Quantum computation of zeta functions of curves

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Abstract

We exhibit a quantum algorithm for determining the zeta function of a genus \( g \) curve over a finite field \( \mathbb{F}_q \), which is polynomial in \( g \) and \( \log(q) \). This amounts to giving an algorithm to produce provably random elements of the class group of a curve, plus a recipe for recovering a Weil polynomial from enough of its cyclic resultants. The latter effectivizes a result of Fried in a restricted setting.

1 Introduction

Given a curve \( C \) (assumed to be smooth, projective and geometrically irreducible) over a finite field \( \mathbb{F}_q \) with \( q = p^a \) for some prime \( p \), the zeta function of \( C \) has the form

\[
Z(C, t) = \exp \left( \sum_{n=1}^{\infty} \frac{T^n}{n} \# C(\mathbb{F}_{q^n}) \right) = \frac{P(t)}{(1 - t)(1 - qt)}
\]

for some polynomial \( P(t) \in \mathbb{Z}[t] \) of degree \( 2g \) with \( P(0) = 1 \). The determination of \( P(t) \) is an active problem in algorithmic number theory, in part because of practical connections to cryptography (especially when \( C \) is an elliptic curve, or more generally a hyperelliptic curve). For \( g \) fixed, the approach introduced by Schoof [22] (compute \( P(t) \) modulo many small primes) gives an algorithm which is polynomial in \( \log(q) \) but exponential in \( g \), as shown by Pila [21] and Adleman-Huang [11]. (A streamlined form of Schoof’s algorithm, incorporating improvements due to Atkin, Elkies, et al., turns out to be usable in practice for \( g = 1 \) and perhaps for \( g = 2 \), but for larger \( g \) the algorithm is highly impractical.) On the other hand, imitating Dwork’s proof of the rationality of zeta functions [4] yields an algorithm which is polynomial in \( p \), \( g \) and \( \log_p(q) \), as observed by Lauder and Wan [15].
(The latter is also not practical, but related “cohomological” techniques have proven more tractable; see [3] for the current state of the art.)

However, a single algorithm for computing $P(t)$ in time polynomial both in $g$ and $\log(q)$ remains elusive. Thus any sign that this problem might be “easy” has some relevance; the main result of this note (originally written as an addendum to [13]) is one such sign, if only an indirect one.

**Theorem 1.** There is a quantum algorithm for computing the numerator $P(t)$ of the zeta function, which is polynomial time in $g, \log(q)$. (See Section 2 for conventions regarding probabilistic algorithms.)

Implicit in the statement of the theorem is the choice of a mechanism for inputting arbitrary curves, such that the length of the input is polynomial in the genus. We will be more explicit about the choice we have in mind in Section 6 however, if the reader prefers to substitute a polynomial time equivalent alternate choice, this will of course not affect the truth of the theorem.

The components of the algorithm specified in Theorem 1 will be described in the subsequent sections of the paper. It may be worth pointing out here some components that may have some interest on their own: a method for producing generators of the Jacobian group of a curve over a finite field with provably high probability (Lemma 10), and a method for recovering a Weil polynomial from a few of its cyclic resultants (Section 8).

## 2 Conventions for probabilistic algorithms

Before proceeding, it will be helpful to fix some conventions about probabilistic algorithms.

Given a real number $b \in (0, 1)$, we define a *Las Vegas algorithm* to be an algorithm that, given a stream of outputs of a “fair coin” (a/k/a a Bernoulli trial with probability $1/2$), accomplishes its specified goal with probability at least $1 - b$ and reports failure with probability $b$. As long as $b$ is fixed, its exact value is not critical, as the success probability of a Las Vegas algorithm can be boosted simply by repeated invocation. This analysis is standard (and easy), but it will be useful for us to record it explicitly: in terms of the success probability $a = 1 - b$, in case $a \leq 1/2$, then after two invocations, the success probability is

$$1 - (1 - a)^2 = 2a - a^2 = a(2 - a) \geq 3a/2.$$ 

In particular, one can boost the success probability from $a$ to $1/2$ with at most

$$2^{\left\lceil \log_{3/2}(2/a) \right\rceil} \leq 2^{2\log_{2}(2/a)+1} = \frac{8}{a^2}$$

invocations, and from there up to any fixed higher value by multiplying the number of invocations by a suitable fixed factor. For instance, to get to success probability $3/4$, it suffices to perform $16/a^2$ invocations. (By the same token, it is sometimes more convenient
to use Bernoulli trials of different probabilities, e.g., to sample uniformly from a finite set; one can simulate such trials with a fair coin up to any fixed failure probability.)

Given a real number $b \in (1/2, 1)$, we define a Monte Carlo algorithm to be an algorithm that, given a stream of outputs of a fair coin, accomplishes its specified goal with probability at least $1 - b$ but may yield any outcome otherwise. Because of the nature of quantum mechanics, all quantum algorithms must be regarded as Monte Carlo algorithms. Again, one can decrease the error probability $b$ below any fixed cutoff, this time by performing a fixed number of invocations and retaining the answer returned most often. This analysis is standard, and it will not be useful for us to record it explicitly, so we omit it.

3 Black box groups

Our quantum algorithm for computing zeta functions reduces the problem to the determination of the order of certain “black box groups”. Before proceeding to the specific groups in question (groups of rational points on Jacobian varieties), we first recall a bit of the formalism of black box groups and cite the result about them we will be using. Note that this formalism makes sense within any of the standard computing paradigms (e.g., deterministic, Las Vegas, Monte Carlo, or quantum).

