The fundamental gap of a kind of two dimensional sub-elliptic operator

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Abstract

This paper is concerned at the minimization fundamental gap problem for a class of two-dimensional degenerate sub-elliptic operators. We establish existence results for weak solutions, Sobolev embedding theorem and spectral theory of sub-elliptic operators. We provide the existence and characterization theorems for extremizing potentials $V(x)$ when $V(x)$ is subject to $L^{\infty}$ norm constraint.

Keywords: sub-elliptic operator, fundamental gap, optimal potentials

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1 Introduction

The eigenvalue extremum problem of Schrödinger operator with Dirichlet/Neumann boundary was initiated at least in the early 1960s\textsuperscript{29}. The difference between the first two eigenvalues, $\lambda_2 - \lambda_1$, is called the fundamental gap, it is of great significance in quantum mechanics, statistical mechanics, and quantum field theory. We give a brief overview of this subject and outline some related work.

Van den Berg\textsuperscript{16} observed that $\lambda_2 - \lambda_1 \geq \frac{3\pi^2}{4d^2}$ on many convex domains, where $d$ is the diameter of domain, this view has also been independently suggested by Yau\textsuperscript{48}, as well as Ashbaugh and Benguria\textsuperscript{6}, which is called fundamental gap conjecture. Ashbaugh and Benguria\textsuperscript{6} proved that the conjecture holds if the potential function is single-well symmetric (not necessarily convex). Later, Horváth\textsuperscript{24} removed the symmetry hypothesis and allowed the potential function to be single-well with the middle transition point on the interval. Lavine\textsuperscript{32} proved the fundamental gap conjecture for Schrödinger operators equipped with homogeneous Dirichlet or Neumann boundary conditions on a bounded interval among the class of convex bounded potentials, and this result was completed for Robin boundary conditions in\textsuperscript{5}. More results could be found in\textsuperscript{22}.

Yau\textsuperscript{45} et al. obtained that the gap is bounded below by $\frac{\pi^2}{4d^2}$ in the high-dimensional case by utilizing the fact that the first eigenfunction is logarithmic concave. Subsequently, many studies

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improved the lower bound of the gap, it was not until 2011 that Andrews and Clutterbuck [3] completely solved this conjecture. The same method has been further exploited in the paper [4]. In particular, Laplacian fundamental gap problem [44, 8, 21, 15] has been extended to particular manifolds and the corresponding detailed characterization is given. The lower bound of eigenvalue gap of vibrating string is also discussed in [25, 10, 41]. For the latest references, we can refer to [30, 31, 2].

For more general elliptic operators, Wolfson [47] estimated the eigenvalue gap for a class of nonsymmetric second-order linear elliptic operators. Cheng et al. [12] considered the eigenvalue gap of the $p$-Laplacian eigenvalue problems, and obtained the minimizer of the eigenvalue gap for the single-well potential function. Significantly, Tan and Liu [46] considered estimates for eigenvalues of a class of fourth order degenerate elliptic operators with a singular potential, Chen et al. [11] provided the lower bounds of Dirichlet eigenvalues for a class of higher order degenerate elliptic operators. However, they mainly provided estimates for each eigenvalue, and need to assume the elliptic operator as a sum of squares of vector fields $\{X^j\}_{j=1}^m$, and the vector fields $\{X^j\}_{j=1}^m$ demand the Hörmander’s condition and Métivier’s condition.

In general, as far as we know, there are relatively few studies on more general elliptic operators involving the fundamental gap. Inspired by the above work and [7, 22], we consider the following degenerate sub-elliptic equation

$$
\begin{aligned}
&- (\partial_{x_1 x_1} u + h(x_1) \partial_{x_2 x_2} u) + V u = \lambda u, \quad x \in \Omega, \\
u = 0, \quad & x \in \partial \Omega,
\end{aligned}
$$

(1.1)

where

$$
\Omega = \{ x = (x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1, 0 < x_2 < 1 \},
$$

$h : [0, 1] \to \mathbb{R}$ is a bounded function with $h(0) = 0$, $h(x_1) > 0$ in $(0, 1]$ and $h(x_1) \in C^1(0, 1]$, moreover there exists a constant $C$ such that

$$
\sup_{\substack{x \in \Omega \ni \theta < r \leq d}} \frac{\int_{B \cap \Omega} h(x_1)^{-1} dx}{|B \cap \Omega|^{\theta}} \leq C \text{ for some } 0 < \theta < 1,
$$

(1.2)

where $B := B(x, r)$ represents the ball with point $x \in \Omega$ as the center of the ball and radius $r$, $d$ is the diameter of $\Omega$. For example, $h(x_1) = x_1^\alpha$ with $\alpha \in (0, 1)$.

Denote

$$
L = - (\partial_{x_1 x_1} + h(x_1) \partial_{x_2 x_2}) + V,
$$

(1.3)

the focus of this paper is to search for the existence and optimality of the minimization problem

$$
\inf_{V \in S} \Gamma(V)
$$

related to operator (1.3) under the condition $V \in S$, where $\Gamma(V) = \lambda_2(V) - \lambda_1(V)$, and

$$
S = \{ V \in L^\infty(\Omega) \mid m \leq V(x) \leq M \text{ a.e.} \},
$$
$m$ and $M$ are non-negative given constants.

It is not hard to see that we do not assume the operator $L$ is written as a sum of squares of vector fields although it allows for this structure. Furthermore, we do not apply the additional metric structures, the background Euclidean metric is employed only in this paper.

### 2 Weighted Sobolev space

In this section, we provide the solvability theory for the degenerate elliptic equation (2.1) on which the spectral theory and subsequent applications are based on it. The purpose of this theory is to establish the weak well-posedness for the problem (2.1):

$$
\begin{cases}
- (\partial_{x_1} u + h(x_1) \partial_{x_2} u) + Vu = f, & x \in \Omega, \\
u = 0, & x \in \partial \Omega,
\end{cases}
$$

(2.1)

we explore ways of showing the existence, uniqueness of a proper notion of weak solution $u$ for each given $f \in L^2(\Omega)$. The space we mentioned below may be involved in many works such as [37, 43, 9, 13], but we will introduce more profound results.

Define

$$
\mathcal{H}^1_h(\Omega) = \left\{ u \in L^2(\Omega) \left| \partial_{x_1} u \in L^2(\Omega), \sqrt{h(x_1)} \partial_{x_2} u \in L^2(\Omega) \right. \right\},
$$

and

$$(u, v)_{\mathcal{H}^1_h(\Omega)} = \int_{\Omega} u v dx + \int_{\Omega} (\partial_{x_1} u)(\partial_{x_1} v) dx + \int_{\Omega} \left( \sqrt{h(x_1)} \partial_{x_2} u \right) \left( \sqrt{h(x_1)} \partial_{x_2} v \right) dx,$$

from which we define

$$
\|u\|_{\mathcal{H}^1_h(\Omega)} = \left( \|u\|_{L^2(\Omega)}^2 + \|\partial_{x_1} u\|_{L^2(\Omega)}^2 + \|\sqrt{h(x_1)} \partial_{x_2} u\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.
$$

It is not hard to find that $\|u\|_{\mathcal{H}^1_h(\Omega)} = \sqrt{(\cdot, \cdot)_{\mathcal{H}^1_h(\Omega)}}$. Furthermore, let $\mathcal{H}^1_{h,0}(\Omega)$ denote the closure of $C^\infty_0(\Omega)$ in the space $\mathcal{H}^1_h(\Omega)$, that is,

$$
\mathcal{H}^1_{h,0}(\Omega) = \overline{C^\infty_0(\Omega)}^{\mathcal{H}^1_h(\Omega)}.
$$

Throughout this paper, let $\| \cdot \|$ denotes the norm, $(\cdot, \cdot)$ denotes the inner product.

**Lemma 2.1.** The space $(\mathcal{H}^1_h(\Omega), (\cdot, \cdot)_{\mathcal{H}^1_h(\Omega)})$ is a Hilbert space.

**Proof.** Firstly, we easily verify that $(\mathcal{H}^1_h(\Omega), (\cdot, \cdot)_{\mathcal{H}^1_h(\Omega)})$ is an inner space. Let $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{H}^1_h(\Omega)$ be a Cauchy sequence, so that $\{u_n\}_{n \in \mathbb{N}}, \{\partial_{x_1} u_n\}_{n \in \mathbb{N}}, \{\sqrt{h(x_1)} \partial_{x_2} u_n\}_{n \in \mathbb{N}}$ are Cauchy sequences in $L^2(\Omega)$. Then there exist $u, v, w \in L^2(\Omega)$ such that

$$
u_n \to u, \partial_{x_1} u_n \to v, \sqrt{h(x_1)} \partial_{x_2} u_n \to w$$

strongly in $L^2(\Omega)$.

For each $\varphi \in C^\infty_0(\Omega)$, we have

$$
\int_{\Omega} w \varphi dx = \int_{\Omega} \left( \sqrt{h(x_1)} \partial_{x_2} u_n \right) \varphi dx = -\int_{\Omega} u_n \left( \sqrt{h(x_1)} \partial_{x_2} \varphi \right) dx \to -\int_{\Omega} u \left( \sqrt{h(x_1)} \partial_{x_2} \varphi \right) dx
$$
in the sense of distribution, which implies that
\[ w = \sqrt{h(x_1)} \partial_{x_2} u \]
in the sense of distribution. Naturally, \( w = \sqrt{h(x_1)} \partial_{x_2} u \) in \( L^2(\Omega) \) since \( w \in L^2(\Omega) \). Implement the same method again then \( v = \partial_{x_1} u \) in \( L^2(\Omega) \) is attained. All these imply that
\[ u_n \to u \text{ strongly in } H^1_h(\Omega). \]

**Proposition 2.2.** The Sobolev space \( H^1(\Omega) \) is a subspace of \( H^1_h(\Omega) \). Generally, \( H^1(\Omega) \subseteq H^1_h(\Omega) \).

**Proof.**

1. Let \( u \in H^1(\Omega) \). Then \( u, \partial_{x_1} u, \partial_{x_2} u \in L^2(\Omega) \). Note that
\[ \|u\|_{H^1_h(\Omega)} = \left( \|u\|_{L^2(\Omega)}^2 + \|\partial_{x_1} u\|_{L^2(\Omega)}^2 + \|\sqrt{h(x_1)} \partial_{x_2} u\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \]
\[ \leq C \left( \|u\|_{L^2(\Omega)}^2 + \|\partial_{x_1} u\|_{L^2(\Omega)}^2 + \|\partial_{x_2} u\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} = C\|u\|_{H^1(\Omega)}, \]

hence \( u \in H^1_h(\Omega) \).

