UNIFORM COMPLEX TIME HEAT KERNEL ESTIMATES
WITHOUT GAUSSIAN BOUNDS

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Abstract. The aim of the paper is twofold. First we study the uniform complex time heat kernel estimates of $e^{-z(-\Delta)^{\frac{\alpha}{2}}}$ for $\alpha > 0, z \in \mathbb{C}^+$. When $\frac{\alpha}{2}$ is not an integer, generally the heat kernel doest not have the Gaussian upper bounds for real time. Thus the Phragmén-Lindelöf methods (for example [6]) fail to give the uniform complex time heat kernel estimates. Instead, we overcome this difficulty by giving the asymptotic estimates for $P(z, x)$ as $z$ tending to the imaginary axis. Then we prove the uniform complex time heat kernel estimates. Secondly, we study the uniform complex time estimates of the analytic semigroup generated by $H = (-\Delta)^{\frac{\alpha}{2}} + V$ where $V$ belongs to higher order Kato class.

1. Introduction

Let $e^{-z(-\Delta)^{\frac{\alpha}{2}}}$ be the analytic semigroup generated by $(-\Delta)^{\frac{\alpha}{2}}$ where $\Delta$ is the Laplace operator on $\mathbb{R}^n$ and $\alpha > 0, z \in \mathbb{C}^+$ with $\mathbb{C}^+ = \{ z \in \mathbb{C} | \Re z > 0 \}$. Denote by $P(z, \cdot)$ the convolution kernel of $e^{-z(-\Delta)^{\frac{\alpha}{2}}}$ on $L^2(\mathbb{R}^n)$. In fact, by the Fourier transform we have

$$P(z, x) = c_n \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{-z|\xi|^\alpha} d\xi, \quad \forall x \in \mathbb{R}^n, z \in \mathbb{C}^+,$$

where $c_n$ is a constant determined by the dimension. Recently, the fractional Laplace operator has been extensively studied due to its wide applications in nonlinear optics, plasma physics and other areas. See for example [6,11,12,15,17,18,20] and references therein.

In this paper, first we focus on the uniform estimates of the heat kernel $P(z, x)$ for $z \in \mathbb{C}^+$. Now we recall some known facts about the heat kernel.

When $z = \Re z$ is real, the estimates for $P(\Re z, x)$ are well known. Indeed, there exists constant $C > 0$ such that (1.1) \cite{1,14,19,22}

$$|P(\Re z, x)| \leq C \Re z^{-\frac{\alpha}{2}} \wedge \frac{\Re z}{|x|^{n+\alpha}}, \quad \forall \alpha, z = \Re z > 0, x \in \mathbb{R}^n.$$  

Throughout this paper, for two functions $f, g$, set $f \wedge g = \min \{ f, g \}$.

Moreover, when $\alpha$ are even numbers, the upper bounds can be improved into the sub-Gaussian type upper bounds in the following sense,

$$|P(\Re z, x)| \leq C_1 \Re z^{-\frac{\alpha}{2}} \exp \left\{ -C_2 \frac{|x|^{\frac{n}{\alpha}}}{\Re z^{\frac{1}{\alpha}}} \right\}, \quad \forall z = \Re z > 0, x \in \mathbb{R}^n,$$  

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for some positive constants $C_1, C_2 > 0$. See, for example [2][10][17].

One important way to deduce the uniform complex time heat kernel estimates from the real time heat kernel estimates is by the Phragmén-Lindelöf theorems. Davies in [8] introduced this method to obtain the uniform complex time heat kernel estimates from the Gaussian upper bounds for the real time. Further, Carron et al in [4] proved the uniform complex time estimates for heat kernels satisfying the sub-Gaussian upper bounds for real time. In particular, by [4, Proposition 4.1] and (1.3), there exist positive constants $C_1, C_2 > 0$ such that

$$\tag{1.4} |P(z)| \leq C_1 \Re z^{-\frac{\alpha}{2}} \exp \left\{ -C_2 \frac{|x|^2}{|z|^n} \cos \theta \right\}, \quad \forall z \in \mathbb{C}^+, x \in \mathbb{R}^n,$$

where $\theta = \arg z$ and $\alpha$ are even numbers. For more results concerning the Phragmén-Lindelöf methods and their applications, we refer the readers to [3][8][9][13][24] and references therein.

However, when $\alpha > 0$ is not an even number, the sub-Gaussian estimates do not hold in general and hence the Phragmén-Lindelöf methods in [5][8] fail to give the uniform complex time heat kernel estimates. On the other hand, by simple calculations, there exists $C > 0$ such that

$$\tag{1.5} |P(z, x)| \leq c_n \int_{\mathbb{R}^n} e^{-|\Re z|^\alpha} d\xi \leq C \Re z^{-\frac{\alpha}{2}}, \quad \forall \alpha > 0, z \in \mathbb{C}^+, x \in \mathbb{R}^n.$$

To the best of our knowledge, we can not find the uniform complex time estimates in the literature for general $\alpha > 0$ except the trivial estimates (1.5), even though the estimates for $P(z, x)$ are well known when $z$ are real numbers or pure imaginary numbers.

To get the desired results without the Gaussian upper bounds, we investigate the asymptotic behavior of $P(z, x)$ as $|x| \to 0$ and $|x| \to \infty$ uniformly for $z$ satisfying $0 < \omega \leq |\arg z| < \frac{\pi}{2}$. Then our first results are as follows.

**Theorem 1.1.** Let $\alpha > 0$ and $P(z, x)$ be defined by (1.1).

(1) When $0 < \alpha < 1$, there exist constants $C_1, C_2 > 0$ such that for all $z \in \mathbb{C}^+, x \in \mathbb{R}^n$, we have,

$$\tag{1.6} |P(z, x)| \leq C_1 \left|z\right|^{-\frac{\alpha}{2}} + \left|z\right| \frac{2(1-\alpha)}{\left|z\right|^n} \exp \left(-C_2 \frac{|x|^2}{\left|z\right|^n} \cos \theta \right) \wedge \frac{|z|}{\left|z\right|^n},$$

where $\theta = \arg z$.

(2) When $\alpha > 1$, there exist constants $C'_1, C'_2 > 0$ such that for all $z \in \mathbb{C}^+, x \in \mathbb{R}^n$, we have,

$$\tag{1.7} |P(z, x)| \leq C'_1 \left|z\right|^{-\frac{\alpha}{2}} \wedge \left[ \frac{|z|}{\left|z\right|^n} \left(\frac{2(1-\alpha)}{\left|z\right|^n} \exp \left(-C'_2 \frac{|x|^2}{\left|z\right|^n} \cos \theta \right) \right) \right]$$

where $\theta = \arg z$.

**Remark 1.2.** Compared with (1.4), (1.5), the estimates are new. When $\alpha$ is an even number, the right hand side of (1.4) tends to infinity as $|\theta| \to \frac{\pi}{2}$. However, according to [24, Proposition 5.1], the upper bounds in (1.7) stay true even for $|\theta| = \frac{\pi}{2}$. Moreover, when $\theta = 0$, the estimates (1.6), (1.7) correspond with (1.2).
Next we consider the heat kernel of $e^{-z((−\Delta)^{\frac{\alpha}{2}} + V)}$ with $V$ belonging to the higher order Kato class $K_{\alpha}(\mathbb{R}^n)$. Recall that, for each $\alpha > 0$, a real valued measurable function $V(x)$ on $\mathbb{R}^n$ is said to lie in $K_{\alpha}(\mathbb{R}^n)$ if

$$\lim \sup_{\delta \to 0} \int_{|x-y|<\delta} w_{\alpha}(x-y)|V(y)|dy = 0, \quad \text{for} \ 0 < \alpha \leq n,$$

and

$$\sup_{x \in \mathbb{R}^n} \int_{|x-y|<1} |V(y)|dy < \infty, \quad \text{for} \ \alpha > n,$$

where

$$w_{\alpha}(x) = \begin{cases} |x|^{-\alpha-n}, & \text{if} \ 0 < \alpha < n, \\ |\ln |x||^{-\alpha-n}, & \text{if} \ \alpha = n. \end{cases}$$

Set $I(t, x) = t^{-\frac{n}{2}} \wedge \frac{1}{|x|^{\alpha-n}}$ and denote the integral kernel of $e^{-z((−\Delta)^{\frac{\alpha}{2}} + V)}$ by $K(z, x, y)$. Then our results concerning $K(z, x, y)$ are as follows.

**Theorem 1.3.** Let $\alpha > 0$ and $V \in K_{\alpha}(\mathbb{R}^n)$.

1. When $0 < \alpha < 1$, then for any $0 < \varepsilon \ll 1$, there exists constant $C > 0$ and $\mu_{\varepsilon, V}$ depending on $V, \varepsilon$, such that

$$|K(z, x)| \leq C e^{\mu_{\varepsilon, V}|z|}(\cos \theta)^{-\frac{n}{2}+\frac{\alpha}{2}} I(|z|, x, y), \quad \forall z \in \mathbb{C}^+, x, y \in \mathbb{R}^n. \quad (1.8)$$

2. When $\alpha > 1$, then for any $0 < \varepsilon \ll 1$, there exists constant $C' > 0$ and $\mu_{\varepsilon, V}'$ depending on $V, \varepsilon$, such that

$$|K(z, x)| \leq C' e^{\mu_{\varepsilon, V}'|z|}(\cos \theta)^{-\frac{n}{2}+1} I(|z|, x, y), \quad \forall z \in \mathbb{C}^+, x, y \in \mathbb{R}^n. \quad (1.9)$$

The paper is organized as follows: In section 2, we will show the asymptotic behavior of $P(z, x)$ as $|x| \to 0$ and $|x| \to \infty$ uniformly for $z$ satisfying $0 < \omega \leq |\arg z| < \frac{\pi}{2}$. The proof relies heavily on properties of Bessel functions and we mainly apply the integration by parts as well as stationary phase methods. The calculation however is complicate. Section 3 is devoted to Theorem 1.1, Theorem 1.3. We will apply the heat kernel estimates in Theorem 1.1 and some global characterizations of $K_{\alpha}(\mathbb{R}^n)$ to show Theorem 1.3. In the appendix, we gather some basic properties of Bessel functions.

Note that the constants $\delta, c, C, C_k, C_k'$ for $k \in \mathbb{N}$ may change from line to line.

### 2. Uniform Asymptotic Behavior of $P(z, x)$

In this section, we denote by $e^{i\theta} = z$ for simplicity.

Recall that, when $z = i\Im z$ is pure imaginary number, the estimates for $P(i\Im z, x)$ are well known. As we shall see, the behaviors of $P(i\Im z, x)$ are quite different from that of the real time heat kernel. To state the results, we make some reduction. By scaling property, we obtain

$$P(z, x) = |z|^{-\frac{n}{2}} P(e^{i\theta}, \frac{x}{|z|^{\frac{n}{2}}}) \quad \forall |z| \neq 0, x \in \mathbb{R}^n,$$

where $\theta = \arg z$. Moreover, since $P(e^{-i\theta}, -y) = \overline{P(e^{i\theta}, y)}$, it is sufficient to consider $P(e^{i\theta}, y)$ for $0 \leq \theta < \frac{\pi}{2}, y \in \mathbb{R}^n$. 

