Analytical formulas for shear and bulk viscosities in relativistic gaseous mixtures with constant cross sections.

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Abstract

Using Mathematica package, we derive analytical closed-form expressions for the shear and the bulk viscosity coefficients in multicomponent relativistic gases with constant cross sections, being the relativistic generalization for the hard spheres model. Some of them are cumbersome and require symbolic manipulations in an algebraic package. The constant cross sections are of the elastic processes, while the inelastic (or number-changing) processes (collisions or decays) are considered only partly. As examples, we find explicit expressions of the correct single-component first-order shear viscosity coefficient and some explicit analytical results for the binary mixture. These formulas have numerous applications in approximate nonequilibrium descriptions of gases of particles or quasiparticles with averaged (temperature dependent) cross sections. In addition to this, we present formulas for collision rates and some other related formulas. This paper is a shortened version of a previous one.

Keywords: bulk viscosity, shear viscosity, kinetics, transport theory

PACS: 47.45.Ab, 51.20.+d, 05.20.Dd

1. Introduction

The bulk and the shear viscosity coefficients are transport coefficients which enter the hydrodynamic equations, and, thus, are important for studying of nonequilibrium evolution of any thermodynamic system. In this regard, another way of dissipative nonequilibrium description can be mentioned [1, 2]. In rarefied gases of particles or quasiparticles with short-range interactions the viscosities can be calculated in a perturbative regime [3]. The leading contribution can be obtained in the framework of the Boltzmann equation (BE), derivable within

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[1] In non-abelian gauge theories there are also contributions from non-perturbative effects [2].
the BBGKY hierarchy with the well-known assumption for correlations \[4\] (Sec. 16). The BE’s justification from first principles and the next-to-leading order corrections have been obtained in calculations for weakly coupled quantum field theories by the Kubo (or Green-Kubo\[4\]) formulas \[7, 8, 9, 10\].

The aim of this paper is to derive analytical closed-form (through special and/or elementary functions) expressions for the shear and the bulk viscosities in multicomponent relativistic gases with constant cross sections, and similarly for the collision rates and some other related quantities. Previously the single-component viscosities were obtained in \[12\] (we correct the shear viscosity), being the relativistic generalization of the ones in the hard spheres model \[4\] (Secs. 8, 10).

The structure of the paper is the following. Sec. 2 contains some comments on cross sections, connection with the hard spheres model, most of the used designations and methodology. In Sec. 4 we present explicit analytical results for the single-component gas, including the ones for the nonequilibrium distribution function (DF) and some analysis for the inelastic processes. We also present some explicit expressions for the binary mixture and the collision rates (and related quantities) in the multicomponent mixture. In Sec. 5 we discuss significance and applications of the obtained formulas. In Sec. 6 we state the conclusions. Transformations of collision brackets, being the 12-dimensional integrals, which enter the viscosities, and some analytical formulas for them can be found in Appendix A.

This paper is a shortened version of the previous one \[25\].

2. Methodology

2.1. Some comments on cross sections and effective radii

For compatibility with previously obtained results and from practical considerations we want to introduce an effective radius \(r\) through the hard core repulsion model or the model of hard spheres. The differential scattering cross section for this model can be inferred from the problem of scattering of point particle on the spherical potential \(U(r) = \infty\) if \(r \leq a\) and \(U(r) = 0\) if \(r > a\) \[13\]. In this model the differential cross section is equal to \(a^2/4\). To apply this result to the gas of hard spheres with the radius \(r\) one can notice that the scattering of any two spheres can be considered as the scattering of the point particle on the sphere of the radius \(2r\), so that one should take \(a = 2r\). The total cross section \(\sigma_{\text{tot}}\) is obtained after integration over the angles of \(r^2d\Omega\), which results in the \(\sigma_{\text{tot}} = 4\pi r^2\). For collisions of hard spheres of different radiuses one should take \(a = r_k + r_l\) or replace the \(r^2\) on the \(\frac{r_k^2 + r_l^2}{2}\):

\[
\sigma_{\text{tot},kl} = \pi(r_k + r_l)^2. \tag{1}
\]

\[4\] The Kubo formulas are distinguished from the Green-Kubo formulas, e. g., in \[3, 6\].

\[5\] There is, however, a special important issue connected with particle number conservation/nonconservation for the bulk viscosity \[11\].
The relativistic generalization of this model is the constant (not dependent on the scattering energy and angle) differential cross sections model.

The hard spheres model is classical, and connection of its cross sections to cross sections, calculated in any quantum theory, is needed. For particles, having a spin, the differential cross sections averaged over the initial spin states and summed over the final ones will be used. If colliding particles are identical and their differential cross section is integrated over the momentums (or the spatial angle to get the total cross section) then it should be multiplied on the factor $\frac{1}{2}$ to cancel double counting of the momentum states. These factors are exactly the factors $\gamma_{kl}$ next to the collision integrals in the BEs (17). The differential cross sections times these factors will be called (adopting the terminology mentioned in [14]) the classical differential cross sections.

2.2. The system of the BEs and its solution

The methodology in this paper goes close to the ones in the monograph [14], though with some differences (including corrections of a couple of typos) and generalizations. We find it very instructive to include compilation of some pieces of the methodology (some of which are absent in the [14]) for convenience and completeness, adding some comments and tacitly answering some questions. We use units $\hbar = c = k_B = 1$ throughout the paper by default. Conversion factors can be found, e. g., in [15]. Let’s start from some definitions. We use the designations of the [14] mostly. Multi-indices $k, l, m, n$ will be used to denote particle species with certain spin states. Indexes $k', l', m', n'$ will be used to denote particle species without regard to their spin states (and run from 1 to the number of the particle species $N'$) and $a, b$ to denote conserved quantum numbers. Quantifiers $\forall$ with respect to the indexes are omitted in the text where they may be needed, which won’t result in a confusion. As nothing depends on spin variables, one has for every sum over the multi-indices

$$\sum_k \ldots = \sum_{k'} g_{k'} \ldots, \quad (2)$$

where $g_{k'}$ is the spin degeneracy factor. The following assignments will be used:

$$n \equiv \sum_k n_k \equiv \sum_{k'} n_{k'}, \quad n_a \equiv \sum_k q_{ak} n_k, \quad x_k \equiv \frac{n_k}{n}, \quad x_a \equiv \frac{n_a}{n},$$

$$\hat{\mu}_k = \frac{\mu_k}{T}, \quad \hat{\mu}_a = \frac{\mu_a}{T}, \quad z_k = \frac{m_k}{T}, \quad \pi^\mu_k = \frac{p^\mu_k}{T}, \quad \tau_k = \frac{p^\mu U_\mu}{T}, \quad (3)$$

6It’s assumed that particle numbers of the same species but with different spin states are equal. If this were not so, then in approximation in which the spin interactions are neglected and probabilities to have certain spin states are equal the numbers of the particles with different spin states would be approximately equal in the mean free time. With equal particle numbers their DFs are equal too. This allows one to use the summed over the final states cross sections in the BEs.

7In systems with only the elastic collisions each particle species have their own “conserved quantum number”, equal to 1.
where $q_{ak}$ denotes values of conserved quantum numbers of the $a$-th kind of the $k$-th particle species. Everywhere the particle number densities are summed the spin degeneracy factor $g_k$ appears and then gets absorbed into the $n_{k'}$ or the $x_{k'}$ by the definition. All other quantities with primed and unprimed indexes don’t differ, except for rates, the mean free times and the mean free paths defined in Sec. 4.3, the $\gamma_{kl}$ commented below, the coefficients $A^{rs}_{kl'}$, $C^{rs}_{kl'}$ and, of course, quantities, whose free indexes set the indexes of the particle number densities $n_k$. The assignment $\int \frac{d^3p_k}{p_k} \equiv \int p_k f_k$ will also be used for compactness somewhere.

The particle number flows are

$$N_k^\mu = \int \frac{d^3p_k}{(2\pi)^3 p_k^\mu} p_k^\mu f_k,$$  \hspace{1cm} (4)

where the assignment $f_k(p_k) \equiv f_k$ is introduced. The energy-momentum tensor is

$$T^{\mu\nu} = \sum_k \int \frac{d^3p_k}{(2\pi)^3 p_k^\mu p_k^\nu} p_k^\mu p_k^\nu f_k.$$  \hspace{1cm} (5)

The local equilibrium DFs are

$$f_k^{(0)} = e^{(\mu_k - p_k^\nu U_\nu)/T},$$  \hspace{1cm} (6)

where $\mu_k$ is the chemical potential of the $k$-th particle species, $T$ is the temperature and $U_\nu$ is the relativistic flow 4-velocity such that $U_\mu U^\mu = 1$ (with a frequently used consequence $U_\mu \partial_\nu U^\mu = 0$). The local equilibrium implies perturbation of the independent thermodynamic variables and the flow velocity over a global equilibrium (see just below) such that they can depend on the space-time coordinate $x^\mu$. We assume chemical equilibrium, which implies that the particle number densities are equal to their global equilibrium values. We call the global equilibrium as the time-independent stationary state with the maximal entropy:

The kinetic equilibrium implies that the momentum distributions are the same as in the global equilibrium. Thus, a state of a system with both the pointwise (for the whole system) kinetic and the pointwise chemical equilibria is the global equilibrium.

\[8\] The $+,-,-,-$ metric signature is used throughout the paper.

\[9\] The kinetic equilibrium implies that the momentum distributions are the same as in the global equilibrium. Thus, a state of a system with both the pointwise (for the whole system) kinetic and the pointwise chemical equilibria is the global equilibrium.
where \(\beta, \lambda\) are the Lagrange coefficients. Equating the first variation to zero, one easily gets the function \(U\) with \(U^\mu = (1, 0, 0, 0)\), \(\beta = \frac{1}{T}\) and

\[
\mu_k = \sum_a q_{ak}\mu_a,
\]

(8)

where \(\mu_a = \lambda_a\) are the independent chemical potentials coupled to the conserved net charges.

With \(f_k = f_k^{(0)}\), substituted in the (14) and the (15), one gets the leading contribution in the gradients expansion of the particle number flow and the energy-momentum tensor:

\[
N_k^{(0)\mu} = n_k U^\mu,
\]

(9)

\[
T^{(0)\mu\nu} = \epsilon U^\mu U^\nu - P \Delta^{\mu\nu},
\]

(10)

where the projector

\[
\Delta^{\mu\nu} \equiv g^{\mu\nu} - U^\mu U^\nu,
\]

(11)

is introduced. Above \(n_k\) is the ideal gas (IG) particle number density, \(\epsilon\) is the IG energy density, \(P\) is the IG pressure [14] (Chap. II, Sec. 4). Also, the following assignments are used:

\[
\hat{e}_k \equiv \frac{e_k}{T} = z_k \frac{K_3(z_k)}{K_2(z_k)} - 1, \quad \hat{e} \equiv \frac{e}{T}, \quad \hat{h}_k \equiv \frac{h_k}{T} = z_k \frac{K_3(z_k)}{K_2(z_k)}, \quad \hat{h} \equiv \frac{h}{T}.
\]

(12)

Above \(h\) is the enthalpy per particle, \(e\) is the energy per particle and \(h_k, e_k\) are the enthalpy and the energy per particle of the \(k\)-th particle species correspondingly, which are well defined in the IG.

