Hadronic Light-by-Light Contribution to Muon $g - 2$: Status and Prospects $^1$

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Abstract

I review the recent calculations and present status of the hadronic light-by-light contribution to muon $g - 2$.

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1 Introduction

Here, I discuss the contribution to the muon $g - 2$ of a hadronic bubble connected to the external static magnetic source through one photon leg and to the muon line with another three photon legs. This corresponds to the so-called hadronic light-by-light contribution to the muon anomaly $a_\mu = (g - 2)/2$. Recent reviews are in [1, 2]. One of the six possible photon momenta configurations is shown in Fig. 1 and its contribution to the vertex $-|e| \bar{u}(p') \Gamma^\beta(p - p') u(p) A_\beta$ is

$$
\Gamma^\beta(p_3) = -e^6 \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \frac{\Pi^{\rho\alpha\beta}(p_1, p_2, p_3)}{q^2 p_1^\rho p_2^\alpha} \gamma_\alpha(p_4 - m)^{-1} \gamma_\nu(p_5 - m)^{-1} \gamma_\rho
$$

where $p_3 \to 0$ is the momentum of the photon that couples to the external magnetic source, $q = p_1 + p_2 + p_3$ and $m$ is the muon mass. The dominant contribution to the hadronic four-point function

$$
\Pi^{\rho\alpha\beta}(p_1, p_2, p_3) = -p_3 \lambda \frac{\partial \Pi^{\rho\alpha\lambda}(p_1, p_2, p_3)}{\partial p_3^\beta} \Big|_{p_3=0}
$$

and therefore one just needs derivatives of the four-point function at $p_3 = 0$. The contribution to $a_\mu$ is

$$
a_\mu^{\text{hl}} = \frac{1}{48m} \text{tr} \left\{ (\not p + m) [\gamma^\beta, \gamma^\lambda] (\not p + m) \frac{\partial \Gamma_\lambda(0)}{\partial p_3^\beta} \right\}.
$$
The four-point function $\Pi^{\rho\alpha\beta}(p_1, p_2, p_3)$ is an extremely difficult object involving many scales and no full first principle calculation of it has been reported yet. Notice that we need momenta $p_1$ and $p_2$ varying from 0 to $\infty$. Unfortunately, there is neither a direct connection of $a^\text{hl}_\mu$ to a measurable quantity. Two lattice groups have started exploratory calculations [3, 4] but the final uncertainty that these calculations can reach is not clear yet.

Attending to a combined large number of colors $N_c$ of QCD and chiral perturbation theory (CHPT) counting one can distinguish four types of contributions [5]. Notice that the CHPT counting is only for organization of the contributions and refers to the lowest order term contributing in each case. The four different types of contributions are:

- Goldstone boson exchanges contribution are $\mathcal{O}(N_c)$ and start at $\mathcal{O}(p^6)$ in CHPT.
- One-meson irreducible vertex contribution and non-Goldstone boson exchanges contribute also at $\mathcal{O}(N_c)$ but start contributing at $\mathcal{O}(p^8)$ in CHPT.
- One-loop of Goldstone bosons contribution are $\mathcal{O}(1/N_c)$ and start at $\mathcal{O}(p^4)$ in CHPT.
- One-loop of non-Goldstone boson contributions which are $\mathcal{O}(1/N_c)$ but start contributing at $\mathcal{O}(p^8)$ in CHPT.

Based on the counting above there are two full calculations [6, 7] and [8,9]. There is also a detailed study of the $\pi^0$ exchange contribution [10] putting emphasis in obtaining analytical expressions for this part.

Using operator product expansion (OPE) in QCD, the authors of [11] pointed out a new short-distance constraint of the reduced full four-point Green function

$$\langle 0| T[V^\nu(p_1)V^\alpha(p_2)V^\rho(-(p_1 + p_2 + p_3))] |\gamma(p_3)\rangle$$

when $p_3 \to 0$ and in the special momenta configuration $-p_1^2 \simeq -p_2^2 \gg -(p_1 + p_2)^2$ Euclidean and large. See also [12]. This short distance constraint was not explicitly imposed in previous calculations.

