Dimer models and quiver gauge theories

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Abstract. $\mathcal{N}=1$ quiver gauge theories on coincident $D3$ branes placed at a tip of a Calabi-Yau singularity $C$ are dual to string theories on $AdS_5 \times X_5$ where $X_5$ are Sasaki-Einstein spaces. We present a neat combinatorial approach called dimer model to understand interrelations between toric quiver gauge theories and toric data representing the Calabi-Yau singularities.

1. Introduction

$\mathcal{N}=1$ quiver gauge theories arise on $D$-branes placed at the tip of the singular Calabi-Yau $C$ and they are dual to string theory on $AdS_5 \times X_5$ [1] where $X_5$ is called the Sasaki-Einstein manifolds having a topology of $S^2 \times S^3$ topology. Mathematically, $C$ is the cone over $X_5$:

$$ds^2(C) = dr^2 + r^2 dX_5^2.$$ (1)

Quiver gauge theory on a $D$-brane placed transverse to orbifold $C^3/\Gamma$ is obtained from $U(r)$ gauge theory subjected to orbifolding action ($r=\text{rank of the orbifold group } \Gamma$). Clearly, orbifolding results in breaking the gauge group $U(r) \rightarrow U(1)^r$. There are possibilities of appearance of bi-fundamental fields and the adjoint fields. The matter content and their interactions of these quiver gauge theories have information about the Calabi-Yau singularities. As indicated in Fig.1, there are systematic algorithms[2] to determine toric data for the Calabi-Yau space from the matter content and gauge groups of quiver gauge theories. Conversely, one could obtain quiver gauge theory data from the toric data. Eventhough, these algorithms are straightforward, the method is tedious. Few years ago, Hanany et al[3] showed that there exists an elegant approach called dimer tiling approach which directly gives both the gauge theory data as well as the toric data.

![Flow chart](image)

**Figure 1.** Flow chart
The plan of the article is as follows: In section 2, we first review quiver gauge theories and the toric diagrams representing toric Calabi-Yau three-folds. Then, we present the salient features of the algorithms. In section 3, we briefly discuss statistical mechanical models called dimer models and their powerfulness in directly determining both the quiver gauge theories and the toric diagram. In the concluding section, we summarise our results.

2. Quiver gauge theories and toric Calabi-Yau spaces

2.1. Quiver gauge theories

The matter content of quiver gauge theories are usually represented by a quiver diagram as shown in Fig.2. The number of nodes \( r \) in the quiver diagram gives the number of gauge groups and the number of edges \( m \) determines the number of adjoint & bi-fundamental fields. The line from node \( A \) to itself denotes the adjoint field. The directed edge from node \( A \) to node \( B \) implies that a bifundamental field \( X_{AB} \) charged \(-1\) with respect to gauge group \( G_A \) and \(+1\) with respect to gauge group \( G_B \). For the example shown in Fig.2, we see that there are three gauge groups \( r = 3 \), one adjoint and six bi-fundamental fields \( (m = 7) \). The diagram directly gives charge matrix \( d_{ai} \) where \( a = (1, 2 \ldots r) \) is the gauge group index and \( i = (1, 2 \ldots m) \) is the matter field index. These \( d_{ai} \)'s appear in the D-term equations: 

\[
D_a = \sum_i d_{ai} |X_i|^2 - \xi_a .
\]

One of the rows of the charge matrix can always be eliminated to give a \((r - 1) \times m\) matrix \( \Delta \). The interaction amongst the fields is given by a superpotential \( W[\{X\}] \). The quiver gauge theories are called toric quiver gauge theories if all the gauge groups have the same rank and \( W \) contains each bifundamental field in two terms with relative signs. For example, the following superpotential

\[
W = X_{BA}X_{AB}X_{BC}X_{CB} - X_{CB}X_{BC}X_{CA}X_{AC} + X_{AC}X_{CA}X_{AA} - X_{AB}X_{BA}X_{AA} ,
\]

plus the quiver diagram in Fig.2 gives the toric quiver gauge theory information. It will be interesting to determine the toric Calabi-Yau corresponding to these quiver gauge theories. We will now briefly present toric diagrams representing toric Calabi-Yau spaces and then discuss the algorithms relating the Calabi-Yau spaces to the quiver gauge theories.

