SOME RESULTS ON MAPS THAT FACTOR THROUGH A TREE

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Abstract. We give a necessary and sufficient condition for a map defined on a simply-connected quasiconvex metric space to factor through a tree. In case the target is the Euclidean plane and the map is Hölder continuous with exponent bigger than 1/2, such maps can be characterized by the vanishing of some integrals over the winding number function. This in particular shows that if the target is the Heisenberg group equipped with the Carnot-Carathéodory metric and the Hölder exponent of the map is bigger than 2/3, the map factors through a tree.

1. Introduction

Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces and \(\varphi : X \to Y\) a continuous map. Depending on some conditions on \((X, d_X)\) we want to characterize those maps \(\varphi\) that factor through a tree. In these notes a tree is a metric space that is uniquely arcwise connected. We say that \(\varphi\) has Property (T) if:

\[
\text{T} \quad \begin{cases} 
\text{For all } x, x' \in X \text{ with } \varphi(x) \neq \varphi(x') \text{ there is a point } y \in Y \\
\text{such that for any curve } \gamma : [0, 1] \to X \text{ connecting } x \text{ and } x', \\
y \text{ is contained in } \varphi \circ \gamma((0, 1)).
\end{cases}
\]

Since a tree is uniquely arcwise connected this property of \(\varphi\) is necessary in order for it to factor through a tree. Depending on some conditions on \(X\), it is also sufficient. To see that this doesn’t work for any \(X\), consider for example the unit circle in the complex plane and the map \(x \mapsto x^2\). This map has Property (T) but it doesn’t factor through a tree. If we additionally assume that the domain is simply-connected, this implication does hold.

Theorem 1.1. Assume that \(X\) is a \(C\)-quasiconvex metric space with \(H_1(X) = 0\) or \(H_1^{Lip}(X) = 0\). Let \(\varphi : X \to Y\) be a map that is \(\sigma\)-continuous and has Property (T). Then there is a tree \((T, \tilde{D})\) and maps \(\psi : X \to T, \tilde{\varphi} : T \to Y\) with \(\varphi = \tilde{\varphi} \circ \psi\) and for all \(x, x' \in X\),

\[
d_Y(\varphi(x), \varphi(x')) \leq \tilde{D}(\psi(x), \psi(x')) \leq \sigma(Cd_X(x, x')).
\]

Further, for any \(p \in T\) there is a contraction \(\pi_p : T \times \mathbb{R}_{\geq 0} \to T\) such that \(\pi(q, t)\) is contained in the arc from \(p\) to \(q\), \(\pi_p(q, 0) = p, \pi_p(q, t) = q\) for \(t \geq V_\sigma([p, q])\) and

\[
\tilde{D}(\pi_p(q, t), \pi_p(q', t')) \leq \tilde{D}(q, q') + \sigma(|t - t'|).
\]

A similar result has been obtained by Wenger and Young for Lipschitz maps in case \(Y\) is purely 2-unrectifiable [12, Theorem 5]. Our construction of the tree is similar to theirs but circumvents any use of geometric measure theory arguments. The tree in the theorem above is a quotient of \(X\). In particular if \(X\) is compact, then

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$T$ consists of the connected components of preimages of points in $Y$, see Lemma 3.2.

In case $Y$ is the Euclidean plane and $\varphi$ is Hölder continuous, we have the following characterization of Property (T) in terms of winding numbers.

**Theorem 1.2.** Let $X$ be a quasiconvex metric space with $\pi^1_{\text{Lip}}(X) = 0$ and $\varphi : X \to \mathbb{R}^2$ be a Hölder continuous map of order $\alpha$. If $\alpha > \frac{2}{3}$, then $\varphi$ has Property (T) if and only if for all closed Lipschitz curves $\gamma : S^1 \to X$,

$$\int_{\mathbb{R}^2} w_{\varphi \circ \gamma}(q) \, dq = 0.$$ 

If $\alpha > \frac{1}{2}$, then $\varphi$ has Property (T) if and only if for all closed Lipschitz curves $\gamma : S^1 \to X$,

$$\int_{\mathbb{R}^2} w_{\varphi \circ \gamma}(q) \, dq = \int_{\mathbb{R}^2} q_x w_{\varphi \circ \gamma}(q) \, dq = \int_{\mathbb{R}^2} q_y w_{\varphi \circ \gamma}(q) \, dq = 0,$$

where $q = (q_x, q_y)$.

It was shown in [14, Proposition 4.6] that for a closed curve $\mu : S^1 \to \mathbb{R}^2$ of Hölder regularity $\alpha > \frac{1}{2}$, the winding number function $q \mapsto w_\mu(q)$ is integrable and hence the integrals in the theorem above are well defined. Further, $\int w_\mu^+$ and $\int w_\mu^-$ are independent of the coordinate system in $\mathbb{R}^2$ in which we evaluate it because for an isometry $A$ of $\mathbb{R}^2$ it is $w_{A \mu}(Aq) = w_\mu(q)$. If we assume that $\int w_\mu(q) \, dq = 0$, then the vector $\int q w_\mu(q) \, dq$ has the geometric interpretation as $(c(w_\mu^+) - c(w_\mu^-)) \int w_\mu^+ \, dq$, where $c(w_\mu^+)$ and $c(w_\mu^-)$ are the centers of mass of the densities $w_\mu^+$ and $w_\mu^-$ respectively.

As such, the length of this vector and in particular the additional assumption for $\alpha > \frac{1}{2}$ in the statement above do not also depend on the coordinate system. These integrals of the winding number function are connected to the signature of paths as demonstrated in [3, Theorem 1] for closed curves with bounded total variation. Actually, they represent the first few nontrivial entries in the logarithmic signature of the closed curves $\varphi \circ \gamma$. For general targets and maps $\varphi \in H^\alpha(X, Y)$ for $\alpha > \frac{1}{2}$, we will also give a characterization of Property (T) in terms of push-forwards of currents, Proposition 4.4.

This result is related to the Hölder problem for the Heisenberg group. Gromov showed in [7] that there is no embedding of an open subset of the plane into the first Heisenberg group $\mathbb{H}$ equipped with the Carnot-Carathéodory metric $d_{cc}$ that is Hölder continuous with exponent $\alpha > \frac{2}{3}$. The two theorems above strengthen this result.

**Theorem 1.3.** Let $X$ be a quasiconvex metric space with $\pi^1_{\text{Lip}}(X) = 0$ and $\varphi : (X, d_X) \to (\mathbb{H}, d_{cc})$ be a Hölder continuous map of order $\alpha > \frac{2}{3}$. Then $\varphi$ factors through a tree. In particular, if $X$ contains a subset bi-Lipschitz equivalent to an open subset of the plane, then $\varphi$ cannot be an embedding.

A natural follow-up question is if the same conclusion also holds for $\alpha > \frac{1}{2}$. This would solve the Hölder problem for the Heisenberg group and show that there is no local homeomorphism $\mathbb{R}^3 \to \mathbb{H}$ of Hölder regularity $\alpha > \frac{1}{2}$. The statement is stronger than that and additionally would characterize these maps as locally factoring through a tree. A solution of this falls short because of the additional vanishing assumptions in Theorem 1.2 for $\alpha > \frac{1}{2}$. 
2. Preliminaries

Here we state some facts we will rely on and hope that the rest of these notes are reasonably self-contained.

A metric space \( X \) is called \( C \)-quasiconvex if for any two points \( x, x' \in X \) there is a curve \( \gamma : [0, 1] \to X \) connecting the two points with

\[
\text{length}(\gamma) \leq Cd(x, x') .
\]

By reparameterizing \( \gamma \) by arc length we can assume that \( \text{Lip}(\gamma) \leq Cd(x, x') \).

We denote by \( \pi_k^{(\text{Lip})}(X, x_0) \) the \( k \)th (Lipschitz) homotopy group of \( X \) and by \( H_k^{(\text{Lip})}(X) \) the \( k \)th singular (Lipschitz) homology group. By the Hurewicz theorem, \( \pi_1^{(\text{Lip})}(X) = 0 \) implies \( H_1^{(\text{Lip})}(X) = 0 \). Moreover we will use that if \( X \) is Lipschitz path connected and \( \pi_1^{\text{Lip}}(X) = 0 \), then for any closed Lipschitz curve \( \gamma : S^1 \to X \) there is a Lipschitz extension \( \Gamma : B^2(0, 1) \to X \).

Let \( \sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) be a continuous, concave, strictly increasing function with \( \sigma(0) = 0 \). A map \( \varphi : X \to Y \) between metric spaces is \( \sigma \)-continuous if for all \( x, x' \in X \),

\[
d_Y(\varphi(x), \varphi(x')) \leq \sigma(d_X(x, x')) .
\]

As a particular instance of this, \( \varphi \) is called \( H \)-\( \text{Lipschitz} \) of regularity \( \alpha > 0 \) if there is some \( C > 0 \) such that for all \( x, x' \in X \),

\[
d_Y(\varphi(x), \varphi(x')) \leq Cd_X(x, x')^\alpha .
\]

The infimum over all such \( C \) is denoted by \( H^\alpha(\varphi) \). \( H^\alpha(X, Y) \) is the set of Hölder maps of regularity \( \alpha \) from \( X \) to \( Y \). In case \( Y = \mathbb{R} \) we abbreviate \( H^\alpha(X) = H^\alpha(X, \mathbb{R}) \).

For a sequence \((\varphi_n)\) in \( H^\alpha(X, Y) \) we write \( \varphi_n \xrightarrow{\alpha} \varphi \) if \( \sup_x d_Y(\varphi_n(x), \varphi(x)) \to 0 \) and \( \sup_n H^\alpha(\varphi_n) < \infty \). It follows immediately from this convergence that the limit satisfies \( H^\alpha(\varphi) \leq \liminf H^\alpha(\varphi_n) \) and in particular \( \varphi \in H^\alpha(X, Y) \). The following result is due to Young [13].

**Theorem 2.1.** Let \( s \leq t \) and \( \alpha, \beta > 0 \) with \( \alpha + \beta > 0 \). If \( f \in H^\alpha([s, t]) \) and \( g \in H^\beta([s, t]) \), then the Riemann-Stieltjes integral \( \int_s^t f \, dg \) exists. Further,

(1) There is a constant \( C = C_{\alpha+\beta} \), such that for all \( u \in [s, t] \),

\[
\left| \int_s^t f \, dg - f(u)(g(t) - g(s)) \right| \leq C H^\alpha(f) H^\beta(g)|t - s|^\alpha+\beta .
\]

(2) If \( f \) and \( g \) are Lipschitz, then

\[
\int_s^t f \, dg = \int_s^t f(x) \, g'(x) \, d\mathcal{L}(x) .
\]

(3) Let \((f_n)\) and \((g_n)\) be sequences of functions on \([s, t]\) with \( f_n \xrightarrow{\alpha} f \) and \( g_n \xrightarrow{\beta} g \). Then

\[
\int_s^t f_n \, dg_n \to \int_s^t f \, dg .
\]

This Riemann-Stieltjes integral over Hölder functions can be generalized to higher dimensions. For a square \( Q \subset \mathbb{R}^2 \) we denote by \( \mathcal{P}_m(Q) \) the partition of \( Q \) into
For $f,g$ we define the functional

$$I_{Q,m}(f,g_1,g_2) := \sum_{R \in \mathcal{P}_m(Q)} f(b_Q) \int_{\partial R} g_1 \, dg_2,$$

in case all the boundary integrals are well defined. These are to be understood as Riemann-Stieltjes integrals running counterclockwise around the boundary of the square indicated and they are defined in particular if $g_1$ and $g_2$ are Hölder continuous as in Theorem 2.1. The following lemma is the two-dimensional case of [15, Theorem 3.2].

