On Coefficients of Some $p$-Valent Starlike Functions

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Abstract. We consider the class $A_p$ of functions $f$ analytic in the unit disk $|z| < 1$ in the complex plane, of the form $f(z) = z^p + \cdots$ such that $\Re\{z f''(z) / f'(z)\} > 0$ in the unit disc. The object of the present paper is to derive some bounds for coefficients in this class and relation with the functions satisfying condition $\Re\{z f''(z) / f'(z)\} > 0$ in the unit disc.

1. Introduction

We denote by $\mathcal{H}$ the class of functions $f(z)$ which are holomorphic in the open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$. A function $f$ analytic in a domain $D \subset \mathbb{C}$ is called $p$-valent in $D$, if for every complex number $w$, the equation $f(z) = w$ has at most $p$ roots in $D$, so that there exists a complex number $w_0$ such that the equation $f(z) = w_0$ has exactly $p$ roots in $D$. The properties of multivalent functions under several operators were established recently in several papers, see for instance [3, 6, 8, 16]. Meromorphic multivalent functions was considered recently in [4, 5, 9]. Denote by $A_p$, $p \in \mathbb{N} = \{1, 2, \ldots\}$, the class of functions $f(z) \in \mathcal{H}$ given by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (z \in D). \quad (1)$$

Let $\mathcal{A} = \mathcal{A}_1$. Let $S$ denote the class of all functions in $\mathcal{A}$ which are univalent. Also let $S_p^*$ and $C_p$ be the subclasses of $\mathcal{A}_p$ defined as follows

$$S_p^* = \left\{ f(z) \in \mathcal{A}_p : \Re\left\{\frac{z f''(z)}{f'(z)}\right\} > 0, \quad z \in D \right\},$$

$$C_p = \left\{ f(z) \in \mathcal{A}_p : \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad z \in D \right\}.$$

The classes $S_p^*$ and $C_p$ will be called the class of $p$-valently starlike functions and the class of $p$-valently convex functions, respectively. Note that $S_1^* = S^*$ and $C_1 = C$, where $S^*$ and $C$ are usual classes of starlike and convex functions respectively.

In this paper we need the following lemmas.

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Lemma 1.1. [13, Theorem 5] If \( f(z) \in \mathcal{A}_p \), then for all \( z \in \mathbb{D} \), we have

\[
\Re \left\{ \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} > 0 \quad \Rightarrow \quad \forall k \in \{1, \ldots, p-1\} : \quad \Re \left\{ \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} > 0.
\]  

(2)

Corollary 1.2. If \( f(z) \in \mathcal{A}_p \), then for \( r \in (0, 1] \), we have

\[
\Re \left\{ \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} > 0, \quad |z| < r \quad \Rightarrow \quad \forall k \in \{1, \ldots, p-1\} : \quad \Re \left\{ \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} > 0, \quad |z| < r.
\]

Lemma 1.3. [14] Let \( p \) be analytic function in \( |z| < 1 \), with \( p(0) = 1 \). If there exists a point \( z_0 \), \( |z_0| < 1 \), such that

\( \Re [p(z)] > 0 \) for \( |z| < |z_0| \)

and

\( p(z_0) = \pm ia \)

for some \( a > 0 \), then we have

\[
\frac{zp'(z_0)}{p(z_0)} = \frac{2k \arg [p(z_0)]}{\pi},
\]

for some \( k \geq (a + a^{-1})/2 \geq 1 \).

Lemma 1.4. [13] If \( f(z) \in \mathcal{A}_p \), and there exists a positive integer \( j \), \( 1 \leq j \leq p \) for which

\[
\Re \left\{ j + \frac{zf^{(j+1)}(z)}{f^{(j)}(z)} \right\} > 0, \quad (z \in \mathbb{D}),
\]

(4)

then for all \( z \in \mathbb{D} \) we have

\[
\forall k \in \{1, \ldots, j\} : \quad \Re \left\{ k - 1 + \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} > 0.
\]

(5)

Corollary 1.5. If \( f(z) \in \mathcal{A}_p \), and there exists a positive integer \( j \), \( 1 \leq j \leq p \) for which

\[
\Re \left\{ j + \frac{zf^{(j+1)}(z)}{f^{(j)}(z)} \right\} > 0, \quad (|z| < r),
\]

(6)

then for \( |z| < r \), we have

\[
\forall k \in \{1, \ldots, j\} : \quad \Re \left\{ k - 1 + \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} > 0, \quad (|z| < r).
\]

(7)

2. Main results

Coefficient bounds for \( p \)-valent functions was considered recently in [15] while the coefficient neighborhoods of certain \( p \)-valently analytic functions with negative coefficients, in [1]. Some convolution (Hadamard product) conditions for starlikeness and convexity of meromorphically multivalent functions one can find in [11].

