Callan-Symanzik-Lifshitz approach to generic competing systems

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Abstract

We present the Callan-Symanzik-Lifshitz method to approaching the critical behaviors of systems with arbitrary competing interactions. Every distinct competition subspace in the anisotropic cases define an independent set of renormalized vertex parts via normalization conditions with nonvanishing distinct masses at zero external momenta. Otherwise, only one mass scale is required in the isotropic behaviors. At the critical dimension, we prove: i) the existence of the Callan-Symanzik-Lifshitz equations and ii) the multiplicative renormalizability of the vertex functions using the inductive method. Away from the critical dimension, we utilize the orthogonal approximation to compute higher loop Feynman integrals, anisotropic as well as isotropic, necessary to get the exponents $\eta_n$ and $\nu_n$ at least up to two-loop level. Moreover, we calculate the latter exactly for isotropic behaviors at the same perturbative order. Similarly to the computation in the massless formalism, the orthogonal approximation is found to be exact at one-loop order. The outcome for all critical exponents matches exactly with those computed using the zero mass field-theoretic description renormalized at nonvanishing external momenta.

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I. INTRODUCTION

Universality classes characterizing the ordinary critical behavior of systems undergoing phase transitions [1] have an interesting parallel with those associated to competing systems of the Lifshitz type [2, 3]. The ordinary universality hypothesis states that all universal amounts characterizing the transition like critical exponents, amplitude ratios of certain thermodynamic potentials above and below the critical temperature, etc., do not depend on the microscopic details of the systems but depend upon the number $N$ of components of the order parameter and the space dimension $d$ of the system. Lifshitz universality classes, on the other hand, depend on an additional parameter: the number $m$ of spatial directions where competition takes place. The simplest competing systems belonging to the $m$-axial Lifshitz universality classes correspond to $m = 1$ (uniaxial) and are examples of complex systems with two ordered phases as well as one disordered phase in the vicinity of the Lifshitz critical point. The $m$ space directions are called the competition axes. Turning off the competing interactions is equivalent to taking the limit $m \rightarrow 0$ and the competing system should turn into an ordinary critical system.

Lifshitz multicritical points appear at the confluence of a disordered phase, a uniformly ordered phase and a modulated ordered phase. The spatially modulated phase is characterized by a fixed equilibrium wavevector, which goes continuously to zero as the Lifshitz point is approached. But how can we realize these ideas in terms of concrete models describing actual physical systems? The language of magnetic systems is particularly convenient to find a simple realization of this critical behavior in terms of a lattice model named ANNNI model [4]. It is a $d$-dimensional Ising model with ferromagnetic couplings between first neighbors together with antiferromagnetic exchange forces between second neighbors along only one space direction. Competition in the interactions arises by varying their strengths. More precisely, the Lifshitz point corresponds to a particular value of the ratio between the ferromagnetic and antiferromagnetic exchange interactions. This situation corresponds to the anisotropic uniaxial criticality ($m = 1$). The ANNNI model can be applied to describe the critical behavior of many systems. However, experiments [3, 6, 7, 8] as well as theoretical investigations [9] have confirmed that the magnetic material $MnP$ yields a simple realization of the three-dimensional ANNNI model: it has a pure uniaxial ($N = 1, d = 3, m = 1$) Lifshitz critical behavior.
The competing axis can be classified according to the number of neighbors which are coupled via competing interactions. The ANNNI model can be generalized to include $m$ space directions with competing interactions. We employ the notation $m_2 \equiv m$ to label this subspace according to the number of neighbors they connect through competing exchange forces. In that case the wavevector characterizing the modulated phase has $m_2$ components. The critical system under study presents an $m_2$-fold (or $m_2$-axial) Lifshitz critical behavior, which can be either anisotropic $m_2 < d$ or isotropic $d = m_2$ (close to 8).

If the field (order parameter) has $N$ components, the Lifshitz universality classes are defined by the triplet $(N, d, m_2)$. These universalities correctly reduce to the Ising-like universality class $(N, d)$ in the limit $m_2 \to 0$ [10]. These criticalities have encountered applications in many real physical systems like liquid crystals [11], ferroelectrics [12], especial polymers [13], microemulsions [14], high-$T_c$ superconductors [15], magnetic materials [5, 6, 7, 8], etc. Furthermore, other aspects have been studied like the formulation of quantum phase transitions in Lifshitz points [16, 17, 18] as well as the connection of Lifshitz type field theories with weighted scale invariant quantum field theories [19].

From the technical point of view, the original calculation of usual critical exponents belonging to the Ising-like universality class (without competition) were performed using the renormalization group and $\epsilon$-expansion methods via diagrammatic perturbation of field-theoretic renormalized massive theories [1]. Some time later this method was reformulated such that the former approach was reduced to the computation of a few diagrams (1PI vertex parts) yielding the same exponents through the use of a renormalized massless scalar field theory at nonvanishing external momenta [20, 21, 22]. Inspired in this massless framework, many calculational schemes in the study of $m$-axial Lifshitz points have been put forward [23, 24, 25]. In particular, the unconventional renormalization group arguments in the anisotropic criticalities along with analytical solution methods to resolve Feynman path integrals have permitted a better comprehension of the critical properties of Lifshitz points using the massless scalar field-theoretic setting renormalized at nonvanishing external momenta together with the renormalization group equations in the large distance infrared regime [26, 27].

The method of Refs. [20, 21, 22] to obtain critical exponents of ordinary critical systems was adapted to include the massive theory renormalized at zero external momenta. It allowed to understand the connection between the infrared behavior of the solution to the
renormalization group equation in the massless theory with the ultraviolet behavior of the solution to the Callan-Symanzik equation \cite{28, 29} in the massive theory \cite{30}. These calculations are more involved, since the massive Feynman integrals are intrinsically more difficult to solve than their counterparts in the massless framework. The benefits of this study is that a more direct connection with quantum field theory can be made in the ultraviolet regime. Besides, this extra information is useful to have a proper understanding of other universal quantities like the equation of state, amplitude ratios of certain thermodynamic potentials above and below the critical temperature, etc. We have shown in a previous work \cite{31} that it is possible to formulate an appropriate renormalized field-theoretic setting for massive scalar fields with quartic self interactions in order to compute the critical exponents associated to $m$-fold Lifshitz points. There were two main steps explicit in that construction. The first one is the appearance of two independent mass parameters necessary to describe the two inequivalent space (or momentum) directions present in anisotropic criticalities in an independent manner. Consequently, the Callan-Symanzik-Lifshitz equations allowed to solve the problem at the repulsive ultraviolet fixed points in the anisotropic cases with two independent mass scales. Second, the isotropic universality class only needs one mass scale. Needless to say, the results for the exponents using either the massless approach or the massive method are the same \cite{27, 31}.

A different generalization of the ANNNI model can be considered to include further alternate competing interactions, for instance, up to third neighbors. From a phenomenological viewpoint, the first nontrivial example of a higher character Lifshitz point (see below) occurs for a uniaxial third character Lifshitz point. The three-dimensional phase diagram consists of two parameters varying with the temperature, i.e., the ratio of exchange interactions between the second and nearest neighbors as well as the ratio of exchange couplings between the third and the second neighbors. When the temperature axis is projected on the plane of these parameters, there is a region of intersection where the different phases associated to the system encounter each other in the uniaxial Lifshitz point of third character, whose existence was established numerically \cite{32}. As before, we should emphasize that the uniaxial third character competing axes connect up to third neighbors with alternate exchange forces. When these space dimensions occur along $m_3$ directions, the critical competing system is said to represent the $m_3$-axial third character Lifshitz critical behavior. These universality classes are characterized by $(N, d, m_3)$. The uniaxial third character Lifshitz criticality is
the particular case of the \( m_3 \)-axial behavior for \( m_3 = 1 \).

Using these ideas, the generalization of competing systems whose competing axes link up to \( L \) neighbors via alternate exchange forces can be easily understood. Consider a \( d \)-dimensional Ising model with competition interactions connecting \( L \) neighbors. Let \( m_L \) be the competition spatial subspace where exchange short range couplings take place with ferromagnetic interactions between first neighbors, antiferromagnetic forces between second neighbors, ferromagnetic interactions between third neighbors and so on, with alternating signs for the exchange forces up to the \( L \)th neighbors. This universality class is characterized by \( (N, d, m_L) \) \cite{33, 34}. Even though the anisotropic \( (m_L < d) \) and isotropic behaviors \( (d = m_L) \) can still be defined, the anisotropic situation can be described with only 2 independent scales. Thus, this model does not correspond to the most general competing system.

Generic competing systems of the Lifshitz type have been introduced recently. A simple realization of their critical properties using the language of magnetic systems can be visualized through a lattice model called CECI model \cite{35, 36}. It is a generalized Ising model with several distinct types of competing axes. Each competition subspace is perpendicular to each other. It describes a complex system, which in a simple situation possesses \( L \) ordered phases, as well as one disordered phase, near the generic higher character Lifshitz point. This is so because the CECI model contains simultaneously independent competing axes whose exchange couplings are independent in each spatial subspace. Consequently, in the anisotropic situation there are \( m_2, m_3, ..., m_L \) types of competing axes, as well as \( (d - m_2 - ... - m_L) \) space directions where there are only ferromagnetic couplings among nearest neighbors. We then define \( m_1 = d - m_2 - ... - m_L \) in order to unify the treatment to all subspaces, the competing and noncompeting ones. The noncompeting subspace can be viewed as the competing axes with only first neighbors interacting ferromagnetically. The ordinary critical behavior can be understood in terms of this generic competing system as the isotropic particular case for \( d = m_1 \) with \( m_2 = ... = m_L = 0 \).

The generic \( L \)-th character Lifshitz anisotropic universality classes are now identified by the grid \( (N, m_1, m_2, ..., m_L) \) (actually the same as \( (N, d, m_2, ..., m_L) \)), therefore generalizing the \( L \)-th character Lifshitz universality class previously discussed, which depends on \( (N, d, m_L) \) \cite{33, 34}. However, can we find physical systems which are realizations of higher character Lifshitz points, or does this model have only academic interest? For instance, higher character Lifshitz points can show up in blends of diblock copolymers. In fact, the
earlier nomenclature regarding the higher character Lifshitz points can be translated into the modern terminology if we identify the $L$-th character Lifshitz point with the (former) Lifshitz point of order $(L - 1)$ \[33\]. Thus, it can be verified that Lifshitz points of up to 6th character are possible in those blends of diblock copolymers \[37\]. This is an encouraging evidence that competing systems represented by the CECI model might find applications in many other examples of actual physical systems yet to be discovered. The ANNNI model can be embedded into the CECI model, such that the former can be retrieved from the latter when we switch off the competing interactions beyond third neighbors by taking $m_2 = 1, m_3 = \ldots = m_L = 0$. Moreover, the isotropic behaviors have their universality classes completely specified by $(N, d = m_L)$, whose critical dimension $d_c = 4L$ depends as well on the number of neighbors coupled via competing interactions. They also reduce to the isotropic $m$-axial case when $L = 2$. Thus, the usual $m$-axial Lifshitz criticality is a particular case of the generic competing system described by the CECI model.

In this paper we generalize the framework of renormalized massive scalar fields previously introduced to compute critical exponents pertaining to the $m$-axial Lifshitz universality classes \[31\] in order to study the most general competing system. It can be mathematically understood in terms of an anisotropic field-theoretic description of a massive renormalized $\lambda \phi^4$ scalar field theory including arbitrary higher order derivatives. Each higher derivative term in the Lagrangian density defines a certain type of competing spatial subspace. There are $L$ different types of competing axes which result in $L$ sets of independent masses and coupling constants. We derive the Callan-Symanzik-Lifshitz equations with $L$ independent mass scales, therefore generalizing the previous massive formulation for anisotropic $m$-axial Lifshitz critical behaviors defined by only two independent mass subspaces \[31\]. We study the solutions of these generalized Callan-Symanzik-Lifshitz equations with several independent mass scales. We show that their ultraviolet behavior at the nonattractive ultraviolet fixed point is completely equivalent to the solutions of the renormalization group equations formulated in the massless theory with several independent momenta scales at the infrared fixed point. We focus on renormalized perturbation theory in order to compute the critical indices by diagrammatic means within this technique. The perfect agreement with those calculated in the massless framework at the same loop order is a clear evidence that the universality hypothesis is obeyed as expected.

The critical indices $\eta_L$ and $\nu_L$ associated to each type of competition axes are computed
in the anisotropic cases using the orthogonal approximation previously introduced in [35, 36] to treat the massless field theory formalism. We utilize several mass scales, one for each competing subspace, and the corresponding renormalized theory is defined at vanishing external momenta. Physically, the masses correspond to the independent correlation lengths $\xi_1, \ldots, \xi_L$ which go critical simultaneously and naturally take place in the anisotropic criticalities.

In addition, we use a similar framework to define the isotropic behaviors with only one mass scale and derive the corresponding Callan-Symanzik equations for them. We compute the critical exponents perturbatively using Feynman graph techniques. The orthogonal approximation is employed to compute the analogous critical exponents in the isotropic behaviors. Besides, we shall demonstrate that in spite of the highly nontrivial feature of the Feynman integrals in the massive theory, they can be computed exactly giving exact results for the critical exponents identical to those found in [35, 36].

The normalization conditions are presented in Sec.II. There we motivate the origin of the several independent mass scales in the anisotropic behaviors and show how they can be understood in terms of the phase diagrams of the CECI model. A simpler treatment will be given for the isotropic behaviors as well with only one mass scale for each type of competing axes.

We discuss the one-loop renormalizability at the critical dimension of the various anisotropic and isotropic situations in Sec. III. Using these concepts, we prove inductively the finiteness of the multiplicatively renormalized vertex parts at all orders in a perturbative expansion and demonstrate that the Callan-Symanzik-Lifshitz equations exist in Sec.IV.

In Sec. V we discuss the Callan-Symanzik-Lifshitz (CSL) equations slightly away from the critical dimensions $(d = d_c - \epsilon_L)$. We derive those equations with several mass scales for the anisotropic criticalities. We present the solution of the CSL equations in the ultraviolet regime and show that at the ultraviolet nonattractive fixed point it has the same scaling form as the solution of the renormalization group equations in the infrared regime. We show that the critical exponents calculated by diagrammatic means can be identified with the anomalous dimension of the field and that of the composite operator at the ultraviolet fixed point.

The computation of the anisotropic critical exponents are presented in Sec.VI. The results for Feynman graphs using the orthogonal approximation are derived in Appendix A. They are extensively used in Sec.VI in the calculation of the critical indices using the orthogonal
approximation.

Section VII describes the calculation of the critical exponents for the isotropic cases utilizing the orthogonal approximation. The corresponding loop integrals are computed in Appendix B.

Section VIII presents the exact computation of the Feynman integrals in the isotropic behaviors. The reason for this explicit computations is that the four-point graphs for arbitrary \( n \) is quite difficult to get in a closed form explicitly. By fixing \( n \) we can compute their contribution and find a recursion formula for arbitrary \( n \). In addition, the two-point vertex part graphs are also computed up to three-loop order. They are shown to be simpler than the four-point contributions.

Section IX is an exposition of the exact results obtained for generic isotropic critical exponents for arbitrary \( n \). They are determined diagrammatically using the results of Section VIII. They are shown to be identical to the exponents previously evaluated using the massless formalism.

We present the discussion of our results along with the conclusions and further possible applications of the present method in Sec.X.

II. NORMALIZATION CONDITIONS FOR THE MASSIVE THEORIES

The functional integral representation of the CECI model was first introduced in Ref.[35]. It corresponds to a \( \lambda \phi^4 \) theory containing higher derivatives. The larger the number of neighbors coupled via competing interactions, the higher is the power of the derivative terms in the bare Lagrangian. In the anisotropic behaviors for generic competing systems, there are many simultaneous types of higher derivative terms. The original bare Lagrangian density reads

\[
L = \frac{1}{2} | \nabla (d- \sum_{n=2}^{L} m_n) \phi_0 |^2 + \sum_{n=2}^{L} \frac{\sigma_n}{2} | \nabla_{m_n} \phi_0 |^2 + \sum_{n=3}^{L-1} \sum_{n'=-2}^{n-1} \frac{1}{2} \tau_{nmn'} | \nabla_{m_n} \phi_0 |^2 + \frac{1}{2} \mu_0^2 \phi_0^2 + \frac{1}{4!} \lambda_0 \phi_0^4. \tag{1}
\]

The first and third summations in the above Lagrangian correspond to the effect of the competition occurring in the system. The parameters \( \sigma_n \) and \( \tau_{nmn'} \) are introduced to guaran-
nee that all terms have the same canonical dimension. At the Lifshitz critical region, the fixed values of the exchange interactions are transliterated into the conditions \( \delta_{0n} = \tau_{nn'} = 0 \) in the above Lagrangian. This simplification permits the decoupling of each subspace in Feynman loop integrals, but produces the apparent complication that all the higher derivative terms become relevant in the free massive propagator. Notice that the Lifshitz point is characterized by \( \mu_0 = 0 \) when the temperature is exactly at the Lifshitz value \( T = T_L \), with \( \delta_{0n} = \tau_{nn'} = 0 \).

It is worthy to separate the momentum subspaces in the form \( p(n) = (p_1, ..., p_L) = (p, k_2, ..., k_L) \) where \( p_1 \equiv p \) is a vector along \( m_1 \) spatial directions connected to the first (non)competing subspace, \( p(2) \equiv k_2 \) where \( k_2 \) is a vector along \( m_2 \) space directions associated to the second neighbor competing subspace, etc., \( p(L) \equiv k_L \) where \( k_L \) is a vector along \( m_L \) directions associated to the \( L \)-th neighbor competing subspace. The variation of \( \kappa_n \) in the renormalized theories whose starting point is the same bare theory are induced by the existence of independent correlation lengths \( \xi_n \). These \( L \) independent flows in the momenta can be implemented through \( L \) independent renormalization group equations for each \( m_n \)-dimensional subspace. Nevertheless, this construction is consistent since the apparent overcounting producing \( L \) independent coupling constants defined in each spatial subset can be overcome, for all of them flow to the same infrared nontrivial fixed point.

As discussed in detail in Ref. [36] in the massless theory, the conditions which define the critical region can be used to perform a dimensional redefinition of momentum scales along each type of competing subspace as follows. Let \( [p] = M \) be the mass dimension of the quadratic momenta corresponding to the noncompeting subspace \( m_1 \). We can get rid of the parameter \( \sigma_n \) appearing in front of each higher derivative term characterizing each competing subspace by redefining the associated momenta scales through the relations \( [k_{(n)}] = M^{\hat{n}} \). This effectively disentangles each momenta subspace such that they can be treated independently. In the anisotropic criticalities there are \( L \) subspaces, each of them are \( m_L \)-dimensional. The masslessness of the theory at the Lifshitz temperature requires that the renormalized 1PI vertex parts (the basic objects in this framework related to the thermodynamical potentials of the critical system) must be computed at nonzero external momenta in order to avoid infrared divergences. For instance, if we consider the vertex functions along the \( L \)-th subspace we use normalization conditions by choosing the symmetry point at nonzero external momenta \( \kappa_L \) along this subspace, whereas all the other momenta
scales perpendicular to the $m_L$ directions are set to zero. Therefore, the nonzero momenta used to renormalize the theory in each individual subspace can be viewed as a label in parameter space defining $L$ independent sets of vertex functions. On the other hand, there is no need to consider more than one subspace for isotropic behaviors.

