Operationalization of Relativistic Motion

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Abstract: We demonstrate the definition of basic observables from physical operations, the key to overcome hidden stumbling blocks and apparent paradoxes from unscrutinized (classical) formalisms. We develop Helmholtz program of basic measurements for relativistic motion. We define the basic observables by direct comparison: "longer than" if one object or process covers the other. To express the spatiotemporal order also numerically (how many times longer) we cover them by a locally regular grid of light clocks. These are the basic physical operations. From their interrelation we derive mathematical relations, e.g. for different observers the formal Lorentz transformation; for accelerating observers we reveal a measurement-methodical view on the apparent Twin paradox.

One usually explains kinematics axiomatically. So one can trace back the whole mathematical formalism to a manageable system of initial propositions which are logically independent from one another. Though this formal bookkeeping of physics already begins in the abstract. The axiomatic formulation assumes scalars, four-vectors, metric etc. as known objects of its description. Without further implicit assumptions it lacks interpretation and physical meaning. Origin, scope and limitations of the variables and algebra remain unclear. The lack of alternative approaches seems to justify the formal path for developing novel theories. For the foundation of elementary kinematics (next also for dynamics [15]) we develop a complementary program; we begin from the primary measurement operations {I}.

Our objective is a foundation of relativistic kinematics from the operationalization of its basic observables. The goal is not to change or improve the mathematical structure, but to gain a deeper physical understanding of kinematics. Like Einstein [5] for the concept of simultaneity we reveal the underlying physical operations. We begin from undisputed natural and measurement principles. We stress the active role of physicists, the interventions to define basic observables, quantification and then derive the four-vector formulation second.

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With Helmholtz’ method (general discussion in retrospect \cite{4}) we will show, how physics generates its own mathematics in empirical practice.

Theodor Hänsch - inventor of the optical frequency comb generator which facilitates the construction of most precise clocks - defines: "Time is what one measures with a clock". In his case a light clock. The origin of basic reference devices and measurement procedures is not in the domain of non-empirical mathematics. We require them (like Einstein's clock postulate and laser ranging technique) before having a theory of matter and as a basis for developing the latter \cite{15}. We develop all protophysical prerequisites from everyday work experience. We make a digression into watchmaking and understand by what actions one provides these reference devices if one did not have them before \cite{1.1}. With classical rulers and clocks one determines the universal motion of light \cite{1.3}, it propagates locally uniform.

For basic measurements we introduce (light) clocks as an unstructured unit \cite{1.4}. Let every observer place them side by side and one after another until the measurement object is covered; for the technical description we introduce measurement termini (simultaneity lines, projections etc.). In the mathematical formulation of the procedures all corresponding terms have physical meaning. From the underlying operations we derive the Lorentz symmetry \cite{2}. Every formal calculation, e.g. in the configuration of the apparent Twin paradox \cite{3}, assumes connected basic measurement operations. For an accelerating observer they become impracticable. From vivid pre-theoretic principles we develop all mathematical variables and operations and finally the relativistic equations.

1 Measurement operations

For the origin of colloquial notions - motion, space and rigid body - from common sensual experience we refer to Poincare \cite{2} and Mach \cite{3}. According to Poincare geometric properties essentially characterize the relative motion between neighboring objects. Leibniz characterizes "space" and "time" as relations between the observable things. Space brings order into things which happen simultaneously. Time brings order into things which happen sequentially. From everyday practice one knows the direct comparison

- \( >_l \) if two extended objects lie on top of each other - one will cover the other
- \( >_t \) if two processes begin simultaneously - one will outlast the other.

The ordering relation is reproducible in an observer independent way. Next one wants to find "how many times" longer.

For reproducible measurements one provides sufficiently constant reference devices and standardized procedures. The construction and the conventions historically developed from daily work experience; we sketch the transition to physical experimentation. One works with natural objects in a natural environment. Their behavior depends on external conditions (some known and others undiscovered). One wants to control the interrelation of work conditions; it pays off. We regard the origin of basic measurements as a standardization in the conduct of reproducible experiments (to specify known work conditions \cite{10}).
With empirical knowledge on feasibility and outcomes of work actions one can probe the objects and rehearse expedient ways of handling. We define all basic observables (length, duration; in \[15\] also impulse, inertial mass, capability to work/energy) and the associated comparison and concatenation operations in the practical domain. The development has a social dimension: a master inherits the demonstrable practice to a student, first simply by pointing a finger and then defines a colloquial and technical language. We presuppose usage of common denominations ("Alice, Bob and Otto move relative to one another.", "They signal with light." etc.) with their common meaning as a known part of work experience.\footnote{We explain the meaning of colloquial expressions by exemplary demonstration (of sufficiently constant phenomena). We can neither demonstrate pure matter in isolation from its behavior nor pure behavior detached from matter. In colloquial speech we express a demonstrable fact by a simple descriptive sentence like "Otto is long". These represent the smallest unit of meaning. The subject Otto $O$ and its attribute length $l$ are distinguishable but inseparably unified \cite{9,11}. The subject terminus "long Otto" emphasizes the subject $O$ which embodies the property long $l$; we symbolize the long object by $O_l$. The predicate terminus "Otto’s length" emphasizes the property which Otto represents; we symbolize the object’s length $l_O$. From elemental operations on tangible things $O_l$ we develop basic measurements for the attribute $l_O$.\footnote{Before Bob can specify the form of Otto’s relative motion he has to find out if his own clock provides a uniform ticking reference. Before Alice can determine whether Otto’s nose is crooked she needs to know if her ruler is straight. The test rules for admissible reference devices do not presuppose an already existent.}

With Lorenzen, Janich \cite{12} we can presuppose (circularity free and without mathematical presuppositions) that every observer can manufacture "straight" "rigid" rulers and "uniform running" clocks \{1.1\}. In a direct measurement one concatenates \(*s\)” the rulers \(\mathcal{R}\)” side by side in a straight way until the layout, symbolized \(\mathcal{R} \ast_s \ldots \ast_s \mathcal{R} \sim_s O\), covers the object. The ordering relation ”longer than” becomes measurable \(s_O = \sharp \{\mathcal{R}\} \cdot s_R\) by the number of connected rulers and their standard length \(s_R\); similarly for durations.

In starting figure \[1] we illustrate objects and observers in motion. Consider a (hidden) railway track along which Alice, Bob and Otto specify their relative motion and the light. After including the historical depth of work experience they are equipped with the local Euclidean metric. We will demonstrate the transition to relativistic kinematics. Each observer measures Otto’s relative motion with classically constructed light clocks. They place them one after the other or side by side; connected by coinciding rays of light \{1.5\}. A regular grid of light clocks covers their relative distances. Each building block is congruent with the next; by counting them they measure the magnitude of the length. They measure their relative motion with (the motion of light in) their reference device.

1.1 Watchmaking

The protophysical foundation of Euclidean geometry \cite{12} explains the standardization of length comparisons circularity free in the categories of purpose and expedient means of everyday work. One works on raw materials and reshapes them for practical needs. One builds rulers and clocks as sufficiently constant representatives of ”length” and ”duration”. The success of (tentative) manufacturing methods is secured by test procedures for the straight form of a constructed ruler and the uniform running of a clock \cite{12} \footnote{Before Bob can specify the form of Otto’s relative motion he has to find out if his own clock provides a uniform ticking reference. Before Alice can determine whether Otto’s nose is crooked she needs to know if her ruler is straight. The test rules for admissible reference devices do not presuppose an already existent.} For this reason mea-
surement instruments are to be understood - not simply as arbitrary designations of natural objects but - as artifacts; our manufacturing actions must realize test norms [13].

