Monte Carlo Study of Correlations Near the Ground State of the Triangular Antiferromagnetic Ising Model

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Abstract

We study the spin-spin correlation function in or near the $T = 0$ ground state of the antiferromagnetic Ising model on a triangular lattice. At zero temperature its modulation on the sublattices gives rise to two Bragg peaks in the structure factor, and a known expression for the algebraic decay of correlations enables us to examine the form of the diffusive scattering. We do so by means of a comparison between exact results and data calculated using standard Monte Carlo techniques. At non-zero temperatures the finite correlation length alters this form, and we account for the change by proposing a generalisation of the zero temperature pair correlation function. The size dependence of our simulation data is investigated through a novel finite-size scaling analysis where $t = e^{-2/T}$ is used as the temperature parameter.

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Since Onsager’s evaluation of the free energy for the Ising-model on a rectangular lattice \[1\] the corresponding model on a triangular lattice has served as an afterthought requiring some modification of the analysis \[2\]. The Triangular Antiferromagnetic Ising (TAI) model has also attracted attention in its own right, it being a simple manifestation of a fully frustrated system \[3\]. Very recently, the interest in the TAI model has been revived through its near-perfect experimental realisation in the yavapaiite layered structure of anhydrous alums such as RbFe(SO$_4$)$_2$ \[4\].

Mathematically the isotropic TAI model to be studied in this paper is defined by the Hamiltonian

\[ \mathcal{H} = J \sum_{\langle i, j \rangle} s_i s_j, \tag{1} \]

where \( J > 0 \) is the antiferromagnetic exchange coupling and \( s_i = \pm 1 \) are Ising spins on the triangular lattice depicted in Fig. 1. The label \( \langle i, j \rangle \) indicates a sum over nearest neighbour pairs each pair being counted only once. Frustration in the ground state arises from the inability of the system to simultaneously satisfy the “local packing rule” that three spins on an elementary triangle are pairwise each other’s nearest neighbour, and the “global packing constraint” that, in order to minimise energy, the ground state must have as many antiferromagnetically satisfied nearest neighbour bonds as possible. In other words, it is impossible to orient three spins in a pairwise antiparallel fashion. It can thus be shown \[2\] that the ground state is macroscopically degenerate with a finite entropy per spin \( s_0 = \frac{2}{\pi} k_B \int_0^{\pi/3} \ln(2 \cos \omega) \, d\omega \simeq 0.323 \, k_B \).

In Wannier’s original approach \[2\] the method of Onsager (diagonalisation of the transfer matrix using representation theory on an associated Lie algebra) was modified to include the diagonal interactions of the TAI model. Due to the technical complexity of this method much work was done to achieve a simplification of the algebra involved (see references in \[3\]), the result being the now well-known technique of reducing the problem to chains of interacting fermions \[5\].

At the same time an alternative scheme, known as the combinatorial method, began to
emerge. Using topological theorems Kasteleyn \cite{6,7} evaluated the configurational generating function of the problem covering a lattice with dimers. The success of this approach relies on the fact that the combinatorics can be lumped into a mathematical entity known as the Pfaffian, which is simply the square root of a determinant. The Ising problem is then addressed by counting dimer configurations on a modified lattice, in which every vertex has been substituted by a suitably oriented polygon in order to ensure a correct counting of the states summed over in the Ising partition function \cite{7,8}.

This scheme (as opposed to the algebraic method) is easily augmented to include the evaluation of correlation functions \cite{8} expressed as Toeplitz determinants. Using asymptotic properties for these objects, Stephenson \cite{9} provided the long-distance behaviour of the two-spin correlations along the three main directions for the TAI model with \cite{10} or without \cite{11} anisotropy.

A recent interest \cite{12–18} has been taken in generalising the Hamiltonian \cite{1} to include ferromagnetic next-nearest neighbour interactions as well as anisotropy. Such additional couplings tend to lift the ground state degeneracy, since, dividing the triangular lattice into the usual three sublattices, next-nearest neighbour pairs belong to the same of these sublattices (see Fig. 1). By applying mappings which are constructed to automatically satisfy the antiferromagnetic nearest neighbour constraint, the ground state ensemble can be investigated within the context of solid-on-solid (SOS) \cite{12,13,15} or domain-wall \cite{16,17} models.

