On the scaling properties of (2+1) directed polymers in the high temperature limit

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In this paper in terms of the replica method we consider the high temperature limit of (2+1) directed polymers in a random potential and propose an approach which allows to compute the scaling exponent \( \theta \) of the free energy fluctuations as well as the left tail of its probability distribution function. It is argued that \( \theta = 1/2 \) which is different from the zero-temperature numerical value which is close to 0.241. This result implies that unlike the (1 + 1) system in the two-dimensional case the free energy scaling exponent is non-universal being temperature dependent.

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I. INTRODUCTION

At present the statistical properties of one-dimensional directed polymers as well as the other systems belonging to the so called KPZ universality class \[1\] are sufficiently well studied (for the reviews see e.g. \[2–4\]). In contrast to that, not so much is achieved in the studies of the so called (2+1) model of directed polymers which describes the fluctuations of an elastic string directed along the time axes which passes through a random medium in the three-dimensional space. Due to extensive numerical simulations it is rather convincingly established that at the zero-temperature and in the limit of large times \( t \) the free energy fluctuations of such directed polymers scale as \( t^\theta \) with the scaling exponent \( \theta \simeq 0.241 \) \[5–7\]. The third digit here is still disputed but there is unanimous agreement that \( \theta \) is not equal to 1/4 (as every theoretician would hope). On the other hand, on the theoretical side, there is a rigorous proof that in the high temperature limit the disorder potential remains relevant \[8–10\]. In other words, there is effective localization of the polymers trajectories around disorder defined favorite “corridors” so that (like in the (1+1) system) even at high temperatures the polymers statistics do not reduce to simple diffusion (as it takes place e.g. in the high temperature phase of the (3+1) model\[11\]).

In this paper in terms of the replica technique we propose an approximate (mean field) method which in the high temperature limit allows to estimate the replica partition function in the limit of large number of replicas. This in turn, makes possible to derive the left tail asymptotics of the free energy fluctuations distribution function. Assuming that this (unknown) distribution function is defined by only one energy scale one eventually finds that the typical value of the disorder defined free energy fluctuations scale as \( \sim \sqrt{u} t^{1/2} \) (where \( u \) is the parameter which describes the strength of the disorder) which means that the time scaling exponent \( \theta = 1/2 \). This value is remarkably different from the zero temperature numerical results. If correct, this statement implies that unlike the (1 + 1) system (where it is rigorously proved that \( \theta = 1/3 \) both at \( T \to 0 \) and at \( T \to \infty \)) in the two-dimensional case the free energy scaling exponent is non-universal being temperature dependent. Note also that with \( \theta = 1/2 \) for the mean square fluctuations of the polymer trajectory \( \phi(\tau) \) \( (0 \leq \tau \leq t) \) simple energy balance arguments yields the scaling \( \bar{\phi}^2 \sim t^{\theta + 1} = t^{3/2} \).

It is also worth to note that comparing the features of the high temperature limits in (1+1) and (2+1) systems one finds one more interesting point. In (1+1) dimension there is the high temperature regime in which the effective temperature parameter is rescaled with time as \( \beta \to \beta t^{-1/4} \) and intermediate exponents are established \[12\] \[13\]. In particular, in the limit \( t \to \infty \) this scaling interpolates between the weak disorder \( (\beta = 0) \) and strong disorder regimes \( (\beta > 0) \). In a sense, this result shares some features of the problem considered in the present paper.

In Section II we define the model and describe the general ideas of the present approach. In Section III we present the systematic mean-field method (hopefully valid in the limit of large number of replicas) which eventually can be reduced to the solution of two-dimensional one-particle non-local (integral) non-linear differential equation \[23\]. In the high temperature limit this equation can be (numerically) solved providing the value of the exponent \( \theta = 1/2 \) (Section IV). Section V is devoted to the brief discussion of the obtained results.

