C-EIGENVALUE INTERVALS FOR PIEZOELECTRIC-TYPE TENSORS VIA SYMMETRIC MATRICES

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Abstract. C-eigenvalues of piezoelectric-type tensors play an crucial role in piezoelectric effect and converse piezoelectric effect. In this paper, by the partial symmetry property of piezoelectric-type tensors, we present three intervals to locate all C-eigenvalues of a given piezoelectric-type tensor. Numerical examples show that our results are better than the existing ones.

1. Introduction. The definition of piezoelectric-type tensors was introduced by Chen et al. [2].

Definition 1.1. [2] Let \( A = (a_{ijk}) \in \mathbb{R}^{n \times n \times n} \) be a third-order \( n \) dimensional real tensor. If the later two indices of \( A \) are symmetric, i.e., \( a_{ijk} = a_{ikj} \) for all \( j,k \in [n] \), where \([n] := \{1, 2, \cdots, n\}\), then \( A \) is called a piezoelectric-type tensor.

Given a piezoelectric-type tensor \( A \), for each \( i \in [n] \), let the matlab multidimensional array notation \( A(i, :, :) \in \mathbb{R}^{n \times n} \) denote the matrix whose \((j, k)\)-entry is \( a_{ijk} \). Then, it follows from Definition 1.1 that \( A(i, :, :) \) is a symmetric matrix.

As we know, third order tensors have many important applications in some fields, such as physics and engineering [3]-[14]. As a special third-order tensor, piezoelectric-type tensors play a crucial role in piezoelectric effect and converse piezoelectric effect [2]. In order to investigate more properties related to piezoelectric effect and converse piezoelectric effect in solid crystal, the authors in [2] gave the definitions of C-eigenvalues and C-eigenvectors for piezoelectric-type tensors.

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Definition 1.2. [2] Let \( A = (a_{ijk}) \in \mathbb{R}^{n \times n \times n} \) be a piezoelectric-type tensor. If there exists a scalar \( \lambda \in \mathbb{R} \), and two vectors \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^n \) satisfying the following system
\[
Ayy = \lambda x, \quad xAy = \lambda y, \quad x^T x = 1, \quad y^T y = 1,
\]
where \( Ayy \in \mathbb{R}^n \) and \( xAy \in \mathbb{R}^n \) with the \( i \)-th entry
\[
(Ayy)_i = \sum_{j,k \in [n]} a_{ijk}y_jy_k \quad \text{and} \quad (xAy)_i = \sum_{j,k \in [n]} a_{ijk}x_jy_k,
\]
respectively, then \( \lambda \) is called a C-eigenvalue of \( A \), and \( x \) and \( y \) are called associated left and right C-eigenvectors, respectively. Also, \((\lambda, x, y)\) can be called a C-eigenpair of \( A \).

Suppose that \( x = (x_1, x_2, \cdots, x_n)^T \) and \( y = (y_1, y_2, \cdots, y_n)^T \) are the left and right C-eigenvectors corresponding to \( \lambda \), then \( \lambda = xAy := \sum_{i,j,k \in [n]} a_{ijk}x_jy_k \).

Moreover, \((\lambda, x, -y), (-\lambda, -x, y), (-\lambda, -x, -y)\) are also C-eigepairs of \( A \). Let \( \lambda^* \) be the largest C-eigenvalue of a piezoelectric type tensor \( A \), then \( \lambda^* = \max\{xAy : x^T x = 1, y^T y = 1\} \). Chen et al. [2] showed that the largest C-eigenvalue of the piezoelectric-type tensor has concrete physical meaning which determines the highest piezoelectric coupling constant.

Due to the importance of the largest C-eigenvalue of the piezoelectric-type tensor, some authors focus their interest on this topic. However, the largest C-eigenvalue \( \lambda^* \) is difficult to determine in practice since computing C-eigenvectors is difficult. Therefore, a valid method is needed to give an interval to locate all the C-eigenvalues. Recently, Li et al. [8] gave an interval to locate all the C-eigenvalues by the S-partition method. Later, Che et al. [1], and Wang et al. [13] also presented some inclusion sets for piezoelectric-type tensors, and Wang et al. [13] proved that their results are tighter than those in [1]. In order to compare them with our results, we will list these existing results in the following. Before that, we introduce some notations.

