GLOBAL ATTRACTOR FOR DAMPED FORCED NONLINEAR LOGARITHMIC SCHRODINGER EQUATIONS

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ABSTRACT. We consider here a damped forced nonlinear logarithmic Schrödinger equation in $\mathbb{R}^N$. We prove the existence of a global attractor in a suitable energy space. We complete this article with some open issues for nonlinear logarithmic Schrödinger equations in the framework of infinite-dimensional dynamical systems.

1. Introduction. We are interested here in the focusing Logarithmic Nonlinear Schrödinger equations (Log-NLS) that read in the conservative case

$$u_t + i\Delta u + iu \log |u|^2 = 0.$$  

(1)

These equations model nonlinear wave mechanics and is also a model in nonlinear optics [15]. The unknown $u$ maps $\mathbb{R}_t \times \mathbb{R}^N_x$ into $\mathbb{C}$. The initial value problem for initial data in $L^2(\mathbb{R}^N)$ or in a suitable energy space was studied in [21], [20]. These Schrödinger equations feature particular standing waves solutions, the so-called “gaussons”. An example of Gausson reads

$$u(t,x) = \exp(iNt - |x|^2/2).$$  

(2)

For properties of these special solutions see [19], [4], [5], [6], [7], [22], [24]. Besides, the dynamics of the defocusing equation, when $+$ is replaced by $-$ above the nonlinearity, was completely described in the recent work [17].

The terminology focusing-defocusing for Log-NLS may be questioned. We may observe that usually focusing NLS equations read

$$u_t + i\Delta u + ig(|u|^2)u = 0,$$  

(3)

with a non negative $g$; here for Log-NLS the function $g(\xi) = \log \xi$ changes sign at $\xi = 1$. Nevertheless, we follow here the literature where (1) is usually quoted as focusing Log-NLS equation.

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Here we are interested in a modified equation with damping and forcing term. These equations read
\[ u_t + \alpha u + i\Delta u + iu \log |u|^2 = f(x), \] (4)
where \( \alpha > 0 \) is the damping parameter and where \( f \) is the external force, that is independent of time and that is square integrable.

We study the equation in the framework of infinite-dimensional dynamical system as described in [37], [34], [32], [35]. Our main result is concerned with the existence of global attractors. In the literature, there are many results concerned with the existence of global attractors for nonlinear Schrödinger equations with pure power nonlinearities that are subcritical in the energy space; these equations read in the focusing case for \( g(\xi) = \xi^p \) as
\[ u_t + \alpha u + i\Delta u + ig(|u|^2)u = f(x). \] (5)
For instance see [27], [28], [29] for cubic NLS in dimension \( N = 1 \); cubic means that \( g(\xi) = \xi \) in (5) above. The critical exponent of the NLS equations for pure power nonlinearities depends on the dimension \( N \), due to Sobolev embeddings and Gagliardo-Nirenberg inequalities.

Here the logarithmic nonlinearity is subcritical for any \( N \), since the growth of the logarithmic function is bounded by above by any pure power nonlinearity. Besides, a major difficulty comes from the fact that the nonlinearity \( u \log |u| \) is not smooth at \( u = 0 \); then some classical methods that work for smooth nonlinearities do not work straightforwardly.

This article outlines as follows. In a second section we follow [21], [20] to first set the functional analysis framework to handle the initial value problem for the equation (4). To guess what is the correct mathematical framework we proceed as follows: let us point out that for the conservative case \( \alpha = 0 \) and \( f = 0 \) the equation (1) has two conserved quantities [19], [20] the mass
\[ M = \int_{\mathbb{R}^N} |u|^2 dx, \]
and the energy
\[ E = ||\nabla u||_{L^2(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} |u|^2 \log |u|^2 dx. \] (6)
This leads to the introduction of the energy space
\[ W = \{ u \in H^1(\mathbb{R}^N); v = |u|^2 \log |u|^2 = 2|u|^2 \log |u| \in L^1(\mathbb{R}^N) \}. \]
Appealing the results in [21], [20] and the references therein, we shall prove that the equation (4) supplemented with initial data in \( L^2(\mathbb{R}^N) \) (respectively in \( W \)) define continuous semigroups respectively in \( L^2(\mathbb{R}^N) \) denoted by \( S_0 \) (respectively \( S_1 \) in \( W \)).

In a third section we prove our main result concerning the existence of a global attractor for (4)

**Theorem 1.1.** The semigroup \( S_1 \) possesses a compact global attractor \( \mathcal{A} \) in \( W \).

In a fourth section we introduce and discuss some open problems related to the dynamical system associated with this Log-NLS equation, namely the issue of the regularity of the global attractor and the question of the fractal and Hausdorff dimension of this global attractor. We also address the issue of the existence of global attractors for other related equation: for the defocusing case, for various boundary conditions and for a regularized version of the Log-NLS equation. We
end the article by addressing the issue of the existence of global attractors for the classical time discretizations of the Log-NLS equation. We complete this introduction by introducing some notations. The letter \(c\) denotes a numerical constant that does not depend on the data \(\alpha, \|f\|_{L^2(\mathbb{R}^N)}\) and that may change from one line to one another. Besides, \(K\) will be a constant that depends on \(\alpha, \|f\|_{L^2(\mathbb{R}^N)}\) and that may change from one line to one another. We deal with real Hilbert space. Hence the scalar product of two functions in \(L^2(\mathbb{R}^N)\) reads

\[
(u, v) = \text{Re} \int_{\mathbb{R}^N} uv \, dx.
\]