II. THE MODEL AND THE REPLICAS APPROACH

We consider the model of directed polymers defined in terms of an elastic string described by the two-dimensional vector \( \phi(\tau) \equiv (\phi_x(\tau), \phi_y(\tau)) \) directed along the \( \tau \)-axes within an interval \([0, t]\) which passes through a random medium...
described by a random potential \( V(\phi, \tau) \). The energy of a given polymer’s trajectory \( \phi(\tau) \) is

\[
H[\phi(\tau); V] = \int_0^t d\tau \left\{ \frac{1}{2} [\partial_\tau \phi(\tau)]^2 + V[\phi(\tau), \tau] \right\};
\]

(1)

Here the disorder potential \( V[\phi, \tau] \) is supposed to be Gaussian distributed with a zero mean \( \overline{V(\phi, \tau)} = 0 \) and the correlation function

\[
\overline{V(\phi, \tau)V(\phi', \tau')} = u \delta(\tau - \tau')U(\phi - \phi')
\]

(2)

The parameter \( u \) is the strength of the disorder and \( U(\phi) \) is a smooth function characterized by the correlation length \( \epsilon \). For simplicity we take

\[
U(\phi) = \frac{1}{2\pi \epsilon^2} \exp\left\{ \frac{\phi^2}{2\epsilon^2} \right\}
\]

(3)

One-dimensional, or the so called (1+1) version of this problem (when instead of the vector we have a scalar field \( \phi(\tau) \)) with the \( \delta \)-correlated random potential has been the focus of intense studies during past three decades [14][33]. At present it is well established that the fluctuations of the free energy of this system are described by the Tracy-Widom (TW) distribution [34] and their typical value scale with time as \( t^{1/3} \).

The general formulation of the considered (2+1) problem in terms of the replica approach looks quite similar to the (1+1) one. For a given realization of the random potential \( V(\phi, \tau) \) the partition function of the considered system (with fixed boundary conditions) is

\[
Z(r, t) = \int_{\phi(0)=0}^{\phi(t)=r} D\phi(\tau) \exp\left\{ -\beta H[\phi(\tau), V] \right\} = \exp\left\{ -\beta F(r, t) \right\}
\]

(4)

where \( \beta \) is the inverse temperature, \( F(r, t) \) is the free energy which is a random quantity and the integration is taken over all trajectories \( \phi(\tau) \) starting at 0 (at \( \tau = 0 \)) and ending at a point \( r \) (at \( \tau = t \)). Note that this problem is equivalent to the KPZ equation [1]

\[
\partial_t F(r, t) = \frac{1}{2\beta} \nabla^2 F(r, t) - \frac{1}{2} \left( \nabla F(r, t) \right)^2 + V(r, t)
\]

(5)

which describe the time evolution of the two-dimensional manifold \( F(r, t) \) in a random potential \( V(r, t) \).

For simplicity, in what follows we are going to consider the problem with the zero boundary conditions: \( \phi(\tau = 0) = \phi(\tau = t) = 0 \). The free energy probability distribution function \( P(F) \) of this system can be studied in terms of the integer moments of the above partition function, eq.(4):

\[
\overline{Z^N} = Z(N, t) = \prod_{a=1}^N \int_{\phi_a(0)=0}^{\phi_a(t)=0} D\phi_a(\tau) \exp\left\{ -\beta \sum_{a=1}^N H[\phi_a(\tau), V] \right\} = \int_{-\infty}^{+\infty} dF P(F) \exp\{ -\beta NF \}
\]

(6)

where \( \overline{(...)} \) denotes the averaging over the random potentials \( V[\phi, \tau] \) Performing this simple Gaussian averaging we get

\[
Z(N, t) = \prod_{a=1}^N \int_{\phi_a(0)=0}^{\phi_a(t)=0} D\phi_a(\tau) \exp\left\{ -\beta H_N[\phi_1(\tau), \phi_2(\tau), ..., \phi_N(\tau)] \right\}
\]

(7)

where

\[
\beta H_N[\phi_1(\tau), \phi_2(\tau), ..., \phi_N(\tau)] = \int_0^t d\tau \left[ \frac{1}{2} \beta \sum_{a=1}^N \left( \partial_\tau \phi_a(\tau) \right)^2 - \frac{1}{2} \beta^2 u \sum_{a,b=1}^N U(\phi_a(\tau) - \phi_b(\tau)) \right]
\]

(8)

is the replica Hamiltonian which describes \( N \) elastic strings \( \{ \phi_1(\tau), \phi_2(\tau), ..., \phi_N(\tau) \} \) with the attractive interactions \( U(\phi_a - \phi_b) \), eq.[3]. To compute the replica partition function \( Z(N, t) \), eq.[7], one introduces the function:

\[
\Psi(r_1, r_2, ..., r_N; t) = \prod_{a=1}^N \int_{\phi_a(0)=0}^{\phi_a(t)=r_a} D\phi_a(\tau) \exp\left\{ -\beta H_N[\phi_1(\tau), \phi_2(\tau), ..., \phi_N(\tau)] \right\}
\]

