Abstract

We study the question of periodic homogenization of a variably scaled reaction-diffusion problem with non-linear drift posed for a domain crossed by a flat composite thin layer. The structure of the non-linearity in the drift was obtained in earlier works as hydrodynamic limit of a totally asymmetric simple exclusion process (TASEP) process for a population of interacting particles crossing a domain with obstacle.

Using energy-type estimates as well as concepts like thin-layer convergence and two-scale convergence, we derive the homogenized evolution equation and the corresponding effective model parameters for a regularized problem. Special attention is paid to the derivation of the effective transmission conditions across the separating limit interface in essentially two different situations: (i) finitely thin layer and (ii) infinitely thin layer.

This study should be seen as a preliminary step needed for the investigation of averaging fast non-linear drifts across material interfaces – a topic with direct applications in the design of thin composite materials meant to be impenetrable to high-velocity impacts.

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1 Introduction

Reaction-diffusion equations posed for thin layers endowed with periodic microstructures arise as mathematical models for a large number of real-world applications. Prominent examples refer, for instance, to blood flow through the blood vessels (here one considers the blood vessel walls as thin membranes with periodic microstructures), membrane filtration (see [23]), passage of oxygen particles through paperboard or through some other paper-based packaging materials (see [31]), formation of fingers in smoldering combustion [17], heat and current flow through thin organic light-emitting diodes (OLEDs) mounted on glass substrates [20].
In this paper, we study the effect of varying scalings on the periodic homogenization and eventual dimension reduction of a perforated thin layer hosting diffusion, chemical reactions, and nonlinear drift \(^1\) as derived earlier as mean-field limit for a totally asymmetric simple exclusion process (TASEP) on a lattice; see \([12]\). As microscopic domain \(\Omega_\varepsilon \subset \mathbb{R}^2\), we have two regions \(\Omega^\varepsilon_L\) and \(\Omega^\varepsilon_R\) glued together through standard transmission conditions via a static flat thin layer \(\Omega^\varepsilon_M\) (see Fig \([4]\)). The thin layer \(\Omega^\varepsilon_M\) is made of an array of periodic microstructures, while the sets \(\Omega^\varepsilon_L\) and \(\Omega^\varepsilon_R\) are in fact non-oscillating. To describe the internal structure of \(\Omega^\varepsilon_M\), we replicate a reference cell \(Z\) (see Fig \([2]\)), whose height is scaled by \(\varepsilon\) and its width by \(\kappa(\varepsilon)\). In our case, the assumed periodicity acts only in vertical direction. In each of the regions \(\Omega^\varepsilon_L\) and \(\Omega^\varepsilon_R\), the coefficients of the partial differential equations are independent of \(\varepsilon\). Instead, within the set \(\Omega^\varepsilon_M\) the coefficients of the evolution equation are assumed to satisfy a variable scaling. To be specific, we consider that both the time derivative term and the production-by-reaction term are scaled by \(\varepsilon^\alpha\), the diffusion coefficient is scaled by \(\varepsilon^\beta\), while the drift term is scaled by \(\varepsilon^\gamma\). The boundary production terms at the oscillating boundaries are proportional to \(\varepsilon^\xi\). Here \(\alpha, \beta, \xi, \gamma \in \mathbb{R}\) are dimensionless parameters. It is worth noting that the factors \(\varepsilon^\alpha, \varepsilon^\beta, \varepsilon^\gamma\) and \(\varepsilon^\xi\) are referred to in the chemical engineering literature as Damköhler numbers, while \(\varepsilon^\xi\) resembles the Thiele modulus (also called surface Damköhler number). They are all ratios of characteristic time scales of pairwise combinations of partial physical processes; see e.g. \([14]\). For instance, \(\varepsilon^\beta\) is of order of \(O\left(\frac{t_{\text{diff}}}{t_{\text{reac}}}\right)\), where \(t_{\text{diff}}\) and \(t_{\text{reac}}\) are the characteristic time scales of diffusion, and respectively, of reaction. The boundary conditions are chosen such that they correspond to the original interacting particle systems scenario. Consequently, we take non-homogeneous Dirichlet boundary conditions on the vertical boundaries of \(\Omega_\varepsilon\) and non-homogeneous Neumann boundary conditions on the rest of the boundaries.

Our main goal is to study how the different choices of the parameters \(\alpha, \beta, \gamma,\) and \(\xi\) affects the structure of the upscaled equations, i.e. when \(\varepsilon \to 0\). From the modeling point of view, the main interest lies in learning which limit transmission conditions correspond to the cases: (i) the finitely thin layer (Fig. \([4]\)) and (ii) the infinitely thin layer (Fig. \([3]\)) and how does depend on the choice of the overall scaling. In this context, we set for (i) \(\kappa(\varepsilon) = \varepsilon\), while for (ii) we consider \(\kappa(\varepsilon)\) to be a constant independent of \(\varepsilon\). Other choices of scaling of the geometry are also possible, especially if we extend the current discussion from 2D to a scenario in 3D. However, we believe that we captured the main ones, especially from the application point of view. This study should be seen as a preliminary step needed for the investigation of averaging fast non-linear drifts across material interfaces – a topic with direct applications in the design of thin composite materials meant to be impenetrable to high-velocity impacts. Most of the upscaled models receive a double-porosity type structure; see \([4]\) for more in this direction.

The main tools used in this context to derive the wanted upscaled evolution equations and corresponding transmission condition for a large variety of choices of scalings include the energy method (see the basic idea of playing with variable scalings in \([33]\) or in \([39]\)) combined with classical two-scale convergence/compactness results (see \([24]\)) and two-scale convergence/compactness for thin layers (see \([29]\)). The current main difficulties lie in deriving \(\varepsilon\)-independent estimates for all scaling options so that passing to the homogenization limit becomes possible in each case, dealing

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\(^1\)The drift term is here the gradient of a bounded, possibly discontinuous polynomial. To keep things simple, we use a suitable mollification of the drift to gain extra regularity. The mollifier function has support within \(\overline{B}(0, \delta)\), where \(\delta > 0\) is independent of \(\varepsilon\). We choose our mollifier such that, as \(\delta \to 0\), the regularized drift converges strongly to the original nonlinear drift in \(L^p(\mathbb{R}^2)\) for all \(p \in (0, \infty)\).
with the the non-linearity of the drift, as well as varying \( \kappa(\varepsilon) \). In this context, we bring in rigorous mathematical analysis results complementing our formal asymptotic calculations reported in [11]. As future step, our investigation will attempt to deal with fast drifts, that is it will be about entering the regime of \( \gamma < 0 \).

For a basic understanding of homogenization theory in the broader context of asymptotic analysis, we refer the reader to the standard monographs [10], [32], [26], and [6], e.g. Classical two-scale convergence and compactness result can be found in [30] and [2]; see also [21]. The earliest result that we know regarding homogenization and dimension reduction for a thin layer including also a drift with a Navier-Stokes-type nonlinearity is [25]; see also [36] for a more recent account. The simultaneous homogenization and dimension reduction of reaction-diffusion equations with non-linear reaction rates posed in a thin heterogeneous layer have been carefully studied in [29]. In loc. cit., the authors introduced a number of new techniques to derive effective transmission conditions along the layer. Our work follows very much the spirit of this paper, as well as of the follow-up investigations for thin channels [7] and [19]. More research is available on the simultaneous homogenization and dimension reduction. We mention here but a few of them which we think are closer to our investigations. Linear reaction-diffusion-convection equations coupled with non-linear surface chemical reactions for infinitely thin layers were studied in [17] in the context of smoldering combustion. In [16], the authors studied pressure-driven Stokes flow through a infinitely thin layer. A double porosity scenario with jumps at sharp heterogeneities was studied in [9]. Further work related to homogenization of infinitely thin layers, sharp interfaces, and other geometric singularities can be found in [21], [38], and [3]. This list of potentially relevant references is not exhaustive.

It is worth mentioning that it is a challenge to approximate numerically the obtained upscaled models (compare e.g. [37]). However, due to the scale separation between the microscopic and the macroscopic characteristic length scales, high performance computing strategies are available to handle efficient approximations of such double-porosity like models (dimensionally-reduced or not). We refer the reader, for instance, to [35] and references cited therein for a possible parallelization strategy.

We organize our paper as follows: In section 2 we describe the model problem, its variable scaling, and introduce the boundary and initial conditions including the perfect transmission conditions. To work with a problem having homogeneous Dirichlet boundary condition on vertical boundaries, we use an affine transformation of the original problem and obtain the transformed problem with homogeneous Dirichlet boundary. The downside of employing the transformation is that we lose the perfect transmission condition on the boundaries between the bulk regions and thin layer. In section 3 we prove the existence and uniqueness of \( \varepsilon \)-dependent weak solution to our microscopic problem via the Galerkin method (see e.g. the standard lines of arguments from p.314 in [15]). In section 4 we prove \( \varepsilon \)-independent energy estimates for the solution of microscopic problem later and point out that we can use the well-established concept of two-scale convergence for thin layers to treat our infinitely thin layer case. By using energy-type estimates and compactness results we derive the two-scale limit equations of the upscaled problem. In section 5 we make choices for \( \alpha, \beta, \gamma, \) and \( \xi \) that we deem as potentially relevant to derive the corresponding upscaled equations and effective coefficients. In the final section, we propose an approximation of solutions of the non-regularized upscaled problem by using a direct method inspired from [34].
2 Microscopic model

2.1 Setting of the problem

In this section, we describe the microscopic reaction-diffusion-drift model we have in mind. The geometry where our equations are posed is sketched in Fig. 1.

Let $\varepsilon, \ell, \kappa(\varepsilon), \kappa, h, T > 0$ with $\frac{h}{\varepsilon} \in \mathbb{N}$, $\kappa(\varepsilon) = \varepsilon$ in the case of infinitely thin layer and $k(\varepsilon) = \kappa$ for the case of finitely thin layer. $\Omega$ be a two dimensional strip defined as $\Omega := [-\ell/2, +\ell/2] \times [0, h]$. Define $Y := (-1, 1) \times (0, 1)$ and the standard cell $Z$ as $Y$ with an impenetrable compact rectangle called obstacle (denote as $Y_0$) with $Y_0 = [a_1, b_1] \times [a_2, b_2]$ which is placed in the center of the $Y$ (i.e $Z := Y \setminus Y_0$ ). Assume that $\partial Y_0$ is Lipchitz boundary and $\partial Y \cap Y_0 = \emptyset$ (see Fig. 2).

We define our microscopic domain $\Omega^\varepsilon \subset \Omega$ as

$$\Omega^\varepsilon := \Omega_L^\varepsilon \cup \Omega_R^\varepsilon \cup \Omega_M^\varepsilon \cup B_{\ell}^\varepsilon \cup B_{\ell}^\varepsilon,$$

where

$$\begin{align*}
\Omega_L^\varepsilon & = (-\ell/2, -\kappa(\varepsilon)) \times (0, h), \\
\Omega_R^\varepsilon & = (\kappa(\varepsilon), \ell/2) \times (0, h), \\
\Omega_M^\varepsilon & = ((-\kappa(\varepsilon), \kappa(\varepsilon)) \times (0, h)) \setminus \Omega_0^\varepsilon
\end{align*}$$

$$\Omega^\varepsilon := \Omega_L^\varepsilon \cup \Omega_R^\varepsilon \cup \Omega_M^\varepsilon \cup B_{\ell}^\varepsilon \cup B_{\ell}^\varepsilon,$$
where $e_1, e_2$ are standard unit vectors in $\mathbb{R}^2$, $k_0 = \frac{b}{\varepsilon}$ and we denote the union of obstacles as $\Omega^\varepsilon_0$

$$\Omega^\varepsilon_0 := \bigcup_{k=0}^{k_0} (ke_2 + (\kappa(\varepsilon))[a_1, b_1] \times \varepsilon[a_2, b_2]).$$

(3)

We refer $\Omega^\varepsilon_M$ as layer and $\Omega^\varepsilon_R, \Omega^\varepsilon_L$ as bulk region, and we cover boundary of $\Omega^\varepsilon$ by the following sets,

$$B^\varepsilon_L := \{-\kappa(\varepsilon)\} \times (0, h),$$

$$B^\varepsilon_R := \{\kappa(\varepsilon)\} \times (0, h),$$

$$\Gamma^\varepsilon_L := \left\{-\frac{\ell}{2}\right\} \times [0, h],$$

$$\Gamma^\varepsilon_R := \left\{\frac{\ell}{2}\right\} \times [0, h],$$

$$\Gamma^\varepsilon_v := \Gamma^\varepsilon_L \cup \Gamma^\varepsilon_R,$$

$$\Gamma^\varepsilon_h := (\partial \Omega^\varepsilon_L \cup \partial \Omega^\varepsilon_R) \setminus (B^\varepsilon_L \cup B^\varepsilon_R \cup \Gamma^\varepsilon_v),$$

$$\Gamma^\varepsilon_0 := \partial \Omega^\varepsilon_M \setminus (B^\varepsilon_L \cup B^\varepsilon_R).$$

Note that $\partial \Omega^\varepsilon = \Gamma^\varepsilon_v \cup \Gamma^\varepsilon_h \cup \Gamma^\varepsilon_0$. The external unit normal vectors at $\partial \Omega^\varepsilon_L, \partial \Omega^\varepsilon_R, \partial \Omega^\varepsilon_M$ are denoted by $n_l, n_r, n_m$ respectively.

We consider the following reaction diffusion problem. Find $(u^\varepsilon_1, u^\varepsilon_m, u^\varepsilon_2)$ satisfying the following equation

$$\frac{\partial u^\varepsilon_1}{\partial t} + \text{div}(-D_L \nabla u^\varepsilon_1 + B_L P_\delta(u^\varepsilon_1)) = f_1 \quad \text{on } \Omega^\varepsilon_L \times (0, T),$$

$$\frac{\partial u^\varepsilon_m}{\partial t} + \text{div}(-D_R \nabla u^\varepsilon_m + B_R P_\delta(u^\varepsilon_m)) = f_m \quad \text{on } \Omega^\varepsilon_M \times (0, T),$$

$$\varepsilon^\alpha \frac{\partial u^\varepsilon_2}{\partial t} + \text{div}(-D^\varepsilon_M \nabla u^\varepsilon_2 + \varepsilon^\gamma B^\varepsilon_M P_\delta(u^\varepsilon_m)) = \varepsilon^\alpha f^\varepsilon_2 \quad \text{on } \Omega^\varepsilon_M \times (0, T),$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and the parameter $\delta > 0$ is fixed, $f_1 : \Omega^\varepsilon_L \rightarrow \mathbb{R}$, $f_m : \Omega^\varepsilon_M \rightarrow \mathbb{R}$, $f_m : \Omega^\varepsilon_M \rightarrow \mathbb{R}$ are given functions, $D_j = \begin{bmatrix} d_j^1 & 0 \\ 0 & d_j^2 \end{bmatrix}$, $B_j = \begin{bmatrix} b_j^1 \\ b_j^2 \end{bmatrix}$, with $d_j^1, d_j^2, b_j^1, b_j^2 > 0$ for $j \in \{L, R\}$, $D^\varepsilon_M(x_1, x_2) = D(x_1/\varepsilon, x_2/\varepsilon)$, $(x_1, x_2) \in \mathbb{R}^2$, $D$ is a $2 \times 2$ diagonal matrix with positive entries defined in standard cell $Z$ and 1-periodic, $B^\varepsilon_M = B(\frac{a_1}{\varepsilon}, \frac{a_2}{\varepsilon})$, $B$ is a $2 \times 1$ matrix with 1-periodic. $P_\delta(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^2$ is defined as

$$P_\delta(r) := \int_\mathbb{R} \rho_\delta(y) P(r - y) dy,$$

where $\rho_\delta(x) := \frac{1}{\sqrt{\pi}} \rho(x/\delta)$ for $x \in \mathbb{R}$, $n_0 \in \mathbb{N}$, $\rho$ is a mollifier, for instance we take

$$\rho(x) := \begin{cases} C e^{\frac{|x|^2}{2^2 - 1}} & |x| < 1 \\
0 & |x| \geq 1, \end{cases}$$

(7)

where the constant $C > 0$ selected so that $\int_{\mathbb{R}^n} \rho dx = 1$ and

$$P(r) = \begin{cases} a_0 + a_1 r + \cdots + a_m r^m & \text{for } r \in [0, 1] \\
0 & \text{otherwise}, \end{cases}$$

(8)

Notice that $P_\delta(r) \rightarrow P(r)$ in $L^p(\mathbb{R})$ as $\delta \rightarrow 0$ for all $p \in [1, \infty)$ (see p.717, [15]).

\[\text{5}\]
with \( a_k \in \mathbb{R} \) for \( k \in \mathbb{N} \).

