EXISTENCE OF POSITIVE SOLUTIONS OF AN ELLIPTIC EQUATION WITH LOCAL AND NONLOCAL VARIABLE DIFFUSION COEFFICIENT

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To Peter Kloeden for his 70th birthday
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Abstract. In this paper we study a stationary problem arising from population dynamics with a local and nonlocal variable diffusion coefficient. We show the existence of an unbounded continuum of positive solutions that bifurcates from the trivial solution. The global structure of this continuum depends on the value of the nonlocal diffusion at infinity and the relative position of the refuge of the species and of the sets where it diffuses locally and not locally, respectively.

1. Introduction. The aim of this paper is the study of the existence of positive solutions for the problem

\[
\begin{cases}
- \left( \chi_{D_1}(x) + \chi_{D_2}(x) a \left( \int_{\Omega} u^p \right) \right) \Delta u = \lambda u - b(x) u^2 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^N, N \geq 1, \) is a bounded and regular domain split into two regular subdomains \( D_1, D_2 \subset \Omega \)

\[ \overline{\Omega} = D_1 \cup D_2, \quad D_1 \cap D_2 = \emptyset, \]

and, as a consequence \( \Omega = D_1 \cup D_2 \cup \Gamma \) with \( \Gamma \) a set with zero measure, \( a : \mathbb{R} \to \mathbb{R} \) is a positive and continuous function, \( p > 0, \lambda \in \mathbb{R}, b \) is a bounded and non-negative function in \( \Omega \) and

\[ \chi_D(x) = \begin{cases} 
1 & \text{if } x \in D, \\
0 & \text{if } x \notin D.
\end{cases} \]

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We define the operator

\[ A : \Omega \times L^\infty(\Omega) \to \mathbb{R}, \]

\[ A(x,u) = \chi_{D_1}(x) + \chi_{D_2}(x)a\left(\int_\Omega u^p\right). \quad (2) \]

This type of equation models the behavior of a species, where \( \Omega \) represents its habitat, \( u \) is the population density, \( \lambda \) is the growth rate, \( b \) represents the crowding effect and \( A \) denotes the diffusion coefficient of the species. Observe that in this case we are assuming that the species diffuses in a random way in the set \( D_1 \) and, on the other hand, its diffusion velocity depends on the total population, in a nonlinear way, in \( D_2 \).

With respect to the function \( b \), we assume that \( b \) verifies

- \( b(x) \geq 0 \) in \( \Omega \) and \( b(x) \equiv 0 \) in \( \Omega_0 \) where

\[ \Omega_0 := \text{int}\{x \in \Omega : b(x) = 0\}, \]

and assume that \( \Omega_0 \subset \subset \Omega \) with \( \Omega_0 \) is a proper subdomain of \( \Omega \).

The set \( \Omega_0 \) represents a region where the species grows freely, hence we call it the refuge of the species.

Problem (1) has been studied for different reaction terms, with \( D_1 = \emptyset \) and when \( A \) is bounded above and below, that is,

\[ 0 < a_0 \leq A(x,u) \leq a_\infty < \infty, \]

see for instance [3], [7], [8], [16], [17] and [18], where the authors use various techniques, such as fixed point theorems, comparison arguments, Bolzano’s Theorem and Leray-Schauder degree theory to obtain their results; see also [6] and references therein for the parabolic counterpart of (1). However, the case in which \( A \) has the form (2) has not been previously analyzed, at least to our knowledge it has been introduced in a slightly different context in [13]. See also [1] for a different diffusion function \( A \).

In order to state our main results and compare them with the previous ones, we need to introduce some notation. First, we define

\[ a(\infty) := \lim_{s \to \infty} a(s). \]

On the other hand, given a domain \( D \subset \Omega \) and two non-negative and non-trivial functions \( f,g \in L^\infty(\Omega) \) with \( f \geq f_0 > 0 \) for some positive constant \( f_0 \), we denote by \( \lambda^D_1[-f \Delta; g] \) the principal eigenvalue of

\[ -f(x)\Delta u = \lambda g(x)u \quad \text{in } D, \quad u = 0 \quad \text{on } \partial D. \]

When \( \Omega = D \), we omit the superscript and we write \( \lambda_1[-f \Delta; g] \) and we also denote

\[ \lambda_1 := \lambda_1[-\Delta; 1], \quad \lambda_1^{\Omega_0} = \lambda_1^{\Omega_0}[-\Delta; 1]. \]

Finally, \( \lambda^D_1[-f \Delta; g] = \infty \) when \( D = \emptyset \) or \( g \equiv 0 \) in \( D \).

We first recall the known-results. First, we consider the case \( b \equiv 0 \) in \( \Omega \):

1. Assume that \( D_2 = \emptyset \). Then, there exists a positive solution if and only if \( \lambda = \lambda_1 \). Moreover, in this case there exist infinite positive solutions.
2. Assume that \( D_1 = \emptyset \). Then, there exist two values \( 0 \leq \lambda < \lambda_1 \leq \infty \) such that if there exists a positive solution of (1) then \( \lambda \in (\lambda_1, \lambda_1^{\Omega_0}) \). Moreover, if

\[ \lambda \in (\lambda_1, \lambda_1^{\Omega_0}), \]
then there exists at least one positive solution of (1). Moreover, if $a(\infty) = \infty$ then $\lambda = \infty$ and if $a(\infty) = 0$, $\lambda = 0$.

The first case ($D_2 = \emptyset$) is the classical eigenvalue problem, and hence the result is well-known. The second one follows by [9].

Here, we apply the bifurcation theory and we prove that from the trivial solution $u \equiv 0$ emanates at

$$\lambda = \lambda_0 := \lambda_1[-(\chi_{D_1} + a(0)\chi_{D_2})\Delta; 1]$$

an unbounded continuum $C$ in $\mathbb{R} \times L^\infty(\Omega)$ of positive solutions of (1). Then, in order to ascertain the structure of the set of positive solutions, we need to study the global behaviour of $C$.

Our main result when $b \equiv 0$ reads:

**Theorem 1.1.** Assume that $b \equiv 0$ in $\Omega$, $D_1 \neq \emptyset \neq D_2$. Then, there exists $0 \leq \lambda_\infty < \infty$ such that (1) possesses at least a positive solution if

$$\lambda \in (\min\{\lambda_0, \lambda_\infty\}, \max\{\lambda_0, \lambda_\infty\}).$$

Moreover,

$$\lambda_\infty = \begin{cases} 
\lambda_1[-(\chi_{D_1} + a(\infty)\chi_{D_2})\Delta; 1] & \text{if } 0 < a(\infty) < \infty, \\
\lambda_1[-\Delta; \chi_{D_1}] & \text{if } a(\infty) = \infty, \\
0 & \text{if } a(\infty) = 0.
\end{cases}$$

In addition, $\lambda_0$ and $\lambda_\infty$ are bifurcation points from the trivial solution and from infinity, respectively.

We remark that $\lambda_0 < \lambda_\infty$ if and only if $a(0) < a(\infty)$. In Figures 1 and 2 we have represented different bifurcation diagrams for the case $b \equiv 0$.

Now, we analyze the case $b \geq 0$ and $b \neq 0$. The main known results are:

1. Assume that $D_2 = \emptyset$. Then, there exists at least a positive solution if and only if

$$\lambda \in (\lambda_1, \lambda_1^{O_0}).$$

In this case, the solution is unique and $\lambda_1$ and $\lambda_1^{O_0}$ are bifurcation points from the trivial solution and from infinity, respectively.

2. Assume that $D_1 = \emptyset$. Then, there exists at least a positive solution of (1) if

$$\lambda \in (\min\{a(0)\lambda_1, a(\infty)\lambda_1^{O_0}\}, \max\{a(0)\lambda_1, a(\infty)\lambda_1^{O_0}\}).$$

The cases $a(\infty) = \infty$ and $a(\infty) = 0$ are allowed. Again, $a(0)\lambda_1$ and $a(\infty)\lambda_1^{O_0}$ are bifurcation points from the trivial solution and from infinity, respectively.

3. Assume that $b(x) \geq b_0 > 0$ for some positive constant $b_0$. Then,

(a) Assume that $D_2 = \emptyset$. Then, there exists at least a positive solution if and only if

$$\lambda \in (\lambda_1, \infty).$$

Moreover, in this case there exists at most a unique positive solution.

(b) Assume that $D_1 = \emptyset$. Then, there exists at least a positive solution of (1) if

$$\lambda \in (a(0)\lambda_1, \infty).$$

The results in the local case, that is $D_2 = \emptyset$, follow by [2], [14], [5] and [12]; this last reference contains a detailed study of this equation. In the pure non-local case, $D_1 = \emptyset$, the results follow by [10] where a more general case is analyzed. Some partial results are obtained in [16], [17] and [7], all of them in the particular case $b \geq b_0 > 0$ in $\Omega$. 
Now, our results depend on the relative position between the sets $D_1$, $D_2$ and $\Omega_0$. We can summarize our results as follows (see Figures 1-2):

**Theorem 1.2.** Assume that $p > 1$. There exists $\lambda_\infty \in [0, \infty]$ such that (1) possesses at least a positive solution for

$$\lambda \in \left( \min\{\lambda_0, \lambda_\infty\}, \max\{\lambda_0, \lambda_\infty\} \right).$$

1. If $0 < a(\infty) < \infty$, then $\lambda_\infty = \lambda_1^{\Omega_0}\left[-(\chi_{D_1} + a(\infty)\chi_{D_2})\Delta; 1\right].$

2. If $\Omega_0 \subseteq D_1$, then $\lambda_\infty = \lambda_1^{\Omega_0}.$

3. If $\Omega_0 \subseteq D_2$, then $\lambda_\infty = +\infty$ if $a(\infty) = \infty$ and $\lambda_\infty = 0$ if $a(\infty) = 0$.