In the relativistic hydrodynamics the flow velocity, \(U^\mu\), needs somewhat extended definition. A convenient condition which can be applied to the \(U^\mu\) is the Landau-Lifshitz condition [17] (Sec. 136). This condition states that in the local rest frame (where the flow velocity is zero though its gradient can have a nonzero value) each imaginary infinitesimal cell of fluid should have zero momentum, and its energy density and the charge density should be related to other thermodynamic quantities through the equilibrium thermodynamic relations (without a contribution of nonequilibrium dissipations). Its covariant mathematical formulation is

\[
(T^{\mu\nu} - T^{(0)\mu\nu})U_\mu = 0, \quad (N_\mu^a - N_a^{(0)\mu})U_\mu = 0.
\]

(13)

The next to leading correction over the gradients expansion to the \(T^{\mu\nu}\) can be written as an expansion over the 1-st order Lorentz covariant gradients, which
are rotationally and space inversion invariant and satisfy the Landau-Lifshitz condition\textsuperscript{[10]}:

\[
T^{(1)\mu\nu} \equiv 2\eta \nabla^\mu U^\nu + \xi \Delta^{\mu\nu} \nabla^\rho U^\rho
\]

(14)

\[
= \eta \left( \Delta^\mu_\nu \Delta^\rho + \Delta^\nu_\mu \Delta^\rho - \frac{2}{3} \Delta^\mu_\nu \Delta^\rho \right) \nabla^\rho U^\rho + \xi \Delta^{\mu\nu} \nabla^\rho U^\rho,
\]

where for any tensor \( a_{\mu\nu} \), the symmetrized traceless tensor assignment is introduced:

\[
\frac{\partial}{\partial a_{\mu\nu}} \equiv \left( \Delta^\mu_\nu \Delta^\rho + \Delta^\nu_\mu \Delta^\rho - \frac{1}{3} \Delta^\mu_\nu \Delta^\rho \right) a^{\rho\tau} \equiv \Delta^{\mu\nu}_{\rho\tau} a^{\rho\tau},
\]

(15)

The equation (14) is the definition of the shear \( \eta \) and the bulk \( \xi \) viscosity coefficients. The \( \xi \Delta^{\mu\nu} \nabla^\rho U^\rho \) term in the (14) can be considered as a nonequilibrium contribution to the pressure, entering the (10).

By means of the projector (11) one can split the space-time derivative \( \partial_\mu \) as

\[
\partial_\mu = U_\mu U^\nu \partial_\nu + \Delta^\nu_\mu \partial_\nu = U_\mu + \nabla_\mu,
\]

(16)

where \( D \equiv U^\nu \partial_\nu \), \( \nabla_\mu \equiv \Delta^\nu_\mu \partial_\nu \). In the local rest frame (where \( U^\mu = (1, 0, 0, 0) \)) the \( D \) becomes the time derivative and the \( \nabla_\mu \) becomes the spatial derivative. Then the BEs can be written in the form

\[
P^\mu_k \partial_\mu f_k = (P^\mu_k U_{\mu} D + P^\mu_k \nabla_\mu)f_k = C^{el}_k[f_k] + C^{inel}_k[f_k],
\]

(17)

where \( C^{inel}_k[f_k] \) represents the inelastic collision integrals (it is omitted in calculations in this paper if the opposite is not stated explicitly) and \( C^{el}_k[f_k] \) is the elastic \( 2 \leftrightarrow 2 \) collision integral. The collision integral \( C^{el}_k[f_k] \) has the form of the sum of positive gain terms and negative loss terms. Its explicit form is\textsuperscript{[11]} (cf.\textsuperscript{[12]})

\[
C^{el}_k[f_k] = \sum_l \gamma_{kl} \frac{1}{2} \int \frac{d^3p_\mu}{(2\pi)^3} \frac{d^3p'_\mu}{(2\pi)^3} \frac{d^3p''_{\mu}}{2\pi^3} \frac{d^3p''_{\mu}}{2\pi^3} \frac{d^3p''_{\mu}}{2\pi^3} (f'_k f''_l - f_k f''_l)
\]

\[
\times |M_{kl}|^2(2\pi)^4 \delta^4(p'_k + p''_l - p_k - p_l),
\]

(18)

where \( \gamma_{kl} = \frac{1}{2} \) if \( k \) and \( l \) denote the same particle species without regard to the spin states and \( \gamma_{kl} = 1 \) otherwise, \( |M_{kl}(p'_k, p''_l; p_k, p_l)|^2 \equiv |M_{kl}|^2 \) is the square of the dimensionless elastic scattering amplitude averaged over the initial spin states and summed over the final ones. Index 1 designates that \( p_k \) and \( p_{1k} \) are different variables. Introducing \( W_{kl} \equiv W_{kl}(p'_k, p''_l; p_k, p_l) \) as

\[
W_{kl} = \frac{|M_{kl}|^2}{64\pi^2} \delta^4(p'_k + p''_l - p_k - p_l),
\]

(19)

\textsuperscript{10} This form of \( T^{(1)\mu\nu} \) also respects the second law of thermodynamics\textsuperscript{[12]} (Sec. 13).

\textsuperscript{11} The factor \( \gamma_{kl} \) cancels double counting in integration over momentums of identical particles. The factor \( \frac{1}{2} \) comes from the relativistic normalization of the scattering amplitudes.
the momentum $p$. This means that the $\phi$ in the l. h. s. should be cancelled by the first nonvanishing contribution to $T$ if the $f$ can easily prove [14] (Chap. II, Sec. 1):}

\begin{equation}
C^{\mu\nu}_{k}[f_k] = (2\pi)^3 \sum_{l} \gamma_{kl} \int_{p_{1l}p_{k}l} \left( \frac{f'_{k}}{(2\pi)^3} - \frac{f_{k}}{(2\pi)^3} \right) W_{kl}. \tag{20}
\end{equation}

The $W_{kl}$ is related to the elastic differential cross section $\sigma_{kl}$ as [14] (Chap. I, Sec. 2)

\begin{equation}
W_{kl} = s\sigma_{kl} \delta^{4}(p'_{k} + p_{1l} - p_{k} - p_{1l}), \tag{21}
\end{equation}

where $s = (p_{k} + p_{1l})^2$ is the usual Mandelstam variable. The $W_{kl}$ has properties $W_{kl}(p'_{k}, p'_{1l}; p_{k}, p_{1l}) = W_{kl}(p_{k}, p_{1l}; p'_{k}, p'_{1l}) = W_{lk}(p'_{1l}, p'_{k}; p_{1l}, p_{k})$ (due to time reversibility and a freedom of relabelling of order numbers of particles taking part in reaction). And, e. g., $W_{kl}(p'_{k}, p'_{1l}; p_{k}, p_{1l}) \neq W_{kl}(p'_{1l}, p'_{k}; p_{1l}, p_{k})$ in the general case. The elastic collision integrals have important properties which one can easily prove [14] (Chap. II, Sec. 1):

\begin{equation}
\int \frac{d^3p_k}{(2\pi)^3 p_k^0} C^{\mu\nu}_{k}[f_k] = 0, \tag{22}
\end{equation}

\begin{equation}
\sum_{k} \int \frac{d^3p_k}{(2\pi)^3 p_k^0} p^\mu_k C^{\mu\nu}_{k}[f_k] = 0. \tag{23}
\end{equation}

Also, the $C^{\mu\nu}_{k}[f_k]$ vanishes if $f_k = f_k^{(0)}$.

The DFs $f_k$ solving the system of the BEs approximately are sought in the form (below the essence of the Chapman-Enskog method is reproduced, see, e. g., [14], Chap. V)

\begin{equation}
f_k = f_k^{(0)} + f_k^{(1)} = f_k^{(0)} + f_k^{(0)} \varphi_k(x, p_k), \tag{24}
\end{equation}

where it’s assumed that $f_k$ depend on the $x^\mu$ entirely through the $T$, $\mu_k$, $U^\mu$ or their space-time derivatives. It is also assumed that $|\varphi_k| \ll 1$. After substitution of $f_k = f_k^{(0)}$ in the (17) the r. h. s. becomes zero and the l. h. s. is zero only if the $T$, $\mu_k$ and $U^\mu$ don’t depend on the $x^\mu$ (provided they don’t depend on the momentum $p_k^0$). The 1-st order space-time derivatives of the $T$, $\mu_k$, $U^\mu$ in the l. h. s. should be cancelled by the first nonvanishing contribution in the r. h. s. This means that the $\varphi_k$ should be proportional to the 1-st order space-time derivatives of the $T$, $\mu_k$, $U^\mu$. The covariant time derivatives $D$ can be expressed through the covariant spacial derivatives by means of approximate hydrodynamic equations, valid at the same order in the gradients expansion. Let’s derive them. Integrating the (17) over the $\frac{d^3p_k}{(2\pi)^3 p_k^0}$ with the $f_k = f_k^{(0)}$ in the l. h. s. with the inelastic collision integrals retained and using the (22) and the (1) one would get (which can be justified using explicit form of the inelastic collision integrals)

\begin{equation}
\partial_\mu N_k^{(0)} = Dn_k + n_k \nabla_\mu U^\mu = I_k, \tag{25}
\end{equation}
where $I_k$ is the sum of the inelastic collision integrals integrated over the momentum. It is responsible for the nonconservation of the total particle number of the $k$-th particle species and has the property $\sum_k q_{ak} I_k = 0$. If $C_k^{inel}[f_k] = 0$, then $I_k = 0$ which results in conservation of the total particle numbers of each particle species. Multiplying the (25) on the $q_{ak}$ and summing over $k$ one gets the continuity equations for the net charge flows (cf. [14], Chap. II, Sec. 1):

$$\partial_\mu N_\alpha^{(0)\mu} = Dn_a + n_a \nabla_\mu U^\mu = 0.$$  

(26)

Also, integrating the (17) over the $p_\mu^k \frac{d^3p_k}{(2\pi)^3 p_k^0}$ with the $f_k = f_k^{(0)}$ in the l. h. s., one gets

$$\partial_\mu T^{(0)\mu} = \partial_\mu (\epsilon^\mu U^\nu - P \Delta^{\mu\nu}) = 0.$$  

(27)

There is zero in the r. h. s. even if the inelastic collision integrals are retained because they respect energy conservation too. Note that the BEs (17) (without any thermal corrections) permit a self-consistent description only if the energy-momentum tensor and the net charge flows of the IG are used. After the convolution of the (27) with the $\Delta_\mu^\nu$ one gets the Euler’s equation:

$$DU^\mu = \frac{1}{\epsilon + P} \nabla^\mu P = \frac{1}{hn} \nabla^\mu P.$$  

(28)

After the convolution of the (27) with the $U_\nu$ one gets equation for the energy density:

$$D\epsilon = -(\epsilon + P) \nabla_\mu U^\mu = -hn \nabla_\mu U^\mu.$$  

(29)

To proceed farther one needs to expand the l. h. s. of the BEs (17) over the gradients of thermodynamic variables and the flow velocity. Let’s choose the $\mu_a$ and the $T$ as the independent thermodynamic variables. Then for the $Df_k^{(0)}$ one can write the expansion

$$Df_k^{(0)} = \sum_a \frac{\partial f_k^{(0)}}{\partial \mu_a} D\mu_a + \frac{\partial f_k^{(0)}}{\partial T} DT + \frac{\partial f_k^{(0)}}{\partial U^\mu} DU^\mu.$$  

(30)

Writing the expansion for the $Dn_a$ and the $D\epsilon$ one gets from the (26) and the (29):

$$Dn_a = \sum_b \frac{\partial n_a}{\partial \mu_b} D\mu_b + \frac{\partial n_a}{\partial T} DT = -n_a \nabla_\mu U^\mu,$$  

