BIRATIONAL PROPERTIES OF SOME MODULI SPACES RELATED TO TETRAGONAL CURVES OF GENUS 7

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ABSTRACT. Let \( M_{7,n} \) be the (coarse) moduli space of smooth curves of genus 7 with \( n \geq 0 \) marked points defined over the complex field \( \mathbb{C} \). We denote by \( M_{17,n;4} \) the locus of points inside \( M_{7,n} \) representing curves carrying a \( g_1^4 \). It is classically known that \( M_{17,n;4} \) is irreducible of dimension \( 17 + n \). We prove in this paper that \( M_{17,n;4} \) is rational for \( 0 \leq n \leq 11 \).

1. INTRODUCTION AND NOTATION

Throughout this paper we will work over the field \( \mathbb{C} \) of complex numbers and we will denote by \( M_{g,n} \) the (coarse) moduli space of smooth curves of genus \( g \geq 2 \) with \( n \geq 0 \) marked points defined over \( \mathbb{C} \).

It is well known that \( M_{g,n} \) is irreducible of dimension \( 3g - 3 + n \). It thus makes sense to deal with its birational properties, such as its rationality, unirationality, Kodaira dimension \( \kappa(M_{g,n}) \) and so on.

For instance, the problem of the rationality of \( M_{g,n} \) has been object of intensive study in the last 30 years (and not only), giving rise to a very long series of papers both in the unpointed and in the pointed case. More precisely, it is known that \( M_{g,n} \) is rational if either \( 2 \leq g \leq 6 \) and \( n = 0 \) (see respectively [23], [28], [37] and [26], [27], [38]), or \( 1 \leq g \leq 6 \) and \( 1 \leq n \leq \sigma(g) \) (see [1], [10], [5]: in this last paper the numerical function \( \sigma(g) \) is defined), while \( \kappa(M_{g,n}) \geq 0 \) when \( g = 1, 4, 5, 6 \) and \( n \geq \tau(g) \) (see again [1] and [29]: for the definition of \( \tau(g) \) see [5]).

When \( g \geq 7 \) the picture is much more complicated. It is known that \( M_{g,n} \) is unirational for \( 7 \leq g \leq 14 \) and \( 0 \leq n \leq \sigma(g) \) (see [1], [3], [35], [11], [39], [29] and again [5], only to quote the most recent references), while \( \kappa(M_{g,n}) \geq 0 \) when either \( 7 \leq g \leq 14 \) and \( n \geq \tau(g) \) (see [29], [17]) or \( g \geq 22 \) without restrictions on \( n \) (see [21], [20], [14], [15], [16], [18]).

Nevertheless \( M_{g,n} \) turns out to contain many interesting (uni)rational loci defined in terms of existence of particular linear series. E.g., one can consider the natural
gonality stratification

\[ \mathcal{M}_{g,n;2}^1 \subseteq \mathcal{M}_{g,n;3}^1 \subseteq \mathcal{M}_{g,n;4}^1 \subseteq \cdots \subseteq \mathcal{M}_{g,n;\left\lceil \frac{g+2}{2} \right\rceil}^1 = \mathcal{M}_{g,n} \]

where \( \mathcal{M}_{g,n;d}^1 \) is the locus of \( d \)-gonal pointed curves, i.e.

\[ \mathcal{M}_{g,n;d}^1 := \{ (C, p_1, \ldots, p_n) \in \mathcal{M}_{g,n} \mid C \text{ is endowed with a } g^1_d \} \]

Such loci are irreducible of dimension \( 2g + 2d - 5 + n \) when \( d < \left\lceil \frac{(g+2)}{2} \right\rceil \) (see [1] and the references therein for \( n = 0 \). When \( n \geq 1 \) the irreducibility of \( \mathcal{M}_{g,n;d}^1 \) is part of folklore: for an easy proof of this fact see [9]). In particular, the general point of \( \mathcal{M}_{g,n;d}^1 \) does not lie in \( \mathcal{M}_{g,n;d-1}^1 \). It follows that the general point of \( \mathcal{M}_{g,n;d}^1 \) carries a base–point–free \( g^1_d \) and no \( g^1_{d-1} \).

The unirationality of these loci is classically known for \( d \leq 5 \) (see e.g. [1] and the references therein: see also [34]). Thus it is quite natural to ask whether the strata of such a stratification are rational.

The rationality of hyperelliptic stratum \( \mathcal{M}_{g,n;2}^1 \) was proved in [24], [25] and [6] when \( n = 0 \) and in [8] when \( 1 \leq n \leq 2g + 8 \). In the case \( n = 0 \), the proof essentially rests on the well–known equivalence between hyperelliptic curves of genus \( g \) and \((2g + 2)\)-tuples of unordered points on in \( \mathbb{P}_C^1 \) up to projective equivalence.

Some other general rationality results are known for the trigonal stratum \( \mathcal{M}_{g,n;3}^1 \). In this case the rationality is known when either \( n = 0 \) and \( g \equiv 2 \) (mod 4) (see [37]) or \( n = 0 \) and \( g \) is odd (see the recent paper [30]) or \( 1 \leq n \leq 2g + 7 \) and every \( g \) (see [9]): in all the cases the key point is the classical representation of a trigonal curve as a trisecant divisor on an embedded ruled surface.

Thus the birational description of \( \mathcal{M}_{7,n;4}^1 \) acquires a particular interest for two different reasons. On the one hand, \( \mathcal{M}_{7,n;4}^1 \) has codimension 1 inside \( \mathcal{M}_{7,n} \), thus its rationality can be viewed as a sort of “suggestion” of the possible rationality of \( \mathcal{M}_{7,n} \). On the other hand, it is the first necessary step for the possible proof of the rationality of the tetragonal stratum \( \mathcal{M}_{g,n;4}^1 \) of the aforementioned gonality stratification.

The main result of the present paper is the following theorem.

**Theorem 1.1.** The tetragonal locus \( \mathcal{M}_{7,n;4}^1 \) is rational for \( 0 \leq n \leq 11 \).

We now quickly describe the content of the paper. In the first section we prove Theorem 1.1 in the pointed case \( n \geq 1 \). The proof is based on the classical representation of a general curve \( C \) of genus 7 as a plane septic \( \overline{C} \) with 8 nodes in general position. When \( C \) is tetragonal, then the eight nodes of \( \overline{C} \) are the base locus of a pencil of cubic curves whose residual ninth intersection again lies on \( \overline{C} \). This remark is at the base of the construction of an easy birational model of \( \mathcal{M}_{7,n;4}^1 \) as quotient of a suitable projective bundle modulo the action of an algebraic group acting on it. The rationality of such a quotient then follows from more or less standard classical results in invariant theory.

In Section 2 we recall some introductory facts about \( g_4^1 \) on a curve of genus 7. In particular, we show that a curve \( C \) of genus 7 represents a general point of the tetragonal stratum if and only if it carries a unique \( g_5^3 \) which is also very ample.
In Section 3, using such a very ample \( g_3^3 \), we are able to describe the general tetragonal curve \( C \) of genus 7 as the residual intersection of a pencil of cubics having a common line. With the help of such an embedded geometric model and a lot of non-trivial representation theory, we are thus able to prove Theorem 1.1 in the case \( n = 0 \) in Section 4.

1.1. Notation. We work over the complex field \( \mathbb{C} \). In particular, all the algebraic groups (\( \text{GL}_k, \text{SL}_k, \text{PGL}_k \) and so on) are always assumed to have coefficients in \( \mathbb{C} \).

If \( g_1, \ldots, g_h \) are elements of a certain group \( G \) then \( \langle g_1, \ldots, g_h \rangle \) denotes the subgroup of \( G \) generated by \( g_1, \ldots, g_h \).

If \( V \) is a vector space, then we denote by \( \mathbb{P}(V) \) the corresponding projective space.

In particular we set \( \mathbb{P}_n^8 := \mathbb{P}(\mathbb{C}^{\oplus n+1}) \).

We denote isomorphisms by \( \cong \) and birational equivalences by \( \approx \).

For other definitions, results and notation we always refer to [22].

