Twisted Heisenberg chain and the six-vertex model with DWBC

W Galleas
Institute for Theoretical Physics and Spinoza Institute, Utrecht University, Leuvenlaan 4, 3584 CE Utrecht, The Netherlands
E-mail: w.galleas@uu.nl

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Abstract. In this work we establish a relation between the six-vertex model with Domain Wall Boundary Conditions (DWBC) and the $XXZ$ spin chain with anti-periodic twisted boundaries. More precisely, we demonstrate a formal relation between the zeroes of the partition function of the six-vertex model with DWBC and the zeroes of the transfer matrix eigenvalues associated with the six-vertex model with a particular non-diagonal boundary twist.

Keywords: integrable spin chains (vertex models), quantum integrability (Bethe ansatz), solvable lattice models

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1. Introduction

One of the remarkable roles played by integrable systems is the establishment of connections between seemingly unrelated topics. For instance, although the relation between one-dimensional quantum spin chains and two-dimensional classical vertex...
Twisted Heisenberg chain and the six-vertex model with DWBC

models is nowadays clear, this remarkable relation has its origins in Lieb’s observation that the ice model transfer matrix and the XXX spin chain Hamiltonian share the same eigenvectors [1]. In fact, this relation was only made clear by Baxter [2], who showed that the logarithmic derivative of a two-dimensional vertex model transfer matrix gives rise to a one-dimensional quantum spin chain Hamiltonian. This correspondence between quantum spin chains and classical vertex models is well established for lattice systems, but we also have further connections emerging in the continuum limit. For instance, it is believed that the massless regimes of vertex models in the continuum are described by the critical properties of Wess–Zumino–Witten field theories [3].

As far as vertex models with Domain Wall Boundary Conditions (DWBC) [4] are concerned, we cannot immediately associate a one-dimensional spin chain along the lines of [5]. Nevertheless, the six-vertex model with domain wall boundaries still exhibits interesting relations with the theory of classical integrable systems [6], special functions [7,8], and enumerative combinatorics [9]. Moreover, in a recent paper [10], we have shown that the partition function of the six-vertex model with DWBC corresponds to the null eigenvalue wave-function of a certain many-body Hamiltonian operator.

On the other hand, the XXZ spin chain with anti-periodic boundary conditions can be embedded in the transfer matrix of a $\mathcal{U}_q[sl(2)]$ invariant six-vertex model with a particular non-diagonal boundary twist along the same lines of [5,11]. This particular spin chain has also been studied in [12–15], and was the first system tackled through the algebraic-functional method used in [16–18] for partition functions with domain wall boundaries. This method has been refined in a series of papers and here we intend to report on a novel connection between the twisted Heisenberg chain and the six-vertex model with DWBC.

This paper is organized as follows. In section 2, we briefly describe the transfer matrix embedding the XXZ chain with anti-periodic boundary conditions and introduce the notation we shall use throughout this paper. In section 3, we explore the Yang–Baxter algebra along the lines of [10], in order to derive a functional equation relating the transfer matrix eigenvalues and the partition function of the six-vertex model with DWBC. The consequences of this functional equation are discussed in section 4 and, in particular, we show how our results can be simplified when the anisotropy parameter is a root of unity. Concluding remarks are then discussed in section 5, and the appendix A is devoted to the derivation of our main result.

### 2. Heisenberg chain and the DWBC partition function

In this section, we shall give a brief description of the anisotropic Heisenberg chain with anti-periodic boundary conditions. This model consists of a spin-$\frac{1}{2}$ system and here we shall mostly adopt the conventions of [13]. The system Hamiltonian $\mathcal{H}$ acts on the tensor product space $V_Q \cong (\mathbb{C}^2)^\otimes L$ and it reads

$$\mathcal{H} := \sum_{i=1}^L \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \cosh(\gamma) \sigma_i^z \sigma_{i+1}^z \right) \in \text{End}(V_Q). \quad (2.1)$$

In (2.1), we have employed the notation $\sigma_i = 1^{\otimes(i-1)} \otimes \sigma \otimes 1^{\otimes(L-i)}$ where $\sigma \in \{\sigma^x, \sigma^y, \sigma^z\}$ denotes the standard Pauli matrices, and 1 stands for the identity matrix in $\text{End}(\mathbb{C}^2)$. 

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As far as boundary terms are concerned, here we consider the following anti-periodic conditions:

\[ \sigma^x_{L+1} := \sigma^x_1, \quad \sigma^y_{L+1} := -\sigma^y_1, \quad \sigma^z_{L+1} := -\sigma^z_1. \quad (2.2) \]

### 2.1. Transfer matrix

The Hamiltonian (2.1) corresponds to the logarithmic derivative of a six-vertex model transfer matrix with a particular boundary twist. Let \( V_A \cong \mathbb{C}_2 \) and consider the element \( G_A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \text{End}(V_A) \). Then consider the operator \( R_A : \mathbb{C} \to \text{End}(V_A \otimes V_j) \) and define the transfer matrix \( T : \mathbb{C} \to \text{End}(V_Q) \) as

\[ T(\lambda) := \text{Tr}_A \left[ G_A \prod_{1 \leq j \leq L} R_A(j)(\lambda - \mu_j) \right], \quad (2.3) \]

where \( \lambda, \mu_j \in \mathbb{C} \). The trace in (2.3) is taken over the space \( V_A \) while the matrix \( R \in \text{End}(V_1 \otimes V_2) \) reads

\[ R(\lambda) := \begin{pmatrix} a(\lambda) & 0 & 0 & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & a(\lambda) \end{pmatrix}. \quad (2.4) \]

The non-null entries of (2.4) correspond to the functions: \( a(\lambda) := \sinh(\lambda + \gamma), \quad b(\lambda) := \sinh(\lambda) \) and \( c(\lambda) := \sinh(\gamma) \) for parameters \( \lambda, \gamma \in \mathbb{C} \). In this way, the Hamiltonian (2.1) is obtained from the relation \( \mathcal{H} \sim \frac{d}{d\lambda} \ln T(\lambda) \big|_{\lambda=0, \mu_j=0} \) and it is also worth mentioning that the \( R \)-matrix (2.4) satisfies the standard Yang–Baxter equation [19]. In addition to that, the matrix \( G_A \) fulfills the property \([R, G_A \otimes G_A] = 0\) ensuring that the transfer matrix (2.3) forms a commutative family.

### 2.2. Monodromy matrix

Let \( T_A : \mathbb{C} \to \text{End}(V_A \otimes V_Q) \) be the following operator

\[ T_A(\lambda) := \prod_{1 \leq j \leq L} R_A(j)(\lambda - \mu_j), \quad (2.5) \]

shall be referred to as monodromy matrix. As the \( R \)-matrix (2.4) satisfies the Yang–Baxter equation, it can be shown that the monodromy matrix (2.5) fulfills the following quadratic identity

\[ R_{12}(\lambda_1 - \lambda_2)T_1(\lambda_1)T_2(\lambda_2) = T_2(\lambda_2)T_1(\lambda_1)R_{12}(\lambda_1 - \lambda_2). \quad (2.6) \]

The relation (2.6) is usually referred to as Yang–Baxter algebra and since \( V_A \cong \mathbb{C}_2 \), the monodromy matrix \( T_A \) can be recast as

\[ T_A(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \quad (2.7) \]

with operators \( A, B, C, D \in \text{End}(V_Q) \). In this way, we find that the transfer matrix (2.3) simply reads \( T(\lambda) = B(\lambda) + C(\lambda) \).

