A Clifford Bundle Approach to the Differential Geometry of Branes

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Abstract

We first recall using the Clifford bundle formalism (CBF) of differential forms and the theory of extensors acting on $\mathcal{C}(M, g)$ (the Clifford bundle of differential forms) the formulation of the intrinsic geometry of a differential manifold $M$ equipped with a metric field $g$ of signature $(p, q)$ and an arbitrary metric compatible connection $\nabla$ introducing the torsion $(2-1)$-extensor field $\tau$, the curvature $(2-2)$ extensor field $R$ and (once fixing a gauge) the connection $(1-2)$-extensor $\omega$ and the Ricci operator $\partial \wedge \partial$ (where $\partial$ is the Dirac operator acting on sections of $\mathcal{C}(M, g)$) which plays an important role in this paper. Next, using the CBF we give a thoughtful presentation the Riemann or the Lorentzian geometry of an orientable submanifold $M$ ($\dim M = m$) living in a manifold $\tilde{M}$ (such that $\tilde{M} \simeq \mathbb{R}^n$ is equipped with a semi-Riemannian metric $\tilde{g}$ with signature $(\tilde{p}, \tilde{q})$ and $\tilde{p} + \tilde{q} = n$ and its Levi-Civita connection $\tilde{D}$) and where there is defined a metric $g = i^*\tilde{g}$, where $i : M \to \tilde{M}$ is the inclusion map. We prove several equivalent forms for the curvature operator $R$ of $M$. Moreover we show a very important result, namely that the Ricci operator of $M$ is the (negative) square of the shape operator $S$ of $M$ (object obtained by applying the restriction on $M$ of the Dirac operator $\tilde{D}$ of $\mathcal{C}(\tilde{M}, \tilde{g})$ to the projection operator $P$). Also we disclose the relationship between the $(1-2)$-extensor $\omega$ and the shape biform $S$ (an object related to $S$). The results obtained are used to give a mathematical formulation to Clifford’s theory of matter. It is hoped that our presentation will be useful for differential geometers and theoretical physicists interested, e.g., in string and brane theories and relativity theory by divulging, improving and expanding very important and so far unfortunately largely ignored results appearing in reference \textsuperscript{[13].}
1 Introduction

In this paper we use the Clifford bundle formalism (CBF) in order to analyze the Riemann or the Lorentzian geometry of an orientable submanifold $M$ (dim $M = m$) living in a manifold $\hat{M}$ such that $\hat{M} \simeq \mathbb{R}^n$ is equipped with a semi-Riemannian metric $\hat{g}$ (with signature $(\hat{p}, \hat{q})$ and $\hat{p} + \hat{q} = n$) and its Levi-Civita connection $\hat{\nabla}$.

In order to achieve our objectives and exhibit some nice results that are not well known (and which, e.g., may possibly be of interest for the description and formulation of branes theories [16] and string theories [2]) we first recall in Section 2 how to formulate using the CBF the intrinsic differential geometry of a structure $(\hat{M}, \hat{g}, \hat{\nabla})$ where $\nabla$ is a general metric compatible Riemann-Cartan connection, i.e., $\nabla g = 0$ and the Riemann and torsion tensors of $\nabla$ are non

2 Curvature and Torsion Extensor of a Riemann-Cartan Connection

3 The Riemannian or Semi-Riemannian Geometry of a Submanifold $M$ of $\hat{M}$

3.1 Motivation

3.1.1 Projection Operator $P$

3.1.2 Shape Operator $S$

3.2 $S(\hat{v}) = S(v) = \partial_u \wedge P_u (\hat{v})$ and $S(\hat{v}) = \partial_u \circ P_u (\hat{v})$

3.2.1 Shape Biform $S$

3.3 Integrability Conditions

3.4 $S(v) = S(\hat{v})$

3.5 $\hat{\nabla} \wedge v = \partial \wedge v + S(v)$ and $\hat{\nabla} v = \partial v$

3.6 $\hat{\nabla} C = \partial C + S(C)$

4 Curvature Biform $\mathcal{R}(u \wedge v)$ Expressed in Terms of the Shape Operator

4.1 Equivalent Expressions for $\mathcal{R}(u \wedge v)$

4.2 $S^2(v) = -\partial \wedge \partial (v)$$$

5 On Clifford’s Little Hills

6 A Maxwell Like Equation for a Brane World with a Killing Vector Field

7 Conclusions

A Some Identities Involving $P$ and $P_u$
null. In our approach we will introduce (once we fix a gauge in the frame bundle) a \((1,2)\)-extensor field \(\omega : \sec \Lambda^1 T^* M \to \Lambda^2 T^* M\) closed related with the connection 1-forms which permits to write a very nice formula for the covariant derivative (see Eq. (2.7)) of any section of the Clifford bundle of the structure \(\langle M, g, \nabla \rangle\). It will be shown that \(\omega\) is related to \(S : \sec \Lambda^1 T^* M \to \Lambda^2 T^* M\) the shape operator biform of the manifold.

Then in Section 3, we suppose that \(M\) is a proper submanifold\(^1\) of \(M\) which \(i : M \to M\) the inclusion map. Introducing natural global coordinates \((x^1, \ldots, x^n)\) for \(M \cong \mathbb{R}^n\) we write \(\hat{g} = \sum_{i,j=1}^n \eta_{ij} dx^i \otimes dx^j \equiv \eta_{ij} dx^i \otimes dx^j\) and equip \(M\) with the pullback metric \(g := i^* \hat{g}\). We then find the relation between the Levi-Civita connection \(D\) of \(g\) and \(\hat{D}\), the Levi-Civita connection of \(\hat{g}\). We suppose that \(g\) is non degenerated of signature \((p,q)\) with \(p + q = m\).

\(\text{Cl}(M, \hat{g})\) and \(\text{Cl}(M, g)\) denote respectively the Clifford bundles of differential forms of \(M\) and \(M\). Moreover, in what follows \(\hat{g} = \sum_{i,j=1}^n \eta_{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}\) is the metric of the cotangent bundle. The Dirac operator\(^2\) of \(\text{Cl}(M, \hat{g})\) and \(\text{Cl}(M, g)\) will be denoted\(^3\) by \(\hat{\theta}\) and \(\theta\). Let \(l = n - m\) and \(\{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_m, \hat{e}_{m+1}, \ldots, \hat{e}_{m+l}\}\) an orthonormal basis for \(TU\) (\(U \subset M\)) such that \(\{e_1, e_2, \ldots, e_m\} = \{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_m\}\) is a basis for \(TU\) (\(U \subset U\)) and if \(\{\hat{\theta}^1, \hat{\theta}^2, \ldots, \hat{\theta}^m, \hat{\theta}^{m+1}, \ldots, \hat{\theta}^{m+l}\}\) is the dual basis of the \(\{e_i\}\) we have that \(\{\theta^1, \theta^2, \ldots, \theta^m\} = \{\hat{\theta}^1, \hat{\theta}^2, \ldots, \hat{\theta}^{m}\}\) is a basis for \(T^* U\) dual to the basis \(\{e_1, e_2, \ldots, e_m\}\) of \(TU\). We have, as well known\(^4\):

\[
\hat{\theta} = \sum_{i=1}^n \hat{\theta}^i D_{e_i} = \hat{\theta}^i D_{e_i}, \quad \theta = \sum_{i=1}^m \theta^i D_{e_i} = \theta^i D_{e_i},
\]

**Remark 1**: Take notice the the bold face sub and superscripts are used to denote bases \(\{e_i\}\) and \(\{\theta^i\}\) of the tangent and cotangent space of \(M\). This notation is conveniently used in what follows.

The dual basis of the natural coordinate basis \(\{\frac{\partial}{\partial x^i}\}\) is denoted in what follows by \(\{\gamma^i\}\) where, of course, \(\gamma^i = dx^i\). Moreover, we denote by \(\{\hat{e}^1, \hat{e}^2, \ldots, \hat{e}^m\}\) the reciprocal frame of \(\{e_i\}\), i.e., \(\hat{g}(\hat{e}^i, \hat{e}_j) = \delta^i_j\) and by \(\{\hat{\theta}^i\}\) the reciprocal basis of \(\{\hat{\theta}^i\}\), i.e., \(\hat{g}(\hat{\theta}^i, \hat{\theta}_j) := \hat{\theta}^i \cdot \hat{\theta}_j = \delta^i_j\). Moreover, take into account that

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\(^1\)By a proper (or regular) submanifold \(M\) of \(M\) we mean a subset \(M \subset M\) such that for every \(x \in M\) in the domain of a chart \((U, \sigma)\) of \(M\) such that \(\sigma : M \cap U \to \mathbb{R}^n \times \{1\}\), \(\sigma(x) = (x^1, \ldots, x^n, 1, \ldots, 1, t_m, \ldots, t_{m-n})\), where \(1 \in \mathbb{R}^{n-m}\).

\(^2\)For all applications in what follows take notice that \(\Lambda^m T^* M = \bigoplus_{r=0}^m \Lambda^r T^* M \hookrightarrow \text{Cl}(M, g)\), where the symbol \(\hookrightarrow\) means that for each \(x \in M\), \(\Lambda^r T^* M\) (the bundle of differential forms) is embedded in \(\text{Cl}(T^*_x M, g_x)\) and \(\Lambda^m T^*_x M \subseteq \text{Cl}(\Lambda^m T^*_x M, g_x)\).

\(^3\)Take notice that the Dirac operators used in this paper is acting on sections of the Clifford bundle. It is not to be confused with the Dirac operator which acts on sections of the spinor bundle (see details in [15]) . This last operator can be used to probe the topology of the brane, as shown in [5].

\(^4\)We follow here, whenever possible, the notation used in [10]. Note that differently from references [13, 14, 23] we use the left and right contractions operators \(\iota\) and \(\iota\) and the scalar product operator (denoted by \(\cdot\)) acting on sections of the Clifford bundle. Also our convention for the Riemann tensor makes some equations to have a different signals than ones appearing in the references just quoted.
for $i, j = 1, \ldots, m$ it is $g(\hat{\theta}^i, \hat{\theta}^j) = g(\hat{\theta}^i, \hat{\theta}^j)$. So we will also write $g(\hat{\theta}^i, \hat{\theta}^j) = \delta^i_j$. The representation of the Dirac operator $\hat{\delta}$ in the natural coordinate basis of $M$ is of course, $\sum_{i=1}^{n} \gamma^i \frac{\partial}{\partial x^i} = \sum_{i=1}^{n} \hat{\theta}^i \hat{D}_e_i$. Note that we have $\hat{\theta}^{m+1} | M = 0, \ldots, \hat{\theta}^{m+l} | M = 0$, i.e., the $\{\hat{\theta}^{m+1}, \ldots, \hat{\theta}^{m+l}\}$ for any vector field $a \in \secTU$ and $d = 1, \ldots, m + l$ we have $\hat{\theta}^{m+l} | M (a) = 0$.

We denote moreover

$$\hat{\delta} = \hat{\delta} \bigg|_M := \theta^i \hat{\delta} = \sum_{i=1}^{m} \theta^i \hat{D}_e_i = \theta^i \hat{D}_e_i,$$

the restriction of $\hat{\delta}$ on the submanifold $M$. The projection operator $P$ (an extensor field\(^6\)) on $M$ and the shape operator $S = \hat{\delta} P$: $\sec Cl(M, \hat{\delta}) \rightarrow \sec Cl(M, \hat{\delta})$ and shape biform operator of the manifold $M$, $S : \sec \Lambda^1 T^*M \rightarrow \sec \Lambda^2 T^*M$, $S(a) := -(a \cdot \partial I_n)I_m^{-1}$ (where $\tau_g = I_m = \theta^i \theta^j \cdots \theta^m$ is the volume form\(^5\) on $U \subset M$) are fundamental objects in this study. The definition of those objects are given in Section 3 and the main algebraic properties of $P$, $S$ and $S$ besides all identities necessary for the present paper are given and proved at the appropriate places.

