“Dynamical” interactions and gauge invariance

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Abstract

Appreciating the classical understanding of the elementary particle the “dynamical” Poincaré algebra is developed. It is shown that the “dynamical” Poincaré algebra and the equations of motion of particles with arbitrary spin are gauge invariant and that gauge invariance and relativistic invariance stand on equal footings. A “dynamical” non-minimal interaction is constructed explicitly and the Rarita–Schwinger equation is considered in the framework of this “dynamical” interaction.
1 Introduction

Understanding the higher-spin interactions is a longstanding problem. However, in spite of its 70 years history, the main goal – the construction of a consistent higher-spin theory, even for the electromagnetic interaction, which ought to be the simplest case – has not been achieved yet.

The investigations of higher-spin fields started in the thirties of the last century with papers by Dirac [2], Wigner [3], Fierz and Pauli [4] and followed by the works of Rarita and Schwinger [5], Bargmann and Wigner [6], and others [7, 8, 9, 10, 11, 12, 13]. The difficulties in higher spin physics revealed themselves when one tried to couple higher-spin fields to an electromagnetic field. In the 1960ies concrete defects of the higher-spin interaction theory were found. Johnson and Sudarshan [14] and Schwinger [15] demonstrated that in the case of minimal electromagnetic coupling some of the anticommutation relations become indefinite. It appeared that the defects were also present on the classical level. Velo and Zwanziger [16] and Shamaly and Capri [17] showed that in an external electromagnetic field there appeared acausal (superluminal) modes of propagation. Afterwards other defects – bad high-energy behaviour of the amplitudes, various algebraic problems etc. – were found. Since the sixties of the last century much work was done to solve the problems, but no result which one can call a breakthrough has been obtained in the framework of ordinary field theory. In case of higher-spin electromagnetic interactions investigations of the last two decades have moved in two directions. One part of community develops the theory on the ground of the minimal electromagnetic coupling, the other part searches for a consistent theory by using non-minimal couplings.

The theory of higher spin interactions has never belonged to the “mainstream” theories. The field has been cultivated by groups of enthusiasts. On the other hand, the theory of higher spin interactions is needed for solving many mainstream problems. It is related to the Standard Model (SM) in several ways. By introducing the massive spin-one gauge bosons into the theory one also introduces the higher-spins problems into the Standard Model. Difficulties appear for instance in scattering processes with the charged gauge bosons $W^\pm$ in the initial or final state, or in constructing three-vertex gauge boson self-interactions. A consistent higher-spin interaction theory is also needed in chromodynamics. Quantum Chromodynamics (QCD) does not yet allow to describe low-energy hadronic processes in terms of underlying quark-gluon dynamics. Due to this one has to use a more

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1In this paper states with spin one and higher are considered as higher-spin states. This concept is not universally accepted. For a part of the investigators “higher spin” means $s \geq 3/2$. The specialists in supergravity updated the convention of the higher spin to be even $s \geq 5/2$ [1]. Nevertheless, at least in the Standard Model the troubles start already from the value $s = 1$. Therefore it seems that the convention $s \geq 1$ as the higher-spin region is more justified than the other ones.

2Earnestly, as shown by Cox [18] the constraint analysis leading to these acausal pathologies is incomplete. On the contrary, in the complete constraint analysis a new tier of constraints occurs for the critical external field values, reducing the pathology to the field-induced change of the degrees of freedom. Because of these field-dependent constraints the analysis of acausal models is very complicated.
phenomenological approach in terms of hadronic fields. However, one of the basic problems here is the treatment of hadrons with higher spins [19].

To understand better the problems of modern theories beyond the SM one also needs a better understanding of ordinary higher-spin field theory. String theory for instance is free of many higher-spin problems and due to this it is believed that it can consistently describe quantum gravity. A reason behind this consistent behaviour is that string theories contain an infinite tower of all spin states. But at the same time there exist serious troubles in the physical interpretation of the string theories. The existence of a consistent higher-spin interaction theory would help to understand better the physics behind the string theory. It is believed that if a breakthrough in understanding the basic problems of the ordinary higher-spin field theory would happen, it might become a fashionable topic [21].

1.1 Problems of higher-spin interactions

The search for a consistent higher-spin interaction theory has been faced with various difficulties. The theory of the relativistic wave equations is based on the representations of the Poincaré group which in the field theory are somewhat specific in their mathematical realization. In addition, the theory of higher-spin fields is altogether rather complicated and due to this the wave functions and Lagrangians proposed have not always been correct. Therefore, it is difficult to understand whether the problems are of technical kind or pertained to the principles. As a matter of fact, the difficulties in higher-spin physics are generic to all field theoretical descriptions of relativistic higher-spin particles. The difficulties are related to the fact that covariant higher-spin field has more components than it is necessary to describe the spin degrees of freedom of the physical particle. To get rid of redundant degrees of freedom one must set up constraints between the field components. Using the language of Lagrangians, one have to construct a free Lagrangian which in addition to the Dirac and Proca type higher-spin equations would yield also constraint equations that reduce the number of degrees of freedom to the physical values. The problem is related to the introduction of interactions. If the interactions are introduced consistently with the free field theory, the number of independent field components remains unchanged. Otherwise the free theory constraints may be violated and unphysical degrees of freedom become involved.

In order to put constraints on the field components it is reasonable to use the symmetry framework. To reduce the number of degrees of freedom of the free field to a physical value certain symmetries have to be imposed in formulating the action. Any higher-spin action has to be invariant under a transformation which leaves only the physical value of $2s + 1$ degrees of freedom. Because of this, the interacting theory has to obey similar symmetry requirements as the corresponding free theory or, even better, preserves the gauge symmetries of the free theory. The possibility to construct consistent higher-spin theories with gauge invariant couplings was first pointed out by Weinberg and Witten [20]. However, the realization of this scenario is beset with difficulties. Even though certain
progress in understanding of a higher-spin interaction theory have been made \cite{21,22}, up to now no general prescription for the construction of a consistent higher-spin field theory for any spin has been found.

1.2 “Dynamical” representation of the Poincaré algebra

In this paper we used a higher-spin electromagnetic interaction theory developed by ourselves, based on the “dynamical” representation of the Poincaré algebra as a dynamical principle which leads to a non-minimal coupling. The “dynamical” representations are built up by introducing a plane electromagnetic field into the free Poincaré algebra. The representations are constructed from the generators of the free Poincaré algebra and the external field in such a way that the new, field-dependent generators obey the commutation relations of free Poincaré algebra. Introducing the interactions in this way preserves the Poincaré symmetry of the free theory and, hopefully, also the number of degrees of freedom of the free theory. The “dynamical” theory had achieved success in constructing causal spin-3/2 equations \cite{23} and for justifying the value of gyromagnetic ratio \( g = 2 \) for any spin \( 24 \).

The paper is organized as follows. In Section 2 we give an introduction to the Lorentz–Poincaré connection. In Section 3 we show how the Poincaré group \( \mathcal{P}_{1,3} \) can help to describe physical states. In Section 4 we show that an external field can be introduced consistently by employing a nonsingular transformation. Using this, in Section 5 we explicitly construct the “dynamical” interaction. Finally, in Section 6 we treat the Rarita–Schwinger equation in the framework of a “dynamical” interaction.

