A NON-LOCAL POISSON BRACKET FOR COXETER–TODA LATTICES

BY

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Abstract. We present a non-local Poisson bracket defined on the phase space $G^{u,v}/H$, where $G^{u,v}$ is a Coxeter double Bruhat cell of $GL_n$ and $H$ is the subgroup of diagonal matrices. The non-local Poisson bracket is written in an appropriate set of coordinates of $G^{u,v}/H$ derived from a set of factorization parameters for $G^{u,v}$. We show that the non-local Poisson bracket corresponds to the Atiyah–Hitchin bracket under the Moser map. As a consequence, the non-local Poisson bracket is compatible with a quadratic Poisson bracket obtained by M. Gekhtman, M. Shapiro and A. Vainshtein (2011).

1. Introduction. One of the main features shared by most finite-dimensional completely integrable Hamiltonian systems is the existence of a bi-Hamiltonian structure, i.e. a pair of compatible Poisson brackets defined on the underlying phase space with respect to each of which the system admits a Hamiltonian description.

Two of the most renowned and well documented examples of finite-dimensional completely integrable Hamiltonian systems are the Toda lattice and the relativistic Toda lattice. The former was discovered in [T] as a special case of the famous Fermi–Pasta–Ulam lattice. The latter was introduced in [Ru]. The bi-Hamiltonian nature of the Toda lattice and the relativistic Toda lattice is one of the most classical results. For instance, Suris [S] has presented both lattices as restrictions of the same Hamiltonian system to different symplectic leaves.

Moser, in his celebrated paper [M], made a fundamental contribution to the study of solutions of the non-periodic Toda lattice. To linearize the Toda lattice, he defined a map from the space of finite Jacobi matrices to the space of proper rational functions of fixed degree. The map, nowadays called the Moser map, associates to each Jacobi matrix its respective Weyl function, which is a matrix entry of the resolvent of the Jacobi matrix. The existence

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of the inverse map of the Moser map played an important role in determining the explicit representation for solutions of the non-periodic Toda lattice.

The Toda and the relativistic Toda lattice were described in [FG1] as particular lattices of a class of integrable lattices derived from the full Kostant–Toda flows on Hessenberg matrices, called elementary Toda lattices. Each of these systems can be linearized using the Moser map. The inverse problem for elementary Toda lattices was solved in [FG3].

In the study of magnetic monopoles a natural Poisson structure, called the Atiyah–Hitchin bracket, on the space of rational functions of fixed degree has been introduced in [AH]. Later, Faybusovich and Gekhtman [FG2] noticed that the Atiyah–Hitchin bracket fits into a finite family of compatible Poisson brackets on the space of rational functions. The latter family along with the Moser map established the multi-Hamiltonian structure of the elementary Toda lattices.

Elementary Toda lattices belong to a broader family of integrable lattices, the so-called Coxeter–Toda lattices. Toda flows on GL_n are commuting Hamiltonian flows generated by conjugation-invariant functions on GL_n with respect to the standard Poisson–Lie structure. Toda flows for an arbitrary standard semisimple Poisson–Lie group were studied in [Re]. A Coxeter–Toda lattice is an induced flow on G^{u,v}/H from the restriction of a Toda flow to G^{u,v}, where G^{u,v} is an instance of a particular class of double Bruhat cells of GL_n, named Coxeter double Bruhat cells, and H is the subgroup of diagonal matrices. The term Coxeter–Toda lattice for an arbitrary simple Lie group was coined in [HKKR]. Double Bruhat cells for any semisimple Lie group along with the concepts of factorization parameters and twisted generalized minors showed up in [FZ] in the context of total positivity. Their connections with integrable systems were explained in [KZ].

In [GSV3], Coxeter–Toda lattices and Bäcklund–Darboux transformations σ_{u,v} : G^{u,v}/H \rightarrow G^{u',v'}/H were described from the cluster algebra and the annular weighted networks perspective using a special set of coordinates derived from the factorization parameters of a Coxeter double Bruhat cell G^{u,v}. These coordinates can be recovered from the Weyl function of generic elements of a Coxeter double Bruhat cell. Thus the Moser map for Coxeter–Toda lattices is invertible. Once again, the Faybusovich–Gekhtman family can be used to establish the multi-Hamiltonian nature of the Coxeter–Toda lattices.

Although the multi-Hamiltonian nature of Coxeter–Toda lattices is well known, apart from the quadratic Poisson bracket (3.5) computed in [GSV3], no other explicit Poisson bracket compatible with (3.5), in the set of coordinates of G^{u,v}/H mentioned above, has appeared in the literature. The non-local Poisson bracket \{\cdot,\cdot\}_{nl} on G^{u,v}/H presented in (4.1)–(4.4) addresses this issue (see Corollary 4.7).
We deduced the non-local Poisson bracket \( \{ \cdot, \cdot \}_\text{nl} \) on \( G^{u,v}/H \) after performing Maple computations with the Moser map \( m_{u,v} : G^{u,v}/H \to W_n \) for small values of \( n \), where \( W_n \) is a subset of the space of rational functions of fixed degree equipped with the Atiyah–Hitchin bracket. Therefore, as expected, we show that the non-local Poisson bracket corresponds to the Atiyah–Hitchin bracket under the Moser map (see Theorem 4.6). To this end, a fundamental part is to show that the generalized Bäcklund–Darboux transformations preserve the non-local Poisson bracket (see Proposition 4.5).

This manuscript is organized as follows. Section 2 is devoted to a review of the following concepts: factorization parameters for double Bruhat cells \( G^{u,v} \), the standard Poisson–Lie structure on \( \text{GL}_n \) and the family of Faybusovich–Gekhtman Poisson brackets. Section 3 treats the Coxeter–Toda lattices and the generalized Bäcklund–Darboux transformations among them. Our main results are in Section 4. First, we present a non-local bracket \( \{ \cdot, \cdot \}_\text{nl} \) on \( G^{u,v}/H \). We claim in Lemma 4.2 that it is a Poisson bracket. Second, when \( u^{-1} = v = s_{n-1} \cdots s_1 \), Proposition 4.4 asserts that the non-local bracket \( \{ \cdot, \cdot \}_\text{nl} \) corresponds to the Atiyah–Hitchin bracket under the Moser map. We then present Proposition 4.5 to draw our main results in Theorem 4.6 and Corollary 4.7. We finish with two problems that we would like to address in the future. The proofs of Lemma 4.2 and Proposition 4.5 are a matter of a direct, though lengthy, computation. They are sketched in Sections 5 and 6, respectively.

2. Preliminaries. Throughout this manuscript, we let \( \text{GL}_n \) indicate the complex general linear group.

2.1. Double Bruhat cells and factorization parameters. Let \( s_i \) denote the elementary transposition \((i, i+1)\) of the symmetric group \( S_n \) acting on the set \([1, n] = \{1, \ldots, n\}\).

Hereafter, \( v \in S_n \) is regarded as the \( n \times n \) matrix \((\delta_{iv(j)})_{i,j=1}^n \) whenever there is no confusion. Let \( B_+ \) and \( B_- \) be the subgroups of upper triangular and lower triangular matrices in \( \text{GL}_n \), respectively. The Bruhat decompositions of \( \text{GL}_n \) with respect to \( B_+ \) and \( B_- \) are defined, respectively, by

\[
\text{GL}_n = \bigcup_{u \in S_n} B_+ u B_+, \quad \text{GL}_n = \bigcup_{v \in S_n} B_- v B_-.
\]

The sets \( B_+ u B_+ \) (respectively, \( B_- v B_- \)) are called Bruhat cells (respectively, opposite Bruhat cells). For any \((u, v) \in S_n \times S_n\), the double Bruhat cells \( G^{u,v} \subseteq \text{GL}_n \), first introduced in [FZ], are defined by

\[
G^{u,v} = B_+ u B_+ \cap B_- v B_-.
\]

Note that any double Bruhat cell \( G^{u,v} \) is invariant under left and right multiplication by elements of the subgroup \( H \) of diagonal matrices.
Example 2.1. Let $u^{-1} = v = s_{n-1} \cdots s_1$. Then

\[ G^{u,v} = \left\{ (x_{ij}) \in \text{GL}_n \bigg| x_{ij} = 0 \text{ if } |i-j| > 1 \text{ and } \prod_{i=1}^{n-1} x_{i,i+1}x_{i+1,i} \neq 0 \right\}. \]

Remark 2.2. It follows from [FZ, Theorem 4.4] that the variety $G^{u,v}$ is biregularly isomorphic to a Zariski open subset of $\mathbb{C}^{l(u)+l(v)+n}$, where $l(w)$ denotes the length of $w \in S_n$. Different birational maps from $\mathbb{C}^{l(u)+l(v)+n}$ to $G^{u,v}$ are constructed by a product of elementary Jacobi matrices

\[ E^{-i}_i(t) = I_n + te_{i+1,i}, \quad E^+_{j,j}(t) = I_n + te_{j,j+1}, \quad E^0_k(t) = I_n + (t-1)e_{k,k}, \]

where $e_{i,j}$ denotes the elementary matrix $(\delta_{i\alpha} \delta_{j\beta})_{\alpha,\beta=1}^n$, and the order of the factors depends on the choice of a shuffle of a reduced word of $(u, v) \in H \times H$ and any rearrangement of a list of $n$ elements. Every such factorization map can be thought of as a system of local coordinates in $G^{u,v}$, called factorization parameters. The factorization map has an interpretation in terms of directed graphs embedded in a disc with weighted edges. These are called perfect networks in a disc (see [GSV1]).