(9)
such that

\[ Z(N, t) = \Psi(r_1, r_2, \ldots r_N; t) \bigg|_{r_a=0} \]  

Here the spatial arguments of this function are \( N \) two-dimensional vectors \( \{r_a\} \). One can easily show that \( \Psi(r_1, r_2, \ldots r_N; t) \) is the wave function of \( N \) quantum bosons defined by the imaginary time Schrödinger equation

\[ \beta \frac{\partial}{\partial t} \Psi(r_1, r_2, \ldots r_N; t) = \frac{1}{2} \sum_{a=1}^{N} \Delta_a \Psi(r_1, r_2, \ldots r_N; t) + \frac{1}{2} \beta^3 u \sum_{a,b=1}^{N} U(r_a - r_b) \Psi(r_1, r_2, \ldots r_N; t) \]  

where \( \Delta_a \) is the two-dimensional Laplacian with respect to the coordinate \( r_a \). The corresponding eigenvalue equation for the eigenfunctions \( \psi(r_1, r_2, \ldots r_N) \), defined by the relation

\[ \Psi(r_1, r_2, \ldots r_N; t) = \psi(r_1, r_2, \ldots r_N) \exp\left\{ -t E_N \right\} \]  

reads:

\[ -2\beta E_N \psi(r_1, r_2, \ldots r_N) = \sum_{a=1}^{N} \Delta_a \psi(r_1, r_2, \ldots r_N) + \beta^3 u \sum_{a,b=1}^{N} U(r_a - r_b) \psi(r_1, r_2, \ldots r_N) \]  

It is at this stage that we are facing the crucial difference of the considered problem with the corresponding \((1+1)\) one. The general solution of the one-dimensional counterpart of eq.\( (13) \) is given by the Bethe ansatz wave function which is valid only for \( U(x) = \delta(x) \) and which is based on the exact two-particle wave functions \((N=2)\) solution exhibiting finite value energy \( E_{N=2} \). It is this fundamental property of \((1+1)\) problem which eventually allows to derive the Tracy-Widom distribution for the free energy fluctuation.

The situation in the \((2+1)\) case, eq.\( (13) \), is much more complicated. First of all, in two dimensions there exists no finite two-particle solution for \( U(r) = \delta(r) \). One can easily construct an approximate ground state solution of the two-particle problem for the finite-size function \( U(r) \), eq.\( (3) \), but then one finds that in the limit \( \epsilon \to 0 \) (when \( U(r) \) turns into the \( \delta \)-function) the ground state energy of this solution \( E_{N=2} \to -\infty \). In other words, in two dimensions (unlike one-dimensional case) one can not consider the problem with \( \delta \)-correlated random potential. We have to study the system with finite size function \( U(r) \) and the value of its spatial size \( \epsilon \) must explicitly enter into the final results.

Second, one can easily demonstrate that in the two-dimensional case the construction of the \( N \)-particle wave function à la Bethe ansatz structure based on the approximate two-particle solution (for finite \( \epsilon \)) doesn’t work. So that, unlike one-dimensional case, here even the ground state energy \( E_N \) as well as \( N \)-particle ground state wave function \( \psi(r_1, r_2, \ldots r_N) \) of eq.\( (13) \) are not known. All that makes the perspective to find the exact solution of the \((2+1)\) problem rather doubtful.

Here we would like to propose somewhat different strategy which, at least in the high-temperature limit, makes possible to estimate the replica partition function \( Z(N, t) \) at \( N \gg 1 \) which in turn allows to derive the scaling exponent of the free energy fluctuations.

### III. MEAN FIELD APPROACH

First, to simplify notations, let us eliminate the parameter \( \epsilon \) of the correlation function \( U(r) \). Redefining

\[ r \to \epsilon r \]  

and

\[ E_N = \frac{1}{\epsilon^2} \bar{E}_N \]  

instead of eq.\( (13) \) we get

\[ -2\beta \bar{E}_N \psi(r_1, r_2, \ldots r_N) = \sum_{a=1}^{N} \Delta_a \psi(r_1, r_2, \ldots r_N) + \beta^3 u \sum_{a,b=1}^{N} U_0(r_a - r_b) \psi(r_1, r_2, \ldots r_N) \]  

where

\[ U_0(r) = \frac{1}{2\pi} \exp\left\{ -\frac{1}{2} \frac{r^2}{2} \right\} \]
It is evident that in a general case eq. (16) can not be solved. However in the limit of large number of particles, \( N \gg 1 \), one hopefully can use the standard trick of the mean field approximation, in which the \( N \)-particle wave function \( \psi(r_1, r_2, \ldots r_N) \) factorizes into the product of \( N \) one-particle functions, namely

\[
\psi(r_1, r_2, \ldots r_N) \simeq \prod_{a=1}^{N} \psi(r_a)
\]  

