Algebraic Geometry

Cubic symmetroids and vector bundles on a quadric surface

Cubiques symétroides et fibrés vectoriels sur une surface quadrique

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Abstract

We investigate the jumping conics of stable vector bundles $E$ of rank 2 on a smooth quadric surface $Q$ with the Chern classes $c_1 = \mathcal{O}_Q(-1, -1)$ and $c_2 = 4$ with respect to the ample line bundle $\mathcal{O}_Q(1, 1)$. As a consequence, we prove that the set of jumping conics $S(E)$ uniquely determines $E$ and that the moduli space of such bundles is rational.

Résumé

Nous étudions les coniques de saut des fibrés vectoriels stables $E$ de rang 2 sur une surface quadrique lisse $Q$ de classes de Chern $c_1 = \mathcal{O}_Q(-1, -1)$ et $c_2 = 4$ relativement au fibré en droites ample $\mathcal{O}_Q(1, 1)$. Nous en déduisons que l'ensemble des coniques de saut $S(E)$ détermine $E$ de manière unique et que l'espace de modules de ce type de fibrés est rationnel.

1. Introduction

Throughout the article, our base field is $\mathbb{C}$, the field of complex numbers.

Let $Q$ be a smooth quadric in $\mathbb{P}_3 = \mathbb{P}(V)$, where $V$ is a 4-dimensional vector space, and $\mathcal{M}(k)$ be the moduli space of stable vector bundles of rank 2 on $Q$ with the Chern classes $c_1 = \mathcal{O}_Q(-1, -1)$ and $c_2 = k$ with respect to the ample line bundle $\mathcal{L} = \mathcal{O}_Q(1, 1)$. $\mathcal{M}(k)$ forms an open Zariski subset of the projective variety $\mathcal{M}(k)$, whose points correspond to the semi-stable sheaves on $Q$ with the same numerical invariants. The Zariski tangent space of $\mathcal{M}(k)$ at $E$ is naturally isomorphic to $H^1(Q, \mathcal{E}nd(E))$ [8] and so the dimension of $\mathcal{M}(k)$ is equal to $h^1(Q, \mathcal{E}nd(E)) = 4k - 5$, since $E$ is simple. In [6], we define the jumping conics of $E \in \mathcal{M}(k)$ as points in $\mathbb{P}_3^2$ and prove that the set of jumping conic is a symmetric determinantal hypersurface of degree $k - 1$ in $\mathbb{P}_3^2$. It enables us to consider a morphism:

$$S: \mathcal{M}(k) \to |\mathcal{O}_{\mathbb{P}_3^2}(k - 1)| \simeq \mathbb{P}_N.$$ 

We conjecture in [6] that the general $E \in \mathcal{M}(k)$ is uniquely determined by $S(E)$ and prove that this map $S$ is generically injective for $k \leq 3$.

In this article, we prove that the conjecture is true when $k = 4$. For $E \in \mathcal{M}(4)$, $S(E)$ is a cubic symmetroid surface, i.e. a symmetric determinantal cubic hypersurface in $\mathbb{P}_3^2$. In terms of short exact sequences that $E$ admits, we can obtain the relation between the singularity of $S(E)$ and the dimension of cohomology of the restriction of $E$ to its hyperplane section.

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It turns out that $S(\mathcal{E})$ has exactly 4 singular points. It enables us to derive the rationality of $\mathcal{M}(4)$, which was proven in a much more general setting in [2]. Lastly, we give a brief description of $S(\mathcal{E})$ for non-general bundles of $\mathcal{M}(4)$. We will denote the dimension of the cohomology $H^i(X, \mathcal{F})$ for a coherent sheaf $\mathcal{F}$ on $X$ by $h^i(X, \mathcal{F})$, or simply by $h^i(\mathcal{F})$ if there is no confusion.