A black box group with unique encodings, in the sense of Babai and Szemerédi [2], consists of an $n$-element subset $T$ of $\{0,1\}^m$ for some $m$ and $n$, and an oracle which has the following properties, for some (unknown) subset $S \subseteq \{0,1\}^m$ containing $T$, and some (unknown) bijective map $f : S \to G$ from $S$ to a group $G$ generated by $f(T)$.

(a) Given $x, y \in S$, the oracle can determine $z \in S$ such that $f(z) = f(x)f(y)$ in $G$.

(b) Given $x \in S$, the oracle can determine $y \in S$ such that $f(y) = f(x)^{-1}$ in $G$.

We may also speak of this data as a “black box presentation of $G$ with unique encodings”; its input length for complexity purposes is taken to be $mn$. Compare this definition with that of a “black box group” without further qualification: in that case $f$ is only required to be surjective, and the oracle is required to be able to determine, given $x \in S$, whether $f(x)$ is the identity element of $G$.

We are now ready to invoke the necessary input from the theory of quantum computing.

Lemma 2. Given a Monte Carlo black box group presentation with unique encodings $f : S \to G$ of an abelian (or even solvable) group $G$, of input length $mn$, there is a quantum algorithm, running in time polynomial in $mn$, for computing the order of $G$.

Proof. See Watrous [27], [28]; the technique extends Shor’s application of Fourier transform methods to the factoring and discrete logarithm problems [23].
4 Algebraic curves

Since our intended reader is not necessarily an expert in algebraic geometry, we include here a synopsis of some relevant facts. For a fuller treatment, see [7 or 9, Chapter IV].

By a curve over a perfect field $k$, we will always mean a smooth, projective, geometrically irreducible variety $C$ of dimension 1 over $k$. To each such curve we can associate the field $K(C)$ of rational functions on $C$; this is a field of transcendence degree 1 over $k$, in which $k$ is relatively algebraically closed. In fact, the functor $C \mapsto K(C)$ is an equivalence between curves and such fields. Let $\overline{k}$ denote the algebraic closure of $k$, and let $C(k)$ and $C(\overline{k})$ denote the sets of $k$-rational and $\overline{k}$-rational points, respectively, on $C$.

A divisor on $C$ is a formal sum

$$D = \sum_{P \in C(\overline{k})} c_P(P) \quad (c_P \in \mathbb{Z}),$$

invariant under the action of $\text{Gal}(\overline{k}/k)$ induced by the Galois action on $C(\overline{k})$, in which $c_P = 0$ for all but finitely many $P$. That last condition means that the sum $\sum_P c_P$ is well-defined; it is called the degree of $D$ and denoted $\deg(D)$.

We point out three special types of divisors. We refer to the sum over a single Galois orbit on $C(\overline{k})$, with all coefficients 1, as a prime divisor; the group of divisors is freely generated by the prime divisors. For $f \in K(C)^*$ and $P \in C(\overline{k})$, let $\text{ord}_P(f)$ denote the order of vanishing (positive, negative, or zero) of $f$ at $P$. Define the divisor $(f) = \sum_P \text{ord}_P(f)(P)$; any divisor of this form is called a principal divisor. Similarly, for $\omega$ a nonzero 1-form on $C$, we may define $\text{ord}_P(\omega)$ as the order of vanishing, and define the divisor $(\omega) = \sum_P \text{ord}_P(\omega)(P)$; any divisor of this form is called a canonical divisor. Note that if $D$ is a principal divisor, then $\deg(D) = 0$, whereas if $D$ is a canonical divisor, then $\deg(D) = 2g - 2$, where $g$ is the genus of $C$ (by the Riemann-Roch theorem; see below). We write $D_1 \sim D_2$ to mean that $D_1 - D_2$ is a principal divisor; this is clearly an equivalence relation. Note that the ratio of two 1-forms is a rational function, so any two canonical divisors are equivalent.

A divisor $D = \sum_P c_P(P)$ is effective if $c_P \geq 0$ for all $P$; we write $D_1 \geq D_2$ to mean that $D_1 - D_2$ is effective. For $D$ effective, we necessarily have $\deg(D) \geq 0$ (but not conversely). For $D$ a divisor on $C$, let $L(D)$ be the set of functions $f \in K(C)$ such that $(f) + D \geq 0$, together with the zero function. The set $L(D)$ is a vector space over $k$; let $\ell(D)$ be the dimension of that space. Note that $\ell(D) = 0$ whenever $\deg(D) < 0$. The main theorem governing $\ell(D)$ is the Riemann-Roch theorem, whose statement is the following.

Proposition 3 (Riemann-Roch theorem). For any divisor $D$ on $C$,

$$\ell(D) = \deg(D) + 1 - g + \ell(K - D).$$

The class group $\text{Cl}(C)$ is defined as the group of divisors of degree zero, modulo the subgroup of principal divisors; it can be identified with the $k$-rational points of a certain $g$-dimensional abelian variety $J$, the so-called Jacobian variety of $C$. Over a finite field, the order of $\text{Cl}(C)$ is closely related to the zeta function, by the following formula (for which see, e.g., [13, Section 14]).

