SINGULAR \( Z_N \) CURVES, RIEMANN-HILBERT PROBLEM AND SCHLESINGER EQUATIONS

V.Z. ENOLSKI AND T.GRAVA

Abstract. We are solving the classical Riemann-Hilbert problem of rank \( N > 1 \) on the extended complex plane punctured in \( 2m + 2 \) points, for \( N \times N \) quasi-permutation monodromy matrices. Our approach is based on the finite gap integration method applied to study the Riemann-Hilbert by Kitaev and Korotkin \[1\], Deift, Its, Kapaev and Zhou \[2\] and Korotkin, \[3\]. This permits us to solve the Riemann-Hilbert problem in terms of the Szegő kernel of certain Riemann surfaces branched over the given \( 2m + 2 \) points. The monodromy group of these Riemann surfaces is determined from the quasi-permutation monodromy matrices of the Riemann-Hilbert problem by setting all their non-zero entries equal to one. In our case, the monodromy group of the Riemann surfaces turns out to be the cyclic subgroup \( \mathbb{Z}_N \) of the symmetric group \( \mathbb{S}_N \) and for this reason these Riemann surfaces of genus \( N(m - 1) \) have \( \mathbb{Z}_N \) symmetry. This fact enables us to write the matrix entries of the solution of the \( N \times N \) Riemann-Hilbert problem as a product of an algebraic function and \( \theta \)-function quotients. The algebraic function is related to the Szegő kernel with zero characteristics.

From the solution of the Riemann-Hilbert problem we automatically obtain a particular solution of the Schlesinger system. The \( \tau \)-function of the Schlesinger system is computed explicitly in terms of \( \theta \)-functions and the holomorphic projective connection of the Riemann surface. In the course of the computation we also derive Thomae-type formulae for a class of non-singular \( 1/N \)-periods.

Finally we study in detail the solution of the rank 3 problem with four singular points \( (\lambda_1, \lambda_2, \lambda_3, \infty) \). The corresponding Riemann surface \( \mathcal{C}_{3,1} \) is of genus two branched at the above four points and admits the dihedral group \( D_3 \) of automorphisms. This implies that \( \mathcal{C}_{3,1} \) is a 2-sheeted cover of two elliptic curves which are 3-isogenous. As a result, the corresponding solution of the Riemann-Hilbert problem and the Schlesinger system is given in terms of Jacobi’s \( \theta \)-function with modulus \( T = T(t) \), \( t = \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1} \) and \( \operatorname{Im} T > 0 \). The inverse function \( t = t(T) \) is automorphic under the action of the subgroup \( \Gamma_0(3) \) of the modular group and generates a solution of a general Halphen system. The analytic counterpart of this picture is given by Goursat’s higher identities for hypergeometric functions.

Contents

1. Introduction 2
2. The \( N \times N \) matrix Riemann-Hilbert problem 9
3. Riemann surface of an algebraic curve 11
   3.1. The curve and differentials 11
   3.2. \( \theta \)-function 11
   3.3. Kernel-forms 13
4. \( Z_N \) curves 14
   4.1. Homologies and periods of \( Z_N \)-curves 15
   4.2. Characteristics supported on branch points 16
   4.3. Szegő kernel for \( \theta \)-periods 19
5. Solution of the Riemann-Hilbert problem for the \( Z_N \)-curve 22
   5.1. Solution of the Schlesinger equations 28

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1. Introduction

The Riemann-Hilbert problem (R-H problem) in its classical formulation consists of deriving a linear differential equation of Fuchsian type with a given set $D$ of singular points and a given monodromy representation

$$M : \pi_1(\mathbb{CP}^1 \setminus D, \lambda_0) \to GL(N, \mathbb{C}), \quad N \geq 2, \quad (1.1)$$

of the fundamental group $\pi_1(\mathbb{CP}^1 \setminus D, \lambda_0)$. An element $\gamma$ of the group $\pi_1(\mathbb{CP}^1 \setminus D, \lambda_0)$ is a loop contained in $\mathbb{CP}^1 \setminus D$ with initial and end point $\lambda_0 \in \mathbb{CP}^1, \lambda_0 \notin D$. Not all the representation (1.1) can be realized as the monodromy representation of a Fuchsian system. For $N = 3$, 4 representations (1.1) for which the R-H problem cannot be solved are given in [6] and [8] respectively. In dimension $N = 2$ the R-H problem is always solvable [9] for an arbitrary number of singular points. For $N \geq 3$, every irreducible representation (1.1) can be realized as the monodromy representation of some Fuchsian system [6], [7], [10]. In general, among the solvable cases, the solution of the matrix R-H problem cannot be computed analytically in terms of known special functions [11], [12]. Nevertheless, there are special cases when the R-H problem can be solved explicitly in terms of $\theta$-functions [1], [2], [3]. We discuss one of these cases.

The method of solution proposed by Plemelj [13] consists of reducing the R-H problem to a homogeneous boundary value problem in the complex plane for a $N \times N$ matrix function $Y(\lambda)$. The boundary can be chosen in the form of a polygon line $\mathcal{L}$, by connecting all the singular points of the set $D := \{\lambda_1, \lambda_2, \ldots, \lambda_{2m+1}, \lambda_{2m+2} = \infty\}$. The line $\mathcal{L}$ divides the complex plane into two domains, $C_-$ and $C_+$ (see Figure 1). Let $\gamma_1, \gamma_2, \ldots, \gamma_{2m+2}$ denote the set of generators of the fundamental group

![Figure 1. The contour $\mathcal{L}$](image-url)
\[ \pi_1(\mathbb{C}P^1 \setminus D, \lambda_0), \text{i.e. the homotopy class } \gamma_k \text{ corresponds to a small clock-wise loop around the point } \lambda_k \text{ (see Figure 1). Then the matrices } M(\gamma_k) = M_k \in SL(N, \mathbb{C}), k = 1, \ldots, 2m + 2, \text{ form a set of generators of the monodromy group. Since the homotopy relation} \]
\[ \gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_{2m+2} \simeq \{ \lambda_0 \}, \]
the generators \( M_k \) satisfy the cyclic relation
\[ M_\infty M_{2m+1} \cdots M_1 = 1_N, \]
where \( M_{2m+2} = M_\infty \). Let us construct the matrices \( G_k \) defined by
\[ G_k = M_k M_{k-1} \cdots M_1, \quad k = 1, \ldots, 2m + 2 \]
\[ G_0 = G_{2m+2} = 1_N. \]

The homogeneous Hilbert boundary value problem formulated by Plemelj is the following [13]: find the \( N \times N \) matrix function \( Y(\lambda) \) which satisfies the following conditions

(i) \( Y(\lambda) \) is analytic in \( \mathbb{C}P^1 \setminus \mathcal{C} \);  
(ii) the limits \( Y_{\pm}(\lambda) \) as \( \lambda \to \mathcal{C}_\pm \) satisfy the jump conditions
\[ Y_- (\lambda) = Y_+ (\lambda) G_k, \quad \lambda \in (\lambda_k, \lambda_{k+1}), \quad k = 0, \ldots, 2m + 1, \quad \lambda_0 = \lambda_{2m+2} = \infty; \]

(iii) for \( 0 \leq \epsilon < 1 \),
\[ Y \left( \frac{1}{\lambda} \right) \left( \frac{1}{\lambda} \right)^\epsilon \to 0 \text{ as } \lambda \to \infty \text{ and } Y_{\pm}(\lambda)(\lambda - \lambda_j)^\epsilon \to 0 \text{ as } \lambda \to \lambda_j, \]
over \( C_+ \) or \( C_- \) respectively;

(iv) \( Y(\lambda_0) = 1_N, \quad \lambda_0 \in \mathbb{C}_+ \setminus D \).

There is always a solution of (i)-(iv) such that \( \det Y(\lambda) \neq 0 \) for \( \lambda \neq D \). The analytic continuation of the solution \( Y(\lambda) \) along a small loop \( \gamma_k \) around \( \lambda_k \) is determined by the matrix \( G_k G_{k-1}^{-1} = M_k \), namely
\[ Y(\gamma_k(\lambda)) = Y(\lambda) M_k, \quad \lambda \in C_+ \setminus D, \quad k = 1, \ldots, 2m + 2. \]

It is possible to show that the solution \( Y(\lambda) \) of the R-H problem (i)-(iv) satisfies a Fuchsian equation
\[ \frac{dY(\lambda)}{d\lambda} = \sum_{k=1}^{2m+1} \frac{A_k}{\lambda - \lambda_k} Y(\lambda), \quad A_k \in Mat(N, \mathbb{C}), \]
if one of the monodromy matrices is diagonalisable [11, 12]. Without this condition, Plemelj original argument does not go through.

Kitaev and Korotkin [11] and Deift, Its, Kapaev, and Zhou [12] solved the \( 2 \times 2 \) matrix R-H problem when all the matrices \( G_{2k} \) are diagonal and all the matrices \( G_{2k-1}, k = 1, \ldots, m + 1 \), are off-diagonal. The idea of the construction in [11, 12] is to consider a hyperelliptic covering \( \mathcal{C} \) over \( \mathbb{C}P^1 \) which is ramified in \( D \) and use the natural monodromy of the hyperelliptic curve. The application of methods of finite-gap integration [14, 15] permits to obtain a \( \theta \)-function solution for the problem, depending on \( 2m \) parameters. A solution of the \( 2 \times 2 \) case appeared also in [16, 17] in the study of asymptotic problems arising in the theory of random matrix models and in [18] in the study of the small dispersion limit of the Korteweg de Vries equation.

The extension of the \( 2 \times 2 \) matrix R-H problem to higher dimensional matrices leads naturally to non-hyperelliptic curves. This fact was pointed out by Zverovich [19], who considered the \( N \times N \) problem
(i)-(iv) when all the matrices $G_{2k}$ are diagonal, and the non-zero entries of the matrices $G_{2k−1}$, $k = 1, \ldots, m + 1$ are
\begin{align}
(G_{2k−1})_{i,i−1} &\neq 0, \quad i = 2, \ldots, N, \\
(G_{2k−1})_{N,1} &\neq 0, \quad k = 1, \ldots, m + 1.
\end{align}

The solvability of the corresponding $N \times N$ matrix R-H problem is proved by lifting it to a scalar problem on the curve
\begin{align}
\mathcal{C}_{N,m} &:= \{ (\lambda, y), \quad y^N = q^{N−1}(\lambda)p(\lambda) \}, \\
q(\lambda) &= \prod_{j=1}^{m}(\lambda − \lambda_{2k}), \quad p(\lambda) = \prod_{j=0}^{m}(\lambda − \lambda_{2k+1}).
\end{align}

The curve $\mathcal{C}_{N,m}$ has singularities at the points $(\lambda_{2k}, 0)$, $k = 1, \ldots, m$. These singularities can be easily resolved \cite{20} to give rise to a compact Riemann surface which we still denote by $\mathcal{C}_{N,m}$. Such surface can be identified with $N$ copies (sheets) of the complex $\lambda$-plane cut along the segments $\mathcal{L}_0 = \cup_{k=1}^{m+1} [\lambda_{2k−1}, \lambda_{2k}]$ and glued together according to the permutation rule \((1\ 2 \ldots \ N−1 \ N \ N \ N−1 \ldots \ 2 \ 3 \ldots \ N−1 \ 1)\), that is the first sheet is pasted to the second, the second to the third and so on. The pre-image $\pi_1^{-1}(\lambda)$, $\lambda \in \mathbb{C}\setminus D$, of the projection $\pi : \mathcal{C}_{N,m} \to \mathbb{C}$, consists of $N$ points $P^{(s)} = (\lambda, e^{2\pi i s/m} y)$. In this paper, we solve explicitly the $N \times N$ matrix R-H problem considered by Zverovich. The algebraic-geometrical approach to the R-H problem was developed further by Korotkin \cite{3}. He showed that for quasi-permutation monodromy matrices (in which each row and each column have only one non-zero element), the R-H problem can be solved in terms of the Szegő kernel of a Riemann surface.

The procedure to obtain the Riemann surface from the monodromy matrices relies on the Riemann existence theorem \cite{20, 21}. In detail the existence theorem associates a permutation representation
\begin{align}
S : \pi_1(\mathbb{C}P^1 \setminus D, \lambda_0) \to S_N,
\end{align}
to a compact ramified cover $\mathcal{C}$ of degree $N$ over the Riemann sphere with a set $D$ of $n$ prescribed branch points in such a way that the product $S(\gamma_1)S(\gamma_2) \ldots S(\gamma_n) = 1_N$. The correspondence is one-to-one between isomorphism classes of covers and equivalent permutation monodromy representations (this latter equivalence relation simply reflects a relabeling of the points in the fiber of the covering over the base point $\lambda_0$). The cover $\mathcal{C}$ is connected if the only invariant subspace of the permutation representation is the $N$-dimensional column vector $(1, 1, \ldots, 1)^t$. The genus $g$ of the surface $\mathcal{C}$ is obtained from the Riemann-Hurwitz relation
\begin{align}
2(N + g - 1) = \sum_{i=1}^{n} \text{Tran}[S(\gamma_i)],
\end{align}
where $\text{Tran}[S(\gamma_i)]$ is the number of transpositions in the permutation $S(\gamma_i)$.

For a given monodromy representation \cite{11}, where the matrices $M(\gamma_i)$ are quasi-permutation, the corresponding elements $S(\gamma_i)$ of the symmetric group $S_N$ are obtained by setting all the non-zero entries of $M(\gamma_i)$, $i = 1, \ldots, n$, equal to unity. However, the Riemann existence theorem is just an existence theorem, that is, it does not produce explicitly algebraic equations for the coverings. In the case under consideration, the permutation representation induced by the monodromy matrices $G_kG_{k−1}^{-1}$, with $G_k$ being defined in \eqref{1.2}, is
\begin{align}
S(\gamma_{2k−1}) &= \begin{pmatrix} 1 & 2 & \cdots & N−1 & N \\ 2 & 3 & \cdots & N & 1 \end{pmatrix}, \quad k = 1, \ldots, m + 1, \\
S(\gamma_{2k}) &= \begin{pmatrix} 1 & 2 & \cdots & N−1 & N \\ N & 1 & \cdots & N−2 & N−1 \end{pmatrix}, \quad k = 1, \ldots, m + 1.
\end{align}
We observe that the points $P_0^{(s)} = (\lambda_0, e^{2\pi i \frac{s-1}{N}}y_0) \in \mathcal{C}_{N,m}$, $s = 1, \ldots, N$, $\lambda \notin D$, belonging to the pre-image $\pi^{-1}(\lambda_0) = (P_0^{(1)}, P_0^{(2)}, \ldots, P_0^{(N)})$, are permuted, when $\lambda_0$ moves along the path $\gamma_k$, according to the rule

$$
(P_0^{(1)}, P_0^{(2)}, \ldots, P_0^{(N)}) \rightarrow S(\gamma_k)(P_0^{(1)}, P_0^{(2)}, \ldots, P_0^{(N)}), \quad k = 1, \ldots, 2m + 2.
$$

Therefore $\mathcal{C}_{N,m}$ given in (1.15) is the Riemann surface associated with the permutation representation (1.16). In general the derivation of an algebraic expression for the cover from the permutation representation is a hard task.

The genus of the surface $\mathcal{C}_{N,m}$ obtained from (1.17) is equal to $g = (N-1)m$. We observe that in our case, the complex dimension of the space of quasi-permutation monodromy matrices $M_k = G_k G_{k-1}^{-1} \in SL(N, \mathbb{C})$, $k = 1, \ldots, 2m + 1$ is equal to $(N-1)(2m + 1)$. In order to solve the R-H problem by using only the Szegö kernel as suggested in [3], the complex dimension of the space of monodromy matrices must be at most equal to $2g = 2(N-1)m$. For this reason one of the monodromy matrices must be fixed as a suitable permutation or quasi-permutation matrix.

Our derivation of the solution of the R-H problem (i)-(iv), incorporates both the method of [2], implemented for hyperelliptic curves and the general treatment of [3]. First, we solve the so-called canonical R-H problem, namely the problem (i)-(iv) when all the matrices $G_{2k}$ are set equal to the identity and all the matrices $G_{2k+1}$ are set equal to the quasi-permutation $\mathcal{P}_N$, where

$$
\mathcal{P}_N = \begin{pmatrix}
0 & 0 & \ldots & 0 & 0 & (-1)^{N-1} \\
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 & 0 \\
0 & 0 & \ldots & 0 & 1 & 0 
\end{pmatrix}.
$$

More precisely the canonical R-H problem consists of finding a matrix valued function $X(\lambda)$ analytic in the complex plane off the segment $\mathcal{L}_0 = \cup_{k=1}^{m+1} [\lambda_{2k-1}, \lambda_{2k}]$ such that

$$
X_-(\lambda) = X_+(\lambda) \mathcal{P}_N, \quad \lambda \in \mathcal{L}_0,
$$

$$
X(\lambda_0) = 1_N, \quad \lambda_0 \in C_+.
$$

The solution of the R-H problem (1.10) can be obtained in an elementary way by diagonalising the matrix $\mathcal{P}_N = U e^{2\pi i \sigma_N} U^{-1}$, where the matrix $\sigma_N$ reads

$$
\sigma_N = \text{Diag} \left( \frac{-N + 1}{2N}, \frac{-N + 3}{2N}, \ldots, \frac{-N - 3}{2N}, \frac{-N - 1}{2N} \right),
$$

and the matrix $U$ is chosen so that the entries $U_{1k} = 1$, $k = 1, \ldots, N$ and $\text{Det}(U) \neq 0$. Then it is quite immediate to verify that

$$
X(\lambda) = U \left( \begin{pmatrix} p(\lambda) & q(\lambda) \\ q(\lambda) & p(\lambda) \end{pmatrix} \right)^{\sigma_N} U^{-1}
$$

solves the R-H problem (1.10). The entries of the matrix $X(\lambda)$ can be expressed in terms of the Szegö kernel with zero characteristics, $S(0)(P, Q)$, defined on $\mathcal{C}_{N,m}$, we show that

$$
S[0](P, Q) = \frac{1}{N} \sqrt{dz(P)dz(Q)} \sum_{k=1}^{N-1} \left( \frac{q(z(P)) p(z(Q))}{p(z(Q)) q(z(P))} \right)^{-\frac{N+1}{2}} dz(P)^{1/2}, \quad P, Q \in \mathcal{C}_{N,m},
$$

where $z(P)$ is a local coordinate near the point $P$ and the polynomials $p$ and $q$ have been defined in (1.14). Then the entries of the matrix $X(\lambda)$ in (1.12) can be written in the form

$$
X_{rs}(\lambda) = S[0](P^{(r)}, P^{(s)}) \frac{z(P) - z(Q)}{\sqrt{dz(P)dz(Q)}}.
$$
Jimbo-Miwa-Ueno τ corresponding to the particular solution (1.15) of the Schlesinger system, has the form

\[
Y = k \frac{e^{2\pi i \frac{r-s}{N}}}{\sqrt{\frac{p(\lambda)}{q(\lambda)}}} \left( e^{-2\pi i \frac{r-s}{N}} \sqrt{\frac{q(\lambda_0)}{p(\lambda_0)}} \right)^{-k+\frac{N-1}{2}}, \quad \lambda_0 \notin D,
\]

where \( P^{(s)} = (\lambda, \rho^{-1}y) \) and \( P^{(r)}_0 = (\lambda_0, \rho^{-1}y_0) \), \( r, s = 1, \ldots, N \), denote the points on the \( s \)-th and \( r \)-th sheet of \( C_{N,m} \) respectively. When \( N = 2 \) and \( \sqrt[4]{\frac{q(\lambda_0)}{p(\lambda_0)}} = 1 \), such a formula coincides with the canonical solution obtained in [16], [2].

The solution \( Y(\lambda) \) of the full R-H problem (i)-(iv), where the constant matrices \( G_k, k = 1, \ldots, 2m+1 \), are parametrised by \( 2(N-1)m \) arbitrary complex constants, is obtained, following [3], using the Szegő kernel with non-zero characteristics. From the relation (1.13), we are able to write the global solution obtained in [16], [2]. If none of the monodromy matrices \( M \) that is if \( \Pi \) is the vector of normalized holomorphic differentials on \( C_{N,m} \) explicitly permits us to evaluate \( \theta \) (i)-(iv) of the full R-H problem, \( \theta \) (i)-(iv), where \( \theta \) is the canonical \( \theta \)-function with characteristics \( \epsilon \) and \( \delta \) determined from the non-zero entries of the matrices \( G_k, k = 1, \ldots, 2m+1 \). The solution (1.14) exists if

\[
\theta \left[ \frac{\delta}{\epsilon} \right] (0; \Pi) \neq 0,
\]

that is if \( \Pi \delta + \epsilon \notin (\Theta) \), where \( (\Theta) \) is the \( \theta \)-divisor in the Jacobian variety \( \text{Jac}(C_{N,m}) \) of the Riemann surface \( C_{N,m} \). This solution coincides with the solution obtained in [2], [1] for \( N = 2 \). The formula (1.14) permits us to evaluate explicitly the characteristics \( \delta \) and \( \epsilon \) in terms of the monodromy matrix entries thus solving the R-H problem effectively.

