The mean tilt of sunspot bipolar regions: theory, simulations and comparison with observations

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ABSTRACT

A theory of the mean tilt of sunspot bipolar regions (the angle between a line connecting the leading and following sunspots and the solar equator) is developed. A mechanism of formation of the mean tilt is related to the effect of Coriolis force on meso-scale motions of super-granular convection and large-scale meridional circulation. The balance between the Coriolis force and the Lorentz force (the magnetic tension) determines a contribution of the large-scale magnetic field to the mean tilt of the sunspot bipolar regions at low latitudes. In addition, the latitudinal dependence of the solar differential rotation affects the mean tilt which can explain deviations from the Joy’s law for the sunspot bipolar regions at high latitudes. The obtained theoretical results and performed numerical simulations based on the nonlinear mean-field dynamo theory which takes into account conservation of the total magnetic helicity are in agreement with observational data of the mean tilt of sunspot bipolar regions over individual solar cycles 15 - 24.

Key words: Sun: dynamo – Sun: activity

1 INTRODUCTION

Origin of solar magnetic field and dynamics of solar activity are the subjects of many studies and discussions (Moffatt 1978; Parkes 1979; Krause & Rädler 1980; Zeldovich et al. 1983; Rüdiger & Hollerbach 2004; Osendrijver 2005; Brandenburg & Subramanian 2005). The solar magnetic fields are observed in the form of sunspots and active regions. One of the characteristics of the solar bipolar region is a tilt defined as the angle between a line connecting the leading and following sunspots and circles of constant latitude parallel to the solar equator plane.

According to the Joy’s law the mean tilt of sunspot bipolar regions increases with the latitude (Hale et al. 1910; Howard 1991; Sivaraman et al. 1996; Pevtsov et al. 2014; McClintock et al. 2013; McClintock & Norton 2016). The reasons for the tilt of sunspot bipolar regions are caused by effect of the Coriolis force (Fisher et al. 2000), direction of the toroidal magnetic field (Babcock 1961; Norton & Gilman 2005), and kink instability (Leighton 1964; Longcope et al. 1999; Holder et al. 2004). In particular, the main effect responsible for the formation of the mean tilt of sunspot bipolar regions is related to the effect of Coriolis force on large-scale motions in super-granular turbulent convection. The Coriolis force is proportional to \( \sin \phi \), where \( \phi \) is the latitude, so that the main dependence of the mean tilt on the latitude is expected to be proportional to \( \sin \phi \).

In spite of various theoretical, numerical and observational studies related to the mean tilt (DSilva & Choudhuri 1993; Kosovichev & Stenflo 1993; Dasi-Espuig et al. 2010; McClintock & Norton 2013; Tlatov et al. 2013; Illarionov et al. 2015; Tlatova et al. 2015), some observational features related to the mean tilt of sunspot bipolar regions are not explained. As follows from observations (Tlatova et al. 2018), the latitudinal dependence of the mean tilt of sunspot bipolar regions is more complicated than a simple dependence as \( \sin \phi \).

In particular, there is a non-zero mean tilt of sunspot bipolar regions at the equator where the Coriolis force vanishes. To investigate the latitudinal dependence of the mean tilt of sunspot bipolar regions and its variations in different solar cycles, Tlatova et al. 2018 used the data of the magnetic field observations of sunspots from Mount Wilson Observatory. They have found that the latitudinal dependence of the tilt varies from one solar cycle to another, and there is a systematic offset in tilt of sunspot bipolar...
regions (non-zero tilts at the equator), with negative offset for odd cycles and positive offset for even cycles.

In the present study we develop a theory of the mean tilt of sunspot bipolar regions, taking into account the effect of the large-scale magnetic field and the solar differential rotation on the mean tilt of sunspot bipolar regions. We have demonstrated that in the balance between the Coriolis force and the magnetic tension determines a contribution of the large-scale magnetic field to the mean tilt of the sunspot bipolar regions at the low latitudes. In addition, we show that the latitudinal dependence of the solar differential rotation can affect the mean tilt. The latter can explain the observed deviations from the Joy’s law for the mean tilt of the sunspot bipolar regions at the higher latitudes. The obtained theoretical results and performed numerical simulations have been compared with the latitudinal dependence of the mean tilt found in observations during the last nine solar cycles.

2 THE THEORY FOR THE MEAN TILT OF SUNSPOT BIPOLAR REGIONS

We consider the boundary between the solar convective zone and the photosphere of the Sun. In the photosphere of the Sun there is no MHD turbulence as well as the dynamo action. The mean tilt of the sunspot bipolar regions is mainly determined by the effect of Coriolis force on meso-scale motions of supergranular convection and large-scale meridional circulation. In the present paper we show that there are two additional contributions to the mean tilt of the sunspot bipolar regions related to (i) the effect of the large-scale magnetic field to the large-scale meridional circulation and (ii) the effect of the latitudinal dependence of the solar differential rotation.

Let us start with the momentum, induction and entropy equations using anelastic approximation and neglecting small dissipation caused by the kinematic viscosity, the magnetic diffusion and the entropy diffusion:

\[
\frac{\partial U}{\partial t} = -\nabla \left( \frac{P_{\text{tot}}}{\rho_0} \right) - g S + \frac{1}{4\pi\rho_0} (B \cdot \nabla) B + \frac{\Lambda_\rho}{8\pi\rho_0} B^2 + U \times (2\Omega + W),
\]

\[
\frac{\partial B}{\partial t} = (B \cdot \nabla) U - (U \cdot \nabla) B - B (\nabla \cdot U),
\]

\[
\frac{\partial S}{\partial t} = -(U \cdot \nabla) S - \frac{\Omega_0^2}{g} U \cdot \hat{r},
\]

\[
\nabla \cdot U = \Lambda_\rho U,
\]

where \(U\) and \(B\) are the velocity and magnetic fields, \(W = \nabla \times U\) is the vorticity, \(P_{\text{tot}} = P + B^2/8\pi + U^2/2\) is the total pressure, \(S\) and \(P\) are the entropy and pressure, \(\Omega_0^2 = -(g \cdot \nabla) S_0, \rho_0\) and \(S_0\) are the plasma density and entropy in the basic reference state, \(\Lambda_\rho = -\nabla \ln \rho_0, \) \(g\) is the acceleration due to the gravity, \(\hat{r} = r/|r|\) is the unit vector in the radial direction, and \(\Omega\) is the angular velocity.

