ON THE EULER NUMBERS AND ITS APPLICATIONS

TAEKYUN KIM

Abstract. Recently, the \(q\)-Euler numbers and polynomials are constructed in [T. Kim, The modified \(q\)-Euler numbers and polynomials, Advanced Studies in Contemporary Mathematics, 16(2008), 161-170]. These \(q\)-Euler numbers and polynomials contain the interesting properties. In this paper we prove Von-Staudt Clausen’s type theorem related to the \(q\)-Euler numbers. That is, we prove that the \(q\)-Euler numbers are \(p\)-adic integers. Finally, we give the proof of Kummer type congruences for the \(q\)-Euler numbers.

§1. Introduction/Definition

Let \(p\) be a fixed odd prime. Throughout this paper \(\mathbb{Z}_p\), \(\mathbb{Q}_p\), \(\mathbb{C}\), and \(\mathbb{C}_p\) will, respectively, denote the ring of \(p\)-adic rational integers, the field of \(p\)-adic rational numbers, the complex number field, and the completion of algebraic closure of \(\mathbb{Q}_p\). Let \(v_p\) be the normalized exponential valuation of \(\mathbb{C}_p\) with \(|p|_p = p^{-v_p(p)} = \frac{1}{p}\). When one talks of \(q\)-extension, \(q\) is variously considered as an indeterminate, a complex \(q \in \mathbb{C}\), or a \(p\)-adic number \(q \in \mathbb{C}_p\), see [9-22]. If \(q \in \mathbb{C}\), then we assume \(|q| < 1\). If \(q \in \mathbb{C}_p\), then we assume \(|1 - q|_p < p^{-\frac{1}{p-1}}\). The ordinary Euler numbers are defined as

\[
\frac{2}{e^t + 1} = e^{Et} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!},
\]

where we use the technique method notation by replacing \(E^n\) by \(E^n\) (\(n \geq 0\)), symbolically (see [1-23]). From this definition, we can derive the following relation.

\[E_0 = 1, \text{ and } (E + 1)^n + E_n = 2\delta_{0,n}, \text{ where } \delta_{0,n} \text{ is Kronecker symbol.}\]

For \(x \in \mathbb{Q}_p \) (or \(\mathbb{R}\)), we use the notation \([x]_q = \frac{1-q^x}{1-q}\), and \([x]_{-q} = \frac{1-(q)^x}{1+(q)^x}\), see [5-6]. In [5], the \(q\)-Euler numbers are defined as

\[
E_{0,q} = \frac{[2]_q}{2}, \text{ and } (qE + 1)^n + E_{n,q} = [2]_q \delta_{0,n},
\]

Key words and phrases. fermionic \(p\)-adic \(q\)-integral, Euler numbers, \(q\)-Volkenborn integral, Kummer congruence.

2000 AMS Subject Classification: 11B68, 11S80
with the usual convention of replacing $E^n$ by $E_{n,q}$. Note that $\lim_{q \to 1} E_{n,q} = E_n$. For a fixed positive integer $d$ with $(p, d) = 1$, let

$$X = X_d = \lim_N \mathbb{Z} / dp^N \mathbb{Z}, \quad X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{0 < a < dp \atop (a, p) = 1} a + dp \mathbb{Z}_p,$$

$$a + dp^N \mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^N} \},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$, (see [4-23]).

We say that $f$ is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$, if the difference quotients $F_f(x, y) = \frac{f(x) - f(y)}{x - y}$ have a limit $l = f'(a)$ as $(x, y) \to (a, a)$. For $f \in UD(\mathbb{Z}_p)$, let us start with the expression

$$\frac{1}{[p^N]_q} \sum_{0 \leq j < p^N} q^j f(j) = \sum_{0 \leq j < p^N} f(j) \mu_q(j + p^N \mathbb{Z}_p),$$

representing a $q$-analogue of Riemann sums for $f$, see [5, 6]. The integral of $f$ on $\mathbb{Z}_p$ will be defined as limit $(n \to \infty)$ of those sums, when it exists. The $q$-deformed bosonic $p$-adic integral of the function $f \in UD(\mathbb{Z}_p)$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[dp^N]_q} \sum_{0 \leq x < dp^N} f(x) q^x, \text{ see [5].}$$