2 The Poincaré group

Relativistic field theories are based on the invariance under the Poincaré group \( \mathcal{P}_{1,3} \) (known also as inhomogeneous Lorentz group \( \mathcal{IL} \) \cite{3,25,26,27,28,29,33,34,35}). This group is obtained by combining Lorentz transformations \( \Lambda \) and space-time translations \( a_\mathcal{T} \),

\[
(a, \Lambda) \equiv a_\mathcal{T} \Lambda : \mathbb{E}_{1,3} \ni x^\mu \rightarrow \Lambda^\mu_\nu x^\nu + a^\mu.
\]  

The group’s composition law \((a_1, \Lambda_1)(a_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2)\) generates the semidirect structure of \( \mathcal{P}_{1,3} \),

\[
\mathcal{P}_{1,3} = \mathcal{T}_{1,3} \circ \mathcal{L}
\]

where \( \mathcal{T}_{1,3} \) is the abelian group of space-time translations (i.e. the additive group \( \mathbb{R}^4 \)) and \( \mathcal{L} = \{ \Lambda : \det \Lambda = +1, \Lambda^0_0 \geq 1 \} \) is the proper orthochronous Lorentz group acting on the Minkowski space \( \mathbb{E}_{1,3} \) with metric

\[
\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1).
\]

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The condition of the metric to be invariant under Lorentz transformations \( \Lambda \) takes the form
\[
\Lambda^{\mu \rho} \eta_{\mu \sigma} \Lambda^\sigma_{\nu} = \eta_{\mu \nu}.
\] (2)

### 2.1 Transformation of covariant functions

Under the Lorentz transformation \( \Lambda \in \mathcal{L} \) the covariant functions \( \psi \) transform according to a representation \( \tau(\Lambda) \) of the Lorentz group \([3, 10, 11, 12, 13, 25, 26, 27, 28, 29]\) where the diagram
\[
\begin{array}{ccc}
\psi : & x \in E_{1,3} & \longrightarrow \psi(x) \\
\tau(\Lambda) & \downarrow \Lambda & \downarrow T(\Lambda) \\
\tau(\Lambda) \psi : & \Lambda x & \longrightarrow (\tau(\Lambda) \psi)(\Lambda x) = T(\Lambda) \psi(x)
\end{array}
\]
is commutative, i.e.
\[
T(\Lambda) \psi(x) = (\tau(\Lambda) \psi)(\Lambda x) \equiv \psi'(x').
\] (3)

The map \( T : \Lambda \rightarrow T(\Lambda) \) is a finite-dimensional representation of \( \mathcal{L} \). If we parametrize the element \( \Lambda \in \mathcal{L} \) by \( \Lambda(\omega) = \exp \left( -\frac{1}{2} \omega_{\mu \nu} e^{\mu \nu} \right) \) where the Lorentz generators are given by
\[
(e_{\mu \nu})^\rho_\sigma = -\eta_{\mu \rho} \eta_{\nu \sigma} + \eta_{\mu \sigma} \eta_{\nu \rho}
\]
and \( \omega^{\mu \nu} = -\omega^{\nu \mu} \) are six independent parameters, the parametrization of \( T \) reads
\[
T(\Lambda(\omega)) = \exp \left( -\frac{i}{2} \omega_{\mu \nu} s^{\mu \nu} \right).
\]

The Lorentz group \( \mathcal{L} \) is non-compact. As a consequence, all unitary representations are infinite dimensional. In order to avoid this, we introduce the concept of \( H \)-unitarity (see e.g. Ref. \([29]\) and references therein). A finite representation \( T \) is called \( H \)-unitary if there exists a non-singular hermitian matrix \( H = H^\dagger \) so that
\[
T^\dagger(\Lambda) H = HT^{-1}(\Lambda) \Leftrightarrow s^\dagger_{\mu \nu} H = H s_{\mu \nu}.
\] (4)

Notice that a \( H \)-unitary metric is always indefinite, so that the inner product \( \langle \, , \, \rangle \) generated by \( H \) is sesquilinear sharing the hermiticity condition \( \langle \psi, \varphi \rangle = \langle \varphi, \psi \rangle^* \). The most famous case of \( H \)-unitarity is given in the Dirac theory of spin-1/2 particles where \( H = \gamma^0 \).

### 2.2 Transformation of operators

The transformation \( \tau(\Lambda) \) in Eq. (3) is a covariant transformation for the operator \( \mathcal{O} \) acting on the \( \psi \)-space of covariant functions\(^3\) if the diagram

\(^3\)We have to impose the action on covariant functions because in case of higher spins the relations between operators we obtain are valid only as weak conditions.
\[ \mathcal{O}\psi : \quad x \quad \rightarrow \quad (\mathcal{O}\psi)(x) \]
\[ \tau(\Lambda) \downarrow \quad \downarrow \quad \Lambda \quad \downarrow \quad T(\Lambda) \]
\[ \tau(\Lambda)(\mathcal{O}\psi) : \quad \Lambda x \quad \rightarrow \quad (\tau(\Lambda)(\mathcal{O}\psi))(\Lambda x) = T(\Lambda)(\mathcal{O}\psi)(x) \]

is commutative, i.e.

\[ (\tau(\Lambda)\mathcal{O})(\Lambda x)(\tau(\Lambda)\psi)(\Lambda x) = T(\Lambda)\mathcal{O}(x)\psi(x). \quad (5) \]

Using Eq. (3) we obtain

\[ (\tau(\Lambda)\mathcal{O})(\Lambda x)T(\Lambda)\psi(x) = T(\Lambda)\mathcal{O}(x)\psi(x). \]

Notice that the covariance of the transformation embodies only the property of equivalence of reference systems. The covariant operator \( \mathcal{O} \) is invariant under the transformation (3) if in addition \( \tau(\Lambda)\mathcal{O} = \mathcal{O} \). As a consequence we obtain

\[ \mathcal{O}(\Lambda x)T(\Lambda)\psi(x) = T(\Lambda)\mathcal{O}(x)\psi(x) \quad (6) \]
or \( \mathcal{O}(\Lambda x)T(\Lambda) = T(\Lambda)\mathcal{O}(x) \) on the \( \psi \)-space. The invariance means the symmetry of the physical system and implies the conservation of currents. In particular, the symmetry transformations leave the equations of motion form-invariant.

