Musical chairs

Yehuda Afek *  Yakov Babichenko †  Uriel Feige ‡  Eli Gafni §  Nati Linial ¶
Benny Sudakov ∥

Abstract

In the *Musical Chairs* game $MC(n, m)$ a team of $n$ players plays against an adversarial scheduler. The scheduler wins if the game proceeds indefinitely, while termination after a finite number of rounds is declared a win of the team. At each round of the game each player occupies one of the $m$ available chairs. Termination (and a win of the team) is declared as soon as each player occupies a unique chair. Two players that simultaneously occupy the same chair are said to be in conflict. In other words, termination (and a win for the team) is reached as soon as there are no conflicts.

The only means of communication throughout the game is this: At every round of the game, the scheduler selects an arbitrary nonempty set of players who are currently in conflict, and notifies each of them separately that it must move. A player who is thus notified changes its chair according to its deterministic program. As we show, for $m \geq 2n - 1$ chairs the team has a winning strategy. Moreover, using topological arguments we show that this bound is tight. For $m \leq 2n - 2$ the scheduler has a strategy that is guaranteed to make the game continue indefinitely and thus win.

We also have some results on additional interesting questions. For example, if $m \geq 2n - 1$ (so that the team can win), how quickly can they achieve victory?

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*The Blavatnik School of Computer Science, Tel-Aviv University, Israel 69978. afek@tau.ac.il
†Department of Mathematics, Hebrew University, Jerusalem 91904, Israel yak@math.huji.ac.il
‡Department of Computer Science and Applied Mathematics Weizmann Institute of Science Rehovot 76100, Israel. uriel.feige@weizmann.ac.il. The author holds the Lawrence G. Horowitz Professorial Chair at the Weizmann Institute. Work supported in part by The Israel Science Foundation (grant No. 873/08).
§Computer Science Department, University of California, CA 95024, eli@cs.ucla.edu
¶School of Computer Science and Engineering, Hebrew University, Jerusalem 91904, Israel nati@cs.huji.ac.il
∥Department of Mathematics, UCLA, Los Angeles, CA, 90095. bsudakov@math.ucla.edu. Research supported in part by NSF grant DMS-1101185 and by a USA-Israeli BSF grant.
1 Introduction

Communication is a crucial ingredient in every kind of collaborative work. But what is the least possible amount of communication required for a team of players to achieve certain goals? Motivated by this question, we consider in this paper the following game.

The Musical Chairs game $MC(n, m)$ involves $m$ chairs numbered $1, \ldots, m$ and a team of $n$ players $P_1, \ldots, P_n$, who are playing against an adversarial scheduler. The scheduler’s goal is to make the game run indefinitely in which case he wins. The termination condition is that each player settles in a different chair. Upon termination the team of players is declared the winner. We say that player $P$ is in conflict if some other player $Q$ is presently occupying the same chair as $P$. Namely, termination and a win of the team is reached if there are no conflicts. The scheduler gets to decide, for each player $P_i$ the chair that $P_i$ occupies at the start of the game. As mentioned above we severely restrict the communication between the players during the game. All such communication is mediated by the scheduler as follows: At every time step, and as long as there are conflicts, the scheduler selects an arbitrary nonempty set of players which are currently in conflict and notifies them that they need to move. A player thus notified to be in conflict changes its chair according to its deterministic program, which he chooses before the game. During the game, each player has no information about the chairs of other players beyond the occasional one bit that tells him that he must move, and we insist that the choice of a player’s next chair be deterministic. Consequently, a player’s action depends only on its current chair, and the sequence of chairs that he had traversed so far. Therefore the sequence of chairs that player $P_i$ traverses is simply an infinite word $\pi_i$ over the alphabet $1, \ldots, m$. Recall that the adversary can start each player on any of the chairs. Consequently we assume that each $\pi_i$ is full, i.e., it contains all the letters in $[m]$. So upon receiving a conflict notification from the scheduler, player $P_i$ occupying chair $\pi_i[k]$ moves to chair $\pi_i[k + 1]$. The scheduler’s freedom in choosing the players’ initial chairs means that for every $i$ he selects an index $k_i$ and the game starts with each $P_i$ occupying chair $\pi_i[k_i]$. A winning strategy for the players is a choice of full words $\pi_i$ with the following property: For every choice of initial positions $k_i$ and for every strategy of the scheduler the game terminates in a finite number of rounds, i.e., the players cannot be beaten by the scheduler.

In this paper we obtain several results about the musical chairs game. Our first theorem determines the minimal $m$ for which the team of players wins the $MC(n, m)$ game.

**Theorem 1** The team of players has a winning strategy in the $MC(n, m)$ game if and only if $m \geq 2n - 1$.

In the winning strategy that we produce, each word $\pi_i$ is periodic, or, what is the same, a finite word that $P_i$ traverses cyclically. We also show that for every $N > n$ there exist $N$ full cyclic words on the alphabet $[m] = [2n - 1]$ such that every set of $n$ out of these $N$ words constitutes a winning strategy for the $MC(n, 2n - 1)$ game.

To prove the lower bound in Theorem 1 we use Sperner’s lemma (see, e.g., [4]) a fundamental tool from combinatorial topology. The use of this tool in proving lower bounds for distributed algorithms was pioneered in [6, 8, 9]. If one is willing to assume a great deal of knowledge in the field of distributed computing, it is possible to deduce our lower bound from known results in this area. Instead, to make
the paper self contained, we chose to include here a direct proof which we think is more illuminating
and somewhat simpler than the one which would result from reductions to the existing literature.

Although the words in Theorem 1 use the least number of chairs, namely \( m = 2n - 1 \), their lengths
are doubly exponential in \( n \). This leads to several interesting questions. Are there winning strategies
for the MC game with much shorter words, say of length \( O(n) \)? Perhaps even of length \( m \)? Can
we provide significantly better upper bounds on the number of rounds till termination? Even if the
scheduling is bound to lose the game, how long can he make the game last? Our next two results give
some answers to these questions. Here we consider an \( MC(n, m) \) winning systems with \( N \) words. This
is a collection of \( N \geq n \) full words on \([m]\), every \( n \) of which constitute a winning strategy for the
players in the \( MC(n, m) \) game.

**Theorem 2** For every \( N \geq n \), almost every choice of \( N \) words of length \( cn \log N \) in an alphabet of
\( m = 7n \) letters is an \( MC(n, m) \) winning system with \( N \) full words. Moreover, every game on these
words terminates in \( O(n \log N) \) steps. Here \( c \) is an absolute constant.

Since we are dealing with full words which we seek to make short, we are ultimately led to consider
the problem under the assumption that each (finite, cyclically traversed) word \( \pi_i \) is a permutation on
\([m] \). We note that the context of distributed computing offers no particular reason for this restric-
tion and that we are motivated to study this question due to its aesthetic appeal. We can design
permutation-based winning strategies for \( MC(n, 2n - 1) \) game for very small \( n \) (provably for \( n = 3 \),
a computer assisted construction and proof for \( n = 4 \)). We suspect that no such constructions are
possible for large values of \( n \), but we are unable at present to show this. We do know, though that

**Theorem 3** For every integer \( d \geq 1 \) there is an \( MC(n, m) \) winning system with \( N = n^d \) permutations
on \( m = cn \) symbols, where \( c \) depends only on \( d \). In fact, this holds for almost every choice of \( N \)
permutations on \([m]\).

We should stress that our proofs of Theorems 2 and 3 are purely existential. The explicit construc-
tion of such systems of words remains largely open, though we have the following result in this
direction.

**Theorem 4** For every integer \( d \geq 1 \) there is an \( MC(n, m) \) winning system with \( N = n^d \) permutations
on \( m = O(d^2n^2) \) symbols.

We conclude this introduction with a discussion of several additional aspects of the subject.

Our work was originally motivated by some questions in distributed computing. In every distri-
buted algorithm each processor must occasionally observe the activities of other processors. This
can be done by reading the messages that other processors send, by inspecting some publicly accessible
memory cells into which they write, or by sensing an effect on the environment due to the actions of
other processors. Hence it is very natural to ask: What is the least possible amount of communication
required to achieve certain goals? To answer it, we consider two severe limitations on the processors’
behavior and ask how this affects the system’s computational power. First, a processor can only post
a proposal for its own output, and second, each processor is “blindfolded” and is only occasionally
provided with the least possible amount of information, namely a single bit that indicates whether its current state is “good” or “bad”. Here “bad/good” indicates whether or not this state conflicts with the global-state desired by all the processors. Moreover, we also impose the requirement that algorithms are deterministic, i.e., use no randomization. This new minimalist model, which we call the oblivious model, was introduced in the conference version of this paper [3]. This model might appear to be significantly weaker than other (deterministic) models studied in distributed computing. Yet, our results show that a very natural distributed problem musical chairs [7], can be solved optimally within the highly limited oblivious model. Further discussion of the oblivious model and additional well-known problems like renaming [1, 2] which we can also solve optimally in this model can be found in [3].

A winning strategy for the \( MC(n, m) \) game cannot include any two identical words. For that allows the scheduler to move the corresponding players together in lock-step, keeping them constantly in a state of conflict. Also for every winning strategy for \( MC(n, m) \), with finite cyclic words, there is a finite upper bound on the number of moves till termination. To see this, let us associate with every state of the system a vector whose \( i \)-th coordinate is the current position of player \( P_i \) on \( \pi_i \). The set of such vectors \( V \) is finite, \( |V| = \prod |\pi_i| \), and in a terminating sequence of moves no vector can be visited twice. In fact, we can associate with every collection of finite words a directed graph on vertex set \( V \), where edges correspond to the possible transitions in response to scheduler’s notifications. The collection of words constitute a winning MC strategy iff this directed graph is acyclic. We note that these observations depend on the assumption that players use no randomness.