2.1 Effect of the large-scale magnetic field on the mean tilt

We decompose the solution of equations (1)–(4) as the sum of the equilibrium fields (with overbars) related to the meridional circulation and perturbations related to both, supergranular motions in the convective zone and rotational motions of sunspots in the photosphere (contributed to the mean tilt of sunspot bipolar regions), i.e., \(U = \overline{U} + \tilde{u}, \)

\(B = \overline{B} + \tilde{b}, S = \overline{S} + \tilde{s}\) and \(P = \overline{P} + \tilde{p},\) where \(\tilde{u} = \partial \xi / \partial t + v^{(c)}\) and \(v^{(c)}\) is the convective velocity related to the supergranular motions. Equations (1)–(4) allow us to obtain an equation for small perturbations \(\xi\) related to the rotational motions of sunspots in the photosphere as

\[
\frac{\partial^2 \xi}{\partial t^2} = -\nabla \left( \frac{P_{\text{tot}}}{\rho_0} \right) - \tilde{r}(\xi \cdot \tilde{r}) \left( \Omega_0^2 + \Lambda_\rho^2 \nabla^2 \right) + 2 \left( \overline{U} \cdot \frac{\partial \xi}{\partial t} + v^{(c)} \right) \times \Omega + \left( \nabla \cdot \nabla \right) \xi + 2 \left( \frac{\partial \xi}{\partial t} + v^{(c)} \right) \times \Omega + \left( \nabla \cdot \nabla \right) \xi + \left( \nabla \cdot \nabla \right) \xi - U_{A} \left( \xi \cdot \tilde{r} \right),
\]

where \(\tilde{P}_{\text{tot}}\) are the perturbations of the total pressure, and \(\nabla_{A} = \nabla / (4\pi\rho_0)^{1/2}\) is the Alfvén speed, \(\Omega_0^2 = \Omega_0^2 + g(\xi \cdot \nabla) \nabla / (\xi \cdot \tilde{r})\) and \(\xi = \xi / |\xi|\) is the unit vector.

Let us introduce the vector \(\delta^w = \nabla \times \xi,\) which is related to the perturbations of vorticity, \(\tilde{w} \equiv (\partial / \partial t) (\nabla \times \xi) \equiv \partial \delta^w / \partial t.\) This implies that the absolute value \(|\delta^w| \approx |\tilde{w}| \delta t\) characterises the small angle in which the magnetic field lines (which are associated with the sunspot bipolar regions and connected the sunspots of the opposite magnetic polarity), are twisted during the small time \(\delta t,\) when the perturbations of the vorticity \(\tilde{w}\) do not vanish. The direction of the vector \(\delta^w\) coincides with that of the vorticity \(\tilde{w},\) and it is perpendicular to the plane of rotation. Therefore, the radial component of the vector \(\delta^w\) at the boundary between the convective zone and the photosphere can characterise the tilt of the sunspot bipolar regions. On the other hand, at this boundary the magnetic field inside the sunspots is preferably directed in the radial direction.

An equation for the mean tilt at the solar surface, \(\gamma \equiv (\delta^w \cdot e_B)_{\text{time}},\) is derived in Appendix A. Here we give the derived expression for the mean tilt of the sunspot bipolar regions at the surface as

\[
\gamma = \frac{\tau_A^2}{2\pi} \left( \nabla \times \left( \left( \overline{U} + v^{(c)} \right) \times \Omega \right) \right)_{\text{time}} \cdot e_B,
\]

where the angular brackets \((\cdots)_{\text{time}}\) denote the averaging over the time that is larger than the maximum Alfvén time \(\tau_A = L_B / \overline{U}_A.\) Here \(e_B = \overline{B} / B\) is the unit vector along the mean large-scale magnetic field, and \(L_B\) is the length of the magnetic field line. In the derivation of equation for the mean tilt of the sunspot bipolar regions, we take into account that \(|\partial \xi / \partial t| \ll |v^{(c)}| / |\overline{U}|.\) We also assume that the source of the tilt of the sunspot bipolar regions, \(I_\gamma = 2 \nabla \times [\left( \overline{U} + v^{(c)} \right) \times \Omega] \cdot e_B,\) is localized at the vicinity of the boundary between the solar convective zone and the photosphere. Calculating the source \(I_\gamma\) and averaging it over the time larger than the maximum Alfvén time, we arrive at the expression for the mean tilt of sunspot bipolar regions as

\[
\gamma = -\delta_0 \left[ \sin \phi - \cos \phi \phi \frac{\tau_A}{R_\odot} \frac{\partial \overline{U}_r}{\partial \phi} \right],\]

where \(\delta_0, R_\odot\) and \(\tau_A\) are the parameters of the model.
where \( \delta_0 = (1 + \tilde{C}) \frac{\tau_{\alpha}}{\pi \tau_c} \Omega / (2\pi \tau_c) \), \( \tau_c = H_e / \nu_e^{eq} \) is the characteristic time associated with convective super-granular motions, \( R_{\odot} \) is the solar radius, and \( \phi \) is the latitude. Here we also took into account that \( \partial \nu_e^{eq} / \partial r = -\tilde{C} \nu_{e, bot}^{eq} / H_e \) and \( \langle \partial \nu_e^{eq} / \partial \phi \rangle = 0 \). The radial mean velocity, \( \bar{U}_r \), is estimated as

\[
\bar{U}_r = \frac{C_\eta}{\kappa} \left( \frac{\ell_{top}}{H_{\odot}} \right) \left( \frac{\rho_{bot}}{\rho_{top}} \right) \left( \frac{u_{bot}^{eq}}{u_{top}^{eq}} \right) \left( \frac{\partial^2 \rho^2 / \partial \phi^2}{B_{eq}^{2}} \right)_{bot},
\]