4. In the general case, $\Omega_0 \cap D_1 \neq \emptyset \neq \Omega_0 \cap D_2$, then $\lambda_\infty = 0$ if $a(\infty) = 0$ and $\lambda_\infty \geq \lambda_1^{\Omega_0 \cup D_2}[-\Delta; \chi_{D_1}]$ when $a(\infty) = \infty$. Moreover, in this case, if one of the following cases holds:
   
   (a) $D_2 \subseteq \Omega_0$,
   
   (b) $p \geq 2$ and $a(s)/s^{1/p}$ bounded above for $s$ large,
   
   (c) $\lim_{s \to \infty} \frac{a(s)}{s^{1/p}} = \infty,$

then

$$\lambda_\infty = \lambda_1^{\Omega_0 \cup D_2}[-\Delta; \chi_{D_1}].$$

In all the cases, $\lambda_0$ and $\lambda_\infty$ are bifurcation points from the trivial solution and from infinity, respectively.

Observe that we are not able to ascertain the bifurcation point from infinity in the general case. We only know that $\lambda_\infty \geq \lambda_1^{\Omega_0 \cup D_2}[-\Delta; \chi_{D_1}]$ and, in some particular cases, that exactly $\lambda_\infty = \lambda_1^{\Omega_0 \cup D_2}[-\Delta; \chi_{D_1}].$

As an immediate consequence we have (taking $\Omega_0 = \emptyset$ in the above result), see Figure 2:

**Corollary 1.** Assume that $b(x) \geq b_0 > 0$ for some positive constant. Then, there exists at least a positive solution of (1) if

$$\lambda \in (a(0)\lambda_1, \infty).$$

Let us compare and interpret our results in the case of the existence of a refuge $\Omega_0$.

- In the local case, $D_2 = \emptyset$, the species tends to the extinction when the growth rate is small enough ($\lambda \leq \lambda_1$), the species survives for intermediate values of $\lambda$ and positive solutions do not exist for $\lambda$ large ($\lambda > \lambda_1^{\Omega_0}$). Indeed, the presence of a refuge causes the species to grow freely in it, and therefore it blows up for large growth rate.

- In the non-local pure case, $D_1 = \emptyset$, the species can coexist even for small growth rate if the diffusion coefficient is small ($a(\infty) = 0$) and for large $\lambda$ if the diffusion coefficient is large ($a(\infty) = \infty$). That is, when the growth rate is small the species survives if the diffusion speed is small when the total population is very large. On the other hand, the presence of a refuge and a large growth do not imply that the species blows up if the diffusion speed is high for large values of the total population.

- Assume now that $D_1 \neq \emptyset \neq D_2$, that is, we have local and non-local diffusion simultaneously. Then,
1. If the non-local coefficient is bounded above and below or the refuge is entirely contained in the region of local diffusion, the results are similar to the pure local case.

2. If the refuge is completely contained in the region of non-local diffusion, the results are similar to the pure non-local case.

3. Assume that the species has part of its refuge contained in the region where it diffuses in a random way. Then, for large growth rate the species does not exist because it blows up. In this case, a large diffusion speed can not avoid the disproportionate growth of population density.

Figure 1. Bifurcation diagrams when $\lambda_0 < \lambda_\infty < \infty$ and $\lambda_\infty < \lambda_0 < \infty$, respectively.

Figure 2. Bifurcation diagrams when $\lambda_\infty = 0$ and $\lambda_\infty = \infty$, respectively. For example, this last diagram appears when $b \geq b_0 > 0$.

The paper is organized as follows. First, in Section 2, we will introduce some notations and results necessary along the paper. Specifically, we study eigenvalue and logistic local equations. In Section 3, we apply the bifurcation method to (1). Section 4 is devoted to the case $b \equiv 0$. In Section 5 we study the structure of set of positive solutions of (1) proving Theorem 1.2.

2. Preliminary results. In this section we introduce some results that will be used throughout the rest of the paper.
2.1. Some local eigenvalue problems. We introduce some properties of the principal eigenvalue of the following problem
\[
\begin{aligned}
&-m_1(x)\Delta u + m_2(x)u = \lambda m_3(x)u \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
(3)
where \(m_1, m_2, m_3 \in L^\infty(\Omega)\), \(m_3 \geq 0\) and non-trivial in \(\Omega\) and there exists a positive constant \(m > 0\) such that \(m_1 \geq m > 0\) in \(\Omega\).

The following result provides us with the main properties of the principal eigenvalue of (3), see for instance [11].

**Proposition 1.** The eigenvalue problem (3) has a principal eigenvalue, \(\lambda \in \mathbb{R}\), denoted by
\[
\lambda_{1}^{\Omega}[-m_1(x)\Delta + m_2(x); m_3(x)],
\]
which has an associated eigenfunction \(\varphi \in W^{2,q}(\Omega)\), for all \(q > 1\), that can be chosen positive. Moreover, it is the only eigenvalue having a sign-defined eigenfunction. Furthermore, the following properties are satisfied:

1. The map \(m_2 \mapsto \lambda_{1}^{\Omega}[-m_1(x)\Delta + m_2(x); m_3(x)]\), from \(L^\infty(\Omega)\) into \(\mathbb{R}\), is continuous and increasing.
2. The map \(m_1 \mapsto \lambda_{1}^{\Omega}[-m_1(x)\Delta + m_2(x); m_3(x)]\), from \(L^\infty(\Omega)\) into \(\mathbb{R}\), is continuous. Moreover, if \(m_2 \leq 0\) in \(\Omega\), the map is also increasing.
3. The principal eigenvalue \(\lambda_{1}^{\Omega}[-m_1(x)\Delta + m_2(x); m_3(x)]\) is continuous and decreasing with respect to \(\Omega\) in the following sense:

(a) If \(\Omega_1 \subset \Omega_2\), then
\[
\lambda_{1}^{\Omega_2}[-m_1(x)\Delta + m_2(x); m_3(x)] < \lambda_{1}^{\Omega_1}[-m_1(x)\Delta + m_2(x); m_3(x)].
\]

(b) If we define the sets
\[
\Omega_\delta := \{ x \in \mathbb{R}^N : \text{dist}(x, \Omega) < \delta \}, \quad \delta > 0,
\]
then
\[
\lim_{\delta \to 0} \lambda_{1}^{\Omega_\delta}[-m_1(x)\Delta + m_2(x); m_3(x)] = \lambda_{1}^{\Omega}[-m_1(x)\Delta + m_2(x); m_3(x)].
\]

**Remark 1.** 1. When \(\Omega = \emptyset\) or \(m_3 \equiv 0\) in \(\Omega\), we write \(\lambda_{1}^{\Omega}[-m_1(x)\Delta + m_2(x); m_3(x)] = \infty\).
2. When no confusion arises, we write
\[
\lambda_{1}^{\Omega}[-m_1(x)\Delta + m_2(x); m_3(x)] = \lambda_1[-m_1(x)\Delta + m_2(x); m_3(x)].
\]
3. Observe that \(\lambda_1[-m_1(x)\Delta + m_2(x); m_3(x)]\) can be written of the following form
\[
\lambda_1[-m_1(x)\Delta + m_2(x); m_3(x)] = \lambda_1 \left[ -\Delta + \frac{m_2(x)}{m_1(x)} \right].
\]

Now, we study a particular problem of (3), specifically we consider
\[
m_1 = \chi_{D_1} + d\chi_{D_2}, \quad m_2 \equiv 0 \quad \text{and} \quad m_3 \equiv 1,
\]
that is the local eigenvalue problem
\[
\begin{aligned}
&- (\chi_{D_1} + d\chi_{D_2}) \Delta u = \lambda u \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
(4)
where \(d > 0\).

We define the functions:
\[
g(d) := \lambda_1 \left[ -\chi_{D_1} + d\chi_{D_2} \right], \quad g_0(d) := \lambda_{1}^{\Omega_0} \left[ -\chi_{D_1} + d\chi_{D_2} \right].
\]
The following result shows the behavior of the function \( g(d) \) with respect to \( d \).

**Proposition 2.**

1. The function \( g : (0, \infty) \to \mathbb{R}_+ \) defined in (5) is continuous and increasing.
2. \( \lim_{d \downarrow 0} g(d) = 0 \).
3. \( \lim_{d \uparrow \infty} g(d) = \lambda_1[\Delta; \chi_{D_1}] \).