Our strategies for the MC game have a number of additional desirable properties. As mentioned, we construct \( N \) full periodic words such that every subset of \( n \) of the \( N \) words constitutes an \( MC(n, m) \) winning system. Hence our strategies are guaranteed to succeed (reach termination against every scheduler’s strategy) in dynamic settings in which the set of players in the system keeps changing. This statement holds provided there are sufficiently long intervals throughout which the set of players remains unchanged. To illustrate this idea, consider a company that manufactures \( N \) communication devices, each of which can use any one of \( m \) frequencies. If several such devices happen to be at the same vicinity, and simultaneously transmit at the same frequency, then interference occurs. Devices can move in or out of the area, hop to a frequency of choice and transmit at this frequency, sense whether there are other transmissions in this frequency. The company wants to provide the following guarantee: If no more than \( n \) devices reside in the same geographical area, then no device will suffer more than a total of \( T \) interference events for some guaranteed bound \( T \). Our strategy for the MC game would yield this by pre-installing in each device a list of frequencies (a word in our terminology), and having the device hop to the next frequency on its list (in a cyclic fashion) in response to any interference it encounters. No communication beyond the ability to sense interference is needed.

In proving the lower bound \( m \geq 2n-1 \) we have to make several assumptions about the setup. The first is the freedom of choice for the scheduler. From the perspective of distributed computing this means that we are dealing with an asynchronous system. In a synchronous setting, in every time step, every player involved in a conflict moves to its next state. One can show that in such a synchronous setup the players have a winning strategy even with \( m = n \) chairs. It is also important that the scheduler can dictate each player’s starting position. If each \( P_i \) starts at the first letter of \( \pi_i \), a trivial winning strategy with \( m = n \) simply sets \( i \) as the first letter of \( \pi_i \) for each \( i \). It is also crucial that our players are deterministic (no randomization). If players are allowed to pick their next state randomly,

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then again \( m = n \) suffices, since in this case with probability 1 eventually a conflict-free configuration will be reached. Hence, this paper is also related to the one of the fascinating questions in computer science, whether and to what extent randomization increases the power of algorithmic procedures. Our results show that without using randomness one can still win an MC game by increasing only slightly the number of chairs (from \( n \) to \( 2n - 1 \)).

2 Simplified Oblivious Model for Musical Chairs

Our general model for oblivious algorithms is specified by providing the rules of possible behavior of the scheduler. Here we consider an immediate scheduler, who enjoys a high degree of freedom in choosing which processor to move. To simplify the design and analysis of oblivious algorithms, it is convenient to consider a more restricted scheduler that has fewer degrees of freedom, but is nevertheless equivalent to the immediate scheduler in their power to win the MC game. In each round an immediate scheduler can select an arbitrary nonempty set of players that are currently in conflict and move them. Below we often refer to a team strategy as an oblivious \( MC(n, m) \) algorithm. It is a winning strategy if the immediate scheduler is forced to reach a conflict-free configuration in finite time. Conversely, an immediate scheduler wins against an oblivious \( MC(n, m) \) algorithm if it can generate an infinite execution without ever reaching a conflict-free configuration.

Terminology. Two schedulers \( \sigma_1, \sigma_2 \) are considered equivalent if for every team strategy scheduler \( \sigma_1 \) has a winning strategy iff so does \( \sigma_2 \).

First we want to limit the number of processors that can be moved in a round.

A pairwise immediate scheduler is similar to the immediate scheduler, except for the following restriction. In every round, the pairwise immediate scheduler can select any two processors \( P \neq Q \) that are currently in conflict with each other, and move either \( P \), or \( Q \), or both. Equivalently, in every round either exactly one processor (that is involved in a conflict) moves, or two processors that share the same chair move.

Proposition 5 The immediate scheduler and the pairwise immediate scheduler are equivalent.

Proof. The pairwise immediate scheduler is more restricted than the immediate scheduler. Hence it remains to show that if the immediate scheduler can win against some team strategy, then the pairwise immediate scheduler can also force an infinite run against it. We prove this last statement by a double induction on the round \( t \) and the number \( k \) of processors that move in round \( t \).

Let us fix an oblivious \( MC(n, m) \) algorithm and an infinite run that is forced by the immediate scheduler. Let \( t \) be the first round in which the moves are not consistent with any pairwise immediate scheduler and let \( k \) be the number of processors that move in round \( t \). There are two cases to consider. In one case \( k \geq 3 \) and all the moving processors occupy the same chair in round \( t \). Break round \( t \) into two rounds, pushing future rounds by one. In the first of them (round \( t \)) move only one of the processors from \( S \) and in the second round (round \( t + 1 \)) move the rest (they can still move because there are at least two of them). This completes the inductive step with respect to \( t \). The other case is that the set \( S \) of processors that moved in round \( t \) collided on at least two different chairs. Pick one of these chairs, say chair \( c \), and let \( S(c) \) be the set of those processors in \( S \) that in round \( t \) occupy
chair $c$. Break round $t$ into two rounds, pushing future rounds by one. In the first of them (round $t$) move only those processors in $S(c)$, and in the second round (round $t + 1$) move those processors in $S - S(c)$. This completes the inductive step (as either $k$ decreased or $t$ increased). □

The use of the pairwise immediate scheduler (which as we showed is equivalent to our original scheduler) helps simplify the proofs of theorems 1 and 3. However, for the proof of Theorem 2 even the pairwise immediate scheduler should be further restricted. It is true that it has to pick only one pair of players to move (and then either move only one or both of them), but it is still free to pick a pair of its choice (among those pairs that are in conflict). We would like to eliminate this degree of freedom.

Canonical Scheduler. The canonical scheduler is similar to the pairwise immediate scheduler but with the following difference. In every round in which there is a conflict, one designates a canonical pair. This is a pair of players currently in conflict with each other, but they are not chosen by the scheduler, but rather dictated to the scheduler. Given the canonical pair $P, Q$, the only choice the scheduler has is whether to move $P$, or $Q$, or both. But how is the canonical pair chosen? In the current paper this does not really matter to us, as long as the choice is deterministic. For concreteness, we shall assume the following procedure. Fix an arbitrary order on the collection of all pairs of players. In a nonterminal configuration, the canonical pair is the first pair of players in the order that share a chair.

Proposition 6 The canonical scheduler and the pairwise immediate scheduler are equivalent.

Proof.

Consider an oblivious $MC(n, m)$ algorithm and an infinite run against the immediate scheduler. Let $t$ be the first round in which the run is inconsistent with a canonical scheduler. That is, the canonical pair at round $t$ consists of two players, say $P_1$ and $P_2$, that occupy the same chair, say chair $c_1$, whereas the immediate scheduler moved at least one player not from the canonical pair. We consider several cases.

Case 1. The immediate scheduler never moves $P_1$ in any round from $t$ onwards. In this case move $P_2$ in round $t$. Note that all moves (except for the move just performed, moving $P_2$ away from $c_1$) performed by the immediate scheduler from round $t$ onwards are still available to this scheduler (because chair $c_1$ remains occupied). Hence the total number of moves in the schedule did not change, whereas $t$ increases by one, completing the inductive step. The same argument can be applied with $P_1$ and $P_2$ exchanged.

Case 2. The immediate scheduler moves $P_2$ away from $c_1$ in a later round than it moves $P_1$. In this case move $P_1$ in round $t$. Again, all moves (except for the move just performed, moving $P_1$ away from $c_1$) performed by the immediate scheduler from round $t$ onwards are still available to this scheduler. The same argument can be applied with $P_1$ and $P_2$ exchanged.

Case 3. The immediate scheduler moves both $P_1$ and $P_2$ out of $c_1$ in the same round $t' \geq t$. There are two subcases to consider. In one, there is no player other than $P_1$ and $P_2$ on chair $c_1$ in any of the rounds $t, \ldots, t'$. In this subcase, move $P_1$ and $P_2$ in round $t$ (pushing future rounds by one). All moves performed by the immediate scheduler from round $t$ to $t'$ are still available to this scheduler.
The other subcase is that there is some round \( t \leq t' \leq t' \) in which some other player say \( P_3 \) is on chair \( c_1 \). Consider the largest such \( t' \). Move \( P_1 \) in round \( t \) (pushing future rounds by one) and \( P_2 \) in round \( t' + 1 \) (the round that previous to the pushing of rounds was round \( t' \)), together with whoever else is moved at that round. □

**Remark.** There are several interesting schedulers which are even more flexible than the immediate scheduler. As we showed in the conference version of this paper [3] all of them are, in fact, equivalent to the canonical scheduler.

### 3 An oblivious MC algorithm with \( 2n - 1 \) chairs

#### 3.1 Preliminaries

In this section we prove the upper bound that is stated in Theorem 1. We start with some preliminaries. The length of a word \( w \) is denoted by \( |w| \). The concatenation of words is denoted by \( \circ \). The \( r \)-th power of \( w \) is denoted by \( w^r = w \circ w \ldots \circ w \) (\( r \) times). Given a word \( \pi \) and a letter \( c \), we denote by \( c \otimes \pi \) the word in which the letters are alternately \( c \) and a letter from \( \pi \) in consecutive order. For example if \( \pi = 2343 \) and \( c = 1 \) then \( c \otimes \pi = 12131413 \). A collection of words \( \pi_1, \pi_2, \ldots, \pi_n \) is called *terminal* if no schedule can fully traverse even one of the \( \pi_i \). Note that we can construct a terminal collection from any MC algorithm just by raising each word to a high enough power.

We now introduce some of our basic machinery in this area. We first show how to extend terminal sets of words.