In this paper we shall concern ourselves with the nature of the correlations in the low-temperature TAI model. Despite the fact that the abundance of entropy prevents the emergence of a long-range order, even at zero temperature \cite{3}, the existence of some kind of short-range order or pattern formation should be evident from a visual inspection of Fig. 2, and the $T = 0$ state is indeed a critical one as witnessed by the algebraic decay of correlations \cite{11}.

We shall study this ordering by means of the structure factor
\[ S(p) = \sum_{r} e^{i p \cdot r} \langle s_0 s_r \rangle, \quad (2) \]

which is proportional to the cross section for quasi-elastic single scattering \[14\]. When scattering, for example, neutrons from a rare-gas monolayer adsorbed on graphite \[20\] the ordering is directly displayed through the positions and shapes of peaks in the structure factor.

By Fourier transforming the known \( T = 0 \) expression for the pair-correlation function \[11\] we find good agreement with numerical data for the structure factor as calculated from standard Monte Carlo (MC) data. Also, simulations performed at non-zero temperature render unchanged values for the Bragg points but a broadening of the line shape, which makes us suggest a generalisation of the expression for the pair-correlation function to the \( T > 0 \) case. Finally, the rôle of finite-size effects in the MC data is discussed by means of a finite-size scaling analysis where \( t = e^{-2/T} \) is used as the temperature parameter.

Stephenson \[11\] made an asymptotic expansion of the spin-spin correlation function for the \( T = 0 \) TAI model

\[ \langle s_0 s_r \rangle \sim \cos \left( \frac{2\pi r}{3} \right) \frac{1}{\sqrt{r}} \quad \text{for} \ r \gg 1, \quad (3) \]

valid along the three main directions of the lattice. Since the derivation of this result depends heavily on certain asymptotic properties of the Toeplitz determinants involved, a similarly stringent result valid for arbitrary lattice directions could hardly be worked out along these lines. However the symmetry between the tree sublattices suggests that the general expression is formed by replacing the factor \( \cos \left( \frac{2\pi r}{3} \right) \) with a weight factor having the value +1 when \( s_0 \) and \( s_r \) are on the same sublattice, and \(-\frac{1}{2}\) otherwise \[13\].

Recently it has been demonstrated that a staggered field on the sublattices can be mapped onto a period-6 spin wave operator within the context of an equivalent SOS-model \[13\]. By renormalisation the latter is reduced to a Gaussian model thus allowing for a determination of the associated critical exponent as \( X_6^{(s)} = \frac{1}{4} \) which in in nice agreement with the algebraic decay \( \sim r^{-2X_6^{(s)}} \) in Eq. (3). Although this of course corroborates the original as-
umption of a staggered field we are not aware of anyone having formerly verified it through a direct study of the TAI model.

Defining \( q_1, q_2 = (\pm \frac{4\pi}{3}, 0) \) it is easily checked that \( \frac{1}{2} (e^{-i q_1 \cdot r} + e^{-i q_2 \cdot r}) \) is the wanted weight factor, and the structure factor is evaluated as

\[
S(p) \sim \sum_{j=1}^{2} \int d^2 r e^{i (p - q_j) \cdot r} f(r),
\]

where we have introduced the radially symmetric function \( f(r) = \frac{1}{\sqrt{r}} \) for the purpose of later generalisation. It is seen that the rôle of the weight factor is to produce two Bragg peaks at wave vectors \( q_1 \) and \( q_2 \), as shown in Fig. 3. Interestingly, these positions are also predicted by a simple mean field theory [20], pleading no knowledge of such exact results as Eq. (3). Furthermore, the knowledge of an algebraic decay makes it possible to determine the shape of the diffusive scattering.

Since the expression for \( \langle s_0 s_r \rangle \) is only valid for \( r \gg 1 \) we expect our evaluation to work for \( |p - q_j| \ll \pi \) (the extent of the first Brillouin zone). With a reasonably fast \( p \)-space decay of the peaks this restriction will have negligible consequences. The conversion of the sum to an integral will cause no trouble for a sufficiently large system, but when making computer simulations on modestly sized lattices we must be on guard for finite-size effects.

We have performed standard Monte Carlo simulations on \( L \times L \) lattices for different sizes up to \( L = 900 \). Periodic boundary conditions are imposed and we demand \( L \) to be divisible by 6 in order to avoid the introduction of screw dislocations in the ground state [15]. After initialising the lattice paramagnetically we equilibrate it at a temperature \( T \) (measured in units of \( k_B/J \)) by performing a suitable number of MC steps per spin (MCSS).