**Proof.**

1. It follows from part 2 of Proposition 1 that \( g \) is continuous and increasing.
2. By definition and part 3 of Proposition 1, we get
   \[
   0 < g(d) = \lambda_1[\chi_{D_1} + d\chi_{D_2}] - d\Delta; 1] < \lambda_1^{D_2}[-d\Delta; 1] = d\lambda_1^{D_2}[-\Delta; 1].
   \]
   We can conclude that \( \lim_{d \downarrow 0} g(d) = 0 \).
3. Let \( \varphi_d \) be the positive eigenfunction associated to \( g(d) \), with \( \|\varphi_d\|_{H^1_0(\Omega)} = 1 \).

By the Sobolev embedding theorem, there exists \( \varphi^*_d \in H^1_0(\Omega), \varphi^*_d \geq 0, \varphi^*_d \neq 0 \) in \( \Omega \) such that

\[
\varphi_d \rightharpoonup \varphi^*_d \text{ in } H^1_0(\Omega),
\varphi_d \rightarrow \varphi^*_d \text{ in } L^2(\Omega) \text{ as } d \rightarrow \infty.
\]

Multiplying (4) by \( \varphi_d \), we obtain

\[
1 = \int_{\Omega} |\nabla \varphi_d|^2 = g(d) \int_{\Omega} \frac{\varphi^2_d}{\chi_{D_1} + d\chi_{D_2}} = g(d) \left[ \int_{D_1} \frac{\varphi^2_d}{d} + \frac{1}{d} \int_{D_2} \varphi^2_d \right].
\]

On the other hand, observe that

\[
0 < g(d) = \lambda_1[\chi_{D_1} + d\chi_{D_2}] - d\Delta; 1] < \lambda_1^{D_1}[-\Delta; 1],
\]
and hence, \( g(d) \) is bounded.

Now, if \( \int_{D_1} \varphi^2_d \rightarrow 0 \) as \( d \rightarrow \infty \), we get a contradiction with (6). Hence, \( \varphi^*_d > 0 \) in \( D_1 \). Then

\[
\int_{\Omega} \nabla \varphi_d \cdot \nabla \varphi = g(d) \left[ \int_{D_1} \varphi_d \varphi + \frac{1}{d} \int_{D_2} \varphi_d \varphi \right], \quad \forall \varphi \in H^1_0(\Omega).
\]

Now, as \( d \) tends to infinity, we obtain

\[
\int_{\Omega} \nabla \varphi^* \cdot \nabla \varphi = g(\infty) \int_{D_1} \varphi^* \varphi = g(\infty) \int_{\Omega} \chi_{D_1} \varphi^* \varphi, \quad \forall \varphi \in H^1_0(\Omega).
\]

Then, \( \varphi^* \) is weak solution of

\[
\begin{aligned}
-\Delta \varphi^* &= g(\infty)\chi_{D_1} \varphi^* \quad \text{in } \Omega, \\
\varphi^* &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
\]

Hence, we derive that \( g(\infty) = \lambda_1[\Delta; \chi_{D_1}] \).

\[\square\]

The following result determines the behaviour of \( g_0 \), that depends basically on the relative position of \( \Omega_0 \) with respect to the sets \( D_1 \) and \( D_2 \).

**Proposition 3.**

1. The function \( g_0 : (0, \infty) \to \mathbb{R}_+ \) defined in (5) is continuous and non-decreasing.
2. If \( \Omega_0 \cap D_2 \neq \emptyset \), then
   \[
   \lim_{d \downarrow 0} g_0(d) = 0.
   \]
3. If $\Omega_0 \cap D_2 = \emptyset$, then
   \[
   \lim_{d \downarrow 0} g_0(d) = \lambda_1^{\Omega_0} [-\Delta; 1].
   \]

4. If $\Omega_0 \cap D_1 \neq \emptyset$, then
   \[
   \lim_{d \uparrow \infty} g_0(d) = \lambda_1^{\Omega_0} [-\Delta; \chi_{D_1}].
   \]

5. If $\Omega_0 \cap D_1 = \emptyset$, then
   \[
   \lim_{d \uparrow \infty} g_0(d) = \infty.
   \]

Proof. 1. The first paragraph follows again from part 2 of Proposition 1.
2. Combining the definition of $g_0(d)$ with part 3 of Proposition 1, we get
   \[
   0 < g_0(d) = \lambda_1^{\Omega_0} [-\chi_{D_1} + d\chi_{D_2}]\Delta u = \lambda_1^{\Omega_0} [-\Delta; d\Delta] = d\lambda_1^{\Omega_0} [-\Delta; 1],
   \]
   whence the result follows directly.
3. Since $\Omega_0 \cap D_2 = \emptyset$, then $\Omega_0 \subseteq D_1$ and we get
   \[
   g_0(d) = \lambda_1^{\Omega_0} [-\Delta; 1],
   \]
   and hence we conclude the paragraph.
4. This paragraph follows exactly as paragraph 3 of Proposition 2.
5. Since $\Omega_0 \subseteq D_2$, we get
   \[
   g_0(d) = d\lambda_1^{\Omega_0} [-\Delta; 1],
   \]
   and the result follows easily.

\[\square\]

2.2. A local logistic problem. In this section we study the following local logistic problem
   \[
   \begin{cases}
   - (\chi_{D_1} + d\chi_{D_2}) \Delta u = \lambda u - b(x)u^2 & \text{in } \Omega, \\
   u = 0 & \text{on } \partial \Omega.
   \end{cases}
   \] (7)

Although equation (7) has been studied extensively in the last years, see for example [12], we include a result showing the existence and uniqueness of positive solution of (7) for the reader’s convenience and we include some necessary estimates.

Proposition 4. There exists at least a positive solution of (7) if and only if
   \[
   \lambda \in (g_0(d), g_0(d)).
   \] (8)

In this case, this is the unique positive solution of (7), and it will be denoted by $\theta_{[\lambda,d]}$.

Moreover, the map $\lambda \in (g_0(d), g_0(d)) \mapsto \theta_{[\lambda,d]} \in C^1(\Omega)$ is continuous, increasing and differentiable. Furthermore, we get that
   \[
   \lim_{\lambda \downarrow g_0(d)} \|\theta_{[\lambda,d]}\|_\infty = 0, \quad \lim_{\lambda \uparrow g_0(d)} \|\theta_{[\lambda,d]}\|_\infty = \infty.
   \]

In addition, if $\Omega_0 \subset \subset D_1$ and we fix $\bar{x} < g_0(d)$, then there exist a positive constant $K(\bar{x})$ and a positive regular function $\varphi$ (both independent of $d$), such that
   \[
   \theta_{[\lambda,d]} \leq K(\bar{x})\varphi \quad \text{for all } \lambda \leq \bar{x}.
   \]
\( g_0(d) = \lambda_1^{\Omega_0} \left[ -(\chi_{D_1} + d\chi_{D_2})\Delta; 1 \right] > \lambda = \lambda_1 \left[ -(\chi_{D_1} + d\chi_{D_2})\Delta + b(x)u; 1 \right] > \lambda_1 \left[ -(\chi_{D_1} + d\chi_{D_2})\Delta; 1 \right] = g(d). \)

(\( \Leftarrow \)) Let us take \( \lambda > g(d) \). Let \( \varphi_d \) be the positive eigenfunction associated to \( g(d) \) with \( \| \varphi_d \|_{\infty} = 1 \). We take as subsolution \( u = \epsilon \varphi_d \), with \( \epsilon \) small to be chosen later. Observe that,
\[-(\chi_{D_1} + d\chi_{D_2})\Delta u \leq \lambda u - b(x)u^2 \iff g(d)\varphi_d \leq \lambda \varphi_d - b(x)\epsilon \varphi_d^2 \iff \epsilon \leq \frac{\lambda - g(d)}{\max_{x \in \Omega} b(x)}.\]

Since \( \lambda > g(d) \), we can choose \( \epsilon > 0 \) such that the above inequality holds.

Take now \( \lambda < g_0(d) \). By the continuity of principal eigenvalue with respect to the domain, see paragraph 3 of Proposition 1, there exists \( \delta > 0 \) small, such that \( \Omega_0 \subset \Omega_{\delta} \) and \( \lambda < g_0(d) < g_0(d) \), where we define
\[ \Omega_{\delta} := \{ x \in \Omega, \text{dist}(x, \Omega_0) < \delta \}, \quad g_0(d) := \lambda_1^{\Omega_{\delta}} \left[ -(\chi_{D_1} + d\chi_{D_2})\Delta; 1 \right]. \]

We define \( \varphi_1^{\Omega_{\delta}} \) as a positive eigenfunction associated to \( \lambda_1^{\Omega_{\delta}} \left[ -(\chi_{D_1} + d\chi_{D_2})\Delta; 1 \right] \).