**Proposition 7** Let \( n, m, N \) be integers with \( 1 < n < m \). Let \( \Pi = \{\pi_1, \ldots, \pi_N\} \) be a collection of \( m \)-full words such that

\[
\text{every } n \text{ of these words form an oblivious } MC(n, m) \text{ algorithm. (1)}
\]

Then \( \Pi \) can be extended to a set of \( N + 1 \) \( m \)-full words that satisfy condition (1).

**Proof.** Suppose that for every choice of \( n \) words from \( \Pi \) and for every initial configuration no schedule lasts more than \( t \) steps. (By the pigeonhole principle \( t \leq L^n \), where \( L \) is the length of the longest word in \( \Pi \).) For a word \( \pi \), let \( \pi' \) be defined as follows: If \( |\pi| \geq t \), then \( \pi' = \pi \). Otherwise it consists of the first \( t \) letters in \( \pi^r \) where \( r > |\pi|/t \). The new word that we introduce is \( \pi_{N+1} = \pi_1' \circ \pi_2' \circ \ldots \circ \pi_n' \). It is a full word, since it contains the full word \( \pi_1 \) as a sub-word.

We need to show that every set \( \Pi' \) of \( n - 1 \) words from \( \Pi \) together with \( \pi_{N+1} \) constitute an oblivious \( MC(n, m) \) algorithm. Observe that in any infinite schedule involving these words, the word \( \pi_{N+1} \) must move infinitely often. Otherwise, if it remains on a letter \( c \) from some point on, replace the word \( \pi_{N+1} \) by an arbitrary word from \( \Pi - \Pi' \) and stay put on the letter \( c \) in this word. This contradicts our assumption concerning \( \Pi \). (Note that this word contains the letter \( c \) by our fullness assumption.) But \( \pi_{N+1} \) moves infinitely often, and it is a concatenation of \( n \) words whereas \( \Pi' \) contains only \( n - 1 \) words. Therefore eventually \( \pi_{N+1} \) must reach the beginning of a word \( \pi_\alpha \) for some \( \pi_\alpha \not\in \Pi' \). From this point onward, \( \pi_{N+1} \) cannot proceed for \( t \) additional steps, contrary to our assumption. □
Note that by repeated application of Proposition 7, we can construct an arbitrarily large collection of \( m \)-full words that satisfy condition (1).

We next deal with the following situation: Suppose that \( \pi_1, \pi_2, \ldots, \pi_m \) is a terminal collection, and we concatenate an arbitrary word \( \sigma \) to one of the words \( \pi_i \). We show that by raising all words to a high enough power we again have a terminal collection in our hands.

**Lemma 8** Let \( \pi_1, \pi_2, \ldots, \pi_p \) be a terminal collection of full words over some alphabet. Let \( \sigma \) be an arbitrary full word over the same alphabet. Then the collection

\[
(\pi_1)^k, (\pi_2)^k, \ldots, (\pi_{i-1})^k, (\pi_i \circ \sigma)^2, (\pi_{i+1})^k, \ldots, (\pi_p)^k
\]

is terminal as well, for every \( 1 \leq i \leq p \), and every \( k \geq |\pi_i| + |\sigma| \).

**Proof.** We split the run of any schedule on these words into *periods* through which we do not move along the word \( (\pi_i \circ \sigma)^2 \). We claim that throughout a single period we do not traverse a full copy of \( \pi_j \) in our progress along the word \( (\pi_j)^k \). The argument is the same as in the proof of Proposition 7. By pasting all these periods together, we conclude that during a time interval in which we advance \( \leq |\pi_i| + |\sigma| - 1 \) positions along the word \( (\pi_i \circ \sigma)^2 \) every other word \( (\pi_j)^k \) traverses at most \( |\pi_i| + |\sigma| - 1 \) copies of \( \pi_j \). In particular, there is a whole \( \pi_j \) in the \( j \)-th word in the collection that is never visited. If the schedule ends in this way, no word is fully traversed, and our claim holds.

So let us consider what happens when a schedule makes \( \geq |\pi_i| + |\sigma| \) steps along the word \( (\pi_i \circ \sigma)^2 \). We must reach at some moment the start of \( \pi_i \) in our traversal of the word \( (\pi_i \circ \sigma)^2 \). But our underlying assumption implies that from here on, no word can fully traverse the corresponding \( \pi_k \) (including \( \pi_i \)). Again, no word is fully traversed, as claimed. \( \Box \)

Lemma \( \S \) yields immediately:

**Corollary 9** Let \( \pi_1, \pi_2, \ldots, \pi_p \) be a terminal collection of full word over some alphabet, and let \( \pi_{p+1}, \pi_{p+2}, \ldots, \pi_n \) be arbitrary full words over the same alphabet. Then the collection

\[
(\pi_1 \circ \pi_2 \circ \ldots \circ \pi_{n})^2, (\pi_1)^k, (\pi_2)^k, \ldots, (\pi_{i-1})^k, (\pi_{i+1})^k, \ldots, (\pi_p)^k
\]

is terminal as well. This holds for every \( 1 \leq i \leq p \) and \( k \geq \sum_{i=1}^{n} |\pi_i| \).

This is a special case of Lemma \( \S \) where \( \sigma = \pi_{i+1} \circ \ldots \pi_n \circ \pi_1 \ldots \circ \pi_{i-1} \).

### 3.2 The MC\((n, 2n - 1)\) upper bound

The proof we present shows somewhat more than Theorem \( \| \) says. A useful observation is that the scheduler can “trade” a player \( P \) for a chair \( c \). Namely, he can keep \( P \) constantly on chair \( c \) and be able, in return to move any other player past \( c \)-chairs. In other words, this effectively means the elimination of chair \( c \) from all other words. This suggests the following definition: If \( \pi \) is a word over alphabet \( C \) and \( B \subseteq C \), we denote by \( \pi(B) \) the word obtained from \( \pi \) by deleting from it the letters from \( C \setminus B \).
Proposition 10 For every integer \( n \geq 1 \)

- There exist full words \( s_1, s_2, \ldots, s_n \) over the alphabet \( \{1, 2, \ldots, 2n - 1\} \) such that \( s_1(A), s_2(A), \ldots, s_p(A) \) is a terminal collection for every \( p \leq n \), and every subset \( A \subseteq \{1, 2, \ldots, 2n - 1\} \) of cardinality \( |A| = 2p - 1 \).

- There exist full words \( w_1, w_2, \ldots, w_n \) over alphabet \( \{1, 2, \ldots, 2n\} \), such that \( w_1(B), w_2(B), \ldots, w_p(B) \) is a terminal collection for every \( p \leq n \), and every subset \( B \subseteq \{1, 2, \ldots, 2n\} \) of cardinality \( |B| = 2p - 1 \).

The words \( s_1, s_2, \ldots, s_n \) in Proposition 10 constitute a terminal collection and are hence an oblivious \( MC(n, 2n - 1) \) algorithm that proves the upper bound part of Theorem 1. In the rest of this section we prove Proposition 10.

Proof.

As mentioned, the proof is by induction on \( n \). For \( n = 1 \) clearly \( s_1 = 11 \) and \( w_1 = 1122 \) satisfy the conditions.

In the induction step we use the existence of \( s_1, s_2, \ldots, s_n \) to construct \( w_1, w_2, \ldots, w_n \). Likewise the construction of \( s_1, s_2, \ldots, s_{n+1} \) builds on the existence of \( w_1, w_2, \ldots, w_n \).

The transition from \( w_1, w_2, \ldots, w_n \) to \( s_1, s_2, \ldots, s_{n+1} \):

To simplify notations we assume that the words \( w_1, w_2, \ldots, w_n \) in the alphabet \( \{2, 3, \ldots, 2n + 1\} \) (rather than \( \{1, 2, \ldots, 2n\} \)) satisfy the proposition. Let \( k := \sum |w_i| \) and define:

\[
\begin{align*}
    s_1 & : = 1 \otimes ((w_1 \circ w_2 \circ \ldots \circ w_n)^{2(2n+1)}) \\
    \forall i = 2, \ldots n + 1 \quad s_i & : = (w_{i-1})^{k(2n+1)} \circ 1
\end{align*}
\]

Fix a subset \( A \subseteq \{1, 2, \ldots, 2n + 1\} \) of cardinality \( |A| = 2p - 1 \) with \( p \leq n + 1 \), and let us show that \( s_1(A), s_2(A), \ldots, s_p(A) \) is a terminal collection. There are two cases to consider:

We first assume \( 1 \not\in A \). This clearly implies that \( p \leq n \) (or else \( A = \{1, 2, \ldots, 2n + 1\} \) and in particular \( 1 \in A \)). In this case the collection is:

\[
\begin{align*}
    s_1(A) & : = ((w_1(A) \circ w_2(A) \circ \ldots \circ w_n(A))^{2(2n+1)}) \\
    \forall i = 2, \ldots p \quad s_i(A) & : = (w_{i-1}(A))^{k(2n+1)}
\end{align*}
\]

By the induction hypothesis, the collection \( w_1(A), w_2(A), \ldots, w_{p-1}(A), w_p(A) \) is terminal. We apply Corollary 9 and conclude that

\[
(w_1(A) \circ w_2(A) \circ \ldots \circ w_n(A))^2, (w_1(A))^k, (w_2(A))^k, \ldots, (w_{p-1}(A))^k
\]
is terminal as well. But the $s_i$ are obtained by taking $(2n + 1)$-th powers of these words, so that $s_1(A), s_2(A), ..., s_p(A)$ is terminal as needed.

We now consider what happens when $1 \in A$.

We define $F_1 := (w_1(A) \circ w_2(A) \circ ... \circ w_n(A))^2$ and for for $j > 1$, let $F_j := (w_{j-1}(A))^k$. We refer to $F_i$ as the $i$-th block. In our construction each word has $2n + 1$ blocks, ignoring chair 1.

