Quasiprobability distribution of Classical solitons

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Abstract
Quasiprobability distributions are useful in formulating the phase space analogue of quantum mechanics. We calculate them for the Kink and Sine-Gordan solitons. The quasiprobability distributions are useful as they can be used to calculate quantities like charge distributions, current density, and the upper bound on the quantum speed limit. We calculate the charge distributions and current densities from the derived quasiprobability distribution for both the types of solitons.

1 Introduction
One of the seminal works in the field of semi-classical physics was carried out by Wigner, who combined the distribution of particle’s position (co-ordinate) and momentum in terms of a wave function. This function which is termed as Wigner function or Wigner distribution [1] shows the phase space formulation of quantum mechanics. Moreover, it acts as a standard tool to study the quantum-classical interface [2]. The classical particle is represented by a point with its position and momentum as coordinates in the phase space [3]. For a given ensemble of particles, the Liouville density [4] gives the probability distribution of that ensemble. The probability distribution is of sheer importance as it helps in finding the trajectory of that entire ensemble. In the case of an ensemble of quantum particles, a similar representation is not possible due to the uncertainty principle [5]. Therefore, Wigner distribution [1], mentioned above, comes for the rescue as it gives a quasiprobability distribution [4] for that ensemble although it does not satisfy all the properties of conventional classical probability distribution and becomes negative [6] in some regions of the phase space. Thus Wigner distribution is helpful in studying the quantum analogue of the classical phase space approach. It has a broad range of applications namely in the field of quantum optics [7, 8, 9, 10, 11], quantum computing [12, 13, 14, 15, 16], signal processing [17, 18, 19, 20, 21, 22], quantum chromodynamics [23, 24, 25, 26], etc.

In this work, we have calculated the Wigner distribution of the classical solitons [27] namely, the kink and the Sine-Gordon solitons. Solitons are the solutions of the classical field equations which are similar to particles sometimes referred to as pseudo particles [28]. We have also calculated the charge and current distributions of the same, using the results obtained from the calculation of the Wigner distribution. The motivation of us to calculate the Wigner distribution of these solitons is due to the behaviour of these solitons as described in [28], which says that soliton forms the solution of semi-classical approximation in the second quantized relativistic field theory. Moreover, the Wigner distribution also emphasises a similar idea as that of a quasiprobability distribution. The study of solitons is restricted in the field of particle physics but has a widespread application in condensed matter physics and quantum computing. Therefore, we have given a short glimpse of how the

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Wigner distribution helps us to find the Classical speed limit time \([29]\), semi-classical speed limit time \([29]\), and Quantum speed limit time \([30]\) in the present context of solitons towards the end, as the quantum speed limit time \([30]\) forms the foundations of quantum information computing.

In section (2) we begin by calculating the Wigner distribution of the Kink soliton and further calculating the charge distribution and current distribution for the same. The contour, Wigner, and the charge distribution are plotted.

In section (3) we have calculated the Wigner distribution of the Sine-Gordan soliton, then we calculate the charge and current distribution for the same. Similar to the previous section we plot the contour, Wigner and the charge distribution.

# 2 Wigner distributions for the kink soliton

One of the quite common methods to represent a quantum mechanical system in phase space is by Wigner representation \([1]\). The general notion of the Wigner distribution \(W(x,p)\) for a given state \(\psi\) can be given by

\[
W(x,p) = \frac{1}{\hbar} \int_{-a}^{b} dy \psi(x+y)\psi^*(x-y)e^{ipy/\hbar}
\]  \(\text{(1)}\)

Here \(a\) and \(b\) represent the bounds within which the particle exists. We have calculated the Wigner distribution of the kink soliton \([27]\). The kink is the simplest topological soliton \([27]\) that arises in the theory of real scalar field in 1 + 1 dimensional space-time. The Lagrangian density for the simple kink soliton is written by

\[
L = \frac{1}{2} \partial_{\mu} \psi \partial_{\mu} \psi + V(\psi)
\]

where \(V(\psi)\) represents the potential energy which can be given by

\[
V(\psi) = -\frac{\mu^2}{2} \psi^2 + \frac{\lambda}{4} \psi^4 + \frac{\mu^2}{4\lambda}
\]

