The group of self-homotopy equivalences of $A^2_n$-polyhedra

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Abstract. Let $X$ be a finite type $A^2_n$-polyhedron, $n \geq 2$. In this paper we study the quotient group $\mathcal{E}(X)/\mathcal{E}_*(X)$, where $\mathcal{E}(X)$ is the group of self-homotopy equivalences of $X$ and $\mathcal{E}_*(X)$ the subgroup of self-homotopy equivalences inducing the identity on the homology groups of $X$.

We show that not every group can be realised as $\mathcal{E}(X)$ or $\mathcal{E}(X)/\mathcal{E}_*(X)$ for $X$ an $A^2_n$-polyhedron, $n \geq 3$, and specific results are obtained for $n = 2$.

1. Introduction

Let $\mathcal{E}(X)$ denote the group of homotopy classes of self-homotopy equivalences of a space $X$ and $\mathcal{E}_*(X)$ denote the normal subgroup of self-homotopy equivalences inducing the identity on the homology groups of $X$. Problems related to $\mathcal{E}(X)$ have been extensively studied, deserving a special mention Kahn’s realisability problem, which has been placed first to solve in [2] (see also [1, 12, 13, 15]). It asks whether an arbitrary group can be realised as $\mathcal{E}(X)$ for some simply connected $X$, and though the general case remains an open question, it has recently been solved for finite groups, [7].

As a way to approach Kahn’s problem, in [10, Problem 19] the question of whether an arbitrary group can appear as the distinguished quotient $\mathcal{E}(X)/\mathcal{E}_*(X)$ is raised.

In this paper we work with $(n-1)$-connected $(n+2)$-dimensional CW-complexes for $n \geq 2$, the so-called $A^2_n$-polyhedra. Homotopy types of these spaces have been classified by Baues in [4, Ch. I, §8] using the long exact sequence of groups associated to simply connected spaces introduced by J. H. C. Whitehead in [16]. In [6], the author uses that classification to study the group of self-homotopy equivalences of an $A^2_n$-polyhedron $X$. He associates to $X$ a group $B^4(X)$ that is isomorphic to $\mathcal{E}(X)/\mathcal{E}_*(X)$ and asks if any group can be realised as such a quotient in this context, that is, if $A^2_n$-polyhedra provide an adequate framework to solve the realisability problem.

Here, in the general setting of an $A^2_n$-polyhedron $X$, $n \geq 2$, we also construct a group $B^{n+2}(X)$ (see Definition 2.4) that is isomorphic to $\mathcal{E}(X)/\mathcal{E}_*(X)$ (see Proposition 2.5). We show that there exist many groups (for example $\mathbb{Z}/p$, $p$ odd, Corollary 1.2) for which the question above does not admit a positive answer. This fact should illustrate that $A^2_n$-polyhedra might not be the right setting to answer [10, Problem 19].

We show, for instance, that under some restrictions on the homology groups of $X$, $B^{n+2}(X)$ is infinite, which in particular implies that $\mathcal{E}(X)$ is infinite (see Proposition 3.4 and Proposition 8.9). Or for example, in many situations the existence of odd order elements in the homology groups of $X$ implies the existence of involutions in $B^{n+2}(X)$ (see Lemma 3.3 and Lemma 3.5).

1991 Mathematics Subject Classification. 20K30, 55P10, 55P15.

Key words and phrases. $A^2_n$-polyhedra, self-homotopy equivalences.

The first author was partially supported by Ministerio de Economía y Competitividad (Spain), grant MTM2016-79661-P.

The second author was partially supported by Ministerio de Educación, Cultura y Deporte grant FPU14/05137, and by Ministerio de Economía y Competitividad (Spain) grants MTM2016-79661-P and MTM2016-78647-P.

The third author was partially supported by by Ministerio de Economía y Competitividad (Spain) grant MTM2016-78647-P.
In this paper we prove the following result:

**Theorem 1.1.** Let $X$ be a finite type $A^2_n$-polyhedron, $n \geq 3$. Then $B^{n+2}(X)$ is either the trivial group or it has elements of even order.

As an immediate corollary, we obtain the following:

**Corollary 1.2.** Let $G$ be a non-trivial group with no elements of even order. Then $G$ is not realisable as $B^{n+2}(X)$ for $X$ a finite type $A^2_n$-polyhedron, $n \geq 3$.

The case $n = 2$ is more complicated. A detailed group theoretical analysis shows that a finite type $A^2_2$-polyhedra might realise finite groups of odd order only under very restrictive conditions. Recall that for a group $G$, rank $G$ is the smallest cardinal of a set of generators for $G$ [14, p. 91]. We have the following result:

**Theorem 1.3.** Suppose that $X$ is a finite type $A^2_2$-polyhedron with a non trivial finite $B^4(X)$ of odd order. Then the following holds:

1. rank $H_1(X) \leq 1$,
2. $\pi_3(X)$ and $H_1(X)$ are 2-groups, and $H_2(X)$ is an elementary abelian 2-group,
3. rank $H_1(X) \leq \frac{1}{2}$ rank $H_2(X) + 1$ - rank $H_4(X) \leq \text{rank } \pi_3(X)$,
4. the natural action of $B^4(X)$ on $H_2(X)$ induces a faithful representation $B^4(X) \leq \text{Aut } H_2(X)$.

All our attempts to find a space satisfying the hypothesis of Theorem 1.3 were unsuccessful. We therefore raise the following conjecture:

**Conjecture 1.4.** Let $X$ be an $A^2_2$-polyhedron. If $B^4(X)$ is a non trivial finite group, then it necessarily has an element of even order.

This paper is organised as follows. In Section 2 we give a brief introduction to Whitehead and Baeus results for the classification of homotopy types of $A^2_n$-polyhedra, or equivalently, isomorphism classes of certain long exact sequences of abelian groups (see Theorem 2.30). In Section 3 we study how restrictions on $X$ affect the group $B^{n+2}(X)$. Finally, Section 4 is devoted to the proof of our main results, Theorem 1.3 and Theorem 1.5.

2. The $\Gamma$-sequence of an $A^2_n$-polyhedron

Let $\text{Ab}$ denote the category of abelian groups. In [16], J.H.C. Whitehead constructed a functor $\Gamma: \text{Ab} \to \text{Ab}$, known as the Whitehead’s universal quadratic functor, and an exact sequence, which are useful to our purposes and we introduce in this section. The $\Gamma$-functor is defined as follows. Let $A$ and $B$ be abelian groups and $\eta: A \to B$ be a map (of sets) between them. The map $\eta$ is said to be quadratic if:

1. $\eta(a) = \eta(-a)$, for all $a \in A$, and
2. the map $A \times A \to B$ taking $(a, a')$ to $\eta(a + a') - \eta(a) - \eta(a')$ is bilinear.