At any moment throughout a schedule we denote by $O_1$ the set of players in $\{P_2, P_3, ..., P_p\}$ that currently occupy chair 1. We show that during a period in which the set $O_1$ remains unchanged, no player can traverse a whole block. The proof splits according to whether $O_1$ is empty or not.

Assume first that $O_1 \neq \emptyset$, and pick some $i > 1$ for which $P_i$ occupies chair 1 during the current period. As long as $O_1$ remains unchanged, $P_i$ stays on chair 1, so the words that the other players repeatedly traverse are as follows: For $P_1$ it is

$$w_1(A\{1\}) \circ w_2(A\{1\}) \circ ... \circ w_n(A\{1\})$$

and for $P_j$ with $p \geq j \neq i \geq 2$ it is

$$w_{j-1}(A\{1\})$$

We now show that no player can traverse a whole block (as defined above). Observe that the collection $\{w_\nu(A\{1\})|\nu = 1, ..., p - 1\}$ (including, in particular the word $w_{i-1}(A\{1\})$) is terminal. This follows from the induction hypothesis, because $|A\{1\}| = 2p - 2$, and because the property of being terminal is maintained under the insertion of new chairs into words. Applying Corollary 9 to this terminal collection implies that this collection of blocks is terminal as well.

We turn to consider the case $O_1 = \emptyset$. In this case player 1 cannot advance from a none-1 chair to the next none-1 chair, since the two are separated by the presently unoccupied chair 1. We henceforth assume that player $P_1$ stays put on chair $c \neq 1$, but our considerations remain valid even if at some moment player $P_1$ moves to chair 1. (If this happens, he will necessarily stay there, since $O_1 = \emptyset$). We are in a situation where players $P_2, P_3, ..., P_p$ traverse the words $w_1(A\{1,c\}), w_2(A\{1,c\}), ..., w_{p-1}(A\{1,c\})$ (chair $c$ which is occupied by player $P_1$ can be safely eliminated from these words). But $|A\{1,c\}| = 2p - 3$, so by the induction hypothesis no player can traverse a whole block $w_i(A\{1,c\})$, so no player can traverse a whole block.

We just saw that during a period in which the set $O_1$ remains unchanged, no player can traverse a whole block.

Finally, assume towards contradiction that $P_j$ fully traverses $s_j$ for some index $j$, and consider the first occurrence of such an event. It follows that $P_j$ has traversed $2n + 1$ blocks, so that the set $O_1$ must have changed at least $2n + 1$ times during the process. However, for $O_1$ to change, some $P_i$ must either move to, or away from a 1-chair in $s_j$. But 1 occurs exactly once in $s_j$, so every $P_i$ can account for at most two changes in $O_1$, a contradiction.

The transition from $s_1, s_2, ..., s_n$ to $w_1, w_2, ..., w_n$:

We assume that the words $s_1, s_2, ..., s_n$ in the alphabet $\{2, 3, ..., 2n\}$ satisfy the proposition. Let $k := \sum |s_i|$ and define:
\[
\begin{align*}
\forall i = 2, \ldots, n & \quad w_i := (s_{i-1})^{k(2n+1)} \circ 1 \\
\forall i = 2, \ldots, n & \quad w_i := 1 \otimes ((s_1 \circ s_2 \circ \ldots \circ s_n)^{2(2n+1)})
\end{align*}
\]

Fix a subset \( B \subseteq \{1, 2, \ldots, 2n\} \) with \(|B| = 2p - 1\). Then

\[
\begin{align*}
w_1(B) &= 1 \otimes ((s_1(B) \circ s_2(B) \circ \ldots \circ s_n(B))^{2(2n+1)}) \\
\forall i = 2, \ldots, p & \quad w_i(B) = (s_{i-1}(B))^{k(2n+1)} \circ 1
\end{align*}
\]

are exactly the same as in the previous transition just by replacing \( s \) with \( w \) and \( A \) with \( B \) (in this case the induction hypothesis is on \( s_i \) and we prove for \( w_i \)). So exactly the same considerations prove that \( w_1(B), w_2(B), \ldots, w_m(B) \) is a terminal collection. \( \square \)

4 Impossibility results

In this section we prove the lower bound of Theorem 1. As it turns out, the situation for \( 2n - 2 \geq m \) and for \( m \geq 2n - 1 \) are dramatically different. As we saw, for \( m \geq 2n - 1 \) the team has a winning strategy, but when \( 2n - 2 \geq m \) not only is it true that the scheduler can win the game. He is guaranteed to have a winning strategy even if we (i) substantially relax the requirement that each word \( \pi_i \) over \([m]\) be full, or (ii) restrict his power to select the players’ starting position on their words. In the next proposition case (i) occurs:

**Proposition 11** Every team strategy \( \tau_1, \ldots, \tau_n \) over \([m] = [2n - 2]\) for which:

- The chair 1 appears in both \( \tau_1, \tau_2 \), and
- For every \( 3 \leq i \leq n \), the word \( \tau_i \) contains both chair \( 2i - 4 \) and \( 2i - 3 \)

is a losing strategy.

Needless to say, this statement is invariant under permuting the player’s names and the indices of the chairs. There are several such arbitrary choices of indices below and we hope that this creates no confusion. In the impossibility results that we prove in this section, the number of chairs \( m \) is always \( 2n - 2 \). We also go beyond the lower bound of Theorem 1 by considering scenarios with a total of \( N \geq n \) players and statements showing that there is a choice of \( n \) out of the \( N \) words that constitute a losing strategy. (Clearly, new words that get added to a losing team strategy make it only easier for the scheduler to win). These deviations from the basic setup (\( N \geq n \) words, weakened fullness conditions, starting points not controlled by the scheduler) give us more flexibility in our arguments and complement each other nicely. Here is one of the main theorems that we prove in this section. It yields exponentially many subsets of \( n \) words that constitute a losing team strategy.
Theorem 12 Let $N = 2n - 2$ and let $\pi_1, \ldots, \pi_N$ be words over $[m] = [2n - 2]$ such that the only equality among the symbols $\pi_1[1], \pi_2[1], \pi_3[1], \ldots, \pi_N[1]$ is $\pi_1[1] = \pi_2[1]$. Then, for every partition of the words $\pi_3, \ldots, \pi_N$ into $n - 2$ pairs, there is a choice of one word from each pair, such that the chosen words together with $\pi_1$ and $\pi_2$ constitute a losing team strategy even when the game starts on each word’s first letter.

While it is obvious that Proposition 11 yields the lower bound of Theorem 1, it is not entirely clear how Theorem 12 fits into the picture. We show next how to derive Proposition 11 from Theorem 12.

Proof. (Theorem 12 implies Proposition 11) Let $\pi_1$ (resp. $\pi_2$) be the suffix of $\tau_1$ (resp. $\tau_2$) starting with the first appearance of the symbol 1. The other words come in pairs. For $3 \leq i \leq n$, we define $\pi_{2i-4}$ to be the suffix of $\tau_i$ starting at chair $2i - 4$, and $\pi_{2i-3}$ is its suffix starting at chair $2i - 3$. Theorem 12 implies that there is a choice of one word from each pair that together with $\pi_1$ and $\pi_2$ is losing when started from the initial chairs. The same scheduler strategy clearly wins the game on $\tau_1, \ldots, \tau_n$ when started from the respective chairs. 

The proof of Theorem 12, which uses some simple topological methods, is presented in Section 4.2. We provide all the necessary background material for this proof in Section 4.1.

What happens if the fullness condition is eliminated altogether but the scheduler maintains his right to select the starting positions? The scheduler clearly loses against the words $\pi_i = (i)$ for $i = 1, \ldots, m$. However, as the following theorem shows once $N > m = 2n - 2$, the scheduler has a winning strategy.

Theorem 13 For every collection of $N = 2n - 1$ words over $[m] = [2n - 2]$, there is a choice of $n$ words and starting locations for which the scheduler wins.

Proof. By pigeonhole, the scheduler wins against every set of words $S$ that together contain fewer than $|S|$ different letters. If such a subcollection $S$ exists with $|S| \leq n$ we are clearly done. So consider such an $S$ of smallest cardinality. By assumption $|S| > n$. By minimality, the total number of letters that appear in the words of $S$ is exactly $|S| - 1$. By the Marriage Theorem, for every word $\pi \in S$ it is possible to mark one letter in every word in $S \setminus \{\pi\}$ so that every letter gets marked at most once. Let $S'$ consist of $\pi$ and the suffix of every other words in $S \setminus \{\pi\}$ starting from the marked letter. If $|S| = |S'|$ is even, then Theorem 12 applies since $S'$ has more words than letters and there is exactly one coincidence among these words’ initial letters. Consequently, $S$ has a subcollection of $|S|/2 + 1 \leq n$ that is a losing team strategy, as claimed. If $|S|$ is odd we first delete a word from $S$ whose marked letter differs from $\pi[1]$ and argue as above. 

4.1 A few words on Sperner’s lemma

In this section we discuss our main topological tool, the Sperner Lemma (see, e.g., [4]). We include all the required background and try to keep our presentation to the minimum that is necessary for a proof of Theorem 12.
Definition 14 A simplicial complex is a collection $X$ of subsets of a finite set of vertices $V$ such that

$$A \in X \text{ and } B \subseteq A \text{ then } B \in X.$$  

A member $A \in X$ is called a face, and its dimension is defined as $\dim A := |A| - 1$. We refer to $d$-dimensional faces as $d$-faces, and define $\dim X$ as the largest dimension of a face in $X$. We note that a vertex is a 0-face, and call a 1-face an edge. The 1-skeleton of $X$ is the graph with vertex set $V$, where $xy$ is an edge iff $\{x, y\}$ is an edge (1-face) of $X$. A face of dimension $\dim X$ is called a facet.

