Left Adjoint for Generalized Multicategories

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The author was partially supported by a grant from the Simons Foundation (Collaboration Grant for Mathematicians number 209092 to Anthony Elmendorf).

Abstract. We construct generalized multicategories associated to an arbitrary operad in $\text{Cat}$ that is $\Sigma$-free. The construction generalizes the passage to symmetric multicategories from permutative categories, which is the case when the operad is the categorical version of the Barratt-Eccles operad. The main theorem is that there is an adjoint pair relating algebras over the operad to this sort of generalized multicategory. The construction is flexible enough to allow for equivariant generalizations.
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Part 1

Introduction and constructions
1. Introduction

This paper describes a new construction for multicategories that generalizes the notions of both symmetric and non-symmetric multicategory in the literature. While not as general as Cruttwell and Shulman’s construction in [1], we are able to give a left adjoint construction to algebras over the operad that controls the construction. This generalizes the free strict monoidal category construction for non-symmetric multicategories and the free permutative category construction for symmetric multicategories. The only hypothesis necessary for the new construction and the left adjoint is the Σ-free property on the controlling categorical operad; that is, the nth category $D_n$ of the operad must have a $\Sigma_n$-action that is free. The special case in which $D_n$ is the discrete category $\Sigma_n$ gives the usual construction of non-symmetric multicategories, and the special case of the categorical version of the Barratt-Eccles operad gives the usual construction of symmetric multicategories.

Our construction is motivated by Leinster’s work in [4], where he showed that monads on certain categories such as sets or categories, subject to a few hypotheses, could give rise to generalized multicategories; however, he did not fit symmetric multicategories into his framework. It is also motivated by the work of Guillou and May in [3], in which they described how algebras over categorical operads in an equivariant setting could give rise to genuine equivariant spectra. However, they did not consider any associated theory of multicategories. Such a theory would be a generalization of the theory given in [2] by Mandell and the author in the non-equivariant setting, in which we showed that the spectrum determined by a permutative category depends only on its underlying (symmetric) multicategory. The present construction does generalize Leinster’s to the symmetric setting, as well as accounting for the context considered by Guillou and May; it also allows for many other examples. Applications to stable homotopy are a subject for future work.

The crucial feature that we exploit in all our constructions is is the presence of an operad of categories that is Σ-free, meaning that the groups $\Sigma_n$ not only act, but act freely on the associated components of the operad. This freeness is critical: it explains the failure of Leinster’s construction when considering strictly commutative monoidal categories, and makes clear that the most natural generalization of his construction to the symmetric case is to permutative categories, since they are precisely the algebras over the Σ-free categorical Barratt-Eccles operad. We make heavy use of the Σ-free hypothesis, but need to assume nothing more about our categorical operad. In particular, we make no use of other properties enjoyed by the Barratt-Eccles operad, such as its $E_\infty$ structure.

Our main results are as follows. Given an operad $D$ of categories that is Σ-free, we construct a category of generalized multicategories associated to the monad $D$ given by the operad $D$. We call these $D$-multicategories. We show that every $D$-algebra has an underlying $D$-multicategory, and that there is a left adjoint to this forgetful functor.
back to \( \mathbb{D} \)-algebras. In the case in which the operad \( \mathcal{D} \) is the categorical Barratt-Eccles operad, the algebras are permutative categories, which in turn all have underlying (symmetric) multicategories; our construction recovers these multicategories and the left adjoint back to permutative categories.

The most novel feature of the paper is a construction that comes within a whisker of being the left adjoint, but fails on exactly one count: the unit map fails to preserve the presheaf structure we impose on the morphisms of our sort of multicategory. We call this the “provisional” left adjoint, since it does most of the heavy lifting in the construction of the actual left adjoint, which is a coequalizer (really a coend) constructed from the provisional version.

The paper is organized as follows. Since the proofs consist almost entirely of lengthy diagram chases, these are deferred to the second part of the paper, which can be ignored by the casual reader. The definition and main constructions, as well as the statements of the main results, are in the first part, consisting of the first six sections. However, these sections, though rather short, contain no proofs at all. The diligent reader will find these by perusing the corresponding sections in the second part.

The results of this paper should be considered as proof of concept theorems, since there are many other questions about the construction that this paper does not address. Perhaps chief among these is the question of producing spectra, especially equivariant spectra, from particular instances of the construction. However, this will definitely require more hypotheses than those we use in this paper, which is long enough as is without investigation of further questions.

It is a pleasure to acknowledge conversations with Mike Shulman on these topics. Conversations with Anna Marie Bohmann and Bjørn Dundas have also been stimulating and useful.

2. Preliminary Constructions

We begin by describing some preliminary constructions which are necessary for our definition of \( \mathbb{D} \)-multicategories, deferring all proofs to later sections. We first need some results about categorical operads, and the consequences we will need that follow from our \( \Sigma \)-free hypothesis.

Given an operad \( \mathcal{D} \) in \( \mathbf{Cat} \), we can form the associated monad \( \mathbb{D} \) in \( \mathbf{Cat} \) by means of the standard construction
\[
\mathbb{D} \mathcal{C} := \coprod_{n \geq 0} \mathcal{D}_n \times_{\Sigma_n} \mathcal{C}^n.
\]
It will be most useful, however, to break this monad up into pieces, using the nerve construction \( N : \mathbf{Cat} \to \mathbf{Ssets} \), where the target is the category of simplicial sets.
Since $N$ is a right adjoint, it preserves products and therefore operad structure, so
$ND$ is an operad of simplicial sets, and since evaluation at any simplicial degree is also
a right adjoint, we get the sequence of (set) operads $ND_0, ND_1, ND_2, \ldots$. Associated
to each we have a monad $D_0, D_1, D_2, \ldots$ in $\text{Set}$. Now given a (small) category $C$, we
also denote by $C_0, C_1, C_2, \ldots$ the simplices of the nerve of $C$, so in particular $C_0$ is the
set of objects of $C$, and $C_1$ its set of morphisms. It is now straightforward to see that
the objects of $D_0 C$ are $D_0 C_0$ and the morphisms are $D_1 C_1$; higher nerve degrees are also
given by $D_n C_n$. It will be useful to consider mixed uses of these monads, however; for
example, we will need to consider both $D_1 C_0$ and $D_0 C_1$. We will make much use of the
following as a calculational tool; its proof makes crucial use of the $\Sigma$-free hypothesis.

**Lemma 2.1.** If $D$ is $\Sigma$-free, the monads $\{D_0, D_1\}$ give a category object in monads
on $\text{Set}$, which in turn determines the remaining monads $D_2, D_3, \ldots$.

We also need the fact that all these monads are what Leinster calls Cartesian. We recall the definition.

**Definition 2.2.** A monad $J$ is Cartesian if it satisfies the following properties:

1. $J$ preserves pullbacks.
2. The naturality diagrams for the product $\mu : J^2 \to J$ and the unit $\eta : \text{id} \to J$
   are all pullbacks.

Our first basic result is the following theorem; note that since the categorical
operad $D$ is assumed to be $\Sigma$-free, so are all the set operads $ND_n$.

**Theorem 2.3.** The monad associated to a $\Sigma$-free operad in $\text{Set}$ is Cartesian.
Consequently, all the monads $D_n$ are Cartesian.

Our construction of generalized multicategories associated to the monad $D$
involved the use of presheaves. We therefore need to introduce our conventions and
notation for presheaves, and give the fundamental lemmas about them that will be
used in the rest of the paper.

When considering a fiber product $X \times_{Z} Y$, we will consider the structure map
from $X$ to $Z$ to be of “source” type, and the structure map from $Y$ to $Z$ to be of
“target” type, where the use of “source” and “target” should be clear from context.
The motto is “source to the right, target to the left.” Our motivating example is to
be able to consider the composition map in a small category $C$ to be given by a map

$$\gamma : C_1 \times_{C_0} C_1 \to C_1;$$

the structure map for the first $C_1$ is the source map, and for the second one the target
map. With this in mind, we have the following characterization of presheaves.
DEFINITION 2.4. Let $X$ be a set, and $\mathcal{C}$ a small category. A presheaf structure on $X$ consists of a map $\varepsilon : X \to \mathcal{C}_0$, together with an action map

$$\xi : X \times_{\mathcal{C}_0} \mathcal{C}_1 \to X.$$ 

Of course, by an action map we mean the usual coherence conditions of associativity and unit must be satisfied. Further, we consider $X \times_{\mathcal{C}_0} \mathcal{C}_1$ to have a structure map given by the source on $\mathcal{C}_1$, and we require the action to preserve structure maps in the sense that the diagram

$$
\begin{array}{ccc}
X \times_{\mathcal{C}_0} \mathcal{C}_1 & \xrightarrow{\xi} & X \\
\downarrow S & & \downarrow \varepsilon \\
\mathcal{C}_0 & \xrightarrow{\varepsilon} & X
\end{array}
$$

commutes; this is necessary in order to make sense of associativity. The fibers in $X$ over the elements of $\mathcal{C}_0$ then form the target objects of a functor from $\mathcal{C}^{\text{op}}$ to $\text{Set}$, and conversely, given such a functor, the disjoint union of the target objects has a presheaf structure as specified above.

The category theorists refer to the following concept as a “discrete fibration.” Since we will also refer to its dual concept, we will use alternate terminology.

DEFINITION 2.5. A functor $F : \mathcal{C} \to \mathcal{C}'$ is a target cover if, given $f' : a' \to b'$ in $\mathcal{C}'_1$ and $b \in \mathcal{C}_0$ such that $F(b) = b'$, there is a unique $f \in \mathcal{C}_1$ such that $F(f) = f'$ and $T(f) = b$, where $T$ refers to the target function.

An alternate way of describing this is to say that the following square is a pullback:

$$
\begin{array}{ccc}
\mathcal{C}_1 & \xrightarrow{T} & \mathcal{C}_0 \\
\downarrow F_1 & & \downarrow F_0 \\
\mathcal{C}'_1 & \xrightarrow{T} & \mathcal{C}'_0.
\end{array}
$$

Replacing $T$ with $S$ (the source map), we get the dual notion of a source cover. If a functor is both a source and target cover, we will say it is simply a cover.

The following theorem is our primary structural tool.

**Theorem 2.6.** Let $\mathbb{D}$ be the monad associated to a $\Sigma$-free $\text{Cat}$-operad. Then $\mathbb{D}$ preserves both target covers and source covers (and consequently covers.)

It will be surprisingly useful to apply this theorem to functors between discrete categories, which are all covers. However, our most fundamental use is to prove the following theorem, whose following corollary is basic to our definition of generalized multicategory. Again, $\mathbb{D}$ is the monad associated to a $\Sigma$-free $\text{Cat}$-operad.
Theorem 2.7. Let $X$ have the structure of presheaf over a category $C$ with structure map $\varepsilon : X \to C_0$. Then $D_0X$ naturally supports the structure of presheaf over $DC$ with structure map $D_0\varepsilon : D_0X \to D_0C_0$.

Corollary 2.8. If $X$ supports the structure of presheaf over $D^k(\ast)$, then $D_0X$ supports the structure of presheaf over $D^{k+1}(\ast)$.

3. The Definition of $D$-multicategories

We are now in a position to define generalized multicategories associated to a $\Sigma$-free operad $D$, which we call $D$-multicategories after the associated monad $D$. To define a $D$-multicategory $M$, we require the following data:

1. A set $M_0$ of objects.
2. A set $M_1$ of morphisms, which must come equipped with a specified presheaf structure over $D(\ast)$.
3. A target map $T : M_1 \to M_0$, which must be a presheaf map over the terminal functor $\varepsilon : D(\ast) \to \ast$.
4. A source map $S : M_1 \to D_0M_0$, which must be a map of presheaves over $D(\ast)$. The presheaf structure on $D_0M_0$ is given by Corollary 2.8.
5. A unit map $I : M_0 \to M_1$ which must be a presheaf map over the monadic unit map $\ast \to D(\ast)$.
6. A composition map, to be described in detail below.

It may be helpful at this point to explain how this works in the case of the categorical Barratt-Eccles operad, whose algebras are the permutative categories. Each permutative category has an underlying symmetric multicategory, which this formalism is designed to capture. The component categories of the operad are the categories $E\Sigma_n$, whose objects are the elements of $\Sigma_n$ and with exactly one morphism for each choice of source and target. The category $D(\ast)$ then has objects the natural numbers (including 0), and the only morphisms are elements of automorphism groups, which are isomorphic to $\Sigma_n$ for each $n$. The presheaf structure on the morphisms of a symmetric multicategory then consists of actions of the $\Sigma_n$ on the morphisms with sources lists of length $n$, with the action permuting the order in which the source objects are listed. Composition involves concatenation of lists. The monad $D_0$ associates to a set the set of lists of elements of the set; that is, it is the free monoid monad. It is now clear how $D_0X$ is a presheaf over $D(\ast)$ in this instance.
The only one of our general data still to be described explicitly is the composition map. We define the \textit{composables} $M_2$ as the pullback in the following diagram:

$$
\begin{array}{ccc}
M_2 & \to & M_1 \\
\downarrow S & & \downarrow S \\
\mathbb{D}_0 M_1 & \to & \mathbb{D}_0 M_0,
\end{array}
$$

so we can also write $M_2 \cong M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1$, with the maps $T$ and $S$ in the diagram given by projection to the first and second factors, respectively. Since Corollary 2.8 tells us that $\mathbb{D}_0 M_1$ is a presheaf over $\mathbb{D}^2(\ast)$, $M_2$ inherits that structure, and $T : M_2 \to M_1$, like $\mathbb{D}_0 T : \mathbb{D}_0 M_1 \to \mathbb{D}_0 M_0$, is a map of presheaves over $\mathbb{D} : \mathbb{D}^2(\ast) \to \mathbb{D}(\ast)$. The composition map we require as part of the data of the multicategory $M$ is a map

$$\gamma : M_2 \to M_1$$

that is a map of presheaves over the monad multiplication $\mu : \mathbb{D}^2(\ast) \to \mathbb{D}(\ast)$. This completes the specification of the data for a $\mathbb{D}$-multicategory $M$.

Of course, we also require properties for these data. For the data other than the composition map, these amount to the commutativity of the diagram

$$
\begin{array}{ccc}
\mathbb{D}_0 M_0 & \stackrel{S}{\to} & M_1 & \to & M_0 \\
\eta & \downarrow & T & \downarrow & = \\
\mathbb{D}_0 M_0 & \leftarrow & M_1 & \leftarrow & M_0,
\end{array}
$$

where $\eta$ is the unit for the monad $\mathbb{D}_0$.

The remaining properties involve the composition map $\gamma$. First, we require that it preserve sources and targets, in the sense that the two diagrams

$$
\begin{array}{ccc}
M_2 & \to & M_1 \\
\downarrow \gamma & & \downarrow T \\
M_1 & \to & M_0
\end{array} \quad \text{and} \quad
\begin{array}{ccc}
M_2 & \to & \mathbb{D}_0 M_1 \\
\downarrow \gamma & & \downarrow \mathbb{D}_0 S \\
M_1 & \to & \mathbb{D}_0 M_0
\end{array}
$$

must commute. Note that the second diagram captures the idea that sources of composites in a multicategory are given by concatenation of sources: this is the role of the monad multiplication map $\mu$ in the diagram.

Next, we require that $\gamma$ be unital, and we can write this most easily by using the expression $M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1$ for $M_2$. The unit conditions now can be expressed by
requiring that the diagram

\[ M_1 \cong M_0 \times_{M_0} M_1 \xrightarrow{I \times_{D_0} \eta} M_1 \times_{D_0} M_1 \xrightarrow{1 \times_{D_0} I} M_1 \times_{D_0} \eta \xrightarrow{\eta} M_0 \xrightarrow{\eta} (M_1 \times_{D_0} \eta) \xrightarrow{\gamma} M_1 \]

commute.

It remains to specify the meaning of associativity for \( \gamma \). To do so, we first introduce the pullback

\[ \begin{array}{ccc}
M_3 & \xrightarrow{T} & M_2 \\
S & & \downarrow{S} \\
D_0 M_2 & \xrightarrow{D_0 T} & D_0 M_1,
\end{array} \]

where we think of \( M_3 \) as giving the associativity data for \( M \). It is a presheaf over \( D^3(*) \) by construction, \( S : M_3 \to D_0 M_2 \) is a map of presheaves over \( D^3(*) \), and \( T : M_3 \to M_2 \) is a map of presheaves over the functor \( D^2 \varepsilon : D^3(*) \to D^2(*) \). We have two induced composition maps \( M_3 \to M_2 \) which we think of as composing in either the first two (target) slots, or the last two (source) slots. For composition in the target slots, the commutative diagram

\[ \begin{array}{ccc}
M_3 & \xrightarrow{T} & M_2 \xrightarrow{\gamma} M_1 \\
S & & \downarrow{S} \\
D_0 M_2 & \xrightarrow{D_0 T} & D_0 M_1 \\
D_0 S & & \downarrow{D_0 S} \\
D_0^2 M_1 & \xrightarrow{D_0^2 T} & D_0^2 M_0 \\
\mu & & \downarrow{\mu} \\
D_0 M_1 & \xrightarrow{D_0 T} & D_0 M_0
\end{array} \]

gives us the induced map \( \gamma_T : M_3 \to M_2 \), which the left vertical maps in the diagram tell us is a map of presheaves over the functor \( \mu : D^2(*) \to D^2(*) \). Next, the
4. The Underlying $\mathbb{D}$-multicategory of a $\mathbb{D}$-algebra

In this section we give the construction of the underlying $\mathbb{D}$-multicategory of a $\mathbb{D}$-algebra. We defer the proofs that it actually satisfies the necessary properties.

Suppose given a $\mathbb{D}$-algebra $\mathcal{C}$, so $\mathcal{C}$ is a category together with an action map $\xi : \mathbb{D}\mathcal{C} \to \mathcal{C}$. Since $\mathbb{D}$ determines (and is determined by) the category object $\{\mathbb{D}_0, \mathbb{D}_1\}$ in monads on $\textbf{Set}$, this is equivalent to having a $\mathbb{D}_0$-algebra structure $\xi_0 : \mathbb{D}_0\mathcal{C}_0 \to \mathcal{C}_0$ and a $\mathbb{D}_1$-algebra structure $\xi_1 : \mathbb{D}_1\mathcal{C}_1 \to \mathcal{C}_1$ which determine a functor. We use this structure to define an underlying $\mathbb{D}$-multicategory $\mathcal{U}\mathcal{C}$.

For the objects $(\mathcal{U}\mathcal{C})_0$, we just use the objects $\mathcal{C}_0$ of $\mathcal{C}$. For the morphisms, we exploit the fact that we have the action map $\xi_0 : \mathbb{D}_0\mathcal{C}_0 \to \mathcal{C}_0$ to define $(\mathcal{U}\mathcal{C})_1$ by means of the following pullback square, which also defines a comparison map $\kappa_1 : (\mathcal{U}\mathcal{C})_1 \to \mathcal{C}_1$:

$$
\begin{array}{ccc}
(\mathcal{U}\mathcal{C})_1 & \xrightarrow{\kappa_1} & \mathcal{C}_1 \\
\downarrow s & & \downarrow s \\
\mathbb{D}_0\mathcal{C}_0 & \xrightarrow{\xi_0} & \mathcal{C}_0.
\end{array}
$$

For the target map $T : (\mathcal{U}\mathcal{C})_1 \to (\mathcal{U}\mathcal{C})_0 = \mathcal{C}_0$, we compose $\kappa_1$ with the target map $T : \mathcal{C}_1 \to \mathcal{C}_0$ in $\mathcal{C}$. In the case of permutative categories, this definition simply
says that a morphism in the underlying multicategory consists of a morphism in the category together with a specified decomposition of the source as a direct sum.

We must provide a presheaf structure on \((UC)_1\) over \(D(\ast)\), and it is not merely the pullback of the presheaf structure on \(D_0M_0\), since that would not preserve the comparison map \(\kappa_1\). We exploit the following basic connection between presheaves and target covers, whose proof appears in Section 8.

**Theorem 4.1.** Let \(F : C \rightarrow C'\) be a target cover. Then a presheaf structure on \(X\) over \(C\) with structure map \(\varepsilon : X \rightarrow C_0\) is equivalent to a presheaf structure on \(X\) over \(C'\) with structure map \(\varepsilon' : X \rightarrow C'_0\) together with an explicit factorization \(\varepsilon' = F_0 \circ \varepsilon\).

The presheaf structure on \(UC_1\) can now be described using the characterization of presheaves from Theorem 4.1. We start with the discrete category \(C_0^\delta\) generated by the set \(C_0\), and observe that the terminal map \(\varepsilon : C_0^\delta \rightarrow \ast\) is a cover. Consequently, by Theorem 2.6 so is the functor \(D\varepsilon : D(C_0^\delta) \rightarrow D(\ast)\). In particular, a presheaf structure on \(UC_1\) over \(D(\ast)\) is equivalent to a presheaf structure over \(D(C_0^\delta)\), which we describe as follows. Note that the objects of \(D(C_0^\delta)\) are \(D_0C_0\) and the morphisms are \(D_1C_0\). Since \(UC_1 := C_1 \times_{C_0} D_0C_0\), we have

\[
UC_1 \times_{D_0C_0} D_1C_0 \cong C_1 \times_{C_0} D_1C_0.
\]

In the following composite defining the presheaf action map, we exploit the fact that the monad action is a functor; in particular, the diagram

\[
\begin{array}{ccc}
D_1C_1 & \xrightarrow{S\delta} & D_0C_0 \\
\downarrow{\xi_1} & & \downarrow{\xi_0} \\
C_1 & \xrightarrow{s} & C_0
\end{array}
\]

induces a map

\[
D_1C_1 \xrightarrow{(\xi_1, S\delta)} C_1 \times_{C_0} D_0C_0
\]

that preserves the structure map of target type to \(C_0\). We now express the presheaf action map on \(UC_1\) as the composite

\[
UC_1 \times_{D_0C_0} D_1C_0 \cong C_1 \times_{C_0} D_1C_0 \xrightarrow{1 \times \xi_1, S\delta} C_1 \times_{C_0} D_0C_0 \cong UC_1.
\]
For the identity map \((UC)_0 = C_0 \to (UC)_1\), we note that the diagram

\[
\begin{array}{ccc}
C_0 & \xrightarrow{I_C} & C_1 \\
\downarrow{\eta} & & \downarrow{S} \\
\mathbb{D}_0C_0 & \xrightarrow{\xi_0} & C_0
\end{array}
\]

commutes, since both composites coincide with the identity on \(C_0\). We get an induced map \(I_{UC} : C_0 \to C_1 \times_{C_0} \mathbb{D}_0C_0 \cong (UC)_1\).

Before constructing the composition map \(\gamma_{UC}\), we pause to introduce the comparison maps \(\kappa_n : (UC)_n \to C_n\); we will need the case \(n = 2\) for the definition of \(\gamma_{UC}\) and the case \(n = 3\) for the verification of associativity. Note first that since \(\{\mathbb{D}_0, \mathbb{D}_1\}\) is a category object in monads on \(\text{Set}\), we have a composite identity natural transformation \(I^n : \mathbb{D}_n \to \mathbb{D}_n\) for each \(n\). We assume for induction that the diagrams

\[
\begin{array}{ccc}
UC_n & \xrightarrow{T} & UC_{n-1} \\
\downarrow{\kappa_n} & & \downarrow{\kappa_{n-1}} \\
C_n & \xrightarrow{T} & C_{n-1}
\end{array}
\hspace{1cm}
\begin{array}{ccc}
UC_n & \xrightarrow{\kappa_n} & C_n \\
\downarrow{S} & & \downarrow{S} \\
\mathbb{D}_0UC_{n-1} \mathbb{D}_0\kappa_{n-1} \mathbb{D}_0C_{n-1} \mathbb{D}_n^{-1} \mathbb{D}_n^{-1} \xi_{n-1} \xi_{n-1} & \xrightarrow{T} & C_{n-1}
\end{array}
\]

both commute; this is true for \(n = 1\) with the convention that \(\kappa_0 = \text{id}_{C_0}\). Now assuming the previous \(\kappa_j\) have been defined for \(j \leq n\), and supposing \(n \geq 1\), we define \(\kappa_{n+1}\) by means of the following commutative diagram, which induces a map to the pullback \(C_{n+1} = C_n \times_{C_{n-1}} C_n\):

\[
\begin{array}{ccc}
UC_{n+1} & \xrightarrow{T} & UC_n & \xrightarrow{\kappa_n} & C_n \\
\downarrow{S} & & \downarrow{S} & & \downarrow{S} \\
\mathbb{D}_0UC_n & \xrightarrow{\mathbb{D}_0T} & \mathbb{D}_0UC_{n-1} \\
\downarrow{\mathbb{D}_0\kappa_n} & & \downarrow{\mathbb{D}_0\kappa_{n-1}} & & \downarrow{S} \\
\mathbb{D}_0C_n & \xrightarrow{\mathbb{D}_0T} & \mathbb{D}_0C_{n-1} \\
\downarrow{I^n} & & \downarrow{I^{n-1}} & & \downarrow{S} \\
\mathbb{D}_nC_n & \xrightarrow{T^2} & \mathbb{D}_n^{-1}C_{n-1} \\
\downarrow{\xi_n} & & \downarrow{\xi_{n-1}} & & \downarrow{\xi_{n-1}} \\
C_n & \xrightarrow{T} & C_{n-1}
\end{array}
\]

The right part of the diagram commutes by induction, and the left stack from top to bottom as follows: the top square defines \(UC_{n+1}\), the next one combines an inductive hypothesis with the fact that \(\mathbb{D}_0\) is Cartesian, the one below it follows from targets
of identities being the object, and the bottom because \( \xi : D C \to C \) is a functor. It is straightforward to see that the inductive hypotheses are preserved.

Now to define the composition map \( \gamma_{UC} : (UC)_2 \to (UC)_1 \), we use the following commutative diagram, which defines a map from \( (UC)_2 \) to \( C_1 \times_{C_0} D_0 C_0 = (UC)_1 \):

\[
\begin{array}{ccccccccc}
(UC)_2 & \xrightarrow{\kappa_2} & C_2 & \xrightarrow{\gamma_{UC}} & C_1 \\
S & & S \\
D_0(UC)_1 & \xrightarrow{D_0 \kappa_1} & D_0 C_1 & \xrightarrow{ID} & D_1 C_1 & \xrightarrow{\xi_1} & C_1 \\
D_0 \xi_0 & \xrightarrow{\xi_0} & D_0 C_0 & \xrightarrow{S^2} & D_0 C_0 & \xrightarrow{S} & D_0 C_0 \\
D_0 C_0 & \xrightarrow{\mu} & D_0 C_0 & \xrightarrow{\xi_0} & C_0.
\end{array}
\]

This completes the specification of the data for the underlying \( D \)-multicategory \( UC \).

5. The provisional left adjoint construction

In this section we introduce the construction of a \( D \)-algebra \( \hat{L}M \) associated to a given \( D \)-multicategory \( M \) that is almost, but not quite, a left adjoint to the forgetful functor. The construction is functorial, and there are unit and counit maps satisfying the triangle identities. The only flaw is that the unit map fails to preserve the presheaf structure. Our actual left adjoint, \( LM \), will be constructed as a quotient of \( \hat{L}M \), and will inherit all the left adjoint properties as well as satisfying the requirement that the unit actually be a map of \( D \)-multicategories.

The idea of \( \hat{L}M \) in the case of ordinary (symmetric) multicategories and their associated free permutative categories is that the objects of \( LM \) (and \( \hat{L}M \)) in this case are given by the free monoid on the objects of \( M \):

\[
\hat{L}M_0 = LM_0 := \coprod_{n \geq 0} M^n_0 = D_0 M_0.
\]

We adopt the principle that for any \( D \) that we are considering, we also want \( LM_0 := D_0 M_0 \). The problem is specifying the morphisms. In the case of classical symmetric multicategories, a morphism from one list \( \langle a_1, \ldots, a_m \rangle \) to another \( \langle b_1, \ldots, b_n \rangle \) is specified by a function \( f : \{1, \ldots, m\} \to \{1, \ldots, n\} \) together with an \( n \)-tuple of morphisms \( \phi_i_{i=1}^n \) of \( M \), where

\[
\phi_i : \langle a_j \rangle_{f(j) = i} \to b_i.
\]

The problem is that the entries in the list \( \langle a_j \rangle_{f(j) = i} \) are given in natural number order, and there is no way to capture this in our formalism. Another way to look at the
problem is that we can certainly capture the idea of a list of morphisms as an element of \( \mathbb{D}_0 M_1 \), but we also need to permute the elements of the source in a compatible way; what the function \( f \) does is permute in such a way that the inputs to the individual morphisms in the list have their order preserved. Instead, we will allow the elements of the source to be permuted arbitrarily as part of the data for a morphism, which means that each morphism in the actual left adjoint will be represented in many different ways. We will then identify the ones that do represent the same morphism by means of a coequalizer construction. The construction of \( \hat{LM} \), however, simply allows the entries to be permuted arbitrarily without identifying morphisms, and this is the essential difference between \( \hat{LM} \) and \( LM \). We turn now to the actual construction of \( \hat{LM} \).

We have already specified the objects \( \hat{LM}_0 \) as \( \mathbb{D}_0 M_0 \). The morphisms are given as the pullback in the following diagram:

\[
\begin{array}{ccc}
\hat{LM}_1 & \xrightarrow{T} & \mathbb{D}_0 M_1 \\
\downarrow & & \downarrow S \\
\hat{S} & \xrightarrow{\mu} & \mathbb{D}_0^2 M_0 \\
\downarrow & & \downarrow \\
\mathbb{D}_1 M_0 & \xrightarrow{T_0} & \mathbb{D}_0 M_0
\end{array}
\]

so we can write

\[
\hat{LM}_1 \cong \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0.
\]

The intuition here is that the elements of \( \mathbb{D}_0 M_1 \) correspond to lists of morphisms, and the elements of \( \mathbb{D}_1 M_0 \) attach permutations to them in a compatible way. This may be clearer if we realize that we have a pullback diagram

\[
\begin{array}{ccc}
\mathbb{D}_1 M_0 & \xrightarrow{T_0} & \mathbb{D}_0 M_0 \\
\downarrow \mathbb{D}_1 \varepsilon & & \downarrow \mathbb{D}_0 \varepsilon \\
\mathbb{D}_1(*) & \xrightarrow{T_0} & \mathbb{D}_0(*)
\end{array}
\]

that can be pasted onto the bottom of the given diagram, giving a pure permutation in the lower left corner. (The diagram is a pullback because we can think of \( \varepsilon \) as giving a cover functor from \( M_0^* \) to \( * \), and then using the fact that \( \mathbb{D} \) preserves covers.) However, the actual diagram given is much more useful for specifying the source of a
morphism. In particular, we may extend the diagram as follows,

\[
\begin{array}{c}
\hat{L}M_1 \xrightarrow{T} D_0M_1 \xrightarrow{D_0T} D_0M_0 \\
\hat{S} \downarrow \quad \downarrow S \\
\hat{D}_0M_0 \\
\end{array}
\]

\[
\begin{array}{c}
\hat{D}_1M_0 \xrightarrow{T_b} D_0M_0 \\
\end{array}
\]

in which the top horizontal composite defines the target map for \( \hat{L}M \) and the left vertical composite defines the source map.

We specify an identity map \( I : D_0M_0 \rightarrow \hat{L}M_1 \) by means of the pullback construction of \( \hat{L}M_1 \): examining the diagram

\[
\begin{array}{c}
D_0M_0 \xrightarrow{D_0I} D_0M_1 \\
\downarrow \quad \downarrow D_0S \\
D_1M_0 \xrightarrow{T_b} D_0M_0 \\
\end{array}
\]

we see that both composites coincide with the identity on \( D_0M_0 \), so the diagram commutes and we get a map to \( \hat{L}M_1 \).

To construct the composition, we start again with

\[
\hat{L}M_1 \cong D_0M_1 \times_{D_0M_0} D_1M_0,
\]

from the pullback diagram defining \( \hat{L}M_1 \). We then have

\[
\hat{L}M_2 := \hat{L}M_1 \times_{D_0M_0} \hat{L}M_1 \cong D_0M_1 \times_{D_0M_0} D_1M_0 \times_{D_0M_0} D_0M_1 \times_{D_0M_0} D_1M_0.
\]

The first step in the composition is the identification of the two terms in the middle as \( D_1M_1 \); this is a consequence of the pullback diagram

\[
\begin{array}{c}
D_1M_1 \xrightarrow{D_1T} D_1M_0 \\
\downarrow S_D \quad \downarrow S_D \\
D_0M_1 \xrightarrow{D_0T} D_0M_0.
\end{array}
\]
This diagram is a pullback because we can think of \( T : M_1 \to M_0 \) as a functor \( M_1^\delta \to M_0^\delta \), which is a cover since the categories are discrete, and then use the fact that \( \mathcal{D} \) preserves covers. We get an isomorphism

\[
\mathcal{D}_1 M_0 \times_{\mathcal{D}_0 M_0} \mathcal{D}_0 M_1 \cong \mathcal{D}_1 M_1
\]

that preserves the structure maps to \( \mathcal{D}_0 M_0 \) of both source and target type. Next, we exploit the commutative diagram

![Diagram](image)

We abbreviate the left vertical composite as \( \theta \), and we obtain a map \( (T_\theta, \theta) : \mathcal{D}_1 M_1 \to \mathcal{D}_0 M_1 \times_{\mathcal{D}_0 M_0} \mathcal{D}_1 M_0 \). This map also preserves the structure maps of both source and target type; the preservation of target type structure maps is a consequence of the commutative diagram

![Diagram](image)

and preservation of source type structure maps follows from the commutative diagram

![Diagram](image)
The net result of these two steps is to produce what we think of as an interchange map \( \chi : \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_0 \mathbb{D}_0 M_1 \rightarrow \mathbb{D}_0 M_1 \times \mathbb{D}_0 M_0 \mathbb{D}_1 M_0 \) defined by the zig-zag involving an isomorphism as follows:

\[
\mathbb{D}_1 M_0 \times \mathbb{D}_0 M_0 \mathbb{D}_0 M_1 \xrightarrow{(\mathbb{D}_1 T, S_b)} \mathbb{D}_1 M_1 \xrightarrow{(T_b, \theta)} \mathbb{D}_0 M_1 \times \mathbb{D}_0 M_0 \mathbb{D}_1 M_0.
\]

We will also need the fact that both squares in the diagram

\[
\begin{array}{ccc}
\mathbb{D}_0 M_2 & \xrightarrow{\mathbb{D}_0 T} & \mathbb{D}_0 M_1 \\
\downarrow S_b & & \downarrow \mathbb{D}_0 S \\
\mathbb{D}_0^2 M_1 & \xrightarrow{\mathbb{D}_0^2 T} & \mathbb{D}_0^2 M_0 \\
\downarrow \mu & & \downarrow \mu \\
\mathbb{D}_0 M_1 & \xrightarrow{\mathbb{D}_0 T} & \mathbb{D}_0 M_0 \\
\end{array}
\]

are pullbacks, since \( \mathbb{D}_0 \) is Cartesian, so

\[
\mathbb{D}_0 M_1 \times \mathbb{D}_0 M_0 \mathbb{D}_0 M_1 \cong \mathbb{D}_0 M_2.
\]

We also need the fact that the diagram

\[
\begin{array}{ccc}
\mathbb{D}_2 M_0 & \xrightarrow{T_b} & \mathbb{D}_1 M_0 \\
\downarrow S_b & & \downarrow S_b \\
\mathbb{D}_1 M_0 & \xrightarrow{T_b} & \mathbb{D}_0 M_0 \\
\end{array}
\]

is a pullback, since it just expresses the composables in \( \mathbb{D}(M_0^2) \), so we have

\[
\mathbb{D}_1 M_0 \times \mathbb{D}_0 M_0 \mathbb{D}_1 M_0 \cong \mathbb{D}_2 M_0.
\]

We can now express the composition in \( \hat{L}M \) as the composite map

\[
\hat{L}M_2 \cong \mathbb{D}_0 M_1 \times \mathbb{D}_0 M_0 \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_0 \mathbb{D}_0 M_1 \times \mathbb{D}_0 M_0 \mathbb{D}_1 M_0 \\
\xrightarrow{1 \times (\mathbb{D}_1 T, S_b) \times 1} \mathbb{D}_0 M_1 \times \mathbb{D}_0 M_0 \mathbb{D}_1 M_1 \times \mathbb{D}_0 M_0 \mathbb{D}_1 M_0 \\
\xrightarrow{1 \times (T_b, \theta) \times 1} \mathbb{D}_0 M_1 \times \mathbb{D}_0 M_0 \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_0 \mathbb{D}_1 M_0 \\
\cong \mathbb{D}_0 M_2 \times \mathbb{D}_0 M_0 \mathbb{D}_2 M_0 \xrightarrow{\mathbb{D}_0 T \gamma M_0 \times \gamma_0} \mathbb{D}_0 M_1 \times \mathbb{D}_0 M_0 \mathbb{D}_1 M_0 \cong \hat{L}M_1.
\]

Note that the middle two arrows can be also written as \( 1 \times \chi \times 1 \). It is not supposed to be obvious that this captures the idea of composition in the intuitive description of \( \hat{L}M \) given above; however, it is the most convenient description for verification of its formal properties.
We now claim the following as the major result about $\hat{LM}$:

**Theorem 5.1.** The category $\hat{LM}$ supports the structure of a $\mathbb{D}$-algebra. There is a natural map of $\mathbb{D}$-algebras $\varepsilon : \hat{LU}C \to C$ for any $\mathbb{D}$-algebra $C$, and there is a map $\eta : M \to U\hat{LM}$ for any $\mathbb{D}$-multicategory $M$ that satisfies all the properties of a map of $\mathbb{D}$-multicategories except preservation of the $\mathbb{D}(\ast)$-presheaf structure on morphisms. Both adjunction triangles commute.

We defer the proof, as with those of all previous claims.

### 6. The actual left adjoint

The actual left adjoint, which we denote by $LM$, is a quotient of $\hat{LM}$. In particular, we define $LM_0 := \hat{LM}_0 = D_0 M_0$, and we define $LM_1$ by means of a coequalizer diagram with target $\hat{LM}_1$, displayed schematically as follows:

$$
\begin{array}{ccc}
D_0 M_1 & \times & D_0 D_0 M_0 \\
\downarrow S & & \downarrow \varepsilon \\
D_0 M_0 & \times & D_0 D_0 M_0 \\
\end{array}
$$

where the target expresses $\hat{LM}_1$. We then define $LM_1$ as the coequalizer of the diagram. The intuition in the case of classical symmetric multicategories is that the middle term in the source of the two arrows displays tuples of permutations that can be attached either to the tuples of morphisms in the front, or the total permutation in the back, and the result should be the same either way they are attached. The two arrows record these two ways of attaching the lists of permutations.