Now, let us take a look at the simplest CECI model, namely, we take $m_2 = m_3 = 1$ with $m_n = 0$ for $n > 3$. The competition are located at the $y$ and $z$ axis. The phase diagram (see Fig.1 of Ref. [36]) can be described by $T, p_z = J_{2z}/J_{1z}, p_{1y} = J_{2y}/J_{1y}$ and $p_{2y} = J_{3y}/J_{1y}$. We can represent the phase diagram in a very simple form through several two-dimensional projections by fixing some parameters and varying only two of them. For instance, the diagram $(T, p_z)$ is associated to the second character behavior provided $p_{1y}$ and $p_{2y}$ are kept fixed. On the other hand, the purely third character behavior can be expressed in terms of a three-dimensional phase diagram with axes $(T, p_{1y}, p_{2y})$ so long as $p_z$ is fixed. But this can be further simplified by considering only the two-dimensional projection of this phase diagram as suggested above if we take instead the variation of $(p_{1y}, p_{2y})$ with $(T, p_z)$ at constant values. Now the superposition of these two two-dimensional phase diagrams yields a point of intersection for particular values of the temperature which is identified with the generic third character Lifshitz point. Notice that the ferromagnetic phase and the two modulated phases named $Helical_2$ and $Helical_3$ (see Fig. 3 of Ref. [36]) encounter themselves at the uniaxial generic third character Lifshitz point. There are two first order lines: one of them separates the ordered-$Helical_2$ regions whereas the other splits the $Helical_2 - Helical_3$ phases.

This situation can be generalized for generic higher character Lifshitz points if we split the corresponding multiparameter(multidimensional) phase diagram in two-dimensional slices and by superposing them together in a single two-dimensional diagram. There are now $L$ modulated phases meeting at the $L$-th generic higher character Lifshitz point. We emphasize that each competing subspace is defined by its own independent correlation length, i.e., $\xi_1$ for the subspace with only ferromagnetic exchange forces coupling first neighbors, ..., $\xi_L$ for the subspace defined by alternate signs in the exchange forces up to $L$-th neighbors.

Roughly speaking, we can associate the (inverse of the) mass to the correlation length. The anisotropic generic higher character universality classes require $L$ independent correlation lengths. In close analogy to what has been done for the $m$-axial Lifshitz critical behavior (actually a second character Lifshitz behavior) we can reexpress the bare Lagrangian density
(1) in terms of $L$ independent bare masses as follows.

It is obvious that we can attain the $L$-th generic higher character Lifshitz point in the phase diagram outlined above by varying the “mass” coming from the noncompeting subspace. This means that we approach the $L$-th generic higher character Lifshitz point from the ferromagnetic phase. Simple inspection of the phase diagram indicates that we can reach this multicritical point coming from any of the several modulated phases. Then, it is possible to introduce $L$ independent bare masses such that they generate the $L$ renormalization group flows in parameter space which are compatible with the $L$ correlation lengths present in these criticalities. Let us describe the introduction of the many independent mass scales beyond these simple phenomenological considerations.

Siegel’s method of dimensional reduction suggests in a simple way how to introduce mass in ordinary quantum field theories with quadratic derivatives in the Lagrangian density[38]. The basic steps for scalar fields are: extend the range of the momenta indices and call the extra index “$-1$”, choose the momentum component associated to this direction equal to the mass $p_{-1} = \mu_0$ and introduce factors of $i$ to re-establish reality $\partial_{-1} = i p_{-1} = i \mu_0$. The operator $p^2$ in the higher dimensional space (including the extra index “$-1$”) results in a massive operator $p^2 + \mu_0^2$ in $d$ spacetime dimensions. In the Lagrangian (1) the metric is Euclidean but the conditions $\delta_{0n} = \tau_{nn'} = 0$ can be used to perform the dimensional redefinition of the momenta characterizing each type of competing axes. The dimensional redefinitions turn out to implement independent dilatation invariance along the $m_1 = d - \sum_{n=2}^{L} m_n$ noncompeting directions, $m_2$ subspace, and so on, up to $m_L$ space directions. The typical momentum combinations which appear in the inverse free propagator is of the form $p_1^2 + \sum_{n=2}^{L} (k_n^2)^n$.

The extension of the massive method to include arbitrary types of competing axes can be understood from the analysis of $m$-axial critical behavior employing massive fields. The introduction of distinct masses using Siegel’s recipe translates itself in the following conditions: extend the range of the vector indices in the $n$ subspace to the “extra direction” ”$-1$”($n$), identify the conjugate momentum component with the mass in that subspace $p_{-1(n)} = \mu_{0n}$ and use factors of $i$ to restore reality $\partial_{-1(n)} = i p_{-1(n)} = i \mu_{0n}$. If we apply this reduction to the $n = 1$ subspace, the operator $p_1^2 + \sum_{n=2}^{L} (k_n^2)^n$ in the higher dimensional space turns out to become $p_1^2 + \sum_{n=2}^{L} (k_n^2)^n + \mu_{01}^2$ in $d$ space dimensions. When we utilize the same procedure to the subspace $n = 2$ with $k_{1(2)} = \mu_{02}$, the simplest situation occurs for uniaxial case $m_2 = 1$. The combination in the inverse free propagator becomes $p_1^2 + k_2^4 + \sum_{n=3}^{L} (k_n^2)^n$. 

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If we define the internal product in the higher dimensional space with index “−1(2)” by $k^2_2 + k^2_{1(2)}$, dimensional reduction yields $p^2_1 + k^2_2 + \sum_{n=3}^L (k^2_n)^n \rightarrow p^2_1 + k^2_2 + \sum_{n=3}^L (k^2_n)^n + \mu^4_0$. Sticking to simplicity, we can extend this procedure to the $n'$ subspace in the uniaxial case $m_{n'} = 1$. Choosing the internal product in the higher dimensional space including the index $-1(n')$ as $k^{2n'}_n + k^{2n'-1}_{-1(n')}$, the combination $p^2_1 + k^{2n'}_n + \sum_{n \neq n'}^{L}(k^2_n)^n$ is reduced to $p^2_1 + k^{2n'}_n + \sum_{n \neq n'}^{L}(k^2_n)^n + \mu^{2n'}_{0n'}$. These arguments suggest that it is possible to choose the mass defining each competing subspace with the same canonical dimension of the momenta along those directions.

The resulting strategy leads us to define the bare masses in the bare Lagrangian density with different powers, depending on the chosen subspace we work with. This makes explicit reference to the fact that those distinct subspaces are inequivalent. Our experience handling labels in the massless theory suggests that when dealing with the renormalized theory, the bare mass in each subspace naturally defines a renormalized mass and coupling constant characterizing that subspace. We can then implement $L$ independent bare masses along with $L$ independent bare coupling constants in the bare Lagrangian density. Bearing in mind these considerations we write the original bare Lagrangian in the form

$$L = \frac{1}{2} |\nabla_{(a-\sum_{n=2}^L m_n)} \phi_0|^2 + \sum_{n=2}^L \frac{\sigma_n}{2} |\nabla_{m_n} \phi_0|^2 + \sum_{n=2}^L \delta_{0n} \frac{1}{2} |\nabla_{m_n} \phi_0|^2 + \sum_{n=3}^{L-1} \sum_{n'=2}^{n-1} \frac{1}{2} \tau_{nn'} |\nabla_{m_n} \phi_0|^2 + \frac{1}{2} \mu^{2n}_{0n} \phi_0^2 + \frac{1}{4!} \lambda_{0n} \phi_0^4.$$  

We shall focus our attention hereafter in the Lifshitz critical region where $\delta_{0n} = \tau_{nn'} = 0$ in Eq.(2). The independent bare mass in each competing subspace naturally prevents the appearance of infrared divergences in calculating Feynman integrals for the associated $1PI$ vertex functions. Thus, the label $n$ can be used to define independent renormalized $1PI$ vertex functions which have the following property: those with arbitrary external momenta $p_{(n)}$ along the $n$-th space directions have a nonvanishing bare mass $\mu_{0n}$ and coupling constant $\lambda_{0n}$. However, they have vanishing external momenta ($p_{n'} = 0$), bare mass ($\mu_{0n'} = 0$) and coupling constant ($\lambda_{0n'} = 0$) along all directions elsewhere (if $n' \neq n$). Moreover, the zero mass limits of the vertex parts are well-defined and reduce to the cases previously investigated in Refs.[35, 36].
One important ingredient to complete the description is the existence of \( L \) independent cutoffs, which are responsible for the independent variations in the mass parameters of the anisotropic cases. Each cutoff has the same canonical dimension as mass and momenta characterizing the competition subspace under consideration. There is a similarity and a difference when we compare the massless and the massive approaches. Unlike the massless case, the massive theories do not flow to fixed points where they become scale invariant: scale invariance is only achieved if the coupling constants in each subspace are set exactly at the eigenvalue conditions \( u_{n\infty} \), i.e., at the non-attractive fixed points. On the other hand, the ultraviolet fixed point value of the coupling constants \( u_{n\infty} \) are independent of \( n \), a feature already encountered in our analysis of the massless theory in the infrared regime. The isotropic behaviors can be described by the Lagrangian (2) with some modifications: the first term does not appear in the isotropic situation \( d = m_n, n = 1, 2, ..., L \) and there is solely one kind of bare (renormalized) mass.

We begin by describing the anisotropic renormalization conditions. Except for minor modifications, we shall follow the conventions adopted in Refs.\[31, 36\]. Let the \( n \)-th subspace \((n = 1, ..., L)\) be defined by a nonvanishing value of bare mass and coupling constant \( \mu_{0n} \neq 0, \lambda_{0n} \neq 0 \) with \( \mu_{0n'} = 0, \lambda_{0n'} = 0 \) for \( n' \neq n \). The nonvanishing bare parameters induce renormalized parameters \( \mu_n \) and \( g_n \). The slight change of notation with respect to Ref.\[31\] in defining the renormalized mass is performed in order to prevent the confusion with the number of space directions \( m_n \) of the \( n \)-th competing subspace. The renormalized 1\( PI \) vertex parts living in the \( m_n \)-dimensional subspace are defined by the following normalization conditions:

\[
\begin{align*}
\Gamma^{(2)}_{R(n)}(0, \mu_n, g_n) &= \mu_n^{2n}, \\
\frac{\partial \Gamma^{(2)}_{R(n)}(p_{(n)}, \mu_n, g_n)}{\partial p_{2n}^2} |_{p_{2n}^2 = 0} &= 1, \\
\Gamma^{(4)}_{R(n)}(0, \mu_n, g_n) &= g_n, \\
\Gamma^{(2,1)}_{R(n)}(0, 0; 0, \mu_n, g_n) &= 1.
\end{align*}
\]

In analogy to the massless case, we can fix the renormalized mass scale in each competing subspace by choosing \( \mu_n^{2n} = 1 \).

The isotropic situations can be described directly by using the above normalization conditions for the renormalized vertex parts where only one type of competing axes are required.
in that case. Although the scaling analysis is a bit different in the isotropic and anisotropic cases, they share the same normalization conditions Eq.(3).

III. ONE-LOOP RENORMALIZABILITY AT THE CRITICAL DIMENSION

In this section we shall investigate the divergence structure of graphs corresponding to primitively divergent bare 1PI vertex parts. In general, the divergences can be expressed in terms of regularized expressions which retain their infinite values provided the regulators used take appropriate limits. We shall show that these divergences can be expressed rather simply in terms of independent cutoffs in the anisotropic cases, when there are many independent competing subspaces appearing simultaneously in the problem. Otherwise, the isotropic cases only require one type of cutoff. We discuss the renormalization of all vertex parts at one-loop level. Many results of this section are going to be useful in proving renormalizability at arbitrary loop order. We shall restrict our attention throughout only to vertex parts which can be renormalized multiplicatively.

A. Anisotropic Sector

Consider the noncompeting subspace corresponding to the label \( n = 1 \). The vertex parts associated to it possess arbitrary external momenta \( p_1 = p \) along the \( m_1 = d - \sum_{n=2}^{L} m_n \) space directions, with nonvanishing bare mass \( \mu_{01} \) and coupling constant \( \lambda_{01} \). The bare primitive divergent vertex parts are \( \Gamma^{(2)}_{R(1)}, \Gamma^{(4)}_{R(1)} \) and \( \Gamma^{(2,1)}_{R(1)} \). Their perturbative expansions up to one-loop order can be written as:

\[
\Gamma^{(2)}_{(1)}(p) = p^2 + \mu_{01}^2 + \lambda_{01}^2 \int \frac{d^{d-\sum_{n=2}^{L} m_n} q \Pi_{n=2}^{L} d^{m_n} k_{(n)}}{\left( \sum_{n=2}^{L} (k_{(n)}^2)^n + q^2 + \mu_{01}^2 \right)}, \quad (4)
\]

\[
\Gamma^{(4)}_{(1)}(p_1) = \lambda_{01}^2 \left( \int \frac{d^{d-\sum_{n=2}^{L} m_n} q \Pi_{n=2}^{L} d^{m_n} k_{(n)}}{\left[ \sum_{n=2}^{L} (k_{(n)}^2)^n + (q + p_1 + p_2)^2 + \mu_{01}^2 \right] \left( \sum_{n=2}^{L} (k_{(n)}^2)^n + q^2 + \mu_{01}^2 \right)} \right)^2, \quad (5)
\]

\[
\Gamma^{(2,1)}_{(1)}(p_1, p_2; p_3) = 1 - \lambda_{01} \int \frac{d^{d-\sum_{n=2}^{L} m_n} q \Pi_{n=2}^{L} d^{m_n} k_{(n)}}{\left[ \sum_{n=2}^{L} (k_{(n)}^2)^n + (q + p_3)^2 + \mu_{01}^2 \right] \left( \sum_{n=2}^{L} (k_{(n)}^2)^n + q^2 + \mu_{01}^2 \right)} \quad (6)
\]
It is easy to see that for $m_2 = \ldots = m_L = 0$ the above integrals reduce to the ordinary noncompeting ones and there is no subintegral involving competing momenta. However, we shall keep the values of $m_n$ unspecified and compute the integrals at the upper critical dimension $d_c = 4 + \sum_{n=2}^{L} \left[ \frac{(n-1)}{n} \right] m_n$.

We begin with the computation of the integral $I_{1(1)}$ with a single propagator appearing in Eq.(4). We may employ the Schwinger’s trick to write the propagator in terms of a parametric integral, namely

$$
\frac{1}{(\sum_{n=2}^{L} (k_{(n)}^2)^n + q^2 + \mu_0^2)} = \int_0^\infty d\alpha \exp[-\alpha (\sum_{n=2}^{L} (k_{(n)}^2)^n + q^2 + \mu_0^2)].
$$

(7)

Making use of the identity \[27\]

$$
\int_{-\infty}^{\infty} dx_1 \ldots dx_m \exp(-a(x_1^2 + \ldots + x_m^2)^n) = \frac{1}{2^n} \Gamma(\frac{m}{2n}) a^{-\frac{m}{2n}} S_{m_n},
$$

(8)

we can solve the integrals over the momenta. The resulting expression for $I_{1(1)}$ is given by

$$
I_{1(1)} = \frac{1}{2} S_{(4-\sum_{n=2}^{L} \frac{m_n}{n})} \Gamma(2 - \sum_{n=2}^{L} \frac{m_n}{2n}) (\prod_{n=2}^{L} \frac{S_{m_n} \Gamma(\frac{m_n}{2n})}{2n}) \int_0^\infty d\alpha \exp[-\alpha \mu_0^2] \alpha^{-2}.
$$

(9)

We introduce the cutoff $\Lambda_1$ in order to express mathematically the ultraviolet divergence implicit in the parametric integral in terms of it. The divergence comes from the region for small values of $\alpha$. We then regularize the integral by suppressing a domain $(0, \Lambda_1^{-2})$ in the small $\alpha$ region of integration. It is very simple to introduce in the integrand a function of $\alpha$ and $\Lambda_1$ $f_{\Lambda_1}(\alpha)$ which vanishes for $\alpha < \Lambda_1^{-2}$ (whose derivatives vanish in the limit $\alpha \rightarrow 0$) and is identically one for $\alpha > \Lambda_1^{-2}$. The Heaviside function is efficient to produce this effect and we choose $f_{\Lambda_1}(\alpha) = \theta(\alpha - \Lambda_1^{-2})$. Integrating by parts twice and letting the cutoff go to infinity, the divergence can be expressed in terms of the cutoff as

$$
I_{1(1)} = \frac{1}{2} S_{(4-\sum_{n=2}^{L} \frac{m_n}{n})} \Gamma(2 - \sum_{n=2}^{L} \frac{m_n}{2n}) (\prod_{n=2}^{L} \frac{S_{m_n} \Gamma(\frac{m_n}{2n})}{2n}) \mu_0^2 (\Lambda_1^2 - \frac{\ln(\Lambda_1^2}{\mu_0^2})).
$$

(10)

The overall angular factor is different from the pure $\phi^4$ field theory and appears in the same way in both cases: it shows up whenever a loop integral is performed. Proceeding in the standard way, it can be absorbed in a redefinition of the coupling constant. Looking at the singularity structure of $I_{1(1)}$ we find that its dependence on $\Lambda_1$ is exactly the same as its counterpart describing ordinary critical behavior has in terms of the cutoff, say $\Lambda$, at the critical dimension $d = 4$ \[21, 39\].
The integral contributing to both $\Gamma^{(4)}_{R(1)}$ and $\Gamma^{(2,1)}_{R(1)}$ denoted by $I_{2(1)}$, namely
\begin{equation}
I_{2(1)}(p) = \int \frac{d^{d-\sum_{n=2}^{L} m_n} q \Pi_{n=2}^{L} d^{m_n} k_{(n)}}{[\sum_{n=2}^{L} (k_{(n)}^2)^n + (q + p)^2 + \mu_{01}^2] \left( \sum_{n=2}^{L} (k_{(n)}^2)^n + q^2 + \mu_{01}^2 \right)},
\end{equation}
can be performed in a similar fashion. Introduce two Schwinger parameters and integrate over the momenta. We are left with two parametric integrals over $\alpha_1$ and $\alpha_2$. Integrate first over $\alpha_1$ by defining a new variable $\alpha' = \alpha_1 + \alpha_2$. The integration limits of the integral over $\alpha'$ turn out to be $(\alpha_2, \infty)$. This integration produces no divergence, but the integral to be done over $\alpha_2$ results in the ultraviolet divergence for small values of $\alpha_2$. We regularize this integral exactly as before by introducing the cutoff function $f_{\Lambda_1} (\alpha_2) = \theta(\alpha_2 - \Lambda_1^{-2})$ inside the integrand. Expanding the integrand in powers of the external momenta, we find out that the divergence is present only in the momentum independent term and taking the limit $\Lambda_1 \to \infty$, we obtain the following divergent result
\begin{equation}
I_{2(1)}(p) = \frac{1}{2} S_{(4-\sum_{n=2}^{L} m_n)} \Gamma(2 - \sum_{n=2}^{L} \frac{m_n}{2n})(\Pi_{n=2}^{L} S_{m_n} \Gamma(\frac{m_n}{2n})) \ln \left( \frac{\Lambda_1^2}{\mu_{01}^2} \right).
\end{equation}

