Simulation of Dirac particles with quantum interference in phase space

Wen Ning,1, * Ri-Hua Zheng,1, * Kai Xu,2, 3 Hekang Li,2 Yan Xia,1 Fan Wu,1, †
Zhen-Biao Yang,1, ‡ Dongning Zheng,2, 3 Heng Fan,2, 3 and Shi-Biao Zheng1, §

1 Fujian Key Laboratory of Quantum Information and Quantum Optics,
College of Physics and Information Engineering,
Fuzhou University, Fuzhou, Fujian 350108, China
2 Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China
3 CAS Center for Excellence in Topological Quantum Computation,
University of Chinese Academy of Sciences, Beijing 100190, China

The Dirac equation predicts the coexistence of the positive and negative eigen-solutions for a relativistic spin-1/2 particle (spinor). Interference between these two parts leads to an unexpected trembling motion—Zitterbewegung, which is, however, difficult to observe. This stimulates its simulations on different platforms, but the unique quantum interference characteristic—that cannot be simulated with classical systems—remains unexplored yet. We here present an on-chip simulation, where a superconducting qubit, together with a microwave field, emulates the spinor. The Zitterbewegung is manifested by oscillations in one quadrature of the field, inferred from the measured quasiprobability distribution in phase space. This distribution, when correlated to the spinor’s internal state, displays negativities originating from quantum interference. We further simulate the Klein tunneling by constructing a linear potential and observing emergent mesoscopic superposition states.

To unify quantum mechanics and relativity, P. A. M. Dirac formulated a famous matter-wave equation for a spin-1/2 particle, known as the Dirac equation [1, 2]

$$ih \frac{\partial \psi}{\partial t} = H_D \psi, \quad (1)$$

where $H_D = c \alpha \cdot p + \beta mc^2$, $c$ denotes the light speed in the vacuum, $\alpha$ and $\beta$ are the four-by-four Dirac matrices, $m$ and $p$ represent the rest mass and momentum of the particle, respectively, and $\psi$ is the wave function with four components. Unlike the Schrödinger equation, the Dirac equation is linear in both the time and space derivatives, satisfying the Lorentz-covariance, and includes the spin degree of freedom at the ab initio level by describing the wave function in terms of a spinor. These features have led to remarkable accomplishments, including predictions of the spin-1/2 feature of electrons and the existence of anti-particles indicated by the negative-energy solution accompanying the positive one, and the accurate description of the spectrum of the hydrogen atom. In addition, the Dirac equation has important implications for understanding exotic phenomena of condensed matter physics [3–6].

Despite the tremendous triumph in the description of relativistic particles, experimental observations of some intriguing phenomena predicted by the Dirac equation remain elusive, i.e., Zitterbewegung (ZB) [7] and Klein’s paradox [8, 9]. Zitterbewegung refers to the oscillatory motion of a free electron described by the Dirac equation, as a result of the interference between the positive and negative energy states, while Klein tunneling is an effect where a relativistic particle can penetrate through a potential barrier without the exponential damping, a nonrelativistic quantum tunneling characteristic. For a free electron, the predicted ZB has an amplitude on the order of the Compton wavelength, $h/mc \sim 10^{-12} \text{ m}$, and a frequency of $2mc^2/h \sim 10^{21} \text{ Hz}$, which are beyond the scope of the presently available experimental techniques.

The difficulties in direct observations of these counterintuitive behaviors for Dirac particles have led to growing interest in the emulation of these phenomena with highly tunable quantum simulators, a concept first proposed by Feynman in 1982 [10] and has been rapidly developed in the last two decades [11]. Simulations of ZB have been reported with different systems, including semiconductor quantum wells [12], ultracold atoms [13, 14], ion traps [15], as well as classical optical systems [16, 17]. Despite these remarkable advancements, the quantum interference characteristic underlying this phenomenon has not been unambiguously characterized yet. When such a nonclassical characteristic is omitted, the ZB phenomenon can be effectively emulated with classical systems, as demonstrated in previous investigations [16–20].

We here present an experimental simulation of the Dirac particle in a circuit quantum electrodynamics (QED) system, featuring pure quantum effects that cannot appear in classical simulations. The spinorial characteristic is encoded in the two lowest energy levels of a superconducting Xmon qubit, while the position and momentum are mapped to the quadratures of the microwave field stored in a resonator coupled to the qubit. With this mapping, ZB is manifested by oscillations of one quadrature of the resonator, which is inferred from the Wigner function (WF) that describes the quasiprobability distribution in position-momentum space, referred to as phase space. The measured WF, when correlated to the qubit state, presents a region with negative values as a result of quantum interference in phase space. The quantum nature of the evolution is further confirmed by
the measured entanglement entropy. In addition to ZB, we simulate the Klein tunneling in a linear potential field and observe mesoscopic superpositions of two separated wavepackets in phase space.

We focus on the simplest case, where the motion of a Dirac Fermion is confined to one dimension (1D). In this case, the Hamiltonian reduces to

$$H_D = c\mathbf{\hat{p}} + mc^2\mathbf{\hat{\sigma}}_z.$$  \hspace{1cm} (2)

We note that here the Pauli operators $\mathbf{\sigma}_y$ and $\mathbf{\sigma}_z$ endow the Dirac particle with a spinor characteristic, manifested by a two-component wave function, where the spatial position and momentum are correlated with the degree of freedom defined in an “internal space”, like isospin. As the Hamiltonian commutes with the momentum operator, it is illustrative to uncover the physics in the momentum representation, where the momentum operator $\hat{p}$ can be taken as a parameter $p$. For a specific value of $p$, $H_D$ has two eigenvalues $\pm E_p = \pm \sqrt{p^2c^2 + m^2c^4}$, with the corresponding eigenstates $|\phi_1(p)\rangle = (\cos\phi_p, i\sin\phi_p)^T$ and $|\phi_2(p)\rangle = (i\sin\phi_p, \cos\phi_p)^T$, where tan$(2\phi_p) = \frac{p}{mc}$.

Suppose that the system is initially in the product state

$$|\psi(0)\rangle = \int dp \, \xi_p |p\rangle |X\rangle,$$

where $|\pm X\rangle = \frac{1}{\sqrt{2}}(1, \pm 1)^T$, $\xi_p$ denotes the wave function in the momentum representation. Under the Dirac Hamiltonian, the system evolves as

$$|\psi(t)\rangle = \int dp |p\rangle \xi_p (\cos \varphi_t |X\rangle - ie^{-2i\varphi_t} \sin \varphi_t |-X\rangle),$$

where $\varphi_t = E pt/\hbar$. This directly yields the average position evolution,

$$\langle x(t)\rangle = \langle x(0)\rangle + \langle\varphi(0)\rangle t + \int dp |\xi_p|^2 \, x_p (1 - \cos \varphi_t),$$

where $x_p = d\phi_p/dp$, $\langle x(0)\rangle$ represents the average value of the initial position, and $\langle\varphi(0)\rangle$ denotes the initial mean velocity. The ZB, manifested by the last term, is observable only in the intermediate regime where $mc$ is comparable with $p$, explained as follows. As shown in Eq. (4), the momentum state can be decomposed into two components, respectively correlated with the symmetric and asymmetric states in the internal space. The ZB originates from interference between these two components. When the spread of the momentum $\delta p$ is much smaller than the mean momentum $\langle p\rangle$, the ZB amplitude and frequency can be roughly estimated as $A = \hbar mc/(2P^2)$ and $f = 2cP/\hbar$, where $P = (m^2c^2 + \langle p\rangle^2)^{1/2}$. This implies that the ZB frequency monotonously increases with the mass, but the amplitude is not a monotonic function of $m$. In the non-relativistic regime $|p| \ll mc$, $A \simeq \lambda_c/2$, where $\lambda_c = \hbar/(mc)$ is the Compton wavelength, which sets the lower bound for the uncertainty of the position. In the far-relativistic regime $|p| \gg mc$, $A \ll \hbar/(2 \langle p\rangle) \ll \delta x$, where $\delta x = \hbar/(2\Delta p)$ is the limitation of precision attainable for any position measurement imposed by the Heisenberg uncertainty relation. Consequently, it is impossible to observe ZB in these two regimes.