2.2. The standard Poisson–Lie bracket. The standard Poisson–Lie bracket on $\text{GL}_n$, denoted by $\{ \cdot, \cdot \}_{\text{GL}_n}$, is defined by

\[ \{x_{ij}, x_{kl}\}_{\text{GL}_n} = \frac{1}{2} (\text{sgn}(k-i) + \text{sgn}(l-j)) x_{il} x_{kj}, \]

where $x_{ij}, x_{kl}, 1 \leq i, j, k, l \leq n$, are the coordinate functions of $\text{GL}_n$.

Remark 2.3. The standard Poisson–Lie bracket is defined, in a systematic way, for Lie groups as a Sklyanin bracket with a particular choice of a classical $R$-matrix (see [ReST]). Kogan and Zelevinsky [KZ, Theorem 2.3] showed that the symplectic leaves of the standard Poisson–Lie bracket for a semisimple Lie group are translations by elements of the Cartan subgroup of a particular symplectic leaf inside of a double Bruhat cell.

In our case, if $\text{GL}_n$ is equipped with $\{ \cdot, \cdot \}_{\text{GL}_n}$ then every double Bruhat cell $G^{u,v} \subset \text{GL}_n$ is a regular Poisson submanifold. Symplectic leaves in $(\text{GL}_n, \{ \cdot, \cdot \}_{\text{GL}_n})$ are of the form $S^{u,v} \cdot a$, where $S^{u,v} \subset G^{u,v}$ is a distinguished symplectic leaf and $a$ is an element of $H$. Furthermore, the dimension of a symplectic leaf in $G^{u,v}$ is equal to $l(u) + l(v) + \text{corank}(uv^{-1} - I_n)$.

2.3. The Faybusovich–Gekhtman Poisson brackets. Consider the space of rational functions

\[ \text{Rat}_n = \left\{ s(\lambda) = \frac{q(\lambda)}{p(\lambda)} \bigg| p(\lambda) \text{ is a monic polynomial, } \deg p = n, \deg q < n \right\}. \]

For fixed $p(\lambda), q(\lambda)$ and $k = 0, \ldots, n-1$, let

\[ q^{[k]}(\lambda) = \lambda^k q(\lambda) \pmod{p(\lambda)} \]
and define a skew-symmetric bracket $\{ \cdot, \cdot \}_k$ on the coefficients of $p(\lambda), q(\lambda)$ by setting

$$\{ p(\lambda), p(\mu) \}_k = \{ q(\lambda), q(\mu) \}_k = 0,$$

(2.2)

$$\{ p(\lambda), q(\mu) \}_k = \frac{p(\lambda)q^{[k]}(\mu) - p(\mu)q^{[k]}(\lambda)}{\lambda - \mu}.$$

The bracket $\{ \cdot, \cdot \}_0$ of the family (2.2) is known as the Atiyah–Hitchin bracket [AH].

**Proposition 2.4 ([FG2 Proposition 2]).** The brackets $\{ \cdot, \cdot \}_k$, $k = 0, \ldots, n-1$, are compatible Poisson brackets on $\text{Rat}_n$.

Consider now the subset of $\text{Rat}_n$ given by

$$\text{Rat}'_n = \left\{ \frac{q(\lambda)}{p(\lambda)} \in \text{Rat}_n \left| q(\lambda) \text{ is monic} \right. \right\}.$$

The Poisson brackets $\{ \cdot, \cdot \}_k$, $k = 0, \ldots, n-1$, given by (2.2) can be restricted to $\text{Rat}'_n$ and take the form

$$\{ p(\lambda), p(\mu) \}_k = \{ q(\lambda), q(\mu) \}_k = 0,$$

(2.3)

$$\{ p(\lambda), q(\mu) \}_k = \frac{p(\lambda)q^{[k]}(\mu) - p(\mu)q^{[k]}(\lambda)}{\lambda - \mu} - q^{[k]}(\lambda)q(\mu).$$

**Example 2.5.** Let $n = 2$, $p(\lambda) = \lambda^2 + \alpha_1 \lambda + \alpha_0$ and $q(\lambda) = \lambda + \beta_0$. In this case

(2.4) $$\{ p(\lambda), q(\mu) \}_0 = \frac{p(\lambda)q(\mu) - p(\mu)q(\lambda)}{\lambda - \mu} - q(\lambda)q(\mu) = \alpha_1 \beta_0 - \alpha_0 - \beta_0^2.$$

Therefore,

$$\{ \alpha_0, \beta_0 \}_0 = \alpha_1 \beta_0 - \alpha_0 - \beta_0^2, \quad \{ \alpha_1, \beta_0 \}_0 = \{ \alpha_1, \alpha_0 \}_0 = 0.$$

**3. Coxeter–Toda lattices on $\text{GL}_n$.** The main reference for this section is [GSV3].

**3.1. Toda flows.** Toda flows on $\text{GL}_n$ are equations of motion generated by $F_k(X) = \frac{1}{k} \text{Trace}(X^k)$ ($k = 1, \ldots, n-1$) and the standard Poisson–Lie structure $\{ \cdot, \cdot \}_{\text{GL}_n}$. The equation of motion generated by $F_k$ has the Lax form

(3.1) $$\dot{X} = \left[ X, -\frac{1}{2}(\pi_+(X^k) - \pi_-(X^k)) \right],$$

where $\pi_+(A)$ and $\pi_-(A)$ denote strictly upper and lower parts of a matrix $A$. The functions $F_1, \ldots, F_{n-1}$ form a maximal family of algebraically independent conjugation-invariant functions of $\text{GL}_n$ and they Poisson commute.

Since the action of $H$ on $\text{GL}_n$ by conjugation is Poisson with respect to the standard Poisson–Lie structure and preserves double Bruhat cells, the standard Poisson–Lie structure induces a Poisson structure on $G^{u,v}/H$. 
Therefore, the Toda hierarchy induces a family of commuting Hamiltonian flows on $G^{u,v}/H$.

**Example 3.1.** Let $u$ and $v$ be as in Example 2.1. Then the space $G^{u,v}/H$ is described as the set of Jacobi matrices of the form

$$L = \begin{pmatrix} b_1 & 1 & 0 & \ldots & 0 \\ a_1 & b_2 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & a_{n-2} & b_{n-1} & 1 \\ 0 & \ldots & 0 & a_{n-1} & b_n \end{pmatrix}, \quad a_1 \ldots a_{n-1} \neq 0, \det L \neq 0.$$

The Lax equations (3.1) then become the equations of the *finite non-periodic Toda hierarchy*

$$\dot{L} = [L, \pi -(L^k)].$$

The case $k = 1$ is the well-known *Toda lattice (3.2)*

$$\begin{align*}
\dot{a}_j &= a_j(b_{j+1} - b_j), \quad j = 1, \ldots, n-1, \\
\dot{b}_j &= a_j - a_{j-1}, \quad j = 1, \ldots, n,
\end{align*}$$

with the boundary conditions $a_n = a_0 = 0$.