(31)

$$D\epsilon = \frac{\partial \epsilon}{\partial T} DT + \sum_a \frac{\partial \epsilon}{\partial \mu_a} D\mu_a = -hn \nabla_\mu U^\mu.$$  

(32)

The solution to the system of equations (31), (32) can be found easily:

$$DT = -RT \nabla_\mu U^\mu.$$  

(33)
\[ D\mu_a = T \sum_b \tilde{A}_{ab}^{-1} (RB_b - x_b) \nabla_\mu U^\mu, \]  

where

\[ R = \frac{\hat{\mathcal{h}} - \sum_{a,b} E_a \tilde{A}_{ab}^{-1} x_b}{C(\mu) - \sum_{a,b} E_a \tilde{A}_{ab}^{-1} B_b}, \]  

and

\[ \frac{\partial n_a}{\partial \mu_b} = \frac{n}{T} \tilde{\mathcal{A}}_{ab}, \quad \frac{\partial n_a}{\partial T} = \frac{n}{T} B_a, \quad \frac{\partial \epsilon}{\partial T} = nC(\mu), \quad \frac{\partial \epsilon}{\partial \mu_a} \equiv nE_a. \]  

Above it is assumed that the matrix \( \tilde{A}_{ab} \) is not degenerate\(^{12}\), which is related to the self-consistency of the statistical description of the system. Using the \( n_k \) and \( \epsilon \) IG formulas one gets

\[ \tilde{A}_{ab} = \sum_k q_a q_b x_k, \quad E_a = \sum_k q_a x_k \hat{e}_k, \quad B_a = E_a - \sum_b \tilde{A}_{ab} \hat{\mu}_b, \]  

\[ C(\mu) = \sum_k x_k (3\hat{h}_k + z_k^2 - \hat{\mu}_k \hat{e}_k) = \sum_k x_k (3\hat{h}_k + z_k^2) - \sum_a E_a \hat{\mu}_a = \tilde{C}(\mu) - \sum_a E_a \hat{\mu}_a, \]  

and simplified expressions for the \( R \) and the \( D\hat{\mu}_a \)

\[ R = \frac{\hat{\mathcal{h}} - \sum_{a,b} E_a \tilde{A}_{ab}^{-1} x_b}{C(\mu) - \sum_{a,b} E_a \tilde{A}_{ab}^{-1} E_b}, \]  

\[ D\hat{\mu}_a = \sum_b \tilde{A}_{ab}^{-1} (RE_b - x_b) \nabla_\mu U^\mu. \]  

For the special case of the vanishing chemical potentials, \( \mu_a \to 0 \), (for a chargeless system the result is the same) the quantities \( n_a, x_a, B_a, E_a \) tend to zero because the contributions from particles and anti-particles cancel each other and the chargeless particles don’t contribute. Then from the \((33)\) and the \((34)\) one finds

\[ DT|_{\mu_a=0} = -\frac{h}{C(\mu)} \nabla_\mu U^\mu, \quad D\mu_a|_{\mu_a=0} = 0. \]  

So if there is an anti-particle for each charged particle (which is so for the exactly conserved charges), vanishing chemical potentials are equivalent to their

\(^{12}\)One can prove that the \( N'' \times N'' \) matrix \( \tilde{A}_{ab} \) in \((37)\) is not degenerate if there are \( N'' \) linearly independent conserved charges. Then one can prove that the denominator in the \((38)\) is not zero.
exclusion from the DFs provided the spacial derivatives of the chemical potentials can be excluded. In systems with only the elastic collisions each particle has its own charge, so that one takes $q_{ak} = \delta_{ak}$ and gets

$$\tilde{A}_{kl} = \delta_{kl}x_k, \quad B_k = x_k(\hat{e}_k - \hat{\mu}_k), \quad E_k = \hat{e}_kx_k, \quad R = \frac{1}{c_v}, \quad (41)$$

$$C_{(\mu)} - \sum_{a,b} E_a \tilde{A}_{ab}^{-1} B_b = \sum_k x_k(-\hat{h}_k^2 + 5\hat{h}_k + z_k^2 - 1) \equiv \sum_k x_k c_{v,k} \equiv c_v.$$  

Then, the equation for the $DT$ (33) remains the same with a new $R$ from the (41), and the equations (34) become

$$D_{\mu k} = \left(\frac{T}{c_v}(\hat{e}_k - \hat{\mu}_k) - T\right) \nabla_{\mu} U^\mu. \quad (42)$$

Note that in systems with only the elastic collisions the $D_{\mu k}$ does not tend to zero for the vanishing chemical potentials so that the $\mu_k$ could not be omitted in the DFs in this case. As the heat conductivity and diffusion are not considered in this paper their nonequilibrium gradients are taken equal to zero, $\nabla_{\nu} P = \nabla_{\nu} T = \nabla_{\nu} \mu_a = 0$. Using the (33), (34) and (28) the l. h. s. of the (17) can be transformed as

$$(p^\mu_k U_\mu D + p^\nu_k \nabla_\nu) f_k^{(0)} = -T f_k^{(0)} \pi^\mu_k \pi^\nu_k \nabla_\mu U^\nu_{\circ} + T f_k^{(0)} \tilde{Q}_k \nabla_\mu U^\mu_{\circ}, \quad (43)$$

where

$$\tilde{Q}_k \equiv \tau_k^2 \left(\frac{4}{3} - R\right) + \tau_k \sum_{a,b} q_{ak} \tilde{A}_{ab}^{-1}(RE_b - x_b) - \frac{1}{3} z_k^2. \quad (44)$$

The $\tilde{Q}_k$ is of a universal and convenient form, see Sec. 5 for discussions. Using the (14) one can notice that the useful (tacitly used) equality $\pi^\mu_k \pi^\nu_k \nabla_\mu U^\nu_{\circ} = \pi^\nu_k \pi^\mu_k \nabla_\nu U^\mu_{\circ}$ holds. In systems with only the elastic collisions the $\tilde{Q}_k$ simplifies in agreement with (14) (Chap. V, Sec. 1):

$$\tilde{Q}_k = \left(\frac{4}{3} - \gamma\right) \tau_k^2 + \tau_k ((\gamma - 1)\hat{h}_k - \gamma) - \frac{1}{3} z_k^2. \quad (45)$$

The $\gamma$ from the (14) can be expressed through the $c_v$ (14) as $\gamma \equiv \frac{1}{c_v} + 1$. The approximate solution of the BEs, which we use in what follows, is connected with inner product, denoted as

$$(F, G)_k \equiv \frac{1}{4\pi z_k^2 K_2(z_k)T^2} \int_{p_k} F(p_k)G(p_k)e^{-\tau_k}. \quad (46)$$

Also, we use assignments

$$\alpha_k^\tau \equiv (\tilde{Q}_k, \tau_k^\tau)_k, \quad \gamma_k^\tau \equiv (\tau_k^\tau \pi_\nu_k \pi^\nu_k, \pi^\mu_k \pi^\nu_k k_{\mu\nu}\tilde{Q}_k)_k, \quad a_k^\tau \equiv (1, \tau_k^\tau)_k. \quad (47)$$
Expressions of the \( \alpha_k^r \) and the \( \gamma_k^r \) through the \( a_k^r \), the recurrence relations for the \( a_k^r \), \( \gamma_k^r \) and \( a_k^r \) can be found in the [14] (Chap. VI, Sec. 1, App.). Using the latter ones, we find for the quantities of a special interest \( \alpha_k^0 \) and \( \alpha_k^1 \) in systems with elastic and inelastic processes

\[
\alpha_k^0 = 1 + \sum_{a,b} q_{ak} \hat{A}_{ab}^{-1} (RE_b - x_b) - \hat{e}_k R, \tag{48}
\]

\[
\alpha_k^1 = \hat{h}_k + \sum_{a,b} \hat{e}_k q_{ak} \hat{A}_{ab}^{-1} (RE_b - x_b) - (3\hat{h}_k + z_k^2) R. \tag{49}
\]

The \( \alpha_k^0 \) and the \( \alpha_k^1 \) should satisfy the equations [14] (Chap. VI, Sec. 3)

\[
\sum_k a_k x_k \alpha_k^0 = 0, \tag{50}
\]

\[
\sum_k x_k \alpha_k^1 = 0, \tag{51}
\]

(for consistency) which can be explicitly checked using the (48) and the (49).

In the partial case of only elastic processes one gets [14] (Chap. VI, Sec. 3)

\[
\alpha_k^0 = 0. \tag{52}
\]

The (50) and the (51) also have relation to the local conservation laws, being warranted by the conservation laws (26) and (27). Analogical quantities \((1, \pi_k^0 \pi_k^0)\) and \((p_k^0, \pi_k^0 \pi_k^0)\) vanish automatically because of the special tensorial structure\(^{13}\) of the \( \pi_k^0 \).

The next step is to transform the r. h. s. of the BEs (17). After the substitution of the (24) in the r. h. s. of the (17) the collision integrals become linear, and one gets

\[
C_{kl}^e[f_k] \approx -f_k^{(0)} \sum_l \mathcal{L}_{kl}^e[\varphi_k], \tag{53}
\]

where

\[
\mathcal{L}_{kl}^e[\varphi_k] = \frac{\gamma_{kl}}{(2\pi)^{\frac{3}{2}}} \int_{p_1, p_2, p_3} f_k^{(0)}(\varphi_k \varphi_{\mu \rho} \varphi_{\nu \rho}) W_{kl}. \tag{54}
\]

\(^{13}\) Direct computation gives \((1, \pi_k^0 \pi_k^0) \propto (C_1 U^\sigma U^\sigma + C_2 \Delta^\sigma \Delta^\mu) \Delta^\mu = 0, (p_k^0, \pi_k^0 \pi_k^0) \propto (C_1 U^\lambda U^\sigma U^\rho + C_2 U^\lambda \Delta^\sigma \Delta^\rho + C_3 \Delta^\lambda \Delta^\rho) \Delta^\mu = 0.\)
The unknown functions $\varphi_k$ are sought in the form

$$\varphi_k = \frac{1}{n\sigma(T)} \left( -A_k(p_k)\nabla_\mu U^\mu + C_k(p_k)\pi^\mu_k \pi^\nu_k \nabla_\mu U_\nu \right),$$  \hspace{1cm} (55)$$

where $\sigma(T)$ is some formal averaged cross section, used to come to dimensionless quantities. Then, using the (43) and the (53), and the fact that the gradients $\nabla_\mu U^\mu$ and $\pi^\mu_k \pi^\nu_k$ are independent, from the BEs independent integral equations follow:

$$\hat{Q}_k = \sum_l x_l L_{kl}^{\varphi}[A_k],$$ \hspace{1cm} (56)$$

$$\pi^\mu_k \pi^\nu_k = \sum_l x_l L_{kl}^{\varphi}[C_k \pi^\mu_k \pi^\nu_k],$$ \hspace{1cm} (57)$$

where the dimensionless collision integrals are introduced:

$$L_{kl}[\chi_k] = \frac{1}{n_l T \sigma(T)} L_{kl}^{\varphi}[\chi_k].$$ \hspace{1cm} (58)$$

In the case of present inelastic processes the l. h. s. of the (56) is set by the source term (44) and the r. h. s. contains the linear inelastic collision integrals. After introduction of inelastic processes the source terms in the (56) become much larger as demonstrated in Sec. 4.1. Using the equations (34) and (33) and the IG formulas (37) one can check that in the zero masses limit the source terms $\hat{Q}_k$ tend to zero (eventually resulting into zero bulk viscosity in the considered approximation) and $D\hat{\mu}_\alpha = 0$, that is the $\hat{\mu}_\alpha$ don’t scale and the DFs become scale invariant in the considered approximation (with similar conclusions being previously made, e. g., in [18]). The shear viscosity source term is much simpler.

