Discrete quantum gravity: The Lorentz invariant weight for the Barrett-Crane model

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Abstract. In a recent paper [1] we have constructed the spin and tensor representations of $SO(4)$ from which the invariant weight can be derived for the Barrett-Crane model in quantum gravity. By analogy with the $SO(4)$ group, we present the complexified Clebsch-Gordan coefficients in order to construct the Biedenharn-Dolginov function for the $SO(3,1)$ group and the spherical function as the Lorentz invariant weight of the model.

1. Review of the Euclidian model

Given a triangulation of a 4-dimensional Riemannian manifold, we assign bivectors to the faces satisfying appropriate constraints [2]. Then we identify the bivectors with Lie algebra elements and associate a representation of $SO(4)$ to each triangle and a tensor to each tetrahedron, invariant under $SO(4)$. The representation chosen is simple, i.e. $j_1 = j_2$.

Now it is easy to construct an amplitude for the quantum 4-simplex. The graph for a spin foam is the 1-complex, dual to the boundary of the 4-simplex having five 4-valent vertices (corresponding to the five tetrahedra) with each of the ten edges connecting two different vertices (corresponding to the ten triangles of the 4-simplex each shared by two tetrahedra). Now we associate to each triangle (the dual of which is an edge) a simple representation of the algebra of $SO(4)$ and to each tetrahedra (the dual of which is a vertex) an intertwiner; and to a 4-simplex the product of the five intertwiners and the sum for all possible representations. The proposed state sum is:

$$Z = \sum_J \prod_{\text{triangle}} A_{tr} \prod_{\text{tetrahedra}} A_{tet} \prod_{\text{4-simplex}} A_{simp}.$$ 

For the simple representations ($j_1 = j_2$) attached to every face of the tetrahedron, we used the elementary spherical functions, that can be calculated from the Biedenharn-Dolginov function in the case $j = m = 0$, namely,

$$d_{0,0}^{(j_1,0)}(\tau) = \frac{\sin(2j_1 + 1)\tau}{(2j_1 + 1)\sin \tau}$$
2. Spinor representation of SL(2, C)

We define the complex valued polynomials

\[ p(z, \bar{z}) = \sum C_{\alpha\beta} z^\alpha \bar{z}^\beta \]

as the basic states of the spinor representations:

\[ T_a p(z, \bar{z}) = (a_{10} z + a_{11})^k (\bar{a}_{10} \bar{z} + \bar{a}_{11})^n p \left( \frac{a_{00} z + a_{01}}{a_{10} z + a_{11}}, \frac{\bar{a}_{00} \bar{z} + \bar{a}_{01}}{\bar{a}_{10} \bar{z} + \bar{a}_{11}} \right) \]

for any \( a \in SL(2, \mathbb{C}) \), \( a \equiv \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \).

This representation with labels \((l_0, l_1) = \left( \frac{k-n}{2}, \frac{k+n}{2} + 1 \right), k, n \in \mathbb{N}\), is irreducible, and finite dimensional. If we enlarge this representation with complex values, \( k, n \), we get

\[ T_a f(z, \bar{z}) = (a_{10} z + a_{11})^{l_0 + l_1 - 1} (\bar{a}_{10} \bar{z} + \bar{a}_{11})^{l_1 - l_0 - 1} f \left( \frac{a_{00} z + a_{01}}{a_{10} z + a_{11}}, \frac{\bar{a}_{00} \bar{z} + \bar{a}_{01}}{\bar{a}_{10} \bar{z} + \bar{a}_{11}} \right), \]

the representation becomes infinite dimensional. For \( l_0 \) integer or half integer and \( l_1 \) complex the representation is irreducible. The principal series of unitary representations for \( SL(2, \mathbb{C}) \) are defined, on the Hilbert space of complex functions, with scalar product \((f_1, f_2) = \int f_1(z) \bar{f}_2(z) \, dz\), as

\[ T_a f(z) = (a_{10} z + a_{11})^{l_0 + l_1 - 1} (\bar{a}_{10} \bar{z} + \bar{a}_{11})^{l_1 - l_0 - 1} f \left( \frac{a_{00} z + a_{01}}{a_{10} z + a_{11}} \right) \]

with \( l_0 = \mu, \) integer or half integer, \( l_1 = i \gamma, \gamma \in \mathbb{R}\).

The complementary series of unitary representations are defined in the Hilbert space of complex functions, with scalar product \((f_1, f_2) = \int \left| z_1 - z_2 \right|^{-2 + 2\sigma} f_1(z_1) \bar{f}_2(z_2) \, dz_1 \, dz_2\), as

\[ T_a f(z) = |a_{10} z + a_{11}|^{2\sigma - 2} f \left( \frac{a_{00} z + a_{01}}{a_{10} z + a_{11}} \right) \]

with \( l_0 = 0, \ l_1 = \sigma, \ \sigma \in \mathbb{R}, \ |\sigma| < 1\).

