$N = 8$ BPS Black Holes with 1/2 or 1/4 supersymmetry and Solvable Lie algebra decompositions

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Abstract

In the context of $N = 8$ supergravity we construct the general form of BPS 0–branes that preserve either 1/2 or 1/4 of the original supersymmetry. We show how such solutions are related to suitable decompositions of the 70 dimensional solvable Lie algebra that describes the scalar field sector. We compare our new results to those obtained in a previous paper for the case of 1/8 supersymmetry preserving black holes. Each of the three cases is based on a different solvable Lie algebra decomposition and leads to a different structure of the scalar field evolution and of their fixed values at the horizon of the black hole.

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# 1 Introduction

Recent attempts to study the non-perturbative properties of gauge and string theories have made an essential use of the low energy supergravity lagrangians encoding their global and local symmetries. In this analysis, the BPS saturated states play an important role, and the interpretation of the classical solutions as states belonging to the non-perturbative string spectrum \[1\],\[2\] has found strong support with the advent of D-branes \[3\], allowing the direct construction of the BPS states.

In virtue of several non-perturbative dualities between different string models, a given supergravity theory is truly specified by the dimension of space-time, the number of unbroken supersymmetries and the massless matter content.

Each supergravity theory can be formulated in terms of a non-compact form of the group \(G = E_{11-D}\) under which the \(p\)-forms in the theory transform in a suitable representation. The scalar fields parametrize the coset \(E_{11-D}/H\), \(H\) being the maximal subgroup of \(E_{11-D}\). It is now well known \[4\] that the discrete version \(E_{11-D}(\mathbb{Z})\), is the U-duality group unifying S and T-dualities and is supposed to be an exact symmetry of non-perturbative string theory.

From an abstract viewpoint, BPS saturated states are characterized by the fact that they preserve a fraction of the original supersymmetries. This means that there is a suitable projection operator \(P^2_{BPS} = P_{BPS}\) acting on the supersymmetry charge \(Q_{SUSY}\), such that:

\[
(P_{BPS} Q_{SUSY}) \mid \text{BPS} >= 0. \tag{1}
\]

Supersymmetry representation theory implies that it is actually an exact state of non-perturbative string theory. Moreover, the classical BPS state is by definition an element of a short supermultiplet and, if supersymmetry is unbroken, it cannot be renormalized to a long supermultiplet.

Since the supersymmetry transformation rules are linear in the first derivatives of the fields, eq.\((1)\) is actually a system of first order differential equations that must be combined with the second order field equations of supergravity. The solutions common to both system of equations are the classical BPS saturated states.

This paper investigates the most general BPS saturated black hole solutions of \(D = 4\) supergravity \[4\],\[5\],\[6\] preserving 1/2 or 1/4 of the \(N = 8\) supersymmetry (thus, in our case the scalar fields span the coset space \(E_{7(7)}/SU(8)\)). This is achieved along the same lines of previous work on the 1/8 case \[8\], where the content of dynamical fields and charges was identified, but an explicit solution was given only by setting to zero the axion fields. Here we will find a complete solution corresponding to a specific configuration of the scalar fields and charges. This particular configuration is obtained by performing \(G\) and \(H\) transformations in such a way that the quantized and central charges are put in the normal frame. Once the solution is obtained, we can recover the most general one by acting on the charges and the scalar fields by the U-duality group.

In terms of the gravitino and dilatino physical fields \(\psi_{A\mu}, \chi_{ABC}, A, B, C = 1, \ldots, 8\), equation \(\text{(1)}\) is equivalent to

\[
\delta_{\epsilon} \psi_{A\mu} = \delta_{\epsilon} \chi_{ABC} = 0 \tag{2}
\]
whose solution is given in terms of the Killing spinor \( \epsilon_A(x) \) subject to the supersymmetry preserving condition

\[
\gamma^0 \epsilon_a = i C_{ab} \epsilon^b \quad ; \quad a, b = 1, \ldots, n_{\text{max}} \\
\epsilon_X = 0 \quad ; \quad X = n_{\text{max}}, \ldots, 8 \\
A = \{a, X\}
\]

where \( n_{\text{max}} \) is twice the number of unbroken supersymmetries.

In the present context, eq. (2) has two main features:

1. It requires an efficient parametrization of the scalar field sector. The use of the rank 7 Solvable Lie Algebra (SLA) \( \text{Solv}_7 \) associated with \( E_{7(7)}/SU(8) \) is of great help in this problem.

2. It breaks the original \( SU(8) \) automorphism group of the supersymmetry algebra to the subgroup \( \hat{H} = \text{Usp}(n_{\text{max}}) \times SU(8 - n_{\text{max}}) \times U(1) \)

Indeed, SLA’s are quite useful in the construction of BPS saturated black hole solutions since they provide us with an efficient tool for decomposing the scalar sector parametrizing \( E_{7(7)}/SU(8) \) in a way appropriate to the decomposition of the isotropy subgroup \( SU(8) \) with respect to the subgroup \( \hat{H} \subset SU(8) \) leaving invariant the Killing condition on the supersymmetry parameters.

The relevant decomposition also yields an answer to the fundamental question: *How many scalar fields are essentially dynamical, namely cannot be set to constants up to U–duality transformations?*

The answer is provided in terms of the SLA description of the scalar fields, whose evolution from arbitrary values at infinity to fixed values at the horizon is best understood and dealt with when the scalars are algebraically characterized as the parameters associated with the generators of the SLA. In fact, an SLA \( \text{Solv}(G/H) \) is associated with every non–compact homogeneous space and so in particular with \( E_{7(7)}/SU(8) \). The scalar fields are in one–to–one correspondence with the generators of the SLA.

Once the charges are reduced to the normal frame, we consider the stabilizer \( G_{\text{stab}} \subset E_{7(7)} \) leaving the charge configuration invariant, and its normalizer group \( G_{\text{norm}}. \) Considering its maximal compact subgroup \( H_{\text{norm}} \subset G_{\text{norm}} \) we obtain a solvable Lie algebra \( \text{Solv}_{\text{norm}}(G_{\text{norm}}/H_{\text{norm}}) \) which is a subalgebra of \( \text{Solv} \left( E_{7(7)}/SU(8) \right) \). The corresponding scalar fields are the essentially dynamical ones up to U–duality transformations. The remaining scalar fields are either gauge–fixed to zero by putting the vector of quantized charges into normal form or they belong to \( H_{\text{norm}} \) and are thus ineffective.

After completing such decomposition in terms of SLA’s we use the group theoretical structure of the Killing spinor equations in order to decompose them into fragments of the \( SU(8) \) subgroup \( \hat{H} \). This analysis is completely consistent with the results based on the SLA’s and we are left with a set of first order equations on the surviving scalar fields and electromagnetic field strengths. At this point one uses the U-duality group theoretical construction of the coset representative and the kinetic matrix of the vectors explained in [8] to compute the geometrical
objects appearing in the Killing spinor equations. This also allows us to compute the truncated bosonic lagrangian for the model reduced to the essential degrees of freedom and enjoying 1/2 or 1/4 of the original supersymmetry. Coupling Killing spinor first order differential equation with the second order equations of motion we find solutions for both. Actually the reduced lagrangian obtained in this way were already known and well studied in the literature and our solution fit nicely in the general scheme studied in ref [9].

The paper is organized as follows:

In section 2 we give a preliminary discussion of the $N = 8$ BPS black holes. Using the results of ref [10] we discuss the stabilizer and normalizer group for the charges reduced to normal form. Furthermore, we give a the SLA decompositions in the 1/2 and 1/4 case and we quote the results obtained from the detailed study of the Killing spinor equations as far as the content of physical fields and charges is concerned. For completeness we also discuss the analogous results obtained in ref. [3] in the 1/8 case.

In section 3 we give a short resume’ of the group theoretical structure of $N = 8$ supergravity in our conventions and the definition of the central and quantized charges.

In section 4 we analyse the decomposition of the Killing spinor equations under the group $\hat{H}$ in the 1/2 case and compute explicitly the $U$- duality coset representative and the associated metric for the surviving vector field. This enables us to recover the reduced Lagrangian and to identify it as a well known model studied in the literature.Furthermore we also give an alternative approach to the 1/2 case using the Dynkin formalism for the determination of the relevant physical quantities.

In section 5 the same kind of analysis is given for the 1/4 case: in particular we show how our Lagrangian and solution fits in the so called ”p-brane taxonomy of ref. [4].

In section 6 we give the conclusions.

2 $N=8$ BPS black holes: a general discussion

The $D = 4$ supersymmetry algebra with $N = 8$ supersymmetry charges is given by

$$\{\overline{Q}_{A\alpha} , \overline{Q}_{B\beta}\} = i (C \gamma^\mu)_{\alpha\beta} P_\mu \delta_{AB} - C_{\alpha\beta} Z_{AB}$$

$$(A, B = 1, \ldots , 8)$$

(3)

where the SUSY charges $\overline{Q}_A \equiv Q_A^\dagger \gamma_0 = Q_A^T C$ are Majorana spinors, $C$ is the charge conjugation matrix, $P_\mu$ is the 4–momentum operator and the antisymmetric tensor $Z_{AB} = -Z_{BA}$ is the central charge operator. It can always be reduced to normal form

$$Z_{AB} = \begin{pmatrix} \epsilon Z_1 & 0 & 0 & 0 \\ 0 & \epsilon Z_2 & 0 & 0 \\ 0 & 0 & \epsilon Z_3 & 0 \\ 0 & 0 & 0 & \epsilon Z_4 \end{pmatrix}$$

(4)

where $\epsilon$ is the $2 \times 2$ antisymmetric matrix, (every zero is a $2 \times 2$ zero matrix) and the four skew eigenvalues $Z_i$ of $Z_{AB}$ are the central charges.
Consider the reduced supercharges:

\[
S_A^\pm = \frac{1}{2} \left( Q_A \gamma_0 \pm i C_{AB} \bar{Q}_B \right)_\alpha \quad ; \quad A, B = 1, \ldots, n_{\text{max}}
\]

\[
S_A^\pm = 0 \quad A > n_{\text{max}}
\]  \hspace{1cm} (5)

where \( C_{AB} \) is the invariant symplectic metric (\( C = -C^T, C^2 = -1 \)). They can be regarded as the result of applying a projection operator to the supersymmetry charges: \( S_A^\pm = Q_B^\pm \Pi_{BA}^\pm \) where \( \Pi_{BA}^\pm = \frac{1}{2} (1 \delta_{BA} \pm i C_{BA} \gamma_0) \). In the rest frame where the four momentum is \( P_\mu = (M, 0, 0, 0) \), we obtain the algebra: \( \{ S_A^\pm, S_B^\pm \} = \pm C_{AC} C_{CB} (M \mp Z_I) \delta_{IJ} \) and the BPS states that saturate the bounds \( (M \pm Z_I) |\text{BPS},i\rangle = 0 \) are those which are annihilated by the corresponding reduced supercharges:

\[
S_A^\pm |\text{BPS},i\rangle = 0
\]  \hspace{1cm} (6)

Eq.\( (6) \) defines \textit{short multiplet representations} of the original algebra \( (3) \) in the following sense: one constructs a linear representation of \( (3) \) where all states are identically annihilated by the operators \( S_A^\pm \) for \( A = 1, \ldots, n_{\text{max}} \). If \( n_{\text{max}} = 2 \) we have the minimum shortening, if \( n_{\text{max}} = 8 \) we have the maximum shortening. On the other hand eq.\( (3) \) can be translated into first order differential equations on the bosonic fields of supergravity whose common solutions with the ordinary field equations are the BPS saturated black hole configurations. In the case of maximum shortening \( n_{\text{max}} = 8 \) the black hole preserves \( 1/2 \) supersymmetry, in the case of intermediate shortening \( n_{\text{max}} = 4 \) it preserves \( 1/4 \), while in the case of minimum shortening it preserves \( 1/8 \).

\section*{2.1 The Killing spinor equation and its covariance group}

In order to translate eq.\( (3) \) into first order differential equations on the bosonic fields of supergravity we consider a configuration where all the fermionic fields are zero and we set to zero the fermionic SUSY rules appropriate to such a background

\[
0 = \delta_{\text{fermions}} = \text{SUSY rule (bosons, } \epsilon_{Ai} \text{)}
\]  \hspace{1cm} (7)

and to a SUSY parameter that satisfies the following conditions:

\[
\xi^\mu \gamma_\mu \epsilon_A = i C_{AB} \epsilon_B \quad ; \quad A, B = 1, \ldots, n_{\text{max}}
\]

\[
\epsilon_A = 0 \quad ; \quad A > n_{\text{max}}
\]  \hspace{1cm} (8)

Here \( \xi^\mu \) is a time–like Killing vector for the space–time metric (in the following we just write \( \xi^\mu = \gamma^0 \) and \( \epsilon_A, \epsilon^A \) denote the two chiral projections of a single Majorana spinor: \( \gamma_5 \epsilon_A = \epsilon_A \) and \( \gamma_5 \epsilon^A = -\epsilon^A \). We name eq.\( (7) \) the \textit{Killing spinor equation} and the investigation of its group–theoretical structure was our main goal in ref \([8]\). There we restricted our attention to the case \( n_{\text{max}} = 2 \): here we consider the other two possibilities.