The simulation is performed with a superconducting qubit (test qubit) that encodes the internal state of the simulated spinor, whose position and momentum are mapped onto the two quadratures of the microwave field stored in a bus resonator, defined as $\tilde{x} = \frac{1}{\sqrt{2}}(a^+ + a)$ and $\tilde{\hat{p}} = \frac{i}{\sqrt{2}}(a^+ - a)$, where $a^+$ and $a$ denote the creation and annihilation operators for the photonic field, respectively. If we take $\hbar = 1$, $x$ and $\hat{p}$ satisfy the same commutation relation as the position and momentum operators. To simulate the 1+1 Dirac equation, the qubit is subjected to two longitudinal parametric modulations and a transverse continuous microwave driving [Fig. 1(a)]. The system Hamiltonian is given by [21–23]

$$H = \omega_\tau a^+ a + [\omega_0 + \varepsilon_1 \cos(\nu_1 t) + \varepsilon_2 \cos(\nu_2 t)]\sigma_z$$

$$+ (\lambda a^+ |g\rangle \langle e| + \Omega e^{i\theta} e^{i\omega_0 t} |g\rangle \langle e| + \text{H.c.}),$$

where $|g\rangle$ and $|e\rangle$ represent the ground and first excited states.
states for the qubit with a mean frequency $\omega_0/(2\pi)$, which is coupled to the resonator of frequency $\omega_r/(2\pi)$ with the coupling strength $\lambda$, $\epsilon_j$ and $\nu_j/(2\pi)$ ($j = 1, 2$) are the corresponding modulation amplitude and frequency, and $\Omega$ and $\theta$ denote the amplitude and phase of the transverse drive, respectively. Under the conditions $\omega_r = \omega_0 + 2\nu_1$ and $\nu_1 \gg \nu_2, \lambda$, the resonator is coupled to the qubit at the second upper sideband of the first modulation with the effective strength $\eta = \lambda J_2(\mu)/2$, where $J_n(\mu)$ is the $n$th Bessel function of the first kind, with $\mu = \epsilon_1/\nu_1$. When $\Omega \gg 2\eta$, the transverse drive effectively transforms the rotating-wave interaction into an equal combination of rotating- and counter-rotating-wave interactions [23], simulating the coupling between the internal and external freedom degrees of the spinor. For $\nu_2 = 2\Omega J_0(\mu)$, the second parametric modulation is on resonance with the rotation produced by the transverse drive, producing an analog of the mass. By discarding the far off-resonant terms, the system Hamiltonian in the interaction picture reduces to (detailed deviation shown in the Supplementary Information)

$$H_R = \eta e^{-i\vartheta} a^\dagger \sigma_\theta + H.c. + \omega \sigma_z, \quad (7)$$

where $\omega = \epsilon_2/4$ and $\sigma_\theta = \cos \theta \sigma_x + \sin \theta \sigma_y$. When $\theta = \pi/2$, this effective Hamiltonian has the same form as Eq. (2), with the correspondence $c^* = \sqrt{2}\eta$ and $m^* e^{i x^2} = \omega$, where $c^*$ and $m^*$ denote the effective light speed and mass of the Dirac particle in the simulation, respectively.

Before the experiment, both the resonator and the spinor qubit are initialized to their ground states. The experiment starts with the application of a pulse to the resonator, translating its state along the $p$-axis in phase space by an amount of $p_0 = 2$, and thus transforming the initial vacuum state to the coherent state $|\sqrt{2}\rangle$. Then a $\pi/2$ rotation is performed on the test qubit, transforming it from $|g\rangle$ to $|X\rangle$ at the operating frequency $\omega_0/(2\pi) = 5.26$ GHz. The initial qubit-resonator state is pictorially shown in Fig. 1(c). After this initial state preparation, two parametric modulations with frequencies $\nu_1/(2\pi) = 160$ MHz and $\nu_2/(2\pi) = 33.4$ MHz are applied to the qubit. These modulations, together with the transverse microwave driving at the frequency $\omega_0/(2\pi)$, couple the qubit to the bus resonator with a fixed frequency of $\omega_r/(2\pi) = 5.58$ GHz, effectively realizing the Dirac Hamiltonian in the rotating frame, with $\eta = 2\pi \times 0.78$ MHz. The ratio between the effective momentum and mass of the Dirac particle is variable by adjusting the amplitude $\epsilon_1$ and/or $\epsilon_2$. Detailed system parameters are shown in the Supplementary Information.

We simulate the ZB behaviors with the choice $\omega = \sqrt{2}\eta p_0$. After a preset evolution time, both the parametric modulations and microwave driving are switched off. This is followed by Wigner tomography, realized by performing a phase-space displacement, $D(\gamma) = e^{\gamma a^\dagger - \gamma a}$, to the resonator and then tuning an ancilla qubit on resonance with the resonator. The photon number population of the displaced resonator field, $P_n(\gamma)$, inferred from the measured Rabi oscillation signals, directly yields the WF [23, 24]

$$W(x, p) = \frac{1}{\pi} \sum_{n=0}^\infty \left(-1\right)^n P_n(\gamma), \quad (8)$$

where $x = \sqrt{2}\Re \gamma$ and $p = \sqrt{2}\Im \gamma$. The mean position is related to the WF by

$$\langle x \rangle = \int \int \ dx dp \ x W(x, p). \quad (9)$$

To infer $\langle x \rangle$ for a preset evolution time $t$, it is necessary to measure the WF in all regions of the phase space with non-negligible quasi-probability distributions. Due to the limitation of the available tomography pulse, the $W(x, p)$ is measured within the region $-3 \leq x \leq 3$ and $0 \leq p \leq 4$. The evolution of the mean position, measured within 330 ns, is presented in Fig. 2(a).

This result well agrees with the simulation (green line), confirming that the simulated particle moves in a ZB pattern within the observed time interval. With the growth

![FIG. 2. Simulation of Zitterbewegung. (a) Evolution of the average value of the resonator’s quadrature $\langle x \rangle$. The value at each point is extracted from the measured WF, (b), (c) Measured WFs correlated with the basis states $|g\rangle$ and $|e\rangle$ of the test qubit; (d) measured WFs irrespective of the test qubit’s state, all measured after an evolution time of 330 ns. Probability distributions $P(x)$ with respect to the quadrature $x$, shown in the lower panels (e-g) are obtained by integrating the WF, $W(x, p)$, over $p$. (b) Entropy evolution of the test qubit. This entropy is directly obtained from the density matrix of the qubit, measured irrespective of the resonator’s state.](image-url)
of the photon number, accurate characterization of $\langle x \rangle$ requires Wigner tomography within a larger phase-space region, which is beyond the capability of our system (see Supplementary Information). With an improvement of the hardware, we could simulate the ZB behavior within a longer time scale.

The nonclassical features of the simulated Dirac particle can be revealed by the WFs of the resonator conditional on the detection of the qubit state. The WFs correlated with the measurement outcomes $|g\rangle$ and $|e\rangle$ are presented in Fig. 2(b) and Fig. 2(c), respectively, and the result irrespective of the test qubit’s state is displayed in Fig. 2(d), all measured after an evolution time of 330 ns. As expected, during the evolution the initial Gaussian wavepacket is split into two parts, propagating towards opposite directions. The WF associated with each qubit state displays a region of negative and positive directions of the measure results imply that the two WFs are independent of the qubit states, all measured after an evolving time $t = 288$ ns. (d), (e) Evolutions of the measured $\langle x \rangle$ and $\langle p \rangle$. The squares and diamonds denote the measured values for wavepackets along the positive and negative directions of the $x$-axis, respectively.