3. Representations of the algebra of SL(2, C)

Given the generators of rotations \((J_1, J_2, J_3) = \vec{J}\) and of pure Lorentz transformations \((K_1, K_2, K_3) = \vec{K}\) satisfying the commutation relations:

\[
\begin{align*}
[J_p, J_q] &= i\epsilon_{pqr} J_r, & J^+ &= \vec{J}, & p, q, r &= 1, 2, 3 \\
[J_p, K_q] &= i\epsilon_{pqr} K_r, & \vec{K}^+ &= \vec{K} \\
[K_p, K_q] &= -i\epsilon_{pqr} J_r
\end{align*}
\]

we obtain the unitary representations of the algebra of \( SL(2, \mathbb{C}) \) in the basis where the operators \( J_3 \) and \( \vec{J}^2 \) are diagonal, namely: \( J_3 \psi_{jm} = m \psi_{jm}, \ \vec{J}^2 \psi_{jm} = j(j+1) \psi_{jm} \)

It is possible also to construct a complexified operators

\[ \vec{A} = \frac{1}{2} (\vec{J} + i\vec{K}), \quad \vec{B} = \frac{1}{2} (\vec{J} - i\vec{K}), \quad \vec{A}^+ = \vec{B} \]
that leads to the commutation relations of two independent angular momenta:

\[
\begin{align*}
[A_p, A_q] &= i\epsilon_{pqr} A_r \\
[B_p, B_q] &= i\epsilon_{pqr} B_r \\
[A_p, B_q] &= 0
\end{align*}
\]

Since \(J_3\) and \(K_3\) commute we construct the representations of these operators in the basis where \(J_3\) and \(K_3\) are diagonal, [3]

\[
\begin{align*}
J_3\phi_{m_1m_2} &= m\phi_{m_1m_2}, & K_3\phi_{m_1m_2} = \lambda\phi_{m_1m_2}, & \text{hence} \\
A_3\phi_{m_1m_2} &= \frac{1}{2}(m + i\lambda)\phi_{m_1m_2} = m\phi_{m_1m_2} \\
B_3\phi_{m_1m_2} &= \frac{1}{2}(m - i\lambda)\phi_{m_1m_2} = m_2\phi_{m_1m_2}
\end{align*}
\]

Notice that \(\lambda\) is a real continuous parameter, but \(m_1\) and \(m_2\) are complex conjugate and \(\bar{m}_1 = m_2\).

In both basis the labels of the representations \((l_0, l_1)\) takes the values \(l_0 = \mu\) (integer or half integer), \(l_1 = i\gamma (\gamma \in R)\) for the principal series, and \(l_0 = 0, l_1 = \sigma, |\sigma| < 1, \sigma \in R\), for the complementary series. For the Casimir operators we have

\[
\begin{align*}
(\vec{J}^2 - K^2)\psi_{jm} &= (l_0^2 + l_1^2 - 1)\psi_{jm} \\
(\vec{J} \cdot \vec{K})\psi_{jm} &= l_0l_1\psi_{jm}
\end{align*}
\]

4. Complexified Clebsch-Gordan coefficients and the representation of the boost operator

In order to connect the basis \(\psi_{jm}\) and \(\phi_{m_1m_2}\) we can use the complexified Clebsch-Gordan coefficients:

\[
\psi_{jm} = \int_{-\infty}^{\infty} d\lambda \langle m_1m_2 | jm \rangle \phi_{m_1m_2}
\]  

(1)

We have used integration because \(\lambda\) is a continuous parameter. It can be proved that these coefficients are related to the Hahn polynomials of imaginary argument [3]

\[
\langle m_1m_2 | jm \rangle = f \sqrt{\rho(\lambda)}p^{(m-\mu,m+\mu)}_{j-m}(\lambda, \gamma), \quad f\bar{f} = 1,
\]  

(2)

for the principal series,

\[
\langle m_1m_2 | jm \rangle = f \sqrt{\rho(\lambda)} q^{-1}_{j-m}^{(m)}(\lambda, \sigma)
\]  

(3)

for the complementary series,

where

\[
\rho(\lambda) = \frac{1}{4\pi} \left| \Gamma \left( \frac{m-\mu+1}{2} + i\lambda \right) \Gamma \left( \frac{m+\mu+1}{2} + i\lambda \right) \right|^2
\]  

(4)

and

\[
d_n^2 = \frac{\Gamma(m-\mu+n+1)\Gamma(m+\mu+n+1)\Gamma(m+i\gamma+n+1)}{n! (2m+2n+1)\Gamma(2m+n+1)}
\]

for the principal series and similar expression for the complementary series.
With the help of equations (1), (2) and (3) we can construct the representation for the boost operator, or the Biedenharn-Dolginov function, namely,

\[
d_{j,j'}^{(\mu,\gamma)}(\tau) = \langle \psi_{jm} | e^{-i\tau K_3} \psi_{j'm} \rangle = \int_{-\infty}^{\infty} d_{j-m}^{-1} p_{j-m}^{(m-\mu,m+\mu)}(\lambda,\gamma) e^{-i\lambda} d_{j'-m}^{-1} p_{j'-m}^{(m-\mu,m+\mu)}(\lambda,\gamma) \rho(\lambda) d\lambda
\]

5. Spherical function for the group SO(3,1)

Given a locally compact group \( G \) with completely irrep. \( T_g \), and a compact subgroup \( K \subset G \), with completely irrep. \( T_k \) (finite dimensional), we define the spherical function

\[
f(g) = T_r \{ E_k T_g \} = T_r \{ T_k \}
\]

where \( E_k \) is the projector \( E_k : T_g \rightarrow T_k \).

