Arbitrary Dimension Convection-Diffusion Schemes for Space-Time Discretizations

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Abstract

This note proposes embedding a time dependent PDE into a convection-diffusion type PDE (in one space dimension higher) with singularity, for which two discretization schemes, the classical streamline-diffusion and the EAFE (edge average finite element) one, are investigated in terms of stability and error analysis. The EAFE scheme, in particular, is extended to be arbitrary order which is of interest on its own. Numerical results, in combined space-time domain demonstrate the feasibility of the proposed approach.

Keywords: space-time formulation, convection-diffusion problems, finite-element method, exponential fitting, streamline-diffusion

1. Introduction

The embedding of time-dependent problems into a one space dimension higher stationary problem is not a new idea. It has many appealing properties, such as: using already existing tools developed for stationary problems;

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using adaptive methods with reliable and efficient error control; the ability to use existing efficient solver libraries developed for stationary problems. There is, however, a drawback: typically, the memory needed to run a simulation using the combined space-time discretization approach is increased by an order of magnitude. One way to keep the memory required by such methods under control is to use time intervals with fixed length. Another, more general, remedy to the extensive use of computer memory in space-time simulations is to employ accurate dimension reduction algorithms, both in space and in time, which can lead to coarser problems with fewer degrees of freedom, also known as upscaled discretizations. Indeed, an accurate coarser problem can replace the expensive, in terms of memory, fine-grid one and still provide a reliable discretization tool. For a general dimension reduction approach by coarsening (in three space dimensions), we refer to [1]. The extension of the technique proposed in [1] to 4D space-time elements is a work in progress. Another feasible approach for dimension reduction in space-time discretizations is to exploit sparse grids, as proposed in [2]. More recently, discrete space-time schemes using B-splines and Non-Uniform Rational Basis Splines (NURBS) have been employed (see [3]) to yield stable isogeometric analysis methods for the numerical solution of parabolic PDEs in fixed and moving spatial domains.

We point out that in the present note we do not consider dimension reduction techniques. Rather, as a first step, we study the accuracy and stability of the proposed embedding. More specifically, for the discretization of the space-time formulation of a parabolic problem we exploit two well-known techniques for convection diffusion equations: the streamline diffusion method [4] (see also [5], [6]) and the EAFE–Edge Average Finite Element scheme [7] (see also [8] and [9]). Let us add that the high order EAFE method developed here provides a novel, high order, exponentially fitted discretization for convection-diffusion problems with suitable stability and approximation properties.

The structure of the remainder of this note is as follows. In Section 2 we introduce the space-time formulation of parabolic problems. Then, in Section 3 we present the streamline diffusion method in our space-time setting. Section 4 contains the derivation of the high order EAFE scheme on simplicial finite element grids in arbitrary spatial dimension. The application details for the lowest order EAFE discretization to parabolic problems is given in Section 5. Finally, in Section 6 we present numerical tests showing the optimality and efficiency of both schemes for space-time formulation of parabolic problems. We conclude this paragraph with remark on the terminology: as the EAFE scheme may be viewed as a multidimensional Scharfetter-Gummel discretization [10], in what follows, we use the terms “EAFE discretization” and “Scharfetter-Gummel discretization” interchangeably.

2. Space time formulation of parabolic problems

We consider the following parabolic problem:

\[
\begin{align*}
    u_t - \text{div}(K(x)\nabla u - \beta \cdot u) + \gamma u &= f, & x \in \Omega_s, \\
    u &= 0, & x \in \Gamma = \partial \Omega_s; \\
    u(x, 0) &= u_0(x), & x \in \Omega_s.
\end{align*}
\]
Here, $\beta$ is a vector field (a velocity) and $K(x)$ is, in general, a scalar (or $d \times d$ tensor valued) function. Let $\Omega_t = (0, t_{\text{max}})$ be the time interval of interest. The space-time domain is $\Omega = \Omega_s \times \Omega_t$. For convenience we have assumed homogeneous Dirichlet boundary conditions $u = 0$ on $\partial \Omega_s \times \Omega_t$. In treating time as a space-like variable, the initial condition at $t = 0$ becomes a Dirichlet boundary condition for the $(d + 1)$ dimensional problem.

In a space-time formulation, introducing a new variable $y = (x, t)$ then gives the following convection diffusion equation: Find $u = u(y)$ such that

$$\begin{align*}
-\text{div}_y(D\nabla_y u + b \cdot \nabla_y u) + \gamma u &= f \quad \text{in} \quad \Omega \\
b = (\beta^t, 1)^t : \Omega \rightarrow \mathbb{R}^{d+1}, \\
u = 0 \quad \text{on} \quad \Gamma = \partial \Omega \times \Omega_t; \\
u = u_0 \quad \text{on} \quad \Gamma_0 = \Omega_s \times \{t = 0\}.
\end{align*}$$

Without loss of generality we may assume that $u_0 = 0$ and we define $H^1_E(\Omega)$ as the subspace of $H^1(\Omega)$ satisfying these homogeneous Dirichlet boundary conditions.

In the following we consider two schemes for discretization of convection diffusion problems and apply them to space-time formulations of (1). These are the Streamline Diffusion and the Scharfetter-Gummel (EAFE) discretizations. For the latter we need a non-singular $D$, while above $D = \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}$ is actually degenerate. To remedy this, we perturb it to make it invertible, i.e., we let

$$D = \begin{bmatrix} K & 0 \\ 0 & \epsilon \end{bmatrix}$$

for a small parameter $\epsilon > 0$.

### 3. Streamline Diffusion

We first consider a simple case when $K = \alpha I$, $\alpha > 0$, $\gamma \geq 0$, and $\beta$ are constant. Then equation (1) has the form:

$$Lu = u_t - \alpha \Delta u + \beta \cdot \nabla u + \gamma u = f$$

The results below generalize to the variable coefficient case in a straightforward and well-studied fashion. Here we consider the constant coefficients case only in an attempt to keep the focus on the important aspect of time discretization. In allowing for different sizes of $\alpha$, $\beta$ and $\gamma$, our analysis covers several scenarios of interest. For simplicity we assume the initial condition $u_0 = 0$.

The weak form of (2) is given by: find $u \in H^1_E$ such that

$$(u_t, v) + \alpha(\nabla u, \nabla v) + (\beta \cdot \nabla u, v) + \gamma(u, v) = f(v)$$

for all $v \in H^1_E$. The space-time bilinear for $B(u,v)$ is given by

$$B(u, v) = \int_0^T (u_t, v) + \alpha(\nabla u, \nabla v) + (\beta \cdot \nabla u, v) + \gamma(u, v) \, dt$$

where

$$(u, v) = \int_{\Omega_s} uv \, dx$$

3
is the usual $L_2$ inner product on $\Omega_s$. The right hand side is given by the linear functional

$$F(v) = \int_0^T f(v) \, dt.$$  

We assume that the space-time domain $\Omega$ is covered by a shape regular quasiuniform tessellation $T_h$ of elements of size $h$. The energy norm for this problem is given by

$$\|u\|_2 = \|u(T)\|^2 + \int_0^T \alpha \|\nabla u\|^2 + h^p \nu |\beta \cdot \nabla u + u_t|^2 + \gamma \|u\|^2 \, dt$$

where

$$\nu = \frac{1}{\sqrt{1 + |\beta|^2}}.$$  

For technical reasons made clear below, we set $p = 1$ for the important case of continuous piecewise linear approximation, or the special case $\alpha = 0$; otherwise we choose $p = 2$.

We make a standard Petrov-Galerkin streamline diffusion discretization for this $d + 1$ dimensional problem. Let $V_h \subset H^1_T$ denote a $C^0$ conforming piecewise polynomial finite element space. The space $V_h$ itself is the trial space. In our Petrov-Galerkin formulation, the test functions are given by $v + \theta h^p \nu (\beta \cdot \nabla v + v_t)$ for $v \in V_h$, where $\theta$ is a parameter to be characterized below. The discrete problem is: find $u_h \in V_h$ such that

$$B_h(u_h, v) \equiv B(u_h, v) + \int_0^T (Lu_h, \theta h^p \nu (\beta \cdot \nabla v + v_t)) \, dt = F(v + \theta h^p \nu (\beta \cdot \nabla v + v_t))$$

for all $v \in V_h$. Because $V_h$ is only $C^0$, the term $(Lu_h, \theta h^p \nu (\beta \cdot \nabla v + v_t))$ is formally interpreted elementwise due to possible discontinuities on inter-element boundaries.

We begin with a basic stability result.

**Lemma 1.** Let $V_h$ be the space of continuous piecewise linear polynomials or $\alpha = 0$. For $v \in V_h$, and $\theta$ sufficiently small, there exists $C > 0$, independent of $h$, such that

$$B_h(v, v) \geq C \|v\|^2.$$  

(3)

**Proof** We first note the term $(\alpha \Delta v, \theta h^p \nu (v_t + \beta \cdot \nabla v)) = 0$.

The term

$$\int_0^T (v_t + \beta \cdot \nabla v, v) \, dt = \frac{|v(T)|^2}{2}$$

and

$$(v_t + \beta \cdot \nabla v, \theta h^p \nu (v_t + \beta \cdot \nabla v)) = \theta h^p \nu |v_t + \beta \cdot \nabla v|^2.$$  

Finally

$$(\gamma v, v + \theta h^p \nu (v_t + \beta \cdot \nabla v)) \geq \gamma |v|^2 - \gamma \theta h^p \nu \|v_t + \beta \cdot \nabla v\|$$

$$\geq \gamma |v|^2 \left(1 - \frac{\gamma \theta h^p \nu}{2}\right) - \frac{\theta h^p \nu}{2} |v_t + \beta \cdot \nabla v|^2.$$
Combining all these estimates, and taking \( \theta \) sufficiently small proves (3). 