The testing method for geometrical shapes originates from grinding practice: A body has a "flat" surface if one can produce two moldings of that body and then fittingly (!) shift the two imprint surfaces against one another. If there is a gap continue grinding them against one another; make two new moldings and check again. Similarly if one has manufactured a body with two flat surfaces which intersect one another, then the intersecting edge represents a "straight" line. The test norms originate from intuitively controlled actions in everyday (technical) work, which is governed by the rationality of purpose and expedient means. Ultimately one explicates expedient work norms as measurement norms. Practicable rules of pre-scientific technical behavior develop into norms for measurement operations [13].

A watchmaker evaluates by test procedures whether a tentative device runs uniform. In the empirical interplay of analyzing manufacturing conditions and examining the respective products the manufacturing method is continually refined until the device realizes the aspired ideal of uniform motion sufficiently precise. In this process we make the practical experience that the ideal is never completely realizable. The closer one wants to approach the more effort and workload is required in the production and also for the conservation of the product (shielding fragile clock). He takes guidance by a test procedure: Take two structurally identical copies of the clock and align them such that their clock hands are running straight into fixed (e.g. perpendicular) directions. Then one can couple the motion of the two clock hands e.g. by a mechanical transmission; draw down the stretch of way of their superposed motion and check its geometric form. The clocks run uniform if - independently from when each was started and coupled together - their superposed stretch of way always has the form of a straight line. As before the path in question represents the ideal of a straight line if any two segments can be fittingly (!) shifted against one another.

A clock is manufactured and tested as a representative for a uniform motion. By metricizing the length of the traversed stretch of way (of the moving clock hand) one obtains ametrical measurement instrument for "durations". In practice (accumulated friction etc.) clocks will run (approximately) uniform for only finite durations. Such "finite duration" measurement standards can be aligned synchronously one after the other to cover longer processes. By this substitution we measure the magnitude of "durations". We arrive at classical Galilei kinematics for space and time. Despite the uniform motion of the clock hand - the clock (housing) can be under acceleration, sitting still or free falling.

1.2 Principle of Inertia

We link uniform motion to the behavior of natural (work) objects by the principle of inertia. Bodies move (without external agent) on their own. "Every body with no (external) forces acting on it remains - as judged from the (inertial) lab - in a state of rest or of uniform rectilinear motion" [6]. In isolation their motion is preserved. We identify the presence of prototype for a straight line and an ur-clock which one can simply copy or transport.

3The protophysical norm for uniform motion originates from a test of the straight shape of rigid objects.
interactions by changing state of motion and the absence from practical reasoning. We postulate an inertial reference as a reproducible experimental prerequisite which we must shield from all external disturbances. We can provide it after probing all empirical conditions of an interaction (e.g. set up a billiard table horizontally and test that a prepared arrangement of object balls does not roll off to any side before the actual experiment). As Galilei and Huygens we develop via principle of inertia and relativity principle elementary dynamical concepts before the latter were transferred by Newton onto gravitational systems. There the (initially practically solved) problem of selecting inertial references is revised; for building and steering machines (for the arising need to mechanize tool use based on the division of labor in industrial revolution) the latter had no practical significance.

Newton could draw on (in top-heavy circles proscribed) "literature of practical (handcraft) mechanics on problems of machine construction and work economy, which is considered to little by historiography of science." For the context of origin of dynamical concepts Wolff’s genetic reconstruction of Impetus theory - mechanics in epoch from 6. to 17. century - provides "plausible arguments for the proposition that the inner conceptual content of mechanics was influenced by motives, which developed during that economic and technical revolution" [4].

1.3 Light principle

In order to give physical meaning to the concept of time Einstein [5] requires the use of some process which establishes relations between distant locations. In principle one could use any type of process. Most favorable for the theory one chooses a process about which we know something certain. For the free propagation of light this holds much more than for any other process.

One measures the motion of light with rulers and clocks. We depict the relative motion between all objects in a spacetime diagram (see figure 1b). Alice, Bob or Otto may move equally or not, but no object can overtake free light. Let Alice and Bob emit light towards Otto. It propagates independently no matter how they move the source. If Alice sends her light to Otto and shortly after it passes Bob he sends his own light to Otto as well, then both rays AO and BO coincide. One measures the "magnitude" of distances and durations and the "form" of motion with classical rulers and clocks. They approximate a straight line and uniform motion. By local comparison with these reference instruments free light propagates in a uniform and straight way. In a spacetime diagram we represent it by a straight line. Locally the light Alice or Bob sends to Otto AO and BO remains parallel.

With the classical metric (in the domain of classical measurements of length and duration) we discover: Locally free light represents a uniform, isotropic and straight form of motion. It provides a universal reference for any intrinsic observer. Based on the light principle we conduct laser ranging measurements. We presuppose this hypothesis also along global paths of light which can be thought of as a connected covering of multiple local segments.
Figure 1: a) moving objects and moving light b) interrelation of corresponding processes

1.4 Light clock

Because of the universal light principle "laser ranging" is a reliable practice of navigation.

Let Alice send out light towards Otto $A_1 \leadsto O \leadsto A_2$ and towards Bob $A_1 \leadsto B \leadsto A_3$ and wait until their reflection returns (see figure 2b). In radar round trips we focus on the distance covered and Alice waiting time. For two ranging cycles $A_1 \leadsto O \leadsto A_2$ and $A_1 \leadsto B \leadsto A_3$ Alice notices the order in which the light returns. By the light principle more waiting time $t_{A_1A_2} > t_{A_1A_3}$ corresponds to a larger distance covered $s_{AO} > s_{AB}$ from Alice to turning point Otto resp. Bob and back.

For quantification Alice constructs a reference device, a light clock $\mathcal{L}$: $\mathcal{L}_I \leadsto \mathcal{L}_{II} \leadsto \mathcal{L}_I \ldots$ with two nearby mirrors $\mathcal{L}_I$ and $\mathcal{L}_{II}$ in a rigid frame. The light constantly oscillates back and forth. Each tick of her measurement unit $\mathcal{L}$ covers the same standard distance $s_{\mathcal{L}}$ and takes the same standard time $t_{\mathcal{L}}$.

Alice light clock substitutes the traditional rulers and clocks. The protophysical test
norms - for manufacturing both, traditional clocks and light clocks - remain the same. In practice one replaces the former by new light clocks because they realize the aspired ideal of uniform running more precisely. The motion of light is not anymore measured with traditional rulers and clocks; instead we determine all other motions with respect to the motion of light in (classical protophysically manufactured) light clocks. The light principle implies a paradigm shift. One abandons the former priority of classical measurement devices in favor for the universal propagation of light. The motion of light becomes a measurement standard itself.\footnote{The definition of standard length $s_L$ is based on a given standard duration $t_L$ and the universal speed of light $c$. Contemporary metrology regards speed of light $c$ as invariant natural constant and introduces optical clocks as frequency standards. Our world’s current time standard (a laser-cooled cesium fountain known as NIST-F1 based on resonant transitions between quantized energy levels in atoms) is accurate to within $\Delta f/f \sim 10^{-16}$. It represents the ultimate reference for time intervals $t_{Cs}$ with accuracy $10^{-16}$ sec. Bureau of Standards defines the atomic second $1\text{sec}^{(86)} := 9192631770 \cdot t_{Cs}$ - on paper - as a multiple of that standard duration. "Cesium provides a 'physical' second that can be realized in laboratories and used for other measurements. ... The basic principle of the atomic oscillator is simple: Since all atoms of a specific element are identical, they should produce the exact same frequency..." \cite{14}. They refer to an intrinsic property of an atom under standardized conditions: "cesium atom at rest at a thermodynamic temperature of 0K". An unperturbed atomic transition is identical from atom to atom (reproducibility).} We use the classical light clock as a new measurement unit.

Figure 2: a) laser ranging b) consecutive and adjacent connection of light clocks
1.5 Direct connections

We essentially refer to the oscillating light inside. In practice the two dimensions "length" and "duration" of a light clock \( \mathcal{L} \) are always addressed unified. Depending on the concatenation:

- adjacently connected \(*_s\) (ticking) light clocks represent a distance unit \( \mathcal{L}_s \) and
- consecutively connected \(*_t\) (light clock) ticks represent a duration unit \( \mathcal{L}_t \).