The spherical function [4] are functions on the homogeneous space \( K \backslash G \) and invariant on right cosets: \( f(kg) = f(g) \)

We apply this definition to the representation of \( SO(3,1) \) given by Biedenharn function, with \( \mu = 0 \), projected into the identity representation of \( SO(3) \), with \( j = m = 0 \) (the elementary spherical function). We have

\[
f(\tau) = T_r d_{0,0}^{(0,\gamma)} = \int_0^{\tau} d_0^{-2} e^{-i\lambda \tau} \left[ p_0^{(0,0)} \right]^2 \rho(\lambda) d\lambda =
\]

\[
= \int_{-\infty}^{\infty} e^{-i\lambda \tau} \left| \frac{1}{|\Gamma(1+i\gamma)|^2} \frac{1}{4\pi} \left| \Gamma \left( \frac{1}{2} + i\frac{\lambda + \gamma}{2} \right) \right|^2 \right| d\lambda =
\]

\[
= \int_{-\infty}^{\infty} e^{-i\lambda \tau} \frac{sh \pi \gamma}{4\gamma} \frac{1}{ch \pi \left( \frac{\lambda + \gamma}{2} \right)} \frac{1}{ch \pi \left( \frac{\lambda - \gamma}{2} \right)} d\lambda
\]

where we have used the properties of \( \Gamma \) functions. From the residue theorem at the poles \( \lambda = \pm \gamma + (2n+1)i \), \( n = 0, 1, 2, \ldots \), we easily obtain

\[
f(\tau) = i e^{i\gamma \tau} - e^{-i\gamma \tau} \sum_{n=0}^{\infty} (e^{2\tau})^n = \frac{1}{\gamma} \sin \gamma \tau
\]

for the principal series with \( l_0 = 0, l_1 = \gamma \). We obtain by the same way

\[
f(\tau) = \frac{1}{\sigma} \frac{sh \sigma \tau}{sh \tau}
\]

for the complementary series, with \( l_0 = 0, l_1 = \sigma, |\sigma| < 1 \)
6. A $SO(3,1)$ invariant for the state sum of spin foam model

As in the case of euclidean $SO(4)$ invariant model, we take a non degenerate finite triangulation of a 4-manifold. We consider the 4-simplices in the homogeneous space $SO(3,1)/SO(3) ∼ H_3$, the hyperboloid $\{x| x \cdot x = 1, x^0 > 0\}$ and define the bivectors $b$ on each face of the 4-simplex, that can be space-like, null or timelike ($\langle b, b \rangle > 0, = 0, < 0$, respectively).

In order to quantize the bivectors, we take the isomorphism $b = \ast L (b_{ab} = \frac{1}{2} \varepsilon^{abcd} L_{de} g_{ec})$ with $g$ a Minkowski metric.

The condition for $b$ to be a simple bivector $\langle b, \ast b \rangle = 0$, gives $C_2 = \langle L, \ast L \rangle = \bar{J} \cdot \bar{K} = \mu \gamma = 0$

We have two cases:
1) $\gamma = 0$, $C_1 = \langle L, L \rangle = \mu^2 - 1 > 0; L$, space time, $b$ time-like

2) $\mu = 0$, $C_1 = J^2 - K^2 = -\gamma^2 - 1 < 0; L$, time like, $b$ space like (remember, the Hodge operator * changes the signature)

In case 2) $b$ is space-like, $\langle b, b \rangle > 0$. Expanding this expression in terms of space like vectors, $x, y$,

$$b_{\mu\nu} b^{\mu\nu} = (x_\mu y_\nu - x_\nu y_\mu) (x^{\mu} y^{\nu} - x^{\nu} y^{\mu}) =$$
$$= |x|^2 |y|^2 - |x| |y| \cos^2 \eta (x, y) = |x|^2 |y|^2 \sin^2 \eta (x, y)$$

where $\eta (x, y)$ is the Lorentzian space-like angle between $x$ and $y$; this result gives a geometric interpretation between the parameter $\gamma$ and the area expanded by the bivector $b = x \wedge y$, namely, $\langle b, b \rangle = \text{area}^2 \{x, y\} = \langle \ast L, \ast L \rangle \cong \gamma^2$. (This result is equivalent to that obtained in [1] where the area of the triangle expanded by the bivector was proportional to the value $(2j + 1)$, $j$ being the spin of the representation).

In order to have the state sum we attach to each face of the 4-simplex the spherical function described in Section 5 and integrate over the whole rank of the parameter of the representation. The resulting partition function is invariant under $SO(3,1)$ and has been proved to be finite [5]. The asymptotic properties of the spherical function are related to the Einstein-Hilbert action, giving a connection of the model with the theory of general relativity.

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