The orthogonality-like relation for the error \( e = u - u_h \) in our approximation is given by

\[
B_h(e, v) = 0
\]

for all \( v \in V_h \).

For \( \chi \in V_h \), let

\[
\phi = u_h - \chi \quad \eta = u - \chi.
\]

Our error relation can be expressed in terms of \( \phi \) and \( \eta \) as

\[
B_h(\phi, v) = B_h(\eta, v)
\]

for all \( v \in V_h \). We take \( v = \phi \in V_h \) and use Lemma 1. Then we have

\[
\|\phi\|^2 \leq CB_h(\phi, \phi) \leq CB_h(\eta, \phi)
\]

(5)

Let \( \delta \) be a sufficiently small parameter to be characterized below. We now estimate all the terms on the right hand side of (5). First,

\[
(\eta_t + \beta \cdot \nabla \eta, \theta h \nu(\phi_t + \beta \cdot \nabla \phi)) \leq C\theta h \nu \|\eta_t + \beta \cdot \nabla \eta\|^2 + \delta \theta h \nu \|\phi_t + \beta \cdot \nabla \phi\|^2
\]

\[
(\gamma \eta, \phi + \theta h \nu(\phi_t + \beta \cdot \nabla \phi)) \leq C\gamma \|\eta\|^2 + \delta \gamma \|\phi\|^2 + \theta h \nu \|\phi_t + \beta \cdot \nabla \phi\|^2
\]

The third term is a bit more involved.

\[
\int_0^T (\eta_t + \beta \cdot \nabla \eta, \phi) \, dt = (\eta(T), \phi(T)) - \int_0^T (\eta, \phi_t + \beta \cdot \nabla \phi) \, dt
\]

\[
\leq C \left( \|\eta(T)\|^2 + (h \nu)^{-1} \int_0^T \|\eta\|^2 \, dt \right) + \delta \left( \|\phi(T)\|^2 + h \nu \int_0^T \|\phi_t + \beta \cdot \nabla \phi\|^2 \, dt \right)
\]

Combining these estimates, and making \( \delta \) sufficiently small, we have

\[
\|\phi\|^2 \leq C \left( \|\eta\|^2 + \int_0^T (h \nu)^{-1} \|\eta\|^2 \, dt \right)
\]

(6)

Using (6) and the triangle inequality, we obtain

**Theorem 2.** Let \( V_h \) be the space of continuous piecewise linear polynomials or \( \alpha = 0 \). Then the error \( e = u - u_h \) satisfies

\[
\|u - u_h\|^2 \leq C \inf_{\chi \in V_h} \left( \|u - \chi\|^2 + \int_0^T (h \nu)^{-1} \|u - \chi\|^2 \, dt \right)
\]

(7)
Suppose $\alpha = O(1)$ and $V_h$ contains piecewise linear polynomials. Then if $u \in H^2(\Omega)$, (7) yields an $O(h^{1/2})$ rate of convergence of the space-time gradient in the streamline direction $b = (\beta^t, 1)^t$, and an optimal $O(h)$ convergence rate for $(\int_0^T \|\nabla (u - u_h)\|^2)^{1/2}$. If $\gamma = O(1)$ we have $O(h)$ convergence for the space-time $L_2$ norm. While not optimal in every norm considered, overall this is in alignment with well-known behavior for the classical streamline diffusion method. If $\alpha = 0$ and $V_h$ contains piecewise polynomials of degree $r$, we lose control of the gradient $\|\nabla p u\|$, but gain improved $O(h^r)$ convergence for the space-time gradient in the streamline direction, and if $\gamma = O(1)$ we have improved $O(h^{r+1/2})$ convergence for the $L_2$ norm. These again correspond with classical results for the streamline diffusion method.

If $\alpha \neq 0$ and $V_h$ contains piecewise polynomials of degree $r > 1$, terms similar to $\alpha \langle \Delta v, v_t \rangle$ become problematic since $\Delta v$ contains no time derivatives and at present forces us to choose $p = 2$. (One might alternatively consider replacing the diffusion term $\Delta u$ with $\Delta u + \epsilon u_{tt}$, and then analyzing as in the standard streamline diffusion scenario, but this dilutes the advantage one obtains through the use of higher order approximation since we lose consistency with the original PDE). Here is the analog of Lemma 1.

**Lemma 3.** Let $V_h$ be the space of continuous piecewise linear polynomials of degree $r \geq 1$ and $\alpha \neq 0$. For $v \in V_h$, and $\theta$ sufficiently small, there exists $C > 0$, independent of $h$, such that

$$B_h(v, v) \geq C\|v\|^2.$$  \hfill (8)

**Proof** Generally the proof follows the same pattern as Lemma 1. The new term is $\alpha \langle \Delta v, v_t \rangle$. On a single element $\tau \in T_h$ we can use a local inverse assumption

$$|(-\Delta v, v_t + \beta \cdot \nabla v)| \leq Ch^{-1}\|\nabla v\|_{\tau}\|v_t + \beta \cdot \nabla v\|_{\tau}$$

Using this estimate, we have

$$\int_0^T \alpha \langle \nabla v, \nabla v \rangle - (\alpha \Delta v, \theta h^2\nu(v_t + \beta \cdot \nabla v)) \, dt$$

$$\geq \int_0^T \alpha (1 - C\alpha \theta)\|\nabla v\|^2 \, dt - \frac{\theta h^2\nu}{4}\|v_t + \beta \cdot \nabla v\|^2.$$ 

The remaining estimates in the proof of Lemma 1 are the same with $h$ replaced by $h^2$. 

We analyze the error similar to proof of Theorem 2. The new terms are

$$\alpha \langle \nabla \eta, \nabla \phi \rangle \leq C\alpha\|\nabla \eta\|^2 + \delta \alpha\|\nabla \phi\|^2$$

and

$$\alpha \langle \Delta \eta, \theta h^2\nu(\phi_t + \beta \cdot \nabla \phi) \rangle \leq C\alpha^2\nu h^2\|\Delta \eta\|^2 + \delta h^2\nu\|\phi_t + \beta \cdot \nabla \phi\|^2.$$ 

The remaining terms are estimated as in Theorem 2 with $h$ replaced by $h^2$, leading to
Theorem 4. Let $V_h$ be the space of continuous piecewise linear polynomials of degree $r > 1$ and $\alpha \neq 0$. Then the error $e = u - u_h$ satisfies

$$
\|u - u_h\| \leq C \inf_{\chi \in V_h} \left( \|u - \chi\|^2 + \int_0^T (h^2 \nu)^{-1} |u - \chi|^2 + \alpha h^2 \|\Delta(u - \chi)\|^2 \, dt \right)^{1/2}.
$$

If $\alpha = O(1)$, $V_h$ is the space of continuous polynomials of degree $r > 1$, and $u$ is sufficiently smooth, we have optimal $O(h^r)$ convergence for $(\int_0^T \|\nabla(u - u_h)\|^2 \, dt)^{1/2}$, but only $O(h^{r-1})$ convergence for the space-time gradient in the streamline direction. If $\gamma = O(1)$ we also obtain $O(h^r)$ convergence in the space-time $L_2$ norm.

A Practical Remark

Suppose that the space domain $\Omega_s$ has a generic length scale $L$. Since the time units for $\Omega_t = [0, T]$ could be completely unrelated to the space units, the space-time domain $\Omega = \Omega_s \times \Omega_t$ could be quite anisotropic. It could be very long if $T \gg L$ or very short if $T \ll L$. Filling such potentially thin domains with a small number of shape regular elements could be problematic from the practical point of view. Therefore it could be useful to rescale the time variable such that it is has a similar scale to the space variables. For example, one could change variables as in

$$
\tilde{t} = \frac{Lt}{T} \equiv \kappa t
$$

for $0 \leq \tilde{t} \leq L$. The modified space-time-domain $\Omega_s \times [0, L]$ is more isotropic, and likely could be tessellated with far fewer shape regular elements. In terms of the partial differential equation,

$$
\frac{\partial u}{\partial \tilde{t}} = \kappa \frac{\partial \tilde{u}}{\partial \tilde{t}}
$$

making the convection in the time direction larger or smaller depending on the value of $\kappa$. In terms of our analysis, we could replace

$$
\alpha \rightarrow \frac{\alpha}{\kappa} \equiv \tilde{\alpha}
$$

$$
\beta \rightarrow \frac{\beta}{\kappa} \equiv \tilde{\beta}
$$

$$
\gamma \rightarrow \frac{\gamma}{\kappa} \equiv \tilde{\gamma}
$$

and directly apply the analysis of the previous section to this modified constant coefficient equation.
4. High Order Scharfetter Gummel discretization

In this section we derive a high order Scharfetter-Gummel scheme on simplicial finite element grids in dimension $d \geq 1$. The original Scharfetter-Gummel difference scheme [10] is a method used in simulating 1-dimensional semiconductor equations. After its discovery, it has been generalized and used for the numerical solution of convection-diffusion equations of the form:

$$-\text{div } J(u) = f, \quad x \in \Omega \subset \mathbb{R}^d$$  \hspace{1cm} (10)

$$J(u) = (D(x)\nabla_x u - bu),$$  \hspace{1cm} (11)

$$u(x) = 0, \quad x \in \Gamma_D, \quad J(u) \cdot \mathbf{n} = 0, \quad x \in \Gamma_N$$  \hspace{1cm} (12)

$$Du \cdot \mathbf{n} = 0, \quad x \in \Gamma_R.$$  \hspace{1cm} (13)

Here, $J(u)$ is the flux variable which plays an important role in approximating the weak form of the equation. We note that the natural boundary condition is the one given on $\Gamma_N$ and the boundary condition (13) is of a Robin type for this problem. The weak form of the equation above is: Find $u \in V$ such that

$$a(u, v) + m_R(u, v) = f(v),$$  \hspace{1cm} (14)

$$a(u, v) = \int_{\Omega} J(u) \cdot \nabla v, \quad f(v) = \int_{\Omega} f v$$  \hspace{1cm} (15)

$$m_R(u, v) = \int_{\Gamma_R} (\mathbf{b} \cdot \mathbf{n})u v$$  \hspace{1cm} (16)

The variational form is obtained after integration by parts and using the fact that on $\Gamma_R$, $Du \cdot \mathbf{n} = J(u) \cdot \mathbf{n} - \mathbf{b} \cdot \mathbf{n} u$.