Let \((x)_n\) denote the Pochhammer symbol which is defined in term of Gamma function \(\Gamma\) as:

\[
(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & \text{for } n = 0, \quad x \neq 0, \\ x(x+1) \ldots (x+n-1) & \text{for } k \in \mathbb{N} = \{1,2,3,\ldots\}. \end{cases}
\]
Theorem 2.1. If \( f(z) \in A_p, \ p \geq 2, \ f(z) = z^p + a_{p+1}z^{p+1} + \cdots, \ z \in \mathbb{D} \) and if
\[
\Re \left\{ \frac{z^{p+1}}{f^{(p-1)}(z)} \right\} > 0, \quad z \in \mathbb{D}, \tag{8}
\]
then for \( n \geq p \), we have
\[
|a_n| \leq \frac{p!(n - p + 1)}{n(n-1)(n-2)\cdots(n-(p-2))} = \frac{p!(n - p + 1)}{(n - p + 2)p-1}.
\]

The result is sharp.

Proof. If a function \( f(z) \) satisfies (8), then \( f^{(p-1)}(z)/p! = z + b_2z^2 + \cdots \) is a starlike function. Therefore, the coefficients of \( f^{(p-1)}(z)/p! \) satisfy
\[
|b_p| \leq n.
\]

From this we can obtain the bound for \( |a_n| \). We have that \( b_{n-p} = n(n-1)(n-2)\cdots(n-(p-2))a_n/p! \), so
\[
|a_n| \leq p!(n - p + 1)\sum_{n-p}(n-1)(n-2)\cdots(n-(p-2)) \leq p!(n - p + 1)\sum_{n-p}(n-1)(n-2)\cdots(n-(p-2)) \leq |a_n|.
\]

To show that the bound is sharp it suffices to prove that the function
\[
f_p(z) = \sum_{p}^{\infty} \frac{p!(n - p + 1)}{n(n-1)(n-2)\cdots(n-(p-2))}z^n, \quad z \in \mathbb{D}, \tag{9}
\]
satisfies (8). We have
\[
f_p^{(p-1)}(z)/p! = \frac{z}{(1 - z)^2}
\]
so (8) holds.
\[\Box\]

It is well known that if \( f(z) \in A_1 \), then \( |a_n| \leq n \). From this and from Theorem 2.1 we the following corollary for \( p \geq 1 \).

Corollary 2.2. If \( f(z) \in A_p, \ p \geq 1, \ f(z) = z^p + a_{p+1}z^{p+1} + \cdots, \ z \in \mathbb{D} \) and if
\[
\Re \left\{ \frac{z^{p+1}}{f^{(p-1)}(z)} \right\} > 0, \quad z \in \mathbb{D},
\]
then we have
\[
|a_{p+1}| \leq \frac{4}{p+1}, \quad |a_{p+2}| \leq \frac{18}{(p+1)(p+2)}, \ldots, \quad |a_{p+k}| \leq (k+1)! \frac{(p+1)(p+2)\cdots(p+k)}{(p+1)(p+2)\cdots(p+k)}.
\]

The result is sharp.

Corollary 2.2 implies that the function (9) may be written as
\[
f_p(z) = z^p + \frac{4z^{p+1}}{p+1} + \frac{18z^{p+2}}{(p+1)(p+2)} + \sum_{k=3}^{\infty} (k+1)! \frac{(p+1)(p+2)\cdots(p+k)}{(p+1)(p+2)\cdots(p+k)} z^{p+k}, \quad z \in \mathbb{D}. \tag{10}
\]

Now we prove an inequality of type Fekete-Szegö type for functions satisfying (8). Fekete-Szegö inequalities for \( p \)-valent starlike and convex functions of complex order was considered recently in [2].
Theorem 2.3. If $f(z) \in \mathcal{A}_p$, $p \geq 1$, $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$, $z \in \mathbb{D}$ and if
\[
\Re\left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} > 0, \quad z \in \mathbb{D},
\]
then for any complex number $\mu$, we have
\[
\left| a_{p+2} - \mu a_{p+1}^2 \right| \leq \frac{6}{(p+1)(p+2)} \max\{1,|2\lambda - 1|\},
\]
where
\[
\lambda = \frac{4\mu(p+2)}{3(p+1)} - 1.
\]
The bound is sharp.