To describe the first arrow, in which we attach the list of permutations to the list of morphisms, we give a presheaf structure on $M_1$ over $\mathbb{D}(M_0^\delta)$ equivalent to the one assumed over $\mathbb{D}(\ast)$ by exploiting Theorem 4.1 and the fact that $\mathbb{S} : M_1 \to D_0 M_0$ is a map of presheaves over $\mathbb{D}(\ast)$. Since $S$ is a map of presheaves, we have a commutative triangle

$$
\begin{array}{ccc}
M_1 & \xrightarrow{S} & D_0 M_0 \\
\downarrow \varepsilon & & \downarrow D_0 \varepsilon \\
D_0 (\ast). & & \\
\end{array}
$$

But $D_0 \varepsilon : D_0 M_0 \to D_0 (\ast)$ is the map on objects of the target cover $D\varepsilon : \mathbb{D}(M_0^\delta) \to \mathbb{D}(\ast)$, so Theorem 4.1 tells us that the presheaf structure on $M_1$ over $\mathbb{D}(\ast)$ is equivalent to one over $\mathbb{D}(M_0^\delta)$ with structure map $S$. In particular, we get a presheaf action map

$$
M_1 \times \mathbb{D}_0 M_0 \xrightarrow{\psi} M_1.
$$

This gives us a composite

$$
\mathbb{D}_0 M_1 \times \mathbb{D}_0 M_0 \xrightarrow{\mathbb{D}_0 \psi} \mathbb{D}_0 M_0.
$$
that induces the first map to be coequalized.

The second arrow is easier to describe: it is induced by the composite
\[
\begin{align*}
\mathbb{D}_0 \mathbb{D}_1 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 & \xrightarrow{I_0 \times 1} \mathbb{D}_1^2 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \\
\mu \times 1 & : \mathbb{D}_1 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \xrightarrow{\gamma^D} \mathbb{D}_1 M_0.
\end{align*}
\]

We now define \( LM_1 \) to be the coequalizer of the two arrows described, and our main theorem is as follows.

**Theorem 6.1.** The coequalizer \( LM_1 \) gives the morphisms for a \( \mathbb{D} \)-multicategory \( LM \), and the construction \( L \) is left adjoint to the underlying \( \mathbb{D} \)-algebra construction \( U \).
Part 2

Proofs
7. Σ-free Categorical Operads and Their Associated Monads

The proofs of our claims start in this section. We begin by explaining the consequences of the Σ-free assumption on a categorical operad that we will use. The Σ-free hypothesis is always used in the context of the following fundamental proposition, which is no doubt well-known to the experts.

**Proposition 7.1.** Let $G$ be a discrete group. Then passage to orbits from the category of $G$-sets to the category of sets sends pullbacks of free $G$-sets to pullbacks of sets.

**Proof.** Suppose given a pullback diagram of free $G$-sets

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{j} & & \downarrow{h} \\
C & \xrightarrow{k} & D.
\end{array}
$$

We wish to show that the diagram of orbits

$$
\begin{array}{ccc}
A/G & \xrightarrow{f/G} & B/G \\
\downarrow{j/G} & & \downarrow{h/G} \\
C/G & \xrightarrow{k/G} & D/G
\end{array}
$$

is also a pullback. So suppose given orbit classes $[c] \in C/G$ and $[b] \in B/G$ such that $[k(c)] = [h(b)]$ in $D/G$; we wish to show that there is a unique orbit class $[a] \in A/G$ such that $[f(a)] = [b]$ and $[j(a)] = [c]$. Since $D$ is a free $G$-set, there is a unique $g \in G$ such that $k(c) = g \cdot h(b) = h(g \cdot b)$. Since the first square is a pullback, there is a unique $a \in A$ such that $f(a) = g \cdot b$ and $j(a) = c$, and consequently $[f(a)] = [b]$ and $[j(a)] = [c]$. This establishes existence. For uniqueness, suppose there is also $a' \in A$ such that $[f(a')] = [b]$ and $[j(a')] = [c]$. We need to show that $[a] = [a']$. Since $[f(a)] = [f(a')]$, there is a unique $g_1 \in G$ such that $f(a) = g_1 \cdot f(a') = f(g_1 \cdot a')$, and since $[j(a)] = [j(a')]$, there is a unique $g_2 \in G$ such that $j(a) = g_2 \cdot j(a') = j(g_2 \cdot a')$. Now we have

$$
g_1 \cdot kj(a') = g_1 \cdot hf(a') = hf(g_1 \cdot a') = hf(a) = kj(a) = kj(g_2 \cdot a') = g_2 \cdot kj(a').
$$

Since the action of $G$ on $D$ is free, it follows that $g_1 = g_2$. We now have

$$
f(a) = f(g_1 \cdot a') \text{ and } j(a) = j(g_2 \cdot a') = j(g_1 \cdot a'),
$$

so since $a$ and $g_1 \cdot a'$ have the same images under both $f$ and $j$, we see that $a = g_1 \cdot a'$. Therefore $[a] = [a']$, and uniqueness is established. □
This allows us to prove Lemma 2.1, which says the set-monads \{D_0, D_1\} give a category object in monads on Set that generates the remaining monads D_2, D_3, \ldots.

**Proof.** Since \(\mathcal{D}\) is an operad of categories, the operad structure maps commute with the category structure maps, so we can just as well consider \(\mathcal{D}\) a category object in operads (in Set.) Since a category object is defined by diagrams and a pullback condition defining the composables, any functor that preserves pullbacks also preserves category objects. The passage from the operad \(\mathcal{D}\) to the monad \(\mathcal{D}\) involves products, which preserve pullbacks, and orbits. However, we are assuming the groups \(\Sigma_n\) act freely on \(\mathcal{D}_n\), and therefore the orbit functor is being taken only on free \(\Sigma_n\) sets, and orbits of free actions do preserve pullbacks, from Proposition 7.1. Consequently \{D_0, D_1\} does give a category object in monads on Set. Since the higher monads D_2, D_3, \ldots are also produced by pullbacks, they are the further components of the nerve of this category object. \(\square\)

We turn next to showing that all these monads are Cartesian. Since all the operads \(\mathcal{D}_n\) are \(\Sigma\)-free, this follows from Theorem 2.3, whose proof is as follows.

**Proof.** Let’s call the operad \(\mathcal{J}\) and its associated monad \(\mathcal{J}\); we must first show that \(\mathcal{J}\) preserves pullbacks. Certainly the assignment
\[
X \mapsto \mathcal{J}_n \times X^n
\]
preserves pullbacks, since it is a left adjoint. Since \(\mathcal{J}\) is \(\Sigma\)-free, Proposition 7.1 now shows that
\[
X \mapsto \mathcal{J}_n \times_{\Sigma_n} X^n
\]
also preserves pullbacks. Finally, coproducts preserve pullbacks in Set, so
\[
X \mapsto \bigsqcup_{n \geq 0} \mathcal{J}_n \times_{\Sigma_n} X^n
\]
also preserves pullbacks. But this is the definition of \(\mathcal{J}X\), so \(\mathcal{J}\) preserves pullbacks.

Next, we observe that the naturality diagrams for the unit are all pullbacks. For a given \(f : X \to Y\), we get the naturality diagram
\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\eta \downarrow & & \downarrow \eta \\
\bigsqcup_{n \geq 0} \mathcal{J}_n \times_{\Sigma_n} X^n & \xrightarrow{\mathcal{J}f} & \bigsqcup_{n \geq 0} \mathcal{J}_n \times_{\Sigma_n} Y^n,
\end{array}
\]
which restricts on the images of the unit maps to

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y,
\end{array}$$

which is trivially a pullback.

To see that the naturality square for the product \(\mu\) is a pullback, we examine the square

$$\begin{array}{ccc}
\prod_{k \geq 0} J_k \times_{\Sigma_k} (\prod_{n \geq 0} J_n \times_{\Sigma_n} X^n)^k & \xrightarrow{\prod_{k \geq 0} J_k \times_{\Sigma_k} (\prod_{n \geq 0} J_n \times_{\Sigma_n} Y^n)^k} \\
\downarrow \mu & & \downarrow \mu \\
\prod_{n \geq 0} J_n \times_{\Sigma_n} X^n & \xrightarrow{\prod_{n \geq 0} J_n \times_{\Sigma_n} Y^n} \\
\end{array}$$

Let’s write a typical element in the upper right corner as \([a; b_1, y_1, \ldots, b_k, y_k]\), and a typical element in the lower left \([c, x]\), with the understanding that \(x\) and all the \(y\)’s are lists of elements. We need to show that if these two elements map to the same one in the lower right, then there is a unique element in the upper left mapping to them. So we assume that the images are the same. But \([a; b_1, y_1, \ldots, b_k, y_k]\) maps to \(\gamma(a; b_1, \ldots, b_k, y)\), where \(y = (y_1, \ldots, y_k)\), while \([c, x]\) maps to \([c, fx]\), with the obvious interpretation of \(fx\). Since the operad is \(\Sigma\)-free, there is a unique element \(\sigma \in \Sigma_n\) such that

\[c = \gamma(a; b_1, \ldots, b_k) \cdot \sigma,\]

so

\[[c, x] = [\gamma(a; b_1 \ldots, b_k), \sigma \cdot x]\]

and

\[[c, fx] = [\gamma(a; b_1, \ldots, b_k), \sigma \cdot fx] = [\gamma(a; b_1, \ldots, b_k), f(\sigma \cdot x)].\]

Now if we partition the entries in \(\sigma \cdot x\) according to the dimensions of \(b_1, \ldots, b_k\) into \(((\sigma \cdot x)_1, \ldots, (\sigma \cdot x)_k)\), we can construct the element

\[[a; b_1, (\sigma \cdot x)_1, \ldots, b_k, (\sigma \cdot x)_k] \in \mathbb{J}X\]

which maps to each of the elements chosen originally. This establishes existence for an element in the upper left corner.

For uniqueness, suppose we have given two elements,

\[[a; b_1, x_1, \ldots, b_k, x_k]\]

and \[[a'; b'_1, x'_1, \ldots, b'_k, x'_k]\]

both mapping to the same pair of elements in the upper right and lower left corners of the diagram. Let’s call the images in the upper right

\[[a; b_1, y_1, \ldots, b_k, y_k]\]

and \[[a'; b'_1, y'_1, \ldots, b'_k, y'_k]\].
Now by freeness, there is a unique \( \sigma \in \Sigma_k \) such that \( a' = a \cdot \sigma \), from which we get
\[
[a'; b'_1, y'_1, \ldots, b'_k, y'_k] = [a; b_{\sigma^{-1}(1)}, y'_{\sigma^{-1}(1)}, \ldots, b_{\sigma^{-1}(k)}, y'_{\sigma^{-1}(k)}].
\]
Next, again by freeness, there is a unique \( \tau_j \in \Sigma_{n_j} \) for each \( 1 \leq j \leq k \) such that \( b_{\sigma^{-1}(i)} = b_i \cdot \tau_j \), which can also be written \( b'_i = b_{\sigma(i)} \cdot \tau_{\sigma(i)} \). Now using May’s notation \( \sigma(j_1, \ldots, j_k) \) from [5] (to be precise, May uses \( \sigma(j_1, \ldots, j_k) \)) for the permutation that permutes blocks of size \( j_1, \ldots, j_k \) in the same way \( \sigma \) permutes letters, we have
\[
\gamma(a'; b'_1, \ldots, b'_k) = \gamma(a \cdot \sigma; b_{\sigma(1)} \cdot \tau_{\sigma(1)}, \ldots, b_{\sigma(k)} \cdot \tau_{\sigma(k)})
\]
\[
= \gamma(a; b_1 \cdot \tau_1, \ldots, b_k \cdot \tau_k) \cdot \sigma(j_1, \ldots, j_k)
\]
\[
= \gamma(a; b_1, \ldots, b_k) \cdot (\tau_1 \oplus \cdots \oplus \tau_k) \cdot \sigma(j_1, \ldots, j_k).
\]
Mapping both elements to the lower left corner, we now see that
\[
[\gamma(a; b_1, \ldots, b_k), x] = [\gamma(a', b'_1, \ldots, b'_k), x']
\]
\[
= [\gamma(a; b_1, \ldots, b_k), (\tau_1 \oplus \cdots \oplus \tau_k) \cdot \sigma(j_1, \ldots, j_k) \cdot x'].
\]
By freeness, this means that
\[
x = (\tau_1 \oplus \cdots \oplus \tau_k) \cdot \sigma(j_1, \ldots, j_k) \cdot x' = (\tau_1 \cdot x'_{\sigma^{-1}(1)}, \ldots, \tau_k \cdot x'_{\sigma^{-1}(k)})
\]
where we have partitioned \( x' \) into blocks of the correct size for \( \sigma(j_1, \ldots, j_k) \) to act. It follows that for \( 1 \leq i \leq k \),
\[
x_i = \tau_i \cdot x'_{\sigma^{-1}(i)}.
\]
Now we can compute:
\[
[a'; b'_1, x'_1, \ldots, b'_k, x'_k] = [a \cdot \sigma, b'_1, x'_1, \ldots, b'_k, x'_k]
\]
\[
= [a; b'_{\sigma^{-1}(1)}, x'_{\sigma^{-1}(1)}, \ldots, b'_{\sigma^{-1}(k)}, x'_{\sigma^{-1}(k)}]
\]
\[
= [a; b_1 \cdot \tau_1, x'_{\sigma^{-1}(1)}, \ldots, b_k \cdot \tau_k, x'_{\sigma^{-1}(k)}]
\]
\[
= [a; b_1, \tau_1 \cdot x'_{\sigma^{-1}(1)}, \ldots, b_k, \tau_k \cdot x'_{\sigma^{-1}(k)}]
\]
\[
= [a, b_1, x_1, \ldots, b_k, x_k].
\]
This establishes uniqueness, and therefore \( \mathbb{J} \) is a Cartesian monad. \( \square \)

8. Presheaves and Cover Functors

This section is devoted to the proofs of all the statements we will need concerning cover functors and their relation to presheaves. Recall that a target cover is a functor \( F: C \to C' \) for which the square
\[
\begin{array}{ccc}
C_1 & \xrightarrow{T} & C_0 \\
F_1 \downarrow & & \downarrow F_0 \\
C'_1 & \xrightarrow{T} & C'_0 
\end{array}
\]
is a pullback, and a source cover is a functor in which the pullback condition applies to the analogous square in which the target maps are replaced with source maps. We have the following consequence for all the squares in the map of nerves. The analogous statement holds for source covers, with the same proof.

**Corollary 8.1.** If \( F : \mathcal{C} \to \mathcal{C}' \) is a target cover, then all the squares

\[
\begin{array}{ccc}
\mathcal{C}_n & \xrightarrow{T} & \mathcal{C}_{n-1} \\
\downarrow{F_n} & & \downarrow{F_{n-1}} \\
\mathcal{C}'_n & \xrightarrow{T} & \mathcal{C}'_{n-1}
\end{array}
\]

are pullbacks for \( n \geq 1 \).

**Proof.** The case \( n = 1 \) is the definition of a target cover. Suppose for induction that the conclusion holds at \( n - 1 \). In the following diagram, the top, central, and bottom subsquares are then pullbacks:

It now follows that the outer square is a pullback, establishing the claim. \( \square \)

Theorem 4.1 gives the key connection between presheaves and target covers: presheaves over the two categories connected by a target cover are essentially equivalent; all they need is to have their structure maps factor through the map on objects. The proof is as follows.

**Proof of Theorem 4.1.** Since \( F \) is a target cover, we have an explicit isomorphism

\[
\mathcal{C}_0 \times_{\mathcal{C}'} \mathcal{C}'_1 \cong \mathcal{C}_1.
\]

Now an explicit factorization \( \varepsilon' = F_0 \circ \varepsilon \) gives us an explicit isomorphism

\[
X \times_{\mathcal{C}_0} \mathcal{C}'_1 \cong \mathcal{C}_1.
\]
and the correspondence between presheaf action maps is given by the requirement that

\[
\begin{align*}
X \times_{C_0} C_1 & \xrightarrow{1 \times F_0 F_1} X \times_{C'_0} C'_1 \\
\xi & \xrightarrow{\cong} \\
X & \xrightarrow{\xi'}
\end{align*}
\]

commute. Equivalence of the unital conditions for the two actions is a consequence of the commutative diagram

\[
\begin{align*}
X \times_{C_0} C_0 & \xrightarrow{\cong} X \times_{C'_0} C'_0 \\
1 \times I & \\
X \times_{C_0} C_1 & \xrightarrow{1 \times F_0 F_1} X \times_{C'_0} C'_1 \\
\xi & \xrightarrow{\cong} \\
X & \xrightarrow{\xi'}
\end{align*}
\]

and equivalence of associativity is a consequence of the commutative cube

\[
\begin{align*}
X \times_{C_0} C_1 \times_{C_0} C_1 & \xrightarrow{1 \times \mu} X \times_{C_0} C_1 \\
\xi \times 1 & \\
X \times_{C_0} C_1 & \xrightarrow{\xi'} \\
1 \times_{F_0} F_1 & \\
X \times_{C_0} C'_1 & \xrightarrow{1 \times \mu} X \times_{C'_0} C'_1 \\
\xi & \\
X & \xrightarrow{\xi'}
\end{align*}
\]

We begin our use of Theorem 4.1 with the following trivial and very useful corollary.

**Corollary 8.2.** Let \( F : C \to C' \) be a target cover. Then \( C_0 \) is naturally a presheaf over \( C' \).

**Proof.** First, \( C_0 \) is a presheaf over \( C \) with structure map \( \text{id}_{C_0} \). It follows that it is also a presheaf over \( C' \) with structure map \( F_0 \). \( \square \)
We have the following converse as well.

**Proposition 8.3.** Let $X$ be a presheaf over a category $C$. Then $X$ is canonically the set of objects of a category $C \int X$, together with a target cover $C \int X \to C$ whose map on objects is the structure map of $X$ as a presheaf.

**Proof.** The category $C \int X$ is just the Grothendieck construction on the composite functor

$$\mathcal{C}^{\text{op}} \overset{X}{\longrightarrow} \text{Set} \overset{\delta}{\longrightarrow} \text{Cat}.$$ 

It has objects $X$, as required, and a morphism $\phi : a \to b$ is a morphism $\phi \in \mathcal{C}(\varepsilon a, \varepsilon b)$ for which $a = b \cdot \phi$. This can also be expressed by saying that

$$(C \int X)_1 := X \times_{C_0} C_1,$$

with target given by the projection to $X$, and source given by the action map. The projection to $C_1$ gives the functor on morphisms, and it is easy to see that this gives a target cover with $\varepsilon$ as the map on objects. □

There is a generalization that is sometimes useful, as follows.

**Lemma 8.4.** If $F : C \to C'$ is a target cover, then $C_1$ is naturally a presheaf over $C'$ consistent with its left action on itself through composition.

**Proof.** First, $C_1$ is a presheaf over $C$ with structure map the source map $S : C_1 \to C_0$, and action given by composition. Therefore $C_1$ is also a presheaf over $C'$ with structure map $F_0 \circ S = S \circ F_1$, and the only issue is consistency with the left action of $C_1$ on itself. But we have the action induced by the composition in $C$, and consistency then follows from associativity of composition via the following diagram:

\[
\begin{array}{c}
\mathcal{C}_1 \times_{C_0} \mathcal{C}_1 \times_{C_0} \mathcal{C}_1' \\
\downarrow_{\mu \times 1} \downarrow_{\mu \times 1} \downarrow_{\mu} \\
\mathcal{C}_1 \times_{C_0} \mathcal{C}_1 \end{array}
\begin{array}{c}
1 \times F_0 \times F_1 \\
\cong \\
1 \times F_0 \\
\cong \\
1 \times F_0 \end{array}
\begin{array}{c}
\mathcal{C}_1 \times_{C_0} \mathcal{C}_1 \times_{C_0} \mathcal{C}_1 \\
\downarrow_{\mu} \\
\mathcal{C}_1 \end{array}
\begin{array}{c}
\mu \\
\mu \\
\mu
\end{array}
\]

The presheaf structure on $C_0$ is just the specialization to identity arrows.

We turn now to the proof of Theorem 2.6 which says that the monads associated to $\Sigma$-free categorical operads preserve target and source covers, and the basic proposition needed is as follows.

**Proposition 8.5.** Let $G$ be a discrete group acting freely on a target cover $F : \mathcal{C} \to \mathcal{C}'$. Then the map on orbit categories $F/G : \mathcal{C}/G \to \mathcal{C}'/G$ is also a target cover. The same is true with “target” replaced by “source.”
Proof. First, we note that since $G$ acts freely on $\mathcal{C}$, the quotient category $\mathcal{C}/G$ has $(\mathcal{C}/G)_0 \cong \mathcal{C}_0/G$ and $(\mathcal{C}/G)_1 \cong \mathcal{C}_1/G$. This is because the composables $\mathcal{C}_2$ are given by a pullback diagram

\[
\begin{array}{ccc}
\mathcal{C}_2 & \xrightarrow{T} & \mathcal{C}_1 \\
\downarrow S & & \downarrow S \\
\mathcal{C}_1 & \xrightarrow{T} & \mathcal{C}_0,
\end{array}
\]

so by Proposition 7.1 we also have a pullback diagram

\[
\begin{array}{ccc}
\mathcal{C}_2/G & \xrightarrow{T/G} & \mathcal{C}_1/G \\
\downarrow S/G & & \downarrow S/G \\
\mathcal{C}_1/G & \xrightarrow{T/G} & \mathcal{C}_0/G.
\end{array}
\]

Consequently the composition map $\mu : \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \to \mathcal{C}_1$ descends to an induced map $\mathcal{C}_1/G \times_{\mathcal{C}_0/G} \mathcal{C}_1/G \to \mathcal{C}_1/G$ that gives us the composition of a category, which clearly has the universal property needed for $\mathcal{C}/G$. Now the pullback diagram that tells us that $F$ is a target cover, namely

\[
\begin{array}{ccc}
\mathcal{C}_1 & \xrightarrow{T} & \mathcal{C}_0 \\
\downarrow F_1 & & \downarrow F_0 \\
\mathcal{C}_1' & \xrightarrow{T} & \mathcal{C}_0',
\end{array}
\]

gives a pullback diagram on orbits

\[
\begin{array}{ccc}
\mathcal{C}_1/G & \xrightarrow{T} & \mathcal{C}_0/G \\
\downarrow F_1/G & & \downarrow F_0/G \\
\mathcal{C}_1'/G & \xrightarrow{T} & \mathcal{C}_0'/G,
\end{array}
\]

which tells us that $F/G$ is also a target cover. Similarly, passage to orbits of free actions preserves source covers. □

Proof of Theorem 2.6. First, since both target and source covers are given by a pullback condition, they are both preserved by products, and it is obvious that the identity functor is a cover. Therefore, if we are given a target (or source) cover $F : \mathcal{C} \to \mathcal{C}'$, the induced functor

$$\mathcal{D}_n \times \mathcal{C}^n \to \mathcal{D}_n \times (\mathcal{C}')^n$$

is a target (or source) cover. Now since $\Sigma_n$ acts freely on $\mathcal{D}_n$, it also acts freely on both $\mathcal{D}_n \times \mathcal{C}^n$ and $\mathcal{D}_n \times (\mathcal{C}')^n$. It now follows from Proposition 8.5 that the induced
functor
\[ D_n \times_{\Sigma_n} C^n \to D_n \times_{\Sigma_n} (C')^n \]
is a target (or source) cover. Further, it is elementary that coproducts preserve both sorts of cover, so the induced map
\[ \prod_{n \geq 0} D_n \times_{\Sigma_n} C^n \to \prod_{n \geq 0} D_n \times_{\Sigma_n} (C')^n \]
is also a target (or source) cover. But this is precisely the functor \( \mathbb{D}F : \mathbb{D}C \to \mathbb{D}C' \), so we have proven Theorem 2.6. \( \square \)

We can now give the proof of our basic Theorem 2.7, which says that whenever \( X \) has a presheaf structure over \( C \), \( \mathbb{D}_0X \) has a presheaf structure over \( \mathbb{D}C \).

**Proof.** Since \( \varepsilon : X \to C_0 \) is a presheaf structure map for \( X \) over \( C \), we have a target cover \( C \int X \to C_1 \) for which \( \varepsilon \) is the map on objects. Since \( \mathbb{D} \) preserves target covers, we have the target cover
\[ \mathbb{D}(C \int X) \to \mathbb{D}C \]
with \( \mathbb{D}\varepsilon \) as the map on objects. Since the objects of \( \mathbb{D}(C \int X) \) are \( \mathbb{D}_0X \), it follows that they are a presheaf over \( \mathbb{D}C \) with structure map \( \mathbb{D}\varepsilon \). \( \square \)

Corollary 2.8 now follows immediately.

### 9. The Underlying \( \mathbb{D} \)-multicategory: Proofs

We gave the structural data for the underlying \( \mathbb{D} \)-multicategory \( UC \) to a \( \mathbb{D} \)-algebra \( C \) in Section 4. In this section, we show that these data do satisfy the necessary properties to define a \( \mathbb{D} \)-multicategory. First, the formula given for the presheaf action really is a presheaf action.

**Theorem 9.1.** The composite
\[ UC_1 \times_{\mathbb{D}_0C_0} \mathbb{D}_1C_0 \cong C_1 \times_{C_0} C_1 \mathbb{D}_1C_0 \overset{1 \times_{\mathbb{D}1} \gamma}{\longrightarrow} C_1 \times_{C_0} \mathbb{D}_1C_0 \]
defines a presheaf action on the morphisms \( UC_1 \) of the underlying \( \mathbb{D} \)-multicategory.

**Proof.** In order to see that this action is unital, we need the diagram
\[ UC_1 \cong UC_1 \times_{\mathbb{D}_0C_0} \mathbb{D}_0C_0 \overset{1 \times_{\mathbb{D}0}}{\longrightarrow} UC_1 \times_{\mathbb{D}_0C_0} \mathbb{D}_1C_0 \]
\[ \sigma \]
\[ UC_1 \]

\[ UC_1 \]
to commute, where $\sigma$ is the presheaf action map. But by expanding the definition of $\sigma$, this becomes

$$C_1 \times C_0 \mathbb{D}_0 C_0 \xrightarrow{1 \times I_0} C_1 \times C_0 \mathbb{D}_1 C_0 \xrightarrow{1 \times \mathbb{D}_1 I} C_1 \times C_0 \mathbb{D}_1 C_0$$

$$= C_1 \times C_0 \mathbb{C}_1 \times C_0 \mathbb{D}_0 C_0$$

$$\xrightarrow{1 \times (\xi, S D S)} C_1 \times C_0 \mathbb{D}_0 C_0$$

$$\xrightarrow{\gamma C \times 1} C_1 \times C_0 \mathbb{D}_0 C_0.$$
and the right half, which is to be glued onto the previous one, is as follows:

\[
\begin{array}{c}
\text{C}_1 \times \text{C}_0 \text{ D}_1 \text{C}_0 \times \text{D}_0 \text{C}_0 \text{ D}_1 \text{C}_0 \\
\downarrow \quad \downarrow \\
\text{C}_1 \times \text{C}_0 \text{ D}_1 \text{C}_1 \\
\downarrow \quad \downarrow \\
\text{C}_1 \times \text{C}_0 \text{ D}_1 \text{C}_0
\end{array}
\]

The top triangle of the left half commutes by definition. The bottom part commutes by examining the projections onto each factor of the target, as follows. The projection onto the first \( \text{C}_1 \) involves only the \( \text{C}_1 \times \text{C}_0 \text{ D}_1 \text{C}_1 \) of the source, and either way around the diagram gives us

\[
\begin{array}{c}
\text{C}_1 \times \text{C}_0 \text{ D}_1 \text{C}_1 \\
\downarrow \quad \downarrow \\
\text{C}_1 \times \text{C}_0 \text{ D}_0 \text{C}_0
\end{array}
\]

Projection to the last two factors \( \text{C}_1 \times \text{C}_0 \text{ D}_0 \text{C}_0 \) involves only the last factor \( \text{D}_1 \text{C}_0 \) of the target, and either way around the diagram gives us

\[
\begin{array}{c}
\text{D}_1 \text{C}_0 \\
\downarrow \quad \downarrow \\
\text{D}_0 \text{C}_0
\end{array}
\]

The left part of the associativity diagram therefore commutes. The right half commutes, from top to bottom, by the identity properties of \( \gamma_C \), because \( \xi : \text{D} \text{C} \to \text{C} \) is a functor, and because \( \gamma_C \) is associative. We may conclude that we have defined a presheaf structure on \( UC_1 \).

We must also show that the source map \( S : UC_1 \to \text{D}_0 \text{C}_0 \) preserves the presheaf structure. The presheaf structure on \( \text{D}_0 \text{C}_0 \) is simply that of the objects of \( \text{D}(\text{C}_0^4) \), given by the composite

\[
\begin{array}{c}
\text{D}_0 \text{C}_0 \times \text{D}_0 \text{C}_0 \\
\downarrow \quad \downarrow \\
\text{D}_1 \text{C}_0 \text{ D}_0 \text{C}_0
\end{array}
\]

Further, the first part of the presheaf structure map on \( UC_1 \) maps to the first part of this composite, using the pullback diagram defining \( UC_1 \), augmented slightly as follows,

\[
\begin{array}{c}
\text{UC}_1 \\
\downarrow \quad \downarrow \\
\text{D}_1 \text{C}_0 \text{ D}_0 \text{C}_0
\end{array}
\]
so we obtain the commuting diagram

\[
\begin{array}{ccc}
UC_1 \times_{D_0C_0} D_1C_0 & \cong & C_1 \times_{C_0} D_1C_0 \\
S \times 1 & \downarrow & \downarrow p_2 \\
D_0C_0 \times_{D_0C_0} D_1C_0 & \cong & D_1C_0.
\end{array}
\]

It now follows that preservation of the presheaf actions by the source map reduces to checking the diagram

\[
\begin{array}{ccc}
C_1 \times_{C_0} D_1C_0 & \xrightarrow{1 \times D_1I} & C_1 \times_{C_0} D_1C_1 \\
p_2 & \downarrow & \downarrow \\
D_1C_0 & \cong & D_0C_0.
\end{array}
\]

But this depends only on the last factors in the fiber products, and that reduces to the commuting diagram

\[
\begin{array}{ccc}
D_1C_0 & \xrightarrow{D_1I} & D_1C_1 \\
\downarrow S_0 & & \downarrow S_0 \\
D_1C_0 & \cong & D_0C_0.
\end{array}
\]

It follows that the source map on UC preserves the presheaf action.

For the unital property of the structure, recall that the identity map of the underlying multicategory is induced from the commutative square

\[
\begin{array}{ccc}
C_0 & \xrightarrow{I_C} & C_1 \\
\eta \downarrow & & \downarrow S \\
D_0C_0 & \xrightarrow{\xi_0} & C_0,
\end{array}
\]

so we get an induced map \(I_{UC} : C_0 \to C_1 \times_{C_0} D_0C_0 \cong (UC)_1\). It is now an easy exercise to see that

\[
\begin{array}{ccc}
C_0 & \xrightarrow{I_{UC}} & C_0 \\
\eta \downarrow & & \downarrow \\
D_0C_0 & \xrightarrow{S} (UC)_1 & \xrightarrow{T} C_0
\end{array}
\]

commutes.
For the remainder of the section, we verify the formal properties of the composition law. We must verify that it preserves sources and targets, is unital, and is associative. To see that the composition preserves targets, we reduce to the corresponding property for \( C \) by means of the following diagram:

For preservation of sources, we have defined \( \gamma_{UC} \) so that the diagram

\[
\begin{align*}
(UC)_2 & \xrightarrow{T} (UC)_1 \\
& \Downarrow \kappa_2 \quad \Downarrow \kappa_1 \quad \Downarrow \gamma_{UC} \\
C_2 & \xrightarrow{T} C_1 \\
& \Downarrow \gamma_C \quad \Downarrow T \\
(UC)_1 & \xrightarrow{T} C_0.
\end{align*}
\]

commutes, so \( \gamma_{UC} \) does preserve sources.

In verifying the unital properties of the composition, we introduce the notations \( I_L : M_1 \rightarrow M_2 \) and \( I_R : M_1 \rightarrow M_2 \) for the composites that appear in the unital conditions for composition in any \( \mathbb{D} \)-multicategory. Specifically, we use \( I_L \) for the left (target) unit map defined by the composite

\[
M_1 \cong M_0 \times_{M_0} M_1 \xrightarrow{1 \times \eta} M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1 \cong M_2,
\]

and \( I_R \) for the right (source) unit map defined by the composite

\[
M_1 \cong M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_0 \xrightarrow{1 \times \mathbb{D}_0 I} M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1 \cong M_2.
\]

Note that both triangles

\[
\begin{align*}
M_1 & \xrightarrow{I_L} M_2 & \text{and} & & M_1 & \xrightarrow{I_R} M_2 \\
& \Downarrow \eta \quad & & \quad & \Downarrow \eta \\
\mathbb{D}_0 M_1 & \xrightarrow{S} & & \quad & \mathbb{D}_0 M_1 & \xrightarrow{S}
\end{align*}
\]
commute. To see that $\gamma_{UC}$ is left (target) unital, we will need to show that

$$\begin{array}{c}
(U\mathcal{C})_1 \xrightarrow{\kappa_1} C_1 \\
I_L \downarrow \quad \quad \quad \quad \quad \quad I_L \\
(U\mathcal{C})_2 \xrightarrow{\kappa_2} C_2
\end{array}$$

commutes, in order to reduce to the analogous property of $\mathcal{C}$. Since $C_2$ is a pullback, the square commutes if and only if it does so after composing with the two maps $S, T : C_2 \to C_1$. Composing with $T$, we obtain

$$\begin{array}{c}
(U\mathcal{C})_1 \xrightarrow{\kappa_1} C_1 \\
I_L \downarrow \quad \quad \quad \quad \quad \quad I_L \\
(U\mathcal{C})_2 \xrightarrow{\kappa_2} C_2 \xrightarrow{T} C_0 \\
T \downarrow \quad \quad \quad \quad \quad \quad T \downarrow \quad \quad \quad \quad \quad \quad I_C \\
(U\mathcal{C})_1 \xrightarrow{\kappa_1} C_1.
\end{array}$$

Since the two new squares commute, this reduces us to verifying the commutativity of the perimeter of the hexagon, which follows from replacing its interior as follows:

$$\begin{array}{c}
(U\mathcal{C})_1 \xrightarrow{\kappa_1} C_1 \\
I_L \downarrow \quad \quad \quad \quad \quad \quad I_L \\
(U\mathcal{C})_2 \xrightarrow{\kappa_2} C_2 \xrightarrow{T} C_0 \\
T \downarrow \quad \quad \quad \quad \quad \quad I_C \downarrow \quad \quad \quad \quad \quad \quad I_C \\
(U\mathcal{C})_1 \xrightarrow{\kappa_1} C_1.
\end{array}$$

Continuing the verification that $\gamma_{UC}$ is left unital, we now compose the desired comparison square with $S$, and obtain

$$\begin{array}{c}
(U\mathcal{C})_1 \xrightarrow{\kappa_1} C_1 \\
I_L \downarrow \quad \quad \quad \quad \quad \quad I_L \\
(U\mathcal{C})_2 \xrightarrow{\kappa_2} C_2 \xrightarrow{S} C_1 \\
\mathbb{D}_0(U\mathcal{C})_1 \xrightarrow{S} \mathbb{D}_0\mathcal{C}_1 \xrightarrow{\xi_1} \mathbb{D}_1\mathcal{C}_1 \xrightarrow{\xi_1} C_1.
\end{array}$$
Again, we have reduced to the question of whether the perimeter commutes, and we rearrange the innards to obtain

\[
\begin{array}{ccc}
(U) C_1 & \xrightarrow{\kappa_1} & C_1 \\
\eta & \downarrow & \eta \\
\mathbb{D}_0(U) C_1 & \xrightarrow{\mathbb{D}_0 \kappa_1} & \mathbb{D}_0 C_1 \\
\end{array}
\]

which does commute. The comparison square therefore commutes. We now use it to show that \( \gamma_{UC} \) is left unital, which says that the diagram

\[
\begin{array}{ccc}
(U) C_1 & \xrightarrow{I_L} & (U) C_2 \\
\gamma_{UC} & \downarrow & \downarrow \\
(U) C_1 & \xrightarrow{\gamma_{UC}} & (U) C_1 \\
\end{array}
\]

commutes. Again, since \((U) C_1\) is defined as a pullback, this diagram commutes if and only if it does so after composition with \(\kappa_1 : (U) C_1 \to C_1\) and \(S : (U) C_1 \to \mathbb{D}_0 C_0\). Composing with \(\kappa_1\), we obtain the diagram

\[
\begin{array}{ccc}
(U) C_1 & \xrightarrow{I_L} & (U) C_2 \\
\kappa_1 & \downarrow & \kappa_2 \\
C_1 & \xrightarrow{\gamma_c} & C_1 \\
\kappa_1 & \downarrow & \kappa_1 \\
C_1 & \xrightarrow{\gamma_c} & C_1 \\
\end{array}
\]

Since the bottom row composes to \(\text{id}_{C_1}\), we just get \(\kappa_1\), as desired. Composing with \(S\), we get

\[
\begin{array}{ccc}
(U) C_1 & \xrightarrow{I_L} & (U) C_2 \\
\eta & \downarrow & \eta \\
\mathbb{D}_0(U) C_1 & \xrightarrow{\mathbb{D}_0 S} & \mathbb{D}_0 C_0 \\
\end{array}
\]

Again, the bottom row composes to the identity, so we just get \(S\), as desired. It follows that \(\gamma_{UC} \circ I_L = \text{id}_{(U) C_1}\), so \(\gamma_{UC}\) is left unital.
To show that \( \gamma_{UC} \) is right (source) unital, we proceed in much the same way. First, we show that the square 

\[ (UC)_1 \xrightarrow{\kappa_1} C_1 \]

\[ I_R \downarrow \quad I_R \downarrow \]

\[ (UC)_2 \xrightarrow{\kappa_2} C_2 \]

commutes, again by showing that the composites coincide after composing with both of \( S, T : C_2 \to C_1 \). Composing with \( T \), we obtain the diagram

whose perimeter obviously commutes. Composing with \( S \), we obtain the diagram

whose perimeter commutes by rearranging the insides as follows:

\[ (UC)_1 \xrightarrow{S} D_0 C_0 \]

\[ \xrightarrow{\xi_0} \]

\[ \xrightarrow{C_0} \]

\[ \xrightarrow{S} \]

\[ \xrightarrow{I_R} \]

\[ \xrightarrow{I_R} \]

\[ \xrightarrow{S} \]

\[ \xrightarrow{I_C} \]

\[ \xrightarrow{C_1} \]

\[ \xrightarrow{\xi_1} \]

\[ \xrightarrow{C_1} \]
The comparison square therefore commutes. We use it to establish the right unital condition for $\gamma_{UC}$, which says that

$$
\begin{array}{ccc}
(UC)_1 & \xrightarrow{IR} & (UC)_2 \\
\downarrow & = & \downarrow \\
\downarrow & \gamma_{UC} & \downarrow \\
(UC)_1
\end{array}
$$

commutes. As with the left unital condition, we compose with the two maps $\kappa_1 : (UC)_1 \to C_1$ and $S : (UC)_1 \to D_0C_0$ and verify that the resulting diagrams commute. Composing first with $\kappa_1$, we get the diagram

$$
\begin{array}{ccc}
(UC)_1 & \xrightarrow{IR} & (UC)_2 & \xrightarrow{\gamma_{UC}} & (UC)_1 \\
\downarrow & \kappa_1 & \downarrow & \kappa_2 & \downarrow & \kappa_1 \\
C_1 & \xrightarrow{IR} & C_2 & \xrightarrow{\gamma_c} & C_1
\end{array}
$$

and since the bottom row composes to $\text{id}_{C_1}$, we get the desired result. Composing with $S$, we get the diagram

$$
\begin{array}{ccc}
(UC)_1 & \xrightarrow{IR} & (UC)_2 & \xrightarrow{\gamma_{UC}} & (UC)_1 \\
\downarrow & \eta & \downarrow & S & \downarrow & \eta \\
\downarrow & \text{D}_0(UC)_1 & \downarrow & \text{D}_0S & \downarrow \\
\downarrow & \text{D}_0C_0 & \downarrow & \text{D}_0^2C_0 & \downarrow \\
\downarrow & \mu & \downarrow & \mu & \downarrow \\
\downarrow & \text{D}_0C_0 & \downarrow & \text{D}_0C_0
\end{array}
$$

in which the bottom row composes to the identity. This completes the check of the right unital condition for $\gamma_{UC}$.

