THE KLEIN-GORDON’S FIELD. A COUNTER-EXAMPLE OF THE CLASSICAL LIMIT

JAUME HARO

Abstract. We will study the Klein-Gordon’s field with an homogeneous external potential, which does not depend on \( \hbar \). We will construct the Fock’s space corresponding to our problem and we will see that there are phenomena of creation and annihilation of pairs particle-antiparticle. Finally, we will see that in dimension 1, when \( \hbar \to 0 \), these phenomena disappear. However, in dimension 2 or 3, when \( \hbar \to 0 \), the creation probability of particle-antiparticle pairs is not zero.

1. Introduction

In this work we will study the classical limit of Klein-Gordon’s field, with an homogeneous potential which does not depend on Planck’s constant.

First we will see that, in this case, the Klein-Gordon’s equation is equivalent to a hamiltonian system, composed by an infinite number of harmonic oscillators with frequencies which depend on time. Once we have seen this equivalence, we will quantize these oscillators and we will obtain the energy and the electric charge operators. With the energy operator, we will obtain the quantum equation of Klein-Gordon’s field. We will also see that we can find all the eigenfunctions of the energy and the electric charge operators. Consequently, with all those eigenfunctions we can construct the Fock’s space.

After that, we will study the quantum dynamic of vacuum state. We will see that, if the space dimension is 1, when \( \hbar \to 0 \), the probability that does not exist any particle-antiparticle pair, converges to 1. However, in dimension 2 or 3, we will prove that, when \( \hbar \to 0 \), this probability does not converge to 1. Consequently, in dimension 2 or 3, the classical limit is not true.

The notation that we are going to use, is the following

\(<,>\) euclidean scalar product.

\(<,>_2\) scalar product of \( L^2 \).

\(||.||_2\) norm \( L^2 \).

\(||.||_2\) norm \( \infty \).

2. The Klein-Gordon’s field with an homogeneous potential

To simplify, we will take \( m = c = e = 1 \).

If we apply the Correspondence Principle \( E \to i\hbar \partial_t, \vec{p} \to -i\hbar \vec{\nabla} \) to the relativistic relation \( E^2 = |\vec{p} + \vec{f}(t)|^2 + 1 \), we obtain the Klein-Gordon’s equation,

\[-\hbar^2 \partial_t^2 \psi = -i\hbar \vec{\nabla} + \vec{f}(t))^2 \psi + \psi.\]

One important property of the K-G’s equation is the electric charge conservation \( \dot{\rho}(t) = 0 \), where \( \rho(t) = \langle i\hbar \partial_t \psi, \psi \rangle >_2 + \langle \psi, i\hbar \partial_t \psi \rangle >_2 \). However there is no norme square conservation \( ||\psi(t)||_2^2 \neq 0 \). Then, to be able to speak about probabilities, we have to consider that the K-G’s field describes an infinity of harmonic oscillators.

1Partially supported by DGESIC (spain), project PB98-0932-C02-01.

Date: 26 February, 2000.
After that, we have to quantize these oscillators to arrive to an equation of this type, \( i\hbar \partial_t \Phi = H\Phi \), where \( H \) is an self-adjoint operator.

### 2.1. The Quantization of Klein-Gordon’s field.

Suppose that the domain is finite. To simplify we take the \( n \)-dimensional interval \([−\pi, \pi]^n\).

The lagrangian and the energy of the system at \( t \) time are:

\[
L(t) = \hbar^2 ||\partial_t \psi||_2^2 - ||(-i\hbar \vec{\nabla} + \vec{f}(t))\psi||_2^2 - ||\psi||_2^2
\]

\[
E(t) = \hbar^2 ||\partial_t \psi||_2^2 + ||(-i\hbar \vec{\nabla} + \vec{f}(t))\psi||_2^2 + ||\psi||_2^2.
\]

We expand \( \psi \) in Fourier’s serie, \( \psi(\vec{x}, t) = \sum_{\vec{k} \in \mathbb{Z}^n} A_{\vec{k}}(t) \psi_{\vec{k}}(\vec{x}) \), with \( \psi_{\vec{k}}(\vec{x}) = \frac{e^{i<\vec{k}, \vec{x}>}}{(2\pi)^{\frac{n}{2}}} \). Then

\[
L(t) = \sum_{\vec{k} \in \mathbb{Z}^n} \frac{\hbar^2}{2}|A_{\vec{k}}|^2 - \epsilon_{\vec{k}}^2(t)|A_{\vec{k}}|^2, \quad \text{where} \quad \epsilon_{\vec{k}}(t) = \sqrt{\hbar^2 + \vec{f}(t)^2} + 1.
\]

With the momenta \( B_{\vec{k}} = \hbar \frac{\partial}{\partial \vec{P}} \), we obtain

\[
E(t) = \sum_{\vec{k} \in \mathbb{Z}^n} \frac{|B_{\vec{k}}|^2}{\hbar^2} + \frac{\epsilon_{\vec{k}}^2(t)}{\hbar^2} |A_{\vec{k}}|^2, \quad \rho(t) = \sum_{\vec{k} \in \mathbb{Z}^n} \frac{i}{\hbar} (A_{\vec{k}}^* B_{\vec{k}} - A_{-\vec{k}}^* B_{-\vec{k}}^*).
\]