The Fuchsian system (1.3) is recovered from the solution (1.14) by evaluating the residue

\[
A_k = A_k(\lambda_1, \ldots, \lambda_{2m+1} | M_1, \ldots, M_{2m+1}) = \text{Res}_{\lambda = \lambda_k} \left[ \frac{dY(\lambda)}{d\lambda} Y^{-1}(\lambda) \right].
\]

If none of the monodromy matrices \( M_i, i = 1, \ldots, 2m+2 \), depends on the position of the singular points \( \lambda_k, k = 1, \ldots, 2m+1 \), then the matrices \( A_k \) satisfy the Schlesinger system (see below 2.16). The Jimbo-Miwa-Ueno \( \tau \)-function (23)

\[
\frac{\partial}{\partial \lambda_k} \log \tau = \frac{1}{2} \text{Res}_{\lambda = \lambda_k} \left( \frac{dY(\lambda)}{d\lambda} Y(\lambda)^{-1} \right)^2,
\]

corresponding to the particular solution (1.15) of the Schlesinger system, has the form

\[
\tau(\lambda_1, \ldots, \lambda_{2m+1}) = \frac{\theta \left[ \frac{\delta}{\epsilon} \right] (0; \Pi)}{\theta(0; \Pi)} \prod_{\substack{i < k \leq m+1 \\lambda_k \neq \lambda_i}} (\lambda_{2k+1} - \lambda_{2i+1})^{\frac{N^2-1}{8N}} \prod_{\substack{i < j \leq m+1 \\lambda_i < \lambda_j}} (\lambda_{2k} - \lambda_{2i})^{\frac{N^2-1}{8N}}.
\]
For $N = 2$ the above expression has been obtained in \[1\]. The $\tau$-function can be written in a different form by using the Thomae-type formula which we derive for the families of curves $\mathcal{C}_{N,m}$

$$\theta^B(0; \Pi) = \frac{\prod_{i=1}^{N-1} \det A_i^2}{(2\pi i)^{4m(N-1)}} \prod_{i<j} (\lambda_{2i} - \lambda_{2j})^{2(N-1)} \prod_{k<l} (\lambda_{2k+1} - \lambda_{2l+1})^{2(N-1)}.$$ 

The form \[(1.17)\] of the solution of the R-H problem enables us to show the following:

1. if the non-singular characteristics $\delta$, $\epsilon$ correspond to a non-special divisor supported on the branch points, then $\delta$, $\epsilon \in (\mathbb{Z}/N\mathbb{Z})^{(N-1)m}$ and the solution of the R-H problem corresponds to a reducible monodromy representation;
2. when two solutions $Y(\lambda)$ and $\tilde{Y}(\lambda)$ have their corresponding characteristics equivalent modulo $(\mathbb{Z}/N\mathbb{Z})^{(N-1)m}$, the matrix entries $Y_{rs}(\lambda)$ and $\tilde{Y}_{rs}(\lambda)$ are related by an algebraic transformation. The corresponding monodromy representations $\mathcal{M} = \{ M_1, M_2, \ldots, M_{2m+1}, M_\infty \}$ of $\mathcal{M} = \{ \lambda_1, \lambda_2, \ldots, \lambda_{2m+1}, \lambda_\infty \}$ are equivalent up to multiplication by $N$-roots of unity. That is $M_k = e^{2\pi i/j_k} M_{k'}$, $j_k$ integer, $\sum_{k=1}^{2m+1} j_k = 0 \mod N$.

We remark that the result in (2) has been suggested by Dubrovin and Mazzocco \[22\] following their investigations of the symmetries of the Schlesinger system. These symmetries generalise the Okamoto symmetries derived in the $2 \times 2$ case \[11\].

Finally we study in detail the solution of the rank 3 problem with four singular points $(\lambda_1, \lambda_2, \lambda_3, \infty)$. The monodromy matrices read

\[(1.17)\]

$$M_1 = \begin{pmatrix} 0 & c_2 & 0 \\ c_1 & 0 & 0 \\ 0 & 1 & c_2 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & c_1d_1 & 0 \\ c_2 & 0 & 2d_2 \\ 0 & 1 & c_1d_1d_2 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 0 & 0 & d_1d_2 \\ 1 & 0 & 0 \\ 0 & 1 & d_2 \end{pmatrix}, \quad M_\infty = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

where $c_1, c_2, d_1, d_2$ are non-zero constants. The solution of the R-H problem is defined in terms of the Szegő kernel of the genus two Riemann surface

$$\mathcal{E}_{3,1} : y^3 = (\lambda - \lambda_1)(\lambda - \lambda_2)^2(\lambda - \lambda_3).$$

The period matrix of the surface has the symmetric form

$$\Pi = \begin{pmatrix} 2T & T \\ T & 2T \end{pmatrix}, \quad \text{Im} T > 0,$$

with

$$T = t\sqrt{3} F \left( \frac{1}{3}, \frac{2}{3}; 1; 1-t \right), \quad t = \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1},$$

where $F \left( \frac{1}{3}, \frac{2}{3}; 1; 1-t \right)$ and $F \left( \frac{1}{3}, \frac{2}{3}; 1; t \right)$ are two independent solutions of the Picard-Fuchs equation

$$t(1-t) \frac{d^2}{dt^2} F + (1 - 2t) \frac{d}{dt} F - \frac{2}{9} F = 0.$$

The inverse function $t = t(T)$ is in general not single valued. For $T$ belonging to Siegel half-space $\mathcal{H}_1$ modulo the sub-group $\Gamma_0(3)$ of the modular group, the function $t = t(T)$ is single-valued and reads

\[(1.18)\]

$$t = 27 \vartheta_3^4(0; 3T)^2 \vartheta_3^4(0; 3T) \left( \frac{\vartheta_3^4(0; 3T) - \vartheta_3^4(0; T)}{3\vartheta_3^4(0; 3T) + \vartheta_3^4(0; T)} \right)^2.$$
Clearly, the above expression is automorphic under the action of the group $\Gamma_0(3)$ and can be expressed in terms of the Dedekind $\eta$-function [26]. From the classical theory of the hypergeometric equation it follows that the function $t = t(T)$ satisfies the Schwarz equation (see for example [25])

$$\{t, T\} + \frac{t^2}{2} \left( \frac{1}{t^2} + \frac{1}{(t-1)^2} - \frac{10}{9t(t-1)} \right) = 0,$$

where $\dot{t} = \frac{dt}{dT}$ and $\{\ , \ \}$ is the Schwarzian derivative

\begin{equation}
\{t, T\} := \frac{\dot{t}}{t} - \frac{3}{2} \left( \frac{\ddot{t}}{\dot{t}} \right)^2.
\end{equation}

From the function $t = t(T)$ it is possible to derive an expression for the solution of the corresponding general Halphen system equivalent to the one derived in [26].

The surface $\mathcal{C}_{3,1}$ is a two-sheeted cover of two elliptic curves that are 3-isogenous. As a result, the solution of the R-H problem and of the Schlesinger equations can be expressed explicitly in terms of Jacobi’s $\vartheta$-functions. The corresponding $\tau$-function of the Schlesinger system reads

\begin{equation}
\tau(\lambda_1, \lambda_2, \lambda_3) = \left( \frac{\lambda_1 - \lambda_3}{\lambda_1 - \lambda_2}(\lambda_2 - \lambda_3) \right)^6 e^{2\pi i (\delta_1 + \delta_2 + \delta_3 + \epsilon_1 \delta_1 + \epsilon_2 \delta_2)} \sum_{k=2}^3 \vartheta_k(\epsilon_1 + \epsilon_2 + 3T(\delta_1 + \delta_2); 6T) \vartheta_k(\epsilon_1 - \epsilon_2 + T(\delta_1 - \delta_2); 2T) \vartheta_3(0; 6T) \vartheta_3(0; 2T) + \vartheta_2(0; 6T) \vartheta_2(0; 2T),
\end{equation}

where $\vartheta_i, i = 2, 3$ are the Jacobi’s $\vartheta$-functions and

$$\epsilon_i = \frac{1}{2\pi i} \log c_i, \quad \delta_i = \frac{1}{2\pi i} \log d_i, \quad i = 1, 2.$$

This paper is organized as follows. In the Section 2 we give some general backgrounds about the theory of R-H problems and we describe the R-H problem we are going to solve. In the Section 3 some backgrounds about classical algebraic geometry of Riemann surfaces and kernel forms are given. We describe in detail the curve $\mathcal{C}_{N,m}$ in the Section 4, namely its homology basis, the characteristics supported on branch points, the Szegő kernel for $1/N$ characteristics and the projective connection. This section contains mainly new material. In the Section 5 we solve the R-H problem for quasi-permutation monodromy matrices and we study the symmetry properties of the solution which are inherited from the symmetries of the curve. We derive the $\tau$-function for the Schlesinger system and Thomae-type formula for the $Z_n$ curve. We describe extensively an example for a $3 \times 3$ matrix R-H problem with four singular points in the sixth Section and we derive the solution of the corresponding $3 \times 3$ Schlesinger system. Interesting relations with the modular surface $\mathcal{H} / \Gamma_0(3)$ are pointed out. We draw our conclusion in the last section.

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2. The $N \times N$ matrix Riemann-Hilbert problem

The method of [13] to solve the R-H problem consists of reducing it to the so-called homogeneous Hilbert boundary value problem of the theory of singular equations [29]. The reduction is carried out in the following way. Let us assume that the set of points $\lambda_1, \ldots, \lambda_{2m+1}$ satisfy the relation

$$\text{Re} \lambda_1 < \text{Re} \lambda_3 < \cdots < \text{Re} \lambda_m < \text{Re} \lambda_{2m+1}.$$  

Let $\mathcal{L}$ be the oriented polygonal line which connects this set of points and infinity

$$\mathcal{L} = (\infty, \lambda_1) \cup (\lambda_1, \lambda_2) \cup (\lambda_2, \lambda_3) \cup \cdots \cup (\lambda_{2m}, \lambda_{2m+1}) \cup (\lambda_{2m+1}, \infty).$$

We denote by $C_+$ and $C_-$ the positive and negative parts of the plane $\mathbb{C}$ with respect to $\mathcal{L}$ (see Figure 1).

Let us consider the set of $2(N-1)m$ non-zero complex constants $c_1, \ldots, c_{(N-1)m}$ and $d_1, \ldots, d_{(N-1)m}$ and define the $N \times N$ quasi-permutation matrices $G_k \in SL(N, \mathbb{C})$ as

$$G_{2k-1} = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & (-1)^{N-1} c_k \\
\frac{c_k}{c_{k+2}} & 0 & \cdots & 0 & 0 \\
0 & \frac{c_{k+2}}{c_{k+4}} & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \frac{c_{k+(N-2)m}}{c_{k+(N-4)m}} & 0 \\
0 & 0 & \cdots & 0 & \frac{1}{c_{k+(N-2)m}}
\end{pmatrix}$$

for $k = 1, \ldots, m$ and $G_{2m+1} = \mathcal{P}_N$, where $\mathcal{P}_N$ has been defined in [13]; the diagonal matrix $G_{2k}$ reads

$$G_{2k} = \text{diag} \left( d_k, d_{k+1}, \ldots, d_{k+(N-2)m}, \frac{1}{\prod_{j=0}^{N-2} d_{k+jm}} \right),$$

for $k = 1, \ldots, m$ and $G_0 = G_{2m+1} = 1_N$. We define the $N \times N$ matrix function $Y(\lambda)$ as the solution of the following R-H problem:

(2.3) $Y(\lambda)$ is analytic in $\mathbb{CP}^1 \setminus \mathcal{L}$,

(2.4) $L_2$-limits $Y_\pm(\lambda)$ as $\lambda \to \mathcal{L}_\pm$ satisfy the jump conditions:

$$Y_-(\lambda) = Y_+(\lambda) G_k, \quad \lambda \in (\lambda_k, \lambda_{k+1}), \quad k = 0, \ldots, 2m + 1, \quad \lambda_0 = \lambda_{2m+2} = \infty$$

(2.5) $Y(\lambda_0) = 1_N, \quad \lambda_0 \in \mathcal{C}_+ \setminus D$.

Assuming the existence of the solution of the R-H problem [20], [23], one can find that the monodromy matrices are obtained from [12] by

$$Y(\gamma_k(\lambda)) = Y(\lambda) M_k,$$

where

$$M_k = G_k (G_{k-1})^{-1}, \quad k = 1, \ldots, 2m + 2,$$

$$Y(\lambda)^\frac{\pi}{2} \mathcal{P}_N = Y(\lambda) M_\infty,$$

where

$$M_\infty = \mathcal{P}_N^{-1}.$$

Remark 2.1. The monodromy representation described by the matrices (2.7) is irreducible if

$$c_{k+sm} \neq \zeta_k^{s+1}, \quad d_{k+sm} \neq \zeta_k, \quad s = 0, \ldots, N - 2,$$
where ξ_k and ζ_k, k = 1, ..., m, are any N-th root of unity and ξ_{m+1} = ζ_{m+1} = 1. Indeed on the contrary, the matrices G_k read

\[ G_{2k} = ξ_k 1_N, \quad G_{2k-1} = ξ_k P_N, \quad k = 1, \ldots, m. \]

The corresponding reducible monodromy representation is given by the matrices

\[ M_{2k} = G_{2k}(G_{2k-1})^{-1} = \frac{ξ_k}{ξ_k} P_N^{-1}, \quad k = 1, \ldots, m + 1, \]
\[ M_{2k-1} = G_{2k-1}(G_{2k-2})^{-1} = \frac{ξ_k}{ξ_k} P_N, \quad k = 1, \ldots, m + 1. \] (2.8)

The matrices M_k can be written in the form

\[ M_k = U_k^{-1} e^{2πiσ_N} U_k, \quad k = 1, \ldots, 2m + 1, \quad U_k ∈ GL(N, C), \]
where the matrix σ_N reads

\[ σ_N = \text{Diag} \left( \frac{-N + 1}{2N}, \frac{-N + 3}{2N}, \ldots, \frac{N - 3}{2N}, \frac{N - 1}{2N} \right). \] (2.9)

The function Y(λ) has regular singularities of the following form near the points λ_k

\[ Y(λ) = \tilde{Y}_k(λ)(λ - λ_k)^{-σ_N} U_k^{±}, \quad λ ∈ C±, \] (2.11)
where the matrices \( \tilde{Y}_k(λ) \) are holomorphic and invertible at \( λ = λ_k, \) \( U_k^+ = U_k \) and \( U_k^- = U_k G_{k-1} \) and the matrices \( G_k \) and \( U_k, \) \( k = 1, \ldots, 2m + 1, \) have been defined in (2.1), (2.2) and (2.9) respectively.

It follows from the above expansion that \( \frac{dY(λ)}{dλ} Y^{-1}(λ) \) is meromorphic in \( \mathbb{CP}^1 \) with simple poles at \( λ_1, λ_2, \ldots, λ_{2m+1} \) and \( ∞. \) Therefore \( Y(λ) \) satisfies the Fuchsian equation

\[ \frac{dY(λ)}{dλ} = \sum_{k=1}^{2m+1} \frac{A_k}{λ - λ_k} Y(λ), \] (2.12)
where

\[ A_k = A_k(λ_1, \ldots, λ_{2m+1} | M_1, \ldots, M_{2m+1}) = \text{Res}_{λ=λ_k} \left[ \frac{dY(λ)}{dλ} Y^{-1}(λ) \right] = \tilde{Y}_k(λ_k)σ_N \tilde{Y}_k^{-1}(λ_k), \quad k = 1, \ldots, 2m + 1, \] (2.13)
which follows from (2.11). If none of the monodromy matrices depend on the position of the singular points \( λ_k, \) \( k = 1, \ldots, 2m + 1, \) the function \( Y(λ; λ_1, \ldots, λ_{2m+1}) \) in addition to (2.12) satisfies the following equations

\[ \frac{∂}{∂λ_k} Y(λ) = \left( \frac{A_k}{λ_0 - λ_k} - \frac{A_k}{λ - λ_k} \right) Y(λ), \quad k = 1, \ldots, 2m + 1. \] (2.14)

Compatibility conditions of (2.12) and (2.14) are described by the system of Schlesinger equations [22]

\[ \frac{∂}{∂λ_j} A_k = \frac{[A_k, A_j]}{λ_k - λ_j} - \frac{[A_k, A_j]}{λ_0 - λ_j}, \quad j ≠ k, \]
\[ \frac{∂}{∂λ_k} A_k = - \sum_{j ≠ k}^{2m+1} \left( \frac{[A_k, A_j]}{λ_k - λ_j} - \frac{[A_k, A_j]}{λ_0 - λ_j} \right). \] (2.15)

Thus the solution of the R-H problem (2.3)-(2.5) leads immediately to the particular solution (2.13) of the Schlesinger system (2.14).
From the solution of the Schlesinger equation (2.13) one can define the corresponding holomorphic \( \tau \)-function given by the formula (2.16)

\[
\frac{\partial}{\partial \lambda_k} \log \tau = \frac{1}{2} \text{Res}_{\lambda=\lambda_k} \text{Tr} \left( \frac{dY(\lambda)}{d\lambda} Y(\lambda)^{-1} \right)^2.
\]

The set of zeros of the \( \tau \)-function in the space of singularities of the R-H problem is called the Malgrange divisor (\( \theta \)). It plays a crucial role in the discussion of the solvability of the R-H problem with the given monodromy data.

3. Riemann surface of an algebraic curve

In order to solve the R-H problem (2.14) - (2.15), we first need to introduce some basic objects on Riemann surfaces.

3.1. The curve and differentials. Let \( \mathcal{C} \) be the Riemann surface of the algebraic equation

\[ y^N + p_1(\lambda)y^{N-1} + \ldots + p_N(\lambda) = 0, \]

where \( p_1, \ldots, p_N \) are polynomials in \( \lambda \). In a neighbourhood \( U_R \) of the point \( R = (\eta, w) \in \mathcal{C} \), a local coordinate \( z(P) \), \( P = (\lambda, y) \in U_R \), is the function defined by

\[
z(P) = \begin{cases} 
\sqrt[\lambda]{\lambda - \eta} & \text{if } R \text{ is an ordinary point,} \\
\sqrt[\lambda]{\lambda - \eta} & \text{if } R \text{ is a finite branch point of order } l, \\
\frac{1}{\sqrt[\lambda]{\lambda - \eta}} & \text{if } R \text{ is an ordinary point at infinity,} \\
\frac{1}{\sqrt[\lambda]{\lambda - \eta}} & \text{if } R \text{ is a branch point at infinity of order } m.
\end{cases}
\]

Let \( dv(P) = (dv_1(P), \ldots, dv_1(P)) \) be the basis of normalized holomorphic differentials, with respect to the canonical homology basis in \( H_1(\mathcal{C}, \mathbb{Z}) \) of \( \alpha \) and \( \beta \)-cycles, \( (\alpha_1, \ldots, \alpha_g; \beta_1, \ldots, \beta_g) \). The matrix of \( \beta \)-periods

\[
\Pi = \left( \int_{\beta_i} dv_k(P) \right)_{i,k=1,\ldots,g}
\]

belongs to the Siegel half space, \( \mathcal{H}_g = \{ \Pi | \Pi^t = \Pi, \text{Im} \Pi > 0 \} \). The Jacobian variety of the curve \( \mathcal{C} \) is denoted by \( \text{Jac}(\mathcal{C}) = \mathbb{C}^g / (1_g \oplus \Pi) \).

We also mention the variation formulas which describe the dependence of the period matrix \( \Pi \) on the branch points. These formula can be already found in the hyperelliptic case in Thomae [31]. For general surfaces the infinitesimal variation of the period matrix with respect to a Beltrami differential is due to Rauch [32] (see also Fay [33]) and Korotkin reduced this deformation to the useful form [3]:

\[
\frac{\partial}{\partial \lambda_k} \Pi_{ij} = 2\pi i \text{Res}_{\lambda=\lambda_k} \left( \frac{1}{(dz(P))^2} \sum_{s=1}^{N} dv_i(P(s))dv_j(P(s)) \right),
\]

where \( i, j = 1, \ldots, 2m, \ k = 1, \ldots, 2m+1 \) and \( P(s) \) is a point on the sheet \( s \) of \( \mathcal{C} \).

3.2. \( \theta \)-function. Any point \( \mathbf{e} \in \mathbb{C}^g \) can be written uniquely as \( \mathbf{e} = (\mathbf{e}, \mathbf{d})^{(1)} \), where \( \mathbf{e}, \mathbf{d} \in \mathbb{R}^g \) are the characteristics of \( \mathbf{e} \). We use the notation \( [\mathbf{e}] = [\mathbf{d}] \).

If \( \mathbf{e} \) and \( \mathbf{d} \) are half integer, then we say that the corresponding characteristics \( [\mathbf{e}] \) are half-integer. The half-integer characteristics are odd or even, whenever \( 4\langle \mathbf{d}, \mathbf{e} \rangle \) is equal to 1 or 0 modulo 2. The bracket \( \langle , \rangle \) denotes the standard Euclidean scalar product.