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2.3. Domain wall boundaries

The $R$-matrix (2.4) encodes the statistical weights of a six-vertex model, as discussed in [19]. However, one still needs to define appropriate boundary conditions in order to have a non-trivial partition function for the model. The case of DWBC for the six-vertex model was introduced on a square lattice of dimensions $L \times L$ by Korepin in [4]. More precisely, in [4] the author derives a recurrence relation for the partition function of the model which was subsequently solved by Izergin [20]. The aforementioned partition function then reads

$$Z(\lambda_1, \ldots, \lambda_L) = \langle \bar{0} | \prod_{1 \leq j \leq L} B(\lambda_j) | 0 \rangle , \quad (2.8)$$

with vectors $|0\rangle$ and $|\bar{0}\rangle$ defined as

$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes L \quad \text{and} \quad |\bar{0}\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes L . \quad (2.9)$$

2.4. Highest/lowest weight vectors

The vectors $|0\rangle$ and $|\bar{0}\rangle$ defined in (2.9) are respectively the $\mathfrak{sl}(2)$ highest and lowest weight vectors. The action of the entries of the monodromy matrix (2.7) on those vectors are given as follows:

$$A(\lambda) |0\rangle = \prod_{j=1}^{L} a(\lambda - \mu_j) |0\rangle \quad D(\lambda) |0\rangle = \prod_{j=1}^{L} b(\lambda - \mu_j) |0\rangle$$
$$A(\lambda) |\bar{0}\rangle = \prod_{j=1}^{L} b(\lambda - \mu_j) |\bar{0}\rangle \quad D(\lambda) |\bar{0}\rangle = \prod_{j=1}^{L} a(\lambda - \mu_j) |\bar{0}\rangle$$
$$B(\lambda) |\bar{0}\rangle = 0 \quad C(\lambda) |0\rangle = 0 . \quad (2.10)$$

3. Functional equations

The spectrum of the anti-periodic Heisenberg chain Hamiltonian (2.1) can be obtained directly from the spectrum of the transfer matrix (2.3). This is due to the fact that the Hamiltonian $H$ is given by the logarithmic derivative of the transfer matrix $T$, in addition to the property $[T(\lambda), T(\mu)] = 0$ ensured by the relation (2.6). Thus, one can shift the attention to the spectral problem associated with the transfer matrix $T$. In its turn, this problem can be approached through the method introduced in [13], and subsequently extended in [16–18, 21]. For that it is convenient to introduce some extra definitions and conventions.

**Definition 1.** Let $B(\lambda_i) \in \mathbb{V}_Q$ be an off-diagonal element of the Yang–Baxter algebra as defined in (2.6). We then introduce the following notation for the product of $n$ generators $B$,

$$[\lambda_1, \ldots, \lambda_n] := \prod_{1 \leq i < j \leq n} B(\lambda_i) . \quad (3.1)$$

doi:10.1088/1742-5468/2014/11/P11028 5
Remark 1. The property $B(\lambda)B(\mu) = B(\mu)B(\lambda)$, encoded in the relation (2.6), ensures that $[\lambda_1, \ldots, \lambda_n]$ is symmetric under the permutation of variables $\lambda_i \leftrightarrow \lambda_j$. Thus, when it is convenient, we shall also employ the simplified notation $[X^{1,n}] := [\lambda_1, \ldots, \lambda_n]$ where $X^{i,j} := \{\lambda_k \mid i \leq k \leq j\}$.

Next, we recall that $T(\lambda) = B(\lambda) + C(\lambda)$ and consider the action of $T(\lambda_0)$ over the element $[X^{1,n}]$. For that, the most lengthy computation is the term $C(\lambda_0) [X^{1,n}]$ which can be evaluated with the help of the commutation relations contained in (2.6). This computation has been performed in [4, 16], so we shall restrict ourselves to presenting only the final results. In this way, we are left with the following expression:

$$T(\lambda_0) [X^{1,n}] = [X^{0,n}] + \sum_{1 \leq i \leq n} [X_{i,j}^{1,n}] \left( \Gamma_{i,k}^i A(\lambda_0) D(\lambda_i) + \Gamma_{i,0}^i A(\lambda_i) D(\lambda_0) \right)$$

$$+ \sum_{1 \leq i < j \leq n} [X_{i,j}^{0,n}] \left( \Omega_{i,j} A(\lambda_i) D(\lambda_j) + \Omega_{j,i} A(\lambda_j) D(\lambda_i) \right),$$

(3.2)

where $X_{i,j}^{1,n} := X^{1,n}\{\lambda_i\}$ and $X_{i,j}^{0,n} := X^{0,n}\{\lambda_i, \lambda_j\}$. The coefficients $\Gamma_{i,k}^i$ and $\Omega_{i,j}$ in (3.2) explicitly read

$$\Gamma_{i,k}^i := \frac{c(\lambda_k - \lambda_i)}{b(\lambda_k - \lambda_j)} \prod_{\lambda \in X_{i,j}^{1,n}} \frac{a(\lambda - \lambda) a(\lambda - \lambda_j)}{b(\lambda_k - \lambda) b(\lambda - \lambda_j)}$$

$$\Omega_{i,j} := \frac{c(\lambda_i - \lambda_j)}{a(\lambda_j - \lambda_i)} \frac{c(\lambda_i - \lambda_j)}{a(\lambda_j - \lambda_i)} \frac{a(\lambda_j - \lambda_i)}{b(\lambda_j - \lambda_i)} \prod_{\lambda \in X_{i,j}^{0,n}} \frac{a(\lambda - \lambda) a(\lambda - \lambda_i)}{b(\lambda - \lambda_i) b(\lambda - \lambda_j)}. \quad (3.3)$$

The interpretation of the Yang–Baxter algebra as a source of functional equations [10] can now be immediately invoked. In this way, one can recognize (3.2) as a Yang–Baxter relation of order $n + 1$ and, in addition to extra properties, this relation will allow us to derive a functional equation describing the spectrum of the transfer matrix $T$. For that, it is also convenient to introduce the following definition.

Definition 2. Let $n \in \mathbb{Z}_{>0}$ be a discrete index and $\mathcal{M}(\lambda) := \{A, B, C, D\}(\lambda)$. Also, let $\mathcal{W}_n := \mathcal{M}(\lambda_1) \times \mathcal{M}(\lambda_2) \times \ldots \times \mathcal{M}(\lambda_n)$ with $n$-tuples $(\xi_1, \xi_2, \ldots, \xi_n)$ be understood as $\prod_{1 \leq i \leq n} \xi_i$. Then, consider the function space $\mathbb{C}[\lambda_1^{\pm1}, \ldots, \lambda_n^{\pm1}]$ of regular complex-valued functions on $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ and define $\hat{\mathcal{W}}_n := \mathbb{C}[\lambda_1^{\pm1}, \ldots, \lambda_n^{\pm1}] \otimes \text{span}_\mathbb{C}(\mathcal{W}_n)$. The map $\pi_n$ is then introduced as the following $n$-additive continuous map

$$\pi_n : \hat{\mathcal{W}}_n \to \mathbb{C}[\lambda_1^{\pm1}, \lambda_2^{\pm1}, \ldots, \lambda_n^{\pm1}] .$$

(3.4)

In other words, the map $\pi_n$ associates a multivariate complex function to any product of $n$ generators of the Yang–Baxter algebra.

The next step within this approach consists of finding a suitable realization of the map $\pi_n$ which is able to convert (3.2) into appropriate functional equations. We shall proceed along the lines of [10], and adopt a particular scalar product as realization of $\pi_n$.