### 2.3 The Lie algebra

While the Lorentz transformation \( T(\Lambda) \) changes the wave function \( \psi \) itself as well as the argument of this function (cf. Eq. (3)), the proper Lorentz transformation \( \tau(\Lambda) \) causes a change of the wave function only. On the ground of infinitesimal transformations, this change is performed by the substancial variation. Starting from an arbitrary infinitesimal coordinate transformation \( \Lambda(\delta\omega) : x^\mu \rightarrow x^\mu + \delta\omega^{\mu \nu}x^\nu \), the substancial variation is given by Ref. [10]

\[ \delta_0\psi(x) \equiv \psi'(x) - \psi(x) = -\frac{i}{2}\delta\omega^{\rho\sigma}M_{\rho\sigma}\psi(x) \]

where \( M_{\rho\sigma} = \ell_{\rho\sigma} + s_{\rho\sigma}, \ell_{\rho\sigma} = i(x_\rho \partial_\sigma - x_\sigma \partial_\rho) \). The corresponding finite proper Lorentz transformation can be written as

\[ \tau(\Lambda(\omega)) = \exp \left( -\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} \right), \]

and the multiplicative structure of the group generates the adjoint action

\[ \text{Ad}_{\tau(\Lambda)} : M_{\mu\nu} \rightarrow \tau^{-1}(\Lambda)M_{\mu\nu}\tau(\Lambda) = \Lambda_\mu^\rho \Lambda_\nu^\sigma M_{\rho\sigma}. \quad (7) \]

Due to Eq. (4) the generators \( s_{\rho\sigma} \) fulfill \( s_{\rho\sigma}^\dagger H = HS_{\rho\sigma} \). They depend on the spin of the field but not on the coordinates \( x_\mu \). Therefore, we have \( [\ell_{\mu\nu}, s_{\rho\sigma}] = 0 \). If a generic element of the translation group is written as

\[ \exp(+i\alpha_{\mu}P^{\mu}), \]
the commutator relations of the Lie algebra are given by

\begin{align*}
[M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}), \\
[M_{\mu\nu}, P_\rho] &= i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu), \\
[P_\mu, P_\nu] &= 0.
\end{align*}

(8)

The Casimir operators of the algebra are \( P^2 = P_\mu P^\mu \) and \( W^2 = W_\mu W^\mu \) where

\[ W^\mu = \frac{1}{2} \epsilon^{\mu\rho\sigma} M_{\nu\rho} P_\sigma \]

is the Pauli-Lubanski pseudovector, \([P_\mu, W_\nu] = 0\). In coordinate representation we have \( P_\mu = i\partial_\mu \), and the finite Poincaré transformation has the form

\[ \tau(a, \Lambda) : \psi(x) \rightarrow (\tau(a, \Lambda)\psi)(x) = T(\Lambda)\psi\left(\Lambda^{-1}(x-a)\right). \]

(9)

This relation constitutes the Lorentz–Poincaré connection [34]. While the representation \( T \) generally generates a reducible representation of \( \mathcal{P}_{1,3} \), the spectra of the Casimir operators \( P^2 \) and \( W^2 \) determine the mass and spin content of the system.

3 Physical states

The natural identification of elementary particle systems is the direct geometric transition from space-time to the system under consideration. In fact, the very definition and characterization of distinct species of elementary particles are provided by the set of inequivalent irreducible projective unitary representations of the space-time symmetry group \( \mathcal{P}_{1,3} \), the Poincaré group. The physical limitations of this identification are found in the description of interacting systems and in the description of internal quantum numbers in composite systems. Space-time symmetry alone does not account for all the characteristics of today’s elementary particles.

According to the conventional understanding of the particle, its physical states of definite mass and spin, labelled by the moment \( p^\mu \) and the helicity \( \lambda \), arise from the irreducible representation of this symmetry group. The irreducible unitary representations of the Poincaré group are characterized by the eigenvalues of the two Casimir operators \( P^2 \) and \( W^2 \) of the Lie algebra \( p_{1,3} \),

\[ P^2|m, s\rangle = m^2|m, s\rangle, \quad W^2|m, s\rangle = -m^2s(s + 1)|m, s\rangle. \]

(10)

The Pauli-Lubanski pseudovector \( W \) has the properties

\[ W^\mu P_\mu = 0, \]

\[ [W^\mu, P^\nu] = 0, \]
\[
[W^\mu, M^{\rho\sigma}] = i(\eta^{\mu\rho} W^\sigma - \eta^{\mu\sigma} W^\rho),
\]
\[
[W^\mu, W^\nu] = -i\epsilon^{\mu\nu\rho\sigma} W_\rho P_\sigma,
\]
\[
W^2 = -\frac{1}{2} M_{\mu\nu} M^{\mu\nu} P^2 + (M_{\mu\rho} P^\rho)(M^{\mu\sigma} P_\sigma).
\] (11)

The independent components of \( W^\mu \) form the Lie algebra of the little group of fixed momentum \( p^\mu \). For every irreducible unitary representation of the little group one can derive a corresponding irreducible induced representation of the Poincaré group labeled by \((m, s)\), i.e. by the eigenvalues of the Casimir operators in Eq. (10). Notice that the procedure of deriving induced representations [29, 36] corresponds very well to the physical idea of first determining the internal degrees of freedom (the helicity) of the system and then all its possible states of motion.

As a matter of fact, all physical variables (like position, momentum, etc.), quantum wave functions and fields transform as finite-dimensional representations of the Lorentz group. The reason is that interactions between fundamental particles (as irreducible representations of the Poincaré group) are most conveniently formulated in terms of field operators (i.e., finite-dimensional representations of the Lorentz group) if the general requirements like covariance, causality, etc. are to be incorporated in a consistent way. The relation between these two groups and their representations is given by the Lorentz-Poincaré connection [34].

As a rule this connection is realized by the relativistic wave equations. If the relativistic wave equation transforms as a finite-dimensional representation of the Lorentz group by Eq. (3), it contains spins exceeding the desired physical spins. In order that the solutions of the field equation correspond to a particle with a definite spin, the equation must act like a projection operator to pick out the desired spin components, i.e. to select the corresponding irreducible representation of the Poincaré group.

### 3.1 The wave function

The wave functions we will consider have the form

\[
\mathcal{D}(x, \psi, \partial) \equiv (\beta^\mu \partial_\mu + i\rho)\psi(x) = 0
\] (12)

where \( \psi \) is an \( N \)-component function, \( \beta^\mu \ (\mu = 0, 1, 2, 3) \), and \( \rho \) are \( N \times N \) matrices independent of \( x \). Following Bhabha’s conception [29], it is “... logical to assume that the fundamental equations of the elementary particles must be first-order equations of the form (12) and that all properties of the particles must be derivable from these without the use of any further subsidiary conditions.”

The principle of relativity states that a change of the reference frame cannot have implications for the motion of the system. This means that Eq. (12) is invariant under Lorentz transformations. Equivalently, the Lorentz symmetry of the system means the
covariance and form-invariance of Eq. (12) under the transformation in Eq. (3), i.e. the transformed wave equation is equivalent to the old one. Therefore, we require that every solution $\psi'(x')$ of the transformed equation

$$D'(x', \psi', \partial') = 0$$

can be obtained as Lorentz transformation of the solution $\psi(x)$ of Eq. (12) in the original system and that the solutions in the original and transformed systems are in one-to-one correspondence. The explicit form of the covariance follows from Eq. (5),

$$\tau(\Lambda)D(\Lambda x, \tau(\Lambda)\psi, \Lambda \partial) = D'(x', \psi', \partial') = T(\Lambda)D(x, \psi, \partial) = 0.$$  \hspace{1cm} (13)

Due to the linearity in $\psi$ one may write

$$D(x, \psi, \partial) \equiv D(\partial)\psi(x) = 0$$

to obtain the explicit Lorentz transformations

$$\beta'^\mu = \Lambda^\mu_\rho T(\Lambda)\beta'^\rho T^{-1}(\Lambda),$$
$$\rho' = T(\Lambda)\rho T^{-1}(\Lambda).$$

The Lorentz invariance is given by the substitution

$$D(\partial)\psi(x) = 0 \overset{\text{Eq. (3)}}{\Rightarrow} D(\partial')\psi'(x) = 0.$$  \hspace{1cm} (14)

An excellent discussion of such matrices $\beta$ can be found in Refs. [37, 7, 8, 10, 38, 39].