**Lemma 2.2.** Let $f \in H^\alpha(Q)$, $g_1 \in H^{\beta_1}(Q)$ and $g_2 \in H^{\beta_2}(Q)$. If $\alpha + \beta_1 + \beta_2 > 2$, then the limit

$$I_Q(f,g_1,g_2) := \lim_{m \to \infty} I_{Q,m}(f,g_1,g_2)$$

exists. Further, $I_Q$ satisfies and is uniquely defined by the following properties:

1. $I_Q$ is linear in each argument.
2. $I_Q(f,g_1,g_2) = \int_Q f \det(D(g_1,g_2)) \, d\mathcal{L}^2$ if all three functions are Lipschitz.
3. $I_Q(f_n,g_{1,n},g_{2,n}) \to I_Q(f,g_1,g_2)$ if $f_n \to f$ and $g_{i,n} \to g_i$ for $i = 1,2$.

We will occasionally use properties of the mapping degree and the winding number respectively. Everything we state here can be found for example in [11]. Another tool we use are currents in metric spaces, see [1] and [9] for a development of this theory. As our currents in connection with Hölder maps may not have finite mass, we will follow the theory presented in [9].

For $w \in L^1(\mathbb{R}^n)$ we write $[w]$ for the current of finite mass in $M_n(\mathbb{R}^n)$ obtained by integrating $n$-forms over $w$. For an oriented submanifold $M^m \subset \mathbb{R}^n$ we also denote by $[M] \in \mathcal{D}_n(\mathbb{R}^n)$ the $m$-dimensional current induced by integrating $m$-forms over $M$. Given locally compact metric spaces $X$ and $Y$, a normal current $T \in N_n(X)$ and a map $\varphi \in H^\alpha(X,Y)$ for some $\alpha > \frac{n}{n+1}$, then $\varphi#T$ is a well defined current in $\mathcal{D}_n(Y)$ by [15, Theorem 4.3]. In case $[S^1]$ is the 1-dimensional current induced by a circle and $Y = \mathbb{R}^2$, there is some connection between the winding number function $q \mapsto w_\gamma(q)$ of $\gamma$ in $\mathbb{R}^2$ and the push-forward $\gamma#[S^1]$, [14, Proposition 4.6].

**Lemma 2.3.** Let $\gamma \in H^\alpha(S^1,\mathbb{R}^2)$ for $\alpha > \frac{1}{2}$. Then $w_\gamma$ is integrable and $[w_\gamma]$ is the unique filling (with compact support) of $\gamma#[S^1]$, i.e.

$$\partial [w_\gamma] = \gamma#[S^1].$$

Respectively, if $g = (g_1,g_2) \in \text{Lip}(\mathbb{R}^2,\mathbb{R}^2)$, then

$$\int_{\mathbb{R}^2} w_{\gamma}(q) \det(Dg_i) \, dq = \int_{S^1} g_1 \circ \gamma \, d(g_2 \circ \gamma).$$

If $\varphi \in H^\alpha(Q,\mathbb{R}^2)$ for a square $Q \subset \mathbb{R}^2$ and $\alpha > \frac{2}{3}$, this can be combined with Lemma 2.2 to obtain,

$$\int_{\mathbb{R}^2} w_{\varphi#Q} f \det(Dg) = \varphi#[Q](f \, dg_1 \wedge dg_2) = I_Q(f \circ \varphi,g \circ \varphi).$$

for $f,g_1,g_2 \in \text{Lip}(\mathbb{R}^2)$ and $g = (g_1,g_2)$.
3. Construction of the tree

Let \( \varphi : X \to Y \) be a continuous map between metric spaces \((X, d_X)\) and \((Y, d_Y)\) as in Theorem [4.1]. Let \( d \) be the intrinsic metric on \( X \). This means, \( d(x, x') \) is the infimal length of all curves connecting \( x \) with \( x' \), see e.g. [4, Chapter 2] for properties of \( d \). Because \((X, d_X)\) is \( C \)-quasiconvex, it is for all \( x, x' \in X \),

\[
(3.1) \quad d_X(x, x') \leq d(x, x') \leq C d_X(x, x').
\]

The resulting space \((X, d)\) is a length space, i.e. any two points \( x, x' \in X \) and any \( \epsilon > 0 \) there is a curve \([x, x']_e\), with length \([x, x']_e \leq d(x, x') + \epsilon \) \( \sigma \) is increasing and therefore \( \varphi \) is also \( \sigma \)-continuous with respect to \( d \). Until the end of this section we work with the inner metric \( d \) instead of \( d_X \). For simplicity we will assume that \( d \) is geodesic, but in all the arguments where this is used we could as well replace the geodesic by a minimizing sequence as above.

For \( x, x' \in X \) define

\[
D(x, x') := \inf \{ \text{diam}(\varphi(C)) : x, x' \in C \text{ and } C \text{ is connected} \}.
\]

**Lemma 3.1.** \( D \) is a pseudo metric on \( X \) and moreover,

\[
d_Y(\varphi(x), \varphi(x')) \leq D(x, x') \leq \sigma(d(x, x')).
\]

**Proof.** It is easy to check that for connected subsets \( A, B \subset X \) with \( x, x' \in A \) and \( x', x'' \in B \) it is \( \text{diam}(\varphi(A \cup B)) \leq \text{diam}(\varphi(A)) \cup \text{diam}(\varphi(B)) \) since \( A \cap B \) is nonempty. By the same reason, \( A \cup B \) is connected and this immediately implies the triangle inequality for \( D \). The first inequality is obvious, the second follows by taking \( C = [x, x'] \) some geodesic segment connecting \( x \) with \( x' \) in \( X \) and the fact that \( \varphi \) is \( \sigma \)-continuous. \( \square \)

We set \( T = X/_{\sim} \), where \( x \sim x' \) if \( D(x, x') = 0 \) and with \( \psi : X \to T \) we denote the quotient map from \( X \) onto \( T \). Further let \( \psi : X \to T \) be the quotient map and define \( \overline{\varphi} : T \to Y \) by \( \overline{\varphi}(\psi([x])) := \varphi(x) \). This is well defined by the lemma above. A metric is defined on \( T \) by \( D(\psi(x), \psi(x')) := D(x, x') \). In the next part we will show that \((T, D)\) is a tree.

3.1. **Proof that \( T \) is a tree.** It follows from Lemma [3.1] that every point \( p \in T \) represents a closed subset of \( \varphi^{-1}(y) \). In particular, any connected component of \( \varphi^{-1}(y) \) is contained in some \( p \in T \). In case \( X \) is compact, this is actually a characterization of \( T \).

**Lemma 3.2.** If \( X \) is compact, then

\[
T = \{ c : c \text{ is a connected component of } \varphi^{-1}(y) \text{ for some } y \in Y \}.
\]

**Proof.** As noted above, any connected component \( c \) like this is contained in some \( p \in T \). On the other side let \( c, c' \) be two such components with \( D(x, x') = 0 \) for some fixed \( x \in c \) and \( x' \in c' \). From Lemma [3.1] it follows that \( \varphi(c) = \varphi(c') = \{ y \} \) for some \( y \in Y \) and from the definition of \( D \) it follows that for any \( \epsilon > 0 \) there is a connected subset \( C_{y} \subset X \) with \( x, x' \in C_{y} \) and \( \varphi(C_{y}) \subset B_Y(y, \epsilon) \). By taking the closure, we can as well assume that \( C_{y} \) is compact. By a theorem of Blaschke [2], the set

\[
\{ K \subset X : K \text{ compact and nonempty} \}
\]
Lemma 3.3. Let $Y$ be Lipschitz path connected. Since $C_t$ is uniformly continuous and by choosing $\delta$ small enough, there is a curve $\gamma_{x,x',\epsilon} : [0, 1] \to X$ with $\gamma_{x,x',\epsilon}(0) = x, \gamma_{x,x',\epsilon}(1) = x'$ and
\[
\text{im}(\varphi \circ \gamma_{x,x',\epsilon}, \epsilon) \subset U_Y(\varphi(x), D(x, x') + \epsilon).
\]

Given two points $x, x' \in X$ with $\varphi(x) \neq \varphi(x')$ let $\mathcal{Y}(x, x')$ be the set of all points $y \in Y$ such that for any curve $\gamma : [0, 1] \to X$ connecting $x$ with $x'$, the point $y$ lies in $\text{im}(\varphi \circ \gamma)$. Property (1) guarantees that apart from $\varphi(x)$ and $\varphi(x')$ the set $\mathcal{Y}(x, x')$ contains additional points.

Lemma 3.3. Let $\gamma_i : [0, 1] \to X$, $i = 1, 2$, be two curves with $\gamma_1(t), \gamma_2(t) \in p_t \in T$ for $t = 0, 1$ and $\overline{\varphi}(p_0) \neq \overline{\varphi}(p_1)$. Then $\mathcal{Y}(\gamma_1(0), \gamma_1(1)) = \mathcal{Y}(\gamma_2(0), \gamma_2(1))$.

Proof. Let $y \in \mathcal{Y}(\gamma_2(0), \gamma_2(1))$ and we want to show that $y$ is also in $\mathcal{Y}(\gamma_1(0), \gamma_1(1))$. If $y = \overline{\varphi}(p_0)$ or $y = \overline{\varphi}(p_1)$ we are done. So assume this is not the case and fix some $\epsilon > 0$ small enough such that
\[
\epsilon < \max\{d_Y(\overline{\varphi}(p_0), y), d_Y(\overline{\varphi}(p_1), y)\}.
\]

Using the curves as used in (3.2), define the curve
\[
\gamma'_1 := \gamma_{\gamma_2(t), \gamma_1(t), \epsilon} \ast \gamma_1 \ast \gamma_{\gamma_1(t), \gamma_2(t), \epsilon}.
\]
$\gamma'_1$ connects $\gamma_2(0)$ with $\gamma_2(1)$ by going through $\gamma_1$. From (3.2) it follows that for $t = 0, 1$,
\[
\text{im}(\gamma_{\gamma_2(t), \gamma_1(t), \epsilon}) \subset U_Y(\overline{\varphi}(p_t), \epsilon) \subset Y \setminus \{y\}.
\]
Since $y \in \text{im}(\varphi \circ \gamma_1)$ by definition, we get that $y \not\in \text{im}(\varphi \circ \gamma_1)$. □

This lemma allows to define $\mathcal{Y}(p, p')$ for $p, p' \in T$ if $\overline{\varphi}(p) \neq \overline{\varphi}(p')$. The next observation is the main reason for this particular definition of $T$ and the reason we assume $X$ to be simply-connected, respectively that $H^{\text{Lip}}_1(X) = 0$. It can be stated in both the continuous and Lipschitz category. Note that for an open set in a locally (Lipschitz) path connected space, components and (Lipschitz) path components are the same.

