BOWDITCH TAUT SPECTRUM AND DIMENSIONS OF GROUPS

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Abstract. For a finitely generated group \( G \), let \( H(G) \) denote Bowditch’s taut loop length spectrum. We prove that if \( G = (A * B)/\langle \langle R \rangle \rangle \) is a \( C'(1/12) \) small cancellation quotient of a free product of finitely generated groups, then \( H(G) \) is equivalent to \( H(A) \cup H(B) \). We use this result together with bounds for cohomological and geometric dimensions, as well as Bowditch’s construction of continuously many non-quasi-isometric \( C'(1/6) \) small cancellation 2-generated groups to obtain our main result: Let \( \mathcal{G} \) denote the class of finitely generated groups. The following subclasses contain continuously many one-ended non-quasi-isometric groups:

1. \( \{ G \in \mathcal{G} : \text{cd}(G) = 2 \text{ and } \text{gd}(G) = 3 \} \)
2. \( \{ G \in \mathcal{G} : \text{cd}(G) = 2 \text{ and } \text{gd}(G) = 3 \} \)
3. \( \{ G \in \mathcal{G} : \text{cd}_2(G) = 2 \text{ and } \text{gd}_2(G) = 3 \} \)

On our way to proving the aforementioned results, we show that the classes defined above are closed under taking relatively finitely presented \( C'(1/12) \) small cancellation quotients of free products, in particular, this produces new examples of groups exhibiting an Eilenberg-Ganea phenomenon for families.

We also show that if there is a finitely presented counter-example to the Eilenberg-Ganea conjecture, then there are continuously many finitely generated one-ended non-quasi-isometric counter-examples.

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1. INTRODUCTION

Let \( G \) be a discrete group. Let \( \text{cd}_R(G) \) denote the cohomological dimension of \( G \) with respect to the ring \( R \). Analogously, let \( \text{cd}(G) \) and \( \text{cd}(G) \) denote the proper cohomological dimension and the virtually cyclic cohomological dimension of \( G \)

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respectively, and let \( \text{gd}(G) \) and \( \text{gd}(G) \) be their geometric counterparts. For explicit definitions see Section 3. The main result of this manuscript is the following.

**Theorem A.** Let \( \mathcal{G} \) denote the class of finitely generated groups. The following subclasses contain continuously many one-ended non-quasi-isometric groups:

1. \( \{ G \in \mathcal{G} : \text{cd}(G) = 2 \text{ and } \text{gd}(G) = 3 \} \)
2. \( \{ G \in \mathcal{G} : \text{cd}(G) = 2 \text{ and } \text{gd}(G) = 3 \} \)
3. \( \{ G \in \mathcal{G} : \text{cd}_{\mathcal{Q}}(G) = 2 \text{ and } \text{cd}_{\mathcal{Z}}(G) = 3 \} \)

The proof of the theorem uses a quasi-isometry invariant of finitely generated groups, introduced by Bowditch [Bow98], known as the **taut loop length spectrum** (or **taut spectrum** for short). For a finitely generated group \( G \) this invariant, denoted as \( H(G) \), takes values on the power set of the natural numbers modulo an equivalence relation, see Section 2. Bowditch observed that if \( A \) and \( B \) are finitely generated quasi-isometric groups, then the taut spectrums \( H(A) \) and \( H(B) \) are equivalent [Bow98, Lemma 3]. The taut spectra of all finitely presented groups are equivalent. In contrast, using small cancellation theory, Bowditch proved the following result on which our main result relies on.

**Theorem B.** [Bow98] There are continuously many torsion-free 2-generator \( C'(1/6) \) small cancellation groups \( \{ B_{\alpha} : \alpha \in \mathbb{R} \} \) such that \( H(B_{\alpha}) \) and \( H(B_{\beta}) \) are equivalent only if \( \alpha = \beta \).

The second ingredient to prove the main result is the following computation of the taut spectrum for certain small cancellation quotients of free products, as defined in the book by Lyndon and Schupp [LS01, Chapter V.11]. Recall that a group of the form \( G = (A \ast_{C} B) / \langle \langle R \rangle \rangle \) where \( R \) is a symmetrized subset of \( A \ast_{C} B \) satisfying the \( C'(1/6) \) small cancellation condition is called a small cancellation product. If \( R \) is finite, we say \( G \) is relatively finitely presented. If no element of \( R \) is a proper power, we say that \( G \) is relatively torsion-free.

**Theorem C (Theorem 2.4).** Let \( A \) and \( B \) be finitely generated groups. Let \( \mathcal{R} \) be a finite symmetrized subset of \( A \ast_{C} B \) that satisfies the \( C'(1/6) \) small cancellation condition, and let \( G = (A \ast_{C} B) / \langle \langle \mathcal{R} \rangle \rangle \). Then \( H(G) \) and \( H(A) \cup H(B) \) are equivalent.

The third ingredient to prove our main result is a collection of dimensional bounds for multi-ended groups. We say a group \( G \) has **small centralizers** if every infinite cyclic subgroup \( C \) of \( G \) has finite index in its centralizer \( C_G(C) \).

**Theorem D (Theorems 3.1 and 3.2).** Let \( G \) be the fundamental group of a graph of groups with finite edge stabilizers, and let \( T \) be the Bass-Serre tree. Then the following inequalities hold

\[
\text{gd}(G) \leq \max\{1, \text{gd}(G_{\sigma})| \sigma \in T \}, \quad \text{cd}(G) \leq \max\{1, \text{cd}(G_{\sigma})| \sigma \in T \}
\]

\[
\text{gd}(G) \leq \max\{2, \text{gd}(G_{\sigma})| \sigma \in T \}, \quad \text{cd}(G) \leq \max\{2, \text{cd}(G_{\sigma})| \sigma \in T \}.
\]

If, in addition, all vertex groups have small centralizers and satisfy the ascending chain condition for finite subgroups, then

\[
\text{gd}(G) \leq \max\{2, \text{gd}(G_{\sigma})| \sigma \in T \} \quad \text{and} \quad \text{cd}(G) \leq \max\{2, \text{cd}(G_{\sigma})| \sigma \in T \}.
\]

The fourth ingredient is a computation of cohomological and geometric dimensions for small cancellation products. It could be interpreted as a generalization of
previous result in the sense that some of those dimensional bounds are preserved under large quotients.

**Theorem E** (Theorem 4.1). Let $A$ and $B$ be groups and let $C$ be a common finite subgroup. Let $\mathcal{R}$ be a finite symmetrized subset of $A \ast_C B$ that satisfies the $C'(1/12)$ small cancellation condition, and $G = (A \ast_C B)/\langle\langle \mathcal{R} \rangle \rangle$. Assume $G$ is not virtually free. Then the following conclusions hold.

1. $\text{gd}(G) = \max\{\text{gd}(A), \text{gd}(B), 2\}$ and $\text{cd}(G) = \max\{\text{cd}(A), \text{cd}(B), 2\}$

2. If $A$ and $B$ are finitely generated, have small centralizers and satisfy the ascending chain condition for finite subgroups, then $G$ has small centralizers, satisfies the ascending chain condition, and

$$\max\{\text{gd}(A), \text{gd}(B), 2\} \leq \text{gd}(G) \leq \max\{\text{gd}(A), \text{gd}(B), 2\}$$

and

$$\max\{\text{cd}(A), \text{cd}(B), 2\} \leq \text{cd}(G) \leq \max\{\text{cd}(A), \text{cd}(B), 2\}.$$

3. For any ring $R$, we have $\text{cd}_R(G) = \max\{\text{cd}_R(A), \text{cd}_R(B), 2\}$.

A notable hypothesis of Theorem E is the $C'(1/12)$ small cancellation condition instead of the standard $C'(1/6)$ condition. While we do not claim that $C'(1/12)$ is the optimal hypothesis, we refer the reader to Remark 4.5 which gives an insight of the difficulty.

The fifth ingredient to prove the main result are the following statements that provide sufficient conditions for certain groups to be one-ended.

**Theorem F** (Theorem 5.1). Let $G = (A \ast_C B)/\langle\langle \mathcal{R} \rangle \rangle$, where $C$ is a common finite subgroup of $A$ and $B$, and $\mathcal{R}$ is a symmetrized subset of $A \ast B$ that satisfies the $C'(1/6)$ small cancellation condition.

- Suppose that for every $r \in \mathcal{R}$, its normal form does not contain elements of finite order of $A$ or $B$.

If $A$ and $B$ are one-ended, then $G$ is one-ended.

In Theorem F, we do not know whether the bulleted hypothesis is necessary. We also use the following proposition, that was communicated to us by Dani Wise, to prove our main result.

**Proposition G** (Proposition 5.3). Let $G$ be a torsion-free, 2-generated group. If $G$ is not a free group, then $G$ is one-ended.

Theorem A will follow from the previous stated results; the fact that the classes of finitely generated groups in the statement are non-empty by results of Brady, Leary and Nucinkis [BLN01], Fluch and Leary [FL14], and Bestvina and Mess [BM91] respectively; and the following abstraction.

**Theorem H.** Let $Q \subset P$ be classes of finitely generated groups that are closed under taking finitely presented subgroups. Suppose

1. $P$ contains the class of torsion-free finitely generated $C'(1/6)$ small cancellation groups.

2. $P$ is closed under taking relatively finitely presented $C'(1/6)$ small cancellation products,

3. $Q$ is closed under taking amalgamated products over finite subgroups and HNN extensions over finite subgroups, and
(4) there is a hyperbolic group $G$ that is in $P$ but not in $Q$.

Then there are continuously many one-ended non-quasi-isometric groups $P$ that are not in $Q$.