We endow (5) with the following boundary and initial conditions

\[
\begin{align*}
    u^e_i &= U_L \text{ on } \Gamma_L \times (0, T), \\
u^e_r &= U_R \text{ on } \Gamma_R \times (0, T), \tag{9}
\end{align*}
\]

\[
\begin{align*}
    -\varepsilon^\beta D^e_M \nabla u^e \cdot n &= \varepsilon^\gamma B^e_M P_\delta(v^e_m) \cdot n_m \text{ on } \Gamma_0^e \times (0, T), \\
    -D_L \nabla u^e_i + B_L P_\delta(u^e_m) &\cdot n_i = g_i \text{ on } (\Gamma_\varepsilon \cap \partial \Omega_L^e) \times [0, T], \\
    -D_R \nabla u^e_r + B_R P_\delta(u^e_i) &\cdot n_r = g_r \text{ on } (\Gamma_\varepsilon \cap \partial \Omega_R^e) \times (0, T), \tag{10}
\end{align*}
\]

\[
\begin{align*}
    u^e_i(0, x) &= h^e_i(x) \text{ for all } x \in \Gamma_L^e, \\
u^e_r(0, x) &= h^e_r(x) \text{ for all } x \in \Gamma_R^e, \\
u^e_m(0, x) &= h^e_m(x) \text{ for all } x \in \Gamma_M^e, \tag{11}
\end{align*}
\]

where \( \xi > 0 \) is a fixed real number, \( U_L : \Gamma_L \times [0, T] \to \mathbb{R}, U_R : \Gamma_R \times [0, T] \to \mathbb{R}, g_i : (\Gamma_\varepsilon \cap \partial \Omega_L^e) \times [0, T] \to \mathbb{R}, g_r : (\Gamma_\varepsilon \cap \partial \Omega_R^e) \times [0, T] \to \mathbb{R}, h^e_i : \Omega_L^e \to \mathbb{R}, h^e_r : \Omega_R^e \to \mathbb{R}, h^e_m : \Omega_M^e \to \mathbb{R} \) are given functions so that \( U_L \in L^\infty(0, T; H^\frac{1}{2}(\Gamma_L)), U_R \in L^\infty(0, T; H^\frac{1}{2}(\Gamma_R)) \). On surface \( \Gamma_L^e \) and \( \Gamma_R^e \) we assume the perfect transmission condition

\[
\begin{align*}
    u^e_i &= u^e_M \text{ on } \Gamma_L^e, \\
u^e_r &= u^e_M \text{ on } \Gamma_R^e, \\
    -\varepsilon^\beta D^e_M \nabla u^e \cdot n &= \varepsilon^\gamma B^e_M P_\delta(v^e_m) \cdot n_m \text{ on } \Gamma_L^e, \\
    -D_L \nabla u^e_i + B_L P_\delta(u^e_m) &\cdot n_i = (D_L \nabla v^e_i + B_L P_\delta(v^e_m)) \cdot n_i \text{ on } \Gamma_L^e, \tag{12}
\end{align*}
\]

\[
\begin{align*}
    -D_R \nabla u^e_r + B_R P_\delta(u^e_i) &\cdot n_r = (D_R \nabla v^e_r + B_R P_\delta(v^e_i)) \cdot n_r \text{ on } \Gamma_R^e.
\end{align*}
\]

### 2.2 Transformation of problem

To obtain the homogenous Dirichlet boundary condition for the problem (5), we use the following transformation

\[
v^e_i := u^e_i + \frac{1}{2}(x_1 - 1)U_L - \frac{1}{2}(x_1 + 1)U_R, \tag{13}
\]

for \( i \in \{l, m, r\} \). Let us denote \( u_b(x, t) := \frac{1}{2}(x_1 - 1)U_L - \frac{1}{2}(x_1 + 1)U_R, \) with \( x = (x_1, x_2) \in \mathbb{R}^2 \) and \( t \in [0, T] \). So,

\[
u^e_i = v^e_i - u_b. \tag{14}
\]

Inserting (14) into (5) gives the transformed reaction diffusion equation

\[
\begin{align*}
    \frac{\partial v^e_i}{\partial t} + \text{div}(-D_L \nabla v^e_i + B_L P_\delta(v^e_m - u_b)) &= f_{\text{bi}} \quad \text{on } \Omega_L^e \times (0, T), \\
    \frac{\partial v^e_r}{\partial t} + \text{div}(-D_R \nabla v^e_r + B_R P_\delta(v^e_i - u_b)) &= f_{\text{br}} \quad \text{on } \Omega_R^e \times (0, T), \\
    \varepsilon^\alpha \frac{\partial v^e_m}{\partial t} + \text{div}(-\varepsilon^\beta D^e_M \nabla v^e_m + \varepsilon^\gamma B^e_M P_\delta(v^e_m - u_b)) &= \varepsilon^\alpha f^\varepsilon_m \\
    + \varepsilon^\beta f^\varepsilon_m \quad \text{on } \Omega_M^e \times (0, T), \tag{15}
\end{align*}
\]

endowed with the following boundary and initial conditions,
\( v^e_t = 0 \) on \( \Gamma_L \times (0, T) \),
\( v^e_r = 0 \) on \( \Gamma_R \times (0, T) \),
\[
\begin{align*}
(-\varepsilon^\beta D_M^e \nabla v^e_m + \varepsilon^\gamma B_M^e P_b(v^e_m - u_b)) \cdot n^e_m &= \varepsilon^\xi g^e_0 + \varepsilon^\delta g^e_0b \text{ on } \Gamma_0^e \times (0, T), \\
(-D_L^\nabla v^e_l + B_L^e P_\delta(v^e_l - u_b)) \cdot n_l &= g^e_{b, l} \text{ on } (\Gamma_h^e \cap \partial \Omega_L^e) \times (0, T), \\
(-D_R^\nabla v^e_r + B_R^e P_\delta(v^e_r - u_b)) \cdot n_r &= g^e_{b, r} \text{ on } (\Gamma_h^e \cap \partial \Omega_R^e) \times (0, T), \\
v^e_l(0, x) &= h^e_{b, l}(x) \text{ for all } x \in \overline{\Omega_L}, \\
v^e_r(0, x) &= h^e_{b, r}(x) \text{ for all } x \in \overline{\Omega_R}, \\
v^e_m(0, x) &= h^e_{b, m}(x) \text{ for all } x \in \overline{\Omega_M},
\end{align*}
\]
where \( f_{b, l} := \partial_t u_b - \text{div}(D_L^\nabla u_b) + f_{l}, \quad f_{b, r} := \partial_t u_b - \text{div}(D_R^\nabla u_b) + f_{r}, \quad f_{b, m} := -\text{div}(D_M^\nabla u_b), \quad g^e_{b, l} := -D_M^e \nabla u_b, \quad g^e_{b, r} = D_L^\nabla u_b, \quad g^e_{b, m} = D_R^\nabla u_b \) and \( h^e_{b, l} = h^e_{b, r} = u_b \) for every \( i \in \{l, m, r\} \) and with the following transmission condition
\[
\begin{align*}
v^e_l &= v^e_m \text{ on } B_L^e, \\
v^e_r &= v^e_m \text{ on } B_R^e,
\end{align*}
\]
\[
\begin{align*}
(-\varepsilon^\beta D_M^e \nabla (v^e_m - u_b) + \varepsilon^\gamma B_M^e P_\delta(v^e_m - u_b)) \cdot n^e_m &= (-D_L^\nabla (v^e_l - u_b) + B_L^e P_\delta(v^e_l - u_b)) \cdot n_l \text{ on } B_L^e, \\
(-\varepsilon^\beta D_M^e \nabla (v^e_m - u_b) + \varepsilon^\gamma B_M^e P_\delta(v^e_m - u_b)) \cdot n^e_m &= (-D_R^\nabla (v^e_r - u_b) + B_R^e P_\delta(v^e_r - u_b)) \cdot n_r \text{ on } B_R^e.
\end{align*}
\]

### 2.3 Assumptions on data

From now on \( C \) denotes a positive real number possibly changing from line to line. When necessary, we will write explicit the parameters on which it will depend.

(A1) **(Ellipticity condition)** For every \( i \in \{L, M, R\} \) with \( j(L) = L, j(R) = R, j(M) = M \) and for every \( \eta \in \mathbb{R}^2 \) there exist a \( \theta > 0 \) such that,
\[
\theta \|\eta\|^2 \leq \eta^t D_\eta \eta,
\]
and
\[
D_\eta \in L^\infty(\Omega^\varepsilon_{j(i)}; \mathbb{R}^4).
\]

(A2) **Concerning the drift coefficient**, we assume
\[
\text{div} B_i \in L^\infty(\Omega^\varepsilon_{j(i)}),
\]
and
\[
B_i \in L^\infty(\Omega^\varepsilon_{j(i)}; \mathbb{R}^2)
\]
for \( i \in \{L, M, R\} \) with \( j(L) = L, j(R) = R, j(M) = M \).

(A3) **For the reaction rate**, we assume \( f_{b, l}, \partial_t f_{b, r} \in L^2(0, T; L^2(\Omega^\varepsilon_L)), \)
\( f_{b, r}, \partial_t f_{b, b} \in L^2(0, T; L^2(\Omega^\varepsilon_R)), f_{b, m}, \partial_t f_{b, m} \in L^2(0, T; L^2(\Omega^\varepsilon_M)) \) and
\[
\varepsilon^\alpha \|f_{b, m}\|_{L^2(0, T; L^2(\Omega^\varepsilon_M))} \leq C,
\]
where \( C \) denotes a positive real number possibly changing from line to line.
for a.e. $t \in (0, T)$. Together we assume there exist $f_{ao} \in L^2((0, T) \times \Sigma \times Z)$ such that

\[ f_{ao}^\varepsilon \overset{2-\varepsilon}{\to} f_{ao}. \]  

(A4) $g_{bo}, \partial_t g_{bo} \in L^\infty(0, T; L^2(\Gamma_h^\varepsilon \cap \partial \Omega_\varepsilon^L)), g_{br}, \partial_t g_{br} \in L^\infty(0, T; L^2(\Gamma_h^\varepsilon \cap \partial \Omega_\varepsilon^R)), g_{bo}, \partial_t g_{bo} \in L^\infty(0, T; L^2(\Gamma_0^\varepsilon))$ and

\[ \varepsilon^{\frac{1}{2}}\|g_{bo}\|_{L^2(\Gamma_0^\varepsilon)} \leq C, \]  
\[ \varepsilon^{\frac{1}{2}}\|g_{br}\|_{L^2(\Gamma_0^\varepsilon)} \leq C, \]  

for a.e. $t \in (0, T)$. Together we assume there exist $g_0 \in L^2((0, T) \times \Sigma \times \partial Y_0)$ such that

\[ g_0^\varepsilon \overset{2-\varepsilon}{\to} g_0. \]  

(A5) For initial conditions, we assume $h_{bi}^\varepsilon \in H^1(\Omega_\varepsilon^L), h_{br}^\varepsilon \in H^1(\Omega_\varepsilon^R), h_{bm}^\varepsilon \in H^1(\Omega_\varepsilon^M)$ with

\[ \|h_{bi}^\varepsilon\|_{L^2(\Omega_\varepsilon^L)}^2 + \|h_{br}^\varepsilon\|_{L^2(\Omega_\varepsilon^R)}^2 + \varepsilon^\alpha \|h_{bm}^\varepsilon\|_{L^2(\Omega_\varepsilon^M)}^2 \leq C, \]  

and

\[ \mathbb{1}_{\Omega_\varepsilon^L} h_{bi}^\varepsilon \to h_{bi}^0 \quad \text{on} \quad L^2((0, T) \times \Omega_\varepsilon^L), \]  
\[ \mathbb{1}_{\Omega_\varepsilon^R} h_{br}^\varepsilon \to h_{br}^0 \quad \text{on} \quad L^2((0, T) \times \Omega_\varepsilon^R), \]  
\[ h_{bm}^\varepsilon \overset{2-\varepsilon}{\to} h_{br}^0. \]  

(A6) On parameter $\alpha, \beta, \gamma, \xi$ we assume, $\beta, \gamma \geq 0, \gamma \geq \beta, \beta \leq \xi + \frac{1}{2}, \alpha + \frac{1}{2} \leq \beta, \alpha + \frac{1}{2} \leq \xi$.  

(A7) $\partial_t u_b \in L^2(0, T; H^1(\Omega_\varepsilon^\varepsilon))$.

Note that we choose $U_L$ and $U_R$ defined in (9) according to satisfy (A1)-(A7).

### 2.4 Weak formulation

In this section we propose the weak formulation of problem (15). We use the following definitions

\[ H^1(\Omega_\varepsilon^L; \Gamma_\varepsilon^L) : = \{ u \in H^1(\Omega_\varepsilon^L) : u = 0 \text{ on } \Gamma_\varepsilon^L \}, \]  
\[ H^1(\Omega_\varepsilon^R; \Gamma_\varepsilon^R) : = \{ u \in H^1(\Omega_\varepsilon^R) : u = 0 \text{ on } \Gamma_\varepsilon^R \}, \]  
\[ V_\varepsilon : = \{ (u_t^\varepsilon, u_m^\varepsilon, u_r^\varepsilon) \in H(\Omega_\varepsilon^L; \Gamma_\varepsilon^L) \times H(\Omega_\varepsilon^R; \Gamma_\varepsilon^R) \times H(\Omega_\varepsilon^M; \Sigma) : \]  
\[ u_t^\varepsilon = u_m^\varepsilon \text{ on } \mathcal{B}_L^\varepsilon, u_r^\varepsilon = u_m^\varepsilon \text{ on } \mathcal{B}_R^\varepsilon \} \]  
\[ v_\varepsilon : = (v_t^\varepsilon, v_m^\varepsilon, v_r^\varepsilon). \]  

(15)
Definition 2.1 The weak formulation of the problem \((15)-(17)\) is to find
\[
(v_1^\varepsilon, v_m^\varepsilon, v_r^\varepsilon) \in L^2(0, T; V_\varepsilon) \cap H^1(0, T; L^2(\Omega_\varepsilon^L) \times L^2(\Omega_\varepsilon^M) \times L^2(\Omega_\varepsilon^R))
\]
such that \((v_1^\varepsilon, v_m^\varepsilon, v_r^\varepsilon)\) satisfies
\[
\begin{align*}
\int_{\Omega_\varepsilon^L} \partial_t v_1^\varepsilon \phi_1 dx + \int_{\Omega_\varepsilon^L} D_L \nabla v_1^\varepsilon \phi_1 dx - \int_{\Omega_\varepsilon^L} B_L P_\varepsilon (v_1^\varepsilon - u_b) \nabla \phi_1 dx \\
= \int_{\Omega_\varepsilon^L} f_{t_1} \phi_1 dx - \int_{\Gamma_{h_1} \cap \partial \Omega_\varepsilon^L} g_{t_1} \phi_1 d\sigma + \int_{B_\varepsilon^L} D_L \nabla u_b \cdot n_1 \phi_1 d\sigma \\
+ \int_{B_\varepsilon^L} (-\varepsilon^{\beta} D_M^\varepsilon \nabla (v_m^\varepsilon - u_b) + \varepsilon^{\gamma} B_M^\varepsilon P_\varepsilon (v_m^\varepsilon - u_b)) \cdot n_m^\varepsilon \phi_1 d\sigma, \quad (36)
\end{align*}
\]
\[
\begin{align*}
\int_{\Omega_\varepsilon^R} \partial_t v_r^\varepsilon \phi_3 dx + \int_{\Omega_\varepsilon^R} D_R \nabla v_r^\varepsilon \phi_3 dx - \int_{\Omega_\varepsilon^R} B_R P_\varepsilon (v_r^\varepsilon - u_b) \nabla \phi_3 dx \\
= \int_{\Omega_\varepsilon^R} f_{t_2} \phi_3 dx - \int_{\Gamma_{h_2} \cap \partial \Omega_\varepsilon^R} g_{t_2} \phi_3 d\sigma + \int_{B_\varepsilon^R} D_R \nabla u_b \cdot n_r \phi_3 d\sigma \\
+ \int_{B_\varepsilon^R} (-\varepsilon^{\beta} D_M^\varepsilon \nabla (v_m^\varepsilon - u_b) + \varepsilon^{\gamma} B_M^\varepsilon P_\varepsilon (v_m^\varepsilon - u_b)) \cdot n_m^\varepsilon \phi_1 d\sigma, \quad (37)
\end{align*}
\]
\[
\begin{align*}
\varepsilon^{\alpha} \int_{\Omega_\varepsilon^M} \partial_t v_m^\varepsilon \phi_2 dx + \varepsilon^{\beta} \int_{\Omega_\varepsilon^M} D_M^\varepsilon \nabla v_m^\varepsilon \phi_2 dx - \varepsilon^{\gamma} \int_{\Omega_\varepsilon^M} B_M^\varepsilon P_\varepsilon (v_m^\varepsilon - u_b) \nabla \phi_2 dx \\
= \varepsilon^{\alpha} \int_{\Omega_\varepsilon^M} f_{m_1} \phi_2 dx + \varepsilon^{\beta} \int_{\Omega_\varepsilon^M} f_{m_2} \phi_2 dx - \varepsilon^{\gamma} \int_{\Gamma_0^\varepsilon} g_{m_1} \phi_2 d\sigma - \varepsilon^{\beta} \int_{\Gamma_0^\varepsilon} g_{m_2} \phi_2 d\sigma \\
+ \varepsilon^{\beta} \int_{B_\varepsilon^L} D_M \nabla u_b \cdot n_m^\varepsilon \phi_2 d\sigma \\
- \int_{B_\varepsilon^L} (-\varepsilon^{\beta} D_M^\varepsilon \nabla (v_m^\varepsilon - u_b) + \varepsilon^{\gamma} B_M^\varepsilon P_\varepsilon (v_m^\varepsilon - u_b)) \cdot n_m^\varepsilon \phi_2 d\sigma \\
+ \varepsilon^{\beta} \int_{B_\varepsilon^R} D_M \nabla u_b \cdot n_m^\varepsilon \phi_2 d\sigma \\
- \int_{B_\varepsilon^R} (-\varepsilon^{\beta} D_M^\varepsilon \nabla (v_m^\varepsilon - u_b) + \varepsilon^{\gamma} B_M^\varepsilon P_\varepsilon (v_m^\varepsilon - u_b)) \cdot n_m^\varepsilon \phi_2 d\sigma, \quad (38)
\end{align*}
\]
for every \(\phi := (\phi_1, \phi_2, \phi_3) \in V_\varepsilon\) and a.e. \(t \in (0, T)\).

Now using, \(n_l = -n_m^\varepsilon\) on \(B_\varepsilon^L\), \(n_r = -n_m^\varepsilon\) on \(B_\varepsilon^R\), \(\phi_1 = \phi_2\) on \(B_\varepsilon^L\), \(\phi_3 = \phi_2\) on \(B_\varepsilon^L\) and adding the equations \((36)-(38)\), we find it is useful to rewrite Definition 2.1 as to find
\[
(v_1^\varepsilon, v_m^\varepsilon, v_r^\varepsilon) \in L^2(0, T; V_\varepsilon) \cap H^1(0, T; L^2(\Omega_\varepsilon^L) \times L^2(\Omega_\varepsilon^M) \times L^2(\Omega_\varepsilon^R))
\]
such that
\[
\begin{align*}
\int_{\Omega_\varepsilon^L} \partial_t v_1^\varepsilon \phi_1 dx + \int_{\Omega_\varepsilon^L} \partial_t v_2^\varepsilon \phi_3 dx + \varepsilon^{\alpha} \int_{\Omega_\varepsilon^M} \partial_t v_m^\varepsilon \phi_2 dx \\
+ \int_{\Omega_\varepsilon^L} D_L \nabla v_1^\varepsilon \phi_1 dx + \int_{\Omega_\varepsilon^R} D_R \nabla v_r^\varepsilon \phi_3 dx + \varepsilon^{\beta} \int_{\Omega_\varepsilon^M} D_M \nabla v_m^\varepsilon \phi_2 dx
\end{align*}
\]
Lemma 1 There exists a priori estimates for the solution $(\nu^\varepsilon_1, \nu^\varepsilon_2, \nu^\varepsilon_3)$ of (39). Then we will use these estimates to get the two-scale limit of $(\nu^\varepsilon_1, \nu^\varepsilon_2, \nu^\varepsilon_3)$. The proof follows via Galerkin method similar to that used in [27].

**Proof:** The proof follows via Galerkin method similar to that used in [27]. □

**3 Weak solvability of the microscopic problem**

The following theorem establishes existence and uniqueness of the weak solution of the problem (39).

**Theorem 1** Under the assumption (A1)-(A6) there exists a unique solution of (39).

**Proof:** The proof follows via Galerkin method similar to that used in [27]. □

**4 Two-scale convergence for thin membrane**

In this section, we first prove $\varepsilon$ independent a priori energy estimates for the solution $(\nu^\varepsilon_1, \nu^\varepsilon_2, \nu^\varepsilon_3)$ of (39). Then we will use these estimates to get the two-scale limit of $(\nu^\varepsilon_1, \nu^\varepsilon_2, \nu^\varepsilon_3)$ as $\varepsilon \to 0$.

