A Natural Renormalization of
the One-Loop Effective Action
for Scalar Field in Curved Space-time

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Abstract

It has been shown that the negative norm states necessarily appear in a covariant
quantization of the free minimally coupled scalar field in de Sitter space [1, 2]. In this
process ultraviolet and infrared divergences have been automatically eliminated [3]. A
natural renormalization of the one-loop interacting quantum field in Minkowski space-
time (\(\lambda\phi^4\) theory) has been achieved through the consideration of the negative norm
states [4]. One-loop effective action for scalar field in a general curved space-time has
been calculated by this method and a natural renormalization procedure in the one-loop
approximation has been established.

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1 Introduction

In the previous papers, we have shown the necessity of keeping the negative norm states, for a fully covariant quantization of the minimally coupled scalar field in de Sitter space (“Krein” QFT) [1, 2, 3]. We have also shown that the effect of these unphysical states appears in the physics of the problem by allowing an automatic renormalization of the theory in the one-loop approximation. We have also shown that, for the physical states (positive norm states), the energy is positive whereas for the negative norm states (called “unphysical” states) the energy is negative.

Consideration of the negative norm states was proposed by Dirac in 1942 [5]. In 1950 Gupta applies the idea in QED [6]. The presence of higher derivative in the Lagrangian also lead to ghosts, states with negative norm [7]. Mathematically for preserving the covariant principle, the auxiliary negative norm states were presented. Their presence has also different consequences for example in QED the negative energy photon disappear [6], and in de Sitter the infrared divergence eliminated [2]. The physical interpretation however is not yet clear and any further interpretation needs far more investigations [8, 9, 10, 11].

In the usual QFT, the one-loop effective action for the scalar field in a general curved space is divergent [12]. One way to remove this divergence is modifying the Einstein’s field equations. Precisely for this reason that one of the most important problems of quantum gravity appears: for every new loop expansion, new terms in the Einstein’s field equations are needed. This means that the theory is not renormalizable. In this paper we have proposed a method of field quantization for calculating of the one-loop effective action for the scalar field in a general curved space-time, which result in a finite one-loop effective action.

2 Usual QFT calculation

Recalling the usual QFT calculation for the one-loop effective action [12], the scalar field in a general curved space-time is defined by:

\[ [\Box + m^2 + \xi R(x)]\phi(x) = 0, \]  

where \( \Box \) is the Laplace-Beltrami operator in curved space-time and \( m \) is the “mass” of the field quanta. \( R(x) \) is the Ricci scalar curvature and \( \xi \) is the coupling constant between the scalar field and the gravitational field. The adiabatic expansion of the Feynman propagator is defined by [12]

\[ G_F^p(x, x') \approx (-g(x))^{-\frac{1}{2}} \int \frac{d^4k}{(2\pi)^4} e^{-ik.y} \left[ \sum_{j=0}^{\infty} a_j(x, x')(\frac{\partial}{\partial m^2})^j \right] \frac{1}{k^2 - m^2 + i\epsilon}, \]  

where symbol \( \approx \) indicates an asymptotic expansion and \( y \) is the Reimann normal coordinates for the point \( x \), with origin at point \( x' \). In the semi classical theory the Einstein’s field equations are:

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G < T_{\mu\nu} >, \]
where \( \langle T_{\mu\nu} \rangle \) is the quantum expectation value of the matter stress-tensor. The effective action \( (W) \) of the quantum matter field is defined by:

\[
\frac{2}{(-g)^{\frac{3}{2}}} \frac{\delta W}{\delta g^{\mu\nu}} = \langle T_{\mu\nu} \rangle. \tag{4}
\]

In the one-loop approximation it is defined by

\[
W = \int d^4x [-g(x)]^{\frac{3}{2}} L_{\text{eff}}^p(x) = -\frac{1}{2} i \int d^4x [-g(x)]^{\frac{3}{2}} \langle x \mid \ln(-G_F^p) \mid x \rangle, \tag{5}
\]

where

\[
\langle x \mid G_F^p \mid x' \rangle = G_F^p(x, x'). \tag{6}
\]

We can also write the effective Lagrangian in the following form

\[
L_{\text{eff}}^p(x) = -\frac{1}{2} i \langle x \mid \ln(-G_F^p) \mid x \rangle = \frac{1}{2} \int_m^\infty \frac{d m^2}{(2\pi)^4} \int k^4 \left( \sum_{j=0}^\infty a_j(x) \frac{-\partial}{\partial m^2} \right)^j \frac{1}{k^2 - m^2 + i\epsilon}. \tag{7}
\]

Replacing \( G_F^p(x, x) \) from equation (2) in equation (7) we have

\[
L_{\text{eff}}^p(x) \approx \frac{i}{2} (-g(x))^{\frac{3}{2}} \int_m^\infty \frac{d m^2}{(2\pi)^4} \left( \sum_{j=0}^\infty a_j(x) \frac{-\partial}{\partial m^2} \right)^j \frac{1}{k^2 - m^2 + i\epsilon}. \tag{8}
\]

In this relation there are three terms which diverge. For eliminating these divergences the left-hand side of the Einstein’s field equation is changed and \( \Lambda \) and \( G \) are also redefined (renormalization procedure) [12]

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} + a^{(1)} H_{\mu\nu} + b^{(2)} H_{\mu\nu}. \tag{9}
\]

A standard technique in QFT indicates that the terms involving higher derivatives of the metric are expected in view of divergence elimination.