2. The rationality of \( \mathcal{M}_{n,4}^1 \), \( 1 \leq n \leq 11 \)

Let us consider \( X_n := (\mathbb{P}^2_{\mathbb{C}})^{n-1} \times X \), where \( X \subseteq S^8 \mathbb{P}^2_{\mathbb{C}} \) is the set of unordered 8-tuple of points \( N := N_1 + \cdots + N_8 \) in \( \mathbb{P}^2_{\mathbb{C}} \) which are base loci of pencils of cubics in the plane. \( X_n \) is open and dense in \( (\mathbb{P}^2_{\mathbb{C}})^{n-1} \times S^8 \mathbb{P}^2_{\mathbb{C}} \). The incidence variety

\[
\mathbb{P}_n := \{ (\overline{C}, A_1, \ldots, A_{n-1}, N_1 + \cdots + N_8) \in |O_{\mathbb{P}^2_{\mathbb{C}}}(7)| \times X_n \ |
\]

\( N_i \) is double on \( \overline{C} \), \( A_j \in \overline{C} \), the pencil of cubics through \( N_1, \ldots, N_8 \)

has its residual base point \( A_{N_1 + \cdots + N_8} \in \overline{C} \}

is naturally endowed with a structure of projective bundle \( p_n : \mathbb{P}_n \rightarrow X_n \) with typical fiber \( \mathbb{P}^{11-n} \). Moreover, there is a natural action of \( \text{PGL}_3 \) on \( \mathbb{P}_n \).

Let us consider the general point \( (\overline{C}, A_1, \ldots, A_{n-1}, N_1 + \cdots + N_8) \in \mathbb{P}_n \). Let \( \nu : \overline{C} \rightarrow \overline{C} \) be the normalization map. \( \overline{C} \) is naturally endowed with an ordered set of \( n \) points \( p_j := \nu^{-1}(A_j), j \leq n - 1, p_n := \nu^{-1}(A_{N_1 + \cdots + N_8}) \). The curve \( \overline{C} \) cannot be hyperelliptic, since it is endowed with a \( g_3^2 \) (see [13], Proposition 2.2 (ii)). Thus, the pencil of cubics through \( N_1, \ldots, N_8 \) cut out the fixed point \( A_{N_1 + \cdots + N_8} \in \overline{C} \) plus a complete \( g_1^1 \) on \( \overline{C} \), say \( [D] \). On the one hand, thanks to [12], Theorem 3.2, the curve \( \overline{C} \) does not carry any other \( g_1^1 \). On the other hand \( \overline{C} \) is the image of \( C \) via a morphism associated to

\[
|\overline{C} \cdot \ell| = |4\overline{C} \cdot \ell - N_1 - \cdots - N_8 - (3\overline{C} \cdot \ell - N_1 - \cdots - N_8 - A_{N_1 + \cdots + N_8}) - A_{N_1 + \cdots + N_8}| = \\
|K_{\overline{C}} - D - p_n|,
\]

i.e. \( \overline{C} \subseteq |K_{\overline{C}} - D - p_n| \cong \mathbb{P}^2_{\mathbb{C}} \).

In particular, we have a natural rational map

\[
m_n : \mathbb{P}_n \dashrightarrow \mathcal{M}_{n,4}.
\]

Lemma 2.1. The map \( m_n \) is dominant and it induces a birational isomorphism

\[
\mathcal{M}_{n,4} \cong \mathbb{P}_n / \text{PGL}_3.
\]
Proof. We first describe the fibers of $m_n$. If

$$(C', A'_1, \ldots, A'_{n-1}, N'_1 + \cdots + N'_8), \quad (C'', A''_1, \ldots, A''_{n-1}, N''_1 + \cdots + N''_8)$$

have the same image $(C, p_1, \ldots, p_n)$, then there is a birational map $\varphi: C' \dashrightarrow C''$.

The corresponding automorphism on $C$ must fix $K_C$, the unique $g^1_3$ and the points $p_1, \ldots, p_n$. Thus $\varphi$ is induced by a projectivity of $|K_C - D - p_n| \cong \mathbb{P}_C^2$. It follows that the fibers of $m_n$ are exactly the orbits of the action of $PGL_3$ on $\mathbb{P}_n$.

In particular $\mathbb{P}_n / PGL_3 \cong \text{im}(m_n) \subseteq M^1_{7,n;4}$. In order to complete the proof it suffices to check that $\dim(\text{im}(m_n)) = \dim(\mathbb{P}_n / PGL_3) = \dim(M^1_{7,n;4})$. We have $17 + n = \dim(M^1_{7,n;4}) \geq \dim(\text{im}(m_n)) = \dim(\mathbb{P}_n) - \dim(PGL_3) = 25 + n - 8 = 17 + n$, thus $\dim(\text{im}(m_n)) = 17 + n = \dim(M^1_{7,n;4})$. It follows that $\text{im}(m_n)$ is dense inside $M^1_{7,n;4}$.

We now go to prove the Theorem 1.1 for $1 \leq n \leq 11$ making use of the aforementioned representation. We first examine the case $5 \leq n \leq 11$ which is the easiest one. To this purpose let $E_1 := [1, 0, 0], E_2 := [0, 1, 0], E_3 := [0, 0, 1], E_4 := [1, 1, 1]$ and consider the subset

$$Y_n := \{ (E_1, E_2, E_3, E_4, A_5, \ldots, A_{n-1}, N_1 + \cdots + N_8) \in X_n \}. $$

It is trivial to check that $Y_n$ is a $(PGL_3, \text{id})$-section of $X_n$ in the sense of [24]. The scheme $p_n^{-1}(Y_n)$ is a projective bundle on $Y_n$, thus it is trivially irreducible and rational. It follows from Proposition 1.2 of [24] that

$$M^1_{7,n;4} \cong \mathbb{P}_n / PGL_3 \cong p_n^{-1}(Y_n),$$

thus $M^1_{7,n;4}$ is also rational for $5 \leq n \leq 11$.

Now we turn our attention to the slightly more difficult case $1 \leq n \leq 4$. Let

$$E := \{ (E, A) \in |O_{\mathbb{P}^2}(3)| \times \mathbb{P}_C^2 \mid A \in E \}. $$

The projection $E \to \mathbb{P}_C^2$ endows $E$ with a natural structure of Zariski locally trivial projective bundle over $\mathbb{P}_C^2$ with fiber isomorphic to $\mathbb{P}_C^8$. Thus $E$ is rational and $\dim(E) = 10$. There exists a natural dominant rational map $q_n: \mathbb{P}_n \to E$ defined by

$$(C, A_1, \ldots, A_{n-1}, N_1 + \cdots + N_8) \mapsto (E_{N_1+\cdots+N_8}, A_{N_1+\cdots+N_8}),$$

where $E_{N_1+\cdots+N_8}$ is the unique cubic tangent to $C$ at $A_{N_1+\cdots+N_8}$ through the points $N_1, \ldots, N_8$. The fiber of $q_n$ over $(E, A)$ is birationally isomorphic to $\mathbb{P}_C^{10-n} \times (\mathbb{P}_C^2)^{n-1} \times |O_E(3E \cdot L - A)|$ (here $L$ is a general line in $\mathbb{P}_C^2$). In particular, the map $q_n$ can be factorized into a sequence of Zariski locally trivial projective bundles.

Trivially $q_n$ is $PGL_3$ equivariant and we have the following more or less classical result.

**Lemma 2.2.** The action of $PGL_3$ on $E$ is almost free.

**Proof.** It is a well–known classical result (see e.g. [7], Theorem 4 of Section II.7.3) that, up to projectivities, the equation of each general cubic curve can be put in its Hesse form, i.e.

$$t_1^3 + t_2^3 + t_3^3 - 3\lambda t_1 t_2 t_3 = 0,$$
where $\lambda \in \mathbb{C}$ satisfies $\lambda^3 \neq 1$. The subgroup of projectivities fixing such a polynomial is the extended Heisenberg group

$$H(3)^e := \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$$

(see Section II.7.3 of [7]). Hence the set $A := \{ A \in \mathbb{P}_C^2 \mid \exists \varphi \in H(3)^e, \varphi \neq \text{id} : \varphi(A) = A \}$ is the union of a finite number of points and lines in $\mathbb{P}_C^2$. Thus $\mathcal{P} := \mathbb{P}_C^2 \setminus A$ is open and dense. Moreover, $\mathcal{P}$ has non-empty intersection with each irreducible curve $E$ in the Hesse pencil.