It remains to verify associativity for $\gamma_{UC}$. We will need the comparison maps $\kappa_2$ and $\kappa_3$ to be compatible with the two composition maps $\gamma_S, \gamma_T$ in the sense that the two squares

$$
\begin{array}{ccc}
(UC)_3 & \xrightarrow{\kappa_3} & C_3 \\
\gamma_S & \downarrow & \gamma_S \\
(UC)_2 & \xrightarrow{\kappa_2} & C_2
\end{array}
\quad\text{and}\quad
\begin{array}{ccc}
(UC)_3 & \xrightarrow{\kappa_3} & C_3 \\
\gamma_T & \downarrow & \gamma_T \\
(UC)_2 & \xrightarrow{\kappa_2} & C_2
\end{array}
$$

both commute. Despite their formal similarity, the second diagram requires more work to verify than the first one, so we begin with the first one. Our strategy in both is to compose with the two maps $S, T : C_2 \to C_1$ and verify the two resulting diagrams.
Composing the compatibility diagram for $\gamma_S$ with $T$, we wish the perimeter of the diagram

$$
\begin{array}{cccccc}
(UC)_3 & \xrightarrow{\kappa_3} & C_3 & \xrightarrow{T} & C_2 \\
\gamma_S & \downarrow & \gamma_S & \downarrow & \gamma_S \\
(UC)_2 & \xrightarrow{\kappa_2} & C_2 & \xrightarrow{T} & C_1 \\
T & \downarrow & T & \downarrow & T \\
(UC)_1 & \xrightarrow{\kappa_1} & C_1
\end{array}
$$

to commute, so we rearrange the interior to get the commutative diagram

$$
\begin{array}{cccccc}
(UC)_3 & \xrightarrow{\kappa_3} & C_3 \\
\gamma_S & \downarrow & (UC)_2 & \xrightarrow{\kappa_2} & C_2 \\
(UC)_2 & \xrightarrow{T} & (UC)_1 & \xrightarrow{\kappa_1} & C_1
\end{array}
$$

Composing instead with $S$, we wish the perimeter of the diagram

$$
\begin{array}{cccccc}
(UC)_3 & \xrightarrow{\kappa_3} & C_3 & \xrightarrow{S} & C_2 \\
\gamma_S & \downarrow & (UC)_2 & \xrightarrow{\kappa_2} & C_2 \\
S & \downarrow & S & \downarrow & \gamma \\
D_0(UC)_1 & \xrightarrow{D_0\kappa_1} & D_0C_1 & \xrightarrow{I_D} & D_1C_1 & \xrightarrow{\xi_1} & D_1
\end{array}
$$

to commute, but here we can rearrange the interior to obtain the commutative diagram

$$
\begin{array}{cccccc}
(UC)_3 & \xrightarrow{\kappa_3} & C_3 \\
\gamma_S & \downarrow & D_0(UC)_2 & \xrightarrow{D_0\kappa_2} & D_0C_2 & \xrightarrow{I_D^2} & D_2C_2 & \xrightarrow{\xi_2} & C_2 \\
\gamma & \downarrow & \gamma & \downarrow & \gamma & \downarrow & \gamma & \downarrow & \gamma \\
D_0(UC)_1 & \xrightarrow{D_0\kappa_1} & D_0C_1 & \xrightarrow{I_D} & D_1C_1 & \xrightarrow{\xi_1} & C_1
\end{array}
$$
Next, we verify the compatibility diagram for $\gamma_T$, and we first compose with $T$ to obtain the diagram

$$
\begin{array}{c}
(UC)_3 \xrightarrow{\kappa_3} C_3 \xrightarrow{T} C_2 \\
\gamma_T \\
(UC)_2 \xrightarrow{\kappa_2} C_2 \\
\gamma \\
(UC)_1 \xrightarrow{\kappa_1} C_1,
\end{array}
$$

whose perimeter we want to commute. However, we again rearrange the insides and get

$$
\begin{array}{c}
(UC)_3 \xrightarrow{\kappa_3} C_3 \\
\gamma_T \\
(UC)_2 \xrightarrow{\kappa_2} C_2 \\
\gamma \\
(UC)_1 \xrightarrow{T} (UC)_1 \xrightarrow{\kappa_1} C_1.
\end{array}
$$

Composing with $S$, we find we wish the perimeter of

$$
\begin{array}{c}
(UC)_3 \xrightarrow{\kappa_3} C_3 \xrightarrow{S} C_2 \\
\gamma_T \\
(UC)_2 \xrightarrow{\kappa_2} C_2 \\
\gamma_T \\
(\mathcal{D}_0(UC))_1 \xrightarrow{\mathcal{D}_0\kappa_1} \mathcal{D}_0C_1 \xrightarrow{I_0} \mathcal{D}_1C_1 \xrightarrow{\xi_1} C_1.
\end{array}
$$
to commute. We can fill in the interior with the following, in which we know all but
the large, irregular sub-diagram at the lower right commutes:

\[
\begin{array}{c}
(U\mathcal{C})_3 \xrightarrow{\kappa_3} C_3 \\
\downarrow S \\
\mathbb{D}_0(U\mathcal{C})_2 \xrightarrow{\mathbb{D}_0\kappa_2} \mathbb{D}_0\mathcal{C}_2 \xrightarrow{I_0^2} \mathbb{D}_2\mathcal{C}_2 \xrightarrow{\xi_2} C_2 \\
\downarrow \gamma_T \\
\mathbb{D}_0^2(U\mathcal{C})_1 \xrightarrow{\mathbb{D}_0^2\kappa_1} \mathbb{D}_0^2\mathcal{C}_1 \\
\downarrow \mu \\
\mathbb{D}_0\mathcal{C}_1 \xrightarrow{\mathbb{D}_0\xi_1} \mathbb{D}_1\mathcal{C}_1 \xrightarrow{\xi_1} C_1.
\end{array}
\]

And the irregular sub-diagram can be expanded and filled in as follows:

\[
\begin{array}{c}
\mathbb{D}_0(U\mathcal{C})_2 \xrightarrow{\mathbb{D}_0\kappa_2} \mathbb{D}_0\mathcal{C}_2 \xrightarrow{I_0^2} \mathbb{D}_2\mathcal{C}_2 \xrightarrow{\xi_2} C_2 \\
\downarrow \mathbb{D}_0S \\
\mathbb{D}_0^2(U\mathcal{C})_1 \\
\downarrow \mathbb{D}_0^2\kappa_1 \\
\mathbb{D}_0^2\mathcal{C}_1 \\
\downarrow \mu \\
\mathbb{D}_0\mathcal{C}_1 \xrightarrow{\mathbb{D}_0\xi_1} \mathbb{D}_1\mathcal{C}_1 \xrightarrow{\xi_1} C_1.
\end{array}
\]

We may conclude that the comparison diagram for \(\gamma_T\) commutes.

Now we can verify the actual associativity diagram, which we do by again recognizing the target of the diagram

\[
\begin{array}{c}
(U\mathcal{C})_3 \xrightarrow{\gamma_T} (U\mathcal{C})_2 \\
\downarrow \gamma \downarrow \gamma \\
(U\mathcal{C})_2 \xrightarrow{\gamma} (U\mathcal{C})_1
\end{array}
\]

as a pullback, and composing with the two maps \(S : (U\mathcal{C})_1 \to \mathbb{D}_0\mathcal{C}_0\) and \(\kappa_1 : (U\mathcal{C})_1 \to \mathcal{C}_1\). Composing with \(\kappa_1\) results in the following diagram, whose perimeter we wish to
commute:

\[
\begin{array}{cccccc}
(UC)_3 & \xrightarrow{\gamma T} & (UC)_2 & \xrightarrow{\kappa_2} & C_2 \\
\gamma_S & \downarrow & \gamma & \downarrow & \\
(UC)_2 & \xrightarrow{\gamma} & (UC)_1 & \xrightarrow{\kappa_1} & \\
\kappa_2 & \downarrow & \gamma & \downarrow & \\
C_2 & \xrightarrow{\gamma} & C_1.
\end{array}
\]

But this follows from the next diagram, in which we use the comparison diagrams just verified:

\[
\begin{array}{cccccc}
(UC)_3 & \xrightarrow{\gamma T} & (UC)_2 \\
\gamma_S & \downarrow & \kappa_2 & \downarrow & \\
C_3 & \xrightarrow{\gamma T} & C_2 \\
\gamma_S & \downarrow & \gamma & \downarrow & \\
(UC)_2 & \xrightarrow{\kappa_2} & C_2 & \xrightarrow{\gamma} & C_1.
\end{array}
\]

Composing instead with \( S \), we get the following diagram, whose perimeter we also wish to commute:

\[
\begin{array}{cccccc}
(UC)_3 & \xrightarrow{\gamma T} & (UC)_2 & \xrightarrow{S} & D_0(UC)_1 \\
\gamma_S & \downarrow & \gamma & \downarrow & D_0 S \\
(UC)_2 & \xrightarrow{\gamma} & (UC)_1 & \xrightarrow{D_0 S} & D_0^2 C_0 \\
S & \downarrow & \mu & \downarrow & \\
D_0 M_1 & \xrightarrow{D_0 S} & D_0^2 C_0 & \xrightarrow{\mu} & D_0 C_0.
\end{array}
\]

However, this follows from the commutativity of

\[
\begin{array}{cccccc}
(UC)_3 & \xrightarrow{S} & \xrightarrow{\gamma T} & (UC)_2 & \xrightarrow{S} \\
\gamma_S & \downarrow & \gamma & \downarrow & \gamma & \downarrow & \gamma \\
D_0(UC)_2 & \xrightarrow{D_0 S} & D_0^2(UC)_1 & \xrightarrow{\mu} & D_0(UC)_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
D_0^2 C_0 & \xrightarrow{D_0 S} & D_0^2 C_0 & \xrightarrow{\mu} & D_0 C_0.
\end{array}
\]
This completes the verification of associativity, and therefore $UC$ has all the necessary properties of a $D$-multicategory.

10. The provisional left adjoint: category structure

In this section we show that the definition given for the provisional left adjoint $\hat{L}M$ for a $D$-multicategory $M$ is actually a category. Recall first that the morphisms $\hat{L}M_1$ are given by a pullback diagram

$$
\begin{array}{ccc}
\hat{L}M_1 & \xrightarrow{T} & D_0M_1 \\
\downarrow & & \downarrow S \\
\hat{S} & \xrightarrow{} & D_0^2M_0 \\
\downarrow & & \downarrow \mu \\
D_1M_0 & \xrightarrow{T_b} & D_0M_0,
\end{array}
$$

so we can write

$$\hat{L}M_1 \cong D_0M_1 \times_{D_0M_0} D_1M_0.$$

We specify an identity map for $\hat{L}M$ by means of the commutative diagram

$$
\begin{array}{ccc}
D_0M_0 & \xrightarrow{D_0I} & D_0M_1 \\
\downarrow I_0 & & \downarrow D_0S \\
D_1M_0 & \xrightarrow{T_b} & D_0M_0.
\end{array}
$$

We see that both composites coincide with the identity on $D_0M_0$, so the diagram commutes and we get a map to $\hat{L}M_1$. Further, both composites

$$
D_0M_0 \xrightarrow{D_0I} D_0M_1 \xrightarrow{D_0T} D_0M_0 \quad \text{and} \quad D_0M_0 \xrightarrow{I_0} D_1M_0 \xrightarrow{S_b} D_0M_0
$$

coincide with the identity on $D_0M_0$, which verifies the necessary properties for an identity map.
The remainder of this section is devoted to the formal properties of the composition on \( \hat{LM} \). Recall that the composition map is defined as the composite

\[
\hat{LM}_2 \cong \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0
\]

\[
\cong \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0
\]

\[
\cong \mathbb{D}_0 M_2 \times_{\mathbb{D}_0 M_0} \mathbb{D}_2 M_0
\]

We will often think of the first part of this construction as being given by an interchange map

\[
\chi : \mathbb{D}_1 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1 \to \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0
\]

given by the composite

\[
\mathbb{D}_1 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1 \cong \mathbb{D}_1 M_1 \mathbb{D}_0 M_1 \mathbb{D}_0 M_0 \mathbb{D}_1 M_0
\]

where the map \( \theta \) is the composite

\[
\mathbb{D}_1 M_1 \mathbb{D}_1 S \mathbb{D}_1 D_0 M_0 \mathbb{D}_1 I_0 \mathbb{D}_1^2 M_0 \mu \mathbb{D}_1 M_0.
\]

We must show that the composition preserves source and target maps, is left and right unital, and is associative. For preservation of source and target maps, we see from the defining diagram for \( \hat{LM} \) that we can write the target as

\[
\hat{LM}_1 \cong \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \mathbb{D}_0 M_1 \mathbb{D}_1 M_0.
\]

From the same diagram, we can write the source as

\[
\hat{LM}_1 \cong \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \mathbb{D}_0 M_1 \mathbb{D}_1 M_0.
\]

We now verify preservation of targets from the commutativity of the following diagram, where the subscript \( \mathbb{D}_0 M_0 \)'s have been suppressed in the interest of space:
Preservation of sources is a similar diagram:

\[
\begin{array}{c}
\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \\
\xrightarrow{1 \times \chi \times 1} \\
\mathbb{D}_0 M_1 \times \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_1 M_0 \\
\xrightarrow{p_{34}} \\
\mathbb{D}_1 M_0 \times \mathbb{D}_1 M_0 \\
\end{array}
\]

The left unit map \( I_L : \hat{L} M_1 \to \hat{L} M_2 \) is given by the composite

\[
\hat{L} M_1 \cong \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \cong \mathbb{D}_0 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \\
\xrightarrow{(\mathbb{D}_0 I, I_2) \times 1} \\
\mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \cong \hat{L} M_2,
\]

and similarly the right unit map \( I_R \) is given by the composite

\[
\hat{L} M_1 \cong \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \cong \mathbb{D}_0 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \cong \hat{L} M_2.
\]

To show that \( \gamma \) is unital, we need to verify commutativity for both triangles in the diagram

\[
\hat{L} M_1 \xrightarrow{I_L} \hat{L} M_2 \xleftarrow{I_R} \hat{L} M_1 \\
\xrightarrow{=} \gamma \xleftarrow{=} \hat{L} M_1.
\]

This requires the following property of the interchange \( \chi : \mathbb{D}_1 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1 \to \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \):

**Lemma 10.1.** The composites

\[
\mathbb{D}_0 M_1 \cong \mathbb{D}_0 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1 \xrightarrow{I_0 \times 1} \mathbb{D}_1 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1 \\
\xrightarrow{\chi} \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \xrightarrow{p_1} \mathbb{D}_0 M_1
\]

and

\[
\mathbb{D}_1 M_0 \cong \mathbb{D}_1 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_0 \xrightarrow{1 \times \mathbb{D}_0 I_1} \mathbb{D}_1 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1 \\
\xrightarrow{\chi} \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \xrightarrow{p_2} \mathbb{D}_1 M_0
\]
both coincide with identity maps on \( \mathbb{D}_0 M_1 \) and \( \mathbb{D}_1 M_0 \), respectively.

**Proof.** The interchange map \( \chi \) factors as

\[
\mathbb{D}_1 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1 \cong \mathbb{D}_1 M_1 \xrightarrow{(T_D, \theta)} \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0,
\]

and examining the pullback diagram giving the isomorphism part of the composite shows easily that

\[
\mathbb{D}_0 M_1 \xrightarrow{(I, 1)} \mathbb{D}_1 M_1 \cong \mathbb{D}_1 M_1
\]

coincides with \( I_D : \mathbb{D}_0 M_1 \to \mathbb{D}_1 M_1 \). Now the composite

\[
\mathbb{D}_1 M_1 \xrightarrow{(T_D, \theta)} \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \xrightarrow{p_1} \mathbb{D}_0 M_1
\]

coinsides with \( T_D \), and since \( T_D \circ I_D = \text{id}_{\mathbb{D}_0 M_1} \), we see that the first composite is as claimed.

For the second composite, we again examine the pullback diagram giving \( \mathbb{D}_1 M_1 \cong \mathbb{D}_1 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1 \) and conclude that

\[
\mathbb{D}_1 M_0 \xrightarrow{(1, \mathbb{D}_0 I_M)} \mathbb{D}_1 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1 \cong \mathbb{D}_1 M_1
\]

coinsides with \( \mathbb{D}_1 I_M \). Now the conclusion about the second composite follows from the commutative diagram

\[
\begin{array}{ccc}
\mathbb{D}_1 M_1 & \xrightarrow{\mathbb{D}_1 I} & \mathbb{D}_1 M_0 \\
\downarrow \mathbb{D}_1 S & & \downarrow \mu \\
\mathbb{D}_1 \mathbb{D}_0 M_0 & \xrightarrow{\mathbb{D}_1 \eta_0} & \mathbb{D}_1 \mathbb{D}_0 M_1
\end{array}
\]

\[
\mathbb{D}_1 \mathbb{D}_0 M_0 \xrightarrow{\mathbb{D}_1 \eta_1} \mathbb{D}_1 \mathbb{D}_0 M_1.
\]

□

Now the left unitality triangle commutes if and only if it does so after composing with \( p_1 : \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \to \mathbb{D}_0 M_1 \) and \( p_2 : \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \to \mathbb{D}_1 M_0 \). Composing with \( p_1 \), we wish the composite

\[
\hat{LM}_1 \cong \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \cong \mathbb{D}_0 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1
\]

\[
\xrightarrow{\mathbb{D}_0 \gamma M \times \gamma} \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \xrightarrow{p_1} \mathbb{D}_0 M_1
\]
to coincide with just $p_1$. The next diagram shows that we can project off the last factor $D_1 M_0$ from the beginning, where we again have suppressed subscript $D_0 M_0$’s in the interest of space:

$$
\begin{align*}
D_0 M_1 \times D_1 M_0 & \xrightarrow{p_1} D_0 M_1 \\
\cong & \\
D_0 M_0 \times D_0 M_1 \times D_1 M_0 & \xrightarrow{p_{12}} D_0 M_0 \times D_0 M_1 \\
(D_0 I, I_0) \times 1 & \\
D_0 M_1 \times D_1 M_0 \times D_0 M_1 \times D_1 M_0 & \xrightarrow{p_{123}} D_0 M_1 \times D_1 M_0 \times D_0 M_1 \\
1 \times 1 \times 1 & \\
D_0 M_1 \times D_0 M_1 \times D_1 M_0 & \xrightarrow{p_{12}} D_0 M_1 \times D_0 M_1 \\
D_0 \gamma \times 1 & \\
D_0 M_1 \times D_1 M_0 & \xrightarrow{p_1} D_0 M_1.
\end{align*}
$$

Now it suffices to have the right column in the above diagram compose to the identity on $D_0 M_1$. But the following diagram shows that the part before the final $D_0 \gamma$ coincides with $D_0 I_L$, since the right hand composite is $id_{D_0 M_1}$ by Lemma 10.1:

Now composing, we have $D_0 \gamma \circ D_0 I_L = id_{D_0 M_1}$ since $M$ is left unital. This shows that $\gamma_{LM}$ is left unital after composing with $p_1$.

Showing that $\gamma_{LM}$ is left unital after composing with $p_2$ follows from the following diagram, in which we still need to verify the irregular sub-diagram in the upper right,
and as before, we suppress the subscript $\mathbb{D}_0 M_0$'s:

The upper right sub-diagram is the product with $\mathbb{D}_1 M_0$ of a diagram that expresses another aspect of the interchange map $\chi$, in which we also reflect across the main diagonal:

We have now shown that the composition in $\hat{L} M$ is left unital.

To show that the composition is right unital, we again compose with both $p_1$ and $p_2$ and verify the resulting diagrams. First composing with $p_2$, we first project off the
first factor by means of the following diagram:

\[
\begin{array}{ccccccccc}
\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 & \xrightarrow{p_2} & \mathbb{D}_1 M_0 \\
\cong & & \downarrow \cong \\
\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_0 & \xrightarrow{p_{23}} & \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_0 \\
1 \times (\mathbb{D}_0 I, I_0) & & & & 1 \times (\mathbb{D}_0 I, I_0) \\
\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 & \xrightarrow{p_{234}} & \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \\
1 \times \chi \times 1 & & & & \chi \times 1 \\
\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 & \xrightarrow{p_{34}} & \mathbb{D}_1 M_0 \times \mathbb{D}_1 M_0 \\
\mathbb{D}_0 \gamma \times \gamma_\mathbb{D} & & & & \gamma_\mathbb{D} \\
\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 & \xrightarrow{p_2} & \mathbb{D}_1 M_0.
\end{array}
\]

It now suffices to show that the right column composes to the identity, but the part before the $\gamma_\mathbb{D}$ coincides with the right unit map for $\mathbb{D}(M_0^\delta)$, because of the following diagram, in which the right composite is the identity on $\mathbb{D}_1 M_0$ by the second part of Lemma 10.1.

The composite with $\gamma_\mathbb{D}$ is therefore the identity, showing that $\gamma_{LM}$ is right unital after composing with $p_2$. 
Composing with \( p_1 \), we get a diagram similar to that for the left unital property composed with \( p_2 \), namely

\[
\begin{array}{ccc}
\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 & \xrightarrow{1 \times T_{\mathbb{D}} = p_1} & \mathbb{D}_0 M_1 \times \mathbb{D}_0 M_0 \\
\cong & & \\
\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_0 \xrightarrow{1 \times (\mathbb{D}_0 I, I_0)} & 1 \times \mathbb{D}_0 I & \\
\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_1 M_0 \xrightarrow{p_{123}} & \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \xrightarrow{1 \times \chi} & \mathbb{D}_0 M_1 \times \mathbb{D}_0 M_1 \\
\mathbb{D}_0 \gamma \times \gamma_0 \downarrow & & \mathbb{D}_0 \gamma \downarrow & \\
\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 & \xrightarrow{p_1} & \mathbb{D}_0 M_1.
\end{array}
\]

Again, we need to verify the large sub-diagram in the upper right, but that follows from the commutativity of

\[
\begin{array}{ccc}
\mathbb{D}_1 M_0 & \cong & \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_0 \mathbb{D}_0 M_0 \\
\cong & & \\
\mathbb{D}_1 M_1 \xrightarrow{\mathbb{D}_1 I} & \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_0 \mathbb{D}_0 M_1 \\
\mathbb{D}_0 \gamma \downarrow & & \mathbb{D}_0 \gamma \downarrow & \\
\mathbb{D}_0 M_0 \xrightarrow{\mathbb{D}_0 I} \mathbb{D}_0 M_1 & \xrightarrow{p_1} & \mathbb{D}_0 M_1 \times \mathbb{D}_0 M_1 \mathbb{D}_1 M_0
\end{array}
\]

after crossing on the left with \( \mathbb{D}_0 M_1 \). We conclude that \( \gamma_{LM} \) is right unital.

It remains to show that \( \gamma_{LM} \) is associative. This reduces to showing commutativity for each of the sub-diagrams of a square diagram with four sub-squares; however, the diagram is too large to fit onto a page, so we display the left half and right half separately. The left half is

\[
\begin{array}{ccc}
(\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0)^3 & \xrightarrow{1 \times \chi \times 1} & (\mathbb{D}_0 M_1)^2 \times (\mathbb{D}_1 M_0)^2 \times \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \\
1 \times \chi \times 1 & & \\
\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times (\mathbb{D}_0 M_1)^2 \times (\mathbb{D}_1 M_0)^2 & \xrightarrow{1 \times \chi^2 \times 1} & (\mathbb{D}_0 M_1)^3 \times (\mathbb{D}_1 M_0)^3 \\
1 \times \mathbb{D}_0 \gamma \times \gamma_0 & & \mathbb{D}_0 \gamma \times (\gamma_0 S) \\
\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 & \xrightarrow{1 \times \chi} & (\mathbb{D}_0 M_1)^2 \times (\mathbb{D}_1 M_0)^2,
\end{array}
\]
where as before we suppress subscript \( D_0 M_0 \)'s, and the right half is

\[
\begin{align*}
(D_0 M_1)^2 \times (D_1 M_0)^2 \times D_0 M_1 \times D_1 M_0 & \xrightarrow{\gamma \times 1 \times 1} (D_0 M_1 \times D_1 M_0)^2 \\
(D_0 M_1)^3 \times (D_1 M_0)^3 & \xrightarrow{D_0 \gamma \times (\gamma_D)^T} (D_0 M_1)^2 \times (D_1 M_0)^2 \\
D_0 \times \gamma & \times \gamma \times \gamma_D \\
(D_0 M_1)^2 \times (D_1 M_0)^2 & \xrightarrow{D_0 \gamma \times \gamma_D} D_0 M_1 \times D_1 M_0.
\end{align*}
\]

The top half of the left diagram reduces to a diagram on the inner four factors, and both ways around the square then reduce to the composite

\[
\begin{align*}
D_1 M_0 \times D_0 M_1 \times D_1 M_0 \times D_0 M_1 & \xrightarrow{\chi \times \chi} D_0 M_1 \times D_1 M_0 \times D_0 M_1 \times D_1 M_0 \\
D_0 M_1 \times D_1 M_0 \times D_0 M_1 \times D_1 M_0 & \xrightarrow{1 \times \chi \times 1} D_0 M_1 \times D_0 M_1 \times D_1 M_0 \times D_0 M_1.
\end{align*}
\]

The bottom half of the right diagram commutes because \( M \) is a multicategory and \( D \) is a category object in monads on \( \text{Set} \). This leaves us with the two other sub-squares.

We will continue to suppress subscript \( D_0 M_0 \)'s for the remainder of this section. For the lower left sub-square, we will need the fact, to be verified, that the interchange \( \chi \) commutes with the composition in \( M \), in the sense that the diagram

\[
\begin{align*}
D_1 M_0 \times D_0 M_1 \times D_0 M_1 & \xrightarrow{\chi \times 1} D_0 M_1 \times D_1 M_0 \times D_0 M_1 \\
1 \times D_0 \gamma & \xrightarrow{1 \times D_0 \gamma} D_1 M_0 \times D_0 M_1 \\
D_0 M_1 \times D_0 M_1 & \xrightarrow{\chi} D_0 M_1 \times D_1 M_0 \\
D_1 M_0 \times D_0 M_1 & \xrightarrow{D_0 \gamma \times 1} D_0 M_0 \times D_1 M_0
\end{align*}
\]

commutes. We need to recall that the interchange is given by the composite of an isomorphism to \( D_1 M_1 \) with the map

\[
D_1 M_1 \xrightarrow{(\gamma_D, \theta)} D_0 M_1 \times_{D_0 M_0} D_1 M_0,
\]

where the map \( \theta \) is the composite

\[
D_1 M_1 \xrightarrow{D_1 \mu} D_1 D_0 M_0 \xrightarrow{D_1 \mu} D_1^3 M_0 \xrightarrow{\mu} D_1 M_0.
\]

We also agree to write \( \theta \) for the related composite

\[
D_1 M_2 \xrightarrow{D_1 \mu} D_1 D_0 M_1 \xrightarrow{D_1 \mu} D_1^3 M_1 \xrightarrow{\mu} D_1 M_1.
\]
Now turning our diagram on its side, we can fill it in as follows with the sub-diagrams still to be explained and verified:

\[
\begin{array}{c}
\mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \times \mathbb{D}_0 M_1 \\
\cong \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \\
\end{array}
\]

The first thing we explain is the isomorphism out of the upper left corner

\[
\mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \times \mathbb{D}_0 M_1 \cong \mathbb{D}_1 M_1,
\]

where we have suppressed subscript \(\mathbb{D}_0 M_0\)'s. This is a consequence of the following diagram, in which all the squares are pullbacks:

Now since the upper right isomorphism \(\mathbb{D}_1 M_1 \cong \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \cong \mathbb{D}_1 M_2\),

the top part of the diagram is a consequence of the commutative squares

\[
\begin{array}{c}
\mathbb{D}_1 M_2 \xrightarrow{\mathbb{D}_1 T} \mathbb{D}_1 M_1 \\
\cong \mathbb{D}_1 M_0 \\
\end{array}
\]

Now since the upper right isomorphism \(\mathbb{D}_1 M_1 \cong \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1\) is given by \((\mathbb{D}_1 T, S_\mathbb{D})\),

the top part of the diagram is a consequence of the commutative squares

\[
\begin{array}{c}
\mathbb{D}_1 M_2 \xrightarrow{\mathbb{D}_1 T} \mathbb{D}_1 M_1 \xrightarrow{\mathbb{D}_1 T} \mathbb{D}_1 M_0 \\
S_\mathbb{D} \downarrow \quad \downarrow \quad \downarrow S_\mathbb{D} \\
\mathbb{D}_0 M_2 \xrightarrow{\mathbb{D}_0 T} \mathbb{D}_0 M_1 \xrightarrow{\mathbb{D}_0 T} \mathbb{D}_0 M_0 \\
\mathbb{D}_0 S \downarrow \quad \downarrow \quad \downarrow \mathbb{D}_0 S \\
\mathbb{D}_0^2 M_1 \xrightarrow{\mathbb{D}_0^2 T} \mathbb{D}_0^2 M_0 \\
\mu \downarrow \quad \downarrow \mu \\
\mathbb{D}_0 M_1 \xrightarrow{\mathbb{D}_0 T} \mathbb{D}_0 M_0.
\end{array}
\]

For the left part of the diagram, the same display of pullback squares shows us that

the inverse of the upper left isomorphism can be expressed as

\[
(\mathbb{D}_1 T, \mu \circ \mathbb{D}_0 S \circ S_\mathbb{D}) : \mathbb{D}_1 M_2 \to \mathbb{D}_1 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1,
\]

The first thing we explain is the isomorphism out of the upper left corner

\[
\mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \times \mathbb{D}_0 M_1 \cong \mathbb{D}_1 M_2,
\]

where we have suppressed subscript \(\mathbb{D}_0 M_0\)'s. This is a consequence of the following diagram, in which all the squares are pullbacks:

Now since the upper right isomorphism \(\mathbb{D}_1 M_1 \cong \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \cong \mathbb{D}_1 M_2\),

the top part of the diagram is a consequence of the commutative squares

\[
\begin{array}{c}
\mathbb{D}_1 M_2 \xrightarrow{\mathbb{D}_1 T} \mathbb{D}_1 M_1 \\
\mathbb{D}_0 M_2 \xrightarrow{\mathbb{D}_0 T} \mathbb{D}_0 M_1 \\
\mathbb{D}_0 M_1 \xrightarrow{\mathbb{D}_0 T} \mathbb{D}_0 M_0 \\
\end{array}
\]

Now since the upper right isomorphism \(\mathbb{D}_1 M_1 \cong \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1\) is given by \((\mathbb{D}_1 T, S_\mathbb{D})\),

the top part of the diagram is a consequence of the commutative squares

\[
\begin{array}{c}
\mathbb{D}_1 M_2 \xrightarrow{\mathbb{D}_1 T} \mathbb{D}_1 M_1 \\
\mathbb{D}_1 M_1 \xrightarrow{\mathbb{D}_1 T} \mathbb{D}_1 M_0 \\
\end{array}
\]

Now since the upper right isomorphism \(\mathbb{D}_1 M_1 \cong \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1\) is given by \((\mathbb{D}_1 T, S_\mathbb{D})\),

the top part of the diagram is a consequence of the commutative squares

\[
\begin{array}{c}
\mathbb{D}_1 M_2 \xrightarrow{\mathbb{D}_1 T} \mathbb{D}_1 M_1 \\
\mathbb{D}_1 M_1 \xrightarrow{\mathbb{D}_1 T} \mathbb{D}_1 M_0 \\
\end{array}
\]

Now since the upper right isomorphism \(\mathbb{D}_1 M_1 \cong \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1\) is given by \((\mathbb{D}_1 T, S_\mathbb{D})\),

the top part of the diagram is a consequence of the commutative squares

\[
\begin{array}{c}
\mathbb{D}_1 M_2 \xrightarrow{\mathbb{D}_1 T} \mathbb{D}_1 M_1 \\
\mathbb{D}_1 M_1 \xrightarrow{\mathbb{D}_1 T} \mathbb{D}_1 M_0 \\
\end{array}
\]

Now since the upper right isomorphism \(\mathbb{D}_1 M_1 \cong \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1\) is given by \((\mathbb{D}_1 T, S_\mathbb{D})\),

the top part of the diagram is a consequence of the commutative squares

\[
\begin{array}{c}
\mathbb{D}_1 M_2 \xrightarrow{\mathbb{D}_1 T} \mathbb{D}_1 M_1 \\
\mathbb{D}_1 M_1 \xrightarrow{\mathbb{D}_1 T} \mathbb{D}_1 M_0 \\
\end{array}
\]
while the inverse of the isomorphism below it is induced by

\[(D_1T, S_D) : D_1M_1 \to D_1M_0 \times_{D_0M_0} D_0M_1.\]

Meanwhile, we recall that the interchange \(\chi\) is given by

\[(T_D, \theta) : D_1M_1 \to D_0M_1 \times_{D_0M_0} D_1M_0.\]

Now proceeding from \(D_1M_2\) both ways to \(D_0M_1 \times D_1M_0 \times D_0M_1\) and projecting to the left \(D_0M_1\), we see that both ways coincide with \(T_D \circ D_1T\). Projecting to \(D_1M_0\), we find we require \(\theta \circ T_D = T_D \circ \theta\), but that is expressed by the commutative diagram

\[
\begin{array}{c}
D_1M_2 \\ \downarrow D_1S \\
D_1D_0M_1 \\ \downarrow D_1\delta \\
D_1^2M_1 \\ \downarrow \mu \\
D_1M_1 \\
\end{array}
\rightarrow
\begin{array}{c}
D_1M_1 \\ \downarrow D_1S \\
D_1D_0M_0 \\ \downarrow D_1\delta \\
D_1^2M_0 \\ \downarrow \mu \\
D_1M_0. \\
\end{array}
\]

And projecting to the right hand \(D_0M_1\) requires

\[\mu \circ D_0S \circ S_D = S_D \circ \theta,\]

but that follows from the following commutative diagram:

\[
\begin{array}{c}
D_1M_2 \\ \downarrow D_1S \\
D_1D_0M_1 \\ \downarrow D_0\delta \\
D_1^2M_1 \\ \downarrow \mu \\
D_1M_1 \\
\end{array}
\rightarrow
\begin{array}{c}
D_0M_2 \\ \downarrow D_0S \\
D_0M_0 \\ \downarrow D_0\delta \\
D_0^2M_1 \\ \downarrow \mu \\
D_0M_1. \\
\end{array}
\]

The left quadrilateral of the diagram therefore commutes.

The bottom left triangle commutes by the definition of \(\chi\).
We come next to the (somewhat distorted) square

\[
\begin{array}{ccc}
\mathbb{D}_1 M_2 & \xrightarrow{(T_D, \theta^2)} & \mathbb{D}_0 M_2 \times \mathbb{D}_1 M_0 \\
\downarrow_{(T_D \circ T, \theta)} & & \downarrow_{\cong} \\
\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_1 & \xrightarrow{1 \times (T_D, \theta)} & \mathbb{D}_0 M_1 \times \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0.
\end{array}
\]

The right vertical isomorphism is a consequence of the diagram of pullback squares

\[
\begin{array}{ccc}
\mathbb{D}_0 M_2 & \xrightarrow{D_0 T} & \mathbb{D}_0 M_1 \\
\downarrow D_0 S & & \downarrow D_0 S \\
\mathbb{D}_0^2 M_1 & \xrightarrow{D_0^2 T} & \mathbb{D}_0^2 M_0 \\
\mu & & \mu \\
\mathbb{D}_0 M_1 & \xrightarrow{D_0 T} & \mathbb{D}_0 M_0,
\end{array}
\]

so can be expressed as

\[(D_0 T, \mu \circ D_0 S) : \mathbb{D}_0 M_2 \rightarrow \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1.\]

We see immediately that the desired square commutes after projecting to the first \(\mathbb{D}_0 M_1\) or the \(\mathbb{D}_1 M_0\). Projecting to the middle \(\mathbb{D}_0 M_1\), we require

\[\mu \circ D_0 S \circ T_D = T_D \circ \theta,
\]

but that follows from the commutativity of

\[
\begin{array}{ccc}
\mathbb{D}_1 M_2 & \xrightarrow{T_D} & \mathbb{D}_0 M_2 \\
\downarrow D_1 S & & \downarrow D_0 S \\
\mathbb{D}_1 \mathbb{D}_0 M_1 & \xrightarrow{T_D} & \mathbb{D}_0 \mathbb{D}_0 M_1 \\
\downarrow D_1 T_D & & \downarrow D_0 T_D \\
\mathbb{D}_1^2 M_1 & \xrightarrow{T_D^2} & \mathbb{D}_0^2 M_1 \\
\mu & & \mu \\
\mathbb{D}_1 M_1 & \xrightarrow{T_D} & \mathbb{D}_0 M_1.
\end{array}
\]

The triangle at the bottom of our desired diagram simply expresses the definition of \(\mathbb{D}_0 \gamma\).
Now we have another somewhat distorted square to verify, namely
\[
\begin{array}{ccc}
\mathbb{D}_1 M_2 & \xrightarrow{\mathbb{D}_1 \gamma} & \mathbb{D}_1 M_1 \\
(T_0, \theta^2) & \downarrow & (T_0, \theta) \\
\mathbb{D}_0 M_2 \times \mathbb{D}_1 M_0 & \xrightarrow{\mathbb{D}_0 \gamma \times 1} & \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0,
\end{array}
\]
as usual with subscript \(\mathbb{D}_0 M_0\)'s suppressed. Projecting to the factor of \(\mathbb{D}_0 M_1\), this is just the naturality of \(T_\mathbb{D}\) with respect to \(\gamma : M_2 \to M_1\). Projecting to the factor of \(\mathbb{D}_1 M_0\), we need to verify that the square
\[
\begin{array}{ccc}
\mathbb{D}_1 M_2 & \xrightarrow{\mathbb{D}_1 \gamma} & \mathbb{D}_1 M_1 \\
\theta & \downarrow & \theta \\
\mathbb{D}_1 M_1 & \xrightarrow{\theta} & \mathbb{D}_1 M_0
\end{array}
\]
commutes. This follows from its expansion as follows, using the definition of \(\theta\):
\[
\begin{array}{ccc}
\mathbb{D}_1 M_2 & \xrightarrow{\mathbb{D}_1 \gamma} & \mathbb{D}_1 M_1 \\
\mathbb{D}_1 S & \downarrow & \mathbb{D}_1 S \\
\mathbb{D}_1 \mathbb{D}_0 M_1 & \xrightarrow{\mathbb{D}_1 \mathbb{D}_0 S} & \mathbb{D}_1 \mathbb{D}_0^2 M_0 \xrightarrow{\mathbb{D}_1 \mu} \mathbb{D}_1 \mathbb{D}_0 M_0 \\
\mathbb{D}_1 I_0 & \downarrow & \mathbb{D}_1 I_0 \\
\mathbb{D}_1^2 M_1 & \xrightarrow{\mathbb{D}_1^2 S} & \mathbb{D}_1^2 M_0 \xrightarrow{\mathbb{D}_1^2 I_0} \mathbb{D}_1^3 M_0 \xrightarrow{\mathbb{D}_1 \mu} \mathbb{D}_1^2 M_0 \\
\mu & \downarrow & \mu & \mu \\
\mathbb{D}_1 M_1 & \xrightarrow{\mathbb{D}_1 S} \mathbb{D}_1 \mathbb{D}_0 M_0 \xrightarrow{\mathbb{D}_1 I_0} \mathbb{D}_1^2 M_0 \xrightarrow{\mu} \mathbb{D}_1 M_0.
\end{array}
\]
The final right hand triangle simply exhibits the definition of \(\chi\). This completes the verification that \(\chi\) respects the multiplication \(\mathbb{D}_0 \gamma\).