(18)

Substituting this into eq. (16), we obtain

\[
-2\beta \tilde{E}_N \prod_{a=1}^{N} \psi(r_a) = \sum_{a=1}^{N} \Delta_a \psi(r_a) \prod_{b \neq a}^{N} \psi(r_b) + \beta^3 u \sum_{a \neq b}^{N} U_0(r_a-r_b) \psi(r_a) \psi(r_b) \prod_{c \neq a, b}^{N} \psi(r_c) + \frac{1}{2\pi} \beta^3 u N \prod_{a=1}^{N} \psi(r_a)
\]  

(19)

Introducing notations

\[
\tilde{E}_N = -\frac{N}{4\beta} \lambda
\]  

(20)

\[
\beta^3 u N = \frac{1}{2} \kappa
\]  

(21)

and integrating eq. (19) over all \( r_1, r_2, \ldots r_N \) (taking into account that \( \int d^2 r \Delta \psi(r) = 0 \)) we get

\[
\kappa(N-1)C_0^{-2} \int d^2 r_1 \int d^2 r_2 U_0(r_1-r_2) \psi(r_1) \psi(r_2) = \lambda N - \frac{1}{2\pi} \kappa
\]  

(22)

where \( C_0 = \int d^2 r \psi(r) \). Now, integrating eq. (19) over \( r_2, \ldots, r_N \), using the above relation, eq. (22), redefining \( \psi(r) \rightarrow C_0 \psi(r) \) and neglecting terms of order \( N^{-1} \) we get the following non-linear mean-field equation for the one-particle function \( \psi(r) \):

\[
\Delta \psi(r) - \lambda \psi(r) + \kappa \psi(r) \int d^2 r' U_0(r-r') \psi(r') = 0
\]  

(23)

where

\[
\int d^2 r \psi(r) = 1
\]  

(24)

and the function \( U_0(r) \) is given in eq. (17).

Further strategy is the following. For given values of the parameters \( \lambda \) and \( \kappa \), eq. (20), we have to find smooth non-negative solution of eq. (23) such that \( \psi(r \rightarrow \infty) \rightarrow 0 \). Next, substituting this solution into the constraint (24) we can find \( \lambda \) as a function of \( \kappa \), which eventually gives us the dependence of the ground state energy, eqs. (15) and (20), on the replica parameter \( N \). First, let us demonstrate how this strategy works in the well studied one-dimensional case.

A. **The example of \((1 + 1)\) system**

The one-dimensional version of eqs. (23)-(24) reads

\[
\psi''(x) - \lambda \psi(x) + \kappa \psi(x) \int dx' U_1(x-x') \psi(x') = 0
\]  

(25)

where

\[
\int_{-\infty}^{+\infty} dx \psi(x) = 1
\]  

(26)

and

\[
U_1(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} x^2\right\}
\]  

(27)
Redefining
\[ \psi(x) = \frac{\lambda}{\kappa} \phi(\sqrt{\lambda}x) \] (28)
and denoting \( \sqrt{\lambda}x = z \), instead of eqs.(25)-(26) we get
\[ \phi''(z) - \phi(z) + \phi(z) \int dz' U_\lambda(z - z') \phi(z') = 0 \] (29)
where the function \( \phi(z) \) satisfy the constraint
\[ \frac{\sqrt{\lambda}}{\kappa} \int_{-\infty}^{+\infty} dz \phi(z) = 1 \] (30)
and
\[ U_\lambda(z) = \frac{1}{\sqrt{2\pi\lambda}} \exp\left\{ -\frac{1}{2\lambda} z^2 \right\} \] (31)
According to eq.(30),
\[ \lambda(\kappa) = I^{-2} \kappa^2 \] (32)
where
\[ I = \int_{-\infty}^{+\infty} dz \phi(z) \] (33)
According to eq.(32) in the high temperature limit both \( \kappa \propto \beta^3 uN \to 0 \) and \( \lambda \to 0 \). Thus, in this limit, according to eq.(31),
\[ \lim_{\beta \to 0} U_\lambda(z) \to \delta(z) \] (34)
so that eq.(29) reduces to
\[ \phi''(z) - \phi(z) + \phi^2(z) = 0 \] (35)
One can easily check (numerically) that this equation has an instanton-like solution with \( \phi(0) \approx 1.50 \), \( \phi'(0) = 0 \) and \( \phi(z \to \infty) \to 0 \) (see Fig.1) Substituting this solution into eq.(33) we find