Proof. We have
\[
z^p f^{(p)}(z) = f^{(p-1)}(z)\left[1 + q_1z + q_2z^2 + \cdots\right],
\]
where $\Re\{1 + q_1z + q_2z^2 + \cdots\} > 0$ in $\mathbb{D}$. This leads us to the conclusion
\[
a_{p+1} = \frac{2q_1}{p+1}, \quad a_{p+2} = \frac{3(q_1^2 + q_2)}{(p+1)(p+2)}.
\]
Thus we have
\[
\left| a_{p+2} - \mu a_{p+1}^2 \right| = \frac{3}{(p+1)(p+2)} \left| q_2 - \frac{4\mu(p+2) - 3(p+1)}{3(p+1)}q_1^2 \right|.
\]
In [10] it was proved that for any complex number $\lambda$ the following sharp estimate holds
\[
|q_2 - \lambda q_1^2| \leq 2 \max\{1,|2\lambda - 1|\}.
\]
Therefore, applying (13) in (12) gives sharp bound (11). 

Corollary 2.4. If $p = 1$, then (11) becomes the known sharp result [10]
\[
|a_3 - \mu a_2^2| \leq \max\{1,|4\mu - 3|\},
\]
for starlike functions, i.e. the solution of Fekete-Szegö problem in the class of starlike functions.

If $\mu = 1$, then (11) becomes the following sharp result.

Corollary 2.5. If $f(z) \in \mathcal{A}_p$, $p \geq 1$, $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$, $z \in \mathbb{D}$ and if
\[
\Re\left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} > 0, \quad z \in \mathbb{D},
\]
then we have
\[
\left| a_{p+2} - a_{p+1}^2 \right| \leq \frac{6}{(p+1)(p+2)} \max\left\{1, \frac{|7 - p|}{3(p+1)} \right\}.
\]
The bound is sharp which show the coefficients of (10).
Theorem 2.6. If \( f(z) \in \mathcal{A}_p, p \geq 2, f(z) = z^p + a_{p+1}z^{p+1} + \cdots, z \in D \) and if
\[
\Re \left\{ \frac{z f^{(p)}(z)}{f^{(p-1)}(z)} \right\} > 0, \quad z \in D,
\]
then for \(|z| = r < 1\), we have
\[
\frac{1}{(1+r)^{2p}} \leq \left| \frac{f(z)}{z^p} \right| \leq \frac{1}{(1-r)^{2p}}.
\]
The bounds are sharp.

Proof. From (14) and from Lemma 1.1, we have
\[
\Re \left\{ \frac{z f'(z)}{f(z)} \right\} > 0, \quad z \in D, \quad \left. \frac{z f'(z)}{f(z)} \right|_{z=0} = p,
\]
and so we have for \(|z| = r < 1\)
\[
\frac{1-r}{1+r} \leq \Re \left\{ \frac{z f'(z)}{p f(z)} \right\} \leq \frac{1+r}{1-r}.
\]
Then it follows that
\[
\log \left| \frac{f(z)}{z^p} \right| = \Re \int_0^\infty \left( \frac{f'(t)}{f(t)} - \frac{p}{t} \right) dt
\]
\[
= \Re \int_0^\infty \frac{p}{t} \left( \frac{f'(t)}{p f(t)} - 1 \right) dt
\]
\[
= \Re \int_0^\infty \frac{p}{\rho} \left( \frac{1}{\rho} - 1 \right) d\rho
\]
\[
= \int_0^\infty \Re \left\{ \frac{p}{\rho} \left( \frac{1}{\rho} - 1 \right) \right\} d\rho
\]
\[
\leq \int_0^\infty \frac{2p}{1-\rho} d\rho = \log \frac{1}{(1-r)^{2p}}.
\]
This shows that for \(|z| = r < 1\)
\[
\left| \frac{f(z)}{z^p} \right| \leq \frac{1}{(1-r)^{2p}}.
\]
Applying the same method as the above, we can obtain for \(|z| = r < 1\)
\[
\frac{1}{(1+r)^{2p}} \leq \left| \frac{f(z)}{z^p} \right|.
\]
The sharpness of (18) shows the function
\[
g(z) = \left[ \frac{z}{(1-z)^2} \right]^p = z^p + \cdots.
\]
This completes the proof of Theorem 2.6.
Theorem 2.7. If $f(z) \in A_p$, $p \geq 2$, $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$, $z \in \mathbb{D}$ and if
\[
\Re \left( \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right) > 0, \quad z \in \mathbb{D},
\]
then for $|z| = r < 1$, we have
\[
\frac{pr^p(1-r)}{1 + (p-1)r} \leq |f'(z)| \leq \frac{pr^p(1+r)}{(1-r)^2}.
\]
The bounds are sharp.

