Tensor product decompositions for $\mathfrak{su}(3)$ of an irreducible representation with itself and with its conjugate

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Abstract. In this article a method to obtain the decomposition, including multiplicities, of an $\mathfrak{su}(3)$ irreducible representation with itself and with its conjugate, is provided. The decomposition of such tensor products plays an important role in physics. The objective is to provide a simple method for performing these decompositions in addition to showing how this method can be used to derive several interesting facts about the decompositions, particularly in regards to multiplicities.

1. Introduction
The decomposition of tensor products for $\mathfrak{su}(3)$ are important in many parts of physics, for example in elementary particle physics [1], nuclear physics [2] and in the analysis of quantum interferometers [3].

The Clebsch-Gordan series for the tensor product of two $\mathfrak{su}(2)$ representations is well documented and has been used extensively (see for example [4]). The literature on the decompositions of two irreducible representations (irreps) of $\mathfrak{su}(3)$ is not as familiar, in part due to multiplicities in the decompositions, arising from the presence of weight multiplicities in $\mathfrak{su}(3)$ representations, but also because closed form expressions are often cumbersome to use.

A lot of attention has been devoted to the $\mathfrak{su}(3)$ decomposition problem; see for instance work by Biedenharn et. al. (e.g. [5]) as well as King (e.g. [6]), but also several others [7]. In addition, a range of computer programs have been developed to compute various types of tensor product decompositions [8]. Some of these methods are used mainly on individual tensor products, making it difficult to see the general structure, or are in other ways not suitable for this particular problem. The method presented in this article takes full advantage of the geometric features of the weight diagram of an irrep and its conjugate, thus making it less complicated and more transparent than the corresponding restrictions of more general results. The methods given in Ref. [9], [10] and [11] are similar to the one discussed here, but either do not provide multiplicities or are only valid for certain tensor products for $\mathfrak{su}(3)$, not including those of interest here.

In a previous paper [12] a simple geometric method for decomposing tensor products for $\mathfrak{su}(3)$ was given together with several applications. Some of the same techniques also appear in Ref. [13]. Here, this geometric method is discussed in the context of the tensor product of an
Figure 1. The boundary of the tensor diagram of \((p, q) \otimes (p, q)\) when \(p \geq q\). The values refer to the number of vertices along each line, and this number will be called the length of the line.

irreducible representation in \(\mathfrak{su}(3)\) with itself, or with its contragredient. These tensor products are especially important because they occur when a particle interacts with an identical particle or antiparticle as well as in the action of \(\mathfrak{su}(3)\) on the density matrix of a system for which states carry representations of \(\mathfrak{su}(3)\). An application to the problem of tomographic reconstructions of density matrices of \(\mathfrak{su}(3)\) systems can be found in Ref. [14].

Although many of the aspects of these tensor products can be deduced from the study of the general case in Ref. [12], they involve, in addition to important applications, several interesting characteristics that justifies studying them further. Contrary to the general case, the symmetry of the highest weights simplifies the analysis, thus allowing for more compact results as well as for additional conclusions to be drawn.

Before describing the method, some notation needs to be established. Let \(\mathfrak{g} = \mathfrak{su}(3)\) and let \((p, q)\) be an irrep with highest weight \(p\lambda_1 + q\lambda_2\), where \(\lambda_i, i = 1, 2\) are the fundamental weights and \(p, q \in \mathbb{Z}_{\geq 0}\). In this article the tensor products \((p, q) \otimes (p, q)\) and \((p, q) \otimes (q, p)\) will be considered.

