Abstract

We study the dynamics of the noncommutative fluid in the Snyder space perturbatively at the first order in powers of the noncommutative parameter. The linearized noncommutative fluid dynamics is described by a system of coupled linear partial differential equations in which the variables are the fluid density and the fluid potentials. We show that these equations admit a set of solutions that are monocromatic plane waves for the fluid density and two of the potentials and a linear function for the third potential. The energy-momentum tensor of the plane waves is calculated.
1 Introduction

Motivated by recent works in which several noncommutative systems of charges were shown to display fluid properties at quantum scale \[1, 2, 3, 4, 5, 6, 7\] as well as cosmological scale \[8\], we have proposed the first field theoretical model of fluid in the canonical noncommutative space \[9\] and the noncommutative Lorentz covariant space \[10\], respectively \[11, 12, 45\]. The noncommutative fluid was constructed as a noncommutative field theory \[13, 14\] in the realization method approach \[15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27\] by generalizing the first order action functional of the commutative perfect relativistic fluid \[28, 29, 30, 31\]. In this approach one obtains the fluid dynamics of the long wavelength degrees of freedom of the system bypassing the statistical analysis of the microscopic degrees of freedom which in the noncommutative spaces is not well understood yet (see for tentative approaches \[32, 33, 34\]). In general, the definition of the noncommutative fluid degrees of freedom is not a trivial task since the phenomenological considerations that led to the fluid equations in the commutative spaces are not allowed. However, by treating the perfect fluid as an effective field theory, the degrees of freedom of the noncommutative fluid are defined by the correspondence principle that requires that in the commutative limit of the noncommutative parameter the known commutative perfect fluid equations be obtained. As was discussed in \[9, 10\], the degrees of freedom of the noncommutative fluid are the noncommutative generalization of the density current and the fluid potentials that parametrize the velocity field \[35, 36\]. The choice of the commutative fluid potentials is not unique. When it is made in terms of real functions \(\theta(x), \alpha(x)\) and \(\beta(x)\) it is called the Clebsch parametrization \[35, 36\] while the fluid potentials given in terms of one real \(\theta(x)\) and two complex functions \(z(x)\) and \(\bar{z}(x)\), respectively, define the so called Kähler parametrization \[37, 38, 39, 40, 41, 42, 43, 44\].

The noncommutative coordinates in the Snyder space \(S\) are the Lie generators of \(so(1, 4)/so(1, 3)\). The degrees of freedom of the noncommutative fluid in \(S\) defined in \[10\] are the natural generalization of the commutative fluid potentials in the Clebsch parametrization. These belong to the set of functions over the Snyder space \(\mathcal{F}(S)\) which can be endowed with the star-product and the co-product constructed in \[16, 47\]. The algebra \(\mathcal{F}(S)\) is isomorphic to the deformed algebra over the Minkowski space-time \((C^\infty(M), \star)\). Since the star-product is nonassociative and noncommutative and the momenta associated to the coordinates do not form a Lie group, understanding the dynamics of the noncommutative fluid in the Snyder space turns to be a very challenging problem. The reason is that the noncommutative fluid equations involve an infinite number of derivatives of the fluid potentials which are multiplied in a nonassociative way. This property makes the equations difficult to analyse in the general case. However, if the deformation parameter of the deformed Poincaré algebra \(s = l_s^2\), where \(l_s\) is the typical length scale of the noncommutative space is small compared to unity, one can attempt to expand the star-product in powers of \(s\). This is certainly the case if the noncommutative structure is the structure of the physical space-time since the phenomenological data and the theoretical arguments suggest that \(l_s\) is of the order of the Planck scale. Another reason for which one should study the terms of the action \[15\] corresponding to the finite order in \(s\) is the following. Due to the nonassociativity of the star-product \(\star\), the highest order of the derivatives of the fluid potentials from the noncommutative functional is infinite. Therefore, the Euler-Lagrange equations of motion cannot be determined for an arbitrary value of \(s\). Moreover, since the theory has just a limited number of symmetries, the equations of motion are not integrable.

