Moments of Heavy Quark Current Correlators at Four-Loop Order in Perturbative QCD

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Abstract

We present the result for the first moment of the scalar and axial-vector current correlator in third order of the strong coupling constant $\alpha_s$ and give the details of a recent evaluation of the pseudo-scalar correlator. The results can be used to reduce the theoretical uncertainty due to higher order corrections for the determination of fundamental parameters of QCD in the context of lattice calculations.

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1 Introduction

The computation of heavy quark current correlators in perturbation theory is of central interest for many phenomenological applications. For instance their low energy expansion can be related to moments, which have been determined with high precision through the calculation of higher order corrections. Moments of the vector current correlator play an important role in combination with sum rules and the experimentally measured ratio \( R(s) = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-) \). They can be used to perform a precise charm- and bottom-quark mass determination. This method has first been suggested in Ref. [1, 2] and has been continuously improved over the years by considering new data from experiment and by including higher order corrections [3–5]. However, also the scalar, pseudo-scalar and axial-vector current correlators can be employed in this context. For example, moments of the pseudo-scalar correlator can be used in combination with lattice calculations to determine fundamental constants of QCD, the strong coupling constant \( \alpha_s \) and the charm-quark mass with high accuracy [6]. Furthermore, the first eight three-loop moments have been used [7–10] as one of several ingredients in order to reconstruct the complete momentum and mass dependence of the scalar, pseudo-scalar, vector and axial-vector polarization functions by means of Padé-approximations. The complete mass and momentum dependence is important in the computation of the \( Z \)-boson decays, in which a combination of the vector and axial-vector density enters or in the calculation of Higgs-boson decays which are related to the scalar or pseudo-scalar correlator.

Recently the low energy expansion at three-loop order has been extended and the first 30 coefficients have been determined for the vector current and for the remaining scalar, pseudo-scalar and axial-vector currents in Ref. [11, 12], where also singlet contributions have been taken into account. In four-loop order only the first moment of the vector current correlator is fully known at present [13, 14]. For double fermionic contributions the first five moments have been computed [15]. Contributions of the order \( \alpha_s^j n_l^{j-1} \) have been calculated to all orders \( j \) in Ref. [16], where \( n_l \) denotes the number of light quarks, considered as massless. Numerical results for two moments of the pseudo-scalar correlator were presented in [6].

Whereas the lowest moments of the vector correlator have been studied up to four-loop order already some time ago, the ones of the scalar, pseudo-scalar and axial-vector current correlators were available up to three-loop order only. The purpose of this paper is to provide the still unknown four-loop QCD corrections to the lowest moments of the scalar and axial-vector current correlators and to present the details of the evaluation for the pseudo-scalar correlator, where first numerical results were presented in [6]. Our discussion will be limited to the non-singlet contributions. These results can be
useful in combination with lattice calculations to reduce the error due to unknown higher order corrections in perturbation theory in the context of the determination of the strong coupling constant and quark mass as demonstrated in [6]. They can also be seen as a first step towards the evaluation of higher moments. Furthermore analytical results of the pseudo-scalar correlator are presented.

The techniques used in this work have already been successfully applied in several other calculations, among which are the calculation of the matching relation of the strong coupling constant at a heavy quark threshold up to four-loop order in perturbative QCD [17, 18] and the computation of the four-loop QCD corrections to the $\rho$-parameter arising from top- and bottom-quark loops [19–22].

The outline of this paper is as follows: In Section 2 we introduce our notations and conventions. In Section 3 we discuss the methods of calculation and give the results for the first moment of the scalar and axial-vector current correlators at four-loop order. For completeness we recall the results for the pseudo-scalar and vector case. Finally in Section 4 we close with a brief summary and our conclusions. The results for the vector and pseudo-scalar correlator are listed in Appendix A, those for the moments with $n = -1$ and $n = 0$ in Appendix B.