In the sense of fermionic, let us define the fermionic $p$-adic $q$-integral as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x) (-q)^x, \text{ see [5-10].}$$

From (3) we note that

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \text{ where } f_1(x) = f(x + 1).$$

In [5], the Witt’s type formula for the $q$-Euler numbers $E_{n,q}$ is given by

$$\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} [x]^n q^{-x} d\mu_{-q}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{[x]q^t} q^{-x} d\mu_{-q}(x) = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}.$$

By comparing the coefficients on both sides in (5), we see that

$$\int_{\mathbb{Z}_p} [x]^n q^{-x} d\mu_{-q}(x) = E_{n,q}, \text{ see [5].}$$
By the definition of the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$, the $q$-Euler polynomials are also defined as

$$\int_{\mathbb{Z}_p} e^{[x+y]q^{-y}} d\mu_q(y) = e^{[x]q^{t}} \int_{\mathbb{Z}_p} e^{q^{x}[y]q^{-y}} d\mu_q(y) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \hspace{1cm} (7)$$

From (6) and (7), we note that

$$E_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k}_q [x]_q^{n-k} q^{k} E_{k,q}, \text{ where } \binom{n}{k}_q = \frac{n(n-1)\cdots(n-k+1)}{k!}. \hspace{1cm} (8)$$

Let $F_q(t, x)$ be the generating function of the $q$-Euler polynomials. Then we have

$$F_q(t, x) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} = [2]_q \sum_{k=0}^{\infty} (-1)^k e^{[k+x]q^{t}}. \hspace{1cm} (9)$$

Let $\chi$ be the Dirichlet’s character with odd conductor $d \in \mathbb{N}$. Then the generalized $q$-Euler numbers attached to $\chi$ are defined as

$$E_{n,\chi,q} = \int_{\chi} [x]_q^n q^{-x} \chi(x) d\mu_q(x) = [d]_q^n \sum_{a=0}^{d-1} \chi(a)(-1)^a E_{n,q^a}(\frac{a}{d}). \hspace{1cm} (10)$$

Let $F_{\chi,q}(t) = \sum_{n=0}^{\infty} E_{n,\chi,q} \frac{t^n}{n!}$. Then we note that

$$F_{\chi,q}(t) = \sum_{n=0}^{\infty} E_{n,\chi,q} \frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty} \chi(n)(-1)^n e^{[n]_q q^{t}}. \hspace{1cm} (11)$$

In this paper we prove the Von-Staudt-Clausen’s type theorem related to the $q$-Euler numbers. That is, we prove that the $q$-Euler numbers are the $p$-adic integers. Finally, we give the proofs of the Kummer congruences for the $q$-Euler numbers.

**§2. q-Euler numbers and polynomials**

From (1) and (6) we derive

$$E_{n,q} = \int_{\mathbb{Z}_p} q^{-x} [x]_q^n d\mu_q(x) = [2]_q \frac{n}{2} \int_{\mathbb{Z}_p} [x]_q^n d\mu_{q-1}(x).$$

Thus, we note that $\lim_{n \to \infty} E_{n,q} = E_n = \int_{\mathbb{Z}_p} x^n d\mu_{q-1}$. For $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$, we have

$$(-1)^j [j]_q - j(-1)^j = (-1)^j \left( \frac{\sum_{l=0}^{j} \binom{j}{l}(q-1)^l - 1}{q-1} - j \right) = (-1)^j \sum_{l=2}^{j} \binom{j}{l} (q-1)^{l-1}.$$
Thus, we see that

\[(12)\quad |(-1)^j([j]_q - j)|_p \leq \max_{2 \leq l \leq j} \left(|(q - 1)|^{l-1}_p\right) = |q - 1|_p.\]