Section 4 is dedicated to find several equivalent expressions for the curvature biform $\mathcal{R}(u, v)$ in terms of the shape operator. There we recall that the square of the Dirac operator $\hat{\delta}$ acting on sections of the Clifford bundle has two different decompositions, namely

$$\hat{\delta}^2 = -(d\delta + \delta d) = \hat{\delta} \cdot \hat{\delta} + \hat{\delta} \wedge \hat{\delta},$$

where $d$ and $\delta$ are respectively the exterior derivative and the Hodge coderivative and $\hat{\delta} \cdot \hat{\delta} + \hat{\delta} \wedge \hat{\delta}$ are respectively the covariant Laplacian and the Ricci operator. The explicit forms of $\hat{\delta} \cdot \hat{\delta}$ and $\hat{\delta} \wedge \hat{\delta}$ are given in [19] where it is shown moreover that $\hat{\delta} \wedge \hat{\delta}$ is an extensorial operator and the remarkable result

$$\hat{\delta} \wedge \hat{\delta} \theta^i = \mathcal{R}^i,$$

where the objects $\mathcal{R}^i = R^i_j \theta^j \in \sec \Lambda^1 T^*M \rightarrow \sec Cl(\hat{M}, \hat{\delta})$ with $R^i_j$ the components of the Ricci tensor associated with $D$ are called the Ricci 1-form fields. One of the main purposes of the present paper is to give (Section 5) a detailed proof of the remarkable equation

$$\hat{\delta} \wedge \hat{\delta} (v) = -S^2(v),$$

\(^6\)For a thoughtful presentation of the theory of extensor fields, see, e.g., [19].

\(^5\)The volume $\tau_g$ for on $U \subset M$ will be denoted by $I_n = \hat{\theta}^1 \hat{\theta}^2 \cdots \hat{\theta}^m$. The volume form $\tau_g$ on $U \subset M$ will be denoted $I_n = \hat{\theta}^1 \hat{\theta}^2 \cdots \hat{\theta}^m \hat{\theta}^{m+1} \cdots \hat{\theta}^{m+l} = I_m \hat{\theta}^m \hat{\theta}^{m+1} \cdots \hat{\theta}^{m+l}$.
which says that the shape biform operator is the negative square root of the Ricci operator\textsuperscript{\footnote{This result appears (with a positive sign on the second member of Eq.(5) in [13]. See also [21]. However, take into account that the methods used in those references use the Clifford algebra of multivectors and thus, comparison of the results there with the standard presentations of modern differential geometry using differential forms are not so obvious, this being probably one of the reasons why some important and beautiful results displayed in [13] are unfortunately ignored.}}. We moreover find the relation between the connection 1-forms $\omega^i$ of the Levi-Civita connection $D$, namely as the angular ‘velocity’ with which the pseudo scalar $I_m$ when it slides on $M$.

We also discuss in Section 5 if the present formalism permits to give a mathematical representation concerning Clifford’s space theory of matter. In Section 6 we show that in a Lorentzian brane containing a Killing vector field Einstein equation can be encoded in a Maxwell like equation whose source is a current given by $J = 2S^2(A)$. The article contains also an Appendix presenting some identities involving the projection operator and its covariant derivative which permit to prove Proposition 16.

In Section 7 we present our conclusions.

2 Curvature and Torsion Extensor of a Riemann-Cartan Connection

Let $u, v, t, z \in \operatorname{sec} T M$ and $u, v, t, z \in \operatorname{sec} \bigwedge^1 T^* M \rightarrow \operatorname{sec} \mathcal{C}(M, g)$ the physically equivalent 1-forms, i.e., $u = g(u, \cdot)$, etc. Let moreover $\{e_a\}$ be an orthonormal basis for $TM$ and $\{\theta^a\}$, $\theta^a \in \operatorname{sec} \bigwedge^1 T^* M \rightarrow \mathcal{C}(M, g)$ the corresponding dual basis and consider the Riemann-Cartan structure $(M, g, \nabla)$.

**Definition 2** *The form derivative of $M$ is the operator*

\[
\partial : \operatorname{sec} \mathcal{C}(M, g) \rightarrow \operatorname{sec} \mathcal{C}(M, g),
\]

\[
\partial C := \theta^a \partial e_a C
\]  

*where $\partial e_a$ is the Pfaff derivative of form fields*

\[
\partial e_a C := \sum_{r=0}^m \partial e_a \langle C \rangle_r
\]  

such that if $\langle C \rangle_r$ is expanded in the basis generated by $\{\theta^a\}$, i.e., $\langle C \rangle_r = C_r = \frac{1}{r!} C_{i_1 \cdots i_r} \theta^{i_1 \cdots i_r} \in \operatorname{sec} \bigwedge^r T^* M \rightarrow \operatorname{sec} \mathcal{C}(M, g)$ it is

\[
\partial e_a \langle C \rangle_r := \frac{1}{r!} e_a (C_{i_1 \cdots i_r} \theta^{i_1 \cdots i_r}) = \frac{1}{r!} e_a (C_{i_1 \cdots i_r}) \theta^{i_1 \cdots i_r}.
\]  

Given two different pairs of basis $\{e_a, \theta^a\}$ and $\{e'_a, \theta'^a\}$ we have that

\[
\theta^a \partial e_a C = \theta'^a \partial e'_a C,
\]
since for all $\mathcal{C}$,
\[
\bar{\partial}'\mathcal{C}_r = \theta^a e_a' \mathcal{C}_r = \theta^a e_a' \left( \frac{1}{r!} C_{i_1 \ldots i_r} \theta^{i_1 \ldots i_r} \right) = \theta^a e_a(\frac{1}{r!} C_{i_1 \ldots i_r} \theta^{i_1 \ldots i_r}). \tag{10}
\]

**Remark 3** We recall also that any biform $B \in \sec \wedge^{2} T^* M \rightarrow \sec \mathcal{Cl}(M, g)$ and any $A_r \in \wedge^{r} T^* M \rightarrow \sec \mathcal{Cl}(M, g)$ with $r \geq 2$ it holds that
\[
BA_r = B \cdot A_r + B \times A_r + B \wedge A_r. \tag{11}
\]
where for any $\mathcal{C}, \mathcal{D} \in \sec \mathcal{Cl}(M, g)$
\[
\mathcal{C} \times \mathcal{D} = \frac{1}{2}(\mathcal{C} \mathcal{D} - \mathcal{D} \mathcal{C}) \tag{12}
\]
We observe that for $v \in \wedge^{1} T^* M \rightarrow \sec \mathcal{Cl}(M, g)$ it is
\[
B \times v = B_v = -v \mathcal{B}. \tag{13}
\]
Call $\bar{\partial} := \theta^a \nabla e_a$ the Dirac operator associated with $\nabla$, a general Riemann-Cartan connection. In [19] it is introduced the Dirac commutator of two 1-form fields $u, v \in \sec \wedge^{1} T^* M \rightarrow \sec \mathcal{Cl}(M, g)$ associated with $\nabla$ by
\[
[u, v] = (u \cdot \bar{\partial})v - (v \cdot \bar{\partial})u - [u, v] \tag{14}
\]
where
\[
[u, v] = (u \cdot \bar{\partial})v - (v \cdot \bar{\partial})u, \tag{15}
\]
which we call the (form) torsion operator.

**Definition 4** For a metric compatible connection $\nabla$, recalling the definition of the torsion operator$^8$ we conveniently write
\[
\tau(u, v) = [u, v], \tag{15}
\]
which we call the (form) torsion operator.

**Remark 5** We recall the action of the operator$^9$ $\partial_u (u \in \sec \wedge^{1} T^* M \rightarrow \sec \mathcal{Cl}(M, g))$ acting on an extensor field $F : \sec \wedge^{r} T^* M \rightarrow \sec \wedge^{r} T^* M, u \rightarrow F(u)$. If $u = u^i \theta_i$, $\partial_u := \theta^k \frac{\partial}{\partial u^k}$ acting on $F(u)$ is given by
\[
\partial_u F(u) := \theta^k \frac{\partial}{\partial u^k} F(u^i \theta_i) := \theta^k \frac{\partial}{\partial u^k} u^i F(\theta_i)
= \theta^k F(\theta_i) = \theta^k \mathcal{B}(\theta_i), \tag{16}
\]

$^8$We have, e.g., that if $[e_a, e_b] = \epsilon^{a}_{bc} e_c$, then $[\theta_a, \theta_b] = \epsilon^{a}_{bc} \theta_c$. $^9$The torsion operator of a connection $\nabla$ is the mapping $\tau : \sec TM \times \sec TM \rightarrow \sec TM$, $(u, v) \rightarrow \tau(u, v) = \nabla_u v - \nabla_v u - [u, v]$. $^{10}$More details on the concept of a general derivative operator $\partial_A (A \in \sec \mathcal{Cl}(M, g))$ acting on a general multiform field $E : \sec \mathcal{Cl}(M, g) \rightarrow \sec \mathcal{Cl}(M, g)$ may be found, e.g., in [19] where several explicit examples are given.
Also the action of the operator \( \partial_u \wedge \partial_v \) performing on an extensor field \( G : \text{sec} \Lambda^1 T^*M \times \text{sec} \Lambda^1 T^*M \mapsto \text{sec} \Lambda^r T^*M \), \((u, v) \mapsto G(u, v)\) is given by

\[
\partial_u \wedge \partial_v G(u, v) = \theta^k \frac{\partial}{\partial u^k} \wedge \theta^l \frac{\partial}{\partial u^l} u^m u^n G(\theta^m, \theta^n)
\]

\[
= \theta^k \wedge \theta^l G(\theta_k, \theta_l). \tag{17}
\]

**Definition 6** The mapping

\[
t : \text{sec} \Lambda^2 T^*M \to \text{sec} \Lambda^1 T^*M, \\
t(B) = \frac{1}{2} B \cdot (\partial_u \wedge \partial_v) \tau(u, v). \tag{18}
\]

is called the \((2, 1)\)-extensorial torsion field and

\[
t(u \wedge v) = \tau(u, v). \tag{19}
\]

Indeed, from Eq. (18) we have taking \( B = a \wedge b \)

\[
t(a \wedge b) = \frac{1}{2}(a \wedge b) \cdot (\partial_u \wedge \partial_v) \tau(u, v). \tag{20}
\]

Now,

\[
(\partial_u \wedge \partial_v) \tau(u, v) = (\theta^k \wedge \theta^l) \tau(\theta_k, \theta_l). \tag{21}
\]

Then,

\[
t(a \wedge b) = \frac{1}{2}(a \wedge b) \cdot (\theta^k \wedge \theta^l) \tau(\theta_k, \theta_l) = \tau(a, b). \tag{20}
\]

**Definition 7** The extensor mapping

\[
\Theta : \text{sec} \Lambda^1 T^*M \to \text{sec} \Lambda^2 T^*M, \\
\Theta(c) = \frac{1}{2} (\partial_u \wedge \partial_v) \tau(u, v) \cdot c, \tag{22}
\]

is called the Cartan torsion field.

We have that

\[
t(u \wedge v) = \partial_c (u \wedge v) \cdot \Theta(c)
\]

and if \( \nabla^c_{ab} \theta^b := -\omega_{ac}^b \cdot \theta^c \) then

\[
z \cdot t(u \wedge v) = z^d u^a v^b T^d_{ab}, \\
T^c_{ab} = \omega^c_{ab} - \omega^c_{ba} - c^c_{ab}. \tag{23}
\]

**Definition 8** The connection \((1, 2)\)-extensor field \( \omega \) in a given gauge is given by \((v = g(v, ))\)

\[
\omega : \text{sec} \Lambda^1 T^*M \to \text{sec} \Lambda^2 T^*M, \\
v \mapsto \omega(v) = \frac{1}{2} v^c \omega^a_{bc} \theta^a \wedge \theta^b. \tag{24}
\]
We also introduce the operator
\[ \omega : \sec \bigwedge^1 T^* M \to \sec \bigwedge^2 T^* M, \]
\[ v \mapsto \omega(v) = \omega_v := \frac{1}{2} v^c \omega^a b \theta^a \wedge \theta^b \]  
(25)
and it is clear that
\[ \omega(v) = \omega_v. \]  
(26)

One can immediately verify that for any \( C \in \sec \mathcal{C}(M, g) \) we have
\[ \nabla_v C = \partial_v C + \frac{1}{2} [\omega_v, C] \]  
(27)
\[ = \partial_v C + \omega_v \times C, \]
where \( \omega_v \times C := \frac{1}{2} (\omega_v C - C \omega_v) \) is the commutator of sections of the Clifford bundle.