2 Generalities and Notation

The polarization functions for the scalar($s$), pseudo-scalar($p$), axial-vector($a$) and vector($v$) current correlator are defined by

$$q^2 \Pi^\delta(q^2) = i \int dx e^{iqx} \langle 0 | T j^\delta(x) j^\delta(0) | 0 \rangle, \quad \text{for } \delta = s, p \quad (1)$$

$$ (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi^\delta(q^2) + q_\mu q_\nu \Pi^\delta_L(q^2) = i \int dx e^{iqx} \langle 0 | T j^\delta_\mu(x) j^\delta_\nu(0) | 0 \rangle, \quad \text{for } \delta = a, v \quad (2)$$

with the currents

$$j^s = \overline{\Psi} \gamma_5 \Psi, \quad j^p = i \overline{\Psi} \gamma_5 \Psi, \quad j^a_\mu = \overline{\Psi} \gamma_\mu \gamma_5 \Psi, \quad j^v_\mu = \overline{\Psi} \gamma_\mu \Psi.$$

The low-energy expansion of the polarization functions in $z = q^2/(2m)^2$ is conveniently written as

$$\Pi^\delta(q^2) = \frac{3}{16 \pi^2} \sum_{n=-1}^{\infty} \overline{C}^\delta_n z^n, \quad (3)$$

where the expansion coefficients $\overline{C}^\delta_n$ are computed up to four-loop order in perturbative QCD. The expansion in the coupling constant $\alpha_s/\pi$ up to four-loop order is given by

$$\overline{C}^\delta_n = \overline{C}^{(0),\delta}_n + \left( \frac{\alpha_s}{\pi} \right) \overline{C}^{(1),\delta}_n + \left( \frac{\alpha_s}{\pi} \right)^2 \overline{C}^{(2),\delta}_n + \left( \frac{\alpha_s}{\pi} \right)^3 \overline{C}^{(3),\delta}_n + \ldots. \quad (4)$$
We decompose the coefficient $C_n^{(i),\delta} \ (i = 0, 1, 2, 3)$ for $n > 0$ into the non-logarithmic and logarithmic parts

$$C_n^{(i),\delta} = \sum_{j=0}^{i} C_n^{(j),\delta} l_m^j,$$

with $l_m = \log \left( \frac{m_\text{MS}^2}{\mu_0^2} \right)$. Furthermore we classify the diagrams with respect to the number of closed fermion loops inserted into a diagram. The symbol $n_l$ denotes the number of light quarks and the symbol $n_h = 1$ denotes a heavy quark with mass $m$. This decomposition at four-loop order is given by

$$C_n^{(3),\delta} = C_{n,0}^{(3),\delta} + C_{n,h}^{(3),\delta} n_h + C_{n,l}^{(3),\delta} n_l + C_{n,hh}^{(3),\delta} n_h^2 + C_{n,h,l}^{(3),\delta} n_h n_l + C_{n,l,l}^{(3),\delta} n_l^2. \quad (6)$$

The bar indicates that renormalization of $m$, $\alpha_s$ and the current has been performed in the $\overline{\text{MS}}$-scheme. We have checked that for $n = 1$ both scalar and axial-vector polarization functions $\Pi^\delta(q^2)$ obey the standard renormalization group equation (RGE). For the vector current correlator the longitudinal part $\Pi_v^L(q^2)$ of the polarization function is zero due to the vector Ward-identity. The longitudinal part of the axial-vector correlator obeys the axial Ward-identity \cite{23–27}

$$q^4 \Pi_v^A(q^2) = (2m)^2 q^2 \Pi^p(q^2) + \text{contact term}. \quad (7)$$

Inserting the expansion of Eq.(3) leads to

$$\sum_{n=-1}^{\infty} C_L^{n} z^n = \sum_{n=-1}^{\infty} C_p^{n} z^{n-1} + \frac{1}{q^4} \text{contact term}. \quad (8)$$

Performing a shift in the summation index $n = k + 1$ on the r.h.s. of Eq.(8) and comparing the coefficients of the different orders in $z^k$ in both sides leads to

$$C_L^{k} = C_p^{k+1}, \quad (9)$$

for $k \geq -1$, which allows to obtain the second moment of the pseudo-scalar correlator from the calculation of the first moment of the longitudinal part of the axial-vector correlator. The contact term in Eq.(8) only contributes to the order $1/z^2$.