Another important feature associated with the simulated particle is the production of quantum entanglement between its internal and external degrees of freedom. To quantitatively characterize the behavior, we present the von Neumann entropy of the test qubit, $S = \text{tr}(\rho_q \log_2 \rho_q)$, measured for different evolution time $t$ in Fig. 2(h), where $\rho_q$ denotes the reduced density operator of this qubit. As expected, the measured entanglement entropy quickly increases and tends to the maximum of 1 with the elapse of evolution time, which agrees with the numerical simulation (blue line), where the small fluctuations are due to the fast Rabi oscillations.

Pushing one step further, we simulate Klein tunneling in a linear potential field [8, 9]. It was first noted by Klein that a relativistic electron may exhibit a counterintuitive behavior when confronted with a semi-infinite step potential with $V = 0$ and $V_0$ for $x < 0$ and $x \geq 0$, respectively. This occurs in the regime $V_0 > E + mc^2$, for which the electron can propagate through the barrier without damping, where it is transformed into a positron, where $E$ denotes the initial kinetic energy. In our setup, it is not easy to engineer the step-shaped potential. However, a linear potential can be added to the Dirac Hamiltonian in situ by applying a continuous microwave to the resonator, given by $V = \sqrt{2}\epsilon x$, where $\epsilon$ is the amplitude of the drive, and set to be $2\pi \times 0.39$ MHz in our experiment. For simplicity, the simulation is performed for a massless Dirac particle, i.e., the Weyl Fermion [25], which corresponds to the choice $\epsilon = 0$. Figures 3(a) and 3(b) showcase the WFs of the resonator correlated to the test qubit of state $|g\rangle$ and $|e\rangle$; (c) measured WF regardless of the test qubit’s state. These WFs are reconstructed for the system evolving from the initial ground state $|g\rangle \otimes |0\rangle$ for a time of 288 ns. (d), (e) Evolutions of the measured $\langle x \rangle$ and $\langle p \rangle$. The squares and diamonds denote the measured values for wavepackets along the positive and negative directions of the $x$-axis, respectively.

To illustrate this phenomenon more clearly, we display the time evolutions of the measured $\langle x \rangle$ and $\langle p \rangle$ in Fig. 3(d) and Fig. 2(e), respectively, where the squares and diamonds denote the results for wavepackets along the positive and negative directions of the $x$-axis, respectively. The measured results imply that the two wavepackets have the same momentum at each moment, but move in opposite directions. This can be explained as follows. For $x > 0$, the momentum of the Dirac particle is given by $p^2 = (E - V)^2 - m^2$ ($c = 1$), with the group velocity $v_g = dE/dp = p/(E - V)$ [9]. When the massless particle moves from left to right with $E < V$, $p$ is assigned with its negative solution, so that $v_g$ is positive.
In conclusion, we have theoretically proposed and experimentally demonstrated a scheme for simulating the 1+1 Dirac equation with a circuit QED device, where a superconducting qubit represents the internal degree of freedom, which is coupled to the microwave field in a resonator that simulates the spatial freedom degree. Our simulation of ZB reveals that the coupling between the internal and external degrees of freedom splits the original Gaussian wavepacket into two parts, which move towards opposite directions in phase space and are continuously deformed. We observe quantum interference effects manifested by negativities via correlating the phase-space distribution to the internal state. We further simulate the Klein tunneling of a massless Fermion in a linear potential and observe a cat-like state involving two separated motional wavepackets entangled with the internal degree of freedom. These results shed new light on relativistic quantum-mechanical effects.

This work was supported by the National Natural Science Foundation of China (Grant No. 12274080, No. 11875108, No. 12204105, No. 11934018, No. 12274080, No. 12204105, No. 11934018, No. 11943493, No. 92065114, and No. T2121001), Innovation Program for Quantum Science and Technology (Grant No. ZDK0102040201), the Strategic Priority Research Program of Chinese Academy of Sciences (Grant No. XDB28000000), the Key-Area Research and Development Program of Guangdong Province, China (Grant No. 2020B0303030001), Beijing Natural Science Foundation (Grant No. Z200009), the Natural Science Funds for Distinguished Young Scholar of Fujian Province under Grant 2020J06011, and Project from Fuzhou University for Distinguished Young Scholar of Fujian Province under Grant No. JG202001-2 and No. 049050011050.

* These authors contribute equally to this work.
† t21060@fzu.edu.cn
‡ zbyang@fzu.edu.cn
‡ t96034@fzu.edu.cn

[1] P. A. M. Dirac, The quantum theory of the electron, Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character 117, 610 (1928).

[2] B. Thaller, The Dirac Equation (Springer Berlin Heidelberg, Berlin, Heidelberg, 1992).

[3] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, The electronic properties of graphene, Rev. Mod. Phys. 81, 109 (2009).

[4] K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, M. I. Katsnelson, I. V. Grigorieva, S. V. Dubonos, and A. A. Firsov, Two-dimensional gas of massless Dirac fermions in graphene, Nature 438, 197 (2005).

[5] C.-H. Park, L. Yang, Y.-W. Son, M. L. Cohen, and S. G. Louie, Anisotropic behaviours of massless Dirac fermions in graphene under periodic potentials, Nat. Phys. 4, 213 (2008).

[6] L. Tarruell, D. Greif, T. Uehlinger, G. Jotzu, and T. Esslinger, Creating, moving and merging Dirac points with a Fermi gas in a tunable honeycomb lattice, Nature 483, 302 (2012).

[7] E. Schrödinger, Über die kräftefreie bewegung in der relativistischen quantenmechanik, Sitz. Preuss. Akad. Wiss. Phys.-Math. Kl. 24, 418 (1930).

[8] O. Klein, Die Reflexion von Elektronen an einem Potentialsprung nach der relativistischen Dynamik von Dirac, Zeit. Phys. 53, 157 (1929).

[9] N. Donbey, Seventy years of the Klein paradox, Phys. Rep. 315, 41 (1999).

[10] R. P. Feynman, Simulating physics with computers, Int. J. Theor. Phys. 21, 467 (1982).

[11] I. M. Georgescu, S. Ashhab, and F. Nori, Quantum simulation, Rev. Mod. Phys. 86, 153 (2014).

[12] J. Schliemann, D. Loss, and R. M. Westervelt, Zitterbewegung of electronic wave packets in III-V zinc-blende semiconductor quantum wells, Phys. Rev. Lett. 94, 206801 (2005).

[13] L. J. LeBlanc, M. C. Beeler, K. Jiménez-García, A. R. Perry, S. Sugawa, R. A. Williams, and I. B. Spielman, Direct observation of Zitterbewegung in a Bose–Einstein condensate, New J. Phys. 15, 073011 (2013).

[14] C. Qu, C. Hammer, M. Gong, C. Zhang, and P. Engels, Observation of Zitterbewegung in a spin-orbit-coupled Bose-Einstein condensate, Phys. Rev. A 88, 021604(R) (2013).

[15] R. Gerritsma, G. Kirchmair, F. Zähringer, E. Solano, R. Blatt, and C. F. Roos, Quantum simulation of the Dirac equation, Nature 463, 68 (2010).

[16] F. Dreisow, M. Heinrich, R. Keil, A. Tönnemann, S. Nolte, S. Longhi, and A. Szameit, Classical simulation of relativistic Zitterbewegung in photonic lattices, Phys. Rev. Lett. 105, 143902 (2010).

[17] T. L. Silva, E. R. F. Taillebois, R. M. Gomes, S. P. Waldorn, and A. T. Avelar, Optical simulation of the free Dirac equation, Phys. Rev. A 99, 022332 (2019).

[18] X. Zhang, Observing Zitterbewegung for photons near the Dirac point of a two-dimensional photonic crystal, Phys. Rev. Lett. 100, 113903 (2008).

