COMPARISON OF VARIOUS CONTINUED FRACTION EXPANSIONS: A LOCHS-TYPE APPROACH

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Abstract

We investigate the efficiency of several types of continued fraction expansions of a number in the unit interval using a generalization of Loch’s theorem from 1964. Thus, we aimed to compare the effectiveness of Chan’s continued fractions, $\theta$–expansions, $N$–continued fractions and Rényi-type continued fractions. A central role in fulfilling our goal is the entropy of the measure preserving transformations which generate these expansions.

1 Introduction

As it is well known, entropy is an important concept of information in physics, chemistry, and information theory [12]. The connection between entropy and the transmission of information was first studied by Shannon in [19]. Thus, the entropy can be seen as a measure of randomness of the system, or the average information acquired under a single application of the underlying map. Entropy also plays an important role in ergodic theory. Thus, in 1958 Kolmogorov [4] imported Shannon’s probabilistic notion of entropy into the theory of dynamical systems and showed how entropy can be used to tell whether two dynamical systems are non-conjugate (i.e., non-isomorphic).

Like Birkhoff’s ergodic theorem [12] the entropy is a fundamental result in ergodic theory. Given a measure preserving system $(X, \mathcal{X}, \mu, T)$, we say $\alpha = \{A_i : i \in I\}$ is a partition of $X$ if $X = \bigcup_{i \in I} A_i$, where $A_i \in \mathcal{X}$ for each $i \in I$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, $i, j \in I$. Here $I$ is a finite or countable index set. For a partition $\alpha$ of $X$, we define the entropy of the partition $\alpha$ as

$$H(\alpha) := \sum_{A \in \alpha} \mu(A) \log \mu(A).$$  \hspace{1cm} (1.1)

In this definition $T$ does not appear. But, as we will see, the entropy of the dynamical system is defined by the entropy of the transformation $T$.

2010 Mathematics Subject Classification. Primary 11J70; Secondary 37A35.
Given a partition $\alpha$, we consider the partition $\alpha_n := n - 1 \bigcap_{i=0}^{n-1} T^{-i} \alpha = \{ n - 1 \bigcap_{i=0}^{n-1} T^{-i} A_i : A_i \in \alpha, i = 0, 1, \ldots, n - 1 \}$ (1.2)

Then the entropy of transformation $T$ w.r.t. $\alpha$ is given by

$$h(\alpha, T) := \lim_{n \to \infty} \frac{1}{n} H(\alpha_n). \quad (1.3)$$

In computation of $h(T)$ the following classical Shannon-McMillan-Breiman Theorem [3] is very useful. Thus, let $(X, \mathcal{X}, \mu, T)$ be an ergodic measure preserving system and let $\alpha$ be a finite or countable partition of $X$ satisfying $H(\alpha) < \infty$. The Shannon-McMillan-Breiman theorem says if $A_n(x)$ denote the unique element $A_n \in \alpha_n$ such that $x \in A_n$, then for almost every $x \in X$ we have:

$$\lim_{n \to \infty} -\frac{1}{n} \log \mu(A_n(x)) = h(\alpha, T). \quad (1.4)$$

In 1964 Rohlin [13] showed that, when Rényi’s condition is satisfied the entropy of a $\mu$-measure preserving operator $T : X \to X$ is given by the formula:

$$h(T) := \int_X \log |T'(x)| \, d\mu(x). \quad (1.5)$$

The Rényi’s condition means that there is a constant $C \geq 1$ such that

$$\sup_{x, y \in T^n(\alpha_n)} \frac{|u_n'(x)|}{|u_n'(y)|} \leq C, \quad (1.6)$$

where $u_n := (T^n|_{\alpha_n})^{-1}$.

### 2 Lochs’ Theorem

In 1964, G. Lochs [11] compared the decimal expansion and the regular continued fraction (RCF) expansion. Thus, roughly 97 RCF digits are determined by 100 decimal digits which indicates that the RCF expansion is slightly more efficient compared to the decimal expansion at representing irrational numbers.

#### 2.1 Decimal expansions

As is well known, any real number $x \in [0, 1)$ can be written as

$$x = \sum_{i=1}^{\infty} \frac{d_i(x)}{10^i}, \quad (2.1)$$

where $d_i = d_i(x) \in \{0, 1, 2, \ldots, 9\}$ for $i \in \mathbb{N}_+ := \{1, 2, 3, \ldots\}$. The representation of $x$ in (2.1), denoted by $x = 0.d_1d_2\ldots$ is called the decimal expansion of $x$. We can generate the decimal expansions by iterating the decimal map

$$T_d : [0, 1) \to [0, 1); \quad T_d(x) = 10x - \lfloor 10x \rfloor, \quad (2.2)$$
where \(\lfloor \cdot \rfloor\) denotes the floor (or entire) function. In other words, \(T_d\) is given by
\[
T_d(x) = 10x - i \text{ if } \frac{1}{10} \leq x < \frac{i + 1}{10}, \quad i = 0, 1, 2, \ldots, 9.
\]
(2.3)
Thus, we obtain:
\[
x = \frac{d_1}{10} + \frac{d_2}{10^2} + \ldots + \frac{d_n}{10^n} + \frac{T_d^n(x)}{10^n},
\]
(2.4)
where
\[
d_1 = d_1(x) = \lfloor 10x \rfloor
\]
(2.5)
and
\[
d_n = d_n(x) = d_1 \left( T_d^{n-1}(x) \right), \quad n \geq 2.
\]
(2.6)
Since \(0 \leq T_d^n(x) < 1\), we obtain
\[
\sum_{i=1}^{n} \frac{d_i}{10^i} \to x \quad \text{as } n \to \infty.
\]
(2.7)