2. Take \( \sqrt{h(x_1)} = x_1^\gamma, \gamma \in (0, \frac{1}{2}) \), consider the function \( u(x_1, x_2) = (x_1^2 + x_2)^{\frac{1}{4}} \) on \( \Omega \). We observe that
\[ \partial_{x_1} u = \frac{x_1}{2} \left( x_1^2 + x_2 \right)^{-\frac{3}{4}}, \quad \partial_{x_2} u = \frac{1}{4} \left( x_1^2 + x_2 \right)^{-\frac{3}{4}}, \]
and
\[ \|u\|_{L^2}^2 = \int_{\Omega} (x_1^2 + x_2)^{\frac{1}{2}} dx \leq \sqrt{2}, \]
\[ \|\partial_{x_1} u\|_{L^2}^2 = \frac{1}{4} \int_{\Omega} \frac{x_1^2}{\left( x_1^2 + x_2 \right)^{\frac{3}{2}}} dx \leq \frac{1}{4} \int_{\Omega} \frac{1}{\sqrt{x_1^2 + x_2}} dx = \frac{1}{4} \int_0^1 \int_{x_1^2}^{x_1^2 + 1} \frac{1}{\sqrt{z}} dz dx_1 \]
\[ = \frac{1}{4} \left( \int_0^1 \frac{1}{2} \sqrt{x_1^2 + 1} dx_1 \right) \leq \sqrt{2}, \]
\[ \|\sqrt{h(x_1)} \partial_{x_2} u\|_{L^2}^2 = \frac{1}{16} \int_{\Omega} \frac{x_1^{2\gamma}}{(x_1^2 + x_2)^{\frac{3}{2}}} dx \leq \frac{1}{16} \int_{\Omega} \frac{(x_1^2 + x_2)^{\gamma}}{(x_1^2 + x_2)^{\frac{3}{2}}} dx = \frac{1}{16} \int_{\Omega} \frac{1}{(x_1^2 + x_2)^{\frac{3}{2} - \gamma}} dx \]
\[ = \frac{1}{16} \int_0^1 \frac{1}{x_1^{\frac{3}{2}}} \frac{1}{z^{\frac{3}{2} - \gamma}} dz dx_1 = \frac{1}{16} \left( -\frac{3}{2} + \gamma \right)^{-1} \int_0^1 z^{-\frac{3}{2} + \gamma} |x_1^{\frac{3}{2} + 1} dx_1 < +\infty, \]

these show that \( u \in H^1_h(\Omega) \). However,
\[ \|\partial_{x_2} u\|_{L^2}^2 = \frac{1}{16} \int_{\Omega} \frac{1}{(x_1^2 + x_2)^{\frac{3}{2}}} dx = \frac{1}{16} \int_0^1 dx_1 \int_0^1 \frac{1}{(x_1^2 + x_2)^{\frac{3}{2}}} dx_2 = \frac{1}{16} \int_0^1 dx_1 \int_{x_1^2}^{x_1^2 + 1} \frac{1}{z^{\frac{3}{2}}} dz \]
\[ = -\frac{1}{8} \int_0^1 z^{-\frac{3}{2}} \bigg|_{x_1^2}^{x_1^2 + 1} dx_1 = -\frac{1}{8} \int_0^1 \left( \frac{1}{\sqrt{x_1^2 + 1}} - \frac{1}{x_1} \right) dx_1 = +\infty, \]

this implies that \( u \notin H^1(\Omega). \)

**Lemma 2.3.** \( H^1_h(\Omega) \) is continuously embedded into \( W^{1,1}(\Omega) \).
Proof. We observe that \( \int_{\Omega} h(x_1)^{-1} \) is bounded from (1.2). For any \( u \in \mathcal{H}_h^1(\Omega) \), thanks to Hölder inequality,

\[
\int_{\Omega} |\partial_{x_2} u| dx = \int_{\Omega} \left( \sqrt{h(x_1)} \right)^{-1} \sqrt{h(x_1)} |\partial_{x_2} u| dx \\
\leq \left( \int_{\Omega} h(x_1)^{-1} dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |h(x_1)\partial_{x_2} u|^2 dx \right)^{\frac{1}{2}} \\
\leq C \|h(x_1)\partial_{x_2} u\|_{L^2(\Omega)}.
\]

Therefore, \( \mathcal{H}_h^1(\Omega) \leftrightarrow W^{1,1}(\Omega) \). \( \square \)

**Lemma 2.4.** \( H_0^1(\Omega) \subset \mathcal{H}_{h,0}^1(\Omega) \subset \mathcal{H}_h^1(\Omega) \cap W_0^{1,1}(\Omega) \).

**Proof.** For \( u \in H_0^1(\Omega) \), there is a sequence \( \{u_n\}_{n \in \mathbb{N}} \subset C_0^\infty(\Omega) \) such that \( \|u_n - u\|_{H^1(\Omega)} \to 0, n \to \infty \), and we observe that

\[
\|u_n - u\|_{\mathcal{H}_h^1(\Omega)} \leq C \|u_n - u\|_{H^1(\Omega)} \to 0, n \to \infty,
\]

therefore \( u \in \mathcal{H}_{h,0}^1(\Omega) \).

For \( u \in \mathcal{H}_{h,0}^1(\Omega) \), there is a sequence \( \{u_n\}_{n \in \mathbb{N}} \subset C_0^\infty(\Omega) \) such that \( \|u_n - u\|_{\mathcal{H}_h^1(\Omega)} \to 0, n \to \infty \). And from Lemma 2.3

\[
\|u_n - u\|_{W^{1,1}(\Omega)} \leq C \|u_n - u\|_{\mathcal{H}_h^1(\Omega)} \to 0, n \to \infty,
\]

so that we have \( \mathcal{H}_{h,0}^1(\Omega) \subset \mathcal{H}_h^1(\Omega) \cap W_0^{1,1}(\Omega) \). \( \square \)

From Lemma 2.4, we know that it makes sense to consider Dirichlet condition problem (2.1).

**Lemma 2.5.** For any \( u \in \mathcal{H}_h^1(\Omega) \),

\[
\tilde{u}(x) = \begin{cases} 
    u(x), & x \in \Omega, \\
    0, & x \in \mathbb{R}^2 \setminus \Omega,
\end{cases}
\]

we have \( \tilde{u} \in \mathcal{H}_h^1(\mathbb{R}^2) \).

**Proof.** In fact, for any \( u \in \mathcal{H}_h^1(\Omega) \), there is a sequence \( \{u_n\}_{n \in \mathbb{N}} \subset C_0^\infty(\Omega) \) such that

\[
u_n \to u \text{ in } \mathcal{H}_h^1(\Omega), n \to \infty.
\]

For any \( \varphi \in C_0^\infty(\mathbb{R}^2) \),

\[
\int_{\mathbb{R}^2} (\sqrt{h(x_1)} \partial_{x_2} \tilde{u}) \varphi dx = - \int_{\mathbb{R}^2} \tilde{u}(\sqrt{h(x_1)} \partial_{x_2} \varphi) dx = - \int_{\Omega} u(\sqrt{h(x_1)} \partial_{x_2} \varphi) dx \\
= - \lim_{n \to \infty} \int_{\Omega} u_n(\sqrt{h(x_1)} \partial_{x_2} \varphi) dx = \lim_{n \to \infty} \int_{\Omega} (\sqrt{h(x_1)} \partial_{x_2} u_n) \varphi dx \\
= \int_{\Omega} (\sqrt{h(x_1)} \partial_{x_2} u) \varphi dx = \int_{\mathbb{R}^2} \left[ \sqrt{h(x_1)} \partial_{x_2} u \right] \varphi dx,
\]

we have

\[
\sqrt{h(x_1)} \partial_{x_2} \tilde{u} = \left[ \sqrt{h(x_1)} \partial_{x_2} u \right] \varphi dx,
\]
where \( \sigma \). This indicates that \( \tilde{u} \in H^1_h(\mathbb{R}^2) \) and \( \| \tilde{u} \|_{H^1_h(\mathbb{R}^2)} = \| u \|_{H^1_h(\Omega)} \).

### 3 Compact embedding theorem

Although the compact embedding problem of degenerate elliptic operators has been discussed a lot, such as [38, 13], it is still difficult to solve in specific problems. Next, we prove that the embedding \( H^1_h(\Omega) \hookrightarrow L^2(\Omega) \) is compact by utilizing different techniques.

We will also utilize the some definitions [33, 39, 26], let \( v(x) > 0 \) a.e. and local integrable in \( \mathbb{R}^2 \) with respect to Lebesgue measure, we shall use the notation \( v(E) = \int_E v(x)dx \) for a measurable set \( E \subset \mathbb{R}^2 \), and ordinary Lebesgue measure will be denoted by \( |E| \). We say that the function \( v(x) \) belongs to the Muckenhoupt class \( A_{\infty}(= A_{\infty}(\mathbb{R}^2, d, dx)) \) if there exist constants \( C \) and \( \sigma > 0 \) such that

\[
\frac{v(E)}{v(B)} \leq C \left( \frac{|E|}{|B|} \right)^{\sigma} \tag{3.1}
\]

for any Euclidean balls \( B \) and any Lebesgue measurable set (for simply, measurable set) \( E \subset B \), where \( d \) and \( dx \) are the standard Euclidean metric and Lebesgue measure in \( \mathbb{R}^2 \) respectively.

On the basis of the results involved in [34, 33, 36, 35], and in combination with the contents of this paper, we develop the following Lemma 3.1.

**Lemma 3.1.** Let \( 1 < p \leq q < \infty, \Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2 \). Suppose that \( v \in A_{\infty}, w_j(x) > 0 \) a.e. and \( w_j(x) \in L^1(\Omega) \). For any ball \( B(x, r) \) having a center \( x \in \Omega, \) \( 0 < r \leq d(\Omega) \), if there is a constant \( A_{pq} \) such that

\[
|B|^{-1}d(B)v(B \cap \Omega)^{\frac{1}{2}} \left[ \omega_j^{-\frac{1}{p-1}}(B \cap \Omega) \right]^{\frac{p-1}{p}} \leq A_{pq}, \quad j = 1, 2, \tag{3.2}
\]

where \( d(\Omega) \) is the diameter of \( \Omega \), then there exists a positive number \( C_0(q, C, \sigma) \) such that for any \( u \in \text{Lip}_0(\Omega) \),

\[
\left( \int_{\Omega} |u(x)|^q v(x)dx \right)^{\frac{1}{q}} \leq C_0 A_{pq} \sum_{j=1}^2 \left( \int_{\Omega} |\partial_{x_j} u|^{q \omega_j(x)}dx \right)^{\frac{1}{q}}, \tag{3.3}
\]

where \( \text{Lip}_0(\Omega) \) is the class of Lipschitz continuous functions which have compact support in \( \Omega, C \) and \( \sigma \) as shown in (3.1).

**Proof.** We carry out this proof by several steps.

**Step 1.** Let \( u(x) \in \text{Lip}_0(\Omega) \). We put \( \Omega^+ = \{ x \in \Omega \mid u(x) > 0 \} \) and \( \Omega^- = \{ x \in \Omega \mid u(x) < 0 \} \). Let \( \Omega^i \) be a connected component of \( \Omega^+ \) \( (i = 1, 2, \ldots) \). For \( \alpha > 0 \), we denote \( \Omega_\alpha = \{ x \in \Omega^i \mid \)}
$u(x) > \alpha$. Since $u(x)$ is continuous, the set $\Omega_\alpha$ is open. Let $\alpha$ be such that $|\Omega_{2\alpha}| > 0$. Then for any fixed point $x \in \Omega_{2\alpha}$, there exists a ball $B = B(x, r(x))$ such that

$$|B(x, r(x)) - \Omega_\alpha| = \gamma|B(x, r(x))|,$$

(3.4)

where $\gamma \in (0, 1)$ is some number independent of $\alpha, x$ and $r(x)$; this $\gamma$ will be specified later.

Indeed, to prove it, consider the function

$$F(t) = |B(x, t) - \Omega_\alpha| - \gamma|B(x, t)|,$$

note that for each $t_1, t_2 \in \mathbb{R}$, we assume $t_1 \leq t_2$ then

$$|F(t_2) - F(t_1)| = \left| |B(x, t_2) - \Omega_\alpha| - |B(x, t_1) - \Omega_\alpha| + \gamma(|B(x, t_1)| - |B(x, t_2)|) \right|$$

$$\leq \left| |B(x, t_2) - \Omega_\alpha| - |B(x, t_1) - \Omega_\alpha| \right| + \gamma(|B(x, t_1)| - |B(x, t_2)|)$$

$$\leq \left| |B(x, t_2)| - |B(x, t_1)| \right| + \gamma(|B(x, t_1)| - |B(x, t_2)|)$$

$$\leq (1 + \gamma)|t_2 - t_1|^2.$$

Therefore, $F(t)$ is a continuous function with respect to $t$. On one hand, $F(t)$ is negative for sufficiently small $t > 0$ since $x$ is an interior point of $\Omega_{2\alpha}$. On the other hand, due to the boundedness of $\Omega$, there exists $t \in (1/\sqrt{\pi}, +\infty)$ such that

$$|B(x, t) - \Omega_\alpha| > |B(x, t) - \Omega| \geq \gamma|B(x, t)|,$$

which implies that the function $F(t)$ be positive for sufficiently large values of $t$. Choose $r(x) \in \mathbb{R}$ such that $F(r(x)) = 0$. This proves (3.4).