Owing to the similarity with the $m$-axial Lifshitz situation, when there is only one type of competing axes, we are going to treat all competing subspaces at once. We select only the competing axes in the $m_n$-dimensional subspace where competing interactions couple up to $n$ neighbors. In that case, we start with $\mu_{0n'r'} \equiv \mu_{0n} \delta_{nn'}$, $\lambda_{0n'} \equiv \lambda_{0n} \delta_{nn'}$, and external momenta $p_i(n') \equiv k'_{i(n')} \delta_{nn'}$. The vertex functions which have primitive divergences can be expanded up to one-loop order as
\begin{equation}
\Gamma^{(2)}_{(n)}(k'_{(n)}) = (k'_{(n)}^2)^n + \mu_{0n}^2 + \frac{\lambda_{0n}}{2} \int \frac{d^{d-\sum_{n=2}^{L} m_n} q \Pi_{n=2}^{L} d^{m_n} k_{(n)}}{(\sum_{n=2}^{L} (k_{(n)}^2)^n + q^2 + \mu_{0n}^2)},
\end{equation}
\begin{equation}
\Gamma^{(4)}_{(n)}(k'_{i(n)}) = \lambda_{0n} - \frac{\lambda_{0n}^2}{2} \int \frac{d^{d-\sum_{n=2}^{L} m_n} q \Pi_{n=2}^{L} d^{m_n} k_{(n)}}{[\sum_{(n' \neq n)}^{L} (k_{(n')}^2)^n + ((k_{(n)} + k'_{i(n)} + k'_{(2(n)}^2)^n + q^2 + \mu_{0n}^2)] (k'_{1(n)}) + k'_{3(n)}) + k'_{2(n)} \to k'_{3(n)}),
\end{equation}
\begin{equation}
\Gamma^{(2,1)}_{(n)}(k'_{1(n)}; k'_{2(n)}; k'_{3(n)}) = 1 - \frac{\lambda_{0n}}{2} \int \frac{d^{d-\sum_{n=2}^{L} m_n} q \Pi_{n=2}^{L} d^{m_n} k_{(n)}}{[\sum_{(n' \neq n)}^{L} (k_{(n')}^2)^n + ((k_{(n)} + k'_{3(n)}^2)^n + q^2 + \mu_{0n}^2)] (k'_{1(n)}) \to k'_{3(n)}; k'_{2(n)} \to k'_{3(n)}),
\end{equation}
\begin{equation}
\times \frac{1}{(\sum_{n=2}^{L} (k_{(n)}^2)^n + q^2 + \mu_{0n}^2)}.\end{equation}
In order to perform these integrals, we recall that the bare mass parameter has $\frac{1}{n}$ of the canonical dimension of the momenta along the noncompeting (quadratic) subspace. We then choose the cutoff $\Lambda_n$ associated to this subspace with the same canonical dimension as $\mu_0$. When using the Schwinger parameters to perform the integrals over the momenta variables, we encounter the divergences implicit over the parametric integrals in the region of small values of $\alpha$. We cutoff the parametric integrals introducing in the integrand the regularization function $f_{\Lambda_n}(\alpha) = \theta(\alpha - \Lambda_n^{-2n})$, which restricts the integration domain to the $\Lambda_n^{-2n} \leq \alpha \leq \infty$. Denote the integrals contributing to the two-point function at one-loop by $I_{1(n)}$ whereas $I_{2(n)}$ represent the contributions to $\Gamma_{(n)}^{(4)}$ (and $\Gamma_{(n)}^{(2,1)}$), respectively. Taking the limit $\Lambda_n \to \infty$ we get to the following expressions for these objects ($n = 2, \ldots, L$)

$$I_{1(n)} = \frac{1}{2} S_{(4-\sum_{n=2}^{L} \frac{m_n}{2n})} \Gamma(2 - \sum_{n=2}^{L} \frac{m_n}{2n})(\Pi_{n=2}^{L} \frac{S_{m_n} \Gamma(m_n/2n)}{2n}) \mu_0 \left( \frac{\Lambda_n^{2n}}{\mu_0^{2n}} - \ln \left( \frac{\Lambda_n^{2n}}{\mu_0^{2n}} \right) \right), \quad (16)$$

$$I_{2(n)}(k'_{(n)}) = \frac{1}{2} S_{(4-\sum_{n=2}^{L} \frac{m_n}{2n})} \Gamma(2 - \sum_{n=2}^{L} \frac{m_n}{2n})(\Pi_{n=2}^{L} \frac{S_{m_n} \Gamma(m_n/2n)}{2n}) \ln \left( \frac{\Lambda_n^{2n}}{\mu_0^{2n}} \right). \quad (17)$$

Comparing the above equations with their counterparts in the noncompeting (quadratic) subspace, it is obvious that we can now unify all subspaces by taking $n = 1, \ldots, L$ in the above formulae. This unification will be useful in our description of the renormalization of masses and coupling constants at one-loop order. The fact of the matter is that when we use the normalization conditions at zero external momenta in each subspace with a set of finite renormalized parameters ($\mu_n, g_n$) starting from infinite bare quantities ($\mu_0, \lambda_0$), we can express the former in terms of the latter via the following equations:

$$\mu_n^{2n} = \mu_0 + \frac{\lambda_0}{2} I_{1(n)}(0), \quad (18)$$

$$g_n = \lambda_0 - \frac{3}{2} I_{2(n)}(0). \quad (19)$$

If we go the other way around by writing the bare parameters in terms of the renormalized ones and getting rid of higher order corrections in the renormalized coupling constants, these simultaneous operations render the original bare vertex parts $\Gamma_{(n)}^{(2)}$ and $\Gamma_{(n)}^{(4)}$ finite. In fact, the corresponding finite vertex parts are given by ($n = 1, \ldots, L$)

$$\Gamma_{(n)}^{(2)}(p_{(n)}) = (p_{(n)})^{2n} + \mu_n^{2n}, \quad (20)$$

$$\Gamma_{(n)}^{(4)}(p_{(n)}) = g_n - \frac{g_n^2}{2} (I_{2(n)}(p_{1(n)} + p_{2(n)}) + I_{2(n)}(p_{1(n)} + p_{3(n)}) + I_{2(n)}(p_{2(n)} + p_{3(n)}) - 3 I_{2(n)}(0)). \quad (21)$$
The bare vertex parts $\Gamma_{(n)}^{(N)}$ with $N > 4$ have skeleton expansions. There are $L$ independent sets of skeleton expansions, one set for each competing subspace. Thus, their diagrammatic expansions result in finite expressions at two-loop level when they are written in terms of the renormalized mass(es) and coupling constant(s) at one-loop order.

In order to study insertion of composite operators and their renormalization, we analyze the bare vertex $\Gamma_{(n)}^{(2,1)}$. They are not automatically finite by the reparametrizations turning the bare mass and coupling constant into those renormalized amounts. We can define the renormalized (finite) vertex part $\Gamma_{R(n)}^{(2,1)}$ as

$$\Gamma_{R(n)}^{(2,1)}(p_1(n); p_2(n); g_n, \mu_n) = Z_{\phi^2(n)} \Gamma_{(n)}^{(2,1)}(p_1(n); p_2(n); p_3(n); \lambda_0(n), \mu_0, \Lambda_n).$$

The normalization conditions (3d) require that, at this order in the loop expansion, the normalization functions are given by

$$Z_{\phi^2(n)} = 1 + \frac{g_n}{2} I_{2(n)}(0).$$

Replacing this expression into the definition of $\Gamma_{R(n)}^{(2,1)}$ turns out to make this vertex function finite, which may be written as

$$\Gamma_{R(n)}^{(2,1)}(p_1(n); p_2(n); g_n, \mu_n) = 1 - \frac{g_n}{2} (I_{2(n)}(p_3(n)) - I_{2(n)}(0)).$$

In most applications we shall describe insertion at zero external momenta, i.e., we refer to the vertex function $\Gamma_{R(n)}^{(2,1)}(p(n); -p(n); 0; g_n, \mu_n)$. In this manner, we can address the multiplicative renormalizability and obtain independent flows in the parameter spaces of vertex parts including arbitrary types of composite operators, irrespective of the competing subspace under scrutiny.

The remaining multiplicatively renormalized vertex parts including composite operators $\Gamma_{R(n)}^{(N,L)}$ with $(N, L) > (2, 1)$ have skeleton expansions. Consequently, they are finite at two-loop level whenever we use $g_n, \mu_n$ and $\Gamma_{R(n)}^{(2,1)}(p_1(n); p_2(n); g_n, \mu_n)$ (or $Z_{\phi^2(n)}$) inside their one-loop subgraphs, due to the following high momentum pattern of the primitively divergent vertex functions

$$|\Gamma_{R(n)}^{(2)}(\rho_n p(n))| \leq \rho_n^{2n} \times \text{power of } \ln \rho_n,$$

$$|\Gamma_{R(n)}^{(4)}(\rho_n p_i(n))| \leq \text{power of } \ln \rho_n,$$

$$|\Gamma_{R(n)}^{(2,1)}(\rho_n p_1(n); p_1(n); \rho_n q(n))| \leq \text{power of } \ln \rho_n.$$
at every finite order in the limit $\rho_n \to \infty$. This large momentum behavior is not going to be derived here, but we shall assume that it is valid henceforth. In other words, the Born values of the various renormalized vertex functions are modified only by powers of logarithms. These modifications are caused by the interactions in every perturbative order at the loop expansion, and we can use a power counting reasoning to approach the multiplicative renormalizability at the critical dimension to all loop orders. This (nonperturbative) proof of multiplicative renormalizability corresponding to the Callan-Symanzik-Lifshitz massive method shall be analyzed after our construction of similar renormalization background for isotropic behaviors at one-loop order, to which we turn our attention next.

B. Isotropic Sector

Recall that for arbitrary higher character isotropic Lifshitz critical behaviors there is only one type of competing subspace. The critical dimension of the isotropic critical behaviors when there are $n$ neighbors along each space dimension interacting via alternate exchange forces is $d = m = 4n$, with $n = 1, ..., L$. The one-loop required vertex parts which are primitively divergent are given by

$$
\Gamma^{(2)}(2)(\mu) = (k^2)^n + \mu^{2n}_0 + \frac{\lambda_0}{2} \int \frac{d^4 k'}{[(k')^2)^n + \mu^{2n}_0]} , \quad (25)
$$

$$
\Gamma^{(4)}(4)(k_i) = \mu_0 - \frac{\lambda_0}{2} \int \frac{d^4 k'}{[(k' + k_1 + k_2)^2)^n + \mu^{2n}_0][(k')^2)^n + \mu^{2n}_0]} + (k_1 \to k_3) + (k_2 \to k_3) \quad , \quad (26)
$$

$$
\Gamma^{(2,1)}(2,1)(k_1, k_2; k_3) = 1 - \frac{\lambda_0}{2} \int \frac{d^4 k'}{[(k' + k_3)^2)^n + \mu^{2n}_0][(k')^2)^n + \mu^{2n}_0]} . \quad (27)
$$

Let $I_{1(n)}$ and $I_{2(n)}$ be the one-loop integrals involved in the calculation of $\Gamma^{(2)}(n)$ and $\Gamma^{(4)}(n)$, respectively. The computation of the integrals are entirely analogous to the $n$th competing subspace of the anisotropic sector: the cutoff $\Lambda_n$ has the same canonical dimension of $\mu_0$ and the removal of small values of the parametric integrals in the Schwinger parameters are implemented via Heaviside’s function. The only difference with respect to the anisotropic case study in the $n$th competing subspace is that the angular factor is different. We then obtain

$$
I_{1(n)}(k) = \frac{1}{2n} \int S_{4n} \mu^{2n}_0 \left( \frac{\Lambda^{2n}_n}{\mu^{2n}_0} - \ln \left( \frac{\Lambda^{2n}_n}{\mu^{2n}_0} \right) \right) , \quad (28)
$$

19
\[ I_{2(n)}(k) = \frac{1}{2n} S_{4n} \ln(\frac{\Lambda^{2n}_{n}}{\mu^{2n}_{0}}). \]  

(29)

Renormalization of the mass and coupling constant can be accomplished through the utilization of the normalization conditions (3a) and (3c). When expressed as functions of the bare amounts (which have infinite values) they take the form

\[ \mu^{2n}_{n} = \mu_{0n} + \frac{\lambda_{0n}}{2} I_{1(n)}(0), \]  

(30)

\[ g_{n} = \lambda_{0n} - \frac{3\lambda_{0n}^{2}}{2} I_{2(n)}(0). \]  

(31)

We could have written the above expressions in the opposite direction. In that case, either the bare functions \( I_{n}^{(X)}(k_{i})(\mu_{n}, g_{n}) \) can be rendered convergent at one-loop order provided \( N \leq 4 \), or they are skeleton expansions. To conclude, whenever we use the normalization condition (3d) along with the appropriate version of Eq. (22), which amounts to define the normalization function \( Z_{\phi^{2}(n)} \), they produce a convergent expression for the renormalized vertex \( \Gamma_{R(n)}^{(2,1)}(k_{1}, k_{2}; k_{3}) \). Consequently, this leads to \( Z_{\phi^{2}(n)} = 1 + \frac{g_{n}}{2} I_{2(n)}(0) \).

Next, let us tackle the issue of multiplicative renormalization in order to extend the one-loop discussion just explained in the present section to arbitrary loop order employing the Callan-Symanzik-Lifshitz equations.

IV. THE CALLAN-SYMANZIK-LIFSHITZ EQUATIONS AT THE CRITICAL DIMENSIONS

Let us establish the inductive proof of multiplicative renormalizability and prove the existence of the Callan-Symanzik-Lifshitz equation in the presence of various simultaneous types of competing axes as well as when solely one type of competing axes take place in the generalized Lifshitz critical behaviors.

Although we shall not make explicit reference, we shall be concerned with vertex parts which are regularized using cutoffs as described in the previous section. It is implicitly assumed that this is the case in the remainder of the present section. Our discussion here is restricted to the critical dimension of the theory.
A. Anisotropic behaviors

We are going to study the physical system at its upper critical dimension \( d_c = 4 + \sum_{n=2}^{L} \left[ \frac{(n-1)}{n} \right] m_n \). The multiplicatively renormalized vertex parts including composite operators are defined by

\[
\Gamma^{(N,L)}_{R(n)}(p_i(n), Q_i(n), g_n, \mu_n) = Z^n_{\phi(n)} Z^n_{\phi^2(n)} \Gamma^{(N,L)}(p_i(n), Q_i(n), \lambda_{0n}, \mu_{0n}, \Lambda_n). \tag{32}
\]

We emphasize that vertex function with \((N, L) = (0, 2)\) which are additively renormalized are precluded from our analysis. In the above expression, \( p_i(n) \) \((i = 1, \ldots, N)\) are the external momenta associated to the \( N \) external legs of \( \phi \) operators, \( Q_i(n) \) \((i = 1, \ldots, L)\) are the external momenta corresponding to the \( L \) insertions of \( \phi^2 \) composite operators and the independent cutoffs \( \Lambda_n \) are required to depict the regularization process in each inequivalent subspace.

In the present section, we shall suppose that every bare diagram at any loop order can be implicitly regularized with the appropriate cutoff.

When we apply the operation \( \frac{\partial}{\partial \mu_{0n}} \) to the bare vertex part \( \Gamma^{(N,L)}_{R(n)}(p_i(n), Q_i(n), \lambda_{0n}, \mu_{0n}, \Lambda_n) \) with fixed \( \lambda_{0n} \) and \( \Lambda_n \), we obtain a zero momentum insertion of the operator \( \phi^2 \) in the vertex part \( \Gamma^{(N,L)}_{R(n)}(p_i(n), Q_i(n), \lambda_{0n}, \mu_{0n}, \Lambda_n) \), namely \( \Gamma^{(N,L+1)}_{R(n)}(p_i(n), Q_i(n), 0, \lambda_{0n}, \mu_{0n}, \Lambda_n) \). Then, we have

\[
\frac{\partial}{\partial \mu_{0n}^{2n}} \Gamma^{(N,L)}_{R(n)}(p_i(n), Q_i(n), \lambda_{0n}, \mu_{0n}, \Lambda_n) = \Gamma^{(N,L+1)}_{R(n)}(p_i(n), Q_i(n), 0, \lambda_{0n}, \mu_{0n}, \Lambda_n). \tag{33}
\]

Eq. (32) can be employed to transform the bare quantities into the renormalized amounts.

Writing the bare mass \( \mu_{0n} \) representing the \( m_n \)-dimensional subspace in terms of the renormalized mass and coupling constant \( \mu_{0n} = \mu_{0n}(\mu_n, g_n) \) and applying the chain rule, the renormalized vertex functions obey the equation

\[
2n \rho_n \frac{\partial}{\partial \mu_n^{2n}} Z_{\phi^2(n)} + \frac{\alpha_n}{\mu_n^{2n}} = Z_{\phi^2(n)} \frac{\partial g_n}{\partial \mu_n^{2n}}, \quad \frac{\alpha_n}{\mu_n^{2n}} = Z_{\phi^2(n)} \frac{\partial g_n}{\partial \mu_n^{2n}}, \quad \frac{\alpha_n}{\mu_n^{2n}} = Z_{\phi^2(n)} \frac{\partial g_n}{\partial \mu_n^{2n}}.
\]

If we define the functions \( \beta_n(= \frac{\alpha_n}{\rho_n}) = \mu_n \frac{\partial g_n}{\partial \mu_n^{2n}} \) and \( \gamma_n(= \frac{\alpha_n}{\rho_n}) = \mu_n \frac{\partial g_n}{\partial \mu_n^{2n}} \), together with the multiplication of the last equation by \( \frac{\alpha_n}{\rho_n} \), yield the result

\[
\langle \mu_n \frac{\partial}{\partial \mu_n^{2n}} + \beta_n \frac{\partial}{\partial g_n} - N \gamma_n \rangle \Gamma^{(N,L+1)}_{R(n)}(p_i(n), Q_i(n), g_n, \mu_n) = 2n \mu_n^{2n} \frac{\partial g_n}{\partial \mu_n^{2n}} Z_{\phi^2(n)}^{-1} \Gamma^{(N,L+1)}_{R(n)}(p_i(n), Q_i(n), 0, g_n, \mu_n). \tag{35}
\]
In order to express last equation purely in terms of renormalized objects we make use of the normalization conditions \( \Gamma^{(2)}_{R(n)}(0) = \mu^{2n}_n, \Gamma^{(2,1)}_{R(n)}(0) = 1 \) for the particular case \((N, L) = (2, 0)\). This results in the Callan-Symanzik-Lifshitz equations (CSLE) for anisotropic Lifshitz points of generic competing systems:

\[
\left( \mu_n \frac{\partial}{\partial \mu_n} + \beta_n \frac{\partial}{\partial g_n} - \frac{N}{2} \gamma_{\phi(n)} + L \gamma_{\phi^2(n)} \right) \Gamma^{(N,L)}_{R(n)}(p_i(n), Q_i(n), g_n, \mu_n) = \mu^{2n}_n (2n - \gamma_{\phi(n)}) \Gamma^{(N,L+1)}_{R(n)}(p_i(n), Q_i(n); 0, g_n, \mu_n).
\]

Some comments are in order [21]. Initially, apply a derivative \( \frac{\partial}{\partial p_{(r)}(n)} \) of the CSLE at zero external momenta for \((N, L) = (2, 0)\) and use the normalization conditions of section II. This means

\[
- \gamma_{\phi(n)} = \mu^{2n}_n (2n - \gamma_{\phi(n)}) \frac{\partial}{\partial p_{(r)}^{2n}_n} \Gamma^{(2,1)}_{R(n)}(p(n), -p(n); 0, g_n, \mu_n) |_{p^{2n}_n = 0} = 0.
\]

At one-loop order we already encountered from Eq.(24) that \( \Gamma^{(2,1)}_{R(n)}(p(n), -p(n); 0, g_n, \mu_n) = 1 + O(g^n_2) \). It is easy to see that \( \gamma_{\phi(n)} \) begins at \( O(g^n_2) \). Next, substitute \((N, L) = (4, 0)\) in the CSLE. Taking advantage of the normalization conditions leads to

\[
\beta_n - 2 \gamma_{\phi(n)} g_n = \mu^{2n}_n (2n - \gamma_{\phi(n)}) \Gamma^{(4,1)}_{R(n)}(0, 0, 0; 0, \mu_n, g_n).
\]

The first contribution to \( \Gamma^{(4,1)}_{R(n)}(0, 0, 0; 0, \mu_n, g_n) \) is \( O(g^n_2) \). From last equation we discover that \( \beta_n \) is \( O(g^n_2) \). Consequently, the operation \( \beta_n \frac{\partial}{\partial g_n} \) is \( O(g^n_2) \), differently from the operation \( \mu_n \frac{\partial}{\partial \mu_n} \) which is \( O(g^n_0) \). Now, taking \( N = 2 \) and \( L = 1 \) into the CSLE along with the normalization condition \( \Gamma^{(2,1)}_{R(n)}(0, 0; 0, g_n, \mu_n) = 1 \), we find

\[
- \gamma_{\phi(n)} + \gamma_{\phi^2(n)} = \mu^{2n}_n (2n - \gamma_{\phi(n)}) \Gamma^{(2,2)}_{R(n)}(0, 0; 0, \mu_n, g_n).
\]

As \( \Gamma^{(2,2)}_{R(n)} \) starts at \( O(g_n) \), we find out directly that \( \gamma_{\phi^2(n)} = O(g_n) \). We have at hand the basic requirements to pursuing the inductive proof of multiplicative renormalizability at all orders of perturbation theory. We turn now our attention to this subject.