3.1. Realization of $\pi_n$

Let $|\Psi\rangle \in \text{span}(\mathcal{W}_Q)$ be an eigenvector of the transfer matrix (2.3) with eigenvalue $\Lambda(\lambda)$. More precisely, we have the action $T(\lambda) |\Psi\rangle = \Lambda(\lambda) |\Psi\rangle$. Then, taking into account the
definition (2.9), we define the map \( \pi \) as
\[
\pi_{n+1}(\mathcal{A}) := \langle \Psi | \mathcal{A} | 0 \rangle \quad \forall \mathcal{A} \in \hat{\mathcal{W}}_{n+1}.
\] (3.5)

At this stage, we have gathered all of the elements required to convert the Yang–Baxter algebra relation (3.2) into a functional equation characterizing the eigenvalues \( \Lambda \). For that, we need only to apply the map (3.4) to the relation (3.2), taking into account the realization (3.5). This procedure can be effectively carried out by acknowledging that the LHS of (3.2) obeys the following reduction property \( \pi_{n+1} \rightarrow \pi_n \):
\[
\pi_{n+1}(T(\lambda_0) [X^{1,n}]) = \Lambda(\lambda_0) \pi_n([X^{1,n}]).
\] (3.6)

On the other hand, by applying the map (3.4) to (3.2), the terms in the RHS are of the following form: \( \pi_{n+1}([X^{0,n}]), \pi_{n+1}([X^{1,n}] A(z_1) D(z_2)) \) for \( (z_1, z_2) \in \text{Sym}(\{\lambda_0, \lambda_i\}) \) and \( \pi_{n+1}([X^{0,n}_i A(z_1) D(z_2)) \) for \( (z_1, z_2) \in \text{Sym}(\{\lambda_i, \lambda_j\}) \). The term \( \pi_{n+1}([X^{0,n}]) \) cannot be significantly simplified, but for \( Y \in \{X^{1,n}_i, X^{0,n}_i\} \) and using (2.10), we find that
\[
\pi_{n+1}([Y]) A(z_1) D(z_2)) = \prod_{k=1}^{L} a(z_1 - \mu_k) b(z_2 - \mu_k) \pi_{n-1}([Y]).
\] (3.7)

Our results so far can be written in a more convenient form with the help of the notation \( \pi_n([X]) := \mathcal{F}_n(X) \) for a given set \( X = \{\lambda_k\} \) of cardinality \( n \). In this way, this procedure yields the following set of functional equations:
\[
\Lambda(\lambda_0) \mathcal{F}_n(X^{1,n}) = \mathcal{F}_{n+1}(X^{0,n}) + \sum_{1 \leq i < n} M_i^{(n)} \mathcal{F}_{n-1}(X^{1,n}_i) + \sum_{1 \leq i < j \leq n} N_{j,i}^{(n)} \mathcal{F}_{n-1}(X^{0,n}_{ij}),
\] (3.8)

with coefficients \( M_i^{(n)} = M_i^{(n)}(\tilde{X}^{0,n}) \) and \( N_{j,i}^{(n)} = N_{j,i}^{(n)}(\tilde{X}^{0,n}) \) given by
\[
M_i^{(n)} := \Gamma_{i,j}^{i} \prod_{k=1}^{L} (a(\lambda_0 - \mu_k) b(\lambda_i - \mu_k) + \Gamma_{i,j}^{i} \prod_{k=1}^{L} (a(\lambda_i - \mu_k) b(\lambda_0 - \mu_k))
\]
\[
N_{j,i}^{(n)} := \Omega_{i,j} \prod_{k=1}^{L} a(\lambda_i - \mu_k) b(\lambda_j - \mu_k) + \Omega_{j,i} \prod_{k=1}^{L} a(\lambda_j - \mu_k) b(\lambda_i - \mu_k).
\] (3.9)

The symbol \( \tilde{X}^{0,n} \) has been introduced in order to emphasize that the functions \( M_i^{(n)} \) and \( N_{j,i}^{(n)} \) are not invariant under the permutation of all variables. Its precise definition is given as follows.

**Definition 3.** Let \( i, j \in \mathbb{Z} \) such that \( i < j \). Then \( \tilde{X}^{ij} \) stands for the vector
\[
\tilde{X}^{i,j} := (\lambda_i, \lambda_{i+1}, \lambda_{i+2}, \ldots, \lambda_j).
\] (3.10)

For convenience, we shall also define the symbols \( \tilde{X}^{i,j}_k \) and \( \tilde{X}^{i,j}_{k,l} \) for \( i \leq k, l \leq j \) such that \( k < l \). They are respectively defined as
\[
\tilde{X}^{i,j}_k := (\lambda_i, \lambda_{i+1}, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_j)
\]
\[
\tilde{X}^{i,j}_{k,l} := (\lambda_i, \lambda_{i+1}, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_l, \lambda_{l+1}, \ldots, \lambda_j).
\] (3.11)

Some remarks are required at this stage. For instance, the functional equation (3.8) consists of an extension of the equation obtained in [13]; it has also been recently described in [15]. Moreover, at algebraic level there is no upper limit for the discrete index \( n \) in equation (3.8). However, the \( \mathfrak{sl}(2) \) highest weight representation theory imposes an upper bound for that index.
4. The eigenvalues $\Lambda$ and the partition function $Z$

In the previous section, we derived a functional equation involving the eigenvalues $\Lambda$ of the transfer matrix (2.3) and a certain set of functions $F_n$. Here we intend to show that the functions $F_n$ can be eliminated from the system of equations (3.8), yielding a single equation for the eigenvalues $\Lambda$. The functional equation for $\Lambda$ obtained in this way will depend explicitly on the partition function $Z$ of the six-vertex model with DWBC.

4.1. Highest weight and domain walls

The highest weight representation theory of the $\mathfrak{sl}(2)$ algebra gives an upper bound for the number of operators $B$ entering the product (3.1), as discussed in [4,16]. This feature is manifested in the following property,

$$[X^{1,L}] |0\rangle = Z(X^{1,L}) |0\rangle .$$

(4.1)

Thus the relations (4.1) and (3.5) imply that $F_L(X^{1,L}) = Z(X^{1,L}) \bar{F}_0$ where $\bar{F}_0 = \langle \Psi | 0 \rangle$. Moreover, the function $F_n$ vanishes for $n > L$ due to (2.10), (3.5) and (4.1). It is worth remarking here that equation (3.8) assumes that $F_n$ vanishes for $n < 0$.