**Step 2.** For a fixed point $x \in \Omega_{2\alpha}$, we denote $B = B(x, r(x))$ for simplicity. There are two possibilities:

1. If

$$|\Omega_{2\alpha} \cap B| < \gamma|B|,$$

(3.5)

using the assumption $v(x) \in A_\infty$, we obtain

$$v(\Omega_{2\alpha} \cap B) \leq C\gamma^\sigma v(B).$$

(3.6)

Using construction (3.4) and assumption $v(x) \in A_\infty$, it follows

$$v(B) = v(B \cap \Omega_\alpha) + v(B - \Omega_\alpha) \leq v(B \cap \Omega_\alpha) + C \left( \frac{|B - \Omega_\alpha|}{|B|} \right)^\sigma v(B) \leq v(B \cap \Omega_\alpha) + C\gamma^\sigma v(B).$$

Choosing $\gamma$ such that $C\gamma^\sigma < 1$, we have

$$v(B) \leq \frac{1}{1 - C\gamma^\sigma} v(B \cap \Omega_\alpha)$$

and from (3.6), it follows that

$$v(B \cap \Omega_{2\alpha}) \leq \frac{C\gamma^\sigma}{1 - C\gamma^\sigma} v(B \cap \Omega_\alpha).$$

(3.7)
(2) If 
\[ |\Omega_{2a} \cap B| \geq \gamma|B|, \]  
then from (3.4) and (3.8), we have
\[ \int_A \left( \int_Z dy \right) dx \geq \gamma^2|B|^2, \]
where \( A = B - \Omega_a, \) \( Z = B \cap \Omega_{2a} \).

Let points \( x \in A, \ y \in Z \) be arbitrarily fixed. The line segment \( \{x + t(y - x) \mid 0 < t < 1\} \) connecting \( x, \ y \) lies in \( B \) necessarily intersects the surfaces \( \{x \in \Omega^i \mid u(x) = \alpha\} \) and \( \{x \in \Omega^i \mid u(x) = 2\alpha\} \) at some points \( x' = x + t_1(y - x), \ x'' = x + t_2(y - x) \) and \( x'' = x + t_2(y - x) \), where \( t_1, t_2 \) be the smallest and largest value of \( t \) when the line segment intersects the surface \( \partial \Omega^{\alpha} \), respectively, and \( t_2 \) denote the first time when the line segment \( \{x + t(y - x) \mid 0 < t < 1\} \) meets the surface \( \partial \Omega_{2a} \), i.e., \( 0 < t_1 \leq t_1' < t_2 < 1 \) are numbers depending on \( x \) and \( y \). Then \( u(x') = u(x') = \alpha \) and \( u(x'') = 2\alpha \).

(i) It is clearly that
\[ \gamma^2|B|^2 \leq \frac{1}{\alpha} \int_A \left( \int_Z \left| u(x') - u(x'') \right| dy \right) dx, \]
whence
\[ \gamma^2|B|^2 \leq \frac{1}{\alpha} \int_A \left( \int_Z \left( \int_{t_1(x,y)}^{t_2(x,y)} \left| u_t(x + t(y - x)) \right| dt \right) dy \right) dx. \]

According to Fubini’s theorem,
\[ \gamma^2|B|^2 \leq \sum_{j=1}^2 \frac{d(B)}{\alpha} \int_A \left( \int_{t_1(x,y)}^{t_2(x,y)} \left( \int_{\{y \in \partial_x | x + t(y - x) \}} \left| \partial_x u(x + t(y - x)) \right| dy \right) dt \right) dx, \]
where \( G = B \cap (\Omega_{\alpha} - \Omega_{2a}) \).

(ii) For each \( a, b \in [t_1, t_1'] \) (\( a < b \)) with \( u(x_t) > \alpha, \forall t \in (a, b) \), where \( u(x_a) = u(x_b) = \alpha \) and \( x_t = x + t(y - x) \). It is also clearly that
\[ 0 \leq \sum_{j=1}^2 \frac{d(B)}{\alpha} \int_A \left( \int_a^b \left( \int_{\{y \in \partial_x | x + t(y - x) \}} \left| \partial_x u(x + t(y - x)) \right| dy \right) dt \right) dx. \]

According to (i) and (ii), we obtain that
\[ \gamma^2|B|^2 \leq \sum_{j=1}^2 \frac{d(B)}{\alpha} \int_A \left( \int_0^1 \left( \int_{\{y \in \partial_x | x + t(y - x) \}} \left| \partial_x u(x + t(y - x)) \right| dy \right) dt \right) dx. \]

Insert a change of variable in the interior integral \( z = x + t(y - x) \) passing from \( y \) to \( z \). Since \( z \in G, \) and \( dy = t^{-2} dz \) we obtain
\[ \gamma^2|B|^2 \leq \sum_{j=1}^2 \frac{d(B)}{\alpha} \int_A \left( \int_0^1 \left( \int_{\{z \in G \mid t^{-1}(z-x) + x \in \Omega \}} \left| \partial_x u(z) \right| dz \right) \frac{dt}{t^2} \right) dx. \]
When \( t \in (0, 1) \), we have \( z \in B \). Then \(|x_j - z_j| < td(B)\), applying Fubini’s formula once again, we get

\[
\gamma^2 |B|^2 \leq \sum_{j=1}^{2} \frac{d(B)}{\alpha} \int_0^1 \left( \int_G |\partial_{z_j} u(z)| \left( \int_{\{x||x_j - z_j| < td(B)\}} dx \right) \right) dt \frac{dt}{t^2} \\
\leq \sum_{j=1}^{2} \frac{2d(B)|B|}{\alpha} \int_G |\partial_{z_j} u(z)| dz,
\]

Therefore, by the inequality \((a + b)^q \leq 2^{q-1}(a^q + b^q)\), we get

\[
1 \leq \sum_{j=1}^{2} \left( \frac{2^{3q-1} d(B)}{\alpha\gamma^2 |B|} \int_G |\partial_{z_j} u(z)| dz \right)^q. \tag{3.9}
\]

By the Hölder inequality, it follows

\[
1 \leq \sum_{j=1}^{2} \left( \frac{2^{3q-1} d(B)}{\alpha\gamma^2 |B|} \int_G |\partial_{z_j} u(z)| dz \right)^q \leq \sum_{j=1}^{2} \left( \int_{B \cap \Omega} \frac{1}{|\partial_{z_j} u(z)|} \right)^{\frac{q(p-1)}{p}} \left( \int_G |\partial_{z_j} u|^p \omega_j(z) dz \right)^\frac{q}{p}. \tag{3.10}
\]

Combining the estimates (3.2) and (3.10), we get

\[
1 \leq \frac{2^{3q-1} A_q}{\gamma^2 \alpha^q} \frac{1}{v(B \cap \Omega)} \sum_{j=1}^{2} \left( \int_{B \cap \Omega} |\partial_{z_j} u|^p \omega_j(z) dz \right)^\frac{q}{p}. \tag{3.11}
\]

Consequently,

\[
v(B \cap \Omega_{2a}) \leq v(B \cap \Omega) \leq \frac{2^{3q-1} A_q}{\gamma^2 \alpha^q} \sum_{j=1}^{2} \left( \int_{B \cap \Omega_{2a}} |\partial_{z_j} u|^p \omega_j(z) dz \right)^\frac{q}{p}. \tag{3.12}
\]

By the estimates (3.7) and (3.12), we obtain

\[
v(B \cap \Omega_{2a}) \leq \frac{C_\gamma}{1 - C_\gamma} v(B \cap \Omega_a) + \frac{2^{3q-1} A_q}{\gamma^2 \alpha^q} \sum_{j=1}^{2} \left( \int_{B \cap (\Omega_a - \Omega_{2a})} |\partial_{z_j} u|^p \omega_j(z) dz \right)^\frac{q}{p}. \tag{3.13}
\]

**Setp 3.** It is evident that the system of balls \( \{B = B(x, r(x)) \mid x \in \Omega_{2a}\} \) covers \( \Omega_{2a} \). Furthermore, by the constructions above, we have \( \sup_{x \in \Omega_{2a}} r(x) < \infty \). Due to the Besicovitch’s covering theorem \([19, 17]\), one can select a finite or countable subcover \( \{B_k\}_{k=1}^{\infty} \) that covers \( \Omega_{2a} \) from \( \{B(x, r(x)) \mid x \in \Omega_{2a}\} \) with finite multiplicity:

\[
\sum_{k=1}^{\infty} \chi_{B_k}(x) \leq K, \tag{3.14}
\]

Writing (3.13) for the system of balls \( B_k \), it follows

\[
v(B_k \cap \Omega_{2a}) \leq \frac{C_\gamma}{1 - C_\gamma} v(B_k \cap \Omega_a) + \frac{2^{3q-1} A_q}{\gamma^2 \alpha^q} \sum_{j=1}^{2} \left( \int_{B_k \cap (\Omega_a - \Omega_{2a})} |\partial_{z_j} u|^p \omega_j(z) dz \right)^\frac{q}{p}. \tag{3.15}
\]
Summing (3.15) over \( k \) and taking into account (3.14), and using elementary inequality for positive numbers \( a_1^{q/p} + a_2^{q/p} + \cdots \leq (a_1 + a_2 + \cdots)^{q/p} \), we arrive at

\[
v(\Omega_\alpha) \leq \frac{KC\gamma^\sigma}{1 - C\gamma^\sigma} v(\Omega_\alpha) + \frac{K2^{3q-1}qA_{pq}}{\gamma^{2q}a^q} \sum_{j=1}^{2} \left( \int_{\Omega_\alpha - \Omega_2^\alpha} |\partial_{x_j} u|^p \omega_j(z)dz \right)^{\frac{q}{p}}.
\]  

(3.16)

We integrate (3.16) over \((0, \infty)\):

\[
\int_0^\infty v(\Omega_\alpha) d\alpha^q \leq \frac{KC\gamma^\sigma}{1 - C\gamma^\sigma} \int_0^\infty v(\Omega_\alpha) d\alpha^q + \sum_{j=1}^{2} \frac{K2^{3q-1}qA_{pq}}{\gamma^{2q}} \int_0^\infty \frac{d\alpha}{\alpha} \left( \int_{\Omega_\alpha - \Omega_2^\alpha} |\partial_{x_j} u|^p \omega_j(z)dz \right)^{\frac{q}{p}}.
\]

(3.17)

Since

\[
\int_0^\infty v(\Omega_\alpha) d\alpha^q = \frac{1}{2^q} \int_{\Omega^2} u^q(x)v(x)dx,
\]

(3.18)

We can apply the Minkowski’s inequality in (3.17) to obtain that

\[
\left( \frac{1}{2^q} - \frac{KC\gamma^\sigma}{1 - C\gamma^\sigma} \right) \int_{\Omega^2} u^q(x)v(x)dx \leq \frac{K2^{3q-1}qA_{pq}}{\gamma^{2q}} \sum_{j=1}^{2} \left( \int_{\Omega^2} |\partial_{x_j} u|^p \omega_j(z) \left( \int_{u(z)/2}^{u(z)} \frac{d\alpha}{\alpha} \right)^{\frac{q}{p}} dz \right)^{\frac{q}{p}}.
\]

(3.19)

Choosing \( \gamma \) so small that

\[
\frac{1}{2^q} - \frac{KC\gamma^\sigma}{1 - C\gamma^\sigma} > 0,
\]

(3.20)