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First of all, we have
\[ P(i, y) = c_n \int_{\mathbb{R}^n} e^{iy \xi} e^{-i|\xi|^\alpha} d\xi \]
\[ = c_n \int_{\mathbb{R}^n} e^{iy \xi} e^{-i|\xi|^\alpha} \varphi(|\xi|) d\xi + c_n \int_{\mathbb{R}^n} e^{iy \xi} e^{-i|\xi|^\alpha} (1 - \varphi(|\xi|)) d\xi \]
\[ \triangleq P_1(i, y) + P_2(i, y), \]
where \( \varphi(t) \) is a smooth cutoff function which equals 1 for \( 0 \leq t \leq \frac{1}{2} \) and 0 for \( t \geq 1 \).

For \( P_1(i, y) \), there exists constant \( C > 0 \) such that
\[ |P_1(i, y)| \leq C(1 + |y|^2)^{-\frac{\alpha+n}{2}} \quad \forall \alpha > 0, \ y \in \mathbb{R}^n. \]

Note that the above estimates are essentially known in various literature. For completeness, we still give a proof below. See for example the proof of (2.1).

For \( P_2(i, y) \), the asymptotic behaviors vary according to different \( \alpha \) and \( y \). By Miyachi (21, Proposition 5.1), Wainger (26, p.41-52), \( [16] \), the properties can be summarized as follows:

When \( 0 < \alpha < 1 \), \( P_2(i, y) \) is smooth in \( \mathbb{R}^n \setminus \{0\} \) and for \( N > 0 \)
\[ P_2(i, y) = O(|y|^{-N}) \quad \text{as } |y| \to +\infty. \]

Moreover, we have
\[ P_2(i, y) = C_1 |y|^{-n \frac{\alpha}{\alpha-n}} \exp(C_2 i|y|^{\frac{\alpha}{\alpha-n}}) + o \left( |y|^{-n \frac{\alpha}{\alpha-n}} \right) + E(y), \quad \text{as } y \to 0, \]
where \( C_2 = \alpha \frac{\alpha}{\alpha-n} (\alpha-1) \), \( C_1 \) is a constant determined by \( \alpha, n \) and \( E(y) \) is a smooth function.

When \( \alpha > 1 \), \( P_2(i, y) \) is smooth in \( \mathbb{R}^n \) and we have
\[ P_2(i, y) = C_1' |y|^{-n \frac{\alpha}{\alpha-n}} \exp(C_2' i|y|^{\frac{\alpha}{\alpha-n}}) + o \left( |y|^{-n \frac{\alpha}{\alpha-n}} \right), \quad \text{as } |y| \to +\infty, \]
where \( C_2 = \alpha \frac{\alpha}{\alpha-n} (\alpha-1) \) and \( C_1' \) is a constant determined by \( \alpha, n \). Then the asymptotic behaviors of \( P(i, y) \) will totally be determined by \( P_1(i, y), P_2(i, y) \).

As we have seen, the asymptotic behaviors of \( P(e^{i\theta} \cdot y) \) are quite different between \( \theta = 0 \) and \( \theta = \frac{\pi}{2} \) as \( |y| \to 0 \) and \( |y| \to \infty \). Moreover, in the case that \( \frac{\pi}{2} \) is integer, the upper bounds in \([13]\) tends to infinity as \( \theta \to \frac{\pi}{2} \). They fail to give the upper bounds for \( P(i, y) \). Thus it is natural to ask the question that how \( P(e^{i\theta} \cdot y) \) changes as \( \theta \to \frac{\pi}{2} \).

Now we give an example which is heuristic for our problems. Consider
\[ I(z) = \int_0^1 e^{-zt} t^{m-1} dt, \]
where \( z \in \mathbb{C}^+ \) and \( m \geq 2 \) is an integer. Integration by parts gives
\[ I(z) = (m-1)! \left( z^{-m} - e^{-z} \sum_{k=0}^{m-1} \frac{z^{k-m}}{k!} \right) = (m-1)! (z^{-m} - e^{-z} z^{-1}) + E(z). \]
It follows that \( I(z) \) has different behaviors between \( z = is \) and \( z = s \) as \( s \to +\infty \). However, the main contribution to \( I(z, m) \) as \( |z| \to +\infty \) is \( z^{-m} - e^{-z} z^{-1} \) uniformly for \( \Re z \geq 0 \).
As we have shown in the example, to determine the asymptotic behavior of \( P(e^{i\theta}, y) \) uniformly for \( 0 < \omega \leq |\theta| < \frac{\pi}{2} \), we need to find a balance between the two cases: \( \theta = 0, \theta = \frac{\pi}{2} \). Set

\[
P(e^{i\theta}, y) = c_n \int_{\mathbb{R}^n} e^{iy \cdot \xi} e^{-e^{i\theta}||\xi||^a} d\xi
\]

\[
= c_n \int_{\mathbb{R}^n} e^{iy \cdot \xi} e^{-e^{i\theta}||\xi||^a} \varphi(|\xi|) d\xi + c_n \int_{\mathbb{R}^n} e^{iy \cdot \xi} e^{-e^{i\theta}||\xi||^a} (1 - \varphi(|\xi|)) d\xi
\]

\[
\triangleq P_1(e^{i\theta}, y) + P_2(e^{i\theta}, y),
\]

where \( \varphi(t) \) is a smooth cutoff function which equals 1 for \( 0 \leq t \leq \frac{1}{2} \) and 0 for \( t \geq 1 \).

Then our first result is as follows.

**Proposition 2.1.** Let \( \alpha > 0 \), and \( P_1(e^{i\theta}, y), P_2(e^{i\theta}, y) \) be defined as above. Then the following hold:

1. There exists positive constant \( C > 0 \) such that

\[
|P_1(e^{i\theta}, y)| \leq C(1 + |y|^2)^{-\frac{n-\alpha}{2}}, \quad \forall \alpha > 0, |\theta| \leq \frac{\pi}{2}, y \in \mathbb{R}^n.
\]

2. When \( 0 < \alpha < 1 \), for \( 0 < \omega \leq |\theta| < \frac{\pi}{2} \), there exists constant \( C_{\theta,1} \) such that

\[
P_2(e^{i\theta}, y) = C_{\theta,1}|y|^{1-\frac{\alpha}{n}} \exp(-s_0 \phi |y|^{\frac{\alpha}{n}} + s_0|y|^{\frac{n-\alpha}{n}}) + E_1(y) + E_2(y),
\]

where \( 0 < \omega < \frac{\pi}{2} \) is a fixed number and \( s_0 = (\alpha \sin \theta)^{\frac{1}{1-n}} \). Furthermore, the following holds uniformly for \( 0 < \omega \leq |\theta| < \frac{\pi}{2} \),

\[
|C_{\theta,1}| \leq C_1;
\]

\[
|E_1(y)| \leq C_2, \quad \forall |y| \leq 1;
\]

\[
|E_2(y)| \leq \begin{cases} C_3|y|^{1-\frac{\alpha}{n}} + \frac{C_4}{n+1} |y|^{\frac{n-\alpha}{n}} e^{-C_4 \cos \theta |y|^{1-\alpha}} & n \geq 2, \quad \forall |y| \leq 1, \\ C_5 \ln |y||y|^{1-\frac{\alpha}{n}} + \frac{C_6}{n+1} |y|^{\frac{n-\alpha}{n}} e^{-C_6 \cos \theta |y|^{1-\alpha}} & n = 1, \end{cases}
\]

where \( C_i > 0 \) for \( 1 \leq i \leq 6 \) are constants determined only by \( \alpha, n, \omega \).

Moreover, for \( N > 0 \) there exists constant \( C_N > 0 \) such that

\[
|P_2(e^{i\theta}, y)| \leq C_N|y|^{-N}, \quad \forall |\theta| \leq \frac{\pi}{2}, |y| \geq 1.
\]

3. When \( \alpha > 1 \), for \( 0 < \omega \leq |\theta| < \frac{\pi}{2} \), there exists constant \( C'_{\theta,1} \) such that

\[
P_2(e^{i\theta}, y) = C'_{\theta,1}|y|^{1-\frac{\alpha}{n}} \exp(-s_0' \phi |y|^{\frac{\alpha}{n}} + s_0'|y|^{\frac{n-\alpha}{n}}) + E'_1(y) + E'_2(y),
\]

where \( 0 < \omega < \frac{\pi}{2} \) is a fixed number. Furthermore, the following holds uniformly for \( 0 < \omega \leq |\theta| < \frac{\pi}{2} \),

\[
|C'_{\theta,1}| \leq C'_1;
\]

\[
|E'_1(y)| \leq C'_2|y|^{-n-\alpha}, \quad \forall |y| \geq 1;
\]

\[
|E'_2(y)| \leq C'_3|y|^{\frac{n-\alpha}{n+1}} + \frac{C'_4}{n+1} |y|^{\frac{n-\alpha}{n}} e^{-C'_4 \cos \theta |y|^{1-\alpha}} \exp(-C'_4 \cos \theta |y|^{1-\alpha}), \quad \forall |y| \geq 1,
\]

where \( C'_i > 0 \) for \( 1 \leq i \leq 4 \) are constants determined only by \( \alpha, n, \omega \).

Moreover, there exists constant \( C > 0 \) such that the following holds uniformly for \( |\theta| \leq \frac{\pi}{2} \)

\[
|P_2(e^{i\theta}, y)| \leq C, \quad \forall |\theta| \leq \frac{\pi}{2}, |y| \leq 1.
\]
Thus integration by parts gives
\[ \phi \]

In the last inequality, we have used the facts that
\[ |y|^{-\alpha} \exp(-s_0 \frac{1}{\sqrt{-1}} + s_0 |y|^{\frac{1}{2}}) \]
dominates the asymptotic behaviors of \( P_2(e^{i\theta}, y) \) for \( 0 < \alpha < 1, |y| \to 0 \) and \( \alpha > 1, |y| \to \infty \) respectively. Moreover, letting \( \theta \to \frac{\pi}{2} \) gives
\[ |y|^{-\alpha} \exp(-s_0 \frac{1}{\sqrt{-1}} + s_0 |y|^{\frac{1}{2}}) \to |y|^{-\alpha} \exp(i \alpha \frac{1}{2} |y|^{\frac{1}{2}}). \]

Thus we have regained the asymptotic behaviors of \( P \).

Remark 2.2. (1) Note that the term
\[ C|y|^{-\frac{1}{2} - \alpha} \exp(-s_0 \frac{1}{\sqrt{-1}} + s_0 |y|^{\frac{1}{2}}), \]
dominate the asymptotic behaviors of \( P(n, y, D) = (2.1) \) are well known and can be found in various literature for example [1, 14, 19, 22, 25].

(2) For \( 0 \leq \theta \leq \omega < \frac{\pi}{2} \), the following estimate holds:
\[ |P(e^{i\theta}, y)| \leq C \wedge |y|^{-\alpha}, \quad \forall \alpha > 0, y \in \mathbb{R}^n. \]

With little modification of the proof for (2.1) will show (2.6). Indeed, the estimates (2.1) are well known and can be found in various literature (for example [1, 14, 19, 22, 25]).

2.1. Proof of Proposition 2.1 (1).

Proof of (2.1). Set
\[ L(y, D) = \frac{y \cdot \nabla \xi}{i|y|^2} \quad \text{and} \quad L^*(y, D) = -\frac{y \cdot \nabla \xi}{i|y|^2}. \]

It is direct to check \( L(y, D)e^{iy \cdot \xi} = e^{iy \cdot \xi} \) and \( L^* \) is the conjugate operator to \( L \).

