(m - i + 1 + (n - \ell - \bar{r})) \geq (\bar{r} - i)(m - i + 1 + (n - \ell - \bar{r} - 1))\) since \( 2\bar{r} \leq m + n - \ell + 1 \).

Thus we may suppose that \( \bar{R} \) is a domain, hence equidimensional. We use the notation of Theorem 2.2 and in addition set \( a_j = 0 \) whenever \( j \leq n - \ell \). For \( 0 \leq j \leq n \) let \( \phi_j \) be the \( j \) by \( m \) matrix consisting of the first \( j \) rows of \( \phi \), and define

\[
t = \min \{ j \mid I_t(\phi) \subset \sqrt{(I_t(\phi_j), I, a_j)_Q} \}.
\]

We may assume that \( i \leq t \). For suppose that \( t < i \). Then \( I_t(\phi_t) = 0 \) so we would have that \( I_t(\phi) \subset \sqrt{(I, a_j)_Q} \). The map from \( R \) to \( S \) is flat, and the \( a_j \) from a regular sequence in \( A_Q/IA_Q \). Hence if \( s \in I_t(\phi) \), then for large \( N \), \( s^N \in (I, a_j)_Q \cap R = I_t(\phi) \), the last equality by flatness. Since \( I_t \) is prime, we obtain that \( s \in I_t \), and then \( I_t(\phi) \subset IR \) and we are done. Henceforth we assume that \( i \leq t \).

We apply Theorem 2.2 with \( \Lambda = \emptyset \) if \( t \leq n - \ell \) and \( \Lambda = \{ t \} \) if \( t > n - \ell + 1 \). Let \( J_t \) be the \( t \)th row ideal of the matrix \( \phi_S \), and write \( S = S/IS, J_t = J_tS \). By Theorem 2.2, \( R \subset S \) and \( \bar{R} \subset \bar{S} \) are flat local extensions, \( S \) and \( \bar{S} \) are equidimensional and catenary, and \( \text{ecodim}(S_P) \leq \epsilon \) for every prime \( P \) of \( S \) not containing \( I_t(\phi) \cdot J_t \). Notice that \( I_t(\phi_S \bar{S}) \subset \sqrt{I_t(\phi_{t-1})S} \) and \( I_t(\phi_S) \nsubseteq \sqrt{I_t(\phi_{t-1})S} \) according to the definition of \( t \). Again by Theorem 2.2, \( \text{ht}(J_t) \leq r - n + \ell \) if \( t \geq n - \ell + 1 \). Furthermore as \( I_t(\phi_{t-1})S + IS \) is extended from \( B \), Lemma 2.3 implies that

\[
\text{ht}(J_t + I_t(\phi_{t-1})S + IS) \leq \text{ht}(J_t) + \text{ht}(I_t(\phi_{t-1})S + IS) + \text{ecodim}(R).
\]

Thus by our equidimensionality conditions,

\[
\text{ht}(\bar{J}_t + I_t(\phi_{t-1})\bar{S}) \leq \text{ht}(J_t) + \text{ht}(I_t(\phi_{t-1})\bar{S}) + \text{ecodim}(R).
\]

Since \( \bar{S} \) is flat over \( \bar{R} \) and

\[
I_t(\varphi_{\bar{S}}) = I_t(\phi_{\bar{S}}) \subset \sqrt{I_t(\phi_{t-1})S} \subset \sqrt{\bar{J}_t + I_t(\phi_{t-1})\bar{S}},
\]

we conclude that

\[
\text{ht}(I_t(\varphi)) = \text{ht}(I_t(\varphi_{\bar{S}})) \leq \text{ht}(J_t) + \text{ht}(I_t(\phi_{t-1})\bar{S}) + \text{ecodim}(R).
\]

To simplify notation we will henceforth write \( \phi, \phi_j, \bar{\phi}, \bar{\phi}_j \) instead of \( \phi_S, (\phi_j)_S, \phi_{\bar{S}}, (\phi_j)_{\bar{S}} \). With this we have

\[
\sqrt{I_t(\phi_{t-1})} \subset \sqrt{I_t(\bar{\phi}_t)}
\]
and

\[
\text{ht}(I_t(\varphi)) = \text{ht}(I_t(\bar{\phi}_t)) \leq \text{ht}(J_t) + \text{ht}(I_t(\phi_{t-1})) + \text{ecodim}(R).
\]
Case 1: \( t \leq n - \ell \). In this case \( I_i(\varphi_S) = I_i(\phi_S) \subseteq \sqrt{I_i(\phi_t)S} \), so that \( \operatorname{ht}(I_i(\varphi)) \leq \operatorname{ht}(I_i(\phi_{n-\ell})) \leq (n - \ell - i + 1)(m - i + 1) \), and according to [B1, Cor. 1],

\[
\operatorname{ht}(I_i(\varphi)) \leq (r - i + 1)(m + n - \ell - r - i + 1)
\]

The first inequality of (1) follows, and the second holds because \( \ell = \ell(M) \geq \operatorname{rank}(M) = n - r \).