Let us consider the subset of $\text{Rat}'_n$ given by

$$\mathcal{W}_n = \left\{ \frac{q(\lambda)}{p(\lambda)} \in \text{Rat}'_n \middle| \deg p = \deg q + 1, \ p \text{ and } q \text{ are coprime, } p(0) \neq 0 \right\}.$$ 

The *Weyl function* of $X \in \text{GL}_n$ is the rational function defined by

$$m(\lambda) = m(\lambda; X) = ((\lambda I - X)^{-1}e_1, e_1) = \frac{\Delta_{\lfloor 2,n \rfloor}(\lambda)}{\Delta_{\lfloor 1,n \rfloor}(\lambda)},$$

where $\Delta_{\lfloor 1,n \rfloor}(\lambda)$ is the characteristic polynomial of $X$, and $\Delta_{\lfloor 2,n \rfloor}(\lambda)$ is the characteristic polynomial of the $(n-1) \times (n-1)$ submatrix of $X$ formed by deleting the first row and column. Since the Weyl function is invariant under the conjugation action of $H$ on $G^{u,v}$, one can consider the *Moser map* given by

$$m_{u,v}: G^{u,v}/H \to \mathcal{W}_n, \quad X \mapsto m_{u,v}(X) = m(\lambda, X).$$

In the tridiagonal case, $v = u^{-1} = s_{n-1} \cdots s_1$, Moser [M] proved that the map $m_{u,v}$ is invertible and the system (3.2) is completely integrable in the sense of Arnold–Liouville. The level sets of the function $\det L$ foliate the set of Jacobi matrices, described in Example 3.1, into $2(n-1)$-dimensional symplectic manifolds.

Gektman, Shapiro and Vainshtein [GSV3] proved that there are other double Bruhat cells which share common features with the tridiagonal case. We present them in the next subsection.
3.2. Coxeter double Bruhat cells on $\GL_n$. A permutation $w \in S_n$ is called Coxeter if it is a product of $n - 1$ distinct transpositions. Given a pair $(u, v)$ of Coxeter elements, $G^{u,v}$ is called a Coxeter double Bruhat cell and in this case $\dim G^{u,v} = 3n - 2$. For any pair $(u, v)$ of Coxeter elements the integrable equations induced on $G^{u,v}/H$ by Toda flows will be called Coxeter–Toda lattices.

We now recall from [GSV3] a combinatorial data useful in the study of Coxeter–Toda lattices. Note that every Coxeter element $v \in S_n$ can be written in the form

$$v = s_{[i_{k-1}, i_k]} \cdots s_{[i_1, i_2]} s_{[i_0, i_1]}$$

for some subset $I = \{1 = i_0 < i_1 < \cdots < i_k = n\} \subset [1, n]$ where $s_{[p,q]} = s_p s_{p+1} \cdots s_{q-1}$ for $1 \leq p < q \leq n$. For a pair $(u, v)$ of Coxeter elements let

$$(3.4) \quad I^+ = \{1 = i_0^+ < \cdots < i_k^+ = n\}, \quad I^- = \{1 = i_0^- < \cdots < i_k^- = n\}$$

be the subsets of $[1, n]$ that correspond to $v$ and $u^{-1}$, respectively, previously described.

Let $c_i^-, c_i^+, i = 1, \ldots, n - 1$, and $d_i, i = 1, \ldots, n$, be a set of complex parameters. Given a pair of Coxeter elements $(u, v)$, generic elements $X \in G^{u,v}$ can be parametrized as follows: $X = X_- X_0 X_+$, where $X_-$ is the product of $E_i^- (c_i^-), i = 1, \ldots, n - 1$, with the order of factors prescribed by $u$, $X_+$ is the product of $E_i^+ (c_i^+), i = 1, \ldots, n - 1$, with the order of factors prescribed by $v$, and $X_0$ is the diagonal matrix $\text{diag}(d_1, \ldots, d_n)$ (see [GSV3, Lemma 3.3]). Then elements in $G^{u,v}/H$ are parametrized by $c_i = c_i^+ c_i^-$ and $d_i$.

**Remark 3.2.** According to [GSV3, Theorem 4.1], for a pair $(u, v)$ of Coxeter elements in $S_n$, the Moser map $m_{u,v} : G^{u,v}/H \to \mathcal{W}_n$ is invertible. Furthermore, the standard Poisson–Lie bracket on $\GL_n$ induces the Poisson bracket $\{\cdot, \cdot\}_1$, defined in (2.3), on the Weyl functions. A proof using the weighted network approach can be found in [GSV3, Proposition 5.3].

Fix a pair $(u, v)$ of Coxeter elements, and hence the sets $I^+, I^-$ given by (3.4), and set

$$\varepsilon_i^\pm = \begin{cases} 0 & \text{if } i = i_j^\pm \text{ for some } 0 < j \leq k_\pm, \\ 1 & \text{otherwise}, \end{cases}$$

and $\varepsilon_i = \varepsilon_i^+ + \varepsilon_i^-$. 

**Proposition 3.3** ([GSV3, Lemma 6.1]). The standard Poisson–Lie structure on $\GL_n$ induces the following Poisson brackets for the variables $c_i, d_i$:

$$(3.5) \quad \{c_i, c_i+1\} = (\varepsilon_{i+1} - 1)c_i c_{i+1}, \quad \{c_i, d_i\} = -c_i d_i, \quad \{c_i, d_{i+1}\} = c_i d_{i+1},$$

and the other brackets are zero.

The Hamiltonians $F_i(X) = \frac{1}{t} \text{Trace}(X^i)$, due to the invariance under conjugation by elements of $H$, when restricted to a Coxeter double Bruhat
cell $G^{u,v}$ can be expressed as functions of $c_i, d_i$. Thus, the functions $F_l, l = 1, \ldots, n - 1$, serve as Hamiltonians for Coxeter–Toda flows on $G^{u,v}/H$. Therefore, the Toda hierarchy defines a completely integrable system on the symplectic leaves (level sets of the determinant) in $G^{u,v}/H$, i.e. Coxeter–Toda lattices are completely integrable systems.

**Example 3.4.** For $l = 1$, the Hamiltonian $F_1$ has the form

$$F_1 = \sum_{i=1}^{n} (d_i + c_{i-1}d_{i-1}) + \sum_{i=3}^{n} \sum_{r=1}^{i-2} d_r c_r \prod_{j=r+1}^{i-1} c_j \epsilon_j^{-1} \epsilon_j^+.$$

We present the Hamiltonian equations of the Coxeter–Toda flow generated by $F_1$ and (3.5) on $G^{u,v}/H$ for some particular cases:

(a) Let $u^{-1} = v = s_{n-1} \cdots s_1$. Then $\epsilon = (2, 0, \ldots, 0)$ and the Hamiltonian equations become

$$\dot{c}_i = \{ c_i, F_1 \} = c_i (d_{i+1} - d_i + c_{i-1}d_{i-1} - c_id_i),$$

$$\dot{d}_i = \{ d_i, F_1 \} = d_i (c_id_i - c_{i-1}d_{i-1}).$$

After the change of variables $a_i = c_id_i^2, b_i = d_i + c_{i-1}d_{i-1}$, the system becomes the Toda lattice (3.2).

(b) Let $u = v = s_{n-1} \cdots s_1$. Then $\epsilon = (2, 1, \ldots, 1, 0)$ and the Hamiltonian equations become

$$\dot{c}_i = c_i (d_{i+1} - d_i + c_{i+1}d_{i+1} - c_id_i), \quad \dot{d}_i = d_i (c_i d_i - c_{i-1}d_{i-1}).$$

After the change of variables $\tilde{c}_i = c_id_i$, this system becomes the relativistic Toda lattice:

$$\dot{\tilde{c}}_i = \tilde{c}_i (d_{i+1} - d_i + \tilde{c}_{i+1} - \tilde{c}_{i-1}), \quad \dot{d}_i = d_i (\tilde{c}_i - \tilde{c}_{i-1}).$$

It is well known that the Toda lattice (3.2) and the relativistic Toda lattice (3.7) are completely integrable bi-Hamiltonian systems [S]. Moreover, one can use the family of compatible Faybusovich–Gekhtman Poisson brackets and the fact that the Moser map is an invertible map to guarantee the multi-Hamiltonian nature of Coxeter–Toda lattices. This fact was exposed in [FG2] for elementary Toda lattices, which are particular Coxeter–Toda lattices $G^{u,v}/H$ by taking $v = s_{n-1} \cdots s_1$ and an arbitrary Coxeter element $u$.

### 3.3. Generalized Bäcklund–Darboux transformations.

We now review the generalized Bäcklund–Darboux transformation. It is a birational automorphism between phase spaces of Coxeter–Toda lattices which preserves the corresponding Coxeter–Toda flows. The cluster algebra interpretation of these automorphisms was exposed in [GSV3].