In this paper, we are going to study perturbatively the dynamics of the noncommutative fluid in the Snyder space by expanding the relevant objects from the algebra \((C^\infty(M), \star)\)
in powers of \( s \). Our main goal is to obtain analytic solutions to the equations of motion of the fluid density current and potentials, respectively, in the linear approximation. The dependence on \( s \) is encoded in the function on two momenta that determines the star-product and the co-product of the deformed Poincaré algebra and the anti-pode of its co-algebra. By truncating the power expansion at first order in \( s \), the noncommutative fluid equations are reduced to a system of coupled linear partial differential equations. We are able to show that these equations admit monocromatic plane wave solutions for the density current \( j^\mu \) and \( \alpha \) and \( \beta \) fluid potentials, respectively and a linear solution for \( \theta \) potential. These are the first solutions of the linearized noncommutative fluid dynamics obtained so far. Also, we calculate the energy-momentum tensor of the monocromatic waves explicitly, which represents one of the few examples of such calculations.

The paper is organized as follows. In Section 2, we give a short review of the recent noncommutative fluid model in the Snyder space obtained in [10] and establish our notations. The perturbative expansion of the model and its first order equations of motion are obtained in Section 3. In Section 4, we show that the equations of motion admit solutions that are monocromatic plane waves of the fluid potentials. These solutions are characterized by the fact that the scalar product of divergence of the potentials \( \alpha \) and \( \beta \) with the current density \( j^\mu \) in the Minkowski space-time is zero. Also, we calculate the energy-momentum tensor for these solutions. In the last section we discuss the properties of the solutions obtained previously.

## 2 Noncommutative fluid in the Snyder space

The Snyder-like geometries can be viewed as realizations of the Snyder algebra which, at its turn, is a deformation of the algebra \( so(1,3) \) with the deformation parameter \( s = l_s^2 \)

\[
[\tilde{x}_\mu, \tilde{x}_\nu] = s M_{\mu\nu},
\]

\[
[p_\mu, p_\nu] = 0,
\]

\[
[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\nu\sigma} M_{\mu\rho},
\]

\[
[M_{\mu\nu}, \tilde{x}_\rho] = \eta_{\rho\nu} \tilde{x}_\mu - \eta_{\rho\mu} \tilde{x}_\nu,
\]

\[
[M_{\mu\nu}, p_\rho] = \eta_{\rho\nu} p_\mu - \eta_{\rho\mu} p_\nu,
\]

where \( \mu, \nu = 0, 3 \) and \( l_s \) has the dimension of length. The generators \( M_{\mu\nu} \) can be expressed in terms of the commutative coordinates and momenta \( \{x_\mu, p_\nu\} \) of the Minkowski space-time \( \mathcal{M} \) in the usual fashion \( M_{\mu\nu} = i(x_\mu p_\nu - x_\nu p_\mu) \). The noncommutative spaces compatible with the Lorentz symmetry can be obtained by realizing geometrically the Snyder algebra (1)-(5) through the so called realization method. In these spaces, the generators \( \tilde{x}_\mu \) are interpreted as noncommutative position operators associated to the sites of a lattice of typical length \( l_s \). The closure of the commutators over \( so(1,3) \) is equivalent to the statement that the lattice space is compatible with the Lorentz symmetry [11]. The functions \( \tilde{x}_\mu(x, p) \) and their commutation relations with the generators \( p_\mu \) are not determined by the Snyder algebra [20]. The realization method allows one to construct noncommutative spaces in which the functions \( \tilde{x}_\mu(x, p) \) are momentum dependent rescalings of the coordinates

\[
\tilde{x}_\mu(x, p) = \Phi_{\mu\nu}(s;p)x_\nu.
\]

It can be shown that the smooth functions take the form \( \Phi_{\mu\nu}(s;p) = \Phi_{\mu\nu}[\varphi(s;p)] \) such that

\[
\tilde{x}_\mu(x, p) = x_\mu \varphi(A) + s \langle xp \rangle p_\mu \left[ 1 + 2 \frac{d\varphi(A)}{dA} \right] \left[ \varphi(A) - 2A \frac{d\varphi(A)}{dA} \right]^{-1},
\]
where $A = \eta^\mu\nu p_\mu p_\nu$ and the commutative scalar product is denoted by $\langle ab \rangle = \eta^\mu\nu a_\mu b_\nu$. From the above equations, one can see that the Snyder geometry is a non-canonical deformation of the commutative phase space of coordinates $\{x_\mu, p_\mu\}$. The realization formalism allows one to work simultaneously with various noncommutative spaces which are characterized by different functions $\varphi(A)$. For example, the Weyl, the Snyder and the Maggiore noncommutative spacetimes can be obtained by choosing $\varphi(A) = \sqrt{A} \cot(A)$, $\varphi(A) = 1$ and $\varphi(A) = \sqrt{1 - sp^2}$, respectively. Interpolations among these spaces are also possible.

