The obstructions for toroidal graphs with no $K_{3,3}$’s

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Abstract

Forbidden minors and subdivisions for toroidal graphs are numerous. We consider the toroidal graphs with no $K_{3,3}$-subdivisions that coincide with the toroidal graphs with no $K_{3,3}$-minors. These graphs admit a unique decomposition into planar components and have short lists of obstructions. We provide the complete lists of four forbidden minors and eleven forbidden subdivisions for the toroidal graphs with no $K_{3,3}$’s and prove that the lists are sufficient.

Key words: toroidal graph, embedding in a surface, forbidden minor, forbidden subdivision

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1 Introduction

We use basic graph-theoretic terminology from Bondy and Murty [6] and Diestel [8]. An undirected graph $G$ consists of a set $V$ of vertices and a set $E$ of edges where each edge is associated with an unordered pair $uv$ of vertices in $V$. A graph $G$ is embeddable in a surface if it can be drawn on the surface with no crossing edges. A topological obstruction for a surface is a graph $G$ with minimum vertex degree three which does not embed on the surface, but $G - e$ is embeddable for any edge $e$. A minor order obstruction is a topological obstruction with the additional property that $G$ with edge $e$ contracted embeds on the surface for all edges $e$.

The purpose of this paper is to characterize all the minor order and topological obstructions for the torus which do not contain a subgraph homeomorphic (homeomorphic means the edges in the figure correspond to paths in the graph) to $K_{3,3}$ pictured in Figure 1. If $H$ is a graph with minimum vertex degree three, then following Diestel [8], an $H$-subdivision denoted by $TH$ is a graph homeomorphic to $H$.

Kuratowski’s theorem [14] states that a graph $G$ is non-planar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or $K_5$ (see Figure 1). Wagner [20] proved that a graph $G$ is non-planar if and only if it contains $K_{3,3}$ or $K_5$ as a minor. We concentrate on the graphs with no $K_{3,3}$-subdivisions. Since $K_{3,3}$ is 3-regular, it is possible to see that a graph does not contain any $K_{3,3}$-subdivision if and only if it does not contain any $K_{3,3}$-minor. Therefore we refer to these graphs as to graphs with no $K_{3,3}$’s.

Wagner [19] and Kelmans [13] gave a recursive decomposition for all non-planar graphs with no $K_{3,3}$’s. Gagarin, Labelle and Leroux [11] explicitly describe the structure of toroidal graphs with no $K_{3,3}$’s in terms of 2-pole planar networks substituted into the edges of non-planar core graphs. The 2-pole planar networks arise from 2-connected planar graphs.

For any surface of fixed genus, Robertson and Seymour theory in [17] indicates that the number of obstructions (topological or minor order) is finite. A proof specifically for orientable surfaces has been independently provided by

![Fig. 1. Minimal non-planar graphs $K_{3,3}$ and $K_5$](image)
Vollmerhaus [18] and Bodendiek and Wagner [5] and for nonorientable surfaces by Archdeacon and Huneke [3]. For the projective plane, there are 103 topological obstructions and 35 minor order obstructions. Glover, Huneke and Wang first proved that these were obstructions [12]. Archdeacon later proved that their list was complete [1].

The torus is a surface shaped like a doughnut. The precise complete lists of obstructions for the torus (both minor order and topological) are still unknown. To date, Myrvold and Chambers [7] have found 239,322 topological obstructions and 16,629 minor order obstructions that include those on up to eleven vertices, the 3-regular ones on up to 24 vertices, the disconnected ones and those with a cut-vertex. Previously, only a few thousand had been determined.

The spindle surface is formed from the sphere by identifying two points that also can be seen as pinching a torus. The two points are usually called the north and south poles, and after identification it is called the pinch point. The spindle surface is a pseudosurface: no neighborhood of the pinch point is homeomorphic to an open disk.