2.2 Regular continued fraction expansions

As is well known, any irrational number \(x \in [0, 1)\) has a unique regular continued fraction expansion
\[
x = \frac{1}{a_1 + \frac{1}{a_2 + \ddots}}.
\]
(2.8)
where \(a_n \in \mathbb{N}_+\) for any \(n \geq 1\). This expansion is obtained by applying repeatedly the Gauss map or the regular continued fraction transformation
\[
T_G : [0, 1) \to [0, 1); \quad T_G(0) = 0, \quad T_G(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.
\]
(2.9)
Therefore, it follows that the digits \(a_1, a_2, \ldots\) are related by
\[
a_n = a_n(x) = a_1 \left( T_G^{n-1}(x) \right), \quad n \geq 2,
\]
(2.10)
where
\[
a_1 = a_1(x) = \left\lfloor \frac{1}{x} \right\rfloor.
\]
(2.11)
If we denote by \([a_1, a_2, \ldots]_G\) the expansion in (2.8), we have
\[
[a_1, a_2, \ldots, a_n]_G \to x \quad \text{as } n \to \infty.
\]
(2.12)
2.3 Comparing the efficiency of decimal expansion and RCF expansion

In [11], Lochs answered the question which of these developments is more efficient, namely, which of the two sequences in (2.7) and (2.12) converges faster to \( x \) as \( n \to \infty \).

Suppose that the irrational number \( x \in (0,1) \) has the decimal expansion \( x = 0.d_1d_2\ldots \) and the RCF expansion \( x = [a_1, a_2, \ldots]_G \). Let \( y \) be the rational number such that \( y = 0.d_1d_2\ldots d_l \) and \( y = [c_1, c_2, \ldots, c_k]_G \). Let

\[
m(n, x) := \max\{i \leq \min(l, k) : \text{for all } j \leq i, a_j = c_j\}.
\] (2.13)

In other words, \( m \) is the largest positive integer for which \( a_i = c_i \) for \( i \leq m \), i.e.,

\[
a_1 = c_1, \ a_2 = c_2, \ldots, a_m = c_m, \ a_{m+1} \neq c_{m+1}.
\]

Lochs [11] proved that, for almost every irrational \( x \in (0,1) \), we have

\[
\lim_{n \to \infty} \frac{m(n, x)}{n} = \frac{6 \log 2 \log 10}{\pi^2} = 0.97027014\ldots
\] (2.14)

For example, Lochs [10] has computed the first 968 RCF digits of \( \pi \) from its first 1000 decimals, and Brent and McMillan [1] have computed the first 29200 RCF digits of Euler-Mascheroni constant from its first 30100 decimals.

2.4 Extended Lochs’ theorem

Dajani and Fieldsteel [2] proved a generalization of Lochs’ theorem showing that we can compare any two expansions of numbers which are generated by surjective interval maps \( S : [0,1) \to [0,1) \) that satisfy the following conditions:

(1) there exists a finite or countable partition of \([0,1)\) into intervals such that \( S \) restricted to each interval is strictly monotonic and continuous,

and

(2) \( S \) is ergodic with respect to Lebesgue measure \( \lambda \), and there exists an \( S \) invariant probability measure \( \mu \) equivalent to \( \lambda \) (i.e., \( \mu(A) = 0 \) if and only if \( \lambda(A) = 0 \) for all Lebesgue sets \( A \)) with bounded density (both \( \frac{d\mu}{d\lambda} \) and \( \frac{d\lambda}{d\mu} \) are bounded).

If \( S_1 \) and \( S_2 \) are any two such maps and if \( m_{S_1,S_2}(n, x) \) is the number of digits in the \( S_2 \)-expansion of \( x \) that can be determined from knowing the first \( n \) digits in the \( S_1 \)-expansion, then

\[
\lim_{n \to \infty} \frac{m_{S_1,S_2}(n, x)}{n} = \frac{h(S_1)}{h(S_2)} \quad \lambda - \text{a.e.}
\] (2.15)

where \( h(S_1) \) and \( h(S_2) \) denote the entropy of \( S_1 \) and \( S_2 \), respectively, with \( h(S_1) > 0 \) and \( h(S_2) > 0 \).
3 Other continued fraction expansions

Apart from the RCF expansions there is a wide variety of continued fraction expansions. Here we mention only a few of the expansions studied from the metrical point of view by the authors over time, namely, Chan’s continued fractions (in \([5, 14]\)), \(\theta\)-expansions (in \([16, 15]\)), \(N\)-continued fraction expansions (in \([6, 7, 18]\)), Rényi-type continued fraction expansions (in \([8, 9, 17]\)).

In this paper, we ask the question which of these expansions is more effective. In order to apply the result of Dajani and Fieldsteel \([2]\) from section 2.4, we briefly present these expansions, calculating the entropy of each map that generates these expansions.

