ASYMPTOTIC SPREADING IN A DELAYED DISPERAL PREDATOR-PREY SYSTEM WITHOUT COMPARISON PRINCIPLE

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Abstract. This paper deals with the initial value problem of a predator-prey system with dispersal and delay, which does not admit the classical comparison principle. When the initial value has nonempty compact support, the initial value problem formulates that two species synchronously invade the same habitat in population dynamics. By constructing proper auxiliary equations and functions, we confirm the faster invasion speed of two species, which equals to the minimal wave speed of traveling wave solutions in earlier works.

1. Introduction

Spatial propagation dynamics of parabolic type systems has been widely investigated in literature, and two important indices on spatial propagation are minimal wave speed and spreading speed. Here, the minimal wave speed is the threshold on the existence of specific traveling wave solutions and the spreading speed of a nonnegative function \( u(x, t) \) for \( x \in \mathbb{R}, t > 0 \) is defined as follows [1].

**Definition 1.1.** Assume that \( u(x, t) \) is a nonnegative function for \( x \in \mathbb{R}, t > 0 \). Then \( \hat{c} \) is called the spreading speed of \( u(x, t) \) if

(a): \( \lim_{t \to \infty} \sup |x| > (\hat{c} + \epsilon) t \, u(x, t) = 0 \) for any given \( \epsilon > 0 \);

(b): \( \lim_{t \to \infty} \inf |x| < (\hat{c} - \epsilon) t \, u(x, t) > 0 \) for any given \( \epsilon \in (0, \hat{c}) \).

From the viewpoint of mathematical biology, the above speed characterizes the spatial expansion of the individuals [28, 34]. In the past decades, some important results on these two thresholds have been established for monotone semiflows, see [12, 18, 26, 37, 38] and a survey paper by Zhao [43]. When some special cooperative systems are concerned, it has been proven that all components governed by a system have the same spreading speed that is also the minimal wave speed of traveling wave solutions [18, 26, 38]. At the same time, it has been shown that different components may have different spreading speeds in several noncooperative systems [19, 20, 21, 22, 30, 33], and at least the spreading speed of one species equals to the minimal wave speed of traveling wave solutions.

Recently, Li et al. [17] investigated the following nonmonotone system

\[
\begin{align*}
\frac{\partial u_1(x, t)}{\partial t} &= d_1 [J_1 * u_1](x, t) + r_1 u_1(x, t) F_1(u_1, u_2)(x, t), \\
\frac{\partial u_2(x, t)}{\partial t} &= d_2 [J_2 * u_2](x, t) + r_2 u_2(x, t) F_2(u_1, u_2)(x, t),
\end{align*}
\]

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where \( x \in \mathbb{R}, t > 0, (u_1, u_2) \in \mathbb{R}^2, r_1, r_2, d_1, d_2 \) are positive constants, \( F_1, F_2 \) are defined by
\[
\begin{align*}
F_1(u_1, u_2)(x, t) &= 1 - u_1(x, t) - b_1 \int_{-\tau}^{0} u_1(x, t + s) d\eta_1(s) - a_1 \int_{-\tau}^{0} u_2(x, t + s) d\eta_2(s), \\
F_2(u_1, u_2)(x, t) &= 1 - u_2(x, t) - b_2 \int_{-\tau}^{0} u_2(x, t + s) d\eta_2(s) + a_2 \int_{-\tau}^{0} u_1(x, t + s) d\eta_1(s)
\end{align*}
\]
with constants \( b_1 \geq 0, b_2 \geq 0, a_1 \geq 0, a_2 \geq 0, \tau > 0 \) and
\[
\eta_{ij}(s) \text{ is nondecreasing on } [-\tau, 0] \text{ and } \eta_{ij}(0) - \eta_{ij}(-\tau) = 1, \ i, j = 1, 2.
\]
In this system, \( [J_1 * u_1](x, t), [J_2 * u_2](x, t) \) reflecting the spatial dispersal indicate the long distance effect and nonadjacent contact among individuals [4, 13, 14], and are defined by
\[
[J_i * u_i](x, t) = \int_{\mathbb{R}} J_i(x - y)[u_i(y, t) - u_i(x, t)] dy, i = 1, 2,
\]
where \( J_i, i = 1, 2, \) play the role of probability kernel functions about the random walk of individuals and satisfy the following assumptions:

- \((J1)\): \( J_i \) is nonnegative and continuous for each \( i = 1, 2; \)
- \((J2)\): for any \( \lambda \in \mathbb{R}, \int_{\mathbb{R}} J_i(y)e^{\lambda y} dy < \infty, i = 1, 2; \)
- \((J3)\): \( \int_{\mathbb{R}} J_i(y) dy = 1, J_i(y) = J_i(-y), y \in \mathbb{R}, i = 1, 2. \)

