Holographic Superconductors with Logarithmic Nonlinear Electrodynamics in an External Magnetic Field

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Abstract: Based on the matching method, we explore the effects of adding an external magnetic field on the $s$-wave holographic superconductor when the gauge field is in the form of the logarithmic nonlinear source. First, we obtain the critical temperature as well as the condensation operator in the presence of logarithmic nonlinear electrodynamics and understand that they depend on the nonlinear parameter $b$. We show that the critical temperature decreases with increasing $b$, which implies that the nonlinear gauge field makes the condensation harder. Then, we turn on the magnetic field in the bulk and find the critical magnetic field, $B_c$, in terms of the temperature, which also depends on the nonlinear parameter $b$. We observe that for temperature smaller than the critical temperature, $T < T_c$, the critical magnetic field increases with increasing $b$ and goes to zero as $T \to T_c$, independent of the nonlinear parameter $b$. In the limiting case where $b \to 0$, all results restore those of the holographic superconductor with magnetic field in Maxwell theory.
1 Introduction

The most successful microscopic theory of superconductivity was proposed by Bardeen, Cooper and Schrieffer (BCS) \[1\]. The BCS theory describes various properties of usual (low temperature) superconducting materials with great accuracy. The gauge/gravity duality \[2, 3\] states that string theory in asymptotic anti-de Sitter (AdS) spacetime can be dual to a conformal field theory (CFT) living on its boundary. This duality provides a well-established method for calculating the properties of the superconductors using a dual classical gravity description. The mechanism of the high temperature superconductors has long been an unsolved mysteries in modern condensed matter physics. Much research is now directed towards solving such worried by helping of different gravitational systems.

A great step toward understanding the strong coupling superconductivity put forwarded by Hartnoll, et. al., in 2008 \[4, 5\] who disclosed that some properties of strongly coupled superconductors can be potentially described by classical general relativity living in one higher dimension. Such strongly coupled superconducting phases of the boundary field theory are termed holographic superconductors in the literatures. The construction described above relates to a s-wave holographic superconductor and typically involves an Abelian Higgs model with a bulk complex scalar field that is charged under the Maxwell field \[6\]. According to the ADS/CFT correspondence, in the gravity side, a Maxwell field and a charged scalar field are introduced to describe the $U(1)$ symmetry and the scalar operator in the dual field theory, respectively. This holographic model undergoes a phase transition from black hole with no hair (normal phase/conductor phase) to the case with scalar hair at low temperatures (superconducting phase). Various aspects of the holographic superconductors have been explored from different perspective \[7–17\].

It is also interesting to investigate the effects of an external magnetic field on the holographic superconductors. One of the major properties of ordinary superconductors is that they exhibit perfect diamagnetism as the temperature is lowered below $T_c$ in the presence of an external magnetic field. In other words, at low temperature, superconductors expel magnetic field line and the phenomenon is known as Meissner effect. It is worthy to explore whether or not such an effect can be seen in the holographic superconductors when
the magnetic field is turned on. Some efforts have been done to disclose the properties of the holographic superconductors in the presence of an external magnetic field using both numerical approaches as well as analytical analysis [18–21]. As an analytical approach for deriving the upper critical magnetic field, an expression was found in the probe limit by extending the matching method first proposed in [22] to the magnetic case [23], which is shown to be consistent with the Ginzburg-Landau theory.

Most of these analysis are carried out in the framework of Maxwell electrodynamics. Along with the conventional Maxwell electrodynamic theory, nonlinear electrodynamic theories, which correspond to the higher derivative corrections to the Abelian gauge fields, have also become interesting topics of research in the past several decades. The primary motivation for introducing nonlinear electrodynamics was to remove the divergences in the self-energy of point-like charged particles [24]. However, they have earned renewed attention over past several years since these theories naturally arise in the low-energy limit of the heterotic string theory [25, 26]. Recently, the response of a holographic superconductor to an external magnetic field in the presence of Born-Infeld nonlinear electrodynamics was investigated by applying a simple analytical approach based on the matching of the solutions to the field equations near the horizon and near the asymptotic AdS region [27]. Unfortunately, the final result obtained for the critical magnetic field in Eq. (48) of Ref. [27] is not correct, unless for the Maxwell case where \( b = 0 \). Indeed, one can easily check that \( B_c \) derived in [27] goes to zero as temperature tends to zero, while we expect \( B_c \) approaches a finite value as \( T \) goes to zero. Besides, \( B_c \) of Ref. [27] does not goes to zero as \( T \to T_c \), while according to the definition, we should have \( B_c \to 0 \) in the limiting case where \( T \to T_c \).