By construction, for each point $A \in \mathcal{P} \cap E$, the pair $(E,A)$ has trivial stabilizer inside $\text{PGL}_3$. □

We can draw the commutative diagram

$$\begin{array}{ccc}
\mathbb{P}_n & \longrightarrow & \mathbb{P}_n/\text{PGL}_3 \approx \mathcal{M}_{1,n;4} \\
\downarrow q_n & & \downarrow \tilde{q}_n \\
\mathcal{E} & \longrightarrow & \mathcal{E}/\text{PGL}_3 \\
\end{array}$$

Let

$$\tilde{\mathcal{E}} := \{ (f, x) \in \text{Sym}^3(\mathbb{C}^3)^\vee \times \mathbb{C}^3 \mid f(x) = 0 \}.$$ 

Thus $\mathcal{E} \cong \tilde{\mathcal{E}}/T$ where $T := (\mathbb{C}^*)^2$ acts almost freely on $\tilde{\mathcal{E}}$ via homotheties. The group $\text{SL}_3$ acts naturally on $\tilde{\mathcal{E}}$. Such an action induces on $\mathcal{E}$ the natural action of $\text{PGL}_3$. It follows that the map $\tilde{\mathcal{E}}/\text{SL}_3 \longrightarrow \mathcal{E}/\text{PGL}_3 \cong \mathcal{E}/\text{PGL}_3$ has a section. Moreover $\text{SL}_3$ is special and acts almost freely on $\tilde{\mathcal{E}}$, thus also the natural projection $\tilde{\mathcal{E}} \longrightarrow \mathcal{E}/\text{SL}_3$ has a section. Composing these two sections with the natural projection $\tilde{\mathcal{E}} \rightarrow \mathcal{E}$ we finally obtain a section $\sigma : \mathcal{E}/\text{PGL}_3 \longrightarrow \mathcal{E}$ of the bottom map of the commutative square above.

Recall that the map $q_n$ is a sequence of Zariski locally trivial projective bundles, so from the existence of $\sigma$ it follows that $\mathbb{P}_n/\text{PGL}_3$ is a tower of Zariski locally trivial projective bundles over $\mathcal{E}/\text{PGL}_3$ too. Since $\mathcal{E}/\text{PGL}_3$ is rational by Castelnuovo’s theorem, we conclude that the same is true also for $\mathbb{P}_n/\text{PGL}_3 \approx \mathcal{M}_{1,1;n;4}$.

Thus we have completed the proof of Theorem 1.1 for $1 \leq n \leq 11$. In order to complete its proof, it remains to analyze the difficult case $n = 0$. We devote the remaining part of this paper to the description of this case.

3. Projective models of tetragonal curves of genus 7

Let $C$ be a curve of genus $g$. Assume the existence of two curves $C_i$ of genera $g_i$ and morphisms $\varphi_i : C \rightarrow C_i$ of respective degrees $d_i$, $i = 1, 2$. If $\varphi_1$ and $\varphi_2$ are not composed with the same pencil, then the following Castelnuovo–Severi formula holds true (see [A–C–G–H], Exercise VIII C 1):

$$g \leq (d_1 - 1)(d_2 - 1) + d_1 g_1 + d_2 g_2.$$
Assume that $C$ is a curve of genus 7 carrying a base–point–free $g^1_3$, say $|D|$. Then $h^0(C, \mathcal{O}_C(2D)) \geq 3$. We say that $|D|$ is of type I if $h^0(C, \mathcal{O}_C(2D)) = 3$, of type II otherwise. Notice that $C$ cannot carry any $g^1_3$. Otherwise, Formula [1] would yield $7 = g \leq 6$, since $|D|$ and the $g^1_3$ cannot be composed with the same pencil. We have the following possible cases.

1. $C$ is hyperelliptic. Thanks to Formula [1] the linear system $|D|$ is composed with the involution: in particular, $C$ is endowed with infinitely many $g^1_3$’s.

2. $C$ is bielliptic. Again Formula [1] yields that the linear system $|D|$ is composed with the involution: in particular, $C$ is endowed with infinitely many $g^1_3$’s.

3. $C$ is neither hyperelliptic nor bielliptic, but it carries more than one $g^1_3$.

4. $C$ carries exactly one $g^1_3$ of type II.

5. $C$ carries exactly one $g^1_3$ of type I.

We now go to characterize the different types of curves in terms of existence of particular $g^3_d$ on them.

**Proposition 3.1.** The curve $C$ carries either a finite number $t \geq 2$ of $g^1_3$’s or exactly one $g^1_3$ of type II if and only if $C$ is birationally isomorphic to a plane sextic with three, possibly infinitely near, double points.

**Proof.** If $C$ is birationally isomorphic to a plane sextic $\overline{C}$ with at most double points as singularities, then $C$ is neither hyperelliptic nor bielliptic due to [13], Proposition 2.2 (ii) and (v). If $\overline{C}$ has at least two double points, then it is clear that $C$ is endowed with at least two $g^1_3$. Assume that $\overline{C}$ has a unique double point, say $N_1$. The arithmetic genus of a plane sextic is 10, hence Clebsch’s formula for the genus of a plane curve yields

$$\sum_{p \in \overline{C}} \mu_p(\overline{C})(\mu_p(\overline{C}) - 1) = 6,$$

$\mu_p(\overline{C})$ being the multiplicity of $p \in \overline{C}$. I follows the existence of a second double point $N_2$ infinitely near to $N_1$. The linear system of conics through $N_1$ and $N_2$ has dimension 3 cut out $|\mathcal{O}_C(2D)|$ on $\overline{C}$ outside $N_1$ and $N_2$. Thus $h^0(C, \mathcal{O}_C(2D)) \geq 4$, i.e. $C$ carries a $g^1_3$ of type II in this case.

Conversely, each pair of distinct $g^1_3$ on $C$ induces a morphism $\varphi: C \rightarrow \mathbb{P}^1_C \times \mathbb{P}^1_C \subseteq \mathbb{P}^3_C$. Its image $\overline{C} \subseteq \mathbb{P}^1_C \times \mathbb{P}^1_C$ is a curve whose bidegree is necessarily either $(2, 2)$ or $(4, 4)$. In the first case $\varphi$ has degree 2, whence $C$ would be hyperelliptic, a contradiction, since $C$ carries a finite number of $g^1_3$. In the second case $\varphi$ would be birational, whence $\overline{C}$ would carry at least a double point. Projecting on a plane from such a double point we obtain a plane sextic birationally isomorphic to $C$. Clebsch’s formula for the genus of a plane curve of degree 6 implies that $\overline{C}$ has either a triple point or it carries three, possibly infinitely near, double points. If $\overline{C}$ carries a triple point, then it would be endowed with a $g^3_3$, a contradiction. We conclude that the singularities of $\overline{C}$ are at most double points.

Finally assume that $C$ is endowed with a unique $g^1_3$ of type II, say $|D|$. In this case, $|2D|$ is a $g^r_5$ with $r \geq 3$. Each such $g^r_5$ is necessarily special, thus Clifford’s Theorem yields $r \leq 4$. If equality holds then $C$ would be hyperelliptic, thus it would carry infinitely many $g^1_3$, a contradiction. It follows that $r = 3$. In particular, if $f_1, f_2 \in \mathbb{P}^1_C$.
Thus we can find two cubic surfaces \( F_1, F_2 \) and a quartic surface \( G \) through \( C \), such that \( C = F_1 \cap F_2 \cap G \). Moreover the general cubic surface through \( C \) is smooth.

**Lemma 3.2.** If \( C \subseteq \mathbb{P}^3_C \) is a curve of degree 8 and genus 7, then there exist two cubic surfaces \( F_1, F_2 \) and a quartic surface \( G \) through \( C \), such that \( C = F_1 \cap F_2 \cap G \). Moreover the general cubic surface through \( C \) is smooth.

**Proof.** If \( C \) lies on a smooth quadric, then its bidegree \( (a, b) \) would satisfy the conditions \( a + b = 8 \) and \( (a - 1)(b - 1) = 7 \). If \( C \) lies on a quadric cone, then we should have \( 8 = 2a \) and \( 7 = (a - 1)^2 \) (see [22], Exercise V.2.9). In both the cases the two equations have no common integral solutions. Thus the minimal degree of a surface through \( C \) is at least 3.

The cohomology of the standard exact sequence \( 0 \to \mathcal{I}_C \to \mathcal{O}_{\mathbb{P}^3_C} \to \mathcal{O}_C \to 0 \) twisted by 3 shows that

\[
h^0(\mathbb{P}^3_C, \mathcal{I}_C(3)) \geq 2.
\]

Thus we can find two cubic surfaces \( F_1 \) and \( F_2 \) such that \( F_1 \cap F_2 = C \cup L \). Due to degree reasons \( L \) is a line, thus \( C \) is aCM (see [31], Theorem 5.3.1). It follows that \( h^0(\mathbb{P}^3_C, \mathcal{I}_C(3)) = 2 \), \( h^0(\mathbb{P}^3_C, \mathcal{I}_C(4)) = 9 \) and the homogeneous ideal of \( C \) is minimally generated by two cubic \( F_1, F_2 \) and a single quartic surfaces \( G \) (see [31], Proposition 5.2.10). In particular, \( C = F_1 \cap F_2 \cap G \).