Clearly, (1) is a predator-prey system in population dynamics. In Li et al. [17], Yu and Yuan [41], Zhang et al. [42], the authors investigated its traveling wave solutions connecting \((0, 0)\) with the positive steady state, which reflect the process that these two species invade a new habitat from the viewpoint of biology invasion. In particular, Li et al. [17] obtained the minimal wave speed defined by \( c^* = \max\{c_1^*, c_2^*\} \) with
\[
c_i^* = \inf_{\lambda > 0} \frac{d_i \left[ \int_{\mathbb{R}} J_i(y)e^{\lambda y} dy - 1 \right] + r_i}{\lambda}, \ i = 1, 2.
\]
From the viewpoint of initial value problem, let any fixed time be the initial time, the traveling wave solutions in [17, 41, 42] indicate the initial size of habitat of both species is infinite, which contradicts to some natural phenomena because the initial invasion often begins in finite domain. The purpose of this paper is to explore the dynamics when the initial habitats of two invaders are finite and investigate the long time behavior of
\[
\begin{align*}
\frac{\partial u_1(x, t)}{\partial t} &= d_1 [J_1 * u_1](x, t) + r_1 u_1(x, t) F_1(u_1, u_2)(x, t), \\
\frac{\partial u_2(x, t)}{\partial t} &= d_2 [J_2 * u_2](x, t) + r_2 u_2(x, t) F_2(u_1, u_2)(x, t), \\
u_i(x, s) &= \phi_i(x, s), x \in \mathbb{R}, t > 0, s \in [-\tau, 0], i = 1, 2,
\end{align*}
\]
in which \( \phi_i(x, s) \) satisfies

- \((1)\): For \( i = 1, 2, \phi_i(x, s), x \in \mathbb{R}, s \in [-\tau, 0], \) is nonnegative, bounded and continuous such that
\[
\phi_i(x, s) = 0, |x| > L, s \in [-\tau, 0], \phi_i(x, 0) > 0
\]
the mild solution of the above initial value problem is denoted as $C$ generates a positive

since (2) involves delay effect of intraspecific competition if $b_1 + b_2 > 0$, it does not satisfy the comparison principle of classical predator-prey systems or monotone semiflows [40]. Therefore, the spreading speeds cannot be investigated by the abstract results mentioned above. In this paper, we shall estimate the spreading speeds of these two species. By constructing proper auxiliary equations and functions, we confirm that either the predator or the prey invades the new habitat at the rough speed $c^*$ while the spreading speed of the other species may be smaller than $c^*$.

2. Main results

In this section, we shall give and prove the main results on (2). Before giving the main results, we first define some positive constants as follows

$$c_1 = \inf_{\lambda > 0} \frac{d_1 \left[ \int_{\mathbb{R}} J_1(y)e^{\lambda y}dy - 1 \right] + r_1(1 - a_1(1 + a_2))}{\lambda},$$

$$c_2 = \inf_{\lambda > 0} \frac{d_2 \left[ \int_{\mathbb{R}} J_2(y)e^{\lambda y}dy - 1 \right] + r_2(1 + a_2)}{\lambda},$$

$$c_2^* = \frac{d_2 \left[ \int_{\mathbb{R}} J_2(y)e^{\lambda y}dy - 1 \right] + r_2}{\lambda_2},$$

in which the existence and uniqueness of $\lambda_2 > 0$ is due to (J2)-(J3) and the convex of $d_2 \left[ \int_{\mathbb{R}} J_2(y)e^{\lambda y}dy - 1 \right] - c\lambda + r_2$ in $\lambda > 0$ for every $c > 0$. Using these constants, we present the following conclusion.

**Theorem 2.1.** Assume that the mild solution $(u_1(x, t), u_2(x, t))$ is defined by (2). Then it satisfies

$$(0, 0) \leq (u_1(x, t), u_2(x, t)) \leq (1, 1 + a_2), x \in \mathbb{R}, t > 0.$$

Moreover, $(u_1(x, t), u_2(x, t))$ satisfies the following properties.

1. If $c_1 > c_2$ is true, then $c^*$ is the spreading speed of $u_1(x, t)$ while the spreading speed of $u_2(x, t)$ is not larger than $c_2$.  

2. Further suppose that

$$d_1 \left[ \int_{\mathbb{R}} J_1(y)e^{\lambda y}dy - 1 \right] - c^*_2 \lambda_2 + r_1 \leq 0.$$

If $c_1^* \leq c_2^*$ is true, then $c^*$ is the spreading speed of $u_2(x, t)$ while the spreading speed of $u_1(x, t)$ is not larger than $c_1^*$.