Thus integration by parts gives
\[
P_1(z, y) = c_n \int_{\mathbb{R}^n} e^{iy \cdot \xi} L^*(e^{-z|\xi|^{\alpha}} \varphi(|\xi|)) d\xi \\
= c_n \int_{\mathbb{R}^n} e^{iy \cdot \xi} \varphi \left( \frac{|\xi|}{\delta} \right) L^*(e^{-z|\xi|^{\alpha}} \varphi(|\xi|)) d\xi \\
+ c_n \int_{\mathbb{R}^n} e^{iy \cdot \xi} \left( 1 - \varphi \left( \frac{|\xi|}{\delta} \right) \right) L^*(e^{-z|\xi|^{\alpha}} \varphi(|\xi|)) d\xi \\
= I + II,
\]

where \( \varphi \) is smooth cutoff and \( \delta > 0 \) will be determined later.

For \( I \), we obtain
\[
|I| \leq C \int_{|\xi| \leq \delta} |L^*(e^{-z|\xi|^{\alpha}} \varphi(|\xi|))| d\xi \\
\leq \frac{C}{|y|} \int_{|\xi| \leq \delta} |\xi|^{\alpha-1} |\varphi(|\xi|)| + |\varphi'(|\xi|)| d\xi \\
\leq \frac{C}{|y|} \int_{|\xi| \leq \delta} |\xi|^{\alpha-1} d\xi = C|y|^{-1}\delta^{\alpha+n-1}.
\]

In the last inequality, we have used the facts that \( \varphi'(|\xi|) \) is supported in \( \frac{1}{2} \leq |\xi| \leq 1 \) and hence \( |\varphi'(|\xi|)| \leq C|\xi|^{\alpha-1} \) for some constant \( C > 0 \).
Letting

\[ \text{Proof of Proposition 2.1 (2).} \]

2.2. \[ \omega \]

It is clear that

\[ \text{Proof of (2.2)} \]

Thus it is sufficient to consider

\[ \text{Since } \varphi(|\xi|) \text{ is supported in } |\xi| \leq 1 \text{ and } \varphi'(|\xi|) \text{ is supported in } \frac{1}{2} \leq |\xi| \leq 1, \text{ then} \]

\[ \frac{1}{2}N |\xi|^{1-n} |y|^{1-n} \]

Therefore

\[ |II| \leq C ||y||^{1-n} |\delta|^{1-n} |N|^{1-n} |y|^{1-n} \]

Combing the estimates of I and II gives

\[ |P_1(z, y)| \leq C(||y||^{1-n} |\delta|^{1-n} |N|^{1-n} |y|^{1-n}) \]

Letting \( \delta = |\xi|^{1-n} \) implies

\[ |P_1(z, y)| \leq C |y|^{-n-a}. \]

As a result, (2.1) follows since \( P_1(z, y) = P(z, -y) \).

2.2. Proof of Proposition 2.1 (2). Note that it is sufficient to consider \( 0 < \omega \leq \theta < \frac{\pi}{2} \), since \( P(z, y) = P(z, -y) \).

Proof of (2.2). Since \( \varphi(|\xi|) \) is supported in \( |\xi| \leq 1 \), we have

(2.7) \[ P_2(z, y) = c_n \int_{|\xi| \leq 1} e^{iy \cdot \xi} e^{-|\xi|^n} (1 - \varphi(|\xi|)) d\xi + c_n \int_{|\xi| \geq 1} e^{iy \cdot \xi} e^{-|\xi|^n} d\xi. \]

It is clear that

\[ \int_{|\xi| \leq 1} e^{iy \cdot \xi} e^{-|\xi|^n} (1 - \varphi(|\xi|)) d\xi \leq C, \quad \forall y \in \mathbb{R}^n, \omega \leq \theta < \frac{\pi}{2}. \]

Thus it is sufficient to consider

\[ \int_{|\xi| \geq 1} e^{iy \cdot \xi} e^{-|\xi|^n} d\xi \]

\[ = C \int_1^{+\infty} e^{-2r^n} r^{n-1} (r|y|) \frac{2r^n}{2r^n - 1} J_{n-1}(r|y|) dr \]

\[ = C' |y|^{1-n} \int_1^{+\infty} e^{-z|y|^n} s^{n-1} s J_{n-1}(s|y|) ds \]

\[ = C' |y|^{1-n} A \int_0^{+\infty} e^{-zA^n} s^{n-1} J_{n-1}(sA) ds, \]

where \( A = |y|^{\frac{1}{n-1}} \) and we have changed the variable \( r = |y|^{\frac{1}{n-1}} s \) in the second equality. Note that \( A \to +\infty \) as \( |y| \to 0 \) when \( 0 < \alpha < 1 \).
Then we have

\begin{equation}
\int_{|\xi| \geq 1} e^{i y \xi} e^{-z|\xi|^\alpha} d\xi \triangleq C |y|^{-\frac{1-\alpha}{\alpha}} A I,
\end{equation}

where

\[ I = \int_{A^{-1}}^{+\infty} e^{-zA^\alpha s^\alpha} s^\frac{\alpha}{2} J_{\frac{\alpha}{2}-1}(sA) ds. \]

To prove (2.2), we need further to estimate \( I \).

(2.9) \[ I = \int_{A^{-1}}^{A^{-1}} + \int_{A^{-1}}^{+\infty} = I_1 + I_2. \]

First we have,

\[ I_1 = \int_{A^{-1}}^{A^{-1}} e^{-zA^\alpha s^\alpha} s^\frac{\alpha}{2} J_{\frac{\alpha}{2}-1}(sA) ds \]
\[ = A^{-\frac{\alpha}{2}-1} \int_{A^{-1}}^{1} e^{-zA^\alpha t^\alpha} t^\frac{\alpha}{2} J_{\frac{\alpha}{2}-1}(t) dt \]
\[ = CA^{-\frac{\alpha}{2}-1} \int_{A^{-\alpha-1}}^{1} e^{-zA^\alpha \tau} \tau^\frac{\alpha}{2} + \frac{1}{2} - 1 J_{\frac{\alpha}{2}-1}(\tau^\alpha) d\tau. \]

According to (11), we have

\[ \int_{A^{-\alpha-1}}^{1} e^{-zA^\alpha \tau} \tau^\frac{\alpha}{2} + \frac{1}{2} - 1 J_{\frac{\alpha}{2}-1}(\tau^\alpha) d\tau \]
\[ = \int_{A^{-\alpha-1}}^{1} e^{-zA^\alpha \tau} \sum_{k \geq 0} a_k \tau^{\frac{1}{2}(n-\alpha+2k)} d\tau \]
\[ = a_0 \int_{A^{-\alpha-1}}^{1} e^{-zA^\alpha \tau} \tau^{\frac{1}{2}(n-\alpha)} d\tau + \sum_{k \geq 1} a_k \int_{A^{-\alpha-1}}^{1} e^{-zA^\alpha \tau} \tau^{\frac{1}{2}(n-\alpha+2k)} d\tau, \]

where \( a_k = \frac{(-1)^k z^{1-2k} A^{\alpha}}{2\pi(k+1)^\alpha}. \)

Note that

\[ \int_{A^{-\alpha-1}}^{1} e^{-zA^\alpha \tau} \tau^{\frac{1}{2}(n-\alpha)} d\tau = \int_{0}^{+\infty} - \int_{0}^{A^{-\alpha-1}} - \int_{1}^{+\infty}. \]

It is clear that

\[ \int_{0}^{+\infty} e^{-zA^\alpha \tau} \tau^{\frac{1}{2}(n-\alpha)} d\tau = \Gamma\left(\frac{n}{\alpha}\right)(z \tau^{-\alpha})^{-\frac{1}{2}} = \Gamma\left(\frac{n}{\alpha}\right) z^{-\frac{1}{2}} A^{\frac{\alpha}{2}(n-\alpha)}, \]

and

\[ \left| \int_{0}^{A^{-\alpha-1}} e^{-zA^\alpha \tau} \tau^{\frac{1}{2}(n-\alpha)} d\tau \right| \leq CA^{\alpha-1} A^{(\alpha-1)\frac{\alpha}{2}(n-\alpha)} = CA^{\frac{\alpha}{2}(n-\alpha)}. \]

Moreover, integration by parts gives

\[ \int_{1}^{+\infty} e^{-zA^\alpha \tau} \tau^{\frac{1}{2}(n-\alpha)} d\tau \]
\[ = \sum_{k=1}^{\left[\frac{n}{\alpha}\right]+1} c_k \frac{e^{-zA^\alpha}}{(zA^{-\alpha})^k} + \frac{c_{\left[\frac{n}{\alpha}\right]+1}}{(zA^{-\alpha})^\left[\frac{n}{\alpha}\right]+1} \int_{1}^{+\infty} e^{-zA^\alpha \tau} \tau^{\frac{1}{2}(n-\alpha)-\left[\frac{n}{\alpha}\right]-2} d\tau. \]
Combing these estimates gives
\[ (2.10) \]
\[
\int_{A^{n-1}}^1 e^{-z A^{1-\alpha} \tau} \frac{1}{\tau^{\frac{n}{\alpha}}(n-\alpha)} d\tau = \Gamma\left(\frac{n}{\alpha}\right) z^{-\frac{n}{\alpha}} A^{\frac{n}{\alpha}(\alpha-1)} + \sum_{k=1}^{[\frac{n}{\alpha}]+1} c_k e^{-z A^{1-\alpha}} \left(\frac{1}{zA^{1-\alpha}}\right)^k + H_1(A),
\]
where \( H_1(A) \) satisfies
\[
|H_1(A)| \leq CA^{\frac{n}{\alpha}(\alpha-1)}, \quad \forall A \geq 1,
\]
for some positive constant \( C > 0 \).

On the other hand, integration by parts gives
\[
\int_{A^{n-1}}^1 e^{-z A^{1-\alpha} \tau} \frac{1}{\tau^{\frac{n}{\alpha}}(n-\alpha+2k)} d\tau = \frac{e^{-z}}{z A^{(\alpha-1)}(n+2k)} - \frac{e^{-z A^{1-\alpha}}}{z A^{1-\alpha}} + \frac{n-\alpha+2k}{A z A^{1-\alpha}} \int_{A^{n-1}}^1 e^{-z A^{1-\alpha} \tau} \frac{1}{\tau^{\frac{n}{\alpha}}(n-\alpha+2k)-1} d\tau.
\]

After \( [\frac{n}{\alpha}] + 1 \) steps of integrating by parts, we obtain
\[
\int_{A^{n-1}}^1 e^{-z A^{1-\alpha} \tau} \frac{1}{\tau^{\frac{n}{\alpha}}(n-\alpha+2k)} d\tau = A^{(\alpha-1)} \Phi(n+2k) e^{-z} \sum_{l=1}^{[\frac{n}{\alpha}]+1} c_l z^{-l} + e^{-z A^{1-\alpha}} \sum_{l=1}^{[\frac{n}{\alpha}]+1} c_l' (z A^{1-\alpha})^{-l}
\]
\[
+ \frac{c^{[\frac{n}{\alpha}]+1}}{(z A^{1-\alpha})^{\frac{n}{\alpha}+1}} \sum_{k \geq 1} a_k \int_{A^{n-1}}^1 e^{-z A^{1-\alpha} \tau} \frac{1}{\tau^{\frac{n}{\alpha}}(n-\alpha+2k)-[\frac{n}{\alpha}]-1} d\tau.
\]

