Spin Particle with a Color Charge
in a Color Field in Riemann–Cartan Space

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Abstract—On the basis of the method of Cartan exterior forms and extended Lie derivatives,
a hydrodynamic equation of the Euler type that describes a perfect spin fluid with an intrinsic color
charge in an external non-Abelian color field in Riemann–Cartan space is derived from the energy-
momentum quasiconservation law. This equation is used to obtain a self-consistent set of equations
of motion for a classical test particle with a spin and a color charge in a color field combined with
a gravitational field characterized by curvature and torsion. The resulting equations generalize the
Wong equation, which describes the motion of a particle with an isospin, and the Tamm–Good and
Bargmann–Michel–Telegdi equations, which describe the evolution of a charged-particle spin in an
electromagnetic field.

1. INTRODUCTION

In our previous study [1], we developed a variational theory of a perfect spin fluid with an
intrinsic non-Abelian color charge in Riemann–Cartan space $U_4$ with curvature and torsion.
This theory takes into account spin-polarization phenomena and chromomagnetic effects.
In particular, we obtained the equations of motion for such a fluid and the rules that govern
the evolution of a color charge and of a spin tensor satisfying the Frenkel condition. We also
derived the expression for the energy-momentum tensor of the fluid. The objective of this
study is to deduce the equation of motion for our color fluid in the form of a generalized
hydrodynamic equation of the Euler type in $U_4$ space and to obtain implications of this
equation.

It is common knowledge that, in the theory of electromagnetic fields, the equations of
motion of charged particles follow from the law of energy-momentum conservation for the

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"particles + field" system and from field equations, the law of energy-momentum conservation being a consequence of the invariance of the physical system in question \[2\], \[3\]. In the general theory of relativity, the equations of motion of matter also follow from the equations for the gravitational field. The reason is that the Einstein equations imply the covariant conservation law for the matter energy-momentum. Conceptually, the same is true for field theory in \(U_4\) space, but the conservation laws have a more complicated structure in that case \[4\]. It was shown in \[5\] that, in field theories formulated in spaces with a more complicated structure (\(U_4\) space with Lagrangians quadratic in curvature and torsion and affine metric space with curvature, torsion, and nonmetricity), the equations of motion for matter are obtained from field equations and conservation laws in just the same way.

This article is organized as follows. In Section 2, the method for obtaining the equations of motion from conservation laws is used to derive the equation of motion for a perfect spin fluid having an intrinsic color charge and interacting with a color field and with a gravitational field associated with the geometry of \(U_4\) space. We employ the method of Cartan exterior forms and extended Lie derivatives that take into account internal gauge symmetries. In \[6\], this problem was solved by a different method. As a limiting case of the equation obtained in Section 2, the equation of motion of a particle with a spin and with a non-Abelian color charge in an external color field in \(U_4\) space is derived in Section 3. This equation, which takes into account spin-chromomagnetic effects, generalizes the well-known Wong equation \[7\] of motion of a particle with an isospin in a Yang–Mills field. In Section 4, the equation from Section 3 is used to obtain the equation that describes the motion of a colored-particle spin in \(U_4\) space and which generalizes the Bargmann–Michel–Telegdi and Tamm–Good equations \[8\], \[9\], \[6\].

In this study, we adopt the notation and conventions introduced in \[1\].

2. HYDRODYNAMIC EQUATION OF MOTION FOR A SPIN FLUID WITH AN INTRINSIC COLOR CHARGE

In Riemann–Cartan \(U_4\) space, we consider a perfect spin fluid with a non-Abelian color charge in a color field. The Lagrangian density of this system is given by

\[
\mathcal{L}_{\text{matter}} = \mathcal{L}_{\text{fluid}} + \mathcal{L}_{\text{field}},
\]

where the Lagrangian densities 4-forms \(\mathcal{L}_{\text{fluid}}\) and \(\mathcal{L}_{\text{field}}\) are presented in \[1\]. The invariance of this Lagrangian under general coordinate transformations and under local Lorentz transformations implies fulfillment of the set of differential identities \[4\]

\[
D\Sigma_a = (\bar{e}_a \lceil T^b \rceil) \wedge \Sigma_b - (\bar{e}_a \rceil R^c_b \lceil) \wedge \Delta^c_b,
\]