The set of minor order and topological obstructions for the spindle surface are not known either. In general, the set of graphs that embed on a pseudosurface can be not closed under the minor order, and there are pseudosurfaces which have an infinite set of obstructions. However, graphs that embed on the spindle surface are closed under the minor order, and hence there is a finite obstruction set. The topological obstructions for cubic graphs on the spindle surface are found by Archdeacon and Bonnington [2].

Section 2 of the present paper describes the structural and algorithmic results of Gagarin and Kocay [10] for graphs containing $K_5$-subdivisions with respect to their embeddability in the torus. In Section 3, we show the lists of four minor order and eleven topological order obstructions and prove they are sufficient for toroidal graphs with no $K_{3,3}$’s. Section 4 contains suggestions for future research on obstructions for surfaces.

2 $K_5$-Subdivisions and toroidality

In this section, we summarize known structural results for graphs containing a $K_5$-subdivision and the corresponding algorithmic results for the torus. A $K_5$-subdivision is denoted by $TK_5$, and a $K_{3,3}$-subdivision is denoted by $TK_{3,3}$. For a graph $H$, the vertices of the $TH$ which have degree three or more are called the corner vertices of the $TH$. 
Let $G$ be a graph and $H$ be a subgraph of $G$. A bridge of $G$ with respect to $H$ is either (a) an edge of $G$ which is not in $H$, but both its endpoints are in $H$ (plus the endpoints), or (b) a connected component $C$ of the graph $G - V(H)$ together with edges connecting a vertex in $C$ to a vertex in $H$ (and their endpoints). Denote by $K$ the set of corner vertices of a $TK_5$. The following theorem is implicit in the work of Asano [4], and has also been stated and used by Fellows and Kaschube [9] and Gagarin and Kocay [10].

**Theorem 1** Let $G$ be a 2-connected graph with a $TK_5$. If a bridge of the set of corner vertices $K$ of the $TK_5$ contains three or more vertices from $K$, then $G$ contains a $TK_{3,3}$. Otherwise, $G$ is decomposable into a collection of bridges of $K$ such that each bridge contains exactly two corner vertices.

Given a $TK_5$ in a graph $G$ such that all of the bridges of $K$ contain exactly two corner vertices, the side component of the $TK_5$ in $G$ corresponding to the pair $a, b$ of corner vertices has its vertex set equal to the union of the vertex sets of the bridges containing these two corners, and the edge set is the union of the edges of the corresponding bridges. If $F$ is a side component corresponding to the corners $a$ and $b$, then the augmented side component is $F$ if the edge $ab$ is already in $F$ and $F + ab$ otherwise. A side component $F$ is special if the edge $ab$ is not in $F$, $F$ is planar, but the augmented side component $F + ab$ is not planar.

Side components of a subdivision of an $M$-graph shown in Figure 2 are defined by analogy with the side components of a $K_5$-subdivision by considering pairs of adjacent vertices of the $M$-graph. The edge $xy$ of an $M$-graph with both endpoints of degree seven is called the central edge of the $M$-graph (see Figure 2).

The graph $K_5$ has six different embeddings in the torus as shown in Figure 3 (the torus is represented as a rectangle with opposite sides identified). To determine if a graph $G$ which has no $TK_{3,3}$'s embeds in the torus, it suffices to examine the ten side components of $TK_5$ in $G$. By analyzing the six embeddings of $K_5$ on the torus, Gagarin and Kocay [10] have proven structural results which imply the following theorem.

**Theorem 2 (Gagarin and Kocay [10])** A graph $G$ with a $TK_5$ but no $TK_{3,3}$'s is toroidal if and only if
Fig. 3. Embeddings of $K_5$ in the torus.

(i) all of the augmented side components of the $TK_5$ are planar, or

(ii) nine of the augmented side components of the $TK_5$ are planar and the remaining side component $S$ is special, or

(iii) $G$ contains a $TM$ where $M$ is as pictured in Figure 2 and all of the augmented side components of the $TM$ are planar graphs.

The special side component $S$ of Theorem 2(ii) must be embedded in a cylinder provided by the face $F$ of embeddings $E_1$ and $E_2$, or, in some particular cases, by the face $F$ of embedding $E_3$, shown in Figure 3. Details can be found in [10].