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Proposition 4. Suppose $k = \mathbb{F}_q$; let $C_n$ denote the base change of $C$ to $\mathbb{F}_{q^n}$. Let $P(t)$ be the numerator of the zeta function of $C$. Then $\deg(P) = 2g$, and if we factor $P(t) = (1 - r_1 t) \cdots (1 - r_{2g} t)$ with $r_1, \ldots, r_{2g} \in \mathbb{C}$, then

$$\# \text{Cl}(C_n) = \prod_{i=1}^{2g} (1 - r_i^n).$$

For this reason, computing the order of $\text{Cl}(C)$ when $k$ is finite is key to our quantum algorithm for computing zeta functions. The order is further controlled by the Riemann hypothesis for curves (see [17, Chapter X] for a not-too-technical treatment).

Proposition 5. With notation as in Proposition 4, $|r_i| = q^{1/2}$ for $i = 1, \ldots, 2g$. In particular,

$$q^n - 2g q^{n/2} \leq \# \text{C}(\mathbb{F}_{q^n}) \leq q^n + 2g q^{n/2}$$

$$q^{ng/2}(\sqrt{q} - 1)^{ng} \leq \# \text{Cl}(C_n) \leq q^{ng/2}(\sqrt{q} + 1)^{ng}.$$ 

We will exploit the Riemann hypothesis via the following lemma.

Lemma 6. For $e$ a positive integer, the number of prime divisors of degree $e$ on $C$ is at least

$$\frac{1}{e}(q^e(1 - q^{-1}) - 4g q^{e/2}).$$

Proof. It suffices to count elements of $\text{C}(\mathbb{F}_{q^e})$, subtract elements of $\text{C}(\mathbb{F}_{q^i})$ for all proper divisors $i$ of $e$, and then divide by $e$. By Proposition 5 this count can be bounded below by

$$\frac{1}{e}(q^e - 2g q^{e/2} - \sum_{i < g, i \mid g} (q^i + 2g q^{i/2})).$$

If $e = 1$, there is no sum at right, so we obtain $q - 2gq^{1/2}$ as the lower bound, which implies the desired bound. If $e = 2$, the bound is

$$\frac{1}{2}(q^2 - 2gq - q - 2gq^{1/2}) \geq \frac{1}{2}(q^2(1 - q^{-1}) - 4gq).$$

Otherwise we may dominate $\sum_{i < g, i \mid g} q^i$ by $\sum_{i=1}^{e-2} q^i \leq q^{e-1}$, and we may dominate $\sum_{i < g, i \mid g} 2gq^{i/2}$ by $\sum_{i=1}^{\lfloor e/2 \rfloor} 2gq^{i/2} \leq 2gq^{e/2}$. This yields the lower bound

$$\frac{1}{e}(q^e - 2g q^{e/2} - q^{e-1} - 2g q^{e/2}) = \frac{1}{e}(q^e(1 - q^{-1}) - 4g q^{e/2}).$$
5 Representing elements of class groups

In the notation of the previous section, we collect here some observations about representing elements of Cl(C).

We first note that elements can be represented in a compact form. Let $U$ be a divisor with $\deg(U) = 1$. Given a divisor $D$ with $\deg(D) = 0$, we have by Riemann-Roch

$$\ell(D + mU) = m + 1 - g + \ell(K - D - mU) \geq m + 1 - g;$$

in particular, if $m \geq g$, then $\ell(D + mU) > 0$, so that $D + mU \sim E$ for some effective divisor $E$. In other words, for any fixed $m \geq g$, every element of Cl(C) can be represented as $E - mU$ for some effective divisor $E$ of degree $m$.

The representations of elements of Cl(C) in the form $E - gU$, for $E$ effective of degree $g$, are unique “generically” but not always; since we will need to generate random elements of Cl(C), it will be useful to have representations which are uniformly distributed across Cl(C). Namely, if $\deg(D) = 0$ and $m \geq 2g - 1$, we have $\deg(K - D - mU) < 0$, so Riemann-Roch yields $\ell(D + mU) = m + 1 - g$. In particular, if $k = \mathbb{F}_q$, then each element of Cl(C) is represented by exactly $q^{m+1-g}$ number of divisors of the form $E - mU$, for $E$ effective of degree $m$.

Finally, we note that in case there exists a rational point $O \in C(k)$, we can represent elements of Cl(C) in a canonical form. Namely, in this case, if $\deg(D) = 0$, then

$$\ell(D + (m - 1)(O)) \leq \ell(D + m(O))$$

$$= m + 1 - g + \ell(K - D - m(O))$$

$$\leq m + 1 - g + \ell(K - D - (m - 1)(O))$$

$$= \ell(D + (m - 1)(O)) + 1.$$ 

Hence if $m$ is the smallest nonnegative integer for which $\ell(D + m(O)) > 0$, then $m \leq g$ (as above) and $\ell(D + m(O)) = 1$. In other words, for this choice of $m$ (which depends on $D$), there is a unique effective divisor $E$ with $D + m(O) \sim E$.

6 Computing in class groups

We now make some remarks about the protocols we have in mind for inputting and computing on algebraic curves, starting with what constraints on these protocols are imposed by the demands of our algorithm. Note that we will make liberal use of factorization of monovariate polynomials over finite fields, so our algorithms will be Las Vegas rather than deterministic.