Now, define the regular function \( \psi \) by
\[ \psi = \begin{cases} \varphi_1^{\Omega_{\delta}} & \text{in } \Omega_{\delta}, \\ \phi & \text{in } \Omega_\delta \setminus \overline{\Omega}_{\delta}, \\ 1 & \text{in } \Omega \setminus \overline{\Omega}_{\delta}, \end{cases} \]

where \( \phi \) is a regular function such that \( \phi \geq \phi_0 > 0 \) in \( \overline{\Omega} \) for some positive constant \( \phi_0 > 0 \). We take as supersolution \( \bar{u} = K\psi \) with \( K > 0 \) to be chosen later. Let us see that \( \bar{u} \) is a supersolution of (7) as \( K \) large. We divide into three situations:

1. In \( \Omega_{\frac{\delta}{2}} \): We get
\[-(\chi_{D_1} + d\chi_{D_2})\Delta \bar{u} \geq \lambda \bar{u} - b(x)\bar{u}^2 \iff g_0(d) - \lambda \geq -b(x)K \varphi_1^{\Omega_{\delta}}.\]

This inequality is true for all \( K > 0 \), because \( \lambda < g_0(d) \).

2. In \( \Omega_\delta \setminus \overline{\Omega}_{\delta} \): We get
\[-(\chi_{D_1} + d\chi_{D_2})\Delta \bar{u} \geq \lambda \bar{u} - b(x)\bar{u}^2 \Leftrightarrow K \geq \frac{\max_{x \in \Omega_\delta \setminus \overline{\Omega}_{\delta}} \left[ \lambda \phi + (\chi_{D_1} + d\chi_{D_2})\Delta \phi \right]}{\min_{x \in \Omega_\delta \setminus \overline{\Omega}_{\delta}} (b(x)\bar{u}^2)}. \quad (9)\]

3. In \( \Omega \setminus \overline{\Omega}_{\delta} \): We get
\[-(\chi_{D_1} + d\chi_{D_2})\Delta \bar{u} \geq \lambda \bar{u} - b(x)\bar{u}^2 \Leftrightarrow 0 \geq -b(x)K \Leftrightarrow K \geq \frac{\lambda}{\min_{\Omega \setminus \Omega_0} b(x)}. \quad (10)\]

Then, we can take \( K \) large enough verifying (9) and (10). Moreover, we can choose \( \epsilon \) small and \( K \) large so that \( \bar{u} \leq \bar{u} \) in \( \Omega \), and we obtain the existence of positive solution of (7).

Since the function
\[ t \mapsto \frac{\lambda - b(x)t}{\chi_{D_1} + d\chi_{D_2}} \]
is non-increasing for all \( x \in \Omega \) and decreasing in \( \Omega \setminus \Omega_0 \), then by [4] we get the uniqueness of positive solution of (7).

The properties of the map \( \lambda \mapsto \theta_{[\lambda,d]} \) follow from Theorem 4.1 in [12].
Assume now that \( \Omega_0 \subset D_1 \) and \( \bar{x} < g_0(d) \), then we can take \( \delta > 0 \) small enough such that
\[
\Omega_0 \subset \Omega_\delta \subset D_1 \quad \text{and} \quad \bar{x} < g_\delta(d),
\]
and then \( g_\delta(d) = \lambda_1^{\Omega_\delta} \left[ - (\chi_D_1 + d\chi_{D_2}) \Delta; 1 \right] = \lambda_1^{\Omega_\delta} \left[ - \Delta; 1 \right] \). Hence, \( \varphi_1^{\Omega_\delta} \) is independent of \( d \). Moreover, if \( \lambda \leq \bar{x} \), the constant \( K \) verifying (9) and (10) must now verify
\[
K \geq \frac{\max_{x \in \Omega_\delta \setminus \Omega_4} \left[ \overline{x} \phi + \chi_D \Delta \phi \right]}{\min_{x \in \Omega_\delta \setminus \Omega_4} (b(x)\phi^2)},
\]
and
\[
K \geq \frac{\overline{x}}{\min_{\Omega \setminus \Omega_4} b(x)},
\]
and then \( K \) is independent of \( d \) and \( \lambda \).

This completes the proof. \( \square \)

In order to study the non-local equation (1), we need to analyze (7) as \( d \to \infty \).

This behaviour is related to the limit equation
\[
\begin{cases}
-\Delta u = \chi_D_1 u(\lambda - b(x)u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega. 
\end{cases}
\]
(11)

We recall that \( g(d) \uparrow \lambda_1 \left[ - \Delta; \chi_D_1 \right] \) as \( d \to \infty \) and \( g_0(d) \uparrow \lambda_1^{\Omega_0} \left[ - \Delta; \chi_D_1 \right] \) or \( g_0(d) \to +\infty \) as \( d \to \infty \) depending on \( \Omega_0 \cap D_1 \neq \emptyset \) or \( \Omega_0 \cap D_1 = \emptyset \).

**Proposition 5.**

1. Assume that \( \lambda < \lambda_1 \left[ - \Delta; \chi_D_1 \right] \), then there exists \( d(\lambda) > 0 \) such that
\[
\theta_{[\lambda, d]}(x) = 0 \quad \text{in } \Omega, \quad \text{for all } d \geq d(\lambda).
\]

2. Assume that \( \lambda \in K \subset \left[ \lambda_1 \left[ - \Delta; \chi_D_1 \right], \lambda_1^{\Omega_0} \right] \), \( \lambda_1^{\Omega_0} \left[ - \Delta; \chi_D_1 \right] \), \( K \) a compact set.
Then, there exist two positive constants \( C \) and \( d(K) \) such that
\[
\| \theta_{[\lambda, d]} \|_{\infty} \leq C \quad \text{for all } d \geq d(K).
\]

3. Assume that \( \lambda_1^{\Omega_0} \left[ - \Delta; \chi_D_1 \right] < \lambda \). Then,
\[
\lim_{d \to \infty} \| \theta_{[\lambda, d]} \|_{\infty} = \infty.
\]

**Proof.**

1. Take \( \lambda < \lambda_1 \left[ - \Delta; \chi_D_1 \right] \). By Proposition 2 there exists \( d(\lambda) > 0 \) such that
\[
\lambda < g(d) \quad \text{for } d > d(\lambda).
\]
We conclude that \( \theta_{[\lambda, d]}(x) = 0 \) in \( \Omega \) by Proposition 4.

2. Take \( \lambda \in K \subset \left[ \lambda_1 \left[ - \Delta; \chi_D_1 \right], \lambda_1^{\Omega_0} \right] \) and then \( \lambda < \lambda_1^{\Omega_0} \left[ - \Delta; \chi_D_1 \right] \).
Define the set
\[
\Omega_\delta := \{ x \in \mathbb{R}^N; dist(x, \Omega_0 \cup D_2) \} < \delta,
\]
and take \( \delta \) small enough such that \( \lambda < \lambda_1^{\Omega_\delta} \left[ - \Delta; \chi_D_1 \right] \), or equivalently,
\[
\mu_\delta(\lambda) := \lambda_1^{\Omega_\delta} \left[ - \Delta - \lambda \chi_D_1; 1 \right] > 0.
\]
(12)
Take \( \varphi_1^\delta \) the positive eigenfunction associated to \( \mu_\delta(\lambda) \) with \( \| \varphi_1^\delta \|_\infty = 1 \) and choose \( \phi \) a regular function such that \( \phi \geq \phi_0 > 0 \) in \( \Omega \) and such that the function \( \psi \) defined by

\[
\psi = \begin{cases} 
\varphi_1^\delta & \text{in } \Omega_1^\delta, \\
\phi & \text{in } \Omega_\delta \setminus \Omega_1^\delta, \\
1 & \text{in } \Omega \setminus \Omega_\delta,
\end{cases}
\]

will be a regular function.

We take as supersolution \( \pi = K \psi \) with \( K > 0 \) to be chosen later. Let us see that \( \pi \) is a supersolution of (7) for \( K \) large. Indeed, again we divide into three situations:

(a) In \( \Omega_2^\delta \): In this case, \( \pi \) is a supersolution of (7) provided that

\[
(\chi_{D_1} + d\chi_{D_2})(\mu_\delta(\lambda) + \lambda\chi_{D_1}) \geq \lambda - b(x)K\varphi_1^\delta \quad \text{in } \Omega_2^\delta.
\]

This inequality is equivalent in \( D_1 \) to

\[
\mu_\delta(\lambda) \geq -b(x)K\varphi_1^\delta,
\]

which is true for all \( K \) by (12). On the other hand, in \( D_2 \) it is sufficient that

\[
d\mu_\delta(\lambda) \geq \lambda,
\]

or equivalently

\[
d \geq \frac{\lambda}{\mu_\delta(\lambda)}.
\]

Observe that for \( \lambda \in \mathcal{K} \), there exists \( d = d(\mathcal{K}) \) such that

\[
\max_{\lambda \in \mathcal{K}} \frac{\lambda}{\mu_\delta(\lambda)} = d(\mathcal{K}).
\]

Hence, taking \( d \geq d(\mathcal{K}) \) we obtain that \( \pi \) is supersolution in \( \Omega_2^\delta \) for any \( K > 0 \).

(b) In \( \Omega_\delta \setminus \Omega_2^\delta \): We get

\[
-(\chi_{D_1} + d\chi_{D_2})\Delta \phi \geq \lambda \phi - K b(x)\phi^2.
\]

By construction, \( \Omega_\delta \setminus \Omega_2^\delta \subset D_1 \), and hence (15) is verified if

\[
K b(x)\phi^2 \geq \lambda \phi + \Delta \phi,
\]

which is true for \( K \) large, independent to \( d \).

