Potentials between D-Branes in a Supersymmetric Model of Space-Time Foam

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Abstract

We study a supersymmetric model of space-time foam with two stacks each of eight D8-branes with equal string tensions, separated by a single bulk dimension containing D0-brane particles that represent quantum fluctuations. The ground-state configuration with static D-branes has zero vacuum energy, but, when they move, the interactions among the D-branes and D-particles due to the exchanges of strings result in a non-trivial, positive vacuum energy. We calculate its explicit form in the limits of small velocities and large or small separations between the D-branes and/or the D-particles. This non-trivial vacuum energy appears as a central charge deficit in the non-critical stringy $\sigma$ model describing perturbative string excitations on a moving D-brane. These calculations enable us to characterise the ground state of the D-brane/D-particle system, and provide a framework for discussing brany inflation and the possibility of residual Dark Energy in the present-day Universe.

CERN-PH-TH/2004-243
December 2004
1 Introduction and Summary

In [1] we have presented a new D-particle model for supersymmetric space-time foam in the context of string theory. The model consists of two stacks of eight D8-branes and their images, each stack being adjacent to an orientifold plane, which compactifies the bulk space [2]. The model has one bulk spatial dimension with a distribution of D0-particles. When the D-branes and D-particles are static, a rigorous perturbative string-theory calculation, valid for weak string couplings $g_s \ll 1$, showed that the vacuum energy of the configuration is zero, as expected in a supersymmetric string vacuum. This result was obtained by calculating, in a $\sigma$-model framework for Dirichlet branes [3], the quantum effective potential between the D-branes and D-particles due to the exchanges of pairs of open strings between them.

In [4], a scenario for brany inflation was proposed, based on collisions between branes in this supersymmetric D-brane model for space-time foam, extending and completing earlier work in this subject [5]. A pair of D-branes was assumed to be moving with a small relative velocity with respect to one another, so that adiabaticity is valid, and hence also string perturbation theory. The collision of the D-branes results in a departure from equilibrium, which, for the perturbative stringy excitations on the observable D-brane world, is quantified by the dynamical generation of a central charge deficit for the respective world-sheet $\sigma$ model. From the space-time point of view, this appears as non-trivial positive vacuum energy. Long after the collision, when string perturbation theory is applicable on the string world sheet, the recoil of the branes is described by appropriate logarithmic conformal field theory deformations, which cause the dark energy to relax adiabatically. This relaxation is described by the underlying dynamics of the super-critical (Liouville) string theory [6, 7] on a recoiling D-brane, in which the Liouville mode is identified with the target time [8]. This identification stems from the appearance of a surplus of central charge in the model, as compared to the critical equilibrium value, and is made manifest by the target-space dynamics, in particular the minimisation of the effective potential in the appropriate low-energy theory describing string dynamics on the D-brane [5].

An important issue in such a scenario is the initial condition for inflation, and in general the characterisation of the non-equilibrium situation following the brane collision. To compute the initial central-charge deficit between the colliding branes, very soon after the collision, lies outside the remit of perturbation theory in general. However, if, for the sake of simplicity, one assumes an adiabatic collision as in [4, 9], then it is possible to estimate the initial central-charge deficit by performing standard critical string/brane theory calculations of annulus amplitudes.
These describe the exchange of open strings when the two branes are close to each other. In general, for adiabatic motion of the branes, such computations are also sufficient to provide estimates of the asymptotic value of the vacuum energy on the observable brane world, when the branes approach equilibrium long after the initial collision.

As was shown in [5], the form of the central-charge deficit at times \( t \) long after the collision of two branes recoiling adiabatically with relative velocity \( v \ll 1 \) (in units of the speed of light \( c = 1 \)), is:

\[
Q^2 = C - C^* \sim Q_\infty^2 + \frac{v^4}{t^2}
\]

where \( Q_\infty \) is the asymptotic (equilibrium) value of the central-charge deficit, and the (relaxing) second contribution is the result of the sudden collision of the branes, which is described by appropriate logarithmic terms [5] in the corresponding conformal field theory. At short times after the collision, one can describe the situation for adiabatic collisions with \( v \ll 1 \) by setting \( Q^2 = Q_{\text{init}}^2(v) \), where \( Q_{\text{init}} \) is independent of time. Both \( Q_{\text{init}} \) and \( Q_\infty \) can be determined by standard (critical) string/brane theory annulus amplitude computations, as they can be identified with the potentials that the brane world feels in the presence of other branes, as a result of the exchange of open string pairs stretched between the branes [9, 13, 3]. The main point of this article is to calculate these potentials explicitly in the supersymmetric D-brane model for space-time foam described in [1]. In this model, there are also contributions to the effective potential due to the presence of the bulk D-particles, which are also be taken into account in our discussion below.

There are various ways in this framework by which branes can collide and produce inflation [4]. According to one scenario, one D-brane from one stack collides with the other stack, and then bounces back to collide again with the stack where it originated, where it eventually stops moving. Another scenario is that two or more D-branes collide, and later return to their initial positions. In these scenarios, inflation on the brane world occurs soon after the first collision [4]. As mentioned above, at early times shortly after the collision, the interaction potential between the colliding D-branes may be expressed as an (approximately) constant central charge deficit in the \( \sigma \) model describing string excitations on the brane world. In such a situation, the (approximately) constant central-charge deficit corresponds to the Hubble parameter of the inflationary era [4]. As the moving D-brane traverses the bulk space, the recoiling brane world traverses a non-trivial distribution of D0-particles, which also contribute to the central charge (Hubble parameter) and affect the D-brane dynamics.

An important point in the analysis of [4] was that the asymptotic state of the configuration,
Figure 1: A model for supersymmetric D-particle foam consisting of two stacks each of eight parallel coincident D8-branes, with orientifold planes (thick dashed lines) attached to them. The space does not extend beyond the orientifold planes. The bulk region of ten-dimensional space in which the D8-branes are embedded is punctured by D0-particles (dark blobs). The two parallel stacks are sufficiently far from each other that any Casimir contribution to the vacuum energy is negligible. Open-string interactions between D0-particles and D8-branes are also depicted (wavy lines). If the D0-particles are stationary or moving parallel to the D8-branes, there is zero vacuum energy on the D8-branes, and the configuration is a consistent supersymmetric string vacuum.

where all the D-branes and D0-particles come to a standstill, is the supersymmetric vacuum of [1], with zero vacuum energy. However, for finite times, the logarithmic conformal field theory analysis has demonstrated the existence of a vacuum energy on the brane that relaxes to zero, scaling with the Robertson-Walker cosmic time $t$ as $t^{-2}$, as seen in (1.1). The vacuum energy is positive. It is crucial in such scenarios to calculate the initial and final values of the central charge, thereby determining the precise relations of the Hubble parameter for inflation and the recoil relative velocities of the branes, as well as the current value of the cosmological constant in our Universe. These relations are essential if one is to make contact with the astrophysical data, and thereby constrain the parameters of the model of [1].

The computations presented here apply critical string theory annulus amplitude calculations to evaluate first the interaction potential between a D0-particle and a D8-brane or O8 orientifold plane in relative motion. We then study D8-brane/D8-brane and D8-brane/O8 orientifold plane potentials. We calculate both long- and short-range limits of the potentials among the
Figure 2: One of the branes in the stacks of Fig. 1 starts moving towards the other stack. As it moves, it crosses D0-particles in the foam, and the (string) interactions with the other branes and D-particles induce a potential on this moving brane.

D8-branes and D0-particles, where the range is always measured relative to the string length $\ell_s = \sqrt{\alpha'}$, where $\alpha'$ is the Regge slope.