We are interested in the decomposition of these tensor products into direct sums of irreps. These, as any other tensor product decompositions, can be described by tensor diagrams [12]. A tensor diagram is a collection of vertices in a triangular lattice (a weight lattice) with each vertex representing the irrep with the corresponding highest weight. Each vertex is assigned a multiplicity according to the number of copies of the relevant irrep that appear in the direct sum decomposition of the tensor product. The irreps not appearing in the direct sum can be thought of as having multiplicity zero, or they can be thought of as not being part of the tensor diagram. Moreover, \(\alpha_i, i = 1, 2\) are the simple roots of \(\mathfrak{su}(3)\) and the \((\alpha_1 + \alpha_2)\)-diagonals refer to the line segments in the tensor diagram, in the direction of \(\alpha_1 + \alpha_2\) (parallel to the dashed line in Figure 1). The centre diagonal is the \((\alpha_1 + \alpha_2)\)-diagonal with the highest vertex as an endpoint.

2. The Tensor Product \((p, q) \otimes (p, q)\)

We are now in a position to describe the decomposition method for \((p, q) \otimes (p, q)\), which involves determining its tensor diagram. W.l.o.g. assume \(p \geq q\); if \(p < q\) it is possible to interchange the coordinates by reflecting all diagrams in the centre diagonal.

(i) The boundary of the tensor diagram of \((p, q) \otimes (p, q)\) is given in Figure 1. All vertices on or inside this boundary have positive multiplicity, and all vertices outside it have multiplicity zero.

(ii) Now find the multiplicities along each \((\alpha_1 + \alpha_2)\)-diagonal as follows: The top vertex of each diagonal has multiplicity one. Moving downwards along the diagonal, the multiplicities then increase by one at a time, until they plateau at some upper bound (described below). Near the end, the multiplicities start decreasing by one at a time, so that they are back at one again.
Figure 2. The tensor decomposition of \((5,2) \otimes (5,2)\). Part (a) shows the boundary adapted from Figure 1, part (b) shows the tensor diagram and part (c) shows the tensor diagram with the irreps labelled by their highest weights.

(iii) The upper bounds along the diagonals behave in a similar way; the leftmost diagonal has maximum one, the next one two and so on until it reaches the upper bound for the entire diagram, which is \(q + 1\). Near the end, the upper bounds decrease by one at a time, so that they are back at one again for the final diagonal. The resulting diagram is the tensor diagram, and it determines the direct sum decomposition.

Remarks: 1. This can be proved using methods similar to those used in proofs in Ref. [12].

2. In the special case \((p,0) \otimes (p,0)\) i.e., where \(q = 0\), this tensor diagram reduces to a horizontal straight line, of length \(p + 1\).

In order to illustrate the procedure, let us consider the tensor product \((5,2) \otimes (5,2)\). The boundary of the tensor diagram for this example is given in part (a) of Figure 2. Using the rules for multiplicities in (ii) and (iii) above gives the tensor diagram in part (b) of Figure 2. The highest highest weight is given by \((2p,2q)\) which in this example is \((10,4)\). Part (c) of Figure 2 shows the tensor diagram with the vertices labelled. In conclusion, the tensor product decomposition is:

\[
(5,2) \otimes (5,2) = (10,4) \oplus (8,5) \oplus (6,6) \oplus 2(9,3) \oplus (11,2) \oplus 2(7,4) \oplus \\
(4,7) \oplus 2(5,5) \oplus 3(8,2) \oplus 2(8) \oplus 3(6,3) \oplus 2(3,6) \oplus 2(10,1) \oplus \\
3(4,4) \oplus (12,0) \oplus 2(5,2) \oplus 3(2,5) \oplus 2(7,1) \oplus 2(1,7) \oplus 2(3,3) \oplus \\
(9,0) \oplus (0,9) \oplus (4,1) \oplus 2(1,4) \oplus (6,0) \oplus (0,6) \oplus (2,2) \oplus (0,3)
\]

In a notation commonly found in physics where representations are labelled by their dimension, we find:

\[
81 \otimes 81 = 440 \oplus 405 \oplus 343 \oplus 2 \cdot 280 \oplus 270 \oplus 2 \cdot 260 \oplus 260 \oplus 2 \cdot 216 \oplus \\
3 \cdot 162 \oplus 162 \oplus 3 \cdot 154 \oplus 2 \cdot 154 \oplus 2 \cdot 143 \oplus 3 \cdot 125 \oplus 91 \oplus 2 \cdot 81 \oplus \\
3 \cdot 81 \oplus 2 \cdot 80 \oplus 2 \cdot 80 \oplus 2 \cdot 64 \oplus 55 \oplus 55 \oplus 35 \oplus 2 \cdot 35 \oplus 28 \oplus 28 \oplus 27 \oplus 10
\]

Further examples are provided in Table A1.