For an abelian group $A$, $\Gamma(A)$ is the only abelian group such that there exists a quadratic map $\gamma: A \to \Gamma(A)$ verifying that every other quadratic map $\eta: A \to B$ factors uniquely through $\gamma$. This means that there is a unique group homomorphism $\eta^\square: \Gamma(A) \to B$ such that $\eta = \eta^\square \gamma$. The $\Gamma$-sequence of an $A^2_n$-polyhedron

The $\Gamma$-functor acts on morphisms as follows: let $f: A \to B$ be a group homomorphism, and $\gamma: A \to \Gamma(A)$ and $\gamma: B \to \Gamma(B)$ the universal quadratic maps. Then, $\gamma f: A \to \Gamma(B)$ is a quadratic map, so there exists a unique group homomorphism $(\gamma f)^\square: \Gamma(A) \to \Gamma(B)$ such that $(\gamma f)^\square \gamma = \gamma f$. Define $\Gamma(f) = (\gamma f)^\square$.

We now list some of its properties that will be used later in this paper:

**Proposition 2.1.** ([5 pp. 16–17]) The $\Gamma$ functor has the following properties:

1. $\Gamma(\mathbb{Z}) = \mathbb{Z}$,
2. $\Gamma(\mathbb{Z}_n)$ is $\mathbb{Z}_{2n}$ if $n$ is even or $\mathbb{Z}_n$ if $n$ is odd,
(3) Let $I$ be an ordered set and $A_i$ be an abelian group, for each $i \in I$. Then,

$$
\Gamma\left( \bigoplus_{i} A_i \right) = \left( \bigoplus_{i} \Gamma(A_i) \right) \oplus \left( \bigoplus_{i \neq j} A_i \otimes A_j \right).
$$

Moreover, the groups $\Gamma(A_i)$ and $A_i \otimes A_j$ are respectively generated by elements $\gamma(a_i)$ and $a_i \otimes a_j$, with $a_i \in A_i$, $a_j \in A_j$, $i < j$, and $\gamma(a_i + a_j) = \gamma(a_i) + \gamma(a_j) + a_i \otimes a_j$, for $a_i \in A_i$, $a_j \in A_j$, $i < j$, [10 §5, §7].

We now introduce Whitehead’s exact sequence. Let $X$ be a simply connected $CW$-complex. For $n \geq 1$, the $n$-th Whitehead $\Gamma$-group of $X$ is defined as

$$
\Gamma_n(X) = \operatorname{Im} (i_* : \pi_n(X^{n-1}) \to \pi_n(X^n)).
$$

Here, $i_* : X^{n-1} \to X^n$ is the inclusion of the $(n - 1)$-skeleton of $X$ into its $n$-skeleton. Then, $\Gamma_n(X)$ is an abelian group for $n \geq 1$. This group can be embedded into a long exact sequence of abelian groups

$$
\cdots \to H_{n+1}(X) \xrightarrow{b_{n+1}} \Gamma_n(X) \xrightarrow{i_{n-1}} \pi_n(X) \xrightarrow{h_n} H_n(X) \to \cdots
$$

where $b_n$ is the Hurewicz homomorphism and $b_{n+1}$ is a boundary representing the attaching maps.

For each $n \geq 2$, a functor $\Gamma_n^1 : \text{Ab} \to \text{Ab}$ is defined as follows. Let $\Gamma^1 = \Gamma$ be the universal quadratic functor, and for $n \geq 3$, $\Gamma^1_n = - \otimes \mathbb{Z}$. It turns out that if $X$ is $(n - 1)$-connected, then $\Gamma^1_n(H_n(X)) \cong \Gamma^1_{n-1}(X)$, [5 Theorem 2.1.22]. Thus, the final part of the long exact sequence (1) can be written as

$$
H_{n+2}(X) \xrightarrow{b_{n+2}} \Gamma^1_n(H_n(X)) \xrightarrow{i_n} \pi_{n+1}(X) \xrightarrow{h_{n+1}} H_n(X) \to 0
$$

Now, for each $n \geq 2$, we define the category of $\mathcal{A}^2_n$-polyhedra as the category whose objects are $(n + 2)$-dimensional $(n - 1)$-connected $CW$-complexes and whose morphisms are continuous maps between objects. Homotopy types of these spaces are classified through isomorphism classes in a category whose objects are sequences like $[2]$, [4 Ch. I, §8]:

**Definition 2.2.** ([3 Ch. IX, §4]) Let $n \geq 2$ be an integer. We define the category of $\Gamma$-sequences$^{n+2}$ as follows. Objects are exact sequences of abelian groups

$$
H_{n+2} \to \Gamma^1_n(H_n) \to \pi_{n+1} \to H_{n+1} \to 0
$$

where $H_{n+2}$ is free abelian. Morphisms are triples of group homomorphisms $f = (f_{n+2}, f_{n+1}, f_n)$, $f_i : H_i \to H'_i$, such that there exists a group homomorphism $\Omega : \pi_{n+1} \to \pi'_{n+1}$ making the following diagram

$$
\begin{array}{ccc}
H_{n+2} & \xrightarrow{f_{n+2}} & \Gamma^1_n(H_n) \xrightarrow{i_n} \pi_{n+1} \xrightarrow{\Omega} H_{n+1} \xrightarrow{f_{n+1}} H'_{n+1} \\
\end{array}
$$

commutative. We say that objects in $\Gamma$-sequences$^{n+2}$ are $\Gamma$-sequences, and morphisms in the category are called $\Gamma$-morphisms.

On the one hand, we can assign to an $\mathcal{A}^2_n$-polyhedron $X$, an object in $\Gamma$-sequences$^{n+2}$ by considering the associated exact sequence, [2]. We call such an object the $\Gamma$-sequence of $X$. On the other hand, to a continuous map $\alpha : X \to X'$ of $\mathcal{A}^2_n$-polyhedra we can assign a morphism between the corresponding $\Gamma$-sequences by considering the induced homomorphisms.
\[
\begin{array}{ccccccc}
H_{n+2}(X) & \longrightarrow & \Gamma^1_n(H_n(X)) & \longrightarrow & \pi_{n+1}(X) & \longrightarrow & H_{n+1}(X) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{n+2}(\alpha) & \longrightarrow & \Gamma^1_n(H_n(\alpha)) & \longrightarrow & \pi_{n+1}(\alpha) & \longrightarrow & H_{n+1}(\alpha) & \longrightarrow & 0
\end{array}
\]

Therefore, we have a functor \( A^n_{\omega^2} \rightarrow \Gamma\)-sequences\(^{n+2} \) which clearly restricts to the homotopy category of \( A^n_{\omega^2} \)-polyhedra, \( \text{Ho}A^n_{\omega^2} \)-polyhedra. It is obvious that this functor sends homotopy equivalences to isomorphisms between the corresponding \( \Gamma \)-sequences. Thus, we can classify homotopy types of \( A^n_{\omega^2} \)-polyhedra through isomorphism classes of \( \Gamma \)-sequences:

**Theorem 2.3** ([4] Ch. I, §[8]). The functor \( \text{Ho}A^n_{\omega^2} \rightarrow \Gamma\)-sequences\(^{n+2} \) previously defined is full. Moreover, for any object in \( \Gamma\)-sequences\(^{n+2} \), there exists an \( A^n_{\omega^2} \)-polyhedron whose \( \Gamma \)-sequence is the given object in \( \Gamma\)-sequences\(^{n+2} \). In fact, there exists a 1-1 correspondence between homotopy types of \( A^n_{\omega^2} \)-polyhedra and isomorphism classes of \( \Gamma \)-sequences.