In order to illustrate our procedure, let us firstly have a closer look at equation (3.8) for the case $L = 2$. In that case, we can set $n = 0, 1, 2, 3$ and by doing so we are left with the following set of equations:

$$\Lambda(\lambda_0) F_0 = F_1(X^{0,0})$$

$$\Lambda(\lambda_0) F_1(X^{1,1}) = Z(X^{0,1}) \bar{F}_0 + M_1^{(1)} F_0$$

$$\Lambda(\lambda_0) Z(X^{1,2}) \bar{F}_0 = \sum_{1 \leq i \leq 2} M_i^{(2)} F_1(X_i^{1,2}) + N_{2,1}^{(2)} F_1(X_{1,2})$$

$$0 = \sum_{1 \leq i \leq 3} M_i^{(3)} Z(X_i^{1,3}) + \sum_{1 \leq i < j \leq 3} N_{j,i}^{(3)} Z(X_{i,j}^{0,3}) .$$

(4.2)

The last equation in (4.2) involves only the partition function $Z$, and has been previously described in [16]. This single equation is fully able to determine the function $Z$, up to an overall constant factor, while the remaining equations relate the eigenvalue $\Lambda$ and the auxiliary function $F_1$. We shall then use the first equation of (4.2) to eliminate $F_1$ from the second and third equations. By doing so we are left with the relations:

$$\Lambda(\lambda_0) \Lambda(\lambda_1) = Z(X^{0,1}) k_0 + M_1^{(1)}$$

$$\Lambda(\lambda_0) \left[ Z(X^{1,2}) k_0 - N_{2,1}^{(2)} \right] = M_1^{(2)} \Lambda(\lambda_2) + M_2^{(2)} \Lambda(\lambda_1) ,$$

(4.3)

where $k_0 = \bar{F}_0/F_0$. In what follows, we shall assume that the partition function $Z$ is already determined, and the only unknown factors in (4.3) are the coefficients $k_0$ and the function $\Lambda$. Both equations in (4.3) are able to determine $k_0$ and the eigenvalues $\Lambda$, however, the first equation is non-linear, while the second is linear. In fact, the direct inspection of (4.3) reveals that $k_0 = \Lambda(\mu_1) \Lambda(\mu_2) [c^2 a(\mu_1 - \mu_2) a(\mu_2 - \mu_1)]^{-1}$.

For the case $L = 3$, we can set $n = 0, 1, 2, 3, 4$ in (3.8). Each choice produces an independent equation and the whole set consists of the following equations:

$$\Lambda(\lambda_0) F_0 = F_1(X^{0,0})$$

$$\Lambda(\lambda_0) F_1(X^{1,1}) = F_2(X^{0,1}) + M_1^{(1)} F_0$$

$$\Lambda(\lambda_0) Z(X^{1,2}) F_0 = \sum_{1 \leq i \leq 2} M_i^{(2)} F_1(X_i^{1,2}) + N_{2,1}^{(2)} F_1(X_{1,2})$$

$$0 = \sum_{1 \leq i \leq 3} M_i^{(3)} Z(X_i^{1,3}) + \sum_{1 \leq i < j \leq 3} N_{j,i}^{(3)} Z(X_{i,j}^{0,3}) .$$

(4.4)
The symbol \( Z \) is defined as

\[
\Lambda(0) Z(X^{1,3}) = Z(X^{0,2}) \bar{\Lambda} + \sum_{1 \leq i < j} M_i^{(3)} \mathcal{F}_2(X_{i,j}^{1,3}) + \sum_{1 \leq i < j < k} N_{i,j}^{(3)} \mathcal{F}_2(X_{i,j}^{1,3}) - \sum_{1 \leq i < j < k} N_{i,j}^{(3)} \mathcal{F}_2(X_{i,j}^{1,3}).
\]

Similarly to the case \( L = 2 \), none of the equations in (4.5) are linear: this is the general behavior for arbitrary \( L \). At this stage it is also worth stressing that for both cases, namely \( L = 2 \) and \( L = 3 \), we have explicitly written two sets of equations relating the eigenvalues \( \Lambda \) and the partition function \( Z \). Each set is formed by two equations, i.e. (4.3) and (4.5), however, there is a dramatic difference between the first and the second equations of each set. For instance, while the first equation runs over the set of variables \( \{ \lambda_k \mid 0 \leq k \leq L - 1 \} \), the second equation is defined over the set \( \{ \lambda_k \mid 0 \leq k \leq L \} \). The partition function \( Z \) can also be described through functional equations, and a similar feature has previously appeared for that problem. If we compare the functional equations for \( Z \) derived in [16] and [18], we can readily see that they are defined over a different number of variables. In what follows, we shall focus on the functional equation relating \( \Lambda \) and \( Z \), and will generalize the first equation of (4.3) and (4.5) for arbitrary values of \( L \). The following definitions will be helpful.

**Definition 4.** The symbol \([x]\) is defined as

\[
[x] := \begin{cases} 
 x & \text{for } x \in 2\mathbb{Z}_{>0} \\
 x - 1 & \text{for } x \in (2\mathbb{Z}_{>0} + 1).
\end{cases}
\]

**Definition 5.** Let \( f(\lambda) \in \mathbb{C}[\lambda] \) and consider the product operator \( \prod_{\lambda}^{i_1, \ldots, i_{2m}} \mathbb{C}[\lambda] \to \mathbb{C}[\lambda_{i_1}] \times \ldots \times \mathbb{C}[\lambda_{i_{L-2m}}] \) for \( \lambda_{i_j} \in X^{0, L-1}_{i_1, \ldots, i_{2m}} \) such that \( \lambda_{i_j} \neq \lambda_{i_k} \) if \( j \neq k \). The relation \( X_{i_1, \ldots, i_{n+1}}^{L,m} = X_{i_1, \ldots, i_n}^{L,m} \setminus \{ \lambda_{i_{n+1}} \} \) generalizes recursively our previous definition and the product operator \( \prod_{\lambda}^{i_1, \ldots, i_{2m}} \) is defined as

\[
\left( \prod_{\lambda}^{i_1, \ldots, i_{2m}} f(\lambda) \right)(\lambda) := \prod_{\lambda \in X^{0, L-1}_{i_1, \ldots, i_{2m}}} f(\lambda) = \prod_{k=0}^{L-1} f(\lambda_k).
\]

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Twisted Heisenberg chain and the six-vertex model with DWBC

**Theorem 1.** The partition function $Z$ can be written in terms of the eigenvalues $\Lambda$ according to the formula

$$Z(X^{0,L-1})_{k_0} = \left\{ \sum_{m=0}^{[L]/2} \sum_{0 \leq i_1 < \ldots < i_{2m} \leq L-1} \Lambda^{(2m)}_{i_{2m-1},i_1} \prod_{\lambda} \Lambda(\lambda) \right\} \Lambda(\lambda), \quad (4.8)$$

where

$$V^{(2m)}_{i_{2m-1},i_1} = \sum_J \prod_{l=1}^{m} \prod_{i=1}^{L} a(\lambda_{ji} - \mu_n) \prod_{\lambda \in X^{0,L-1}_{i_1 \ldots i_{2m}}} a(\lambda - \lambda_{ji}) b(\lambda - \lambda_{ji})$$

$$\times \prod_{K(J)} \prod_{l=1}^{m} b(\lambda_{ki} - \mu_n) c(\lambda_{ji} - \lambda_{ki}) b(\lambda_{ji} - \lambda_{ki}) \prod_{\lambda \in X^{0,L-1}_{i_1 \ldots i_{2m}}} a(\lambda_{ki} - \lambda) b(\lambda_{ki} - \lambda)$$

$$\times \prod_{1 \leq r < s \leq m} a(\lambda_{kr} - \lambda_{ks}) a(\lambda_{kr} - \lambda_{js}) a(\lambda_{ks} - \lambda_{js} + \gamma) b(\lambda_{kr} - \lambda_{ks}) b(\lambda_{kr} - \lambda_{js}) b(\lambda_{ks} - \lambda_{js}), \quad (4.9)$$

with summation symbols defined as $\sum_J := \sum_{j_1 \ldots j_m \in J_{2m}}$ and $\sum_{K(J)} := \sum_{k_1 \ldots k_m \in J_{2m}}$.