Proof. First we argue that there is a group $A$ in $P$ that is not $Q$ which is one-ended, finitely presented, and contains elements of infinite order. This is a direct consequence Dunwoody’s accessibility [Dun85]. Since $G$ is hyperbolic, it is finitely presented and hence it splits as the fundamental group of a graph of groups with one-ended finitely presented vertex groups and finite edge groups. In fact, each vertex group is a quasi-convex subgroup of $G$, and hence it is hyperbolic. By hypothesis on $P$, all vertex groups are in $P$, but not all can be in $Q$, else $G$ would belong to $Q$ by hypothesis (4), which is a contradiction. Therefore there is a one-ended hyperbolic group $A$ that is in $P$ but not in $Q$, and such a group always contains elements of infinite order.

By Theorem B, there is a continuous collection of groups $\{B_\alpha : \alpha \in \mathbb{R}\}$ where each $B_\alpha$ is a torsion-free 2-generated $C'(1/6)$ small cancellation group such that $B_\alpha$ is not quasi-isometric to $B_\beta$ if $\alpha \neq \beta$. In particular, each $B_\alpha$ is a one-ended group (see Proposition G).

For each $\alpha \in \mathbb{R}$, let $R_\alpha$ be a finite symmetrized subset of $A * B_\alpha$ that satisfies the $C'(1/12)$ small cancellation condition and such that each $r \in R_\alpha$ only involves elements of infinite order of $A$ and $B_\alpha$. For example, if $a \in A$ and $b_1, b_2 \in B_\alpha$ are distinct elements of infinite order, let $R_\alpha$ be the symmetrized subset generated by $r \in A * B_\alpha$ given by

$$r = (ab_1)ab_2(ab_1)^2ab_2(ab_1)^3 \cdots ab_2(ab_1)^{12}ab_2.$$  

Let $G_\alpha$ be the quotient group $(A * B_\alpha)/\langle \langle R \rangle \rangle$. By the second assumption on $P$, the group $G_\alpha \in P$ for each $\alpha \in \mathbb{R}$. Since $G_\alpha$ contains a subgroup isomorphic to $A$ (see for example [LS01, Ch. V. Cor. 9.4]), and $A \not\in Q$, it follows that $G_\alpha \not\in Q$. Since $A$ and $B_\alpha$ are one-ended, it follows that $G_\alpha$ is one-ended as well (see Theorem F).

Since $A$ is finitely presented, Theorem C implies that $H(G_\alpha)$ is equivalent to $H(B_\alpha)$ and hence $G_\alpha$ and $G_\beta$ are quasi-isometric only if $\alpha = \beta$.

We are now ready to prove our main result:

Proof of Theorem A. For the first item, let $P$ be the class of finitely generated groups $G$ such that $cd(G) \leq 2$ and $gd(G) \leq 3$. Let $Q$ be the subclass of $P$ defined by groups $G$ with $gd(G) \leq 2$. Note that both $P$ and $Q$ are closed under taking finitely presented subgroups. Since torsion-free $C'(1/6)$ small cancellation groups have geometric dimension at most two, they belong to $P$. By Theorem E(1), the class $P$ is closed under taking $C'(1/12)$ small cancellation products. From Theorem D we get that $Q$ is closed under taking amalgamated products over finite groups and HNN extensions over finite subgroups. Note that a group $G$ is in $P$ but not in $Q$ if and only if $cd(G) = 2$ and $gd(G) = 3$. By a result of Brady, Leary and Nucinkis [BLN01], there is a hyperbolic group $\Gamma$ in $P$ that is not in $Q$. Then Theorem H implies the first statement of the theorem.

For the second statement of the theorem, let $P$ be the class of finitely generated groups $G$ such that

- $G$ has the small centralizers property, that is, the centralizer of any infinite cyclic subgroup of $G$ is virtually cyclic,
- $G$ satisfies the ascending chain condition for finite subgroups, and
\[ \text{cd}(G) \leq 2, \, \text{gd}(G) \leq 3, \, \text{cd}(G) \leq 2, \text{ and } \text{gd}(G) \leq 3. \]

Define \( Q \) as the subclass of groups \( G \) that additionally satisfy \( \text{gd}(G) \leq 2 \). It is not difficult to see that both \( P \) and \( Q \) are closed under taking subgroups.

Any torsion-free finitely generated \( C'(1/6) \) small cancellation group \( G \) has the small centralizers property, see [Tru74] or [Sey74], and trivially satisfies the ascending chain condition for finite subgroups. Now, \( G \) has geometric dimension two, so \( \text{cd}(G) \leq 2 \). Since \( G \) has small centralizers and satisfies the ascending chain condition,

\[ \text{gd}(G) \leq \max\{2, \text{gd}(G)\} \text{ and } \text{cd}(G) \leq \max\{2, \text{cd}(G)\}, \]

see Proposition 3.6. Hence \( \text{cd}(G) \leq 2 \) and \( \text{gd}(G) \leq 3 \), and therefore \( G \) belongs to the class \( P \). That \( P \) is closed under taking \( C'(1/6) \) small cancellation products is a direct consequence of Theorem E(2). A result of Fluch and Leary [FL14] shows that there is a hyperbolic group \( \Gamma \) which satisfies \( \text{cd}(\Gamma) = 2, \text{gd}(\Gamma) = 3, \text{cd}(\Gamma) = 2, \) and \( \text{gd}(\Gamma) = 3 \). It is well-known that hyperbolic groups have small centralizers and satisfy the ascending chain condition for finite subgroups, see for example [BH99]. Thus \( \Gamma \) is in \( P \) but not in \( Q \). By Theorem D, the class \( Q \) is closed under taking amalgamated products under over finite groups and HNN extensions over finite subgroups. Therefore the second statement follows from Theorem H.

For the last statement, let \( P \) be the class of finitely generated groups \( G \) such that \( \text{cd}_2(G) \leq 2 \) and \( \text{cd}_2(G) \leq 3 \). Let \( Q \) be the subclass of \( P \) defined by groups \( G \) with \( \text{gd}_2(G) \leq 2 \). Then the argument follows from Theorem H in an analogous way as in the previous paragraphs, in this case, invoking Theorem E(3) and the existence of a hyperbolic group \( G \) such that \( \text{cd}_2(G) = 2 \) and \( \text{cd}_2(G) = 3 \). The existence of such a group was described by Bestvina and Mess [BM91, Paragraph before Corollary 1.4] and it uses their dimension formula and a technique of Gromov and Davis-Januszkiewicz [DJ91]; there are other examples described by Dranishnikov [Dra99, Corollary 2.3]. \( \Box \)

**Other results.** From the proof of Theorem H, observe that hypothesis (4) of this theorem could be replaced with there is a finitely presented torsion-free group that is in \( P \) but not in \( Q \). Hence the same type of argument proving Theorem A proves the following statement on the Eilenberg-Ganea conjecture [EG57].

**Corollary I.** If the class of groups \( \{G \in \text{Groups}: \text{cd}(G) = 2 \text{ and } \text{gd}(G) = 3\} \) contains a finitely presented group, then it contains continuously many finitely generated one-ended non-quasi-isometric groups.

We record the following corollary which is a direct consequence of Theorem E. This result describes a way to construct new groups that exhibit an Eilenberg-Ganea phenomenon for the families of finite and virtually cyclic subgroups, as well as (hyperbolic) groups of rational cohomological dimension two and integral cohomological dimension three. Let \( G \) be a group and \( \mathcal{F} \) be a family of subgroups of \( G \). We say that \( G \) is an \( \mathcal{F} \)-Eilenberg-Ganea group if \( \text{gd}_\mathcal{F}(G) = 3 \) and \( \text{cd}_\mathcal{F}(G) = 2 \).

**Corollary J.** Let \( A \) and \( B \) be groups. Let \( \mathcal{R} \) be a finite symmetrized subset of \( A*\mathcal{B} \) that satisfies the \( C'(1/12) \) small cancellation condition, and \( G = (A*\mathcal{B})/\langle\langle \mathcal{R} \rangle\rangle \).

1. If \( A \) is a \( \text{FIN-Eilenberg-Ganea} \) group, and \( \text{cd}(B) \leq 2 \), then \( G \) is a \( \text{FIN-Eilenberg-Ganea} \) group.
(2) If $A$ is an $\mathcal{F}$-Eilenberg-Ganea group for both $\mathcal{F} = F_{IN}$ and $\mathcal{V}_{CYC}$, $\text{cd}_\mathcal{F}(B) \leq 2$ for both $\mathcal{F} = F_{IN}$ and $\mathcal{V}_{CYC}$, and $A$ and $B$ are finitely generated, have small centralizers, and satisfy the chain ascending condition for finite subgroups, then $G$ is an $\mathcal{V}_{CYC}$-Eilenberg-Ganea group.

(3) If $\text{cd}_2(A) = 3$, $\text{cd}_2(B) \leq 3$, and $\text{cd}_3(G) = 2$. If, in addition, $A$ and $B$ are one-ended and $\mathcal{R}$ satisfies the hypothesis of Theorem $F$, then $G$ is one-ended. Similarly if $A$ and $B$ are finitely presented, then $G$ is finitely presented.

A couple of questions. Since the taut spectrum of any two finitely presented groups are equivalent, it can only distinguish between quasi-isometry types of infinitely presented groups. Thus the following question is out of the scope of the present article.

**Question K.** For each of the classes in Theorem A, does the class contain infinitely many non-quasi-isometric finitely presented groups?

Let $n \geq 0$ be an integer. A group is said to be virtually $\mathbb{Z}^n$ if it contains a subgroup of finite index isomorphic to $\mathbb{Z}^n$. For any group $G$, define the family

$$\mathcal{F}_n = \{H \leq \Gamma | H \text{ is virtually } \mathbb{Z}^r \text{ for some } 0 \leq r \leq n\}.$$ 

Observe that $\mathcal{F}_0 = F_{IN}$ is the family of finite subgroups and $\mathcal{F}_1 = \mathcal{V}_{CYC}$ is the family of virtually cyclic subgroups, these are particularly relevant due to their connection with the Farrell-Jones and Baum-Connes isomorphism conjectures [FJ93, Lüc05].