The Scharfetter-Gummel scheme was extended to more than 1 spatial dimension as the Edge Average Finite Element (EAFE) Scheme. A priori error estimates in any dimension were shown in [7]. This work only considered scalar valued diffusion coefficients (although in any spatial dimension); a discretization for matrix valued diffusion coefficients was proposed and analyzed in [9]. Related work on exponential fitting in discretizing convection-diffusion equations via mixed finite element methods is [11]. More recently, the techniques from [7] have been utilized to yield a second order gauge invariant discretizations for Pauli and Schrödinger equations (see [12]).

Here, we provide a novel approach which gives a Scharfetter-Gummel discretization for finite element spaces of order $r \geq 1$. Our approach follows the ideas in [7] and [9]. The extension to $r \geq 1$, however is not at all straightforward and requires results from the recently developed Finite Element Exterior Calculus. The rationale of constructing the high order Scharfetter-Gummel scheme is:

(i) approximate the flux $J(u)$ via the Nédélec elements (discrete differential 1-forms with polynomial coefficients);

(ii) eliminate the flux variable and write the resulting discrete problem in terms of the scalar valued finite element approximation of the solution of (1) $u$ (a 0-form).
To set up the finite element approximation, we let us itemize some of the ingredients and the main assumptions needed for the discretization.

- We assume that $\Omega$ is covered by a conforming, simplicial, shape-regular mesh $T_h$. We have $\Omega = \cup \{T \mid T \in T_h\}$.

- The space $V_h$ is the space of conforming Lagrange finite elements of degree $r$ and for the derivation of the scheme, we also need the 1st-kind-Nédélec polynomial spaces on a fixed element (cf.eg. [13, 14, 15, 16]). The details are described below in §4.1.

- We assume that the flux $J$ and $u$ are smooth enough so that all the norms of functions below make sense. In particular $J \in W^{1,p}(T)$, for all $T \in T_h$ and for some $p > d$. The solution $u$ is at least continuous, so that its Lagrange interpolant is well defined.

- We assume that the coefficients $D$, $b$ are piece-wise constants with discontinuities aligned with $T_h$.

**Remark 1.** The assumption $J \in W^{1,p}(\Omega)$ needs some comments. One important feature of the Scharfetter–Gummel scheme is that the estimates on $\|u_I - u_h\|_{1,\Omega}$ are in terms of norms of the flux $J(u)$. We thus approximate more accurately the interpolant $u_I \in V_h$ of the solution if the flux is smooth, while both the solution and the coefficients can be rough functions. For example, if we look at the 1D problem on $(0, 1)$:

$$-(u' - \beta u)' = 0, \quad u(0) = 0, \quad u(1) = 1,$$

we observe that $J(u) = (u' - \beta u)$ is a constant, i.e. smooth, while the solution $u$ may exhibit a sharp boundary layer, depending on $\beta$. In fact, in this idealized situation in 1D, the estimate in Theorem 8 implies that $u_I = u_h$, i.e. we have captured the exact solution at the vertices.

Next, we show that (i) and (ii) in the rationale given earlier are computationally feasible steps.

### 4.1. Notation and Nédélec spaces

Consider the Nédélec space $P^N$, which restricted to any element $T$ is the following polynomial space

$$P^N = (P_{j-1})^d \oplus S_r, \quad P_{j-1}^d \subseteq P^N \subseteq (P_r)^d,$$

where $P_j$, $j = (r - 1)$, $r$ is the space of polynomials of degree $\leq j$ on $T$, and $S_r$ is a subspace of the space $H_r$ of vector valued homogeneous polynomials of degree $r$ defined as

$$S_r = \{s \in H_r \mid s \cdot x = 0\}.$$

By definition, the inclusion relations given in equation (17) hold on any element $T \in T_h$. From now on we fix this element. We refer to [13, 14, 17, 18, 19].
for the classical and the modern description of these spaces and studies of their properties. In what follows we use some of the tools from [17] and [18]. In our notation, the lowest order of such polynomials corresponds to $r = 1$.

Further, let $M = \text{dim} P^N$ be the dimension of the Nédélec polynomial space on $T$. The elements of the basis in the dual space of $P^N$ are known as *degrees of freedom* and we denote them by $\{\eta_j\}_{j=1}^M$. Next, the basis in $P^N$, dual to the degrees of freedom we denote by $\{\varphi_j\}_{j=1}^M$. For general simplex in $\mathbb{R}^d$, the explicit form of the degrees of freedom and their dual basis is found in [17]. For our purposes it is sufficient to note that the functionals $\eta_j$ can be thought as integrals of traces of functions over sub-simplices. For the lowest order case, we have

$$\langle \eta_e, v \rangle = \int_e v \cdot \tau_e, \quad \varphi_e = \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i,$$

for every edge $e = (i, j)$ of $T$ (there are $\binom{d+1}{2}$ edges). Here, $\tau_e$ is the tangent for edge $e$, and $\{\lambda_i\}$ are the usual barycentric coordinates for element $T$ (cf., e.g., [20]). Using this notation, we have that any function $v \in P^N$ can be written as

$$v = \sum_{j=1}^M \langle \eta_j, v \rangle \varphi_j(x). \quad (18)$$

We stress that this representation is unique and provides a canonical interpolation operator, which for sufficiently smooth vector valued $v$ is defined as

$$\Pi^N v = \sum_{j=1}^M \langle \eta_j, v \rangle \varphi_j(x). \quad (19)$$

The smoothness of $v$ must be such that the linear forms $\langle \eta_j, \cdot \rangle$ are bounded.

Consider now the space $V_h$ of Lagrange finite elements of order $r$. The standard set of the degrees of freedom in such case are point evaluations (see [20, Theorem 2.2.1]) and we denote them by $\{\mu_j\}$. Further, the polynomial basis, dual to these degrees of freedom, we denote by $\{\xi_j\}$. We then have a canonical interpolation operator, well defined for any continuous $v$. The image of $v \in C^0(\Omega)$ under this interpolation is denoted by $v_I$ and we have

$$v_I = \sum_{j=1}^{N_h} \langle \mu_j, v \rangle \xi_j(x). \quad (20)$$

There is no need to distinguish the global interpolation operator (on $\Omega$) and the local one (on $T \in \mathcal{T}_h$) for our considerations and we use the same notation for both. Let us note, however, that when working on fixed $T \in \mathcal{T}_h$ we will use $N = \text{dim} P_r = (r+d)$, instead of $N_h = \text{dim} V_h$.

As is well known [17], we have commutative diagrams linking the Nédélec elements and the Lagrange elements of matching orders (order $r$ here), and on every element $T$ we have

$$\Pi^N \nabla v = \nabla v_I.$$
This relation is in fact a relation between degrees of freedom, namely
\[ \langle \eta_j, \nabla v \rangle = \langle \eta_j, \nabla v_I \rangle. \tag{21} \]
This is obvious by using the definition of \( \Pi^N \), the fact that \( \nabla v_I \in P^N \), and the uniqueness of the representation in [18].

4.2. Derivation of a high order Scharfetter-Gummel scheme
Let us fix \( T \in T_h \) and we start with the definition of \( J \) and use that \( D \) and \( b \) are constant on \( T \).

\[ J_p = u_p - b \exp(-q \cdot x) \nabla \exp(-q \cdot x) u, \quad q = D^{-1} b. \]

Hence, we have
\[ \exp(-q \cdot x) D^{-1} J(u) = \nabla (\exp(-q \cdot x) u). \tag{22} \]

If we apply now \( \langle \eta_j, \cdot \rangle, \ j = 1 : M \) on both sides, and then use [21] we get
\[ \langle \eta_j, e^{-q \cdot x} D^{-1} J(u) \rangle = \langle \eta_j, \nabla (\exp(-q \cdot x) u) \rangle, \]
\[ = \langle \eta_j, \nabla (\exp(-q \cdot x) u_I) \rangle, \]
\[ = \langle \eta_j, \nabla (\exp(-q \cdot x) u_I) \rangle. \]

The latter identity on the right hand side above, uses the fact that the "\( I \)"-interpolant is based on the functionals \( \mu_j \) that are based on nodal evaluation. As expected, the right hand side is a gradient of a function in \( V_h \) and in summary we have
\[ \langle \eta_j, e^{-q \cdot x} D^{-1} J(u) \rangle = \langle \eta_j, \nabla (\exp(-q \cdot x) u_I) \rangle, \quad j = 1, \ldots, M. \tag{23} \]

Introducing now \( G(J(u)) \in \mathbb{R}^M \) and \( d(u) \in \mathbb{R}^M \) by
\[ [G(J(u))]_j = \langle \eta_j, e^{-q \cdot x} D^{-1} J(u) \rangle, \quad j = 1, \ldots, m \tag{24} \]
\[ [d(u)]_j = \langle \eta_j, \nabla (\exp(-q \cdot x) u_I) \rangle, \quad j = 1, \ldots, M. \tag{25} \]
and we can write (23) as
\[ G(J(u)) = d(u), \tag{26} \]
Note that both \( G \) and \( d \) are linear operators, mapping vector fields and functions to \( \mathbb{R}^M \). We remark that the relation [26] is used later in the definition of the approximate bilinear form, and, in the proof of the error estimates, and, we further stress on the fact that \( d(u) = d(u_I) \), by definition. The advantage of the exponential weighting in [23] is that it provides a way to accurately approximate the flux by polynomials. Note that the right side of (23) are the values of \( \eta_j \) evaluated on a polynomial, while the left side contains the "true" flux \( J(u) \). As we shall see later, it is advantageous to use discretizations schemes based on (23) when the flux is piece-wise smooth. This not only includes the
case when $u$ and the PDE coefficients are piece-wise smooth, but also includes many other cases (see Remark 1 for a simple 1D example on this).

