AdS$_3 \times_w (S^3 \times S^3 \times S^1)$ solutions of type IIB string theory

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Abstract

We analyse a recently constructed class of local solutions of type IIB supergravity that consist of a warped product of AdS$_3$ with a seven-dimensional internal space. In one duality frame the only other non-vanishing fields are the NS 3-form and the dilaton. We analyse in detail how these local solutions can be extended to give infinite families of globally well-defined solutions of type IIB string theory, with the internal space having topology $S^3 \times S^3 \times S^1$ and with properly quantized 3-form flux.

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1. Introduction

Supersymmetric solutions of string or M-theory that contain AdS$_{d+1}$ factors are dual to supersymmetric conformal field theories in $d$ spacetime dimensions. Starting with the work of [1], general characterizations of the geometries underlying such solutions, using $G$-structure techniques [2, 3], have been achieved for various $d$ and for various amounts of supersymmetry [4–23]. With a few exceptions, mostly with 16 supersymmetries, many of these geometries are still poorly understood, and it has proved difficult to find explicit solutions.

One notable exception is the class of AdS$_3$ solutions of type IIB string theory with non-vanishing 5-form flux, dual to $d = 2$ conformal field theories with (0, 2) supersymmetry, that were classified in [7]. It was shown that the seven-dimensional internal space has a Killing vector which is dual to the $R$-symmetry of the dual SCFT. The Killing vector defines a foliation and the solution is completely determined, locally, by a Kähler metric on the six-dimensional leaf space whose Ricci tensor satisfies an additional differential condition. Moreover, a rich set of explicit solutions have been constructed in [14, 24, 25] and the corresponding central charges of the dual SCFTs have also been calculated.

More recently, it was understood how to generalize this class of type IIB AdS$_3$ solutions to also include 3-form flux [26]. The solutions are again locally determined by a six-dimensional
Kähler metric and a choice of a closed, primitive \((1, 2)\)-form on the Kähler space. Once again additional explicit solutions were constructed with the six-dimensional Kähler space having a 2-torus factor and the 3-form flux being parametrized by a real parameter \(Q\). After two \(T\)-dualities on the 2-torus it was also shown that these explicit solutions give type IIB AdS\(_3\) solutions with non-vanishing dilaton and RR 3-form flux only. After an additional \(S\)-duality the solutions only involve NS fields.

In \([26]\) these explicit solutions were examined in more detail for the special case of \(Q = 0\). It was shown that the parameters and ranges of the coordinates could be chosen to give globally defined supergravity solutions consisting of a warped product of AdS\(_3\) with a seven-dimensional internal manifold that is diffeomorphic to \(S^3 \times S^3 \times T^2\). It was shown that the solutions, with properly quantized 3-form flux, are specified by a pair of positive coprime integers \(p, q\).

The purpose of this paper is to carry out a similar analysis when we switch on the parameter \(Q\). We will find that we are led to infinite classes of solutions, with the seven-dimensional internal space being diffeomorphic to \(S^3 \times S^3 \times S^1\) and with properly quantized fluxes.

While the final topology of the solutions is simple, it is not easy to see this in the local coordinates in which the solutions are presented. When \(Q = 0\) the \(S^2 \times S^1\) factor is realized in a manner very similar to the \(Y^{p,q}\) Sasaki–Einstein spaces \([27]\). When \(Q \neq 0\) one of the circles in the \(T^2\) factor is fibred over the \(S^2 \times S^3\) and we need to carefully check that the circle fibration is globally well defined, leading to \(S^3 \times S^3\). Furthermore we need to check that the 3-form flux is properly quantized. This is not straightforward since it is not clear ‘where’ the two \(S^3\) factors are in the local coordinates. After some false starts we developed a workable prescription for ensuring that the 3-form is properly quantized, as we shall explain.

The plan of the paper is as follows. In section 2, we begin by recalling the local solutions of \([26]\) and then discuss how, after suitable choices of parameters and periods for the coordinates, the seven-dimensional internal manifold has topology \(S^3 \times S^3 \times S^1\). We discuss some aspects of the topology in detail, leading to a prescription for carrying out flux quantization which is dealt with in section 3. Our method uses a quotient construction, which is explained in section 2, as well as explicit coordinate patches. In sections 2 and 3, the solutions depend on a pair of coprime positive integers \(p, q\), the electric 3-form flux, \(n_1\), the magnetic 3-form flux through each of the two \(S^3\) factors, \(M_1\) and \(M_2\), and the parameter \(Q\). For these solutions, it turns out that \(M_1\) and \(M_2\) are not independent and are given by \(M_1 = M(p + q)^2\) and \(M_2 = Mq^2\), where \(M\) is an integer. We calculate the central charge and show that it is given by the simple formula

\[
c = 6n_1 \frac{(M_1 - M_2)M_3}{M_1}. \tag{1.1}
\]

In particular it is independent of \(Q\), and since the solutions with \(Q \neq 0\) and other parameters fixed are all smoothly connected with each other, we conclude that when \(Q \neq 0\), the parameter \(Q\) corresponds to an exactly marginal deformation in the dual \((0, 2)\) SCFT. It is interesting to observe that the value of the central charge is precisely the same as for the \(Q = 0\) solutions studied in \([26]\). However, as we shall explain in section 3.3 taking the limit \(Q \to 0\) does not smoothly lead to the \(Q = 0\) solutions and so it is not clear whether or not the \(Q \neq 0\) solutions correspond to exactly marginal deformations of the \(Q = 0\) solutions.

In section 4, we generalize our construction by making more general identifications on the coordinates, obtaining solutions that involve more parameters. We show that the central charge has exactly the same form as in (1.1), but now, however, the integers \(M_1\) and \(M_2\) labelling the 3-form flux through the two \(S^3\)’s are no longer constrained. Thus not all of these more general solutions correspond to exactly marginal deformations of those that we consider in sections 2 and 3. We conclude in section 5.
We noted above that the $S^2 \times S^3$ factor in the AdS$_3$ solutions constructed in [26], with $Q = 0$, is realized in a similar way to the $Y_{p,q}$ Sasaki–Einstein spaces found in [27]. In particular, in both cases the metrics on $S^2 \times S^3$ are cohomogeneity one. Given that the $Y_{p,q}$ metrics can be generalized to cohomogeneity two Sasaki–Einstein metrics $L^{ab,c}$ on $S^2 \times S^3$ [28] (see also [29]), it is natural to suspect that there are analogous AdS$_3$ solutions, with 5-form flux only, with internal space having topology $S^2 \times S^3 \times T^2$ and with the metric on the $S^2 \times S^3$ factor having cohomogeneity two. This is indeed possible, and moreover it is also possible to find generalizations with non-zero 3-form flux and with the internal manifold having topology $S^3 \times S^3 \times S^1$. We will present such solutions in appendix C, but we will leave a detailed analysis of the regularity and flux quantization conditions for future work.

2. The AdS$_3$ solutions

2.1. The local solutions

We start with the explicit class of AdS$_3$ solutions of section 4.3 of [26]. The string frame metric is given by

$$\frac{1}{L^2} \, ds^2 = \frac{\beta}{y^{1/2}} [ds^2(\text{AdS}_3) + ds^2(X_7)],$$

(2.1)

where

$$ds^2(X_7) = \frac{\beta^2}{4} \left[ 1 + \frac{2y - Q^2 y^2}{4\beta^2} Dz^2 + \frac{U(y)}{4(\beta^2 - 1 + 2y - Q^2 y^2)} D\psi^2 + \frac{dy^2}{4\beta^2 y^2 U(y)} + \frac{1}{4\beta^2} ds^2(S^2) + (du^1 - \frac{Q}{2\beta} [(1 - g) D\psi - Dz])^2 + (du^2)^2. \right]$$

(2.2)

where $\beta, Q$ are positive constants, $L$ is an arbitrary length scale and

$$U(y) = 1 - \frac{1}{\beta^2} (1 - y)^2 - Q^2 y^2.$$

(2.3)

In addition, $ds^2(S^2)$ is the standard metric on a 2-sphere, $ds^2(S^2) = d\theta^2 + \sin^2 \theta d\phi^2$, and we have defined

$$D\psi = d\psi + P$$

(2.4)

with

$$dP = \text{Vol}(S^2) = \sin \theta \, d\theta \wedge d\phi \equiv J.$$  

(2.5)

Note that $P$ is only a locally defined 1-form on $S^2$. In fact, more precisely, $P$ is a connection 1-form on the $U(1)$ principal bundle associated with the tangent bundle of $S^2$. The 2-form $J$ introduced in (2.5) may be regarded as a Kähler form on $S^2$. We also have

$$Dz = dz - g(y) D\psi$$

(2.6)

with

$$g(y) = \frac{y(1 - Q^2 y)}{\beta^2 - 1 + 2y - Q^2 y^2}.$$  

(2.7)

The only other non-trivial type IIB supergravity fields are the dilaton and the RR 3-form. The dilaton is given by

$$e^{2\phi} = \frac{\beta^2}{y},$$  

(2.8)

Note that we have rescaled the metric on $S^2$ appearing in [26] by a factor of 4.
While the RR 3-form field strength is given by

$$\frac{1}{L^2} F^{(3)} = -\frac{1}{4\beta^2} dy \wedge D\psi \wedge Dz - \frac{y}{4\beta^2} J \wedge Dz + \left[ \frac{1 - y g}{4\beta^2} \right] J \wedge D\psi$$

$$+ \frac{Q}{2\beta} du^1 \wedge \left[ dy \wedge Dz - y J - (1 - g) dy \wedge D\psi \right] + 2\text{Vol}(\text{AdS}_3). \quad (2.9)$$

This is closed. After a further S-duality transformation we obtain AdS$_3$ solutions with only NS fields non-vanishing, but we will continue to work with the above solution.

In order to simplify some of the formulae it will be helpful to introduce

$$Z \equiv 1 - \sqrt{1 + Q^2(\beta^2 - 1)}. \quad (2.10)$$

We next change coordinates via

$$dz = dw + \frac{2Q \beta}{Z - 2} dv, \quad du^1 = dv + \frac{Q(1 - \beta^2)}{2\beta(Z - 2)} dw \quad (2.11)$$

to bring the metric to the form

$$ds^2(X_7) = \frac{2(1 - Z)(1 - \beta^2 - yZ)}{(2 - Z)(1 - \beta^2)} Dv^2 + \frac{(1 - Z)(2y - yZ - 1 + \beta^2)}{2\beta^2(2 - Z)} Dw^2$$

$$+ \frac{(1 - \beta^2)U(y)}{4(1 - \beta^2 - yZ)(\beta^2 - 1 + yZ)} D\psi^2$$

$$+ \frac{dy^2}{4\beta^2 y^2 U(y)} + \frac{1}{4\beta^2} ds^2(S^2) + (du^2)^2, \quad (2.12)$$

where

$$Dv = dv - A_v D\psi, \quad Dw = dw - A_w D\psi$$

and

$$A_v = \frac{Q(1 - \beta^2)y}{4\beta(1 - \beta^2 - yZ)}, \quad A_w = \frac{(2 - Z)y}{2(2y - yZ - 1 + \beta^2)}. \quad (2.13)$$

The 3-form in the new coordinates is given by

$$\frac{1}{L^2} F^{(3)} = 2\text{Vol}(\text{AdS}_3) + \frac{(1 - \beta^2)U(y)}{4(1 - \beta^2 - yZ)(\beta^2 - 1 + yZ)} J \wedge D\psi$$

$$- Dw \wedge \left( \frac{(Z - 1)(1 - \beta^2)}{4\beta^2(2y - yZ - 1 + \beta^2)} dy \wedge D\psi + \frac{y(1 - Z)}{4\beta^2} J \right)$$

$$- Q Dw \wedge \left( \frac{(1 - Z)(1 - \beta^2)}{2\beta(2 - Z)(2y - yZ - 1 + \beta^2 + 2y)} dy \wedge D\psi + \frac{y(1 - Z)}{2\beta(2 - Z)} J \right)$$

$$- \frac{Q(1 - Z)}{\beta(Z - 2)} dy \wedge Dw \wedge D\psi. \quad (2.14)$$

Finally, we note that the canonical Killing vector related to supersymmetry is given by

$$\partial_\psi + \partial_z. \quad (2.15)$$

In the new coordinates this reads

$$\partial_\psi + \frac{(Z - 2)}{2(Z - 1)} \partial_w + \frac{Q(\beta^2 - 1)}{4\beta(\beta^2 - 1)} \partial_v. \quad (2.16)$$

We now would like to find the restrictions on the parameters $\beta, Q$ so that these local solutions extend to global solutions on a globally well-defined manifold $X_7$. Having achieved
that goal, we will analyse the additional constraints imposed by ensuring that the 3-form is properly quantized. Note that when \( Q = 0 \) the corresponding analysis was carried out in \cite{26} and in particular it was shown that there were infinite classes of solutions, labelled by a pair of positive coprime integers, \( p, q \), with \( X_7 \) having the topology of \( S^3 \times S^2 \times T^2 \).