We make the real canonical change

\[
B_{\vec{k}} = \frac{\hbar}{\sqrt{2}} (P_{\vec{k}} + i Q_{\vec{k}}), \quad A_{\vec{k}} = \frac{1}{\hbar \sqrt{2}} (Q_{\vec{k}} + i P_{\vec{k}}),
\]

and let \( \omega_{\vec{k}}(t) = \frac{\epsilon_{\vec{k}}(t)}{\hbar} \) be the corresponding frequency, then \( E(t) \) and \( \rho(t) \) take the form

\[
E(t) = \frac{1}{2} \sum_{\vec{k} \in \mathbb{Z}^n} \left( P_{\vec{k}}^2 + \omega_{\vec{k}}^2(t) Q_{\vec{k}}^2 \right) \left( P_{\vec{k}}^2 + \omega_{\vec{k}}^2(t) Q_{\vec{k}}^2 \right)
\]

\[
\rho(t) = \frac{1}{\hbar} \sum_{\vec{k} \in \mathbb{Z}^n} (Q_{\vec{k}} P_{\vec{k}} - Q_{\vec{k}} P_{\vec{k}}).
\]

That is the energy decomposition in oscillators. Notice, the K-G’s equation is equivalent to the hamiltonian system

\[
\begin{cases}
\dot{Q}_{\vec{k}} = P_{\vec{k}} \\
\dot{P}_{\vec{k}} = -\omega_{\vec{k}}^2(t)Q_{\vec{k}}
\end{cases}
\]

\[
\begin{cases}
\dot{Q}_{\vec{k}} = P_{\vec{k}} \\
\dot{P}_{\vec{k}} = -\omega_{\vec{k}}^2(t)\bar{Q}_{\vec{k}}
\end{cases}
\]

Now, to obtain the quantum theory, what we have to do is to quantize these oscillators, i.e. \( P_{\vec{k}} \rightarrow -i\hbar \partial_{\bar{Q}_{\vec{k}}}, P_{\vec{k}} \rightarrow -i\hbar \partial_{Q_{\vec{k}}} \), and the equation will be

\[
i\hbar \partial_t \Phi = \frac{1}{2} \sum_{\vec{k} \in \mathbb{Z}^n} \left[ (-\hbar^2 \partial^2_{\bar{Q}_{\vec{k}}} + \omega_{\vec{k}}^2(t)Q_{\vec{k}}^2) + \frac{(-\hbar^2 \partial^2_{Q_{\vec{k}}} + \omega_{\vec{k}}^2(t)\bar{Q}_{\vec{k}}^2)}{2} \right] \Phi - \sum_{\vec{k} \in \mathbb{Z}^n} \omega_{\vec{k}}(t)\Phi.
\]

Now, we will look for the eigenfunctions of the energy and of the electric charge operators. First, we have to introduce the creation and anihilation operators for particles and antiparticles
Because consequently,\n
Then \n
We construct the vacuum state at \n
Let’s consider \( \phi_{k}^{0,0}(Q_{k}, \bar{Q}_{k}, t) = \sqrt{\frac{\omega_{k}(t)}{\pi \hbar}} e^{-\frac{\omega_{k}(t)}{2\hbar}(Q_{k}^{2} + \bar{Q}_{k}^{2})} \), then the vacuum state at \( t \) time, \( |0 > (t) \), is

because

Starting from this state we will define all the others. In fact, the state \( |1_{k}^{+} > (t) = a_{k}^{+}(t)|0 > (t) \), verifies

consequently, \( |1_{k}^{+} > (t) \) is the state of a particle with energy \( \epsilon_{k}(t) \) at \( t \) time.

The state \( |1_{k}^{-} > (t) = b_{k}^{+}(t)|0 > (t) \), verifies

consequently, \( |1_{k}^{-} > (t) \) is the state of an antiparticle with energy \( \epsilon_{-k}(t) \) at \( t \) time.

In general, we consider series

\[ \{n_{k}\} : \mathbb{Z}^{n} \rightarrow \mathbb{N} \]

and let

\[ |\{n_{k}\}; \{m_{k}\} > (t) = \prod_{k \in \mathbb{Z}^{n}} \frac{(\alpha_{k}(t))^{n_{k}} (b_{k}^{+}(t))^{m_{k}}}{\sqrt{n_{k}! \sqrt{m_{k}!}}} |0 > (t). \]

Then \( |\{n_{k}\}; \{m_{k}\} > (t) \), verifies

\[ E(t)|\{n_{k}\}; \{m_{k}\} > (t) = \sum_{l \in \mathbb{Z}^{n}} \epsilon_{l}(t)(n_{l} + m_{l})|\{n_{k}\}; \{m_{k}\} > (t) \]
\[ \rho(t)|\{n_\vec{k}\}; \{m_\vec{k}\} > (t) = \sum_{\vec{i} \in \mathbb{Z}^n} (n_{\vec{i}} - m_{\vec{i}})|\{n_\vec{k}\}; \{m_\vec{k}\} > (t). \]

Consequently, the state \(|\{n_\vec{k}\}; \{m_\vec{k}\} > (t)\) contains, at \(t\) time, \(n_\vec{k}\) particles and \(m_\vec{k}\) antiparticles with energy \(\epsilon_\vec{k}(t)\), for each \(\vec{k} \in \mathbb{Z}^n\).