It follows that
\[
\sum_{k \geq 1} a_k \int_{A^{n-1}}^1 e^{-z A^{1-\alpha} \tau} \frac{1}{\tau^{\frac{n}{\alpha}}(n-\alpha+2k)} d\tau
\]
\[
= e^{-z} \sum_{l=1}^{[\frac{n}{\alpha}]+1} c_l z^{-l} \sum_{k \geq 1} a_k A^{(\alpha-1)}(n+2k) + e^{-z A^{1-\alpha}} \sum_{l=1}^{[\frac{n}{\alpha}]+1} c_l' (z A^{1-\alpha})^{-l} \sum_{k \geq 1} a_k
\]
\[
+ \frac{c^{[\frac{n}{\alpha}]+1}}{(z A^{1-\alpha})^{\frac{n}{\alpha}+1}} \sum_{k \geq 1} a_k \int_{A^{n-1}}^1 e^{-z A^{1-\alpha} \tau} \frac{1}{\tau^{\frac{n}{\alpha}}(n-\alpha+2k)-[\frac{n}{\alpha}]-1} d\tau.
\]

Furthermore there exists \( C > 0 \) determined by \( \alpha, n \) such that
\[
\left| \int_{A^{n-1}}^1 e^{-z A^{1-\alpha} \tau} \frac{1}{\tau^{\frac{n}{\alpha}}(n-\alpha+2k)-[\frac{n}{\alpha}]-1} d\tau \right| \leq C, \quad \forall k \geq 1, \ A \geq 1.
\]

Since \( \sum_{k \geq 1} a_k < \infty \), we conclude
\[ (2.11) \]
\[
\sum_{k \geq 1} a_k \int_{A^{n-1}}^1 e^{-z A^{1-\alpha} \tau} \frac{1}{\tau^{\frac{n}{\alpha}}(n-\alpha+2k)} d\tau
\]
\[
= e^{-z} \sum_{l=1}^{[\frac{n}{\alpha}]+1} c_l z^{-l} \sum_{k \geq 1} a_k A^{(\alpha-1)}(n+2k) + e^{-z A^{1-\alpha}} \sum_{l=1}^{[\frac{n}{\alpha}]+1} c_l (z A^{1-\alpha})^{-l} + H_2(A),
\]
where $H_2(A)$ satisfies
\[ |H_2(A)| \leq CA^{\frac{\alpha}{n-1}}, \hspace{1cm} \forall A \geq 1. \]
Thus (2.10) and (2.11) imply
\[
I_1 = Cz^{-\frac{2}{n}}A^{-\frac{2}{n}-1} + B_1(z)A^{\frac{2}{n}-\frac{3}{n}-1} \sum_{k \geq 1} a_k A^{(\alpha-1)\frac{\Delta}{A}} \\
+ e^{-zA^{1-\alpha}} A^{-\frac{2}{n}-1} \sum_{k=1}^{[\frac{n}{2}]+1} c_k (zA^{1-\alpha})^{-l} + A^{-\frac{2}{n}-1}(H_1(A) + H_2(A)),
\]

where
\[
B_1(z) = e^{-2} \sum_{l=1}^{\infty} c_l z^{-l},
\]
as in (2.11).

By the definition, $A = |y|^\frac{n-\omega}{n-1}$, we have $|y|^\frac{1-\omega}{n-1} = A^{\frac{n}{n-2}}$ and hence
\[ |y|^{\frac{1-\omega}{n-1}} AI_1 = \tilde{E}_1(y) + \tilde{E}_2(y), \]
where
\[ |\tilde{E}_1(y)| \leq C_1, \hspace{1cm} |\tilde{E}_2(y)| \leq C_2 |y|^{\frac{1-\omega}{n-1} + \frac{\alpha}{n-1} - \frac{\omega}{n-1}} H(y) \]
for some constants $C_1, C_2, C_3 > 0$ determined only by $n, \alpha, \omega$.

Next we will employ the oscillatory integrals theory to deal with $I_2$. By (1.2), we have
\[
I_2 = \int_{A^{-1}}^{+\infty} e^{-zA^{\alpha}} s^{\frac{n}{n-1}} J_{\frac{2}{n}-1}(sA) ds \\
= A^{-\frac{2}{n}} \int_{A^{-1}}^{+\infty} e^{-zA^{\alpha} + isA} s^{\frac{n-1}{n}} L_1(sA) ds \\
+ A^{-\frac{2}{n}} \int_{A^{-1}}^{+\infty} e^{-zA^{\alpha} - isA} s^{\frac{n-1}{n}} L_2(sA) ds,
\]
where
\[ L_1(sA) = \sum_{k \geq 0} b_k (sA)^{-k}, \hspace{1cm} L_2(sA) = \sum_{k \geq 0} b_k'(sA)^{-k}, \]
as in (1.2). To proceed, consider
\[ \int_{A^{-1}}^{+\infty} e^{-zA^{\alpha} + isA} s^{\frac{n-1}{n}} L_1(sA) ds = \int_{A^{-1}}^{s_0} + \int_{s_0}^{\frac{s_0}{\delta}} + \int_{\frac{s_0}{\delta}}^{+\infty} \Delta J_1 + J_2 + J_3, \]
where $s_0 = (\alpha \sin \theta)^\frac{1}{\omega - 1}$, $\delta > 0$ is close enough to 1 and will be determined later.

According to the definition of $L_1(sA)$, it follows
\[ J_1 = b_0 \int_{A^{-1}}^{s_0} e^{-zA^{\alpha} + isA} s^{\frac{n-1}{n}} ds + A^{-1} \int_{A^{-1}}^{s_0} e^{-zA^{\alpha} + isA} s^{\frac{n-1}{n}} \sum_{k \geq 1} b_k (sA)^{-k+1} ds. \]
Note that for $A^{-1} \leq s \leq s_0$, we have $1 \leq sA$ and hence
\[ \left| A^{-1} \int_{A^{-1}}^{s_0} e^{-zA^{\alpha} + isA} s^{\frac{n-1}{n}} \sum_{k \geq 1} b_k (sA)^{-k+1} ds \right| \leq CA^{-1} e^{-\cos \theta A^{\alpha-1}} \int_{A^{-1}}^{s_0} s^{\frac{n-1}{n}} ds. \]
Then we obtain, for $n \geq 2$

\begin{equation}
(2.15) \quad \left| A^{-1} \int_{A^{-1}}^{\delta s_0} e^{-zAs^\alpha + isA} s^{\frac{n-2}{2}} \sum_{k \geq 1} b_k(sA)^{-k+1} ds \right| \leq CA^{-1} e^{-\cos \theta A^{1-\alpha}},
\end{equation}

and for $n = 1$,

\begin{equation}
(2.16) \quad \left| A^{-1} \int_{A^{-1}}^{\delta s_0} e^{-zAs^\alpha + isA} s^{\frac{n-1}{2}} \sum_{k \geq 1} b_k(sA)^{-k+1} ds \right| \leq C' \ln \frac{A}{A} e^{-\cos \theta A^{1-\alpha}}.
\end{equation}

On the other hand, integration by parts gives for $n \geq 2$

\[
\int_{A^{-1}}^{\delta s_0} e^{-zAs^\alpha + isA} s^{\frac{n-1}{2}} ds
\]

\[
= A^{-1} \int_{A^{-1}}^{\delta s_0} s^{\frac{n-1}{2}} h(s) \, ds + A^{-1} \left[(\delta s_0) s^{\frac{n-1}{2}} h(\delta s_0) e^{-zA(\delta s_0)^\alpha + i\delta s_0 A} - A^{-\frac{n-1}{2}} h(A^{-1}) e^{-zA^{1-\alpha} + i]}ight.
\]

\[
- A^{-1} \int_{A^{-1}}^{\delta s_0} e^{-zAs^\alpha + isA} \left[\frac{n-1}{2} s^{\frac{n-3}{2}} h(s) + s^{\frac{n-1}{2}} h'(s)\right] ds,
\]

where $h(s) = (-e^{i \alpha s^{\alpha-1}} + i)^{-1}$. By Lemma 4.1, we have

\[
\left| A^{-1} \int_{A^{-1}}^{\delta s_0} e^{-zAs^\alpha + isA} \left[\frac{n-1}{2} s^{\frac{n-3}{2}} h(s) + s^{\frac{n-1}{2}} h'(s)\right] ds \right|
\]

\[
\leq CA^{-1} e^{-\cos \theta A^{1-\alpha}} \int_{A^{-1}}^{\delta s_0} s^{\frac{n-1}{2}} \left|h(s)\right| + s^{\frac{n-1}{2}} \left|h'(s)\right| ds
\]

\[
\leq C' A^{-1} e^{-\cos \theta A^{1-\alpha}}.
\]

Then we conclude for $n \geq 2$

\begin{equation}
(2.17) \quad \left| A^{-1} \int_{A^{-1}}^{\delta s_0} e^{-zAs^\alpha + isA} s^{\frac{n-1}{2}} ds \right| \leq CA^{-1} e^{-\cos \theta A^{1-\alpha}}.
\end{equation}

When $n = 1$, similarly we have

\[
\int_{A^{-1}}^{\delta s_0} e^{-zAs^\alpha + isA} ds
\]

\[
= A^{-1} \int_{A^{-1}}^{\delta s_0} h(s) e^{-zAs^\alpha + isA} \, ds + A^{-1} \left[h(\delta s_0) e^{-zA(\delta s_0)^\alpha + i\delta s_0 A} - h(A^{-1}) e^{-zA^{1-\alpha} + i]\right]
\]

\[
- A^{-1} \int_{A^{-1}}^{\delta s_0} e^{-zAs^\alpha + isA} h'(s) ds.
\]

Since $|h'(s)| \leq cs^{-\alpha}$, we conclude for $n = 1$

\begin{equation}
(2.18) \quad \left| A^{-1} \int_{A^{-1}}^{\delta s_0} e^{-zAs^\alpha + isA} ds \right| \leq C' A^{-1} e^{-\cos \theta A^{1-\alpha}}.
\end{equation}
By (2.15), (2.16), (2.17), (2.18), it follows
\begin{equation}
|J_1| \leq \begin{cases} 
CA^{-1}e^{-\cos \theta A^{1-n}} n \geq 2; \\
C'\ln A e^{-\cos \theta A^{1-n}} n = 1,
\end{cases}
\end{equation}
for some constants $C, C' > 0$ only determined by $n, \alpha, \omega$.