The symbols $\mathcal{I}_{2m}$ and $\mathcal{J}_{2m}$ stand respectively for the sets $\mathcal{I}_{2m} := \{i_1, \ldots, i_{2m}\}$ and $\mathcal{J}_{2m} := \mathcal{I}_{2m}\setminus\{j_1, \ldots, j_m\}$. For clarity’s sake, we stress here that $V^{(0)} := 1$.

**Proof.** The proof follows from the extension of the derivation presented for the cases $L = 2$ and $L = 3$. These cases are respectively covered by formulae (4.3) and (4.5). The derivation of formula (4.9) for arbitrary $L$ is discussed in appendix A. \[\Box\]

**Example 1.** The RHS of (4.8) for $L = 2$ reads

$$\Lambda(\lambda_0)\Lambda(\lambda_1) + V^{(2)}_{1,0} , \quad (4.10)$$

while for $L = 3$ we have

$$\Lambda(\lambda_0)\Lambda(\lambda_1)\Lambda(\lambda_2) + \sum_{0 \leq i_1 < i_2 \leq 2} V^{(2)}_{i_2,i_1} \prod_{k=0}^{2} \Lambda(\lambda_k) . \quad (4.11)$$

The structure of the relation (4.8) is quite appealing and some remarks are in order. For instance, the relation (4.8) converts the problem of evaluating the partition function $Z$ into the diagonalisation of the transfer matrix of the six-vertex model with a non-diagonal boundary twist. This situation is analogous to the case of the six-vertex model defined on a torus, where the model partition function is given in terms of the eigenvalues of the standard transfer matrix of the six-vertex model with periodic boundary conditions [19,22]. Furthermore, if one assumes that the eigenvalues $\Lambda$ are parameterized by solutions of Bethe ansatz-like equations, as obtained in [12, 14], then we could expect the relation (4.8) to offer access to thermodynamic properties of the six-vertex model with DWBC in the same fashion as for the case with toroidal boundary conditions.
4.2. The zeroes \( w_j \)

The eigenvalues \( \Lambda(\lambda) \) are essentially a polynomial of order \( L - 1 \) in the variable \( x := e^{2\lambda} \), as demonstrated in [13]. Thus, it can be written in terms of its zeroes \( w_j \) as

\[
\Lambda(\lambda) = \Lambda(0) \prod_{j=1}^{L-1} \frac{\sinh (w_j - \lambda)}{\sinh (w_j)}.
\]

(4.12)

We shall assume that the zeroes \( w_j \) are all distinct and the following corollary will allow us to determine the zeroes \( w_j \).

**Corollary 1.** The relation (4.8) under the specialization \( \lambda_j = w_j \) for \( 1 \leq j \leq L - 1 \) implies the following constraints:

\[
\frac{Z(\lambda_0, w_1, \ldots, w_{L-1})}{V_{L-1, \ldots, 0}^{(L)}(\lambda_0, w_1, \ldots, w_{L-1})} \bigg|_{k_0} = \begin{cases} 
\frac{(-1)^{\frac{L}{2}}}{2^{L-1} (L-1)!} & \text{for } L \in 2\mathbb{Z}_{>0} \\
\Lambda(0) & \text{for } L \in (2\mathbb{Z}_{>0} + 1).
\end{cases}
\]

(4.13)

Now we can use the analytic properties of the functions \( Z \) and \( V_{L-1, \ldots, 0}^{(L)} \), in addition to the relation (4.13), to determine the set of zeroes \( \{w_j\} \). For that it is important to highlight that the variable \( \lambda_0 \) in (4.13) is still an arbitrary complex variable. Moreover, the partition function \( Z \) is a symmetric multivariate polynomial [4,17], while the function \( V_{L-1, \ldots, 0}^{(L)} \) for \( L \in 2\mathbb{Z}_{>0} \) consists of a polynomial of order \( L - 1 \) in the variable \( x_0 := e^{2\lambda_0} \). On the other hand, for \( L \in 2\mathbb{Z}_{>0} + 1 \) we have

\[
V_{L-1, \ldots, 0}^{(L-1)}(\lambda_0, w_1, \ldots, w_{L-1}) = \frac{\hat{V}_{L-1, \ldots, 0}^{(L-1)}(\lambda_0, w_1, \ldots, w_{L-1})}{\prod_{j=1}^{L-1} b(\lambda_0 - w_j)},
\]

(4.14)

where the function \( \hat{V}_{L-1, \ldots, 0}^{(L-1)}(\lambda_0, w_1, \ldots, w_{L-1}) \) is a polynomial of order \( L - 1 \) in the variable \( x_0 \).

4.2.1. The case \( L \in 2\mathbb{Z}_{>0} \). This case corresponds to the even values of \( L \), and we can use the analytical properties of (4.13) to characterize the set of variables \( \{w_j\} \). For this analysis, it is important to recall that both functions \( Z \) and \( V_{L-1, \ldots, 0}^{(L)} \) in the LHS of (4.13) are polynomials of the same degree in the variable \( x_0 \). Consequently, since the RHS of (4.13) is a constant, we can conclude that the residues of the LHS must vanish at the zeroes of \( V_{L-1, \ldots, 0}^{(L)} \). In other words, the zeroes of \( Z \) and \( V_{L-1, \ldots, 0}^{(L)} \) must coincide. This analysis yields a formal condition determining the set \( \{w_j\} \) which is summarized in corollary 2.

**Corollary 2.** Consider \( L \in 2\mathbb{Z}_{>0} \) and let \( \lambda^Z_k \in \{ \lambda \in \mathbb{C} \mid Z(\lambda, w_1, \ldots, w_{L-1}) = 0 \} \) while \( \lambda^V_k \in \{ \lambda \in \mathbb{C} \mid V_{L-1, \ldots, 0}^{(L)}(\lambda, w_1, \ldots, w_{L-1}) = 0 \} \). The zeroes \( \lambda^Z_k \) and \( \lambda^V_k \) shall depend on the set of parameters \( \{w_j\} \), and we can conclude that

\[
\lambda^Z_k(\{w_j\}) = \lambda^V_k(\{w_j\}) \quad 1 \leq k \leq L - 1.
\]

(4.15)

The direct inspection of (4.15) for small values of \( L \) reveals that the variables \( w_j \) are completely fixed by the aforementioned constraints.

4.2.2. The case \( L \in (2\mathbb{Z}_{>0} + 1) \). The situation for \( L \) odd requires a slightly more elaborate analysis due to the presence of the eigenvalue \( \Lambda \) in the RHS of (4.13). In this case, we also
need to consider (4.14), and it is revealed that (4.13) simplifies to
\[
\frac{\Lambda(0)}{k_0} \prod_{j=1}^{L-1} b(-w_j)^{-1} = \frac{Z(\lambda_0, w_1, \ldots, w_{L-1})}{V_{L-1,\ldots,0}^{(L-1)}(\lambda_0, w_1, \ldots, w_{L-1})}.
\] (4.16)

The LHS of (4.16) is a constant with respect to the variable \(\lambda_0\), while the RHS is a rational function. Thus, the polynomials \(Z\) and \(V_{L-1,\ldots,0}^{(L-1)}\) must share the same zeroes. Similarly to the \(L\) even case, this statement can be formulated more precisely as the following corollary.