We return now to the lower part of the left half of the associativity diagram,
\[
\begin{array}{ccc}
\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times (\mathbb{D}_0 M_1)^2 \times (\mathbb{D}_1 M_0)^2 \xrightarrow{1 \times \chi \times 1 \times 1} (\mathbb{D}_0 M_1)^3 \times (\mathbb{D}_1 M_0)^3 \\
\mathbb{D}_0 \gamma S \times (\gamma) S & \downarrow & \mathbb{D}_0 \gamma S \times (\gamma) S \\
\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \xrightarrow{1 \times \chi \times 1} (\mathbb{D}_0 M_1)^2 \times (\mathbb{D}_1 M_0)^2,
\end{array}
\]
and note that \(\mathbb{D}_0 \gamma S\) is really just
\[
1 \times \mathbb{D}_0 \gamma : \mathbb{D}_0 M_1 \times (\mathbb{D}_0 M_1)^2 \to \mathbb{D}_0 M_1 \times \mathbb{D}_0 M_1,
\]
where we have again suppressed subscript \(\mathbb{D}_0 M_0\)'s. Similarly, \((\gamma) S\) is also really just
\[
1 \times \gamma_\mathbb{D},
\]
so this diagram is just the identity on the left \(\mathbb{D}_0 M_1, \gamma_\mathbb{D}\) on the right two
\(\mathbb{D}_1 M_0\)'s, and the rest is the diagram we just verified. This concludes the verification that the lower left square of the associativity diagram commutes.

For the upper right square,

\[
(\mathbb{D}_0 M_1)^2 \times (\mathbb{D}_1 M_0)^2 \times \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \xrightarrow{\mathbb{D}_0 \gamma \times \mathbb{D}_0 \times 1} (\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0)^2
\]

we note that, similarly to the above,

\[
\mathbb{D}_0 \gamma_T = \mathbb{D}_0 \gamma \times 1 \quad \text{and} \quad (\gamma_T)\mathbb{D} = \gamma_D \times 1.
\]

Consequently the diagram decomposes to just \(\mathbb{D}_0 \gamma\) on the first two factors, and \(\text{id}_{\mathbb{D}_1 M_0}\) on the last one, leaving us with the diagram

\[
\begin{array}{ccc}
\mathbb{D}_1 M_0 \times \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 & \xrightarrow{\gamma_0 \times 1} & \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \\
\text{id} \times \chi & & \chi \\
\mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 & x & \\
\chi \times \chi & & \\
\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_1 M_0 & \xrightarrow{1 \times \gamma_D} & \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0
\end{array}
\]

to verify. We fill it in with an interior analogous to the one we used for the lower left square:

Here we use the notation \(\hat{\theta}\) for the composite

\[
\mathbb{D}_2 M_1 \xrightarrow{\mathbb{D}_2 S} \mathbb{D}_2 \mathbb{D}_0 M_0 \xrightarrow{\mathbb{D}_2 \mathbb{I}_0^2} \mathbb{D}_2^2 M_0 \xrightarrow{\mu} \mathbb{D}_2 M_0,
\]
and the map $I_D^2 : \mathcal{D}_0 M_0 \to \mathcal{D}_2 M_0$ means either of the coincident composites

\[
\begin{array}{c}
\mathcal{D}_0 M_0 \xrightarrow{I_D} \mathcal{D}_1 M_0 \xrightarrow{(I_D)_L} \mathcal{D}_2 M_0 \quad \text{or} \quad \mathcal{D}_0 M_0 \xrightarrow{I_D} \mathcal{D}_1 M_0 \xrightarrow{(I_D)_R} \mathcal{D}_2 M_0.
\end{array}
\]

(The composites coincide since $\mathcal{D}$ is a category object.)

We proceed to verify the sub-diagrams. The upper left isomorphism is a consequence of the diagram of pullback squares

\[
\begin{array}{c}
\mathcal{D}_2 M_1 \xrightarrow{D_2 T} \mathcal{D}_2 M_0 \xrightarrow{T_D} \mathcal{D}_1 M_0 \\
\downarrow S_D \quad \downarrow S_D \quad \downarrow S_D \\
\mathcal{D}_1 M_1 \xrightarrow{D_1 T} \mathcal{D}_1 M_0 \xrightarrow{T_D} \mathcal{D}_0 M_0 \\
\downarrow S_D \quad \downarrow S_D \quad \downarrow S_D \\
\mathcal{D}_0 M_1 \xrightarrow{D_0 T} \mathcal{D}_0 M_0,
\end{array}
\]

where the right hand square is a pullback since $\mathcal{D}$ is a category object, the lower left square is a pullback since $T : M_1^\delta \to M_0^\delta$ is a cover and $\mathcal{D}$ preserves covers, and the upper left square is a pullback from an application of the source cover version of Corollary 8.1 to the cover $\mathcal{D} T : \mathcal{D}(M_1^\delta) \to \mathcal{D}(M_0^\delta)$.

Now the top part of our desired diagram is a consequence of the two commutative squares

\[
\begin{array}{c}
\mathcal{D}_2 M_1 \xrightarrow{D_2 T} \mathcal{D}_2 M_0 \quad \text{and} \quad \mathcal{D}_2 M_1 \xrightarrow{S_D} \mathcal{D}_1 M_1 \\
\downarrow \gamma_D \quad \downarrow \gamma_D \\
\mathcal{D}_1 M_1 \xrightarrow{D_1 T} \mathcal{D}_1 M_0 \\
\downarrow \gamma_D \\
\mathcal{D}_0 M_1.
\end{array}
\]

For the left part of the diagram, we first use the display of pullbacks giving the upper left isomorphism to rewrite it as

\[(T_D \circ D_2 T, S_D) : \mathcal{D}_2 M_1 \to \mathcal{D}_1 M_0 \times_{\mathcal{D}_0 M_0} \mathcal{D}_1 M_1.\]

Also recalling again that $\chi$ is induced by the map

\[(T_D, \theta) : \mathcal{D}_1 M_0 \to \mathcal{D}_0 M_1 \times_{\mathcal{D}_0 M_0} \mathcal{D}_1 M_0,
\]

we verify the left part after projecting to each of the three factors in the lower left corner. Projecting to the right hand $\mathcal{D}_1 M_0$ gives in each case $\theta \circ S_D$, so that checks. Projecting to the left hand $\mathcal{D}_1 M_1$ requires

\[T_D \circ D_2 T = D_1 T \circ S_D,
\]

but that follows from the naturality of $S_D$. And projecting to $\mathcal{D}_0 M_1$, we require $T_D \circ S_D = S_D \circ T_D$, but that is a consequence of the defining diagram for $\mathcal{D}_2 M_1$. The left part of the diagram therefore commutes.
For the square

\[
\begin{array}{cccc}
\mathbb{D}_2 M_1 & \xrightarrow{(T_2^2, \hat{\theta})} & \mathbb{D}_0 M_1 \times \mathbb{D}_2 M_0 \\
\downarrow^{(T_0, \theta \circ S_D)} & & \downarrow^{\cong} \\
\mathbb{D}_1 M_1 \times \mathbb{D}_1 M_0 & \xrightarrow{(T_0, \theta) \times 1} & \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_1 M_0,
\end{array}
\]

the right hand isomorphism is induced by the isomorphism

\[(T_D, S_D) : \mathbb{D}_2 M_0 \to \mathbb{D}_1 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0,
\]

so we can again verify commutativity by checking after projection to each of the three factors on the lower right. Projecting to the \(\mathbb{D}_0 M_1\), both composites are \(T_2^2\). Projecting to the right hand \(\mathbb{D}_1 M_0\), we require \(\theta \circ S_D = S_D \circ \hat{\theta}\). This follows from the following diagram, in which we are careful to use \((I_D)_L\) rather than \((I_D)_R\) so that the inner triangle commutes:
And projecting to the middle $D_1 M_0$ requires us to verify $\theta \circ T_D = T_D \circ \hat{\theta}$. This follows from the following diagram, in which we now use $(I_D)_R$ so the triangle commutes:

\[
\begin{array}{c}
D_2 M_1 \xrightarrow{T_D} D_1 M_1 \\
\downarrow \quad \downarrow \\
D_2 D_0 M_0 \xrightarrow{T_D} D_1 D_0 M_0 \\
\downarrow \quad \downarrow \\
D_2 I_D \quad D_2 D_1 M_0 \xrightarrow{T_D} D_1 I_D \\
\downarrow \quad \downarrow \\
D_2 (I_D)_R \quad D_2 D_1 M_0 \xrightarrow{T_D} D_1 D_1 M_0 \\
\downarrow \quad \downarrow \\
D_2^2 M_0 \xrightarrow{T_D^2} D_1^2 M_0 \\
\downarrow \quad \downarrow \\
\mu \quad \mu \\
D_2 M_0 \xrightarrow{T_D} D_1 M_0.
\end{array}
\]

We next have to verify the square

\[
\begin{array}{c}
D_2 M_1 \xrightarrow{\gamma_D} D_1 M_1 \\
\downarrow (T_D^2, \theta) \quad \downarrow (T_D, \theta) \\
D_0 M_1 \times D_2 M_0 \xrightarrow{1 \times \gamma_D} D_0 M_1 \times D_1 M_0.
\end{array}
\]

Projecting to $D_0 M_1$ reduces to the commutative square

\[
\begin{array}{c}
D_2 M_1 \xrightarrow{\gamma_D} D_1 M_1 \\
\downarrow T_D \quad \downarrow T_D \\
D_1 M_1 \xrightarrow{T_D} D_0 M_1.
\end{array}
\]

while projecting to $D_1 M_0$ requires $\theta \circ \gamma_D = \gamma_D \circ \hat{\theta}$. This follows from the following diagram, in which the triangle commutes since $\gamma_D$ is unital (we could use either $(I_D)_L$ or $(I_D)_R$), and the bottom square is a result of $\mathbb{D}$ being a category object in monads.
The left triangle simply records the definition of $\chi$. We have completed the verification of the upper right square of the associativity diagram, and therefore completed the verification that the composition on $\hat{LM}$ is associative.

11. The provisional left adjoint: $\mathbb{D}$-algebra structure

In this section we specify a $\mathbb{D}$-algebra structure on $\hat{LM}$ by specifying a $\mathbb{D}_0$-action on $\hat{LM}_0 = \mathbb{D}_0 M_0$ and a $\mathbb{D}_1$-action on $\hat{LM}_1$. We then verify that the structure maps for a category are preserved, so we actually get an action $\mathbb{D}\hat{LM} \to \hat{LM}$ that is a functor.

First, since $\hat{LM}_0 = \mathbb{D}_0 M_0$ is the free $\mathbb{D}_0$-algebra on $M_0$, we use that as its algebra structure over $\mathbb{D}_0$. Explicitly, we have an action map given by

$$\mathbb{D}_0(\hat{LM}_0) = \mathbb{D}_0^2 M_0 \xrightarrow{\mu} \mathbb{D}_0 M_0.$$
This does give us a map to \( \hat{LM}_1 = D_0 M_1 \times_{D_0 M_0} D_1 M_0 \) because of the following commuting diagram; note that the left rectangle is simply \( D_1 \) applied to the pullback defining \( \hat{LM}_1 \):

We wish to show that this really is an action, that is, that \( \xi_1 \) is unital and associative. To be unital, the diagram

\[
\begin{array}{ccc}
\hat{L}M_1 & \xrightarrow{\eta} & D_1 \hat{L}M_1 \\
\downarrow & = & \downarrow \xi_1 \\
\hat{L}M_1 & \xrightarrow{\xi_1} & D_1 M_1 \\
\end{array}
\]

must commute, but that will follow if the diagram commutes after composing with both projections \( p_1 \) and \( p_2 \) with targets \( D_0 M_1 \) and \( D_1 M_0 \) respectively. For composition with \( p_1 \), we have the diagram

\[
\begin{array}{ccc}
\hat{L}M_1 & \xrightarrow{\eta} & D_1 \hat{L}M_1 \\
& \downarrow p_1 & \downarrow p_1 \eta_{D_1 D_0} \\
D_0 M_1 & \xrightarrow{\epsilon_0} & D_0^2 M_1 \\
\end{array}
\]
where the central triangle commutes since $T_D : D_1 \to D_0$ is a map of monads. For composition with $p_2$, we have the commuting diagram

\[
\begin{array}{ccc}
\hat{L}M_1 & \xrightarrow{\eta} & \hat{D}_1 \hat{L}M_1 \\
p_2 & & \downarrow \hat{D}_1 p_2 \\
\hat{D}_1 M_0 & \xrightarrow{\eta} & \hat{D}_1^2 M_0 \\
& \searrow {\overset{\mu}{\Rightarrow}} & \\
& & \hat{D}_1 M_0.
\end{array}
\]

and it follows that $\xi_1$ is unital.

To show that the action is associative, we need to show that the diagram

\[
\begin{array}{ccc}
\hat{D}_1^2 \hat{L}M_1 & \xrightarrow{\hat{D}_1 \xi_1} & \hat{D}_1 \hat{L}M_1 \\
\downarrow \mu & & \downarrow \xi_1 \\
\hat{D}_1 \hat{L}M_1 & \xrightarrow{\xi_1} & LM_1
\end{array}
\]

commutes, which again happens if and only if it does after composing with $p_1$ and $p_2$. Composing with $p_1$, we find the diagram commutes as a result of the following:
And composing with $p_2$, we have the following commutative diagram:

$$
\begin{array}{c}
\begin{array}{ccc}
\mathbb{D}_2 \hat{LM}_1 & \xrightarrow{\mathbb{D}_1 \xi_1} & \mathbb{D}_1 \hat{LM}_1 \\
\mathbb{D}_2 \hat{LM}_0 & \xrightarrow{\mathbb{D}_1 p_2} & \mathbb{D}_1 M_0 \\
\mathbb{D}_1 \hat{LM}_1 & \xrightarrow{\mathbb{D}_1 p_2} & \mathbb{D}_1 M_0 \\
\end{array}
\end{array}
\end{array}
$$

It now follows that $\xi_1$ is an action of the monad $\mathbb{D}_1$ on $\hat{LM}_1$.

We need to show that the actions $\xi_0$ and $\xi_1$ preserve the identity, source, target, and composition maps, so that we actually get a functor $\mathbb{D} \hat{LM} \to \hat{LM}$. For preservation of $I$, we need the diagram

$$
\begin{array}{c}
\begin{array}{ccc}
\mathbb{D}_0 \hat{LM}_0 & \xrightarrow{\xi_0} & \hat{LM}_0 \\
\mathbb{D}_1 \hat{LM}_1 & \xrightarrow{\mathbb{D}_1 p_2} & \mathbb{D}_1 M_0 \\
\end{array}
\end{array}
$$

to commute. Since the target of the diagram is the pullback $\hat{LM}_1 = \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0$, we project onto each factor and verify the resulting diagrams. Composing with $p_1$ and using the defining property that $p_1 \circ \xi_1 = \mu \circ T_D \circ p_1$, as well as $\mathbb{D}_0 \hat{LM}_0 = \mathbb{D}_0^2 M_0$ and $p_1 \circ I_{\hat{LM}} = \mathbb{D}_0 I$, the result for $p_1$ follows from the commutative diagram

$$
\begin{array}{c}
\begin{array}{ccc}
\mathbb{D}_0^2 M_0 & \xrightarrow{\mathbb{D}_0^2 I} & \mathbb{D}_0 M_0 \\
\mathbb{D}_1 \mathbb{D}_0 M_0 & \xrightarrow{T_D} & \mathbb{D}_0^2 M_0 \\
\mathbb{D}_1 \hat{LM}_1 & \xrightarrow{\mathbb{D}_1 p_1} & \mathbb{D}_1 \mathbb{D}_0 M_1 \\
\end{array}
\end{array}
$$
Composing with \( p_2 \) and recalling that \( p_2 \circ \xi_1 = \mu \circ p_2 \), we wish the perimeter of to commute, but that is simply a property of a category object in monads on \( \text{Cat} \).

It follows that our action maps preserve the identity structure maps.

To show that our action maps preserve the source structure maps, we recall that the source map for \( \hat{\Lambda}M \) is given by

\[
\hat{\Lambda}M \cong \mathbb{D}_0M_1 \times_{\mathbb{D}_0M_0} \mathbb{D}_1M_0 \xrightarrow{p_2} \mathbb{D}_1M_0 \xrightarrow{S_0} \mathbb{D}_0M_0,
\]

and we wish

\[
\mathbb{D}_1\hat{\Lambda}M_1 \xrightarrow{\xi_1} \hat{\Lambda}M_1
\]

\[
\mathbb{D}_0^2M_0 \xrightarrow{\mu} \mathbb{D}_0M_0
\]

to commute. But this expands to

\[
\mathbb{D}_1\hat{\Lambda}M_1 \xrightarrow{\xi_1} \hat{\Lambda}M_1
\]

\[
\mathbb{D}_1^2M_0 \xrightarrow{\mu} \mathbb{D}_1M_0
\]

\[
\mathbb{D}_1\mathbb{D}_0M_0 \xrightarrow{S_0} \mathbb{D}_0^2M_0 \xrightarrow{\mu} \mathbb{D}_0M_0,
\]

in which the top square commutes by the definition of \( \xi_1 \) and the bottom rectangle because we have a category object in monads on \( \text{Cat} \).

For preservation of the target structure maps, recall that the target on \( \hat{\Lambda}M \) is given by the composite

\[
\hat{\Lambda}M_1 \cong \mathbb{D}_0M_1 \times_{\mathbb{D}_0M_0} \mathbb{D}_0M_1 \xrightarrow{p_1} \mathbb{D}_0M_1 \xrightarrow{D_0T} \mathbb{D}_0M_0
\]
Then the desired square

\[
\begin{array}{ccc}
\mathbb{D}_1 \hat{L}M_1 & \xrightarrow{\xi_1} & \hat{L}M_1 \\
\mathbb{D}_0 \hat{M}_0 & \xrightarrow{\mu} & \mathbb{D}_0 M_0
\end{array}
\]

expands to

\[
\begin{array}{ccc}
\mathbb{D}_1 \hat{L}M_1 & \xrightarrow{\xi_1} & \hat{L}M_1 \\
\mathbb{D}_1 D_0 M_1 & \xrightarrow{\hat{T}_D} & \mathbb{D}_0 M_0 \\
\mathbb{D}_1 D_0 T & \xrightarrow{\hat{D}_T} & \mathbb{D}_0 T
\end{array}
\]

in which the top part follows from our definition of \(\xi_1\), and the bottom from having a category object in monads in \(\mathbf{Cat}\).

It remains to show that the actions commute with the composition map \(\gamma_{LM} : \hat{L}M_2 \to \hat{L}M_1\). In order for this to make sense, we need an action of \(\mathbb{D}_2\) on \(\hat{L}M_2\); this is given by a map \(\xi_2 : \mathbb{D}_2 \hat{L}M_2 \to \hat{L}M_2\) such that the two diagrams

\[
\begin{array}{ccc}
\mathbb{D}_2 \hat{L}M_2 & \xrightarrow{\xi_2} & \hat{L}M_2 \\
\mathbb{D}_1 \hat{L}M_1 & \xrightarrow{\xi_1} & \hat{L}M_1
\end{array}
\]

commute. We then want the diagram

\[
\begin{array}{ccc}
\mathbb{D}_2 \hat{L}M_2 & \xrightarrow{\xi_2} & \hat{L}M_2 \\
\mathbb{D}_1 \hat{L}M_1 & \xrightarrow{\xi_1} & \hat{L}M_1
\end{array}
\]

\[\gamma_0 \gamma_{LM} \quad \gamma_{LM}\]

\[
\begin{array}{ccc}
\mathbb{D}_2 \hat{L}M_2 & \xrightarrow{\xi_2} & \hat{L}M_2 \\
\mathbb{D}_1 \hat{L}M_1 & \xrightarrow{\xi_1} & \hat{L}M_1
\end{array}
\]

for this purpose, we want an explicit expression for \(\xi_2\), which is given by the following lemma.
Lemma 11.1. The action map $\xi_2 : \mathbb{D}_2 \hat{L}M_2 \to \hat{L}M_2$ can be expressed as the composite
\[
\mathbb{D}_2 \mathbb{D}_0 M_1 \times_{\mathbb{D}_2 D_0 M_0} \mathbb{D}_2 D_1 M_1 \times_{\mathbb{D}_2 D_0 M_0} \mathbb{D}_2 D_1 M_0 \\
\xrightarrow{T_0^2 \times T_0^2 T_0^2 \times T_0^2 S_0 S_0}
\mathbb{D}_0^2 M_1 \times_{\mathbb{D}_0^2 M_0} \mathbb{D}_1^2 M_1 \times_{\mathbb{D}_0^2 M_0} \mathbb{D}_1^2 M_0 \\
\xrightarrow{\mu \times \mu \times \mu \times \mu}
\mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0.
\]

Proof. This is true if and only if the two diagrams defining $\xi_2$ commute with this expression in place of $\xi_2$, and in turn, each of those diagrams commute if and only if they commute when composed with each of $p_1 : \hat{L}M_1 \to \mathbb{D}_0 M_1$ and $p_2 : \hat{L}M_1 \to \mathbb{D}_1 M_0$.

We can expand $S, T : \hat{L}M_2 \to \hat{L}M_1$ as in the following two diagrams:

\[
\begin{array}{c}
\mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \xrightarrow{=} \hat{L}M_2 \\
\xrightarrow{p_{23}}
\mathbb{D}_1 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \\
\xrightarrow{S \times 1}
\mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \xrightarrow{=} \hat{L}M_1
\end{array}
\]

and

\[
\begin{array}{c}
\mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \xrightarrow{=} \hat{L}M_2 \\
\xrightarrow{p_{12}}
\mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_1 \\
\xrightarrow{1 \times T}
\mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \xrightarrow{=} \hat{L}M_1.
\end{array}
\]

Consequently, we can express four of our composites of interest as follows:

\[
\begin{align*}
p_1 \circ T : \hat{L}M_2 & \xrightarrow{p_1} \mathbb{D}_0 M_1, \\
p_2 \circ T : \hat{L}M_2 & \xrightarrow{p_2} \mathbb{D}_1 M_1 \xrightarrow{\mathbb{D}_1 T} \mathbb{D}_1 M_0, \\
p_1 \circ S : \hat{L}M_2 & \xrightarrow{p_2} \mathbb{D}_1 M_1 \xrightarrow{S_0} \mathbb{D}_0 M_1, \\
p_2 \circ S : \hat{L}M_2 & \xrightarrow{p_2} \mathbb{D}_1 M_0.
\end{align*}
\]
Further, we already have expressions for $p_1 \circ \xi_1$ and $p_2 \circ \xi_1$, namely

$$p_1 \circ \xi_1 : \mathbb{D}_1 \hat{\mathcal{L}} \mathfrak{M}_1 \xrightarrow{\mathbb{D}_1 p_1} \mathbb{D}_1 \mathfrak{M}_0 \mathfrak{M}_1 \xrightarrow{T_\mathbb{D}} \mathfrak{M}_0 \mathfrak{M}_1 \xrightarrow{\mu} \mathfrak{M}_0 \mathfrak{M}_1,$$

$$p_2 \circ \xi_1 : \mathbb{D}_1 \hat{\mathcal{L}} \mathfrak{M}_1 \xrightarrow{\mathbb{D}_1 p_2} \mathfrak{M}_0 \mathfrak{M}_1 \xrightarrow{\mu} \mathfrak{M}_0 \mathfrak{M}_1.$$

These expressions allow us to verify the four diagrams we want, which we take in order. First, the one for $p_2$ and $S$ becomes

$$\begin{array}{c}
\mathbb{D}_2 \hat{\mathcal{L}} \mathfrak{M}_2 \\
\mathbb{D}_2 \mathfrak{M}_1 \times_{\mathbb{D}_2 \mathfrak{M}_0} \mathbb{D}_2 \mathfrak{M}_0 \xrightarrow{p_2} \mathbb{D}_2 \mathfrak{M}_0
\end{array}$$

The one for $p_1$ and $S$ becomes

$$\begin{array}{c}
\mathbb{D}_2 \hat{\mathcal{L}} \mathfrak{M}_2 \\
\mathbb{D}_2 \mathfrak{M}_1 \times_{\mathbb{D}_2 \mathfrak{M}_0} \mathbb{D}_2 \mathfrak{M}_0 \xrightarrow{p_2} \mathbb{D}_2 \mathfrak{M}_1 \xrightarrow{T_\mathbb{D}} \mathfrak{M}_0 \mathfrak{M}_1 \xrightarrow{\mu} \mathfrak{M}_0 \mathfrak{M}_1
\end{array}$$
The one for $p_2$ and $T$ becomes

![Diagram](image)

And the diagram for $p_1$ and $T$ becomes

![Diagram](image)

Since all four diagrams commute, the expression for $\xi_2$ is correct. □

It will also be convenient for us to realize that the expression we already have in place for $\xi_1 : D_1 \hat{L}M_1 \to \hat{L}M_1$ can be written as the composite

$$D_1D_0M_1 \times D_1D_0M_0 \xrightarrow{T_b \times T_0T_b} D_0^2M_1 \times D_0^2M_0 \xrightarrow{\mu \times \mu \mu} D_0M_1 \times D_0M_0 \xrightarrow{\gamma_{\hat{L}M}} D_0M_1.$$

We now verify our desired square

$$D_2\hat{L}M_2 \xrightarrow{\xi_2} \hat{L}M_2$$

by composing with $p_1 : \hat{L}M_1 \to D_0M_1$ and $p_2 : \hat{L}M_1 \to D_1M_0$ and checking commutativity of the resulting diagrams. Composing with $p_1$, we can rewrite $p_1 \circ \gamma_{\hat{L}M}$ as the
11. THE PROVISIONAL LEFT ADJOINT: $B$-ALGEBRA STRUCTURE

composite

$$
\begin{array}{c}
\mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \\
\xrightarrow{p_{12}} \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_1 \\
\xrightarrow{1 \times T_B} \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1 \cong \mathbb{D}_0 M_2 \xrightarrow{\mathbb{D}_0 \gamma M} \mathbb{D}_0 M_1.
\end{array}
$$

Consequently, we can express $p_1 \circ \gamma_{LM} \circ \xi_2$ as

$$
\begin{array}{c}
\mathbb{D}_2 \mathbb{D}_0 M_1 \times_{\mathbb{D}_2 \mathbb{D}_0 M_0} \mathbb{D}_2 \mathbb{D}_1 M_1 \times_{\mathbb{D}_2 \mathbb{D}_0 M_0} \mathbb{D}_2 \mathbb{D}_1 M_0 \\
\xrightarrow{p_{12}} \mathbb{D}_2 \mathbb{D}_0 M_1 \times_{\mathbb{D}_2 \mathbb{D}_0 M_0} \mathbb{D}_2 \mathbb{D}_1 M_1 \\
\xrightarrow{T^2_B \times T^2_B} \mathbb{D}_2^2 M_1 \times_{\mathbb{D}_0^2 M_0} \mathbb{D}_2^2 M_1 \xrightarrow{\mu \times \mu} \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_1 \\
\xrightarrow{1 \times T} \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1 \cong \mathbb{D}_0 M_2 \xrightarrow{\mathbb{D}_0 \gamma M} \mathbb{D}_0 M_1.
\end{array}
$$

In the other direction, we can write $p_1 \circ \xi_1$ as the composite

$$
\begin{array}{c}
\mathbb{D}_1 \hat{L} M_1 \cong \mathbb{D}_1 \mathbb{D}_0 M_1 \times_{\mathbb{D}_1 \mathbb{D}_0 M_0} \mathbb{D}_1^2 M_0 \\
\xrightarrow{p_1} \mathbb{D}_1 \mathbb{D}_0 M_1 \\
\xrightarrow{T_B} \mathbb{D}_0^2 M_1 \xrightarrow{\mu} \mathbb{D}_0 M_1,
\end{array}
$$

so we can express $p_1 \circ \xi_1 \circ \gamma_{B \gamma_{LM}}$ as

$$
\begin{array}{c}
\mathbb{D}_2 \mathbb{D}_0 M_1 \times_{\mathbb{D}_2 \mathbb{D}_0 M_0} \mathbb{D}_2 \mathbb{D}_1 M_1 \times_{\mathbb{D}_2 \mathbb{D}_0 M_0} \mathbb{D}_2 \mathbb{D}_1 M_0 \\
\xrightarrow{p_{12}} \mathbb{D}_2 \mathbb{D}_0 M_1 \times_{\mathbb{D}_2 \mathbb{D}_0 M_0} \mathbb{D}_2 \mathbb{D}_1 M_1 \\
\xrightarrow{\gamma_0 \times \gamma_0 \gamma_0} \mathbb{D}_1 \mathbb{D}_0 M_1 \times_{\mathbb{D}_1 \mathbb{D}_0 M_0} \mathbb{D}_1^2 M_1 \xrightarrow{1 \times D^1 T} \mathbb{D}_1 (\mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1) \cong \mathbb{D}_1 \mathbb{D}_0 M_2 \\
\xrightarrow{D_1 \mathbb{D}_0 \gamma M} \mathbb{D}_1 \mathbb{D}_0 M_1 \xrightarrow{T_B} \mathbb{D}_0^2 M_1 \xrightarrow{\mu} \mathbb{D}_0 M_1.
\end{array}
$$

Since both expressions we wish to coincide begin with $p_{12}$, we can simply ask that the rest of the expressions coincide. This follows from the following commutative
Consequently, we can write $p_2$ as the composite

$$\begin{align*}
&D_2D_0M_1 \times_{D_2D_0M_0} D_2D_1M_1 \xrightarrow{T^2_B \times T^2_B} D^2_0M_1 \times_{D^2_0M_0} D^2_1M_1 \xrightarrow{\mu \times \mu} D_0M_1 \times_{D_0M_0} D_1M_1 \\
&\cong D_1D_0M_1 \times_{D_1D_0M_0} D_1D_0M_1 \xrightarrow{T_B \times T_B} D_1^2M_1 \times_{D_1^2M_0} D_1^2M_1 \xrightarrow{\mu} D_1M_1.
\end{align*}$$

For the composition with $p_2$, we can first rewrite $p_2 \circ \gamma_{LM}$ as the composite

$$\begin{align*}
&D_0M_1 \times_{D_0M_0} D_1M_1 \times_{D_0M_0} D_1M_0 \xrightarrow{p_2^{23}} D_1M_1 \times_{D_0M_0} D_1M_0 \\
&\cong D_1D_0M_1 \times_{D_1D_0M_0} D_1D_0M_1 \xrightarrow{D_1I_0 \times 1} D_1^2M_0 \times_{D_0M_0} D_1M_0 \\
&\xrightarrow{\mu \times 1} D_1M_0 \times_{D_0M_0} D_1M_0 \cong D_2M_0 \xrightarrow{\gamma_0} D_1M_0.
\end{align*}$$

Consequently, we can write $p_2 \circ \gamma_{LM} \circ \xi_2$ as the composite

$$\begin{align*}
&D_2D_0M_1 \times_{D_2D_0M_0} D_2D_1M_1 \times_{D_2D_0M_0} D_2D_1M_0 \xrightarrow{p_2^{23}} D_2D_1M_1 \times_{D_2D_0M_0} D_2D_1M_0 \\
&\xrightarrow{T_B \times T_B \times T_B} D_1^2M_1 \times_{D_1^2M_0} D_1^2M_0 \xrightarrow{\mu \times \mu} D_1M_1 \times_{D_0M_0} D_1M_0 \\
&\xrightarrow{\mu \times 1} D_1M_0 \times_{D_0M_0} D_1M_0 \cong D_2M_0 \xrightarrow{\gamma_0} D_1M_0.
\end{align*}$$

In the other direction, we can write $p_2 \circ \xi_1$ as the composite

$$\begin{align*}
&D_1\hat{L}M_1 \cong D_1D_0M_1 \times_{D_1D_0M_0} D_1^2M_0 \xrightarrow{p_2} D_1^2M_0 \xrightarrow{\mu} D_1M_0,
\end{align*}$$
and consequently we can write $p_2 \circ \xi_1 \circ \gamma_{\mathcal{D}}\gamma_{LM}$ as the composite

$$
\begin{array}{c}
\mathcal{D}_2\mathcal{D}_0M_1 \times \mathcal{D}_2\mathcal{D}_0M_0 \xrightarrow{\gamma_0 \times \gamma_0 \gamma_0} \mathcal{D}_1^2M_1 \times \mathcal{D}_1\mathcal{D}_0M_0 \xrightarrow{\mathcal{D}_1^{1S} \times 1} \mathcal{D}_2^2\mathcal{D}_1M_1 \times \mathcal{D}_2\mathcal{D}_0M_0 \xrightarrow{p_{23}} \mathcal{D}_2\mathcal{D}_1M_1 \times \mathcal{D}_2\mathcal{D}_0M_0 \mathcal{D}_2\mathcal{D}_1M_0
\end{array}
$$

$$
\begin{array}{c}
\mathcal{D}_1^2M_0 \times \mathcal{D}_1\mathcal{D}_0M_0 \xrightarrow{\mathcal{D}_1^1\mu \times 1} \mathcal{D}_1^2\mathcal{D}_1M_0 \times \mathcal{D}_1\mathcal{D}_0M_0 \xrightarrow{\mathcal{D}_1^2M_0 \times \mathcal{D}_1\mathcal{D}_0M_0 \mathcal{D}_1^2M_0}
\end{array}
$$

Adopting our previous notation $\theta$ for the composite of source type

$$
\begin{array}{c}
\mathcal{D}_1M_1 \xrightarrow{\mathcal{D}_1S} \mathcal{D}_1\mathcal{D}_0M_0 \xrightarrow{\mathcal{D}_1\mathcal{D}_0M_0 \mathcal{D}_1^2M_0} \mathcal{D}_1M_0,
\end{array}
$$

we can see that these coincide by means of the following commutative diagram:

$$
\begin{array}{c}
\mathcal{D}_2\mathcal{D}_1M_1 \times \mathcal{D}_2\mathcal{D}_0M_0 \xrightarrow{\mathcal{D}_2\mathcal{D}_1M_1 \times \mathcal{D}_2\mathcal{D}_0M_0 \mathcal{D}_2\mathcal{D}_1M_0} \mathcal{D}_2^2\mathcal{D}_1M_1 \times \mathcal{D}_2\mathcal{D}_0M_0 \mathcal{D}_2\mathcal{D}_1M_0
\end{array}
$$

$$
\begin{array}{c}
\mathcal{D}_2\mathcal{D}_1M_0 \times \mathcal{D}_2\mathcal{D}_0M_0 \mathcal{D}_2\mathcal{D}_1M_0 \xrightarrow{\mathcal{D}_2\mathcal{D}_1M_0 \times \mathcal{D}_2\mathcal{D}_0M_0 \mathcal{D}_2\mathcal{D}_1M_0} \mathcal{D}_2^2\mathcal{D}_1M_0 \times \mathcal{D}_2\mathcal{D}_0M_0 \mathcal{D}_2\mathcal{D}_1M_0
\end{array}
$$

Consequently, our action maps preserve the composition, and we have defined a $\mathcal{D}$-algebra structure on $\hat{L}M$.

### 12. The adjunction structure I: the unit map

This section is devoted to constructing a unit map $M \to U\hat{L}M$ for a $\mathcal{D}$-multicategory $M$ and verifying its properties. Together with the counit described in the next section, this map almost (but not quite) determines an adjunction between $U$ and $\hat{L}$. They will do most of the heavy lifting for the actual adjunction once we construct the actual left adjoint $L$. The reason we don’t get an adjunction immediately is that the unit map of this section isn’t quite a map of $\mathcal{D}$-multicategories: it doesn’t preserve the presheaf structure on the sets of morphisms.

The unit map consists of maps $\eta_0 : M_0 \to U\hat{L}M_0$ and $\eta_1 : M_1 \to U\hat{L}M_1$. Since $(UC)_0 = C_0$ for a $\mathcal{D}$-algebra $C$ and $\hat{L}M_0 = \mathcal{D}_0M_0$, we just use the unit of the monad

$$
\begin{array}{c}
\mathcal{D}_1\mathcal{D}_0M_0 \xrightarrow{\mathcal{D}_1\gamma_0} \mathcal{D}_1\mathcal{D}_0M_0 \xrightarrow{\mathcal{D}_1\gamma_0 \gamma_0} \mathcal{D}_2\mathcal{D}_1M_0 \xrightarrow{\mathcal{D}_2\mathcal{D}_1M_0 \mathcal{D}_2\mathcal{D}_1M_0} \mathcal{D}_2\mathcal{D}_1M_0 \xrightarrow{\mathcal{D}_2\mathcal{D}_1M_0 \mathcal{D}_2\mathcal{D}_1M_0}
\end{array}
$$
$D_0$

\[ \eta_0 = \eta : M_0 \to D_0M_0 \]

as the map on objects of our unit.