These calculations are then applied to the case of a D8-brane moving between a stack of eight D8-branes and an orientifold on one side, and seven D8-branes and the other orientifold on the other side, as shown in the supersymmetric space-time foam configuration of Fig. 2, as well as scenarios with two moving D8-branes, as seen in Fig. 3. As we show, the computations are functions of even powers of the relative velocity, so that the direction of motion is irrelevant. These calculations determine the initial values of the vacuum energy for the different colliding-brane scenarios, and thus the initial values of the central-charge deficit in the respective $\sigma$ models. These values should be matched smoothly with the relaxing value of the vacuum energy, long after the collision, and determine the way in which the vacuum energy approaches zero after reheating. The computations also shed light on the possible value of the vacuum energy near a second collision, where the moving D-brane world may eventually stop, in a model with of two moving D-branes, cf, Fig. 3.

We perform the relevant calculations case by case, following the conventions of [9, 11], examining first the short- and long-range potentials in various simplified cases, and then combining the results to obtain the effective interaction felt by the moving D-brane world in the scenarios of Figs. 2 and 3, with their respective stacks of D-branes, orientifolds and bulk D0-particles.
Figure 3: A scenario with asymmetric colliding branes. We estimate the effective potential felt by the D8-brane world $A$ in the environment of the other D8-branes, D0-particles and O8 orientifold planes.

2 D0-Brane Potentials

The potentials are evaluated by means of annulus world-sheet calculations, representing a stretched pair of open strings emanating from a point of one D-brane and ending up on another point on the other brane. As explained in detail in [13], to leading order in a weak string coupling $g_s$, to which we restrict ourselves throughout this work, in the open-string channel the annulus is equivalent to a trace over all open-string states

$$V_{\text{lightest}}^n \sim v^n \int d^{p+1}k \int_0^{\infty} dt t^{n-1} e^{-t(k^2 + m^2)},$$

where $p$ is the number of Neumann-Neumann coordinates, leading to an expansion in powers of $v^2/r^4$, where $r$ is the distance between the D-branes, for small velocities $v \ll 1$. Convergence of this series implies the existence of a new characteristic minimum length $r_M$, shorter than the string length $\ell_s$ [13]:

$$r_M \approx \sqrt{v\ell_s}, \quad v \ll 1 \quad (2.1)$$

which we use later in the article, when we discuss the short-distance behaviours of the various potentials.

In what follows, we calculate the potentials between moving branes in both short- ($\ell_s \gg r \geq r_M, v \ll 1$) and long-distance regimes ($r \gg \ell_s$), as compared to the string length $\ell_s$, in the context of the supersymmetric model of space-time foam described in [1]. We start with simple cases that pave the way to the final configuration, which involves moving D8-branes in a bulk
filled with supersymmetric D0-particles, that may themselves be moving \(^1\).

### 2.1 D8-Brane/D0-Particle Potential

#### 2.1.1 Short Range

We remind the reader that the annulus amplitude \( \mathcal{A} \) is the integral over target time of the potential between the branes, \( \mathcal{V} \). In what follows, we suppress (for brevity) volume factors \( V_p \), where \( p \) is the number of Neumann coordinates, as well as an overall proportionality factor of \( g_s^2 \), where \( g_s \) is the (weak) string coupling, which appears in front of the world-sheet graphs. These factors will be re-instated at the very end of our computations.

The static D-brane potential between a D0-particle and a D8-brane is \(^{[15,1]}\)

\[
\mathcal{V} = -\frac{1}{2} T_0 R [1 - 1].
\]

(2.2)

When the D0-particle is moving with velocity \( v \) transverse to the D8-brane, this potential is modified to

\[
\mathcal{V} = -\int \frac{dt}{2t} e^{-R^2 t / (2 \pi \alpha')} \left\{ \left( \frac{\vartheta_3(0,q)}{\vartheta_1'(0,q)} \right)^{-1} \left( \frac{\vartheta_2(0,q)}{\vartheta_4'(0,q)} \right)^4 \left( i \vartheta_3(vt,q) / \vartheta_1(vt,q) \right)_{\text{NS}} - \left( \frac{\vartheta_2(0,q)}{\vartheta_4'(0,q)} \right)^{-1} \left( \frac{\vartheta_3(0,q)}{\vartheta_1'(0,q)} \right)^4 \left( i \vartheta_2(vt,q) / \vartheta_1(vt,q) \right)_{\text{R}} \right\},
\]

(2.3)

where \( \Delta = 8 \). The \((-1)^F R\) sector is divergent because the velocity changes the ghost and superghost zero-mode properties. This divergence is cancelled later on by the potential between the D0-brane and a O8 orientifold plane.

To find the potential \( \mathcal{V} \) between the two branes, it is convenient to re-express the annulus world-sheet diagram \( \mathcal{A} \) in terms of a proper time \( (\tau) \) integral:

\[
\mathcal{A} = \int_{-\infty}^{\infty} d\tau \mathcal{V}(v, r^2)
\]

\(^1\)It is a general feature of such calculations that motion of a brane (or a D0-particle) parallel to the direction of another brane does not lead to any force between the two. Therefore, we always consider transverse relative motions.
\[ V(v, r^2) = - \int_0^\infty \frac{dt}{2t} \sqrt{tv} e^{-r^2t/2\pi \alpha'} \{ \ldots \}, \quad (2.4) \]

where the quantity in the \{ \} brackets is the same as the corresponding one in (2.3), \( r^2 \equiv R^2 + v^2r^2 \), and the additional factors are to counteract the effect of the integral

\[ \int_{-\infty}^{\infty} dt \exp [-t[R^2 + v^2r^2]/2\pi \alpha'] = e^{-tR^2 \sqrt{2\pi^2 \alpha'/v}} [\text{erf}(\infty) - \text{erf}(-\infty)] = e^{-tR^2 \sqrt{2\pi^2 \alpha'/v}}. \quad (2.5) \]

The short-range potential corresponds to the case \( r \ll \ell_s \), which formally corresponds to taking the \( t \to \infty \) limit, keeping \( r^2t \) finite but arbitrary. Notice that the minimum-distance condition (2.1) is mathematically compatible with this limit, assuming small velocities, \( v \ll 1 \).

In that case, expanding the factor in curly brackets to the lowest order in \( v \) gives

\[ V(v, r^2) = - \int_0^\infty \frac{dt}{4t^{3/2}} e^{-r^2t/2\pi \alpha'} [-1 + \frac{1}{3} i^2 v^2] \quad (2.6) \]

giving

\[ V = -\frac{r}{4\pi \alpha'} - \frac{\pi \alpha' v^2}{12r^3}. \quad (2.7) \]

Some remarks are now in order. We first notice that the velocity-independent term is due to the fact that an isolated D8-brane is not permitted, due to flux conservation. As we discuss in the next subsection, this term will indeed be cancelled when we consider the case of a D8-brane in the presence of an orientifold. This is consistent from the point of view of flux conservation because the overall RR charge in a compact space must be zero. Secondly, we observe that the minimum short-distance condition (2.1), guarantees that the second term in (2.7) is less than \( 1/\ell_s \), which renders well defined an effective low-energy field theory with such a potential on the brane.

