CONSTANT MEAN CURVATURE FOLIATIONS OF
3-DIMENSIONAL TWISTED PRODUCT SPACETIMES

A. DIRMEIER

Abstract. For 2+1 spacetime dimensions, we derive sufficient conditions for
the twisting function in a twisted product spacetime, such that there is a global
foliation by spacelike CMC surfaces.

1. Introduction

Foliations of spacetimes by hypersurfaces of constant mean curvature are of wide
interest in mathematical relativity and cosmology (see, e.g., [7], [2]). Hereby, the
mean curvature of the hypersurfaces (slices) of the foliation is assumed constant over
the respective hypersurface but may vary over time. Twisted product manifolds are
generalisations of warped product manifolds. Such manifolds of arbitrary signature
were investigated in [6] and they have many applications in differential geometry
for example also in relation to foliations (see, e.g., [8]). We investigate here CMC
foliations of twisted product spacetime (i.e., oriented Lorentzian manifolds) only
in 2+1-spacetime dimensions. The reason is a particularly simple formula for the
mean curvature of the slices in this case (see equation (2) below).

2. Preliminaries

Let \((N, \gamma)\) be a 2-dimensional Riemannian manifold, and \((\mathbb{R}, -dt^2)\) the real line
doved with the negative definite standard metric. Then the twisted product
\(M = \mathbb{R} \times_s N\) is a 3-dimensional Lorentzian manifold with the metric
\[ g = -dt^2 + s^2 \gamma \]
with \(s: \mathbb{R} \times N \to \mathbb{R}^+\) being the twisting function. We denote points in the product
space \(M\) by \((t, x) \in \mathbb{R} \times M\). Hence the hypersurfaces \(N_t := \{t\} \times N \subset M\) for \(t \in \mathbb{R}\)
form a spacelike foliation of \(M\). The vector field \(V = \partial_t\) is the lift of the canonical
vector field \(\partial_t\) on \(\mathbb{R}\) to \(M = \mathbb{R} \times N\), which we denote be a slight abuse of notation
with the same symbol. The vector field \(V\) is everywhere perpendicular to the slices
\(N_t\) and has unit length, i.e., \(g(V, V) = -1\).

Let \(h(t, x) = s^2(t, x)\gamma(x)\) be the induced metric on the slices \(N_t\). We denote by
\(u = g(V, \cdot)\) the one-form metrically associated with \(V\). Then the symmetric part of

---

2010 Mathematics Subject Classification. Primary 53B30, 53A10; Secondary 53C12.

Key words and phrases. Twisted Product Spacetimes, Constant Mean Curvature Foliations.
the projection of the covariant derivative of \( u \) onto the slices \( N_t \) can be decomposed into its trace-free part \( \sigma \) and its trace \( \theta \):
\[
h(\text{sym}(\nabla u)) = \frac{\theta}{2} h + \sigma.
\]
The trace \( \theta \) is called expansion of \( V \) and \( \sigma \) is called shear of \( V \) (cf. [4, Sec. 4.1], [3]). The expansion is given by
\[
\theta = \text{div}(V) = 2 \frac{\dot{s}}{s};
\]
where the dot denotes the derivative with respect to \( t \). Furthermore, we denote by
\[
\dot{u} = g(\nabla_V V, \cdot)
\]
the acceleration of the vector field \( V \). Thus the second fundamental form \( K \) of the slices \( N_t \) is given by
\[
K = -\frac{1}{2} L_V h = -\frac{\theta}{2} h - \sigma + \dot{u} \otimes u + u \otimes \dot{u}.
\]
Because of \( h(V, \cdot) = 0 \), we get for the mean curvature \( k \) of the slices \( N_t \)
\[
k = \text{trace}(K) = -\theta = -2 \frac{\dot{s}}{s}.
\]
So obviously, if \( M \) is a warped product, i.e., \( s = s(t) \) (or if \( s(t,x) = s_1(t) \cdot s_2(x) \), which can be regarded a warped product with Riemannian metric \( s_2 \gamma \) on \( N \)), then the slices \( N_t \) form a CMC foliation of \( M \). Hence, we can assume from now on that \( M \) is a generic twisted product where the twisting function does not decompose as a product function.

Any change of the foliation of \( M = \mathbb{R} \times N \) by 2-dimensional spacelike slices is given in terms of a transformation
\[
\psi: \mathbb{R} \times N \to \mathbb{R} \times N, \quad \psi: (t, x) \to (\tau(t, x), x) = (t + f(x), x)
\]
with \( f \) being an arbitrary (smooth) function on \( N \). Note that such a transformation can be regarded as a global gauge transformation in the trivial principal fibre bundle \( \mathbb{R} \times N \), i.e., \( \psi \) is an automorphism of the principal fibre bundle or a change of the trivialisation.