**4.1 A priori estimates**

**Lemma 1** There exists $C > 0$ independent of $\varepsilon$, such that

\[
\|\nu^\varepsilon_1\|^2_{L^2(\Gamma^\varepsilon_1 \cap \Omega^\varepsilon)} \leq C\|\nabla \nu^\varepsilon_1\|^2_{L^2(\Omega^\varepsilon_1)},
\]

\[
\|\nu^\varepsilon_2\|^2_{L^2(\Omega^\varepsilon_2)} \leq C\|\nabla \nu^\varepsilon_2\|^2_{L^2(\Omega^\varepsilon_2)},
\]

\[
\|\nu^\varepsilon_3\|^2_{L^2(\Omega^\varepsilon_3)} \leq C\|\nabla \nu^\varepsilon_3\|^2_{L^2(\Omega^\varepsilon_3)},
\]

\[
\|\nu^\varepsilon_4\|^2_{L^2(\Omega^\varepsilon_4)} \leq C\|\nabla \nu^\varepsilon_4\|^2_{L^2(\Omega^\varepsilon_4)}.
\]
Proof: The proof of (41) is a simple application of Theorem 3.3 from [22]. For each $\varepsilon > 0$, we can calculate the best trace constant $C(\varepsilon)$ exactly for $\Omega_\varepsilon$ (see similar case in Example 4.3 of [22]). Then we bound the trace constants $C(\varepsilon)$ by a general constant $C > 0$.

The proof of (42) is application of Proposition 2.1.1 from [28].

Remark 1 The inequalities similar to (41) - (44) are not valid in the case of $v_m^\varepsilon$. Since in the case of $\Omega_\varepsilon$ or $\Omega_\varepsilon^\alpha$ we have nested set i.e for every $\varepsilon_1 \leq \varepsilon_2$ we have $\Omega_\varepsilon^{\varepsilon_2} \subset \Omega_\varepsilon^{\varepsilon_1}$ and $\Omega_\varepsilon^{\varepsilon_2} \subset \Omega_\varepsilon^{\varepsilon_1}$. But in the case of $v_m^\varepsilon$, for $\varepsilon_1 \neq \varepsilon_2$ we can not say either $\Omega_\varepsilon^{\varepsilon_1,M} \subset \Omega_\varepsilon^{\varepsilon_2,M}$ or $\Omega_\varepsilon^{\varepsilon_1,M} \subset \Omega_\varepsilon^{\varepsilon_2,M}$.

Lemma 2 There exists $C > 0$ independent of $\varepsilon$, such that

$$\|v_1^\varepsilon\|^2_{L^2(\Omega_\varepsilon^{\varepsilon_1})} \leq C \|\nabla v_1^\varepsilon\|^2_{L^2(\Omega_\varepsilon^{\varepsilon_1})},$$

(45)

or $\|v_1^\varepsilon\|^2_{L^2(\Omega_\varepsilon^{\varepsilon_1})} \leq C \|\nabla v_1^\varepsilon\|^2_{L^2(\Omega_\varepsilon^{\varepsilon_1})}$.

(46)

Proof: Proof follows same lines of proof of Lemma [7].

Next we state Lemma 3.1 from [7].

Lemma 3 For all $v_m^\varepsilon \in H^1(\Omega_\varepsilon^{\varepsilon_1,M})$, there exist $C > 0$ satisfying the following inequality

$$\|v_m^\varepsilon\|_{L^2(\Omega_\varepsilon^{\varepsilon_1})} \leq C \left(\varepsilon^{\frac{1}{2}} \|v_m^\varepsilon\|_{L^2(\Omega_\varepsilon^{\varepsilon_1,M})} + \varepsilon^{\frac{1}{2}} \|\nabla v_m^\varepsilon\|_{L^2(\Omega_\varepsilon^{\varepsilon_1,M})}\right)$$

(47)

Proof: For proof we refer Lemma 3.1 of [7].

Theorem 2 The weak solution $v^\varepsilon$ to the problem (39) satisfies the following energy estimates with $C > 0$

$$\|v_1^\varepsilon\|^2_{L^2(\Omega_\varepsilon^{\varepsilon_1})} + \|v_1^\varepsilon\|^2_{L^2(\Omega_\varepsilon^{\varepsilon_1})} + \varepsilon^\alpha \|v_m^\varepsilon\|^2_{L^2(\Omega_\varepsilon^{\varepsilon_1,M})} \leq C,$$

(48)

$$\|\nabla v_1^\varepsilon\|^2_{L^2(\Omega_\varepsilon^{\varepsilon_1})} + \|\nabla v_1^\varepsilon\|^2_{L^2(\Omega_\varepsilon^{\varepsilon_1})} + \varepsilon^\beta \|\nabla v_m^\varepsilon\|^2_{L^2(\Omega_\varepsilon^{\varepsilon_1,M})} \leq C,$$

(49)

$$\|P_\delta(v_1^\varepsilon - u_b)^2\|^2_{L^2(\Omega_\varepsilon^{\varepsilon_1})} + \|P_\delta(v_m^\varepsilon - u_b)^2\|^2_{L^2(\Omega_\varepsilon^{\varepsilon_1,M})} \leq C,$$

(50)

$$\|\partial_t v_1^\varepsilon\|^2_{L^2(\Omega_\varepsilon^{\varepsilon_1})} + \|\partial_t v_1^\varepsilon\|^2_{L^2(\Omega_\varepsilon^{\varepsilon_1})} + \varepsilon^\alpha \|\partial_t v_m^\varepsilon\|^2_{L^2(\Omega_\varepsilon^{\varepsilon_1,M})} \leq C.$$

(51)

Proof: We prove the estimate (48) by choosing the test function $\phi = (v_1^\varepsilon, v_m^\varepsilon, v_r^\varepsilon)$ in the weak formulation (39), we get

$$\int_{\Omega_\varepsilon^{\varepsilon_1}} \partial_t v_1^\varepsilon v_1^\varepsilon dx + \int_{\Omega_\varepsilon^{\varepsilon_1}} \partial_t v_1^\varepsilon v_1^\varepsilon dx + \varepsilon^\alpha \int_{\Omega_\varepsilon^{\varepsilon_1}} \partial_t v_m^\varepsilon \phi_2 dx +$$

$$\int_{\Omega_\varepsilon^{\varepsilon_1}} D_L \nabla v_1^\varepsilon \nabla v_1^\varepsilon dx + \int_{\Omega_\varepsilon^{\varepsilon_1}} D_R \nabla v_1^\varepsilon \nabla v_1^\varepsilon dx + \varepsilon^\beta \int_{\Omega_\varepsilon^{\varepsilon_1}} D_m \nabla v_m^\varepsilon \nabla v_m^\varepsilon dx$$

$$- \int_{\Omega_\varepsilon^{\varepsilon_1}} B_L P_\delta(v_1^\varepsilon - u_b) \nabla v_1^\varepsilon dx - \int_{\Omega_\varepsilon^{\varepsilon_1}} B_R P_\delta(v_r^\varepsilon - u_b) \nabla v_r^\varepsilon dx$$

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where the constant $C$ in (53) depends on $\|B_L\|_{L^\infty(\Omega_2^c,\mathbb{R}^2)}$ and $\|P_\delta(u)\|_{L^\infty(\mathbb{R})}$, similarly $C$ in (54) depend on $\|B_L\|_{L^\infty(\Omega_2^c,\mathbb{R}^2)}$ and $\|P_\delta(u)\|_{L^\infty(\mathbb{R})}$, $C$ in (55) depends on $\|B_L\|_{L^\infty(\Omega_2^c,\mathbb{R}^2)}$ and $\|P_\delta(u)\|_{L^\infty(\mathbb{R})}$.

Now using (18) together with the estimates (53), (54), (55), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|v_0^\varepsilon\|_{L^2(\Omega_2^c)}^2 + \frac{d}{dt} \|v_0^\varepsilon\|_{L^2(\Omega_2^r)}^2 + \varepsilon \frac{d}{dt} \|v_m^\varepsilon\|_{L^2(\Omega_2^m)}^2 \geq \frac{\theta}{2} \int_{\Omega_2^c} |\nabla v_0^\varepsilon|^2 dx + \theta \int_{\Omega_2^c} |\nabla v_0^\varepsilon|^2 dx + \varepsilon \frac{\theta}{2} \int_{\Omega_2^m} |\nabla v_m^\varepsilon|^2 dx + \frac{\theta}{2} \int_{\Omega_2^c} |\nabla v_0^\varepsilon|^2 dx + \varepsilon \frac{\theta}{2} \int_{\Omega_2^m} |\nabla v_m^\varepsilon|^2 dx
\]

Now, by applying Young’s Inequality and using the assumption [A6] for any $\zeta > 0$ we have

\[
\int_{\Omega_2^c} |\nabla v_0^\varepsilon|^2 dx \leq \zeta \int_{\Omega_2^c} |\nabla v_0^\varepsilon|^2 dx + C(\zeta),
\]

\[
\int_{\Omega_2^r} |\nabla v_0^\varepsilon|^2 dx \leq \zeta \int_{\Omega_2^r} |\nabla v_0^\varepsilon|^2 dx + C(\zeta),
\]
Using Cauchy Schwarz’s inequality, Lemma 3, Young’s inequality and (A6), we get
\[ \varepsilon^\gamma \int_{\Omega_M^\epsilon} |\nabla v_m^\epsilon| dx \leq \varepsilon^\gamma \zeta \int_{\Omega_M^\epsilon} |\nabla v_m^\epsilon|^2 dx + \varepsilon^\gamma C(\zeta) |\Omega_M^\epsilon|^{\frac{1}{2}} \]
\[ \leq \varepsilon^\gamma \zeta \int_{\Omega_M^\epsilon} |\nabla v_m^\epsilon|^2 dx + \varepsilon^{\gamma + \frac{1}{2}} \sqrt{2} h C(\zeta) \]
\[ \leq \varepsilon^\gamma \zeta \int_{\Omega_M^\epsilon} |\nabla v_m^\epsilon|^2 dx + C(\zeta), \]  
(59)

\[ \int_{\Omega_E^\epsilon} |f_b v_f^\epsilon| dx \leq \frac{1}{2} \int_{\Omega_E^\epsilon} |f_b|^2 dx + \frac{1}{2} \int_{\Omega_E^\epsilon} |v_f^\epsilon|^2 dx, \]  
(60)

\[ \int_{\Omega_R^\epsilon} |f_b v_r^\epsilon| dx \leq \frac{1}{2} \int_{\Omega_R^\epsilon} |f_b|^2 dx + \frac{1}{2} \int_{\Omega_R^\epsilon} |v_r^\epsilon|^2 dx, \]  
(61)

\[ \varepsilon^\alpha \int_{\Omega_M^\epsilon} |f_{am} v_m^\epsilon| dx \leq \frac{1}{2} \varepsilon^\alpha \int_{\Omega_M^\epsilon} |f_{am}|^2 dx + \frac{1}{2} \varepsilon^\alpha \int_{\Omega_M^\epsilon} |v_m^\epsilon|^2 dx, \]  
(62)

\[ \varepsilon^\beta \int_{\Omega_M^\epsilon} |f_{bm} v_m^\epsilon| dx \leq \frac{1}{2} \varepsilon^\beta \int_{\Omega_M^\epsilon} |f_{bm}|^2 dx + \frac{1}{2} \varepsilon^\beta \int_{\Omega_M^\epsilon} |v_m^\epsilon|^2 dx, \]  
(63)

Using Cauchy Schwarz’s inequality, Lemma 3, Young’s inequality, we have
\[ \int_{\Gamma^\epsilon_h \cap \partial \Omega_E^\epsilon} |g_b||v_f^\epsilon| d\sigma \leq \|g_b\|_{L^2(\Gamma^\epsilon_h \cap \partial \Omega_E^\epsilon)} \|v_f^\epsilon\|_{L^2(\Gamma^\epsilon_h \cap \partial \Omega_E^\epsilon)} \]
\[ \leq C \|g_b\|_{L^2(\Gamma^\epsilon_h \cap \partial \Omega_E^\epsilon)} \|
abla v_f^\epsilon\|_{L^2(\Omega_E^\epsilon)} \]
\[ \leq C(\zeta) \|g_b\|^2_{L^2(\Gamma^\epsilon_h \cap \partial \Omega_E^\epsilon)} + C\zeta \|
abla v_f^\epsilon\|^2_{L^2(\Omega_E^\epsilon)}, \]  
(64)

By similar argument as for (64), we get
\[ \int_{\Gamma^\epsilon_h \cap \partial \Omega_R^\epsilon} |g_b||v_r^\epsilon| d\sigma \leq C(\zeta) \|g_b\|^2_{L^2(\Gamma^\epsilon_h \cap \partial \Omega_R^\epsilon)} + C\zeta \|
abla v_r^\epsilon\|^2_{L^2(\Omega_R^\epsilon)}. \]  
(65)

Using Cauchy Schwarz’s inequality, Lemma 3, Young’s inequality and (A6), we get
\[ \varepsilon^\beta \int_{\Gamma^\epsilon_0} |g_{u_0}||v_m^\epsilon| d\sigma \leq \varepsilon^\beta \|g_{u_0}\|_{L^2(\Gamma^\epsilon_0)} \|v_m^\epsilon\|_{L^2(\Gamma^\epsilon_0)} \]
\[ \leq C \varepsilon^{\beta - \frac{1}{2}} \|g_{u_0}\|_{L^2(\Gamma^\epsilon_0)} \|v_m^\epsilon\|_{L^2(\Omega_M^\epsilon)} \]
\[ + C \varepsilon^{\beta + \frac{1}{2}} \|g_{u_0}\|_{L^2(\Gamma^\epsilon_0)} \|
abla v_m^\epsilon\|_{L^2(\Omega_M^\epsilon)} \]
\[ \leq C \varepsilon^{\beta - \frac{1}{2}} \|g_{u_0}\|^2_{L^2(\Gamma^\epsilon_0)} \]
\[ + \varepsilon^{\beta - \frac{1}{2}} C \|v_m^\epsilon\|^2_{L^2(\Omega_M^\epsilon)} + C(\zeta) \varepsilon^{\beta + \frac{1}{2}} \|g_{u_0}\|^2_{L^2(\Gamma^\epsilon_0)} \]
\[ + C(\zeta) \varepsilon^{\beta + \frac{1}{2}} \|g_{u_0}\|^2_{L^2(\Omega_M^\epsilon)} + \zeta \varepsilon^\beta \|
abla v_m^\epsilon\|^2_{L^2(\Omega_M^\epsilon)} \]
\[ \leq C \varepsilon^{\beta - \frac{1}{2}} \|g_{u_0}\|^2_{L^2(\Gamma^\epsilon_0)} + \varepsilon^\alpha C \|v_m^\epsilon\|^2_{L^2(\Omega_M^\epsilon)} \]
\[ + C(\zeta) \varepsilon^{\beta + \frac{1}{2}} \|g_{u_0}\|^2_{L^2(\Omega_M^\epsilon)} + \zeta \varepsilon^\beta \|
abla v_m^\epsilon\|^2_{L^2(\Omega_M^\epsilon)}, \]  
(66)
\[
\varepsilon^k \int_{\Gamma^0} |g_0^\varepsilon| v_m^\varepsilon \, d\sigma \leq \varepsilon^k \frac{1}{2} C \|g_0^\varepsilon\|^2_{L^2(\Gamma^0)} + \varepsilon^\alpha C \|v_m^\varepsilon\|^2_{L^2(\Omega^\varepsilon_M)} \\
+ C(\zeta) \varepsilon^{k+\frac{1}{2}} \|g_0^\varepsilon\|^2_{L^2(\Gamma_0)} + \zeta \varepsilon^\beta \|\nabla v_m^\varepsilon\|^2_{L^2(\Omega^\varepsilon_M)}
\]

(67)

Using (19), (A6) Cauchy Schwarz’s inequality, the trace inequality and Young’s Inequality, we have

\[
\int_{\mathcal{B}^\varepsilon_L} |(D_L - \varepsilon^\beta D_M) \nabla u_b||v_b^\varepsilon| \, d\sigma \leq C \int_{\mathcal{B}^\varepsilon_L} |\nabla u_b| |v_b^\varepsilon| \, d\sigma \leq C \|\nabla u_b\|_{L^2(\mathcal{B}_L)} \|v_b^\varepsilon\|_{L^2(\mathcal{B}_L)} \\
\leq C \|\nabla u_b\|_{L^2(\mathcal{B}_L)} \|v_b^\varepsilon\|_{L^2(\Omega^\varepsilon_L)}
\]

(68)

similarly,

\[
\int_{\mathcal{B}^\varepsilon_R} |(D_R - \varepsilon^\beta D_M) \nabla u_b||v_r^\varepsilon| \, d\sigma \leq C(\zeta) + \zeta \|\nabla v_r^\varepsilon\|^2_{L^2(\Omega^\varepsilon_R)}.
\]

(69)

Now, we use assumption (A6), (57)-(69) in (56), and for \( \zeta \) small enough, we control the gradient terms appearing on the right-hand side using the analogous nonnegative terms on the left-hand side, thus we get

\[
\frac{1}{2} \frac{d}{dt} \|v_f^\varepsilon\|^2_{L^2(\Omega^\varepsilon_L)} + \frac{1}{2} \frac{d}{dt} \|v_r^\varepsilon\|^2_{L^2(\Omega^\varepsilon_R)} + \varepsilon^\alpha \frac{1}{2} \frac{d}{dt} \|v_m^\varepsilon\|^2_{L^2(\Omega^\varepsilon_M)} \\
\leq C \left( 1 + \|f_b\|_{L^2(\Omega^\varepsilon_L)} + \|f_b\|_{L^2(\Omega^\varepsilon_R)} + \varepsilon^\alpha \|f_{a_m}\|_{L^2(\Omega^\varepsilon_M)} + \varepsilon^\beta \|f_{b_m}\|_{L^2(\Omega^\varepsilon_M)} \\
+ \|g_b\|_{L^2(\Gamma_{\varepsilon_L} \cap \partial \Omega^\varepsilon_L)} + \|g_b\|_{L^2(\Gamma_{\varepsilon_R} \cap \partial \Omega^\varepsilon_R)} + \varepsilon^\beta \frac{1}{2} \|g_{a_0}\|_{L^2(\Gamma_{\varepsilon_R})} + \varepsilon^\beta \frac{1}{2} C \|g_{b_0}\|_{L^2(\Gamma_{\varepsilon_R})} \\
+ \|v_f^\varepsilon\|_{L^2(\Omega^\varepsilon_L)} + \|v_r^\varepsilon\|_{L^2(\Omega^\varepsilon_R)} + \varepsilon^\alpha \|v_m^\varepsilon\|_{L^2(\Omega^\varepsilon_M)} \right).
\]

(70)

By using (A3), (A4) in (70) and using Gronwall’s inequality with (A5), we obtain

\[
\|v_f^\varepsilon\|^2_{L^2(\Omega^\varepsilon_L)} + \|v_r^\varepsilon\|^2_{L^2(\Omega^\varepsilon_R)} + \varepsilon^\alpha \|v_m^\varepsilon\|^2_{L^2(\Omega^\varepsilon_M)} \leq C.
\]

(71)

Hence we have finished the proof of the estimate (48).