**Question L.** Does the class of finitely generated groups

$$\{G \in \mathcal{G} : \text{cd}_{\mathcal{F}_k}(G) = 2 \text{ and } \text{gd}_{\mathcal{F}_k}(G) = 3\}$$

satisfy the conclusion of Theorem A? This class is non-empty by [SSn20, Rem. 4.6].

Organization. The rest of article proves the results stated above. Each section corresponds to one of the results.

Section 2 is on the Taut spectrum of small cancellation quotients of free products. It contains an argument that shows that the taut spectrum of a free product of finitely generated groups is equivalent to the union of the spectrums of the factors, $H(A * B) = H(A) \cup H(B)$, see Proposition 2.2. It concludes with the proof of Theorem $C$.

Section 3 is on the proper and virtually cyclic dimensions of multi-ended groups, and in particular, it discusses the proof of Theorem $D$.

Section 4 uses some of the results of the previous section to obtain upper bounds on the proper and virtually cyclic dimensions of small cancellation products of free groups, and in particular discusses the proof of Theorem $E$.

Section 5 is on one-ended small cancellation quotients of free products, and contains the proofs of Theorem $F$ and Proposition $G$.

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2. Taut spectrum of small cancellation quotients of free products

2.1. The taut loop length spectrum. Let us recall the definition of Bowditch’s taut loop length spectrum (or taut spectrum for shorter) introduced in [Bow98], using the approach described in [KLS20].

Let $\Gamma$ be a connected, simplicial graph. An edge loop of length $l$ in $\Gamma$ is a sequence of $v_0, \ldots, v_l$ of vertices such that $v_0 = v_l$ and $\{v_{i-1}, v_i\}$ is an edge for $1 \leq i \leq l$. For a graph $\Gamma$ and a fixed integer $k$, let $\Gamma_k$ denote the 2-complex whose 1-skeleton is the geometric realization of $\Gamma$, with one 2-cell attached to each edge loop in $\Gamma$ of length strictly less than $l$. An edge loop of length $l$ is said to be taut if it is not null-homotopic in $\Gamma_i$. The taut loop length spectrum of $\Gamma$ is by definition

$$H(\Gamma) = \{n \in \mathbb{N} | \text{there is a taut loop in } \Gamma \text{ of length } n\}.$$  

As observed in [KLS20, Lemma 2.2], an equivalent definition of the taut loop length spectrum of $\Gamma$ is

$$H(\Gamma) = \{l \in \mathbb{N} | \pi_1(\Gamma_l) \to \pi_1(\Gamma_{l+1}) \text{ is not an isomorphism}\}$$

where the map between fundamental groups is the one induced by inclusion.

The vertex set of the graph $\Gamma$ is endowed with the combinatorial path metric $d_\Gamma$, that is, the distance between two vertices is the length of the shortest edge path between them. For $k > 0$, a function $f: X \to Y$ between metric spaces is $k$-Lipschitz if $d_Y(f(x), f(x')) \leq k d_X(x, x')$ for all $x, x' \in X$. Two graphs $\Gamma$ and $\Lambda$ are $k$-quasi-isometric if there exist a pair of $k$-Lipschitz maps between vertex sets $\phi: V(\Gamma) \to V(\Lambda)$ and $\psi: V(\Lambda) \to V(\Gamma)$ such that $d_\Gamma(x, \psi \circ \phi(x)) \leq k$ and $d_\Gamma(y, \phi \circ \psi(y)) \leq k$ for all $x \in V(\Gamma)$ and $y \in V(\Lambda)$. Two simplicial connected graphs are quasi-isometric if their vertex sets with the combinatorial path metrics are $k$-quasi-isometric metric spaces for some $k > 0$.

Two subsets $H$ and $H'$ of $\mathbb{N}$ are said to be $k$-related, and write $H \sim_k H'$, if for all $l \geq k^2 + 2k + 2$, whenever $l \in H$ then there exists $l' \in H'$ such that $l/k \leq l' \leq lk$ and vice-versa. Two subsets of $\mathbb{N}$ are equivalent if they are $k$-related for some $k$.

**Lemma 2.1.** [Bow98, Lemma 3] If two connected simplicial graphs $\Gamma$ and $\Lambda$ are $k$-quasi-isometric, then $H(\Gamma)$ and $H(\Lambda)$ are $k$-related.

Let $G$ be a group and $S$ a finite generating set. Then we denote $H(G, S) = H(\Gamma(G, S))$ where $\Gamma(G, S)$ is the Cayley graph of $G$ with respect to $S$. Hence the taut spectrum of a group $G$ is well defined up to equivalence. If the finite generating set of $G$ is clear from the context we just write $H(G)$ for the taut spectrum.

2.2. The taut spectrum of a free product. This subsection computes the taut spectrum of a free product. It is a warm-up for the next subsection where we compute the taut spectrum of certain small cancellation quotients of free products.

**Proposition 2.2.** Let $A$ and $B$ be groups with finite generating sets $S_A$ and $S_B$ respectively. Then

$$H(A \ast B, S_A \sqcup S_B) = H(A, S_A) \cup H(B, S_B).$$

**Proof.** A cyclic word $w$ in the alphabet $S_A \sqcup S_B$ represents the identity in $G$ only if there is a syllable (a maximal subword in $S_A$ or $S_B$) of $w$ that represents the identity. Therefore if $w$ is a reduced cyclic word of length $m$ that represents the identity then either

1. $w$ is an $A$-word or a $B$-word; or
(2) $w$ is equal to a product $\prod r_i^{q_i}$ in $G$ where each $r_i$ is an $A$-relation or $B$-relation of length strictly less than $m$.

In the Cayley graph $\Gamma(G)$, a cycle is called homogeneous if its label is either an $A$-word or a $B$-word. Let $\Gamma'_\ell(G)$ be the complex with 1-skeleton the Cayley graph of $G$ and a 2-cell for each homogenous cycle of length less than $\ell$. A consequence of the above observations is that the inclusion

$$\Gamma'_\ell(G) \hookrightarrow \Gamma_\ell(G)$$

is $\pi_1$-injective (and hence $\pi_1$-isomorphism). Indeed, if $\gamma$ is a cycle of length less than $\ell$ in $\Gamma_\ell(G)$ then there is disk diagram in $\Gamma'_\ell(G)$ for $\gamma$.

Let $\ell > 0$ and consider the inclusion

$$\Gamma_\ell(B) \rightarrow \Gamma'_\ell(G).$$

Let us show that this inclusion is $\pi_1$-injective. Let $\gamma: S^1 \rightarrow \Gamma_\ell(B)$ be a closed path labeled by a $B$-word, and suppose that $\gamma$ that is null homotopic in $\Gamma'_\ell(G)$. We claim that $\gamma$ is null-homotop in $\Gamma_\ell(B)$. Observe that the quotient complex $\Gamma'_\ell(G)/G$ is isomorphic to a wedge $X_\ell(A) \vee X_\ell(B)$, where $X_\ell(A) = \Gamma_\ell(A)/A$ and $X_\ell(B) = \Gamma_\ell(B)/B$. Then the closed path $\gamma: S^1 \rightarrow X_\ell(A) \vee X_\ell(B)$ factors through the $\pi_1$-injective inclusion $X_\ell(B) \hookrightarrow X_\ell(A) \vee X_\ell(B)$. It follows that there is a disk diagram $D \rightarrow X_\ell(B)$ for $\gamma: S^1 \rightarrow X_\ell(B)$ which can be lifted to a map $D \rightarrow \Gamma_\ell(B)$. Hence $\gamma$ is null-homotopic in $\Gamma_\ell(B)$. It follows that the inclusion

$$\Gamma_\ell(B) \rightarrow \Gamma_\ell(G)$$

is a $\pi_1$-injective map. Thus, for each $\ell$, we have the following commutative diagram

$$\begin{array}{ccc}
\pi_1(\Gamma_\ell(B)) & \longrightarrow & \pi_1(\Gamma_{\ell+1}(B)) \\
\downarrow & & \downarrow \\
\pi_1(\Gamma_\ell(G)) & \longrightarrow & \pi_1(\Gamma_{\ell+1}(G))
\end{array}$$

with injective vertical maps. The diagram shows that $H(B) \subseteq H(G)$. A completely analogous argument shows that $H(A) \subseteq H(G)$, and therefore $H(A) \cup H(B) \subseteq H(G)$.

Next we prove $H(A) \cup H(B) \supseteq H(G)$. Let $\ell \in H(G)$. Let $\gamma$ be a closed path in $\Gamma_\ell(G)$ of length $\ell$ that is not null-homotopic. Suppose that the label of $\gamma$ is a word $w$. First let us argue that $\gamma$ is homogeneous, i.e, $w$ has only one syllable. Suppose that $w$ has more than one syllable. If $w$ represents the identity in $A * B$, then it contains a syllable that represents the identity. This gives a proper subpath of $\gamma$ that is a closed loop of length less than $\ell$. It follows that $\gamma$ is homotopic to a closed path of shorter length in $\Gamma_\ell(G)$, and therefore $\gamma$ is null-homotopic, a contradiction. We have that $\gamma$ is homogeneous. Suppose $\gamma$ has label a $B$-word of length $\ell$. Since $\Gamma_\ell(B) \hookrightarrow \Gamma_\ell(G)$, by translating with an element of $G$, assume that $\gamma$ is a path in $\Gamma_\ell(B)$. Then $\gamma$ is a closed path of length $\ell$ that is not null homotopic in $\Gamma_\ell(B)$ and hence $\ell \in H(B)$. Analogously we prove that if $\gamma$ has label an $A$-word, then $\ell \in H(A)$.