The main idea of the Scharfetter-Gummel and EAFE schemes is to approximate $J(u)$,

$$J(u) \approx J_T(u) \in P^N,$$

or equivalently, we seek

$$J_T(u) = \sum_{j=1}^{M} c_j \varphi_j,$$

for some coefficient vector $c = (c_j)$. The coefficient $c$ is chosen so that the relation (23) still holds for the approximation. An important question is whether this is possible. If $r = 1$, and we use the lowest order Nédélec elements, this is definitely the case as shown in the earlier works [7, 9].

A construction of exponentially-fitted discretizations with higher order polynomial spaces is a bit more intricate. In general, we would like to find $J_T(u) \in P^N$. A key observation is that in order to derive our scheme, we use the weak form of the equation (10) and we will aim to approximate the weak form as follows:

$$\int_T J(u_h) \cdot \nabla v_h \approx \int_T J_T(u_h) \cdot \nabla v_h,$$

for functions $v_h \in V_h$ and $J_T(u_h) \approx J(u_h)$. As on $T$, $\nabla v_h \in (P_{r-1})^d$, it is sufficient to look for approximations $J_T(u) \in (P_{r-1})^d \subset P^N$.

We now explore this observation and look at how it affects the identity (26). Let $P$ be the matrix representation of the embedding $(P_{r-1})^d \subset P^N$. To define this matrix, let $\{\varphi_j\}_{j=1}^{M_0}$ be a basis in $(P_{r-1})^d$, with $M_0 = \dim(P_{r-1})^d$ and let $\{\varphi_j\}_{j=1}^M$ be the basis in $P^N$, dual to the degrees of freedom $\{\eta_m\}_{m=1}^M$. Then the entries of $P$ are the coefficients in the expansion of $\psi_m$ in terms of $\{\varphi_j\}_{j=1}^M$, and we have,

$$\psi_k = \sum_{j=1}^{M} p_{jk} \varphi_j, \quad \text{with} \quad p_{mk} = \langle \eta_m, \psi_k \rangle. \quad (27)$$

Note that $\nabla(\exp(-\mathbf{q} \cdot \mathbf{x}) u_I)$ is an element $(P_{r-1})^d$, and, as such, it can be written as a linear combination via $\{\psi_k\}_{k=1}^{M_0}$. Recalling the definition of $\mathbf{d}(u)$ in (25) then leads to the following useful relations:

$$\nabla(\exp(-\mathbf{q} \cdot \mathbf{x}) u_I) = \sum_{k=1}^{M_0} \tilde{d}_k \psi_k, \quad [\mathbf{d}(u)]_j = \sum_{k=1}^{M_0} \langle \eta_j, \psi_k \rangle \tilde{d}_k,$$

$$\mathbf{d}(u) = P \tilde{\mathbf{d}}.$$

As a consequence, to define the approximation $J_T$, we need to find a solution of the following problem

$$P^* Z P \tilde{\mathbf{c}} = P^* \tilde{\mathbf{d}}, \quad (28)$$

where we have set $\mathbf{c} = P \tilde{\mathbf{c}}$, and, as we have shown, $\mathbf{d} = \tilde{\mathbf{d}}$. Above the matrix $Z \in \mathbb{R}^{M \times M}$ has entries

$$Z_{jk} = \langle \eta_j, e^{-b \cdot D^{-1} \times D^{-1} \varphi_k} \rangle.$$
The following remark is in order. In general, we may have tried to solve the following system of equations for the coefficients $c$:

$$Zc = d.$$  \hspace{1cm} (29)

Clearly, if this is a well posed problem, then we can find the approximation $J_T(u)$. However, as it described above, we only need the solution in subspace, i.e., to solve problem (28). To show that the subspace problem (28) is solvable it is sufficient to show that $ZP$ is injective, and since $P$ is injective, it is sufficient to show that $Z$ is injective on the range of $P$.

We let

$$Z^\dagger = P (P^* Z P)^{-1} P^*.$$  \hspace{1cm} (30)

In the following, we will simply denote $c = P\hat{c}$ by $c = Z^\dagger d$, or, by (26), by $Z^\dagger G(J(u))$.

The following lemma follows by the construction of the approximation $J_T(u)$.

Lemma 5. If $J(u)$ is polynomial of degree $r - 1$, then its approximation $J_T(u)$ defined by $Z^\dagger G(J(u))$ coincides with $J(u)$.

4.2.1. A unisolvence result

Note that, when $b = 0$, the solvability of such system follows from the fact that the Nédélec degrees of freedom form a unisolvent set of functionals on $P^N \supset (P_{r-1})^d$. Multiplying by the exponent changes the game, and, we need to prove some of the basic results on unisolvence of Nédélec degrees of freedom for quasi-polynomials which we state in the following lemma.

Lemma 6. The matrix $ZP$ is injective, or, equivalently, if $p \in (P_{r-1})^d$ and $\langle \eta_j, e^{(-b D^{-1})x} p \rangle = 0$, for all $j = 1 : M$, then $p = 0$.

Proof This proof follows exactly the lines of the proofs of [18, Lemma 4.5, Lemma 4.6]. The only modifications involve the weighting with exponential function, which is positive everywhere. The rest of the arguments carry over without changes. A proof in terms of vector proxies is given in Appendix A.

4.2.2. Derivation of the discrete problem

Since now the approximation $J_T(u_h)$, for $u_h \in V_h$ is well defined, due Lemma 5 we have a natural approximating bilinear form. For $u_h \in V_h$ and $v_h \in V_h$ we set

$$a_h(u_h, v_h) = \sum_T \int_T J_T(u_h) \cdot \nabla v_h$$

$$= \sum_T \sum_{j=1}^M \int_T [Z_T^\dagger d_T(u_h)]_j \int_T \varphi_j \cdot \nabla v_h.$$  \hspace{1cm} (31)
The coefficients in $J_T(u_h)$ are determined by $Z_T^j d_T(u_h)$, for all $T \in T_h$, which in turn indeed makes the right side of (31) to depend only on the degrees of freedom of $u_h$.

We then define the following discrete problem: Find $u_h \in V_h$ such that

$$a_h(u_h, v) = f(v), \quad \text{for all } v \in V_h. \quad (32)$$

We note another useful relation which follows from the derivation above and is used in the error estimates below. It is an analogue of [7, Equation (3.16)], and [9, Equation (3.8)] and it plays a crucial role in the a priori error estimates. In particular it is useful to estimate the deviation of the derived discrete scheme from the standard Galerkin one (with bilinear form $a(u, v) = \int_\Omega (D \nabla u - b u) \cdot \nabla v$).

**Lemma 7.** For any continuous $u$, and sufficiently smooth $J$, such that $\Pi^N J(u)$ is well defined we have:

$$a_h(u_I, v_h) = \sum_{T} \sum_{j} [Z_T^j G_T(J(u))] \varphi_j \cdot \nabla v_h. \quad (33)$$

**Proof** Recalling that $d_T(u_I) = d_T(u)$, and substituting (26) in (31) gives the desired result.

Next, we show that, under certain conditions, this is a well posed problem, and we also prove an a priori error estimate.

### 4.3. Stability and error analysis

Error estimates and other properties of such discretization schemes are found in [7, 9]. Here we give an estimate for higher order Scharfetter-Gummel discretization and assume for simplicity, and without loss of any generality that we have Dirichlet boundary conditions. We have the following theorem:

**Theorem 8.** Assume that $a(\cdot, \cdot)$ is invertible on $V_h$. Then, for sufficiently small $h$, the discrete variational problem (32) is well posed and the following error estimate holds:

$$|u_I - u_h|_{1, \Omega} \leq c h^{r+p} |J(u)|_{r,p,\Omega}. \quad (34)$$

**Proof** From the definition of $a_h(\cdot, \cdot)$, for all $v \in V_h$ we have

$$|a(u, v) - a_h(u_I, v)| = \left| \sum_{T} a_T(u, v) - a_h(u_I, v) \right|$$

$$\leq \sum_{T} \left| \int_T J(u) \cdot \nabla v - \sum_{j} [Z_T^j G_T(J(u))] \varphi_j \cdot \nabla v \right|. \quad (35)$$

Note that from Lemmas [7 and 5] the right side vanishes for all $J(u)$ that are polynomials of degree $(r-1)$, and, standard scaling argument shows the estimate

$$|a(u, v) - a_h(u_I, v)| \leq c h^{r} |J(u)|_{r,p,\Omega} |v|_{1,q,\Omega}, \quad p^{-1} + q^{-1} = 1. \quad (35)$$

14
The solvability of the discrete problem then follows from the fact that by assumption \( a(\cdot, \cdot) \) provides a solvable problem, and hence it satisfies an inf-sup condition on \( V_h \). According to (35) \( a(\cdot, \cdot) \) and \( a_h(\cdot, \cdot) \) are close when \( h \to 0 \), and, hence, \( a(\cdot, \cdot) \) also satisfies an inf-sup condition for sufficiently small \( h \). This in turn implies that the discrete problem (32) is well posed. The error estimate (34) then follows from the inequality (35).