Our strategy is to build \( X_7 \) in stages. The \( u^2 \) coordinate is taken to parametrize an \( S^1 \); for now the period of \( u^2 \) is arbitrary but it will later be fixed by flux quantization. We therefore write \( X_7 = M_6 \times S^1 \) with

\[
\text{d}s^2(M_6) = \frac{2(1 - Z)(1 - \beta^2 - yZ)}{(2 - Z)(1 - \beta^2)} \, \text{d}u^2 + \text{d}s^2(M_3),
\]

where

\[
\text{d}s^2(M_3) = \frac{(1 - Z)(2y - yZ + 1 + \beta^2)}{2\beta^2(2 - Z)} \, \text{d}w^2 + \text{d}s^2(B_4)
\]

and

\[
\text{d}s^2(B_4) = \frac{(1 - \beta^2)U(y)}{4(1 - \beta^2 - yZ)(\beta^2 - 1 + 2y - yZ)} \, \text{d}y^2 + \frac{\text{d}y^2}{4\beta^2 y^2 U(y)} + \frac{1}{4\beta^2} \, \text{d}s^2(S^2).
\]

We will first analyse \( \text{d}s^2(B_4) \), showing that, by taking \( \psi \) to be a periodic coordinate with period \( 2\pi \), \( B_4 \) is a smooth manifold diffeomorphic to \( S^2 \times S^2 \). We then show that, by taking \( u \) to be a periodic coordinate with a suitable period, with the parameter \( \beta \) fixed by two relatively prime positive integers \( p, q \), \( M_5 \) is the total space of a circle fibration over \( B_4 \) and has topology \( S^3 \times S^2 \). Here, \( p \) and \( q \) have a topological interpretation as Chern numbers of the circle bundle over \( B_4 \). These steps are familiar from the construction of the Sasaki–Einstein manifolds \( Y^{p,q} \) \cite{27} (and also for the \( Q = 0 \) solutions studied in \cite{26}). The final step is to show that, by taking \( v \) to be periodic with a suitable period, \( M_6 \) is the total space of a circle fibration over \( M_5 \), and has topology \( S^3 \times S^1 \).

It will be useful in the following to observe that the function \( U(y) \) is a quadratic function of \( y \) with roots \( y_1 \) and \( y_2 \) given by

\[
y_1 = \frac{1 - \beta^2}{1 + \beta(1 - Z)}, \quad y_2 = \frac{1 - \beta^2}{1 - \beta(1 - Z)}.
\]

It will also be useful to know the values of the functions \( A_w \) and \( A_v \) appearing in \( \text{d}s^2(M_6) \) at \( y_1 \) and \( y_2 \). We find

\[
A_w(y_1) = \frac{2 - Z}{2(1 - Z)(1 - \beta)}, \quad A_w(y_2) = \frac{2 - Z}{2(1 - Z)(1 + \beta)}
\]

\[
A_v(y_1) = \frac{Q(1 - \beta)}{4\beta(1 - Z)}, \quad A_v(y_2) = \frac{Q(1 + \beta)}{4\beta(1 - Z)}.
\]

2.2. \( B_4 = S^2 \times S^2 \)

\( B_4 \) is parametrized by \( \theta, \phi, y \) and \( \psi \). We take the coordinate \( y \) to lie in the interval \( y \in [y_1, y_2] \) where \( y_1 \) are the two distinct positive\(^6 \) roots of \( U(y) \), given by (2.21). This requires that we demand

\[
0 < \beta < 1, \quad 0 \leq Z < 1.
\]

We next observe that if we choose the period of \( \psi \) to be \( 2\pi \), then \( y, \psi \) parametrize a smooth 2-sphere, with \( y \) being a polar coordinate and \( \psi \) an azimuthal coordinate on the metrically

\(^6\) We need \( y \) to be positive to ensure that the warp factor is real.
We next construct squashed $S^3$ fibre. In particular, fixing a point on the round 2-sphere, one can check that $ds^2(B_4)$ is free from conical singularities at the poles $y = y_1$ and $y = y_2$. $B_4$ is then a smooth $S^3$ bundle over the round $S^2$. The transition functions are in $U(1)$, acting in the obvious way on the fibre. The first Chern number of the $U(1)$ fibration is $-2$ and thus, as explained in [27], $B_4$ is diffeomorphic to $S^2 \times S^2$.

We have $H_2(B_4, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Three obvious 2-spheres in $B_4$ are the sections $\Sigma_1 = \{ y = y_1 \}$ and $\Sigma_2 = \{ y = y_2 \}$, each a copy of the 2-sphere base, and a copy of the fibre $\Sigma_f$ at some point on the 2-sphere base (for concreteness, say, the north pole $\{ \theta = 0 \}$). Call the corresponding homology classes $[\Sigma_1], [\Sigma_2]$ and $[\Sigma_f]$, respectively. We can take $[\Sigma_2]$ and $[\Sigma_f]$ to generate $H_2(B_4, \mathbb{Z})$, but we note that this is not the natural basis of $S^2 \times S^2$. In particular, the intersections of the 2-cycles are

$$[\Sigma_f] \cap [\Sigma_f] = 0, \quad [\Sigma_f] \cap [\Sigma_2] = 1, \quad [\Sigma_2] \cap [\Sigma_2] = 2. \quad (2.24)$$

The only non-obvious equality above is the last. This follows since the self-intersection of a 2-cycle in a 4-manifold is equal to the Chern number of the normal bundle. Similar calculations show that

$$[\Sigma_1] = [\Sigma_2] - 2[\Sigma_f]. \quad (2.25)$$

Later it will be useful to use a more natural basis given by $[C_1] = [\Sigma_2] - [\Sigma_f]$ and $[C_2] = [\Sigma_f]$; indeed one can then check that $[C_1] \cap [C_1] = [C_2] \cap [C_2] = 0$ and $[C_1] \cap [C_2] = 1$.

By Poincaré duality we have $H_2(B_4, \mathbb{Z}) \cong H^2(B_4, \mathbb{Z})$. Recall that, by definition, the Poincaré dual $\eta_{\Sigma}$ of a submanifold $\Sigma \subset M$ satisfies

$$\int_{\Sigma} \omega = \int_{M} \omega \wedge \eta_{\Sigma} \quad (2.26)$$

for any closed form $\omega$. We introduce the closed 2-forms on $B_4$

$$\sigma_2 = \frac{1}{4\pi} J, \quad \sigma_f = \frac{1}{2\pi} [(A_w(y_1) - A_w(y_2))^2] \gamma_{\Sigma_f}(A_w(y_1) - A_w(y_2)) dy \wedge D\psi. \quad (2.27)$$

These forms satisfy

$$\int_{\Sigma_f} \sigma_2 = \int_{\Sigma_f} \sigma_f = 1, \quad \int_{\Sigma_2} \sigma_f = \int_{\Sigma_2} \sigma_2 = 0, \quad (2.28)$$

and one finds that Poincaré duality maps $\Sigma_f \mapsto \sigma_2$ and $\Sigma_2 \mapsto \sigma_f + 2\sigma_2$.

### 2.3. $M_5 = S^1 \times S^2$

We next construct $M_5$ as the total space of a circle bundle over $B_4$, by letting $w$ be periodic with period $2\pi l_w$, for a suitably chosen $l_w$. We begin by observing from (2.12) that the norm of the Killing vector $\partial_w$ is nowhere vanishing, and so the size of the $S^1$ fibre does not degenerate anywhere. Recalling that $Dw = dw - A_w D\psi$, we require that $l_w^{-1} A_w D\psi$ is a connection on a bona fide $U(1)$ fibration with first Chern class represented by $(2\pi l_w)^{-1} d(A_w D\psi)$.

It is straightforward to first check that $(2\pi l_w)^{-1} d(A_w D\psi)$ is indeed a globally defined 2-form on $B_4$. We next impose that it has integer valued periods:

$$\frac{1}{2\pi l_w} \int_{\Sigma_2} d(A_w D\psi) = 2 l_w A_w(y_1) = p$$

$$\frac{1}{2\pi l_w} \int_{\Sigma_f} d(A_w D\psi) = l_w [A_w(y_2) - A_w(y_1)] = -q. \quad (2.29)$$
where \( p, q \) are positive integers. One can then calculate
\[
\frac{1}{2\pi l_w} \int_{\Sigma_1} d(A_w D\psi) = \frac{2}{l_w} A_w(y_1) = p + 2q
\]
as expected from (2.25). We then deduce that
\[
\beta = \frac{q}{p+q}
\]
which, remarkably, is independent of \( Q \), and
\[
l_w = \frac{2-Z}{p(1-Z)(1+\beta)}.
\]
With these choices we have that \( M_5 \) is the total space of a circle bundle with first Chern class given by
\[
c_1 = p[\sigma_2] - q[\sigma_f] \in H^2(B_4, Z).
\]
As in [27], taking \( p \) and \( q \) to be relatively prime, as we shall henceforth do, one can show that \( M_5 \) is simply connected with
\[
H^2(M_5, \mathbb{Z}) \cong \mathbb{Z},
\]
and for \( H^3(M_5, \mathbb{Z}) \cong \mathbb{Z} \), which will be useful both for integration using Poincaré duality and also for quantizing the 3-form flux. We also find representatives of the generating 2-cycle and 3-cycle in \( H^2(M_5, \mathbb{Z}) \cong \mathbb{Z} \) and \( H^3(M_5, \mathbb{Z}) \cong \mathbb{Z} \), respectively.

The generator of \( H^2(M_5, \mathbb{Z}) \cong \mathbb{Z} \) may be taken to be the pull-back of the class
\[
\tau = b[\sigma_2] + a[\sigma_f] \in H^2(B_4, \mathbb{Z})
\]
under the projection
\[
\pi : M_5 \to B_4,
\]
where \( a \) and \( b \) are (any) integers satisfying
\[
pa + qb = 1.
\]
These exist and are unique up to \( b \to b + mp, a \to a - mq, \) for any integer \( m \), by Bezout’s lemma. The non-uniqueness simply corresponds to the fact that the Chern class \( c_1 = p\sigma_2 - q\sigma_f \) of the circle bundle over \( B_4 \) is trivial when pulled back to \( M_5 \), as is the Chern class of any tensor power of this circle bundle (the power corresponds to the integer \( m \) above).

To see that \( \pi^*\tau \) is the generator of \( H^2(M_5, \mathbb{Z}) \) as claimed, note that, a priori, \( \pi^*\tau \) is necessarily \( \delta \) times the generator, for some integer \( \delta \in \mathbb{Z} \). Thus we write \( \pi^*\tau = \delta \in H^2(M_5, \mathbb{Z}) \cong \mathbb{Z} \). Next note that the circle bundle \( \pi \) trivializes over any \( S \subset B_4 \) that represents the cycle
\[
[S] = q[\Sigma_2] + p[\Sigma_f].
\]
This is simply because the first Chern class \( c_1 \) evaluated on \( [S] \) is zero, as one sees using (2.28). Hence we may take a section \( s \) of \( \pi \) over \( S \):
\[
s : S \to M_5.
\]

Although \( S \) certainly exists, in practice it is not easy to define such a smooth submanifold in the above coordinate system.
This defines a 2-cycle \([s(S)]\) in \(H_2(M_5, \mathbb{Z}) \cong \mathbb{Z}\), which we may take to be \(\alpha\) times the generator, for some integer \(\alpha\). But then by construction
\[
\int_{s(S)} \pi^* \tau = \int_S \tau = 1, \tag{2.39}
\]
implying that \(\alpha \delta = 1\), and thus \(\alpha\) and \(\delta\) are both \(\pm 1\). Hence \(\pi^* \tau\) generates \(H^2(M_5, \mathbb{Z})\), and \(s(S)\) generates \(H_2(M_5, \mathbb{Z})\).

The only other non-trivial homology group is \(H_1(M_5, \mathbb{Z}) \cong \mathbb{Z}\). There are three natural 3-submanifolds of \(M_5\), which we call \(E_1\), \(E_2\) and \(E_f\). These are the restriction of the circle bundle \(\pi\) to the submanifolds \(\Sigma_1\), \(\Sigma_2\) and \(\Sigma_f\) of \(B_4\), respectively. These 3-manifolds are all Lens spaces\(^8\). Indeed, \(\Sigma_1\), \(\Sigma_2\), \(\Sigma_f\) are all 2-spheres. The Chern numbers are easily read off from \(c_1\) above to be \(p + 2q\), \(p\) and \(-q\). Thus
\[
E_1 \cong \mathbb{S}^3/\mathbb{Z}_{p+2q}, \quad E_2 \cong \mathbb{S}^3/\mathbb{Z}_p, \quad E_f \cong \mathbb{S}^3/\mathbb{Z}_q. \tag{2.40}
\]
We may take the generator of \(H_3(M_5, \mathbb{Z})\) to be
\[
E = k[E_1] + l[E_f]
\]
where \(k\) and \(l\) are (any) integers satisfying
\[
 pk + ql = 1. \tag{2.41}
\]
Note this is the same as (2.36), so one could choose \(k = a\) and \(l = b\). A simple way to check this is to note that the generator has intersection number 1 with \([s(S)]\). One computes
\[
[s(S)] \cap E = pk + ql = 1 \tag{2.43}
\]
which uniquely identifies \(E\) as the generator. We then have
\[
[E_1] = pE, \quad [E_2] = (p + 2q)E, \quad [E_f] = qE, \tag{2.44}
\]
which again can be shown by taking intersection numbers with \([s(S)]\).

Finally, we may also write down a representative \(\Phi\) of the generator of \(H^3(M_5, \mathbb{Z})\). By definition this is a closed 3-form on \(M_5\) that integrates to 1 over \(E\). We choose
\[
\Phi = \frac{1}{(2\pi)^2 l_{v_0}^2} \left\{ D w \wedge \left[ (A_w(y_1) + A_w(y_2) - A_w(y)) J \right] - \partial_y A_w \, dy \wedge D \psi \right\} - \left[ A_w'(y) - A_w(y_1) A_w'(y_1) + A_w(y_1) A_w(y_2) + A_w(y_1) A_w(y_2) \right] J \wedge D \psi \}. \tag{2.45}
\]
The 3-form \(\Phi\) is Poincaré dual to the non-trivial 2-cycle in \(M_5\).