To estimates $J_3$, we separate the integral into two parts
\begin{align*}
J_3 &= \sum_{k=0}^{[n+1]+1} b_k A^{-k} \int_0^{\infty} e^{-z A s^{\alpha} + i s} s^{-n-1-k} ds \\
&+ A^{-[n+1]+1} \int_0^{\infty} e^{-z A s^{\alpha} + i s} s^{-n-1-[n+1]-1} \sum_{k \geq [n+1]+1} b_k (s A)^{-k+[n+1]+1} ds.
\end{align*}
It is clear that
\begin{align*}
&\left| A^{-[n+1]+1} \int_0^{\infty} e^{-z A s^{\alpha} + i s} s^{-n-1-[n+1]-1} \sum_{k \geq [n+1]+1} b_k (s A)^{-k+[n+1]+1} \right| \\
&\leq CA^{-[n+1]+1} e^{-\cos \theta A^{1-n}} \int_0^{\infty} s^{-n-1-[n+1]-1} ds \\
&\leq C' A^{-[n+1]+1} e^{-\cos \theta A^{1-n}}.
\end{align*}
Integration by parts for $N = \frac{n+1}{2} + 1$ times gives
\begin{align*}
\int_0^{+\infty} e^{-z A s^{\alpha} + i s} s^{-n-1} A^{-k} ds &= e^{-z A^{\left(\frac{n}{2}\right)^\alpha} + i A^{\frac{n}{2}}} \sum_{k=1}^{N} c_k A^{-k} \\
&- A^{-N} \int_0^{+\infty} e^{-z A s^{\alpha} + i s} \sum_{\beta_1, \ldots, \beta_{N+1}} C_{\beta_1, \ldots, \beta_{N+1}} h(\beta_1)(s) \cdots h(\beta_N)(s) s^{-n-1-k-\beta_{N+1}} ds,
\end{align*}
where $\beta_k \geq 0$ are integers satisfying $\beta_1 + \cdots + \beta_{N+1} = N$. By Lemma 4.1, we obtain
\begin{align*}
&\left| A^{-N} \int_0^{+\infty} e^{-z A s^{\alpha} + i s} \sum_{\beta_1, \ldots, \beta_{N+1}} C_{\beta_1, \ldots, \beta_{N+1}} h(\beta_1)(s) \cdots h(\beta_N)(s) s^{-n-1-k-\beta_{N+1}} ds \right| \\
&\leq CA^{-N} e^{-\cos \theta A^{1-n}} \int_0^{+\infty} s^{-n-1-N-k} ds \\
&\leq C' A^{-N} e^{-\cos \theta A^{1-n}}.
\end{align*}
Therefore,
\begin{equation}
|J_3| \leq CA^{-1}e^{-c \cos \theta A},
\end{equation}
where $C, c > 0$ only determined by $n, \alpha, \omega$. 

For \( J_2 \), we will apply the oscillatory integral theories. For this purpose, \( J_2 \) can be written as

\[
J_2 = b_0 \int_{\delta s_0}^{s_0} e^{-iA(\sin \theta s^n - s)} e^{-\cos \theta A s^n} s^{\frac{n-1}{2}} ds + A^{-1} \int_{\delta s_0}^{s_0} e^{-z A s^n + is A s^n} s^{\frac{n-3}{2}} \sum_{k \geq 1} b_k(s A)^{-k+1} ds.
\]

It is clear that

\[
\left| A^{-1} \int_{\delta s_0}^{s_0} e^{-z A s^n + is A s^n} s^{\frac{n-3}{2}} \sum_{k \geq 1} b_k(s A)^{-k} ds \right| \leq CA^{-1} e^{-(\delta s_0)\alpha \cos \theta A}.
\]

On the other hand,

\[
\int_{\delta s_0}^{s_0} e^{-iA(\sin \theta s^n - s)} e^{-\cos \theta A s^n} s^{\frac{n-1}{2}} ds = \int_{\delta s_0}^{s_0} e^{-iA(\sin \theta s^n - s)} e^{-\cos \theta A s^n} s^{\frac{n-1}{2}} (\eta_1(s) + \eta_2(s)) ds,
\]

where \( \eta_1(s) \) is smooth, supported in \( [\delta s_0, s_0] \) and equals 1 for \( s \in [\delta' s_0, s_0] \) with \( \delta < \delta' \); \( \eta_2(s) = 1 - \eta_1(s) \).

By stationary phase method (\cite{25}, Proposition 3, p.334), letting \( \delta' \) close enough to 1 implies

\[
\int_{\delta s_0}^{s_0} e^{-iA(\sin \theta s^n - s)} e^{-\cos \theta A s^n} s^{\frac{n-1}{2}} \eta_1(s) ds = e^{-iA(\sin \theta s_0^n - s_0)} e^{-\cos \theta A(\delta s_0)\alpha} \\
\times \int_{\delta s_0}^{s_0} e^{-iA(\sin \theta s_0^n - s_0)} e^{-\cos \theta A(\delta s_0)\alpha} s^{\frac{n-1}{2}} \eta_1(s) ds = e^{-iA(\sin \theta s_0^n - s_0)} e^{-\cos \theta A(\delta s_0)\alpha} A^{-\frac{1}{2}} d_0 + H_3(A),
\]

where

\[
d_0 = \left( \frac{2\pi}{-i\alpha(\alpha - 1) \sin \theta s_0^{\alpha - 2}} \right)^{-\frac{1}{2}} s_0^{\frac{n-1}{2}} e^{-\cos \theta A s_0^n + \cos \theta A(\delta s_0)\alpha},
\]

and

\[
|H_3(A)| \leq CA^{-1} e^{-c \cos \theta A}.
\]

We have used the facts for \( k \geq 0 \), there exists \( C_k \) such that

\[
\left| \frac{d}{ds} e^{-\cos \theta A s^n + \cos \theta A(\delta s_0)} \right| \leq C_k, \quad \forall A \geq 1, \quad 0 < \omega \leq \theta < \frac{\pi}{2}.
\]
Moreover, we have (Corollary. p.334])
\[
\left| \int_{\delta s_0}^{\tilde{s}} e^{-iA \sin \theta s^\alpha - s} e^{-\cos \theta As^\alpha} s^{n-1} \eta_2(s) ds \right|
\leq C A^{-1} \left[ e^{-\cos \theta A (\tilde{s} / s)^n} \left( \frac{\tilde{s}}{s} \right)^{\frac{n-1}{2}} + \right.
\int_{\delta s_0}^{\tilde{s}} e^{-\cos \theta As^n} s^{n-1} \left( \cos \theta As^{\alpha-1} \eta_2(s) + s^{-1} \eta_2'(s) + \eta_2''(s) \right) ds \bigg] 
\leq C A^{-1} \left( e^{-\cos \theta A (\frac{\tilde{s}}{s})^n} + e^{-\cos \theta A (\delta s_0)^n} \cos \theta A \right) 
\leq C A^{-1} e^{-\frac{1}{4} \cos \theta A (\delta s_0)^n}.
\]
As a result, we have
\[
(2.24)
\]
Moreover, we have (Corollary. p.334])
\[
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\]
\[\text{Following the arguments for (2.22), (2.23), we conclude that}
\]
\[
(2.21)
J_2 = C_{\theta,1} A^{-\frac{1}{2}} e^{-z As^\alpha + iAs} + H_4(A), \quad \text{with } |H_4(A)| \leq C A^{-1} e^{-c \cos \theta A},
\]
where \(C, c > 0\) are determined by \(n, \alpha, \omega\).

Since there is no critical point, i.e. \(|i \sin \theta As^\alpha + iA| \geq \sin \omega A^{1-\alpha} + 1 > 0\) for \(s \geq A^{-1}\), we can use integration by parts to estimate
\[
\int_{A^{-1}}^{+\infty} e^{-z As^\alpha - iA} s^{n-1} L_2 s Ad s = \int_{A^{-1}}^{1} + \int_{1}^{+\infty} \triangle J_1 + J_2.
\]
Following the arguments for \(J_1, J_3\), similarly we obtain
\[
|J_1| \leq \begin{cases} C_1 A^{-1} e^{-\cos \theta A^{1-\alpha}} & n \geq 2; \\ C_2 {\ln \frac{A}{\delta}} A^{-1} e^{-\cos \theta A^{1-\alpha}} & n = 1, \end{cases}
\]
as well as
\[
|J_2| \leq C_3 A^{-1} e^{-c \cos \theta A},
\]
where \(C_1, C_2, C_3, c > 0\) determined only by \(n, \alpha\).

As a result, by (2.24), (2.23), (2.23), (2.22), (2.22), (2.23), we conclude that
\[
|y|^{n-\frac{2}{\alpha+1}} A I_2 = C_{\theta,1} |y|^{n-\frac{2}{\alpha+1}} e^{-z |y|^{\frac{2}{\alpha+1}} s_0 + i|y|^{\frac{2}{\alpha+1}} s_0} + \bar{E}_3(y) + \bar{E}_4(y),
\]
where
\[
|\bar{E}_3(y)| \leq C_1 |y|^{n-\frac{2}{\alpha+1}} + \frac{1}{\alpha+1} e^{-c_1 \cos \theta |y|^{\frac{2}{\alpha+1}}}, \quad |y| \leq 1,
\]
and
\[
|\bar{E}_4(y)| \leq \begin{cases} C_3 |y|^{n-\frac{2}{\alpha+1}} + \frac{1}{\alpha+1} e^{-c_1 \cos \theta |y|^{\frac{2}{\alpha+1}}} & n \geq 2; \\ C_4 |\ln |y||y|^{n-\frac{2}{\alpha+1}} + \frac{1}{\alpha+1} e^{-c_1 \cos \theta |y|^{\frac{2}{\alpha+1}}} & n = 1, \end{cases}
\]
Finally we have shown (2.22) through (2.22), (2.23), (2.24), (2.25), (2.26).

\[\square\]

**Proof of (2.3).** Indeed, (2.3) follows easily from the arguments in (26) p.52. To be more precious, since
\[
[(1 - \varphi(|\xi|)) e^{-z |\xi|^2}]^\vee (x) = c |x|^{-2} \Delta (1 - \varphi(|\xi|)) e^{-z |\xi|^2} (x),
\]
and for $0 < \alpha < 1$, $k > \frac{n}{2(1-\alpha)}$

\[ \int_{\mathbb{R}^n} \left| \Delta^k (1 - \varphi(|\xi|)) e^{-z|\xi|^{\alpha}} \right| d\xi < +\infty, \]

we have proved \(2.3\). \(\square\)

### 2.3. Proof of Proposition 2.1 (3).

**Proof of 2.3.** For simplicity, set $\psi(|\xi|) = 1 - \varphi(|\xi|)$ and $P_2(z, y)$ can be written as

\[ P_2(z, y) = c_n \int_{|\xi| \geq \frac{2}{3}} e^{iy \cdot \xi} e^{-z|\xi|^{\alpha}} \psi(|\xi|) d\xi. \]

Note that we can not separate the integral into $\int_{\frac{2}{3} \leq |\xi| \leq 1} + \int_{|\xi| \geq 1}$ as in 2.7 to simplify our proof. This is because the integrand at $|\xi| = 1$ does not decay as $|y| \to +\infty$ and hence the endpoint is hard to deal with after integrating by parts. For our purposes,

\[
\int_{|\xi| \geq \frac{2}{3}} e^{iy \cdot \xi} e^{-z|\xi|^{\alpha}} \psi(|\xi|) d\xi \\
= C |y|^{\frac{1-\alpha}{\alpha}} A \int_{\frac{3}{2}A - \frac{1}{3}}^{+\infty} \psi(sA^{\frac{1}{\alpha}}) e^{-zA^{n} s^{\frac{n}{\alpha}}} J_{\frac{n}{\alpha} - 1}(sA) ds \\
= C |y|^{\frac{1-\alpha}{\alpha}} A \int_{\frac{3}{2}A - \frac{1}{3}}^{+\infty} \psi(sA^{\frac{1}{\alpha}}) e^{-zA^{n} s^{\frac{n}{\alpha}}} e^{isA} s^{\frac{n}{\alpha} - 1} L_1(sA) ds \\
+ C |y|^{\frac{1-\alpha}{\alpha}} A \int_{\frac{3}{2}A - \frac{1}{3}}^{+\infty} \psi(sA^{\frac{1}{\alpha}}) e^{-zA^{n} s^{\frac{n}{\alpha}}} e^{-isA} s^{\frac{n}{\alpha} - 1} L_2(sA) ds,
\]

where $A = |y|^{\frac{\alpha}{n-\alpha}}$ and

\[ L_1(sA) = \sum_{k \geq 0} b_k(sA)^{-k}, \quad L_2(sA) = \sum_{k \geq 0} b_k'(sA)^{-k}. \]

Observe that $A \to +\infty$ as $|y| \to +\infty$ for $\alpha > 1$.