3. The transport coefficients

After substitution of the $f_k^{(1)}$ with the $\varphi_k$ (55) into the (5) and comparison with the (14) one finds the formula for the bulk viscosity (cf. [14], Chap. VI, Sec. 1)

$$\xi = -\frac{1}{3} \frac{T}{\sigma(T)} \sum_k x_k (\Delta^\mu \pi^\nu_k A_k)_k = \frac{T}{\sigma(T)} \sum_k x_k (\hat{Q}_k, A_k)_k,$$ \hspace{1cm} (59)$$

and for the shear viscosity

$$\eta = \frac{1}{10} \frac{T}{\sigma(T)} \sum_k x_k (\pi^\mu_k \pi^\nu_k, C_k \pi^\mu_k \pi^\nu_k)_k,$$ \hspace{1cm} (60)$$
where the relation $\Delta_{\mu\tau}^\nu \Delta_{\rho}^\sigma \Delta_{\sigma}^\tau = 5$ is used (cf. [4], Sec. 8). In the matching conditions are used, see just below.

In kinetics the conditions that the nonequilibrium perturbations of the DFs do not contribute to the net charge and the energy-momentum densities are used as a convenient choice and are called the matching conditions, implying also some choice of the flow velocity. They can reproduce the Landau-Lifshitz condition [13] [14] (Chap. V, Sec. 1). The matching conditions for the net charge densities can be written as

$$\sum_k q_{ak} \int \frac{d^3p_k}{(2\pi)^3} p_\mu^k U_\mu f^{(0)}_k \varphi_k = 0,$$

(61)

and for the energy-momentum density can be written as

$$\sum_k \int \frac{d^3p_k}{(2\pi)^3} p_\mu^k p_\nu^k U_\nu f^{(0)}_k \varphi_k = 0.$$

(62)

For the special tensorial functions $C_{k,\mu\nu,\alpha\beta\gamma}$ in the (55) they are satisfied automatically and for the scalar functions $A_k$ they can be rewritten in the form (the 3-vector part of the (62) is automatically satisfied)

$$\sum_k q_{ak} x_k(\tau_k, A_k) = 0, \quad \sum_k x_k(\tau_k^2, A_k) = 0.$$  

(63)

The conditions (61) and (62) exclude the nonphysical solutions [4] $A_{nk,ph.}^a = \sum C_a q_{ak} + C \tau_k$ ($C_a$ and $C$ are some constants) of the linearized equations (56) (the (57) don’t have nonphysical solutions). One can show explicitly essential positiveness of the $\xi$ (with help of these matching conditions) and the $\eta$ (within the underlying assumptions of the BEs), see [14] (Chap. VI, Sec. 1).

We consider variational (or Ritz) method [14] (Chap. VI, Sec. 3) allowing to find an approximate solution of the integral equations (56) and (57) in the form of a linear combination of test-functions. The coefficients next to the test-functions are found from the condition to deliver extremum to some functional, the first variation of which can reproduce the equations (56) and (57). One could take this functional in the form of some special norm, as in [14]. Or one can take somewhat different functional, like in [19], which is a little more convenient, and get the same result. The approximate values of the viscosities are smaller than the precise ones, being hinted by the applicability of the variational methods [4] (Sec. 10), [19]. Questions concerning the uniqueness and existence of the solution and the convergence of the approximate solution to the precise one are covered in [14] (Chap. IX, Secs. 1-2).

[14] Which cannot be solutions in inhomogeneous systems and are produced just due to shifts in the $T, \mu_a$ [8] (Sec. 6).
The approximate solution of the system of the integral equations (56) and (57) are sought in the form

\[ A_k = \sum_{r=0}^{n_1} A^r_k \tau^r_k, \]  
(64)

\[ C_k = \sum_{r=0}^{n_2} C^r_k \tau^r_k, \]  
(65)

where \( n_1 \) and \( n_2 \) set the number of the used test-functions. Test-functions used in [19] would cause less significant digit cancellation in numerical calculations, but there is a need to reduce the dimension of the 12-dimensional integrals from these test-functions as more as possible to perform the calculations in a reasonable time. The test-functions in the form of just powers of the \( \tau_k \) seem to be the most convenient for this purpose. As long as particles of the same particle species and different spin states are undistinguishable, their functions \( \varphi_k \) are equal, and the variational problem is reduced to the variation of the coefficients \( A^r_k \) and \( C^r_k \), and the bulk (59) and the shear (60) viscosities can be rewritten as

\[ \xi = \frac{T}{\sigma(T)} \sum_{k'=1}^{N'} \sum_{r=0}^{n_1} x_{k', r} A^r_{k'}, \]  
(66)

\[ \eta = \frac{1}{10} \frac{T}{\sigma(T)} \sum_{k'=1}^{N'} \sum_{r=0}^{n_2} x_{k', r} \gamma^r_{k'} C^r_{k'}. \]  
(67)

Applying the variational method one gets the following matrix equations (with the multi-indexes \((l', s)\) and \((k', r)\)) for the bulk and the shear viscosities corresponding to [14] (Chap. VI, Sec. 3 and Chap. XIII, Sec. 2)

\[ x_{k', r} A^r_{k'} = \sum_{l'=1}^{N'} \sum_{s=0}^{n_1} A^{r}_{l'k'} A^s_{l'}, \]  
(68)

\[ x_{k', r} \gamma^r_{k'} = \sum_{l'=1}^{N'} \sum_{s=0}^{n_2} C^{r}_{l'k'} C^s_{l'}. \]  
(69)

\[ \text{One can first derive the same equations for the } A_k \text{ and } C_k, \text{ treating them as different functions for all } k, \text{ with the coefficients } A^{rs}_{k'} \text{ and } C^{rs}_{k'} \text{ having the same form as the } A^{r}_{l'k'} \text{ and } C^{r}_{l'k'}. \text{ Then, after summation of the equations over the spin states of identical particles, and taking } A_k = A_{k'} \text{ and } C_k = C_{k'}, \text{ one reproduces the system of equations for the } A_{k'} \text{ and } C_{k'}. \]
where the introduced coefficients $A^s_{k'l'}$ and $C^s_{k'l'}$ are

$$A^s_{k'l'} = x_{k'}x_l'\left[\tau^r, \tau^s_1\right]_{k'l'} + \delta_{k'l'} x_{k'} \sum_{m'=1}^{N'} x_{m'} \left[\tau^r, \tau^s\right]_{k'm'},$$

(70)

$$C^s_{k'l'} = x_{k'}x_l'\tau^r_{1\mu\pi_{\mu\nu}}, \tau^s_{1\mu\pi_{\mu\nu}}\left|k'l'\right| + \delta_{k'l'} x_{k'} \sum_{m'=1}^{N'} x_{m'} \tau^r_{1\mu\pi_{\mu\nu}}, \tau^s_{1\mu\pi_{\mu\nu}}\left|k'm'\right|. $$

(71)

They are expressed through the collision brackets

$$[F, G]_{kl} = \frac{\gamma_{kl}}{T^6(4\pi)^2 z_k^2 z_l^2 K_2(z_k) K_2(z_l) \sigma(T)} \int_{\mathbf{p}_k, \mathbf{p}_{l'}, \mathbf{p}_{k'}, \mathbf{p}_{l}} e^{\tau_k^{-} - \tau_l^{+}} (F_k - F'_k) G_{1l} W_{kl}. $$

(72)

The collision brackets $[F, G]_{kl}$ are obtained from the last formula by the replacement of the $G_{1l}$ on the $G_k$. Due to the time reversibility property of the $W_{kl}$ one can replace the $G_{1l}$ on the $\frac{1}{2}(G_{1l} - G'_{1l})$ in the (72). Note that

$$\left[\tau^r, \varrho\right]_{kl} > 0. $$

(73)

It’s easy to notice the following symmetries

$$[F, G]_{kl} = [G, F]_{lk}, \quad [F, G]_{kl} = [G, F]_{lk}. $$

(74)

They result in the following symmetric properties: $A^s_{k'l'} = A^r_{l'k'}$, $C^s_{k'l'} = C^r_{l'k'}$. Also, the microscopical particle number and energy conservation laws imply for the $A^r_{l'k'}$ (Chap. VI, Sec. 3):

$$A^0_{k'l'} = 0, $$

(75)

$$\sum_{k'=1}^{N'} A^r_{k'l'} = 0. $$

(76)

The (75) together with the $\alpha^0_{k'} = 0$ (52) means that the equations with $r = 0$ in the (68) are excluded. From the (75) and (51) it follows that each one equation with $r = 1$ in the (68) can be expressed through the sum of the other ones, reducing the rank of the matrix on 1. To solve the matrix equation (68) one eliminates one equation, for example with $k' = 1, r = 1$. One of coefficients of $A^r_{1l'}$ is independent; for example, let it be $A^r_{1l'}$. Using the (75), we rewrite the matrix equation (68) in a reduced form as

$$x_{k'} \alpha^r_{k'} = \sum_{l'=2}^{N'} A^r_{l'k'} (A^r_{l'-1} - A^r_{l'-1}) + \sum_{l'=1}^{n_1} \sum_{s=2}^{n_1} A^r_{n_1 l'} A^r_{n_1 s}. $$

(77)
Then, using the (52) and the (51), we present the bulk viscosity (66) in the form

$$\xi = \frac{T}{\sigma(T)} \sum_{k'=2}^{N'} x_{k'} \alpha_{k'} (A_{k'} - A_1^1) + \frac{T}{\sigma(T)} \sum_{k'=1}^{N'} \sum_{r=2}^{n_1} x_{k'} \alpha_{k'} A_{k'}.$$  \hspace{1cm} (78)

Then, the coefficient $A_1^1$ can be eliminated by shifting of other $A_{k'}^l$ and be implicitly used to satisfy one energy conservation matching condition. The particle number conservation matching conditions are implicitly satisfied by means of the coefficients $A_{k'}^0$. The first term in the (78) is present only in mixtures. That’s why it is small in gases with close to each other masses (and the considered framework of the BEs) of particles of different species. In gases with very different masses contribution of the first term in the (78) can become dominant.