[19] J. Otterbach, R. G. Unanyan, and M. Fleischhauer, Confining stationary light: Dirac dynamics and Klein tunneling, Phys. Rev. Lett. 102, 063602 (2009).

[20] S. Longhi, Photonic analog of Zitterbewegung in binary waveguide arrays, Optics Letters 35, 235 (2010).

[21] D.-W. Wang, C. Song, W. Feng, H. Cai, D. Xu, H. Deng, H. Li, D. Zheng, X. Zhu, H. Wang, S.-Y. Zhu, and M. O. Scully, Synthesis of antisymmetric spin exchange interaction and chiral spin clusters in superconducting circuits, Nat. Phys. 15, 382 (2019).

[22] Z. Wang, Z. Xiang, T. Liu, X. Song, P. Song, X. Guo, L. Su, H. Zhang, Y. Du, and D. Zheng, Observation of the exceptional point in superconducting qubit with dissipation controlled by parametric modulation, Chin. Phys. B 30, 100309 (2021).

[23] R.-H. Zheng, W. Ning, Y.-H. Chen, J.-H. Lii, L.-T. Shen, K. Xu, Y.-R. Zhang, D. Xu, H. Li, Y. Xia, F. Wu, Z.-B. Yang, A. Miranowicz, N. Lambert, D. Zheng, H. Fan, F. Nori, and S.-B. Zheng, Emergent Schrödinger cat states during superradiant phase transitions, arXiv:2207.05512 (2022).

[24] M. Hofheinz, H. Wang, M. Ansmann, R. C. Bialczak, E. Lucero, M. Neeley, A. D. O’Connell, D. Sank, J. Wenner, J. M. Martinis, and A. N. Cleland, Synthesizing ar-
bitary quantum states in a superconducting resonator, *Nature* **459**, 546 (2009).

[25] H. Weyl, Electron and gravitation. I, *Zeit. Phys.* **56**, 330 (1929).

[26] J. S. Pedernales, R. Di Candia, D. Ballester, and E. Solano, Quantum simulations of relativistic quantum physics in circuit QED, *New J. Phys.* **15**, 055008 (2013).

[27] A. Bermudez, M. A. Martin-Delgado, and E. Solano, Mesoscopic superposition states in relativistic Landau levels, *Phys. Rev. Lett.* **99**, 123602 (2007).

[28] R. Gerritsma, B. P. Lanyon, G. Kirchmair, F. Zähringer, C. Hempel, J. Casanova, J. J. García-Ripoll, E. Solano, R. Blatt, and C. F. Roos, Quantum simulation of the Klein paradox with trapped ions, *Phys. Rev. Lett.* **106**, 060503 (2011).

[29] T. Salger, C. Grossert, S. Kling, and M. Weitz, Klein tunneling of a quasirelativistic Bose-Einstein condensate in an optical lattice, *Phys. Rev. Lett.* **107**, 240401 (2011).

[30] Y. Jiang, M.-L. Cai, Y.-K. Wu, Q.-X. Mei, W.-D. Zhao, X.-Y. Chang, L. Yao, L. He, Z.-C. Zhou, and L.-M. Duan, Quantum simulation of the two-dimensional Weyl equation in a magnetic field, *Phys. Rev. Lett.* **128**, 200502 (2022).

[31] B. Q. Lv, H. M. Weng, B. B. Fu, X. P. Wang, H. Miao, J. Ma, P. Richard, X. C. Huang, L. X. Zhao, G. F. Chen, Z. Fang, X. Dai, T. Qian, and H. Ding, Experimental discovery of Weyl semimetal TaAs, *Phys. Rev. X* **5**, 031013 (2015).

[32] L. Lu, Z. Wang, D. Ye, L. Ran, L. Fu, J. D. Joannopoulos, and M. Soljačić, Experimental observation of Weyl points, *Science* **349**, 622 (2015).

[33] S.-Y. Xu, I. Belopolski, N. Alidoust, M. Neupane, G. Bian, C. Zhang, R. Sankar, G. Chang, Z. Yuan, C.-C. Lee, S.-M. Huang, H. Zheng, J. Ma, D. S. Sanchez, B. Wang, A. Bansil, F. Chou, P. P. Shibayev, H. Lin, S. Jia, and M. Z. Hasan, Discovery of a Weyl fermion semimetal and topological Fermi arcs, *Science* **349**, 613 (2015).
Supplementary Information for
“Simulation of Dirac particles with quantum interference in phase space”

Wen Ning,1,∗ Ri-Hua Zheng,1,∗ Kai Xu,2,3 Hekang Li,2 Yan Xia,1 Fan Wu,1,† Zhen-Biao Yang,1,‡ Dongning Zheng,2,3 Heng Fan,2,3 and Shi-Biao Zheng1,§

1Fujian Key Laboratory of Quantum Information and Quantum Optics,
  College of Physics and Information Engineering,
  Fuzhou University, Fuzhou, Fujian 350108, China
2Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China
3CAS Center for Excellence in Topological Quantum Computation,
  University of Chinese Academy of Sciences, Beijing 100190, China

CONTENTS

S1. Experimental Device and System Parameters 1
S2. Derivation of the Effective Dirac Hamiltonian 3
  A. Effective Coupling Between the Qubit and One Quadrature of the Resonator 3
  B. Stark Shifts due to Far Off-Resonant Interactions 4
S3. Experimental Synthesis of the Dirac Hamiltonian 5
  A. Optimized Qubit Driving 5
  B. Bus Resonator Driving 5
  C. Full Pulse Sequence 6
S4. Characterization of the Qubit-Resonator State 6
  A. Photon-Number Distribution 6
  B. Wigner Matrix Elements 7
S5. Correction of Phase-Space Rotations due to Stark Shifts 8
S6. Distinction of Positive and Negative Wavepackets in Klein Tunneling 8
References 9

S1. EXPERIMENTAL DEVICE AND SYSTEM PARAMETERS

The whole electronics and wiring of the used superconducting 5-qubit sample [S1, S2] are shown in Fig. S1. Each frequency-tunable Xmon qubit has an individual flux line for its dynamic tuning and a microwave drive for controllably flipping its states. All the qubits are capacitively coupled to a bus resonator with coupling strength \( \lambda_j \approx 2\pi \times 20 \text{ MHz} \) (\( j = 1, 2 \)) and every qubit has a readout resonator for reading out its states. The bus resonator \( R \) is a superconducting coplanar waveguide resonator with fixed frequency \( \omega_r/(2\pi) = 5.5835 \text{ GHz} \), which is measured when all used qubits stay in ground states at their respective idle frequencies (See below, unused qubits are always at their sweet points \( \sim 6 \text{ GHz} \)). More detailed parameters of the sample are listed in Tab. S1.

In this experiment, we use two qubits (\( Q_1, Q_2 \)) and the bus resonator (\( R \)). One qubit (\( Q_1 \)) is used to realize the effective Dirac model, and the other qubit (\( Q_2 \)) is used to measure the photon number distribution of the bus resonator by resonant population exchange (see Sec. S4A for details). For initialization, we wait for 300 \( \mu \text{s} \) to make sure all qubits return to their ground states, and then bias them to their idle frequencies, where single qubit gates and qubit states measurements are performed.