To start with, consider

\[
\int_{\frac{3}{2}A - \frac{1}{3}}^{+\infty} \psi(sA^{\frac{1}{\alpha}}) e^{-zA^{n} + isA} s^{\frac{n-1}{\alpha}} L_1(sA) ds = \int_{\frac{3}{2}A - \frac{1}{3}}^{\delta s_0} + \int_{\delta s_0}^{\delta s_0 + \delta} + \int_{\delta s_0 + \delta}^{+\infty} \triangleq I_1 + I_2 + I_3,
\]

where $s_0 = (\alpha \sin \theta)^{\frac{n-1}{\alpha}}$ and $\delta$ will be determined later. Set $N_1 = \lceil \alpha + \frac{n-1}{2} \rceil + 1$ and we have

\[
I_1 = \sum_{k=0}^{N_1} b_k A^{-k} \int_{\frac{3}{2}A - \frac{1}{3}}^{\delta s_0} \psi(sA^{\frac{1}{\alpha}}) e^{-zA^{n} + isA} s^{\frac{n-1}{\alpha} - k} ds \\
+ A^{-N_1} \int_{\frac{3}{2}A - \frac{1}{3}}^{\delta s_0} \psi(sA^{\frac{1}{\alpha}}) e^{-zA^{n} + isA} s^{\frac{n-1}{\alpha} - N_1} \sum_{k \geq N_1} b_k(sA)^{-k+N_1} ds.
\]
For $0 \leq k \leq N_1$, integrating by parts $N_1$ times gives

$$A^{-k} \int_{\frac{1}{2}A^{-\frac{1}{\beta}}}^{\delta s_0} \psi(sA^{\frac{1}{\beta}}) e^{-zA^\alpha + isA^{\frac{n+1}{2} - k}} ds$$

$$= A^{-k} e^{-zA(\delta s_0)^\alpha + iA\delta s_0} \sum_{l=1}^{N_1} C_l A^{-l}$$

$$+ C'_{N_1} A^{-k - N_1} \int_{\frac{1}{2}A^{-\frac{1}{\beta}}}^{\delta s_0} \sum_{\beta_1, \ldots, \beta_{N_1+2}} C_{\beta_1, \ldots, \beta_{N_1+2}} A_{\frac{d}{\alpha}} \psi(\beta_1) (sA^{\frac{1}{\beta}})^{s_{\frac{n-1}{2} - k - \beta_2} - \cdots - \beta_{N_1+2}} ds$$

where $\beta_k \geq 0$ are integers satisfying $\beta_1 + \cdots + \beta_{N_1+2} = N_1$ and $h(s) = (-\alpha s^{\alpha-1} + i)^{-1}$. By Lemma 4.1, we obtain

$$|H_4(A)| \leq CA^{-k - N_1 + \frac{\beta_1}{\alpha}} \int_{\frac{1}{2}A^{-\frac{1}{\beta}}}^{\delta s_0} \sum_{\beta_1, \ldots, \beta_{N_1+2}} \left| \psi(\beta_1) (sA^{\frac{1}{\beta}})^{s_{\frac{n-1}{2} - k - \beta_2} - \cdots - \beta_{N_1+2}} ds.\right.$$

When $\beta_1 = 0$, it implies

$$|H_4(A)| \leq CA^{-k - N_1} \int_{\frac{1}{2}A^{-\frac{1}{\beta}}}^{\delta s_0} \sum_{\beta_1 = 0} A_{\frac{d}{\alpha}} \psi(\beta_1) (sA^{\frac{1}{\beta}})^{s_{\frac{n-1}{2} - k - \beta_2} - \cdots - \beta_{N_1+2}} ds$$

$$\leq CA^{-k - N_1} A^{\frac{1}{\beta}} \int_{\frac{1}{2}}^{\delta s_0} A^{\frac{d}{\alpha}} \left( \frac{n-1}{2} - k - N_1 + \beta_1 \right) ds$$

$$\leq CA^{-\frac{n+1}{2} + \left( 1 - \frac{1}{\alpha} \right) (\frac{n+1}{2} - k - N_1)}$$

$$\leq CA^{-\frac{n-1}{2} - \alpha}.$$
Together with the following estimates
\[
A^{-N_1} \left| \int_{\frac{1}{2}A^{-\frac{n}{2}}}^{\delta s_0} \psi(sA^\frac{1}{n}) e^{-zA^n + iA_s} s^{\frac{a}{2} - N_1} \sum_{k \geq N_1} b_k(sA)^{-k + N_1} \, ds \right| 
\]
\[
\leq CA^{-N_1} \int_{\frac{1}{2}A^{-\frac{n}{2}}}^{\delta s_0} s^{\frac{a}{2} - N_1} \, ds 
\]
\[
\leq CA^{(\frac{1}{2})} N_1 - \frac{a+1}{2\alpha} \leq CA^{-\frac{a+1}{2\alpha}} ,
\]
\text{(2.27)} \implies \]
\[
I_1 = e^{-zA(\delta s_0)^n + iA\delta s_0} \sum_{k=1}^{N_1} C_k A^{-k} + H_5(A), \]
and
\[
|H_5(A)| \leq CA^{-\frac{a+1}{2\alpha}} , \quad \forall A \geq 1.
\]
In turn, we obtain that
\[
|y|^{\frac{1}{n-1}} A^\frac{1}{n} I_1 = |y|^{\frac{1}{n-1}} A^\frac{1}{n} e^{-zA(\delta s_0)^n + iA\delta s_0} + \mathcal{E}_1(A) + \mathcal{E}_2(A),
\]
where
\[
|\mathcal{E}_1(A)| \leq C_1 |y|^{-\frac{n}{n-1}} \sum_{k=1}^{\infty} C_k A^{-k} e^{-zA(\delta s_0)^n + iA\delta s_0} \mathcal{E}_1(A), \quad \forall |y| \geq 1,
\]
and
\[
|\mathcal{E}_2(A)| \leq C_3 |y|^{-n-\alpha} \quad \forall |y| \geq 1,
\]
where \(C_1, C_2, C_3 > 0\) are only determined by \(n, \alpha, \omega\).

Since \(\psi(sA^\frac{1}{n}) = 1\) for \(s \geq A^{-\frac{1}{n}}\), then for \(|A| \gg 1\) we have
\[
I_2 = \int_{\delta s_0}^{+\infty} e^{-zA^n + iA_s} s^{\frac{a}{2} - 1} L_1(sA) \, ds \quad \text{and} \quad I_3 = \int_{\delta s_0}^{+\infty} e^{-zA^n + iA_s} s^{\frac{a}{2} - 1} L_1(sA) \, ds.
\]
Then the proof are almost the same as in the case \(0 < \alpha < 1\) and we omit the details. It follows that
\[
|I_2| \leq C_1 A^{-1} e^{-C_2 \cos \theta A},
\]
\[
|I_2| = C_{\theta,1} A^{-\frac{1}{2}} e^{-zA(\delta s_0)^n + iA\delta s_0} + H_5(A) \quad \text{with} \quad |H_5(A)| \leq C_3 A^{-1} e^{-C_4 \cos \theta A},
\]
for some constants \(C_1, C_2, C_3, C_4 > 0\) only determined by \(n, \alpha, \omega\).

The estimates for
\[
C|y|^{\frac{1}{n-1}} A^\frac{1}{n} \int_{\frac{1}{2}A^{-\frac{1}{n}}}^{+\infty} \psi(sA^\frac{1}{n}) e^{-zA^n} s^{\frac{a}{2} - 1} L_2(sA) \, ds,
\]
are easier than the above proof due to the facts there is no critical points. The proof are minor correction to the above arguments and we omit the detail. Combing \(\text{(2.28), (2.29), (2.30), (2.31), (2.32)}\) implies \(\text{(2.4)}\).

\text{Proof of (2.25).} \text{ In fact, (2.25) can be shown by Laplace transform. Firstly,}
\[
P(z, y) = c_n |y|^{1-\frac{a}{2}} \int_{0}^{+\infty} e^{-zr^n} r^{\frac{a}{2}-1} J_{\frac{a}{2}-1}(r|y|) \, dr
\]
\[
= C|y|^{1-\frac{a}{2}} \int_{0}^{+\infty} e^{-zA^n} s^{\frac{a}{2} - 1} J_{\frac{a}{2}-1}(s\frac{a}{2} |y|) \, ds.
\]
By (2.1), we have

$$P(z, y) = C|y|^{-\frac{\alpha}{n}} \int_0^{+\infty} e^{-zs} s^{\frac{\alpha}{2n} - 1} \sum_{k \geq 0} \frac{(-1)^k}{k! \Gamma(k + \frac{\alpha}{2})} \left( \frac{s^{\frac{\alpha}{2}}|y|}{2} \right)^{2k + \frac{\alpha}{2} - 1} \, ds$$

$$= C \sum_{k \geq 0} \frac{(-1)^k 2^{-2k - \frac{n}{2} + 1}}{k! \Gamma(k + \frac{n}{2})} |y|^{2k} \int_0^{+\infty} e^{-zs} s^{\frac{\alpha + 2k}{n} - 1} \, ds$$

$$= Cz^{-\frac{n}{\alpha}} \sum_{k \geq 0} \frac{(-1)^k \Gamma\left(\frac{n + 2k}{\alpha}\right)}{4^k k! \Gamma(k + \frac{n}{2})} z^{-\frac{2k}{\alpha}} |y|^{2k}.$$

The converge radius of above series is $0, +\infty$ for $\alpha > 1$. Together with (2.1), the above implies (2.5).

3. PROOF OF THEOREM 1.1 AND THEOREM 1.3

Proof of Theorem 1.1. Set $\omega = \frac{\pi}{4}$. In view of Proposition 2.1, for $0 < \alpha < 1$, $|y| \geq 1$, $\frac{\pi}{4} \leq |\theta| < \frac{\pi}{2}$, we have

$$|P(e^{i\theta}, y)| \leq C|y|^{-n - \alpha}.$$

On the other hand, by (2.2) and (2.3), we obtain for $0 < \alpha < 1$, $|y| \leq 1$, $\frac{\pi}{4} \leq |\theta| < \frac{\pi}{2}$

$$|P(e^{i\theta}, y)| \leq C(1 + |y|^{-1 - \frac{\alpha}{n}} e^{-c |\cos \theta| |y|^n})$$

Therefore, by (2.6) we obtain for $0 < \alpha < 1$, $0 \leq |\theta| < \frac{\pi}{2}$,

$$|P(e^{i\theta}, y)| \leq \begin{cases} C_1 (1 + |y|^{-1 - \frac{\alpha}{n}} e^{-c |\cos \theta| |y|^n}), & |y| \leq 1; \\ C_2 |y|^{-n - \alpha}, & |y| > 1. \end{cases}$$

Since $P(z, x) = |z|^{-\frac{n}{\alpha}} P(e^{i\theta}, y)$ with $y = \frac{z}{|z|^\frac{n}{\alpha}}$, (1.6) follows.