2. Functional analysis.

2.1. The energy space. For reader convenience, we recall here some results concerning the space \(W\). Following Cazenave [19, 20] we first define a nonnegative increasing convex function \(A\) that belongs to \(C^1((0, \infty)) \cap C^2(0, \infty)\) as

\[
A(s) = -s^2 \log s^2 \text{ if } s \leq e^{-3},
\]

and

\[
A(s) = 3s^2 + 4e^{-3}s - e^{-6} \text{ if } s > e^{-3}.
\]

We now define the Orlicz space (see [3])

\[
L_A(\mathbb{R}^N) = \{ u \in L^1_{\text{loc}}(\mathbb{R}^N); \int_{\mathbb{R}^N} A(|u|) \, dx < \infty \},
\]

equipped with the Luxemburg norm that is defined as

\[
||u||_{L_A(\mathbb{R}^N)} = \inf \{ \lambda > 0; \int_{\mathbb{R}^N} A\left(\frac{|u|}{\lambda}\right) \, dx \leq 1 \}.
\]

Then \(L_A(\mathbb{R}^N)\) is a Banach space. We have \(W = H^1(\mathbb{R}^N) \cap L_A(\mathbb{R}^N)\) equipped with the norm

\[
||u||_W = \||\nabla u||_{L^2(\mathbb{R}^N)} + ||u||_{L_A(\mathbb{R}^N)}.
\]

Hence \(W\) is a uniformly convex space, thus a reflexive separable Banach space. The dual space of \(W\) is \(W^* = H^{-1}(\mathbb{R}^N) + L_A(\mathbb{R}^N)\) where \(A\) can be computed from \(A\) (see [20] and the references therein).

Remark 1. Let us observe that for small \(s\) the function \(A(|s|)\) behaves like the nonlinear logarithmic term in the energy functional, while on the other hand for large \(s\) the function \(A(|s|)\) is quadratic. For large \(|u|\) the nonlinearity \(|u|^2 \log |u|^2\) can be bounded using the Sobolev embedding \(H^1(\mathbb{R}^N) \subset L^{\frac{2N}{N-2}}(\mathbb{R}^N)\), for \(N \geq 3\), and in any \(L^p(\mathbb{R}^N)\), \(p < +\infty\) for \(N = 1, 2\).

We then set \(B(s) = A(s) + s^2 \log s^2\). A set of useful properties lists as follows (see [19], [4]).

Lemma 2.1. If a sequence \(u_j\) is bounded in \(W\), converges strongly in \(L^2(\mathbb{R}^N)\) and almost everywhere then \(B(u_j)\) converges strongly to \(B(u)\) in \(L^1(\mathbb{R}^N)\).

Lemma 2.2. The following statements hold true

- If \(u_n \to u\) in \(L_A(\mathbb{R}^N)\) then \(A(|u_n|) \to A(|u|)\) in \(L^1(\mathbb{R}^N)\).
- If \(u_n \to u\) a.e. and if \(\int_{\mathbb{R}^N} A(|u_n|) \, dx \to \int_{\mathbb{R}^N} A(|u|) \, dx\) then \(u_n \to u\) in \(L_A(\mathbb{R}^N)\).
- There exists \(c > 0\) such that

\[
c \min(||u||_{L_A(\mathbb{R}^N)}, ||u||^2_{L_A(\mathbb{R}^N)}) \leq \int_{\mathbb{R}^N} A(|u|) \leq \frac{1}{c} \max(||u||_{L_A(\mathbb{R}^N)}, ||u||^2_{L_A(\mathbb{R}^N)}).
\]
We now state and prove a useful lemma

**Lemma 2.3.** Consider a sequence $u_j$ that converges towards $u$ weakly in $W$, strongly in $L^2(\mathbb{R}^N)$ and almost everywhere. Then if

$$\limsup \left( \|\nabla u_j\|^2_{L^2(\mathbb{R}^N)} - \int_{\mathbb{R}^N} |u_j|^2 \log |u_j|^2 \, dx \right) \leq \|\nabla u\|^2_{L^2(\mathbb{R}^N)} - \int_{\mathbb{R}^N} |u|^2 \log |u|^2 \, dx,$$

then $u_j$ converges strongly towards $u$ in $W$.

**Proof.** Applying Lemma 2.1 we know that if $u_j$ converges strongly in $L^2(\mathbb{R}^N)$ then, recalling $-s^2 \log s^2 = A(s) - B(s)$, we then have $B(u_j) \to B(u)$ in $L^1(\mathbb{R}^N)$. We then infer from this

$$\limsup \left( \|\nabla u_j\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} A(|u_j|) \, dx \right) \leq \|\nabla u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} A(|u|) \, dx. \quad (10)$$

Therefore, by weak convergence and (10)

$$\limsup \|\nabla u_j - \nabla u\|^2_{L^2(\mathbb{R}^N)} \leq \int_{\mathbb{R}^N} A(|u|) \, dx - \liminf \int_{\mathbb{R}^N} A(|u_j|) \, dx. \quad (11)$$

Thanks to Fatou’s Lemma

$$\int_{\mathbb{R}^N} A(|u|) \, dx - \liminf \int_{\mathbb{R}^N} A(|u_j|) \, dx \leq 0,$$

and $u_j$ converges strongly towards $u$ in $H^1(\mathbb{R}^N)$. We moreover have

$$\limsup \int_{\mathbb{R}^N} A(|u_j|) \, dx \leq \int_{\mathbb{R}^N} A(|u|) \, dx \leq \liminf \int_{\mathbb{R}^N} A(|u_j|) \, dx, \quad (12)$$

and the strong convergence in $L^2_A(\mathbb{R}^N)$ follows from Lemma 2.2. \hfill \qed