Lemma 3.4. Let $Z$ be a connected and locally (Lipschitz) path connected space with $H^{\text{Lip}}_1(Z) = 0$. Assume that $A \subset Z$ is a closed set that disconnects $z$ and $z'$ in $Z$. Then there is a connected component of $A$ that disconnects $z$ and $z'$. 
Proof. We will formulate the proof in the continuous category, since the arguments in the Lipschitz case are essentially the same. Consider the set $\mathcal{A}$ of closed subsets of $A$ that disconnect $z$ and $z'$. The inclusion gives a partial order on $\mathcal{A}$. We want to show that there is a minimal element in $\mathcal{A}$. By the axiom of choice it suffices to show that any chain $\mathcal{A}' \subset \mathcal{A}$ has a lower bound in $\mathcal{A}$. Let $B := \bigcap \mathcal{A}'$ and $C \subset X$ be a path that contains both $z$ and $z'$. By definition, $B$ is closed and the intersection $C \cap A_1 \cap \cdots \cap A_n$ is nonempty for every finite collection $A_1, \ldots, A_n \in \mathcal{A}'$. Since $C$ is compact, $C \cap B$ is nonempty too and hence $B \in \mathcal{A}$.

So let $M$ be a minimal element of $\mathcal{A}$. This $M$ has to be connected. Assume by contradiction that it is not and let $M_1$ and $M_2$ be a partition of $M$ into disjoint, nonempty, closed subsets. Set $U = X \setminus M_1$ and $V = X \setminus M_2$. Clearly, $X = U \cup V$ and the tail of the Mayer-Vietoris sequence reads as follows,

$$0 = H_1(X) \xrightarrow{\partial} H_0(U \cap V) \xrightarrow{(i_*, j_*)} H_0(U) \oplus H_0(V) \xrightarrow{k_* - l_*} H_0(X) = \mathbb{Z},$$

where $i : U \cap V \to U$, $j : U \cap V \to V$, $k : U \to X$ and $l : V \to X$ are the inclusions. Because $H_1(X) = 0$ and this sequence is exact, the homomorphism $(i_*, j_*)$ is injective. Since $z$ and $z'$ are disconnected by $M$, they represent different elements $[z]$ and $[z']$ in $H_0(U \cap V)$. It follows, $(i_*[z], j_*[z]) \neq (i_*[z'], j_*[z'])$ and hence $i_*[z] \neq i_*[z']$ or $j_*[z] \neq j_*[z']$. This means that $z$ and $z'$ are in different path components of $U$ or $V$, respectively, $M_1$ or $M_2$ disconnects $z$ and $z'$. This contradicts the minimality of $M$. Hence $M$ is connected and therefore contained in some connected component of $A$.

This result is used in the following lemma.

Lemma 3.5. Let $p, p' \in T$ with $\varphi(p) \neq \varphi(p')$. Then for every $y \in \mathcal{Y}(p, p') \setminus \{\varphi(p), \varphi(p')\}$ there is some $q \in P$ with $\varphi(q) = y$ that disconnects $p$ from $p'$ inside $X$, i.e. every curve from $p$ to $p'$ in $X$ intersects $q$.

Proof. Since $(X, d)$ is a length space, it is of course locally Lipschitz path connected. Fix some points $x, x' \in X$ with $\psi(x) = p$ and $\psi(x') = p'$. By Lemma 3.3 the set $\varphi^{-1}(y)$ disconnects $x$ from $x'$ whereas it follows from the construction of $T$ that $p$ as well as $p'$ are contained in the same component of $X \setminus \varphi^{-1}(y)$ as $x$ and $x'$ respectively. From Lemma 3.3 it follows that there is a component $c$ of $\varphi^{-1}(y)$ that disconnects $x$ and $x'$. By the construction of $T$, this set $c$ is contained in some $q \in T$ with $\varphi(q) = y$. Hence any curve connecting $p$ with $p'$ in $X$ intersects $q$.

For all $p, p' \in T$ with $\varphi(p) \neq \varphi(p')$ let $C(p, p')$ be the collection of all $q \in T$ for which any curve connecting $p$ with $p'$ in $X$ intersects $q$. The lemma above guarantees that there is some $q \in C(p, p') \setminus \{p, p'\}$. This result can be applied to curves in $T$.

Lemma 3.6. Let $\gamma : [0, 1] \to T$ be a (continuous) curve connecting $p$ with $p'$ in $T$. Then there is a sequence of Lipschitz curves $\eta_n : [0, 1] \to X$ in $X$ with $\psi \circ \eta_n(t) = \gamma(t)$ for $t = 0, 1$ and $\psi \circ \eta_n$ converges uniformly to $\gamma$.

Proof. Since $\gamma$ is uniformly continuous, we can find for every $\delta > 0$ some $m \in \mathbb{N}$ such that $D(\gamma(t), \gamma(t')) < \delta$ if $|t - t'| \leq \frac{\delta}{m}$. In particular, for every $0 \leq i \leq m - 1$ it is $D(\gamma\left(\frac{i}{m}\right), \gamma\left(\frac{i+1}{m}\right)) < \delta$. Fix some points $x_{i,0} \in \gamma(\frac{i}{m})$ and $x_{i,1} \in \gamma(\frac{i+1}{m})$ and consider the curve $\eta_b : [0, 1] \to X$ defined by

$$\eta_b := \left\{\begin{array}{ll}
\gamma_{x_{0,0},x_{0,1},\delta} & \text{on } [0, \frac{1}{m}]
\gamma_{x_{i-1,1},x_{i,0},\delta} \ast \gamma_{x_{i,0},x_{i,1},\delta} & \text{on } \left[\frac{i}{m}, \frac{i+1}{m}\right]
\end{array}\right. \text{ for } 1 \leq i \leq m - 1.$$
Where this curves are understood to be linear reparameterizations of the ones in \([3.2]\). There it is stated that \(\text{im}(\varphi \circ \gamma_{x_{i-1},x_{i},0,\delta})\) is contained in \(U_Y(\overline{\varphi(\gamma(\delta))},\delta)\) and \(\text{im}(\varphi \circ \gamma_{x_{i},0,x_{i+1},\delta})\) in \(U_Y(\overline{\varphi(\gamma(\frac{1}{m}))},2\delta)\). By the definition of \(D\) this translates to

\[
\text{im}(\overline{\varphi} \circ \gamma_{x_{i-1},x_{i},0,\delta}) \subset U_T(\varphi(\gamma(\frac{1}{m})),2\delta) \quad \text{and} \quad \text{im}(\varphi \circ \gamma_{x_{i},0,x_{i+1},\delta}) \subset U_T(\gamma(\frac{1}{m}),4\delta) .
\]

Hence for \(t \in \left[\frac{1}{m},\frac{1}{m} + \frac{1}{m}, \ldots, \frac{1}{m}\right]\), \(0 \leq t \leq m - 1\),

\[
D(\overline{\varphi} \circ \eta(t), \gamma(t)) \leq D(\overline{\varphi} \circ \eta(t), \gamma(\frac{1}{m})) + D(\gamma(\frac{1}{m}), \gamma(t)) \leq 5\delta .
\]

Choosing \(\delta = \frac{1}{n}\) finishes the lemma. \(\square\)

Next we check that curves in \(T\) are not completely degenerate in some sense.

**Lemma 3.7.** If \(\gamma : [0,1] \to T\) is a nonconstant curve, then there is a \(t \in [0,1]\) such that \(\overline{\varphi}(\gamma(0)) \neq \overline{\varphi}(\gamma(t))\).

**Proof.** Assume that \(y = \overline{\varphi}(\gamma(0)) = \overline{\varphi}(\gamma(t))\) for all \(t\). Fix some \(\epsilon > 0\). From Lemma \([3.4]\) it follows that there is a sequence of curves \(\eta_n : [0,1] \to X\) with \(\psi(\eta_n(t)) = \gamma(t)\) for \(t = 0,1\) and \(\psi \circ \eta_n\) converges uniformly to \(\gamma\). Since \(\overline{\varphi} : T \to Y\) is uniformly continuous we get for any \(\epsilon > 0\) that \(\text{im}(\varphi \circ \eta_n) \subset U_Y(y,\epsilon)\) for large enough \(n\). By the definition of \(D\) we get \(D(\eta_n(s),\eta_n(t)) \leq 2\epsilon\) and hence

\[
D(\gamma(0), \gamma(t)) \leq \limsup_{n \to \infty} D(\psi \circ \eta_n(0), \psi \circ \eta_n(t)) + D(\psi \circ \eta_n(t), \gamma(t)) \leq 2\epsilon .
\]

This is true for any \(\epsilon\) and it follows that \(\gamma\) is constant. \(\square\)

Now we are ready to prove that \((T, D)\) is a tree.

**Proposition 3.8.** \((T, D)\) is an arc-connected metric space. Further, if \(\gamma_1, \gamma_2 : [0,1] \to T\) are two injective curves with the same endpoints, then the curves are reparameterizations of each other.

**Proof.** As an image of a path-connected space, \(T\) is clearly also path-connected. It is a standard fact that such a space is arc-connected. Let \(\gamma_1\) and \(\gamma_2\) be two arcs as in the statement. We will show that \(\text{im}(\gamma_1) = \text{im}(\gamma_2)\). Assume by contradiction that there is a \(t \in [0,1]\) such that \(\gamma_1(t) \notin \text{im}(\gamma_2)\). By continuity there are \(t_1 < t < t_2\) such that \(\gamma_1(t) \notin \text{im}(\gamma_2)\) for all \(t \in (t_1, t_2)\) but \(\gamma_1(t_i) \in \text{im}(\gamma_2)\) for \(i = 1,2\). Concatenating the arc from \(\gamma_1\) with the arc connecting \(\gamma_1(t_1)\) and \(\gamma_1(t_2)\) through \(\gamma_2\) we get a simple closed curve \(\gamma : S^1 \to T\). By Lemma \([3.7]\) there are two points \(s, s' \in S^1\) with \(\overline{\varphi}(\gamma(s)) \neq \overline{\varphi}(\gamma(s'))\). But by Lemma \([3.6]\) there is some \(a \in \mathcal{C}(\gamma(s), \gamma(s'))\)\(\} \{\gamma(s), \gamma(s')\}\) and \(\gamma\) has to go twice through \(a\), a contradiction. Hence, \(\text{im}(\gamma_1) \supset \text{im}(\gamma_2)\) and vice versa. This shows that \(\gamma_1 \circ \gamma_2^{-1}\) is a homeomorphism of \([0,1]\) and we get the desired reparameterization. \(\square\)

### 3.2. Monotone metric and sigma-variation.

For \(p, p' \in T\) we denote by \([p, p']\) a parameterization of the arc in \(T\) connecting \(p\) with \(p'\). By abuse of notation we also use \([p, p']\) for the image of this curve. The metric \(D\) may not be monotone on arcs. For this reason we introduce a new metric \(\tilde{D}\) on \(T\) defined by

\[
\tilde{D}(p, p') := \sup\{D(q, q') : [q, q'] \subset [p, p']\} .
\]

It is not so hard to check that \(\tilde{D}\) is indeed a metric on \(T\). This follows from the fact that for all \(p, p', p'' \in T\) the arc \([p, p'']\) is contained in the union \([p, p'] \cup [p', p'']\).
Lemma 3.9. For all \(x, x' \in X\),
\[d_Y(\varphi(x), \varphi(x')) \leq \hat{D}(\psi(x), \psi(x')) \leq \sigma(d(x, x')) .\]

Further, \((T, \hat{D})\) is a tree.