From (50)-(69) and (A6) we have

\[
\frac{1}{2} \frac{d}{dt} \|v_f^\varepsilon\|^2_{L^2(\Omega^\varepsilon_L)} + \frac{1}{2} \frac{d}{dt} \|v_r^\varepsilon\|^2_{L^2(\Omega^\varepsilon_R)} + \varepsilon^\alpha \frac{1}{2} \frac{d}{dt} \|v_m^\varepsilon\|^2_{L^2(\Omega^\varepsilon_M)} \\
+ \theta \int_{\Omega^\varepsilon_L} |\nabla v_f^\varepsilon|^2 \, dx + \theta \int_{\Omega^\varepsilon_R} |\nabla v_r^\varepsilon|^2 \, dx + \varepsilon^\beta \theta \int_{\Omega^\varepsilon_M} |\nabla v_m^\varepsilon|^2 \, dx \\
\leq C(C(\zeta) + \|f_b\|_{L^2(\Omega^\varepsilon_L)} + \|f_b\|_{L^2(\Omega^\varepsilon_R)} + \varepsilon^\alpha \|f_{a_m}\|_{L^2(\Omega^\varepsilon_M)} + \varepsilon^\beta \|f_{b_m}\|_{L^2(\Omega^\varepsilon_M)} \\
+ C(\zeta) \|g_b\|_{L^2(\Gamma_{\varepsilon_L} \cap \partial \Omega^\varepsilon_L)} + C(\zeta) \|g_b\|_{L^2(\Gamma_{\varepsilon_R} \cap \partial \Omega^\varepsilon_R)} + \varepsilon^\beta \frac{1}{2} \|g_{a_0}\|_{L^2(\Gamma_{\varepsilon_R})} + \varepsilon^\beta \frac{1}{2} \|g_{b_0}\|_{L^2(\Gamma_{\varepsilon_R})} \\
+ (C(\zeta) \varepsilon^{k+\frac{1}{2}} \|g_{a_0}\|_{L^2(\Gamma_0)} + \varepsilon^{k+\frac{1}{2}} \|g_{b_0}\|_{L^2(\Gamma_0)} + C(\zeta) \varepsilon^{k+\frac{1}{2}} \|g_{a_0}\|_{L^2(\Gamma_0)} + C(\zeta) \varepsilon^{k+\frac{1}{2}} \|g_{b_0}\|_{L^2(\Gamma_0)})
\]

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Remark 2 from (76) we can see that we can relax the assumption on \( \gamma \) stated in (A6) by \( \gamma \in (-1, \infty) \) for the estimate (50).

To prove the estimate (51), we approximate the weak solution of (39) using Galerkin scheme and take \((\partial_t v^e_t, \partial_t v^e_m, \partial_t v^e_{\gamma})\) as test function.

Let \( \{w^e_{l,k}\}_{k=1}^\infty \) be an orthogonal basis of \( H^1(\Omega^e_L; \Gamma_L) \) and orthonormal basis of \( L^2(\Omega^e_L) \). Similarly, we define \( \{w^e_{r,k}\}_{k=1}^\infty \) and \( \{w^e_{m,k}\}_{k=1}^\infty \) as basis of \( H^1(\Omega^e_R; \Gamma_R) \) and \( H^1(\Omega^e_M) \) respectively. Then the Galerkin scheme problem will be to find \( d^e_{l,1}, d^e_{l,2}, \ldots, d^e_{l,n}, d^e_{m,1}, d^e_{m,2} \)
\[ v_{i,n}^\varepsilon = \sum_{k=0}^{n} d_{i,k}^\varepsilon w_{i,k}^\varepsilon, \]
\[ v_{m,n}^\varepsilon = \sum_{k=0}^{n} d_{m,k}^\varepsilon w_{m,k}^\varepsilon, \]
\[ v_{r,n}^\varepsilon = \sum_{k=0}^{n} d_{r,k}^\varepsilon w_{r,k}^\varepsilon, \]

satisfying the weak formulation corresponding to (39) with the initial condition (40). The existence and uniqueness of such \( d_{i,1}^\varepsilon, d_{i,2}^\varepsilon, \ldots, d_{i,n}^\varepsilon, d_{m,1}^\varepsilon, d_{m,2}^\varepsilon, \ldots, d_{m,n}^\varepsilon, d_{r,1}^\varepsilon, d_{r,2}^\varepsilon, \ldots, d_{r,n}^\varepsilon \) follows by direct application of Picard-Lindelöf Theorem (see [13]). Moreover \( d_{i,1}^\varepsilon, d_{i,2}^\varepsilon, \ldots, d_{i,n}^\varepsilon, d_{m,1}^\varepsilon, d_{m,2}^\varepsilon, \ldots, d_{m,n}^\varepsilon, d_{r,1}^\varepsilon, d_{r,2}^\varepsilon, \ldots, d_{r,n}^\varepsilon \) lie in \( C^1(0,T) \cap C[0,T] \) as a consequence of Picard-Lindelöf Theorem. So, we have

\[
\int_{\Omega_L^\varepsilon} \partial_t v_{i,n}^\varepsilon w_{i,k}^\varepsilon dx + \int_{\Omega_R^\varepsilon} \partial_t v_{r,n}^\varepsilon w_{r,k}^\varepsilon dx + \varepsilon^\alpha \int_{\Omega_M^\varepsilon} \partial_t v_{m,n}^\varepsilon w_{m,k}^\varepsilon dx + \varepsilon^\beta \int_{\Omega_M^\varepsilon} D_m^\varepsilon \nabla v_{m,n}^\varepsilon \nabla w_{m,k}^\varepsilon dx
\]
\[
- \int_{\Omega_L^\varepsilon} B_L P_\delta (v_{i,n}^\varepsilon - u_b) \nabla w_{i,k}^\varepsilon dx - \int_{\Omega_R^\varepsilon} B_R P_\delta (v_{r,n}^\varepsilon - u_b) \nabla w_{r,k}^\varepsilon dx
\]
\[
- \varepsilon^\gamma \int_{\Omega_M^\varepsilon} B_M^\varepsilon P_\delta (v_{m,n}^\varepsilon - u_b) \nabla w_{m,k}^\varepsilon dx
\]
\[
= \int_{\Omega_L^\varepsilon} f_{i,k}^\varepsilon w_{i,k}^\varepsilon dx + \int_{\Omega_R^\varepsilon} f_{r,k}^\varepsilon w_{r,k}^\varepsilon dx + \varepsilon^\alpha \int_{\Omega_M^\varepsilon} f_{m,k}^\varepsilon w_{m,k}^\varepsilon dx + \varepsilon^\beta \int_{\Omega_M^\varepsilon} f_{m,k}^\varepsilon w_{m,k}^\varepsilon dx
\]
\[
- \int_{\Gamma_{L,k}^\varepsilon} g_{b,k}^\varepsilon w_{i,k}^\varepsilon d\sigma - \int_{\Gamma_{R,k}^\varepsilon} g_{b,k}^\varepsilon w_{r,k}^\varepsilon d\sigma - \varepsilon^\alpha \int_{\Gamma_0^\varepsilon} g_{b,k}^\varepsilon w_{m,k}^\varepsilon d\sigma - \varepsilon^\beta \int_{\Gamma_0^\varepsilon} g_{b,k}^\varepsilon w_{m,k}^\varepsilon d\sigma
\]
\[
+ \int_{\Gamma_0^\varepsilon} (D_L - \varepsilon^\beta D_M) \nabla u_b \cdot n_L w_{i,k}^\varepsilon d\sigma + \int_{\Gamma_0^\varepsilon} (D_R - \varepsilon^\beta D_M) \nabla u_b \cdot n_L w_{r,k}^\varepsilon d\sigma, \quad (78)
\]

for all \( k \in \{1, 2, \ldots, n\}. \)

Now, multiplying (78) with \( 1_{\Omega_L^\varepsilon} \partial_t d_{i,k}^\varepsilon, 1_{\Omega_R^\varepsilon} \partial_t d_{r,k}^\varepsilon, 1_{\Omega_M^\varepsilon} \partial_t d_{m,k}^\varepsilon \) for \( k \in \{1, 2, \ldots, n\} \) and summing over \( k = 1, 2, \ldots, n \), we get

\[
\int_{\Omega_L^\varepsilon} \partial_t v_{i,n}^\varepsilon \partial_t v_{i,n}^\varepsilon dx + \int_{\Omega_R^\varepsilon} \partial_t v_{r,n}^\varepsilon \partial_t v_{r,n}^\varepsilon dx + \varepsilon^\alpha \int_{\Omega_M^\varepsilon} \partial_t v_{m,n}^\varepsilon \partial_t v_{m,n}^\varepsilon dx + \varepsilon^\beta \int_{\Omega_M^\varepsilon} D_m^\varepsilon \nabla v_{m,n}^\varepsilon \nabla v_{m,n}^\varepsilon dx
\]
\[
- \int_{\Omega_L^\varepsilon} B_L P_\delta (v_{i,n}^\varepsilon - u_b) \partial_t v_{i,n}^\varepsilon dx - \int_{\Omega_R^\varepsilon} B_R P_\delta (v_{r,n}^\varepsilon - u_b) \partial_t v_{r,n}^\varepsilon dx
\]

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To derive the desired estimates, we use the following identities,

\[
- \varepsilon \gamma \int_{\Omega_M^b} B_M^b P_\delta (v_{m,n}^\varepsilon - u_b) \nabla \partial_t v_{m,n}^\varepsilon \, dx \\
= \int_{\Omega_B^b} f_b \partial_t v_{l,n}^\varepsilon \, dx + \int_{\Omega_R} f_b \partial_t v_{r,n}^\varepsilon \, dx + \varepsilon \alpha \int_{\Omega_M} f_m \partial_t v_{m,n}^\varepsilon \, dx \\
+ \varepsilon \beta \int_{\Omega_M} f_m \partial_t v_{m,n}^\varepsilon \, dx \\
- \int_{\Gamma_h^b \cap \partial \Omega_L^b} g_b \partial_t v_{l,n}^\varepsilon \, d\sigma - \int_{\Gamma_R^b \cap \partial \Omega_R} g_b \partial_t v_{r,n}^\varepsilon \, d\sigma - \varepsilon \varepsilon \int_{\Gamma_b^0} \delta v_{m,n}^\varepsilon \, d\sigma \\
- \varepsilon \beta \int_{\Gamma_b} \delta v_{m,n}^\varepsilon \, d\sigma \\
+ \int_{\Omega_L^b} (D_L - \varepsilon \beta D_M) \nabla u_b \cdot n_t \partial_t v_{l,n}^\varepsilon \, d\sigma + \int_{\Omega_R} (D_R - \varepsilon \beta D_M) \nabla u_b \cdot n_r \partial_t v_{r,n}^\varepsilon \, d\sigma. \quad (79)
\]

To derive the desired estimates, we use the following identities,

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega_L^b} D_L \nabla v_{l,n}^\varepsilon \nabla v_{l,n}^\varepsilon \, dx = \int_{\Omega_L^b} D_L \nabla v_{l,n}^\varepsilon \nabla \partial_t v_{l,n}^\varepsilon \, dx, \quad (80)
\]

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega_R} D_R \nabla v_{r,n}^\varepsilon \nabla v_{r,n}^\varepsilon \, dx = \int_{\Omega_R} D_R \nabla v_{r,n}^\varepsilon \nabla \partial_t v_{r,n}^\varepsilon \, dx, \quad (81)
\]

\[
\frac{\varepsilon \beta}{2} \frac{d}{dt} \int_{\Omega_M} D_M \nabla v_{m,n}^\varepsilon \nabla v_{m,n}^\varepsilon \, dx = \varepsilon \beta \int_{\Omega_M} D_M \nabla v_{m,n}^\varepsilon \nabla \partial_t v_{m,n}^\varepsilon \, dx, \quad (82)
\]

\[
\int_{\Omega_L^b} B_L P_\delta (v_{l,n}^\varepsilon - u_b) \nabla \partial_t v_{l,n}^\varepsilon \, dx = \frac{d}{dt} \int_{\Omega_L^b} B_L P_\delta (v_{l,n}^\varepsilon - u_b) \nabla v_{l,n}^\varepsilon \, dx \\
- \int_{\Omega_L^b} B_L P_\delta (v_{l,n}^\varepsilon - u_b) \partial_t (v_{l,n}^\varepsilon - u_b) \nabla v_{l,n}^\varepsilon \, dx, \quad (83)
\]

\[
\int_{\Omega_R} B_R P_\delta (v_{r,n}^\varepsilon - u_b) \nabla \partial_t v_{r,n}^\varepsilon \, dx = \frac{d}{dt} \int_{\Omega_R} B_R P_\delta (v_{r,n}^\varepsilon - u_b) \nabla v_{r,n}^\varepsilon \, dx \\
- \int_{\Omega_R} B_R P_\delta (v_{r,n}^\varepsilon - u_b) \partial_t (v_{r,n}^\varepsilon - u_b) \nabla v_{r,n}^\varepsilon \, dx, \quad (84)
\]

\[
\varepsilon \gamma \int_{\Omega_M} B_M P_\delta (v_{m,n}^\varepsilon - u_b) \nabla \partial_t v_{m,n}^\varepsilon \, dx = \varepsilon \gamma \frac{d}{dt} \int_{\Omega_M} B_M P_\delta (v_{m,n}^\varepsilon - u_b) \nabla v_{m,n}^\varepsilon \, dx \\
- \varepsilon \gamma \int_{\Omega_M} B_M P_\delta (v_{m,n}^\varepsilon - u_b) \partial_t (v_{m,n}^\varepsilon - u_b) \nabla v_{m,n}^\varepsilon \, dx, \quad (85)
\]

\[
\int_{\Gamma_h^b \cap \partial \Omega_L^b} g_b \partial_t v_{l,n}^\varepsilon \, d\sigma = \frac{d}{dt} \int_{\Gamma_h^b \cap \partial \Omega_L^b} g_b v_{l,n}^\varepsilon \, d\sigma - \int_{\Gamma_h^b \cap \partial \Omega_L^b} \partial_t g_b v_{l,n}^\varepsilon \, d\sigma \quad (86)
\]
\[ \int_{\Gamma_h} g_b \partial_t v_{t,n}^\varepsilon d\sigma = \frac{d}{dt} \int_{\Gamma_h} g_b \partial_t v_{t,n}^\varepsilon d\sigma - \int_{\Gamma_h} \partial_t g_b \partial_t v_{t,n}^\varepsilon d\sigma \] (87)

\[ \varepsilon \int_{\Gamma_0} g_0 \partial_t v_{m,n}^\varepsilon d\sigma = \frac{d}{dt} \int_{\Gamma_0} g_0 \partial_t v_{m,n}^\varepsilon d\sigma - \varepsilon \int_{\Gamma_0} \partial_t g_0 \partial_t v_{m,n}^\varepsilon d\sigma \] (88)

\[ \varepsilon \beta \int_{\Gamma_0} g_b \partial_t v_{m,n}^\varepsilon d\sigma = \frac{d}{dt} \int_{\Gamma_0} g_b \partial_t v_{m,n}^\varepsilon d\sigma - \varepsilon \beta \int_{\Gamma_0} \partial_t g_b \partial_t v_{m,n}^\varepsilon d\sigma. \] (89)

We obtain

\[
\| \partial_t v_{t,n}^\varepsilon \|^2_{L^2(\Omega^\varepsilon)} + \| \partial_t v_{r,n}^\varepsilon \|^2_{L^2(\Omega^\varepsilon)} + \varepsilon^\alpha \| \partial_t v_{m,n}^\varepsilon \|^2_{L^2(\Omega^\varepsilon)} + \\
\frac{1}{2} \frac{d}{dt} \int_{\Omega^\varepsilon} D_L \nabla v_{t,n}^\varepsilon \nabla v_{t,n}^\varepsilon dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega^\varepsilon} D_R \nabla v_{r,n}^\varepsilon \nabla v_{r,n}^\varepsilon dx \\
+ \frac{\varepsilon \beta}{2} \frac{d}{dt} \int_{\Omega^\varepsilon} D_M \nabla v_{m,n}^\varepsilon \nabla v_{m,n}^\varepsilon dx \\
= \frac{d}{dt} \int_{\Omega^\varepsilon} B_L P_\delta(v_{t,n}^\varepsilon - u_b) \nabla v_{t,n}^\varepsilon dx + \frac{d}{dt} \int_{\Omega^\varepsilon} B_R P_\delta(v_{r,n}^\varepsilon - u_b) \nabla v_{r,n}^\varepsilon dx \\
+ \varepsilon \gamma \frac{d}{dt} \int_{\Omega^\varepsilon} B_M P_\delta(v_{m,n}^\varepsilon - u_b) \nabla v_{m,n}^\varepsilon dx \\
- \int_{\Omega^\varepsilon} B_L P_\delta(v_{t,n}^\varepsilon - u_b) \partial_t(v_{t,n}^\varepsilon) \nabla v_{t,n}^\varepsilon dx \\
- \int_{\Omega^\varepsilon} B_R P_\delta(v_{r,n}^\varepsilon - u_b) \partial_t(v_{r,n}^\varepsilon) \nabla v_{r,n}^\varepsilon dx \\
- \varepsilon \gamma \int_{\Omega^\varepsilon} B_M P_\delta(v_{m,n}^\varepsilon - u_b) \partial_t(v_{m,n}^\varepsilon) \nabla v_{m,n}^\varepsilon dx \\
+ \int_{\Omega^\varepsilon} f_b \partial_t v_{t,n}^\varepsilon dx + \int_{\Omega^\varepsilon} f_R \partial_t v_{r,n}^\varepsilon dx + \varepsilon \alpha \int_{\Omega^\varepsilon} f_M \partial_t v_{m,n}^\varepsilon dx \\
+ \varepsilon \beta \int_{\Omega^\varepsilon} f_{b,n} \partial_t v_{m,n}^\varepsilon dx - \frac{d}{dt} \int_{\Gamma^\varepsilon_h \cap \partial \Omega^\varepsilon} g_b \partial_t v_{t,n}^\varepsilon d\sigma - \frac{d}{dt} \int_{\Gamma^\varepsilon_h \cap \partial \Omega^\varepsilon} g_b \partial_t v_{r,n}^\varepsilon d\sigma \\
- \varepsilon \frac{d}{dt} \int_{\Gamma^\varepsilon_h \cap \partial \Omega^\varepsilon} g_0 \partial_t v_{m,n}^\varepsilon d\sigma - \frac{d}{dt} \int_{\Gamma^\varepsilon_h \cap \partial \Omega^\varepsilon} \partial_t g_b \partial_t v_{m,n}^\varepsilon d\sigma \\
+ \int_{\Gamma^\varepsilon_h \cap \partial \Omega^\varepsilon} \partial_t g_b \partial_t v_{m,n}^\varepsilon d\sigma + \varepsilon \beta \int_{\Gamma^\varepsilon_h \cap \partial \Omega^\varepsilon} \partial_t g_b \partial_t v_{m,n}^\varepsilon d\sigma + \varepsilon \beta \int_{\Gamma^\varepsilon_h \cap \partial \Omega^\varepsilon} \partial_t g_b \partial_t v_{m,n}^\varepsilon d\sigma \\
+ \int_{\Gamma^\varepsilon_h \cap \partial \Omega^\varepsilon} (D_L - \varepsilon \beta D_M) \nabla u_b \cdot n, \partial_t v_{t,n}^\varepsilon d\sigma + \int_{\Gamma^\varepsilon_h \cap \partial \Omega^\varepsilon} (D_R - \varepsilon \beta D_M) \nabla u_b \cdot n, \partial_t v_{r,n}^\varepsilon d\sigma + \int_{\Gamma^\varepsilon_h \cap \partial \Omega^\varepsilon} (D_R - \varepsilon \beta D_M) \nabla u_b \cdot n, \partial_t v_{r,n}^\varepsilon d\sigma. \] (90)

