Numerical computations in cobordism categories

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Abstract. The sequence 2, 5, 15, 51, 187,... with the form $(2^n + 1)(2^n - 1 + 1)/3$ has two interpretations in terms of the density of a language with four letters [MR05] and the cardinality of the quotient of $\mathbb{Z}_2^n \times \mathbb{Z}_2^n$ under the action of the special linear group $\text{SL}(2, \mathbb{Z})$. The last interpretation follows the rank of the fundamental group of the $\mathbb{Z}_2^n$-cobordism category in dimension 1+1, see [Seg12]. This article presents how to pass from one side to another between these two approaches.

Introduction

It was a knock out when after writing the sequence 2, 5, 15, 51, 187,... in the page [oei], we found a great variety of different interpretations for it. This sequence was presented to the author for first time in some numerical computations in his PhD dissertation, see [Seg12]. There exists more that four contrasted approaches, each one inside a totally dissimilar subject and we use the following two paragraphs to give a description of two of them. Finally, in the next section we give a natural connection between them in terms of the binary numeral system and we state an open question.

The density of a language with four letters is defined as follows. Take the number of words of length $n$ made with letters 1, 2, 3, 4 with the property that numbered from left to right each letter satisfies $0 < a_i \leq \max_{j\leq i} \{a_j\} + 1$. Thus the first letter is always 1, so we can dismiss it. For example, for $n = 2$ there are two words 11 and 2, for $n = 3$ the words are 111, 112, 121, 122, 123, while for $n = 4$ we have 15 words

$$
\begin{align*}
111 & \quad 112 & \quad 121 & \quad 122 & \quad 123 \\
211 & \quad 212 & \quad 221 & \quad 222 & \\
223 & \quad 231 & \quad 232 & \quad 233 & \quad 234.
\end{align*}
$$

The second point considers the cardinality of the quotient of $\mathbb{Z}_2^n \times \mathbb{Z}_2^n$ under the action of the special linear group $\text{SL}(2, \mathbb{Z})$. This group is generated by two matrices which produce essentially two basic equations $(g, k) \sim (k, -g)$ and $(g, k) \sim (g, k + mg)$. The orbits of this quotient gives a set of generators for the monoid of principal $\mathbb{Z}_2^n$-bundles over closed surfaces with two boundary circles up to a homeomorphism identification, see [Seg12]. If you do not understand the last commentary you could go ahead and ignore it. For $n = 1$, we get two orbits $(0, 0)$ and $(0, 1) \sim (1, 0) \sim (1, 1)$. For $n = 2$, we get 5 orbits

1. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
   $\sim \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
   $\sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
   $\sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
   $\sim \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

2. $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$
   $\sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
   $\sim \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$
   $\sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
   $\sim \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Note that the easiest way to define the sequence 2, 5, 15, 51, 187,... is as the density of a language with four letters.

1\textsuperscript{The group $\mathbb{Z}_2$ is the only group with just two elements.}
1. Main constructions

Denote the two approaches of the last section as follows:

(1) the density of a language with four letters, and
(2) the cardinality of the quotient of \( \mathbb{Z}_n^4 \times \mathbb{Z}_n^4 \) under the action of the special linear group \( \text{SL}(2, \mathbb{Z}) \).

Now we define an application from [1] to [2]. This is defined for a word \( a_1a_2...a_n \) by the binary representation for \( a_i = 2, 3 \mapsto 10, 11 \): while for \( a_i = 1, 4 \) we should take the assignations 00, 01 respectively. As an illustration for \( n = 2 \), we have the assignations

\[
\begin{align*}
1 & \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & 2 & \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} & 2 & \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & 3 & \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\end{align*}
\]

When we have words with a letter with value 4 we forget the first two zeros as follows

\[
\begin{align*}
2 & \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & 3 & \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 4 & \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\end{align*}
\]

This application is surjective since the identification in [2] is just column operations, which is used to transform every element by one which is the image of a word from [1]. In [MR05] it is proved that the density of a language with four letters has the value \( (2^n + 1)(2^{n-1} + 1)/3 \). We prove in theorem 1.2 that for the second approach we have the same value. Thus this application between these two approaches should be a bijection.

**Definition 1.1.** For an abelian group \( G \) denote \( r(G) \) the cardinality of the quotient of \( G \times G \) up to the identification generated by the equations \((g, k) \sim (k, -g)\) and \((g, k) \sim (g, k + mg)\).