For the purpose of this note, we avoid the cluster algebra approach and simply refer the interested reader to [GSV3]. However, relying on [GSV3], we
define these transformations in a more axiomatic way as a finite composition of some elementary transformations.

| ε    | Mutation | ε′   |
|------|----------|------|
| ε_1  | 0, i     | ε_1′ | 1, ε_1′+1 = 1 |
| ε_1+1| 0, i     | ε_1′ | 0, ε_1′+1 = 1 |
| ε_{n-1} | (1, n-1) | ε_{n-1}′ | 1 |
| ε_{n-1} | (1, n-1) | ε_{n-1}′ | 2 |

Table 1. Mutations

Consider two pairs \((u, v)\) and \((u′, v′)\) of Coxeter elements such that the entries of their corresponding \(n\)-tuples \(ε = (ε_i)_{i=1}^n\) and \(ε′ = (ε′_i)_{i=1}^n\) satisfy one of the possible situations listed in Table 1. Here the entries of the \(n\)-tuples \(ε\) and \(ε′\) are all equal except for the entries specified in Table 1. In the first two rows, \(i\) is assumed to be less than \(n-1\).

We denote by \((s, i)\), \(s = 0, 1\), \(i \in [1, n-1]\), as indicated in Table 1, the transformation, called mutation, that takes the \(n\)-tuple \(ε\) to the \(n\)-tuple \(ε′\). If \((s, i)\), \(s = 0, 1\), \(i \in [1, n-1]\), denotes the mutation of \(n\)-tuple \(ε\) to the

| ε′   | Transformation | Inverse |
|------|----------------|---------|
| ε_1′ = 1 | \(c_{i-1}′ = c_{i-1}(1 + c_i)\) | \(c_{i-1} = \frac{c_{i-1}′ - d_{i-1}d_{i-1}′}{d_{i-1} + d_{i-1}′}\) |
| ε_1′+1 = 1 | \(c_{i+1}′ = c_{i+1}(1 + c_i)\) | \(c_{i+1} = \frac{c_{i+1}′d_{i+1} + c_{i+1}d_{i+1}′}{c_{i+1}d_{i+1} + c_{i+1}′d_{i+1}}\) |
| ε_{i+1}′ = 1 | \(d_i′ = d_i(1 + c_i)\) | \(d_i = \frac{d_i′ - c_i′d_i′}{d_i′ + c_i′}\) |
| ε_{i+1}′+1 = 1 | \(d_{i+1}′ = d_{i+1}(1 + c_i)\) | \(d_{i+1} = \frac{d_{i+1}′ - c_i′d_{i+1}′}{d_{i+1}′ + c_i′}\) |
| ε_{n-1}′ = 1 | \(d_{n-1}′ = \frac{d_{n-1}d_{n-1}′}{d_n + c_{n-1}d_{n-1}}\) | \(d_{n-1} = \frac{d_{n-1}′}{d_n + c_{n-1}d_{n-1}}\) |
| ε_{n-1}′+1 = 2 | \(d_n′ = d_n + c_{n-1}d_{n-1}\) | \(d_n = \frac{d_n′}{d_n + c_{n-1}d_{n-1}}\) |

Table 2. Elementary transformations
$n$-tuple $\varepsilon'$ specified in Table 1 then we denote by $(1 - s, i)$, $s = 0, 1$, the inverse mutation from the $n$-tuple $\varepsilon'$ to the $n$-tuple $\varepsilon$.

There exist birational transformations between $G^{u,v}/H$ and $G^{u',v'}/H$, called **elementary transformations**, corresponding to the mutations of Table 1. These transformations are listed in Table 2.

Now, fix two arbitrary pairs $(u, v)$ and $(u', v')$ of Coxeter elements and let $\varepsilon = (\varepsilon_i)_{i=1}^n$, $\varepsilon' = (\varepsilon'_i)_{i=1}^n$ be their corresponding $n$-tuples. It was noted in [GSV3, Lemma 5.2] that we can transform the $n$-tuple $\varepsilon$ to the $n$-tuple $\varepsilon'$ via a finite sequence of mutations listed in Table 1.

**Example 3.5.** Let $n = 4$. The diagram below shows the mutations of Table 1 among all possible values of $\varepsilon$ corresponding to a pair of Coxeter elements $(u, v)$ of $S_4$.

![Diagram showing mutations of $\varepsilon$](image)

The generalized Bäcklund–Darboux transformation $\sigma_{u,v}^{u',v'} : G^{u,v}/H \rightarrow G^{u',v'}/H$ is the corresponding finite composition of the elementary transformations, listed in Table 2, associated to the finite sequence of mutations, listed in Table 1 that takes the $n$-tuple $\varepsilon$ into $\varepsilon'$.

**4. Main results.** Despite the fact that Coxeter–Toda lattices are completely integrable multi-Hamiltonian systems, no other Poisson bracket in the $c_i, d_i$ coordinates and compatible with (3.5) is known in the literature. Corollary 4.7 will address this issue.
4.1. Non-local Poisson bracket. Fix a pair \((u, v)\) of Coxeter elements. We consider the following non-local bracket \(\{\cdot, \cdot\}_{nl}\) on \(G_{u,v}/H\) defined in the \(c_i, d_i\) coordinates:

\[
\{d_i, d_{i+k}\}_{nl} = \begin{cases} 
  c_i d_i & \text{if } k = 1, \\
  d_i \prod_{j=i}^{i+k-1} c_j & \text{if } k > 1, \varepsilon_{i+1} = \ldots = \varepsilon_{i+k-1} = 0,
\end{cases}
\]

\[
\{c_i, d_{i+k}\}_{nl} = \begin{cases} 
  c_i & \text{if } k = 0, \\
  -c_i (c_i + 1) & \text{if } k = 1, \\
  -(c_i + 1) \prod_{j=i}^{i+k-1} c_j & \text{if } k > 1, \varepsilon_{i+1} = \ldots = \varepsilon_{i+k-1} = 0, \\
  \prod_{j=i}^{i+k-1} c_j & \text{if } k > 1, \varepsilon_{i+1} = 2, \varepsilon_{i+2} = \ldots = \varepsilon_{i+k-1} = 0,
\end{cases}
\]

\[
\{c_{i+k}, d_i\}_{nl} = \begin{cases} 
  (2 - \varepsilon_{i+1}) c_i c_{i+1} \frac{d_i}{d_{i+1}} & \text{if } k = 1, \\
  (2 - \varepsilon_{i+k}) \frac{d_i}{d_{i+k}} \prod_{j=i}^{i+k} c_j & \text{if } k > 1, \varepsilon_{i+1} = \ldots = \varepsilon_{i+k-1} = 0,
\end{cases}
\]

\[
\{c_i, c_{i+k}\}_{nl} = \begin{cases} 
  (2 - \varepsilon_{i+1}) c_i \frac{c_i + 1}{d_{i+1}} c_{i+1} & \text{if } k = 1, \\
  (2 - \varepsilon_{i+k}) \frac{c_i + 1}{d_{i+k}} \prod_{j=i}^{i+k} c_j & \text{if } k > 1, \varepsilon_{i+1} = \ldots = \varepsilon_{i+k-1} = 0, \\
  (\varepsilon_{i+k} - 2) \frac{\prod_{j=i}^{i+k} c_j}{d_{i+k}} & \text{if } k > 1, \varepsilon_{i+1} = 2, \varepsilon_{i+2} = \ldots = \varepsilon_{i+k-1} = 0,
\end{cases}
\]

and the other brackets are zero.

Remark 4.1. We deduced the non-local bracket (4.1)–(4.4) by computing the pull-back bracket from \((\mathcal{W}_n, \{\cdot, \cdot\}_0)\) via the Moser map \(m_{u,v} : G_{u,v}/H \to \mathcal{W}_n\) for small values of \(n\), where \(\{\cdot, \cdot\}_0\) is the Atiyah–Hitchin bracket. For instance, the case \(n = 3\) is shown in Table 3.

To achieve our main results we need the following technical lemma. The proof is sketched in Section 5.