The hermiticity of the representation $T$ in Eq. (4) implies the hermiticity of Eq. (12).

Including a still unspecified hermitian matrix $H$ the hermiticity condition reads $D(\partial)^\dagger H = (D(\partial)H)^\dagger = -HD(-\partial)$ or

$$\beta^{\mu\dagger}H = H\beta^\mu, \quad \rho H = H\rho.$$  \hspace{1cm} (15)

Writing $\bar{\psi} = \psi^\dagger H$, one obtains the adjoint equation

$$\bar{\psi}D(-\bar{\partial}) = \bar{\psi}(-\beta^\mu \bar{\partial}_\mu + i\rho) = -(HD(\partial)\psi)^\dagger = 0.$$  \hspace{1cm} (16)
3.2 Conserved currents

The physical meaning of hermiticity is the particle-antiparticle symmetry and the conservation laws in elementary particle processes. The technique of Takahashi and Umezawa \[40\] for deriving conservation laws directly from equations is based upon the differential operator $\Gamma\mu(\bar{\partial})$ with $[\partial\mu, \Gamma\nu] = 0$ and

$$ (\bar{\partial}_\mu + \tilde{\partial}_\mu)\Gamma^\mu(\bar{\partial}) = \mathcal{D}(\bar{\partial}) - \mathcal{D}(\tilde{\partial}). $$

(17)

Since the general form of a linear differential equation can be written as

$$ \mathcal{D}(\bar{\partial}) = \sum_{j=0}^{N} \beta_{\mu_1...\mu_j} \partial^{\mu_1}...\partial^{\mu_j} $$

with the coefficients $\beta_{\mu_1...\mu_j}$ symmetric in all indices for $j > 1$, the operators $\Gamma\mu$ become

$$ \Gamma\mu(\bar{\partial}) = \beta_\mu + \beta_{\mu\nu}(\bar{\partial}^\nu - \tilde{\partial}^\nu) + \beta_{\mu\nu\rho}(\bar{\partial}^\nu \partial^\rho - \tilde{\partial}^\nu \partial^\rho + \tilde{\partial}^\nu \tilde{\partial}^\rho) +... $$

(18)

Thanks to identity (17) one can derive a general current continuity equation caused by a continuous infinitesimal transformation

$$ x \to x' = x + \delta x, \quad \psi \to \psi' = \psi + \delta \psi. $$

(19)

If we find two functions $F_L[x]$ and $F_R[x]$ of $\psi$ and $\delta \psi$ and their derivatives with

$$ F_L \mathcal{D}(\bar{\partial}) F_R - F_L \mathcal{D}(\tilde{\partial}) F_R = \partial_\mu \Omega^\mu $$

(20)

with $\Omega \neq F_L \Gamma_\mu F_R$, the current

$$ j_\mu = F_L \Gamma_\mu(\bar{\partial}) F_R - \Omega_\mu $$

(21)

is conserved,

$$ \partial_\mu j^\mu = 0. $$

(22)

Notice that this algebraic technique is very useful in the application to the case of interacting fields and discrete symmetries.

4 Introduction of an external field

The identification of elementary particle systems and irreducible representations of the Poincaré group find its physical limitations in the description of interacting systems and internal quantum numbers of composite systems. Since gauge symmetry is a fundamental concept in Quantum Electrodynamics, all physical quantities and dynamical equations of particles have to be gauge invariant. However, if gauge invariance is realized by minimal coupling, Poincaré invariance is broken at least for the theory of higher-spin fields ($s \geq 1$).
The deficits occur both on the classical level (acausality and algebraic inconsistency) as well as on the quantum level (indefiniteness of antimutation relations). A lot of work has been done to solve these problems.

If the problem is investigated by group theoretical methods of space-time symmetries of interacting systems, symmetries of interacting systems lead to the general covariance group in case of a charged particle moving in an external electromagnetic field. As a consequence, the group theoretical definition of an elementary particle can be extended to the case where an external field is present. Even though the Poincaré group is not a subgroup of the general covariance group [31, 41], this point of view is of help to solve the problem.

4.1 A nonsingular transformation

It may be reasonable to introduce an external field directly into the Poincaré algebra which can be applied to classically understand the elementary particle. To do so one has to transform the generators of the Poincaré group to be dependent on the external field in such a way that the new, field-dependent generators obey the commutation relations [3]. As it was proposed by Chakrabarti [42] and Beers and Nickle [43], the simplest way to build such a field dependent algebra is to introduce the external field by a nonsingular transformation

\[ V : p_{1,3} \rightarrow p_{1,3}^d = \mathcal{V} p_{1,3} \mathcal{V}^{-1} = p_{1,3} + [\mathcal{V}, p_{1,3}] \mathcal{V}^{-1}. \]

More explicitly, the transformed operators

\[
\Pi^\mu &= P^\mu + [\mathcal{V}, P^\mu] \mathcal{V}^{-1}, \\
\xi^\mu &= x^\mu + [\mathcal{V}, x^\mu] \mathcal{V}^{-1}, \\
\sigma^{\mu\nu} &= s^{\mu\nu} + [\mathcal{V}, s^{\mu\nu}] \mathcal{V}^{-1}, \\
\mu^{\mu\nu} &= \xi^\mu \Pi^\nu - \xi^\nu \Pi^\mu + \sigma^{\mu\nu}
\]

must satisfy the commutation relations of the Poincaré algebra. The concept of Lorentz covariance raises the requirement that the operator \( V \) has to be of Lorentz type for the generator \( s^{\mu\nu} \), i.e.

\[ \mathcal{V} s^{\mu\nu} \mathcal{V}^{-1} = V^\mu_\rho V^\nu_\sigma s^{\rho\sigma} \]  

which is a local extension of Eq. (7). \( V = V(x, A) \) is the local Lorentz transformation depending on the external field \( A \) and obeying

\[ V\mu_\rho V^\nu_\sigma = V\rho_\mu V^\mu_\sigma = \eta_{\rho\sigma}. \]

If such a local Lorentz transformation exists, the problem is solved. Therefore, in the following we make the attempt to find explicit realizations of the local Lorentz transformation.
There is no way to construct the Lorentz transformation \( V_{\mu\nu} \) in general. However, as first shown by Taub [44], in the case of a plane-wave field we obtain

\[
V_{\mu\nu} = \eta_{\mu\nu} - \frac{q}{k_P}(k_\mu A_\nu - k_\nu A_\mu) - \frac{q^2}{2k_P^2}A^2 k_\mu k_\nu
\]

(26)

where \( q \) is the electric charge of the particle. The plane wave is characterized by its lightlike propagation vector \( k_\mu, k^2 = 0 \), and it polarization vector \( a^\mu \) such that

\[
a^2 = -1, \quad k_a = 0.
\]

(27)

The operator \( k_P \equiv k_\mu P^\mu \) commutes with any other and has a special role in the theory. For particles with nonzero mass one has \( k_\mu P^\mu \neq 0 \). Therefore, for the plane wave the inverse operator \( 1/k_P \) is well-defined for the plane-wave solution \( \psi_P \) of the Klein–Gordon equation. In all other cases, \( 1/k_P \) is assumed to exist.