3. The counter-example

3.1. Quantum dynamic. First, we study the case \(\vec{f}(t) \equiv 0\), then

\[
E = \frac{1}{2} \left[ \sum_{\vec{k} \in \mathbb{Z}^n} (-\hbar^2 \partial^2_\vec{k} + \omega_\vec{k}^2 Q_\vec{k}^2) + (-\hbar^2 \partial^2_\vec{k} + \omega_\vec{k}^2 Q_\vec{k}^2) \right] - \sum_{\vec{k} \in \mathbb{Z}^n} \omega_\vec{k},
\]

where \(\omega_\vec{k} = \sqrt{(\hbar \vec{k})^2 + 1}\). Notice that the energy does not depend on time, then the eigenvalues \(|\{n_\vec{k}\}; \{m_\vec{k}\} > (t)\equiv |\{n_\vec{k}\}; \{m_\vec{k}\} > (0)\) do not depend on time. Therefore, the solution of the problem

\[
\begin{cases}
  i\hbar \partial_t \Psi > = E \Psi >, \\
  |\Psi > (0) = |\{n_\vec{k}\}; \{m_\vec{k}\} >,
\end{cases}
\]

is

\[ T^t_q |\{n_\vec{k}\}; \{m_\vec{k}\} > = e^{-\sum_{\vec{k} \in \mathbb{Z}^n} \epsilon_\vec{k}(n_{\vec{k}} + m_{\vec{k}}) t}|\{n_\vec{k}\}; \{m_\vec{k}\} > . \]

In particular, \(T^t_q |0 > = |0 >\), i.e., when \(\vec{f}(t) \equiv 0\), the vacuum state is invariant for the quantum dynamic, and there is no creation and anihilation particle-antiparticle pairs.

We now study the vacuum dynamic when \(\vec{f}(t) \neq 0\). Let \(T^t_q |0 > (0)\) be the solution of the problem

\[
\begin{cases}
  i\hbar \partial_t \Psi > = E(t) \Psi >, \\
  |\Psi > (0) = |0 > (0),
\end{cases}
\]

then \(T^t_q |0 > (0) = \prod_{\vec{k} \in \mathbb{Z}^n} T^t_{h^2 \vec{k}} \Phi^{0,0}_{\vec{k}} (Q_\vec{k}, 0, 0)\), where \(T^t_{h^2 \vec{k}} \Phi^{0,0}_{\vec{k}} (Q_\vec{k}, 0, 0)\) is the solution of problem

\[
\begin{cases}
  i\hbar \partial_t \phi = \frac{1}{2} (-\hbar^2 \partial^2_\vec{k} + \omega_\vec{k}^2 Q_\vec{k}^2 - \hbar^2 \partial^2_\vec{k} + \omega_\vec{k}^2 Q_\vec{k}^2) \phi(t) - \omega_\vec{k}(t) \phi(t) + \omega_\vec{k}^2(0) \Phi_{\vec{k}}^{0,0} (Q_\vec{k}, 0, 0)
\end{cases}
\]

(1)

Denote by \(P^0_h (t) = |\langle t | < 0|T^t_q |0 > (0)\rangle^2\), the probability that it does not exist any particle-antiparticle pair at \(t\) time.

Then, we have the

**Theorem 3.1.** Let \(n\) be the dimension of the space and suppose that \(\vec{f} \in C^\infty_0 (0, T)\), then:

If \(n = 1\) we have

\[ \lim_{h \to 0} P^0_h (t) = 1 \quad \forall t \in \mathbb{R} . \]
If \( n = 2 \) or \( 3 \), at \( t \) time such that \( \dot{f}(t) \neq 0 \), we have
\[
\lim_{\hbar \to 0} P_0^0(t) \neq 1.
\]

Consequently, in the case \( n = 2 \) or \( 3 \), at \( t \) time such that \( \dot{f}(t) \neq 0 \), we do not obtain the classical limit.

**Remark.** In dimension 1 the result is valid for periodic potentials, i.e., for \( f(x, t) = \sum_{k=0}^{N} [f_k(t) \sin(kx) + g_k(t) \cos(kx)] \), we have \( \lim_{\hbar \to 0} P_0^0(t) = 1 \).

**Remark.** Le Theorem 3.1 is valid for the Dirac’s field.

4. **Proofs**

To make the proof of theorem we need the following

**Lemma 4.1.** The solution of the problem (4) is
\[
T^i_{\hbar} \phi_{k,0,0}^0(0) = A_k(t) \phi_{k,0,0}^0(t) + \left( -\frac{i \hbar \dot{\phi}(t)}{4e_k(t)} + \hbar^2 B_k(t) \right) \phi_{k,1,1}^0(t) + e^2 \gamma_k(t)
\]
with
\[
|1 - |A_k(t)|^2| \leq \frac{\hbar^2 K}{e_k^2},
\]
\[
|B_k(t)|^2, ||\gamma_k(t)||^2 \leq \frac{K}{e_k^2}; \quad \gamma_k(t) \perp \phi_{k,0,0}^0(t), \phi_{k,1,1}^0(t).
\]
Where,
\[
\phi_{k,1,1}^0(t) = a_k^+(t) b_k^-(t) \phi_{k,0,0}^0(t).
\]
\( K \) is a constant independent on \( \vec{k} \), \( \hbar \) and \( t \).
\[
e_k = \sqrt{|\hbar \vec{k}|^2 + 1}.
\]

With this lemma we can make the

**Proof of Theorem 3.1.**
If \( n = 1 \), \( P_0^0(t) = \prod_{k \in \mathbb{Z}} |A_k(t)|^2 \). We write, \( |A_k(t)|^2 = 1 + \tilde{A}_k(t) \), then
\[
P_0^0(t) = 1 + \frac{1}{2^n} \sum_{k \in \mathbb{Z}} \tilde{A}_k(t) + \frac{1}{2^n} \sum_{\substack{k_1, k_2 \in \mathbb{Z} \\kappa_1 \neq k_2}} \tilde{A}_{k_1}(t) \tilde{A}_{k_2}(t) + \frac{1}{3!} \sum_{\substack{k_1, k_2, k_3 \in \mathbb{Z} \\kappa_i \neq k_j, i \neq j \\kappa_j \neq k_i \\kappa_i \neq k_j}} \tilde{A}_{k_1}(t) \tilde{A}_{k_2}(t) \tilde{A}_{k_3}(t) + \cdots
\]