The Riemann \( \theta \)-function with characteristics \( [\mathbf{d}] \) is given on \( \mathcal{H}_g \times \text{Jac}(\mathcal{C}) \) as the Fourier series

\[
\theta_{[\mathbf{d}]}(z; \Pi) = \sum_{\mathbf{n} \in \mathbb{Z}^g} \exp(\pi i (\Pi \mathbf{n} + \Pi \mathbf{d}, \mathbf{n} + \mathbf{d}) + 2\pi i (z + \mathbf{e}, \mathbf{n} + \mathbf{d})).
\]
The \( \theta \)-function is an entire function in the variable \( z \) with periodicity properties:

\[
\theta^\delta_{\epsilon}(z + e_k; \Pi) = e^{2\pi i \delta_k} \theta^\delta_{\epsilon}(z; \Pi),
\]

(3.5)

\[
\theta^\delta_{\epsilon + n'}(z; \Pi) = e^{2\pi i (\epsilon, n')} \theta^\delta_{\epsilon}(z; \Pi),
\]

(3.6)

\[
\theta^\delta_{\epsilon + n''}(z; \Pi) = e^{2\pi i (\epsilon, n'')} \theta^\delta_{\epsilon}(z; \Pi),
\]

(3.7)

where \( e_k = (0, \ldots, 1, \ldots, 0) \) is the standard basis in \( \mathbb{C}^g \), \( n' \) and \( n'' \) integer vectors. When \( \theta^\delta_{\epsilon}(z; \Pi) \) is equal to zero we write \( \theta^\delta_{\epsilon}(z; \Pi) = \theta(z; \Pi) \). The function \( \theta(z; \Pi) \) is even and clearly satisfies the relation

\[
\frac{\partial}{\partial z_i} \theta(z; \Pi) \bigg|_{z=0} = 0, \quad i = 1, \ldots, g.
\]

The \( \theta \)-function with arbitrary characteristics satisfies the heat equation

\[
\frac{\partial^2}{\partial z_k \partial z_l} \theta^\delta_{\epsilon}(z; \Pi) = 2\pi i (1 + 2\delta_{k,l}) \frac{\partial}{\partial \Pi_{kl}} \theta^\delta_{\epsilon}(z; \Pi), \quad k, l = 1, \ldots, g.
\]

(3.8)

The zeros of the \( \theta \)-function are described by the Riemann vanishing theorem.

**Theorem 3.1.** Let \( e \in \text{Jac}(\mathcal{C}) \) be an arbitrary vector. Then the multi-valued function

\[
P \to \theta \left( \int_{P_0}^P \frac{dv}{e - \Pi} \right)
\]

has on \( \mathcal{C} \) exactly \( g \) zeros \( Q_1, Q_2, \ldots, Q_g \) provided it does not vanish identically. There is a one-to-one correspondence between \( e \in \text{Jac}(\mathcal{C}) \) and the non-special divisor \( \sum_{i=1}^g Q_i e = g \sum_{i=1}^g \int_{Q_0}^{Q_i} dv - K_{Q_0} \),

where \( K_{Q_0} \) is the vector of Riemann constants

\[
(K_{Q_0})_j = \frac{1 + \Pi_{jj}}{2} - \sum_{i=1, i \neq j}^g \int_{Q_0}^{Q_i} dv_i(P) \int_{Q_0}^{P} dv_j.
\]

(3.10)

**Remark 3.2.** The vector of Riemann constants depends on the homology basis \( (\alpha_1, \ldots, \alpha_g; \beta_1, \ldots, \beta_g) \in H_1(\mathcal{C}, \mathbb{Z}) \) and the base point \( Q_0 \in \mathcal{C} \).

For a point \( P \in \mathcal{C} \), we define the Abel map \( \mathfrak{A} : \mathcal{C} \to \text{Jac}(\mathcal{C}) \) by setting

\[
\mathfrak{A}(P) = \int_{P_0}^P dv,
\]

(3.11)

for some base point \( P_0 \in \mathcal{C} \). For a positive divisor \( D \) of degree \( n \) the Abel map reads

\[
\mathfrak{A}(D) = \int_{nP_0}^D dv.
\]

There exists a non-positive divisor \( \Delta \) of degree \( g - 1 \) such that

\[
\mathfrak{A}(\Delta - (g - 1)Q_0) = K_{Q_0},
\]

(3.12)

where \( K_{Q_0} \) has been defined in (3.10). The divisor \( \Delta \) is called the Riemann divisor and satisfies the condition \( 2\Delta = \mathcal{K} \), where \( \mathcal{K} \) is the canonical class (that is the class of divisors of Abelian differentials).
Definition 3.1. The characteristic $[\delta_\epsilon]$ of a point $e = \epsilon + \delta \Pi$ is called singular if
\[ \theta(e; \Pi) = 0. \]
The odd half-integer characteristics $[\hat{\delta} \hat{\epsilon}]$ of a point $\gamma = \hat{\epsilon} + \hat{\delta} \Pi$ is non-singular if among the derivatives
\[ \left. \frac{\partial}{\partial z_j} \theta[\gamma](z; \Pi) \right|_{z=0}, \quad j = 1, \ldots, g, \]
there is at least one non-vanishing.

3.3. Kernel-forms. The Schottky-Klein prime form $E(P, Q)$, $P, Q \in \mathbb{C}$ is a skew-symmetric $(-\frac{1}{2}, -\frac{1}{2})$-form on $\mathbb{C} \times \mathbb{C}$ [34]

\[ E(P, Q) = \frac{\theta[\gamma] \left( \int_P^Q dv; \Pi \right)}{h(P)h(Q)}, \tag{3.13} \]

where $[\gamma]$ is a non-singular odd half-integer characteristics and
\[ h^2(P) = \sum_{j=1}^{g} \frac{\partial}{\partial z_j} \theta[\gamma](0; \Pi)dv_j(P). \]

The prime form does not depend on the point $\gamma$. The automorphic factors of the prime form along all cycles $\alpha_k$ are trivial; the automorphic factor along each $\beta_k$ cycle in the $Q$ variable equals $\exp\{-\pi \Pi_{kk} - 2\pi i \int_P^Q dv_k\}$. If the points $P$ and $Q$ are placed in the vicinity of the point $R$ with local coordinate $z$, $z(R) = 0$, then the prime form has the following local behaviour as $P \to Q$

\[ E(P, Q) \sim \frac{z(P) - z(Q)}{\sqrt{dz(P)\sqrt{dz(Q)}}} (1 + O(1)). \tag{3.14} \]

The prime form $E(P, Q)$ is the generating form of the Bergmann and Szegő kernels. Let $P = (\lambda, y)$ and $Q = (\mu, w)$. Then the Bergmann kernel $\omega(P, Q)$ is defined as a symmetric 2-differential,

\[ \omega(P, Q) = d_\lambda d_\mu \log E(P, Q). \tag{3.15} \]

All the $\alpha$-periods of $\omega(P, Q)$ with respect to any of its two variables vanish. The period of the Bergmann kernel with respect to the variable $P$ or $Q$, along the $\beta_k$ cycle, is equal to $2\pi i dv_k(Q)$ or $2\pi i dv_k(P)$ respectively. The Bergmann kernel has a double pole along the diagonal with the following local behaviour

\[ \omega(P, Q) = \frac{1}{(z(P) - z(Q))^2} + H(z(P), z(Q)) + \text{higher order terms} \right) dz(P)dz(Q), \tag{3.16} \]

where $H(z(P), z(Q))dz(P)dz(Q)$ is the non-singular part of $\omega(P, Q)$ in each coordinate chart. The restriction of $H$ on the diagonal is the projective connection (see for example [35])

\[ R(z(P)) = 6H(z(P), z(P)) \tag{3.17} \]

which depends non-trivially on the chosen system of local coordinates. Namely the projective connection transforms as follows with respect to a change of local coordinates $z \to f(z)$

\[ R(z) \to R(f(z))[f'(z)]^2 + \{f(z), z\}, \]

where $\{, \}$ is the Schwarzian derivative [14].
The Szegő kernel $S^{[\delta]}(P,Q)$ is defined for all non-singular characteristics $[\delta]$ as the $(\frac{1}{2},\frac{1}{2})$-form on $\mathcal{C} \times \mathcal{C}$.

\begin{equation}
S^{[\delta]}(P,Q) = \frac{\theta^{[\delta]}(P;\Pi)}{\theta^{[\delta]}(0;\Pi)E(P,Q)}.
\end{equation}

The local behaviour of the Szegő kernel when $P \to Q$ is

\begin{equation}
S^{[\delta]}(P,Q) = \frac{\sqrt{dz(P) \sqrt{dz(Q)}}}{z(P) - z(Q)} \left[ 1 + T(z(P))(z(P) - z(Q)) + O((z(P) - z(Q))^2) \right],
\end{equation}

where

\begin{equation}
T(z(P))dz(P) = \sum_{k=1}^{g} \frac{\partial}{\partial z_k} \log \theta^{[\delta]}(0;\Pi)dv_k(P).
\end{equation}

The Szegő kernel transforms when the variable $P$ goes around $\alpha_k$ and $\beta_k$-cycles as follows

\begin{align}
S^{[\delta]}(P + \alpha_k, Q) &= e^{2\pi i \delta_k}S^{[\delta]}(P,Q), \\
S^{[\delta]}(P + \beta_k, Q) &= e^{-2\pi i \kappa_k}S^{[\delta]}(P,Q), \quad k = 1, \ldots, g.
\end{align}

The Riemann divisor $\Delta$ is the divisor class of the Szegő kernel with zero characteristics $[0]$ (34), p. 7).

Another important relation (34), Cor. 2.12, connects the Szegő and Bergmann kernels

\begin{equation}
S^{[\delta]}(P,Q)S^{[-\delta]}(P,Q) = \omega(P,Q) + \sum_{k,l=1}^{g} \frac{\partial^2}{\partial z_k \partial z_l} \log \theta^{[\delta]}(0;\Pi)dv_k(P)dv_l(Q).
\end{equation}

Finally we point the following equality, (34), Cor. 2.19,

\begin{equation}
\det \left( \left( S^{[\delta]}(P_j,Q_k) \right)_{j,k=1,\ldots,n} \right) = \frac{\theta^{[\delta]} \left( \sum_{j=1}^{n} \int_{Q_j}^{P_j} dv; \Pi \right) \prod_{1 \leq j < k \leq n} E(P_j,P_k)E(Q_k,Q_j)}{\theta^{[\delta]}(0;\Pi) \prod_{j,k=1}^{n} E(P_j,Q_k)},
\end{equation}

for any two sets of points $P_1, \ldots, P_n$, and $Q_1, \ldots, Q_n$, $n \geq g$, and non-singular characteristics $[\delta]$.

4. Z\textsubscript{N} CURVES

In order to solve the R-H problem (2.3)–(2.5) explicitly we need to study in detail the Riemann surface $\mathcal{C}_{N,m}$ of the curve

\begin{equation}
y^N = p(\lambda)q(\lambda)^{N-1},
\end{equation}

where $p(\lambda)$ and $q(\lambda)$ have been defined in (1.4). The curve (1.4) has singularities at the points $(\lambda_{2k}, 0)$, $k = 1, \ldots, m$. These singularities can be resolved (17) to give rise to a compact Riemann surface which we denote by $\mathcal{C}_{N,m}$. The genus $g$ of the curve (1.4) can be computed from (1.7) and is equal to $(N-1)m$.

The branch points of the curve are $(\lambda_1, 0), \ldots, (\lambda_{2m+1}, 0)$ and $(\infty, \infty)$. The projection $\pi : (\lambda, y) \to \lambda$, defines $\mathcal{C}_{N,m}$ as a $N$–sheeted covering of the complex plane $\mathbb{CP}^1$. Therefore the pre-image of an ordinary point $\lambda \in \mathbb{CP}^1$ consists of $N$ points. The $N$-cyclic automorphism $J$ of $\mathcal{C}_{N,m}$ is given by the action

\begin{equation}
J : (\lambda, y) \to (\lambda, \rho y),
\end{equation}

where $\rho$ is the $N$-primitive root of unity, namely $\rho = e^{2\pi i/N}$. In a neighbourhood $U_R$ of the point $R = (\eta, w) \in \mathcal{C}_{N,m}$, a local coordinate $z(P)$, $P = (\lambda, y) \in U_R$, is the function defined by

\begin{equation}
z(P) = \begin{cases} 
\lambda - \eta, & \text{if } R \text{ is an ordinary point,} \\
\sqrt[N]{\lambda - \eta}, & \text{if } R = (\lambda_k, 0), \ k = 1, \ldots, 2m + 1, \\
\frac{1}{\sqrt[N]{\lambda}}, & \text{if } R = (\infty, \infty).
\end{cases}
\end{equation}
4.1. **Homologies and periods of $Z_N$-curves.** The canonical homology basis,

$$(\alpha_1, \ldots, \alpha_{(N-1)m}; \beta_1, \ldots, \beta_{(N-1)m}) \in H(C, \mathbb{Z})$$

of $C_{N,m}$ is shown in the Figure 2. Namely the cycles $\alpha_{j+km}$, $j = 1, \ldots, m$ lie on the $k+1$ sheet,

$$k = 0, \ldots, N-2.$$ The cycles $\beta_{j+km}$, $j = 1, \ldots, m$, $k = 0, \ldots, N-2$, emerges on the $(k+1)$th sheet on the cut $(\lambda_{2j-1}, \lambda_{2j})$, pass anti-clockwise to the Nth sheet through the cut $(\lambda_{2j+1}, \lambda_{2j+2})$ and return to the initial point through the Nth sheet.

**Remark 4.1.** We remark that on Figure 3, when $N > 3$, the $\beta$-cycles placed from the second to the $(N-2)$th sheet should intersect the cuts only on the branch points. If we drop this requirement we need to draw a more complicated but equivalent homology basis.

The action of the automorphism $J$ on the basis of cycles is given by

$$J\alpha_{i+sm} = \alpha_{i+(s+1)m}, \quad i = 1, \ldots, m, \quad s = 0, \ldots, N-3,$$

$$J\alpha_{i+(N-2)m} = -\sum_{s=0}^{N-2} \alpha_{i+sm}, \quad i = 1, \ldots, m,$$

$$J\beta_{i+sm} = \beta_{i+(s+1)m} - \beta_i, \quad s = 0, \ldots, N-3, \quad J\beta_{i+(N-2)m} = -\beta_i, \quad i = 1, \ldots, m.$$ The basis of canonical holomorphic differentials reads

$$du_{j+sm}(P) = \frac{\lambda^{j-1} q(\lambda)^s}{y^{s+1}} d\lambda, \quad j = 1, \ldots, m, \quad s = 0, \ldots, N-2.$$ The $(N-1)m \times (N-1)m$ matrices $A$ of $\alpha$-periods and $B$ of $\beta$-periods are expressible in terms of $m \times m$-matrices

$$(A_{s+1})_{kj} = \oint_{\alpha_j} du_{k+ms}, \quad (B_{s+1})_{kj} = \oint_{\beta_j} du_{k+ms}, \quad j, k = 1, \ldots, m, \quad s = 0, \ldots, N-2,$$ in the following way. Let us introduce the $(N-1)m \times (N-1)m$ dimensional matrices

$$\mathcal{R}_A = \left( \frac{\rho^{-i(k-1)} - \rho^{-ik}}{1 - \rho^{-1}} \right)_{i,k=1,\ldots,N-1} \otimes 1_m,$$

$$\mathcal{R}_B = \left( \frac{\rho^{-i(k-1)} - \rho^{-i(N-1)}}{1 - \rho^{-(N-1)i}} \right)_{i,k=1,\ldots,N-1} \otimes 1_m.$$
Then

\[
A = \left( \oint_{\alpha_j} \frac{du_k}{\partial_j} \right)_{k,j=1,\ldots,(N-1)m} = \text{Diag}(A_1, A_2, \ldots, A_{N-1}) R_A,
\]

\[
B = \left( \oint_{\beta_j} \frac{du_k}{\partial_j} \right)_{k,j=1,\ldots,(N-1)m} = \text{Diag}(B_1, B_2, \ldots, B_{N-1}) R_B,
\]

where

\[
\text{Diag}(A_1, A_2, \ldots, A_{N-1}), \quad \text{Diag}(B_1, B_2, \ldots, B_{N-1})
\]

are the block diagonal \((N-1)m \times (N-1)m\) dimensional matrices having as entries the matrices \(A_s\) and \(B_s\), \(s = 1, \ldots, N - 1\), respectively.

The basis of normalized holomorphic differentials

\[ dv = (dv_1, \ldots, dv_{(N-1)m}), \quad \oint_{\alpha_j} dv_k = \delta_{jk}, \]

is written as

\[ dv_j = \sum_{k=1}^{g} (A^{-1})^{-1}_{jk} du_k, \quad j = 1, \ldots, (N-1)m. \]

The period matrix \(\Pi\),

\[ \Pi_{j,k} = \oint_{\beta_j} dv_j, \quad j, k = 1, \ldots, (N-1)m \]

is given by

\[
\Pi = R^{-1}_A \text{Diag}(A_1^{-1}, A_2^{-1}, A_{N-1}^{-1}) R_B
\]

with \(R_A\) and \(R_B\) defined in (4.8) and (4.9) respectively.

4.2. Characteristics supported on branch points. In this section we are going to compute the integrals of the form

\[ \int_{P_k}^{P_{\infty}} dv_j, \quad j, k = 1, \ldots, (N-1)m, \]

in terms of the period matrix \(\Pi\), where \(P_k = (\lambda_k, 0)\) and \(P_{\infty} = (\infty, \infty)\). Below we shall omit the second coordinate of the points \(P_k\) and \(P_{\infty}\).

**Lemma 4.2.** The following relations are satisfied for \(k = 1, \ldots, m, s = 0, \ldots, N - 2\),

\[
\int_{\lambda_{2k}}^{\lambda_{2k-1}} dv_{k+sm} = \frac{N-1-s}{N},
\]

(4.13)

\[
\int_{\lambda_{2k+2}}^{\lambda_{2k+1}} dv_{k+sm} = -\frac{N-1-s}{N},
\]

(4.14)

\[ \int_{\lambda_{2k+2}}^{\lambda_{2k+1}} dv_{j+sm} = 0, \quad j \neq k, k+1, j = 1, \ldots, m \]

(4.15)

and

\[
\int_{\lambda_{2j}}^{\lambda_{2j+1}} dv_{k+sm} = \frac{N-1}{N} \Pi_{k+sm,j} - \frac{1}{N} \sum_{r=1}^{N-2} \Pi_{k+sm,j+r+m},
\]

for \(k, j = 1, \ldots, m, s = 0, \ldots, N - 2\).
Proof. To prove (4.13) we observe that for \( r, s = 0, \ldots, N - 2 \),
\[
\oint_{\alpha_{k+r}} dv_{k+s} = 0 = \sum_{j=1}^{j<k} \int_{\lambda_{2j-1}}^{\lambda_{2j}} (J^{(r)}(dv_{k+s}) - J^{s+1}(dv_{k+s})), \quad j < k.
\]
Since
\[
\sum_{r=0}^{N-1} J^{(r)}(dv) = 0,
\]
the above two equations imply that
\[
\int_{\lambda_{2j-1}}^{\lambda_{2j}} J^{(r)}(dv_{k+s}) = 0, \quad j < k, \quad r, s = 0, \ldots, N - 2.
\]
Therefore for \( k = 1, \ldots, m \) and \( s = 0, \ldots, N - 2 \),
\[
\oint_{\alpha_{k+r}} dv_{k+s} = 1 = \int_{\lambda_{2k-1}}^{\lambda_{2k}} (J^{(s)}(dv_{k+s}) - J^{(s+1)}(dv_{k+s})),
\]
\[
\oint_{\alpha_{k+r}} dv_{k+s} = 0 = \int_{\lambda_{2k-1}}^{\lambda_{2k}} (J^{(r)}(dv_{k+s}) - J^{(r+1)}(dv_{k+s})), \quad r \neq s.
\]
Combining (4.17) and (4.19) we can write the system
\[
\begin{pmatrix}
1 -1 0 \ldots 0 0 \\
0 1 -1 \ldots 0 0 \\
\vdots \vdots \vdots \ldots \vdots \\
0 0 0 \ldots 1 -1 \\
1 1 1 \ldots 1 2
\end{pmatrix}
\begin{pmatrix}
\int_{\lambda_{2k-1}}^{\lambda_{2k}} dv_{k+s} \\
\int_{\lambda_{2k-1}}^{\lambda_{2k}} J(dv_{k+s}) \\
\vdots \\
\int_{\lambda_{2k-1}}^{\lambda_{2k}} J^{(N-3)}(dv_{k+s}) \\
\int_{\lambda_{2k-1}}^{\lambda_{2k}} J^{(N-2)}(dv_{k+s})
\end{pmatrix}
= \begin{pmatrix}
0 \\
1 \\
\vdots \\
0
\end{pmatrix}
\]
for \( s = 0, \ldots, N - 2 \) and \( k = 1, \ldots, m \),
which leads to (4.13). The relation (4.14) follows from the combination of (4.13), (4.20) and the fact that
\[
\oint_{\alpha_{k+r}} dv_{k+s} = 0 \quad \text{for} \quad r, s = 0, \ldots, N - 2.
\]
The relation (4.15) follows from (4.13), (4.14) and the fact that \( dv \) are normalized differentials.
Finally to prove (4.16) we observe that
\[
\int_{\beta_{j+r}} dv_{k+s} = \Pi_{k+s, j+r} = \int_{\lambda_{2j+1}}^{\lambda_{2j+1}} J^{(r)}(dv_{k+s}) - J^{(N-1)}(dv_{k+s}), \quad r = 0, \ldots, N - 2.
\]
Writing the above equation in matrix form and using (4.17) we obtain
\[
\begin{pmatrix}
2 & 1 & 1 & \ldots & 1 & 1 \\
1 & 2 & 1 & \ldots & 1 & 1 \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
1 & 1 & 1 & \ldots & 2 & 1 \\
1 & 1 & 1 & \ldots & 1 & 2
\end{pmatrix}
\begin{pmatrix}
\int_{\lambda_{2j+1}}^{\lambda_{2j+1}} dv_{k+s} \\
\int_{\lambda_{2j+1}}^{\lambda_{2j+1}} J(dv_{k+s}) \\
\vdots \\
\int_{\lambda_{2j+1}}^{\lambda_{2j+1}} J^{(N-3)}(dv_{k+s}) \\
\int_{\lambda_{2j+1}}^{\lambda_{2j+1}} J^{(N-2)}(dv_{k+s})
\end{pmatrix}
= \begin{pmatrix}
\Pi_{k+s, j} \\
\Pi_{k+s, j+m} \\
\Pi_{k+s, j+(N-3)m} \\
\Pi_{k+s, j+(N-2)m}
\end{pmatrix}
\]
for \( s = 0, \ldots, N - 2 \) and \( k = 1, \ldots, m \),
which is equivalent to (4.16). \( \square \)
We observe that the quantities in (4.14) and (4.16) satisfy
\[ \frac{N - 1}{N} \Pi_{k sm,j} = -\frac{1}{N} \Pi_{k+sm,j} \text{ modulo lattice,} \]
\[ -\frac{N - 1 - s}{N} = \frac{s + 1}{N} \text{ modulo lattice.} \]