\[
D\Delta_{ab} = -\theta^{[a} \wedge \Sigma_{b]},
\]

where \(\wedge\) denotes exterior multiplication; \(\lceil\) denotes inner multiplication (contraction); \(\theta^a\) are basis 1-forms; \(D\) represents exterior covariant differentiation; \(T^b\) is the 2-form of torsion; \(R^c_b\) is the 2-form of the curvature of \(U_4\) space; \(\Sigma_a\) is the 3-form of the energy-momentum; \(\Delta_{ab}\) is the 3-form of the spin. Equation (2.2) follows from the invariance of the Lagrangian under spacetime diffeomorphisms – it generalizes the law of energy-momentum conservation – while equation (2.3) appears to be a consequence of the invariance of the
Lagrangian under local Lorentz transformations, generalizing the law of spin-momentum conservation.

If the equations of motion for the physical system being considered are satisfied in $U_4$ space, equations (2.2) and (2.3) hold identically. As a matter of fact, they offer an alternative representation of the equations of motion. We will now use relation (2.2) to obtain the equation that describes the motion of a perfect spin fluid with an intrinsic color charge and which has a form generalizing the hydrodynamic Euler equation. To do this, the expression obtained in [1] for the 3-form of the energy-momentum of the system in question {equation (5.9) in [1]} is substituted into equation (2.2). This 3-form can be represented as

$$
\Sigma_a = p(\tilde{e}_a | \eta) + n \left( \pi_a + \frac{p}{nc^2} u_a \right) u + \chi n(\tilde{e}_a | F^m) \wedge J_m * S + \frac{\alpha}{2} \left( (\tilde{e}_a | F^m) \wedge * F_m - F^m \wedge (\tilde{e}_a | * F_m) \right),
$$

(2.4)

where $\eta$ is the 4-form of volume; $n$ is the particle-number density of the fluid; $u$ is the 3-form of the particle velocity; $J_m$ is the color charge of a fluid element {see equation (2.4) in [1]}; $S = (1/2)S_{ab} \theta^a \wedge \theta^b$ is the 2-form of the spin of a fluid element ($S_{ab}$ is the spin tensor); and $\pi_a$ is the dynamical momentum of a fluid element. According to equation (5.7) from [1], this momentum is given by

$$
\pi_a = \frac{1}{c^2} \varepsilon^* u_a - \frac{1}{c^2} S_a c \left( \dot{u}_c + u^b (\chi J_m F^m_{bc} + \omega_{bc}) \right),
$$

(2.5)

$$
\varepsilon^* \eta = \frac{\varepsilon}{n} \eta + \chi J_m F^m \wedge * S,
$$

(2.6)

where $\varepsilon$ is the energy density of the fluid in the reference frame comoving with it, while $\varepsilon^*$ is the effective energy per a particle of the fluid [6]. In these formulas, an overdot denotes differentiation defined for an arbitrary tensor $\Phi$ as

$$
\dot{\Phi}^a_b := \dot{\Phi}^a_b.
$$

(2.7)

In order that the Bianchi identities $D F^m = 0$ for the 2-form of the strength tensor of the color field could be used explicitly, the operation of exterior covariant differentiation $D$ must be defined with respect to the two gauge groups [Lorentz group and $SU(3)$ color group] according to equation (3.2) from [1]. In calculating covariant derivatives of contractions, this implies the use of the Lie derivative $L_{\tilde{v}}$ generalized to arbitrary local gauge groups [including the $SU(3)$ color group]; that is,

$$
D \circ i_{\tilde{v}} = L_{\tilde{v}} - i_{\tilde{v}} \circ D,
$$

(2.8)

where $i_{\tilde{v}}$ is the operator of inner multiplication with respect to the vector $\tilde{v}$. The calculations must rely on the continuity equation $d(nu) = 0$; on the relation

$$
D(\chi n J_m * S) = n J_m u - D(\alpha * F_m),
$$

(2.9)

which follows from the equation for the color gauge field {see equation (4.3) in [1]}; and on the equations

$$
D(\tilde{e}_a | \eta) = D \eta_a = (\tilde{e}_a | T^b) \wedge \eta_b.
$$

(2.10)

which are valid in $U_4$ space.
In the calculations, there arises the differential operator