We bound
\[
|P_0^0(t) - 1| \leq \frac{1}{2^n} \sum_{k \in \mathbb{Z}} |\tilde{A}_k(t)| + \frac{1}{2^n} \sum_{\substack{k_1, k_2 \in \mathbb{Z} \\kappa_1 \neq k_2}} |\tilde{A}_{k_1}(t)||\tilde{A}_{k_2}(t)| + \cdots \leq \sum_{n=1}^{\infty} \frac{1}{n!} \left( \sum_{k \in \mathbb{Z}} |\tilde{A}_k(t)| \right)^n
\]
\[
\leq \sum_{n=1}^{\infty} \frac{1}{n!} \left( K \hbar \sum_{k \in \mathbb{Z}} \frac{\hbar}{e_k} \right)^n \leq \sum_{n=1}^{\infty} \frac{1}{n!} \left( K \hbar \sum_{k \in \mathbb{Z}} \frac{\hbar}{e_k^2} \right)^n,
\]
since \( \sum_{k \in \mathbb{Z}} \frac{\hbar}{e_k} \leq \int_{\mathbb{R}} \frac{dx}{x^2 + 1} + \hbar = \pi + \hbar \), we have
\[
|P_0^0(t) - 1| \leq \sum_{n=1}^{\infty} \frac{1}{n!} (K \hbar (\pi + \hbar))^n = e^{K \hbar (\pi + \hbar)} - 1.
\]
therefore
\[ \lim_{\hbar \to 0} P^0_h(t) = 1. \]

We now study the case \( n = 2 \). The case \( n = 3 \) is analogous.

Denote by \( P^1_h(t) = \sum_{k \in \mathbb{Z}^2} |t | 1 \leq 1^+ \sum_{k | T_q | 0 > (0) |^2, \) the probability that at \( t \) time, does exist a particle-antiparticle pair, then

\[ P^1_h(t) = \sum_{k \in \mathbb{Z}^2} \left| \frac{i\hbar \dot{e}_k(t)}{4c_k^2(t)} + \hbar^2 B_k(t) \right|^2 \prod_{\substack{i \in \mathbb{Z}^2 \atop i \neq k}} |A_i(t)|^2. \]

We calculate

\[ \lim_{\hbar \to 0} \left| P^1_h(t) - \sum_{k \in \mathbb{Z}^2} \frac{\hbar^2 \dot{e}_k^2(t)}{16 c_k^2(t)} P^0_h(t) \right| \leq \lim_{\hbar \to 0} \left| P^1_h(t) - \sum_{k \in \mathbb{Z}^2} \frac{\hbar^2 \dot{e}_k^2(t)}{16 c_k^2(t)} \prod_{\substack{i \in \mathbb{Z}^2 \atop i \neq k}} |A_i(t)|^2 \right| \]

\[ + \lim_{\hbar \to 0} \left| \sum_{k \in \mathbb{Z}^2} \frac{\hbar^2 \dot{e}_k^2(t)}{16 c_k^2(t)} (1 - |A_k(t)|^2) \prod_{\substack{i \in \mathbb{Z}^2 \atop i \neq k}} |A_i(t)|^2 \right| \]

\[ \leq \lim_{\hbar \to 0} \sum_{k \in \mathbb{Z}^2} \left( \frac{\hbar^3 |\dot{e}_k^2(t)|}{2} |B_k(t)| + \hbar^4 |B_k(t)|^2 + \frac{\hbar^2 \dot{e}_k^2(t)}{16 c_k^2(t)} |1 - |A_k(t)|^2| \right). \]

Because of the lemma (4.1) and the relation \( c_k^2 \leq C \dot{e}_k^2(t) \) where \( C = 2(1 + ||\vec{f}||_\infty^2) \), we obtain

\[ \lim_{\hbar \to 0} \left| P^1_h(t) - \sum_{k \in \mathbb{Z}^2} \frac{\hbar^2 \dot{e}_k^2(t)}{16 c_k^2(t)} P_h(t) \right| \]

\[ \leq \lim_{\hbar \to 0} \sum_{k \in \mathbb{Z}^2} \frac{\hbar^2 \dot{e}_k^2(t)}{16 c_k^2(t)} \left( \sqrt{C}||\vec{f}||_\infty + K \hbar(||\vec{f}||_\infty^2 + 1) \right) = 0, \]

because \( \lim_{\hbar \to 0} \sum_{k \in \mathbb{Z}^2} \frac{\hbar^2 \dot{e}_k^2(t)}{16 c_k^2(t)} = \int_{\mathbb{R}^2} \frac{d^2 \vec{x}}{(\vec{x}^2 + 1)^2} = \pi. \)

Therefore, we have proved that

\[ \lim_{\hbar \to 0} P^1_h(t) = \lim_{\hbar \to 0} \sum_{k \in \mathbb{Z}^2} \frac{\hbar^2 \dot{e}_k^2(t)}{16 c_k^2(t)} P^0_h(t). \]

