Embedding simply connected
2-complexes in 3-space
IV. Dual matroids

Johannes Carmesin
University of Cambridge
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Abstract

We introduce dual matroids of 2-dimensional simplicial complexes. Under certain necessary conditions, duals matroids are used to characterise embeddability in 3-space in a way analogous to Whitney’s planarity criterion.

We further use dual matroids to extend a 3-dimensional analogue of Kuratowski’s theorem to the class of 2-dimensional simplicial complexes obtained from simply connected ones by identifying vertices or edges.

1 Introduction

A well-known characterisation of planarity of graphs is Whitney’s theorem from 1932. It states that a graph can be embedded in the plane if and only if its dual matroid is graphic (that is, it is the cycle matroid of a graph) [10].

In this paper we define dual matroids of (2-dimensional) simplicial complexes. We prove under certain necessary assumptions an analogue of Whitney’s characterisation for embedding simplicial complexes in 3-space. More precisely, under these assumptions a simplicial complex can be embedded in 3-space if and only if its dual matroid is graphic.

Our definition of dual matroid is inspired by the following fact.

Theorem 1.1. Let C be a directed 2-dimensional simplicial complex embedded into $\mathbb{S}^3$. Then the edge/face incidence matrix of C represents over the integers\footnote{See Section 2 for a definition.} a matroid M which is equal to the cycle matroid of the dual graph.
Indeed, we define the dual matroid of a simplicial complex \( C \) to be the matroid represented by the edge/face incidence matrix of \( C \) over the finite field \( \mathbb{F}_3 \).

Although the cone over \( K_5 \) does not embed in 3-space, its dual matroid just consists of a bunch of loops, and thus is graphic. In order to exclude examples like the cone over \( K_5 \) we restrict our attention to simplicial complexes \( C \) whose dual matroid captures the local structure at all vertices of \( C \). We call such dual matroids local, see Section 3 for a precise definition. Examples of simplicial complex whose dual matroid is local are those where every edge is incident with precisely three faces and the dual matroid has no loops. Another example is the 3-dimensional grid whose faces are the 4-cycles.

Furthermore matroids (of graphs and also of simplicial complexes) do not depend on the orderings of edges on cycles. Hence it can be shown that dual matroids cannot distinguish triangulations of homology spheres from triangulations of the 3-sphere. While the later ones are always embeddable, this is not true for triangulations of homology spheres in general. Thus we restrict our attention to simply connected simplicial complexes. Under these necessary restrictions we obtain the following 3-dimensional analogue of Whitney’s theorem.

**Theorem 1.2.** Let \( C \) be a simply connected 2-dimensional simplicial complex whose dual matroid \( M \) is local.

Then \( C \) is embeddable in 3-space if and only if \( M \) is graphic.

Tutte’s characterisation of graphic matroids \([9]\) yields the following consequence.

**Corollary 1.3.** Let \( C \) be a simply connected simplicial complex whose dual matroid \( M \) is local.

Then \( C \) is embeddable in 3-space if and only if \( M \) has no minor isomorphic to \( U^*_{2,4} \), the fano plane, the dual of the fano plane or the duals of either \( M(K_5) \) or \( M(K_3,3) \). □

\(^2\)The choice of \( \mathbb{F}_3 \) is a bit arbitrary. Indeed any other field \( \mathbb{F}_p \) with \( p \) a prime different from 2 works.

\(^3\)See for example \([1]\).

\(^4\)These are compact connected 3-manifolds whose homology groups are trivial. Unlike in the 2-dimensional case, this does not imply that the fundamental group is trivial.
We further apply dual matroids to study embeddings in 3-space of – not necessarily simply connected – simplicial complexes with locally small separators as follows.

Given a 2-dimensional simplicial complex \( C \), the link graph, denoted by \( L(v) \), at a vertex \( v \) of \( C \) is the graph whose vertices are the edges incident with \( v \) and whose edges are the faces incident with \( v \) and their incidence relation is as in \( C \). If the link graph at \( v \) is not connected, we can split \( v \) into one vertex for each connected component. There is a similar splitting operation at edges of \( C \). It can be shown that no matter in which order one does all these splittings, one always ends up with the same simplicial complex, the split complex of \( C \).

It can be shown that if a simplicial complex embeds topologically into \( \mathbb{S}^3 \), then so does its split complexes. However, the converse is not true. For an example see Figure 1. Here we give a characterisation of when certain simplicial complexes embed, where one of the conditions is that the split complex embeds.

**Theorem 1.4.** Let \( C \) be a globally 3-connected simplicial complex and \( \hat{C} \) be its split complex. Then \( C \) embeds into \( \mathbb{S}^3 \) if and only if \( \hat{C} \) embeds into \( \mathbb{S}^3 \) and the dual matroid of \( C \) is the cycle matroid of a graph \( G \) and for any vertex or edge of \( C \) the set of faces incident with it is a connected edge set of \( G \).

Figure 1: The \( 4 \times 2 \times 1 \)-grid whose faces are the 4-cycles. It can be shown that the complex obtained by identifying the two edges coloured red cannot be embedded in 3-space.

Here a simplicial complex \( C \) is **globally 3-connected** if its dual matroid is 3-connected. For an extension of Theorem 1.4 to simplicial complexes that are not globally 3-connected, see Theorem 4.19 below.

\(^5\)In Appendix A we give an equivalent definition directly in terms of \( C \).
The condition that a given set of elements of the dual matroid is connected (in some graph representing that matroid) can be characterised by a finite list of obstructions as follows. Given a matroid $M$ and a set $X$ of its elements, a constraint minor of $(M, X)$ is obtained by contracting arbitrary elements or deleting elements not in $X$. In [3], we prove for any 3-connected graphic matroid $M$ (that is a 3-connected graph) with an edge set $X$ that $X$ is connected in $M$ if and only if $(M, X)$ has no constraint minor from the finite list depicted in Figure 2.

![Figure 2: The six obstructions characterising connectedness of $X$. In these graphs we depicted the edge set $X$ in grey.](image)

In [1], we introduced space minors of simplicial complexes and proved that a simply connected locally 3-connected simplicial complex $C$ embeds in 3-space if and only if it does not have a space minor from a finite list $L$ of obstructions. Using Theorem 1.4 we can further extend this characterisation from simply connected simplicial complexes to those whose split complex is simply connected.

**Theorem 1.5.** Let $C$ be a globally 3-connected simplicial complex such that the split complex is simply connected and locally 3-connected. Then $C$ embeds into $S^3$ if and only if its split complex has no space minor from $L$ and the dual matroid has no constraint minor from the list of Figure 2.

If we do not require global 3-connectivity in Theorem 1.5, there are infinitely many obstructions to embeddability, see Section 5. We remark that Theorem 1.2 can be extended from simply connected simplicial complexes to those whose split complex is simply connected.

The paper is structured as follows. In Section 2 we prove Theorem 1.1 which is used in the proof of Theorem 1.2 and Theorem 1.4. In Section 3 we prove Theorem 1.2. In Section 4 we prove Theorem 1.4 and Theorem 1.5. Finally in Section 5 we construct infinitely many obstructions to embed-

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6In [4] we discuss how this result can be extended to simplicial complexes whose split complexes are not local 3-connected.
dability in 3-space (inside the class of simplicial complexes with a simply
connected and locally 3-connected split complex).

For graph we follow the notations of [5] and for matroids [7]. Beyond
that we rely on some definitions of [2].

2 Dual matroids

In this section we prove Theorem 1.1 and the fact that a simplicial complex
and its split complexes have the same dual matroid, which are used in the
proofs of Theorem 1.2 and Theorem 1.4.

A directed simplicial complex is a simplicial complex $C$ together with an
assignment of a direction to each edge of $C$ and together with an assignment
of a cyclic orientation to each face of $C$. A signed incidence vector of an edge
$e$ of $C$ has one entry for every face $f$; this entry is zero if $e$ is not incident
with $f$, it is plus one if $f$ traverses $e$ positively and minus one otherwise.

The matrix given by all signed incidence vectors is called the (signed)
edge/face incidence matrix. The dual matroid of a simplicial complex is the
matroid represented by the edge/face incidence matrix of $C$ over the finite
field $\mathbb{F}_3$.

Although in this paper we work with directed simplicial complexes, dual
matroids do not depend on the chosen directions. Indeed, changing a di-
rection of an edge or of a face of $C$ changes the linear representation of the
dual matroid but not the matroid itself.

A matrix $A$ is a regular representation (or representation over the in-
tegers) of a matroid $M$ if all its entries are integers and the columns are
indexed with the elements of $M$. Furthermore for every circuit $o$ of $M$ there
is a $\{0, -1, +1\}$-valued vector\[^7\] $v_o$ in the span over $\mathbb{Z}$ of the rows of $A$ whose
support is $o$. And the vectors $v_o$ span over $\mathbb{Z}$ all row vectors of $A$.

2.1 Proof of Theorem 1.1

Let $C$ be a directed simplicial complex embedded into $S^3$, the dual digraph
of the embedding is the following. Its vertex set is the set of components
of $S^3 \setminus C$. It has one edge for every face of $C$. This face touches one or two
components of $S^3 \setminus C$. If it touches two components, the edge for that face
joins the vertices for these two components. The edge is directed from the

\[^7\] A vector is an element of a vector space $k^S$, where $k$ is a field and $S$ is a set. In a
slight abuse of notation, in this paper we also call elements of modules of the form $\mathbb{Z}^S$
vectors.
vertex whose complement touches the chosen orientation of the face to the other component. If the face touches just one component, its edge is a loop attached at the vertex corresponding to that component.

Let \((\sigma(e)|e \in E(C))\) be the planar rotation system of \(C\) induced by the topological embedding of \(C\). It is not hard to check that \(\sigma(e)\) is a closed trail in the dual graph. The dual complex of the embedding is the directed simplicial complex obtained from the dual digraph by adding for each edge of \(C\) the cyclic orderings of the cyclic orientations \(\sigma(e)\) as faces and we choose their orientations to be \(\sigma(e)\).

**Observation 2.1.** Let \(C\) be a connected and locally connected 9 simplicial complex embedded in \(S^3\) with induced planar rotation system \(\Sigma\). Then the dual complex of the embedding is equal to the dual complex of \((C, \Sigma)\).

**Proof.** By [\cite{2}, Lemma 3.4,10], the local surfaces for \((C, \Sigma)\) agree with the local surfaces of the embedding. Hence these two complexes have the vertex set. As they also have the same incidence relations between edges and vertices and edges and faces, they must coincide. \(\Box\)

By Observation 2.1 and the definition of ‘generated over the integers’ and by Theorem B.6 in order to prove Theorem 1.1 it suffices to show that the dual complex for \((C, \Sigma)\) is nullhomologous.

First we prove this in the special case when \(C\) is nullhomologous and locally connected.

**Lemma 2.2.** Let \(C\) be a nullhomologous locally connected simplicial complex together with a planar rotation system \(\Sigma\) such that local surfaces for \((C, \Sigma)\) are spheres. Then the dual complex \(D\) of \((C, \Sigma)\) is nullhomologous.

**Proof.** By [\cite{2}, Lemmas 6.3,6.5,6.7] the complexes \(C\) and \(D\) satisfy euler’s formula, that is:

\[
|V(C)| - |E| + |F| - |V(D)| = 0
\]

8A trail is sequence \((e_i|i \leq n)\) of distinct edges such that the endvertex of \(e_i\) is the starting vertex of \(e_{i+1}\) for all \(i < n\). A trail is closed if the starting vertex of \(e_1\) is equal to the endvertex of \(e_n\).
9A simplicial complex \(C\) is locally connected if all its link graphs are connected.
10Local surfaces of embeddings are defined in [\cite{2}].
11A simplicial complex \(C\) is nullhomologous if the face boundaries of \(C\) generate all cycles over the integers. This is equivalent to the condition that the face boundaries of \(C\) generate all cycles over the field \(\mathbb{F}_p\) for every prime \(p\).
12This last property follows from the first two if we additionally assume that \(\Sigma\) is induced by a topological embedding in \(S^3\) by [\cite{2}, Theorem 6.1].
Hence we deduce that $D$ nullhomologous by applying the ‘Moreover’-part of [2, Lemma 6.3] for every prime $p$.