Following the ideas of [6], we introduce the following:

**Definition 2.4.** Let \( X \) be an \( A^n_{\omega^2} \)-polyhedron. We denote by \( B^{n+2}(X) \) the group of \( \Gamma \)-isomorphisms of the \( \Gamma \)-sequence of \( X \).

Let \( \Psi: \mathcal{E}(X) \rightarrow B^{n+2}(X) \) be the map that associates to \( \alpha \in \mathcal{E}(X) \) the \( \Gamma \)-isomorphism \( \Psi(\alpha) = (H_{n+2}(\alpha), H_{n+1}(\alpha), b_n(\alpha)) \). Then \( \Psi \) is a group homomorphism: its kernel is the subgroup of self-homotopy equivalences inducing the identity map on the homology groups of \( X \), that is, \( \mathcal{E}_s(X) \). Also, \( \Psi \) is onto as a consequence of Theorem 2.3. Hence, we immediately obtain the following result.

**Proposition 2.5.** Let \( X \) be an \( A^n_{\omega^2} \)-polyhedron, \( n \geq 2 \). Then \( B^{n+2}(X) \cong \mathcal{E}(X)/\mathcal{E}_s(X) \).

### 3. Self-homotopy equivalences of finite type \( A^n_{\omega^2} \)-polyhedra

Henceforth, an \( A^n_{\omega^2} \)-polyhedron will mean an \((n - 1)\)-connected, \((n + 2)\)-dimensional CW-complex of finite type. Recall that for simply connected and finite type spaces, the homology and homotopy groups \( H_n(X) \) and \( \pi_n(X) \) are finitely generated and abelian for \( n \geq 1 \).

The \( \Gamma \)-sequence tool introduced in Section 2 will help us to illustrate, from an algebraic point of view, how different restrictions on an \( A^n_{\omega^2} \)-polyhedron \( X \) affect the quotient group \( \mathcal{E}(X)/\mathcal{E}_s(X) \). We devote this section to that matter. We also obtain several results that are needed in the proof of Theorem 1.1 and Theorem 1.3. The following result is a generalisation of [6] Theorem 4.5.

**Proposition 3.1.** Let \( X \) be an \( A^n_{\omega^2} \)-polyhedron and suppose that the Hurewicz homomorphism \( b_{n+2}: \pi_{n+2}(X) \rightarrow H_{n+2}(X) \) is onto. Then, every automorphism of \( H_{n+2}(X) \) is realised by a self-homotopy equivalence of \( X \).

**Proof.** As part of the exact sequence (1) for \( X \) we have:

\[
\cdots \rightarrow \pi_{n+2}(X) \xrightarrow{h_{n+2}} H_{n+2}(X) \xrightarrow{b_{n+2}} \Gamma^1_n(H_n(X)) \rightarrow \pi_{n+1}(X) \rightarrow \cdots
\]

Then, since \( h_{n+2} \) is onto by hypothesis, \( b_{n+2} \) is the trivial homomorphism. Thus, for every \( f_{n+2} \in \text{Aut}(H_{n+2}(X)) \), \( b_{n+2}f_{n+2} = b_{n+2} = 0 \), so if \( \Omega = \text{id} \), \( (f_{n+2}, \text{id}, \text{id}) \in B^{n+2}(X) \). Then there exists \( f \in \mathcal{E}(X) \) with \( H_{n+2}(f) = f_{n+2}, H_{n+1}(f) = \text{id}, H_n(f) = \text{id} \).

We can easily prove that automorphism groups can be realised, a result that can also be obtained as a consequence of [15] Theorem 2.1):

**Example 3.2.** Let \( G \) be a group isomorphic to \( \text{Aut}(H) \) for some finitely generated abelian group \( H \). Then, for any integer \( n \geq 2 \), there exists an \( A^n_{\omega^2} \)-polyhedron \( X \) such that \( G \cong B^{n+2}(X) \): take the Moore space \( X = M(H, n+1) \), which in particular is an \( A^n_{\omega^2} \)-polyhedron. The \( \Gamma \)-sequence of \( X \) is

\[
H_{n+2}(X) = 0 \rightarrow \Gamma^1_n(H_n(X)) = 0 \rightarrow H = \longrightarrow H = 0.
\]
Then, for every \( f \in \text{Aut}(H) \), by taking \( \Omega = f \) we see that \((\text{id}, f, \text{id}) \in B^{n+2}(X)\), and those are the only possible \( \Gamma \)-isomorphisms. Thus \( B^{n+2}(X) \cong \text{Aut}(H) \cong G \).

The use of Moore spaces is not required in the \( n = 2 \) case:

**Example 3.3.** Let \( G \) be a group isomorphic to \( \text{Aut}(H) \) for some finitely generated abelian group \( H \). Consider the following object in \( \Gamma \)-sequences\(^4\)

\[
Z \xrightarrow{b_2} \Gamma(Z_2) = \mathbb{Z}_4 \to H \xrightarrow{\pi_3} H \to 0.
\]

By Theorem 2.3 there exists an \( A^2_\mathbb{Z}\)-polyhedron \( X \) realising this object. In particular, \( H_3(X) = \mathbb{Z}, H_4(X) = \pi_3(X) = H \) and \( H_2(X) = \mathbb{Z}_2 \). It is clear from (3) that \((\text{id}, f, \text{id})\) is a \( \Gamma \)-isomorphism for every \( f \in \text{Aut}(H) \). Now \( \text{Aut}(\mathbb{Z}_2) \) is the trivial group while \( \text{Aut}(\mathbb{Z}) = \{\pm \text{id}\} \). It is immediate to check that \((-\text{id}, f, \text{id})\) is not a \( \Gamma \)-isomorphism since \( b_4 \neq b_4(-\text{id}) \). Then, we obtain that \( B^4(X) \cong \text{Aut}(H) \).

Observe that not every group \( G \) is isomorphic to the automorphism group of an abelian group (for example \( \mathbb{Z}_p \) if \( p \) is odd). Hence, examples from above only provide a partial positive answer to the realisability problem for \( B^{n+2}(X) \). Indeed, the automorphism group of an abelian group (other than \( \mathbb{Z}_2 \)) has elements of even order. The following results go in that direction:

**Lemma 3.4.** Let \( X \) be an \( A^n_\mathbb{Z} \)-polyhedron, \( n \geq 2 \). If \( H_n(X) \) is not an elementary abelian 2-group, then \( B^{n+2}(X) \) has an element of order 2.

**Proof.** Since \( H_n(X) \) is not an elementary abelian 2-group, it admits a non-trivial involution \(-\text{id}: H_n(X) \to H_n(X)\). But \( \Gamma^1_n(-\text{id}) = \text{id} \) for every \( n \geq 2 \), so \((\text{id}, \text{id}, -\text{id}) \in B^{n+2}(X)\) and the result follows.