# 2 Leading in the $1/N_c$ Expansion Contribution

Using effective field theory techniques, the authors of [13] shown that leading contribution to $a^\text{hl}_\mu$ contains a term enhanced by a $\log^2(\mu/m)$ factor where $\mu$ is an ultraviolet scale and the muon mass $m$ provides the infrared scale. This leading logarithm is generated by the Goldstone boson exchange contributions and is fixed by the Wess–Zumino–Witten (WZW) vertex $\pi^0\gamma\gamma$. In the chiral limit where quark masses are neglected and at large $N_c$, the coefficient of this double
Table 1: Results for the $\pi^0$, $\eta$ and $\eta'$ exchange contributions.

| Reference | $10^{10} \times a_\mu$ | $\pi^0$ only | $\pi^0$, $\eta$ and $\eta'$ |
|-----------|------------------------|--------------|-------------------------------|
| [6, 7]    | 5.7                    | 8.3 $\pm$ 0.6 |                              |
| [8, 9]    | 5.6                    | 8.5 $\pm$ 1.3 |                              |
| [10] with $h_2 = 0$ | 5.8                    | 8.3 $\pm$ 1.2 |                              |
| [10] with $h_2 = -10$ GeV$^2$ | 6.3                    |              |                              |
| [16]      | 6.3 $\sim$ 6.7        |              |                              |
| [11]      | 7.65                   | 11.4$\pm$1.0 |                              |

logarithm is model independent and has been calculated and shown to be positive in [13]. All the calculations we discuss here agree with these leading behaviour and its coefficient including the sign. A global sign mistake in the $\pi^0$ exchange in [6, 8] was found by [10, 13] and confirmed by [7, 9] and by others [14, 15]. The subleading $\mu$-dependent terms [13], namely, $\log(\mu/m)$ and a non-logarithmic term $\kappa(\mu)$, are model dependent and calculations of them are implicit in the results presented in [6–9, 11]. In particular, $\kappa(\mu)$ contains the large $N_c$ contributions from the one-meson irreducible vertex and non-Goldstone boson exchanges. In the next section we review the recent model calculations of the full leading in the $1/N_c$ expansion contributions.

2.1 Model Calculations

The $\pi^0$ exchange contribution was calculated in [6–10, 16] by constructing the relevant four-point function in terms of the off-shell $\pi^0\gamma^*(p_1)\gamma^*(p_2)$ form factor $\mathcal{F}(p_1^2, p_2^2)$ and the off-shell $\pi^0\gamma^*(q)\gamma(p_3 = 0)$ form factor $\mathcal{F}(q^2, 0)$ modulating each a WZW $\pi^0\gamma\gamma$ vertex. In all cases several short-distance QCD constraints were imposed on these form-factors. In particular, they all have the correct QCD short-distance behaviour

$$\mathcal{F}(Q^2, Q^2) \to \frac{A}{Q^2} \quad \text{and} \quad \mathcal{F}(Q^2, 0) \to \frac{B}{Q^2}$$

(6)

when $Q^2$ is Euclidean and are in agreement with $\pi^0\gamma^*\gamma$ data. They differ slightly in shape due to the different model assumptions (VMD, ENJL, Large $N_c$, $N_\chi QM$) but they produce small numerical differences always compatible within quoted uncertainty $\sim 1 \times 10^{-10}$ –see Table 1.

Within the models used in [6–10, 16], to get the full contribution at leading in $1/N_c$ one needs to add the one-meson irreducible vertex contribution and the non-Goldstone boson exchanges. In particular, in [8, 9] the one-meson irreducible vertex contribution below some scale $\Lambda$ was identified with the ENJL quark loop contribution while a loop of a heavy quark with mass $\Lambda$ was used to mimic the
Table 2: Sum of the short- and long-distance quark loop contributions [8] as a function of the matching scale Λ.

| Λ [GeV] | 0.7 | 1.0 | 2.0 | 4.0 |
|---------|-----|-----|-----|-----|
| $10^{10} \times a_\mu$ | 2.2 | 2.0 | 1.9 | 2.0 |

Table 3: Results for the axial-vector exchange contributions from [6, 7] and [8, 9].

| References | $10^{10} \times a_\mu$ |
|------------|------------------------|
| [6, 7]     | 0.17 ± 0.10            |
| [8, 9]     | 0.25 ± 0.10            |

correlation massless QCD quark loop above Λ. The results are in Table 2 where one can see a very nice stability region when Λ is in the interval [0.7, 4.0] GeV. Within the ENJL model, the ENJL quark loop is related through Ward identities to the scalar exchange which we discuss below and both have to be included [8, 9]. Similar results for the quark loop below Λ were obtained in [6, 7] though these authors didn’t discuss the short-distance long-distance matching.