We now prove the above theorem by several lemmas. Let $X$ be the Banach space of uniformly continuous and bounded functions equipped with supremum norm. For each $i \in \{1, 2\}$, we see that $d_i[J_i * v](x) : X \to X$ is a bounded linear operator by (J1), so $d_i[J_i * v](x) : X \to X$ generates a positive $C_0$ semigroup $T_i(t) : X \to X, t \geq 0$ (see [39, Lemma 3.1]), and the mild solution of the above initial value problem is denoted as $u_i(x, t) = T_i(t)u_i(x, 0) = T_i(t - s)u_i(x, s)$
for any $0 \leq s < t < \infty$. Moreover, for any given kernel function $J$ satisfying (J1)-(J3), it also generates a positive $C_0$ semigroup $T(t): X \to X, t \geq 0$. Consider the following initial value problem

$$
\begin{align*}
\frac{\partial u(x,t)}{\partial t} &= d[J \ast u](x,t) + u(x,t) [r - u(x,t)] , \\
u(x,0) &= \chi(x), x \in \mathbb{R},
\end{align*}
$$

where $J$ satisfies (J1)-(J3), $d > 0$ and $r > 0$ are constants. Also define

$$c' = \inf_{\lambda > 0} \frac{d \left[ \int_{\mathbb{R}} J(y)e^{\lambda y} dy - 1 \right] + r}{\lambda}.
$$

By Jin and Zhao [15], we have the following conclusion.

**Lemma 2.2.** Assume that $0 \leq \chi(x) \leq 1$. Then (5) admits a solution $u(\cdot, t) \in X$ for all $t > 0$, which also satisfies

$$u(x,t) = T(t-s)u(x,s) + \int_s^t T(t-\theta)[u(x,\theta)[r - u(x,\theta)]]d\theta$$

for $x \in \mathbb{R}, 0 \leq s < t < \infty$. If $w(\cdot, t) \in X, t \geq 0$ is nonnegative and bounded such that

$$\begin{align*}
\frac{\partial w(x,t)}{\partial t} &\geq (\leq)d[J \ast w](x,t) + w(x,t) [r - w(x,t)] , x \in \mathbb{R}, t > 0, \\
w(x,0) &\geq (\leq)\chi(x), x \in \mathbb{R},
\end{align*}
$$

or

$$w(x,t) \geq (\leq)T(t-s)w(x,s) + \int_s^t T(t-\theta)[w(x,\theta)[r - w(x,\theta)]]d\theta$$

for $x \in \mathbb{R}, 0 \leq s < t < \infty$, then

$$w(x,t) \geq (\leq)u(x,t), x \in \mathbb{R}, t > 0.$$

If $\chi(x)$ has nonempty support, then for any $c < c'$, we have

$$\liminf_{t \to \infty} \inf_{|x| < ct} u(x,t) = \limsup_{t \to \infty} \sup_{|x| < ct} u(x,t) = r.$$

If $\chi(x)$ has compact support, then

$$\lim_{t \to \infty} \sup_{|x| > ct} u(x,t) = 0, c > c'.$$

**Remark 1.** By the positivity of the semigroup, if

$$\begin{align*}
\frac{\partial w_i(x,t)}{\partial t} &\geq d[J \ast w_i](x,t) + w_i(x,t) [r - w_i(x,t)] , x \in \mathbb{R}, t > 0, \\
w_i(x,0) &\geq \chi(x), x \in \mathbb{R}
\end{align*}
$$

for $i \in \{1, 2, 3\}$, then

$$\min\{w_1(x,t), w_2(x,t), w_3(x,t)\} \geq u(x,t), x \in \mathbb{R}, t > 0.$$

On the existence of mild solution of (1), we have the following result.

**Lemma 2.3.** The positive mild solution $(u_1(\cdot,t), u_2(\cdot,t)) \in X^2$ exists for all $t > 0$ and satisfies (3).
Proof. The local existence is evident by the theory of abstract functional differential equations [27], here the mild solution is defined by

\[ u_1(x, t) = T_1(t - \theta)u_1(x, \theta) + \int_0^t T_1(t - s)[r_1u_1(x, s)F_1(u_1, u_2)(x, s)]ds, \]

\[ u_2(x, t) = T_2(t - \theta)u_2(x, \theta) + \int_0^t T_2(t - s)[r_2u_2(x, s)F_2(u_1, u_2)(x, s)]ds \]

for \( 0 \leq \theta < t < T \) with some \( T \in (0, \infty) \). If \( T = \infty \), then the global existence is true.