To define the unit map on morphisms, we recall that $\hat{LM}_1$ is given by the pullback diagram

\[
\begin{array}{c}
\hat{LM}_1 \xrightarrow{p_1} D_0M_1 \\
p_2 \downarrow \downarrow \downarrow p_2 \\
D_1M_0 \xrightarrow{T_b} D_0M_0,
\end{array}
\]

that is, $\hat{LM}_1 \cong D_0M_1 \times_{D_0M_0} D_1M_0$. Further, the underlying multicategory $UC$ of a $D$-algebra $C$ (such as $\hat{LM}$) is given by the pullback

\[
\begin{array}{c}
(U\hat{C})_1 \xrightarrow{p_1} C_1 \\
S \downarrow \downarrow \downarrow S \\
D_0C_0 \xrightarrow{\xi_0} C_0,
\end{array}
\]

where $\xi_0 : D_0C_0 \to C_0$ is the restriction to objects of the action of $D$ on its algebra $C$, and $p_1$ coincides with the comparison map $\kappa_1$. Combining the two diagrams, we see that $(U\hat{LM})_1$ is given by the pullback in the diagram

\[
\begin{array}{c}
U\hat{LM}_1 \xrightarrow{p_1} \hat{LM}_1 \xrightarrow{p_1} D_0M_1 \\
p_2 \downarrow \downarrow \downarrow p_2 \\
D_1M_0 \xrightarrow{T_b} D_0M_0 \\
S \downarrow \downarrow \downarrow S \\
D_0^2M_0 \xrightarrow{\mu} D_0M_0,
\end{array}
\]

so we can write

\[ U\hat{LM}_1 \cong D_0M_1 \times_{D_0M_0} D_1M_0 \times_{D_0M_0} D_0^2M_0. \]

Consequently, any map to $U\hat{LM}_1$ is given by maps to $D_0^2M_0$, to $D_1M_0$, and to $D_0M_1$, subject to compatibility relations. We agree to define $\eta : M_1 \to U\hat{LM}_1$ by the following maps:
We will show this is almost a map of $\mathbb{D}$-multicategories: the only defect is that the map on morphisms doesn’t preserve the presheaf structure.

First, we observe that we have a well-defined map. For this, we have the two diagrams

\[ \begin{array}{ccc}
M_1 & \xrightarrow{S} & \mathbb{D}_0 M_0 \\
\downarrow{s} & & \downarrow{I_D} \\
\mathbb{D}_0 M_0 & \xrightarrow{I_D} & \mathbb{D}_1 M_0 \\
\downarrow{\mathbb{D}_0 \eta} & & \downarrow{\mathbb{D}_0 \eta} \\
\mathbb{D}_0^2 M_0 & \xrightarrow{\mu} & \mathbb{D}_0 M_0
\end{array} \quad \text{and} \quad \begin{array}{ccc}
M_1 & \xrightarrow{\eta} & \mathbb{D}_0 M_1 \\
\downarrow{s} & & \downarrow{\mathbb{D}_0 \eta} \\
\mathbb{D}_0 M_0 & \xrightarrow{\eta} & \mathbb{D}_0^2 M_0 \\
\downarrow{I_D} & & \downarrow{I_D} \\
\mathbb{D}_1 M_0 & \xrightarrow{\mu} & \mathbb{D}_0 M_0
\end{array} \]

which verify that we have actually defined a map to $U \hat{L} M_1$.

We will show that this map satisfies all the rest of the properties of a map of $\mathbb{D}$-multicategories, namely, that it preserves $I$, $S$, $T$, and $\gamma$. For preservation of $I$, we need the diagram

\[ \begin{array}{ccc}
M_0 & \xrightarrow{\eta} & \mathbb{D}_0 M_0 \\
\downarrow{I} & & \downarrow{I} \\
M_1 & \xrightarrow{\eta} & U \hat{L} M_1
\end{array} \]

to commute, which we show by composing to the three components $\mathbb{D}_0^2 M_0$, $\mathbb{D}_1 M_0$, and $\mathbb{D}_0 M_1$. First, we recall that $I : \mathbb{D}_0 M_0 \to U \hat{L} M_1$ is a special case of the more general $I : C_0 \to UC_1$ for a $\mathbb{D}$-algebra $C$, which is given by the commutative diagram

\[ \begin{array}{ccc}
C_0 & \xrightarrow{I} & C_1 \\
\downarrow{\eta} & & \downarrow{\eta} \\
\mathbb{D}_0 C_0 & \xrightarrow{=} & \mathbb{D}_0 C_0 \\
\downarrow{s} & & \downarrow{s} \\
C_0 & \xrightarrow{\xi_0} & C_0
\end{array} \]
which defines a map to the pullback \( UC_1 \) of the square. When \( C = \hat{LM} \), we have the identity map of \( \hat{LM} \) induced by the diagram

\[
\begin{array}{c}
\begin{array}{c}
\mathbb{D}_0 M_0 \xrightarrow{E_0 I} \mathbb{D}_0 M_1 \\
\mathbb{D}_0 \eta \quad \mathbb{D}_0 S
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\mathbb{D}_1 M_0 \xrightarrow{T} \mathbb{D}_0 M_0,
\end{array}
\end{array}
\]

and then \( \eta : \mathbb{D}_0 M_0 \to \mathbb{D}_0^2 M_0 \) satisfies commutativity in

\[
\begin{array}{c}
\begin{array}{c}
\mathbb{D}_0 M_0 \xrightarrow{I_0} \mathbb{D}_1 M_0 \\
\eta \quad \mathbb{D}_0 M_0 \xrightarrow{S} \mathbb{D}_0 M_0,
\end{array}
\end{array}
\]

and so induces the identity map \( I : \mathbb{D}_0 M_0 \to U \hat{LM}_1 \). We can now verify preservation of \( I \) by checking the composites to \( \mathbb{D}_0^2 M_0, \mathbb{D}_0 M_0, \) and \( \mathbb{D}_0 M_1 \). To \( \mathbb{D}_0^2 M_0 \), we check that

\[
\begin{array}{c}
\begin{array}{c}
M_0 \xrightarrow{\eta} \mathbb{D}_0 M_0 \\
I \quad \mathbb{D}_0 M_0 \xrightarrow{\eta} \mathbb{D}_0^2 M_0
\end{array}
\end{array}
\]

commutes by naturality of \( \eta \). To \( \mathbb{D}_1 M_0 \), we have the trivial diagram

\[
\begin{array}{c}
\begin{array}{c}
M_0 \xrightarrow{\eta} \mathbb{D}_0 M_0 \\
I \quad \mathbb{D}_0 M_0 \xrightarrow{I} \mathbb{D}_2 M_0,
\end{array}
\end{array}
\]

and to \( \mathbb{D}_0 M_1 \), we have

\[
\begin{array}{c}
\begin{array}{c}
M_0 \xrightarrow{\eta} \mathbb{D}_0 M_0 \\
I \quad \mathbb{D}_0 M_0 \xrightarrow{\eta} \mathbb{D}_0 M_1
\end{array}
\end{array}
\]

which commutes by naturality of \( \eta \). Consequently, \( I \) is preserved.
For preservation of $S$, we have

\[
\begin{array}{c}
M_1 \xrightarrow{\eta} \hat{U} \hat{L} M_1 \\
S \\
\hat{D}_0 M_0 \xrightarrow{\hat{D}_0 \eta} \hat{D}_0^2 M_0,
\end{array}
\]

which is why $S \circ \eta$ is defined the way it is, and for preservation of $T$, we have

\[
\begin{array}{ccc}
M_1 & \xrightarrow{T} & M_0 \\
\eta & & \downarrow \eta \\
\hat{U} \hat{L} M_1 & \xrightarrow{p_1} & \hat{L} M_1 \\
p_1 & & \downarrow \eta \\
\hat{D}_0 M_1 & \xrightarrow{\hat{D}_0 T} & \hat{D}_0 M_0,
\end{array}
\]

which is another application of naturality of $\eta$. This leaves preservation of $\gamma$ to be verified.

For preservation of $\gamma$, what we need is commutativity of the diagram

\[
\begin{array}{c}
M_1 \times \hat{D}_0 M_0 \xrightarrow{\gamma^M} M_1 \\
\eta \times \hat{D}_0 \eta \xrightarrow{\gamma^M} \eta \\
\hat{U} \hat{L} M_1 \times \hat{D}_0 \hat{U} \hat{L} M_0 \xrightarrow{\gamma^M} \hat{U} \hat{L} M_1.
\end{array}
\]

We already have the expression $\hat{D}_0 M_1 \times \hat{D}_0 M_0 \xrightarrow{\gamma^M} \hat{D}_0 M_1 \times \hat{D}_0 M_0$ for $\hat{U} \hat{L} M_1$, so expanding out the lower left corner, which is really $\hat{U} \hat{L} M_2$, we get the following isomorphism by canceling the $\hat{D}_0^2 M_0$'s in the middle:

\[
\begin{align*}
\hat{D}_0 M_1 \times \hat{D}_0 M_0 & \xrightarrow{\gamma^M} \hat{D}_0 M_1 \times \hat{D}_0 M_0 \\
& \cong \hat{D}_0 M_1 \times \hat{D}_0 M_0 \xrightarrow{\gamma^M} \hat{D}_0 M_1 \times \hat{D}_0 M_0.
\end{align*}
\]

Now the definition of composition for $UC$ is in terms of the comparison map $\kappa_2 : UC_2 \to C_2$, which can be expressed as follows. First, we have

\[
UC_2 := UC_1 \times_{\hat{D}_0 UC_0} \hat{D}_0 UC_1
\]

\[
\cong C_1 \times_{c_0} \hat{D}_0 C_0 \times_{\hat{D}_0 C_0} \hat{D}_0 C_1 \times_{\hat{D}_0 C_0} \hat{D}_0^2 C_0
\]

\[
\cong C_1 \times_{c_0} \hat{D}_0 C_1 \times_{\hat{D}_0 C_0} \hat{D}_0^2 C_0
\]

and then the comparison map $\kappa_2$ can be expressed as the composite

\[
C_1 \times_{c_0} \hat{D}_0 C_1 \times_{\hat{D}_0 C_0} \hat{D}_0^2 C_0 \xrightarrow{p_{12}} C_1 \times_{c_0} \hat{D}_0 C_1 \xrightarrow{1 \times \hat{D}_0} C_1 \times_{c_0} \hat{D}_1 C_1 \xrightarrow{1 \times \xi_1} C_1 \times_{c_0} C_1 = C_2.
\]

Now we can write the composition in $UC$ in terms of its projections to the factors of $UC_1 = C_1 \times_{c_0} \hat{D}_0 C_0$. The projection to $C_1$ is just $\kappa_2$ as expressed above, composed with $\gamma_C : C_2 \to C_1$. The projection to $\hat{D}_0 C_0$ consists of projection to the last factor.
$\mathbb{D}_0^2 \mathcal{C}_0$ of $U \mathcal{C}_2$, composed with $\mu : \mathbb{D}_0^2 \mathcal{C}_0 \rightarrow \mathbb{D}_0 \mathcal{C}_0$. This now has to be interpreted in the case where $\mathcal{C} = \hat{LM}$, in which case we have the actual composition taking place in the composite

\[
\begin{array}{c}
\mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0^2 M_1 \times_{\mathbb{D}_0^2 M_0} \mathbb{D}_0 \mathbb{D}_1 M_0 \times_{\mathbb{D}_0^2 M_0} \mathbb{D}_0^3 M_0 \\
p_{1234} \downarrow \\
\mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0^2 M_1 \times_{\mathbb{D}_0^2 M_0} \mathbb{D}_0 \mathbb{D}_1 M_0 \\
1 \times 1 \times I_0 \times I_D \\
\mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0^2 M_1 \times_{\mathbb{D}_0^2 M_0} \mathbb{D}_0^2 M_0 \\
1 \times 1 \times T_0 \times T_0 \\
\mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \\
1 \times (D_1 T \cdot S_0) \times 1 \\
\mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \\
1 \times (T_0 \cdot \theta) \times 1 \\
\mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \\
\mathbb{D}_0 \gamma \times \gamma_0 \\
\mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0.
\end{array}
\]

The other factor is much more straightforward: it is just

\[
\begin{array}{c}
U \hat{L} M_1 \quad \xrightarrow{p_5} \quad \mathbb{D}_0^3 M_0 \quad \xrightarrow{\mu} \quad \mathbb{D}_0^2 M_0.
\end{array}
\]

We verify preservation of $\gamma$ by projecting to each of the three factors of $U \hat{L} M_1 = \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0^2 M_0$, and the last factor of $\mathbb{D}_0^2 M_0$ doesn’t present much difficulty: it’s a consequence of the diagram

\[
\begin{array}{c}
M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1 \quad \xrightarrow{\gamma \times M} \quad M_1 \\
p_2 = S \\
\mathbb{D}_0 M_1 \quad \xrightarrow{D_0 S} \quad \mathbb{D}_0^2 M_0 \quad \xrightarrow{\mu} \quad \mathbb{D}_0 M_0 \\
\mathbb{D}_0 \eta \\
\mathbb{D}_0^3 M_0 \quad \xrightarrow{\mu} \quad \mathbb{D}_0^2 M_0.
\end{array}
\]

We next consider the projection to the first factor $\mathbb{D}_0 M_1$, and observe first that the long composite above, projected to $\mathbb{D}_0 M_1$, depends only on the first three factors
of $U \hat{LM}_2$, and after projecting onto those, we can express the map as follows:

$$
\begin{array}{c}
\mathbb{D}_0M_1 \times_{\mathbb{D}_0M_0} \mathbb{D}_1M_0 \times_{\mathbb{D}_0M_0} \mathbb{D}_2^2M_1 \\
\mathbb{D}_0M_1 \times_{\mathbb{D}_0M_0} \mathbb{D}_1M_0 \times_{\mathbb{D}_0M_0} \mathbb{D}_0M_1 \\
\mathbb{D}_0M_1 \times_{\mathbb{D}_0M_0} \mathbb{D}_0M_1 \times_{\mathbb{D}_0M_0} \mathbb{D}_1M_0 \\
\mathbb{D}_0M_0 \mathbb{D}_1M_0 \\
\mathbb{D}_0M_1.
\end{array}
$$

Next, the portion of the unit map $\eta_2 = \eta_1 \times_{\mathbb{D}_0M_0} \mathbb{D}_0\eta_1 : M_1 \times_{\mathbb{D}_0M_0} \mathbb{D}_0M_1 \to U \hat{LM}_2$
that actually maps to the first three factors consists of

$$(\eta, I_D \circ S, \mathbb{D}_0\eta \circ S) : M_1 \to \mathbb{D}_0M_1 \times_{\mathbb{D}_0M_0} \mathbb{D}_1M_0 \times_{\mathbb{D}_0M_0} \mathbb{D}_2^2M_0$$

and

$$\mathbb{D}_0\eta : \mathbb{D}_0M_1 \to \mathbb{D}_0^2M_1.$$ 

However, since we’re taking a fiber product over $\mathbb{D}_0^2M_0$ of these two maps, the result is just $$(\eta, I_D \circ S) \times \mathbb{D}_0\eta.$$ 

Further, the diagram

allows us to simplify and write the material part of the composite of $\eta$ and $\gamma_{LM}$ with $p_1$ as follows:

$$
\begin{array}{c}
\mathbb{D}_0M_1 \xrightarrow{\mathbb{D}_0\eta} \mathbb{D}_0^2M_1 \\
\mathbb{D}_0M_1 \xrightarrow{\mathbb{D}_0\eta} \mathbb{D}_0^2M_1 \\
\mathbb{D}_0M_1 \xrightarrow{\mathbb{D}_0\eta} \mathbb{D}_0^2M_1 \\
\mathbb{D}_0M_1 \xrightarrow{\mathbb{D}_0\eta} \mathbb{D}_0M_1.
\end{array}
$$
The diagram we wish to commute then becomes the perimeter of the following one, in which as usual we suppress subscript $D_0$'s:

The lower left triangle records the definition of $\chi$, and all the rest of the sub-diagrams are clear with the exception of the central slanted rectangle. We claim that the following diagram commutes, which will allow us to rewrite the upper right composite in the remaining sub-diagram:

The main reason this is true is that the unit map $\eta : M_1 \to D_0 M_1$ preserves the structure map of source type, as seen in the following diagram:
Consequently, the following cube commutes, with the front and back faces being pullbacks:

\[ \array{c \quad M_2 \quad T \quad M_1 \quad \eta \quad \eta \\ S \quad \downarrow \quad \downarrow \quad \downarrow \\ \mathbb{D}_0 M_2 \quad \mathbb{D}_0 T \quad \mathbb{D}_0 M_1 \quad S \\ \mathbb{D}_0 M_1 \quad \mathbb{D}_0 T \quad \mathbb{D}_0 M_0 \quad \mathbb{D}_0 T \quad \mathbb{D}_0 M_0. } \]

This implies that the claimed square commutes. We can now replace the desired sub-diagram with the following one:

\[ \array{c \quad M \times \mathbb{D}_0 M_1 \\ (\eta, I \circ S) \times 1 \quad \eta \times 1 \\ \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \xrightarrow{\cong} \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_1 \xrightarrow{1 \times 1 \times T} \mathbb{D}_0 M_1 \times \mathbb{D}_0 M_1. } \]

After composition with projection to the first \( \mathbb{D}_0 M_1 \), both composites coincide with \( \eta \), so we can concentrate on the composition with the second factor of \( \mathbb{D}_0 M_1 \). Here we find the diagram commutes as a consequence of the following filling:

\[ \array{c \quad M_1 \times \mathbb{D}_0 M_1 \\ S \times 1 \quad \eta \times 1 \\ \mathbb{D}_0 M_0 \times \mathbb{D}_0 M_1 \xrightarrow{\cong} \mathbb{D}_0 M_1 \xrightarrow{(D_0 T, id)} \mathbb{D}_0 M_1 \xrightarrow{1 \times T} \mathbb{D}_0 M_1 \times \mathbb{D}_0 M_1. } \]

We may conclude that the unit maps preserve \( \gamma \) after projection onto the first factor.

It remains to verify preservation of \( \gamma \) after projection onto the second factor, \( \mathbb{D}_1 M_0 \). Looking again at the long composite giving the composition in \( \hat{L} M \), we see that the projection onto \( \mathbb{D}_1 M_0 \) depends only on the middle three factors. Further, the portion of the unit mapping to these three factors consists of the following, in which we again suppress the subscripts for the fiber products:

\[ \array{c \quad M_1 \times \mathbb{D}_0 M_1 \\ (I \circ S) \times (D_0 \circ (I \circ S)) \quad \mathbb{D}_1 M_0 \times \mathbb{D}_0^2 M_1 \times \mathbb{D}_0 \mathbb{D}_1 M_0. } \]
We now simplify as follows:

**Lemma 12.1.** The composite of the previous part of the unit with the portion of the composition in $\hat{LM}$ ending at $\mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0$ coincides with

$$(I_{\mathbb{D}} \circ S) \times (1, I_D \circ \mu \circ D_0S).$$

**Proof.** Tracing the projections to each factor of the target, the first one is clear: both coincide with $I_{\mathbb{D}} \circ S$. For the middle factor, we do just get the identity on $\mathbb{D}_0 M_1$ since

commutes. The expressions for the last $\mathbb{D}_1 M_0$ coincide by the commutativity of

Now we simplify the composite of $p_2 \circ \gamma_{LM}$ with the unit map even further by means of the following commutative diagram, in which it is extremely important to
realize that the subscript \(D_0M_0\)'s have been suppressed:

\[
\begin{array}{c}
\begin{array}{c}
M_1 \times D_0M_1 \\
S \times 1 \\
D_0M_0 \times D_0M_1 \xrightarrow{(D_0T,1)} D_0M_1 \\
& \xrightarrow{p_2} D_0M_1 \\
\end{array}
\end{array}
\]

Next, we claim the following diagram commutes:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
D_1M_0 \times D_0M_1 \times D_0^2M_0 \\
1 \times 1 \times \mu \\
D_1M_0 \times D_0M_1 \times D_0M_0 \xrightarrow{(D_1T,S_0,\mu \circ D_0S \circ S_0)} D_1M_1 \\
& \xrightarrow{1 \times 1 \times \mu} D_1M_0 \times D_0M_0 \xrightarrow{(D_1T,S_0) \times 1} D_1M_1 \times D_1M_0 \\
\times \chi \times 1 \\
D_0M_1 \times D_0M_0 \times D_1M_0 \xrightarrow{p_{23}} D_1M_0 \times D_1M_0 \xrightarrow{\gamma_D} D_1M_0.
\end{array}
\end{array}
\end{array}
\]

The lower triangle commutes because \(\gamma_D\) is right unital, and the upper one commutes because of the commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
D_1M_1 \xrightarrow{\theta} D_1M_0 \xrightarrow{(1,S_0)} D_1M_0 \times D_0M_0 \xrightarrow{(1,I_D)} D_1M_0 \times D_1M_0 \\
& \xrightarrow{(\theta,\mu \circ D_0S \circ S_0)} D_1M_0 \times D_0M_0 \xrightarrow{(1,I_D)} D_1M_0 \times D_1M_0 \\
& \xrightarrow{\gamma_D} D_1M_0.
\end{array}
\end{array}
\end{array}
\]

The right square commutes because \(D_1S_D\) splits \(D_1I_D\) and the square commutes when starting at \(D_1^2M_0\). We can now replace the composite in the previous large diagram starting at \(D_1M_1\) with just \(\theta : D_1M_1 \to D_1M_0\), which expands to \(\mu_D \circ D_1I_D \circ D_1S\). The
whole diagram we wish to commute then reduces to the perimeter of the following diagram, which does commute:

\[
\begin{array}{ccc}
M_1 \times_{D_0 M_0} D_0 M_1 & \xrightarrow{\gamma_1} & M_2 \\
\downarrow p^2 & & \downarrow \gamma M \\
D_0 M_1 \xrightarrow{D_0 S} D_0^2 M_0 & \xrightarrow{\mu} & D_0 M_0 \\
\downarrow I_0 & & \downarrow I_0 \\
D_1 M_1 \xrightarrow{D_1 S} D_1 D_0 M_0 & \xrightarrow{\varphi_1} & D_1 M_0. \\
\end{array}
\]

This completes the verification that the unit map preserves the compositions. The only thing missing in showing that it is a map of \(D\)-multicategories is to show that \(\eta_1 : M_1 \to \hat{L}M_1\) is a map of \(D(\ast)\)-presheaves. However, this is in general false, and is the reason we need to descend to a quotient, which will give us the actual left adjoint.

13. The adjunction structure II: the counit map

Next we define the counit map \(\varepsilon : \hat{L}UC \to C\), and verify its properties. On objects, we have

\[
\hat{L}UC_0 = D_0 C_0,
\]

so for the counit \(\varepsilon_0\) we use the action map \(\xi_0 : D_0 C_0 \to C_0\) given by the algebra structure on \(C\) over \(D\). The significant part is the definition of the map on morphisms, \(\varepsilon_1 : (\hat{L}UC)_1 \to C_1\). First, we can apply \(D_0\) to the pullback diagram defining \((UC)_1\) to get another pullback diagram

\[
\begin{array}{ccc}
D_0 (UC)_1 & \xrightarrow{D_0 \varphi_1} & D_0 C_1 \\
\downarrow D_0 S & & \downarrow D_0 S \\
D_0^2 C_0 & \xrightarrow{D_0 \xi_0} & D_0 C_0.
\end{array}
\]
Next, the definition of morphisms in $\hat{LM}$ applied to $M = UC$ gives us a pullback diagram

\[
\begin{array}{c}
(\hat{L}UC)_1 \to D_0 UC_1 \\
\downarrow \quad \downarrow \\
D_0 S \quad D_0 S \\
\downarrow \quad \downarrow \\
D_0 C_0 \quad D_0 C_0 \\
\downarrow \quad \downarrow \\
D_1 C_0 \quad D_0 C_0 \\
\end{array}
\]

These paste together to form the core of the following diagram, which allows us to form a composable pair in $C_1$ from an element of $\hat{L}UC_1$; we define the counit on morphisms by the resulting composite:

\[
\begin{array}{c}
(\hat{L}UC)_1 \to D_0 UC_1 \to D_0 C_1 \\
\downarrow \quad \downarrow \quad \downarrow \\
D_0 S \quad D_0 S \quad I_D \\
\downarrow \quad \downarrow \quad \downarrow \\
D_0 C_0 \quad D_0 C_0 \quad D_1 C_1 \\
\downarrow \quad \downarrow \quad \downarrow \\
D_1 C_0 \quad D_0 C_0 \quad C_0 \\
\downarrow \quad \downarrow \quad \downarrow \\
D_1 I \quad I \quad C_1 \\
\end{array}
\]

\[
\begin{array}{c}
(\hat{L}UC)_1 \to D_0 UC_1 \to D_0 C_1 \to D_1 C_1 \\
\end{array}
\]

These paste together to form the core of the following diagram, which allows us to form a composable pair in $C_1$ from an element of $\hat{L}UC_1$; we define the counit on morphisms by the resulting composite:
We can further augment this diagram to verify that the resulting composite has the correct source and target:

\[
\begin{array}{ccccc}
(\hat{LUC})_1 & \mathbb{D}_0UC_1 & \rightarrow & \mathbb{D}_0C_1 \\
\downarrow \hat{LUC} & \downarrow \hat{DU} & \rightarrow & \downarrow \hat{DC} \\
\mathbb{D}_0C_0 & \rightarrow & \mathbb{D}_1C_0 \\
\downarrow S & \downarrow \xi_0 & \rightarrow & \downarrow \xi_0 \\
\mathbb{D}_0C_1 & \rightarrow & \mathbb{D}_1C_1 \\
\downarrow T & \downarrow T & \rightarrow & \downarrow T \\
\mathbb{D}_0C_0 & \rightarrow & \mathbb{D}_1C_1 \\
\downarrow \xi_1 & \downarrow \gamma & \rightarrow & \downarrow \gamma \\
\mathbb{D}_0C_0 & \rightarrow & \mathbb{D}_1C_1 \\
\end{array}
\]

To complete showing that \( \varepsilon \) is a functor, we need to show that it preserves identity and compositions. We then wish to show that it is a map of \( \mathbb{D} \)-algebras, for which we need to show that \( \varepsilon \) preserves \( \xi_0 \) and \( \xi_1 \).

For preservation of identities, we wish

\[
\begin{array}{ccc}
\hat{LC}_0 & \xrightarrow{I_{\hat{LUC}}} & \hat{LUC}_1 \\
\downarrow \varepsilon_0 & \downarrow \varepsilon_1 & \downarrow I_{C} \\
\mathbb{C}_0 & \rightarrow & \mathbb{C}_1 \\
\end{array}
\]

to commute. The identity map for \( \hat{L}M \) is given by

\[
\hat{L}M_0 = \mathbb{D}_0M_0 \xrightarrow{(\hat{D}_0I_M, I_D)} \mathbb{D}_0M_1 \times_{\mathbb{D}_0M_0} \mathbb{D}_1M_0 = \hat{L}M_1,
\]

so the identity map for \( \hat{LUC} \) is given by

\[
\hat{LUC}_0 = \mathbb{D}_0C_0 \xrightarrow{(\hat{D}_0I_{\hat{LUC}}, I_D)} \mathbb{D}_0UC_1 \times_{\mathbb{D}_0C_0} \mathbb{D}_1C_0.
\]

Now the counit map \( \varepsilon_1 : \hat{LUC}_1 \rightarrow \mathbb{C}_1 \) is a composite of a map to \( \mathbb{C}_1 \times_{\mathbb{C}_0} \mathbb{C}_1 \) with the composition \( \gamma_C \). Deferring the composition for the time being, the projection to the first factor of \( \mathbb{C}_1 \) can be expressed as the composite

\[
\hat{LUC}_1 \cong \mathbb{D}_0UC_1 \times_{\mathbb{D}_0C_0} \mathbb{D}_1C_0 \xrightarrow{p_1} \mathbb{D}_0UC_1 \xrightarrow{\hat{D}_0\xi_1} \mathbb{D}_0C_1 \xrightarrow{I_D} \mathbb{D}_1C_1 \xrightarrow{\xi_1} \mathbb{C}_1.
\]
Since this depends only on the first factor $D_0UC_1$ of $\hat{LUC}_1$, we can compose just with the first term for the identity map, and we get the following commutative diagram:

\[ \begin{array}{ccc}
D_0C_0 & \xrightarrow{\xi_0} & C_0 \\
D_0I_C & \downarrow & \downarrow I_C \\
D_0UC_1 & \xrightarrow{\delta_1} & D_1C_1 & \xrightarrow{\xi_1} & C_1.
\end{array} \]

The projection to the second factor of $C_1$ is given by the composite

\[ \hat{LUC}_1 \cong D_0UC_1 \times_{D_0C_0} D_1C_0 \xrightarrow{p_2} D_1C_0 \xrightarrow{D_1I_C} D_1C_1 \xrightarrow{\xi_1} C_1. \]

This only depends on the second factor $D_1C_0$ of $\hat{LUC}_1$, so composing with just the second term of the identity map for $\hat{LUC}$, we get the commutative diagram

\[ \begin{array}{ccc}
D_0C_0 & \xrightarrow{\xi_0} & C_0 \\
I_C & \downarrow & \downarrow I_C \\
D_1C_0 & \xrightarrow{D_1I_C} & D_1C_1 & \xrightarrow{\xi_1} & C_1.
\end{array} \]

Consequently, both composites to $C_1$ consist of $I_C \circ \xi_0$. Now the diagram we wish to commute collapses simply to

\[ \begin{array}{ccc}
D_0C_0 & \xrightarrow{\xi_0} & C_0 \\
I_C & \downarrow & \downarrow \gamma_C \\
D_1C_1 & \xrightarrow{\gamma_C} & C_1.
\end{array} \]

The triangle commutes because $\gamma_C$ is unital, so the counit does preserve the identity maps.

To show that the counit preserves compositions, we need to show that the diagram

\[ \begin{array}{ccc}
\hat{LUC}_1 \times_{\hat{LUC}_0} \hat{LUC}_1 & \xrightarrow{\xi_1 \times_{\gamma_0} \xi_1} & C_1 \times_{C_0} C_1 \\
\gamma_{\hat{LUC}} & \downarrow & \downarrow \gamma_C \\
\hat{LUC}_1 & \xrightarrow{\xi_1} & C_1
\end{array} \]

commutes. It will be convenient for us to have a formula for the counit $\xi_1$ on morphisms, and examination of the defining diagram shows that it can be expressed as
follows:

\[
\hat{L}UC_1 \cong D_0U \times D_0C_0 \times D_1C_0 \xrightarrow{D_0 \times \xi_0} D_0C_1 \times D_0 \times D_1C_0
\]

Meanwhile, the definition gives the following expression for the composition in \( \hat{L}UC \):

\[
\begin{align*}
&\xrightarrow{1 \times (T_1, T_0) \times 1} D_0U \times D_0C_0 \times D_0U \times D_0C_0 \times D_0U \times D_0C_0 \times D_0U \times D_0C_0 \times D_0U \times D_0C_0 \times D_0U \times D_0C_0 \\
&\cong \xrightarrow{1 \times (T_1, T_0) \times 1} D_0U \times D_0C_0 \times D_0U \times D_0C_0 \times D_0U \times D_0C_0 \times D_0U \times D_0C_0 \\
&\cong \xrightarrow{1 \times (T_1, T_0) \times 1} (D_0U, T_0, D_0U) \times (T_1, T_0)
\end{align*}
\]

The total diagram we need in order to show that the counit preserves composition is too large to display as one piece, so we break it up into three pieces. The first one is the left part, and looks like this, where the unadorned products have a suppressed subscript \( D_0C_0 \):

\[
\begin{align*}
&\xrightarrow{1 \times (T_1, T_0) \times 1} D_0U \times D_0C_0 \times D_0U \times D_0C_0 \times D_0U \times D_0C_0 \times D_0U \times D_0C_0 \\
&\cong \xrightarrow{1 \times (T_1, T_0) \times 1} D_0U \times D_0C_0 \times D_0U \times D_0C_0 \times D_0U \times D_0C_0 \\
&\cong \xrightarrow{1 \times (T_1, T_0) \times 1} (D_0U, T_0, D_0U) \times (T_1, T_0)
\end{align*}
\]
The central piece, which is to be glued to the right of the previous one, is as follows:

$$
\begin{align*}
\mathbb{D}_0 C_1 \times \mathbb{D}_1 C_0 \times \mathbb{D}_0 C_1 \times \mathbb{D}_1 C_0 & \xrightarrow{(I_\mathbb{D} \times I_{\mathbb{D} 1})^2} (\mathbb{D}_1 C_1)^4 \xrightarrow{\xi_1^4 \times \xi_0 \xi_1^4} C_1^4 \\
\mathbb{D}_0 C_1 \times \mathbb{D}_0 C_1 \times \mathbb{D}_1 C_0 \times \mathbb{D}_1 C_0 & \xrightarrow{I_0 \times I_{\mathbb{D} 1} \times I_1^2 \times (I_{\mathbb{D} 1})^2} (\mathbb{D}_1 C_1)^4 \xrightarrow{\xi_1^4} C_1^4 \\
\mathbb{D}_0 C_2 \times \mathbb{D}_0 C_0 \times (\mathbb{D}_0 T \times \mathbb{D}_0 S) & \xrightarrow{(I_\mathbb{D} T \times I_{\mathbb{D} 0} S \times (I_{\mathbb{D} 1} S) \times (I_{\mathbb{D} 0} T \times I_{\mathbb{D} 1} S) \times \gamma \times \gamma_0} \xrightarrow{(\mathbb{D}_1 C_1)^2 \times \mathbb{C}_0 \times (\mathbb{D}_1 C_0)^2 \times \gamma \times \gamma \times (\mathbb{D}_1 C_1)^4 \\
\mathbb{D}_0 C_1 \times \mathbb{D}_1 C_0 & \xrightarrow{I_0 \times I_{\mathbb{D} 1} \times I_1^2 \times (I_{\mathbb{D} 1})^2} (\mathbb{D}_1 C_1)^4 \xrightarrow{\xi_1^4 \times \xi_1} C_1^4 \times \mathbb{D}_0 C_1.
\end{align*}
$$

And the right hand piece, to be glued to the right of the previous one, is

$$
\begin{align*}
&\xrightarrow{\gamma \times \gamma} C_1^4 \\
&\xrightarrow{\gamma \times \gamma} C_1^3 \\
&\xrightarrow{\gamma \times \gamma} C_1^2 \\
&\xrightarrow{\gamma \times \gamma} C_1^1 \\
&\xrightarrow{\gamma \times \gamma} C_1.
\end{align*}
$$

We proceed to verify that all the subdiagrams commute, therefore verifying that
the counit preserves compositions. We begin in the upper left corner, so at the top of
the left portion displayed, in which the only part that needs explanation is the part
originating with the $\mathbb{D}_1 UC_1$ in the lower left and ending in the upper right,. But both
components agree because first $T_{UC} := T_e \circ \kappa_1$ by definition, and second $S_0$ is natural.

Proceeding to the rectangle next to the right, so at the top of the middle portion
displayed, we find that commutation only needs to be explained starting with the
middle $\mathbb{D}_1 C_1$ in the lower left corner. This is equivalent to checking commutativity of
the diagram

\[
\begin{array}{c}
\mathbb{D}_1C_1 \\
\downarrow \cong \\
\mathbb{D}_1C_0 \times_{\mathbb{D}_0C_0} \mathbb{D}_0C_1 \\
\downarrow \xi_1 \times \xi_1 \\
C_1 \times_{C_0} C_1 \rightarrow C_1,
\end{array}
\]

in which the bottom square commutes because \( \xi : \mathbb{D}C \rightarrow C \) is a functor. We still need to check the top triangle. This amounts to checking the following diagram:

\[
\begin{array}{c}
\mathbb{D}_1C_0 \times_{\mathbb{D}_0C_0} \mathbb{D}_0C_1 \rightarrow_{D_1(T,S_0)} \mathbb{D}_1C_1 \\
\downarrow \mathbb{D}_1I \times 1 \\
\mathbb{D}_1C_1 \times_{\mathbb{D}_0C_0} \mathbb{D}_0C_1 \rightarrow_{(D_1I L)} \mathbb{D}_1C_2 \rightarrow_{D_1\gamma C} \mathbb{D}_1C_1 \\
\downarrow 1 \times I_0 \\
\mathbb{D}_1C_1 \times_{\mathbb{D}_0C_0} \mathbb{D}_1C_1 \rightarrow_{(I R)_D} \mathbb{D}_2C_2 \rightarrow_{D_1\gamma C} \mathbb{D}_1C_1.
\end{array}
\]

The right side of the diagram commutes by the unital properties of composition in \( C \) and the category object \( \mathbb{D} \). The top left square expands to the commutative diagram

\[
\begin{array}{c}
\mathbb{D}_1C_0 \times_{\mathbb{D}_0C_0} \mathbb{D}_0C_1 \rightarrow_{D_1(T,S_0)} \mathbb{D}_1C_1 \\
\downarrow \mathbb{D}_1I \times 1 \\
\mathbb{D}_1C_1 \times_{\mathbb{D}_0C_0} \mathbb{D}_1C_1 \rightarrow_{(D_1I L)} \mathbb{D}_1C_2 \rightarrow_{D_1\gamma C} \mathbb{D}_1C_1 \\
\downarrow \mathbb{D}_1I \times 1 \\
\mathbb{D}_1C_1 \times_{\mathbb{D}_0C_0} \mathbb{D}_1C_1 \rightarrow_{(D_1I L)} \mathbb{D}_1C_2 \rightarrow_{D_1\gamma C} \mathbb{D}_1C_1.
\end{array}
\]

The left triangle commutes by the definition of \( I_L \), and the rest of the diagram is straightforward. The lower left square commutes as a result of its expansion as follows:

\[
\begin{array}{c}
\mathbb{D}_1C_2 \rightarrow_{(D_1T,S_0 \circ D_1S)} \mathbb{D}_1C_1 \times_{\mathbb{D}_0C_0} \mathbb{D}_0C_1 \\
\downarrow (I R)_D \\
\mathbb{D}_1C_2 \times_{\mathbb{D}_0C_2} \mathbb{D}_1C_1 \rightarrow_{(T_0, S_0)} \mathbb{D}_1C_1 \times_{\mathbb{D}_0C_0} \mathbb{D}_1C_1 \rightarrow_{1 \times I_0} \mathbb{D}_2C_2.
\end{array}
\]
where now the left triangle commutes as a consequence of the definition of \( I_R \) for the category object \( D \).

Returning to the large multi-part diagram, we proceed next to the second rectangle down on the left, which extends through both the left and middle portions. When restricted to the outer factors, both ways of traversing the rectangle coincide directly, leaving us with the part originating with \( D_1 U C_1 \). To verify this part, we begin with the following diagram:

\[
\begin{array}{c}
D_1 U C_1 \xrightarrow{D_1 \kappa_1} D_1 C_1 \\
(T_b, \theta) \\
\downarrow (T_b, D_1 S) \\
\downarrow \\
D_0 C_1 \times_{D_0 C_0} D_1 C_0 \xrightarrow{I_0 \times_{D_1} I} D_1 C_1 \times_{D_0 C_0} D_1 C_1 \\
\downarrow 1 \times_{\xi_0} 1 \\
\downarrow \\
D_0 C_1 \times_{D_0 C_0} D_1 C_0 \xrightarrow{I_0 \times D_1} D_1 C_1 \times_{D_1 C_0} D_1 C_0 \\
\downarrow \xi_1 \times \xi_0 \xi_1 \\
\downarrow \\
D_1 C_1 \times_{D_0 C_0} D_1 C_1 \xrightarrow{\xi_1 \times \xi_1} C_1 \times_{D_0 C_0} C_1.
\end{array}
\]

The reader is warned that with the central dotted arrow included, the left rectangle does not commute, however tempting it may be to claim it does. The problem is that when projected to the right factor \( D_1 C_0 \) of the target of the dotted arrow, there is an ambiguity in the structure map for \( D_1 D_0 C_0 \). The use of the map \( \theta \) dictates the use of \( \mu \circ D_1 I' \), while the use of \( \kappa_1 \) dictates the use of \( D_1 \xi_0 \). They aren’t the same. However, the entire large diagram does commute. The projection onto the left factor causes no problems; we can even use the dotted arrow, although it is easy enough to see that the composites coincide. Projection onto the right factor is a consequence of the following diagram:

\[
\begin{array}{c}
D_1 U C_1 \xrightarrow{D_1 \kappa_1} D_1 C_1 \\
D_1 S \\
\downarrow \\
D_1 D_0 C_0 \xrightarrow{D_1 \xi_0} D_1 C_0 \\
D_1 I \\
\downarrow \\
D_1 C_0 \xrightarrow{\mu} D_1 C_1 \xrightarrow{\xi_1} C_1
\end{array}
\]
The left vertical composite expresses $\theta$, so the diagram says that although $\theta \neq \mathbb{D}_1 S \circ \mathbb{D}_1 \kappa_1$, they do coincide after composing with $\xi_1 \circ \mathbb{D}_1 I$. This is the principle reason the left part of the previous diagram doesn’t commute with the dotted arrow included.