2.4. \(M_6 = S^3 \times S^3\)

We now construct \(M_6\) as a circle bundle over \(M_5\). Since \(H^2(M_5, \mathbb{Z}) \cong \mathbb{Z}\), such circle bundles are determined, up to isomorphism, by an integer. Since \(M_5 \cong S^3 \times S^2\), taking this integer to be 1 (or \(-1\)) gives a total space \(M_6 \cong S^3 \times S^3\). Taking the Chern number to be \(n\) would instead lead to an \(M_6\) with \(\pi_1(M_6) \cong \mathbb{Z}_n\), which we may always lift to the simply connected cover with \(n = \pm 1\). So, we will do this. However, as we shall see later, in fixing the 3-form flux quantization it will be helpful to consider such quotients of \(M_6\).

Observe from (2.12) that the norm of the Killing vector \(\alpha_\nu\) is nowhere vanishing, and so the size of the \(S^1\) fibre does not degenerate anywhere. The period of \(v\) is taken to be \(2\pi l_v\), where \(l_v\) will be fixed shortly. Recalling that \(Dv = dv - A_v D\psi\), we require that \(l_v^{-1} A_v D\psi\) is a connection on a \(U(1)\) fibration with first Chern class represented by \((2\pi l_v)^{-1} d(A_v D\psi)\). It is straightforward to check that \((2\pi l_v)^{-1} d(A_v D\psi)\) is a globally defined 2-form on \(M_6\). We

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See appendix A for some discussion.

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Footnote:

8 See appendix A for some discussion.
next impose that it has unit period. To do this we would like to integrate \((2\pi l_v)^{-1} d(A_v D\psi)\) over a smooth submanifold in the same homology class as \(s(S)\), the generator of \(H_2(M_5, \mathbb{Z})\). However, as we have already noted, finding such a smooth submanifold is not so easy. Luckily, we can use Poincaré duality to calculate the period instead. Recalling that \([\Phi]\) is Poincaré dual to \([s(S)]\), we demand that

\[
\frac{1}{2\pi l_v} \int_{s(S)} d(A_v D\psi) = \frac{1}{2\pi l_v} \int_{M_5} d(A_v D\psi) \wedge \Phi
\]

\[
= \frac{2}{l_v} [A_v(y_2)A_w(y_1) - A_v(y_1)A_w(y_2)]
\]

\[
= \frac{1}{l_v} [2q A_v(y_2) - p(A_v(y_1) - A_v(y_2))] = 1,
\]

so that the circle bundle has Chern number 1, which can be achieved by setting

\[
l_v = \frac{Q(p + q)}{1 - Z}.
\]

Let us denote this circle bundle over \(M_5\) by \(L\), with corresponding projection

\[
\Pi: M_6 \to M_5.
\]

Recalling that the generator of \(H^2(B_4, \mathbb{Z})\) may be taken to be the pull-back of \(\tau\) in (2.34) under the projection \(\pi: M_5 \to B_4\), we see that \(L\) may be regarded as the pull-back of the circle bundle \(L_5\) over \(B_4\), with first Chern class given by \(\tau \in H^2(B_4, \mathbb{Z})\). We write this as \(L = \pi^*L_5\).

Since \(M_6 \cong S^3 \times S^3\), it follows that the only non-trivial homology group is \(H_3(M_6, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}\). The two generators are clearly the two copies of \(S^3\), at a fixed point on the other copy. However, because of the way we have constructed \(M_6\) above, it is not easy to see the diffeomorphism of \(M_6\) with \(S^3 \times S^3\) explicitly. Nevertheless, we observe that one 3-cycle is represented by the total space of the circle bundle \(L\) over the \(S^2\) in \(M_5 \cong S^3 \times S^2\).

Since \(s(S)\) is homologous to the \(S^2\) in \(M_5\), it follows\(^9\) that taking the total space of \(L\) over both submanifolds gives homologous 3-submanifolds of \(M_6\), which is the total space of \(L\). Thus the total space of the \(L\) circle bundle over \(s(S)\) is one of the generators of the homology of \(M_6\). It should be pointed out, though, that finding a smooth representative of this generator is not straightforward. For the other generator, the obvious thing to try is to take a representative for \(E\), which after all is represented by \(S^3 \subset M_5\), and then try to take a section of \(\Pi\) over this representative. However, unfortunately just because two submanifolds are homologous in \(M_5\), with \(L\) trivial over one of them, this does not necessarily guarantee that the circle bundle \(L\) is trivial over the other submanifold\(^10\). So, we cannot necessarily do this. An additional observation is that, while a section of \(\Pi\) exists over \(E\), it does not exist, in general, over the submanifolds \(E_1, E_2\) and \(E_f\), as we explain in the appendix.

In order to carry out the flux quantization of the 3-form in the supergravity solutions, we need a prescription to integrate 3-forms over a basis of \(H_3(M_6, \mathbb{Z})\). The comments in the

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\(^9\) Being homologous in \(M_6\) means there is a three-dimensional chain in \(M_6\) with boundary \(s(S) - S^2\). By taking the total space of \(L\) over this chain, one obtains a chain in \(M_6\) with boundary given by the total space of \(L\) over \(s(S) - S^2\).

\(^10\) As a simple example, consider the 5-manifold \(T^{1,1} \cong S^2 \times S^1\), which recall is naturally a circle bundle over \(S^3 \times S^2\). For our two 3-submanifolds we take a contractible \(S^1\), say the equatorial \(S^0\) on a contractible \(S^0\) that links a point, and the ‘diagonally embedded’ Lens space \(S^1/\mathbb{Z}_2\). Since \(T^{1,1}\) is a circle bundle over \(S^3 \times S^2\), we may describe the latter 3-submanifold more precisely as the restriction of this circle bundle to the diagonal \(S^2 \times S^2\), which is the easiest way to see that the topology is indeed \(S^3/\mathbb{Z}_2\). Both 3-cycles are trivial—to see this for the latter construct the generator of \(H^2(T^{1,1}, \mathbb{Z})\) and integrate over the \(3\)-cycle. However, if we pull-back the complex line bundle \(O(1, 0)\) on \(S^2 \times S^2\) with winding numbers 1 and 0 on \(S^3 \times S^2\) to \(T^{1,1}\), this is trivial over the \(S^3\) but non-trivial over the contractible \(S^3\) \((\mathbb{Z}_2\) the latter follows using arguments similar to those in appendix B).
last paragraph indicate that this is not as straightforward as it might seem. Our approach, employing a quotient construction\textsuperscript{11}, will be explained in the following subsection.

2.5. A quotient of $M_6$ and integral 3-forms

In this section we want to explain how considering the periods of the 3-form on the quotient $\hat{M}_6 = M_6/\mathbb{Z}(p+q)$ leads to a practical procedure for ensuring that a 3-form, such as the suitably normalized RR 3-form, has integral periods.

In order to obtain more insight into the topology of $M_6$, it will be helpful to think of it as a group manifold, $M_6 = S^3 \times S^3 \cong SU(2) \times SU(2)$, and observe that taking the quotient by the maximal torus $T_2 \subset SU(2) \times SU(2)$ leads to $B_4$:

\begin{equation}
M_6/T^2 = S^2 \times S^2 = B_4. \tag{2.50}
\end{equation}

Now, recall that we constructed $M_5$ as the total space of a circle bundle over $B_4$ with winding numbers $p$ and $-q$ over $\Sigma_1$ and $\Sigma_f$, respectively. With respect to the natural basis $[C_1] = [\Sigma_2] - [\Sigma_f]$ and $[C_2] = [\Sigma_f]$ of $B_4 \cong S^2 \times S^2$ introduced in section 2.2, we thus have Chern numbers

\begin{align}
\int_{[C_1]} c_1 &= p + q, \\
\int_{[C_2]} c_1 &= -q. \tag{2.51}
\end{align}

In this section we make the $U(1)$ fibration structure of $M_5$ explicit in the notation by denoting the latter as $M_5(p, q)$.

The key observation is that we may realize $M_5(p, q)$ as a quotient by the $U(1)$ subgroup of $T^2$ with charges $(q, p+q)$, as illustrated in the following diagram:

\begin{equation}
\begin{array}{ccc}
U(1)_{q, p+q} & \hookrightarrow & M_6 \\
\downarrow & & \downarrow \\
T^2/\mathbb{Z}(p+q) & \hookrightarrow & M_5(p, q) \rightarrow B_4.
\end{array} \tag{2.52}
\end{equation}

To see this more explicitly, we introduce Euler angles, $\psi_1, \theta_1, \phi_1$ and $\psi_2, \theta_2, \phi_2$ for each of the two $SU(2)$ factors. We also introduce the corresponding left-invariant 1-forms $\sigma_\alpha^i$ for each factor, respectively, where $\alpha = 1, 2; i = 1, 2, 3$. Thus

\begin{align}
\sigma_1^1 &= \cos \psi_1 \, d\theta_1 + \sin \theta_1 \sin \psi_1 \, d\phi_1 \\
\sigma_1^2 &= -\sin \psi_1 \, d\theta_1 + \sin \theta_1 \cos \psi_1 \, d\phi_1 \\
\sigma_1^3 &= d\psi_1 + \cos \theta_1 \, d\phi_1.
\end{align} \tag{2.53}

Now $\psi_1, \psi_2 \in [0, 4\pi)$ parametrize the $T^2$. The $U(1)_{q, p+q}$ circle action is then given explicitly by

\begin{equation}
(\psi_1, \psi_2) \mapsto (\psi_1 + q \psi, \psi_2 + (p + q) \psi), \tag{2.54}
\end{equation}

where $\psi \in [0, 4\pi)$ parametrizes the circle subgroup. If we introduce coordinates $\tilde{v}, \tilde{w}$ defined by

\begin{equation}
\tilde{v} = -\frac{1}{q}\psi_1, \quad \tilde{w} = (p + q)\psi_1 - q\psi_2 \tag{2.55}
\end{equation}

\textsuperscript{11} We thank Dominic Joyce for suggesting this approach.
then the $T^2$ is parametrized by taking $\tilde{v}, \tilde{w} \in [0, 4\pi)$. In these coordinates the $U(1)_{q,p+q}$ circle action reads
\begin{equation}
(\tilde{v}, \tilde{w}) \mapsto (\tilde{v} - \psi, \tilde{w})
\end{equation}
and hence $\tilde{w}$ parametrizes the circle $T^2/U(1)_{q,p+q}$. The globally defined connection 1-form on the total space of the circle bundle on the bottom line of (2.52) is given by
\begin{align}
\eta &= \frac{1}{2}((p+q)\sigma_1^1 - q\sigma_2^1) \\
&= \frac{1}{2}(d\tilde{w} + (p+q)\cos\theta_1 \, d\phi_1 - q \cos\theta_2 \, d\phi_2).
\end{align}
We can define two natural copies of $S^2$ in $B_4$ to be $C_1$ and $C_2$, which are round $S^2$s at the north pole of the other. So, $C_1 = \{\theta_2 = 0\}$, $C_2 = \{\theta_1 = 0\}$. We observe that (2.57) gives rise to Chern numbers $p+q$ and $-q$ for $C_1$ and $C_2$, respectively, as required for $M_5(p, q)$.

Let us denote the total space over each sphere $C_1$ and $C_2$ in $M_5(p, q)$ to be $F_1$ and $F_2$, respectively. Then by following similar arguments as in (2.40)–(2.44) we deduce that
\begin{equation}
F_1 \cong S^3/\mathbb{Z}_{p+q}, \quad F_2 \cong S^3/\mathbb{Z}_q
\end{equation}
and also the homology relations
\begin{equation}
[F_1] = (p+q)[S^3], \quad [F_2] = q[S^3].
\end{equation}
In fact one can see (2.58) rather explicitly from the above quotient construction. We define $W_1 \cong S^3$ and $W_2 \cong S^3$ to be the two natural copies of $S^3$ in $M_6$ given by $W_1 = \{\theta_2 = 0, \psi_2 = 0\}$, $W_2 = \{\theta_1 = 0, \psi_1 = 0\}$. Consider now $\{\theta_2 = 0\} \subset M_6$. This is $W_1 \times S^1 \cong S^3 \times S^1$,
\begin{equation}
\end{equation}
where the $S^1$ is parametrized by $\psi_2$. When we take the quotient by the $U(1)_{q,p+q}$ circle action (2.54) we may set $\psi_2 = 0$. However, there is then a remaining gauge freedom given by setting
\begin{equation}
\psi = \frac{4\pi k}{p+q},
\end{equation}
with $k = 1, \ldots, p+q$, since this also fixes $\psi_2 = 0$. This then acts on $\psi_1$, which is the Hopf fibre of $W_1$ realized as an $S^1$ bundle over $S^2$, and we see explicitly that $F_1 \cong S^3/\mathbb{Z}_{p+q}$. A similar argument applies to $F_2$.

We next observe that
\begin{equation}
\Phi = \frac{1}{8\pi^2}[(p+q)\eta \wedge \sigma_1^1 \wedge \sigma_2^1 + q\eta \wedge \sigma_1^2 \wedge \sigma_2^1]
\end{equation}
is a closed globally defined 3-form on $M_5(p,q)$. We see explicitly that
\begin{equation}
\int_{F_1} \Phi = \frac{p+q}{8\pi^2} \int_{F_1} \eta \wedge \sigma_1^1 \wedge \sigma_2^1 = p+q.
\end{equation}
which shows that $\Phi$ generates $H^3(M_5(p,q), \mathbb{Z})$.