Case 2: \( t \geq n - \ell + 1 \). In this case \( \operatorname{ht}(J_t) \leq r - n + \ell \), and therefore (3.3) yields

\[
\operatorname{ht}(I_i(\varphi)) \leq r - n + \ell + \operatorname{ht}(I_i(\overline{\phi}_{t-1})) + \operatorname{ecodim}(R).
\]

By (3.2) there exists a prime ideal \( P \) of \( S \) with \( I_i(\phi_{t-1}) + IS \subset P \) and \( I_i(\phi_t) \not\subset P \). Since \( I_i(\phi_t) \) is contained in \( I_{t-1}(\phi_{t-1}) \), in \( I_i(\varphi)S \), and in \( J_t + I_i(\phi_{t-1}) \), one automatically has \( I_{t-1}(\phi_{t-1}) \not\subset P \) as well as \( I_i(\varphi) \cdot J_t \not\subset P \). By the latter, \( \operatorname{ecodim}(S_P) \leq \epsilon \). Set \( s = \max\{j|I_j(\phi) \not\subset P\} \). Clearly \( 1 \leq i \leq s \). Recall that \( I_{t-1}(\phi_{t-1})_P = S_P \) and \( I_i(\overline{\phi}_{t-1})_P \neq S_P \). Thus without changing the ideal \( I_i(\overline{\phi}_{t-1})_P \), we may perform elementary row and column operations over \( S_P \) to assume that

\[
\phi_P = \begin{bmatrix}
1_{i-1} & 0 \\
0 & \phi' & \phi'' \\
0 & 1_{s-i+1}
\end{bmatrix}
\]

where \( \phi', \phi'' \) have entries in the maximal ideal of \( S_P \). Notice that the \( n - s \) by \( m - s \) matrix \( \phi' \) has rank \( r - s \) and \( \overline{\phi}' \) has rank \( \overline{r} - s \), with \( \overline{\phi}', \overline{\phi}'' \) standing for \( \phi'_{S_P}, \phi''_{S_P} \).

Since \( I_i(\overline{\phi}_{t-1})_P \subset I_1(\overline{\phi}') + I_1(\overline{\phi}'') \neq S_P \) and \( S_P \) is equidimensional, we obtain

\[
\operatorname{ht}(I_i(\overline{\phi}_{t-1})) \leq \operatorname{ht}(I_i(\overline{\phi}_{t-1})_P) \leq \mu(I_1(\overline{\phi}'')) + \operatorname{ht}(I_1(\overline{\phi}')) 
\]

\[
\leq (s - i + 1)(n - s) + \operatorname{ht}(I_1(\overline{\phi}')).
\]

Thus by (3.4),

\[
\operatorname{ht}(I_i(\overline{\varphi})) \leq r - n + \ell + (s - i + 1)(n - s) + \operatorname{ht}(I_1(\overline{\phi}')) + \operatorname{ecodim}(R).
\]
Applying the induction hypothesis to the matrix \( \phi' \) yields
\[
\text{ht}(I_1(\phi')) \leq ((\bar{r} - s) - 1)(\max\{m - s, (n - s) + \epsilon\} - 1 + \delta + 1) \\
+ \max\{(m - s) - 1 + 1, (n - s) + \epsilon\} + \delta \\
= (\bar{r} - s)(\max\{m, n + \epsilon\} - s + \delta).
\]
Hence by (3.5),
\[
\text{ht}(I_i(\bar{\varphi})) \leq r - n + \ell + (s - i + 1)(n - s) + (\bar{r} - s)(\max\{m, n + \epsilon\} - s + \delta) + \text{ecodim}(R) \\
\leq r - n + \ell + n - i + (\bar{r} - i)(\max\{m, n + \epsilon\} - i + \delta) + \text{ecodim}(R),
\]
because \( i \leq s \) and \( n - s \leq \max\{m, n + \epsilon\} - i + \delta \). It follows that
\[
\text{ht}(I_i(\bar{\varphi})) \leq (\bar{r} - i)(\max\{m, n + \epsilon\} - i + \delta + 1) + \ell + \delta + \text{ecodim}(R),
\]
proving (1) in Case 2 as well.