**Remark 9** Note for future reference that if \( v = g(v,v) \) then
\[ v \times C = v_c C. \]  
(28)

Also take notice that
\[ \nabla_v C = v^c \partial_c \]  
(29)

**Definition 10** The form curvature operator is the mapping
\[ \rho : \sec (\bigwedge^1 T^* M \times \bigwedge^1 T^* M) \to \text{End} \bigwedge^1 T^* M, \]
\[ \rho(u, v) = [u^c \partial_c, v^c \partial_c] - [u, v]^c \partial_c \]
\[ = [\nabla_u, \nabla_v] - \nabla_{[u,v]} \]
with \( u = g(u, ), v = g(v, ) \), \( u, v \in \sec TU \subset \sec TM \)

**Definition 11** The form curvature extensor is the mapping
\[ \rho : \sec (\bigwedge^1 T^* M \times \bigwedge^1 T^* M \times \bigwedge^1 T^* M) \to \sec \bigwedge^1 T^* M, \]
\[ \rho(u, v, w) = [u^c \partial_c, v^c \partial_c, w^c \partial_c] - [u, v]^c \partial_c w \]
\[ = [\nabla_u, \nabla_v] w - \nabla_{[u,v]} w \]

\[ ^{11}\text{For a rigorous derivation of this formula using the concept of connections as 1-forms on a principal bundle with values in a given Lie algebra see }[17]. \]

\[ ^{12}\text{As well known the curvature operator of a general connection } \nabla \text{ is the mapping } \rho : \sec (TM \times TM) \to \text{EndTM, } \rho(u, v) = [\nabla_u, \nabla_v] - \nabla_{[u,v]} \text{.} \]
with \( u = g(u, ) \), \( v = g(v, ) \), \( w = g(w, ) \), \( u, v, w \in \text{sec} \, TU \subset TM \).

It is obvious that for any Riemann-Cartan connection we have

\[
\rho(u, v, w) = -\rho(v, u, w), \tag{30}
\]

One can easily verify that for a Levi-Civita connection we have

\[
\rho(u, v, w) + \rho(v, w, u) + \rho(w, u, v) = 0. \tag{31}
\]

Note however that Eq. (31) is not true for a general connection.

**Definition 12** The mapping

\[
R : \text{sec} \left( \bigwedge^1 T^* M \right)^4 \rightarrow \bigwedge^0 T^* M,
\]

\[
R(a, b, c, w) = -\rho(a, b, c) \cdot w, \tag{32}
\]

with \( a = g(a, ) \), \( b = g(b, ) \), \( c = g(c, ) \), \( w = g(w, ) \) and \( u, v, w, c \in \text{sec} \, TU \subset TM \) is called the curvature tensor.

One can show immediately that for the connection \( \nabla \)

\[
R(a, b, c, w) = -R(b, a, c, w), \tag{33}
\]

\[
R(a, b, c, w) = -R(a, b, w, c), \tag{34}
\]

and that for a Levi-Civita connection

\[
R(a, b, c, w) = R(c, w, a, b), \tag{35}
\]

\[
R(a, b, c, w) + R(b, c, a, w) + R(c, a, b, w) = 0, \tag{36}
\]

Equation (36) is known as the first Bianchi identity.

**Proposition 13** There exists a smooth \((2,2)\)-extensor field,

\[
\mathcal{R} : \text{sec} \bigwedge^2 T^* M \rightarrow \bigwedge^2 T^* M, \tag{37}
\]

\[
B \mapsto \mathcal{R}(B)
\]

called the curvature biform such that for any \( a, v, c, d \in \text{sec} \, \bigwedge^1 T^* M \) we have

\[
R(a, b, c, d) = \mathcal{R}(a \wedge b) \cdot (c \wedge d) = -(c \wedge d) \cdot \mathcal{R}(a \wedge b) \tag{38}
\]

Such \( B \mapsto \mathcal{R}(B) \) is given by

\[
\mathcal{R}(B) = -\frac{1}{4} B \cdot (\partial_a \wedge \partial_b) \partial_c \wedge \partial_d \rho(a, b, c) \cdot d, \tag{39}
\]

and we also have

\[
\mathcal{R}(a \wedge b) = -\frac{1}{2} \partial_c \wedge \partial_d \rho(a, b, c) \cdot d. \tag{40}
\]
Proof. First, we verify that Eq. (39) and Eq. (40) are indeed equivalent. Indeed, Eq. (39) implies Eq. (40), since we have

\[ \mathfrak{R}(a \wedge b) = -\frac{1}{4}(a \wedge b) \cdot (\partial_p \wedge \partial_q) \partial_c \wedge \partial_d \rho(p, q, c) \cdot d \]
\[ = -\frac{1}{4} \det \begin{bmatrix} a \cdot \partial_p & a \cdot \partial_q \\ b \cdot \partial_p & b \cdot \partial_q \end{bmatrix} \partial_c \wedge \partial_d \rho(p, q, c) \cdot d \]
\[ = -\frac{1}{4} (a \cdot \partial_p b \cdot \partial_q - a \cdot \partial_q b \cdot \partial_p) \partial_c \wedge \partial_d \rho(p, q, c) \cdot d \]
\[ = -\frac{1}{2} (a \cdot \partial_p b \cdot \partial_q) \partial_c \wedge \partial_d \rho(p, q, c) \cdot d \]
\[ = -\frac{1}{2} \partial_c \wedge \partial_d \rho(a, b, c) \cdot d . \]

Also, Eq. (40) implies Eq. (39) since taking into account that

\[ B = \frac{1}{2} B \cdot (\partial_a \wedge \partial_b)a \wedge b \]

we have

\[ \mathfrak{R}(B) = \mathfrak{R}(\frac{1}{2} B \cdot (\partial_a \wedge \partial_b)a \wedge b) \]
\[ = \frac{1}{2} B \cdot (\partial_a \wedge \partial_b) \mathfrak{R}(a \wedge b) \]
\[ = -\frac{1}{4} B \cdot (\partial_a \wedge \partial_b) \partial_c \wedge \partial_d \rho(a, b, c) \cdot d . \]

Now, we show the validity of Eq. (38). We have taking into account Eq. (40)

\[ \mathfrak{R}(a \wedge b) \wedge (c \wedge d) = -\frac{1}{2} (c \wedge d) \cdot (\partial_p \wedge \partial_q) \rho(a, b, p) \cdot q \]
\[ = -\frac{1}{2} \det \begin{bmatrix} c \cdot \partial_p & c \cdot \partial_q \\ d \cdot \partial_p & d \cdot \partial_q \end{bmatrix} \rho(a, b, p) \cdot q \]
\[ = -\frac{1}{2} (c \cdot \partial_p d \cdot \partial_q - c \cdot \partial_q d \cdot \partial_p) \rho(a, b, p) \cdot q \]
\[ = -c \cdot \partial_p d \cdot \partial_q \rho(a, b, p) \cdot q \]
\[ = -\rho(a, b, c) \cdot d = \mathfrak{R}(a, b, c, d), \]

and the proposition is proved. □

**Proposition 14** The curvature biform \( \mathfrak{R}(u \wedge v) \) is given by\(^{13}\)

\[ \mathfrak{R}(u \wedge v) = u \cdot \partial \omega (v) - v \cdot \partial \omega (u) + \omega(u) \times \omega(v). \quad (41) \]

\(^{13}\)Note that in Eq. (12) \( [u, v] \) is the standard Lie bracket of vector fields \( u \) and \( v \) and \( [u, v] := u \cdot \partial v - v \cdot \partial u \) is the commutator of the 1-form fields \( u \) and \( v \). More details if necessary may be found in \[^{19}\].
Proof. The proof is given in three steps (a), (b) and (c).

(a) We first show that Eq. (41) can be written as

\[ R(u \wedge v) = u \cdot \delta \omega_v - v \cdot \delta \omega_u - \frac{1}{2} [\omega_v(u), \omega_u] \]

\[ = \nabla_u \omega_v - \nabla_v \omega_u - \frac{1}{2} [\omega_v(u), \omega_u] - \omega_{[u,v]}, \]  

(42)

with \( u = g(u, \cdot) \), \( v = g(v, \cdot) \). Indeed, we have

\[ u \cdot \delta \omega(v) = u \cdot \delta \omega(v) + \frac{1}{2} [\omega(u), \omega(v)]. \]  

(43)

and recalling the definition of the derivative of an extensor field, it is:

\[ (u \cdot \delta \omega)(v) \equiv u \cdot \delta \omega(v) := u \cdot \delta \omega(v) - \omega(u \cdot \delta v). \]  

(44)

we have,

\[ u \cdot \delta \omega(v) - v \cdot \delta \omega(u) = u \cdot \delta \omega(v) - v \cdot \delta \omega(u) + \omega(u \cdot \delta v) - \omega(v \cdot \delta u) \]

\[ = u \cdot \delta \omega(v) - v \cdot \delta \omega(u) + \omega([u,v]) \]

\[ = u \cdot \delta \omega_v - v \cdot \delta \omega_u - \omega_{[u,v]}, \]  

(45)

and using the above equations in Eq. (41) we arrive at Eq. (42).

(b) Next we show (by finite induction) that for any \( C \in \sec \ell(M, g) \) we have

\[ ([\nabla_u, \nabla_v] - \nabla_{[u,v]})C = \frac{1}{2} [R(u \wedge v), C], \]  

(46)

with \( R(u \wedge v) \) given by Eq. (42). Given that any \( C \in \sec \ell(M, g) \) is a sum of nonhomogeneous differential forms, i.e. \( C = \sum_{p=0}^{n} C_p \) with \( C_p \in \sec \wedge^p T^* M \rightarrow \sec \ell(M, g) \) and taking into account that \( C_p = \frac{1}{p!} C_{i_1 \cdots i_p} \theta_{i_1} \cdots \theta_{i_p} \) it is easy to verify the formula for \( p \)-forms. We first verify the validity of the formula for a 1-form \( \theta^i \in \sec \wedge^1 T^* M \rightarrow \sec \ell(M, g). \) Using Eq. (42) and the Jacobi identity

\[ [\omega_v, [\omega_u, \theta_i]] + [\omega_u, [\theta_i, \omega_v]] + [\theta_i, [\omega_v, \omega_u]] = 0, \]  

(47)

we have that

\[ \frac{1}{2} [R(u \wedge v), \theta^i] \]  

(48)

\[ = \frac{1}{2} \left\{ (\nabla_u \omega_v \theta^i - \theta^i \nabla_u \omega_v + \frac{1}{2} [\omega_v(u), \theta^i]) - (\nabla_v \omega_u \theta^i - \theta^i \nabla_v \omega_u - [\omega_v(u), \theta^i]) \right\} \]

\[ = \frac{1}{2} \left\{ (\nabla_u \omega_v \theta^i + \frac{1}{2} [\omega_v, \omega_u] \theta^i) - (\nabla_v \omega_u \theta^i) - \frac{1}{2} [\omega_v(u), \theta^i] \right\} \]

\[ = \frac{1}{2} \left\{ (\nabla_u \omega_v \theta^i + \frac{1}{2} [\omega_v, \omega_u] \theta^i) - (\nabla_v \omega_u \theta^i) - \frac{1}{2} [\omega_v(u), \theta^i] \right\} \]

\[ = \nabla_u \omega_v \theta^i - \nabla_v \omega_u \theta^i - \nabla_{[u,v]} \theta^i \]. \]  

(49)
Now, suppose the formula is valid for $p$-forms. Let us calculate the first member of Eq. (40) for the $(r + 1)$-form $\theta^{1\cdots r+1} = \theta^1 \theta^2 \cdots \theta^{r+1}$. We have:

$$\nabla_u \nabla_v (\theta^{1\cdots r+1}) = \nabla_v \nabla_u (\theta^{1\cdots r+1}) - \nabla_{[u,v]} (\theta^{1\cdots r+1})$$

$$= \nabla_u ((\nabla_v \theta^1) \theta^{2\cdots r+1} + \theta^1 \nabla_v \theta^{2\cdots r+1}) - \nabla_v ((\nabla_u \theta^1) \theta^{2\cdots r+1} + \theta^1 \nabla_u \theta^{2\cdots r+1})$$

$$- (\nabla_{[u,v]} \theta^1) \theta^{2\cdots r+1} - \theta^1 \nabla_{[u,v]} \theta^{2\cdots r+1}$$

$$= (\nabla_u \nabla_v \theta^1) \theta^{2\cdots r+1} + \nabla_v \theta^1 \nabla_u \theta^{2\cdots r+1} + \nabla_v \theta^1 \nabla_u \theta^{2\cdots r+1} + \theta^1 \nabla_v \nabla_u \theta^{2\cdots r+1}$$

$$- (\nabla_{[u,v]} \theta^1) \theta^{2\cdots r+1} - \theta^1 \nabla_{[u,v]} \theta^{2\cdots r+1}$$

$$= \theta^1 (1/2 [\mathfrak{R}(u \wedge v), \theta^{2\cdots r+1}]) + (1/2 [\mathfrak{R}(u \wedge v), \theta^1]) \theta^{2\cdots r+1}$$

$$= 1/2 [\mathfrak{R}(u \wedge v), \theta^{1\cdots r+1}]$$, \hspace{1cm} (50)

where the last line of Eq. (50) is the second member of Eq. (40) evaluated for $\theta^1 \theta^2 \cdots \theta^{r+1}$.