Next it is convenient to define $M_6$ to be
\begin{equation}
M_6 = M_6/\mathbb{Z}_{p+q}\mathbb{Z}_q
\end{equation}
where we embed $\mathbb{Z}_{p+q}\mathbb{Z}_q$ along $U(1)_{q,p+q}$. This defines a quotient
\begin{equation}
f : M_6 \rightarrow M_6.
\end{equation}
The action on the Euler angles is
\begin{equation}
(\psi_1, \psi_2) \mapsto \left( \psi_1 + \frac{4\pi kq}{(p+q)q}, \psi_2 + \frac{4\pi k(p+q)}{(p+q)q} \right)
\end{equation}
\begin{equation}
= \left( \psi_1 + \frac{4\pi k}{p+q}, \psi_2 + \frac{4\pi kq}{q} \right).
\end{equation}
Here $k = 1, \ldots, (p + q)q$. This realizes the $\mathbb{Z}_{(p+q)q}$ action as a $\mathbb{Z}_{p+q} \times \mathbb{Z}_q$ action (the groups are isomorphic as $p + q$ and $q$ are coprime) and we have

$$M_6 \cong (S^1/\mathbb{Z}_{p+q}) \times (S^3/\mathbb{Z}_q). \quad (2.67)$$

In terms of $\hat{v}$, $\hat{w}$ we have

$$(\hat{v}, \hat{w}) \mapsto \left( \hat{v} = \frac{4\pi k}{(p + q)q}, \hat{w} \right). \quad (2.68)$$

Thus on $\hat{M}_6$ we can introduce a new coordinate $\hat{v} = (p + q)q \tilde{v}$ with period $4\pi$ and we also have

$$\hat{M}_6 \cong (S^1/\mathbb{Z}_{p+q}) \times S^3. \quad (2.69)$$

A key point is that the $\tilde{v}$ circle bundle trivializes over both $F_1$ and $F_2$. One way to see this is to observe that the $\tilde{v}$ circle bundle has first Chern class being $q(p + q)$ times the generator of $H^2(M_3(p, q), \mathbb{Z})$ and then following the arguments in the appendices. We can also see this directly. Consider again

$$W_1 \times S^1 \quad (2.70)$$

where the $S^1$ is coordinatized by $\psi_2$. The action of $\mathbb{Z}_{p+q}$ is given by $(2.66)$. We first set $k = nq$, with $n = 1, \ldots, p + q$. This defines a $\mathbb{Z}_{p+q}$ subgroup that acts trivially on $\psi_2$, but acts non-trivially on $W_1$, with quotient $W_1/\mathbb{Z}_{p+q} \cong S^1/\mathbb{Z}_{p+q} = F_1$. We may then set $k = 1, \ldots, q$ in the identification. This now acts trivially on $W_1/\mathbb{Z}_{p+q}$, but acts non-trivially on $S^1$ to give $S^1/\mathbb{Z}_q \cong S^1$. This shows explicitly that

$$(W_1 \times S^1)/\mathbb{Z}_{p+q} \cong F_1 \times S^1 \quad (2.71)$$

which in turn shows that the $\hat{v}$ bundle restricted to $F_1$ is trivial, as it is manifestly a product. Obviously, similar reasoning applies to $F_2$.

Let us now define $V_1$ and $V_2$ to be the obvious two factors of $\hat{M}_6$ in $(2.67)$. Because of the discrete identification $(2.66)$, $W_1$ is a $(p + q)$-fold cover of $V_1$, and $W_2$ is a $q$-fold cover of $V_2$. Thus for any 3-form $\Psi$ on $\hat{M}_6$ we have

$$\int_{W_1} f^* \Psi = (p + q) \int_{V_1} \Psi, \quad \int_{W_2} f^* \Psi = q \int_{V_2} \Psi. \quad (2.72)$$

Here $f^* \Psi$ is obtained by simply replacing $\hat{v}$ in $\Psi$ with $(p + q)q \tilde{v}$.

For example, if we let $\Pi : M_6 \to M_5(p, q)$ be the projection for the fibration in the second column in $(2.52)$, then $\Pi^* \Phi$ is a 3-form on $\hat{M}_6$ that is invariant under $f$ (it has no dependence on the coordinate $\hat{v}$). It is therefore obviously the pull-back of a 3-form on the quotient $\hat{M}_6$, and hence we may use $(2.72)$ to calculate

$$\int_{W_1} \Pi^* \Phi = (p + q)^2, \quad \int_{W_2} \Pi^* \Phi = q^2. \quad (2.73)$$

Finally, we are in a position to provide our prescription for quantizing the RR flux. We first observe that while we may take $C_2 = \Sigma_f$, we cannot quite take $C_1$ to be $\Sigma_2 \cup (-\Sigma_f)$, because the two submanifolds intersect at a point and we do not have a smooth submanifold. We may remedy this by cutting out a small neighbourhood of the intersection point and gluing in a cylinder. This results in a 2-sphere, which we can take to be $C_1$. We may then identify

$$F_1 = E_2 \cup (-E_f), \quad F_2 = E_f \cong S^1/\mathbb{Z}_q. \quad (2.74)$$

12 A point we shall return to later, in passing, is that the above arguments show that for the quotient $M_6/\mathbb{Z}_{p+q}$ the corresponding circle bundle trivializes over $F_1$, while for $M_6/\mathbb{Z}_q$ it trivializes over $F_2$. We consider $M_6/\mathbb{Z}_{p+q}$ as it trivializes over both.
with the understanding that $F_1$ is to be smoothed out into $S^3/\mathbb{Z}_q$, rather than the union of $S^3/\mathbb{Z}_q$ with $S^3/\mathbb{Z}_p$ over the circle where they intersect. As we have shown, on $M_6$ the $\hat{v}$ circle fibration trivializes over $F_1$ and $F_2$, and hence we may take sections giving submanifolds $V_1$ and $V_2$. The correct quantization condition for an integral 3-form on $M_6$ (such as our appropriately normalized RR 3-form), in a workable form, is then given by \((2.72)\), where the integrals over $W_1$ and $W_2$ are integers $M_1, M_2$.

3. Flux quantization and the central charge

In order to obtain a good solution to string theory, we need to impose that both the electric and magnetic RR 3-form charges are properly quantized. In this section, we analyse this in detail and then derive the central charge of the corresponding \((0, 2)\) SCFT. We conclude the section with a discussion of taking the limit $Q \to 0$ in our solutions and the relationship to the $Q = 0$ solutions of [26].

3.1. Electric and magnetic charges

For the electric charge we require

$$n_1 = \frac{1}{(2\pi l_s)^2 g_s} \int_{X_7} \ast F^{(3)} \in \mathbb{Z}. \quad (3.1)$$

Since

$$\frac{1}{L_6} \ast F^{(3)} = \frac{(Z - 1)}{8(Z - 2)\beta^2 y^2} J \wedge dy \wedge D\psi \wedge Dw \wedge dv \wedge da^2 + \text{Vol}(\text{AdS}_3) \wedge (\cdots) \quad (3.2)$$

we have

$$n_1 = \left(\frac{L}{l_s}\right)^6 \frac{1}{g_s 8\pi^2 p^2 q(p + 2q)^2} \Delta u^2, \quad (3.3)$$

which we interpret as fixing the period, $\Delta u^2$, of the $u^2$ circle.

We next turn to the magnetic 3-form charge. We require that

$$\frac{1}{(2\pi l_s)^2 g_s} \int_{W} F^{(3)} \in \mathbb{Z} \quad (3.4)$$

when integrated over any 3-cycle $W \subset X_7 = M_6 \times S^1$. The relevant 3-cycles are in $M_6$, and so the quantization condition amounts to quantizing the restriction of $F^{(3)}$ to $M_6$ at a point on the $S^1$ coordinatized by $u^2$. In the previous subsection we gave a prescription for performing such integrals by instead calculating integrals on submanifolds of the quotient space $\hat{M}_6$. In the following subsection we will calculate these integrals by introducing explicit coordinate patches. This will illuminate and confirm many of our observations about the topology in the previous section. Furthermore, the techniques will be essential for the generalization that we consider in section 4.

In the present case, however, there is a much simpler way to impose flux quantization. The key observation is that, remarkably, the relevant part of $F^{(3)}$ is in the same cohomology class as $\Phi$. Indeed we have

$$\frac{1}{L^2} F^{(3)} - 2\text{Vol}(\text{AdS}_3) = \frac{(2\pi l_s)^2 (1 - Z)}{(Z - 2)q\beta} \Phi + d\{K_1 Dv \wedge Dw + K_2 Dw \wedge D\psi\} \quad (3.5)$$

\(^{13}\) Here we are not distinguishing between $\Phi$ and $\Pi^* \Phi$. 

13 Here we are not distinguishing between $\Phi$ and $\Pi^* \Phi$. 

13
where

$$K_1 = \frac{Q(-2y + 3Zy - Z^2y + 1 - \beta^2 - Z + Z\beta^2)}{\beta(Z - 2)^2}$$

$$K_2 = \frac{(1 - \beta^2)(1 - Z)U(y)}{(-1 + \beta^2 + 2y - Zy)(-1 + \beta^2 + Zy)(2 - Z)}.$$  \hspace{1cm} (3.6)

Note in particular that the function $K_2$ vanishes at $y_1$ and $y_2$, ensuring that the 2-form $K_1 Dv \wedge Dw + K_2 Dw \wedge D\psi$ is globally defined. We thus conclude that

$$\frac{1}{(2\pi l_s)^2 g_s} \int_W F^{(3)} = -\frac{L^2}{l_s^2 g_s} \frac{(p + q)^2}{pq^2(p + 2q)} \int_\Phi.$$  \hspace{1cm} (3.7)

Furthermore, we have already calculated the periods of $\Phi$ (more precisely, $\Pi^* \Phi$) over a basis of 3-cycles on $M_6$ in (2.73). We find that if the length scale is taken to be

$$L^2 = \frac{pq^2(p + 2q)M}{(p + q)^2}$$  \hspace{1cm} (3.8)

for some positive integer $M$, then

$$M_1 = -\frac{1}{(2\pi l_s)^2 g_s} \int_W F^{(3)} = M(p + q)^2, \quad M_2 = -\frac{1}{(2\pi l_s)^2 g_s} \int_W F^{(3)} = Mq^2.$$  \hspace{1cm} (3.9)

We may now calculate the central charge of the dual SCFT. It is given by [31]

$$c = \frac{3R_{AdS_3}}{2G_{(3)}}$$  \hspace{1cm} (3.10)

where $G_{(3)}$ is the three-dimensional Newton’s constant and $R_{AdS_3}$ is radius of the AdS3 space. In our conventions the type IIB supergravity Lagrangian has the form

$$\frac{1}{(2\pi)^2 g_s^2 l_s^2} \sqrt{-\det g} e^{-2\phi} R + \ldots$$  \hspace{1cm} (3.11)

and after a short calculation we find

$$c = 6n_1 \left( \frac{L}{l_s} \right)^2 \frac{1}{g_s} = 6n_1 \frac{pq^2(p + 2q)M}{(p + q)^2} = 6n_1 \frac{(M_1 - M_2)M_2}{M_1}.$$  \hspace{1cm} (3.12)

This result is independent of $Q$ and since the solutions with $Q \neq 0$ and all other parameters fixed are smoothly connected, we conclude that the parameter $Q$ corresponds to an exactly marginal deformation in the dual SCFT. It is interesting to observe that the central charge is the same as that of the $Q = 0$ solutions studied in [26]. However, as we shall explain in section 3.3, taking the limit $Q \to 0$ does not smoothly give the $Q = 0$ solutions of [26] and so it is not clear whether or not the $Q \neq 0$ solutions are exactly marginal deformations of those with $Q = 0$.

### 3.2. Computing periods using coordinate patches

In this subsection we directly compute the flux of $F^{(3)}$ through the two 3-cycles of $M_6$ using coordinate patches. This provides a nice cross-check on various calculations carried out so far. Furthermore, we will use this method in the following section when we construct more general type IIB string theory solutions—there we will not be able to use the approach in the last subsection since the 3-form flux will no longer be in the same cohomology class as $\Pi^* \Phi$. 

14
Recall from section 2.5 that instead of considering the circle bundle $L$ over $M_5$ with total space $M_6$, we should consider the circle bundle $\hat{L} = L^{(p\times q)q}$ with total space $\hat{M}_6 = M_6/\mathbb{Z}(p\times q)q$. This is useful since $\hat{L}$ trivializes over both the submanifolds $F_1$, a smoothed out version of $E_2 \cup -E_f$, and $F_2 \equiv E_f$ of $M_5$. We may thus take sections of $\hat{L}$ over these submanifolds to obtain submanifolds $V_1$ and $V_2$ of $\hat{M}_6$. Then the quantization of the 3-form flux on $M_6$, through the two 3-cycles $W_1$, $W_2$, is related to that on $\hat{M}_6$ via the general formulae (2.72).

In particular, this procedure involves trivializing the circle bundle $\hat{L}$ over $F_1$ and $F_2$. Concretely, this means that the corresponding connection 1-form is a globally defined 1-form over $F_1$ and $F_2$. However, to see this requires carefully covering the manifold with coordinate patches, so that the connection form is represented by a globally defined 1-form on each patch, and then gluing these forms together on overlaps using $U(1)$ transition functions. Only when one has picked a gauge where the connection 1-form is globally defined on $F_1$, $F_2$ can one then represent a section by taking the (appropriately gauge transformed) $\nu$ coordinate to be constant in the 3-form flux $F^{(3)}$. This might sound overly technical, but if one does not follow this carefully one obtains incorrect periods for the flux.

We begin by covering $M_5$ with four coordinate patches: $U_{1N}$, $U_{2N}$, $U_{1S}$ and $U_{2S}$. Here, for example, $U_{1N}$ is defined by removing $\{y = y_2\}$ and $\{\theta = \pi\}$, while $U_{1S}$ is defined by removing $\{y = y_2\}$ and $\{\theta = 0\}$. On $B_4$ the points we remove in each case are two $S^2$s that intersect over a point. It follows that, regarded as defining subsets of $B_4$, the above conditions give four patches diffeomorphic to $\mathbb{R}^4$. On $M_5$ we thus obtain patches diffeomorphic to $S^1 \times \mathbb{R}^4$, with the $S^1$ in each patch parametrized by a coordinate $w_{1N}$, $w_{2N}$, $w_{1S}$, $w_{2S}$, respectively.