To appreciate the distinction among the three types of \( N = 8 \) black–hole solutions we need to recall the results of \([10]\) where a classification was given of the 56–vectors of quantized electric
and magnetic charges \( \vec{Q} \) characterizing such solutions. The basic argument is provided by the reduction of the central charge skew–symmetric tensor \( Z_{AB} \) to normal form. The reduction can always be obtained by means of local \( SU(8) \) transformations, but the structure of the skew eigenvalues depends on the orbit–type of the 56–dimensional charge vector which can be described by means of its stabilizer subgroup \( G_{stab}(\vec{Q}) \subset E_7(7) \):

\[
g \in G_{stab}(\vec{Q}) \subset E_7(7) \iff g \vec{Q} = \vec{Q}
\]

There are three possibilities:

| SUSY | Central Charge | Stabilizer \( \equiv G_{stab} \) | Normalizer \( \equiv G_{norm} \) |
|------|----------------|---------------------|---------------------|
| 1/2  | \( Z_1 = Z_2 = Z_3 = Z_4 \) | \( E_6(6) \) | \( O(1, 1) \) |
| 1/4  | \( Z_1 = Z_2 \neq Z_3 = Z_4 \) | \( SO(5, 5) \) | \( SL(2, \mathbb{R}) \times O(1, 1) \) |
| 1/8  | \( Z_1 \neq Z_2 \neq Z_3 \neq Z_4 \) | \( SO(4, 4) \) | \( SL(2, \mathbb{R})^3 \) |

where the normalizer \( G_{norm}(\vec{Q}) \) is defined as the subgroup of \( E_7(7) \) that commutes with the stabilizer:

\[
[G_{norm}, G_{stab}] = 0
\]

The main result of [8] is that the most general 1/8 black–hole solution of \( N = 8 \) supergravity is related to the normalizer group \( SL(2, \mathbb{R})^3 \). In this paper we show that analogous relations apply also to the other cases.

### 2.1.1 The 1/2 SUSY case

Here we have \( n_{max} = 8 \) and correspondingly the covariance subgroup of the Killing spinor equation is \( Usp(8) \subset SU(8) \). Indeed condition (8) can be rewritten as follows:

\[
\gamma^0 \epsilon_A = i C_{AB} \epsilon^B ; \quad A, B = 1, \ldots, 8
\]

where \( C_{AB} = -C_{BA} \) denotes an \( 8 \times 8 \) antisymmetric matrix satisfying \( C^2 = -I \). The group \( Usp(8) \) is the subgroup of unimodular, unitary \( 8 \times 8 \) matrices that are also symplectic, namely that preserve the matrix \( C \). Relying on eq. (11) we see that in the present case \( G_{stab} = E_6(6) \) and \( G_{norm} = O(1, 1) \). Furthermore we have the following decomposition of the 70 irreducible representation of \( SU(8) \) into irreducible representations of \( Usp(8) \):

\[
70 \overset{Usp(8)}{\rightarrow} 42 \oplus 1 \oplus 27
\]

We are accordingly lead to decompose the solvable Lie algebra as

\[
Solv_7 = Solv_6 \oplus O(1, 1) \oplus \mathbb{D}_6
\]

\[
70 = 42 + 1 + 27
\]
where, following the notation established in [11, 12]:

\[ \text{Solv}_7 \equiv \text{Solv} \left( \frac{E_{7(7)}}{SU(8)} \right) \]
\[ \text{Solv}_6 \equiv \text{Solv} \left( \frac{E_{6(6)}}{USp(8)} \right) \]

\[ \dim \text{Solv}_7 = 70 \quad ; \quad \text{rank } \text{Solv}_7 = 7 \]
\[ \dim \text{Solv}_6 = 42 \quad ; \quad \text{rank } \text{Solv}_6 = 6 \]

In eq. (14) \( \text{Solv}_6 \) is the solvable Lie algebra that describes the scalar sector of \( D = 5, N = 8 \) supergravity, while the 27–dimensional abelian ideal \( \mathbb{D}_6 \) corresponds to those \( D = 4 \) scalars that originate from the 27–vectors of supergravity one–dimension above [12]. Furthermore, we may also decompose the 56 charge representation of \( E_{7(7)} \) with respect to \( O(1,1) \times E_{6(6)} \) obtaining

\[ 56 \xrightarrow{USp(8)} (1, 27) \oplus (1, 27) \oplus (2, 1) \]  \hspace{1cm} (17)

In order to single out the content of the first order Killing spinor equations we need to decompose them into irreducible \( USp(8) \) representations. The gravitino equation is an 8 of \( SU(8) \) that remains irreducible under \( USp(8) \) reduction. On the other hand the dilatino equation is a 56 of \( SU(8) \) that reduces as follows:

\[ 56 \xrightarrow{USp(8)} 48 \oplus 8 \]  \hspace{1cm} (18)

Hence altogether we have that 3 Killing spinor equations in the representations 8, 8’, 48 constraining the scalar fields parametrizing the three subalgebras 42, 1 and 27. Working out the consequences of these constraints and deciding which scalars are set to constants, which are instead evolving and how many charges are different from zero is the content of section 4. As it will be seen explicitly there the content of the Killing spinor equations after \( USp(8) \) decomposition, is such as to set a constant 69 scalar fields parametrizing \( \text{Solv}_6 \oplus \mathbb{D}_6 \) thus confirming the SLA analysis discussed in the introduction: indeed in this case \( G_{\text{norm}} = O(1,1) \) and \( H_{\text{norm}} = 1 \), so that there is just one surviving field parametrising \( G_{\text{norm}} = O(1,1) \). Moreover, the same Killing spinor equations tell us that the 54 belonging to the two \((1,27)\) representation of eq. (17) are actually zero, leaving only two non–vanishing charges transforming as a doublet of \( O(1,1) \).

2.1.2 The 1/4 SUSY case

Here we have \( n_{\text{max}} = 4 \) and correspondingly the covariance subgroup of the Killing spinor equation is \( USp(4) \times SU(4) \times U(1) \subset SU(8) \). Indeed condition (8) can be rewritten as follows:

\[ \gamma^0 \epsilon_a = i C_{ab} \epsilon^b \quad ; \quad a, b = 1, \ldots, 4 \]
\[ \epsilon_X = 0 \quad ; \quad X = 5, \ldots, 8 \]

\hspace{1cm} (19)
where $C_{ab} = -C_{ba}$ denotes a $4 \times 4$ antisymmetric matrix satisfying $C^2 = -\mathbb{I}$. The group $Usp(4)$ is the subgroup of unimodular, unitary $4 \times 4$ matrices that are also symplectic, namely that preserve the matrix $C$.

We are accordingly lead to decompose the solvable Lie algebra as follows. Naming:

$$\text{Solv}_S \equiv \text{Solv} \left( \frac{SL(2,R)}{U(1)} \right)$$

$$\text{Solv}_T \equiv \text{Solv} \left( \frac{SO(6,6)}{SO(6) \times SO(6)} \right)$$

$$\dim \text{Solv}_S = 2 ; \quad \rank \text{Solv}_S = 1$$

$$\dim \text{Solv}_T = 36 ; \quad \rank \text{Solv}_T = 6$$

we can write:

$$\text{Solv}_7 = \text{Solv}_S \oplus \text{Solv}_T \oplus \mathcal{W}_{32}$$

$$70 = 2 + 36 + 32.$$  (21)

As shown in ref. [11],[12], the SLA’s $\text{Solv}_S$ and $\text{Solv}_T$ describe the dilaton–axion sector and the six torus moduli, respectively, in the interpretation of $N = 8$ supergravity as the compactification of Type IIA theory on a six–torus $T^6$ [12]. The rank zero abelian subalgebra $\mathcal{W}_{32}$ is instead composed by Ramond-Ramond scalars.

Introducing the decomposition (21), (22) we have succeeded in singling out a holonomy subgroup $SU(4) \times SU(4) \times U(1) \subset SU(8)$. Indeed we have $SO(6) \equiv SU(4)$. This is a step forward but it is not yet the end of the story since we actually need a subgroup $Usp(4) \times SU(4) \times U(1)$ corresponding to the invariance group of equation (12). This means that we must further decompose the SLA $\text{Solv}_T$. This latter is the manifold of the scalar fields associated with vector multiplets in an $N = 4$ decomposition of the $N = 8$ theory. Indeed the decomposition (21) with respect to the S–T–duality subalgebra is the appropriate decomposition of the scalar sector according to $N = 4$ multiplets.

The further SLA decomposition we need is:

$$\text{Solv}_T = \text{Solv}_{T5} \oplus \text{Solv}_{T1}$$

$$\text{Solv}_{T5} \equiv \text{Solv} \left( \frac{SO(5,6)}{SO(5) \times SO(6)} \right)$$

$$\text{Solv}_{T1} \equiv \text{Solv} \left( \frac{SO(1,6)}{SO(6)} \right)$$

where we rely on the isomorphism $Usp(4) \equiv SO(5)$ and we have taken into account that the $70$ irreducible representation of $SU(8)$ decomposes with respect to $Usp(4) \times SU(4) \times U(1)$ as follows.
\[ 70 \ Usp(4) \times SU(4) \times U(1) \rightarrow (1, 1, 1 + 1) \oplus (5, 6, 1) \oplus (1, 6, 1) \oplus (4, 4, 1) \oplus (4, 4, 1) \]  

(24)

Hence, altogether we can write:

\[
Solv_7 = Solv_S \oplus Solv_{TS} \oplus Solv_{T1} \oplus W_{32}
\]

(25)

Just as in the previous case we should now single out the content of the first order Killing spinor equations by decomposing them into irreducible \( Usp(4) \times SU(4) \times U(1) \) representations. The dilatino equations \( \delta \chi_{ABC} = 0 \), and the gravitino equation \( \delta \psi_A = 0 \), \( A, B, C = 1, \ldots, 8 \) (\( SU(8) \) indices), decompose as follows

\[
\begin{align*}
56 & \ Usp(4) \times SU(4) \rightarrow (4, 1) \oplus 2(1, 4, ) \oplus (5, 4) \oplus (4, 6) \\
8 & \ Usp(4) \times SU(4) \rightarrow (4, 1) \oplus (1, 4, )
\end{align*}
\]

(26)

(27)

As we shall see explicitly in section 5, the content of the reduced Killing spinor equations is such that only two scalar fields are essentially dynamical all the other being set to constant up to \( U \)-duality transformations. Moreover 52 charges are set to zero leaving 4 charges transforming the \( (2, 2) \) representation of \( SL(2,\mathbb{R}) \times O(1,1) \). Note that in the present case on the basis of the SLA analysis given in the introduction, one would expect 3 scalar fields parametrising \( G_{norm}/H_{norm} = SL(2,\mathbb{R})/U(1,1) \); however the relevant Killing spinor equation gives an extra reality constraint on the \( SL(2,\mathbb{R})/U(1,1) \) field thus reducing the number of non trivial scalar fields to two.

2.1.3 The 1/8 SUSY case

Here we have \( n_{\text{max}} = 2 \) and \( Solv_7 \) must be decomposed according to the decomposition of the isotropy subgroup: \( SU(8) \rightarrow SU(2) \times U(6) \). We showed in § that the corresponding decomposition of the solvable Lie algebra is the following one:

\[
Solv_7 = Solv_3 \oplus Solv_4
\]

(28)

\[\begin{align*}
\text{rank } Solv_3 & = 3 \\
\text{dim } Solv_3 & = 30 \\
\text{rank } Solv_4 & = 4 \\
\text{dim } Solv_4 & = 40
\end{align*}\]

(29)

The rank three Lie algebra \( Solv_3 \) defined above describes the thirty dimensional scalar sector of \( N = 6 \) supergravity, while the rank four solvable Lie algebra \( Solv_4 \) contains the remaining forty scalars belonging to \( N = 6 \) spin 3/2 multiplets. It should be noted that, individually, both manifolds \( \exp[Solv_3] \) and \( \exp[Solv_4] \) have also an \( N = 2 \) interpretation since we have:

\[\begin{align*}
\exp[Solv_3] & = \text{homogeneous special Kähler} \\
\exp[Solv_4] & = \text{homogeneous quaternionic}
\end{align*}\]

(30)
so that the first manifold can describe the interaction of 15 vector multiplets, while the second can describe the interaction of 10 hypermultiplets. Indeed if we decompose the $N = 8$ graviton multiplet in $N = 2$ representations we find:

$$N=8 \text{ spin } 2 \rightarrow \text{ spin } 2 + 6 \times \text{ spin } 3/2 + 15 \times \text{ vect. mult. } + 10 \times \text{ hypermult.}$$  \hspace{1cm} (31)

Introducing the decomposition (28) we found in [8] that the 40 scalars belonging to $\text{Solv}_4$ are constants independent of the radial variable $r$. Only the 30 scalars in the Kähler algebra $\text{Solv}_3$ can be radial dependent. In fact their radial dependence is governed by a first order differential equation that can be extracted from a suitable component of the Killing spinor equation. The result in this case is that 64 of the scalar fields are actually constant while 6 are dynamical. Moreover 48 charges are annihilated leaving 6 nonzero charges transforming in the representation $(2, 2, 2)$ of the normalizer $G_{\text{norm}} = [\text{SL}(2, \mathbb{R})]^3$. More precisely we obtained the following result. Up to $U$–duality transformations the most general $N = 8$ black–hole is actually an $N = 2$ black–hole corresponding to a very specific choice of the special Kähler manifold, namely $\text{exp}[\text{Solv}_3]$ as in eq.(30),(29). Furthermore up to the duality rotations of $SO^{\star}(12)$ this general solution is actually determined by the so called $\text{STU}$ model studied in [13] and based on the solvable subalgebra:

$$\text{Solv} \left( \frac{\text{SL}(2, \mathbb{R})^3}{U(1)^3} \right) \subset \text{Solv}_3$$ \hspace{1cm} (32)

In other words the only truly independent degrees of freedom of the black hole solution are given by three complex scalar fields, $S, T, U$. The real parts of these scalar fields correspond to the three Cartan generators of $\text{Solv}_3$ and have the physical interpretation of radii of the torus compactification from $D = 10$ to $D = 4$. The imaginary parts of these complex fields are generalised theta angles.