Next we shall extend Lemma 2.2 to simplicial complexes that are only locally connected.

**Lemma 2.3.** Let $C$ be a locally connected simplicial complex together with a planar rotation system $\Sigma$ that is induced by a topological embedding $\iota$ in $S^3$. Then the dual complex $D$ of $(C, \Sigma)$ is nullhomologous.

**Proof.** By [2, Theorem 7.1] there is a simplicial complex $C'$ that is obtained from $C$ by subdividing edges, baricentric subdivisions of faces and adding faces along closed trails. And $C'$ is nullhomotopic and has an embedding $\iota'$ into $S^3$ that induces $\iota$. Let $D'$ be the dual of $\iota'$. By Lemma 2.2, $D'$ is nullhomologous.

We shall deduce that $D$ is nullhomologous by showing that reversing each of the operations in the construction of $C'$ from $C$ preserves being nullhomologous in the dual. We call such an operation preserving.

**Sublemma 2.4.** Subdividing an edge is preserving.

**Proof.** Subdividing an edge in the primal corresponds to adding a copy of a face in the dual. Clearly, the deletion of the copy preserves being nullhomologous for the dual.

**Sublemma 2.5.** A baricentric subdivision of a face is preserving.

**Proof.** It suffices to show that the subdivision by a single edge is preserving. Subdividing a face by an edge in the primal corresponds to replacing an edge in the dual by two edges in parallel and adding a face containing precisely these two edges. Reversing this operation preserves being nullhomologous.

**Sublemma 2.6.** Adding a face is preserving.

**Proof.** Adding a face in the primal corresponds to coadding an edge in the dual. Contracting that edge preserves being nullhomologous.

By Sublemma 2.4, Sublemma 2.5 and Sublemma 2.6, the fact that $D'$ is nullhomologous implies that $D$ is nullhomologous.

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13 This means that we obtain $\iota$ from $\iota'$ by deleting the newly added faces, contracting the newly added subdivision edges and undoing the baricentric subdivisions.

14 A complex $A$ is obtained from a complex $A'$ by *coadding* an edge $e$ if $A'$ is obtained from $A$ by contracting the edge $e$. 

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It remains to prove Theorem 1.1 for simplicial complexes $C$ that are not locally connected. First we need some preparation.

Given a simplicial complex $C$, its vertical split complex is obtained from $C$ by replacing each vertex $v$ by one vertex for each connected component of $L(v)$, where the edges and faces incident with that vertex are those in its connected component. We refer to these new vertices as the clones of $v$.

**Observation 2.7.** The vertical split complex of any simplicial complex is locally connected.

**Observation 2.8.** A simplicial complex and its vertical split complex have the same dual matroid.

*Proof.* A simplicial complex and its vertical split complex have the same edge/face incidence matrix.

Given an embedding $\iota$ of a simplicial complex $C$ into $S^3$, we will define what an induced embedding of the vertical split complex is.

For that we need some preparation. Let $v$ be a vertex of $C$ whose link graph is not connected. By changing $\iota$ a little bit locally (but not its induced planar rotation system) if necessary, we may assume that there is a 2-ball $B$ of small radius around $v$ such that firstly $v$ is the only vertex of $C$ contained in the inside of $B$. And secondly its boundary $\partial B$ intersects each edge incident with $v$ in a point and each face incident with $v$ in a line. In other words, the intersection of $C$ with the boundary is the link graph at $v$. As the link graph is disconnected, there is a circle (homeomorphic image of $S^1$) $\gamma$ in the boundary such that the two components of $B \setminus \gamma$ both contain vertices of the link graph, see Figure 3.

![Figure 3: The link graph at $v$ embedded into $\partial B$.](image)

The simplicial complex $C_\gamma$ is obtained from $C$ by replacing the vertex $v$ by two vertices, one for each connected component of $\partial B \setminus \gamma$ that is incident
with the edges and faces whose vertices and edges, respectively, are in that
connected component.

The embedding $\iota$ induces the following embedding $\iota_\gamma$ of $C_\gamma$ into $\mathbb{S}^3$. We pick a disc contained in $B$ with boundary $\gamma$ that intersects $C$ only in $v$. We replace $v$ by its two clones – both with tiny distance from $v$ and one above that disc and the other below. We only need to change faces and edges incident with $v$ in a tiny neighbourhood around $v$. Faces and edges above and below do not interfere.

It is easy to see that $\iota$ and $\iota_C$ have the same planar rotation system and that $C$ and $C_\gamma$ have the same vertical split complex.

A topological embedding of the vertical split complex of $C$ into $\mathbb{S}^3$ is (vertically) induced by $\iota$ if it is obtained by applying the above procedure iteratively until $C_\gamma$ is equal to the vertical split complex of $C$. It is clear that if $\iota$ is a topological embedding of a simplicial complex $C$ into $\mathbb{S}^3$, then its vertical split complex has a topological embedding into $\mathbb{S}^3$ that is induced by $\iota$.

**Observation 2.9.** Let $\iota$ be an embedding of a simplicial complex into $\mathbb{S}^3$ and let $\iota'$ be an induced embedding of $\iota$ of the vertical split complex. Then $\iota$ and $\iota'$ have the same dual complex.

*Proof.* In both embeddings, the incidence relation between the local surfaces and the faces is the same. Hence both dual complexes have the same vertex/edge incidence relation. They also have the same sets of faces as $\iota$ and $\iota'$ have the same rotation system. \hfill \Box

A set $S$ of vertices in a simplicial complex $C$ is a vertex separator if $C$ can be obtained from two disjoint simplicial complexes that each have at least one face by gluing them together at the vertex set $S$. As the empty set might also be a vertex separator, any simplicial complex with no vertex separator is connected.

**Lemma 2.10.** Let $C$ be a simplicial complex without a vertex separator. Assume that $C$ has an embedding $\iota$ into $\mathbb{S}^3$. Then the dual complex $D$ of $\iota$ is nullhomologous.

*Proof.* Let $C'$ be the vertical split complex of $C$. By Observation 2.7 $C'$ is locally connected. By assumption $C$ has no vertex separator. Thus $C'$ is

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15 The construction of $\iota_C$ depends on the choice of $B$. Still we use the term ‘induced’ in this context since in this paper we consider topological embeddings equivalent if they have the same planar rotation system.
connected. Let \( \iota' \) be the embedding of \( C' \) induced by \( \iota \). Let \( \Sigma' \) be the planar rotation system induced by \( \iota' \).

By Lemma 2.3 the dual \( D' \) for \( (C', \Sigma') \) is nullhomologous. By Observation 2.1 \( D' \) is the dual complex of \( \iota' \). By Observation 2.9 \( D' \) is equal to \( D \). So \( D \) is nullhomologous.

Lemma 2.11. Let \( C \) be a simplicial complex embedded into \( \mathbb{S}^3 \) that is obtained from two simplicial complexes \( C_1 \) and \( C_2 \) by gluing them together at a set of vertices. Assume that \( C_2 \) has no separating vertex set. Let \( G_i \) be the dual graph of the embedding restricted to \( C_i \) for \( i = 1, 2 \). Then the dual graph of the embedding of \( C \) is equal to a graph obtained by gluing together \( G_1 \) and \( G_2 \) at a single vertex.

Proof. We denote the embedding of \( C \) into \( \mathbb{S}^3 \) by \( \iota \) and the restricted embedding of \( C_1 \) by \( \iota_1 \). Suppose for a contradiction that \( \iota \) maps interior points of faces of \( C_2 \) to interior points of different local surfaces of \( \iota_1 \). Let \( \ell \) be a local surface of \( \iota_1 \) to which an interior point of a face of \( C_2 \) is mapped by \( \iota \). Let \( C'_2 \) be the subcomplex of \( C_2 \) that contains all faces whose interior points are mapped to interior points of \( \ell \). Its edges and vertices are those of \( C_2 \) that are incident with these faces. Note that if one interior point of a face is mapped to \( \ell \), then all are. Hence the subcomplex \( C''_2 \) that contains all other faces and their incident vertices and edges contains a face. The subcomplexes \( C'_2 \) and \( C''_2 \) of \( C_2 \) can only intersect in points of \( C_1 \). Hence they only can intersect in vertices. Thus \( C'_2 \) and \( C''_2 \) witness that \( C_2 \) has a separating vertex set contrary to our assumption.

Thus there is a single local surface of \( \iota_1 \) to which all interior points of faces of \( C_2 \) are mapped by \( \iota \). Hence the dual graph of \( \iota \) is equal to the graph obtained by gluing together \( G_1 \) and \( G_2 \) at that vertex.

Proof of Theorem 1.1. By applying Lemma 2.11 recursively, we may assume that \( C \) has no separating vertex set. Recall that the dual graph of the embedding is the 1-skeleton of the dual complex of the embedding. By Lemma 2.10 the edge/face incidence matrix is a representation over the integers of the cycle matroid of the dual graph of the embedding.

2.2 Split complexes

A naive way to define splittings of edges might be to consider the incidences at one of their endvertices and split according to that. We shall show that when using this notion of splitting, split complexes will not have all nice
properties we want them to have, see Appendix A. A more refined definition takes into account the incidences at both endvertices, defined as follows.

Given a simplicial complex $C$ and an edge $e$ with two endvertices $v$ and $w$, two faces incident with $e$ are $v$-related if - when considered as edges of $e$, they have endvertices in the same connected component of the link graph $L(v) - e$ with the vertex $e$ removed. Analogously, we define $w$-related. Two faces $f_1$ and $f_2$ incident with $e$ are in the same connected component at $e$ if there is a chain of faces incident with $e$ from $f_1$ to $f_2$ such that adjacent faces in the chain are $v$-related or $w$-related. Note that ‘being in the same connected component at $e$’ is the equivalence relation generated from the union of ‘$v$-related’ and ‘$w$’-related.

The simplicial complex obtained from $C$ by splitting the edge $e$ is obtained from $C$ by replacing the edge $e$ by one copy $e_X$ for every connected component $X$ at $e$. The faces incident with $e_X$ are those in $X$.

We refer to the edges $e_X$ as the clones of $e$. If we apply several splittings, we extend the notion of cloning iteratively so that each edge of the resulting simplicial complex is cloned from a unique edge of $C$.

If we split an edge in a nontrivial way, then the resulting simplicial complex has the same number of faces but at least one edge more. As in a simplicial complex every edge is incident with a face, we can only split edges a bounded number of times. A simplicial complex obtained from $C$ by splitting edges such that for every edge there is only one component at $e$ is called an edge split complex of $C$. As explained above, every simplicial complex has an edge split complex.

Since splitting edges, does not change the 2-blocks of the link graphs, splittings of edges commute. In particular, edge split complexes are unique. In the following we will talk about ‘the edge split complex’.

The split complex of a simplicial complex $C$ is the vertical split complex of its edge split complex. Clearly, splitting a vertex does not change the edge split complex.

**Example 2.12.** A simplicial complex, its vertical split complex and its edge split complex have the same split complex. Locally 2-connected\textsuperscript{16} simplicial complexes are equal to their split complex.

**Lemma 2.13.** A simplicial complex and its edge split complex have the same dual matroid.

\textsuperscript{16}A simplicial complex is locally 2-connected if its link graphs are connected and have no cutvertices.
Proof. We shall show that a simplicial complex \( C \) and a simplicial complex \( C' \) have the same dual matroid, where we obtain \( C' \) from \( C \) by splitting an edge \( e \). Once this is shown, the lemma follows inductively as an edge split complex is obtained by a sequence of edge splittings.

Clearly, \( C \) and \( C' \) have the same set of faces. Hence their dual matroids have the same ground sets.