Lemma 4.2. The bracket given by (4.1)–(4.4) is a Poisson bracket.
Table 3. Non-local bracket for $n = 3$

\[
\begin{array}{cccc}
\{c_1, c_2\}_{\text{nl}} & 0 & \frac{c_1 c_2 (c_1 + 1)}{d_2} & \frac{2c_1 c_2 (c_1 + 1)}{d_2} \\
\{c_1, d_1\}_{\text{nl}} & c_1 & 0 & c_1 \\
\{c_1, d_2\}_{\text{nl}} & -c_1(c_1 + 1) & -c_1(c_1 + 1) & -c_1(c_1 + 1) \\
\{c_1, d_3\}_{\text{nl}} & c_1 c_2 & 0 & -c_1 c_2 (c_1 + 1) \\
\{c_2, d_1\}_{\text{nl}} & 0 & \frac{c_1 c_3 d_2}{d_2} & \frac{2c_1 c_3 d_2}{d_2} \\
\{c_2, d_2\}_{\text{nl}} & c_2 & 0 & c_2 \\
\{c_2, d_3\}_{\text{nl}} & -c_2 (c_2 + 1) & -c_2 (c_2 + 1) & -c_2 (c_2 + 1) \\
\{d_1, d_2\}_{\text{nl}} & c_1 d_1 & 0 & c_1 d_1 \\
\{d_1, d_3\}_{\text{nl}} & 0 & 0 & c_1 c_2 d_1 \\
\{d_2, d_3\}_{\text{nl}} & c_2 d_2 & c_2 d_2 & c_2 d_2 \\
\varepsilon_1 & 2 & 2 & 2 \\
\varepsilon_2 & 2 & 1 & 0 \\
\varepsilon_3 & 0 & 0 & 0 \\
\end{array}
\]

Example 4.3. Let $u = v = s_{n-1} \cdots s_1$. Then $\varepsilon = (2, 1, \ldots, 1, 0)$ and the Poisson bracket (4.1)–(4.4) becomes

\[
\{d_i, d_{i+1}\}_{\text{nl}} = d_i c_i, \quad \{c_i, c_{i+1}\}_{\text{nl}} = \frac{(c_i + 1) c_i c_{i+1}}{d_{i+1}},
\]

(4.5)

\[
\{c_{i+1}, d_i\}_{\text{nl}} = \frac{d_i c_i c_{i+1}}{d_{i+1}}, \quad \{c_i, d_i\}_{\text{nl}} = c_i,
\]

\[
\{c_i, d_{i+1}\}_{\text{nl}} = -c_i (c_i + 1).
\]

After the change of variables $\tilde{c}_i = c_i d_i$, the Poisson bracket (4.5) becomes

\[
\{\tilde{c}_i, d_i\}_{\text{nl}} = \tilde{c}_i = \{d_i, d_{i+1}\}_{\text{nl}} = -\{\tilde{c}_i, d_{i+1}\}_{\text{nl}},
\]

which is a linear Poisson bracket for the relativistic Toda lattice (3.7) (see [S]).

In the next three claims we will focus on the particular case $u^{-1} = v = s_{n-1} \cdots s_1$. Then $\varepsilon = (2, 0, \ldots, 0)$ and generic elements $L \in G^{u,v}/H$ are described as the set of Jacobi matrices

\[
L = \begin{pmatrix}
    d_1 & 1 & 0 & \cdots & 0 \\
    c_1 d_1^2 & d_2 + c_1 d_1 & 1 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    0 & \cdots & * & * & 1 \\
    0 & \cdots & 0 & c_{n-1} d_{n-1}^2 & d_n + c_{n-1} d_{n-1}
\end{pmatrix}.
\]

(4.6)
Moreover, the non-local Poisson bracket \((4.1)–(4.4)\) becomes
\[
\{d_i, d_j\}_nl = d_i \prod_{l=i}^{j-1} c_l, \quad \{c_i, c_j\}_nl = \frac{2(c_i + 1)}{d_j} \prod_{l=i}^{j} c_l, \quad \{c_i, d_i\}_nl = c_i,
\]
\[(4.7)\]
\[
\{c_j, d_i\}_nl = \frac{2d_i}{d_j} \prod_{l=i}^{j} c_l, \quad \{c_i, d_j\}_nl = -(c_i + 1) \prod_{l=i}^{j-1} c_l.
\]

By means of the change of variables \(a_i = c_id_i^2, b_i = d_i + c_i-1d_i-1\), it is not hard to show that the Poisson bracket \((4.7)\) becomes
\[
\{a_i, b_i\}_nl = -\{a_i, b_{i+1}\}_nl = a_i.
\]

The latter linear relations \((4.8)\) and the fact that \(\Delta_{[i+1,n]}(\lambda)\) involves only \(a_j, b_j, j = i+k, \ldots, n\).

**Claim 1.** For a generic element \(L \in G^{u,v}/H\), described in the variables \(a_i, b_i\), we have
\[
\{b_i, \Delta_{[i+1,n]}(\lambda)\}_nl = 0, \quad \{a_i, \Delta_{[i+1,n]}(\lambda)\}_nl = a_i \Delta_{[i+2,n]}(\lambda),
\]
\[(4.9)\]

**Proof.** The claim follows from
\[
\Delta_{[i+1,n]}(\lambda) = (\lambda - b_{i+1}) \Delta_{[i+2,n]}(\lambda) - a_{i+1} \Delta_{[i+3,n]}(\lambda),
\]
the linear relations \((4.8)\) and the fact that \(\Delta_{[i+k,n]}(\lambda)\) involves only \(a_j, b_j, j = i+k, \ldots, n\).

**Claim 2.** For \(n = 2\), the Moser map \(m_{u,v} : (G^{u,v}/H, \{\cdot, \cdot\}_nl) \to (\mathcal{W}_2, \{\cdot, \cdot\}_0)\) is a Poisson map.

**Proof.** Let
\[
L_i = \begin{pmatrix} b_i & 1 \\ a_i & b_{i+1} \end{pmatrix}.
\]
Its Weyl function is
\[
m(\lambda) = m(\lambda; L_i) = \frac{\Delta_{[2,2]}(\lambda)}{\Delta_{[1,2]}(\lambda)} = \frac{\lambda - b_{i+1}}{\lambda^2 - (b_i + b_{i+1})\lambda + b_i b_{i+1} - a_i}.
\]
The claim follows, by using the linear Poisson bracket \((4.8)\), from the identities
\[
-\{\Delta_{[1,2]}(\lambda), \Delta_{[2,2]}(\mu)\}_nl = -\{a_i, b_{i+1}\}_nl = a_i
\]
\[
= \frac{\Delta_{[1,2]}(\lambda)\Delta_{[1,1]}(\mu) - \Delta_{[1,2]}(\mu)\Delta_{[1,1]}(\lambda)}{\lambda - \mu} = \Delta_{[2,2]}(\lambda)\Delta_{[2,2]}(\mu),
\]
\[
\{\Delta_{[1,2]}(\lambda), \Delta_{[1,2]}(\mu)\}_nl = \{b_i + b_{i+1}, a_i\}_nl\lambda + \{a_i, b_i + b_{i+1}\}_nl\mu = 0,
\]
\[
\{\Delta_{[2,2]}(\lambda), \Delta_{[2,2]}(\mu)\}_nl = 0.
\]
Claim 3. Let $n = 3$. Then the Moser map $m_{u,v} : (G^{u,v}/H, - \{ \cdot, \cdot \}_{nl}) \to (\mathcal{W}_3, \{ \cdot, \cdot \}_{0})$ is a Poisson map.

Proof. Let

$$L_i = \begin{pmatrix} b_i & 1 & 0 \\ a_i & b_{i+1} & 1 \\ 0 & a_{i+1} & b_{i+2} \end{pmatrix}. $$

Its Weyl function is

$$m(\lambda) = m(\lambda; L_i) = \frac{\Delta_{[2,3]}(\lambda)}{\Delta_{[1,3]}(\lambda)}. $$

From Claims 1 and 2 and $\Delta_{[1,3]}(\lambda) = (\lambda - b_i)\Delta_{[2,3]}(\lambda) - a_i\Delta_{[3,3]}(\lambda)$, we get

$$\{ \Delta_{[1,3]}(\lambda), \Delta_{[2,3]}(\mu) \}_{nl} = -a_i \frac{\Delta_{[2,3]}(\mu)\Delta_{[3,3]}(\lambda) - \Delta_{[2,3]}(\lambda)\Delta_{[3,3]}(\mu)}{\mu - \lambda}. $$

On the other hand, using the Atiyah–Hitchin bracket $(2.3)$, we get

$$\{ \Delta_{[1,3]}(\lambda), \Delta_{[2,3]}(\mu) \}_0 = \frac{\Delta_{[1,3]}(\lambda)\Delta_{[2,3]}(\mu) - \Delta_{[1,3]}(\mu)\Delta_{[2,3]}(\lambda)}{\lambda - \mu}$$

$$= a_i \frac{\Delta_{[2,3]}(\lambda)\Delta_{[3,3]}(\mu) - \Delta_{[2,3]}(\mu)\Delta_{[3,3]}(\lambda)}{\lambda - \mu}. $$

Therefore, $-\{ \Delta_{[1,3]}(\lambda), \Delta_{[2,3]}(\mu) \}_{nl} = \{ \Delta_{[1,3]}(\lambda), \Delta_{[2,3]}(\mu) \}_0$. In a similar way, simple computations show that

$$\{ \Delta_{[1,3]}(\lambda), \Delta_{[1,3]}(\mu) \}_{nl} = \{ \Delta_{[2,3]}(\lambda), \Delta_{[2,3]}(\mu) \}_{nl} = 0,$$

and thus the claim follows. $lacksquare$

Claims 1–3 indicate that we can proceed by induction on $n$ to get the next result.