The general cubic surface \( F \) through \( C \) is smooth outside \( C \cup L \) due to Bertini’s theorem. Since \( C \) and \( L \) are smooth and they are exactly the intersection of \( F_1 \) and \( F_2 \) outside \( C \cap L \), it also follows that such a general cubic \( F \) is smooth outside \( C \cap L \). At the points of \( C \cap L \), \( C \) is smooth, thus it is the intersection of two surfaces at those points. The two cubics \( F_1 \) and \( F_2 \) are tangent at the points of \( C \cap L \), thus one can always take \( G \) and one of the cubics. In particular, at least one among the cubics \( F_1 \) and \( F_2 \) is smooth at the points of \( C \cap L \). We conclude that we can assume that the general cubic \( F \) through \( C \) is smooth everywhere. \( \square \)

We are now ready to characterize curves carrying a unique \( g_1^1 \) of type \( I \).

**Proposition 3.3.** The curve \( C \) carries exactly one \( g_1^1 \) of type \( I \) if and only if it is endowed with a very ample \( g_3^3 \).

**Proof.** Consider a divisor \( D \) on \( C \) and let \( K_C \) a canonical divisor on \( C \). Then \(|D|\) is a \( g_1^1 \) (resp. \( g_3^3 \)) if and only if \(|K_C - D|\) is a \( g_3^3 \) (resp. \( g_1^1 \)). Thus \( C \) carries a unique \( g_1^1 \) if and only if it carries a unique \( g_3^3 \).

Now assume that \( C \) is endowed with a unique base–point–free \( g_1^1 \) of type \( I \), say \(|D|\). We will show that \(|K_C - D|\) is very ample. To this purpose we have first to
show that it is base–point–free. For each $p \in C$, we have
\[ h^0(C, O_C(K_C - D - p)) = h^0(C, O_C(K_C - D)) \]
if and only if $|D + p|$ is a $g^2_3$. Thus $C$ would have a plane model of degree at most 5, hence its genus would be at most 6, a contradiction. Now we have to show that $|K_C - D|$ separates points and tangent vectors. For possibly coinciding points $p_1, p_2 \in C$, we have
\[ h^0(C, O_C(K_C - D - p_1 - p_2)) = h^0(C, O_C(K_C - D)) - 1 \]
if and only if $|D + p_1 + p_2|$ is a $g^2_6$. Consider the associated morphism $\varphi: C \to \overline{C} \subseteq \mathbb{P}^2_C$. We have three possible cases for the pair $(\deg(\varphi), \deg(\overline{C}))$, namely $(1, 6)$, $(2, 3)$, $(3, 2)$. In the last case $C$ would be trigonal. In the second case $C$ would be either bielliptic or hyperelliptic. The first case cannot occur due to Proposition 3.1 above, since $C$ is endowed with a unique $g^1_4$ of type $I$.

Conversely let us assume that $C$ is endowed with a very ample $g^3_8$. Thus $C$ carries a $g^1_4$, say $|D|$. We know that the very ample $g^3_8$ is $|K_C - D|$. We have to show that such a $g^1_4$ is of type $I$ and it is unique.

Assume that $|D|$ is of type $II$. Let us identify $C$ with its image via the map associated to the $g^3_8$ in $\mathbb{P}^3_C$. We know from Proposition 3.3 that there is a line $L \subseteq \mathbb{P}^3_C$ such that $C \cup L$ is the complete intersection of a smooth cubic surface $F$ with another cubic surface. Let us consider the standard representation of $F$ as the blow up of $\mathbb{P}^2_C$ at six general points. Let $\ell$ be the strict transform on $F$ of a general line of $\mathbb{P}^2_C$ and let $e_1, \ldots, e_6$ be the exceptional divisors in the blow up. We can always assume that $L = e_1$ so that
\[ C \in |9\ell - 4e_1 - 3e_2 - \cdots - 3e_6| . \]

By adjunction on $S$ the canonical system $|K_C|$ on $C$ is cut out by $|6\ell - 3e_1 - 2e_2 - \cdots - 2e_6|$, thus $|D|$ is cut out on $C$ by the pencil $|3\ell - 2e_1 - e_2 - \cdots - e_6|$, which coincides with the pencil cut out on $S$ by the planes through $L$.

Notice that $|2D|$ is cut out on $C$ by $|6\ell - 4e_1 - 2e_2 - \cdots - 2e_6|$. The cohomology of the standard exact sequence $0 \to O_S(-C) \to O_S \to O_C \to 0$ twisted by $O_S(6\ell - 4e_1 - 2e_2 - \cdots - 2e_6)$, gives the exact sequence
\[ 0 \to H^0(S, O_S(-3\ell + e_2 + \cdots + e_6)) \to H^0(S, O_S(6\ell - 4e_1 - 2e_2 - \cdots - 2e_6)) \to H^0(C, O_C(2D)) \to H^1(S, O_S(-3\ell + e_2 + \cdots + e_6)). \]

Since $| - 3\ell + e_2 - \cdots + e_6|$ cannot be effective, the first space is zero. Thanks to Serre’s duality $h^2(S, O_S(-3\ell + e_2 + \cdots + e_6)) = h^0(S, O_S(e_1)) = 1$. Riemann–Roch theorem finally yields $h^1(S, O_S(-3\ell + e_2 + \cdots + e_6)) = 0$.

Trivially $|3\ell - 2e_1 - e_2 - \cdots - e_6|$ contains a smooth integral curve $A$. Since $(3\ell - 2e_1 - e_2 - \cdots - e_6)^2 = 0$, it follows that $O_A(A) \cong O_A$. Thus the cohomology of $0 \to O_S \to O_S(A) \to O_A(A) \to 0$, yields $h^0(S, O_S(3\ell - 2e_1 - e_2 - \cdots - e_6)) = 2$.

Since
\[ (6\ell - 4e_1 - 2e_2 - \cdots - 2e_6) \cdot (3\ell - 2e_1 - e_2 - \cdots - e_6) = 0 \]
we deduce that each effective divisor in $|6\ell - 4e_1 - 2e_2 - \cdots - 2e_6|$ is the sum of two elements in $|3\ell - 2e_1 - e_2 + \cdots - e_6|$. It follows that

$$h^0(C, O_C(2D)) = h^0(S, O_S(6\ell - 4e_1 - 2e_2 - \cdots - 2e_6)) = 3$$

i.e. \(|D|\) is of type \(I\).

Assume that \(|D|\) is not unique and let \(|D'|\) be a distinct \(g^1_4\) on \(C\). Since \(C\) carries a \(g^3_8\), due to \([13]\), Proposition 2.2 (ii) and (v), it follows that \(C\) is not hyperelliptic nor bielliptic. As in the proof of Proposition 3.1 on checks that \(\dim(\im(\alpha_{|\mathbb{P}_C^2/\mathbb{P}_C^2}) = 4\). Then \(|K_C - D - D'|\) is a \(g^1_4\), say \(|D''|\). In particular, the linear system of planes of \(\mathbb{P}_C^2\) cut out on \(C\) exactly the linear system \(|D' + D''|\). It follows that the planes through the divisor \(D' \in |D'|\) cut out \(|D''|\) on \(C\). It follows that the support of \(D'\) must be contained in a line \(L'\). Since \(\deg(D') = 4\) such a line is contained in each cubic through \(C\), thus \(L' = L\), whence \(|D''| = |D|\). Thus \(h^0(C, O_C(2D)) = 2 + h^0(C, O_C(D')) = 4\), i.e. \(|D|\) should be of type \(II\). But this contradicts what we proved above. \(\square\)

4. The rationality of \(\mathcal{M}^1_{7,0,4}\)

We fix a line in \(L \subset \mathbb{P}_C^2\) and we consider the set of pencils of cubic surfaces through \(L\) which is a Grassmannian \(G(2, 16)\). If \(\Sigma \in G(2, 16)\), then its base locus contains \(L\). If \(\Sigma\) is general, then it contains a smooth cubic surface \(F\) and the residual intersection is a smooth curve \(C\) on \(F\), whence its genus is 7. Thus \(C\) is endowed with a base-point-free \(g^3_8\). In particular we have a natural rational map

$$m_0: G(2, 16) \dashrightarrow \mathcal{M}^1_{7,0,4}$$

whose image is the locus of tetragonal curves of genus 7 endowed with a unique \(g^1_4\) of type \(I\), thanks to Proposition 3.3. Let \(\text{PGL}_{4,L}\) be the stabilizer of \(L\) inside \(\text{PGL}_4\). It is clear that there is a natural action of \(\text{PGL}_{4,L}\) on \(G(2, 16)\). The orbits of such an action are trivially contained in the fibers of \(m_0\).