Using the matching method in the probe limit, some properties of holographic superconductor in Gauss-Bonnet gravity with Born-Infeld electrodynamics and in the presence of a critical magnetic field were investigated in [28]. Performing explicit analytic computations, with and without magnetic field, the effects of higher order corrections to gravity as well as gauge field on the holographic s-wave condensation and Meissner-like effect have been explored in [29, 30]. In recent years, other types of nonlinear electrodynamics in the context of gravitational field have also been introduced, which can also remove the divergence of the electric field at the origin, similar to Born-Infeld nonlinear electrodynamics. Two well-known nonlinear Lagrangian for electrodynamics are logarithmic [31] and exponential [32] Lagrangian. In this paper, we shall provide the matching method to investigate the effects of the logarithmic nonlinear electrodynamics (LNE) on the holographic superconductors with an external magnetic field. The case with exponential nonlinear electrodynamics will be addressed in our future investigation.

This paper is organized as follows. In the next section, we introduce the basic field equations of holographic superconductors with LNE and obtain the critical temperature as well as the critical exponent. In section III, we turn on an external magnetic field and investigate the effects of it as well as nonlinear corrections to the gauge field on the properties of the holographic superconductors. In particular, we obtain the critical magnetic field in terms of the temperature which is affected by the nonlinear gauge field. We finish our paper with conclusions and discussions in section IV.
2 Basic Equations of Holographic Superconductors with LNE

Our starting point is a planar Schwarzschild AdS black holes, which is described by the line element

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2 + dy^2), \]

where

\[ f(r) = r^2 - \frac{r^3}{l}, \]

in units in which the AdS radius is unity, i.e. \( l = 1 \). In the above relation \( r_+ \) is the horizon radius. The Hawking temperature associated with the horizon is

\[ T = \frac{f'(r_+)}{4\pi} = \frac{3r_+}{4\pi}. \]

Let us now consider an electric field and a charged complex scalar field in this fixed background. The corresponding Lagrangian density can be expressed as

\[ L = L_{LN} - |\nabla_\mu \psi - iqA_\mu \psi|^2 - m^2 |\psi|^2, \]

where \( A_\mu \) and \( \psi \) are the gauge and scalar field, respectively. The term \( L_{LN} \) in (2.4) corresponds to the Lagrangian of the logarithmic nonlinear electrodynamics, which we take it in the following form [31]

\[ L_{LN} = -\frac{1}{b} \ln \left( 1 + \frac{bF}{4} \right), \]

where \( F \equiv F_{\mu\nu}F^{\mu\nu} \) and \( F^{\mu\nu} \) is the electromagnetic field tensor. The constant \( b \) is the nonlinear parameter which indicates the strength of the nonlinearity. In the limiting case where \( b \to 0 \), the Lagrangian \( L_{LN} \) reduces to the standard Maxwell Lagrangian, \( L_M = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \). We consider the following ansatz for the gauge and scalar fields [4]

\[ A_\mu = (\phi(r), 0, 0, 0), \quad \psi = \psi(r), \]