We now return to the general discussion. From the decomposition method above it is clear that the upper bound for the multiplicities in the tensor diagram of \((p,q) \otimes (p,q)\) is \(q + 1\). Calculations similar to those in Ref. [12] show that this upper bound is always attained, i.e., there is always at least one vertex with this multiplicity. It also follows that the vertices with maximum multiplicity are arranged in a horizontal line of length \(p - q + 1\). This is illustrated in Figure 3.
Figure 3. Tensor diagram of $(p, q) \otimes (p, q)$. The lined region is where the diagonals are longest. The multiplicities are stable along each of these indicated lines, and the thicker dashed line is the line where multiplicities are maximal.

Note also that the length of the centre diagonal is $2q + 1$, so the maximum multiplicity must occur $q$ vertices below the highest vertex (which is at $(2p, 2q)$), i.e., at $(2p - q, q)$. The lengths and directions of the lines of maximum multiplicity are already known, so the complete list of irreps with maximum multiplicity can be found. It is:

$$(2p - q, q), (2p - q - 2, q + 1), \ldots, (q, p) \text{ (} p - q + 1 \text{ vertices)}$$

With a similar argument, using the decomposition method it can be shown that the irrep with lowest highest weight is $(0, p - q)$. It is also clear from the decomposition method that the vertices of multiplicity one are exactly those on the boundary of the convex hull of the tensor diagram. In particular the irrep with lowest highest weight has multiplicity one.

Note that although all vertices on the boundary have multiplicity one, this does not imply that the first/last vertex on any given line has multiplicity one, as these vertices may not lie on the boundary of the convex hull. For example, there are many horizontal lines where the first vertex has multiplicity two. This is the case in the example shown in Figure 2. See also Ref. [12].

Figure 3 shows the tensor diagram of $(p, q) \otimes (p, q)$. The thick dashed line indicates the line where the multiplicities are maximal. This falls within the lined region, which is where the diagonals are longest. In this region, the multiplicities are stable along each of the horizontal lines. The lowest highest weight is in agreement with the one stated above.

Let $\lambda = p\lambda_1 + q\lambda_2$ i.e., $\lambda$ is the highest weight in $(p, q)$. Consider the weight diagram of $(p, q)$ shifted by $\lambda$. The purpose of this is to ensure that the highest vertex of the weight diagram is shifted to coincide with the highest vertex of the tensor diagram, in order to place the tensor diagram in the context of shifted weight diagram. It is known that the tensor diagram of $(p, q) \otimes (p, q)$ is contained in this shifted weight diagram of $(p, q)$.

Near the upper right corner, the vertices in the tensor diagram have the same multiplicities as those in the shifted weight diagram of $(p, q)$. The region where the multiplicities coincide, is illustrated in Figure 4. Inside or on the boundary of this region, the multiplicities are the same for both diagrams, and outside this region the multiplicities are strictly less for the tensor diagram. This can be proved using the techniques developed in Ref. [12].

Note that a result similar to this was proved in Ref. [12]. However, in the general case only part of this region was given, so that outside, the tensor multiplicities were less than or equal to the weight multiplicities. In the case of $(p, q) \otimes (p, q)$ we are able to give the exact region.