To show part (2) first notice that the \( \bar{R} \)-module \( \bar{M} \) has a rank, as can be seen from the Abhyankar-Hartshorne connectedness lemma (see Hartshorne [Ha2]). The natural map \( \text{Sym}(M) \to \text{Sym}(\bar{M}) \) induces an epimorphism \( \mathcal{R}(M) \to \mathcal{R}(\bar{M}) \) since \( M \) is of linear type locally at every associated prime of \( I \). Therefore \( \ell(M) \geq \ell(\bar{M}) \). On the other hand \( \ell(\bar{M}) \geq \text{rank}(\bar{M}) + 1 \) by Proposition 1.3. Therefore \( \ell \geq n - \bar{r} + 1 \), and (2) follows from (1).

\[\square\]

**Corollary 3.6.** Let \( R \) be an equidimensional universally catenary Noetherian local ring and let \( \varphi \) be an \( n \) by \( m \) matrix of rank \( r \) with entries in \( R \). Assume that \( M = \text{Coker}(\varphi) \) has a rank and write \( \ell = \ell(M) \). Let \( i \leq r \) be an integer such that \( I_i(\varphi) \neq R \). Set \( \epsilon = \max_P\{\text{ecodim}(R_P)\} \) where the maximum is taken over all prime ideals \( P \) of \( R \) not containing \( I_i(\varphi) \).

1. \[
\text{ht}(I_i(\varphi)) \leq \max\{(n - \ell - i + 1)(m - i + 1), (r - i)(\max\{m, n + \epsilon\} - i + 1) + \ell + \text{ecodim}(R)\}
\]
\[
\leq (r - i)(\max\{m, n + \epsilon\} - i + 1) + \max\{m - i + 1, \ell + \text{ecodim}(R)\}.
\]

2. If \( M \) is not a direct sum of a torsion module and a free module then
\[
\text{ht}(I_i(\varphi)) \leq (r - i)(\max\{m, n + \epsilon\} - i + 1) + \ell + \text{ecodim}(R).
\]

**Proof.** Apply Theorem 3.1 with \( I = 0 \) and use that \( \ell \geq \text{rank}(M) \). \[\square\]

In the setting of Corollary 3.6, part (1) could also be deduced from (2): for if \( M \) is a direct sum of a torsion module and a free module then obviously \( \text{ht}(I_i(\varphi)) \leq (r - i + 1)(m - i + 1) \).
Corollary 3.7. Let $R$ be an equidimensional universally catenary Noetherian local ring and let $\varphi$ be an $n$ by $m$ matrix of rank $r$ with entries in $R$. Assume that $M = \text{Coker}(\varphi)$ has a rank and write $\ell = \ell(M)$.

1. ([B1, Cor. 1]) If $M$ is not free then $\text{ht}(I_r(\varphi)) \leq \max\{m - r + 1, \ell + \text{ecodim}(R)\}$.
2. ([F, Kor. 1]) If $M$ is not a direct sum of a torsion module and a free module then $\text{ht}(I_r(\varphi)) \leq \ell + \text{ecodim}(R)$.

Proof. Set $i = r$ in Corollary 3.6. □

Corollary 3.8. Let $R$ be an equidimensional universally catenary Noetherian local ring of dimension $d$ and let $M$ be a finitely generated $R$-module having a rank. Let $\Lambda$ be the set of all prime ideals $Q$ such that the $R_Q$-module $M_Q$ is not a direct sum of a torsion module and a free module. If $\Lambda$ is nonempty then

$$d \leq \max_{Q \in \Lambda}\{\mu_Q(M) + \text{ecodim}(R_Q) + \dim(R/Q)\}.$$ 

Proof. We may factor out the torsion of $M$ to assume that $M$ is torsionfree. Notice this does not change the set $\Lambda$. Choose $Q$ minimal in $\Lambda$. Then $M_P$ is free for all primes $P \subsetneq Q$. If $\varphi$ is a matrix minimally presenting $M_Q$ we let $r$ be the rank of $\varphi$. Our choice of $Q$ shows that $\sqrt{I_r(\varphi)} = QR_Q$. Corollary 3.7.2 then gives $\text{ht}(I_r(\varphi)R_Q) \leq \mu_Q(M) + \text{ecodim}(R_Q)$. Hence $d - \dim(R/Q) = \dim(R_Q) = \text{ht}(I_r(\varphi)R_Q) \leq \mu_Q(M) + \text{ecodim}(R_Q)$, from which the corollary follows. □

Corollary 3.9. Let $R$ be an equidimensional universally catenary Noetherian local ring and let $\varphi$ be an $n$ by $m$ matrix of rank $r$ with entries in $R$. Assume that $M = \text{Coker}(\varphi)$ has a rank and write $\ell = \ell(M)$. Let $i \leq r$ be an integer such that $I_i(\varphi) \neq R$.