(c) Now, it remains to verify that

$$\mathbf{R}(u, v, t, z) = (t \wedge z) \cdot \mathfrak{R}(u \wedge v)$$

with $\mathfrak{R}(u \wedge v)$ given by Eq. (42). Indeed, from a well known identity, we have that for any $t, z \in \sec \Lambda^1 T^*M$, and $\mathfrak{R}(u \wedge v) \in \sec \Lambda^2 T^*M$ it is

$$(z \wedge t) \cdot \mathfrak{R}(u \wedge v) = -z \cdot (t \mathfrak{R}(u \wedge v))$$

$$= z \cdot (\mathfrak{R}(u \wedge v)_{\wedge t})$$

$$= 1/2 z \cdot [\mathfrak{R}(u \wedge v), t]$$

$$\hspace{1cm} \text{Eq. (40)}$$

$$z \cdot (\nabla_u \nabla_v t - \nabla_v \nabla_u t - \nabla_{[u,v]} t)$$

and the proposition is proved. \hspace{1cm} \Box

In particular we have:

$$\mathbf{R}(u, v, t, z) = z e^c_a u^a v^b R^c_{dab},$$

$$R^d_{c;ab} = \varepsilon_a (\omega^d_{bc}) - \varepsilon_b (\omega^d_{ac}) + \omega^a_{.ak} \omega^k_{bc} - \omega^b_{.bk} \omega^k_{ac} - e^c_a \omega^d_{kc}$$

and

$$\mathbf{R}(\theta^a, \theta^b, \theta_a, \theta_b) = (\theta^a \wedge \theta^b) \cdot \mathfrak{R}(\theta_a \wedge \theta_b) = R,$$

where $R$ is the curvature scalar.

**Proposition 15** For any $v \in \sec \Lambda^1 T^*M \hookrightarrow \sec \mathcal{L}(M, g)$

$$[\nabla_{v_a}, \nabla_{v_b}] v = \mathfrak{R}(\theta_a \wedge \theta_b)_{\wedge} v - (T^c_{ab} - \omega^c_{ab}) \omega^c_{ab} \nabla_{v_a} v.$$ \hspace{1cm} (54)
Proof. From Eq. (46) we can write
\[
[\nabla_{e_a}, \nabla_{e_b}]v = \frac{1}{2} [\mathcal{R}(\theta_a \wedge \theta_b), v] + \nabla_{[e_a, e_b]}v
\]
\[
= \mathcal{R}(\theta_a \wedge \theta_b)_{\alpha} v + \nabla_{[(e_a \cdot e_b) - \nabla_{e_a} e_b + \nabla_{e_b} e_a]}v + \nabla_{e_a e_b v} - \nabla_{e_b e_a v}
\]
\[
= \mathcal{R}(\theta_a \wedge \theta_b)_{\alpha} v - (T^c_{ab} - \omega^c_{ab} + \omega^c_{ba}) \nabla_{e_c} v
\]
which proves the proposition. ■

Proposition 16
\[
\mathcal{R}(\theta^a \wedge \theta^b) = \mathcal{R}^a_{\cdot b} = d\omega^a_{\cdot b} + \omega^c_{\cdot b} \wedge e_c
\]

Proof. Recall that using
\[
([\nabla_{e_k}, \nabla_{e_l}] - \nabla_{[e_k, e_l]})\theta^j = \rho(e_k, e_l)\theta^j = -R^{kijl}_{\cdot \cdot \cdot} e_i
\]
we have
\[
\mathcal{R}(\theta_a \wedge \theta_b)_{\cdot} v = v^m \rho(e_a, e_b)\theta_m = v^m R^{i\cdot \cdot \cdot}_{mab} \theta_i
\]
On the other hand, for a general connection, we must write
\[
\mathcal{R}_{ab} := \frac{1}{2} R_{klab} \theta^k \wedge \theta^l
\]
and then
\[
\mathcal{R}_{ab\cdot} v = \frac{1}{2} v^m R_{klab} (\theta^k \wedge \theta^l)_{\cdot} \theta_m = -v^m R_{mlab} \theta^l = v^m R_{mlab} \theta^l = v^m R^{l\cdot \cdot \cdot}_{mab} \theta_l
\]
and the proposition is proved. ■

Proposition 17 The Ricci 1-form \[\mathcal{R}^d := R^d_{\cdot b} \theta^b\] and the curvature biform \[\mathcal{R}(\theta_a \wedge \theta_b)\] for the Levi-Civita connection \(D\) of \(g\) are related by,
\[
\mathcal{R}^d = \frac{1}{2} (\theta^a \wedge \theta^b) (\mathcal{R}(\theta_a \wedge \theta_b))_{\cdot} \theta^d
\]

Proof. Recalling that the Ricci operator is given by\[15\]
\[
\theta^{l\cdot \cdot \cdot} \theta^d = \frac{1}{2} (\theta^a \wedge \theta^b) \left( [D_{e_a}, D_{e_b}] \theta^d - c^{c\cdot \cdot \cdot}_{ab} D_{e_c} \theta^d \right)
\]

\[\text{Fig.} 14\] The \(R^d_{\cdot b} \theta^b\) are the components of the Ricci tensor.

\[\text{See Chapter 4 of [19].}\]
and moreover taking into account that by the first Bianchi identity it is $\mathcal{R}_{cd}\wedge\theta_e = 0$, we have

$$
\frac{1}{2}(\theta^a \wedge \theta^b) ([D_{ea}, D_{eb}] \theta^d - \epsilon_{a}^{ab} D_{ec} \theta^d) = -\frac{1}{2}(\theta^a \wedge \theta^b) \mathcal{R}^{cd} \theta^e = \mathcal{R}^{cd} \theta_e = \mathcal{R}^{cd} \|ightharpoonup \theta_e = -1/2 \mathcal{R}^{cd} \theta^a \wedge \theta^b \mathcal{R}^{ab}
$$

which proves the proposition. ■

Proposition 17 suggests the

**Definition 18** The Ricci extensor is the mapping

$$
\mathcal{R} : \sec \bigwedge^1 T^* M \to \sec \bigwedge^1 T^* M,

\mathcal{R}(v) = \partial_u \mathcal{R}(u \wedge v).
$$

**Remark 19** Of course, we must have $\mathcal{R}(\theta^d) = \mathcal{R}^d$. Moreover, we have

$$
\partial_u \mathcal{R}(u \wedge v) = \theta^b \frac{\partial}{\partial u^b} \mathcal{R}(u_k \theta^k \wedge v) = \theta^b \frac{\partial}{\partial u^b} u^k \mathcal{R}(\theta_k \wedge v)

= \theta^b \mathcal{R}(\delta^b_k \theta_k \wedge v) = \theta^b \mathcal{R}(\theta_b \wedge v)

= \theta^b \wedge \mathcal{R}(\theta_b \wedge v)

= \theta^b \wedge \mathcal{R}(\theta_b \wedge v).
$$

So,

$$
\partial_u \mathcal{R}(u \wedge v) = \partial_u \mathcal{R}(u \wedge v) \text{ and } \partial_u \wedge \mathcal{R}(u \wedge v) = 0.
$$

3 The Riemannian or Semi-Riemannian Geometry of a Submanifold $M$ of $\tilde{M}$

3.1 Motivation

Any manifold $M$, $\dim M = m$, according to Whitney’s theorem (see, e.g., [1]), can be realized as a submanifold of $\mathbb{R}^n$, with $n = 2m$. However, if $M$ carries additional structure the number $n$ in general must be greater than $2m$. Indeed, it has been shown by Eddington [9] that if $\dim M = 4$ and if $M$ carries a Lorentzian metric $g$ and which moreover satisfies Einstein’s equations, then $M$ can be locally embedded in a (pseudo)Euclidean space $\mathbb{R}^{1,9}$. Also, isometric embeddings of general Lorentzian spacetimes would require a lot of extra dimensions [5]. Indeed, a compact Lorentzian manifold can be embedded isometrically in $\mathbb{R}^{2,46}$ and a non-compact one can be embedded isometrically in $\mathbb{R}^{2,87}$! In particular this last result shows that the spacetime of M-theory [4, 15] may not be large.
enough to contain 4-dimensional branes with arbitrary metric tensors. In what
follows we show how to relate the intrinsic differential geometry of a structure
\((M, g, D)\) where \(g\) is a metric of signature \((p, q)\), \(D\) is its Levi-Civita connection
and \(M\) is an orientable proper submanifold of \(\hat{M}\), i.e., there is defined on \(M\) a
global volume element \(\tau_g = I_m\) whose expression on \(U \subset M\) is given by
\[
I_m = \theta^1 \theta^2 \cdots \theta^m. \tag{64}
\]
We suppose moreover that \(\hat{M} \simeq \mathbb{R}^n\) and it is equipped with a metric \(\hat{g}\)
of signature \((\hat{p}, \hat{q}) = n\). However, take notice that our presentation in the form of
a local theory is easily adapted for a general manifold \(\hat{M}\).

3.1.1 Projection Operator \(P\)

Definition 20 Let \(C = \sum_{r=0}^{n} C_r\), with \(C_r \in \sec \Lambda^r T^* \hat{M} \hookrightarrow \sec \mathcal{C}(\hat{M}, \hat{g})\). The
Projection operator on \(M\) is the extensor field
\[
P : \sec \mathcal{C}(\hat{M}, \hat{g}) \rightarrow \sec \mathcal{C}(M, g),
\]
\[
P(C) = (C \cdot I_m) I_m^{-1}. \tag{65}
\]

Remark 21 Note that for all \(C_k \in \sec \Lambda^k T^* \hat{M} \hookrightarrow \sec \mathcal{C}(\hat{M}, \hat{g})\), if \(k > m\) then
\(P(C_k) = 0\), but of course, it may happen that even if \(A_r \in \sec \Lambda^r T^* \hat{M} \hookrightarrow \sec \mathcal{C}(\hat{M}, \hat{g})\) with \(r \leq m\) we may have \(P(A_r) = 0\).

We define the complement of \(P\) by
\[
P_\perp(C) = C - P(C) \tag{66}
\]
and it is clear that \(P_\perp(C)\) have only components lying outside \(\mathcal{C}(M, g)\). It is
quite clear also that any \(C\) with components not all belonging to \(\sec \mathcal{C}(M, g)\)
will satisfy \(C \cdot I_m = 0\).

Having introduced in Section 1 the derivative operators \(\hat{\partial}\) and its restriction
\(\hat{\delta} = \hat{\partial} \big|_{\hat{M}}\) (Eq. (1) and Eq. (2)) we extend the action of \(P\) to act on the operator
\(\hat{\delta}\), defining:
\[
P(\hat{\delta}) = \sum_{k=1}^{m} P(\theta^k \hat{D} e_k) := \sum_{k=1}^{m} P(\theta^k) \hat{D} e_k = \sum_{k=1}^{m} \theta^k \hat{D} e_k = \hat{\delta}. \tag{67}
\]

3.1.2 Shape Operator \(S\)

Definition 22 Given \(C \in \sec \mathcal{C}(\hat{M}, \hat{g})\) we define the shape operator
\[
S : \sec \mathcal{C}(\hat{M}, \hat{g}) \rightarrow \sec \mathcal{C}(\hat{M}, \hat{g}),
\]
\[
S(C) = \hat{\delta} P \ (C) = \hat{\delta}(P(C)) - P(\hat{\delta} C). \tag{68}
\]

For any \(C \in \sec \mathcal{C}(M, g)\) and \(v \in \sec TM\) we write as usual [3][12]
\[
\hat{D}_v C = (\hat{D}_v C)_\parallel + (\hat{D}_v C)_\perp \tag{69}
\]
where \((\tilde{D}_v C)_{\parallel} \in \sec \mathcal{Cl}(M, g)\) and \((\tilde{D}_v C)_{\perp} \in \sec \mathcal{Cl}(M, g)_{\perp}\) where \([\mathcal{Cl}(M, g)]_{\perp}\) is the orthogonal complement of \(\mathcal{Cl}(M, \hat{g})\) in \(\mathcal{Cl}(M, g)\).