2.2. Non-compact toric Calabi–Yau spaces

Non-compact toric Calabi-Yau spaces are spaces described as \( T^3 \) fibration over \( R^3 \). These spaces can also be obtained from partial resolutions of orbifolds of \( C^3 \). There is an equivalent description of these toric Calabi-Yau spaces in Witten’s gauged linear sigma model. Every toric Calabi-Yau spaces can be represented by a toric diagram. The toric diagram corresponding to \( C^3 \) and orbifolds of \( C^3 \) are drawn in Fig.3. We call these orbifolds as parent spaces. The numbers against every node \( i \) in the toric diagram indicate multiplicity of the sigma model fields. The
Figure 3. Toric diagrams for \( C^3 \) and their orbifolds

Partial resolutions of these parent spaces gives non-orbifold toric Calabi-Yau spaces (daughter spaces) as shown in Fig. 4. There is a systematic algorithm of relating a quiver gauge theories to the corresponding Calabi-Yau space which we call forward algorithm \([2]\). In the following subsection, we elaborate the essential details of the algorithm.

2.3. Forward Algorithm
(Quiver gauge theory → toric diagrams)
Given a gauge theory data (quiver diagram+W)\([2]\), the \( F \)-term equations
\[
\frac{\partial W}{\partial X_i} = 0 ,
\]
are all not independent. These can be solved by introducing \( r + 2 \) new fields \( v_j \) as:
\[
X_i = \prod_j v_j^{K_{ij}}
\]
where \( i \in (1, m) \) and \( j \in (1, r + 2) \). For \( W \) \((2)\) we can determine the \( K \) matrix (analogue of charge matrix \( \Delta \)) in the following way: First we choose the \( v_i \)'s to be
\[
v_1 = X_{BA}, v_2 = X_{CA}, v_3 = X_{CB}, v_4 = X_{BC},
\]
and then determine the other matter fields $X_i$'s using the $F$-term equations. We find $X_{AA} = v_2v_4$, $X_{AC} = v_1v_5$ and $X_{AB} = v_2v_5$. Hence, the $K$ matrix for this example is

$$K = \begin{pmatrix}
v_1 & v_2 & v_3 & v_4 & v_5 \\
X_{BA} & 1 & 0 & 0 & 0 \\
X_{CA} & 0 & 1 & 0 & 0 \\
X_{AA} & 0 & 0 & 1 & 0 \\
X_{AB} & 0 & 1 & 0 & 0 \\
X_{BC} & 0 & 0 & 0 & 1 \\
X_{AC} & 1 & 0 & 0 & 0 \\
X_{CB} & 0 & 0 & 1 & 0 
\end{pmatrix}$$

Using this $K$ matrix, we determine another rectangular matrix $(r + 2) \times c$ matrix $T$ satisfying $K.T \geq 0$. Here $c$ is known only from the evaluation of the $T$ matrix. Similar to $K$ matrix relating $X_i$'s to $v_j$'s, the $T$ matrix relates $v_j$ fields to $c$ new fields $p_\alpha$'s in the following way:

$$v_j = \prod_\alpha p_\alpha^{T_{j\alpha}}.$$}

Clearly, the extra $c-(r-2)$ fields are redundant fields which are eliminated using a $(c-r-2) \times c$ matrix $Q_F$ such that $TQ_F = 0$. We say such a $Q_F$ as the co-kernel of $T$. Similarly we could write a $Q_D$ matrix related to the charge matrix $\Delta$. Rewriting $X_i$'s in terms of $p_\alpha$'s

$$X_i = \prod_\alpha \sum_\alpha K_{ij}T_{j\alpha},$$

we can see that $(K.T)Q_D^T = \Delta^t$. Concatenating $Q_D$ and $Q_F$ gives a matrix $Q$. This $Q$ plays a crucial role to obtain the toric data. For the toric quiver gauge theories, we can find a rectangular matrix $G$ such that $Q.G^t = 0$. Each column of $G$ is a vector which can be plotted to give a toric diagram.

