Integral Sliding Mode Controller Design for the Global Chaos Synchronization of a New Finance Chaotic System with Three Balance Points and Multi-Stability

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Abstract. This paper conveys new results of a new finance chaotic system with three unstable balance points of which one balance point is a hyperbolic saddle while the other balance points are saddle-foci. The new finance chaotic system involves three nonlinearities of which one is quadratic, the other is quartic and the third is absolute function nonlinearity. As an application of the new finance chaotic system, integral sliding mode control is invoked to achieve synchronization of master-slave finance chaotic systems. MATLAB simulations are carried out to explain the main results of this research report.

Keywords: Finance system, chaos, chaotic system, sliding mode control, balance points, etc.

1. Introduction

Chaos applications of dynamical models arise in various engineering domains such as nonlinear oscillatory systems [1-6], biological models [7-8], circuit devices [9-12], ANN models [13-14], chemical models [15-16], fuzzy models [17-18], robotics [19-20], etc.

Dynamical systems with $\tau = 1$ are called chaotic dynamical systems, where $\tau$ is the number of positive Lyapunov index numbers. Chaotic systems have high sensitivity to changes in the initial states.

Chaotic systems possess engineering applications such as secure devices [21-23], crypto-communication devices [24-25], memristor devices [26-28], neural networks [29-30], etc.
Chaos theory has also many applications in other areas such as finance systems [31-34], ecology [35], epidemics [36-37], biological systems [38-40], etc.
As an application of the new finance chaotic system, we obtain new results for the global chaos synchronization of master-slave finance systems using integral sliding mode control. Sliding mode control (SMC) is a useful control technique which has advantages like robustness [41-43]. SMC is a popular technique applied for synchronization of chaotic attractors [44-48]. MATLAB plots are shown to visualize the SMC results for the synchronization of the master-slave finance systems.

2. A New Finance Chaos System with Three Balance Points

A novel three-dimensional finance chaotic system with three nonlinear terms is proposed in this work. The three-dimensional vector \( \mathbf{K} = (\xi, \eta, \zeta) \) designates the state of the finance system (1). It is remarked that there are three nonlinear terms in the dynamics (1). The first differential equation in (1) has a quadratic nonlinearity \( \eta \xi \), the second differential equation in (1) has an absolute function nonlinearity \( \xi \) and a quartic nonlinearity \( \xi^4 \).

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In the finance model (1), the phase \( \xi \) represents the interest rate, the phase \( \eta \) represents the investment demand and the phase \( \zeta \) stands for the price exponent. The finance constant \( a \) stands for the household savings and the finance constant \( b \) stands for the investment cost. Also, the constant \( c \) signifies the demand elasticity of commercial markets while the constant \( d \) represents a real parametric uncertainty. We assume that all system constants \( a, b, c, d \) are positive.

It will be established using Lyapunov Index (LI) values spectrum analysis in MATLAB that chaos exists in the finance system (1) when the parameters assume the values

\[
a = 0.9, \quad b = 0.1, \quad c = 1.1, \quad d = 0.1
\]

In order for the time-series analysis of the state \( \mathbf{K} \) of the system (1), we assume the initial state as

\[
\xi(0) = 0.3, \quad \eta(0) = 0.2, \quad \zeta(0) = 0.3
\]

Using Wolf’s procedure [49] for Lyapunov index numbers, the LI values of the 3-D finance model (1) are numerically found for \( T = 1e4 \) seconds (see Figure 1) as follows:

\[
\mu_1 = 0.1407, \quad \mu_2 = 0, \quad \mu_3 = -0.6070
\]

The occurrence of a positive Lyapunov index value \( \mu_1 \) signifies that the model (1) has chaos behaviour. As the total of all LI values in the LI spectrum is seen to be negative, the model (1) has also dissipative motion of all its trajectories converging to the strange chaotic attractor.

The Kaplan-Yorke dimension of the finance chaotic system (1) is found as follows:

\[
D_{KY} = 2 + \frac{\mu_1 + \mu_2}{|\mu_3|} = 2.2334
\]

Figures 1 and 2 show the LI values and signal plots of the model (1) simulated in MATLAB for the parameter set \( (a, b, c, d) = (0.9, 0.1, 1.1, 0.1) \) and \( \mathbf{K}_0 = (0.3, 0.2, 0.3) \).