**FIG. 1**: Instanton solution of eq.(35)
I = \int_{-\infty}^{+\infty} dz \, \phi(z) \simeq 6.00 \quad (36)

Thus, according to eqs. (20), (21), (32) and (36) we find

$$E_N \simeq -\frac{1}{36} \beta^3 u^2 N^3 \propto -N^3$$

(37)

This result, except for the numerical prefactor, perfectly fits with the exact value of ground state energy $-\frac{1}{24} \beta^3 u^2 N^3$ of the one-dimensional $N$-particle boson system (see e.g. [4]) and correspondingly provide the well known value of the free energy scaling exponent $\theta = 1/3$.

**B. (2 + 1) directed polymers**

The situation in the (2 + 1) case is more complicated. As there are no reasons to expect that the ground state solution of the original Schrödinger equation (13) is anisotropic, in what follows it will be assumed that the function $\psi(r)$ is radially symmetric: $\psi(r) = \psi(|r|) \equiv \psi(r)$. In this case equations (23)-(24) take the form

$$\psi''(r) + \frac{1}{r} \psi'(r) - \lambda \psi(r) + \kappa \psi(r) \int d^2 r' U_0(|r - r'|) \psi(r') = 0$$

(38)

$$2\pi \int_0^{+\infty} dr \, r \psi(r) = 1$$

(39)

Redefining

$$\psi(r) = \frac{\lambda}{\kappa} \phi(\sqrt{\lambda} r)$$

(40)

and denoting $\sqrt{\lambda} r = z$, instead of eqs. (38)-(39) we get

$$\phi''(z) + \frac{1}{z} \phi'(z) - \phi(z) + \phi(z) \int d^2 z' U_\lambda(|z - z'|) \phi(z') = 0$$

(41)

$$2\pi \int_0^{+\infty} dz \, z \phi(z) = \kappa$$

(42)

where

$$U_\lambda(|z|) = \frac{1}{2\pi \lambda} \exp\left\{ -\frac{1}{2\lambda} |z|^2 \right\}$$

(43)

The main difference with the one-dimensional case is that now in the limit $\lambda \to 0$ the value of $\kappa$, eq. (42) remains finite. Indeed, according to eq. (43),

$$\lim_{\lambda \to 0} U_\lambda(|z|) = \delta(z)$$

(44)

In this case eq. (41) reduces to

$$\phi''(z) + \frac{1}{z} \phi'(z) - \phi(z) + \phi^2(z) = 0$$

(45)

This equation has an instanton-like solution with $\phi(0) \simeq 2.39$, $\phi'(0) = 0$ and $\phi(z \to \infty) \to 0$ (see Fig.2). Substituting this solution into eq. (42) we find that at $\lambda = 0$

$$\kappa(\lambda = 0) \equiv \kappa_0 \simeq 31.00$$

(46)

At non-zero $\lambda \ll 1$, for $\kappa > \kappa_0$ numerical solution of eqs. (41)-(42) demonstrate perfect linear dependence (see Fig.3)

$$\lambda(\kappa) = \gamma (\kappa - \kappa_0)$$

(47)

with

$$\gamma \simeq 0.050$$

(48)

On the other hand, in the region $\kappa < \kappa_0$ by numerical methods we find no non-negative solutions of equations (41)-(42) (such that $\psi(r \to \infty) \to 0$) for any value of $\lambda$. 
Substituting eqs (15) and (20) into eq.(47) for the mean-field ground state energy of considered two-dimensional $N$-particle boson system we find

$$E_N \simeq -\frac{\gamma}{4\beta\epsilon^2} N \left(2\beta^3 u N - \kappa_0\right)$$