Let $C$ be a curve (which as usual is smooth, projective, and geometrically irreducible) of genus $g$ over $\mathbb{F}_q$; for $n$ a positive integer, let $C_n$ be the base change of $C$ to $\mathbb{F}_{q^n}$. For the proof of Theorem 1 we will need an algorithm to compute $\# \text{Cl}(C)$ in time polynomial in $g$ and $\log(q)$. Using Lemma 2 we see that it is enough to exhibit a Monte Carlo black box presentation with unique encodings of $\# \text{Cl}(C)$, of input length bounded by a polynomial in
that polynomiality can be measured in terms of \( d \) by Riemann-Roch, any curve of degree \( g \) admits a singular plane model of degree \( g \), so can be properly input into our algorithm.

We now proceed to describing our input protocol and the construction of the black box presentation of \( \# \text{Cl}(C) \), except for producing a generating set; we defer that construction to the next section. To begin with, we will input \( C \) by specifying a homogeneous polynomial in three variables over \( \mathbb{F}_q \) cutting out a possibly singular plane model of \( C \) within the projective plane \( \mathbb{P}^2 \), i.e., a projective, geometrically irreducible one-dimensional scheme \( C' \) whose normalization is isomorphic to \( C \). Let \( d \) be the degree of the polynomial; then by Plücker's adjunction formula, the genus \( g \) of \( C \) is at most \((d-1)(d-2)/2\). That is, \( g \) is bounded by a polynomial in \( d \). We will assume also conversely that \( d \) is bounded by a polynomial in \( g \), so that polynomiality can be measured in terms of \( d \) rather than \( g \). This is no real restriction: by Riemann-Roch, any curve of degree \( g \) admits a singular plane model of degree \( g \), so can be properly input into our algorithm.

We need to explicitly describe the singularities of \( C' \) and the sequence of blowups of \( \mathbb{P}^2 \) that resolves these singularities. Straightforward algorithms for doing this require passing to extensions of \( \mathbb{F}_q \) whose degree is not polynomial in the input length (e.g., an extension over which all singular points become rational). However, there exist methods that perform the resolution of singularities in polynomial time, e.g., that of Kozen [14]. Note that the number of \( \mathbb{F}_q \)-rational points of \( C \) lying over singular points of \( C' \) is at most \((d-1)(d-2)/2\), since each one contributes at least one to the discrepancy between the Plücker bound and \( g \).

Put \( m = \lceil 2\log_q(d) \rceil \). Since there are at most \((d-1)(d-2)/2\) geometric points of \( C \) lying above singular points on \( C' \), we can draw an \( \mathbb{F}_q \)-rational line in \( \mathbb{P}^2 \) not meeting any of the singular points. Pick such a line, let \( F \) be the divisor in which the line meets \( C' \), and choose an \( \mathbb{F}_q \)-point \( O \) of \( F \); then \( O \) is defined over \( \mathbb{F}_{q^n} \) for some \( n \leq d \).

The key to constructing a black box presentation of \( \text{Cl}(C) \) is that the Riemann-Roch theorem on \( C_{mn} \) can be made (Las Vegas) polynomial time effective; in other words, given a divisor \( D \) on \( C_{mn} \), one can efficiently test functions for membership in \( L(D) \), write down a basis of \( L(D) \), and express elements of \( L(D) \) as linear combinations of that basis. See for instance Huang and Ierardi [12 §2] for an explicit construction; see also Volcheck [25, 26] for a somewhat more practical construction. (Note that Huang and Ierardi assume that all singular points are rational, but they also point out that this restriction is only needed to ensure that resolution of singularities can be performed efficiently. Thanks to the argument of Kozen from [14], this restriction can be lifted.)

Let \( S \) be the set of effective divisors \( E \) on \( C_{mn} \), with \( \deg(E) \leq g \) and \( \ell(E) = 1 \), represented as bit strings by listing the \( \mathbb{F}_q \)-points on \( E \) (on the blowup of \( \mathbb{P}^2 \) chosen to resolve the singularities of \( C' \)). Then given a divisor \( D \) of degree 0, we can describe a reduction procedure to produce \( E \in S \) with \( D \sim E - \deg(E)(O) \) as follows. Apply effective Riemann-Roch to produce \( E_0 \) of degree \( g \) with \( E_0 \sim D + g(O) \). Then repeatedly apply effective Riemann-Roch to find divisors \( E_1, E_2, \ldots \) with \( \deg(E_i) = g - i \) and \( E_i - (g - i)(O) \sim E_{i+1} - (g - i - 1)(O) \), until it is no longer possible to do so. If this stops at \( E_i \), then \( E_i \in S \) and \( E_i - (g - i)(O) \sim D \).
To add \(D_1, D_2 \in S\), we may apply the reduction procedure to \(D_1 + D_2 - \deg(D_1 + D_2)(O)\). To negate \(D \in S\), we may apply the reduction procedure to \(-D + \deg(D)(O)\). Hence we have produced a black box presentation with unique encodings \(f : S \to \text{Cl}(C_{mn})\), modulo the problem of exhibiting a generating set. We discuss generating sets in the next section.

7 Finding generators of class groups

With notation as in the previous section, let \(T\) be the subset of \(S\) corresponding to elements of \(\text{Cl}(C)\). In order to have a black box presentation with unique encodings \(f : T \to \text{Cl}(C)\), so that we can apply Watrous’s algorithm to compute \(\#\text{Cl}(C)\), we need to exhibit with high probability a subset of \(T\) which generates \(\text{Cl}(C)\); to do this provably (without too much headache), we will have to assume that \(q\) is “not too small” compared to \(g\). It may be possible to lift this restriction with an even more elaborate argument than the already involved procedure given below.