3.1 Chan’s continued fractions

Fix an integer \(\ell \geq 2\). Then any \(x \in [0, 1)\) can be written in the form

\[
x = \frac{\ell - a_1(x)}{1 + \frac{(\ell - 1)\ell - a_2(x)}{1 + \frac{(\ell - 1)\ell - a_3(x)}{1 + \ldots}}},
\]

(3.1)

where \(a_n(x)\)'s are non-negative integers. Define \(p_{\ell,n}(x) := [a_1(x), a_2(x), \ldots, a_n(x)]\ell\) the \(n\)-convergent of \(x\) by truncating the expansion on the right-hand side of (3.1), that is,

\[
[a_1(x), a_2(x), \ldots, a_n(x)]\ell \to x \quad (n \to \infty).
\]

(3.2)

This continued fraction is associated with the following transformation \(T_\ell\) on \([0, 1]\):

\[
T_\ell(x) := \begin{cases} 
\frac{\log x^{-1}}{\ell} - \left\lfloor \frac{\log x^{-1}}{\log \ell} \right\rfloor - 1, & \text{if } x \neq 0 \\
0, & \text{if } x = 0.
\end{cases}
\]

(3.3)

It is easy to see that \(T_\ell\) maps the set of irrationals in \([0, 1]\) into itself. For any \(x \in (0, 1)\) put

\[
a_n = a_n(x) = a_1\left(T_\ell^{n-1}(x)\right), \quad n \in \mathbb{N}_+,
\]

(3.4)

with \(T_\ell^0(x) = x\) and

\[
a_1 = a_1(x) = \begin{cases} 
[\log x^{-1}/\log \ell], & \text{if } x \neq 0 \\
\infty, & \text{if } x = 0.
\end{cases}
\]

(3.5)

The transformation \(T_\ell\) which generates the continued fraction expansion (3.1) is ergodic with respect to an invariant probability measure, \(G_\ell\), where

\[
G_\ell(A) := k_\ell \int_A \frac{dx}{((\ell - 1)x + 1)((\ell - 1)x + \ell)}, \quad A \in \mathcal{B}_{[0,1]},
\]

(3.6)

with

\[
k_\ell := \frac{(\ell - 1)^2}{\log (\ell^2/(2\ell - 1))}
\]

(3.7)
and $\mathcal{B}_{[0,1]}$ is the $\sigma$-algebra of Borel subsets of $[0,1]$.

An $n$-block $(a_1, a_2, \ldots, a_n)$ is said to be admissible for the expansion in (3.1) if there exists $x \in [0,1)$ such that $a_i(x) = a_i$ for all $1 \leq i \leq n$. If $(a_1, a_2, \ldots, a_n)$ is an admissible sequence, we call the set

$$I_\ell(a_1, a_2, \ldots, a_n) = \{x \in [0,1] : a_1(x) = a_1, a_2(x) = a_2, \ldots, a_n(x) = a_n\},$$

the $n$-th order cylinder.

Define $(u_{\ell,i})_{i \in \mathbb{N}}$ by

$$u_{\ell,i} : [0,1] \to [0,1]; \quad u_{\ell,i}(x) := \frac{\ell^{-i}}{1 + (\ell - 1)x}.$$  \hspace{1cm} (3.9)

For each $i \in \mathbb{N}$, $u_{\ell,i} = \left( T_\ell I_\ell(a_1) \right)^{-1}$. Let

$$u_{\ell,a_1a_2\ldots a_n}(t) := (u_{\ell,a_1} \circ u_{\ell,a_2} \circ \ldots \circ u_{\ell,a_n})(t) = \frac{\ell^{-a_1(x)}}{1 + \frac{(\ell - 1)\ell^{-a_2(x)}}{1 + \ldots + \frac{(\ell - 1)\ell^{-a_n}}{1 + (\ell - 1)\ell}}}. \hspace{1cm} (3.10)$$

We observe that $u_{\ell,a_1a_2\ldots a_n} = \left( T_\ell^n | I_\ell(a_1, a_2, \ldots, a_n) \right)^{-1}$. Therefore

$$I_\ell(a_1, a_2, \ldots, a_n) = \{u_{\ell,a_1\ldots a_n}(t) : t \in [0,1]\} \hspace{1cm} (3.11)$$

which is an interval with the endpoints $\frac{p_{\ell,a}}{q_{\ell,n}}$ and $\frac{p_{\ell,a} + (\ell - 1)\ell^{a_n}p_{\ell,a-1}}{q_{\ell,n} + (\ell - 1)\ell^{a_n}q_{\ell,n-1}}$ and which form a partition of $[0,1]$.

Before applying Rohlin’s formula, we must check Rényi condition (1.6). We will use directly, without mentioning them here, some properties proved in [5]. Thus, we have

$$\left| \frac{u'_{\ell,a_1\ldots a_n}(\ell)}{u'_{\ell,a_1\ldots a_n}(r)} \right| = \left( \frac{q_{\ell,n} + r(\ell - 1)\ell^{a_n}q_{\ell,n-1}}{q_{\ell,n} + t(\ell - 1)\ell^{a_n}q_{\ell,n-1}} \right)^2 \leq \left( \frac{q_{\ell,n} + (\ell - 1)\ell^{a_n}q_{\ell,n-1}}{q_{\ell,n}} \right)^2 \leq \ell^2. \hspace{1cm} (3.12)$$