We say that $X$ is pure if every face of $X$ is contained in some facet. Finally, a $d$-pseudomanifold is a pure $n$-dimensional complex $X$ such that

$$\text{Every face of dimension } d - 1 \text{ is contained in exactly two facets.} \quad (2)$$

A good simple example of a 2-dimensional pseudomanifold is provided by a planar graph in which all faces including the outer face are triangles. The vertices and the edges of the complex are just the vertices and the edges of the graph. The facets (2-simplices) of the complex are the faces of the planar graph, including the outer face. This is clearly a pure complex and every edge is contained in exactly two facets. Note that such a graph drawn on a torus or on another 2-manifold works just as well. The pseudo part of the definition comes since we are allowing to carry out identifications such as the following: Take a set of vertices that forms an anticlique in the graph and identify all of them to a single vertex. The result is still a 2-dimensional pseudomanifold. At any event, the uninitiated reader is encouraged to use planar triangulations as a good mental model for a pseudomanifold. Henceforth we shorten pseudomanifold to psm.

Let $X$ be a psm on vertex set $V$. A $k$-coloring of $X$ is a mapping $\varphi : V \to \{1, \ldots, k\}$. A face of $X$ on which $\varphi$ is 1 : 1 is said to be $\varphi$-rainbow (we only say rainbow, when it is clear what coloring is involved). We are now ready to state and prove a special case of Sperner’s lemma that suffices for our purposes.

Lemma 15 Let $X$ be an $n$-dimensional psm. Then for every $(n + 1)$-coloring $\varphi$ of $X$, the number of $\varphi$-rainbow facets of $X$ is even.

Proof. Consider all pairs $A \supset B$ with $A$ is a facet of $X$, $B$ being $(n - 1)$-dimensional and $\varphi$-rainbow, where on the vertices of $B$, $\varphi$ takes all values except for $n + 1$. We count the number of such pairs in two different ways.

Each $(n - 1)$-simplex $B$ that $\varphi$ maps onto $\{1, \ldots, n\}$ participates in exactly two such pairs, once with each of the two facets that contain it. Hence the total count is even.

Each rainbow facet $A$ participates in exactly one good pair $A \supset B$, where $B = A \setminus \varphi^{-1}(n + 1)$. The claim follows. \hfill \Box

Thus, in particular, if we 3-color the vertices of a triangulated planar graph, so that the outer face is rainbow, then there must be at least one more rainbow face in the triangulated planar graph.

We say that an $n$-dimensional psm is colorable if it has a $(n + 1)$-coloring for which no edge is monochromatic. In other words, an $(n + 1)$-coloring for which all facets are rainbow.
Lemma 16  Let $\delta$ be a 2-coloring of a colorable psm $X$. Then the number of $\delta$-monochromatic facets of $X$ is even.

Proof. By assumption $X$ is colorable, so let $\chi$ be some proper $(n+1)$-coloring of $X$. Define next a new $(n+1)$-coloring $\varphi$ via $\varphi := \chi + \delta \mod (n+1)$. By assumption, every facet is $\chi$-rainbow and the addition (mod$(n+1)$) of a constant value of a monochromatic $\delta$ does not change this property. In other words, every $\delta$-monochromatic facet is $\varphi$-rainbow. We claim the reverse implication holds as well. Indeed, if $\delta$ is not constant on the facet $A$, then we can find two vertices $x, y \in A$ for which $\delta(x) = 1$, $\delta(y) = 2$ and $\chi(y) = \chi(x) + 1 \mod (n+1)$. But then no vertex $z \in A$ satisfies $\varphi(z) = \chi(x) + 2 \mod (n+1)$. In other words, a facet is $\varphi$-rainbow, iff it is $\delta$-monochromatic. By Lemma 15 the proof is complete. □

4.2 MC as a pseudomanifold

Here we prove Theorem 12 by using psm’s and Lemma 16. While it is true that psm’s can be realized geometrically, we do not refer to such realizations. Still, as mentioned above, planar triangulations can be useful in guiding one’s intuition in this area.

Given the $N$ words, our plan is to construct a psm $X$ that encodes certain possible executions of the MC algorithm. Vertices of $X$ correspond to states of individual players, and facets correspond to reachable configurations. Since we limit ourselves to schedules that involve only $n$ out of the $N$ available players, every facet has $n$ vertices, so that dim $X = n - 1$.

In the setting of Theorem 12 the scheduler selects one player from each of the $n - 2$ pairs (and adds in players $P_1, P_2$). This gives $2^{n-2}$ possible initial configurations, which we call initial facets. Note, however, that these $2^{n-2}$ sets do not constitute the collection of facets of a psm’s. An $(n-2)$-dimensional face that contains one player from each of the $n-2$ pairs and exactly one of $P_1, P_2$ is covered by exactly one initial facet, in violation of condition (2). To overcome this difficulty we add two auxiliary vertices called $A_1$ and $A_2$, where $A_1$ is viewed as being paired with $P_1$, and $A_2$ with $P_2$. This yields the $2^n$ sets that are obtained by making all possible choices of one vertex from each of the $n$ pairs. It is easily verified that this collection constitutes the set of facets of an $(n-1)$-dimensional psm. These $2^n$ facets include the $2^{n-2}$ initial configurations, and $2^n - 2^{n-2}$ auxiliary facets, those that include at least one of $A_1, A_2$. Figure 4 illustrates the situation for $n = 3$. There are six vertices, which correspond to $N = 2n - 2 = 4$ players plus two auxiliary players. The vertices are 3-colored, where each pair of players (say $P_1$ and $A_1$) are equally colored. The planar graph has eight faces (including the outer face), which are the $2^3 = 8$ facets.

Let us now introduce a 2-coloring of the vertices. We partition the $2n - 2$ chairs into two subsets of cardinality $n - 1$ each, called the 0-chairs and the 1-chairs. The initial chair of $P_1, P_2$ is a 1-chair whereas the initial chair of $A_1, A_2$ is a 0-chair. Also, within each pair of players (out of the $n - 2$ original pairs), one starts at a 0-chair and the other at a 1-chair. These requirements are consistent, since by assumption the only coincidence among initial chairs is that $\pi_1[1] = \pi_2[1]$.

Proposition 17  The collection of all subsets of the $2^n$ initial sets is a colorable $(n-1)$-dimensional psm. In the above-described 2-coloring there is exactly one monochromatic auxiliary cell.
Proof. We already noticed that this is indeed a psm. Let us associate a unique color to each pair of paired vertices. This makes every facet rainbow, so the psm is indeed colorable.

In the above-described 2-coloring there is indeed a unique monochromatic auxiliary facet. This the facet that contains $A_1$, $A_2$, and the $n - 2$ players (one from each pair) who start from a 0-chair. □

Starting from the initial system, the rules of MC allow the scheduler to generate new psm’s whose facets represent reachable configurations. We remark that unlike the initial system, it may happen that several facets correspond to the same configuration. This fact will cause no harm to us.

Let us consider a move of the scheduler relative to some pseudomanifold $PSM$. At a given configuration corresponding to a facet in $PSM$, if some players are in conflict, the scheduler may select two such players that occupy the same chair, and move either one of them or both. Hence given the two players and their states (say, corresponding to vertices $v_1$ and $v_2$ in $PSM$), two new states are exposed by this choice of three possible moves. These correspond to two new vertices (say $v_1'$ and $v_2'$) in a new psm. The given configuration can be moved to one of three new configurations, which in our psm representation corresponds to splitting the facet $\sigma$ that corresponds to the given configuration to three new facets. These new facets are obtained as $v_1$ or $v_2$ or both in $\sigma$ are replaced by $v'_1$, $v'_2$ or both, respectively. However, we are not done yet. The edge $\{v_1, v_2\}$ may be contained in several facets of $PSM$. Each such facet corresponds to a configuration in which the scheduler can apply any of the same three types of moves (moving $v_1$ to $v'_1$, moving $v_2$ to $v'_2$, or moving both). Hence every such facet is split in three as described above. This completes the description of the new pseudomanifold $PSM'$.

We say that the above process subdivides the edge $\{v_1, v_2\}$. Figure 2 illustrates the subdivision process when $n = 3$. (It is convenient to have $A_1$ and $A_2$ on the outer faces of such drawings, so that edges correspond to straight line segments.)

Proposition 18 No move of the scheduler can subdivide an auxiliary cell.
Figure 2: The cell system when $n = 3$ after one step by the scheduler. The auxiliary players are the red and green vertices on the outer face.

**Proof.** Observe that $A_1$ and $A_2$ are never involved in a subdivided edge (they are not true players and the scheduler cannot move them). As to the rest of the vertices in an auxiliary cell, they all correspond to different chairs, and hence cannot pair up to create a subdivided edge. □

We can extend the 2-coloring of $PSM$ to that of $PSM'$ by giving the two new vertices $v_1'$ and $v_2'$ the 0/1 color of the chairs corresponding to their respective states.

**Proposition 19** The simplicial complex $PSM'$ described above is a colorable pseudomanifold. The 2-coloring described for it has exactly one monochromatic auxiliary cell.

**Proof.** By induction and using Proposition 17 as the base case, we may assume that $PSM$ that gives rise to $PSM'$ is a psm. We need to show that edge subdivision (and the implied cell subdivisions) maintains the property that each $(n - 1)$-face is covered by exactly two facets. This amounts to a simple case analysis, presented for completeness in Section 4.3.

The colorability property of $PSM'$ is inherited from $PSM$, because every new facet contains the same set of players as its parent facet that was subdivided.