Further by the quasipositivity in \( u_1 F_1, u_2 F_2 \), we see the mild solution is nonnegative. If \( u_1(x, t) \) only exists for \( t \in [0, T) \) with bounded \( T \) such that

\[ \lim_{t \to T} \sup_{x \in \mathbb{R}} u_1(x, t) = \infty, \]

then

\[ u_1(x, t) \leq T_1(t - \theta)u_1(x, \theta) + \int_0^t T_1(t - s)[r_1u_1(x, s)[1 - u_1(x, s)]]ds \]

for \( 0 \leq \theta < t < T \), and the comparison principle (Lemma 2.2) implies

\[ 0 \leq u_1(x, t) \leq 1, x \in \mathbb{R}, t \in [0, T). \]

A contradiction occurs. The proof is complete by similar discussion on \( u_2(x, t) \).

To continue the discussion, we investigate the following scalar equation

\[ \begin{cases} \frac{\partial v(x, t)}{\partial t} = d[J \ast v](x, t) + rv(x, t) \left[ 1 - v(x, t) - b \int_{-\tau}^0 v(x, t + s)d\eta(s) \right], \\ v(x, s) = \nu(x, s), \end{cases} \tag{7} \]

where \( x \in \mathbb{R}, t > 0, s \in [-\tau, 0], J \) satisfies (J1)-(J3), \( d > 0, r > 0, b \geq 0, \eta(s) \) is nondecreasing on \([-\tau, 0]\) such that \( \eta(0) - \eta(-\tau) = 1 \).

Furthermore, \( \nu(x, s) \geq 0 \) is uniformly continuous and bounded. Evidently, the global existence of mild solution of (7) is true by Lemma 2.3.

Lemma 2.4. Assume that \( v(x, t) \) is the mild solution defined by (7). If \( \nu(x, 0) \) admits nonempty compact support such that \( 0 \leq \nu(x, 0) \leq 1, x \in \mathbb{R} \), then its spreading speed is \( c' \) defined by (6).

Proof. We now prove it by the idea in Liu and Pan [25]. If

\[ b \int_{-\tau}^0 v(x, t + s)d\eta(s) = bv(x, t), \]

then the result is clear by Lemma 2.2. Otherwise, the positivity implies that

\[ v(x, t) \leq T(t - s)v(x, s) + \int_s^t T(t - \theta)[rv(x, \theta)[1 - v(x, \theta)]]d\theta \]

for any \( 0 \leq s < t < \infty \), then Lemma 2.2 implies \( v(x, t) \leq 1 \) and

\[ \lim_{t \to \infty} \sup_{|x| > ct} v(x, t) = 0, c > c'. \]

For any fixed \( c < c' \), it suffices to prove that

\[ \lim_{t \to \infty} \inf_{|x| < ct} v(x, t) > 0. \]
For the purpose, we select \( \epsilon > 0 \) such that
\[
c < \inf_{\lambda > 0} \frac{d}{\lambda} [\int_{\mathbb{R}} J(y)e^{\lambda y}dy - 1] + r(1 - 4\epsilon),
\]
and \( \tau' \in (0, \tau) \) such that
\[
b \int_{-\tau'}^{0} d\eta(s) < \epsilon.
\]
If \( b \int_{-\tau'}^{-\tau} v(x, t + s)d\eta(s) \leq 2\epsilon \), then
\[
rv(x, t) \left[ 1 - v(x, t) - b \int_{-\tau}^{0} v(x, t + s)d\eta(s) \right] \geq rv(x, t) \left[ 1 - 3\epsilon - v(x, t) \right].
\]
When \( b \int_{-\tau'}^{-\tau} v(x, t + s)d\eta(s) > 2\epsilon \), there exists \( s_0 \in [-\tau, -\tau'] \) such that
\[
v(x, s_0) \geq \frac{2\epsilon}{b\tau},
\]
and the uniform continuity implies
\[
v(y, s_0) \geq c, |x - y| \leq \sigma
\]
for some \( \sigma > 0 \). Consider the initial value problem
\[
\begin{cases}
\frac{\partial v(x, t)}{\partial t} = d [J * v](x, t) + rv(x, t) \left[ 1 - b - v(x, t) \right], \\
v(x, 0) = \varphi(x),
\end{cases}
\]
where \( \varphi(x) \) is a continuous function satisfying
\begin{enumerate}
\item \( \varphi(x) = \frac{c}{\rho}, |x| \leq \sigma/2; \)
\item \( \varphi(x) = 0, |x| \geq \sigma; \)
\item \( \varphi(x) \) is even and decreasing in \( x \in [\sigma/2, \sigma] \).
\end{enumerate}
By the positivity of \( T(t) \) and the property of continuous functions, we see that \( \varphi(0, t) \) is positive in \( t > 0 \), and there exists \( \rho > 0 \) such that \( \varphi(0, t) > \rho, t \in [\tau', \tau] \)
and so
\[
b \int_{-\tau}^{0} v(x, t + s)d\eta(s) \leq b = \frac{b\rho}{\rho} \leq \frac{b}{\rho}v(x, t).
\]
That is, \( v(x, t) \) satisfies
\[
v(x, t) \geq T(t - s)v(x, s) + \int_{s}^{t} T(t - \theta)rv(x, \theta)[1 - 3\epsilon - (1 + b/\rho)v(x, \theta)]d\theta
\]
for all \( 0 \leq s < t < \infty \). The proof is complete by Lemma 2.2.