We write \( A_\mu(\xi) = a_\mu f(\xi) \) where the variable \( \xi = k_\mu x^\mu \) can be used in place of the proper time. From Eq. (27) one obtains the conditions

\[
\partial_\mu A^\mu = k_\mu \frac{dA^\mu(\xi)}{d\xi} = k_\mu A^\mu(\xi) = 0, \quad k_\mu A^\mu = 0
\]

(28)

where we used \( A'_\mu(\xi) = dA_\mu(\xi)/d\xi \), while the field

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = k_\mu A'_\nu(\xi) - k_\nu A'_\mu(\xi) = F_{\mu\nu}(\xi)
\]

satisfies

\[
\begin{align*}
\partial_\mu F^{\mu\nu} &= k_\mu F^{\mu\nu}(\xi) = 0, \\
F_{\mu\nu}F^{\nu\rho} &= -k_\mu k_\rho (A'(\xi))^2.
\end{align*}
\]

(29)

It turns out that Eq. (26) can be written as

\[
V_{\mu\nu} = \exp \left( -\frac{q}{k_P}G \right)_{\mu\nu}
\]

(30)

where \( G_{\mu\nu} = k_\mu A_\nu - k_\nu A_\mu \). Note that the exponential series truncates after the second order term. In addition one obtains

\[
\begin{align*}
V_{\mu\nu} &= V_{\nu\mu} + \frac{2q}{k_P}G_{\nu\mu}, \\
V_{\mu\nu}k^\nu &= V_{\nu\mu}k^\nu = k_\mu, \\
[P_\mu, V_{\rho\sigma}] &= -i\frac{q}{k_P}k_\mu F_{\rho\sigma} - i\frac{q^2}{k_P^2}(AA')k_\mu k_\rho k_\sigma.
\end{align*}
\]

(31)

From the second equation in (31) one concludes that \( V_{\mu\nu} \) is an element of the (local) little group \( Lg(\xi) \) of the propagation vector \( k_\mu \). It is easy and interesting to see that \( V_{\mu\nu} \) generates a gauge transformation on \( A_\mu \),

\[
V_{\mu\nu}A^\nu = A_\mu + \partial_\mu \lambda V(\xi), \quad \lambda V(\xi) = -\frac{q}{k_P} \int_\xi^{\xi_0} d\xi' A^2(\xi'),
\]

(32)
and that the field $F_{\mu\nu}$ is invariant under this gauge transformation,

$$V_{\rho} V_{\sigma} F^{\rho\sigma} = F^{\mu\nu}.$$  \hspace{1cm} \text{(33)}

Therefore, the local Lorentz transformation $V$ is a symmetry. Notice that the local Lorentz transformation (26) has been rederived many times [45, 46, 47] and widely exploited often in the context of its physical implications. In particular, at the classical level the solutions of the Lorentz form equation can be expressed in terms of these local transformations (26). Therefore, in the plane-wave case $V_{\mu\nu}$ plays the role of an evolution operator.

According to the line of thought presented by the relations (24) and (25), the realization of (27) can be achieved by the singular transformation

$$V = \mathcal{V}_0 \mathcal{V}_s$$

with

$$\mathcal{V}_0 = \exp \left\{ - \int \frac{d\xi}{2k_p} (2q(\xi^2) - q^2 A^2) \right\}, \\
\mathcal{V}_s = \exp \left\{ - \frac{i q}{2k_p} G_{\mu\nu} s^{\mu\nu} \right\}.$$  \hspace{1cm} \text{(34)}

Collecting the results obtained, the generators of the interacting Poincaré algebra $p_{1,3}$ have the form

$$\Pi_{\mu} = P_{\mu} + k_{\mu} \frac{q}{2k_p} (qA^2 - 2Ap - \mathcal{F}),$$

$$\sigma_{\mu\nu} = s_{\mu\nu} - \frac{q}{2k_p} \left[ \frac{q}{2k_p} A^2 (\eta_{\mu\nu} k_{\rho} - \eta_{\mu\rho} k_{\nu}) k_{\sigma} + \eta_{\mu\rho} (k_{\nu} A_{\sigma} - k_{\sigma} A_{\nu}) - \eta_{\nu\rho} (k_{\mu} A_{\sigma} - k_{\sigma} A_{\mu}) + \frac{q}{k_p} (k_{\mu} A_{\nu} - k_{\nu} A_{\mu}) k_{\rho} A_{\sigma} \right] s^{\rho\sigma},$$

$$\xi_{\mu} = x_{\mu} - \frac{q}{2k_p} \left[ x_{\mu}, \int d\xi (qA^2 - 2AP - \mathcal{G}) \right]$$

where $\mathcal{F} \equiv F_{\mu\nu} s^{\mu\nu}$ and $\mathcal{G} \equiv G_{\mu\nu} s^{\mu\nu}$. The transformed first Casimir operator $\Pi^2$ reads

$$\Pi^2 = D^2 - q \mathcal{F},$$  \hspace{1cm} \text{(36)}

where $D_{\mu} = P_{\mu} - qA_{\mu}$. The explicit form of the transformed Pauli–Lubanski vector $\Omega_{\mu}$ is

$$\Omega_{\mu} = W_{\mu} - \frac{q}{2k_p} \epsilon_{\mu\rho\sigma} \left\{ \eta^{\rho\alpha} \left( \frac{q}{2k_p} A^2 k^{\rho} k^{\beta} + G^{\rho\beta} \right) - \frac{q}{2k_p} G^{\rho\beta} G^{\alpha\beta} \right\} s_{\alpha\beta} P^{\sigma} + \frac{q}{4k_p} \epsilon_{\mu\rho\sigma} k^{\rho} \eta^{\alpha\beta} \left( \eta^{\rho\beta} - \frac{2q}{k_p} G^{\rho\beta} \right) s_{\alpha\beta} (qA^2 - 2AP - \mathcal{F})$$  \hspace{1cm} \text{(37)}
which yields the transformed second Casimir operator

\[ \Omega^2 = -\frac{1}{2} s^2 D^2 + s^{\alpha} s_{\beta} D_{\alpha} D_{\beta} + \frac{1}{2} q s^2 F + \]

\[ - \frac{q}{2k_P} \{ k_\alpha s^{\alpha} F \} s_{\beta} + s_{\beta} (k_\alpha s^{\alpha} F) \} D^2 + \frac{q^2}{4k_P} (k_\alpha s^{\alpha} F) (k_\beta s_{\beta} F) + \]

\[ - \frac{iq}{2k_P} (k^\alpha F s_{\alpha \beta}) D^\beta - \frac{iq}{2k_P} (k_\alpha s^{\alpha}) (k_\beta s_{\beta} F'). \] (38)

4.2 Gauge symmetry of the interacting Poincaré algebra

It is now due time to study the gauge symmetry of the interacting Poincaré algebra \( p^{d}_{1,3} \).