**Corollary 3.** Assume that \(L \in (2\mathbb{Z}_{>0} + 1)\) and let \(\lambda_k^Z \in \{\lambda \in \mathbb{C} \mid Z(\lambda, w_1, \ldots, w_{L-1}) = 0\}\) as previously defined. Also, let \(\lambda_k^V \in \{\lambda \in \mathbb{C} \mid \tilde{V}_{L-1,\ldots,0}^{(L-1)}(\lambda, w_1, \ldots, w_{L-1}) = 0\}\). Thus, we have the following conditions determining the set of variables \(\{w_j\}\),
\[
\lambda_k^Z(\{w_j\}) = \lambda_k^V(\{w_j\}) \quad 1 \leq k \leq L - 1.
\] (4.17)

Both corollaries 2 and 3 state that the zeroes of the partition function \(Z\), with respect to one of its variables, coincide with the zeroes of the function \(V_{L-1,\ldots,0}^{(L)}\) when the remaining variables correspond to zeroes of the transfer matrix eigenvalues \(\Lambda\).

### 4.2.3. Wronskian condition

The constraints (4.15) and (4.17) are given in terms of the zeroes of certain polynomials whose explicit evaluation might still be a particularly non-trivial problem. Alternatively, one can also obtain equations determining the set of zeroes \(\{w_j\}\) in terms of the coefficients of the polynomial part of \(V_{L-1,\ldots,0}^{(L)}\) and \(Z\). This analysis can be performed for \(L \in 2\mathbb{Z}_{>0}\) and \(L \in (2\mathbb{Z}_{>0} + 1)\) in a unified manner with the help of the function \(F\) defined as
\[
F := \begin{cases} 
V_{L-1,\ldots,0}^{(L)} & \text{for } L \in 2\mathbb{Z}_{>0} \\
\tilde{V}_{L-1,\ldots,0}^{(L-1)} & \text{for } L \in (2\mathbb{Z}_{>0} + 1) 
\end{cases}.
\] (4.18)

The term \(\tilde{V}_{L-1,\ldots,0}^{(L-1)}\) in (4.18) has been previously defined in (4.14). Thus, the function \(F(\lambda_0, w_1, \ldots, w_{L-1})\) is a polynomial of order \(L - 1\) in the variable \(x_0\) for \(L \in \mathbb{Z}_{>0}\).

The equations fixing the zeroes \(w_j\) from the coefficients of \(Z\) and \(F\) can be directly read from the relations (4.13) and (4.16). However, this approach would leave us with an overall constant factor. We can avoid this drawback by simply demanding that the Wronskian determinant between \(Z\) and \(F\) vanishes. This is justified by the fact that (4.13) and (4.16) tell us that \(Z\) and \(F\) are two linearly dependent functions. In this way, we are left with the condition:
\[
P(x_0) := Z(x_0, \{w_j\}) F'(x_0, \{w_j\}) - F(x_0, \{w_j\}) Z'(x_0, \{w_j\}) = 0, \quad (4.19)
\]
where the symbol \(\cdot\) denotes differentiation with respect to the variable \(x_0\).

The function \(P\) is a polynomial of order \([L]\) in the variable \(x_0\) which must vanish in the entire complex domain according to the Wronskian condition (4.19). The coefficients of \(P\) are given by
\[
C_k = \frac{1}{k!} \left. \frac{\partial^k P}{\partial x_0^k} \right|_{x_0=0},
\] (4.20)
and we demand that these coefficients must vanish in order to satisfy (4.19). Thus, we end up with the following formal condition fixing the zeroes \(w_j\),
\[
C_k(\{w_j\}) = 0 \quad 0 \leq k \leq [L].
\] (4.21)
It is important to remark here that (4.21) provides one or two more equations than variables \( w_j \) which need to be determined. For \( L \) even, we have one more equation, while for \( L \) odd, we have two additional equations. Each equation is a non-linear algebraic equation, and consequently we have a large number of solutions. This feature is similar to that found when solving standard Bethe ansatz equations. However, the direct inspection of the solutions of (4.21) for small values of \( L \) reveals that these extra equations play the role of a filter, keeping only solutions which actually describe the spectrum of the transfer matrix (2.3).

4.3. Truncation at roots of unity

Vertex models based on solutions of the Yang–Baxter equation can exhibit special properties when their anisotropy parameter satisfies certain root of unity conditions. For instance, Tarasov demonstrated in [23] that the property

\[
\prod_{0 \leq k \leq l-1} B(\lambda - k\gamma) = 0
\]  

holds for the \( \mathcal{U}_q[\hat{sl}(2)] \) invariant six-vertex model when the anisotropy parameter \( \gamma \) obeys the condition \( e^{2\gamma} = 1 \). The case \( l = 1 \) is not illuminating for our present discussion, as we can see from definitions (2.5) and (2.7) that both operators \( B(\lambda) \) and \( C(\lambda) \) are proportional to the factor \( (e^{2\gamma} - 1) \). Consequently, the transfer matrix (2.3) is also proportional to that same quantity, and this implies that its eigenvalues trivially vanish when we set \( l = 1 \). The free-fermion point \( \gamma = i\pi/2 \) is, in its turn, covered by the case \( l = 2 \), and we shall start our analysis with this case, despite its triviality. We shall then consider the cases \( l = 3 \) and \( l = 4 \) separately before discussing the general case.

4.3.1. \( l = 2 \). Our goal here is to analyze the system of equations (3.8), taking into account the representation theoretic properties of the functions \( F_n(X^{1,n}) = \pi_n(\lambda_1, \ldots, \lambda_n) \). The property (4.22) can then be used in a very natural way: for \( l = 2 \) we have \( [\lambda, \lambda - \gamma] = 0 \). More precisely, we can exploit this property by looking at (4.3) under the specialization \( \lambda_0 = \lambda \) and \( \lambda_1 = \lambda - \gamma \). By doing so, we are left with the following functional relation,

\[
\Lambda(\lambda)\Lambda(\lambda - \gamma) = M^{(1)}_l(\lambda, \lambda - \gamma)
\]

\[
= \prod_{k=1}^{L} \sinh (\lambda - \mu_k)^2 - \prod_{k=1}^{L} \sinh (\lambda - \mu_k + \gamma) \sinh (\lambda - \mu_k - \gamma) .
\]  

(4.23)

It is worth remarking here that (4.23) corresponds to an analogous inversion relation proposed by Stroganov in [24]. Next, we consider the representation (4.12) and set \( \lambda = w_i \) in (4.23). This procedure then yields the following equation which determines the set of zeroes \( \{w_j\} \),

\[
\prod_{k=1}^{L} \frac{\sinh (w_i - \mu_k + \gamma) \sinh (w_i - \mu_k - \gamma)}{\sinh (w_i - \mu_k) \sinh (w_i - \mu_k)} = 1 .
\]  

(4.24)

As previously mentioned, the free-fermion point \( \gamma = i\pi/2 \) fits in the case \( l = 2 \), and at this particular point we find that (4.24) simplifies to

\[
\prod_{k=1}^{L} \coth (w_i - \mu_k)^2 = 1 .
\]  

(4.25)
We can now readily see that (4.25) generalizes the proposal of [15] in the presence of inhomogeneities $\mu_k$.