Now we paste onto the right of the double rectangle diagram the following one, which completes the rectangle we wish to verify:

\[
\begin{array}{c}
\mathbb{D}_1 C_1 \\
\downarrow (T_h, \mathbb{D}_1 S) \\
\mathbb{D}_0 C_1 \times_{\mathbb{D}_0 C_0} \mathbb{D}_1 C_0 
\end{array}
\xrightarrow{I_0 \times \mathbb{D}_1 I}
\begin{array}{c}
\mathbb{D}_1 C_1 \times_{\mathbb{D}_0 C_0} \mathbb{D}_1 C_0 \\
\underline{\xi_1 \times \xi_1} \\
C_1 \times_{C_0} C_1
\end{array}
\xrightarrow{\gamma_C} 
\begin{array}{c}
\mathbb{D}_1 C_1 \\
\underline{\xi_1} \\
C_1
\end{array}
\]

The square is one we’ve checked before, and the triangle is analogous to the one we checked above. If the reader wants the details, here they are: the triangle amounts to checking commutativity of

\[
\begin{array}{c}
\mathbb{D}_0 C_1 \times_{\mathbb{D}_0 C_0} \mathbb{D}_1 C_0 \\
\downarrow 1 \times \mathbb{D}_1 I \\
\mathbb{D}_0 C_1 \times_{\mathbb{D}_0 C_0} \mathbb{D}_1 C_1 \\
\downarrow I_0 \times 1 \\
\mathbb{D}_1 C_1 \times_{\mathbb{D}_0 C_0} \mathbb{D}_1 C_1
\end{array}
\xrightarrow{(T_h \circ \mathbb{D}_1 T, \mathbb{D}_1 S)}
\begin{array}{c}
\mathbb{D}_1 C_1 \\
\downarrow \mathbb{D}_1 I_R \\
\mathbb{D}_1 C_2 \\
\downarrow (I_L)_D \\
\mathbb{D}_1 C_1
\end{array}
\xrightarrow{(T_h \circ \mathbb{D}_2 T, \mathbb{D}_2 S)}
\begin{array}{c}
\mathbb{D}_1 C_2 \\
\downarrow \gamma_D \\
\mathbb{D}_1 C_2
\end{array}
\]

The top left square follows from being filled in as follows:

\[
\begin{array}{c}
\mathbb{D}_1 C_1 \\
\downarrow (T_h, \mathbb{D}_1 S) \\
\mathbb{D}_0 C_1 \times_{\mathbb{D}_0 C_0} \mathbb{D}_1 C_0 \\
\downarrow (1, \mathbb{D}_1 (I \circ S)) \\
\mathbb{D}_1 C_1 \\
\downarrow \mathbb{D}_1 I_R \\
\mathbb{D}_1 C_1 \\
\downarrow (\mathbb{D}_1 T, \mathbb{D}_1 S) \\
\mathbb{D}_1 C_2 \\
\downarrow (T_h \circ \mathbb{D}_1 T, \mathbb{D}_1 S) \\
\mathbb{D}_0 C_1 \times_{\mathbb{D}_0 C_0} \mathbb{D}_1 C_1 \\
\downarrow 1 \times \mathbb{D}_1 I \\
\mathbb{D}_1 C_2
\end{array}
\xrightarrow{T_h \times T_h 1}
\begin{array}{c}
\mathbb{D}_0 C_1 \times_{\mathbb{D}_0 C_0} \mathbb{D}_1 C_1
\end{array}
\]
The bottom left square fills as follows,

\[
\begin{array}{c}
\mathbb{D}_1 \mathbb{C}_2 \rightarrow \mathbb{D}_0 \mathbb{C}_1 \\
\downarrow \mathbb{D}_0(\mathbb{C}_0 \mathbb{D}_1) \downarrow \mathbb{D}_0(\mathbb{C}_0) \\
\mathbb{D}_2 \mathbb{C}_2 \rightarrow \mathbb{D}_1 \mathbb{C}_1 \\
\end{array}
\]

and the rest follows from unital properties of \( \gamma_C \) and \( \gamma_D \).

We return again to the large multi-part diagram, and proceed to the next rectangle down on the left: the second one that spans both the left and middle displays. The portion starting with \( \mathbb{D}_0 \mathbb{C}_0 \) is unproblematic: both composites coincide explicitly. For the portion starting with \( \mathbb{D}_0 \mathbb{U} \mathbb{C}_2 \), the part proceeding first via \( \mathbb{D}_0 \mathbb{T} \) commutes because of the square

\[
\begin{array}{c}
\mathbb{D}_0 \mathbb{U} \mathbb{C}_2 \mathbb{D}_0 \mathbb{T} \rightarrow \mathbb{D}_0 \mathbb{U} \mathbb{C}_1 \\
\downarrow \mathbb{D}_0 \mathbb{S} \downarrow \mathbb{D}_1 \mathbb{T} \downarrow \mathbb{D}_0 \mathbb{S} \\
\mathbb{D}_0 \mathbb{C}_2 \mathbb{D}_0 \mathbb{T} \rightarrow \mathbb{D}_1 \mathbb{C}_1 \\
\end{array}
\]

to which \( \xi_1 \circ \mathbb{I}_\mathbb{D} \) is then appended. The part proceeding from \( \mathbb{D}_0 \mathbb{U} \mathbb{C}_2 \) via \( \mu \circ \mathbb{D}_0 \mathbb{S} \) is more of a challenge, but it too commutes as a result of the following diagram:

\[
\begin{array}{c}
\mathbb{D}_0 \mathbb{U} \mathbb{C}_2 \mathbb{D}_0 \mathbb{S} \rightarrow \mathbb{D}_0 \mathbb{C}_2 \\
\downarrow \mathbb{D}_0 \mathbb{S} \downarrow \mathbb{D}_0 \mathbb{S} \downarrow \mathbb{D}_0 \mathbb{S} \\
\mathbb{D}_0 \mathbb{U} \mathbb{C}_1 \mathbb{D}_0 \mathbb{S} \rightarrow \mathbb{D}_1 \mathbb{C}_1 \\
\end{array}
\]

In particular, this diagram exhibits the fact that although \( \mathbb{D}_0 \mathbb{K}_1 \circ \mu \circ \mathbb{D}_0 \mathbb{S} \neq \mathbb{D}_0 \mathbb{S} \circ \mathbb{D}_0 \mathbb{K}_2 \), they do coincide when composed with \( \xi_1 \circ \mathbb{I}_\mathbb{D} \). This is the reason the rectangle has no shorter fill.

Returning again to the large, multi-part diagram, we have arrived at the bottom of the left hand portion, which commutes because of the definition of \( \gamma_{\mathbb{U} \mathbb{C}} \). The subdiagram to its right, which includes portions in both the middle and right displays,
is the product of two diagrams, one with source $\mathbb{D}_0C_2$ and the other with source $\mathbb{D}_2C_0$. The one with source $\mathbb{D}_0C_2$ commutes as a result of the following diagram:

And the one starting with $\mathbb{D}_2C_0$ commutes as a result of the following analogous diagram:

The remaining parts of the large, multi-part diagram are both in the right hand display, and are a consequence of associativity for $\gamma_C$. We may conclude that the counit preserves composition, and is therefore a functor.

We still must show that the counit respects the $\mathbb{D}$-algebra structure on $\hat{LM}$, that is, that the diagram
commutes. We expand using the definitions of $\varepsilon$ and $\xi_1$ and fill as follows:

\[
\begin{array}{c}
\xymatrix{
D_1D_0U_1C_1 \times_D_1D_0C_0 \ar[r]^-{T_b \times T_b} & D_0U_1C_1 \times_D_0C_0 \ar[r]^-{\mu \times \mu} & D_0U_1C_1 \times_D_0C_0 \ar[d]^-{D_0 \times \xi_0} & D_0 \times D_1C_0 \\
D_1D_0 \times D_0C_0 \ar[r]^-{T_b \times T_b} & D_0 \times D_0C_0 \ar[r]^-{\mu \times \mu} & D_0 \times D_0C_0 \\
D_1C_1 \times D_1C_1 \ar[r]^-{(I_b \circ T_b) \times T_b} & D_1C_1 \times D_1C_1 \\
D_1 \times D_1C_1 \ar[r]^-{(I_b \circ T_b) \times T_b} & D_1 \times D_1C_1 \\
D_1C_1 \ar[r]^-{\varepsilon} & C_1.
}\end{array}
\]

The one part of this fill that may be less than clear is the triangle in the lower left corner, but this is a consequence of the left unital property of the category structure on $D$. In particular, we can expand the triangle as follows:

\[
\begin{array}{c}
\xymatrix{
D_1C_1 \times_D_1C_1 \ar[r]^-{(I_b \circ T_b) \times T_b} & D_1C_1 \times_D_1C_1 \\
D_1C_1 \ar[r]^-{(I_b \circ T_b) \times T_b} & D_1C_1 \\
D_1C_1 \ar[r]^-{\varepsilon} & C_1.
}\end{array}
\]

Here the lower part of the diagram captures the left unital property for the category structure on $D$, and therefore the diagram commutes. It follows that the counit preserves the $D$-algebra structure on $\hat{L}M$.

14. The adjunction structure III: commuting triangles

We wish to verify the adjunction triangles

\[
\begin{array}{c}
U \Upsilon \xRightarrow{\eta} U \Upsilon \hat{L}U \Upsilon & \text{and} & \hat{L} \Upsilon \xRightarrow{\hat{L} \eta} \hat{L}U \hat{L}U \Upsilon \\
= \xRightarrow{\Upsilon \varepsilon} & \xRightarrow{\varepsilon} & = \xRightarrow{\varepsilon} \hat{L} \Upsilon \Upsilon.
\end{array}
\]
Both of these are straightforward when restricted to objects: we have the actual adjunction between sets and $D_0$-algebras. We therefore concentrate on the maps on morphisms, and start with the first triangle.

In general we have $UC_1 \cong C_1 \times_{C_0} D_0C_0$, so replacing $C$ with $\hat{L}UC$, we have

$$U\hat{L}UC_1 \cong D_0UC_1 \times_{D_0C_0} D_1C_0 \times_{D_0C_0} D_0^2C_0,$$

where the structure map on the $D_0^2C_0$ on the end is given by $\mu$. Next, in general we have

$$U\hat{L}M_1 \cong D_0M_1 \times_{D_0M_0} D_1M_0 \times_{D_0M_0} D_0^2M_0,$$

and the unit $\eta_1 : M_1 \rightarrow U\hat{L}M_1$ is given by $(\eta_D, I_D \circ S, D_0\eta \circ S)$. Replacing $M_1$ with $UC_1$ and rewriting $D_0UC_1$ as $D_0C_1 \times_{D_0C_0} D_0^2C_0$, we see that the unit $\eta_1 : UC_1 \rightarrow U\hat{L}UC_1$ is a map

$$C_1 \times_{C_0} D_0C_0 \rightarrow D_0C_1 \times_{D_0C_0} D_0^2C_0 \times_{D_0C_0} D_1C_0 \times_{D_0C_0} D_0^2C_0$$

given by the four components $(\eta_D \circ p_1, \eta_D \circ p_2, I_D \circ p_2, D_0\eta \circ p_2)$. The structure maps on the two copies of $D_0^2C_0$ need particular care: the first one has structure map of target type $D_0\xi_0$ and source type $\mu$, while the second one has structure map of target type $\mu$.

Next, the counit $\varepsilon_1 : \hat{L}UC_1 \rightarrow C_1$ consists of the following composite:

$$\hat{L}UC_1 \xrightarrow{\cong} D_0UC_1 \times_{D_0C_0} D_1C_0 \xrightarrow{D_0\kappa_1 \times \xi_0} D_0C_1 \times_{C_0} D_1C_0 \xrightarrow{I_D \times D_1I} D_1C_1 \times_{C_0} D_1C_1 \xrightarrow{\xi_1 \times \xi_1} C_1 \times_{C_0} C_1 \xrightarrow{\gamma_c} C_1.$$

Since $U\hat{L}UC_1 \cong \hat{L}UC_1 \times_{\hat{L}UC_0} D_0\hat{L}UC_0 = \hat{L}UC_1 \times_{D_0C_0} D_0^2C_0$, what $U\varepsilon_1$ does is $\varepsilon_1$ on the first factor, and $D_0\varepsilon_0 = D_0\xi_0$ on the second factor. Writing $\xi_0^2$ for either way of composing in the commutative square

$$\begin{array}{ccc}
D_0^2C_0 & \xrightarrow{D_0\xi_0} & D_0C_0 \\
\mu \downarrow & & \downarrow \xi_0 \\
D_0C_0 & \xrightarrow{\xi_0} & C_0,
\end{array}$$
we can express the induced map $U \varepsilon_1$ as the following composite, in which the $\xi_0^2$ simply has the effect of deleting the $D^2_0C_0$ term:

\[
\begin{array}{c}
D_0C_1 \times_{D_0C_0} D^2_0C_0 \times_{D_0C_0} D_1C_0 \times_{D_0C_0} D^2_0C_0 \\
\downarrow 1 \times \iota_0 \iota_0^2 \times \iota_0 \downarrow \downarrow 1 \downarrow \\
D_0C_1 \times_{D_0C_0} D_1C_0 \times_{D_0C_0} D^2_0C_0 \\
\downarrow I_0 \times D_1 I \downarrow I_1 \\
D_1C_1 \times_{D_0C_0} D_1C_1 \times_{D_0C_0} D^2_0C_0 \\
\downarrow \xi_0 \xi_1 \downarrow I_0 \xi_0 \xi_0 \\
C_1 \times_{D_0C_0} C_1 \times_{D_0C_0} D_0C_0 \\
\downarrow \gamma \downarrow 1 \\
C_1 \times_{D_0C_0} D_0C_0 = UC_1.
\end{array}
\]

We proceed by identifying the composite up to, but not including, the very last step, which is the use of $\gamma \times 1$, so the target is $C_1 \times_{D_0C_0} C_1 \times_{D_0C_0} D_0C_0$. If we project to the first factor of $C_1$, tracing through shows that the composite depends only on the factor of $C_1$ in the initial source, and consists of either way around the triangle

\[
\begin{array}{c}
C_1 \xrightarrow{\eta_0} D_0C_1 \\
\downarrow \eta_{D_1} \downarrow I_0 \\
D_1C_1 \\
\downarrow \xi_1 \\
C_1.
\end{array}
\]

The top part commutes because the unit $\eta : 1 \to D$ is natural in the simplicial structure of $D$, and the bottom because $C_1$ is a $D_1$-algebra.

If we project to the second factor of $C_1$, we see that the map depends only on the factor of $D_0C_0$ in the initial source, and consists of either way around the commutative rectangle

\[
\begin{array}{c}
D_0C_0 \xrightarrow{I_0} D_1C_0 \xrightarrow{D_1I} D_1C_1 \\
\downarrow \xi_0 \downarrow I \\
C_0 \xrightarrow{I} C_1.
\end{array}
\]

Consequently, the image in this factor of $C_1$ consists entirely of identity maps, so won’t change the first factor upon composing. We see therefore that the composite including $\gamma \times 1$ is the identity on the factor of $C_1$. 
Projecting to the factor of $\mathbb{D}_0\mathcal{C}_0$, the composite consists of either way around the commuting triangle

$$\begin{array}{ccc}
\mathbb{D}_0\mathcal{C}_0 & \xrightarrow{\mathbb{D}_0\eta} & \mathbb{D}_0^2\mathcal{C}_0 \\
\downarrow & \searrow & \downarrow \\
\mathbb{D}_0\mathcal{C}_0 & \xrightarrow{\mathbb{D}_0\xi} & \mathbb{D}_0\mathcal{C}_0.
\end{array}$$

The entire composite is the identity, and we see that the first adjunction triangle commutes.

For the second adjunction triangle, we need explicit expressions for both $\hat{L}\eta_1 : \hat{L}M_1 \rightarrow \hat{L}\hat{U}\hat{L}M_1$ and $\varepsilon_1 : \hat{L}\hat{U}\hat{L}M_1 \rightarrow \hat{L}M_1$. We start by recalling $\hat{L}M_1 \cong \mathbb{D}_0M_1 \times_{\mathbb{D}_0M_0} \mathbb{D}_1M_0$, and also $\hat{L}M_0 := \mathbb{D}_0M_0$. Then since in general $U\mathcal{C}_1 = \mathcal{C}_1 \times_{\mathcal{C}_0} \mathbb{D}_0\mathcal{C}_0$, we have

$$U\hat{L}\hat{M}_1 \cong \hat{L}M_1 \times_{\mathbb{D}_0M_0} \mathbb{D}_0^2M_0 \cong \mathbb{D}_0M_1 \times_{\mathbb{D}_0M_0} \mathbb{D}_1M_0 \times_{\mathbb{D}_0M_0} \mathbb{D}_0^2M_0,$$

where the structure map for $\mathbb{D}_0^2M_0$ is $\mu$. We also have $U\hat{L}\hat{M}_0 = \hat{L}M_0 = \mathbb{D}_0M_0$. Applying $\hat{L}$ to all this, we get

$$\hat{L}\hat{U}\hat{L}M_1 = \mathbb{D}_0\hat{U}\hat{L}M_1 \times_{\mathbb{D}_0\hat{U}\hat{L}M_0} \mathbb{D}_1\hat{U}\hat{L}M_0 \cong \mathbb{D}_0^2M_1 \times_{\mathbb{D}_0^2M_0} \mathbb{D}_0\mathbb{D}_1M_0 \times_{\mathbb{D}_0^2M_0} \mathbb{D}_0^3M_0 \times_{\mathbb{D}_0^2M_0} \mathbb{D}_1\mathbb{D}_0M_0.$$

(We must take care to note that the structure maps on $\mathbb{D}_0^2M_0$ are $\mu$ for target type, but $\mathbb{D}_0\mu$ for source type.) Now $\eta : 1 \rightarrow U\hat{L}$ has components $\eta_0 : M_0 \rightarrow U\hat{L}M_0 = \mathbb{D}_0M_0$ given by $\eta_{\mathbb{D}_0}$, the unit for the monad $\mathbb{D}_0$, and $\eta_1 : M_1 \rightarrow U\hat{L}M_1$ given by

$$M_1 \rightarrow \mathbb{D}_0M_1 \times_{\mathbb{D}_0M_0} \mathbb{D}_1M_0 \times_{\mathbb{D}_0M_0} \mathbb{D}_0^2M_0$$

given by component maps $(\eta_{\mathbb{D}_0}, I_{\mathbb{D}_0} \circ S, \mathbb{D}_0\eta \circ S)$. Applying $\hat{L}$ to these data, we find the induced map $\hat{L}\eta_1 : \hat{L}M_1 \rightarrow \hat{L}\hat{U}\hat{L}M_1$ is given by the map

$$\mathbb{D}_0M_1 \times_{\mathbb{D}_0M_0} \mathbb{D}_1M_0 \rightarrow \mathbb{D}_0^2M_1 \times_{\mathbb{D}_0^2M_0} \mathbb{D}_0\mathbb{D}_1M_0 \times_{\mathbb{D}_0^2M_0} \mathbb{D}_0^3M_0 \times_{\mathbb{D}_0^2M_0} \mathbb{D}_1\mathbb{D}_0M_0$$

with components $(\mathbb{D}_0\eta \circ p_1, \mathbb{D}_0(I_{\mathbb{D}_0} \circ S) \circ p_1, \mathbb{D}_0(\mathbb{D}_0\eta \circ S) \circ p_1, \mathbb{D}_1\eta_{\mathbb{D}_0} \circ p_2)$. In particular, the projection to the first three terms depend only on the first term of the source, and the projection to the fourth depends only on the last term in the source.

For the counit map, we have in general the composite given above, namely

$$\begin{array}{ccc}
\hat{L}\mathcal{C}_1 & \xrightarrow{\cong} & \mathbb{D}_0\mathcal{C}_1 \times_{\mathbb{D}_0\mathcal{C}_0} \mathbb{D}_1\mathcal{C}_0 \\
\xrightarrow{I_0 \times_{\mathbb{D}_1} I} & \mathbb{D}_1\mathcal{C}_1 \times_{\mathbb{D}_0\mathcal{C}_0} \mathbb{D}_1\mathcal{C}_1 \\
\xrightarrow{\xi_1 \times \xi_1} & \mathbb{D}_1\mathcal{C}_1 \times_{\mathbb{D}_0\mathcal{C}_0} \mathbb{D}_1\mathcal{C}_1 \\
\xrightarrow{\gamma c} & \mathcal{C}_1.
\end{array}$$

When we have $\mathcal{C} = \hat{L}M$, this becomes in particular the following composite, in which $\mu^2 : \mathbb{D}_0^2M_0 \rightarrow \mathbb{D}_0M_0$ is either way around the evident square, and simply has the effect
of deleting the $D_0^3 M_0$ term:

\[
\begin{align*}
D_0^2 M_1 \times D_0^2 M_0 & \xrightarrow{1 \times 1 \times \mu^2 \times \mu} D_0^2 M_1 \times D_0^2 M_0 \times D_0^2 M_0 \times D_0^2 M_0 \\
\end{align*}
\]

\[
\begin{align*}
I_0 \times I_0 \times (D_1 D_0 I_0) & \xrightarrow{[\mu T_0 \times \nu T_0 \mu]^2} D_1 D_0 M_1 \times D_1 D_0 M_0 \times D_1 D_0 M_0 \times D_1 D_0 M_0 \\
\end{align*}
\]

\[
\begin{align*}
1 \times (D_1 T, S_0) \times 1 & \xrightarrow{1 \times (T_0, \theta) \times 1} D_0 M_1 \times D_0 M_0 \times D_1 M_1 \times D_0 M_0 \times D_0 M_0 \times D_1 M_0 \\
\end{align*}
\]

\[
\begin{align*}
D_0 \gamma M \times \gamma D_0 & \xrightarrow{D_0 \gamma M \times \gamma D_0} D_0 M_1 \times D_0 M_0 \times D_1 M_0 = \hat{L} M_1.
\end{align*}
\]

We analyze the composite of this with $\hat{L} \eta$ by first stopping halfway through the counit at the term that gives $\hat{L} M_2$, namely

\[
D_0 M_1 \times D_0 M_0 \times D_1 M_0 \times D_0 M_1 \times D_0 M_0 \times D_1 M_0.
\]

Projecting to the first term, and restricting to the first term in the source of $\hat{L} \eta$, we get the following commutative diagram:

\[
\begin{array}{ccc}
D_0 M_1 & \xrightarrow{D_0 \eta} & D_0^2 M_1 \\
& \searrow & \downarrow T_0 \\
& & D_0^2 M_1 \\
& \downarrow & \downarrow \mu \\
& & D_0 M_1.
\end{array}
\]

Projecting to the second term, and again restricting to $D_0 M_1$ in the source, we have the diagram

\[
\begin{array}{ccc}
D_0 M_1 & \xrightarrow{D_0 S} & D_0^2 M_0 \\
& \searrow & \downarrow D_0 I_0 \\
& & D_0^2 M_0 \\
& \downarrow \mu & \downarrow \mu \\
& & D_0 M_0 \\
& \downarrow I_0 & \downarrow I_0 \\
& & D_1 M_0.
\end{array}
\]
Notice that $\mu \circ \mathbb{D}_0 S$ is the structure map for $\mathbb{D}_0 M_1$ of source type. Projecting to the third term, and now restricting to $\mathbb{D}_1 M_0$, we get the commuting diagram

\[
\begin{array}{ccc}
\mathbb{D}_1 M_0 & \xrightarrow{\mathbb{D}_1 T_0} & \mathbb{D}_1 \mathbb{D}_0 M_0 \\
| & & | \\
\mathbb{D}_0 M_0 & \xrightarrow{\mu} & \mathbb{D}_0^2 M_0 \\
\end{array}
\]

Notice that $T_\mathbb{D}$ is the structure map for $\mathbb{D}_1 M_1$ of target type. Projecting to the fourth term, and still restricting to $\mathbb{D}_1 M_0$, we get

\[
\begin{array}{ccc}
\mathbb{D}_1 M_0 & \xrightarrow{\mathbb{D}_1 \mathbb{D}_0 T_0} & \mathbb{D}_1 \mathbb{D}_0 M_0 \\
| & & | \\
\mathbb{D}_0 M_0 & \xrightarrow{\mu} & \mathbb{D}_1^2 M_0 \\
\end{array}
\]

The net result of these calculations is that the composite map to $\hat{L} M_2$ can be written as the map $1 \times (\mu \circ \mathbb{D}_0 S, T_\mathbb{D}) \times 1$, as follows:

\[
\mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0
\]

\[
\mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0,
\]

where the source of the middle terms is the $\mathbb{D}_0 M_0$ over which the pullback defining the overall source is defined. We examine what happens to these middle terms as we go on in the counit: the next term is the backwards arrow (which is however an isomorphism) and the forwards one, both from $\mathbb{D}_1 M_1$. We obtain the following diagram, which we claim commutes:

\[
\begin{array}{ccc}
\mathbb{D}_0 M_0 & \xrightarrow{(I_0, \mathbb{D}_0 I)} & \mathbb{D}_1 M_0 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1 \\
| & & | \\
\mathbb{D}_0 M_0 & \xrightarrow{(\mathbb{D}_0 I, I_0)} & \mathbb{D}_1 M_1 \\
\end{array}
\]

\[
\mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0.
\]
The only part of the diagram that isn’t immediate from properties of categories and
multicategories is the part involving $\theta$, but that follows from the following diagram,
in which the bottom row displays $\theta$ explicitly:

```
\[ \begin{array}{c}
\text{D}_0 M_0 \\
\text{D}_1 M_1 \rightarrow \text{D}_1 \text{D}_0 M_0 \rightarrow \text{D}_1^2 M_0 \rightarrow \text{D}_1 M_0,
\end{array} \]
```

It now follows that the middle terms in the part of the counit just before the compo-
sition at the end,

\[ \text{D}_0 M_1 \times_{\text{D}_0 M_0} \text{D}_0 M_1 \times \text{D}_1 M_0 \times_{\text{D}_0 M_0} \text{D}_1 M_0 \]

\[ \text{D}_0 \gamma M \times \gamma D \rightarrow \text{D}_0 M_1 \times_{\text{D}_0 M_0} \text{D}_1 M_0, \]

consist only of identity elements, and so contribute nothing after composition. Since
the terms on the end are given by identity maps in the total composition, we conclude
that the second adjunction triangle commutes.

## 15. The actual left adjoint: category structure

In this section we begin the proof of Theorem 6.1. Recall that the morphism set
$LM_1$ of the actual left adjoint is given by a coequalizer of two arrows to the morphism
set $\hat{LM}_1$, while the objects are just the objects $\text{D}_0 M_0$ of $\hat{LM}$.

We first give the category structure of $LM$. The unit map $I : LM_0 \rightarrow LM_1$ is just
the unit map $\hat{LM}_0 \rightarrow \hat{LM}_1$ composed with the quotient map $\hat{LM}_1 \rightarrow LM_1$. Source
and target maps are also inherited from $\hat{LM}$, since the coequalization takes place in
the middle of the two terms defining $\hat{LM}$, while the source and target maps are on
the “outside.” The significant problem is verifying that the composition map on $\hat{LM}$
duces a map on $LM$. For this, we need to see that composition in $\hat{LM}$ preserves
equivalence classes. Our strategy for doing so is to lift the composition in $\hat{LM}$ to the
source of the coequalizer defining $LM_1$ in two different ways so that the composition
in $\hat{LM}$ then descends to the coequalizer. Here are the details.

We abbreviate the coequalizer diagram as

\[ QM \xrightarrow{\psi_*} \hat{LM}_1 \longrightarrow LM_1, \]

where we have written $QM$ for $\text{D}_0 M_1 \times_{\text{D}_0 M_0} \text{D}_0 \text{D}_1 M_0 \times_{\text{D}_0 M_0} \text{D}_1 M_0$, $\psi_*$ for the map
$\text{D}_0 \psi_1 \times 1$, and $\gamma_*$ for the map $1 \times \gamma_D ((\mu \circ I_D) \times 1)$. We will construct a map $\Gamma_R$:
\(QM \times_{D_0 M_0} \hat{LM}_1 \to QM\) for which both top and bottom choices of horizontal arrow in the following diagram commute:

\[
\begin{array}{cccccc}
QM \times \hat{LM}_1 & \xrightarrow{\psi_\ast \times 1} & \hat{LM}_1 \times \hat{LM}_1 \\
\downarrow \Gamma_R & & \downarrow \gamma_{LM} \\
QM & \xrightarrow{\gamma_\ast} & \hat{LM}_1 \\
\end{array}
\]

This will show that equivalent morphisms in the left slot give rise to equivalent composites. We also construct a map \(\Gamma_L : \hat{LM}_1 \times_{D_0 M_0} QM \to QM\) for which both top and bottom choices of horizontal arrow in the diagram commute. This will show that equivalent morphisms in the right slot give rise to equivalent composites, and therefore the composition descends to \(LM\).

We define \(\Gamma_R\) as the following composite, which requires some explanation about the map \(\bar{\chi}\):

\[
\begin{array}{cccccc}
D_0 M_1 \times_{D_0^2 M_0} D_0 D_1 M_0 & \xrightarrow{1^2 \times \gamma_\ast \times 1} & D_0 M_1 \times_{D_0^2 M_0} D_0 D_1 M_0 \\
D_0 M_1 \times_{D_0^2 M_0} D_0 D_1 M_0 & \xrightarrow{1 \times \bar{\chi} \times 1^2} & D_0 M_1 \times_{D_0^2 M_0} D_0 D_1 M_0 \\
D_0 M_1 \times_{D_0^2 M_0} D_0^2 M_1 & \xrightarrow{D_0 \gamma M \times 1 \times \gamma D} & D_0 M_1 \times_{D_0^2 M_0} D_0 D_1 M_0
\end{array}
\]
To explain $\tilde{\chi}$, we observe that both squares in the diagram

\[
\begin{array}{ccc}
D_0D_1M_1 & \xrightarrow{D_0D_1T} & D_0D_1M_0 \\
\downarrow D_0S_0 & & \downarrow D_0S_0 \\
D_0^2M_1 & \xrightarrow{D_0'^T} & D_0^2M_0 \\
\mu & \downarrow & \mu \\
D_0M_1 & \xrightarrow{D_0T} & D_0M_0
\end{array}
\]

are pullbacks, the bottom one since $D_0$ is Cartesian, and the top one since $D_0$ preserves pullbacks (being Cartesian.) Consequently, we can define $\tilde{\chi}$ as the composite

\[
D_0D_1M_0 \times_{D_0M_0} D_0M_1 \xrightarrow{(D_0D_1T, \mu \circ D_0S_0)} D_0D_1M_1 \xrightarrow{(D_0T, D_0\theta)} D_0^2M_1 \times_{D_0^2M_0} D_0D_1M_0.
\]

There are now two diagrams we wish to commute involving $\Gamma_R$. The first one, using the arrows $\psi \times 1$ and $\psi_*$ in the diagram given above, doesn’t involve the final factor of $D_1M_0$ before it is multiplied, so can be omitted from the diagram. With subscripts of $D_0M_0$ again suppressed, we wish the following to commute:

\[
\begin{array}{ccc}
D_0M_1 \times_{D_0^2M_0} D_0D_1M_0 \times D_1M_0 \times D_0M_1 & \xrightarrow{D_0\psi \times 1^2} & D_0M_1 \times D_1M_0 \times D_0M_1 \\
1^2 \times \chi & \downarrow & 1 \times \chi \\
D_0M_1 \times_{D_0^2M_0} D_0D_1M_0 \times D_0M_1 \times D_1M_0 & \xrightarrow{D_0\psi \times 1^2} & D_0M_1 \times D_0M_1 \times D_1M_0 \\
1 \times \tilde{\chi} \times 1 & \downarrow & \quad \\
D_0M_1 \times_{D_0^2M_0} D_0D_1M_0 \times D_1M_0 & \xrightarrow{D_0\gamma \times 1} & D_0M_1 \times D_0M_1 \times D_1M_0 \\
D_0\gamma \times 1^2 & \downarrow & \quad \\
D_0M_1 \times_{D_0^2M_0} D_0D_1M_0 \times D_1M_0 & \xrightarrow{D_0\psi \times 1} & D_0M_1 \times D_1M_0.
\end{array}
\]

However, the top rectangle commutes by naturality, so we are left with the bottom rectangle, in which the $D_1M_0$ at the end plays no role. We are left with wanting the
to commute. In order to verify this, we need some observations about the presheaf actions on $M_1$ and $M_2$.

First, the target map $M_2 \to M_1$ is supposed to be a map of presheaves over the functor $D\varepsilon : D^2(*) \to D(*)$. In order to lift this to the actions by $D^2 M_0$ and $D M_0$, we need the following diagram, in which the dotted arrow labeled $\varepsilon_*$ is induced as a map of pullbacks from the left tall rectangle to the right one:

We can glue the following pullback cube to the top of the diagram, where the map $1 \times \varepsilon_*$ is so named since it really expresses the composite

$$M_2 \times_{D^0 M_0} D^1 M_0 \xrightarrow{\cong} M_1 \times_{D^0 M_0} D^1 M_1 \times_{D^0 M_0} D^1 M_0 \xrightarrow{1 \times \varepsilon_*} M_1 \times_{D^0 M_0} D^1 M_0 :$$
15. THE ACTUAL LEFT ADJOINT: CATEGORY STRUCTURE

We can express the fact that the target map \( T : M_2 \rightarrow M_1 \) is a map of presheaves over \( \mathbb{D} \) by the diagram

\[
\begin{array}{c}
\xymatrix{
M_2 \times_{\mathbb{D}_0^2 M_0} \mathbb{D}_1^2 M_0 \ar[r]^{1 \times \varepsilon_*} & M_1 \times \mathbb{D}_1 M_0 \\
S \ar@{|->}[u] & S \\
\mathbb{D}_0 M_1 \times_{\mathbb{D}_0^2 M_0} \mathbb{D}_1^2 M_0 \ar[r]_{\varepsilon^*} & \mathbb{D}_1 M_0 \\
\mathbb{D}_0 M_1 \ar[u]^{p_1} \ar[r] & \mathbb{D}_0 M_0.
\end{array}
\]

But the previous two cubes allow us to rewrite the top arrow to give the equivalent diagram

\[
\begin{array}{c}
\xymatrix{
M_2 \times_{\mathbb{D}_0^2 M_0} \mathbb{D}_1^2 M_0 \ar[r]^T & M_1 \times \mathbb{D}_1 M_0 \\
M_2 \ar[r]^T & M_1.
\end{array}
\]

Meanwhile, the equivariance of \( S : M_2 \rightarrow \mathbb{D}_0 M_1 \) as a presheaf over \( \mathbb{D}^2 M_0 \) can be expressed by the commutative rectangle

\[
\begin{array}{c}
\xymatrix{
M_2 \times_{\mathbb{D}_0^2 M_0} \mathbb{D}_1^2 M_0 \ar[r]^{S \times 1} & \mathbb{D}_0 M_1 \times_{\mathbb{D}_0^2 M_0} \mathbb{D}_1^2 M_0 \\
\mathbb{D}_0 M_1 \ar[r]_{\mathbb{D} \psi_1} & \mathbb{D}_0 M_1.
\end{array}
\]

and the equivariance of \( \gamma_M : M_2 \rightarrow M_1 \) as a map over \( \mu : \mathbb{D}^2(*) \rightarrow \mathbb{D}(*) \) lifts directly to \( \mu : \mathbb{D}^2 M_0 \rightarrow \mathbb{D} M_0 \) in the rectangle

\[
\begin{array}{c}
\xymatrix{
M_2 \times_{\mathbb{D}_0^2 M_0} \mathbb{D}_1^2 M_0 \ar[r]^{\gamma_M \times \mu} & M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 \\
M_2 \ar[r]_{\gamma_M} & M_1.
\end{array}
\]
We return now to our previous reduction of the first diagram, which we expand and augment as follows:

All the sub-diagrams here already commute, with the exception of the second one down: the one with the second $\mathbb{D}_0^{\psi_1} \times 1$ as its top arrow. This sub-diagram can be turned upside down, and have a $\mathbb{D}_0$ stripped off, to become the following diagram of our desire, in which, as usual, we suppress subscript $\mathbb{D}_0$'s on products:

This diagram commutes precisely when it does so after projecting onto each of the factors in the target $M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 M_1$ (which is the same thing as $M_2$), to which we
can then apply the equivariance diagrams we already know about $\psi_2$, namely

$$M_1 \times \mathbb{D}_0 M_1 \times \mathbb{D}_0^2 M_0 \xrightarrow{1 \times \varepsilon_*} M_1 \times \mathbb{D}_1 M_0$$

$$\psi_2 \downarrow \quad \psi_1$$

$$M_1 \times \mathbb{D}_0 M_1 \xrightarrow{T = p_1} M_1$$

and

$$M_1 \times \mathbb{D}_0 M_1 \times \mathbb{D}_0^2 M_0 \xrightarrow{p_{23}} \mathbb{D}_0 M_1 \times \mathbb{D}_0^2 M_0$$

$$\psi_2 \downarrow \quad \psi_1$$

$$M_1 \times \mathbb{D}_0 M_1 \xrightarrow{S = p_2} \mathbb{D}_0 M_1,$$

where $\mathbb{D}_{\psi_1}$ indicates the action induced by the $\mathbb{D}_{\psi}^2(*)$ action on $\mathbb{D}_0 M_1$. Combining these with the previous diagram, we find that the diagram for $\Gamma_R$ with the top arrows commutes as a consequence of the following two lemmas.

**Lemma 15.1.** The diagram

$$M \times \mathbb{D}_1 M_1 \xrightarrow{1 \times \varepsilon_*(T_{\mathbb{D}_1} (I_D \circ S))} M \times \mathbb{D}_1 M_0$$

$$1 \times \mathbb{D}_1 T \downarrow \quad \psi_1$$

$$M \times \mathbb{D}_1 M_0 \xrightarrow{\psi_1} M_1$$

commutes.