Let \( \mathbb{C} \) and \( \mathbb{R} \) be the complex and real field, and \( \mathbb{C}^{m \times n} \) denote the set of all \( m \times n \) matrices over the complex field \( \mathbb{C} \). For a matrix \( A = (a_{ij}) \in \mathbb{C}^{m \times n} \), let \( ||A||_{v_1} = \sum_{i \in [m], j \in [n]} |a_{ij}| \), which is a compatible matrix norm. If the matrix \( A \) is positive semidefinite, we denote it by \( A \geq 0 \) for short. Hence, \( A \geq B \) means that both matrices \( A \) and \( B \) are Hermitian and \( A - B \) is positive semidefinite. In addition, the symbol \( \otimes \) stands for Kronecker product.

Theorem 1.3. [8] Let \( A = (a_{ijk}) \in \mathbb{R}^{n \times n \times n} \) be a piezoelectric-type tensor, and \( \lambda \) be a C-eigenvalue of \( A \). And let \( S \) be a subset of \([n]\). Then
\[
\lambda \in [-\rho_S, \rho_S],
\]
where
\[
\rho_S = \max_{i,j \in [n]} \frac{1}{2} \left\{ R_j^{\Delta_S, (3)}(A) + \left[ (R_j^{\Delta_S, (3)}(A))^2 + 4R_i^{(1)}(A)R_j^{\Delta_S, (3)}(A) \right]^\frac{1}{2} \right\},
\]
\[
\Delta_S = \{(i,j) : i \in S \text{ or } j \in S\}, \quad \Delta = \{(i,j) : i \notin S \text{ and } j \notin S\},
\]
and
\[
R_i^{(1)}(A) = \sum_{j,k \in [n]} |a_{ijk}|, \quad R_j^{\Delta_S, (3)}(A) = \sum_{l,k \in \Delta_S} |a_{lkj}|, \quad R_j^{\Delta_S, (3)}(A) = \sum_{l,k \in \Delta_S} |a_{lkj}|.
\]
Furthermore, 
\[ \lambda \in [-\rho_{\text{min}}, \rho_{\text{min}}], \]
where \( \rho_{\text{min}} = \min_{S \subseteq [n]} \rho_S \).

**Theorem 1.4.** [13] Let \( A = (a_{ijk}) \in \mathbb{R}^{n \times n \times n} \) be a piezoelectric-type tensor. Then

\[ \sigma(A) \subseteq \gamma(A) = \bigcup_{i \in [n]} \left( \bigcap_{j \in [n], j \neq i} \gamma_{i,j}(A) \right), \quad \gamma_{i,j}(A) = \{ \hat{\gamma}_{i,j}(A) \cup (\tilde{\gamma}_{i,j}(A) \cap \Gamma_i(A)) \}, \]

where

\[ \hat{\gamma}_{i,j}(A) = \left\{ z \in \mathbb{R} : |z| < R_i(A) - R_j^i(A), \quad \text{and} \quad |z| < R_j^i(A) \right\}, \]

\[ \tilde{\gamma}_{i,j}(A) = \left\{ z \in \mathbb{R} : |z - R_i(A) - R_j^i(A)| \leq R_j^i(A) \left( R_j(A) - R_j^j(A) \right) \right\}, \]

\[ \Gamma_i(A) = \{ z \in \mathbb{R} : |z| \leq R_i(A) \}, \]
\[ R_j(A) = \sum_{i,k \in [n]} |a_{ikj}|, \quad R_j^i(A) = \sum_{i \in [n]} |a_{ikj}|. \]

Numerical examples show that Theorem 1.4 is tighter than Theorem 1.3 and the results in [1].