With this result, we can prove that for \( n = 2 \), if \( \vec{f}(t) \neq \vec{0} \), then \( \lim_{\hbar \to 0} P^0_h(t) \neq 1 \). In fact, we take \( t_0 \) such that \( \vec{f}(t_0) \neq \vec{0} \) and we assume that \( \lim_{\hbar \to 0} P^0_h(t_0) = 1. \) Thus, \( \lim_{\hbar \to 0} P^1_h(t_0) = 0. \)
However

\[
\lim_{\hbar \to 0} P^1_{\hbar}(t_0) = \lim_{\hbar \to 0} \sum_{k \in \mathbb{Z}^2} \frac{\hbar^2 \epsilon_k^2(t_0)}{16 \epsilon_k^2(t_0)} \lim_{\hbar \to 0} P^0_{\hbar}(t_0) = (\text{for hypothesis}) = \frac{1}{16} \int_{\mathbb{R}^2} \frac{\langle \hat{f}'(t_0), \vec{x} \rangle^2}{(|\vec{x}|^2 + 1)^3} d\vec{x}^2 \neq 0,
\]

because \( \hat{f}'(t_0) \neq 0 \). Therefore, we have a contradiction and in consequence, \( \lim_{\hbar \to 0} P^0_{\hbar}(t_0) \neq 1 \).

Now, we make the

**Proof of lemma 4.1:**

First, we will construct a semi-classical solution of the problem \([4] \). To search a semi-classical solution, we have to consider the functions \( \phi^{s,s}_k(t) = \frac{(\alpha^{+}_{\vec{k}}(t))^{s} (b^{+}_{-\vec{k}}(t))^{s}}{s!} \phi^{0,0}_k(t) \) with \( s \in \mathbb{N} \).

We write the problem \([4] \) in the form

\[
\begin{align*}
&i\hbar \partial_t \phi = H_\vec{k}(t) \phi \\
&\phi(0) = \phi^{0,0}_k(0),
\end{align*}
\]

where \( H_\vec{k}(t) = \epsilon_\vec{k}(t) (a^{+}_{\vec{k}}(t) a_{\vec{k}}(t) + b^{+}_{-\vec{k}}(t) b_{-\vec{k}}(t)) \). We expand the solution in powers serie of \( \hbar \), in the following form, \( T_\hbar \phi^{0,0}_k(0) = \sum_{s \in \mathbb{N}} \hbar^{s+1} A^{s}_{s,k}(t) \phi^{s,s}_k(t) \). Then, because of following

**Lemma 4.2.**

\[
\phi^{s,s}_k(t) = \frac{\epsilon_\vec{k}(t)}{2 \epsilon_\vec{k}(t)} (s \phi^{s-1,s-1}_k(t) - (s+1) \phi^{s+1,s+1}_k(t)),
\]

we obtain, after having equalized the powers of \( \hbar \), the system:

If \( s = 0 \)

\[
A^{0}_{0,k} = 0; \quad \dot{A}^j_{0,k} + \frac{\epsilon_\vec{k}(t)}{2 \epsilon_\vec{k}(t)} A^{j-1}_{1,k} = 0, \text{ for } j > 0.
\]

If \( s > 0 \)

\[
\begin{align*}
-i \frac{\epsilon_\vec{k}(t)}{2 \epsilon_\vec{k}(t)} A^0_{s-1,k} - 2 \epsilon_\vec{k}(t) A^0_{s,k} &= 0, \\
i \dot{A}^0_{s,k} - i \frac{\epsilon_\vec{k}(t)}{2 \epsilon_\vec{k}(t)} s \epsilon_\vec{k}(t) A^1_{s-1,k} - 2 s \epsilon_\vec{k}(t) A^1_{s,k} &= 0, \\
i \dot{A}^{j-1}_{s,k} + \frac{\epsilon_\vec{k}(t)}{2 \epsilon_\vec{k}(t)} ((s+1) A^{j-2}_{s+1,k} - s A^j_{s-1,k}) - 2 s \epsilon_\vec{k}(t) A^j_{s,k} &= 0, \text{ for } j > 1.
\end{align*}
\]

We obtain the solution of the system by recurrence. In fact

\[
\begin{align*}
A^0_{0,k}(t) &\equiv 1; \quad A^0_{1,k}(t) = -i \frac{\epsilon_\vec{k}(t)}{4 \epsilon_\vec{k}(t)}; \quad A^1_{0,k}(t) = \int_0^t i \frac{\epsilon_\vec{k}(\tau)}{8 \epsilon_\vec{k}(\tau)} d\tau, \\
A^1_{1,k}(t) &\equiv \frac{1}{2 \epsilon_\vec{k}(t)} (i A^0_{1,k} - i \frac{\epsilon_\vec{k}(t)}{2 \epsilon_\vec{k}(t)} A^1_{0,k}); \quad A^0_{2,k}(t) = -i \frac{\epsilon_\vec{k}(t)}{4 \epsilon_\vec{k}(t)} A^0_{1,k}(t), \\
A^2_{0,k}(t) &= -\int_0^t \frac{\epsilon_\vec{k}(\tau)}{2 \epsilon_\vec{k}(\tau)} A^1_{1,k}(\tau) d\tau; \quad A^2_{1,k}(t) = \frac{1}{2 \epsilon_\vec{k}(t)} (i A^1_{1,k}(t) + i \frac{\epsilon_\vec{k}(t)}{2 \epsilon_\vec{k}(t)} (2 A^2_{2,k} - A^0_{2,k})).
\end{align*}
\]
is a semi-classical solution. In fact, we calculate Lemma 4.3. If $\bar{\bar{\epsilon}}_k \leq C \epsilon_k(t)$, we obtain the

\begin{equation}
A^0_{3,k}(t) = -i \frac{\dot{\epsilon}_k(t)}{4 \epsilon_k(t)} A^0_{2,k}(t); \quad A^3_{0,k}(t) = - \int_0^t \frac{\dot{\epsilon}_k(\tau)}{2 \epsilon_k(\tau)} A^2_{1,k}(\tau) d\tau \\
A^1_{2,k}(t) = \frac{1}{4 \epsilon_k(t)} (i \dot{A}^0_{2,k} - i \frac{\dot{\epsilon}_k(t)}{\epsilon_k(t)} A^1_{1,k}); \quad \text{etc.}
\end{equation}