Therefore, (3.18) implies that

\[
\int_{\Omega^2} u^q(x)v(x)dx \leq \left( \frac{1}{2^q} - \frac{KC\gamma^\sigma}{1 - C\gamma^\sigma} \right)^{-1} \frac{K2^{3q-1}q\ln 2}{\gamma^{2q}} A_{pq} \sum_{j=1}^{2} \left( \int_{\Omega^2} |\partial_{x_j} u|^p \omega_j(z)dz \right)^{\frac{q}{p}}.
\]

(3.21)

Summing the inequalities (3.20) for all \( \Omega^i \) and using elementary inequality for positive numbers \( a_1^{q/p} + a_2^{q/p} + \cdots \leq (a_1 + a_2 + \cdots)^{q/p} \), it follows

\[
\int_{\Omega^2} u^q(x)v(x)dx \leq C_0 A_{pq} \sum_{j=1}^{2} \left( \int_{\Omega^2} |\partial_{x_j} u|^p \omega_j(z)dz \right)^{\frac{q}{p}},
\]

(3.22)

with

\[
C_0 = \left( \frac{1}{2^q} - \frac{KC\gamma^\sigma}{1 - C\gamma^\sigma} \right)^{-1} \frac{K2^{3q-1}q\ln 2}{\gamma^{2q}}.
\]

Similarly, we can prove the inequality is true in \( \Omega^- \) for the function \(-u(x)\):

\[
\int_{\Omega^-} (-u(x))^q v(x)dx \leq C_0 A_{pq} \sum_{j=1}^{2} \left( \int_{\Omega^-} |\partial_{x_j} u|^p \omega_j(z)dz \right)^{\frac{q}{p}}.
\]

(3.23)

Combining estimates (3.21) and (3.22), we obtain the inequality:

\[
\left( \int_{\Omega} |u(x)|^q v(x)dx \right)^{\frac{1}{q}} \leq C_0 A_{pq} \sum_{j=1}^{2} \left( \int_{\Omega} |\partial_{x_j} u|^p \omega_j(x)dx \right)^{\frac{1}{p}},
\]

(3.24)

so far, the proof is completed. \( \Box \)
Lemma 3.2. For any \( u \in H_{h,0}^1(\Omega) \) and \( 2 \le q \le \frac{2}{1-\theta} \), we have
\[
\|u\|_{L^q(\Omega)} \le C\|u\|_{H_{h,0}^1}(\Omega). \tag{3.24}
\]

Proof. We see that \( H_{h,0}^1(\Omega) = \text{Lip}_0(\Omega)^H_{h}^1(\Omega) \). Indeed, we shall only to prove that \( \text{Lip}_0(\Omega)^H_{h}^1(\Omega) \subset C_0^\infty(\Omega) \text{Lip}_0(\Omega)^H_{h}^1(\Omega) \subset \text{Lip}_0(\Omega)^H_{h}^1(\Omega) \). Indeed, we shall only prove that \( \text{Lip}_0(\Omega)^H_{h}^1(\Omega) \subset \text{Lip}_0(\Omega)^H_{h}^1(\Omega) \), there exists a sequence \( \{u_n\}_{n \in \mathbb{N}} \subset \text{Lip}_0(\Omega) \) such that \( \|u_n - u\|_{H_{h}^1(\Omega)} \to 0 \) as \( n \to \infty \) and \( |Du_n| \le C \) a.e. on \( \Omega \), which shows that \( u_n \in H_{h}^1(\Omega) \) with compact support on \( \Omega \), then there exists \( \{v_n\}_{n \in \mathbb{N}} \subset C_0^\infty(\Omega) \) such that \( \|v_n - u_n\|_{H_{h}^1(\Omega)} \to 0, \ n \to \infty \), and we have
\[
\|v_n - u_n\|_{H_{h}^1(\Omega)} \le C\|v_n - u_n\|_{H_{h}^1(\Omega)} \to 0, \ n \to \infty
\]
in view of the definition of \( h(x_1) \), in addition
\[
\|v_n - u\|_{H_{h}^1(\Omega)} \le \|v_n - u_n\|_{H_{h}^1(\Omega)} + \|u_n - u\|_{H_{h}^1(\Omega)} \to 0, \ n \to \infty,
\]
so that \( H_{h,0}^1(\Omega) = \text{Lip}_0(\Omega)^H_{h}^1(\Omega) \), \( \text{Lip}_0(\Omega) \) is dense in space \( H_{h,0}^1(\Omega) \).

It is not hard to verify that \( v(x) = 1 \) meets the conditions of \( A_\infty(\Omega) \). Consider \( w_1 = 1 \) and \( w_2 = h(x_1) \), and \( p = 2 \) in (3.2), therefore for \( p = 2, 2 \le q \le \frac{2}{1-\theta} \), we immediately receive that
\[
|B|^{-1}d(B)v(B \cap \Omega)^{\frac{1}{q}} \left( \int_{B \cap \Omega} w_1(x)^{-1}dx \right)^{\frac{1}{2}} \le C r^{\frac{2}{q}} \le C,
\]
\[
|B|^{-1}d(B)v(B \cap \Omega)^{\frac{1}{q}} \left( \int_{B \cap \Omega} w_2(x)^{-1}dx \right)^{\frac{1}{2}} \le C r^{\theta-1+\frac{2}{q}} \le C,
\]
furthermore, by Lemma 3.1 for any \( u \in H_{h,0}^1(\Omega) \)
\[
\left( \int_{\Omega} |u|^qdx \right)^{\frac{1}{q}} \le C \left[ \left( \int_{\Omega} |\partial_{x_1}u|^2dx \right)^{\frac{1}{2}} + \left( \int_{\Omega} h(x_1)|\partial_{x_2}u|^2dx \right)^{\frac{1}{2}} \right].
\]
By Jensen inequality \( \frac{a_1^t + a_2^t}{2} \le \left( \frac{a_1 + a_2}{2} \right)^t \) for \( t \in (0, 1), a_1, a_2 \in \mathbb{R}^+ \), we have
\[
\left( \int_{\Omega} |\partial_{x_1}u|^2dx \right)^{\frac{1}{2}} + \left( \int_{\Omega} h(x_1)|\partial_{x_2}u|^2dx \right)^{\frac{1}{2}} \le C \left( \int_{\Omega} |\partial_{x_1}u|^2 + h(x_1)|\partial_{x_2}u|^2dx \right)^{\frac{1}{2}},
\]
so that the desired result is proved.

\[\square\]

Remark 3.3. In particular, Theorem 3.2 implies Poincaré inequality, i.e.
\[
\|u\|_{L^2(\Omega)} \le C \left( \int_{\Omega} |\partial_{x_1}u|^2 + h(x_1)|\partial_{x_2}u|^2dx \right)^{\frac{1}{2}}, \ \forall u \in H_{h,0}^1(\Omega). \tag{3.25}
\]

Therefore, the equivalent norm of \( H_{h,0}^1(\Omega) \) could be written as:
\[
\|u\|_{H_{h,0}^1}(\Omega) = \left( \int_{\Omega} |\partial_{x_1}u|^2 + h(x_1)|\partial_{x_2}u|^2dx \right)^{\frac{1}{2}}.
\]

Theorem 3.4. The embedding \( H_{h,0}^1(\Omega) \hookrightarrow L^2(\Omega) \) is compact.
Proof. Firstly, let \( \{u_n\}_{n \in \mathbb{N}} \subset \mathcal{H}^1_{h,0}(\Omega) \) be a bounded sequence, we assert that the embedding \( \mathcal{H}^1_{h,0}(\Omega) \hookrightarrow L^1(\Omega) \) is compact, so \( \{u_n\}_{n \in \mathbb{N}} \) is Cauchy in \( L^1(\Omega) \). Set

\[
\bar{u}_n(x) = \begin{cases} 
  u_n(x), & x \in \Omega, \\
  0, & x \in \mathbb{R}^2 \setminus \Omega,
\end{cases}
\]

then \( \bar{u}_n \in \mathcal{H}^1_{h,0}(\mathbb{R}^2) \) according to Lemma 2.5. In order to show \( \{\bar{u}_n\}_{n \in \mathbb{N}} \) is paracompact in \( L^1(\Omega) \), we shall show that [1, Theorem 2.32]: for every \( \epsilon > 0 \), there exists \( \delta > 0 \) and a subset \( G \subset \subset \Omega \), such that

\[
\int_\Omega |\bar{u}_n(x + \xi) - \bar{u}_n(x)| \, dx < \epsilon, \quad \int_{\Omega \setminus G} |\bar{u}_n(x)| \, dx < \epsilon
\]

for each \( n \in \mathbb{N} \) and \( \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \), \( |\xi| < \delta \). Let \( \epsilon \in (0, 1) \) be arbitrary, for each \( n \in \mathbb{N} \), there exists \( \{\tilde{u}_n\}_{n \in \mathbb{N}} \subset C_0^\infty(\Omega) \), such that

\[
\|u_n - \tilde{u}_n\|_{\mathcal{H}^1_p(\Omega)} < \epsilon,
\]

still denoted by \( \widehat{(\tilde{u}_n)} = \tilde{u}_n \) for each \( \tilde{u}_n \in C_0^\infty(\Omega) \). Obviously, \( \{\tilde{u}_n\}_{n \in \mathbb{N}} \subset C_0^\infty(\Omega) \) is bounded in \( \mathcal{H}^1_{h,0}(\mathbb{R}^2) \), note that

\[
\tilde{u}_n(x_1, x_2) = \int_0^{x_2} (\partial_{x_2} \tilde{u}_n)(x_1, t) \, dt = \int_0^{x_2} \sqrt{h^{-1}(x_1)} \left( \sqrt{h(x_1)} \partial_{x_2} \tilde{u}_n \right)(x_1, t) \, dt,
\]

which imply

\[
\int_0^1 |\tilde{u}_n(s, x_2)| \, ds \leq \int_0^1 \left| \int_0^{x_2} \sqrt{h^{-1}(s)} \sqrt{h(s)} \partial_{x_2} \tilde{u}_n(s, t) \, dt \right| \, ds \\
\leq \|h^{-1}\|_{L^1(0,1)} \|\sqrt{h(\cdot)} \partial_{x_2} \tilde{u}_n\|_{L^2(\Omega)} \frac{1}{2},
\]

we deduce that

\[
\int_{(0,1) \times (0, j^{-1})} |\tilde{u}_n(x)| \, dx \leq \|h^{-1}\|_{L^1(0,1)} \|\sqrt{h(\cdot)} \partial_{x_2} \tilde{u}_n\|_{L^2(\Omega)} j^{-\frac{3}{2}}
\]

with \( j \in \mathbb{N} \). The same argument is developed again, we have

\[
\int_{(0,1) \times (1-j^{-1}, 1)} |\tilde{u}_n(x)| \, dx \leq \frac{2}{3} \|h^{-1}\|_{L^1(0,1)} \|\sqrt{h(\cdot)} \partial_{x_2} \tilde{u}_n\|_{L^2(\Omega)} \left( 1 - (1 - \frac{1}{j})^{\frac{3}{2}} \right),
\]

\[
\int_{(0,j^{-1}) \times (0,1)} |\tilde{u}_n(x)| \, dx \leq \frac{2}{3} \|\partial_{x_1} \tilde{u}_n\|_{L^2(\Omega)} j^{-\frac{3}{2}},
\]

\[
\int_{(1-j^{-1},1) \times (0,1)} |\tilde{u}_n(x)| \, dx \leq \frac{2}{3} \|\partial_{x_1} \tilde{u}_n\|_{L^2(\Omega)} \left( 1 - (1 - \frac{1}{j})^{\frac{3}{2}} \right).
\]

Set

\[
\Omega_j = \{(x_1, x_2) \in \Omega \mid x_1, x_2 \in (0, j^{-1}) \cup (1-j^{-1}, 1)\}, j \in \mathbb{N},
\]

then

\[
\int_{\Omega_j} |\tilde{u}_n(x)| \, dx \leq C \|\tilde{u}_n\|_{\mathcal{H}^1_{h,0}(\Omega)} \max \left\{ j^{-\frac{3}{2}}, \left( 1 - (1 - \frac{1}{j})^{\frac{3}{2}} \right) \right\},
\]