Remark 2. For the case \( r = 1 \) our proof here is analogous to the one given in [7, Lemma 6.2 and Theorem 6.3].

5. Application to parabolic problems

In this section, we recall the parabolic equation (1):

\[
\begin{align*}
    u_t - \text{div}(K(x)\nabla_x u - \beta u) &= f, \\
    u(x, 0) &= u_0(x), \quad \text{for} \quad t = 0; \\
    u(x, t) &= 0, \quad x \in \Gamma = \partial \Omega \times \{(0, t_{\text{max}})\}. 
\end{align*}
\]

(36)

This equation and the equation discretized by the streamline diffusion method match, if \( \text{div}_x \beta = 0 \), which we assume to hold. In general, the divergence form comes from a material law and many mathematical models of physical phenomena (if not all) are in divergence form.

The space-time formulation, (written in terms of a flux \( J_0 \), and with \( y = (x, t) \)) then is:

\[
\begin{align*}
    -\text{div}_y J_0(u) &= f, \quad \tilde{J}_0(u) = D_0(x)\nabla_x u - bu \\
    u(y) &= 0, \quad x \in \Gamma = \partial \Omega \times \{(0, t_{\text{max}})\}, \\
    u(y) &= u_0(x), \quad x \in \Gamma_0 = \partial \Omega \times \{t = 0\}. 
\end{align*}
\]

(37)

Here we have introduced the semidefinite, tensor valued function \( D_0(x) \), and more generally, we denote, \( D_\varepsilon(x) : \Omega \to \mathbb{R}^{(d+1) \times (d+1)} \):

\[
D_\varepsilon = \begin{pmatrix} K(x) & 0 \\ 0 & \varepsilon \end{pmatrix}, \quad J_\varepsilon = D_\varepsilon \nabla_y u - bu. \tag{38}
\]

The well known heat equation, \( u_t - \Delta u = f \), corresponds to \( K(x) = I \) and \( \beta = 0 \) and \( \varepsilon = 0 \).

The technique described in the previous section does not work in a straightforward fashion in the case of space-time formulation, because \( D_0 \) is a singular matrix. In fact, there is no obvious construction that works in the case of singular \( D_0 \). We consider then a formulation using perturbation of the diffusion tensor \( D_\varepsilon \) and the flux \( J_\varepsilon \). Thus, for the parabolic problem we set

\[
J_\varepsilon(u) = D_\varepsilon \nabla_y u - bu, \quad b = (\beta^T, 1)^T,
\]

(39)
5.1. Lowest order discretization for parabolic equations

In this section we discuss the Scharfetter-Gummel discretization when applied to space-time formulation of a parabolic equation, in the lowest order case. As a simple, but important example, we consider the simple case of heat equation, i.e. \( \beta_K = 0 \), which implies that \( b = e_{d+1} \), and \( e_{d+1} = (0, \ldots, 0, 1)^T \).

We next compute the action of the local stiffness matrix corresponding to a parabolic problem on a vector of degrees of freedom \( u \) representing a function in \( V_h \). We fix an element \((d+1)\) dimensional simplex \( T \in \mathcal{T}_h \) and we denote its barycentric coordinates by \( \{\lambda_i\}_{i=1}^{d+2} \) and the space-time coordinates of its vertices are \( \{y_i\}_{i=1}^{d+2} = \{(x_i, t_i)\}_{i=1}^{d+2} \). The degrees of freedom of a linear polynomial \( u \in V_h \) restricted to \( T \) are \( \{u_i\}_{i=1}^{d+2} = \{u(y_i)\}_{i=1}^{d+2} \) and we have \( u(y) = \sum_{i=1}^{d+2} u_i \lambda_i(y) \). For an edge \( E \in T, \ E = (y_i, y_j), \ i = 1, \ldots, (d+2), \ j = 1, \ldots, (d+2) \), we denote

\[
\tau_{ij} = \tau_E = \frac{(y_i - y_j)}{|y_i - y_j|}, \quad |r| = \sqrt{\sum_{i=1}^{d+1} r_i^2}, \quad \text{for all} \ r \in \mathbb{R}^{d+1}.
\]

We note that \( \tau_{ij} = -\tau_{ji} \), but as we shall see, this is of no consequence for the final form of the local stiffness matrix. To avoid complications in the presentation coming from unnecessary subscripts we will write \( D \) (resp. \( J \)) instead of \( D_e \) and (resp. \( J_e \)).

For any \( u \in V_h \), as \( D \) and \( J(u) \) are constants on \( T \), we have the following obvious identities from the definition of \( J \):

\[
D^{-1}J \cdot \tau_E = e^{t_i/\varepsilon} \nabla_y \left(e^{-t_j/\varepsilon} u \right),
\]

\[
\int_E e^{-t_j/\varepsilon} D^{-1}J \cdot \tau_E dE = \int_E \nabla_y \left(e^{-t_j/\varepsilon} u \right) \cdot \tau_E dE,
\]

\[
(D^{-1}J \cdot \tau_E) \int_E e^{-t_j/\varepsilon} dE = [e^{-t_i/\varepsilon} u(x_i, t_i) - e^{-t_j/\varepsilon} u(x_j, t_j)].
\]

Computing the integral on the left side gives

\[
\int_E e^{-t_j/\varepsilon} dE = |E| \int_0^1 \exp \left(-\frac{t_j + s(t_i - t_j)}{\varepsilon} \right) ds
\]

\[
= \frac{|E|}{t_j - t_i} \left[ e^{t_i/\varepsilon} - e^{-t_j/\varepsilon} \right] B \left( \frac{t_i - t_j}{\varepsilon}, \frac{t_j - t_i}{\varepsilon} \right),
\]

where \( B(s, t) = e^{-s} B(-s) \) is the Bernoulli function (\( B(0) = 1 \)). Note that, \( B(s) = e^{-s} B(-s) \) and \( B(s) > 0 \) for all \( s \in \mathbb{R} \). We then conclude that on every edge \( E \) in \( T \) we have:

\[
|E|(D^{-1}J \cdot \tau_E) = B \left( \frac{t_i - t_j}{\varepsilon} \right) u(y_i) - B \left( \frac{t_j - t_i}{\varepsilon} \right) u(y_j).
\]

(39)
In the derivation for general order of polynomials we needed the Nédélec basis and spaces. In the lowest order case, we can take a route that does not use these spaces explicitly. In the evaluation of the stiffness matrix entries, we need to compute integrals of the form
\[
\int_T (J \cdot \nabla_y \lambda_j) = |T|(J \cdot \nabla_y \lambda_j).
\]

We note that since \( D^{-1}J \) is a constant on \( T \), we can write it as a gradient of a linear function, namely
\[
J = D(D^{-1}J) = D \nabla_y (D^{-1}J \cdot y) = \sum_{i=1}^{d+2} (D^{-1}J \cdot y_i) D \nabla_y \lambda_i. \quad (40)
\]

Since \( \sum_{i=1}^{d+2} \nabla_y \lambda_i \equiv 0 \) on \( T \), we have that
\[
0 = (D^{-1}J \cdot \mathbf{y}_j)(D \sum_{i=1}^{d+2} \nabla_y \lambda_i \cdot \nabla_y \lambda_j) = \sum_{i=1}^{d+2} (D^{-1}J \cdot \mathbf{y}_j)(D \nabla_y \lambda_i \cdot \nabla_y \lambda_j)
\]

Hence,
\[
|T|(J \cdot \nabla_y \lambda_j) = \sum_{i=1}^{d+2} (D^{-1}J \cdot \mathbf{y}_i)(D \nabla_y \lambda_i \cdot \nabla_y \lambda_j) = \sum_{i \neq j} (D^{-1}J \cdot (\mathbf{y}_i - \mathbf{y}_j))(D \nabla_y \lambda_i \cdot \nabla_y \lambda_j) = \sum_{i \neq j} |E|(D^{-1}J \cdot \tau_{ij})(D \nabla_y \lambda_i \cdot \nabla_y \lambda_j).
\]

We have computed earlier (see (39)) the quantity \(|E|(D^{-1}J \cdot \tau_{ij})\) for all \( E \subset \partial T \). Therefore,
\[
|T|(J \cdot \nabla_y \lambda_j) = \sum_{i=1}^{d+2} d_{ji}^T \left[ B \left( \frac{t_i - t_j}{\varepsilon} \right) u_i - B \left( \frac{t_j - t_i}{\varepsilon} \right) u_j \right]. \quad (41)
\]

Here \( d_{ji}^T = \int_T D \nabla_y \lambda_i \cdot \nabla_y \lambda_j \) are the entries of the local stiffness matrix corresponding to the discretization of \(( - \text{div} \, D \nabla )\) with linear elements on \( T \). Therefore on \( T \) we get
\[
[A_T]_{jj} = - \sum_{i=1}^{d+2} d_{ji}^T B \left( \frac{t_j - t_i}{\varepsilon} \right), \quad [A_T]_{ji} = d_{ji}^T B \left( \frac{t_i - t_j}{\varepsilon} \right). \quad (42)
\]

The global stiffness matrix is assembled from \( A_T \). It is invertible for sufficiently small mesh size, invertible whenever the assembly of \( d_{ij}^T \) gives an \( M \)-matrix.

For more detailed discussions about sufficient conditions which lead to a stiffness matrix which is an \( M \)-matrix, as well as relations to finite volume
methods we refer to [8]. More importantly, the work [8] provides techniques for consistent modification of the local stiffness matrices, leading to solvable linear systems for wide range of meshes. In 2 dimensions, a sufficient condition for the stiffness matrix to be an $M$-matrix is that the triangulation is a Delaunay triangulation which is easily achieved by any standard mesh generator. For spatial dimensions greater than 2, meshes satisfying the condition given in [7, Lemma 2.1] yields discretization with $M$-stiffness matrix. If this condition is violated by the mesh, then the techniques proposed in [8] can be used to modify the consistently the local stiffness matrices so that the resulting global stiffness matrix is an $M$-matrix.