3 Calculations and Results

In a first step the program QGRAF \cite{28} has been used to generate all necessary diagrams. Subsequently all appearing integrals have been mapped on a small set of 13 master integrals with the traditional Integration-By-Parts (IBP) method \cite{29} in combination with Laporta’s algorithm \cite{30, 31}. This procedure has been coded with the help of the
programs FORM [32–34] and FERMAT [35]. The remaining master integrals are shown in Fig. 1 in the standard basis. They have first been determined in Ref. [36] with the method of difference equation [31, 37] and subsequently in Ref. [38], where the method of ε-finite basis has been introduced. Also other authors [19, 39–44] have contributed in this connection with analytical or numerical results.

Figure 1: Master integrals in the standard basis. The solid (dashed) lines denote massive (massless) propagators.

The four-loop moment with $n = 1$ is known analytically for all four correlators. For the scalar correlator it is given by

\[
\begin{align*}
\mathcal{C}_{1,0}^{(30),s} &= \frac{240320}{1701} a_5 + \frac{513923}{5103} a_4 + \frac{32995}{1701} \zeta_5 - \frac{1707578737}{3265920} \zeta_3 - \frac{6008}{5103} \log^5(2) \\
&\quad + \frac{513923}{122472} \log^4(2) + \frac{30040}{15309} \log^3(2) \pi^2 - \frac{122472}{5103} \log^2(2) \pi^2 \\
&\quad + \frac{20122}{15309} \log(2) \pi^4 + \frac{11458913}{2939328} \pi^4 + \frac{183424051}{4898880}, \\
\mathcal{C}_{1,h}^{(30),s} &= \frac{13326713}{34020} a_4 + \frac{4}{9} \zeta_5 + \frac{609136933177}{2514758400} \zeta_3 + \frac{13326713}{816480} \log^4(2) \\
&\quad - \frac{816480}{13217613} \log^2(2) \pi^2 - \frac{97977600}{2939328} \pi^4 + \frac{91614310199}{3772137600}, \\
\mathcal{C}_{1,t}^{(30),s} &= \frac{14}{81} a_4 + \frac{6278503}{233280} \zeta_3 + \frac{7}{972} \log^4(2) - \frac{7}{972} \log^2(2) \pi^2 \\
&\quad - \frac{35927}{116640} \pi^4 - \frac{2197597}{349920}, \\
\mathcal{C}_{1,hl}^{(30),s} &= \frac{205}{324} a_4 - \frac{38171}{62208} \zeta_3 - \frac{205}{7776} \log^4(2) + \frac{205}{7776} \log^2(2) \pi^2 \\
\mathcal{C}_{1,hh}^{(30),s} &= \frac{27479}{34020} \zeta_3 + \frac{1729337}{1377810},
\end{align*}
\]
\[
\mathcal{T}^{(30),s}_{1,hl} = \frac{+2009 \pi^4 + 1937539}{186624} - n_t n_h \left( \frac{806681}{4199040} - \frac{10493}{62208} \zeta_3 \right) - n_h^2 \left( \frac{806681}{4199040} - \frac{10493}{62208} \zeta_3 \right) \]

\[
\mathcal{T}^{(31),s}_1 = \frac{197329 \zeta_3 - 2573}{1296} + n_h \left( \frac{2541989 - 429089 \zeta_3}{233280 - 31104 \zeta_3} \right)
+ n_l \left( \frac{5843}{1215} - \frac{17939}{1944} \zeta_3 \right) - n_h^2 \left( \frac{6059}{349920} - \frac{1435}{5184} \zeta_3 \right)
+ n_t n_h \left( \frac{1597}{69984} + \frac{1435}{5184} \zeta_3 \right) + n_t^2 \frac{238}{1215},
\]