From (12), we can derive

\[(13)\quad \left|\sum_{j=0}^{p-1} (-1)^j [j]_q \right|_p = \left|\sum_{j=0}^{p-1} (-1)^j ([j]_q - j) + \sum_{j=0}^{p-1} (-1)^j j \right|_p = \left|\sum_{j=0}^{p-1} (-1)^j ([j]_q - j) + \frac{p-1}{2} \right|_p \leq 1.\]

For \(k \geq 1\), let

\[(14)\quad T_n(k) = \sum_{x=0}^{p^k - 1} (-1)^x [x]_q^n = [0]_q^n - [1]_q^n + \cdots + [p^k - 1]_q^n.\]

Note that \(\lim_{k \to \infty} T_n(k) = \frac{2}{[2]_q} E_{n,q}\). From (14), we can derive

\[(15)\quad T_n(k + 1) = \sum_{x=0}^{p^{k+1} - 1} (-1)^x [x]_q^n = \sum_{i=0}^{p^k - 1} \sum_{j=0}^{p-1} [i + j p^k]_q^n (-1)^{i+j} p^k = \sum_{i=0}^{p^k - 1} \sum_{j=0}^{p-1} \sum_{l=0}^{n} \binom{n}{l} [i]_q^{n-l} q^l [j]_q^{l} (-1)^{i+j} p^k\]

\[
= \sum_{i=0}^{p^k - 1} \sum_{j=0}^{p-1} \sum_{l=0}^{n} \binom{n}{l} [i]_q^{n-l} q^l [j]_q^{l} (-1)^{i+j} p^k
\]

\[
= \sum_{i=0}^{p^k - 1} \sum_{j=0}^{p-1} \sum_{l=0}^{n} \binom{n}{l} [i]_q^{n-l} q^l [j]_q^{l} (-1)^{i+j} p^k.
\]

Thus, we have

\[(16)\quad T_n(k + 1) \equiv \sum_{i=0}^{p^k - 1} [i]_q^n (-1)^i \pmod{[p^k]_q}.\]
From (15) we note that

\[ \begin{align*}
\sum_{x=0}^{p^{k+1}-1} [x]_q^n (-1)^x &= \sum_{j=0}^{p-1} \sum_{i=0}^{p-1} [j + ip]_q^n (-1)^{j+ip} \\
&= \sum_{j=0}^{p-1} (-1)^j \sum_{i=0}^{p-1} (-1)^i \sum_{l=0}^{n} \binom{n}{l} [j]_q^{n-l} q^{jl} [p]_q^l [i]_q^l \\
&= \sum_{j=0}^{p-1} (-1)^j [j]_q^n + \sum_{j=0}^{p-1} (-1)^j \sum_{i=0}^{p-1} (-1)^i \sum_{l=1}^{n} \binom{n}{l} [j]_q^{n-l} q^{jl} [p]_q^l [i]_q^l \\
&\equiv \sum_{j=0}^{p-1} (-1)^j [j]_q^n \pmod{[p]_q}.
\end{align*} \]

By (17), we obtain

\[ \begin{align*}
\sum_{x=0}^{p^{k+1}-1} (-1)^x [x]_q^n &\equiv \sum_{j=0}^{p-1} (-1)^j [j]_q^n \pmod{[p]_q}.
\end{align*} \]

From (16) and (19), we can also derive

\[ \sum_{j=0}^{p-1} (-1)^j [j]_q^n = \frac{2}{[2]_q} \int_X [x]_q^n q^{-x} d\mu_q(x) = \frac{2}{[2]_q} E_{n,q} \pmod{[p]_q}. \]

Thus, we note that

\[ \sum_{j=0}^{p-1} (-1)^j [j]_q^n \equiv \frac{2}{[2]_q} E_{n,q} \pmod{[p]_q}. \]

Therefore we obtain the following theorem.