When $\alpha > 1$, by Proposition 2.1, we have for $\alpha > 1$, $|y| \geq 1$, $\frac{\pi}{4} \leq |\theta| < \frac{\pi}{2}$

$$|P(e^{i\theta}, y)| \leq C(|y|^{-n - \alpha} + |y|^{-1 - \frac{\alpha}{n}} e^{-c |\cos \theta| |y|^n}).$$

In turn, combining the estimates (2.6) we conclude for $\alpha > 1$, $0 \leq |\theta| < \frac{\pi}{2}$

$$|P(e^{i\theta}, y)| \leq \begin{cases} C_1, & |y| \leq 1; \\ C_2 (|y|^{-n - \alpha} + |y|^{-1 - \frac{\alpha}{n}} e^{-c |\cos \theta| |y|^n}), & |y| > 1. \end{cases}$$

And hence (1.7) follows. $\square$

Now we are ready to consider the fractional Schrödinger operator with Kato potentials. We adopt the methods in [3, 17] to prove Theorem 1.3. Set

$$I(|z|, x) = |z|^{-\frac{n}{\alpha}} \wedge \frac{|z|}{|x|^n}.$$
In the second step, we have used the facts

\[ D \]

Then there exist constants \( D_1, D_2 > 0 \) depending only on \( n, \alpha \) such that

\[
|P(z, x)| \leq D_1 (\cos \theta)^{-\frac{n}{2} + \frac{\alpha}{2}} I(|z|, x), \quad 0 < \alpha < 1, \quad \forall z \in \mathbb{C}^+, x \in \mathbb{R}^n,
\]

and

\[
|P(z, x)| \leq D_2 (\cos \theta)^{-\frac{n}{2} - \alpha + 1} I(|z|, x), \quad \alpha > 1, \quad \forall z \in \mathbb{C}^+, x \in \mathbb{R}^n.
\]

Next we only prove (1.3) in details cause minor correction of the proof will show (1.9).

Following [17], we need some characterizations of Kato potentials.

**Lemma 3.1.** \( V \in K_\alpha(\mathbb{R}^n) \) if and only if \( \lim_{t \to 0} K_V(t) = 0 \), where

\[
K_V(t) = \sup_x \int_{\mathbb{R}^n} J(t, x - y)|V(y)|dy,
\]

and

\[
J(t, x) = \begin{cases}
|x|^{\alpha - n} \wedge t^2 |x|^{-n - \alpha}, & 0 < \alpha < n, \\
(1 \vee \ln(t|x|^{-n})) \wedge t^2 |x|^{-2n}, & \alpha = n, \\
t^{1 - n/\alpha} \wedge t^2 |x|^{-n - \alpha}, & \alpha > n.
\end{cases}
\]

**Proof.** The proof can be found in [17].

Denote by \( \tilde{H} = e^{i\theta}(-\Delta)^{\frac{\alpha}{2}} + e^{i\theta}V \). Then we have \( e^{-z((-\Delta)^{\frac{\alpha}{2}} + V)} = e^{-|z|/\tilde{H}} \). To start with, set

\[
\tilde{K}_j(|z|, x, y) = \int_{\mathbb{R}^n} \int_0^{[z]} \tilde{K}_{j-1}(|z| - s, x, \zeta)e^{i\theta}V(\zeta)\tilde{K}_0(s, \zeta, y)dsd\zeta, \quad j \in \mathbb{N}^*,
\]

where \( \tilde{K}_0(|z|, x, y) = P(z, x - y) \).

Then we have the following estimate for \( \tilde{K}_j(|z|, x, y) \).

**Lemma 3.2.** Let \( 0 < \alpha < 1 \). There exists a constant \( \omega \) depending on \( n, \alpha \) such that the following holds for \( j \in \mathbb{N}^* \)

\[
|\tilde{K}_j(|z|, x, y)| \leq D_1 (w\tilde{K}_V(|z|))^2 \tilde{I}(|z|, x - y),
\]

where \( \tilde{I}(|z|, x - y) = \eta I(|z|, x), \quad \tilde{K}_V(|z|) = \eta K_V(|z|) \) for \( \eta = (\cos \theta)^{-\frac{n}{2} + \frac{\alpha}{2}} \) and \( D_1 \) is the constant in (3.1).
Proof. When \( j = 0 \), it is just (5.1).

Note first that

\[
\tilde{I}(|z|, x) \wedge \tilde{I}(s, y) = \eta(I(|z|, x) \wedge I(s, y)) 
\leq D_3 \eta I(|z| + s, x + y) = D_3 \tilde{I}(|z| + s, x + y),
\]

where \( D_3 = 2^{\alpha - 1} \vee 2^{\alpha \cdot z} \). And hence

\[
\tilde{I}(|z|, x) \tilde{I}(s, y) = (\tilde{I}(|z|, x) \wedge \tilde{I}(s, y))(\tilde{I}(|z|, x) \vee \tilde{I}(s, y)) 
\leq D_3 \tilde{I}(|z| + s, x + y)(\tilde{I}(|z|, x) \vee \tilde{I}(s, y)).
\]

Moreover, we have

\[
\int_0^{|z|} \tilde{I}(|z| - s, x)ds = \int_0^{|z|} \tilde{I}(s, x)ds \leq e \int_0^\infty e^{-\frac{\pi}{4} I} I(s, x)ds 
= e\eta \int_0^\infty e^{-\frac{\pi}{4} I} I(s, x)ds 
\leq eD_4 \eta J(|z|, x).
\]

The proof of the last inequality can be found in [17]. Then we have by induction,

\[
\left| \tilde{K}_j(|z|, x, y) \right| 
\leq D_3^j (w \tilde{K}_V)^{j-1} \int_{\mathbb{R}^n} \int_0^{|z|} \tilde{I}(|z| - s, x - \zeta) \wedge \tilde{I}(s, \zeta - y) |V(\zeta)|dsd\zeta 
\leq D_3^j D_4 (w \tilde{K}_V)^{j-1} \tilde{I}(|z|, x - y) \int_{\mathbb{R}^n} \int_0^{|z|} \tilde{I}(|z| - s, x - \zeta) \vee \tilde{I}(s, \zeta - y) |V(\zeta)|dsd\zeta 
\leq eD_3^j D_4 |V(\zeta)|d\zeta.
\]

Let \( \omega = \epsilon D_3 D_4 \) and by the definition of \( \tilde{K}_V(\zeta) \) we get the desired result. \( \square \)

To proceed, let

\[
T_j(|z|)f(x) = \int_{\mathbb{R}^n} \tilde{K}_j(|z|, x, y)f(y)dy,
\]

where \( f \in L^1 \). Then we have the following lemma.

Lemma 3.3. Let \( 0 < \alpha < 1 \) and \( \tilde{H} = e^{i\theta}(-\Delta)^{\frac{\alpha}{2}} + e^{i\theta}V \) for \( 0 \leq |\theta| < \frac{\pi}{2} \) where \( V \in K_\alpha(\mathbb{R}^n) \). Then the following holds for every \( |z| > 0 \)

\[
\lim_{N \to \infty} \| e^{-|z|\tilde{H}} - \sum_{j=0}^N (-1)^j T_j(|z|) \|_{L^1, L^1} = 0.
\]

Proof. Note first that \( e^{i\theta}(-\Delta)^{\frac{\alpha}{2}} \) generates an analytic semigroup of angle \( \frac{\pi}{2} - |\theta| \) on \( L^1(\mathbb{R}^n) \). Since \( V \in K_\alpha(\mathbb{R}^n) \), then for each \( \epsilon > 0 \), there exists \( C_\epsilon > 0 \) such that (27)

\[
\|e^{i\theta}V\phi\|_{L^1} \leq \epsilon \|e^{i\theta}(-\Delta)^{\frac{\alpha}{2}} + C_\epsilon \|\phi\|_{L^1} \quad \forall \phi \in L^{2\alpha,1}(\mathbb{R}^n).
\]

Then \( \tilde{H} \) generates an analytic semigroup and hence can be represented as for certain proper path \( \Gamma \)

\[
e^{-|z|\tilde{H}} = \frac{1}{2\pi i} \int_{\Gamma} e^{i|z|}(-\mu \tilde{H})^{-1}d\mu.
\]
Moreover there exist large enough \( \omega > 0 \) and \( \varepsilon > 0 \) such that the following holds for \( \mu \in \omega + \Sigma_{\pi - |\theta|} = \{ z : \arg z < \pi - |\theta| \} \)

\[
|e^{i\theta}V(\mu + e^{i\theta}(-\Delta)^{1/2})^{-1}|_{L^1,L^1} \\
\leq e^{|e^{i\theta}(-\Delta)^{1/2}(\mu + e^{i\theta}(-\Delta)^{1/2})^{-1}|_{L^1,L^1} + C_{\varepsilon}(\mu + e^{i\theta}(-\Delta)^{1/2})^{-1}|_{L^1,L^1} \\
< \frac{1}{2}.
\]

As a result, for \( \mu \in \omega + \Sigma_{\pi - |\theta|} \) we have

\[
(\mu + \overline{H})^{-1} = \sum_{j=0}^{\infty} (-1)^j (\mu + e^{i\theta}(-\Delta)^{1/2})^{-1}(e^{i\theta}V(\mu + e^{i\theta}(-\Delta)^{1/2})^{-1})^j,
\]

and

\[
(\mu + \overline{H})^{-1} - \sum_{j=0}^{N} (-1)^j (\mu + e^{i\theta}(-\Delta)^{1/2})^{-1}(e^{i\theta}V(\mu + e^{i\theta}(-\Delta)^{1/2})^{-1})^j = (-1)^{N+1}r_N(\mu),
\]

where \( r_N(\mu) = (\mu + e^{i\theta}(-\Delta)^{1/2})^{-1}(e^{i\theta}V(\mu + e^{i\theta}(-\Delta)^{1/2})^{-1})^N e^{i\theta}V(\mu + \overline{H})^{-1}. \)

Then \( r_N(\mu) \) is an analytic function satisfying

\[
\sup \{ (\| \omega - \omega_0 \|_{L^1,L^1} : \mu \in \omega + \Sigma_{\pi - |\theta|} \} \leq C 2^{-N}.
\]

It follows that

\[
\left\| \int_\Gamma e^{it\mu} r_N(\mu) \mu \|_{L^1,L^1} \leq C 2^{-N} e^{i\omega|z|} \to 0 \quad \text{as} \quad N \to \infty,
\]

where \( \Gamma = \Gamma_0 + \Gamma_\pm, \Gamma_0 = \{ \mu : \mu = w + \delta e^{i\psi}, |\psi| \leq \theta_1 + \frac{\pi}{2} \} \) and \( \Gamma_\pm = \{ \mu : \mu = w + re^{i(\theta_1 + \frac{\pi}{2})}, r \geq \delta \} (0 < \theta_1 < \theta, \delta > 0) \). Then we obtain

\[
\sum_{j=0}^{N} (-1)^j \int_\Gamma e^{it\mu} (\mu + e^{i\theta}(-\Delta)^{1/2})^{-1}(e^{i\theta}V(\mu + e^{i\theta}(-\Delta)^{1/2})^{-1})^j \mu \to \int_\Gamma e^{it\mu} (\mu + \overline{H})^{-1} \mu,
\]

in operator norm on \( L^1(\mathbb{R}^n) \) as \( N \) goes to infinity. By the uniqueness of Laplace transforms, it is sufficient to prove \( (\mu + e^{i\theta}(-\Delta)^{1/2})^{-1}(e^{i\theta}V(\mu + e^{i\theta}(-\Delta)^{1/2})^{-1})^j \) and the Laplace transform of \( T_j(|z|) \) coincide.