Recall that $B_4$ is constructed as an $S^2$ bundle over $S^1$, where the fibre $S^2$ has poles $\{y = y_1\}, \{y = y_2\}$. Removing these, one can define a global 1-form:

$$D\psi = D\psi_N = dy_N + (1 - \cos \theta) d\phi$$
$$= D\psi_S = dy_S - (1 + \cos \theta) d\phi.$$  \(\text{(3.13)}\)

The corresponding space is an $I \times S^1$ bundle over $S^2$, where $I = (y_1, y_2)$ is an open interval, and the circle $S^1$ is parametrized by $\psi_N$ and $\psi_S$, each with period $2\pi$. Here the first expression is valid on the complement of the south pole $\{\theta = \pi\}$, while the second is valid on the complement of the north pole $\{\theta = 0\}$. This is because the azimuthal coordinate $\phi$ degenerates at the poles of the base $S^2$. On the overlap one has

$$\psi_S - \psi_N = 2\phi$$  \(\text{(3.14)}\)

which shows that the $S^1$ bundle has Chern number $-2$. This is because the connection form is locally $\cos \theta \, d\phi$, and so has curvature form $-\sin \theta \, d\theta \wedge d\phi$, which integrates to $-2 \times 2\pi$ over the $S^2$. It is important that $D\psi$ is not defined at $\{y = y_1\}$, since these are coordinate singularities.

Recalling (2.13), we next define the global 1-form on $M_5$:

$$Du = Dw_{1N} = dw_{1N} + A_u(y_1) \, d\psi_N - A_u \, D\psi_N$$
$$= dw_{2N} + A_u(y_2) \, d\psi_N - A_u \, D\psi_N$$
$$= dw_{1S} + A_u(y_1) \, d\psi_S - A_u \, D\psi_S$$
$$= dw_{2S} + A_u(y_2) \, d\psi_S - A_u \, D\psi_S.$$  \(\text{(3.15)}\)

These are defined on the four patches $U_{1N}$, $U_{2N}$, $U_{1S}$, $U_{2S}$, respectively. Take, for example, $Dw_{1N}$. $\psi_N$ is a coordinate on the complement of the south pole of the base $S^2$, although it degenerates at $y = y_1$. However, at $y = y_1$ we have

$$Dw_{1N}|_{y = y_1} = dw_{1N} - A_u(y_1)(1 - \cos \theta) \, d\phi,$$  \(\text{(3.16)}\)

and we see that $w_{1N}$ is indeed a good coordinate on the $S^1$ of $U_{1N} \equiv S^1 \times \mathbb{R}^4$. The period of all the $w$ coordinates above is $2\pi l_w$. 

15
One can immediately see the fibration structure of the $w$ circle bundle, with total space $M_5$, from the above formulæ. For example, on the overlap region where both are defined, we have
\begin{equation}
\frac{1}{l_v} (w_{2N} - w_{1N}) = q \psi_N.
\end{equation}
In particular, restricting to $\{ \theta = 0 \}$, which is $E_f$, we see that the circle bundle has Chern number $-q$ and thus $E_f \cong S^3/Z_q$. Similarly,
\begin{equation}
\frac{1}{l_v} (w_{2S} - w_{2N}) = -p \phi
\end{equation}
showing that the Chern number over $E_2 = \{ y = y_2 \}$ is $p$, thus proving that $E_2 \cong S^3/Z_p$.

In each of the patches we define the connection 1-form that appears in the $v$ circle fibration over $M_5$ to give $M_v$. Recalling (2.13) we write $Dv \equiv dv - A'$ and define
\begin{align}
A'_1 &= -A_v(y_1) d\psi_N + A_v D\psi_N + l_v \lambda_{1N} \frac{dw_{1N}}{l_v} \\
A'_2 &= -A_v(y_2) d\psi_N + A_v D\psi_N + l_v \lambda_{2N} \frac{dw_{2N}}{l_v} \\
A'_5 &= -A_v(y_1) d\psi_S + A_v D\psi_S + l_v \lambda_{1S} \frac{dw_{1S}}{l_v} \\
A'_8 &= -A_v(y_2) d\psi_S + A_v D\psi_S + l_v \lambda_{2S} \frac{dw_{2S}}{l_v}.
\end{align}

Here $\lambda_{1N}, \lambda_{2N}, \lambda_{1S}, \lambda_{2S}$ are constants to be fixed by the requirement that the $(1/l_v)A'$ patch together to give a connection 1-form. We choose $\lambda_{1N} = \lambda_{2N} = \lambda_{1S} = \lambda_{2S} = \lambda$ with
\begin{equation}
\frac{A_v(y_1) - A_v(y_2)}{l_v} + \lambda q = -a, \quad \frac{2A_v(y_2)}{l_v} + \lambda p = b,
\end{equation}
where $a, b$ are integers satisfying $ap + bq = 1$, which is possible because of (2.46). Consider first the overlap of $U_{1N}$ with $U_{2N}$. On this overlap we have
\begin{equation}
\frac{1}{l_v} [A'_{2N} - A'_{1N}] = -a \, d\psi_N.
\end{equation}
Since $\psi_N$ has period $2\pi$ and $a$ is an integer, we see that the two connections do indeed differ by a $U(1)$ gauge transformation. Next consider the overlap of $U_{1S}$ with $U_{2N}$. Here we have
\begin{equation}
\frac{1}{l_v} [A'_{2S} - A'_{1N}] = -b \, d\phi.
\end{equation}
It is illuminating to compare with equations (2.36) and (B.4), (B.7) in appendix B. In particular, we see that (3.21) and (3.22) give the torsion Chern classes over $E_f$ and $E_2$, respectively.

As a further check, we compute
\begin{equation}
\frac{1}{l_v} [A'_{1S} - A'_{1N}] = -(b - 2a) \, d\phi,
\end{equation}
which is equivalent to the Chern number of the $w$-fibration over $\Sigma_4$ being $p + 2q$ and agrees with (B.6). Note that, conversely, if one allows general $\lambda$ in (3.19) and instead imposes that the connections differ by $U(1)$ gauge transformations (3.21), (3.22) on the overlaps, then one finds the solution (3.20).

For a more detailed explanation of the relation between the transition functions (3.21), (3.22) and torsion Chern classes, we refer to appendix A.
Now consider $\hat{M}_2$, where we divide the period of $v$ by $q(p + q)$. Note immediately that the connection form on $U_{2N} \cap U_{1N}$ is

$$\frac{q(p + q)}{l_v} [A'_{2N} - A'_{1N}] = -a(p + q) \left[ \frac{dw_{2N}}{l_v} - \frac{dw_{1N}}{l_v} \right].$$

(3.24)

Thus we may define

$$\frac{q(p + q)}{l_v} \hat{A}'_{1N} = \frac{q(p + q)}{l_v} A'_{1N} + a(p + q) \frac{dw_{1N}}{l_v},$$

$$\frac{q(p + q)}{l_v} \hat{A}'_{2N} = \frac{q(p + q)}{l_v} A'_{2N} + a(p + q) \frac{dw_{2N}}{l_v}.$$  

(3.25)

These are good gauge transformations on each patch. We see that $\hat{A}'_{1N}$ and $\hat{A}'_{2N}$ agree on the overlap, and thus define a globally defined 1-form on the complement of $\{ \theta = \pi \}$. In particular, this shows explicitly that the $\hat{L}$ circle bundle over $E_f$, i.e. the $v$ bundle over $E_f$ (with the period above), is trivial\(^{15}\). A globally defined connection 1-form is provided by \( \frac{q(p + q)}{l_v} \hat{A}' \) above, restricted to $\{ \theta = 0 \}$.

Remarkably, the factors of $a$ and $b$ in $\hat{A}'$ now cancel, and the connection form reduces to

$$\hat{A}'_{1N} = -A_v(y_1) d\psi_N + A_v D\psi_N + \frac{l_v}{2q(p + q)} \frac{dw_{1N}}{l_v}.$$  

(3.26)

We are now in a position to calculate the integral of the 3-form flux over $V_2$. Recall that the submanifold $V_2$ is obtained as a section of the $\hat{L}$ circle bundle over $F_2 \equiv E_f$. In $F^{(3)}$ we therefore set $\theta = 0$ and replace $Dv = dv - A'$ with $-\hat{A}'_{1N}$. After some calculation we obtain

$$\frac{1}{(2\pi l_v)^2 g_s} \int_{V_2} F^{(3)} = \frac{L^2}{l_v^2 g_s} \frac{(p + q)^2}{pq(p + 2q)} = -\frac{M_2}{q}.$$  

(3.27)

where $M_2$ is a positive integer.

It remains to calculate the integral over the submanifold $V_1$, obtained as a section of the $\hat{L}$ circle bundle over the submanifold $F_1$ obtained by smoothing out $E_2 \cup -E_f$. We cover $V_1$ by three patches: $U_{1N}, U_{2N}$ and $U_{2S}$. These will cover the northern hemisphere, equatorial strip and southern hemisphere patches, respectively, of the $S^2$ we get by gluing $\Sigma_2$ to $-\Sigma_f$. This is illustrated in figure 1. To be more precise we will cover most of $\Sigma_f$ in $U_{1N}$ by setting $\theta = 0$, letting $y \in [y_1, y_2 - \epsilon]$ with $\psi_N$ the azimuthal angle. We will cover most of $\Sigma_2$ in $U_{2S}$ by setting $y = y_2$, letting $\theta \in [\delta, \pi]$ with $\phi$ the azimuthal angle. Here $\epsilon, \delta > 0$ are small. On the overlap in $U_{2N}$ the equatorial strip, $E_{qf}$, is the line in the $\delta, y$ plane stretching from $(\theta, y) = (\delta, y_2)$ to $(\theta, y) = (0, y_2 - \epsilon)$, over which there is an azimuthal angle—at the first end of this line it is $\phi$ and at the other end it is $\psi_N$. In fact, on this strip the azimuthal angles get identified via

$$\phi = -\psi_N,$$

(3.28)

with the sign corresponding to an orientation flip.

We first examine the overlaps

$$\frac{q(p + q)}{l_v} [A'_{2N} - A'_{1N}] = -aq(p + q) \, d\psi_N = q((b - a)q - 1) \, d\psi_N$$

(3.29)

$$\frac{q(p + q)}{l_v} [A'_{2S} - A'_{2N}] = -bq(p + q) \, d\phi = -q((b - a)p + 1) \, d\phi.$$

\(^{15}\) Note that we only need to quotient the period of $v$ by $q$ to able to do this, not $q(p + q)$, as expected from the comment in footnote 8.
This leads us to define

\[
\frac{q(p+q)}{l_v} \tilde{\lambda}'_{2N} = \frac{q(p+q)}{l_v} \lambda'_{2N} + q \, d\psi_N - q(b-a) \frac{dw_{2N}}{l_w},
\]

(3.30)

which is obtained via a good gauge transformation on this patch. We then find

\[
\frac{q(p+q)}{l_v} [\tilde{\lambda}'_{2N} - \lambda'_{1N}] = q^2 (b-a) \, d\psi_N - q(b-a) \frac{dw_{2N}}{l_w} = -q(b-a) \frac{dw_{1N}}{l_w} \]

(3.31)

\[
\frac{q(p+q)}{l_v} [A'_{2S} - \tilde{\lambda}'_{2S}] = -pq(b-a) \, d\phi + q(b-a) \frac{dw_{2S}}{l_w} = q(b-a) \frac{dw_{2S}}{l_w}.
\]

(3.32)

This prompts us to define

\[
\frac{q(p+q)}{l_v} \tilde{\lambda}'_{1N} = \frac{q(p+q)}{l_v} \lambda'_{1N} - q(b-a) \frac{dw_{1N}}{l_w},
\]

(3.33)

\[
\frac{q(p+q)}{l_v} \tilde{\lambda}'_{2S} = \frac{q(p+q)}{l_v} \lambda'_{2S} - q(b-a) \frac{dw_{2S}}{l_w},
\]

which are again obtained via good gauge transformations on the patches. After all this, \(\tilde{\lambda}'\) is a globally defined 1-form on \(F_1\), and thus we see explicitly that the \(v\) bundle trivializes over it since we have divided the period by \((p+q)q\). In particular, all of the above gauge transformations are well defined.

16 Note that to obtain this result we only needed to quotient the period of \(v\) by \(p+q\) here, not \((p+q)q\). In particular, all of the above gauge transformations are well defined.
By taking $\epsilon, \delta \to 0$ we effectively use the gauge $\tilde{A}_{1N}$ over $E_2$ and $\tilde{A}_{2N}$ over $E_3$ and then consider the result for $E_3$ minus the result for $E_2$. After some calculation this gives the period

$$\frac{1}{(2\pi l_s)^2 g_s} \int_{V_1} F^{(3)} = -\frac{L^2}{l_s^2 g_s} \frac{(p + q)^3}{pq^2(p + 2q)} \equiv -\frac{M_1}{p + q},$$

(3.35)

where $M_1$ is a positive integer.

Consistency of (3.27) and (3.35) implies that we choose

$$M_1 = M(p + q)^2, \quad M_2 = Mq^2$$

(3.36)

for some positive integer $M$ and

$$\frac{L^2}{l_s^2 g_s} = \frac{pq^2(p + 2q)M}{(p + q)^2}.$$

(3.37)

We have thus recovered the results (3.8) and (3.9), which is very satisfying.

3.3. Taking the limit $Q \to 0$

We have noted that the parameter $Q \neq 0$ corresponds to an exactly marginal deformation of the underlying dual $(0, 2)$ SCFT. In particular, the solutions, with the internal manifold having topology $S^3 \times S^3 \times S^1$, and all other parameters fixed, are smoothly connected with each other as $Q$ varies and they all have the same central charge. It is interesting to observe that the value of the central charge is precisely the same as for the solutions with $Q = 0$ that were analysed in [26]. It is therefore tempting to conclude that the solutions with $Q \neq 0$ correspond to exactly marginal deformations of the SCFTS dual to the $Q = 0$ solutions. While this may in fact be the correct interpretation, it seems difficult to draw this conclusion based on the results of this paper. In particular, taking the limit $Q \to 0$ does not simply lead to the $Q = 0$ solutions discussed in [26], as we now explain.