Applying Rohlin’s formula (1.5) we obtain:

$$h(T_\ell) = \int_0^1 \log |T_\ell'(x)| \, dG_\ell = \int_0^1 \log \left( \frac{\ell^{-a_1(x)}}{(\ell - 1)x^2} \right) \, dG_\ell$$

$$= \int_0^1 \left( 2 \log(1/x) - a_1(x) \log \ell - \log(\ell - 1) \right) \, dG_\ell$$

$$= 2k_\ell \int_0^1 \frac{\log(1/x)}{((\ell - 1)x + 1)((\ell - 1)x + \ell)} \, dx$$

$$- k_\ell \log \ell \int_0^1 \frac{a_1(x)}{((\ell - 1)x + 1)((\ell - 1)x + \ell)} \, dx - \log(\ell - 1). \hspace{1cm} (3.13)$$

As examples, we have:
| ℓ  | \( h(T_ℓ) \) |
|-----|---------------|
| 2   | 1.62258       |
| 3   | 1.26775       |
| 5   | 0.996315      |
| 10  | 0.765943      |
| 50  | 0.476521      |
| 100 | 0.406218      |
| 200 | 0.350849      |

Table 1: Entropies \( h(T_ℓ) \) for different values of \( ℓ \)

### 3.2 \( \theta \)-expansions

For a fixed irrational \( \theta \in (0, 1) \), we consider a generalization of the Gauss map, \( T_θ : [0, \theta] \rightarrow [0, \theta] \) defined as

\[
T_θ(x) := \begin{cases} 
\frac{1}{x} - \theta \left\lfloor \frac{1}{xθ} \right\rfloor, & \text{if } x \in (0, \theta], \\
0, & \text{if } x = 0.
\end{cases}
\]

The transformation \( T_θ \) is connected with the \( \theta \)-expansion for a number in \((0, \theta)\) as follows. The numbers \( \theta \left\lfloor \frac{1}{yθ} \right\rfloor \) obtained by taking \( y \) successively equal to \( x, T_θ(x), T_2θ(x), \ldots \), lead to the \( \theta \)-expansion of \( x \) as

\[
x = \frac{1}{\vartheta_1 \theta + \frac{1}{\vartheta_2 \theta + \frac{1}{\vartheta_3 \theta + \cdots}}},
\]

where \( \vartheta_n \in \mathbb{N}_+ \) for all \( n \in \mathbb{N}_+ \). The positive integers \( \vartheta_n = \vartheta_n(x) = \vartheta_1 \left( T_{\theta}^{n-1}(x) \right) \), \( n \in \mathbb{N}_+ \), with \( T_{\theta}^0(x) = x \) and \( \vartheta_1 = \vartheta_1(x) = \left\lfloor \frac{1}{xθ} \right\rfloor \) are called the digits of \( x \) with respect to the \( \theta \)-expansion in (3.15), and we have that the finite truncation of (3.15), \( p_{θ,n}/q_{θ,n} := [\vartheta_1, \vartheta_2, \ldots, \vartheta_n]_θ \), tends to \( x \) as \( n \to \infty \).

If \( \theta^2 = 1/s, s \in \mathbb{N}_+ \), the digits \( \vartheta_n \)'s take values greater or equal to \( s \), and the transformation \( T_θ \) is ergodic with respect to an absolutely continuous invariant probability measure

\[
G_θ(A) := \frac{1}{\log (1 + \theta^2)} \int_A \frac{\theta dx}{1 + xθ}, \quad A \in \mathcal{B}_{[0,1]}.
\]

An \( n \)-block \((\vartheta_1, \vartheta_2, \ldots, \vartheta_n)\) is said to be admissible for the expansion in (3.15) if there exists \( x \in [0, \theta) \) such that \( \vartheta_i(x) = \vartheta_i \) for all \( 1 \leq i \leq n \). If \((\vartheta_1, \vartheta_2, \ldots, \vartheta_n)\) is an admissible sequence, we call the set

\[
I_θ(\vartheta_1, \vartheta_2, \ldots, \vartheta_n) = \{x \in [0, \theta] : \vartheta_1(x) = \vartheta_1, \vartheta_2(x) = \vartheta_2, \ldots, \vartheta_n(x) = \vartheta_n\},
\]

the \( n \)-th order cylinder.
Define \((u_{\theta,i})_{i \geq s}\) by
\[
 u_{\theta,i} : [0, \theta] \to [0, \theta]; \quad u_{\theta,i}(x) := \frac{1}{i \theta + x}. \tag{3.18}
\]
For each \(i \geq s\), \(u_{\theta,i} = \left( T_{\theta}^{a_1} \right)^{-1} \). Let
\[
 u_{\theta,\vartheta_1 \vartheta_2 \ldots \vartheta_n}(t) := (u_{\theta,\vartheta_1} \circ u_{\theta,\vartheta_2} \circ \ldots \circ u_{\theta,\vartheta_n})(t) = \frac{1}{\vartheta_1 \theta + \frac{1}{\vartheta_2 \theta + \ldots + \frac{1}{\vartheta_n \theta + t}}}. \tag{3.19}
\]
We observe that \(u_{\theta,\vartheta_1 \vartheta_2 \ldots \vartheta_n} = \left( T_{\theta}^{\vartheta_1} \right)^{-1} \). Therefore
\[
 I_{\theta}(\vartheta_1, \vartheta_2, \ldots, \vartheta_n) = \{u_{\theta,\vartheta_1 \vartheta_2 \ldots \vartheta_n}(t) : t \in [0, \theta]\} \tag{3.20}
\]
which is an interval with the endpoints \(\frac{q_{\theta,n}}{q_{\theta,n}}\) and \(\frac{q_{\theta,n} + \theta q_{\theta,n-1}}{q_{\theta,n} + \theta q_{\theta,n-1}}\) and which form a partition of \([0, \theta]\).