This paper is organized as follows. In Section 2, we first give an improved conclusion of Theorem 1.3; then, we derive two new intervals to locate all C-eigenvalues for the piezoelectric-type tensor. Illustrative numerical examples are presented in Section 3.

To continue the following section, we first list some useful results as follows.

**Lemma 1.5.** Let \( A \in \mathbb{C}^{n \times n} \) be an orthogonal projector, i.e., \( A^2 = A = A^* \), and \( B \in \mathbb{C}^{m \times n} \). Then \( BAB^* \leq BB^* \).

**Proof.** Since \( BB^* - BAB^* = B(I - A)B^* \), noting that \( I - A \) is also an orthogonal projector and \( I - A \geq 0 \), we have \( B(I - A)B^* \geq 0 \). Then the proof is completed. \( \square \)

**Lemma 1.6.** Let \( A = (a_{ij}) \in \mathbb{C}^{n \times n} \) be a positive semidefinite matrix, and \( y = (y_1, y_2, \ldots, y_n)^T \neq 0 \). Then,

\[ y^* Ay \leq \|A\|_{v_1} |y_q|^2, \]

where \( |y_q| = \max_{i \in [n]} |y_i| \).

**Proof.** A simple computation shows that

\[ 0 \leq y^* Ay = \sum_{i,j \in [n]} a_{ij} y_i y_j = \left| \sum_{i,j \in [n]} a_{ij} y_i y_j \right| \leq \sum_{i,j \in [n]} |a_{ij}| |y_i| |y_j| \leq \|A\|_{v_1} |y_q|^2. \]

Then the proof is completed. \( \square \)
2. **Main results.** In this section, we present three intervals to locate all C-eigenvalue for a piezoelectric-type tensor, and the comparisons are also established.

**Theorem 2.1.** Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$ be a piezoelectric-type tensor. Then

$$\lambda \in [-\tilde{\rho}_S, \tilde{\rho}_S],$$

where

$$\tilde{\rho}_S = \max_{i,j \in [n]} \left\{ \frac{1}{2} \left[ R_j^{S,(3)}(\mathcal{A}) + \left( R_j^{\Delta S,(3)}(\mathcal{A}) \right)^2 + 4 \left\| A^2(i,.;.) \right\|_{v_1}^2 R_j^{\Delta S,(3)}(\mathcal{A}) \right] \right\}.$$  

Furthermore,

$$\lambda \in [-\tilde{\rho}_{\min}, \tilde{\rho}_{\min}],$$

where $\tilde{\rho}_{\min} = \min_{S \subseteq [n]} \tilde{\rho}_S$.

**Proof.** Suppose that $(\lambda, x, y)$ is a C-eigenpairs of $\mathcal{A}$, where $x = (x_1, x_2, \cdots, x_n)^T$ and $y = (y_1, y_2, \cdots, y_n)^T$. Let $|x_p| = \max_{i \in [n]} |x_i|$ and $|y_q| = \max_{i \in [n]} |y_i|$. By Definition 1.1, we have

$$\lambda x_p = \sum_{j,k \in [n]} a_{pjk} y_j y_k = y^T A(p,.;.) y.$$  

Note that $yy^T$ is an orthogonal projector, and $A^2(p,.;.)$ is positive semidefinite. Hence, applying Lemma 1.5 and Lemma 1.6, we get

$$(\lambda x_p)^2 \leq y^T A(p,.;.) y \leq \left\| A^2(p,.;.) \right\|_{v_1}^2 |y_q|^2,$$

therefore,

$$|\lambda| |x_p| \leq \left\| A^2(p,.;.) \right\|_{v_1}^\frac{1}{2} |y_q|.$$  

(1)

Li et al. [8] proved that

$$\left( |\lambda| - R_j^{\Delta S,(3)}(\mathcal{A}) \right) |y_q| \leq R_{\Delta q}^{\Delta S,(3)}(\mathcal{A}) |x_p|.$$  