(see Appendix B), where \( \ell_{top} \) is the integral scale of turbulent motions in the upper part of the convective zone, \( \rho_{bot} \) and \( \rho_{top} \) are the plasma densities in the bottom and upper parts of the convective zone, respectively, \( u_{bot}^{eq} \) and \( u_{top}^{eq} \) are the characteristic turbulent velocity and the turbulent viscosity, respectively, in the upper part of the convective zone, and \( B_{eq} = u \sqrt{4\pi \rho} \) is the equipartition magnetic field. The parameter \( \kappa \approx 0.3 - 0.4 \) characterises a fraction of the large-scale radial momentum of plasma which is lost during crossing the boundary between the convective zone and photosphere. The constant \( C_\eta \) in equation (8) varies from 0.7 to 1 depending on the radial profile of the mean magnetic field. Substituting equation (8) into equation (4), we obtain the expression for the mean tilt of the sunspot bipolar regions as

\[
\gamma = -\delta_0 \left[ \sin \phi - \delta_M \cos \phi \right],
\]

(9)

where \( \delta_M = C_M \left( \frac{\ell_{bot}}{H_{\odot}} \right)^2 \left( \frac{\rho_{bot}}{\rho_{top}} \right) \left( \frac{\eta_{top}}{\eta_{bot}} \right) \left( \frac{\tau_c}{\tau_{bot}} \right) \times \left( \frac{\partial^2 \rho^2 / \partial \phi^2}{B_{eq}^{2}} \right)_{bot}
\]

(10)

and \( \tau_{bot} = \ell_{bot} / u_{bot} \) is the characteristic turbulent time in the bottom part of the convective zone, \( C_M = 3C_{\nu_{e, bot}} / (\kappa \nu_T) \approx 10 \), \( \nu_T = \nu_e / \eta_e \) is the turbulent Prandtl number, and \( \eta_T \) is the turbulent magnetic diffusion coefficient.

The parameter \( \delta_M \) describes the magnetic contribution to the mean tilt of the sunspot bipolar regions. The mechanism related to the magnetic contribution to the mean tilt is as follows. The Coriolis force results in the rotational motions of sunspots in the photosphere, and the balance between the Coriolis force and the magnetic tension determines the large-scale magnetic field contribution to the mean tilt of the sunspot bipolar regions.

To estimate the mean tilt of the sunspot bipolar regions, we use the values of governing parameters taken from models of the solar convective zone (see, e.g., Baker & Temesvary (1963); Spruit (1972); more modern treatments make little difference to these estimates). In particular, at depth \( H \approx 2 \times 10^{15} \) cm (i.e., at the bottom of the convective zone), the magnetic Reynolds number \( R_{m, bot} = u_{bot} \ell_{bot} / \eta \approx 2 \times 10^9 \) (where \( \eta \) is the magnetic diffusion coefficient due to electrical conductivity of plasma), the turbulent velocity \( u_{bot} \approx 2 \times 10^7 \) cm s\(^{-1}\), the integral scale of turbulence \( \ell_{bot} \approx 8 \times 10^9 \) cm, the plasma density \( \rho_{bot} \approx 2 \times 10^{-1} \) g cm\(^{-3}\), and the turbulent diffusion coefficient \( \eta_{bot} \approx 5 \times 10^{12} \) cm\(^2\) s\(^{-1}\). The density stratification scale is estimated here as \( H_{\rho, bot} = \rho / |\nabla \rho| \approx 6.5 \times 10^9 \) cm and the equipartition mean magnetic field \( B_{eq, bot} = 3000 \) G. In the upper part of the convective zone, say at depth \( H \approx 2 \times 10^7 \) cm, these parameters are \( R_{m, top} = u_{top} / \ell_{top} / \eta \approx 10^7 \), \( u_{top} \approx 9 \times 10^4 \) cm s\(^{-1}\), \( \ell_{top} = 2.6 \times 10^{10} \) cm, \( \rho_{top} = 4.5 \times 10^{-7} \) g cm\(^{-3}\), \( \eta_{top} = 0.8 \times 10^{12} \) cm\(^2\) s\(^{-1}\), \( H_{\rho, top} = 3.6 \times 10^7 \) cm, and the equipartition mean magnetic field is \( B_{eq, top} = 220 \) G.

Using these estimates, we calculate the parameters \( \delta_0 \) and \( \delta_M \) which determine the mean tilt of the sunspot bipolar region. Taking the Alfvén speed \( U_A = 5 \times 10^7 \) cm s\(^{-1}\), the length the magnetic field line \( L_B = 4H_e = 4 \times 10^8 \) cm, we obtain the Alfvén time \( \tau_A = L_B / U_A = 10^5 \) s. Taking the convective velocity \( u_c = (3 - 5) \times 10^4 \) cm s\(^{-1}\), we obtain the convective time as \( \tau_c = (2 - 3) \times 10^8 \) s. This yields \( \delta_0 = 0.3 - 0.5 \) (in radians) and \( \delta_M = 0.05 - 0.2 \). This implies that the magnetic contribution \( \delta_M \) to the mean tilt \( \gamma \) is essential only in the low latitude region where \( \sin \phi \) is small.

The mean tilt of sunspot bipolar regions

In this section we take into account an effect of latitudinal dependence of the solar differential rotation on the tilt of the sunspot bipolar regions. In particular, the latitudinal dependence of the solar rotation can be approximated by

\[
\Omega = \Omega_0 \left( 1 - C_2 \sin^2 \phi - C_4 \sin^4 \phi \right),
\]

(11)

[see LaBonte & Howard (1982)], where \( \Omega_0 = 2.83 \times 10^{-6} \) s\(^{-1}\), \( C_2 = 0.121 \) and \( C_4 = 0.173 \). Substituting equation (11) into equation (9) with \( \delta_0 = (1 + \tilde{C}) \tau_A / (2\pi \tau_c) \), we obtain

\[
\gamma = -\tilde{\delta}_0 \left[ \sin \phi + \delta_M \sin 3\phi - \delta_3 \sin 5\phi \right.