In the following, we will use indices \( i, j \in \{1, 2\} \) for local coordinates \( x^i \) of \( N \) and indices \( a, b \in \{0, 1, 2\} \) for local coordinates \( x^a \) of \( M \). The transformation \( \psi \) results in \( dt = d\tau - df = d\tau - f_i dx^i \). Hence the decomposition of the metric \( g \) with respect to the new foliation is given by
\[
g = -d\tau^2 + 2df d\tau + h - df \otimes df = -d\tau^2 + 2f_i dx^i d\tau + (h_{ij} - f_i f_j)dx^i dx^j =
\]
\[
= -d\tau^2 + 2df d\tau + s^2 \gamma - df \otimes df = -d\tau^2 + 2f_i dx^i d\tau + (s^2 \gamma_{ij} - f_i f_j)dx^i dx^j.
\]
Obviously, it is necessary to have
\[
(1) \quad \|df\|^h = \frac{\|df\|^\gamma}{s^2} = h^{ij} f_i f_j = \frac{s^2f_i f_j}{s^2} < 1,
\]
so that the new slicing \( \tilde{N}_\tau := \{\tau\} \times N \) is spacelike, too.

Now we compute the normal vector field for \( \tilde{N}_\tau \). This is given by
\[
X = (X^a) = (g^{ab} d\tau_b).
\]
Using the shorthand notation $\partial_i := \frac{\partial}{\partial x_i}$ for a local basis of one-forms, a tedious, but straightforward, calculation shows that

$$g^{-1} = (\|df\|^h - 1)\partial_t^2 + 2h^{ij}f_{,j}\partial_i + h^{ij}\partial_i\partial_j.$$ 

Thus we have

$$X = (\|df\|^h - 1)V + h^{ij}f_{,j}\partial_i$$

and now the expansion of $X$ and therefore the mean curvature $\tilde{k}$ of $\tilde{N}_\tau$ is given by

$$-\tilde{k} = \text{div}(X) = \partial_t(\|df\|^h) + (\|df\|^h - 1)\theta + \Delta_h f =$$

$$\partial_t\frac{\|df\|^\gamma}{s^2} + \frac{\|df\|^\gamma}{s^2} \cdot 2\frac{\dot{s}}{s} - 2\frac{\dot{s}}{s} + \Delta_h f =$$

$$= \Delta_h f - 2\frac{\dot{s}}{s},$$

where $\Delta_h$ denotes the Laplace-Beltrami operator with respect to $h$ on the slices $N_t$. Note that the terms containing $\|df\|^\gamma$ do cancel out exactly because $\dim(N) = 2$.

As we have

$$\Delta_h f = \frac{1}{\sqrt{\det h}}\partial_t\left(\sqrt{\det hh^{ij}\partial_j f}\right)$$

and $h^{ij} = s^{-2}\gamma^{ij}$ it follows that

$$\Delta_h f = \frac{1}{s^2}\Delta_\gamma f.$$ 

Note that $t$ is a time function for the spacetime $(M,g)$ and so is $\tau$. Hence $(M,g)$ is stably causal (see, e.g., [3] for an overview of causality theory for Lorentzian manifolds). The preimages of $t$ (resp. $\tau$) are the spacelike slices $N_t$ (resp. $N_\tau$). If the preimages of a time function are hypersurfaces of constant mean curvature, it is called a CMC time function (cf., e.g., [1]).

Also note that due to $\tau = t + f(x)$, we have $\partial_t = \partial_\tau$, i.e., the dot denotes derivatives by $t$ and $\tau$.

3. Main Result

Now we are able to state the following

**Theorem.** A $2+1$-dimensional twisted product spacetime

$$(M, g) = (\mathbb{R} \times N, -dt^2 + s^2\gamma)$$

with $s: M \to \mathbb{R}^+$ a function and $\gamma$ a Riemannian metric on $N$, has a foliation by spacelike surfaces of constant mean curvature if there are functions $\alpha: \mathbb{R} \to \mathbb{R}^+$, $\beta: N \to \mathbb{R}$, $\xi: N \to \mathbb{R}$ and $f: N \to \mathbb{R}^+$, such that the following three conditions hold

(i) $s^2(\tau, x) = \frac{\alpha(\tau)\beta(x) + \xi(x)}{\alpha(\tau)}$,  
(ii) $\Delta_\gamma f = \beta$ and  
(iii) $\|df\|^\gamma < s^2$,

with $\tau = t + f(x)$. Moreover, in this case, $\tau$ is a CMC time function.
Proof: Using (i) and (ii), we can calculate the mean curvature \( \tilde{k} \) to be

\[
\tilde{k} = \frac{1}{s^2} \Delta \gamma f - 2 \frac{\dot{s}}{s} \frac{\dot{\alpha}}{\alpha} - \frac{1}{\alpha^2} \left( \frac{\beta - \frac{\dot{\alpha}}{\alpha^2} (\alpha \beta + \xi)}{\alpha} \right) = \frac{\tilde{\alpha}}{\alpha} (\tau),
\]

which is constant on any slice \( \tilde{N}_\tau \). Condition (iii) assures the CMC foliation to be spacelike, as can be seen by comparison to (I), hence \( \tau \) is a time function. And \( \tau \) is even a CMC time function, because its level sets \( \tilde{N}_\tau \) are all spacelike CMC surfaces.

\[\square\]