∗ These authors contribute equally to this work.
† t21060@fzu.edu.cn
‡ zbyang@fzu.edu.cn
§ t96034@fzu.edu.cn

arXiv:2211.12779v1 [quant-ph] 23 Nov 2022
We use two independent XY signal channels controlled by Digital-to-Analog converters (DACs), to generate two microwave sequences. These two low-frequency microwave sequences are mixed with a continuous microwave by the In-phase and Quadrature (IQ) mixer to generate two tone pulses. This continuous microwave is a carrier microwave emitted by the microwave source (MS) and has a frequency of 5.21 GHz. One-tone pulse is used to drive the test qubit to prepare the initial states, the other is used to trigger the resonator by crosstalk instead of the resonator line to achieve the displacement operation on its states in phase space, and to provide continuous driving pulses for Klein tunneling. For the Z signal, we use two independent channels on the DACs to control, which allows us to flexibly adjust the qubit frequency. In addition to the DACs described above, we need XY signal from the DACs to output the readout pulse with multiple tones implemented by sideband mixing. Then, the output pulse passes through the circulator, impedance-transformed Josephson parametric amplifier (JPA), cryogenic amplifier, and room temperature amplifier to improve signal-to-noise ratio. Finally, it is captured by Analog-to-Digital converters (ADCs) after passing through the readout resonator, and thus the multiple tones are demodulated to return the IQ information of each tone. The readout of the qubit states is performed at its idle frequency with the duration time of 1 µs. Both DACs and ADCs are supplied with the carrier microwave from the same MS. The frequency of the readout resonator $\omega_{ro}/(2\pi)$ ranges from 6.65 to 6.86 GHz, right in the bandwidth of our JPA with a pump frequency about 13.39 GHz.

\[ F_j = \begin{pmatrix} F_{g,j} & 1 - F_{e,j} \\ 1 - F_{g,j} & F_{e,j} \end{pmatrix}, \]  
\quad \text{(S1)}

where $F_{g,j}$ and $F_{e,j}$ are the readout fidelities of qubits $Q_j$ (see Tab. S1), to reconstruct the readout results. Defining $F = F_1 \otimes F_2$ as a two-qubit calibration matrix. We rewrite the measurement results of two qubits into a column vector $P_m$, and the calibrated measurement result is then $P = F^{-1} \cdot P_m$.  

FIG. S1. **Schematic diagram of the experimental setup.** Note there is a low-pass filter inverted with the others because it connects to the readout line. The number inside the attenuator icon indicates the magnitude of the attenuator in dBm.

During the procedure of qubit’s initial state preparation and measurement, some errors are caused by decoherence. To calibrate these errors, we use a calibration matrix, defined as
TABLE S1. Qubit and resonator characteristics. The test qubit, auxiliary qubit, and bus resonator are denoted by Q1, Q2, and R, respectively. The idle frequencies of Qj (j = 1, 2) are marked by \( \omega_{10}/(2\pi) \), where the pulses for initial state preparation and tomographic pulses for qubit measurement are applied. The energy relaxation time, the Ramsey Gaussian dephasing time, and the spin echo Gaussian dephasing time are measured at the idle frequency of each qubit. The coupling strength \( \lambda_{j} \) between the qubits Qj and bus resonator R is measured by their population exchange rate at resonance. The symbol \( \alpha_{j} \) is the anharmonicity of the qubits, \( \omega_{ro} \) is the bare frequency of qubit’s readout resonator, and \( F_{g} \) (\( F_{c} \)) is the probability of reading out Qj in \(|g\rangle \) (\(|c\rangle \)) when it is prepared in \(|g\rangle \) (\(|c\rangle \)).

### S2. DERIVATION OF THE EFFECTIVE DIRAC HAMILTONIAN

#### A. Effective Coupling Between the Qubit and One Quadrature of the Resonator

To realize the spinor-momentum coupling, the excitation energy of the test qubit is periodically modulated as (\( \hbar = 1 \) hereafter)

\[
\omega_{q}(t) = \omega_{0} + \varepsilon_{1}\cos(\nu_{1}t) + \varepsilon_{2}\cos(\nu_{2}t),
\]

where \( \omega_{0} \) is the mean excitation energy, and \( \varepsilon_{j} \) and \( \nu_{j} \) (\( j = 1, 2 \)) are the corresponding modulation amplitude and angular frequency, respectively. In addition to interaction with the resonator, the qubit is driven by a continuous microwave with frequency \( \omega_{0} \). The system Hamiltonian is given by

\[
H = H_{0} + H_{I},
\]

where

\[
H_{0} = \omega_{r} a^\dagger a + \left[ \omega_{0} + \varepsilon_{1}\cos(\nu_{1}t) \right] |e\rangle \langle e|,
\]

\[
H_{I} = \varepsilon_{2}\cos(\nu_{2}t) |e\rangle \langle e| + \left( \lambda a^\dagger |g\rangle \langle e| + \Omega e^{i\theta} e^{i\omega_{r}t} |g\rangle \langle e| + \text{H.c.} \right),
\]

\( \lambda \) is the qubit-resonator coupling strength, and \( \Omega (\theta) \) denote the amplitude (phase) of the classical driving field. We here assume \( \omega_{r} = \omega_{0} + 2\nu_{1} \). With this setting, the resonator interacts with the qubit at the second sideband associated with the first modulation, while the microwave drive works at the carrier. Performing the transformation

\[
U_{0} = \exp \left( i \int_{0}^{t} H_{0} \, dt \right),
\]

we obtain the system Hamiltonian in the interaction picture

\[
H'_{I} = \varepsilon_{2}\cos(\nu_{2}t) |e\rangle \langle e| + \exp \left[ -i\mu\sin(\nu_{1}t) \right] \left( \lambda \exp(2i\nu_{1}t) a^\dagger + \Omega e^{i\theta} \right) |g\rangle \langle e| + \text{H.c.},
\]

where \( \mu = \varepsilon_{1}/\nu_{1} \). To clarify the underlying physics clearly, we use the Jacobi-Anger expansion

\[
\exp \left[ i\mu\sin(\nu_{1}t) \right] = \sum_{m=-\infty}^{\infty} J_{m}(\mu) \exp \left( im\nu_{1}t \right),
\]

with \( J_{m}(\mu) \) being the \( m \)th Bessel function of the first kind, we obtain

\[
H'_{I} = \varepsilon_{2}\cos(\nu_{2}t) |e\rangle \langle e| + \sum_{m=-\infty}^{\infty} J_{m}(\mu) \left\{ \lambda \exp(-i(m-2\nu_{1}t))a^\dagger + \Omega e^{i\theta} \exp(-im\nu_{1}t) \right\} |g\rangle \langle e| + \text{H.c.}
\]

The qubit is resonantly coupled to the resonator by the upper second-order sideband modulation with \( m = 2 \), and driven by the external microwave at the carrier. For \( \lambda, \Omega J_{2}(\mu) \ll \nu_{1} \), the qubit interacts with the resonator at the upper second-order sideband modulation with \( m = 2 \), and is driven at the carrier with \( m = 0 \). With the fast oscillating terms being discarded, \( H'_{I} \) reduces to

\[
H'_{I} = K\sigma_{z} + \frac{1}{2} \varepsilon_{2}\cos(\nu_{2}t)\sigma_{z} + \eta e^{-i\theta} a^\dagger (\sigma_{\theta} - i\sigma_{\theta+\pi/2}) + \text{H.c.},
\]
where $K = \Omega J_0(\mu)$, $\eta = \lambda J_2(\mu)/2$, $\sigma_\theta = e^{i\theta} |g\rangle \langle e| + e^{-i\theta} |e\rangle \langle g|$, and $\sigma_z = |e\rangle \langle e| - |g\rangle \langle g|$. With the assumption $\nu_2 = 2K \gg \eta, \varepsilon_2/2$ and under the further transformation $\exp(iK\sigma_\theta t)$, the system Hamiltonian can be well approximated by

$$H''_I = \eta e^{-i\theta} a^\dagger \sigma_\theta + \text{H.c.} + \omega \sigma_z,$$

where $\omega = \varepsilon_2/4$. When $\theta = \pi/2$, this effective Hamiltonian has the same form as the 1+1 Dirac equation of a spin-1/2 particle, with the correspondence $c^* = \sqrt{2}\eta$ and $m^*c^2 = \omega$. Here in $c^*$ and $m^*$ denote the effective light speed and mass of the Dirac particle in the simulation, respectively.