Proof. It is \(D \leq \hat{D}\) by the definition of \(\hat{D}\). Hence \(id : (T, \hat{D}) \to (T, D)\) is continuous and the first inequality follows from Lemma 3.1. To obtain the second, let \([x, x']\) be a geodesic in \(X\) connecting \(x\) with \(x'\) and let \(p, p' \in T\) be such that \([p, p'] \subset [\psi(x), \psi(x')]\) and \(D(p, p') = \hat{D}(\psi(x), \psi(x'))\). Because \((T, D)\) is a tree, the curve \(\psi \circ [x, x']\) goes through both \(p\) and \(p'\). Therefore, \(d(p, p') \leq d(x, x')\) and by Lemma 3.1
\[\hat{D}(\psi(x), \psi(x')) = D(p, p') \leq \sigma(d(p, p')) \leq \sigma(d(x, x')) .\]

Since \(X\) is path connected and \(\hat{D}(\psi(x), \psi(x')) \leq \sigma(d(x, x'))\) any two points in \((T, D)\) can be connected by an arc. Because of \(D \leq \hat{D}\) any arc in \((T, \hat{D})\) is also an arc in \((T, D)\), hence it has to be unique. \(\square\)

Since \(\sigma\) is strictly increasing, the \(\sigma\)-variation of a curve \(\gamma : [a, b] \to (Z, d_Z)\) can be defined by
\[V_\sigma(\gamma) := \sup_P \sum_{i=0}^{n-1} \sigma^{-1}(d_Z(\gamma(t_{i+1}), \gamma(t_i))) ,\]
where the supremum is taken over all partitions \(P\) of \([a, b]\) given by \(a = t_0 < t_1 < \ldots t_n = b\). This definition is independent of the parameterization of \(\gamma\). One can show that for any \(\gamma\) with \(V_\sigma(\gamma) < \infty\) there is a reparameterization \(\tilde{\gamma} : [0, V_\sigma(\gamma)] \to Z\) of \(\gamma\) with \(t = V_\sigma(\gamma|_{[0, t]}).\) This is a standard result and uses the fact that \(\tau(t) := V_\sigma(\gamma|_{[0, t]}))\) is continuous. For the readers convenience we include a proof here.

Lemma 3.10. Let \(\nu : [0, 1]^2 \to \mathbb{R}_{\geq 0}\) and define
\[\varpi(t) := \sup_{i=0}^{n-1} \nu(t_{i+1}, t_i) ,\]
where the supremum is taken over all partitions of \([0, t]\) given by \(0 \leq t_0 \leq t_1 \leq \ldots t_n \leq t\). If \(\nu\) is continuous, \(\nu(t, t) = 0\) for all \(t\) and \(\varpi(1) < \infty\), then \(\varpi\) is continuous.

Proof. We will first show continuity at \(0\). Obviously, \(\varpi(0) = 0\) and \(\varpi\) is non-decreasing by definition. So it is enough to find a decreasing sequence \((t_n)\) with \(\lim_{n} t_n = 0 = \lim_{n} \nu(t_n)\). W.l.o.g. we assume that \(\varpi(t) > 0\) for all \(t > 0\). We start with \(t_1 = 1\) and proceed recursively as follows. Given \(t_n\), let \(0 = t_0^n < \ldots < t_k^n = t_n\) be a partition of \([0, t_n]\) with
\[\sum_{i=0}^{k_n-1} \nu(t_{i+1}^n, t_i^n) > \frac{\varpi(t_n)}{2} .\]

Because \(\lim_{a \to 0} \nu(b, a) + \nu(a, 0) = \nu(b, 0)\) for all \(b\) we can assume that in the partition above we have \(0 < t_i^n < \frac{t_n}{2}\) and
\[\sum_{i=1}^{k_n-1} \nu(t_{i+1}^n, t_i^n) > \frac{\varpi(t_n)}{2} .\]
Note that the sum here starts from \( i = 1 \). Set \( t_{n+1} := t_n^l \). Obviously, \((t_n)\) converges to 0 by construction and further for all \( l \in \mathbb{N} \)

\[
\sum_{n=1}^{l} \nu(t_n) < 2 \sum_{n=1}^{l} \sum_{i=1}^{k_n-1} \nu(t_n^{i+1}, t_n^i) \leq 2\nu(1) .
\]

Because \( \nu(1) < \infty \) we get that \( \nu(t_n) \to 0 \) for \( n \to \infty \).

Now let \( t > 0 \) and we will show that \( \nu \) is continuous from below at \( t \). For any \( n \in \mathbb{N} \) let \( 0 = t_0 < \cdots < t_{k_n} = t \) be a partition of \([0, t]\) with

\[
\sum_{i=0}^{k_n-1} \nu(t_{i+1}, t_i) > \nu(t) - \frac{1}{n} .
\]

Because \( \lim_{n \to \infty} \nu(b, a) + \nu(a, t) = \nu(b, t) \) for all \( b \) we can assume that \( t_{k_n-1} > t - \frac{1}{n} \) and \( \nu(t, t_{k_n-1}) < \frac{1}{n} \). Hence,

\[
\nu(t_{k_n-1}) \geq \sum_{i=0}^{k_n-2} \nu(t_{i+1}, t_i) > \nu(t) - \frac{2}{n} .
\]

Hence, \( \lim_{n} t_{k_n-1} = t \) and \( \lim_{n} \nu(t_{k_n-1}) = \nu(t) \). Since \( \nu \) is non-decreasing, this shows that it is continuous from below at \( t \).

To see continuity from above, let \( t < 1 \) and \((t_n)\) be a decreasing sequence with \( \lim_n t_n = t \). W.l.o.g. we assume that \( \nu(t_n) > 0 \). Let \( 0 = t_0^n < \cdots < t_{k_n}^n = t_n \) be a partition such that

\[
(3.3) \quad \sum_{i=0}^{k_n-1} \nu(t_n^{i+1}, t_n^i) > \nu(t_n) - \frac{1}{n} .
\]

For each \( n \) let \( 0 = i_n < k_n \) be the index with \( t \in [t_{i_n}^n, t_{i_n+1}^n) \). By assumption it is \( t_{i_n+1}^n \to t \) and since \( \nu \) is continuous,

\[
\lim_{n \to \infty} \nu(t_{i_n+1}^n, t) + \nu(t, t_{i_n}^n) - \nu(t_{i_n}^{n+1}, t_{i_n+1}^n) = 0 .
\]

For big \( n \) we can therefore assume that \((3.3)\) is satisfied with \( t \) being part of the partition, say \( t = t_{i_n}^n \) for some \( i_n \). But then

\[
\nu(t_n) < \frac{1}{n} + \nu(t) + \sum_{i=i_n}^{k_n-1} \nu(t_n^{i+1}, t_n^i) .
\]

This latter sum is over a partition of \([t, t_n]\) and as such tends to zero for \( t_n \to t \) as we have already seen in the first part of the proof. \( \square \)

Let \( \gamma : [a, b] \to Z \) be a continuous curve that satisfies \( V_\sigma(\gamma) < \infty \). Then \( \tau : [a, b] \to [0, V_\sigma(\gamma)] \) defined by \( \tau(t) := V_\sigma(\gamma|_{[0, t]} \) is non-decreasing and continuous by the lemma above. For \( 0 \leq s \leq t \leq V_\sigma(\gamma) \) let \( a \leq u \leq v \leq b \) be such that \( \tau(u) = s \) and \( \tau(v) = t \). This is possible, precisely because \( \tau \) is continuous. Then

\[
\sigma^{-1}(d_Z(\gamma(v), \gamma(u))) \leq V_\sigma(\gamma|_{[u, v]}) \leq V_\sigma(\gamma|_{[0, v]}) - V_\sigma(\gamma|_{[0, u]}) = \tau(v) - \tau(u) = t - s .
\]

This allows to define \( \tilde{\gamma} : [0, V_\sigma(\gamma)] \to Z \) by \( \tilde{\gamma}(\tau(v)) = \gamma(v) \) and moreover it holds

\[
d_Z(\tilde{\gamma}(t), \tilde{\gamma}(s)) \leq \sigma(t - s) .
\]

It follows that \( \tilde{\gamma} \) is continuous reparameterization of \( \gamma \) and thus we also obtain

\[
V_\sigma(\tilde{\gamma}|_{[0, t]}) = V_\sigma(\gamma|_{[0, \tau(t)]}) = \tau(t) = t .
\]
For a curve $\gamma : [0,1] \to Z$ let $S_\gamma$ be the collection of intervals $[a,b] \subset [0,1]$ such that $a < b$, $\gamma(a) = \gamma(b)$ and there is no interval $[a',b']$ strictly containing $[a,b]$ with $\gamma(a') = \gamma(b')$. For a collection of disjoint intervals $S \subset S_\gamma$ define $\gamma_S : [0,1] \to Z$ by

$$\gamma_S(t) := \begin{cases} \gamma(b) & \text{if } t \in [a,b] \in S \\ \gamma(t) & \text{otherwise} \end{cases}.$$

**Lemma 3.11.** Let $\gamma : [0,1] \to Z$ be a $\sigma$-continuous curve and $S \subset S_\gamma$ some maximal subset of disjoint intervals. Then $\gamma_S$ is $\sigma$-continuous and $\gamma_S(s) = \gamma_S(t)$ for $0 \leq s < t \leq 1$ implies that $\gamma_S$ is constant on $[s, t]$.

**Proof.** Let $S := \bigcup S \subset [0,1]$. For a point $u \in S$ denote by $[a_u,b_u] \in S$ the unique interval with $u \in [a_u,b_u]$. Now let $s,t \in [0,1]$ with $s < t$. If both $s$ and $t$ are not in $S$, then

$$d_Z(\gamma_S(t), \gamma_S(s)) = d_Z(\gamma(t), \gamma(s)) \leq \sigma(t-s).$$

If $s, t \in S$ we have two cases. If the two intervals $[a_s,b_s]$ and $[a_t,b_t]$ intersect they are the same and $\gamma_S(t) = \gamma_S(s)$. Otherwise, $b_s < a_t$ and hence

$$d_Z(\gamma_S(t), \gamma_S(s)) = d_Z(\gamma(b_s), \gamma(a_t)) \leq \sigma(a_t - b_s) \leq \sigma(t-s).$$

If $s \in S$ and $t \notin S$, then

$$d_Z(\gamma_S(t), \gamma_S(s)) = d_Z(\gamma(t), \gamma(b_s)) \leq \sigma(t - b_s) \leq \sigma(t-s).$$

The case $s \notin S$ and $t \in S$ is treated analogously.