(c) In \( \Omega \setminus \Omega_\delta \): \( \pi \) is supersolution provided

\[
K \geq \frac{\lambda}{\min_{\Omega \setminus \Omega_\delta} b(x)},
\]

again a constant independent to \( d \).

This completes the proof of the second paragraph.

3. Take now \( \lambda > \lambda_1^{\Omega_\delta \cup D_2}[\Omega \setminus \chi_{D_1}, \chi_{D_1}] \). We are going to build a subsolution of (7) growing up to infinity as \( d \to \infty \). Since \( \lambda > \lambda_1^{\Omega_\delta \cup D_2}[\Omega \setminus \chi_{D_1}, \chi_{D_1}] \) we have that

\[
\mu(\lambda) = \lambda_1^{\Omega_\delta \cup D_2}[\Omega \setminus \chi_{D_1}, \chi_{D_1}] < 0.
\]
Denote by $\varphi_1$ the positive eigenfunction associated to $\mu(\lambda)$ such that $\|\varphi_1\|_\infty = 1$. Define
\[
\psi := \begin{cases} 
\varphi_1 & \text{in } \Omega_0 \cup D_2, \\
0 & \text{in } \Omega \setminus (\Omega_0 \cup D_2).
\end{cases}
\]
Observe that $\psi \in H^1_0(\Omega)$. Then, $u := \epsilon \psi$ is subsolution of (7) if
\[
(\chi_{D_1} + d\chi_{D_2})(\mu(\lambda) + \lambda \chi_{D_1}) + b(x)\epsilon \varphi_1 \leq \lambda \quad \text{in } \Omega_0 \cup D_2.
\]
In $D_2$, it is sufficient that $\epsilon = \lambda - d\mu(\lambda) b_M$, where $b_M = \sup_{x \in D_2} b(x)$.
On the other hand, in $\Omega_0 \setminus D_2$, it suffices that $\mu(\lambda) \leq 0$, which is true for any $\epsilon > 0$. Then,
\[
\epsilon \psi \leq \theta_{[\lambda, d]} \quad \text{in } \Omega,
\]
in particular,
\[
\frac{\lambda - d\mu(\lambda)}{b_M} \varphi_1 \leq \theta_{[\lambda, d]} \quad \text{in } \Omega_0 \cup D_2,
\]
and hence $\|\theta_{[\lambda, d]}\|_\infty \to \infty$ as $d \to \infty$.

This result is obviously true for the case $\Omega = \Omega_0 \cup D_2$. In this case, the result reads:

**Corollary 2.** Assume that $\Omega = \Omega_0 \cup D_2$. Then,
\begin{enumerate}
\item If $\lambda < \lambda_1[-\Delta; \chi_{D_1}]$ there exists $d(\lambda) > 0$ such that $\theta_{[\lambda, d]} \equiv 0$ in $\Omega$ for $d \geq d(\lambda)$.
\item If $\lambda > \lambda_1[-\Delta; \chi_{D_1}]$ we have that $\|\theta_{[\lambda, d]}\|_\infty \to \infty$ as $d \to \infty$.
\end{enumerate}

3. **Bifurcation results.** In the current section we apply the bifurcation theory to prove the existence of an unbounded continuum of positive solutions of (1).

We say that $\lambda_0 \in \mathbb{R}$ is a bifurcation point from the trivial solution of (1) if there exists a sequence $(\lambda_n, u_n)$ of positive solutions of (1) such that $\lambda_n \to \lambda_0$ and $\|u_n\|_\infty \to 0$. Analogously, we say that $\lambda_\infty \in \mathbb{R}$ is a bifurcation point from infinity if there exists a sequence $(\lambda_n, u_n)$ of positive solutions of (1) such that $\lambda_n \to \lambda_\infty$ and $\|u_n\|_\infty \to \infty$.

In order to write (1) as a fixed point equation, we introduce the operator $\mathcal{L} : L^\infty(\Omega) \to L^\infty(\Omega)$, defined by
\[
\mathcal{L}(f) := u,
\]
where $u$ is the unique solution of the linear elliptic equation
\[
\begin{cases} 
-\Delta u = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

The following result is well-known.

**Lemma 3.1.** The operator $\mathcal{L}$ is compact and strictly positive. Also, if $f \in L^\infty(\Omega)$, then there exists a positive constant $C > 0$ such that
\[
\|\mathcal{L}f\|_\infty \leq C\|f\|_\infty.
\]

In the following result, we prove that the unique bifurcation point of (1) from the trivial solution is $\lambda = \lambda_1[-A(x, 0)\Delta; 1]$. For convenience, we denote
\[
\lambda_0 := \lambda_1[-A(x, 0)\Delta; 1] = \lambda_1[-(\chi_{D_1} + a(0)\chi_{D_2})\Delta; 1].
\]
Lemma 3.2. Let \((\lambda_n, u_n)\) be a sequence of positive solutions of (1). If \(\|u_n\|_\infty \to 0\), then \(\lambda_n \to \lambda_0\).

Proof. Since \(\|u_n\|_\infty \to 0\), then \(\int_\Omega u_n^p \to 0\). Moreover, by the continuity of function \(a\), we get
\[
A(x, u_n) \to A(x, 0) \text{ uniformly in } \Omega.
\]
Hence, by the continuity of the principal eigenvalue, see Proposition 1,
\[
\lambda_n = \lambda_1[-A(x, u_n)\Delta + b(x)u_n; 1] \to \lambda_0.
\]

Next, we prove the existence of an unbounded continuum of positive solutions of (1) that bifurcates from the trivial solution \(u \equiv 0\) at \(\lambda = \lambda_0\).

Theorem 3.3. There exists an unbounded continuum \(C\) in \(\mathbb{R} \times C(\Omega)\) of positive solutions of (1) emanating from \((\lambda, u) = (\lambda_0, 0)\).

Proof. Observe that (1) is equivalent to
\[
u = T_{\lambda}(u) := \frac{\lambda}{A(x, 0)} Lu + h(\lambda, u),
\]
where
\[
h(\lambda, u) = \begin{cases} 0 & \text{if } u \leq 0, \\ \lambda \mathcal{L} \left( \left( \frac{1}{A(x, u)} - \frac{1}{A(x, 0)} \right) u \right) - \mathcal{L} \left( \frac{b(x)}{A(x, u)} u^2 \right) & \text{if } u > 0,
\end{cases}
\]
and \(u^+ = \max\{u, 0\}\). Indeed, observe that \(u\) is a non-negative and non-trivial solution of (1) if and only if \(u = T_{\lambda}(u)\).

By Lemma 3.1 and the continuity of function \(a\), it yields that
\[
\frac{\|h(\lambda, u)\|_\infty}{\|u\|_\infty} \leq \left\| \lambda \mathcal{L} \left( \left( \frac{1}{A(x, u^+)} - \frac{1}{A(x, 0)} \right) u \right) \right\|_\infty \frac{1}{\|u\|_\infty} + \left\| \mathcal{L} \left( \frac{b(x)}{A(x, u^+)} u^2 \right) \right\|_\infty \frac{1}{\|u\|_\infty} \leq C \left( \frac{\lambda}{A(x, u^+)} - \frac{1}{A(x, 0)} \right) \frac{\|b(x)\|_\infty}{\|u\|_\infty} \|u\|_\infty \to 0,
\]
when \(\|u\|_\infty \to 0\). Hence, we can apply [15] and conclude that there exists a connected component \(C\) of positive solutions that emanates from \((\lambda_0, 0)\). By Lemma 3.2, we can conclude the \(C\) is unbounded.

Observe that, by the strong maximum principle, any non-negative and non-trivial solution of (1) is in fact positive in all \(\Omega\). On the other hand, by elliptic regularity, a bounded \(u\) solution of (1) belongs to \(W^{2,q}(\Omega)\) for all \(q > 1\), and hence \(u \in C^1(\Omega)\).

4. A nonlocal eigenvalue problem. In this section we study the existence of positive solution of
\[
\begin{cases}
-\Delta u = \lambda u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Proposition 6. If \(\lambda \leq 0\) then (17) does not possess positive solutions.
Proof. Let $u$ be a positive solution of (17). We suppose that $\lambda \leq 0$, then
\[-A(x,u)\Delta u = \lambda u \leq 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega.\]
A straightforward application of the Maximum Principle entails that $u \leq 0$ in $\Omega$, a contradiction. \hfill \Box

First we have the following result regarding the non-existence of positive solution of (17) for $\lambda$ large.

**Proposition 7.** If there exist positive solutions of (17), then $\lambda < \lambda_1^{D_1}[-\Delta;1]$.

**Proof.** If $u$ is a positive solution of (17), then $\lambda = \lambda_1[-A(x,u)\Delta;1] < \lambda_1^{D_1}[-A(x,u)\Delta;1] = \lambda_1^{D_1}[-\Delta;1]$. \hfill \Box

In the next result, we show that $\int_{\Omega} u^p$ is unbounded when $\|u\|_{\infty} \to \infty$.