MATLAB plots (Figures 1 and 2) show the high complexity of the finance chaos system (1).
Figure 1. Lyapunov index values spectrum for the new finance chaos system (1)

(a)

(b)

(c)

(d)

Figure 2. Signal plots of the new finance chaotic system (1) simulated in MATLAB
The balance points of the finance system (1) are got by seeking the roots of the following system:

\[ \zeta + (\eta - a)\zeta = 0 \]  
\[ 1 - b\eta - |\zeta| - d\zeta^4 = 0 \]  
\[ -\zeta - c\zeta = 0 \]

The parameter values are taken as in the chaotic case, viz. \((a, b, c, d) = (0.9, 0.1, 1.1, 0.1)\).

Solving the system (6) numerically, three balance points are derived as given below:

1. \[ \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix} \quad \Omega_1 = \begin{bmatrix} 0.7817 \\ 1.8091 \\ -0.7107 \end{bmatrix} \quad \Omega_2 = \begin{bmatrix} -0.7817 \\ 1.8091 \\ 0.7107 \end{bmatrix} \quad \Omega_3 = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} \]

The Jacobian matrix of the finance system (1) at \(\Omega_1\) has the following spectral values:

\[ \alpha_1 = -0.1, \quad \alpha_2 = -1, \quad \alpha_3 = 9.001 \]

Thus, it is concluded that the balance point \(\Omega_1\) is a saddle-point and unstable.

The Jacobian matrix of the finance system (1) at \(\Omega_2\) has the following spectral values:

\[ \alpha_1 = -0.7737, \quad \alpha_{2,3} = 0.2414 \pm 1.1249i \]

The Jacobian matrix of the finance system (1) at \(\Omega_3\) has the following spectral values:

\[ \alpha_1 = -0.7737, \quad \alpha_{2,3} = 0.2414 \pm 1.1249i \]

From (9) and (10), it is concluded that the balance points \(\Omega_2\) and \(\Omega_3\) are saddle-foci and unstable points. Since all the three balance points of the finance system (1) are established as unstable points, the chaotic attractor exhibited by the finance system (1) for the chaotic case \((a, b, c, d) = (0.9, 0.1, 1.1, 0.1)\) is a self-excited chaotic attractor.

The finance chaotic system (1) remains the same under the transformation of coordinates

\[ (\xi, \eta, \zeta) \mapsto (-\xi, \eta, -\zeta) \]

This shows that the finance chaotic system (1) possesses rotation symmetry about the \(\eta\) – axis.

As a consequence of the rotation symmetry about the \(\eta\) – axis, every non-trivial trajectory of the finance chaotic system (1) must have a twin trajectory. The rotation symmetry is also observed in the balance points \(\Omega_2\) and \(\Omega_3\).

Multistability means two or more attractors coexist together with different initial conditions and has been found in many nonlinear systems [11].

Let the parameters be fixed as \(a = 0.9, \quad b = 0.1, \quad c = 1.1\) and \(d = 0.1\).

We suppose that the initial states of the finance chaotic system (1) are picked as \(K_0 = (0.3, 0.2, 0.3)\) and \(\Lambda_0 = (-0.6, -0.6, 0.6)\).

Figure 3 shows multistability of the finance chaotic system (1) with two coexisting chaotic attractors emanating from \(K_0\) (blue color) and \(\Lambda_0\) (red color), respectively.
Figure 3. Phase portraits of the coexisting chaos attractors of the finance system (1) for the initial states $K_0 = (0.3, 0.2, 0.3)$ (blue color) and $\Lambda_0 = (-0.6, -0.6, 0.6)$ (red color).

3. Global Synchronization of Master-Slave Finance Chaotic Systems

This section applies Integral Sliding Mode Control (ISMC) to achieve global synchronization of master-slave finance chaotic systems and we use Lyapunov stability theory [50] to prove the main control result.

As the master system, we consider the finance chaotic system

$$
\begin{align*}
\dot{\xi}_1 &= \xi_1 + (\eta_1 - a)\xi_1 \\
\eta_1 &= 1 - b\xi_1 - |\xi_1| - d\xi_1^4 \\
\dot{\eta}_1 &= -\xi_1 - c\xi_1
\end{align*}
$$

(12)

As the slave system, we consider the finance chaotic system equipped with three controls

$$
\begin{align*}
\dot{\xi}_2 &= \xi_2 + (\eta_2 - a)\xi_2 + u_x \\
\eta_2 &= 1 - b\xi_2 - |\xi_2| - d\xi_2^4 + u_y \\
\dot{\xi}_2 &= -\xi_2 - c\xi_2 + u_z
\end{align*}
$$

(13)

In the system (12), $K = (\xi_1, \eta_1, \zeta_1)$ is the state and $a, b, c, d$ are constant, positive, parameters. In the system (13), $\Omega = (\xi_2, \eta_2, \zeta_2)$ is the state and $u_x, u_y, u_z$ are controls.