The vectors indexed by clones of the edge \( e \) of the edge/face incidence matrix \( A' \) of \( C' \) sum up to the vector indexed by \( e \) of the edge/face incidence matrix \( A \) of \( C \). Hence the vectors indexed by edges of \( A' \) generate the vectors indexed by edges of \( A \). So it remains to show that any vector indexed by a clone \( e' \) of \( e \) of \( A' \) is generated by the vectors indexed by edges of \( A \).

Let \( v \) be an endvertex of \( e \). Let \( K \) be the connected component of the link graph \( L(v) \) of \( C \) at \( v \) that contains \( e \). Let \( Y \) be the union of the components \( Y' \) of \( K - e \) such that faces incident with \( e' \) – when considered as edges of \( L(v) \) – have an endvertex in \( Y' \). The sum over all vectors indexed by edges \( y \in V(Y) \) of \( A \) is the vector indexed by \( e' \) of \( A' \). Since \( e' \) was an arbitrary clone, the vectors indexed by edges of \( A \) generate the vectors indexed by edges of \( A' \).

We have shown that splitting a single edge preserves the dual matroid. Since the edge split complex is obtained by splitting edges, it must have the same dual matroid as the original complex. \( \square \)

**Corollary 2.14.** A simplicial complex and its split complex have the same dual matroid.

**Proof.** A simplicial complex and its vertical split complex have the same incidence relations between edges and faces. Hence this is a consequence of *Lemma 2.13* \( \square \)

### 3 A Whitney type theorem

In this section we prove *Theorem 1.2*.

In general the dual matroid of a simplicial complex \( C \) does not contain enough information to decide whether \( C \) is embeddable in 3-space. For example, the dual matroid of the cone over \( K_5 \) consists of a bunch of loops. So it cannot distinguish this non-embeddable simplicial complex from other embeddable ones. The following fact gives an explanation of this phenomenon (in the notation of that fact: from the graph \( G \) we can in general not reconstruct the matroid \( M[v] \)). Given a vertex \( v \) of a simplicial complex, we denote the dual matroid of the link graph at \( v \) by \( M[v] \).
**Fact 3.1.** Let $C$ be a simplicial complex embedded in $S^3$. Then the dual matroid $M$ restricted to the faces incident with $v$ is represented by a graph $G$. Moreover, $G$ can be obtained from some graph representing $M[v]$ by identifying vertices.

**Proof.** By [Theorem 1.1] $M$ is the cycle matroid of the dual graph of the embedding of $C$. So $G$ is the restriction of that graph to the faces incident with $v$.

By $G'$ be denote the ‘local dual graph’ of $C$ at $v$. This is defined as the ‘dual graph’ but with ‘$S^3$’ replaced by ‘a small neighbourhood $U$ around $v$’ in the embedding. Clearly, $G'$ represents $M[v]$. We obtain the vertices of $G$ from those of $G'$ by identifying those vertices for components of $U \setminus C$ that lie in the same component of $S^3 \setminus C$. The ‘Moreover’-part follows.

To exclude the phenomenon described in [Fact 3.1] we restrict our attention to simplicial complexes $C$ whose dual matroid captures the local structure at all vertices of $C$, defined as follows. Given a simplicial complex $C$ with dual matroid $M$, we say that $M$ is local if for every vertex $v$ the matroid $M[v]$ is equal to $M$ restricted to the faces incident with $v$.

Furthermore matroids (of graphs and also of simplicial complexes) do not depend on the orderings of edges on cycles. Hence it can be shown that dual matroids cannot distinguish triangulations of homology spheres from triangulations of the 3-sphere. While the later ones are always embeddable, this is not true for triangulations of homology spheres. Thus we restrict our attention to simply connected simplicial complexes.

If we exclude these two phenomenons, [Theorem 1.2] stated in the Introduction, characterises when a simplicial complex is embeddable just in terms of its dual matroid.

**Remark 3.2.** The assumptions of [Theorem 1.2] can be interpreted as some face maximality assumption. By [2, Theorem 7.1] this is true for being simply connected. For locality, let $C$ be any embeddable simplicial complex embeddable. By [Fact 3.1] we can add faces until for every vertex $v$ the matroid $M[v]$ is equal to $M$ restricted to the faces incident with $v$. This preserves being simply connected.

Now we prepare for the proof of [Theorem 1.2].

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17 These are compact connected 3-manifolds whose homology groups are trivial. Unlike in the 2-dimensional case, this does not imply that the fundamental group is trivial.
**Lemma 3.3.** Let $H$ be a graph whose cycle matroid is the dual matroid $M$ of a simplicial complex $C$. There is a directed graph $\vec{H}$ with underlying graph $H$ such that for all edges $e$ of $C$ the signed vectors are 3-flows$^{18}$.

Proof. First we consider the case when $H$ is 2-connected. We start with an arbitrarily directed graph $\vec{H}$ with underlying graph $H$ some of whose directions of the edges we might reverse later on in the argument. Since $H$ is 2-connected, the set of edges incident with a vertex is a bond of $H$, which is called the atomic bond of $v$. By elementary properties of representations, there is a vector $b_v$ with all entries $-1$, $+1$ or $0$ that has the same support$^{19}$ as the atomic bond at $v$.

Given an edge $e$ of $H$ and one of its endvertices $v$, we say that $e$ is effectively directed towards $v$ with respect to a vector $b$ with entries in $\mathbb{Z}$ if $\vec{e}$ is directed towards $v$ and $b(e)$ is positive or $\vec{e}$ is directed away from $v$ and $b(e)$ is negative. First we shall prove that we can modify the directions of the edges of $\vec{H}$ such that all edges $e$ of $H$ are directed such that for some endvertex $v$ they are effectively directed towards $v$ with respect to the at $b_v$.

Let $T$ be a spanning tree of $H$. Since $T$ does not contain any cycle, we can pick the $b_v$ such that if $vw$ is an edge of $T$, then $b_v(vw) = -b_w(vw)$. Hence an edge $vw$ of $T$ is effectively directed towards $v$ with respect to $b_v$ if and only if it is effectively directed towards $w$ with respect to $b_w$. So by reversing the direction of an edge if necessary$^{20}$ we may assume that every edge $vw$ of $T$ is effectively directed towards $v$ with respect to $b_v$ and also effectively directed towards $w$ with respect to $b_w$.

Next let $xy$ be an edge not in $T$. By reversing the direction of $xy$ if necessary we may assume that $xy$ is effectively directed towards $x$ with respect to $b_x$. Our aim is to show that $xy$ is effectively directed towards $y$ with respect to $b_y$. Let $C$ be the fundamental circuit of $xy$ with respect to $T$. By elementary properties of representations, there is a vector $v_C$ with support $C$ that is orthogonal over $\mathbb{F}_3$ to all the vectors $b_z$ for vertices $z$ on $C$. At all vertices $z$ of $C$ except possibly $y$, the two edges on $C$ incident with $z$ are effectively directed towards $z$ with respect to the vector $b_z$. Hence for $v_C$ to be orthogonal, precisely one of these edges must be effectively directed towards $z$ with respect to $v_C$. Using this property inductively along $C$, we deduce that of the two edges on $C$ incident with $y$ also precisely one is

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$^{18}$A 3-flow in a directed graph $\vec{H}$ is an assignment of integers to the edges of $\vec{H}$ that satisfies Kirchhoff’s first law modulo three at every vertex of $\vec{H}$.

$^{19}$The support of a vector is the set of coordinates with nonzero values.

$^{20}$To be very formal, we delete the edge from the graph and glue it back the other way round. Note that we do not change the director.
effectively directed towards $y$ with respect to $v_C$. Since $b_y$ is orthogonal to $v_C$ and the edge incident with $y$ that is on $T$ and $C$ is effectively directed towards $y$ with respect to $b_y$, also $xy$ must be effectively directed towards $y$ with respect to $b_y$.

Hence our final directed graph $\vec{H}$ has the property that all edges $e$ of $H$ are effectively directed towards any of their endvertices $v$ with respect to $b_v$. Since signed vectors of edges $e$ of $C$ are orthogonal at to $b_v$, it follows that it accumulates 0 (mod 3) at all vertices $v$. So the signed vectors of $C$ are 3-flows for $\vec{H}$. This completes the proof if $H$ is 2-connected. If $H$ is not 2-connected, we do the same construction independently in every 2-connected component and the result follows.

First we prove Theorem 1.2 under the additional assumption that $C$ is locally 2-connected:

**Lemma 3.4.** Let $C$ be a simply connected locally 2-connected simplicial complex whose dual matroid is local. Then $C$ is embeddable in 3-space if and only if $M$ is graphic.

*Proof.* Assume that $C$ is embeddable and let $D$ by its dual complex. Then by Theorem 1.1 $M$ is equal to the cycle matroid of the 1-skeleton of $D$. In particular $M$ is graphic.

Now conversely assume that $C$ is a simply connected simplicial complex such for every vertex $v$ the matroid $M[v]$ is equal to dual matroid $M$ restricted to the faces incident with $v$; and that there is a graph $G$ whose cycle matroid is $M$. We pick an arbitrary direction at each edge of $G$ and an arbitrary orientation at each face of $C$. Our aim is to construct a planar rotation system $\Sigma$ of $C$ and apply \[2, \text{Theorem 1.1}\] to deduce that $C$ is embeddable.

By Lemma 3.3 there is a direction $\vec{G}$ of $G$ such that the signed incidence vector $v_e$ for each edge $e$ of $C$ is a 3-flow in $\vec{G}$. As the link graph $L(v)$ at each vertex $v$ is 2-connected, none of its vertices $e$ is a cutvertex. Hence the edges incident with $e$ in $L(v)$ form a bond. So they form a circuit in the dual matroid $M[v]$. Thus by assumption the support of $v_e$ is a circuit in the matroid $M$. By the construction of $\vec{G}$, the signed vector $v_e$ is a directed cycle$^{21}$ in $\vec{G}$. This directed cycle defines a cyclic orientation $\sigma(e)$. In terms of $C$ this is a cyclic orientation of the oriented faces incident with the directed edge $\vec{e}$. Put another way $\Sigma = (\sigma(e)|e \in E(C))$ is a rotation system.

$^{21}$A vector $v$ whose entries are in $\{0, +1, -1\}$ is a directed cycle if its support is a cycle and it satisfies Kirchhoff’s first law at every vertex, see [5].
Our aim is to prove that $\Sigma$ is planar. So let $v$ be a vertex of $C$ and let $\Sigma_v$ be the rotation system of the link graph $L(v)$ induced by $\Sigma$. This rotation system of $L(v)$ defines an embedding of $L(v)$ in a 2-dimensional oriented surface $S_v$ in the sense of [6][22]. It remains to show the following.

**Sublemma 3.5.** $S_v$ is a sphere.

*Proof.* As the graph $L = L(v)$ is connected, $S_v$ is connected. Thus it suffices to show that it has Euler genus two, that is:

$$V_L - E_L + F_L = 2$$  \hspace{1cm} (1)

Here we abbreviate: $|V(L)| = V_L$, $|E(L)| = E_L$ and $F_L$ denotes the faces of the embedding of $L(v)$ in $S_v$.

We denote the dual graph of the embedding of $L$ in $S_v$ by $H$. Our aim is to show that $H$ is equal to the restriction $R$ of $G$ to the faces incident with $v$. We obtain $S'$ from $R$ by gluing on each directed cycle $v_e$ the face $\sigma(e)$. Similarly as in [2] we use the Edmonds-Hefter-Ringel rotation principle [6, Theorem 3.2.4] to deduce that $L$ is the surface dual of $R$ with respect to the embedding into $S'$. In particular $S' = S$ and $R$ is equal to $H$.

Having shown that $R$ is the surface dual of $L$, we conclude our proof of Equation 1 as follows. We denote the dimension of the cycle space of $L$ by $d$. We have $V_L - E_L = -d + 1$ and $F_L = V_R$ (where $V_R$ is the number of vertices of $V_R$). Hence in order to prove Equation 1 it suffices to show that $d = V_R - 1$. This follows from the assumption that the cycle matroid of $R$ is the dual of the cycle matroid of $L$. Indeed, the cycle matroid of $L$ is 2-connected by assumption.