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2. Preliminaries

Let $Q$ be a smooth quadric surface isomorphic to $\mathbb{P}(V_1) \times \mathbb{P}(V_2)$ for two 2-dimensional vector spaces $V_1$ and $V_2$. Then $Q$ is embedded into $\mathbb{P}_3 = \mathbb{P}(V)$ by the Segre map, where $V = V_1 \otimes V_2$. Let us denote $f^*O_P(1) = g^*O_P(1)$ by $O_Q(a, b)$ and $E \otimes O_Q(a, b)$ by $E(a, b)$ for coherent sheaves $E$ on $Q$, where $f$ and $g$ are the projections from $Q$ to each factors. Then the canonical line bundle $K_Q$ of $Q$ is $O_Q(-2, -2)$. As a direct consequence of the Kunneth formula, we have:

$$H^i(P, O_Q(a + b)) = \begin{cases} 0, & \text{if } a = -1; \\ H^i(P, O_P(a + b)^{\oplus(a+1)}), & \text{if } a \geq 0. \end{cases}$$

Now let us denote the ample line bundle $O_D(1, 1)$ by $L$ and let $\mathcal{M}(k)$ be the moduli space of semi-stable sheaves of rank 2 on $Q$ with the Chern classes $c_1 = O_Q(-1, -1)$ and $c_2 = k$ with respect to $L$. The existence and projectivity of $\mathcal{M}(k)$ are shown in [4] and it has an open Zariski subset $\mathcal{M}(k)$ consisting of the stable vector bundles with the given numeric invariants. By Bogomolov’s inequality [8], $\mathcal{M}(k)$ is empty if $4k < c_1^2 = 2$ and so we consider only the case of $k \geq 1$. The dimension of $\mathcal{M}(k)$ can be computed to be $h^1(Q, E\text{nd}(E)) = 4k - 5$. Note that $E \cong E^*(-1, -1)$ and by the Riemann–Roch theorem [5], we have $\chi(E(m, m)) = 2m^2 + 2m + 1 - k$ for $E \in \mathcal{M}(k)$. For a hyperplane $H$ in $\mathbb{P}_3$, let $C_H := Q \cap H$ be the corresponding hyperplane section on $Q$.

Definition 2.1. The conic $C \subset Q$ is called a jumping conic if $h^0(E|_C) \geq 1$.

Remark 2.2. Since any conic $C \subset Q$ is a hyperplane section, we define the set $S(\mathcal{E})$ of jumping conics of $\mathcal{E}$ as a subset of $\mathbb{P}_3^*$. More precisely,

$$S(\mathcal{E}) := \{ H \in \mathbb{P}_3^* \mid h^0(E|_H) \geq 1 \}.$$ 

When $C_H$ is smooth, it is a jumping conic if the vector bundle $E$ splits non-generically over it.

Theorem 2.3. (See [6].) For a Hulsbergen bundle $E \in \mathcal{M}(k)$, $S(\mathcal{E})$ is a symmetric determinantal hypersurface of degree $k - 1$ in $\mathbb{P}_3^*$ and it has a singular point at $H \in \mathbb{P}_3^*$ if $h^0(E|_H) \geq 2$.

Remark 2.4. The referee pointed out that the converse might not be true in general. Indeed, the determinant of the following matrix is singular along a line but the ideal of $2 \times 2$ minors has length 4:

$$\begin{pmatrix} t_0 & t_1 & t_3 \\ t_1 & t_0 + t_3 & t_2 \\ t_3 & t_2 & 0 \end{pmatrix}.$$

Theorem 2.3 enables us to consider a morphism $S : \mathcal{M}(k) \rightarrow |O_{P^3}(k - 1)| \cong \mathbb{P}_N$ with $N = (k + 2)^2 - 1$. In [6] and [7], the cases of $k = 2, 3$ are dealt in detail. For example, when $k = 2$, the morphism $S$ extends to an isomorphism from $\mathcal{M}(2) \rightarrow \mathbb{P}_3$ and $\mathcal{M}(2)$ is isomorphic to $\mathbb{P}_3 \setminus Q$. In particular, $S(\mathcal{E})$ determines uniquely $E \in \mathcal{M}(2)$. A similar result also holds for $k = 3$.