The synchronization error between the finance systems (12) and (13) is quantified as

$$
\varepsilon = \Omega - K
$$

(14)

Mathematically, the error can be represented as follows:

$$
\begin{align*}
\varepsilon_x &= \xi_2 - \xi_1 \\
\varepsilon_y &= \eta_2 - \eta_1 \\
\varepsilon_z &= \xi_2 - \zeta_1
\end{align*}
$$

(15)

The synchronization error dynamics is obtained with a simple calculation as follows:

$$
\begin{align*}
\dot{\varepsilon}_x &= \varepsilon_x - a\varepsilon_x + \eta_2\xi_2 - \eta_1\xi_1 + u_x \\
\dot{\varepsilon}_y &= -b\varepsilon_y - |\xi_2| + |\xi_1| - d(\xi_2^4 - \xi_1^4) + u_y \\
\dot{\varepsilon}_z &= -c\varepsilon_z + u_z
\end{align*}
$$

(16)
An integral sliding surface connected with each error variable can be defined as follows:

\[
\begin{align*}
\alpha_x &= \varepsilon_x + \beta_x \int_0^t e_x(\tau) d\tau \\
\alpha_y &= \varepsilon_y + \beta_y \int_0^t e_y(\tau) d\tau \\
\alpha_z &= \varepsilon_z + \beta_z \int_0^t e_z(\tau) d\tau 
\end{align*}
\]  

(17)

Using differentiation, we obtain the following set of differential equations:

\[
\begin{align*}
\dot{\alpha}_x &= \dot{e}_x + \beta_x e_x \\
\dot{\alpha}_y &= \dot{e}_y + \beta_y e_y \\
\dot{\alpha}_z &= \dot{e}_z + \beta_z e_z 
\end{align*}
\]  

(18)

Combining (16) and (18) leads to the following dynamics:

\[
\begin{align*}
\dot{\alpha}_x &= \dot{e}_x - a e_x + \eta_2 \xi_2 - \eta_1 \xi_1 + \beta_x e_x + u_x \\
\dot{\alpha}_y &= \dot{e}_y - b e_y + \xi_2 |\xi_2| - d (\xi_2^4 - \xi_1^4) + \beta_y e_y + u_y \\
\dot{\alpha}_z &= \dot{e}_z - c e_z + \beta_z e_z + u_z 
\end{align*}
\]  

(19)

Thus, we derive the sliding mode controls as follows:

\[
\begin{align*}
u_x &= -e_x + a e_x - \eta_2 \xi_2 - \eta_1 \xi_1 - \beta_x e_x - \kappa_x \text{sgn}(\alpha_x) - \theta_\alpha \alpha_x \\
u_y &= b e_y + \xi_2 |\xi_2| - d (\xi_2^4 - \xi_1^4) - \beta_y e_y - \kappa_y \text{sgn}(\alpha_y) - \theta_\alpha \alpha_y \\
u_z &= e_z + c e_z - \beta_z e_z - \kappa_z \text{sgn}(\alpha_z) - \theta_\alpha \alpha_z 
\end{align*}
\]  

(20)

In (20), we take the constants \(\beta_x, \beta_y, \beta_z, \kappa_x, \kappa_y, \kappa_z, \theta_x, \theta_y, \theta_z\) to be all positive.

By plugging in (20) into (19), we get the closed-loop (CL) dynamics

\[
\begin{align*}
\dot{\alpha}_x &= -\kappa_x \text{sgn}(\alpha_x) - \theta_\alpha \alpha_x \\
\dot{\alpha}_y &= -\kappa_y \text{sgn}(\alpha_y) - \theta_\alpha \alpha_y \\
\dot{\alpha}_z &= -\kappa_z \text{sgn}(\alpha_z) - \theta_\alpha \alpha_z 
\end{align*}
\]  

(21)

Next, we establish the following global synchronization result for the new finance chaotic systems (12) and (13).