**Lemma 4.1.** The map \(m_0\) is dominant and it induces a birational isomorphism

$$\mathcal{M}^1_{7,0,4} \cong G(2, 16)/\text{PGL}_{4,L}.\$$

**Proof.** We first describe the fibers of \(m_0\). Let \(\Sigma'\) and \(\Sigma''\) be two general pencils of cubic surfaces with base loci \(C' \cup L\) and \(C'' \cup L\) respectively. Since \(\Sigma'\) and \(\Sigma''\) are general, \(C'\) and \(C''\) are smooth curve of genus 7. If \(m_0(\Sigma') = m_0(\Sigma'')\), then the base loci of \(\Sigma'\) and \(\Sigma''\) must be abstractly isomorphic. Such an isomorphism must induces an isomorphism \(\varphi: C' \to C''\). If \(|D'|\) and \(|D''|\) are the unique \(g^1_4\) on \(C'\) and \(C''\) respectively, then we have \(\varphi^*|D'| = |D''|\). Since also \(\varphi^*|K_{C'} = |K_{C'} - D'|\) holds, we finally deduce that \(\varphi^*|K_{C'} - D''| = |K_{C'} - D'|\). In particular \(\varphi\) sends the very ample \(g^3_8\) on \(C''\) into the very ample \(g^3_8\) on \(C'\), thus it induces a projectivity of the whole \(\mathbb{P}_C^2\) transforming \(\Sigma'\) in \(\Sigma''\). Such a projectivity must fix \(L\). It follows that the fibers of \(m_0\) are exactly the orbits of the action of \(\text{PGL}_{4,L}\) on \(G(2, 16)\).

In particular \(G(2, 16)/\text{PGL}_{4,L} \approx \im(m_0) \subseteq \mathcal{M}^1_{7,0,4}\). In order to complete the proof it suffices to check that \(\dim(\im(m_0)) = \dim(G(2, 16)/\text{PGL}_{4,L}) = \dim(M^1_{7,0,4})\). We
have
\[17 = \dim(\mathcal{M}_{7,0;4}^1) \geq \dim(\text{im}(m_0)) \geq \dim(G(2, 16)) - \dim(\text{PGL}_{4,L}) = 28 - 11 = 17,\]
thus \( \dim(\text{im}(m_0)) = 17 = \dim(\mathcal{M}_{7,0;4}^1). \) We conclude that \( \text{im}(m_0) \) is dense inside \( \mathcal{M}_{7,0;4}^1. \)
\[\Box\]

Let \( \text{SL}_{4,L} \) be the stabilizer of \( L \) in \( \text{SL}_4 \). We are interested in the rationality properties of the quotient \( G(2, 16) / \text{SL}_{4,L} \cong G(2, 16) / \text{PGL}_{4,L} \).

**Theorem 4.2.** The space \( X = G(2, 16) / \text{SL}_{4,L} \) is rational.

The proof requires several preparatory results. The next lemma shows that \( X \) is birational to a linear group quotient and collects all the facts about this representation which are needed in the sequel.

**Lemma 4.3.** The space \( X = G(2, 16) / \text{SL}_{4,L} \) is birational to the quotient \( R/G \) where \( R \) is a linear representation of the linear algebraic group \( G \), and \( R, G \) and the action are defined below:

1. **(Group structure).** One has
   \[ G = G_R \ltimes U \]
   where \( G_R = \text{GL}_2 \times G'_R \), the group \( G'_R \subseteq \text{SL}_4 \) being the subgroup consisting of matrices
   \[ \begin{pmatrix} A_2 & 0 \\ 0 & A_3 \end{pmatrix}, \quad A_2, A_3 \in \text{Mat}_{2 \times 2}(\mathbb{C}), \]
   hence as an abstract group a central product \( (\mathbb{C}^*) \cdot (\text{SL}_2 \times \text{SL}_2) \). Here the torus \( \mathbb{C}^* \) is embedded in \( \text{SL}_4 \) via
   \[ \lambda \mapsto \text{diag}(\lambda, \lambda, \lambda^{-1}, \lambda^{-1}). \]
   The group \( U \) is given by \( U = (\mathbb{C}^2)^{\vee} \otimes \mathbb{C}^2 = \text{Hom}(\mathbb{C}^2, \mathbb{C}^2) \), viewed as an abelian algebraic group under addition of homomorphisms. As a normal subgroup in the semidirect product, elements \( u \in U \) are acted on by \( (A_1, A_2, A_3) \in G_R \), where \( A_1 \in \text{GL}_2 \), and \( A_2, A_3 \) are as before, in the following way:
   \[ (A_1, A_2, A_3) \cdot u = A_2 u A_3^{-1}. \]

2. **(Structure of the representation).** The representation \( R \) is a representation with a Jordan-Hölder filtration
   \[ R_0 := 0 \subset R_1 \subset R_2 \subset R_3 = R \]
   with completely reducible quotients \( Q_i = R_i / R_{i-1} \) which as representations of \( G_R \) are given by
   \[ Q_1 = \mathbb{C}^2 \otimes \text{Sym}^3 \mathbb{C}^2 \otimes \mathbb{C}, \]
   \[ Q_2 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \text{Sym}^3 \mathbb{C}^2, \]
   \[ Q_3 = \mathbb{C}^2 \otimes \text{Sym}^2 \mathbb{C}^2 \otimes \mathbb{C}^2, \]
with the action of $G_R$ being given by the tensor product action of $(A_1, A_2, A_3)$ in the three factors. The action of $U$ is induced by the standard $G_R$-equivariant maps

$$(\mathbb{C}^2)^\vee \otimes \mathbb{C}^2 \to \text{Hom}((\mathbb{C}^2 \otimes \text{Sym}^2 \mathbb{C}^2) \otimes \mathbb{C}^2, \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \text{Sym}^2 \mathbb{C}^2),$$

$$(\mathbb{C}^2)^\vee \otimes \mathbb{C}^2 \to \text{Hom}((\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \text{Sym}^2 \mathbb{C}^2, \mathbb{C}^2 \otimes \text{Sym}^3 \mathbb{C}^2 \otimes \mathbb{C}^2)$$

given by contraction and multiplication in the second and third factors.

(3) \textbf{(Ineffectivity kernel)} The ineffectivity kernel $I$ of the action of $G$ on $R$ is as an abstract group $I \simeq \mathbb{Z}/4\mathbb{Z}$, and generated by

$$(A_1, A_2, A_3) = \left( \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \right).$$

The group acting effectively will be denoted by $G = G/(\mathbb{Z}/4\mathbb{Z})$.

\textbf{Proof}. Note that for $X = G(2, 16)/\text{SL}_{4,L}$, the group $\text{SL}_{4,L}$ is nothing but $G_R \ltimes U$, and $X$ is birational to (we view the stabilized line $L \subset \mathbb{C}^4$ as a two-plane)

$$(\text{Hom}(\mathbb{C}^2, \text{Sym}^3(\mathbb{C}^4)^\vee/\text{Sym}^3(L)^\vee)) / \text{GL}_2 \times \text{SL}_{4,L},$$

which in turn is birational to

$$(\mathbb{C}^2 \otimes \text{Sym}^3(\mathbb{C}^4)^\vee/\text{Sym}^3(L)^\vee)) / \text{GL}_2 \times \text{SL}_{4,L}$$

(because of the self-duality of $\mathbb{C}^2$ as $\text{SL}_2$-representation, we get birational quotients if we choose as $\text{GL}_2$-representation in the first factor of the tensor product $\mathbb{C}^2$ or $(\mathbb{C}^2)^\vee$; we prefer $\mathbb{C}^2$ for simplicity, but this is not essential for the following). The representation $\mathbb{C}^2 \otimes \text{Sym}^3(\mathbb{C}^4)^\vee/\text{Sym}^3(L)^\vee$ has a filtration as a $G$-module with quotients $Q_1, Q_2, Q_3, Q_3$ as claimed in (2) above. This proves (1) and (2).