### 2.1.2 Long Range

To find the velocity dependence of the long-range potentials, the \( t \to 0 \) limit must be taken, by applying the converse arguments to the short-range case, as before. The exponential terms in the Jacobi Theta functions are \( q = e^{-\pi t} \) thus a modular transformation taking \( t \to 1/t \) is required. Properties of the Jacobi Theta functions, including modular transformations, can be found in the appendix. Expanding the potentials before after the modular transformation gives

\[ \gamma_{long}^{D0-D8} = - \int \frac{d\tau}{4\sqrt{r}} (-1 + v^2/2) e^{-r^2/2\pi \alpha'} = -\frac{r}{4\pi \alpha'} + \frac{rv^2}{8\pi \alpha'}. \quad (2.8) \]
2.2 O8-Plane/D0-Particle Potential

2.2.1 Long Range

The orientifold potential is less straightforward than the annulus. The velocity dependent amplitude is most easily constructed in the closed string channel, and then modular transformed to the open string channel to recover the short range behaviour. In the closed string channel, the orientifold interaction takes the form of a cross-cap, which complexifies the distance and twists the oscillators.

The static open string potential between a D0-particle and an O8-plane [15] involves the interaction of the D0-particle with its image, thus the potential is the sum of an annulus and a M"obius graph. In the open string channel this is

\[
V = -\int \frac{dt}{4t} (8\pi^2 \alpha' t)^{-1/2} e^{-4R^2 t/2\pi\alpha'} \left[ \frac{f_3^8(q) - f_4^8(q) - f_2^8(q)}{f_1^8(q)} + 16 \times \frac{-f_4^8(iq) + f_3^8(iq) + f_2^8(iq)}{f_2^8(iq)} \right]
\]

\[
= 8T_0 R [1 + 1]. \tag{2.9}
\]

The difference in dimension between the brane and the orientifold gives the potential the same form as a $Dp - Dp'$ potential but with the significant difference that the boundary conditions are no longer $NN$, $DD$ or $ND$ but twisted versions, thus the theta functions within are different.

To obtain the form of the potential for the moving case, it is simplest to start with the open string annulus interaction and then move to the closed string channel. From there it is straightforward to transform the amplitude into a cross-cap (giving long distance behaviour) and then modular transform back into the open string channel to obtain the short distance behaviour.

The open string annulus oscillators have the form

\[
\frac{f_3^8(q) - f_4^8(q) - f_2^8(q)}{f_1^8(q)}. \tag{2.10}
\]

where the terms correspond to NS, NS($-1)^F$ and R respectively. Substituting $t \rightarrow 1/t$ and modular transforming into the closed string channel gives the corresponding cylinder amplitude

\[
\frac{f_3^8(r) - f_2^8(r) - f_1^8(r)}{f_1^8(r)}, \tag{2.11}
\]

where $r = e^{-\pi \tau}$ and the closed string channel NS-NS sector (first two terms) corresponds to the open string NS and NS($-1)^F$ terms and the closed string R-R sector corresponds to the open string R term.
The action of the orientifold on the oscillator trace manifests as a twist in the oscillators and a modification of the argument of the functions \( r \to ir \):

\[
\begin{align*}
\alpha_{n}^{0.9} & \quad \rightarrow \quad (-1)^{n} \alpha_{n}^{0.9} \\
\alpha_{n-8}^{1} & \quad \rightarrow \quad -(-1)^{n} \alpha_{n-8}^{1},
\end{align*}
\]

(2.12)
giving

\[
\frac{-f_{4}^{8}(ir) + f_{3}^{8}(ir) + f_{2}^{8}(ir)}{f_{2}^{8}(ir)}.
\]

(2.13)

where the last term is the R-R sector contribution \( R((-1)^{F} \Omega I_{9}) \), which is divergent in the moving case for the same reasons as the D8-D0 case. As will be seen below, when one considers the combination of 16 D8-branes and an O8-plane, an exact cancellation of terms in the individual potentials occurs, including the divergences from the O8-D0 and D8-D0 interactions. This is a demonstration of why Type IIA string theory has the form that it does, requiring two O8-planes and 32 D8-branes for stability.

When the D0-brane is moving transverse to the orientifold, the static case becomes

\[
\mathcal{V}(v, r^{2}) = -\frac{2v}{\sqrt{2\pi^{2}\alpha'}} \int_{0}^{\pi} \frac{d\tau}{\sqrt{\tau}} e^{-4v^{2}/(2\pi \alpha')} \left( \frac{\vartheta_{4}(0|i\tau + 1/2)}{\vartheta_{3}(0|i\tau + 1/2)} \right) \times 
\left\{ \left( \frac{\vartheta_{3}(0|i\tau + 1/2)}{\vartheta_{2}(0|i\tau + 1/2)} \right)^{4} \frac{\vartheta_{4}(v|i\tau + 1/2)}{\vartheta_{4}(0|i\tau + 1/2)} - \left( \frac{\vartheta_{4}(0|i\tau + 1/2)}{\vartheta_{2}(0|i\tau + 1/2)} \right)^{4} \frac{\vartheta_{3}(v|i\tau + 1/2)}{\vartheta_{3}(0|i\tau + 1/2)} \right\},
\]

(2.14)

As the D0-brane and O8-plane have different dimensions, the velocity dependence of the potential will be \( \mathcal{O}(v^{2}) \), thus when considering these interactions in isolation the annulus graph corresponding to the interaction of the D-brane with its image does not contribute. Due to the specific arrangement of branes under consideration, the cancellations mentioned above mean that this term solely determines the form of the potential. Expanding and integrating (2.14) gives

\[
\mathcal{V}_{long}^{D0-O8} = \frac{4r}{\pi \alpha'} - \frac{2rv^{2}}{\pi \alpha'}
\]

(2.15)

Summing the contributions from the 16 D8-branes with that from the orientifold plane, the overall potential cancels exactly (the cancellation occurs at the level before the expansion in

\[\text{[2,13]}\]

The Möbius amplitude has contributions from NS(\( \Omega I_{9} \)), NS((-1)^{F} \Omega I_{9}) and R((-1)^{F} \Omega I_{9}) [15], so, using the argument above, the cylinder R term changes to R((-1)^{F} \Omega I_{9}). The (-1)^{F} projection is combined with the \( \Omega I_{9} \) projection, leaving it unchanged.
powers of the velocity). The annulus graph coming from the interaction of the D0-brane with its image is now the only overall non-zero contribution, giving

\[
V_{D0-D0}^{long} = -\frac{15\pi^3\alpha'^3v^4}{2r^7}.
\] (2.16)

### 2.2.2 Short Range

As before, to find the short-range potential, a modular transformation must be performed. To convert a cross-cap into a Möbius strip requires the transformation \( \tau = 1/4t \), details of which can be found in the Appendix. Using this transformation and writing in terms of a proper-time integral gives

\[
V(v, r^2) = -\frac{4v}{\sqrt{2\pi^2\alpha'}} \int \frac{dt}{\pi \sqrt{t}} e^{-4t/(2\pi\alpha')} \left( \frac{\partial_1'(0|it + 1/2)}{\partial_1'(2it|it + 1/2)} \right) \times \\
\left\{ \left( \frac{\partial_2(0|it + 1/2)}{\partial_2(0|it + 1/2)} \right)^4 \frac{\partial_3(2vt|it + 1/2)}{\partial_3(0|it + 1/2)} - \left( \frac{\partial_2(0|it + 1/2)}{\partial_2(0|it + 1/2)} \right)^4 \frac{\partial_4(2it|it + 1/2)}{\partial_4(0|it + 1/2)} \right\}.
\] (2.17)

Expanding and integrating, we find

\[
V_{D0-08}^{short} = \frac{4r}{\pi\alpha'} - \frac{2\pi\alpha'v^2}{3r^3}.
\] (2.18)

Summing the contribution from the 16 D8-branes with this result shows that the velocity-independent terms are cancelled:

\[
V_{short} = 16 \left( -\frac{r}{4\pi\alpha'} - \frac{\pi\alpha'v^2}{12r^3} \right) + \frac{4r}{\pi\alpha'} - \frac{2\pi\alpha'v^2}{3r^3} = -\frac{2\pi\alpha'v^2}{r^3},
\] (2.19)

in agreement with [12, 10].