Thus, multiplying (1) by this inequality produces

$$|\lambda| \left( |\lambda| - R_{\Delta q}^{\Delta S,(3)}(\mathcal{A}) \right) |x_p| |y_q| \leq \left\| A^2(p,.;.) \right\|_{v_1}^\frac{1}{2} R_{\Delta q}^{\Delta S,(3)}(\mathcal{A}) |x_p| |y_q|,$$

consequently,

$$|\lambda| \left( |\lambda| - R_{\Delta q}^{\Delta S,(3)}(\mathcal{A}) \right) \leq \left\| A^2(p,.;.) \right\|_{v_1}^\frac{1}{2} R_{\Delta q}^{\Delta S,(3)}(\mathcal{A}).$$  

(2)

Inequality (2) yields

$$|\lambda| \leq \frac{1}{2} \left\{ R_j^{\Delta S,(3)}(\mathcal{A}) + \left( R_j^{\Delta S,(3)}(\mathcal{A}) \right)^2 + 4 \left\| A^2(p,.;.) \right\|_{v_1}^2 R_j^{\Delta S,(3)}(\mathcal{A}) \right\}.$$  

The remaining proof is similar to that for Theorem 2 in [8], so we omit the details. \qed

Comparing Theorem 2.1 with Theorem 1.3, we have the following result.

**Theorem 2.2.** Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$ be a piezoelectric-type tensor. Then

$$[-\tilde{\rho}_{\min}, \tilde{\rho}_{\min}] \subseteq [-\rho_{\min}, \rho_{\min}].$$

**Proof.** Note that

$$\left\| A^2(i,.;.) \right\|_{v_1}^\frac{1}{2} \leq \left( \left\| A(i,.;.) \right\|_{v_1}^2 \right)^\frac{1}{2} = \left\| A(i,.;.) \right\|_{v_1} R_i^{(1)}(\mathcal{A}),$$

which implies that $\tilde{\rho}_S \leq \rho_S$, consequently, $\tilde{\rho}_{\min} \leq \rho_{\min}$. The conclusion follows. \qed
Theorem 2.3. Let $A = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$ be a piezoelectric-type tensor. Then
\[
\lambda \in [-\rho_\sigma, \rho_\sigma],
\]
where $\rho_\sigma = \sqrt{\sum_{i \in [n]} \rho_i^2}$, and $\rho_i$ (i ∈ [n]) is the spectral radius of $A(i, :, :)$.

Proof. Suppose that $(\lambda, x, y)$ is a C-eigenpairs of $A$, where $x = (x_1, x_2, \ldots, x_n)^T$ and $y = (y_1, y_2, \ldots, y_n)^T$. Then
\[
|\lambda| = |xAy| = \left| \sum_{i,j,k \in [n]} a_{ijk} x_i y_j y_k \right| = \left| \sum_{i \in [n]} x_i [y^T A(i, :, :) y] \right| \leq \sum_{i \in [n]} |x_i| |y^T A(i, :, :) y| \leq \sum_{i \in [n]} |x_i| \rho_i \leq \sqrt{\sum_{i \in [n]} \rho_i^2},
\]
which completes the proof. \qed

Theorem 2.4. Let $A = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$ be a piezoelectric-type tensor. Then
\[
\lambda \in [-\rho_A, \rho_A],
\]
where $\rho_A = \|A\|_2$, and $A = \left( A(1, :, :) \cdots A(n, :, :) \right) \in \mathbb{R}^{n \times n^2}$ is a column block matrix with the i-th block is $A(i, :, :)$.

Proof. Suppose that $(\lambda, x, y)$ is a C-eigenpairs of $A$, where $x = (x_1, x_2, \ldots, x_n)^T$ and $y = (y_1, y_2, \ldots, y_n)^T$. Then, by Lemma 1.5, we get
\[
\lambda^2 = \left( \sum_{i \in [n]} x_i [y^T A(i, :, :) y] \right)^2 = \sum_{i,j \in [n]} (x_i y_j A(i, :, :) y_j) (x_j y_i)
= \sum_{i,j \in [n]} (x \otimes y)^T (A(1, :, :) \cdots A(n, :, :) ) (x \otimes y)
\leq \|A\|^2.
\]
Then the conclusion follows. \qed

Comparing Theorem 2.3 and Theorem 2.4, we have the following relationship between $\rho_\sigma$ and $\rho_A$.

