Hopf instantons, Chern-Simons vortices and Heisenberg ferromagnets

P. A. HORVÁTHY *
Laboratoire de Mathématiques et de Physique Théorique
Université de Tours
Parc de Grandmont, F-37200 TOUPS (France)

Abstract. The dimensional reduction of the three-dimensional model (related to Hopf maps) of Adam et al. is shown to be equivalent to (i) either the static, fixed–chirality sector of the non-relativistic spinor-Chern-Simons model in 2+1 dimensions, (ii) or a particular Heisenberg ferromagnet in the plane.

1. Scalar Chern-Simons vortices and Hopf instantons

In the non-relativistic Chern-Simons model of Jackiw and Pi [1], one considers a scalar field $\Phi$ which satisfies a second–order non-linear Schrödinger equation,

$$iD_t\Psi = \frac{D_iD^i}{2m}\Psi - g|\Psi|^2\Psi = 0,$$

while the dynamics of the gauge field is governed by the Chern-Simons field/current identities. When the coupling constant $g$ is minus or plus the inverse of the Chern-Simons coupling constant $\kappa$, static solutions arise by solving instead the self-duality equations,

$$D^\pm \Psi \equiv (D_1 \pm D_2)\Psi = 0, \quad (D_k = \partial_k - iA_k),$$

supplemented with one of the Chern-Simons equations, namely

$$\kappa B \equiv \kappa \epsilon^{ij}\partial_iA^j = -\varrho,$$

where $\varrho = \Phi^\dagger\Phi$ is the particle density. Expressing the gauge potential from (1.2) one finds that the other Chern-Simons equations, $\kappa E^i \equiv -\kappa(\partial_iA^0 + \partial_0A^i) = \epsilon^{ij}J_j$, merely fixes $A_0$. Then, inserting into (1.3) yields the Liouville equation, whose well-known solutions provide us with Chern-Simons vortices which carry electric and magnetic fields. The self-dual solutions represent furthermore the absolute minima of the energy, cf. [1].

In a recent paper, Adams, Muratori and Nash [2] consider instead a massless two-spinor $\Phi = \begin{pmatrix} \Phi_+ \\ \Phi_- \end{pmatrix}$ on ordinary 3-space, coupled to a (euclidean) Chern-Simons field. Their field equations read

$$D_i\sigma_i\Phi = 0,$$

$$\Phi^\dagger\sigma_i\Phi = B_i.$$

Note that this model only contains a (three–dimensional) magnetic but no electric field. These authors also mention that assuming independence of $x_3$ and setting $A_3 = 0$, their model will reduce to the planar self-dual Jackiw-Pi system, (1.2-3). The third component of (1.5) requires in fact

$$|\Phi_+|^2 - |\Phi_-|^2 = B;$$

* e-mail: horvathy@univ-tours.fr
the two other components imply, however, that either $\Phi_+$ or $\Phi_-$ has to vanish. Therefore, the reduced equations read finally one or the other of

\begin{equation}
D_\pm \Phi_\mp = 0, \quad B = \pm |\Phi_\pm|^2, \quad \text{and} \quad \Phi_\mp = 0.
\end{equation}

Fixing up the sign problem by including a Chern-Simons coupling constant $\kappa$, these equations look indeed formally the same as in the self-dual Jackiw-Pi case. They have, however, a slightly different interpretation: they are purely magnetic, while those of Jackiw and Pi have a non-vanishing electric field. Let us underline that the equations (1.7) differ from the second-order field equation (1.1).