3 Kuratowski theorem for the toroidal graphs with no $K_{3,3}$’s

In this section we prove the characterization of the toroidal graphs with no $K_{3,3}$’s in terms of forbidden minors. Also we show the corresponding list of eleven topological obstructions.

**Theorem 3** A graph $G$ with no $K_{3,3}$’s is toroidal if and only if $G$ does not contain any of $G_1$, $G_2$, $G_3$ or $G_4$ of Figure 4 as a minor.

**Proof** (Necessity). The program from [16] was used to check that none of the graphs $G_1$, $G_2$, $G_3$, or $G_4$ of Figure 4 is toroidal and also that deleting or contracting any edge of $G_1$, $G_2$, $G_3$, or $G_4$ results in a toroidal graph, i.e. $G_1$, $G_2$, $G_3$ and $G_4$ are non-toroidal minor order minimal. This can also be
Fig. 4. *The minor-minimal non-toroidal graphs* \( G_1, G_2, G_3 \) and \( G_4 \) containing no \( K_{3,3} \)'s.

verified by hand for \( G_1, G_2 \) and \( G_3 \) without too much difficulty (see Lemma 4.4 in [10]).

Graph \( G_4 \) of Figure 4 can be considered as obtained from the \( M \)-graph of Figure 2 by substituting side component \( K_5 - e \) for the central edge: the endpoints \( u \) and \( v \) of the deleted edge \( e = uv \) are identified respectively with the end-points \( x \) and \( y \) of the central edge \( xy \) of the \( M \)-graph. Clearly, the corresponding augmented side component is \( K_5 \) which is non-planar, and, by Theorem 2(iii), the whole graph \( G_4 \) is not toroidal. Deletion or contraction of any edge \( e \) of \( G_4 \) results in an \( M \)-graph subdivision with all augmented side components planar. Therefore, by Theorem 2(iii), \( G_4 \) is a minor order minimal non-toroidal graph.

Since none of \( G_1, G_2, G_3, \) or \( G_4 \) is toroidal, it is obvious that any graph which contains one of them as a minor is not toroidal. Therefore we need to prove that these are the only minor order minimal obstructions.

*(Sufficiency).* If \( v \) is a cut-vertex of \( G \), then Miller [15] has proven that the genus of \( G \) equals the sum of the genera of the bridges of \( G \) with respect to \( v \). This implies that a graph \( G \) is toroidal if and only if it contains at most one 2-connected non-planar toroidal component and all the other 2-connected components of \( G \) are planar. Suppose \( G \) does not contain any of \( G_1, G_2, G_3, G_4, \) or \( K_{3,3} \) as a minor. Since \( G \) has no \( K_{3,3} \)-minors, \( G \) does not contain any \( K_{3,3} \)-subdivision as a subgraph. It is necessary to prove that \( G \) is toroidal.

Since \( G \) does not contain minors \( G_1, G_2, \) or \( K_{3,3} \), \( G \) can contain at most one non-planar 2-connected component. If all the 2-connected components of \( G \) are planar, \( G \) is toroidal. Therefore suppose \( G \) is 2-connected and non-planar. Since \( G \) does not contain a \( K_{3,3} \)-subdivision, it contains a \( K_5 \)-subdivision \( TK_5 \).
such that (by Theorem 1) each bridge of the set of corner vertices \( K \) of \( TK_5 \)
contains exactly two corner vertices from \( K \), and it is possible to decompose \( G \) into the side components of a \( TK_5 \).

