State Selection in Accelerated Systems

Martin B. Tarlie and K. R. Elder
1James Franck Institute, University of Chicago, 5640 South Ellis Avenue, Chicago, IL 60637
2Department of Physics, Oakland University, Rochester, MI, 48309-4487
(March 24, 2022)

The problem of state selection when multiple metastable states compete for occupation is considered for systems that are accelerated far from equilibrium. The dynamics of the supercurrent in a narrow superconducting ring under the influence of an external electric field is used to illustrate the general phenomenology.

PACS: 02.50.Ey, 05.20.-y, 05.40.+j

Many systems when driven far from equilibrium encounter instabilities that lead towards new states or phases. Frequently there exist multiple states that can be selected following the onset of the instability. The determination of the particular state that is selected is a complex problem of fundamental interest in a wide variety of fields. In addition to the complexity associated with the presence of multiple competing states, if the system is accelerated so that the environment evolves in time, then the state that is selected can depend in an important way on the driving force. The focus of this paper is on state-selection in “accelerated” systems when multiple metastable states compete for occupation.

For the purposes of this paper an accelerated system is defined as one for which a control parameter is varied in time so that the system gradually progresses from stable to unstable regimes. For example, in a narrow superconducting ring under the influence of a constant electromagnetic force, the superconducting electrons are accelerated by the electric field and the supercurrent increases with time until the critical current is reached and the system becomes (Eckhaus) unstable. Similar behavior could occur in direction solidification if the solidification cell is accelerated slowly, rather than pulled at a constant velocity, through a temperature gradient which an infinitely long solenoid, carrying a current that increases linearly with time, passes through the center of a narrow superconducting ring of cross-sectional area A and circumference L ≡ ξ(T)ℓ, where ξ(T) is the temperature dependent correlation length. By Faraday’s law of induction, a constant electromotive force (emf) V is induced in the superconductor, thereby accelerating the superconducting electrons. The dynamics of the (dimensionless) superconducting order parameter ψ(x,t), where x is the longitudinal spatial coordinate and t is time, is described by the stochastic time-dependent Ginzburg-Landau equation:

\[ \partial_t \psi = \partial_x^2 \psi + \psi - |\psi|^2 + i\ell^{-1}xωψ + η \tag{1} \]

where ω ≡ τGL(2eV/ℏ) is a dimensionless measure of the strength of the induced emf. Throughout this paper the regime where ω ≪ 1 is considered. In Eq. (1) τGL is the Ginzburg-Landau relaxation time, and ψ satisfies the twisted-periodic boundary condition ψ(t + x, t) = ψ(t + x', t') = exp(iωt)ψ(x, t). This equation ignores the influence of the normal current and is valid for a voltage driven system in the limit of low normal state resistivity. The variable η is a Gaussian random variable, with expectation values \( \langle η(x,t) \rangle = 0 \), and \( \langle η(x,t)η^*(x',t') \rangle = 2Dδ(x - x')δ(t - t') \), where D is determined by the fluctuation-dissipation theorem.

For ω ≪ 1, the relevant current-carrying states of the superconductor are uniformly twisted plane wave solutions given by \( \tilde{ψ} = \sqrt{1 - q^2} \exp(iq x) \), where \( q = mK + ωt/\ell \), and \( K ≡ 2π/\ell \). The dimensionless current density J of these states is given by \( J = \langle \tilde{ψ}^* \partial_x \tilde{ψ} - \bar{ψ} \partial_x \tilde{ψ}^* \rangle /2i = q/(1 - q^2) \). Thus the effect of the induced emf (which increases q linearly with time) is to wind the order parameter, or equivalently, to accelerate the superconducting electrons. However, this acceleration cannot continue in-
the superconducting transition temperature
thermally activated phase slip occurring is exceedingly
become positive. As illustrated in Fig. 1, the
bility as the eigenvalues of each Fourier mode eventually
activation over an energy barrier and this process has re-
by an integral multiple of $2\pi$. Physically, the supercur-
ψ = \sqrt{\psi^2 + \phi^2}$.