3. Results

From now on, we will investigate $S(\mathcal{E})$ for $E \in \mathcal{M}(4)$, which is now a cubic symmetroid surface, i.e. a symmetric determinantal cubic surface in $\mathbb{P}_3^*$. Note that a nonsingular cubic surface cannot be symmetrically determinantal [3]. Since $\chi(E(1, 1)) = 1$ and $E$ is stable, it admits an exact sequence:

$$0 \rightarrow O_Q \rightarrow E(1, 1) \rightarrow I_Z(1, 1) \rightarrow 0,$$

where $Z$ is a zero-dimensional subscheme of $Q$ with length 4 and $I_Z(1, 1)$ is the tensor product of the ideal sheaf of $Z$ and $O_Q(1)$. Let us assume that $Z$ is in general position and then we have $h^0(E(1, 1)) = 1$, which leads us to conclude that for $k = 4$, a general $E$ is a Hulsbergen bundle. In particular, $Z$ is uniquely determined by $E$. Note that $\mathbb{P} \text{ Ext}^1(Z(1, 1), O_Q) \cong \mathbb{P} H^0(O_Z)^* \cong \mathbb{P}^3$. A general point in this family of extensions corresponds to a stable vector bundle [1] and so $\mathcal{M}(4)$ is
birational to a $\mathbb{P}_3$-bundle over the Hilbert scheme $\mathfrak{M}^{[4]}$ of zero-dimensional subscheme of $Q$ with length 4. It is consistent with the fact that the dimension of $\mathfrak{M}(4)$ is 11. Note that $\mathfrak{M}^{[4]}$ is a resolution of singularity of $S^4 Q$, the fourth symmetric power of $Q$, and in particular it is 8-dimensional [9].

Assume that $Z$ is not contained in any hyperplane section. If $|Z \cap H| = 3$ for a hyperplane section $H$ of $\mathbb{P}^4_3$, we can tensor the sequence (1) with $\mathcal{O}_{C_H}$ to obtain:

$$0 \to \mathcal{O}_{C_H} \to \mathcal{E}(1, 1)|_{C_H} \to \mathcal{O}_{C_H}(-p) \oplus \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_3 \to 0,$$

where $p$ is a point on $C_H$. The last surjection gives a surjective map $\mathcal{E}(1, 1)|_{C_H} \to \mathcal{O}_{C_H}(-p)$ and its kernel is $\mathcal{O}_{C_H}(3p)$ for degree reason. Twisting by $\mathcal{O}_{C_H}(-2p)$, we obtain:

$$0 \to \mathcal{O}_{C_H}(p) \to \mathcal{E}|_{C_H} \to \mathcal{O}_{C_H}(-3p) \to 0.$$

Since $h^0(\mathcal{E}|_{C_H}) = 2$, $H$ is a singular point of $S(\mathcal{E})$ by Theorem 2.3 and so $S(\mathcal{E})$ has at least 4 singular points.

**Proposition 3.1.** For a general vector bundle $\mathcal{E}$ in $\mathfrak{M}(4)$, there are exactly 4 singular points and 6 lines in $S(\mathcal{E})$, i.e. $S(\mathcal{E})$ is a Cayley surface.