\[
\mathcal{T}^{(32),s}_1 = \frac{7381}{1620} - \frac{n_h}{1215} - \frac{n_l}{1215} + \frac{n_h^2}{3645} + \frac{n_l n_h}{3645} + \frac{n_t^2}{3645},
\]

where \( \zeta_n \) denotes the Riemann zeta-function and \( a_n = \text{Li}_n(1/2) \). The result for the axial-vector correlator is given by

\[
\mathcal{T}^{(30),a}_{1,0} = -\frac{3996704}{25515} a_5 + \frac{5966100779}{12247200} a_4 + \frac{499588}{382725} \log^5(2) + \frac{5966100779}{293932800} \zeta_5,
\]

\[
\mathcal{T}^{(30),a}_{1,h} = -\frac{453463328653}{4115059200} a_5 + \frac{920203769}{680400} a_4 + \frac{920203769}{16329600} \log^4(2) - \frac{2277007338343}{2743372800} \zeta_3,
\]

\[
\mathcal{T}^{(30),a}_{1,l} = -\frac{839808}{16384897} \pi^4 + \frac{3326873}{313282529} \zeta_3,
\]

\[
\mathcal{T}^{(30),a}_{1,hl} = -\frac{16796160}{15134719} \pi^4 + \frac{303799}{29160} \zeta_3,
\]

\[
\mathcal{T}^{(30),a}_{1,hh} = -\frac{16533720}{408240} \zeta_3,
\]

\[
\mathcal{T}^{(30),a}_{1,hl} = -\frac{21592349}{50388480} a_4 - \frac{1499}{39312} \log^4(2) + \frac{1499}{93312} \pi^2 - \frac{3732480}{73451} \zeta_3,
\]

\[
\mathcal{T}^{(30),a}_{1,hl} = -\frac{11197440}{42133} \pi^4 - \frac{56}{405} \zeta_3,
\]

\[
\mathcal{T}^{(31),a}_1 = -\frac{96167813}{311040} \zeta_3 - \frac{164649889}{466560} - \frac{n_t^2}{3645} \left( \frac{806681}{4199040} - \frac{10493}{62208} \zeta_3 \right)
+ \frac{349337}{4199040} \left( \frac{10493}{62208} \zeta_3 \right) + \frac{397}{3645} n_h + \frac{33280729}{933120}.
\]
\[
\begin{align*}
C^{(32),a}_1 &= -\frac{59917211}{1866240} \zeta_3 + n_l \left( \frac{22748503}{699840} - \frac{14152979}{466560} \zeta_3 \right), \\
C^{(33),a}_1 &= \frac{7}{15} - n_h \frac{4}{27} - n_l \frac{4}{27} + n_h^2 \frac{4}{405} + n_l n_h \frac{8}{405} + n_l^2 \frac{4}{405}.
\end{align*}
\]

Numerical results for the pseudo-scalar case have first been presented in Ref. [6]. The analytical result is given in Appendix A, together with the corresponding one for the vector current correlator, taken from Ref. [13, 14]. The coefficients \(C^\delta_1 (\delta = s, a, p, v)\) and \(C^\delta_2\) are listed in numerical form in Table 1.