or, which amounts to the same thing

\[
V(\psi) = \frac{\lambda}{4} (\psi^2 - a^2)^2
\]

where, \(a = \frac{\mu}{\lambda}\), the value of \(\lambda\) depends on the particular system. The minimum of the potential occurs when \(\frac{dV}{d\psi} = 0\). The classical vacua occur at the minima of the potential. Therefore, they are at \(\pm a\). \(\psi\) is continuous and a transition region is observed between the two vacua. This is called the Domain wall \([27]\) and the symmetry is spontaneously broken upon the transformation, \(\psi \rightarrow -\psi\) in this domain wall. Thus the kink soliton can be considered as a static solution of the field equations which is interpolating between two vacuum solutions. The wave function \([27]\) of the simplest kink soliton can be given by

\[
\psi(x) = \text{atanh}\left(\frac{\sqrt{\lambda}a}{2}x\right)
\]  \(\text{(2)}\)

Since the soliton exists between its vacuum solutions we define the bound of the solitons between \(-a\) and \(+a\). Substituting the value of the wave function i.e., eq.\((2)\) in eq.\((1)\) we get

\[
W(x,p) = \frac{1}{\hbar} \int_{-a}^{a} dy \left[ \text{atanh}\left(\frac{\sqrt{\lambda}a}{2}x+y\right)\text{atanh}\left(\frac{\sqrt{\lambda}a}{2}x-y\right)e^{ipy/\hbar}\right]
\]  \(\text{(3)}\)

By using the series expansion \(^1\) of \(\text{tanh}(\frac{\sqrt{\lambda}a}{2}x+y)\), \(\text{tanh}(\frac{\sqrt{\lambda}a}{2}x-y)\), \(e^{ipy/\hbar}\) and simplifying we get

\[
W(x,p) = \frac{a^2}{\hbar} \int_{-a}^{a} dy \left[ \text{tanh}^2\left(\frac{\sqrt{\lambda}a}{2}x\right) + \frac{2ipy}{\hbar}\text{tanh}^2\left(\frac{\sqrt{\lambda}a}{2}x\right) - \frac{4p^2}{2!}\frac{1}{\hbar^2}\text{tanh}^2\left(\frac{\sqrt{\lambda}a}{2}x\right)y^2\right]
\]  \(\text{(4)}\)

\(^1\)Series expansion is given in Appendix A
By integrating eq. (4) we get

\[
W(x, p) = \frac{a^2}{\hbar} \left[ \text{tanh}^2 \left( \sqrt{\frac{\lambda}{2}ax} \right) + \frac{2ipy^2}{2\hbar} \text{tanh}^2 \left( \sqrt{\frac{\lambda}{2}ax} \right) - \frac{4p^2}{2!} \frac{1}{\hbar} \text{tanh}^2 \left( \sqrt{\frac{\lambda}{2}ax} \right) \frac{y^3}{3} \right]_{-a}^a
\]

Therefore, upon substituting the boundary values we obtain the Wigner distribution of the kink soliton as

\[
W(x, p) = \frac{2a^3}{\hbar} \left[ \text{tanh}^2 \left( \sqrt{\frac{\lambda}{2}ax} \right) - \frac{2p^2a^2}{9\hbar} \text{tanh}^2 \left( \sqrt{\frac{\lambda}{2}ax} \right) \right]
\]

Figure 1: Wigner distribution for Kink soliton; For \(-a < x < a\) and \(a = 10^{-10} m\)

2.1 Calculation of charge and current density from Wigner distribution of Kink soliton

The charge distribution \((Q_W(x)) [31]\) is obtained by

\[
Q_W(x) = \frac{1}{\hbar} \int dp \int dy \left[ \psi(x + y)\psi^*(x - y)e^{ipx} \right]
\]

From eq.(1) we can write eq.(6) as

\[
Q_W(x) = \int dpW(x, p)
\]

Let us assume the arbitrary bounds for momentum, say \(p_1\) and \(p_2\). Thus we get

\[
Q_W(x) = \int_{p_1}^{p_2} dpW(x, p)
\]