**Theorem 1.** The integral SMC law (20) globally synchronizes the states of the new finance chaotic systems (12) and (13), where we take the control constants \(\beta_x, \beta_y, \beta_z, \kappa_x, \kappa_y, \kappa_z, \theta_x, \theta_y, \theta_z\) to be all positive.

**Proof.** We use Lyapunov stability theory to establish the proposed result. We consider the closed-loop system (21), which is obtained by plugging in the integral SMC law (20) into the sliding dynamics (19).

We use the Lyapunov function candidate given by

\[
V(\alpha_x, \alpha_y, \alpha_z) = \frac{1}{2} \alpha_x^2 + \frac{1}{2} \alpha_y^2 + \frac{1}{2} \alpha_z^2
\]  

(22)

By calculating the time-derivative of (22) along the CL dynamics (21), we get

\[
\dot{V} = -\kappa_x |\alpha_x| - \kappa_y |\alpha_y| - \kappa_z |\alpha_z| - \theta_x \alpha_x^2 - \theta_y \alpha_y^2 - \theta_z \alpha_z^2
\]  

(23)
Since $V$ is a negative definite function on $\mathbb{R}^3$, we conclude from Lyapunov stability theory that the CL dynamics (21) is globally asymptotically stable. Hence, it follows that $\epsilon_x(t) \to 0$, $\epsilon_y(t) \to 0$, $\epsilon_z(t) \to 0$ as $t \to \infty$ for all values of $\epsilon_x(0)$, $\epsilon_y(0)$, and $\epsilon_z(0)$. This ends the proof. ■

**Numerical Simulations:**
We consider the system parameter values as $a = 0.9$, $b = 0.1$, $c = 1.1$, $d = 0.1$.

We take $\beta_1 = \beta_2 = \beta_3 = 10$. We also take $\alpha_1 = \alpha_2 = \alpha_3 = 0.2$ and $\theta_1 = \theta_2 = \theta_3 = 8$.

We regard the initial values of the master and slave finance chaotic systems represented by (12) and (13) as follows.

$\xi_1(0) = 2.4$, $\eta_1(0) = 3.9$, $\zeta_1(0) = 6.5$

$\xi_2(0) = 1.9$, $\eta_2(0) = 7.2$, $\zeta_2(0) = 3.1$

Figure 4 shows the global synchronization between the master and slave finance chaotic systems represented by the equations (12) and (13). The integral sliding mode controller is very efficient as we notice that the synchronization between the finance chaotic systems is achieved in less than 5 seconds.

![Figure 4](image_url)

**Figure 4.** Global synchronization between the master and slave finance chaotic systems (12) and (13)
4. Conclusions
In this research paper, new results of a new finance chaotic system with three unstable balance points are reported. Multistability of the new finance chaotic system was proved by exhibiting two coexisting chaotic attractors for the same parameter set but different initial states. As an application of the new finance chaotic system, integral sliding mode control was applied to achieve global chaos synchronization of the master-slave finance chaotic systems. Lyapunov stability theory was invoked to ascertain the main control result of this research work. MATLAB simulations were shown to explain the main results of this research report.