Proposition 4.4. If $u^{-1} = v = s_{n-1} \cdots s_1 \in S_n$ then the Moser map

$$m_{u,v} : (G^{u,v}/H, - \{ \cdot, \cdot \}_{nl}) \to (\mathcal{W}_n, \{ \cdot, \cdot \}_{0})$$

is a Poisson map.

To obtain the previous result for any pair $(u, v)$ of Coxeter elements of $S_n$, we will need the next proposition. The proof is sketched in Section 6.

Proposition 4.5. The map $a_{u,v}^w : G^{u,v}/H \to G^{u',v'}/H$ is Poisson with respect to the Poisson structures $(4.1)–(4.4)$ on $G^{u,v}/H$ and $G^{u',v'}/H$.

As an immediate consequence we get our main result.

Theorem 4.6. For any pair of Coxeter elements $u, v$, the Moser map

$$m_{u,v} : (G^{u,v}/H, - \{ \cdot, \cdot \}_{nl}) \to (\mathcal{W}_n, \{ \cdot, \cdot \}_{0})$$

is a Poisson map.
Proof. This is a consequence of Propositions 4.4 and 4.5 and the fact that the generalized Bäcklund–Darboux map \( \sigma_{u',v'} \) preserves the Weyl function.

Combining Propositions 2.4 and 3.3 with the fact that the standard Poisson–Lie bracket \( \{\cdot,\cdot\}_{GL_n} \) induces \( \{\cdot,\cdot\}_1 \) on Weyl functions (see Remark 3.2), we obtain the following corollary of Theorem 4.6.

**Corollary 4.7.** The non-local Poisson structure (4.1)–(4.4) and the quadratic Poisson bracket (3.5) are compatible.

**4.2. Problems**

(1) **Other compatible Poisson brackets.** A natural question is to compute explicit Poisson brackets in the \( c_i, d_i \) coordinates corresponding to the other members of the family of compatible Faybusovich–Gekhtman Poisson brackets for \( k = 2, \ldots, n - 1 \). It would be interesting to determine them in a systematic way.

(2) **Poisson relations in factorization parameters.** In \([KZ, (4.1)]\), a general formula for Poisson relations, using factorization parameters \( t_k, a^\gamma \), is induced from the standard Poisson–Lie structure on a simple Lie group \( G \). The Poisson bracket (4.1)–(4.4) can suggest the existence of another Poisson bracket in factorization parameters \( t_k, a^\gamma \) which will be compatible with the quadratic one presented in \([KZ]\).

**5. Proof of Lemma 4.2** The Jacobi property for the bracket (4.1)–(4.4) follows from verifying it for eight possible triples of functions \( d_i, d_{i+r}, d_{i+r+s}; d_i, d_{i+r}, c_{i+r+s}; d_i, c_{i+r}, d_{i+r+s}; c_i, d_{i+r}, d_{i+r+s}, \) etc., where \( r, s \geq 1 \). Each case is verified separately depending on the values of \( r \) and \( s \). We will only provide a justification for the triple \( d_i, d_{i+r}, d_{i+r+s} \). The remaining seven cases can be treated similarly (see \([C]\)).

Consider the expression

\[
\{d_i, \{d_{i+r}, d_{i+r+s}\}_{nl}\}_{nl} + \{d_{i+r}, \{d_{i+r+s}, d_i\}_{nl}\}_{nl} + \{d_{i+r+s}, \{d_i, d_{i+r}\}_{nl}\}_{nl}.
\]

We have to show that (5.1) is equal to zero for any \( r, s \geq 1 \). To this end, we consider the following three cases.

**Case 1: \( r, s > 1 \).** Let us introduce two conditions:

\[
(A1) \quad \varepsilon_{i+1} = \cdots = \varepsilon_{i+r-1} = 0,
\]

\[
(B1) \quad \varepsilon_{i+r+1} = \cdots = \varepsilon_{i+r+s-1} = 0.
\]

**Subcase 1.1:** (A1) is false. We have two subcases.

**Subcase 1.1.1:** (B1) is false. Then it follows immediately that (5.1) is equal to zero.
SUBCASE 1.1.2: (B1) is true. Then (5.1) is equal to
\[
\left\{ d_i, d_{i+r} \prod_{j=i+r}^{i+r+s-1} c_j \right\}_{nl} = \left\{ d_i, d_{i+r} \right\}_{nl} \prod_{j=i+r}^{i+r+s-1} c_j + d_{i+r} \left\{ d_i, \prod_{j=i+r}^{i+r+s-1} c_j \right\}_{nl} = 0.
\]

SUBCASE 1.2: (A1) is true. We have three subcases.

SUBCASE 1.2.1: (B1) is false. Then (5.1) is equal to
\[
\left\{ d_{i+r+s}, d_i \prod_{j=i}^{i+r-1} c_j \right\}_{nl} = d_i \left\{ d_{i+r+s}, \prod_{j=i}^{i+r-1} c_j \right\}_{nl} = 0.
\]

The last equality follows from \( \left\{ d_{i+r+s}, c_i \right\}_{nl} = 0, l = i, \ldots, i + r - 1, \) since (B1) is false.

SUBCASE 1.2.2: Condition (B1) is true and \( \varepsilon_{i+r} \neq 0. \) Then (5.1) is equal to
\[
(5.2) \quad \left\{ d_i, d_{i+r} \right\}_{nl} \prod_{j=i+r}^{i+r+s-1} c_j + d_{i+r} \left\{ d_i, \prod_{j=i+r}^{i+r+s-1} c_j \right\}_{nl} + d_i \left\{ d_{i+r+s}, \prod_{j=i}^{i+r-1} c_j \right\}_{nl}.
\]

Since \( \varepsilon_{i+r} \neq 0, \) we have
\[
\left\{ d_i, c_i \right\}_{nl} = 0, \quad l = i + r + 1, \ldots, i + r + s - 1,
\]
\[
\left\{ d_{i+r+s}, c_i \right\}_{nl} = 0, \quad l = i, \ldots, i + r - 2.
\]

Therefore, (5.2) is equal to
\[
d_i \prod_{j=i}^{i+r+s-1} c_j + (\varepsilon_{i+r} - 2)d_i \prod_{j=i}^{i+r+s-1} c_j + \left\{ d_{i+r+s}, c_{i+r-1} \right\}_{nl} d_i \prod_{j=i}^{i+r-2} c_j,
\]
and hence, equal to zero for \( \varepsilon_{i+r} = 1, 2, \) since
\[
\left\{ d_{i+r+s}, c_{i+r-1} \right\}_{nl} = \begin{cases} -\prod_{j=i+r-1}^{i+r+s-1} c_j & \text{if } \varepsilon_{i+r} = 2, \\ 0 & \text{if } \varepsilon_{i+r} = 1. \end{cases}
\]

SUBCASE 1.2.3: (B1) is true and \( \varepsilon_{i+r} = 0. \) Then (5.1) is equal to
\[
(5.3) \quad d_i \prod_{j=i}^{i+r+s-1} c_j + d_{i+r} \left\{ d_i, \prod_{j=i+r}^{i+r+s-1} c_j \right\} - d_i \prod_{j=i+r}^{i+r+s-1} c_j \left\{ d_{i+r}, \prod_{j=i}^{i+r-1} c_j \right\}
\]
\[
- d_i \prod_{l=i}^{i+r-1} c_i \left\{ d_{i+r}, \prod_{j=i+r}^{i+r+s-1} c_j \right\} + d_i \left\{ d_{i+r+s}, \prod_{j=i}^{i+r-1} c_j \right\}.
\]

Using the identities
\[
\left\{ d_i, \prod_{j=i+r}^{i+r+s-1} c_j \right\}_{nl} = -2d_i \prod_{j=i+r}^{i+r+s-1} c_j \sum_{p=i+r}^{i+r+s-1} \frac{1}{d_p} \prod_{j=i}^{p-1} c_j,
\]
\[
\begin{align*}
\{d_{i+r}, \prod_{j=i}^{i+r-1} c_j\}_{nl} &= \prod_{j=i}^{i+r-1} c_j \left( \sum_{p=i}^{i+r-2} (c_p + 1) \prod_{j=p+1}^{i+r-1} c_j + c_{i+r-1} + 1 \right), \\
\{d_{i+r}, \prod_{j=i+r}^{i+r+s-1} c_j\}_{nl} &= -\prod_{j=i+r}^{i+r+s-1} c_j \left( 2d_{i+r} \sum_{p=i+r+2}^{i+r+s-1} \prod_{j=i+r}^{p-1} \prod_{j=p+1}^{i+r+s-1} c_j + \frac{2c_{i+r}d_{i+r}}{d_{i+r+1}} + 1 \right), \\
\{d_{i+r+s}, \prod_{j=i}^{i+r-1} c_j\}_{nl} &= \prod_{j=i}^{i+r-1} c_j \sum_{p=i}^{i+r-1} (c_p + 1) \prod_{j=p+1}^{i+r+s-1} c_j,
\end{align*}
\]

we find that (5.3) is equal to zero.