For the quiver theory example, $G$ is

$$G = \begin{pmatrix}
p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \\
0 & 1 & 0 & 0 & -1 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 
\end{pmatrix}$$

which represents the suspended pinch point toric diagram in Fig.4. So far, we briefly presented the methodology of obtaining the toric data corresponding to a quiver gauge theory. It appears that each step is invertible and hence we should be able to tackle the converse problem of obtaining a quiver gauge theory from the toric data matrix $G$. Unlike the forward algorithm which gives a unique toric data, the inverse algorithm results in more than one quiver gauge theory corresponding to a toric diagram[2]. The inverse algorithm does not give the adjoint matter fields. Also, the algorithm is not practical to implement for large toric diagrams. Interestingly, an elegant combinatorial approach involving a statistical mechanical model called dimer models determines both the gauge-theory data as well as the Calabi-Yau geometry toric data[3]. We describe dimer models in the following section.

3. Dimer models

Dimer model refers to a bipartite graph with vertices colored either black or white such that no two adjacent vertices have the same color. Each edge is called a dimer. These graphs are on a torus and hence periodic as indicated. For such a graph, we can draw a fundamental domain as
Similarly, matchings they line(s) introduce R corresponds on the perfect and dimensional for a perfect matching. Perfect matching is defined as a subset of edges such that every vertex in the fundamental domain is an endpoint of only one edge. For instance, in Fig.5, the dotted line(s) gives one perfect matching and dashed line(s) gives another perfect matching and so on. Similarly, there are six perfect matchings for the C^3/Z_3 case as indicated by blue edges in Fig.7. We can draw the edges representing difference between any two perfect matchings and see that they give closed curves enclosing a face of the dimer diagram. The total number of perfect matchings \( p_n \)'s enumerated for a dimer diagram will determine \( C \) automatically without working out the \( T \) matrix. In order to determine the toric diagram associated with a dimer diagram, we introduce height function \( h = (h_W, h_Z) \) to each perfect matching. These height functions plotted on a two dimensional plane gives a toric diagram. In Fig.7, the height function for matchings (1),(5) and (6) are (0,0), and the matchings (2), (3) and (4) will have heights (−1,0), (1,−1) and (0,1) respectively. Plotting the six height functions, we can see that the internal point in the toric diagram is (0,0) and its multiplicity is 3. There is another method of determining R-charge for every perfect matching in closed string theory[4]. Using basis vectors (1/3, 0, 0), (0, 1/3, 0), (0, 0, 1/3)[4], the perfect matchings 1, 5, 6 will have R-charge as (1/3, 1/3, 1/3) which corresponds to a marginal twisted sector. It can seen that the internal point in the toric diagram

**Figure 5.** Dimer tiling for \( C^3 \) and \( C^3/Z_2 \)

**Figure 6.** Dimer tiling for \( C^3/Z_3 \)

**Figure 7.** Perfect matchings for \( C^3/Z_3 \)
Table 1. Quiver gauge theory and dimer diagram correspondence

| Quiver theory | dimer diagram |
|---------------|---------------|
| Gauge groups  | Faces         |
| bi-fundamental fields | edges       |
| superpotential W | vertices     |

corresponds to a marginal twisted sector state. We have seen that every perfect matching can be given a height function and represents a point in the toric diagram. Interestingly, all the faces of the dimer diagram can be obtained from a subset of perfect matchings[5] which motivated us to conjecture the following:

All the face symmetries of the dimer diagram can be written in terms of perfect matchings corresponding to an internal point in the toric diagram whenever present.

We have verified the conjecture for many non-orbifold toric Calabi-Yau spaces(daughter spaces).

For the dimer diagram and perfect matchings as shown in Fig.6 and Fig.7, we see that the three faces $F_i$'s can be obtained using perfect matchings $p_5, p_6$ and $p_1$ corresponding to the internal point as follows:

$F_1 = p_5 - p_6$, $F_2 = p_1 - p_5$ and $F_3 = p_6 - p_1$.