**Theorem 1.** For \( n \geq 0 \), we have

\[ \sum_{j=0}^{p-1} (-1)^j [j]_q^n \equiv \frac{2}{[2]_q} E_{n,q} \pmod{[p]_q}. \]
**Corollary 2.** For \( n \geq 0 \), we have

\[
\frac{2}{[2]_q} E_{n,q} + \sum_{j=0}^{p-1} (-1)^{j+1} [j]_q^n \in \mathbb{Z}_p.
\]

For \( n \geq 0 \), we note that

\[
\left| \frac{2}{[2]_q} E_{n,q} \right|_p = \left| \frac{2}{[2]_q} E_{n,q} - \sum_{j=0}^{p-1} (-1)^j [j]_q^n + \sum_{j=1}^{p-1} (-1)^j [j]_q^n \right|_p
\]

\[
\leq \max \left( \left| \frac{2}{[2]_q} E_{n,q} - \sum_{j=0}^{p-1} (-1)^j [j]_q^n \right|_p, \left| \sum_{j=1}^{p-1} (-1)^j [j]_q^n \right|_p \right).
\]

By (13) and Corollary 2, we obtain the following corollary.

**Corollary 3.** For \( n \geq 0 \), we have

\[
\frac{2}{[2]_q} E_{n,q} \in \mathbb{Z}_p.
\]

Let \( \chi \) be the Dirichlet’s character with odd conductor \( d(\in \mathbb{N}) \). Then the generalized \( q \)-Euler numbers attached to \( \chi \) as follows.

\[
\sum_{n=0}^{\infty} E_{n,\chi,q} \frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty} \chi(n)(-1)^n e^{[n]_q t} = \int_X \chi(x)e^{[n]_q t}q^{-x}d\mu_{-q}(x).
\]

We denote \( \bar{d} = [d,p] \) the least common multiple of the conductor \( d \) of \( \chi \) and \( p \). From (21), we derive

\[
\sum_{n=0}^{\infty} E_{n,\chi,q} \frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty} \chi(n)(-1)^n e^{[n]_q t} = \int_X \chi(x)e^{[x]_q t}q^{-x}d\mu_{-q}(x).
\]

By (22), we see that

\[
\frac{2}{[2]_q} E_{n,\chi,q} = \lim_{\rho \to \infty} \sum_{1 \leq x \leq \bar{d}\rho^\rho \atop (x,p)=1} \chi(x)(-1)^x [x]_q^n + \lim_{\rho \to \infty} \sum_{y=1}^{\bar{d}\rho^\rho-1} \chi(p)\chi(y)[y]_{q^p}^n(-1)^y
\]

\[
= \lim_{\rho \to \infty} \sum_{1 \leq x \leq \bar{d}\rho^\rho \atop (x,p)=1} \chi(x)(-1)^x [x]_q^n + \chi(p)[y]_{q^p}^n \lim_{\rho \to \infty} \sum_{y=1}^{\bar{d}\rho^\rho-1} \chi(y)[y]_{q^p}^n(-1)^y.
\]
Thus, we have

\[ (23) \frac{2}{[2]_q} E_{n,\chi,q} - \chi(p)[p]_q^n \frac{2}{[2]_q} E_{n,\chi,q^p} = \lim_{\rho \to \infty} \sum_{1 \leq x \leq dp^\rho \atop (x,p)=1} \chi(x)(-1)^x [x]_q^n. \]

Let \( w \) denote the Teichmüller character \( \text{mod } p \). For \( x \in X^* \), we set \( < x > = \frac{|x|}{w(x)} \). Note that \( | < x > - 1| < p^{-\frac{1}{p-1}} \), where \( < x >^s = \exp(s \log_p < x >) \) for \( s \in \mathbb{Z}_p \). For \( s \in \mathbb{Z}_p \), we define the \( p \)-adic \( q \)-L-function related to \( E_{n,\chi,q} \) as follows.

\[ (24) L_{p,q,E}(s, \chi) = \lim_{\rho \to \infty} \sum_{1 \leq x \leq dp^\rho \atop (x,p)=1} \chi(x)(-1)^x < x >^s. \]