As it is very well known [3, 12] if \(g := i^* \hat{g}\) and \(v \in \sec TM\) (and \(v = g(v, )\) and \(C \in \sec \mathcal{Cl}(M, \hat{g})\) the Levi-Civita connection \(D\) of \(\hat{g}\) is given by

\[
D_v C := (\tilde{D}_v C)_{\parallel}
\]  

and of course

\[
D_v C := (v \cdot \tilde{\partial} C)_{\parallel}
\]

Moreover, note that we can write for any \(C \in \sec \mathcal{Cl}(M, g)\)

\[
v \cdot \partial C = (v \cdot \tilde{\partial} C)_{\parallel} = P(v \cdot \tilde{\partial} C)
\]

Also, writing

\[
(\tilde{D}_v C)_{\perp} := P_{\perp}(v \cdot \tilde{\partial} C)
\]

we have

\[
v \cdot \partial = P(v \cdot \tilde{\partial}) = (v \cdot \tilde{\partial})_{\parallel} = (v \cdot \tilde{\partial})_{\parallel} = v \cdot \tilde{\partial} - P_{\perp}(v \cdot \tilde{\partial}) = v \cdot \tilde{\partial} - P_{\perp}(v \cdot \tilde{\partial}).
\]

So, it is

\[
v \cdot \tilde{\partial} I_m I_m^{-1} = \sum_{j=1}^m \theta^j \cdots (D_v \theta^j + P_{\perp}(v \cdot \tilde{\partial} \theta^j)) \cdots \theta^m I_m^{-1}
\]

\[
= D_v I_m I_m^{-1} + P_{\perp}(v \cdot \tilde{\partial} \theta^j) \wedge \theta^j.
\]

Now, \(D_v I_m \in \sec \wedge^m TM \hookrightarrow \sec \mathcal{Cl}(M, g)\) is a multiple of \(I_m\) and since \(I_m^2 = \pm 1\) depending on the signature of the metric \(g\) we have that \(D_v I_m = 0\).

Indeed,

\[
0 = D_v I_m^2 = 2(D_v I_m)I_m
\]

and so

\[
0 = (D_v I_m)I_m I_m^{-1} = D_v I_m.
\]

In any Clifford algebra bundle, in particular \(\mathcal{Cl}(M, g)\) we can build multiples of the \(I_m\) not only multiplying it by a scalar function, but also multiplying it by a an appropriated biform. This result will be used below to define the shape biform.

### 3.2 \(S(\tilde{v}) = S(v) = \partial_u \wedge P_u (\tilde{v})\) and \(S(\tilde{v}_{\perp}) = \partial_u P_u (\tilde{v})\)

For any \(C \in \sec \mathcal{Cl}(M, g)\) it is \(C = P(C)\) we have (with \(u \in \sec \wedge^1 T^* M \hookrightarrow \sec \mathcal{Cl}(M, \hat{g})\))

\[
\partial(P(C)) = \partial P(C) - P(\partial C)
\]

\[
= \partial_u P_u (C) - P(\partial C)
\]

\[
= \partial_u \wedge P_u (C) + \partial_u P_u (C) - P(\partial C)
\]  

\[
S(\tilde{v}) = S(v) = \partial_u \wedge P_u (\tilde{v})\]  

\[
S(\tilde{v}_{\perp}) = \partial_u P_u (\tilde{v})
\]
where

\[ P_u(C) := u \cdot \dot{\delta} P(C) = u \cdot \dot{\delta}(P(C)) - (P(u \cdot \dot{\delta} C)). \]  

(78)

Recall that for \( \dot{v} \in \sec \wedge^1 T^* M \hookrightarrow \sec C\ell(M, \g) \) we can write

\[ S(\dot{v}) = \dot{\delta} P(\dot{v}) = \partial_u \wedge P_u(\dot{v}) + \partial_u \cdot P_u(\dot{v}) \]  

(79)

where we used that for any \( \dot{\mathcal{C}} \in \sec C\ell(M, \g) \) it is

\[ \partial_u P_u(\dot{\mathcal{C}}) := \sum_{i=1}^m \theta^i \partial_u \wedge \dot{\delta} P(\mathcal{C}) = \sum_{i=1}^m \theta^i \dot{D}_{\mathcal{E}_i} P(\dot{\mathcal{C}}) = \dot{\delta} P(\dot{\mathcal{C}}) \]  

(80)

Putting \( \ddot{v} = \ddot{v}_\parallel + \ddot{v}_\perp = v + \dot{v}_\perp \) we have the

**Proposition 23**

\[ S(\ddot{v}) = S(v) = \partial_u \wedge P_u(\ddot{v}), \quad S(\ddot{v}_\perp) = \partial_u \cdot P_u(\ddot{v}). \]  

(81)

**Proof.** Indeed,

\[ S(\ddot{v}) = S(\ddot{v}_\parallel) + S(\ddot{v}_\perp) = \partial_u \wedge P_u(\ddot{v}) + \partial_u \cdot P_u(\ddot{v}) \]  

(82)

and so, it is enough to show that

\[ \partial_u \cdot P_u(\ddot{v}_\parallel) = 0 \quad \text{and} \quad \partial_u \wedge P_u(\ddot{v}_\perp) = 0, \]  

(83)

From \( P^2(\ddot{v}) = P(\ddot{v}) \) we get

\[ P_u P(\ddot{v}_\parallel) + PP_u(\ddot{v}_\parallel) = P_u(\ddot{v}_\parallel). \]  

(84)

So, for \( \ddot{v}_\parallel \) and \( \ddot{v}_\perp \) it is

\[ PP_u(\ddot{v}_\parallel) = 0 \quad \text{and} \quad PP_u(\ddot{v}_\perp) = P_u(\ddot{v}_\perp). \]  

(85)

Since \( PP_u(\ddot{v}_\parallel) = 0 \) we have that

\[ \partial_u \cdot PP_u(\ddot{v}_\parallel) = P(\partial_u) \cdot P_u(\ddot{v}_\parallel) = \partial_u \cdot P_u(\ddot{v}_\parallel) = 0. \]  

(86)

From \( PP_u(\ddot{v}_\perp) = P_u(\ddot{v}_\perp) \) we can write

\[ \partial_u \wedge P_u(\ddot{v}_\perp) = \partial_u \wedge PP_u(\ddot{v}_\perp) = P(\partial_u) \wedge PP_u(\ddot{v}_\perp) = P(\partial_u \wedge P_u(\ddot{v}_\perp)). \]  

(87)

Now, take \( t, y \in \sec \wedge^1 T^* M \hookrightarrow \sec C\ell(M, g) \). We have

\[ (t \wedge y) \cdot (\partial_u \wedge PP_u(\ddot{v}_\perp)) = (t \wedge y) \cdot (\partial_u \wedge P_u(\ddot{v}_\perp)) \]

\[ = t \wedge (y \cdot \partial_u \wedge P_u(\ddot{v}_\perp) - \partial_u \wedge (y \cdot PP_u(\ddot{v}_\perp))) \]

\[ = t \wedge (P(y \cdot \dot{\delta} \ddot{v}_\perp) - \theta^i \wedge (y \cdot \theta_1 \cdot \dot{\delta}(P(\ddot{v}_\perp)) - P(\theta_1 \cdot \dot{\delta} \ddot{v}_\perp))) \]

\[ = t \wedge (D_y \ddot{v}_\perp - \theta^i \wedge (y \cdot D_{\theta_1 \cdot \dot{\delta} \ddot{v}_\perp})) = 0 \]  

(88)

from where it follows that

\[ \partial_u \wedge P_u(\ddot{v}_\perp) = 0 \]  

(89)

and the proposition is proved. \( \blacksquare \)
3.2.1 Shape Biform $S$

**Definition 24** The shape biform (a $(1,2)$-extensor field) is the mapping

$$ S : \text{sec} \wedge^1 T^* M \to \wedge^2 T^* M, $$

such that

$$ v \mapsto S(v), $$

such that

$$ v \cdot \tilde{d}I_m = -S(v)I_m. $$

From Eq. (91) it follows that $S(v)I_m = 0$ and $S(v) \wedge I_m = 0$.

Since $S(v)I_m = 0$ it follows from Remark 21 that $P(S(v)) = 0$.

Now, using the fact that $D_vI_m = 0$ and Eq. (91) it follows from Eq. (75) that

$$ v \cdot \tilde{d}I_m = (P_\perp (v \cdot \tilde{d}\theta_j) \wedge \theta^j)I_m = -S(v)I_m, $$

i.e.,

$$ S(v) = -P_\perp (v \cdot \tilde{d}\theta_j) \wedge \theta^j. $$

**Proposition 25** For any $C \in \text{sec} C\ell(M,g)$ we have

$$ D_vC = v \cdot \tilde{d}C + S(v) \times C = \hat{D}_vC + S(v) \times C. $$

**Proof.** Taking into account Eq. (94) we have for any $v, w \in \text{sec} \wedge^1 T^* M \mapsto \text{sec} C\ell(M,g)$

$$ v \cdot S(w) = -v \cdot (P_\perp (w \cdot \tilde{d}\theta_j) \wedge \theta^j) $n

$$ = -(v \cdot (P_\perp (w \cdot \tilde{d}\theta_j)) \theta^j + v^j P_\perp (w \cdot \tilde{d}\theta_j) $n

$$ = v^j P_\perp (w \cdot \tilde{d}\theta_j) = P_\perp (w \cdot \tilde{d}v) - P_\perp [(w \cdot \tilde{d}v^j) \theta_j] $n

$$ = P_\perp (w \cdot \tilde{d}v). $$

So,

$$ v \cdot \tilde{d}w = P(v \cdot \tilde{d}w) + P_\perp (v \cdot \tilde{d}w) $n

$$ = D_v w + w \cdot S(v) $n

$$ = D_v w - S(v) \cdot w. $$

Now, for $v, w \in \text{sec} \wedge^1 T^* M \mapsto \text{sec} C\ell(M,g)$ we have

$$ D_v(wu) = (D_vw)u + wD_vu = (\hat{D}_v w)u + (S(v) \times w)u + w(\hat{D}_v u) + w(S(v) \times u) $n

$$ = (\hat{D}_v w)u + (S(v) \times w)u - w(u \times S(v)) $n

$$ = (\hat{D}_v w)u + (S(v) \times wu), $$

18
from where the proposition follows trivially by finite induction. □

Of course, it is

\[ D_{e_i}C = \hat{D}_{e_i}C + S(v) \times C \]  

(99)

Now, recalling Eq. (27) we have

\[ \hat{D}_{e_i}C = \partial_{e_i}C + \hat{\omega}_{e_i} \times C \]  

(100)

where for \( i, j = 1, ..., m \), \( \hat{D}_{e_i} \theta^j = \hat{\nabla}_i \theta^j = -\sum_{k=1}^n \hat{\omega}^j_{ik} \hat{\theta}^k \), it is

\[ \hat{\omega}_v = \frac{1}{2} v_c \hat{\omega}_a b \theta^a \wedge \theta^b \]  

(101)

So, we get

\[ D_v C = v \cdot \hat{\nabla}C + S(v) \times C \]

\[ = \partial_v C + (\hat{\omega}_v + S(v)) \times C \]  

(102)

and in particular

\[ D_{e_i}C = \partial_{e_i}C + (\hat{\omega}_{e_i} + S(e_i)) \times C. \]  

(103)

Comparison of Eq. (103) with Eq. (25) (valid for any metric compatible connection) implies the important result

\[ \omega_v = (\hat{\omega}_v + S(v)) \]  

(104)

We can easily find by direct calculation that in a gauge where \( \hat{\omega}_v \neq 0 \),

\[ \omega_v = P(\hat{\omega}_v) \]  

(105)

which is consistent with the fact that from Eq. (99), it is \( P(S(v)) = 0 \).

Let \( (x^1, ..., x^n) \) be the natural orthogonal coordinate functions of \( \hat{M} \simeq \mathbb{R}^n \).

**Corollary 26** For \( C \in \text{sec} \ Cl(M, g) \)

\[ D_v C = v^i \frac{\partial}{\partial x^i} C + S(v) \times C. \]  

(106)

**Proof.** Taking into account that \( D_{\frac{\partial}{\partial x^j}}dx^j = 0 \) follows that \( \hat{\omega}_{\frac{\partial}{\partial x^j}} = \frac{1}{2} (\Omega_{kij}) dx^k \wedge dx^j = 0 \). Using this result in Eq. (115) with \( e_i \rightarrow \frac{\partial}{\partial x^i} \) gives the desired result. □

**Remark 27** Comparison of Eq. (102) and Eq. (106) shows that \( S(v) \) cannot always be identified with \( \omega(v) \) which is a gauge dependent operator.