As a consequence of the explicit form (34) the associated transformation of the evolution operator \( V(A) \) under the local gauge transformation

\[ A_\mu(\xi) \rightarrow A_\mu(\xi) + \partial_\mu \lambda(\xi) \] (39)

becomes

\[ V(A) \rightarrow V(A + \partial \lambda) = e^{-iq\lambda} V(A). \] (40)

On substituting these relations into the interaction algebra one arrives at

\[ p^d_{1,3}(A) \rightarrow p^d_{1,3}(A + \partial \lambda) = e^{-iq\lambda} p^d_{1,3}(A) e^{iq\lambda}. \] (41)

In particular, one obtains

\[ \Pi_\mu(A + \partial \lambda) = \Pi_\mu(A) - k_\mu q\lambda', \]

\[ \Pi^2(A + \partial \lambda) = \Pi^2(A) - 2k_P q\lambda'. \] (42)

It has to be mentioned that the evolution operator \( V(A) \) may be chosen to be \( H \)-unitary according to the representation \( T \) in Eq. (4), i.e.

\[ V^\dagger(A) H = H V^{-1}(A). \]

5 The “dynamical” interaction

We have shown that in the case of a particular external electromagnetic field \( A \) there exists an evolution operator \( V(A) \) which transforms the free Poincaré algebra \( p^d_{1,3} \) into the interacting algebra \( p^d_{1,3}(A) \), called the “dynamical” representation. By analogy with the free particle case one can realize this representation on the solution space of relativistically invariant equations. Expressing for the “dynamical” representation of the equation the operators explicitly in terms of free-field operators, one obtains “dynamical” interactions. Applying for instance the operator \( V \) to Eq. (12) one obtains

\[ V(A) : (P_{\mu} \beta^\mu - m) \psi(x) = 0 \rightarrow (\Pi_\mu \Gamma^\mu - m) \Psi(x, A) = 0 \] (43)
where $\Gamma^\mu \equiv \mathcal{V}(A)\beta^\mu \mathcal{V}^{-1}(A)$ and

$$\Psi(x, A) = \mathcal{V}(A)\psi(x). \quad (44)$$

The transformed matrices $\Gamma^\mu$ satisfy the requirement of relativistic invariance with respect to the “dynamical” representation,

$$[\Gamma_\mu, \sigma_{\rho\sigma}] = i(\eta_{\mu\rho}\Gamma_\sigma - \eta_{\mu\sigma}\Gamma_\rho) \quad (45)$$

just as in the free-field case \cite{14}. On substituting (35) into (43) and (44) one arrives at

$$\Gamma_\mu = V_{\mu\rho}\beta^\rho = \beta_\mu - \frac{q}{k_P} \left( \frac{q}{2k_P} A^2 k_\mu k_\rho + G_{\mu\rho} \right) \beta^\rho \quad (46)$$

and

$$D^d(A)\Psi \equiv \left( D_\mu \beta^\mu - \frac{q}{2k_P} k_\mu F - m \right) \Psi = 0 \quad (47)$$

where $k \equiv k_\mu \beta^\mu$ and $F \equiv F_{\mu\nu} S^{\mu\nu}$. It is important to notice that for the “dynamical” interaction the “minimal” replacement $P_\mu \rightarrow D_\mu = P_\mu - qA_\mu$ is modified by

$$P_\mu \rightarrow P_\mu - qA_\mu - \frac{q}{2k_P} k_\mu F = D_\mu - \frac{q}{2k_P} k_\mu F. \quad (48)$$

The additional term $-qk_\mu F/2k_P$ in this non-minimal replacement is nontrivial in the higher spin cases ($s \geq 1$) while in the lower spin cases ($s = 0, 1/2$) it vanishes identically. In these cases the “dynamical” interaction coincides with the “minimal” coupling. This is investigated in much detail \cite{23}. Note also that the substitution (48) does not coincide with the formal replacement $P_\mu \rightarrow \Pi_\mu$ in Eq. (35).

### 5.1 Local gauge invariance

Having found the “dynamical” interaction as a result of the Lorentz–Poincaré invariance (cf. Eqs. \cite{23} and \cite{24}), it is interesting to ask about the local gauge invariance. The usual way is to use the local gauge invariance to conjecture the field equation on the ground of Lorentz covariance. As a matter of fact, in the “dynamical” model the Lorentz typeness \cite{24} yields the gauge covariance, i.e. the diagram

$$\Psi : A \quad \longrightarrow \quad \Psi(A)$$

$$\downarrow \text{gauge} \quad \downarrow \text{gauge} \quad \downarrow \text{gauge} \quad (49)$$

$$\Psi^g : A + \partial \lambda \quad \rightarrow \quad \Psi^g(A + \partial \lambda) = e^{-iq\lambda} \Psi(A)$$

is commutative (here and in the following we skip the argument $x$ for $\Psi$). By virtue of Eq. (40) one has

$$\Psi(A + \partial \lambda) = e^{-iq\lambda} \Psi(A)$$

which implies $\Psi^g \equiv \Psi$. We conclude that for the “dynamical” model the relativistic covariance in Eq. (3) and the local gauge covariance are on equal footings.
According to the general idea of the invariance (6) it is clear that the “dynamical” interaction in Eq. (47) is gauge invariant under the transformation $A \rightarrow A + \partial \lambda$, i.e. the diagram

$$
\begin{align*}
\mathcal{D}^d \Psi : A &\rightarrow \mathcal{D}^d(A)\Psi(A) \\
\downarrow \quad \downarrow \text{gauge} &\quad \downarrow \text{gauge} \\
\mathcal{D}^d \Psi : A + \partial \lambda &\rightarrow e^{-i\lambda} \mathcal{D}^d(A)\Psi(A)
\end{align*}
$$

(50)
is commutative,

$$
\mathcal{D}^d(A + \partial \lambda)\Psi(A + \partial \lambda) = e^{-i\lambda} \mathcal{D}^d(A)\Psi(A).
$$

It should be pointed out that the promotion of the local gauge invariance stands for the specified transformation $A \rightarrow A + \partial \lambda$ of the electromagnetic field only. The transformation $\Psi \rightarrow e^{-i\lambda}\Psi$ is a consequence of Eq. (40). Moreover, due to Eq. (47) the gauge invariance does not determine the interaction uniquely and does not demand the minimal substitution only; rather the other principles of symmetry may be reconciled.

For the physical quantities $k_\mu$ and $F_{\mu\nu}$ in the model introduced before in Eqs. (31) and (33) the external (unquantized) field is acting on the particle without reaction of the particle on the field. The identification of the elementary particle system with the Poincaré group invariants in Eqs. (36) and (38) lead to the equations

\begin{align*}
(P^2 - m^2)\psi = 0 &\rightarrow (\Pi^2 - m^2)\Psi = (D^2 - qF_{\mu\nu}S^{\mu\nu} - m^2)\Psi = 0, \\
(W^2 + m^2s(s + 1))\psi = 0 &\rightarrow (\Omega^2 + m^2s(s + 1))\Psi = 0.
\end{align*}

(51) (52)

These two equations must be satisfied by any field in the presence of the plane-wave field. As a consequence of Eq. (51) the gyromagnetic factor is $g = 2$ and the Bargmann-Michel-Telegdi equation for the four-polarization vector of the particle takes its simplest form in the proper time frame of the particle [24].