For \( k \geq 0 \), we have

\[
(25) L_{p,q,E}(-k, \chi w^k) = \lim_{\rho \to \infty} \sum_{1 \leq x \leq dp^\rho \atop (x,p)=1} \chi(x)(-1)^x [x]_q^k \\
= \frac{2}{[2]_q} \int_X [x]_q^k \chi(x) q^{-x} d\mu_{-q}(x) - \frac{2}{[2]_q} \int_{pX} [x]_q^k \chi(x) q^{-x} d\mu_{-q}(x) \\
= \frac{2}{[2]_q} \int_X [x]_q^k \chi(x) q^{-x} d\mu_{-q}(x) - \chi(p)[p]_q^n \frac{2}{[2]_q} \int_X [x]_q^k \chi(x) q^{-x} d\mu_{-q^p}(x) \\
= \frac{2}{[2]_q} E_{n,\chi,q} - \chi(p)[p]_q^n \frac{2}{[2]_q} E_{n,\chi,q^p}. 
\]

It is easy to see that \( < x >^{p^n} \equiv 1 \pmod{p^n} \). From the definition of \( L_{p,q,E}(s, \chi) \), we can derive

\[
L_{p,q,E}(-k, \chi) = \lim_{\rho \to \infty} \sum_{1 \leq x \leq dp^\rho \atop (x,p)=1} \chi(x)(-1)^x < x >^k \\
\equiv \lim_{\rho \to \infty} \sum_{1 \leq x \leq dp^\rho \atop (x,p)=1} \chi(x)(-1)^x < x >^{k'} \pmod{p^n},
\]

whenever \( k \equiv k' \pmod{p^n(p-1)} \). That is, \( L_{p,q,E}(-k, \chi w^k) \equiv L_{p,q,E}(-k', \chi w') \pmod{p^n} \).

Therefore we obtain the following theorem.

**Theorem 4. (Kummer Congruence)** For \( k \equiv k' \pmod{p^n(p-1)} \), we have

\[
\frac{2}{[2]_q} E_{k,\chi,q} - \frac{2}{[2]_q} E_{k,\chi,q^p} \equiv \frac{2}{[2]_q} E_{k',\chi,q} - \frac{2}{[2]_q} E_{k',\chi,q^p} \pmod{p^n}.
\]
Let $\chi$ be the primitive Dirichlet’s character with conductor $p$. Then we have

$$p^{N+1} \sum_{x=0} \chi(x)(-1)^x \lfloor x \rfloor_q^n = p^{-1} \sum_{a=0}^{p-1} \chi(a)(a + px)(-1)^{a+px} \lfloor a + px \rfloor_q^n$$

$$= \sum_{a=0}^{p-1} \chi(a)(-1)^a \sum_{x=0}^{p-1} (-1)^x \left( [a]_q + q^a [p]_q [x]_q^p \right)^n$$

$$= \sum_{a=0}^{p-1} \chi(a)(-1)^a \sum_{x=0}^{p-1} (-1)^x \sum_{l=0}^{n} \binom{n}{l} [a]_q^{n-l} q^{al} [p]_q [l]_q^l [x]_q^p$$

$$\equiv \sum_{a=0}^{p-1} \chi(a)(-1)^a [a]_q^n \pmod{[p]_q}.$$ 

If $\rho \to \infty$, then we have

$$\frac{2}{[2]_q} \int_X \chi(x)(-1)^x \lfloor x \rfloor_q^n q^{-x} d\mu_q(x) \equiv \sum_{a=0}^{p-1} \chi(a)(-1)^a [a]_q^n \pmod{[p]_q}.$$ 

Thus, we can obtain the following. Let $\chi$ be the primitive Dirichlet’s character with conductor $p$. Then we have

$$\int_X \chi(x)(-1)^x \lfloor x \rfloor_q^n q^{-x} d\mu_q(x) \equiv \sum_{a=0}^{p-1} \chi(a)(-1)^a [a]_q^n \pmod{[p]_q}.$$ 

The Eq. (26) also seems to be the new interesting formula. As $q \to 1$, we can also obtain

$$E_{n,\chi} \equiv \sum_{a=0}^{p-1} \chi(a)(-1)^a a^n \pmod{p}.$$ 

References

[1] M. Cenkci, M. Can and V. Kurt, $p$-adic interpolation functions and Kummer-type congruences for $q$-twisted Euler numbers, Adv. Stud. Contemp. Math. 9 (2004), 203–216.