\[ \mathcal{L}_{\vec{e}_a} F^m - (\vec{e}_a \mathcal{T}^b) \wedge (\vec{e}_b \mathcal{F}^m) =: (\vec{e}_a \nabla) \mathcal{F}^m, \]

which represents the total covariant derivative of tensor-valued forms and which takes into account all the types of indices [spinorial and tensorial indices associated with the representations of the Lorentz and SU(3) groups, as well as indices associated with the space of p-forms]. It can be shown that the operator in (2.11) satisfies the condition

\[ * (\vec{e}_a \nabla) \mathcal{F}^m = (\vec{e}_a \nabla) F^m. \]

Substituting (2.4) into (2.2), using equations (2.8)–(2.12), and considering that \{see equation (5.1) in [1]\} the expression for the 3-form of the spin momentum is

\[ \Delta_{ab} = \frac{1}{2} S_{ab} u, \]

we arrive at the equation of motion for the fluid in the form

\[ u \wedge D (\pi_a + \frac{p}{nc^2} u_b) = \frac{1}{n} \eta (\vec{e}_a \nabla) p - (\vec{e}_a \mathcal{T}^b) \wedge (\pi_b + \frac{p}{nc^2} u_b) u - \frac{1}{2} (\vec{e}_a \mathcal{R}^{bc}) \wedge S_{bc} u - (\vec{e}_a \mathcal{F}^m) \wedge J_m u + \chi (\vec{e}_a \nabla) \mathcal{F}^m \wedge J_m * S, \]

which represents a generalization of the well-known hydrodynamic Euler equation to the case of a perfect spin fluid with a color charge.

### 3. EQUATIONS OF MOTION OF A PARTICLE WITH A SPIN AND A COLOR CHARGE

By going over to the limit of zero pressure in equation (2.14), we find that, in \( U_4 \) space, the equation of motion of a particle with a spin and a color charge in an external non-Abelian color gauge field has the form

\[ u \wedge D \pi_a = - (\vec{e}_a \mathcal{F}^m) \wedge J_m u + \chi (\vec{e}_a \nabla) \mathcal{F}^m \wedge J_m * S - \frac{1}{2} (\vec{e}_a \mathcal{R}^{bc}) \wedge S_{bc} u - (\vec{e}_a \mathcal{T}^b) \wedge \pi_b. \]

The first term on the right-hand side of this equation is a generalization of the Lorentz force to the case of a non-Abelian gauge field. The second term is a chromo-magnetic analog of the Stern–Gerlach force acting on a magnetic moment in an electromagnetic field (this force is generated by the additional potential energy of a magnetic moment in a magnetic field \[9\]). The third term on the right-hand side of equation (3.1) represents the Mathisson force arising from the interaction of the particle spin with the curvature of space, while the fourth term is the so-called translational force, which is due to the interaction of the particle dynamical momentum with the torsion of space. The emergence of this force is peculiar to \( U_4 \) space.

Equation (3.1) must be supplemented with the law that governs spin evolution \{see equation (3.18) in [1]\},

\[ u \wedge D S_{ab} = 2 \pi_{[a} u_{b]} \eta - 2 S_{[c} (\chi \eta_{0} c) \wedge \mathcal{F}^m J_m + \omega_{[c} \eta) \],

which is the equation of motion for the spin.

4
and with the law that governs the evolution of the particle color charge \{see equation (4.4) in [1]\},

\[ u \land D J_m = -c_{m^p} J_p (\chi F^m \land *S + \omega^n \eta) , \tag{3.3} \]

where \( \eta_{bc} = \varepsilon_c^b \eta = \varepsilon_c^b \varepsilon_b^c \eta = *(\theta_b \land \theta_c) \), and \( c_{m^p} \) are the structure constants of the \( SU(3) \) group.

Equations (3.1)–(3.3) describe the motion of a particle with a spin and a color charge in the presence of spin-chromomagnetic interaction. They generalize the well-known Wong equations \([7]\) to the case of the \( SU(3) \) color group and take into account the particle spin, which may be responsible for the possible interaction between the spin and the chromo-
magnetic component of the color field and for the additional effect of the gravitational field associated with the geometry of Riemann–Cartan space on the motion of the particle being considered.