6. Numerical tests

We consider the 2D heat equation with Dirichlet boundary conditions on the unit square $(0,1) \times (0,1)$. We test both schemes: StreamLineDiffusion (SLDI) and EAFE on a uniform triangulation of the unit square. The exact solution is

$$U(x, t) = e^{-t} \sin \pi x \sin \pi y.$$ 

The domain is the unit square in 2D, and the space-time problem is solved as fully coupled 3D convection diffusion problem.

![Figure 1: (a) Trace of the solution on the plane $x = \frac{1}{2}$. (b) Error reduction in $L^2$-norm. Quadratic convergence is clearly observed.](image)

We have tested the lowest order streamline diffusion scheme which has the same number of degrees of freedom as the EAFE scheme. The convergence behavior of both discretizations is shown in Figure 1.

We have tested the convergence on a family of successively refined triangulations. The coarsest one has a mesh size $h_0 \approx \frac{1}{2}$ and the finest $2^{-8}$ in 3D. The parameter $\varepsilon$ in the diffusion tensor $D_{\varepsilon}$ for the EAFE scheme was $10^{-5}$ on all grids. The parameter $\theta$ in the streamline diffusion method was set to $10^{-2}$ on
all grids. Such pool of tests corresponds to mesh with 27 vertices on the coarsest grid, and, \( \approx 2.1 \times 10^6 \) vertices on the finest grid. In Figure 1a we have plotted the trace of the approximate solution on the plane \( x = 0 \). The approximate solution obtained via the EAFE scheme looks exactly the same, as is also the exact solution.

We next show a plot of a solution to an equation with convection depending on time. The equation is

\[
  u_t - \text{div}(K(x)\nabla u - bu) = 1, \quad x \in \Omega, \quad b = \begin{pmatrix} 100 \sin(6\pi t) \\ 0 \end{pmatrix}
\]

(43)

and the boundary and initial conditions are homogeneous, the domain is the unit square and the time interval is \((0, 1)\). The solution via the Scharfetter-Gummel (EAFE) scheme is shown in Figure 2. Note that with such convection term, the convection is 0 for \( t = k/6 \) and \( k \) integer; it is, however, convection dominated for other values of \( t \).

![Figure 2: Trace of the numerical solution of equation (43) on the plane \( x = \frac{1}{2} \). The effect of the time dependent convection is clearly seen in the plot.

We have mentioned already the software used in performing the tests. In summary, we have used the C++ library and examples from the mfem package \cite{mfem}

(discretization); The solutions of the resulting linear systems are done using the Algebraic Multigraph Multilevel ILU algorithm by Bank and Smith \cite{banksmith1, banksmith2}

found at \url{http://ccom.ucsd.edu/~reb/software.html}. The visualization was done using the glvis tool \cite{glvis}.

7. Concluding remarks

We introduced a class of numerical methods for convection diffusion equations in arbitrary spatial dimensions. In principle, these schemes can be applied
to a wide range of problems, such as linearization of the Nernst-Plank equations for transport of species in a charged media and the space-time discretizations of such equations. We have derived novel exponentially fitted (higher order Scharfetter-Gummel and streamline diffusion) discretizations for convection diffusion equations. Distinctive features of the proposed Scharfetter-Gummel discretization are: (1) its monotonicity in the lowest order case; (2) its applicability in any spatial dimension; and (3) the a priori estimates are in terms of the flux only. For order higher than 1, the derivation and the analysis of this scheme is new, and its implementation for space-time formulation of parabolic problems is a subject of a current research.

Appendix A. Remarks on unisolvence for the quasi-polynomial Nédélec space

To keep the presentation self contained, in this appendix, we give a proof of Lemma 6 using vector proxies. Recall that Lemma 6 states that the Nédélec degrees of freedom are unisolvent set of functionals for the quasi-polynomial space

\((P_{r-1})^d = \{ e^{(-q \cdot x)} D^{-1} p \mid p \in (P_{r-1})^d \} \).

To prove Lemma 6 using vector proxies in \(\mathbb{R}^n\) and with arbitrary polynomial degree \(r\) we recast some of the results from [18, Section 4] for the quasi-polynomial spaces in terms of vector proxies. This is done with the aim to keep the notation in accordance with what we used before.

Quasi-polynomial spaces: A two dimensional example

To illustrate the difficulties that occur when using quasi-polynomial spaces instead of polynomial spaces, let us look closely at the case \(\Omega \subset \mathbb{R}^2\), i.e. \(\mathcal{T}_h\) is a triangulation. The Nédélec degrees of freedom are the 0-th moments on the edges of the triangulation, namely \(\langle \eta_e, v \rangle = \int_e v \cdot \tau_e\). The polynomial space \(P^N\) on a \(T \in \mathcal{T}_h\) is the space of vector valued linear polynomials of the form:

\(\mathbf{p}(\mathbf{x}) = \mathbf{a} + \beta \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \).

Clearly, we have that \((P_1)^d \subset P^N \subset (P_1)^d\).

Next, to show that the set of Nédélec degrees of freedom is a unisolvent set of degrees of freedom for \(P^N\) it is sufficient to show that: If \(p \in P^N\) and \(\langle \eta_e, p \rangle = 0\) for all \(e \in \partial T\), then \(p \equiv 0\). The proof follows by integration by parts:

\[0 = \sum_{e \in \partial T} \int_{e} p \cdot \tau_e = - \int_T \text{rot} \ p = |T| \beta.\]

This implies that \(\beta = 0\), and hence \(p = \mathbf{a}\) is a constant vector. As the degrees of freedom vanish we have that

\[0 = \int_{e} p \cdot \tau_e = |e| (p \cdot \tau_e), \quad \text{for all} \quad e \in \partial T.\]
This implies that $p \equiv 0$, because any two of the edges of $T$ are linearly independent for a non-degenerate simplex $T \subset \mathbb{R}^2$ and $p$ is a constant vector orthogonal to them.

Let us now look at the case of quasi-polynomial space. We set for simplicity $D = I$ and consider the quasi-polynomial space

$$e P^N = \{ e^{-(q \cdot x)} p \mid p \in P^N \}.$$

It is easy to see that first part of the argument given above fails. Indeed, if $p \in e P^N$ we have

$$0 = \sum_{e \in \partial T} \int_e e^{-(q \cdot x)} p \cdot \tau_e = - \int_T \text{rot}(e^{-(q \cdot x)} p). \quad (A.1)$$

In general, the above relation does not imply that $\beta = 0$ and we cannot conclude from (A.1) that $p$ is a constant.

If we reduce the degree of the polynomial space, however, the argument works. Indeed, let us take $s \in (e P_0)^2$ such that $s = e^{-(q \cdot x)} p$ for some $p \in (P_0)^2$ and assume $\langle \eta_e, s \rangle = 0$ for all $e \in \partial T$. We then have

$$0 = \int_e e^{-(q \cdot x)} p \cdot \tau_e = (p \cdot \tau_e) \int_e e^{-(q \cdot x)},$$

and this clearly gives $(p \cdot \tau_e)$ for all $e \in \partial T$, which tells us that $p \equiv 0$ and hence $s \equiv 0$.

As a conclusion we have shown that the Nédélec degrees of freedom are unisolvent set of degrees of freedom (functionals) for $e P_0^N$, but not necessarily for $e P^N$. Generalizing such statement to arbitrary spatial dimension and polynomial degrees requires a bit more work and is detailed in what follows.

**Simplices and barycentric coordinates**

As $D$ is a constant matrix, we may assume $D = I$, because this does not change the degree of the quasi-polynomial space.

The following itemized list contains well known definitions and facts related to the geometry of simplices in $\mathbb{R}^d$.

- The vertices of $T$ are denoted by $[x_0, \ldots, x_d]$. We assume that we have ordered the vertices in such a way that $\{ \tau_m = (x_m - x_0) \}_{m=1}^d$ form a positively oriented basis in $\mathbb{R}^d$.

- For $k$-vectors $\{ v_j \}_{j=1}^k$ in $\mathbb{R}^n$, we introduce the volume of the $k$-simplex with one vertex at the origin and the other vertices at the coordinates of the $k$-points. We set

$$\text{vol}(v_1, \ldots, v_k) = \pm \frac{1}{k!} \sqrt{\det(V^T V)}, \quad V = (v_1, \ldots, v_k) \in \mathbb{R}^{n \times k}.$$

The plus sign is chosen if $\{ v_1, \ldots, v_k \}$ is a positively oriented frame. The volume is obviously zero if the vectors are linearly dependent.
The set of all sub-simplices of $T$ of dimension $j$ can be identified by the monotone set of indices $\sigma = [i_0, \ldots, i_j]$, with $i_0 < \ldots < i_j$. For any such set of indices, we denote by $\sigma^c$ its complement in $\{0, \ldots, d\}$. The elements of $\sigma$ and $\sigma^c$ are always assumed to be in increasing order. In general $s_\sigma$ will denote a sub-simplex of $T$ of dimension $|\sigma|$ (with $s_{\{1:d\}} = T$), and every such simplex is the convex hull of any $(|\sigma| + 1)$ vertices of $T$, i.e. $s_\sigma = \text{chull}(x_\sigma) = \text{chull}(x_{i_0}, \ldots, x_{i_j})$, with $i_0 < \ldots < i_j$.