| \(\delta\) | \(n\) | \(C^0_1\) | \(C^{10}_1\) | \(C^{11}_1\) | \(C^{20}_1\) | \(C^{21}_1\) | \(C^{22}_1\) | \(C^{30}_1\) | \(C^{31}_1\) | \(C^{32}_1\) | \(C^{33}_1\) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \(s\) | 1 | 0.8000 | 0.6025 | 0.0000 | -7.7402 | -1.2551 | 0.0000 | -5.4135 | 32.8284 | 2.6149 | 0.0000 |
| \(a\) | 1 | 0.5333 | 0.8461 | 1.0667 | -1.0171 | 1.6925 | -0.0444 | -2.4297 | 5.2981 | 0.4406 | 0.0321 |
| \(p\) | 1 | 1.3333 | 3.1111 | 0.0000 | 0.1154 | -6.4815 | 0.0000 | -1.2224 | 2.5008 | 13.5031 | 0.0000 |
| \(p\) | 2 | 0.5333 | 2.0642 | 1.0667 | 7.2362 | 1.5909 | -0.0444 | 7.0659 | -7.5852 | 0.5505 | 0.0321 |
| \(v\) | 1 | 1.0667 | 2.5547 | 2.1333 | 2.4967 | 3.3130 | -0.0889 | -5.6404 | 4.0669 | 0.9590 | 0.0642 |

Table 1: Numerical values for \(C^\delta_1 (\delta = s, a, p, v)\) and \(C^\delta_2\) for \(n_l = 3\) which corresponds to the case of the charmed quark.

In Appendix B we provide in addition the expansion coefficients for \(n = 0, -1\). Except for \(C^0_1\) they are only known numerically, however, with high precision. A completely analytical result would require the analytical determination of the constants \(T_{54,2}, T_{64,2}, T_{61,2}, T_{62,3}, T_{72,1}, T_{71,1}, T_{81,1}, T_{91,1}\), where \(T_{n,i}\) are the coefficients of the \(\varepsilon\)-expansion \((\varepsilon = (4 - d)/2)\) of the master integrals \(T_n\) as defined in Fig. 1

\[
T_n = \sum_{i=n_{\text{min}}}^{\infty} \varepsilon^i T_{n,i}.
\]

## 4 Summary and Conclusion

We have computed the lowest coefficients in the low-energy expansion of the scalar and axial-vector current correlators at four-loop order in perturbative QCD. All appearing loop-integrals have been reduced to known master integrals, using Laporta’s algorithm. We also gave the details of the calculation as well as the analytical results for the moments of the pseudo-scalar correlator. These results allow to reduce the theoretical error originating from higher order corrections in the determination of fundamental
constants of QCD, like the strong coupling constant and the charm-quark mass in the context of lattice calculations.

Acknowledgments:
C.S. would like to thank K.G. Chetyrkin, J.H. Kühn and M. Steinhauser for their support and many discussions. This research was supported by U.S. Department of Energy contract No.DE-AC02-98CH10886. Our results are also available in computer readable form under the URL http://arxiv.org by downloading the source of this article.

A Moments for the pseudo-scalar and vector correlator

The analytical result for the first two moments of the pseudo-scalar correlator presented in numerical form in Ref. [6] is given by