---

16 Take notice that this formula being gauge dependent is not valid if \( e_i \rightarrow x_i \) where the \( x_i \) coordinate vector fields. See Corollary 26.
3.3 Integrability Conditions

Remark 28 Take into account that the commutator of Pfaff derivatives acting on any $C \in \sec \mathcal{C}(\mathcal{M}, g)$ is in general non null, i.e.,

$$[\partial e_i, \partial e_j]C = \sum_{r=0}^{m} c_{ij}^k e_k(C_{i_1 \cdots i_m}) e^{i_1 \cdots i_m} \neq 0,$$

(107)

unless $e_i$ are coordinate vector fields, i.e., $e_i \mapsto x_i$.

Remark 29 Also, since the torsion of $\tilde{D}$ is null we have in general

$$[\theta_i \cdot \tilde{\theta}, \theta_j \cdot \tilde{\theta}]C = [\theta_i, \theta_j] \cdot \tilde{D}_e C = c_{ij}^k \theta_k \cdot \tilde{D}_e C \neq 0,$$

(108)

unless $e_i$ are coordinate vector fields. Moreover, for the case of orthonormal vector fields

$$[\theta_i \cdot \tilde{\theta}, \theta_j \cdot \tilde{\theta}]C \neq [\partial e_i, \partial e_j]C.$$

(109)

Remark 30 The integrability condition for the connection $\tilde{D}$ is expressed, given the previous results, by

$$\tilde{\partial} \land \tilde{\partial} = 0$$

(110)

which means that for any $\tilde{C} \in \sec \mathcal{C}(\tilde{\mathcal{M}}, \tilde{g})$ it is

$$\tilde{\partial} \land \tilde{\partial} \tilde{C} = 0$$

For the manifold $M$ recalling that $x_i \equiv \partial x_i$ is a Pfaff derivative we have for any $C \in \sec \mathcal{C}(\mathcal{M}, g)$

$$(\partial x_i \cdot \tilde{\partial} x_j - \tilde{\partial} x_i \cdot \partial x_j)C = 0.$$  

(111)

If we recall the definition of the form derivative (Eq.(6)), putting

$$\tilde{\partial} := \varphi i \partial x_i$$

(112)

we can express the ‘integrability’ condition in $M$ by

$$\tilde{\partial} \land \tilde{\partial} = 0.$$  

(113)

Finally recalling Eqs.(60) and (61) for $v \in \sec \mathcal{C}_\bot T^* M \rightarrow \sec \mathcal{C}(\mathcal{M}, g)$ it is

$$\partial \land \partial v = v_i \mathcal{R}^i$$

(114)

where $\mathcal{R}^i$ are the Ricci 1-form fields.

3.4 $\mathbf{S}(v) = \mathbf{S}(v)$

Proposition 31 Let $C = v \in \sec \mathcal{C}_\bot T^* M \rightarrow \sec \mathcal{C}(\mathcal{M}, g)$ we have

$$\mathbf{S}(v) = \mathbf{S}(v).$$

(115)
**Proof.** We have

\[
S(v) = \delta(P(v)) - P(\delta v) \\
= \delta v - P(\delta v). \tag{116}
\]

Now, \(\delta v = \delta \wedge v + \delta \cdot v\) and since \(P(\delta \cdot v) = \delta \cdot v\) we have

\[
S(v) = \delta \wedge v - P(\delta \wedge v) \tag{117}
\]

It is only necessary due to the linearity \(S\) of to show Eq.\((115)\) for \(v = \theta_d, d=1, \ldots, m\). We then evaluate

\[
\delta \wedge \theta_d = \sum_{k=1}^{m} \theta_k \tilde{D}_{ek} \theta_d \tag{118}
\]

from where it follows that

\[
S(\theta_d) = \sum_{k=1}^{m} \sum_{t=m+1}^{m+l} \omega_{tkd} \theta^k \wedge \tilde{\theta}^t = \frac{1}{2} \sum_{k=1}^{m} \sum_{t=m+1}^{m+l} (\omega_{tkd} - \tilde{\omega}_{tkd}) \theta^k \wedge \tilde{\theta}^t \tag{119}
\]

On the other hand

\[
\theta_d \cdot \tilde{D}_m = \eta^{11} \cdots \eta^{mm} \tilde{D}_{ea}(\theta_1 \wedge \cdots \wedge \theta_m) \\
= \alpha \tilde{D}_{ea}(\theta_1 \cdots \theta_m) \\
= \alpha \sum_{k=1}^{m} \sum_{t=1}^{n} \omega_{tkd} \theta_1 \cdots \underbrace{\tilde{\theta}^t}_{k\text{-position}} \cdots \theta_m \\
= \alpha \sum_{k=1}^{m} \sum_{t=1}^{n} \omega_{tkd} \theta_1 \cdots \underbrace{\tilde{\theta}^t}_{k\text{-position}} \cdots \theta_m \\
+ \alpha \sum_{k=1}^{m} \sum_{t=m+1}^{m+l} \omega_{tkd} \theta_1 \cdots \underbrace{\tilde{\theta}^t}_{k\text{-position}} \cdots \theta_m \tag{120}
\]

and now we can easily see that

\[
S(\theta_d) \times I_m = \theta_d \cdot \tilde{D}_m \tag{121}
\]

and it follows that \(S(\theta_d) = S(\theta_d)\). □

We also have the

**Proposition 32** Let \(v, w \in \text{sec} \bigwedge^1 T^* M \hookrightarrow \text{sec} \ Cl(M, g)\) we have

\[
v \cdot S(w) = w \cdot S(v) \tag{122}
\]

**Proof.** Recalling Eq.\((119)\) we can write

\[
v \cdot S(w) = \sum_{i,d=1}^{m} v^i w^d \theta_i \cdot \sum_{k=1}^{m} \sum_{t=m+1}^{m+l} \omega_{tkd} \theta^k \wedge \tilde{\theta}^t \\
= \frac{1}{2} \sum_{i,d=1}^{m} v^i w^d (\omega_{id} - \tilde{\omega}_{id}) \tilde{\theta}^i = w \cdot S(v)
\]

and the proposition is proved. □
3.5 \( \check{\partial} \wedge v = \partial \wedge v + S(v) \) and \( \check{\partial} \cdot v = \partial \cdot v \)

We first observe that since torsion is null for the Levi-Civita connection \( \check{\partial} \) we have for any \( u, v \in \text{sec} T^*M \leftrightarrow \text{sec} \mathcal{C}(M, g) \) we have

\[
u \cdot \check{\partial}v = v \cdot \partial u + [u, v]
\]

from where it follows \( \mathbf{P}_\perp([u, v]) = 0 \) when \( u, v \in \text{sec} T^*M \leftrightarrow \text{sec} \mathcal{C}(M, g) \) since calculating \([u, v] \) with \( \check{\partial} \) expressed in the natural coordinates of \( M \) we find that \( [u, v] \in \text{sec} T^*M \leftrightarrow \text{sec} \mathcal{C}(M, g) \). From this it follows that

\[
\mathbf{P}_\perp(u \cdot \check{\partial}v) = \mathbf{P}_\perp(v \cdot \check{\partial}u).
\]

Then we can write

\[
\check{\partial} \wedge v = \sum_{r,k=1}^m \theta^r \wedge \check{D}_{er}(v^k \theta_k)
\]

\[
= \sum_{r,k=1}^m \theta^r \wedge \{ e_r(v^k) + v^k \sum_{s=1}^m \ell_{rk}^s \theta_s \} + \sum_{r,k=1}^m \theta^r \wedge v^k \sum_{s=m+1}^{m+\ell} L_{rk}^s \theta_s
\]

\[
= \partial \wedge v + \sum_{m,k=1}^m \theta^r \wedge v^k \mathbf{P}_\perp(\check{D}_r \theta_k)
\]

\[
= \partial \wedge v + \sum_{m,k=1}^m \theta^r \wedge v^k \mathbf{P}_\perp(\check{D}_k \theta_r)
\]

\[
= \partial \wedge v + \theta^r \wedge \mathbf{P}_\perp(\check{D}_r \theta_r)
\]

and recalling that \( S(v) = -\mathbf{P}_\perp(v \cdot \check{\partial} \theta_r) \wedge \theta^r \) we finally have

\[
\check{\partial} \wedge v = \partial \wedge v + S(v)
\]

and the proposition is proved.

Also, from (Eq.(83)) we know that \( \partial \cdot \mathbf{P}_u (\check{v}_\parallel) = 0 \). So,

\[
\check{\partial}v = \check{\partial}(\mathbf{P}(v)) = \check{\partial} \mathbf{P} (v) + \mathbf{P}(\check{\partial}v)
\]

\[
= \partial_u \wedge \mathbf{P}_u (v) + \partial_u \cdot \mathbf{P}_u (v) + \check{\partial}v
\]

\[
= \partial_u \wedge \mathbf{P}_u (v) + \check{\partial} \wedge v + \partial_u \cdot \partial \cdot v
\]

\[
= S(v) + \partial \wedge v + \partial \cdot v
\]

and thus we see that

\[
\check{\partial} \cdot v = \partial \cdot v
\]

We then can write

\[
\check{\partial}v = \partial v + S(v).
\]
3.6 \( \dd C = \partial C + S(C) \)

We can generalize Eq. (128), i.e., we have the

**Proposition 33** For any \( C \in \sec \ell(M, g) \) we have

\[
\dd C = \partial C + S(C), \\
\dd \wedge C = \partial \wedge C + S(C), \\
\dd \ll C = \partial \ll C.
\] (129)

**Proof.** (i) From the fact that for any \( A, B \in \sec \ell(M, g) \) it is \( P(A \wedge B) = P(A) \wedge P(B) \) we have differentiating with respect to \( u \in \sec \wedge^1 T^* M \into \sec \ell(M, g) \)

\[
P_u(A \wedge B) = P_u(A) \wedge B + A \wedge P_u(B) \quad (130)
\]

and of course

\[
P_u(A \perp \wedge B \parallel) = P_u(A \perp) \wedge B \parallel, \\
P_u(A \perp \wedge B \perp) = P_u(A \perp) \wedge B \perp + A \parallel \wedge P_u(B \parallel). \quad (131)
\]

(ii) For \( C \in \sec \ell(M, g) \) it is \( C = P(C) \) and we have using Eq. (68)

\[
\dd C = \dd P(C) - P(\dd C)
\]

\[
= \dd \wedge P(C) + \dd \ll P(C) + \partial C
\]

(132)

(iii) Now, we can verify recalling that \( S(C) = S(C \parallel + C \perp) = S(C \parallel) + S(C \perp) \) and following steps analogous to the ones used in the proof of Proposition 23 that

\[
S(C \parallel) = S(P(C \parallel)) = \dd \wedge P(C \parallel) \\
S(C \perp) = P(S(C \perp)) = \dd \ll P(C \parallel).
\]

(133)

(iv) Using Eq. (133) in Eq. (131) we have

\[
\dd \wedge C + \dd \ll C = \dd \wedge P(C) + \dd \ll P(C) + \partial \wedge C + \partial \ll C
\]

or

\[
\dd \wedge C + \dd \ll C = S(C) + S(C \perp) + \partial \wedge C + \partial \ll C
\]

\[
= S(C) + \partial \wedge C + \partial \ll C,
\]

(134)

which provides the proof of the proposition. \( \blacksquare \)

**Proposition 34** For any \( C \in \sec \ell(M, g) \) we have:

\[
\partial C = P(\dd C)
\] (135)

**Proof.** From Eq. (68) when \( C \in \sec \ell(M, g) \) it is

\[
P(S(C)) = 0.
\]

Then, applying \( P \) to both members of the first line of Eq. (128) we have

\[
P(\dd C) = P(\dd C) + P^2(S(C)) = P(\partial C) = \partial C
\]

(136)

and the proposition is proved. \( \blacksquare \)
4 Curvature Biform $\mathcal{R}(u \wedge v)$ Expressed in Terms of the Shape Operator

4.1 Equivalent Expressions for $\mathcal{R}(u \wedge v)$

In this section we suppose that the structure $(M, g, D)$ is such that $M$ a submanifold of $M \simeq \mathbb{R}^n$ and $D$ the Levi-Civita connection of $g = i^* \hat{g}$. We obtained in Section 1 a formula (Eq.(41)) for the curvature biform $\mathcal{R}$ of a general Riemann-Cartan connection. Of course, taking into account the fact that $\mathcal{R}$ is an intrinsic object, the evaluation of $\mathcal{R}(u \wedge v)$ does not depend on the coordinate chart and basis for vector and form fields used for its calculation. In what follows we take advantage of this fact choosing the basis $\{\partial/\partial x^i\}$ as introduced above for which $\omega(u) = 0$. Thus, we have, recalling Eq.(41) and Eq.(12) that

$$\mathcal{R}(u \wedge v) = D_u \omega(v) - D_v \omega(u) + \omega(u) \times \omega(v) - \omega_{[u,v]}$$