We now check Rényi condition (1.6). We will use directly, without mentioning them here, some properties proved in [16]. Thus, we have
\[
 \left| \frac{u_{\theta,\vartheta_1 \vartheta_2 \ldots \vartheta_n}(t)}{u_{\theta,\vartheta_1 \vartheta_2 \ldots \vartheta_n}(r)} \right| = \left( \frac{q_{\theta,n} + \theta q_{\theta,n-1}}{q_{\theta,n} + t q_{\theta,n-1}} \right)^2 \leq \left( \frac{q_{\theta,n} + \theta q_{\theta,n-1}}{q_{\theta,n}} \right)^2 \leq (1 + \theta^2)^2. \tag{3.21}
\]
We compute the entropy \(h(T_{\theta})\) by the Rohlin’s formula (1.5):
\[
 h(T_{\theta}) = \int_0^\theta \log |T_{\theta}'(x)| \, dG_{\theta}(x) = \int_0^\theta -\frac{\log x^2}{\log(1 + \theta^2)} \frac{\theta dx}{1 + \theta x} = \frac{-2 \theta}{\log(1 + \theta^2)} \int_0^\theta \frac{\log x}{1 + \theta x} \, dx. \tag{3.22}
\]
As examples, we have:

| \(s\) | \(h(T_{\theta})\) |
|------|----------------|
| 1 | 2.37314 |
| 3 | 3.24705 |
| 5 | 3.70244 |
| 10 | 4.35074 |
| 50 | 5.92195 |
| 100 | 6.61015 |
| 1000 | 8.90825 |

Table 2: Entropies \(h(T_{\theta})\) for different values of \(s = 1/\theta^2\)
3.3 \( N \)-continued fraction expansions

Fix an integer \( N \geq 1 \). The measure-theoretical dynamical system \(([0, 1], B_{[0,1]}, T_N, G_N)\) is defined as follows:

\[
T_N : [0, 1] \to [0, 1]; \quad T_N(x) := \begin{cases} 
\frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor, & \text{if } x \in (0, 1], \\
0, & \text{if } x = 0
\end{cases}
\]  

(3.23)

and

\[
G_N(A) := \frac{1}{\log N} \int_A \frac{dx}{x + N}, \quad A \in B_{[0,1]}.
\]  

(3.24)

The probability measure \( G_N \) is \( T_N \)-invariant, and the dynamical system \(([0, 1], B_{[0,1]}, T_N, G_N)\) is ergodic.

For any \( 0 < x < 1 \) put \( \varepsilon_1(x) = \lfloor N/x \rfloor \) and \( \varepsilon_n(x) = \varepsilon_1(T_{N-1}^{-1}(x)), n \in \mathbb{N}_+, \) with \( T_N^0(x) = x \). Then every irrational \( 0 < x < 1 \) can be written in the form

\[
x = \frac{N}{\varepsilon_1 + \frac{N}{\varepsilon_2 + \cdots + \frac{N}{\varepsilon_3 + \cdots}}},
\]

(3.25)

where \( \varepsilon_n \)'s are non-negative integers, \( \varepsilon_n \geq N, n \in \mathbb{N}_+ \). We call (3.25) the \( N \)-continued fraction expansion of \( x \) and \( p_{N,n}(x)/q_{N,n}(x) := [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n]_N \) the \( n \)-th order convergent of \( x \in [0, 1] \). Then \( p_{N,n}(x)/q_{N,n}(x) \to x, n \to \infty \).

An \( n \)-block \((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)\) is said to be admissible for the expansion in (3.25) if there exists \( x \in [0, 1) \) such that \( \varepsilon_i(x) = \varepsilon_i \) for all \( 1 \leq i \leq n \). If \((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)\) is an admissible sequence, we call the set

\[
I_N(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = \{ x \in [0, 1] : \varepsilon_1(x) = \varepsilon_1, \varepsilon_2(x) = \varepsilon_2, \ldots, \varepsilon_n(x) = \varepsilon_n \},
\]

(3.26)

the \( n \)-th order cylinder.

Define \((u_{N,i})_{i \geq N}\) by

\[
u_{N,i} : [0, 1] \to [0, 1]; \quad u_{N,i}(x) := \frac{N}{i + x}.
\]  

(3.27)

For each \( i \geq N \), \( u_{N,i} = (T_N|_{I_N(\varepsilon_1)})^{-1} \). Let

\[
u_{N,\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n}(t) := (u_{N,\varepsilon_1} \circ u_{N,\varepsilon_2} \circ \cdots \circ u_{N,\varepsilon_n})(t) = \frac{N}{\varepsilon_1 + \frac{N}{\varepsilon_2 + \cdots + \frac{N}{\varepsilon_n + t}}}.
\]  

(3.28)

We observe that \( u_{N,\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n} = (T_N|_{I_N(\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n)})^{-1} \). Therefore

\[
I_N(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = \{ u_{N,\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n}(t) : t \in [0, 1] \}
\]  

(3.29)
which is an interval with the endpoints \( \frac{p_{N,n}}{q_{N,n}} \) and \( \frac{p_{N,n} + q_{N,n-1}}{q_{N,n} + t q_{N,n-1}} \) and which form a partition of \([0, 1]\).