It is certainly the case that setting $Q$ to zero in the local solutions given by (2.1), (2.2), (2.8) and (2.9) one obtains the local solutions with $Q = 0$ studied in [26]. However, we also need to check what happens to the global identifications we have made on the coordinates, and also the quantization of the flux. Recall that in the solutions of [26] with $Q = 0$ the internal manifold has topology $S^2 \times S^3 \times T^2$, with the $S^2 \times S^3$ factor realized in the same way as the $M_3$ factor for the $Q \neq 0$ solutions studied here. It is simple to check that taking the limit $Q \to 0$ in the spaces $B_1$ and $M_3$, labelled by $p, q$, studied in sections 2.2 and 2.3, respectively, leads to the corresponding spaces in the $Q = 0$ solutions of [26]. The problem arises in taking the limit of the $v$ circle fibration. In (2.46) we impose\(^\text{17}\) that this has twist 1 over the base $M_3 = S^2 \times S^3$. This is impossible as $Q \to 0$, because the connection $A_v \to 0$ and correspondingly, the period $l_v$ goes to zero in (2.47). From (2.11) we therefore conclude that the period of the $u^1$ circle goes to zero in this limit and we do not match onto the $Q = 0$ solutions of [26] where the $u^1$ circle in the $T^3$ factor had finite period.

It is worth pointing out that in the $Q = 0$ solutions of [26], with topology $S^2 \times S^3 \times T^2$, only the volume of the $T^3$ was fixed by the flux quantization. The shape of the $T^3$ therefore corresponds to exactly marginal deformations. It might be the case that going to the limit where the period of the $u^1$ circle goes to zero is connected to the $Q \to 0$ limiting solutions. However, in this limit the supergravity approximation is clearly breaking down and one needs to analyse string corrections before any definite conclusions can be drawn. It would be interesting to study this further.

\(^{17}\) If one instead imposes that this has twist 0 then (2.46), (2.22) and (2.31) imply that $q = 0$ or $p = -q$. Returning to (2.29) we see that imposing $q = 0$ and using (2.22) would imply $\beta = 0$ which is excluded. Similarly imposing $p = -q$ is not compatible with (2.22).
4. More general identifications

In this section we will generalize the class of solutions that we have already constructed. We return to the local solution (2.12), (2.8), (2.15) and then employ the general linear coordinate transformation
\[ w = hw' + rQZ' , \quad v = sw' + \frac{t}{2\beta}v' \]  
for constant \( r, t, s, h \) with
\[ \Delta = h \frac{t}{2\beta} - r \frac{Q}{Z} \neq 0. \]  
The idea is to now make appropriate periodic identifications of the new coordinates \( v', w' \). As we shall see this will embed our solutions of type IIB string theory of the last two sections into larger families.

We first observe that
\[ Dw = h Dw' + rQZv' , \quad Dv = s Dw' + \frac{t}{2\beta} Dv' , \]
where we have defined
\[ Dv' = dv' - A_v D\psi , \quad Dw' = dw' - A_w D\psi \]
with
\[ A_w = \frac{t}{2\beta \Delta} A_w - r \frac{Q}{Z \Delta} A_v , \quad A_v = \frac{h}{\Delta} A_v - \frac{s}{\Delta} A_w . \]

We now construct \( M_5 \) as a circle fibration, with circle parametrized by \( w' \), over \( B_4 \) and then construct \( M_6 \) as a circle fibration, with circle parametrized by \( v' \), over \( M_5 \). It is straightforward to write the metric in the primed coordinates and then appropriately ‘complete the square’ to make this fibration structure manifest in the metric. However, we will not need the explicit details. Observe that what will become the globally defined angular 1-form on \( M_5 \) for the \( w' \) circle fibration is \( Dw' \). After completing the square in the metric on \( M_6 \) we obtain an expression for what will become the globally defined angular 1-form corresponding to the \( v' \) circle fibration and it has the form
\[ dv = A_v D\psi - k(y) Dw' \]
for some smooth function \( k(y) \) that can easily be determined. The connection 1-form on \( M_5 \) for this circle fibration is thus \( A_v D\psi + k(y) Dw' \). This will turn out to be a local connection 1-form on the same circle bundle as that for the connection 1-form \( A_w D\psi \), since \( kDw' \) will be globally defined on \( M_5 \) (in particular, the corresponding curvature 2-forms are in the same cohomology class on \( M_5 \)). Below, for convenience, we will use the connection 1-form \( A_v D\psi \).

The analysis now proceeds in an almost identical fashion as in the last sections, so we can be brief. We choose the period of the \( w' \) circle to be \( 2\pi l_w' \) so that \( l_w' A_w D\psi \) is a connection on a \( U(1) \) fibration. We demand that \( (2\pi l_w')^{-1} d(A_v D\psi) \) has integer periods on \( B_4 \), as in (2.29), with primes on all \( w \), for some integers \( p, q \), now not necessarily positive. When \( r + t = 0 \) we have \( q = 0 \), while when \( r - t = 0 \) we have \( p + q = 0 \) and these cases require a separate analysis which we will return to later. We thus continue here with \( r \neq \pm t \) and conclude that
\[ \beta = \frac{t - r}{t + r} \frac{q}{p + q} , \quad l_w' = \frac{(2 - Z)(r + t)}{2q(1 - Z)(1 - \beta^2)\Delta} . \]

Note that if we choose \( t = \beta(Z - 2)/(Z - 1) \), \( r = -\beta Z/(Z - 1) \), \( h = (Z - 2)/2(Z - 1) \) and \( s = Z(Z - 2)/4\beta(Z - 1) \), then we have \( w' = z, v' = u \), where \( z, u \) are the coordinates that we started with in (2.2). In this case equation (4.7) becomes \( \beta = (1 - Z)/(1 + X) \) and \( l_w' = 2(1 + X)/q(X + Z)(2 + X - Z) \), where \( X = p/q \) and this agrees with the results in equation (4.22) of [26].
The topology of $M_5$ is again $S^3 \times S^2$. For the generator of $H^3 (M_5, \mathbb{Z})$ we can use the primed version of (2.45).

We now turn to the $v'$ circle fibration over $M_5$ to give $M_6$. We let $v'$ be a periodic coordinate with period $2\pi l v'$, and the connection 1-form is given by $l^{-1} A_{v'} D\psi$. To ensure that the circle fibration is well defined and that $M_6 = S^3 \times S^3$ we impose the primed version of (2.46) to conclude that

$$l v' = \frac{2q Q}{(r + t) (1 - Z)}. \quad (4.8)$$

Now we determine the flux quantization conditions. The electric flux quantization condition (3.1) fixes the period of $u^2$ as before:

$$n_1 = \frac{L^6}{l_s^8} \frac{Q}{8\pi^2 \beta (1 - \beta^2)^2} \Delta u^2. \quad (4.9)$$

For the magnetic flux quantization, we follow the same procedure as before, by introducing explicit coordinate patches and considering integrals on submanifolds of $M_6 = M_6/\mathbb{Z}_{\nu(p+q)\rho}.$

By following the same steps as in section 3.2 we find that

$$\frac{1}{(2\pi l)^2 l_s^8} \int_{v_1} F^{(3)} \equiv \frac{L^2}{l_s^8} \frac{1}{q \beta^2 - 1} \equiv -\frac{M_2}{q} \quad (4.10)$$

for integers $M_2, M_1$. Consistency implies that we must have

$$\frac{M_2}{M_1} = \beta^2 = \frac{(r - t)^2}{(r + t)^2} \frac{q^2}{(p + q)^2}, \quad (4.11)$$

which implies that $(r + t)^2/(r - t)^2$ must be rational, and that the length scale is fixed by

$$\frac{L^2}{l_s^8} \equiv (1 - \beta^2) M_2 = \frac{(M_1 - M_2) M_2}{M_1}. \quad (4.12)$$

The central charge can now be calculated, and we find that it can be expressed as

$$c = 6n_1 \frac{(M_1 - M_2) M_2}{M_1}. \quad (4.13)$$

In particular we note that, in addition to $Q$, there is also no dependence on the parameters $r, s, t$ and $h$. We note that the only restrictions on these parameters are (4.2), (4.11) with $\beta$ given in (4.7) satisfying $0 < \beta < 1$. We have thus constructed large continuous families of solutions that are dual to SCFTs. Note that, in general, the solutions of this section are not exactly marginal deformations of those in section 3: for example, in section 3 we saw that the magnetic 3-form flux quantum numbers were constrained to be of the form (3.36), whereas here there is no such constraint.

4.1. When $r = -t$;

When $r = -t$, we have $A_{w'}(y_1) = A_{w'}(y_2)$ and hence in considering the $w'$ circle fibration over $B_4$ to construct $M_5$ we find that the period over $C_2 = \Sigma_f$ vanishes, $q = 0$. We choose $p = 1$, so that the period over $C_1 = \Sigma_2 - \Sigma_f$ is one, and hence $M_5 = S^3 \times S^2$, which implies that

$$l v' = 2A_{w'}(y_2) = \frac{r(Z - 2)}{\beta (\beta^2 - 1)(Z - 1)} \Delta. \quad (4.14)$$
At this stage, there is no restriction on the parameter $\beta$ (apart from the usual $0 < \beta < 1$). We now find on $M_6$ that $E_1, E_2 \cong S^3$, and $[E_1] = [E_2]$ generate $H_3(M_6, \mathbb{Z})$. On the other hand, now $E_f \cong S^1 \times S^3$ (and hence there is a section of the $u^i$ circle fibration over $\Sigma_f$). The generator of $H^3(M_5, \mathbb{Z})$ is $\tau = \sigma_f$ i.e. $a = 1, b = 0$, in the notation of section 2.3.

In order to construct $M_6 = S^3 \times S^3$, we can again fix the period of the $u^i$ circle using $\Phi$ as in (2.46) and we find that

$$l_v = A_v(y_2) - A_v(y_1) = -\frac{\beta Q}{(Z - 1)t}.$$  \hspace{1cm} (4.15)

It is now easier to find representatives of the two generators of $H_3(M_6, \mathbb{Z})$, and we would not have to consider a quotient of $M_6$ in order to impose the flux quantization conditions. In particular, one generator of $H_3(M_6, \mathbb{Z})$, $W_1$, can be taken to be, as above, the section of the $u^i$ circle fibration over a desingularized version of $E_2 \cup -E_f$. For the other generator, $W_2$, we can take the $u^i$ circle bundle over the section $s(S)$ on $M_5$ where $S = \Sigma_f$. We note that two other 3-cycles $W', W''$ are obtained by considering a section of the $u^i$ circle fibration over $E_1$, $E_2$, respectively: we shall show that $[W'] = [W''] = [W_1] + [W_2]$.

We now introduce patches in exactly the same way as section 3.2. The analogue of (3.17) now reads $w_{2N} = w_{1N}$ and we explicitly see that the $u^i$ circle fibration is indeed trivial over $\Sigma_f$. To obtain the section $s(\Sigma_f)$ we can simply set $w_{1N} = \text{constant}$.

Moving to $M_6$, we have the analogue of the connection 1-forms as in (3.19), (3.20) with $a = 1, b = 0, p = 1, q = 0$. Equation (3.22) shows that the $u^i$ circle fibration is indeed trivial over $E_2$ and we can take a section to obtain the 3-cycle $W''$. One can then obtain the integral of the 3-form flux over $W''$ by using the connection 1-form $A_{2N}'$, and after a calculation we find

$$\frac{1}{(2\pi l_s)^2 g_s} \int_{W''} F^{(3)} = -\frac{L^2}{l_s^2 g_s} \frac{1}{\beta^2}.$$ \hspace{1cm} (4.16)

The $u^i$ circle fibration is also trivial over $E_1$. Indeed, after considering (3.22) we can see that the connection 1-form

$$A_{1S}' + 2l_v \frac{dw_{1N}}{l_w}$$ \hspace{1cm} (4.17)

is a globally defined 1-form on $E_1$. We can use this gauge to calculate the integral over $W'$ and we find exactly the same result as for $W''$.

To calculate the integral of the flux over the 3-cycle $W_2$, the $u^i$ circle bundle over the section $s(\Sigma_f)$, we just need to set $w_{2N}' = \text{constant}$ in the expression for the 3-form and then integrate. We therefore impose

$$\frac{1}{(2\pi l_s)^2 g_s} \int_{W_2} F^{(3)} = \frac{L^2}{l_s^2 g_s} \frac{1}{(1 - \beta^2)} = M_2.$$ \hspace{1cm} (4.18)

To carry out the flux integral over $W_1$, a section of the $u^i$ circle fibration over $E_2 \cup -E_f$, we define

$$\frac{1}{l_v} A_{2N}' = \frac{1}{l_v} A_{2N} + d\psi_N + \frac{d w_{2N}'}{l_w}$$
$$\frac{1}{l_v} A_{1N}' = \frac{1}{l_v} A_{1N} + \frac{d w_{1N}'}{l_w}$$
$$\frac{1}{l_v} A_{2S}' = \frac{1}{l_v} A_{2S} + \frac{d w_{2S}'}{l_w}.$$ \hspace{1cm} (4.19)

$^{19}$ Before, when $[S] = q[S_1] + p[S_f]$ it was not clear how to take a smooth representative for $S$. 