**Lemma 2.5.** If \( c_1 > c_2 \) is true, then \( c^* \) is the spreading speed of \( u_1(x, t) \) while the spreading speed of \( u_2(x, t) \) is not larger than \( c_2 \).

**Proof.** By (3), \( u_2 \) satisfies
\[
u_2(x, t) \leq T_2(t - \theta)u_2(x, \theta) + \int_{\theta}^{t} T_2(t - s) [r_2u_2(x, s)[1 + a_2 - u_2(x, s)]] ds
\]
for \( 0 \leq \theta < t < \infty \), so Lemma 2.2 implies that the spreading speed of \( u_2(x, t) \) is not larger than \( c_2 \), which also leads to
\[
\limsup_{t \to \infty} \sup_{3|x| > (2c_2 + c_1)T} u_2(x, t) = 0.
\]
Again by (3), we see that
\[
\begin{aligned}
u_1(x, t) &\geq T_1(t - \theta)u_1(x, \theta) + \int_{\theta}^{t} T_1(t - s) \left[ r_1 u_1(x, s) \times [1 - a_1(1 + a_2) - u_1(x, s) - b_1 \int_{-\tau}^{0} u_1(x, s + \gamma) d\eta_1(\gamma)] \right] ds \\
&\text{for all } 0 \leq \theta < t < \infty, \text{ and Lemma 2.4 or its proof implies}
\end{aligned}
\]
\[
\liminf_{t \to \infty} \inf_{|x| < c} u_1(x, t) > 0.
\]

We now verify that \( c^* \) is the spreading speed of \( u_1(x, t) \). Because of Lemma 2.2
\[
u_1(x, t) \leq T_1(t - \theta)u_1(x, \theta) + \int_{\theta}^{t} T_1(t - s) \left[ r_1 u_1(x, s)[1 - u_1(x, s)] \right] ds
\]
for all \( 0 \leq \theta < t < \infty \), it suffices to confirm that
\[
\liminf_{t \to \infty} \inf_{|x| < c} u_1(x, t) > 0
\]
for any given \( c < c^* \). We now fix \( 3c > c_2 + 2c_1 \) and \( \epsilon > 0 \) such that
\[
c < \inf_{\lambda > 0} d \left[ \int_{\mathbb{R}} J(y) e^{\lambda y} dy - 1 \right] + r(1 - 2\epsilon).
\]
By (9) and (10), there exists \( T > 0 \) such that
\[
\sup_{2|x| > (c_2 + c_1) t} u_2(x, t) < \epsilon, t \geq T
\]
and
\[
\inf_{t > T, 2|x| \leq (c_2 + c_1) t} u_1(x, t) > 0,
\]
which implies that there exists \( M > 0 \) depending on \( \epsilon \) such that
\[
u_1(x, t) \geq T_1(t - \theta)u_1(x, \theta) + \int_{\theta}^{t} T_1(t - s) \left[ r_1 u_1(x, s)(1 - \epsilon - Mu_1(x, s) - b_1 \int_{-\tau}^{0} u_1(x, s + \gamma) d\eta_1(\gamma)] \right] ds
\]
for any \( T \leq \theta < t \). Clearly, the spreading speed is not less than that of
\[
\frac{\partial v(x, t)}{\partial t} = d[J * v](x, t) + rv(x, t) \left[ 1 - \epsilon - Mv(x, t) - b \int_{-\tau}^{0} v(x, t + s) d\eta(s) \right]
\]
by repeating the proof of Lemma 2.4. The proof is complete. \( \Box \)

Lemma 2.6. Assume that (4) is true. If \( c_1^* \leq c_2^* \) is true, then \( c^* \) is the spreading speed of \( u_2(x, t) \) while the spreading speed of \( u_1(x, t) \) is not larger than \( c_1^* \).