For \( j \geq 1 \), let

\[
R_j(\mu, x, y) = \int_{\mathbb{R}^n} R_{j-1}(\mu, x, y)e^{it\mu} R_0(\mu, x, y)dz,
\]

where \( R_0(\mu, x, y) = R(\mu, x, y) = (\mu - e^{i\theta}(-\Delta)^{1/2})^{-1}. \)

To start with, we have

\[
|R_0(\mu, x, y)| \leq \int_0^\infty e^{-\mu t} \overline{K}_0(t, x, y)dt \leq \int_0^\infty e^{-\mu t} \overline{I}(t, x, y)dt \leq D_3 \eta J(\mu^{-1}, x - y).
\]

Therefore by induction

\[
|R_j(\mu, x, y)| \leq C \overline{K}_V(\mu^{-1})^j \eta J(\mu^{-1}, x - y).
\]
Finally we obtain

\[
\int_0^\infty e^{-t\mu} \tilde{K}_{j+1}(t, x, y)dt
\]

\[
= \int_0^\infty e^{-t\mu} \int_{\mathbb{R}^n} \tilde{K}_j(t-s, x, z)e^{i\theta}V(z)\tilde{K}_0(s, z, y)dsdzdt
\]

\[
= \int_{\mathbb{R}^n} e^{i\theta}V(z)dz \int_0^\infty e^{-t\mu} \tilde{K}_j(t, x, z)dt \int_0^\infty e^{-s\mu} \tilde{K}_0(s, z, y)ds
\]

\[
= \int_{\mathbb{R}^n} R_j(\mu, x, z)e^{i\theta}V(z)R(\mu, z, y)dz = R_{j+1}(\mu, x, y).
\]

We have used the Fubini’s Theorem in the second step which is due to the fact

\[
\int_0^\infty e^{-t\mu}|\tilde{K}_j(t, x, y)|dt \leq C\tilde{K}_V(\mu^{-1})\eta|J(\mu^{-1}, x - y)|.
\]

Finally we obtain

\[
\int_0^\infty e^{-t\mu}T_j(t)f(x)dt = \int_0^\infty e^{-t\mu} \int_{\mathbb{R}^n} \tilde{K}_j(t, x, y)f(y)dydt
\]

\[
= \int_{\mathbb{R}^n} R_j(\mu, x, y)f(y)dy
\]

\[
= (\mu + e^{i\theta}(-\Delta)^{\frac{\alpha}{2}})^{-1}(e^{i\theta}V(\mu + e^{i\theta}(-\Delta)^{\frac{\alpha}{2}})^{-1}f(x).
\]

We have used the fact in the second step

\[
\int_{\mathbb{R}^n} \int_0^\infty |e^{-t\mu} \tilde{K}_j(t, x, y)|dtdy \leq C\tilde{K}_V(\mu^{-1})^{2}\eta \int_{\mathbb{R}^n} |J(\mu^{-1}, x - y)|dy
\]

\[
\leq C\tilde{K}_V(\mu^{-1})^{2}\eta \mu.
\]

Thus we have proved the lemma. \(\square\)

Now we are ready to prove Theorem 1.3 for \(0 < \alpha < 1\).

Proof of (1) of Theorem 1.3. For \(0 < \varepsilon < 1\), set

\[
V^\varepsilon = \sup\{\sigma \leq 1 : t \in (0, \sigma), \omega \tilde{K}_V(t) \leq \varepsilon\}.
\]

Denote

\[
T(|z|)f(x) = \int_{\mathbb{R}^n} \tilde{K}(|z|, x, y)f(y)dy,
\]

where \(\tilde{K}(|z|, x, y) = \sum_{j\geq 0} \tilde{K}_j(|z|, x, y)\). Thus by Lemma 3.2, we have

\[
|\tilde{K}(|z|, x, y)| \leq \sum_{j=0}^\infty D_2(\omega \tilde{K}_V(|z|))^{\frac{1}{2}}I(|z|, x - y) \leq \frac{D_2}{1 - \varepsilon}I(|z|, x - y),
\]

and for \(0 < |z| < V^\varepsilon\)

\[
\lim_{N \to \infty} \frac{1}{|z|} \int_{|z|}^\infty \frac{1}{|z|} |T(|z|) - \sum_{j=0}^N (-1)^j T_j(|z|)||_{L^1} = 0.
\]

Then by Lemma 3.3 we conclude that \(\tilde{K}(|z|, x, y)\) coincides with \(K(z, x, y)\) which is the kernel of \(e^{-z((-\Delta)^{\frac{\alpha}{2}} + V)}\) for \(0 < |z| < V^\varepsilon\). Now we will pass the estimates
above the general case \(|z| > 0\). Then for \(|z| \in (V^\varepsilon, 2V^\varepsilon)\) we have by semigroup property
\[
K(z, x, y) = \int_{\mathbb{R}^n} \tilde{K} \left( \frac{|z|}{2}, x, \zeta \right) \tilde{K} \left( \frac{|z|}{2}, \zeta, y \right) d\zeta.
\]
It follows that
\[
|K(z, x, y)| \leq \left( \frac{D_1}{1 - \varepsilon} \right)^2 \tilde{I}(|z|, x - y) \int_{\mathbb{R}^n} \left| \tilde{K} \left( \frac{|z|}{2}, x, \zeta \right) \right| + \left| \tilde{K} \left( \frac{|z|}{2}, \zeta, y \right) \right| d\zeta
\leq 2D_3D_5 \left( \frac{D_1}{1 - \varepsilon} \right)^2 \tilde{I}(|z|, x - y),
\]
where \(D_5 = \int_{\mathbb{R}^n} \tilde{I}(|z|, x - y) dy\) is independent of \(|z|\) and \(x\).

By inductive argument, we have for \(|z| \in (2^{n-1}V^\varepsilon, 2^nV^\varepsilon)\)
\[
|K(z, x, y)| \leq \frac{1}{2D_3D_5} \left( \frac{2D_1D_3D_5}{1 - \varepsilon} \right)^2 \tilde{I}(|z|, x - y).
\]
Let \(\mu_{\varepsilon, V} = \frac{2\pi A}{4}\) where \(A = \frac{2D_1D_3D_5}{1 - \varepsilon}\) and we obtain
\[
|K(z, x, y)| \leq \frac{1}{2D_3D_5} e^{\mu_{\varepsilon, V}|z|} \tilde{I}(|z|, x - y).
\]
Thus we have completed the proof. \(\square\)

4. **Appendix**

In this section, we gather some facts about the Bessel functions as well as the auxiliary functions which are frequently used.

Denote by \(J_\nu(z)\) the bessel function for \(\Re \nu > -\frac{1}{2}\) and \(|\arg z| < \pi\) which can be defined by (\[23\ p.211\])
\[
J_\nu(z) = \sum_{k \geq 0} a_k z^{\nu+2k}, \quad \text{with } a_k = \frac{(-1)^k 2^{1-2k-\frac{\nu}{2}}}{k! \Gamma(k + \frac{\nu}{2})}.
\]
Moreover, we have the asymptotic development of \(J_\nu(z)\) as \(z \to \infty\) (\[23\ p.209\])
\[
J_\nu(z) = \frac{1}{2} [H^{(1)}_\nu(z) + H^{(2)}_\nu(z)] \sim z^{-\frac{1}{2}} e^{iz} \sum_{k \geq 0} b_k z^{-k} + z^{\frac{1}{2}} e^{-iz} \sum_{k \geq 0} b'_k z^{-k},
\]
where \(b_k = (\frac{1}{2\pi})^{\frac{1}{2}} e^{-i\left(\frac{\nu}{2} + \frac{k}{2}\right)} \frac{\Gamma(\nu + \frac{k+1}{2})}{\Gamma(\nu - \frac{k}{2})}\) and \(b'_k = (\frac{1}{2\pi})^{\frac{1}{2}} e^{i\left(\frac{\nu}{2} + \frac{k}{2}\right)} \frac{\Gamma(\nu + \frac{k+1}{2})}{\Gamma(\nu - \frac{k}{2})}\).

The above expansion holds in the sense that
\[
\sum_{k \geq N} b_k z^{-k} \equiv \frac{1}{2} z^{\frac{1}{2}} e^{-iz} H^{(1)}_\nu(z) - \sum_{k=0}^{N-1} b_k z^{-k} = O(z^{-N}) \text{ as } |z| \to \infty;
\]
\[
\sum_{k \geq N} b'_k z^{-k} \equiv \frac{1}{2} z^{-\frac{1}{2}} e^{iz} H^{(2)}_\nu(z) - \sum_{k=0}^{N-1} b'_k z^{-k} = O(z^{-N}) \text{ as } |z| \to \infty.
\]

In our proof, the following properties of the auxiliary functions have been used.

**Lemma 4.1.** Set \(h(s) = \left(-e^{i\theta} \alpha s^{\alpha-1} + i\right)^{-1}\) for \(s, \alpha > 0, 0 < \omega < \theta < \frac{\pi}{2}\). Then for nonnegative integer \(\gamma\) there exists constant \(C_\gamma > 0\) such that
\[
|h^{(\gamma)}(s)| \leq C_\gamma s^{-\gamma}, \quad \forall 0 < s < \delta s_0, \text{ and } s > \frac{s_0}{\delta},
\]
where $0 < \delta < 1$ and $s_0 = (\alpha \sin \theta) \frac{1}{\alpha}$. 

**Proof.** It is direct to check that 

$$|h(s)| \leq |\alpha \sin \theta s^{\alpha-1} - 1|^{-1} \leq C_0$$

for each $0 < s < \delta s_0$, $s > \frac{\alpha}{\delta}$ and $\alpha > 0$, $\alpha \neq 1$. Since $h'(s) = e^{i\theta} \alpha (\alpha - 1) \alpha^{\alpha-2} h^2(s)$, we obtain for $\alpha > 0$, $\alpha \neq 1$

$$|h'(s)| \leq |\alpha| (\alpha - 1) |s^{-1} h(s)||s^{\alpha-1} h(s)| \leq C'_1 s^{-1} |h(s)| \leq C_1 s^{-1}$$

where $0 < s < \delta s_0$ or $s > \frac{\alpha}{\delta}$. Then for $\gamma \geq 2$ we have

$$h^{(\gamma)}(s) = (h')^{(\gamma-1)}(s) = \sum_{k_1,k_2,k_3} c_{k_1,k_2,k_3} s^{\alpha-2-k_1} h^{(k_2)}(s) h^{(k_3)}(s)$$

where $k_1, k_2, k_3 \geq 0$ and $k_1 + k_2 + k_3 = \gamma - 1$. Since we have proved $|h'(s)| \leq C'_1 s^{-1} |h(s)|$, by induction, we have

$$|h^{(\gamma)}(s)| \leq C'_\gamma s^{-\gamma} |h(s)|$$

for $0 < s < \delta s_0$, $s > \frac{\alpha}{\delta}$ and $\alpha > 0$, $\alpha \neq 1$. Thus the result follows. \[\square\]

Specifically when $\gamma = 1$, $0 < \alpha < 1$, we also have for $0 < s < \delta s_0$

$$|h'(s)| \leq C s^{-\alpha} s^{2(\alpha-1)} |h^2(s)| \leq C'_\alpha s^{-\alpha}.$$

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