To understand the Eckhaus instability for finite size
systems it is necessary to perform a linear stability analy-
sis about the state $\psi$, as the previous analysis only applies
when $\ell = \infty$. Standard linear stability analysis gives one
potentially positive eigenvalue that takes the form $[10]$

$$\lambda_n(q) = -1 + q^2 - k_n^2 + \sqrt{(1 - q^2)^2 + 4q^2k_n^2}. \quad (2)$$

The eigenvector associated with this eigenvalue is a linear
combination of Fourier modes with wavevector $q \pm k_n$
and amplitude $A_n$, where $k_n = nK$. The interesting feature
of this eigenvalue is that it can become positive when $q > \kappa_1 > q_c$, where $\kappa_n \equiv \sqrt{\frac{1}{2}}[1 + m^2K^2/2]^{1/2}$. Thus, for
finite size systems the instability is pushed to wavevectors
greater than $q_c$ by an amount that depends on $\ell$ $[11]$. In
particular, for $\kappa_n > q > \kappa_1$, $\lambda_n$ is positive for all values
of $k_n < mK$ $[11]$. The dependence of $\lambda$ on $k_n$ is shown in
the inset of Fig. 1 for several values of $q$.

The growth of a single Fourier mode (with amplitude
$A_n$) of wavevector $q - k_n$, and simultaneous decay of
$A_0$, corresponds to a decrease of the winding number
$W = (2\pi)^{-1} \int_0^L \phi(x)/dx$, where $\phi$ is the phase of $\psi$, by
an amount $n$. This phenomenon is known as a “phase-
slip” as the total phase of the order parameter changes
$$\left| \langle A_n(t)^2 \rangle = \frac{2D}{\ell} e^{2\sigma_n(t)} \int_0^t dt' e^{-2\sigma_n(t')} \right., \quad (3)$$

where $\sigma_n(t) \equiv \int_0^t dt' \lambda_n(q(t'))$, and angular brackets
denote a noise average. After the onset of the in-
\begin{align*}
\lambda_n(q) & = -1 + q^2 - k_n^2 + \sqrt{(1 - q^2)^2 + 4q^2k_n^2}. \quad (2) \\
\lambda_n(q) & = -1 + q^2 - k_n^2 + \sqrt{(1 - q^2)^2 + 4q^2k_n^2}. \quad (2) \\
\end{align*}

double phase-slip processes dominate. As $\omega$ is increased
further, double phase-slip processes should give way to
triple phase-slip processes, and so on.

The generic features displayed in Fig. 2 are common to
many systems and come under the general classification
scheme of Cross and Hohenberg $[1]$ as type $\Pi$. Thus, the
dynamic competition between unstable modes discussed
above is a phenomena that has relevance to many sys-
tems. The precise determination of which of the modes
will initially be selected following the onset of the insta-
\begin{align*}
\lambda_n(q) & = -1 + q^2 - k_n^2 + \sqrt{(1 - q^2)^2 + 4q^2k_n^2}. \quad (2) \\
\end{align*}

due to an analysis that is based on the properties of the growth
rates $\lambda$.

To evaluate the probability of the occurrence of a given
phase slip as a function of $\omega$, Eq. $[10]$ was numerically
integrated in time for a noise strength of $D = 10^{-3}$ and
a length corresponding to $n \ell = \ell q_c/(2\pi) = 5$ $[11]$. In
Fig. 2b, the probability of a type-$n$ phase slip is plotted
as a function of $\omega$. As expected, for small $\omega$, single phase-
slips dominate. As $\omega$ increases further there is a crossover
to a regime in which double phase-slips dominate. Fur-
ther increase of $\omega$ results in a subsequent crossover to a
regime in which triple phase-slips dominate, and so on.

An example of the dynamics that lead to such results is
shown in Fig. 3. In this figure, the winding number and
current are plotted as a function of time for $\omega = 5 \times 10^{-4}$.
This value of $\omega$ is in the crossover region between the
single and double phase-slip dominated regimes. Clearly
evident in this figure are the single and double phase slips
in which $W$ changes by one or two, respectively. Also seen
in Fig. 3 are the discrete jumps of the supercurrent.