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where $C > 0$ is a constant dependent only on $h$, which implies that there exists $j_0 \in \mathbb{N}$, when $j \geq j_0$, we have

$$\int_{\Omega} |\tilde{u}_n(x)|dx < \epsilon,$$

taking $G = \Omega - \overline{\Omega}_{j_0}$, we have proved that

$$\int_{\Omega - \overline{\Omega}} |\tilde{u}_n(x)|dx < \epsilon. \quad (3.29)$$

Let $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, without loss of generality, we assume $\xi_1 > 0, \xi_2 > 0$, since

$$|\tilde{u}_n(x_1 + \xi_1, x_2 + \xi_2) - \tilde{u}_n(x_1, x_2 + \xi_2)| = \left| \int_{x_1}^{x_1 + \xi_1} \partial_{x_1} \tilde{u}_n(s, x_2 + \xi_2) ds \right|
\leq \xi_1^\frac{1}{2} \left( \int_{x_1}^{x_1 + \xi_1} |\partial_{x_1} \tilde{u}_n(s, x_2 + \xi_2)|^2 ds \right)^{\frac{1}{2}}
\leq \xi_1^\frac{1}{2} \left( \int_{\mathbb{R}} |\partial_{x_1} \tilde{u}_n(s, x_2 + \xi_2)|^2 ds \right)^{\frac{1}{2}},$$

we obtain that

$$\int_{\mathbb{R}} |\tilde{u}_n(x_1 + \xi_1, x_2 + \xi_2) - \tilde{u}_n(x_1, x_2 + \xi_2)|dx_2 \leq \xi_1^\frac{1}{2} \|\partial_{x_1} \tilde{u}_n\|_{L^2(\Omega)}.$$

hence

$$\int_{\mathbb{R}^2} |\tilde{u}_n(x_1 + \xi_1, x_2 + \xi_2) - \tilde{u}_n(x_1, x_2 + \xi_2)|dx_2 \leq \xi_1^\frac{1}{2} \|\partial_{x_1} \tilde{u}_n\|_{L^2(\Omega)}. \quad (3.30)$$

By the same way as (3.28), we obtain that

$$\int_{\mathbb{R}} |\tilde{u}_n(x_1, x_2 + \xi_2) - \tilde{u}_n(x_1, x_2)|dx_2 \leq \xi_2^\frac{1}{2} \left( \int_0^1 \left( \int_{\mathbb{R}} \frac{1}{\sqrt{h'}} \|\partial_{x_2} \tilde{u}_n\|_{L^2(\Omega)} \right) ds \right)^{\frac{1}{2}}.$$

therefore

$$\int_{\mathbb{R}^2} |\tilde{u}_n(x_1, x_2 + \xi_2) - \tilde{u}_n(x_1, x_2)|dx_2 \leq \xi_2^\frac{1}{2} \left( \int_0^1 \left( \int_{\mathbb{R}} \frac{1}{\sqrt{h'}} \|\partial_{x_2} \tilde{u}_n\|_{L^2(\Omega)} \right) ds \right)^{\frac{1}{2}}. \quad (3.31)$$

From

$$|\tilde{u}_n(x + \xi) - \tilde{u}_n(x)| \leq |\tilde{u}_n(x_1 + \xi_1, x_2 + \xi_2) - \tilde{u}_n(x_1, x_2 + \xi_2)| + |\tilde{u}_n(x_1, x_2 + \xi_2) - \tilde{u}_n(x_1, x_2)|$$

and (3.30), (3.31) to lead

$$\int_{\mathbb{R}^2} |\tilde{u}_n(x + \xi) - \tilde{u}_n(x)|dx \leq \xi_1^\frac{1}{2} \|\partial_{x_1} \tilde{u}_n\|_{L^2(\Omega)} + \xi_2^\frac{1}{2} \left( \int_0^1 \left( \int_{\mathbb{R}} \frac{1}{\sqrt{h'}} \|\partial_{x_2} \tilde{u}_n\|_{L^2(\Omega)} \right) ds \right)^{\frac{1}{2}},$$

this implies that

$$\int_{\mathbb{R}^2} |\tilde{u}_n(x + \xi) - \tilde{u}_n(x)|dx < \epsilon \quad (3.32)$$
for $|\xi| = \sqrt{\xi_1^2 + \xi_2^2}$ small enough. Combining inequalities (3.27), (3.29) and (3.32), we can get
\[
\int_{\mathbb{R}^2} |\hat{u}_n(x_1 + \xi_1, x_2 + \xi_2) - \hat{u}_n(x_1, x_2)|dx \\
= \int_{\mathbb{R}^2} |\hat{u}_n(x_1 + \xi_1, x_2 + \xi_2) - \hat{u}_n(x_1 + \xi_1, x_2 + \xi_2)|dx + \int_{\mathbb{R}^2} |\hat{u}_n(x_1 + \xi_1, x_2 + \xi_2) - \hat{u}_n(x_1, x_2)|dx \\
+ \int_{\mathbb{R}^2} |\hat{u}_n(x_1, x_2) - \hat{u}_n(x_1, x_2)|dx \\
\leq 2\|\hat{u}_n - \hat{u}_n\|_{L^2(\mathbb{R}^2)} + \int_{\mathbb{R}^2} |\hat{u}_n(x_1 + \xi_1, x_2 + \xi_2) - \hat{u}_n(x_1, x_2)|dx \\
\leq 3\varepsilon,
\]
and
\[
\int_{\Omega-\mathbf{C}} |\hat{u}_n(x_1, x_2)|dx \leq \int_{\Omega-\mathbf{C}} |\hat{u}_n(x_1, x_2) - \hat{u}_n(x_1, x_2)|dx + \int_{\Omega-\mathbf{C}} |\hat{u}_n(x_1, x_2)|dx \leq 2\varepsilon.
\]
so far the inequality (3.26) is verified, then we know that $\{u_n\}_{n \in \mathbb{N}}$ is Cauchy in $L^1(\Omega)$.

According to Lemma 3.2, for $q = \frac{2}{1-\theta} > 2$, we have $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $L^q(\Omega)$. By interpolation inequality:
\[
\|u_n - u_m\|_{L^2(\Omega)} \leq \|u_n - u_m\|_{L^q(\Omega)}^{\frac{\alpha}{\alpha}} \|u_n - u_m\|_{L^1(\Omega)}^{1-\frac{\alpha}{\alpha}} \leq (2C)^{\frac{1-\alpha}{\alpha}} \|u_n - u_m\|_{L^1(\Omega)}^{\alpha} \quad (3.33)
\]
for some $\alpha \in (0, 1)$, given that $\{u_n\}_{n \in \mathbb{N}}$ is Cauchy in $L^1(\Omega)$, choosing $n, m$ sufficiently large shows that $\{u_n\}_{n \in \mathbb{N}}$ is Cauchy in $L^2(\Omega)$ by (3.33), this argument completes the proof of Theorem 3.4. □

4 Weak solution and its regularity

**Definition 4.1.** (i) The bilinear form $B[\cdot, \cdot]$ associated with the sub-elliptic operator $L$ is
\[
B[u, v] = \int_{\Omega} (\partial_{x_1} u) (\partial_{x_1} v) + (\sqrt{h(x_1)} \partial_{x_2} u) (\sqrt{h(x_1)} \partial_{x_2} v) + Vu \quad (\ast)
\]
for $u, v \in H_{h,0}^{1}(\Omega)$.

(ii) We call $u \in H_{h,0}^{1}(\Omega)$ is a weak solution to the sub-elliptic equation (2.1), if
\[
B[u, v] = (f, v)_{L^2(\Omega)}
\]
for all $v \in H_{h,0}^{1}(\Omega)$.

**Theorem 4.2.** Let $f \in L^2(\Omega)$ be arbitrary given function, then the sub-elliptic equation (2.1) has a unique solution $u \in H_{h,0}^{1}(\Omega)$.

**Proof.** From Lemma 2.1 we know that $(H_{h,0}^{1}(\Omega), (\cdot, \cdot)_{H_{h,0}^{1}(\Omega)})$ is a Hilbert space. Given that Remark 3.3 we have
\[
B[u, u] = \int_{\Omega} |\partial_{x_1} u|^2 + |\sqrt{h(x_1)} \partial_{x_2} u|^2 + Vu^2 dx \geq \frac{1}{2} \|u\|_{H_{h,0}^{1}(\Omega)}^{2}
\]
for all $u \in \mathcal{H}^1_{h,0}(\Omega)$. Moreover, thanks to the Hölder inequality, we deduce that

$$
|B[u, v]| \\
\leq \|\partial_{x_1} u\|_{L^2(\Omega)}\|\partial_{x_1} v\|_{L^2(\Omega)} + \|\sqrt{h(\cdot)}\partial_{x_2} u\|_{L^2(\Omega)}\|\sqrt{h(\cdot)}\partial_{x_2} v\|_{L^2(\Omega)} + M\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\
\leq \max\{1, M\} \left(\|\partial_{x_1} u\|_{L^2(\Omega)}\|\partial_{x_1} v\|_{L^2(\Omega)} + \|\sqrt{h(\cdot)}\partial_{x_2} u\|_{L^2(\Omega)}\|\sqrt{h(\cdot)}\partial_{x_2} v\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\right) \\
\leq C \left(\|u\|_{L^2(\Omega)}^2 + \|\partial_{x_1} u\|_{L^2(\Omega)}^2 + \|\sqrt{h(\cdot)}\partial_{x_2} u\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \left(\|v\|_{L^2(\Omega)}^2 + \|\partial_{x_1} v\|_{L^2(\Omega)}^2 + \|\sqrt{h(\cdot)}\partial_{x_2} v\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \\
= C\|u\|_{\mathcal{H}^1_{h,0}(\Omega)}\|v\|_{\mathcal{H}^1_{h,0}(\Omega)}
$$

for all $u, v \in \mathcal{H}^1_{h,0}(\Omega)$. For each $v \in \mathcal{H}^1_{h,0}(\Omega)$, it is briefly found that $f : \mathcal{H}^1_{h,0}(\Omega) \to \mathbb{R}$ is a linear functional, and note that

$$(f, v)_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}\|v\|_{\mathcal{H}^1_{h,0}(\Omega)},$$

therefore $f$ is a bounded linear functional on $\mathcal{H}^1_{h,0}(\Omega)$. Applying the Lax-Milgram theorem [18], the desired result is proved.

We denote $\Omega' \subset \subset \Omega$ by $\Omega' \subset \Omega$ is an open subset of $\Omega$ and $\overline{\Omega'}$ is a compact subset of $\Omega$.

**Theorem 4.3.** (1) Let $D_i^l u(x) = \frac{u(x+le_i) - u(x)}{l}$ ($i = 1, 2$) denote the $i$-th difference quotient of size $l$ for $x \in \Omega'$, $l \in \mathbb{R}$, $0 < |l| < \text{dist}(\Omega', \partial \Omega)$. Suppose $u \in \mathcal{H}^1_{h,0}(\Omega)$, then for any $\Omega' \subset \subset \Omega$, we have

$$
\|D_i^l u\|_{L^2(\Omega')} \leq C\sqrt{h(x_1)}\|D_l u\|_{L^2(\Omega)} \leq C\|u\|_{\mathcal{H}^1_{h,0}(\Omega)}
$$

for some constant $C$ and all $0 < |l| < \frac{1}{2} \text{dist}(\Omega', \partial \Omega)$, where $D_i^l u = (D_1^l u, D_2^l u)$.

(2) Let $D_i^l u$ be defined as shown in (1). Suppose $u \in L^2(\Omega')$, and there is a constant $C$ such that

$$
\|D_i^l u\|_{L^2(\Omega')} \leq C
$$

for all $0 < |l| < \frac{1}{2} \text{dist}(\Omega', \partial \Omega)$, then $u \in H^1(\Omega')$ with $\|Du\|_{L^2(\Omega')} \leq C$.