#### 2.2.3 Discussion

When there are equal numbers of D8-branes on either side of the D0-particle, the terms linear in \( r \) are cancelled, leaving an overall long-range potential

\[
V_{long} = -\frac{15\pi^3\alpha'^3v^4}{r^7}.
\] (2.20)

For the dynamical situations under consideration in later sections, the distribution of D8-branes is asymmetric, thus this cancellation will not occur. For an asymmetric case, terms independent of the velocity would be present in the potentials.
Figure 4: Single string creation due to a D0-particle traversing a D8-brane in the presence of orientifolds, in the model of [1]. (a) The D0-particle (circle) is on the right side of an isolated D8-brane (thin vertical line), which itself is on the right side of a stack of D8-branes (thick line) and an orientifold O8-plane (dashed line). (b) The D0-particle (circle) crosses the D8-brane, creating an elementary string, each half of which is attached on one side of the brane. (c) The interaction of the D0-particle with its image, as it approaches the stack, due to the presence of the orientifold O8-plane. Images of D8-branes and D0-particles with respect to the O8-plane are denoted by grey shading.

This would appear to be a problem, since velocity-independent terms mean that one cannot have a vacuum configuration. The solution is via the mechanism of string creation [16, 17], which is related to the fact that between two equal sets of D8-branes the effective low-energy theory is Type IIA supergravity, whereas between an asymmetric distribution of branes there is massive Type IIA supergravity.

We recall that the system we consider [1] is Type IIA string theory [3], with the following configurations of branes: one orientifold plane located at each of the fixed points of the $S_1/Z_2$ orbifold, $X^9 = 0$ and $X^9 = R$, and eight D8-branes and their images sitting on each orientifold. In the bulk there are some D0-particles, distributed so that interactions between them are negligible. A moving D0-particle in this situation feels a force $\propto v^2$ or $v^4$, and hence zero force at zero velocity.

If one of the D8-branes were to move adiabatically across a D0-particle, there would be an unequal number of D8-branes on either side of the D0-particle. Naively, using only string perturbation theory, the potential felt by the D0-particle would then include linear terms as in
In Type IIA supergravity the D8-brane is the source of a 10-form field strength, with dual $F = \star F_{10}$ that couples to the D0-brane as

$$\mu_0 \int d\tau F A_0.$$  \hfill (2.21)

Here $F$ is piece-wise constant, which means that when the D0-particle traverses the D8-brane it jumps by $\mu_8$, and so $\mu_0 F$ jumps by $\mu_0 \mu_8 = 1/2\pi \alpha'$. This is interpreted as the creation of a string between the branes, which cancels the linear force from the annulus diagram, see Fig. 4. As discussed in [15], when a D0-brane is in a region of massive supergravity with fundamental strings between the D0-brane and D8-branes, the D0-brane can only move adiabatically in the transverse direction. Thus D0-branes in between an asymmetric configuration of D8-branes give no contribution to the vacuum energy; only those in massless Type IIA backgrounds can move. The string creation mechanism ensures that the overall energy of the system is velocity dependent and that there always is a supersymmetric vacuum in the static case.

The maximum distance between the orientifold planes is defined by the harmonic function in the D8-brane metric. When the distribution of branes on the orientifold planes is not symmetric the harmonic function blows up at $|X^9| = |X^9_{\text{critical}}| = 1/(g_s|m|)$, where $g_s$ is the string coupling and $|m|$ is the mass parameter associated with the massive supergravity present in the bulk. When the situation is symmetric, the bulk supergravity returns to normal Type IIA, thus $|m|$ is zero and the critical distance goes to infinity. In physical terms, this gives a critical distance $R < 2\pi \ell_s/(n - 8) g_s$, where there are $16 - 2n$ D8-branes at $X = 0$ and $2n$ D8-branes at $X = \pi R$, which for the symmetric case $n = 8$ yields no upper bound on $R$.

### 3 Interactions of a Moving D8-Brane

We now consider the situation where a D8-brane is moving towards the stack of 8 D8-branes and the O8-plane. The same approach as for the D0-O8 interaction is used, with the main difference being that as the D8-brane and O8-plane have the same dimension the annulus contribution of $O(v^4)$ cannot be discarded.
3.1 D8-Brane/D8-Brane Interaction

3.1.1 Short Distance

The static potential between two D8-branes vanishes

\[ V = - \int \frac{dt}{2t} (8\pi^2 \alpha' t)^{-9/2} e^{-R^2 t/2\pi \alpha'} \left[ \frac{f_3^8(q) - f_2^8(q) - f_1^8(q)}{f_1^8(q)} \right] = 0. \] (3.1)

The potential for the case of a moving brane is similarly

\[ V = - \int \frac{dt}{2t} (8\pi^2 \alpha' t)^{-8/2} e^{-R^2 t/2\pi \alpha'} (2\pi)^3 \times \left\{ \left( \frac{\partial_3(0, q)}{\partial_1(0, q)} \right)^3 \left( \frac{i\partial_3(\nu t, q)}{\partial_1(\nu t, q)} \right) - \left( \frac{\partial_2(0, q)}{\partial_1(0, q)} \right)^3 \left( \frac{i\partial_2(\nu t, q)}{\partial_1(\nu t, q)} \right) - \left( \frac{\partial_4(0, q)}{\partial_1(0, q)} \right)^3 \left( \frac{i\partial_4(\nu t, q)}{\partial_1(\nu t, q)} \right) \right\} , \] (3.2)

which expands to

\[ V = \frac{r v^4}{2^{13}\pi^9 \alpha'^5} \] (3.3)

at short distances and also the same for long distances

\[ V(v, r^2)_{D8-D8}^{long} = - \int_0^{\infty} \frac{dt}{2t^{3/2}} (8\pi^2 \alpha')^{-4} \frac{v^4}{\sqrt{2\pi^2 \alpha'}} e^{-r^2 t/(2\pi \alpha')} \]
\[ = \frac{r v^4}{2^{13}\pi^9 \alpha'^5}. \] (3.4)

3.2 D8-Brane/O8-Plane Interaction

3.2.1 Long Range

The extension of the D8-brane/D8-brane static interaction to the D8-brane/O8-plane case is a simple extension of the previous result:

\[ V = - \int \frac{dt}{4t} (8\pi^2 \alpha' t)^{-8/2} e^{-4R^2 t/2\pi \alpha'} \left[ \frac{f_3^8(q) - f_4^8(q) - f_1^8(q)}{f_1^8(q)} + 16 \times \frac{f_4^8(iq) + f_3^8(iq) - f_2^8(iq)}{f_1^8(iq)} \right] \] (3.5)

Performing the same procedure as before to find the velocity dependent orientifold contribution yields

\[ V(v, r^2) = -\frac{v}{\sqrt{2\pi^2 \alpha'}} \int \frac{d\tau}{4\sqrt{\tau}} e^{-4\tau^2/2\pi \alpha'} (8\pi^2 \alpha')^{-4}(2\pi)^3 \times \]
Expanding,
\[
\mathcal{V}(v, r^2)_{D8-\text{O8}}^{\text{long}} = - \int_0^\infty \frac{d\tau}{4 \sqrt{\tau}} (8 \pi^2 \alpha')^4 \frac{-15 v^4}{\sqrt{2 \pi^2 \alpha'}} e^{-4r^2/2\pi \alpha'}
= \frac{-15 rv^4}{2^{13/2} \pi^9 \alpha'^5},
\]
the form of the potential reflecting the fact that it is interacting with an object of similar dimension with negative RR charge.