4. Results

4.1. The single-component gas

In the single-component gas, using one test-function, the matrix equations can be easily solved, and the shear (67) and the bulk (78) viscosities become (indexes ”1” of the particle species are omitted)

$$\eta = \frac{1}{10} \frac{T}{\sigma(T)} \left( \gamma^0 \right)^2 C_{00},$$  \hspace{1cm} (79)

$$\xi = \frac{T}{\sigma(T)} \left( \alpha^2 \right)^2 A_{22}.$$  \hspace{1cm} (80)

In this approximation the explicit closed-form relativistic formulas for the bulk and the shear viscosities were obtained in the [12]. There the parameter $a = 2r$. In [14] (Chap. XI, Sec. 1) they are written through the parameter $\sigma = 2r^2$. The results are

$$\eta = \frac{15}{64\pi} \frac{T}{r^2} \frac{z^2 K^2_2(z) \hat{h}^2}{(5z^2 + 2)K_2(2z) + (3z^3 + 49z)K_3(2z)},$$  \hspace{1cm} (81)

$$\xi = \frac{1}{64\pi} \frac{T}{r^2} \frac{z^2 K^2_2(z) [(5 - 3\gamma) \hat{h} - 3\gamma]^2}{2K_2(2z) + zK_3(2z)},$$  \hspace{1cm} (82)

where $\gamma = \frac{1}{c_c} + 1 = \frac{z^2 + 5\hat{h} - k^2}{z^2 + 5\hat{h} - k^2 - 1}$. Though the correct result for the shear viscosity is

$$\eta = \frac{15}{64\pi} \frac{T}{r^2} \frac{z^2 K^2_2(z) \hat{h}^2}{(15z^2 + 2)K_2(2z) + (3z^3 + 49z)K_3(2z)},$$  \hspace{1cm} (83)

\footnote{It is the differential cross section for identical particles. The total cross section is $\sigma_{tot} = \int \frac{d\Omega}{2\pi} 2\pi = 4\pi r^2$.}
This result is numerically in agreement with the result in [20]. To get the C0 and the A2 from Appendix A with z_k = z_l = z and the γ^0 and α^2 are defined in the (47). The discrepancies in the nonrelativistic (z ≫ 1) and the ultrarelativistic (z ≪ 1) expansions between the (81) and the (83) appear starting from the first correction. Though the expansions we obtain and the ones in the [12] or the [14] (Chap. XI, Sec. 1) coincide because they were previously obtained in some other calculations.

The perturbation of the DF \( \varphi \) (53) can be easily found too. We don’t know whether this was done previously, but one may be interested in this result, so we present it too:

\[
\varphi = \frac{1}{n\sigma(T)} \left( -(A^0 + A^1\tau + A^2\tau^2) \nabla_\mu U^\mu + C_0^0 \pi^\nu \pi^\sigma \nabla_\mu U^\mu \right),
\]

where the \( C_0^0 \) is equal to

\[
C_0^0 = \frac{15}{64\pi} \frac{\sigma(T) z^2 K_2^2(z) \hbar}{r^2 \left( 15z^2 + 2 \right) K_2(2z) + (3z^3 + 49z) K_3(2z)},
\]

and the \( A^2 \) is equal to

\[
A^2 = \frac{1}{64\pi} \frac{\sigma(T) z^2 K_2^2(z) \left[ (5 - 3\gamma) \hbar - 3\gamma \right]}{r^2 \left( 2K_2(2z) + zK_3(2z) \right)}.
\]

The \( A^0 \) and the \( A^1 \) are used to satisfy the matching conditions (88) and are equal to

\[
A^0 = A^2 a^2 a^4 - (a^3)^2, \quad A^1 = A^2 a^2 a^3 - a^1 a^4, \quad \Delta_A = a^1 a^3 - (a^2)^2,
\]

where the \( a^r \) are defined in the (47). In the nonrelativistic limit (z ≫ 1) one has

\[
\varphi = \frac{5\pi e^{-\mu}}{32\sqrt{2T^3}z^2\tau^2} \left( - (\tau^2 + 2z\tau - z^2) \nabla_\mu U^\mu + 2\pi^\nu \pi^\sigma \nabla_\mu U^\mu \right).
\]

In the ultrarelativistic limit (z ≪ 1) one has

\[
\varphi = \frac{\pi e^{-\mu}}{480T^3\tau^2} \left( -5z^2(\tau^2 + 8\tau - 12) \nabla_\mu U^\mu + 36\pi^\nu \pi^\sigma \nabla_\mu U^\mu \right).
\]

---

\(^{17}\) This reproduces the result of Chapman and Enskog in the nonrelativistic theory for the shear viscosity. The vanishing value of the bulk viscosity is obtained in the limit \( m \rightarrow \infty \) [3] (Secs. 8, 10). The result of the vanishing bulk viscosity of a monoatomic classical gas in the nonrelativistic theory is attributed to James Clerk Maxwell, see [15].

\(^{18}\) The vanishing value of the bulk viscosity of a monoatomic classical gas in the ultrarelativistic limit is attributed to I. M. Khalatnikov, see [4] (Sec. 8).
Note that, although the shear viscosity diverges for $T \to \infty$, the perturbative expansion over the gradients does not break down because the $\varphi$ does not diverge (it tends to zero, conversely).

The phenomenological formula, coming from the momentum transfer considerations in the kinetic-molecular theory, for the shear viscosity is $\eta_{ph} \propto \ln \langle |\vec{p}| \rangle$ (with the coefficient of proportionality $\sim 1$), where $\langle |\vec{p}| \rangle$ is the average relativistic momentum, $l$ is the mean free path. It gives the correct leading $m$ and $T$ parameter dependence of the (83) with quite a precise coefficient $^{19}$.

The mean free path can be estimated as $l \sim 1/(\sigma_{tot} n)$ (see Sec. 4.3). Choosing the coefficient of proportionality to match the nonrelativistic limit one gets $^{24}$

$$\eta_{ph} = \frac{5}{64\sqrt{\pi}} \frac{\sqrt{mT} K_{5/2}(m/T)}{r^2 K_2(m/T)}.$$  \hspace{1cm} (90)

If the bulk viscosity is expressed as $\xi_{ph} \propto \ln \langle |\vec{p}| \rangle$, the coefficient of proportionality is not $\sim 1$. In the nonrelativistic limit it is $25/(512\sqrt{2}z^2)$ and in the ultrarelativistic limit it is $z^4/(864\pi)$. To reproduce these asymptotical dependencies the bulk viscosity should be proportional to the second power of the averaged product of the source term $\hat{Q}$ and the $\tau$, that is to $(\alpha^2)^2$.

If a system has no charges, then terms proportional to the $\tau_k$ in the (44) are absent, and the $R$ quantity gets another form. This results in quite different values of the $\alpha^2_k$. In particular, for the single-component gas in the case $z \gg 1$ one gets

$$\frac{(\alpha^2)^2}{\alpha^2_{|q_{11}=0}} = \frac{4z^4}{25} + ..., \hspace{1cm} (91)$$

and in the case $z \ll 1$ one gets

$$\frac{(\alpha^2)^2}{\alpha^2_{|q_{11}=1}} = 4 + .... \hspace{1cm} (92)$$

In both cases these estimates suppose enhancement of the bulk viscosity $^{80}$ if the number-changing processes are not negligible.

4.2. The binary mixture

The mixture of two species with masses $m_1$, $m_2$ and the different classical elastic differential constant cross sections $\sigma_{11}^e$, $\sigma_{12}^e = \sigma_{21}^e$, $\sigma_{22}^e$ is considered in this section (only formal expressions can be found in the $^{14}$, Chap. VI, Sec. 3). Using the (67) with $n_2 = 0$ and solving the matrix equation (69) one gets for the shear viscosity

$$\eta = \frac{T}{10\sigma(T)} \Delta \eta \left[ (x_{11}0^0)^2 C_{12}^{00} - 2x_{11}x_{22}^0 \gamma_{12}^0 C_{11}^{00} + (x_{22}^0)^2 C_{11}^{00} \right], \hspace{1cm} (93)$$

$^{19}$This formula is justified only for rarefied systems where the IG equation of state is applicable.
where $\Delta_\eta = C_{11}^{00}, C_{22}^{00} - (C_{12}^{00})^2$. The collision brackets for the $C_{k,l}^{00}$ can be found in Appendix A and the $\gamma_k^0$ are defined in the (47).

In the important limiting case when one mass is large $z_2 \gg 1$ ($g_2$ and $\bar{\mu}_2$ are finite so that $x_2' \ll 1$) and another mass is finite one can perform asymptotic expansion of the special functions. Then, one has $x_1' \propto O(1)$, $\gamma_1^0 \propto O(1)$, $x_2' \propto O(e^{-z_2 z_2^{3/2}})$, $\gamma_2^0 \propto O(z_2)$. The collisions of light and heavy particles dominate over the collisions of heavy and heavy particles in the $C_{22}^{00}$, and one has $\left[\frac{\pi^s}{\pi^r}, \frac{\pi^s}{\pi^r}\right]_1 \propto O(z_2)$, $C_{22}^{00} \propto O(e^{-z_2 z_2^{5/2}})$. In the $C_{11}^{00}$, the collisions of light and light particles dominate, and one gets $C_{11}^{00} \propto O(1)$. And $\left[\frac{\pi^s}{\pi^r}, \frac{\pi^s}{\pi^r}\right]_2 \propto O(1)$, $C_{11}^{00} \propto O(e^{-z_2 z_2^{3/2}})$. In the shear viscosity the first nonvanishing contribution is the single-component shear viscosity $\eta_{kk}$, where one should take $r^3 = \sigma_{11}^3$ and $z = z_1$. The next correction is

$$\Delta \eta = \frac{5}{2} e^{-z_2} \frac{3 T g_2 e^{z_1 - \hat{\mu}_1 + \bar{\mu}_2}}{64 \sqrt{2 \pi} (3 + 3 z_1 + z_1^2) g_1 \sigma_{11}^3}. \quad (94)$$

The approximate formula \[ \eta = \sum_k \eta_k x_k, \quad (95) \]

where $\eta_k$ is given by the (83) or the (90) with mass $m_k$ and cross section $\sigma_{kk}^3$, would give somewhat different heavy mass power dependence $O(e^{-z_2 z_2^3})$.

Using the (83) with $n_1 = 1$ and solving the matrix equation (77) one gets for the bulk viscosity

$$\xi = \frac{T}{\sigma(T)} \left( x_2 \alpha_1^2 \right) = \frac{T}{\sigma(T)} \frac{x_1' x_2' \alpha_1^2}{A_{12}^{11}}. \quad (96)$$

Using the definition of the $A_{12}^{11}$ (70) and the fact $[\tau, \tau]_{kl} + [\tau, \tau]_{kl} = 0$ (A.21) one gets $A_{12}^{11} = x_1' x_2' [\tau, \tau]_{12}$. Using the (83) one gets $[\tau, \tau]_{12} > 0$. Then, using $x_1' \alpha_1^2 + x_2' \alpha_2^2 = 0$, coming from the (51), the bulk viscosity can be rewritten as

$$\xi = \frac{T}{\sigma(T)} \frac{x_2 (\alpha_1^2)}{x_1' [\tau, \tau]_{12}} = \frac{T}{\sigma(T)} \frac{x_1' (\alpha_1^2)}{x_2' [\tau, \tau]_{12}} > 0. \quad (97)$$

The collision bracket $[\tau, \tau]_{12}$ can be found in Appendix A and the $\alpha_k$ are defined in the (47). In the limiting case $z_2 \gg 1$ one has $x_1' \propto O(1)$, $x_2' \propto O(e^{-z_2 z_2^{3/2}})$, $\alpha_1^2 \propto O(e^{-z_2 z_2^{5/2}})$, $\alpha_2^2 \propto O(1)$, $A_{13}^{13} \propto \frac{1}{2} A_{12}^{12} \propto O(e^{-z_2 z_2^{3/2}})$, $[\tau, \tau]_{12} \propto O(z_2^{-1})$. Then, for the bulk viscosity one gets

$$\xi = e^{-z_2 z_2^{3/2}} \frac{g_2 T e^{-\hat{\mu}_1 + \bar{\mu}_2 + z_1} [z_1^2 - 5 - 2 h_1^2 + 10 h_1]}{128 \sqrt{2 \pi} g_1 \sigma_{11}^3 (3 z_1 + 3) [z_1^2 - 1 - h_1^2 + 5 h_1]} + \ldots. \quad (98)$$
4.3. The collision rates and the mean free paths