### 3 Summary of $N = 8$ supergravity.

We first establish the relevant definition and notation (for further details see [14], [8] and for a general review on the duality formalism [15]) We introduce the coset representative $I_L$ of $\frac{E_{(7)7}}{SU(8)}$ in the 56 representation of $E_{(7)7}$:

$$I_L = \frac{1}{\sqrt{2}} \left( \begin{array}{c|c} f + ih & f - ih \\ \hline f - ih & f + ih \end{array} \right)$$ \hspace{1cm} (33)

where the submatrices $(h, f)$ are $28 \times 28$ matrices indexed by antisymmetric pairs $\Lambda, \Sigma, A, B$ ($\Lambda, \Sigma = 1, \ldots, 8$; $A, B = 1, \ldots, 8$) the first pair transforming under $E_{(7)7}$ and the second one under $SU(8)$:

$$(h, f) = (h_{\Lambda \Sigma | AB}, f^{\Lambda \Sigma}_{\ AB})$$ \hspace{1cm} (34)
Note that $\mathbb{L} \in Usp(28,28)$. The vielbein $P_{ABCD}$ and the $SU(8)$ connection $\Omega^A_B$ of $E_{SU(8)}$ are computed from the left invariant 1-form $\mathbb{L}^{-1}d\mathbb{L}$:

$$\mathbb{L}^{-1}d\mathbb{L} = \begin{pmatrix} \delta^{[A}_{[C} \Omega^B]_D] & P^{ABCD} \\ P_{ABCD} & \delta_{[A}^{[C} \Omega_B]_D] \end{pmatrix}$$

(35)

where $P_{ABCD} \equiv P_{ABCD,\alpha}d\Phi^\alpha$ ($\alpha = 1, \ldots, 70$) is completely antisymmetric and satisfies the reality condition

$$P_{ABCD} = \frac{1}{24} \epsilon_{ABCD} F_{EFGH}$$

(36)

The bosonic lagrangian of $N = 8$ supergravity is

$$\mathcal{L} = \int \sqrt{-g} d^4x \left( 2R + \text{Im} \mathcal{N}_{\Lambda\Sigma|\Gamma\Delta} F_{\mu\nu}^{\Lambda\Sigma} F^{\Gamma\Delta|\mu\nu} + \frac{1}{6} P_{ABCD,ij} \mathcal{P}^{ABCD}_j \partial_\mu \Phi^i \partial^\mu \Phi^j \right) + \frac{1}{2} \text{Re} \mathcal{N}_{\Lambda\Sigma|\Gamma\Delta} e^{\mu_0\rho_0} \sqrt{-g} F_{\mu_0}^{\Lambda\Sigma} F^{\Gamma\Delta}_{\rho_0}$$

(37)

where the curvature two-form is defined as

$$R^{ab} = d\omega^a - \omega^a_c \wedge \omega_c^b.$$  

(38)

and the kinetic matrix $\mathcal{N}_{\Lambda\Sigma|\Gamma\Delta}$ is given by:

$$\mathcal{N} = h f^{-1} \rightarrow \mathcal{N}_{\Lambda\Sigma|\Gamma\Delta} = h_{\Lambda\Sigma|AB} f^{-1} \frac{AB}{\Gamma\Delta}.$$  

(39)

The same matrix relates the (anti)self-dual electric and magnetic 2-form field strengths, namely, setting

$$F^{\pm \Lambda\Sigma} = \frac{1}{2} (F \pm i \ast F)^{\Lambda\Sigma}$$

(40)

one has

$$G^{-}_{\Lambda\Sigma} = \overline{\mathcal{N}}_{\Lambda\Sigma|\Gamma\Delta} F^{- \Gamma\Delta}$$  

$$G^+_{\Lambda\Sigma} = \mathcal{N}_{\Lambda\Sigma|\Gamma\Delta} F^{+ \Gamma\Delta}$$

(41)

where the ”dual” field strengths $G^\pm_{\Lambda\Sigma}$ are defined as $G^\pm_{\Lambda\Sigma} = \frac{i}{2} \delta^\pm \delta_{\Lambda\Sigma}$. Note that the 56 dimensional (anti)self-dual vector $(F^{\pm \Lambda\Sigma}, G^\pm_{\Lambda\Sigma})$ transforms covariantly under $U \in Sp(56,\mathbb{R})$

$$U \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} F' \\ G' \end{pmatrix}$$  

$$A^t C - C^t A = 0$$

$$B^t D - D^t B = 0$$

$$A^t D - C^t B = 1$$

(42)
The matrix transforming the coset representative $I_L$ from the $Usp(28,28)$ basis, eq.(33), to the real $Sp(56,\mathbb{R})$ basis is the Cayley matrix:

\[
I_{Usp} = CI_{Sp}C^{-1} \quad C = \left( \begin{array}{cc} I & iI \\ I & -iI \end{array} \right)
\]

implying

\[
f = \frac{1}{\sqrt{2}} (A - iB) \quad h = \frac{1}{\sqrt{2}} (C - iD)
\]

Having established our definitions and notations, let us now write down the Killing spinor equations obtained by equating to zero the SUSY transformation laws of the gravitino $\psi_{A\mu}$ and dilatino $\chi_{ABC}$ fields of $N = 8$ supergravity:

\[
\delta \chi_{ABC} = 4i P_{ABCD} \partial_\mu \Phi^i \gamma^\mu \epsilon^D - 3T_{AB|\rho\sigma} \gamma^{\rho\sigma} \epsilon_C + \ldots = 0
\]

\[
\delta \psi_{A\mu} = \nabla_\mu \epsilon_A - \frac{1}{4} T_{AB|\rho\sigma} \gamma^{\rho\sigma} \gamma_\mu \epsilon_B + \ldots = 0
\]

where the dots denote unessential trilinear fermion terms, $\nabla_\mu$ denotes the derivative covariant both with respect to Lorentz and $SU(8)$ local transformations

\[
\nabla_\mu \epsilon_A = \partial_\mu \epsilon_A - \frac{1}{4} \gamma_{ab} \omega^{ab}_\mu \epsilon_A - \Omega_{AB} \epsilon_B
\]

and where $T_{AB}^{(-)}$ is the ”dressed graviphoton”2–form, defined as follows

\[
T_{AB}^{(-)} = \left( h_{\Lambda \Sigma AB} (\Phi) F_{\Lambda \Sigma} - f_{\Lambda \Sigma AB} (\Phi) G^{(-)} \right)
\]

From equations (39), (41) we have the following identities:

\[
T_{AB}^+ = 0 \rightarrow T_{AB}^- = T_{AB} \quad \overline{T}_{AB}^+ = 0 \rightarrow \overline{T}_{AB}^- = \overline{T}_{AB}
\]

Note that the central charge is defined as

\[
Z_{AB} = \int_{S^2} T_{AB} = h_{\Lambda \Sigma AB} g^{\Lambda \Sigma} - f_{\Lambda \Sigma AB} e_{\Lambda \Sigma}
\]

where the integral of the two-form $T_{AB}$ is evaluated on a large two-sphere at infinity and the quantized charges ($e_{\Lambda \Sigma}$, $g^{\Lambda \Sigma}$) are defined by

\[
g^{\Lambda \Sigma} = \int_{S^2} F^{\Lambda \Sigma} \\
e_{\Lambda \Sigma} = \int_{S^2} \mathcal{N}_{\Lambda \Sigma|\Gamma \Delta} \star F^{\Gamma \Delta}
\]
4 Detailed study of the 1/2 case

As established in [10], the $N = 1/2$ SUSY preserving black hole solution of $N = 8$ supergravity has 4 equal skew eigenvalues in the normal frame for the central charges. The stabilizer of the normal form is $E_{(6,6)}$ and the normalizer of this latter in $E_{(7,7)}$ is $O(1,1)$:

$$E_{(7,7)} \supset E_{(6,6)} \times O(1,1)$$

(51)

According to our previous discussion, the relevant subgroup of the $SU(8)$ holonomy group is $Usp(8)$, since the BPS Killing spinor conditions involve supersymmetry parameters $\epsilon_A$, $\epsilon^A$ satisfying eq. (12). Relying on this information, we can write the solvable Lie algebra decomposition (14), (15) of the $\sigma$-model scalar coset $E_{SU(8)}$.

As discussed in the introduction, it is natural to guess that modulo $U$–duality transformations the complete solution is given in terms of a single scalar field parametrizing $O(1,1)$. Indeed, we can now demonstrate that according to the previous discussion there is just one scalar field, parametrizing the normalizer $O(1,1)$, which appear in the final lagrangian, since the Killing spinor equations imply that 69 out of the 70 scalar fields are actually constants. In order to achieve this result, we have to decompose the $SU(8)$ tensors appearing in the equations (55), (56) with respect to $Usp(8)$ irreducible representations. According to the decompositions

$$
\begin{align*}
70 & \cong Usp(8) = 42 \oplus 27 \oplus 1 \\
28 & \cong Usp(8) = 27 \oplus 1
\end{align*}
$$

(52)

we have

$$
\begin{align*}
P_{ABCD} & = \tilde{P}_{ABCD} + \frac{3}{2} C_{[AB} \tilde{P}_{CD]} + \frac{1}{16} C_{[AB} C_{CD]} P \\
T_{AB} & = \tilde{T}_{AB} + \frac{1}{8} C_{AB} T
\end{align*}
$$

(53)

where the notation $\tilde{t}_{A_1 \ldots A_n}$ means that the antisymmetric tensor is $Usp(8)$ irreducible, namely has vanishing $C$-traces: $C^{A_1 A_2} \tilde{t}_{A_1 A_2 \ldots A_n} = 0$.

Starting from equation (13) and using equation (12) we easily find:

$$
4 P_{,a} \gamma^a \gamma^0 - 6 T_{ab} \gamma^{ab} = 0,
$$

(54)

where we have twice contracted the free $Usp(8)$ indices with the $Usp(8)$ metric $C_{AB}$. Next, using the decomposition (53), eq. (13) reduces to

$$
-4 \left( \tilde{P}_{ABCD, a} + \frac{3}{2} \tilde{P}_{[CD, a} C_{AB]} \right) C^{DL} \gamma^a \gamma^0 - 3 \tilde{T}_{[AB} \delta^L_{C]} \gamma^{ab} = 0.
$$

(55)

Now we may alternatively contract equation (22) with $C^{AB}$ or $\delta^L_C$ obtaining two relations on $\tilde{P}_{AB}$ and $\tilde{T}_{AB}$ which imply that they are separately zero:

$$
\begin{align*}
\tilde{P}_{AB} & = \tilde{T}_{AB} = 0,
\end{align*}
$$

(56)
which also imply, taking into account \[57\]

\[\hat{P}_{ABCD} = 0. \tag{57}\]

Thus we have reached the conclusion

\[\hat{P}_{ABCD|i} \partial_{\mu} \Phi^{i}_{} \gamma^{\mu}_{\rho} e^{D} = 0 \]
\[\hat{P}_{AB|i} \partial_{\mu} \Phi^{i}_{} \gamma^{\mu}_{\rho} e^{B} = 0 \tag{58}\]
\[\dot{T}_{AB} = 0 \tag{59}\]

implying that 69 out of the 70 scalar fields are actually constant, while the only surviving central charge is that associated with the singlet two-form \(T\). Since \(T_{AB}\) is a complex combination of the electric and magnetic field strengths \[48\], it is clear that eq. \[59\] implies the vanishing of 54 of the quantized charges \(g_{\Lambda/\Sigma}, e_{\Lambda/\Sigma}\), the surviving two charges transforming as a doublet of \(O(1, 1)\) according to eq. \[17\]. The only non-trivial evolution equation relates \(P\) and \(T\) as follows:

\[\left( \hat{P} \partial_{\mu} \Phi \gamma^{\mu}_{\rho} - \frac{3}{2} i T^{(-)} \gamma^{\rho}_{\sigma} \gamma^{0} \right) \epsilon_{A} = 0 \tag{60}\]

where we have set \(P = \hat{P} d\Phi\) and \(\Phi\) is the unique non-trivial scalar field parametrizing \(O(1, 1)\).