\[
C_{1,0}^{(30),p} = \frac{10304}{243} a_5 - \frac{130535}{1458} a_4 - \frac{35189}{243} \zeta_5 + \frac{465367}{15552} \zeta_3 - \frac{1288}{3645} \log^5(2)
\]
\[
- \frac{130535}{34992} \log^4(2) + \frac{1288}{2187} \log^3(2) \pi^2 + \frac{130535}{34992} \log^2(2) \pi^2
\]
\[
+ \frac{10094}{10935} \log(2) \pi^4 - \frac{5686729}{4199040} \pi^4 - \frac{34992}{34992}
\]
\[
C_{1,h}^{(30),p} = \frac{294727}{2430} a_4 + \frac{80}{9} \zeta_5 - \frac{7656133}{85050} \zeta_3 - \frac{294727}{58320} \log^4(2)
\]
\[
+ \frac{58320}{58320} \log^2(2) \pi^2 + \frac{9593011}{6998400} \pi^4 - \frac{373843}{453600}
\]
\[
C_{1,l}^{(30),p} = \frac{115}{243} a_4 - \frac{157783}{7776} \zeta_3 + \frac{115}{5832} \log^4(2) - \frac{115}{5832} \log^2(2) \pi^2
\]
\[
+ \frac{115709}{699840} \pi^4 - \frac{7381}{5832}
\]
\[
C_{1,hl}^{(30),p} = \frac{403}{630} \zeta_3 - \frac{1777}{25515}
\]
\[
C_{1,hl}^{(30),p} = \frac{-4}{9} a_4 - \frac{1}{54} \log^4(2) + \frac{1}{54} \log^2(2) \pi^2 + \frac{49}{6480} \pi^4 - \frac{35}{96} \zeta_3 - \frac{3457}{11664}
\]
\[
C_{1,ll}^{(30),p} = \frac{277}{729}
\]
\[
C_{1}^{(31),p} = \frac{-71203}{864} \zeta_3 + \frac{143465}{1296} - n_h \left( \frac{12439}{1944} - \frac{2315}{1296} \zeta_3 \right) - n_l \left( \frac{17191}{1944} \right)
\]
\[
- \frac{6473}{1296} \zeta_3 + \frac{n_h^2}{243} + \frac{7}{36} \zeta_3 + n_l n_h \left( \frac{64}{243} + \frac{7}{36} \zeta_3 \right) + n_l^2 \frac{50}{243}
\]
\[
C_{1}^{(32),p} = \frac{847}{36} - n_h \frac{77}{27} + n_l \frac{77}{27} + n_h^2 \frac{7}{81} + n_l n_h \frac{14}{81} + n_l^2 \frac{7}{81}
\]
\begin{align}
\mathcal{C}_{2,0}^{(30),p} &= \frac{278048}{945} a_5 - \frac{3509250197}{4082400} a_4 - \frac{45178393}{34020} \zeta_5 + \frac{2871407869129}{1306368000} \zeta_3 \\
&\quad - \frac{34756}{14175} \log^5(2) - \frac{3509250197}{97977600} \log^4(2) + \frac{3509250197}{97977600} \log^2(2) \pi^2 \\
&\quad + \frac{8505}{14175} \log^3(2) \pi^2 + \frac{180277}{28350} \log(2) \pi^4 - \frac{97977600}{1175731200} \pi^4 \\
&\quad - \frac{653184000}{127853600}, \\
\mathcal{C}_{2,h}^{(30),p} &= -\frac{978527581}{680400} a_4 + \frac{62}{3} \zeta_5 - \frac{8948001289387}{1005903600} \zeta_3 - \frac{978527581}{16329600} \log^4(2) \\
&\quad + \frac{16329600}{16329600} \log^2(2) \pi^2 + \frac{351736938533}{391910400} \pi^4 - \frac{3017710080}{3017710080}, \\
\mathcal{C}_{2,l}^{(30),p} &= -\frac{493}{3240} a_4 - \frac{151413217}{933120} \zeta_3 - \frac{493}{77760} \log^4(2) + \frac{493}{77760} \log^2(2) \pi^2 \\
&\quad + \frac{622080}{8402929} \pi^4 + \frac{279936}{279936}, \\
\mathcal{C}_{2,hl}^{(30),p} &= -\frac{179}{1296} a_4 + \frac{17839}{1244160} \zeta_3 - \frac{179}{31104} \log^4(2) \\
&\quad + \frac{179}{31104} \log^2(2) \pi^2 + \frac{8771}{3732480} \pi^4 - \frac{1951867}{16796160}, \\
\mathcal{C}_{2,l}^{(30),p} &= -\frac{56}{405} \zeta_3 + \frac{15511}{65610}, \\
\mathcal{C}_{2}^{(31),p} &= -\frac{54646039}{103680} \zeta_3 + \frac{97431227}{155520} - n_h \left( \frac{18258607}{311040} - \frac{30278653}{622080} \zeta_3 \right) \\
&\quad - n_l \left( \frac{13992849}{233280} - \frac{7668337}{155520} \zeta_3 \right) - n_h n_l \left( \frac{39137}{1399680} - \frac{1253}{20736} \zeta_3 \right) \\
&\quad + n_h n_l \left( \frac{5511}{1399680} + \frac{247}{20736} \zeta_3 \right) + n_l^2 \frac{247}{3645}, \\
\mathcal{C}_{2}^{(32),p} &= -\frac{257}{540} - n_h \frac{136}{1215} - n_l \frac{136}{1215} + n_h^2 \frac{119}{3645} + n_h n_l \frac{238}{3645} + n_l^2 \frac{119}{3645}, \\
\mathcal{C}_{2}^{(33),p} &= \frac{7}{15} - n_h \frac{4}{27} - n_l \frac{4}{27} + n_h^2 \frac{4}{405} + n_h n_l \frac{8}{405} + n_l^2 \frac{4}{405}. 
\end{align}