Monitoring the excess energy per spin at \( T = 0 \) as a function of MCSS (see Fig. 4) we find that the system after a short transient time enters a regime of linear relaxation where it looses most of its excitational energy. Qualitatively this regime corresponds to a ‘trivial’ relaxation during which the energy can be efficiently lowered by updating the spin configuration locally. After a certain cross-over time which, as demonstated on the inset of Fig. 4, is proportional to \( L \) a new regime of roughly algebraic decay is entered.
as the annealing defects generated during the quench are slowly eliminated and long-range correlations are built up. Although the system at this point is energetically very close to the ground state, we must take the length of our runs to be considerably longer than this cross-over time in order to obtain reliable data for the Bragg peaks in the structure factor. To ensure that the MC data for different lattice sizes are of a comparable quality we shall thus take the number of MCSS to be proportional to $L$ and fix MCSS at 10,000 for $L = 900$.

After thermalisation we calculate the structure factor $S(p)$ for the wave vectors in the vicinity of $p = q_1$ which are compatible with translational invariance, reinitialize, and make a new quench. (Since only a small fraction of the simulation time is spent in the regime of linear relaxation, as mentioned above, it would not be advantageous to perform the simulations as sequential heating runs. Instead, the method of repeated thermal quenches employed here guarantees the independence of the different runs.) For each value of $T$ and $L$ we average $S(p)$ over 20 quenches. Interference with the finite system size, however, manifests itself as a kind of noise on the scale of the least allowable wave vector, even after calculating the ensemble average. This ruggedness tends to obscure the shape of the peak, wherefore we dispose of it by doing a mild coarse-graining replacing each value of $S(p)$ with the average of itself and its four nearest neighbours. The result is a smooth peak as shown in Fig. 5.

The position of the Bragg peaks as calculated from their first moment compares favourably with the theoretical $T = 0$ result. For $T = 0.5$ and $L = 900$ we find $q_1 = \pi (1.33340(6), -0.00010(9)) \simeq \left(\frac{4\pi}{3}, 0\right)$ with similar results for other temperatures and lattice sizes.

To avoid a circumstantial notation we focus our attention on one of the two peaks in $S(p)$, picking out the $j = 1$ term of Eq. (4) in the following. Shifting the $p$-space origin to the centre of the peak we have

$$S(p) \sim \int d^2r e^{ip \cdot r} f(r) = 2\pi \int_0^\infty dr r f(r) J_0(pr),$$

where $J_0(x)$ is a spherical Bessel function of order zero.
Consider first the case of $T = 0$ where Stephenson’s result is represented by $f(r) = 1/\sqrt{r}$. The integral can be written in terms of gamma functions \[23\] as $S(p) \sim 2\pi \sqrt{2} p^{-3/2} \Gamma \left( \frac{3}{4} \right) / \Gamma \left( \frac{1}{4} \right)$ or

$$S(p) \sim 3.0033 p^{-(2-\eta)}$$

with $\eta = \frac{1}{2}$.

This prediction is readily verified from our simulation data by computing a circular average of $S(p)$. The result for the $L = 900$ lattice at $T = 0$ is displayed in Fig. 6. As expected we find a power law behaviour $S(p) \sim p^{-(2-\eta_{900})}$, valid in all but the immediate vicinity of the peak. The exponent is $\eta_{900} = 0.45$.

Finite-size effects enter in two different ways. Firstly, the exponent deviates slightly from the theoretical value $\eta = \frac{1}{2}$. Secondly, the impossibility of reproducing exact divergences on a finite lattice means that the power law fit breaks down near the peak centre $p = 0$. This can be interpreted as the effect of a large but finite correlation length as described below.

At finite temperatures $T > 0$ the system is no longer critical and is, as opposed to the zero-temperature case, characterised by a finite correlation length. The Stephenson expression for the decay of correlations in this case \[11, Eq. (1.31)\] can be rewritten as

$$f(r) = \frac{e^{-r/\xi}}{\sqrt{r}}$$

where the correlation length is given by $\xi^{-1} = -\ln \tanh \frac{1}{T}$ thus implying that $\nu = 1$ \[21\]. Actually the oscillatory factor is now proportional to $\cos \left( c(T) \frac{2\pi r}{3} \right)$, but since $c(T) \to 1$ as $T \to 0$ the deviation from $\cos \left( \frac{2\pi r}{3} \right)$ is of no consequence for sufficiently low temperatures \[22\]. Note that Eq. (6) reduces to the $T = 0$ result for $\xi(T) \to \infty$ as it should.