Proof. By the positivity of (3), \( u_1(x, t) \) satisfies
\[
u_1(x, t) \leq T_1(t - \theta)u_1(x, \theta) + \int_{\theta}^{t} T_1(t - s) \left[ r_1 u_1(x, s)[1 - u_1(x, s)] \right] ds
\]
for any \( 0 \leq s < t \), the spreading speed of \( u_1(x, t) \) is not larger than \( c_1^* \) by Lemma 2.2.
On the other hand, \( u_2(x, t) \) satisfies
\[
    u_2(x, t) \geq T_2(t - \theta)u_2(x, \theta) + \\
    \int_0^t T_2(t - s) \left[ r_2u_2(x, s) \left( 1 - u_2(x, s) - b_2 \int_{-\tau}^0 u_2(x, s + \gamma)d\eta_{22}(\gamma) \right) \right] ds
\]
for any \( 0 \leq s < t \), and Lemma 2.4 implies that the spreading speed of \( u_2(x, t) \) is not less than \( c^* \). Now, we shall prove that
\[
    \limsup_{t \to \infty} \sup_{|x| > ct} u_2(x, t) = 0
\]
for any fixed \( c > c^* \). By the positivity, we obtain
\[
    u_2(x, t) \leq T_2(t - \theta)u_2(x, \theta) + \\
    \int_0^t T_2(t - s) \left[ r_2u_2(x, s) \left( 1 - u_2(x, s) + a_2 \int_{-\tau}^0 u_1(x, s + \gamma)d\eta_{21}(\gamma) \right) \right] ds,
\]
and the result is true if the spreading speed of the following equation is \( c^* \)
\[
    \frac{\partial u_2(x, t)}{\partial t} = d_2[J_2 * u_2](x, t) \\
    + r_2u_2(x, t) \left[ 1 - u_2(x, t) + a_2 \int_{-\tau}^0 \pi_1(x, s)d\eta_{21}(s) \right],
\]
where \( u_2(x, 0) = u_2(x, 0), \pi_1(x, s) = \phi_1(x, s), s \leq 0, \) and \( \pi_1(x, t), t > 0 \) is defined by
\[
    \pi_1(x, t) = T_1(t)\pi_1(x, 0) + \int_0^t T_1(t - s)[r_1\pi_1(x, s)[1 - \pi_1(x, s)]]ds.
\]
The main reason why the above claim is true is that the above equation (12) is monotone and admits comparison principle.

By Remark 1 and (4), we see that there exists \( T_1 > 0 \) such that
\[
    \pi_1(x, t) \leq \min \left\{ e^{\lambda_2(x + c_2t + T_1)}, e^{\lambda_2(-x + c_2t + T_1)}, 1 \right\}, x \in \mathbb{R}, t \geq 0.
\]
In fact, let \( \tilde{u}_1(x, t) = e^{\lambda_2(x + c_2t + T_1)} \) or \( e^{\lambda_2(-x + c_2t + T_1)} \), then
\[
    \frac{\partial \tilde{u}_1(x, t)}{\partial t} \geq d_1[J_1 * \tilde{u}_1](x, t) + r_1\tilde{u}_1(x, t),
\]
and
\[
    \frac{\partial \tilde{u}_1(x, t)}{\partial t} \geq d_1[J_1 * \tilde{u}_1](x, t) + r_1\tilde{u}_1(x, t)[1 - \tilde{u}_1(x, t)]
\]
if \( \tilde{u}_1(x, t) = 1 \). Further select \( T_2 \geq T_1 \) such that
\[
    e^{\lambda_2(T_2 - T_1)} > a_2
\]
and
\[
    \min \left\{ e^{\lambda_2(x + T_2 - c_2\tau)}, e^{\lambda_2(-x + T_2 - c_2\tau)}, 1 + a_2 \right\} \geq \sup_{s \in [-\tau, 0]} \phi_2(x, s), x \in \mathbb{R}.
\]
Then Remark 1 implies that
\[
    u_2(x, t) \leq \min \left\{ e^{\lambda_2(x + c_2t + T_2)}, e^{\lambda_2(-x + c_2t + T_2)}, 1 + a_2 \right\}, x \in \mathbb{R}, t \geq 0
\]
because \( \tilde{u}_2(x, t) = e^{\lambda_2(x + c_2t + T_2)} \) \( (e^{\lambda_2(-x + c_2t + T_2)}, 1 + a_2) \) implies
\[
    \frac{\partial \tilde{u}_2(x, t)}{\partial t} \geq d_2[J_2 * \tilde{u}_2](x, t) + r_2\tilde{u}_2(x, t) \left[ 1 - \tilde{u}_2(x, t) + a_2 \int_{-\tau}^0 \pi_1(x, s)d\eta_{21}(s) \right].
\]
3. Discussion

The propagation dynamics of predator-prey systems has important ecology background, one typical case is the evolution of insect herbivores and lupins on Mount St Helens, see [11, 29]. Another related topic is the asymptotic spreading in epidemic models because of the similar monotone conditions. In literature, much attention has been paid to the traveling wave solutions since the work of Dunbar [10]. However, there are a few results on asymptotic spreading of predator-prey systems, see part results by Ducrot [7, 8], Ducrot et al. [9], Lin [19], Lin et al. [23], Pan [30, 32], Wang and Zhang [36].