We point out a key difference between the \( n = 2 \) and the \( n \geq 3 \) cases: \( \Gamma^1_n(A) = \Gamma(A) \) is never an elementary abelian 2-group when \( A \) is finitely generated and abelian, as it can be deduced from Proposition 2.1. However, for \( n \geq 3 \), \( \Gamma^1_n(A) = A \oplus \mathbb{Z}_2 \) is always an elementary abelian 2-group. Taking advantage of this fact we can prove the following result:

**Lemma 3.5.** Let \( X \) be an \( A^n_\mathbb{Z} \)-polyhedron, \( n \geq 3 \). If any of the homology groups of \( X \) is not an elementary abelian 2-group (in particular, if \( H_n+2(X) \neq 0 \)), then \( B^{n+2}(X) \) contains a non trivial element of order 2.

**Proof.** Under our assumptions, \( \Gamma^1_n(H_n(X)) \) is an elementary abelian 2-group. For \( \Omega = -\text{id} \), the triple \((-\text{id}, -\text{id}, -\text{id})\) is a \( \Gamma \)-isomorphism of order 2 unless \( H_{n+2}(X) \), \( H_{n+1}(X) \) and \( H_n(X) \) are all elementary abelian 2-groups.

We remark that this result does not hold for \( A^n_\mathbb{Z} \)-polyhedra. Indeed, if we consider the construction in Example 3.3 for \( H = \mathbb{Z}_2 \), then \( B^3(X) \cong \text{Aut}(\mathbb{Z}_2) = \{e\} \) does not contain a non trivial element of order 2 although \( H_4(X) = \mathbb{Z} \) is not an elementary abelian 2-group.

We now prove some results regarding the finiteness of \( B^{n+2}(X) \):

**Proposition 3.6.** Let \( X \) be an \( A^n_\mathbb{Z} \)-polyhedron, \( n \geq 2 \), with rank \( H_{n+2}(X) \geq 2 \) and every element of \( \Gamma^1_n(H_n(X)) \) of finite order. Then \( B^{n+2}(X) \) is an infinite group.

**Proof.** Since rank \( H_{n+2}(X) \geq 2 \), we may write \( H_{n+2}(X) = \mathbb{Z}^2 \oplus G \), \( G \) a (possibly trivial) free abelian group. Consider the \( \Gamma \)-sequence of \( X \):

\[
\mathbb{Z}^2 \oplus G \xrightarrow{b_{n+2}} \Gamma^1_n(H_n(X)) \xrightarrow{\pi_n} \pi_{n+1}(X) \xrightarrow{b_{n+1}} H_{n+1}(X) \to 0.
\]

Since \( b_{n+2}(\mathbb{Z}^2) \leq \Gamma^1_n(H_n(X)) \) is a finitely generated \( \mathbb{Z} \)-module with finite order generators, it is a finite group. Define \( k = \exp(b_{n+2}(\mathbb{Z}^2)) \) and consider the automorphism of \( \mathbb{Z}^2 \) given by the matrix

\[
\begin{pmatrix}
1 & k \\
0 & 1
\end{pmatrix} \in \text{GL}_2(\mathbb{Z}),
\]

which is of infinite order. If we take \( f \oplus \text{id}_G \in \text{Aut}(\mathbb{Z}^2 \oplus G) \), then \( b_{n+2}(f \oplus \text{id}) = b_{n+2} \), thus \((f \oplus \text{id}_G, \text{id}, \text{id}) \in B^{n+2}(X)\), which is an element of infinite order. \(\square\)
As we have previously mentioned, $\Gamma_n^1(H_n(X))$ is an elementary abelian 2-group, for $n \geq 3$. Hence, from Proposition 5.4 we get:

**Corollary 3.7.** Let $X$ be an $A^2_n$-polyhedron, $n \geq 3$, with rank $H_{n+2}(X) \geq 2$. Then $B^{n+2}(X)$ is an infinite group.

This result does not hold, in general, for $n = 2$. However, if $A$ is a finite group, Proposition 2.1 implies that $\Gamma(A)$ is finite as well so from Proposition 4.3 we get:

**Corollary 3.8.** Let $X$ be an $A^2_2$-polyhedron with rank $H_4(X) \geq 2$ and $H_3(X)$ finite. Then $B^4(X)$ is an infinite group.

We end this section with a more result on the infiniteness of $B^{n+2}(X)$:

**Proposition 3.9.** Let $X$ be an $A^2_n$-polyhedron, $n \geq 3$. If $H_n(X) = \mathbb{Z}^2 \oplus G$ for a certain abelian group $G$, then $B^{n+2}(X)$ is an infinite group.

**Proof.** If $H_n(X) = \mathbb{Z}^2 \oplus G$, then $\Gamma_n^1(H_n(X)) = H_n(X) \oplus \mathbb{Z}_2 = \mathbb{Z}_2^2 \oplus (G \oplus \mathbb{Z}_2)$. Hence $\text{GL}_2(\mathbb{Z}) \leq \text{Aut}(H_n(X))$ and $\text{GL}_2(\mathbb{Z}_2) \leq \text{Aut}(H_n(X) \otimes \mathbb{Z}_2)$. Moreover, for every $f \in \text{GL}_2(\mathbb{Z})$ we have $f \oplus \text{id}_{\mathbb{Z}_2} \in \text{Aut}(H_n(X))$ which yields, through $\Gamma_n^1$, an automorphism $(f \oplus \text{id}_{\mathbb{Z}_2}) \otimes \mathbb{Z}_2 = (f \otimes \mathbb{Z}_2) \oplus \text{id}_{G \otimes \mathbb{Z}_2} \in \text{Aut}(H_n(X) \otimes \mathbb{Z}_2)$. This means that the functor $\Gamma_n^1$ restricts to $\text{GL}_2(\mathbb{Z}) \rightarrow \text{GL}_2(\mathbb{Z}_2)$. Moreover, $- \otimes \mathbb{Z}_2$: $\text{GL}_2(\mathbb{Z}) \rightarrow \text{GL}_2(\mathbb{Z}_2)$ has an infinite kernel. Hence, there are infinitely many morphisms $f \in \text{Aut}(H_n(X))$ such that $f \otimes \mathbb{Z}_2 = \text{id}$. For any such a morphism $f$, $(\text{id}, \text{id}, f)$ is an element of $B^{n+2}(X)$. Therefore $B^{n+2}(X)$ is infinite. □

4. Obstructions to the realisability of groups

We have seen in Section 3 that the group $B^{n+2}(X)$ contains elements of even order unless strong restrictions are imposed on the homology groups of the $A^2_n$-polyhedron $X$. Since we are interested in realising an arbitrary group $G$ as $B^{n+2}(X)$ for $X$ a finite-type $A^2_n$-polyhedron, in this section we focus our attention on the remaining situations and prove Theorems 1.1 and 1.3.