In order to make this equation explicit we perform the usual static ansaetze:

**Black hole metric:**

\[d s^{2} = e^{2U(r)} d t^{2} - e^{-2U(r)} d \bar{x}^{2} \quad (r^{2} = \bar{x}^{2}) \tag{61}\]

**Matter fields:**

\[\Phi = \Phi (r) \tag{62}\]
\[F^{(-)}_{\Lambda/\Sigma} = \frac{1}{4 \pi} t^{\Lambda/\Sigma} (r) E^{(-)} \tag{63}\]
\[E^{(-)} = \frac{i}{r^{3}} e^{2U} d t \wedge x^{i} d x^{i} + \frac{x^{i}}{2 r^{3}} d x^{j} \wedge d x^{k} \epsilon_{ijk} \tag{64}\]
\[t^{\Lambda/\Sigma} (r) = 2 \pi (g + i q (r)) \Lambda/\Sigma \tag{65}\]

Using the definitions \[10\] , \[11\] , \[18\] , \[34\] , \[63\] we have

\[T_{ab} = i t^{\Lambda/\Sigma} (r) E_{a}^{+} C_{AB}^{+} \text{Im} N_{\Lambda/\Sigma, \Gamma/\Delta} f^{\Gamma/\Delta}_{AB} \tag{66}\]

A simple gamma matrix manipulation gives further

\[\gamma_{ab} E_{a}^{+} = 2 i \frac{e^{2U}}{r^{3}} x^{i} \gamma^{0} \gamma^{i} \left( \frac{\pm 1 + \gamma_{5}}{2} \right) \tag{67}\]

and we arrive at the final equation

\[\frac{d \Phi}{d r} = - \frac{\sqrt{3}}{4} q (r)^{\Lambda/\Sigma} \text{Im} N_{\Lambda/\Sigma, \Gamma/\Delta} f^{\Gamma/\Delta}_{AB} \frac{e^{U}}{r^{2}}. \tag{68}\]
In eq. (68), we have set $g^{\Lambda \Sigma} = 0$ since reality of the l.h.s. and of $f^\Gamma_{AB}$ (see eq. (83)) imply the vanishing of the magnetic charge. Furthermore, we have normalized the vielbein component of the $Usp(8)$ singlet as follows

$$\hat{P} = 4\sqrt{3}$$

(69)

which corresponds to normalizing the $Usp(8)$ vielbein as

$$P_{ABCD}^{\text{(singlet)}} = \frac{1}{16} PC_{[AB}C_{CD]} = \frac{\sqrt{3}}{4} C_{[AB}C_{CD]} d\Phi.$$  

(70)

This choice agrees with the normalization of the scalar fields existing in the current literature. Let us now consider the gravitino equation (46). Computing the spin connection $\omega^a_b$ from equation (61), we find

$$\omega^0_i = \frac{dU \, x^i}{dr} e^{\ellU(r)} V^0$$

$$\omega^{ij} = 2 \frac{dU \, x^k}{dr} \gamma^{[i} V^{j]} e^{\ellU}$$

(71)

where $V^0 = e^{\ellU} dt$, $V^i = e^{-\ellU} dx^i$. Setting $\epsilon_A = e^{f(r)} \zeta_A$, where $\zeta_A$ is a constant chiral spinor, we get

$$\left\{ \frac{df \, x^i}{dr} e^{\ellU} \delta^B_A \gamma^i V^i + \Omega^B_{A,\alpha} \partial_\alpha \Phi^A e^{f} V^i \right.$$  

$$- \frac{1}{4} \left( 2 \frac{dU \, x^i}{dr} e^{\ellU} \gamma^{0} \gamma^i V^0 + \gamma^{ij} V^i \right) \delta^B_A + \delta^B_A T_{ab} \gamma^a \gamma^c \gamma^0 V^i \right\} \zeta_B = 0$$

(72)

where we have used eqs. (10), (17), (33). This equation has two sectors; setting to zero the coefficient of $V^0$ or of $V^i \gamma^{ij}$ and tracing over the $A, B$ indices we find two identical equations, namely:

$$\frac{dU}{dr} = -\frac{1}{8} g(r)^{\Lambda \Sigma} e^{\ellU} r^2 C^{AB} \text{Im} \mathcal{N}_{\Lambda \Sigma, \Gamma \Delta} f^{\Gamma \Delta}_{AB}.$$  

(73)

Instead, if we set to zero the coefficient of $V^i$, we find a differential equation for the function $f (r)$, which is uninteresting for our purposes. Comparing now equations (68) and (73) we immediately find

$$\Phi = 2\sqrt{3} \ellU.$$  

(74)

### 4.1 Explicit computation of the Killing equations and of the reduced Lagrangian in the 1/2 case

In order to compute the l.h.s. of eqs. (68), (73) and the lagrangian of the 1/2 model, we need the explicit form of the coset representative $\mathbb{L}$ given in equation (33). This will also enable us to compute explicitly the r.h.s. of equations (68), (73). In the present case the explicit form of $\mathbb{L}$ can be retrieved by exponentiating the $Usp(8)$ singlet generator. As stated in equation
the scalar vielbein in the $Usp(28, 28)$ basis is given by the off diagonal block elements of $\mathbb{I}^{-1} d\mathbb{I}$, namely

$$\mathbb{I}^P = \begin{pmatrix} 0 & \mathcal{P}_{ABCD} \\ P_{ABCD} & 0 \end{pmatrix}. \quad (75)$$

From equation (70), we see that the $Usp(8)$ singlet corresponds to the generator

$$\mathbb{I} = \frac{\sqrt{3}}{4} \begin{pmatrix} 0 & C^{[AB}C_{HL]} \\ C_{[CD}C_{RS]} & 0 \end{pmatrix} \quad (76)$$

and therefore, in order to construct the coset representative of the $O(1, 1)$ subgroup of $E_{7(7)}$, we need only to exponentiate $\Phi \mathbb{I}$. Note that $\mathbb{I}$ is a $Usp(8)$ singlet in the 70 representation of $SU(8)$, but it acts non-trivially in the 28 representation of the quantized charges $(e_{AB}, g^{AB})$. It follows that the various powers of $\mathbb{I}$ are proportional to the projection operators onto the irreducible $Usp(8)$ representations 1 and 27 of the charges:

$$\mathbb{I}^P_1 = \frac{1}{8} C^{AB} C_{RS} \quad (77)$$

$$\mathbb{I}^P_{27} = (\delta^{AB}_{RS} - \frac{1}{8} C^{AB} C_{RS}) \quad (78)$$

Straightforward exponentiation gives

$$\exp(\Phi \mathbb{I}) = \cosh \left( \frac{1}{2\sqrt{3}} \Phi \right) \mathbb{I}^P_{27} + \frac{3}{2} \sinh \left( \frac{1}{2\sqrt{3}} \Phi \right) \mathbb{I}^P_{27} \mathbb{I} \mathbb{I}^P_{27} +$$

$$+ \cosh \left( \frac{\sqrt{3}}{2} \Phi \right) \mathbb{I}^P_1 + \frac{1}{2} \sinh \left( \frac{\sqrt{3}}{2} \Phi \right) \mathbb{I}^P_1 \mathbb{I} \mathbb{I}^P_1 \quad (79)$$

Since we are interested only in the singlet subspace

$$\mathbb{I}^P_1 \exp(\Phi \mathbb{I}) \mathbb{I}^P_1 = \cosh\left( \frac{\sqrt{3}}{2} \Phi \right) \mathbb{I}^P_1 + \frac{1}{2} \sinh\left( \frac{\sqrt{3}}{2} \Phi \right) \mathbb{I}^P_1 \mathbb{I} \mathbb{I}^P_1 \quad (81)$$

$$\mathbb{I}^P_{singlet} = \frac{1}{8} \begin{pmatrix} \cosh\left( \frac{\sqrt{3}}{2} \Phi \right) C^{AB} C_{CD} & \sinh\left( \frac{\sqrt{3}}{2} \Phi \right) C^{AB} C^{FG} \\ \sinh\left( \frac{\sqrt{3}}{2} \Phi \right) C_{CD} C_{LM} & \cosh\left( \frac{\sqrt{3}}{2} \Phi \right) C_{CM} C^{FG} \end{pmatrix} \quad (82)$$

Comparing (82) with the equation (33), we find $^1$

$$f = \frac{1}{8\sqrt{2}} e^{\frac{\sqrt{3}}{2} \Phi} C^{AB} C_{CD} \quad (83)$$

$$h = -i \frac{1}{8\sqrt{2}} e^{-\frac{\sqrt{3}}{2} \Phi} C_{AB} C_{CD} \quad (84)$$

$^1$Note that we are writing the coset matrix with the same couple of indices $AB, CD, \ldots$ without distinction between the couples $\Lambda \Sigma$ and $AB$ as was done in sect.(3)
and hence, using $\mathcal{N} = hf^{-1}$, we find

$$\mathcal{N}_{ABCD} = -\frac{1}{8} e^{-\sqrt{3}\Phi} C_{AB}C_{CD}$$

(85)

so that we can compute the r.h.s. of (88), (73). Using the relation (74) we find a single equation for the unknown functions $U(r)$, $q(r) = C_{\lambda\Sigma}q^{\lambda\Sigma}(r)$

$$\frac{dU}{dr} = \frac{1}{8\sqrt{2}} \frac{q(r)}{r^2} \exp(-2U)$$

(86)

At this point to solve the problem completely we have to consider also the second order field equation obtained from the lagrangian. The bosonic supersymmetric lagrangian of the 1/2 preserving supersymmetry case is obtained from equation (37) by substituting the values of $P_{ABCD}$ and $\mathcal{N}_{\Lambda\Sigma|\Gamma\Delta}$ given in equations (70) and (39) into equation (37). We find

$$\mathcal{L} = 2R - e^{-\sqrt{3}\Phi} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} \partial_{\mu}\Phi \partial^{\mu}\Phi$$

(87)

4.2 The 1/2 solution

The resulting field equations are

$\underline{\text{Einstein equations:}}$

$$U'' + \frac{2}{r}U' - (U)^2 = \frac{1}{4} (\Phi')^2$$

(88)

$\underline{\text{Maxwell equations:}}$

$$\frac{d}{dr} (e^{-\sqrt{3}\Phi} q(r)) = 0$$

(89)

$\underline{\text{Dilaton equation:}}$

$$\Phi'' + \frac{2}{r} \Phi' = -e^{-\sqrt{3}\Phi} q(r)^2 \frac{1}{r^4}.$$  

(90)

From Maxwell equations one immediately finds

$$q(r) = e^{\sqrt{3}\Phi(r)}.$$  

(91)

Taking into account (74), the second order field equation and the first order Killing spinor equation have the common solution

$$U = -\frac{1}{4} \log H(x)$$

$$\Phi = -\frac{\sqrt{3}}{2} \log H(x)$$

$$q = H(x)^{-\frac{3}{4}}$$

(92)

where:

$$H(x) \equiv 1 + \sum_{\ell} \frac{e_{\ell}}{x - x_{\ell}}$$

(93)
is a harmonic function describing 0–branes located at $\vec{x}_0^\ell$ for $\ell = 1, 2, \ldots$, each brane carrying a charge $e_\ell$. In particular for a single 0–brane we have:

$$H(x) = 1 + \frac{k}{r}$$  \hspace{1cm} (94)

Note that the lagrangian (87) and the solution (92) agree with the well known case studied in the literature describing a single scalar field and a single vector field strength with the peculiar value $a = \sqrt{3}$ as coefficient of $\Phi$ in the exponential in front of the vector kinetic term. In the notations of [2] an elementary $p$–brane solution in space–time dimensions $D$ corresponds to a metric of the form:

$$ds^2 = H(x_\perp)^{-4\frac{d}{D-3-2}} dx_\parallel^\mu \otimes dx_\parallel^\nu \eta_{\mu\nu} + H(x_\perp)^{4\frac{d}{D-3}} dx_\perp^I \otimes dx_\perp^J \delta_{IJ}$$  \hspace{1cm} (95)

where $H(x_\perp)$ is a harmonic function of the transverse coordinates, $x_\parallel$ are the parallel coordinates $d = p + 1$ is the dimension of the $p$–brane volume, $\tilde{d} = D - 3 - p$ is the dimension of a dual magnetic brane and the dimension reduction invariant $\Delta$ is defined as follows:

$$\Delta = a^2 + 2 \frac{d \tilde{d}}{D-2}$$  \hspace{1cm} (96)

It is a common wisdom that an elementary $p$–brane preserving $1/2$ of the original string or M–theory supersymmetry has:

$$\Delta = 4$$  \hspace{1cm} (97)

implying for 0–branes ($p = 0$) in four–dimensions ($D = 4$):

$$a = \pm \sqrt{3}$$  \hspace{1cm} (98)

Our derivation completely confirms this result. What we have shown is that the most general BPS-saturated black hole preserving $1/2$ of the $N = 8$ supersymmetry is actually described by the lagrangian (87) with the solution given by (92), in the sense that any other solution with the same property can be obtained from the present one by an $E(7)\mathcal{R}$ (U-duality) transformation.