The width of the light clock \( \mathcal{L} \) becomes our unit length \( \mathcal{L}_s \) and each tick lasts unit time \( \mathcal{L}_t \).

1.5.1 Time-like concatenation

Let Alice join together light clock ticks \( \mathcal{L} \) one after another until the sequence - symbolized by \( \mathcal{L} *_t \ldots *_t \mathcal{L} \) - covers the waiting interval of her laser ranging cycle \( A_1 \bowtie B \bowtie A_2 \):

\[
\overline{A_1A_2} \sim_t \mathcal{L} *_t \ldots *_t \mathcal{L}. \tag{1}
\]

In her material model Alice can count the number of ticks, symbolized \( \# \{ \mathcal{L}_t \} =: t_{(A)} \mathcal{L} \). The ordering relation "longer than" becomes quantified. Alice measures the duration of her laser ranging interval

\[
t_{\overline{A_1A_2}} \mathcal{L} *_t \ldots *_t \mathcal{L} \overset{\text{(Congr.)}}{=} t_{(A)} \mathcal{L} \mathcal{L} \cdot t_{\mathcal{L}} \tag{2}
\]

by the sequence of ticks and the latter according to the congruence principle by the number of congruent (light clock) ticks and its reference duration \( t_{\mathcal{L}} \).

1.5.2 Space-like concatenation

Furthermore Alice can place ticking light clocks \( \mathcal{L} \) literally side by side. She utilizes the same units \( \mathcal{L} \) to produce an adjacent layout of comoving light clocks.

**Lemma 1** It represents Alice simultaneous straight measurement path towards Bob \( \overline{AB} \).

**Proof:** Imagine a swarm of identically constituted light clocks \( \mathcal{L}^{(A_i)} \). Beginning with her own in moment \( A \), Alice successively places pairs of light clocks \( \mathcal{L}^{(A_i)} *_s \mathcal{L}^{(A_{i+1})} \) next to one another by letting their inner light rays overlap. She builds a locally regular grid of light clocks in an intrinsically simultaneous and straight way (see figure 2b):

(a) Suppose we have successively laid out light clocks from \( \mathcal{L}^{(A_1)} \) all the way to \( \mathcal{L}^{(A_n)} \). Consider the two ticks of light clock \( \mathcal{L}^{(A_n)} *_{A_n} \mathcal{L}^{(A_n)} \) around the moment \( A_n \).

(b) The next comoving light clock \( \mathcal{L}^{(A_{n+1})} \) is placed so that the extended (dashed) light ray from \( \mathcal{L}^{(A_n)} *_{A_n} \mathcal{L}^{(A_n)} \) coincides with the light ray from \( \mathcal{L}^{(A_{n+1})} \).

(c) Starting from \( A_{n+1} \) - by isotropy - light travels in identical round trip duration \( t_{\mathcal{L}^{(A_{n+1})}} \) the same distance back to left light clock \( \mathcal{L}^{(A_n)} \) as to the right light clock \( \mathcal{L}^{(A_{n+1})} \). \[6\]

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6Let two synchronously ticking light clocks \( \mathcal{L} *_s \mathcal{L} \) sit side by side. We assume that two-way light cycle on the left covers same standard distance \( s_{\mathcal{L}} \) to its turning point as the other two-way light cycle on the right.
(d) Consider a series of three (preceding and following) ticks of light clock \(L^{(A_{n+1})}\).

(e) Place light clock \(L^{(A_{n+2})}\) so that the extended (dotted) light rays from \(L^{(A_{n+1})}\) coincide with the two ticks of light clock \(L^{(A_{n+2})}\) around the moment \(A_{n+2}\).

(f) By analogous induction steps \(L^{(A_n)}\) \(L^{(A_{n+1})}\) \(L^{(A_{n+2})}\) \(\forall n\)

Alice proceeds towards Bob.

In every step the extended (dashed resp. dotted) light rays coincide. In her straight comoving connection \(L^{(A)}|A_n \star L^{(A_n)} \Rightarrow L^{(A_{n+1})} \Rightarrow L^{(A_{n+2})} \forall n\) all light clocks tick synchronized along connecting moments \(A_1, A_2, \ldots, A_n, A_{n+2}, \ldots B\).

The construction steps do not depend on the scale of light clock \(L\) (e.g. refining the locally regular layout with twice the light clocks of half the size coincides with the original pattern). They are locally well-defined; the global measurement path \(L \star s \ldots s \star L\) is universal.

\[\square\]

Alice covers the laser ranging path to Bob by an adjacent layout of light clocks

\[
\overline{AB} \sim s L \star s \ldots s L. \tag{3}
\]

Each represents a length unit \(L_s\). Alice measures the length along her laser ranging path

\[
s_{\overline{AB}} = s_{L \star s \ldots s L} \overset{\text{(Congr.)}}{=} s_{L^{(A)}} \cdot s_{L} \tag{4}
\]

by the adjacent layout \(L \star s \ldots s L\) and the latter according to the congruence principle by the number \(\sharp \{L_s\} = s_{L^{(A)}}\) of congruent clocks \(L_s\) and its standard length \(s_L\).

### 1.5.3 Spacetime-like concatenation

With every laser ranging ping \(A_1 \approx B \approx A_2\) Alice measures the position of Bob at the moment \(B\) when her signal reflects (see figure 2a). Alice covers the outgoing light ray \(\overline{A_1B}\) by a swarm of light clocks in both space-like and time-like way: She connects a consecutive sequence until ”half-time” moment \(A\) (after waiting half of her laser ranging interval)

\[
\overline{A_1A} \sim t L|A_1 \star t \ldots \star t L|A
\]

in light clock \(L|A\) to an adjacent layout of (ticking) light clocks that reaches to moment \(B\)

\[
\overline{AB} \sim s L|A \star s \ldots s L|B. \tag{5}
\]

The collective motion of light inside the composite of ticking light clocks - symbolized by \(L|A_1 \star t \ldots \star t L|A \star s \ldots s L|B\) - covers the light ray from Alice towards Bob

\[
\overline{A_1B} \sim_{t,s} L \star t \ldots \star t L \star s \ldots s L. \tag{5}
\]
Figure 3: (local) indirect characterization of simultaneous straight measurement paths

Alice utilizes copies of the same light clock $L$ as spatiotemporal units. Along a consecutive segment $L_{\ast t_1} \ldots \ast t_i L$ each represents a time unit $L_t$ and along an adjacent segment $L_{\ast s_1} \ldots \ast s_s L$ a distance unit $L_s$. In both segments she counts the congruent ticks $\sharp \{ L_t \} =: t^{(A)}_{A_1 B}$ and the congruent clocks $\sharp \{ L_s \} =: s^{(A)}_{A_1 B}$. Alice measures the spatiotemporal distance towards Bob

$$
(t, s)_{A_1 B} \equiv (t, s)_{L_{\ast t_1} \ldots \ast t_i L_{\ast s_1} \ldots \ast s_s L} \equiv \left( \frac{t^{(A)}_{A_1 B}}{s^{(A)}_{A_1 B}} \cdot t_L, \frac{s^{(A)}_{A_1 B}}{t^{(A)}_{A_1 B}} \cdot s_L \right)
$$

by her composite layout. It is reproducible from the number of congruent light clocks, the consecutive or adjacent way of their connection and their standard length $s_L$ and duration $t_L$.