By substituting the value of Wigner distribution \([1]\) from eq.(5) we get

\[
Q_W(x) = \frac{2a^3}{\hbar} \int_{p_1}^{p_2} dp \left[ \text{tanh}^2 \left( \sqrt{\frac{\lambda}{2}ax} \right) - \frac{2p^2a^2}{9\hbar} \text{tanh}^2 \left( \sqrt{\frac{\lambda}{2}ax} \right) \right]
\]
Integration leads us to

\[ Q_W(x) = \frac{2a^3}{\hbar} \left[ \tanh^2 \left( \sqrt{\frac{\lambda}{2}} ax \right) p - \frac{2p^3 a^2}{27\hbar} \tanh^2 \left( \sqrt{\frac{\lambda}{2}} ax \right) \right] \]

Therefore the charge distribution is given by

\[ Q_W(x) = \frac{2a^3}{\hbar} \left[ \tanh^2 \left( \sqrt{\frac{\lambda}{2}} ax \right) (p_2 - p_1) - \frac{2a^2}{27\hbar} \tanh^2 \left( \sqrt{\frac{\lambda}{2}} ax \right) (p_3^2 - p_1^2) \right] \]  

(9)

If we take the value of \( p_1, p_2 \) to be \(-p_0, p_0\) respectively. Then eq. (9) reduces to

\[ Q_W(x) = \frac{2a^3}{\hbar} \left[ 2p_0 \tanh^2 \left( \sqrt{\frac{\lambda}{2}} ax \right) - \frac{4a^3 p_0}{27\hbar} \tanh^2 \left( \sqrt{\frac{\lambda}{2}} ax \right) \right] \]

(10)

By neglecting the higher order powers of \( p_0^3 \) we obtain

\[ Q_W(x) = \frac{4a^3 p_0}{\hbar} \tanh^2 \left( \frac{\lambda}{2} ax \right) \approx |\psi(x)|^2 \]  

(11)

The computation of current density [31] from Wigner distribution is given by

\[ J_W(x) = \int dp.p.W(x, p) \]

From eq. (1) we get

\[ J_W(x) = \frac{1}{\hbar} \int dp.p \int dy \left[ \psi(x + y)\psi^*(x - y)e^{\frac{i\mathcal{R}}{\hbar}} \right] \]  

(12)
Let us assume the arbitrary bounds for momentum, say $p_1$ and $p_2$. So,

$$J_W(x) = \frac{2a^3}{\hbar} \int_{p_1}^{p_2} dp \cdot p \left[ \tanh^2 \left( \sqrt{\frac{\lambda}{2}} ax \right) - \frac{2p^2}{9\hbar} \tanh^2 \left( \sqrt{\frac{\lambda}{2}} ax \right) \right]$$

Integrating and substituting the boundary values gives

$$J_W(x) = \frac{2a^3}{\hbar} \left[ \frac{1}{2} \tanh^2 \left( \sqrt{\frac{\lambda}{2}} ax \right) (p_2^2 - p_1^2) - \frac{2a^2}{36\hbar} \tanh^2 \left( \sqrt{\frac{\lambda}{2}} ax \right) (p_2^4 - p_1^4) \right]$$  \hspace{1cm} (13)

If we take the value of $p_1, p_2$ to be $-p_0, p_0$ respectively, then eq.(13) reduces to

$$J_W(x) = 0$$

3  \hspace{1cm} **Wigner distribution for the Sine-Gordan soliton**

Let us consider an another simple classical soliton [27] which is a non-linear hyperbolic differential equation that is the Euler-Lagrange equation of the following Lagrangian density,

$$\mathcal{L}_{SG} = \frac{1}{2} \partial^\mu \psi \partial_\mu \psi + V(\psi)$$

where $V(\psi)$ represents the potential energy which can be given by

$$V(\psi) = \alpha (\cos \beta \phi - 1)$$

where $\alpha, \beta$ depend on the system which we consider. The function of the Sine-Gordan soliton can be given by

$$\psi_{SG} = \frac{4}{\beta} \tan^{-1} \left[ e^{\sqrt{\alpha} \beta x} \right]$$  \hspace{1cm} (14)

where, the value of $\alpha$ and $\beta$ depend on the system which we take in to consideration. Using eq.(1) we write the corresponding Wigner distribution for the Sine-Gordan soliton as