**Proof of Theorem 1.2.** As in the proof of Lemma 3.4 by Theorem 1.1 it suffices to show that any simply connected simplicial complex $C$ whose dual matroid $M$ is graphic and local can be embedded in 3-space.

We prove this in two steps. First we prove it for locally connected simplicial complexes. We prove this by induction. The base case is when $C$ is locally 2-connected and this is dealt with in Lemma 3.4. So now we assume that $C$ has a vertex $v$ such that the link graph $L(v)$ has a cut vertex\footnote{23}{A vertex $v$ of a graph is a *cut vertex* if the component of the graph containing $v$ with $v$ removed is disconnected.}, and that we proved the statement for every simplicial complex as above such
that it has a fewer number of cutvertices – summed over all link graphs. Let $e$ be an edge of $C$ that is a cutvertex in $L(v)$.

**Sublemma 3.6.** The simplicial complex $C$ is obtained from a simplicial complex $C'$ by identifying two vertex-disjoint edges $e_1$ and $e_2$ onto $e$.

**Proof.** In the link graph $L(v)$, let $f_1$ and $f_2$ be two edges incident with $e$ that are in different 2-blocks of $L(v)$. Hence $L(v)$ has a 1-separation $(X_1, X_2)$ with cutvertex $e$ such that $f_i$ is in the side $X_i$ for $i = 1, 2$.

Let $w$ be the endvertex of $e$ in $C$ different from $v$. Our aim is to construct a 1-separation $(Y_1, Y_2)$ with cutvertex $e$ of $L(w)$ such that $X_i$ and $Y_i$ agree when restricted to the edges incident with $e$ for $i = 1, 2$. For that we have to show that if two such edges are in different $X_i$ then they do not lie in the same 2-block of $L(w)$. That is, in the matroid $M[w]$ they do not lie in a common circuit consisting of edges incident with $e$. By the assumption, this property is true in $M[v]$ if and only if it is true in $M$ if and only if it is true in $M[w]$, which it is not true as $(X_1, X_2)$ is a 1-separation.

We obtain $C'$ from $C$ by replacing $v$ by two new vertices $v_1$ and $v_2$ and $w$ by two new vertices $w_1$ and $w_2$. A face or edge incident with $v$ is in $v_i$ if and only if it is in $X_i$. Similarly, a face or edge incident with $w$ is incident with $w_i$ if and only if it is in $Y_i$. Thus every edge or face incident with $v$ is incident with precisely one of $v_1$ and $v_2$ except for the edge $e$ for which we introduce two copies, which we denote by $e_1$ and $e_2$. The same is holds with ‘$w$’ in place of ‘$v$’. Clearly, the edge $e_i$ joins $v_i$ and $w_i$. Hence $C'$ has the desired properties.

**Sublemma 3.7.** The edges $e_1$ and $e_2$ lie in different connected components of $C'$.

**Proof.** The simplicial complex $C/e$ is simply connected and obtained from $C'/\{e_1, e_2\}$ by identifying the vertices $e_1$ and $e_2$ onto $e$. Since $C/e$ is not locally connected at $e$ we can apply [2, Lemma 5.1] to deduce that $e$ has to be a cutvertex of $C/e$.

Since the link graph $L(e)$ of $C/e$ is a disjoint union of the connected link graphs $L(e_1)$ and $L(e_2)$ of $C'/\{e_1, e_2\}$, two faces incident with the same edge $e_i$ in $C'$ cannot be cut off by $e$ in $C/e$. Hence the only way $e$ can cut $C/e$ is that $e_1$ and $e_2$ are cut off from one another. Put another way, $e_1$ and $e_2$ lie in different connected components of $C'$.

For $i = 1, 2$, let $C_i$ be the component of $C'$ containing $e_i$ and $M_i$ the dual matroid of $C_i$. We may assume that $C$ is connected. Hence $C'$ is the
disjoint union of the $C_i$. By Sublemma 3.7 and Lemma 2.13, the dual matroid $M$ of $S$ is the disjoint union of the matroids $M_i$. So we can apply the induction hypothesis to each simplicial complex $C_i$. So all $C_i$ are embeddable. Analogously to [2, Lemma 5.2] one proves that $C$ is embeddable in 3-space.\footnote{An alternative is the following: it is easy to see that a simplicial complex $S$ is embeddable if and only if $S/e$ is embeddable for some nonloop $e$. So the $C_i/e_i$ are embeddable. Then by [2, Lemma 5.2] $C/e$ is embeddable. So $C$ is embeddable.}

Finally, we prove the statement for arbitrary simplicial complexes. Again, we prove it by induction. This time the locally connected case is the base case. So now we assume that $C$ has a vertex $v$ such that the link graph $L(v)$ is disconnected; and that we proved the statement for every simplicial complex as above such that the number of components of link graphs minus the total number of link graphs is smaller. As $C$ is simply connected, by [2, Lemma 5.1] the vertex $v$ is a cutvertex of $C$. That is, $C$ is obtained from gluing together two simplicial complexes $C'$ and $C''$ at the vertex $v$. Since splitting vertices preserves dual matroids, the dual matroid of $C$ is the disjoint union of the dual matroid of $C'$ and the dual matroid of $C''$. Thus the simplicial complexes $C'$ and $C''$ are embeddable in $S^3$ by induction. Hence by [2, Lemma 5.2] $C$ is embeddable.

\begin{remark}
The proof of Theorem 1.2 works also if we change the definition of dual matroid in that we replace ‘$F_3$’ by ‘$F_p$ with $p$ prime and $p > 2$’. By Theorem 1.1, if $C$ is embeddable, the signed incidence vectors of the edges of $C$ generate the same matroid over any field $F_p$ with $p$ prime. So if $C$ is embeddable all these definitions of dual matroids coincide.

The special role of $p = 2$ is visible in Corollary 1.3, where we have to exclude the matroid $U_{2,4}$, which is representable over any field $F_p$ with $p$ prime and $p > 2$ but not over $F_2$.
\end{remark}

4 Constructing embeddings from embeddings of split complexes

In this section we prove Theorem 1.4. We subdivide this proof in four subsections.

4.1 Constructing embeddings from vertical split complexes

\begin{lemma}
Let $C$ be a simplicial complex obtained from a simplicial complex $C'$ by identifying two vertices $v$ and $w$. Let $i'$ be a topological embedding

\end{lemma}
of $C'$ into $\mathbb{S}^3$. Assume that there is a local surface of $\iota'$ that contains both $v$ and $w$. Then there is a topological embedding of $C$ into $\mathbb{S}^3$ that has the same dual graph as $\iota'$.

Proof. We join $v$ and $w$ by a copy of the unit interval $I$ inside the local surface of $\iota'$ that contains them both. We may assume that there is an open cylinder around $I$ that does not intersect $C'$. We obtain a topological embedding $\iota$ of $C$ from $\iota'$ by moving $v$ along $I$ to $w$. We do this in such a way that we change the edges and faces incident with $v$ only inside the small cylinder. It is clear that $\iota'$ and $\iota$ have the same dual graph.

**Lemma 4.2.** Let $x$ be a vertex or edge of a simplicial complex $C$ embedded into $\mathbb{S}^3$. The set of faces incident with $x$ is a connected edge set of the dual graph of the embedding.

Proof. If $x$ is an edge, then the set of faces incident with $x$ is a closed trail, and hence connected. Hence it remains to consider the case that $x$ is a vertex. Let $H_x$ be the dual graph of the link graph at $x$ with respect to the embedding in the 2-sphere given by the embedding of $C$. The restriction $R_x$ of the dual graph of the embedding of $C$ to the faces incident with $x$ is obtained from $H_x$ by identifying vertices. Since $H_x$ is connected, also $R_x$ is connected. This completes the proof.

Given a simplicial complex $C$ and a topological embedding $\iota$ of its vertical split complex into $\mathbb{S}^3$, we say that $\iota$ satisfies the **vertical dual graph connectivity constraints** if for any vertex $x$ of $C$, the set of faces incident with $x$ is a connected edge set of the dual graph of $\iota$.

**Theorem 4.3.** Let $C$ be a simplicial complex. Then $C$ embeds into $\mathbb{S}^3$ if and only if its vertical split complex $\hat{C}$ has an embedding into $\mathbb{S}^3$ that satisfies the vertical dual graph connectivity constraints.

Proof. First assume that $C$ has a topological embedding $\iota$ in $\mathbb{S}^3$. Let $\iota'$ be the embedding induced by $\iota$ of $\hat{C}$. By Observation 2.9, $\iota$ and $\iota'$ have the same dual graph. Hence by Lemma 4.2, $\iota'$ satisfies the vertical dual graph connectivity constraints.

Now conversely assume that $\iota'$ is an embedding into $\mathbb{S}^3$ of $\hat{C}$ that satisfies the vertical dual graph connectivity constraints. Let $G$ be the dual graph of $\iota'$. We shall recursively construct a sequence $(C_n)$ of simplicial complexes by identifying vertices that belong to the same vertex of $C$ that all have the vertical split complex $\hat{C}$ and topological embeddings $\iota_n$ of $C_n$ into $\mathbb{S}^3$ that all have the same dual graph $G$. 

19
If \( C_n = C \), we stop and are done. So there is a vertex \( v \) of \( C \) such that \( C_n \) has at least two vertices cloned from \( v \). The set of faces incident with \( v \) is a connected edge set of \( G \). So there are two distinct vertices \( v_1 \) and \( v_2 \) of \( C_n \) cloned from \( v \) whose incident faces share a vertex when considered as edge sets of \( G \). Hence there is a local surface of \( \iota_n \) that contains \( v_1 \) and \( v_2 \). We obtain \( C_{n+1} \) from \( C_n \) by identifying \( v_1 \) and \( v_2 \). The existence of a suitable embedding \( \iota_{n+1} \) follows from [Lemma 4.1](#).

Since this recursion cannot continue forever, we must eventually have that \( C_n = C \). Then \( \iota_n \) is the desired embedding of \( C \) and we are done. \( \square \)

### 4.2 Constructing embeddings from edge split complexes

Our next step is to prove the following lemma analogously to one of the implications of [Theorem 4.3](#). Given a simplicial complex \( C \) and a topological embedding \( \iota \) into \( S^3 \) of any of its split complex \( \hat{C} \) into \( S^3 \), we say that \( \iota \) satisfies the dual graph connectivity constraints (with respect to \( C \)) if for any vertex or edge \( x \) of \( C \), the set of faces incident with \( x \) is a connected edge set of the dual graph of \( \iota \).

**Lemma 4.4.** Let \( C \) be a locally connected simplicial complex. Assume that the split complex of \( C \) has an embedding \( \iota' \) into \( S^3 \) that satisfies the dual graph connectivity constraints. Then \( C \) has an embedding in \( S^3 \) that has the same dual graph as \( \iota' \).

Working with a strip instead of a unit interval, one shows the following analogously to [Lemma 4.1](#).

**Lemma 4.5.** Let \( C \) be a simplicial complex obtained from a simplicial complex \( C' \) by identifying two edges \( e \) and \( e' \) with disjoint sets of endvertices. Let \( \iota' \) be a topological embedding of \( C' \) into \( S^3 \). Assume that there is a local surface of \( \iota' \) that contains both \( e \) and \( e' \). Then there is a topological embedding of \( C \) into \( S^3 \) that has the same dual graph as \( \iota' \). \( \square \)

**Proof of Lemma 4.4.** Since the split complex is independent of the ordering in which we do splittings, the split complex \( C' \) of \( C \) is obtained by a sequence of the following operations: first we split an edge. Then we split the two endvertices of that edge. After that the complex is again locally connected. So we eventually derive at the split complex.

We make an inductive argument similar as in the proof of [Theorem 4.3](#). Thus it suffices to show that if a complex embeds and satisfies the dual graph connectivity constraints at the clones of some edge, we can reverse the splitting at that edge within the embedding.
After such a splitting operation the original edge is split into a set of vertex-disjoint edges. By the dual graph connectivity constraints, there are two of these edges in a common local surface of the embedding. So we can apply Lemma 4.5 to identify them. Arguing inductively, we can identify them all recursively. This shows why one such splitting can be reversed. Hence we can argue inductively as in the proof of Theorem 4.3 to complete the proof.

