Abstract

It is proved that the special fibre of the Nash transform of an irreducible isolated singularity has maximal possible dimension.

In the paper [2] by A. Simis, K. Smith and B. Ulrich it is asked (in slightly different terms) as a question, whether at the Nash transformation of an isolated singularity the special fibre has always the maximal dimension. In the particular case of a Cohen-Macaulay singularity a positive answer is given there, using a result of E. Kunz and R. Waldi [1]. Here, we propose a simple proof for the general case.

Let $(X, 0) \subseteq (\mathbb{C}^N, 0)$ be a reduced isolated singularity of dimension $d \geq 2$. We suppose $X$ to be irreducible and non-smooth. The smooth locus will be denoted by $X^*$. The Nash transform $\nu: \tilde{X} \rightarrow X$ can be described geometrically as the closure of the graph of the Gauss map $X^* \rightarrow G$, $x \mapsto T_{x,X}$, which assigns to every smooth point its tangent space considered as a point of the Grassmann manifold of $d$-dimensional subspaces of $\mathbb{C}^N$. In this way $\tilde{X} \subseteq X \times G$ is an analytic subset, and $\nu$, which is induced from the first projection, is a proper modification. The special fibre $\nu^{-1}(0) \subseteq G$ is just the set of all limits of tangent spaces at smooth points. The universal bundle on $\tilde{X}$ provides $\tilde{X}$ with a (dual) locally free sheaf $\Omega$ and a surjection $\nu^*\Omega \rightarrow \Omega$, which is isomorphic off $\nu^{-1}(0)$. (Cf. e.g. [3])

Let now $n: Y \rightarrow \tilde{X}$ be the normalisation and $\pi: Y \rightarrow X$ the composition with the Nash transformation. The sheaf $\tilde{\omega} := n^*\Lambda^d\tilde{\Omega}$ is invertible and outside $\pi^{-1}(0)$ it coincides with the canonical sheaf $\omega_Y$. We note that for a normal complex space the canonical sheaf is the sheaf of all forms of top degree which need only be defined and regular at smooth points.

**Theorem:** The fibre $\pi^{-1}(0)$ has pure dimension $d - 1$. In particular the same holds for the fibre $\nu^{-1}(0)$ of $\tilde{X}$.

Proof: By its construction as a closure of a manifold of dimension $d$ the Nash transform is pure dimensional. This property is inherited by the normalisation $Y$. Therefore it is sufficient to disprove, that there is a point $P \in \pi^{-1}(0)$ at which $\pi^{-1}(0)$ has codimension $\geq 2$. Under this assumption however, taking into account that $\tilde{\omega}$ is invertible, we must have equality $\tilde{\omega}_P = \omega_{Y,P}$. This means that we have a surjective map $(\pi^*\Omega^d_{\tilde{X}})_P \rightarrow \omega_{Y,P}$. Our aim is to show that this is impossible.

Case 1: $P$ is a smooth point in $Y$. The map $\pi: Y \rightarrow X \rightarrow \mathbb{C}^N$ has in local coordinates $(y_1, \ldots, y_d)$ around $P$ the form $\pi: (\mathbb{C}^d, 0) \rightarrow (\mathbb{C}^N, 0)$, and the image of the map

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\[(\pi^*\Omega_X^d)_p \to \omega_{Y,p} \cong \mathcal{O}_{Y,p}\]
is the ideal of maximal minors \(I_d(\partial \pi / \partial y)\) of the Jacobian. From this fact we see that the map can only be surjective when \(\pi\) is regular at \(P\). But then \(X\) would have a smooth component.

Case 2: \(P\) is a singular point in \(Y\). We consider the decomposition:
\[(\pi^*\Omega_X^d)_p \to \Omega_Y^d, \to \omega_{Y,p}\]
The map at right is of course also surjective. To get a contradiction, let \(y_1, \ldots, y_m\) be a general linear coordinate system on some embedding space \((\mathbb{C}^m, 0) \supseteq (Y, P)\). As \(\omega_{Y,p} \cong \mathcal{O}_{Y,p}\) and \(\Omega_Y^d, p\) is generated by all \(dy_{i_1} \wedge \ldots \wedge dy_{i_d}\) \((1 \leq i_1 < \ldots < i_d \leq m)\), we can suppose that \(\omega_{Y,p} = \mathcal{O}_{Y,p}(dy_{i_1} \wedge \ldots \wedge dy_{i_d})\). By the general choice of coordinates, the mapping
\[(y_1, \ldots, y_d): (Y, P) \to (\mathbb{C}^d, 0)\]
is finite. However, this map has an empty ramification divisor. \(Y\) being normal, this is impossible.

**Remark:** The above proof is also valid for non-isolated singularities. In this case one obtains a lower estimate of the fibre dimension by \(d - 1\) minus the dimension of the singular locus. This can also be obtained by taking hyperplane sections.

**References**
1. Kunz, E., R. Waldi: Regular Differential Forms. Contemporary Mathematics Vol. 79 AMS (1988).
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3. Teissier, B.: Variétés polaires II. In: Algebraic Geometry. Lecture Notes in Mathematics 961. Springer (1982), 314-491.