**Proof.** (1) Suppose $u \in C^\infty_0(\Omega)$, for any $x \in \Omega'$, $i = 1, 2$, $0 < |l| < \frac{1}{2} \text{dist}(\Omega', \partial \Omega)$, we see that

$$
|u(x + le_i) - u(x)| = \int_0^1 \partial_{x_i} u(x + tle_i)dt \cdot le_i,
$$

so that

$$
|D_i^l u(x)| \leq |l| \int_0^1 |\partial_{x_i} u(x + tle_i)|dt,
$$

i.e.

$$
|D_i^l u(x)| \leq \int_0^1 |\partial_{x_i} u(x + tle_i)|dt.
$$

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Therefore for any $\delta \in (0, 1)$, by Cauchy inequality, we obtain that

$$\inf_{x_1 \in [0,1-\delta]} h(x_1) \int_{\Omega'} |D^l u(x)|^2 dx \leq \int_{\Omega'} h(x_1) |D^l u(x)|^2 dx = \int_{\Omega'} h(x_1) \left( |D_1^l u(x)|^2 + |D_2^l u(x)|^2 \right) dx \leq \int_{\Omega'} h(x_1) \sum_{i=1}^{2} \left( \int_{0}^{1} |\partial x_i u(x+t\ell e_i)| dt \right)^2 dx \leq \sum_{i=1}^{2} \int_{\Omega'} h(x_1) \int_{0}^{1} |\partial x_i u(x+t\ell e_i)|^2 dt dx$$

$$= \int_{0}^{1} \int_{\Omega'} h(x_1) |\partial x_i u(x_1 + t\ell, x_2)|^2 + h(x_1) |\partial x_2 u(x_1, x_2 + t\ell)|^2 dx dt \leq C \left( \int_{\Omega} |\partial x_1 u(x)|^2 dx + \int_{\Omega} |\sqrt{h(x_1)} \partial x_2 u(x)|^2 dx \right) = C \|u\|^2_{H_{h,0}^1(\Omega)}.$$

Since $C_0^\infty(\Omega)$ is dense in $H_{h,0}^1(\Omega)$, the above inequality is established for any $u \in H_{h,0}^1(\Omega)$.

(2) For this proof, we can refer to Theorem 3 [18, Chapter 5.8.2]. \qed

**Theorem 4.4.** Assuming that $h(x_1)$ and $V(x)$ are defined as given in the Introduction (Section 1), and $f \in L^2(\Omega)$. Suppose that $u \in H_{h,0}^1(\Omega)$ is the weak solution for the problem $Lu = f$ on $\Omega$, then $u \in H_{\text{loc}}^2(\Omega)$.

**Proof.** 1. For any subset $\Omega' \subset \subset \Omega$, we may choose an open set $W$ such that $\Omega' \subset \subset W \subset \subset \Omega$. In addition, we select a function $\eta \in C_0^\infty(\mathbb{R}^2)$ such that $\eta \equiv 1$ on $\Omega'$, $\eta \equiv 0$ on $\mathbb{R}^2 - W$ and $0 \leq \eta \leq 1$. Note that $u$ is the weak solution for $Lu = f$, then for any $v \in H_{h}^1(\Omega)$:

$$\sum_{i,j=1}^{2} \int_{\Omega} a_{ij}(x) \partial x_i u \partial x_j v dx = \int_{\Omega} f v - Vuv dx,$$

where $a_{11} = 1, a_{12} = a_{21} = 0, a_{22} = h(x_1)$. Let

$$A = \sum_{i,j=1}^{2} \int_{\Omega} a_{ij}(x) \partial x_i u \partial x_j v dx, \quad B = \int_{\Omega} f v - Vuv dx,$$

then $A = B$.

2. Let $0 < |l| < \frac{1}{2} \min\{\text{dist}(\Omega', \partial W), \text{dist}(W, \partial \Omega)\}$ and consider that $|l|$ be sufficiently small, then substitute $v = -D_k^{-l}(\eta^2 D_k^l u)$ into (4.1), $k \in \{1, 2\}$. Then

$$A = -\sum_{i,j=1}^{2} \int_{\Omega} a_{ij}(x) \partial x_i u \partial x_j \left( D_k^{-l}(\eta^2 D_k^l u) \right) dx$$

$$= \sum_{i,j=1}^{2} \int_{\Omega} D_k^l (a_{ij} \partial x_i u) \partial x_j (\eta^2 D_k^l u) dx$$

$$= \sum_{i,j=1}^{2} \int_{\Omega} a'_{ij}(D_k^l \partial x_i u) \partial x_j (\eta^2 D_k^l u) + (D_k^l a_{ij}) \partial x_i u \partial x_j (\eta^2 D_k^l u) dx$$

$$= \sum_{i,j=1}^{2} \int_{\Omega} a'_{ij}(D_k^l \partial x_i u)(D_k^l \partial x_j u) \eta^2 dx + \sum_{i,j=1}^{2} \int_{\Omega} \left( 2\eta(\partial x_j, \eta) a'_{ij}(D_k^l \partial x_i u) D_k^l u \right) dx.$$
\[ (D^t_k a_{ij})(\partial_x u)(D^t_k \partial_x u)\eta^2 + 2\eta(\partial_x \eta)(D^t_k a_{ij})(\partial_x u)D^t_k u \right) dx \\
= A_1 + A_2, \]

where \( a^t_{ij}(x) = a_{ij}(x + le_k) \). According to the definitions of \( h(x_1) \) and \( \eta \), we have

\[ A_1 \geq \min \left\{ 1, \inf_{x_1 \in [\delta, 1-\delta]} h(x_1) \right\} \int_{\Omega} \eta^2 |D^t_k Du|^2 dx \geq \theta \int_{\Omega} \eta^2 |D^t_k Du|^2 dx \]  

for some proper constant \( \theta \) and \( \delta \in (0, 1) \), and

\[ |A_2| \leq C \int_{\Omega} \left( \eta |D^t_k Du| |D^t_k u| + \eta |D^t_k Du| |Du| + \eta |D^t_k u| |Du| \right) dx. \]

Furthermore, by \( \text{supp} \eta \subset W \) and Cauchy’s inequality with \( \epsilon \), then

\[ |A_2| \leq \epsilon \int_{\Omega} \eta^2 |D^t_k Du|^2 dx + \frac{C}{\epsilon} \int_W \left( |D^t_k u|^2 + |Du|^2 \right) dx. \]

By invoking the result (1) of Theorem 4.3, we have \( \int_W |D^t_k u|^2 \leq C \|u\|_{H^{1\delta,0}(\Omega)}^2 \). It is not hard to find that

\[ \int_W |Du|^2 dx \leq \left( \min\{1, \inf_{x_1 \in [\delta, 1-\delta]} h(x_1) \} \right)^{-1} \|u\|_{H^{1\delta,0}(\Omega)}^2 \leq C \|u\|_{H^{1\delta,0}(\Omega)}^2 \]

for \( \delta \in (0, 1) \). We may choose \( \epsilon = \frac{\theta}{2} \), hence

\[ |A_2| \leq \frac{\theta}{2} \int_{\Omega} \eta^2 |D^t_k Du|^2 dx + C \|u\|_{H^{1\delta,0}(\Omega)}^2. \]  

Combining (4.2) with (4.3), we can get

\[ A \geq \frac{\theta}{2} \int_{\Omega} \eta^2 |D^t_k Du|^2 dx - C \|u\|_{H^{1\delta,0}(\Omega)}^2. \]  

3. Since \( v = -D^{-t}_k(\eta^2 D^t_k u) \), by the (1) of Theorem 4.3,

\[ \int_{\Omega} |v|^2 dx \leq C \int_{\Omega} \left( h(x_1)D^t_k \left( \eta^2 D^t_k u \right) \right)^2 dx \]

\[ \leq C \int_{W} |D^t_k u|^2 + \eta^2 |D^t_k Du|^2 dx \]

\[ \leq C \|u\|_{H^{1\delta,0}(\Omega)}^2 + C \int_{\Omega} \eta^2 |D^t_k Du|^2 dx. \]

By the Cauchy’s inequality with \( \epsilon \), we have

\[ |B| \leq C \int_{\Omega} |f| |v| + |u| |v| dx \]

\[ \leq \epsilon \int_{\Omega} \eta^2 |D^t_k Du|^2 dx + \frac{C}{\epsilon} \int_{\Omega} |f|^2 + |u|^2 dx + \frac{C}{\epsilon} \|u\|_{H^{1\delta,0}(\Omega)}^2. \]

Similarly, we choose \( \epsilon = \frac{\theta}{4} \), then

\[ |B| \leq \frac{\theta}{4} \int_{\Omega} \eta^2 |D^t_k Du|^2 dx + C \int_{\Omega} |f|^2 + |u|^2 dx + C \|u\|_{H^{1\delta,0}(\Omega)}^2. \]  

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4. According to (4.4) and (4.5), we obtain that
\[ \frac{\theta}{4} \int_{\Omega} \eta^2 |D_k^1 Du|^2 \, dx \leq C \int_{\Omega} |f|^2 + |u|^2 \, dx + C \|u\|_{H_{h,0}^1(\Omega)}^2, \]
therefore
\[ \int_{\Omega'} |D_k^1 Du|^2 \, dx \leq C \int_{\Omega} |f|^2 + |u|^2 \, dx + C \|u\|_{H_{h,0}^1(\Omega)}^2, \quad k = 1, 2, \]
for any sufficiently small |l| \neq 0. Using the result (2) of Theorem 4.3, we know Du \in H^1(\Omega'), hence
u \in H^2_{\text{loc}}(\Omega).

5 Fundamental gap

This section is mainly to characterize the optimal potentials over the class S, some of ideas were developed in [7, 23, 40, 28, 27]. Before that, we will obtain basic properties of spectral theory for the boundary-value problem:
\[ \begin{cases}
Lu = \lambda u, & x \in \Omega, \\
u = 0, & x \in \partial \Omega,
\end{cases} \tag{5.1} \]
where the operator L is defined as (1.3).

**Lemma 5.1.** For any given f \in L^2(\Omega), define
\[ S : L^2(\Omega) \to L^2(\Omega), \quad f \mapsto u, \tag{5.2} \]
where u \in H_{h,0}^1(\Omega) is the unique solution for differential equation (2.1), then S is a bounded self-adjoint compact operator.

**Proof.** For any f, g \in L^2(\Omega), let u and v be the solutions corresponding to f and g, respectively, we observe that
\[ (Sf, g)_{L^2(\Omega)} = (u, g)_{L^2(\Omega)} = B[u, v] = (f, v)_{L^2(\Omega)} = (f, Sg)_{L^2(\Omega)}. \]
And
\[ B[u, u] = (f, u)_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|u\|_{H_{h,0}^1(\Omega)}, \]
this implies that \|u\|_{H_{h,0}^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}, therefore
\[ \|Sf\|_{L^2(\Omega)} = \|u\|_{L^2(\Omega)} \leq \|u\|_{H_{h,0}^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \]
To make a connection with the result obtained in Theorem 3.4, the Lemma 5.1 is proved.

**Remark 5.2.** It is not hard to find that \dim H_{h,0}^1(\Omega) = \infty, so that 0 \in \sigma(S), \sigma(S) - \{0\} = \sigma_p(S) - \{0\}, and \sigma(S) - \{0\} is a sequence tending to 0, where \sigma_p(S) is recorded as the point spectrum of S and \sigma(S) is considered as the spectrum of S.
Lemma 5.3. (1) All the eigenvalues of $L$ is real and can be arranged in a monotone sequence on the basis of its (finite) multiplicity:

$$
\sigma(L) = \{\lambda_k\}_{k=1}^{\infty}, \quad 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \cdots \to \infty, \quad k \to \infty.
$$

(2) There exists an orthonormal basis $\{w_k\}_{k=1}^{\infty} \subset L^2(\Omega)$, where $w_k \in H^1_{h,0}(\Omega)$ is an eigenfunction with respect to $\lambda_k$

\[
\begin{cases}
Lw_k = \lambda_k w_k, & x \in \Omega, \\
u = 0, & x \in \partial \Omega,
\end{cases}
\]
for $k = 1, 2 \cdots$.