- \tilde{\delta}_M \left( \cos \phi + \delta_M \cos 3\phi - \delta_3 \cos 5\phi \right)
\]

(12)

where \( \tilde{\delta}_0 = C_D \delta_0 \), \( \delta_M = \delta_M \tilde{C}_D / 16C_D \), \( C_D = 1 - (3C_2 + 5C_4) / 4 \approx 0.693 \), \( \tilde{C}_D = 1 - 4C_2 - 2C_4 \approx 0.17 \) and \( \delta_3 = (4C_2 + 5C_4) / 16C_D \approx 0.122 \), \( \delta_3 = C_4 / 16C_D \approx 1.56 \times 10^{-2} \), \( \delta_3 = (4C_2 + 3C_4) / C_D \approx 4.48 \), and \( \delta_3 = C_4 / 3C_D \approx 1.02 \). For the derivation of equation (12) we take into account that \( \partial \Omega / \partial \phi \rangle \approx 0 \), and \( \Omega / \partial \phi \rangle \approx r^{-1} \partial \Omega / \partial \theta \), and we also use identities given in Appendix C.

In Figure 4 we show the mean tilt \( -\gamma \) (solid line) versus latitude \( \phi \) given by equation (12) of our theory, where \( \gamma \) and \( \phi \) are measured in degrees, and we use the following values of parameters: \( \tilde{\delta}_0 = 0.35 \), \( \delta_3 = 0.12 \), \( \delta_2 = 1.6 \times 10^{-2} \).
We consider the mean-field dynamo in a thin convective shell, taking into account strong variation of the plasma density in the radial direction by averaging the equations for the mean toroidal field $B_\varphi$ and the magnetic potential $\mathbf{A}$ of the mean poloidal field over the depth of the convective shell. We neglect the curvature of the convective shell and replace it by a flat slab. The terms describing turbulent diffusion of the mean magnetic field in the radial direction in equations (13) and (14) in the framework of the no-z model are given as $-\mu^2 B_\varphi$ and $-\mu^2 A$ [Kleeorin et al. 2016; Safiullin et al. 2018], and see also references therein. The differential rotation is characterised by parameter $G = \partial \Omega / \partial r$, and the parameter $\mu$ is determined by the following equation: \[ \int_0^\infty (\partial^2 B_\varphi/\partial r^2)^2 \, dr = -(\mu^2/3) B_\varphi. \]

Equations (13) and (14) are written in a non-dimensional form so that the length is measured in units of the solar radius $R_\odot$, time is measured in units of the turbulent magnetic diffusion time $R_\odot^2/\eta$, the differential rotation $\Omega$ is measured in units of the maximal value of the angular velocity $\Omega$, and $\alpha$ is measured in units of the maximum value of the kinetic part of the $\alpha$-effect. The toroidal mean magnetic field, $B_\varphi$, is measured in the units of the equipartition field $B_{eq} = u\sqrt{4\pi\rho_{sat}}$, and the vector potential of the mean poloidal field $A$ is measured in units of $R_\odot, R_\odot, B_{eq}$. The density $\rho_0$ is normalized by its value $\rho_{sat}$ at the bottom of the convective zone, and the integral scale of the turbulent motions $\ell$ and turbulent velocity $u$ at the scale $\ell$ are measured in units of their maximum values through the convective region. The magnetic Reynolds number $Rm = \ell u/\eta$ is defined using the maximal values of the integral scale $\ell$ and the characteristic turbulent velocity $u$, and the turbulent magnetic diffusion coefficient is $\eta = \ell u/3$.

We use the standard profile of the kinetic part of the $\alpha$-effect: $\alpha(\theta) = \alpha_0 \sin^2 \theta \cos \theta$. The dynamo number is defined as $D = R_\odot, R_\odot$, where $R_\odot = \alpha_0 R_\odot/\eta$, and $\omega = (3G) R_\odot^2/\eta$. Equations (13) and (14) describe the dynamo waves propagating from the central latitudes towards the equator when the dynamo number is negative. The radius $r$ varies from 2/3 to 1 inside the convective shell, so that the value $\mu = 3$ corresponds to a convective zone with a thickness of about 1/3 of the radius.

The total $\alpha$ effect is defined as the sum of the kinetic and magnetic parts: $\alpha(\theta) = \chi_\alpha \Phi_\alpha(B) + \sigma \chi_\alpha \Phi_\alpha(B)$ [Kleeorin et al. 2016; Safiullin et al. 2018], where $\chi_\alpha = -(r/3) u \cdot (\nabla \times \mathbf{u})$, $\chi_\alpha = (\tau/12\pi) b \cdot (\nabla \times b)$, $\tau$ is the correlation time of the turbulent velocity field, $\mathbf{u}$ and $\mathbf{b}$ are the velocity and magnetic fluctuations, and $\sigma = \int_0^1 (\rho_0(\tau)/\rho_{sat})^{-1} \, d\tau$. The magnetic part of the $\alpha$ effect [Frisch et al. 1975; Pouquet et al. 1976] and density of the magnetic helicity are related to the density of the current helicity $b \cdot (\nabla \times b)$ in the approximation of weakly inhomogeneous turbulent convection [Kleeorin & Rogachevskii 1994]. The quenching functions $\Phi_\alpha(B)$ and $\Phi_\beta(B)$ in equation for the total $\alpha$ effect are given by: $\Phi_\alpha(B) = (1/4) \Phi_{in}(B) + 3 \Phi_{in}(B)$ and $\Phi_\beta(B) = (3/8B^2) [1 - \arctan(\sqrt{B}/\sqrt{3})]^{1/2}$ [Rogachevskii & Kleeorin 2004, 2005], where $\Phi_{in}(B) = 1 - 16B^2/127B^2 + 16B^2/127B^2$ and $\Phi_{in}$ and $\chi$ and $\chi$ are measured in units of maximal value of the $\alpha$-effect. The function $\Phi_\alpha$ describes the algebraic quenching of the kinetic part of the $\alpha$ effect that is caused by the effects of the mean magnetic field on the electromotive force. The densities of the helicities and quenching functions are associated with a middle part of the convective zone. The parameter $\sigma > 1$ is a free parameter.