$$W(x, p) = \frac{1}{\hbar} \int_a^b dy \left[ \frac{4}{\beta} \left( \tan^{-1} \left( e^{\sqrt{\alpha} \beta (x+y)} \right) \right) \frac{4}{\beta} \left( \tan^{-1} \left( e^{\sqrt{\alpha} \beta (x-y)} \right) \right) e^{\frac{ipy}{\hbar}} \right]$$  \hspace{1cm} (15)
Here $a$ and $b$ can be fixed by the bounds in which the solitons exist. For convention, let us take their values to be finite values with the length dimension. We do the series expansion\(^2\) of $\tan^{-1}(e^{\sqrt{\alpha \beta}(x+y)}), \tan^{-1}(e^{\sqrt{\alpha \beta}(x-y)}), e^{iPy}$ and simplifying we get

$$W(x, p) = \frac{1}{\hbar} \int_a^b dy \left[ \left( \tan^{-1}(e^{\sqrt{\alpha \beta}x}) \right)^2 - \frac{ip}{\hbar} \left( \tan^{-1}(e^{\sqrt{\alpha \beta}x}) \right)^2 y - \frac{\alpha \beta^2 e^{2\sqrt{\alpha \beta}x}y^2}{(e^{2\sqrt{\alpha \beta}x} + 1)^2} - \frac{ip \alpha \beta^2 e^{2\sqrt{\alpha \beta}x}y^3}{\hbar^2 e^{2\sqrt{\alpha \beta}x (x+y)}} \right]$$ (16)

Integrating eq. (16) we get

$$W(x, p) = \frac{1}{\hbar} \left[ \tan^{-1}(e^{\sqrt{\alpha \beta}x})^2 y + \frac{ip}{2\hbar} \tan^{-1}(e^{\sqrt{\alpha \beta}x})^2 y^2 - \frac{p^2}{3\hbar^2} \left( \tan^{-1}(e^{\sqrt{\alpha \beta}x})^2 y^3 - \frac{\alpha \beta^2 e^{2\sqrt{\alpha \beta}x}y^3}{3(e^{2\sqrt{\alpha \beta}x} + 1)^2} \right) + O(y^4) \right]$$

Substituting the limits and neglecting the higher orders we get

$$W(x, p) = \frac{1}{\hbar} \left[ \left( \tan^{-1}(e^{\sqrt{\alpha \beta}x})^2 (b-a) + \frac{ip}{2\hbar} \tan^{-1}(e^{\sqrt{\alpha \beta}x})^2 (b^2-a^2) - \frac{p^2}{3\hbar^2} \left( \tan^{-1}(e^{\sqrt{\alpha \beta}x})^2 + \frac{\alpha \beta^2 e^{2\sqrt{\alpha \beta}x}y^3}{3(e^{2\sqrt{\alpha \beta}x} + 1)^2} \right) (b^3-a^3) \right] + O(y^4) \right]$$

Let us consider $a = -b$ we obtain the Wigner distribution of the Sine-Gordon soliton as

$$W(x, p) = \frac{2b}{\hbar} \left[ \left( \tan^{-1}(e^{\sqrt{\alpha \beta}x})^2 - b^2 + \frac{p^2}{3\hbar^2} \left( \tan^{-1}(e^{\sqrt{\alpha \beta}x})^2 + \frac{\alpha \beta^2 e^{2\sqrt{\alpha \beta}x}y^3}{3(e^{2\sqrt{\alpha \beta}x} + 1)^2} \right) + O(b^3) \right]$$ (17)

![Figure 4: Wigner distribution for Sine-Gordan soliton; For $-b < x < b$ and $b = 10^{-10}m$](image)

### 3.1 Calculation of charge and current density from Wigner distribution of Sine Gordan soliton

The charge distribution ($Q_W(x)$) \([31]\) is obtained by

$$Q_W(x) = \frac{1}{\hbar} \int dp \int dy \left[ \psi(x+y)\psi^*(x-y)e^{iPx} \right]$$ (18)

\(^2\)Series expansion is given in Appendix A
From eq. (1), we can write eq. (6) as

$$Q_W(x) = \int dp W(x, p)$$  \hspace{1cm} (19)