$\square$

2.3. The taut spectrum of a small cancellation product. Consider groups $A$ and $B$ and let $R$ be a subset of $A *_C B$ where $C$ is a common finite subgroup. We use the definitions, language and conventions by Lyndon and Schupp [LS01, Ch. V.]. In particular, a symmetrized subset $R$ of $A *_C B$ that satisfies the $C'(1/6)$
small cancellation is non-empty and every \( r \in \mathcal{R} \) has normal form with at least seven alternating letters from \( A \) and \( B \), see [LS01, Page 286]. We will use the following well known result in this section:

**Theorem 2.3.** [LS01, Ch. V. Theorem 11.2] Let \( A \) and \( B \) be groups, let \( C \) be a common finite group, and let \( R \) be a finite symmetrized subset of \( \langle A \ast_C B \rangle \) that satisfies the \( C'(1/6) \) small cancellation condition. Let \( G = \langle A \ast_C B \rangle / \langle \langle R \rangle \rangle \). The natural homomorphisms \( A \to G \) and \( B \to G \) are injective, and we regard \( A \) and \( B \) as subgroups.

In the rest of the section we prove the following statement. We divert the proof to the end of the section.

**Theorem 2.4.** Let \( A \) and \( B \) groups and let \( \mathcal{R} \) be a finite set of elements of \( A \ast B \) satisfying the \( C'(1/6) \) small cancellation condition. Let \( G = \langle A \ast B \rangle / \langle \langle \mathcal{R} \rangle \rangle \), then \( H(A) \cup H(B) \) and \( H(G) \) are k-related for some \( k \).

The small cancellation conditions over free products are expressed in terms of the length of normal forms. On the other hand, the definition of the taut spectrum is in terms of the word metric with respect to finite generating sets. Lemma 2.5 below makes the connection between the two norms. It is essentially a translation of Theorem 9.3 in [LS01, p. 293].

Let \( A \) and \( B \) be groups with finite symmetric generating sets \( X_A \) and \( X_B \), and consider the generating set \( X = X_A \sqcup X_B \) of \( A \ast B \). We will be considering words over the alphabet \( X \). For \( g \in A \ast B \), denote by \( |g| \) the length of its normal form. If \( g \in A \ast B \) has normal form \( g_1 g_2 \cdots g_n \) then each element \( g_i \) is called a syllable of \( g \). If \( f, g \in A \ast B \) the product \( fg \) is reduced if the last syllable of \( f \) and the first syllable of \( g \) are in distinct factors.

For a word \( W \) over \( X \), let \( |W|_X \) denote the length of the word. A word \( W \) over the alphabet \( X \) is cyclically reduced if when considered as a cyclic word, it is reduced. A word over the alphabet \( X_A \) (resp. \( X_B \)) is geodesic if there is no shorter word over \( X_A \) (resp. \( X_B \)) that represents the same element of \( A \) (resp. \( B \)). An \( A \)-syllable (resp. \( B \)-syllable) of a cyclically reduced word is a maximal subword that uses only letters from \( X_A \) (resp. \( X_B \)). An \( A \)-syllable is geodesic if it is geodesic as a word over \( X_A \). A geodesic \( B \)-syllable is defined analogously. A word over \( X \) is geodesic if all its syllables are geodesic.

If \( U \) and \( V \) are reduced words, we say that their concatenation \( U \cdot V \) does not combine syllables if the last letter of \( U \) and the first letter of \( V \) are not both letters in \( X_A \) or \( X_B \). In particular, if \( u, v \in A \ast B \) are the elements represented by \( U \) and \( V \), then the expression \( uv \) is reduced.

Let \( \mathcal{R} \) be a nonempty symmetrized finite subset of \( A \ast B \) that satisfies the \( C'(\lambda) \) small cancellation condition with \( \lambda \leq 1/6 \), and let \( N = \langle \langle \mathcal{R} \rangle \rangle \) be the normal subgroup generated by \( \mathcal{R} \). Let

\[
M = \max\{|W|_X : W \text{ is a syllable of a geodesic word representing } r \in \mathcal{R}\}.
\]

Observe that \( M \geq 1 \) is well defined since \( X_A, X_B \) and \( \mathcal{R} \neq \emptyset \) are finite sets.

**Lemma 2.5** (Sufficient condition for Dehn Algorithm). Suppose that \( 1 \geq 3\lambda(M + 1) \). Let \( W \) be a non-trivial reduced cyclic \( X \)-word that represents an element of \( N \). Suppose that each syllable of \( W \) is a geodesic word. Then there is a cyclically reduced word \( R \) representing an element of \( \mathcal{R} \) such that
(1) every syllable of $R$ is geodesic,
(2) $W$ equals the concatenation of three words $U \cdot S \cdot V$ and the concatenation does not combine syllables,
(3) $R$ equals the concatenation $S \cdot T$, where $S$ is the word of the previous item, and the concatenation does not combine syllables, and
(4) $|S|_X > |T|_X$.

Proof. Observe that $W$ is neither an $A$-word nor a $B$-word, since $\langle \langle R \rangle \rangle$ intersects trivially both $A$ and $B$.

Since $W$ represents a non-trivial element $w$ of $\langle \langle R \rangle \rangle$, then there is $r \in R$ such that $r = st$ in reduced form, $w = usv$ in reduced form, and $|s| > (1 - 3\lambda)|r|$, see [LS01, Chapter 5, Theorem 9.3]. Since $|r| = |s| + |t|$, the last inequality implies

$$|s| > \frac{1 - 3\lambda}{3\lambda}|t|.$$  

On the other hand, $W$ is a concatenation of words $W = U \cdot S \cdot V$ that does not combine syllables and $U, S, V$ represent the elements $u, s, v$ of $A \ast B$ respectively. Since $r = st$ is a reduced expression, then for any geodesic word $T$ representing $t$, the concatenation $R := S \cdot T$ does not combine syllables and is a word that represents $r$. Since $1 \geq 3\lambda(M + 1)$, we have that $1 - 3\lambda \geq 3\lambda M$, and therefore

$$|S|_X \geq |s| > \frac{1 - 3\lambda}{3\lambda}|t| \geq \frac{1 - 3\lambda}{3\lambda |T|_X} \geq |T|_X.$$  

\hfill \Box

From now on, we assume $\mathcal{R}$ satisfies $C'(\lambda)$ with $\lambda \leq 1/6$ and moreover that $1 \geq 3\lambda(M + 1)$, as in the hypothesis of Lemma 2.5. Note that this second assumption can be obtained since $\mathcal{R}$ is finite and hence one can add all syllables of the elements of $\mathcal{R}$ to $X_A$ and $X_B$ which would make $M = 1$.

Let $G = (A \ast B)/\langle \langle \mathcal{R} \rangle \rangle$, and let

$$\ell_0 = M \max\{|r| : r \in \mathcal{R}\}.$$  

Hence $\ell_0$ is an upper bound on the maximal length of a geodesic word in $X_A \cup X_B$ representing an element of $\mathcal{R}$.

Lemma 2.6. Let $\ell > \ell_0$ and suppose $\ell \in H(G)$. Then there is $k \in H(A) \cup H(B)$ such that $\ell < k < 2\ell$.

Proof. There is a closed path $\gamma$ of length $\ell$ that is not null-homotopic in $\Gamma_{\ell}(G)$. Observe that $\gamma$ is an embedded path, and hence it is labeled by a cyclically reduced word $W$ such that each syllable represents a non-trivial element in the corresponding factor.

If $W$ is a $B$-word then, up to a translate, $\gamma$ lifts to a path in $\Gamma_{\ell}(B)$ that is not null-homotopic and hence $\ell \in H(B)$. We proceed analogously when $W$ is an $A$-word.

Suppose that $W$ has at least two syllables, i.e., $W$ is neither an $A$-word nor a $B$-word.

Let us argue that there is a syllable of $W$ that is not geodesic. Suppose that all syllables of $W$ are geodesic. Then $W$ is a non-trivial cyclically reduced geodesic word, and Lemma 2.5 implies that there is a geodesic word $R = ST$ representing an element of $\mathcal{R}$ such that $W = U \cdot S \cdot V$ and $|S| > |T|$. Since $R$ is geodesic, it follows that $|R| \leq \ell_0 < \ell$ and hence there is a path $\gamma'$ in $\Gamma_{\ell}(G)$ labeled by the word $U \cdot T \cdot V$ that is homotopic to $\gamma$. Since $|\gamma'| < |\gamma| = \ell$, it follows that $\gamma'$ is null.
homotopic in $\Gamma_{\ell}(G)$ and therefore $\gamma$ as well. This is a contradiction. Therefore not all syllables of $W$ are geodesic.

Suppose $S$ is a $B$-syllable of $W$ that is not geodesic. Let $T$ be a geodesic $B$-word that represents the same element as $S$. Note that $S \cdot T$ does not label a null homotopic path in $\Gamma_{\ell}(G)$ since the otherwise the $\ell$-path $\gamma$ would be homotopic to a shorter closed path in $\Gamma_{\ell}(G)$ and then $\gamma$ would be null homotopic which is not the case. Since $|T| < |S| < |W| = \ell$, it follows that the $B$-word $S \cdot T$ has length strictly smaller than $2\ell - 2$. Let $\beta$ be a path in $\Gamma_{\ell}(B)$ labeled by the $B$-word $S \cdot T$. Since $\beta$ is not null homotopic in $\Gamma_{\ell}(B)$ and $|\beta| < 2\ell$, it follows that there there is $\ell < k < 2\ell$ such that $k \in H(B)$. We proceed analogously when $S$ is an $A$ syllable of $W$ that is not geodesic.