Then $A'$ is a global 1-form on $E_2 \cup -E_f$. To calculate the integral of flux over the section over the $v'$ circle bundle over $E_2 \cup -E_f$ we use $A'_{2\Sigma}$ on $E_2$ and $A'_{1\Sigma}$ on $E_f$. We find
\[
\frac{1}{(2 \pi l_s)^2 g_s} \int_{W_1} F^{(3)} = -\frac{L^2}{l_s^2 g_s} \frac{1}{\beta^2 (1 - \beta^2)} = -M_1.
\] (4.20)
Comparing (4.16) with (4.18) and (4.20), we can deduce the homology relation $[W'] = [W_1] = [W_2]$, as mentioned above.

Consistency of (4.18) and (4.20) implies that the length scale of the solution is again as in (4.12) and that $\beta^2$ is rational
\[
\beta^2 = \frac{M_2}{M_1}.
\] (4.21)
The electric flux quantization condition is given again by (4.9) and the central charge takes the form (4.13).

4.2. When $r = t$:

When $r = t$, we have $A_{v}(y_1) = -A_{v}(y_2)$ and hence in considering the $w'$ circle fibration over $B_t$ to construct $M_5$ we find that the period over $C_1 = \Sigma_2 - \Sigma_f$ vanishes, $p + q = 0$. We choose $q = 1$ so that the period over $C_2 = \Sigma_f$ is one, and hence $M_5 = S^3 \times S^2$, which implies that
\[
l_{w'} = \frac{-t(Z - 2)}{(\beta^2 - 1)(Z - 1)\Delta}.
\] (4.22)
with no restriction on the parameter $\beta$. We now find $E_1, E_2, E_f \cong S^3$, and $- [E_1] = [E_2] = [E_f]$ generate $H^3(M_5, \mathbb{Z})$. The generator of $H^2(M_5, \mathbb{Z})$ is $\tau = b_\sigma + a_\sigma_f$ with $b - a = 1$.

In order to construct $M_6 = S^3 \times S^2$, we find that the period of the $v'$ circle is
\[
l_{v'} = A_{v}(y_1) + A_{v}(y_2) = -\frac{Q}{(Z - 1)t}.
\] (4.23)
For the generators of $H_3(M_6, \mathbb{Z})$ we can take $W_1$ to be the $v'$ circle fibration over the a representative of the section $s(S)$ of $M_5$, with $[S] = [\Sigma_2] - [\Sigma_f]$. For $W_2$ we take a section of the $v'$ circle fibration over $E_f$. We note that we can also obtain 3-cycles $W', W''$ which are obtained by considering sections of the $v'$ circle fibration over $E_1, E_2$ respectively: we shall see that $- [W'] = [W''] = [W_1] + [W_2]$.

We again introduce patches in exactly the same way as section 3.2. The connection 1-forms are as in (3.19), (3.20) with $b - a = 1$ and $q = -p = 1$. By taking $a = 0, b = 1$, we see from (3.21) that $A'_{2N}$ is a globally defined connection 1-form on $E_f$. Calculating the flux integral we find that we should impose
\[
\frac{1}{(2 \pi l_s)^2 g_s} \int_{W_1} F^{(3)} = -\frac{L^2}{l_s^2 g_s} \frac{1}{\beta^2 (1 - \beta^2)} = -M_2.
\] (4.24)
To integrate the flux integrals for $W'$ one should take $b = 2, a = 1$ while for $W''$ we should take $b = 0, a = -1$ and we find
\[
-\frac{1}{(2 \pi l_s)^2 g_s} \int_{W'} F^{(3)} = -\frac{1}{(2 \pi l_s)^2 g_s} \int_{W''} F^{(3)} = \frac{L^2}{l_s^2 g_s} \frac{1}{\beta^2}.
\] (4.25)
We now turn to the flux integral over $W_1$. For $S$ we desingularize $\Sigma_2 - \Sigma_f$ as in figure 1. By making the gauge transformation $w_{2N}^l \rightarrow w_{2N}^l - l_{w'} d\psi_N$ in (3.15), we find that we obtain
a globally defined connection 1-form on $S \subset M_5$ and hence we can take a section. $W_1$ is obtained by considering the $v'$ circle fibration over this section. Thus to calculate the flux integral, one should set $w_{1N}' = \text{constant}$ in $Du_{1N}'$ for the $\Sigma_2$ piece and $w_{1S}' = \text{constant}$ in $Du_{1S}'$ for the $\Sigma_f$ piece. After doing this we find

$$\frac{1}{(2\pi l_s)^2 g_s} \int_{w_1} F^{(3)} = -\frac{L^2}{l_s^2 g_s} \frac{1}{\beta^2(1 - \beta^2)} = -M_1.$$ (4.26)

We thus find the same conditions as for the $r = -t$ case above.

5. Final comments

We have analysed in detail some local supersymmetric AdS$_3$ solutions of type IIB supergravity, first found in [26], that have non-vanishing dilaton and RR 3-form flux. We have shown that the parameters can be chosen and coordinates identified in such a way that the solutions extend to give rich classes of globally defined solutions of the form $\text{AdS}_3 \times (S^3 \times S^3 \times S^1)$ with properly quantized flux. We have shown that the solutions depend on continuous parameters and are hence dual to continuous families of SCFTs in two spacetime dimensions with $(0, 2)$ supersymmetry.

Although the internal compact spaces are diffeomorphic to $S^3 \times S^3 \times S^1$, the diffeomorphisms are far from apparent in the local coordinates that the solutions are presented in. It seems unlikely to us that there is a simple change of coordinates that will make the topology more manifest. In this paper we used a number of techniques to illuminate various aspects of the topology which, in particular, allowed us to find a workable procedure to impose flux quantization. It seems likely that our approach, or generalizations thereof, will be very useful in other contexts.

In section 4 we considered identifications on the coordinates after we made a general linear transformation on the $v, w$ coordinates. It is worth pointing out that we could consider more general linear coordinate transformations that also involve the $u^2$ coordinate. This will lead to larger families of solutions that would be worth exploring. It seems possible that some of these solutions can be obtained as $\beta$-deformations using the techniques of [38]. In fact returning to the solutions in section 2 and 3, one might wonder if $Q$ corresponds to a $\beta$-deformation. One way to see that it is not is to return to the local solutions as written down at the beginning of section 4 of [26], which are obtained after two T-dualities on the solutions we have discussed in this paper. In this duality frame only the metric and the self-dual 5-form are non-trivial for any $Q$, and in particular the dilaton is constant. However, looking at equation (A.16) of [38] we see that the $\beta$-deformation activates a non-trivial dilaton and 3-form.

It is an important outstanding issue to identify the dual $(0, 2)$ SCFTs for the solutions discussed here and in [14, 24–26]. In the duality frame that we have used in this paper, the amount of supersymmetry that is preserved combined with the fluxes that are active suggests that the dual SCFTs might arise on a D1–D5 brane system that is wrapped on a holomorphic 4-cycle in a Calabi–Yau 4-fold. While we remain hopeful that progress will be made in this direction, we note that the SCFTs dual to the much simpler type IIB $\text{AdS}_3 \times S^3 \times S^1 \times S^1$ solutions of [32], which have $(4, 4)$ supersymmetry, are still not well understood, despite interesting progress [33–36].

The AdS$_3$ solutions with $Q = 0$, that were analysed in [26], and with $Q \neq 0$ that we have discussed here, can be generalized further and we have presented some details in appendix C. It will be interesting to carry out a complete analysis of the conditions for regularity and flux quantization conditions for these more general solutions.
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Appendix A. $U(1)$ bundles over lens spaces

In this section we briefly review the lens spaces $S^3/\mathbb{Z}_q$, which appear throughout the main text, and also the construction of $U(1)$ principal bundles over these manifolds.

We construct $S^3/\mathbb{Z}_q$ as the total space of a $U(1)$ bundle over $S^2$ with Chern number $q$. Let $\theta, \phi$ be standard coordinates on $S^2$, and cover the $S^2$ with two patches: $V_N$ which excludes the south pole $\theta = \pi$, and $V_S$ which excludes the north pole $\theta = 0$. We then consider the products $S^1 \times V_N, S^1 \times V_S$, and on each space define the 1-forms
\[ D_{\nu N} = d\nu_N - \frac{q}{2} (1 - \cos \theta) \, d\phi, \quad D_{\nu S} = d\nu_S + \frac{q}{2} (1 + \cos \theta) \, d\phi. \] (A.1)

Here $\nu_N$ and $\nu_S$ are coordinates on the $S^1$s, each with period $2\pi$. If we now glue the two patches together via
\[ \nu_S - \nu_N = -q\phi \] (A.2)
on the overlap then note that
\[ D\nu = D_{\nu N} = D_{\nu S} \] (A.3)
extends to a global 1-form on the whole manifold, because the two 1-forms agree on the overlap. This is a global connection form on the total space of the $U(1)$ principal bundle $p : S^3/\mathbb{Z}_q \to S^2$ with $U(1)$ fibre parametrized by $\nu$, and is sometimes also called the global angular form.

Now consider the connection form
\[ A_N = \frac{a}{2} (1 - \cos \theta) \, d\phi, \quad A_S = -\frac{a}{2} (1 + \cos \theta) \, d\phi \] (A.4)
on the base $S^2$. This has Chern number $a \in \mathbb{Z}$ over the base $S^2$. We denote the corresponding $U(1)$ principal bundle by $P$. We may pull-back $P$ to a $U(1)$ bundle $p^*P$ over $S^3/\mathbb{Z}_q$. Pulling back the connection (A.4), on the overlap one finds
\[ A_S - A_N = -a \, d\phi = \frac{a}{q} \, d(\nu_S - \nu_N). \] (A.5)

Note that $a\nu_S/q$ is a multi-valued $U(1)$ function on the patch $S^1 \times V_S$ unless $a/q \in \mathbb{Z}$. If $a/q \in \mathbb{Z}$ then in each patch we can define the new connection 1-forms $A_S = -a \, d\nu_S/q$ and $A_N = -a \, d\nu_N/q$, and since they agree on the overlap, this defines a globally defined connection 1-form and hence $p^*P$ is trivial.

Thus $p^*P$ is trivial if and only if $a \equiv 0 \mod q$. One sees this in more a abstract way by recalling that $U(1)$ principal bundles are classified by $H^2(S^3/\mathbb{Z}_q, \mathbb{Z}) \cong H_1(S^3/\mathbb{Z}_q, \mathbb{Z}) \cong \mathbb{Z}_q$. Thus $a \in \mathbb{Z}_q$ is precisely the Chern number of $p^*P$, and the latter bundle is torsion. Because of this, the topology cannot be measured by integrating the curvature of a connection $A$ over a 2-cycle—to see torsion classes using the connection is more subtle. This is explained in
general in the paper [37]. The latter reference implies that the torsion first Chern class may be computed by picking a flat connection on \( p^*P \), and then computing the log of the holonomy of this flat connection around the 1-cycles that generate \( H_1(S^3/\mathbb{Z}_q, \mathbb{Z}) \). We may shift to a flat connection here by defining

\[
A_S^{\text{flat}} = A_S + \frac{a}{q} dv_S, \quad A_N^{\text{flat}} = A_N + \frac{a}{q} dv_N.
\]  

(A.6)

Here we have added a global 1-form \( (a/q) dv \) to the original connection—we are simply picking a different connection on the same bundle. Then \( H_1(S^3/\mathbb{Z}_q, \mathbb{Z}) \) is generated by, for example, the \( \psi_N \) circle at \( \theta = 0 \). Thus the log of the holonomy is

\[
i \int_{S'} A_N^{\text{flat}} = \frac{2\pi ia}{q} \mod 2\pi i.
\]  

(A.7)

This implies that our connection above is \( a \) times the generator of \( \mathbb{Z}_q \).

Finally, we make a comment about quotients. First note that quotienting the period of the \( U(1) \) fibre coordinate of \( P \) by \( q \) is the same as taking the \( q \)th power of \( P \). In particular, the \( \mathbb{Z}_q \) quotient of the bundle \( p^*P \) over \( S^3/\mathbb{Z}_q \) is then trivial. This follows simply because the connection on this bundle in the two patches is \( qA_S \) and \( qA_N \), or after a gauge transformation \( qA_S - a dv_S \) and \( qA_N - a dv_N \), and from (A.5) we see that this is a globally defined connection 1-form, and hence the bundle is trivial.

Appendix B. More on the topology of \( M_5 \)

Recall that, in the main text, \( M_5 \) is constructed as the total space of a circle bundle \( L \) over \( M_5 \cong S^3 \times S^3 \). Here \( c_1(L) \in H^2(M_5, \mathbb{Z}) \cong \mathbb{Z} \) is the generator, so that \( M_5 \cong S^3 \times S^3 \). Although this is straightforward as stated, the issue is that we have infinitely many coordinate systems on \( M_5 \), labelled by the integers \( p \) and \( q \), and the diffeomorphism \( M_5 \cong S^3 \times S^3 \) is not explicit for general \( p \) and \( q \). For each \( p \) and \( q \) there are different naturally defined 3-submanifolds of \( M_5 \)—we are especially interested in 3-submanifolds since we would like to quantize the RR 3-form flux. In this appendix we consider these submanifolds in more detail, and in particular determine the topology of \( L \) restricted to them.