The noncommutative fluid constructed in [10] generalizes the perfect relativistic fluid models in the Clebsch parametrization. The dynamics of the commutative fluid can be derived from an action functional $S[\phi(x)]$ that depends on the density current and three real fluid potentials $\phi(x) = \{j^\mu(x), \theta(x), \alpha(x), \beta(x)\}$ by applying field theoretical methods [35]. The action of the noncommutative fluid is determined by a correspondence principle that constraints the possible noncommutative functionals such that the equations of the relativistic fluid are obtained in the commutative limit

$$\lim_{s \to 0} S_s[\hat{\phi}(\tilde{x})] = S[\phi(x)].$$

It follows that the fluid potentials must be generalized to the functions $\hat{\phi}(\tilde{x}) = \{\hat{j}^\mu(\tilde{x}), \hat{\theta}(\tilde{x}), \hat{\alpha}(\tilde{x}), \hat{\beta}(\tilde{x})\}$ from $\mathcal{F}(S)$ that should be identified with the degrees of freedom of the noncommutative fluid. It is useful to map the noncommutative and nonassociative algebra $\mathcal{F}(S)$ into the deformed algebra of the Minkowski space-time $(C^\infty(M), *)$ as follows. If $\hat{\phi}(\tilde{x})$ is a noncommutative function and 1 is the identity element of the algebra of commutative functions over $x_\mu$ then

$$\phi(x) \triangleright 1 = \psi(x),$$

where $\psi(x)$, in general, differs from $\phi(x)$. The star-product, the co-product and the anti-pode $S$ of the Poincaré co-algebra are defined by the following relations

$$e^{i(K_1^{-1}(k_1)\tilde{x})} \ast e^{i(K_2^{-1}(k_2)\tilde{x})} = e^{i(D^{(2)}(k_2, k_1)x)},$$

$$\Delta p_\mu = D^{(2)}_\mu(p \otimes 1, 1 \otimes p),$$

$$D^{(2)}_\mu (g, S(g)) = 0,$$

for any element $g$ of the deformed Poincaré group which has its co-algebra structure deformed according to the equation (11) while its Poincaré algebra is undeformed as given by the equations (11) - (13). The non-commutative functions $e^{i(k\tilde{x})}$ depend on the deformed momentum $\tilde{K}_\mu = K_\mu(k)$ and are defined by the following relation

$$e^{i(k\tilde{x})} \triangleright 1 = e^{i(K\tilde{x})}.$$

It follows from the equations (10) - (12) that the two-functions $D^{(2)}(k_2, k_1)$ determine completely the algebraic structure of the deformed algebra. By choosing the differential representation of the generators $p_\mu = -i\partial_\mu$ the star-product can be written as [18]

$$(f \ast g)(x) = \lim_{y \to x} \lim_{z \to x} \exp \left[ i \left\langle \left\{ D^{(2)}(p_y, p_z) - p_y - p_z, x \right\} \right\rangle \right].$$

As was shown in [10] the action of the noncommutative fluid that obeys the correspondence principle [35] has the following form

$$S_s[j^\mu(x), \theta(x), \alpha(x), \beta(x)] = \int d^4x \tilde{L}[\hat{\theta}(\tilde{x}), \hat{\alpha}(\tilde{x}), \hat{\beta}(\tilde{x})] \triangleright 1$$

$$= \int d^4x \left[ -j^\mu(x) \ast [\partial_\mu \theta(x) + \alpha(x) \ast \partial_\mu \beta(x)] - f_s \left( \sqrt{-j^\mu(x) \ast j_\mu(x)} \right) \right],$$

(15)
where the last equality defines a functional over \((C^\infty(M), \star)\). The function \(f_s\) from \((C^\infty(M), \star)\) is the image under the map \([\mathcal{F}]\) of an arbitrary function \(\tilde{f}\) from \(\mathcal{F}(S)\). It follows that the action \((15)\) describes a class of noncommutative fluids parameterized by \(f_s\) for any given value of \(s\).

