Renormalization for free harmonic oscillators

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Abstract

We introduce a model of free harmonic oscillators that requires renormalization. The model is similar to but simpler than the soluble Lee model. We introduce two concrete examples: the first, resembling the three dimensional $\phi^4$ theory, needs only mass renormalization, and the second, resembling the four dimensional $\phi^4$ theory and the Lee model, needs additional renormalization of a coupling and a wave function.

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I. INTRODUCTION

The purpose of this short paper is to introduce simple examples of renormalization using a model of free harmonic oscillators. The model was originally introduced by Dirac [1] for his explanation of resonance scattering. We have simplified Dirac’s model slightly by transcribing it in terms of harmonic oscillators.

We consider two concrete examples. The first example requires renormalization of only a frequency, and it resembles the $\phi^4$ theory in three dimensions. The second example requires additional renormalization of a dimensionless coupling, and it resembles the $\phi^4$ theory in four dimensions. The latter example is particularly illuminating since it suffers from the Landau pole just as the four dimensional $\phi^4$ theory.

Since our model is free, we can compute its Green function by summing a geometric series. In this sense the model captures the essence of the soluble Lee model [2] and the large $N$ limit of the $O(N)$ linear sigma model. Our model is simpler thanks to the use of harmonic oscillators.

The implication of the Landau pole for our second example is exactly the same as for the four dimensional $\phi^4$ theory. To keep a non-vanishing interaction, we must keep the ultraviolet cutoff of the theory finite. We will derive the exact cutoff dependence of the renormalized coupling as for the Lee model and the large $N$ limit of the $O(N)$ linear sigma model.

In Appendix [3] we show that part of the Lee model can be reproduced exactly with a judicious choice of frequency dependence of the coupling in our free model.

II. THE MODEL

We consider the hamiltonian for a collection of harmonic oscillators:

$$H = H_F + H_I$$

where

$$\begin{cases} H_F & = \Omega a^\dagger a + \sum_n \omega_n a_n^\dagger a_n \\ H_I & = -\sum_n g_n (a_n^\dagger a + a_n^\dagger a_n) \end{cases}$$

This is a model of coupled oscillators. Since the hamiltonian is quadratic in oscillators, the model is free. But a model as simple as this can be interesting and useful, since it admits a
variety of interpretations. Here are three examples:

1. $a$ stands for a charged oscillator, and $a_n$ for modes of radiation. Thus, the model mimics an atom unstable under a radiative decay.

2. $a$ stands for a meson in its center of mass system, and $a_n$ for a pair of decay products whose relative momentum is oriented in a direction denoted by $n$.

3. $a$ stands for a mode mediating an attractive force between a Cooper pair of electrons denoted by $n$. (The model of Cooper [3] is reproduced in the limit $\Omega \to \infty$, where $g_n^2/\Omega$ is a fixed frequency. See Appendix A)

The model reduces to the Jaynes-Cummings model [4] if we single out a particular mode $n$.

III. GREEN FUNCTION

Let us consider a complex valued Green function defined by

$$G(z) \equiv \langle 0 | a \frac{1}{z - H} a^\dagger | 0 \rangle$$

We can sum the geometric series given by perturbation theory as

$$G(z) = \frac{1}{z - \Omega} \sum_{L=0}^{\infty} \left( \sum_n g_n^2 \frac{1}{z - \omega_n} \frac{1}{z - \Omega} \right)^L = \frac{1}{z - \Omega - \sum_n g_n^2 \frac{1}{z - \omega_n}}$$

If we denote the physical size (or “volume”) of the system by $V$, the number of modes $n$ contained in a finite frequency interval is proportional to $V$. Hence, we can define the density of states per unit volume by

$$\frac{dn}{d\omega} \equiv \lim_{V \to \infty} \frac{1}{V} \sum_n \delta(\omega - \omega_n)$$

Assuming that $g_n^2$ is of order $\frac{1}{V}$, we obtain a finite non-negative function

$$g^2_\omega \equiv \lim_{V \to \infty} \sum_n g_n^2 \delta(\omega - \omega_n)$$

in this thermodynamic limit. Note $g^2_\omega$ has the dimension of a frequency. Hence, in the limit we obtain

$$G(z) = \frac{1}{z - \Omega - \int d\omega \ g^2_\omega \frac{1}{z - \omega}}$$
Let us suppose \( g_2^2 \) is non-vanishing only in a finite range \([\omega_L, \omega_H]\) of \( \omega \). For example, if \( a_n \) denotes a pair of electrons, we may take \( \omega_L \) to be twice the electron mass. The choice of the band width \( \omega_H - \omega_L \) depends on the model. It may be a finite Debye temperature as in Cooper’s model, or we may wish to take \( \omega_H \) to infinity as in the case of a meson decay.