From the relations (4.13)-(4.16) and the above observation we are able to write the characteristics \([U_k]\) of the vectors

\[ U_k = \int_{\lambda_k}^{\lambda_k + \lambda} d\nu \]

in the form

\[ [U_{2m+1}] = \begin{bmatrix} 0 & \cdots & 0 & m_{\downarrow} \cdots & 0 & \cdots & 0 & s_{m_{\downarrow}} \cdots & 0 & \cdots & 0 & 0 \end{bmatrix} \],

which immediately follows from (4.14). To pass from \([U_{2m+1}]\) to \([U_{2m}]\), and in general from \([U_{2k+1}]\) to \([U_{2k}]\) we use (4.15), while for passing from \([U_{2k}]\) to \([U_{2k-1}]\) we use (4.13) and (4.14), thus obtaining

\[ [U_{2m}] = \begin{bmatrix} 0 & \cdots & 0 & m_{\downarrow} \cdots & 0 & \cdots & 0 & s_{m_{\downarrow}} \cdots & 0 & \cdots & 0 & 0 \end{bmatrix} \]

\[ [U_{2k}] = \begin{bmatrix} 0 & \cdots & 0 & k_{\downarrow} \cdots & 0 & \cdots & 0 & k_{s_{m_{\downarrow}}} \cdots & 0 & \cdots & 0 \end{bmatrix} \]

\[ [U_{2k+1}] = \begin{bmatrix} 0 & \cdots & 0 & \frac{1}{N} \cdots & 0 & \cdots & 0 & \frac{1}{N} \cdots & 0 & \cdots & 0 \end{bmatrix} \]

\[ [U_{2}] = \begin{bmatrix} 0 & \cdots & 0 & \frac{1}{N} \cdots & 0 & \cdots & 0 & \frac{1}{N} \cdots & 0 & \cdots & 0 \end{bmatrix} \]
\[ U_1 = \begin{bmatrix} \frac{-1}{N} & \frac{-1}{N} & \cdots & \frac{-1}{N} \\ \frac{-1}{N} & \frac{-1}{N} & \cdots & \frac{-1}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-1}{N} & \frac{-1}{N} & \cdots & \frac{-1}{N} \end{bmatrix} \cdot \]

These formulas will be useful for the construction of $1/N$ non-singular characteristics.

4.3. Szegő kernel for $1/N$-periods. In this section we construct the Szegő kernel for $1/N$ characteristics. For the purpose we first need to determine the vector of Riemann constants of the curve $C_{N,m}$.

**Lemma 4.3.** The vector of Riemann constants computed in the homology basis described in Figure 2 and with base point $\infty$ equals

\[ K_\infty = (N - 1) \sum_{k=1}^{m} P_{2k} \int_{\infty}^{d\nu}, \]

where $P_{2k} = (\lambda_{2k}, 0)$.

The proof of the above relation is obtained by direct calculations from the definition (3.10) following the lines of the proof of the Lemma 4.2.

**Lemma 4.4.** The Riemann divisor $\Delta$ of the curve $C_{N,m}$ in the homology basis described in Figure 2 is equivalent to

\[ \Delta = (N - 1) \sum_{k=1}^{m} P_{2k} - \infty. \]

**Proof.** The above relation follows immediately from (4.22).

The canonical divisor $K_C$ is the divisor class of any Abelian differential on $C_{N,m}$. Choosing $d\lambda$ as representative differential, we have according to (4.2) that

\[ K_C = (N - 1) \sum_{i=1}^{2m+1} P_i - (N + 1) \infty. \]

From (4.23) and (4.24) it is possible to verify that $2\Delta = K_C$. According to the results of Section 4.2 the formula (3.11) when $Q_0$ is a branch point, put into correspondence divisors $D$ consisting of branch points with $1/N$-periods.

Following Diez [37], we describe a family of non-special divisors on $C_{N,m}$ supported on the branch points. For $m \leq l \leq 2m + 1$, let $s_1, \ldots, s_l$ be positive integers such that

\[ \sum_{i=1}^{l} s_i = (N - 1)m, \quad s_i \leq N - 1, \]

i.e. when $l = m$ all $s_i = N - 1$. For each $l$ let us define the divisor class $D_l$ supported on the branch points

\[ D_l = s_1 P_{i_1} + \ldots + s_l P_{i_l}, \quad \{i_1, \ldots, i_l\} \in \{1, \ldots, 2m + 1\}. \]

In particular, the divisor class $D_m$ contains \( \binom{2m+1}{m} \) divisors

\[ D_m = (N - 1)P_{i_1} + \ldots + (N - 1)P_{i_{m-1}} + (N - 1)P_{i_m}. \]

Among the divisors with $m + 1$ branch points we consider the divisor class $D_{m+1,1}$ which contains $rac{1}{2} (m + 1) m \binom{2m+1}{m+1}$ divisors

\[ D_{m+1,1} = (N - 1)P_{i_1} + \ldots + (N - 1)P_{i_{m-1}} + (N - 2)P_{i_m} + P_{i_{m+1}}. \]
It is out of the scope of the present manuscript to classify all the non-singular divisors of the form \((4.26)\). However we can single out two families of non-special divisors.

**Lemma 4.5.** The divisors \(D_m\) defined in \((4.27)\) are non-special and the divisors \(D_{m+1,1}\) defined in \((4.28)\) are non-special for \(N > 3\). At \(N = 3\) the divisors

\[
D_m = 2P_{i_1} + \ldots + 2P_{i_{m-1}} + P_{i_m} + P_{i_{m+1}},
\]

\[i_m \in \{1, 3, 5, \ldots, 2m+1\}, \quad i_{m+1} \in \{2, 4, 6, \ldots, 2m\},\]

are non-special.

The proof is given in the Appendix.

**Remark 4.6.** The importance of the divisor classes \(D_m\) and \(D_{m+1,1}\) is due to the fact that one can construct meromorphic functions with zeros and poles in prescribed branch points. Indeed let \(D_m\) and \(D_{m+1,1}\) be the divisors defined in \((4.27)\) and \((4.28)\). Then the function

\[
f(P) = C \left( \frac{e^{P} \left( \int_{\infty}^{P} dv - \int_{g_{\infty}}^{P} dv + K_{\infty}; \Pi \right)}{e^{P} \left( \int_{\infty}^{P} dv - \int_{g_{\infty}}^{P} dv + K_{\infty}; \Pi \right)} \right)^N,
\]

with \(C\) a constant, has the only zero of \(N\)-th order at the point \(P_m\) and the only pole of \(N\)-th order at the point \(P_{i_{m+1}}\). When the normalising constant \(C\) is chosen in an appropriate way, the function \(f(P)\) can be identified with the coordinate \(\lambda\) of the curve.

Now we associate to the non-special divisors \(D_m\) the Szegő kernel corresponding to such divisors. The Szegő kernel for the complete class of divisors \((4.26)\) will be considered in a separate publication.

For the purpose we define the divisor class \(\mathcal{D}\) as

\[
\mathcal{D} = P + J(P) + J^{(2)}(P) + \cdots + J^{(N-1)}(P),
\]

which is independent from the point \(P \in C_{N,m}\). The following relations hold

\[
\mathcal{D} = NP_i, \quad i = 1, \ldots, 2m+1, \quad \mathcal{D} = NP_{\infty}.
\]

Let us associate to the divisor \(D_m\) the divisor of degree \(m(N-1) - 1\)

\[
\tilde{D}_m = D_m + (N-1)P_{\infty} - \mathcal{D},
\]

and let \([\tilde{D}_m]\) be the corresponding \(1/N\) period of the divisor \(\tilde{D}_m\), that is

\[
[\tilde{D}_m] = \mathfrak{A} (D_m + (N-1)P_{\infty} - \mathcal{D} - \Delta),
\]

where \(\Delta\) is the Riemann divisor. We observe that when the base point is at infinity then \((4.33)\) reads

\[
[\tilde{D}_m] = -\int_{(N-1)mP_{\infty}}^{D_m} dv - K_{\infty},
\]

therefore, the characteristics \([\tilde{D}_m]\) is non-singular because of Lemma \((4.33)\). Let us define the function

\[
\psi_k(P, Q) = \frac{z(P) - \lambda_k}{z(Q) - \lambda_k}, \quad k = 1, \ldots, 2m+1.
\]

and agree to omit the arguments \((P, Q)\) if no ambiguities appear.
Theorem 4.7. The Szegő kernel associated to the characteristics $[\tilde{D}_m]$ reads

$$S[\tilde{D}_m](P, Q) = \frac{1}{N} \sqrt{\frac{dPdz(Q)}{z(P) - z(Q)}} \sum_{s=0}^{N-1} \left( \prod_{k=1}^{m+1} \psi_{i_k}(P, Q) \prod_{j=1}^{m+1} \psi_{j_k}(P, Q) \right)^{-\frac{1}{N} + \frac{N-1}{2N}},$$

where $I_m = \{i_1, \ldots, i_m\} \subset \{1, 2, \ldots, 2m+1\}$ and $J_{m+1} = \{1, 2, \ldots, 2m+1\} \setminus \{i_1, \ldots, i_m\}$. In particular, the Szegő kernel with zero characteristics reads

$$S[0](P, Q) = \frac{1}{N} \sqrt{\frac{dPdz(Q)}{z(P) - z(Q)}} \sum_{s=0}^{N-1} \left( \frac{q(z(P)) p(z(Q))}{p(z(P)) q(z(Q))} \right)^{-\frac{1}{N} + \frac{N-1}{2N}},$$

where the polynomials $p(\lambda)$ and $q(\lambda)$ have been defined in (4.3). 

Proof. The Szegő kernel $S[\tilde{D}_m](P, Q)$ is the unique, up to a constant, $(\frac{1}{2}, \frac{1}{2})$-form on $C_{N,m} \times C_{N,m}$ that has a simple pole along the diagonal $P = Q$ and divisor $K_{C} - \tilde{D}_m$ in the variable $P$ and $\tilde{D}_m$ in the variable $Q$ (see e.g. Narasimhan [88]). Here $K_C$ is the canonical divisor and $\tilde{D}_m$ has been defined in (4.3). Therefore we just need to verify that the right hand sides of the expressions (4.35) and (3.18) have the same divisor. It is enough to show this by setting $Q = P_{j_1}$. Regarding the formula (3.18) we have

$$\text{Div} \left( \frac{\theta[\tilde{D}_m]\left(P^{\int \frac{1}{P_{j_1}} \ d\nu}\right)}{E(P, P_{j_1})} \right) = (N-1) \sum_{k=2}^{m+1} P_{j_k} - P_{j_1} = K_C - \tilde{D}_m.$$

Next putting $Q = P_{j_1}$ into the expression (4.35) we obtain

$$\text{div} \left( \frac{1}{N} \sqrt{\frac{dPdz(Q)}{z(P) - z(Q)}} \sum_{s=0}^{N-1} \left( \prod_{k=1}^{m+1} \psi_{i_k}(P, Q) \prod_{k=1}^{m+1} \psi_{j_k}(P, Q) \right)^{-\frac{1}{N} + \frac{N-1}{2N}} \right)_{Q=P_{j_1}} = \frac{1}{\sqrt{N}} \left( \prod_{k=1}^{m+1} (\lambda_{j_1} - \lambda_{i_k}) \right)^{\frac{N-1}{2N}} \times$$

$$\times \text{div} \left( \sqrt{\frac{dP}{z(P) - \lambda_{j_1}}} \prod_{k=1}^{m+1} \frac{(z(P) - \lambda_{j_k})}{(z(P) - \lambda_{i_k})} \right) = (N-1) \sum_{k=2}^{m+1} P_{j_k} - P_{j_1} = K_C - \tilde{D}_m,$$

which shows that the two expressions (4.35) and (3.18) have the same divisor class in $P$. In the same way one can check the divisor class in $Q$. Therefore the expressions (4.3) and (4.35) of the Szegő kernel differ at most from a multiplicative constant. This constant is equal to one because when $P \to Q$ the expression (4.35) has the following expansion

$$S[\tilde{D}_m](P, Q) = \sqrt{\frac{dPdz(Q)}{z(P) - z(Q)}} [1 + O((z(P) - z(Q))],$$

which coincides with the leading coefficient of the expansion (3.19). \[ \square \]

Example 4.8. In particular for $N = 3$, the above families of Szegő kernels read

$$S\{2P_{i_1} + \ldots + 2P_{i_m}\}(P, Q) = \frac{1}{3} \left( \sqrt[3]{\psi_{i_1} \cdots \psi_{i_m}} + 1 + \sqrt[3]{\psi_{j_1} \cdots \psi_{j_{m+1}}} \right) \sqrt{\frac{dPdz(Q)}{z(P) - z(Q)}},$$

$$S\{3P_{i_1} + \ldots + 3P_{i_m}\}(P, Q) = \frac{1}{3} \left( \sqrt[3]{\psi_{i_1} \cdots \psi_{i_m}} + 1 + \sqrt[3]{\psi_{j_1} \cdots \psi_{j_{m+1}}} \right) \sqrt{\frac{dPdz(Q)}{z(P) - z(Q)}},$$

$$S\{P_{i_1} + \ldots + P_{i_m}\}(P, Q) = \sqrt{\frac{dPdz(Q)}{z(P) - z(Q)}} [1 + O((z(P) - z(Q))].$$
for \( N = 4 \),
\[
S\{3P_1 + \ldots + 3P_m\}(P, Q) = \frac{1}{4} \sqrt{dz(P)dz(Q)} \left( \sqrt{\psi_{i_1}^3 \psi_{j_1}^3 \psi_{j_{m+1}}^3} + \sqrt{\psi_{i_1} \psi_{i_m} \psi_{j_{m+1}}} + \sqrt{\psi_{j_1} \psi_{j_m} \psi_{i_m}} + \sqrt{\psi_{j_1} \psi_{j_{m+1}}^3 \psi_{i_m}} \right).
\]

The following corollary can be checked in a straightforward manner.

**Corollary 4.9.** The expansion of the Szegö kernel with zero characteristics as \( P \to Q \) reads
\[
S[0](P, Q) = \frac{\sqrt{dz(P)dz(Q)}}{z(P) - z(Q)} \times \left\{ 1 + \left( \frac{1}{6} \{z(P), P\} + \frac{N^2 - 1}{24N^2} \left[ \frac{d}{dz} \log \frac{p(z(P))}{q(z(P))} \right]^2 \right) (z(P) - z(Q))^2 + \ldots \right\},
\]
where \( \{z(P), P\} \) is the Schwarzian derivative (1.19).

5. **Solution of the Riemann-Hilbert Problem for the \( Z_N \)-curve**

Now we are ready to solve the canonical R-H problem (1.10), that is we determine a \( N \times N \) matrix valued function \( X(\lambda) \) that satisfies
\[
\begin{align*}
X_-(\lambda) &= X_+(\lambda)P_N, \quad \lambda \in \bigcup_{k=0}^{m}(\lambda_{2k+1}, \lambda_{2k+2}), \\
X(\lambda_0) &= 1_N, \quad \lambda_0 \in C_+,
\end{align*}
\]
where \( P_N \) has been defined in (1.9). The quasi-permutation monodromy matrix \( P_N \) can be diagonalised to the form
\[
P_N = UE^{2\pi i \sigma_N}U^{-1},
\]
where the diagonal matrix \( \sigma_N \) is defined in (2.10) and the matrix \( U \) can be chosen with entries \( U_{1k} = 1 \), \( k = 1, \ldots, N \), \( \det(U) \neq 0 \). In this way the canonical R-H problem (5.1) is reduced to the form
\[
(U^{-1}XU)_- = (U^{-1}XU)_+e^{2\pi i \sigma_N} \quad \lambda \in \bigcup_{k=0}^{m}(\lambda_{2k+1}, \lambda_{2k+2}),
\]
\[
U^{-1}X(\lambda_0)U = 1_N, \quad \lambda_0 \in C_+.
\]

It is easy to verify that the diagonal matrix
\[
U^{-1}X(\lambda)U = \left( \frac{p(\lambda)}{q(\lambda)} \right)_{\lambda^{-2N}}^{\lambda^{-2N}},
\]
where the polynomials \( p(\lambda) \) and \( q(\lambda) \) has been defined in (1.6), solves the above R-H problem. Indeed choosing the function
\[
\left( \frac{p(\lambda)}{q(\lambda)} \right)_{\lambda^{-2N}}^{\lambda^{-2N}} \to \lambda^{-\frac{N-1}{2N}}, \quad \lambda \to \pm \infty, \quad \arg \lambda = \frac{\pi}{2},
\]
it follows that
\[
\left( \frac{p(\lambda)}{q(\lambda)} \right)_{(-k \pm \frac{N-1}{2N})} = e^{2\pi i (-k \pm \frac{N-1}{2N})} \left( \frac{p(\lambda)}{q(\lambda)} \right)_{(-k \pm \frac{N-1}{2N})}, \quad k = 0, \ldots, N - 1.
\]
Furthermore
\[
\det(U^{-1}X(\lambda)U) = \det \left( \left( \frac{p(\lambda)}{q(\lambda)} \right)_{\lambda^{-2N}}^{\lambda^{-2N}} \right) = 1, \quad \lambda \in \mathbb{C} \cup \infty
\]
and clearly \(U^{-1}X(\lambda_0)U = 1_N\). Therefore the matrix function
\[
X(\lambda) = U \left( \frac{p(\lambda)}{q(\lambda)} \right)_{\lambda_0}^{\sigma_N} U^{-1}
\]
solves the canonical R-H problem \(5.1\). The entries of the matrix \(X(\lambda)\) can be expressed in terms of the Szegö kernel with zero characteristics, \(S[0](P, Q)\), defined on \(C_{N,m}\) and derived in \(4.30\). Indeed it turns out that the entries \(X_{rs}(\lambda)\), \(r, s = 1, \ldots, N\), of \(X(\lambda)\) are also equal to
\[
X_{rs}(\lambda) = S[0](P^{(s)}, P^{(r)}_0) \frac{z(P) - z(Q)}{\sqrt{dz(P)dz(P_0)}} = \frac{1}{N} \sum_{k=0}^{N-1} \left( \frac{p(\lambda)}{q(\lambda)} \right)^{\frac{k+\frac{N-1}{2}}{N}}, \quad \lambda_0 \notin D,
\]
where \(P^{(s)} = (\lambda, \rho^{s-1}y)\) and \(P^{(r)}_0 = (\lambda_0, \rho^{-r}y_0)\), \(r, s = 1, \ldots, N\), denote the points on the \(s\)-th and \(r\)-th sheet of \(C_{N,m}\) respectively. When \(N = 2\) and \(\sqrt{\frac{q(\lambda_0)}{p(\lambda_0)}} = 1\), such formula coincides with the canonical solution obtained in \(2\).