1. $\text{ht}(I_i(\varphi)/I_{i+1}(\varphi)) \leq \max\{m - i + 1, \ell + \text{ecodim}(R)\} + r - i$.
2. If $i \geq n - \ell + 1$, then

$$\text{ht}(I_i(\varphi)/I_{i+1}(\varphi)) \leq \ell + r - i + \text{ecodim}(R)$$

and in particular

$$\text{ht}(I_i(\varphi)) \leq (r - i + 1)(\ell + \text{ecodim}(R)) + \left(\frac{r - i + 1}{2}\right).$$

Proof. Apply Theorem 3.1 with $I = I_{i+1}(\varphi)$. Notice that $r = i$ and $n - \ell \leq r$. Iterate to get the second statement. □

The reader may want to compare Corollary 3.9.2 to Corollary 2.7. The significance of both formulas is that they do not involve $m$.

The above result leads to improved height bounds for $I_i(\varphi)$ if one knows a priori that for some $j \geq i$, the height of $I_j(\varphi)$ is “smaller than expected”. Applying this observation to ideals one obtains:
Corollary 3.10. Let $R$ be an equidimensional universally catenary Noetherian local ring with residue field $k$ and let $J$ be an $R$-ideal with $\text{grade}(J) > 0$. Write $g = \text{ht}(J)$, $\ell = \ell(J)$, $n = \mu(J)$, and $m = \dim_k \text{Tor}_1^R(k, J)$. Let $i$ be an integer with $g - 1 \leq i \leq n - 1$.

1. If $i \leq \ell - 1$ then

$$\text{ht}(\text{Fitt}_i(J)) \leq (i - g + 1)(\ell + g - 1 + \text{ecodim}(R)) + \binom{i - g + 1}{2} + g.$$  

2. If $i \geq \ell$, then

$$\text{ht}(\text{Fitt}_i(J)) \leq (\ell - g)(\ell + g - 1 + \text{ecodim}(R)) + \binom{\ell - g}{2} + g + (i - \ell + 1) \max\{m - n + \ell + i, \frac{3\ell + i}{2} - 1 + \text{ecodim}(R)\}.$$  

Proof. Notice that $\text{ht}(\text{Fitt}_{g-1}(I)) \leq g$ and apply Corollary 3.9. □

Example 3.11. Let $R$ be a regular local ring and let $J$ be a proper $R$-ideal with $g = \text{ht}(J)$ and $\ell = \ell(J)$.

1. (Non-complete-intersection locus, [Hu, Thm. 1.1]) If $J$ is not a complete intersection then $\text{ht}(\text{Fitt}_g(J)) \leq \ell + 2g - 1$.

2. (Non-almost-complete-intersection locus) If $\text{Ext}_R^g(J, R) = 0$, $J_Q$ is a complete intersection for every prime $Q$ containing $J$ with $\dim(R_Q) = g$, and $J$ is not an almost complete intersection, then $\text{ht}(\text{Fitt}_{g+1}(J)) \leq 2\ell + 3g - 1$.

Proof. In (1) we may suppose that $\text{ht}(\text{Fitt}_g(J)) \geq g + 1$. But then $J$ satisfies $G_{g+1}$, and hence $\ell \geq g + 1$ by [CN]. The assertion follows from Corollary 3.10.1. Likewise in (2) one can assume that $\text{ht}(\text{Fitt}_{g+1}(J)) \geq g + 2$. Thus $J$ satisfies $G_{g+2}$, and therefore $\ell \geq g + 2$ according to [CEU, 4.4 and 3.4(a)] and Proposition 1.2. We may apply Corollary 3.10.1. □

4. A Family of Examples

We present a class of $n$ by $m$ matrices of rank $r$ which show that the inequalities of Corollary 3.6.2 are fairly sharp for all values of $i, r, m, n$. Unlike the examples given in the introduction, these matrices have no generalized zeros.