Theorem 2.5. Let $A = (a_{ijk}) \in \mathbb{R}^{n \times n \times n}$ be a piezoelectric-type tensor. Then
\[
[-\rho_A, \rho_A] \subseteq [-\rho_\sigma, \rho_\sigma].
\]

Proof. According to the definition of spectral norm, we have
\[
\rho_A^2 = \lambda_{\max}(AA^T) = \lambda_{\max}[A(1, :, :)A^T(1, :, :) + \cdots + A(n, :, :)A^T(n, :, :)]
\leq \lambda_{\max}[A(1, :, :)A^T(1, :, :) + \cdots + \lambda_{\max}[A(n, :, :)A^T(n, :, :)]
= \sum_{i \in [n]} \rho_i^2 = \rho_\sigma^2.
\]
Now, the conclusion is evident. \qed
Chen et al. [2] showed that C-eigenvalues of a piezoelectric-type tensor are invariant under orthogonal transformations, i.e., if \( \lambda, x \) and \( y \) are a C-eigenvalue and its associated C-eigenvectors of a piezoelectric-type tensor \( A \), then \( AQ^3 \) is also a piezoelectric-type tensor, and \( \lambda, Q^T x \) and \( Q^T y \) are a C-eigenvalue and its associated C-eigenvectors of \( AQ^3 \) whose elements are

\[
(AQ^3)_{rst} = \sum_{i,j,k} a_{ijk} q_{ir} q_{js} q_{kt},
\]

where \( Q = (q_{ij}) \in \mathbb{R}^{n \times n} \) is an orthogonal matrix.

Note that Theorem 2.1 is obtained by a similar method used by Theorem 2 in [8] which is not invariant under an orthogonal transformation, so Theorem 2.1 is not invariant under an orthogonal transformation either.

Denote \( B = AQ^3 \), then, we have

\[
B(r,;,:) = Q^T \left( \sum_i q_{ir} A(i,;,:) \right) Q, \ r = 1, 2, \cdots, n.
\]

That is,

\[
B = \begin{pmatrix} B(1,;,:) & \cdots & B(n,;:) \end{pmatrix} \\
= Q^T \left( \sum_i q_{i1} A(i,;,:) \cdots \sum_i q_{in} A(i,;,:) \right) (I_n \otimes Q) \\
= Q^T \left( A(1,;,:) \cdots A(n,;:) \right) (Q \otimes I_n) (I_n \otimes Q) = Q^T A(Q \otimes Q).
\]

Hence, \( \rho_B = \rho_A \), which means that Theorem 2.4 is invariant under an orthogonal transformation.

Specially, by taking the following orthogonal matrix generated by the MATLAB,

\[
Q = \begin{pmatrix}
-0.661224606225391 & -0.412065352084474 & -0.626884491540169 \\
-0.674207337631308 & -0.040035772924967 & 0.737456170067350 \\
-0.328977941519156 & 0.910274289704922 & -0.251344845771740
\end{pmatrix}.
\]

Consider the piezoelectric tensor \( A_{VFeSb} \) given by Example 1, we have

\[
\rho_{\sigma}(A) = 6.3771 < 6.8899 = \rho_{\sigma}(AQ^3).
\]

By taking other orthogonal matrices generated by the MATLAB, we can also find that Theorem 2.3 is not invariant under an orthogonal transformation.

3. Numerical examples. In this section, we present eight piezoelectric tensors from practical problems in [2, 5] to illustrate the results obtained above. The comparison results are listed in Table 1, which shows that our results are tighter than the ones in [8, 13].