The quantity \( \frac{W_{k'\ell'}^0}{\pi^3 p^0_{k'} p^0_{1'\ell'} d^3 p^0_{k'} d^3 p^0_{1'\ell'}} \), which enters the elastic collision integral \(\{20\}\), represents the probability of scattering per unit time times volume for two particles which had momentums \(\tilde{p}_{k'}\) and \(\tilde{p}_{1'\ell'}\) before scattering and momentums in the ranges \((\tilde{p}^0_{k'}, \tilde{p}^0_{k'} + d\tilde{p}^0_{k'})\) and \((\tilde{p}^0_{1'\ell'}, \tilde{p}^0_{1'\ell'} + d\tilde{p}^0_{1'\ell'})\) after the scattering. The quantity \(g_k \frac{d^3 p_k}{(2\pi)^3} f_k\) represents the number of particles per unit volume, which have momentums in the range \((\tilde{p}_k, \tilde{p}_k + d\tilde{p}_k)\). The number of collisions of particles of the \(k\)-th species with particles of the \(l\)-th species per unit time per unit volume is then\(^2\)

\[
\bar{R}_{k'\ell'}^e = g_{k'} g_{l'} \gamma_{k'\ell'} \int \frac{d^3 p_{k'}}{(2\pi)^3} \frac{d^3 p_{1'\ell'}}{(2\pi)^3} \frac{d^3 p'_{k'}}{(2\pi)^3} \frac{d^3 p'_{1'\ell'}}{(2\pi)^3} \frac{f_{k'}(0)}{f_{l'}(0)} W_{kl}.
\]  

(99)

To get the corresponding number of collisions of particles of the \(k\)-th species with particles of the \(l\)-th species per unit time per particle of the \(k\)-th species, \(R_{k'\ell'}^e\), one has to divide the \(\{99\}\) on the \(\gamma_{k'\ell'} n_{k'}\) (recall that \(n_{k'} \propto g_{k'}\) by definition), which is the number of particles of the \(k\)-th species per unit volume divided on the number of particles of the \(k\)-th species taking part in the given type of reaction \((2\) for binary elastic collisions, if particles are identical, and \(1\) otherwise). This rate can be directly obtained averaging the collision rate with fixed momentum \(p_k\) of the \(k\)-th particle species

\[
R_{k'\ell'}^e = g_{k'} g_{l'} \gamma_{k'\ell'} \int \frac{d^3 p_{1'\ell'}}{(2\pi)^3} \frac{d^3 p_{1'\ell'}}{(2\pi)^3} \frac{d^3 p'_{k'}}{(2\pi)^3} \frac{d^3 p'_{1'\ell'}}{(2\pi)^3} \frac{f_{k'}(0)}{f_{l'}(0)} W_{kl}.
\]  

(100)

over the momentum with the probability distribution \(\frac{d^3 n_k}{(2\pi)^3} f_k\) (and spin states, which is trivial):

\[
R_{k'\ell'}^e = g_{k'} g_{l'} \gamma_{k'\ell'}^2 \int \frac{d^3 p_{k'}}{(2\pi)^3} \frac{d^3 p_{1'\ell'}}{(2\pi)^3} \frac{d^3 p'_{k'}}{(2\pi)^3} \frac{d^3 p'_{1'\ell'}}{(2\pi)^3} \frac{f_{k'}(0)}{f_{l'}(0)} W_{kl} = \bar{R}_{k'\ell'}^e \gamma_{k'\ell'} n_{k'}.
\]  

(101)

So that to get the mean rate of the elastic collisions per particle of the \(k\)-th species with all particles in the system one can just integrate the sum of the gain terms in the collision integral \(\{20\}\) over \(\frac{d^3 p_k}{(2\pi)^3} n_k\) and average it over spin:

\[
\bar{R}_{k'\ell'}^e = \sum_{\ell'} R_{k'\ell'}^e.
\]  

(102)

One can express the \(\bar{R}_{k'\ell'}^e\) through the \(J_{kl}^{(0,0,0,0,0,0)}\) integrals from Appendix A as

\[
\bar{R}_{k'\ell'}^e = \gamma_{k'\ell'} \sigma(T) n_{k'} n_{\ell'} J_{k'\ell'}^{(0,0,0,0,0,0)}.
\]  

(103)

\(^2\)It represents some sum over all possible collisions. In the case of the same species one factor \(\gamma_{k'\ell'}\) just cancels the double counting in momentum states after scattering and another factor \(\gamma_{k'\ell'}\) also reflects the fact that scattering takes place for \((\tilde{p}_k, \tilde{p}_k + d\tilde{p}_k)\) pairs of undistinguishable particles in a given unit volume.
Then, the (103) becomes

\[
\bar{R}_{k'v}^{el} = g_{k'} g_v \gamma_{k'v} \frac{2 \sigma_{k'v}^{el} T^6}{\pi^3} \left[ (z_{k'} - z_v)^2 K_2(z_{k'} + z_v) + z_{k'} z_v (z_{k'} + z_v) K_3(z_{k'} + z_v) \right],
\]

where \( \sigma_{k'v}^{el} \) is the classical elastic differential constant cross section of scattering of a particle of the \( k' \)-th species on particles of the \( l' \)-th species. For the case of large temperature or when both masses are small, \( z_{k'} \ll 1 \) and \( z_v \ll 1 \), one has expansion

\[
\bar{R}_{k'v}^{el} = g_{k'} g_v \gamma_{k'v} \frac{4 \sigma_{k'v}^{el} T^6}{\pi^3} \left( 1 - \frac{1}{4} (z_{k'}^2 + z_v^2) + ... \right).
\]  
(105)

For the case of small temperature or when both masses are large, \( z_{k'} \gg 1 \) and \( z_v \gg 1 \), one has expansion

\[
\bar{R}_{k'v}^{el} = g_{k'} g_v \gamma_{k'v} \sqrt{2 \sigma_{k'v}^{el} T^6 z_{k'} z_v} \sqrt{z_{k'} + z_v e^{-z_{k'} - z_v}} \times \left( 1 + \frac{8 z_{k'}^2 + 19 z_{k'} z_v + 8 z_v^2}{8 z_{k'} z_v (z_{k'} + z_v) + ...} \right).
\]  
(106)

For the case when only one mass is large, \( z_v \gg 1 \), one has somewhat different expansion

\[
\bar{R}_{k'v}^{el} = g_{k'} g_v \gamma_{k'v} \sqrt{2 \sigma_{k'v}^{el} T^6 (z_{k'} + 1) z_v^{3/2} e^{-z_{k'} - z_v}} \times \left( 1 + \frac{4 z_{k'}^2 + 15 z_{k'} + 15}{8 z_{k'} + 8 z_v^{-1} ...} \right).
\]  
(107)

The \( \sigma(T) f_{k'v}^{el} \) in the (103) can be replaced in the ultrarelativistic limit with \( 4 \pi \sigma_{k'v}^{el} (|\tilde{v}_{k'}|) \) and in the nonrelativistic limit with \( 4 \pi \sigma_{k'v}^{el} (|\tilde{v}_{rel,k'v}|) = 4 \pi \sigma_{k'v}^{el} (|\tilde{v}_{k'}|) \sqrt{1 + m_{k'}/m_v} \), where \( |\tilde{v}_{k'}| \) is the mean modulus of particle’s velocity of the \( k' \)-th species,

\[
\langle |\tilde{v}_{k'}| \rangle = \frac{\int d^3 p_{k'} \tilde{v}_{k'} f_{k'}^{el} (p_{k'})}{\int d^3 p_{k'} f_{k'}^{el} (p_{k'})} = \frac{2 e^{-z_v} (1 + z_{k'})}{z_{k'} K_2(z_{k'})} = \frac{\sqrt{8 z_{k'}} K_{3/2}(z_{k'})}{\pi z_{k'} K_2(z_{k'})},
\]  
(108)

and \( \langle |\tilde{v}_{rel,k'v}| \rangle \) is the mean modulus of the relative velocity, which coincides with the \( \langle |\tilde{v}_{k'}| \rangle \) at high temperatures. Then, the resultant collision rate \( R_{k'v}^{el} \) would reproduce simple nonrelativistic collision rates know in the kinetic-molecular theory \([21]\) (Sec. 9). To get a (approximate) mean free time one has just to invert the \( R_{k'v}^{el} \), \( t_{k'v}^{el} = 1/R_{k'v}^{el} \). A (approximate) mean free path \( l_{k'v}^{el} \) can be obtained after multiplication of it on the \( \langle |\tilde{v}_{k'}| \rangle \):

\[
l_{k'v}^{el} = \frac{\langle |\tilde{v}_{k'}| \rangle}{R_{k'v}^{el}}.
\]  
(109)
For the single-component gas one gets

\[ l_{11}^{el} = \frac{\langle |\vec{v}_1| \rangle}{R_{11}^{el}} = \frac{\pi e^{-z_1}(z_1 + 1)}{g_1\sigma_{11}^2 T^3 z_1^2 K_3(2z_1)}. \]  

(110)

The nonrelativistic limit of the (110) with the \( g_1 = 1 \) coincides with the same limit of the formula

\[ l_{1}^{el} = \frac{\langle |\vec{v}_1| \rangle}{4\pi\sigma_{11}^2 n_1 \langle |\vec{v}_{rel}| \rangle} = \frac{1}{4\pi\sigma_{11}^2 n_1 \sqrt{2}}, \]  

(111)

which is the mean free path formula coming from the nonrelativistic kinetic-molecular theory, obtained by Maxwell \[21\] (Secs. 10, 9). The ultrarelativistic limit of the (110) with the \( g_1 = 1 \) coincides with the same limit of the formula

\[ l_{1}^{el} = \frac{1}{4\pi\sigma_{11}^2 n_1}. \]  

(112)

5. Discussion

The presented formulas can be considered as quite precise ones for rarefied gases with short-range interactions. However, for the bulk viscosity one should be sure that the approximation of only the elastic processes is a good one. Our estimations indicate the considerable increase of the bulk viscosity if inelastic processes are introduced. With the choice of the chemical potentials and the temperature as independent thermodynamic variables we find the bulk viscosity source term (41) to have a more convenient and universal form. We believe this piece of methodology is a new result. There are anal ogical expressions for multicomponent gases, e.g., as in \[22\], but our expression does not require further transformations if the thermodynamic functions are known as functions of the temperature and the chemical potentials (which is usually so). Also any sort of analysis or obtaining of a partial case, like the one of zero chemical potentials, might be more easily done having the chemical potentials and the temperature as independent variables. Switching between the two qualitatively different cases of maximal particle number conservation and nonconservation (and intermediate ones) is realized through one quantity, the matrix \( q_{ab} \).

The approximation of the Maxwell-Boltzmann (classical) statistics and only the elastic processes allows one to obtain relatively simple analytical closed-form expressions. We present the correct form of the single-component shear viscosity, though the deviations are small and appear only in the corrections of the nonrelativistic and the ultrarelativistic expansions. We have used Mathematica package \[23\], which allowed to avoid such omissions in this and more bulky expressions. The single-component and the binary mixture formulas can be used for estimations or checkups of the viscosities in mixtures, besides their direct applications. We also present simple expressions for the collision rates and approximate formulas for the mean free paths and times, which one might need for justification of the applicability of the hydrodynamical description. We
believe that all these formulas (except for the single-component bulk viscosity) are new too.