Varying Lagrangian (2.4) with respect to \( \phi(r) \) and \( \psi(r) \), the scalar and gauge field equations become

\[ \left( 2 + b\phi'^2(r) \right) \phi''(r) + \frac{2}{r} \left( 2 - b\phi'^2(r) \right) \phi'(r) - \frac{\phi(r)}{f(r)} \psi^2(r) \left( 2 - b\phi'^2(r) \right)^2 = 0, \]

\[ \psi''(r) + \left( \frac{f'}{f} + \frac{2}{r} \right) \psi'(r) + \left( \frac{\phi^2(r)}{f^2} - \frac{m^2}{f} \right) \psi(r) = 0, \]

where prime denotes derivative with respect to \( r \). One may note that for \( b \to 0 \), the field equation (2.7) for gauge field, restore the field equation of holographic superconductor in Maxwell theory [5]. In order to solve the non-linear equations (2.7) and (2.8), we should find the boundary condition for \( \phi \) and \( \psi \) near the black hole horizon \( r \sim r_+ \) as well as at
the spatial infinite $r \to \infty$. Using Eq. (2.8) and the fact $f(r_+) = 0$ and $f'(r_+) = 3r_+$, one can show that the regularity condition at the horizon leads to the boundary conditions

$$\phi(r_+) = 0 \quad \text{and} \quad \psi'(r_+) = -\frac{z^4}{3r_+^3}\psi(r_+).$$

Next, we transform the coordinate as $r \to z = r^1_+$ and also set $m^2 = -2$. In the new coordinate, the horizon $(r = r_+)$ and the boundary $(r \to \infty)$, are correspond to $z = 1$ and $z = 0$, respectively. The equations of motion for the scalar field $\psi(z)$ and the gauge field $A_\mu$ translate to

$$
\left(2 + \frac{z^4}{r_+^3}\phi'(z)\right)\phi''(z) + \frac{4b^2}{r_+^2}\phi'(z) - \frac{2}{r_+^2}\phi'(z) - \frac{2}{r_+^2}\phi'(z) = 0,
$$

$$
\psi''(z) + \frac{f'(z)}{f(z)}\psi'(z) + \frac{2}{r_+^2}\phi'(z) - \frac{r_+^2}{r_+^2}\phi'(z)\psi(z) = 0,
$$

where prime now denotes derivative with respect to $z$ and $f'(1) = -3r_+^2$. Let us now take a look at the boundary conditions in the new coordinate $z$. It is easy to check that the regularity at the horizon $z = 1$ implies,

$$\phi(1) = 0, \quad \psi'(1) = 0 \quad \psi(1).$$

While, in the asymptotic region ($z \to 0$) the solutions may be written as,

$$\phi(z) \approx \mu - \frac{\rho}{r_+ z}, \quad \psi(z) \approx J_- z + J_+ z^2,$$

where $\mu$ and $\rho$ are interpreted as a chemical potential and charge density, respectively. In what follows we set $J_+ = 0$. The condensation operator $<\mathcal{O}_->$ is related to $J_-$ as $<\mathcal{O}_-> = \sqrt{2}r_+ J_- [27]$.

Now we are in a position to obtain expressions for the critical temperature and values of the condensation, analytically. Let us first consider the solutions of the gauge field, $\phi(r)$, and the scalar field, $\psi(z)$, near the horizon ($z = 1$). Using the Taylor expansion for $\phi(z)$ and $\psi(z)$ near the horizon, we can write

$$
\phi(z) = \phi(1) - \phi'(1)(1 - z) + \frac{1}{2}\phi''(1)(1 - z)^2 + ...
$$

$$
\approx -\phi'(1)(1 - z) + \frac{1}{2}\phi''(1)(1 - z)^2,
$$

$$
\psi(z) = \psi(1) - \psi'(1)(1 - z) + \frac{1}{2}\psi''(1)(1 - z)^2 + ..., \quad (2.14)
$$

where we have used the boundary condition $\phi(1) = 0$ in the second line of Eq. (2.13). We further assume $\phi'(1) < 0$ and $\psi(1) > 0$ in order to make $\phi(z)$ and $\psi(z)$ positive. This can be done without any loss of generality. Near the horizon, $z = 1$, and using the relations $f'(1) = -3r_+^2$ and $f''(1) = 6r_+^2$, one can show that

