Correlation length of the two-dimensional Ising spin glass with bimodal interactions

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We study the correlation length of the two-dimensional Edwards-Anderson Ising spin glass with bimodal interactions using a combination of parallel tempering Monte Carlo updates and a rejection-free cluster algorithm in order to speed up equilibration. Our results show that the correlation length grows \( \sim \exp(2J/T) \) suggesting through hyperscaling that the degenerate ground state is separated from the first excited state by an energy gap \( \sim 4J \), as would naively be expected.

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I. INTRODUCTION

Despite intense research\textsuperscript{1,2} spin glasses still pose many unanswered questions. Thus it is of great interest to understand the nature of the spin-glass state for realistic short-range spin glasses. A model that has proven to be a good “workhorse” due to its simplicity and ease of implementation is the Edwards-Anderson Ising spin glass\textsuperscript{3,4} with bimodal interactions. Here we consider this model in two dimensions, for which the spin-glass transition occurs at zero temperature. Nonetheless the low-temperature properties of the model remain controversial: There are different predictions on the energy gap between the ground state and the first excited state, and the critical behavior of the correlation length is not fully understood.

Wang and Swendsen\textsuperscript{5} first suggested an exponential scaling of the specific heat of the two-dimensional bimodal Ising spin glass:

\[
C_V \sim T^{-2} \exp(-\beta J),
\]

where \( \beta = 1/T \) represents the inverse temperature, \( J \) is the mean interaction strength of the (random) bonds, and \( A \) is a numerical prefactor. Although they argue analytically that the energy gap should be \( 4J \), i.e., \( A = 4 \), their final estimate from their numerical work is \( A = 2 \). In addition, they estimated the correlation length exponent \( \nu \) to be finite and positive, implying power-law scaling of the correlation length. Unfortunately, their simulations were done on small system sizes with few disorder realizations, suggesting that their results are governed by corrections to scaling.

Saul and Kardar\textsuperscript{6,7} by means of an exact integer algorithm to estimate the partition function of the two-dimensional \( \pm J \) spin glass, later argued that \( A = 4 \) and, in addition, proposed an exponential scaling form for the correlation length, i.e., \( \xi \sim \exp(n\beta J) \) with \( n = 2 \). The exponential scaling present in the bimodal glass was later verified by Houdayer\textsuperscript{8} by analyzing the scaling properties of the spin-glass susceptibility and the Binder cumulant\textsuperscript{9} for large system sizes using a novel cluster algorithm. Recently, Lukic \textit{et al.}\textsuperscript{10} reconsidered the problem by computing the exact partition function of the system for system sizes up to \( 50 \times 50 \) spins. They conclude that \( A = 2 \). According to hyperscaling, the singular part of the free energy scales as \( \xi^{-2} \) (in two dimensions) and so, if \( \xi \) diverges exponentially, the leading exponential dependence of the specific heat will also be given by \( \xi^{-2} \). Therefore, one expects \( A = 2n \), and in the case of Lukic \textit{et al.} this would mean that \( \xi \sim \exp(n\beta J) \) with \( n = 1 \). For a summary of all predictions of \( A \) and \( n \), the prefactors to \( \beta J \) in the exponential scaling of specific heat and correlation length, respectively, see Table I.

In this work we compute the finite-size correlation length\textsuperscript{11} directly via Monte Carlo simulations for large system sizes and show that \( n \) changes continuously for intermediate system sizes, yet saturates at \( n \approx 2 \) for large enough systems. This suggests via hyperscaling that \( A = 4 \), i.e., the energy gap in the two-dimensional Ising spin glass is \( \sim 4J \).

The paper is structured as follows. In Sec. II we introduce the model and observables and in Sec. III we present our results. Conclusions are summarized in Sec. IV.

II. MODEL AND OBSERVABLES

The Hamiltonian of the two-dimensional Ising spin glass is given by

\[
\mathcal{H} = - \sum_{(i,j)} J_{ij} S_i S_j,
\]

TABLE I: Different estimates for \( n \) and \( A \). From finite-size scaling arguments in two dimensions, we expect that the specific heat scales as \( C_V \sim \xi^{-2} \) where \( \xi \) is the correlation length. Therefore, one expects \( A = 2n \), and in the case of Lukic \textit{et al.} this would mean that \( \xi \sim \exp(n\beta J) \) with \( n = 1 \). For a summary of all predictions of \( A \) and \( n \), the prefactors to \( \beta J \) in the exponential scaling of specific heat and correlation length, respectively, see Table I.