We first give some previous results:

**Lemma 4.1.** For $G$ an elementary abelian 2-group, $\Gamma(-): \text{Aut}(G) \rightarrow \text{Aut}(\Gamma(G))$ is injective.

**Proof.** Let us show that the kernel of $\Gamma(-)$ is trivial. Assume that $G$ is generated by \{ $e_j \mid j \in J$, $J$ an ordered set. If $f \in \text{Aut}(G)$ is in the kernel of $\Gamma(-)$, then for each $j \in J$, there exists a finite subset $I_j \subset J$ such that $f(e_j) = \sum_{i \in I_j} e_i$, and

$$\gamma(e_j) = \Gamma(f)\gamma(e_j) = \gamma(f(e_j)) = \gamma\left(\sum_{i \in I_j} e_i\right) = \sum_{i \in I_j} \gamma(e_i) + \sum_{i < k} e_i \otimes e_k,$$

as a consequence of Proposition 2.1(3), so $I_j = \{ j \}$ and $f(e_j) = e_j$ for every $j \in J$. □

**Lemma 4.2.** Let $H_2 = \oplus_{i=1}^n \mathbb{Z}_2$ and $\chi \in \Gamma(H_2)$ be an element of order 4. If there exists a non trivial automorphism of odd order $f \in \text{Aut}(H_2)$ such that $\Gamma(f)(\chi) = \chi$, then there exists $g \in \text{Aut}(H_2)$ of order 2 such that $\Gamma(g)(\chi) = \chi$.

**Proof.** Notice that according to [16, p. 66], we can write $h \otimes h = 2\gamma(h)$, for any element $h \in H_2$. Therefore, given a basis $\{ h_1, h_2, \ldots, h_n \}$ of $H_2$, and replacing $3\gamma(h_i)$ by $\gamma(h_i) + h_i \otimes h_i$, if needed, we can write

$$\chi = \sum_{i=1}^n a(i)\gamma(h_i) + \sum_{i,j=1}^n a(i,j)h_i \otimes h_j,$$

where every coefficient $a(i)$, $a(i,j)$ is either 0 or 1. We now construct inductively a basis $\{ e_1, e_2, \ldots, e_n \}$ of $H_2$ as follows. Without loss of generality, assume $a(1) = 1$ and define $e_1 = \sum_{i=1}^n a(i)h_i$. Then $\{ e_1, h_2, \ldots, h_n \}$ is again a basis of $H_2$ and

$$\chi = \gamma(e_1) + \alpha_1 e_1 \otimes e_1 + \beta_1 e_1 \otimes (\sum_{s=2}^n b(1,s)h_s) + \sum_{i,j>1} a_1(i,j)h_i \otimes h_j,$$
where every coefficient in the equation is either 0 or 1. Assume a basis \( \{e_1, \ldots, e_r, h_{r+1}, \ldots, h_n\} \) has been constructed such that
\[
\chi = \gamma(e_1) + \sum_{j=1}^{r} \alpha_j e_j \otimes e_j + \sum_{j=1}^{r-1} \beta_j e_j \otimes e_{j+1} + \beta_r e_r \otimes \left( \sum_{s=r+1}^{n} b(r, s) h_s \right) + \sum_{i,j>r} \alpha_{r}(i,j) h_i \otimes h_j,
\]
where every coefficient in the equation is either 0 or 1. We may assume \( b(r, r + 1) = 1 \) and define \( e_{r+1} = \sum_{s=r+1}^{n} b(r, s) h_s \). Thus \( \{e_1, \ldots, e_{r+1}, h_{r+2}, \ldots, h_n\} \) is again a basis of \( H_2 \) and
\[
\chi = \gamma(e_1) + \sum_{j=1}^{r+1} \alpha_j e_j \otimes e_j + \sum_{j=1}^{r} \beta_j e_j \otimes e_{j+1} + \beta_{r+1} e_{r+1} \otimes \left( \sum_{s=r+2}^{n} b(r + 1, s) h_s \right) + \sum_{i,j>r+1} \alpha_{r+1}(i,j) h_i \otimes h_j.
\]
Finally, we obtain a basis \( \{e_1, e_2, \ldots, e_n\} \) of \( H_2 \) such that
\[
\chi = \gamma(e_1) + \sum_{j=1}^{n} \alpha_j e_j \otimes e_j + \sum_{j=1}^{n-1} \beta_j e_j \otimes e_{j+1},
\]
for some coefficients \( \alpha_j \in \{0, 1\} \) and \( \beta_j \in \{0, 1\} \).

Now, for \( n = 1 \), \( H_2 = \mathbb{Z}_2 \) has a trivial group of automorphisms, so the result holds. For \( n = 2 \), assume that there exists \( f \in \text{Aut}(H_2) \) such that \( \Gamma(f)(\chi) = \chi \). From Equation (4), \( \chi = \Gamma(f)(\gamma(e_1)) + \Gamma(f)(P) \), where \( P \in \Omega_{1}(\Gamma(H_2)) = \{ h \in \Gamma(H_2) : \text{ord}(h)2 \} \). Then \( \Gamma(f)(\gamma(e_1)) \) has a multiple of \( \gamma(e_1) \) as its only summand of order 4, which implies that \( f(e_1) = e_1 \). Then either \( f(e_2) = e_2 \), so \( f \) is trivial, or \( f(e_2) = e_1 + e_2 \), so \( f \) has order 2.

For \( n \geq 3 \), we define \( g \in \text{Aut}(H_2) \) by \( g(e_j) = e_j \), for \( j = 1, 2, \ldots, n-2 \), and \( g(e_{n-1}) \) and \( g(e_n) \), depending on \( \alpha_{n-1} \) and \( \beta_{n-1} \), for \( j = 0, 1 \), in Equation (4), according to the following table:

| \( \alpha_n \) | \( \beta_{n-1} \) | \( \alpha_{n-1} \) | \( \beta_{n-2} \) | \( g(e_{n-1}) \) | \( g(e_n) \) |
|---|---|---|---|---|---|
| 0 | 0 | 0 or 1 | 0 or 1 | \( e_{n-1} \) | \( e_{n-1} + e_n \) |
| 0 | 1 | 0 | 0 | \( e_n \) | \( e_{n-1} \) |
| 0 | 1 | 0 | 1 | \( e_{n-2} + e_n \) | \( e_{n-2} + e_{n-1} \) |
| 0 | 1 | 1 | 0 | \( e_{n-1} + e_n \) | \( e_n \) |
| 0 | 1 | 1 | 1 | \( e_{n-2} + e_{n-1} + e_n \) | \( e_n \) |
| 1 | 0 | 0 | 0 | \( e_{n-2} + e_{n-1} \) | \( e_n \) |
| 1 | 0 | 0 | 1 | \( e_{n-2} + e_{n-1} \) | \( e_{n-2} + e_n \) |
| 1 | 0 | 1 | 0 | \( e_n \) | \( e_{n-1} \) |
| 1 | 0 | 1 | 1 | \( e_{n-2} + e_{n-1} \) | \( e_n \) |
| 1 | 1 | 0 or 1 | 0 or 1 | \( e_{n-1} \) | \( e_{n-1} + e_n \) |

A simple computation shows that in all cases \( g \) has order 2 and \( \Gamma(g)(\chi) = \chi \), so the result follows. \( \square \)

**Definition 4.3.** Let \( f : H \to K \) be a morphism of abelian groups. We say that a non-trivial subgroup \( A \leq K \) is \( f \)-split if there exist groups \( B \leq H \) and \( C \leq K \) such that \( H \cong A \oplus B \), \( K = A \oplus C \) and \( f \) can be written as \( \text{id}_A \oplus g : A \oplus B \to A \oplus C \) for some \( g : B \to C \).