The magnetic part $\alpha_m$ of the $\alpha$ effect is based on
with the spot formation threshold, \( \sigma = 0.4 B_0 \). The function \( \chi(B) \) is determined by a non-dimensional dynamical equation (Kleeorin & Ruzmaikin 1982; Kleeorin & Rogachevskii 1999; Kleeorin et al. 1995, 2000, 2002, 2003a, b; Brandenburg & Subramanian 2003; Zhang et al. 2004, 2012):

\[
\frac{\partial \chi_c}{\partial t} + (\tau_c^{-1} + \kappa_T \mu^2) \chi_c = 2 \left( \frac{\partial A}{\partial \theta} \frac{\partial B_v}{\partial \phi} + \mu^2 \frac{\partial A_{\phi}}{\partial \phi} \right) - \left( \frac{R_c^2}{2 \ell^2} \right) \alpha B_v - \frac{\partial}{\partial \theta} \left( B_v \frac{\partial \chi_c}{\partial \theta} - \kappa_T \frac{\partial \chi_c}{\partial \phi} \right),
\]

where \( F_c = -\kappa_T \nabla \chi_c \) is the turbulent diffusion flux of the density of the magnetic helicity, \( \kappa_T \) is the coefficient of the turbulent diffusion of the magnetic helicity, \( \tau_c \) is the relaxation time of magnetic helicity. This equation is derived from the conservation law for magnetic helicity. The average value of \( \tau_c = 0 \) is given by

\[
\tau_c^{-1} = H^{-1} \int_{2/3}^1 \tau_c^{-1}(r) dr \sim \frac{H_I R_c^2 \eta}{H \ell^2 \eta_T},
\]

where \( H \) is the depth of the convective zone, \( H_I \) is the characteristic scale of variations \( \ell \), and \( \tau_c(r) = (\eta \overline{R_c^2}(\ell^2/\eta) \) is the non-dimensional relaxation time of the density of the magnetic helicity. The values \( H, \eta, \ell \) in equation (16) are associated with the upper part of the convective zone. The mean magnetic field is given by

\[
\frac{B^2}{R^2} = \frac{2\ell^2}{R^2} \left[ \frac{B_v^2}{R^2} + R_{\alpha} \left( \mu^2 + \frac{\partial A}{\partial \theta} \right)^2 \right].
\]

The solar activity is characterized by the Wolf number (Gibson 1973; Stix 1983), that is defined as \( W = 10 g_w + f_w \), where \( g_w \) is the number of sunspot groups and \( f_w \) is the total number of sunspots in the visible part of the Sun. In the framework of the nonlinear mean-field dynamo model by Kleeorin et al. (2016); Safiullin et al. (2018), the phenomenological budget equation for the surface density of the Wolf number is given by equation (17) in Appendix D.

We solve numerically Eqs. (15), (16), (17) and (18).

We use the following initial conditions: \( B_\phi(t = 0, \theta) = S_1 \sin \theta + S_2 \sin(2\theta) \) and \( A(t = 0, \theta) = 0 \). The parameters of the numerical simulation are as follows: \( D = -8450, G = 1, \sigma = 3, \mu = 3, \kappa_T = 0.1, R_\alpha = 2, \tau_c = 6.3, S_1 = 0.051 \) and \( S_2 = 0.95 \). The choice of these parameters in the numerical simulations is caused by the following reasons. In our previous studies (Kleeorin et al. 2016; Safiullin et al. 2018) we performed a parameter scan using about \( 10^7 \) runs with different sets of parameters to find an optimal set of parameters to reach a high level of correlation between the dynamo model results and observations of the Wolf numbers. The variations of the parameters affect the results as follows. There are two crucial parameters which strongly affect the dynamics of the nonlinear dynamo system: the dynamo number \( D \) and the initial field \( B^\text{init} \). A proper choice of the initial field \( B^\text{init} \) allows to avoid very long transient regimes.

Comparing the results of the dynamo model with observations, we determine the correlation between the numerical simulation results for the evolution of the Wolf number and the observational data. To find the maximum correlation between the dynamo model results and the observed Wolf numbers, the following parameter scan has been performed: \( -8800 \leq D \leq -8200 \) and \( 0.85 \leq S_2 \leq 0.95 \) [see Fig. 12 in Kleeorin et al. (2016)]. The maximum correlation is obtained when the parameters are \( D = -8450 \) and \( S_2 = 0.95 \). The parameter \( \mu \) determines the critical dynamo number, \( |D| \), for the excitation of the large-scale dynamo instability. The flux of the magnetic helicity [see Eq. (13)], characterised by the parameter \( \kappa_T \), cannot be very small to avoid the catastrophic quenching of the \( \alpha \) effect [Kleeorin et al. 2004, 2002, 2003a, b]. The optimal value for this parameter is \( \kappa_T \approx 0.1 \). The variations of the other parameters only weakly affect the obtained results [Kleeorin et al. (2016)].

Using results of these numerical simulations, we plot in Fig. 3 the butterfly diagram of the normalised mean tilt \(-\gamma/\delta_0\) given by equation (9) with the magnetic contribution to the mean tilt as

\[
\delta_M^B = C(\frac{\partial^3 B^2}{\partial \phi^3} \overline{B^2})_{\text{bot}},
\]

where \( \phi = \pi/2 - \theta \) is the latitude, and the exact value of the free parameter \( C \) is determined to obtain good agreement with observational data on the mean tilt of sunspot bipolar regions. For comparison, in Fig. 4 we show the butterfly diagram of the surface density of the Wolf numbers using the threshold \( 0.4 B_0 \) for the magnetic spot formation. First, the butterfly diagram of the normalised mean tilt of sunspot bipolar regions is essentially different from that of the surface density of the Wolf numbers. In particular, the mean tilt distribution in every hemisphere is nearly homogeneous.
i.e., it depends weakly on the phase of the solar cycle except for small regions for the low latitudes where the mean tilt has opposite signs in every hemisphere. On the other hand, the distribution of the surface density of the Wolf number is strongly inhomogeneous, i.e., it strongly depends on the phase of the solar cycle.