In general, due to the lack of phenomenological information about the noncommutative fluid, it is difficult to define the relevant physical quantities such as the energy and momentum densities of the fluid. The functional approach to the noncommutative fluid has the advantage of making the definition of these quantities conceptually simpler, although their computation is diffcultuated by the nonassociativity of the star-product. The energy and momentum can be defined by the variation of the action \((15)\) under infinitesimal deformed translations \(\delta \varepsilon x_\mu\) that can be obtained from the deformed Poincaré transformations. It was shown in \([10]\) that the variation of the action under the deformed translations results in the following energy-momentum tensor

\[
T_{\mu\nu} = \Theta^\sigma_{\mu\sigma}(\phi) - \mathcal{L}_{\mu\nu},
\]

where \(\Theta^\mu\nu(\phi)\) is a functional of the fluid potentials and their derivatives up to the third order. An outstandingly difficult problem created by the noncommutativity and the nonassociativity of the star-product is to determine and to solve the equations of motion which, in general, form a system of nonlinear partial differential equations of arbitrarily high order and to obtain an analytic formula for \(T_{\mu\nu}\). This task becomes tractable at finite order in the powers of the noncommutative parameter \(s\).

3 Lower order expansion in \(s\)

In this section, we are going to determine the equations of motion and the variation of the action under the deformed Poincaré transformations at the first order in \(s\).

3.1 First order action and equations of motion

The nonassociative exponential from the star-product contains infinitely many derivatives of the fluid potentials. Therefore, the truncation of the star-product to some finite order in the powers of \(s\) is needed. The \(s\) dependence of the \(\star\)-product is encoded in the two-functions \(D^{(2)}_{\mu}(k_1, k_2)\) \([17]\)

\[
D^{(2)}_{\mu}(k_1, k_2) = \sum_{n=1}^{\infty} s^n D^{(2)n}_{\mu}(k_1, k_2).
\]

In order to obtain the linearized action in \(s\), we consider only the first two terms from the above series

\[
D^{(2)0}_{\mu}(k_1, k_2) = k_{1,\mu} + k_{2,\mu},
\]

\[
D^{(2)1}_{\mu}(k_1, k_2) = A(k_1, k_2)k_{1,\mu} + B(k_1, k_2)k_{2,\mu},
\]

where the functions \(A(k_1, k_2)\) and \(B(k_1, k_2)\) have the following form

\[
A(k_1, k_2) = c \left(k_2^2 + 2k_1k_2\right),
\]

\[
B(k_1, k_2) = \left(c - \frac{1}{2}\right) k_1^2 + \left(2c - \frac{1}{2}\right) k_1 k_2,
\]

\[
c = \frac{2c_1 + \frac{1}{2}}{2}.
\]
The real constant $c_1$ is realization dependent and has the following values: $c_1 = -1/2$ for the Maggiore, $c_1 = 0$ for the Snyder and $c_1 = -1/3$ for the Weyl realizations, respectively. The first order action can be obtained by linearizing simultaneously the star-product and the two-functions with respect to $s$. Some algebra shows that the linearized action takes the following form

$$S_s = -\int d^4x \left\{ j^\mu(x) \partial_\mu \theta(x) + ix^\mu \left[ K^s_\mu(y, z) j^\nu(y) \partial_\nu \theta(z) \right] \right|_{y=z=x} \right\}$$

$$- \int d^4x \left\{ j^\mu(x) \alpha(x) \partial_\mu \beta(x) + ix^\mu \left[ K^s_\mu(w, x) j^\nu(w) \alpha(x) \partial_\nu \beta(x) \right] \right|_{w=x} \right\}$$

$$- i \int d^4x j^\nu(x) x^\mu \left[ K^s_\mu(y, z) j^\nu(y) \partial_\nu \theta(z) \right] \alpha(y) \partial_\nu \beta(z) \big|_{y=z=x}$$

$$+ \int d^4x x^\mu x^\rho K^s_\mu(w, x) K^s_\rho(y, z) j^\nu(w) \alpha(y) \partial_\nu \beta(z) \big|_{w=y=z=x}$$