The model in this paper admits nonlocal dispersal and time delays, the mechanism has significant biological reasons and other backgrounds [4, 5, 13, 14, 28], and the monotone case has been widely studied, see some recent works by [2, 3, 6]. Similar to that in [16, 24], the model in this paper is not a monotone system [35] and time delay is not small enough. When the diffusion is of the classical Ficker type in (1), Pan [31] studied its minimal wave speed of traveling wave solutions connecting trivial steady state with the positive one.

In this paper, we show that $c^*$ may be the spreading speed of $u_1$ or $u_2$, which is the minimal wave speed in [17]. It is possible to study the asymptotic spreading of the model in [31] by our idea. Both thresholds in this paper and [17] formulate that two species invade a new habitat. From our results, we see that the predator and the prey may have different spreading speeds, but it is difficult to estimate these spreading speeds. To answer these questions, more properties on the nonlocal operator and delayed systems are necessary. We shall further investigate these questions in our future research.

References

[1] D. G. Aronson and H. F. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation, In Partial Differential Equations and Related Topics, J.A. Goldstein Eds., Lecture Notes in Mathematics, Vol. 446. Springer, Berlin, German, (1975), 5–49. MR 0427837
[2] X. Bao and W.-T. Li, Propagation phenomena for partially degenerate nonlocal dispersal models in time and space periodic habitats, Nonlinear Anal. Real World Appl., 51 (2020), 102975, 26 pp. MR 3985608
[3] X. Bao, W.-T. Li, W. Shen and Z.-C. Wang, Spreading speeds and linear determinacy of time dependent diffusive cooperative/competitive systems, J. Differential Equations, 265 (2018), 3048–3091. MR 3812221
[4] P. W. Bates, On some nonlocal evolution equations arising in materials science, In: Nonlinear Dynamics and Evolution Equations (Ed. by H. Brunner, X.Q. Zhao, X. Zou), Fields Inst. Commun., 48 (2006), 13–52, AMS, Providence. MR 2223347
[5] J. Coville and L. Dupaigne, On a non-local equation arising in population dynamics, Proc. Roy. Soc. Edinburgh Sect. A, 137 (2007), 725–755. MR 2345778
[6] W. Ding and X. Liang, Principal eigenvalues of generalized convolution operators on the circle and spreading speeds of noncompact evolution systems in periodic media, SIAM J. Math. Anal., 47 (2015), 855–896. MR 3313826
[7] A. Ducrot, Convergence to generalized transition waves for some Holling-Tanner prey-predator reaction-diffusion system, J. Math. Pures Appl., 100 (2013), 1–15. MR 3057299
[8] A. Ducrot, Spatial propagation for a two component reaction-diffusion system arising in population dynamics, J. Differ. Equations, 260 (2016), 8316–8357. MR 3482685
[9] A. Ducrot, J. S. Guo, G. Lin and S. Pan, The spreading speed and the minimal wave speed of a predator-prey system with nonlocal dispersal, Z. Angew. Math. Phys., 70 (2019), Art. 146, 25 pp. MR 3999344
[10] S. R. Dunbar, *Travelling wave solutions of diffusive Lotka-Volterra equations*, *J. Math. Biol.*, 17 (1983), 11–32. MR 707221

[11] W. F. Fagan and J. G. Bishop, *Trophic interactions during primary succession: Herbivores slow a plant reinvasion at Mount St. Helens*, *Amer. Nat.*, 155 (2000), 238–251.

[12] J. Fang and X. Q. Zhao, *Traveling waves for monotone semiflows with weak compactness*, *SIAM J. Math. Anal.*, 46 (2014), 3678–3704. MR 3274366

[13] P. Fife, *Some nonclassical trends in parabolic and parabolic-like evolutions*, In: *Trends in Nonlinear Analysis* (Ed. by M. Kirkilionis, S. Krömker, R. Rannacher, F. Tomi), 153–191, Springer: Berlin, 2003. MR 1999098

[14] L. Hopf, *Introduction to Differential Equations of Physics*, Dover: New York, 1948. MR 0025035

[15] Y. Jin and X. Q. Zhao, *Spatial dynamics of a periodic population model with dispersal*, *Nonlinearity*, 22 (2009), 1167–1189. MR 2501040

[16] X. Li and S. Pan, *Traveling wave solutions of a delayed cooperative system*, *Mathematics*, 7 (2019), ID: 269.