We first observe that it suffices to somehow generate uniformly random elements of \(\text{Cl}(C)\).

**Lemma 7.** Let \(G\) be a finite abelian group of order \(\leq 2^h\). Then for any nonnegative integer \(i\), if one chooses \(h + i\) elements of \(G\) uniformly at random (with replacement), the probability that the chosen elements generate \(G\) is at least \(1 - 2^{-i}\).

**Proof.** As stated, this is [16, Theorem D.1]; the argument therein is due to Pak [19], [20]. (Roughly, one checks that the probability is minimized by elementary 2-groups, then verifies the bound explicitly in that case.) An older but weaker result in the same spirit (which only yields the desired probability \(1 - 2^{-i}\) after sampling on the order of \(2h + i\) elements, rather than \(m + i\)) is due to Erdős and Rényi [5, Theorem 1].

By a \(b\)-uniform oracle on a finite set \(V\), we will mean an oracle which either fails to return an answer with probability at most \(1/4\), or returns an element of \(V\) according to a probability distribution \(p : S \to [0,1]\) such that for any \(x, y \in V\), \(p(x) \leq bp(y)\). (The constant \(1/4\) is chosen merely for definiteness; as in Section 2, there is no harm in replacing \(1/4\) by any other fixed constant between 0 and 1.)

**Lemma 8.** Given a positive integer \(e\) such that \(16g < q^{e/2}\), let \(V\) be the set of prime divisors on \(C\) of degree \(e\). Then there exists a \((1 + (2g - 2 + d)/e)\)-uniform oracle on \(V\), running in time polynomial in \(g\) and \(\log(q)\).

**Proof.** Put \(j = \lceil(2g - 1 + e)/d\rceil\). Consider an oracle that performs the following operation: select a random homogeneous polynomial over \(\mathbb{F}_q\) of degree \(j\), then extract uniformly at random an \(\mathbb{F}_q\)-rational point of \(C\) on which this polynomial vanishes, and return the divisor consisting of the Galois orbit of that point. (Here failure occur if there is no such point, if the chosen polynomial restricts to zero on \(C\), or if Las Vegas univariate polynomial factorization fails.)
To analyze this oracle, we first note that the homogeneous polynomials of degree \( j \) give rise to \( q^{jd+1-g} \) distinct functions on \( C \), by Riemann-Roch (and each occurs the same number of times). Also by Riemann-Roch, each prime divisor \( E \) of degree \( e \) occurs in the zero locus of \( q^{jd-e+1-g} \) such functions: namely, if \( F \) is the divisor along which \( C \) meets some line, we have \( \ell(jF - E) = jd - e + 1 - g + \ell(K - jF + E) = jd - e + 1 - g \) since \( \deg(K - jF + E) = 2g - 2 - jd + e < 0 \).

Note that each nonzero homogeneous polynomial of degree \( j \) can give rise to at most \( \lfloor jd/e \rfloor \) distinct divisors. This means that on one hand, the ratio between the probabilities of producing any two prime divisors of degree \( e \) is at most \( \lfloor jd/e \rfloor \leq \frac{1}{e} \frac{q^e(1 - q^{-1}) - 4gq^{e/2}}{q^{jd+1-g}} > \frac{1}{e} \frac{q^e(1 - q^{-1}) - 4gq^{e/2}}{q^e} \geq \frac{1}{4} \frac{q^e - 8gq^{e/2}}{q^e} > \frac{1}{8e} \).

With \( 1024e^2 \) invocations of this oracle (as in Section 2), we can boost this probability to \( 3/4 \), yielding the desired result.

Note that one cannot state the previous lemma as written without some lower bound on \( q \) with respect to \( g \); otherwise it might happen that \( V \) is empty, and one certainly cannot construct the desired oracle in that case! This complication is the reason we will be limited to the case where \( q \) is “not too small” below.

Next, we give a “simulation” conversion from a \( b \)-uniform oracle into a 1-uniform oracle. (The “simulation” qualifier refers to the fact that one must explicitly know the probability distribution on the initial oracle, which is too strong an assumption to make in practice.)

**Lemma 9.** Suppose we are given a \( b \)-uniform oracle on a finite set \( V \) with known distribution and error probability. Then we can construct a 1-uniform oracle on \( V \) requiring at most \( 16b^2 \) invocations of the initial oracle.

**Proof.** Let \( p : V \rightarrow [0,1] \) be the probability distribution of the initial oracle, and put \( p_0 = \min_{x \in V} \{ p(x) \} \); note that

\[
p_0 \geq \frac{1}{b\#V}
\]

since the initial oracle is \( b \)-uniform. Consider the following operation: invoke the initial oracle once to produce \( x \), then return \( x \) with probability \( p_0/p(x) \) and fail otherwise. This operation is equally likely to return any element of \( V \), and succeeds with probability \( p_0\#V \geq 1/b \). Performing the operation \( 16b^2 \) times (as in Section 2) gives a new oracle with failure probability at most \( 1/4 \), as desired. \( \square \)
We now put together the previous lemmas. It should be cautioned that the awkward intricacy of the resulting Lemma 10 is caused by our desire to have a fully unconditional complexity analysis; in practice, one is quite likely to obtain a generating set by selecting divisors by any reasonably arbitrary process!

**Lemma 10.** Under the assumption $16g < q^{1/2}$, there exists a Monte Carlo algorithm that produces a subset of $T$ generating $\text{Cl}(C)$ in time polynomial in $g, \log(q)$.