References
[1] Vaidyanathan S 2015 Kyungpook Mathematical Journal 55 563-586
[2] Sundarapandian V 2013 Lecture Notes in Electrical Engineering 131 319-327
[3] Vaidyanathan S 2011 Communications in Computer and Information Science 133 98-107
[4] Vaidyanathan S 2011 Communications in Computer and Information Science 198 1-9
[5] Vaidyanathan S 2015 International Journal of PharmTech Research 8 156-166
[6] Vaidyanathan S 2015 International Journal of PharmTech Research 8 956-963
[7] Awal N M and Epstein I R 2020 Physical Review E 101 042222
[8] Vaidyanathan S 2015 International Journal of PharmTech Research 8 117-127
[9] Sambas A, Vaidyanathan S, Tlelo-Cuautle E, Zhang S, Guillon-Fernandez O, Sukono, Hidayat Y and Gundara G 2019 Electronics 8 1211
[10] Sambas A, Vaidyanathan S, Tlelo-Cuautle E, Abd-El-Atty B, El-Latif, A A A, Guillon-Fernandez O, Sukono, Hidayat Y and Gundara G 2020 IEEE Access 8 137116-137132
[11] Sambas A, Vaidyanathan S, Zhang S, Zeng Y, Mohamed M A and Mamat M 2019 IEEE Access 7 115454-115462
[12] Volos C, Maaita J O, Vaidyanathan S, Pham V T, Stouboulos I and Kyprianidis I 2017 IEEE Transactions on Circuits and Systems II: Express Briefs 64 339-343.
[13] Xiu C, Zhou R and Liu Y 2020 Chaos Solitons and Fractals 141 110316
[14] Vaidyanathan S 2015 International Journal of PharmTech Research 8 61-73
[15] Vaidyanathan S 2015 International Journal of ChemTech Research 8 209-221
[16] Vaidyanathan S and Azar A T 2016 International Journal of Intelligent Engineering Informatics 4 135-150
[17] Sukono, Sambas A, He S, Liu H, Vaidyanathan S, Hidayat Y and Saputra J 2020 Advances in Difference Equations 2020 674
[18] Wali W A 2021 International Journal of Electrical and Computer Engineering 11 328-335
[19] Bazzi S and Sernad D 2020 Advanced Robotics 34 1137-1155
[20] Moysis L, Petzatzizis E, Marwan M, Volos C, Nistazakis H and Ahmad S 2020 Complexity 2020 2826850
[21] Faragallah O S, El-sayed H S, Afifi A and El-Shafai W 2021 Optics and Lasers in Engineering 137 106333
[22] Zhong C and Pan M S 2020 Chinese Journal of Liquid Crystals and Displays 35 91-97
[23] Li Z, Jiang A and Mu Y 2020 Lecture Notes in Electrical Engineering 571 796-810
[24] Ding L and Ding Q 2020 Electronics 9 1-20
[25] Shakiba A 2020 Multimedia Tools and Applications 79 32575-32605
[26] Vaidyanathan S, Azar A T, Akgul A, Lien C H, Kacar S and Cavusoglu U 2019 International Journal of Automation and Control 13 644-667
[27] Raj B and Vaidyanathan S 2017 Studies in Computational Intelligence 701 449-476
[28] Vaidyanathan S, Pham V T and Volos C 2017 Studies in Computational Intelligence 701 101-130
[29] Swathy P S and Thamilmaran K 2014 Nonlinear Dynamics 78 2639-2650
[30] Lin H, Wang C and Tan Y 2020 Nonlinear Dynamics 99 2369-2386
[31] Gao Q and Ma J 2009 Nonlinear Dynamics 58 209-216
[32] Vaidyanathan S, Volos C K, Tacha O I, Kyprianidis I M, Stouboulos I N and Pham V T 2016 Studies in Computational Intelligence 636 495-512
[33] Zhao X S, Li Z B and Li S A 2011 Applied mathematics and Computation 217 6031-6039
[34] Pereira-Pinto F H I and Savi M A 2020 Nonlinear Dynamics 102 1151-1171
[35] Mahmoud E E, Trikha P, Jahanzaib L S and Almaghrabi O A 2020 Chaos, Solitons and Fractals 141 110348
[36] Wang X, Wang Z, Lu J and Meng B 2021 Mathematics and Computers in Simulation 182 182-194
[37] Postavaru O, Anton S R and Toma A 2021 Mathematics and Computers in Simulation 181 138-149
[38] Khan A Q and Khalique T 2020 International Journal of Biomathematics 13 2050022
[39] Yamaguchi H Q, Ode K L and Ueda H R 2021 iScience 24 101946
[40] Vandermeer J 2020 Theoretical Ecology 13 177-182
[41] Teng Q, Xu D, Yang W, Li J and Shi P 2021 Energy Reports 7 1-9
[42] Liu Y-A, Tang S, Liu Y, Kong Q and Wang J 2021 Applied Mathematics and Computation 396 125901
[43] Fujimoto K, Sakata N, Maruta I and Ferguson J 2021 IEEE Control Systems Letters 5 839-844
[44] Takhi H, Kemih K, Moyis L and Volos C 2021 Mathematics and Computers in Simulation 181 150-169
[45] Wu X, Bao H and Cao J 2021 Journal of the Franklin Institute 358 1002-1020
[46] Almatroud A O 2020 Advances in Difference Equations 2020 78
[47] Jia H, Guo C, Zhao L and Xu Z 2020 Algorithms 13 346
[48] Li W, Bai G and Imani Marrani H 2020 Journal of Control, Automation and Electrical Systems 31 1375-1385
[49] Wolf A, Swift J B, Swinney H L and Vastano J A 1985 Physica D 16 285-317
[50] Khalil H K 2002 Nonlinear Systems Pearson Education (New York: USA)