For $\ell > 0$, define $\Gamma'_{\ell}(G)$ as the 2-complex with 1-skeleton the Cayley graph of $G$, and a 2-cell for each homogeneous cycle of length less than $\ell$ and each closed path labeled by a geodesic word representing an element of $R$. Note that $G$ acts on $\Gamma'_{\ell}(G)$.

**Lemma 2.7.** Let $\ell > 0$. The map $\Gamma_{\ell}(B) \to \Gamma'_{\ell}(G)$ is $\pi_1$-injective.

**Proof.** Note that $\Gamma'_{\ell}(G)/G$ is obtained from the wedge of $\Gamma_{\ell}(A)/A$ and $\Gamma_{\ell}(B)/B$ by attaching 2-cells using the elements of $R$. Thus

$$\pi_1(\Gamma'_{\ell}(G)/G) = \langle \pi_1(\Gamma_{\ell}(A)/A) * \pi_1(\Gamma_{\ell}(B)/B) \rangle / \langle \langle R \rangle \rangle.$$ 

Moreover, the relations still satisfy the $C'(1/6)$ small cancellation condition. Therefore $\Gamma_{\ell}(B)/B \to \Gamma'_{\ell}(G)/G$ is $\pi_1$-injective. Now the proof is analogous to the one in the free product case.

**Lemma 2.8.** Any closed path in $\Gamma'_{2\ell}(G)$ of length less than $\ell$ is null homotopic.

**Proof.** It is enough to prove the statement for embedded closed paths. Let $\alpha$ be an embedded closed path of length less than $\ell$ in $\Gamma'_{2\ell}(G)$. Let $W$ be the reduced cyclic word that labels the path $\alpha$.

If $W$ is a monosyllable word then $\alpha$ is a null-homotopic closed path in $\Gamma'_{2\ell}(G)$ by definition.

If $W$ is not a monosyllable word and all syllables are geodesic then Lemma 2.5 implies that $\alpha$ is homotopic in $\Gamma'_{2\ell}(G)$ to a shorter closed path.

If $W$ is not a monosyllable word and is not geodesic. Then any syllable has length less than $|W| < \ell$. Hence replacing each syllable by a geodesic syllable implies $\alpha$ is homotopic in $\Gamma'_{2\ell}(G)$ to a shorter closed path.

The result of the lemma follows by induction on the length of the closed path. □

**Lemma 2.9.** If $\gamma: S^1 \to \Gamma_{\ell}(G)$ is null homotopic, then $\gamma: S^1 \to \Gamma'_{2\ell}(G)$ is null homotopic.

**Proof.** Since $\gamma: S^1 \to \Gamma_{\ell}(G)$ is null homotopic, there is a cellular disk diagram $D \to \Gamma_{\ell}(G)$ whose boundary path is $\gamma$. Then the boundary path of any 2-cell of $D$ maps to the boundary path of a 2-cell of $\Gamma_{\ell}(G)$ and hence it has length less than $\ell$.

By Lemma 2.8, each 2-cell of $D$ can be replaced by a disk diagram that maps into $\Gamma'_{2\ell}(G)$. Hence there is a disk diagram $D' \to \Gamma'_{2\ell}(G)$ whose boundary path is $\gamma$. □

**Lemma 2.10.** Let $\ell > \ell_0$ and suppose that $\ell \in H(A) \cup H(B)$. Then there is $k \in H(G)$ such that $\ell/2 \leq k \leq \ell + 1$. 
Lemma 2.7 implies that $\gamma$ is not null homotopic in $\Gamma'_2(G)$. Then Lemma 2.9 implies that $\gamma$ is not null homotopic in $\Gamma_{\ell/2}(G)$. Hence $\gamma$ is path of length $\ell$ that is not null homotopic in $\Gamma_{\ell/2}(G)$. It follows there is $k \in H(G)$ such that $\ell/2 \leq k \leq \ell$. We proceed in an analogous way when $\gamma$ is a closed path in $\Gamma(A)$ of length $\ell$ that is not null homotopic.

Proof Theorem 2.4. The statement of the theorem follows from Lemmas 2.6 and 2.10.

3. Proper and virtually cyclic dimensions of multi-ended groups

Let $G$ be a discrete group. Given a CW-complex $X$ with a cellular $G$-action, we say $X$ is a $G$-CW-complex if every element of $G$ that fixes setwise a cell, it fixes the cell pointwise. A non-empty collection $\mathcal{F}$ of subgroups of $G$ is called a family if it is closed under conjugation and under taking subgroups. A $G$-CW-complex $X$ is a model for the classifying space $E_{\mathcal{F}}G$ if the following conditions are satisfied:

1. For all $x \in X$, the isotropy group $G_x$ belongs to $\mathcal{F}$.
2. For all $H \in \mathcal{F}$ the subcomplex $X^H$ of $X$, consisting of points in $X$ that are fixed under all elements of $H$, is contractible. In particular $X^H$ is non-empty.

The $\mathcal{F}$-geometric dimension of $G$, denoted $\text{gd}_{\mathcal{F}}(G)$, is the minimum $n$ for which there exists an $n$-dimensional model for $E_{\mathcal{F}}G$, see Section 4.1. The orbit category $\mathcal{O}_{\mathcal{F}}G$ has as objects the homogeneous $G$-spaces $G/H$, $H \in \mathcal{F}$, and morphisms are $G$-maps. A $\mathcal{O}_{\mathcal{F}}G$-module is a contravariant functor from $\mathcal{O}_{\mathcal{F}}G$ to the category of abelian groups, and a morphism between two $\mathcal{O}_{\mathcal{F}}G$-modules is a natural transformation of the underlying functors. Denote by $\mathcal{O}_{\mathcal{F}}G$-$\text{mod}$ the category of $\mathcal{O}_{\mathcal{F}}G$-modules. It turns out that $\mathcal{O}_{\mathcal{F}}G$-$\text{mod}$ is an abelian category with enough projectives (see [MV03, pg. 9]). The $\mathcal{F}$-cohomological dimension of $G$, denoted $\text{cd}_{\mathcal{F}}(G)$, is the length of the shortest projective resolution of the constant $\mathcal{O}_{\mathcal{F}}G$-module $\mathbb{Z}_{\mathcal{F}}$, where $\mathbb{Z}_{\mathcal{F}}$ is given by $\mathbb{Z}_{\mathcal{F}}(G/H) = \mathbb{Z}$, for all $H \in \mathcal{F}$, and every morphism of $\mathcal{O}_{\mathcal{F}}G$ goes to the identity function. The following inequality is a standard result

$$\text{cd}_{\mathcal{F}}(G) \leq \text{gd}_{\mathcal{F}}(G),$$

see for instance [LM00]. We denote by $\text{cd}(G)$ and $\text{gd}(G)$ the cohomological dimensions for the families of finite and virtually cyclic subgroups respectively, and analogously for geometric dimensions. We also denote by $\underline{EG}$ and $\underline{EG}$ the classifying spaces of $G$ for the families of finite and virtually cyclic subgroups respectively. In this section we prove the following results.

Theorem 3.1. Let $G$ be the fundamental group of a graph of groups with finite edge stabilizers, and let $T$ be the Bass-Serre tree. Then the following inequalities hold

$$\text{gd}(G) \leq \max \{1, \text{gd}(G_\sigma) | \sigma \in T\}, \quad \text{cd}(G) \leq \max \{1, \text{cd}(G_\sigma) | \sigma \in T\}$$

$$\text{gd}(G) \leq \max \{2, \text{gd}(G_\sigma) | \sigma \in T\}, \quad \text{cd}(G) \leq \max \{2, \text{cd}(G_\sigma) | \sigma \in T\}. $$

Theorem 3.2. Under the assumptions of Theorem 3.1, if all vertex groups have small centralizers and satisfy the ascending chain condition for finite subgroups, then

$$\text{gd}(G) \leq \max \{2, \text{gd}(G_\sigma) | \sigma \in T\} \text{ and } \text{cd}(G) \leq \max \{2, \text{cd}(G_\sigma) | \sigma \in T\}. $$
3.1. **An Auxiliary Result.** The following proposition is a classical result in homology theory, see for instance the Proof 1 of Proposition 1.1 in [Bro87].

**Proposition 3.3.** Let $G$ be a group and let $R$ be a ring. Assume $G$ acts on an $R$-acyclic $G$-CW-complex $X$. Then

$$cd_R(G) \leq \max\{cd_R(G_\sigma) + \dim(\sigma) \mid \sigma \text{ is a cell of } X\}.$$ 

In this section we prove the following generalization of the above inequality for cohomological and geometric dimension relative to families of subgroups. The proof of the cohomological part of this is adapted from an argument by Fluch and Nucinkis [FN13, Proof of Proposition 4.1]. The geometric part is implicit in [Lüc00, Proof of Theorem 3.1].

**Proposition 3.4.** Let $G$ be a group. Let $\mathcal{F}$ and $\mathcal{G}$ be families of subgroups of $G$ such that $\mathcal{F} \subseteq \mathcal{G}$. Let $\mathcal{F}$ be a model for $E_\mathcal{G}G$, then

$$gd_\mathcal{F}(G) \leq \max\{|gd_{\mathcal{F} \cap G_\sigma}(G_\sigma) + \dim(\sigma)| \sigma \text{ is a cell of } X\}$$

and

$$cd_\mathcal{F}(G) \leq \max\{|cd_{\mathcal{F} \cap G_\sigma}(G_\sigma) + \dim(\sigma)| \sigma \text{ is a cell of } X\}.$$ 

The next theorem taken from [MS20, Theorem 2.3] will be used to prove Proposition 3.4. Given a group $G$, a family of subgroups $\mathcal{G}$ and a $G$-CW-complex, we say that $X$ is a $\mathcal{G}$-CW-complex if all isotropy groups of $X$ belong to $\mathcal{G}$.