**Proof.** Put $h = \lceil \log_2(q^{g/2}(\sqrt{q} + 1)^g) \rceil$, so that $\# \text{Cl}(C) \leq 2^h$ by Proposition 5. Put $N = \lceil (1 + (2g - 2 + d)/e) \rceil$. By repeated use of Lemma 8 together with Lemma 9 we can produce, for each of $i = 1, \ldots, 2g + 1$, a list of $32N^2(2g - 1)^2(h + 3)$ prime divisors of degree $i$, each produced by an $N$-uniform oracle. Moreover, we can do this with overall probability of failure at most $1/16$.

Apply Lemma 8 to produce a divisor $U$ of degree 1, then convert each divisor $E$ in the list into an element of $T$ by reducing $E - \deg(E)U$ via effective Riemann-Roch. We now verify that the resulting elements of $\text{Cl}(C)$ generate $\text{Cl}(C)$ with probability at least $3/4$ by a “simulation” argument; namely, we exhibit the existence of another random process which necessarily produces a sublist of our given list, but which also produces a generating set for $\text{Cl}(C)$ with probability at least $3/4$.

In this context, we may assume that we know the distribution of the oracle produced by Lemma 8. (In the context of constructing the algorithm, we cannot use this knowledge, as it would amount to already knowing the zeta function of $C$. The point is that we do not use the information to perform any algorithmic steps, only to verify the error bound.) By Lemma 9 we may then extract from the given data a list of $(2g - 1)(h + 3)$ prime divisors of degree $i$, with failure probability at most $1/16$. (The factor of $16N^2$ is shed in the application of Lemma 9; shedding the factor of $2(2g - 1)$ allows us to shrink the failure probability to $1/16^{2g-1}$, so that the combined failure probability after producing all $2g - 1$ lists is at most $1/16$.)

From these new lists, we can in turn simulate the uniform random choice of $h + 3$ divisors of degree $2g - 1$. We do this assuming knowledge of the number of prime divisors of degrees $1, \ldots, 2g - 1$ (again, this amounts to knowing the desired zeta function, but this is okay for proving an error bound). With that knowledge, we may choose a “shape” of a degree $2g - 1$ divisor (i.e., the information of how many prime divisors occur with a given degree and multiplicity) according to the distribution which is uniform for individual divisors. (That is, each shape has probability proportional to the number of divisors taking that shape.) Given a shape, we may then read off from our lists uniformly random prime divisors of the appropriate lengths; we cannot use more than $2g - 1$ divisors of any one length at a time, so we have enough data to do this $h + 3$ times (with no additional failure probability at this step).

Finally, with $h + 3$ uniformly random divisors of degree $2g - 1$ in hand, we obtain by reduction $h + 3$ uniformly random elements of $\text{Cl}(C)$ (by the calculations of Section 5). By Lemma 7 these generate $\text{Cl}(C)$ with probability at least $1 - 1/8$. Since the divisors we produced were synthesized from the original list we produced, that list also generates $\text{Cl}(C)$.
with probability at least $1 - 1/8$. Totaling the failure and error probabilities yields an error probability in the Monte Carlo algorithm of $1/4$, as desired. (Note that the only step which is Monte Carlo rather than Las Vegas is the last one, since we do not check whether the random elements we produced actually do generate $\text{Cl}(C)$.)

We now may combine all of our efforts so far to obtain the following result.

**Proposition 11.** For $e$ such that $16g < q^{e/2}$, there exists a quantum algorithm to compute $\# \text{Cl}(C_e)$ in time polynomial in $g, \log(q), e$.

**Proof.** The construction of the previous section exhibits a black box presentation with unique encodings for $\text{Cl}(C_e)$, minus the construction of a set of generators; these are furnished by Lemma 10. Now Lemma 2 applies to yield the desired algorithm. \qed

## 8 Computing the zeta function

Retain notation as in Section 6. By Proposition 11, we can exhibit a quantum algorithm to compute the order of the group $\# \text{Cl}(C_n)$ in time polynomial in $g, \log(q), n$, as long as $16g < q^n/2$. With this quantum input in hand, we now establish Theorem 1.

**Proof of Theorem 1.** We first proceed under the assumption that $16g < q^{1/2}$, so that we may apply Proposition 11 for any $e$. Note that this assumption only intervenes via the invocation of Proposition 11; if one were to prove a form of that proposition without the lower bound on $q$, this restriction would drop out of the proof of Theorem 1.

Recall that by the Weil conjectures (see Proposition 5 and also [9, Appendix C]), we can factor $P(t)$ over $\mathbb{C}$ as

$$(1 - r_1 t) \cdots (1 - r_{2g} t),$$

where each $r_i$ is an algebraic integer of absolute value $q^{1/2}$, and $r_i r_{g+i} = q$ for $i = 1, \ldots, g$. Write $P(t) = a_0 + a_1 t + \cdots + a_{2g} t^{2g}$ with $a_0 = 1$; then the symmetry $r_i r_{g+i} = q$ implies that $a_{g+i} = q^i a_{g-i}$ for $i = 1, \ldots, 2g$, so to determine $P(t)$ it is enough to determine the integers $a_1, \ldots, a_g$.