We now check Rényi condition (1.6). We will use directly, without mentioning them here, some properties proved in [7]. Thus, we have

\[
\left| \frac{u'_{N,\varepsilon_1\ldots\varepsilon_n}(t)}{u'_{N,\varepsilon_1\ldots\varepsilon_n}(r)} \right| = \left( \frac{q_{N,n} + r q_{N,n-1}}{q_{N,n} + t q_{N,n-1}} \right)^2 \leq \left( \frac{q_{N,n} + q_{N,n-1}}{q_{N,n}} \right)^2 \leq \left( \frac{N+1}{N} \right)^2.
\] (3.30)

Using Rohlin’s entropy formula (1.5), we have:

\[
h(T_N) = \int_0^1 \log |T'_N(x)| \, dG_N(x) = \frac{1}{\log \frac{N+1}{N}} \int_0^1 \log \left( N^\frac{1}{x} \right) \, dx
= \frac{\pi^2}{3} + 2 \text{Li}_2(N+1) + \log(N+1) \log \frac{N+1}{N},
\] (3.31)

where \( \text{Li}_2 \) denotes the dilogarithm function, defined by

\[
\text{Li}_2(x) = \int_0^x \frac{\ln t}{1-t} \, dt \quad \text{or} \quad \text{Li}_2(x) = \sum_{k=1}^\infty \frac{x^k}{k^2}.
\] (3.32)

As examples, we have:

| \( N \) | \( h(T_N) \) |
|-----|-----|
| 1   | 2.37314 |
| 3   | 3.24705 |
| 5   | 3.70244 |
| 10  | 4.35074 |
| 50  | 5.92195 |
| 100 | 6.61015 |
| 1000| 8.90825 |

Table 3: Entropies \( h(T_N) \) for different values of \( N \)

### 3.4 Rényi-type continued fraction expansions

Fix an integer \( N \geq 2 \). Let the Rényi-type continued fraction transformation \( R_N : [0, 1] \to [0, 1] \) be given by

\[
R_N(x) := \begin{cases} 
\frac{N}{1-x} - \left\lfloor \frac{N}{1-x} \right\rfloor, & \text{if } x \in [0, 1), \\
0, & \text{if } x = 1.
\end{cases}
\] (3.33)

For any irrational \( x \in [0, 1] \), \( R_N \) generates a new continued fraction expansion of \( x \) of the form

\[
x = 1 - \frac{N}{1 + r_1 - \frac{N}{1 + r_2 - \frac{N}{1 + r_3 - \ddots}}} =: [r_1, r_2, r_3, \ldots]_R.
\] (3.34)
Here, \( r_n \)'s are non-negative integers greater than or equal to \( N \) defined by

\[
r_1 := r_1(x) = \left\lfloor \frac{N}{1-x} \right\rfloor, \quad x \neq 1; \quad r_1(1) = \infty
\]  
(3.35)

and

\[
r_n := r_n(x) = r_1 \left( R_N^{n-1}(x) \right), \quad n \geq 2,
\]  
(3.36)

with \( R_N^0(x) = x \). The sequence of rationals \( \{ p_{R,n}/q_{R,n} \} := [r_1, r_2, \ldots, r_n]_R, n \in \mathbb{N}_+ \) are the convergents to \( x \) in \([0, 1]\).

The dynamical system \( ([0, 1], \mathcal{B}_{[0, 1]}, R_N, \rho_N) \) is measure preserving and ergodic, where the probability measure \( \rho_N \) is defined by

\[
\rho_N(A) := \frac{1}{\log \left( \frac{N}{N-1} \right)} \int_A \frac{dx}{x + N - 1}, \quad A \in \mathcal{B}_{[0, 1]}.
\]  
(3.37)

An \( n \)-block \((r_1, r_2, \ldots, r_n)\) is said to be \emph{admissible} for the expansion in (3.34) if there exists \( x \in [0, 1) \) such that \( r_i(x) = r_i \) for all \( 1 \leq i \leq n \). If \((r_1, r_2, \ldots, r_n)\) is an admissible sequence, we call the set

\[
I_R(r_1, r_2, \ldots, r_n) = \{ x \in [0, 1] : r_1(x) = r_1, r_2(x) = r_2, \ldots, r_n(x) = r_n \},
\]  
(3.38)

the \( n \)-th order cylinder.

Define \((u_{R,i})_{i \geq N}\) by

\[
u_{R,i} : [0, 1] \rightarrow [0, 1]; \quad u_{R,i}(x) := 1 - \frac{N}{i + x},
\]  
(3.39)

For each \( i \geq N \), \( u_{R,i} = (R_N|_{I_R(r_1)})^{-1} \). Let

\[
u_{R,r_1r_2\ldots r_n}(t) := (u_{R,r_1} \circ u_{R,r_2} \circ \cdots \circ u_{R,r_n})(t) = 1 - \frac{N}{1 + r_1 - \frac{N}{1 + r_2 - \cdots - \frac{N}{r_n + t}}.}
\]  
(3.40)

We observe that \( u_{R,r_1r_2\ldots r_n} = (R_N|_{I_R(r_1, r_2, \ldots, r_n)})^{-1} \). Therefore

\[
I_R(r_1, r_2, \ldots, r_n) = \{ u_{R,r_1\ldots r_n}(t) : t \in [0, 1] \}
\]  
(3.41)

which is the interval \([p_{R,n}/q_{R,n-1}, p_{R,n}/q_{R,n}]\) and which form a partition of \([0, 1]\).