**Theorem 3.5 (The Haefliger-Lück construction for families).** Let $G$ be a group. Let $\mathcal{F}$ and $\mathcal{G}$ be families of subgroups of $G$ such that $\mathcal{F} \subseteq \mathcal{G}$. Consider a $\mathcal{G}$-$G$-CW-complex $X$. For each cell $\sigma$ of $X$, fix models $X_\sigma$ for $E_{\mathcal{F} \cap G_\sigma}G_\sigma$ so that for any two cells in the same $G$-orbit the same model has been chosen. Then, for each $n \geq 0$ there exists an $\mathcal{F}$-$G$-CW-complex $\hat{X}_n$ and a $G$-map $f_n : \hat{X}_n \to X^{(n)}$ such that:

1. We have $\hat{X}_{n-1} \subseteq \hat{X}_n$ and $f_n$ restricted to $\hat{X}_{n-1}$ is $f_{n-1}$.
2. For every open simplex $\sigma$ of $(X^{(n)})^H$ such that $f^{-1}_n(\sigma)$ is a model for $E_{\mathcal{F} \cap G_\sigma}G_\sigma$.
3. For all $H \in \mathcal{F}$, $f_n^H : \hat{X}_n^H \to (X^{(n)})^H$ is a (nonequivariant) homotopy equivalence.
4. There is a $G$-equivariant homeomorphism between $f_n^{-1}(\sigma)$ and $X_\sigma \times \sigma$.

**Proof of Proposition 3.4.** For each $n \geq 0$ let $\hat{X}_n$ be the spaces provided by Theorem 3.5 with $X_\sigma$ models of minimal dimension for each cell $\sigma$ of $X$, i.e. $\dim(X_\sigma) = \max\{|gd_{\mathcal{F} \cap G_\sigma}(G_\sigma) + \dim(\sigma)| \sigma \text{ is a cell of } X\}$. Denote $\hat{X} = \bigcup_{n=0}^{\infty} \hat{X}_n$. As consequence of hypothesis (3) of Theorem 3.5 we have that $\hat{X}^K$ is contractible for every $K \in \mathcal{F}$, therefore $\hat{X}$ is a model for $E_\mathcal{F}G$. By hypothesis (4) of Theorem 3.5 we have that

$$\dim(\hat{X}) = \max\{|gd_{\mathcal{F} \cap G_\sigma}(G_\sigma) + \dim(\sigma)| \sigma \text{ is a cell of } X\}$$

and the statement for $\mathcal{F}$-geometric dimension follows.

Now we prove the statement for the $\mathcal{F}$-cohomological dimension. Denote by $\text{res}_\mathcal{F} : \mathcal{O}_0G\text{-mod} \to \mathcal{O}_\mathcal{F}G\text{-mod}$ the natural restriction functor. For $P$ a $\mathcal{O}_\mathcal{F}$-module, denote by $\text{pd}_\mathcal{F}(P)$ the projective dimension of $P$, i.e. the smallest number $n$ such that $P$ admits a $\mathcal{O}_\mathcal{F}G$-projective resolution of length $n$.

The augmented Breeden cellular complex of $\hat{X}$

$$\cdots \to \bigoplus \mathbb{Z}[\sigma, G/G_\sigma] \to \bigoplus \mathbb{Z}[\sigma, G/G_{\sigma_1}] \to \bigoplus \mathbb{Z}[\sigma, G/G_{\sigma_0}] \to \mathbb{Z}_G \to 0$$

2-cells \hspace{1cm} 1-cells \hspace{1cm} 0-cells
is a free resolution of $\mathbb{Z}_G$ of length $\dim(\hat{X})$. Following the proof of Proposition 4.1 in [FN13] we have that

$$
\cdots \rightarrow \bigoplus_{1\text{-cells}} \text{res}^G_{F} \mathbb{Z}[-, G/G_{\sigma^1}] \rightarrow \bigoplus_{0\text{-cells}} \text{res}^G_{F} \mathbb{Z}[-, G/G_{\sigma^0}] \rightarrow \text{res}^G_{F}(\mathbb{Z}_G) = \mathbb{Z}_F \rightarrow 0
$$

is a resolution for $\mathbb{Z}_F$ in $O_F G$-mod, and $\text{pd}_F(\text{res}^G_{F} \mathbb{Z}[-, G/G_{\sigma}]) = \text{cd}_{F \cap G_{\sigma}}(G_{\sigma})$ for each cell $\sigma$ of $\hat{X}$. Now, a standard argument let us assemble all the projective resolutions of the $\text{res}^G_{F} \mathbb{Z}[-, G/G_{\sigma}]$ to obtain to obtain a projective resolution of $\mathbb{Z}_F$ of dimension

$$
\max\{\text{cd}_{F \cap G_{\sigma}}(G_{\sigma}) + \dim(\sigma) \mid \sigma \text{ is a cell of } X\}\}.
$$

3.2. Proof of Theorem 3.1.

Proof. If $G$ is the fundamental group of a graph of groups with finite edge groups, then $G$ acts on its Bass-Serre tree $T$ with finite edge stabilizers. For every subgroup $H$ of $G$, the fixed point set $T^H$ is either empty or it is convex (and therefore contractible), thus $T$ is a classifying space for $G$ with respect to the smallest family $\mathcal{G}$ of $G$ that contains all vertex subgroups. It is well-known that every finite group acting on a tree fixes at least one point, thus the family $\mathcal{F}$ of finite subgroups of $G$ is contained in $\mathcal{G}$. By Proposition 3.4 and the fact that $\text{gd}(F) = 0$ for every finite group $F$, we obtain $\text{gd}(G) \leq \max\{1, \text{gd}(G_{\sigma}) \mid \sigma \text{ is a vertex of } X\}$.

For the inequalities for cohomological dimension we run the same proof using the cohomological conclusions of Proposition 3.4.

3.3. Proof of Theorem 3.2. The result is obtained by putting together the inequalities for $\text{gd}(G)$ and $\text{cd}_G$ given by Theorem 3.1 and Proposition 3.6 below.

Let us remark that the geometric part of Proposition 3.6 is based on an argument by Juan-Pineda and Leary [JPL06] for the particular case that $G$ is a hyperbolic group. The cohomological part of Proposition 3.6 takes ideas from Fluch and Leary in [FL14].

**Proposition 3.6.** Let $G$ be a group that has small centralizers, and satisfies the ascending chain condition for finite subgroups. Then

$$
\text{gd}(G) \leq \max\{\text{gd}(G), 2\},
$$

□
and
\[ \text{cd}(G) \leq \max\{\text{cd}(G), 2\}. \]

Proof. By our hypothesis on \( G \) and [LW12, Theorem 3.1] we have that every infinite virtually cyclic subgroup of \( G \) is contained in a unique maximal virtually cyclic subgroup of \( G \), and \( N_G(V) = V \) for every infinite maximal virtually cyclic subgroup of \( G \). Thus by [LW12, Corollary 2.11] we obtain a model for \( EG \) as the \( G \)-CW-complex \( Y \) defined by the following homotopy pushout

\[
\begin{array}{ccc}
G \times_V EV & \xrightarrow{g} & EG \\
\downarrow f & & \downarrow \ \\
G/V & \xrightarrow{\bar{g}} & Y
\end{array}
\]

where \( M \) is a complete system of representatives of the conjugacy classes of maximal infinite virtually cyclic subgroups of \( G \), \( g \) is the inclusion map, and \( f \) is the disjoint union of the projection maps \( G \times_V EV \to G/V \). Hence if we take each \( EV \) to be the real line, and \( EG \) a space of dimension \( gd(G) \), then \( Y \) is obtained from \( EG \) by coning-off certain lines (the images of the maps \( EV \to EG \)). Since \( Y \) is already a model for \( EG \) of dimension \( \max\{gd(G), 2\} \) the first claim follows.

For the second claim, we can run verbatim the argument in the proof of Theorem 7 in [FL14]. We include this argument for the sake of completeness. Let \( F \) and \( G \) be the families of finite subgroups and virtually cyclic subgroups of \( G \) respectively. Consider the following short exact sequence of \( O_G \)-modules

\[ 1 \to \bar{Z}_F \to Z_G \to Q \to 1 \]

where \( Q \) is the \( O_G \)-module given by \( Q(G/H) = \mathbb{Z} \) if \( H \notin F \) and \( Q(G/H) = 0 \) if \( H \in F \), and \( \bar{Z}_F \) is the \( O_G \)-module given by \( \bar{Z}_F(G/H) = \mathbb{Z} \) if \( H \in F \) and \( \bar{Z}_F(G/H) = 0 \) if \( H \notin F \). Since \( Y \) is a model for \( EG \) obtained from a model \( X \) for \( EG \) by attaching cells of dimension at most \( 2 \), the Bredon cellular chain complex \( C_*(Y, X) \) yields a projective resolution for \( Q \) of length 2. Let \( P_* \to Z_G \) be a projective resolution of length \( cd(G) \), and promote it to a projective resolution \( \bar{P}_* \) for \( \bar{Z}_F \) with the same length by setting \( \bar{P}_*(G/H) = 0 \) for every \( H \notin F \). We can apply the Horseshoe lemma to the resolutions \( C_*(Y, X) \) and \( \bar{P}_* \) to obtain a projective resolution for \( Z_G \) of length \( \max\{cd(G), 2\} \). This concludes the proof. \( \square \)

4. PROPER AND VIRTUALLY CYCLIC DIMENSIONS OF SMALL CANCELLATION QUOTIENTS OF FREE PRODUCTS

This section contains the proof of the following statement.