We also observe that the matrix \(X(\lambda)\) satisfies the differential equation
\[
\frac{dX(\lambda)}{d\lambda} = \sum_{k=1}^{2m+1} A_k \frac{X(\lambda)}{\lambda - \lambda_k}
\]
where the matrices \(A_k\) are given by
\[
A_k = (-1)^{k-1}U\sigma_NU^{-1}, \quad k = 1, \ldots, 2m + 1,
\]
with \(\sigma_N\) defined \(2.10\). Therefore the canonical R-H problem gives a constant solution of the Schlesinger system \(2.15\).

We are now ready to derive the solution of the R-H problem \(2.4, 2.5\) for arbitrary non-zero values of the constants \(c_k\) and \(d_k\), \(k = 1, \ldots, (N-1)m\).

**Theorem 5.1** (Main Theorem). Let the characteristics \(\epsilon, \delta \in C^{(N-1)m}\) be
\[
\epsilon_{k+sm} = \frac{1}{2\pi i} \log \frac{c_{k+sm}}{c_{k+1+sm}}, \quad s = 0, \ldots, N-2, \quad k = 1, \ldots, m-1, \quad (5.3)
\]
\[
\epsilon_{sm} = \frac{1}{2\pi i} \log c_{sm}, \quad s = 1, \ldots, N-1, \quad (5.4)
\]
\[
\delta_k = \frac{1}{2\pi i} \log d_k \quad k = 1, \ldots, (N-1)m.
\]
Suppose that \(\theta[\delta]\) \(0: \Pi\) \(\neq 0\). Then the matrix valued function \(Y(\lambda) = (Y_{rs}(\lambda))_{r,s = 1, \ldots, N}\)
\[
(5.5)
\]
solves the R-H problem \(2.4, 2.5\) and \(\det Y(\lambda) \neq 0\) for \(\lambda \neq \lambda_k, \ k = 1, \ldots, 2m + 2\).

**Proof.** First of all we show that matrix \(5.5\) is holomorphic outside the singular set \(\lambda \neq \lambda_1, \ldots, \lambda_{2m+1}, \infty\). Indeed combining \(4.18\) and \(4.30\), the entries of the matrix \(5.5\) can be written in the form given in
\begin{equation}
Y_r(\lambda) = S \left[ \Phi \right] (P_0^{(r)}, P^{(s)}) \frac{z(P^{(s)}) - z(P_0^{(r)})}{\sqrt{dz(P^{(s)})dz(P_0^{(r)})}}, \quad r, s = 1, \ldots, N.
\end{equation}

From the properties of the Szegö kernel, the matrix \( P \) is clearly holomorphic for \( \lambda \neq \lambda_1, \ldots, \lambda_{2m+1}, \infty \). Furthermore, using the formula \( 5.6 \) for the entries of \( Y(\lambda) \) and applying the relation \( 5.5 \) we conclude that

\begin{equation}
\det Y(\lambda) = \left( \frac{z(P^{(1)}) - z(P_0^{(1)})}{\sqrt{dz(P^{(1)})dz(P_0^{(1)})}} \right)^N \prod_{1 \leq r < s \leq N} E(P_0^{(r)}, P_0^{(s)}), \quad \prod_{r,s=1}^N E(P_0^{(r)}, P^{(s)}) \neq 0,
\end{equation}

for \( P \neq (\lambda_k, 0) \) or \( (\infty, \infty) \). In the above formula we have used the relation \( z(P^{(r)}) = z(P^{(1)}) \) and \( z(P_0^{(r)}) = z(P_0^{(1)}) \), \( r = 1, \ldots, N \). Evidently we have that

\[ \det Y(\lambda_0) = 1_N. \]

In order to prove that \( 5.5 \) does indeed satisfy the R-H problem \( 2.4, 2.5 \) the following considerations are needed. The action of the automorphism \( J \) on \( dv_j \) is given by the relation

\begin{equation}
J(dv_j(\lambda, y)) = \sum_{k=1}^m \gamma_{j,k} \lambda^{m-k} \frac{(q(\lambda)^{r-1}}{\rho^j y^r} d\lambda,
\end{equation}

where \( \gamma_{j,k} \) are the normalisation constants of the holomorphic differentials and \( \rho \) is the \( N \)th root of unity. Let us consider the Abelian integral

\begin{equation}
v(P) = \int_P dv.
\end{equation}

The action of the automorphism \( J \) on \( v(P) \) is naturally given by

\begin{equation}
J(v(P)) = \int_J(v(P)) dv = \int_J(v) dv.
\end{equation}

When \( P^{(s)} = (\lambda, \rho^{s-1} y) \) is on the \( s \)-th sheet, we denote by \( J^{(s-1)}(v(\lambda)) \) the natural restriction of the integral \( v(P^{(s)}) \) on \( C_+ \cup C_- \):

\[ J^{(s-1)}(v(\lambda)) := \int_{(\lambda, y)} J^{(s-1)}(dv) = \sum_{k=1}^m \gamma_{j,k} \int_{(\lambda, y)} \xi^{m-k} \frac{(q(\xi)^{r-1}}{\rho^j y^r} d\lambda, \quad (\xi, \eta) \in \mathbb{C}_{N,m} \]

The integral itself is taken on the first sheet of \( \mathbb{C}_{N,m} \) and the integration path lies in \( C_+ \) and \( C_- \) for \( \lambda \in C_+ \) or \( \lambda \in C_- \) respectively. The integral in \( 5.5 \) is defined as

\[ \int_{P_0^{(r)}} dv := J^{(s-1)}(v(\lambda)) - J^{(r-1)}(v(\lambda_0)). \]

If \( \lambda \in \mathcal{L} \) the integrals \( v_\pm(\lambda) \), are shown on Figure \( 8 \), namely the integration path of \( v_\pm(\lambda) \), lies in \( C_\pm \) respectively. From the properties of the homology basis \( 4.1, 4.2 \) the following relations can be easily derived:

\begin{equation}
\left[ J^{(s-1)}v_-^{(s)}(\lambda) - J^{(s)}v_+(\lambda) \right]_{[\lambda_{2k-1}, \lambda_{2k}]} = \sum_{j=k}^m \left( \int_{\beta_{j-(s-1)m}}^\phi dv - \int_{\beta_{j-(s-1)m}}^\phi dv \right)
\end{equation}
for \( s = 1, \ldots, N - 2 \) and \( k = 1, \ldots, m, \)

\[
\left[ J^{(N-2)}v_-(\lambda) - J^{(N-1)}v_+(\lambda) \right]_{[\lambda_{2k-1}, \lambda_{2k}]} = \sum_{j=k}^{m-1} \oint_{c_{\beta_j}} dv, 
\]

\[
\left[ J^{(N-1)}v_-(\lambda) - v_+(\lambda) \right]_{[\lambda_{2k-1}, \lambda_{2k}]} = -\sum_{j=k}^{m-1} \oint_{c_{\beta_j}} dv 
\]

for \( k = 1, \ldots, m \) and

\[
\left[ J^{(s-1)}v_-(\lambda) - J^sv_+(\lambda) \right]_{[\lambda_{2m+1}, \infty]} = 0, \quad s = 1, \ldots, N. 
\]

In the same way we obtain

\[
\left[ J^{s}v_-(\lambda) - J^sv_+(\lambda) \right]_{[\lambda_{2k-1}, \lambda_{2k+1}]} = \oint_{c_{\alpha_k+sm}} dv, \quad s = 0, \ldots, N - 2, 
\]

\[
\left[ J^{(N-1)}v_-(\lambda) - J^{(N-1)}v_+(\lambda) \right]_{[\lambda_{2k-1}, \lambda_{2k+1}]} = -\sum_{s=0}^{N-2} \oint_{c_{\alpha_k+sm}} dv, 
\]

for \( k = 1, \ldots, m \) and

\[
\left[ J^{s}v_-(\lambda) - J^sv_+(\lambda) \right]_{[\infty, \lambda_1]} = 0, \quad s = 0, \ldots, N - 1. 
\]

Now let us suppose that \( \lambda \in [\lambda_{2k-1}, \lambda_{2k}] \). Then for \( s = 1, \ldots, N - 2 \) and \( r = 1, \ldots, N \) we have

\[
(Y_-)_{rs} = (X_-)_{rs} \left( \frac{\theta (\delta^s)}{\theta (\delta^s)} \frac{(J^{(s-1)}v_-(\lambda) - J^{(r-1)}v(\lambda_0); \Pi)}{\theta (\delta^s)} \frac{\theta (0; \Pi)}{\theta (\delta^s)} \right) 
\]

\[
= (X_+ (\lambda)^N)_{rs} \left( \frac{\theta (\delta^s)}{\theta (\delta^s)} \frac{\int_{P_{\delta}}^p dv_+ + \int_{P_{\delta}}^p dv_- + \int_{P_{\delta}}^p dv_+; \Pi}{\theta (\delta^s)} \right) \frac{\theta (0; \Pi)}{\theta (\delta^s)} 
\]

\[
\theta (\delta^s) \frac{\int_{P_{\delta}}^p dv_+ + \int_{P_{\delta}}^p dv_- + \int_{P_{\delta}}^p dv_+; \Pi}{\theta (\delta^s)} \frac{\theta (0; \Pi)}{\theta (\delta^s)} 
\]
where the quantities \((Y_-(\lambda))_{r,s}\), \((X_-(\lambda))_{r,s}\), and \((X_-(\lambda))_{r,s}\) denote the \(r,s\) entry of the matrix \(Y_-(\lambda), X_-(\lambda), \) and \(X_-(\lambda))^\circ\) respectively, and in the last identity we have used the relation \((5.11)-(5.13)\) and the periodicity property \((3.6)\) of the \(\theta\)-function. From \((5.18)\) it is immediate to verify that the constants \(\epsilon_1, \ldots, \epsilon_{(N-1)m}\) and \(c_1, \ldots, c_{(N-1)m}\) are related by \((5.3)\) if and only if the matrix \(Y(\lambda)\) satisfies
\[
Y_-(\lambda) = Y_+(\lambda)G_{2k-1}, \quad \lambda \in [\lambda_{2k-1}, \lambda_{2k}], \quad k = 1, \ldots, m + 1, \quad \lambda_{2m+2} = \infty,
\]
where the matrix \(G_{2k-1}\) has been defined in \((1.2)\). Repeating the same procedure for \(\lambda \in [\lambda_{2k}, \lambda_{2k+1}]\) and using \((5.15)-(5.17)\) and the periodicity properties \((3.5)\) of the \(\theta\)-function, we derive \((5.4)\) if and only if the matrix valued function \(Y(\lambda)\) satisfies
\[
Y_-(\lambda) = Y_+(\lambda)G_{2k}, \quad \lambda \in [\lambda_{2k}, \lambda_{2k+1}], \quad k = 0, \ldots, m, \quad \lambda_0 = \infty,
\]
where \(G_{2k}\) has been defined in \((2.2)\). We conclude that the matrix \((5.5)\) satisfies the R-H problem \((2.4)-(2.5)\).

The form of the solution \((5.5)\) and the \(Z_N\) symmetry of the curve \(\mathcal{C}_N,m\) enable us to prove the following.

**Proposition 5.2.** Let \(\delta_N, \epsilon_N \in (\mathbb{Z/NZ})^{(N-1)m}\) be the characteristics associated to the non-singular divisor \(\mathcal{D}_1\) supported on the branch points, that is
\[
(5.19) \quad \epsilon_N + \delta_N \Pi = \sum_{i} s_i \int_{P_i} dv - K_\infty, \quad \sum_{i} s_i = (N - 1)m,
\]
where \(i \in \{1, 2, \ldots, 2m + 1\}\) and \(K_\infty\) is the vector of Riemann constants \((4.22)\). Then the matrix \(Y(\lambda)\) with entries
\[
Y_{rs}(\lambda) = X_{rs}(\lambda) \frac{\theta [\delta_N \epsilon_N] (\int_{P_0} dv; \Pi)}{\theta (\int_{P_0} dv; \Pi)} \frac{\theta (0; \Pi)}{\theta [\epsilon_N \delta_N] (0; \Pi)}, \quad r, s = 1, \ldots, N,
\]
solves the R-H problem \((2.4)-(2.5)\) with reducible monodromy representation \((2.3)\).

**Proof.** Using the notations of Section 4.2 we write \((5.19)\) in the form
\[
(5.20) \quad \epsilon_N + \delta_N \Pi = \sum_{i} s_i [U_i] - (N - 1) \sum_{l=1}^{m} [U_l],
\]
where \([U_k]\) are the characteristics defined in Sect. 4.2. Because of the relations \((5.11)-(5.13)\), the operation of summing two characteristics is equivalent to the operation of multiplying two different sets of constants \(c_j^{(k)}, d_j^{(k)}, k = 1, 2, j = 1, \ldots, (N - 1)m\). For this reason, we associate to the characteristics \([U_k]\), the constants \(c_j^{(k)}, d_j^{(k)}, j = 1, \ldots, (N - 1)m\) according to the rule \((5.3)-(5.4)\), that is
\[
[U_i] \leftrightarrow c_j^{(1)} = 1, \quad d_j^{(1)} = e^{\frac{2i\pi j}{N}}, \quad j = 1, \ldots, (N - 1)m;
\]
for \( k = 2, \ldots, m \)
\[
[U_{2k-1}] \leftrightarrow c^{(2k-1)}_{k-1+m} = e^{\frac{2\pi i (s+1)}{N}}, \quad c^{(2k-1)}_{j+m} = 1, \quad j \neq k-1, \quad j = 1, \ldots, m,
\]
\[
d^{(2k-1)}_{j+m} = e^{-\frac{2\pi i}{N}}, \quad k \leq j \leq m, \quad d^{(2k-1)}_{j+m} = 1, \quad 1 \leq j < k, \quad s = 0, \ldots, N-2;
\]
for \( k = 1, \ldots, m \)
\[
[U_{2k}] \leftrightarrow c^{(2k)}_{k+m} = e^{\frac{2\pi i (s+1)}{N}}, \quad c^{(2k)}_{j+m} = 1, \quad j \neq k, \quad j = 1, \ldots, m,
\]
\[
d^{(2k)}_{j+m} = e^{-\frac{2\pi i}{N}}, \quad k \leq j \leq m, \quad d^{(2k)}_{j+m} = 1, \quad 1 \leq j < k, \quad s = 0, \ldots, N-2.
\]
Combining the above relations it is possible to verify that the monodromy representation associated to the non-singular characteristics is
\[
\begin{align*}
M_k &= \prod_{t \in \mathbb{Z}/N\mathbb{Z}} \xi_{\lambda_t}^{s_{\lambda_t}} P^{(-1)^{k-1}}, \quad k = 1, \ldots, 2m + 1, \quad M_\infty = \mathcal{P}_N^{-1}, \quad 2m+1 \mathcal{P}_N^{-1} = 1, \\
\prod_{k \in \mathbb{N}_{\text{even}}} \xi_{\lambda_k}^{s_{\lambda_k}} (N-1) = 1,
\end{align*}
\]
\[
\xi_{\lambda_k}, \xi_{\lambda_k} \in \{1, e^{\frac{2\pi i}{N}}, \ldots, e^{\frac{2\pi i (N-1)}{N}}\}, \quad \sum_{\lambda_k} s_{\lambda_k} = (N-1)m.
\]
According to remark \[2.4\] the above monodromy representation is reducible. \(\square\)

In the following we consider a couple of divisors whose difference is a non-singular divisor supported on the branch points. If \(\delta, e_N\) is a non-singular characteristics corresponding to a divisor in general position and if \(e_N, \delta_N \in \mathbb{Z}/N\mathbb{Z}\) is a characteristics corresponding to the non-special divisors \(\mathcal{D}_l\) defined in \[2.26\] then \(\delta + e_N\) is a non-singular characteristics. Indeed the characteristics \(\delta + e_N\) correspond to a divisor of degree \(2g\). But all divisors \(\mathcal{D}\) of degree \(\deg \mathcal{D} > 2g - 2\) are non-special \[3.6\].

**Theorem 5.3.** Let \(\delta, e_N\) be as above. Then the entries \(Y_{rs}(\lambda)\) and \(\overline{Y}_{rs}(\lambda)\) of the solutions \(Y(\lambda)\) and \(\overline{Y}(\lambda)\) of the R-H problem \[2.3\] with characteristics \(\delta, e_N\), are equivalent up to an algebraic transformation. The monodromy representation \(\mathcal{M} = \{M_1, M_2, \ldots, M_{2m+1}, M_\infty\}\) and \(\overline{\mathcal{M}} = \{\overline{M}_1, \overline{M}_2, \ldots, \overline{M}_{2m+1}, \overline{M}_\infty\}\) associated to the solutions \(Y(\lambda)\) and \(\overline{Y}(\lambda)\) respectively, are equivalent up to multiplication by \(N\)th roots of unity. That is \(\overline{M}_k = e^{\frac{2\pi i nk}{N}} M_k, jk \text{ integer, } \sum_{k=1}^{2m+1} jk = 0 \text{ mod } N\).

**Proof.** If \(\delta, e_N \in \mathbb{Z}/N\mathbb{Z}\) then by \[3.5, 3.6\], the ratio
\[
\mathcal{F}(P(s), P_0^{(r)}) := \left( \frac{\theta^{\delta + e_N}}{\theta^{e_N}} \left( \int_{P_0^{(r)}} P(s) \theta^{e_N} \right) \right)^N \left( \frac{\theta^{\delta}}{\theta} \left( \int_{P_0^{(r)}} P(s) \right) \right)
\]
is a single-valued function on \(\mathbb{C}_{N,m}\) in both the arguments \(P(s)\) and \(P_0^{(r)}\). Hence \(\mathcal{F}(P(s), P_0^{(r)})\) is a meromorphic function. This means that \(\mathcal{F}(P(s), P_0^{(r)})\) is a rational expression in \(\lambda, y, \lambda_0, y_0\), therefore \(\mathcal{F}(P(s), P_0^{(r)})\) is algebraic. Hence, from \[1.5, 1.22\]
\[
\overline{Y}_{rs}(\lambda) = \sqrt[N]{\mathcal{F}(P(s), P_0^{(r)} Y_{rs}(\lambda)), \quad r, s = 1, \ldots, N,
\]
which is the first statement of the theorem. The equivalence of the corresponding monodromy representation \(\mathcal{M}\) and \(\overline{\mathcal{M}}\) up to multiplication by \(N\)th roots of unity, follows from the proof of Proposition \[5.2\] \(\square\).
Example 5.4. We consider the case \( N = 3 \) and \( m = 1 \) when the Riemann surface \( \mathcal{C}_{3,1} \ni \{(\lambda, y), \quad y^3 = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)\} \) is of genus 2. Let \( \epsilon \) and \( \delta \) be a non-singular characteristics. We consider the non-singular characteristics \( \epsilon_3, \delta_3 \) supported on the branch points given by

\[
\epsilon_3 + \delta_3 \Pi = 2[U_3] - 2[U_2] = \left(-\frac{4}{3}, \frac{2}{3}\right) \Pi,
\]

as follows from the relations derived in Section \( \text{(4.2)} \). If \( \{M_1, M_2, M_3, M_\infty\} \) are the monodromy matrices associated to the characteristics \( \left[\delta \right], \) the monodromy matrices associated to the characteristics \( \left[\epsilon + \delta \right] \) are

\[
\{M_1, e^{\frac{4\pi}{3}} M_2, e^{\frac{2\pi}{3}} M_3, M_\infty\}.
\]

5.1. Solution of the Schlesinger equations. From the solution of the R-H problem \( \text{(5.8)} \) one can derive a particular solution of the Schlesinger system. In this sub-section we denote by \( Y(\lambda, \lambda_0) \) the solution \( \text{(5.5)} \) of the R-H problem \( \text{(2.3)-(2.5)} \) with base point \( \lambda_0 \), that is \( Y(\lambda_0, \lambda_0) = 1_N \). According to the results in \( \cite{B3} \), the solution of the Schlesinger system can be derived from the relation \( \text{(2.14)} \) that can be re-written in the form

\[
(\lambda - \lambda_k) \frac{\partial}{\partial \lambda_k} Y(\lambda, \lambda_0) Y^{-1}(\lambda, \lambda_0) = (\lambda - \lambda_k) \left( \frac{A_k}{\lambda_0 - \lambda_k} - \frac{A_k}{\lambda - \lambda_k} \right).
\]

Taking the derivative with respect to \( \lambda \) of both sides of the above relation and setting \( \lambda = \lambda_0 \) it follows that

\[
A_k = (\lambda_0 - \lambda_k)^2 \frac{\partial^2}{\partial \lambda \partial \lambda_k} Y(\lambda) \bigg|_{\lambda = \lambda_0}, \quad k = 1, \ldots, 2m + 1,
\]

or, equivalently, \( \cite{B1} \)

\[
A_k = (\lambda_0 - \lambda_k)^2 \frac{\partial^2}{\partial \lambda \partial \lambda_k} Y(\lambda) \bigg|_{\lambda = \lambda_0}, \quad k = 1, \ldots, 2m + 1.
\]

Using the formula \( \text{(5.5)} \) for the solution \( Y(\lambda, \lambda_0) \) of the R-H problem and the expansion \( \text{(3.20)} \) of the Szegö kernel, the above relation can be written in the form

\[
(5.23) \quad (A_k)_{ss} = (\lambda_0 - \lambda_k)^2 \frac{\partial}{\partial \lambda_k} \left( \sum_{l=1}^{(N-1)m} \frac{\partial}{\partial z_l} \log \theta \left[\frac{\delta}{\epsilon}\right] (0; \Pi) \left| \frac{d\nu(P)}{dz(P)} \right|_{P = P_0^{(s)}} \right), \quad s = 1, \ldots, N,
\]

\[
(5.24) \quad (A_k)_{rs} = (\lambda_0 - \lambda_k)^2 \sum_{l=0}^{N-1} \left( \frac{l}{N^2} \exp \left[ 2\pi i \frac{r - s}{2N} (2l - N + 1) \right] \right) \times
\]

\[
\times \frac{\partial}{\partial \lambda_k} \left[ \theta \left[\frac{\delta}{\epsilon}\right] (P_0^{(s)}) \left( \frac{P_0^{(s)}}{P_0^{(r)}} \right) \right] \left( \theta \left[\frac{\delta}{\epsilon}\right] (0; \Pi) \right), \quad r \neq s, \quad r, s = 1, \ldots, N.
\]

The matrix \( A_\infty \) is obtained from the relation

\[
A_\infty = -\sum_{k=1}^{2m+1} A_k.
\]

From the relation \( \text{(5.23)} \) and \( \text{(1.17)} \) we can immediately verify that

\[
\text{Trace}(A_k) = 0, \quad k = 1, \ldots, 2m + 1.
\]
Because of (2.13), the eigenvalues of the matrices \( A_k, k = 1, \ldots, 2m + 1 \), are
\[
\text{Eigenvalues}(A_k) = \left( \frac{-N + 1}{2N}, \frac{-N + 3}{2N}, \ldots, \frac{N - 3}{2N}, \frac{N - 1}{2N} \right).
\]
The substitution (5.23) and (5.24) into the above equation leads to non-trivial \( \tau \)-function equivalence.