2. Spinor vortices

Here we point out that the model of Adam et al. reduces rather more naturally to a particular case of our spinor model in $2 + 1$ dimensions [3]. In this theory, the 4–component Dirac spinor with components $\Phi_\mp, \chi_\mp, \chi_+ \text{ and } \Phi_+$ satisfies the Lévy-Leblond equations [4]

\begin{equation}
\begin{cases}
(\bar{\sigma} \cdot \vec{D}) \Phi + 2m \chi = 0, \\
D_t \Phi + i(\bar{\sigma} \cdot \vec{D}) \chi = 0,
\end{cases}
\end{equation}

where $\Phi$ and $\chi$ are two-component ‘Pauli’ spinors $\Phi = \begin{pmatrix} \Phi_- \\ \Phi_+ \end{pmatrix}$ and $\chi = \begin{pmatrix} \chi_- \\ \chi_+ \end{pmatrix}$. This non-relativistic Dirac-type equation is completed with the Chern-Simons equations

\begin{equation}
B = (-1/\kappa)(|\Phi_+|^2 + |\Phi_-|^2), \\
E_i = (1/\kappa)\epsilon_{ij}J_j, \quad J_j = i(\Phi_+^\dagger \sigma_j \chi - \chi_+^\dagger \sigma_j \Phi).
\end{equation}

In the static and purely magnetic case, $A_t = 0$, and choosing $\chi_+ = \chi_- = 0$, the second equation in (2.2) is identically satisfied, leaving us with the coupled system

\begin{equation}
\begin{cases}
D_+ \Phi_- = 0, \\
D_- \Phi_+ = 0, \\
B = (-1/\kappa)(|\Phi_+|^2 + |\Phi_-|^2).
\end{cases}
\end{equation}

Choosing a fixed chirality, $\Phi_- \equiv 0$ or $\Phi_+ \equiv 0$, yields furthermore either of the two systems

\begin{equation}
\begin{cases}
D_\pm \Phi_\mp = 0, \\
B = (-1/\kappa)|\Phi_\mp|^2,
\end{cases}
\end{equation}

which, for $\kappa = 1$, are precisely (1.7). For both signs, the equations (2.4) reduce to the Liouville equation; regular solutions were obtained for $\Phi_+$ when $\kappa < 0$, and for $\Phi_-$ when $\kappa > 0$. They are again purely magnetic, and carry non-zero spin.

It would be easy keep both terms in (1.7) by allowing a non-vanishing (but still $x_3$-independent) $A_3$. Then one would loose the equations $D_\pm \Phi_\mp = 0$, however. The impossibility to having both components in (2.3) but not in (1.7) comes from the type of reduction performed: while for spinors one eliminates non-relativistic time, (1.7) comes from a spacelike reduction. The difference is also related to the structure of the Lévy-Leblond equation (2.1), which can be obtained by lightlike reduction from a massless Dirac equation in 4-dimensions, while (1.4) comes by spacelike reduction [3].

It is interesting to observe that eliminating $\chi$ in favor of $\Phi$ in the Lévy-Leblond equation (2.3) yields

\begin{equation}
iD_t \Phi = \left[ -\frac{1}{2m} D_i D^i + \frac{1}{2m\kappa} (|\Phi_+|^2 + |\Phi_-|^2) \sigma_3 \right] \Phi.
\end{equation}
For both chiralities, we get hence a second-order equation of the Jackiw-Pi form (1.1), but with opposite signs i.e., with attractive/repulsive coupling.

It is worth noting that minima of the energy correspond to the coupled equations (2.3) and not to (2.4). In fact, the identity

\[ |\bar{D}\Phi|^2 = |D_+\Phi_-|^2 + |D_-\Phi_+|^2 - (1/2m\kappa)|\Phi|^2\Phi^\dagger\sigma_3\Phi + \text{surface terms} \]

shows that the energy of a field configuration,

\[ H = \int \left\{ (1/2m)|\bar{D}\Phi|^2 + (1/2m\kappa)|\Phi|^2\Phi^\dagger\sigma_3\Phi \right\} d^2\vec{x}, \]

is actually

\[ H = \frac{1}{2m} \int d^2\vec{r} \left\{ |D_+\Phi_-|^2 + |D_-\Phi_+|^2 \right\}, \]

which is positive definite, \( H \geq 0 \), provided the currents vanish at infinity. The “Bogomolny” bound is furthermore saturated precisely when (2.3) holds. Its solutions are therefore stable; (2.3) should be considered as the true self-duality condition.