In the direct measurement we cover the object or process by a grid of light clocks. Now consider the measurement of a ray of light. In figure 3 Alice covers the smallest segment $A_1 B \sim_{t,s} L_{\ast t} \ast ( L_{\ast s} L)$ with one light clock tick and two adjacent light clocks. Step by step she covers the uniform motion of the outgoing light ray $A_1 B$ by a (locally) regular pattern of light clocks (and the same for the returning light ray $B A_2$). No matter to what extent Alice covers the departing light ray $A_1 B \subset A_1 B$ by similarity any pair of durations scales...
has the proportionality constant
\[ \frac{s_{AB}}{s_{AB'}} = \frac{t_{AB}}{t_{AB'}}. \] (7)

For a generic segment \((t_c, s_c)\) of the uniform light ray we express the proportionality relation \( \frac{s_c}{2s_L} = \frac{t_c}{t_L} \) between pairs of \textit{same} basic observables by a proportionality constant
\[
\left\{ \frac{s_c}{s_L} \right\} \equiv \underbrace{2}_{c(L)} \cdot \left\{ \frac{t_c}{t_L} \right\}. \] (8)

We define the \textit{velocity} of light \( c(L) := \frac{s_c(L)}{t_c(L)} = 2 \) (in standard light clock dimensions \( s_L, t_L \)) as a derived physical quantity. From known ”distance for each time unit“ \( 2 \cdot s_L \) and the ”number of time units“ \( \frac{t_c}{t_L} \) along the way one gets the total distance as a product of velocity \( c(L) \) and time of flight
\[
s_c = 2 \cdot s_L \cdot \frac{t_c}{t_L} =: \left( \frac{2 \cdot s_c}{t_L} \right) \cdot \frac{t_c}{t_L} \equiv c. \] (9)

1.6 Indirect laser ranging

In direct laser ranging \( A_1 \rightsquigarrow B \rightsquigarrow A_2 \) Alice measures the distance \( s_{AB} \) to Bob \( B \) by counting units along the (potentially global) simultaneous measurement path \( \overline{AB} \). From a direct measurement the departing and returning ray of light \( c(L) \) cover in the same duration

\footnote{The division in \( s_1/s_2 \) resp. \( t_1/t_2 \) symbolizes Alice dividing operation. The formulation \( s_{AB}/s_L := n \) means: by connecting \( n \) congruent reference paths \( L \sim \ldots \sim L \sim \overline{AB} \) she will cover the path \( \overline{AB} \).}

\footnote{The formal reduction of fractions, that ”same dimensions (unit length, unit mass etc.) cancel one another”, gives back the relation \( s_c/s_L \) between quantities (ratios) which can all be measured directly by concatenation operations.}

\footnote{For measuring \( c = c(L) \cdot \frac{t_c}{t_L} \) we utilize a light clock with dimensions: width \( s_L \) and cycle length \( t_L \). Basic dimensions (unit length, unit time etc.) are ”arbitrarily chosen constant reference measures“. Contemporary metrology refers to units based on the standard duration \( t_{Cs} \) of an atomic Cesium period and the invariant speed of light \( c \). The atomic second \( \text{sec}_{\text{SI}} := 9192631770 \cdot t_{Cs} \) is a multiple of that standard duration (factor chosen to match traditional calendrical second). The (multiple of the) standard meter \( 299792458 \cdot \text{m}_{\text{SI}} := (c \cdot \text{sec}_{\text{SI}}) \) is the distance of free light in one atomic second of flight. The numerical factor is fixed by convention (to cover \( 1 \text{m}_{\text{SI}} \approx 1 \text{m}_{\text{bar}} \) the traditional platinum-iridium standard in the Bureau of Weights and Measures). One refers to the traditional units (meter bar and fraction of a tropical year) one last time, to match the conversion factors. Now one defines the international unit measures \( \text{sec}_{\text{SI}}, \text{m}_{\text{SI}} \) independently and more precise from invariant natural processes (intrinsic Cs-period and speed of light) and the fixed numerical factors; the old prototypes stay in the museum.

A light clock with SI-unit period \( t_L := t_{Cs} \) and corresponding width \( 2 \cdot s_L = (c \cdot 1\text{sec}_{\text{SI}}) = 299792458 \cdot \text{m}_{\text{SI}} \) has the proportionality constant \( \left\{ \frac{s_m}{m} \right\} \equiv 299792458 \cdot \left\{ \frac{s_c}{c} \right\} \); thus measures speed of light \( c = 299792458 \cdot \frac{m}{\text{sec}} \).

\footnote{Locally regular pattern of light clocks (see figure \ref{fig:lightclocks}) covers laser ranging waiting interval \( \overline{A_1A_2} \) by same number of consecutive (light clock) ticks as there are adjacent (ticking) clocks along laser ranging route \( \overline{AB} \).}
Figure 4: combination of two elementary laser ranging measurements

\[ t_{\overline{AB}} = t_{\overline{BA}} = \frac{1}{2} \cdot t_{\overline{A_1A_2}} \] the proportional distance \( s_{\overline{AB}} \) to \( c \cdot t_{\overline{A_1B}} = \frac{c}{2} \cdot t_{\overline{A_1A_2}} \). Thus (locally) Alice can also indirectly compute that length

\[ (t, s)_{\overline{A_1B}} = \left( \frac{1}{2} \cdot t_{\overline{A_1A_2}}, \frac{c}{2} \cdot t_{\overline{A_1A_2}} \right) \] (10)

from measuring the round trip time \( t_{\overline{A_1A_2}} \); the familiar principle of indirect laser ranging.

For the resulting equation Alice must obey a measurement condition (underlying the direct measurement of light rays in figure 3), that during the radar waiting interval \( \overline{A_1A_2} \) her motion is preserved. After emitting the light pulse \( \overline{A_1B} \) she neither accelerates away \( \overline{A_2'} \) nor towards \( \overline{A_2''} \) the returning light pulse \( \overline{BA_1} \). In local laser ranging practice accelerations are negligible; for larger configurations the effects accumulate. Then Alice can characterize all elements of her simultaneity line \( A_n \in \overline{AB} \) by moments \( A', A'' \in \overline{A_1A_2} \) along the waiting interval (see figure 3). In local laser ranging \( A' \sim A_n \sim A'' \) the preceding emission and subsequent reception are symmetric \( t_{\overline{A'A}} = t_{\overline{A''A}} \) with respect to Alice moment \( A \).

With two elementary laser rangings \( A_1 \sim B \sim A_2 \) and \( A_1 \sim \tilde{B} \sim A_2 \) towards the two consecutive moments \( B \) and \( \tilde{B} \) (see figure 4). Alice measures the relative motion of Bob \( \overline{BB} \)\(^{11} \)

Her indirect laser ranging involves three steps:

1. construct her straight simultaneous measurement paths towards Bob

\(^{11}\)In a direct measurement her layout of light clocks \( \mathcal{L}_{\overline{B} \sim \overline{A}} \) covers a segment of his motion.
2. *enclose* the measurement object $\overline{BB}$ by her simultaneity lines $\overline{AB}$ and $\overline{BA}$

$$\overline{BB} \sim_{t,s} \overline{BA} * \overline{AA} * \overline{AB}$$  \hspace{1cm} (11)

3. *project* between and along the simultaneity lines for temporal and spatial components

$$\begin{align*}
(t, s)_{\overline{BB}} &= (t, s)_{\overline{BA} * \overline{AA} * \overline{AB}} \\
&= (t, s)_{\overline{BA}} + (t, s)_{\overline{AA}} + (t, s)_{\overline{AB}} \\
&= (t_{\overline{A}}, s_{\overline{AB}} - s_{\overline{AB}})
\end{align*}$$  \hspace{1cm} (12)

The vectorial addition of components $(t, s)_{\overline{BA}} + (t, s)_{\overline{AB}} = (t, s)_{\overline{AB}}$ corresponds with direct measurement operations. In the underlying material model we concatenate a number of light clocks $\mathcal{L} \ast_t \ldots \ast_t \mathcal{L}$ and $\mathcal{L} \ast_s \ldots \ast_s \mathcal{L}$ to create a composite layout $\mathcal{L} \ast_t \ldots \ast_t \mathcal{L} \ast_s \ldots \ast_s \mathcal{L}$.