6 The Rarita–Schwinger equation in the framework of a “dynamical” interaction

The spin-3/2 field may be described entirely in terms of the vector-bispinor $\Psi_\mu$ corresponding to the representation of the proper Lorentz group

$$
\left(\frac{1}{2}, \frac{1}{2}\right) \otimes \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right) = \left(1, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 1\right) \oplus \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right).
$$

(53)
The transformation rule according to Eq. (3) is

$$
(\tau(\lambda)\psi)_\mu(p) = \Lambda_{\mu\nu}T_D(\Lambda)\psi^\nu(\Lambda^{-1}p)
$$

(54)

where $T_D(\Lambda)$ is the Dirac representation of the Lorentz group. The generators of the representation are

\begin{align*}
s_{\mu\nu} &= -ie_{\mu\nu} \otimes 1_D + 1_P \otimes s_D^{\mu\nu} = \\
&= i\left(-\frac{1}{2}\eta_{\mu\nu} + E_{\mu\nu} \otimes 1_D - E_{\nu\mu} \otimes 1_D + \frac{1}{2}1_P \otimes \gamma_{\mu}\gamma_{\nu}\right)
\end{align*}

(55)
where the indices $P$ and $D$ stand for the Proca and Dirac part of the direct product in Eq. (53). Here the 16 matrices $E_{\mu\nu}$ generate the Weyl’s basis of the set of $4 \times 4$ matrices,

$$(E_{\mu\nu})_{\rho\sigma} = \eta_{\mu\rho} \eta_{\nu\sigma}, \quad E_{\mu\nu} E_{\rho\sigma} = \eta_{\nu\rho} E_{\mu\sigma},$$

and $e_{\mu\nu} = -E_{\mu\nu} + E_{\nu\mu}$ for the Lorentz generators of the vector representation. The $SO_3$ decomposition of the representation (53) is

$$2D^{(3/2)} \oplus 4D^{(1/2)}. \quad (56)$$

Therefore, the representation of the Poincaré group contains spins $3/2$ and $1/2$. The Pauli–Lubanski vector reads

$$W_{\mu} = i \epsilon_{\mu\rho\sigma\nu} \left( E^{\rho\sigma} \otimes 1_D + \frac{1}{4} 1_P \otimes \gamma^\rho \gamma^\sigma \right) P^\nu \quad (57)$$

and its root

$$W^2 = -\frac{15}{4} P^2 + P^2 (E^{\mu\nu} \otimes \gamma_\mu \gamma_\nu) + P_\mu P^\nu (E^{\mu\rho} \otimes \gamma_\rho \gamma_\nu + E^{\rho\mu} \otimes \gamma_\nu \gamma_\rho). \quad (58)$$

Note that

$$(W^2)^2 = -\frac{9}{4} P^2 \left\{ - \frac{15}{4} P^2 + P^2 (E^{\mu\nu} \otimes \gamma_\mu \gamma_\nu) + 
+ P_\mu P^\nu (E^{\mu\rho} \otimes \gamma_\rho \gamma_\nu) + P_\mu P^\nu (E^{\rho\mu} \otimes \gamma_\nu \gamma_\rho) + \frac{5}{8} P^2 \right\} = 
= -2s^2 P^2 \left( W^2 + \frac{s^2 - 1}{2} P^2 \right) \bigg|_{s=3/2} \quad (59)$$

is a pure spin-$3/2$ object which enables us to construct the Poincaré covariant mass $(m)$ and spin $(j)$ projectors $(j = 3/2, 1/2)$. The free spin-$3/2$ particle Rarita–Schwinger equation is given as

$$(P_\nu \gamma^\nu - m) \psi^\mu = 0, \quad (60)$$

$$\gamma_\mu \psi^\mu = 0. \quad (61)$$

The other constraints

$$(P^2 - m^2) \psi^\mu = 0, \quad (62)$$

$$P_\mu \psi^\mu = 0 \quad (63)$$

are a consequence of Eqs. (60) and (61). It is interesting to note that the static condition (61) and the dynamic condition (63) together eliminate the spin-$1/2$ state completely, i.e. the equations

$$(P^2 - m^2) \psi^\mu = 0, \quad \gamma_\mu \psi^\mu = P_\mu \psi^\mu = 0$$

are a consequence of Eqs. (60) and (61). It is interesting to note that the static condition (61) and the dynamic condition (63) together eliminate the spin-$1/2$ state completely, i.e. the equations

$$(P^2 - m^2) \psi^\mu = 0, \quad \gamma_\mu \psi^\mu = P_\mu \psi^\mu = 0$$
with $\psi_\mu$ transforming according to Eq. (54) gives a theory for spin-3/2 states.

Using the explicit form of $W^2$ in Eq. (58) it is easy to see that under the constraints (61) and (63) we obtain

$$W^2\psi = -\left(\frac{15}{4} P^2 \psi = -s(s+1)P^2\psi\right)_{s=3/2}. \quad (64)$$

Therefore, Eqs. (60) and (61) describe indeed a single particle of mass $m$ and spin 3/2.

The “dynamical” interaction is obtained in the way described in Sec. 3. Taking into account the explicit form (55) of the generators $s_{\mu\nu}$, the transformation $V$ in Eq. (34) becomes

$$V_{RS} = \exp\left(-\frac{iq}{k_P} \int (AP - \frac{q}{2} A^2) (1_P \otimes 1_D) \times \{1_P - \frac{q}{k_P} (G_{\rho\sigma} - \frac{q}{2k_P} (G^2)_{\rho\sigma}) E^{\rho\sigma}\} \otimes \{1_D + \frac{q}{4k_P} G^{\rho\gamma} \gamma_\rho, \gamma_\sigma}\right). \quad (65)$$

A straightforward calculation yields

$$P_\mu \rightarrow \Pi_\mu = \left(P_\mu + k_\mu \frac{q}{2k_P} (qA^2 - 2AP)\right) (1_P \otimes 1_D) +$$

$$- k_\mu \frac{iq}{k_P} F_{\rho\sigma} (E^{\rho\sigma} \otimes 1_D) - k_\mu \frac{iq}{4k_P} F_{\rho\sigma} (1_P \otimes \gamma^\rho \gamma^\sigma),$$

$$s_{\mu\nu} \rightarrow \sigma_{\mu\nu} = -i \left(\frac{1}{2} \eta_{\mu\nu} + \frac{q}{k_P} G_{\mu\nu}\right) (1_P \otimes 1_D) +$$

$$+ i \left\{-\eta_{\mu\rho} \eta_{\nu\sigma} + \frac{q}{k_P} (\eta_{\mu\rho} G_{\nu\sigma} - \eta_{\nu\rho} G_{\mu\sigma}) +$$

$$- \frac{q^2}{2k_P} \left(\eta_{\mu\rho} (G^2)_{\nu\sigma} - \eta_{\nu\rho} (G^2)_{\mu\sigma} + G_{\mu\nu} G_{\rho\sigma}\right)\right\} (1_D + \frac{1}{2} 1_P \otimes \gamma^\rho \gamma^\sigma),$$