**Proposition 8.** Let $(\lambda_n, u_n)$ be a sequence of positive solutions of (17). If $\|u_n\|_{\infty} \to \infty$, then
\[\int_{\Omega} u_n^p \to \infty \quad \text{for all } p > 0.\]

**Proof.** We assume by contradiction that $\int_{\Omega} u_n^p \leq C$ for some positive constant $C$, then
\[A(x,u_n) \leq C_1. \tag{18}\]
We define
\[v_n = \frac{u_n}{\|u_n\|_{\infty}}.\]
Consequently, $v_n$ is solution of
\[-\Delta v_n = \frac{\lambda_n}{A(x,u_n)} v_n \text{ in } \Omega, \quad v_n = 0 \text{ on } \partial \Omega.\]
By Proposition 7, we get that $\lambda_n < \lambda_1^{D_1}[-\Delta;1]$. Hence, taking advantage of (18) we can conclude that $\|v_n\|_{W^{2,q}} \leq C_2$ for all $q > 1$. Hence, there exists $v \in C^1(\overline{\Omega})$ such that $v_n \to v_\ast \geq 0$ in $C^1(\overline{\Omega})$.

Since,
\[\int_{\Omega} u_n^p = \|u_n\|_{\infty}^p \int_{\Omega} v_n^p \to \infty.\]
We arrive at a contradiction. \hfill \Box

Now, we are ready to prove the main result related to (17):
4.1. **Proof of Theorem 1.1:** By Theorem 3.3, there exists an unbounded connected component $C$ of positive solutions of (4) that emanates from $(\lambda, u) = (\lambda_0, 0)$. By Propositions 6 and 7, we obtain that

$$\text{Proj}_{\mathbb{R}}(C) \subset [0, \lambda_1^{D_1}[-\Delta; 1]].$$

Hence, there exists a sequence $(\lambda_n, u_n)$ of positive solutions of (4) with $\lambda_n < \lambda_1^{D_1}[-\Delta; 1]$ and there exists $\lambda^* \in [0, \lambda_1^{D_1}[-\Delta; 1]]$ such that $\lambda_n \to \lambda^* < \infty$ and $\|u_n\|_{\infty} \to \infty$.

Observe that

$$\lambda_n = \lambda_1 \left[ - \left( \chi_{D_1} + a \left( \int_{\Omega} u_n^p \right) \chi_{D_2} \right) \Delta; 1 \right],$$

and then, denoting by

$$d_n = a \left( \int_{\Omega} u_n^p \right),$$

we get that

$$\lambda_n = g(d_n),$$

where the function $g$ is defined in (5).

Now, by Propositions 8 we have that $\int_{\Omega} u_n^p \to \infty$ and then $d_n \to a(\infty)$. Then, the result follows by Proposition 2.

Finally, observe that $\lambda_0 = g(a(0)) < \lim_{d \to \infty} g(d) = \lambda_1[-\Delta; \chi_{D_1}]$. This completes the proof.

5. **A nonlocal logistic problem.** In this section our aim is to study the existence of positive solution of the nonlocal logistic problem (1).

The first result is an easy consequence of Proposition 4.

**Proposition 9.** If there exists a positive solution of (1), then

$$\lambda < \lambda_1^{\Omega_0}[-\Delta; \chi_{D_1}] \quad \text{and} \quad \lambda < \lambda_1^{\Omega_0}[-(\chi_{D_1} + a_M \chi_{D_2}) \Delta; 1],$$

where $a_M := \sup_{s \in [0, \infty)} a(s)$.

**Proof.** Let $u$ be a positive solution of (1), then

$$\lambda = \lambda_1 [-A(x, u) \Delta + b(x)u; 1] < \lambda_1^{\Omega_0}[-A(x, u) \Delta; 1] = g_0 \left( \int_{\Omega} u^p \right) < \lim_{d \to \infty} g_0(d) = \lambda_1^{\Omega_0}[-\Delta; \chi_{D_1}].$$

On the other hand,

$$\lambda = \lambda_1 \left[ - \left( \chi_{D_1} + a \left( \int_{\Omega} u^p \right) \chi_{D_2} \right) \Delta + b(x)u; 1 \right] < \lambda_1^{\Omega_0}[-(\chi_{D_1} + a_M \chi_{D_2}) \Delta; 1].$$

**Remark 2.** Observe that $\lambda_1^{\Omega_0}[-\Delta; \chi_{D_1}] = \infty$ if $\Omega_0 \subset D_2$. On the other hand, if $a(\infty) = \infty$, then $a_M = \infty$. Hence, $\lambda_1^{\Omega_0}[-(\chi_{D_1} + a_M \chi_{D_2}) \Delta; 1] = \lambda_1^{\Omega_0}[-\Delta; \chi_{D_1}]$.

In the following result, we prove that $\int_{\Omega} u^p$ is unbounded when $\|u\|_{\infty}$ is unbounded, for that, we mainly use a bootstrap argument.
Lemma 5.1. Let \( p > 1 \). If there exists \((\lambda_n, u_n)\) a sequence of positive solutions of (1), such that \( \|u_n\|_{\infty} \to \infty \) and \( \lambda_n \to \lambda^* < \infty \), then

\[
\int_{\Omega} u_n^p \to \infty.
\]

Proof. We assume that

\[
\int_{\Omega} u_n^p < \infty,
\]  

(19)

hence \( A(x, u_n) \) is bounded. We multiply (1) by \( u_n^{p-1} \) and we obtain

\[
\int_{\Omega} \nabla u_n \cdot \nabla u_n^{p-1} = \int_{\Omega} \frac{\lambda_n u_n^p - b(x)u_n^{p+1}}{\chi_{D_1} + a \left( \int_{\Omega} u_n^p \right) \chi_{D_2}}.
\]

Hence,

\[
C(p) \int_{\Omega} |\nabla u_n^\frac{p}{2}|^2 = \int_{\Omega} \frac{\lambda_n u_n^p - b(x)u_n^{p+1}}{\chi_{D_1} + a \left( \int_{\Omega} u_n^p \right) \chi_{D_2}} \leq \lambda_n \int_{\Omega} \frac{u_n^p}{\chi_{D_1} + a \left( \int_{\Omega} u_n^p \right) \chi_{D_2}},
\]

where

\[
C(p) = \frac{4(p-1)}{p^2}.
\]

According to (19), we get

\[
u_n^\frac{p}{2}\]

is bounded in \( H^1_0(\Omega) \).

By continuous imbedding, we have that

\[ u_n \]

is bounded in \( L^{2t_1}(\Omega) = L^{t_1}(\Omega) \),

where

\[ t_1 = \frac{N}{N-2} p. \]

Now, we multiply (1) by \( u_n^{t_1-1} \) and then we get

\[
C(t_1) \int_{\Omega} |\nabla u_n^\frac{p}{t_1}|^2 \leq C \int_{\Omega} u_n^{t_1},
\]

where \( C(t_1) \) and \( C \) are positive constants and thus, \( u_n^{t_1} \) is bounded in \( H^1_0(\Omega) \). By continuous imbedding, we have that

\[ u_n \]

is bounded in \( L^{t_2}(\Omega) \),

with

\[ t_2 = \left( \frac{N}{N-2} \right)^2 p. \]

Repeating this argument \( k \) times, we obtain that

\[ u_n \]

is bounded in \( L^{t_k}(\Omega) \),

with

\[ t_k = \left( \frac{N}{N-2} \right)^k p. \]

Hence,

\[ u_n^2 \]

is bounded in \( L^{t_k}(\Omega) \),
and therefore,
\[
\lambda_n u_n - b(x)u_n^2, \quad \chi_{D_1} + a \left( \int_{\Omega} u_n^p \right) \chi_{D_2}
\]
is bounded in \( L^{\frac{t_k}{2}}(\Omega) \),
and by elliptic regularity, we obtain that
\[
u_n \in W^{2, \frac{t_k}{2}}(\Omega).
\]
Taking \( k \) large such that \( t_k/2 > N/2 \), that is,
\[
\left( \frac{N}{N - 2} \right)^k > \frac{N}{p},
\]
we have that \( \|u_n\|_\infty \leq C \), an absurdum.

The next result characterizes, in some particular cases, the possible bifurcation points from infinity. The argument of the proof is rather similar to Lemma 2.4 in [2].

**Lemma 5.2.** Assume \( a(\infty) > 0 \) and \( p > 1 \). Assume that there exists a sequence of positive solutions \((\lambda_n, u_n)\) of (1), such that \( \|u_n\|_\infty \to \infty \) and \( \lambda_n \to \lambda^* < \infty \).