Performing the $r$-integral \[23\] we arrive at

$$S(p) \sim 3.0033 p^{-(2-\eta)} g(p\xi),$$

where the function $g(x)$, shown in Fig. 7, can be expressed as a hypergeometric function \[24\]. Since $g(x) \to 1$ for $x \to \infty$ the consistency with Eq. (8) is evident.
For high values of $\xi$ the expression for $S(p)$ retains its pure power-law form for all wave vectors $p$, since those allowed are bounded from below by $\frac{2\pi}{L}$. On the other hand we have $g(x) \sim x^{2-\eta}$ for $x \to 0$, whence a lower value of $\xi$ makes $S(p)$ tend towards a constant for small wave vectors.

Our data for the circular averages of the diffusive scattering at non-zero temperatures are shown in Fig. 8, and the agreement with the predictions of Eq. (8) is seen to be quite good. The curves shown are parametrised by the values of $\xi(T)$ obtained from the exact expression given above, but as shown in Table 1 these values are within 10% of the optimum parameters found from a non-linear least-square fit. However, at very low temperatures ($T \leq 0.25$) when the deviation from a pure power law decay becomes minute we are unable to reproduce the $T \to 0$ divergence in $\xi(T)$ due to the finite size of our lattices.

We now turn our attention to a scaling analysis of the data obtained for various lattices. According to the standard theory of finite-size scaling [25] the spin-spin correlation function should scale with system size $L$ as $\langle s_0 s_r \rangle = L^{-2\beta/\nu} f \left( \frac{r}{L}, t L^{1/\nu} \right)$, where $t = \frac{T - T_c}{T_c}$ is the reduced temperature. When applying this scaling ansatz to the TAI model two caveats must be taken into account. Firstly, $t$ is not a suitable measure of the dimensionless temperature in a model with $T_c = 0$. Secondly, the critical exponent $\beta$, which is defined in terms of the spontaneous magnetisation below the critical temperature, is a very doubtful parameter indeed, since its domain of definition simply does not exist.

We shall dispose of the first complication by appealing to the Coulomb gas description given in Ref. [13] according to which a temperature induced excitation from the TAI model ground state is equivalent to the formation of a vortex in the associated SOS-model. These vortices correspond to all three spins on an elementary triangle being aligned and accordingly have the Boltzmann weight $e^{-2/\beta}$ when expressed by our dimensionless variables. The second problem is easily eliminated, since an inspection of Kadanoff’s block spin argument [26] reveals that the factor $L^{-2\beta/\nu}$ is only conventional and may be replaced by $L^{-\eta}$ without ever referring to $\beta$.

Finally, we define a sublattice independent correlation function $\Gamma(r, T) = \langle s_0 s_r \rangle (3 \delta_{r, A} -$
where \( i_r \) is the label of the sublattice (we fix the origo so that \( i_0 = A \)), i.e., by eliminating the dependence on the weight factor. The finite-size scaling hypothesis now assumes the form

\[
\Gamma(r, T) = L^{-\eta} f \left( \frac{r}{L}, e^{-2/T} L^{1/\nu} \right) .
\]

(9)

Note that \( \eta \) is the correlation function exponent pertinent to the staggered field. Albeit the scenario of Ref. [13] opens the possibility of three different spin waves, each with its own value of \( \eta \), the period-2 wave would only be present if we were to include an external magnetic field, and the period-3 wave would only be relevant if the coupling constants were sublattice dependent.

At zero temperature we have, from Eq. (3), that \( \Gamma(r, 0) \sim r^{-\eta} \) for \( r \gg 1 \), whence

\[
f(x, 0) \sim x^{-\eta}
\]

(10)

for values of \( x = \frac{r}{L} \) satisfying \( 0 \ll x \ll 1 \). As witnessed by Fig. 3 this asymptotic behaviour of \( f(x, 0) \) is in fact nicely brought out for all values of \( x \) shown on the graph, at least for the small lattices. When increasing the lattice size we first see an unexpected fall-off for large \( x \) and eventually, for \( L = 900 \), even a clear tendency for \( \Gamma \) to ramify in three individual curves corresponding to the three sublattices.