We begin the inductive proof of multiplicative renormalizability with the hypothesis that the renormalized vertices defined by Eq.(32) have been transformed into convergent expressions up to the \( L \)th-loop order at fixed \( \mu_n \) and \( g_n \) in the limit \( \Lambda_n \to \infty \). Thus, we commence with the claim that in the infinite cutoff limit \( \Gamma^{(2)}_{R(n)}, \Gamma^{(4)}_{R(n)} \) and \( \Gamma^{(2,1)}_{R(n)} \) are finite at order \( g^n_n, g^{n+1}_n \) and \( g^n_2 \), respectively. We suppress the arguments of the renormalized vertex functions in order to simplify the succeeding discussion.
The preliminary observations considered above at one-loop can be generalized to $L$-loop level. Using Eq. (37) at this order in the coupling constant $g_n$, we find out that $\gamma_{\phi(n)}$ is finite at $O(g_n^\mu)$. From Eq. (38) at $(L + 1)$-loop order, $\Gamma^{(4)}_{R(n)}$ in the right-hand side (rhs) is already finite at $(L + 1)$-loop order for it has a skeleton expansion. On the other hand, at $(L + 1)$-loop order this vertex part is of order $g_n^{L+2}$. In other words, the combination $(\beta_n - 2\gamma_{\phi(n)}g_n)$ is convergent at $O(g_n^{L+2})$. As $\gamma_{\phi(n)}$ is finite at $O(g_n^L)$, this implies that $\beta_n$ is finite at $O(g_n^{L+1})$.

We can write the CSLE alternatively as

$$
(\mu_n \frac{\partial \Gamma^{(N,M)}_{R(n)}}{\partial m_n}) = (-\beta_n \frac{\partial}{\partial g_n} + \frac{N}{2} \gamma_{\phi(n)} - M \gamma_{\phi^2(n)}) \Gamma^{(N,M)}_{R(n)} + \mu_n^2 (2n - \gamma_{\phi(n)}) \Gamma^{(N,M+1)}_{R(n)}.
$$

Our aim is to prove that we can find a finite result for the rhs of this equation at $(L + 1)$-loop order. Note that $\Gamma^{(N,M)}_{R(n)}$ in the rhs is only needed at $L$th loop order, since its coefficients $\beta_n \frac{\partial}{\partial g_n}$ and $\gamma_{\phi^2(n)}$ are at least of order $g_n$. In the last piece of the rhs, either $\Gamma^{(N,M+1)}_{R(n)}$ has a skeleton expansion (convergent at $(L + 1)$-loop order) or the CSLE must be iterated.

Let us fix our attention in the case $N = 4, M = 0$ in Eq. (40). Both terms in the rhs are finite at $(L + 1)$-loop order. We then conclude that $(\mu_n \frac{\partial \Gamma^{(4)}_{R(n)}}{\partial \mu_n})$ is also convergent at this order.

The proof that $(\mu_n \frac{\partial \Gamma^{(4)}_{R(n)}}{\partial \mu_n})$ is finite at $(L + 1)$-loop order as well can be achieved by considering the perturbative integration of $\Gamma^{(4)}_{R(n)}$. At the critical dimension $\Gamma^{(4)}_{R(n)}$ is dimensionless; it gets unchanged under independent dilatations in their dimensionful parameters:

$$(p_n, \mu_n, \Lambda_n, g_n) \rightarrow (\rho_n p_n, \rho_n \mu_n, \rho_n \Lambda_n, g_n).$$

The choice $\rho_n = \frac{1}{\mu_n}$ together with the dilatation invariance stated above can be written as

$$\Gamma^{(4)}_{R(n)}(p_n, \mu_n, \Lambda_n, g_n) = \Gamma^{(4)}_{R(n)}(\frac{p_n}{\mu_n}, \frac{\Lambda_n}{\mu_n}, g_n).$$

Taking $N = 4, M = 0$ in Eq. (40) for arbitrary external momenta at order $g_n^{L+2}$, it can be reexpressed in the following manner in terms of a running variable $\mu_n'$:

$$(\mu_n' \frac{\partial \Gamma^{(4)}_{R(n)}}{\partial \mu_n'}(\frac{p_n}{\mu_n'}, \frac{\Lambda_n}{\mu_n'}, g_n))|_{L+2} = f^{(4)}(\frac{p_n}{\mu_n'}, \frac{\Lambda_n}{\mu_n'}, g_n))|_{L+2}. \tag{43}$$

First, write the running mass as a function of a dimensionless variable $\mu_n' = \frac{\mu_n}{\alpha}$. The integration of $\mu_n'$ in the interval $[\infty, \mu_n]$ amounts to integrate over $\alpha$ in the region $(0, 1)$. We
then employ the normalization conditions (3c) at the limit \( \mu'_n = \infty \) as boundary conditions. We encounter the following result

\[
[\Gamma^{(4)}_{R(n)}(\frac{p_n}{\mu'_n}, \frac{\Lambda_n}{\mu'_n}, g_n)]_{L+2} = g_n - \int_0^1 \frac{d\alpha}{\alpha} [f^{(4)}_{(n)}(\alpha \frac{p_n}{\mu'_n}, \alpha \frac{\Lambda_n}{\mu'_n}, g_n)]_{L+2}.
\] (44)

From what we have been discussing, \([f^{(4)}_{(n)}]|_{L+2}\) can be written entirely in terms of lower order amounts which, by assumption, all possess a finite value at \( \Lambda_n \to \infty \). Since \([f^{(4)}_{(n)}]|_{L+2}\) is analytic for small momenta, the limit of zero external momenta can be taken safely. Indeed, this does not introduce any difficulty into the integral over \( \alpha \). Hence, we are led to

\[
[\Gamma^{(4)}_{R(n)}(\frac{p_n}{\mu'_n}, \infty, g_n)]_{L+2} = g_n - \int_0^1 \frac{d\alpha}{\alpha} f^{(4)}_{(n)}(\alpha \frac{p_n}{\mu'_n}, \infty, g_n)|_{L+2},
\] (45)

which demonstrates that the renormalized vertex \( \Gamma^{(4)}_{R(n)} \) can be successfully related to a finite integral over lower order renormalized vertex functions.

Considering Eq.(40) for the values \( N = 2, M = 1 \) at order \( g_n^{L+1} \) yields

\[
(\mu_n \frac{\partial \Gamma^{(2,1)}_{R(n)}}{\partial \mu_n})|_{L+1} = [-\beta_n \frac{\partial}{\partial g_n} \Gamma^{(2,1)}_{R(n)}]|_{L+1} + [(\gamma_{\phi(n)} - \gamma_{\phi^2(n)}) \Gamma^{(2,1)}_{R(n)}]|_{L+1} + \mu_n^{2\tau} [(2n - \gamma_{\phi(n)}) \Gamma^{(2,2)}_{R(n)}]|_{L+1}.
\] (46)

The first term in the rhs is a combination of \( \beta_n \) at \( O(g_n^{L+1}) \) and \( \Gamma^{(2,1)}_{R(n)} \) at \( O(g_n^L) \), for \( \beta_n \frac{\partial}{\partial g_n} \) is \( O(g_n) \). The two contributions coming from this term are finite at this loop order in the above equation. The last term has to do with \( \Gamma^{(2,2)}_{R(n)} \) which has a skeleton expansion. Since the coupling constants and masses are actually finite at \( O(g_n^L) \), this implies that the last term is automatically finite at \( O(g_n^{L+1}) \).

The second term in Eq.(46) has two contributions: \( \Gamma^{(2,1)}_{R(n)} \) at \( O(g_n^L) \) which is hypothetically finite (\( \gamma_{\phi^2(n)} \) starts at \( O(g_n) \)) as well as (\( \gamma_{\phi(n)} - \gamma_{\phi^2(n)} \)) at \( O(g_n^{L+1}) \). The proof that (\( \gamma_{\phi(n)} - \gamma_{\phi^2(n)} \)) is finite at \( O(g_n^{L+1}) \) still has to be demonstrated. The calculation of this expression at zero momentum is identical to Eq.(39) computed at \( (L + 1) \)-loop order. The rhs of Eq.(39) involves the skeleton \( \Gamma^{(2,2)}_{R(n)} \) at this loop order, which is finite. This shows that (\( \gamma_{\phi(n)} - \gamma_{\phi^2(n)} \)) is finite at \( O(g_n^{L+1}) \). Consequently, this concludes the proof that the rhs of Eq.(46) is finite.

Integration of the left hand side of Eq.(46) for arbitrary external momenta gives the result

\[
(\mu_n \frac{\partial \Gamma^{(2,1)}_{R(n)}}{\partial \mu_n})|_{L+1} = f^{(2,1)}_{(n)}.
\] (47)

Owing to the dimensionlessness of \( \Gamma^{(2,1)}_{R(n)} \) at the critical dimension, we can perform a scaling transformation following similar steps to what was done for the vertex \( \Gamma^{(4)}_{R(n)} \). By requiring
analyticity of \( f_{(n)}^{(2,1)} \) for small \( p_n \) and using the normalization conditions at zero external momenta, the proof that \( \Gamma_{R(n)}^{(2,1)} \) is manifestly finite at \( O(g_n^{L+1}) \) follows at once. Using Eq.(37), we learn that \( \gamma_{\phi(n)} \) is finite at \( O(g_n^{L+1}) \) as well.

The conclusion of the inductive proof of multiplicative renormalizability can be achieved by turning our attention to the case \( N = 2, M = 0 \) at order \( g_n^{L+1} \) in (40), which reads

\[
(\mu_n \frac{\partial \Gamma_{R(n)}^{(2)}}{\partial \mu_n})|_{L+1} = \left[ -\beta_n \frac{\partial}{\partial g_n} \Gamma_{R(n)}^{(2)} \right]|_{L+1} + [\gamma_{\phi(n)} \Gamma_{R(n)}^{(2,1)}]|_{L+1} + [\mu_n^{2n}(2n - \gamma_{\phi(n)}) \Gamma_{R(n)}^{(2,1)}]|_{L+1}.
\]  

(48)

Now, \( \Gamma_{R(n)}^{(2)} \) is required solely at \( O(g_n^{L}) \) (finite by hypothesis) since \( \beta_n \) starts at \( O(g_n) \) and we have already proven that \( \beta_n \) and \( \gamma_{\phi(n)} \) are finite at \( O(g_n^{L+1}) \). This is sufficient to prove that the first two terms of the rhs in the above equation are manifestly finite. For the last term, we have just proved that both contributions inside it are finite at \( O(g_n^{L+1}) \). Since the rhs has a finite limit, we proceed to integrating the CSLE for this vertex function with arbitrary momenta. Perform the redefinition

\[
\tilde{\Gamma}_{R(n)}^{(2)}(p_n) = \Gamma_{R(n)}^{(2)}(p_n) - p_n^{2n} - \mu_n^{2n}.
\]

(49)

The redefined vertex has mass dimension \( \mu_n^{2n} \) and only deviates from \( \Gamma_{R(n)}^{(2)}(p_n) \) by higher powers of \( p_n^{2\tau} \) and \( \mu_n^{2n} \), which turn out to be cancelled by negative powers of the cutoffs. Normalization conditions require that it vanishes with \( (p_n^{2n})^2 \) for small \( |p_n| \). Thus, collecting all the information contained in our previous discussions, \( \tilde{\Gamma}_{R(n)}^{(2)}(p_n) \) satisfies an equation of the type

\[
[(\mu_n' \frac{\partial \tilde{\Gamma}_{R(n)}^{(2)}}{\partial \mu_n'})|_{\mu_n = \mu_n', \pm_n = \pm_n'}|_{L+1} = \mu_n^{2n} f_{(n)}^{(2)}(\mu_n, \pm_n, g_n)|_{L+1},
\]

(50)

with \( f_{(n)}^{(2)} = O((p_n^{2n})) \) for small \( |p_n| \). Introduce the change of variables \( \mu_n' = \frac{\mu_n}{\alpha} \). Integrate over the variable \( \mu_n' \) in the interval \((\infty, \mu_n)\). Using the normalization condition at zero external momenta as a boundary condition to the solution, i.e., \( \tilde{\Gamma}_{R(n)}^{(2)}(0, 0, g_n) = 0 \), and taking the limit of infinite cutoffs, we get to

\[
[\tilde{\Gamma}_{R(n)}^{(2)}(\frac{p_n}{\mu_n}, \infty, g_n)|_{L+1} = - \int_{0}^{1} \frac{d\alpha}{\alpha^{2n+1}} f_{(n)}^{(2)}(\alpha \frac{p_n}{\mu_n}, \infty, g_n)|_{L+1}.
\]

(51)

We are left with the task of proving that the integral is finite. But this is straightforward from the behavior of the integrand for small \( p_n \): it goes like \( f_{(n)}^{(2)}(\alpha \frac{p_n}{\mu_n}, \infty, g_n) = O((\alpha p_n)^{2n}) \), which is free of singularities in the lower integration limit \( \alpha \to 0 \).

We have thus succeeded in demonstrating the multiplicative renormalizability by the inductive method as well as the existence of the Callan-Symanzik-Lifshitz Equations (36).
B. Isotropic behaviors

The isotropic situation has a simple parallel with the \( n \)th competition subspace of the anisotropic case as explicated before. We just have to keep in mind that the critical dimension is different \( d_c = 4n \) and there is only one type of competition along all space directions. Then it follows in a straightforward manner that the isotropic vertex parts satisfy exactly the Callan-Symanzik-Lifshitz equation associated to the \( n \)th competing subspace of the anisotropic behaviors discussed above, namely

\[
\left( \mu_n \frac{\partial}{\partial \mu_n} + \beta_n \frac{\partial}{\partial g_n} - \frac{N}{2} \gamma_\phi(n) + L \gamma_\phi^2(n) \right) \Gamma_{R(n)}^{(N,L)}(p_{i(n)}, Q_{i(n)}, g_n, \mu_n) = \mu_n^{2n} (2n - \gamma_\phi(n)) \Gamma_{R(n)}^{(N,L+1)}(p_{i(n)}, Q_{i(n)}; 0, g_n, \mu_n). \tag{52}
\]

In addition, the arguments presented above in the inductive proof of renormalizability for the \( n \)th subspace in the anisotropic cases can also be used to prove inductively the finiteness of all vertex parts of the isotropic cases. The situation is identical to the proof furnished in the particular isotropic \( m \)-axial case \( (n = 2) \) \[31\]. By the same token, the existence of the Callan-Symanzik-Lifshitz equations (52) for arbitrary isotropic critical behaviors follows directly. This concludes the formal proof of multiplicative renormalizability of arbitrary competing systems of the Lifshitz type to all orders in perturbation theory.

V. THE CALLAN-SYMANZIK-LIFSHITZ EQUATIONS AT \( d = d_c - \epsilon_L \)

We will restrict our considerations only to vertex parts including composite operators which can be renormalized multiplicatively. The renormalized 1PI vertex parts are defined with respect to the bare functions in the following way

\[
\Gamma_{R(n)}^{(N,L)}(p_{i(n)}, Q_{i(n)}, g_n, \mu_n) = Z_{\phi(n)}^{\frac{N}{L}} Z_{\phi^2(n)}^{L} \Gamma_{R(n)}^{(N,L)}(p_{i(n)}, Q_{i(n)}, \lambda_{0n}, \mu_{0n}, \Lambda_n). \tag{53}
\]

We shall make reference to the cutoffs implicitly from now on. Although the regularization method utilizing cutoffs explained in the previous section is useful in many instances, we shall use dimensional regularization. Away from the critical dimension, the ultraviolet divergences of the original bare vertex parts can be represented by inverse powers (poles) in the variable \( \epsilon_L \) in dimensional regularization. This manner of manipulating infinities will be shown in a moment to be particularly simple in the calculation of critical exponents.
The inductive proof of renormalizability of Eq.(53) at all orders in perturbation theory at the critical dimension (where the theory is renormalizable) was already demonstrated in the last section. Below the critical dimension, at \( d = d_c - \epsilon_L \), the theory is less divergent (superrenormalizable) than its formulation at the critical dimension. Thus, we do not have to worry about the details of the rigorous proof of multiplicative renormalizability of the theory at \( d = d_c - \epsilon_L \) in analogy with our discussion in the previous section at \( d = d_c \), but take on this property as valid in that case as well. Instead, we shall give indirect evidence of this renormalizability by proving explicitly that the \( \beta \)-functions and Wilson functions are all finite at the specific perturbative order we are interested in the present work.

It is interesting to express the renormalized and bare coupling constants in terms of the dimensionless couplings \( u_n \), that is, \( g_n = u_n(\mu_n^{2n})^{\frac{d}{d_c}} \), and \( \lambda_{0n} = u_{0n}(\mu_n^{2n})^{\frac{d}{d_c}} \). Moreover, we can write all the renormalization functions in terms of \( u_n \).

The dimensionless bare coupling constants \( u_{0n} \) and the renormalization functions \( Z_{\phi(n)} \), \( \bar{Z}_{\phi^2(n)} = Z_{\phi(n)}Z_{\phi^2(n)} \) can be expanded in terms of the dimensionless parameter \( u_n \) up to two-loop order as

\[
\begin{align*}
    u_{0n} &= u_n(1 + a_{1n}u_n + a_{2n}u_n^2), \\
    Z_{\phi(n)} &= 1 + b_{2n}u_n^2 + b_{3n}u_n^3, \\
    \bar{Z}_{\phi^2(n)} &= 1 + c_{1n}u_n + c_{2n}u_n^2.
\end{align*}
\]

These ingredients can be used to discuss the Callan-Symanzik-Lifshitz equations analogously to the analysis performed at the critical dimension. They shall provide a better comprehension of the scaling behavior of the solutions in the anisotropic and isotropic sectors at the ultraviolet fixed points.