The first moment of the vector current correlator has been obtained in Ref. [13, 14]

\begin{align}
\mathcal{C}_{1,0}^{(30),v} &= -\frac{1019840}{5103} a_5 - \frac{84951877}{306180} a_4 - \frac{3655}{10206} \zeta_5 + \frac{17554601717}{32659200} \zeta_3 + \frac{25496}{15309} \log^5(2) \\
&\quad - \frac{84951877}{7348320} \log^4(2) - \frac{84951877}{45927} \log^3(2) \pi^2 + \frac{84951877}{7348320} \log^2(2) \pi^2 \\
&\quad - \frac{299635}{881798400} \log(2) \pi^4 - \frac{5397779543}{146966400}, \\
\mathcal{C}_{1,h}^{(30),v} &= -\frac{1394804}{8505} a_4 + \frac{128}{27} \zeta_5 - \frac{95617883401}{943034400} \zeta_3 - \frac{348701}{51030} \log^4(2)
\end{align}
\[ \mathcal{C}_{(30),v}^{(30),v} = -\frac{328701}{7290} \log(2) \pi^2 + \frac{48350497}{1399680} \zeta_3 - \frac{4793}{174960} \log^4(2) + \frac{372689}{839808} \pi^4 - \frac{9338899}{2099520}, \]  

\[ \mathcal{C}_{(30),v}^{(30),v} = -\frac{116}{243} a_4 - \frac{38909}{58320} \zeta_3 - \frac{29}{1458} \log^4(2) + \frac{262877}{787320} \pi^4 + \frac{29}{1458} \log^2(2) \pi^2 + \frac{2}{787320}, \]  

\[ \mathcal{C}_{(30),v}^{(30),v} = \frac{1}{7290} \zeta_3 + \frac{163868}{295245}, \]  

\[ C_{(30),v}^{(30),v} = \frac{42173}{98415} \frac{1}{405} \zeta_3, \]  

\[ C_{(31),v}^{(30),v} = \frac{7236859}{38880} - \frac{10589033}{77760} \zeta_3 - n_h \left( \frac{520823}{34992} - \frac{1049579}{116640} \zeta_3 \right) \]  

\[ -n_l \left( \frac{1103117}{58320} - \frac{1305359}{116640} \zeta_3 \right) - n_h^2 \left( \frac{14483}{65610} - \frac{203}{972} \zeta_3 \right) \]  

\[ -n_h n_l \left( \frac{3779}{65610} - \frac{203}{972} \zeta_3 \right) + n_h^2 \frac{1784}{10935}, \]  

\[ C_{(32),v}^{(30),v} = -\frac{451}{405} + n_h \frac{1574}{3645} + n_l \frac{1574}{3645} + n_h^2 \frac{236}{10935} + n_h n_l \frac{472}{10935} + n_l^2 \frac{236}{10935}. \]  

\[ C_{(33),v}^{(30),v} = \frac{14}{15} - n_h \frac{8}{27} - n_l \frac{8}{27} + n_h^2 \frac{8}{405} + n_h n_l \frac{16}{405} + n_l^2 \frac{8}{405}. \]  