On the other hand since in the gauge where $\omega(u) = 0$ we have that $\omega(u) = S(u)$ and thus we can also write

$$\mathcal{R}(u \wedge v) = \hat{D}_u \omega(v) - \hat{D}_v \omega(u) + S(u) \times S(v) - S([u,v]).$$

Now, putting $x_i = \partial/\partial x^i$ we have

$$\hat{D}_u \omega(v) - \hat{D}_v \omega(u)$$

$$= -u^j \delta^i \{D_{x_i} S(\hat{v})_j - D_{x_j} S(\hat{v})_i\}$$

$$= -u^j \delta^i \{D_{x_i} (\hat{D}_{x_j} I^1) - \hat{D}_{x_j} (\hat{D}_{x_i} I^1)\}$$

$$= -u^j \delta^i \{(D_{x_i} \hat{D}_{x_j} I^1) - (\hat{D}_{x_i} \hat{D}_{x_j} I^1)\}$$

$$= -u^j \delta^i \{(\hat{D}_{x_i} I^1) + (\hat{D}_{x_j} I^1)\}$$

$$= -u^j \delta^i S(\hat{v})_j S(\hat{v})_i$$

$$= -S(u) S(v) + S(v) S(u) = -2S(u) \times S(v).$$

Thus, we get

$$\mathcal{R}(u \wedge v) = -S(u) \times S(v) - S([u,v]).$$
Now take into account that since $\mathfrak{R}(u \wedge v) \in \sec \bigwedge^2 T^* M \hookrightarrow \sec \mathcal{C}(M, g)$ we must have, of course, $-S(u) \times S(v) - S([u, v]) = P(-S(u) \times S(v) - S([u, v]))$ and since Eq. (93) tells us that $P(S([u, v])) = 0$ we have the nice formula\[\text{(141)}\]

$$\mathfrak{R}(u \wedge v) = -P(S(u) \times S(v))$$

which express the curvature biform in terms of the shape biform.

4.2 $S^2(v) = -\partial \wedge \partial (v)$

In this subsection we want to show the

**Proposition 35** Let $v \in \sec \bigwedge^1 T^* M \hookrightarrow \sec \mathcal{C}(M, g)$. Then,

$$S^2(v) = -\partial \wedge \partial (v). \tag{142}$$

Eq. (142) tell us that the square of the shape operator applied to a 1-form field $v$ is equal to the Ricci operator applied to $v$. We will comment more on the significance of this result in the conclusions.

Now, to prove the Proposition 35 we need the following lemmas

**Lemma 36** Let $C \in \sec \mathcal{C}(\hat{M}, g)$ and $v \in \sec \bigwedge^1 T^* M \hookrightarrow \sec \mathcal{C}(M, g)$. Then

$$P_v(C) = P(C) \times S(v) - P(C \times S(v)). \tag{143}$$

**Proof.** Indeed,

$$P_v(C) = \dot{D}_v(P(C)) - P(\dot{D}_v C) = D_v(P(C)) - S(v) \times P(C) - P(D_v C - S(v) \times C) = D_v C - S(v) \times P(C) - D_v C + P(S(v) \times C) = P(C) \times S(v) - P(C \times S(v)) \tag{144}$$

which proves the lemma. \[\blacksquare\]

**Lemma 37** Let $C \in \sec \mathcal{C}(M, g)$ and $v \in \sec \bigwedge^1 T^* M \hookrightarrow \sec \mathcal{C}(M, g)$. Then

$$D_v C = \dot{D}_v C - P_v(C). \tag{145}$$

**Proof.** Follows from the first line in Eq. (144). \[\blacksquare\]

**Lemma 38** Let $C \in \sec \mathcal{C}(M, g)$ and $u, v \in \sec \bigwedge^1 T^* M \hookrightarrow \sec \mathcal{C}(M, g)$. Then

$$D_u D_v C = P(\dot{D}_u \dot{D}_v C) + P_u P_v(C). \tag{146}$$

\[\text{Note that in} [13, 21] \text{the second member of Eq. (141) is the negative of what we found. Our result agrees with the one in} [7].\]

25
Proof. Using Eq. (145) we have
\[ D_u(D_v C) = D_u(\tilde{D}_v C - \mathbf{P}_v(C)) \]
\[ = \tilde{D}_u \tilde{D}_v C - \mathbf{P}_u(\tilde{D}_v C) - \tilde{D}_u(\mathbf{P}_v(C)) + \mathbf{P}_u \mathbf{P}_v(C). \]  \hfill (147)

On the other hand we have
\[ \mathbf{P}(\tilde{D}_u \tilde{D}_v C) = -\mathbf{P}_u(\tilde{D}_v C) + \tilde{D}_u(\mathbf{P}(\tilde{D}_v C)) \]
\[ = -\mathbf{P}_u(\tilde{D}_v C) + \tilde{D}_u(\tilde{D}_v C - \mathbf{P}_v(C)) \]
\[ = \tilde{D}_u D_v C - \mathbf{P}_u(D_v C) - \tilde{D}_u(\mathbf{P}_v(C)). \]  \hfill (148)

Putting Eq. (148) in Eq. (146) gives the desired result. \hfill \qed

Lemma 39 Let \( C \in \text{sec Cl}(M, g) \) and \( u, v \in \text{sec} \wedge^1 T^* M \mapsto \text{sec Cl}(M, g). \) Then
\[ \mathcal{R}(u \wedge v) \times C = [\mathbf{P}_u, \mathbf{P}_v] C. \]  \hfill (149)

Proof. Using Eq. (146) we have
\[ [D_u, D_v]C = \mathbf{P}([\tilde{D}_u, \tilde{D}_v]C) + [\mathbf{P}_u, \mathbf{P}_v]C \]
\[ = \mathbf{P}(\tilde{D}_{[u, v]}C) + [\mathbf{P}_u, \mathbf{P}_v]C \]
\[ = D_{[u, v]}C + [\mathbf{P}_u, \mathbf{P}_v]C \]  \hfill (150)

Thus we get that
\[ ([D_u, D_v] - D_{[u, v]}C = [\mathbf{P}_u, \mathbf{P}_v]C \]  \hfill (151)

Recalling now Eq. (146) we have
\[ \mathcal{R}(u \wedge v) \times C = [\mathbf{P}_u, \mathbf{P}_v] C \]
and the lemma is proved. \hfill \qed

Lemma 40 Let \( C \in \text{sec Cl}(M, g) \) and \( u, v \in \text{sec} \wedge^1 T^* M \mapsto \text{sec Cl}(M, g). \) Then,
\[ \mathcal{R}(u \wedge v) \times C = -\mathbf{P}(\mathcal{S}(u) \times \mathcal{S}(v)) \times C \]  \hfill (152)

Proof. This follows directly from Eq. (141). \hfill \qed

Remark 41 We shall now evaluate directly the first member of Eq. (153) to get Eq. (156) which when compared with Eq. (141) will furnish identities given by Eq. (157).
\[ ([D_u, D_v] - D_{[u, v]})C = \mathcal{R}(u \wedge v) \times C. \]  \hfill (153)

Given the linearity of \( \mathcal{R}(u \wedge v) \) we calculate the first member of Eq. (153) for the case \( u = x_i, \quad v = x_j. \) Taking into account that \( D_u C = D_u C + \mathcal{S}(u) \times C \) we get with calculations analogous to the ones in Eq. (153) that

\[ \_\_\_\_\_\_\_\_\_\_\]
\[ [D_{x_i}, D_{x_j}]\mathcal{C} = -\mathcal{S}(\vartheta_i) \times \mathcal{S}(\vartheta_j) \times \mathcal{C} \]  
(154)

and taking into account that it is \([x_i, x_j] = 0\) we can write the last equation as

\[ ([D_{x_i}, D_{x_j}] - D_{[x_i, x_j]}]\mathcal{C} = \mathfrak{R}(\vartheta_i \wedge \vartheta_j) \times \mathcal{C} = -\mathcal{S}(\vartheta_i) \times \mathcal{S}(\vartheta_j) \times \mathcal{C} \]  
(155)

and so it follows that

\[ \mathfrak{R}(u \wedge v) \times \mathcal{C} = -\mathcal{S}(u) \times \mathcal{S}(v) \times \mathcal{C}. \]  
(156)

Of course, we must have \(\mathfrak{P}(\mathcal{S}(u) \times \mathcal{S}(v) \times \mathcal{C}) \in \text{sec} \mathcal{C}(M, g)\). Since \(\mathcal{S}(u) \times \mathcal{S}(v) = \mathfrak{P}(\mathcal{S}(u) \times \mathcal{S}(v)) + \mathfrak{P}_\perp(\mathcal{S}(u) \times \mathcal{S}(v))\) and we already know that \(\mathfrak{R}(u \wedge v) = -\mathfrak{P}(\mathcal{S}(u) \times \mathcal{S}(v))\) it follows that \(\mathfrak{P}_\perp(\mathcal{S}(u) \times \mathcal{S}(v)) = 0\) and moreover we get that

\[ \mathfrak{P}(\mathcal{S}(u) \times \mathcal{S}(v) \times \mathcal{C}) = \mathfrak{P}(\mathcal{S}(u) \times \mathcal{S}(v)) \times \mathcal{C} = \mathfrak{P}(\mathcal{S}(u) \times \mathcal{S}(v)) \times \mathfrak{P}(\mathcal{C}). \]  
(157)

**Lemma 42** Let \(\mathcal{C} \in \text{sec} \mathcal{C}(M, g)\) and \(u, v \in \text{sec} \Lambda^1 T^*M \hookrightarrow \in \text{sec} \mathcal{C}(M, g)\).

\[ \mathfrak{R}(u \wedge v) = \mathfrak{P}_v(\mathcal{S}(u)) \]  
(158)

**Proof.** Taking \(\mathcal{C} = \mathcal{S}(u)\) in Eq.\(\text{(157)}\) and recalling Eq.\(\text{(154)}\) \(\mathfrak{P}(\mathcal{S}(u)) = 0\) we get

\[ \mathfrak{P}_v(\mathcal{S}(u)) = -\mathfrak{P}(\mathcal{S}(u) \times \mathcal{S}(v)) \]  
(159)

which proves the lemma. ■

**Remark 43** From Eq.\(\text{(159)}\) we immediately have

\[ \mathfrak{P}_u(\mathcal{S}(v)) = -\mathfrak{P}(\mathcal{S}(v) \times \mathcal{S}(u)) = \mathfrak{P}(\mathcal{S}(u) \times \mathcal{S}(v)) = -\mathfrak{P}_v(\mathcal{S}(u)) = -\mathfrak{P}(\mathcal{S}(u)) \]  
(160)

where the last term follows from the fact that \(\mathcal{S}(u) = \mathcal{S}(u)\).

**Proof.** (of Proposition \(\text{35}\)) We know that \(\mathfrak{R}(v) = \partial_v \mathfrak{R}(u \wedge v)\). Thus using Eq.\(\text{(159)}\) and recalling Eq.\(\text{(168)}\) we can write

\[ \mathfrak{R}(v) = \partial_v \mathfrak{P}_v(\mathcal{S}(u)) = -\partial_v \mathfrak{P}_u(\mathcal{S}(u)) \]  
(161)

\[ = -\delta \mathfrak{P}(\mathcal{S}(v)) = -\mathcal{S}(\mathcal{S}(v)) = -\mathcal{S}^2(v). \]

Since we already showed that \(\mathfrak{R}(v) = \partial \wedge \partial \ (v)\) we get

\[ \partial \wedge \partial \ (v) = -\mathcal{S}^2(v) \]

and the proposition is proved. ■

**Remark 44** Note that whereas \(\mathcal{S}(v)\) is a section of \(\mathfrak{C}(\mathfrak{m}, \mathfrak{g})\), \(\mathcal{S}^2(v) \in \text{sec} \Lambda^1 T^*M \hookrightarrow \text{sec} \mathcal{C}(M, g)\)

**Proposition 45** Let \(u, v, w \in \text{sec} \Lambda^1 T^*M \hookrightarrow \text{sec} \mathcal{C}(M, g)\). Then,

\[ \mathfrak{R}(u \wedge v) = \frac{1}{2} \partial_w \wedge [\mathfrak{P}_v, \mathfrak{P}_u](w). \]  
(162)

27
Proof. From Eq. (149) with $C = w \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}(M, g)$ we have

$$
\mathcal{R}(u \wedge v) \times w = [P_u, P_v](w). \tag{163}
$$

Now, the first member of Eq. (163) is

$$
\mathcal{R}(u \wedge v) \times w = -w \cdot \mathcal{R}(u \wedge v)
$$

Now, writing $\mathcal{R}(u \wedge v) = \frac{1}{2} u^i v^j R^k l^m \theta_k \wedge \theta_l$ we have

$$
\frac{1}{2} u^i v^j \theta_r \wedge (\theta_r \wedge \mathcal{R}(u \wedge v) = -2 \mathcal{R}(u \wedge v).
$$

(164)

Taking into account Eq. (163) and Eq. (164) the proof follows.