It is interesting to see the tensor diagram of $(p, q) \otimes (p, q)$ in the context of the weight diagram of $(p, q)$ shifted by $\lambda$, and in particular how the regions where the multiplicities are maximal are
Figure 5. Tensor diagram of $(p, q) \otimes (p, q)$ in the context of the shifted weight diagram of $(p, q)$. The thin dashed lines show the boundary of the shifted weight diagram of $(p, q)$, and the lines forming the triangle where the multiplicity is largest. The thick dashed line indicates the line where the multiplicities are maximal in the tensor diagram. The shaded region shows where the multiplicities of the tensor and weight diagrams coincide.

Figure 6. The boundary of the tensor diagram of $(p, q) \otimes (q, p)$ when $p \geq q$. The dashed line is the centre diagonal, and it has length $2q + 1$.

related (see Figure 5). As expected, the tensor diagram is contained within the shifted weight diagram. The shaded region shows where the multiplicities of the tensor and weight diagrams coincide, as discussed above. Although the tensor diagram is larger than this region, the tensor multiplicities are strictly less than the weight multiplicities outside the shaded area.

The maximum multiplicity is $q + 1$ in both the weight and tensor diagrams. The centre triangle formed by the thin dashed lines show the region where the multiplicities are maximal in the weight diagram. Figure 5 shows that the line of maximum multiplicities in the tensor diagram is one side of this triangle.

3. The Tensor Product $(p, q) \otimes (q, p)$

Now consider the decomposition of the tensor product $(p, q) \otimes (q, p)$. To ease notation assume w.l.o.g. that $p \geq q$ (otherwise interchange the two irreps). Many of the same properties that were considered for $(p, q) \otimes (p, q)$ in Section 2 will be considered here but now for $(p, q) \otimes (q, p)$.

First the decomposition method for $(p, q) \otimes (p, q)$ is adjusted to fit $(p, q) \otimes (q, p)$.

i) The boundary of the tensor diagram of $(p, q) \otimes (q, p)$ is given in Figure 6. All vertices on or inside this boundary have positive multiplicity, and all vertices outside it have multiplicity zero.

ii) and iii) The multiplicities along each $(\alpha_1 + \alpha_2)$-diagonal satisfy the same properties as in the method in Section 2.

Remarks: 1. This tensor diagram is symmetric about the centre $(\alpha_1 + \alpha_2)$-line. This is to be expected because interchanging the first and second coordinates in $(p, q) \otimes (q, p)$ would result in the same tensor product.

2. In the special case $(p, 0) \otimes (0, p)$ i.e., where $q = 0$, this tensor diagram reduces to a straight line in the $(\alpha_1 + \alpha_2)$-direction, of length $p + 1$. See Ref. [14] for an application of decompositions of these particular tensor products.

As an example, consider the tensor product $(5, 2) \otimes (2, 5)$. (Cf. the example of $(5, 2) \otimes (5, 2)$ in Section 2.) Following the procedure with this example, Figure 6 becomes part (a) of Figure 7. Using the rules for multiplicities in (ii) and (iii) above gives the tensor diagram in part (b) of Figure 7. The highest highest weight is given by $(p + q, p + q)$ which in this example is $(7, 7)$. Part (c) of Figure 7 shows the tensor diagram with the vertices labelled. In conclusion, the
Figure 7. The tensor decomposition of $(5, 2) \otimes (2, 5)$. Part (a) shows the boundary adapted from Figure 6, part (b) shows the tensor diagram and part (c) shows the tensor diagram with the irreps labelled by their highest weights.