For \( \omega \in [\omega_L, \omega_H] \), we obtain the imaginary part:

\[
\Im G(\omega + i\epsilon) = \frac{-\pi g_2^2}{b_\omega^2 + \pi^2 g_4^2} \tag{8}
\]

where

\[
b_\omega \equiv \Re G(\omega + i\epsilon)^{-1} = \omega - \Omega - \int_{\omega_L}^{\omega_H} d\omega' \frac{g_2^2}{\omega - \omega'} P \tag{9}
\]

Hence, we obtain the dispersion relation

\[
G(z) = \sum_i \frac{r_i}{z - \omega_i} + \int_{\omega_L}^{\omega_H} d\omega \frac{1}{z - \omega} \rho(\omega) \tag{10}
\]

where \( \omega_i \) are isolated poles with positive residues \( r_i \), and the spectral function is defined by

\[
\rho(\omega) \equiv \frac{1}{\pi} (-) \Im G(\omega + i\epsilon) = \frac{g_2^2}{b_\omega^2 + \pi^2 g_4^2} \quad (\omega_L < \omega < \omega_H) \tag{11}
\]

The asymptotic behavior

\[
G(z) \mid_{|z| \to \infty} \xrightarrow[]{} \frac{1}{z} \tag{12}
\]

gives the sum rule

\[
\sum_i r_i + \int_{\omega_L}^{\omega_H} d\omega \rho(\omega) = 1 \tag{13}
\]

IV. FIRST EXAMPLE

The first example is given by

\[
g_2^2 = g^2 > 0 \quad (\omega_L < \omega < \omega_H) \tag{14}
\]

where \( g^2 \) is a constant frequency. Since

\[
\int_{\omega_L}^{\omega_H} d\omega \frac{1}{z - \omega} = \ln \frac{z - \omega_L}{z - \omega_H} \tag{15}
\]

we obtain

\[
G(z)^{-1} = z - \Omega - g^2 \ln \frac{z - \omega_L}{z - \omega_H} \tag{16}
\]
By plotting $\omega - g^2 \ln \frac{\omega - \omega_L}{\omega - \omega_H}$ for $\omega < \omega_L$ and $\omega > \omega_H$ (Fig. 1), we find two isolated states, one below $\omega_L$ and another above $\omega_H$. Hence, the force mediated by the $\Omega$ mode is attractive for the state below $\omega_L$, and repulsive for that above $\omega_H$. This is easy to understand. The second order perturbation theory gives the correction to the energy of the mode $\omega_n$ as

$$\Delta \omega_n = g^2 n \frac{1}{\omega_n - \Omega}$$

This is negative for $\omega_n < \Omega$, and positive for $\omega_n > \Omega$.

We now take $\omega_H$ large. We then obtain

$$G(z)^{-1} = z - \Omega - g^2 \ln \frac{\omega_L - z}{\mu} + g^2 \ln \frac{\omega_H}{\mu}$$

(17)

By defining a renormalized frequency

$$\Omega_r \equiv \Omega - g^2 \ln \frac{\omega_H}{\mu}$$

(18)

where $\mu$ is a renormalization scale, we obtain the renormalized Green function as

$$G_r(z) \equiv \lim_{\omega_H \to \infty} G(z) = \frac{1}{z - \Omega_r - g^2 \ln \frac{\omega_L - z}{\mu}}$$

(19)

This satisfies the renormalization group equation

$$\left( \frac{\partial}{\partial \mu} + g^2 \frac{\partial}{\partial \Omega_r} \right) G_r(z) = 0$$

(20)

$G_r$ has only one pole at $\omega = \omega_b$, which satisfies

$$\omega_b - \Omega_r - g^2 \ln \frac{\omega_L - \omega_b}{\mu} = 0$$

(21)