As described earlier, the essential features shown in
Fig. 2 can be understood using the properties of the
growth rates $\lambda_n$. This idea can be made more concrete in
the following way. Ignoring the nonlinear interactions
between the different modes, the expectation of $|A_n|^2$ is
given by $[12]$

\begin{align*}
\langle |A_n(t)|^2 \rangle & = \frac{2D}{\ell} e^{2\sigma_n(t)} \int_0^t dt' e^{-2\sigma_n(t')}, \quad (3)
\end{align*}

where $\sigma_n(t) \equiv \int_0^t dt' \lambda_n(q(t'))$, and angular brackets
denote a noise average. After the onset of the in-
\begin{align*}
\lambda_n(q) & = -1 + q^2 - k_n^2 + \sqrt{(1 - q^2)^2 + 4q^2k_n^2}. \quad (2) \\
\lambda_n(q) & = -1 + q^2 - k_n^2 + \sqrt{(1 - q^2)^2 + 4q^2k_n^2}. \quad (2) \\
\end{align*}

due to an analysis that is based on the properties of the growth
rates $\lambda$.

To evaluate the probability of the occurrence of a given
phase slip as a function of $\omega$, Eq. $[10]$ was numerically
integrated in time for a noise strength of $D = 10^{-3}$ and
\begin{align*}
\lambda_n(q) & = -1 + q^2 - k_n^2 + \sqrt{(1 - q^2)^2 + 4q^2k_n^2}. \quad (2) \\
\lambda_n(q) & = -1 + q^2 - k_n^2 + \sqrt{(1 - q^2)^2 + 4q^2k_n^2}. \quad (2) \\
\end{align*}

due to an analysis that is based on the properties of the growth
rates $\lambda$.

To evaluate the probability of the occurrence of a given
phase slip as a function of $\omega$, Eq. $[10]$ was numerically
integrated in time for a noise strength of $D = 10^{-3}$ and

\begin{align*}
\lambda_n(q) & = -1 + q^2 - k_n^2 + \sqrt{(1 - q^2)^2 + 4q^2k_n^2}. \quad (2) \\
\lambda_n(q) & = -1 + q^2 - k_n^2 + \sqrt{(1 - q^2)^2 + 4q^2k_n^2}. \quad (2) \\
\end{align*}

due to an analysis that is based on the properties of the growth
rates $\lambda$.

To evaluate the probability of the occurrence of a given
phase slip as a function of $\omega$, Eq. $[10]$ was numerically
integrated in time for a noise strength of $D = 10^{-3}$ and

\begin{align*}
\lambda_n(q) & = -1 + q^2 - k_n^2 + \sqrt{(1 - q^2)^2 + 4q^2k_n^2}. \quad (2) \\
\lambda_n(q) & = -1 + q^2 - k_n^2 + \sqrt{(1 - q^2)^2 + 4q^2k_n^2}. \quad (2) \\
\end{align*}

due to an analysis that is based on the properties of the growth
rates $\lambda$.

To evaluate the probability of the occurrence of a given
phase slip as a function of $\omega$, Eq. $[10]$ was numerically
integrated in time for a noise strength of $D = 10^{-3}$ and

\begin{align*}
\lambda_n(q) & = -1 + q^2 - k_n^2 + \sqrt{(1 - q^2)^2 + 4q^2k_n^2}. \quad (2) \\
\lambda_n(q) & = -1 + q^2 - k_n^2 + \sqrt{(1 - q^2)^2 + 4q^2k_n^2}. \quad (2) \\
\end{align*}

due to an analysis that is based on the properties of the growth
rates $\lambda$.

To evaluate the probability of the occurrence of a given
phase slip as a function of $\omega$, Eq. $[10]$ was numerically
integrated in time for a noise strength of $D = 10^{-3}$ and

\begin{align*}
\lambda_n(q) & = -1 + q^2 - k_n^2 + \sqrt{(1 - q^2)^2 + 4q^2k_n^2}. \quad (2) \\
\lambda_n(q) & = -1 + q^2 - k_n^2 + \sqrt{(1 - q^2)^2 + 4q^2k_n^2}. \quad (2) \\
\end{align*}

due to an analysis that is based on the properties of the growth
rates $\lambda$.