**2.2. The initial value problem.** We follow here the ideas in [21]. To begin with we have (see Lemma 9.3.5. in [20])

$$|\Im \int_{\mathbb{R}^N} v - u(v \log |v|^2 - u \log |u|^2) \, dx| \leq 2 \|v - u\|^2_{L^2(\mathbb{R}^N)}. \quad (13)$$

As a consequence of this inequality we have

**Lemma 2.4.** For $\lambda > 2$, the nonlinear operator $M_\lambda : u \mapsto \lambda u + i\Delta u + iu \log |u|^2$ is maximal monotone in $L^2(\mathbb{R}^N)$. Moreover $M_\lambda$ is a bounded operator from $W$ to $W^*$, and the domain of the nonlinear operator $M_\lambda$ is

$$D(M_\lambda) = \{ u \in H^2(\mathbb{R}^N); \int_{\mathbb{R}^N} |u|^2 \log |u|^2 \, dx < +\infty \}.$$

We set $D(M_\lambda) = D(M)$ since it does not depend on $\lambda$. Let us recall that the nonlinear operator is monotone means that for any pair $u, v$ in $D(M)$ then

$$(M_\lambda u - M_\lambda v, u - v) \geq 0, \quad (14)$$

and maximal means that $M_\lambda : D(M) \to L^2(\mathbb{R})$ is onto.

To define the notion of solution to (4), we use the theory of “bonnes solutions” (as known as mild solutions [13]; see also [16] for alternate arguments). Assume that $t$ belongs to $[0, T]$. We split this interval into $N$ intervals of length $\tau$. Using that the operator is monotone, we now solve recursively the implicit scheme

$$\frac{u^{n+1} - u^n}{\tau} + M_{2+\alpha}(u^{n+1}) = 2u^n + f. \quad (15)$$
Since $u \mapsto M_{2+\alpha}(u)$ is maximal monotone then we have a discrete semigroup
\[ \Sigma_{\tau} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N), \]
\[ u^n \mapsto u^{n+1}. \] \hspace{1cm} (16)

Besides, equation (15) leads to
\[ (u^{n+1} - u^n) - (1 + 2\tau)(u^n - u^{n-1}) + \tau(M_{2+\alpha}(u^{n+1}) - M_{2+\alpha}(u^n)) = 0. \] \hspace{1cm} (17)

Considering the scalar product of (17) with $u^{n+1} - u^n$ leads to, thanks to monotonicity property (14) and to Cauchy-Schwarz inequality
\[ ||u^{n+1} - u^n||_{L^2(\mathbb{R}^N)} \leq (1 + 2\tau)||u^n - u^{n-1}||_{L^2(\mathbb{R}^N)}. \] \hspace{1cm} (18)

This leads to, for initial data $u_0$ in $D(M)$, $n \leq N - 1$,
\[ ||u^{n+1} - u^n||_{L^2(\mathbb{R}^N)} \leq (1 + 2\tau)^n||u^1 - u^0||_{L^2(\mathbb{R}^N)} \leq e^{2\tau} \tau(||M_{2+\alpha}(u^1)||_{L^2(\mathbb{R}^N)} + 2||u^0||_{L^2(\mathbb{R}^N)} + ||f||_{L^2(\mathbb{R}^N)}) \] \hspace{1cm} (19)

Consider now
\[ u_\tau : [0, T] \to L^2(\mathbb{R}^N), \]
\[ t \mapsto u_\tau(t), \]

the function that is piecewise linear, continuous in $t$ and such that $u_\tau(n\tau) = u^n$. From (19) we infer that this family of functions are equicontinuous in $C(0, T; L^2(\mathbb{R}^N))$.

Going back to (15) and thanks to (19) we have that $M_0(u^{n+1})$ remains in a bounded set of $L^2(\mathbb{R}^N)$. Let us check that this yields that the family $u_\tau$ remains bounded in $L^\infty(0, T; D(M))$. This is a consequence of the following statement

**Lemma 2.5.** The set
\[ \{ v \in L^2(\mathbb{R}^N); ||v||_{L^2(\mathbb{R}^N)} + ||M_0(v)||_{L^2(\mathbb{R}^N)} \leq K \}, \]

is a bounded set of $D(M)$.

**Proof.** Actually, if for a function $v$ in $L^2(\mathbb{R}^N)$
\[ ||\Delta v + v \log |v|^2||_{L^2(\mathbb{R}^N)} \leq K, \] \hspace{1cm} (20)

then
\[ ||\Delta v||^2_{L^2(\mathbb{R}^N)} + |||v \log |v|^2||^2_{L^2(\mathbb{R}^N)} \leq K^2 + 2\text{Re} \int_{\mathbb{R}^N} \nabla v, \nabla (v \log |v|^2) dx. \] \hspace{1cm} (21)

This leads to, setting $a+ = \frac{|a|+a}{2}$, and dropping some non positive terms
\[ ||\Delta v||^2_{L^2(\mathbb{R}^N)} + |||v \log |v|^2||^2_{L^2(\mathbb{R}^N)} \leq K^2 + 2 \int_{\mathbb{R}^N} |\nabla v|^2 (1 + \log_+ |v|^2) dx. \] \hspace{1cm} (22)

On the one hand
\[ ||\nabla v||^2_{L^2(\mathbb{R}^N)} \leq ||v||_{L^2(\mathbb{R}^N)} ||\Delta v||_{L^2(\mathbb{R}^N)} \leq K ||\Delta v||_{L^2(\mathbb{R}^N)}. \]

On the other hand, by Holder inequality, for any small $\varepsilon > 0$ there exists $p_\varepsilon$ such that
\[ \int_{\mathbb{R}^N} |\nabla v|^2 \log_+ |v|^2 dx \leq c_\varepsilon ||\nabla v||^2_{L^{2+\varepsilon}(\mathbb{R}^N)} ||v||^p_{L^2(\mathbb{R}^N)}. \] \hspace{1cm} (23)
Choosing $\varepsilon$ small enough in order to ensure $H^2(\mathbb{R}^N) \subset W^{1,2+2\varepsilon}(\mathbb{R}^N)$ we infer from (23) that there exists $\gamma_\varepsilon < 2$ such that
\[
\int_{\mathbb{R}^N} |\nabla v|^2 \log_+ |v|^2 \, dx \leq \tilde{K} ||u||_{H^2(\mathbb{R}^N)}^\gamma_\varepsilon.
\]
The conclusion follows promptly.