4 COMPARISON WITH OBSERVATIONS

In this section we compare our numerical results with observational data of the mean tilt $-\gamma$ of sunspot bipolar regions.

We use the observational data which have been obtained by Tlatova et al. (2018) from daily sunspot drawings taken at Mount Wilson Observatory (MWO). The data cover a century long period. The original MWO drawings were digitized using software package developed by Tlatova et al. (2015), see also references therein. The digitization includes the date and time of observations, heliographic coordinates of each umbra, its area, the strength, and polarity of its magnetic field. The overall digitized dataset used by us contains the total number of 20,318 days of observations, from 1917 to October 2016. The method of Tlatova et al. (2018) enables to identify clusters of sunspots of positive and negative polarity, from which bipolar pairs have been formed.

There has been the total of 441,973 measurements of the magnetic field of individual nuclei and pores of sunspots carried out, and the total number of 51,413 bipolar regions allocated. Initially, clusters of active regions of positive and negative polarity were searched for. For achieving this, the sunspots were sorted by area for each day of observation, and kernels of the same polarity located at a distance of no more than 10 degrees in longitude and 7 degrees in latitude from the spot of maximum area were selected. For each cluster, the average coordinates were found, which were computed using the weight function over the area. Next, a bipole counterpart through clusters of sunspot negative polarity was found.

The observational data are two-fold. The first group of the data used in the present study to produce Figures 1 and 2 (see Section 2), is the result of averaging of bipolar pairs of all sizes. This group of the data is presented in Tables 1 and 2 in Tlatova et al. (2018), where the mean value and the standard deviation of Gaussian fittings have been computed. We use the data to compare with the theoretical results and to check our theory. We have shown that the theoretical results fit the observations very well. The data have been filtered out by the bipolar regions smaller in length than 3 degrees. In total there were 18,547 bipolar regions in the even and 17,435 in the odd solar cycles. We used the bipolar regions greater than 3 degrees because smaller bipolar regions almost do not possess a certain tilt angles.

The second group of the data used below to produce Figure 5 is comprised of the all data on the tilts of all bipolar regions filtered by the small sized bipolar pairs, so that only the bipolar regions larger by size than 3 degrees were retained. The cut-off area of those pairs was set to several $\mu H$ ($4\pi \times 10^{-6}$ of steradian). We have used those data to confront with our theoretical results based on the time evolution of the mean tilt of sunspot bipolar regions computed from dynamo models and related to the magnetic contribution to the mean tilt.

Figure 5. The mean tilt $-\gamma$ (in degrees) versus the latitude $\phi$ (in degrees): numerical simulations (solid line) and observations of sunspot bipolar regions (dashed line) averaged over individual cycles 15 - 24.

from dynamo models and related to the magnetic contribution to the mean tilt.

Note that both these samplings are very different from what was published for statistics of bipolar regions earlier by Tlatov et al. (2013); Tlatov (2015). In earlier works the bipolar regions have been composed from individual sunspot nuclei, while in our studies they are formed from the clusters of sunspots. Thus, our results may be qualitatively very different from those of Kosovichev & Stenflo (2008); Dasi-Espuig et al. (2010). Since the nuclei of spots are formed of the two opposite polarities, the technique and the results are significantly different.

In Figure 5 we show the mean tilt $-\gamma$ (in degrees) versus the latitude $\phi$ obtained in this numerical simulation (see solid line in Figure 5 for $C_1 = 0.8$, $\delta_b = 0.29$, $\delta_l = 0.122$, $\delta_2 = 1.56 \times 10^{-2}$, $\delta_3 = 4.48$, and $\delta_5 = 1.02$. These numerical results are also compared with the observational data of the mean tilt $-\gamma$ of sunspot bipolar regions. The observational data have been averaged over individual solar cycles (from the cycle 15 to 24). The numerical results are also averaged over the same cycles. It follows from Figure 5 that there is an asymmetry between the northern and southern hemisphere. We stress that we have taken into account here an effect of the latitudinal dependence of the solar differential rotation on the mean tilt of the sunspot bipolar regions as well as the contribution to the mean tilt caused by the large-scale magnetic field. The obtained theoretical results and performed numerical simulations for the mean tilt of sunspot bipolar regions are in an agreement with the observational data.

5 CONCLUSIONS

We have developed a theory of the mean tilt of sunspot bipolar regions. The formation of the mean tilt is caused by the effect of the Coriolis force on meso-scale motions of supergranular convection and large-scale meridional circulation.

We have demonstrated that at low latitudes the joint action of the Coriolis force and the magnetic tension results in an additional contribution to the mean tilt of the sunspot bipolar regions which depends on the large-scale magnetic field. We have also found an additional contribution to the mean tilt of the sunspot bipolar regions which is caused by an effect of the latitudinal dependence of the solar differential rotation on the mean tilt. The latter can explain the devia-
tions from the Joy’s law for the mean tilt at high latitudes. The obtained theoretical results and performed numerical simulations for the mean tilt are in an agreement with the observational data of the mean tilt of the sunspot bipolar regions.

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APPENDIX A: DERIVATION OF EQ. (6) FOR THE MEAN TILT

To derive equation (6) for the mean tilt of sunspot bipolar regions, we exclude the pressure term from equation (5) for \( \xi \) by applying \( \text{curl} \) to this equation. This yields

\[
\begin{align*}
\frac{\partial^2 \tilde{\gamma}}{\partial t^2} &= 2 \left[ \nabla \times \left( \left( \mathbf{T} + \frac{\partial \xi}{\partial t} + \mathbf{v}^c \right) \times \Omega \right) \right] \cdot \mathbf{e}_B \\
&\quad + \left( \mathbf{U}_A \times \nabla \right)^2 \delta_B, \\
\end{align*}
\]

where \( \tilde{\gamma} = \delta^w \cdot \mathbf{e}_B \) is the tilt, \( \delta^w = \nabla \times \xi \) and \( \mathbf{e}_B = \mathbf{B}/|\mathbf{B}|. \) We seek for the solution of equation (A1) in the form of standing Alfvén waves as

\[
\tilde{\gamma} = \sum_{m=0}^{\infty} A_m \cos \left( \frac{2(m+1)\pi \xi}{L_B} \right) \cos \left( \frac{2\pi t}{T_m} + \varphi \right),
\]