For the determination of the ineffectivity kernel in (3), we refer to the stronger statement Lemma 4.7 below proven with the help of computer algebra, and which implies in particular part (3) of the present lemma. \hfill $\square$

\textbf{Remark 4.4.} We have to recall the representation theory of $\mathfrak{S}_4$ viewed as the group of permutations of four letters \{a, b, c, d\} for subsequent use. The character table is (cf. [33]).

| 1 | (ab) | (ab)(cd) | (abc) | (abcd) |
|---|------|---------|-------|--------|
| $\chi_0$ | 1 | 1 | 1 | 1 | 1 |
| $\epsilon$ | 1 | -1 | 1 | 1 | -1 |
| $\theta$ | 2 | 0 | 2 | -1 | 0 |
| $\psi$ | 3 | 1 | -1 | 0 | -1 |
| $\epsilon\psi$ | 3 | -1 | -1 | 0 | 1 |

$V_{\chi_0}$ is the trivial 1-dimensional representation, $V_\epsilon$ is the 1-dimensional representation where $\epsilon(g)$ is the sign of the permutation $g$; $\mathfrak{S}_4$ being the semidirect product of $\mathfrak{S}_3$ by the Klein four group, $V_\theta$ is the irreducible two-dimensional representation induced from the representation of $\mathfrak{S}_3$ acting on the elements of $\mathbb{C}^3$ which satisfy $x + y + z = 0$ by permutation of coordinates. $V_\psi$ is the representation on the elements of $\mathbb{C}^4$ with $x + y + z + w = 0$ by permutation of coordinates; finally, $V_{\epsilon\psi} = V_\epsilon \otimes V_\psi$. 
The $G_R$-representation $Q_3$ in item (2) of Lemma 4.3 is one with a nontrivial stabilizer in general position $H$. The determination of $H$ and its normalizer $N(H)$ in $G_R$ is carried out in the next lemma. It gives us a $(G, N(H) \ltimes U)$-section in $R$.

**Lemma 4.5.** Consider the action of $G_R$ on $Q_3 = R_3/R_2 = \mathbb{C}^2 \otimes \text{Sym}^2 \mathbb{C}^2 \otimes \mathbb{C}^2$.

1. **(Structure of the stabilizer in general position).** The stabilizer in general position $H$ of the representation $Q_3$, consisting of matrices $(A_1, A_2, A_3)$ as in Lemma 4.3, can be described as follows: put 
   \[ A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

   Then $A$ and $B$ generate inside $SL_2$ a nontrivial central extension of the Klein four group $\mathbb{Z}/2\mathbb{Z}^2 \subset PSL_2$, which is a finite Heisenberg group (or extraspecial $2$-group in different terminology) which we denote by $\tilde{H}$.

   \[ 0 \to \mathbb{Z}/2\mathbb{Z} \simeq \{ \pm 1 \} \to \tilde{H} \to (\mathbb{Z}/2\mathbb{Z})^2 = \langle \overline{A}, \overline{B} \rangle \to 0 \]

   (here $\overline{A}$ and $\overline{B}$ denote the classes of $A$ and $B$ in $PSL_2$ respectively).

   Then the stabilizer in general position $H \subset G_R$ (well defined up to conjugacy) has a representative given by the subgroup of matrices 
   \[ (A_1, A_2, A_3) = (\lambda h, \pm \lambda^{-1} h, \lambda h), \quad \lambda \in \mathbb{C}^*, \ h \in \tilde{H}. \]

2. **(Structure of the normalizer).** The normalizer $N(H)$ of $H$ in $G_R$ can be described as follows: define matrices $(\tau, \tau, \tau)$ and $(\sigma, \sigma, \sigma)$ by
   \[ \tau = \begin{pmatrix} \theta^{-1} & 0 \\ 0 & \theta \end{pmatrix}, \quad \sigma := \frac{1}{\sqrt{2}} \begin{pmatrix} \theta^3 & \theta^5 \\ \theta^5 & \theta^3 \end{pmatrix}. \]
   and $\theta = \exp(2\pi i/8)$. Their classes in $PSL_2$ generate the symmetric group $S_4$ normalizing the Klein four group, but inside $SL_2$, they generate a nontrivial central extension
   \[ 1 \to \mathbb{Z}/2\mathbb{Z} \simeq \{ \pm 1 \} \to \tilde{S}_4 \to S_4 \to 1. \]

   Then $N(H)$ contains a subgroup abstractly isomorphic to $\tilde{S}_4 \ltimes \tilde{H}$ where: $\tilde{S}_4$ is embedded in $N(H)$ diagonally as the matrices
   \[ (A_1, A_2, A_3) = (x, x, x), \ x \in \tilde{S}_4; \]

   and the additional copy of $\tilde{H}$ in the semidirect product $\tilde{S}_4 \ltimes \tilde{H}$ is embedded in the second factor as matrices
   \[ (A_1, A_2, A_3) = (\text{id}, y, \text{id}), \ y \in \tilde{H}. \]

   The complete normalizer $N(H)$ is generated by $\tilde{S}_4 \ltimes \tilde{H}$ and the center of $G_R$ consisting of matrices
   \[ (A_1, A_2, A_3) = \left( \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} \pm \mu & 0 \\ 0 & \pm \mu \end{pmatrix}, \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} \right). \]
(3) **(Structure of the \((G, \ N(H) \ltimes U)\)-section).** By (1) and (2) there is a \((G, \ N(H) \ltimes U)\)-section \(R'\) in \(R\) which is a linear representation of \(N(H) \ltimes U\) with a Jordan-Hölder filtration \(0 = R'_0 \subset R'_1 \subset R'_2 \subset R' = R'\) with completely reducible quotients \(Q'_i = R'_i/R'_{i-1}\) given by

\[
Q'_1 = Q_1 = \mathbb{C}^2 \otimes \text{Sym}^3 \mathbb{C}^2, \quad Q'_2 = Q_2 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \text{Sym}^2 \mathbb{C}^2, \quad Q'_3 = Q_3^H.
\]

The space of \(H\)-invariants in \(R_3/R_2\) is a \(N(H)\)-section for the action of \(\prod_{i=1}^3 \text{SL}_2\). This space \((R_3/R_2)^H\) has dimension 3. The action of the copy of \(\tilde{S}_4\) in \(N(H)\) on \(Q_3^H\) is via the standard representation of \(\tilde{S}_4\). That is, in the notation introduced in Remark [4.4],

\[
Q_3^H = V_0 \oplus V_{\theta}.
\]

All assertions of Lemma 4.5 have been double-checked independently via computer algebra by the second author to avoid mistakes. The Macaulay 2 scripts can be found at [http://xwww.uni-math.gwdg.de/bothmer/tetragonal/](http://xwww.uni-math.gwdg.de/bothmer/tetragonal/) . Below we give a proof not relying on this.

*Proof of Lemma 4.5. Step 1. Determination of the stabilizer in general position.* Recall the accidental isomorphisms of Lie groups \(\text{Spin}_3 \simeq \text{SL}_2\) and \(\text{Spin}_4 \simeq \text{SL}_2 \times \text{SL}_2\). Under these isomorphisms we have the correspondences of representations

\[
\mathbb{C}^3 \simeq \text{Sym}^2 \mathbb{C}^2, \quad \mathbb{C}^4 \simeq \mathbb{C}^2 \otimes \mathbb{C}^2.
\]

Thus \(R_3/R_2\) is isomorphic to \(\mathbb{C}^3 \otimes \mathbb{C}^4\) and the group \(\text{Spin}_3 \times \text{Spin}_4\) acts as \(\text{SO}_3 \times \text{SO}_4\) in this representation. The table in [33] shows firstly that the stabilizer in general position inside \(\text{SO}_3 \times \text{SO}_4\) of this representation is isomorphic to \((\mathbb{Z}/2\mathbb{Z})^2\); and secondly, that it maps isomorphically to the stabilizer in general position for the action of \((\text{SO}_3 \times \text{SO}_4)/(\text{center})\) on \(\mathbb{P}(\mathbb{C}^3 \otimes \mathbb{C}^4)\). Hence, to prove (1) of Lemma 4.5 it suffices to show that the subgroup of matrices

\[
(A_1, \ A_2, \ A_3) = (h, \pm h, \ h), \ h \in \tilde{H}
\]

of order 16 coincides with the preimage of the stabilizer in general position inside \(\text{SO}_3 \times \text{SO}_4\) in \(\text{SL}_2 \times \text{SL}_2 \times \text{SL}_2\). Note that a stabilizer in general position is only well-defined up to conjugacy, and we are in particular interested in finding a good model for it and the associated invariant subspace.

Every general \(3 \times 4\) complex matrix can be brought to the normal form

\[
\begin{pmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 \\
0 & 0 & \lambda_3 & 0
\end{pmatrix},
\]

the \(\lambda_i\) being pairwise different. This follows from the polar decomposition for complex matrices and, afterwards, the fact that symmetric complex matrices which are similar are orthogonally similar. See [19], Chapter XI, §2, Thm. 3 and Thm. 4.