Consider restricting this circle bundle \( L \) over \( M_5 \) to one of the 3-submanifolds of \( M_5 \): \( E_1, E_2 \) or \( E_f \). For example, take \( E_f \cong S^3/\mathbb{Z}_q \). Recall this is itself a circle bundle over \( \Sigma_f \cong S^2 \) with Chern class \( q \). There is an inclusion map \( i_f : E_f \hookrightarrow M_5 \), and we can define a circle bundle \( L_f \) over \( E_f \) by pulling back

\[
L_f \equiv i_f^* L.
\]  

(B.1)

Since \( E_f \) is a lens space, \( E_f \cong S^3/\mathbb{Z}_q \), circle bundles over \( E_f \) are classified up to isomorphism by

\[
c_1(L_f) \in H^2(E_f, \mathbb{Z}) \cong \mathbb{Z}_q.
\]  

(B.2)

To compute this Chern class, recall that \( c_1(L) = \pi^* \tau \), where \( \tau \in H^2(B_4, \mathbb{Z}) \) was defined in (2.34). Hence to compute \( c_1(L_f) = i_f^* \pi^* (\tau) \) we may instead first restrict \( \tau \) to \( \Sigma_f \), and then pull-back using \( \pi^* \) the corresponding circle bundle to \( E_f \). This is summarized by the following commutative square:

\[
\begin{array}{ccc}
H^2(M_5, \mathbb{Z}) & \xrightarrow{i_f^*} & H^2(E_f, \mathbb{Z}) \\
\pi^* \uparrow & & \uparrow \pi^* \\
H^2(B_4, \mathbb{Z}) & \xrightarrow{i_f^*} & H^2(\Sigma_f, \mathbb{Z})
\end{array}
\]  

(B.3)
Here we have denoted the embedding of $\Sigma_f$ into $B_4$ by $\iota_f : \Sigma_f \to B_4$. Then $\iota_f^*\tau$ defines an integer class in $H^2(\Sigma_f, \mathbb{Z}) \cong \mathbb{Z}$. This in turn defines a circle bundle with Chern number $a$, using (2.34). Using the results in appendix A, lifting this circle bundle to $E_f$ then gives a bundle with Chern number

$$a = c_1(L_f) \in H^2(E_f, \mathbb{Z}) \cong \mathbb{Z}_q.$$  

Thus the bundle $L$ restricted to $E_f$ is trivializable only if $a = 0 \mod q$; in other words, if $a = mq$ for some integer $m$. But if this were the case, then we would have

$$(mp + b)q = 1.$$

This is only possible if $q = \pm 1$. Thus we see that for general $q$ it is not possible to take a section of $L$ over $E_f$ to obtain a 3-submanifold of $M_6$.

One can do similar computations for the 3-submanifolds $E_1$ and $E_2$, with similar conclusions. We have

$$L_1 \equiv i_1^* L,$$  
$$c_1(L_1) = b - 2a \in H^2(E_1, \mathbb{Z}) \cong \mathbb{Z}_{p^2 2q},$$

and

$$L_2 \equiv i_2^* L,$$  
$$c_1(L_2) = b \in H^2(E_2, \mathbb{Z}) \cong \mathbb{Z}_p.$$  

Thus the corresponding bundles are trivial if and only if $b = mp, b - 2a = m_1(p + 2q)$, respectively, where $m_1, m_2 \in \mathbb{Z}$, which implies

$$p(a + q m_2) = 1, \quad (p + 2q)(a + m_1 q) = 1$$

respectively. These equations imply in particular that $p = \pm 1$ and $(p + 2q) = \pm 1$.

We thus conclude that, for generic $p$ and $q$, the circle bundle $L$ restricted to $E_1$, $E_2$ and $E_f$ is non-trivial, and thus we cannot globally take a section of $L$. This means that these natural 3-submanifolds of $M_5$ cannot be used to construct natural 3-submanifolds of $M_6$.

**Appendix C. More general AdS$_3$ solutions**

We first recall from [7, 26] the local data that is sufficient to construct supersymmetric AdS$_3$ solutions of type IIB supergravity with non-vanishing 5-form flux and complex 3-form flux $G$. We require a six-dimensional local Kähler metric $ds^2_6$ whose Ricci tensor satisfies

$$\Box R - \frac{1}{2} R^2 + R^{ij} R_{ij} + \frac{3}{2} G^{ijk} G_{ijk} = 0$$  

and $G$ must be a closed, primitive and $(-1,1)$-form on the six-dimensional space. We refer to [7, 26] for details of how the full ten-dimensional solution is constructed from this data.

For the solutions that we have discussed in this paper, which we will now generalize, the local six-dimensional Kähler metric has the form

$$ds^2_6 = ds^2_4 + ds^2(T^2)$$  

where $ds^2(T^2) = (du^1)^2 + (du^2)^2$ is the standard metric on a 2-torus, $ds^2_4$ is a four-dimensional local Kähler metric, and

$$G = d\bar{u} \wedge W$$

where $u = u^1 + i u^2$ and $W$ is a closed, primitive $(1,1)$-form on the four-dimensional Kähler space.

---

20 This analysis assumes that $p, p + 2q, q$ are non-zero.
21 Changing the sign of the last term leads to type IIB bubble solutions, as explained in [26]. The construction in this appendix can easily be adapted to construct bubble solutions.
Inspired by the six-dimensional Kähler metrics discussed in equation (5.10) of [25], we start with the ansatz for a four-dimensional Kähler metric given by
\[ ds_4^2 = \frac{Y}{4F} dw^2 + \sum_{i=1}^{2} (w + q_i) (\mu_i^2 + \mu_i^2 d\phi_i^2) + \frac{F - 1}{Y} \left( \sum_{i=1}^{2} \mu_i^2 d\phi_i \right)^2 \] (C.4)
with
\[ \sum_{i=1}^{2} \mu_i^2 = 1, \quad Y = \sum_{i=1}^{2} \frac{\mu_i^2}{w + q_i} \] (C.5)
and \( F \) an arbitrary function of \( w \). To show that the metric is Kähler we introduce the orthonormal frame
\[ e_i = \frac{1}{2\sqrt{F}} \frac{\mu_i}{\sqrt{w + q_i}} \left( dw + \sqrt{w + q_i} d\mu_i \right) \]
\[ \bar{e}_i = \frac{\sqrt{F} - 1}{Y} \frac{\mu_i}{\sqrt{w + q_i}} \sum_{j=1}^{2} \mu_j^2 d\phi_j + \sqrt{w + q_i} \mu_i d\phi_i \] (C.6)
with
\[ ds_4^2 = \sum_{i=1}^{2} (e_i \otimes e_i + \bar{e}_i \otimes \bar{e}_i) \] (C.7)
The Kähler form can be written
\[ J = \frac{i}{2} \sum_{i=1}^{2} (e_i - ie_i) \wedge (e_i + ie_i) = - \sum_{i=1}^{2} e_i \wedge \bar{e}_i \]
\[ = -\frac{1}{2} dw \wedge \sum_{i=1}^{2} \mu_i^2 d\phi_i - \sum_{i=1}^{2} (w + q_i) \mu_i d\mu_i \wedge d\phi_i \] (C.8)
which is clearly closed for any choice of \( F \).
The holomorphic \((2,0)\)-form \( \Omega \) is given by
\[ \Omega = \prod_{i=1}^{2} (e_i - ie_i) \]
\[ = \sqrt{w + q_1} \sqrt{w + q_2} \left[ \frac{Y}{2\sqrt{F}} dw \wedge d\theta - \sqrt{F} \cos \theta \sin \theta d\phi_1 \wedge d\phi_2 \right] \]
\[ - i \sqrt{w + q_1} \sqrt{w + q_2} \frac{1}{2\sqrt{F}} \cos \theta \sin \theta dw \wedge \left( \frac{d\phi_2}{w + q_1} - \frac{d\phi_1}{w + q_2} \right) \]
\[ + i \sqrt{w + q_1} \sqrt{w + q_2} \sqrt{F} d\theta \wedge (\cos^2 \theta d\phi_1 + \sin^2 \theta d\phi_2) \] (C.9)
where we have introduced \( \mu_1 = \cos \theta, \mu_2 = \sin \theta, 0 < \theta < \frac{\pi}{2} \). A calculation now shows that
\[ d\Omega = iP \wedge \Omega \] (C.10)
with
\[ P = \frac{2\sqrt{F}}{Y \sqrt{w + q_1} \sqrt{w + q_2}} \partial_w \left( \sqrt{F} \sqrt{w + q_1} \sqrt{w + q_2} (\cos^2 \theta d\phi_1 + \sin^2 \theta d\phi_2) \right) \]
\[ + \frac{1}{Y} \cos 2\theta \left( \frac{d\phi_2}{w + q_1} - \frac{d\phi_1}{w + q_2} \right). \] (C.11)

\[ \text{One can consider the scaling } \mu_3 \rightarrow \epsilon \rho, q_3 \rightarrow 1/\epsilon^2, \lambda \rightarrow \lambda/\epsilon^2 \text{ in equation (5.10) of [25] and then take } \epsilon \rightarrow 0. \]
From this we deduce that the complex structure is integrable, and thus we do indeed have a local Kähler metric with Ricci form given by $dP$. It is helpful to observe that we can also write

$$P = \partial_w [(F - 1)(w + q_1)(w + q_2)] \frac{\sum_{i=1}^2 \mu_i^2 d\phi_i}{Y(w + q_1)(w + q_2)} + d\phi_1 + d\phi_2. \quad (C.12)$$

We now construct a closed 2-form $W$ which satisfies

$$\Omega \wedge W = 0, \quad (C.13)$$

which is the condition for it to be a $(1,1)$-form, and also

$$J \wedge W = 0, \quad (C.14)$$

which is the condition for it to be a primitive 2-form. We make the ansatz

$$W = d \left[ f(w) \frac{\sum_{i=1}^2 \mu_i^2 d\phi_i}{Y(w + q_1)(w + q_2)} \right] \quad (C.15)$$

which satisfies the first equation. The second equation reads

$$J \wedge W = -\frac{\partial_w f}{Y(w + q_1)(w + q_2)} J \wedge J = 0 \quad (C.16)$$

and so we take

$$W = Q d \left[ \frac{\sum_{i=1}^2 \mu_i^2 d\phi_i}{Y(w + q_1)(w + q_2)} \right] \quad (C.17)$$

where $Q$ is a constant. The 2-form $W$ is anti-self-dual and we note that

$$W^{ij} W_{ij} = \frac{16 Q^2}{[Y(w + q_1)(w + q_2)]^2}. \quad (C.18)$$

Having fixed $W$, and hence the 3-form flux $G$, we just need to fix the function $F$ to obtain the Kähler metric $ds^2_4$ by solving (C.1) which reads

$$\Box R - \frac{1}{2} R^2 + R^{ij} R_{ij} + 4 W^{ij} W_{ij} = 0. \quad (C.19)$$

We consider the ansatz

$$F = 1 + \lambda w^2 \prod_{i=1}^2 \frac{1}{w + q_i} + \Lambda \prod_{i=1}^2 \frac{1}{w + q_i}, \quad (C.20)$$

observing from (C.12) that the constant $\Lambda$ does not enter the Ricci potential. A calculation shows that the Ricci scalar is given by

$$R = -\frac{8\lambda}{Y(w + q_1)(w + q_2)}. \quad (C.21)$$

and that (C.19) boils down to solving

$$\frac{\Lambda}{Y(w + q_1)(w + q_2)} \partial_w^2 R + W^{ij} W_{ij} = 0 \quad (C.22)$$

which implies that $\Lambda = \frac{Q^2}{\lambda}$. In summary, supersymmetric AdS$_3$ solutions of type IIB supergravity can be constructed from the six-dimensional Kähler metric (C.2), with the four-dimensional Kähler metric.
given by
\[ \text{d}s_4^2 = \frac{Y}{4F} \text{d}u^2 + \sum_{i=1}^{2} (w + q_i) \left( \text{d}\mu_i^2 + \mu_i^2 \text{d}\phi^2 \right) + \frac{F - 1}{Y} \left( \sum_{i=1}^{2} \mu_i^2 \text{d}\phi_i \right)^2 \]  
(C.23)
and
\[ F = 1 + \left( \frac{\lambda w^2 + Q^2}{\lambda} \right) \frac{1}{(w + q_1)(w + q_2)}. \]  
(C.24)
The 3-form flux is given by (C.3) with the closed, primitive and (1, 1)-form W given by
\[ W = Q \text{d} \left[ \sum_{i=1}^{2} \mu_i^2 \text{d}\phi_i \right]. \]  
(C.25)
Observe that when \( q_1 = q_2 = q \), the metric is precisely of the form found in [26] leading to the AdS_3 solutions that we have analysed in detail in this paper. To see this we let \( w + q = 1/x \) and we also introduce Euler angles via
\[ \mu_1 e^{i\phi_1} = \cos \frac{\theta}{2} e^{i\frac{\psi}{2}}, \quad \mu_2 e^{i\phi_2} = \sin \frac{\theta}{2} e^{i\frac{\psi}{2}}. \]  
(C.26)
We then find that
\[ \text{d}s_4^2 = \frac{\text{d}x^2}{4x^3 U} + \frac{1}{4x} (\text{d}\theta^2 + \sin^2 \theta \text{d} \phi^2) + \frac{U}{4x} (\text{d}\psi + \cos \theta \text{d} \phi)^2 \]  
(C.27)
with
\[ U = 1 + \lambda (1 - qx)^2 + \frac{Q^2}{\lambda} x^2 \]  
(C.28)
which should be compared with equations (C.1) and (C.7) of [26]. Furthermore,
\[ W = \frac{Q}{2} \text{d}[x (\text{d}\psi + \cos \theta \text{d} \phi)] \]  
(C.29)
which should be compared with equation (C.5) of [26]. When \( q_1 = q_2 \), the metric \( \text{d}s_4^2 \) has local isometry group \( SU(2) \times U(1) \) and the metric is cohomogeneity one. In the more general solutions with \( q_1 \neq q_2 \), the local isometry group is \( U(1) \times U(1) \) and the metric is cohomogeneity two.

It will be interesting to analyse these more general AdS_3 solutions with \( q_1 \neq q_2 \) in more detail. When \( Q = 0 \) the internal space will have topology \( S^2 \times S^3 \times T^2 \) and when \( Q \neq 0 \) it will have topology \( S^3 \times S^3 \times S^1 \). This can be shown using the techniques used in [28] and in this paper. When \( Q \neq 0 \), one will also need to check the flux quantization conditions and this will require generalizing the techniques that we have used in this paper. We leave this for the future.

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