This is solved explicitly as

$$\omega_b = \omega_L - g^2 W_0 \left( \frac{\mu}{g^2 e^{-\frac{\Omega_r}{g}}} \right)$$

(22)
where $W_0(x)$ is the main branch of the Lambert W function,[5] satisfying

$$W_0(x) \exp(W_0(x)) = x$$  \hspace{1cm} (23)

The dispersion relation for the renormalized Green function is given by

$$G_r(z) = \frac{r_b}{z - \omega_b} + \int_{\omega_L}^{\infty} d\omega \frac{1}{z - \omega} \rho(\omega)$$  \hspace{1cm} (24)

where

$$\rho(\omega) \equiv \frac{g^2}{\left(\omega - \Omega_r - g^2 \ln \frac{\omega - \omega_L}{\mu}\right)^2 + \pi^2 g^4}$$  \hspace{1cm} (25)

For $g^2 \ll \Omega_r$, we find $r_b \ll 1$, and the spectral function $\rho(\omega)$ is sharply peaked at $\Omega_r$ with width $\pi g^2$. (In fact there is an additional peak of an extremely narrow width $\mu e^{\Omega_r - \omega_L - g^2 \ln \omega_L}$ just above $\omega_L$.) We obtain the approximate sum rule

$$\int_{\omega_L}^{\infty} d\omega \rho(\omega) \simeq 1$$  \hspace{1cm} (26)

V. SECOND EXAMPLE

The second example is given by

$$g^2_\omega = \omega \bar{g}^2 \quad (\omega_L < \omega < \omega_H)$$  \hspace{1cm} (27)

where $\bar{g}^2$ is a dimensionless positive constant. This model has a stronger coupling toward the high frequencies.

The Green function is obtained as

$$G(z)^{-1} = z - \Omega + \bar{g}^2 \left\{ \omega_H - \omega_L - z \ln \frac{z - \omega_L}{z - \omega_H} \right\}$$  \hspace{1cm} (28)

As in the first example, there are two isolated states, one below $\omega_L$ and another above $\omega_H$. (Fig. 2)

Let us now consider the limit $\omega_H \to \infty$. To get a limit, we must renormalize not only $\Omega$ but also $\bar{g}^2$. We define renormalized parameters by

$$\Omega_r \equiv \frac{1}{Z} \left( \Omega - \bar{g}^2(\omega_H - \omega_L) \right)$$  \hspace{1cm} (29)

$$\bar{g}^2_r \equiv \frac{\bar{g}^2}{Z}$$  \hspace{1cm} (30)
FIG. 2. The dark curves give \( \omega \left( 1 - \bar{g}^2 \ln \frac{\omega - \omega}{\omega - \omega_H} \right) \) for \( \omega < \omega_L \) and \( \omega > \omega_H \)

where

\[
Z \equiv 1 + \bar{g}^2 \ln \frac{\omega}{\mu} \tag{31}
\]

We then obtain the renormalized Green function as

\[
G_r(z) \equiv \lim_{\omega_H \to \infty} Z \cdot G(z) = \frac{1}{z - \Omega_r - \bar{g}^2 z \ln \frac{\omega - \omega_H}{\mu}} \tag{32}
\]

Note the necessity of a wave function renormalization by the factor \( Z \). The renormalized Green function satisfies the renormalization group equation

\[
\left( \mu \frac{\partial}{\partial \mu} + \bar{g}_r^2 \Omega_r \frac{\partial}{\partial \Omega_r} + \bar{g}_r^4 \frac{\partial}{\partial \bar{g}_r^2} \right) G_r(z) = -\bar{g}_r^2 G_r(z) \tag{33}
\]

implying the anomalous dimension \( \bar{g}_r^2 \) of the Green function.