Thus the stabilizer \((\mathbb{Z}/2\mathbb{Z})^2\) in \(\text{SO}_3 \times \text{SO}_4\) consists of pairs of matrices

\[
(\text{diag}(\epsilon_1, \epsilon_2, \epsilon_3), \ \text{diag}(\epsilon_1, \epsilon_2, \epsilon_3, 1))
\]
where the $\epsilon_i$ are either $+1$ or $-1$ subject to the condition that the determinant of $\text{diag}(\epsilon_1, \epsilon_2, \epsilon_3)$ is 1. The invariant subspace of this $(\mathbb{Z}/2\mathbb{Z})^2$ has dimension 3.

Note that the quadratic form that $\text{SL}_2 \times \text{SL}_2$ fixes on $\mathbb{C}^2 \otimes \mathbb{C}^2$ is the determinant of a two by two matrix. An orthogonal system of three vectors for the associated bilinear form can thus be given by the matrices

$$m_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad m_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. $$

The bilinear form that $\text{SL}_2$ fixes on $\text{Sym}^2(\mathbb{C}^2)$ is given by contraction of conics (interpret one of them as a dual conic via the isomorphism $\text{Sym}^2\mathbb{C}^2 \simeq \text{Sym}^2(\mathbb{C}^2)^{\vee}$). Hence an orthogonal basis can be given by

$$q_1 = x^2 + y^2, \quad q_2 = x^2 - y^2, \quad q_3 = xy.$$ 

We thus have to find the stabilizer of

$$\lambda_1 m_1 \otimes q_1 + \lambda_2 m_2 \otimes q_2 + \lambda_3 m_3 \otimes q_3$$

inside $\text{SL}_2 \times \text{SL}_2 \times \text{SL}_2$ for general $\lambda_1, \lambda_2, \lambda_3$. Let

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. $$

Then

$$A \begin{pmatrix} a & b \\ c & d \end{pmatrix} A^t = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix}, \quad B \begin{pmatrix} a & b \\ c & d \end{pmatrix} B^t = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

hence we see that the elements

$$(A, A, A), \ (B, B, B) \in \text{SL}_2 \times \text{SL}_2 \times \text{SL}_2$$

stabilize the matrix $\lambda_1 m_1 \otimes q_1 + \lambda_2 m_2 \otimes q_2 + \lambda_3 m_3 \otimes q_3$ given above! These generate a Klein four group in $(\text{SL}_2 \times \text{SL}_2 \times \text{SL}_2)/(-\text{id}, -\text{id}, -\text{id})$, but as they only commute up to the central element $z := (-\text{id}, -\text{id}, -\text{id})$ in $\text{SL}_2 \times \text{SL}_2 \times \text{SL}_2$ they generate there a nontrivial central extension

$$0 \to \mathbb{Z}/2\mathbb{Z} = \langle z \rangle \to \bar{H} \to (\mathbb{Z}/2\mathbb{Z})^2 = \langle [A, A], [B, B, B] \rangle \to 0$$

which is a particular (finite) Heisenberg group in the sense of Mumford’s Tata Lectures on Theta III [32]. There is one further $\mathbb{Z}/2\mathbb{Z}$ in the stabilizer in general position: the center of the copy of $\text{SL}_2$ acting on $\text{Sym}^2(\mathbb{C})$; this establishes (1) of the Lemma.

**Step 2. Determination of the normalizer.** Note first that $G_H$ does contain a copy of $\tilde{S}_4$ as indicated in (2), normalizing $H$, and that the full automorphism group of the Klein four group coincides with $\tilde{S}_4$. Hence it suffices to determine the matrices
\((A_1, A_2, A_3) \in G_R\) which, via conjugation, act as the identity on the Heisenberg group of matrices

\[(h, h, h), h \in \tilde{H}\]

inside \(G_R\) modulo the subgroup of \(G_R\) generated by elements \((\lambda, \pm \lambda^{-1}, \lambda)\). But the conditions for a matrix \(M\) to commute with \(A\) above up to some nonzero constant factor \(\lambda\) imply that \(M\) equals

\[
\begin{pmatrix}
m_1 & 0 \\
0 & m_4
\end{pmatrix}, \lambda = +1, \quad \text{or} \quad \begin{pmatrix}
0 & m_2 \\
m_3 & 0
\end{pmatrix}, \lambda = -1
\]

and if one of the matrices of the preceding two shapes also commutes with \(B\) up to a (possibly different) nonzero factor \(\lambda'\), we must have

\[
M = \begin{pmatrix}
m_1 & 0 \\
0 & m_1
\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}
0 & 0 \\
0 & -m_1
\end{pmatrix}
\]

or

\[
M = \begin{pmatrix}
0 & m_2 \\
m_2 & 0
\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}
0 & m_2 \\
-m_2 & 0
\end{pmatrix},
\]

that is- modulo the center of \(G_R\)- we must have that \((A_1, A_2, A_3)\) is in \(\tilde{H} \times \tilde{H} \times \tilde{H} \subset \text{SL}_2 \times \text{SL}_2 \times \text{SL}_2\). But then the assertion of (2) follows at once, as elements in \(\tilde{H}\) commute up to a sign, but the sign change must be the same in the first and third factors, and an element in \(\tilde{H}\) anticommutes with every other element in \(\tilde{H}\) except itself.

**Step 3. Structure of \((R_3/R_2)^H\).** The only statement of Lemma 4.5, (3) which still needs some explanation is that as \(\mathfrak{S}_4\)-representation \((R_3/R_2)^H = V_{\chi_0} \oplus V_{\theta}\).

This is most readily seen by going back to the \(\text{SO}_3 \times \text{SO}_4\)-picture introduced at the beginning of the proof, where it is obvious, namely \(\mathfrak{S}_4\) acts as \(\mathfrak{S}_3\) via ordinary permutations on \(\lambda_1, \lambda_2, \lambda_3\).

We compute the decomposition of the Jordan-Hölder quotients as \(\mathfrak{S}_4\)-representations.

**Lemma 4.6.** As \(\mathfrak{S}_4\)-representations we have

\[
\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \text{Sym}^2 \mathbb{C}^2 = V_{\chi_0} \oplus V_{\theta} \oplus V_{\psi} \oplus 2V_{e\psi},
\]

\[
\mathbb{C}^2 \otimes \text{Sym}^3 \mathbb{C}^2 = V_{\theta} \oplus V_{\psi} \oplus V_{e\psi}.
\]

Here we view these as \(\mathfrak{S}_4\)-representations via the embedding \(\tilde{\mathfrak{S}}_4 \subset \text{SL}_2 \times \text{SL}_2 \times \text{SL}_2\) described in Lemma 4.5.

**Proof.** We start by decomposing \(\mathbb{C}^2 \otimes \mathbb{C}^2\) viewed as two by two matrices

\[
M = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]

under the action of \(\mathfrak{S}_4\). Using the formulas for \(A\) and \(B\) in Lemma 4.5, we find that \(M\) can only be invariant under the Klein four group if it is antisymmetric. But an antisymmetric matrix \(M\) is also invariant under \(\sigma\) and \(\tau\). Thus the space of \(\mathfrak{S}_4\)-invariants in \(\mathbb{C}^2 \otimes \mathbb{C}^2\) is one-dimensional. The three-dimensional subspace of
matrices spanned by $m_1$, $m_2$, $m_3$ (notation as in the proof of Lemma 4.5) is $S_4$-invariant. The trace of an element in the Klein four-group, $A$ say, on this space is easily calculated to be $-1$, and the trace of the four-cycle $\tau$ on this space (remark $\tau^4 = \sigma^3 = 1$) is similarly found to be $+1$. Thus $$C^2 \otimes C^2 = V_{\chi_0} \oplus V_{\epsilon \psi}.$$ Now $\text{Sym}^2 C^2$ has no $A$-invariants. Hence it is either $V_{\psi}$ or $V_{\epsilon \psi}$. The trace of $\tau$ on it is $+1$. So $\text{Sym}^2 C^2 = V_{\epsilon \psi}$. Checking characters on both sides shows the decomposition $V_{\epsilon \psi} \otimes V_{\epsilon \psi} = V_{\chi_0} \oplus V_{\psi} \oplus (V_{\psi} \oplus V_{\epsilon \psi})$.