4.3 Comparison with the Dynkin basis formalism

The computation of the kinetic matrix $\mathcal{N}_{\Lambda\Sigma,\Gamma\Delta}$ and of the scalar kinetic term has been explicitly performed using the structure of the $Usp(8)$ singlet in the so called Young basis, that is in the basis where the generators of the coset are written in terms of four index antisymmetric tensors. Alternatively, we could have used the intrinsic Lie algebra basis in the Weyl–Cartan formalism, which we name the Dynkin basis. Here the $Usp(8)$ singlet generator $H$ among the 70 generators of the $F(E_{6(7)})$ coset is given as a suitable linear combination of the noncompact Cartan generators $H_{\alpha_i}$, $i = 1, \ldots, 7$, dual to the simple roots $\alpha_i$. The explicit form of such a linear combination is as follows:

$$H = a \left( \frac{3}{2} H_{\alpha_1} + 2 H_{\alpha_2} + \frac{5}{2} H_{\alpha_3} + 3 H_{\alpha_4} + \frac{3}{2} H_{\alpha_5} + 2 H_{\alpha_6} + H_{\alpha_7} \right)$$  \hspace{1cm} (99)
\( a \) being a normalization constant. In the Dynkin Basis \( H \) is a diagonal matrix whose elements are given by

\[
H = \text{diag} (\vec{s}, -\vec{s}),
\]

\[
\vec{s} = \left( \begin{array}{ccccccc}
-\frac{1}{2} & -\frac{1}{2} & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & -1 \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
 \end{array} \right)
\]

\[
(100)
\]

In this basis we immediately get

\[
I_L D = e^{\Phi H} = Hd \Phi
\]

\[
(103)
\]

where the coset representative \( I_L D \) is written in the \( Sp(56) \) real basis:

\[
I_L D = \left( \begin{array}{cc}
A & B \\
C & D \\
\end{array} \right)
\]

\[
(105)
\]

The transformation of \( I_L D \) from the real \( Sp(56) \) basis to the complex \( Usp(28, 28) \) basis (see eq. (33)) is performed by the the Cayley matrix (see eq.(43) ) and we get:

\[
f = \frac{1}{\sqrt{2}} (A - iB)
\]

\[
h = \frac{1}{\sqrt{2}} (C - iD)
\]

\[
(106)
\]

\[
(107)
\]

where in our case \( B = C = 0 \). Therefore, in the Dynkin basis, the kinetic vector matrix is

\[
\mathcal{N} = hf^{-1} = -iDA^{-1}
\]

\[
(108)
\]

The explicit listing of the simple roots of \( E_7(7) \), of the weights of its \( 56 \)-dimensional representation and their identification with the scalar fields of \( M \)-theory or of type IIA string theory compactified on a torus was given in [8]. We use those conventions and in particular we refer to tables 1, 2 of such a paper. Expressing the vector of the electric and magnetic field strengths in this basis we find that the singlet (determined through computation of the Casimir operator) corresponds to a single electric field strength at the 23\(^rd\) entry plus a magnetic field strength at the 28 + 23 entry of the general 56\(-\)dimensional vector, all the other entries being zero. Therefore we get

\[
\mathcal{N}_A \Sigma_\Gamma \Delta F^{A \Sigma}_\mu F^{\Gamma \Delta \mu} = e^{3a \Phi} F^{23}_\mu F^{23 \mu}.
\]

\[
(109)
\]

In order to agree with our previous normalization we set \( a = \frac{1}{\sqrt{3}} \). In an analogous way we can compute the scalar kinetic term. The vielbein is defined as

\[
P = Tr (I_L^{-1} dI_L H) = Tr (H^2 d\Phi) = 18a^2 d\Phi
\]

\[
(110)
\]

Therefore

\[
\mathcal{L}_{scal} = k^{11} P_\mu P^\mu = 18a^2 \partial_\mu \Phi \partial^\mu \Phi
\]

\[
(111)
\]
where \( k^{11} \) is the one dimensional inverse metric in the 70 representation. In our case \( k_{11} = Tr \left( HH \right) = 18a^2 \). This computation enables us to relate the term \( \frac{1}{6} P_{ABCD\mu} P^{ABCD\mu} \) to the analogue expression computed in Dynkin basis. Indeed, in full generality we have:

\[
P_{ABCD\mu} P^{ABCD\mu} = \alpha Tr \left( \mathbb{L}^{-1} d\mathbb{L} K_i \right) Tr \left( \mathbb{L}^{-1} d\mathbb{L} K_j \right) k^{ij}
\]

where \( K_i, K_j \) are \( \frac{E_{(7,7)}}{SU(8)} \) coset generators. In our case we find that, in order to obtain the scalar lagrangian normalized as in equation (87), \( \alpha = \frac{1}{2} \) (once we have fixed \( a = \frac{1}{\sqrt{3}} \)). The normalization \( \alpha = \frac{1}{2} \) will be confirmed in next section by the same comparison between Young and Dynkin formalism at the level of the 1/4 solution.

5 Detailed study of the 1/4 case

Solutions preserving 1/4 of \( N = 8 \) supersimmetry have two pairs of identical skew eigenvalues in the normal frame for the central charges. In this case the stability subgroup preserving the normal form is \( O(5,5) \) with normalizer subgroup in \( E_{(7,7)} \) given by \( SL(2,\mathbb{R}) \times O(1,1) \) (see [10]), according to the decomposition

\[
E_{(7,7)} \supset O(5,5) \times SL(2,\mathbb{R}) \times O(1,1) = G_{stab} \times G_{norm}
\]

The relevant fields parametrize \( SL(2,\mathbb{R}) \times O(1,1) \) while the surviving charges transform in the representation \( (2,2) \) of \( SL(2,\mathbb{R}) \times O(1,1) \). The group \( SL(2,\mathbb{R}) \) rotates electric into electric and magnetic into magnetic charges while \( O(1,1) \) mixes them. \( O(1,1) \) is therefore a true electromagnetic duality group.

5.1 Killing spinor equations in the 1/4 case:surviving fields and charges.

The holonomy subgroup \( SU(8) \) decomposes in our case as

\[
SU(8) \rightarrow Usp(4) \times SU(4) \times U(1)
\]

indeed in this case the killing spinors satisfy (19) where we recall the index convention:

\[
\begin{align*}
A, B &= 1 \ldots 8 \quad SU(8) \text{ indices} \\
a, b &= 1 \ldots 4 \quad Usp(4) \quad \text{indices} \\
X, Y &= 5 \ldots 8 \quad SU(4) \quad \text{indices}
\end{align*}
\]

and \( C_{ab} \) is the invariant metric of \( Usp(4) \). With respect to the holonomy subgroup \( SU(4) \times Usp(4) \), \( P_{ABCD} \) and \( T_{AB} \) appearing in the equations (15), (14) decompose as follows:

\[
\begin{align*}
70 \quad & Usp(4) \times SU(4) \rightarrow (1, 1) \oplus (4, 4) \oplus (5, 6) \oplus (1, 6) \oplus (1, 1) \\
28 \quad & Usp(4) \times SU(4) \rightarrow (1, 6) \oplus (4, 4) \oplus (5, 1) \oplus (1, 1)
\end{align*}
\]
We decompose eq. (45) according to eq. (116). We obtain:

\[ \delta \chi_{XYZ} = 0 \]  
\[ \delta \chi_{aXY} = 0 \]  
\[ \delta \chi_{abX} = C^{ab} \delta \chi_{abX} = 0 \]  
\[ \delta \chi_{abc} = C_{[ab} \delta \chi_{c]} = 0. \]  

\[ \therefore \text{From } \delta \chi_{XYZ} = 0 \text{ we immediately get:} \]

\[ P_{XYZa,\alpha} \partial^\mu \Phi_\alpha^\gamma \mu \gamma_0 = 0 \]  

by means of which we recognize that 16 scalar fields are actually constant in the solution.

\[ \therefore \text{From the reality condition of the vielbein } P_{ABCD} \text{ (equation (36)) we may also conclude} \]

\[ P_{Xabc} \equiv P_{X[a} C_{bc]} = 0 \]  

so that there are 16 more scalar fields set to constants.

\[ \therefore \text{From } \delta \chi_{aXY} = 0 \text{ we find} \]

\[ P_{XY,i} \partial^\mu \Phi^i \gamma^0 \epsilon_a = T_{XY} \epsilon^\mu \gamma^\mu_0 \epsilon_a \]  
\[ \therefore P_{XY,ab,i} \partial^\mu \Phi^i = 0 \]  

where we have set

\[ P_{XYab} = \frac{1}{4} C_{ab} P_{XY} \]  

Note that equation (124) sets 30 extra scalar fields to constant.

\[ \therefore \text{From } \delta \chi_{Xab} = 0, \text{ using (122), one finds that also } T_{Xa} = 0. \]

Finally, setting

\[ P_{abcd} = C_{[ab} C_{cd]} P \]  
\[ T_{ab} = \frac{1}{4} C_{ab} T \]  

the Killing spinor equation \( \delta \chi_{abc} \equiv C_{[ab} \delta \chi_{c]} = 0 \) yields:

\[ \therefore P_{i} \partial_\mu \Phi^i \gamma^\mu \gamma_0 = \frac{3}{16} T_{\mu \nu} \gamma^\mu \epsilon_a = 0 \]

Performing the gamma matrix algebra and using equation (67), the relevant evolution equations \( \delta \chi_{i} \) become

\[ P_{i} \frac{d \Phi^i}{dr} = \frac{3}{8} (g + i q(r))^\Lambda \Sigma \text{ Im} \mathcal{N}_{\Lambda \Sigma, \Gamma \Delta} f_{\Gamma \Delta}^{AB} C_{AB} \epsilon^\mu \frac{e^\mu}{r^2} \]  
\[ P_{XY,i} \frac{d \Phi^i}{dr} = 2i (g + i q(r))^\Lambda \Sigma \text{ Im} \mathcal{N}_{\Lambda \Sigma, \Gamma \Delta} f_{\Gamma \Delta}^{XY} \epsilon^\mu \frac{e^\mu}{r^2} \]  

20
According to our previous discussion, $P_{XY,i}$ is the vielbein of the coset $O^{(1,6)} / SU(4)$, which can be reduced to depend on 6 real fields $\Phi$ since, in force of the $SU(8)$ pseudo–reality condition, $P_{XY,i}$ satisfies an analogous pseudo–reality condition. On the other hand $P_i$ is the vielbein of $SL(2,\mathbb{R}) \times U(1)$, and it is intrinsically complex. Indeed the $SU(8)$ pseudo–reality condition relates the $SU(4)$ singlet $P_{XY ZW}$ to the $Usp(4)$ singlet $P_{abcd}$. Hence $P_i$ depends on a complex scalar field. In conclusion we find that equations (130) are evolution equations for 8 real fields, the 6 on which $P_{XY,i}$ depends plus the 2 real fields sitting in $P_i$. However, according to the discussion given in section 2, we expect that only three scalar fields, parametrizing $SL(2,\mathbb{R}) \times U(1)$ should be physically relevant. To retrieve this number we note that $O(1,1)$ is the subgroup of $O(1,6)$ which commutes with the stability subgroup $O(5,5)$, and hence also with its maximal compact subgroup $Usp(4) \times Usp(4)$. Therefore out of the 6 fields of $O(1,1)$ we restrict our attention to the real field parametrizing $O(1,1)$, whose corresponding vielbein is $C_{XY} P^i_{XY} = P_1 d\Phi_1$. Thus the second of equations (130) can be reduced to the evolution equation for the single scalar field $\Phi_1$, namely:

$$P_1 \frac{d\Phi_1}{dr} = -2 q(r)^{\Lambda \Sigma} \text{Im} N_{\Lambda \Sigma, \Gamma \Delta} f_{X Y}^{\Gamma \Delta} \frac{e^H}{r^2}$$

(131)