**B Moments for n = -1, 0**

The moments for \( n = -1 \) and \( n = 0 \) exhibit an overall UV divergence. The corresponding \( 1/\varepsilon \)-poles are dropped by definition in the \( \overline{\text{MS}} \)-scheme. The finite parts are given by

\[ C_{(3),s}^{(3),s} = -325.6276432 + 16.39537650 n_h + 19.76434509 n_l \]  

\[ -1.670198265 n_h^2 - 0.9856898698 n_h n_l + 0.7103788267 n_l^2, \]  

\[ C_0^{(3),s} = -82.24477459 + 14.55555905 n_h + 22.86448444 n_l \]  

\[ -0.6046449347 n_h^2 - 1.620853371 n_h n_l - 1.047986065 n_l^2, \]  

\[ C_{(3),p}^{(3),p} = -358.5973626 + 30.02655475 n_h + 35.65136262 n_l \]  

\[ -1.485773643 n_h^2 - 2.003379176 n_h n_l - 0.2645810947 n_l^2, \]  

\[ C_0^{(3),p} = -25.62696915 + 8.147150256 n_h + 6.938402913 n_l \]  

\[ +0.08246295965 n_h^2 - 0.5283433562 n_h n_l - 0.4972293123 n_l^2, \]  

\[ C_{(3),a}^{(3),a} = 25.62696915 - 8.147150256 n_h - 6.938402913 n_l. \]
\begin{equation}
C^{(30),a}_{0,0} = \frac{-0.08246295965 n_h^2 + 0.5283433562 n_l n_h + 0.4972293123 n_l^2}{59584} a_5 + \frac{362533}{4374} a_4 + \frac{668057}{5832} \zeta_5 - \frac{29132419}{233280} \zeta_3 + \frac{7448}{362533} \log^5(2) + \frac{362533}{104976} \log^4(2) - \frac{7448}{6561} \log^3(2) \pi^2
+ \frac{362533}{104976} \log^2(2) \pi^2 - \frac{35584}{32805} \log(2) \pi^4
- \frac{309377}{2519424} \pi^4 + \frac{402928129}{4199040},
\end{equation}

\begin{equation}
C^{(30),a}_{0,0l} = \frac{-31}{162} \zeta_3 - \frac{1655}{69984},
\end{equation}

\begin{equation}
C^{(30),a}_{0,0l} = \frac{2}{27} a_4 - \frac{1919}{2592} \zeta_3 + \frac{1}{324} \log^4(2) - \frac{1}{324} \log^2(2) \pi^2
+ \frac{49}{38880} \pi^4 - \frac{8748}{4255},
\end{equation}

\begin{equation}
C^{(30),a}_{0,0h} = \frac{-913}{1134} \zeta_3 + \frac{317311}{489888},
\end{equation}

\begin{equation}
C^{(30),a}_{0,0h} = \frac{-1025}{729} a_4 + \frac{310001}{23328} \zeta_3 - \frac{1025}{17496} \log^4(2) + \frac{1025}{17496} \log^2(2) \pi^2
- \frac{18635}{419904} \pi^4 - \frac{339551}{139968},
\end{equation}

\begin{equation}
C^{(30),a}_{0,0h} = \frac{859259}{7290} a_4 - \frac{85}{27} \zeta_5 + \frac{12520907}{145800} \zeta_3 + \frac{859259}{174960} \log^4(2)
- \frac{859259}{174960} \log^2(2) \pi^2 - \frac{28913567}{20995200} \pi^4 - \frac{556867}{1166400},
\end{equation}

where for compactness \( \mu = m \) has been chosen. The logarithms \( \log(\mu^2/m^2) \) can be reconstructed with the help of the RGE.

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