We now prove that $u_{\tau}$ remains bounded in a compact set of $L^2(\mathbb{R}^N)$. Consider a smooth function $\theta : \mathbb{R}^+ \to [0,1]$ such that $\theta(\xi) = 1$ if $\xi \leq 1$ and $\theta(\xi) = 0$ if $\xi \geq 2$. Introduce $q_R(u^n) = \int_{\mathbb{R}^N} |u^n|^2 (1 - \theta(|x|/R)) \, dx$. Then considering the scalar product of (15) with $(1 - \theta(|u|/R)) u^{n+1}$ we have, after some mere computations
\[
\frac{1}{2\tau} (q_R(u^{n+1}) + q_R(u^{n+1} - u^n) - q_R(u^n)) + 2\alpha q_R(u^{n+1}) = \frac{2}{R} \sum_{j=1}^{N} \int_{\mathbb{R}^N} \theta' \left( \frac{|x|}{R} \right) u^{n+1} \frac{x_j \partial_j u^{n+1}}{|x|} \, dx + 2\text{Re} \int_{\mathbb{R}^N} \tilde{f}(1 - \theta(|x|/R)) u^{n+1} \, dx.
\]
On the one hand, by Cauchy-Schwarz and Young inequalities
\[
2\text{Re} \int_{\mathbb{R}^N} \tilde{f}(1 - \theta(|x|/R)) u^{n+1} \, dx \leq \frac{q_R(f)}{\alpha} + \alpha q_R(u^{n+1}).
\]
On the other hand, using that $u^{n+1}$ remains trapped in $H^1(\mathbb{R}^N)$ in a ball of radius $K_0$ we have
\[
\frac{2}{R} \sum_{j=1}^{N} \int_{\mathbb{R}^N} \theta' \left( \frac{|x|}{R} \right) u^{n+1} \frac{x_j \partial_j u^{n+1}}{|x|} \, dx \leq \frac{2}{R} ||\theta'||_{\infty} K_0^2.
\]
Gathering these inequalities leads to
\[
1 + 2\tau + \alpha \tau q_R(u^{n+1}) \leq q_R(u^n) + \frac{K_0 \tau}{R} + \frac{\tau q_R(f)}{\alpha}.
\]
The discrete Gronwall Lemma yields that
\[
q_R(u^{n+1}) \leq q_R(u^0) + c \left( \frac{K_0}{R} + \frac{q_R(f)}{\alpha} \right).
\]

Therefore, for any $\varepsilon > 0$, for $R$ large enough the functions $u^{n+1}(1 - \theta(|x|/R))^{1/2}$ remains trapped in a small ball $B(0,\varepsilon)$ in $L^2(\mathbb{R}^N)$. On the other hand, $u^{n+1} \sqrt{\theta(|x|/R)}$ remains trapped in $H^1_0(B(0,2R))$ that is a compact subset of $L^2(\mathbb{R}^N)$. Hence the sequence $u^{n+1}$ is trapped into a totally bounded subset of $L^2(\mathbb{R}^N)$ and then the sequence $u_{\tau}$ is bounded in $C(0,T;\mathcal{C})$ where $\mathcal{C}$ is a compact subset of $L^2(\mathbb{R}^N)$.

We then apply the Ascoli Theorem and for initial data in $D(M)$ the approximated sequence $u_{\tau}$ converges towards $u$ that is a solution to (4).

We now observe that, using once again the monotonicity of $M_{2+\alpha}$, the difference between two solutions $u$ and $v$ of (4) satisfy
\[
||u(t) - v(t)||_{L^2(\mathbb{R}^N)} \leq c^2 ||u_0 - v_0||_{L^2(\mathbb{R}^N)}.
\]
Considering for $u_0$ in $L^2(\mathbb{R}^N)$ a sequence $u_0^k$ in $D(M)$ that converges towards $u_0$ in $L^2(\mathbb{R}^N)$, then the corresponding solution $u^k(t)$ is a Cauchy sequence in $C(0,T;L^2(\mathbb{R}^N))$ that converges towards a mild (or “bonne”) abstract solution of the equation. Then we have defined a semigroup $S_0$, such that $S_0(t)$ is $c^2t$-Lipschitzian.

It is worth to point out that this solution is an abstract mild solution and not a PDE solution in a weak sense.
If the initial data $u_0$ belongs to $W$, then the corresponding solution remains bounded in $W$ (see the computations for the absorbing sets in the next section for instance); then we have an abstract semigroup $S_1$ in $W$. Moreover we can prove that the corresponding mild solution is a weak solution of the PDE since $M$ maps $W$ into $W^*$. Then we have also defined the semigroup $S_1$ acting on $W$.

3. The global attractor. We now consider the semigroup $S_1$ in $W$. We recall the very definition of a global attractor.