(A2)

where \( T_m = 2\tau_A/(2m + 1) \) is the period of non-dissipating oscillations, \( \tau_A = L_B/\mathbf{U}_A \) is the Alfvén time, \( \xi \) is the coordinate along the magnetic field line of the length \( L_B. \) Now we take into account that \( T_m \Omega \ll 1, \) which implies that \( |\partial \xi/\partial t| \ll |\mathbf{v}^c|/|\mathbf{U}|. \) We also assume that the source of the tilt \( I_s = 2 \nabla \times \left[ \left( \mathbf{U} + \mathbf{v}^c \right) \times \Omega \right] \cdot \mathbf{e}_B \) in equation (A1) is localized near the boundary between the solar convective zone and the photosphere. This source can be modelled as the combination of two Dirac delta-functions:

\[
I_s(\xi) = 2 \left[ \nabla \times \left[ \left( \mathbf{U} + \mathbf{v}^c \right) \times \Omega \right] \right] \cdot \mathbf{e}_B \\
\times \left[ \delta(\xi/L_B) - \delta(\xi/L_B - 1) \right],
\]

(A3)

where \( \delta(x) \) is the Dirac delta-function.

We substitute equation (A2) into equation (A1) and after the Fourier transformation of the source term (A3), we obtain equation for the amplitude \( A_m(t) \) as

\[
\frac{\partial^2 A_m}{\partial t^2} = \frac{2I_s}{\pi} - \left( \mathbf{U}_A \times \frac{2(m+1)\pi \xi}{L_B} \right)^2 A_m.
\]

(A4)

This equation with initial condition \( A_m(t=0) = 0 \) has the following solution:

\[
A_m(t) = \frac{2I_s \tau_A^2}{\pi^2(2m+1)^2} \left\{ 1 - \cos \left( \frac{(2m+1)\pi t}{\tau_A} \right) \right\},
\]

(A5)

which yields the expression for \( \tilde{\gamma} \) as

\[
\tilde{\gamma} = \frac{2I_s \tau_A^2}{\pi^2} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \cos \left( \frac{(2m+1)\pi \xi}{L_B} \right) \left\{ 1 - \cos \left( \frac{(2m+1)\pi t}{\tau_A} \right) \right\}.
\]

(A6)

Averaging equation (A6) over the time that is larger than the maximum Alfvén time \( \tau_A, \) we obtain equation (6) for the mean tilt \( \gamma = \langle \tilde{\gamma} \rangle_{t=0} \) of sunspot bipolar regions at the surface of the sun.

APPENDIX B: EQUATION FOR THE RADIAL MEAN VELOCITY

The momentum equation (11) with additional force caused by the eddy viscosity in a steady state in spherical coordinates reads:

\[
\frac{\partial \mathbf{P}_{\text{tot}}}{\partial t} + \frac{\partial}{\partial r} \left( r^2 \mathbf{P}_{\text{tot}} \right) = \frac{\partial}{\partial r} \left( \frac{2\rho_0 \mathbf{U}_r \mathbf{v}_r}{H_r} \right) - \frac{B_r^2}{4\pi} + 2\rho_0 \mathbf{U}_r \Omega \sin \theta \\
+ \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \rho_0 \mathbf{U}_\theta \mathbf{v}_\theta \right),
\]

(B1)

\[
\frac{\partial \mathbf{P}_{\text{tot}}}{\partial \theta} = \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{3\rho_0 \mathbf{U}_\theta \mathbf{v}_r}{H_r} \right) - \frac{B_r^2}{4\pi} \cot \theta \\
+ 2\rho_0 \mathbf{U}_r \Omega r \cos \theta,
\]

(B2)

Here \( \mathbf{P}_{\text{tot}} = \mathbf{P} + \mathbf{B}^2/8\pi + \mathbf{v}^2/2 \) is the total pressure, \( H_r \) is the density height scale, and \( \mathbf{v}_r \) is the eddy viscosity.

We exclude the total pressure term, use the continuity equation \( \nabla \cdot (\rho_0 \mathbf{U}) = 0, \) and introduce the stream function \( \psi: \)

\[
\rho_0 \nabla^2 \mathbf{U} = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}, \quad \rho_0 \mathbf{U}_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \theta}.
\]

(B4)

After neglecting a week dependence of \( \mathbf{v}_r/H_r \) on radius \( r, \) equations (B1)–(B3) are reduced to

\[
\frac{\partial^2 \mathbf{Y}}{\partial X^2} + \frac{\partial}{\partial X} \left( \frac{1}{Y} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (Y \sin \theta) \right) \right) = f(X, \theta),
\]

(B5)

where \( X = r^3, \quad Y = X \rho_0 \mathbf{U}_r \mathbf{v}_r/H_r \), and

\[
f(X, \theta) = \frac{1}{3\pi} \left( \frac{1}{X} \frac{\partial}{\partial \theta} - \frac{3}{\tan \theta} \frac{\partial}{\partial X} \right) \mathbf{B}^2/4\pi.
\]

(B6)

Here we take into account that the contribution of the Coriolis force into the function \( f(X, \theta) \) under condition of the slow rotation is small [Kleedorin & Zuzmakina 1991, Kleedorin et al. 1996]. The solution of equation (B5) with the boundary condition

\[
\left[ 1 - \kappa \frac{\partial (\rho_0 \mathbf{U}_r)}{\partial r} \right]_{r=R_\odot} = 0,
\]

(B7)
is given by
\[ U_r = \frac{\ell_0^2}{4 \pi k_F \rho_{\text{op}} R_\odot} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left[ \sin \theta F(\theta) \right]. \tag{B8} \]
where the parameter \( \kappa \approx 0.3 - 0.4 \) characterises a fraction of the large-scale radial momentum of plasma which is lost during crossing the boundary between the convective zone and photosphere, and
\[ F(\theta) \approx \int_{R_{\odot} - L}^{R_\odot} \left( 1 + \frac{R_{\odot} - r}{L - \ell_0} \right) \left( \frac{\partial B}{\partial \theta} \right) \frac{dr}{r} \approx C_u \left( \frac{\partial B}{\partial \theta} \right) \text{bot}, \tag{B9} \]
where the constant \( C_u \) varies from 0.7 to 1 depending on the radial profile of the mean magnetic field. Therefore, equations \([35]-[39]\) yield equation \([3] \).