$$
\phi''(1) = -\frac{2b\phi'(1)}{r_+^3} - \frac{2}{3}\phi'(1)\psi'(1)\left(1 - \frac{3b}{2r_+^2}\phi'(1)\right) + O(b^2), \quad (2.15)
$$
where we have only kept the expression up to the linear term of the nonlinear parameter $b$, by assuming that $b$ is small. Finally, substituting Eqs. (2.15) and (2.16) in the Taylor expansion (2.13), (2.14), one gets

$$
\phi(z) \approx \psi(1) - \psi'(1)(1-z) - \left[ b \frac{\beta^2}{r_+^2} + \frac{1}{3} \psi^2(1) \left( 1 - \frac{3b}{2r_+^2} \phi^2(1) \right) \right] \phi'(1)(1-z)^2 + O(b^2),
$$

(2.17)

$$
\psi(z) \approx \psi(1) - \psi'(1)(1-z) - \left[ \frac{2}{9} \psi(1) + \frac{\phi^2(1)\psi(1)}{36r_+^2} \right] (1-z)^2
$$

$$
= \frac{1}{3} \psi(1) + \frac{2}{3} \psi'(1)z - \left[ \frac{2}{9} \psi(1) + \frac{\phi^2(1)\psi(1)}{36r_+^2} \right] (1-z)^2,
$$

(2.18)

where in the last step of (2.18), we have also used the regularity conditions for $\psi$ at the horizon $z = 1$. Now, using the method prescribed by the matching technique [22], we match the solutions (2.12) with solutions (2.17) and (2.18) at some intermediate point $z = z_m$. It is easy to check that the matching of the two asymptotic solutions smoothly at $z = z_m$ leads to the following four conditions

$$
\mu - \frac{\rho}{r_m} = \beta(1 - z_m) + \left[ b \frac{\beta^2}{r_+^2} + \frac{1}{3} \alpha^2 \left( 1 - \frac{3b}{2r_+^2} \beta^2 \right) \right] \beta(1 - z_m)^2,
$$

(2.19)

$$
\frac{\rho}{r_+} = \beta + 2\beta \left[ b \frac{\beta^2}{r_+^2} + \frac{1}{3} \alpha^2 \left( 1 - \frac{3b}{2r_+^2} \beta^2 \right) \right] (1 - z_m),
$$

(2.20)

$$
J_+ z_m = \frac{\alpha}{3} + \frac{2\alpha}{3} z_m - \frac{\alpha}{9} \left[ 2 + \frac{3\beta^2}{4r_+^2} \right] (1 - z_m)^2,
$$

(2.21)

$$
J_+ = \frac{2\alpha}{3} + \frac{\alpha}{9} \left[ 4 + \frac{3\beta^2}{2r_+^2} \right] (1 - z_m),
$$

(2.22)

where we have defined $\alpha \equiv \psi(1)$, $\beta \equiv -\psi'(1)$ ($\alpha, \beta > 0$). From Eq. (2.20), we find

$$
\alpha^2 = \frac{3}{2(1 - z_m)} \left[ \frac{\rho}{\beta r_+^2} + \frac{3b\beta}{2r_+^2} - \frac{7}{2} b\beta^2 + 2b\beta^2 z_m - 1 \right] + O(b^2),
$$

(2.23)

which by using (2.3), can be rewritten

$$
\alpha^2 = \frac{3}{2(1 - z_m)} \left[ 1 + \frac{b\beta^2}{2(7 - 4z_m)} \right] \frac{T_c^2}{T^2} \left( 1 - \frac{T^2}{T_c^2} \right) + O(b^2),
$$

(2.24)

where $\tilde{\beta} = \beta/r_+$ and the quantity $T_c$ may be identified as the critical temperature for condensation and is given by,

$$
T_c = \kappa \sqrt{\rho},
$$

(2.25)
where
\[
\kappa = \frac{3}{4\pi \sqrt{\tilde{\beta}}} \sqrt{1 - 2b\tilde{\beta}^2(1 - z_m)}. \tag{2.26}
\]