**Definition 3.1.** The set $A$ is the global attractor for $S_1(t)$ if

- $A$ is a compact subset of $W$
- $A$ is invariant by the flow i.e. $S_1(t)A = A$, for any $t$
- $A$ attracts all the trajectories for the strong topology of $W$, uniformly on bounded sets in $W$, i.e. for any $B$ bounded in $W$,

$$\lim_{t \to +\infty} d(S_1(t)B, A) = 0,$$

where $d$ is the Hausdorff semi-distance between sets (see [37]).

The strategy of the proof is first to establish the existence of an absorbing set in $W$, and to consider the omega-limit set of this absorbing set (for various topologies). We prove first that this set is attracting for the strong topology in $L^2(\mathbb{R}^N)$ and for the weak topology in $W$. We then prove that this set is actually attracting for the strong topology in $W$, and then that it is a compact subset of $W$.

3.1. Absorbing sets.

**Proposition 1.** There exists $K$ such that for any $u_0$ in $W$ there exists $T_0$ that depends on $||u_0||_W$ such that for $t \geq T_0$ then $||u(t)||_W \leq K$.

**Proof.** We begin with the existence of an absorbing set in $L^2(\mathbb{R}^N)$. Consider the scalar product of (4) with $u$. This leads to

$$\frac{1}{2} \frac{d}{dt} ||u||^2_{L^2(\mathbb{R}^N)} + \alpha ||u||^2_{L^2(\mathbb{R}^N)} = \text{Re} \int_{\mathbb{R}^N} f \bar{u} dx,$$

and then by Cauchy-Schwarz inequality and Gronwall lemma

$$||u(t)||^2_{L^2(\mathbb{R}^N)} \leq ||u_0||^2_{L^2(\mathbb{R}^N)} \exp(-\alpha t) + (1 - \exp(-\alpha t)) \frac{||f||^2_{L^2(\mathbb{R}^N)}}{\alpha^2}. \quad (32)$$

Then for $t \geq T(||u_0||_{L^2(\mathbb{R}^N)})$ we have that

$$||u(t)||^2_{L^2(\mathbb{R}^N)} \leq \frac{2||f||^2_{L^2(\mathbb{R}^N)}}{\alpha^2},$$

that leads to the existence of an absorbing set in $L^2(\mathbb{R}^N)$.

Introduce now the energy functional

$$J(u) = ||\nabla u||^2_{L^2(\mathbb{R}^N)} + ||u||^2_{L^2(\mathbb{R}^N)} - \int_{\mathbb{R}^N} |u|^2 \log(|u|^2) dx + 2 \text{Im} \int_{\mathbb{R}^N} f \bar{u} dx. \quad (33)$$

Consider the scalar product of (4) with $i(u_0 + \alpha u)$. This leads to

$$\dot{J} + 2\alpha J = 2\alpha \text{Im} \int_{\mathbb{R}^N} f \bar{u} dx + 2\alpha ||u||^2_{L^2(\mathbb{R}^N)}; \quad (34)$$

here we have used

$$\text{Re} \int_{\mathbb{R}^N} u_0 \bar{u} \log(|u|^2) dx = \frac{1}{2} \frac{d}{dt} \left( \int_{\mathbb{R}^N} |u|^2 \log(|u|^2) dx - \int_{\mathbb{R}^N} |u|^2 dx \right). \quad (35)$$
Using the $L^2(\mathbb{R}^N)$ bound above we can prove that after a transient time $T_0$ that depends on $u_0$ we have that for $t \geq T_0$ then $J(u(t)) \leq K$. Then

$$||\nabla u||^2_{L^2(\mathbb{R}^N)} - \int_{|u| \leq 1} |u|^2 \log(|u|^2)dx \leq K + \int_{|u| \geq 1} |u|^2 \log(|u|^2)dx.$$  (36)

We now recall the logarithmic Sobolev inequality (see [8])

**Lemma 3.2.** For any $\beta > 0$ and for any $v$ in $H^1(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} |v|^2 \log(|v|^2)dx \leq \frac{\beta^2}{\pi} ||\nabla v||^2_{L^2(\mathbb{R}^N)} + \frac{\log(\|v\|_{L^2(\mathbb{R}^N)})}{\log(1 + \beta)} ||v||^2_{L^2(\mathbb{R}^N)}.$$  (37)

This leads to, choosing $\beta = \sqrt{\frac{2}{s}}$ and combining with (36)

$$||\nabla u||^2_{L^2(\mathbb{R}^N)} \leq K + \frac{1}{2} ||\nabla u||^2_{L^2(\mathbb{R}^N)}.$$  (38)

Then $\nabla u$ remains bounded in $L^2(\mathbb{R}^N)^N$. Due to Sobolev embedding $u$ is now bounded in $L^{2s,\infty}(\mathbb{R}^N)$ if $N \geq 3$ (and in any $L^p(\mathbb{R}^N)$, $p < +\infty$, if $N \leq 2$). Hence since for $s \geq 1$ then $s \log s \leq C_N s^{\frac{2s-2}{N}}$ then

$$\int_{|u| \geq 1} |u|^2 \log |u|^2dx \leq K.$$

We now infer from (36) the following inequality

$$-\int_{|u| < 1} |u|^2 \log |u|^2dx \leq K.$$

It remains to prove that $\int_{\mathbb{R}^N} A(|u|)dx$ remains bounded. Since for $s \leq e^{-3}$ then $A(s) = -s^2 \log s^2$ and for $s \geq e^{-3}$ then $A(s) \leq Cs^2$ the results follows (35) and the bound on $-\int_{|u| < 1} |u|^2 \log |u|^2dx$.

**Lemma 3.3.** The semigroup $S(t)$ is continuous in $L^2(\mathbb{R}^N)$ on bounded sets of $W$.

The next lemma is concerned an asymptotic compactness property.