We present the lowest orders collision brackets for the viscosities in the mixtures and the \( J \)-integrals through which one can obtain the collision brackets of other orders (with ease using an algebraic manipulation package). To calculate any transport coefficient using a variational method (allowing to control precision) in the lowest order approximation in a mixture with a very large number of components \( N' \) one would need to calculate \( O(N'^2) \) 12-dimensional \( J \)-integrals, being the main difficulty, if only the elastic collisions are considered. Fortunately, it’s possible to simplify these integrals considerably, expressing them through the known special functions. This allows to perform calculations of the viscosities in gases with many particle species in a reasonable time. We consider this the main result.

If there are nonconstant cross sections, then one can describe the viscosities approximately using averaged (temperature dependent) cross sections with the formulas mentioned above. We postpone to the future our possible investigations of the deviations due to application of the averaged cross sections. If for some energy dependencies of the cross sections the deviations are large, improved definitions of the averaged cross sections may exist. The next-to-leading order corrections in the form of thermal masses and thermal cross sections, as well as the beta function contribution to the bulk viscosity source term at high enough temperatures, as in the [3] can be relatively easily taken into account if they are calculated. The main applications of the results of this paper are to the hydrodynamical description of the hadronic phase created in heavy ion collisions [24, 25, 26], where the particles are relativistic ones and there are many species of them, or in the same but simplified description [27, 28, 29]. The formulas mentioned above may have similar applications to other fields of physics, like cosmology [18, 30, 31, 32], solid state physics [33, 34], studies of atmosphere [35], etc.

6. Conclusions

We have obtained formulas for the shear and the bulk viscosities in mixtures with constant cross sections. They allow to conduct fast computations for mixtures with large number of particle species. These expressions can be used in many fields of physics. These formulas represent the relativistic generalization of the formulas for the nonrelativistic hard spheres model. Additionally, we have presented explicit relatively simple expressions (which is a benefit for applications) for different auxiliary quantities, like the collision rates.

7. Acknowledgments

The author is grateful to the referees for comments on the improvement of this paper’s form.
Appendix A. The calculation of the collision brackets

The momentum parametrization and the most of transformations of the 12-dimensional integrals considered below are taken from the [14] (Chap. XI and XIII). There the reader can find some assignments appearing below.

There is a need to calculate the following integrals

\[
J_{kl}^{(a,b,d,e,f|q,r)} \equiv \frac{\gamma_{kl} T^6 (4\pi)^2 z_k^2 z_l^2 K_2(z_k) K_2(z_l) \sigma(T)}{p_{k1} p_{l1} p'_{k1} p'_{l1}} \times e^{-P \cdot U / T} (1 + \alpha_{kl})^q (1 - \alpha_{kl})^r \left( \frac{p^2_2}{T^2} \right)^a \left( \frac{P \cdot U}{T} \right)^b \left( \frac{Q \cdot U}{T} \right)^d \times \left( \frac{Q' \cdot U}{T} \right)^e \left( -Q' \cdot Q \right)^f W_{kl}.
\]

(A.1)

Using some nontrivial transformations, described in more details in [14] (Chap. XI and XIII), we arrive at

\[
J_{kl}^{(a,b,d,e,f|q,r)} = \frac{\pi (d + e + 1)!! \gamma_{kl}^{(d,e,f)}}{z_k^2 z_l^2 K_2(z_k) K_2(z_l)} \sum_{q_1=0}^{q} \sum_{r_1=0}^{r} \sum_{k_2=0}^{d+e+f+1} \sum_{k_3=0}^{d+e+f+1} \sum_{h=0}^{b/2} \times (z_k + z_l)^2 (q_1 + r_1 + k_2 + k_3) \left( \frac{z_k - z_l}{z_k + z_l} \right)^{q_1 + r_1 + 2k_3} (-1)^{r_1 + k_2 + k_3 + h} (2h - 1)!! \times \binom{b}{2h} \binom{q}{q_1} \binom{r}{r_1} \binom{d+e+f+1}{k_2} \binom{d+e+f+1}{k_3} \times I \left( 2(a + f - q_1 - r_1 - k_2 - k_3) + 3, b + \frac{d + e}{2} - h + 1, z_k + z_l \right),
\]

(A.2)

where

\[
\sigma_{kl}^{(d,e,f)} = \frac{\sigma_{kl}^{cl}}{\sigma(T)} \sum_{g=0}^{\min(d,e)} \sigma(f,g) K(d, e, g),
\]

(A.3)

where \(\sigma_{kl}^{cl} = \gamma_{kl} \sigma_{kl}^{cl}\) is the classical elastic differential constant cross section. The \(\sigma(f,g)\) is equal to the real, nonzero and non-diverging value (for any non-negative integer \(g\))

\[
\sigma(f,g) = \frac{2g + 1}{2} \int_{-1}^{1} dx x^f P_g(x) = (2g + 1) \frac{f!}{(f - g)!!(f + g + 1)!!},
\]

(A.4)

if the difference \(f - g\) is even and \(g \leq f\). Above the \(P_g(x)\) is the Legendre polynomial. The \(K(d, e, g)\) is equal to the real, nonzero and non-diverging quantity (for any non-negative integer \(g\))

\[
K(d, e, g) = \frac{d! e!}{(d - g)!!(d + e + 1)!!(e + g + 1)!!},
\]

(A.5)
if $g \leq \min(d, e)$ and both the $d - g$ and the $e - g$ are even (which also implies that $d + e$ is even). The $[...]$ denotes the integer part. The integral $I$ is

$$I(r, n, x) \equiv x^{r+n+1} \int_1^\infty du u^{r+n} K_n(xu).$$

(A.6)

Also, there is the following frequently used combination of the $J$ integrals

$$J_{kl}^{(a,b,d,e,f|q,r)} \equiv \sum_{u=0}^f (-1)^u \left( \frac{f}{u} \right) (2z_k)^{2(f-u)} J_{kl}^{(a+k,b,d+e,0,0|q+2u,r)}$$

(A.7)

The first term in the difference is obtained by the replacement of the $Q'$ on the $Q$ everywhere except for the $W_{kl}$. Using this fact, the $J'$ can be rewritten in the form

$$J_{kl}^{(a,b,d,e,f|q,r)} = \frac{\sigma_{kl}^{(d,e,f)}}{(d + e + 1)\sigma_{1kl}^{(d,e,f)}} J_{kl}^{(a,b,d,e,f|q,r)},$$

(A.8)

where

$$\sigma_{kl}^{(d,e,f)} = \frac{\sigma_{kl}^{(d,e)}}{\sigma(T)} (d + e + 1) \left( K(d + e, 0, 0)\sigma^{(0,0)} - \sum_{g=0}^{\min(d,e)} K(d, e, g)\sigma^{(f,g)} \right)$$

(A.9)

There is a recurrence relation for the integral $I$ (Chap. XI, Sec. 1):

$$I(r, n, x) = (r - 1)(r + 2n - 1)I(r - 2, n, x) + (r - 1)x^{r+n-1}K_n(x) + x^{r+n}K_{n+1}(x).$$

(A.10)

For the calculations one needs only the integrals $I(r, n, x)$ with the positive values of the $n$ and the odd values of the $r$. If $r \geq -2n + 1$, the $I$ integrals can be expressed through the Bessel functions $K_n(x)$, using the (A.10), when $r = 1$ or $r = -2n + 1$. Then, using the recurrence relation for the $K_n(x)$ (A.11),

$$K_{n+1}(x) = K_{n-1}(x) + \frac{2n}{x} K_n(x),$$

(A.11)

the final result can be expressed through a couple of Bessel functions. If $r \leq -2n - 1$, then the recurrence relation (A.10) becomes singular if one tries to express the $I(r, n, x)$ through the $I(-2n + 1, n, x)$. Using the (A.10), the $I$ integrals with $r \leq -2n - 1$ can be expressed through the integrals $G(n, x)$

$$G(n, x) \equiv I(-2n - 1, n, x) = x^{-n} \int_1^\infty du u^{-n-1} K_n(xu).$$

(A.12)
There is a recurrence relation for the $G(n, x)$:

$$G(n, x) = -\frac{1}{2n}(G(n - 1, x) - x^{-n}K_n(x)). \quad (A.13)$$

It can be easily proved by the integration by parts of the (A.12) and using the following relation for the $K_n(x)$:

$$\frac{\partial}{\partial x}K_n(x) = -\frac{n}{x}K_n(x) - K_{n-1}(x). \quad (A.14)$$

It is found that collision brackets have the simplest form if they are expressed through $G(n, x)$ with $n = 3$ or $n = 2$ and the Bessel functions $K_3(x)$ and $K_2(x)$ or $K_2(x)$ and $K_1(x)$. It was chosen to take $G(x) \equiv G(3, x)$ and $K_3(x)$, $K_2(x)$. The $G(x)$ can be expressed through the Meijer function:

$$G(x) = \frac{1}{32}G_{3,0}^{3,0}\left((x/2)^2 \mid -3, 0, 0 \right). \quad (A.15)$$

The needed scalar collision brackets can be expressed through the $J^r$ as

$$[r^r, \tau^s_k]kl = \frac{1}{2^{r+s}} \sum_{u=1}^{r} \sum_{v=1}^{s} (-1)^v \binom{r}{u} \binom{s}{v} J^{(0, r+s-u-v, u, v, 0)}_{kl} \equiv \nu, \nu, \nu, \nu, (A.16)$$

and the needed tensorial collision brackets can be expressed as

$$[r^r, \tau^s_1]kl = \frac{1}{2^{r+s-4}} \sum_{n_1=0}^{r} \sum_{n_2=0}^{s} \binom{s}{n_2} \binom{r}{n_1} (-1)^{s-n_2} (A.18)$$

$$\times (J^{(2, n_1+n_2, r-n_1-s-n_2, 0)+2+n_1, 2+n_2}_{kl} + 2J^{(1, n_1+n_2, r-n_1-s-n_2, 1)+n_1, 1+n_2}_{kl})$$

$$+ J^{(0, n_1+n_2, r-n_1-s-n_2, 2|n_1, n_2)}_{kl} - \frac{1}{2^{r+s+3}} \sum_{n_1=0}^{r+1} \sum_{n_2=0}^{s+1} \binom{s+1}{n_2} \binom{r+1}{n_1}$$

$$\times (-1)^{s+1-n_2}(J^{(1, n_1+n_2, r+1-n_1-s+1-n_2, 0)+n_1, 1+n_2}_{kl})$$

$$+ J^{(0, n_1+n_2, r+1-n_1-s+1-n_2, 1|n_1, n_2)}_{kl} + \frac{2}{3} [r^{r+2}, \tau^s_1]kl$$

$$+ \frac{1}{3} z^2 [r^{r+2}, \tau^s_1]kl + \frac{1}{3} z^2 [r^r, \tau^s_1]kl - \frac{1}{3} z^2 z^2 [r^r, \tau^s_1]kl,$$
\[
[r^s_i \bar{\Pi}^s, r^s_{\mu} n_{kl}^s]_{kl} = \frac{1}{2r+s+4} \sum_{n_1=0}^{r} \sum_{n_2=0}^{s} \binom{s}{n_2} \binom{r}{n_1} \quad (A.19)
\]
\[
\times \left( J^r_{kl}^{(2,n_1+n_2,r-n_1,s-n_2,0)} + 2J^r_{kl}^{(1,n_1+n_2,r-n_1,s-n_2,2)} \right) + \frac{1}{2r+s+3} \sum_{n_1=0}^{r+1} \sum_{n_2=0}^{s+1} \binom{s+1}{n_2} \binom{r+1}{n_1} \quad (A.20)
\]