There is a canonical affine map $\Phi : \hat{T} \mapsto T$ with domain the simplex $\hat{T} \subseteq \mathbb{R}^d$ with vertices $[0, e_1, \ldots, e_d]$. This map is described by a $(d \times d)$ matrix $B$, and

$$\Phi(\hat{x}) = x_0 + B\hat{x}, \quad B = (\tau_1 | \ldots | \tau_d).$$

If $\hat{s}_j$ is the reference simplex in $\mathbb{R}^j$ with vertices $[0, e_1, \ldots, e_j]$, then we also have a map (induced by $\Phi$), $\Phi_\sigma : \hat{s}_j \mapsto s_\sigma$

$$\Phi_\sigma(\hat{x}) = x_0 + B_\sigma\hat{x}, \quad B_\sigma = (\tau_1 | \ldots | \tau_j), \quad \hat{x} \in \hat{s}_j.$$

$$\{\lambda_j\}_{j=0}^d$$ denote the barycentric coordinates of $T$. Note that $\{\nabla \lambda_m\}_{m=1}^d$ form a basis for $\mathbb{R}^d$. In fact, with the definitions above it is easily verified that

$$(\nabla \lambda_m, \tau_j)_{\mathbb{R}^d} = (B^{-T} \hat{\nabla} \hat{\lambda}_m, B e_j)_{\mathbb{R}^d} = (\hat{\nabla} \hat{\lambda}_m, e_j)_{\mathbb{R}^d} = \delta_{m,j}.$$

In addition, for any vector field $u(x) \in (P_{r-1})^n$ and any $n$, we can use the basis $\{\nabla \lambda_j\}_{j=1}^n$ to write

$$u(x) = \sum_{j=1}^n u_j \nabla \lambda_j, \quad u_j(x) \in P_{r-1}. \quad \text{(A.2)}$$

Let $s_\sigma$, $\sigma = (l_1, \ldots, l_k)$ be a subsimplex of $T$. Let $\nabla_\sigma \lambda_j$, $j \in \sigma$ be the surface gradient of $\lambda_j$ when on the hyperplane spanned containing $s_\sigma$. This is another basis for $\text{span}\{\tau_\sigma\}$ as is easily verified. We note that $\nabla_\sigma \lambda_j = \pi_\sigma \nabla \lambda_j$, where $\pi_\sigma$ is an orthogonal projection on the span of $(\tau_\sigma)$. We have

$$\pi_\sigma \nabla \lambda_j = B_\sigma (B_\sigma^T B_\sigma)^{-1} B_\sigma^T \nabla \lambda_j = B_\sigma (B_\sigma^T B_\sigma)^{-1} \nabla \hat{\lambda}_j.$$

Note that $\pi_\sigma \nabla \lambda_k = 0$ if $k \notin \sigma$ (the gradient is normal to the zero level set of $\lambda_k$), and, we also have, $\pi_\sigma \nabla \lambda_j = B_\sigma (B_\sigma^T B_\sigma)^{-1} e_j$ for $j = 1, \ldots, k$, where $e_j \in \mathbb{R}^k$.

By trace of a vector field $v$ on a sub-simplex $\sigma$ we mean

$$\text{Tr}_\sigma(v) = \pi_\sigma v(x), \quad x \in s_\sigma.$$
• For an \( n \)-dimensional simplex \( s \subset \mathbb{R}^n \), \( s = [x_1, \ldots, x_n] \), the zero level sets of \( \lambda_j \) we denote by \( \{ F_j(s) \}_{j=0}^n \) are often also referred to as “faces” of \( s \), and are the \((n-1)\) dimensional simplices opposite to the 0-dimensional simplices (vertices) of \( s \). In short, \( F_j(s) = s_\sigma \) faces the vertex \( x_j \). Here, for a fixed \( j \), \( \sigma = \{ 0, \ldots, j - 1, j + 1, \ldots, n \} \).

Next, using the facts above, we now prove a well known result [18, Lemma 4.5] and use the language of vector proxies. The changes in the proof of [18, Lemma 4.5] needed to account for the fact that we have a quasi-polynomial space, instead of polynomial space, are obvious. Note that the face \( F_0 \) is excluded from the statement of the lemma.

**Lemma 9.** Let \( s \) be a simplex in \( \mathbb{R}^n \) and \( u \in (eP_{r-1})^n \) be such that its traces on the faces of \( s \), \( \{ F_j(s) \}_{j=1}^n \) vanish. Then,

\[
u = \sum_{j=1}^{n} \left( \prod_{k \neq j} \lambda_k \right) \tilde{u}_j \nabla \lambda_j,
\]

where \( \tilde{u}_j \in eP_{r-n} \).

**Proof** We use (A.2) to write

\[
u(x) = \exp(-q \cdot x) \left( \sum_{j=1}^{n} u_j \nabla \lambda_j \right), \tag{A.3}
\]

with \( u_j \in P_{r-1} \). Let us fix a \( k \) and recall that the trace of \( \nu \) on \( F_k(s) = \{ x \mid \lambda_k = 0 \} \) vanishes. As we pointed out, \( \pi_{P_k} \nabla \lambda_j \), \( j \neq k \), form a basis on the hyperplane containing \( F_k \). Further, for \( x \in F_k \), by the definition of the trace of \( \nu \) on \( F_k \), we have

\[
\text{Tr}_{F_k} \nu(x) = \exp(-q \cdot x) \sum_{j=1}^{n} u_j \pi_{P_k} \nabla \lambda_j = \exp(-q \cdot x) \sum_{j \neq k} u_j \pi_{P_k} \pi_{P_k} \nabla \lambda_j,
\]

As \( \{ \pi_{P_k} \nabla \lambda_j \}_{j \neq k} \) form a basis for the hyperplane containing \( F_k \), we have that \( \nu_j = 0 \) for \( x \in F_k \) for all \( j \neq k \) and hence \( u_j = \lambda_k v_j \), where \( v_j \in P_{r-2} \). Applying this argument for \( k = 1, \ldots, n \) concludes the proof. \( \square \)

**Basic facts about differential forms**

When next introduce some basic facts about differential forms. We need these to define the action of the functionals \( \eta_j \), which we introduced in §4.1. In \( \mathbb{R}^3 \) these were introduced by Nédélec [13, 14] (see also [15, 16]). Our goal here is to prove unisolvence for any spatial dimension and any polynomial degree. That brings us into the framework of Finite Element Exterior Calculus, in the spirit of [17, 18, 19]. Of course, we restrict our discussion to vector valued functions (a flux \( J(u) \)), and, hence we can drop some of the generalities since we consider differential 1-forms only.
The space of differential \( k \)-forms on \( T \) we denote by \( F\Lambda^k \)
\[
F\Lambda^k = \{ \omega \mid \omega = \sum_{\sigma} f_{\sigma}(\mathbf{x})dx_{\sigma}, \; f_{\sigma}(\mathbf{x}) \in C^\infty(T) \).
\]

\( \sigma = \{ i_1, \ldots, i_l \} \subset \{ 1, \ldots, d \} \), \( dx_{\sigma} = dx_{i_1} \wedge \ldots \wedge dx_{i_l} \),

The summation is over all subsets \( \sigma \subset \{ 1, \ldots, d \} \) of length \( |\sigma| = k \).

If the coefficients \( f_{\sigma} \) are polynomials of degree at most \( (r-1) \), i.e. \( f_{\sigma} \in P_{r-1} \) (resp. \( f_{\sigma} \in eP_{r-1} \)), we denote the polynomial space of such \( k \)-forms by \( P_{r-1}\Lambda^k \) (resp. \( eP_{r}\Lambda^k \)). By convention, all these spaces are trivial if \( k < 0 \) or \( r < 1 \). The exterior product is denoted by "\( \wedge \)" and the symbol "\( d\omega \in D\Lambda^{k+1} \)" denotes the exterior derivative of a form \( \omega \in F\Lambda^k \). By definition we have

\[
\text{If } \omega = w(\mathbf{x})dx_{\sigma}, \text{ then } d\omega = \sum_{j=1}^d \partial_j w(\mathbf{x})dx_j \wedge dx_{\sigma}.
\]

A basis in the space of \( k \)-forms is given by \( dx_{\sigma} = dx_{i_1} \wedge \ldots \wedge dx_{i_k} \). On a simplex \( T \) such basis is also provided by \( d\lambda_j \), where \( \lambda_j, j = 1, \ldots, d \) are the barycentric coordinates in \( T \) as described earlier.

Let us note, though, that we do not need the exterior derivative to prove the unisolvence result. We introduced it to clarify some differences between using quasi-polynomial spaces, instead of regular polynomials. The functionals \( \eta_j \) are dual to the Nédélec basis, which spans the classical Nédélec space of degree \( r \): \( P_r^N = P_{r-1}\Lambda^1 \oplus \tilde{P}_r\Lambda^1 \), where \( \tilde{P}_r \) denotes the space of all homogenous polynomials of degree \( r \). Clearly, from this we see that \( \dim P_r^N > \dim P_{r-1}\Lambda^1 \).

Next, when proving unisolvence for regular polynomial spaces, we have that the exterior differentiation decreases the degree of the polynomials by 1, namely, \( dP_{r-1}\Lambda^k \subset P_{r-2}\Lambda^{k+1} \). For the quasi-polynomial space this is not the case, and we only have \( dP_{r-1}\Lambda^k \subset P_{r-1}\Lambda^{k+1} \). As a consequence, the proofs of unisolvence of \( \eta_j \) on the space of polynomials do not extend directly to the case of quasi-polynomials generated by \( P_r^N \). This explains our choice in \( \text{4.1} \) of \( eP_{r-1}\Lambda^1 \) instead of \( eP_r^N \).