Example 4.1. Let $i, r, m, n$ be integers with $1 \leq i \leq r \leq n \leq m$ and let $\varphi$ be the product of a generic $n$ by $r$ matrix with a generic $r$ by $m$ matrix. One has

$$\text{ht}(I_i(\varphi)) = \begin{cases} 
(r-i+1)(n-i+1) & \text{if } m \geq n + r - i + 1 \\
(r-i+1)(n-i+1) - \frac{(r+n-m-i+1)^2}{4} & \text{if } r + n - m - i + 1 > 0 \text{ and even} \\
(r-i+1)(n-i+1) - \frac{(r+n-m-i+1)^2-1}{4} & \text{if } r + n - m - i + 1 > 0 \text{ and odd.}
\end{cases}$$
Proof. We may assume that the ambient ring $R$ is obtained by adjoining the entries of the two generic matrices to a ring $k$. The height of $I_i(\varphi)$ cannot decrease when $k$ is replaced by the residue field of any minimal prime of $k$, and it cannot increase if we pass to the residue field of $P \cap k$ for some minimal prime $P$ of $I_i(\varphi)$ having minimal height. Thus it suffices to consider the case where $k$ is a field, and we may even assume that $k$ is algebraically closed.

Let $X$ be the closed subset of $\mathbb{P}^{P(m+n)-1} = \mathbb{P}(\text{Hom}_k(k^m,k^r) \times \text{Hom}_k(k^r,k^n))$ defined by the homogeneous ideal $I_i(\varphi)$. Notice that $X = \{[(\alpha, \beta)] | \text{rank}(\alpha) \leq i - 1\}$, where $\alpha \in \text{Hom}_k(k^m,k^r)$ and $\beta \in \text{Hom}_k(k^r,k^n)$. For $0 \leq s \leq r - i + 1$ set $X_s = \{[(\alpha, \beta)] | \text{rank}(\alpha) \leq s + i - 1, \text{rank}(\beta) \leq r - s, \text{rank}(\beta \alpha) \leq i - 1\}$. As $X$ is the union of the closed subsets $X_s$, our formula will follow once we have shown that

$$\dim X_s = (s + i - 1)(r + m - s - i + 1) + (r - s)n + (i - 1)s - 1.$$ 

In doing so we even show that $X_s$ is irreducible and we construct an explicit desingularization (see also Huneke and Ulrich [1987, the proof of 3.16], and Arbarello, Cornalba, Griffiths and Harris [1985, Chapter II, Section 2]). Let $Y$ be the flag variety $\text{Fl}(s, s + i - 1; k^r) = \{(U, V) | U \subset V \subset k^r\}$, where $U$ and $V$ are subspaces of dimension $s$ and $s + i - 1$, respectively. In $Y \times \mathbb{P}^{r(m+n)-1}$ consider the closed subset $Z = \{((U, V), [(\alpha, \beta)]) | \text{Image}(\alpha) \subset V, \text{Ker}(\beta) \supset U\}$. The projections onto the first and second factor of $Y \times \mathbb{P}^{r(m+n)-1}$ yield surjective morphisms

$$\begin{array}{ccc}
Z & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
Y & \xrightarrow{g} & X_s
\end{array}$$

$Y$ is irreducible of dimension $(s + i - 1)(r - s - i + 1) + (i - 1)s$. The fibers of $f$ over all closed points $(U, V)$ of $Y$ are isomorphic to $\mathbb{P}(\text{Hom}_k(k^m, V) \times \mathbb{P}(k^r/U, k^n)) \cong \mathbb{P}^{m(s+i-1)+n(r-s)-1}$, hence are irreducible of constant dimension. Since, furthermore, $Z \subset Y \times \mathbb{P}^{r(m+n)-1}$, it follows that $Z$ is irreducible (see Eisenbud [1995, Exercise 14.3]). One necessarily has

$$\dim Z = \dim Y + \dim \mathbb{P}^{m(s+i-1)+n(r-s)-1} = (s + i - 1)(r + m - s - i + 1) + (r - s)n + (i - 1)s - 1,$$

as can be seen, for instance, from the lemma of generic flatness (see Eisenbud [1995, 14.4]). On the other hand, since $Z$ is irreducible and $g$ is surjective, $X_s$ is irreducible as well. As $\{[(\alpha, \beta)] | \text{rank}(\alpha) \leq s + i - 2 \text{ or } \text{rank}(\beta) \leq r - s - 1\} \cap X_s$ is empty or a closed proper subset of $X_s$, it follows that for every closed point $(\alpha, \beta)$ in some dense open subsets of $X_s$, the fiber of $g$ over $(\alpha, \beta)$ consists of the single point $((\text{Ker}(\beta), \text{Image}(\alpha)), [(\alpha, \beta)])$. Thus again by generic flatness, $\dim X_s = \dim Z$, which proves our assertion. \qed

5. Some Results on Symmetric Matrices
We prove the conjecture of Problem 1 in the extremal cases \( i = 1 \) and \( i = n - 1 \) if the ring is regular.