Since \( G \) does not contain minors \( G_1 \), or \( G_2 \), or \( K_{3,3} \), at most one side component, say \( H \), of \( TK_5 \) in \( G \) can be non-planar. Figure 5 shows why there can be no pair of non-planar augmented side components of \( TK_5 \) in \( G \): otherwise the side components \( H_1, H_2 \) and paths \( ae, eb, de, ec \) in \( G \) of Figure 5(a) would contain a forbidden minor \( G_2 \), or the side components \( H_1, H_2 \) and paths \( bd, da, ae, ec \) in \( G \) of Figure 5(b) would contain a forbidden minor \( G_2 \). The cases of other pairs of side components of \( TK_5 \) in \( G \) are equivalent to the two cases of Figure 5 by symmetry. Therefore there is at most one non-planar augmented side component \( H' \) of \( TK_5 \) in \( G \). If \( H' \) is planar, by Theorem 2(i), \( G \) is toroidal. Suppose \( H' \) is the unique non-planar augmented side component of \( TK_5 \). Reasoning similar to that shown in Figure 5 demonstrates that the non-planar augmented side component \( H' \) must be obtained from the planar or non-planar side component \( H \). If \( H \) is planar, by Theorem 2(ii), \( G \) is toroidal.

![Fig. 5. Two side components \( H_1 \) and \( H_2 \) of \( TK_5 \) in \( G \).](image)

Suppose \( H \) is the unique non-planar side component and \( TK'_5 \) is a \( K_5 \)-subdivision in \( H \). Because \( G \) does not contain minors \( G_1, G_2 \) and \( G_3 \), the two subdivisions \( TK_5 \) and \( TK'_5 \) must share two common corners, say \( x \) and \( y \). Therefore \( G \) contains a subdivision \( TM \) of the \( M \)-graph (a detailed proof of this fact can be found in Theorem 4.7 of [10]).

Now, because \( G \) has no \( K_{3,3} \)-subdivision, we can decompose \( G \) into the side components of the \( M \)-graph subdivision \( TM \). Since \( G \) does not contain minors \( G_1, G_2 \) and \( K_{3,3} \), only the side component \( H_{xy} \) of \( TM \) in \( G \) that corresponds to the central edge \( xy \) of \( M \)-graph can be non-planar (see Figure 2): all the other side components of \( TM \) in \( G \) must be planar even when augmented.

Since \( G \) does not contain minors \( G_1, G_2, G_3, \) or \( K_{3,3} \), the augmented side component \( H_{xy} + xy \) of \( TM \) can contain just a \( K_5 \)-subdivision \( TK''_5 \) that shares the same two common corners \( x \) and \( y \) with \( TK_5 \) and \( TK'_5 \) in \( G \). However, because \( G \) does not contain minor \( G_4 \), the side component \( H_{xy} \) of \( TM \) in \( G \) must be planar even when augmented, and, by Theorem 2(iii), the whole graph \( G \) is toroidal. □
The remaining seven topological obstructions that are not minor order obstructions are depicted in Figure 6. They were obtained by splitting the vertices of $G_1, G_2, G_3,$ or $G_4$ in all possible ways by the first author, and were also determined as a part of a much larger project done by Chambers and Myrvold [7].

![Fig. 6. Topological obstructions $G_5, G_6, \ldots, G_{11}$ for the torus.]

4 Final Remarks

The results in this paper completely characterize the torus obstructions for all graphs with no $K_{3,3}$’s. Characterizing the complete set of torus obstructions seems to be a very difficult task. Such a characterization could help in finding a better understanding of the torus and, perhaps, the surfaces of higher genus. The obstructions for the toroidal graphs with no $K_{3,3}$’s suggest that a feasible approach to characterizing the complete set of torus obstructions would be to break the set into tractable subclasses.

Note that the torus can also be represented as a hexagon with its three pairs of opposite sides identified. This representation may help give a better understanding of the toroidal graphs as well. For example, the unique embedding of $K_7$ on the torus is very complicated in the rectangular representation of the torus. However there exists a simple symmetric drawing of $K_7$ on the torus represented as a hexagon (see Figure 7). The drawing of Figure 7 was found by the first author.
After the torus, the next most likely avenues of attack are determining complete obstruction sets for the Klein bottle, the spindle surface and the two-holed torus. Since a characterization in terms of forbidden minors and subdivisions can be too complicated, one may try to find a characterization of graphs on these surfaces in terms of forbidden substructures of a different kind.

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