4.3 Embeddings induce embeddings of split complexes

The goal of this subsection is to prove the following.

**Lemma 4.6.** Let $C$ be a locally connected simplicial complex with an embedding $\iota$ in $S^3$. Then its split complex has an embedding into $S^3$ that satisfies the dual graph connectivity constraints and has the same dual graph as $\iota$.

Before we can prove this, we need some preparation. We start with the following lemma very similar to Lemma 4.5. We define ‘determined’ and reveal the definition in the proof of the next lemma.

**Lemma 4.7.** Let $C$ be a simplicial complex obtained from a simplicial complex $C'$ by identifying two edges $e$ and $e'$ that only share the vertex $v$. Let $\iota'$ be a topological embedding of $C'$ into $S^3$. Assume that the embedding of $L(v)$ in the plane induced by $\iota'$ has a region $R^{25}$ that contains both $e$ and $e'$. Then there is a topological embedding of $C$ into $S^3$ that has the same dual graph as $\iota'$. The cyclic orientation at the new edge is determined.

**Proof.** We image that the link graph at $v$ is embedded in a small ball around $v$. Then the region $R$ containing $e$ and $e'$ is included in a unique local surface of $\iota'$. We call that local surface $\ell$. We obtain $\bar{C}$ from $C'$ by adding a face $f$ at the edges $e$, $e'$ and one new edge. The embedding $\iota$ induces an embedding of $\bar{C}$ as follows. We embed $C'$ as prescribed by $\iota'$ and embed $f$ in $\ell$. It remains to specify the faces just before or just after $f$ at $e$ and $e'$. The face $f'$ just before $f$ at $e$ corresponds to some edge of $L(v)$ that has the region $R$ on its left, when directed towards $e$. Similarly, the face $f''$ just after $f$ at $e'$ corresponds to some edge of $L(v)$ that has the region $R$ on its right, when directed towards $e'$. This embedding of $\bar{C}$ induces some embedding of $C$ by first contracting the third edge of $f$, the one not equal to $e$ or $e'$ and then contracting the face $f$, that is, we identify $e$ and $e'$ along $f$. Clearly this embedding has the same dual graph as $\iota'$.

$^{25}$Component of $S^2$ without $L(v)$
It remains to show that the cyclic orientation of the incident faces induced by the embedding at the new edge is determined. For that we reveal the definition of determined. It means that the cyclic ordering at the new edge is obtained by concatenating the cyclic orientations of $e$ and $e'$ induced by $\iota'$ so that $f'$ is followed by $f''$.

For the rest of this subsection we fix a topological embedding $\iota$ of a locally connected simplicial complex $C$ into $S^3$. Our aim is to explain how $\iota$ gives rise to an embedding of any split complex of $C$. First we need some preparation. Let $\Sigma = (\sigma(e)| e \in E(C))$ be the combinatorial embedding induced by $\iota$.

Let $e$ be an edge of $C$ and $I$ a subinterval of $\sigma(e)$. Let $\bar{C}$ be the simplicial complex obtained from $C$ by replacing $e$ by two edges, one that is incident with the faces in $I$ and the other that is incident with the faces incident with $e$ but not in $I$. We call $\bar{C}$ the simplicial complex obtained from $C$ by \textit{opening} the edge $e$ \textit{along} $I$. We refer to the two new edges as the \textit{opening clones} of $e$. If we apply several openings, we extend the notion of opening cloning iteratively so that each edge of the resulting simplicial complex is opening cloned from a unique edge of $C$.

Let $C'$ be a simplicial complex obtained from a simplicial complex $C$ by splitting edges. Given a rotation system $\Sigma$ of $C$, we obtain the \textit{induced} rotation system of $C'$ by restricting for each $e'$ of $C'$ cloned from an edge $e$ of $C$ the cyclic ordering $\sigma(e)$ to the faces incident with $e'$. We define also an induced rotation system if $C'$ is obtained from $C$ by opening edges. This is as above with ‘clone’ replaced by ‘opening clone’.

Let $\bar{C}$ be a simplicial complex obtained from $C$ by opening an edge and let $\bar{\Sigma}$ be the rotation system induced by $\Sigma$.

**Lemma 4.8.** The simplicial complex $\bar{C}$ has a topological embedding $\bar{\iota}$ into $S^3$ whose induced planar rotation system is $\bar{\Sigma}$.

The dual graph of $\bar{\iota}$ is obtained from the dual graph $G$ of $\iota$ by identifying the two endvertices of $I$ when considered as a trail in $G$.

In particular, if $I$ is a closed trail in $G$, then $G$ is the dual graph of $\bar{\iota}$.

**Proof of Lemma 4.8.** We can modify the embedding of $C$ such that there is an open cylinder around $e$ that does not intersect any edge except for $e$ or any face not incident with $e$. And all faces in $I$ intersect that cylinder only in the left half of the cylinder and the others only in the right half. Now we replace $e$ by two copies - one in the left half, the other in the right half. It is straightforward to check that the dual graph of the embedding has the desired property. $\Box$
We fix an edge $e$ of $C$ with endvertices $v$ and $w$.

**Lemma 4.9.** There is an embedding $\iota'$ of $C$ in $S^3$ that has the same dual graph as $\iota$ such that there is some connected component $X$ at $e$ that is a subinterval of the cyclic orientation $\sigma'(e)$, where $\Sigma' = (\sigma'(e)|e \in E(C))$ is the induced rotation system of $\iota'$.

**Example 4.10.** The following example demonstrates that in Lemma 4.9 we cannot always pick $\iota' = \iota$. In the embedding in 3-space indicated in Figure 4 no component at the edge $e$ is a subinterval of the cyclic orientation of the faces incident with $e$ induced by the embedding.

![Figure 4: This complex is obtained by gluing together two discs, each with four faces, at the edge $e$.](image)

Before we can prove Lemma 4.9 we need some preparation.

Given a cyclic orientation $\sigma$ and a subset $X$, we say that two elements $y_1$ and $y_2$ of $\sigma$ separate $X$ for $\sigma$ if they are both not in $X$ and the two intervals $y_1 \sigma y_2$ and $y_2 \sigma y_1$ both contain elements of $X$.

**Lemma 4.11.** Let $\sigma$ be a cyclic orientation and $(P_i|i \in [n])$ be a partition of the elements of $\sigma$ such that no two elements of the same $P_i$ separate some other $P_j$. Then there is some $P_k$ that is a subinterval of $\sigma$.

**Proof.** We pick an arbitrary element $a$ of $P_1$. We may assume that a partition class $P_2$ exists. For any $P_i$ not containing $a$, we define its first element to first element of $P_i$ after $a$ in $\sigma$, and its last element to first element of $P_i$ before $a$ in $\sigma$. The closure of $P_i$ consists of those elements of $\sigma$ between its first and last element (including the first and the last one). We denote the closure of $P_i$ by $\overline{P_i}$.

By assumption any two such closures $\overline{P_i}$ and $\overline{P_j}$ are either disjoint or contained in one another, that is, $\overline{P_i} \subseteq \overline{P_j}$ or vice versa. Let $P_k$ be such

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26By $y_1 \sigma y_2$ we denote the subinterval of $\sigma$ starting at $y_1$ and ending with $y_2$. 

23
that its closure is inclusion-wise minimal. Then \( P_k \) is equal to its closure and hence a subinterval of \( \sigma \).

Given \( e \in \sigma \), we denote the element just before \( e \) by \( e - 1 \) and the element just after \( e \) by \( e + 1 \). Given a cyclic orientation \( \sigma \) and four of its elements \( x_1, x_2, x_3, x_4 \) such that \((x_1x_2x_3x_4)\) is a cyclic subordering of \( \sigma \), the exchange of \( \sigma \) with respect to \( x_1, x_2, x_3, x_4 \) is the following cyclic orientation on the same elements as \( \sigma \). We concatenate the two cyclic orientations obtained from \( \sigma \) by deleting \( x_1\sigma x_3 - x_1 - x_3 \) and \( x_3\sigma x_1 \) such that the immediate successor of \( x_4 \) is \( x_2 \); see Figure 5, formally, it is

\[
x_3\sigma x_4(x_2\sigma x_3 - x_3)(x_1\sigma x_2 - x_1 - x_2)(x_4 + 1)\sigma x_3
\]

Figure 5: The cyclic orientation \( \sigma \) is depicted as a cycle. The four segments between the element \( x_i \) are labelled with the elements of \( \mathbb{Z}_4 \). This describes the ordering in which these segments are traversed by the exchanged cyclic orientation.

Let \((P_i|i \in I)\) be a partition of the elements of \( \sigma \), the fluctuation of \( \sigma \) with respect to \((P_i|i \in I)\) is the number of adjacent elements of \( \sigma \) in different \( P_i \). Given a partition \( \mathcal{P} = (P_i|i \in I) \) of \( \sigma \), an exchange is \( \mathcal{P} \)-improving if \( x_2 \) and \( x_4 \) are in the same \( P_i \) but none of the following four pairs is in the same \( P_i \): \((x_4, x_4 + 1), (x_2, x_2 - 1), (x_1, x_1 + 1), (x_3, x_3 - 1)\).

**Lemma 4.12.** A cyclic orientation \( \sigma' \) obtained from \( \sigma \) by an exchange that is \( \mathcal{P} \)-improving has strictly smaller fluctuation.

**Proof.** The adjacent elements of \( \sigma \) and \( \sigma' \) are the same except for four pairs involving \( x_1, x_2, x_3, x_4 \). For \( \sigma \) these pairs are those mentioned in the definition of ‘\( \mathcal{P} \)-improving’. All these four pairs contribute to the fluctuation by the definition of \( \mathcal{P} \)-improving. For \( \sigma' \) the pair \((x_4, x_2)\) does not contribute to the fluctuation. \( \square \)
One way to partition \( \sigma(e) \) is to put two elements of \( \sigma(e) \) in the same class if – when considered as edges of \( L(v) \) – they have endvertices in the same component of \( L(v) - e \). An exchange is \( v \)-improving for \( \sigma(e) \) if it is \( \mathcal{P} \)-improving for that particular partition.

For the next lemma we fix the following notation. Let \( X \) be the set of edges between \( e \) and a connected component of \( L(v) - e \). Let \( Y \) be the set of edges between \( e \) and a connected component of \( L(w) - e \). Assume that no connected component at \( e \) includes both \( X \) and \( Y \).

**Lemma 4.13.** Assume that two elements of \( Y \) separate \( X \) in the cyclic orientation \( \sigma(e) \). Then there is an embedding \( \iota' \) of \( C \) in \( S^3 \) that has the same dual graph as \( e \) such that \( \sigma'(e) \) is obtained from \( \sigma(e) \) by a \( v \)-improving exchange, where \( \Sigma' = (\sigma'(e)|e \in E(C)) \) is the induced rotation system of \( \iota' \).

**Proof.** We abbreviate \( \sigma(e) \) by \( \sigma \). We denote the connected component at \( e \) including \( Y \) by \( c(Y) \).

**Sublemma 4.14.** There are edges \( f_1 \) and \( f_3 \) of \( c(Y) \) that separate \( X \) such that the region of \( L(w) \) just after \( f_1 \) is equal to the region just before \( f_3 \). And \( f_1 + 1 \) and \( f_3 - 1 \) are not in \( c(Y) \).

**Proof.** Let \( f'_1 \) and \( f'_3 \) be two elements of \( Y \) that separate \( X \). We fix two elements \( x_1 \) and \( x_2 \) of \( X \) such that \( x_1 \) is in \( f'_1 \sigma f'_3 \) and \( x_2 \) is in \( f'_3 \sigma f'_1 \). By choosing \( f'_1 \) and \( f'_3 \) as near to \( x_1 \) as possible, we ensure that the region just after \( f'_1 \) is equal to the region just before \( f'_3 \). We denote this region by \( R \). Let \( Y' \) be the set of edges between \( e \) and a connected component of \( L(w) - e \) that is included in \( c(Y') \). The set of all such \( Y' \) is denoted by \( \mathcal{Y} \). By replacing \( Y \) by any \( Y' \in \mathcal{Y} \) if necessary, we may assume that no set \( Y' \in \mathcal{Y} \) contains elements both before and after \( x_1 \) on \( f'_1 \sigma f'_3 \); indeed, by any such replacement \( f'_1 \sigma f'_3 \) strictly decreases.