(49)

where $\epsilon$ is the correlation size of the random potential, eq.(3), and the numerical (approximate) values of the factors $\gamma$ and $\kappa_0$ are given in eqs.(46) and (48). Note that the above result is valid only for

$$N > N_\ast \equiv \frac{\kappa_0}{2\beta^3 u} \gg 1$$

(50)

Thus, according to eqs.(49), (10) and (12), in the limit $t \to \infty$ for the replica partition function we find the following estimate

$$Z(N,t) \sim \exp\left\{\frac{\gamma}{4\beta\epsilon^2} N \left(2\beta^3 u N - \kappa_0\right) t\right\}$$

(51)
Correspondingly, we see that in the high-temperature limit and at large \( N \) eq. (6) reads
\[
\int_{-\infty}^{+\infty} dF P(F) \exp\{-\beta NF\} \sim \exp\left\{ \frac{\gamma u}{2\epsilon^2} (\beta N)^2 t - \frac{\gamma \kappa_0}{4\beta^2 \epsilon^2} \beta N t \right\}
\]
(52)

As in the exponential in r.h.s of the above relation there is linear in \( N \) term the total free energy \( F \) can be redefined such that it splits into two independent parts: \( F = F + \tilde{F} \), where \( \tilde{F} = \gamma \kappa_0 t / (4\beta^2 \epsilon^2) \) is an extensive non-random (selfaveraging) part while \( \tilde{F} \) is the fluctuating contribution described by a distribution function \( \tilde{P}(\tilde{F}) \) which according to eq. (52) is defined by the relation
\[
\int_{-\infty}^{+\infty} d\tilde{F} \tilde{P}(\tilde{F}) \exp\{-\beta N \tilde{F}\} \sim \exp\left\{ \frac{\gamma u}{2\epsilon^2} t (\beta N)^2 \right\}
\]
(53)

In the case the above relation would be valid for any \( N \) we would find that \( \tilde{P}(\tilde{F}) \) is just simple Gaussian distribution function. In fact, as eq. (53) is valid only for \( N > N_* \gg 1 \), eq. (50), it gives us only the left tail of this distribution:
\[
\tilde{P}(\tilde{F} \to -\infty) \sim \exp\left\{ -\frac{\epsilon^2}{2\gamma u} \tilde{F}^2 \right\}
\]
(54)

Note that this asymptotics sets in at \( |\tilde{F}| \gg (\beta N_*) \gamma u t / \epsilon^2 \sim t / (\beta^2 \epsilon^2) \).

It would be natural to suppose that the entire (unknown) probability distribution \( \tilde{P}(\tilde{F}) \) is a universal function in a sense that it is defined by the only energy scale of the free energy fluctuations. It is supposed that this form holds for all scales of \( \tilde{F} \) including far tails where \( |\tilde{F}| \gg F_0(t) \). If the above hypothesis is correct then using the explicit expression for far left tail of the distribution function \( \tilde{P}(\tilde{F}) \), eq. (54), one immediately finds that in the considered high-temperature limit the typical value of the free energy fluctuations scale as
\[
\tilde{F} \sim F_0(t) = \frac{\sqrt{\gamma u}}{\epsilon} t^{1/2}
\]
(55)

where \( \gamma \simeq 0.05 \), eq. (48), and \( u \) and \( \epsilon \) are the strength and the correlation length of the random potential, eqs. (2)-(3).

The above eqs. (54)-(55) constitute the main results of the present research.

V. CONCLUSIONS

Regardless of their somewhat "trivial" form, the results presented in this paper, eqs. (54) and (55), imply several rather non-trivial conclusions.

First of all, it should be noted that the prefactor in the time scaling of the free energy fluctuations, eq. (55), is defined by the parameters of the disorder potential. Thus the estimate, eq. (55), is in a full agreement with the rigorous proof \[8–10\] which states that even in the high temperature limit the statistical properties of (2+1) directed polymers are defined by the random potential.

Second, the fact that in the high temperature limit the free energy time scaling exponent \( \theta = 1/2 \) is different from the one at the zero temperature (which is close to 0.241) means that in two dimensions this scaling exponent is temperature dependent. Maybe this takes place because the dimension \( d = 2 \) is critical for \( (d+1) \) directed polymers.

Third, it is also interesting to note that in the high-temperature limit the prefactor in time scaling of the free energy fluctuations, eq. (55), turns out to be temperature independent (unlike the one-dimensional case, where it is proportional to \( \beta^{\beta/3} \)).