$$+ s \int d^4x x^\mu x^\rho \left[ D^{(2)0}_\mu (-i \partial^\mu_\mu - i \partial^\mu_\rho) + i \partial^\mu_\rho \right] D^{(2)1}_\mu (-i \partial^\mu_\mu - i \partial^\mu_\rho) j^\nu(w) \alpha(y) \partial_\nu \beta(z) \big|_{w=y=z=x}$$

$$+ \int d^4x f \left( - \left( 1 + ix^\mu \left[ K^s_\mu(y, z) j^\nu(y) j_\nu(z) \right] \right|_{y=z=x} \right)^{1/2} \right) ,$$

where we have used the momentum representation $k_\mu = -i \partial_\mu$ and the following notation

$$K^s_\mu(y, z) = D^{(2)0}_\mu (-i \partial^\mu_\mu - i \partial^\mu_\rho) + s D^{(2)1}_\mu (-i \partial^\mu_\mu - i \partial^\mu_\rho) + i \partial^\mu_\rho + i \partial^\mu_\sigma .$$

By direct and lengthy calculations one can show that the equation of motion of $j^\mu$ has the following form

$$- \partial_\mu \theta - \alpha \partial_\mu \beta + \partial_\nu \left( \left( f^\prime \right) \frac{f}{\rho_0} \right) x^\nu j_\mu + \frac{5}{2} s \left[ \partial^2 \partial_\mu \theta + \partial^2 (\alpha \partial_\mu \beta) \right]$$

$$+ \frac{1}{2} s x^\nu \left[ \partial^2 \alpha \partial_\mu \partial_\mu \beta + \partial_\nu \partial_\mu \partial_\rho \partial_\nu \partial_\rho \partial_\mu \beta + \partial_\nu \partial_\rho \alpha \partial^\rho \partial_\mu \beta \right]$$

$$+ i s \partial_\nu \left[ \frac{f^\prime}{2 \rho_0} \left( 2c - 1 \right) \delta^{\nu\sigma} x_\sigma \partial^2 j_\mu \right.$$\n
$$+ i s \delta^{\nu\rho} \left[ \frac{f^\prime}{2 \rho_0} x_\sigma \left( 2c - 1 \right) \eta^{\rho\sigma} \partial^2 j_\mu \right] \right] = 0 .$$

Here, $f^\prime$ represents the derivative of $f$ with respect to its argument $\rho_0 = \sqrt{-j^\mu(x) * j_\mu(x)}$. In a similar way, one can derive the equation of motion for the fluid potential $\theta$

$$\partial_\mu j^\mu + \frac{1}{2} s \partial^2 \partial_\mu j^\mu = 0 .$$

The equation of motion of the potential $\alpha$ is given by the following relation

$$- j^\mu \partial_\mu \beta + s \left( \partial^2 j^\mu \partial_\mu \beta + 2 \partial_\mu j^\nu \partial^\mu \partial_\nu \partial_\beta + j^\mu \partial^2 \partial_\mu \beta \right)$$

$$+ \frac{s}{2} x^\mu \left( \partial^2 j^\nu \partial_\mu \partial_\nu \beta + \partial_\mu j^\nu \partial^2 \partial_\nu \beta + \partial_\mu \theta \partial_\nu j^\sigma \partial^\sigma \partial_\nu \beta + \partial_\nu j^\sigma \partial_\mu \theta \partial^\nu \partial_\sigma \beta \right) = 0 .$$

Finally, one can show the equation of motion of the potential $\beta$ takes the following form

$$\partial_\mu \left( j^\mu \alpha \right) + s \left( 3 \partial_\mu j^\mu \partial^2 \alpha + 4 \partial_\mu \partial_\nu j^\nu \partial^\mu \alpha + \frac{5}{2} \partial_\mu j^\nu \partial_\nu \partial^\mu \alpha + \partial^2 j^\mu \partial_\mu \alpha + \partial^2 \partial_\mu j^\mu \alpha + j^\mu \partial^2 \partial_\mu \alpha \right) = 0 .$$
We note that the zeroth order terms of the equation of motion (25) differ from the corresponding ones from the commutative equation of the current $j^\mu$ by a term proportional to $x^\nu j^\mu$ which can be cancelled by choosing proper boundary conditions. The equation (26) shows that the current density is not conserved in the noncommutative case which is a consequence of a lack of translation symmetry. The violation of the translation invariance has been previously analysed in [48]. Nevertheless, the conservation is restored in the commutative limit.