The exchange of axial-vectors and scalars in nonet symmetry –this symmetry is exact in the large $N_c$ limit, was also included in [8, 9] while only the axial-vector exchange was included in [6, 7]. The result of the scalar exchange obtained in [8] is

$$a_\mu(\text{Scalar}) = -(0.7 \pm 0.2) \times 10^{-10}.$$  \hspace{1cm} (7)

The result of the axial-vector exchanges in [6, 7] and [8, 9] can be found in Table 3.

Melnikov and Vainshtein used a model that saturates the hadronic four-point function in (2) at leading order (LO) in the $1/N_c$ expansion with $\pi^0$ and axial-vector exchanges. In that model, the new OPE constraint of the reduced four-point function found in [11] forces the $\pi^0\gamma^*(q)\gamma(p_3 = 0)$ vertex to be point-like rather than including a $F(q^2, 0)$ form factor. There are also OPE constraints for other momenta regions which are not satisfied by the model in [11] though they argued that this makes only a small numerical difference of the order of $0.05 \times 10^{-10}$. In fact, within the large $N_c$ framework, it has been shown [17] that in general for other than two-point functions, to satisfy fully the QCD short-distance properties requires the inclusion of an infinite number of narrow states.

The results in [11] for the Goldstone boson exchanges and for the axial-vector exchanges can be found in Table 1 and 3, respectively.
Table 4: Results quoted in Ref. [11] for the axial-vector exchange depending of the $f_1(1285)$ and $f_1(1420)$ resonances mass mixing.

| Mass Mixing                                                                 | $10^{10} \times a_\mu$ |
|----------------------------------------------------------------------------|------------------------|
| No New OPE and Nonet Symmetry with M=1.3 GeV                                | 0.3                    |
| New OPE and Nonet Symmetry with M=1.3 GeV                                  | 0.7                    |
| New OPE and Nonet Symmetry with M= $M_\rho$                                 | 2.8                    |
| New OPE and Ideal Mixing with Experimental Masses                           | 2.2 $\pm$ 0.5          |

Table 5: Full hadronic light-by-light contribution to $a_\mu$ at $\mathcal{O}(N_c)$. The difference between the two results of Refs. [8] and [9] is the contribution of the scalar exchange $-(0.7 \pm 0.1) \times 10^{-10}$. This contribution is not included in Refs. [6,7] and [11].

| Hadronic light-by-light at $\mathcal{O}(N_c)$                               | $10^{10} \times a_\mu$ |
|----------------------------------------------------------------------------|------------------------|
| Nonet Symmetry [6,7]                                                       | 9.4 $\pm$ 1.6          |
| Nonet Symmetry + Scalar [8,9]                                              | 10.2 $\pm$ 1.9         |
| Nonet Symmetry [8,9]                                                       | 10.9 $\pm$ 1.9         |
| New OPE and Nonet Symmetry [11]                                            | 12.1 $\pm$ 1.0         |
| New OPE and Ideal Mixing [11]                                              | 13.6 $\pm$ 1.5         |
3 Next-to-Leading in the $1/N_c$ Expansion Contributions

At next-to-leading (NLO) in the $1/N_c$ expansion, the pion loop is the dominant one and because the pion mass is not much larger than the muon mass $m$, one expects a contribution of the order of $10^{-10}$. To dress the photon interacting with pions, a particular Hidden Gauge Symmetry (HGS) model was used in [6,7] while a full VMD was used in [8,9]. The results obtained are $-(0.45 \pm 0.85) \times 10^{-10}$ in [6] and $-(1.9 \pm 0.5) \times 10^{-10}$ in [8]. Both models satisfy the known constraints though start differing at $O(p^6)$ in CHPT. It is also known that the full VMD does rather well reproducing higher order terms of CHPT while the special version of the HGS used in [6] does not give the correct QCD high energy behavior in some two-point functions, in particular it does not fulfill the Weinberg Sum Rules, see [8] for more comments. Some studies of the cut-off dependence of the pion loop using the full VMD model was done in [8] and showed that their final number comes from fairly low energies where the model dependence should be smaller.