**Integration of forms**

A \( k \)-form can be integrated on a subsimplex of dimension \( k \) (only), and, for any such subsimplex \( s_\sigma \subset T \), and any \( \omega = w(\mathbf{x})dx_L \), where \( L = \{ l_1, \ldots, l_k \} \subset \{ 1, 2, \ldots, d \} \), \( dx_L = dx_{l_1} \wedge \ldots \wedge dx_{l_k} \) we define (see, for example, Rudin [25]):

\[
\int_{s_\sigma} \omega := \int_{s_k} w(\Phi_\sigma(\mathbf{x})) \frac{\partial L \Phi_\sigma}{\partial (x_1, \ldots, x_1)} dx_1 \ldots dx_k.
\]

This is just a definition of an integral of a differential form, and, the integral on the right side is to be understood as ordinary integral over the reference \( k \)-simplex \( \hat{s}_k \subset \mathbb{R}^k \). The map \( \Phi_\sigma : \hat{s}_\sigma \rightarrow \mathbb{R}^d \) was defined earlier and \( \frac{\partial L \Phi_\sigma}{\partial (x_1, \ldots, x_1)} \) denotes the determinant of a \( k \times l \) submatrix of its Jacobi matrix corresponding
to the indices in $L$:
\[
\frac{\partial L \Phi_\sigma}{\partial (\tilde{x}_1, \ldots, \tilde{x}_l)} = \det\left( \begin{array}{ccc}
\partial L \Phi_{\sigma, i_1} & \cdots & \partial L \Phi_{\sigma, i_1} \\
\vdots & \ddots & \vdots \\
\partial L \Phi_{\sigma, i_l} & \cdots & \partial L \Phi_{\sigma, i_l}
\end{array} \right), \quad \partial L \Phi_{\sigma, m} = \frac{\partial \Phi_{\sigma, m}}{\partial \tilde{x}_j}.
\]

In our case $\Phi_\sigma$ is an affine map, so the Jacobi matrix and its determinant are independent of $\tilde{x}_j$. Let $B$ be the matrix that maps $e_j \mapsto \tau_j$, $j = 1 : n$. For a monotone $L \subset \{1, \ldots, d\}$, $\sigma = \{i_1, \ldots, i_k\}$ we define $I_L \in \mathbb{R}^{d \times k}$ to be the matrix whose range is $\text{span}\{e_{i_1}, \ldots, e_{i_k}\}$. The definition is

\[
I_L := \sum_{j=1}^k e_{i_j} [e_j(\mathbb{R}^k)]^T.
\]

Note that $e_j(\mathbb{R}^k)$ are the canonical basis vectors in $\mathbb{R}^k$. Next, if $\omega = \sum_L w_L dx_L$, is a differential $k$-form (the summation is over all monotone subsets $L$ of $\{1, \ldots, n\}$, and $L = \{l_1, \ldots, l_k\}$) we can write the definiton of the integral above as

\[
\int_{s_\sigma} \omega := \int_{s_\sigma} \sum_L w_L(\Phi_\sigma(\tilde{x})) \det(B_{\sigma, L}) d\tilde{x}_1 \ldots d\tilde{x}_k,
\]

where $B_{\sigma, L} = I_L^T B_{\sigma, L}$. As we pointed out earlier, we have $e_j = B^{-T} \nabla \lambda_j$ and $e_j = B \tau_j$, and hence, the vectors tangent to $T$ and $\nabla \lambda_i$ form a bi-orthogonal basis. It is often convenient to represent differential form via the basis $d\lambda_1, \ldots, d\lambda_d$. Thus for a differential form we will have two representations

\[
\omega = \sum_L w(\mathbf{x}) dx_L, \quad \omega = \sum_L v_L(\mathbf{x})(d\lambda)_L, \quad v_L = \sum \det(B_{L, \sigma}) w_\sigma.
\]

The identity between the coefficients is easy to verify using that

\[
d\lambda_j = \sum_{m=1}^d (\nabla \lambda_j)_m dx_m = \sum_{m=1}^d (B^{-1})_{m,j} dx_m.
\]

Differential $k$-forms are elusive objects, they only manifest themselves when integrated over $k$-simplices. Otherwise they mean nothing. The fact that $\lambda_j$ and the trace of $\nabla \lambda_j$ vanishes on any sub-simplex where $\lambda_j = 0$ shows that $\int_{s_\sigma} v_L(\mathbf{x})(d\lambda)_L = 0$ if $L \neq \sigma$. This is a nice property, because the change of variable formula gives us that

\[
\int_{s_\sigma} \omega = \int_{s_\sigma} v_\sigma(\mathbf{x})d\text{vol}(s),
\]

where the right side is just the ordinary integral over the simplex $s_\sigma$,

\[
d\text{vol}(s) = \sqrt{\det(\Lambda^T \Lambda)} dx, \quad \Lambda = (\pi_i \nabla \lambda_{i_1}, \ldots, \nabla \lambda_{i_k}) \in \mathbb{R}^{d \times k}.
\]
Exterior product of two differential forms is also used below for defining degrees of freedom. If \( \omega, \eta \) are \( p \) and \( q \) form respectively, then \( (\omega \wedge \eta) \) is a \((p + q)\) form. The definition and properties of the exterior product “\( \wedge \)” are found in standard texts.

**Degrees of freedom**

Since we only consider 1-forms, let us focus on this case here and the degrees of freedom of the classical Nédélec space in terms of differential 1-forms, and later we convert to definitions involving vector proxies. Following [18, Section 4] and [17, Definition 9], the degrees of freedom for polynomial differential 1-form \( \omega \in P_r^N \) are:

\[
\langle \eta_\sigma, \omega \rangle = \int_{s_\sigma} \omega \wedge \tilde{\eta}_\sigma, \quad \tilde{\eta}_\sigma \in P_{r-k}^{k-1}(s_\sigma),
\]

(A.4)

Here, the notation is as follows: \( \sigma \) ranges over all monotone subsets of \( \{1, \ldots, d\} \) of cardinality \( k \); \( s_\sigma \) are the corresponding sub-simplices of \( T \) of dimension \( k \); \( \tilde{\eta}_\sigma \) forms a basis in \( P_{r-k}^{k-1} \) and \( \eta_\sigma \) is the corresponding functional it generates, as defined by the integral on the right side. As we pointed out, we consider here only 1-forms, which makes things easier. In such case, any \( \eta \in P_{r-n}^{n-1} \) can be identified with a proxy-vector, \( p \in (P_{r-n})^n \) by identifying every \((n - 1)\) dimensional \( F_i \) simplex with the vertex \( i \) it opposes.

\[
\tilde{\eta} = \sum_{i=1}^{n} p_i(x)(d\lambda_i)_{F_i} \rightarrow (p_i)_{i=1}^{n}.
\]

Integration then of \( u \wedge \tilde{\eta} \) amounts to:

\[
\sum_j \int_s u_j p_j \text{vol}(\nabla \lambda_j, \nabla \lambda_1, \ldots, \nabla \lambda_j, \ldots, \nabla \lambda_n)dx.
\]

Here the coefficients \( u_j \) are the coefficients in the expansion \( \text{(A.2)} \).

Next we prove a result analogous to the [18 Lemma 4.6]

**Lemma 10.** Let \( s \) be an \( n \)-dimensional simplex and let \( u \in eP_{r-1} \Lambda^1 \) be a given vector field with vanishing traces on the faces \( F_i \subset \partial s \), \( F_i = \{x \mid \lambda_i(x) = 0\} \), \( i = 1, \ldots, n \). Assume also that

\[
\sum_j \int_s \exp(-q \cdot x)u_j p_j \text{vol}(\nabla \lambda_j, (\pi_{F_j} \nabla \lambda_m)_{m \neq j}) = 0, \quad \text{for all } p \in (P_{r-n}(s))^n.
\]

Then, \( u = 0 \).

**Proof** Since the traces of \( u \) vanish on the faces of \( s \), from Lemma [9] we know that

\[
u = \sum_{j=1}^{n} \left( \prod_{m \neq j} \lambda_m \right) \tilde{u}_j \nabla \lambda_j, \]

where \( \tilde{u}_j \in eP_{r-n} \). Choosing \( p_j = \tilde{u}_j \) then concludes the proof of the lemma because \( \prod_{m \neq j} \lambda_m \) is a strictly positive function in the interior of \( s \). \( \square \)
Unisolvence result

After all these long introductory remarks, let us recall that our goal is to show that $ZP$ defined in §4.2 is injective. This is equivalent to showing that if the functionals defined in (A.4) vanish on a differential 1-form $\omega$ with coefficients in $(eP_{k-1})^d$, then $\omega = 0$. Equivalently we can state this in terms of vector fields as we did in Lemma 10. We have the following unisolvence result.

**Lemma 11.** If $u = \sum_{j=1}^d \exp(-q \cdot x)u_j \nabla \lambda_j$ is a vector field for which

$$\sum_{j=1}^n \int_{s_\sigma} \exp(-q \cdot x)u_j p_j \text{vol}(\pi_\sigma \nabla \lambda_j, (\pi_\sigma \nabla \lambda_m)_{m \neq j}) = 0, \quad \text{for all } p_j \in (P_{r-n})^n,$$

where $n = 1, \ldots, d$, $|\sigma| = n$, then $u = 0$.

**Proof** The proof is just an iteration using the result from Lemma 10. Indeed taking $n = 0$ and we have that

$$0 = \frac{1}{|s|} \int_s \exp(-q \cdot x)u \cdot \tau_s p, \quad \text{for all } p \in P_{r-1},$$

where $s$ is any of the 1-dimensional sub-simplices (edges) of $T$. This clearly tells us that $\pi_s u = 0$ on every edge. Since two dimensional simplices have as boundary edges 1-dimensional simplices, an application of Lemma 10 shows that $\pi_s u = 0$ also on all two-dimensional sub-simplices $s$. Formalizing this as an induction argument is trivial and left to the reader. The proof is complete.

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