**Sublemma 4.15.** The interval \( f'_1 \sigma x_1 \) contains some \( f_1 \in c(Y) \) such that the region just after \( f_1 \) is \( R \) and \( f_1 + 1 \) is not in \( c(Y) \).

**Proof.** We recursively define a sequence \( f^n_1 \) of elements of \( f'_1 \sigma x_1 \). They are strictly increasing and contained in \( c(Y) \). We start with \( f^1_1 = f'_1 \). Assume that we already constructed \( f^n_1 \). If \( f^n_1 + 1 \) is not in \( c(Y) \) we stop and let \( f_1 = f^n_1 \). Otherwise \( f^n_1 + 1 \) is in \( c(Y) \). Let \( Y' \in \mathcal{Y} \) so that \( f^n_1 + 1 \in Y' \).

We prove inductively during this construction that any set \( Y'' \in \mathcal{Y} \) that contains an element of \( (f^n_1 + 1)\sigma x_1 \) contains no element of \( f'_1 \sigma f^n_1 \).

By the induction hypothesis, \( Y' \) is a subset of \( (f^n_1 + 1)\sigma x_1 \). Let \( f^{n+1}_1 \) be the maximal element of \( Y' \) in \( (f^n_1 + 1)\sigma x_1 \). By construction \( f^{n+1}_1 \in c(Y) \).
and \( f_1^{n+1} \) is strictly larger than \( f_1^n \). The region \( R \) is just before \( f_1^n + 1 \), the first element of \( Y' \). Thus the region after \( f_1^{n+1} \), the last element of \( Y' \), must also be \( R \). The induction step follows from the planarity of \( L(w) - e \) as there is a component of \( L(w) - e \) that is adjacent to the set \( Y' \), and the induction hypothesis.

This process has to stop as \( f'_1 \sigma x_1 \) is finite and the \( f_1^n \) are strictly increasing. Thus we eventually find an \( f_1 \). \( \square \)

Similarly as \( \text{Sublemma 4.15} \) one shows that the interval \( x_1 \sigma f'_3 \) contains some \( f_3 \in c(Y) \) such that the region just before \( f_3 \) is \( R \) and \( f_3 - 1 \) is not in \( c(Y) \). So \( f_1 \) and \( f_3 \) have the desired properties. \( \square \)

We obtain \( C_1 \) from \( C \) by opening the edge \( e \) at the subinterval \( f_1 \sigma f_3 \) of \( \sigma \). By \( \iota_1 \) we denote the embedding of \( C_1 \) induced by \( \iota \). By the choice of \( f_1 \) and \( f_3 \), the local surface just after \( f_1 \) is equal to the local surface just before \( f_3 \). Hence by \( \text{Lemma 4.8} \) the embeddings \( \iota_1 \) and \( \iota \) have the same dual graph.

By \( \text{Sublemma 4.14} \) the link graph at \( w \) of \( C_1 \) has two connected components. We obtain \( C_2 \) from \( C_1 \) by splitting the vertex \( w \). By \( \iota_2 \) we denote the embedding of \( C_2 \) induced by \( \iota_1 \). As splitting vertices does not change the dual graph by \( \text{Observation 2.9} \) the embeddings \( \iota_2 \) and \( \iota_1 \) have the same dual graph. Summing up, \( \iota_2 \) and \( \iota \) have the same dual graph.

We denote the copy of \( e \) incident with \( f_1 \) by \( e' \) and the other copy by \( e'' \). Since \( e' \) and \( e'' \) are both incident with edges of \( X \), the component of \( L(v) - e \) adjacent to the edges of \( X \) has in the link graph of \( C_2 \) the two vertices \( e' \) and \( e'' \) in the neighbourhood. Thus the vertices \( e' \) and \( e'' \) share a face in the link graph at \( v \) of \( C_2 \).

By \( \text{Lemma 4.7} \) \( \iota_2 \) induces an embedding \( \iota' \) of \( C \) in \( S^3 \) that has the same dual graph as \( \iota_2 \). Let \( \Sigma' = (\sigma'(e) | e \in E(C)) \) is the induced rotation system of \( \iota' \). We denote the element of \( X \) in \( f_1 \sigma f_3 \) nearest to \( f_1 \) by \( f_2 \). Similarly, by \( f_4 \) we denote the element of \( X \) in \( f_3 \sigma f_1 \) nearest to \( f_1 \). As \( \sigma'(e) \) is determined by \( \text{Lemma 4.7} \) it is obtained by concatenating the cyclic orientations at \( e' \) and \( e'' \) so that \( f_4 \) is followed by \( f_2 \). That is, \( \sigma'(e) \) is obtained from \( \sigma(e) \) by exchanging with respect to \( f_1, f_2, f_3, f_4 \).

It remains to check that this exchange is \( v \)-improving. Both \( f_2 \) and \( f_4 \) are in \( X \). On the other hand \( f_1 \) and \( f_3 \) are in \( c(Y) \) but \( f_1 + 1 \) and \( f_3 - 1 \) are not in \( c(Y) \). In particular, they are in different \( P_i \). Whilst \( f_2 \) and \( f_4 \) are in \( X \), the two elements \( f_2 - 1 \) and \( f_4 + 1 \) are not in \( X \). Thus this exchange is \( v \)-improving. \( \square \)

\textit{Proof of Lemma 4.11} \( \text{By (} R_k | k \in K \text{) we denote the partition of the faces incident with } e \text{ into the connected components at } e \text{. If no two elements of} \)
the same \( R_a \) separate some other \( R_b \), then by Lemma 4.11 there is some \( R_a \) that is a subinterval of \( \sigma(e) \). In this case we can just pick \( \iota' = \iota \) and are done.

We define the partition \((P_i|i \in I)\) of the faces incident with \( e \) as follows. Two faces incident with \( e \) are in the same partition if – when considered as edges of \( L(v) \) – they have endvertices in the same component of \( L(v) - e \). We define the partition \((Q_j|j \in J)\) the same with ‘\( w \)’ in place of ‘\( v \)’. If some \( P_i \) contains two elements separating some \( Q_j \) for the cyclic orientation at \( e \), we can apply Lemma 4.13 to construct a new embedding of \( C \). We do this until there are no longer such pairs \((P_i, Q_j)\). This has to stop after finitely many steps as by Lemma 4.12 the fluctuation – which is a non-negative constant only defined in terms of \((P_i|i \in I)\) – of the cyclic orientation at \( e \) strictly decreases in each step. So there is an embedding \( \iota' \) of \( C \) in \( \mathbb{S}^3 \) such that no \( P_i \) contains two elements separating some \( Q_j \) for the cyclic orientation \( \sigma'(e) \) and such that \( \iota' \) has the same dual graph as \( \iota \); here we denote by \( \Sigma' = (\sigma'(e)|e \in E(C)) \) is the induced rotation system of \( \iota' \). Hence by applying Lemma 4.11, it suffices to prove the following.

Sublemma 4.16. For \( \sigma'(e) \), either there is some \( P_i \) containing two elements separating some \( Q_j \) or no two elements of the same \( R_a \) separate some other \( R_b \).

Proof. We assume that there is some \( R_a \) that contains two elements \( r_1 \) and \( r_2 \) that separate some other \( R_b \). The set \( R_b \) is a disjoint union of sets \( P_i \). Either \( r_1 \) and \( r_2 \) separate one of these \( P_i \) or by the definition of connected component at \( e \), there is some \( Q_j \) included in \( R_b \) that contains elements of different \( P_i \), one included in \( r_1\sigma'(e)r_2 \) and the other in \( r_2\sigma'(e)r_1 \). Summing up there is some \( P_i \) or \( Q_j \) included in \( R_b \) that is separated by \( r_1 \) and \( r_2 \).

First we consider the case that there is a set \( P_i \). So two elements of that set \( P_i \) separate \( R_a \). By an argument as above we conclude that there is some \( P_m \) or \( Q_n \) included in \( R_a \) that is separated by two elements of \( P_i \).

Since the sets \( P_m \) are defined from components of \( L(v) - e \) and \( \Sigma' \) induces an embedding of \( L(v) \) in the plane, these components cannot attach at \( e \) in a ‘crossing way’, that no two elements of some \( P_i \) can separate some other \( P_m \). Thus there has to be such a set \( Q_n \).

Summing up, if there is a set \( P_i \) separated by \( r_1 \) and \( r_2 \), then it contains two elements separating some \( Q_n \). Analogously one shows that otherwise the set \( Q_j \) separated by \( r_1 \) and \( r_2 \) contains two elements separating some \( P_n \). But then two elements of \( P_n \) separate \( Q_j \). This completes the proof. \( \square \)

By the construction of \( \iota' \), no two elements of the same \( R_a \) separate
some other $R_b$ for $\sigma'(e)$. Then by Lemma 4.11 there is some $R_a$ that is a subinterval of $\sigma'(e)$, as desired.

Let $C'$ be a simplicial complex obtained from the locally connected simplicial complex $C$ by splitting the edge $e$.

**Lemma 4.17.** There is a topological embedding $\iota'$ of $C'$ whose induced planar rotation system is the rotation system induced by $\Sigma$.

Moreover $\iota$ and $\iota'$ have the same dual graph.

*Proof.* We denote the dual graph of $\iota$ by $G$. We prove this lemma by induction on the number of connected components at $e$. If there is only one such component, then $C' = C$ and the lemma is trivially true. So we may assume that there are at least two components. By changing the embedding if necessary, by Lemma 4.9 we may assume that there is a component $J$ at $e$ that is a subinterval of $\sigma(e)$. As $J$ is a subinterval of the closed trail $\sigma(e)$ of $G$, it is a trail in $G$. Next we show that it is a closed one:

**Sublemma 4.18.** The interval $J$ is a closed trail in $G$.

*Proof.* We are to show that the local surface of the embedding just before the first face $f_1$ of $J$ is the same as the local surface just after the last edge $f_2$ of $J$. For that it suffices to show that in the embedding of the link graph $L(v)$ of $v$ induced by $\Sigma$, the region just before the edge $f_1$ is the same as the region just after the edge $f_2$. This follows from the fact that $J$ is the set of edges out of a set of connected components of $L(v) - e$. Indeed, the first and last edge out of every component are always in the same region.

We obtain $\bar{C}$ from $C$ by opening the edge $e$ along $J$. By Lemma 4.8, $\bar{C}$ has a topological embedding $\bar{\iota}$ into $S^3$ whose induced planar rotation system is induced by $\Sigma$. By Sublemma 4.18 and Lemma 4.8, the dual graph of $\bar{\iota}$ is $G$.

We observe that $C'$ is obtained from $\bar{C}$ by splitting the clone of $e$ that corresponds to the subinterval $\sigma(e) \setminus J$. Thus the lemma follows by applying induction on $\bar{C}$ and $\bar{\iota}$.

*Proof of Lemma 4.6.* The split complex of $C$ is obtained from $C$ by a sequence of edge splittings and vertex splittings. By changing the order of the splittings if necessary, we may assume that the complex is always locally connected before we perform an edge splitting. Hence we can apply Lemma 4.17 and Theorem 4.3 recursively to construct an embedding of the split complex. Since in each splitting step the dual graph is preserved, it
satisfies the dual graph connectivity constraints by Lemma 4.2 applied to the dual graph of $\iota$.

4.4 Proof of Theorem 1.4

We summarise the results of the earlier subsections in the following.

Theorem 4.19. Let $C$ be a simplicial complex and $\hat{C}$ be its split complex. Then $C$ embeds into $S^3$ if and only if $\hat{C}$ has an embedding into $S^3$ that satisfies the dual graph connectivity constraints.