The 2-coloring property is a direct consequence of Proposition 17 and Proposition 18. □

We now reach a crucial part of our proof. Consider a psm $PSM$ generated by the process described above (starting from the initial psm and subdividing edges and their respective cells). By Proposition 19 $PSM$ is colorable. Hence for the 2-coloring that we associate with it, there are an even number of monochromatic facets, by Lemma 16. As exactly one of these facets is auxiliary (by Proposition 19), there is at least one non-auxiliary monochromatic facet, and this facet corresponds to a reachable configuration. Since there are only $n - 1$ 0-chairs and $n - 1$ 1-chairs, by pigeonhole there are
at least two players on the same chair in this configuration. Therefore the scheduler can subdivide the respective edge. It follows that the scheduler can continue to subdivide the psm indefinitely, creating arbitrarily large psm’s. Together with the next proposition this completes the proof of Theorem 12.

**Proposition 20** If the pseudomanifold that is generated by the subdivision process has \(2^n3^{t-1} + 1\) facets, then there is a scheduler strategy that makes the game last for at least \(t\) steps.

**Proof.** We describe an iterative process in which we construct a rooted tree \(T\) the reflects the process of generating the psm. The \(2^n\) facets of the initial psm are \(2^n\) children of the root. To represent a step in which we subdivide a facet \(\sigma\), we let the leaf corresponding to \(\sigma\) have three children. Thus every internal (neither root nor leaf) node of \(T\) has exactly three children. The leaves of \(T\) correspond to facets, of which \(2^n - 2^{n-2}\) are auxiliary, and the others corresponding to reachable configurations. (These need not be distinct reachable configurations, and not every reachable configuration is necessarily represented by the subdivision process). If \(T\) has \(2^n3^{t-1} + 1\) leaves, then there must be a leaf at depth at least \(t + 1\). The root to leaf path describes a possible schedule of \(t\) moves, as claimed. □

4.3 A case analysis

Here we present the case analysis used for the proof of Proposition 19.

We first consider various options for an \((n-1)\)-face \(F\) of \(PSM'\).

1. If \(F\) contains none of \(v_1, v_2, v'_1, v'_2\), then it must be a face in \(PSM\) as well. The facets that contain \(F\) were not subdivided (since they could contain at most one of \(v_1\) and \(v_2\), and hence also in \(CS'\) there are exactly two facets containing \(F\).

2. If \(v_1 \in F\) the \(F\) must be an \((n-1)\)-face in \(PSM\) as well. Possibly some facet \(C\) in \(PSM\) contains \(F\) and was subdivided in going from \(PSM\) to \(PSM'\), if \(C\) contained \(v_2\). But then \(PSM\) has the facet \(C'\) with \(v'_2\) replacing \(v_2\), and the number of facets in \(PSM'\) that contain \(F\) is the same as their number in \(PSM\).

3. If \(v_2 \in F\) of \(CS'\) the same argument applies.

4. Note that \(v_1, v_2 \in F\) is impossible, since edge \(\{v_1, v_2\}\) was subdivided).

We next turn to analyze various options for an \((n-1)\)-face \(F'\) of \(PSM'\) that is not a face in \(PSM\), where, say \(v'_1 \in F'\). Indeed \(F'\) is not a face in \(PSM\) (where \(v'_1\) does not exist), and it results from a subdivision step. We need to show that it is covered exactly twice following the subdivision. Consider two subcases.

- \(F'\) contains neither \(v_2\) nor \(v'_2\). Then one facet containing \(F'\) has \(v_2\) as its remaining vertex, and the other cell has \(v'_2\).
• \( F' \) contains either \( v_2 \) or \( v'_2 \) (note that it cannot contain both). Consider the unique \((n - 1)\)-face \( F \) of \( PSM \) that is derived from \( F' \) by replacing \( v'_1 \) by \( v_1 \), and also \( v'_2 \) by \( v_2 \) if \( v'_2 \in F' \) (this last replacement is not required if \( v_2 \in F \)). Then \( F \) is covered twice in \( PSM \), say by \( C_1 \) and \( C_2 \). Each of them is subdivided (because they contain both \( v_1 \) and \( v_2 \)), giving rise to two facets in \( PSM' \) that contain \( F' \). No other facet in \( PSM' \) can contain \( F' \).

The case where \( v'_2 \in F' \) is handled similarly.

5 Oblivious MC algorithms via the probabilistic method

We start with an observation that puts Theorems 2 and 3 (as well as Theorem 1) in an interesting perspective. The expected number of pairwise conflicts in a random configuration is exactly \( \binom{n}{2}/m \). In particular, when \( m \gg n^2 \), most configurations are safe (namely, have no conflicts). Therefore, it is not surprising that in this range of parameters \( n \) random words would yield an oblivious \( MC(n, m) \) algorithm. However, when \( m = O(n) \), only an exponentially small fraction of configurations are safe, and the existence of oblivious \( MC(n, m) \) algorithms is far from obvious.

5.1 Full words with \( O(n) \) chairs, allowing repetitions

Theorem 2 can be viewed as a (non-constructive) derandomization of the randomized MC algorithm in which players choose their next chair at random (and future random decisions of players are not accessible to the scheduler). Standard techniques for derandomizing random processes involve taking a union bound over all possible bad events, which in our case corresponds to a union bound over all possible schedules. The immediate scheduler has too many options (and so does the pairwise immediate scheduler), making it infeasible to apply a union bound. For this reason, we shall consider in this section the canonical scheduler, which is just as powerful (see Section 2). In every unsafe configuration, the canonical scheduler has just three possible moves to choose from. This allows us to use a union bound. We now prove Theorem 2.

**Proof.** Each of the \( N \) words is chosen independently at random as a sequence of \( L \) chairs, where each chair in the sequence is chosen independently at random. We show that with high probability (probability tending to 1 as the constant \( c \) grows), this choice satisfies Theorem 2.

It is easy to verify that in this random construction, with high probability, all words are full. To see this note that the probability that chair \( j \) is missing from such a random word is \( (m - 1)/m \). Consequently, the probability that a word chosen this way is not full is \( \leq m((m - 1)/m)^L \). Therefore, the expected number of non-full words is \( \leq m \cdot N \cdot ((m - 1)/m)^L \). But with our choice of parameters \( m = 7n \) and \( L = cn \log N \), we see that \( m \cdot N \cdot ((m - 1)/m)^L = o(1) \), provided that \( c \) is large enough.

In our approach to the proof we keep track of all possible schedules. To this end we use “a logbook” that is the complete ternary tree \( T \) of depth \( L \) rooted at \( r \). Associated with every node \( v \) of \( T \) is a random variable \( X_v \). The values taken by \( X_v \) are system configurations. For a given choice of words and an initial system configuration we define the value of \( X_r \) to be the chosen initial configuration. Every node \( v \) has three children corresponding to the three possible next configurations that are available to the canonical scheduler at configuration \( X_v \).
Another important ingredient of the proof is a potential function (defined below) that maps system configurations to the nonnegative reals. It is also convenient to define an (artificial) “empty” configuration of 0 potential. Every safe configuration has potential 1, and every non-empty unsafe configuration has potential $> 10$. If the node $u$ is a descendant of $v$ and the system configuration $X_v$ is safe, then we define $X_u$ to be the empty configuration.

We thus also associate with every node of $T$ a nonnegative random variable $P = P_v$ that is the potential of the (random) configuration $X_v$. The main step of the proof is to show that if $v_1, v_2, v_3$ are the three children of $v$, then $\sum_{i=1}^{3} \mathbb{E}(P_{v_i}) \leq r \mathbb{E}(P_v)$ for some constant $r \leq 0.99$. (Note that this inequality holds as well if $X_v$ is either safe or empty). This exponential drop implies that

$$\mathbb{E}(\sum_{v \text{ is a leaf of } \tau} (P_v)) = \sum_{v \text{ is a leaf of } \tau} \mathbb{E}(P_v) = o(1)$$

provided that $L$ is large enough. This implies that with probability $1 - o(1)$ (over the choice of random words) all leaves of $T$ correspond to an empty configuration. In other words every schedule terminates in fewer than $L$ steps.

We turn to the details of the proof. A configuration with $i$ occupied chairs is defined to have potential $x^{n-i}$, where $x > 1$ is a constant to be chosen later. In a nonempty configuration the potential can vary between 1 and $x^{n-1}$, and it equals 1 iff the configuration is safe.

Consider a configuration of potential $x^{n-i}$ (with $i < n$), where the canonical pair is $(\alpha, \beta)$. It has three children representing the move of either $\alpha$ or $\beta$ or both. Let us denote $\rho = i/m$ and $\rho' = (i-1)/m$. When a single player moves, the number of occupied chairs can stay unchanged, which happens with probability $\rho$. With probability $1 - \rho$ one more chair will be occupied and the potential gets divided by $x$. Consider next what happens when both players move. Here the possible outcomes (in terms of number of occupied chairs) depend on whether there is an additional player $\gamma$ currently co-occupying the same chair as $\alpha$ and $\beta$. It suffices to perform the analysis in the less favorable case in which there is no such player $\gamma$, as this provides an upper bound on the potential also for the case that there is such a player. With probability $(\rho')^2$ both $\alpha$ and $\beta$ move to occupied chairs and the potential gets multiplied by $x$. With probability $\rho'(1-\rho') + (1-\rho')\rho = (\rho + \rho')(1-\rho')$ the number of occupied chairs (and hence the potential) does not change. With probability $(1-\rho')(1-\rho)$ the number of occupied chairs grows by one and the potential gets divided by $x$.