When the temperature is very close to the critical temperature \(T \sim T_c\), from Eq. (2.24) we obtain,
\[
\alpha = \sqrt{\frac{3}{1 - z_m}} \left(1 + \frac{b\tilde{\beta}^2}{4}(7 - 4z_m)\right) \sqrt{1 - \frac{T}{T_c}} + O(b^2). \tag{2.27}
\]

It is worth noting that in the expressions for \(\alpha\) and \(T_c\) we again kept up to the linear terms in \(b\). From Eqs. (2.21), (2.22), we obtain
\[
\tilde{\beta} = 2\sqrt{\frac{1 + 2z_m^2}{1 - z_m}}, \quad J_- = \frac{2\alpha(2 + z_m)}{3(1 + z_m)}. \tag{2.28}
\]

Note that one may arrive at the above value for \(\tilde{\beta}\) by using the fact that \(J_+ = 0\). Indeed, from Eq. (2.12), we have
\[
\psi(z \to 0) \approx \psi_0 = J_- z + J_+ z^2, \tag{2.29}
\]
which leads to
\[
J_+ = -\frac{\psi_0}{z^2} + \frac{\psi_0'}{z} = 0. \tag{2.30}
\]

Combining the above condition with
\[
\psi_0 = \frac{\alpha}{3} + \frac{2\alpha}{3}z_m - \frac{\alpha}{9} \left[2 + \frac{\beta^2}{4r^2_+}\right](1 - z_m)^2, \tag{2.31}
\]
we arrive at value (2.28) for \(\tilde{\beta}\). Now, we compute the condensation operator from Eqs. (2.27), (2.28),
\[
< O_- > = \sqrt{2r_+ J_-} = \gamma T_c \sqrt{1 - \frac{T}{T_c}}, \tag{2.32}
\]
where we have defined
\[
\gamma = \frac{8\sqrt{2}\pi(2 + z_m)}{9(1 + z_m)} \sqrt{\frac{3}{(1 - z_m)} \left(\frac{7}{2}b\tilde{\beta}^2 - 2b\tilde{\beta}^2 z_m + 1\right)}. \tag{2.33}
\]

In is notable to mention that there is an upper bound on the nonlinear parameter \(b\). Indeed, from Eq. (2.26) we see that the nonlinear parameter should be satisfied in the following upper bound
\[
b < \frac{1}{2\beta^2(1 - z_m)} = \frac{1 + z_m}{8(1 + 2z_m^2)}. \tag{2.34}
\]

Since \(0 \leq z_m \leq 1\), therefore for \(z_m = 0\), we have for the upper bound \(b < 1/8\), and for \(z_m = 1\) we have \(b < 1/12\). In the limiting case where \(b \to 0\), all the above relations reduce to the results of holographic superconductor in external magnetic field in Maxwell theory [23]. Let us summarize the result for critical temperature in table 1. We see that the critical
temperature decreases with increasing $b$, which implies that the nonlinear electrodynamics makes the condensation harder. This is consistent with the results of [27, 33]. Comparing the results obtained in this table from matching method, with those obtained from the numerical method [34] and those obtained from Sturm-Liouville variational method for Born-Infeld nonlinear electrodynamics [35] shows that the Sturm-Liouville method yields better results than the matching method.

| $b$ | 0   | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 |
|-----|-----|------|------|------|------|------|
| $T_c/\sqrt{\rho}$ | 0.168 | 0.161 | 0.155 | 0.148 | 0.141 | 0.133 |

Table 1: Critical temperature for the LN holographic superconductors by using the matching method for different values of nonlinear parameter $b$ at $z_m = 0.1$.