**Lemma 3.4.** Consider $u_j$ that is bounded in $W$ and $t_j \rightarrow +\infty$ then the sequence $S(t_j)u_j$ is relatively compact in $L^2(\mathbb{R}^N)$.

**Proof.** We proceed as in [33]. Consider a smooth function $\theta : \mathbb{R}^+ \rightarrow [0, 1]$ such that $\theta(\xi) = 1$ if $\xi \leq 1$ and $\theta(\xi) = 0$ if $\xi \geq 2$. Introduce $q_R = \int_{\mathbb{R}^N} |u|^2 (1 - \theta(\frac{|x|}{R}))dx$. Then considering the scalar product of (4) with $\frac{1}{2} (1 - \theta(\frac{|x|}{R}))u$ we have

$$\frac{1}{2} \frac{d}{dt} q_R + \alpha q_R = \text{Re} \int_{\mathbb{R}^N} (1 - \theta(\frac{|x|}{R}))(\bar{u} \Delta u + \frac{1}{R} \sum_{j=1}^{N} \theta'(\frac{|x|}{R}) \text{Im}(\frac{x_j \bar{u} \partial_j u}{|x|}) dx) .$$  (38)

On the one hand,

$$\text{Im} \int_{\mathbb{R}^N} (1 - \theta(\frac{|x|}{R}))(\bar{u} \Delta u) dx = \frac{1}{R} \sum_{j=1}^{N} \int_{\mathbb{R}^N} \theta'(\frac{|x|}{R}) \text{Im}(x_j \bar{u} \partial_j u) |x| dx .$$  (39)
Then assuming that the trajectory is trapped into the $W$ absorbing set for $t \geq 0$ we have that
\[
|\text{Im} \int_{\mathbb{R}^N} (1 - \theta |x|/R) \bar{u} \Delta u dx| \leq \frac{||\theta'||_{L^\infty}}{2R} \int_{R \leq |x| \leq 2R} (|\nabla u|^2 + N|u|^2) dx \leq \frac{K}{R}. \tag{40}
\]
On the other hand, by Cauchy-Schwarz inequality
\[
|\int \bar{u} f(1 - \theta |x|/R) dx| \leq \frac{1}{2} q_R + \frac{1}{2\alpha} \int_{|x| > R} |f|^2 dx, \tag{41}
\]
and then
\[
\frac{d}{dt} q_R + \alpha q_R \leq \frac{1}{\alpha} \int_{|x| > R} |f|^2 dx + \frac{K}{R}. \tag{42}
\]
Therefore we can chose $R$ large enough such that $q_R(t) \leq \varepsilon$ for $t \geq T$. On the other hand the sequence $u\sqrt{\theta(|x|/R)}$ remains bounded in $H_0^1(B(0,2R))$ that is compactly embedded in $L^2(\mathbb{R}^N)$. Hence $S(t_j)u_j$ is trapped into a compact set of $L^2(\mathbb{R}^N)$ for $t \geq T$.

We now complete the proof of the Theorem. Consider $B$ the absorbing set in $W$. Consider $\mathcal{A} = \cap_{a > 0} \cup_{t > s} S(t)B$ where the closure is for the strong topology of $L^2(\mathbb{R}^N)$.

**Lemma 3.5.** The set $\mathcal{A}$ is a compact invariant subset of $L^2(\mathbb{R}^N)$ that attracts the trajectories strongly in $L^2(\mathbb{R}^N)$ and weakly in $W$.

**Proof.** Due to the previous lemmata, the semigroup is asymptotically compact in $L^2(\mathbb{R}^N)$ and the existence of $\mathcal{A}$ that is a compact attracting set in $L^2(\mathbb{R}^N)$ is proved (applying Theorem 1.1.1. in [37]). It remains to check that the set is invariant. If $a = \lim_j S(t_j)u_j$ in $L^2(\mathbb{R}^N)$, since the semigroup is continuous in $L^2(\mathbb{R}^N)$ then $S(t)a = \lim_j S(t_j + t)u_j$ in $L^2(\mathbb{R}^N)$ and $S(t)\mathcal{A} \subset \mathcal{A}$. Similarly $\mathcal{A} \subset S(t)\mathcal{A}$. Moreover for any sequence $t_j$ that diverges to $+\infty$ and $u_j$ that is trapped into the $W$ absorbing ball, then $S(t_j)u_j$ converges to a point in $\mathcal{A}$ strongly in $L^2(\mathbb{R}^N)$ and weakly in $W$.

We now complete the proof of the Theorem thanks to the J. Ball’s argument [9] proving that in fact $\mathcal{A}$ is attracting for the strong topology of $W$. Consider $u_j$ that is bounded in $W$ and $t_j \to +\infty$ such that $S(t_j)u_j$ converges towards $a$ strongly in $L^2(\mathbb{R}^N)$ and weakly in $W$; we may also assume without loss of generality that $S(t_j)u_j$ converges almost everywhere. We set $G(u) = 2\alpha \text{Im} \int_{\mathbb{R}^N} f \bar{u} + 2\alpha ||u||_{L^2(\mathbb{R}^N)}^2$, that is continuous for the strong topology in $L^2(\mathbb{R}^N)$. Appealing (34) we have, for any $t > 0$,
\[
J(S(t_j)u_j) = e^{-2\alpha t} J(S(t_j - t)u_j) + \int_0^t e^{-2\alpha (t-s)} G(S(t_j + s - t)u_j) ds. \tag{43}
\]
Using that $\mathcal{A}$ is bounded in $W$ we have that
\[
J(S(t_j)u_j) \leq Ke^{-2\alpha t} + \int_0^t e^{-2\alpha (t-s)} G(S(t_j + s - t)u_j) ds. \tag{44}
\]
We let $j \to +\infty$ and we use the strong convergence in $L^2(\mathbb{R}^N)$ and the Lebesgue dominated convergence theorem to get
\[
\limsup J(S(t_j)u_j) \leq Ke^{-2\alpha t} + \int_0^t e^{-2\alpha (t-s)} G(S(s-t)a) ds. \tag{45}
\]
We also have
\[ J(a) = e^{-2\alpha t} J(S(-t)a) + \int_0^t e^{-2\alpha (t-s)} G(S(s-t)a) ds. \] (46)