The latter observation could be taken to imply that the largest lattices have been insufficiently thermalised to correctly bring out the \( T = 0 \) correlations for distances above some one hundred lattice constants. To check this hypothesis we have produced some extra runs for the \( L = 300 \) lattice, using a different number of MCSS than in our main series of data. In Fig. 4 we have redisplayed the \( L = 300 \) data as above along with the results obtained when using 10%, 30% and 300% of the nominal number of MCSS. The results clearly corroborate our suspicion, that both the anomalous fall-off and the sublattice upsplitting are indeed effects of an insufficient thermalisation time. Considerations on currently available CPU time, however, prevented us from performing longer runs than the ones reported here.

The dependence of the scaling function \( f(x, y) \) on its second variable can be examined by plotting correlation data for a range of non-zero temperatures and different lattice sizes.
in a series of curves with a fixed value $x = x_0$ [27]. In Fig. 11 we show the plot of $f(x_0, y)$ versus $y = e^{-2/T} L^{1/\nu}$, and it is found that the choice $\nu = 1.0 \pm 0.1$ makes the data collapse on distinct curves parametrised by the different values of $x_0$. As expected from the above discussion on insufficient thermalisation the quality of the $x_0 = \frac{1}{6}$ curve is inferior to that of the other branches of the graph.

Note, that for sufficiently low $y$, i.e., at low temperatures, $f(x_0, y)$ is constant for a fixed value of $\frac{r}{L}$. Thus $\Gamma(r, T) \sim L^{-\eta} \sim r^{-\eta}$, whence the zero temperature algebraic decay $\Gamma(r, 0) \sim r^{-\eta}$ is in fact valid even for small finite values of $e^{-2/T} L^{1/\nu}$.

In summary, we have shown that Stephenson’s expression (Eq. 3) can indeed be applied to all lattice directions in the way suggested by Ref. [13]. Moreover, we have confirmed the validity of the appropriate generalisation to small non-zero temperatures by exhibiting the agreement with our Monte Carlo data. Finally, a finite-size scaling analysis demonstrated that the correct way of approaching the $T = 0$ critical point is through the temperature parameter $t = e^{-2/T}$.

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This form is well known for the ferromagnetic Ising model; see chapter 11 of B. M. McCoy and T. T. Wu, *The Two-Dimensional Ising Model* (Harvard University Press, Cambridge, Massachusetts, 1973). In this context it can be thought of as emerging from the form valid at criticality by multiplication by a factor of the form \((\xi r)^a\) when \(r \gg \xi\). In our case, however, \(a = 0\).

For \(T = 0.6\) which is the highest temperature used in our simulations \(c(T) = 0.99930\) implying that sublattices begin to mix up for \(r \geq 1425\). Since our largest lattices are of size \(L = 900\) the deviation of \(c(T)\) from 1 is indeed negligible.

From [24] we have \(g(x) = \frac{1}{4\sqrt{\pi}} \Gamma^2 \left( \frac{1}{4} \right) \left( 1 + x^{-2} \right)^{-3/4} F \left( \frac{3}{4}, -\frac{1}{4}; 1; \frac{1}{1+x^{-2}} \right)\), where the Gaussian hypergeometric function \(F(\alpha, \beta; \gamma; z)\) is defined by the series \(\sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} z^n\), which is convergent for \(|z| \leq 1\). Since the convergence of \(F\) is extremely slow for the values \(z \simeq 1\) useful at low temperatures, it is advantageous to employ a translation formula for the hypergeometric function [23] yielding \(g(x) = (1 + x^{-2})^{-3/4} \left( F \left( \frac{3}{4}, -\frac{1}{4}; \frac{1}{2}; \frac{1}{1+x^{-2}} \right) + \frac{k}{\sqrt{1+x^2}} F \left( \frac{1}{4}, \frac{5}{4}; \frac{3}{2}; \frac{1}{1+x^2} \right) \right)\), where \(k = 2\Gamma^2 \left( \frac{1}{4} \right) / \Gamma^2 \left( -\frac{1}{4} \right) \simeq 1.094\), whereafter the two new hypergeometric functions can be expanded to low order.