4.3.2. $l = 3$. In that case, the property (4.22) reads $[\lambda, \lambda - \gamma, \lambda - 2\gamma] = 0$ and we can immediately substitute it in equations (4.4) and (4.5) under the specializations $\lambda_j = \lambda - j\gamma$. By doing so, we are left with the following relation,

$$
\Lambda(\lambda)\Lambda(\lambda - \gamma)\Lambda(\lambda - 2\gamma) = \Lambda(\lambda) \left[ M_1^{(1)}(\lambda, \lambda - \gamma, \lambda - 2\gamma) + N_2^{(2)}(\lambda, \lambda - \gamma, \lambda - 2\gamma) \right] 
+ \Lambda(\lambda - \gamma)M_2^{(2)}(\lambda, \lambda - \gamma, \lambda - 2\gamma) 
+ \Lambda(\lambda - 2\gamma)M_1^{(2)}(\lambda, \lambda - \gamma, \lambda - 2\gamma),
$$

(4.26)

which simplifies to

$$
\Lambda(\lambda)\Lambda(\lambda - \gamma)\Lambda(\lambda - 2\gamma) = -\Lambda(\lambda) \prod_{j=1}^{\ell} \sinh(\lambda - \mu_j) \sinh(\lambda - \mu_j - 2\gamma) 
+ \Lambda(\lambda - \gamma)2 \cosh(\gamma) \prod_{j=1}^{\ell} \sinh(\lambda - \mu_j) \sinh(\lambda - \mu_j - \gamma) 
- \Lambda(\lambda - 2\gamma) \prod_{j=1}^{\ell} \sinh(\lambda - \mu_j + \gamma) \sinh(\lambda - \mu_j - \gamma)
$$

(4.27)

upon the use of (3.9). The representation (4.12) can now be used in (4.27). In this way, we set $\lambda = w_i + \gamma$ in (4.27), and for this particular specialization we notice that the term $\Lambda(\lambda - \gamma)|_{\lambda=w_i+\gamma}$ vanishes. This procedure then yields the following equation which determines the set of zeroes $\{w_j\}$,

$$
\prod_{k=1}^{\ell} \frac{\sinh(w_i - \mu_k + \gamma) \sinh(w_i - \mu_k - \gamma)}{\sinh(w_i - \mu_k + 2\gamma) \sinh(w_i - \mu_k)} = -\prod_{j=1}^{\ell-1} \frac{\sinh(w_j - w_i + \gamma)}{\sinh(w_j - w_i - \gamma)}.
$$

(4.28)

Although the equation (4.28) has been derived using the property (4.22) for the case $l = 3$, we notice that (4.28) reduces to (4.24) for values of $\gamma$ belonging to $l = 2$. Thus, our results so far show that equation (4.28) is valid for for both the cases $l = 2$ and $l = 3$.

4.3.3. $l = 4$. This particular root of unity condition also truncates the system of equations (3.8). For $l = 4$, we are then left with the relation

$$
\left\{ \sum_{m=0}^{2} \sum_{0 \leq i_1 < \ldots < i_{2m} \leq 3} V_{i_2m, \ldots, i_1}^{(2m)} \prod_{\lambda} \lambda_{i_1, \ldots, i_{2m}} \right\} \Lambda(\lambda)|_{\lambda_j = \lambda - j\gamma} = 0,
$$

(4.29)

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where the form of the functions $V_{2m,...,t_1}^{(2m)}$ are given by (4.9). By making explicit use of (4.9), we find that (4.29) simplifies to

$$
\Lambda(\lambda)\Lambda(\lambda - \gamma)\Lambda(\lambda - 2\gamma)\Lambda(\lambda - 3\gamma)
= \Lambda(\lambda - \gamma)\Lambda(\lambda - 2\gamma)\frac{\sinh(3\gamma)}{\sinh(\gamma)} \prod_{k=1}^{L} \sinh(\lambda - \mu_k) \sinh(\lambda - \mu_k - 2\gamma)
- \Lambda(\lambda - 2\gamma)\Lambda(\lambda - 3\gamma) \prod_{k=1}^{L} \sinh(\lambda - \mu_k + \gamma) \sinh(\lambda - \mu_k - \gamma)
- \Lambda(\lambda)\Lambda(\lambda - \gamma) \prod_{k=1}^{L} \sinh(\lambda - \mu_k - \gamma) \sinh(\lambda - \mu_k - 3\gamma)
- \Lambda(\lambda)\Lambda(\lambda - 3\gamma) \prod_{k=1}^{L} \sinh(\lambda - \mu_k) \sinh(\lambda - \mu_k - 2\gamma) + Q(\lambda),
$$

(4.30)

where the function $Q(\lambda)$ is given by

$$
Q(\lambda) = \frac{\sinh(3\gamma)}{\sinh(\gamma)} \prod_{k=1}^{L} \sinh(\lambda - \mu_k)^2 \sinh(\lambda - \mu_k - 2\gamma)^2
- \prod_{k=1}^{L} \sinh(\lambda - \mu_k + \gamma) \sinh(\lambda - \mu_k - 3\gamma) \sinh(\lambda - \mu_k - \gamma)^2
- 2 \cosh(2\gamma) \prod_{k=1}^{L} \sinh(\lambda - \mu_k) \sinh(\lambda - \mu_k - 2\gamma) \sinh(\lambda - \mu_k - \gamma)^2.
$$

(4.31)

In what follows, we shall describe how one can extract a set of equations determining the set of zeroes $\{w_j\}$ from the functional relation (4.30). For that, we assume the representation (4.12) and set $\lambda = w_i + \gamma$ in (4.30). Under this specialization, the term $\Lambda(\lambda - \gamma)\big|_{\lambda=w_i+\gamma}$ vanishes and we are left with a relation depending on the function $Q(w_i+\gamma)$. Next, we consider the specialization $\lambda = w_i + 2\gamma$ such that $\Lambda(\lambda - 2\gamma)\big|_{\lambda=w_i+2\gamma} = 0$. This procedure yields two equations: one involving the function $Q(w_i + \gamma)$, and another depending on $Q(w_i + 2\gamma)$. However, we can readily verify that $Q(\lambda) = Q(\lambda + \gamma)$ under the root of unity condition $l = 4$. This property allows us to eliminate the functions $Q$ from our equations, leaving us with the following relation:

$$
\prod_{k=1}^{L} \frac{\sinh(w_i - \mu_k + \gamma) \sinh(w_i - \mu_k - \gamma)}{\sinh(w_i - \mu_k + 2\gamma) \sinh(w_i - \mu_k)} = -\frac{\Lambda(w_i - \gamma)}{\Lambda(w_i + \gamma)}.
$$

(4.32)

By substituting the representation (4.12) into (4.32), we immediately recognize equation (4.28). Thus, our analysis so far shows that the set of equations (4.28) is valid for $l = 2, 3, 4$. In fact, equation (4.28) seems to be valid for arbitrary roots of unity, as we shall discuss.

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4.3.4. General case. For arbitrary values of $l$, the property (4.22) truncates the system of functional relations (3.8) and we only need to consider the following equation:

$$\left\{ \sum_{m=0}^{[l]/2} \sum_{0 \leq i_1 < \ldots < i_{2m} \leq l-1} V_{i_{2m}, \ldots, i_1}^{(2m)} \prod_{\lambda} \Lambda(\lambda) |_{\lambda_j = \lambda - j\gamma} = 0 \right. \right\} \Lambda(\lambda) = 0 . \quad (4.33)$$

Then we assume the representation (4.12), and consider the sequence of specializations $\lambda = w_i + p\gamma$ for $1 \leq p \leq p-2$. This procedure yields one equation at each level of specialization. Similarly to the case $l = 4$, the resulting system of equations can be manipulated in order to find compact equations determining the roots $w_j$. Although this last step involves the use of non-trivial properties satisfied by the functions $V_{i_{2m}, \ldots, i_1}^{(2m)}$, the procedure described above holds in general, and its implementation for particular values of $l$ leads to the very same equation (4.28). A rigorous proof of (4.28) for arbitrary values of $l$ is still missing, but our analysis so far leads us to conjecture that (4.28) is valid for general roots of unity.