## 3 Effects of an External Magnetic Field with LNE

In this section we would like to investigate the effects of an external magnetic field on the holographic superconductors in the presence of LNE. For this purpose, we consider a magnetic field in the bulk. The asymptotic value of this magnetic field corresponds to a magnetic field added to the boundary field theory by the gauge gravity correspondence. This allows us to adopt the following ansatz for the gauge field and the scalar field [20]

$$A_t = \phi(z), \quad A_y = Bx, \quad A_x = A_z = 0, \quad \psi = \psi(x, z).$$

Therefore, the equation for the scalar field $\psi$ becomes

$$\psi''(x, z) + \frac{f'(z)}{f(z)} \psi'(x, z) + \frac{2r^2}{z^4f(z)} \psi(x, z) + \frac{r^2 \phi^2(z) \psi(x, z)}{z^4f^2(z)} + \frac{1}{z^2f(z)} (\partial_x^2 \psi - B^2 x^2 \psi) = 0.$$  \hspace{1cm} (3.2)

In order to solve Eq. (3.2), we shall use the method of separation of variables. Let us consider the solution of the following form

$$\psi(x, z) = X(x)R(z).$$

Substituting (3.3) into Eq. (3.2), we obtain the following equation

$$z^2f(z) \left[ \frac{R''}{R} + \frac{f' R'}{f R} + \frac{r^2 \phi^2}{z^4f^2} + \frac{2r^2}{z^4f} \right] - \frac{X''}{X} + B^2 x^2 = 0.$$  \hspace{1cm} (3.4)

Clearly, the equations for two variables, $x$ and $z$ are separable. The equation for $X(x)$ is readily identified as the Schrodinger equation in one dimension with a $B$ dependent frequency [20]

$$-X''(x) + B^2 x^2 X(x) = \lambda_n B X(x),$$  \hspace{1cm} (3.5)

where $\lambda_n = 2n + 1$ ($n =$ integer). Since the most stable solution corresponds to $n = 0$, the $z$ dependent part of (3.4) may be expressed as [20]

$$R''(z) + \frac{f'(z)}{f(z)} R'(z) + \frac{2r^2 R(z)}{z^4f(z)} + \frac{r^2 \phi^2(z) R(z)}{z^4f^2(z)} = BR(z).$$  \hspace{1cm} (3.6)
Now at the horizon, $z = 1$, using (3.6) and $f'(1) = 3r_+^2$, we can write the following equation

$$R'(1) = \left(\frac{2}{3} - \frac{B}{3r_+^2}\right) R(1). \tag{3.7}$$

On the other hand, at the asymptotic infinity, $z \to 0$, the solution of Eq. (3.6) can be written as

$$R(z) = J_- z + J_+ z^2. \tag{3.8}$$

Note that in our analysis we shall choose $J_+ = 0$. Our aim here is to find the value of the (critical) magnetic field for which the condensation vanishes. Again, we employ the matching method [22]. The Taylor expansion of $R(z)$ around $z = 1$ leads to

$$R(z) = R(1) - R'(1)(1 - z) + \frac{1}{2} R''(1)(1 - z)^2 + ... \tag{3.9}$$

Considering Eq. (3.6) near the horizon ($z = 1$) and using the fact that $f'(1) = -3r_+^2$, $f''(1) = 6r_+^2$ as well as the regularity condition for $\phi$ and Eq. (3.7), we arrive at

$$R''(1) = \left[-\frac{4}{9} - \frac{2B}{9r_+^2} + \frac{B^2}{18r_+^4} - \frac{\phi'^2(1)}{18r_+^4}\right] R(1), \tag{3.10}$$

Combining Eq. (3.10) with Eqs. (3.7) and (3.9), we find the following expression

$$R(z) \approx \frac{1}{3} R(1) + \frac{2}{3} R(1) z + \frac{BR(1)}{3r_+^2}(1 - z) + \frac{1}{2} \left[-\frac{4}{9} - \frac{2B}{9r_+^2} + \frac{B^2}{18r_+^4} - \frac{\phi'^2(1)}{18r_+^4}\right] R(1)(1 - z)^2. \tag{3.11}$$