With these solutions, and the relation $\epsilon_k^2 \leq C \epsilon_k(t)$, we obtain the

Lemma 4.3. If $s, j \leq 3$ we have

\begin{align*}
|A^j_{s,k}(t)| &\leq \frac{\bar{C}}{\epsilon_k^{2s+j}} \quad \text{for } s > 0; \quad |A^j_{0,k}(t)| \leq \frac{\bar{C}}{\epsilon_k^j} \quad \text{for } j > 0, \\
|\dot{A}^j_{s,k}(t)| &\leq \frac{g(t)}{\epsilon_k^{2s+j}} \quad \text{for } s > 0; \quad |\dot{A}^j_{0,k}(t)| \leq \frac{g(t)}{\epsilon_k^{2+j}} \quad \text{for } j > 0,
\end{align*}

where $\bar{C}$ is a constant independent on $k$, and $g(t) \in C^\infty_0(0,T)$ is a function independent on $k$.

Now, we show that the function

\begin{align*}
(\hbar \partial_t - H_k) \tilde{\phi}_k(t) &= -2i \hbar^4 \frac{\dot{\epsilon}_k(t)}{\epsilon_k(t)} A^0_{3,k} \phi^{4,4}_k(t) + \hbar^4 (i \dot{A}^0_{3,k} - i \frac{3 \dot{\epsilon}_k(t)}{2 \epsilon_k(t)} A^1_{2,k}) \phi^{3,3}_k(t) \\
&\quad + \hbar^4 (i \dot{A}^1_{2,k} + i \frac{\dot{\epsilon}_k(t)}{2 \epsilon_k(t)} (3A^0_{3,k} - 2A^2_{1,k})) \phi^{2,2}_k(t) \\
&\quad + \hbar^4 (i \dot{A}^2_{1,k} + i \frac{\dot{\epsilon}_k(t)}{2 \epsilon_k(t)} (2A^2_{3,k} - A^3_{0,k})) \phi^{1,1}_k(t).
\end{align*}

We deduce from the lemma [1,3], that

\begin{equation}
||((\hbar \partial_t - H_k) \tilde{\phi}_k(t))||_2^2 \leq \frac{2 \hbar^8}{\epsilon_k^4} (3g^2(t) + 14C \bar{C}^2 \tilde{f}(t)^2).
\end{equation}

Furthermore, if using that

\begin{equation}
||T_k^\hbar \phi^{0,0}_k(0) - \tilde{\phi}_k(t)|| \leq \frac{1}{\hbar} \int_0^t ||((\hbar \partial_t - H_k) \tilde{\phi}_k(\tau))||_2 d\tau,
\end{equation}

we obtain

\begin{equation}
||T_k^\hbar \phi^{0,0}_k(0) - \tilde{\phi}_k(t)|| \leq \frac{1}{\epsilon_k^4} \int_0^t \sqrt{3g^2(\tau) + 14C \bar{C}^2 \tilde{f}(\tau)^2} d\tau \\
\leq \frac{1}{\epsilon_k^4} \int_0^T \sqrt{3g^2(\tau) + 14C \bar{C}^2 \tilde{f}(\tau)^2} d\tau \equiv \frac{\hbar^3 \bar{C}}{\epsilon_k^4}.
\end{equation}

Therefore, $T_k^\hbar \phi^{0,0}_k(0)$ has the form
To make this calculation we will use the following:

Another proof of theorem \[3.1\]:

To finish the work we will make

THE KLEIN–GORDON’S FIELD. A COUNTER-EXAMPLE OF THE CLASSICAL AL LIMIT 9

Finally, if we take

\[ A_\vec{k}(t) = A^0_{0,\vec{k}}(t) + hA^1_{0,\vec{k}}(t) + h^2A^2_{0,\vec{k}}(t) + h^3A^3_{0,\vec{k}}(t) + h^3F_{\vec{k}}(t) \]

\[ B_\vec{k}(t) = A^1_{1,\vec{k}}(t) + hA^2_{1,\vec{k}}(t) + hG_{\vec{k}}(t) \]

\[ \gamma_\vec{k}(t) = (A^0_{2,\vec{k}}(t) + hA^1_{2,\vec{k}}(t) + hI_\vec{k}(t))\phi_\vec{k}^{2,2}(t) + h\beta_\vec{k}(t), \]

and \( K = 4(1 + \vec{C} + \vec{C})^2 \), we obtain the proof of lemma \[1.1\].

To finish the work we will make

**Another proof of theorem \[3.1\]:**

First, we study the case of dimension 2. Since

\[ A^2_{0,\vec{k}}(t) = -\int_0^t \frac{\dot{\epsilon}_\vec{k}(\tau)}{4\epsilon^2_\vec{k}(\tau)}(iA^0_{1,\vec{k}} - \frac{i}{2}\dot{\epsilon}_\vec{k}(\tau)A^1_{0,\vec{k}})d\tau = -\frac{\dot{\epsilon}^2_\vec{k}(t)}{32\epsilon^4_\vec{k}(t)} - \frac{1}{2} \left( \int_0^t \frac{\dot{\epsilon}^2_\vec{k}(\tau)}{8\epsilon^4_\vec{k}(\tau)}d\tau \right)^2, \]

and \( A^3_{0,\vec{k}}(t) \) is imaginary, we have

\[ |A_\vec{k}(t)|^2 = 1 - \frac{\dot{\epsilon}^2_\vec{k}(t)}{16\epsilon^4_\vec{k}(t)} + h^4J_\vec{k}(t), \]

with \( |J_\vec{k}(t)| \leq \frac{K}{t^2} \), where \( K \) is a constant independent on \( \vec{k} \) and \( \hbar \).