**Theorem 4.1.** Let \( A \) and \( B \) be groups and let \( C \) be a common finite subgroup. Let \( R \) be a finite symmetrized subset of \( A \ast_C B \) that satisfies the \( C'(1/12) \) small cancellation condition, and \( G = (A \ast_C B)/\langle \! \langle R \rangle \! \rangle \). Assume \( G \) is not virtually free. Then the following conclusions hold.

1. \( gd(G) = \max\{gd(A), gd(B), 2\} \) and \( cd(G) = \max\{cd(A), cd(B), 2\} \)
(2) If $A$ and $B$ are finitely generated, have small centralizers and satisfy the ascending chain condition for finite subgroups, then $G$ has small centralizers, satisfies the ascending chain condition, and

$$\text{max}\{\text{cd}(A), \text{cd}(B), 2\} \leq \text{cd}(G) \leq \text{max}\{\text{cd}(A), \text{cd}(B), 2\}.$$ 

and

$$\text{max}\{\text{gd}(A), \text{gd}(B), 2\} \leq \text{gd}(G) \leq \text{max}\{\text{gd}(A), \text{gd}(B), 2\}.$$ 

(3) For any ring $R$, we have $\text{cd}_R(G) = \text{max}\{\text{cd}_R(A), \text{cd}_R(B), 2\}.$

4.1. A Cocompact Model for Small Cancellation Quotients. For a small cancellation product $G = (A * B)/\langle\langle R \rangle\rangle$ there is a natural family of subgroups to consider, namely, the smallest family $F$ that contains $\{A, B\}$. In this section we describe a canonical 2-dimensional model for $E_XG$.

**Definition 4.2** (Coned-off Cayley complex $\hat{X}$). Let $A$ and $B$ be groups, let $C$ be a common finite subgroup, and let $\mathcal{R}$ be a finite subset of $A *_C B$. Let $G = (A *_C B)/\langle\langle \mathcal{R} \rangle\rangle$. Suppose that the natural homomorphisms $A \to G$ and $B \to G$ are injective, and regard $A$ and $B$ as subgroups of $G$.

Let $\hat{\Gamma}$ be the $G$-graph with vertex set the $G$-set of left cosets of $A$ and $B$ in $G$, and edge set $\{\{gA, gB\} : g \in G\}$. Equivalently, if $T$ is the Bass-Serre tree of $A *_C B$, then $\hat{\Gamma} = T/\langle\langle \mathcal{R} \rangle\rangle$. Note that $\hat{\Gamma}$ is a cocompact connected $G$-graph with finite edge stabilizers. Moreover, the stabilizer of every vertex is either $A$ or a conjugate of $A$ or of $B$.

The **coned-off Cayley complex** $\hat{X}$ of $G$ is a 2-dimensional $G$-complex with 1-skeleton $\hat{\Gamma}$ defined as follows. Since the natural morphisms $A \to G$ and $B \to G$ are injective, the subgroup $\langle\langle \mathcal{R} \rangle\rangle$ intersects trivially with $A$ and $B$. Thus the action of $N = \langle\langle \mathcal{R} \rangle\rangle$ on the Bass-Serre tree $T$ is free, and the quotient map $\rho: T \to T/N$ is a covering map. Fix a vertex $x_0$ of $T$ and consider it as a base point. Then any element $g$ of $\langle\langle \mathcal{R} \rangle\rangle$ induces a unique path $\alpha_g$ from $x_0$ to $gx_0$. Let $\gamma_g = \rho \circ \alpha_g$ be the closed path in $\hat{\Gamma}$ induced by $\alpha_g$ based at $\rho(x_0)$. This induces an isomorphism $N \to \pi_1(\hat{\Gamma}, \rho(x_0))$ defined by $g \mapsto \gamma_g$. For $g \in G$ and $h \in N$, let $g \cdot \gamma_h$ be the translated closed path in $\hat{\Gamma}$ without an initial point, i.e., these are cellular maps from $S^1 \to \hat{\Gamma}$. Consider the $G$-set $\Omega = \{g, \gamma_r \mid r \in \mathcal{R}, g \in G\}$ of closed paths in $\hat{\Gamma}$. The complex $\hat{X}$ is then obtained by attaching a 2-cell to $\hat{\Gamma}$ for every closed path in $\Omega$. In particular, no pair of distinct 2-cells of $\hat{X}$ have the same boundary path, and the pointwise $G$-stabilizer of a 2-cell of $\hat{X}$ coincides with the pointwise $G$-stabilizer of its boundary path (in particular they are finite subgroups). The natural isomorphism from $N$ to $\pi_1(X^{(1)}, \rho(x_0))$ implies that $\hat{X}$ is simply connected. Moreover, the $G$-action is cocompact since $\mathcal{R}$ is finite.

**Proposition 4.3.** Let $A$ and $B$ be groups, let $C$ be a common finite subgroup, and let $\mathcal{R}$ be a finite symmetrized subset of $A *_C B$ that satisfies the C’(1/12) small cancellation condition. Let $G = (A *_C B)/\langle\langle \mathcal{R} \rangle\rangle$. If $\mathcal{F}$ is the family of subgroups generated by the $G$-stabilizers of cells of $\hat{X}$, then $\hat{X}$ is a 2-dimensional cocompact model for $E_XG$ such that the stabilizers of 1 and 2-cells are finite, and the vertex stabilizers are conjugates of $A$ or $B$.

Recall that a collection of subgroups $\mathcal{P}$ of a group $G$ is **almost malnormal** if for any $P, P' \in \mathcal{P}$ and $g \in G$, either $gPg^{-1} \cap P'$ is finite, or $P = P'$ and $g \in P$. 

Proof of Proposition 4.3. The argument that if \( R \) satisfies the \( C'(1/12) \) small cancellation condition, then \( \hat{X} \) is a contractible \( C'(1/6) \) small cancellation complex can be found in [ACCCMP0, Proof of Theorem 7.1] in a slightly more general context; see Remark 4.5 for an additional comment.

Before showing that \( \hat{X} \) is a model of \( E_FG \) let us make a remark that small cancellation products are standard examples of relatively hyperbolic groups [Osi06, Page 4]. In particular, since \( G \) is hyperbolic relative to the collection of subgroups \( \{A, B\}\), [Osi06, Proposition 2.36] implies that \( \{A, B\} \) is an almost malnormal collection in \( G \).

Let \( K \) be a subgroup in \( \mathcal{F} \). If \( K \) is a finite subgroup, then \( \hat{X}^K \) is a contractible subcomplex by [HMP14, Proposition 5.7]. Suppose \( K \) is infinite. Then the fixed point set \( \hat{X}^K \) consists only of vertices, since 1-cells and 2-cells have finite stabilizers. Since \( \{A, B\} \) is an almost malnormal collection, the \( G \)-stabilizers of any two distinct vertices of \( \hat{X} \) have finite intersection. It follows that \( \hat{X}^K \) consists of a single vertex and hence it is a contractible subcomplex.

\[ \square \]

Remark 4.4 (Alternative argument proving Proposition 4.3). Let \( \hat{\Gamma} \) be the one-skeleton of \( \hat{X} \) and note that this is the coned-off Cayley graph of \( G \) with respect to \( \{A, B\} \). Since \( G \) is hyperbolic relative to \( \{A, B\} \), we have that \( \hat{\Gamma} \) is a fine graph in the sense of Bowditch [Bow12], see for example [MPW11b, Proposition 4.3]. Then it is a direct consequence of [AM21, Corollary 1.6] that the coned-off Cayley complex \( \hat{X} \) is a cocompact model for \( E_FG \).

Remark 4.5 (On the \( C'(1/12) \) hypothesis of Proposition 4.3). A notable hypothesis in the statement of Proposition 4.3 is the \( C'(1/12) \) small cancellation condition. This comes from the fact that \( (A \ast_C B)/\langle\langle R\rangle\rangle \) being a \( C'(\lambda) \)-small cancellation quotient does not imply that the coned-off Cayley complex \( \hat{X} \) is a \( C'(\lambda) \) small cancellation complex. This phenomenon regarding a difference between algebraic and geometric versions of small cancellation conditions in the case of quotients of free products have been observed in the literature. For example, in [MPW11a, Page 2404], this issue is illustrated by considering the free product \( \mathbb{Z}_2 \ast \mathbb{Z}_3 \) and the relation \( (ab)^7 \) where \( a \) and \( b \) are generators of \( \mathbb{Z}_2 \) and \( \mathbb{Z}_3 \) respectively. Then \( (\mathbb{Z}_2 \ast \mathbb{Z}_3)/\langle\langle (ab)^7\rangle\rangle \) is a \( C'(1/7) \) small cancellation quotient of the free product \( (\mathbb{Z}_2 \ast \mathbb{Z}_3) \), the coned-off Cayley complex \( \hat{X} \) is \( C'(1/7 + \varepsilon) \) for any \( \varepsilon > 0 \), but it is not \( C'(1/7) \). In this particular case, \( \hat{X} \) is a subdivision of the \{7, 3\} tiling of the hyperbolic plane.

4.2. Proof of Theorem 4.1. The proof relies on the following lemma.

Lemma 4.6. Let \( A \) and \( B \) be groups and let \( C \) be a common finite subgroup. Let \( \mathcal{R} \) be a finite symmetric subset of \( A \ast_C B \) that satisfies the \( C'(1/12) \) small cancellation condition, and \( G = (A \ast_C B)/\langle\langle \mathcal{R}\rangle\rangle \). If \( A \) and \( B \) have small centralizers, then \( G \) has small centralizers.