The authors of [11] analyzed the model used in [6,7] and showed that there is a large cancellation between the first three terms of an expansion in powers of $(m_\pi/M_\rho)^2$ and with large higher order corrections when expanded in CHPT orders but the same applies to the $\pi^0$ exchange as can be seen from Table 6 in the first reference in [1] by comparing the WZW column with the others. The authors of [11] took $(0 \pm 1) \times 10^{-10}$ as a guess estimate of the total NLO in $1/N_c$ contribution. This seems too simply and certainly with underestimated uncertainty.

4 Comparison Between Different Calculations

The comparison of individual contributions in [6–10,16] and in [11] has to be done with care because they come from different model assumptions to construct the full relevant four-point function. In fact, the authors of [16] have shown that their constituent quark loop provides the correct asymptotics and in particular the new OPE found in [11]. It has more sense to compare results for $a_\mu^{\text{bl}}$ either at leading order or at next-to-leading order in the $1/N_c$ expansion. The recent results for $a_\mu^{\text{bl}}$ at LO in the $1/N_c$ expansion is what is shown in Table 5. The nice agreement between them within the quoted uncertainty leads us [1] to take

$$a_\mu^{\text{bl},N_c} = (11 \pm 4) \times 10^{-10}$$

as a robust result for the hadronic light-by-light contribution to muon anomaly $a_\mu$ at LO in the $1/N_c$ expansion.

The results for the final hadronic light-by-light contribution to $a_\mu$ quoted in [6,7], [8,9] and [11] are in Table 6. The apparent agreement between [6,7]
and [8, 9] hides non-negligible differences which numerically almost compensate between the quark-loop and charged pion and kaon loops. Notice also that [6, 7] didn’t include the scalar exchange. Comparing the results of [8, 9] and [11], as discussed above, we have found several differences of order $1.5 \times 10^{-10}$ which are not related to the new short-distance constraint used in [11]. The different axial-vector mass mixing accounts for $-0.7 \times 10^{-10}$ and the absence of the scalar exchange in [11] accounts for $-1.9 \times 10^{-10}$. These model dependent differences add up to $-4.1 \times 10^{-10}$ out of the final $-5.3 \times 10^{-10}$ difference between [8, 9] and [11]. Clearly, the new OPE constraint used in [11] accounts only for a small part of the large numerical final difference.

5 Conclusions

We observe a nice agreement, see Table 5, between the recent model calculations of the hadronic light-by-light contribution to $a_\mu$ at LO in the $1/N_c$ expansion, hence concluding that

$$a_{\mu, N_c}^{l, N_c} = (11 \pm 4) \times 10^{-10}$$

is a very solid result. We also understand the origin of the final numerical difference between the results quoted in [11] and [8, 9]. Its origin is not dominated by the new OPE constraint found in [11] and it rather comes from the addition of several model dependent differences of order $1.5 \times 10^{-10}$ as discussed above.

It is possible and desirable to make a new calculation of $a_{\mu}^{l, N_c}$ using the techniques developed in [13, 17, 18] and the new OPE results [11].

The authors of [2] have done a conservative analysis of the present situation of the hadronic light-by-light contribution to $a_\mu$ including the NLO in the $1/N_c$ expansion contribution.

Very valuable information about various pieces of the theoretical models used to calculate the hadronic light-by-light contribution to $a_\mu$ can be obtained by measuring the \( \pi^0 \rightarrow \gamma\gamma^* \), \( \pi^0 \rightarrow \gamma^*\gamma^* \) and \( \pi^0 \rightarrow e^+e^- \) decays which constrain the off-shell \( \pi^0\gamma^*\gamma^* \) and \( \pi^0\gamma^*\gamma \) form factors and the subleading $\mu$-dependent terms discussed in Section 2 and by measuring the \( \gamma^*\gamma^* \rightarrow \pi^+\pi^- \), \( e^+e^- \rightarrow \pi^+\pi^- \) processes which constrain the \( \pi^+\pi^-\gamma^*\gamma^* \) vertex which dominates the uncertainty.

| Full Hadronic Light-by-Light | $10^{10} \times a_\mu$ |
|-----------------------------|-------------------|
| [6, 7]                      | $8.9 \pm 1.7$    |
| [8, 9]                      | $8.9 \pm 3.2$    |
| [11]                        | $13.6 \pm 2.5$   |
of the pion loop contribution. The $\gamma\gamma$ programme at the upgraded DAΦNE-2 facility at Frascati is very well suited for these measurements.

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