Below some lowest orders collision brackets are presented with the following notations:
\[
\bar{K}_1 = \frac{K_3(z_k + z_l)}{K_2(z_k)K_2(z_l)), \quad \bar{K}_2 = \frac{K_2(z_k + z_l)}{K_2(z_k)K_2(z_l)}, \quad \bar{K}_3 = G(z_k + z_l) \quad (A.21)
\]

For the scalar collision brackets one has:
\[
-[\tau, \tau_1]_{kl} = [\tau, \tau]_{kl} = \frac{\sigma_{kl}^{cl}}{\sigma(T)} \frac{\pi}{8z_kz_l^2} Z^2_{kl}(P^{(1,1)}_{s1} \bar{K}_1 + P^{(1,1)}_{s2} \bar{K}_2 + P^{(1,1)}_{s3} \bar{K}_3), \quad (A.22)
\]
\[
P^{(1,1)}_{s1} = -2Z_{kl}(z^4_k + 4z^2_k Z^2_{kl} - 2Z^4_{kl}), \quad (A.23)
\]
\[
P^{(1,1)}_{s2} = z^4_k(3Z^2_{kl} + 8) + 32z^2_k Z^2_{kl} + 8Z^4_{kl}, \quad (A.24)
\]
\[
P^{(1,1)}_{s3} = -3z^4_k Z^2_{kl}, \quad (A.25)
\]

and
\[
[r^2 \bar{\Pi}^{2}, r^2]_{kl} = \frac{\sigma_{kl}^{cl}}{\sigma(T)} \frac{\pi}{4z_k^3z_l^3} Z^2_{kl}(P^{(1,2)}_{s11} \bar{K}_1 + P^{(1,2)}_{s12} \bar{K}_2 + P^{(1,2)}_{s13} \bar{K}_3), \quad (A.26)
\]
\[
P^{(1,2)}_{s11} = 2Z_{kl}(z^5_k Z_{kl} + 8z^4_k + 16z^3_k Z_{kl} + 32z^2_k Z^2_{kl} + 16z_k Z^3_{kl} - 40Z^4_{kl}), \quad (A.27)
\]
\[
P^{(1,2)}_{s12} = -z^5_k Z_{kl}(Z^2_{kl} + 8) - 8z^4_k (Z^2_{kl} + 8) - 16z^3_k Z_{kl}(Z^2_{kl} + 8) \quad (A.28)
\]
\[
+ 16z^2_k Z^2_{kl}(Z^2_{kl} - 16) + 8z_k Z^3_{kl}(Z^2_{kl} - 16) - 8Z^4_{kl}(Z^2_{kl} + 8),
\]
\[ P_{s_{13}}^{(1,2)} = z_{kl}^7 Z_{kl}, \]  
(A.28)

and

\[
[\tau, \tau^2]_{kl} = [\tau^2, \tau]_{kl} = \frac{\sigma_{kl}^{\text{cl}}}{\sigma(T)} \frac{\pi}{4 z_{kl}^2 z_{kl}^6 Z_{kl}^4} (P_{s_{21}}^{(1,2)} \bar{K}_1 + P_{s_{22}}^{(1,2)} \bar{K}_2 + P_{s_{23}}^{(1,2)} \bar{K}_3), \]  
(A.29)

where

\[
P_{s_{21}}^{(1,2)} = 2Z_{kl}(z_{kl}^6 Z_{kl} - 8 z_{kl}^4 + 16 z_{kl}^3 Z_{kl} - 32 z_{kl}^2 Z_{kl}^2 + 16 z_{kl} Z_{kl}^3 + 40 Z_{kl}^4), \]  
(A.30)

\[
P_{s_{22}}^{(1,2)} = -z_{kl}^5 Z_{kl}(Z_{kl}^4 + 8) + 8 z_{kl}^3(Z_{kl}^2 + 8) - 16 z_{kl}^2 Z_{kl}(Z_{kl}^3 + 8) \]  
\[- 16 z_{kl}^2 Z_{kl}^3(Z_{kl}^2 - 16) + 8 z_{kl} Z_{kl}^4(Z_{kl}^2 - 16) + 8 Z_{kl}^4(Z_{kl}^2 + 8), \]  
(A.31)

\[
P_{s_{23}}^{(1,2)} = z_{kl}^5 Z_{kl}, \]  
(A.32)

and

\[
[\tau^2, \tau^2]_{kl} = \frac{\sigma_{kl}^{\text{cl}}}{\sigma(T)} \frac{\pi}{24 z_{kl}^2 z_{kl}^6 Z_{kl}^4} (P_{s_{11}}^{(2,2)} \bar{K}_1 + P_{s_{12}}^{(2,2)} \bar{K}_2 + P_{s_{13}}^{(2,2)} \bar{K}_3), \]  
(A.33)

where

\[
P_{s_{11}}^{(2,2)} = -2Z_{kl}[z_{kl}^6(Z_{kl}^2 + 2) + 6 z_{kl}^4(11 Z_{kl}^2 - 32) - 72 z_{kl}^2 Z_{kl}(Z_{kl}^2 + 8) \]  
\[+ 24 Z_{kl}^4(Z_{kl}^2 + 96)], \]  
(A.34)

\[
P_{s_{12}}^{(2,2)} = z_{kl}^6(Z_{kl}^4 + 10 Z_{kl}^2 + 16) - 6 z_{kl}^4(Z_{kl}^2 - 56 Z_{kl}^2 + 256) \]  
\[+ 144 z_{kl}^2 Z_{kl}^3(5 Z_{kl}^2 - 32) - 48 Z_{kl}^4(13 Z_{kl}^2 + 32), \]  
(A.35)

\[
P_{s_{13}}^{(2,2)} = -4 z_{kl}^4 \sigma_{kl}^6[z_{kl}^2(Z_{kl}^2 - 6) - 6 Z_{kl}^2], \]  
(A.36)

and

\[
[\tau^2, \tau^2]_{kl} = \frac{\sigma_{kl}^{\text{cl}}}{\sigma(T)} \frac{\pi}{24 z_{kl}^2 z_{kl}^6 Z_{kl}^4} (P_{s_{21}}^{(2,2)} \bar{K}_1 + P_{s_{22}}^{(2,2)} \bar{K}_2 + P_{s_{23}}^{(2,2)} \bar{K}_3), \]  
(A.37)

where

\[
P_{s_{21}}^{(2,2)} = -2Z_{kl}[z_{kl}^6(Z_{kl}^2 + 2) - 36 z_{kl}^4 Z_{kl} + 18 z_{kl}^4(Z_{kl}^2 + 16) \]  
\[+ 96 z_{kl}^3 Z_{kl}(Z_{kl}^2 - 10) + 24 z_{kl}^2 Z_{kl}^2(Z_{kl}^2 + 56) \]  
\[+ 48 z_{kl} Z_{kl}^3(Z_{kl}^2 + 20) - 24 Z_{kl}^4(Z_{kl}^2 + 100)], \]  
(A.38)

\[
P_{s_{22}}^{(2,2)} = z_{kl}^6(Z_{kl}^4 + 10 Z_{kl}^2 + 16) + 12 z_{kl}^5 Z_{kl}(Z_{kl}^2 - 24) \]  
\[- 6 z_{kl}^4(Z_{kl}^4 - 72 Z_{kl}^2 - 384) - 192 z_{kl}^3 Z_{kl}(Z_{kl}^2 + 40) \]  
\[- 48 z_{kl}^2 Z_{kl}^2(13 Z_{kl}^2 - 224) + 96 Z_{kl}^4(Z_{kl}^2 - 80) \]  
\[+ 48 Z_{kl}^4(13 Z_{kl}^2 + 48), \]  
(A.39)
\[ P_{s23}^{(2,2)} = -z_k^4 Z_{kl}^2 \{ 12 z_k^2 Z_{kl} - 6 \} + 12 z_k Z_{kl} - 6 Z_{kl} \}. \] (A.40)

And for the tensor collision brackets one has:

\[ \frac{\sigma_{cl}}{\sigma(T) 72 z_k^4 Z_{kl}^2} \left( P_{T11}^{(0,0)} \bar{K}_1 + P_{T12}^{(0,0)} \bar{K}_2 + P_{T13}^{(0,0)} \bar{K}_3 \right), \] (A.41)

where

\[ P_{T11}^{(0,0)} = -2 Z_{kl} \left\{ z_k^4 (5Z_{kl}^2 - 8) + 24 z_k^4 (Z_{kl}^2 - 16) - 144 z_k^2 Z_{kl}^2 (Z_{kl}^2 + 8) + 48 Z_{kl}^4 (Z_{kl}^2 + 72) \right\}, \] (A.42)

\[ P_{T12}^{(0,0)} = z_k^6 (5Z_{kl}^4 - 40Z_{kl}^2 - 64) - 24 z_k^4 (5Z_{kl}^4 + 8Z_{kl}^2 + 128) + 576 z_k^2 Z_{kl}^2 (Z_{kl}^2 - 16) - 192 Z_{kl}^4 (5Z_{kl}^2 + 16), \] (A.43)

\[ P_{T13}^{(0,0)} = -5 z_k^4 Z_{kl}^2 \{ z_k^2 (Z_{kl}^2 - 24) - 24 Z_{kl}^2 \}, \] (A.44)

and

\[ \frac{\sigma_{cl}}{\sigma(T) 72 z_k^4 Z_{kl}^2} \left( P_{T21}^{(0,0)} \bar{K}_1 + P_{T22}^{(0,0)} \bar{K}_2 + P_{T23}^{(0,0)} \bar{K}_3 \right), \] (A.45)

where

\[ P_{T21}^{(0,0)} = 2 Z_{kl} \left\{ z_k^6 (8 - 5Z_{kl}^2) + 72 z_k^4 (3Z_{kl}^2 - 8) - 480 z_k^2 Z_{kl} (Z_{kl}^2 - 4) \right\} \] - 336 z_k^2 Z_{kl}^2 (Z_{kl}^2 + 8) + 240 z_k Z_{kl}^3 (Z_{kl}^2 + 8) + 192 Z_{kl}^4 (Z_{kl}^2 + 67), \] (A.46)

\[ P_{T22}^{(0,0)} = z_k^6 (5Z_{kl}^4 - 40Z_{kl}^2 - 64) + 240 z_k^4 Z_{kl} (Z_{kl}^2 - 8) - 24 z_k^2 (5Z_{kl}^4 + 8Z_{kl}^2 - 192) + 1920 z_k^3 Z_{kl}^2 (Z_{kl}^2 - 8) - 192 Z_{kl}^4 (17Z_{kl}^2 - 112) + 1920 z_k Z_{kl}^3 (Z_{kl}^2 - 8) + 768 Z_{kl}^4 (5Z_{kl}^2 + 6), \] (A.47)

\[ P_{T23}^{(0,0)} = -5 z_k^4 Z_{kl}^2 \{ z_k^2 (Z_{kl}^2 - 24) + 48 z_k Z_{kl}^2 - 24 Z_{kl}^2 \}. \] (A.48)

If \( z_k = z_l \), then the \( G(x) \) function is eliminated everywhere and the collision brackets simplify considerably.

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