Now assume that $\gamma_S(s) = \gamma_S(t)$. If both $s$ and $t$ are not in $S$, then $\gamma(s) = \gamma(t)$ and there is some $[a,b] \in S$ that intersects $[s,t]$ by the maximality of $S$. Neither $s$ or $t$ can be contained in $[a,b]$ because $s,t \notin S$, so $[a,b]$ is a proper subset of $[s,t]$, but this is not possible by the definition of $S_\gamma$. If $s \in S$ and $t \notin S$, then $b_s < t$ and

$$\gamma(a_s) = \gamma_S(s) = \gamma_S(t) = \gamma(t),$$

contradicting $[a_s,b_s] \in S_\gamma$. The case $s \notin S$ and $t \in S$ is treated analogously. If $s, t \in S$ then $\gamma(a_s) = \gamma(b_t)$ and hence $[a_s,b_s] = [a_t,b_t]$ by the maximality of these intervals. \hfill $\square$

Let $p, p' \in T$ and consider a geodesic $\mu : [0,1] \to X$ parameterized by arclength connecting the subset $p$ with $p'$ in $X$. This means that $d(p,p')|t-s| = d(\mu(t), \mu(s))$.

By Lemma 3.11 $\gamma := \psi \circ \mu$ satisfies

$$\tilde{D}(\gamma(t), \gamma(s)) \leq \sigma(d(\mu(t), \mu(s))) \leq \sigma(d(p,p')|t-s|).$$

This shows that the curve $\gamma_S : [0,1] \to (T, \tilde{D})$ constructed in the lemma above satisfies $V_\sigma(\gamma_S) \leq d(p,p')$. With $(3.4)$, $(3.5)$ and $(3.6)$, the parameterization of an arc $[p,p'] : [0, V_\sigma([p,p'])] \to T$ with respect to the $\sigma$-variation has the following properties:

$$(3.7) \begin{cases} \tilde{D}([p,p'](t), [p,p'](s)) \leq \sigma(t-s), \\ V_\sigma([p,p']|[u,v]) = t, \\ V_\sigma([p,p']) \leq d(p,p'), \\ [p,p'] \text{ is injective}. \end{cases}$$

To see the last statement, let $s \leq t$ with $[p,p'](s) = [p,p'](t)$ and consider $s = \tau(u)$ and $t = \tau(v)$, where $\tau(w) = V_\sigma(\gamma_S|[0,w])$ as before. Then $\gamma_S(u) = \gamma_S(v)$ and from Lemma 3.11 it follows that $\gamma_S$ is constant on $[u,v]$. This implies

$$s = \tau(u) = V_\sigma(\gamma_S|[0,u]) = V_\sigma(\gamma_S|[0,v]) = \tau(v) = t.$$
Proof of Theorem 1.1. Combining Proposition 5.8 and Lemma 3.9 we get that $(T, \bar{D})$ is a tree and that the maps $\psi$ and $\varphi$ have the right continuity properties with respect to the inner metric $d$ on $X$. By translating the estimates in Lemma 3.9 back to the original metric $d_X$ using (3.1) we obtain the first part of the theorem. Next we will construct the contractions.

Fix a point $p \in T$ and consider the map $\pi_p : T \times \mathbb{R}_{\geq 0} \to T$ defined by

$$\pi_p(q, t) := [p, q](\min\{V_q([p, q]), t\}) .$$

By (3.7), $\pi_p(q, t) = q$ if $t \geq d(p, q)$. Set $V_q := V_{\sigma}([p, q])$. On an arc through $p$ we have again with (3.7),

$$\bar{D}(\pi_p(q, t), \pi_p(q, t')) \leq \sigma(|\min\{V_q, t\} - \min\{V_q, t'\}|) \leq \sigma(|t - t'|) .$$

Because $T$ is a tree, there is a unique point $q'' \in T$ in the intersection of the images of the arcs $[q, q']$, $[q, p]$ and $[q', p]$ for all choices of $q, q' \in T$. At equal times we have $\pi_p(q, t') = \pi_p(q', t')$ if $t' \leq V_{q''}$ and $[\pi_p(q, t'), \pi_p(q', t')]$ is contained in $[q, q']$ otherwise. Because $\bar{D}$ is monotone on arcs, this leads to

$$\bar{D}(\pi_p(q, t'), \pi_p(q', t')) \leq \bar{D}(q, q') .$$

Combining the two estimates we get

$$\bar{D}(\pi_p(q, t), \pi_p(q', t')) \leq \bar{D}(q, q') + \sigma(|t - t'|) .$$

This finishes the proof of Theorem 1.1. □

4. Hölder maps

In this section we want to prove Theorem 1.2. First we show a different result that connects Property (11) with currents and the winding number.

Proposition 4.1. Let $X$ be a quasiconvex metric space with $H_1(X) = 0$ or $H_1^{sup}(X) = 0$ and $\varphi : X \to Y$ be a Hölder continuous map of order $\alpha > \frac{1}{2}$. Then $\varphi$ has Property (11) if and only if $(\varphi \circ \gamma)_\# [S^1] = 0$ for all closed Lipschitz curves $\gamma : S^1 \to X$.

Moreover, if $Y = \mathbb{R}^2$, then $\varphi$ has Property (11) if and only if for all closed Lipschitz curves $\gamma : S^1 \to X$ there holds $w_{\varphi \circ \gamma}(q) = 0$ for almost every $q$.

Proof. First assume that $(\varphi \circ \gamma)_\# [S^1] = 0$ for all closed Lipschitz curves $\gamma : S^1 \to X$. Note that since $X$ is quasiconvex, any curve in $X$ can be uniformly approximated by Lipschitz curves. So if we show Property (11) for Lipschitz curves, we have it for all continuous curves. Fix two points $x, x' \in X$ with $\varphi(x) \neq \varphi(x')$. Let $\mu : [0, 1] \to X$ be a Lipschitz curve connecting $x$ with $x'$. By [13] Theorem 4.3], the current $(\varphi \circ \mu)_\# [0, 1] \in D_1(Y)$ is well defined and

$$\partial((\varphi \circ \mu)_\# [0, 1]) = (\varphi \circ \mu)_\# (\partial[0, 1]) = [\varphi(x')] - [\varphi(x)] \neq 0 .$$

This shows that $(\varphi \circ \mu)_\# [0, 1] \neq 0$. A nonzero metric current $T \in D_1(Y)$ as defined in [9] can't be supported on finitely many points because $T(\emptyset, g) = 0$ if $g$ is locally constant on spt$(T)$, [9] Lemma 3.2]. For another argument, a finite metric space has Nagata dimension zero, but the Nagata dimension of spt$(T)$ has to be at least $\dim(T)$ by [14] Proposition 2.5]. Therefore we can find a point $y \in \text{spt}((\varphi \circ \mu)_\# [0, 1]) \setminus \{\varphi(x), \varphi(x')\}$. Let $\mu' : [0, 1] \to X$ be any other Lipschitz curve connecting $x$ with $x'$. We define the closed Lipschitz curve $\gamma := \mu \ast \mu'^{-1} : S^1 \to X$.

By assumption,

$$0 = (\varphi \circ \gamma)_\# [S^1] = (\varphi \circ \mu)_\# [0, 1] - (\varphi \circ \mu')_\# [0, 1] .$$
In particular, $y \in \text{spt}((\varphi \circ \mu)_{\#}[0,1]) = \text{spt}((\varphi \circ \mu')_{\#}[0,1])$. By the definition of the push-forward and the support of currents it is clear that $y \in \text{spt}((\varphi \circ \mu)_{\#}[0,1]) \subset \text{im}(\varphi \circ \mu)$ and also $y \in \text{im}(\varphi \circ \mu')$. Since $\mu'$ was arbitrary, this shows Property (1) for $\varphi$.

Now assume that $Y = \mathbb{R}^2$ and $w_{\varphi \circ \gamma} = 0$ almost everywhere for all Lipschitz curves $\gamma : S^1 \to X$. Lemma 2.3 implies that $0 = \partial[\text{im}(\varphi \circ \gamma)] = (\varphi \circ \gamma)_{\#}[S^1]$ and hence $\varphi$ has Property (1). On the other side, if $\varphi : X \to \mathbb{R}^2$ has Property (1), it factors through a tree as in Theorem 1.1. So there are $\psi : X \to T$, $\varphi : T \to \mathbb{R}^2$ with $\varphi = \varphi \circ \psi$. Let $\gamma : S^1 \to X$ be a closed Lipschitz curve and fix some $p \in \text{im}(\psi \circ \gamma) \subset T$. Since $T$ is uniquely arcwise connected and $\text{im}(\psi \circ \gamma)$ is connected, any arc $[p, c]$ with $c \in \text{im}(\psi \circ \gamma)$ satisfies $\text{im}([p, c]) \subset \text{im}(\psi \circ \gamma)$. Using $\pi_p$, the contraction of Theorem 1.1 we define $\Gamma : B^2(0,1) \to T$ by

$$\Gamma(st) := \pi_p(\gamma(s), Rt),$$

for $s \in S^1$, $t \in [0,1]$ and $R = C \text{diam}(X)$, where $C$ is the constant for the quasiconvexity of $X$. $\varphi \circ \Gamma$ is a continuous extension of $\varphi \circ \gamma$. By construction, $\text{im}(\Gamma) \subset \text{im}(\psi \circ \gamma)$ and hence $\text{im}(\varphi \circ \Gamma) \subset \text{im}(\varphi \circ \gamma)$. By a property of the mapping degree, $\deg(\varphi \circ \Gamma, U^2(0,1), q) \neq 0$ implies that $q$ is in the image of $\varphi \circ \Gamma$. Since $H^2(\text{im}(\varphi \circ \Gamma)) \leq H^2(\text{im}(\varphi \circ \gamma)) = 0$ it follows

$$w_{\varphi \circ \gamma}(q) = \deg(\varphi \circ \Gamma, U^2(0,1), q) = 0,$$

for almost every $q$ (for all $q$ not in $\text{im}(\psi \circ \gamma)$).