Contrary to our expectation that the renormalized Green function has only one pole just below \( \omega_L \), we find an additional pole \( \omega_t \) which is very negative. (Fig. 3) We call this a tachyon since the residue \( r_t \) of the pole at \( \omega_t \) is negative:

\[
G_r(z) \xrightarrow{z \to \omega_t} \frac{r_t}{z - \omega_t} \quad (r_t < 0) \tag{34}
\]

FIG. 3. The dark curve gives \( \omega \left( 1 - \bar{g}_t^2 \ln \frac{\omega - \omega}{\omega - \omega_L} \right) \). The tachyon pole lies below \( \omega_L - \mu e^{1/\bar{g}_t^2} \)
This tachyon pole is reminiscent of the tachyon pole in the large $N$ limit of the $O(N)$ linear sigma model in four dimensions.\(^{[6]}\)

The tachyon pole arises since we cannot really take $\omega_H$ all the way to infinity. In the limit $\omega_H \to \infty$, we get a trivial result:

$$\bar{g}^2 = \frac{1}{\bar{g}^2} + \ln \frac{\omega_H}{\mu} \xrightarrow{\omega_H \to \infty} 0$$

To find the largest possible $\omega_H$, we use $1/\bar{g}^2 \geq 0$ to obtain

$$\bar{g}^2 \leq \frac{1}{\ln \frac{\omega_H}{\mu}}$$

Hence,

$$\frac{\omega_H}{\mu} \leq \exp \left( \frac{1}{\bar{g}^2} \right)$$

The equality corresponds to the Landau pole

$$\bar{g}^2 = +\infty$$

Thus, to be rid of the tachyon, we must keep $\omega_H$ large but finite. The same resolution works for the Lee model and the large $N$ limit of the four dimensional scalar theory.

**Appendix A: Renormalized Cooper’s model**

We consider the strong coupling limit of the first example. Let

$$g^2 = \bar{g}^2 \Omega_r$$

and take the limit $\Omega_r \to \infty$. We obtain

$$\lim_{\Omega_r \to \infty} \frac{1}{\Omega_r} G_r(z)^{-1} = -1 - \bar{g}^2 \ln \frac{\omega_L - z}{\mu}$$

Hence, the bound state energy is given by

$$\omega_L - \omega_b = \mu \exp \left( -\frac{1}{\bar{g}^2} \right)$$
Appendix B: Lee’s model

The Lee model [2] is a non-relativistic model describing an interaction of a fermion with a meson of mass $m_{\theta}$. The fermion comes in two flavors: a V-particle with mass $m_V$ and an N-particle with mass $m_N$. They only interact via

$$V \leftrightarrow N + \theta$$

so that the number of V-particles plus N-particles, $N_V + N_N$, and the number of V-particles plus $\theta$-mesons, $N_V + N_\theta$, are conserved. Using our free field model, we can reproduce exactly the spectrum of the states satisfying $N_V + N_N = 1 \& N_V + N_\theta = 1$ (one V or a pair of N and $\theta$). The V-particle corresponds to the mode $\Omega = m_V - m_N + \delta m_V$, and the pair of N and $\theta$ of momentum $k$ corresponds to the mode $\omega_n = \sqrt{k^2 + m_{\theta}^2}$. $m_{\theta}$ plays the role of our $\omega_L$. The hamiltonian of this subspace is identical with that of our free model with the choice

$$g_\omega^2 = \frac{g^2}{4\pi^2} \sqrt{\omega^2 - m_{\theta}^2} \quad (B1)$$

For large $\omega \gg m_{\theta}$, we find $g_\omega^2 \propto \omega$, and our second model shares the same renormalization properties as the Lee model.

In our notation Eq. (8) of Lee is given by

$$\delta m_V = -\int_{m_\theta}^{\omega_H} d\omega \frac{g_\omega^2}{m_V - m_N - \omega} \quad (B2)$$

and Eq. (10) of Lee for the renormalization constant is given by

$$Z_2^{-1} = 1 + \int_{m_\theta}^{\omega_H} d\omega \frac{g_\omega^2}{(m_V - m_N - \omega)^2} \quad (B3)$$

We note that $\delta m_V$ is linearly divergent, and that $Z_2^{-1}$ is logarithmically divergent, as $\omega_H \to \infty$.

Finally, the phase shift $\delta$, defined by the phase of the Green function $G(\omega + i\epsilon)$, is given by

$$\tan \delta \equiv \frac{\pi g_\omega^2}{b_\omega} = \frac{\pi g_\omega^2}{m_V - m_N - \omega} \left(1 + \int_\omega^{\omega_H} \frac{g_{\omega'}^2}{m_V - m_N - \omega'} P \frac{1}{\omega - \omega'} \right)^{-1} \quad (B4)$$

which agrees with Lee’s (16). ($\omega_0$ is replaced by $\omega$ here.)
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