A. Anisotropic

There are at least two ways to derive the CSLE (at the and) away from the critical dimension \( d = 4 + \sum_{n=2}^{L} \frac{1}{n} m_n - \epsilon_L \). In the first manner, one can take a total derivative of the renormalized vertex part (including composite operators) with respect to the logarithm of the renormalized mass \( \mu_n \) characterizing the \( m_n \)-dimensional subspace at fixed bare coupling \( \lambda_n \) and cutoff \( \Lambda_n \), in conjunction with the normalization conditions. The second trend is to take a derivative of an arbitrary bare vertex part with relation to the bare parameter.
µ²n, and expressing everything solely in terms of renormalized quantities via normalization
conditions, with bare coupling λn and cutoff Λn kept fixed. The latter is almost identical
to our procedure at the critical dimension d_c = 4 + ∑ₙ₌₂ⁿ(1/ₙ−1) mn explained above. Both
strategies lead to the corresponding Callan-Symanzik-Lifshitz equations:

\[
\left(\mu_n \frac{\partial}{\partial \mu_n} + \beta_n(g_n, \mu_n) \frac{\partial}{\partial g_n} - \frac{N}{2} \gamma_\phi(n) + L\gamma_{\phi^2}(n)\right)\Gamma^{(N,L)}_{R(n)}(p_i(n), Q_i(n), g_n, \mu_n) = (55)
\]

\[
\mu_n^2 (2n - \gamma_\phi(n)) \Gamma^{(N,L+1)}_{R(n)}(p_i(n), Q_i(n); 0, g_n, \mu_n).
\]

These equations resemble the CSLE at the critical dimension, but now the renormalized
coupling constants gn are dimensionful. This provokes a discrepancy between the beta
functions defined at the critical dimension and its corresponding version at \( \epsilon L \neq 0 \). When
expressed in terms of the renormalized parameters, the coefficients appearing in the CSLE
equations satisfy the following relations:

\[
\beta_n(g_n, \mu_n) = \mu_n \frac{\partial g_n}{\partial \mu_n},
\]

(56)

\[
\gamma_\phi(n)(g_n, \mu_n) = \mu_n \frac{\partial ln Z_\phi(n)}{\partial \mu_n},
\]

(57)

\[
\gamma_{\phi^2}(n)(g_n, \mu_n) = -\mu_n \frac{\partial ln Z_{\phi^2}(n)}{\partial \mu_n}.
\]

(58)

Replacing the definition \( g_n = u_n(\mu_n^2 \lambda^\frac{1}{2}) \) into the above equation for \( \beta_n(g_n, \mu_n) \) we find

\[
\beta_n(g_n, \mu_n) \frac{\partial}{\partial g_n} = (\mu_n \frac{\partial u_n}{\partial \mu_n})\lambda_{\lambda_0} \frac{\partial}{\partial u_n} + n\epsilon_L g_n \frac{\partial}{\partial g_n},
\]

(59)

where the derivative in the first term is calculated at fixed bare coupling constant \( \lambda_{\lambda_0} \). It is
equivalent to the dimensionless function \( \beta_n(u_n) = (\mu_n \frac{\partial u_n}{\partial \mu_n})\lambda_{\lambda_0} \). Now define the Gell-Mann-
Low function for \( \epsilon L \neq 0 \) in terms of the dimensionless coupling according to \( 40, 41 \)

\[
[\beta_n(g_n, \mu_n)]_{GL} = -n\epsilon_L g_n + \beta_n(g_n, \mu_n),
\]

(60)

where the last term can be identified with the value of the function at the critical dimension.
The solution of the Callan-Symanzik-Lifshitz equations for the vertex parts away from the
critical dimension will possess a scaling limit provided it can be expressed entirely in terms
of dimensionless quantities. We can get rid of the undesirable contribution coming from
dimensionful couplings if we start from scratch with the Gell-Mann-Low function in Eq.(64).
In that case, Eq.(59) turns out to be

\[
[\beta_n]_{GL}(g_n) \frac{\partial}{\partial g_n} = \beta_n(u_n) \frac{\partial}{\partial u_n}.
\]

(61)
Consequently, the Callan-Symanzik-Lifshitz equations away from the critical dimension can be rewritten as

\[
\frac{\partial}{\partial \mu_n} \ln u_n - \frac{N}{2} \frac{\gamma_n}{\mu_n} + L \frac{\gamma_{\phi^2(n)}}{\mu_n} \Gamma_{R(n)}^{(N,L)}(p_i(n), Q_i(n), u_n, \mu_n) = 0.
\]

Let us take a closer look at the dimensionless functions \( \beta_n(u_n) \). Since the derivatives of the renormalized masses in terms of dimensionless renormalized coupling constants are taken at fixed bare coupling constants, we can use the identity

\[
\mu_n \frac{\partial u_n}{\partial \mu_n} = -\mu_n \frac{\partial \lambda_0}{\partial \mu_n} \frac{\partial u_n}{\partial \lambda_0} \beta_n,
\]

in order to trade partial derivatives. Using last equation, we find the following result for the flow functions in terms of dimensionless parameters

\[
\beta_n(u_n) = -n \epsilon_L \left( \frac{\partial \ln u_0}{\partial u_n} \right)^{-1}.
\]

Employing this result, the remaining renormalization functions can be rewritten in terms of dimensionless quantities in the form

\[
\gamma_{\phi(n)}(u_n) = \beta_n \frac{\partial \ln Z_{\phi(n)}}{\partial u_n},
\]

\[
\gamma_{\phi^2(n)}(u_n) = -\beta_n \frac{\partial \ln Z_{\phi^2(n)}}{\partial u_n},
\]

\[
\tilde{\gamma}_{\phi^2(n)}(u_n) = -\beta_n \frac{\partial \ln Z_{\phi^2(n)}}{\partial u_n}.
\]

The equations (62) are the Callan-Symanzik-Lifshitz equations for anisotropic vertex parts describing generic competing systems of Lifshitz type with arbitrary composite operators which are multiplicatively renormalizable. Note that \( \Gamma_{R(n)}^{(0,2)} \) is precluded from this discussion. The independence of renormalization flows in each subspace characterized by their corresponding bare(/{renormalized}) masses and coupling constants are manifest in this construction of anisotropic criticalities.

The main obstacle to encounter a general solution of the CSL Eq.(62) is the appearance of the inhomogeneous term \( \Gamma_{R(n)}^{(N,L+1)}(p_i(n), Q_i(n), 0, g_n, \mu_n) \) in its right hand side (rhs). It corresponds to the insertion at zero momentum in the vertex \( \Gamma_{R(n)}^{(N,L)}(p_i(n), Q_i(n), g_n, \mu_n) \) due to the action of the derivative with respect to the mass parameter characterizing the \( n \)-th competing
subspace in the anisotropic cases. For the sake of simplicity, take $L = 0$. Generally speaking, this derivative produces an additional propagator in $\Gamma^{(N,L+1)}_{R(n)}(p_i(n), Q_i(n), 0, g_n, \mu_n)$ when we compare it with $\Gamma^{(N,L)}_{R(n)}(p_i(n), Q_i(n), g_n, \mu_n)$. The limit $p_i(n) \rightarrow \infty$ describes the ultraviolet behavior of such vertex parts. Thus, $\Gamma^{(N,1)}_{R(n)}(p_i(n), 0, g_n, \mu_n)$ is of order $p_i^{-2n} \Gamma^{(N)}_{R(n)}(p_i(n), g_n, \mu_n)$ for large $p_n$, up to powers of $\ln p_n$, at every order in perturbation theory. We make the assumption that these logarithms do not add up to compensate the factor $p_i^{-2n}$.

All the momenta involved in a given Feynman diagram are Euclidean. Taking a fixed nonvanishing Euclidean momentum $k_{i(n)}$, we can reach the ultraviolet region of any diagram by performing a scale transformation like $p_i(n) = \rho_n k_{i(n)}$, where $\rho_n$ is a dimensionless flow variable in the limit $\rho_n \rightarrow \infty$. Interesting anisotropic configurations have the following structure: there are $n$ independent subsets of momenta which go to infinity independently.

Since the $n$ independent mass subspaces have a well defined zero mass limit and all the momenta appearing in $\Gamma^{(N)}_{R(n)}(p_i(n), g_n, \mu_n)$ are nonexceptional in the ultraviolet regime (except for the zero momentum of the inserted $\phi^2$ operator in the vertex part $\Gamma^{(N,1)}_{R(n)}(p_i(n), 0, g_n, \mu_n)$), we can apply the Weinberg’s theorem in each subspace, which guarantees that the rhs of Eq.(62) can be neglected order by order in the perturbative expansion.

The limit $\rho_n \rightarrow \infty$ implies to take all internal momenta in a given diagram as large as possible. This goal can be scored by regulating the integral with cutoffs $\Lambda_n$ in the limit $\Lambda_n \rightarrow \infty$. Hence, the integral becomes a homogeneous function of the mass $\mu_n$. The regions in momentum space where the rhs in Eq. (62) can be neglected when compared to the left-hand side (lhs) come from the limit $\frac{p_i(n)}{\mu_n} \rightarrow \infty$ which is identical to the ultraviolet limit. This is indeed the case, as we have already shown in Sec. III using cutoffs. Moreover, it will be demonstrated in the Appendixes via dimensional regularization that the ultraviolet divergences are associated to this region in momenta space where the loop integrals shall be computed.

Recall that several dimensional redefinitions of the momentum components along the various competing subspaces have been performed as discussed in Sec. II (see also [36]). If $d$ is the spatial dimension of the system, these redefinitions along with the Lifshitz conditions translate themselves into an “effective” space dimension for the anisotropic situations of generic competing systems, i.e., $d_{eff} = d - \sum_{n=2}^{L} \left[ \frac{n-1}{n} \right] m_n$. Elementary dimensional analysis implies that, under scaling, the vertex parts possess the following behavior
\[ \Gamma^{(N)}_{R(n)}(\rho_n k_i(n), u_n, \mu_n) = \rho_n^{n(N+(d-d-\sum_{n=2}^L[\frac{a-1}{n}])m_n) - N(d-\sum_{n=2}^L[\frac{a-1}{n}])m_n)} \Gamma^{(N)}_{R(n)}(k_i(n), u_n(\rho_n), \frac{\mu_n}{\rho_n}). \]

The solution of the asymptotic part associated to the vertex function satisfying the homogeneous CSL equations has the property

\[ \Gamma^{(N)}_{as R(n)}(k_i(n), u_n, \frac{\mu_n}{\rho_n}) = \exp \left[ \frac{N}{2} \int_{u_n}^{u_n(\rho_n)} \gamma_{\phi(n)}(u_n(\rho_n)) \frac{du_n'}{\beta_n(u_n')} \right] \Gamma^{(N)}_{as R(n)}(k_i(n), u_n(\rho_n), \mu_n), \] (66)

where

\[ \rho_n = \int_{u_n}^{u_n(\rho_n)} \frac{du_n'}{\beta_n(u_n')} . \] (67)

From now on we omit the subscript for asymptotic in the vertex parts satisfying the homogeneous Callan-Symanzik-Lifshitz equations. The eigenvalue conditions \( \beta_n(u_{n\infty}) = 0 \) yields the ultraviolet nontrivial fixed point. Exactly at this value of the coupling, which is independent of the competing subspace under consideration in the anisotropic cases, the solutions of the CSL Eq.(62) under a scale in the external momenta can be written as

\[ \Gamma^{(N)}_{R(n)}(\rho_n k_i(n), u_{n\infty}, \mu_n) = \rho_n^{n(N+(d-d-\sum_{n=2}^L[\frac{a-1}{n}])m_n) - N(d-\sum_{n=2}^L[\frac{a-1}{n}])m_n)} \frac{N\gamma_{\phi(n)}(u_{n\infty})}{2} \Gamma^{(N)}_{R(n)}(k_i(n), u_{n\infty}, \mu_n). \] (68)

The dimension of the field can be defined by

\[ \Gamma^{(N)}_{(n)}(\rho_n k_{i(n)}) = \rho_n^{n[d-\sum_{n=2}^L[\frac{a-1}{n}])m_n) - Nd_{\phi(n)}]} \Gamma^{(N)}_{(n)}(k_i). \] (69)

Due to the interactions, the field develop an anomalous term, which can also be defined through the relation \( d_{\phi(n)} = \frac{d-\sum_{n=2}^L[\frac{a-1}{n}])m_n}{2} - 1 + \frac{m_n}{2n} \). Comparing Eqs. (68) and (69), one can easily verify that the anomalous dimension \( \eta_n \) can be identified with \( \gamma_{\phi(n)}(u_{n\infty}) \), i.e.,

\[ \eta_n = \gamma_{\phi(n)}(u_{n\infty}). \]

Plus, the anomalous dimension of the composite operator \( \phi^2 \) can be extracted by analyzing the vertex parts including composite operators. The asymptotic behavior of the vertex parts at the fixed point \( u_n = u_{n\infty} \) coming from the solution of the CSL Eq.(62) has the following simple scaling property \( ((N, L) \neq (0, 2)) \)

\[ \Gamma^{(N,L)}_{R(n)}(\rho_n k_i(n), \rho_n p_i(n), u_{n\infty}, \mu_n) = \rho_n^{n(N+(d-d-\sum_{n=2}^L[\frac{a-1}{n}])m_n) - N(d-\sum_{n=2}^L[\frac{a-1}{n}])m_n) - 2L) - \frac{N\gamma_{\phi(n)}(u_{n\infty})}{2} \] \times \rho_n^{L\gamma_{\phi^2}(u_{n\infty})} \Gamma^{(N)}_{R(n)}(k_i(n), p_i(n), u_{n\infty}, \mu_n). \] (70)
If we write the coefficient in the rhs as \( \rho_n \), we discover that 
\[ d_{\phi^2(n)} = -2n + \gamma_{\phi^2(n)}(u_{n\infty}) \]. The correlation length exponents can be computed through the identification 
\[ \nu^{-1} = -d_{\phi^2(n)} = 2n - \gamma_{\phi^2(n)}(u_{n\infty}) \].

This method with several independent mass scales generalizes, in a nontrivial way, the previous method developed for \( m \)-axial anisotropic Lifshitz points where only two mass scales are present.

### B. Isotropic

The existence of only one type of competing axis in the isotropic criticalities make the analysis simpler for the cases \( d = m_n \). As before, we shall briefly discuss this case by focusing on the analogy with the competing sector labeled by \( n \) of the anisotropic treatment already discussed above. We start with the mass scale \( \mu_n \), dimensionless coupling constant \( u_n \) and subscript \( n \) in all vertex functions. The critical dimension is \( 4n \), whereas the expansion parameter is \( \epsilon_L = 4n - d \). The effective space dimension is \( m_n \). At the (eigenvalue condition \( \beta_n(u_{n\infty}) = 0 \)) fixed point \( u_{n\infty} \), a scale transformation in the external momenta implies that the vertex functions present the following property:

\[ \Gamma^{(N)}(\beta_n k_i(n), u_{n\infty}, \mu_n) = \rho_n^{n(N + \frac{m_n}{2n} - N \epsilon_L) - \frac{N \gamma_{\phi^2(n)}(u_{n\infty})}{2}} \Gamma^{(N)}(k_i, u_{n\infty}, \mu_n) \] (71)

The dimension of the field is defined by

\[ \Gamma^{(N)}(n) \left( \rho_n k_i(n) \right) = \rho_n^{n \left( \frac{m_n}{2n} - N \epsilon_L \right)} \Gamma^{(N)}(k_i) \] (72)

Then, it follows that \( d_{\phi(n)} = \frac{m_n}{2n} - 1 + \frac{2\epsilon_L}{2n} \), which in turn implies that the anomalous dimension of the field \( (\eta_n \equiv \eta_{ln}) \) satisfies the identity \( \eta_n = \gamma_{\phi(n)}(u_{n\infty}) \).

Now, let us consider the vertex parts including composite fields. The identification of the anomalous dimension of the composite operator \( \phi^2 \) can be done through the following steps: at the fixed point \( u_n = u_{n\infty} \) the asymptotic behavior of the vertex parts under scale transformation can be written as

\[ \Gamma^{(N, L)}(\rho_n k_i(n), \rho_n p_i(n), u_{n\infty}, \mu_n) = \rho_n^{n(N + \frac{m_n}{2n} - N \epsilon_L) - \frac{N \gamma_{\phi^2(n)}(u_{n\infty})}{2} + L \gamma_{\phi^2(n)}(u_{n\infty})} \times \Gamma^{(N)}(k_i, p_i, u_{n\infty}, m_n) \] (73)
Then, write the coefficient in the rhs as $\rho_n^{[\frac{m_n}{n}-Nd\phi(n)]+Ld\phi^2(n)}$. One learns that $d\phi^2(n) = -2n + \gamma\phi^2(n)(u_n\infty)$ and the correlation length exponent ($\nu_n \equiv \nu_{Ln}$) in the isotropic case can be identified with $d\phi^2(n)$ through $\nu_n^{-1} = -d\phi^2(n) = 2n - \gamma\phi^2(n)(u_n\infty)$.

Finally, the beta function does not have the global factor of $n$ as in the competition directions of the anisotropic case. Rather, in terms of dimensionless parameters $\beta_n = -\epsilon_L(\frac{\partial m_{nm}}{\partial u_n})^{-1}$.

These are all informations required for the calculations of the critical exponents by diagrammatic means. This theme will be developed in the following remaining sections.

VI. CRITICAL EXONENTS IN THE ANISOTROPIC CASES

In order to calculate the critical exponents for anisotropic critical behaviors of arbitrary competing systems, we shall use the results in Appendix A for the massive Feynman graphs using the generalized orthogonal approximation. As already demonstrated in $[35, 36]$, this approximation is the most general one consistent with homogeneity in the several independent scales of external momenta. In the massive case introduced in the present work, the orthogonal approximation is also consistent with homogeneity in the various mass scales associated to each competing subspace. We shall employ the normalization conditions setup of Sec.II in our computational procedure.

To begin with, we use Eqs.(3) and (53) to fixing the normalization functions $Z\phi(n), \bar{Z}\phi^2(n)$ in powers of $u_n$ in conjunction with the appropriate Feynman diagrams at the loop order needed. In the expressions of the one- two and three-loop level of the integrals $I_2, I_3, I_4, I_5'$ given in Appendix A, we emphasize that a geometric angular factor appears every time we perform a loop integral. It is explicitly given by $[S_{(d-\sum_{n=2}^{L}m_n)}(2-\sum_{n=2}^{L}m_n^{2n})\Pi_{n=2}^{L}\frac{S_{m_n}\Gamma\left(\frac{m_n}{2n}\right)}{2n}]$ and can be absorbed in a redefinition of the coupling constants. We can perform those redefinitions in order to discard it from our considerations henceforth. We then obtain the
renormalization functions in terms of those loop integrals in the following form

\[ u_{0n} = u_n \left[ 1 + \frac{(N + 8)}{6} I_2 u_n + \left( \frac{(N + 8) I_2^2}{18} \right) \right] - \left( \frac{(N^2 + 6N + 20) I_2^2}{36} + \frac{(5N + 22) I_4}{9} - \frac{(N + 2) I_3'}{9} \right) u_n^2, \]  

(74a)

\[ Z_{\phi(n)} = 1 + \frac{(N + 2) I_3'}{18} u_n + \left( \frac{(N + 2)(N + 8)(I_2 I_3' - \frac{I_4}{2})}{54} \right) u_n^2, \]  

(74b)

\[ Z_{\phi^2(n)} = 1 + \frac{(N + 2) I_2}{6} u_n + \left\{ \frac{(N^2 + 7N + 10) I_2}{18} - \frac{(N + 2)}{6} \left( \frac{(N + 2) I_2^2}{6} + I_4 \right) \right\} u_n^2. \]  

(74c)

In dimensional regularization, the coefficients of the various terms in powers of \( u_n \) present poles in \( \epsilon_L \). They cancel in the evaluation of \( \beta_n \) and the critical exponents. When we combine Eqs.(54), (64) and (65) together, we obtain the \( \beta_n \) and Wilson functions in every subspace. We can write them through the following expressions

\[ \beta_n = -n \epsilon_L u_n \left[ 1 - a_{1n} u_n + 2(a_{1n}^2 - a_{2n}) u_n^2 \right], \]  

(75a)

\[ \gamma_{\phi(n)} = -n \epsilon_L u_n \left[ 2b_{2n} u_n + (3b_{3n} - 2b_{2n} a_{1n}) u_n^2 \right], \]  

(75b)

\[ \bar{\gamma}_{\phi^2(n)} = n \epsilon_L u_n \left[ c_{1n} + (2c_{2n} - c_{1n}^2 - a_{1n} c_{1n}) u_n \right]. \]  

(75c)

Using the explicit values of the integrals presented in Appendix A, we can determine the above mentioned coefficients by using the normalization conditions as functions of those integrals calculated at zero external momenta. When the results in Appendix A are combined with Eqs.(54) and (74), we conclude therefore that

\[ a_{1n} = \frac{(N + 8) \epsilon_L}{6} \left[ 1 + (h_{mL} - 1) \epsilon_L \right], \]  

(76a)

\[ a_{2n} = \frac{(N + 8)^2 \epsilon_L}{6} \left[ \frac{(N + 8)^2}{18} (h_{mL} - 1) - \frac{(3N + 14)}{24} \right] \epsilon_L, \]  

(76b)

\[ b_{2n} = -\frac{(N + 2) \epsilon_L}{144} \left[ (h_{mL} - \frac{5}{4}) \epsilon_L \right] - \frac{(N + 2) \epsilon_L}{144} I, \]  

(76c)

\[ b_{3n} = -\frac{(N + 2)(N + 8)}{1296 \epsilon_L^2} + \frac{(N + 2)(N + 8)}{108 \epsilon_L} \left( -\frac{1}{4} h_{mL} + \frac{13}{48} \right), \]  

(76d)

\[ c_{1n} = \frac{(N + 2) \epsilon_L}{6} \left[ 1 + (h_{mL} - 1) \epsilon_L \right], \]  

(76e)

\[ c_{2n} = \frac{(N + 2)(N + 5) \epsilon_L}{36 \epsilon_L^2} + \frac{(N + 2)(N + 5)}{18 \epsilon_L} (h_{mL} - 2) + \frac{(4N^2 + 25N + 34)}{72 \epsilon_L}. \]  

(76f)

Just as in the case of noncompeting and \( m \)-axial universality classes, the integral I encountered in Appendix Eq.(A9) appears explicitly in the coefficient \( b_{2n} \) Eq. (76c). We emphasize
that these expressions differ from Eqs.(67) of Ref.[36] obtained in the massless theory at nonvanishing external momenta in each subspace. Since the normalization conditions in the present work are defined at zero external momenta as well as nonvanishing mass in the specific subspace labeled by \( n \) and recalling that these coefficients depend upon the normalization conditions, this disagreement in their values using either renormalization scheme should not be surprising.