As noted earlier (Proposition 4), we then have

$$\# \text{Cl}(C_n) = \prod_{i=1}^{2g} (1 - r_i^n) = q^{gn} \prod_{i=1}^{2g} (1 - r_i^{-n}).$$

Put

$$c_n = q^{-gn} \# \text{Cl}(C_n), \quad s_n = q^{-n} \frac{1}{n} \sum_{i=1}^{2g} r_i^n = \frac{1}{n} \sum_{i=1}^{2g} r_i^{-n},$$

then we can write

$$-\frac{\log c_n}{n} = \sum_{j=1}^{\infty} s_{nj}.$$
By the Newton-Girard formulae,

\[ nq^n s_n + a_1(n-1)q^{n-1}s_{n-1} + \cdots + a_{n-1}qs_1 + na_n = 0 \quad (n = 1, \ldots, g); \]

in particular, it is enough to determine \( s_1, \ldots, s_g \), as we can then recover \( a_1, \ldots, a_g \).

Using Proposition 11, we can compute \( c_n \) in suitable time for \( n = 1, \ldots, m \) with \( m = \max\{18, 2g\} \). We can then compute \( s_1, \ldots, s_g \) exactly as follows. Suppose \( n \leq g \) and that \( s_i \) has been computed exactly for \( i = 1, \ldots, n-1 \). By the Newton-Girard formulae, the residue modulo \( n \) of the integer \( nq^n s_n \) is determined by \( s_1, \ldots, s_{n-1} \). Hence we can recover the exact value of \( s_n \) if we can compute \( q^n s_n \) to within an error of less than 0.05.

Let \( \mu(n) \) denote the Möbius function, put \( k = \lfloor m/n \rfloor \), and compute

\[ q^ne_n = \sum_{i=1}^{k} -q^n \mu(i) \frac{\log c_{ni}}{ni} = q^n s_n + \sum_{j=k+1}^{\infty} q^n a_{n,j} s_{nj} \]

to an error of less than 0.005. Here

\[ a_{n,j} = \sum_{1 \leq i \leq k, i|j} \mu(i) \]

is an integer of absolute value at most \( k \), so

\[ \left| \sum_{j>k} q^n a_{n,j} s_{nj} \right| \leq q^n \sum_{j=k+1}^{\infty} \frac{2gkq^{-nj/2}}{nj} \leq q^n \frac{2gk}{n} \sum_{j=k+1}^{\infty} \frac{q^{-nj/2}}{k+1} = \frac{2g q^{-n(k-1)/2}}{n} \frac{q^{-n/2}}{1-q^{-n/2}} \leq \frac{(k+1)q^{-n(k-1)/2}}{1-q^{-n/2}}. \]

This last expression is less than 0.495 if \( k \geq k_0 \) and \( q^n \geq q_0 \) for each of

\[ (k_0, q_0) \in \{(2, 50), (3, 14), (4, 7), (5, 5), (6, 4), (8, 3), (15, 2)\}. \]

Note that \( 18 \geq (k_0 + 1) \log_2(q_0) \) for each pair \((k_0, q_0)\) in the above list. Since \( m \geq 18 \) and \( q \geq 2 \), for any pair \((k, n)\) with \( k \geq 2 \) and \( k = \lfloor m/n \rfloor \), we then have \( k \geq k_0 \) and \( q^n \geq q_0 \) for some pair \((k_0, q_0)\). Thus the computed value of \( q^ne_n \) differs from \( q^n s_n \) by less than 0.5, so we may determine \( s_n \) exactly. We may thus recover the zeta function in this fashion.

To recap, we have proved that we can recover the zeta function of \( C \) provided that \( 16g < q^{1/2} \); it remains to relax this restriction. Given arbitrary \( g \) and \( q \), choose \( m_1, m_2 \) subject to the following conditions.
• $m_1 < m_2$.

• For $i = 1, 2$, $m_i$ is prime and $m_i - 1$ is divisible by some prime greater than $2g$.

• $16g < q^{m_1/2}$.

The existence of such $m_1, m_2$ of size bounded by a polynomial in $g, \log(q)$ is guaranteed, e.g., by a theorem of Harman [8, Theorem 1.2], which asserts that for any fixed $\theta \leq 0.610$, there exist effectively computable constants $\delta > 0$ and $x_0 \in \mathbb{R}$ such that for $x \geq x_0$, there are at least $\delta x/\log(x)$ primes $p \in \{1, \ldots, x\}$ such that $p - 1$ has greatest prime factor bigger than $x^\theta$. (Many results of this ilk exist in the analytic number theory literature, but the effective computability of the constants seems to be new to [8].)

Apply the previous argument to compute the zeta functions of $C_{m_1}, C_{m_2}$. We thus have the lists $r_1^{m_1}, \ldots, r_2^{m_2}$ and $r_1^{m_2}, \ldots, r_2^{m_2}$. By the construction of $m_1$ and $m_2$, the field extension $\mathbb{Q}(r_1, \ldots, r_{2g})$ cannot contain a nontrivial $m_1$-st or $m_2$-nd root of unity (else such a root of unity would generate a field whose degree contains a prime factor greater than $2g$, whereas the degree of $\mathbb{Q}(r_1, \ldots, r_{2g})$ divides $(2g)!$). Thus we have $(r_j^{m_1})^{m_2} = (r_j^{m_2})^{m_1}$ if and only if $r_j = r_l$.

If we now pick out an element $A$ of the first list, there is only one value (possibly repeated) $B$ occurring in the second list with $A^{m_2} = B^{m_1}$. We can thus unambiguously (up to interchanging identical values) pair off each $r_j^{m_1}$ with its corresponding $r_j^{m_2}$, and then recover the $r_j$. This completes the proof.