Now, integrating (90) from 0 to t, we get

\[
\| \partial_t v_{t,n}^\varepsilon \|^2_{L^2(0,t;L^2(\Omega^\varepsilon))} + \| \partial_t v_{r,n}^\varepsilon \|^2_{L^2(0,t;L^2(\Omega^\varepsilon))} + \varepsilon^\alpha \| \partial_t v_{m,n}^\varepsilon \|^2_{L^2(0,t;L^2(\Omega^\varepsilon))}
\]
\begin{align*}
&+ \frac{1}{2} \int_{\Omega_L} D_L \nabla v_{l,n} \cdot \nabla v_{l,n} \, dx + \frac{1}{2} \int_{\Omega_R} D_R \nabla v_{r,n} \cdot \nabla v_{r,n} \, dx \\
&\quad + \frac{\varepsilon \beta}{2} \int_{\Omega_M} D_M^{\varepsilon} \nabla v_{m,n} \cdot \nabla v_{m,n} \, dx \\
&= \int_{\Omega_L} B_L P_{\delta}(v_{l,n} - u_b) \nabla v_{l,n} \, dx + \int_{\Omega_R} B_R P_{\delta}(v_{r,n} - u_b) \nabla v_{r,n} \, dx \\
&\quad + \varepsilon^\gamma \int_{\Omega_M} B_M P_{\delta}(v_{m,n} - u_b) \nabla v_{m,n} \, dx - \int_0^t \int_{\Omega_L} B_L P_{\delta}^\prime(v_{l,n} - u_b) \partial_t (v_{l,n} - u_b) \nabla v_{l,n} \, dx dt \\
&\quad - \int_0^t \int_{\Omega_R} B_R P_{\delta}^\prime(v_{r,n} - u_b) \partial_t (v_{r,n} - u_b) \nabla v_{r,n} \, dx dt \\
&\quad - \varepsilon^\gamma \int_0^t \int_{\Omega_M} B_M P_{\delta}^\prime(v_{m,n} - u_b) \partial_t (v_{m,n} - u_b) \nabla v_{m,n} \, dx dt \\
&+ \int_0^t \int_{\Omega_L} f_{bl} \partial_t v_{l,n} \, dx dt + \int_0^t \int_{\Omega_R} f_{br} \partial_t v_{r,n} \, dx dt + \varepsilon^{\alpha} \int_0^t \int_{\Omega_M} f_{bm,n} \partial_t v_{m,n} \, dx dt \\
&\quad + \varepsilon^\beta \int_0^t \int_{\Omega_M} f_{bm,n} \partial_t v_{m,n} \, dx dt \\
&- \int_{\Gamma_h \cap \partial \Omega_L} g_{b_l} v_{l,n} \, d\sigma - \int_{\Gamma_h \cap \partial \Omega_R} g_{b_r} v_{r,n} \, d\sigma - \varepsilon^\xi \int_{\Gamma_0} g_{0b}^e v_{m,n} \, d\sigma - \varepsilon^\beta \int_{\Gamma_0} g_{b_0}^e v_{m,n} \, d\sigma \\
&\quad + \int_0^t \int_{\Gamma_h \cap \partial \Omega_L} \partial_t g_{b_l} v_{l,n} \, d\sigma dt + \int_0^t \int_{\Gamma_h \cap \partial \Omega_R} \partial_t g_{b_r} v_{r,n} \, d\sigma dt \\
&\quad + \varepsilon^\xi \int_0^t \int_{\Gamma_0} \partial_t g_{0b}^e v_{m,n} \, d\sigma dt + \varepsilon^\beta \int_0^t \int_{\Gamma_0} \partial_t g_{b_0}^e v_{m,n} \, d\sigma dt \\
&+ \int_0^t \int_{\Omega_L} (D_L - \varepsilon^{\beta} D_M) \nabla u_b \cdot n_r \partial_t v_{r,n} \, d\sigma dt + \int_0^t \int_{\Omega_R} (D_R - \varepsilon^{\beta} D_M) \nabla u_b \cdot n_r \partial_t v_{r,n} \, d\sigma dt \\
&\quad + \frac{1}{2} \int_{\Omega_L} D_L \nabla v_{l,n} (0) \cdot \nabla v_{l,n} (0) \, dx + \frac{1}{2} \int_{\Omega_R} D_R \nabla v_{r,n} (0) \cdot \nabla v_{r,n} (0) \, dx \\
&\quad + \frac{\varepsilon^{\beta}}{2} \int_{\Omega_M} D_M^{\varepsilon} \nabla v_{m,n} (0) \cdot \nabla v_{m,n} (0) \, dx - \int_{\Omega_L} B_L P_{\delta}(v_{l,n} (0) - u_b (0)) \nabla v_{l,n} (0) \, dx \\
&\quad - \int_{\Omega_R} B_R P_{\delta}(v_{r,n} (0) - u_b (0)) \nabla v_{r,n} (0) \, dx - \varepsilon^\gamma \int_{\Omega_M} B_M P_{\delta}(v_{m,n} (0) - u_b (0)) \nabla v_{m,n} (0) \, dx \\
&\quad + \int_{\Gamma_h \cap \partial \Omega_L} g_{b_l} (0) v_{l,n} (0) \, d\sigma + \int_{\Gamma_h \cap \partial \Omega_R} g_{b_r} (0) v_{r,n} (0) \, d\sigma + \varepsilon^\xi \int_{\Gamma_0} g_{0b}^e (0) v_{m,n} (0) \, d\sigma + \varepsilon^\beta \int_{\Gamma_0} g_{b_0}^e (0) v_{m,n} (0) \, d\sigma. \tag{91}
\end{align*}
By (21), Cauchy-Schwarz’s inequality, (50) and Young’s inequality, we obtain

$$
\int_{\Omega_{\varepsilon}^c} |B_L P_\delta(v^\varepsilon_{l,n} - u_b) \nabla v^\varepsilon_{l,n}| \, dx \leq C \int_{\Omega_{\varepsilon}^c} |P_\delta(v^\varepsilon_{l,n} - u_b)| |\nabla v^\varepsilon_{l,n}| \, dx \\
\leq C \|P_\delta(v^\varepsilon_{l,n} - u_b)\|_{L^2(\Omega_{\varepsilon})} \|\nabla v^\varepsilon_{l,n}\|_{L^2(\Omega_{\varepsilon})} \\
\leq C \|P_\delta(v^\varepsilon_{l,n} - u_b)\|_{L^2(\Omega_{\varepsilon})}^2 + C \|\nabla v^\varepsilon_{l,n}\|_{L^2(\Omega_{\varepsilon})}^2 \\
\leq C + C \|\nabla v^\varepsilon_{l,n}\|_{L^2(\Omega_{\varepsilon})}^2.
$$

(92)

Similarly it holds

$$
\int_{\Omega_{\varepsilon_R}^c} |B_L P_\delta(v^\varepsilon_{r,n} - u_b) \nabla v^\varepsilon_{r,n}| \, dx \leq C + C \|\nabla v^\varepsilon_{r,n}\|_{L^2(\Omega_{\varepsilon_R})}^2,
$$

(93)

and

$$
\varepsilon^\gamma \int_{\Omega_{\varepsilon_m}^c} |B_M P_\delta(v^\varepsilon_{m,n} - u_b) \nabla v^\varepsilon_{m,n}| \, dx \leq C + C \varepsilon^\beta \|\nabla v^\varepsilon_{m,n}\|_{L^2(\Omega_{\varepsilon_m})}^2,
$$

(94)

for (91) we used the assumption [A6] Using Cauchy Schwarz’s inequality, for \( \eta > 0 \) and \( C(\eta) > 0 \)

$$
\int_0^t \int_{\Omega_{\varepsilon}^c} |f_b \partial_t v^\varepsilon_{l,n}| \, dx \, dt \leq C(\eta) \|f_b\|_{L^2(0,t;L^2(\Omega_{\varepsilon}^c))}^2 + \eta \|\partial_t v^\varepsilon_{l,n}\|_{L^2(0,t;L^2(\Omega_{\varepsilon}^c))}^2,
$$

(95)

$$
\int_0^t \int_{\Omega_{\varepsilon_R}^c} |f_b \partial_t v^\varepsilon_{r,n}| \, dx \, dt \leq C(\eta) \|f_b\|_{L^2(0,t;L^2(\Omega_{\varepsilon_R}^c))}^2 + \eta \|\partial_t v^\varepsilon_{r,n}\|_{L^2(0,t;L^2(\Omega_{\varepsilon_R}^c))}^2,
$$

(96)

$$
\varepsilon^\alpha \int_0^t \int_{\Omega_{\varepsilon_m}^c} |f_{a_m} \partial_t v^\varepsilon_{m,n}| \, dx \, dt \leq \varepsilon^\alpha C(\eta) \|f_{a_m}\|_{L^2(0,t;L^2(\Omega_{\varepsilon_m}^c))}^2 + \varepsilon^\alpha \eta \|\partial_t v^\varepsilon_{m,n}\|_{L^2(0,t;L^2(\Omega_{\varepsilon_m}^c))}^2,
$$

(97)

$$
\varepsilon^\beta \int_0^t \int_{\Omega_{\varepsilon_m}^c} |f^\varepsilon_{b_m} \partial_t v^\varepsilon_{m,n}| \, dx \, dt \leq \varepsilon^\beta C(\eta) \|f^\varepsilon_{b_m}\|_{L^2(0,t;L^2(\Omega_{\varepsilon_m}^c))}^2 + \varepsilon^\beta \eta \|\partial_t v^\varepsilon_{m,n}\|_{L^2(0,t;L^2(\Omega_{\varepsilon_m}^c))}^2.
$$

(98)

Using the regularity and the structure of \( P_\delta(\cdot) \) together with Cauchy-Schwarz’s inequality, Young’s inequality, (21), (99) and (A7) we see that

$$
\int_0^t \int_{\Omega_{\varepsilon}^c} |B_L P_\delta(v^\varepsilon_{l,n} - u_b) \partial_t(v^\varepsilon_{l,n} - u_b) \nabla v^\varepsilon_{l,n}| \, dx \, dt \\
\leq C \int_0^t \int_{\Omega_{\varepsilon}^c} |\partial_t(v^\varepsilon_{l,n} - u_b) \nabla v^\varepsilon_{l,n}| \, dx \, dt \\
\leq C \|\partial_t v^\varepsilon_{l,n}\|_{L^2(0,t;L^2(\Omega_{\varepsilon}^c))} \|\nabla v^\varepsilon_{l,n}\|_{L^2(0,t;L^2(\Omega_{\varepsilon}^c))} \\
+ C \|\partial_t u_b\|_{L^2(0,t;L^2(\Omega_{\varepsilon}^c))} \|\nabla v^\varepsilon_{l,n}\|_{L^2(0,t;L^2(\Omega_{\varepsilon}^c))} \\
\leq C \eta \|\partial_t v^\varepsilon_{l,n}\|_{L^2(0,t;L^2(\Omega_{\varepsilon}^c))} + C(\eta) \|\nabla v^\varepsilon_{l,n}\|_{L^2(0,t;L^2(\Omega_{\varepsilon}^c))} \\
+ C + C \|\nabla v^\varepsilon_{l,n}\|_{L^2(0,t;L^2(\Omega_{\varepsilon}^c))}^2 \\
\leq \eta C \|\partial_t v^\varepsilon_{l,n}\|_{L^2(0,t;L^2(\Omega_{\varepsilon}^c))} + C(\eta).
$$

(99)
Similarly, we have
\[ \int_0^t \int_{\Omega_R^\varepsilon} |B_R P_\varepsilon^0 (v_{r,n}^\varepsilon - u_b) \partial_t (v_{r,n}^\varepsilon - u_b) \nabla v_{r,n}^\varepsilon| \, dx \, dt \leq C(\eta) + \eta C \| \partial_t v_{r,n}^\varepsilon \|_{L^2(0,t;L^2(\Omega_R^\varepsilon))}, \]  
(100) 
\[ \varepsilon \gamma \int_0^t \int_{\Omega_M^\varepsilon} |B_M P_\delta^0 (v_{m,n}^\varepsilon - u_b) \partial_t (v_{m,n}^\varepsilon - u_b) \nabla v_{m,n}^\varepsilon| \, dx \, dt \leq C(\eta) + \eta C \varepsilon \| \partial_t v_{m,n}^\varepsilon \|_{L^2(0,t;L^2(\Omega_M^\varepsilon))}, \]  
(101) 
Using (A4), (41) and Young’s inequality, we get
\[ \int_{\Gamma_k \cap \partial \Omega_E^\varepsilon} |g_{b_\varepsilon} v_{l,n}^\varepsilon| \, d\sigma \leq C(\varepsilon) + C(\varepsilon) \| \nabla v_{l,n}^\varepsilon \|_{L^2(\Omega_E^\varepsilon)}, \]  
(102) 
Similarly, we have
\[ \int_{\Gamma_k \cap \partial \Omega_E^\varepsilon} |g_{b_\varepsilon} v_{r,n}^\varepsilon| \, d\sigma \leq C(\varepsilon) + C(\varepsilon) \| \nabla v_{r,n}^\varepsilon \|_{L^2(\Omega_E^\varepsilon)}, \]  
(103) 
Using similar arguments as those used in the proof of (66), namely, Cauchy Schwarz’s inequality, Lemma 3, Young’s inequality and (A6), we get
\[ \varepsilon \xi \int_{\Gamma_0^\varepsilon} |g_{b_\varepsilon} v_{m,n}^\varepsilon| \, d\sigma \leq \varepsilon^{1/2} \xi C \| g_{b_0} \|_{L^2(\Gamma_0^\varepsilon)} \| v_{m,n}^\varepsilon \|_{L^2(\Omega_M^\varepsilon)} + \varepsilon \xi \| \nabla v_{m,n}^\varepsilon \|_{L^2(\Omega_M^\varepsilon)} \]  
(104) 
\[ + C \varepsilon^{1/2} \xi \| g_{b_0} \|_{L^2(\Gamma_0^\varepsilon)} \| v_{m,n}^\varepsilon \|_{L^2(\Omega_M^\varepsilon)} \leq C(t) \varepsilon^{1/2} \xi \| \nabla v_{m,n}^\varepsilon \|_{L^2(\Omega_M^\varepsilon)}. \]  
Similarly, we get
\[ \varepsilon \beta \int_{\Gamma_0^\varepsilon} |g_{b_\varepsilon} v_{m,n}^\varepsilon| \, d\sigma \leq C(\varepsilon) + \varepsilon \beta \| \nabla v_{m,n}^\varepsilon \|_{L^2(\Omega_M^\varepsilon)} \]  
(105) 
By using similar arguments as in (102), (103), (104) and (105) together with (49), we get
\[ \int_0^t \int_{\Gamma_k \cap \partial \Omega_E^\varepsilon} |\partial_t g_{b_\varepsilon} v_{l,n}^\varepsilon| \, d\sigma dt \leq C, \]  
(106) 
\[ \int_0^t \int_{\Gamma_k \cap \partial \Omega_E^\varepsilon} |\partial_t g_{b_\varepsilon} v_{r,n}^\varepsilon| \, d\sigma dt \leq C, \]  
(107) 
\[ \varepsilon \xi \int_0^t \int_{\Gamma_0^\varepsilon} |\partial_t g_{b_\varepsilon} v_{m,n}^\varepsilon| \, d\sigma dt \leq C, \]  
(108) 
\[ \varepsilon \beta \int_0^t \int_{\Gamma_0^\varepsilon} |\partial_t g_{b_\varepsilon} v_{m,n}^\varepsilon| \, d\sigma dt \leq C. \]  
(109)
Furthermore, we have
\[ \int_0^t \int_{B^c_\varepsilon} (D_L - \varepsilon^3 D_M) \nabla u_b \cdot n_{l_t} v_{l,n}^\varepsilon \, d\sigma dt = \int_{B^c_\varepsilon} (D_L - \varepsilon^3 D_M) \nabla u_b \cdot n_{l_1} v_{l,n}^\varepsilon \, d\sigma \]
- \int_0^t \int_{B^c_\varepsilon} (D_L - \varepsilon^3 D_M) \nabla \partial_t u_b \cdot n_{l} v_{l,n}^{\varepsilon} \, d\sigma dt 
- \int_{B^c_\varepsilon} (D_L - \varepsilon^3 D_M) \nabla u_b(0) \cdot n_{l_{1}} v_{l,n}^{\varepsilon}(0) \, d\sigma, \tag{110} \]
and
\[ \int_0^t \int_{B^c_\varepsilon} (D_R - \varepsilon^3 D_M) \nabla u_b \cdot n_{r} v_{r,n}^{\varepsilon} \, d\sigma dt = \int_{B^c_\varepsilon} (D_R - \varepsilon^3 D_M) \nabla u_b \cdot n_{r} v_{r,n}^{\varepsilon} \, d\sigma \]
- \int_0^t \int_{B^c_\varepsilon} (D_R - \varepsilon^3 D_M) \nabla \partial_r u_b \cdot n_{r} v_{r,n}^{\varepsilon} \, d\sigma dt 
- \int_{B^c_\varepsilon} (D_R - \varepsilon^3 D_M) \nabla u_b(0) \cdot n_{r} v_{r,n}^{\varepsilon}(0) \, d\sigma. \tag{111} \]

Now, using (19), Cauchy-Schwarz’s inequality, Young’s inequality (A6) and (45), we get
\[ \int_{B^c_\varepsilon} |(D_L - \varepsilon^3 D_M) \nabla u_b \cdot n_{l_1} v_{l,n}^{\varepsilon}| \, d\sigma \leq C \int_{B^c_\varepsilon} |\nabla u_b| |v_{l,n}^{\varepsilon}| \, d\sigma \]
\[ \leq C(t) + C \|v_{l,n}^{\varepsilon}\|_{L^2(\Omega^c_\varepsilon)}^2 \tag{112} \]
\[ \int_{B^c_R} |(D_R - \varepsilon^3 D_M) \nabla u_b \cdot n_{r} v_{r,n}^{\varepsilon}| \, d\sigma \leq C(t) + C \|v_{r,n}^{\varepsilon}\|_{L^2(\Omega_R^c)}^2 \tag{113} \]
\[ \int_0^t \int_{B^c_\varepsilon} |(D_L - \varepsilon^3 D_M) \nabla \partial_t u_b \cdot n_{l} v_{l,n}^{\varepsilon}| \, d\sigma dt \]
\[ \leq C \|\nabla \partial_t u_b\|_{L^2(0,t;L^2(\Omega^c_\varepsilon))} \|v_{l,n}^{\varepsilon}\|_{L^2(0,t;L^2(\Omega^c_\varepsilon))} \]
\[ \leq C \|\nabla \partial_t u_b\|_{L^2(0,t;L^2(\Omega^c_\varepsilon))}^2 + C \|v_{l,n}^{\varepsilon}\|_{L^2(0,t;L^2(\Omega^c_\varepsilon))}^2 \]
\[ \leq C, \tag{114} \]
and
\[ \int_0^t \int_{B^c_R} |(D_R - \varepsilon^3 D_M) \nabla \partial_r u_b \cdot n_{r} v_{r,n}^{\varepsilon}| \, d\sigma dt \leq C. \tag{115} \]