The lowest order expansion of the energy-momentum tensor can be computed by expanding $T^\mu_\nu$ from the equation (16) in powers of $s$.

4 Plane waves in noncommutative fluids in Snyder space

The equations of motion (25), (26), (27) and (28) obtained in the previous section describe the dynamics of the noncommutative fluid at first order in $s$. Therefore, in order to understand the dynamics of the noncommutative fluid, it is important to look for analytic solutions to the system (25) - (28). In this section, we are going to determine a particular set of solutions that are monochromatic waves of fluid potentials. To this end, we observe that the equations (26), (27) and (28) form a subsystem of coupled linear equations in each argument that involve the fluid potentials $j^\mu(x)$, $\alpha(x)$ and $\beta(x)$ which belong to the algebra $(C^\infty(M), \langle \cdot \rangle)$ with the usual commutative product.

Let us solve the system by starting with the equation (26) which is independent of the potentials $\alpha(x)$ and $\beta(x)$ and is second order and linear. By a simple field redefinition which lowers the degree of the equation by one and for $s \neq 0$ the equation (26) is equivalent with the Klein-Gordon equation with the mass parameter

$$m_\alpha^2 = \frac{2}{s}. \quad (29)$$

It follows that the equation of motion of the current density admits plane wave solutions independent of the other fluid potentials

$$j^\mu_\pm(x) = \mp i A_\pm \frac{k^\mu}{k^2} e^{\pm i \langle kx \rangle}, \quad (30)$$

where $A_\pm$ is an arbitrary constant that determines the boundary value of the density current and $k^2 = \langle kk \rangle$. As can be seen from the equation (27), the solutions $j^\mu_\pm(x)$ enter as coefficients in the equation of motion of the potential $\beta(x)$. One can also lower the degree of (27) by one and after somewhat lengthy but straightforward calculations, one can show that (27) has plane wave solutions $\beta^{\pm}(x)$ which have the property that the scalar products between their gradients with the current density in the Minkowski space-time is zero

$$\langle j(x) \partial \beta^\pm(x) \rangle = 0. \quad (31)$$

The explicit form of the $\beta^{\pm}$ - waves is given by the following equation

$$\beta^{\pm}(x) = B^{\pm} e^{\mp i \langle kx \rangle}, \quad (32)$$

where $B^{\pm}$ are arbitrary constants specified by the boundary conditions for the potential $\beta^\pm(x)$. The $\beta^{\pm}$ - waves move in opposed direction to $j^\mu_\pm$ - waves. By following the same steps as above,
it is possible to show that the equation (28) has plane wave solutions of gradients that satisfy the same condition
\[ \langle j(x) \partial \alpha^\pm (x) \rangle = 0. \]  

The \( \alpha^\pm \) - wave solutions have the following form
\[ \alpha^\pm (x) = C^\pm e^{\pm i \langle kx \rangle}, \]  

where \( C^\pm \) are arbitrary constants determined by the boundary value of the field \( \alpha^\pm (x) \).

Let us focus now on the equation (25) that describes the dynamics of the potential \( \theta^\pm \) as a function of the current density and the other two fluid potentials. In order to find its solution, one needs to plug the previous wave solutions (30), (32) and (34) into (25). After doing that and after some more algebraic manipulations, we obtain the following equation
\[ \left( \partial^2 - \frac{2}{5s} \right) \partial_\mu \theta^\pm = \pm \frac{2i}{5s} \mathbf{A}^\pm k_\mu x^\nu \partial_\nu \left( \frac{f'}{\rho_0} \right) e^{\pm i \langle kx \rangle} \]
\[ - \frac{2i}{5} \left[ (\Omega^\pm_\mu (k) \eta^\nu + \Lambda^\omega_\pm_\mu (k)) \partial_\sigma + (\Gamma^\omega_\pm_\mu (k) + \Upsilon^\omega_\pm_\mu (k)) \partial^2 \sigma \right] \left( \frac{f'}{\rho_0} x^\nu e^{\pm i \langle kx \rangle} \right) \]
\[ - \frac{2i}{5s} \mathbf{D}^\pm k_\mu. \]  