Before applying Rohlin’s formula, we must check Rényi condition (1.6). We will use directly, without mentioning them here, some properties proved in [8]. Thus, we have

\[
\frac{|u_{R,r_1\ldots r_n}'(t)|}{|u_{R,r_1\ldots r_n}'(r)|} = \left( \frac{q_{R,n} + (r-1)q_{R,n-1}}{q_{R,n} + (t-1)q_{R,n-1}} \right)^2 \leq \left( \frac{q_{R,n}}{q_{R,n} - q_{R,n-1}} \right)^2 \leq \left( \frac{N}{N-1} \right)^2.
\]  
(3.42)
The entropy \( h(R_N) \) is given by

\[
h(R_N) = \int_0^1 \log |R'_N(x)| \, d\rho_N(x) = \frac{1}{\log \frac{N}{N-1}} \int_0^1 \log \frac{N}{(1-x)^2} \, dx
\]

\[
= \log N + \frac{2\text{Li}_2 \left( \frac{1}{N} \right)}{\log \frac{N}{N-1}},
\]

(3.43)

where \( \text{Li}_2 \) is as in (3.32). As examples, we have:

| \( N \) | \( h(R_N) \) |
|---|---|
| 2 | 2.37314 |
| 3 | 2.905 |
| 5 | 3.50063 |
| 10 | 4.25052 |
| 50 | 5.90194 |
| 100 | 6.60015 |
| 1000 | 8.90726 |

Table 4: Entropies \( h(R_N) \) for different values of \( N \)

### 4 Comparing the efficiency of some expansions

In this section we apply the Extended Lochs’ theorem presented in Section 2.4 and compare two by two the expansions presented in the previous section. First of all, we observe that for various values of the parameters involved, the entropies \( h(T_\theta) \) and \( h(T_N) \) are equal. Since entropy is an isomorphism invariant, we conjecture the following result.

**Conjecture 4.1.** For an irrational \( \theta \in (0, 1) \) and a non-negative integer \( N \geq 2 \) with \( 1/\theta^2 = N \), the transformations \( T_\theta \) in (3.14) and \( T_N \) in (3.23) are isomorphic.

For this reason, we make only the following pairs: \( N \)--continued fractions and Chan’s continued fractions, \( N \)--continued fractions and Rényi-type continued fractions, Rényi-type continued fractions and Chan’s continued fractions.

We observe that the transformations \( T_N, T_\ell \) and \( R_N \) satisfy the two conditions from Extended Lochs’ theorem (see Section 2.4).

#### 4.1 \( N \)--continued fractions and Chan’s continued fractions

Let \( I_N^\varepsilon(x, \varepsilon_2, \ldots, \varepsilon_n) \) denotes the \( n \)--order cylinder of the \( N \)--continued fraction that contains \( x \), and \( I_\ell^\varepsilon(a_1, a_2, \ldots, a_m) \) denotes the \( m \)--order cylinder of the Chan’ continued fraction that contains \( x \). Then

\[
m_{N\ell}(n, x) := \sup \left\{ m : I_N^\varepsilon(\varepsilon_2, \ldots, \varepsilon_n) \subset I_\ell^\varepsilon(a_1, a_2, \ldots, a_m) \right\}
\]

(4.1)
represents the number of digits of the \( N \)-continued fraction in (3.25) that are determined by the first \( n \) digits in the Chan’ continued fraction in (3.1). Therefore, applying (2.15) we have

\[
\lim_{n \to \infty} \frac{m_N(n, x)}{n} = \frac{h(T_N)}{h(T_\ell)} \quad (4.2)
\]

where \( h(T_N) \) and \( h(T_\ell) \) are as in (3.31) and (3.13), respectively. Given the values in tables 1 and 3, we observe that \( N \)-continued fraction expansion is more effective than Chan’ continued fraction expansion regardless of the values taken by the parameters \( N \) and \( \ell \), respectively.

As examples, we have,

\[
\lim_{n \to \infty} \frac{m_{12}(n, x)}{n} = 1.462571953 \ldots \quad \text{or} \quad \lim_{n \to \infty} \frac{m_{32}(n, x)}{n} = 2.001164812 \ldots \quad (4.3)
\]

**4.2 \( N \)-continued fractions and Rényi-type continued fractions**

Let \( I_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n}(x) \) denotes the \( n \)-order cylinder of the \( N \)-continued fraction that contains \( x \), and \( I_{r_1, r_2, \ldots, r_m}(x) \) denotes the \( m \)-order cylinder of the Rényi-type continued fraction that contains \( x \). Then

\[
m_{NR}(n, x) := \sup \{ m : I_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n}(x) \subset I_{r_1, r_2, \ldots, r_m}(x) \} \quad (4.4)
\]

represents the number of digits of the \( N \)-continued fraction in (3.25) that are determined by the first \( n \) digits in the Rényi-type continued fraction in (3.34). Therefore, applying (2.15) we have

\[
\lim_{n \to \infty} \frac{m_{NR}(n, x)}{n} = \frac{h(T_N)}{h(R_N)} \quad (4.5)
\]

where \( h(T_N) \) and \( h(R_N) \) are as in (3.31) and (3.43), respectively. Given the values in tables 3 and 4, we observe that \( N \)-continued fraction expansion is more effective than Rényi-type continued fraction expansion regardless of the values taken by the parameter \( N \).