Let *Alice* and *Bob* coincide (without loss of generality) in the initial moment $\mathcal{P}$. Now the first laser ranging configuration becomes trivial and we are left with $\mathcal{P} \rightarrow \mathcal{A}_1 \sim \mathcal{B} \sim \mathcal{A}_2$ (see figure 5). *Alice* measures the spatiotemporal interval of *Bob’s motion*

$$(t, s)_{\mathcal{P}O} = \underbrace{(t_{\mathcal{P}A_1} + \frac{1}{2} \cdot t_{\overline{A_1}} , \frac{c}{2} \cdot t_{\overline{A_1}})}_{(12,10)}.$$  \hspace{1cm} (13)

2 Lorentz transformation

We have defined the termini of *Alice* laser ranging measurements towards *Otto* $\mathcal{P} \rightarrow \mathcal{A}_1 \sim \mathcal{O} \sim \mathcal{A}_3$. Let another observer *Bob* measure the same segment $\mathcal{P} \mathcal{O}$ of *Otto’s motion* (see figure 5). *Bob* conducts laser ranging $\mathcal{P} \rightarrow \mathcal{B}_1 \sim \mathcal{O} \sim \mathcal{B}_2$ in the same way as *Alice*. Following protophysical principles he manufactures his own light clock $\mathcal{L}^{(B)}$ and uses it in a standardized way. Step by step *Bob* develops analogous measurement termini \(\{1\}\).

*Bob* constructs his (dotted) simultaneity lines towards *Otto* $\mathcal{P} \mathcal{O}$ (or back to *Alice* $\mathcal{B} \mathcal{A}$). Directly, by adjacent connection of comoving light clocks $\mathcal{L}^{(B)} \ast_t \ldots \ast_t \mathcal{L}$ or $\mathcal{L}^{(B)} \ast_s \ldots \ast_s \mathcal{L}$, or indirectly, from round trip signaling times. Though, the *same measurement principle* and intrinsic operations (independent propagation of light and intrinsic construction and connection of their respective light clocks) lead not to the same results. *Alice* constructs simultaneity lines $\overline{AB}, \overline{AO}$ with different orientation than *Bob’s simultaneity lines $\overline{BA}, \overline{BO}$ (see figure 3).

Next *Bob* encloses measurement object *Otto* $\mathcal{P} \mathcal{O}$ in between his simultaneity lines $\overline{BO}$

$$\mathcal{P} \mathcal{O} \sim_{t,s} \overline{BB} * \overline{BO}$$

and projects *Otto’s relative motion onto the spatial and temporal components

$$\begin{align*}
(t, s)_{\mathcal{P}O} &= (t, s)_{\mathcal{P}B} * \overline{BB} * \overline{BO} \\
&= (t_{\mathcal{P}B}, s_{\overline{BO}}) \quad \hspace{1cm} (14)
\end{align*}$$
The intrinsic procedure is the same. By consecutive and adjacent connection of her light clocks $L^{(A)}$ Alice covers same segment of Ottó’s motion as Bob with his light clocks $L^{(B)}$

$$\overline{PO} \sim_{t,s} \left( t^{(A)}_{\overline{PO}} , s^{(A)}_{\overline{PO}} \right) \cdot L^{(A)}$$

$$\sim_{t,s} \left( t^{(B)}_{\overline{PO}}, s^{(B)}_{\overline{PO}} \right) \cdot L^{(B)} .$$

Let both also measure the same segment of Bob’s and of Alice’ motion

$$\overline{PB}_1 \sim_{t,s} \left( t^{(A)}_{\overline{PB}_1}, s^{(A)}_{\overline{PB}_1} \right) \cdot L^{(A)}$$

$$\sim_{t,s} \left( t^{(B)}_{\overline{PB}_1}, 0 \right) \cdot L^{(B)}$$

$$\overline{PA}_1 \sim_{t,s} \left( t^{(A)}_{\overline{PA}_1}, 0 \right) \cdot L^{(A)}$$

$$\sim_{t,s} \left( t^{(B)}_{\overline{PA}_1}, s^{(B)}_{\overline{PA}_1} \right) \cdot L^{(B)} .$$

The coinciding light rays in their laser ranging processes are depicted in figure 5.

From the interrelation of their physical prerequisites and the same measurement principle we derive the transformation

$$\begin{aligned}
\overline{PO} & \leftrightarrow \left( t^{(A)}_{\overline{PO}}, s^{(A)}_{\overline{PO}} \right) \leftrightarrow \left( t^{(B)}_{\overline{PO}}, s^{(B)}_{\overline{PO}} \right) \leftrightarrow \left( t^{(B)}_{\overline{PB}_1}, s^{(B)}_{\overline{PB}_1} \right)
\end{aligned}$$

between Alice and Bob’s measured values of the same measurement object: Ottó’s motion $\overline{PO}$. Provided measurements of their own motion $\overline{PA}_1$, $\overline{PB}_1$ it follows by successive substitution in three steps:

\[ I \left( t^{(B)}_{\overline{PO}} , s^{(B)}_{\overline{PO}} \right) \quad (M, N) \]

\[ II \quad (M, N) \quad (A, B, C) \]

\[ III \quad (A, B, C) \quad \left( \left( t^{(A)}_{\overline{PO}}, s^{(A)}_{\overline{PO}} \right) \& \left( t^{(A)}_{\overline{PB}_1}, s^{(A)}_{\overline{PB}_1} \right) \right) \]

where we use the following abbreviations for indirect laser ranging measurements by Alice

$$ A := t^{(A)}_{\overline{PA}_1} \quad B := \frac{1}{2} \cdot t^{(A)}_{\overline{A}_1\overline{A}_2} \quad C := \frac{1}{2} \cdot t^{(A)}_{\overline{A}_1\overline{A}_3}$$

and by Bob

$$ M := t^{(B)}_{\overline{PB}_1} \quad N := \frac{1}{2} \cdot t^{(B)}_{\overline{B}_1\overline{B}_2} .$$

In step I we express Bob’s indirectly determined values of Ottó’s motion $(t, s)^{(B)}_{\overline{PO}}$ in terms of Bob’s direct measurements of round-trip signaling durations $M, N$

$$ t^{(B)}_{\overline{PO}} \equiv M + N \quad (15)$$

$$ s^{(B)}_{\overline{PO}} \equiv c \cdot N . \quad (16)$$
Figure 5: coinciding light rays in intrinsic measurements by Alice and Bob
In step II we express Bob’s measured durations $M$, $N$ in terms of Alice’ duration measurements $A$, $B$, $C$. In order to substitute the two physical quantities $M$, $N$ in terms of physical quantities $A$, $B$, $C$ we need two relations between corresponding measurements.

Alice’ and Bob’s laser ranging processes overlap in figure 5. The outgoing light rays $\overline{A_1B_1}$ and $\overline{A_1O}$ partially coincide and the reflected light rays $\overline{B_1A_2}$ and $\overline{B_2A_3}$ are parallel \{13\}. From similar triangles $\triangle \overline{PB_1A_2}$ and $\triangle \overline{PB_2A_3}$ (all sides are pairwise parallel) we get one relation

$$\frac{M}{A + B + B} = \frac{M + N + N}{A + C + C}.$$ \hspace{1cm} (17)

For a second relation we analyze the two triangles $\triangle \overline{PA_1B_1}$ and $\triangle \overline{PB_1A_2}$. We can regard each as ”degenerate trapezoid”, as a calibration procedure by means of which Alice and Bob can compare their light clocks $L^{(A)}$ and $L^{(B)}$:

- In $\triangle \overline{PA_1B_1}$ Alice sends out two light signals along $\overline{PA_1} = A \cdot \mathcal{L}^{(A)}$ - the first at moment $P$ and the second at moment $A_1$ after $\frac{A}{2}$ ticks of her light clock - which Bob receives along $\overline{PB_1} = M \cdot \mathcal{L}^{(B)}$ - at moments $P$ and $B_1$ after $\frac{M}{2}$ ticks of his light clock.