It follows that if $v$ is a node of $T$ with children $v_1, v_2, v_3$ and if the configuration $X_v$ is unsafe and nonempty then $\sum_{i=1}^{3} \mathbb{E}(P_{v_i}) \leq \mathbb{E}(P) (2\rho + 2(1-\rho)/x + (\rho')^2 x + (\rho + \rho')(1-\rho') + (1-\rho)(1-\rho')/x)$. Recall that $x > 1$ and $\rho' < \rho < 1$. This implies that the last expression increases if $\rho'$ is replaced by $\rho$, and thereafter it is maximized when $\rho$ attains its largest possible value $q = (n-1)/m$. We conclude that

$$\sum_{i=1}^{3} \mathbb{E}(P_{v_i}) \leq \mathbb{E}(P)(2q + 2(1-q)/x + q^2 x + 2q(1-q) + (1-q)^2/x).$$

We can choose $q = 1/7$ and $x = 23/2$ to obtain $\sum_{i=1}^{3} \mathbb{E}(P_{v_i}) \leq r \mathbb{E}(P_v)$ for $r < 0.99$. This guarantees an exponential decrease in the expected sum of potentials and hence termination, as we now explain.

It follows that for every initial configuration the expected sum of potentials of all leaves at depth $L$ does not exceed $x^{n-1}$ (the largest possible potential) times $r^L$. On the other hand, if there is at
least one leaf $v$ for which the configuration $X_v$ is neither safe nor empty, then the sum of potentials at depth $L$ is at least $x > 1$. Our aim is to show that with high probability (over the choice of $N$ words), all runs have length $< L$: (i) For every choice of $n$ out of the $N$ words, (ii) Each selection of an initial configuration, and (iii) Every canonical scheduler’s strategy. The $n$ words can be chosen in $\binom{N}{n}$ ways.

For every $n$ words, there are $L^n$ possible initial configurations. The probability of length-$L$ run from a given configuration is at most $x^{n-1}r^L$, where $x = 23/2$ and $r < 0.99$. Therefore our claim is proved if $\binom{N}{n} \cdot x^{n-1}r^L \leq o(1)$. This inequality clearly holds if we let $L = cn \log N$ with $c$ a sufficiently large constant. This completes the proof of Theorem 2.

A careful analysis of the proof of Theorem 2 shows that it actually works as long as $\frac{m}{n} > 4+2\sqrt{2} = 6.828...$ It would be interesting to determine the value of $\liminf_{n \to \infty} \frac{m}{n}$ for which $n$ long enough random words over an $m$-letter alphabet constitute, with high probability, an oblivious $MC(n, m)$ algorithm.

### 5.2 Permutations over $O(n)$ chairs

The argument we used to prove Theorem 2 is inappropriate for the proof of Theorem 3. Theorem 3 deals with random permutations, whereas in the proof of Theorem 2 we use words of length $\Omega(n \log n)$. (Longer words are crucial there for two main reasons: To guarantee that words are full and to avoid wrap-around. The latter property is needed to guarantee independence.) Indeed in proving Theorem 3 our arguments are substantially different. In particular, we work with a pairwise immediate scheduler, and unlike the proof of Theorem 2 there does not appear to be any significant benefit (e.g., no significant reduction in the ratio $\frac{m}{n}$) if a canonical scheduler is used instead.

We first prove the special case $N = n$ of Theorem 3.

**Theorem 21** If $m \geq cn$ where $c > 0$ is a sufficiently large constant, then there is a family of $n$ permutations on $[m]$ which constitute an oblivious $MC(n, m)$ algorithm.

We actually show that with high probability, a set of random permutations $\pi_1, \ldots, \pi_n$ has the property that in every possible schedule the players visit at most $L = O(m \log m)$ chairs. Our analysis uses the approach of deferring random decisions until they are actually needed. For each of the $m^n$ possible initial configuration, we consider all possible sequences of $L$ locations. For each such sequence we fill in the chairs in the locations in the sequence at random, and prove that the probability that this sequence represents a possible schedule is extremely small – so small that even if we take a union bound over all initial configurations and over all sequences of length $L$, we are left with a probability much smaller than 1.

The main difficulty in the proof is that since $L \gg m$, some players may completely traverse their permutation (even more than once) and therefore the chairs in these locations are no longer random. To address this, we partition the sequence of moves into $L/t$ blocks, where in each block players visit a total of $t$ locations. We can and will assume that $t$ divides $L$. We take $t = \delta m$ for some sufficiently small constant $\delta$, and $n = em$, where $\epsilon$ is a constant much smaller than $\delta$. This choice of parameters implies that within a block, chairs are essentially random and independent. To deal with dependencies among different blocks, we classify players (and their corresponding permutations) as light or heavy.
A player is light if during the whole schedule (of length $L$) it visits at most $t / \log m = o(t)$ locations. A player that visits more than $t / \log m$ locations during the whole sequence is heavy. Observe that for light players, the probability of encountering a particular chair in some given location is at most $\frac{1}{m - o(t)} \leq \frac{1 + o(1)}{m}$. Hence, the chairs encountered by light players are essentially random and independent (up to negligible error terms). Thus it is the heavy players that introduce dependencies among blocks. Every heavy player visits at least $t / \log m$ locations, so that $n_h$, the number of heavy players does not exceed $n_h \leq (L \log m) / t = O(\log^2 m)$. The fact that the number of heavy players is small is used in our proof to limit the dependencies among blocks.

The following lemma is used to show that in every block of length $t$ the number of locations that are visited by heavy players is not too large. Consequently, sufficiently many locations are visited by light players. In the lemma we use the following notation. A segment of $k$ locations in a permutation is said to have volume $k - 1$. Given a collection of locations, a chair is unique if it appears exactly once in these locations.

**Lemma 22** Let $n_h \leq m / \log^2 m$ and let $\delta > 0$ be a sufficiently small constant. Consider $n$ random permutations over $[m]$. Select any $n_h$ of the permutations and a starting location in each of them. Choose next intervals in the selected permutations with total volume $t'$ for some $t / 10 \leq t' \leq t$. With probability $1 - o(1)$ for every such set of choices at least $4t' / 5$ of the chairs in the chosen intervals are unique.

**Proof.** We first note that we will be using the lemma with $n_h = O(\log^2 n)$. Also, if a list of letters contains $u$ unique letters (i.e., they appear exactly once) and $r$ repeated letter (i.e., appearing at least twice), then it has $d = u + r$ distinct letters and length $\lambda \geq u + 2r$. In particular $d \leq (\lambda + u) / 2$.

There are $\binom{n}{n_h}$ ways of choosing $n_h$ of the permutations. Then, there are $m^{n_h}$ choices for the initial configuration. We denote by $s_i$ the volume of the $i$-th interval, so that $\sum_{i=1}^{n_h} s_i = t'$. Therefore there are $\binom{t' + n_h - 1}{n_h - 1} \leq m^{n_h}$ ways of choosing the intervals with total volume $t'$. Since the volume of every interval is at most $t'$ we have that the probability that a particular chair resides at a particular location in this interval is at most $1 / (m - t')$. This is because the permutation is random and at most $t'$ chairs appeared so far in this interval. Therefore the probability that a sequence of $t'$ labels involves less than $0.9t'$ distinct chairs is at most

$$\left( \frac{m}{0.9t'} \right)^t \left( \frac{0.9t'}{m-t'} \right)^t \leq \left( \frac{em}{0.9t'} \right)^{0.9t'} \left( \frac{0.9t'}{m-t'} \right)^t \leq e^{t' \left( \frac{m}{m-t'} \right)^{0.9t'} \left( \frac{t'}{m-t'} \right)^{0.1t'}} \leq 4^t (2\delta)^{0.1t'} \ll e^{-t'}.$$  

Explanation: The set of chairs that appear in these intervals can be chosen in $\binom{m}{0.9t'}$ ways. The probability that a particular location in this union of intervals is assigned to a chair from the chosen set does not exceed $\frac{0.9t'}{m-t'}$. In addition $m / (m - t') \leq (1 + \delta)$, $t' / (m - t') \leq 2\delta$ and $\delta$ is a very small constant.

Now we take a union bound over all choices of $n_h$ permutations, all starting locations and all collection of intervals with total volume $t'$. It follows that the probability that there is a choice of intervals of volume $t'$ that span $\leq n_h$ permutations and contain fewer than $9t' / 10$ distinct chairs is at most

$$m^{3n_h} e^{-t'} = o(1).$$
Since the conclusion of this lemma holds with probability $1 - o(1)$ we can assume that our set of permutations satisfies it. In particular, in every collection of intervals in these permutations with total volume $\frac{t}{10} \leq t' \leq t$ that reside in $O(\log^2 m)$ permutations there are at least $4t'/5$ unique chairs.

As already mentioned, we break the sequence of $L$ locations visited by players into blocks of $t$ locations each. We analyze the possible runs by considering first the breakpoints profile, namely where each block starts and ends on each of the $n$ words. There are $m^n$ possible choices for the starting locations. If, in a particular block player $i$ visits $s_i$ chairs, then $\sum_{i=1}^{n} s_i = t$. Consequently the parameters $s_1, \ldots, s_n$ can be chosen in $\binom{t + n - 1}{n} \leq 2^{t + n}$ ways. There are $L/t$ blocks, so that the total number of possible breakpoints profiles is at most $m^n(2^{t + n})^{L/t} \leq m^n 2^{2L}$ (here we used the fact that $t > n$). Clearly, by observing the breakpoints profile we can tell which players are light and which are heavy. We recall that there are at most $O(\log^2 m)$ heavy players, and that the premise of Lemma 22 can be assumed to hold.

Let us fix an arbitrary particular breakpoints profile $\beta$. We wish to estimate the probability (over the random choice of chairs) that some legal sequence of moves by the pairwise immediate scheduler yields this breakpoints profile $\beta$. Let $B$ be an arbitrary block in $\beta$. Let $p(B)$ denote the probability over choice of random chairs and conditioned over contents of all previous blocks in $\beta$ that there is a legal sequence of moves by the pairwise immediate scheduler that produces this block $B$.