It should be stressed however, that all the results presented in this paper are based on two crucial assumptions. The first one is pure heuristic mean-field ansatz, eq. (18). It looks quite reasonable in the limit of large number of replicas (which according to eq. (50) corresponds to the high-temperature limit). Moreover, it works very well in the one-dimensional case (see Section III.A). The second one is the hypothesis that in the considered system the entire probability distribution function of the free energy fluctuations \( \tilde{P}(\tilde{F}) \) reduces to a universal function: \( \tilde{P}(\tilde{F}) = G(\tilde{F}/F_0(t)) \) and this is valid for all energy scales including far tails where \( |\tilde{F}| \gg F_0(t) \). For the moment the only support for this hypothesis is that usually in physical systems exhibiting scaling phenomena it is correct. In particular, it is certainly correct for (1+1) polymers where the exact solution provide us with the universal function \( G(x) \) which is the Tracy-Widom distribution function. On the other hand, although both assumptions looks quite reasonable this does not guarantee that they are correct... In view of that, further analytic studies as well as numerical simulations of the considered system at finite (or high) temperatures would be extremely helpful.
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[1] Kardar M, Parisi G, Zhang Y-C, Phys. Rev. Lett. 56, 889 (1986)
[2] Corwin I, Random Matrices: Theory Appl. 1, 1130001 (2012)
[3] Borodin A, Corwin I and Ferrari P, Comm. Pure Appl. Math. 67, 1129–1214 (2014)
[4] V. Dotsenko Statistical properties of one-dimensional directed polymers in a random potential, arXiv:1703.04305 (2017)
[5] Halpin-Healy T, Phys. Rev. Lett. 109, 170602 (2012)
[6] Halpin-Healy T, Phys. Rev. E 109, 042118 (2013)
[7] Halpin-Healy T and Pulasantas G, EPL, 105, 50001 (2013)
[8] Lacoin H, Commun. Math. Phys. 294 471 (2020)
[9] Berger Q and Lacoin H, Ann. Inst. Henri Poincaré: Probab. Stat. 53 430 (2017)
[10] Comets F, Lecture notes from the 46th Probability Summer School, Saint-Flour, 2016, Springer International Publishing (2017)
[11] Bolthausen E, Commun. Math. Phys. 123 529 (1989)
[12] Alberts T, Khanin K and Quastel J, Phys. Rev. Lett. 105 090603 (2010)
[13] Alberts T, Khanin K and Quastel J, Ann. Probab. 43 1212 (2014)
[14] Huse D A, Henley C L and Fisher D S, Phys. Rev. Lett. 55, 2924 (1985)
[15] Huse D A and Henley C L, Phys. Rev. Lett. 54, 2708 (1985)
[16] Kardar M and Zhang Y-C, Phys. Rev. Lett. 58, 2087 (1987)
[17] Kardar M, Nucl. Phys. B 290, 582 (1987)
[18] Bouchaud J P and Orland H, J. Stat. Phys. 61, 877 (1990)
[19] Halpin-Healy T and Zhang Y-C, Phys. Rep. 254, 215 (1995)
[20] Johansson K, Comm. Math. Phys. 209, 437 (2000)
[21] Prahofer M and Spohn H, J. Stat. Phys. 108, 1071 (2002)
[22] Ferrari P L and Spohn H, Comm. Math. Phys. 265, 1 (2006)
[23] Sasamoto T and Spohn H, Phys. Rev. Lett. 104, 230602 (2010)
[24] Sasamoto T and Spohn H, Nucl. Phys. B834, 523 (2010)
[25] Sasamoto T and Spohn H, J. Stat. Phys. 140, 209 (2010)
[26] Amir G, Corwin I and Quastel J, Comm. Pure Appl. Math. 64, 466 (2011)
[27] Calabrese P, Le Doussal P and Rosso A, EPL, 90, 20002 (2010)
[28] Dotsenko V, EPL, 90, 20003 (2010)
[29] Dotsenko V, J.Stat.Mech. P07010 (2010)
[30] Calabrese P and Le Doussal P, Phys. Rev. Lett. 106, 250603 (2011)
[31] Dotsenko V, J. Stat. Mech. P11014 (2012)
[32] Dotsenko V, J. Stat. Mech., P02012 (2013)
[33] Gueudré T and Le Doussal P, EPL, 100, 26006 (2012)
[34] Tracy C A and Widom H, Commun.Math.Phys., 159, 151 (1994)