### B. Stark Shifts due to Far Off-Resonant Interactions

The Xmon qubit used in this experiment is not an ideal two-level system. The system Hamiltonian, with all the high-energy levels being considered, can be written as

$$H = H_0 + H_I,$$

where

$$H_0 = \omega, a^\dagger a + [\omega_0 + \varepsilon_1 \cos(\nu_1 t)] q^1 q - \frac{\alpha^2}{2} q^2 q^2,$$

$$H_I = \varepsilon_2 \cos(\nu_2 t) q^1 q^2 + (\lambda a^\dagger q + \Omega e^{i\theta} e^{i\omega_0 t} q + \text{H.c.}),$$

where $q = |g\rangle \langle e| + \sqrt{2}|e\rangle \langle f| + \ldots$ denotes the annihilation operator for the Xmon mode and $\alpha$ is the anharmonicity. For simplicity, here we mainly consider the effects of microwave driving frequency $\omega_0$, the first frequency modulation amplitude $\varepsilon_1$, and the lowest three levels ($\{|g\rangle, |e\rangle, |f\rangle\}$) of the Xmon. In the rotating frame, the Hamiltonian of the system is

$$H'_I = \exp[-i\mu \sin(\nu_1 t)] \left( \lambda \exp(2i\nu_1 t) a^\dagger + \Omega e^{i\theta} \right) \left[ |g\rangle \langle e| + \sqrt{2} \exp(i\omega t) |f\rangle \langle e| \right] + \text{H.c.}.$$  

Using the Jacobi-Anger expansion, and assuming $|\lambda J_0(\mu)| \ll \nu_1$, we reduce $H'_I$ to the effective Hamiltonian $H_e$

$$H_e = [J_2(\mu) \lambda a^\dagger + \Omega e^{i\theta}] |g\rangle \langle e| + \text{H.c.} + S_{11} (|g\rangle \langle g| - |e\rangle \langle e|) a^\dagger a - S_{11} |e\rangle \langle e| + S_{21} |e\rangle \langle e| a^\dagger a$$

$$+ S_{12} (|g\rangle \langle g| - |e\rangle \langle e|) (ae^{i\theta} + a^\dagger e^{-i\theta}) + S_{22} |e\rangle \langle e| (ae^{i\theta} + a^\dagger e^{-i\theta}),$$

with

$$S_{11} \simeq \frac{|J_0(\mu)|^2}{2\nu_1} + \frac{|J_1(\mu)|^2}{\nu_1} + \frac{|J_{-1}(\mu)|^2}{3\nu_1},$$

$$S_{21} \simeq \frac{2|J_0(\mu)|^2}{2\nu_1 + \alpha} + \frac{2|J_1(\mu)|^2}{\nu_1 + \alpha} + \frac{2|J_{-1}(\mu)|^2}{3\nu_1 + \alpha},$$

$$S_{12} \simeq \frac{J_0(\mu) J_{-2}(\mu) \Omega \lambda}{2\nu_1} + \frac{J_1(\mu) J_{-1}(\mu) \Omega \lambda}{\nu_1} + \frac{J_{-1}(\mu) J_{-3}(\mu) \Omega \lambda}{3\nu_1},$$

$$S_{22} \simeq \frac{2J_0(\mu) J_{-2}(\mu) \Omega \lambda}{2\nu_1 + \alpha} + \frac{2J_1(\mu) J_{-1}(\mu) \Omega \lambda}{\nu_1 + \alpha} + \frac{2J_{-1}(\mu) J_{-3}(\mu) \Omega \lambda}{3\nu_1 + \alpha}.$$  

Discarding the constant term, we can rewrite $H_e = H_{e,1} + H_{e,2}$, where

$$H_{e,1} = [J_2(\mu) \lambda a^\dagger + \Omega e^{i\theta}] |g\rangle \langle e| + \text{H.c.},$$

$$H_{e,2} = \left( S_{11} - \frac{1}{2} S_{21} \right) (|g\rangle \langle g| - |e\rangle \langle e|) a^\dagger a - \frac{1}{2} S_{11} (|g\rangle \langle g| - |e\rangle \langle e|) + \frac{1}{2} S_{21} a^\dagger a$$

$$+ \left( S_{12} - \frac{1}{2} S_{22} \right) (|g\rangle \langle g| - |e\rangle \langle e|) (ae^{i\theta} + a^\dagger e^{-i\theta}) + \frac{1}{2} S_{21} (ae^{i\theta} + a^\dagger e^{-i\theta}).$$

When the microwave drive is applied, $H_{e,2}$ becomes

$$H'_{e,2} = \left( S_{11} - \frac{1}{2} S_{21} \right)^2 \sigma_x (a^\dagger a)^2 + \frac{1}{2} S_{11}^2 \sigma_x + \frac{1}{2} S_{21} a^\dagger a$$

$$+ \left( S_{21} - \frac{1}{2} S_{22} \right)^2 \sigma_x (ae^{i\theta} + a^\dagger e^{-i\theta})^2 + \frac{1}{2} S_{21}^2 \sigma_x + \frac{1}{2} S_{22} (ae^{i\theta} + a^\dagger e^{-i\theta}).$$
S3. EXPERIMENTAL SYNTHESIS OF THE DIRAC HAMILTONIAN

A. Optimized Qubit Driving

To achieve a periodic modulation of the qubit frequency, we first measure the resonant frequency $\omega_{10}$ versus the Z-line pulse amplitude (ZPA) for test qubit, the results are shown in Fig. S2 [S3]. Here we set the detuning between the idle frequency of the $Q_1$ and the resonator frequency being $\Delta/(2\pi) \approx 320$ MHz, and the frequency of the first modulation pulse $\nu_1/(2\pi) = 160$ MHz. In such a case, the working frequency of the qubit is the idle frequency point, avoiding a large rising edge at the beginning of the pulse. Considering the frequency of the second frequency modulation pulse $\nu_2/(2\pi) = 33.4$ MHz, this parameter setting will effectively reduce the high-energy level excitation caused by the modulation pulse. Subsequently, we modulate the qubit in the frequency with amplitude $\varepsilon_1 = 2\pi \times 130$ MHz, centering around its idle frequency according to the $\omega_q$ versus ZPA curves obtained previously.

Considering that the nonlinear curve of frequency versus ZPA and the limited DACs bandwidth lead to imperfect waveforms of the periodically modulated excitation energy, we modify the frequency-modulated pulse as

$$\omega_q(t) = \omega_0 + \delta + \varepsilon_1 \cos(\nu_1 t + \phi_1) + \varepsilon_2 \cos(\nu_2 t + \phi_2),$$

(S23)

making the experiment dynamics closer to the Dirac dynamics. We traverse these two phases $(\phi_1, \phi_2)$ between 0 and $2\pi$, set the initial state of the system to be $|\psi\rangle = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle) \otimes |0\rangle$ in both the ZB and Klein tunneling simulations, and finally obtain the one which is closest to the ideal situation. Here we use the 2-norm as an index to measure the agreement between the experimental population data of qubit state $|e\rangle$ and the theoretical values. Then, we optimize the $\delta$ and other parameters in the same way. In this case, the optimized population is shown in Fig. S3, and the experimental data are in good agreement with the ideal ones, which proves the correctness of our method.

For the transverse field driving the qubit, we want the microwave to arrive at the qubit with an initial phase $\pi/2$. Therefore, we use a similar approach to optimize this phase. In addition, in the application of modulated pulses, the oscillation center frequency of the qubit may deviate slightly from the working frequency $\omega_0$, so we mildly tune the microwave frequency (about $1 \sim 2$ MHz) of the transverse field to make the results more predictable.

![Fig. S2. Frequency of the test qubit versus Z line bias.](image_url)

**FIG. S2.** Frequency of the test qubit versus Z line bias. The desired modulation of $\omega_q(t)$ can be realized by mapping it to the modulation of Z bias.