Proof. Recall that by Theorem 2.3 we can regard \( A \) and \( B \) as subgroups of \( G \). Let \( g \in A \) be an element of infinite order, and let \( x \in C_G(\langle g \rangle) \). Since \( G \) is hyperbolic relative to \( \{A, B\} \), we have that \( \{A, B\} \) is an almost malnormal collection. Since \( A \) is infinite, \( g \in xA^{-1} \cap A \) implies \( x \in A \). Therefore \( C_G(\langle g \rangle) = C_A(\langle g \rangle) \) and hence \( \langle g \rangle \) has finite index in \( C_G(\langle g \rangle) \). For \( g \in B \) we proceed analogously. Finally if \( g \) is neither subconjugate to \( A \) nor \( B \), then by [Osi06, Theorem 1.14] we have that \( \langle g \rangle \) has finite index in its centralizer.

\[ \square \]
Proof of Theorem 4.1. (1) Consider the family $\mathcal{F}$ of subgroups generated by the $G$-stabilizers of cells of the coned-off Cayley complex $\hat{X}$. Proposition 4.3 implies that $\hat{X}$ is a 2-dimensional model of $E_F G$ such that the stabilizers of 1 and 2-cells are finite, and the vertex stabilizers are conjugates of $A$ or $B$. By [HMP14, Proposition 5.7] every finite subgroup of $G$ fixes a point of $\hat{X}$, therefore $F_{IN} \subseteq \mathcal{F}$ where $F_{IN}$ is the family of finite subgroups of $G$. By Proposition 3.4 applied for the case $\mathcal{F} = \mathcal{G}$, our hypothesis on the dimensions of $A$ and $B$ and the fact that the stabilizers of all 1-cells and 2-cells of $\hat{X}$ are finite, we have

$$\overline{\text{gd}}(G) \leq \max\{\overline{\text{gd}}(G_\sigma) + \dim(\sigma) | \sigma \text{ is a cell of } \hat{X}\}$$

$$\leq \max\{\overline{\text{gd}}(A), \overline{\text{gd}}(B), 2\}$$

On the other hand, since $A \leq G$, we have $\overline{\text{gd}}(G) \geq \overline{\text{gd}}(A)$. A completely analogous argument using the cohomological dimension part of Proposition 3.4 let us conclude that $\overline{\text{cd}}(G) \leq \max\{\overline{\text{gd}}(A), \overline{\text{gd}}(B), 2\}$. Since $G$ is not finite nor virtually free, we have that both $\overline{\text{gd}}(G)$ and $\overline{\text{cd}}(G)$ are greater than or equal to 2. This concludes the proof.

(2) If $A$ and $B$ satisfy the ascending chain condition for finite subgroups, then $G$ also satisfies the ascending chain condition for finite subgroups. Indeed, since $G$ is a small cancellation product of $A$ and $B$, every finite subgroup of $G$ is subconjugate to $A$ or $B$ or is a cyclic subgroup arising from a proper power in $R$. Since $R$ is finite, the ascending chain condition for finite subgroups holds for $G$. By our hypothesis and part (1) of this Theorem, we have $\overline{\text{gd}}(G) = \max\{\overline{\text{gd}}(A), \overline{\text{gd}}(B), 2\}$ and $\overline{\text{cd}}(G) = \max\{\overline{\text{cd}}(A), \overline{\text{cd}}(B), 2\}$. By Lemma 4.6, $G$ has small centralizers. Thus by Proposition 3.6 we have

$$\overline{\text{gd}}(G) \leq \max\{\overline{\text{gd}}(G), 2\}$$

$$\leq \max\{\overline{\text{gd}}(A), \overline{\text{gd}}(B), 2\}$$

Since $A$ and $B$ are finitely generated, then we can conclude $\overline{\text{gd}}(G) \geq 2$, see for instance [LS21, Lemma 2.2]. Since $\overline{\text{gd}}(A) \leq \overline{\text{gd}}(G)$ and $\overline{\text{gd}}(B) \leq \overline{\text{gd}}(G)$ we conclude

$$\max\{\overline{\text{gd}}(A), \overline{\text{gd}}(B), 2\} \leq \overline{\text{gd}}(G) \leq \max\{\overline{\text{gd}}(A), \overline{\text{gd}}(B), 2\}.$$  

The analogous statement for cohomological dimension follows in exactly the same way.

(3) To obtain that $\overline{\text{cd}}_R(G) = \max\{\overline{\text{cd}}_R(A), \overline{\text{cd}}_R(B), 2\}$ for any ring $R$, we run the same argument used in (1) considering the action of $G$ on $\hat{X}$ and using Proposition 3.3.

\[\square\]

5. One-ended small cancellation quotients of free products

Theorem 5.1. Let $G = (A *_C B)/\langle\langle R \rangle\rangle$, where $C$ is a common finite subgroup of $A$ and $B$, and $R$ is a symmetrized subset of $A *_C B$ that satisfies the $C'(1/6)$ small cancellation condition.

- Suppose that for every $r \in R$, its normal form does not contain elements of finite order of $A$ or $B$.

If $A$ and $B$ are one-ended, then $G$ is one-ended.
We are not aware of an example that shows that the bulleted hypothesis of Proposition 5.1 is necessary. The proof of Theorem 5.1 relies on the following lemma:

**Lemma 5.2** (Ping-pong for elliptic elements acting on trees). Let $A$ and $B$ be subgroups of the $\text{Aut}(T)$ where $T$ is a simplicial tree. Suppose

1. $A$ and $B$ act with finite edge stabilizers on $T$.
2. $A$ has a global fixed point $v_A$, $B$ has a global fixed point $v_B$, and $v_A \neq v_B$.

Suppose that $C$ is a finite common subgroup of $A$ and $B$. Consider the natural map $\varphi: A \ast_C B \to \text{Aut}(T)$. If $r \in \ker(\varphi)$, then either $r$ is the identity element, or its normal form contains elements of finite order of $A$ or $B$. In particular, if $A$ and $B$ are torsion-free, then $A \ast B \to \text{Aut}(T)$ is injective.

**Proof.** Consider $T$ as metric space with the path metric arising by regarding each edge as an interval of length one. Then the groups $A$ and $B$ act faithfully by isometries on $T$. For an arbitrary point $v$ of $T$, we say that $v$ points to $A$ (resp. $B$) if the geodesic from $v$ to $v_B$ (resp. $v_A$) contains $v_A$ (resp. $v_B$). In particular, if $v$ points to $A$ (resp. $B$) then $\text{dist}(v, \{v_A, v_B\})$ equals $\text{dist}(v, v_A)$ (resp. $\text{dist}(v, v_B)$).

Observe that if $v$ points to $A$ and $b \in B$ has infinite order, then $b.v$ points to $B$ and

$$\text{dist}(v, \{v_A, v_B\}) < \text{dist}(b.v, \{v_B, v_B\}).$$

This is a direct consequence of $T$ being a tree and that the fixed point set of $b$ is the single vertex $v_B$ (since $b$ has infinite order it cannot fix an edge of the tree). Analogously, if $v$ points to $B$ and $a \in A$ has infinite order then $a.v$ points to $A$ and

$$\text{dist}(v, \{v_A, v_B\}) < \text{dist}(a.v, \{v_A, v_B\}).$$

Let $r \in A \ast_C B$ be a non-trivial element such that its normal form $r = g_1g_2 \cdots g_k$ only involves elements of infinite order of $A$ and $B$. Let $v_0$ be the midpoint of the geodesic path from $v_A$ to $v_B$. Note that $v_0$ neither points to $A$ nor $B$. Note that if $g_k \in A$ then $g_k.v_0$ points to $A$; and if $g_k \in B$ then $g_k.v_0$ points to $B$. By the statement of the previous paragraph, $r.v_0$ points either to $A$ or $B$, and hence $r.v_0 \neq v_0$. Hence $r \notin \ker(\varphi)$. \qed

**Proof of Theorem 5.1.** Suppose that $G$ is multiended. By Stallings, $G$ admits an non-trivial action $G \to \text{Aut}(T)$ on a simplicial tree $T$ with finite edge stabilizers and without inversions. Since $A$ is one-ended and edge stabilizers are finite, it follows that the fixed point set of $A$ in $T$ is a single vertex $v_A$, and by symmetry $B$ fixes a single vertex $v_B$. Since $A$ and $B$ generate $G$, and the $G$-action on $T$ is non-trivial, we have that $v_A \neq v_B$. Restrictions of the $G$-action induce maps $A \to \text{Aut}(T)$ and $B \to \text{Aut}(T)$; and these two morphisms induced a morphism $A \ast_C B \to \text{Aut}(T)$ such that

$$\begin{array}{ccc}
A \ast_C B & \longrightarrow & \text{Aut}(T) \\
& \downarrow & \\
& G & \\
\end{array}$$

is a commutative diagram. Let $r \in \mathbb{R}$. By definition of $C'(1/6)$ small cancellation condition $r$ is non-trivial, hence by Lemma 5.2, $r$ contains elements of finite order in its normal form, which is a contradiction. \qed
The proof of the following proposition was communicated to us by Dani Wise.

**Proposition 5.3.** Let $G$ be a torsion-free, 2-generated group. If $G$ is not a free group, then $G$ is one-ended.

**Proof.** Suppose $G$ is multi-ended. Then $G$ splits as a free product or an HNN extension over a trivial group. First, suppose $G$ is a free product $A \ast B$. Since $G$ is 2-generated, Grushko’s theorem implies that $A$ and $B$ are cyclic. Then $G$ being torsion-free implies that $A$ and $B$ are infinite cyclic groups. It follows that $G$ is a free group of rank two. Suppose that $G$ is an HNN extension with a trivial edge group. Then $G = H \ast Z$ for some subgroup $H$ of $G$. If $H$ is non-trivial, we are in the previous case, and $G$ is a free group of rank two. If $H$ is trivial then $G$ is a free group of rank one.

□

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