Here, we have introduced the following notations for the momenta depending coefficients in an arbitrary realization
\[ \Omega^\pm_\mu (k) = \pm \frac{i}{2} \left( 2c - \frac{1}{2} \right) \mathbf{A}_\pm k_\mu, \]  
\[ \Lambda^\omega_\pm_\mu (k) = \pm \frac{i}{2} \left( 4c - \frac{1}{2} \right) \mathbf{A}_\pm k_\mu k^\nu k^\sigma \frac{1}{k^2}, \]  
\[ \Gamma^\omega_\pm_\mu (k) = \left( 2c - \frac{1}{2} \right) \mathbf{A}_\pm \eta^\omega_\mu k_\sigma \frac{1}{k^2}, \]  
\[ \Upsilon^\omega_\pm_\mu (k) = \left( 4c - \frac{1}{2} \right) \mathbf{A}_\pm \delta^\omega_\sigma k_\mu k^\nu \frac{1}{k^2}. \]  

We note that while \( j^\mu_- \), \( \alpha^\pm \) - and \( \beta^\pm \) - waves, respectively, are common to all noncommutative fluids from the class described by the linearized action (23), the potentials \( \theta^\pm (x) \) depend on the details of each particular model. As was mentioned in the previous sections, a model can be specified by choosing the arbitrary function \( f(\rho_0) \) and the fluid density function \( \rho_0 (x) \).

A simple class of models is given by \( f(\rho_0) = \lambda \rho_0^2 / 2 \) where \( \lambda \) is a real parameter and \( \rho_0 \) is an arbitrary function. This type of commutative fluids has been discussed in [37] in the Kähler parametrization, while in [43] it was shown that they can be quantized using canonical methods. Then the equation (35) takes the following simpler form
\[ \left( \partial^2 - \frac{2}{5s} \right) \partial_\mu \theta^\pm = \pm \frac{2i}{5s} \mathbf{D}^\pm k_\mu, \]  

The above equation can be integrated by standard methods or the solutions can be simply guessed. In either way, we find that the solutions \( \theta^\pm \) that correspond to \( j^\mu_- \), \( \alpha^\pm \) - and \( \beta^\pm \) - waves are linear functions on \( x \)
\[ \theta^\pm (x) = \theta^\pm \pm i \mathbf{D}^\pm \langle kx \rangle, \]  

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where $\vartheta^\pm$ are arbitrary integration constants. The fact that the $\theta^\pm$ - potentials are linear implies that they contribute by constant pieces to the fluid velocity $v^\pm_\mu \sim \partial_\mu \theta^\pm + \ldots$ in the commutative limit.

The interpretation of the above solutions is that of the noncommutatively deformed monocromatic waves propagating in the relativistic perfect fluid with a simple geometry of fields determined by the orthogonality of the gradients of the fluid potentials on the density current. These waves are solutions to the dynamics of the long wavelength degrees of freedom of a system on the noncommutative space-time which reduces in the commutative limit to the relativistic perfect fluid.

### 4.1 The energy-momentum tensor

An important quantity that contains information about the thermodynamics of the relativistic fluid is the energy-momentum tensor. For the effective field theory that describes the noncommutative fluid, it has the general form given by the equation (16). By expanding in powers of $s$ and retaining only the linear terms we obtain the linearized energy-momentum tensor $T^\mu_\nu (j, \theta, \alpha, \beta)$. From it, one can calculate the energy-momentum tensor of the wave potentials $T^\mu_\nu (j^\pm, \theta^\pm, \alpha^\pm, \beta^\pm)$. After lengthy computations, one can show that the energy-momentum tensor takes the following form

$$
T^\mu_\nu (j^\pm, \theta^\pm, \alpha^\pm, \beta^\pm) = A^\pm (D^\pm - C^\pm B^\pm) e^{\pm i (kx)} \left( \eta^\mu_\nu - \frac{k^\mu k_\nu}{k^2} \right)
$$

$$
+ \frac{s}{4} (A^\pm)^2 \frac{e^{\pm 2i (kx)}}{k^4} \left[ \left( \frac{7}{2} \pm 2i \langle kx \rangle \right) k^\mu k_\nu \pm 2ik^2 x^\mu k_\nu \right]
$$

$$
+ \frac{s}{2} A^\pm D_\pm e^{\pm i (kx)} \left[ (1 \mp i \langle kx \rangle) k^\mu k_\nu \pm ik^2 \langle kx \rangle \eta^\mu_\nu \right]
$$

$$
\mp \frac{is}{2} A^\pm B^\pm C^\pm e^{\mp i (kx)} \left[ - (8 \mp 7i) \langle kx \rangle k^\mu k_\nu + (3 \pm 5i) k^\mu k_\nu + (2 \mp i) k^2 x^\mu k_\nu \right]
$$