Proof. Assume that $C$ embeds into $S^3$. Then by Theorem 4.3 its vertical split complex embeds into $S^3$ and satisfies the vertical graph connectivity constraints. Since the vertical split complex is locally connected, we can apply Lemma 4.6 to get the desired embedding of the split complex. Note that this embedding has the same dual graph as the vertical split complex. Hence it also satisfies the connectivity constraints for the vertices.

Now conversely assume that the split complex has an embedding $\iota'$ that satisfies the dual graph connectivity constraints. By Lemma 4.4 the vertical split complex has an embedding in $S^3$. As this embedding has the same dual graph as $\iota'$, it satisfies the vertical dual graph connectivity constraints. So we can apply Theorem 4.3. This completes the proof.

Now we show how Theorem 4.19 implies Theorem 1.4.

Proof of Theorem 1.4. Let $C$ be a globally 3-connected simplicial complex and let $\hat{C}$ be its split complex. If $C$ embeds into $S^3$, then $\hat{C}$ has an embedding into $S^3$ whose dual graph $G$ satisfies the dual graph connectivity constraints by Theorem 4.19. By Corollary 2.14 the two simplicial complexes $C$ and $\hat{C}$ have the same dual matroid. So by Theorem 1.1 the cycle matroid of $G$ is the dual matroid of $C$. This completes the proof of the ‘only if’-implication.

Conversely assume that a split complex $\hat{C}$ of a simplicial complex $C$ has an embedding $\iota$ into $S^3$ and the dual matroid $M$ of $C$ is the cycle matroid of a graph $G$ and the set of faces incident with any vertex or edge of $C$ is a connected edge set of $G$. By Corollary 2.14 $M$ is the dual matroid of $\hat{C}$. Let $G'$ be the dual graph of the embedding $\iota$ of $\hat{C}$. By Theorem 1.1 the cycle matroid of $G'$ is equal to $M$. Since $M$ is 3-connected by assumption, by a theorem of Whitney [11], the graphs $G$ and $G'$ are identical. Hence $G'$ satisfies the connectivity constraints. So we can apply the ‘if’-implication of Theorem 4.19 to deduce the ‘if’-implication of Theorem 1.4.
Proof of Theorem 1.5. By [1], it suffices to show that a simplicial complex \( C \) whose split complex is embeddable has an embedding if and only if its dual matroid has no constraint minor in the list of Figure 2. Since the split complex is embeddable, its dual matroid is the cycle matroid of a graph \( G \). By Corollary 2.14 the dual matroid of \( C \) is the cycle matroid of \( G \). By Theorem 1.4, \( C \) is embeddable if and only if \( G \) satisfies the graph connectivity constraints. The latter is true if and only if there is no vertex or edge such that the set \( X \) of incident faces is disconnected in \( G \). By the main result of [3], \( X \) is disconnected in \( G \) if and only if \( (G, X) \) has a constraint minor in the list of Figure 2. \( \square \)

5 Infinitely many obstructions to embeddability into 3-space

In this section we construct an infinite sequence \( (A_n|n \in \mathbb{N}) \) of minimal obstructions to embeddability. More precisely, \( A_n \) will have the property that its split complex is simply connected and embeddable, its dual matroid \( M_n \) is the cycle matroid of a graph but no such graph will satisfy the connectivity constraints. However, if we remove a constraint or contract or delete an element from the dual matroid, then there is such a graph.

The dual matroid \( M_n \) of \( A_n \) will be the disjoint union of a cycle \( C_n \) of length \( n \) and a loop \( \ell \), see Figure 6.

Figure 6: The matroid \( M_8 \). For each of the eight vertices on the cycle, there is a connectivity constraint forbidding that the loop is attached at that vertex.

The connectivity constraints are as follows. Fix a cyclic orientation \( \{e_i|i \in Z_i\} \) of the edges on \( C_n \). We have a connectivity constraint for every \( i \in [n] \), namely that \( X[i, n] = C_n - e_i - e_{i+1} + \ell \) is a connected set.

**Fact 5.1.** There is no graph whose cycle matroid is \( M_n \) that meets all the connectivity constraints \( X[i, n] \).
Proof. By \( \overline{C_n} \) we denote the graph that is a cycle of length \( n \) whose edges have the cyclic ordering \( \{e_i | i \in \mathbb{Z}\} \). It is straightforward to see that \( \overline{C_n} \) is the unique graph whose cycle matroid is \( C_n \) that meets all the connectivity constraints \( X[i, n] \rightarrow e \).

Now suppose for a contradiction that there is a graph \( G \) whose cycle matroid is \( M_n \) that meets all the connectivity constraints \( X[i, n] \). Then \( G \) is obtained from \( \overline{C_n} \) by attaching a loop. Since each \( X[i, n] \) contains \( e \), we have to attach the loop at some vertex of \( \overline{C_n} \). The connectivity constraint \( X[i, n] \), however, forbids us to attach the loop at the vertex incident with \( e_i \) and \( e_{i+1} \). Hence \( G \) does not exist. \( \square \)

A careful analysis of this proof yields the following simple facts.

**Fact 5.2.**

1. There is a graph whose cycle matroid is \( M_n \) that meets all the connectivity constraints \( X[i, n] \) but one.

2. for every element \( e \), there is a graph whose cycle matroid is \( M_n - e \) that meets all the connectivity constraints \( X[i, n] - e \);

3. for every element \( e \), there is a graph whose cycle matroid is \( M_n/e \) that meets all the connectivity constraints \( X[i, n] - e \).

\( \square \)

Hence it remains to construct \( A_n \) such that its dual matroids is \( M_n \) and so that the nontrivial connectivity constraints are the \( X[i, n] \). We remark that we allow the faces of \( A_n \) to be arbitrary closed walks. (One obtains a simplicial complexes from \( A_n \) by applying baricentric subdivisions to the faces.)

We start the construction of \( A_n \) with a cycle \( C \) of length \( n \). We attach \( n \) faces, which we call \( e_1, ..., e_n \). For each \( e_i \), and each vertex \( v_k \) of \( C \) except for the \( i \)-th vertex \( v_i \), we attach \( n - 1 \) edges and let \( e_i \) traverse them in between the two edges incident with \( v_k \). We denote the endvertices of the new edges not on \( C \) by \( x(i, k, j) \) where \( (k, j \leq n; k, j \neq i) \), see Figure 7.

Next we disjointly add a copy of the original cycle \( C \) and only attach a single face to it which we denote by \( \ell \). Call the resulting walk-complex\(^{27} \) \( A'_n \). We finally obtain \( A_n \) from \( A'_n \) by identifying for each \( i \in [n] \) the \( i \)-th vertex \( v_i \) on the new copy of \( C \) with all vertices \( x(i', i, i) \) with \( i' \neq i \).

\(^{27}\)A **walk-complex** is a graph together with a family of closed walks, which we call its faces. Every simplicial complex is a walk-complex. Conversely, from every walk complex we can build a simplicial complex by attaching at each face a cone over that walk.
By construction, the split complex of $A_n$ is $A'_n$. Hence by Corollary 2.14 above, the dual matroid of $A_n$ is $M_n$. By construction, the nontrivial connectivity constraints are the $X[i,n]$. Clearly, the split complex $A'_n$ is simply connected and embeddable.

This completes the construction of the $A_n$. By Fact 5.1 and Fact 5.2 they have the desired properties.

A Appendix I

First we give a definition of ‘globally 3-connected’ directly in terms of the simplicial complex without referring to its dual matroid. Given a simplicial complex $C$, its edge/face incidence matrix $A$ and a subset $L$ of the faces of $C$, we denote by $r(L)$ the rank over $\mathbb{F}_3$ of the submatrix of $A$ induced by the vectors whose faces are in $L$. A 2-separation of a simplicial complex $C$ is a partition of its set $F$ of faces into two sets $L$ and $R$ both of size at least two such that $r(L) + r(R) \leq r(F) + 1$. It is straightforward it check that a simplicial complex is globally 3-connected if and only if it has no 2-separation.

When defining ‘edge split complexes’, we mentioned a related more naive definition. Here we give this definition. In Example A.1 and Example A.2 we show that this notion lacks two important features of edge split complexes. Splitting an edge $e$ at an endvertex $v$ is defined like ‘splitting $e$’ but with ‘in the same connected component at $e$’ replaced by ‘$v$-related’. A lazy edge split complex is defined as ‘edge split complex’ but with ‘for every edge there is only one component at $e$’ replaced by ‘it is locally 2-connected’. lazy split complex is defined like ‘split complex’ with ‘lazy edge split complex’ in place of ‘edge split complex’.
Example A.1. In this example we construct a simplicial complex $C$ that has two distinct lazy edge split complexes. We will construct $C$ such that it has two vertices $v$ and $w$; these vertices are joined by five edges $e$, $e_1$, $e_2$, $e_3$ and $e_4$. The edge $e$ is a cut vertex in the link graphs at $v$ and $w$. And splitting $e$ at one endvertex will make the link graph at the other endvertex 2-connected, see Figure 8.

![Link graph at v](image1)

![Link graph at w](image2)

Figure 8: If we split one of these link graphs at $e$, the other becomes a six-cycle.

Next we construct $C$ with the above properties. We obtained $C$ from four triangular faces $f_1$, $f_2$, $f_3$ and $f_4$ glued together at a single edge $e$. Let $v$ and $w$ be the two endvertices of that edge. Let $e_i[v]$ be the edge of $f_i$ incident with $v$ different from $e$. Let $e_i[w]$ be the edge of $f_i$ incident with $w$ different from $e$. Let $v_i$ be the vertex incident with $f_i$ that is not incident with $e$. We add the edges $e_k$ between $v_k$ and $v_{k+1}$ for any $k \in \mathbb{Z}_4$. We add the four faces: $e_1[v]e_1e_2[v]$, $e_3[v]e_3e_4[v]$, $e_2[w]e_3[w]$ and $e_4[w]e_4e_1[w]$. This completes the construction of $C$.

Example A.2. In this example we show that Theorem 4.19 with ‘split complex’ replaced by ‘lazy split complex’ is false. Let $H$ be a planar graph with vertices $v$ and $w$ such that the graph $H'$ obtained from $H$ by identifying the vertices $v$ and $w$ is not planar. Let $C$ be the cone over $H$. We obtain $C'$ from $C$ by identifying the two edges corresponding to $v$ and $w$. Whilst the link at the top of $C$ is $H$, the complex $C'$ has the link $H'$ and is hence not embeddable. By choosing $v$ and $w$ far apart in $H$, one ensures that $C'$ is a simplicial complex.

The lazy split complex of $C'$ is unique and equal to $C$. Unlike $C'$, the simplicial complex $C$ is embeddable. The dual graph of every embedding consists of a single vertex, and so trivially satisfies the graph connectivity constraints. This completes the example.

Concerning Theorem 1.4, it is straightforward to modify the example to make the dual graph of the embedding 3-connected.
B Appendix II: Matrices representing matroids over the integers

Matroids representable over the integers are well-studied [7]. In this appendix, we study something very related but slightly different, namely matrices that represent matroids over the integers. Our aim in this appendix is to prove Theorem B.6 below, which is a characterisation of certain matrices representing matroids over the integers.

A matrix $A$ is a representation of a matroid $M$ over a field $k$ if all its entries are in $k$ and the columns are indexed with the elements of $M$. Furthermore for every circuit $o$ of $M$ there is a vector $v_o$ in the span over $k$ of the rows of $A$ whose support is $o$. And the vectors $v_o$ span over $k$ all row vectors of $A$.

The following is well-known.

**Lemma B.1.** Let $A$ be a matrix representing a matroid $M$ over some field $k$. Let $I$ an element set that is independent in $M$. Then the matrix obtained from $A$ by deleting all columns belonging to elements of $I$ represents the matroid $M/I$ over $k$.