Proof. By the same reason as in the proof of Theorem 2.6, we have for $|z| = r < 1$
\[
\frac{1-r}{1+r} \leq \Re \left( \frac{zf'(z)}{pf(z)} \right) \leq \frac{1+r}{1-r}.
\]
Applying Theorem 2.6 we easily have the proof of Theorem 2.7. The sharpness of (20) shows the function (16). \(\square\)

Theorem 2.8. If $f(z) \in A_p$, $p \geq 2$, $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$, $z \in \mathbb{D}$ and if
\[
\Re \left( \frac{f^{(p-1)}(z)}{z} \right) > 0, \quad z \in \mathbb{D},
\]
then, we have
\[
\Re \left( \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right) > 0, \quad |z| < \sqrt{2} - 1.
\]
The bound is sharp.

Proof. Let us put
\[
q(z) = \frac{f^{(p-1)}(z)}{zp!}, \quad q(0) = 1.
\]
From the hypothesis (19), we have
\[
\Re |q(z)| > 0, \quad z \in \mathbb{D}.
\]
Applying [7, p.186], [12, Th.2], we have
\[
\left| \frac{qf'(z)}{q(z)} \right| = \left| \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - 1 \right| \leq \frac{2|z|}{1-|z|^2}, \quad z \in \mathbb{D}.
\]
Therefore, we have
\[
\left| \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - 1 \right| < 1 \quad \text{for} \quad |z| < \sqrt{2} - 1
\]
and so
\[
\Re \left( \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right) > 0 \quad \text{for} \quad |z| < \sqrt{2} - 1.
\]
It is easy to check that the function $f_1(z)$ such that
\[
f_1^{(p-1)}(z) = \frac{z(1+z)}{1-z}
\]
gives
\[
\begin{align*}
\frac{zf_1^{(p)}(z)}{f_1^{(p-1)}(z)} &\bigg|_{z=1-\sqrt{2}} = \frac{1+2z-z^2}{1-z^2} \\
&= 0
\end{align*}
\]
which shows the sharpness of (20).

\[\square\]

\textbf{Corollary 2.9.} If $f(z) \in \mathcal{A}_p$, $p \geq 2$, $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$, $z \in \mathbb{D}$ and if
\[
\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in \mathbb{D},
\]
then, $f(z)$ is $p$-valently starlike in $|z| < \sqrt{2} - 1$. The bound is sharp.

\textbf{Proof.} From Theorem 2.8, we have (19). Then from Corollary 1.2 we have
\[
\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad |z| < \sqrt{2} - 1.
\]
\[\square\]

\textbf{Theorem 2.10.} Let $f(z) \in \mathcal{A}_p$, $p \geq 2$, $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$, $z \in \mathbb{D}$ and let
\[
\Re\left\{\frac{f^{(k)}(z)}{z^{p-k}}\right\} > 0, \quad z \in \mathbb{D}
\]
for some integer $k \in [0,p]$. Then $f(z)$ is $p$-valently convex in $(\sqrt{1+p^2} - 1)/p$ i.e.
\[
1 + \Re\left\{\frac{zf''(z)}{f'(z)}\right\} > 0, \quad |z| < (\sqrt{1+p^2} - 1)/p.
\]
The result is sharp.

\textbf{Proof.} Let us put
\[
q(z) = \frac{f^{(k)}(z)}{(p-k)z^{p-k}}, \quad q(0) = 1.
\]
Then it follows that
\[
\frac{zq'(z)}{q(z)} = \frac{zf^{(k+1)}(z)}{f^{(k)}(z)} - (p-k) = k + \frac{zf^{(k+1)}(z)}{f^{(k)}(z)} - p, \quad z \in \mathbb{D}.
\]
And so from the hypothesis (22), applying [7, p.186], [12, Th.2], we have
\[
\left|\frac{zq'(z)}{q(z)}\right| = \left|k + \frac{zf^{(k+1)}(z)}{f^{(k)}(z)} - p\right| \leq \frac{2|z|}{1-|z|^2}, \quad z \in \mathbb{D}.
\]
Therefore, we have
\[ \Re \left\{ k + \frac{zf^{(k+1)}(z)}{f^{(k)}(z)} \right\} > 0, \quad |z| < \left( \sqrt{1 + p^2} - 1 \right)/p. \]
Applying Corollary 1.5, we have
\[ \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, \quad |z| < \left( \sqrt{1 + p^2} - 1 \right)/p. \]
Further, taking the function \( f(z) \) given by
\[
f(z) = \left( \frac{1 + z}{1 - z} \right)^{z^p}, \quad z \in \mathbb{D},
\]
we see that the result is sharp.

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