To evaluate the probability of the occurrence of a given
phase slip as a function of $\omega$, Eq. $[10]$ was numerically
integrated in time for a noise strength of $D = 10^{-3}$ and

\begin{align*}
\lambda_n(q) & = -1 + q^2 - k_n^2 + \sqrt{(1 - q^2)^2 + 4q^2k_n^2}. \quad (2) \\
\lambda_n(q) & = -1 + q^2 - k_n^2 + \sqrt{(1 - q^2)^2 + 4q^2k_n^2}. \quad (2) \\
\end{align*}

due to an analysis that is based on the properties of the growth
rates $\lambda$.
\[ \sum_{n=1}^{\infty} \frac{\langle |A_n(t)|^2 \rangle}{A_n^2}. \] After onset of the Eckhaus instability, this sum increases rapidly and reaches unity at a time \( t^* \). Assuming that at \( t = t^* \) a phase-slip has occurred with probability one it is natural to interpret \( \langle |A_n(t^*)|^2 \rangle / A_n^2 \) as the relative probability of the occurrence of a type-\( n \) phase-slip. The probabilities calculated using this procedure are shown in Fig. 2b.

It is clear from Fig. 2 that the preceding analysis provides a qualitatively accurate description of the state-selection probabilities, and their dependence on the driving force \( \omega \). Most notably, the values of \( \omega \) at the peak positions agree very well with the numerical results. Nevertheless it is important to point out that the preceding analysis is only a plausible argument and is not systematic. A quantitative description must include the subtle non-linear interactions that are an important element in determining state selection. Even at the present level of ignorance, however, the analysis presented here provides a qualitatively useful description of the state-selection probabilities, and their dependence on the driving force.

The growth rates \( \lambda \) are an extremely important factor in determining the state selection probabilities. The preceding analysis accounts for these growth rates and therefore provides a qualitatively accurate description. The analysis also provides predictions for the dependence of \( P_n \) on the noise strength \( D \), which may be more convenient to vary in some experiments. Plotted in Fig. 3 are the probabilities of a type-\( n \) phase slip as a function of \( D \), for a fixed value of \( \omega \), obtained from a numerical simulation of Eq. (3). In Fig. 4, the corresponding \( P_n \)'s obtained from the growth-rate analysis are plotted for comparison. Once again, it is seen that the simple analysis provides an accurate qualitative picture.

For the smallest values of \( D \) considered triple phase-slip processes dominate. This is because the time required for a given mode to grow to saturation diverges logarithmically as \( D \to 0 \). Consequently, if \( D \) is very small, the mode amplitudes \( A_1 \) and \( A_2 \), for example, may still be very small by the time the growth rate of \( A_3 \) is significantly larger than the growth rates for \( A_1 \) or \( A_2 \).

One of the most interesting aspects of the phenomena exposed here is that the selection rules depend on both the intrinsic properties of the system and the external parameters. To understand this connection more deeply, it is instructive to consider the characteristic growth times for individual modes. Typically, the characteristic time associated with the initial growth of an unstable mode is taken to be the inverse of the growth rate. However, for accelerated systems the growth rate \( \lambda \) starts out negative and passes through zero. Thus, \( |\lambda^{-1}| \) is not a relevant quantity as it diverges at the instability. To determine the characteristic time, consider Eq. (6) for \( \langle |A_n(t)|^2 \rangle \). The quantity \( \sigma(t) \) achieves a local minimum at \( t = t_n = \ell \omega / \kappa_n \) so that a second order expansion about \( t_n \) yields \( \sigma_n(t) \approx \sigma_n(t_n) + \frac{1}{2} \frac{\lambda_n^2}{\kappa_n^2}(t - t_n)^2 \), where \( \lambda_n^2 \equiv \partial \lambda_\omega / \partial q |_{q = \omega \kappa_n} \). Inserting this expansion into Eq. (8) and assuming that \( \omega < 1 \) gives

\[ \langle |A_n(t)|^2 \rangle = 2D \tau_n \ell^{-1} \exp \left( z_n^2(t) \right) \left[ \text{erf} \left( z_n(t) \right) + 1 \right], \]

where \( z_n(t) = (t - t_n) / \tau_n \) and \( \tau_n = \sqrt{\ell / \lambda_n^2 \omega} \). The quantity \( \tau_n \) is the characteristic time for the growth of mode-\( n \), and is interesting because it depends on the geometric mean of \( \lambda_n^2 \) and \( \omega \). Thus, the time scale \( \tau_n \) embodies in a natural way the importance of the combination of the intrinsic dynamics (\( \lambda_n^2 \)) and the external driving force (\( \omega \)).