- In $\triangle \overline{PB_1A_2}$ Bob sends out two light signals along $\overline{PB_1} = M \cdot \mathcal{L}^{(B)}$ - at moments $P$ and $B_1$ after $\frac{M}{2}$ ticks of his light clock - which Alice receives along $\overline{PA_2} = (A + B + B) \cdot \mathcal{L}^{(A)}$ - the first in $P$ and the second in $A_2$ after $\frac{(A + B + B)}{2}$ ticks of her light clock.

If Alice and Bob use identically constituted reference devices (light clocks made from same material) then both encounter the same dilation effect for each others relative motion. According to the relativity principle both configurations are intrinsically similar. Alice and Bob have no way to specify absolute motion. By means of intrinsic measurements both determine the same ratio between the two durations for receiving both signals (heard from the other) and the duration of the sending interval (measured by themselves)

$$\frac{M}{A} = \frac{A + B + B}{M}.$$ \hspace{1cm} (18)

In step III we express Alice laser ranging durations $A$, $B$, $C$ in terms of Alice indirect determined values (13) for Otto’s motion $(t, s_{\overline{PO}}^{(A)})$ and for Bob’s motion $(t, s_{\overline{PO}}^{(B)})$

$$A = t_{\overline{PO}}^{(A)} - \frac{1}{c} \cdot s_{\overline{PO}}^{(A)}$$ \hspace{1cm} (19)

$$B = t_{\overline{PO}}^{(B)} - \frac{1}{c} \cdot s_{\overline{PO}}^{(A)}$$ \hspace{1cm} (20)

$$C = \frac{1}{c} \cdot s_{\overline{PO}}^{(A)}$$ \hspace{1cm} (21)

$$C = \frac{1}{c} \cdot s_{\overline{PO}}^{(B)}$$ \hspace{1cm} (22)

After successive insertion of these three steps (see appendix A) we can express Bob’s physical quantities of Otto’s motion $(t, s_{\overline{PO}}^{(B)})$ in terms of Alice measurements of Otto’s motion...
where Alice determines Bob’s relative velocity from \( v_B := \frac{s_B^{(A)}}{t_B^{(A)}} \).

We derive the Lorentz transformation \( \Lambda_{AB} : (t, s)^{(A)}_{\mathcal{P}_A} \mapsto (t, s)^{(B)}_{\mathcal{P}_B} \) from Alice and Bob’s intrinsic construction of physical quantities of Otto’s motion (step I and III) and from their overlapping laser ranging operations (step II) (see figure 6). While the formal approach assumes vectors, Lorentz symmetry for its description; we form the mathematical framework. From simple measurement-methodical principles - without mathematical presuppositions - we generate physical quantities \((t^{(A)}_{\mathcal{P}_A}, s^{(A)}_{\mathcal{P}_A})\). They specify the layout and number of building blocks in the material model \( \mathcal{L} \ast \ldots \ast \mathcal{L} \) which Alice assembles to cover the ”duration” and ”length” of measurement object \( \mathcal{P}_A \). From the interrelation of the underlying practical operations we derive the ”local Lorentz symmetry”.

From classical measurement practice we get Galilei kinematics. Einstein analyzed mutual measurements of moving objects and recognized the need to establish a physical connection.
between clocks at different locations and speeds. Intrinsic operations with light clocks represent the classical metric locally (Euclidean geometry). For their connection Einstein chose the universal motion of light. By including the (local) light principle and the relativity principle we derive Poincare kinematics. Our locally regular composable grid of light clocks can potentially grow into every direction. It is our metric connection between distant measurements. Then the formerly isolated and local notions of the classical metric (absolute time, space, local flatness etc.) will reveal new intricate interrelations.

3 Twin paradox

In the Twin configuration Alice and Bob explore their mutual time dilation in a round trip experiment. They depart at moment \( P \). While Alice remains at rest Bob rides with uniform motion \( v^A_B \) to a distant turning point \( U \) and returns with same velocity \(-v^A_B\) to reunite with Alice in future moment \( R \) (see figure 7).

Throughout the whole round trip Alice can observe Bob. Vice versa Bob will receive all light signals which Alice sends from departure until return. The time during which Alice measures Bob’s journey \( t^A_B \equiv t^A_A \) coincides with her (proper) waiting time; then Bob spends less time on tour
\[
t^B_B = \sqrt{1 - \frac{v^2_B}{c^2}} \cdot t^A_A < 1\cdot t^A_A
\]
than Alice waiting and watching (the previous measurement object \( PO \equiv B \) coincides with Bob’s ride \( (t, s)^B_B = \sqrt{1 - \frac{v^2_B}{c^2}} \cdot t^A_A, 0 \)). She observes her own clock ticks faster than the moving clock of Bob and vice versa by the symmetry of their relative motion \( v^A_A = -v^A_B \). During his trip Bob can see all of Alice; if we assume his observation time \( t^B_B \)
\[
t^B_B \triangleq t^A_A
\]
(24) is the time to measure all of Alice \( t^B_B \), then she spends less time waiting
\[
t^A_A = \sqrt{1 - \frac{v^2_B}{c^2}} \cdot t^B_B
\]
\(< 1\) than the ride takes for Bob himself (using the reverse Lorentz transformation of Alice waiting interval \( (t, s)^A_A = \sqrt{1 - \frac{v^2_B}{c^2}} \cdot t^B_B, 0 \)). The combined conclusion is a contradiction; the so called Twin paradox. The assumption (24) was incorrect; Bob’s observation period \( t^B_B \neq t^A_A \) is not the time for measuring Alice. The Lorentz transformation (23) between physical measures does not refer to durations of observations; it refers to durations of their measurements. For a physically meaningful calculation we remember that basic physical quantities \((s, t)\) originate from tangible operations. The implicit conditions (for constructing the underlying grid of light clock units) must be fulfilled before one can apply the Lorentz transformation between results of supposed measurements. Our completed view prevents unreflective calculations in
Figure 7: connected laser rangings in the Twin configuration of Alice and Bob$_{1,2}$
the Lorentz formalism and provides an explanation of the apparent Twin paradox from a measurement-methodical perspective.

Alice can observe and measure Bob throughout the whole trip, unlike Bob. He can receive all light signals from Alice. Though the middle segment of Alice motion $\overline{A_1A_2}$ is 
observable but not measurable for him. Bob’s light clocks $\mathcal{L}^{(B_1)}$ and $\mathcal{L}^{(B_2)}$ function properly on the way out and back. Though on each leg Bob cannot cover measurement objects beyond the $U$-turn point. Bob’s last indirect laser ranging $B_1 \sim A_1 \sim B_1$ reaches Alice at moment $A_1$ so the reflection (to analyze round trip times) returns before he changes his motion at moment $U$ (violating measurement condition (1.6)). His direct laser ranging by consecutive and adjacent connection of his light clocks covers Alice up to moment $A_1$. Later Bob$_2$ assembles his light clocks $\mathcal{L}^{(B_2)}$ next to one another forming simultaneity lines with respect to his new state of motion. His direct and indirect laser ranging measurements cover Alice from moment $A_2$ resp. $A_2$ up to the point of Return. Bob$_1$ and Bob$_2$ cannot measure segment $\overline{A_1A_2}$ of Alice motion by intrinsic use of their light clock units.