**Theorem 1.2.** For the group \( \mathbb{Z}_n^4 \), with \( p \) a prime number, we have the identity

\[
r(\mathbb{Z}_n^4) = \frac{p^{2n-1} + p^{n+1} - p^{n-1} + p^2 - p - 1}{p^2 - 1}.
\]

**Proof.** For \( r_p^n := r(\mathbb{Z}_p^n) \), let \( F(n) \) be the number \( r_p^{n+1} - r_p^n \). We will prove that

\[
F(n) = p^{n-1}(p^n + p - 1).
\]

Since \( r_p^n = (r_p^n - r_p^{n-1}) + (r_p^{n-1} - r_p^{n-2}) + ... + (r_p^2 - r_p^1) + (r_p^1 - r_p^0) + r_p^1 \), where \( r_p^1 = 2 \). Thus \( r_p^n = p^{n-2}(p^2 + p - 1) + p^{n-3}(p^2 + 2 + 1) + ... + p^2(p^2 + p - 1) + (p + p - 1) + 2 \) and as a consequence we have the following equations

\[
r_p^n = \sum_{i=0}^{n-2} p^{2i+1} + (p - 1) \sum_{i=0}^{n-2} p^i + 2
\]

\[
= p \frac{p^{n-1} - 1}{p^2 - 1} + (p - 1) \frac{p^{n-1} - 1}{p - 1} + 2
\]

\[
= \frac{p^{n-1} - p + (p^{n-1} - 1)(p^2 - 1)}{p^2 - 1}
\]

\[
= \frac{p^{2n-1} + p^{n+1} - p^{n-1} + p^2 - p - 1}{p^2 - 1}.
\]

We will prove the formula (1.1) by induction, where we use the following

\[
F(n) = pF(n - 1) + p^{n-2}(p - 1).
\]

We end with the proof of the formula (1.2). By definition \( F(n) \) consists of elements in \( \mathbb{Z}_p^{n+1} \times \mathbb{Z}_p^{n+1} \) such that the last row is different from the zero row. There are three cases to consider:
1) the representatives of the classes have zeros in the \( n \)-coordinate, i.e. of the form
\[
\begin{pmatrix}
0 & 0 \\
\vdots & \vdots \\
i & j
\end{pmatrix},
\]
with \( i, j \neq 0 \) at least one, and for this case the number of classes is \( F(n - 1) \);

2) the representatives of the classes that have zeros in the second column for the last two rows, i.e. of the form
\[
\begin{pmatrix}
0 & 0 \\
\vdots & \vdots \\
i & 0 \\
j & 0
\end{pmatrix},
\]
with \( i \neq 0 \) and \( j \neq 0 \). For these elements the stabilizer group is the same, before and after erasing the last column, so we have \((p - 1)F(n - 1)\) classes, where we multiply by \( p - 1 \) since we can not take the zero value for \( j \); and

3) the last case is composed by classes with a representative of the form
\[
(1.3) \quad \begin{pmatrix}
\vdots & \vdots \\
i & 0 \\
0 & j
\end{pmatrix},
\]
with \( i \neq 0 \) and \( j \neq 0 \). Every stabilizer of an element of the form \( (1.3) \) has to be the identity, then the classes have as cardinality the order of \( \text{SL}(2, \mathbb{Z}_p) \), which is \( p(p^2 - 1) \). Thus the number of classes is given by the product of \( p^{n-1}p^{n-1} \) (given by the first \( n - 1 \)-elements of the two columns) product with \( |\text{GL}(2, \mathbb{Z}_p)| \) (given by the part \( \begin{pmatrix} i & 0 \\ 0 & j \end{pmatrix} \) \in \( \text{GL}(2, \mathbb{Z}_p) \) in \( (1.3) \)) and divided by \( |\text{SL}(2, \mathbb{Z}_p)| \). Therefore, the number of classes is \( p^{2n-2}(p - 1) \). Finally, the sum of the numbers associated to these three cases, gives equation \( (1.2) \) and we have the proof of the theorem.

\[ \square \]

**Question 1.3.** What is the analog of the approach \( (1) \) for every prime number?

**References**

[MR05] Nelma Moreira and Rogério Reis, *On the density of languages representing finite set partitions*, Journal of Integer Sequences 8 (2005), 1–11.

[oei] *The on-line encyclopedia of integer sequences.*

[Seg12] Carlos Segovia, *The classifying space of the 1+1 dimensional g-cobordism category*, http://arxiv.org/abs/1211.2144, Nov 2012.