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FIGURES

FIG. 1. The $L \times L$ triangular lattice spanned by $\mathbf{a}_{1,2} = (\pm \frac{1}{2}, \frac{\sqrt{3}}{2})$ can be divided into three interpenetrating sublattices labelled $A$, $B$ and $C$. Sublattice $A$ is spanned by $\mathbf{a}_{1,2}^A = (\pm \frac{3}{2}, \frac{\sqrt{3}}{2})$. When imposing periodic boundary conditions, $L$ must be divisible by six in order to preserve this sublattice structure and avoid the introduction of screw dislocations in the ground state. Note, that next-nearest neighbours belong to the same sublattice, whence a ferromagnetic coupling between them tends to align the spins on each sublattice and thus lift the ground state degeneracy.

FIG. 2. MC simulation of the TAI model on a $90 \times 90$ lattice clearly shows the difference between the appearance of the paramagnetic initial state (left) and the labyrinthine patterns of the locally ordered ground state (right), which is reached after 1000 MC steps per spin. Here, spin up (down) is represented by the presence (absence) of a dot.

FIG. 3. A point in the reciprocal lattice spanned by $\mathbf{b}_{1,2} = 2\pi (\pm 1, \pm \frac{1}{\sqrt{3}})$, with its six nearest neighbours. The inner hexagon is the first Brillouin zone. Variational mean field theory predicts that the Bragg points describing the continuous transition to an ordered phase be located according to the Lifshitz criterion, i.e., that they be points of high symmetry in the first Brillouin zone. Apart from the origo, which characterises a genuine long-range order, the possibilities satisfying this symmetry demand are marked by a dot in the figure. Imposing the additional condition that the Landau free energy be minimal, we are left with the two Bragg peaks at $\mathbf{q}_1$ and $\mathbf{q}_2$. Fourier transformation of the pair-correlation function confirms the mean field scenario.

FIG. 4. The main graph shows the excess energy per spin at $T = 0$, measured relative to the ground state, as a function of Monte Carlo time for a range of different lattice sizes $L = 300, 600, 900, 1200$ and $1500$. After a short transient time the system enters a regime of linear relaxation terminating in a cross-over to an algebraic decay as the ground state is approached. As demonstrated in the inset this cross-over time is proportional to $L$. 
FIG. 5. Close-up of the structure factor in the vicinity of the Bragg point $q_1 = \left( \frac{4\pi}{3}, 0 \right)$ for a system of size $L = 900$ at temperature $T = 0.5$. An average over 20 independent quenches as well as a mild coarse-graining have been performed.

FIG. 6. Circular average showing the line shape of the $T = 0$ Bragg peak simulated on a $L = 900$ lattice. The exponent of the algebraic decay is $-(2 - \eta_{900}) = -1.55$.

FIG. 7. The function $g(x)$ controlling the deviation of the diffusive scattering from a pure power law form. From the asymptotic behaviour we infer the constancy of $S(p)$ at small values of $x = p\xi$ (using $g(x) \sim x^{2-\eta}$ for $x \to 0$) and the unchanged form at large ones (using $g(x) \to 1$ for $x \to \infty$).

FIG. 8. Fits to the shape of the diffusive scattering at temperatures $T = 0.25, 0.35, 0.40, 0.45, 0.50$ and $0.60$ as indicated by the labels. For clarity the graphs have been shifted along the $S(p)$-axis.

FIG. 9. Finite-size scaling at $T = 0$. The graphs for $f(x, 0)$ versus $x$ for different system sizes ($L = 90, 150, 300, 600$ and $900$ as labelled) collapse on a universal curve. The breakdown of scaling at large $x$ is due to an insufficient thermalisation time.

FIG. 10. Replotting the $L = 300$ data for different thermalisation times (the label indicates the percentage of the nominal number of MCSS) we infer that the deviation from the universal scaling function $f(x, 0) \sim x^{-\eta}$ could be ameliorated by making longer runs.

FIG. 11. Finite-size scaling at $T > 0$. The graphs for $f(x_0, y)$ versus $y$ fall on distinct branches according to the value of the parameter $x_0 = r/L$. For clarity the different branches have been shifted along the ordinate.
TABLES

| $T$  | 0.25 | 0.35 | 0.40 | 0.45 | 0.50 | 0.60 |
|------|------|------|------|------|------|------|
| $\xi$ (fit) | 172  | 124  | 82   | 48   | 29   | 15   |
| $\xi$ (exact) | 1490 | 152  | 74   | 43   | 27   | 14   |

TABLE I. Correlation lengths $\xi$ (in units of the lattice constant) for a $L = 900$ lattice as a function of temperature $T$. 

15
Slope $-0.5$