**Proof.** It suffices to show that

$$\mathbb{D}_1 T = \varepsilon_*(T_{\mathbb{D}_1} (I_D \circ S)) : \mathbb{D}_1 M_1 \to \mathbb{D}_1 M_0.$$

From the diagram defining $\varepsilon_*$, since its target is the pullback

$$\mathbb{D}_1 M_0 \cong \mathbb{D}_0 M_0 \times_{\mathbb{D}_0(*)} \mathbb{D}_1 (*),$$

we see that $\varepsilon_*$ is completely determined by the two composites

$$T_{\mathbb{D}_1} \circ \varepsilon_* = \mathbb{D}_0 T \circ p_1 \quad \text{and} \quad (\mathbb{D}_1 \varepsilon) \circ \varepsilon_* = \mathbb{D}_1 \varepsilon \circ \mathbb{D}_1^2 \varepsilon \circ p_2.$$

We can therefore compose our desired equality with $T_{\mathbb{D}_1}$ and $\mathbb{D}_1 \varepsilon$ to see if it is true. Composing with $T_{\mathbb{D}_1}$, we find that

$$T_{\mathbb{D}_1} \circ \varepsilon_*(T_{\mathbb{D}_1} (I_D \circ S))$$

$$= \mathbb{D}_0 T \circ p_1 (T_{\mathbb{D}_1} (I_D \circ S))$$

$$= \mathbb{D}_0 T \circ T_{\mathbb{D}_1} = T_{\mathbb{D}_1} \circ \mathbb{D}_1 T.$$
Composing with $D_1\varepsilon$, we have the diagram

\[
\begin{array}{c}
\xymatrix{
D_1M_1 \ar[r]^{D_1S} \ar[d]_{D_1\varepsilon} & D_1D_0M_0 \ar[r]^{D_1I_D} \ar[d]_{D_1\varepsilon} & D_1^2M_0 \\
D_1T & D_1^2(\ast) \\
D_1M_0 \ar[r]_{D_1\varepsilon} & D_1(\ast),
}
\end{array}
\]

which commutes since $\varepsilon : M_1 \to \ast$ is terminal. We may conclude that the diagram commutes. \qed

**Lemma 15.2.** The diagram

\[
\begin{array}{c}
\xymatrix{
D_1M_1 \ar[r]^{(T_b, D_1(I_D^0S))} \ar[d]_{S_D} & D_0M_1 \times_{D_0^2M_0} D_1^2M_0 \\
D_0M_1 \ar[r]_{D\psi_1} & D_0M_1
}
\end{array}
\]

commutes.

**Proof.** We need some generalities about how the action of $D^2(M_0^\delta)$ on $D_0M_1$ can be expressed. Whenever we have a presheaf $X$ over a category $C$, we know that $D_0X$ is a presheaf over $DC$. This is because $D$ preserves target covers, which are equivalent to presheaf structures on the objects of the source category of a target cover. In particular, we may conclude that

\[
\xymatrix{
[D(C \int X)]_1 \ar[r]^T \ar[d] & [D(C \int X)]_0 \\
(DC)_1 \ar[r]^T & (DC)_0
}
\]

is a pullback diagram, which with a little unpacking becomes

\[
\begin{array}{c}
\xymatrix{
D_1X \times_{D_1C_0} D_1C_1 \ar[r]^{T} \ar[d] & D_0X \\
D_1C_1 \ar[r]_{T_bT} & D_0C_0.
}
\end{array}
\]
This in turn expands to

\[
\begin{align*}
\mathbb{D}_1 X \times_{\mathbb{D}_0 C_0} \mathbb{D}_1 C_1 & \xrightarrow{T_b} \mathbb{D}_0 X \times_{\mathbb{D}_0 C_0} \mathbb{D}_0 C_1 \xrightarrow{D_0 T} \mathbb{D}_0 X \\
\mathbb{D}_1 C_1 & \xrightarrow{T_b} \mathbb{D}_0 C_1 \xrightarrow{D_0 T} \mathbb{D}_0 C_0,
\end{align*}
\]

both of whose squares are pullbacks, the left one because of the general principle that whenever we have a function \( f : X \to Y \), there is a canonical isomorphism

\[
\mathbb{D}_0 X \times_{\mathbb{D}_0 Y} \mathbb{D}_1 Y \cong \mathbb{D}_1 X.
\]

(This is a consequence of the fact that \( f^\delta : X^\delta \to Y^\delta \) is a cover, and therefore so is \( \mathbb{D}(f^\delta) \).) Now tracing definitions shows that the action of \( \mathbb{D}_1 C_1 \) on \( \mathbb{D}_0 X \) is given by the maps

\[
\begin{align*}
\mathbb{D}_0 X \times_{\mathbb{D}_0 C_0} \mathbb{D}_1 C_1 & \cong \mathbb{D}_0 X \times_{\mathbb{D}_0 C_0} \mathbb{D}_0 C_1 \times_{\mathbb{D}_0 C_1} \mathbb{D}_1 C_1 \\
& \cong \mathbb{D}_1 X \times_{\mathbb{D}_1 C_0} \mathbb{D}_1 C_1 \cong \mathbb{D}_1 (X \times_{C_0} C_1)
\end{align*}
\]

\[
\xrightarrow{D_1 \xi} \mathbb{D}_1 X \xrightarrow{S_0} \mathbb{D}_0 X.
\]

In our case, we have \( M_1 \) in the role of \( X \) and \( \mathbb{D}(M_0^\delta) \) in the role of \( C \), so we can express the action of \( \mathbb{D}_1 M_0 \) on \( \mathbb{D}_0 M_1 \) as the composite

\[
\begin{align*}
\mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0 & \cong \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_0 \mathbb{D}_1 M_0 \times_{\mathbb{D}_0 \mathbb{D}_1 M_0} \mathbb{D}_1^2 M_0 \\
& \cong \mathbb{D}_1 M_1 \times_{\mathbb{D}_1 \mathbb{D}_0 M_0} \mathbb{D}_1^2 M_0 \cong \mathbb{D}_1 (M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0)
\end{align*}
\]

\[
\xrightarrow{D_1 \psi_1} \mathbb{D}_1 M_1 \xrightarrow{S_0} \mathbb{D}_0 M_1,
\]

which is the map denoted \( \mathbb{D}\psi_1 \) in the diagram we are currently trying to verify. We can now do so by means of the following diagram:

\[
\begin{align*}
\mathbb{D}_1 M_1 \xrightarrow{(T_b, D_1 S)} & \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 \mathbb{D}_0 M_0 \xrightarrow{1 \times_{D_1 I_0}} \mathbb{D}_0 M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1^2 M_0 \\
\cong & \mathbb{D}_1 M_1 \times_{\mathbb{D}_1 \mathbb{D}_0 M_0} \mathbb{D}_1 \mathbb{D}_0 M_0 \xrightarrow{1 \times_{D_1 I_0}} \mathbb{D}_1 M_1 \times_{\mathbb{D}_1 \mathbb{D}_0 M_0} \mathbb{D}_1^2 M_0 \\
\cong & \mathbb{D}_1 M_1 \xrightarrow{D_1 \psi_1} \mathbb{D}_0 M_1 \xrightarrow{S_0} \mathbb{D}_0 M_1.
\end{align*}
\]

The two diagonal arrows are the canonical isomorphisms, so compose to the identity, and the only nontrivial sub-diagram is the lower triangle, which commutes because.
of the unit property of the action $\psi_1$. We have completed verifying coherence of the first of our four directions for descent of the product on $\hat{L}M$ to $LM$. □

The diagram involving $\Gamma_R$ with the bottom choice of horizontal arrows again doesn’t involve the last factor of $\mathbb{D}_1 M_0$, but omitting it still results in a diagram that is too wide to fit on the page. The left half is the following,

$$
\begin{align*}
\mathbb{D}_0 M_1 \times_{\mathbb{D}_0^2 M_0} \mathbb{D}_0 \mathbb{D}_1 M_0 & \xrightarrow{1 \times_\mu (\mu \circ I_0) \times 1^2} \mathbb{D}_0 M_1 \times (\mathbb{D}_1 M_0)^2 \times \mathbb{D}_0 M_1 \\
\mathbb{D}_0 M_1 \times_{\mathbb{D}_0^2 M_0} \mathbb{D}_0 \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 & \xrightarrow{1 \times_\mu (\mu \circ I_0) \times 1^2} \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \\
\mathbb{D}_0 M_1 \times_{\mathbb{D}_0^2 M_0} \mathbb{D}_0^2 M_1 \times_{\mathbb{D}_0^2 M_0} \mathbb{D}_0 \mathbb{D}_1 M_0 \times \mathbb{D}_1 M_0 & \xrightarrow{1 \times_\mu (\mu \circ I_0) \times 1} (\mathbb{D}_0 M_1)^2 \times (\mathbb{D}_1 M_0)^2 \\
\mathbb{D}_0 \gamma M \times 1^2 & \downarrow 1 \times _\chi \times 1 \\
\mathbb{D}_0 \gamma M \times 1^2 & \downarrow 1 \times _\chi \times 1 \\
\mathbb{D}_0 \gamma M \times 1^2 & \downarrow 1 \times _\chi \times 1
\end{align*}
$$

and the right half, to be pasted to the right of the above part, is

$$
\begin{align*}
\mathbb{D}_0 M_1 \times (\mathbb{D}_1 M_0)^2 \times \mathbb{D}_0 M_1 & \xrightarrow{1 \times_\gamma_0 \times 1} \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \\
\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 & \xrightarrow{1 \times_\chi} \\
(\mathbb{D}_0 M_1)^2 \times (\mathbb{D}_1 M_0)^2 & \xrightarrow{1 \times_\gamma_0} (\mathbb{D}_0 M_1)^2 \times \mathbb{D}_1 M_0 \\
\mathbb{D}_0 \gamma M \times 1 & \downarrow 1 \times _\chi \\
\mathbb{D}_0 \gamma M \times 1 & \downarrow 1 \times _\chi \\
\mathbb{D}_0 \gamma M \times 1 & \downarrow 1 \times _\chi
\end{align*}
$$

The top part of the right half is the product on the right with $\mathbb{D}_0 M_1$ of a diagram we verified as part of showing that $\gamma_{LM}$ was associative, and the bottom part is a simple naturality diagram. The top part of the left half is also a naturality diagram, and the rest of the left half doesn’t involve the last factor of $\mathbb{D}_1 M_0$. Deleting it, we are
left with verifying the following diagram:

\[
\begin{array}{ccc}
\mathbb{D}_0 M_1 \times \mathbb{D}_0^2 M_0 & \overset{1 \times \mu (\mu \circ I_0) \times 1}{\longrightarrow} & \mathbb{D}_0 M_1 \times D_1 M_0 \times D_1 M_1 \\
\mathbb{D}_0 M_1 \times \mathbb{D}_0^2 M_0 & \overset{1 \times \chi}{\longrightarrow} & \mathbb{D}_0 M_1 \times D_1 M_0 \times D_1 M_0 \\
\mathbb{D}_0 M_1 \times \mathbb{D}_0^2 M_0 & \overset{1 \times \chi}{\longrightarrow} & \mathbb{D}_0 M_1 \times D_1 M_0 \\
\mathbb{D}_0 M_1 \times \mathbb{D}_0^2 M_0 & \overset{1 \times \mu (\mu \circ I_0)}{\longrightarrow} & \mathbb{D}_0 M_1 \times D_1 M_0 \\
\mathbb{D}_0 M_1 \times \mathbb{D}_0^2 M_0 & \overset{1 \times \mu (\mu \circ I_0)}{\longrightarrow} & \mathbb{D}_0 M_1 \times D_1 M_0 \\
\mathbb{D}_0 M_1 \times \mathbb{D}_0^2 M_0 & \overset{1 \times \mu (\mu \circ I_0)}{\longrightarrow} & \mathbb{D}_0 M_1 \times D_1 M_0.
\end{array}
\]

The bottom half of the diagram follows from the fact that the map

\[
\mathbb{D}_0 M_1 \times \mathbb{D}_0^2 M_0 \xrightarrow{1 \times \mu \mu} \mathbb{D}_0 M_1 \times D_1 M_0
\]

is a canonical isomorphism between two expressions for \(\mathbb{D}_0 M_2\): the relevant diagram is as follows, and consists of two stacked pullbacks:

\[
\begin{array}{ccc}
\mathbb{D}_0 M_2 & \overset{\mathbb{D}_0 T}{\longrightarrow} & \mathbb{D}_0 M_1 \\
\mathbb{D}_0 S & \downarrow & \mathbb{D}_0 S \\
\mathbb{D}_0^2 M_1 & \overset{\mathbb{D}_0^2 T}{\longrightarrow} & \mathbb{D}_0^2 M_0 \\
\mu & \downarrow & \mu \\
\mathbb{D}_0 M_1 & \overset{\mathbb{D}_0 T}{\longrightarrow} & \mathbb{D}_0 M_0.
\end{array}
\]

Expanding the top half of the previous diagram by deleting the initial \(\mathbb{D}_0 M_1\), which plays no role, and using the definitions of \(\chi\) and \(\chi\), we get the following fill:

\[
\begin{array}{ccc}
\mathbb{D}_0 \mathbb{D}_1 M_0 & \overset{I_0 \times 1}{\longrightarrow} & \mathbb{D}_0^2 M_0 \times \mathbb{D}_0 M_1 \overset{\mu \times 1}{\longrightarrow} \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \\
(\mathbb{D}_0 \mathbb{D}_1 T, \mathbb{D}_0 S_0) & \cong & (\mathbb{D}_0^2 T, \mathbb{D}_0 S_0 \circ \mu) \cong (\mathbb{D}_1 T, S_0) \\
\mathbb{D}_0 \mathbb{D}_1 M_1 & \overset{I_0}{\longrightarrow} & \mathbb{D}_0^2 M_1 \overset{\mu}{\longrightarrow} \mathbb{D}_1 M_1 \\
(\mathbb{D}_0 T_0, \mathbb{D}_0 \theta) & \cong & (\mathbb{D}_0^2 T, \mathbb{D}_0 \theta) \cong (T_0, \theta) \\
\mathbb{D}_0^2 M_1 \times \mathbb{D}_0^2 M_0 & \overset{1 \times I_0}{\longrightarrow} \mathbb{D}_0^2 M_1 \times \mathbb{D}_0^2 M_0 \overset{\mu \times \mu}{\longrightarrow} \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0.
\end{array}
\]
Here the projection of the top left square to the factor of $\mathbb{D}_1^2 M_0$ is just a naturality diagram, and the projection to $\mathbb{D}_0 M_1$ follows from the following:

$$
\begin{array}{c}
\mathbb{D}_0 \mathbb{D}_1 M_1 \xrightarrow{I_D} \mathbb{D}_1^2 M_1 \\
\downarrow \quad \downarrow \mu \\
\mathbb{D}_0 S \mathbb{D}_1 M_1 \xrightarrow{S^2_D} \mathbb{D}_1 M_1 \\
\downarrow \quad \downarrow S_D \\
\mathbb{D}_0^2 M_1 \xrightarrow{\mu} \mathbb{D}_0 M_1.
\end{array}
$$

The upper right square, projected to $\mathbb{D}_1 M_0$, is again a naturality diagram, and the projection to $\mathbb{D}_0 M_1$ is trivial: both composites are $S_D \circ \mu$. The lower left square, projected to $\mathbb{D}_0^2 M_1$, follows from the fact that $T_D \circ I_D = 1$, and the projection to $\mathbb{D}_1^2 M_0$ follows from the fact that $I_D$ is a natural map from $\mathbb{D}_0$ to $\mathbb{D}_1$; this is part of the category structure on $\{\mathbb{D}_0, \mathbb{D}_1\}$. In the lower right square, projection to $\mathbb{D}_0 M_1$ is again a naturality diagram, while projection to $\mathbb{D}_1 M_0$ uses the expansion of $\theta$ as the composite

$$
\begin{array}{c}
\mathbb{D}_1 M_1 \xrightarrow{D_1 S} \mathbb{D}_1 \mathbb{D}_0 M_0 \xrightarrow{D_1 I_D} \mathbb{D}_1^2 M_0 \xrightarrow{\mu} \mathbb{D}_1 M_0 \\
\end{array}
$$

to give the following fill:

$$
\begin{array}{c}
\mathbb{D}_1^2 M_1 \xrightarrow{D_1^2 S} \mathbb{D}_1^2 \mathbb{D}_0 M_0 \xrightarrow{D_1^2 I_D} \mathbb{D}_1^2 M_0 \xrightarrow{D_1^1 \mu} \mathbb{D}_1^2 M_0 \\
\downarrow \mu \quad \downarrow \mu \quad \downarrow \mu \\
\mathbb{D}_1 M_1 \xrightarrow{D_1 S} \mathbb{D}_1 \mathbb{D}_0 M_0 \xrightarrow{D_1 I_D} \mathbb{D}_1^2 M_0 \xrightarrow{D_1^1 \mu} \mathbb{D}_1 M_0.
\end{array}
$$

This completes the verification of the diagram involving $\Gamma_R$ with the bottom choice of horizontal arrows. It follows that composition in $\hat{L} M$ preserves equivalence classes in the left slot.

We display $\Gamma_L$ as the composite in figure [1] where the unadorned products have the usual suppressed subscript $\mathbb{D}_0 M_0$. Some care is necessary in reading this picture; in particular, the temptation to use the composition $\mathbb{D}_0^2 \gamma_M : \mathbb{D}_0 M_1 \times \mathbb{D}_0 M_1 \to \mathbb{D}_0 M_1$, suggested by the fact that $\mathbb{D}_0 M_1 \times \mathbb{D}_0 M_1 \cong \mathbb{D}_0 M_2$, must be avoided, since the map does not preserve the structure map of source type to $\mathbb{D}_0^2 M_0$. Instead, we must use the somewhat more complicated formalism displayed. The preservation of structure maps along the $\mu$ in the middle of the next to the last arrow of the figure is a consequence
of the following diagram:

\[
\begin{array}{c}
\text{Figure 1.}
\end{array}
\]
The diagram with $\Gamma_L$ and the top choice of horizontal arrows now consists of a left column as in figure [I] connected to a right column as below by an arrow $1^2 \times D_0\psi_1 \times 1$ on top and one on the bottom labeled $D_0\psi_1 \times 1$. The right column expresses the multiplication in $\hat{L}M_1$, as in the first diagram, but we expand it a bit for the purposes of filling in; it appears as follows:

\[
\begin{align*}
D_0M_1 \times D_1M_0 \times D_0M_1 \times D_1M_0 & \cong 1 \times (D_1T,S_0) \times 1 \\
D_0M_1 \times D_1M_1 \times D_1M_0 & \cong 1 \times (T_b,D_1S) \times 1 \\
D_0M_1 \times D_0M_1 \times D_0M_0 \times D_1M_0 & \cong 1 \times \mu \times 1^2 \\
D_0M_1 \times D_0M_0 \times D_2M_1 \times D_0M_0 \times D_1M_0 & \cong D_0\gamma_M \times (\mu \circ T_0) \times 1 \\
D_0M_1 \times D_1M_0 & \cong 1 \times \gamma_0 \\
D_0M_1 \times D_1M_0 &
\end{align*}
\]

We now proceed to fill in the diagram with $\Gamma_L$ and the top choice of horizontal arrows, and start at the top, where the first part of the fill doesn’t involve either the first factor of $D_0M_1$ or the last factor of $D_1M_0$, so we omit them. The claim is now that the diagram

\[
\begin{align*}
D_1M_1 \times D_0M_0 \times D_0D_1M_0 & \cong 1 \times s_0S_0 \\
\cong D_1M_1 \times D_1D_0M_0 \times D_0M_0 \times D_1M_0 & \cong D_1M_1 \\
\cong (D_1T,S_0) \\
D_1M_0 \times D_0M_1 \times D_0M_0 & \cong 1 \times D_0\psi_1 \\
\cong D_1M_0 \times D_0M_1 &
\end{align*}
\]

commutes; it has to be turned upside down in order to fit into the larger diagram (which we haven’t actually written down completely, for reasons of size.) The projection to the first factor of the lower right corner commutes since it’s $D_1$ applied to the following diagram, which expresses the fact that $\psi_1$ preserves targets:

\[
\begin{align*}
M_1 \times D_1M_0 & \xrightarrow{\psi_1} M_1 \\
\xrightarrow{T_{\psi_1}} & \\
M_0 &
\end{align*}
\]
Projection to the second factor amounts just to commutativity of the naturality diagram

\[
\begin{array}{c}
\mathbb{D}_1(M_1 \times \mathbb{D}_0 M_0) \xrightarrow{D_1 \psi_1} \mathbb{D}_1 M_1 \\
\downarrow S_0 \\
\mathbb{D}_0(M_1 \times \mathbb{D}_1 M_0) \xrightarrow{D_0 \psi_1} \mathbb{D}_0 M_1.
\end{array}
\]

We may conclude that the first part of the fill commutes.

The next part of the fill also doesn’t need the first or last factors, and consists of the following diagram:

\[
\begin{array}{c}
\mathbb{D}_1 M_1 \times \mathbb{D}_2^2 M_0 \\
\mathbb{D}_0 \mathbb{D}_1 M_0 \xrightarrow{1 \times S_0 S_0} \mathbb{D}_1 M_1 \times \mathbb{D}_1 \mathbb{D}_0 M_0 \xrightarrow{D_1 \psi_1} \mathbb{D}_1 M_1 \\
\mathbb{D}_0 M_1 \times \mathbb{D}_0^2 M_0 \xrightarrow{T_b \times T_b} \mathbb{D}_0 M_1 \times \mathbb{D}_0^2 M_0 \xrightarrow{D_0 \psi_1 \times 1} \mathbb{D}_0 M_1 \times \mathbb{D}_0^2 M_0.
\end{array}
\]

The upper left portion consists entirely of isomorphisms, and commutes by projecting to each of the three factors of the target, which is second down on the left column. The projections to the outer two factors commute trivially, and the one to the central \(\mathbb{D}_1 \mathbb{D}_0 M_0\) commutes since they coincide with the structure maps in the pullback \(\mathbb{D}_1 M_1 \times \mathbb{D}_1 \mathbb{D}_0 M_0 \mathbb{D}_2^2 M_0\) that gives the source of the sub-diagram. We are left for this portion of the fill with the diagram

\[
\begin{array}{c}
\mathbb{D}_1 M_1 \times \mathbb{D}_1 \mathbb{D}_0 M_0 \xrightarrow{T_b \times T_b} \mathbb{D}_0 \mathbb{D}_1 M_0 \xrightarrow{1 \times (T_b, D_1 S_b)} \mathbb{D}_0 M_1 \times \mathbb{D}_0^2 M_0 \xrightarrow{D_0 \psi_1 \times 1} \mathbb{D}_0 M_1 \times \mathbb{D}_0^2 M_0.
\end{array}
\]
Projecting to the factor of \( D_0 M_1 \) in the target, we see that the diagram reduces to the naturality diagram

\[
\begin{array}{ccc}
D_1(M_1 \times D_1 M_0) & \xrightarrow{D_1 \psi_1} & D_1 M_1 \\
T_b & \downarrow & T_b \\
D_0(M_1 \times D_1 M_0) & \xrightarrow{D_0 \psi_1} & D_0 M_1.
\end{array}
\]

And projection to the factor of \( D_1 D_0 M_0 \) consists of \( D_1 \) applied to the diagram

\[
\begin{array}{ccc}
M_1 \times D_1 M_0 & \xrightarrow{\psi_1} & M_1 \\
\mu \times \mu & \downarrow & \mu \\
D_1 M_0 & \xrightarrow{\mu} & D_0 M_0,
\end{array}
\]

which records the fact that \( \psi_1 \) preserves source structure maps. This part of the fill is now complete.

The next part of the fill is just the naturality diagram

\[
\begin{array}{ccc}
D_0^2 M_1 \times D_0^3 M_0 & \xrightarrow{D_0^2 \psi_1} & D_0^2 M_1 \\
\mu \times \mu & \downarrow & \mu \\
D_0 M_1 \times D_0^2 M_0 & \xrightarrow{D_0 \psi_1} & D_0 M_1
\end{array}
\]

turned upside down and with \( D_0 M_1 \) attached at the front, and \( D_1 D_0 M_0 \times D_1 M_0 \) at the back.

We come next to the following diagram, in which a \( D_1 M_0 \) at the back has been suppressed:
While it is clear that the part involving the last factor commutes, since it is just the composite
\[
\begin{array}{c}
\mathcal{D}_1 \mathcal{D}_0 M_0 \\
\mathcal{D}_1 I_0 \\
\mathcal{D}_1^2 M_0 \\
\mathcal{D}_1 M_0
\end{array}
\]
either way around the diagram, it is displayed here to emphasize that the structure maps are preserved by the arrows in the diagram, especially the second one in the left column. In the source of the arrow, the structure map of source type for \(\mathcal{D}_0 \mathcal{D}_1 M_0\) is given, by examining the previous fill diagram, by the composite
\[
\begin{array}{c}
\mathcal{D}_0 \mathcal{D}_1^2 M_0 \\
\mathcal{D}_0 S_0^2 \\
\mathcal{D}_0^2 M_0 \\
\mathcal{D}_0^2 M_0
\end{array}
\]
This does map to the structure map for \(\mathcal{D}_0 \mathcal{D}_1 M_0\) in the next term because of the commutativity of
\[
\begin{array}{c}
\mathcal{D}_0 \mathcal{D}_1^2 M_0 \\
\mathcal{D}_0 S_0^2 \\
\mathcal{D}_0^2 M_0 \\
\mathcal{D}_0^2 M_0
\end{array}
\]
in which the structure map for \(\mathcal{D}_0 \mathcal{D}_1 M_0\) now appears at a right angle.

Stripping off the last term, we wish to see that
\[
\begin{array}{c}
\mathcal{D}_0 M_1 \times_{\mathcal{D}_1^2 M_0} \mathcal{D}_0^2 M_1 \\
\mathcal{D}_0 \mathcal{D}_1 M_0 \\
\mathcal{D}_0 \mathcal{D}_1 M_0
\end{array}
\]
commutes. However, this can now have a \(\mathcal{D}_0\) stripped off, resulting in the diagram of our desire looking like
\[
\begin{array}{c}
M_1 \times \mathcal{D}_0 M_1 \times_{\mathcal{D}_1^2 M_0} \mathcal{D}_0 \mathcal{D}_1 M_0 \\
M_1 \times \mathcal{D}_0 M_1 \times_{\mathcal{D}_1^2 M_0} \mathcal{D}_1 M_0 \\
\gamma M \times_{\mu \mu}
\end{array}
\]
Filling this diagram requires the following lemma connecting the presheaf actions on $M_2$ and $M_1$.

**Lemma 15.3.** The following diagram commutes.

\[
\begin{array}{ccc}
M_2 \times \mathbb{D}^2_0 M_0 & \xrightarrow{(T,S) \times 1} & M_1 \times \mathbb{D}_0 M_1 \times \mathbb{D}^2_0 M_0 \times \mathbb{D}_0 \mathbb{D}_1 M_0 \\
1 \times I_0 & \downarrow & \approx \\
M_2 \times \mathbb{D}^2_0 M_0 & \times \mathbb{D}^2_0 M_0 & M_1 \times \mathbb{D}_0 (M_1 \times \mathbb{D}_1 M_0) \\
\psi_2 & \downarrow & \approx \\
M_2 & \approx & M_1 \times \mathbb{D}_0 M_1.
\end{array}
\]

**Proof.** If we project onto the second factor, the resulting diagram commutes as a consequence of the following one:

\[
\begin{array}{ccc}
M_2 \times \mathbb{D}^2_0 M_0 & \mathbb{D}_0 \mathbb{D}_1 M_0 & M_1 \times \mathbb{D}_0 M_1 \times \mathbb{D}^2_0 M_0 \times \mathbb{D}_0 \mathbb{D}_1 M_0 \\
1 \times I_0 & \downarrow \times S \times 1 & \approx \\
M_2 \times \mathbb{D}^2_0 M_0 & \mathbb{D}^2_0 M_0 & M_1 \times \mathbb{D}_0 M_1 \times \mathbb{D}^2_0 M_0 \\
\psi_2 & \downarrow \approx & \mathbb{D}_1 (M_1 \times \mathbb{D}_1 M_0) \\
M_2 & \approx \mathbb{D}_0 (M_1 \times \mathbb{D}_1 M_0) \\
\end{array}
\]

Here the lower right rectangle commutes because of the requirement that $S : M_2 \to \mathbb{D}_0 M_1$ be a map of $\mathbb{D}^2(M_0^2)$ presheaves, along with our previous identification of the presheaf action on $\mathbb{D}_0 M_1$. The only other part of the diagram that isn’t immediate is the upper left (somewhat distorted) square, but that is a consequence of the fact that $I_0$ splits $T_0$ in the following naturality diagram, in which the right square is a pullback:

\[
\begin{array}{cccc}
\mathbb{D}_0 (M_1 \times \mathbb{D}_1 M_0) & \xrightarrow{I_0} & \mathbb{D}_1 (M_1 \times \mathbb{D}_1 M_0) & \xrightarrow{T_0} \mathbb{D}_0 (M_1 \times \mathbb{D}_1 M_0) \\
\mathbb{D}_0 p_2 & \downarrow & \mathbb{D}_1 p_2 & \downarrow \mathbb{D}_0 p_2 \\
\mathbb{D}_0 \mathbb{D}_1 M_0 & \xrightarrow{I_0} & \mathbb{D}_1^2 M_0 & \xrightarrow{T_0} \mathbb{D}_0 \mathbb{D}_1 M_0.
\end{array}
\]
Projection onto the first factor requires us to see that the diagram

\[
\begin{array}{c}
M_2 \times \mathbb{D}_0^2 M_0 & \xrightarrow{p_1} & M_2 \\
1 \times I_D & \downarrow \cong & \downarrow T \\
M_2 \times \mathbb{D}_0^2 M_0 & \xrightarrow{\psi_2} & M_2 & \xrightarrow{T} & M_1
\end{array}
\]

commutes. However, this follows as a result of the following larger expansion of the diagram:

\[
\begin{array}{c}
M_2 \times \mathbb{D}_0^2 M_0 & \xrightarrow{(T,S) \times 1} & M_1 \times \mathbb{D}_0 M_1 \times \mathbb{D}_0^2 M_0 & \xrightarrow{1 \times \psi_1} & M_1 \times \mathbb{D}_0 M_1 \\
1 \times I_D & \downarrow \cong & \downarrow 1 \times I_D & \downarrow 1 \times \varepsilon_* \\
M_2 \times \mathbb{D}_0^2 M_0 & \xrightarrow{(T,S) \times 1} & M_1 \times \mathbb{D}_0 M_1 \times \mathbb{D}_0^2 M_0 & \xrightarrow{p_{12}} & M_1 \times \mathbb{D}_0 M_0 \\
\psi_2 & \downarrow \cong & \downarrow \psi_1 & \downarrow 1 \times I_D & \downarrow 1 \times I_D & \downarrow 1 \times \varepsilon_*
\end{array}
\]

The only part of this expansion that hasn’t been previously established is the pentagon incorporated in the right hand column. However, we can expand the diagram defining \( \varepsilon_* \) to include \( I_D \)'s as follows:

\[
\begin{array}{c}
\mathbb{D}_0 M_1 \times \mathbb{D}_0^2 M_0 & \xrightarrow{1 \times I_D} & \mathbb{D}_0 M_1 \times \mathbb{D}_0^2 M_0 & \xrightarrow{p_1} & \mathbb{D}_0 M_1 \\
\mathbb{D}_0 M_0 & \xrightarrow{I_D} & \mathbb{D}_1 M_0 & \xrightarrow{T_D} & \mathbb{D}_0 T \\
\mathbb{D}_0 \mathbb{D}_1 M_0 & \xrightarrow{I_D} & \mathbb{D}_1^2 M_0 & \xrightarrow{T_D} & \mathbb{D}_0^2 M_0 \\
\mathbb{D}_0 \mathbb{D}_1 (\ast) & \xrightarrow{I_D} & \mathbb{D}_1^2 (\ast) & \xrightarrow{T_D} & \mathbb{D}_0^2 (\ast) \\
\mathbb{D}_0 (\ast) & \xrightarrow{I_D} & \mathbb{D}_1 (\ast) & \xrightarrow{T_D} & \mathbb{D}_0 (\ast).
\end{array}
\]

In this diagram, the left \( \varepsilon_* \) does have \( \mathbb{D}_0 M_0 \) as its target, since both the front squares are pullbacks: the right one since \( \varepsilon : M_0^3 \to \ast \) is a target cover, and the left one since \( T_D \circ I_D = \text{id} \). Consequently, the top of the diagram tells us that \( \varepsilon_* = \mathbb{D}_0 T \circ p_1 \), as
necessary in the previous diagram. This establishes projection onto the first factor in the diagram of the lemma, concluding its proof. □

We can now proceed with the current fill diagram, above the statement of the lemma, which follows from its expansion as follows:

\[
\begin{array}{c}
M_1 \times D_0 M_1 \times D_0^2 M_0 \Rightarrow D_0 D_1 M_0 \\
\downarrow 1^2 \times I_0 \\
M_1 \times D_0 M_1 \times D_0^2 M_0 \Rightarrow D_1 M_0 \\
\downarrow \gamma_M \times \mu \mu \\
M_1 \times D_1 M_0 \\
\end{array}
\]

\[
\begin{array}{c}
M_2 \times D_0^2 M_0 \Rightarrow D_0 D_1 M_0 \\
\downarrow (T,S) \times 1 \\
M_2 \times D_0^2 M_0 \Rightarrow D_0^2 M_0 \\
\downarrow \psi_2 \\
M_2 \\
\end{array}
\]

Here the irregular pentagon in the upper right is a consequence of the previous lemma, and the distorted square at the bottom expresses the equivariance of \( \gamma : M_2 \to M_1 \) as a presheaf map over the monad multiplication \( \mu : D^2 \to D \).

Now the final part of the fill is just the naturality diagram

\[
\begin{array}{c}
D_0 M_1 \times D_0^2 M_0 \Rightarrow D_0 D_1 M_0 \\
\downarrow 1^2 \times \gamma_0 \\
D_0 M_1 \times D_0^2 M_0 \Rightarrow D_0 D_1 M_0 \\
\downarrow \psi_1 \\
D_0 M_1 \times D_0^2 M_0 \\
\end{array}
\]

\[
\begin{array}{c}
D_0 M_1 \times D_0^2 M_0 \Rightarrow D_0 D_1 M_0 \\
\downarrow 1^2 \times \psi_1 \\
D_0 M_1 \times D_0^2 M_0 \\
\end{array}
\]

We may conclude that the diagram with \( \Gamma_L \) and the top choice of horizontal arrows does in fact commute.

The diagram with \( \Gamma_L \) and the bottom choices of horizontal arrows is as with the top choices, but with the two horizontal arrows replaced with \( 1^2 \times \gamma_0 \circ ((\mu \circ I_D) \times 1) \) on top, and similarly on the bottom without the square on the 1. In pursuit of this diagram, which amounts to a glorified associativity diagram, we need a few lemmas.

**Lemma 15.4.** We have a natural isomorphism of natural transformations

\[
D_2^2 \Rightarrow D_1^2 \times D_0^2.
\]
PROOF. There are at least three diagrams that will demonstrate this; the one we want is the following one:

The upper right and lower left squares are pullbacks by definition. The upper left square is a pullback since $S_0 : D_2 X^\delta \to D_1 X^\delta$ is a target cover for any set $X$, which $D_2$ preserves, and the lower right square is a pullback since $T_0 : D_1 X^\delta \to D_0 X^\delta$ is a source cover, which $D_1$ preserves. The total diagram is therefore a pullback. \qed

From this, we can now proceed to the following lemma.

**Lemma 15.5.** The diagram

\[
\begin{array}{c}
D_1^2 \\
\downarrow^{(D_1 T, S_0)}
\end{array}
\xrightarrow{\cong} \begin{array}{c}
D_1 D_0 \times_{E_0^2} D_0 D_1 \\
\downarrow^{D_1 I_0 \times I_0}
\end{array}
\xrightarrow{D_1 I_0 \times I_0}
\begin{array}{c}
D_1^2 \times_{E_0^2} D_1^2 \\
\downarrow^{(T_0^2, S_0^2)}
\end{array}
\xrightarrow{\cong}
\begin{array}{c}
D_2^2 \\
\downarrow^{\gamma_0^2}
\end{array}
\xrightarrow{\cong}
\begin{array}{c}
D_1^2
\end{array}
\xrightarrow{\cong}
\begin{array}{c}
D_1 D_0 \times_{D_0} D_1
\end{array}
\xrightarrow{1 \times I_0}
\begin{array}{c}
D_1 D_0 \times_{E_0^2} D_1^2 \\
\downarrow^{D_1 I_0 \times I_0}
\end{array}
\xrightarrow{D_1 I_0 \times I_0}
\begin{array}{c}
D_1^2 \times_{D_0} D_1^2 \\
\downarrow^{(T_0^2, S_0^2)}
\end{array}
\xrightarrow{\cong}
\begin{array}{c}
D_2^2 \\
\downarrow^{\gamma_0}
\end{array}
\xrightarrow{\cong}
\begin{array}{c}
D_2 D_2
\end{array}
\xrightarrow{\gamma_0^2}
\begin{array}{c}
D_2 D_2
\end{array}
\]

commutes.