5.2. \( \tau \)-function of the Schlesinger equations. From the solution of the R-H problem one can derive the \( \tau \) function for the Schlesinger system defined by (2.16).

**Theorem 5.5.** The \( \tau \)-function for the Schlesinger system reads
\[
\tau(\lambda_1, \lambda_2, \ldots, \lambda_{2m+1}) = \frac{\theta[\delta]}{\theta(0; \Pi)} \prod_{\substack{k < i \leq m+1 \\text{i.e.}}}(\lambda_{2k+1} - \lambda_{2i+1})^{\frac{N^2 - i}{2N}} \prod_{\substack{k < i \leq m+1}}(\lambda_{2k} - \lambda_{2i})^{\frac{N^2 - i}{2N}}.
\]

**Proof.** We define the \( \tau \)-function by the formula (2.10), where \( Y(\lambda) \) is the solution (5.3) of the R-H problem (2.4)–(2.5). It follows from the definition that the \( \tau \)-function does not depend on the normalisation point \( \lambda_0 \). In order to obtain the explicit expression of the residue in the r.h.s. of (2.16), we use the relation obtained in [3], namely
\[
\frac{\partial}{\partial \lambda_k} \log \tau = \frac{1}{2} \mathfrak{Res}_{\lambda = \lambda_k} \text{Tr} \left( \frac{dY(\lambda)}{d\lambda} Y(\lambda)^{-1} \right)^2
\]
(5.26)
\[
= \frac{\partial}{\partial \lambda_k} \log \theta[\delta](0; \Pi) - \mathfrak{Res}_{p = (\lambda_k, 0)} \left\{ \sum_{r<s} \frac{d\omega(p^{(r)}, p^{(s)})}{(dz(P)^2)} \right\},
\]
where \( \omega(P, Q) \) is the Bergmann kernel and \( P^{(s)} \) is on the \( s \)-th sheet of \( \mathcal{C}_{N,m} \), \( s = 1, \ldots, N \). Since
\[
\mathfrak{Res}_{p = (\lambda_k, 0)} \left\{ \sum_{r<s} \frac{d\omega(p^{(r)}, p^{(s)})}{(dz(P)^2)} \right\} = -\frac{1}{2} \mathfrak{Res}_{p = (\lambda_k, 0)} \left\{ \sum_{s=1}^{N} \frac{d\omega(p^{(s)}, p^{(s)})}{(dz(P)^2)} \right\},
\]
we can write (5.26) in the form
\[
\frac{1}{2} \mathfrak{Res}_{\lambda = \lambda_k} \left\{ \text{Tr} \left( \frac{dY(\lambda)}{d\lambda} Y(\lambda)^{-1} \right)^2 \right\} = \frac{\partial}{\partial \lambda_k} \log \theta[\delta](0; \Pi) + \frac{1}{2} \mathfrak{Res}_{\lambda = \lambda_k} \left\{ \sum_{s=1}^{N} \frac{d\omega(p^{(s)}, p^{(s)})}{(dz(P)^2)} \right\}.
\]
From the identity (3.26) and the expansion (3.31) we express (5.27) in the form
\[
\frac{1}{2} \mathfrak{Res}_{\lambda = \lambda_k} \left\{ \text{Tr} \left( \frac{dY(\lambda)}{d\lambda} Y(\lambda)^{-1} \right)^2 \right\} = \frac{\partial}{\partial \lambda_k} \log \theta[\delta](0; \Pi) + \frac{N^2 - 1}{24N} \mathfrak{Res}_{\lambda = \lambda_k} \left\{ \frac{d}{d\lambda} \log p(\lambda) \right\}^2 - \frac{1}{2} \sum_{i,j=1}^{g} \frac{\partial^2}{\partial z_i \partial z_j} \log \theta(0; \Pi) \left\{ \mathfrak{Res}_{p = (\lambda_k, 0)} \sum_{s=1}^{N} \left[ \frac{d\nu_i(P^{(s)})}{d\nu_j(P^{(s)})} \right] \right\},
\]
Using the relation \( \theta \), the heat equation \( \theta \) and the property \( \theta \), the above formula can be reduced to the form

\[
\frac{1}{2} \text{Res}_{\lambda = \lambda_k} \left\{ \text{Tr} \left( \frac{dY(\lambda)}{d\lambda} Y(\lambda)^{-1} \right)^2 \right\} = \frac{\partial}{\partial \lambda_k} \log \theta \left[ \frac{\delta}{\epsilon} \right] (0; \Pi) + \frac{N^2 - 1}{24N} \text{Res}_{\lambda = \lambda_k} \left\{ \left[ \frac{d}{d\lambda} \log \frac{p(\lambda)}{q(\lambda)} \right] \right\}^2 - \frac{\partial}{\partial \lambda_k} \log \theta(0; \Pi).
\]

From the above formula, we can easily obtain the \( \tau \) function defined in (5.25).

We remark that the formula for the \( \tau \)-function obtained in [3] for the case of a general \( N \)-sheeted Riemann surface reads \( \tau(\lambda_1, \ldots, \lambda_{2m+1}) = F(\lambda_1, \ldots, \lambda_{2m+1}) \theta \left[ \frac{\delta}{\epsilon} \right] (0; \Pi) \), where the function \( F \) depends only on the Bergmann projective connection of the Riemann surface. In the formula (5.25) the term derived from the projective connection is explicitly evaluated.

The set of zeros of the \( \tau \)-function in the space of singularities of the R-H problem, that is the set

\[
\{(\lambda_1, \ldots, \lambda_{2m+1}), \quad \lambda_i \neq \lambda_j \neq \infty, \quad i, j = 1, \ldots, 2m + 1, \quad \tau(\lambda_1, \ldots, \lambda_{2m+1}) = 0\}
\]

is called the Malgrange divisor \( (\theta) \). From the expression (5.25) it follows that the \( \tau \)-function vanishes when

\[
\theta \left[ \frac{\delta}{\epsilon} \right] (0; \Pi) = 0,
\]

that is when \( \Pi \delta + \epsilon \notin (\Theta) \). Therefore the set of singularities \( (\lambda_1, \ldots, \lambda_{2m+1}) \) belongs to the Malgrange divisor \( (\theta) \) if the vector \( \epsilon + \delta \Pi \) belongs to the \( (\Theta) \)-divisor.

The expression for the \( \tau \) function can be written in a different form substituting the Thomae formula for the \( \theta \)-constant.

**Theorem 5.6.** The Thomae-type formula for \( \theta(0; \Pi) \) reads

\[
\theta^\Theta(0; \Pi) = \prod_{i=1}^{N-1} \frac{\det A_i^4}{(2\pi i)^4 (N-1)!} \prod_{i < j} (\lambda_{2i} - \lambda_{2j})^{2(N-1)} \prod_{k < \ell} (\lambda_{2k+1} - \lambda_{2\ell+1})^{2(N-1)},
\]

where the matrices \( A_s \), \( s = 1, \ldots, N - 1 \), are defined in (4.7).

The proof of the theorem is shown in the Appendix.

**Remark 5.7.** We remark that the Thomae formulae for \( Z_N \) curve \( y^N = \prod_{k=1}^{m} (\lambda - \lambda_k) \), \( \lambda_i \neq \lambda_j \) was discovered by Bershadsky and Radul [39] and Knizhnik [40] and rigorously proved by Nakayashiki [41]. The formula (5.25) is written for singular \( Z_N \) curves and it does not follow from the results in [39, 41].

Combining (5.25) and (5.28) we have

\[
\tau(\lambda_1, \lambda_2, \ldots, \lambda_{2m+1}) = \xi (2\pi i)^{(N-1)} \frac{\theta \left[ \frac{\delta}{\epsilon} \right] (0; \Pi)}{\prod_{s=1}^{N-1} \det A_s^2} \prod_{i < j}^{2m+1} (\lambda_i - \lambda_j)^{-\frac{N^2 - 1}{12N}} \times \prod_{k < \ell}^{m} (\lambda_{2k+1} - \lambda_{2\ell+1})^{-\frac{(N-1)(N-2)}{12N}} \prod_{k < \ell}^{m} (\lambda_{2k} - \lambda_{2\ell})^{\frac{(N-1)(N-2)}{12N}},
\]

where \( \xi^8 = 1 \). For \( N = 2 \) \([40]\) coincides with the expression derived in [41].
6. Example: R-H problem with four singular points

Consider the class of curves (1.5) for \( m = 1 \)

\[
\mathcal{C}_{N,1} : \quad y^N = (\lambda - \lambda_1)(\lambda - \lambda_3)(\lambda - \lambda_2)^{N-1}.
\]

The curve (6.1) is a non-ramified covering over the hyperelliptic curve

\[
\mathcal{C}_{\text{hyperel}} : \quad w^2 = \xi^{2N} + 2(\lambda_1 + \lambda_3 - 2\lambda_2)\xi^N + (\lambda_1 - \lambda_3)^2.
\]

The coordinates of the cover \( \psi : \mathcal{C}_{N,1} \to \mathcal{C}_{\text{hyperel}} \) are

\[
\xi = \frac{y}{\lambda - \lambda_2}, \quad w = \frac{\lambda^2 - 2\lambda_3\lambda + 2\lambda_2(\lambda_1 + \lambda_3) - \lambda_1\lambda_3}{\lambda - \lambda_2}.
\]

The canonical holomorphic differentials of both curves,

\[
dU_k(\xi, w) = \frac{-(\lambda - \lambda_2)^{k-1}}{y^k}d\lambda, \quad dU_k(\xi, w) = \xi^{k-1}\frac{d\xi}{w}, \quad k = 1, \ldots, N - 1
\]

are linked under the action of \( \psi \) as

\[
dU_k(\lambda, y) = N dU_{N-k}(\xi, w).
\]

The curve (6.2) admits two automorphisms \( f_\pm \) of order two different from the hyperelliptic involution \( J \):

\[
f_+ (\xi, w) = \left( \frac{(\lambda_3 - \lambda_1)\frac{w}{\xi}}{\xi}, (\lambda_3 - \lambda_1)\frac{w}{\xi^N} \right), \quad f_- (\xi, w) = (f_+ \circ J)(\xi, w).
\]

We observe that the automorphism group of the surfaces (6.2) is

\[
\{Id, J, f_+, J\}, \quad J(\xi, w) = (e^{\frac{2\pi i}{N}}, w).
\]

The quotient of the above automorphism group by \( \{Id, J\} \) is isomorphic to the dihedral group \( D_N \) of symmetries of the \( N \)-sided regular polygon. For \( N = 3 \) this result was pointed out in [42].

For \( N \) odd each of the maps \( f_+ \) and \( f_- \) fixes exactly two points of \( \mathcal{C}_{\text{hyperel}} \). The automorphism \( f_+ \) fixes the two points were \( \xi = (\lambda_3 - \lambda_1)^\frac{1}{N} \) while \( f_- \) fixes the two points were \( \xi = -(\lambda_3 - \lambda_1)^\frac{1}{N} \). According to Riemann-Hurwitz formula the quotient surfaces:

\[
\mathcal{C}_+ = \mathcal{C}_{\text{hyperel}}/\{Id, f_+\}
\]

have genus equal to \( \frac{N-1}{2} \).

For \( N \) even, the map \( f_+ \) fixes the four points were \( \xi = \pm(\lambda_3 - \lambda_1)^\frac{1}{N} \) while \( f_- \) has no fixed points. Therefore the quotient surfaces \( \mathcal{C}_+ = \mathcal{C}_{\text{hyperel}}/\{Id, f_\pm\} \) have genus equal to \( \frac{N}{2} - 1 \) and \( \frac{N}{2} \) respectively.

The Jacobian varieties \( \text{Jac}(\mathcal{C}_{N,1}) \) and \( \text{Jac}(\mathcal{C}_+) \times \text{Jac}(\mathcal{C}_-) \) are complex tori of dimension \( N - 1 \). Following [43] it is possible to show that these tori are isomorphic. From the factorisation of the Jacobian variety \( \text{Jac}(\mathcal{C}_{\text{hyperel}}) \), the \( \theta \)-functions defined on the surface of genus \( N - 1 \) can be expressed in terms of \( \theta \)-functions defined on two surfaces of genus \( \frac{N-1}{2} \) for \( N \) odd and of genus \( N/2 - 1 \) and \( N/2 \) for \( N \) even. The procedure for obtaining the period matrix of the two quotient surfaces and the factorisation of the \( \theta \)-function is illustrated, for automorphisms of order two, in [44] and [34].

We are going to study in detail the case \( N = 3 \) when \( \theta \)-functions decomposes as product of Jacobi’s \( \vartheta \)-functions.

Remark 6.1. We remark that the curves of the form (6.1) are hyperelliptic only in the case \( m = 1 \).
6.1. Decomposition of two-dimensional Jacobian to elliptic curves: \( N = 3 \) and \( m = 1 \). We restrict ourselves to the curve

\[
\mathcal{C}_{3,1}: \quad y^3 = (\lambda - \lambda_1)(\lambda - \lambda_3)(\lambda - \lambda_2)^2.
\]

Its holomorphic differentials are

\[
du_1(\lambda, y) = \frac{d\lambda}{y}, \quad dv_2(\lambda, y) = \frac{(\lambda - \lambda_2)d\lambda}{y^2},
\]

The matrices of \( \alpha \) and \( \beta \)-periods in the homology basis given on the Figure 3, are

\[
A = \begin{pmatrix}
A_1 & \rho_2 A_1 \\
A_2 & \rho A_2
\end{pmatrix}, \quad B = \begin{pmatrix}
B_1 & -\rho B_1 \\
B_2 & -\rho^2 B_2
\end{pmatrix}, \quad \rho = e^{\frac{2\pi i}{3}},
\]

where \( \oint_{\alpha_1} du_i = A_i, \oint_{\beta_i} du_i = B_i, \; i = 1, 2 \). Evaluating explicitly the integrals we obtain

\[
A_1 = \int_0^{\lambda_2} d\xi = \frac{1}{\sqrt{\lambda_3 - \lambda_1}} \int_0^{\frac{1}{\sqrt{\lambda_3 - \lambda_1}}} \frac{d\xi}{\sqrt{(\xi - \lambda_1)(\lambda_3 - \xi)(\lambda_2 - \xi)^2}} = \frac{(1 - \rho^2)}{\sqrt{\lambda_3 - \lambda_1}} \int_0^{\frac{1}{\sqrt{\lambda_3 - \lambda_1}}} \frac{d\xi}{\sqrt{\xi(1 - \xi)(1 - \frac{\lambda_3 - \lambda_1}{\lambda_2 - \xi})}} = \frac{2\pi}{\sqrt{3}} \frac{(1 - \rho^2)}{\sqrt{\lambda_3 - \lambda_1}} F\left(\frac{1}{3}, \frac{2}{3}; 1; t\right),
\]

where \( F(a, b, c, \lambda) \) is the standard hypergeometric function and

\[
t = \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1},
\]

and, analogously,

\[
B_1 = \frac{2\pi i}{\sqrt{\lambda_3 - \lambda_1}} F\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - t\right) = \frac{1}{\sqrt{3}} F\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{\lambda_3 - \lambda_1}{\lambda_3 - \lambda_2}\right).
\]

From (6.7) and (6.8) the normalized holomorphic differentials read

\[
du_1 = \frac{1}{A_1(1 - \rho)} (du_1 + \sqrt{\lambda_3 - \lambda_1} dv_2), \quad dv_2 = \frac{1}{A_1(1 - \rho^2)} (\rho du_1 + \sqrt{\lambda_3 - \lambda_1} dv_2).
\]

From (6.7)-(6.10), the Riemann \( \Pi \)-matrix has the form

\[
\Pi = \begin{pmatrix}
2T & T \\
T & 2T
\end{pmatrix}, \quad \text{Im} \; T > 0,
\]

where

\[
T = \frac{1}{1 - \rho A_1} = \frac{i}{\sqrt{3}} F\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{\lambda_3 - \lambda_1}{\lambda_3 - \lambda_2}\right).
\]

The curve \( \mathcal{C}_{3,1} \) covers the hyperelliptic curve of genus two

\[
\mathcal{C}_{\text{hyperel}}: \quad w^2 = \xi^6 + 2(\lambda_1 + \lambda_3 - 2\lambda_2)\xi^3 + (\lambda_1 - \lambda_3)^2.
\]

Bolza \cite{12}, Igusa \cite{47} and Lange \cite{19} have classified the curves of genus two with automorphism and in particular the curves with involutions. The moduli of such curves describe a 2-dimensional sub-variety of the moduli space of curves of genus two. The automorphism group of the curve (6.14) is generated by \( \{Id, \; J, \; f_+\} \) where now \( J(\xi, w) = (e^{\frac{2\pi i}{3}}, \xi, w) \) and

\[
f_+(\xi, w) = \left(\frac{\lambda_3 - \lambda_1}{\xi}, (\lambda_3 - \lambda_1)\frac{w}{\xi^3}\right).
\]
The reduced group of automorphism is isomorphic to the dihedral group $D_3$ \[12] and such curves describes a one-dimensional variety in the moduli space of curves of genus two. The quotient surfaces $C_\pm = C_{\text{hyperel}}/\{Id, f_\pm \}$ with $f_- = f_+ \circ \varphi$ are elliptic surfaces.