Example 1. Consider the piezoelectric tensor \( A_{VFeSb} \) defined by

\[
a_{ijk} = \begin{cases} 
 a_{123} = a_{213} = a_{312} = -3.68180677, \\
 a_{ijk} = 0, otherwise. 
\end{cases}
\]

Example 2. Consider the piezoelectric tensor \( A_{SiO_2} \) defined by

\[
a_{ijk} = \begin{cases} 
 a_{111} = -a_{122} = -a_{212} = -0.13685, \\
 a_{123} = a_{213} = -0.009715, \\
 a_{ijk} = 0, otherwise. 
\end{cases}
\]
Example 3. Consider the piezoelectric tensor $A_{Cr_2AgBiO_6}$ defined by

$$a_{ijk} = \begin{cases} 
  a_{123} = a_{213} = -0.22163, & a_{113} = -a_{223} = 2.608665, \\
  a_{311} = -a_{322} = 0.152485, & a_{312} = -0.37153, \\
  a_{ijk} = 0, & \text{otherwise}.
\end{cases}$$

Example 4. Consider the piezoelectric tensor $A_{BKTaO_3}$ defined by

$$a_{ijk} = \begin{cases} 
  a_{113} = a_{223} = -8.40955, & a_{222} = -a_{212} = -a_{211} = -5.412525, \\
  a_{311} = -a_{322} = -4.3031, & a_{333} = -5.14766, \\
  a_{ijk} = 0, & \text{otherwise}.
\end{cases}$$

Example 5. Consider the piezoelectric tensor $A_{NaBiS_2}$ defined by

$$a_{ijk} = \begin{cases} 
  a_{113} = -8.90808, & a_{223} = -0.00842, & a_{311} = -7.11526, \\
  a_{322} = -0.6222, & a_{333} = -7.93831, \\
  a_{ijk} = 0, & \text{otherwise}.
\end{cases}$$

Example 6. Consider the piezoelectric tensor $A_{LiBiB_2O_5}$ defined by

$$a_{ijk} = \begin{cases} 
  a_{123} = 2.35682, & a_{112} = 0.34929, & a_{211} = 0.16101, & a_{222} = 0.12562, \\
  a_{233} = 0.1361, & a_{213} = -0.05587, & a_{323} = 6.91074, & a_{312} = 2.57812, \\
  a_{ijk} = 0, & \text{otherwise}.
\end{cases}$$

Example 7. Consider the piezoelectric tensor $A_{KBi_F_7}$ defined by

$$a_{ijk} = \begin{cases} 
  a_{111} = 12.64393, & a_{122} = 1.08802, & a_{133} = 4.14350, & a_{123} = 1.59052, \\
  a_{113} = 1.96801, & a_{112} = 0.22465, & a_{211} = 2.59187, & a_{222} = 0.08263, \\
  a_{233} = 0.81041, & a_{223} = 0.511165, & a_{213} = 0.71432, & a_{212} = 0.10570, \\
  a_{311} = 1.51254, & a_{322} = 0.68235, & a_{333} = -0.23019, & a_{323} = 0.19013, \\
  a_{313} = 0.39030, & a_{312} = 0.08381, \\
  a_{ijk} = 0, & \text{otherwise}.
\end{cases}$$

Example 8. Consider the piezoelectric tensor $A_{BaNiO_3}$ defined by

$$a_{ijk} = \begin{cases} 
  a_{113} = a_{223} = 0.038385, & a_{311} = a_{322} = 6.89822, & a_{333} = 27.4628, \\
  a_{ijk} = 0, & \text{otherwise}.
\end{cases}$$