1. If \( a(\infty) < \infty \), then
\[
\lambda^* = \chi^\Omega_1 \left[ - (\chi_{D_1} + a(\infty)\chi_{D_2})\Delta; 1 \right].
\]
2. If \( \Omega \setminus \Omega_0 \subset D_1 \) and \( a(\infty) = \infty \), then
\[
\lambda^* = \chi^\Omega_1 \left[ - \Delta; \chi_{D_1} \right].
\]
3. If \( p \geq 2 \), \( a(\infty) = \infty \) and there exists a positive constant \( C \) such that
\[
a(s) \leq C, \quad s \geq 1 \]
then
\[
\lambda^* = \chi^\Omega_1 \left[ - \Delta; \chi_{D_1} \right].
\]

**Proof.** First, by Lemma 5.1 we have that \( \int_{\Omega} u_n^p \to \infty \). By the continuity of the function \( a \), we get
\[
a \left( \int_{\Omega} u_n^p \right) \to a(\infty).
\]
By a similar argument to the used in Lemma 5.1, we can conclude that
\[
\int_{\Omega} u_n^2 \to \infty,
\]
Define
\[
z_n = \frac{u_n}{\|u_n\|_2}.
\]
We multiply (1) by \(z_n\) and since \(a(\infty) > 0\), we obtain
\[
\int_{\Omega} |\nabla z_n|^2 = \lambda_n \int_{\Omega} \left( \chi_{D_1} + \frac{1}{a(\int_{\Omega} u_n^p)} \chi_{D_2} \right) z_n^2 - \\
- \int_{\Omega} b(x) \left( \chi_{D_1} + \frac{1}{a(\int_{\Omega} u_n^p)} \chi_{D_2} \right) z_n^3 \|u_n\|_2
\]
\[
\leq \lambda_n \int_{\Omega} \left( \chi_{D_1} + \frac{1}{a(\int_{\Omega} u_n^p)} \chi_{D_2} \right) z_n^2 \leq C \int_{\Omega} z_n^2 = C.
\]
Hence, there exists \(z \in H_0^1(\Omega), z \geq 0\) and \(z \neq 0\) such that
\[
z_n \rightharpoonup z \quad \text{in} \quad H_0^1(\Omega),
\]
\[
z_n \rightarrow z \quad \text{in} \quad L^s(\Omega), \forall s \in \left[1, 2^* = \frac{2N}{N-2}\right).
\]
Assume 1. or 2. We claim that
\[
z \equiv 0 \quad \text{in} \quad \Omega \setminus \Omega_0.
\] (20)
We argue by contradiction. Assume that there exists \(D \subset \Omega \setminus \Omega_0\) such that \(z > 0\) in \(D\). Take \(\psi \in C_0^\infty(D)\), then we get
\[
\int_D \nabla z_n \cdot \nabla \psi = \lambda_n \int_D \left( \chi_{D_1} + \frac{1}{a(\int_{\Omega} u_n^p)} \chi_{D_2} \right) z_n \psi
\]
\[
- \int_D b(x) \left( \chi_{D_1} + \frac{1}{a(\int_{\Omega} u_n^p)} \chi_{D_2} \right) z_n^2 \psi \|u_n\|_2.
\] (21)
Observe that since \(z > 0\) in \(D\) we have that \(u_n(x) = z_n(x)\|u_n\|_2 \rightarrow \infty\) and hence, if \(a(\infty) < \infty\) we get
\[
\int_D b(x) \left( \chi_{D_1} + \frac{1}{a(\int_{\Omega} u_n^p)} \chi_{D_2} \right) z_n^2 \psi \|u_n\|_2 \rightarrow \infty.
\]
On the other hand, if \(\Omega \setminus \Omega_0 \subset D_1\), we have that \(D \subset D_1\) and hence
\[
\int_D b(x) \left( \chi_{D_1} + \frac{1}{a(\int_{\Omega} u_n^p)} \chi_{D_2} \right) z_n^2 \psi \|u_n\|_2 > \int_D b(x) \chi_{D_1} z_n^2 \psi \|u_n\|_2 \rightarrow \infty.
\]
Consequently, in both cases we deduce that
\[
\int_D \nabla z_n \cdot \nabla \psi \rightarrow -\infty.
\]
We arrive at a contradiction. Therefore by virtue of (20) we have that \( z \in H^1_0(\Omega_0) \).
Hence, we obtain
\[
\int_{\Omega_0} \nabla z_n \cdot \nabla \varphi = \lambda_n \int_{\Omega_0} \left( \chi_{D_1} + \frac{1}{a \left( \int_{\Omega} u_n^p \right)} \chi_{D_2} \right) z_n \varphi \quad \forall \varphi \in H^1_0(\Omega_0).
\] (22)

When \( a(\infty) < \infty \), we take the limit in (22) and obtain
\[
\int_{\Omega_0} \nabla z \cdot \nabla \varphi = \lambda^* \int_{\Omega_0} \left( \chi_{D_1} + \frac{1}{a(\infty)} \chi_{D_2} \right) z \varphi \quad \forall \varphi \in H^1_0(\Omega_0),
\]
which means that
\[
\lambda^* = \lambda^{1\Omega_0} \left[ - \Delta; \chi_{D_1} \right].
\]

On the other hand, if \( a(\infty) = \infty \), taking limit in (22) we arrive at
\[
\int_{\Omega_0} \nabla z \cdot \nabla \varphi = \lambda^* \int_{\Omega_0} \left( \chi_{D_1} + \frac{1}{a(\infty)} \chi_{D_2} \right) z \varphi \quad \forall \varphi \in H^1_0(\Omega_0),
\]
whence we deduce that
\[
\lambda^* = \lambda^{1\Omega_0} \left[ - \Delta; \chi_{D_1} \right].
\]

Now, we assume 3. In this case, we can prove that \( z \equiv 0 \) in \( \Omega \setminus (\Omega_0 \cup D_2) \).
Indeed, if \( z > 0 \) in \( D \subset \Omega \setminus (\Omega_0 \cup D_2) \), then
\[
\int_D b(x) \left( \chi_{D_1} + \frac{1}{a \left( \int_{\Omega} u_n^p \right)} \chi_{D_2} \right) z_n^2 \psi \| u_n \|_2 = \int_D b(x) \chi_{D_1} z_n^2 \psi \| u_n \|_2 \to \infty.
\]
Again, a contradiction. Then, \( z \in H^1_0(\Omega_0 \cup D_2) \). Take \( \psi \in H^1_0(\Omega_0 \cup D_2) \), then we get
\[
\int_{\Omega_0 \cup D_2} \nabla z_n \cdot \nabla \psi = \lambda_n \int_{\Omega_0 \cup D_2} \left( \chi_{D_1} + \frac{1}{a \left( \int_{\Omega} u_n^p \right)} \chi_{D_2} \right) z_n \psi
\]
\[
- \int_{D_2} b(x) \left( \frac{1}{a \left( \int_{\Omega} u_n^p \right)} \right) z_n^2 \psi \| u_n \|_2.
\] (23)

Since \( p \geq 2 \) and \( a(s)/s^{1/p} \geq C \) for large \( s \), we have that
\[
\frac{\| u_n \|_2}{a \left( \int_{\Omega} u_n^p \right)} \leq C \frac{\| u_n \|_p}{a \left( \int_{\Omega} u_n^p \right)} \leq C.
\]
Hence, since \( z_n \to 0 \) in \( \Omega_0 \cup D_2 \), we conclude that
\[
\int_{D_2} b(x) \left( \frac{1}{a \left( \int_{\Omega} u_n^p \right)} \right) z_n^2 \psi \| u_n \|_2 \to 0.
\]
Taking limit in (23) we deduce that \( \lambda^* = \lambda^{1\Omega_0 \cup D_2} \left[ - \Delta; \chi_{D_1} \right]. \) \( \square \)
5.1. **Proof of Theorem 1.2.** We know that from \((\lambda, u) = (\lambda_0, 0)\) emanates an unbounded continuum \(C\) of positive solutions of \((1)\).

1. **Case 0 \(<\ a(\infty) \<\ \infty\):**

   In this case, the results do not depend on the position of the sets \(\Omega_0, D_1\) and \(D_2\). Indeed, by Proposition 9, \(\text{Proj}_{\mathbb{R}}(C)\) is bounded in \(\mathbb{R}\). Hence, there exists a sequence of positive solutions \((\lambda_n, u_n) \in C\) of \((1)\) such that \(\lambda_n \to \lambda_\infty < \infty\) and \(\|u_n\|_\infty \to \infty\). By Lemma 5.2, we know that
   \[
   \lambda_\infty = \lambda_1^{\Omega_0\Omega}[-(\chi_{D_1} + a(\infty)\chi_{D_2})\Delta; 1],
   \]

2. **Case \(\Omega_0 \subset\subset D_1\):**

   In this case, the result does not depend on the value of \(a(\infty)\). Indeed, by Proposition 9, since \(\Omega_0 \cap D_1 \neq \emptyset\), we get that \(\text{Proj}_{\mathbb{R}}(C)\) is bounded in \(\mathbb{R}\), and hence there exists a sequence \((\lambda_n, u_n) \in C\) of positive solutions of \((1)\) such that \(\lambda_n \to \lambda_\infty < \infty\) and \(\|u_n\|_\infty \to \infty\).