We are going to construct the covering maps $\phi_\pm := h_\pm \circ \psi$ \[33\].

$$C_{N,1} \xrightarrow{\psi} C_{\text{hyperel}} \xrightarrow{h_\pm} C_\pm$$

where $\psi$ is defined in \[6.3\]. Let $(a_1,a_2;b_1,b_2)$ be the canonical homology basis defined on $C_{\text{hyperel}}$ so that $f_+(a_1) = a_2$ and $f_+(b_1) = b_2$. Then $f_-(a_1) = -a_2$ and $f_-(b_1) = -b_2$. It is easy to verify that $a_1 = \psi(a_1)$, $a_2 = \psi(a_1) - \psi(a_2)$ and $b_1 = \psi(b_1) - \psi(b_2)$, $b_2 = -\psi(b_2)$, where $\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ is the canonical homology basis defined on $C_{3,1}$. We fix $h_+(a_1) = \alpha_\pm$ and $h_+(b_1) = \beta_\pm$ where $\{\alpha_\pm, \beta_\pm\}$ is the canonical homology basis on $C_\pm$ respectively. It follows that $h_+(a_2) = \pm \alpha_\pm$ and $h_+(b_2) = \pm \beta_\pm$ so that

$$\phi_+(\alpha_1) = \alpha_+, \phi_+(\alpha_2) = -2\alpha_+, \phi_+(\beta_1) = 0, \phi_+(\beta_2) = -\beta_+, \phi_-(\alpha_1) = \alpha_-, \phi_+(\alpha_2) = 0, \phi_+(\beta_1) = 0, \phi_+(\beta_2) = \beta_-. \tag{6.15}$$

The action of $f_\pm$ on the holomorphic differentials $dU_k$, $k = 1, 2$, of the curve $C_{\text{hyperel}}$ is given by

$$f_\pm(dU_1(\xi, w)) = \mp \frac{1}{\sqrt{\lambda_3 - \lambda_1}} dU_2(\xi, w), \quad f_\pm(dU_2(\xi, w)) = \mp \sqrt{\lambda_3 - \lambda_1} dU_1(\xi, w).$$

Therefore the differentials $\sqrt{\lambda_3 - \lambda_1} dU_1 \mp dU_2$ are invariant under the action of $f_\pm$. It follows that the differentials $\sqrt{\lambda_3 - \lambda_1} dU_1 \mp dU_2 = \frac{1}{3}(\sqrt{\lambda_3 - \lambda_1} dU_2 \mp dU_1)$ project to the holomorphic differentials of the curve $C_\pm$. From the above considerations and from \[6.14\] we conclude that

$$dV_\pm = dV_1 - 2dV_2, \quad dV_- = dV_1, \tag{6.17}$$

where $dV_\pm$ are the holomorphic differentials of the curve $C_\pm$. From the relations \[6.15\] and \[6.16\] we deduce that $dV_\pm$ are the normalized holomorphic differentials of $C_\pm$ with periods

$$\int_{\beta_-} dV_- = T, \quad \int_{\beta_+} dV_+ = 3T.$$ 

Therefore the elliptic curves $C_\pm$ are 3-isogenous. Let us write the equations of the two elliptic curves $C_\pm$ in the Legendre form:

$$C_\pm : z_\pm^2 = \eta(1 - \eta)(1 - k_\pm^2 \eta). \tag{6.18}$$

The Jacobi’s moduli $k_\pm$ are related by a third order transformation and are parametrised as

$$k^2 = \frac{1}{16p}(p + 1)^3(3 - p), \quad k_\pm^2 = \frac{1}{16p^3}(p + 1)(3 - p)^3, \tag{6.19}$$

where the parameter $p$ can be expressed in terms of $\vartheta$-constants (see e.g. \[51\])

$$p = \frac{3\vartheta_3^2(0; 3T)}{\vartheta_3^2(0; T)}. \tag{6.20}$$

The holomorphic differentials $dV_\pm$ reads

$$dV_\pm = \frac{1}{4K_\pm} \frac{dy}{z_\pm}, \quad K_+ = \frac{\pi}{2} \vartheta_3^2(0; 3T), \quad K_- = \frac{\pi}{2} \vartheta_3^2(0; T). \tag{6.21}$$

From the relation \[6.17\] and \[6.20\] we construct the coordinates of the covers $\phi_\pm : C_{3,1} \rightarrow C_\pm$

$$\eta = \frac{p^2(y - (\lambda - \lambda_2)\sqrt{\lambda_3 - \lambda_1})^2 + 3(y + (\lambda - \lambda_2)\sqrt{\lambda_3 - \lambda_1})^2}{k_\pm^2(y - (\lambda - \lambda_2)\sqrt{\lambda_3 - \lambda_1})^2 + 3k_\pm^2(y + (\lambda - \lambda_2)\sqrt{\lambda_3 - \lambda_1})^2}, \quad \frac{y^2}{\lambda} = \frac{A_1(1 \pm \rho)}{4K_\pm} \frac{dy}{y \mp (\lambda - \lambda_2)\sqrt{\lambda_3 - \lambda_1}} \frac{d\eta}{d\lambda}. \tag{6.22}$$
Constructing the covering maps \( f_{\pm} = h_{\pm} \circ \psi \) as in \( \ref{12}, \ref{15} \), by mapping the branch points

\[
\xi_0 = \sqrt[3]{2 \lambda_2 - \lambda_1 - \lambda_3 + 2\sqrt[3]{2 \lambda_2^2 - \lambda_2 (\lambda_1 + \lambda_3) + \lambda_1 \lambda_3}}, \quad \rho \xi_0, \quad \rho^2 \xi_0, \quad \frac{1}{\xi_0}, \quad \frac{\rho}{\xi_0}, \quad \frac{\rho^2}{\xi_0},
\]

of the hyperelliptic curve \( \mathcal{C}_{\text{hyperel}} \) to \((0,0), (\infty, \infty), (1,0) \in \mathcal{C}_{\pm} \) according to the rule

\[
((\xi_0)^{\pm 1}, 0) \rightarrow (0,0), \quad ((\rho \xi_0)^{\pm 1}, 0) \rightarrow (\infty, \infty), \quad ((\rho^2 \xi_0)^{\pm 1}, 0) \rightarrow (1,0),
\]

we derive the algebraic dependence of the parameter \( p \) with \( C \) of the hyperelliptic curve

\[
F_t \quad \text{or} \quad \text{equality (6.23) follows from the superposition of the transformations [52], (126), p. 140, (118), p. 138, (35), p. 119.}
\]

\textbf{Remark 6.2.} The parameter \( T \) defined in \( \ref{6.15} \) reads

\[
T = \frac{2\pi}{\sqrt{3}} \frac{(1 - \rho^2)}{\sqrt[3]{\lambda_3 - \lambda_1}} F \left( \frac{1}{3}, \frac{2}{3}, 1; 1 - t \right), \quad t = \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1}
\]

where \( F \left( \frac{1}{3}, \frac{2}{3}, 1; 1 - t \right) \) and \( F \left( \frac{1}{3}, \frac{2}{3}, 1; t \right) \) are two independent solutions of the Gauss hypergeometric equation

\[
t(1 - t)F'' + (1 - 2t)F' - \frac{2}{9}F = 0.
\]

For \( T \) belonging to Siegel half-space \( \mathcal{H}_1 \) modulo the sub-group \( \Gamma_0(3) \) [44, 46], the expression \( \ref{6.24} \) is invertible and the inverse function is given in \( \ref{6.22} \) and reads

\[
t = 27 \vartheta_3(0; 3T) \left( \frac{\vartheta_3^4(0; 3T) - \vartheta_3^4(0; T)^3}{3 \vartheta_3^2(0; 3T) + \vartheta_3^2(0; T)^3} \right)^2.
\]

We recall that the sub-group \( \Gamma_0(3) \) of the modular group is defined by the matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \) with \( c = 0 \mod 3 \). One can show, by comparing \( q \)-expansions, that

\[
t(T/2)f(T) = 1,
\]

where \( f(T) \) is the automorphic function of \( \Gamma_0(3) \) found in \( \ref{20} \), Table on p. 12,

\[
f(T) = 1 + \frac{1}{27} \frac{\eta(T)}{\eta(3T)},
\]

and \( \eta \) is the Dedekind \( \eta \)-function \( \ref{24} \). Alternatively, one can express \( t \) in terms of \( \theta \)-functions with \( 1/3 \)-characteristics as

\[
t = 1 - \left( \frac{\theta \left[ \begin{array}{c} 0 \\ \frac{a}{3} \end{array} \right] (0; \Pi)}{\theta(0; \Pi)} \right)^3.
\]
The equivalence of the above expression and (6.25) involves non-trivial \( \theta \)-function identities. The function \( t(T) \) in (6.25) gives a solution of the Schwarzian equation \[53, 54\]

\[
\{t, T\} + \frac{j^2}{2}V(t) = 0,
\]

where \( \{ , \} \) is the Schwarzian derivative \[152\] and the potential \( V(t) \) is given by (see for example \[53\])

\[
V(t) = \frac{1 - \beta^2}{t^2} + \frac{1 - \gamma^2}{(t - 1)^2} + \frac{\beta^2 + \gamma^2 - \alpha^2 - 1}{t(t - 1)}, \quad \alpha = \frac{1}{3}, \beta = \gamma = 0.
\]

It follows that the functions

\[
\omega_1 = -\frac{1}{2} \frac{d}{dT} \ln \frac{i}{t(t-1)}, \quad \omega_2 = -\frac{1}{2} \frac{d}{dT} \ln \frac{i}{t-1}, \quad \omega_3 = -\frac{1}{2} \frac{d}{dT} \ln \frac{i}{t},
\]

solve the general Halphen system

\[
\dot{\omega}_1 = \omega_2 \omega_3 - \omega_1 (\omega_2 + \omega_3) + R,
\]

\[
\dot{\omega}_2 = \omega_1 \omega_3 - \omega_2 (\omega_1 + \omega_3) + R,
\]

\[
\dot{\omega}_3 = \omega_1 \omega_2 - \omega_1 (\omega_1 + \omega_2) + R,
\]

where

\[
R = \alpha^2 (\omega_1 - \omega_2)(\omega_3 - \omega_1) + \beta^2 (\omega_2 - \omega_3)(\omega_1 - \omega_2) + \gamma^2 (\omega_3 - \omega_1)(\omega_2 - \omega_3).
\]

When \( R = 0 \) the above system coincides with the classical Halphen system. The solution of the classical and general Halphen system has been investigated by many authors \[26, 55, 56, 57, 58\]. The expression (6.25) gives a formula for the solution of the general Halphen system with parameters \( \alpha = \frac{1}{3}, \beta = \gamma = 0 \) equivalent to the one derived in \[26\].

In the following we derive the decomposition of the genus two \( \theta \)-functions in terms of Jacobi’s \( \vartheta \)-functions.

**Lemma 6.3.** The \( \theta \)-function of the curve \( C_{3,1} \) is decomposed in terms of Jacobi's \( \vartheta \)-functions of the curves \( C_{\pm} \) as

\[
\theta_{[\delta]}(z_1, z_2; \Pi) = e^{\pi i (\delta, \Pi) + 2\pi i (z + \epsilon, \delta)} \left[ \vartheta_3(e_1; 6T) \vartheta_3(e_2; 2T) \vartheta_3(e_1; 6T) \vartheta_3(e_2; 2T) \right]
\]

where

\[
e_1 = z_1 + z_2 + \epsilon_1 + \epsilon_2 + 3T(\delta_1 + \delta_2), \quad e_2 = z_1 - z_2 + \epsilon_1 - \epsilon_2 + T(\delta_1 - \delta_2).
\]

**Proof.** By definition of \( \theta \)-function we obtain

\[
\theta_{[\delta]}(z_1, z_2; \Pi) = e^{\pi i (\delta, \Pi) + 2\pi i (z + \epsilon, \delta)} \sum_{n_1, n_2 \in \mathbb{Z}} \exp \left[ \pi i \left( 2T(n_1^2 + n_2^2 + n_1 n_2) + 3T(\delta_1 + \delta_2)(n_1 + n_2) + T(\delta_1 - \delta_2)(n_1 - n_2) + 2(z_1 + \epsilon_1) n_1 + 2(z_2 + \epsilon_1) n_2 \right) \right].
\]
Substituting in the above $m_1 = n_1 + n_2$ and $m_2 = n_1 - n_2$ where $m_i = 2k_i + r$, $i = 1, 2$, $r = 0, 1$ we obtain

$$
\theta^{[\delta]}_{[\varepsilon]}(z_1, z_2; \Pi) = e^{\pi i (\delta \Pi) + 2\pi i (z + \varepsilon \delta)} \sum_{r=0,1} \sum_{k_1, k_2 \in \mathbb{Z}} \exp \left[ \pi i \left( 6T \left( k_1 + \frac{r}{2} \right)^2 + 2T \left( k_2 + \frac{r}{2} \right)^2 + 6T(\delta_1 - \delta_2)(k_1 + \frac{r}{2}) + 2(k_1 + \frac{r}{2})(z_1 + \epsilon_1 + z_2 + \epsilon_2) + 2(k_2 + \frac{r}{2})(z_1 - z_2 + \epsilon_1 - \epsilon_2) \right) \right] =
$$

$$
e^{\pi i (\delta \Pi) + 2\pi i (z + \varepsilon \delta)} \sum_{k=2}^{3} \theta_k(z_1 + z_2 + \epsilon_1 + \epsilon_2 + 3T(\delta_1 + \delta_2); 6T) \times \theta_k(z_1 - z_2 + \epsilon_1 - \epsilon_2 + T(\delta_1 - \delta_2); 2T),
$$

which is equivalent to (6.26).

\[ \square \]

6.2. Solution of the $3 \times 3$ matrix R-H problem with four singular points. Let us consider the R-H problem with four singular points $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4 = \infty$ and with monodromy matrices

$$
(6.27) \quad M_1 = \begin{pmatrix} 0 & 0 & c_1 \\ c_2 & 0 & 0 \\ 0 & \frac{1}{c_2} & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & c_1 d_1 & 0 \\ 0 & c_2 & c_2 d_2 \\ \frac{1}{c_1 d_1 d_2} & 0 & 0 \end{pmatrix},
$$

$$
M_3 = \begin{pmatrix} 0 & 0 & d_1 d_2 \\ \frac{1}{d_1} & 0 & 0 \\ 0 & \frac{1}{d_2} & 0 \end{pmatrix}, \quad M_\infty = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},
$$

where $c_1, c_2, d_1, d_2$ are non-zero constants. The solution of this R-H problem is given in (6.25) and read

$$
(6.28) \quad Y_{rs}(\lambda) = X_{rs}(\lambda) \frac{\theta^{[\delta]}_{[\varepsilon]} \left( \int_{P^* \nu} dv; \Pi \right)}{\theta \left( \int_{P^* \nu} dv; \Pi \right)} \theta(0; \Pi) \frac{\theta(0; \Pi)}{\theta^{[\delta]}_{[\varepsilon]}(0; \Pi)}.
$$

with $dv$ and $\Pi$ defined in (6.11) and (6.12) respectively and

$$
\delta_i = \frac{1}{2\pi i} \log d_i, \quad \epsilon_i = \frac{1}{2\pi i} \log c_i, \quad i = 1, 2.
$$

The entries of the matrix $X(\lambda)$ in the expression (6.28) read

$$
X_{rs}(\lambda) = \frac{1}{3} \left( e^{2\pi i \frac{r}{2}} \sqrt{(\lambda - \lambda_1)(\lambda - \lambda_3)} \sqrt{\frac{\lambda_0 - \lambda_2}{(\lambda_0 - \lambda_1)(\lambda_0 - \lambda_3)}} + 1 + e^{-2\pi i \frac{r}{2}} \sqrt{(\lambda - \lambda_2)(\lambda - \lambda_3)} \sqrt{\frac{\lambda_0 - \lambda_1}{(\lambda_0 - \lambda_1)(\lambda_0 - \lambda_3)}} \right).
$$
Using the reduction formula (6.20) it is possible to write the solution (6.28) in terms of Jacobi's $\vartheta$-functions

\begin{equation}
Y_{rs}(\lambda) = X_{rs}(\lambda) \frac{e^{2\pi i (c, d)} \vartheta_3(0; 6T) \vartheta_3(0; 2T) + \vartheta_2(0; 6T) \vartheta_2(0; 2T)}{\sum_{k=2}^{3} \vartheta_k(1) \log \left( \frac{3T}{2\pi i} \log d_2; 6T \right) \vartheta_k(1) \log \left( \frac{T}{2\pi i} \log d_2^2; 2T \right)} 
\end{equation}

\begin{align*}
&\times \sum_{k=2}^{3} \vartheta_k \left( \int \frac{1}{\varphi_+(P_0')} dv_+ + \frac{1}{2\pi i} \log \left( \frac{3T}{2\pi i} \log d_2; 6T \right) \vartheta_k \left( \int \frac{1}{\varphi_-(P_0')} dv_- + \frac{T}{2\pi i} \log d_2^2; 2T \right) \right) \\
&\times \sum_{k=2}^{3} \vartheta_k \left( \int \frac{1}{\varphi_+(P_0')} dv_+; 6T \right) \vartheta_k \left( \int \frac{1}{\varphi_-(P_0')} dv_-; 2T \right) ^{-1},
\end{align*}

where $dv_\pm$ have been defined in (6.17), the covering maps $\varphi_\pm$ have been described in the previous section and

\begin{alignat*}{2}
z_1 &= \int \frac{1}{\varphi_-(P_0')} dv_+, \\
z_2 &= \int \frac{1}{\varphi_-(P_0')} dv_- - \int \frac{1}{\varphi_+(P_0')} dv_+.
\end{alignat*}

The expression (6.29) has been obtained after performing a modular transformation of the $\theta$-function under the action of the following symplectic transformation

\begin{equation}
\begin{pmatrix}
C^2 & 0 \\
0 & C^t
\end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad C^3 = 1,
\end{equation}

induced by the automorphism $J^2$.

6.3. Solution of the 3 × 3 Schlesinger system. From (6.28) and (6.21) we obtain the following expressions for the solution of the 3 × 3 Schlesinger system (6.15)

\begin{align*}
(A_k)_{ss} &= (\lambda_0 - \lambda_k)^2 \frac{\partial}{\partial \lambda_k} \left( \sum_{l=1}^{2} \frac{\partial}{\partial z_l} \log \frac{z_l}{2} \left( 0; \Pi \right) \frac{dv_l (P)}{dz_l (P)} \bigg|_{P=P_0} \right), \quad s = 1, 2, 3, \\
(A_k)_{rs} &= (s - r)(-1)^{(s-r)} \sqrt{3} i \left( \lambda_0 - \lambda_k \right)^2 \frac{\partial}{\partial \lambda_k} \left[ \sum_{l=1}^{3} \lambda_0 - \lambda_l \right] \frac{\theta_2 \left( e_{sr}; \Pi \right) - \theta_2 \left( e_{rs}; \Pi \right)}{\theta_2 \left( 0; \Pi \right)},
\end{align*}

where $s \neq r, r, s = 1, 2, 3$ and the vectors

\begin{equation}
e_{sr} = \int_{P_0}^{P_0} dv,
\end{equation}

satisfy the relation $e_{12} + e_{23} + e_{31} = 0$.

The matrix $A_\infty$ is determined from the condition

\begin{equation}
A_\infty = -A_1 - A_2 - A_3.
\end{equation}
The τ function corresponding to $3 \times 3$ solution of the Schlesinger system can be written in terms of Jacobi’s $\vartheta$-functions. According to the formula (6.30), we obtain

\[ \tau(\lambda_1, \lambda_2, \lambda_3) = \left( \frac{\lambda_1 - \lambda_3}{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)} \right)^{\frac{3}{2}} e^{\pi i (\delta, \delta') + 2\pi i (\epsilon, \delta')} \times \]

\[
\sum_{k=1}^{3} \vartheta_k \left( \frac{1}{2\pi i} \log c_1 c_2 + \frac{3T}{2\pi i} \log d_1 d_2; 6T \right) \vartheta_k \left( \frac{1}{2\pi i} \log c_1 c_2 + \frac{T}{2\pi i} \log d_2; 2T \right)^{\frac{3}{2}}.
\]

If the non-singular characteristics $\epsilon$ and $\delta$ are shifted by 1/3 periods, the corresponding constants $d_i$ and $c_i$ are shifted by the third roots of unity and the corresponding $\tau$ function is expressed by the above formula with the Jacobi’s $\vartheta$-function shifted by 1/6 periods. As an example we consider the shift

\[ \epsilon + \delta \Pi \rightarrow \epsilon + \delta \Pi + 2 \Pi(P_2 + P_3 - 2P_2), \quad P_2 = (\lambda_2, 0), \quad P_3 = (\lambda_3, 0), \]

where the vector $2 \Pi(P_2 + P_3 - 2P_2) = (-\frac{2}{3}, \frac{1}{3})$ is non-singular. The corresponding constants $d_i$ and $c_i$, $i = 1, 2$ transform to

\[ d_1 \rightarrow d_1 e^{-\frac{4\pi i}{3}}, \quad d_2 \rightarrow d_2 e^{\frac{2\pi i}{3}}, \quad c_1 \rightarrow c_i, \quad i = 1, 2. \]

If \{M_1, M_2, M_3, M_{\infty}\} are the monodromy matrices associated to the characteristics $\epsilon$ and $\delta$, the monodromy matrices associated to the characteristics $\epsilon$ and $\delta + (-\frac{2}{3}, \frac{1}{3})$ are

\[ \{M_1, e^{-\frac{4\pi i}{3}} M_2, e^{-\frac{2\pi i}{3}} M_3, M_{\infty}\}. \]

7. Conclusion

In this manuscript we have studied the solution of the R-H problem for a particular class of quasi-permutation monodromy matrices and for a given set of $2m + 2$ singular points. The dimension of the space of monodromy matrices is $2m(N - 1)$. Inspired by [3] we have solved the problem using the Szegő kernel of a Riemann surface. The monodromy of Riemann surface is obtained from the reduction of the monodromy representation of the R-H problem to a permutation representation of the symmetric group $S_N$. The form of the monodromy matrices considered, is such that the permutation representation obtained, generates the cyclic subgroup $Z_N$ of the permutation group. For this reason the family of Riemann surfaces $\mathcal{S}_{N,m}$ have $Z_N$ symmetry and genus $N(m - 1)$. The symmetry in our problem has enabled us to write the entries of the $N \times N$ matrix solution of the R-H problem as a product of an algebraic function and $\theta$-quotients. The algebraic function turns out to be related to the Szegő kernel with zero characteristics. The $2N(m - 1)$ monodromy parameters are in one to one correspondence with the $2N(m - 1)$ characteristics of the $\theta$-quotients. The R-H problem is solvable if the corresponding characteristics are non-singular.