**Case 2: r = 1, s > 1.** In this case, (5.1) is equal to

(5.4) \( \{d_i, \{d_{i+1}, d_{i+1+s}\}_{nl}\} + \{d_{i+1}, \{d_{i+1+s}, d_i\}_{nl}\} + \{d_{i+1+s}, c_id_i\}_{nl} \).

Let us introduce the condition

(B3) \( \varepsilon_{i+2} = \cdots = \varepsilon_{i+s} = 0 \).

**Subcase 2.1:** (B3) is false. Then \( \{d_j, d_{i+1+s}\}_{nl} = 0, j = i, i + 1 \) and \( \{c_i, d_{i+1+s}\}_{nl} = 0 \). Therefore, (5.4) is equal to zero.

**Subcase 2.2:** (B3) is true. Then (5.4) is equal to

(5.5) \[ c_id_i \prod_{j=i+1}^{i+s} c_j + d_{i+1} \sum_{k=i+1}^{i+s} \prod_{j\neq k}^{j=i+1} c_j \{d_i, c_k\}_{nl} + \{d_{i+1}, \{d_{i+1+s}, d_i\}_{nl}\}_{nl} \\
+ c_i \{d_{i+1+s}, d_i\}_{nl} + d_i \{d_{i+1+s}, c_i\}_{nl} \]

**Subcase 2.2.1:** \( \varepsilon_{i+1} \neq 0 \). Then \( \{d_i, d_{i+1+s}\}_{nl} = 0, \{d_i, c_k\}_{nl} = 0 \) for \( k = i + 2, \ldots, i + s \), and the relations

\[
\begin{align*}
\{d_i, c_{i+1}\}_{nl} &= \begin{cases} 
0 & \text{if } \varepsilon_{i+1} = 2, \\
-\frac{c_ic_{i+1}d_i}{d_{i+1}} & \text{if } \varepsilon_{i+1} = 1,
\end{cases} \\
\{d_{i+1+s}, c_i\}_{nl} &= \begin{cases} 
-\prod_{j=i}^{i+s} c_j & \text{if } \varepsilon_{i+1} = 2, \\
0 & \text{if } \varepsilon_{i+1} = 1,
\end{cases}
\end{align*}
\]

imply that (5.5) is equal to zero.

**Subcase 2.2.2:** \( \varepsilon_{i+1} = 0 \). Then (5.5) is equal to

\[ 2d_i \prod_{j=i}^{i+s} c_j + d_{i+1} \sum_{k=i+1}^{i+s} \prod_{j\neq k}^{j=i+1} c_j \{d_i, c_k\}_{nl} + c_id_i \prod_{j=i}^{i+s} c_j - d_i \sum_{k=i}^{i+s} \prod_{j\neq k}^{j=i} c_j \{d_{i+1}, c_k\}_{nl}. \]
We see that the last expression is equal to zero, since
\[
\sum_{k=i+1}^{i+s} \prod_{j=i+1}^{i+s} c_j \{d_i, c_k\}_n l = - \frac{d_i}{d_{i+1}} \prod_{j=i+1}^{i+s} c_j + \sum_{k=i+2}^{i+s} \prod_{j=i}^{i+s} c_j \{d_i, c_k\}_n l,
\]
\[
\sum_{k=i}^{i+s} \prod_{j=i}^{i+s} c_j \{d_{i+1}, c_k\}_n l = -(c_i + 1) \prod_{j=i}^{i+s} c_j - \sum_{k=i+2}^{i+s} \prod_{j=i}^{i+s} c_j \{d_{i+1}, c_k\}_n l.
\]

**Case 3:** \( r = s = 1 \). In this case, (5.1) is equal to
\[(5.6) \quad d_{i+1} \{d_i, c_{i+1}\}_n l + c_i c_{i+1} d_i + \{d_{i+1}, \{d_{i+2}, d_i\}_n l\}_n l + c_i \{d_{i+2}, d_i\}_n l + d_i \{d_{i+2}, c_i\}_n l.
\]
We have two subcases: \( \varepsilon_{i+1} \neq 0 \) and \( \varepsilon_{i+1} = 0 \). For both, a simple computation shows that (5.6) is zero. ■

**Remark 5.1.** We will need additional conditions to prove the Jacobi identity for some of the remaining seven triples. For example, for the triple \( c_i, c_{i+r}, c_{i+r+s}, \) besides conditions (A1) and (B1), we introduce
(A2) \( \varepsilon_{i+1} = 2, \varepsilon_{i+2} = \cdots = \varepsilon_{i+r-1} = 0, \)
(B2) \( \varepsilon_{i+r+1} = 2, \varepsilon_{i+r+2} = \cdots = \varepsilon_{i+r+s-1} = 0. \)
Detailed computations can be found in [C].

**6. Proof of Proposition 4.5.** Because of the definition of \( \sigma_{u,v}^{u',v'} \), it is enough to verify that each elementary transformation listed in Table 2 is a Poisson map.

We outline the proof for the first row of Table 2 i.e. the map that corresponds to the mutation \( \varepsilon \xrightarrow{(0,i)} \varepsilon' \), where \( \varepsilon_i = 2, \varepsilon_{i+1} = 0, \varepsilon'_i = 1, \varepsilon'_{i+1} = 1 \) and the remaining entries of the \( n \)-tuples \( \varepsilon, \varepsilon' \) are equal. To this end, we have to prove relations (4.1)–(4.4) with respect to the variables \( c'_j, d'_j \) and the \( n \)-tuple \( \varepsilon' \). We will only provide justification for the relation (4.2). Similar computations, though lengthy, can be done for the brackets \( \{d'_j, d_{j+k}'\}_n l \), \( \{c'_j, d_{j+k}'\}_n l \) and \( \{c'_j, c_{j+k}'\}_n l \).

**Case \( \{c'_j, d_{j+k}'\}_n l \).** We have three subcases.

**Subcase 1:** \( k = 0 \). Depending on the value of \( j \), we have four subcases.

**Subcase 1.1:** \( j = i - 1 \). Then \( c'_j = c_{i-1}(1 + c_i) \) and \( d'_j = d_{i-1} \). We have
\[
\{c'_j, d'_j\}_n l = (1 + c_i)\{c_{i-1}, d_{i-1}\}_n l = c_{i-1}(1 + c_i) = c'_j.
\]
Subcase 1.2: \(j = i\). Then \(c'_j = \frac{c_i d_{i+1}}{d_i (1 + c_i)^2}\) and \(d'_j = d_i (1 + c_i)\). We have

\[
\{c'_j, d'_j\}_{nl} = \frac{c_i (1 + c_i) \{d_{i+1}, d_i\}_{nl} + c_i d_i \{d_{i+1}, c_i\}_{nl} + d_{i+1} \{c_i, d_i\}_{nl}}{d_i (1 + c_i)^2} = \frac{c_i d_i}{d_i (1 + c_i)^2} = c'_j.
\]

Subcase 1.3: \(j = i + 1\). Then \(c'_j = c_{i+1} (1 + c_i)\) and \(d'_j = \frac{d_{i+1}}{1 + c_i}\). We have

\[
\{c'_j, d'_j\}_{nl} = \{c_{i+1}, d_{i+1}\}_{nl} - \frac{d_{i+1}}{1 + c_i} \{c_{i+1}, c_i\}_{nl} + \frac{c_{i+1}}{1 + c_i} \{c_i, d_{i+1}\}_{nl}
\]

\[
\varepsilon_{i+1} = 0 \quad \text{c}_{i+1} (1 + c_i) = c'_j.
\]

Subcase 1.4: \(j < i - 1\) or \(j > i + 1\). Then \(\{c'_j, d'_j\}_{nl} = \{c_j, d_j\}_{nl} = c_j = c'_j\). Therefore, for any \(j = 1, \ldots, n - 1\), we have \(\{c'_j, d'_j\}_{nl} = c'_j\).

Subcase 2: \(k = 1\). Depending on the value of \(j\), we have four subcases.