In summary, the problem of state selection in accelerated systems has been shown to contain unique and rich phenomenology. Although the focus of this work has been on the dynamics of quasi-one-dimensional superconducting rings, the essential features of this particular system are generic and should be observable in a diverse array of experimentally realizable situations. Despite the success of the linear analysis, it is clear that new methods must be developed to explore this complex and important area of research in nonequilibrium statistical mechanics. Recent work [14] on state selection in non-accelerated marginally stable systems suggests a possible systematic framework that could be extended to address the phenomena considered here.

We thank Paul Goldbart for suggesting the problem of nonequilibrium superconductivity, and Paul Goldbart and Alan McKane for useful discussions. This work was supported in part by the MRSEC program of the NSF (DMR-9400379) (MBT), Research Corporation Grant #CC4181 (KRE), and grant NSF-DMR-8920538 administered through the U. of Illinois Materials Research Laboratory (MBT, KRE).

[1] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. 65, 851 (1993).
[2] L. Kramer, H. R. Schober and W. Zimmerman, Physica D 31, 212 (1988).
[3] J. S. Langer and V. Ambegaokar, Phys. Rev. 164, 498 (1967).
[4] D. E. McCumber and B. I. Halperin, Phys. Rev. B 1, 1054 (1970).
[5] M. B. Tarlie, Nonequilibrium Properties of Mesoscopic Superconducting Rings, PhD. Thesis, University of Illinois at Urbana Champaign, 1995.
[6] M. Tinkham, Introduction to Superconductivity (Krieger, Malabar, FL, 1980).
[7] W. W. Mullins and R. F. Sekerka, J. Appl. Phys. 35, 444 (1964).
[8] J. S. Langer, Rev. Mod. Phys. 52 1 (1980).
[9] Taking typical experimental parameters, such as \( \sqrt{A} \approx 1000A \), and \( T = 0.93T_c \), with \( T_c = 3K \), gives \( D \approx 10^{-3} \) and \( \omega \approx \sqrt{V/(23\mu V)} \).
[10] L. Kramer and W. Zimmerman, Physica 16D, 221 (1985).
[11] Because all modes with \( q > \kappa_1 \) are unstable, the modes
of wavevector $q + k$, which grow initially according to linear analysis, will be damped by the nonlinear terms. Therefore, these modes act as catalysts in that they play an active role in the growth of the $q - k$ modes, but they themselves cannot compete for occupation as they are unstable.

[12] M. B. Tarlie, E. Shimshoni and P. M. Goldbart, Phys. Rev. B 49, 494 (1994).

[13] The parameter $n_\ell$ gives the winding number of $\psi$ when $q = q_c$ and is therefore a useful way to parameterize the circumference of the wire.

[14] M. B. Tarlie and A. J. McKane, (submitted for publication).

FIG. 1. $\lambda$ as a function of $q = \omega t/\ell$ for $k = K, 2K, 3K$. Inset: $\lambda$ as a function of $k$ for two values of $q > \kappa_1$ such that the upper curve corresponds to the larger value of $q$.

FIG. 2. State selection probabilities as a function of the driving force $\omega$. Open squares, solid squares, open circles, solid circles and open triangles correspond to the probabilities $P_1, P_2, P_3, P_4$ and $P_5$, respectively. Results of the numerical integration of Eq. (1) and those of the linear analysis described in the text are shown in Figs. (a) and (b), respectively.

FIG. 3. Dynamics of winding number (a) and supercurrent (b), for $\omega = 5 \times 10^{-4}$ and $D = 10^{-3}$.

FIG. 4. State selection probabilities as a function of the noise strength $D$. The symbols in this figure are identical to those in Fig. 2. Results of the numerical integration of Eq. (1) and those of the linear analysis described in the text are shown in Figs. (a) and (b), respectively.