Following the arguments of matching technique, we match Eq. (3.11) with Eq. (3.8) at some intermediate point $z = z_m$ which may be put as,

$$J_- z_m = \frac{1}{3} R(1) + \frac{2}{3} R(1) z_m + \frac{BR(1)}{3r_+^2}(1 - z_m) + \frac{1}{2} \left[-\frac{4}{9} - \frac{2B}{9r_+^2} + \frac{B^2}{18r_+^4} - \frac{\phi'^2(1)}{18r_+^4}\right] R(1)(1 - z_m)^2, \tag{3.12}$$

$$J_+ = \frac{2}{3} R(1) - \frac{BR(1)}{3r_+^2} + \left[\frac{4}{9} + \frac{2B}{9r_+^2} - \frac{B^2}{18r_+^4} + \frac{\phi'^2(1)}{18r_+^4}\right] R(1)(1 - z_m). \tag{3.13}$$

It is a matter of calculations to show that the above set of equations, lead to the following quadratic equation for $B$

$$B^2 + 4r_+^2 \left(\frac{2 + z_m^2}{1 - z_m^2}\right) B + 4r_+^4 \left(\frac{1 + 2z_m^2}{1 - z_m^2}\right) - \phi'^2(1)r_+^2 = 0. \tag{3.14}$$

Again, it is a matter of calculation to show that Eq. (3.14) can by obtained by combining Eq. (3.8) with condition $J_+ = 0$ at the asymptotic infinity $z \to 0$. Eq. (3.14) has a solution of the form

$$B = \sqrt{\phi'^2(1)r_+^2 + 12r_+^4 \left(\frac{1 + z_m^2 + z_m^4}{1 - z_m^2}\right) - 2r_+^2 \left(\frac{2 + z_m^2}{1 - z_m^2}\right)}, \tag{3.15}$$
We consider the case in which the value of the external magnetic field \((B)\) is very close to the upper critical value i.e, \(B \sim B_c\). Thus, one can neglect all the quadratic terms in \(\psi\). Under this condition, Eq. (2.9) simplifies to

\[
\left(2 + b \frac{z^4}{r_+^2} \phi'^2(z)\right) \phi''(z) + \frac{4bz^3}{r_+^2} \phi'^3(z) = 0. \tag{3.16}
\]

Next, we integrate the above equation in the interval \([1, z]\), after using \(\phi'(1) < 0\) and the asymptotic boundary condition for \(\phi\) given in Eq. (2.12), we arrive at

\[
\phi'(z) = \frac{r_+(1 - \sqrt{1 + 2bz^4 \lambda^2})}{bz^4 \lambda}, \tag{3.17}
\]

where \(\lambda = \rho/r_+^2\). Expanding (3.17) for small value of \(b\) leads to the following expression at the horizon \((z = 1)\),

\[
\phi'(1) = -\frac{\rho}{r_+}. \tag{3.18}
\]

From Eqs. (2.25), (2.26) and invoking the value of \(\bar{\beta}\) from Eq. (2.28), we have

\[
\rho = \frac{16\pi^2 \bar{\beta}}{9} T_c^2(0) \left[1 + 2b\bar{\beta}^2(1 - z_m)\right], \tag{3.19}
\]

where \(T_c(0)\) is the critical temperature at zero magnetic field. Combining Eqs. (3.15), (3.18) and (3.19), we find the expression of the critical magnetic field as

\[
\frac{B_c}{T_c^2(0)} = \frac{16}{9} \pi^2 \left\{ \frac{\bar{\beta}}{\sqrt{\left(1 + 2b\bar{\beta}^2(1 - z_m)\right)^2 + \frac{12(1 + z_m^2 + z_m^4)}{b^2(1 - z_m^2)^2} \left(\frac{T}{T_c(0)}\right)^4 - \frac{2(2 + z_m^2)}{(1 - z_m^2) \left(\frac{T}{T_c(0)}\right)^2}} \right\} + \mathcal{O}(b^3). \tag{3.20}
\]