Using following integral representation in (2)

\[
\frac{1}{k^2 - m^2 + i\epsilon} = -i \int_0^\infty ds e^{is(k^2 - m^2 + i\epsilon)},
\]

permuting the \( d^4k \) integration with \( ds \) integration in (2), and performing the former, the Green’s function can be written in terms of the Bessel functions [12]

\[
G_F^p(x, x') \approx \Delta^\frac{3}{2}(x, x') \left[ \sum_{j=0}^\infty a_j(x, x') \left( -\frac{\partial}{\partial m^2} \right)^j \right] G_F^{p(M)}(x, x'),
\]

where \( \Delta \) is the Van Vleck determinant and \( G_F^{p(M)}(x, x') \) is the Feynman Green function in the Minkowski space

\[
G_F^{p(M)}(x, x') = -\frac{1}{8\pi} \delta(\sigma) + \frac{m^2}{8\pi} \theta(\sigma) J_1\left(\frac{\sqrt{2m^2\sigma}}{\sqrt{2m^2}}\right) - iN_1 \left(\frac{\sqrt{2m^2\sigma}}{\sqrt{2m^2}}\right)
\]

\[
- \frac{im^2}{4\pi^2} \theta(-\sigma) K_1\left(\frac{\sqrt{2m^2\sigma}}{\sqrt{2m^2}}\right), \quad \sigma = \frac{1}{2}(x - x')^2.
\]

3
3 Krein QFT calculation

The origin of divergence lies in the singular character of Green’s function at short relative distances. It has been shown in [1, 2, 3, 4] that if the unphysical negative norm states are taken into account in the field quantization, the time-ordered product of fields or the “Feynman” propagator

\[ iG_T(x, x') = \langle 0 \mid T\phi(x)\phi(x') \mid 0 \rangle, \]

is defined by the following relation

\[ G_T(x, x') = \frac{1}{2}[G_F(x, x') + (G_F(x, x'))^*], \]

where \( G_F(x, x') \) is usual Feynman propagator. Similar to the quantization of the electromagnetic field in Minkowski space [13], insofar as only average values are observed, we see that the unphysical negative norm states disappeared when restricting ourselves to physical states although the Green’s function is changed due to presence of negative norm states.

In this method the time-ordered product two-point function is

\[ G_T(x, x') \approx \Delta^\sharp(x, x') \left[ \sum_{j=0}^\infty a_j(x, x')(-\frac{\partial}{\partial m^2})^j \right] \left[ \frac{m^2}{8\pi} \theta(\sigma) \frac{J_1(\sqrt{2m^2}\sigma)}{\sqrt{2m^2}\sigma} - \frac{1}{8\pi} \delta(\sigma) \right] \]

\[ = \Delta^\sharp(x, x') \left( -\frac{a_0}{8\pi} \delta(\sigma) + \sum_{j=0}^\infty a_j(x, x')(-\frac{\partial}{\partial m^2})^j \right) \left[ \frac{m^2}{8\pi} \theta(\sigma) \frac{J_1(\sqrt{2m^2}\sigma)}{\sqrt{2m^2}\sigma} \right]. \]

This expression in the limit \( x \to x' \) and \( \sigma > 0 \), simplifies to

\[ \lim_{x \to x'} G_T(x, x') \approx \frac{1}{16\pi} [a_0(x)m^2 - a_1(x)] \int_0^\infty s e^{-s} ds, \]

where the integral \( \int_0^\infty s e^{-s} ds = 1 \) is presented in view of the following calculation of the effective action. The divergence of the delta function form \( (\delta(\sigma = 0)) \), is ignored. This term produce a constant term in the effective action.

If we use eq. (13) and the procedure which is used when only positive norm states are involved [12], the following expression for the effective Lagrangian is obtained

\[ L_{eff}(x) = -\frac{i}{2} \lim_{x \to x'} < x \mid \ln(-G_T) \mid x' > = -\frac{i}{2} \lim_{x \to x'} \int_0^\infty < x \mid e^{-iKs} \mid x' > (is)^{-1} ds, \]

where

\[ \lim_{x \to x'} < x \mid e^{-iKs} \mid x' > \approx \frac{i}{16\pi} [a_0(x)m^2 - a_1(x)]m^4 s e^{-m^2 s}. \]

Then the effective Lagrangian in the one-loop approximation reads:

\[ L_{eff}(x) \approx \frac{1}{32\pi} [a_0(x)m^2 - a_1(x)]m^4 \int_0^\infty e^{-m^2 s} ds = \frac{1}{32\pi} [a_0(x)m^2 - a_1(x)]m^2. \]

By using the following relations [12]:

\[ a_0(x) = 1 , \quad a_1(x) = (\frac{1}{6} - \xi)R(x), \]
the effective action give
\[ L_{\text{eff}}(x) \approx \frac{m^4}{32\pi} - \left(\frac{1}{6} - \xi\right)\frac{m^2}{32\pi}R(x). \] \hspace{1cm}(18)

One of the interesting issue of this calculation is that, in the one-loop approximation, the effective action is similar to the Einstein-Hilbert action with a cosmological constant (de Sitter background).

4 Conclusion

’t Hooft and Veltman have shown that all one-loop divergencies of pure gravity can be absorbed in a field renormalization. However, in the case of interaction of gravity and scalar field, divergencies in physical quantities are not eliminated [14], unless new terms are introduced to the Einstein’s field equation [12]. In this paper, it is proved that if the unphysical negative norm states are considered in QFT formalism, non-divergent one loop-approximation of the interaction of gravity with a scalar field can be archived i.e. the effective action for scalar field is naturally convergent. Thus the Einstein’s field equation is not altered. The quantum gravity in the one-loop approximation behaves in a non anomalous way.

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