**APPENDIX C: IDENTITIES USED FOR THE DERIVATION OF EQUATION \([12]\)**

For the derivation of equation \([12]\) we used identities given below:

\[ \sin^3 \phi = \frac{1}{4} [3 \sin \phi - \sin 3\phi], \tag{C1} \]
\[ \sin^5 \phi = \frac{1}{16} [10 \sin \phi - 5 \sin 3\phi + \sin 5\phi], \tag{C2} \]
\[ \sin^2 \phi \cos \phi = \frac{1}{4} [\cos \phi - \cos 3\phi], \tag{C3} \]
\[ \sin^4 \phi \cos \phi = \frac{1}{16} [2 \cos \phi - 3 \cos 3\phi + \cos 5\phi]. \tag{C4} \]

**APPENDIX D: THE EVOLUTION OF THE WOLF NUMBER**

In the framework of the nonlinear mean-field dynamo model by Klecorin et al. (2016) and Safiullin et al. (2018), the phenomenological budget equation for the surface density of the Wolf number is given by
\[ \frac{\partial \overline{W}}{\partial t} = I_w(t, \theta) - \frac{\overline{W}}{\tau_s(\overline{B})}, \tag{D1} \]
where the rate of production of the surface density of the Wolf number caused by the formation of sunspots is
\[ I_w(t, \theta) = \frac{\gamma_{\text{inst}} \left| \overline{B} - \overline{B}_{\text{cr}} \right|}{\Phi_s} \Theta(\overline{B} - \overline{B}_{\text{cr}}), \tag{D2} \]
and the rate of decay of sunspots is \( \dot{\overline{W}}/\tau_s(\overline{B}) \) with the decay time, \( \tau_s(\overline{B}) \), of sunspots and \( \Theta(x) \) is the \( \Theta \) function, defined as \( \Theta(x) = 1 \) for \( x > 0 \), and \( \Theta(x) = 0 \) for \( x \leq 0 \). Here \( \overline{B}_{\text{cr}} \) is the threshold for the sunspot formation (see below). The phenomenological budget equation \([D1]\) is derived on the idea of negative effective magnetic pressure instability \([Klecorin et al. 1989, 1990, 1993, 1994; Klecorin & Rogachevskii 1994]\) resulting in formation of magnetic spots \([Brandenburg et al. 2011, 2013, 2014; Käpylä et al. 2012, 2016]\) and bipolar active regions

\[ \text{The mean tilt of sunspot bipolar regions} \]

\[ \text{[Warnecke et al. 2014, 2016].} \]

The growth rate \( \gamma_{\text{inst}} \) of the negative effective magnetic pressure instability is given by
\[ \gamma_{\text{inst}} = \left( \frac{2 \ell_0^2 k_F^2}{H_2^2 k^2} \frac{dP_{\text{eff}}}{d\beta^2} - 4 \left( \frac{\Omega}{k} \right)^2 \right)^{1/2} - \eta_r \left( k^2 + \frac{1}{(2H_2)^2} \right), \tag{D3} \]
where \( k \) is the wave number, \( P_{\text{eff}} = \frac{1}{4} [1 - q_0(\beta)] \beta^2 \) is the effective magnetic pressure, the nonlinear function \( q_0(\beta) \) is the turbulence contribution to the mean magnetic pressure and \( \beta = \overline{B}/\overline{B}_{\text{eq}} \). We assume here that the characteristic time of the Wolf number variations is of the order of the characteristic time for excitation of the instability, \( \gamma_{\text{inst}}^{-1} \).

When the instability is not excited (\( \gamma_{\text{inst}} < 0 \)), the production rate of sunspots, \( I_w(t, \theta) \to 0 \), which means that the function \( I_w(t, \theta) \propto |\gamma_{\text{inst}}| \Theta(\overline{B} - \overline{B}_{\text{cr}}) \). The production term of sunspots is also proportional to the maximum number of sunspots per unit area, which is estimated as \( \sim (\overline{B} - \overline{B}_{\text{cr}})/\Phi_s \), where \( (\overline{B} - \overline{B}_{\text{cr}}) \) is the magnetic flux per unit area that contributes to the sunspot formation and \( \Phi_s \) is the magnetic flux inside a magnetic spot. This instability is excited when the mean magnetic field is larger than a critical value, \( \overline{B} > \overline{B}_{\text{crit}} \):

\[ \overline{B}_{\text{crit}} = \frac{\ell_0}{50 H_2} \left[ 1 + \left( \frac{10 \Omega H_2^2}{\ell_0^2} \right) \right]^{1/2}, \tag{D4} \]

This instability is excited in the upper part of the convective zone, where the Coriolis number \( \text{Co} = 2\Omega \tau \) is small. The decay time \( \tau_s(\overline{B}) \) varies from several weeks to a couple of month, while the solar cycle period is about 11 years. This allows us use the steady-state solution of Eq. \([D1]\), \( \dot{\overline{W}} = \tau_s(\overline{B}) I_w(t, \theta) \). The Wolf number is defined as a surface integral as \( \overline{W} = \overline{R}_2^2 \int \overline{W}(t, \theta) \sin \theta \, d\theta \, d\varphi = 2\pi \overline{R}_2^2 \int \tau_s(\overline{B}) I(t, \theta) \sin \theta \, d\theta \, d\varphi \). The function \( \tau_s(\overline{B}) \) is given by \( \tau_s(\overline{B}) = \tau_s \exp \left( C_s \frac{\partial \overline{B}}{\partial t} \right) \), where \( C_s = 1.8 \times 10^{-3} \) and \( \gamma_{\text{inst}} \tau_s \sim 1 \).