We notice that \( h(T_1) = h(R_2) \). We also have

\[
\lim_{n \to \infty} \frac{m_{33}(n, x)}{n} = 1.117745267 \ldots \quad \text{or} \quad \lim_{n \to \infty} \frac{m_{55}(n, x)}{n} = 1.057649623 \ldots \quad (4.6)
\]

As \( N \) grows, the entropies are very close, which means that the efficiency of the two continued fraction expansions are about the same.

**4.3 Rényi-type continued fractions and Chan’s continued fractions**

Let \( I_{r_1, r_2, \ldots, r_n}(x) \) denotes the \( n \)-order cylinder of the Rényi-type continued fraction that contains \( x \), and \( I_{a_1, a_2, \ldots, a_m}(x) \) denotes the \( m \)-order cylinder of the Chan’s continued fraction that contains \( x \). Then

\[
m_{Ra}(n, x) := \sup \{ m : I_{r_1, r_2, \ldots, r_n}(x) \subset I_{a_1, a_2, \ldots, a_m}(x) \} \quad (4.7)
\]
represents the number of digits of the Rényi-type continued fraction in (3.34) that are determined by the first \( n \) digits in the Chan’s continued fraction in (3.1). Therefore, applying (2.15) we have
\[
\lim_{n \to \infty} \frac{m_{R\ell}(n, x)}{n} = \frac{h(R_N)}{h(T_\ell)}
\] (4.8)
where \( h(R_N) \) and \( h(T_\ell) \) are as in (3.33) and (3.13), respectively. Given the values in tables 4 and 1, we observe that Rényi-type continued fraction expansion is more effective than Chan’s continued fraction expansion regardless of the values taken by the parameters \( N \) and \( \ell \), respectively.

As examples, we have,
\[
\lim_{n \to \infty} \frac{m_{22}(n, x)}{n} = 1.462571953 \ldots \quad \text{or} \quad \lim_{n \to \infty} \frac{m_{23}(n, x)}{n} = 1.871930586 \ldots
\] (4.9)

5 Final remarks

As we have already mentioned, even if we have studied only four expansions, RCFs are the most efficient at representing a number in the unit interval, with a very close efficiency being Rényi-type continued fractions. Thus, although RCFs are the oldest and the most studied, they remain the most useful tool in representing real numbers.

References

[1] R.P. Brent, E.M. McMillan, Some new algorithms for highprecision computation of Euler’s constant, Math. Comp. 34 (1980), 305–312

[2] K. Dajani, A. Fieldsteel, Equipartition of Interval Partitions and an Application to Number Theory, Proc. Amer. Math. Soc. 129(12) (2001), 3453–3460

[3] K. Dajani, C. Kraaikamp, Ergodic theory of numbers, Cambridge University Press, 2002

[4] A.N. Kolmogorov, A new metric invariant of transient dynamical systems and automorphisms in Lebesgue spaces, Doklady Akademii Nauk SSSR (N.S.) 119 (1958), 861–864.

[5] D. Lascu, On a Gauss-Kuzmin-type problem for a family of continued fraction expansions, J. Number Theory 133(7) (2013), 2153–2181.

[6] D. Lascu, Dependence with complete connections and the Gauss-Kuzmin theorem for \( N \)-continued fractions. J. Math. Anal. Appl. 444 (2016), 610–623.

[7] D. Lascu, Metric properties of \( N \)-continued fractions. Math. Reports 19(69), 2 (2017), 165–181.

[8] D. Lascu, G. I. Sebe, A dependence with complete connections approach to generalized Rényi continued fractions, Acta Math. Hungar. 160(2) (2020), 292–313
[9] D. Lascu, G. I. Sebe, *A Gauss-Kuzmin-Lévy theorem for Rényi-type continued fractions*, Acta Arith. **193**(3) (2020), 283–292

[10] G. Lochs, *Die ersten 968 Kettenbruchnenner von π*, Monatsh. Math. **67** (1963), 311–316

[11] G. Lochs, *Vergleich der Genauigkeit von Dezimalbruch und Kettenbruch*, Abh. Math. Sem. Hamburg **27** (1964), 142–144

[12] M. Pollicott, M. Yuri, Dynamical Systems and Ergodic Theory, Cambridge University Press, New York, 1998.

[13] V.A. Rohlin, *Exact endomorphisms of a Lebesgue space*, Amer. Math. Soc. Transl.II. Ser. **39** (1964), 1–36.

[14] G.I. Sebe, *Convergence rate for a continued fraction expansion related to Fibonacci type sequences*, Tokyo J. Math. **33**(2) (2010), 487–497.

[15] G.I. Sebe, *A near-optimal solution to the GaussKuzminLévy problem for θ-expansions*, J. Number Theory **171** (2017), 43–55.

[16] G.I. Sebe, D. Lascu, A Gauss-Kuzmin theorem and related questions for θ-expansions, Journal of Function Spaces, vol. **2014** (2014), 12 pages.

[17] G. I. Sebe, D. Lascu, *Convergence rate for Rényi-type continued fraction expansions*, Period. Math. Hung. (2020), https://doi.org/10.1007/s10998-020-00325-2

[18] G. I. Sebe, D. Lascu, *A two-dimensional GaussKuzmin theorem for N-continued fraction expansions*, Publ. Math. Debrecen **96**(3-4) (2020) http://publi.math.unideb.hu/contents.php

[19] C. Shannon, *A mathematical theory of communication*, Bell System Tech.J. **27** (1948), 379–423.

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