4. EVOLUTION OF THE PARTICLE SPIN IN A COLOR FIELD IN RIEMANN–CARTAN SPACE

The particle-spin vector (Tamm–Pauli–Lyubanski vector) is defined as

\[ \sigma^a := \frac{1}{2c^2} \eta^{abcd} S_{bc} u_d \, , \quad \sigma := \sqrt{\sigma^a \sigma_a} \tag{4.1} \]

where \( \eta^{abcd} \) are the components of the Levi-Civita antisymmetric tensor (4-form of volume \( \eta \)). In Minkowski spacetime, the evolution of this vector in a slowly varying external electromagnetic field is governed by the Bargmann–Michel–Telegdi equation \([8]\). However, this equation takes no account of the effect of the spin on the trajectory of the particle.

A more precise equation that describes the motion of the particle spin in a non-uniform electromagnetic field and which takes into account the effect of the spin on the motion of the particle was derived by Good who extended the Tamm equation (see \([8, 9]\)).

By using equations (3.1) and (3.2), which describe the motion of a particle with a spin and a color charge, we will now extend the Tamm–Good and Bargmann–Michel–Telegdi equations to the case of the motion of such a particle in an external color (generally nonuniform) field in \( U_4 \) Riemann–Cartan space \([6]\). To this end, we recast the dynamical momentum of the particle [it is given by (2.5)] into the form

\[ \pi_a = m^* u_a - \lambda_a \, , \quad m^* := \frac{\varepsilon^*}{c^2} \, , \quad \lambda_a := \frac{1}{c^2} S^c a w_c \, , \quad w_c := \dot{u}_c + u^b (\chi J_m F^m F_{bc} + \omega_{bc}) \tag{4.2} \]

Using definitions (4.1) and (4.3) and considering that the spin tensor, which is the inverse of that in (4.1), is given by

\[ S^a b = \frac{1}{c^2} \eta_{abcd} \sigma^c u^d \tag{4.4} \]

we can easily prove that the vectors \( u^a, \sigma^a \), and \( \lambda^a \) are mutually orthogonal.

Differentiating the spin vector (4.1), we arrive at

\[ u \land D \sigma^a = \frac{1}{m^* c^2} u^a \sigma^b u \land D (\pi_b - \lambda_b) + \sigma^b \Pi^{ac} (\chi \eta_{bc} \land \mathcal{F}^m J_m + \omega_{bc} \eta) \tag{4.5} \]
where $\Pi^{ac} = \rho^{ac} + (1/2c^2)u^au^c$ is the projection tensor. To derive this relation, we used equation (3.2), which governs the evolution of the spin tensor; the Frenkel condition $S_{ab}u^b = 0$; expression (4.1) for the spin tensor in terms of the spin vector; the prescription $\eta_{abcd}\eta^{ijkl}=-6\delta^i_{[a}\delta^j_{b}\delta^k_{c]}\delta^l_{d]}$ for contracting indices of the Levi–Civita tensor; and the result obtained by differentiating expression (4.2) for the dynamical momentum,

$$m^*u \wedge Du_a = u \wedge D(\pi_a + \lambda_a) + u_a u \wedge Dm^*.$$  \hspace{1cm} (4.6)

Differentiating expression (4.3) for $\lambda_a$ and taking into account equation (3.2) and the orthogonality conditions $\sigma^a u_a = 0$, $\sigma^a \pi_a = 0$ and $\sigma^a S_{ab} = 0$, we obtain

$$\sigma^a u \wedge D\lambda_a = \lambda^a \sigma^b(\chi \eta_{ab} \wedge F^m J_m + \omega_{ab} \eta).$$  \hspace{1cm} (4.7)

Substituting this equation and the equation (3.1) of the motion of a particle into (4.5) and taking into account equation (4.2), we arrive at

$$u \wedge D\sigma^a = -\sigma^b(\chi \eta^a_{b} \wedge F^m J_m + \omega^a_{b} \eta) -$$

$$-\frac{u^a \sigma^b}{m^*c^2}[\left(\epsilon_{b} \right)|F^m \wedge J_m u - \pi^c(\chi \eta_{bc} \wedge F^m J_m + \omega_{bc} \eta) -$$

$$- \chi(\epsilon_{b} \nabla)F^m \wedge J_m *S + \frac{1}{2}(\epsilon_{b} |R^{cd}) \wedge S_{cd} u + (\epsilon_{b} |T^c) \wedge \pi_c u].$$  \hspace{1cm} (4.8)

Equation (4.8) represents the sought extension of the Tamm–Good and Bargmann–Michel–Telegdi equations to the case of a particle having a spin and a color charge and moving in an external nonuniform non-Abelian color field in Riemann–Cartan space.