Let us assume the arbitrary bounds for momentum, say $p_1$ and $p_2$. Thus we get

$$Q_W(x) = \int_{p_1}^{p_2} dp W(x, p)$$

By substituting the value of Wigner distribution from eq.(17) we get

$$Q_W(x) = \frac{2b}{\hbar} \int_{p_1}^{p_2} dp \left[ \tan^{-1}(e^{\sqrt{\alpha} \beta x})^2 - b^2 \left( \frac{p^2}{3\hbar^2} \tan^{-1}(e^{\sqrt{\alpha} \beta x})^2 + \frac{\alpha \beta^2 e^{2\sqrt{\alpha} \beta x}}{3(e^{2\sqrt{\alpha} \beta x} + 1)} \right) \right] + \mathcal{O}(b^3)$$  \hspace{1cm} (20)

By integrating, substituting the limits and neglecting the higher orders we get

$$Q_W(x) = \frac{2b}{\hbar} \left[ (p_2 - p_1) \left( \tan^{-1}(e^{\sqrt{\alpha} \beta x})^2 - \frac{\alpha \beta^2 b^2 e^{2\sqrt{\alpha} \beta x}}{3(e^{2\sqrt{\alpha} \beta x} + 1)^2} \right) - \frac{b^2 (p_2^3 - p_1^3)}{9\hbar^2} \left( \tan^{-1}(e^{\sqrt{\alpha} \beta x}) \right)^2 \right]$$

If we take the value of $p_1, p_2$ to be $-p_0, p_0$ respectively. Then we get

$$Q_W(x) = \frac{4b}{\hbar} \left[ p_0 \left( \tan^{-1}(e^{\sqrt{\alpha} \beta x})^2 - \frac{\alpha \beta^2 b^2 e^{2\sqrt{\alpha} \beta x}}{3(e^{2\sqrt{\alpha} \beta x} + 1)^2} \right) - \frac{b^2 p_0^3}{9\hbar^2} \left( \tan^{-1}(e^{\sqrt{\alpha} \beta x}) \right)^2 \right] \approx |\psi(x)|^2$$  \hspace{1cm} (21)
Figure 6: Charge distribution for Sine-Gordan soliton

The computation of current density [31] from Wigner distribution is given by

\[ J_w(x) = \int dp.p.W(x,p) \]  

(22)

With the arbitrary bounds for momentum, say \( p_1 \) and \( p_2 \), we get

\[ J_w(x) = \frac{2b}{h}\left[\left(\frac{p_2^2}{2}\{\text{tan}^{-1}(\frac{e^{2\sqrt{\alpha\beta}x}}{\sqrt{\alpha\beta}x})\}\right) - b^2\left(\frac{p_4}{12h^2}\{\text{tan}^{-1}(\frac{e^{\sqrt{\alpha\beta}x}}{\sqrt{\alpha\beta}x})\}\right) + \frac{\alpha\beta^2}{6}[e^{2\sqrt{\alpha\beta}x} + 1]^{-2}\right]_{p_1}^{p_2} \]  

Substituting limits we get

\[ J_w(x) = \frac{2b}{h}\left[\left(\frac{p_2^2 - p_1^2}{2}\{\text{tan}^{-1}(\frac{e^{2\sqrt{\alpha\beta}x}}{\sqrt{\alpha\beta}x})\}\right) - b^2\left(\frac{p_4^2 - p_1^4}{12h^2}\{\text{tan}^{-1}(\frac{e^{\sqrt{\alpha\beta}x}}{\sqrt{\alpha\beta}x})\}\right) + \frac{\alpha\beta^2}{6}[e^{2\sqrt{\alpha\beta}x} + 1]^{-2}\right]_{p_1}^{p_2} \]  

(23)

Assuming the bounds of ‘p’ to be symmetric, (i.e.) \( p_1 = -p_0 \& p_2 = p_0 \) we get the current density

\[ J_w(x) = 0 \]

Since it is an even function.

