Perfect IFG-Formulas

Allen L. Mann

Abstract. IFG logic is a variant of the independence-friendly logic of Hintikka and Sandu. We answer the question: “Which IFG-formulas are equivalent to ordinary first-order formulas?” We use the answer to prove the ordinary cylindric set algebra over a structure can be embedded into a reduct of the IFG-cylindric set algebra over the structure.

Mathematics Subject Classification (2000). Primary 03G25; Secondary 03B60, 03G15.

Keywords. Independence-friendly logic, cylindric algebra.

1. IFG logic

Independence-friendly logic (IF logic) is a conservative extension of first-order logic that has the same expressive power as existential second-order logic [6, 7]. In IF logic the truth of a sentence is defined via a game between two players, Abélard (\(\forall\)) and Eloïse (\(\exists\)). The additional expressivity is obtained by modifying the quantifiers and connectives of an ordinary first-order sentence in order to restrict the information available to the existential player, Eloïse, in the associated semantic game.

In IF logic, only the information available to Eloïse is restricted, which means existential quantifiers are not dual to universal quantifiers. To compensate, negation symbols are only allowed before atomic formulas. Generalized independence-friendly logic (IFG logic) is a variant of independence-friendly logic in which the information available to both players can be restricted, making existential quantifiers dual to universal quantifiers and allowing any formula to be negated [2].

Definition 1.1. Given a first-order signature \(\sigma\), an atomic IFG-formula is a pair \((\phi, X)\) where \(\phi\) is an atomic first-order formula and \(X\) is a finite set of variables that includes every variable that appears in \(\phi\) (and possibly more).

Definition 1.2. Given a first-order signature \(\sigma\), the language \(L_{\text{IFG}}^\sigma\) is the smallest set of formulas such that:
(a) Every atomic IFG-formula is in $L_{\text{IFG}}$.
(b) If $\langle \phi, Y \rangle$ is in $L_{\text{IFG}}$ and $Y \subseteq X$, then $\langle \phi, X \rangle$ is in $L_{\text{IFG}}$.
(c) If $\langle \phi, X \rangle$ is in $L_{\text{IFG}}$, then $\langle \neg \phi, X \rangle$ is in $L_{\text{IFG}}$.
(d) If $\langle \phi, X \rangle$ and $\langle \psi, X \rangle$ are in $L_{\text{IFG}}$, and $Y \subseteq X$, then $\langle \phi \lor_{Y} \psi, X \rangle$ is in $L_{\text{IFG}}$.
(e) If $\langle \phi, X \rangle$ is in $L_{\text{IFG}}$, $x \in X$, and $Y \subseteq X$, then $\langle \exists x_{/Y} \phi, X \rangle$ is in $L_{\text{IFG}}$.

Above $X$ and $Y$ are finite sets of variables.

From now on we will make certain assumptions about IFG-formulas that will allow us to simplify our notation. First, we will assume that the set of variables of $L_{\text{IFG}}$ is $\{v_{n} \mid n \in \omega\}$. Second, since it does not matter much which particular variables appear in a formula, we will assume that variables with smaller indices are used before variables with larger indices. More precisely, if $\langle \phi, X \rangle$ is a formula, $v_{j} \in X$, and $i \leq j$, then $v_{i} \in X$. By abuse of notation, if $\langle \phi, X \rangle$ is a formula and $|X| = N$, then we will say that $\phi$ has $N$ variables and write $\phi$ for $\langle \phi, X \rangle$. As a shorthand, we will call $\phi$ an IFG-$N$-formula. Let $L_{\text{IFG}}^{\phi} = \{ \phi \in L_{\text{IFG}} \mid \phi \text{ has } N \text{ variables} \}$.

Third, sometimes we will write $\phi \lor_{J} \psi$ instead of $\phi \lor_{Y} \psi$ and $\exists v_{n/J} \phi$ instead of $\exists v_{n/Y} \phi$, where $J = \{j \mid v_{j} \in Y\}$. Finally, we will use $\phi \land_{J} \psi$ to abbreviate $\neg(\neg \phi \lor_{J} \neg \psi)$ and $\forall v_{n/J} \phi$ to abbreviate $\neg \exists v_{n/J} (\neg \phi)$.

Truth and falsity for IFG-sentences are defined in terms of a two-player, win-loss game of imperfect information. Given an IFG-sentence, Eloïse’s goal is to verify the sentence, and Abélard’s goal is to falsify it. The sentence is true if Eloïse has a winning strategy, and it is false if Abélard has a winning strategy. For example, consider a structure $A$ with universe $A$ and the ordinary first-order sentence

$$\forall v_{0} \exists v_{1} \left[ v_{0} \neq v_{1} \right].$$

First Abélard chooses an element $a$ to be the value of the variable $v_{0}$, then Eloïse chooses an element $b$ to be the value of the variable $v_{1}$. If $a \neq b$, Eloïse wins; otherwise, Abélard wins. If $A$ has more than one element, Eloïse can win every play of the game; hence the sentence is true. If $A$ has only one element, then Abélard will win every play; hence the sentence is false. Now consider the IFG$_{2}$-sentence

$$\forall v_{0/0} \exists v_{1/(v_{0})} \left[ v_{0} \neq v_{1} \right].$$

The subscripts indicate what information is unavailable to the players at each move. The game begins as before with Abélard choosing an $a \in A$ to be the value of $v_{0}$. Next Eloïse chooses an element $b \in A$ to be the value of $v_{1}$, but this time she must make her choice in ignorance of the value of $v_{0}$. Let us assume $A$ has more than one element. On the one hand, Eloïse does not have a winning strategy because she might blindly choose the same element as Abélard. Therefore the sentence is not true. On the other hand, Abélard does not have a winning strategy either, because Eloïse might get lucky and choose a different element than the one he chose. Therefore the sentence is not false.