We can also prove the proposition as follows: directly from Eq. (151) we can write

$$
[P_u, P_v](w) = \frac{1}{2} \theta^m \frac{\partial}{\partial w} \wedge (u^k v^l R^i j^m \theta_i).
$$

(165)

Thus

$$
\frac{1}{2} \partial_w \wedge [P_u, P_v](w) = \frac{1}{2} \theta^m \frac{\partial}{\partial w} \wedge (u^k v^l R^i j^m \theta_i)
$$

$$
= \frac{1}{2} u^k v^l R^i j^m \theta^m \wedge \theta^i
$$

$$
= -u^k v^l \mathcal{R}_{kl} = -\mathcal{R}(u \wedge v)
$$

(166)

and the proof is complete. ■

Proposition 46 Let $u, v, w \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}(M, g)$. Then,

$$
\mathcal{R}(u \wedge v) = \partial_w \wedge P_v P_u(w). \tag{167}
$$

Proof. Recall that we proved (Eq. (158)) that $\mathcal{R}(u \wedge v) = P_v(S(u))$. Also Eq. (81) says that $S(u) = S(u) = \partial_w \wedge P_v P_u(u)$ for any $u, w \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}(M, g)$. Now from Eq. (181) we have

$$
P_w(u) = P_u(w) = u \cdot \hat{\partial} P(w) = \hat{D}_u P(w) = \hat{D}_u(P(w)) - P(\hat{D}_u w) = (\hat{D}_u w) \perp
$$

(168)

which means that

$$
P_w(u) = (P_w(u)) \perp. \tag{169}
$$

Then, we have that

$$
\mathcal{R}(u \wedge v) = P_v(S(u)) = P_v(\partial_w \wedge P_w(u))
$$

$$
= P_v(\partial_w \wedge P_u(w)) = P_v((\partial_w) \perp \wedge (P_u(w)) \perp)
$$

(158)

$$
\partial_w \wedge P_v P_u(w) \perp
$$

and the proof is complete. ■
Remark 47 Since \( R(u \wedge v) = -R(v \wedge u) \), Eq. (167) implies that \( u, v, w \in \text{sec} \Lambda T^*M \hookrightarrow \text{secCl}(M, g) \)
\[
\partial_w \wedge P_u P_v(w) = -\partial_w \wedge P_u P_v(w),
\]
thus exhibiting the consistency of Eq. (167) with Eq. (162).

5 On Clifford’s Little Hills

One could think that the fact that \( \partial \wedge \partial (v) = R(v) = -S^2(v) \) when applied to General Relativity coupled with brane theory permits to give a mathematical formalization to Clifford’s intuition\(^{18}\) presented in [6], namely that:

1. That small portions of space are in fact of a nature analogous to little hills on a surface which is on the average flat; namely, that the ordinary laws of geometry are not valid in them.
2. That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave.
3. That this variation of the curvature of space is what really happens in that phenomenon which we call the motion of matter, whether ponderable or ethereal.
4. That in the physical world nothing else takes place but this variation, subject (possibly) to the law of continuity.

Let us see how to proceed. Let \((M, g, D, \tau_g, \uparrow)\) be a model of a gravitational field generated by an energy momentum tensor \( T^a := T^a_b \theta^b \otimes \theta^a \) describing all matter of the universe according to General Relativity theory. As well known Einstein equation can be written as
\[
\partial \wedge \partial \theta^a = -T^a + \frac{1}{2} \mathcal{T}^a,
\]
where \( \mathcal{T}^a := T^a_b \theta^b \) and \( \mathcal{T} := T^a_a \), with \( T^a_a \). If we suppose that the structure \((M, g)\) is a submanifold of \((\tilde{M} \simeq \mathbb{R}^n, \tilde{g})\) for \( n \) large enough as discussed in the beginning of Section 3 we can write Eq. (171) taking into account Eq. (142) as
\[
S^2(\theta^a) = \mathcal{T}^a - \frac{1}{2} \mathcal{T} \theta^a.
\]
So, in a region where there is no matter \( S^2(\theta^a) = 0 \), despite the fact that \( S(\theta^a) = S(\theta^a) \) may be non null. So, a being living in the hyperspace \( \mathbb{R}^n \) and looking at our brane world will see the little hills (i.e., “matter”) are special shapes in \( M \), places where the \( S^2(\theta^a) \neq 0 \) which act as sources for \( P(\partial \lrcorner S(\theta^a)) \) since \( P(\partial \lrcorner S(\theta^a)) = -S^2(\theta^a) \).

\(^{18}\)Taking into account, of course, that differently from Clifford’s idea, instead of a space theory of matter, we must talk about a spacetime theory of matter.

\(^{19}\)The symbol \( \uparrow \) means that the Lorentzian manifold \((M, g)\) is time orientable. Details in [19].
Remark 48 To properly appreciate the above argument one must take in mind that the shape extensor depends for its definition on the metric $\mathbf{g}$ and the Levi-Civita $D$ connection of $\mathbf{g}$ used in $M$. So, a different choice of metric in $M$ will imply in Clifford’s little hills to be represented by different shape extensors. Despite this fact, it seems to us that shape is most appealing than the curvature biform or the Ricci $1$-form fields $R^a = \nabla \partial \theta^a$ as indicator of the presence of matter as distortions in the world brane $M$. Indeed, inner observers living in $M$ in general may not have enough skills and technology to discover the topology of $M$ and so cannot know if their brane world is a bended surface in the hyperspace (i.e., $\mathbb{M}$) or even if a open set $U \subset M$ is a part of an hyperplane or not. Moreover, those inner observers that have learned a little bit of differential geometry know that they cannot say that their manifold is curved based on the fact that the curvature biform is non null, for they know that the curvature biform is a property of the connection (parallelism rule) that they decide to use by convention in $M$ and not an intrinsic property of $M$. They know that if they choose a different connection it may happen that its curvature biform may be null and their connection (not their manifold) may have torsion and even a non null nonmetricity tensor. So, with their knowledge of differential geometry they infer that little hills (as seems for beings living in $\mathbb{M}$) can only be associated to the shape extensor if they use Levi-Civita connection of $g$ in $M$.

6 A Maxwell Like Equation for a Brane World with a Killing Vector Field

When $(M,g)$ admits a Killing vector field $A \in \text{sec}TM$ then it follows [20] that $\delta A = 0$, where $A = g(A, \ ) \in \text{sec} \wedge^1T^*M \mapsto \text{sec}\mathcal{C}(M,g)$. In this case we can show that the Ricci operator applied to $A$ is equal to the covariant D’Alembertian operator applied to $A$, i.e.,

$$\nabla \wedge \partial A = \nabla \cdot \partial A$$  \hfill (173)

Now, recalling Eq.(3) that the square of the Dirac operator $\nabla^2$ can be decomposed in two ways, i.e.,

$$\nabla \wedge \partial A + \nabla \cdot \partial A = \nabla^2 A = -d\delta A - \delta dA$$  \hfill (174)

we have writing $F = dA$ and taking into account that $\delta A = 0$ that Einstein equation can be written as

$$\delta F = 2S^2(A)$$ \hfill (175)

and since $dF = ddA = 0$ we can write Einstein equation as:

$$\nabla F = -2S^2(A).$$  \hfill (176)

20Recall that $R$ is in general non null even in vacuum.
21Details about these possibilities are discussed in [10] where a theory of the gravitational field on a brane diffeomorphic to $R^4$ is discussed.
Eq. (176) shows that in a Lorentzian brane $M$ of dim 4 which contains a Killing vector field $A$, Einstein equation is encoded in an “electromagnetic like field” $F$ having as source a current $J = 2S^2(A) \in \sec C\ell(M, g)$.

7 Conclusions

In this paper, we gave a thoughtful presentation of the geometry of vector manifolds using the Clifford bundle formalism, hopping to provide a useful text for people (who know the Cartan theory of differential forms) and are interested in the differential geometry of submanifolds $M$ (of dimension $m$ equipped with a metric $g$ of signature $(p, q)$ and its Levi-Civita connection $D$) of a manifold $M \simeq \mathbb{R}^n$ (of dimension $n$ and equipped with a metric $\hat{g}$ of signature $(\hat{p}, \hat{q})$ and its Levi-Civita connection $\hat{D}$). We proved in details several equivalent expressions for the curvature biforms $\mathcal{R}(u \wedge v)$ and moreover proved that the Ricci operator $\partial \wedge \partial$ when applied to a 1-form field $v$ is such that $\partial \wedge \partial(v) = \mathcal{R}(v) = -S^2(v)$ ($\mathcal{R}(v) = R^a_{\ b}v^b$) is the negative of the square of the shape operator $S$. We showed in Section 5 that when this result is applied to General Relativity it permits to give a mathematical realization to Clifford’s theory of matter. Moreover in Section 6 we show that in a Lorentzian brane containing a Killing vector field Einstein equation can be encoded in a Maxwell like equation whose source is a current given by $J = 2S^2(A)$.

To end we observe that although some (but not all) of the results in this paper appear in [13, 14, 21], our methodology and many proofs differs considerably. We use of the Clifford bundle of differential forms $C\ell(M, g)$ and give detailed and (we hope) intelligible proofs of all formulas, clarifying some important issues, presenting, e.g., the precise relation between the shape biform $S$ evaluate at $v$ (a 1-form field) and the connection extensor $\omega$ evaluated at $v$ (Eq. (104)). In particular, our approach also generalizes for a general Riemann-Cartan connection the results in [14] which are valid only for the Levi-Civita connection $D$ of a Lorentzian metric of signature $(1, 3)$. Moreover our approach makes rigorous the results in [14] which are valid only for 4-dimensional Lorentzian spacetimes admitting spinor structures since in [14] it is postulated that the frame bundle of $M$ has a global section (there called a fiducial frame).

A Some Identities Involving $P$ and $P_u$

The projection operator $P$ has been defined by Eq. (65) and its covariant derivative $P_u := u \cdot \delta P$ has been defined by Eq. (78). Let $\mathcal{C}, \mathcal{D} \in \sec C\ell(M, \hat{g})$. Since

$$P(\mathcal{C} \wedge \mathcal{D}) = P(\mathcal{C}) \wedge P(\mathcal{D}),$$

(177)

---

22 This includes people, we think, interested in string and brane theories and General Relativity.

23 $i : M \to M$ is the inclusion map.

24 See [11] for details.
we have for any \( u \in \text{sec} \bigwedge^1 T^* M \hookrightarrow \text{sec} \mathcal{C} \ell(M, g) \) that
\[
P_u(C \wedge D) = P_u(C) \wedge P(D) + P(C) \wedge P_u(D). \tag{178}
\]

From \( P^2(C) = P(C) \) we have that
\[
P_uP(C) + PP_u(C) = P_u(C). \tag{179}
\]

Now, we easily verify that
\[
P_u(w) = P_\perp(u \cdot \hat{s}w), \quad P_w(u) = P_\perp(w \cdot \hat{s}u). \tag{180}
\]

Now, we already know from Eq.(124) that \( P_\perp(u \cdot \hat{s}w) \) and \( P_\perp(w \cdot \hat{s}u) \) are equal and thus
\[
P_u(w) = P_w(u). \tag{181}
\]

From this equation also follows immediately that
\[
P_uP(w) = P_uP(u). \tag{182}
\]

Now given that each \( X \in \text{sec} \mathcal{C} \ell(\hat{M}, \hat{g}) \) can be written as \( X = X_\parallel + X_\perp \), with \( X_\parallel = P(X) \) we get from Eq.(179) that
\[
PP_u(X_\parallel) = 0, \quad PP_u(X_\perp) = P_u(X_\perp). \tag{183}
\]

Also, from Eq.(178) we have immediately taking into account Eq.(183) for any \( C, D \in \text{sec} \mathcal{C} \ell(M, g) \) that
\[
P_u(C_\parallel \wedge D_\perp) = C_\parallel \wedge P_u(D_\perp). \tag{184}
\]

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