[17] X. Li, S. Pan and H. B. Shi, *Minimal wave speed in a dispersal predator-prey system with delays*, *Boundary Value Problems*, 2018 (2018), Paper No. 49, 26 pp. MR 3782680

[18] X. Liang and X. Q. Zhao, *Asymptotic speeds of spread and traveling waves for monotone semiflows with applications*, *Comm. Pure Appl. Math.*, 60 (2007), 1–40. MR 2270161

[19] G. Lin, *Spreading speeds of a Lotka-Volterra predator-prey system: the role of the predator*, *Nonlinear Analysis*, 74 (2011), 2448–2461. MR 2776497

[20] G. Lin, *Asymptotic spreading fastened by inter-specific coupled nonlinearities: A cooperative system*, *Physica D*, 241 (2012), 705–710. MR 2881108

[21] G. Lin and W. T. Li, *Asymptotic spreading of competition diffusion systems: The role of interspecific competitions*, *Eur. J. Appl. Math.*, 23 (2012), 669–689. MR 2994495

[22] G. Lin, W. T. Li and S. Ruan, *Spreading speeds and traveling waves of a competitive recursion*, *J. Math. Biol.*, 62 (2011), 165–201. MR 2765597

[23] G. Lin, S. Pan and X. P. Yan, *Spreading speeds of epidemic models with nonlocal delays*, *Math. Biosci. Eng.*, 16 (2019), 7562–7588.

[24] X. L. Liu and S. Pan, *Traveling wave solutions for delayed reaction-diffusion systems and applications to Lotka-Volterra competition-diffusion models with distributed delays*, *J. Dyn. Differ. Equ.*, 26 (2014), 583–605. MR 3274433

[25] C. Lin and S. Pan, *Spreading speed in a nonmonotone equation with dispersal and delay*, *Mathematics*, 7 (2019), 291.

[26] M. R. Owen and M. A. Lewis, *How predation can slow, stop or reverse a prey invasion*, *Bull. Math. Biol.*, 63 (2001), 655–684. MR 3363426

[27] S. Pan, *Convergence and traveling wave solutions for a predator-prey system with distributed delays*, *Mediterr. J. Math.*, 14 (2017), Art. 103, 15 pp. MR 3633363

[28] S. Pan, *Invasion speed of a predator-prey system*, *Appl. Math. Lett.*, 74 (2017), 46–51. MR 3677840

[29] S. Pan, G. Lin and J. Wang, *Propagation thresholds of competitive integrodifference systems*, *J. Difference Equ. Appl.*, 25 (2019), 1680–1705. MR 4041297

[30] N. Shibesada and K. Kawasaki, *Biological Invasions: Theory and Practice*, Oxford University Press: Oxford, UK, 1997.

[31] H. L. Smith, *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*, AMS: Providence, RI, USA, 1995. MR 1319817

[32] M. Wang and Y. Zhang, *Dynamics for a diffusive prey-predator model with different free boundaries*, *J. Differential Equations*, 264 (2018), 3527–3558. MR 3741397

[33] H. F. Weinberger, *Long-time behavior of a class of biological model*, *SIAM J. Math. Anal.*, 13 (1982), 353–396. MR 653463
[38] H. F. Weinberger, M. A. Lewis and B. Li, Analysis of linear determinacy for spread in cooperative models, *J. Math. Biol.*, 45 (2002), 183–218. MR 1930974

[39] P. Weng and X. Q. Zhao, Spreading speed and traveling waves for a multi-type SIS epidemic model, *J. Differential Equations*, 229 (2006), 270–296. MR 2265628

[40] Q. Ye, Z. Li, M. Wang and Y. Wu, *Introduction to Reaction Diffusion Equations*, Science Press, Beijing, 2011.

[41] Z. Yu and R. Yuan, Travelling wave solutions in non-local convolution diffusive competitive-cooperative systems, *IMA J. Appl. Math.*, 76 (2011), 493–513. MR 2824593

[42] G. Zhang, W. T. Li and G. Lin, Traveling waves in delayed predator-prey systems with nonlocal diffusion and stage structure, *Math. Comput. Model.*, 49 (2009), 1021–1029. MR 2495017

[43] X. Q. Zhao, Spatial dynamics of some evolution systems in biology, In *Recent Progress on Reaction-Diffusion Systems and Viscosity Solutions*, Y. Du, H. Ishii, W.Y. Lin, Eds.; World Scientific: Singapore, 2009, 332–363. MR 2532932

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