5. Concluding remarks

The main result of this work is the formula (4.8), which states a relation between the six-vertex model with DWBC and the anti-periodic Heisenberg chain. This relation is a direct consequence of the algebraic-functional approach introduced in [13] and refined in the series of works [16–18]. The Yang–Baxter algebra is the main ingredient for the derivation of (4.8), which allows us to establish a non-trivial relation between the zeroes of certain quantities related to six-vertex models with different boundary conditions. On the one hand, we have the zeroes of the partition function of the six-vertex model with DWBC. On the other, we have the zeroes of the transfer matrix eigenvalues associated with the six-vertex model with a non-diagonal boundary twist. The relation between those zeroes is then made clear in (4.15) and (4.17).

In this work, we have also analyzed the cases where the six-vertex model anisotropy parameter satisfies a root of unity condition. In that case, we found a compact set of equations, namely (4.28), characterizing the zeroes of the eigenvalues $\Lambda$.

Boundary conditions of domain wall type can be formulated for a variety of lattice integrable systems. In particular, the so called 8V-SOS model also admits domain wall boundary conditions, and has been studied through this algebraic-functional approach in [17, 18]. The latter consists of an elliptic integrable system and one might wonder if there exists a twisted transfer matrix such that a relation analogous to (4.8) holds. This problem has eluded us so far and its investigation would probably bring further insights into the structure of integrable solid-on-solid models.

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Appendix A. The function $V_{2m\ldots,h}^{(2m)}$

In this appendix, we aim to discuss the derivation of formulae (4.8) and (4.9). The function $V_{2m\ldots,h}^{(2m)}$ given by (4.9) follows straightforwardly from the functions $M_i^{(n)}$ and $N_j^{(n)}$ defined in (3.9). We shall start by reviewing the cases $L = 2$ and $L = 3$ already discussed in section 4.

A.1. $L = 2$

The first equation of (4.3) can be rewritten as

$$Z(X^{0,1})k_0 = \Lambda(\lambda_0)\Lambda(\lambda_1) - M_1^{(1)}(\vec{X}^{0,1}) , \quad (A.1)$$

and we can compare its RHS with (4.10). In this way, we require that

$$V_{1,0}^{(2)} = -M_1^{(1)}(\vec{X}^{0,1}).$$

The explicit evaluation of (4.9) then yields the following expression

$$V_{1,0}^{(2)} = \frac{c(\lambda_0 - \lambda_1)}{b(\lambda_0 - \lambda_1)} \prod_{k=1}^2 a(\lambda_0 - \mu_k)b(\lambda_1 - \mu_k) + \frac{c(\lambda_1 - \lambda_0)}{b(\lambda_1 - \lambda_0)} \prod_{k=1}^2 a(\lambda_1 - \mu_k)b(\lambda_0 - \mu_k) ,$$

which corresponds to $-M_1^{(1)}(\vec{X}^{0,1})$ according to (3.9).

A.2. $L = 3$

Similarly to the previous case, we firstly rewrite the first equation of (4.5) as

$$Z(X^{0,2})k_0 = \Lambda(\lambda_0)\Lambda(\lambda_1)\Lambda(\lambda_2) - \Lambda(\lambda_0) \left[ M_1^{(1)}(\vec{X}^{1,2}) + N_2^{(2)}(\vec{X}^{0,2}) \right] - \Lambda(\lambda_1)M_2^{(2)}(\vec{X}^{0,2}) - \Lambda(\lambda_2)M_1^{(2)}(\vec{X}^{0,2}) . \quad (A.2)$$

We can now compare the RHS of (A.2) with (4.11). By doing so, we find the following conditions:

$$V_{1,0}^{(2)} = -M_1^{(2)}(\vec{X}^{0,2})$$
$$V_{2,0}^{(2)} = -M_2^{(2)}(\vec{X}^{0,2})$$
$$V_{2,1}^{(2)} = -M_1^{(1)}(\vec{X}^{1,2}) - N_{2,1}^{(2)}(\vec{X}^{0,2}) . \quad (A.3)$$

It is now a straightforward computation to verify that the functions $V_{1,0}^{(2)}$, $V_{2,0}^{(2)}$ and $V_{2,1}^{(2)}$ obtained from (4.9) satisfy the conditions (A.3) with functions $M_i^{(n)}$ and $N_j^{(n)}$ given by (3.9). It is also worth remarking that the verification of the third condition of (A.3) involves the simplification of functions as it typically occurs in algebraic Bethe ansatz framework.
A.3. L = 4

We start from (3.8) and for the case L = 4 we set n = 0, 1, 2, 3, 4. In this way, we are left with a total of 5 equations for the functions $F_n$ which can be solved in favor of the eigenvalue Λ. The resulting equation will then depend on the functions $M_i^{(n)}$ and $N_{j,i}^{(n)}$ defined in (3.9); it can be directly compared with (4.8). By doing so we find the following conditions:

\begin{align*}
V_{3,0}^{(2)} &= -M_3^{(3)}(\vec{X}^{0,3}) & V_{3,2}^{(2)} &= -M_1^{(1)}(\vec{X}^{2,3}) - N_{2,1}^{(2)}(\vec{X}^{1,3}) - N_{3,2}^{(3)}(\vec{X}^{0,3}) \\
V_{2,0}^{(2)} &= -M_2^{(3)}(\vec{X}^{0,3}) & V_{3,1}^{(2)} &= -M_2^{(2)}(\vec{X}^{1,3}) - N_{3,1}^{(3)}(\vec{X}^{0,3}) \\
V_{1,0}^{(2)} &= -M_1^{(3)}(\vec{X}^{0,3}) & V_{2,1}^{(2)} &= -M_1^{(2)}(\vec{X}^{1,3}) - N_{2,1}^{(3)}(\vec{X}^{0,3})
\end{align*}

in addition to

\begin{align*}
V_{3,2,1,0}^{(4)} &= M_1^{(3)}(\vec{X}^{0,3}) M_1^{(1)}(\vec{X}^{1,3}) + M_2^{(2)}(\vec{X}^{0,3}) M_1^{(1)}(\vec{X}^{2,3}) + M_3^{(3)}(\vec{X}^{0,3}) M_1^{(1)}(\vec{X}^{3,3}) \\
&\quad + N_{2,1}^{(3)}(\vec{X}^{0,3}) M_1^{(1)}(\vec{X}^{1,3}) + N_{3,1}^{(3)}(\vec{X}^{0,3}) M_1^{(1)}(\vec{X}^{2,3}) + N_{3,2}^{(3)}(\vec{X}^{0,3}) M_1^{(1)}(\vec{X}^{3,3})
\end{align*}

Both relations (A.4) and (A.5) can be readily verified with the help of (4.9) and (3.9).

A.4. General L

The structure of the function $V_{i_{2m},...,i_1}^{(2m)}$ for arbitrary values of L is obtained from particular combinations of the functions $M_i^{(n)}$ and $N_{j,i}^{(n)}$. These combinations are built by eliminating the functions $F_n$ from the system of equations (3.8) in favor of the eigenvalue Λ. By carrying out this procedure, we find the relation (4.8). The function $V_{i_{2m},...,i_1}^{(2m)}$, as defined in (4.9), captures the aforementioned combinations of $M_i^{(n)}$ and $N_{j,i}^{(n)}$ which are explicitly given by (3.9). Although it is a cumbersome computation, the simplifications required to arrive at formula (4.9) are performed in much the same spirit of the algebraic Bethe ansatz.

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