We have studied the set of non-singular divisors supported on the branch points and we have shown that the corresponding non-singular characteristics are rational numbers of the form $k/N$, $k = 1, \ldots, N - 1$. We have shown that the solution of the R-H problem for reducible monodromy representation is expressed in terms of $\theta$-quotients with $k/N$ characteristics. Furthermore we have shown that if two monodromy representations are equivalent up to multiplication by $N$-th roots of unity, then the corresponding solutions of the R-H problem have characteristics that differ by $1/N$.

From the solution of the R-H problem we have straightforwardly obtained a particular solution of the Schlesinger equations. The Jimbo-Miwa-Ueno $\tau$-function corresponding to this particular solution of the Schlesinger system is derived in a complete form by the explicit evaluation of the projective connection associated to the Riemann surfaces $\mathcal{S}_{N,m}$.

Finally we have investigated in detail the case of a $3 \times 3$ matrix R-H problem with four singular points, $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4 = \infty$. The monodromy matrices are parametrized by four parameters. The R-H problem is solved in terms of the Szegő kernel defined on a trigonal curve of genus two admitting the dihedral group.
D_3 of automorphisms. For this reason the trigonal curve is a covering over two elliptic curves which are 3-isogenous. This fact enables us to write the solution of the R-H problem in terms of Jacobi’s ð-functions with modulus T = T(t), t = \(\frac{\lambda_3 - \lambda_1}{\lambda_3 - \lambda_2}\). The inverse function t = t(T) is in general not single valued. For T belonging to Siegel half space \(\mathcal{H}\) modulo the group \(\Gamma_0(3)\), the function t = t(T) is single valued and the explicit formula is given in [30, 34]. From this formula we have derived an expression for the solution of the corresponding general Halphen system equivalent to the one derived in [20]. From the solution of the R-H problem we have derived a four parameter family of solutions of the Schlesinger system. We suppose that these solutions would be the analogous of the elliptic solution of the Painlevé VI equation [59]. The study of the analytic continuation of the solutions of the above 3 x 3 Schlesinger system in the spirit of [30, 31] remains one of the subjects of our further investigation. Our first observations show that the analytic continuation of the solution of the Schlesinger system is induced by the action of \(\Gamma_0(3)\) on the characteristics \(\delta_1, \delta_2, \epsilon_1, \epsilon_2\). We are also interested to single out algebraic solutions and derive explicit algebraic expression for the 3 x 3 Schlesinger system as in [32, 33].

Finally, the R-H problem corresponding to the non-singular \(Z_N\) curves

\[
y^N = \prod_{k=1}^{mN} (\lambda - \lambda_k),
\]

should be investigated. This case is similar to the one treated in the present manuscript. The main technical difficulty of the above case is the determination of the explicit correspondence between monodromy data and ð-characteristics. However, for the family of non-singular \(Z_N\) surfaces, the fundamental quantities defined on the surfaces like Bergmann kernel, projective connection, Szegö kernel and Thomae type formula for 1/N characteristics can be found in the literature [39, 41].

8. Appendix

8.1. Proof of Lemma [41,5] Assume the opposite: suppose that the divisor \(\mathcal{D}_m\) or \(\mathcal{D}_{m+1, 1}\) are special, this means that there exists a non-constant meromorphic function \(f(\lambda, y)\) whose divisor of poles is \(\mathcal{D}_m\) or \(\mathcal{D}_{m+1, 1}\). Then the function

\[
\phi(\lambda, y) = f(\lambda, y) \prod_{i_j \in I_i} (\lambda - \lambda_{i_j}), \quad l = m, m + 1,
\]

has poles only at infinity. It follows from the Weierstrass gap theorem, that the ring of meromorphic functions with poles at infinity is generated in the case of the curve \(y^N = p(\lambda)q^{N-1}(\lambda)\) by powers of \(\lambda\) and functions \(y^i/q(\lambda)^i\), \(i = 0, \ldots, N - 1\). Therefore the function \(\phi(\lambda, y)\) can be written in the form

(8.1)

\[
\phi(\lambda, y) = R_0(\lambda) + \sum_{i=1}^{N-1} R_i(\lambda) \frac{y^i}{q^{i-1}(\lambda)},
\]

where \(R_i(\lambda)\) are polynomials in \(\lambda\) and \(q(\lambda) = \prod_{k=1}^{m}(\lambda - \lambda_k)^l.\)

We remark that \(\text{ord}_\infty \left( R_i(\lambda) \frac{y^i}{q^{i-1}} \right) \neq \text{ord}_\infty \left( R_j(\lambda) \frac{y^j}{q^{j-1}} \right)\) for \(i \neq j\) because otherwise

(8.2)

\[
\text{Nord}_\lambda R_i(\lambda) + i \text{deg} y - \text{Nim} = \text{Nord}_\lambda R_j(\lambda) + j \text{deg} y - Njm,
\]

and \(N\) and \(\text{deg} y\) would not be relatively prime. This observation implies that

(8.3)

\[
\text{ord}_\infty (f(\lambda, y) \prod_{i_j \in I_i} (\lambda - \lambda_{i_j})) = \text{ord}_\infty \left( R_j(\lambda) \frac{y^j}{q^{j-1}(\lambda)} \right)
\]

In this point our proof differs from that given in [27] which is working for Galois covers of the form \(y^N = \prod_{i=1}^{mN} (\lambda - \lambda_i)\) where the ansatz for the function \(\phi(\lambda, y)\) can be written as \(\sum R_i y^i\).
for some $0 \leq j \leq N - 1$. Moreover
\[
\operatorname{ord}_\infty(f(\lambda, y) \prod_{i_j \in I_l} (\lambda - \lambda_{i_j})) = -N|I_l| + k_l, \quad l = m, m + 1,
\]
where $k_l, l = m, m + 1$ is the order at infinity of $f(\lambda, y)$ and the number of elements $|I_m| = m$, $|I_{m+1}| = m + 1$. From the equation of the curve we get $\deg y = mN + 1$. Therefore the equality (8.3) can be written as
\[
N|I_l| - k_l = N(r_j + m) + j
\]
so that
\[
j = N(|I_l| - r_j - m) - k_l \geq 0.
\]
When $l = m$ that is $|I_l| = m$ it follow that $r_j = 0, j = 0, k_m = 0$ and
\[
f(\lambda, y) = \frac{1}{\prod_{i_n \in I_m} (\lambda - \lambda_{i_n})}
\]
which has divisor
\[
\operatorname{Div} f(\lambda, y) = -N \sum_{i_n \in I_{m+1}} P_{i_n} + (N - k_{m+1}) \sum_{j=1}^{m+1} P_{2j+1} + k_{m+1} \sum_{j=1}^{m} P_{2j}.
\]
Namely the divisors of poles of $f(\lambda, y)$ is
\[
\operatorname{Div}_{\text{poles}} f(\lambda, y) = (N - k_{m+1}) \sum_{i_n \in I_{m+1}, i_n \text{ even}} P_{i_n} + k_{m+1} \sum_{i_n \in I_{m+1}, i_n \text{ odd}} P_{i_n}
\]
and for $N > 3$, differs from $\mathcal{D}_{m+1,1}$. This contradicts the assumption unless $f$ is constant. For $N = 3$ the divisor of poles of $f(\lambda, y)$ coincides with $\mathcal{D}_{m+1,1}$ in the following two cases:
\[
\mathcal{D}_{m+1,1} = 2 \sum_{k=1}^{m-1} P_{i_k} + P_{i_m} + P_{i_{m+1}},
\]
with
\[
i_m, i_{m+1} \in \{2, 4, 6 \ldots, 2m\}, \quad i_k \in \{1, 3, 5, \ldots, 2m + 1\}, \quad k = 1, \ldots, m - 1
\]
or
\[
i_m, i_{m+1} \in \{1, 3, 5 \ldots, 2m + 1\}, \quad i_k \in \{2, 4, 6 \ldots, 2m\}, \quad k = 1, \ldots, m - 1.
\]
We conclude that the divisors $\{4.29\}$ where $i_m$ and $i_{m+1}$ have different parity are non-special. \hfill \Box

8.2. Derivation of the Thomae formula. We prove here Theorem 5.6 that is the formula
\[
\theta^S(\mathbf{0}; \Pi) = \prod_{i,j} \frac{N-1}{(2\pi i)^4} \prod_{i,j} (\lambda_{2i} - \lambda_{2j})^{2(N-1)} \prod_{k<l} (\lambda_{2k+1} - \lambda_{2l+1})^{2(N-1)},
\]
where the matrices $A_s, s = 1, \ldots, N - 1$, are defined in (4.27).
Proof. The proof of the theorem, consists of several steps. First we use Fay relation for zero characteristics, namely

\[
S(P, Q)^2 = \omega(P, Q) + \sum_{k,l=1}^{q} \frac{\partial^2}{\partial z_k \partial z_l} \log \theta(0; \Pi) dv_k(P) dv_l(Q).
\]

We derive Thomae formula by evaluating the residues of \[8.5\] at \( P = Q = (\lambda_i, 0) \). The residue of the term containing the derivatives of \( \theta(0; \Pi) \), can be obtained combining the heat equation \[8.4\], the variation formula \[8.3\] and the fact that the function \( \theta(0; \Pi) \) is even, which gives

\[
\begin{align*}
\text{Res}_{P=(\lambda,0)} \left[ \sum_{s=1}^{N} \sum_{k,l=1}^{(N-1)m} \frac{\partial^2}{\partial z_k \partial z_l} \log \theta(0; \Pi) \frac{dv_k(P^{(s)}) dv_l(P^{(s)})}{(dz(P))^2} \right] \\
= \sum_{k,l=1}^{(N-1)m} (1 + 2\delta_{kl}) \frac{\partial}{\partial \Pi_{k,l}} \log \theta(0; \Pi) \frac{\partial \Pi_{k,l}}{\partial \lambda_i} = 2 \frac{\partial}{\partial \lambda_i} \log \theta(0; \Pi).
\end{align*}
\]

From the expansion of the Szegö kernel given in \[8.2\], we obtain

\[
\text{Res}_{P=(\lambda,0)} \left[ \sum_{s=1}^{N} \frac{(S[0]((P^{(s)}, Q^{(s)})^2)}{dz(P)dz(Q)} \right] = \frac{N^2 - 1}{12N} \text{Res}_{\lambda=\lambda_i} \left[ \frac{p'(\lambda)}{p(\lambda)} - \frac{q'(\lambda)}{q(\lambda)} \right]^2.
\]

Now let us consider the Bergmann kernel \( \omega(P, Q) \). In order to write the explicit expression for \( \omega(P, Q) \), we follow \[8.3\]. The first step consists of constructing the normalized meromorphic differential of the third kind \( \omega_{Q,0}(P) \) with simple poles at the points \( Q = (\nu, w) \) and \( Q = (\nu_0, w_0) \), with residues \( \pm 1 \) respectively, that is, for \( P = (\lambda, y) \),

\[
\begin{align*}
\Omega_{Q,0}(P) &= \frac{d\lambda}{N(\lambda - \nu)} \left( 1 + \sum_{s=1}^{N-1} \frac{w^s q(\lambda)^{s-1}}{y^s q(\nu)^{s-1}} \right) - \frac{d\lambda}{N(\lambda - \nu_0)} \left( 1 + \sum_{s=1}^{N-1} \frac{w_0^s q(\lambda)^{s-1}}{y_0^s q(\nu_0)^{s-1}} \right) \\
&- \frac{1}{N} \sum_{j=1}^{(N-1)m} \int_{y_j} \frac{d\nu}{\nu} \int_{\alpha_j} \frac{d\xi}{\xi} \frac{\left( 1 + \sum_{s=1}^{N-1} \frac{w^s q(\xi)^{s-1}}{y^s q(\nu)^{s-1}} \right) - \left( 1 + \sum_{s=1}^{N-1} \frac{w_0^s q(\xi)^{s-1}}{y_0^s q(\nu_0)^{s-1}} \right)}{(\xi - \nu)}
\end{align*}
\]

where \( d\nu_j, j = 1, \ldots, (N-1)m \) is the basis of normalized holomorphic differentials and the point \( (\xi, y_0) \in \mathbb{C}_{N,m} \). The differential \( \Omega_{Q,0}(P) \) as a function of \( Q \) is an Abelian integral with periods given by the relations

\[
\oint_{\alpha_j} d_\nu \Omega_{Q,0}(P) = 0, \quad \oint_{\beta_j} d_\nu \Omega_{Q,0}(P) = 2\pi i dv_j(P), \quad j = 1, \ldots, (N-1)m.
\]

Furthermore the differential \( \Omega_{Q,0}(P) \) satisfies the symmetry property \( d_\nu \Omega_{Q,0}(P) = d_\lambda \Omega_{P;P_0}(Q) \), for \( P_0 \neq P \). Therefore the 2-differential, \( \omega(P, Q) := d_\nu \Omega_{Q,0}(P) \),

\[
(1) \text{ is symmetric in } P \text{ and } Q; \\
(2) \text{ is holomorphic everywhere except for a double pole along } P = Q, \text{ where} \\
\omega(P, Q) = d\lambda d\nu \left( \frac{1}{(\lambda - \nu)^2} + \text{regular terms} \right); \\
(3) \text{ for any fixed } P, \text{ it satisfies } \omega(P, Q) = \text{Bergmann kernel given alternatively in the form } \omega(P, Q).
\]

Therefore, \( \omega(P, Q) \) is the Bergmann kernel given alternatively in the form \[8.2\].
In order to write more explicitly the Bergmann kernel, let us introduce the Abelian differentials \( \sigma_{r,j}(\nu, w) \) of the second kind having the only pole at infinity of order \( N(j + 1) - r + 1 \), that is

\[
\sigma_{r,j}(\nu, w) = \frac{q(\nu)^{r-1}}{w^r} Q_{r,j}(\nu) d\nu, \quad r = 1, \ldots, N - 1, \quad j \geq 0,
\]

where \( Q_{r,j}(\nu) \) are polynomials in \( \nu \) of degree \( m + j \). The coefficients of the polynomials \( Q_{r,j}(\nu) \), \( r = 1, \ldots, N - 1, \quad j \geq 0 \), are uniquely determined by the conditions

\[
\int_{\nu_s} \sigma_{r,j}(\nu, w) = 0, \quad s = 1, \ldots, m,
\]

\[
\sigma_{r,j}(\nu, w) \simeq \left( \nu^j + O \left( \frac{1}{\nu^1} \right) \right) d\nu, \quad (\nu, w) \rightarrow (\infty, \infty).
\]

From the Riemann bilinear relations we obtain the identities

\[
\int_{Q_0}^Q \sigma_{r,j}(P) + \text{Res}_{P=(\infty,\infty)} \left[ \Omega_{Q,Q_0}(P) \sigma_{r,j}(P) \right] = 0, \quad r = 1, \ldots, N - 1, \quad j = 0, \ldots, m - 1,
\]

so that we can reduce the expression of \( \omega(P, Q) = d\nu \Omega_{Q,Q_0}(P) \) to the form

\[
\omega(P, Q) = \frac{d\nu d\lambda}{N(\lambda - \nu)^2} \left( 1 + \sum_{s=1}^{N-1} \frac{w^s q(\lambda)^{s-1}}{y^s q(\nu)^{s-1}} \right) + \frac{d\nu d\lambda}{N(\lambda - \nu)} \frac{d}{d\nu} \left( \sum_{s=1}^{N-1} \frac{w^s q(\lambda)^{s-1}}{y^s q(\nu)^{s-1}} \right) +
\]

\[
- \frac{1}{N} \sum_{s=1}^{N-1} \sum_{j=1}^m \lambda^j q(\lambda)^{s-1} q^{N-s-1}(\nu)^{\tilde{Q}_{s,j}(\nu)} \frac{d}{d\nu} \log \det A_s + \frac{1}{N} \sum_{j=1}^m \lambda^j \tilde{Q}_{s,j}(\lambda), \quad i = 1, \ldots, m + 1,
\]

where \( \tilde{Q}_{s,j}(\nu) \) is a polynomial depending on \( \Omega_{N-s,0}(\nu), \Omega_{N-s,1}(\nu), \ldots, \Omega_{N-s,m-j}(\nu), \quad j = 1, \ldots, m, \quad s = 1, \ldots, N - 1 \).

**Proposition 8.1.** For \( s = 1, \ldots, N - 1 \), the following identities are satisfied:

\[
\frac{\partial}{\partial \lambda_i} \log \det A_s = \frac{1}{\prod_{l \neq i}(\lambda_i - \lambda_l)} \sum_{j=1}^m \lambda^j \tilde{Q}_{s,j}(\lambda_i), \quad i = 1, \ldots, m + 1,
\]

where the matrix \( A_s \) is defined in (4.7).

**Sketch of the proof.** The integral of \( \omega(P, Q) \) in the \( P \) variable along the \( \alpha_j \) periods is identically zero. Therefore, substituting the local coordinate \( \nu - \lambda_i = t^N \) in \( \omega(P, Q) \) and imposing that the terms of order \( dt, \ldots, t^{N-2} dt \) of the integral

\[
\int_{\alpha_i} \omega(P, Q) = 0,
\]

are identically zero, we obtain the statement.

Combining all the above relations we can derive the explicit expression of the projective connection

\[
\frac{1}{6} R(z(P)) = \lim_{P \rightarrow Q} \left[ \frac{\omega(P, Q)}{dz(P) \ dz(Q)} - \frac{1}{(z(P) - z(Q))^2} \right].
\]

**Proposition 8.2.** The projective connection \( R(z(P)) \) can be obtained from (8.11) and reads

\[
\frac{1}{6} R(z(P)) = - \frac{1}{N} \sum_{s=1}^{N-1} \sum_{j=1}^m \frac{z(P)^{s-1} \tilde{Q}_{s,j}(z(P))}{p(z(P)) q(z(P))} + \frac{N^2 - 1}{12N^2} \left[ \frac{q'(z(P))}{q(z(P))} \right]^2 -
\]

\[
- \frac{N - 1}{4N} \left[ \frac{q''(z(P))}{q(z(P))} + \frac{p''(z(P))}{p(z(P))} \right],
\]
where the prime denotes the derivative $\frac{d}{dz}(P)$.

Combining (8.12) and (8.13), we evaluate the residue of the Bergmann kernel at the branch points, namely,

$$
\begin{align*}
\text{Res}_{P=Q=(\lambda, \theta)} \left[ \sum_{s=1}^{N} \omega(P(s), Q(s)) \right] &= \frac{N^2 - 1}{12N} \left[ \frac{p'(\lambda)}{p(\lambda)} - \frac{q'(\lambda)}{q(\lambda)} \right] - \\
&- \frac{N - 1}{4} \left[ \frac{q''(\lambda)}{q(\lambda)} + \frac{p''(\lambda)}{p(\lambda)} \right] - \sum_{k=1}^{N-1} \frac{\partial}{\partial \lambda_i} \log \det A_k.
\end{align*}
$$

(8.14)

Substituting (8.6), (8.7) and (8.14) in (8.5) and simplifying, we obtain

$$
2 \frac{\partial}{\partial \lambda_i} \log \theta(0; \Pi) = \frac{N - 1}{4} \left[ \frac{q''(\lambda)}{q(\lambda)} + \frac{p''(\lambda)}{p(\lambda)} \right] + \sum_{k=1}^{N-1} \frac{\partial}{\partial \lambda_i} \log \det A_k,
$$

(8.15)

which gives (8.4) up to a constant $C$.

To compute $C$ we pinch the branch points in the following way

$$
\lambda_{2k} = e_k + \epsilon, \quad \lambda_{2k-1} = e_k - \epsilon, \quad k = 1, \ldots, m, \quad 0 < \epsilon \ll 1.
$$

In this case the l.h.s of (8.4) becomes equal to one as $\epsilon \to 0$, more precisely $\theta(0; \Pi) = 1 + O(\epsilon)$. Regarding the r.h.s the following relations are needed:

$$
\lim_{\epsilon \to 0} (A_s)_{ij} = 2\pi i \frac{\epsilon^{j-1}}{\prod_{k \neq i} \prod_{k=1}^{m} (e_i - e_k)(e_i - e_{2m+1})},
$$

so that

$$
\lim_{\epsilon \to 0} (\det A_s) = (2\pi i)^m \frac{1}{\prod_{k < j} \prod_{k,j=1}^{m} (e_k - e_j)(e_k - e_{2m+1})}. \tag{8.16}
$$

Substituting (8.16) into (8.4) and letting $\epsilon \to 0$ in all the terms of (8.4), we obtain

$$
1 = C (2\pi i)^{4m(N-1)}, \tag{8.17}
$$

and the expression for $C$ follows. □

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