Henceforward we will make extensive use of this notation applied to \( h_{n+1} : \pi_{n+1}(X) \to H_{n+1}(X) \), the Hurewicz morphism. We prove the following:

**Lemma 4.4.** Let \( X \) be an \( A^2 \)-polyhedron, \( n \geq 2 \). Let \( A \leq H_{n+1}(X) \) be an \( h_{n+1} \)-split subgroup, thus \( H_{n+1}(X) = A \oplus C \) for some abelian group \( C \). Then, for every \( f_A \in \text{Aut}(A) \) there exists \( f \in \mathcal{E}(X) \) inducing \( (\text{id}, f_A \oplus \text{id}_C, \text{id}) \in B^{n+2}(X) \).
Proof. By hypothesis $H_{n+1}(X) = A \oplus C$, $\pi_{n+1}(X) \cong A \oplus B$, for some abelian group $B$, and $h_{n+1}$ can be written as $\id_A \oplus g$ for some morphism $g : B \to C$. Thus, for every $f_A \in \text{Aut}(A)$ we have a commutative diagram

$$
\begin{array}{cccccc}
H_{n+2}(X) & \xrightarrow{b_{n+2}} & \Gamma^1_n(H_n(X)) & \xrightarrow{f_A \oplus \id_B} & A \oplus B & h_{n+1} & A \oplus C & 0 \\
\id & & \id & & f_A \oplus \id_B & & f_A \oplus \id_C \\
H_{n+2}(X) & \xrightarrow{b_{n+2}} & \Gamma^1_n(H_n(X)) & & & & & \\
\end{array}
$$

Hence $(\id, f_A \oplus \id_C, \id) \in B^{n+2}(X)$, and by Theorem \ref{thm:2.3} there exists $f \in \mathcal{E}(X)$ such that $H_{n+1}(f) = f_A \oplus \id_C$, $H_{n+2}(f) = \id$ and $H_n(f) = \id$. \hfill \Box

The following lemma is crucial in the proof of Theorems \ref{thm:1.1} and \ref{thm:1.3}.

Lemma 4.5. Let $X$ be an $A^n_2$-polyhedron, $n \geq 2$. Suppose that there exist $h_{n+1}$-split subgroups of $H_{n+1}(X)$. Then:

1. If $n \geq 3$, $B^{n+2}(X)$ is either trivial or it has elements of even order.
2. If $B^4(X)$ is finite and non-trivial, then it has elements of even order.

Proof. First of all, observe that we just need to consider when $H_n(X)$ is an elementary abelian 2-group. In other case, the result is a consequence of Lemma \ref{lem:3.4}.

Let $A$ be an arbitrary $h_{n+1}$-split subgroup of $H_{n+1}(X)$. If $A \neq \mathbb{Z}_2$, there is an involution $\iota \in \text{Aut}(A)$ that induces, by Lemma \ref{lem:4.3}, an element $(\id, \iota \oplus \id, \id) \in B^{n+2}(X)$ of order 2, and the result follows. Hence we can assume that every $h_{n+1}$-split subgroup of $H_{n+1}(X)$ is $\mathbb{Z}_2$.

Both assumptions, $H_n(X)$ being an elementary abelian 2-group and every $h_{n+1}$-split subgroup of $H_{n+1}(X)$ being $\mathbb{Z}_2$, imply that $H_{n+1}(X)$ is a finite 2-group. Indeed, since $H_n(X)$ is finitely generated, $\Gamma^1_n(H_n(X))$ is a finite 2-group and so is $\text{coker} b_{n+2}$. Then, since $H_{n+1}(X)$ is also finitely generated, any direct summand of $H_{n+1}(X)$ which is not a 2-group would be $h_{n+1}$-split, contradicting our assumption that every $h_{n+1}$-split subgroup of $H_{n+1}(X)$ is $\mathbb{Z}_2$.

To prove our lemma, we start with the case $A = H_{n+1}(X)$ is $h_{n+1}$-split.

When $H_{n+2}(X) = 0$, the $\Gamma$-sequence of $X$ becomes then the short exact sequence

$$
0 \to \Gamma^1_n(H_n(X)) \to \Gamma^1_n(H_n(X)) \oplus \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0.
$$

Notice that any automorphism of order 2 in $H_n(X)$ yields an automorphism of order 2 in $\Gamma^1_n(H_n(X))$. Since $\Gamma^1_n(H_n(X))$ is injective on morphisms: it is immediate for $n \geq 3$, and for $n = 2$ we apply Lemma \ref{lem:4.1}. As our sequence is split, any $f \in \text{Aut}(H_n(X))$ induces the $\Gamma$-isomorphism $(\id, \iota, f)$ of the same order. Hence, for $H_n(X) \neq \mathbb{Z}_2$ it suffices to consider an involution. For $H_n(X) = \mathbb{Z}_2$, since by hypothesis $H_{n+1}(X) = \mathbb{Z}_2$ and $H_{n+2}(X) = 0$, the only $\Gamma$-isomorphism is $(\id, \iota, \id)$ and therefore $B^{n+2}(X)$ is trivial as claimed.

When $H_{n+2}(X) \neq 0$, for $n \geq 3$ the result follows directly from Lemma \ref{lem:3.5}. For $n = 2$ we also assume that $B^4(X)$ is finite and non-trivial. Hence, since $H_2(X)$ is an elementary abelian 2-group, Proposition \ref{prop:3.3} implies that $H_4(X) = \mathbb{Z}$. Then, if a $\Gamma$-isomorphism of the form $(-, f, \iota, \id)$ exists, it is of even order. In particular, if $\text{Im} b_4$ is a subgroup of $\Gamma(H_2(X))$ of order 2, $(-, \id, \id, \id)$ is a $\Gamma$-isomorphism of even order.

Assume otherwise that $\text{Im} b_4$ is a group of order 4. If a $\Gamma$-isomorphism $(\id, f, \iota, \id)$ of odd order exists, then $\Gamma(f) \circ b_4 = b_4$. In this situation, by Lemma \ref{lem:4.2} for $\chi = b_4(1)$, there exists $g \in \text{Aut}(H_2(X))$ an automorphism of order 2 such that $\Gamma(g)b_4(1) = b_4(1)$. Moreover, as we are in the case $A = H_2(X)$ being $h_2$-split, $(\id, g, \id) \in B^3(X)$ is a $\Gamma$-isomorphism of order 2.