Inserting the values of the coefficients given in Eqs. (76) into (75a), \( \beta_n \) can be written as

\[
\beta_n = -nu_n[\epsilon_L - \frac{(N + 8)}{6}(1 + (h_{nL} - 1))u_n - \frac{(3N + 14)}{12}u_n^2] + O(u_n^4). \tag{77}
\]

Using last equation, the eigenvalue conditions \( \beta_n(u_{n\infty}) = 0 \) lead to two solutions: a trivial zero as well as a nontrivial zero of order \( \epsilon_L \) of each \( \beta_n \) characterizing the independent noncompeting \( (n = 1) \) and competing \( (n = 2, ..., L) \) subspaces. Remarkably, they correspond to the same value of the coupling constant \( (u_{1\infty} = u_{2\infty} = ... = u_{L\infty} \equiv u_{n\infty}) \), i.e.,

\[
u_{n\infty} = \frac{6}{8 + N} \epsilon_L \left[ 1 + \epsilon_L \left( -1 + \frac{(9N + 42)}{(8 + N)^2} \right) \right]. \tag{78}\]

Substitution of this value in the functions \( \gamma_{\phi(n)} \) and \( \bar{\gamma}_{\phi^2(n)} \) together with the coefficients given in Eqs.(76) allows us to find the critical exponents \( \eta_n \) and \( \nu_n \). Indeed proceeding as indicated, we first find

\[
\gamma_{\phi(n)}(u_{n\infty}) = \frac{n}{2} \epsilon_L^2 \frac{N + 2}{(N + 8)^2}[1 + \epsilon_L \left( \frac{6(3N + 14)}{(N + 8)^2} - \frac{1}{4} \right)]. \tag{79}\]

Note that these expressions are exactly the same of those obtained previously in [36], Eq.(73) therein, upon the identification \( \gamma_{\phi(n)}(u_{n\infty}) = \eta_n \).

The exponents \( \nu_n \) can be encountered from the expression of the anomalous dimension of the composite operator \( \phi^2 \). As we know, \( \gamma_{\phi^2(n)} = \bar{\gamma}_{\phi^2(n)} + \gamma_{\phi(n)} \). Thus, it follows that \( \nu_n^{-1} = -d_{\phi^2(n)} = 2n - \gamma_{\phi^2(n)}(u_{n\infty}) - \gamma_{\phi(n)}(u_{n\infty}) \). Using Eqs.(76) once more, the following intermediate result can be verified:

\[
\bar{\gamma}_{\phi^2(n)}(u_n) = \frac{n(N + 2)}{6}u_n[1 + \epsilon_L(h_{nL} - 1) - \frac{1}{2}u_n]. \tag{80}\]

Using the value \( u_{n\infty} \) into the last expression along with \( \gamma_{\phi(n)}(u_{n\infty}) \) from Eq.(79), we get

\[
\nu_n = \frac{1}{2n} + \frac{(N + 2)}{4n(N + 8)} \epsilon_L + \frac{1}{8n} \frac{(N + 2)(N^2 + 23N + 60)}{(N + 8)^3} \epsilon_L^2. \tag{81}\]
These exponents are in exact agreement with those calculated previously using a massless theory \[36\]. Note that the properties of strong anisotropic scaling \[43\], \( \eta_n = n\eta_1 \) and \( \nu_n = \left(\frac{1}{n}\right)\nu_1 \) already encountered in the massless case are reproduced using quite a different method and generalizes the more restricted \( m \)-axial Lifshitz universality class \[27, 31\].

**VII. ISOTROPIC CRITICAL EXponents IN THE ORTHOGONAL APPROXIMATION**

Although the isotropic diagrams can be calculated exactly without the need of any sort of approximation, in this section we shall evaluate the isotropic exponents for two reasons. Firstly, for completeness: pursuing the analogy with the anisotropic cases within an approximation which keeps the desirable concept of homogeneity in the mass(es). Secondly, simplicity is a key ingredient, since possible applications in other field theories with higher derivatives might be worthwhile using this simple setting of computation. Due to nontrivial features of the exact computation, we postpone the discussion of the calculation of the integrals and of the exact critical exponents to the next sections.

We now have only one type of subspace to be integrated over, along with only one type of mass parameter for each isotropic behavior labeled by the number of neighbors \( n \) coupled via competing interactions according to the CECI model introduced in \[35\] to understanding the most general magnetic systems with competing interactions. We can take advantage of the discussion in the previous section: we focus on the \( nth \) subspace taking into account of course the particularities of the isotropic renormalization functions as pointed out in Sec. V. In addition, we shall use the results in Appendix B for the Feynman integrals in order to compute the exponents \( \eta_n \) and \( \nu_n \) via perturbative expansion. Each loop integral originates an angular factor of \( S_{m_n} \), the area of an \( m_n \)-dimensional unit sphere, which shall be absorbed in a redefinition of the coupling constant. In order to make a clear distinction among the various isotropic behaviors and the \( nth \) competing subspace of the anisotropic behaviors, we shall replace the perturbative parameter in the isotropic case, i.e., \( \epsilon_L \rightarrow \epsilon_n \). Furthermore, this slight change of notation shall be useful to perform a comparison with the critical exponents already evaluated in the massless theory in Ref.\[36\].

First we list the analogues of Eqs.(75) for the renormalization functions in the isotropic
\[ \beta_n = -\epsilon_n u_n [1 - a_{1n} u_n + 2(a_{1n}^2 - a_{2n}) u_{n}^2], \]  
\(82a\)

\[ \gamma_{\phi(n)} = -\epsilon_n u_n [2b_{2n} u_n + (3b_{3n} - 2b_{2n} a_{1n}) u_{n}^2], \]  
\(82b\)

\[ \bar{\gamma}_{\phi^2(n)} = \epsilon_n u_n [c_{1n} + (2c_{2n} - c_{1n}^2 - a_{1n} c_{1n}) u_n]. \]  
\(82c\)

Employing the results of the previous section along with the output obtained in Appendix B utilizing the orthogonal approximation in the computation of the appropriate loop integrals for the isotropic behaviors, the various coefficients can be expressed in the form

\[ a_{1n} = \frac{N + 8}{6\epsilon_n} [1 - \frac{1}{2n} \epsilon_n], \]  
\(83a\)

\[ a_{2n} = \left(\frac{N + 8}{6\epsilon_n}\right)^2 - \frac{2N^2 + 41N + 170}{72n \epsilon_n}, \]  
\(83b\)

\[ b_{2n} = -\frac{(N + 2)}{144n \epsilon_n} [1 - \frac{1}{4n} \epsilon_n] - \frac{(N + 2)}{144n^2} I, \]  
\(83c\)

\[ b_{3n} = -\frac{(N + 2)(N + 8)}{1296n \epsilon_n^2} + \frac{7(N + 2)(N + 8)}{5184n^2 \epsilon_n}, \]  
\(83d\)

\[ c_{1n} = \frac{(N + 2)}{6\epsilon_n} [1 - \frac{1}{2n} \epsilon_n], \]  
\(83e\)

\[ c_{2n} = \frac{(N + 2)(N + 5)}{36\epsilon_n^2} - \frac{(N + 2)(2N + 13)}{72n \epsilon_n}. \]  
\(83f\)

Note that these expressions are different from Eqs.(90) derived in Ref.[36] using the orthogonal approximation in the massless limit. Since \( \epsilon_n = 4n - d \) and substituting the above results in the expression for \( \beta_n \), we find

\[
\beta_n = -u_n[\epsilon_n - \frac{(N + 8)}{6}(1 - \frac{1}{2n} \epsilon_n)u_n + \frac{(3N + 14)}{12n} u_{n}^2] + O(u_{n}^4). 
\]  
\(84\)

The eigenvalue condition \( \beta_n(u_{n\infty}) = 0 \) permits us to find out the nontrivial fixed point of the dimensionless coupling constant, whose value is given by

\[
u_{n\infty} = \frac{6}{8 + N} \epsilon_n \left\{ 1 + \epsilon_n \left[ \frac{1}{2} + \frac{(9N + 42)}{(8 + N)^2} \right] \right\}.
\]  
\(85\)

Replacing this fixed point in the expression for \( \gamma_{\phi(n)} \) together with Eqs.(83), we then have

\[
\gamma_{\phi(n)}(u_{n\infty}) = \epsilon_n^2 \frac{N + 2}{2n(N + 8)^2} [1 + \epsilon_n \left( \frac{6(3N + 14)}{(N + 8)^2} - \frac{1}{4} \right)].
\]  
\(86\)
This value corresponds exactly to the exponent $\eta_n$ previously obtained using the renormalization group equation in the massless theory \[35,36\]. Furthermore, using again the results (83) in the definition of $\bar{\gamma}_{\phi^2(n)}$, we obtain

$$\bar{\gamma}_{\phi^2(n)}(u_n) = \frac{(N + 2)}{6} u_n [1 - \frac{1}{2n} \epsilon_n - \frac{1}{2n} u_n]. \tag{87}$$

Replacing Eq.(85) into last equation, we get to the following result

$$\bar{\gamma}_{\phi^2(n)}(u_{n\infty}) = \frac{(N + 2)}{(N + 8)} \epsilon_n [1 + \frac{6(N + 3)}{n(N + 8)^2} \epsilon_n]. \tag{88}$$

Using the relation $\nu_n = (2n - \bar{\gamma}_{\phi^2(n)}(u_{n\infty}) - \gamma_{\phi(n)}(u_{n\infty}))^{-1}$, we find out the correlation length critical exponent

$$\nu_n = \frac{1}{2n} + \frac{(N + 2)}{4n^2(N + 8)} \epsilon_n + \frac{1}{8n^3} \frac{(N + 2)(N^2 + 23N + 60)}{(N + 8)^3} \epsilon_n^2. \tag{89}$$

The expression for $\bar{\gamma}_{\phi^2(n)}(u_{n\infty})$ in Eq.(88) is the same as the one associated to a scalar theory in the massless limit, computed at the fixed point using the renormalization group equation. Besides, Eq.(89) corresponds to Eq.(95) for this exponent in the orthogonal approximation using the massless method of Ref.[36].

We can compare the results given in Eqs.(86) and (89) of the exponents $\eta_n$ and $\nu_n$ for generic $n$ with the previous massive method recently introduced to treat the $n = 2$ isotropic case corresponding to $m$-axial Lifshitz points in Ref.[31] employing the orthogonal approximation. Two misprints in the expressions for the critical exponents $\eta_2 \equiv \eta_{LA}$ and $\nu_2 \equiv \nu_{LA}$, namely Eqs.(95) and (98) in [31], respectively, took place in that paper. A wrong factor of 2 occurred in Eq.(95) as the coefficient of the $N$ dependent fraction of the $O(\epsilon_L^2)$ contribution. In Eq.(98) an incorrect factor of $\frac{1}{4}$ also appeared in the $O(\epsilon_L^2)$ contribution therein. We emphasize that the correct expressions for those exponents are given by Eqs.(86) and (89) for $n = 2$ with $\epsilon_L$ in [31] identified with $\epsilon_2$ herein.

**VIII. ISOTROPIC INTEGRALS IN THE EXACT CALCULATION**

In the present section we shall discuss and compute the isotropic integrals for arbitrary $n$ in order to calculate the critical exponents in Sec.IX. This computation in the present massive method is by far much more complicated than its counterpart in the massless framework of Ref.[36]. Since this setting for the case $n = 2$ was not explicitly demonstrated in our previous
work [31] (the solution to the integrals were only quoted in one of the appendixes to that paper), we take this opportunity to fill this gap by working out more complicated cases along the same lines of reasoning. Here we shall describe in great detail all the technicalities involved in the evaluation of these integrals.

As we are going to see, the two- and three-loop contributions for the two-point functions represented by the integrals $I_3'$ and $I_5'$, respectively, present no problem in their calculation. In fact, they can be solved in terms of the expected poles and regular terms accompanied by multiparametric integrals which do not need to be computed explicitly, but cancell out in the renormalization algorithm. The issue is the computation of one- and two-loop contributions to the (four-point vertex part) coupling constant.

We start by considering the integral $I_2$. Since it always appears as a subdiagram in higher loop contributions of arbitrary 1PI vertex parts, we first attempt to compute it at nonvanishing external momenta. In this example we shall have a precise idea of the difficulty in this computation.

At nonvanishing external momenta $K$, $I_2$ is given by

$$I_2 = \int \frac{d^{n+2}k}{((k+K)^2)^n + 1} \quad (90)$$

From elementary complex algebra, we can use the identity $(k^2)^n + 1 = (k^2 - r_1)(k^2 - r_2)...(k^2 - r_n)$, where $r_l$ is the $l$th complex root of the equation $(k^2)^n + 1 = 0$.

To see this procedure at work, we apply it to the simplest nontrivial case which occurs for $n = 2$. In that case, using Feynman parameters, the propagator can be written as

$$\frac{1}{(k^2 - r_1)(k^2 - r_2)} = \Gamma(2) \int_0^1 \frac{dx}{(k^2 + m_x^2)^2}, \quad (91)$$

where $m_x^2 = (r_2 - r_1)x - r_2$. In particular, using Feynman parameters to fold denominators with different roots amounts to write the last expression for $m_x^2$ in different ways as functions of $r_1$ and $r_2$. Analogous remarks are valid for multiple roots. The final answer for the specific integral we are interested in is naturally independent of these maneuvers. For the case $n = 3$, a similar reasoning leads to the following representation for the propagator

$$\frac{1}{(k^2 - r_1)(k^2 - r_2)(k^2 - r_3)} = \Gamma(3) \int_0^1 \int_0^1 \frac{dx_1 dx_2 x_2}{(k^2 + m_{x_2}^2)^3}, \quad (92)$$

with $m_{x_2}^2 = (m_{x_1}^2 + r_3)x_2 - r_3$ and so on. For arbitrary positive integer $n$, we find by the
same token the following result:

$$\frac{1}{(k^2 - r_1)(k^2 - r_2)\cdots(k^2 - r_n)} = \Gamma(n) \int_0^1 \cdots \int_0^1 dx_1 dx_2 \cdots dx_{n-1} x_{n-1}^{n-2} \frac{1}{(k^2 + m_{x_{n-1}}^2)^n}, \quad (93)$$

where $m_{x_{n-1}}^2 = (m_{x_{n-2}}^2 + r_n)x_{n-1} - r_n$. Now let us insert Eq.(93) inside Eq.(90). Consequently $I_2$ becomes

$$I_2 = \Gamma(n)^2 \int_0^1 \cdots \int_0^1 dx_1 dx_2 \cdots dx_{n-1} x_{n-1}^{n-2} dy_1 dy_2 \cdots dy_{n-1} y_{n-1}^{n-2}$$

$$\times \int d^{m_n} k \frac{1}{((k + K)^2 + m_{x_{n-1}}^2)^n(k^2 + m_{y_{n-1}}^2)^n}. \quad (94)$$

We then utilize another Feynman parameter, say $t$, in order to fold the two denominators. Integrating over the loop momenta, we get an angular factor $S_{mn}$ which can be discarded/omitted through the redefinition of the coupling constant just as before. The reader should be warned that we shall use this fact in all loop integrals henceforward. Performing the $\epsilon_n$-expansion of the prefactors (which are simple Gamma functions), $I_2$ turns out to be

$$I_2 = \frac{\Gamma(2n)}{\epsilon_n}[1 + \frac{\epsilon_n}{2}(\psi(1) - \psi(2n))] \int_0^1 \cdots \int_0^1 dx_1 dx_2 \cdots dx_{n-1} x_{n-1}^{n-2} dy_1 dy_2 \cdots dy_{n-1} y_{n-1}^{n-2}$$

$$\times dt[3(1 - t)]^{n-1}[t(1 - t)K^2 + (m_{y_{n-1}}^2 - m_{x_{n-1}}^2)t + m_{x_{n-1}}^2]^{-\epsilon_n}, \quad (95)$$

where $\psi(z) = \frac{d\ln\Gamma(z)}{dz}$ is the digamma function. One could think that matters get simple to grasp if we set $K = 0$ (which is actually the symmetry point we shall be concerned in this massive setting), namely

$$I_{2SP} = \frac{\Gamma(2n)}{\epsilon_n}[1 + \frac{\epsilon_n}{2}(\psi(1) - \psi(2n))] \int_0^1 \cdots \int_0^1 dx_1 dx_2 \cdots dx_{n-1} x_{n-1}^{n-2} dy_1 dy_2 \cdots dy_{n-1} y_{n-1}^{n-2}$$

$$\times dt[3(1 - t)]^{n-1}[(m_{y_{n-1}}^2 - m_{x_{n-1}}^2)t + m_{x_{n-1}}^2]^{-\epsilon_n}, \quad (96)$$

but this is not so. The reason for the difficulty even for $K^2 = 0$ is that the complex roots of the polynomial in $k$ corresponding to the propagator depends explicitly on the value of $n$: we can not just go on, within this method, without specifying the value of $n$. This is the main obstruction to find the result of the integral in terms of a pole in $\epsilon_n$ together with a simple regular term for general $n$.

The simplest way to find the values of the integral for arbitrary $n$ is trying to discover a recurrence formula for this integral by analyzing the cases with fixed $n$ at the above symmetry point. Two cases are already known. The case $n = 1$ which corresponds to the
standard quadratic $\phi^4$ theory, whose integral is given by $I_{2SP} = \frac{1}{\epsilon_1}(1 - \frac{4}{\epsilon_1})$. The case $n = 2$, previously derived in Ref. [31], is given by $I_{2SP} = \frac{1}{\epsilon_2}(1 - \frac{4}{\epsilon_2})$. It would be interesting to compute this integral for higher values of $n$ in order to see if the method can be trusted for arbitrary $n$, provided the value of $n$ is fixed.

With this idea in mind, let us compute the integral for $n = 3$ in the first place. The three complex roots are $r_1 = -1$, $r_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2} \equiv A$ and $r_3 = A^*$. We find useful to melt firstly the contributions of $r_2, r_3$ and folding the resulting expression with $r_1$ term afterward.

Expansion of the digamma functions in the prefactors, we find

$$I_{2SP} = \frac{120}{\epsilon_3}[1 - \frac{137}{120}\epsilon_3] \int_{0}^{1} dx \int_{0}^{1} dz \int_{0}^{1} dy \int_{0}^{1} dw \int_{0}^{1} dt (t(1-t))^2 \times [(i\sqrt{3}x - \frac{3}{2} - i\frac{\sqrt{3}}{2})y(1-t) + (i\sqrt{3}z - \frac{3}{2} - i\frac{\sqrt{3}}{2})wt + 1]^{-\frac{3}{2}}. \quad (97)$$

When we perform the parametric integrals, we set $\epsilon_3 = 0$ in the powers of the resulting complex numbers which remain. The above expression is thus equal to

$$I_{2SP} = \frac{1}{\epsilon_3}(1 - \frac{\epsilon_3}{6}). \quad (98)$$

For good measure, let us analyze the case $n = 4$. Since this case is more involved and the calculations are rather long (though straightforward), we begin with the most basic facts.