**Proposition 5.1.** Let \((R, m)\) be a regular local ring and let \( \varphi \) be a symmetric \( n \times n \) matrix of rank \( r \) with entries in \( m \).

1. \( \operatorname{ht}(I_1(\varphi)) \leq rn - \binom{r}{2} \).
2. If \( 2 \) is a non zerodivisor on \( R \) and \( r = n - 1 \), then
   
   \[
   \operatorname{ht}(I_{n-1}(\varphi)) \leq 2.
   \]

**Proof.** To prove (1) we apply Theorem 2.1 to the module \( M = \operatorname{Coker}(\varphi) \). One has \( \ell(M) \geq \operatorname{rank}(M) = n - r \). By the theorem there exists a local homomorphism \( R \to S \) with \( S \) a localization of a polynomial ring over \( R \), and an invertible \( n \times n \) matrix \( T \) over \( S \) so that \( n - r \) row ideals \( J_1, \ldots, J_{n-r} \) of \( \Psi = T\varphi T^* \) have height at most \( r \). By the symmetry of \( \varphi \), \( \mu(I_1(\Psi)/(J_1 + \cdots + J_{n-r})) \leq \binom{r+1}{2} \). Therefore \( \operatorname{ht}(I_1(\varphi)) = \operatorname{ht}(I_1(\varphi)S) = \operatorname{ht}(I_1(\Psi)) \leq \operatorname{ht}(J_1 + \cdots + J_{n-r}) + \binom{r+1}{2} \leq (n-r)r + \binom{r+1}{2} = rn - \binom{r}{2} \).

To prove (2) we suppose that \( \operatorname{ht}(I_{n-1}(\varphi)) \geq 3 \). Since \( 2 \) is a non zerodivisor we may assume that \( \varphi_{11} \), the \((1,1)\) entry of \( \varphi \), does not lie in \( mI_1(\varphi) \). Having rank \( n - 1 \), the matrix \( \varphi \) fits into an exact sequence

\[
0 \to R \xrightarrow{\psi} R^n \xrightarrow{\varphi} R^{n*}.
\]

As \( \operatorname{ht}(I_1(\psi)) \geq \operatorname{ht}(I_{n-1}(\varphi)) \geq 3 \), the complex

\[
F. : 0 \to R \xrightarrow{\psi} R^n \xrightarrow{\varphi} R^{n*} \xrightarrow{\psi^*} R^*
\]

is exact by the Buchsbaum-Eisenbud acyclicity criterion, see Buchsbaum and Eisenbud [BE1, Theorem]. Thus \( I_1(\psi) = I_1(\psi^*) \subset I_1(\varphi) \). Furthermore \( I_1(\psi^*) \) is a Gorenstein ideal of height 3, and hence by Buchsbaum and Eisenbud [BE2, Theorem 2.1], there is an exact sequence

\[
G. : 0 \to R \xrightarrow{\psi} R^n \xrightarrow{\chi} R^{n*} \xrightarrow{\psi^*} R^*
\]

with \( \chi \) alternating.

The identity map on \( \operatorname{Ker}(\psi^*) \) lifts to a morphism of complexes \( \alpha. : F. \to G. \) where \( \alpha_0 = id \) and \( \alpha_1 = id \). Notice that \( \alpha_3 \) is multiplication by some \( u \in R \). Thus, since \( \alpha^*: G.* \to F.* \) is a morphism of acyclic complexes of free modules, \( \alpha.* \) is homotopic to multiplication by \( u \). Consequently \( \alpha. \) has the same property. It follows that

\[
\varphi \equiv u\chi \mod (I_1(\chi)I_1(\varphi) + I_1(\chi)I_1(\psi)),
\]

where \( u \) is a unit in \( R \).
hence
\[ \varphi \equiv u\chi \mod I_1(\varphi)^2. \]

But this is impossible because \( \chi_{11} = 0 \), whereas \( \varphi_{11} \not\in mI_1(\varphi) \). \( \square \)

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