Lemma 23 For $p(B)$ as defined above we have that $p(B) \leq 8^{-t}$.

Proof. The total number of chairs encountered in block $B$ is $n \ll t$ (for the initial locations) plus $t$ (for the moves). Recall that the set of heavy players is determined by the block-sequence $\beta$. Hence within block $B$ it is clear which are the heavy players and which are the light players. Let $t_h$ (resp. $t_l = t - t_h$) be the number of chairs visited by heavy (resp. light) players in this block. The proof now breaks into two cases, depending on the value of $t_h$.

Case 1: $t_h \leq 0.1t$. Light players altogether visit $n + t_l$ chairs ($n$ initial locations plus $t_l$ moves). If $u$ of these chair are unique, then they visit at most $(n + t_l + u)/2$ distinct chairs. But a chair in this collection that is unique is either: (i) One of the $n$ chairs where a player terminates his walk, or, (ii) A chair that a light player traverses due to a conflict with a heavy player, and there are at most $t_h$ of those. Consequently, the number of distinct chairs visited by light players does not exceed $(n + t_l + n + t_h)/2 = t/2 + n$.

Fix the set $S$ of $t/2 + n$ distinct chairs that we are allowed to use. There are $\binom{m}{n + t/2}$ choices for $S$. Now assign chairs to the locations one by one, in an arbitrary order. Each location has probability of at most $(1 + o(1))^{n + t/2} m \choose m$ of receiving a chair in $S$. Since we are dealing here with light players, we have exposed only $o(m)$ chairs for each of them (in $B$ and in previous blocks of $\beta$), and as mentioned above, this can increase the probability by no more that a $1 + o(1)$ factor.

Hence the probability that the segments traversed by the light players contain only $n + t/2$ chairs is at most
\[
\binom{m}{n+\ell/2} \left( (1 + o(1)) \frac{n+\ell/2}{m} \right)^t \leq \left( \frac{em}{n+t/2} \right)^{n+t/2} 2^\ell \left( \frac{n+t/2}{m} \right)^{t \ell} \\
\leq (2e)^\ell \left( \frac{n+t/2}{m} \right)^{(t\ell-t_h)/2-n} \leq (2e)^\ell \left( \frac{t}{m} \right)^{t/4} < 8^{-t}.
\]

Here we used that \( t_h + t_\ell = t, t_h \leq 0.1t, t_\ell \geq 0.9t \) and \( n \ll t \ll m \).

**Case 2:** \( t_h \geq 0.1t \). Let us reveal first the chairs visited by the heavy players. By Lemma 22 we find there at least \( 4t_h/5 \) unique chairs. In order that the heavy players traverse these chairs, they must be visited by light players as well. Hence the \( t_\ell \) locations visited by light players must include all these \( 0.8t_h \) pre-specified chairs. We bound the probability of this as follows. First choose for each of the \( 0.8t_h \) pre-specified chairs a particular location where it should appear in the intervals of light players. The number of such choices is \( t_\ell^{0.8t_h} \). As mentioned above the probability that a particular chair is assigned to some specific location is \( (1 + o(1))/m \). Therefore the probability that \( 0.8t_h \) pre-specified chairs appear in the light intervals is at most \( t_\ell^{0.8t_h} \left( (1 + o(1))/m \right)^{0.8t_h} \). Thus the probability that a schedule satisfying the condition of the lemma exists is at most

\[
t_\ell^{0.8t_h} \left( (1 + o(1))/m \right)^{0.8t_h} \leq (2t/m)^{0.8t_h} \leq (2t/m)^{t/15} < 8^{-t},
\]

where we used that \( n \ll t \ll m \). □

Lemma 23 implies an upper bound of \( p(B)^{L/t} = 8^{-L} \) on the probability there is a legal sequence of moves by the pairwise immediate scheduler that gives rise to breakpoints profile \( \beta \). Taking a union bound over all block sequences (whose number is at most \( m^n 2^{2L} \leq 6^L \), by our choice of \( L = Cm \log m \) for a sufficiently large constant \( C \), Theorem 21 is proved.

Observe that the proof of Theorem 21 easily extends to the case that there are \( N = m^{O(1)} \) random permutations out of which one chooses \( n \). We simply need to multiply the number of possibilities by \( N^n \), a term that can be absorbed by increasing \( m \), similar to the way the term \( m^n \) is absorbed. In Lemma 22 we need to replace \( \binom{n}{n_h} \) by \( \binom{N}{n_h} \), and the proof goes through without any change (because \( n_h \) is so small). This proves Theorem 3.

### 5.3 Explicit construction with permutations and \( m = O(n^2) \)

In this section we present for every integer \( d \geq 1 \) an explicit collection of \( n^d \) permutations on \( m = O(d^2 n^2) \) such that every \( n \) of these permutations constitute an oblivious \( MC(n, m) \) algorithm. This proves Theorem 4.

We let \( LCS(\pi, \sigma) \) stand for the length of the longest common subsequence of the two permutations \( \pi \) and \( \sigma \), considered cyclically. (That is, we may rotate \( \pi \) and \( \sigma \) arbitrarily to maximize the length of the resulting longest common subsequence.) The following easy claim is useful.

**Proposition 24** Let \( \pi_1, \ldots, \pi_n \) be permutations of \( \{1, \ldots, m\} \) such that \( LCS(\pi_i, \pi_j) \leq r \) for all \( i \neq j \). If \( m > (n-1)r \), then in every schedule none of the \( \pi_i \) is fully traversed.

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Proof. By contradiction. Consider a schedule in which one of the permutations is fully traversed, say \( \pi_1 \) is the first permutation to be fully traversed. Each move along \( \pi_1 \) reflects a conflict with some other permutation. Hence there is a permutation \( \pi_i, i > 1 \) that has at least \( m/(n-1) \) agreements with \( \pi_1 \). Consequently, \( r \geq LCS(\pi_1, \pi_i) \geq \frac{m}{n-1} \), a contradiction. \( \square \)

This yields an inexplicit oblivious \( MC(n, m) \) algorithm with \( m = O(n^2) \), since (even exponentially) large families of permutations in \([m]\) exist where every two permutations have an LCS of only \( O(\sqrt{m}) \).

We omit the easy details. On the other hand, we should notice that by [5] this approach is inherently limited and can, at best yield bounds of the form \( m \leq O(n^{3/2}) \).

We now present an explicit construction that uses some algebra.

Lemma 25 Let \( p \) be a prime power, let \( d \) be a positive integer and let \( m = p^2 \). Then there is an explicit family of \( (1 - o(1))m^d \) permutations of an \( m \)-element set, where the LCS of every two permutations is at most \( 4d\sqrt{m} \).

Proof. Let \( \mathbb{F} \) be the finite field of order \( p \). Let \( \mathcal{M} := \mathbb{F} \times \mathbb{F} \), and \( m = p^2 = |\mathcal{M}| \). Let \( f \) be a polynomial of degree \( 2d \) over \( \mathbb{F} \) with vanishing constant term, and let \( j \in \mathbb{F} \). We call the set \( B_{f,j} = \{(x, f(x) + j)|x \in \mathbb{F}\} \) a block. We associate with \( f \) the following permutation \( \pi_f \) of \( \mathcal{M} \): It starts with an arbitrary ordering of the elements in \( B_{f,0} \) followed by \( B_{f,1} \) arbitrarily ordered, then of \( B_{f,2} \) etc. A polynomial of degree \( r \) over a field has at most \( r \) roots. It follows that for every two polynomials \( f \neq g \) as above and any \( i, j \in \mathbb{F} \), the blocks \( B_{f,i} \) and \( B_{g,j} \) have at most \( 2d \) elements in common. There are \( (p-1) \cdot p^{2d-1} = (1 - o(1))m^d \) such polynomials. There are \( p \) blocks in \( \pi_f \) and in \( \pi_g \), so that \( LCS(\pi_f, \pi_g) \leq 4dp \), as claimed. \( \square \)

6 Discussion and Open Problems

This work originated with the introduction of the concept of oblivious distributed algorithms. In the present paper we concentrated on oblivious MC algorithms, a topic which yields a number of interesting mathematical challenges. We showed that \( m \geq 2n - 1 \) chairs are necessary and sufficient for the existence of an oblivious MC algorithm with \( n \) processors. Still, our construction involves very long words. It is interesting to find explicit constructions with \( m = 2n - 1 \) chairs and substantially shorter words.

In other ranges of the problem we can show, using the probabilistic method, that oblivious \( MC(n, m) \) algorithms exist with \( m = O(n) \) and relatively short full words. We still do not have explicit constructions with comparable properties. We would also like to determine \( \liminf \frac{m}{n} \) such that \( n \) random words over an \( m \) letter alphabet typically constitute an oblivious \( MC(n, m) \) algorithm.

Computer simulations strongly suggest that for random permutations, a value of \( m = 2n - 1 \) does not suffice. On the other hand, we have constructed (details omitted from this manuscript) oblivious \( MC(n, 2n - 1) \) algorithms using permutations for \( n = 3 \) and \( n = 4 \) (for the latter the proof
of correctness is computer-assisted). For \( n \geq 5 \) we have neither been able to find such systems (not even in a fairly extensive computer search) nor to rule out their existence.

We do not know how hard it is to recognize whether a given collection of words constitute an oblivious \( MC \) algorithm. This can be viewed as the problem whether some digraph contains a directed cycle or not. The point is that the digraph is presented in a very compact form. It is not hard to place this problem in \( \text{PSPACE} \), but is it in a lower complexity class, such as \( \text{co-NP} \) or \( \text{P} \)?

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