The tensor product decomposition is:

$$(5, 2) \otimes (2, 5) = (7, 7) \oplus (8, 5) \oplus (5, 8) \oplus 2(6, 6) \oplus (9, 3) \oplus (3, 9) \oplus 2(4, 7) \oplus 3(5, 5) \oplus (8, 2) \oplus 2(2, 8) \oplus 2(6, 3) \oplus 2(3, 6) \oplus 3(4, 4) \oplus 2(5, 2) \oplus 2(2, 5) \oplus (7, 1) \oplus (1, 7) \oplus (3, 3) \oplus 2(4, 1) \oplus 2(1, 4) \oplus (6, 0) \oplus (0, 6) \oplus 3(2, 2) \oplus (3, 0) \oplus (0, 3) \oplus 2(1, 1) \oplus (0, 0)$$

In dimension notation this becomes:

$$81 \otimes 81 = 512 \oplus 405 \oplus 405 \oplus 2 \cdot 343 \oplus 280 \oplus 280 \oplus 2 \cdot 260 \oplus 2 \cdot 260 \oplus 3 \cdot 216 \oplus 162 \oplus 162 \oplus 2 \cdot 154 \oplus 3 \cdot 154 \oplus 2 \cdot 154 \oplus 2 \cdot 154 \oplus 2 \cdot 125 \oplus 2 \cdot 125 \oplus 2 \cdot 81 \oplus 2 \cdot 81 \oplus 80 \oplus 80 \oplus 3 \cdot 64 \oplus 2 \cdot 35 \oplus 2 \cdot 35 \oplus 28 \oplus 28 \oplus 28 \oplus 3 \cdot 27 \oplus 10 \oplus 10 \oplus 2 \cdot 8 \oplus 1$$

More examples can be found in Table A1.

We can now discuss multiplicities. The upper bound for the multiplicities in the tensor diagram of $(p, q) \otimes (q, p)$ is $q + 1$ and, as was the case for $(p, q) \otimes (p, q)$, this upper bound is always attained. The vertices with maximum multiplicity are again arranged in a straight line of length $p - q + 1$, but for $(p, q) \otimes (q, p)$ the line goes along the centre diagonal. The length of the centre diagonal is $2q + 1$, so the top vertex with maximum multiplicity must be $q$ vertices below the highest vertex (which is at $(p + q, p + q)$), i.e., at $(p, p)$. The subsequent $p - q$ vertices along the centre diagonal have maximum multiplicity as well; hence the irreps with maximum multiplicity are exactly:

$$(p, p), (p - 1, p - 1), ..., (p - q, p - q) \quad (p - q + 1 \text{ vertices})$$

From the decomposition method it also follows that the vertices of multiplicity one are exactly those on the convex hull of the tensor diagram. Also, the irrep with lowest highest weight is $(0, 0)$ and it has multiplicity one, as it is on the boundary.

As in the case of $(p, q) \otimes (p, q)$, even though the vertices on the boundary have multiplicity one, this does not imply that the first/last vertex in any given line has multiplicity one.

Now let $\lambda = p\lambda_1 + q\lambda_2$ and $\lambda = q\lambda_1 + p\lambda_2$ i.e., $\lambda$ and $\lambda'$ are the highest weights in $(p, q)$ and $(q, p)$, respectively. Shifting the weight diagram of $(p, q)$ by $\lambda$ and the weight diagram of
Figure 8. Boundary of region where the vertices have the same multiplicities in the tensor
diagram as in the shifted weight diagrams $(p, q)$ and $(q, p)$ respectively. The highest
weight is indicated by a dot. See also Figure 9.

$(q, p)$ by $\lambda'$ brings the highest vertex of each weight diagram to the highest vertex of the tensor
diagram so that part of the diagrams coincide.

In a region near the upper right corner, the vertices in the tensor diagram have the same
multiplicities as those in the shifted weight diagram of $(p, q)$ and/or $(q, p)$. This region is shown
in Figure 8.

As in the case of $(p, q) \otimes (p, q)$, it is interesting to see the tensor diagram of $(p, q) \otimes (q, p)$
in the context of the shifted weight diagram $(p, q)$ although here the shifted weight diagram of
$(q, p)$ is also relevant (see Figure 9). The given highest and lowest vertices refer to the tensor
diagram (and coincide with those given above).

The shaded region shows where the multiplicities of the tensor and shifted weight diagrams
coincide, as discussed above. Outside this region the tensor multiplicities are strictly less than
the weight multiplicities.