Using (A4) leads to
\[ \int_{B^c_R} |(D_R - \varepsilon^3 D_M) \nabla u_b(0) \cdot n_{r} v_{r,n}^{\varepsilon}(0)| \, d\sigma \leq C \int_{B^c_R} |\nabla u_b(0)| |h_{r,n}^{\varepsilon}| \, d\sigma \]
\[ \leq C + C \|h_{r,n}^{\varepsilon}\|_{L^2(\Omega_R^c)}^2 \tag{116} \]
\[ \leq C, \]
where \( h_{b_r,n}^\varepsilon := \sum_{k=0}^n d_{r,k}^\varepsilon(0)w_{r,k} \) and by using \( d_{r,k}^\varepsilon(0) = \int_{\Omega_R^\varepsilon} h_{b_r,n}^\varepsilon w_{r,k}^\varepsilon dx \), we get \( \|h_{b_r,n}^\varepsilon\|_{H^1(\Omega_R^\varepsilon)} \leq \|h_{b_r}^\varepsilon\|_{H^1(\Omega_R^\varepsilon)} \),

\[
\int_{B_L^\varepsilon} |(D_L - \varepsilon\beta D_M)\nabla u_b(0) \cdot n_l v_{m,n}(0)|d\sigma \leq C. \tag{117}
\]

Using (19), (A4) and (A6) allow us to write

\[
\frac{1}{2} \int_{\Omega_L^\varepsilon} |D_L \nabla v_{l,n}(0) \cdot \nabla v_{l,n}^\varepsilon(0)|dx \leq C \int_{\Omega_L^\varepsilon} |\nabla h_{b_r,n}^\varepsilon|dx \leq C, \tag{118}
\]

\[
\frac{1}{2} \int_{\Omega_R^\varepsilon} |D_R \nabla v_{r,n}(0) \cdot \nabla v_{r,n}^\varepsilon(0)|dx \leq C. \tag{119}
\]

Using the structure of \( P_\delta(\cdot) \), there exist a \( k \in \mathbb{R} \) such that \( P_\delta(k) = 0 \). Now we use Mean Value Theorem, (21), we get

\[
\varepsilon^\beta \frac{1}{2} \int_{\Omega_M^\varepsilon} D_M^\varepsilon \nabla v_{m,n}(0) \cdot \nabla v_{m,n}^\varepsilon(0)dx \leq C, \tag{120}
\]

\[
\int_{\Omega_R^\varepsilon} |B_R P_\delta(v_{r,n}^\varepsilon(0) - u_b(0))\nabla v_{r,n}^\varepsilon(0)|dx = \int_{\Omega_R^\varepsilon} |B_L P_\delta(h_{b_r}^\varepsilon - u_b(0))\nabla h_{b_r,n}^\varepsilon|dx \leq C \int_{\Omega_R^\varepsilon} |h_{b_r}^\varepsilon - u_b(0) - k\|\nabla h_{b_r,n}^\varepsilon|dx \leq C, \tag{121}
\]

and

\[
\int_{\Omega_L^\varepsilon} |B_L P_\delta(v_{l,n}^\varepsilon(0) - u_b(0))\nabla v_{l,n}^\varepsilon(0)|dx \leq C, \tag{122}
\]

\[
\varepsilon^\gamma \int_{\Omega_M^\varepsilon} |B_M P_\delta(v_{m,n}^\varepsilon(0) - u_b(0))\nabla v_{m,n}^\varepsilon(0)|dx \leq C. \tag{123}
\]

By using similar arguments of (102), (103), (104) and (105) together with (A5), we get

\[
\int_{\Gamma_{h_r} \cap \partial \Omega_L^\varepsilon} |g_b(0)v_{l,n}^\varepsilon(0)|d\sigma \leq C, \tag{124}
\]

\[
\int_{\Gamma_{h_r} \cap \partial \Omega_K^\varepsilon} |g_b(0)v_{r,n}^\varepsilon(0)|d\sigma \leq C, \tag{125}
\]

\[
\varepsilon^\xi \int_{\Gamma_0^\varepsilon} |g_b^\varepsilon(0)v_{m,n}^\varepsilon(0)|d\sigma \leq C, \tag{126}
\]

\[
\varepsilon^\beta \int_{\Gamma_0^\varepsilon} |g_b(0)v_{m,n}^\varepsilon(0)|d\sigma \leq C. \tag{127}
\]

Choosing \( \eta > 0 \) small enough and using ellipticity condition together with (91)-(127), we obtain for \( t = T \) the bound

23
\[ \| \partial_t v_{\varepsilon,n} \|_{L^2(0,T;L^2(\Omega_{\varepsilon}^Z))}^2 + \| \partial_t v_{\varepsilon,n} \|_{L^2(0,T;L^2(\Omega_{\varepsilon}^M))}^2 \leq C + C(t) + C \left( \| \nabla v_{\varepsilon} \|_{L^2(\Omega_{\varepsilon}^Z)}^2 + \| \nabla v_{\varepsilon,n} \|_{L^2(\Omega_{\varepsilon}^M)}^2 \right). \quad (128) \]

From (128), we can observe that, the time dependent constant \( C(t) \) is a consequence of (103), (104), (112) and (113). So, by using (A3) and (A4) we get \( \int_0^T C(t) \leq C \). Now, integrating again from 0 to \( T \) with respect to \( t \), and using (128), we get

\[ \| \partial_t v_{\varepsilon,n} \|_{L^2(0,T;L^2(\Omega_{\varepsilon}^Z))}^2 + \| \partial_t v_{\varepsilon,n} \|_{L^2(0,T;L^2(\Omega_{\varepsilon}^M))}^2 \leq C. \quad (129) \]

Now as an application of Aubin-Lions compactness lemma, we get

\[ \| \partial_t v_{\varepsilon} \|_{L^2(0,T;L^2(\Omega_{\varepsilon}^Z))}^2 + \| \partial_t v_{\varepsilon} \|_{L^2(0,T;L^2(\Omega_{\varepsilon}^M))}^2 \leq C. \quad (130) \]

Hence we proved (51)

\[ \square \]

4.2 Extension to fixed domain

**Lemma 4** If \( v_{\varepsilon} \in H^1(\Omega_{\varepsilon}^M) \), then there exists an extension of \( v_{\varepsilon} \) to \( H^1((-\varepsilon,\varepsilon) \times (0,h)) \) denote as \( \tilde{v}_{\varepsilon} \) satisfying the following inequality

\[ \| \tilde{v}_{\varepsilon} \|_{H^1((-\varepsilon,\varepsilon) \times (0,h))} \leq C \| v_{\varepsilon} \|_{H^1(\Omega_{\varepsilon}^M)}. \quad (131) \]

**Proof:** By using Theorem 9.7 of [5] we can easily obtain the extension result for standard cell \( Z \) with the inequality

\[ \| \tilde{v}_{\varepsilon} \|_{H^1(Y)} \leq C \| v \|_{H^1(Z)}, \quad (132) \]

for some constant \( C \).

Now, using (132), we have

\[ \| \tilde{v}_{\varepsilon} \|_{H^1((-\varepsilon,\varepsilon) \times (0,h))} = \int_{(-\varepsilon,\varepsilon) \times (0,h)} |\tilde{v}_{\varepsilon}|^2 + |\nabla \tilde{v}_{\varepsilon}|^2 \, dx \quad (133) \]

\[ = \sum_{k=0}^{h/\varepsilon} \int_{\varepsilon(x) \times (0,h)} |\tilde{v}_{\varepsilon}|^2 + |\nabla \tilde{v}_{\varepsilon}|^2 \, dx \quad (134) \]

\[ = \sum_{k=0}^{h/\varepsilon} \varepsilon^2 \int_{\varepsilon(x) \times (0,h)} |\tilde{v}_{\varepsilon}(\varepsilon(x))|^2 + |\nabla \tilde{v}_{\varepsilon}(\varepsilon(x))|^2 \, dx \quad (135) \]

\[ \leq C \sum_{k=0}^{h/\varepsilon} \varepsilon^2 \int_{\varepsilon(x) \times (0,h)} |v_{\varepsilon}(\varepsilon(x))|^2 + |\nabla v_{\varepsilon}(\varepsilon(x))|^2 \, dx \quad (136) \]

\[ = C \| v_{\varepsilon} \|_{L^2(\Omega_{\varepsilon}^M)} \quad (137) \]

To prove the above result we used a similar technique as used in [1].

24
4.3 Two scale convergence for thin layer

Here we use Theorem 2 and obtain two scale limit of $v_m^\varepsilon$ as $\varepsilon \to 0$ for the layer. We use two scale limit for layer definition similar to definition defined in [29] which is motivated from [24].

Definition 4.1 We define sequence of functions $v_m^\varepsilon \in L^2((0,T) \times \Omega_{\mathcal{M}}^{\varepsilon})$ two-scale converges to $v_0(t,\bar{x},y) \in L^2((0,T) \times \Sigma \times Z)$, if

$$
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^T \int_{\Omega_{\mathcal{M}}^{\varepsilon}} v_\varepsilon(t,x)\psi(t,\bar{x},\frac{x}{\varepsilon})dxdt = \int_0^T \int_{\Sigma} \int_{Z} v_0(t,\bar{x},y)\psi(t,\bar{x},y)dydxdt,
$$

(139)

for all $\psi \in L^2((0,T) \times Z; C_\#(\bar{Z}))$, where $\Sigma := \{(0,x_2) \in \Omega : x_2 \in (0,h)\}$ and we denote the two-scale convergence of $v_m^\varepsilon$ to $v_m^0$ as $v_m^0 \rightarrow v_m^\varepsilon$.

Definition 4.2 We define sequence of functions $v_m^\varepsilon \in L^2((0,T) \times \Gamma_0^{\varepsilon})$ two-scale converges to $v_0(t,\bar{x},y) \in L^2((0,T) \times \Sigma \times \partial Y_0)$, if

$$
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^T \int_{\Gamma_0^{\varepsilon}} v_\varepsilon(t,x)\psi(t,\bar{x},\frac{x}{\varepsilon})dxdt = \int_0^T \int_{\Sigma} \int_{\partial Y_0} v_0(t,\bar{x},y)\psi(t,\bar{x},y)d\sigma_yd\bar{x}dt
$$

(140)

for all $\psi \in L^2((0,T) \times \Sigma; C_\#(\partial Y_0))$.

Theorem 3 For any sequence $v_m^\varepsilon \in L^2((0,T) \times \Omega_{\mathcal{M}}^{\varepsilon})$ with the condition

$$
\frac{1}{\varepsilon} \left\| v_m^\varepsilon \right\|^2_{L^2((0,T) \times \Omega_{\mathcal{M}}^{\varepsilon})} \leq C,
$$

(141)

for a constant $C$, we can find a subsequence, again denoted as $v_m^\varepsilon$, such that $v_m^\varepsilon$ two-scale converges to $v_0^m \in L^2((0,T) \times \Sigma \times Z)$.

Theorem 4 For any sequence $v_m^\varepsilon \in L^2(\Gamma_0^{\varepsilon} \times (0,T))$ with the condition

$$
\left\| v_m^\varepsilon \right\|^2_{L^2(\Gamma_0^{\varepsilon} \times (0,T))} \leq C,
$$

(142)

for a constant $C$, we can find a subsequence, again denoted as $v_m^\varepsilon$, such that $v_m^\varepsilon$ two-scale converges to $v_0^m \in L^2(\Sigma \times \partial Y_0 \times (0,T))$.

Proof: For proof of Theorem 3 and Theorem 4, refer proof of Theorem 4.4 of [7] and Proposition 4.2 of [29].

Theorem 5 Let $(v_0^f, v_0^r, v_0^\varepsilon)$ be the weak solution of (33), then there exist $(v_1^f, v_1^0) \in (L^2(0,T; H^1(\Omega_\mathcal{L})), L^2(0,T; H^1(\Omega_\mathcal{R})))$ such that

$$
1_{\Omega_\mathcal{L}} v_\varepsilon^f \to v_1^f \quad \text{on} \quad L^2((0,T) \times \Omega_\mathcal{L}), \quad (143)
$$

$$
1_{\Omega_\mathcal{R}} v_\varepsilon^r \to v_1^r \quad \text{on} \quad L^2((0,T) \times \Omega_\mathcal{R}), \quad (144)
$$

$$
1_{\Omega_\mathcal{L}} v_\varepsilon^f(t,x_1,-\varepsilon) \to v_1^0(t,x_1,0) \quad \text{on} \quad L^2((0,T) \times \Omega_\mathcal{L}), \quad (145)
$$

$$
1_{\Omega_\mathcal{R}} v_\varepsilon^r(t,x_1,\varepsilon) \to v_1^0(t,x_1,0) \quad \text{on} \quad L^2((0,T) \times \Omega_\mathcal{R}), \quad (146)
$$

$$
1_{\Omega_\mathcal{L}} \nabla v_\varepsilon^f \overset{w}{\to} \nabla v_1^0 \quad \text{on} \quad L^2((0,T) \times \Omega_\mathcal{L}), \quad (147)
$$
Proof: To prove (155) and (157) we use Theorem 2 and Theorem 3. For details we refer Proposition 2.1 and Proposition 2.2 of [29]. Using Theorem 2 we have

\[ \| \bar{\Omega}_\varepsilon \nabla v^\varepsilon - \nabla v^0_r \|_{L^2(0,T \times \Omega_R)}, \]
\[ \| \bar{\Omega}_\varepsilon \partial_t v^\varepsilon - \partial_t v^0_r \|_{L^2(0,T;L^2(\Omega_L))}, \]
\[ \| \bar{\Omega}_\varepsilon \partial_t v^\varepsilon - \partial_t v^0_r \|_{L^2(0,T;L^2(\Omega_R))}, \]
\[ \| \bar{\Omega}_\varepsilon P_\delta (v^\varepsilon_r - u_b) \rightarrow P_\delta (v^0_r - u_b) \|_{L^2((0,T) \times \Omega_L)}, \]
\[ \| \bar{\Omega}_\varepsilon P_\delta (v^\varepsilon_r - u_b) \rightarrow P_\delta (v^0_r - u_b) \|_{L^2((0,T) \times \Omega_R)}, \]

as \( \varepsilon \rightarrow 0 \).

Proof of convergence (143), (144), (147)-(150) is application of Lemma 2, Lemma 1, Theorem 2 and Lions-Aubin’s compactness lemma (see [5]). For details of the proof see Proposition 2.1 in [29]. To prove convergence result (151) we use the following estimate

\[ \| \bar{\Omega}_\varepsilon P_\delta (v^\varepsilon_r - u_b) \rightarrow P_\delta (v^0_r - u_b) \|_{L^2(0,T;L^2(\Omega_L))} \leq \| \bar{\Omega}_\varepsilon P_\delta (v^\varepsilon_r - u_b) \rightarrow \bar{\Omega}_\varepsilon P_\delta (v^0_r - u_b) \|_{L^2(0,T;L^2(\Omega_L))} \]

(153)

to get the inequality (153) we used the structure of \( P_\delta \) operator, Mean Value Theorem and Minkowski’s inequality. As a consequence of Monotone Convergence Theorem, we have

\[ \| \bar{\Omega}_\varepsilon \nabla v^\varepsilon - \nabla v^0_r \|_{L^2(0,T;L^2(\Omega_L))} \rightarrow 0 \]

(154)

as \( \varepsilon \rightarrow 0 \). Now, using (143), (153) and (154) as \( \varepsilon \rightarrow 0 \) we can conclude \( \bar{\Omega}_\varepsilon P_\delta (v^\varepsilon_r - u_b) \rightarrow P_\delta (v^0_r - u_b) \)

strongly in \( L^2((0,T) \times \Omega_L) \) as \( \varepsilon \rightarrow 0 \).

Similarly we can prove (152).

**Theorem 6** Let \( (v^\varepsilon_l, v^\varepsilon_m, v^\varepsilon_r) \) be a weak solution of (39). Then there exists \( v^0_m \in L^2((0,T) \times \Sigma; H^1_{+\delta}(Z)) \) such that up to a subsequence, it holds

\[ v^\varepsilon_m \rightarrow v^0_m, \]
\[ \partial_t v^\varepsilon_m \rightarrow \partial_t v^0_m, \]
\[ \varepsilon \nabla v^\varepsilon_m \rightarrow \nabla v^0_m, \]
\[ \varepsilon P_\delta (v^\varepsilon_r - u_b) \rightarrow 0. \]

(158)

as \( \varepsilon \rightarrow 0 \).

Proof: To prove (155) and (157) we use Theorem 2 and Theorem 3. For details we refer Proposition 2.1 and Proposition 2.2 of [29]. Using Theorem 2 we have

\[ \frac{1}{\varepsilon} \| P_\delta (v^\varepsilon_r - u_b) \|_{L^2(\Omega_{\delta R})} \leq C. \]

(159)
Now using Theorem 3 for (159), we have

\[ P_\delta(v_m^\varepsilon - w_b)^{2-\varepsilon} w, \]  

where \( w \in L^2((0,T) \times \Sigma \times Z) \). Consequently, we get \( \varepsilon P_\delta(v_m^\varepsilon - w_b)^{2-\varepsilon} \) 0.

\[ \square \]

5 Macroscopic model

In this section we derive upscaled equations and effective transmission conditions and coefficients for a variable selection of scalings depending on the small parameter \( \varepsilon \); see Table 1.

| Scaling options for infinitely thin layer | Scaling options for finitely thin layer |
|---------------------------|---------------------------------------|
| Choice S1                 | Choice S2                             |
| \( \alpha = -1 \)        | \( \alpha = -1 \)                    |
| \( \beta = 1 \)          | \( \beta \in (0,1) \)                |
| \( \gamma \geq 1 \)      | \( \gamma \geq \beta \)              |
| \( \xi \geq \frac{1}{\alpha} \) | \( \xi \geq \min\{\beta - \frac{1}{\alpha}, 0\} \) |
| Choice S3                 | Choice S4                             |
| \( \alpha \in (-1, \infty) \) | \( \alpha \in (-1, \infty) \)        |
| \( \beta - \alpha = 2 \) | \( \beta - \alpha \in (1, \infty) \setminus \{2\} \) |
| \( \gamma - \alpha \geq 1 \) | \( \gamma - \alpha \geq 1 \)        |
| \( \xi - \alpha \geq 1 \) | \( \xi - \alpha \geq 1 \)          |

In Fig. 3 and Fig. 4 we sketch the basic thin layers geometries we are handling here.