During his entire round trip Bob measures two segments $\overline{PA_1}$ and $\overline{A_2R}$ of Alice motion. He sees all processes for moving Alice run slower by the same factor $t_A^{(A)} = \sqrt{1 - \frac{v^2}{c^2}} \cdot t_A^{(B)}$ as in Alice reverse perspective. Though Bob$_1$ covers a shorter segment of Alice relative motion

$$\overline{PA_1} \sim_{t,s} \left( t_{B_1}^{(B_1)} \cdot v_A \cdot t_{B_1}^{(B_1)} \right) \cdot \mathcal{L}^{(B_1)}$$

with proper duration $t_A^{(A)}$.

$$\overline{PA_1} \equiv \sqrt{1 - \frac{v^2}{c^2}} \cdot t_{B_1}^{(B_1)} \equiv \left( 1 - \frac{v^2}{c^2} \right) \cdot t_A^{(A)}$$. Similarly during his return Bob$_2$ measures a segment of Alice with same proper duration $t_A^{(A)}$. With regard to measurability by Bob the waiting process of Alice $\overline{PR}$ splits into measurable segments $\overline{PA_1}$, $\overline{A_2R}$ and non-measurable segment $\overline{A_1A_2}$. Her waiting time divides accordingly

\[
t_A = \frac{t_{PA_1}}{\overline{PA_1}} + \frac{t_{A_2R}}{\overline{A_2R}} + \frac{t_{A_1A_2}}{\overline{A_1A_2}}
\]

\[
t_A = \left( 1 - \frac{v^2}{c^2} \right) \cdot t_A + \frac{v^2}{c^2} \cdot t_A.
\]

In the limit where Bob approaches speed of light $v_B \rightarrow c$ his total trip duration vanishes

$$t_B + t_B \equiv \sqrt{1 - \frac{v^2}{c^2}} \cdot t_A \rightarrow 0$$

while during (observable but) unmeasurable part of her waiting process Alice grows older

$$t_{A_1A_2} = \frac{v^2}{c^2} \cdot t_A \rightarrow t_A.$$

While the formal explanation of the Twin paradox assumes four-vectors $x^\mu$, Minkowski metric $\eta_{\mu\nu}$ and "integrates proper time along a curved worldline" our measurement-methodical foundation reveals the physical reason and it derives the rules of the calculus as well.
4 Mathematical formulation of physical operations

We present a novel approach to the foundation of the physical theory, which begins with questions on measurement practice. We are long familiar with basic measurements, as in the case of "known, very old procedure of length measurements by repeated placement of unit sticks one after the other" [8]. Wallot [7] defines "physical measures are never tangible things, but always attributes of things, properties, which we can notice on the things of our experience". Helmholtz regards attributes of objects, which in a comparison allow the difference of larger, equal or smaller. We want to express their value also numerically (how many times more). The procedure for finding these values is the measurement.

Helmholtz [1] starts from counting same objects. He begins with fundamental questions about two aspects of basic measurement operations:

1. "What is the physical meaning if we declare two objects as equal in a certain relation?"

2. "Which character must the physical concatenation of two objects have, that we may consider comparable attributes thereof as connected additively?"

In this way Helmholtz elucidates familiar examples like the weight \( m_\Omega \) of a material \( \Omega \) object, the length \( s_{\overline{AB}} \) of a straight line \( \overline{AB} \), the duration \( t_P \) of a physical \( P \)rocess etc. Thereby we notice, that the way of concatenation generally depends on the kind of measure. "We add e.g. weights, by simply placing them on the same weighing-pan. We add time periods, by letting the second begin at exactly the moment, where the first stops; we add lengths, by placing them next to each other in a certain way, namely in a straight line etc." A basic measurement therefore requires knowledge of the method of comparison (of a particular attribute of both bodies) "\(~_m\)" and of the method of their physical concatenation "\(*_m\)".

We have applied Helmholtz program of direct measurements to relativistic kinematics. We can compare the spatiotemporal order of two objects by the classical probe, whether one of them covers the other. Without one word of mathematics one can manufacture identically constituted light clocks \( \mathcal{L} \) and place them literally side by side or one after the other. Alice concatenates these measurement units by a physical process, by letting their inner light rays overlap. Her measurements are based on light principle, classical construction of light clocks \( \mathcal{L} \) and their direct and indirect connection by the independent motion of light.

Mathematics comes into being at the moment we introduce units and count, how many congruent building blocks it takes for assembling a regular grid \( \mathcal{L} \ast \cdots \ast \mathcal{L} \sim_s \mathcal{O} \) which covers the measurement object. If both are interchangeable in the comparison they have same length (up to a non-vanishing but practically admissible measurement error)

\[
s_\mathcal{O} = s_\mathcal{L} \ast \cdots \ast s_\mathcal{L} + \Delta s .
\]

\[\text{We define derived quantities by equations. Helmholtz calls them more accurately coefficients. "Basic quantities cannot be deduced by equations onto other already explained quantities."}\]
Remark 1 Helmholtz way of basic measurement involves a pair comparison. Measurement object and material model are natural objects. Alice covers the spatiotemporal interval of e.g. relative motion of Bob by a regular grid of ticking light clocks. It is built of solely congruent building blocks $\mathcal{L}$ and it is (locally) invariant under permuting their order; by counting them Alice finds "how many times" further and longer Bob’s relative motion spreads than the (universal) light in her reference device.

Alice builds wide layouts of (ticking) light clocks $\mathcal{L} \ast_s \ldots \ast_s \mathcal{L}$ and enduring sequences of (light clock) ticks $\mathcal{L} \ast_t \ldots \ast_t \mathcal{L}$. All steps of her procedure (assembling the material model and conducting length comparison $\sim_s$) are reproducible by any other physicist – the practical purpose of standardizing measurements (of the magnitude of durations and lengths). In starting figure 1 we illustrate only observed objects and observers in motion. Provided the construction of light clocks (measurement unit $\mathcal{L}$) and their connection in consecutive and adjacent ways (concatenation $\ast_t, \ast_s$) we introduce increasingly complex material models which where not yet assembled in the uncultivated beginning. As a colloquial expression we introduce measurement termini. Along figures 2, 3, 4 we define physical notions based on measurement operations with light clocks (simultaneous straight measurement path, spatial and temporal projection etc.). Step by step we introduce operational denominations which specify aspects of Alice measurement practice precisely.

From the underlying operational definitions their interrelation becomes transparent. Their common origin inherits a genetic interrelation between measurement termini and the corresponding terms in the mathematical formulation. Based on measurement-methodical principles of their formation we can avoid apparent paradoxes in blind calculations. In retrospect of practical operations with light clocks we emphasize, that all our assertions on the one-way propagation of light strictly come from closed two-way cycles. In laser ranging configuration $A_1 \leadsto B \leadsto A_2$ Alice has no way to measure whether the departing light ray $A_1B$ towards Bob takes more time than the returning ray $BA_2$. Similarly we cannot figure what happens inside the measurement unit $\mathcal{L}: \mathcal{L}_I \leadsto \mathcal{L}_{II} \leadsto \mathcal{L}_I \ldots$ (when light travels between both mirrors to the right $\mathcal{L}_I \leadsto \mathcal{L}_{II}$ vs. to the left $\mathcal{L}_{II} \leadsto \mathcal{L}_I$).

Direct and indirect laser ranging with light clocks $\mathcal{L}$ always involves both, outgoing and returning pulse. In practice we solely deal with two-way light cycles: (i) inside individual light clocks $\mathcal{L}$ and (ii) in suitably connected configurations of light clocks. Our measurement unit $\mathcal{L}$ comprises both dimensions, width $s_\mathcal{L}$ and duration $t_\mathcal{L}$ of an elementary two-way light cycle. For the measurement we generate complex configurations of two-way light cycles in suitable layouts of ticking light clocks $\mathcal{L} \ast_t \ldots \ast_t \mathcal{L} \ast_s \ldots \ast_s \mathcal{L}$. Thus Alice covers one-way light rays $A_1B$ and $BA_2$ by a layout of two-way light cycles (in light clock grid figure 3). By counting ticks along her waiting interval $A_1A_2$ and the light clocks sitting side by side to cover all of laser ranging path $A_1A_2$ Alice measures the magnitude of their length and duration

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13Her technique is preserved until in empirical practice one oversteps unforeseen conditions, physically specifies them further and thus evolves - in a continual historic process - measurement practice and its (physically determined) mathematical formulation. – Remembering Feynman’s motto: *Yesterdays sensation is todays calibration and tomorrows background.*

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Remark 2 The meaning of basic physical quantities arises - not by chopping measurement units \( L \) into pieces but instead - by concatenating many congruent measurement units \( L \) (each taken as inseparable unity) to construct material models \( L \ast \cdots \ast L \).