The thick dashed line shows where irreps in the tensor diagram has maximum multiplicity.
As can be seen in the figure, this line is one side of the triangle where the weight multiplicities
are maximal.

Figure 9. The tensor diagram of $(p, q) \otimes (q, p)$ in the context of the shifted weight diagrams
of $(p, q)$ and $(q, p)$. The dashed lines show the boundary of the shifted weight diagram, and
the lines forming the triangle where the weight multiplicities are largest. The thick dashed line
indicates the line where the multiplicities are maximal in the tensor diagram. Note that when
$p = q$ the two diagrams coincide.
Remark: When $p = q$, $(p, q) \otimes (q, p)$ coincides with $(p, q) \otimes (p, q)$, and in particular the lines of maximum multiplicity reduce to a single point in both cases; some examples are provided in Table A1.

4. Conclusions

The $\mathfrak{su}(3)$ tensor products $(p, q) \otimes (q, p)$ and $(p, q) \otimes (p, q)$ were considered from a geometric perspective, and a method for decomposing them into direct sums was given. This geometric approach makes it particularly easy to rapidly obtain multiplicities of irreps. In particular, irreps of maximum multiplicity were discussed.

The work in this article can be applied to many different physical problems and there are several directions in which further work should be considered.

One possible application is the missing label problem, where geometrical constructions such as those presented here could be used to find additional arguments that justify the choice of labeling operators. The information regarding multiplicities given in this article provide new insights into the problem of deciding whether the appearing degeneracies can be solved by means of the given Casimir operators, or whether the full number of missing label operators are needed.

Also, tensor products for $\mathfrak{su}(3)$ are related to affine $\hat{\mathfrak{su}}(3)_k$ fusions [15] and when $k$ is large the fusion coefficients coincide with a triple tensor product decomposition [16]. It is therefore natural to ask whether a similar method can be developed for triple tensor products.

Other potential further developments include constructing a similar procedure for the analysis of the symmetric and antisymmetric tensor products of multiplets, as well as the decomposition of tensor products into symmetric and antisymmetric components. These have applications in the problem of conflicting symmetries in atomic spectroscopy [17], where they are studied in the context of vanishing matrix elements.

The procedure could perhaps also be extended to find decompositions in the asymptotic limit [18], which are important in the study of the non-compact rigid rotor algebra [19].

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### Appendix A. Table

**Table A1.** Some small examples of decompositions of tensor products of the form \((p, q) \otimes (p, q)\) and/or \((p, q) \otimes (q, p)\). Note that the tensor diagram of e.g. \((1, 2) \otimes (1, 2)\) is the diagram of \((2, 1) \otimes (2, 1)\) reflected in the centre diagonal.