Finally assume that $\varphi : X \to Y$ has Property (1) for a general metric space $Y$. Theorem 1.1 gives again a factorization through a tree as in the case $Y = \mathbb{R}^2$ above. Consider a closed Lipschitz curve $\gamma : S^1 \to X$. Since the quotient map $\psi : X \to T$ is Hölder continuous of order $\alpha > \frac{1}{2}$ we can consider the current $(\psi \circ \gamma)_{\#}[S^1] \in \mathcal{D}_1(T)$. Assume by contradiction that this current is nonzero. By the definition of metric currents this means that there are Lipschitz functions $g_1, g_2 : T \to \mathbb{R}$ with $0 \neq (\psi \circ \gamma)_{\#}[S^1](g_1, g_2)$. Using the Lipschitz map $g : T \to \mathbb{R}^2$ this implies with Lemma 2.3

$$0 \neq (\psi \circ \gamma)_{\#}[S^1](g_1, g_2) = (g \circ \psi \circ \gamma)_{\#}[S^1](x \, dy) = \int_{\mathbb{R}^2} w_{g \circ \psi \circ \gamma}(q) \, dq.$$

Hence the map $g \circ \psi : X \to \mathbb{R}^2$ does not have Property (1) by the case $Y = \mathbb{R}^2$ considered above. But $g \circ \psi$ factors though a tree by definition and therefore has property (1), a contradiction.

The assumption $\alpha > \frac{1}{2}$ is optimal in the sense that for $\mu \in H^n(S^1, \mathbb{R}^2)$ the winding number $w_\mu(q)$ is defined for almost every $q \in \mathbb{R}^2$ precisely because $\text{im}(\mu)$ is a set of Lebesgue measure zero. For $\alpha \leq \frac{1}{2}$ there are Peano curves $\mu$ covering $[0,1]^2$ for example and as such $w_\mu(q)$ is not defined for any $q \in [0,1]^2$. It is also optimal for defining continuous extensions for currents to Hölder functions. For such an extension one wishes the continuity property as in Theorem 2.1. But it was already noticed by Young [13], that for $\alpha \leq \frac{1}{2}$ there are sequences of smooth functions $f_n \xrightarrow{\alpha} f$ and $g_n \xrightarrow{\alpha} g$ such that $\int f_n \, dg_n$ doesn’t converge. Proposition 1.1 has some immediate consequences in combination with Theorem 1.1. In particular we can recover [12] Theorem 5.

**Corollary 4.2.** Let $\varphi : X \to Y$. 
(1) If $X$ is a quasiconvex metric space with $H_1(X) = 0$ or $H_1^{\text{Lip}}(X) = 0$, $Y = \mathbb{R}^2$, $\varphi$ is $\alpha$-Hölder continuous for $\alpha > \frac{1}{2}$ and $\mathcal{L}^2(\text{im}(\varphi)) = 0$, then $\varphi$ factors through a tree.

(2) If $X$ is a quasiconvex metric space with $\pi_1^{\text{Lip}}(X) = 0$, $Y$ is purely 2-unrectifiable and $\varphi$ is Lipschitz continuous, then $\varphi$ factors through a geodesic tree via Lipschitz maps.

Proof. The first statement is obvious because any winding number function considered in Proposition 1.1 vanishes outside the image of $\varphi$. To see the second, let $\gamma : S^1 \to X$ be a closed Lipschitz curve. Because $\pi_1^{\text{Lip}}(X) = 0$, there is a Lipschitz extension $\Gamma : B^2(0,1) \to X$. The current $(\varphi \circ \Gamma)_# [B^2(0,1)]$ is a 2-dimensional integral current in $Y$. Since $Y$ is purely 2-unrectifiable, $(\varphi \circ \Gamma)_# [B^2(0,1)] = 0$ and hence also,

$$0 = \partial ((\varphi \circ \Gamma)_# [B^2(0,1)]) = (\varphi \circ \gamma)_# [S^1].$$

From the estimates of the distances in Theorem 1.1, the maps $\psi$ and $\varphi$ are Lipschitz and by switching to the path metric we can also assume $T$ to be a geodesic tree. □

With 2.4 we can give a proof of Theorem 1.2 for the case $\alpha > \frac{3}{4}$. As noted before, $\pi_1^{\text{Lip}}(X) = 0$ implies $H_1^{\text{Lip}}(X) = 0$. Let $Q = [0,1]^2$ and assume $\varphi : X \to \mathbb{R}^2$ satisfies $\int w_{\varphi \gamma} = 0$ for all closed Lipschitz curves $\gamma : \partial Q \to X$. Pick some such curve $\gamma : \partial Q \to X$. Since $\pi_1^{\text{Lip}}(X) = 0$ there is a Lipschitz extension $\Gamma : Q \to X$. We want to show that $w_{\varphi \gamma} = 0$ almost everywhere and the statement then follows from Proposition 1.1 above. For any square $R \subset Q$, Lemma 2.3 implies

$$\int_{\partial R} (\varphi \circ \Gamma)_x d(\varphi \circ \Gamma)_y = \int_{w_{\varphi \gamma} \partial R} = 0.$$  

Hence for any $f \in \text{Lip}(\mathbb{R}^2),$

$$\int_{\partial R} w_{\varphi \gamma}(q)f(q) dq = (\varphi \circ \Gamma)_# [Q] (f dx \wedge dy) = I_Q(f \circ \varphi \circ \Gamma, (\varphi \circ \Gamma)_x, (\varphi \circ \Gamma)_y)$$

$$= \lim_{m \to \infty} \sum_{R \in \mathcal{P}_m(Q)} f \circ \varphi \circ \Gamma(b_R) \int_{\partial R} (\varphi \circ \Gamma)_x (\varphi \circ \Gamma)_y = 0.$$  

Since $w_{\varphi \gamma}(q)$ is locally constant outside of $\text{im}(\varphi \circ \gamma)$ it follows immediately that $w_{\varphi \gamma}(q) = 0$ for almost all $q$ (those in the complement of $\text{im}(\varphi \circ \gamma)$).

This argument doesn’t work for $\frac{3}{4} < \alpha \leq \frac{3}{2}$ because we can’t define the two-dimensional current $(\varphi \circ \Gamma)_# [Q]$ if $\varphi$ has this lower regularity. To circumvent this problem, we will define a functional close in spirit to the definition of $I_Q$ that makes sense also for this range of $\alpha$ and allows for a smooth test-function $f$ similar to the calculation above.

4.1. A formula for the second moments. In this subsection we consider a Hölder map $\varphi : Q \to \mathbb{R}^2$ defined on some square $Q \subset \mathbb{R}^2$. We first fix some notation.

For a function $f \in C^{k+1}(\mathbb{R}^n)$ consider the Taylor polynomial representation of degree $k$ at a point $w$,

$$f(v) = T_{f,k,w}(v) + R_{f,k,w}(v) = \sum_{|\alpha| \leq k} \frac{\partial^\alpha f(v)}{\alpha!} (v-w)^\alpha + R_{f,k,w}(v),$$

where $T_{f,k,w} = \sum_{|\alpha| \leq k} \frac{\partial^\alpha f(w)}{\alpha!} (v-w)^\alpha$.
where \( a = (a_1, \ldots, a_n) \in \mathbb{N}^n \) is some multi-index and \( |a| = |a_1| + \cdots + |a_n| \) as well as \((v - w)^a = (v_1 - w_1)^{a_1} \cdots (v_n - w_n)^{a_n}\). From the definition it is obvious that \( R_{f,k,w} \) is in \( C^{k+1}(\mathbb{R}^n) \). Further, it is well known that

\[
R_{f,k,w}(v) = (k + 1) \sum_{|a|=k+1} \frac{(v - w)^a}{a!} \int_0^1 (1 - t)^k \partial^a f(w + t(v - w)) \, dt.
\]

So, if \( f \in C^{k+2}(\mathbb{R}^n) \), then

\[
R_{f,k,w}(v) = \sum_{|a|=k+1} (v - w)^a R_{f,k,a,w}(v),
\]

for some functions \( R_{f,k,a,w} \) that are Lipschitz continuous with

\[
\text{Lip}(R_{f,k,a,w}) \leq C \|D^{k+2}f\|_\infty.
\]

for some constant \( C \) depending only on \( k \). Because \( R_{f,k,a,w}(w) = 0 \) (if \( |a| > 0 \)) we get \( \|R_{f,k,a,w}\|_{B^1(\mathbb{R}^n)} \leq C r \|D^{k+2}f\|_\infty \). We denote \( \|f\|_k := \sum_{|a|=k} \|D^a f\|_\infty \). From now on we will drop the \( \infty \) index if we take the sup-norm. Using the Taylor polynomial representation we have the following approximation of the Riemann-Stieltjes integrals around squares.

**Lemma 4.3.** Let \( \varphi \in H^\alpha(Q, \mathbb{R}^2) \) for some \( \alpha > \frac{1}{2} \) and \( g_1, g_2 \in C^\infty(\mathbb{R}^2) \). Then there is a constant \( C > 0 \) such that for any square \( R \subset Q \) with side length \( s(R) \leq 1 \),

\[
\left| \int_{\partial R} g_1 \circ \varphi \, d(g_2 \circ \varphi) - \int_{\partial R} T_{g_1,2,\varphi(b_R)} \circ \varphi \, d(T_{g_2,2,\varphi(b_R)} \circ \varphi) \right| \leq C \|g_1\|_4 \|g_2\|_4 s(R)^{4\alpha},
\]

for some constant \( C \) depending only on \( \alpha \) and \( H^\alpha(\varphi) \). Further, for multiindices \( a \) and \( c \),

\[
\left| \int_{\partial R} (\varphi - \varphi(b_R))^a \, d(\varphi - \varphi(b_R))^c \right| \leq C_n |a||c| H^\alpha(\varphi)^{|a|+|c|} s(R)^{(|a|+|c|)\alpha}.
\]

**Proof.** We will restrict the function \( \varphi \) to \( R \), \( g_1 \) and \( g_2 \) to \( B(b_R, r) \) where \( r = H^\alpha(\varphi) s(R)^\alpha \) since this ball contains \( \varphi(R) \). By Theorem 2.1(1) we know that

\[
\left| \int_{\partial R} (h_1 \circ \varphi) \, d(h_2 \circ \varphi) \right| \leq C_\alpha H^\alpha(h_1 \circ \varphi) H^\alpha(h_2 \circ \varphi) s(R)^{2\alpha},
\]

for any \( h = (h_1, h_2) \in \text{Lip}(\mathbb{R}^2, \mathbb{R}^2) \) and some constant \( C_\alpha \) depending only on \( \alpha \). For \( \psi_1, \psi_2 \in H^\alpha(R, \mathbb{R}^2) \), \( u \in \text{Lip}(\mathbb{R}^2) \) the following estimates are easy to obtain,

\[
H^\alpha(u \circ \varphi) \leq \text{Lip}(u) H^\alpha(\varphi),
\]

\[
H^\alpha(\psi_1 \psi_2) \leq \|\psi_1\|_2 H^\alpha(\psi_2) + \|\psi_2\|_2 H^\alpha(\psi_1).
\]