In this equation we have set the magnetic charge $g^{\Lambda \Sigma} = 0$ since, as we show explicitly later, the quantity $\text{Im} N_{\Lambda \Sigma, \Gamma \Delta} f_{X Y}^{\Gamma \Delta}$ is actually real. Hence, since the left hand side of equation (131) is real, we are forced to set the corresponding magnetic charge to zero. On the other hand, as we now show, inspection of the gravitino Killing spinor equation, together with the first of equations (130), further reduces the number of fields to two. Indeed, let us consider the $\delta \psi_a = 0$ Killing spinor equation. The starting equation is the same as (72), (48). In the present case, however, the indices $A,B,\ldots$ are $SU(8)$ indices, which have to be decomposed with respect to $SU(4) \times Usp(4) \times U(1)$. Then, from $\delta \Psi_X = 0$, we obtain

$$\Omega_X^a = 0; \quad T_{Xa} = 0$$

(132)

From $\delta \psi_a = 0$ we obtain an equation identical to (72) with $SU(8)$ indices replaced by $SU(4)$ indices. With the same computations performed in the $Usp(8)$ case we obtain the final equation

$$\frac{dU}{dr} = -\frac{1}{4} q(r)^{\Lambda \Sigma} \frac{e^H}{r^2} C^{ab} \text{Im} N_{\Lambda \Sigma, \Gamma \Delta} f_{ab}^{\Gamma \Delta},$$

(133)

where we have taken into account that $C^{ab} \text{Im} N_{\Lambda \Sigma, \Gamma \Delta} f_{ab}^{\Gamma \Delta}$ implying the vanishing of the magnetic charge corresponding to the singlet of $U(1) \times SU(4) \times Usp(4)$. Furthermore, since the right hand side of the equation (130) is proportional to the right hand side of the gravitino equation, it turns out that the vielbein $P_1$ must also be real. Let us name $\Phi_2$ the scalar field appearing in left hand side of the equation (130), and $P_2$ the corresponding vielbein component. Equation (130) can be rewritten as:

$$P_2 \frac{d\Phi_2}{dr} = -\frac{3}{8} q(r)^{\Lambda \Sigma} \text{Im} N_{\Lambda \Sigma, \Gamma \Delta} f_{X Y}^{\Gamma \Delta} C_{XY}^{X Y} \frac{e^H}{r^2}$$

(134)

In conclusion, we see that the most general model describing BPS–saturated solutions preserving $\frac{1}{4}$ of $N = 8$ supersymmetry is given, modulo $E(7,7)$ transformations, in terms of two scalar fields and two electric charges.
5.2 Derivation of the 1/4 reduced lagrangian in Young basis

Our next step is to write down the lagrangian for this model. This implies the construction of
the coset representative of $\frac{SL(2, \mathbb{R})}{U(1)} \times O(1, 1)$ in terms of which the kinetic matrix of the vector
fields and the $\sigma$–model metric of the scalar fields is constructed.

Once again we begin by considering such a construction in the Young basis where the
field strenghts are labeled as antisymmetric tensors and the $E_{7(7)}$ generators are written as
$Usp(28, 28)$ matrices.

The basic steps in order to construct the desired lagrangian consist of

1. Embedding of the appropriate $SL(2, \mathbb{R}) \times O(1, 1)$ Lie algebra in the $Usp_Y(28, 28)$ basis
   for the 56 representation of $E_{7(7)}$

2. Performing the explicit exponentiation of the two commuting Cartan generators of the
   above algebra

3. Calculating the restriction of the $\mathbb{I}$ coset representative to the 4–dimensional space
   spanned by the $Usp(4) \times Usp(4)$ singlet field strenghts and by their magnetic duals

4. Deriving the restriction of the matrix $N_{\Lambda \Sigma}$ to the above 4–dimensional space.

5. Calculating the explicit form of the scalar vielbein $P^{ABCD}$ and hence of the scalar kinetic
terms.

Let us begin with the first issue. To this effect we consider the following two antisymmetric
$8 \times 8$ matrices:

$$\omega_{AB} = -\omega_{BA} = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} ; \quad \Omega_{AB} = -\Omega_{BA} = \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}$$

(135)

where each block is $4 \times 4$ and the non vanishing block $C$ satisfies

$$C^T = -C \quad ; \quad C^2 = -\mathbb{I}$$

(136)

The subgroup $Usp(4) \times Usp(4) \subset SU(8)$ is defined as the set of unitary unimodular matrices
that preserve simultaneously $\omega$ and $\Omega$:

$$A \in Usp(4) \times Usp(4) \subset SU(8) \quad \leftrightarrow \quad A^\dagger \omega A = \omega \quad \text{and} \quad A^\dagger \Omega A = \Omega$$

(137)

Obviously any other linear combinations of these two matrices is also preserved by the same
subgroup so that we can also consider:

$$\tau^\pm_{AB} \equiv \frac{1}{2} (\omega_{AB} \pm \Omega_{AB}) = \begin{pmatrix} C & 0 \\ 0 & \pm C \end{pmatrix}$$

(138)

2The upper and lower matrices appearing in $\omega_{AB}$ and in $\Omega_{AB}$ are actually the matrices $C_{ab}, a, b = 1, \ldots, 4$
and $C_{XY} \quad X, Y = 1, \ldots, 4$ used in the previous section.
Introducing also the matrices:

\[ \pi_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad ; \quad \Pi_{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]

we have the obvious relations:

\[ \pi_{AB} = -\varpi_{AC} \varpi_{CB} \quad ; \quad \Pi_{AB} = -\Omega_{AC} \Omega_{CB} \] (139)

In terms of these matrices we can easily construct the projection operators that single out from
the 28 of SU(8) its Usp(4) × Usp(4) irreducible components according to:

\[ 28 \rightarrow Usp(4) \times Usp(4) \rightarrow (1, 0) \oplus (0, 1) \oplus (5, 0) \oplus (0, 5) \oplus (4, 4) \] (141)

These projection operators are matrices mapping antisymmetric 2–tensors into antisymmetric
2–tensors and read as follows:

\[ P^{(1,0)}_{AB \ RS} = \frac{1}{4} \varpi_{AB} \varpi_{RS} \]
\[ P^{(0,1)}_{AB \ RS} = \frac{1}{4} \Omega_{AB} \Omega_{RS} \]
\[ P^{(5,0)}_{AB \ RS} = \frac{1}{2} (\pi_{AR} \pi_{BS} - \pi_{AS} \pi_{BR}) - \frac{1}{4} \varpi_{AB} \varpi_{RS} \]
\[ P^{(0,5)}_{AB \ RS} = \frac{1}{2} (\Pi_{AR} \Pi_{BS} - \Pi_{AS} \Pi_{BR}) - \frac{1}{4} \Omega_{AB} \Omega_{RS} \]
\[ P^{(4,4)}_{AB \ RS} = \frac{1}{8} (\pi_{AR} \Pi_{BS} + \Pi_{AR} \pi_{BS} - \pi_{AS} \Pi_{BR} - \Pi_{AS} \pi_{BR}) \] (142)

We also introduce the following shorthand notations:

\[ \ell_{RS}^{AB} \equiv \frac{1}{2} (\pi_{AR} \pi_{BS} - \pi_{AS} \pi_{BR}) \]
\[ L_{RS}^{AB} \equiv \frac{1}{2} (\Pi_{AR} \Pi_{BS} - \Pi_{AS} \Pi_{BR}) \]

\[ U^{ABCD} \equiv \varpi^{[AB} \varpi^{CD]} = \frac{1}{3} [\varpi^{AB} \varpi^{CD} + \varpi^{AC} \varpi^{DB} + \varpi^{AD} \varpi^{BC}] \]
\[ W^{ABCD} \equiv \Omega^{[AB} \Omega^{CD]} = \frac{1}{3} [\Omega^{AB} \Omega^{CD} + \Omega^{AC} \Omega^{DB} + \Omega^{AD} \Omega^{BC}] \]
\[ Z^{ABCD} \equiv \varpi^{[AB} \Omega^{CD]} = \frac{1}{6} [\varpi^{AB} \Omega^{CD} + \varpi^{AC} \Omega^{DB} + \varpi^{AD} \Omega^{BC} + \Omega^{AB} \varpi^{CD} + \Omega^{AC} \varpi^{DB} + \Omega^{AD} \varpi^{BC}] \] (143)

Then by direct calculation we can verify the following relations:

\[ Z_{ABRS} Z_{RSUV} = \frac{4}{9} \left( P^{(1,0)}_{AB \ UV} + P^{(0,1)}_{AB \ UV} + P^{(4,4)}_{AB \ UV} \right) \]
\[ U_{ABRS} U_{RSUV} = \frac{4}{9} \ell_{UV}^{AB} \]
\[ W_{ABRS} W_{RSUV} = \frac{4}{9} L_{UV}^{AB} \] (144)
Using the above identities we can write the explicit embedding of the relevant $SL(2, \mathbb{R}) \times O(1, 1)$ Lie algebra into the $E_{7(7)}$ Lie algebra, realized in the Young basis, namely in terms of $USp(28, 28)$ matrices. Abstractly we have:

$$
SL(2, \mathbb{R}) \text{ algebra} \quad \rightarrow \quad \begin{cases} 
[L_+, L_-] = 2 L_0 \\
[L_0, L_\pm] = \pm L_\pm
\end{cases}
$$

$$
O(1, 1) \text{ algebra} \quad \rightarrow \quad \mathcal{C}
$$

and they commute $$[\mathcal{C}, L_\pm] = [\mathcal{C}, L_0] = 0$$

The corresponding $E_{7(7)}$ generators in the 56 Young basis representation are:

$$
L_0 = \begin{pmatrix}
0 & \frac{3}{4} \left( U_{LMCD} + W_{LMCD} \right) \\
\frac{3}{4} \left( U_{LMCD} + W_{LMCD} \right) & 0
\end{pmatrix},
$$

$$
L_\pm = \begin{pmatrix}
\pm \frac{1}{2} \left( \ell_{AB} - L_{AB} \right) & \frac{3}{4} \left( U_{ABFG} - W_{ABFG} \right) \\
-\frac{3}{4} \left( U_{LMCD} - W_{LMCD} \right) & \mp \frac{1}{2} \left( \ell_{LMFG} - L_{LMFG} \right)
\end{pmatrix},
$$

$$
\mathcal{C} = \begin{pmatrix}
0 & \frac{3}{4} \mathcal{Z}_{ABFG} \\
\frac{3}{4} \mathcal{Z}_{LMCD} & 0
\end{pmatrix}
$$

The non-compact Cartan subalgebra of $SL(2, \mathbb{R}) \times O(1, 1)$, spanned by $L_0$, $\mathcal{C}$ is a 2-dimensional subalgebra of the full $E_{7(7)}$ Cartan subalgebra. As such this abelian algebra is also a subalgebra of the 70-dimensional solvable Lie algebra $\text{Solv}_7$ defined in eq. (16). The scalar fields associated with $L_0$ and $\mathcal{C}$ are the two dilatons parametrizing the reduced bosonic lagrangian we want to construct. Hence our programme is to construct the coset representative:

$$
\mathbb{I}L(\Phi_1, \Phi_2) \equiv \exp [\Phi_1 \mathcal{C} + \Phi_2 L_0]
$$

and consider its restriction to the 4-dimensional space spanned by the $USp(4) \times USp(4)$ singlets $\varpi_{AB}$ and $\Omega_{AB}$. Using the definitions (142) and (143) we can easily verify that:

$$
P_{(1,0)}^{AB \ RS} \frac{3}{4} Z_{RSUV}^{UV \ PQ} P_{(1,0)}^{(0, 1)} = 0
$$

$$
P_{(0,1)}^{AB \ RS} \frac{3}{4} Z_{RSUV}^{UV \ PQ} P_{(0,1)}^{(1,0)} = 0
$$

$$
P_{(1,0)}^{AB \ RS} \frac{3}{4} Z_{RSUV}^{UV \ PQ} P_{(1,0)}^{(0,1)} = \frac{1}{8} \varpi_{AB} \Omega_{PQ}
$$

$$
P_{(0,1)}^{AB \ RS} \frac{3}{4} Z_{RSUV}^{UV \ PQ} P_{(0,1)}^{(1,0)} = \frac{1}{8} \Omega_{AB} \varpi_{PQ}
$$

and similarly:

$$
P_{(1,0)}^{AB \ RS} \frac{3}{4} \left( U_{RSUV} + W_{RSUV} \right) P_{(1,0)}^{(1,0)} = \frac{1}{2} P_{(1,0)}^{AB \ PQ}
$$

and

$$
P_{(0,1)}^{AB \ RS} \frac{3}{4} \left( U_{RSUV} + W_{RSUV} \right) P_{(0,1)}^{(1,0)} = \frac{1}{2} P_{(1,0)}^{AB \ PQ}
$$

and

$$
P_{(1,0)}^{AB \ RS} \frac{3}{4} \left( U_{RSUV} + W_{RSUV} \right) P_{(1,0)}^{(1,0)} = \frac{1}{2} P_{(1,0)}^{AB \ PQ}
$$
This means that in the 4–dimensional space spanned by the \( Usp(4) \times Usp(4) \) singlets, using also the definition (138) and the shorthand notation

\[
\Phi^\pm = \frac{\Phi_2 \pm \Phi_1}{2}
\]  

the coset representative can be written as follows

\[
\exp \left[ \Phi_1 C + \Phi_2 L_0 \right] = 
\begin{pmatrix}
\cosh \Phi^+ \frac{1}{2} \tau_{AB} \tau_{CD} + \cosh \Phi^- \frac{1}{2} \tau_{AB} \tau_{CD} \\
\sinh \Phi^+ \frac{1}{2} \tau_{AB} \tau_{CD} + \sinh \Phi^- \frac{1}{2} \tau_{AB} \tau_{CD}
\end{pmatrix}
\begin{pmatrix}
\sinh \Phi^+ \frac{1}{2} \tau_{AB} \tau_{CD} + \sinh \Phi^- \frac{1}{2} \tau_{AB} \tau_{CD} \\
\cosh \Phi^+ \frac{1}{2} \tau_{AB} \tau_{CD} + \cosh \Phi^- \frac{1}{2} \tau_{AB} \tau_{CD}
\end{pmatrix}
\]  