Then
\[ \limsup J(S(t_j)u_j) \leq Ke^{-2\alpha t} + J(a). \] (47)

Letting \( t \to +\infty \) we infer that \( \limsup J(S(t_j)u_j) \leq J(a) \). This leads to, setting \( a_j = S(t_j)u_j \), and using the strong convergence in \( L^2(\mathbb{R}^N) \), due to the very definition of the energy functional \( J \) (see (33))
\[ \limsup \left( \| \nabla a_j \|_{L^2(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} |a_j|^2 \log |a_j|^2 \right) \leq \| \nabla a \|_{L^2(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} |a|^2 \log |a|^2. \] (48)

We then conclude by Lemma 2.3.

The compactness of \( \mathcal{A} \) relies also on J. Ball’s argument. Consider a sequence \( a_j \) that belongs to the global attractor. Up to a subsequence extraction we may assume that \( a_j \) converges weakly in \( W \), strongly in \( L^2(\mathbb{R}^N) \) and almost everywhere towards \( a \) that belongs to \( \mathcal{A} \). We now have to prove that the convergence holds for the strong topology in \( W \). We start with, for \( t > 0 \)
\[ J(a_j) = e^{-2\alpha t} J(S(-t)a_j) + \int_0^t e^{2\alpha (t-s)} G(S(s-t)a_j) ds. \] (49)

We also have
\[ J(a) = e^{-2\alpha t} J(S(-t)a) + \int_0^t e^{2\alpha (t-s)} G(S(s-t)a) ds. \] (50)

By Lebesgue dominated convergence theorem, using the strong convergence in \( L^2(\mathbb{R}^N) \) we know that
\[ \lim_{j \to +\infty} \int_0^t e^{2\alpha (t-s)} G(S(s-t)a_j) ds = \int_0^t e^{2\alpha (t-s)} G(S(s-t)a) ds. \] (51)

Therefore
\[ \limsup_{j \to +\infty} J(a_j) \leq Ke^{-2\alpha t} + J(a). \] (52)

Letting \( t \) diverges towards \( +\infty \) and using the strong convergence in \( L^2(\mathbb{R}^N) \) we infer from (33) and (52) that
\[ \limsup \left( \| \nabla a_j \|_{L^2(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} |a_j|^2 \log |a_j|^2 \right) \leq \| \nabla a \|_{L^2(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} |a|^2 \log |a|^2. \] (53)

Applying Lemma 2.3 as above completes the proof.

4. Directions for further research.

4.1. The global attractor issue in the defocusing case. We may address the attractor issues for defocusing damped forced Log-NLS equations that read
\[ u_t + \alpha u + i\Delta u - iu \log |u|^2 = f, \] (54)
where \( \alpha > 0 \) is the damping parameter and where \( f \) is a time independent forcing term that is square integrable. Here we have to use the methodology introduced in [17] for the defocusing Log-NLS equation.
4.2. Studying the equation with other boundary conditions. When addressing the initial value problem for (1) in standard Sobolev-Orlicz space, we miss some particular solutions that do not converge towards to 0 when $|x| \to +\infty$. For instance the function $u(t, x) = 1$ is a particular solution to the equation that does not belong to the energy space $W$. One first direction of research could be to study the initial value problem (and then the attractor issue in the damped forced case) in the so-called Zhidkov spaces. To define Zhidkov space you do not assume that $u$ has finite mass, you still assume that the energy $E$ of solutions is finite and add another condition as $u$ belongs to $L^\infty(\mathbb{R}^N)$ (See [25], see also [26] for another approach). Hence you can have particular solutions such that $|u|$ converge to 1 when $|x|$ diverges to $+\infty$.

We can also study the Log-NLS equations with other boundary conditions as periodic boundary conditions. In this case we have more stationary solutions than in the case $x \in \mathbb{R}^N$.

4.3. Studying $S_0$ the semigroup in $L^2(\mathbb{R}^N)$. For classical damped forced dispersive evolution equations, the existence of global attractors for low regularity solutions is known. For cubic NLS equations in one dimension, there exists a global attractor for the solution flow in $L^2(\mathbb{R})$ (see [30]). For Korteweg-de Vries equations, the results is true also for solutions in $H^{-\delta}(\mathbb{R})$ for $\delta < \frac{3}{4}$ small enough (see [38]).

For Log-NLS equation, from the theory developed above, we also have and abstract semigroup that acts in $L^2(\mathbb{R}^N)$ and that possesses an absorbing set in $L^2(\mathbb{R}^N)$. If we knew that $S_0$ is continuous for the weak topology in $L^2(\mathbb{R}^N)$, then we would have a global attractor $\mathcal{A}_0$ in $L^2(\mathbb{R}^N)$ using J. Ball’s argument as in [1], [27], [28] and the references therein. Here the main difficulty is that we just have a mild solution that is not a weak solution of the associated PDE; we cannot try to address the weak continuity issue as in [30] and in the references therein.