   Denote by
   \[
   d_n = a \left( \int_{\Omega} u_n^p \right),
   \]
   then \(u_n = \theta_{[\lambda_n, d_n]}\) is the unique positive solution of \((7)\) and hence
   \[
   g(d_n) < \lambda_n < g_0(d_n).
   \]

   Since \(\Omega_0 \subset\subset D_1\) we get that
   \[
   g_0(d_n) = \lambda_1^{\Omega_0\Omega}[-\Delta; 1].
   \]

   We will prove that \(\lambda_\infty = \lambda_1^{\Omega_0\Omega}[-\Delta; 1]\). We assume by contradiction that, at least for a subsequence, \(\lambda_n \to \lambda_\infty < \lambda_1^{\Omega_0\Omega}[-\Delta; 1]\). Then, there exists \(\overline{\lambda}\) such that \(\lambda_n \leq \overline{\lambda} < \lambda_1^{\Omega_0\Omega}[-\Delta; 1]\). Then, since \(\Omega_0 \subset\subset D_1\) by Proposition 4 there exist a positive constant \(K(\overline{\lambda})\) and a positive regular function \(\varphi\) (both independent of \(d_n\)), such that
   \[
   u_n = \theta_{[\lambda_n, d_n]} \leq K(\overline{\lambda})\varphi \quad \text{for all } \lambda_n \leq \overline{\lambda}.
   \]

   Hence, \(\|u_n\|_\infty \leq C\). A contradiction.

3. **Case \(\Omega_0 \subset D_2\):**

   With the similar notation above, \(d_n = a \left( \int_{\Omega} u_n^p \right),\) then \(u_n = \theta_{[\lambda_n, d_n]}\) is the unique positive solution of \((7)\) and hence
   \[
   g(d_n) < \lambda_n < g_0(d_n) = d_n \lambda_1^{\Omega_0\Omega}[-\Delta; 1].
   \]

   Observe that
   \[
   d_n \to a(\infty),
   \]
   and hence, it is clear that when \(a(\infty) = 0\), then \(\lambda_n \to 0\).

   Now assume that \(a(\infty) = \infty\) and then \(d_n \to \infty\). We claim that \(\lambda_\infty = \infty\).

   For this purpose, we assume by contradiction that \(\lambda_n \to \lambda_\infty < \infty\). We will build a supersolution of \((7)\). Indeed, take a subdomain \(D\) such that \(\Omega \subset\subset D\) and consider \(\lambda_1^D[-\Delta; 1]\) and its positive eigenfunction associated \(\varphi^D_1\) with \(\|\varphi^D_1\|_\infty = 1\). Then, \(\pi = K\varphi^D_1\) is supersolution of \((7)\) if
   \[
   \lambda_1^D[-\Delta; 1](\chi_{D_1} + d_n\chi_{D_2}) + b(x)K\varphi^D_1 \geq \lambda_n \quad \text{in } \Omega.
   \]

   It is clear that, since \(\Omega_0 \subset D_2\), in \(D_2\), the inequality \((24)\) holds if
   \[
   d_n\lambda_1^D[-\Delta; 1] > \lambda_n,
   \]
   which is verified due to \(\lambda_n\) is bounded and \(d_n \to \infty\).
On the other hand, in $D_1$, (24) holds if
\[ \lambda_n^D[-\Delta;1] + b(x)K\varphi^D_n \geq \lambda_n \text{ in } D_1. \]
Since $\lambda_{\infty} < \infty$, $\varphi^D_1 \geq b_1 > 0$ in $\Omega$ and $b(x) \geq b_0 > 0$ in $D_1$ for some positive constants $b_0, b_1$, we have that this inequality holds for $K$ is large (independent of $n$). Hence, for $n$ large we obtain
\[ u_n \leq K\varphi^D_n. \]
This leads to a contradiction.

4. General case: $\Omega \cap \Omega_0 \neq \emptyset$ and $\Omega_0 \cap D_1 \neq \emptyset$.

Since $\Omega_0 \cap D_1 \neq \emptyset$, we can apply Proposition 9, and then $\text{Proj}_B(\mathcal{C})$ is bounded in $\mathbb{R}$. Hence, there exists $(\lambda_n, u_n) \in \mathcal{C}$ a sequence of positive solutions of (1) such that $\lambda_n \to \lambda_{\infty} < \infty$ and $\|u_n\|_{\infty} \to \infty$.

First, assume that $a(\infty) = 0$. We get
\[ \lambda_n = \lambda_1[-(\chi_{D_1} + d_n\chi_{D_2})\Delta + b(x)u_n;1] < \lambda_1^{\Omega_0 \cup D_2}[-(\chi_{D_1} + d_n\chi_{D_2})\Delta;1] < \lambda_1^B[-(\chi_{D_1} + d_n\chi_{D_2})\Delta;1] = d_n\lambda_1^B[-\Delta;1], \]
where $B \subset \Omega_0 \cap D_2$. Taking limit in $n$, we get
\[ \lambda_{\infty} = 0. \]
Assume now that $a(\infty) = \infty$. We will prove that
\[ \lambda_{\infty} \geq \lambda_1^{\Omega_0 \cup D_2}[-\Delta;\chi_{D_1}]. \]
We suppose $\lambda_{\infty} < \lambda_1^{\Omega_0 \cup D_2}[-\Delta;\chi_{D_1}]$. Then, there exists a compact set $\mathcal{K} \subset [0, \lambda_1^{\Omega_0 \cup D_2}[-\Delta;\chi_{D_1}]]$ such that $\lambda_n \in \mathcal{K}$. Then, by Proposition 5, there exist two positive constants $d(\mathcal{K})$ and $C > 0$ such that
\[ \|u_n\|_{\infty} = \|\theta_{[\lambda_n, d_n]}\|_{\infty} \leq C, \quad \text{for } d_n \geq d(\mathcal{K}) \text{ and for all } \lambda_n. \]
This contradicts our assumption.

We now consider several situations where we can determine that $\lambda_{\infty} = \lambda_1^{\Omega_0 \cup D_2}[-\Delta;\chi_{D_1}]$.

(a) Assume that $D_2 \subset \Omega_0$. Observe that in this case $\Omega \setminus \Omega_0 \subset D_1$, and then applying Lemma 5.2 we obtain that
\[ \lambda_{\infty} = \lambda_1^{\Omega_0 \cup D_2}[-\Delta;\chi_{D_1}] = \lambda_1^{\Omega_0 \cup D_2}[-\Delta;\chi_{D_1}]. \]

(b) Assume that $p \geq 2$ and there exists a positive constant $C$ such that
\[ \frac{a(s)}{s^{1/p}} \geq C \quad \text{for large } s. \]
Then, making use again of Lemma 5.2 we obtain that
\[ \lambda_{\infty} = \lambda_1^{\Omega_0 \cup D_2}[-\Delta;\chi_{D_1}]. \]

(c) Assume that
\[ \lim_{s \to \infty} \frac{a(s)}{s^{1/p}} = \infty, \]
and that $\lambda_{\infty} > \lambda_1^{\Omega_0 \cup D_2}[-\Delta;\chi_{D_1}]$. Then, at least for a subsequence
\[ \lambda_n \in \mathcal{K} := [\lambda_{\infty} - \epsilon, \lambda_{\infty} + \epsilon], \quad (25) \]
with $\epsilon > 0$ small enough such that $\lambda_\infty - \epsilon > \lambda_1^{\Omega_0 \cup D_2}[-\Delta; \chi_{D_1}]$. Then, by (16) we derive that
\[
\lambda_n - d_n \mu(\lambda_n) b_M \varphi_n \leq u_n \quad \text{in } \Omega_0 \cup D_2,
\]
where $\mu(\lambda_n) = \lambda_1^{\Omega_0 \cup D_2}[-\Delta - \lambda_n \chi_{D_1}; 1] < 0$ and $\varphi_n$ is the positive eigenfunction associated to $\mu(\lambda_n)$ such that $\|\varphi_n\|_\infty = 1$, that is
\[-\Delta \varphi_n - \lambda_n \chi_{D_1} \varphi_n = \mu(\lambda_n) \varphi_n \quad \text{in } \Omega_0 \cup D_2, \quad \varphi_n = 0 \quad \text{on } \partial(\Omega_0 \cup D_2).
\]
By elliptic regularity, $\|\varphi_n\|_{W^{2,q}} \leq C$ for all $q > 1$, and then $\varphi_n \rightharpoonup \varphi_\infty$ in $C^1(\Omega)$ for some $\varphi_\infty > 0$. Moreover, thanks to (25) there exist positive constants $a, b$ such that
\[
(a + bd_n) \varphi_n \leq u_n \quad \text{in } \Omega_0 \cup D_2.
\]
Thus,
\[
(a + bd_n) \left( \int_{\Omega_0 \cup D_2} \varphi_n^p \right)^{1/p} \leq \left( \int_{\Omega} u_n^p \right)^{1/p},
\]
which yields
\[
\left( \frac{a}{d_n} + b \right) \left( \int_{\Omega_0 \cup D_2} \varphi_n^p \right)^{1/p} \leq \frac{\left( \int_{\Omega} u_n^p \right)^{1/p}}{\frac{1}{a} \left( \int_{\Omega} u_n^p \right)} \to 0,
\]
an absurdum. This completes the proof.

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