The complex roots are $r_1 = \frac{1}{2} + i\frac{\sqrt{3}}{2} \equiv B$, $r_2 = -B^*$, $r_3 = -B$ and $r_4 = B^*$. The propagator can be written as

$$\frac{1}{(k^2)^4 + 1} = \Gamma(4) \int_{0}^{1} dx dy dz \frac{1}{(k^2 + m_z^2)^4}, \quad (99)$$

where $m_z^2 = ([(2x - 1)B^* - B|y + 2B)z - B$. After using the Feynman parameters, the integral $I_{2SP}$ then becomes

$$I_{2SP} = \Gamma(4)^2 \int_{0}^{1} dx dy dz \frac{1}{s}^2 \frac{1}{(k^2 + m_z^2)^4} \int \frac{d^m k}{(k^2 + m_z^2)^4(k^2 + m_z^2)^4}, \quad (100)$$

with $m_s^2 = ([(2w - 1)B^* - B|r + 2B)s - B$. Using another Feynman parameter $t$, integrating over the momenta and absorbing the angular factor $S_{m_s}$ in the redefinition of the coupling constant, we obtain

$$I_{2SP} = \frac{\Gamma(8)}{\epsilon_4} \int_{0}^{1} dx dy dz \frac{1}{s}^2 \frac{1}{(k^2 + m_z^2)^4} \int t(1-t)^3 \times [(m_s^2 - m_z^2)t + m_z^2]^{-\frac{3}{2}}. \quad (101)$$

We are thus left with the task of calculating a large number of elementary integrals, which makes the whole process a tedious one. Obviously, the order of the integrations chosen was
the following: integrate over $x, w, y, r, z, s$ and $t$. In the end of the day, set $\epsilon_4 = 0$ in the powers of the remaining complex numbers to get to the result

$$I_{2SP} = \frac{1}{\epsilon_4}(1 - \frac{\epsilon_4}{8}).$$

(102)

From the cases $n = 1, 2, 3, 4$, we discover that the method is reliable for arbitrary high values of $n$. There is no change of pattern in the calculations for different higher values of $n$. Perhaps the only undesirable feature is the proliferation of elementary parametric integrals whose number increases with increasing $n$. Therefore, for arbitrary $n$ we conclude that

$$I_{2SP} = \frac{1}{\epsilon_n}(1 - \frac{\epsilon_n}{2n}).$$

(103)

We then find out that the orthogonal approximation discussed in Appendix B is exact at one-loop level for the massive method.

Let us turn our attention to the integral $I_3$. In a preliminary stage, we leave the external momenta arbitrary and in that case it is given by:

$$I_3 = \int \frac{d^m n k_1 d^m n k_2}{((k_1 + k_2 + K)^2)^n + 1}.$$  

(104)

Now, performing the integral over the loop momenta $k_2$ by conjugating the results expressed in Eqs.(93) and (95), it is easy to see that

$$I_3 = \frac{\Gamma(2n)\Gamma(n)}{\epsilon_n}[1 + \frac{\epsilon_n}{2}(\psi(1) - \psi(2n))] \int_0^1 \ldots \int_0^1 dx_1 dx_2 x_2 \ldots dx_{n-1} x_{n-1}^{n-2} dy_1 dy_2 y_2 \ldots dy_{n-1} y_{n-1}^{n-2} \times dz_1 dz_2 z_2 \ldots dz_{n-1} z_{n-1}^{n-2} dt [t(1-t)]^{n-1-\frac{\epsilon_n}{2}} \int \frac{d^m n k_1}{(k_1^2 + m^2_{z_{n-1}})^n [(k_1 + K)^2 + m^2_t]^\frac{\epsilon_n}{2},}$$

(105)

where

$$m^2_t = \frac{(m^2_{y_{n-1}} - m^2_{z_{n-1}}) t + m^2_{z_{n-1}}}{t(1-t)}$$

and $m^2_{z_{n-1}} = (m^2_{z_{n-2}} + r_n) z_{n-1} - r_n$. We then use another Feynman parameter, say $u$, in order to integrate over the loop momenta $k_1$ to find

$$I_3 = \frac{\Gamma(2n)\Gamma(2n-\frac{\epsilon_n}{2})\Gamma(-n + \epsilon_n)}{2\Gamma(\frac{n+1}{2}) \epsilon_n}[1 + \frac{\epsilon_n}{2}(\psi(1) - \psi(2n))] \int_0^1 dx_1 dx_2 x_2 \ldots dx_{n-1} x_{n-1}^{n-2} \times dy_1 dy_2 y_2 \ldots dy_{n-1} y_{n-1}^{n-2} dz_1 dz_2 z_2 \ldots dz_{n-1} z_{n-1}^{n-2} dt [t(1-t)]^{n-1-\frac{\epsilon_n}{2}} \times du u^{\frac{\epsilon_n}{2}-1} (1-u)^{n-1}[u(1-u)K^2 + (m^2_t - m^2_{z_{n-1}}) u + m^2_{z_{n-1}}]^\frac{\epsilon_n}{2}. \quad (106)$$

From this expression we can compute the derivative at zero external momenta, i.e., $I_3' = \frac{\partial}{\partial K^{2n}}|_{K^{2n}=0}$. First, we expand the $\Gamma$-functions in the prefactor using the appropriate identities
for these functions. Second, after taking the derivative and setting the external momenta to zero, the first term in the last bracket contributes to the integral over $u$, whereas the powers of the remaining parameters are expanded in $\epsilon_n$ through the elementary identity $a^{-\epsilon_n}(u, t, z_{n-1}) = 1 - \epsilon_n \ln a(u, t, z_{n-1})$. Thirdly, the identity [44]

$$\int_0^1 dx x^{\mu-1}(1 - x^r)^{\nu-1} \ln x = \frac{1}{r^2} B\left(\frac{\mu}{r}, \nu\right)[\psi\left(\frac{\mu}{r}\right) - \psi\left(\frac{\mu}{r} + \nu\right),]$$

where $B(x, y)$ is the Euler beta function, will be useful to our purposes. It allows to separate the pole and a regular term by solving all the parametric integrals together with a regular term which depends on only one multiparametric integral. Collecting together this set of steps, $I_3'$ turns out to be

$$I_3' = \frac{(-1)^n \Gamma(2n)^2}{4\Gamma(n + 1)^2} [1 + \epsilon_n(-\frac{3}{4} - \sum_{p=3}^{3n-1} \frac{1}{2p} + \sum_{p=1}^{n-1} \frac{1}{2p})]$$

$$+ (-1)^{n+1} \frac{\Gamma(2n)^2}{4\Gamma(n + 1)} H',$$

where the remaining multiparametric integral $H'$ is given by

$$H' = \int_0^1 dx_1 dx_2 x_2 \cdots dx_{n-1} x_1^{n-2} dy_1 dy_2 dy_3 \cdots dy_{n-1} y_1^{n-2} dz_1 dz_2 \cdots dz_{n-1} z_1^{n-2} dt [t(1 - t)]^{n-1}$$

$$\times du u^{n-1} (1 - u)^{2n-1} \ln \left[\left(\frac{(m_2 y_{n-1} - m_2 x_{n-1}) t + m_2^2 x_{n-1}}{t(1 - t)}\right) - m_2^2 z_{n-1}\right] u + m_2^2 z_{n-1}\right] .$$

The above integral is the generalization of the integral $H$ Eq.(C5) from Ref.[31] appearing in the exact computation of the analogous two-point diagram for $m$-axial Lifshitz points.

After this discussion, it is straightforward to perform the three-loop integral contributing to the two-point function, namely

$$I_5 = \int \frac{d^m k_1 d^m k_2 d^m k_3}{((k_1 + k_2 + K)^2)^n + 1} \frac{((k_1 + k_3 + K)^2)^n + 1) ((k_1^2)^n + 1)((k_2^2)^n + 1)((k_3^2)^n + 1)}{(110)} .$$

Note that the internal subdiagram of the four-point function appears quadratically in the above integral. Using Eq.(93) in conjunction with Eq.(95), we get to the following intermediate result

$$I_5 = \frac{\Gamma(2n) \Gamma(n)}{\epsilon_n^2} [1 + \epsilon_n(\psi(1) - \psi(2n))] \int_0^1 \cdots \int_0^1 dx_1 dx_2 x_2 \cdots dx_{n-1} x_1^{n-2} dy_1 dy_2 dy_3 \cdots dy_{n-1} y_1^{n-2}$$

$$\times dz_1 dz_2 \cdots dz_{n-1} z_1^{n-2} dt [t(1 - t)]^{n-1-\epsilon_n} \int \frac{d^m k_1}{[k_1^2 + m_2^2 z_{n-1}^2][(k_1 + K)^2 + m_2^2]^\epsilon_n} .$$

(111)
Proceeding as before, integrate over the remaining loop momenta utilizing another Feynman parameter. This leads to

\[
I_5 = \frac{\Gamma(2n)\Gamma(2n - \frac{\alpha_n}{2})\Gamma(-n + \frac{3\alpha_n}{2})}{2\Gamma(\epsilon_n)\epsilon_n^2}[1 + \epsilon_n(\psi(1) - \psi(2n))] \int_0^1 dx_1dx_2x_2 \ldots dx_{n-1}x_{n-1}^{n-2} \times dy_1dy_2y_2 \ldots dy_{n-1}y_{n-1}^{n-2}dz_1dz_2z_2 \ldots dz_{n-1}z_{n-1}^{n-2}dt[t(1-t)]^{n-1-\epsilon_n} \\
\times duu^{\epsilon_n-1}(1-u)^{n-1}[u(1-u)K^2 + \left(m_i^2 - m_{\tilde z_{n-1}}^2\right)u + m_{\tilde z_{n-1}}^2]^{n-\frac{3\alpha_n}{2}}.
\]  

(112)

We then compute \(I'_5 = \frac{\partial I_5}{\partial K^{2n}}\). Taking the derivative, performing the expansion of the \(\Gamma\)-functions, using \(w\epsilon_n = 1 + \epsilon_n\ln u\) along with a similar expansion for the brackets (naturally taken at \(K^{2n} = 0\)) and employing the identity (107), the final result for the integral is

\[
I'_5 = \frac{(-1)^n\Gamma(2n)^2}{3\Gamma(n+1)\Gamma(3n)\epsilon_n^2}[1 + \epsilon_n(-1 - \sum_{p=3}^{n-1} \frac{1}{p} + \sum_{p=1}^{n-1} \frac{1}{2p} + \sum_{p=1}^{2n-1} \frac{1}{2p} + \frac{1}{2p})]
\]

\[+ (-1)^{n+1} \frac{\Gamma(2n)^2}{3\Gamma(n+1)\epsilon_n}H',
\]

(113)

where \(H'\) is given by eq.(109).

Finally, let us compute \(I_4\) at zero external momenta, namely

\[
I_4 = \int \frac{d^{mn}k_1d^{mn}k_2}{[(k_1^2)^n + 1][2][(k_2^2)^n + 1][(k_1 + k_2)^2]^n + 1]}.
\]

(114)

The same kind of limitation taking place for \(I_2\) occurs for \(I_4\): the lack of a solution for general \(n\) due to the fact that the roots of the polynomial in the momenta in our new representation of the propagator depends explicitly on \(n\). To see this, integrate first over the loop momenta \(k_2\) just as explained above, that is, use Feynman parameters to rewrite \(I_4\) as

\[
I_4 = \frac{\Gamma(2n)\Gamma(n)}{\epsilon_n}[1 + \frac{\epsilon_n}{2} (\psi(1) - \psi(2n))] \int_0^1 \ldots \int_0^1 dx_1dx_2x_2 \ldots dx_{n-1}x_{n-1}^{n-2}dy_1dy_2y_2 \ldots dy_{n-1}y_{n-1}^{n-2} \times dt[t(1-t)]^{n-1-\frac{\alpha_n}{2}} \int \frac{d^{mn}k_1}{[(k_1^2)^n + 1][2][(k_1^2)^2 + m_t^2]^\frac{1}{4}}.
\]

(115)

If we use new sets of Feynman parameters \(z_i\) to work out the first propagator, this operation produces the expression

\[
I_4 = \frac{\Gamma(2n)\Gamma(n)}{\epsilon_n}[1 + \frac{\epsilon_n}{2} (\psi(1) - \psi(2n))] \int_0^1 \ldots \int_0^1 dx_1dx_2x_2 \ldots dx_{n-1}x_{n-1}^{n-2}dy_1dy_2y_2 \ldots dy_{n-1}y_{n-1}^{n-2} \times dt[t(1-t)]^{n-1-\frac{\alpha_n}{4}} dz_1z_1(1-z_1)dz_2z_2(1-z_2) \ldots dz_{n-1}z_{n-1}^{2n-3}(1-z_{n-1}) \int \frac{d^{mn}k_1}{[k_1^2 + m_{z_{n-1}}^2]^{2n}[2][(k_1^2)^2 + m_t^2]^\frac{1}{4}}.
\]

(116)
Performing another integral over the remaining loop momenta the integral can be entirely represented by parametric integrals, that is

\[ I_4 = \frac{\Gamma(2n)}{2\Gamma(\frac{2n}{2})}\epsilon_n [1 + \frac{\epsilon_n}{2}(\psi(1) - \psi(2n))] \int_0^1 dx_1 dx_2 x_2 \ldots dx_{n-1} x_{n-1}^{n-2} \]

\[ \times dy_1 dy_2 y_2 \ldots dy_n \frac{\Gamma(2n-2)}{2\Gamma(2n)} \epsilon_n [1 + \frac{\epsilon_n}{2}(\psi(1) - \psi(2n)) - \psi(2n)] \]

\[ \int_0^1 dx_1 dx_2 x_2 \ldots dx_{n-1} x_{n-1}^{n-2} \]

\[ \times \int_0^1 dt [t(1-t)]^{n-1-\frac{\epsilon_n}{2}} du u^{\frac{\epsilon_n}{2}-1}(1-u)^{2n-1} \left( m_t^2 - m_{z_{n-1}}^2 \right) u + m_{z_{n-1}}^2 \]

\[ \times \left[ (m_t^2 - m_{z_{n-1}}^2) \right]^{n-\epsilon_n}. \] (117)

A standard procedure is to compute the integrand at \( u = 0 \): by summing and subtracting the integrand at this value, the difference between the integral at \( u \neq 0 \) and at \( u = 0 \) is higher order in \( \epsilon_n \) and can be neglected [22, 45]. We emphasize the factor \( m_{z_{n-1}}^2 \) depends upon the variables \( z_i \). Therefore, we can integrate over the other parameters and performing the \( \epsilon_n \)-expansion of the \( \Gamma \)-functions, we obtain the following expression:

\[ I_4 = \frac{\Gamma(2n)}{2\epsilon_n^2} \left[ 1 + \epsilon_n \left( \frac{3}{2} \psi(1) - \frac{1}{2} \psi(2n) - \psi(n) \right) \right] \]

\[ \times \int_0^1 d\epsilon_n [1 + \epsilon_n (D(n) - \frac{1}{n})]. \] (118)

In order to proceed from this point, we must specify again the values of \( n \) in order to determine the roots explicitly in order to solve the remaining parametric integrals. We already know this integral for \( n = 1, 2 \), namely

\[ I_4 = \frac{1}{2\epsilon_1^2} \left( 1 - \frac{\epsilon_1}{2} \right), \] (119)

\[ I_4 = \frac{1}{2\epsilon_2^2} \left( 1 - \frac{7\epsilon_2}{12} \right). \] (120)

The cases \( n = 3, 4 \) can be worked out just as we did in the one-loop case. With the resources furnished so far, it is not difficult to obtain the following results

\[ I_4 = \frac{1}{2\epsilon_3^2} \left( 1 - \frac{83\epsilon_3}{120} \right), \] (121)

\[ I_4 = \frac{1}{2\epsilon_4^2} \left( 1 - \frac{661\epsilon_4}{840} \right). \] (122)

The value of \( I_4 \) for general \( n \) which correctly reduces to the above particular cases is then given by

\[ I_4 = \frac{1}{2\epsilon_n^2} \left( 1 + \epsilon_n (D(n) - \frac{1}{n}) \right). \] (123)

where \( D(n) = \frac{1}{2} \psi(1) - \psi(n) + \frac{1}{2} \psi(2n). \)

This completes our task of calculating explicitly the exact isotropic integrals in the massive case [46] necessary to compute the critical exponents. This aim shall be tackled in the next section.
IX. ISOTROPIC EXPONENTS IN THE EXACT COMPUTATION

Now we apply the normalization conditions of the massive theory in pretty much the same way as worked out in the anisotropic and isotropic cases. In other words, the algorithm to determine the critical exponents is the same, but now we have to replace the values of the integrals calculated in last section.

The coefficients of the several renormalization functions can be easily found. Indeed, using the Wilson functions appropriate to the isotropic cases the coefficients have the following expressions:

\[ a_{1n} = \frac{N + 8}{6\epsilon_n} [1 - \frac{1}{2n}\epsilon_n], \]  
\[ a_{2n} = \left(\frac{N + 8}{6\epsilon_n}\right)^2 - \left[\frac{N^2 + 26N + 108}{36n\epsilon_n}\right] + \frac{5N + 22}{18\epsilon_n} \left(\frac{1}{n} - D(n)\right) \]  
\[ -(-1)^n \frac{\Gamma(2n)^2(N + 2)}{36\Gamma(n + 1)\Gamma(3n)\epsilon_n}, \]  
\[ b_{2n} = (-1)^n \frac{\Gamma(2n)^2(N + 2)}{72\Gamma(n + 1)\Gamma(3n)\epsilon_n} [1 + \epsilon_n(-\frac{3}{4} + \sum_{p=1}^{n-1} \frac{1}{2p} - \sum_{p=3}^{3n-1} \frac{1}{2p} - \Gamma(3n)H')] \]  
\[ + \sum_{p=3}^{3n-1} \frac{1}{2p} - \frac{3}{2n})], \]  
\[ b_{3n} = (-1)^n \frac{\Gamma(2n)^2(N + 2)(N + 8)}{648\Gamma(3n)\epsilon_n^2} \left(1 + \epsilon_n(-\frac{1}{4} + \sum_{p=1}^{n-1} \frac{1}{2p} - \sum_{p=3}^{3n-1} \frac{1}{2p} - \Gamma(3n)H') \right) \]  
\[ + \sum_{p=3}^{3n-1} \frac{1}{2p} - \frac{3}{2n})] \]  
\[ b_{3n} = \frac{(N + 2)}{6\epsilon_n} [1 - \frac{1}{2n}\epsilon_n], \]  
\[ c_{2n} = \frac{(N + 2)}{6\epsilon_n^2} \frac{(N + 5)}{6} (1 - \epsilon_n\frac{1}{n}) - \frac{1}{2} D(n)\epsilon_n]. \]

In practice these results determine the several renormalization functions. They are given by:

\[ \beta_n(u_n) = -u_n[\epsilon_n - \frac{N + 8}{6} (1 - \frac{\epsilon_n}{2n}) u_n + \left(\frac{5N + 22}{9} D(n) + (-1)^n \frac{\Gamma(2n)^2(N + 2)}{18\Gamma(n + 1)\Gamma(3n)}\right) \]  
\times \left[ u_n^2 \right], \]  
\[ \gamma_{\phi(n)}(u(n)) = (-1)^{n+1} \frac{\Gamma(2n)^2(N + 2)}{36\Gamma(n + 1)\Gamma(3n)\epsilon_n} u_n^2 \left[1 + \epsilon_n(-\frac{3}{4} + \sum_{p=1}^{n-1} \frac{1}{2p} - \sum_{p=3}^{3n-1} \frac{1}{2p} - \Gamma(3n)H') \right] \]  
\[ + \frac{N + 8}{6} \left(\frac{1}{2} - \sum_{p=2}^{2n-1} \frac{1}{2p} + \sum_{p=3}^{3n-1} \frac{1}{2p} - \frac{1}{n} + \Gamma(3n)H')u_n \right], \]  
\[ \gamma_{\phi^2(n)}(u_n) = \frac{N + 2}{6} u_n \left[1 - \frac{\epsilon_n}{n} - D(n)u_n \right]. \]
The fixed points $u_{n\infty}$ are obtained from the eigenvalue conditions $\beta_n(u_{n\infty}) = 0$, which yield

$$u_{n\infty} = \frac{6}{N+8}\epsilon_n \left[1 + \epsilon_n \left[\left(20N + 88\right)D(n) + \left(-1\right)^n \frac{\Gamma(2n)^2(2N + 4)}{\Gamma(n+1)\Gamma(3n)} \frac{1}{(N+8)^2} + \frac{1}{2n}\right]\right]. \quad (126)$$

It should be pointed out that neither the coefficients/renormalization functions nor the fixed points are equal to those obtained in Sec.VII of Ref.\[36\] for the massless theory: the normalization conditions are different and the difference is due to the nonuniversal feature of these functions away from the fixed points. Incidentally, we make the observation that even the fixed points are not universal but actually vary with the renormalization scheme employed.