4.4. Regularity of the attractor. We may address the regularity issue for the semigroup $S_1$ as for the classical Schrödinger equation (see [29]). Is the global attractor included in $D(M)$? We guess that this is indeed the case. A major technical difficulty to handle for this Log-NLS equation it that the nonlinearity is not smooth at $u = 0$. Therefore the machinery in [29] does not work straightforwardly. Pretending that we also have a semigroup in $L^2(\mathbb{R}^N)$, that is we were able to solve the issue raised in the previous subsection, we may also wonder if the semigroup $S_0$ has the same attractor than the semigroup $S_1$.

4.5. Fractal dimension of the global attractor. Assuming moreover that the external force $f$ belongs to some weighted space to ensure some extra decay at the infinity like for instance

$$\{ f \in L^2(\mathbb{R}^N); \int_{\mathbb{R}^N} |xf(x)|^2 dx < +\infty \},$$

we may bet that the global attractor has finite fractal and Hausdorff dimension. Unfortunately, the classical methods to establish this does not work once again due to the lack of regularity of the nonlinearity at $u = 0$. We expect also in some special case to derive some lower bound on the dimension of the global attractor as in [27], [28].
Actually, due to Lemma 13, if we consider the difference \( w = u - \tilde{u} \) of two solutions we have that
\[
\frac{1}{2} \frac{d}{dt} ||w||^2_{L^2(\mathbb{R}^N)} + \alpha ||w||^2_{L^2(\mathbb{R}^N)} \leq 2 ||w||^2_{L^2(\mathbb{R}^N)}.
\]
(55)

Therefore for large \( \alpha > 2 \), we can easily prove that the nonlinear semigroup is a contraction in \( L^2(\mathbb{R}^N) \), and that all solutions converge towards the unique stationary solution. We can imagine to have similar result for \( \alpha > 1 \) but this is not the case for small \( \alpha \). Let us describe below an example. Consider in dimension 1 the Gaussian function \( g(x) = \exp(-\frac{|x|^2}{2}) \). A mere computations gives
\[
\Delta g + 2g \log g = -g.
\]

Then for \( f(x) = (\alpha + i)g(x) \), the function \( g(x) \) is a stationary solution to the equations, that is a fixed point to the semigroup.

Let us linearize the flow at a neighborhood of this stationary solution. Then the differential \( w \) is solution to the linear evolution equation
\[
w_t + \alpha w + i\Delta w - i|x|^2w + 2i\text{Re} w = 0.
\]
(56)

Considering \( e(x) = g(x) \) that is an eigenvector for the harmonic operator, that is \( -\Delta e + |x|^2e = e \). We have that the two dimensional space spanned by \( e(x) \) and \( ie(x) \) is stable under the linearized flow and that the corresponding linearized linear operator has two eigenvalues whose product is \( \alpha^2 - 1 \). Hence for \( \alpha \) small enough there is one negative eigenvector and the dimension of the unstable manifold at \( g \) has dimension more than 1. Then the global attractor is not trivial.

4.6. **Regularization of Log-NLS equations.** In order to proceed to numerical computations, a regularized version of the Log-NLS equation was introduced in [10], [11]. In order to avoid round-off errors, the following equations were proposed as a substitute for Log-NLS equations
\[
u_t + i\Delta u + \text{in} \log(\varepsilon + |u|) = 0,
\]
(57)

where \( \varepsilon \) is a very small positive parameter. When adding to this equation damping and forcing term, one can address the attractor issue with a smooth non linear term. Actually the solution to (57) approximates the solution to the Log-NLS equation in \( L^2(\mathbb{R}^N) \) when \( \varepsilon \) converges to 0. This approximation process can provide new informations on the original semigroup in the damped forced case.

4.7. **Dynamics, analysis for large time of numerical schemes.** In the last decades, considerable efforts have been made to understand the long-time behaviour of solutions to dissipative evolution equations in terms of global attractors. The discrete counterpart of PDEs are numerical schemes. Considering semi-discrete in time schemes as discrete dynamical systems in infinite-dimensional Banach spaces lead to the issue of the existence of properties of global attractors for these discrete dynamical systems as in [2], [31], [23], [18]. To our knowledge, these issues are open for Log-NLS equations.

We may address the implicit Euler scheme that was instrumental to derive the existence of the semigroup as in (15), or other schemes that are more suitable for NLS equations as the Crank-Nicolson scheme as in [36],
\[
\frac{u^{n+1} - u^n}{\tau} + M_\alpha(u^{n+1} + u^n) = f,
\]
(58)
supplemented with initial data \( u^0 = u_0 \), or the C. Besse relaxation scheme \([14]\) that reads on staggered grids

\[
\frac{u^{n+1} - u^n}{\tau} + \alpha \left( \frac{u^{n+1} + u^n}{2} \right) + i\Delta \left( \frac{u^{n+1} + u^n}{2} \right) + i\varphi^{n+\frac{1}{2}} \left( \frac{u^{n+1} + u^n}{2} \right) = f,
\]

(59)

\[
\varphi^{n+\frac{1}{2}} + \varphi^{n-\frac{1}{2}} = \log |u^n|^2,
\]

supplemented with initial data \((u_0, \varphi^{-\frac{1}{2}})\), or the so-called splitting scheme \([12]\)

\[
\frac{u^{n+\frac{1}{2}} - u^n}{\tau} + \alpha \left( \frac{u^{n+\frac{1}{2}} + u^n}{2} \right) + i\Delta \left( \frac{u^{n+\frac{1}{2}} + u^n}{2} \right) = f,
\]

(60)

\[
u^{n+1} = \exp(-i\tau \log |u^{n+\frac{1}{2}}|^2)u^{n+\frac{1}{2}},
\]

supplemented with initial data \( u^0 = u_0 \).

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