![Figure 3: Schematic representation of the macroscopic model for infinitely thin layer.](image)

5.1 Macroscopic model for infinitely thin layer

Theorem 7 Assume [A1]-[A7]. Then for scaling choice S1,

\( (v^\varepsilon_l, v^\varepsilon_m, v^\varepsilon_r) \in L^2(0,T; V_\varepsilon) \cap H^1(0,T; L^2(\Omega^\varepsilon_L) \times L^2(\Omega^\varepsilon_M) \times L^2(\Omega^\varepsilon_R)), \)
satisfying $(P_\varepsilon)$ in the sense of Definition 2.1 converges to 

$$(v_1^0, v_m^0, v_r^0) \in (L^2(0,T; H^1(\Omega_L)), L^2((0,T) \times \Sigma; H^1_\#(Z)), L^2(0,T; H^1(\Omega_R)))$$

which satisfies the identity

\begin{align}
\int_0^T \int_{\Omega_L} \partial_t v_1^0 \phi_1 dx dt + \int_0^T \int_{\Omega_L} D_L \nabla v_1^0 \nabla \phi_1 dx dt \\
- \int_0^T \int_{\Omega_L} D_L B_L P_b (v_1^0 - u_b) \nabla \phi_1 dx dt \\
+ \int_0^T \int_{\Omega_R} \partial_t v_r^0 \phi_3 dx dt + \int_0^T \int_{\Omega_R} D_R \nabla v_1^0 \nabla \phi_1 dx dt \\
- \int_0^T \int_{\Omega_R} B_R P_b (v_r^0 - u_b) \phi_3 dx dt \\
+ \int_0^T \int_{\Sigma} \int_Z \partial_t v_m^0 (t, \bar{x}, y) \phi_2 (t, \bar{x}, y) dy d\bar{x} dt \\
+ \int_0^T \int_{\Sigma} \int_Z D_M(y) \nabla_y v_m^0 (t, \bar{x}, y) \nabla_y \phi_2 (t, \bar{x}, y) dy d\bar{x} dt
\end{align}

\begin{align}
= \int_0^t \int_{\Omega_L} f_b \phi_1 dx dt - \int_0^t \int_{\Gamma_h \cap \partial \Omega_L} g_b \phi_1 d\sigma dt + \int_0^t \int_{\Omega_R} f_b \phi_3 dx dt \\
- \int_0^t \int_{\Gamma_h \cap \partial \Omega_L} g_b \phi_3 d\sigma dt + \int_0^T \int_{\Sigma} \int_Z f_{a_0} (t, \bar{x}, y) \phi_2 (t, \bar{x}, y) dy d\bar{x} dt \\
\int_0^T \int_{\Sigma} \int_Z D_L \nabla_x u_b (t, \bar{x}, 0) \cdot n_l \phi_2 (t, \bar{x}, \bar{y}, -1) dy d\bar{x} dt \\
- \int_0^T \int_{\Sigma} \int_Z D_R \nabla_x u_b (t, \bar{x}, 0) \cdot n_l \phi_2 (t, \bar{x}, \bar{y}, +1) dy d\bar{x} dt
\end{align}  

(161)
for all \((\phi_1, \phi_3) \in L^2((0, T); H^1(\Omega_L; \Gamma_L)) \times L^2((0, T); H^1(\Omega_R; \Gamma_R))\) and \(\phi_2 \in L^2((0, T) \times \Sigma \times Z)\), along with the initial condition

\[
\begin{align*}
v^0_i(0, x) &= h^0_{b_i}(x) \text{ for all } x \in \Omega_L, \\
v^0_i(0, x) &= h^0_{b_i}(x) \text{ for all } x \in \Omega_R, \\
v^0_m(0, x, y) &= h^0_{b_m}(x, y) \text{ for all } (x, y) \in \Sigma \times \mathcal{Z},
\end{align*}
\]

where the limit function \((v^0_i, v^0_m, v^0_r)\) are given in Theorem 5 and Theorem 6.

**Proof:** We integrate the weak formulation (39) from 0 to \(T\) and choose

\[
(\phi_1, \phi_3) \in L^2((0, T); H^1(\Omega_L; \Gamma_L)) \times L^2((0, T); H^1(\Omega_R; \Gamma_R))
\]

and \(\phi_2 = \psi^\varepsilon(t, x) := \psi(t, \bar{x}, \frac{x}{\varepsilon}) \in L^2\left((0, T) \times \Sigma; C^\infty_0(\mathcal{Z})\right)\) with \(\psi = \psi^\varepsilon\) on \(\mathcal{B}^L_{\varepsilon}\), \(\phi_3 = \psi^\varepsilon\) on \(\mathcal{B}^R_{\varepsilon}\), we get

\[
\begin{align*}
\int_0^T \int_{\Omega_L} \partial_t v_i \phi_1 dx \, dt + &\int_0^T \int_{\Omega_L} D_L \nabla v_i \nabla \phi_1 dx \, dt - \int_0^T \int_{\Omega_L} B_L p_\delta (v_i - u_b) \nabla \phi_1 dx \, dt \\
+ &\int_0^T \int_{\Omega_R} \partial_t v_i \phi_3 dx \, dt + \int_0^T \int_{\Omega_R} D_R \nabla v_i \nabla \phi_3 dx \, dt - \int_0^T \int_{\Omega_R} B_R p_\delta (v_i - u_b) \nabla \phi_3 dx \, dt \\
+ &\varepsilon^\alpha_0 \int_0^T \int_{\Omega_M} \partial_t \phi_m \phi_i \psi(t, \bar{x}, \frac{x}{\varepsilon}) dx \, dt \\
+ &\varepsilon^\beta \int_0^T \int_{\Omega_M} D_M \left(\frac{x}{\varepsilon}\right) \nabla \phi_m \left(\nabla_x \psi(t, \bar{x}, \frac{x}{\varepsilon}) + \frac{1}{\varepsilon} \nabla_y \psi(t, \bar{x}, \frac{x}{\varepsilon})\right) dx \, dt \\
- &\varepsilon^\gamma \int_0^T \int_{\Omega_M} B^\varepsilon \left(\frac{x}{\varepsilon}\right) p_\delta (v_m - u_b) \left(\nabla_x \psi(t, \bar{x}, \frac{x}{\varepsilon}) + \frac{1}{\varepsilon} \nabla_y \psi(t, \bar{x}, \frac{x}{\varepsilon})\right) dx \, dt \\
= &\int_0^T \int_{\Omega_L} f_{b_i} \phi_1 dx \, dt - \int_0^T \int_{\Gamma_h \cap \partial \Omega^\varepsilon_L} g_{b_i} \phi_1 ds \, dt + \int_0^T \int_{\Omega_R} f_{b_i} \phi_3 dx \, dt \\
- &\int_0^T \int_{\Gamma_h \cap \partial \Omega^\varepsilon_R} g_{b_i} \phi_3 ds \, dt \\
+ &\varepsilon^\alpha \int_0^T \int_{\Omega_M} f^\varepsilon \psi(t, \bar{x}, \frac{x}{\varepsilon}) dx \, dt + \varepsilon^\beta \int_0^T \int_{\Omega_M} f_{b_m} \psi(t, \bar{x}, \frac{x}{\varepsilon}) dx \, dt \\
- &\varepsilon^\gamma \int_0^T \int_{\Gamma_0^L} g^\varepsilon_0 \psi(t, \bar{x}, \frac{x}{\varepsilon}) ds \, dt - \varepsilon^\beta \int_0^T \int_{\Gamma_0^R} g_{b_0} \psi(t, \bar{x}, \frac{x}{\varepsilon}) ds \, dt \\
+ &\int_0^T \int_{\mathcal{B}^L_{\varepsilon}} (D_L - \varepsilon^\beta D_M) \nabla u_b \cdot n_i \phi_1 ds \, dt + \int_0^T \int_{\mathcal{B}^R_{\varepsilon}} (D_R - \varepsilon^\beta D_M) \nabla u_b \cdot n_r \phi_3 ds \, dt. \quad (163)
\end{align*}
\]

Now, using (143), (147), (149) and (151) for \(\varepsilon \to 0\), we obtain

\[
\int_0^T \int_{\Omega_L} \partial_t v_i \phi_1 dx \, dt + \int_0^T \int_{\Omega_L} D_L \nabla v_i \nabla \phi_1 dx \, dt
\]
Similarly, using (144), (148), (150) and (152) for \( \varepsilon \to 0 \), we get

\[
\int_0^T \int_{\Omega_R} \partial_t \nu^\varepsilon \phi_3 dxdt + \int_0^T \int_{\Omega_R} D_R \nabla \nu^\varepsilon \nabla \phi_3 dxdt - \int_0^T \int_{\Omega_R} B_R P_b(v^\varepsilon - u_b) \nabla \phi_3 dxdt \\
- \int_0^T \int_{\Omega_R} f_b \phi_3 dxdt + \int_0^T \int_{\Gamma_h \cap \partial \Omega_R} g_b \phi_3 d\sigma dt \\
= \int_0^T \int_{\Omega_R} \partial_t \nu^\varepsilon \phi_3 dxdt + \int_0^T \int_{\Omega_R} D_R \nabla \nu^\varepsilon \nabla \phi_3 dxdt \\
- \int_0^T \int_{\Omega_R} f_b \phi_3 dxdt + \int_0^T \int_{\Gamma_h \cap \partial \Omega_R} g_b \phi_3 d\sigma dt. \quad (165)
\]

Now for \( \alpha = -1, \beta = 1, \gamma \geq 1, \xi \geq \frac{1}{2} \) and \( \varepsilon \to 0 \), we use Theorem 6 and obtain

\[
\frac{1}{\varepsilon} \int_0^T \int_{\Theta_M} \partial_t \psi^\varepsilon(t, \bar{x}, \frac{x}{\varepsilon}) dxdt \to \int_0^T \int_{\Sigma} \partial_t \psi^0(t, \bar{x}, y) \psi(t, \bar{x}, y) dy d\bar{x} dt, \quad (166)
\]

\[
\frac{1}{\varepsilon} \int_0^T \int_{\Theta_M} D_M^\varepsilon(\frac{x}{\varepsilon}) \varepsilon^2 \nabla \nu^\varepsilon \nabla \psi(t, \bar{x}, \frac{x}{\varepsilon}) dxdt \to 0, \quad (167)
\]

\[
\frac{1}{\varepsilon} \int_0^T \int_{\Theta_M} D_M^\varepsilon(\frac{x}{\varepsilon}) \varepsilon \nabla \nabla \psi(t, \bar{x}, \frac{x}{\varepsilon}) dxdt \\
\to \int_0^T \int_{\Sigma} D_M(y) \nabla \nu^0(t, \bar{x}, y) \nabla \psi(t, \bar{x}, y) dy d\bar{x} dt, \quad (168)
\]

\[
\varepsilon^{1+\gamma} \int_0^T \int_{\Theta_M} B_M^\varepsilon(\frac{x}{\varepsilon}) P_b(v^\varepsilon - u_b) \nabla \psi(t, \bar{x}, \frac{x}{\varepsilon}) dxdt \to 0, \quad (169)
\]

\[
\varepsilon^{-\gamma} \int_0^T \int_{\Theta_M} D_M^\varepsilon(\frac{x}{\varepsilon}) P_b(v^\varepsilon - u_b) \nabla y \psi(t, \bar{x}, \frac{x}{\varepsilon}) dxdt \to 0. \quad (170)
\]
Using (23), we have

\[
\frac{1}{\varepsilon} \int_{T} \int_{\Omega_{\varepsilon}} f_{a_{m}}(t, \bar{x}, \frac{x}{\varepsilon}) dt + \int_{0}^{T} \int_{Z} f_{a_{0}}(t, \bar{x}, y) \psi(t, \bar{x}, y) dy d\bar{x} dt, \tag{171}
\]

\[
\varepsilon \int_{T} \int_{\Omega_{\varepsilon}} f_{b_{m}}(t, \bar{x}, \frac{x}{\varepsilon}) dt = \varepsilon \int_{T} \int_{\Omega_{\varepsilon}} -\text{div}(D_{M}(\frac{x}{\varepsilon}) \nabla \psi(t, \bar{x}, \frac{x}{\varepsilon})) dt \\
= \varepsilon \int_{T} \int_{\Omega_{\varepsilon}} D_{M}(\frac{x}{\varepsilon}) \nabla \psi(t, \bar{x}, \frac{x}{\varepsilon}) + \frac{1}{\varepsilon} \nabla \psi(t, \bar{x}, \frac{x}{\varepsilon}) dt \\
\rightarrow 0. \tag{172}
\]

Using (26), we obtain

\[
\varepsilon^{\xi} \int_{0}^{T} \int_{\Gamma_{0}} g_{0}(t, \bar{x}, \frac{x}{\varepsilon}) d\sigma dt \rightarrow 0, \tag{173}
\]

\[
\varepsilon \int_{0}^{T} \int_{\Gamma_{0}} g_{b_{0}}(t, \bar{x}, \frac{x}{\varepsilon}) d\sigma dt = \varepsilon \int_{0}^{T} \int_{\Gamma_{0}} -\nabla \psi \left( D_{M}(\frac{x}{\varepsilon}) \right)^{t} \psi(t, \bar{x}, \frac{x}{\varepsilon}) d\sigma dt \\
\rightarrow 0. \tag{174}
\]

\[
\int_{B_{L}} (D_{L} - \varepsilon D_{M}) \nabla \psi_{1} \cdot n_{f} d\sigma + \int_{B_{R}} (D_{R} - \varepsilon D_{M}) \nabla \psi_{3} \cdot n_{f} d\sigma \\
= \int_{B_{L}} (D_{L} - \varepsilon D_{M}) \nabla \psi_{1} \cdot n_{f} d\sigma - \int_{B_{R}} (D_{R} - \varepsilon D_{M}) \nabla \psi_{3} \cdot n_{f} d\sigma. \tag{175}
\]

\[
\int_{0}^{T} \int_{B_{L}} (D_{L} - \varepsilon D_{M}(\frac{x}{\varepsilon})) \nabla \psi_{1} \cdot n_{f}(t, \bar{x}, \frac{x}{\varepsilon}) d\sigma dt \\
- \int_{0}^{T} \int_{B_{R}} (D_{R} - \varepsilon D_{M}(\frac{x}{\varepsilon})) \nabla \psi_{3} \cdot n_{f}(t, \bar{x}, \frac{x}{\varepsilon}) d\sigma dt \\
\rightarrow \int_{0}^{T} \int_{Z} D_{L} \nabla \bar{x} u_{b}(t, \bar{x}, 0) \cdot n_{f}(t, \bar{x}, \bar{y}, -1) d\bar{x} dt \\
- \int_{0}^{T} \int_{Z} D_{R} \nabla \bar{x} u_{b}(t, \bar{x}, 0) \cdot n_{f}(t, \bar{x}, \bar{y}, +1) d\bar{x} dt. \tag{176}
\]

Combining (165)-(175) yields the desired result (161).

For deriving initial conditions, first we choose \( \phi_{1} \in C_{c}^{\infty}(\Omega_{L}) \) and
\[ \Theta(t) \in C^\infty([0, T]) \text{ with } \Theta(T) = 0, \text{ then} \]

\[ \int_{\Omega} v_l^0(0, x) \phi_1(x) \Theta(0) dx = - \int_0^T \int_{\Omega} \partial_t v_l^0(t, x) \phi_1(x) \Theta(t) dx dt \]

\[ - \int_0^T \int_{\Omega} v_l^0(t, x) \partial_t \Theta(t) dx dt + \int_{\Omega} v_l^0(T, x) \phi_1(x) \Theta(T) dx \]

\[ = - \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega} \partial_t v_l^\varepsilon(t, x) \phi_1(x) \Theta(t) dx dt \]

\[ - \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega} v_l^\varepsilon(t, x) \phi_1(x) \partial_t \Theta(t) dx dt \]

\[ = \lim_{\varepsilon \to 0} \int_{\Omega} h_l^\varepsilon(x) \phi_1(x) \Theta(0) dx \]

\[ = \int_{\Omega} h_l^0(x) \phi_1(x) \Theta(0) dx, \]

here we used the assumption (28).

Similarly, for \( \phi_3 \in C_c^\infty(\Omega_R) \) and using (29), we get

\[ \int_{\Omega} v_r^0(0, x) \phi_3(x) \Theta(0) dx = \int_{\Omega_R} h_r^0(x) \phi_3(x) \Theta(0) dx, \quad (178) \]

and for \( \psi \in C_c^\infty(\Sigma; C_c^\#(Z)) \) and using (30), we get

\[ \int_{\Sigma} \int_{Z} v_m^0(0, \bar{x}, y) \psi(\bar{x}, y) \Theta(0) d\bar{x} dy = \int_{\Sigma} \int_{Z} h_m^0(\bar{x}, y) \psi(\bar{x}, y) \Theta(0) d\bar{x} dy. \quad (179) \]

From (177), (178) and (179) we get the desired result (162).

**Theorem 8** Assume \((A1)-(A7)\). Then for scaling choice S1, the limit functions \( (v_l^0, v_m^0, v_r^0) \) which are given in Theorem 5 and Theorem 6 satisfies the following boundary conditions

\[ v_l^0(t, \bar{x}, 0) = v_m^0(t, \bar{x}, y) \quad \text{for a.e } (t, \bar{x}, y) \in (0, T) \times \Sigma \times Z_L \quad (180) \]

\[ v_r^0(t, \bar{x}, 0) = v_m^0(t, \bar{x}, y) \quad \text{for a.e } (t, \bar{x}, y) \in (0, T) \times \Sigma \times Z_R. \quad (181) \]

**Proof:** To prove Theorem 8 we use same technique of Theorem 4.2 of [19]. To prove (180), we choose \( \psi \in C^\infty ((0, T) \times \Sigma \times C_c^\#(Z)) \) such that \( \psi(t, x, \cdot) \) has compact support in \( Z_L \cup Z \) Now, using
integration by parts, Theorem 6 and (17), we have

\[ \int_0^T \int_{\Sigma} \int_Z \nabla_y v_m^0 \psi dyd\bar{x}d\bar{t} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^T \int_{\Omega_{\varepsilon}} \varepsilon \nabla_{\varepsilon} v_m^0 \psi(t, x, \frac{x}{\varepsilon}) dxdt \]

\[ = \lim_{\varepsilon \to 0} \left( -\frac{1}{\varepsilon} \int_0^T \int_{\Omega_{\varepsilon}} v_m^0 \left( \nabla_y \psi(t, \bar{x}, \frac{x}{\varepsilon}) + \varepsilon \nabla_{\varepsilon} \phi(t, x, \frac{x}{\varepsilon}) \right) dt \right) \]

\[ + \int_0^T \int_{\mathcal{B}_L} v_m^0 \psi(t, x, \frac{x}{\varepsilon}) \cdot nd\sigma dt \]

\[ = - \int_0^T \int_{\Sigma} \int_Z v_m^0 \nabla_y \psi dyd\bar{x}d\bar{t} \]

\[ + \lim_{\varepsilon \to 0} \int_0^T \int_{\mathcal{B}_L} v_m^0 \psi(t, x, \frac{x}{\varepsilon}) \cdot nd\sigma dt \quad (182) \]

\[ = - \int_0^T \int_{\Sigma} \int_Z v_m^0 \nabla_y \psi dyd\bar{x}d\bar{t} \]

\[ + \int_0^T \int_{\Sigma} \int_{Z_L} v_m^0 \psi(t, \bar{x}, y) \cdot nd\sigma d\bar{x}d\bar{t} \]

\[ = \int_0^T \int_{\Sigma} \int_Z \nabla_y v_m^0 \psi dyd\bar{x}d\bar{t} - \int_0^T \int_{\Sigma} \int_{Z_L} v_m^0 \psi(t, \bar{x}, y) \cdot nd\sigma d\bar{x}d\bar{t} \]

\[ + \int_0^T \int_{\Sigma} \int_{Z_L} v_m^0 \psi(t, \bar{x}, y) \cdot nd\sigma d\bar{x}d\bar{t}. \]