(41)

Note that the energy-momentum tensor of the single mode wave fluids in the noncommutative fluid model studied above is complex as it is in the case of spinor fields and $N = 1$ chiral supergravity in the commutative space-time. As can be seen from the equation (41), the imaginary part is a consequence of the noncommutativity of space-time. The noncommutative deformation of the energy-momentum tensor of the relativistic fluid contains off-diagonal terms that are not proportional to the fluid velocity component as is the case in the commutative space-time, but rather with the space-time coordinates. This shows that as the wave moves in the space, there is an exchange of energy and momentum of the different terms from (41) with the noncommutative space-time. A similar phenomenon occurs with nonlocal systems in the gravitational field where the normal modes exchange energy with the gravitational field viewed as a fixed (non-trivial) background which can lead to thermalization.

### 5 Concluding remarks

In this paper we have addressed the question of the dynamics of the noncommutative fluid in the Snyder space at first order in the expansion of the star-product, co-product and the anti-pode in powers of the noncommutative parameter $s$ and have shown that the linearized equations of motion of the fluid density and the fluid potentials form a system of linear partial
differential equations. We have determined a class of perturbative analytic solutions of these equations that describe monocromatic plane waves of \( j^\mu \) and \( \alpha \) and \( \beta \) potentials. For the class of fluids characterized by \( f(\rho_0) = \lambda \rho_0^2 / 2 \) the equation of motion of the potential \( \theta \) admits a linear solution. We have calculated the energy-momentum tensor of these solutions and shown that it has a complex structure determined by the noncommutativity of space-time at first order in \( s \). The physical interpretation of the diagonal terms that are proportional to \( s \) is that they are proportional to the noncommutative corrections to the energy density and pressure density of the monocromatic waves propagating in the relativistic perfect fluid. However, there are non-zero off-diagonal terms which are not proportional with the velocity of the fluid particles and are not present in the perfect commutative fluid. These terms arise at the long wave scale from the gradients of the fluid potentials and density current along the noncommutative directions of space-time. From the form of the energy-momentum tensor we conclude that its components proportional to \( s \) describe the exchange of energy and momentum between the monocromatic waves and the noncommutative space-time similarly to the phenomenon that occurs in the gravitational field.

We note that the noncommutative parameter determines at first order the dispersion relations of the monocromatic wave potentials as follows

\[
\begin{align*}
  k_j^2 - \frac{2}{s} &= 0, \\
  k_\alpha^2 &= \frac{1 - s}{3s}, \\
  k_\beta^2 &= 0,
\end{align*}
\]

which can be used to determine the on-shell solutions and energy-momentum tensor. In the commutative limit \( s \to 0 \) the dispersion relations for the density of current density and \( \alpha \) - potential diverge as do all terms analogous to the masses in the equations of motion. This shows that the noncommutative waves behave as infinitely massive fields in the commutative limit. Their dynamics in this case is no longer described by wave - equations since the formal step taken to derive these equations is invalid in this limit. The proper equations of motion of the commutative fluid are defined by the commutative limit of the equations (25) - (28). This render, for example, a conservation equation for the current density. Another important property of the solutions obtained in (30), (32), (34) and (40) is that the geometry of the waves is such that the gradients of \( \alpha \) and \( \beta \) potentials is normal to the direction of \( j^\mu \). However, the plane waves do not create vorticity in the linear approximation of the noncommutative fluid regardless the choice of the parameters \( f \) and \( \rho_0 \) which is an expected result since the vorticity is not a well defined concept in the Snyder space. Nevertheless, the model can still produce in the commutative limit \( s \to 0 \) a rotational fluid.

The results presented in this paper provide a new insight in the dynamics of the noncommutative fluids. They are interesting by themselves, as well as for offering a novel explicit construction of noncommutative effective field theories in which the energy-momentum tensor is calculated from the scratch in the linear approximation. It is interesting to study further the possible solutions of the noncommutative fluid dynamics and their thermodynamics as well as to extend it to incorporate charges, which could be useful for developing phenomenological models. We are going to report on these aspects in future.

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