| Reference | \( n \) | \( A \) |
|-----------|-------|-------|
| Wang and Swendsen (Ref. 4) | 2 | 2 |
| Saul and Kardar (Ref. 5) | 2 | 4 |
| Houdayer (Ref. 8) | 2 | |
| Lukic \textit{et al.} (Ref. 10) | 1 | 2 |
| Katzgraber and Lee (this work) | 2 |
where the sum ranges over nearest neighbors on a square lattice with periodic boundary conditions. $S_i$ represent Ising spins taking values $\pm 1$, and the interactions $J_{ij}$ are bimodally distributed, i.e., $J_{ij} \in \pm 1$. For the Monte Carlo simulations we use a combination of single-spin flips, parallel tempering updates, and rejection-free cluster moves\cite{Simons90}, in order to speed up equilibration. To ensure that the system is equilibrated, we perform a logarithmic data binning of all observables (energy, spin-glass susceptibility, and correlation length) and require that the last three bins agree within error bars and are independent of the number of Monte Carlo sweeps $N_{\text{sweep}}$.

The parameters of the simulation are listed in Table II.

As mentioned in Sec. III according to Saul and Kardar, the correlation length of the two-dimensional $\pm J$ spin glass scales as

$$\xi \sim e^{n\beta J},$$

where $\beta = 1/T$ and $n = 2$. In this paper we determine the value of $n$ from Monte Carlo simulations.

In order to compute the correlation length $\xi$, as well as estimate the critical exponent $\eta$, we compute the spin-glass susceptibility

$$\chi_{SG} = N[\langle q^2 \rangle]_{av},$$

where $[\cdots]_{av}$ represents a disorder average and $\langle \cdots \rangle$ a thermal average, and

$$q = \frac{1}{N} \sum_{i=1}^{N} S_i^\alpha S_i^\beta$$

is the Edwards-Anderson spin-glass order parameter. In Eq. (5) $N = L^2$ represents the number of spins and $\alpha$ and $\beta$ represent two replicas of the system with the same disorder. Due to the exponential scaling\footnote{The exponential scaling of the correlation length we expect.} of the correlation length we expect

$$\chi_{SG}(T, L) = L^{2-\eta} \tilde{C} \left[ \beta - \frac{1}{n} \ln L \right].$$

The finite-size correlation length\cite{Kim90,10,15,16,17} $\xi_L$ is given by

$$\xi_L = \frac{1}{2 \sin(k_{min}/2)} \left[ \frac{\chi_{SG}(0)}{\chi_{SG}(k_{min})} - 1 \right]^{1/2},$$

where $k_{min} = (2\pi/L, 0, 0)$ is the smallest nonzero wave vector, and $\chi_{SG}(k)$ is the wave-vector-dependent spin-glass susceptibility:

$$\chi_{SG}(k) = \frac{1}{N} \sum_{i,j}[\langle S_i S_j \rangle]_{av} e^{i\mathbf{k} \cdot (\mathbf{R}_i - \mathbf{R}_j)}. \tag{8}$$

Because $\xi_L/L$ is dimensionless, we expect\cite{10,15,16,17}

$$\xi_L/L = \tilde{X} \left[ \beta - \frac{1}{n} \ln L \right] \quad (T_c = 0).$$

Here $\tilde{X}$ is a scaling function. Note that in Eqs. (3) and (4) we have explicitly left $n$ as a “variable.”

To determine the asymptotic value of $n$, we compute the bulk correlation length $\xi_{\infty}$ at low temperatures using the method of Kim\cite{Kim90} (first introduced in Ref. [18]). The finite-size scaling relation for the correlation length can be written as

$$\frac{\xi_{\infty}}{L} = f \left( \frac{\xi_L}{L} \right),$$

where we determine $f(x)$ by fitting to data in the range $0.500 < T < 1.391$ where we have data for the correlation length in both the bulk and finite-size regimes. Using $f(x)$, we then determine $\xi_{\infty}$ from Eq. (10) using data for $L = 96$ and 128 in the range $0.396 < T < 1.391$.