Let \( a = (a_x, a_y) \) be some multi-index with \( |a| = |a_x| + |a_y| \geq 1 \). Using \( s(R) \leq 1 \) and setting \( H := \max\{1, H^\alpha(\varphi)\} \) these estimates lead to,

\[
H^\alpha((u \circ \varphi)(\varphi - \varphi(b_R))^a)
\leq \|u\|_2 H^\alpha((\varphi - \varphi(b_R))^a) + \|((\varphi - \varphi(b_R))^a\| \text{Lip}(u) H^\alpha(\varphi)
\leq \|u\|_2 H^\alpha(\varphi)^{|a|} |a| s(R)^{|a|-1}\alpha + \text{Lip}(u) H^\alpha(\varphi)^{|a|+1} s(R)^{|a|\alpha}
\leq (\|u\| + \text{Lip}(u)) H^{|a|+1} s(R)^{|a|-1}\alpha.
\]
further implies,
\[
\left| \int_{\partial R} (h_1 \circ \varphi)(\varphi - \varphi(b_R))^a \, d(h_2 \circ \varphi) \right| 
\leq C_\alpha |a| \, \text{Lip}(h_2)(\|h_1\| + \text{Lip}(h_1))H^{\alpha+3}s(R)^{|a|+1}\alpha .
\]

(4.2)

The Taylor polynomial representation of order two for \(g_1\) and \(g_2\) around \(\varphi(b_R)\) are given by,
\[
g_i(v) = T_{g_i,2,\varphi(b_R)}(v) + \sum_{|a|=3} R_{g_i,2,a,\varphi(b_R)}(v)(v-b_R)^a .
\]

By (4.2),
\[
\left| \int_{\partial R} (g_1 - T_{g_1,2,\varphi(b_R)} \circ \varphi) \, d(g_2 \circ \varphi) \right| 
\leq \sum_{|a|=3} \left| \int_{\partial R} (R_{g_1,2,a,\varphi(b_R)} \circ \varphi)(\varphi - \varphi(b_R))^a \, d(g_2 \circ \varphi) \right| 
\leq \sum_{|a|=3} C_\alpha \text{Lip}(g_2)(\text{Lip}(R_{g_1,2,a,\varphi(b_R)}) + \|R_{g_1,2,a,\varphi(b_R)}\|_{\varphi(R)})\|H^6s(R)^{4\alpha}
\leq C' \text{Lip}(g_2)\|D^4g_1\|s(R)^{4\alpha} ,
\]

for some \(C'\) depending only on \(\alpha\) and \(H^\alpha(\varphi)\). Similarly,
\[
\left| \int_{\partial R} T_{g_1,2,\varphi(b_R)} \circ \varphi \, d((g_2 - T_{g_2,2,\varphi(b_R)}) \circ \varphi) \right| 
= \left| \int_{\partial R} (g_2 - T_{g_2,2,\varphi(b_R)}) \circ \varphi \, d(T_{g_1,2,\varphi(b_R)} \circ \varphi) \right| 
\leq \sum_{|a|=3} \left| \int_{\partial R} (R_{g_2,2,a,\varphi(b_R)} \circ \varphi)(\varphi - \varphi(b_R))^a \, d(T_{g_1,2,\varphi(b_R)} \circ \varphi) \right| 
\leq C' \text{Lip}(T_{g_1,2,b_R})\|D^4g_2\|s(R)^{4\alpha} .
\]

Note that \(\text{Lip}(T_{g_1,2,\varphi(b_R)}) \leq \|Dg_1\| + \|D^2g_1\|r\). Summing the two estimates above gives the first estimate of the lemma.

The second follows from (4.1). If \(|a| = 0\) or \(|c| = 0\) the integral vanishes, and if \(|a|, |c| \geq 1\),
\[
\left| \int_{\partial R} (\varphi - \varphi(b_R))^a d(\varphi - \varphi(b_R))^c \right| 
\leq C_\alpha H^\alpha((\varphi - \varphi(b_R))^a) H^\alpha((\varphi - \varphi(b_R))^c)s(R)^{2\alpha}
\leq C_\alpha |a||c| H^\alpha(\varphi)^{|a|+|c|}s(R)^{|a|+|c|}\alpha .
\]

If we have good enough bounds on the Riemann-Stieltjes integrals over the boundary of squares, then we can construct some multilinear functional that has similar behaviour to a 2-dimensional current. The statement and the proof of the following lemma is very similar to Lemma [2.2]
Lemma 4.4. Let $\varphi \in C^\alpha(Q, \mathbb{R}^2)$ for $\alpha > \frac{1}{2}$ where $Q$ is some square and let $g, h \in \text{Lip}(\mathbb{R}^2)$. Assume that

$$\left| \int_{\partial R} g \circ \varphi d(h \circ \varphi) \right| \leq C \text{Lip}(R)^c,$$

for some constants $C > 0$, $c > 2 - \alpha$ and all squares $R \subset Q$. Then the limit

$$A_\varphi(f, g, h) := \lim_{m \to \infty} \sum_{R \in P_m(Q)} f(\varphi(b_R)) \int_{\partial R} g \circ \varphi d(h \circ \varphi)$$

exists for all $f \in \text{Lip}(\mathbb{R}^2)$.

Proof. Let $R \subset Q$ be some square. $R$ is the union of the 4 similar squares $R_1, \ldots, R_4$ half the size. By the assumptions of the lemma we have,

$$\left| f(\varphi(b_R)) \int_{\partial R} g_1 \circ \varphi d(g_2 \circ \varphi) - \sum_{i=1}^4 f(\varphi(b_{R_i})) \int_{\partial R_i} g \circ \varphi d(g \circ \varphi) \right|$$

$$= \left| \sum_{i=1}^4 (f(\varphi(b_R)) - f(\varphi(b_{R_i}))) \int_{\partial R_i} g_1 \circ \varphi d(g_2 \circ \varphi) \right|$$

$$\leq 4 \text{Lip}(f) \text{H}^\alpha(\varphi) s(R)^\alpha C \text{Lip}(R)^c$$

$$=: C' s(R)^{c+\alpha}.$$

Let $A_m$ be the term in the approximating sequence where we sum over squares of side length $2^{-m}s(Q)$. By the estimate above,

$$|A_m - A_{m+1}| \leq 4^m C'(2^{-m}s(Q))^{c+\alpha} = C' 2^{m(2-\alpha-c)} s(Q)^{c+\alpha}.$$ 

Because $c > 2 - \alpha$, $(A_m)$ is a Cauchy sequence in $\mathbb{R}$ and hence converges. \qed

From Lemmas 4.3 and 4.4 we get:

Proposition 4.5. Let $Q \subset \mathbb{R}^2$ be a square and $\varphi \in C^\alpha(Q, \mathbb{R}^2)$ for some $\alpha > \frac{1}{2}$ and assume that $\int_{\partial R} \varphi_x d\varphi_y = 0$ for all squares $R \subset Q$. Then $A_\varphi(f, g, h)$ is well defined if $f \in \text{Lip}(\mathbb{R}^2)$ and $(g, h) \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$. Further,

$$2A_\varphi(f, g, h) = A_\varphi(f \partial^x \det(D(g, h)), x^2, y) + A_\varphi(f \partial^y \det(D(g, h)), x, y^2),$$

and in particular if $f \in C^\infty(\mathbb{R}^2)$,

$$2 \int_{\mathbb{R}^2} f(q) \deg(\varphi, Q, q) \, dq = A_\varphi(\partial^x f, x^2, y) + A_\varphi(\partial^y f, x, y^2).$$
Proof. We use the notation \( \psi_a(R) := \partial^a \psi(\varphi(b_R)) \) for a smooth function \( \psi : \mathbb{R}^2 \to \mathbb{R} \) and a multi-index \( a \). By Lemma 4.3 we have,

\[
\int_{\partial R} T_{g,2,\varphi(b_R)} \circ \varphi d(T_{h,2,\varphi(b_R)} \circ \varphi) = \\
+ \frac{1}{2} g_x(R) h_y(R) \int_{\partial R} (\varphi_x - \varphi_x(b_R)) d(\varphi_y - \varphi_y(b_R))^2 \\
+ g_x(R) h_{xy}(R) \int_{\partial R} (\varphi_x - \varphi_x(b_R)) d(\varphi_x - \varphi_x(b_R))(\varphi_y - \varphi_y(b_R)) \\
+ \frac{1}{2} g_y(R) h_{xx}(R) \int_{\partial R} (\varphi_y - \varphi_y(b_R)) d(\varphi_x - \varphi_x(b_R))^2 \\
+ g_y(R) h_{xy}(R) \int_{\partial R} (\varphi_y - \varphi_y(b_R)) d(\varphi_x - \varphi_x(b_R))(\varphi_y - \varphi_y(b_R)) \\
- (g \leftrightarrow h) + o(s(R)^2) .
\]

These terms can be simplified. For example it is,

\[
\int_{\partial R} \varphi_x d\varphi_y^2 = \int_{\partial R} \varphi_x d(\varphi_y^2 - 2 \varphi_y \cdot \varphi_y(b_R) + \varphi_y(b_R)^2) \\
= \int_{\partial R} (\varphi_x - \varphi_x(b_R)) d(\varphi_y - \varphi_y(b_R))^2 .
\] (4.3)

Integration by parts leads to,

\[
\int_{\partial R} \varphi_x d\varphi_y = - \int_{\partial R} \varphi_x \varphi_y d\varphi_x = - \frac{1}{2} \int_{\partial R} \varphi_y d\varphi_x^2 = \frac{1}{2} \int_{\partial R} \varphi_x d\varphi_y^2 .
\]

Consider the two evaluations on subsquares \( R \subset Q \),

\[
e_y(R) := \frac{1}{2} \int_{\partial R} \varphi_x^2 d\varphi_y \quad \text{and} \quad e_x(R) := \frac{1}{2} \int_{\partial R} \varphi_x d\varphi_y^2 .
\]

Using the notation \( f(R) := f(b_R) \) for functions on \( \mathbb{R}^2 \) and apply the first estimate of Lemma 4.3 we can express,

\[
\int_{\partial R} g \circ \varphi d(h \circ \varphi) = o(s(R)^2) \\
+ (g_x h_{yy} e_x)(R) + (g_x h_{xy} e_y)(R) - (g_y h_{xx} e_y)(R) - (g_y h_{xy} e_x)(R) \\
- (h_x g_{yy} e_x)(R) - (h_x g_{xy} e_y)(R) + (h_y g_{xx} e_y)(R) + (h_y g_{xy} e_x)(R) \\
= o(s(R)^2) + (\det(D(g,h)) e_x)(R) + (\det(D(g,h)) e_y)(R) .
\]

The second estimate of Lemma 4.3 together with (4.3) shows that \( |e_x(R)| \leq Cs(R)^{3\alpha} \) and similarly for \( e_y \). The estimate above then shows that,

\[
(4.4) \quad \left| \int_{\partial R} g \circ \varphi d(h \circ \varphi) \right| \leq C' s(R)^{3\alpha} ,
\]
Assume $g$ is well defined and in particular, \[ f \] holds. To see the last equation note that if \( \alpha \) is satisfied in Theorem 1.2 for some $C'$ depending only on $\alpha$, $g$ and $h$. Lemma 1.3 implies that $A_\varphi(f,g,h)$ is well defined and in particular,

\[
2A_\varphi(f,g,h) = 2 \lim_{m \to \infty} \sum_{R \in \mathcal{P}_m(Q)} f(\varphi(b_R)) \int_{\partial R} g \circ \varphi \, d(h \circ \varphi)
\]

\[
= \lim_{m \to \infty} \sum_{R \in \mathcal{P}_m(Q)} o(s(R)^2) + (f \det(D(g,h)))_x(b_R) \int_{\partial R} \varphi_x^2 d\varphi_y
\]

\[
+ (f \det(D(g,h)))_y(b_R) \int_{\partial R} \varphi_x d\varphi_y
\]

\[
= A_\varphi(f \det(D(g,h))_x, x^2, y) + A_\varphi(f \det(D(g,h))_y, x, y^2).
\]