For a face/closed curve, we require the $R$-charge to be zero which is another proof for the conjecture. For a general dimer diagram, it may not be practical to draw all the perfect matchings. There is an algebraic method of enumerating the perfect matchings by looking at the terms in the determinant of a matrix called Kastelyne matrix $K(Z,W)$. This algebraic approach is really useful for large toric diagrams. Let $a_{ij}^k$ represent an edge-weight for edge $k$ between white vertex $i$ and black vertex $j$. Then $K_{ij}(Z,W) = \sum_k a_{ij}^k Z^{\langle k, \gamma_Z \rangle} W^{\langle k, \gamma_W \rangle}$ where $\langle k, \gamma_Z \rangle$ gives the intersection number of edge $k$ with the oriented $\gamma_Z$ cycle of the torus which will be $\pm 1, 0$. Taking the row index as white-vertex numbers and column index as black-vertex index, the matrix $K$ will be:

$$K(Z,W) = \begin{pmatrix}
  a_{11} & a_{12}W & a_{13}Z \\
  a_{21} & a_{22} & a_{23} \\
  \frac{a_3}{Z} & \frac{a_2}{W} & \frac{a_1}{W}
\end{pmatrix}$$

The determinant of $K(Z,W)$ will give six terms corresponding to the six perfect matchings in Fig.7:

$$\det K = -a_{13}a_{22}a_{31} + a_{12}a_{23}a_{31} \frac{W}{Z} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32}Z + a_{11}a_{22}a_{33} \frac{1}{W}.$$
which is 1 if the edge $e_i$ is in the perfect matching $p_\alpha$. $Q_F$ is given by

$$Q_F \mathcal{M}^t = 0.$$ 

In terms of matching matrix, the matter fields $X_i$’s and the charge matrix can be written as

$$X_i = \prod_\alpha p_{M_{i\alpha}}$$ and $d = Q_D \mathcal{M}^t.$ \hfill (3)

Having presented the essential ingredients relating dimer model to the toric and gauge theory data, we will now present the inverse algorithm and indicate when adjoint fields must be introduced \cite{5}.

### 3.1. Our results

Starting from the dimer diagram for the parent space $C^3/(Z_m \times Z_n)$, we can remove one or more edges (this is called Higgsing), we get daughter theories and exotic theories. For these theories, we have verified the conjecture that the face symmetries are obtainable from the perfect matchings corresponding to any internal point in the toric diagram\cite{5}.

In order to see whether the theory requires adjoint fields, we first check whether the matching matrix is consistent: For instance, the number of edges participating in any perfect matching are same (see Fig.7). This means that every column (representing perfect matching) in the matching matrix must have equal number of 1’s.

**Figure 8.** Superpotential terms

Matching matrix for $C^3/(Z_2 \times Z_3)$ is

$$
\begin{pmatrix}
X_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
X_2 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
X_3 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
X_4 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
X_5 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
X_6 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
X_7 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
X_8 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
X_9 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
Z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
Z_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
Z_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
Z_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
Z_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
Z_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
\end{pmatrix}
$$
We will briefly outline our approach of obtaining quiver gauge theory corresponding to non-orbifold Calabi-Yau spaces. Starting from the matching matrix $\mathcal{M}$ of a parent space, we would like to obtain the matching matrix for a daughter theory. This corresponds to deleting the row corresponding to an edge representing a bifundamental field which is given a vev. Also, delete the columns representing those perfect matchings which contains the specific edge to obtain the reduced matching matrix $\mathcal{M}_r$. Then, we determine the kernel $Q_{F(r)}$ for the reduced matching matrix $\mathcal{M}_r$. For $Q_{F(r)}$, we obtain the kernel $T_r$ and hence the dual of $T_r$ giving $K_r$.

If the new matching matrix $K_rT_r$ is consistent, then we can obtain the superpotential from $K_r$ and draw the dimer diagram for the daughter space. In case $K_rT_r$ is not consistent, we add one or more rows to $K_r$ with entries 0 or 1 so that the number of 1’s in every column is equal. The number of additional rows will indicate the number of adjoint fields which will give a consistent $\mathcal{M}$ and $W$.

4. Conclusions
Dimer models gives an elegant method of obtaining quiver gauge theories on daughter spaces (non-orbifold toric CY$_3$). Perfect matchings of the dimer diagram will determine the toric data including the multiplicity of the sigma model fields. Also, the charge matrix, matter fields and the superpotential of the quiver gauge theories can be directly obtained from the dimer diagram. Imposing a consistency requirement on the matching matrix gives a methodical way of incorporating adjoint fields in the quiver gauge theory.

We hope to look at counting of gauge invariant operators from the dimer model approach. There are interesting works on dimer tiling approaches to understand Chern-Simons gauge theories on coincident $M2$ branes. We are presently looking at some of these problems.

Acknowledgments
The author would like to thank IRCC, IIT Bombay for the grant.

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