5. CONCLUSION

On the basis of the method of Cartan exterior forms and generalized Lie derivatives, a hydrodynamic equation of the Euler type that describes the motion of an element of a perfect spin fluid with a color charge in a non-Abelian color gauge field associated with the SU(3) group and a gravitational field described by the geometry of Riemann–Cartan space with curvature and torsion has been derived from the law of energy-momentum quasiconservation. The ”fluid + field” system being considered is closed because the gravitational field is generated by the energy-momentum tensor of the fluid and because the color field is induced by the current of non-Abelian color charge of the fluid.

From the resulting hydrodynamic equation, we have deduced an equation that describes the motion of a particle with a spin and a color charge. We have shown that the forces acting on such a particle represent generalizations of forces of well-known types. These are a Lorentz force acting on the non-Abelian color charge; the Stern–Gerlach force, which is proportional to the gradient of the color field; the Mathisson force resulting from the interaction between the particle spin and the curvature of spacetime; and the force of the translational type caused by the interaction between the particle momentum and the torsion of spacetime. It is noteworthy that forces analogous to the Mathisson force and to the translational force appear in modern gauge theories of plasticity that employ gauge groups to describe defects in crystals, disclinations in the case of the gauge group of rotations and dislocations in the case of the gauge group of translations [10], [11].

Finally, the equation of motion of a particle has been used to deduce an equation that describes the evolution of the particle-spin vector in external color and gravitational
(with curvature and torsion) fields and which generalizes, to this case, the known equations that describe motion of the vector of the charged-particle spin in an electromagnetic field (Bargmann–Michel–Telegdi equation for a uniform field and Tamm–Good equation for a nonuniform field) [8], [9], [6].

It is appropriate to compare our equations of motion of a classical particle and of a classical spin vector with the equations representing the limiting case of the corresponding quantum equations obtained by using QED methods. In QED, neither the Bargmann–Michel–Telegdi equation nor the equation of motion of a particle (Lorentz equation) involves gradient terms in the original forms of these equations. That there are no such terms is due to the use of the procedure that derives these equations for constant fields. Nevertheless, terms of this type arise in the equations of motion if a consistent procedure based on the Maslov method [12] and on its ensuing development [13] is invoked to deduce the classical equations of motion for particles and the equations of motion for spin and isospin vectors from the quantum Dirac equation with allowance for external fields [14], [15]. Therefore, these terms make it possible to demonstrate nontrivial quantum effects at the classical level.

Note that, apart from terms involving field gradients, our equations of motion for a fluid element coincide with the equations of motion that were obtained by the path-integral method [16]. They also coincide in form with the corresponding equations of motion that follow from the string action functional [17]. This suggests that our equations of motion may prove to be valid not only for a fluid element but also for extended objects like strings. This gives reason to hope that the Weyssenhoff–Raabe model of a fluid can be extended to nonlocal objects and that some additional analogous structures can be recognized in doing this. An important point is that the Weyssenhoff-Raabe model is a classical model of a fluid, but it describes the properties of a second-quantized fermion field rather than of Dirac electrons associated with a nonquantized spinor field, as is often erroneously assumed. This property of the Weyssenhoff–Raabe model stems from the well-known fact that the energy of the spinor field becomes positive definite only after Fermi–Dirac second quantization.

To develop this theory further, it is advisable to study the effect of a dilaton field on the motion of a particle with a spin and a color charge. For this, a particle must be endowed with a dilaton charge, and the theory developed in [18] to describe the motion of a particle with a spin and a dilaton charge in Riemann–Cartan and Weyl–Cartan spaces must be taken into consideration. A self-consistent theory constructed in this way can be used to study a nonperturbative gluon condensate up to the point of the phase transition from hadron matter to quark-gluon plasma.

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