### 3. Heisenberg ferromagnets

The relative minus sign of the component densities in the “provisional” formula (1.6) differs from ours in (2.3), and is rather that in the 2-dimensional Heisenberg model studied by Martina et al. [5]. Here the spin, represented by a unit vector \( \mathbf{S} \), satisfies the Landau-Lifschitz equation

\[ \partial_t \mathbf{S} = \mathbf{S} \times \nabla \mathbf{S}. \]

In the so-called tangent-space representation, \( \mathbf{S} \) is replaced by two complex fields, \( \Psi_+ \) and \( \Psi_- \), each of which satisfies a (second-order) non-linear Schrödinger equation,

\[ iD_t \Psi_{\pm} = -\left[ D_iD^i + 8|\Psi_{\pm}|^2 \right] \Psi_{\pm}, \]

as well as a geometric constraint, \( D_+\Psi_- = D_-\Psi_+ \). The covariant derivatives here refer to a Chern-Simons-type abelian gauge field,

\[ B = -8(|\Psi_+|^2 - |\Psi_-|^2), \]

\[ E_i = 8\epsilon_{ij}J_j, \quad J_i = (\Psi_+^*D_i\Psi_+ - (D_i\Psi_+)^*) - (\Psi_-^* D_i\Psi_- - (D_i\Psi_-)^*). \]

It is now easy to check that in the static and purely magnetic case, these equations can be solved by the first-order coupled system

\[ D_\pm \Psi_{\mp} = 0, \]

\[ B = -8(|\Psi_+|^2 - |\Psi_-|^2). \]

For \( \Psi_+ = 0 \) or \( \Psi_- = 0 \), we get once again the equation of Adams et al.. In the general case, (3.3) leads to an interesting generalization of the Liouville equation : making use of its conformal properties, Martina et al. have shown that it can be transformed into the “sinh-Gordon” form

\[ \Delta \sigma = -\sinh\sigma, \]

where \( \sigma \) is suitably defined from \( \Psi_+ \) and \( \Psi_- \). Although this equation has no finite-energy regular solution defined over the whole plane [6], it admits doubly-periodic solutions i.e. solutions defined in cells with
periodic boundary conditions on the boundary [7]. This generalises the results of Olesen [8] in the scalar case. A similar calculation applied to the general SD equations, (2.3), of our spinor model would yield

\[ \Delta \sigma = -\cosh \sigma, \]

whose (doubly periodic) solutions could be interpreted as non-linear superpositions of the chiral vortices in [DHP].

References

[1] R. Jackiw and S-Y. Pi, Phys. Rev. Lett. 64, 2969 (1990); Phys. Rev. D42, 3500 (1990); For reviews, see, e. g., R. Jackiw and S-Y. Pi, Prog. Theor. Phys. Suppl. 107, 1 (1992) or G. Dunne, Self-Dual Chern-Simons solitons. Springer Lecture Notes in Physics (New Series) 36, (1995).

[2] C. Adam, B. Muratori, and C. Nash, Phys. Lett. B 479, 329 (2000).

[3] C. Duval, P. A. Horváthy and L. Palla, Phys. Rev. D52, 4700 (1995); Ann. Phys. (N.Y.) 249, 265 (1996). The same self-dual equations arise in the relativistic model of Y. M. Cho, J. W. Kim, and D. H. Park, Phys. Rev. D45, 3802 (1992).

[4] J-M. Lévy-Leblond, Comm. Math. Phys. 6, 286 (1967).

[5] L. Martina, O. K. Pashaev, G. Soliani, Phys. Rev. B48, 15787 (1993).

[6] G. Dunne, R. Jackiw, S.-Y. Pi, Trugenberger, Phys. Rev. D43, 1332 (1991).

[7] A. C. Ting, H. H. Chen and Y. C. Lee, Phys. Rev. Lett. 53, 1348 (1984); Physica 26D, 37 (1987).

[8] P. Olesen, Phys. Lett. B265, 361 (1991); ibid. B268, 389 (1991).