To see the last equation note that if $f = 1$, then by Lemma 2.3 \[ A_\varphi(1, g, h) = \int_{\partial Q} (g \circ \varphi) \, d(h \circ \varphi) = \int_{\mathbb{R}^2} \det(D(g,h)) \deg(\varphi, Q, q) \, dq. \]

Assume $g$ does not depend on $y$, then

\[
\int_{\mathbb{R}^2} g_x(h_y(q)) \deg(\varphi, Q, q) \, dq = \int_{\mathbb{R}^2} \det(D(g,h)_y) \deg(\varphi, Q, q) \, dq
\]

\[
= A_\varphi(1, g, h)
\]

\[
= \frac{1}{2} A_\varphi(g_x h_{yy}, x, y^2) + \frac{1}{2} A_\varphi(g_x h_{xy} + h_y g_{xx}, x^2, y).
\]

For $g(x) = x$ this leads to

\[
2 \int_{\mathbb{R}^2} h_y(q) \deg(\varphi, Q, q) \, dq = A_\varphi(h_{xy}, x^2, y) + A_\varphi(h_{yy}, x, y^2),
\]

for any $h \in C_c^\infty$. If $h \in C_c^\infty$ satisfies $\int_{-\infty}^{\infty} h(x,t) \, dt = 0$ for all $x$, there is a $H \in C_c^\infty$ with $H_y = h$. Hence

\[
2 \int_{\mathbb{R}^2} h(q) \deg(\varphi, Q, q) \, dq = A_\varphi(h_x, x^2, y) + A_\varphi(h_y, x, y^2).
\]

Because, both sides of the equation are not affected by changing $h$ outside the image of $\varphi$, the identity above is true for all $h \in C_c^\infty$. \qed

With this proposition at hand we can prove Theorem 1.2 in the introduction.

**Proof of Theorem 1.2.** As noted in the beginning of this section, \( \pi_1^{\text{Lip}}(X) = 0 \) implies $H_1^{\text{Lip}}(X) = 0$, so all the assumptions for $X$ are satisfied in order to apply Theorem 1.1. Fix $Q = [0,1]^2$ and let $\gamma: \partial Q \to X$ be some closed Lipschitz curve. With Proposition 4.1 it remains to show that the assumptions in Theorem 1.2 for $\alpha > \frac{2}{3}$ and $\alpha > \frac{3}{4}$ respectively imply that \( w_{\varphi \circ \gamma} = 0 \) almost everywhere. Let \( \Gamma: Q \to \mathbb{R}^2 \) be a Lipschitz extension and set $\varphi' := \varphi \circ \Gamma: Q \to \mathbb{R}^2$. By assumption we have for all squares $R \subset Q$,

\[
\int_{\partial R} \varphi'_x \, d\varphi'_y = 0,
\]

in case $\alpha > \frac{2}{3}$ and

\[
\int_{\partial R} \varphi'_x \, d\varphi'_y = \sum_{R} \varphi'_y \, d\varphi'_y = \int_{\partial R} \varphi'_x \, d\varphi'_y = 0,
\]
in case $\alpha > \frac{1}{2}$. Proposition 4.3 gives for smooth $f$,

$$2 \int_{\mathbb{R}^2} f(q) w_{\varphi \gamma}(q) \, dq = A_{\gamma} (\partial_x^2 f, x^2, y) + A_{\gamma} (\partial_y^2 f, x, y^2).$$

In case $\alpha > \frac{2}{3}$, Lemma 4.3 implies,

$$\left| \int_{\partial R} \varphi_x^2 \, d\varphi_y \right|, \quad \left| \int_{\partial R} \varphi_x \, d\varphi_y^2 \right| \leq C s(R)^{3\alpha} = o(s(R)^2),$$

and hence $A_{\gamma} (\partial_x f, x^2, y) = A_{\gamma} (\partial_y f, x, y^2) = 0$. In case $\alpha > \frac{1}{2}$, the additional assumptions imply the same conclusion and therefore for all smooth $f$,

$$\int_{\mathbb{R}^2} f(q) w_{\varphi \gamma}(q) \, dq = 0.$$ 

This shows that $w_{\varphi \gamma}$ vanishes almost everywhere and Proposition 4.1 finishes the theorem. \qed

4.2. Heisenberg group target. The metric space $(\mathbb{H}, d_{cc})$ is bi-Lipschitz equivalent to $\mathbb{R}^3$ equipped with the Korányi metric, see e.g. [5],

$$d_K(p, q) := \left[ |q_x - p_x|^2 + |q_y - p_y|^4 + 16|q_z - p_z - 2^{-1}(p_x q_y - p_y q_x)|^2 \right]^\frac{1}{2}.$$ 

Since the statements of Theorem 1.3 do not depend on a change to a bi-Lipschitz equivalent space, we will work with $(\mathbb{R}^3, d_K)$ instead of $(\mathbb{H}, d_{cc})$. It is rather direct to check that for any bounded subset $B \subset \mathbb{R}^3$ there is a constant $C_B \geq 0$ such that for all $p, q \in B$,

$$C_B^{-1} d_K(p, q)^2 \leq d_{\text{Eucl}}(p, q) \leq C_B d_K(p, q).$$

Along the proof of [10] Lemma 3.2 one can show that for any curve $\gamma : [a, b] \to (\mathbb{R}^3, d_K)$ and any $f : [a, b] \to \mathbb{R}$ that are $\alpha$-Hölder continuous for $\alpha > 1/2$,

$$(4.5) \quad \int_a^b f \, d\gamma_z = \frac{1}{2} \left[ \int_a^b f \, d\gamma_y \, d\gamma_x - \int_a^b f \, d\gamma_x \, d\gamma_y \right].$$

If $\gamma$ is closed, this implies

$$(4.6) \quad \int \gamma_x \, d\gamma_z = \frac{3}{4} \int \gamma_x^2 \, d\gamma_y, \quad \int \gamma_y \, d\gamma_z = \frac{3}{4} \int \gamma_x \, d\gamma_y^2.$$ 

Here is a derivation of the first identity,

$$\int \gamma_x \, d\gamma_z = \frac{1}{2} \left[ \int \gamma_x^2 \, d\gamma_y - \int \gamma_x \, d\gamma_x \right] = \frac{1}{2} \left[ \int \gamma_x^2 \, d\gamma_y - \frac{1}{2} \int \gamma_y \, d\gamma_z^2 \right] = \frac{3}{4} \int \gamma_x \, d\gamma_y^2.$$ 

Note that terms $\int \gamma_y \, d\gamma_z$ and $\int \gamma_x \, d\gamma_x$ that appear in (4.6) are precisely those that are assumed to vanish in Theorem 1.2 in case $\frac{1}{2} < \alpha \leq \frac{2}{3}$. First we show the following lemma.

**Lemma 4.6.** Let $Q \subset \mathbb{R}^2$ be a square and $\varphi : (Q, d_{\text{Eucl}}) \to (\mathbb{R}^3, d_K)$ be Hölder continuous of order $\alpha > \frac{2}{3}$. Then $\tilde{\varphi}_Q[Q] = 0$ for the Hölder map $\tilde{\varphi} : (Q, d_{\text{Eucl}}) \to (\mathbb{R}^3, d_{\text{Eucl}})$ obtained by changing the metric on $\mathbb{R}^3$. 
Proof. By a smoothing argument it is enough to show that \( \hat{\varphi}_\# [Q](\omega) = 0 \) for any smooth differential form \( \omega \in \Omega^2(\mathbb{R}^3) \). Using (2.1) and since \( \omega \) can be written as \( \sum_{i<j} \mu_{ij} \, dx_i \wedge dx_j \) for smooth functions \( \mu_{ij} \) on \( \mathbb{R}^3 \), it is enough to show that
\[
\hat{\varphi}_\# [Q](\mu_{ij} \, dx_i \wedge dx_j) = I_Q(\mu_{ij} \circ \varphi, \varphi_i, \varphi_j) = 0,
\]
for all \( i < j \). By the definition of \( I_Q \) and (4.6),
\[
\hat{\varphi}_\# [Q](\mu \, dx \wedge dz) = I_Q(\mu \circ \varphi, \varphi_x, \varphi_z)
\]
\[
= \lim_{m \to \infty} \sum_{R \in P_m(Q)} \mu \circ \varphi(b_Q) \int_{\partial R} \varphi_x \, d\varphi_z
\]
\[
= \lim_{m \to \infty} \frac{3}{4} \sum_{R \in P_m(Q)} \mu \circ \varphi(b_Q) \int_{\partial R} \varphi_y^2 \, d\varphi_y
\]
\[
= \frac{3}{4} \hat{\varphi}_\# [Q](\mu \, dx^2 \wedge dy)
\]
\[
= \frac{3}{2} \hat{\varphi}_\# [Q](x \mu \, dx \wedge dy).
\]
Similarly, \( \hat{\varphi}_\# [Q](\mu \, dx \wedge dy) = \frac{3}{2} \hat{\varphi}_\# [Q](y \mu \, dx \wedge dy) \) and hence it remains to show that \( \hat{\varphi}_\# [Q](\mu \, dx \wedge dy) = 0 \) for all smooth \( \mu : \mathbb{R}^3 \to \mathbb{R} \). By setting \( f = 1 \) in (4.5), we get \( \int_{\partial R} \varphi_x \, d\varphi_y = 0 \) for all squares \( R \subset Q \), and therefore \( \hat{\varphi}_\# [Q](\mu \, dx \wedge dy) = 0 \) follows from the definition of \( I_Q \).

With this preparation we can give a proof of the remaining theorem in the introduction.

Proof of Theorem 1.3. Let \( \varphi : (X, d) \to (\mathbb{H}, d_{cc}) \) be a Hölder map as in Theorem 1.3. In order to apply Theorem 1.1 we will show that \( \hat{\varphi} : X \to \mathbb{R}^3 \) as defined in the lemma above has Property T. Let \( \gamma : \partial Q \to X \) be any closed Lipschitz curve defined on the boundary of some square \( Q \subset \mathbb{R}^2 \). Because \( \pi_1^{\text{Lip}}(X) = 0 \), there is a Lipschitz extension \( \Gamma : Q \to X \) of \( \gamma \). By Lemma 1.6
\[
0 = \partial(\hat{\varphi} \circ \Gamma) \# [Q] = (\hat{\varphi} \circ \gamma) \# [\partial Q].
\]
Proposition 4.1 now implies that \( \hat{\varphi} \) has Property T. Since Property T is purely topological, the same holds for \( \varphi \) and Theorem 1.3 applies.

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