(3) We have

$$
\lambda_k = \inf_{E \subset H^1_{h,0}(\Omega), \|u\|_{L^2(\Omega)} = 1} \sup_{u \in E} B[u, u].
$$

(5.3)

In particular, assuming that we have already computed $u_1, u_2, \cdots, u_{k-1}$ the $(k-1)$-th first eigenfunctions, we also have: $\lambda_k = \inf \{B[u, u] \mid u \in H^1_{h,0}(\Omega), u \perp V_{k-1}, \|u\|_{L^2(\Omega)} = 1\}$, where $V_{k-1} = \operatorname{span}\{u_1, u_2 \cdots, u_{k-1}\}$, the equality holds if and only if $u = w_k$.

(4) The eigenvalue $\lambda_1$ is simple and the first eigenfunction $u_1$ is positive on $\Omega$.

Proof. Let $S = L^{-1}$, $S$ is a self-adjoint compact linear operator owing to Lemma 5.1. Note

$$(Lf, f) = (u, f) = B[u, u] \geq 0$$

for any given $f \in L^2(0,1)$. According to Theorem D7 (pp. 728) [18], we know that the eigenvalues of $S$ is real and positive, and there exists a countable orthonormal basis of $L^2(\Omega)$ consisting of eigenvectors of $S$. Moreover, for $\eta \neq 0$, $Sw = \eta w$ if and only if $Lw = \lambda w$ for $\lambda = \frac{1}{\eta}$. Then (1)(2) is proved.

By (2), we have

\[
\begin{cases}
B[w_k, w_k] = \lambda_k(w_k, w_k) = \lambda_k, \\
B[w_k, w_l] = \lambda_k(w_k, w_l) = 0, k, l = 1, 2 \cdots, k \neq l.
\end{cases}
\]

(5.4)

Since $\{w_k\}_{k=1}^{\infty}$ is the orthogonal basis of $L^2(\Omega)$, if $u \in H^1_{h,0}(\Omega)$ and $\|u\|_{L^2(\Omega)} = 1$, then

$$
u = \sum_{k=1}^{\infty} d_k w_k, \quad d_k = (u, w_k), \quad \sum_{k=1}^{\infty} d_k^2 = \|u\|^2_{L^2(\Omega)} = 1,
$$

(5.5)

the series converging in $L^2(\Omega)$. By (5.4), $\frac{w_k}{\sqrt{\lambda_k}}$ is an orthonormal subset of $H^1_{h,0}(\Omega)$, endowed with the new inner product $B[., .]$. Actually, due to $V(\varphi) \in S$,

$$
c_1\|u\|^2_{H^1_{h,0}(\Omega)} \leq B[u, u] \leq c_2\|u\|^2_{H^1_{h,0}(\Omega)}.
$$

(5.6)

Besides, if $B[w_k, u] = \lambda_k(w_k, u) = 0$ for $k = 1, 2 \cdots$, we deduce that $u \equiv 0$ due to $\lambda_k > 0$ and $\{w_k\}_{k=1}^{\infty}$ is the orthonormal basis of $L^2(\Omega)$. Hence $u = \sum_{k=1}^{\infty} \mu_k \frac{w_k}{\sqrt{\lambda_k}}$ for $\mu_k = B[u, \frac{w_k}{\sqrt{\lambda_k}}]$, the series
converging in $\mathcal{H}^1_{h,0}(\Omega)$. Combining with (5.5), we obtain that $\mu_k = d_k \sqrt{\lambda_k}$ and $u = \sum_{k=1}^{\infty} d_k w_k$ is convergent in $\mathcal{H}^1_{h,0}(\Omega)$.

Let $E$ denote any $k$-dimensional subspace of $\mathcal{H}^1_{h,0}(\Omega)$, and let $U = \text{span}\{w_k, w_{k+1}, \cdots\}$, clearly, $E \cap U \neq \emptyset$. For any $u \in E \cap U$ with $\|u\|_{L^2(\Omega)} = 1$, we have $u = \sum_{i=k}^{\infty} d_i w_i$ with $\sum_{i=k}^{\infty} d_i^2 = 1$, furthermore,

$$B[u, u] = \sum_{i=k}^{\infty} d_i^2 B(w_i, w_i) = \sum_{i=k}^{\infty} d_i^2 \lambda_i \geq \lambda_k,$$

in other words, $\sup_{u \in E \cap U} B[u, u] \geq \lambda_k$ is established for any subspace $E \subset \mathcal{H}^1_{h,0}(\Omega)$, i.e.

$$\inf_{E \subset \mathcal{H}^1_{h,0}(\Omega)} \sup_{d(E) = k \text{ and } \|u\|_{L^2(\Omega)} = 1} B[u, u] \geq \lambda_k. \quad (5.7)$$

On the other side, consider $V = \text{span}\{w_1, w_2, \cdots, w_k\}$, we find that $V \subset E$. For any $u \in V$ with $\|u\|_{L^2(\Omega)} = 1$, we obtain

$$B[u, u] = \sum_{i=1}^{k} d_i^2 B[w_i, w_i] = \sum_{i=1}^{k} d_i^2 \lambda_i \leq \lambda_k,$$

that is, $\sup_{u \in V \text{ with } \|u\|_{L^2(\Omega)} = 1} B[u, u] \leq \lambda_k$, given that $V$ is one of the subspaces of $E$, then

$$\inf_{E \subset \mathcal{H}^1_{h,0}(\Omega)} \sup_{d(E) = k \text{ and } \|u\|_{L^2(\Omega)} = 1} B[u, u] \leq \lambda_k. \quad (5.8)$$

By employing the equalities (5.7) and (5.8), then (5.3) is proved.

Suppose we have got $u_1, u_2, \cdots, u_{k-1}$, for any $u \in \mathcal{H}^1_{h,0}(\Omega)$ and $u \perp V_{k-1}$ with $\|u\|_{L^2(\Omega)} = 1$, where $V_{k-1} = \text{span}\{u_1, u_2, \cdots, u_{k-1}\}$, then $u = \sum_{i=1}^{\infty} (u, w_i) w_i = \sum_{i=k}^{\infty} (u, w_i) w_i = \sum_{i=k}^{\infty} d_i w_i$ and $\sum_{i=k}^{\infty} d_i^2 = 1$.

Moreover, $B[u, u] = \sum_{i=k}^{\infty} B[w_i, w_i] = \sum_{i=k}^{\infty} d_i^2 \lambda_i \geq \lambda_k$, which implies that

$$\inf_{u \in \mathcal{H}^1_{h,0}(\Omega), u \perp V_{k-1}} B[u, u] \geq \lambda_k.$$

By (5.4), we see that $B[w_k, w_k] = \lambda_k$, then the result (3) is confirmed.

For the weak solution $u \in \mathcal{H}^1_{h,0}(\Omega)$, actually $u \in W^{1,2}(\Omega')$ and the elliptic operator $L$ are uniform for any $\Omega' \subset \subset \Omega$ by the definition of $h(x_1)$, according to Chapter 8.6 and Chapter 8.8 on [20], we can get the Harnack inequality $\sup_{x \in \Omega'} u(x) \leq C \inf_{x \in \Omega'} u(x)$. According to (3), if $u$ is the eigenfunction for $\lambda_1$, so is $|u|$. Since $|u|$ is non-negative in $\Omega$, we further obtain that $|u|$ is a positive eigenfunction taking advantage of Harnack inequality. This shows that the eigenfunctions of $\lambda_1$ are either positive or negative and thereby it is impossible that two of them are orthogonal, therefore $\lambda_1$ is simple.

\[\square\]

Next, we commence by investigating the existence of the minimum fundamental gap when $V(x)$ is limited to the set $S$, and then further characterize the optimal function and express its behavior.
Theorem 5.4. The fundamental gap $\Gamma(V)$ attains its minimum in the classes of $S$ by $V^*$.

Proof. Let $\{V^k\}_{k \in N} \in S$ be the minimization sequence of $\Gamma(V)$, i.e.

$$\Gamma(V^k) \downarrow \inf_{V \in S} \Gamma(V).$$

Since $S$ is a bounded closed convex set, there is a sequence $\{V^k\}_{k \in N}$ and $V^* \in S$, such that

$$V^k \to V^* \text{ weakly star in } L^\infty(\Omega).$$

Let $\{(\lambda^k_j, u^k_j)\}_{k \in N}$ be a sequence of eigenpairs of fixed index of the degenerate elliptic equation (5.1) related to $V^k$, where $u^k_j$ has unit $L^2$ norm, $j = 1, 2$. We may assume $h(x_1) \leq H$ for $x_1 \in (0, 1)$ a.e. so that

$$\lambda^k_j = \inf_{E \subseteq H^1_{h,0}(\Omega)} \sup_{u \in E, \|u\|_{L^2(\Omega)} = 1} \int_{\Omega} |\partial_{x_1} u|^2 + |\sqrt{h(x_1)} \partial_{x_2} u|^2 + V^k |u|^2 \, dx$$

$$\leq \inf_{E \subseteq H^1_{h,0}(\Omega)} \sup_{u \in E, \|u\|_{L^2(\Omega)} = 1} \int_{\Omega} |\partial_{x_1} u|^2 + H |\partial_{x_2} u|^2 + M |u|^2 \, dx$$

$$= \int_{\Omega} |\partial_{x_1} u_j|^2 + H |\partial_{x_2} u_j|^2 + M |\bar{u}_j|^2 \, dx \leq C,$$

where $\|\bar{u}_j\|_{L^2(\Omega)}$ is the corresponding normalized eigenfunction of (5.1) involving with $h(x_1) \equiv H$ and $V(x) \equiv M$ on $\Omega$, $j = 1, 2$. In fact,

$$\frac{1}{2} \|u^k_j\|^2_{H^1_{h,0}(\Omega)} \leq \int_{\Omega} |\partial_{x_1} u^k_j|^2 + |\sqrt{h(x_1)} \partial_{x_2} u^k_j|^2 + V|u^k_j|^2 \, dx = \lambda^k_j \leq C,$$

therefore

$$\|u^k_j\|_{H^1_{h,0}(\Omega)} \leq C.$$

So, there exists a subsequence of $\{u^k_j\}_{k \in N}$ by (5.11), still preserved this index, such that

$$u^k_j \to u^*_j \text{ weakly in } H^1_{h,0}(\Omega), \quad j = 1, 2,$$

and by Theorem 3.4

$$u^k_j \to u^*_j \text{ strongly in } L^2(\Omega), \quad j = 1, 2.$$  

Now we can extract a further subsequence such that

$$\lambda^k_j \to \lambda^*_j, \quad j = 1, 2,$$

for each $j$ we know for all $v \in H^1_{h,0}(\Omega)$

$$\int_{\Omega} (\partial_{x_1} u^k_j)(\partial_{x_1} v) + (\sqrt{h(x_1)} \partial_{x_2} u^k_j)(\sqrt{h(x_1)} \partial_{x_2} v) + V^k u^k_j v \, dx = \lambda^k_j \int_{\Omega} u^k_j v \, dx,$$

choosing $k$ sufficiently large, according to (5.9) (5.12) and (5.13) (5.14),

$$\int_{\Omega} (\partial_{x_1} u^*_j)(\partial_{x_1} v) + (\sqrt{h(x_1)} \partial_{x_2} u^*_j)(\sqrt{h(x_1)} \partial_{x_2} v) + V^*_u u^*_j v \, dx = \lambda^*_j \int_{\Omega} u^*_j v \, dx.$$  

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This indicates that \( \lambda_j^k \) converges to an element of the spectrum of the problem (5.1) given by \( V^* \). In particular, we may extract a subsequence of \( \{u_k^j\}_{k \in \mathbb{N}} \subset L^2(\Omega) \) such that \( u_k^j \) is converge to \( u_1^j \) a.e. by (5.13), based on the non-negativity of \( u_k^j \) (see Lemma 5.3 (4)), then \( u_1^j \) must be the first eigenfunction and \( \lambda_1^j = \lambda_1(V^*) \). Considering that \( u_j^* \) and \( u_1^* \) are orthogonal on \( \Omega \), we know that \( u_2^* \) must change the sign on \( \Omega \), which means that \( u_2^* \) not to be the first eigenfunction of \( \lambda_1(V^*) \) by Lemma 5.3. Therefore, we have \( \lambda_2^* \geq \lambda_2(V^*) \), furthermore,

\[
\Gamma(V^k) \to \lambda_2^* - \lambda_1^j \geq \Gamma(V^*), k \to \infty,
\]

(5.17) that is, \( \Gamma(V^*) \leq \inf_{V \in S} \Gamma(V) \). Meanwhile, \( \Gamma(V^*) \geq \inf_{V \in S} \Gamma(V) \), then we conclude that the minimum value of \( \Gamma(V) \) can be reached.