B. Bus Resonator Driving

The resonator is driven by using a flattop envelope with a resonance frequency, described as $\Omega_r(a + a^\dagger)$, where $\Omega_r$ is the Rabi frequency of the pulse. During the preparation of the initial state in the ZB simulation experiment, the presence of a dispersion coupling (form of $\sigma_y a^\dagger a$) between the test qubit and the bus resonator is $\lambda_f^2/\Delta \approx 2\pi \times 1.25$ MHz. So we reduce the pulse time and increase the Rabi frequency of the pulse to avoid the initial state rotation...
FIG. S3. **Validity check of the Rabi model.** For the present Rabi model, the effective frequency of the qubit and the effective coupling strength are \( \omega/(2\pi) = 2.2 \text{ MHz} \) and \( \eta = 2\pi \times 0.78 \text{ MHz} \), respectively. We fix the initial state as \( |\psi_0\rangle = \frac{1}{\sqrt{2}}(|g\rangle + |e\rangle) \otimes |0\rangle \).

After optimizing parameters \( \varepsilon_1, \varepsilon_2, \phi_1, \phi_2, \) and \( \delta \) appropriately, in the experiment, we measure the population of the test qubit state \( |e\rangle \), marked as \( P_e^t \), at different times, and compare it with the numerical results.

caused by the dispersion interaction. Lengths of pulse for initial state preparation and tomography pulse are 24 ns and 60 ns, respectively.

For pulses of a specific length, we apply them with different amplitudes to the resonator and measure the photon number to fit the slope between the Rabi frequency (\( \Omega_r \)) and the pulse amplitude. Up to now, together with the initial phase of the microwave pulse, we can implement arbitrary displacement operators \( D(\gamma) \) for Wigner measurements.

The Klein tunneling experiment requires a continuous pulse on the resonator during the dynamics. There is a fixed resonator frequency shift induced by the non-resonant coupling (see Sec. S2B), thus we adjust the microwave frequency to about \( \frac{1}{2} S_{21}/(2\pi) \approx 0.75 \text{ MHz} \) to get better results.

The qubit-state-dependent resonator frequency shift in the dynamical process is still unavoidable, therefore, additional correction of the results is necessary so as to make the results more intuitive (see Sec. S5).

### C. Full Pulse Sequence

The pulse sequence is shown in Fig. S4, including three steps: 1. Initial state preparation; 2. Dirac dynamics; 3. Quantum state measurement. For clarity of reading, the real-time scales are not used in the figure.

### S4. CHARACTERIZATION OF THE QUBIT-RESONATOR STATE

#### A. Photon-Number Distribution

All the Wigner function (WF) values in the main text are deduced from the photon number distribution. We use the same method as in Ref. [S4]. After the Dirac dynamics and displacement operation, we bias the auxiliary qubit \( Q_2 \) to the frequency of the bus resonator for a given time \( \tau \). Then, we bias it to the idle frequency and measure. The excited state population \( P_e^a(\tau) \) of \( Q_2 \) is defined as

\[
P_{e}^{a}(\tau) = \frac{1}{2} \left[ 1 - P_{g}^{a}(0) \sum_{n=0}^{n_{\text{max}}} P_n e^{-\kappa_n \tau} \cos \left( 2\sqrt{n}\lambda_2 \tau \right) \right],
\]

where \( P_n \) denotes the photon-number distribution probability, \( n_{\text{max}} \) is the cutoff of the photon number, and \( \kappa_n = n^l/T_{1,p} \) \( (l = 1) \) [S5–S9] is the empirical decay rate of the \( n \)-photon state, \( \lambda_2 \) is the coupling strength between auxiliary qubit \( Q_2 \) and resonator. It is worth noting that due to the finite detuning \( \Delta = 2\pi \times 470 \text{ MHz} \) between the auxiliary qubit \( Q_2 \) and bus resonator \( R \), as the number of photons increases, ancilla qubit \( Q_2 \) inevitably interacts with the resonator during the Dirac dynamics, the ground state population \( P_{g}^{a}(0) \) may not be 1 at \( \tau = 0 \). Referring to [S5],
FIG. S4. Sketch of the Pulse Sequences. The test qubit $Q_1$ is driven by a transverse microwave field with the Rabi frequency $\Omega/(2\pi) \approx 20.03$ MHz through the XY-line (red), and two Sine longitudinal modulations apply to the Z-line (purple) with the modulation frequencies $\nu_{1(2)}/(2\pi) = 160$ (33.4) MHz and the amplitudes $\varepsilon_{1(2)} = 2\pi \times 130$ (8.8) MHz in the ZB simulation. While in the Klein tunneling simulation, we set $\varepsilon_2 = 0$ to simulate massless particles. The orange dashed box represents the pulse sequence of $R_{xy}$ used in the ZB simulation, which is replaced by the series of yellow dashed boxes in the Klein tunneling simulation, and outside the dashed box there are the same pulse sequences for these two simulations. The evolution times $t$ of the two experiments are 330 ns and 288 ns, respectively. The auxiliary qubit $Q_2$ is used to measure the resonator photon number. After the displacement operation pulse, we bias $Q_2$ to the resonator frequency for a given time $\tau$ and then deduce the photon number distribution of the resonator according to the result of the Rabi oscillation. Finally, a multiplexing tone is applied to the readin line to simultaneously measure two qubits’ states.

The potentially slight excitation can be ignored and thus $P_{a}^{n}(0)$ is introduced to make the fit more accurate. In the experiment, $P_{a}^{n}(0) \leq 0.05$, $\tau$ is taken every 2 ns from 0 to 200 ns. We use the least square method to fit the experimental data according to Eq. (S24) to obtain the actual photon number distribution and further obtain the value of the WF. However, with the further increase of the photon number, the entanglement between the auxiliary qubit $Q_2$ and the resonator can not be ignored, and the photon number and its distribution in the resonator can not be obtained accurately. It is also worth noting that although ancilla qubits can be biased further down to increase the detuning in the Dirac dynamics, the performance of the qubits deteriorates and the frequency change more widely, affecting the measurement results. Ergo, the photon number capability of the present system is about 20, resulting in the boundary of phase-space in the main text.

B. Wigner Matrix Elements

The WF is given by

$$W(x, p) = \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n P_n(\gamma),$$

(S25)

$$P_n(\gamma) = \langle n| D(-\gamma)|\rho D(\gamma)|n \rangle,$$

(S26)

where $x = \sqrt{2}\text{Re}(\gamma)$, $p = \sqrt{2}\text{Im}(\gamma)$, $D(\gamma) = \exp(\gamma a^\dagger - \gamma^* a)$. The photon number distribution $P_n$ is inferred by the Rabi oscillation signal \[S9\]. And then we calculate the Wigner matrix according to Eq. (S25). We adjust the Wigner tomography pulse based on $\gamma$ to realize the measurement of WF value at different positions in phase-space. The WF conditional on test qubit states $|e\rangle$ and $|g\rangle$ are given by

$$W_{e(g)}(x, p) = \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n P_{n}^{e(g)}(\gamma),$$

(S27)

$$P_{n}^{e(g)}(\gamma) = \frac{1}{P_k} \langle n| \otimes \langle e(g)| D(-\gamma)|\rho D(\gamma)|e(g) \rangle \otimes |n \rangle,$$

(S28)

with $P_k^t$ is the population of the test qubit $Q_1$ in $|k\rangle$ state at time $t$. 

$\omega_{\alpha}/(2\pi) = 160$ (33.4) MHz for the ZB simulation, $\varepsilon_{1(2)} = 2\pi \times 130$ (8.8) MHz for the Klein tunneling simulation.
In this way, we can get all the information about the resonator state. Then, we use the CVX toolbox based on MATLAB [S10] to reconstruct the corresponding density matrix $\rho_k$ to calculate the average position $\langle x \rangle$ and average momentum $\langle p \rangle$ of the resonator.