[2] M. Cenkci, The $p$-adic generalized twisted ($h, q$)-Euler-$l$-function and its applications, Adv. Stud. Contemp. Math 15 (2007), 37-47.

[3] L. Comtet, Advanced combinatories, Reidel, Dordrecht, 1974.

[4] E.Deeba, D.Rodriguez, Stirling’s series and Bernoulli numbers, Amer. Math. Monthly 98 (1991), 423-426.

[5] T. Kim, The modified $q$-Euler numbers and polynomials, Adv. Stud. Contemp. Math. 16 (2008), 161-170.

[6] T. Kim, Euler numbers and polynomials associated with zeta functions, Abstract and Applied Analysis 2008 (2008), 11 pages(Article ID 581582).

[7] T. Kim, $q$–Volkenborn integration, Russ. J. Math. Phys. 9 (2002), 288–299.
[8] T. Kim, A Note on p-Adic q-integral on $\mathbb{Z}_p$ Associated with q-Euler Numbers, Adv. Stud. Contemp. Math. 15 (2007), 133–138.
[9] T. Kim, A note on the q-Genocchi numbers and polynomials, J. Inequal. Appl. 2007 (2007), Art. ID 71452, 8 pp.
[10] T. Kim, q-Extension of the Euler formula and trigonometric functions, Russ. J. Math. Phys. 14 (2007), 275–278.
[11] T. Kim, Power series and asymptotic series associated with the $q$-analog of the two-variable $p$-adic $L$-function, Russ. J. Math. Phys. 12 (2005), 186–196.
[12] T. Kim, Non-Archimedean $q$-integrals associated with multiple Changhee $q$-Bernoulli polynomials, Russ. J. Math. Phys. 10 (2003), 91–98.
[13] T. Kim, $q$-Euler numbers and polynomials associated with $p$-adic $q$-integrals, J. Nonlinear Math. Phys. 14 (2007), 15–27.
[14] T. Kim, Some formulae for the q-Bernoulli and Euler polynomials of higher order, J. Math. Anal. Appl. 273 (2002), 236–242.
[15] B. A. Kupershmidt, Reflection symmetries of $q$-Bernoulli polynomials, J. Nonlinear Math. Phys. 12 (2005), 412–422.
[16] H. Ozden, I.N. Cangul, Y. Simsek, Multivariate interpolation functions of higher-order $q$-Euler numbers and their applications, Abstract and Applied Analysis 2008 (2008), Art. ID 390857, 16 pp.
[17] M. Schork., Ward’s "calculus of sequences", $q$-calculus and the limit $q \to -1$, Adv. Stud. Contemp. Math. 13 (2006), 131–141.
[18] M. Schork, Combinatorial aspects of normal ordering and its connection to $q$-calculus, Adv. Stud. Contemp. Math. 15 (2007), 49-57.
[19] K. Shiratani, S. Yamamoto, On a $p$-adic interpolation function for the Euler numbers and its derivatives, Mem. Fac. Sci., Kyushu University Ser. A 39 (1985), 113-125.
[20] Y. Simsek, On $p$-adic twisted $q$-$L$-functions related to generalized twisted Bernoulli numbers, Russ. J. Math. Phys. 13 (2006), 340–348.
[21] Y. Simsek, Theorems on twisted $L$-function and twisted Bernoulli numbers, Advan. Stud. Contemp. Math. 11 (2005), 205–218.
[22] Y. Simsek, $q$-Dedekind type sums related to $q$-zeta function and basic $L$-series, J. Math. Anal. Appl. 318 (2006), 333-351.
[23] H.J.H. Tuenter, A Symmetry of power sum polynomials and Bernoulli numbers, Amer. Math. Monthly 108 (2001), 258-261.

Taekyun Kim
Division of General Education-Mathematics, Kwangwoon University, Seoul 139-701, S. Korea e-mail: tkkim@kw.ac.kr