**Definition 5.5.** A real-valued, measurable and bounded function \( P(x) \) on \( \Omega \) is called an admissible perturbation of \( V(x) \) provided \( V(x) + tP(x) \in S \) for any sufficiently small \( t \in (-\epsilon, \epsilon) \). The function \( P(x) \) is called a left-admissible (right-admissible) perturbation provided \( V(x) + tP(x) \in S \) for non-positive (non-negative) \( t \).

Consider the operator \( L \) as in (1.3) on the Hilbert space \( H^1_{h,0}(\Omega) \) and a family of operators \( L_t \) defined by replacing \( V \) by \( V + tP(x) \) in \( L \), \( t \in (-\epsilon, \epsilon) \) for small \( \epsilon > 0 \). Let \( \lambda_0 \) be a discrete eigenvalue of \( L_0 \), then there exist families \( \lambda_l(t) \) \( (l = 1, 2, \ldots, r) \) of discrete eigenvalues of \( L_t \) such that \( \lambda_l(0) = \lambda_0 \), the total multiplicity of the eigenvalues \( \lambda_l(t) \) \( (l = 1, 2, \ldots, r) \) is equal to the multiplicity of \( \lambda_0 \), and the eigenvalues are analytic in \( t \). Especially, if the discrete eigenvalue \( \lambda_j(L_0) \) is simple \([23, 40, 28] \), we have

\[
\frac{d\lambda_j(V + tP(x))}{dt}\bigg|_{t=0} = \int_{\Omega} P(x)u_j^2 dx,
\]

(5.18)

where \( u_j \) is the normalized eigenfunction with respect to \( L_0 \). If the discrete eigenvalue \( \lambda_j(L_0) \) is degenerate (the multiplicity is \( r \) \([23, 40, 28, 14, 42] \), then it splits into several eigenvalue branches \( \lambda_{j,m} \) under a perturbation, these are the eigenvalues of \( L_t \) which converge to \( \lambda_j \) as \( t \to 0 \), and each branch is an analytic function for small \( t \), but those functions do not ordinarily correspond to the ordering of eigenvalues given by the min-max principle (Lemma 5.3 (3)). For example, the lowest one for \( t < 0 \) will be the highest for \( t > 0 \). Moreover, we

\[
\frac{d\lambda_{j,m}(V + tP(x))}{dt}\bigg|_{t=0} = \int_{\Omega} P(x)u_{j,m}^2 dx,
\]

(5.19)

where the orthonormal eigenfunctions \( u_{j,m} \) are chosen so that

\[
\int_{\Omega} u_{j,i} P(x)u_{j,m} = 0, \ i \neq m.
\]

**Theorem 5.6.** If a function \( V^* \) minimizes \( \Gamma(V) \) in the class \( S \) for the eigenvalue problem (5.1), then \( \lambda_2(V^*) \) is non-degenerate and there exists a subset \( \omega \subset \Omega \) such that

\[
V^* = m\chi_{\omega} + M\chi_{\omega^c},
\]

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where $u_1^*$ and $u_2^*$ represent the first and second normalized eigenfunctions with respect to $V^*$, respectively.

Proof. Step 1. Suppose $\lambda_2(V^*)$ is simple, we claim that $T := \{ x \in \Omega \mid m < V^*(x) < M \}$ is a subset with zero measure.

By contradictory, we suppose $|T| > 0$. Let $T_k = \{ x \in \Omega \mid m + \frac{1}{k} < V^*(x) < M - \frac{1}{k} \}$ for all $k \in \mathbb{N}$. Since $T = \bigcup_{k=1}^{\infty} T_k$ and $T_k \subset T_{k+1}$ for all $k \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that $|T_{k_0}| > 0$.

Applying the Lebesgue Differential Theorem, we have $|u_2^*|^2 - |u_1^*|^2 = 0$ on $T_{k_0}$ and hence on $T$.

Step 2. Next, we show that $|u_2^*|^2 = |u_1^*|^2$ can only appear on the set with zero measure. Otherwise, based on (4) of Lemma 5.3, we have $T = T^+ \cup T^-$, $T^+ = \{ x \in T \mid u_2^*(x) > 0 \}$, $T^- = \{ x \in T \mid u_2^*(x) < 0 \}$, without loss of generality, we assume the set $T^+$ is a set of positive measure. Then we have $u_1^* = u_2^*$ on $T^+$, note that $u_1^* - u_2^* \in H^2_{\text{loc}}(\Omega)$ by Theorem 4.4, then $\Delta(u_1^* - u_2^*) = 0$ a.e. on $T^+$ [26, (Corollary 1.21, Chapter 1)]. Furthermore, we can get $Lu_1^* = Lu_2^*$ a.e. on $T^+$. Together with equation (5.1), we obtain that $(\lambda_2 - \lambda_1)u_1^* = 0$ a.e. on $T^+$, in view of result (4) of Lemma 5.3, then $\lambda_2 = \lambda_1$, it is a contradictory.

Step 3. If $\lambda_2(V^*)$ is simple, for any Lebesgue point $x_0 \in \omega := \{ x \in \Omega \mid V^*(x) = m \}$ and $G_j$ is a sequence of sets containing $x_0$, we see that $P(x) = \chi_{G_j}$ is a right admissible perturbation for sufficiently small non-negative $t$, and repeating the proof of Step 1 then we obtain $|u_2^*|^2 - |u_1^*|^2 \geq 0$ on $\omega$. Similarly, we can receive the properties of $\omega^c$ employing the same technique for $V(x) = M$.

According to the normalization condition $\|u_1^*\|^2_{L^2(\Omega)} = \|u_2^*\|^2_{L^2(\Omega)} = 1$, we see that

$$\int_{\Omega} |u_2^*|^2 - |u_1^*|^2 dx = 0 \quad (5.21)$$

we observe that $\omega$ and $\omega^c$ are non-empty. Indeed, if the set $\omega^c$ is empty, we must have $|u_2^*|^2 - |u_1^*|^2 \geq 0$ on $\Omega$, this is in contradiction with (5.21). Similarly, if $\omega$ is empty, it will also cause contradiction.

Step 4. Suppose $\lambda_2(V^*)$ is $r$-fold degenerate, we first realize that the set $T$ mentioned in Step 1 is a subset of zero measure, and we will give a proof by contradiction. In fact, $\lambda_2(V^*)$ will split into several eigenvalue branches $\lambda_2,m$ under an admissible perturbation $P(x)$, each branch is an analytic function for small $t$ at $t = 0$, and by (5.19)

$$\frac{d\Gamma_m(V + tP(x))}{dt}\bigg|_{t=0} = \int_{\Omega} P(x) \left( |u_{2,m}^*|^2 - |u_1^*|^2 \right) dx, \quad (5.22)$$
where \( \Gamma_m = \lambda_{2,m} - \lambda_1 \) and the orthonormal eigenfunctions \( u_{2,m}^* \) are specially chosen so that \( \int_\Omega u_{2,j}^* P(x) u_{2,m}^* dx = 0 \) for \( j \neq m, m = 1, 2, \ldots, r \).

For any given proper admissible perturbation \( P(x) \), if there exists \( m \) such that
\[
\frac{d \Gamma_m(V + tP(x))}{dt} \bigg|_{t=0} > 0,
\]
we would obtain that
\[
\Gamma(L_{t_0}) \leq \Gamma_m(t_0) < \Gamma_m(0) = \Gamma(0)
\]
(5.23)
for some negative \( t_0 \), however, \( \Gamma(L_0) \) is a minimum, so this situation is excluded. Analogously, if there exists \( m \) such that \( \frac{d \Gamma_m(V + tP(x))}{dt} \bigg|_{t=0} < 0 \), we would obtain that (5.23) for some positive \( t_0 \), this situation was also ruled out. Therefore, the integral (5.22) must vanish.

Suppose \( u \) is any normalized eigenfunction in the eigenspace for \( \lambda_2(V^*) \), then
\[
\begin{align*}
  u^* &= \sum_{i=1}^r \alpha_i u_{2,i}^*, \\
  \sum_{i=1}^r |\alpha_i|^2 &= 1,
\end{align*}
\]
where \( \alpha_i \in \mathbb{R} \). According to the above conclusion, we can see that
\[
\int_\Omega P(x) \left( |u^*|^2 - |u_1^*|^2 \right) dx = \int_\Omega P(x) \left( \sum_{i=1}^r |\alpha_i|^2 |u_{2,i}^*|^2 - |u_1^*|^2 \right) dx
\]
\[
= \int_\Omega P(x) \left( \sum_{i=1}^r |\alpha_i|^2 |u_{2,i}^*|^2 - \sum_{i=1}^r |\alpha_i|^2 |u_1^*|^2 \right) dx
\]
\[
= \sum_{i=1}^r |\alpha_i|^2 \int_\Omega P(x) \left( |u_{2,i}^*|^2 - |u_1^*|^2 \right) dx = 0.
\]
As demonstrated in Step 1 and Step 2, we know the set \( T \) is a subset of zero measure from (5.24) by selecting the appropriate \( P(x) \), that is, \( V^* = m\chi_\omega + M\chi_{\omega^c} \). By the same way, we have
\[
|u^*|^2 \geq |u_1^*|^2 \text{ on } \omega, \quad |u^*|^2 \leq |u_1^*|^2 \text{ on } \omega^c,
\]
(5.25)
and \( \omega \) and \( \omega^c \) are non-empty.

Take advantage of the Theorem 4.4, we find that the weak solution of problem (5.1) belongs to \( H^2_{\text{loc}}(\Omega) \), furthermore, the weak solution is continuous for any subset \( \Omega' \subset \subset \Omega \) by Sobolev Embedding Theorem [18, 20]. Then we may find a interior point \( x_0 \) on \( \omega \) such that \( u_{2,1}(x_0) \neq 0 \) and \( u_{2,2}(x_0) \neq 0 \), where \( u_{2,1} \) and \( u_{2,2} \) are orthogonal on \( \Omega \). Consider a nonzero vector \( (\alpha_1, \alpha_2) \in \mathbb{R}^2 \), \( \alpha_1^2 + \alpha_2^2 = 1, \alpha_1 \neq 0, \alpha_2 \neq 0 \) and \( \alpha_1 u_{21}(x_0) + \alpha_2 u_{22}(x_0) = 0 \). Consider the normalized eigenfunction \( u = \alpha_1 u_{21} + \alpha_2 u_{22} \), we have \( u(x_0) = 0 \) inside \( \omega \), this is incompatible with the fact of (5.25). Therefore, the second eigenvalue \( \lambda_2(V^*) \) cannot be degenerate. □

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Competing interests declaration

We declare that we have no conflict of interest and we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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