Starting from this relation, we have

\[ \lim_{\hbar \to 0} P_{\vec{k}}^6(t) = \lim_{\hbar \to 0} \prod_{\vec{k} \in \mathbb{Z}^2} |A_\vec{k}(t)|^2 = \lim_{\hbar \to 0} \prod_{\vec{k} \in \mathbb{Z}^2} \left( 1 - \frac{h^2 \dot{\epsilon}^2_\vec{k}(t)}{16\epsilon^4_\vec{k}(t)} \right). \]

We now calculate

\[ \prod_{\vec{k} \in \mathbb{Z}^2} \left( 1 - \frac{h^2 \dot{\epsilon}^2_\vec{k}(t)}{16\epsilon^4_\vec{k}(t)} \right) = 1 - \frac{h^2}{1!} \sum_{\vec{k} \in \mathbb{Z}^2} \frac{\dot{\epsilon}^2_\vec{k}(t)}{16\epsilon^4_\vec{k}(t)} + \frac{h^4}{2!} \sum_{\vec{k}_1, \vec{k}_2 \in \mathbb{Z}^2} \frac{\dot{\epsilon}^2_{\vec{k}_1}(t)}{16\epsilon^4_{\vec{k}_1}(t)} \frac{\dot{\epsilon}^2_{\vec{k}_2}(t)}{16\epsilon^4_{\vec{k}_2}(t)} - \cdots \]

To make this calculation we will use the following

**Lemma 4.4.** If \( n \geq 2 \) and \( f_{\vec{k}} \geq 0 \ \forall \vec{k} \in \mathbb{Z}^n \), then

\[ \left( \sum_{\vec{k}} f_{\vec{k}} \right)^n - \sum_{\vec{k}_1, \ldots, \vec{k}_n} f_{\vec{k}_1} \cdots f_{\vec{k}_n} \leq \frac{n(n-1)}{2} \left( \sum_{\vec{k}} f_{\vec{k}} \right)^{n-2} \sum_{\vec{k}} f^2_{\vec{k}}, \]
consequently,
\[
\left| \prod_{k \in \mathbb{Z}^2} \left( 1 - \hbar^2 \frac{\epsilon_k^2(t)}{16 \epsilon_k^4(t)} \right) - \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\hbar^2 \sum_{k \in \mathbb{Z}^2} \frac{\epsilon_k^2(t)}{16 \epsilon_k^4(t)} \right)^n \right| \\
\leq \sum_{n=2}^{\infty} \frac{n(n-1)}{2} \left( \sum_{k \in \mathbb{Z}^2} \frac{\hbar^2 \epsilon_k^2(t)}{16 \epsilon_k^4(t)} \right)^{n-2} \sum_{k \in \mathbb{Z}^2} \frac{\hbar^4 \epsilon_k^4(t)}{16 \epsilon_k^4(t)} \frac{1}{n!} \\
\leq \frac{\hbar^2 \|\tilde{f}\|_\infty^2}{32} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \left( \sum_{k \in \mathbb{Z}^2} \frac{\hbar^2 \epsilon_k^2(t)}{16 \epsilon_k^4(t)} \right)^n.
\]

We use that, \(\lim_{h \to 0} \sum_{k \in \mathbb{Z}^2} \frac{\hbar^2 \epsilon_k^2(t)}{16 \epsilon_k^4(t)} = \frac{1}{16} \int_{\mathbb{R}^2} \frac{<\tilde{f}(t), \vec{x}>^2}{(|\vec{x}|^2 + 1)^3} d\vec{x}^2\) and \(\sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} = xe^x\), then we obtain
\[\lim_{h \to 0} \left| \prod_{k \in \mathbb{Z}^2} \left( 1 - \hbar^2 \frac{\epsilon_k^2(t)}{16 \epsilon_k^4(t)} \right) - \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\hbar^2 \sum_{k \in \mathbb{Z}^2} \frac{\epsilon_k^2(t)}{16 \epsilon_k^4(t)} \right)^n \right| \leq \lim_{h \to 0} \frac{\hbar^2 \|\tilde{f}\|_\infty^2}{32} \frac{1}{16} \int_{\mathbb{R}^2} \frac{<\tilde{f}(t), \vec{x}>^2}{(|\vec{x}|^2 + 1)^3} d\vec{x}^2 e^{-\frac{\hbar^2 \|\tilde{f}\|_\infty^2}{(|\vec{x}|^2 + 1)^3} d\vec{x}^2} = 0.
\]

By virtue of this result, we have
\[\lim_{h \to 0} P^0_h(t) = \lim_{h \to 0} \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\hbar^2 \sum_{k \in \mathbb{Z}^2} \frac{\epsilon_k^2(t)}{16 \epsilon_k^4(t)} \right)^n = e^{-\frac{\hbar^2 \|\tilde{f}\|_\infty^2}{(|\vec{x}|^2 + 1)^3} d\vec{x}^2}.
\]

Therefore, \(\lim_{h \to 0} P_h(t) < 1\) if \(\tilde{f}(t) \neq 0\).