So, we obtain

\[ \int_0^T \int_{\Sigma} \int_{Z_L} v_m^0 \psi(t, \bar{x}, y) \cdot nd\sigma d\bar{x}d\bar{t} = \int_0^T \int_{\Sigma} \int_{Z_L} v_m^0 \psi(t, \bar{x}, y) \cdot nd\sigma d\bar{x}d\bar{t}. \quad (183) \]

which is equivalent to (180). Similarly by choosing test function from \( C^\infty ((0, T) \times \Sigma \times C_\#(Z)) \) such that \( \psi(t, x, \cdot) \) has compact support in \( Z_R \cup Z \) gives (181). \( \square \)

**Theorem 9** Assume (A1)-(A7). Then for scaling choice S1, the limit function \((v_1^0, v_m^0, v_r^0)\) given in Theorem 8 and Theorem 9 is the weak solution of the following problem:

\[ v_1^0 \in L^2((0, T); H^1(\Omega_L)) \cap H^1((0, T); L^2(\Omega_L)) \quad (184) \]
\[ v_m^0 \in L^2((0, T) \times \Sigma; H^1_\#(Z)) \cap H^1((0, T) \times \Sigma; L^2_\#(Z)) \quad (185) \]
\[ v_r^0 \in L^2((0, T); H^1(\Omega_R)) \cap H^1((0, T); L^2(\Omega_R)) \quad (186) \]

satisfying

\[ \frac{\partial v_1^0}{\partial t} + \text{div}(-D_L \nabla v_1^0 + B_L P_\delta (v_1^0 - u_b)) = f_{b_1} \quad \text{on} \quad (0, T) \times \Omega_L, \]
\[ \frac{\partial v_r^0}{\partial t} + \text{div}(-D_R \nabla v_r^0 + B_R P_\delta (v_r^0 - u_b)) = f_{b_r} \quad \text{on} \quad (0, T) \times \Omega_R, \quad (187) \]

\[ v_1^0 = 0 \quad \text{on} \quad (0, T) \times \Gamma_L \]
\[ v_r^0 = 0 \quad \text{on} \quad (0, T) \times \Gamma_R \quad (188) \]
satisfying and solves the following cell problem

\[ \begin{align*}
\frac{\partial v^0}{\partial t} + \nabla \cdot (P \nabla v^0) &= n_l = g_{b_l} \text{ on } (\Gamma_h \cap \partial \Omega_L) \times (0,T), \\
\frac{\partial v^0}{\partial t} + \nabla \cdot (R \nabla v^0) &= n_r = g_{b_r} \text{ on } (\Gamma_h \cap \partial \Omega_R) \times (0,T), \\
v^0(0,x) &= h_{b_l}^0 \text{ on } \overline{\Omega_L}, \\
v^0(0,x) &= h_{b_r}^0 \text{ on } \overline{\Omega_R}.
\end{align*} \]

\[ \begin{align*}
(-D_L \nabla v^0_l + B_L P_b(v^0_l - u_b)) \cdot n_l = 0 & \text{ on } \Gamma_h \cap \partial \Omega_L, \\
(-D_R \nabla v^0_r + B_R P_b(v^0_r - u_b)) \cdot n_r = 0 & \text{ on } \Gamma_h \cap \partial \Omega_R.
\end{align*} \]

\[ \begin{align*}
(-D_L \nabla v^0_l + B_L P_b(v^0_l - u_b) + D_R \nabla v^0_r - B_R P_b(v^0_r - u_b)) \cdot n_l \\
= \int_{Z_L} D_M \nabla y v^0_m \cdot n_l + D_L \nabla \bar{u}_b(t, \bar{x}, 0) \cdot n_l \\
- \int_{Z_R} D_M \nabla y v^0_m \cdot n_l + D_R \nabla \bar{u}_b(t, \bar{x}, 0) \cdot n_l \\
on (0,T) \times Z.
\end{align*} \]

and \( v^m_0 \) solves the following cell problem

\[ \begin{align*}
\frac{\partial v^0_m}{\partial t} + \nabla \cdot (D_M \nabla y v^0_m) &= f_{a_0} \text{ on } (0,T) \times \Sigma \times Z, \\
(-D_M \nabla y v^0_m) \cdot n &= 0 \text{ on } (0,T) \times \Sigma \times (\partial Z \setminus (Z_L \cup Z_R)), \\
v^0_l(0,x) &= h_{b_l}^0 \text{ on } \Sigma \times Z.
\end{align*} \]

**Proof:** The proof follows directly from Theorem [1] and Theorem [5].

**Theorem 10** Assume [(A1)] [(A7)] Then for scaling choice S2, the macroscopic equation for \( P_\varepsilon \) problem is:

\[ \begin{align*}
v^0_l \in L^2((0,T); H^1(\Omega_L)) \cap H^1((0,T); L^2(\Omega_L)) \\
v^0_m \in L^2((0,T); \Sigma) \cap H^1((0,T); \Sigma) \\
v^0_r \in L^2((0,T); H^1(\Omega_R)) \cap H^1((0,T); L^2(\Omega_R))
\end{align*} \]

satisfying

\[ \begin{align*}
\frac{\partial v^0_l}{\partial t} + \nabla \cdot (-D_L \nabla v^0_l + B_L P_b(v^0_l - u_b)) &= f_{b_l} \text{ on } (0,T) \times \Omega_L, \\
\frac{\partial v^0_r}{\partial t} + \nabla \cdot (-D_R \nabla v^0_r + B_R P_b(v^0_r - u_b)) &= f_{b_r} \text{ on } (0,T) \times \Omega_R, \\
v^0_l = 0 & \text{ on } (0,T) \times \Gamma_L \\
v^0_r = 0 & \text{ on } (0,T) \times \Gamma_R \\
v^0_l(t, \bar{x}, 0) &= v^0_m(t, \bar{x}) \text{ for a.e } (t, \bar{x}) \in (0,T) \times \Sigma, \\
v^0_r(t, \bar{x}, 0) &= v^0_m(t, \bar{x}) \text{ for a.e } (t, \bar{x}) \in (0,T) \times \Sigma,
\end{align*} \]

\[ \begin{align*}
(-D_L \nabla v^0_l + B_L P_b(v^0_l - u_b)) \cdot n_l = g_{b_l} \text{ on } (\Gamma_h \cap \partial \Omega_L) \times (0,T), \\
(-D_R \nabla v^0_r + B_R P_b(v^0_r - u_b)) \cdot n_r = g_{b_r} \text{ on } (\Gamma_h \cap \partial \Omega_R) \times (0,T),
\end{align*} \]

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Proof: Proof is application of Theorem 5 and Theorem 6 and follows via similar technique of proof of Theorem 9 and convergence results from [7].

5.2 Macroscopic equation for finitely thin layer

To derive macroscopic equation for finitely thin layer we use the following assumption (B1), (B2) and (B3) instead of (A3), (A4) and (A5)

(B1) For the reaction rate, we assume \( f_{b_t}, \partial_t f_{b_t} \in L^2(0, T; L^2(\Omega_L)), \)
\( f_{b_r}, \partial_r f_{b_r} \in L^2(0, T; L^2(\Omega_R)), \)
\( f_{b_m}, \partial_t f_{b_m} \in L^2(0, T; L^2(\Omega_{\delta M})) \)
and
\[ \varepsilon^\alpha \| f_{b_m} \|_{L^2(0, T; L^2(\Omega_{\delta M}))} \leq C, \]
for a.e. \( t \in (0, T). \) Together we assume there exist \( f_{a_0} \in L^2((0, T) \times \Omega_M \times Z) \) such that
\[ f_{a_m}^{\varepsilon} \rightarrow f_{a_0}. \]

(B2) \( g_{b_t}, \partial_t g_{b_t} \in L^\infty(0, T; L^2(\Gamma_h \cap \partial \Omega_L)), \) \( g_{b_r}, \partial_r g_{b_r} \in L^\infty(0, T; L^2(\Gamma_h \cap \partial \Omega_R)), \) \( g_{b_m}, \partial_t g_{b_m} \in L^\infty(0, T; L^2(\Gamma_{\delta M})) \) and
\[ \varepsilon^{\frac{1}{2}} \| g_{b_m} \|^2_{L^2(\Gamma_{\delta M})} \leq C, \]
\[ \varepsilon^{\beta+\frac{1}{2}} \| g_{b_m} \|^2_{L^2(\Gamma_{\delta M})} \leq C, \]
for a.e. \( t \in (0, T). \) Together we assume there exist \( g_0 \in L^2((0, T) \times \Omega_M \times \partial Y_0) \) such that
\[ g_{b_m}^{\varepsilon} \rightarrow g_0. \]

(B3) For initial conditions, we assume \( h_{b_t}^{\varepsilon} \in H^1(\Omega_L), h_{b_r}^{\varepsilon} \in H^1(\Omega_R), h_{b_m}^{\varepsilon} \in H^1(\Omega_{\delta M}) \) with
\[ \| h_{b_t}^{\varepsilon} \|^2_{L^2(\Omega_L)} + \| h_{b_r}^{\varepsilon} \|^2_{L^2(\Omega_R)} + \varepsilon^\alpha \| h_{b_m}^{\varepsilon} \|^2_{L^2(\Omega_{\delta M})} \leq C, \]
and
\[ 1_{\Omega_L} h_{b_t}^{\varepsilon} \rightarrow h_{b_t}^0 \quad \text{on} \quad L^2((0, T) \times \Omega_L), \]
\[ 1_{\Omega_R} h_{b_r}^{\varepsilon} \rightarrow h_{b_r}^0 \quad \text{on} \quad L^2((0, T) \times \Omega_R), \]
\[ h_{b_m}^{\varepsilon} \rightarrow h_{b_m}^0. \]
On assumption (B1), (B2) and (B3) we use two scale convergence definition from [24].

**Theorem 11** Assume (A1), (A2), (A6), (A7) and (B1)-(B3). Then for scaling choice S3, the macroscopic equation for $(P_\varepsilon)$ problem is:

\begin{align}
    v^0_l & \in L^2((0,T);H^1(\Omega_L)) \cap H^1((0,T);L^2(\Omega_L)) \\
    v^0_m & \in L^2((0,T) \times \Omega_M;H^1_{\#}(Z)) \cap H^1((0,T) \times \Omega_M;L^2_{\#}(Z)) \\
    v^0_r & \in L^2((0,T);H^1(\Omega_R)) \cap H^1((0,T);L^2(\Omega_R))
\end{align}

satisfying

\begin{align}
    \frac{\partial v^0_l}{\partial t} + \text{div}(D_L \nabla v^0_l + B_L P_\delta(v^0_l - u_b)) & = f_{bi} \quad \text{on } (0,T) \times \Omega_L, \\
    \frac{\partial v^0_r}{\partial t} + \text{div}(D_R \nabla v^0_r + B_R P_\delta(v^0_r - u_b)) & = f_{br} \quad \text{on } (0,T) \times \Omega_R, \\
    v^0_l & = 0 \quad \text{on } (0,T) \times \Gamma_L \\
    v^0_r & = 0 \quad \text{on } (0,T) \times \Gamma_R \\
    (-D_L \nabla v^0_l + B_L P_\delta(v^0_l - u_b)) \cdot n_l & = g_{bl} \quad \text{on } (\Gamma_h \cap \partial \Omega_L) \times (0,T), \\
    (-D_R \nabla v^0_r + B_R P_\delta(v^0_r - u_b)) \cdot n_r & = g_{br} \quad \text{on } (\Gamma_h \cap \partial \Omega_R) \times (0,T), \\
    \frac{\partial v^0_m}{\partial t} + \text{div}((\lambda_1 D_M \nabla y_2 v^0_m + \lambda_2 B_M P_\delta(v^0_m - u_b)) & = f_{am} \quad \text{on } (0,T) \times \Omega_M \times Z, \\
    (-\lambda_1 D_M \nabla y_2 v^0_m + \lambda_2 B_M P_\delta(v^0_m - u_b)) \cdot n_m & = g_0 \quad \text{on } (0,T) \times \Omega_M \times \partial Y_0. \\
    v^0_l(t,\bar{x},0) & = v^0_m(t,\bar{x},y) \quad \text{for a.e } (t,\bar{x}) \in (0,T) \times B_L \times Z_L \\
    v^0_r(t,\bar{x},0) & = v^0_m(t,\bar{x},y) \quad \text{for a.e } (t,\bar{x}) \in (0,T) \times B_R \times Z_R.
\end{align}

\begin{align}
    (-D_L \nabla v^0_l + B_L P_\delta(v^0_l - u_b)) \cdot n_l & \quad \text{on } (0,T) \times \Omega_L \\
    = \int_{Z_L} (-\lambda_1 D_M \nabla y_2 v^0_m + \lambda_2 B_M P_\delta(v^0_m - u_b)) \cdot n_l dy_2 - D_L \nabla u_b \cdot n_l \\
    (-D_R \nabla v^0_l + B_R P_\delta(v^0_l - u_b)) \cdot n_r & \quad \text{on } (0,T) \times \Omega_R \\
    = \int_{Z_R} (-D_M \nabla y_2 v^0_m + B_M P_\delta(v^0_m - u_b)) \cdot n_r dy_2 - D_R \nabla u_b \cdot n_r \\
    \quad \text{on } (0,T) \times B_R, \\
    v^0_l(0,x) & = h^0_b(x) \quad \text{on } \bar{\Omega_L} \\
    v^0_r(0,x) & = h^0_b(x) \quad \text{on } \bar{\Omega_R} \\
    v^0_m(0,x,y) & = h^0_{bm}(x,y) \quad \text{on } \bar{\Omega_M} \times \bar{Z},
\end{align}

where $\lambda_1 = 1$, and $\lambda_2 = 1$ if $\gamma - \alpha = 1$ and $\lambda_2 = 0$ if $\gamma - \alpha > 1$. 

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Proof: The proof follows from Theorem 2 and the two scale compactness result from [24]. □

Remark 3 The working strategy to obtain macroscopic equation for choice $S_4$ is similar to that used to obtain the macroscopic equation for choice $S_3$. The only difference in macroscopic equations for the choice $S_3$ and choice $S_4$ is the value of $\lambda_1$. We obtain $\lambda_1 = 1$ for the choice $S_3$ while we obtain $\lambda_1 = 0$ for the choice $S_4$.

6 Approximation of non-regularized problem

In this section, we propose a strategy that allows the vanishing of the parameter $\varepsilon$ arising in our regularized nonlinear reaction-diffusion-convection problem, i.e. we replace the nonlinear operator $P_\delta(\cdot)$ by $P(\cdot)$ as defined in [8] and comment on what possibilities are available to handle a fully nonlinear oscillating drift.

Within this section, we refer to the jointly $\varepsilon$- and $\delta$-dependent problem (39) and (40) as problem $(P^{\delta}_\varepsilon)$. Similarly, the $\delta$ independent problem where $P_\delta(\cdot)$ replaced by $P(\cdot)$ in (39) and (40) is referred to as the $(P^{\delta}_0)$ problem. What concerns the macroscopic equation (161) with initial condition (162), we call it the $(P^{\delta}_0)$ problem. Finally, we denote the $\varepsilon$ and $\delta$ independent macroscopic equation as the $(P^0)$ problem. It appears anytime $P_\delta(\cdot)$ is replaced in (161) with (162) by $P(\cdot)$.

The hypothesis on data and parameters needed for the solvability of problems $(P^{\delta}_\varepsilon)$, $(P^0_\varepsilon)$, $(P^{\delta}_0)$, and $(P^0)$ are assumed to hold. In such case, the following approximation results hold:

**Theorem 12** If $(v_1^{\varepsilon,0} , v_m^{\varepsilon,0} , v_r^{\varepsilon,0})$ is the weak solution of $(P^{\delta}_\varepsilon)$ and $(v_1^{\varepsilon,0} , v_m^{\varepsilon,0} , v_r^{\varepsilon,0})$ is the weak solution of $(P^0_\varepsilon)$ problem, then as $\delta \to 0$, $(v_1^{\varepsilon,0} , v_m^{\varepsilon,0} , v_r^{\varepsilon,0})$ weakly in $L^2((0,T);H^1(\Omega^\varepsilon;\Gamma_L)) \times L^2((0,T);H^1(\Omega^\varepsilon;\Gamma_R))$.

**Proof:** To prove this result we rely on the basic working ideas from [34]. The proof follows via a direct application of the convolution property (see Theorem 4.22 from [8]). We take $\delta \to 0$ in $(P^{\delta}_0)$ and apply the property of convolution which is $P_\delta(r) \to P(r)$ in $L^2(\mathbb{R})$ strongly as $\delta \to 0$. See [34] for related arguments. □

**Theorem 13** If $(v_1^{0,0} , v_m^{0,0} , v_r^{0,0})$ is the weak solution of $(P^0_0)$ problem, then as $\delta \to 0$, $(v_1^{0,0} , v_m^{0,0} , v_r^{0,0})$ weakly in $L^2((0,T);H^1(\Omega_L)) \times L^2((0,T);H^1(\Omega_R))$.

**Proof:** The proof follows similar lines as when proving Theorem 12. □

Combining Theorem 12 and Theorem 13, we conclude that the weak solution to $(P^0_\varepsilon)$ can be approximated in terms of the weak solution to $(P^0_0)$. We indicate this fact in flowchart shown in Fig [5].

7 Conclusion

Starting off from a setting involving reaction-diffusion with nonlinear drift crossing a periodically perforated layer, we derived upscaled equations, some of them reduced dimensionally, as well as effective transmission conditions for different choice of scalings in terms of a small heterogeneity parameter called $\varepsilon$ for the diffusion and drift transport terms as well as for the microscopic surface
reaction rates. To pass to the homogenization limit \( \varepsilon \to 0 \), we used both the classical concept of two-scale convergence (see e.g. [30], [2]) as well as the concept of two-scale layer convergence (see [29]), depending on the used parametric scaling. The second type of convergence is able to handle simultaneously periodic homogenization and dimension reduction limits.

A number of distinct limit upscaled model equations have been obtained in this framework. It is worth noting that our list is not exhaustive. Some more cases can be added. However, we believe that these options are potentially the most relevant ones if one has in mind the physical problem. At this moment, we are unable to classify, in the spirit of Occam’s razor, which of these models is best. A robust multiscale numerical approach as well as access to flux measurements for a given flat membrane with controlled regular internal structure are ingredients needed to make such comparisons. This is yet to be done.

We studied here only the 2D case. We did that because the derivation of the original problem has been done for an interacting particle system in 2D. Our convergence results extend to higher dimensions without additional mathematical difficulties. However, we expect that eventual numerical approximations of the proposed problems are harder in 3D compared to 2D. Notice also the fact that our rectangular microstructures can be replaced in theory by any other type of inclusion having Lipschitz boundary and satisfying the restriction \( \partial Y \cap Y_0 = \emptyset \).

We expect that the diagram shown in Figure 5 is commutative. However, more mathematical results still need to be obtained to support such statement. The main issue is that, currently, we do not control in a parameter-independent way the non-regularized nonlinear drift. We believe that the way of working proposed in [25] will turn to be useful to clarify this matter. It is also worth to study the corrector estimates of our problem since it can give an idea about how good our approximation is. We expect that the method proposed e.g. in [39] and in [18] can be used to derive corrector estimates for our problem.

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