9 Cyclic resultants

The above argument can also be described as follows. Given a polynomial $P(t)$ with roots $r_1, \ldots, r_d$, the $m$-th cyclic resultant of $P(t)$ is defined as

$$\text{Res}(P(t), t^m - 1) = \prod_{i=1}^{d}(r_i^m - 1).$$

These arise in a number of applications; see [10] for further discussion. A theorem of Fried [6] asserts that if $P(t)$ has even degree and is reciprocal (i.e., $P(t) = t^dP(1/t)$), then $P$ is uniquely determined by its sequence of cyclic resultants. This is precisely the situation in which we are in, which is not surprising: Fried arrived at this situation by counting fixed points of the powers of an endomorphism of a topological torus in terms of the Lefschetz trace formula on cohomology, and we are doing the same with the Frobenius endomorphism on an abelian variety.

Unfortunately, Fried’s theorem does not give an effective bound on the number of cyclic resultants needed to recover $P(t)$, nor an algorithm for doing so. A conjecture of Sturmfels and Zworski asserts that the first $d/2 + 1$ cyclic resultants should suffice for $P$ generic (if $P$ is not reciprocal, they conjecture that generically $d + 1$ resultants suffice). A theorem of Hillar and Levine [11] states that the first $2^{d+1}$ cyclic resultants determine $P$; what we have
done is show that for very special reciprocal $P$, we can explicitly recover $P$ from only $d$ cyclic resultants.

Whether one can bring $d$ down any closer to the theoretical lower bound $d/2$, i.e., whether one can compute the zeta function of a curve of genus $g$ using fewer than $2g$ calls to the quantum oracle, is a tantalizing question. Our current approach fails to accomplish this because, for instance, we recover $s_g$ from $s_g + s_{2g} + \cdots$, and the term $s_{2g}$ is of exactly the same order as the size of the interval in which we must bound $s_g$ in order to determine it exactly, namely $q^{-g}$. Thus breaking the $2g$ barrier would seem to require a fundamental new idea.

Incidentally, this barrier may be of interest even in the absence of quantum computers, as it may be possible to use the proof of Theorem 1 to obtain a probabilistic polynomial time algorithm for verifying the zeta function of a curve, which verifies the orders of the first few Jacobian groups. Unfortunately, while it is easy to efficiently verify the exponent of a black box group, it is less clear how to efficiently verify its order. (Thanks to Dan Bernstein for this remark.)

10 Further comments

It should be noted that the problem of giving an efficient quantum algorithm to compute the zeta function of an arbitrary variety $X$ over a finite field $\mathbb{F}_q$ is now effectively solved in dimension $\leq 1$. For $\dim(X) = 0$, i.e., for $X$ a finite union of closed points, computing the zeta function of $X$ amounts to finding the distinct-degree factorization of a monovariate polynomial, so this can even be done in deterministic polynomial time. For $\dim(X) = 1$, if $X$ is geometrically irreducible, one can find the unique smooth projective curve $C$ birational to $X$, compute its zeta function, then express the discrepancy between the zeta functions of $X$ and $C$ in terms of the zeta functions of zero-dimensional varieties. If $X$ is not geometrically irreducible, one can split it over an extension of degree at most its genus and proceed as above.

However, considering varieties of a fixed higher dimension seems to pose more serious challenges. (Allowing the dimension to vary brings us dangerously close to the $P = NP$ problem, which we prefer to stay well clear of.) Things are well understood, at least theoretically, if the characteristic $p$ of $\mathbb{F}_q$ is fixed; as noted earlier, Lauder and Wan [15] give a deterministic algorithm for computing the zeta function of a singular hypersurface of degree $d$ in $\mathbb{P}^n$, in time polynomial in $p, \log_p(q), d$. (Again, one can reduce to this case by induction on dimension, since any irreducible variety is birational to a hypersurface.)

On the other hand, if $p$ is allowed to vary, then even the following question remains somewhat mysterious, except in some cases related to modular forms (as demonstrated by ongoing work of Bas Edixhoven and his collaborators on efficient computation of the values of Ramanujan’s $\tau$ function).

Question 12. Let $X$ be a fixed variety over $\mathbb{Q}$ (or better, fix a model over $\mathbb{Z}$) of dimension greater than 1. Does there necessarily exist a deterministic, random, or quantum polynomial
time algorithm in \( \log(p) \) to determine the zeta function of \( X \) over \( \mathbb{F}_p \), for \( p \) a varying prime?

For \( X \) of dimension 1, Schoof-Pila gives a deterministic affirmative answer. However, the approach used there breaks down in higher dimensions; briefly put, there is no “geometric” realization of the higher étale cohomology groups analogous to the realization of the first étale cohomology group in the Tate module of the Jacobian. The work of Edixhoven suggests such a realization in case the relevant cohomology group is “modular”, by comparing the higher étale cohomologies to first étale cohomologies on other spaces. However, already the case when \( X \) is a (fixed) surface of general type, without any special structure, seems to require a new idea.

We also point out a related but markedly different investigation initiated by van Dam [24], who looks for “efficient” quantum circuits for computing the zeta functions of varieties, mostly in dimensions greater than 1. The emphasis there is on directly realizing Frobenius eigenvalues within easy-to-construct Hermitian operators; this is done in [24] for some diagonal hypersurfaces (where the relevant eigenvalues are Gauss sums) but seems quite difficult in general.

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