We deal now with the case $A \leq H_{n+1}(X)$. Since $A = \mathbb{Z}_2$ is a proper $h_{n+1}$-split subgroup of $H_{n+1}(X)$, there exist non-trivial groups $B$ and $C$ such that

$$
\begin{align*}
\pi_{n+1}(X) &= \mathbb{Z}_2 \oplus B & h_{n+1} & \mathbb{Z}_2 \oplus C & H_{n+1}(X) \\
(t, b) &\mapsto & (t, g(b))
\end{align*}
$$
We can also assume that there are no subgroups in $A_n$. By construction, $h_{n+1} = fh_{n+1}$, and if $(t, b) \in \ker b_{n+2} = \ker h_{n+1}$ (thus $g(b) = 0$), then $\Omega(t, b) = (t, b)$. In other words, $(\id, f, \id) \in B^{n+2}(X)$ and it has order 2.

We now prove our main results.

**Proof of Theorem 3.3.** Assume that $H_n(X)$ and $H_{n+1}(X)$ are elementary abelian 2-groups, and $H_{n+2}(X) = 0$. Otherwise, there would already be elements of order 2 in $B^{n+2}(X)$ as a consequence of Lemma 3.5.

Write $H_n(X) = \oplus I \mathbb{Z}_2$, $I$ an ordered set. Since $n \geq 3$, $\Gamma_n^1 = - \otimes \mathbb{Z}_2$, so $\Gamma_n^1(H_n(X)) = H_n(X)$. We can also assume that there are no subgroups in $H_{n+1}(X)$ that are $h_{n+1}$-split. In other case, we would deduce from Lemma 3.5 that there are elements of order 2 in $B^{n+2}(X)$. Thus $H_{n+1}(X) = \oplus J \mathbb{Z}_2$ with $J \subset I$, and the $\Gamma$-sequence corresponding to $X$ is

$$0 \to \bigoplus J \mathbb{Z}_2 \xrightarrow{b} \left( \bigoplus I - J \mathbb{Z}_2 \right) \oplus \left( \bigoplus J \mathbb{Z}_4 \right) \xrightarrow{b} \bigoplus J \mathbb{Z}_2 \to 0.$$ 

We may rewrite the sequence as

$$0 \to \left( \bigoplus I - J \mathbb{Z}_2 \right) \oplus \left( \bigoplus J \mathbb{Z}_2 \right) \xrightarrow{b} \left( \bigoplus I - J \mathbb{Z}_2 \right) \oplus \left( \bigoplus J \mathbb{Z}_4 \right) \xrightarrow{b} \bigoplus J \mathbb{Z}_2 \to 0$$

and assume that $b(x, y) = (x, 2y)$ and $h(x, y) = y \mod 2$. It is clear that any $f \in \text{Aut} \left( \bigoplus I - J \mathbb{Z}_2 \right)$ induces a $\Gamma$-isomorphism $(0, \id, f \oplus \id)$ of the same order.

On the one hand, for $|I - J| \geq 2$, $\bigoplus I - J \mathbb{Z}_2$ has an involution and therefore $B^{n+2}(X)$ has elements of even order. On the other hand, for $|I - J| < 2$, we consider the remaining possibilities.

Suppose that $|I - J| = 1$. Then, $\pi_{n+1}(X) = \mathbb{Z}_2 \oplus (\bigoplus J \mathbb{Z}_4)$. If $J$ is trivial, $B^{n+2}(X)$ is clearly trivial as well. Otherwise, suppose that $I - J = \{i\}$ and choose $j \in J$. Define $f \in \text{Aut} \left( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus (\bigoplus I - \{i,j\} \mathbb{Z}_2) \right)$ by $f(x, y, z) = (x, x + y, z)$ and $g \in \text{Aut} \left( \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus (\bigoplus I - \{i,j\} \mathbb{Z}_4) \right)$ by $g(x, y, z) = (x, 2x + y, z)$. Then $(\id, \id, f)$ is a $\Gamma$-isomorphism of order 2 since we have a commutative diagram

$$\begin{array}{c}
0 \to Z \oplus Z \oplus (\bigoplus_{I \setminus \{i,j\}} Z) & \xrightarrow{f} & Z \oplus Z \oplus (\bigoplus_{I \setminus \{i,j\}} Z) & \xrightarrow{g} & Z \oplus Z \oplus (\bigoplus_{I \setminus \{i,j\}} Z) \to 0 \\
& & & & \\
0 \to Z \oplus Z \oplus (\bigoplus_{I \setminus \{i,j\}} Z) & \xrightarrow{g} & Z \oplus Z \oplus (\bigoplus_{I \setminus \{i,j\}} Z) & \xrightarrow{id} & Z \oplus Z \oplus (\bigoplus_{I \setminus \{i,j\}} Z) \to 0.
\end{array}$$

Suppose that $I = J$. If $H_n(X) = H_{n+1}(X) = \mathbb{Z}_2$, $B^{n+2}(X)$ is trivial. If not, choose $i, j \in I$ and define maps $f \in \text{Aut} \left( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus (\bigoplus_{I \setminus \{i,j\}} \mathbb{Z}_2) \right)$ by $f(x, y, z) = (y, x, z)$, and $g \in \text{Aut} \left( \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus (\bigoplus_{I \setminus \{i,j\}} \mathbb{Z}_4) \right)$ by $g(x, y, z) = (y, x, z)$. We have the following commutative diagram

$$\begin{array}{c}
0 \to Z \oplus Z \oplus (\bigoplus_{I \setminus \{i,j\}} Z) & \xrightarrow{f} & Z \oplus Z \oplus (\bigoplus_{I \setminus \{i,j\}} Z) & \xrightarrow{g} & Z \oplus Z \oplus (\bigoplus_{I \setminus \{i,j\}} Z) \to 0 \\
& & & & \\
0 \to Z \oplus Z \oplus (\bigoplus_{I \setminus \{i,j\}} Z) & \xrightarrow{g} & Z \oplus Z \oplus (\bigoplus_{I \setminus \{i,j\}} Z) & \xrightarrow{f} & Z \oplus Z \oplus (\bigoplus_{I \setminus \{i,j\}} Z) \to 0.
\end{array}$$

Then, $(0, f, f)$ is a $\Gamma$-isomorphism of order 2.

As a consequence, we obtain a negative answer to the problem of realising groups as self-homotopy equivalences of $A^2_0$-polyhedra:

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**THE GROUP OF SELF-HOMOLOGY EQUIVALENCES OF $A^2_0$-POLYHEDRA.**
COROLLARY 4.6. Let $G$ be a non nilpotent finite group of odd order. Then, for any $n \geq 3$ and for any $A^n$-polyhedron $X$, $G \not\cong \mathcal{E}(X)$.

PROOF. Assume that there exists an $A^n$-polyhedron $X$ such that $\mathcal{E}(X) \cong G$. Then, if $\mathcal{E}(X) \neq \mathcal{E}_i(X)$, the quotient $\mathcal{E}(X)/\mathcal{E}_i(X)$ is a finite group of odd order, which contradicts Theorem 1.3. Thus $G \cong \mathcal{E}(X) = \mathcal{E}_i(X)$. However, since $X$ is a 1-connected and finite-dimensional CW-complex, $\mathcal{E}_i(X)$ is a nilpotent group, [8, Theorem D], which contradicts the fact that $G$ is non nilpotent.