**Proof.** We can fill the diagram as follows.

The upper left triangle commutes since $I_R$ can be expressed as the composite

\[
D_1 \xrightarrow{\cong} D_0 \times_{D_0} D_1 \xrightarrow{I_0 \times 1} D_1 \times_{D_0} D_1 \xrightarrow{\cong} D_2,
\]
and similarly the upper right square commutes since $I_L$ can be expressed as the composite
\[
\mathbb{D}_1 \cong \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_0 \xrightarrow{1 \times I_D} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \cong \mathbb{D}_2.
\]
The rest of the diagram expresses the unit conditions for $\gamma_D$. □

The next lemma is the one we will really need for the diagram with $\Gamma_L$ and the bottom choice of horizontal arrows:

**Lemma 15.6.** The following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{D}_1^{(D,b,S_b)} & \xrightarrow{(\mu \circ D_1 \circ I_D) \times (\mu \circ I_D)} & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \\
\mu & \downarrow & \\
\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 & \xrightarrow{\gamma_D} & \mathbb{D}_1.
\end{array}
\]

**Proof.** We can expand and fill in the diagram as follows:

\[
\begin{array}{ccc}
\mathbb{D}_1^{(D,b,S_b)} & \xrightarrow{\cong} & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \\
\cong & \downarrow & \\
\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 & \xrightarrow{\cong} & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1
\end{array}
\]

The left triangle commutes by the previous lemma, the upper right square because $\mu$ commutes with $T_D$ and $S_D$, and the lower right square because $\mu$ commutes with $\gamma_D$; all of these are aspects of the fact that $\mathbb{D}$ is a category object in monads on $\textbf{Set}$. □

We can now begin filling in the diagram with $\Gamma_L$ and the bottom choice of horizontal arrows, which has its left column the expression for $\Gamma_L$ displayed in figure 11 connected to a right column we may need to display after filling, and connected across the top using the action $\phi : \mathbb{D}_0 \mathbb{D}_1 M_0 \times \mathbb{D}_1 M_0 \to \mathbb{D}_1 M_0$ given by the composite

\[
\mathbb{D}_0 \mathbb{D}_1 M_0 \times \mathbb{D}_1 M_0 \xrightarrow{I_0 \times 1} \mathbb{D}_1^2 M_0 \times \mathbb{D}_1 M_0 \xrightarrow{\mu \times 1} \mathbb{D}_1 M_0 \times \mathbb{D}_1 M_0 \xrightarrow{\gamma_D} \mathbb{D}_1 M_0.
\]
The first part of the fill consists of the following diagram, from which an initial $\mathbb{D}_0 \mathbb{D}_1$ has been suppressed, since all its morphisms are the identity:

$$
\begin{align*}
\mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \times D_0 M_0 & \to \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \\
\mathbb{D}_1 M_1 \times D_0 M_0 & \to \mathbb{D}_1 M_1 \times \mathbb{D}_1 M_0 \\
D_0 M_1 \times D_0 M_0 & \to \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_1 M_0
\end{align*}
$$

For the rest of the fill, we can delete all terms involving $M_1$, since they just get multiplied together and have no effect on the rest of the diagram. Since the rest of the diagram involves only $M_0$, we can suppress it, and now wish to verify the following diagram:
The upper left part commutes because of lemma 15.6, the part below it by its analogue switching $T_\mathcal{D}$ and $S_\mathcal{D}$, and the only other part of the diagram requiring verification is the third part down on the left. That diagram, however, easily commutes on projection to the second and third factors of $\mathcal{D}_1$ in the target, leaving us only with projection to the first factor. That in turn falls to commutativity of the following diagram:

\[
\begin{array}{c}
\mathcal{D}_2 \mathcal{D}_1 & \xrightarrow{\mu} \mathcal{D}_0 \mathcal{D}_1 \\
\downarrow \mathcal{D}_2 I_\mathcal{D} & \quad & \downarrow I_\mathcal{D} \\
\mathcal{D}_0 \mathcal{D}_1 & \xrightarrow{I_\mathcal{D}} \mathcal{D}_1 \mathcal{D}_2 \\
\downarrow \mathcal{D}_0 \mu & \quad & \downarrow \mu \\
\mathcal{D}_0 \mathcal{D}_1 & \xrightarrow{I_\mathcal{D}} \mathcal{D}_1 \mathcal{D}_2 & \xrightarrow{\mu} \mathcal{D}_1.
\end{array}
\]

We have verified all four diagrams we needed in order to see that composition in $\hat{\mathcal{L}} \mathcal{M}$ descends to composition in $LM$. It follows that $LM$ inherits a category structure from $\hat{LM}$.

16. The actual left adjoint: $\mathcal{D}$-algebra structure

We next need to show that $LM$ inherits a $\mathcal{D}$-algebra structure from $\hat{LM}$. For this purpose, it is convenient to recognize that the composite

\[
\begin{align*}
\mathcal{D}_0 M_1 \times_{\mathcal{D}_0 M_0} & \mathcal{D}_1 M_0 \\
\cong & \\
\mathcal{D}_0 M_1 \times_{\mathcal{D}_2 M_0} \mathcal{D}_2 M_0 \times_{\mathcal{D}_0 M_0} \mathcal{D}_1 M_0 \\
\downarrow 1 \times \mathcal{D}_0 I_\mathcal{D} & \\
\mathcal{D}_0 M_1 \times_{\mathcal{D}_2 M_0} \mathcal{D}_0 \mathcal{D}_1 M_0 \times_{\mathcal{D}_0 M_0} \mathcal{D}_1 M_0
\end{align*}
\]

provides a common right inverse for the two maps coequalized by the map from $\hat{LM}_1$ to $LM_1$. Consequently, the coequalizer is reflexive, and so is preserved by products. This is most of the proof of the following lemma.

**Lemma 16.1.** Let $\mathcal{D}$ be the monad associated to an operad. Then $\mathcal{D}$ preserves reflexive coequalizers.
16. THE ACTUAL LEFT ADJOINT: $\mathbb{D}$-ALGEBRA STRUCTURE

**Proof.** Let $\xymatrix{A \ar[r]^f & B \ar[r]^h & C}$ be a reflexive coequalizer. Then since products preserve reflexive coequalizers,

\[
\xymatrix{A^n \ar[r]^{f^n} & B^n \ar[r]^{h^n} & C^n}
\]

is also a reflexive coequalizer. The identity coequalizer on $D_n$ is also reflexive, so

\[
\xymatrix{D_n \times A^n \ar[r]^{1 \times f^n} & D_n \times B^n \ar[r]^{1 \times h^n} & D_n \times C^n}
\]

is also a reflexive coequalizer. Now passage to orbits, being a colimit, commutes with coequalizers, and the section map is also preserved. \qed

We wish to show that the $\mathbb{D}$-algebra structure on $\hat{LM}$ descends to one on $LM$, and since the objects of $\hat{LM}$ and of $LM$ coincide (both are $\mathbb{D}_0M_0$), and we have shown that the quotient map preserves the category structure, it suffices to show that the $\mathbb{D}_1$-algebra structure on $\hat{LM}_1$ descends to $LM_1$. Since $LM_1$ is defined as the coequalizer of a reflexive fork, we can use the previous lemma to express $\mathbb{D}_1LM_1$ as the coequalizer of $\mathbb{D}_1$ applied to that same fork. Now we want the action map

\[
\mathbb{D}_1\hat{LM}_1 \xrightarrow{\xi_1} \hat{LM}_1
\]

to descend to one on $LM_1$, that is, we want to define an action map on $LM_1$ so that the diagram

\[
\xymatrix{\mathbb{D}_1\hat{LM}_1 \ar[r]^{\xi_1} & \hat{LM}_1 \ar[d] \cr \mathbb{D}_1LM_1 \ar[r]_{\xi_1} & LM_1 \ar[u]}
\]

commutes, where the vertical arrows are given by the descent map from $\hat{LM}_1$ to $LM_1$. In order to do so, we recall that the action map on $\hat{LM}_1$ is given by the map

\[
\mathbb{D}_1(\mathbb{D}_0M_1 \times \mathbb{D}_1M_0) \cong \mathbb{D}_1\mathbb{D}_0M_1 \times_{\mathbb{D}_1\mathbb{D}_0M_0} \mathbb{D}_1^2M_0 \xrightarrow{(\mu_0I_0) \times (\mu_0I_0)I_0} \mathbb{D}_0M_1 \times \mathbb{D}_1M_0.
\]
What we will use is an action map \( \hat{\xi} \) on the source of the coequalizer, and show that the resulting two diagrams

\[
\begin{align*}
D_1(D_0 M_1 \times D_0^2 M_0) \xrightarrow{\xi} D_0 M_1 \times D_0^2 M_0 \xrightarrow{\mu} D_1 M_0,
\end{align*}
\]

one each for choice of left vertical arrows or right vertical arrows, commute. It will then follow that \( \xi_1 \) induces an action map on \( LM_1 \), as desired.

For \( \hat{\xi} \) we choose the map

\[
\begin{align*}
D_1 D_0 M_1 \times D_1 D_0^2 M_0 \xrightarrow{(\mu \circ I_2) \times \mu \circ I_2 (\mu \circ I_2) \times \mu \circ I_2} D_0 M_1 \times D_0^2 M_0 \xrightarrow{\mu} D_1 M_0.
\end{align*}
\]

In order to see that this map is well-defined, we call the first two factors in the source \( P_{12} \) and the second two \( P_{23} \), and display the commuting diagrams

\[
\begin{align*}
P_{12} & \\
& \xrightarrow{p_1} D_1 D_0 M_1 \xrightarrow{T_B} D_0^2 M_1 \xrightarrow{\mu} D_0 M_1 \\
& \xrightarrow{r_2} D_1 D_0 S_B \xrightarrow{D_1 D_0 S_B} D_0^2 S_B \xrightarrow{D_0 S_B} D_0 M_1,
\end{align*}
\]

and

\[
\begin{align*}
P_{23} & \\
& \xrightarrow{p_1} D_1 D_0 D_1 M_0 \xrightarrow{T_B} D_0^2 D_1 M_0 \xrightarrow{\mu} D_0 D_1 M_0 \\
& \xrightarrow{r_2} D_1 D_0 S_B \xrightarrow{D_1 D_0 S_B} D_0^2 S_B \xrightarrow{D_0 S_B} D_0 D_1 M_0.
\end{align*}
\]

Now the diagram we want with the left choice of vertical arrows doesn’t involve the terms \( D_1^2 M_0 \xrightarrow{\mu} D_1 M_0 \), but the rest of the diagram now reduces to the naturality
17. The actual left adjoint: the counit

We need to show that the counit \( \varepsilon : \hat{L}U \to C \) factors through \( LUC \). Since the quotient map \( \hat{L}U \to LUC \) is the identity on objects, this amounts to checking that the map on morphisms factors. Since the quotient map is a coequalizer on morphisms, this happens if and only if the counit \( \varepsilon : LUC \to C \) coequalizes the maps defining \( LUC \). We begin by recalling that the action \( \psi_1 : UC_1 \times D_0C_0 \to D_1C_0 \) can be expressed as
the composite

\[ UC_1 \times_{D_0 C_0} D_1 C_0 \xrightarrow{\kappa_1 \times \xi_0} C_1 \times_{C_0} D_1 C_0 \xrightarrow{1 \times D_1 I} C_1 \times_{C_0} D_1 C_1 \]

\[ \xrightarrow{1 \times (\xi_1, S \delta S)} C_1 \times_{C_0} C_1 \times_{C_0} D_0 C_0 \xrightarrow{\gamma_0 \times 1} C_1 \times_{C_0} D_0 C_0 \cong UC_1. \]

Now the counit \( \varepsilon_1 : \hat{L}UC_1 \rightarrow C_1 \) expands as the composite

\[ D_0 UC_1 \times_{D_0 C_0} D_1 C_0 \xrightarrow{D_0 \kappa_1 \times \xi_0} D_0 C_1 \times_{C_0} D_1 C_0 \]

\[ \xrightarrow{I_0 \times D_1 I} D_1 C_1 \times_{C_0} D_1 C_1 \xrightarrow{\xi_1 \times \xi_1} C_1 \times_{C_0} C_1 \xrightarrow{\gamma_0} C_1. \]

Since the first step of the counit is \( D_0 \kappa_1 \) on the factor of \( D_0 UC_1 \cong D_0 C_1 \times_{D_0 C_0} D_0^2 C_0 \), and \( D_0 \kappa_1 \) just discards the \( D_0^2 C_0 \), we can rewrite the composite \( \varepsilon_1 \circ (D_0 \psi_1 \times 1) \) as follows, in which it should be noted carefully that the structure maps from \( D_0 D_1 C_0 \) to \( D_0 C_0 \) are not the same: to the left it’s \( D_0 \otimes_0 (T_D) \), and to the right it’s \( \mu \circ D_0 S_D \):

\[ D_0 UC_1 \times_{D_0 C_0} D_0 D_1 C_0 \times_{D_0 C_0} D_1 C_0 \xrightarrow{D_0 \kappa_1 \times \xi_0} D_0 C_1 \times_{D_0 C_0} D_0 D_1 C_0 \times_{D_0 C_0} D_1 C_0 \]

\[ \xrightarrow{1 \times D_0 D_1 I \times 1} D_0 C_1 \times_{D_0 C_0} D_0 D_1 C_1 \times_{D_0 C_0} D_1 C_0 \]

\[ \xrightarrow{1 \times D_0 \xi_1 \times \xi_0} D_0 C_1 \times_{C_0} D_1 C_1 \xrightarrow{\xi_1 \times \xi_1} C_1 \times_{C_0} C_1 \xrightarrow{\gamma_0} C_1. \]

We can now rewrite this using the following diagram, which starts at the second term of the above composite:

The bottom rectangle simply records the fact that the composition of identities is the identity in the category structure on \( D \). We can now express the composite
the composite:

\[ \xi_1 \circ (D_0 \psi_1 \times 1) \]

as follows:

\[
\begin{array}{c}
D_0 U C_1 \times D_0 C_0 \quad D_0 D_1 C_0 \times D_0 C_0 \quad D_1 C_0 \\
\downarrow I_0 \times I_0 \times D_1 I \quad \downarrow \mu \times D_0 C_0 \\
D_1 C_1 \times D_0 C_0 \quad D_1 C_1 \times D_0 C_0 \quad D_1 C_1
\end{array}
\]

Next, we appeal to the diagram

in which the upper left and lower right rectangles are associativity diagrams, and the upper right and lower left follow from the requirement that \( \xi : D C \to C \) be a functor; in particular, the diagram

\[
\begin{array}{c}
(D C)_2 \quad \xi_2 \\
\downarrow \gamma_{D C} \quad \downarrow \gamma_C \\
(D C)_1 \quad \xi_1 
\end{array}
\]

must commute. Note that the action map \( \xi_2 \) can be rewritten as \( \xi_1 \times_{D_0} \xi_1 \); the rectangles in the previous diagram now follow. We can now rewrite our composite:

\[
\xi_1 \circ (D_0 \psi_1 \times 1) \]

coincides with

\[
\begin{array}{c}
D_0 U C_1 \times D_0 C_0 \quad D_0 D_1 C_0 \times D_0 C_0 \quad D_1 C_0 \\
\downarrow I_0 \times I_0 \times D_1 I \quad \downarrow \mu \times D_0 C_0 \\
D_1 C_1 \times D_0 C_0 \quad D_1 C_1 \times D_0 C_0 \quad D_1 C_1
\end{array}
\]
We now appeal to the diagram

\[
\begin{array}{c}
\mathbb{D}_0 C_1 \times \mathbb{D}_0^2 C_0 \times \mathbb{D}_1 C_0 \xrightarrow{I_0 \times I_0 \times I_1} \mathbb{D}_1 C_1 \times \mathbb{D}_1^2 C_1 \times C_1 C_1 \\
\mathbb{D}_0 C_1 \times \mathbb{D}_1^2 C_0 \times \mathbb{D}_1 C_0 \xrightarrow{I_0 \times \mathbb{D}_1^2 I \times I_1} \mathbb{D}_1 C_1 \times \mathbb{D}_1^2 C_1 \times C_1 C_1 \\
\mathbb{D}_0 C_1 \times C_0 \mathbb{D}_1 C_0 \times \mathbb{D}_1 C_0 \xrightarrow{I_0 \times \mathbb{D}_1 I \times I_1} \mathbb{D}_1 C_1 \times C_0 \mathbb{D}_1 C_0 \times \mathbb{D}_1 C_0 \\
\mathbb{D}_0 C_1 \times \mathbb{D}_1 C_0 \times \mathbb{D}_1 C_0 \xrightarrow{I_0 \times \mathbb{D}_1 I \times I_1} \mathbb{D}_1 C_1 \times C_0 \mathbb{D}_1 C_0 \times \mathbb{D}_1 C_0 \\
\mathbb{D}_0 C_1 \times \mathbb{D}_1^2 C_0 \times \mathbb{D}_1 C_0 \xrightarrow{I_0 \times \mathbb{D}_1^2 I \times I_1} \mathbb{D}_1 C_1 \times C_0 \mathbb{D}_1 C_0 \times \mathbb{D}_1 C_0 \\
\mathbb{D}_0 C_1 \times \mathbb{D}_1 C_0 \times \mathbb{D}_1 C_0 \xrightarrow{I_0 \times \mathbb{D}_1 I \times I_1} \mathbb{D}_1 C_1 \times C_0 \mathbb{D}_1 C_0 \times \mathbb{D}_1 C_0 \\
\end{array}
\]

in which the lower rectangle commutes since composition of identities in \( C \) give identities, and we can now rewrite \( \varepsilon_1 \circ (\mathbb{D}_0 \psi_1 \times 1) \) as follows:

\[
\begin{array}{c}
\mathbb{D}_0 U C_1 \times \mathbb{D}_0^2 C_0 \mathbb{D}_0 D C_0 \times \mathbb{D}_0 C_0 \mathbb{D}_1 C_0 \xrightarrow{\mathbb{D}_0 \kappa_1 \times \mathbb{D}_0 \xi_0 \times 1} \mathbb{D}_0 C_1 \times \mathbb{D}_0 \mathbb{D}_0 C_0 \mathbb{D}_0 C_0 \mathbb{D}_1 C_0 \\
\mathbb{D}_0 U C_1 \times \mathbb{D}_0^2 C_0 \mathbb{D}_0 D C_0 \times \mathbb{D}_0 C_0 \mathbb{D}_1 C_0 \xrightarrow{\mathbb{D}_0 \kappa_1 \times \mathbb{D}_0 \xi_0 \times 1} \mathbb{D}_0 C_1 \times \mathbb{D}_0 \mathbb{D}_0 C_0 \mathbb{D}_0 C_0 \mathbb{D}_1 C_0 \\
\mathbb{D}_0 U C_1 \times \mathbb{D}_0^2 C_0 \mathbb{D}_0 D C_0 \times \mathbb{D}_0 C_0 \mathbb{D}_1 C_0 \xrightarrow{\mathbb{D}_0 \kappa_1 \times \mathbb{D}_0 \xi_0 \times 1} \mathbb{D}_0 C_1 \times \mathbb{D}_0 \mathbb{D}_0 C_0 \mathbb{D}_0 C_0 \mathbb{D}_1 C_0 \\
\mathbb{D}_0 U C_1 \times \mathbb{D}_0^2 C_0 \mathbb{D}_0 D C_0 \times \mathbb{D}_0 C_0 \mathbb{D}_1 C_0 \xrightarrow{\mathbb{D}_0 \kappa_1 \times \mathbb{D}_0 \xi_0 \times 1} \mathbb{D}_0 C_1 \times \mathbb{D}_0 \mathbb{D}_0 C_0 \mathbb{D}_0 C_0 \mathbb{D}_1 C_0 \\
\mathbb{D}_0 U C_1 \times \mathbb{D}_0^2 C_0 \mathbb{D}_0 D C_0 \times \mathbb{D}_0 C_0 \mathbb{D}_1 C_0 \xrightarrow{\mathbb{D}_0 \kappa_1 \times \mathbb{D}_0 \xi_0 \times 1} \mathbb{D}_0 C_1 \times \mathbb{D}_0 \mathbb{D}_0 C_0 \mathbb{D}_0 C_0 \mathbb{D}_1 C_0 \\
\end{array}
\]

However, we can now appeal to the diagram

\[
\begin{array}{c}
\mathbb{D}_0 U C_1 \times \mathbb{D}_0^2 C_0 \mathbb{D}_0 D C_0 \times \mathbb{D}_0 C_0 \mathbb{D}_1 C_0 \xrightarrow{\mathbb{D}_0 \kappa_1 \times \mathbb{D}_0 \xi_0 \times 1} \mathbb{D}_0 C_1 \times \mathbb{D}_0 \mathbb{D}_0 C_0 \mathbb{D}_0 C_0 \mathbb{D}_1 C_0 \\
\mathbb{D}_0 U C_1 \times \mathbb{D}_0^2 C_0 \mathbb{D}_0 D C_0 \times \mathbb{D}_0 C_0 \mathbb{D}_1 C_0 \xrightarrow{\mathbb{D}_0 \kappa_1 \times \mathbb{D}_0 \xi_0 \times 1} \mathbb{D}_0 C_1 \times \mathbb{D}_0 \mathbb{D}_0 C_0 \mathbb{D}_0 C_0 \mathbb{D}_1 C_0 \\
\mathbb{D}_0 U C_1 \times \mathbb{D}_0^2 C_0 \mathbb{D}_0 D C_0 \times \mathbb{D}_0 C_0 \mathbb{D}_1 C_0 \xrightarrow{\mathbb{D}_0 \kappa_1 \times \mathbb{D}_0 \xi_0 \times 1} \mathbb{D}_0 C_1 \times \mathbb{D}_0 \mathbb{D}_0 C_0 \mathbb{D}_0 C_0 \mathbb{D}_1 C_0 \\
\mathbb{D}_0 U C_1 \times \mathbb{D}_0^2 C_0 \mathbb{D}_0 D C_0 \times \mathbb{D}_0 C_0 \mathbb{D}_1 C_0 \xrightarrow{\mathbb{D}_0 \kappa_1 \times \mathbb{D}_0 \xi_0 \times 1} \mathbb{D}_0 C_1 \times \mathbb{D}_0 \mathbb{D}_0 C_0 \mathbb{D}_0 C_0 \mathbb{D}_1 C_0 \\
\mathbb{D}_0 U C_1 \times \mathbb{D}_0^2 C_0 \mathbb{D}_0 D C_0 \times \mathbb{D}_0 C_0 \mathbb{D}_1 C_0 \xrightarrow{\mathbb{D}_0 \kappa_1 \times \mathbb{D}_0 \xi_0 \times 1} \mathbb{D}_0 C_1 \times \mathbb{D}_0 \mathbb{D}_0 C_0 \mathbb{D}_0 C_0 \mathbb{D}_1 C_0 \\
\end{array}
\]
18. The actual left adjoint: the unit

Our last piece of business is to verify that the unit map preserves the presheaf structure on morphism sets; this was not done for the unit to $\hat{L}M$, since that is the one piece of the adjunction framework that fails, and therefore $\hat{L}$ is not the actual left adjoint. If we write $\pi : \hat{L}M_1 \to LM_1$ for the map on morphisms given by the coequalizer construction, the unit map is given by the composite

$$M_1 \xrightarrow{\eta_1} \hat{L}M_1 \xrightarrow{\pi} LM_1.$$  

Given a map $f : M \to N$ of $\mathbb{D}$-multicategories, the condition for preservation of presheaf structure can be expressed by the requirement that the following diagram commutes:

$$
\begin{array}{ccc}
M_1 \times_{\mathbb{D}_0M_0} \mathbb{D}_1M_0 & \xrightarrow{f_1 \times_{\mathbb{D}_0N_0} f_0} & N_1 \times_{\mathbb{D}_0N_0} \mathbb{D}_1N_0 \\
\psi_M & & \psi_N \\
M_1 & \xrightarrow{f_1} & N_1.
\end{array}
$$

In our case, what we want is for the perimeter of the diagram

$$
\begin{array}{ccc}
M_1 \times_{\mathbb{D}_0M_0} \mathbb{D}_1M_0 & \xrightarrow{\eta_1 \times_{\mathbb{D}_0\hat{L}M_0} \eta_0} & U\hat{L}M_1 \times_{\mathbb{D}_0\hat{L}M_0} \mathbb{D}_1\hat{L}M_0 \\
\psi_1 & & \psi_{ULM} \\
M_1 & \xrightarrow{\eta_1} & U\hat{L}M_1
\end{array}
\xrightarrow{U\pi \times 1} 
\begin{array}{ccc}
ULM_1 \times_{\mathbb{D}_0M_0} \mathbb{D}_1M_0 & \xrightarrow{U\pi \times 1} & ULM_1 \times_{\mathbb{D}_0M_0} \mathbb{D}_1M_0 \\
\psi_{ULM} & & \psi_{ULM} \\
ULM_1 & \xrightarrow{U\pi} & ULM_1
\end{array}
$$

to commute. However, although the right square does commute since we’ve shown that the $\mathbb{D}$-algebra structure on $\hat{L}M$ descends to one on $LM$, the left square emphatically does not commute.

Our strategy is first to express $U\pi$ as a coequalizer, and then show that the two ways around the left square factor through the two maps that are coequalized by $U\pi$. We need the following elementary lemma.
Lemma 18.1. Suppose

\[ A \rightarrow B \rightarrow C \]

is a coequalizer diagram in \( \text{Set}/Z \) for some set \( Z \), and let \( X \) be a set over \( Z \). Then the induced diagram

\[ A \times_Z X \rightarrow B \times_Z X \rightarrow C \times_Z X \]

is also a coequalizer diagram.

Proof. The coequalizer diagram can be decomposed as a coproduct of coequalizer diagrams of fibers over the elements of \( Z \),

\[ \coprod_{z \in Z} A_z \rightarrow \coprod_{z \in Z} B_z \rightarrow \coprod_{z \in Z} C_z. \]

The conclusion now follows easily. \( \square \)

In our application, we have \( LM_1 \) displayed in the coequalizer diagram

\[ D_0 M_1 \times D_0^2 M_0 D_0 D_1 M_0 \times D_0 M_0 D_1 M_0 \Rightarrow D_0 M_1 \times D_0 M_0 D_1 M_0 \rightarrow LM_1, \]

and using the structure maps of source type throughout, all maps are over \( D_0 M_0 \). Since in general for a \( D \)-algebra \( C \) we have

\[ UC_1 \cong C_1 \times C_0 D_0 C_0, \]

and in our case \( C_0 = LM_0 = D_0 M_0 \), we get

\[ ULM_1 \cong LM_1 \times D_0 M_0 D_0^2 M_0. \]

Now the lemma tells us that \( ULM_1 \) arises as the coequalizer in the following diagram, where in the interest of space we have adopted our usual convention of suppressing \( D_0 M_0 \)'s in the subscripts on pullbacks:

\[ D_0 M_1 \times D_0^2 M_0 D_0 D_1 M_0 \times D_1 M_0 \times D_0^2 M_0 \Rightarrow D_0 M_1 \times D_1 M_0 \times D_0^2 M_0 \rightarrow ULM_1. \]

Note that the middle term is \( ULM_1 \), and therefore the coequalizer map is \( U \pi \). The two maps for which \( U \pi \) is the coequalizer can be written as \( D_0 \psi_1 \times 1^2 \) and \( 1 \times \phi \times 1 \), since they also are induced from the maps defining \( LM_1 \) as a coequalizer.

We next introduce a map

\[ M_1 \times D_0 M_0 D_1 M_0 \xrightarrow{\eta} D_0 M_1 \times D_0^2 M_0 D_0 D_1 M_0 \times D_0 M_0 D_1 M_0 \times D_0 M_0 D_0^2 M_0 \]

given as \( \eta \times \eta (\eta, I_D \circ S_D, D_0 \eta \circ S_D) \). We claim that the diagrams

\[ M_1 \times D_0 M_0 D_1 M_0 \xrightarrow{\psi_1} M_1 \]

\[ D_0 M_1 \times D_0^2 M_0 D_0 D_1 M_0 \times D_1 M_0 \times D_0^2 M_0 \xrightarrow{D_0 \psi_1 \times 1^2} D_0 M_1 \times D_1 M_0 \times D_0^2 M_0 \]
and

\[
\begin{array}{ccc}
M_1 \times \mathbb{D}_0 M_0 & \longrightarrow & \eta \times \mathbb{D}_0 \mathbb{D}_1 \eta \\
\downarrow \psi_1 & \downarrow \eta & \downarrow \psi_{ULM} \\
\mathbb{D}_0 M_1 \times \mathbb{D}_0^2 M_0 & \longrightarrow & \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_0^2 M_0
\end{array}
\]

both commute, that is, that both ways around the left square above that does not commute instead factor through the two maps that are coequalized by \( U\pi \), with the factorization provided by the map \( \tilde{\eta} \). This will establish that the perimeter of the rectangle does commute, verifying that the unit is in fact a map of \( \mathbb{D} \)-multicategories.

We must still verify the two claimed diagrams.

For the first diagram, we first expand the right vertical arrow as \((\eta, I_{\mathbb{D}} \circ S, \mathbb{D}_0 \eta \circ S)\), from the definition of the unit map to \( U\tilde{L}M_1 \). Now projecting to the first factor \( \mathbb{D}_0 M_1 \) of the target gives us the diagram

\[
\begin{array}{ccc}
M_1 \times \mathbb{D}_0 M_0 & \longrightarrow & \mathbb{D}_1 M_0 \\
\downarrow \eta \times \eta & \downarrow \eta & \downarrow \psi_{ULM} \\
\mathbb{D}_0 M_1 \times \mathbb{D}_0^2 M_0 & \longrightarrow & \mathbb{D}_0^2 M_0
\end{array}
\]

which is just a naturality diagram for \( \eta : 1 \to \mathbb{D}_0 \), since the lower left corner is isomorphic to \( \mathbb{D}_0(M_1 \times \mathbb{D}_0 M_0 \mathbb{D}_1 M_0) \). Projecting to the second factor \( \mathbb{D}_1 M_0 \) of the target gives us the diagram

\[
\begin{array}{ccc}
M_1 \times \mathbb{D}_0 M_0 & \longrightarrow & M_1 \\
\downarrow p_2 & & \downarrow S \\
\mathbb{D}_1 M_0 & \longrightarrow & \mathbb{D}_0 M_0 \mathbb{D}_1 M_0,
\end{array}
\]

which is a consequence of the requirement that the presheaf action \( \psi_1 \) preserve the structure maps of source type. Projection to the third factor is almost the same diagram: we just compose with \( \mathbb{D}_0 \eta \) instead of \( I_{\mathbb{D}} \):

\[
\begin{array}{ccc}
M_1 \times \mathbb{D}_0 M_0 & \longrightarrow & M_1 \\
\downarrow p_2 & & \downarrow S \\
\mathbb{D}_1 M_0 & \longrightarrow & \mathbb{D}_0 M_0 \mathbb{D}_0^2 M_0.
\end{array}
\]

This completes verification of the first claimed diagram.
For the second one, we claim that both ways around the square coincide with the map $\eta \times (1, 1, D_0 \eta \circ S_D)$. For the lower left composite, unpacking $1 \times \phi \times 1$ results in the claim that

$$
\eta \times (1, 1, D_0 \eta \circ S_D) \\
\downarrow \\
\eta \times (1, 1, D_0 \eta \circ S_D)
$$

commutes. To check this, we first simplify the left column by checking the projections to each factor in the target. The first $D_0 \ M_1$ clearly is the result of just $\eta_0 : M_1 \to D_0 \ M_1$ from the first factor of the source. Everything else arises from the factor $D_1 \ M_0$ in the source. The map to the second factor $D_1 \ M_0$ in the bottom left corner is the identity, because of the diagram

$$
\eta_0 \\
\downarrow \\
\eta_1 \\
\downarrow \\
\eta_0
$$

The map to the third factor, also $D_1 \ M_0$, is simply $I_D \circ S_D$, and therefore the composite with the multiplication $\gamma_D$ in the bottom arrow gives us the identity because of the commuting triangle

$$
\gamma_D \\
\downarrow \\
\gamma_D
$$

We conclude that the left column simplifies as claimed.
What remains is to show that the upper right composite also simplifies to $\eta \times (1, D_0 \eta \circ S_D)$. Here we begin by simplifying the top arrow as follows:

Now we need to discuss the presheaf action map on $U \hat{L}M_1$. The presheaf action on an underlying $D$-multicategory $UC$ is given by the composite

$$UC_1 \times_{D_0UC_0} D_1UC_0 = UC_1 \times_{D_0C_0} D_1C_0 \cong C_1 \times_{C_0} D_0C_0 \times_{D_0C_0} D_1C_0 \xrightarrow{\kappa_1 \times \kappa_0 \times 1} C_1 \times_{C_0} D_1C_0$$

Next, replacing $C$ with $\hat{L}M$, we can replace $C_1$ with $D_0M_1 \times_{D_0M_0} D_1M_0$, and $C_0$ with $D_0M_0$. Furthermore, the identity arrow $I_{\hat{L}M} : D_0M_0 \to D_0M_1 \times_{D_0M_0} D_1M_0$ is given by the components $(D_0I, I_D)$. Starting with the term

$$C_1 \times_{C_0} D_1C_0 \cong D_0M_1 \times_{D_0M_0} D_1M_0 \times_{D_0M_0} D_1D_0M_0,$$

and suppressing the subscript $D_0M_0$'s as usual, we can express the presheaf action on $U \hat{L}M_1$ as the composite

Precomposing this with

$$M_1 \times D_1M_0 \xrightarrow{(\eta, I_D) \circ D_1\eta} D_0M_1 \times D_1M_0 \times D_1D_0M_0,$$
we wish to show that the entire composite ends up being $\eta \times (1, \mathbb{D}_0 \eta \circ S_D)$. We proceed in steps, since the overall diagram is a bit large. First, we have

$$M_1 \times \mathbb{D}_1 M_0 \xrightarrow{(\eta, \mathbb{D}_0 \circ S) \times \mathbb{D}_1 \eta} \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_1 \mathbb{D}_0 M_0 \xrightarrow{1^2 \times (\mathbb{D}_1 \mathbb{D}_0 I, \mathbb{D}_1 I_0)} \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_1 \mathbb{D}_0 M_1 \times \mathbb{D}_1 \mathbb{D}_0 M_0 \times \mathbb{D}_1^2 M_0.$$ 

This diagram commutes by projecting to each factor of the target, and the only one that isn’t immediate is the last one, which is a consequence of the category structure on $\mathbb{D}$, and in particular the diagram

$$M_0 \xrightarrow{\eta_0} \mathbb{D}_0 M_0 \xrightarrow{I_0} \mathbb{D}_1 M_0.$$ 

Next, we have the following diagram, which can be pasted to the previous step,

$$\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_1 \mathbb{D}_0 M_1 \times \mathbb{D}_1 \mathbb{D}_0 M_0 \times \mathbb{D}_1^2 M_0 \xrightarrow{(\eta, \mathbb{D}_0 \mathbb{D}_0 S) \times (\mathbb{D}_1 \mathbb{D}_0 I \circ \eta) \times \mathbb{D}_1 \eta_1) \times (\mathbb{D}_1 \mathbb{D}_0 \circ S) \times (\mathbb{D}_1 \mathbb{D}_0 \circ T \mathbb{D}_0 \circ S) \times (\mathbb{D}_0 \mathbb{D}_0 \circ S \mathbb{D}_0 \circ S) \times (\mathbb{D}_0 \mathbb{D}_0 \circ S \mathbb{D}_0 \circ S)} \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_0^2 M_0.$$ 

Again, this commutes by projecting to each factor of the target, and the first two are immediate. The third one, to the second $\mathbb{D}_0 M_1$, commutes because of the following diagram:

$$\mathbb{D}_0 M_0 \xrightarrow{\mathbb{D}_0 \eta} \mathbb{D}_0^2 M_0 \xrightarrow{\mathbb{D}_0^2 I} \mathbb{D}_0^2 M_1 \xrightarrow{\mu} \mathbb{D}_0 M_1.$$ 

Projection to the fourth factor, the second $\mathbb{D}_1 M_0$, merely expresses the monad identity

$$\mathbb{D}_1 M_0 \xrightarrow{\mathbb{D}_1 \eta} \mathbb{D}_1^2 M_0 \xrightarrow{\mu} \mathbb{D}_1 M_0.$$
And the last one, to $\mathbb{D}_0^2 M_0$, is the elementary diagram

$$
\begin{array}{ccc}
\mathbb{D}_1 M_0 & \xrightarrow{D_1 \eta} & \mathbb{D}_0^2 M_0 \\
S_0 & & S_0 \\
\mathbb{D}_0 M_0 & \xrightarrow{D_0 \eta} & \mathbb{D}_0 \mathbb{D}_1 M_0 & \xrightarrow{D_0 S} & \mathbb{D}_0^2 M_0.
\end{array}
$$

Next, to be pasted to the previous step, is the diagram

$$
\begin{array}{ccc}
\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_0^2 M_0 & \cong & 1 \times (\mathbb{D}_1 T, S_0) \times 1^2 \\
M_1 \times \mathbb{D}_1 M_0 & \xrightarrow{(\eta, I_0 \circ S) \times (D_0 I_0 \circ T_0, 1, D_0 \eta \circ S_0)} & \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_0^2 M_0.
\end{array}
$$

Projection to the first factor is clear, and to the second factor, $\mathbb{D}_1 M_0$, is a consequence of

$$
\begin{array}{ccc}
M_1 & \xrightarrow{S} & \mathbb{D}_0 M_0 & \xrightarrow{I_0} & \mathbb{D}_1 M_0 \\
& & I_0 I & \xrightarrow{D_1 I} & D_1 T \\
& & \mathbb{D}_1 M_1 & \xrightarrow{D_1 I} & \mathbb{D}_1 M_0.
\end{array}
$$

Projection to the third factor relies on the fact that the map factors through the common projection to $\mathbb{D}_0 M_0$ in the pullback $M_1 \times_{\mathbb{D}_0 M_0} \mathbb{D}_1 M_0$ from which all these maps emanate. The relevant diagram is as follows:

$$
\begin{array}{ccc}
M_1 \times \mathbb{D}_1 M_0 & \xrightarrow{p_1} & M_1 \\
p_2 \downarrow & & \downarrow S \\
\mathbb{D}_1 M_0 & \xrightarrow{T_0} & \mathbb{D}_0 M_0 & \xrightarrow{D_0 I} & \mathbb{D}_0 M_1 \\
& & I_0 I & \xrightarrow{I_0} & \mathbb{D}_1 M_1 & \xrightarrow{S_0} & \mathbb{D}_0 M_1.
\end{array}
$$
The other two factors are clear. We next paste on the diagram

\[
\begin{array}{c}
\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_0^2 M_0 \\
\rightarrow \\
\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_1 M_0 \times \mathbb{D}_0^2 M_0.
\end{array}
\]

Here projection to the first factor is clear, and to the second is a consequence of

\[
\begin{array}{c}
\mathbb{D}_0 M_0 \xrightarrow{D_0 I} \mathbb{D}_0 M_1 \\
\downarrow I_0 \\
\mathbb{D}_1 M_1 \xrightarrow{T_0} \mathbb{D}_0 M_1.
\end{array}
\]

For the third projection, to the first factor of \(\mathbb{D}_1 M_0\), we appeal to the fact that the map factors through \(S : M_1 \rightarrow \mathbb{D}_0 M_0\), and so can just as well be expressed factoring through \(T_D : \mathbb{D}_1 M_0 \rightarrow \mathbb{D}_0 M_0\), since the source is the pullback along these two maps. The upper map can therefore be considered \(I_D I \circ T_D : \mathbb{D}_1 M_0 \rightarrow \mathbb{D}_1 M_1\). We then expand the definition of \(\theta\) as the composite

\[
\begin{array}{c}
\mathbb{D}_1 M_1 \xrightarrow{D_1 S} \mathbb{D}_1 \mathbb{D}_0 M_0 \xrightarrow{D_1 I_0} \mathbb{D}_1^2 M_0 \xrightarrow{\mu} \mathbb{D}_1 M_0,
\end{array}
\]

and appeal to the following diagram:

\[
\begin{array}{c}
\mathbb{D}_1 M_0 \xrightarrow{T_0} \mathbb{D}_0 M_0 \xrightarrow{I_0} \mathbb{D}_1 M_0 \\
\downarrow I_0 \downarrow \xrightarrow{D_1 I_0} \\
\mathbb{D}_1 M_1 \xrightarrow{D_1 S} \mathbb{D}_1 \mathbb{D}_0 M_0 \xrightarrow{D_1 I_0} \mathbb{D}_1^2 M_0 \xrightarrow{\mu} \mathbb{D}_1 M_0.
\end{array}
\]

Projection to the last two factors is trivial. Finally, we paste in the diagram

\[
\begin{array}{c}
\mathbb{D}_0 M_1 \times \mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_1 M_0 \times \mathbb{D}_0^2 M_0 \\
\rightarrow \\
\mathbb{D}_0 M_1 \times \mathbb{D}_1 M_0 \times \mathbb{D}_0^2 M_0.
\end{array}
\]

Here projection to the first factor is a consequence of the right unital property of \(\gamma_M\), and projection to the second follows from the left unital property for \(\gamma_D\). This
concludes the proof that the unit map preserves the presheaf structure, and therefore that we have, in fact, constructed a left adjoint to the forgetful functor.
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