A matrix $A$ is a regular representation (or representation over the integers) of a matroid $M$ if all its entries are integers and the columns are indexed with the elements of $M$. Furthermore for every circuit $o$ of $M$ there is a $\{0, -1, +1\}$-valued vector $v_o$ in the span over $\mathbb{Z}$ of the rows of $A$ whose support is $o$. And the vectors $v_o$ span over $\mathbb{Z}$ all row vectors of $A$. The following is well-known.

**Lemma B.2.** Assume that a matrix $A$ regularly represents a matroid $M$. Then for every cocircuit $d$ of $M$, there is a $\{0, -1, +1\}$-valued vector $w_d$ whose support is equal to $d$ that is orthogonal over $\mathbb{Z}$ to all row vectors of $A$. These vectors $w_d$ generate over $\mathbb{Z}$ all vectors that are orthogonal over $\mathbb{Z}$ to every row vector.

The following is well-known.

**Lemma B.3.** Let $M$ be a matroid regularly represented by a matrix $A$. Let $v$ be a sum of row vectors of $A$ with integer coefficients. If the support of $v$ is nonempty, then it includes a circuit of $M$.

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28 A vector is an element of a vector space $k^S$, where $k$ is a field and $S$ is a set. In a slight abuse of notation, in this paper we also call elements of modules of the form $\mathbb{Z}^S$ vectors.

29 Two vectors $a$ and $b$ in $k^S$ are orthogonal if $\sum_{s \in S} a(s) \cdot b(s)$ is identically zero over $k$. 
Example B.4. A matrix is **unimodular** if it is \{0, -1, +1\}-valued and the determinant of every quadratic submatrix is \{0, -1, +1\}-valued\(^{30}\). Every unimodular matrix is a regular representation of some matroid, see for example [8]. For example, the vertex/edge incidence matrix of a graph \(G\) is a regular representation of the graphic matroid of \(G\).

There also exist regular representations that are not totally unimodular:

**Example B.5.**

\[
\begin{pmatrix}
1 & 1 \\
1 & -1 \\
1 & 0 \\
0 & -1
\end{pmatrix}
\]

This matrix is a regular representation of the matroid consisting of two elements in parallel but it is not totally unimodular.

A matroid is **regular** if it can be regularly represented by some matrix. The class of regular matroids has many equivalent characterisations [7]. For example, a matroid has a regular representation (in fact a totally unimodular one) if and only if it has a representation over every field. In this paper, we need the following related fact, which focuses on the matrices instead of the matroids:

**Theorem B.6.** Let \(A\) be a matrix whose entries are \(-1, +1\) or 0. Then \(A\) regularly represents a matroid if and only if there is a single matroid \(M\) such that \(A\) represents \(M\) over any field.

Whilst the 'only if'-implication is immediate, the other implication is less obvious. To prove it we rely on the following.

**Lemma B.7.** Let \((v_i| i \in I)\) be a family of integer valued vectors of \(\mathbb{Z}^S\), where \(S\) is a finite set. Assume that the family \((v_i| i \in I)\) considered as vectors of the vector space \(\mathbb{Q}^S\) spans the whole of \(\mathbb{Q}^S\) over \(\mathbb{Q}\). Additionally, assume that for every prime number \(p\), the same assumption is true with the finite field \(\mathbb{F}_p\) in place of \(\mathbb{Q}\). Then the family \((v_i| i \in I)\) spans over \(\mathbb{Z}\) all integer valued vectors in \(\mathbb{Z}^S\).

**Proof that Lemma B.7 implies Theorem B.6.** Assume that \(A\) is an integer valued matrix that represents the matroid \(M\) over \(\mathbb{Q}\) and over all finite fields \(\mathbb{F}_p\) for every prime number \(p\), when we interpret\(^{31}\) the entries of \(A\)

---

\(^{30}\)Here we evaluate the determinate over \(\mathbb{Z}\)

\(^{31}\)Here in \(\mathbb{F}_p\) we interpret the integer \(m\) as its remainder after division by \(p\).
as elements of the appropriate field. Our aim is to show that \( A \) regularly represents the matroid \( M \).

Let \( b \) be a base of \( M \). Let \( A' \) be the matrix obtained from \( A \) by deleting all columns belonging to elements of \( b \). We denote by \( M' \) the matroid \( M/b \), in which every element is a loop. By Lemma B.1, \( A' \) represents the matroid \( M' \) over \( \mathbb{Q} \) and over all finite fields \( \mathbb{F}_p \). Let \( (v_i | i \in I) \) be the family of row vectors of \( A' \). Since every element of \( M' \) is a loop, we can apply Lemma B.7 and deduce that the family \( (v_i | i \in I) \) spans over \( \mathbb{Z} \) all integer valued vectors in \( \mathbb{Z}^{E'} \), where \( E' \) is the set of elements of \( M' \).

Let \( v \) be any integer valued vector that is generated by the rows of \( A \) over \( \mathbb{Q} \). We show that \( v \) is also generated by the rows of \( A \) with integer coefficients. By the above, there is a vector \( w \) generated from the row vectors of \( A \) over \( \mathbb{Z} \) that agrees with \( v \) in all coordinates of \( E' \). Hence \( v - w \) is generated by the row vectors over \( \mathbb{Q} \). So if \( v - w \) is nonzero, its support must contain a circuit of \( M \) by Lemma B.3. Since the support of \( v - w \) is contained in the base \( b \), the support does not contains a circuit of \( M \). Hence \( v \) must be equal to \( w \). Thus \( v \) is in the span of the row vectors with coefficients in \( \mathbb{Z} \).

Now let \( o \) be a circuit of \( M \). Since \( A \) is a regular representation of \( M \) over \( \mathbb{Q} \), there is a vector \( v_o \) with entries in \( \mathbb{Q} \) generated by the row vectors of \( A \) over \( \mathbb{Q} \) whose support is \( o \). We multiplying all entries with a suitable rational number if necessary, we may assume that additionally all entries of \( v_o \) are integers and that the greatest common divisor of the entries is one. By the above \( v_o \) is in the span of the row vectors with coefficients in \( \mathbb{Z} \).

Next we show that all entries of \( v_o \) are zero, plus one, or minus one. Suppose for a contradiction that there is some prime number \( p \) that divides some entry of \( v_o \). If we interpret the entries of \( v_o \) as elements of \( \mathbb{F}_p \), then \( v_o \) is also in the span of the row vectors with coefficients in \( \mathbb{F}_p \). Indeed, the coefficients are just the integer coefficients we have in the representation over \( \mathbb{Z} \) interpreted as elements of \( \mathbb{F}_p \). Since the greatest common divisor of the entries of \( v_o \) is one, \( v_o \) when interpreted over \( \mathbb{F}_p \) is nonzero but its support is properly contained in \( o \). Since in \( M \) the circuit \( o \) does not include another circuit, we get a contraction to the assumption that \( A \) represents \( M \) over \( \mathbb{F}_p \). Thus all entries of \( v_o \) are zero, plus one, or minus one.

It remains to show that the set of vectors \( v_o \) where \( o \) is a fundamental circuit of \( b \) generates every row vector \( x \) of \( A \). Since for every element not in \( b \), there is a unique \( v_o \) which takes the value plus one or minus one at that element and zero at every other elements not in \( b \), there is a vector \( x' \) generated over \( \mathbb{Z} \) by the \( v_o \) that agrees with \( x \) when restricted to \( E' \). As above we deduce that \( x' = x \), and hence \( x \) is generated by the \( v_o \) over \( \mathbb{Z} \).
Thus $A$ regularly represents $M$.  

In order to prove Lemma B.7, we rely on the following well-known lemma.

**Lemma B.8.** Let $m$ and $n$ be integer and let $d$ be their greatest common divisor. Then there are integers $\alpha$ and $\beta$ such that $\alpha \cdot m - \beta \cdot n = d$.  

**Proof of Lemma B.7**. Let $s \in S$ be arbitrary. By $e_s$ we denote the vector which in coordinate $s$ has the entry one and otherwise the entry zero. Since the family $(v_i | i \in I)$ spans $e_s$ over $\mathbb{Q}$, there is some positive natural number $\gamma_s$ so that the family $(v_i | i \in I)$ spans $\gamma_s \cdot e_s$ over $\mathbb{Z}$. Let $\delta_s$ be the least possible value for $\gamma_s$. Our aim is to show that all $\delta_s$ are equal to one. Suppose not for a contradiction. Then there is some prime number $p$ that divides some $\delta_s$. Let $\bar{s}$ be the index so that in the factorisation of $\delta_{\bar{s}}$ the prime number $p$ has the highest multiplicity, say $k$.

**Sublemma B.9.** There is some nonzero integer $\epsilon$ such that $p$ has the multiplicity at most $k - 1$ in the factorisation of $\epsilon$ and such that $\epsilon \cdot e_{\bar{s}}$ is spanned by the family $(v_i | i \in I)$ over $\mathbb{Z}$.

Let us first see how we finish the proof assuming Sublemma B.9. By Lemma B.8, there are $\alpha$ and $\beta$ such that $\alpha \cdot \delta_{\bar{s}} - \beta \cdot \epsilon$ is equal to the greatest common divisor $D$ of $\delta_{\bar{s}}$ and $\epsilon$. Hence by Sublemma B.9, $D \cdot e_{\bar{s}}$ is generated by the family $(v_i | i \in I)$ over $\mathbb{Z}$. Since $p$ has the multiplicity at most $k - 1$ in the factorisation of $D$, the number $D$ is strictly smaller than $\delta_{\bar{s}}$. This contradicts the choice of $\delta_{\bar{s}}$. Hence all $\delta_s$ are equal to one. It remains so show that the following.

**Proof of Sublemma B.9.** Since the family $(v_i | i \in I)$ spans $e_s$ over $\mathbb{F}_p$, there is an integer valued vector $w$ such that the family $(v_i | i \in I)$ spans $e_s + p \cdot w$ over $\mathbb{Z}$. For a subset $T$ of $S$ we denote by $w_T$ the vector which takes the value $w(s)$ in coordinate $s$ if $s \in T$ and zero otherwise. We denote the multiplicity of $p$ in the factorisation of an integer $n$ by $\sharp_p(n)$.

We shall show inductively for every subset $T$ of $S$ that there is some nonzero natural number $\epsilon_T$ with $\sharp_p(\epsilon_T) \leq k - 1$ such that $\epsilon_T \cdot (e_s + p \cdot w_T)$ is spanned by the family $(v_i | i \in I)$ over $\mathbb{Z}$. We start the induction with $T = S$ and $\epsilon_T = 1$ and so $w_T = w$. Assume that we already proved the induction hypothesis for a nonempty subset $T$ of $S$. Let $t \in T$ be arbitrary. We denote the greatest common divisor of $\epsilon_T \cdot p \cdot w(t)$ and $d_t$ by $d_t$. We let $\epsilon_{T \setminus t} = \epsilon_T \cdot \frac{\delta_t}{d_t}$. We have

\[
\sharp_p(\epsilon_{T \setminus t}) = \sharp_p(\epsilon_T) + \sharp_p(\delta_t) - \sharp_p(d_t) \leq \sharp_p(\epsilon_T) + \sharp_p(\delta_t) - \min\{\sharp_p(\epsilon_T) + 1, \sharp_p(\delta_t)\} = \Box
\]
\[ = \max\{\#p(\delta t), \#p(\epsilon_T) - 1\} \]

Hence by the choice of \(\bar{t}\) and by induction \(\#p(\epsilon_{T-t}) \leq k-1\). Furthermore:

\[ \frac{\delta_t}{d_t} \cdot \epsilon_T \cdot (e_{\bar{t}} + p \cdot w_T) - \frac{\epsilon_T \cdot p \cdot w(t)}{d_t} \cdot \delta_t e_{\bar{t}} = \epsilon_{T-t} \cdot (e_{\bar{t}} + p \cdot w_{T-t}) \]

Note that all fractions in the above equation are integers. This completes the induction step. Hence the vector \(\epsilon_0 \cdot e_{\bar{t}}\) is spanned by the family \((v_i | i \in I)\) over \(\mathbb{Z}\), which completes the proof.

\[ \square \]

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