**Proof.** Similarly as above, we can prove that $H$ is a point of $S(\mathcal{E})$ if $|Z \cap H| = 2$, and not a point of $S(\mathcal{E})$ if $|Z \cap H| = 1$. Thus the intersection of $S(\mathcal{E})$ with the hyperplane containing a singular point above is the union of three distinct lines, and in particular $S(\mathcal{E})$ contains 6 lines. Let $Z' = \{p_1, \ldots, p_4\} \subset S(\mathcal{E})$ be the set of 4 singular points above and denote the line connecting $p_1, p_2$ by $l_{ij}$. For an arbitrary line $l \subset S(\mathcal{E})$ which is different from $l_{ij}$, let us assume that $l$ does not intersect with $l_{ij}$. If $\pi : \mathbb{P}_3^3 \to \mathbb{P}_3^2$ is the projection from $p_1$, then the images of $l$ and $l_{ij}$, $i, j \neq 1$ intersect. It implies that $l$ and $l_{ij}$ intersect for $i, j \neq 2$. It is impossible, since the plane containing $p_2, p_3, p_4$ would contain $l$. The case of $l$ meeting $l_{ij}$ can be shown impossible similarly. Thus $S(\mathcal{E})$ contains exactly the 6 lines above and in particular $S(\mathcal{E})$ is not a cone over a plane cubic curve. If $S(\mathcal{E})$ is not normal, then its singular locus would have a 1-dimensional part of degree $d$ and multiplicity $m$. Its intersection with a generic hyperplane section is a plane cubic curve, and so we have $d = 1$ and $m = 2$. In other words, the singular locus of $S(\mathcal{E})$ would be a line, which is one of the 6 lines above. It is impossible, since its multiplicity must be 1, and thus $S(\mathcal{E})$ is normal. We can also easily check that $S(\mathcal{E})$ is irreducible, and so the singularities of $S(\mathcal{E})$ are rational double points. Now, without loss of generality, let us assume that $p_1 = [1, 0, 0, 0]$ and write the equation $f$ of $S(\mathcal{E})$ by $f = t_0 f_2(t_1, t_2, t_3) + f_3(t_1, t_2, t_3)$, where $f_i$ is a homogeneous polynomial of degree $i$. It is easy to check that if $p = [a_0, a_1, a_2, a_3] \in S(\mathcal{E})$ is a singular point of $S(\mathcal{E})$, then the conic $V(f_2)$ and the cubic $V(f_3)$ intersect at $[a_1, a_2, a_3]$ with multiplicity at least 2. From the irreducibility of $S(\mathcal{E})$, $V(f_2)$ and $V(f_3)$ do not share common components. So the other singular points than $p_1$ must be contained in the 6 lines above and, by the Bézout theorem, they must be the remaining points in $Z'$. Hence $S(\mathcal{E})$ contains exactly 4 singular points and 6 lines connecting them.

**Remark 3.2.** Considering a $\mathbb{P}_2$-family of hyperplanes of $\mathbb{P}_3$ that contains a point of $Z$, the intersection of $\mathbb{P}_2$ with $S(\mathcal{E})$ is a cubic plane curve. Since there are 3 hyperplanes in this family, that contain 3 points of $Z$, so the intersection of the $\mathbb{P}_2$-family with $S(\mathcal{E})$ is the union of three lines.

Conversely, let us consider a cubic hypersurface $S_3$ in $\mathbb{P}^4_3$ with exactly 4 singular points, say $H_1, \ldots, H_4 \subset \mathbb{P}_3$. Then $H_i$’s are 4 hyperplanes of $\mathbb{P}^4_3$ in general position. If $S_3$ is equal to $S(\mathcal{E})$ for some $\mathcal{E} \in \mathfrak{M}(4)$ with the exact sequence (1), then there are 3 points of $Z$ on each $H_i$. The intersection of $C_{H_i}$ with $H_i$, $i = 2, 3, 4$ is two points of $Z$ and so 3 points of $Z$ are determined. The last point is just the intersection of $H_2, H_3$ and $H_4$.

**Theorem 3.3.** The morphism $S : \mathfrak{M}(4) \to |\mathcal{O}_{\mathbb{P}^4_3}(3)|$ is generically injective. In other words, the set of jumping conics of $\mathcal{E} \in \mathfrak{M}(4)$ uniquely determines $\mathcal{E}$ in general.

**Proof.** It is enough to check that for two different stable vector bundles $\mathcal{E}$ and $\mathcal{E}'$ that fit into the sequence (1) with the same $Z$, $S(\mathcal{E})$ and $S(\mathcal{E}')$ are different. From the previous argument, they have the same singular points. Now, $\mathcal{E}$ and $\mathcal{E}'$ are in the extension family $\mathcal{Ext}^1(\mathcal{I}_Z(1, 1), \mathcal{O}_Q)$, which is isomorphic to $H^1(\mathcal{I}_Z(-1, -1))^*$. From the short exact sequence $0 \to \mathcal{I}_Z(-1, -1) \to \mathcal{I}_Z \to \mathcal{O}_{C_H} \to 0$, where $C_H$ is a smooth conic that does not intersect with $Z$, we have:

$$0 \to H^1(\mathcal{I}_Z)^* \to H^1(\mathcal{I}_Z(-1, -1))^* \xrightarrow{\text{res}} H^0(\mathcal{O}_{C_H})^* \to 0.$$  

Here, the map ‘res’ sends $\mathcal{E}$ to $\mathcal{E}|_{C_H}$. Note that $H^1(\mathcal{I}_Z)^*$ is a corank 1-subspace of $H^1(\mathcal{I}_Z(-1, -1))^*$. If we choose $H$ properly so that the image of $H^1(\mathcal{I}_Z)^*$ contains $\mathcal{E}$, but not $\mathcal{E}'$, then their splitting will be different. To be precise, we have $\mathcal{E}|_{C_H} = \mathcal{O}_{C_H}(-2p) \oplus \mathcal{O}_{C_H}$ and $\mathcal{E}'|_{C_H} = \mathcal{O}_{C_H}(-p)^{\oplus 2}$, where $p$ is a point on $C_H$. In particular, $S(\mathcal{E})$ and $S(\mathcal{E}')$ are different.