4 Concluding Remarks

We plotted the Wigner distribution for Kink and Sine-Gordan solitons in Fig. (1) and Fig. (4) respectively. Fig. (2) and Fig. (5) are the contour plots of the Wigner distribution of Kink and Sine-gordan solitons. Fig. (3) and Fig. (6) show the variation of charge distribution of Kink and Sine-Gordan solitons. In the figures (1), (2), (3), the value of \( a \) represents the bound in which the Kink soliton exists. In the figures (4), (5), (6), the value of \( a \) and \( b \) represent the bound in which the Sine-Gordan soliton exists. From the contour plot of the kink soliton (Fig. (2)) we infer, for a given point P from \( -x \) to \( x \), the Wigner distribution remains the same. We also infer from the plot that as we move away from the origin the Wigner distribution shows the variation, specifically along the diagonals. In the case of Sine-Gordan soliton (Fig. (5)), the Wigner distribution obtained is the real part of the Wigner distribution, as the complex part of the distribution has vanished, which can be observed from the calculations performed. The Wigner distributions of both the Kink soliton and the Sine-Gordan solitons are calculated in their respective pure states. These can be used to study the classical, semi-classical, and quantum speed limit time [30] which forms the foundations of quantum computing. The
quantum speed limit time gives us the value of the rate at which two quantum states are evolved [30]. It can be calculated by calculating the value of Quantum Fidelity (QF) [32] - which measures the closeness of two states. Upon calculating the QF, we get the value of the quantum speed limit time as

$$\tau_{QSL} = \frac{1 - F(t)}{\Delta E}$$

where, $F(t)$ represents the Quantum fidelity [32], which is given by

$$F(t) = 2\pi\hbar \int dqdpW_0W_t$$

where $W_0,W_t$ represent the Wigner distribution in the initial state and in the time dependent state respectively. The quantum speed limit time [30] which we have written above in eq. (24) represents the Mandelstam-Tamm speed limit time [33]. In the current work we have calculated the Wigner distributions of the kink and Sine-Gordan solitons. We will extend this work to calculate the same for the tunneling instantons [27] and also try to find the rate of evolution of two quantum states in case of instantons.

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6 Appendix A - Series expansion

The series expansion of $tanh(\sqrt{2a(x+y)})$, $tanh(\sqrt{2a(x-y)})$, $e^{ipy}$ is as follows:

$$tanh\left(\sqrt{\frac{a}{2}}(x+y)\right) = tanh\left(\sqrt{\frac{a}{2}}x\right) + ysech^2\left(\sqrt{\frac{a}{2}}x\right) - y^2tanh\left(\sqrt{\frac{a}{2}}x\right)sech^2\left(\sqrt{\frac{a}{2}}x\right) + O(y^3)$$

$$tanh\left(\sqrt{\frac{a}{2}}(x-y)\right) = tanh\left(\sqrt{\frac{a}{2}}x\right) - ysech^2\left(\sqrt{\frac{a}{2}}x\right) - y^2tanh\left(\sqrt{\frac{a}{2}}x\right)sech^2\left(\sqrt{\frac{a}{2}}x\right) - O(y^3)$$

$$e^{ipy} = 1 + \left(\frac{ip}{\hbar}\right)y - \frac{y^2}{2!} + ..$$

The series expansion of $\tan^{-1}(e^{\sqrt{a}}(x+y))$, $\tan^{-1}(e^{\sqrt{a}}(x-y))$, $e^{ipy}$ is as follows:

$$\tan^{-1}\left(e^{\sqrt{a}}(x+y)\right) = \tan^{-1}\left(e^{\sqrt{a}}x\right) + \frac{\sqrt{a}ye^{\sqrt{a}x}}{e^{2\sqrt{a}x} + 1} - \frac{y^2(\alpha^2e^{\sqrt{a}x}(e^{2\sqrt{a}x} - 1))}{2!(e^{2\sqrt{a}x} + 1)^2} + O(y^3)$$

$$\tan^{-1}\left(e^{\sqrt{a}}(x-y)\right) = \tan^{-1}\left(e^{\sqrt{a}}x\right) - \frac{\sqrt{a}ye^{\sqrt{a}x}}{e^{2\sqrt{a}x} + 1} - \frac{y^2(\alpha^2e^{\sqrt{a}x}(e^{2\sqrt{a}x} - 1))}{2!(e^{2\sqrt{a}x} + 1)^2} + O(y^3)$$

$$e^{ipy} = 1 + \left(\frac{ip}{\hbar}\right)y - \frac{p^2y^2}{h^22!} - \frac{ip^3x^3}{3!h^3} + ..$$
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