Starting from eq.(152) we can easily write down the matrices \( f_{AB \ CD}, h_{AB \ CD} \) and \( N_{AB \ CD} \). We immediately find:

\[
f_{AB \ CD} = \frac{1}{\sqrt{2}} \left( \exp[\Phi^+] \frac{1}{2} \tau_{AB} \tau_{CD} + \exp[\Phi^-] \frac{1}{2} \tau_{AB} \tau_{CD} \right)
\]

\[
h_{AB \ CD} = -\frac{i}{\sqrt{2}} \left( \exp[-\Phi^+] \frac{1}{2} \tau_{AB} \tau_{CD} + \exp[-\Phi^-] \frac{1}{2} \tau_{AB} \tau_{CD} \right)
\]

\[
N_{AB \ CD} = -\frac{i}{4} \left( \exp[-2 \Phi^+] \frac{1}{2} \tau_{AB} \tau_{CD} + \exp[-2 \Phi^-] \frac{1}{2} \tau_{AB} \tau_{CD} \right)
\]

To complete our programme, the last point we have to deal with is the calculation of the scalar vielbein \( P^{ABCD} \). We have:

\[
\frac{3}{4} \begin{pmatrix}
0 \\
d\Phi_1 \ Z_{LMCD} + d\Phi_2 \left( U_{LMCD} + W_{LMCD} \right)
\end{pmatrix}
\begin{pmatrix}
d\Phi_1 \ Z^{ABFG} + d\Phi_2 \left( U^{ABFG} + W^{ABFG} \right) \\
0
\end{pmatrix}
\]

so that we get:

\[
P^{ABCD} = d\Phi_1 \frac{3}{4} \ Z^{ABCD} + d\Phi_2 \frac{3}{4} \left( U^{ABCD} + W^{ABCD} \right)
\]

and with a straightforward calculation:

\[
P_{\mu}^{ABCD} P_{\mu}^{ABCD} = \frac{3}{2} \partial_{\mu} \Phi_1 \partial^{\mu} \Phi_1 + 3 \partial_{\mu} \Phi_2 \partial^{\mu} \Phi_2
\]
Hence recalling the normalizations of the supersymmetric $N = 8$ lagrangian (34), and introducing the two $USp(4) \times USp(4)$ singlet electromagnetic fields:

$$A^{AB}_\mu = \tau^+_{AB} \frac{1}{2\sqrt{2}} A_\mu^1 + \tau^+_{AB} \frac{1}{2\sqrt{2}} A_\mu^2 + 26 \text{ non singlet fields} \quad (157)$$

we get the following reduced Lagrangian:

$$\mathcal{L}^{1/4}_{\text{red}} = \sqrt{-g} \left[ 2 R[g] + \frac{1}{4} \partial_\mu \Phi_1 \partial^\mu \Phi_1 + \frac{1}{2} \partial_\mu \Phi_2 \partial^\mu \Phi_2 
- \exp \left[ -\Phi_1 - \Phi_2 \right] \left( F_{\mu\nu}^1 \right)^2 
- \exp \left[ \Phi_1 - \Phi_2 \right] \left( F_{\mu\nu}^1 \right)^2 \right] \quad (158)$$

Redefining:

$$\Phi_1 = \sqrt{2} h_1 \quad ; \quad \Phi_2 = h_2 \quad (159)$$

we obtain the final standard form for the reduced lagrangian

$$\mathcal{L}^{1/4}_{\text{red}} = \sqrt{-g} \left[ 2 R[g] + \frac{1}{2} \partial_\mu h_1 \partial^\mu h_1 + \frac{1}{2} \partial_\mu h_2 \partial^\mu h_2 
- \exp \left[ -\sqrt{2} h_1 - h_2 \right] \left( F_{\mu\nu}^1 \right)^2 
- \exp \left[ \sqrt{2} h_1 - h_2 \right] \left( F_{\mu\nu}^2 \right)^2 \right] \quad (160)$$

### 5.3 Solution of the reduced field equations

We can easily solve the field equations for the reduced lagrangian (160). The Einstein equation is

$$\mathcal{U}'' + \frac{2}{r} \mathcal{U}' \left( \mathcal{U}' \right)^2 = \frac{1}{4} \left( h_1' \right)^2 + \frac{1}{4} \left( h_2' \right)^2 \quad (161)$$

the Maxwell equations are

$$\sqrt{2} h_1' - h_2' = -\frac{q_2'}{q_2}$$
$$-\sqrt{2} h_1' - h_2' = -\frac{q_1'}{q_1} \quad (162)$$

the scalar equations are

$$h_1'' + \frac{2}{r} h_1' = \frac{1}{2r^4} \left( e^{\sqrt{2} h_1 - h_2 + 2t} q_2^2 - e^{-\sqrt{2} h_1 - h_2 + 2t} q_1^2 \right)$$
$$h_2'' + \frac{2}{r} h_2' = \frac{1}{2r^4} \left( e^{\sqrt{2} h_1 - h_2 + 2t} q_2^2 + e^{-\sqrt{2} h_1 - h_2 + 2t} q_1^2 \right) \quad (163)$$

The solution of the Maxwell equations is

$$q_1(r) = q_1 e^{\sqrt{2} h_1 + h_2}$$
$$q_2(r) = q_2 e^{-\sqrt{2} h_1 + h_2} \quad (164)$$
where

\[ q_1 \equiv q_1(\infty) \]
\[ q_2 \equiv q_2(\infty) \] (165)

\[ h_1 = -\frac{1}{\sqrt{2}} \log \frac{H_1}{H_2} \]
\[ h_2 = -\frac{1}{2} \log H_1 H_2 \]
\[ U = -\frac{1}{4} \ln H_1 H_2 \] (166)

and

\[ H_1 (r) = 1 + \sum_\ell \frac{q_1^\ell}{x - x^{(1)}_\ell} \]
\[ H_2 (r) = 1 + \sum_\ell \frac{q_2^\ell}{x - x^{(2)}_\ell} \] (167)

are a pair of harmonic functions.

### 5.4 Derivation of the 1/4 reduced lagrangian in Dynkin basis

The 1/4 reduced lagrangian can be alternatively computed in the Dynkin basis. According to the previous group theoretical analysis, the lagrangian can depend only on the scalars that parametrize the normalizer \( \frac{SL(2, \mathbb{R})}{U(1)} \times O(1, 1) \). Figures 1 and 2 show graphically the splitting

![Diagram](image)

**figure 1**

\( E_{7(7)} \to O(5, 5) \times SL(2, \mathbb{R}) \times O(1, 1) \) dictated by the SLA decomposition (21) (see also [12]). The generator of \( O(1, 1) \) is the Cartan generator contained in the \( O(6, 6) \) subspace commuting with \( SL(2, \mathbb{R}) \), that is

\[ u_1 = a \left( H_{a_1} + H_{a_2} + H_{a_3} + H_{a_4} + \frac{1}{2} H_{a_5} + \frac{1}{2} H_{a_6} \right) \] (168)
On the other hand, the two solvable \( \frac{SL(2,R)}{U(1)} \) generators are the Cartan generator and the positive root generator corresponding to the exceptional root \( \beta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 2\alpha_5 + 3\alpha_6 + 2\alpha_7 \), which commute with \( O(6,6) \) (see [12]). As we have seen from the Killing spinor equations that one of the two scalars corresponding to these two generators is constant, so we can consider only one of these two generators.

The correct choice of the surviving generator corresponds to the choice of the local \( SU(8) \) gauge fixing. Indeed, since the \( SL(2,R) \) subalgebra commutes with \( O(5,5) \times O(1,1) \), once the generators of the noncompact part of \( SL(2,R) \) are given, we are free to select which of them is identified with \( H_\beta \) and which with \( E_\beta + E_{-\beta} \). This choice affects our results only when we define the solvable parametrization of \( \frac{SL(2,R)}{U(1)} \), which is given in terms of \( H_\beta \) and \( E_\beta \). This is achieved by adding the compact generator \( E_\beta - E_{-\beta} \) to \( E_\beta + E_{-\beta} \). In this way the coset parametrization given in terms of \( \{ H_\beta, E_\beta \} \) becomes the solvable parametrization in terms of \( H_\beta, E_\beta \). This procedure amounts to singling out a particular parametrization of the coset, which implies the \( SU(8) \) gauge fixing. We choose the Cartan generator

\[
\alpha_2 = s (H_{\alpha_1} + 2H_{\alpha_2} + 3H_{\alpha_3} + 4H_{\alpha_4} + 2H_{\alpha_5} + 3H_{\alpha_6} + 2H_{\alpha_7}).
\]

which corresponds to take as scalar field a dilaton instead of an axion field.

In our Dynkin basis the coset representative is

\[
\mathbb{I}_D = e^{\Phi_1 u_1} e^{\Phi_2 u_2} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

and, since the off-diagonal submatrices \( B, C \) are zero, the kinetic matrix (see equation (108)) becomes

\[
\mathcal{N} = -iDA^{-1}.
\]

The previous analysis of the Killing spinor equations shows that there are no magnetic charges and only two quantized electric charges corresponding to the \( O(5,5) \) singlets (recall that the quantized charges belong to the 56 of \( E_{(7)} \)). Looking at the quadratic Casimir operator of \( O(5,5) \), we find that the singlets are at the positions 17, 23, 45, 51 of the 56-dimensional vector, two of them being electric and the other two magnetic. However, we know that the Killing spinor equations are fulfilled only if both the charges are electric. To select which of these four charges are electric and which magnetic we resort to the equation of motion following
from the lagrangian. It turns out that the equation of motion give a solution with two electric charges only choosing the singlets corresponding to the positions 1, 7. Alternatively, if we have chosen the charges at positions 17, 23, we would have obtained equations of motion admitting configuration with one electric and one magnetic charge (and not preserving one fourth supersymmetry). We now show that the correct choice gives the same lagrangian previously obtained via the Young method. We set

\[
F = \begin{cases} 
F_{17} \equiv F_1 \\
F_{51} \equiv F_2 \\
\text{others zero}
\end{cases}
\]

and furthermore we fix the normalizations of the scalar fields setting

\[
a = -\frac{1}{2}, \quad s = \frac{1}{2}
\]

Then we find:

\[
\text{Im} \mathcal{N}_{\Lambda \Sigma} |\Delta F_{\mu \nu}^{\Lambda \Sigma} F^{\Gamma \Delta}_{\mu \nu} = - \left( e^{-\Phi_1 - \Phi_2} F_1^2 + e^{\Phi_1 - \Phi_2} F_2^2 \right),
\]

\[
\frac{1}{6} P_{ABCD|\mu} P^{ABCD|\mu} = \frac{1}{12} \text{Tr} \left( \mathbb{I}^{-1} dI \mathbb{I} \right) \text{Tr} \left( \mathbb{I}^{-1} dI \mathbb{I} \right) k^{ij} = \frac{1}{4} \partial_\mu \Phi_1 \partial^\nu \Phi_1 + \frac{1}{2} \partial_\mu \Phi_2 \partial^\nu \Phi_2
\]

reproducing the lagrangian (160), provided the redefinitions (159) are performed. In the previous formula \(k^{ij}\) is the inverse of the metric of the representation 70 \(k^{ij} = \text{Tr} \left( I K_i I K_j \right)\). Actually \(k^{ij}\) is restricted to the generators \(u_1, u_2\) and is given by

\[
k^{ij} = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}
\]