Subcase 2.1: \(j = i - 1\). Then \(c'_j = c_{i-1} (1 + c_i)\) and \(d'_{j+1} = d_i (1 + c_i)\). We have

\[
\{c'_j, d'_{j+1}\}_{nl} = \frac{1}{d_i (1 + c_i)^2} \left[ \{c_i, d_{i+1}\}_{nl} - \frac{c_i}{d_i (1 + c_i)^2} \{d_i (1 + c_i), d_{i+1}\}_{nl} \right.
\]

\[
\left. + \frac{c_i d_{i+1}}{d_i (1 + c_i)^2} \{d_i, c_i\}_{nl} \right] = -\frac{c_i d_{i+1}}{d_i (1 + c_i)^2} \left( 1 + \frac{c_i d_{i+1}}{d_i (1 + c_i)^2} \right) = -c'_j (c'_j + 1).
\]

Subcase 2.2: \(j = i\). Then \(c'_j = \frac{c_i d_{i+1}}{d_i (1 + c_i)^2}\) and \(d'_{j+1} = \frac{d_{i+1}}{1 + c_i}\). We have

\[
\{c'_j, d'_{j+1}\}_{nl} = \frac{1}{d_i (1 + c_i)^2} \left[ \{c_i, d_{i+1}\}_{nl} - \frac{c_i}{d_i (1 + c_i)^2} \{d_i (1 + c_i), d_{i+1}\}_{nl} \right.
\]

\[
\left. + \frac{c_i d_{i+1}}{d_i (1 + c_i)^2} \{d_i, c_i\}_{nl} \right] = -\frac{c_i d_{i+1}}{d_i (1 + c_i)^2} \left( 1 + \frac{c_i d_{i+1}}{d_i (1 + c_i)^2} \right) = -c'_j (c'_j + 1).
\]

Subcase 2.3: \(j = i + 1\). Then \(c'_j = c_{i+1} (1 + c_i)\) and \(d'_{j+1} = d_{i+2}\). We have

\[
\{c'_j, d'_{j+1}\}_{nl} = \frac{c_{i+1} (1 + c_i) (1 + c_{i+1}) (1 + c_i)}{d_i (1 + c_i)^2} = -c'_j (c'_j + 1).
\]

Subcase 2.4: \(j < i - 1\) or \(j > i + 1\). Then \(c'_j = c_j\) and \(d'_{j+1} = d_{j+1}\). Thus, \(\{c'_j, d'_{j+1}\}_{nl} = -c'_j (c'_j + 1)\).

Therefore, for any \(j = 1, \ldots, n - 1\), we have \(\{c'_j, d'_{j+1}\}_{nl} = -c'_j (c'_j + 1)\).

Subcase 3: \(k > 1\). Depending on the value of \(j\), we have five subcases.

Subcase 3.1: \(j > i + 1\). Then \(c'_j = c_j\) and \(d'_{j+k} = d_{j+k}\). It is straightforward that \(\{c'_j, d'_{j+k}\}_{nl}\) satisfies, depending on the \(n\)-tuple \(\varepsilon'\), one of the last two cases of (4.2) or is equal to zero.
Subcase 3.2: \( j = i + 1 \). Then \( c_j' = c_{i+1}(1 + c_i) \) and \( d'_{j+k} = d_{i+k+1} \). We have

\[
\{c_j', d'_{j+k}\}_{nl}^{\varepsilon_{i+1}=0} \equiv \begin{cases} 
-(1 + c_i)(c_{i+1}(c_i + 1) + 1) \prod_{l=i+1}^{i+k} c_l 
& \text{if } \varepsilon_{i+2} = \cdots = \varepsilon_{i+k} = 0, \\
(1 + c_i) \prod_{l=i+1}^{i+k} c_l 
& \text{if } \varepsilon_{i+2} = 2, \varepsilon_{i+3} = \cdots = \varepsilon_{i+k} = 0, \\
nl 
& \text{otherwise,}
\end{cases}
\]

\[
= \begin{cases} 
-(c_j' + 1) \prod_{l=j}^{j+k-1} c_l' 
& \text{if } \varepsilon_{j+1} = \cdots = \varepsilon_{j+k-1} = 0, \\
\prod_{l=j}^{j+k-1} c_l' 
& \text{if } \varepsilon_{j+1} = 2, \varepsilon_{j+2} = \cdots = \varepsilon_{j+k-1} = 0, \\
nl 
& \text{otherwise.}
\end{cases}
\]

which is consistent with (4.2) because of \( \varepsilon_{i+1}' = 1 \).

Subcase 3.3: \( j = i \). Then \( c_j' = \frac{c_i d_{i+1}}{d_i (1 + c_i)^2} \) and \( d'_{j+k} = d_{i+k} \). We have

\[
\{c_j', d'_{j+k}\}_{nl}^{\varepsilon_{i+1}=0} \equiv \begin{cases} 
- \frac{d_{i+1}}{d_i (1 + c_i)^2} (1 - c_i - 1 + c_i) \prod_{l=i}^{i+k-1} c_l 
& \text{if } \varepsilon_{i+2} = \cdots = \varepsilon_{i+k-1} = 0, \\
= 0, \\
& \text{otherwise,}
\end{cases}
\]

Subcase 3.4: \( j = i - 1 \). Then \( \{c_j', d'_{j+k}\}_{nl} = 0 \), which is consistent with (4.2) because of \( \varepsilon_{i}' = 1 \). In fact:

(a) If \( k = 2 \) then

\[
\{c_j', d'_{j+k}\}_{nl} = \left\{ c_{i-1}(1 + c_i), \frac{d_{i+1}}{1 + c_i} \right\}_{nl}^{\varepsilon_{i+2}=0} \equiv 0.
\]

(b) If \( k > 2 \) then \( c_j' = c_{i-1}(1 + c_i) \) and \( d'_{j+k} = d_{i-1+k} \). We have

\[
\{c_j', d'_{j+k}\}_{nl} = c_{i-1}\{c_i, d_{i-1+k}\}_{nl} + (1 + c_i)\{c_{i-1}, d_{i-1+k}\}_{nl}^{\varepsilon_{i+2}=0} \equiv 0.
\]

Subcase 3.5: \( j < i - 1 \). Then we have three possibilities.

(a) If \( j + k = i \) then \( c_j' = c_j \) and \( d'_{j+k} = d_i(1 + c_i) \). We have

\[
\{c_j', d'_{j+k}\}_{nl}^{\varepsilon_{i+2}=0} \equiv \begin{cases} 
(1 + c_i) \prod_{l=j}^{i-1} c_l & \text{if } \varepsilon_{j+1} = \cdots = \varepsilon_{i-1} = 0, \\
= (1 + c_i) \prod_{l=j}^{i-1} c_l & \text{if } \varepsilon_{j+1} = 2, \varepsilon_{j+2} = \cdots = \varepsilon_{i-1} = 0, \\
nl 
& \text{otherwise},
\end{cases}
\]

\[
= \begin{cases} 
-(c_j' + 1) \prod_{l=j}^{j+k-1} c_l' 
& \text{if } \varepsilon_{j+1} = \cdots = \varepsilon_{j+k-1} = 0, \\
\prod_{l=j}^{j+k-1} c_l' 
& \text{if } \varepsilon_{j+1} = 2, \varepsilon_{j+2} = \cdots = \varepsilon_{j+k-1} = 0, \\
nl 
& \text{otherwise.}
\end{cases}
\]
(b) If \( j + k = i + 1 \) then \( c_j' = c_j \) and \( d_{j+k}' = \frac{d_{i+1}}{1+c_i} \). We have

\[
\{ c_j', d_{j+k}' \}_{\text{nl}} = \frac{\{ c_j, d_{i+1} \}_{\text{nl}}}{1 + c_i} - \frac{d_{i+1}}{(1 + c_i)^2} \{ c_j, c_i \}_{\text{nl}} \varepsilon_i = 2 \leq 0,
\]

which is consistent with (4.2) because of \( \varepsilon_i' = 1 \).

(c) If \( j + k < i - 1 \) or \( j + k > i + 1 \) then \( c_j' = c_j \) and \( d_{j+k}' = d_{j+k} \). It is straightforward that \( \{ c_j', d_{j+k}' \}_{\text{nl}} \) satisfies, depending on the \( n \)-tuple \( \varepsilon' \), one of the last two cases of (4.2) or is equal to zero.

Summarizing, the bracket \( \{ c_j', d_{j+k}' \}_{\text{nl}} \) satisfies the equation (4.2) with respect to variables \( c_j', d_j' \) and the \( n \)-tuple \( \varepsilon' \). A similar reasoning can be applied to the remaining elementary transformations of Table 2.

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