Now, it is easy to calculate \(\lim_{h \to 0} P^1_h(t)\). In fact, in the first proof of theorem 3.1, we have obtained
\[\lim_{h \to 0} P^1_h(t) = \lim_{h \to 0} \hbar^2 \sum_{k \in \mathbb{Z}^2} \frac{\epsilon_k^2(t)}{16 \epsilon_k^4(t)} P^0_h(t),
\]
then
\[\lim_{h \to 0} P^1_h(t) = \frac{1}{16} \int_{\mathbb{R}^2} \frac{<\tilde{f}(t), \vec{x}>^2}{(|\vec{x}|^2 + 1)^3} d\vec{x}^2 e^{-\frac{\hbar^2 \|\tilde{f}\|_\infty^2}{(|\vec{x}|^2 + 1)^3} d\vec{x}^2}.
\]

In general, let
\[P^*_h(t) = \frac{1}{n!} \sum_{k_1, \ldots, k_n \in \mathbb{Z}^2} |<t| 1^+_{k_1} 1^-_{k_1} \cdots 1^+_{k_n} 1^-_{k_n} |T^*_q|0 > (0)|^2,
\]
be the probability, that at \(t\) time, does exist \(n\) particle-antiparticle pairs. Then we have
\[\lim_{h \to 0} P^*_h(t) = \frac{1}{n!} \left( \frac{1}{16} \int_{\mathbb{R}^2} \frac{<\tilde{f}(t), \vec{x}>^2}{(|\vec{x}|^2 + 1)^3} d\vec{x}^2 \right)^n e^{-\frac{\hbar^2 \|\tilde{f}\|_\infty^2}{(|\vec{x}|^2 + 1)^3} d\vec{x}^2}.
\]
To finish the proof, we have to consider the case of dimension 3. We can prove, proceeding as the case of dimension 2, that

$$\lim_{\hbar \to 0} P_0^1(t) = \lim_{\hbar \to 0} \hbar^2 \sum_{\vec{k} \in \mathbb{Z}^3} \frac{\dot{\epsilon}_k^2(t)}{16 \epsilon_k^4(t)} P_0^0(t).$$

However, $\lim_{\hbar \to 0} \hbar^2 \sum_{\vec{k} \in \mathbb{Z}^3} \frac{\dot{\epsilon}_k^2(t)}{16 \epsilon_k^4(t)} = \infty$ if $\dot{\vec{f}}(t) \neq \vec{0}$, whence we conclude that,

$$\lim_{\hbar \to 0} P_0^0(t) = 0$$

if $\dot{\vec{f}}(t) \neq \vec{0}$.

Another proof of last result, is the following

$$\lim_{\hbar \to 0} P_0^0(t) = \lim_{\hbar \to 0} \prod_{\vec{k} \in \mathbb{Z}^3} \left( 1 - \hbar^2 \frac{\dot{\epsilon}_k^2(t)}{16 \epsilon_k^4(t)} \right).$$

Since $\prod_{\vec{k} \in \mathbb{Z}^3} \left( 1 - \hbar^2 \frac{\dot{\epsilon}_k^2(t)}{16 \epsilon_k^4(t)} \right) \leq \prod_{\vec{k} \in \mathbb{Z}^3} \left( 1 - \frac{L \hbar^3 \dot{\epsilon}_k^2(t)}{16 \epsilon_k^4(t)} \right)$ if $L \hbar \leq 1$, we obtain

$$\lim_{\hbar \to 0} P_0^0(t) \leq \lim_{L \to \infty} \lim_{\hbar \to 0} \prod_{\vec{k} \in \mathbb{Z}^3} \left( 1 - \frac{L \hbar^3 \dot{\epsilon}_k^2(t)}{16 \epsilon_k^4(t)} \right) = \lim_{L \to \infty} e^{-\frac{L \hbar^3}{16} \int_{\mathbb{R}^3} \frac{\dot{\vec{f}}(t,x)^2}{(|x|^2+1)^3} \, dx^2}
= \begin{cases} 
0 & \text{if } \quad \dot{\vec{f}}(t) \neq \vec{0} \\
1 & \text{if } \quad \dot{\vec{f}}(t) = \vec{0}.
\end{cases}$$

5. References

[1] J.D. BJORKEN and S.D. DRELL, Relativistic Quantum Fields; McGraw-Hill Book Co., New York (1965).
[2] P.A. M. DIRAC, The principles of quantum mechanics; Oxford University Press (1958).
[3] S.A. FULLING, Aspects of Quantum Field Theory in Curved Spacetime; Cambridge University Press (1989).
[4] W. GREINER, B. MÜLLER, J. RAFELSKI, Quantum Electrodynamics of Strong Fields; Springer-Verlag (1985).
[5] G.A. HAGEDORN, Semiclassical quantum mechanics I. The $\hbar \to 0$ limit for coherent states; Comm. Math. Phys. 71, no. 1, pag. 77-93, (1980).
[6] J. HARO, El límite clàssic de la mecànica quàntica; Tesi Doctoral, U.A.B. (1997).
[7] J. HARO, Étude classique de l’équation de Dirac; Ann. Fond. Louis de Broglie 23, no. 3-4, pag. 166-172, (1998).
[8] J. HARTHONG, Études sur la mécanique quantique; Asterisque, 111, (1984).
[9] V.P. MASLOV and M.V. FEDORIUK, Semi-classical approximation in quantum mechanics; D. Riedel Publishing Compay, Dordrecht, Holland (1981).

DEPARTAMENT DE MATEMÀTICA APPLICADA I, E.T.S.E.I.B., UNIVERSITAT POLITECNICA DE CATALUNYA
E-mail address: haro@ma1.upc.es