Considering the fact that the nonlinear parameter \(b\) is small, and following [29, 30], we can write the above relation as

\[
\frac{B_c}{T_c^2(0)} \approx \frac{16}{9} \pi^2 \left\{ \frac{2b\bar{\beta}^2(1 - z_m)}{1 + \frac{12(1 + z_m^2 + z_m^4)}{b^2(1 - z_m^2)^2} \left(\frac{T}{T_c(0)}\right)^4} \right\} \left(\frac{z}{\bar{\beta}}\right) \sqrt{\left(1 + \frac{12(1 + z_m^2 + z_m^4)}{b^2(1 - z_m^2)^2} \left(\frac{T}{T_c(0)}\right)^4 - \frac{2(2 + z_m^2)}{(1 - z_m^2) \left(\frac{T}{T_c(0)}\right)^2}} \right) + \mathcal{O}(b^3). \tag{3.21}
\]

The above result clearly reveals the dependence of the critical magnetic field on the nonlinear parameter \(b\). Note that this result is valid for small values of the nonlinear parameter \(b\). In the limiting case where \(b \to 0\), the results of holographic superconductor in Maxwell theory in the presence of an external magnetic field are restored [23]. We have plotted the behavior of the critical magnetic field in terms of temperature in Figs. 1 – 3.
Figure 1. The behaviour of $B_c/T_c^2(0)$ in terms of $T/T_c(0)$ at $z_m = 0.5$ for different value of nonlinear parameters $b$.

From Figs. 1 and 2 we see that $B_c$ is larger for LN holographic superconductor ($b \neq 0$) compared to the Maxwell case ($b = 0$). Also, we find out that for temperature smaller than the critical temperature, $T < T_c$, the critical magnetic field increases with increasing $b$ and goes to zero as $T \rightarrow T_c$, independent of the nonlinear parameter $b$. Finally, from Fig. 3, we observe that for a fixed value of $b$, the critical temperature increases with increasing the intermediate point $z_m$.

4 Conclusions

In this paper, based on the matching technique, we have investigated various properties of the holographic s-wave superconductors in an external magnetic field and in the presence of logarithmic nonlinear electrodynamics (LNE). Considering the probe limit in which the scalar and gauge field do not affect on the background metric, we have presented a detailed analysis of solving the coupled equations of motion for the scalar and the LNE gauge field. We have obtained the relationship between the critical temperature and the charge density. We kept the terms up to the linear order in the nonlinear parameter $b$ and neglected $O(b^2)$ and higher order terms by assuming that $b$ is small. We have summarized our result for the critical temperature in table 1. From this table, we understand that the critical temperature for condensation, $T_c$, decreases with increasing the values of the nonlinear parameter $b$. The variation of the order operator with temperature, $< O_- > \sim \sqrt{1 - \frac{T}{T_c}}$, exhibits a mean field
behavior with critical exponent $1/2$. It is worth mentioning that the expressions for the critical temperature and the condensation values were obtained up to the linear order in $b$.

We have also disclosed the effects of an external magnetic field on the holographic superconductor in the presence of LNE. We have found an expression for the critical magnetic field $B_c$ in terms of the temperature. In the limiting case where $b \to 0$, all results reduce to those of the holographic superconductor with magnetic field in Maxwell theory \[23\]. We have investigated the behavior of the critical magnetic field in terms of the temperature by plotting $B_c/T_c^2(0)$ versus $T/T_c(0)$. We observed that for $T < T_c$, the critical magnetic field increases with increasing the nonlinear parameter $b$ and goes to zero at $T = T_c$ independent of the value of $b$ (see Figs. 1 and 2). This implies that the nonlinear electrodynamics makes condensation harder. Finally, we found out that for a fixed value of $b$, the critical magnetic field increases as the intermediate point $z_m$ approaches to the boundary ($z = 1$) (Fig 3).

Note that in this paper, we have limited our study to the case where the scalar and gauge fields do not back react on the metric background. It is interesting to generalize this study by exploring the effects of external magnetic field as well as LNE on the holographic superconductor away from the probe limit. One may also consider the case with exponential nonlinear electrodynamics. These issues are currently under investigation and the results will be reported subsequently.
Figure 3. The behaviour of $B_c/T_c^2(0)$ in terms of $T/T_c(0)$ for $b = 0.02$ and different values of $z_m$.

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