As mentioned in the main text, a larger number of photons requires Wigner tomography within a larger phase-space region. Due to the hardware-limited pulse amplitude, we can increase the size of $|\gamma|$ by increasing the length of the tomography pulse, at the cost of increasing the dispersive interaction time. Considering the finite detuning between the auxiliary qubit and the resonator, our ZB simulation ends at 330 ns. With the improvement of the hardware, such as larger pulse amplitude, or better qubits performance, we could simulate the ZB behavior within a longer time scale.

S5. CORRECTION OF PHASE-SPACE ROTATIONS DUE TO STARK SHIFTS

After reconstructing the density matrix $\rho_k$ from the Wigner tomography $\mathcal{W}_k$, we add a rotation operator on the corresponding density matrix numerically. The rotation operator is defined as

$$U_k = \exp \left[ -i(\theta_{k,0} + \theta_k t/t_f) a^\dagger a \right], \quad (S29)$$

$$\rho'_k = U_k \rho_k U_k^\dagger, \quad (k = e, g) \quad (S30)$$

where $\theta_{k,0}$ is used to counteract the resonator states rotation due to the initial state preparation when the test qubit in $|k\rangle$ state, $\theta_k$ cancels out the rotation caused by the difference between the experimental frame in Eq. (S22) and the effective Hamiltonian frame in Eq. (S10), whose form is $\frac{1}{2}S_{21}a^\dagger a$ [see Eq. (S22)], $t_f$ is the end time of each simulation, and $\rho'_k$ is the resonator density matrix after rotation when the test qubit in $|k\rangle$ state. In the Klein tunneling simulation experiment, because the initial state of the resonator is a vacuum state, we set $\theta_{k,0} = 0$. Here we choose $(\theta_{e,0}, \theta_{g,0}, \theta_e, \theta_g, t_f) = (0.260, 0.045, 1.510, 1.217, 330 \text{ ns})$ in the ZB simulation and $(\theta_{e,0}, \theta_{g,0}, \theta_e, \theta_g, t_f) = (0, 0, 1.402, 1.162, 280 \text{ ns})$ in the Klein tunneling simulation. Note $\theta_e$ and $\theta_g$ are slightly different because the dispersive interaction during Wigner tomography (time scale 60 ns, causing phase difference $2\lambda_f^2/\Delta \times 60 \text{ ns} \sim 0.3$). While both $\theta_e$ and $\theta_g$ are around $\frac{1}{2}S_{21}t_f \approx 1.3$, offsetting the resonator phase induced by the frame difference, $\frac{1}{2}S_{21}a^\dagger a$, and therefore demonstrating that the rotation of our data is reasonable. The unconditional resonator WF is obtained by adding two rotated WFs according to the corresponding qubits state population:

$$\mathcal{W}'(t) = P^e_k \mathcal{W}'_e(t) + P^g_k \mathcal{W}'_g(t), \quad (S31)$$

where $\mathcal{W}'_k(t)$ is calculated from $\rho'_k$. The complete Wigner data are shown in Fig. S5 for ZB simulation and Fig. S6 for Klein tunneling simulation. For the selection of the time point, we try to make the population of states $|e\rangle$ and $|g\rangle$ of the test qubit close to 0.5, making the normalized data of the auxiliary qubit accurate and convenient to fit the photon number distribution of the resonator.

S6. DISTINCTION OF POSITIVE AND NEGATIVE WAVEPACKETS IN KLEIN TUNNELING

In the Klein tunneling simulation, we need to show the measured of $\langle x \rangle$ and $\langle p \rangle$ for the positive and negative wavepackets. Here, we simply use $x = 0$ as a boundary to distinguish positive and negative wavepackets. The measured position average $\langle x \rangle$ and momentum average $\langle p \rangle$ for positive wavepackets can be calculated by

$$\langle x \rangle = \frac{\int_{-\infty}^{+\infty} \int_{0}^{+\infty} x \mathcal{W}(x,p) \, dx \, dp}{\int_{-\infty}^{+\infty} \int_{0}^{+\infty} \mathcal{W}(x,p) \, dx \, dp},$$

$$\langle p \rangle = \frac{\int_{-\infty}^{+\infty} \int_{0}^{+\infty} p \mathcal{W}(x,p) \, dx \, dp}{\int_{-\infty}^{+\infty} \int_{0}^{+\infty} \mathcal{W}(x,p) \, dx \, dp},$$

where $\mathcal{W}(x,p)$ is shown in Fig. S6(c). At $t = 0$ and 76 ns, the positive and negative wavepackets cannot be fully distinguished, so the data for these two mean positions $\langle x \rangle$ are abandoned in Fig. 3(d) of the main text. The negative wavepacket can be obtained in the same way,

$$\langle x \rangle = \frac{\int_{-\infty}^{+\infty} \int_{0}^{-\infty} x \mathcal{W}(x,p) \, dx \, dp}{\int_{-\infty}^{+\infty} \int_{0}^{-\infty} \mathcal{W}(x,p) \, dx \, dp},$$

$$\langle p \rangle = \frac{\int_{-\infty}^{+\infty} \int_{0}^{-\infty} p \mathcal{W}(x,p) \, dx \, dp}{\int_{-\infty}^{+\infty} \int_{0}^{-\infty} \mathcal{W}(x,p) \, dx \, dp}. $$

(S32)

(S33)

(S34)

(S35)
FIG. S5. **Wigner functions for Zitterbewegung.** The WFs correlated with the basis states $|g\rangle$ and $|e\rangle$ of the test qubit displayed in (a) and (b), respectively, and (c) show the result irrespective of the test qubit’s state. These five columns show the WFs at different times: 0, 90, 178, 240, and 330 ns.

FIG. S6. **Wigner function for Klein tunneling.** The WFs correlated with the basis states $|g\rangle$ and $|e\rangle$ of the test qubit displayed in (a) and (b), respectively, and (c) show the result irrespective of the test qubit’s state. These five columns show the WFs at different times: 0, 76, 140, 216, and 288 ns.

[S1] C. Song et al., “Continuous-variable geometric phase and its manipulation for quantum computation in a superconducting circuit,” *Nat. Commun.* 8, 1061 (2017).
[S2] W. Ning et al., “Deterministic entanglement swapping in a superconducting circuit,” *Phys. Rev. Lett.* 123, 060502 (2019).
[S3] W. Liu et al., “Synthesizing Three-Body Interaction of Spin Chirality with Superconducting Qubits,” *Appl. Phys. Lett.* 116, 114001 (2020).
[S4] R.-H. Zheng et al., “Emergent Schrödinger Cat States during Superradiant Phase Transitions.” arXiv:2207.05512 (2022).

[S5] M. Hofheinz et al., “Synthesizing arbitrary quantum states in a superconducting resonator,” Nature 459, 546–549 (2009).

[S6] D. M. Meekhof, C. Monroe, B. E. King, W. M. Itano, D. J. Wineland, “Generation of nonclassical motional states of a trapped atom,” Phys. Rev. Lett. 76, 1796–1799 (1996).

[S7] D. Leibfried, R. Blatt, C. Monroe, D. Wineland, “Quantum dynamics of single trapped ions,” Rev. Mod. Phys. 75, 281–324 (2003).

[S8] D. Lv et al., “Quantum simulation of the quantum Rabi model in a trapped ion,” Phys. Rev. X 8, 021027 (2018).

[S9] M.-L. Cai et al., “Observation of a quantum phase transition in the quantum Rabi model with a single trapped ion,” Nat. Commun. 12, 1126 (2021).

[S10] M. Grant, S. Boyd, CVX: Matlab software for disciplined convex programming, version 2.0 beta. http://cvxr.com/cvx, September (2013).