### 3.2.2 Short Range

Applying the modular transform $\tau \to 1/4t$, the short range potential is
\[
\mathcal{V} = - \frac{v}{\sqrt{2 \pi^2 \alpha'}} \int \frac{dt}{4\sqrt{t}} e^{-4r^2/2\pi \alpha'} (8 \pi^2 \alpha' t)^{-4}(2\pi)^3 \times
\left\{ \left( \frac{\vartheta_3(0|it)}{\vartheta_1'(0|it)} \right)^3 \left( \frac{\vartheta_3(\nu t|it)}{\vartheta_1(\nu t|it)} \right) - \left( \frac{\vartheta_4(0|it)}{\vartheta_1'(0|it)} \right)^3 \left( \frac{\vartheta_4(\nu t|it)}{\vartheta_1(\nu t|it)} \right) - \left( \frac{\vartheta_2(0|it)}{\vartheta_1'(0|it)} \right)^3 \left( \frac{\vartheta_2(\nu t|it)}{\vartheta_1(\nu t|it)} \right)
+ \frac{16}{8} \left( \frac{\vartheta_3(0|it + 1/2)}{\vartheta_1'(0|it + 1/2)} \right)^3 \left( \frac{\vartheta_3(2\nu t|it + 1/2)}{\vartheta_1(2\nu t|it + 1/2)} \right) + \left( \frac{\vartheta_4(0|it + 1/2)}{\vartheta_1'(0|it + 1/2)} \right)^3 \left( \frac{\vartheta_4(2\nu t|it + 1/2)}{\vartheta_1(2\nu t|it + 1/2)} \right)
+ \left( \frac{\vartheta_2(0|it + 1/2)}{\vartheta_1'(0|it + 1/2)} \right)^3 \left( \frac{\vartheta_2(2\nu t|it + 1/2)}{\vartheta_1(2\nu t|it + 1/2)} \right) \right\}
\]

which upon expansion gives
\[
\mathcal{V}_{D8-\text{O8}}^{\text{short}} = \frac{-15 rv^4}{2^{13/2} \pi^9 \alpha'^5}.
\]

Summing the contributions from the 16 D8-branes in the stack and the orientifold contribution gives the result
\[
\mathcal{V}_{16 \times D8-\text{D8}} + \mathcal{V}_{D8-\text{O8}} = \frac{+rv^4}{2^{13/2} \pi^9 \alpha'^5}.
\]

It is interesting to note that the brane-image interaction has an important effect in both the long- and short-range cases.
4 Space-Time Foam Configurations

In light of the calculations presented above, a number of interesting situations can be considered, which are linked to the supersymmetric space-time foam construction of [1]. This construction includes two stacks of D8-branes and their images, due to orientifold planes that are attached to each stack, as well as a gas of D0-particles in the bulk dimension.

4.1 Vacuum Configurations

We consider Type-IA string theory in the configuration where two orientifold planes sit on the fixed points of the \( S_1/Z_2 \) orbifold, at \( X^9 = 0 \) and \( X^9 = R \). Eight D8-branes and their images sit at \( X^9 = 0 \), seven (plus images) at \( X^9 = R \). Clearly, the D8-branes can have no relative motion in the vacuum state. The bulk contains a gas of D0-particles which is sufficiently dilute that the interactions among the D0-particles are negligible. However, the D0-particles may, in principle, be moving. If all are moving parallel to the D8-branes, then, as mentioned previously, they exert no force on the D8-branes. But, in the case of intersecting (or bent [1]) D-brane configurations, which are required in certain constructions in order to obtain chiral matter localised at intersections (or foldings), there is no longer a parallel direction. In this case, there would be a unique vacuum configuration, in which the D0-particles are completely static [1].

4.2 D8-Branes Moving in a Dilute Gas of D0-Particles

We now consider a configuration in which one D8-brane has separated itself from the stack and moves adiabatically into the bulk.

As described above, when the moving D8-brane passes by a D0-particle, charge conservation requires [16] that a string be attached between the D8-brane and the D0-particle. When the D8-brane passes by a D0-particle, the D0-particle enters a region of massive Type-IIA supergravity, as there are \( 2 \times 9 \) D8-branes to the left and \( 7 \times 2 \) to the right. From the R-R field version of Gauss’ Law, for charge to be conserved a fundamental string must be created between the D8-brane and the D0-particle. When the D8-brane moves further away from the D0-particle, it is energetically favourable for the fundamental string to be replaced by one stretching between the D0-brane and its image, as seen in Fig. 4. The potential of the configuration is

\[
V_{\text{Total}} = 14 \times D8(r,v) + 16 \times D8(R - r,v) + O8(r,v) + O8(R - r,v)
\]
\[ (R - 2r)v^4 \]
\[ \frac{2^{13} \pi^9 \alpha'^5}{2^{13} \pi^9 \alpha'^5}, \quad (4.1) \]

where \( D8 \) denotes the interaction between two D8-branes, with relative velocity \( v \), and \( O8 \) the interaction between the D8-brane and the orientifold. We note from the above equation that the potential energy is positive provided \( r < R/2 \). This indicates an instability of the configuration, which has a tendency to relax to its equilibrium position, in which the moving brane returns to its original stack.

In the model of [1], the D8-brane interacts with the gas of D0-particles at both long and short range. As shown before, the long-range D0/D8-O8 potential may be represented as the interaction of the D0-particle with its image, and falls off like \( r^{-7} \), and so is negligible at large distances. When a D8-brane moves into the bulk, the cancellation due to the orientifold plane still occurs, as the D8-branes centre of mass is still over the orientifold [12]. Hence the potential felt by the D0-brane close to the D8-brane is the repulsive force (2.19):

\[ V = \frac{2\pi \alpha' v^2}{r^3}. \quad (4.2) \]

As mentioned at the beginning of Section 2, there is a shortest effective length scale [13, 11, 8] (2.1): \( r \approx \alpha'^{1/2} v^{1/2} \). Using this, the D0-particle potential becomes

\[ V \approx \frac{2\pi \alpha' v^2}{\alpha'^{3/2} v^{3/2}} \approx -\left( \frac{v}{\alpha'} \right)^{1/2}. \quad (4.3) \]

The total potential is then estimated to be (re-instating, for completeness, the eight-brane volume factors \( V_8 \) where appropriate)

\[ V_{TOTAL} \approx -N \left( \frac{v}{\alpha'} \right)^{1/2} + \frac{V_8}{2^{13} \pi^9 \alpha'^5} \left[ v^4 (R - 2r) \right], \quad (4.4) \]

where \( N \) denotes the number of D0-particles near the D8-brane, and \( r \) denotes the distance between the D8-brane and the stack of seven D8-branes.