5.5 Fitting the 1/4 reduced action into the general framework of \(p\)-brane taxonomy

The reduced action we have found for the \(N = 8\) black holes with 1/4 supersymmetry is a particular case of a general action involving \(\ell\) scalar fields \(h^i\) (\(i = 1, \ldots, \ell\)) and the field strengths of \(\ell\) \(p+1\)-forms \(A^\alpha\) (\(\alpha = 1, \ldots, \ell\)), studied by Pope and Lu [9]. The action reads

\[
S_{PL} = \int \mathcal{L}_{PL} d^D X
\]

\[
\mathcal{L}_{PL} = \sqrt{-g} \left[ 2 R[g] + \frac{1}{2} \sum_{i=1}^{\ell} \partial_\mu h^i \partial^\mu h^i - \frac{(-1)^{p+1}}{2(p+2)!} \sum_{\alpha=1}^\ell \exp \left[ -2 \Lambda_\alpha \cdot \vec{h} \right] \left( F_{\mu_1,\ldots,\mu_{p+2}}^\alpha \right)^2 \right]
\]

where \(\Lambda_\alpha\) are \(\ell\) constant vectors, each with \(\ell\)-components. In our interpretation \(\Lambda_\alpha\) are weights, but what is crucial in the present discussion is that these weights have a number of effective

\[3\] The normalizations used in this section are the same as we used in the rest of the paper, except that the field strength \(F_{\mu \nu}\) which in the rest of the paper is defined as \(F = F_\mu dx^\mu \wedge dx^\nu \rightarrow F_{\mu \nu} = \frac{1}{2} \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right)\) is now defined as \(F_{\mu \nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)\). The same observation applies to the \((p + 2)\)-forms \(F_{\mu_1,\ldots,\mu_{p+2}}\) which are defined as \(F_{\mu_1,\ldots,\mu_{p+2}} = \partial_\mu A_{\mu_2,\ldots,\mu_{p+2}} + \ldots \) + permutations. This explains the presence of the extra factor \(\frac{1}{(p+2)!}\) in the \((p + 2)\)-form kinetic lagrangian in (177).
components equal to the number of field strenghts in the game. Our lagrangian (160) is of the form (177) with:

\[
D = 4 \quad ; \quad p = 0 \quad ; \quad \ell = 2
\]

\[
\Lambda_1 = (2\sqrt{2}, 2)
\]

\[
\Lambda_2 = (-2\sqrt{2}, 2)
\]

(178)

The field equations derived from the action (177) are the following ones:

**Einstein equation:**

\[
-2 R_{\mu\nu} = \frac{1}{2} \partial_{\mu} h^i \partial_{\nu} h^i + S_{\mu\nu}
\]

where

\[
S_{\mu\nu} = \frac{(-1)^{p+1}}{2(p + 1)!} \sum_{\alpha=1}^{\ell} \exp \left[ -2\bar{\Lambda}_\alpha \cdot \vec{h} \right] \left( F_\alpha^{\mu...F_\alpha^{\nu...} - \frac{p + 1}{(D - 2)(p + 2)} g_{\mu\nu} F_\alpha^{\alpha} F_\alpha^{\alpha} \right)
\]

(181)

**dilaton equation:**

\[
\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu h^i \right) = \frac{(-1)^{p+1}}{2(p + 2)!} \sum_{\alpha=1}^{\ell} \left( -2\bar{\Lambda}_\alpha \cdot \vec{h} \right) \exp \left[ -2\bar{\Lambda}_\alpha \cdot \vec{h} \right] \left( F_\alpha^{\alpha} F_\alpha^{\alpha} \right)^2
\]

(182)

**Maxwell equation:**

\[
\partial_\mu \left( \sqrt{-g} \exp \left[ -2\bar{\Lambda}_\alpha \cdot \vec{h} \right] F_\alpha^{\mu\nu...F_\alpha^{\nu...}} \right) = 0
\]

(184)

Writing for the metric the usual p–brane ansatz:

\[
ds^2 = \exp[2A(r)] dx^a \otimes dx^b \eta_{ab} - \exp[2B(r)] dy^n \otimes dy^m \delta_{nm}
\]

(185)

where \(x^a\) are the coordinates on the world–volume (\(a = 0, 1, \ldots, p\)), \(y^n\) are the transverse coordinates (\(n = p + 1, \ldots, D\)) and \(A(y), B(y)\) are two suitable function depending only of the transverse coordinates, the crucial observation made by Pope and Lu is that, provided a certain algebraic condition on the *weight vectors* \(\Lambda_\alpha\) is satisfied, then one can write a general solution of the differential system (180), (182), (184) in terms of exactly \(\ell\) harmonic functions:

\[
H_\alpha(y) = 1 + \sum_m^k \frac{k_{am}}{(y - y_{am})^{D-p-3}}
\]

(186)

the constants \(y_{(a m)}\) denoting the location of the p–brane singularities and the constants \(k_{(a m)}\) denoting the charges concentrated on such singular loci.

The algebraic condition is as follows. Define the \(\ell \times \ell\) matrix:

\[
M_{\alpha\beta} \equiv 4 \Lambda_\alpha \cdot \Lambda_\beta
\]

(187)

and search for a set of \(\ell\) sign choices:

\[
\varepsilon_\alpha = \begin{cases} +1 & \text{or} \\ -1 \end{cases}
\]

(188)
such that the following equation is satisfied:

\[ M_{\alpha\beta} = 4 \delta_{\alpha\beta} - 2 \frac{(p+1)(D-p-3)}{D-2} \varepsilon_\alpha \varepsilon_\beta \]  

(189)

If a solution to eq.(189) exists, then in terms of the corresponding \( \varepsilon_\alpha \) and of the \( \ell \) harmonic functions (186) the complete BPS solution of the equations of motion can be written in a universal form. Setting:

\[ \varphi_\alpha \equiv -2 \Lambda_\alpha \cdot h \]  

(190)

we have

metric:

\[ A = -\frac{D-p-3}{p+1} B = -\frac{1}{2} \frac{D-p-3}{D-2} \sum_{\alpha=1}^{\ell} \log H_\alpha \]  

(191)

dilatons:

\[ \varepsilon_\alpha \varphi_\alpha = \log \left[H_\alpha^2\right] - \frac{(p+1)(D-p-3)}{D-2} \log \left[\prod_{\gamma} H_\gamma\right] \]  

(192)

Maxwell fields:

\[ F^\alpha = \frac{(-)^{p+1}}{2} (1 + \varepsilon_\alpha) \; dx^{a_1} \wedge \ldots \wedge dx^{a_{p+1}} \epsilon_{a_1 \ldots a_{p+1}} \; dy^m \frac{\partial}{\partial y^m} H_\alpha^{-\varepsilon_\alpha} \]

\[ + \frac{1}{2} (1 - \varepsilon_\alpha) \; dy^{m_1} \wedge \ldots \wedge dy^{m_{p+2}} \epsilon_{m_1 \ldots m_{p+2}} \; \frac{\partial}{\partial y^n} H_\alpha^{-\varepsilon_\alpha} \]  

(193)

As one can see the choice of the \( \varepsilon_\alpha \) sign decides whether the field strength \( F^\alpha \) is electric or magnetic. In dimensions \( D \) and for a generic value of \( p \) only the electric possibility is allowed so that in order to get a solution the \( \varepsilon_\alpha \) satisfying eq.(189) should all be positive. Indeed, in order for the two addends in eq.(193) to make simultaneous sense it is necessary that

\[ D - p - 1 = p + 2 + 1 \quad \longrightarrow \quad D = 2(p+2) \]  

(194)

which is satisfied only in even dimensions and by \( \frac{D-4}{2} \)–branes. In this special case which is precisely that relevant to us, since we have \( D = 4, \; p = 0 \), there are solutions with both positive and negative \( \varepsilon_\alpha \). These are generalized dyons involving both electric and magnetic charges.

If we apply these general formulae to the case of the lagrangian (160) we can immediately obtain the BPS solution that via U–duality transformations generates the most general 1/4 SUSY preserving black–hole. Inserting the weight vectors (178) into eq.(187) we obtain:

\[ M = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \]  

(195)

which by comparison with eq.(189) leads to the solutions

\[ \varepsilon_1 = \varepsilon_2 = 1 \quad \text{or} \quad \varepsilon_1 = \varepsilon_2 = -1 \]  

(196)
So we either have a purely electric or a purely magnetic solution. If we choose the electric one, specializing the general formulae (191), (192), (193), we obtain:

$$ds^2 = (H_1 H_2)^{-1/2} dt^2 - (H_1 H_2)^{1/2} \left( dr^2 + r^2 d\Omega^2 \right)$$

$$h_1 = -\frac{1}{\sqrt{2}} \log \frac{H_1}{H_2}$$

$$h_2 = -\frac{1}{2} \log [H_1 H_2]$$

$$F^{1,2} = -dt \wedge dx \cdot \frac{\partial}{\partial x} (H_{1,2})^{-1}$$

which is nothing else but the same solution we have already directly derived from eq.s (166), (163), (164). Hence we have a perfect fit of the general 1/2 and 1/4 supersymmetry preserving solutions into the general taxonomy of $p$–branes devised by Pope and Lu.

### 5.6 Comparison with the Killing spinor equations

In this section we show that the Killing spinor equations (131), (134) are identically satisfied by the solution (197) giving no further restriction on the harmonic function $H_1, H_2$. Using the formalism developed in section 5.2 the equations (131), (134) can be combined as follows:

$$\frac{16}{3} P_2 \frac{d\Phi_2}{dr} \pm P_1 \frac{d\Phi_1}{dr} = -2q(r)^{\Lambda \Sigma} \tau^{(+)}_{\Lambda \Sigma} f^{\Gamma \Delta} \frac{e^\Phi}{r^2}$$

where

$$P_2 d\Phi_2 = \frac{3}{8} C^{ab} C_{cd} P_{abcd} = \frac{3}{8} U^{ABCD} P_{ABCD} = \frac{3}{4} d\Phi_2$$

$$P_1 d\Phi_1 = C^{XY} C_{ab} P_{XY ab} = Z^{ABCD} P_{ABCD} = 2d\Phi_1$$

Furthermore, using equations (153), we have

$$\text{Im} N_{\Lambda \Sigma \Gamma \Delta} f^{\Gamma \Delta}_{AB} = -\frac{1}{2\sqrt{2}} \left( e^{-\Phi_+} \tau^{(+)}_{\Lambda \Sigma} \tau^{(+)}_{\Lambda \Sigma} + e^{-\Phi_-} \tau^{(-)}_{\Lambda \Sigma} \tau^{(-)}_{\Lambda \Sigma} \right).$$

Therefore, the equation (198) becomes

$$4 \frac{d\Phi_2}{dr} \pm 2 \frac{d\Phi_1}{dr} = 4\sqrt{2} \tau^{(\pm)}_{\Lambda \Sigma} q^{\Lambda \Sigma} \frac{e^\Phi}{r^2}$$

Using equation (157) we also have

$$\tau^{(+)}_{\Lambda \Sigma} q^{\Lambda \Sigma} = \frac{1}{\sqrt{2}} q_1 (r)$$

$$\tau^{(-)}_{\Lambda \Sigma} q^{\Lambda \Sigma} = \frac{1}{\sqrt{2}} q_2 (r).$$

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Comparing equation (197) with (83), (64), (65), we obtain

\[
q_1 = -r^2 \frac{H'_1}{H_1} \sqrt{H_1 H_2}
\]
\[
q_2 = -r^2 \frac{H'_2}{H_2} \sqrt{H_1 H_2}.
\]

(203)

Using all these informations, a straightforward computation shows that the Killing spinor equations are identically satisfied.

6 Conclusions

This paper extends the analysis of ref. [8] to BPS saturated black holes preserving 1/2 or 1/4 of the \( N = 8 \) supersymmetry. Relying on the SLA approach to the non compact coset space spanned by the scalar fields we have determined by purely group theoretical tools how many of the scalar fields are essentially dynamical, in the sense that by an appropriate \( U \)-duality rotation all the other can be set to a constant value, in particular to zero. The same analysis also fixes the number of non–zero charges and how they transform under the relevant normalizer group of the stabilizer of the charge in the normal frame.

The Killing spinor equations were also analysed in a group theoretical fashion and they confirm the predictions of the SLA approach giving in the 1/4 case the extra information that one of the scalar fields is actually zero. We were able to solve explicitly the differential equations of the Killing spinor equations coupled to the Lagrangian of the theory reduced to the relevant scalar fields and electromagnetic field strengths allowed by our analysis. These solutions fit nicely in the general framework of “p-brane taxonomy” studied in [9].

In conclusion we have now a solution and a thorough group-theoretical understanding of the 1/2 and 1/4 preserving BPS saturated black holes. The most general solution is obtained by acting on it by a \( U \)-duality transformation on the normal frame configuration. Note that in ref [8] the same SLA analysis was given for the 1/8 preserving solution yielding the result that the STU model is the most general 1/8 model modulo \( U \)-duality transformations. However, the most general solution could not be derived explicitly, but only one where the axion fields are set to zero. It would be nice also in this case to obtain the most general solution. We know that work is progressing on this issue [23].


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