Replacing the fixed points back into $\gamma_{\phi(n)}(u_n)$, the result is just the anomalous dimension $\eta_n = \gamma_{\phi(n)}(u_{n\infty})$, which is given by

$$\eta_n = (-1)^{n+1} \frac{(N + 2)\Gamma(2n)^2}{(N + 8)^2\Gamma(n+1)\Gamma(3n)} \epsilon_n^2 + (-1)^{n+1} \frac{(N + 2)\Gamma(2n)^2 F(N,n)}{(N + 8)^2\Gamma(n+1)\Gamma(3n)} \epsilon_n^3, \quad (127)$$

where

$$F(N,n) = \left[\left((-1)^n \frac{\Gamma(2n)^2(4N + 8)}{\Gamma(n+1)\Gamma(3n)} + (40N + 176)D(n)\right) \frac{1}{(N + 8)^2}ight] - \sum_{p=1}^{2n-1} \frac{1}{p} + \frac{1}{2} \sum_{p=1}^{n-1} \frac{1}{p} + \frac{1}{2} \sum_{p=1}^{3n-1} \frac{1}{p}. \quad (128)$$

Comparing this expression with Eq.(111) from Ref.\[36\], we detected a misprint therein: an extra factor of $\frac{3}{4}$ appearing there should be disconsidered. The correct expression for $F(N,n)$ is given by the last equation shown above.

Note that the integral $H'$ cancelled out in the computation of $\eta_n$. Finally let us compute the exponent $\nu_n$. First, we write $\bar{\gamma}_{\phi^2(n)}(u_n)$ at the fixed point, and then use the scaling relation among $\nu_n, \bar{\gamma}_{\phi^2(n)}(u_{n\infty})$ and $\eta_n$ to get the following result:

$$\nu_n = \frac{1}{2n} + \frac{(N + 2)}{4n^2(N + 8)} \epsilon_n + \frac{(N + 2)}{4n^2(N + 8)^3} \epsilon_n^2 \left((-1)^n \frac{\Gamma(2n)^2}{\Gamma(n+1)\Gamma(3n)} \frac{(N + 4)}{(N + 8)^2}\right)$$

$$+ \frac{(N + 2)(N + 8)}{2n} + (14N + 40)D(n). \quad (129)$$

These universal quantities are the same as those obtained in Ref.\[36\] in the zero mass approach. The other critical exponents can be obtained by scaling relations from the two exponents above and are also the same as those calculated from the zero mass setting. A rather interesting point is the following: while the exponents in the zero mass limit are
obtained analyzing the behavior of the vertex parts in the infrared, the exponents here are obtained in the ultraviolet regime since the fixed points $u_{n,\infty}$ are ultraviolet fixed points. We thus have complete equivalence of the zero mass theory renormalized at nonzero external momenta with the massive theory renormalized at vanishing external momenta. Universality is obeyed as expected.

X. FINAL COMMENTS

The introduction of a massive method to calculating critical exponents of generic competing systems is another step forward to a better comprehension of Lifshitz criticalities for several reasons. First, the massive method along with the Callan-Symanzik-Lifshitz equations is appropriate to proving the multiplicative renormalizability to all orders in perturbation theory as demonstrated herein. This proof for generic competing systems is a nontrivial generalization of that already performed for $m$-axial Lifshitz points [31]. Besides, the massive framework with its complete equivalence to the zero mass treatment reflected in identical critical exponents with those in Refs. [36] whether we use approximations for the appropriate cases or approach the calculations exactly is another exact manifestation of universality. Let us discuss some properties of anisotropic and isotropic cases separately and how they can show up in other field-theoretic contexts.

It is worthy to emphasize some especial features that occur in both massless and massive renormalized perturbation theories constructed to describe the generic higher character Lifshitz universality classes. As we have discussed, the computations in the massive theory is much more elaborate than in the massless method. Even though the Wilson functions determining the renormalized theory are explicitly different in both cases, they converge to the same value at the respective fixed points, which make them responsible for the same values of the critical exponents (and all universal quantities) in either scheme. This is not the only difference: while the massless 1PI vertex parts are scale invariant at the infrared attractive fixed point $u^*_n$, their counterparts in the massive theory are scale invariant precisely at the ultraviolet nonattractive fixed point $u_{n,\infty}$. The same remark was already pointed out in the $m$-axial Lifshitz universality [31] following closely the behavior of the pure $\phi^4$ noncompeting case argument of BLZ [30]. The several independent mass scales implementing independent renormalization group transformations in each type of competing subspace correspond to the
several correlation lengths appearing in each modulated ordered phase in the CECI model.

We can consider direct applications of the present formalism in the first place. We can use the method to compute critical amplitude ratios of various thermodynamic potentials (as well as other universal observables constructed from the renormalized $1PI$ vertex functions) above an below the Lifshitz temperature. Some uniaxial $m$-fold Lifshitz critical amplitudes have already been calculated at one-loop level \[10, 47\]. Even in this simpler case, not all amplitudes were computed and much more can be done. The motivation is to compare the results with new materials which might present Lifshitz points of generic higher character involving several independent length scales. Indeed, the comparison of the specific heat amplitude ratio of Ref.\[47\] with the one obtained experimentally for the material $MnP$ \[48\] yielded a remarkable agreement between the two results. Our hope is that the present method can help to unveil all the critical amplitude ratios, therefore generalizing the analysis for the $m$-axial Lifshitz with just two independent length scales. Moreover, we hope that numerical calculations such as those studied in high temperature series for the uniaxial Lifshitz case at fixed values of the space dimension and number of components of the order parameter \[49\] shall be put forth in order to improve our understanding of the universality classes of the most general competing systems.

Beyond the problem of critical phenomena occurring in generic competing critical systems, the mathematical apparatus developed in the present paper might be useful to attack the perturbative aspects of many quantum field theories with higher derivatives that have been studied recently. It has been pointed out that field theories with higher space derivatives and some sort of mechanism like the Lifshitz condition which suppresses the second space derivatives eliminate the lack of unitarity and produces a well behaved theory without ghosts \[19\]. It shares a similar renormalization group treatment: the time scale behaves in the same way as the noncompeting direction since it is quadratic in the derivatives, whereas the space directions have different scaling dimensions analogous to the Lifshitz competing directions. These Lorentz violating models have also been constructed in the cases of gauge theories \[50\] with implications in the standard model and the analysis of neutrino masses \[51\].

Although not studied yet within our present investigation, the issue of quantum critical behaviors of Lifshitz points might also be worthwhile. An instigating aspect which appeared recently in the literature is the application of these ideas in examining the quantum crit-
icality of membranes, which is analogous to the anisotropic $m$-axial $n = 2$ case with two length scales [52]. This description corresponds to a new class of gravity models. It has been further speculated that the case $n = 3$ of third character anisotropic Lifshitz points can be studied in this framework and explains the basic features of quantum gravity at these Lifshitz points [53]. This study adds more ingredients to understand the infrared modifications of gravity described by the ghost condensate previously introduced in Ref. [54], which has a more transparent analogy with the Lifshitz CECI model: gravity could have attractive character at “short” distances, but might develop repulsive properties in the long distance limit.

Furthermore, the conjectures put forward in [31] can be further extended to the most general cases of arbitrary independent mass scales: each scale is characterized by an independent Compton wavelength. This is kind of bizarre, since now more than one mass scale could characterize the corresponding “elementary particle”. This anisotropy could reflect the way it is distributed over space(time). In other words, if space(time) would be anisotropic (as suggested by these many independent mass scales) this feature could leave a mark on particles propagating on it. The connection with the model discussed here can be easily viewed in considering a scalar quantum field with a Lifshitz-like condition which only keeps higher space derivatives and maintains solely second order time derivatives. A Wick rotation brings this theory to the statistical mechanics form of the Lagrangian density Eq.(2). In this “generalized Lifshitz space”, the Laplacian operator is originally defined in $d$ dimensions, but due to distinct types of competition axes (caused by forces with alternate signals) as realized in the CECI model, only $(d - \sum_{n=2}^{L} m_n)$ of its components (e.g., the time components) remain. The residual information that the field lived in $d$-dimensions in the first place before dynamical effects break these pattern is encoded in the higher derivative terms in the Lagrangian density.

Minkowski space corresponds to the limit of zero gravity, where quantum fields propagate in. Competition between repulsive and attractive components of gravity in the large distance limit could then produce a tiny effect, which could result in this Lifshitz space. When Wick rotated back, this would produce precisely another type of flat space limit where gravity is small but has an observable effect on it: Lorentz invariance is broken in the bare propagator of the quantum fields, since now the timelike components of the momentum is quadratic as before, but the spacelike components have higher powers. Choosing independent metrics in
each competing subspace in the Lagrangian (2), the higher order term corresponding to the $m_n$ dimensional competing subspace could be written as $g_{rs} (\partial_r)^n (\partial_s)^n$, with $r, s = 1, \ldots, n$ for $g_{rs} = \delta_{rs}$. With this choice, the introduction of masses in dimensional reduction using the Siegel method would follow simply as discussed for each uniaxial subspace $m_n = 1$ without any further complication. We believe that the perturbative analysis of the examples above mentioned incorporating these new ideas might be worthwhile after the study described in the present work.

Most of the remarks done in the anisotropic cases can be extended to isotropic points. From a realistic point of view, however, it is much more difficult to visualize examples in quantum field theories: as pointed out before, more than two time derivatives in the Lagrangian density leads to trouble with unitarity. We can make remarks concerning the method itself in comparison with the zero mass case previously discussed in Ref. [36].

The massive integrals computed using the orthogonal approximation are very simple and can be computed explicitly for generic number of neighbors coupled via competing interactions $n$. On the other hand, the exact massive integrals are rather involved, even at the one-loop order, within the proposal presented. Just as happens in the massless case, the one-loop massive diagram explicit computation with general $n$ is identical to the result using the orthogonal approximation. The latter is exact at one loop. Deviations start at two-loop order and beyond. What is really remarkable using either the massless or the massive approach is that usual systems without competition are particular cases of generic isotropic criticalities with $n = 1$, i.e., they are Lifshitz behaviors of first character.

In summary, we evaluated the critical indices of generic higher character Lifshitz points using massive fields along with normalization conditions defining the renormalized theories at zero external momenta with many independent masses responsible for independent renormalization group transformations in each competing subspace. The results turn out to be identical to those obtained previously using a massless framework. Thus, universality is corroborated once again.

XI. ACKNOWLEDGMENTS

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APPENDIX A: FEYNMAN GRAPHS FOR ANISOTROPIC BEHAVIORS

The required diagrams corresponding to one-, two- and three-loop integrals in momenta space shall be computed using dimensional regularization and Feynman parameters as our main tools in their solution. Solely the one-loop integral associated to the four-point $1PI$ vertex part can be solved exactly for anisotropic behaviors. Multiloop graphs can be evaluated through the generalized orthogonal approximation. The main step is actually quite simple: the loop momenta characterizing a certain competition subspace in a given bubble (subdiagram) do not mix to all loop momenta not belonging to that bubble. In other words, the loop momenta in a given subdiagram is orthogonal to all loop/external momenta appearing in other subdiagrams. The normalization conditions defined in the text indicates that the integrals to be worked out should be determined at zero external momenta. The minimal set of Feynman integrals to be solved are presented with increasing order in the number of propagators, namely the one-loop integral contributing to the four-point function

$$I_2 = \int \frac{d^{d-\Sigma_{n=2}^{L} m_n} q_1 \prod_{n=2}^{L} d^{m_n} k_{(n)}^{L}}{[\sum_{n=2}^{L} (k_{(n)}^2)^n + (q)^2 + 1]^2} ,$$

the two-loop contribution to the two-point function $I_3' = \frac{\partial I_3(P,K_{(n)})}{\partial P^2}|_{P=K_{(n)}=0}$ ($= \frac{\partial I_3(P,K_{(n)})}{\partial K_{(n)}^2}|_{P=K_{(n)}=0}$), where $I_3(P, K_{(n)})$ is the integral

$$I_3 = \int \frac{d^{d-\Sigma_{n=2}^{L} m_n} q_1 d^{d-\Sigma_{n=2}^{L} m_n} q_2 \prod_{n=2}^{L} d^{m_n} k_{1(n)}^{L} \prod_{n=2}^{L} d^{m_n} k_{2(n)}^{L}}{(q_1^2 + \sum_{n=2}^{L} (k_{1(n)}^2)^n + 1) \left( q_2^2 + \sum_{n=2}^{L} (k_{2(n)}^2)^n + 1 \right)} \times \frac{1}{[(q_1 + q_2 + P)^2 + \sum_{n=2}^{L} ((k_{1(n)} + k_{2(n)} + K_{(n)})^2)^n + 1]} ,$$

the two-loop contribution to the coupling constant

$$I_4 = \int \frac{d^{d-\Sigma_{n=2}^{L} m_n} q_1 d^{d-\Sigma_{n=2}^{L} m_n} q_2 \prod_{n=2}^{L} d^{m_n} k_{1(n)}^{L} \prod_{n=2}^{L} d^{m_n} k_{2(n)}^{L}}{(q_1^2 + \sum_{n=2}^{L} (k_{1(n)}^2)^n + 1)^2} \times \frac{1}{\left( q_2^2 + \sum_{n=2}^{L} (k_{2(n)}^2)^n + 1 \right) [(q_1 + q_2)^2 + \sum_{n=2}^{L} ((k_{1(n)} + k_{2(n)})^2)^n + 1]} ,$$

(A1)
and finally the three-loop integral
\[ I_5' = \frac{\partial I_3(P,K_{(n)})}{\partial P} |_{P=K_{(n)}} = \frac{\partial I_5(P,K_{(n)})}{\partial K_{(n)}} |_{P=K_{(n)}} \]
with \(I_5(P,K_{(n)})\) representing the graph
\[
I_5 = \int d^{d-\sum_{n=2}^L m_n}q_1d^{d-\sum_{n=2}^L m_n}q_2d^{d-\sum_{n=2}^L m_n}q_3\Pi_{n=2}^L d^{m_n}k_{1(n)}
\]
\[
\times \frac{\Pi_{n=2}^L d^{m_n}k_{2(n)}\Pi_{n=2}^L d^{m_n}k_{3(n)}\Pi_{n=2}^L d^{m_n}k_{1(n)}}{[(q_1+q_2-P)^2+\sum_{n=2}^L (k_{1(n)}+k_{2(n)}-K_{(n)})^2]^{n+1}}.
\]
We stress that in the expressions for \(I_3\) and \(I_5\), \(P\) is the external momenta perpendicular to the several types of competing axes (i.e., it belongs to the \(m_1\) subspace) whereas \(K_{(n)}\) \((n = 2, ..., L)\) is the external momenta representing the \(n\)th \(m_n\)-dimensional competing subspace. They are only needed to compute the derivative of these integrals, but shall be set to zero after that operation.

One should keep in mind that whenever a loop integral is performed, the geometric angular factor \(S_{(d-\sum_{n=2}^L m_n)} \Gamma(2 - \sum_{n=2}^L \frac{m_n}{2})/(\Pi_{n=2}^L S_m \Gamma(\frac{m_n}{2}))\) appears but can be omitted in the final result of the integrals by a redefinition of the coupling constant. Performing this redefinition and applying the mathematical tools explained above, we learn that those divergent integrals have the following representation in terms of dimensional poles corresponding to their \(\epsilon_L\)-expansion values:

\[ I_2 = \frac{1}{\epsilon_L} \left[1 + (h_{m_L} - 1)\epsilon_L\right] + O(\epsilon_L), \quad (A5) \]
\[ I'_3 = \frac{-1}{8\epsilon_L} \left[1 + \left(2h_{m_L} - \frac{5}{4}\right)\epsilon_L - \frac{1}{8} I + O(\epsilon_L)\right], \quad (A6) \]
\[ I_4 = \frac{1}{2\epsilon_L^2} \left(1 + (2h_{m_L} - \frac{3}{2})\epsilon_L + O(\epsilon_L^2)\right), \quad (A7) \]
\[ I'_5 = \frac{-1}{6\epsilon_L^2} \left[1 + (3h_{m_L} - \frac{7}{4})\epsilon_L + O(\epsilon_L^2) - \frac{1}{4\epsilon_L} I\right], \quad (A8) \]

where \(h_{m_L} = 1 + (\psi(1) - \psi(2 - \sum_{n=2}^L \frac{m_n}{2}))\) and
\[ I = \int_0^1 dx \left(\frac{1}{1-x(1-x)} + \frac{ln[x(1-x)]}{[1-x(1-x)]^2}\right), \quad (A9) \]
is the same integral occurring in the original work by BLZ and in the \(m\)-axial Lifshitz anisotropic integrals. This integral drops out in the calculation of the critical exponents by extensive cancellations in the renormalization functions at the fixed point. As pointed out in
the previous massive approach for the \( m \)-fold anisotropic Lifshitz criticality, its appearing is a peculiarity of the orthogonal approximation. The normalization conditions and our choice of the subtraction point are responsible for this nonuniversal function, since it does not show up in the \( \epsilon_L \)-expansion of these integrals in the massless case.

**APPENDIX B: ISOTROPIC GRAPHS IN THE GENERALIZED ORTHOGONAL APPROXIMATION**

We shall pursue the analogy with the anisotropic case in order to compute the isotropic integrals \((d = m_n)\) in the orthogonal approximation. There is only one type of subspace (competition axes) to be integrated over and the parameter is now \( \epsilon_n = 4n - d \). The minimal set of loop integrals now read:

\[
I_2 = \int \frac{d^{m_n} k}{((k^2)^n + 1)^2},
\]

\[
I_3 = \int \frac{d^{m_n} k_1 d^{m_n} k_2}{(((k_1 + k_2 + K)^2)^n + 1)\left[((k_1^2)^n + 1))(1 + 1)\right]}.
\]

\[
I_4 = \int \frac{d^{m_n} k_1 d^{m_n} k_2}{((k_1^2 + 1)^2)((k_2^2 + 1)^n)}\left((k_1 + k_2)^2)^n + 1)\right].
\]

\[
I_5 = \int \frac{d^{m_n} k_1 d^{m_n} k_2 d^{m_n} k_3}{(((k_1 + k_2 + K)^2)^n + 1)\left(((k_1 + k_3 + K)^2)^n + 1)\right)(1 + 1)\right).}
\]

One should keep in mind that the derivatives with respect to \( K^{2n} \) of the two-point functions, namely \( I_3' \) and \( I_5' \), are actually the objects of interest in our discussion. Recall that these integrals are evaluated at \( K = 0 \). Therefore, using the same technology as before, the calculation is even simpler than in the anisotropic case. Again, every time a loop integral is performed an angular factor takes place. The geometric angular factor to be absorbed in a redefinition of the coupling constant in the isotropic integrals is \( S_{m_n} \), which corresponds to the area of the unit sphere in \( d = m_n \) dimensions. Omitting this factor, we encounter the following results for the required integrals:

\[
I_2 = \frac{1}{\epsilon_n} \left[1 - \frac{1}{2n} \epsilon_n\right] + O(\epsilon_n),
\]

\[\text{(B5)}\]
\( I_3' = \frac{-1}{8n\epsilon_n} [1 - \frac{1}{4n} \epsilon_n] - \frac{1}{8n^2} I + O(\epsilon_n), \)  
(B6)

\[ I_4 = \frac{1}{2\epsilon_n^2} \left( 1 - \frac{1}{2n} \epsilon_n + O(\epsilon_n^2) \right), \]  
(B7)

\[ I_5' = \frac{-1}{6n^2\epsilon_n^2} [1 - \frac{1}{4n} \epsilon_n + O(\epsilon_n^2)] - \frac{1}{4n^2\epsilon_n^2} I, \]  
(B8)

where \( I \) is the same integral taking place in Appendix A. Similarly to the situation found in the anisotropic cases, it does not contribute to the isotropic critical exponents as well.
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