The behaviour of this system is physically in agreement with the stability of Type IA string theory. In the initial situation, there are eight D8-branes and their images on top of each orientifold. In an orientifold compactification models with two stacks of \( n \) branes each, one obtains the theoretical constraint \( R < 2\pi \ell_s/(n - 8) g_s \), which for our case \( n = 8 \) yields no upper bound on \( R \). For the case where there are seven branes on one orientifold and nine on the other, the critical distance is \( R < 2\pi \ell_s/g_s \approx \mathcal{O}(10\ell_s) \) for weak string coupling. Thus if we start with a symmetric configuration with large \( R \), one cannot move a brane to the opposite orientifold without decreasing the separation, which would be unphysical.
4.3 Two Moving D8-Branes in the Foam

We now consider the case when two D8-branes, one from each stack, move asymmetrically into
the bulk with non-relativistic velocities \( v_1, v_2 \), in a direction transverse to their planes, as seen
in Fig. 3. In this case, more complicated potentials occur:

\[
\mathcal{V} = 14D8(r, v_1) + 14D8(R - r, v_1) + 14D8(r_2, v_2) + 14D8(R - r_2, v_2) + O8(r, v_1) + O8(R - r, v_1) + O8(r_2, v_2) + O8(R - r_2, v_2) + 2D8D8(r_1, v_1 + v_2) + O8(r, v_1) + O8(R - r, v_1) + O8(r_2, v_2) + O8(R - r_2, v_2) + 2D8D8(r_1, v_1 + v_2)
\]

\[
= -v_1^4 R - v_1^4 R + 2r_1(v_1 - v_2)^4.
\]

(4.5)

where \( R = r + r_1 + r_2 \). In general, the potential does not have a definite sign as the branes move
in the bulk, indicating an instability of the configuration, which relaxes to equilibrium when the
two moving branes return to their original stacks. In the symmetric case where \( v_1 = -v_2 = v \),
we have (re-instating eight-brane volume factors)

\[
\mathcal{V}_{\text{sym}} = V_8(30R - 64r)v^4,
\]

(4.6)

which is positive provided \( r \) is less than \( 15R/32 \).

In the presence of D0-particles close to the moving D8-branes, short-distance potentials
occur, which are the dominant contributions due to the fact that long-range D0-particle/D8-
brane potentials fall like \( r^{-7} \) (2.16). Using the short-distance substitutions above, an overall
contribution of order

\[
\sim -N \left( \frac{v}{\alpha'} \right)^{1/2}
\]

(4.7)

is added to the D8-brane/D8-O8 potential. When the D8-branes move at different velocities,
the simplification present in (4.5) does not occur but the results are qualitively the same.

The total energy of the configuration, therefore, is given by summing the results (4.5) and
(4.7):

\[
\mathcal{V}_{\text{total}} \simeq -N \left( \frac{v}{\alpha'} \right)^{1/2} + V_8 \frac{-v_1^4 R - v_1^4 R + 2r_1(v_1 - v_2)^4}{2^{13} \pi^9 \alpha'^5}.
\]

(4.8)

In the symmetric case which we concentrate on here for simplicity, this reduces to:

\[
\mathcal{V}_{\text{total}} \simeq -N \left( \frac{v}{\alpha'} \right)^{1/2} + V_8 \frac{(30R - 64r)v^4}{2^{13} \pi^9 \alpha'^5}.
\]

(4.9)

As in the previous subsection, the total potential is positive, and thus the configuration is
stable, for a sufficiently small eight-density of D0-particles near the D8-brane.
5 Physical Applications: Inflation and Dark Energy on the Brane World

The calculation in the previous Section gave the potential energy of the configuration of the space-time foam model [1] in the case of moving D8-branes. As mentioned in the Introduction, such computations provide information on the energy density that an observer on the (physical) brane world would detect. For string excitations which are confined on this brane, this would imply that their perturbative dynamics is described by a non-critical string $\sigma$ model, with a central-charge surplus computed from the above potential energies.

In particular, in the colliding-brane scenario for inflation presented in [4], the total energy of the configuration is given by (4.4) in the scenario when only one brane moves, collides with the other stack and bounces back. Alternatively, when two branes from opposite stacks collide and bounce back, one would have (4.8) (or (4.9) in the symmetric case), to which we restrict ourselves from now on for simplicity. For an observer on the physical brane world, taken for definiteness to be the brane A in Fig. 3, the expressions (4.4) (or (4.9)) lead to bulk contributions to the effective Dark Energy observed on the brane.

The total energy measured by an observer on the brane located at a position $X^9 = r$ (see Fig. 3) is:

$$V_{\text{brane}} = \int_0^R dX^9 V^{(8)} \delta(X^9 - r) + \int_0^R dX^9 \rho(X^9, v),$$

where $V^{(8)}$ is the D8-brane tension $^3$, and $\rho$ is the bulk energy density, such that:

$$\int_0^R dX^9 \rho(x^9, v) = V_{\text{total}}$$

with $V_{\text{total}}$ given by (4.4) (or (4.9)) in the symmetric two-brane case), in the situation in which the energy is positive.

For a $\sigma$ model describing both open-string excitations on the brane world and closed-string excitations propagating in the bulk, the quantity $\rho(x^9, v)/V_8$, where $V_8$ is the eight-volume, defines the central-charge surplus $Q^2 = C - C^* > 0$, as measured with reference to the critical value $C^*$. In our case (4.4) (or (4.9)) this is positive, under the conditions specified previously, and Liouville dressing [6] is necessary to restore world-sheet conformal invariance. The

$^3$This arises from possible quantum corrections of strings on the D8-brane, and may be assumed to vanish, when a sufficient number of supersymmetries exist on the brane.
resulting string theory on the moving brane world is then supercritical \[7\], so we can apply the considerations of \[8\] \[4\] \[5\] and identify the Liouville zero mode with the target time. In fact, following the precise scenario of \[5\] \[4\] we may apply target-space dynamical arguments, supporting energetically this identification.

The central charge surplus, then, is nothing other than the vacuum energy density of the effective low-energy ten-dimensional bulk field theory, which is given by:

\[ Q^2 = \rho / V_8. \]  

(5.3)

To calculate \( \rho(X^9, v) \), we first re-write \( N = nV_8 \), where \( n \) is the eight-dimensional density of defects near the D8-brane world, and then use (4.2) for the D0-particle contributions to the total bulk energy in (4.4) or (4.9). Taking these into account, we may write, cf. (4.2):

\[ \rho(X^9, v) = V_8 \left( \frac{c_1 v^4}{2^{13/3} \pi^2 \alpha'^5} - \Theta(r - X^9) \frac{c_2 v^4}{2^{13/3} \pi^2 \alpha'^5} + n \delta_{r_\ell} \frac{6 \pi \alpha' v^2}{r^4} \right), \]  

(5.4)

where \( c_1 = 1(30) \) and \( c_2 = 2(64) \) in the case (4.4) (4.9), and \( \Theta(r - X^9) \) is the Heaviside step function, meaning that the result is non-zero (and unity) only for \( 0 \leq X^9 \leq r \) (see Fig. 3), and \( \delta_{r_\ell} \) is non-zero and unity only when \( r + r_\ell \geq X^9 \geq r - r_\ell \), where \( r_\ell \) is the characteristic short-distance scale (2.1).

We notice that in the limit of very small velocities, \( v \to 0 \), such that the \( v^4 \) term is not dominant, it is the D0-brane contributions that make the leading-order contribution to the central-charge deficit. For consistency of the Liouville conformal field theory \[6\], one should insist that \( Q^2 \) never changes sign, as the theory flows to a fixed point on the world-sheet renormalization-group trajectory. In our case, the model theory is supercritical: \( Q^2 > 0 \) \[6\] \[7\].