$$W_\mu \rightarrow \Omega_\mu = -iq \frac{2k_P}{k_P} \epsilon_{\mu\nu\rho\sigma} k^\nu A^\rho P^\sigma (1_P \otimes 1_D) +$$

$$- i \frac{q}{2k_P} \epsilon_{\mu\nu\rho\sigma} \left\{\left(\eta^\rho_{\alpha} \eta_{\beta} - \frac{q}{k_P} \eta^\rho_{\alpha} G_{\beta} + \frac{q^2}{2k_P^2} G_{\mu\rho} G_{\alpha\beta} - \frac{q^2 A^2}{2k_P^2} k^\kappa k_\beta (2q)\right) P^\sigma +$$

$$+ \frac{q}{2k_P} (qA^2 - 2AP) k^\kappa \eta^\rho_{\alpha} \left(\eta^\rho_{\beta} + \frac{2q}{k_P} A^\rho k_\beta\right)\right\} (1_D + \frac{1}{2} 1_P \otimes \gamma^\alpha \gamma^\beta) +$$

$$+ \frac{q}{2k_P} \epsilon_{\mu\nu\rho\sigma} k^\rho \eta^\nu_{\alpha} \left(\eta^\nu_{\beta} + \frac{2q}{k_P} A^\nu k_\beta\right) F_{\lambda\tau} \times$$

$$\times \left\{e^{\alpha\beta} e^{\lambda\tau} \otimes 1_P - \frac{1}{2} e^{\alpha\beta} \otimes \gamma^\lambda \gamma^\tau - \frac{1}{2} e^{\lambda\tau} \otimes \gamma^\alpha \gamma^\beta + \frac{1}{4} 1_P \otimes \gamma^\alpha \gamma^\beta \gamma^\lambda \gamma^\tau\right\}. \quad (66)$$

The two Casimir invariants of the “dynamical” Poincaré algebra are

$$P^2 \rightarrow \Pi^2 = D^2 - 2iqF^{\rho\sigma} \left\{(E_{\rho\sigma} \otimes 1_D) + \frac{1}{4} (1_P \otimes \gamma_\sigma)\right\}. \quad (67)$$
and

\[
W^2 \rightarrow \Omega^2 = \left( \frac{9}{2} + (\epsilon^\rho\sigma \otimes \gamma^\rho\sigma) \right) D^2 + \\
+ \frac{1}{2} \left( -4(E^\alpha\beta \otimes 1_D) + e^\alpha\rho \otimes \gamma^\rho\beta + e^\beta\rho \otimes \gamma^\rho\alpha \right) D^\alpha D^\beta + \\
- \frac{iq}{2k_P} k^\tau F^{\rho\sigma} \left\{ - \frac{3}{2}(h^\tau\beta \otimes \gamma^\rho\alpha) + (h^\tau\rho \otimes \gamma^\rho\beta) + \\
- \eta^\sigma\beta(h^\rho\alpha \otimes \gamma^\alpha\tau) - \frac{i}{2} \epsilon^\rho\sigma\alpha\beta(e^\tau\alpha \times \gamma^5) \right\} D^\beta + \\
+ iq F^{\rho\sigma} \left\{ - 16(E^\rho\sigma \otimes 1_D) - \frac{29}{8}(1_P \otimes \gamma^\rho\sigma) + \\
- 6(e^\alpha\sigma \otimes \gamma^\rho\alpha) - i\epsilon^\rho\sigma\alpha\beta(E^\alpha\beta \otimes \gamma^\alpha\tau) \right\} + \\
- \frac{q}{k_P} k^\alpha k^\beta F^{\rho\sigma}(E^\alpha\beta \otimes \gamma^\rho\sigma) \right\} \tag{68}
\]

where we used the abbreviations \( \gamma^\mu\nu \equiv \gamma^\mu\gamma^\nu \) and \( h^\mu\nu = E^\mu\nu + E^\nu\mu \). Applying the operator \( \mathcal{V}_{RS} \) to the Rarita–Schwinger equation one obtains

\[
\left\{ (D^\mu \gamma^\mu - m)\eta^\rho\sigma - \frac{iq}{k_P} (k^\mu \gamma^\mu) F^{\rho\sigma} \right\} \Psi^\sigma = 0, \tag{69}
\]

\[
\gamma^\mu \Psi^\mu = 0 \tag{70}
\]

where \( \Psi(x, A) = \mathcal{V}(x, A)\psi(x) \).

The first equation is the true equation of motion containing all derivatives \( D^\mu \Psi^\sigma \). The static constraint (70) survives the “dynamical” interaction and eliminates all superfluous spin-1/2 components. As a consequence the other constraints are the Feynman–Gell-Mann equation

\[
\left\{ (\Psi^2 - m^2)\eta^\mu\rho - 2iqF^\mu\rho \right\} \Psi^\rho = 0 \tag{71}
\]

and the kinematical constraint

\[
\left\{ D^\mu - \frac{iq}{4k_P}(F^{\rho\sigma}\gamma^\mu\gamma^\sigma)k^\mu \right\} \Psi^\mu = 0. \tag{72}
\]

Note that as in the free case the “dynamical” interaction is algebraically consistent. Moreover, the second order equation (71) describes the causal propagation of waves (assuming the continuity of the first order derivatives of \( \Psi \)).
7 Conclusions

Starting from the Lorentz–Poincaré connection presented in Sec. 2, applied to physical states in Sec. 3, in this paper we constructed a non-minimal interaction by developing a field dependent algebra. For this purpose we introduced a transformation $V$ based on a Lorentz type matrix $V$. We showed that the application of this transformation on the gauge field $A_\mu$ generates a gauge transformation. On the other hand, if a local gauge transformation is performed, the transformation $V$ generates a phase transformation for the covariant fields. The difference to the standard procedure can be formulated as follows:

In order to make quantum mechanics consistent with Maxwell equations, we usually have to impose the local gauge transformation given by the scheme

\[
\begin{align*}
A_\mu(x) &\to A'_\mu(x) = A_\mu(x) + \partial_\mu \lambda(x) \\
\psi(x) &\to \psi'(x) = e^{-i q \lambda} \psi(x)
\end{align*}
\\Rightarrow \quad D'_\mu \psi' = e^{-i q \lambda} D_\mu \psi. \tag{73}
\]

In the “dynamical” model we have a chain of related transformations,

\[
\begin{align*}
A_\mu(x) &\to A'_\mu(x) = A_\mu(x) + \partial_\mu \lambda(x) \\
\psi(x) &\to \Psi(x, A + \partial \lambda) = e^{-i q \lambda} \Psi(x, A) \\
D^d(A + \partial \lambda) \Psi(x, A + \partial \lambda) &\to e^{-i q \lambda} D^d(A) \psi(x, A), \tag{74}
\end{align*}
\]

motivated by the Lorentz–Poincaré symmetry, where the “dynamical” extension $D \to D^d$ is a minimal extension of the minimal coupling.

Applying to the generalized Dirac equation for arbitrary spin we showed explicitly how the transformation $V$ changes the minimal interaction to a non-minimal one. We applied the formalism to the Rarita–Schwinger equation and derived explicit expressions for the field-dependent generators of the algebra.

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