### III. RESULTS

In Fig. II we show data for the natural logarithm of the finite-size correlation length as a function of $1/T$ for different system sizes. According to Eq. (13), we expect data for $\ln(\xi_L)$ vs $1/T$ to asymptotically approach a slope of $n$. The data presented in Fig. II show good agreement with $n \approx 2$, which in turn agrees with the results of Refs. [3] and [4]. We have also used the extrapolation method by Kim\cite{Kim90} and show data for the bulk correlation length extrapolated from data for the largest system sizes. These data also agree well with the naive scaling with $n = 2$, and not with $n = 1$ (dotted line in Fig. II), as predicted by Lukic et al.\cite{18} from specific heat studies. Corrections to scaling in the two-dimensional bimodal spin glass are strong for system sizes $L$ up to $\sim 64$. Hence the results of
FIG. 1: (Color online) Data for the finite-size correlation length: \( \ln(\xi_L) \) vs \( 1/T \). The slope of the curves in the thermodynamic limit determine \( n \). For small values of \( 1/T \) the data are independent of system size \( L \), but the data “peel off” from this common curve at a value of \( 1/T \) which increases with increasing size. The slope of the common curve asymptotically seems to approach the value \( n = 2 \) (dashed line). This is supported by data for the bulk correlation length (●) which shows an asymptotic slope also compatible with \( \xi \sim \exp(2\beta J) \), i.e., \( n = 2 \). (The dotted diagonal line has slope 1.)

Lukic et al. for \( L \leq 50 \) could be influenced by corrections to scaling that mask the true critical behavior.

In order to test this hypothesis, we show a finite-size scaling plot of the data of the finite-size correlation length according to Eq. (9) with \( n = 2 \) (Fig. 2). We see that the data for \( L \lesssim 64 \) are not in the asymptotic limit, whereas the data for \( L \gtrsim 64 \) scale reasonably well. This shows that in order to obtain precise estimates of \( n \), very large system sizes are required. In addition, the exponential scaling means that, for the two-dimensional bimodal spin glass, the critical exponent for the correlation length is infinite, i.e.,

\[
\nu = \infty. \tag{11}
\]

We have also tried to scale the data according to Eq. (9) using \( n = 1 \) in accordance with the prediction of Lukic et al. As shown in Fig. 3, the data do not scale well for any system size and any temperature suggesting that \( n = 1 \) is not probable.

Data for the spin-glass susceptibility show similar finite-size effects as in the case of the correlation length. In Fig. 4 we show a finite-size scaling plot of the spin-glass susceptibility \( \chi_{SG} \) assuming \( n = 2 \) and \( \eta = 0.138 \).

Note that scaling works “reasonably well” for a large range of \( \eta \) values (not shown) thus illustrating again that the spin-glass susceptibility is not a good observable with which to study critical properties of glassy systems.

At \( T = T_c \) (\( = 0 \)) we expect, according to Eq. (6),

\[
\chi_{SG} \sim L^{2-\eta} \quad (T = T_c). \tag{12}
\]

In order to estimate the critical exponent \( \eta \), we plot \( \ln(L^{-2}\chi_{SG}) \) vs \( \ln(L) \) and extract the slope of the
strong statement regarding lowest temperatures (see Table II), we cannot make a result which is slightly smaller than a recent estimate by Houdayer. These results are in agreement with previous work by Saul and Kardar and Houdayer. However, if one assumes the hyperscaling result $A = 2n$, our results disagree with specific heat studies by Lukic et al. as well as Wang and Swendsen who find $A = 2$. Using the hyperscaling relation, our results also imply that the excitation gap for the bimodal spin glass is $\approx 4J$. Exponential scaling of the correlation length means that the critical exponent for the correlation length is infinite ($\nu = \infty$) for the two-dimensional bimodal spin glass. In addition, we estimate the critical exponent $\eta$ and find that $\eta \approx 0.138(5)$.

Although in this work we were able to present data that can be extrapolated to the bulk regime for a certain temperature range therefore allowing us to draw conclusions in the thermodynamic limit for the scaling of the correlation length, finite-size effects are very strong in this system and so an analysis with yet larger system sizes at lower temperatures is desirable. While it is unlikely, a change in the asymptotic behavior at larger $L$ cannot be ruled out completely.

**IV. CONCLUSIONS**

To conclude, we have shown that the correlation length of the two-dimensional Ising spin glass with bimodal interactions scales exponentially $\sim \exp(n3J)$ with $n = 2$. These results are in agreement with previous work by Saul and Kardar and Houdayer. However, if one assumes the hyperscaling result $A = 2n$, our results disagree with specific heat studies by Lukic et al. as well as Wang and Swendsen who find $A = 2$. Using the hyperscaling relation, our results also imply that the excitation gap for the bimodal spin glass is $\approx 4J$. Exponential scaling of the correlation length means that the critical exponent for the correlation length is infinite ($\nu = \infty$) for the two-dimensional bimodal spin glass. In addition, we estimate the critical exponent $\eta$ and find that $\eta \approx 0.138(5)$.

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