We now remark that, in the cosmological model of \[4\], inflation occurs relatively soon after the initial collision, which corresponds to the case considered in (5.4), with the stack of branes far away from the moving D8-brane world. It was assumed in \[4\] that the bulk D0-particle gas is sufficiently dilute during the inflationary era for the dominant contributions to (5.4) to come from the D8-brane interactions. We also remark that, in the model of \[4\], compactification of the extra dimensions on the branes is necessary, in order to arrive at physically realistic situations. This is a separate delicate issue, especially because of the presence of the orientifolds. Moreover, chiral matter is usually achieved in such scenarios by intersecting brane configurations, with chiral string matter localised on the intersection. We leave these detailed model issues to future works.
Within the uncertainties in numerical factors associated with these important model details, the inflationary era is then described by (5.4). It is not unreasonable to assume that, for adiabatic motion of the D-branes during inflation, the density of D0-branes near the moving D8-brane world adjusts itself so that the central charge surplus of the corresponding $\sigma$ model is approximately constant, dominated by the first term on the right-hand-side of (5.4), i.e.,

$$Q_{\text{init}}^2 \sim 4 \cdot 10^{-9} c_1 v^4 g_s^2.$$  \hspace{1cm} (5.5)

where $c_1 = 1(30)$ for the single (two) moving brane scenario, and $g_s$ is the weak string coupling constant, which we have re-instated here for completeness.

In the inflationary model of [4], one considers an effective gravitational field theory on a brane world with a four-dimensional space-time, in which the effective central charge will acquire an appropriate compactification volume factor $V_5 R \sim R$. For this, we assume an orientifold compactification of one large bulk dimension of size $R$, and five small, i.e., of size $\ell_s$, compactified D8-brane dimensions. In such a scenario, $QR^{1/2}/3$ (expressed in string units with $\ell_s = 1/M_s$) was identified in [4], with the Hubble parameter $H_I$ during inflation, assumed to be constant. From WMAP data [18] one obtains at the 2-$\sigma$ level:

$$H_I = \frac{QR^{1/2}}{3} \leq 1.48 \times 10^{-5} M_P,$$  \hspace{1cm} (5.6)

where $M_P \sim 10^{19}$ GeV is the four-dimensional Planck mass, which is in general different from the bulk string scale $M_s = 1/\ell_s$.

From (5.5) this implies an upper bound on the relative velocity of the moving branes in the symmetric case (1.6):

$$v < 0.8 \cdot c_1^{-1/4} R^{-1/4} \sqrt{M_P \ell_s / g_s}.$$  \hspace{1cm} (5.7)

In realistic string theories, and also in the models of D0-particle foam we consider here [11][14] in which there is one large bulk dimension of size $R$ which undergoes orientifold compactification, and five small, i.e., of size $\ell_s$, compactified D8-brane dimensions, one may assume $M_P \ell_s \sim 2\sqrt{2} g_s^{-1} R^{1/2}$, yielding

$$v \leq 1.4 \cdot c_1^{-1/4} g_s^{-1}.$$  \hspace{1cm} (5.8)

For weakly-coupled strings with $g_s^2 \sim 1/2$, which is a value commonly considered as it leads to acceptable grand-unification-scale gauge couplings in phenomenological (supersymmetric) effective low-energy field theories derived from strings, we have $v \leq 1.98 \cdot c_1^{-1/4}$, indicating that this value of string coupling is incompatible with the single moving-brane model, $c_1 = 1$. 20
On the other hand, $g_s^2 \sim 0.5$ leads to velocities which are not very non-relativistic, namely $v \sim 0.8$ for the model of two branes moving relative to the orientifold planes ($c_1 = 30$). Our approximations in the previous Sections assumed $v$ to be small enough that $\sin v \sim v$. The above value of $v$ satisfies this relation to within 10%, which may be acceptable. One gets compatibility for larger values of the string coupling (still less than one), but then the validity of the weak coupling is in jeopardy. Moreover, in the colliding-branes scenario of [4], there is the following relation between the spectral index $n_S$ for scalar perturbations and the number of e-foldings $N$:

$$n_S - 1 = -\frac{3}{N}. \quad (5.9)$$

WMAP data [18] yield $n_s - 1 \simeq -4 \cdot 10^{-2}$, which, on account of (5.9), implies $N \simeq 75$.

Assuming adiabatic motion of branes during inflation, which lasts for a period of $t_I \sim x \ell_s/v$, with $R > x > 1$ characterising the brane separation, we have $H_I \frac{x^2}{v^3} = N \simeq 75$, from which we obtain, using (5.8), a large separation of the branes at the end of inflation for the case of the two moving branes [4]:

$$R > x > 2.54 \cdot 10^6 \cdot c_1^{-1/4} \cdot R^{-1/2}. \quad (5.10)$$

This implies that $R > 1.06 \cdot 10^4$. We do not discuss cosmological inflation further here, intending to return in a future publication [19] to possible scenarios for the end of inflation and reheating.

Before closing, we remark that the relaxation phenomena discussed in [4], which result from the recoil of the brane world after the collision, may be continuing into the present era, in which case they could contribute to the present-day cosmological Dark Energy. The relaxation of the recoil-induced vacuum energy density on our brane world would then receive contributions [5] [4] [20] of the form $\rho \sim v^4/t^2$, where $t \sim 10^{60} t_{\text{Planck}}$: $t_{\text{Planck}} = 10^{-43}$ sec is the present cosmic time, and $v$ is the recoil velocity of the brane just after the collision, which may be of order $v \simeq 0.8$ in our model.

We are far from claiming a detailed understanding of inflation and Dark Energy in the above framework. Nevertheless, we believe that our work provides useful steps towards a consistent mathematical formulation of inflationary scenarios in the context of non-equilibrium (non-critical) effective string theories on excited brane worlds. The use of non-critical strings is highly appropriate for the study of the non-equilibrium phenomena that dominate Early Universe physics, and may also control the current evolution of the Universe.
Acknowledgements

N.E.M. wishes to thank Juan Fuster and IFIC-University of Valencia (Spain) for their interest and support, and P. Sodano and INFN-Perugia (Italy) for the hospitality and support during the last stages of this work. M.W. thanks CERN, Physics Department (Theory) for their hospitality. The work of M.W. is supported by an EPSRC (U.K.) Research Studentship.

6 Appendix - Properties of Jacobi Theta Functions

Definitions

\[ \vartheta_1(v|it) \equiv \vartheta_{11}(v|it) = -2q^{\frac{1}{8}} \sin[\pi v] \prod_{n=1}^{\infty} (1 - q^{2n})(1 - e^{2i\pi v}q^{2n})(1 - e^{-2i\pi v}q^{2n}) \] (6.1)

\[ \vartheta_2(v|it) \equiv \vartheta_{10}(v|it) = 2q^{\frac{1}{8}} \cos[\pi v] \prod_{n=1}^{\infty} (1 - q^{2n})(1 + e^{2i\pi v}q^{2n})(1 + e^{-2i\pi v}q^{2n}) \] (6.2)

\[ \vartheta_3(v|it) \equiv \vartheta_{00}(v|it) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + e^{2i\pi v}q^{2n-1})(1 + e^{-2i\pi v}q^{2n-1}) \] (6.3)

\[ \vartheta_4(v|it) \equiv \vartheta_{01}(v|it) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 - e^{2i\pi v}q^{2n-1})(1 - e^{-2i\pi v}q^{2n-1}) \] (6.4)

where \( q = e^{-\pi t} \). The sum representation is also useful

\[ \vartheta \left[ \begin{array}{c} \frac{a}{b} \\ \frac{c}{d} \end{array} \right] (v|t) = \sum_{n=-\infty}^{\infty} \exp \left( 2\pi i \left[ \frac{1}{2}(n + \frac{a}{2})^2 t + (n + \frac{a}{2})(v + \frac{b}{2}) \right] \right) \] (6.5)

as well as the identity

\[ \vartheta \left[ \begin{array}{c} \frac{a}{b} \\ \frac{c}{d} \end{array} \right] (v + \frac{\epsilon_1}{2} t + \frac{\epsilon_2}{2} |t) = e^{-\frac{i\pi \epsilon_1 \epsilon_2}{4} t} e^{-\frac{i\pi \epsilon_1}{4} (2v+b)} e^{-\frac{i\pi \epsilon_2}{4} v} \vartheta \left[ \begin{array}{c} \frac{a + \epsilon_1}{b + \epsilon_2} \\ \frac{c}{d} \end{array} \right] (v|t). \] (6.6)