Note by the way that this also implies $$(C^2 \otimes C^2) \otimes (R_3/R_1)^H = C^2 \otimes \text{Sym}^2 C^2 \otimes C^2$$ as $S_4$-representations as it should be. This checks the computation. Hence we have seen that $$C^2 \otimes \text{Sym}^2 C^2 \otimes C^2 = V_{\chi_0} \oplus V_{\theta} \oplus V_{\psi} \oplus 2V_{\epsilon \psi}.$$ The $S_4$-representation $C^2 \otimes \text{Sym}^3 C^2$ can be decomposed as follows. One has $$C^2 \otimes C^2 \otimes \text{Sym}^2 C^2 = C^2 \otimes (C^2 + \text{Sym}^3 C^2) = C^2 \otimes C^2 + C^2 \otimes \text{Sym}^3 C^2$$ as $S_4$-representations. Hence $$C^2 \otimes \text{Sym}^3 C^2 = V_{\theta} \oplus V_{\psi} \oplus V_{\epsilon \psi}.$$ □

The proof of the main Theorem 4.2 now requires some further lemmas. In what follows we say that a variety $X$ is stably rational of level $t$ if $X \times \mathbb{P}^t$ is rational.

**Lemma 4.7.** The action of $(N(H) \ltimes U)/I$, where $I$ is the ineffectivity kernel $I \simeq \mathbb{Z}/4\mathbb{Z}$ of the action described in lemma 4.3, item (3), is already generically free on the two step quotient $R'_3/R'_1$ of $R'$. Hence, by the no-name lemma, $R'/(N(H) \ltimes U)$ is generically a vector bundle of rank 8 over $(R'_3/R'_1)/(N(H) \ltimes U)$. Thus the proof of Theorem 4.2 reduces to the proof that $$(R'_3/R'_1)/(N(H) \ltimes U)$$ is stably rational of level at most 8.

**Proof.** The only point to check is the generic freeness of the action of $N(H) \ltimes U$ on $R'_3/R'_1$. This can been done by a computer algebra calculation, the commented Macaulay 2 scripts are at http://xwww.uni-math.gwdg.de/bothmer/tetragonal/.

□

The following Lemma contains a standard trick to reduce from $N(H) \ltimes U$ to $N(H)$.

**Lemma 4.8.** The quotient $(R'_3/R'_1)/(N(H) \ltimes U)$ is stably rational of level 8 if $(R'_3/R'_1)/N(H)$ is stably rational of level 4.
Proof. Consider the representation $E$ of $N(H) \rtimes U$ which is a two-step extension (of
dimension 5)

$$0 \to (\mathbb{C}^2)^\vee \otimes \mathbb{C}^2 \to E \to \mathbb{C}.$$ 

Then $(R_3'/R_1')(N(H) \rtimes U) \times \mathbb{C}^5 \simeq ((R_3'/R_1') \oplus E)/(N(H) \rtimes U) \simeq (R_3'/R_1')/N(H) \times \mathbb{C}$. \hfill \Box

Lemma 4.9. The group $G_R$ has a natural representation $V$ which is

$$\mathbb{C}^2 \otimes \mathbb{C}^2 + \mathbb{C}^2 \otimes \mathbb{C}^2 + \mathbb{C}^2 \otimes \text{Sym}^2 \mathbb{C}^2 \otimes \mathbb{C}^2$$

and $(A_1, A_2, A_3) \in G_R$ acts via

$$(A_1, A_2, A_3) \cdot (M_1, M_2, x) = (A_1 M_1 A_2^{-1}, A_1 M_2 A_3^{-1}, (A_1, A_2, A_3) \cdot x)$$

where $M_1$ and $M_2$ are interpreted as matrices in $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $x$ is some element in
$\mathbb{C}^2 \otimes \text{Sym}^2 \mathbb{C}^2 \otimes \mathbb{C}^2$, and the action there is the same action considered already above.
Then the quotient $V/G_R$ is rational. Moreover the representation of $N(H)/I$ given by

$$V' = \mathbb{C}^2 \otimes \mathbb{C}^2 + \mathbb{C}^2 \otimes \mathbb{C}^2 + Q_3^H$$

(obtained by taking a section in $V$) is generically free for $N(H)/I$ (and the quotient is birationally equivalent to the former, hence also rational). Hence $N(H)/I$ has a
generically free representation of dimension 11 with rational quotient.

Proof. There is a $(G_R, G')$-section in the representation $V$ which is $\{\text{id}\} \times \{\text{id}\} \times$
$\mathbb{C}^2 \otimes \text{Sym}^2 \mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathbb{C}^2 \otimes \text{Sym}^2 \mathbb{C}^2 \otimes \mathbb{C}^2$ where id $\in \mathbb{C}^2 \otimes \mathbb{C}^2$ is the identity $2 \times 2$
matrix, and $G'$ is the subgroup of $G_R$ consisting of matrices $(A_1, A_2, A_3)$ with
$A_1 = A_2 = A_3$. That is, $G'$ is the subgroup of $\text{GL}_2$ generated by $\text{SL}_2$ and $\text{diag}(i, i)$.
It follows firstly that $V/G_R$ is rational as $\mathbb{P}(\mathbb{C}^2 \otimes \text{Sym}^2 \mathbb{C}^2 \otimes \mathbb{C}^2)/\text{SL}_2$ is rational (recall
that each linear representation of $\text{SL}_2 \times \mathbb{C}^*$ has rational quotient, thanks to some
well-known theorems by Bogomolov and Kastylo: e.g., see [6, 24, 25] and the other
references cited therein); and secondly, that $V$ is generically free for $G_R/I$, hence
that $V'$ is generically free for $N(H)/I$, because $\mathbb{C}^2 \otimes \text{Sym}^2 \mathbb{C}^2 \otimes \mathbb{C}^2$ is generically free for
$G'/\langle \text{diag}(i, i) \rangle$. This concludes the proof of the lemma. \hfill \Box

Hence we can use the following.

Proof of Theorem 4.2. It suffices, by Lemma 4.8, to establish stable rationality of
level 4 of $(R_3'/R_1')/(N(H)/I)$. By Lemma 4.9 it suffices to find a generically free
$N(H)/I$-subrepresentation of $R_3'/R_1'$ with a complement in $R_3'/R_1'$ of dimension $\geq 7$,
using the no-name Lemma. Now

$$R_3'/R_1' = Q_3^H + \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \text{Sym}^2 \mathbb{C}^2.$$ 

The representation $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \text{Sym}^2 \mathbb{C}^2$ has an $N(H)$-invariant summand which is $\mathbb{C}^2 \otimes \mathbb{C}^2$, an $N(H)$-invariant complement being $\mathbb{C}^2 \otimes \text{Sym}^3 \mathbb{C}^2$. In fact, the representation
$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \text{Sym}^2 \mathbb{C}^2$ is, as an $\hat{\mathfrak{S}}_4 \ltimes \hat{H}$-representation a tensor product of two $\hat{\mathfrak{S}}_4 \ltimes \hat{H}$-
representations (recall also from Lemma 4.5 item (2), that $N(H)$ is generated by
$\hat{\mathfrak{S}}_4 \ltimes \hat{H}$ and the center of $G_R$): the one representation is a representation of $\hat{\mathfrak{S}}_4 \ltimes \hat{H}/\hat{H} \simeq \hat{\mathfrak{S}}_4$, namely $\mathbb{C}^2 \otimes \text{Sym}^2 \mathbb{C}^2 \simeq \mathbb{C}^2 + \text{Sym}^3 \mathbb{C}^2$ corresponding to grouping the first
and third factor in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \text{Sym}^2 \mathbb{C}^2$; the other representation is $\mathbb{C}^2$ (corresponding
to the factor in $C^2 \otimes C^2 \otimes \text{Sym}^2 C^2$ in the middle), which is an $\tilde{S}_4 \rtimes \tilde{H}$-representation via the inclusions

$$\tilde{S}_4 \rtimes \tilde{H} \subset N(H) \subset \text{SL}_2 \times \text{SL}_2 \times \text{SL}_2$$

followed by the projection $\text{pr}_2 : \text{SL}_2 \times \text{SL}_2 \times \text{SL}_2 \rightarrow \text{SL}_2$ to the second factor. Hence, as every subspace of $C^2 \otimes C^2 \otimes \text{Sym}^2 C^2$ is stabilized by the center of $G_R$, we indeed have an $N(H)$-invariant splitting

$$C^2 \otimes C^2 \otimes \text{Sym}^2 C^2 \simeq C^2 \otimes C^2 + C^2 \otimes \text{Sym}^3 C^2$$

Then $(R_3/R_2)^H + C^2 \otimes C^2$ is generically free for $N(H)/I$, and the complement $C^2 \otimes \text{Sym}^3 C^2$ has dimension $\geq 7$. To check the generic freeness, which can be done by hand, note that in Lemma 4.6 we saw that $C^2 \otimes C^2 \simeq V_{\chi_0} \oplus V_{\psi}$, and the Klein four group $\tilde{H}/\{\pm 1\}$, which is contained in the stabilizer $H/I$ of a general point in $Q^H_3$, consequently acts effectively in $\mathbb{P}(C^2 \otimes C^2)$. $\square$

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