| $A_{V_{Pb,Sb}}$ | $A_{B_{O_2}}$ | $A_{Cr_2AgBiO_6}$ | $A_{BKTaO_3}$ | $A_{NaBiS_2}$ | $A_{LiBiB_2O_5}$ | $A_{KBi_F_7}$ | $A_{BaNiO_3}$ |
|----------------|--------------|-------------------|---------------|--------------|-----------------|--------------|--------------|
| $\lambda^*$   | 4.2514       | 0.1375            | 2.6258        | 12.4234      | 11.6674         | 7.7376       | 13.5021      | 27.4628      |
| $\rho_{\min}$ | 7.3636       | 0.2834            | 5.6606        | 23.5377      | 16.8548         | 12.3206      | 20.2351      | 27.5398      |
| $\rho_{V}$    | 7.3636       | 0.2834            | 5.6666        | 23.5377      | 16.8548         | 12.3206      | 20.2351      | 27.5398      |
| $\rho_{C}$    | 7.3636       | 0.2744            | 4.8058        | 23.5377      | 16.5640         | 11.0127      | 18.8793      | 27.5109      |
| $\rho_{M}$    | 7.3636       | 0.2834            | 4.7861        | 23.5377      | 16.8464         | 11.0038      | 19.8830      | 27.5013      |
| $\rho_{\min}$ | 7.3636       | 0.2393            | 4.6717        | 22.7163      | 14.5723         | 12.1694      | 18.7025      | 27.5398      |
| $\rho_{\sigma}$| 6.3771       | 0.1943            | 3.7242        | 16.0259      | 11.9319         | 7.7540       | 15.5113      | 27.4629      |
| $\rho_{A}$    | 5.2069       | 0.1938            | 3.7625        | 14.9344      | 11.9319         | 7.7523       | 15.5054      | 27.4629      |

From Table 1, we can see that the improved result $\hat{\rho}_{\min}$ is more effective than $\rho_{\min}$; and $\rho_{A}$ is best among the existing results except Example 3.

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REFERENCES

[1] H. T. Che, H. B. Chen and Y. J. Wang, C-eigenvalue inclusion theorems for piezoelectric-type tensors, *Applied Mathematics Letters*, **89** (2019), 41–49.
[2] Y. Chen, A. Jákli and L. Qi, Spectral analysis of piezoelectric tensors, preprint, arXiv:1703.07937v1.
[3] Y. N. Chen, L. Q. Qi and E. G. Virga, Octupolar tensors for liquid crystals, *J. Phys. A*, **51** (2018), 025206, 20 pp.
[4] J. Curie and P. Curie, Développement, par compression de l’électricité polaire dans les cristaux hémisphères à faces inclinées, *Bulletin de Minéralogie*, **3**, 4 (1880), 90–93.
[5] M. De Jong, W. Chen, H. Geerlings, M. Asta and K. A. Persson, A database to enable discovery and design of piezoelectric materials, *Scientific Data*, **2** (2015), 150053.
[6] S. Haussl, *Physical Properties of Crystals: An Introduction*, Wiley-VCH Verlag, Weinheim, 2007.
[7] A. Kholkin, N. Pertsev and A. Goltsev, *Piezoelectricity and Crystal Symmetry*, Piezoelectric and Acoustic Materials, Springer, New York, 2008.
[8] C. Q. Li, Y. J. Liu and Y. T. Li, C-eigenvalues intervals for piezoelectric-type tensors, *Applied Mathematics and Computation*, **358** (2019), 244–250.
[9] D. Lovett, *Tensor Properties of Crystals*, 2nd edition, Institute of Physics Publishing, Bristol, 1989.
[10] J. F. Nye, Physical properties of crystals: Their representation by tensors and matrices, *Physics Today*, **10** (1957), 26 pp.
[11] L. Qi, Transposes, L-eigenvalues and invariants of third order tensors, preprint, (2017), arXiv:1704.01327.
[12] L. Q. Qi and Z. Y. Luo, *Tensor Analysis: Spectral Theory and Special Tensors*, Society for Industrial and Applied Mathematics, Philadelphia, PA, 2017.
[13] W. J. Wang, H. B. Chen and Y. J. Wang, A new C-eigenvalue interval for piezoelectric-type tensors, *Applied Mathematics Letters*, **100** (2020), 106635, 6 pp.
[14] W.-N. Zou, C.-X. Tang and E. Pan, Symmetry types of the piezoelectric tensor and their identification, *Proceedings of The Royal Society A: Mathematical Physical and Engineering Sciences*, **469** (2013), 20120755.

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