In fact, the argument after Proposition 3.1 can be applied to any symmetric determinantal cubic hypersurface with 4 singular points; we obtain the following:
Corollary 3.4. \( M(4) \) is birational to the variety of the symmetric determinantal cubic hypersurfaces \( \mathbb{P}^3 \) with 4 singular points whose corresponding hyperplanes in \( P_3 \) satisfy the property that any three hyperplanes among them have the intersection point on \( Q \).

**Proof.** It is known in [3] that cubic surfaces with 4 rational double points are projectively isomorphic to the Cayley 4-nodal cubic surface, which is a cubic surface with 4 nodal points defined by:

\[
t_0t_1t_2 + t_0t_1t_3 + t_0t_2t_3 + t_1t_2t_3 = \det \begin{pmatrix} t_0 & 0 & t_2 \\ 0 & t_1 & -t_2 \\ -t_3 & t_2 & t_3 \end{pmatrix},
\]

which has 4 nodal points \([1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0] \) and \([0, 0, 0, 1] \). It means that we have a 3-dimensional family of cubic symmetroids for each fixed 4 points as singularities. Here \( 3 = \text{dim} \text{PGL}(4) - \text{dim}(\mathbb{P}^3) \). So the assertion follows automatically from the previous theorem, because the dimension of the variety of the cubic symmetroids in the assertion is \( 11 = \text{dim}(\text{PGL}(4)) - 4 \), which is the dimension of \( M(4) \). \( \square \)

Corollary 3.5. (See Theorem 4.7 in [2].) \( M(4) \) is rational.

**Proof.** Let us prove that the variety \( Y \) of the cubic symmetroids with 4 singular points whose corresponding hyperplanes have 4 intersection points on \( Q \) is rational. First of all, the variety \( X \) of cubic symmetroids with 4 singular points generically has a \( P_3 \)-bundle structure over \( \mathbb{P}^3 \) and it is transitively acted by \( \text{PGL}(4) \). Thus \( X \) is rational and we have a dominant map \( \pi : X \rightarrow \mathbb{P}^3 \) to a rational variety \( \mathbb{P}^3 \). Since \( Y \) is a subvariety of \( X \) that is generically a \( P_3 \)-bundle over \( \mathbb{P}^3 \) from \( \pi \) and \( \mathbb{P}^3 \) is rational, so \( Y \) is a rational variety. \( \square \)

Now let us consider a special case when \( Z \) is coplanar. In this case, \( S(\mathcal{E}) \) is a cubic surface with a unique singular point corresponding to the hyperplane containing \( Z \), say \( H \). Note that \( h^0(\mathcal{E}(1, 1)) = 2 \). Then there is a 1-dimensional family of zero-dimensional subscheme \( Z \) for which \( \mathcal{E} \) fits into the sequence \((1)\). Such \( Z \) should be contained in \( C_H \). For each \( Z \), we can consider the \( P_1 \)-family of hyperplanes that contain two points of \( Z \), and this corresponds to a line contained in \( S(\mathcal{E}) \). So we can find 6 lines contained in \( S(\mathcal{E}) \) out of one such \( Z \). As we vary \( Z \) in the 1-dimensional family, we have infinitely many lines through \( H \) contained in \( S(\mathcal{E}) \). Thus we obtain the following statement:

**Proposition 3.6.** For the vector bundle \( \mathcal{E} \) fitted into the sequence \((1)\) with coplanar \( Z \), \( S(\mathcal{E}) \) is a cone over a cubic curve in \( \mathbb{P}^2 \) with the vertex point contained in the hyperplane containing \( Z \).

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