We introduce all arithmetic operations between measures "+", "−", "\( \frac{1}{2} \)" etc. via underlying connection of congruent light clocks. Basic physical quantities specify a reproducible layout of reference devices which covers the measurement objects sufficiently precise.

Our objective is a definition of basic observables from physical operations (what one does in measurement practice). In absence of interactions we have developed Helmholtz method for basic measurements of relativistic motion. In this approach, which derives the mathematical formalism of kinematics from this operationalization of length and duration, one can address scope and limitations of the formalism. It can be taken as a basis for our next step, basic measurements of interactions. Next we develop Helmholtz method for the foundation of classical [15] and relativistic dynamics [16].

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Appendix A: Successive substitution

In step I of our series of substitutions we express the space and time component of Bob’s physical measure \( \left( t^{(B)}_{PO}, s^{(B)}_{PO} \right) \) in terms of his laser ranging duration measurements \( M, N \)

\[
t^{(B)}_{PO} = M + N \quad \text{(15)}
\]

\[
s^{(B)}_{PO} = c \cdot N \quad \text{(16)}
\]

In step II we substitute Bob’s laser ranging durations \( M, N \) - due to the interrelation of their measurement conditions - with Alice laser ranging durations \( A, B, C \)

\[
M = \sqrt{A \cdot (A + B + B)} \quad \text{(17)}
\]

\[
N = \frac{1}{2} \cdot \sqrt{A \cdot (A + B + B)} \cdot \frac{A + C + C}{A + B + B} - \frac{1}{2} \cdot \sqrt{A \cdot (A + B + B)} \quad \text{(18)}
\]

In step III finally we reformulate Alice laser ranging durations \( A, B, C \) in terms of the space and time components of Alice’s physical measures \( \left( t^{(A)}_{PO}, s^{(A)}_{PO} \right) \) and \( \left( t^{(A)}_{PB}, s^{(A)}_{PB} \right) \). We
successively insert all substitutions for the space and time component separately

\[ s^{(B)}_{\mathcal{PO}} = c \cdot N \]

\[ = c \cdot \left[ \frac{1}{2} \cdot \sqrt{A \cdot (A + B + B)} \cdot \frac{A + C + C}{A + B + B} - \frac{1}{2} \cdot \sqrt{A \cdot (A + B + B)} \cdot \frac{\sqrt{A + B + B}}{1} \right] \]

\[ = c \cdot \frac{1}{2} \cdot \frac{\sqrt{A}}{\sqrt{A + B + B}} \cdot (A + C + C) - c \cdot \frac{1}{2} \cdot \frac{\sqrt{A}}{\sqrt{A + B + B}} \cdot (A + B + B) \]

\[ = c \cdot \frac{\sqrt{A}}{\sqrt{A + B + B}} \cdot (C - B) \]

\[ = \frac{1}{\sqrt{A \cdot (A + B + B)}} \cdot \left( -c \cdot A \cdot B + c \cdot A \cdot C \right) \]

\[ = \frac{1}{\sqrt{A \cdot (A + B + B)}} \cdot \left( -c \cdot A \cdot B \cdot C + c \cdot B \cdot C + c \cdot A \cdot C \right) \]

\[ = \frac{1}{\sqrt{A \cdot (A + B + B)}} \cdot \left( -c \cdot B \cdot (A + C) + c \cdot C \cdot (A + B) \right) \]

\[ \begin{align*}
\text{10) } t_{PB} & = \frac{1}{\sqrt{\left( \frac{t(A)_{PB}}{T_{PB}} - \frac{1}{c} \cdot s_{\mathcal{PO}}(A)_{PB} \right) \left( \frac{t(A)_{PB}}{T_{PB}} + \frac{1}{c} \cdot s_{\mathcal{PO}}(A)_{PB} \right) \cdot \left( -s_{\mathcal{PO}}(A)_{PB} \cdot i_{\mathcal{PO}}(A)_{PB} + s_{\mathcal{PO}}(A)_{PB} \cdot i_{\mathcal{PO}}(A)_{PB} \right) } \\
& = \frac{t_{PB}}{\sqrt{t_{PB}^2 - \frac{1}{c^2} \cdot s_{PB}^2}} \cdot \left( -s_{PB} \cdot t_{\mathcal{PO}} + s_{\mathcal{PO}} \right) \\
& = \frac{1}{\sqrt{t_{PB}^2 - \frac{1}{c^2} \cdot s_{PB}^2}} \cdot \left( -v_B \cdot t_{\mathcal{PO}} + s_{\mathcal{PO}} \right) \\
\end{align*} \]

\[ s^{(B)}_{\mathcal{PO}} = -\frac{1}{\sqrt{1 - \frac{v_B^2}{c^2}}} \cdot v_B \cdot i_{\mathcal{PO}}(A)_{PB} + \frac{1}{\sqrt{1 - \frac{v_B^2}{c^2}}} \cdot s_{\mathcal{PO}}(A)_{PB} \]  \( (27) \)
where Alice has determined the velocity of the relative motion of Bob \( v_B := \frac{t_A^{(A)}}{t_B^{(A)}} \) and with notation simplified in last steps on Alice right hand side by suppressing her indices \( ^{(A)} \).

\[
\begin{align*}
t_B^{(B)} \underline{T_5} & \equiv N + M \\
\underline{25} & \equiv \frac{1}{2} \cdot \sqrt{A \cdot (A + B + B)} \cdot \frac{A + C + C}{A + B + B} + \frac{1}{2} \cdot \sqrt{A \cdot (A + B + B)} \cdot \frac{\sqrt{A + B + B}}{1} \\
= & \frac{1}{2} \cdot \frac{\sqrt{A}}{\sqrt{A + B + B}} \cdot (A + C + C) + \frac{1}{2} \cdot \frac{\sqrt{A}}{\sqrt{A + B + B}} \cdot (A + B + B) \\
= & \frac{\sqrt{A}}{\sqrt{A + B + B}} \cdot (A + B + C) \\
\underline{19} & \equiv \frac{1}{\sqrt{A \cdot (A + B + B)}} \cdot \left( A \cdot A + A \cdot B + A \cdot C + B \cdot C - B \cdot C \right) = 0 \\
= & \frac{1}{\sqrt{A \cdot (A + B + B)}} \cdot (A + C) \cdot (A + B) - B \cdot C \\
\underline{22} & \equiv \frac{1}{\sqrt{t_B^{(A)} - \frac{1}{c^2} \cdot s_B^{(A)}}} \cdot \frac{t_A^{(A)}}{t_B^{(A)}} + \frac{1}{c} \cdot s_B^{(A)} \cdot \frac{1}{c} \cdot s^{(A)}_{PO} \\
= & \frac{t_B^{(A)}}{t_B^{(A)} - \frac{1}{c^2} \cdot s_B^{(A)}} \cdot \left( t_{PO} - \frac{1}{c^2} \cdot s_{PO} \cdot s_{PO} \right) \\
t_B^{(B)} \underline{T_8} & \equiv \frac{1}{\sqrt{1 - \frac{v_B^2}{c^2}}} \cdot \frac{t_A^{(A)}}{t_B^{(A)}} - \frac{1}{\sqrt{1 - \frac{v_B^2}{c^2}}} \cdot \frac{v_B}{c^2} \cdot s^{(A)}_{PO} \tag{28}
\end{align*}
\]

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