We end this paper by proving our second main result:

PROOF OF THEOREM 1.3. By hypothesis $B^4(X)$ is a finite group of odd order. From Lemma 3.4 we deduce that $H_2(X)$ is an elementary abelian 2-group and from Proposition 2.1 that $\Gamma(H_2(X))$ is a 2-group. In particular, every element of $\Gamma(H_2(X))$ is of finite order, and therefore, by Proposition 3.6 rank $H_4(X) \leq 1$ so we have Theorem 1.3(1). Now, any element in $B^4(X)$ is of the form $(0, f_2, f_3)$ if $H_4(X) = 0$ or $(id, f_2, f_3)$ if $H_4(X) = \mathbb{Z}$. Notice that a $\Gamma$-morphism of the form $(-, id, f_2, f_3)$ has even order thus it cannot be a $\Gamma$-isomorphism under our hypothesis. Therefore, if $H_4(X) = \mathbb{Z}$, then $b_4(1)$ generates a $\mathbb{Z}_4$ factor in $\Gamma(H_2(X))$, and under our hypothesis the equation

$$\text{rank } \Gamma(H_2(X)) = \text{rank } H_4(X) + \text{rank}(\text{coker } b_4)$$

holds for rank $H_4(X) \leq 1$.

Observe that any $\Gamma$-isomorphism of $X$ induces a chain morphism of the short exact sequence

$$0 \to \text{coker } b_4 \to \pi_3(X) \xrightarrow{b_3} H_3(X) \to 0.$$ 

We will draw our conclusions from this induced morphism, which can be seen as an automorphism of $\pi_3(X)$ that maps the subgroup $i_2(\text{coker } b_4)$ to itself, thus inducing an isomorphism on the quotient, $H_3(X)$.

As we mentioned above, $\Gamma(H_2(X))$ is a 2-group. Then coker $b_4$ is a quotient of a 2-group so a 2-group itself. We claim that $H_3(X)$ is also a 2-group: otherwise, $H_3(X)$ has a summand whose order is either infinite or odd and therefore this summand would be $h_3$-split, which from Lemma 4.5 implies that $B^4(X)$ has elements of even order, leading to a contradiction. Since coker $b_4$ and $H_3(X)$ are 2-groups, so is $\pi_3(X)$, proving thus Theorem 1.3(2).

Moreover, no subgroup of $H_3(X)$ can be $h_3$-split as a consequence of Lemma 4.5 and thus, rank $H_3(X) \leq \text{rank}(\text{coker } b_4) = \text{rank} \Gamma(H_2(X)) - \text{rank } H_4(X)$. We can compute rank $\Gamma(H_2(X))$ using Proposition 2.1 and immediately obtain Theorem 1.3(3).

Now for a 2-group $G$, define the subgroup $\Omega_1(G) = \{g \in G : \text{ord}(g) \equiv 2\}$. One can easily check that $\Omega_1(\pi_3(X)) \leq i_2(\text{coker } b_4)$ and, from [11, Ch. 5, Theorem 2.4], we obtain that any automorphism of odd order of $\pi_3(X)$ acting as the identity on $i_2(\text{coker } b_4)$ must be the identity.

Then, if $(id, f_3, f_2) \in B^4(X)$ is a $\Gamma$-morphism with $f_3$ non-trivial, $f_2$ has odd order, so we may assume that $\Omega : \pi_3(X) \to \pi_3(X)$ (see Definition 2.2) has odd order too. By the argument above, it must induce a non-trivial homomorphism on $i_2(\text{coker } b_4)$ and therefore $f_2$ is non-trivial as well. Thus, the natural action of $B^4(X)$ on $H_2(X)$ must be faithful, since any $\Gamma$-automorphism $(id, f_3, f_2) \in B^4(X)$ induces a non-trivial $f_2 \in \text{Aut } (H_2(X))$. Then, Theorem 1.3(4) follows. □

References

[1] Martin Arkowitz, The group of self-homotopy equivalences—a survey, Groups of self-equivalences and related topics (Montreal, PQ, 1988), Lecture Notes in Math., vol. 1425, Springer, Berlin, 1990, pp. 170–203.

[2] ———, Problems on self-homotopy equivalences, Groups of homotopy self-equivalences and related topics (Gargnano, 1999), Contemp. Math., vol. 274, Amer. Math. Soc., Providence, RI, 2001, pp. 309–315.

[3] Hans-Joachim Baues, Algebraic homotopy, Cambridge Studies in Advanced Mathematics, vol. 15, Cambridge University Press, Cambridge, 1989.

[4] ———, Combinatorial homotopy and 4-dimensional complexes, De Gruyter Expositions in Mathematics, vol. 2, Walter de Gruyter & Co., Berlin, 1991, With a preface by Ronald Brown.

[5] ———, Homotopy type and homology, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1996, Oxford Science Publications.
[6] Mahmoud Benkhalifa, *The group of self-homotopy equivalences of a simply connected and 4-dimensional CW-complex*. Int. Electron. J. Algebra 19 (2016), 19–34.

[7] Cristina Costoya and Antonio Viruel, *Every finite group is the group of self-homotopy equivalences of an elliptic space*, Acta Math. 213 (2014), no. 1, 49–62.

[8] Emmanuel Dror and Alexander Zabrodsky, *Unipotency and nilpotency in homotopy equivalences*, Topology 18 (1979), no. 3, 187–197.

[9] Julien Federinov and Yves Félix, *Realization of 2-solvable nilpotent groups as groups of classes of homotopy self-equivalences*, Topology Appl. 154 (2007), no. 12, 2425–2433.

[10] Yves Félix, *Problems on mapping spaces and related subjects*, Homotopy theory of function spaces and related topics, Contemp. Math., vol. 519, Amer. Math. Soc., Providence, RI, 2010, pp. 217–230.

[11] Daniel Gorenstein, *Finite groups*, Harper & Row, Publishers, New York-London, 1968.

[12] Donald W. Kahn, *Realization problems for the group of homotopy classes of self-equivalences*, Math. Ann. 220 (1976), no. 1, 37–46.

[13] _, *Some research problems on homotopy-self-equivalences*, Groups of self-equivalences and related topics (Montreal, PQ, 1988), Lecture Notes in Math., vol. 1425, Springer, Berlin, 1990, pp. 204–207.

[14] Roger C. Lyndon, and Paul E. Schupp, *Combinatorial group theory*. Reprint of the 1977 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001. xiv+339 pp. ISBN: 3-540-41158-5.

[15] John W. Rutter, *Spaces of homotopy self-equivalences*, Lecture Notes in Mathematics, vol. 1662, Springer-Verlag, Berlin, 1997, A survey.

[16] John H. C. Whitehead, *A certain exact sequence*, Ann. of Math. (2) 52 (1950), 51–110.

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