| Decomposition in highest weight notation | Decomposition in dimension notation | Tensor diagram |
|------------------------------------------|-----------------------------------|---------------|
| \((1, 0) \otimes (1, 0) = (2, 0) \oplus (0, 1)\) | \(3 \otimes 3 = 6 \oplus 3\) | · · · |
| \((1, 0) \otimes (0, 1) = (1, 1) \oplus (0, 0)\) | \(3 \otimes 3 = 8 \oplus 1\) | · |
| \((1, 1) \otimes (1, 1) = (2, 2)\) | \(8 \otimes 8 = 27 \oplus 10\oplus\) | · · · |
| \((3, 0) \oplus (0, 3) \oplus 2(1, 1) \oplus (0, 0)\) | \(10 \oplus 2 \cdot 8 \oplus 1\) | · |
| \((2, 0) \otimes (2, 0) = (4, 0) \oplus (2, 1) \oplus (0, 2)\) | \(6 \otimes 6 = 15 \oplus 15 \oplus 6\) | · · · |
| \((2, 0) \otimes (0, 2) = (2, 2) \oplus (1, 1) \oplus (0, 0)\) | \(6 \otimes 6 = 27 \oplus 8 \oplus 1\) | · |
| \((2, 1) \otimes (2, 1) = (4, 2)\) | \(15 \otimes 15 = 60\oplus\) | · · · · |
| \((2, 3) \oplus 2(3, 1) \oplus (5, 0)\) | \(42 \oplus 2 \cdot 24 \oplus 21\oplus\) | · · · · |
| \((2, 1, 2) \oplus (0, 4) \oplus (2, 0) \oplus (0, 1)\) | \(2 \cdot 15 \oplus 15 \oplus 6 \oplus 3\) | · · · |
| \((2, 1) \otimes (1, 2) = (3, 3)\) | \(15 \otimes 15 = 64\oplus\) | · · · |
| \((4, 1) \oplus (1, 4) \oplus 2(2, 2)\) | \(35 \oplus 35 \oplus 2 \cdot 27\oplus\) | · · · |
| \((3, 0) \oplus (0, 3) \oplus 2(1, 1) \oplus (0, 0)\) | \(10 \oplus 10 \oplus 2 \cdot 8 \oplus 1\) | · · |
| \((2, 2) \otimes (2, 2) = (4, 4) \oplus (5, 2)\) | \(27 \otimes 27 = 125 \oplus 81\oplus\) | · · · · |
| \((2, 5) \oplus 2(3, 3) \oplus 2(4, 1)\) | \(81 \oplus 2 \cdot 64 \oplus 2 \cdot 35\oplus\) | · · · · |
| \((2, 1, 4) \oplus (6, 0) \oplus (0, 6) \oplus 3(2, 2)\) | \(2 \cdot 35 \oplus 28 \oplus 28 \oplus 3 \cdot 27\oplus\) | · · · · |
| \((3, 0) \oplus (0, 3) \oplus 2(1, 1) \oplus (0, 0)\) | \(10 \oplus 10 \oplus 2 \cdot 8 \oplus 1\) | · · |
| \((3, 0) \otimes (3, 0) = (6, 0)\) | \(10 \otimes 10 = 28\oplus\) | · · · · |
| \((4, 1) \oplus (2, 2) \oplus (0, 3)\) | \(35 \oplus 27 \oplus 10\) | · · · · |
| \((3, 0) \otimes (0, 3) = (3, 3)\) | \(10 \otimes 10 = 64\oplus\) | · · |
| \((2, 2) \oplus (1, 1) \oplus (0, 0)\) | \(27 \oplus 8 \oplus 1\) | · · |
| \((3, 1) \otimes (3, 1) = (6, 2)\) | \(24 \otimes 24 = 105 \oplus 90\oplus\) | · · · · |
| \((4, 3) \oplus (2, 4) \oplus 2(5, 1)\) | \(60 \oplus 2 \cdot 48 \oplus 2 \cdot 42 \oplus 36\oplus\) | · · · · |
| \((2, 3, 2) \oplus (7, 0) \oplus 2(1, 3)\) | \(2 \cdot 24 \oplus 21 \oplus 15 \oplus 15 \oplus 6\oplus\) | · · · · |
| \((4, 5) \oplus (4, 0) \oplus (2, 1) \oplus (0, 2)\) | \(2 \cdot 24 \oplus 21 \oplus 15 \oplus 15 \oplus 6\oplus\) | · · · · |
| \((3, 1) \otimes (1, 3) = (4, 4)\) | \(24 \otimes 24 = 125 \oplus 81\oplus\) | · · · · |
| \((5, 2) \oplus (2, 5) \oplus 2(3, 3)\) | \(81 \oplus 2 \cdot 64 \oplus 35 \oplus 35\oplus\) | · · · · |
| \((4, 1) \oplus (1, 4) \oplus 2(2, 2)\) | \(2 \cdot 27 \oplus 10 \oplus 10 \oplus 2 \cdot 8 \oplus 1\) | · · · · |
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