Modular Transformations

Annulus

\[ \eta(\tau) = (-i\tau)^{-1/2} \eta \left( \frac{-1}{\tau} \right) \]
\[ \vartheta_1(\nu|\tau) = -(-i\tau)^{-1/2}e^{-\pi i\nu^2/\tau} \vartheta_1 \left( \frac{\nu}{\tau} - \frac{1}{\tau} \right) \]
\[ \vartheta_2(\nu|\tau) = (-i\tau)^{-1/2}e^{-\pi i\nu^2/\tau} \vartheta_4 \left( \frac{\nu}{\tau} - \frac{1}{\tau} \right) \]
\[ \vartheta_3(\nu|\tau) = (-i\tau)^{-1/2}e^{-\pi i\nu^2/\tau} \vartheta_3 \left( \frac{\nu}{\tau} - \frac{1}{\tau} \right) \]
\[ \vartheta_4(\nu|\tau) = (-i\tau)^{-1/2}e^{-\pi i\nu^2/\tau} \vartheta_2 \left( \frac{\nu}{\tau} - \frac{1}{\tau} \right) \]

(6.7)

Orientifold

The orientifold modular transformations can be easily derived by following the Appendix of [11].

\[ \vartheta'_1(0|i/4t + 1/2) = (2it)^{3/2}\vartheta'_1(0|i + 1/2) \]
\[ \vartheta_1(v|i/4t + 1/2) = (2it)^{1/2}e^{-4\pi\nu^2t}\vartheta_1(2iv|it + 1/2) \]
\[ \vartheta_2(v|i/4t + 1/2) = i(2it)^{1/2}e^{-4\pi\nu^2t}\vartheta_2(2iv|it + 1/2) \]
\[ \vartheta_3(v|i/4t + 1/2) = e^{-i\pi/2}(2it)^{1/2}e^{-4\pi\nu^2t}\vartheta_3(2iv|it + 1/2) \]
\[ \vartheta_4(v|i/4t + 1/2) = e^{i\pi}(2it)^{1/2}e^{-4\pi\nu^2t}\vartheta_4(2iv|it + 1/2) \]

(6.8)

References

[1] J. R. Ellis, N. E. Mavromatos and M. Westmuckett, Phys. Rev. D 70, 044036 (2004) [arXiv:gr-qc/0405066].

[2] see, for instance: L. E. Ibanez, R. Rabadan and A. M. Uranga, Nucl. Phys. B 576, 285 (2000) [arXiv:hep-th/9905098]; L. E. Ibanez, [arXiv:hep-ph/9905349]; C. A. Scrucca and M. Serone, JHEP 9912, 024 (1999) [arXiv:hep-th/9912108]; D. M. Ghilencea and G. G. Ross, Nucl. Phys. B 595, 277 (2001) [arXiv:hep-ph/0006318], and references therein.

[3] J. Polchinski, *String theory*, Vol. 2 (Cambridge University Press, 1998); J. H. Schwarz, [arXiv:hep-th/9907061].

[4] J. Ellis, N. E. Mavromatos, D. V. Nanopoulos and A. Sakharov, [arXiv:gr-qc/0407089]. New J. Phys. 6, 171 (2004).
[5] E. Gravanis and N. E. Mavromatos, Phys. Lett. B 547, 117 (2002) [arXiv:hep-th/0205298]; N. E. Mavromatos, [arXiv:hep-th/0210079] (published in Oulu 2002 (Finland), Beyond the desert (ed. H.V. Klapdor-Kleingrothaus, IoP 2003)), 3.

[6] F. David, Mod. Phys. Lett. A 3, 1651 (1988); J. Distler and H. Kawai, Nucl. Phys. B 321, 509 (1989); J. Distler, Z. Hlousek and H. Kawai, Int. J. Mod. Phys. A 5, 391 (1990); see also: N. E. Mavromatos and J. L. Miramontes, Mod. Phys. Lett. A 4, 1847 (1989); E. D’Hoker and P. S. Kurzepa, Mod. Phys. Lett. A 5, 1411 (1990).

[7] I. Antoniadis, C. Bachas, J. R. Ellis and D. V. Nanopoulos, Nucl. Phys. B 328, 117 (1989); Phys. Lett. B 257, 278 (1991).

[8] J. R. Ellis, N. E. Mavromatos and D. V. Nanopoulos, Phys. Lett. B 293, 37 (1992) [arXiv:hep-th/9207103]; Invited review for the special Issue of J. Chaos Solitons Fractals, Vol. 10, (eds. C. Castro amd M.S. El Naschie, Elsevier Science, Pergamon 1999) 345 [arXiv:hep-th/9805120]; Phys. Rev. D 63, 024024 (2001) [arXiv:gr-qc/0007044].

[9] C. Bachas, Phys. Lett. B 374, 37 (1996) [arXiv:hep-th/9511043]; [arXiv:hep-th/9806199]

[10] J. R. David, A. Dhar and G. Mandal, Phys. Lett. B 415 (1997) 135 [arXiv:hep-th/9707132].

[11] P. Di Vecchia, A. Liccardo, R. Marotta and F. Pezzella, JHEP 0409 (2004) 050 [arXiv:hep-th/0407038].

[12] U. H. Danielsson and G. Ferretti, Int. J. Mod. Phys. A 12, 4581 (1997) [arXiv:hep-th/9610082].

[13] M. R. Douglas, D. Kabat, P. Pouliot and S. H. Shenker, Nucl. Phys. B 485, 85 (1997) [arXiv:hep-th/9608024].

[14] E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest and A. Van Proeyen, Class. Quant. Grav. 18, 3359 (2001) [arXiv:hep-th/0103233].

[15] O. Bergman, M. R. Gaberdiel and G. Lifschytz, Nucl. Phys. B 509 (1998) 194 [arXiv:hep-th/9705130].

[16] O. Bergman, M. R. Gaberdiel and G. Lifschytz, Nucl. Phys. B 524, 524 (1998) [arXiv:hep-th/9711098].

24
[17] U. H. Danielsson and G. Ferretti, Nucl. Phys. Proc. Suppl. 68, 78 (1998) arXiv:hep-th/9709171; C. P. Bachas, M. B. Green and A. Schwimmer, JHEP 9801, 006 (1998) arXiv:hep-th/9712086; J. H. Brodie, JHEP 0111, 014 (2001) arXiv:hep-th/0012068.

[18] D. N. Spergel et al. [WMAP Collaboration], Astrophys. J. Suppl. 148, 175 (2003) arXiv:astro-ph/0302209.

[19] J. R. Ellis, N. E. Mavromatos, D. V. Nanopoulos and M. Westmuckett, in preparation.

[20] J. Ellis, N. E. Mavromatos and D. V. Nanopoulos, arXiv:hep-th/0412240.