DARCY-FORCHHEIMER RELATION IN MAGNETOHYDRODYNAMIC JEFFREY NANOFLUID FLOW OVER STRETCHING SURFACE

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Abstract. Present article aims to investigate the heat and mass transfer developments in boundary layer Jeffrey nanofluid flow via Darcy-Forchheimer relation over a stretching surface. A viscous Jeffrey nanofluid saturates the porous medium under Darcy-Forchheimer relation. A variable magnetic effect normal to the flow direction is applied to reinforce the electro-magnetic conductivity of the nanofluid. However, small magnetic Reynolds is considered to dismiss the induced magnetic influence. The so-formulated set of governing equations is converted into set of nonlinear ODEs using transformations. Homotopy approach is implemented for convergent relations of velocity field, temperature distribution and the concentration of nanoparticles. Impact of assorted fluid parameters such as local inertial force, Porosity factor, Lewis and Prandtl factors, Brownian diffusion and Thermophoresis on the flow profiles is analyzed diagrammatically. The drag force (skin-friction) and heat-mass flux is especially analyzed through numerical information compiled in tabular form. It has been noticed that the inertial force and porosity factor result in decline of momentum boundary layer but, the scenario is opposite for thermal profile and solute boundary layer. The concentration of nanoparticles increases with increased porosity and inertial effect however, a significant reduction is detected in mass flux.

1. Introduction. Formulation of nanofluids has introduced an entirely new direction to the researchers working in fluid mechanics. This formulation is basically a mixture of ultra-fine nano-sized metallic particles having powerful thermal characteristics in a conventional typical base fluid like water, oil or glycol etc. The characteristics of nano-sized particles play a vital role in this formulation for enhancement of thermo-physical properties of the base fluid. Consequently, a saturated liquid is formulated that is more powerful in various thermo-physical properties as compared to the original base liquid. The prime improvement is noticed in the thermal conductivity of the fluid. Paper production, printing, coating, painting, power and electricity generation, drugs, cancer therapies and various food production processing are dependent on nanofluids. MHD pumps, hypothermia, 2010 Mathematics Subject Classification. Primary: 58F15, 58F17; Secondary: 53C35.

Key words and phrases. Jeffrey nanofluid, nanoparticles, Magnetohydrodynamics (MHD), Darcy-Forchheimer relation, stretching sheet.

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tumor treatment etc. are the applications of MHD nanofluids. Choi and Eastman [14] introduced the world with nanofluids. Their study concluded that pouring metallic nano-sized particles in typical base liquid can drastically improve its thermo-physical properties. Later on, Buongiorno [12] introduced a model explaining the convective nanofluids transport. This study ended-up with a conclusion that the Brownian factor and thermo-phoretic factor are the two very prominent phenomenon in nanofluid convection. Turkyilmazoglu and Pop [46] reported an unsteady nanofluid flow passing a vertically fixed thermally radiative plate. Rasool et al. [37] disclosed the impact of vertical Riga plate and convective heating on nanofluid flow and heat transfer phenomena. Ibrahim and Makinde [25] interpreted the influence of pertinent fluid parameters on a boundary layer nanofluid flow passing a vertically fixed plate. Rasool and Zhang [38] reported convective flow of nanofluids past a radiative Riga plate. In another article, Rasool et al. [39] discussed the properties of MHD and Darcy relation in nanofluid flow over non-linearly stretching surface. For further understanding of nanofluids one can read [27],[48],[16],[28],[17],[40],[41],[30],[31],[19],[44],[45],[9],[33],[34] and cross references cited therein.

The flow initiated by a stretching flat surface has a profound influence in various industrial and engineering applications of nanofluids. These applications include copper wire drawing, extrusion and die forging of various polymers in melt spinning, stretching the plastic film/sheets, process of condensation, paper and fiber production etc. Numerous article are found in literature on flow caused by stretching sheets. Cortell [15] reported heat-flux in flow past a flat stretching sheet involving two thermally dissimilar boundary conditions. Hydro-magnetic fluid flow initiated by a stretching factor in a flat sheet via modified A-domian PADE method is reported by Hayat et al. [20]. Mustafa et al. [35] reported axi-symmetric nanofluid that flows over flat linearly stretching sheet. Magnetohydrodynamic nanofluidic flow initiated by non-linearly variable thick stretching surface/sheet has been reported by Mabood et al. [36] using viscous dissipation. MHD nanofluidic flow past a nonlinearly stretching surface/sheet is presented by Hayat et al. [21]. Khan and Pop [29] reported a nanofluid convection through a non-linear variably thick stretching sheet analyzing the results graphically.

Recent developments in non-Newtonian nanofluids have achieved significant attention in engineering and industrial applications. Such as the process of plastic making, food processing, copper wire annealing and thinning, aero-dynamical plastic film production/extrusion, etc. To deal with such cases, various relations are introduced in the literature. In this communication, we are dealing with Jeffrey nanofluid model [32],[10],[47],[22],[23],[24] that involves the retardation-relaxation of time parameters ($\lambda_1, \lambda_2$). These parameters play significant role in controlling the heat-mass flux and help in describing the flow profiles.

The wide range of applications of heat flux through porous medium can not be neglected in the recent developments of fluid mechanics. It has become a subject of utmost interest in the heat-mass flux and fluid flow analysis. It has variety in environmental and chemical industry. For example, Cheng et al. [13] reported mixed convection of nanofluids through porous medium. Gbadeyan et al. [18] reported the Soret-Dufour effects on heat-mass flux in mixed convective flow of nanofluids over a stretching sheet through porous medium. Imran et al. [26] reported an unsteady mixed-convective fluid flow saturated through porous media using heated semi-infinite vertical stretching surface involving a heat source/sink. Aly and Elbaid
[11] reported boundary layer formulation on mixed-convective nanofluid flow using inclined plate with porosity factor. Rasool and Zhang [42] reported Darcy-Forchheimer relation in nanofluids manifested with Cattaneo-Christov theory of heat and mass flux over nonlinearly stretching surface. Narayana [43] disclosed the impact of radiation factor on chemically reactive nanofluid flow past a porous medium between vertical channels apart at an infinite distance. Some interesting articles can be seen in [1], [2], [3], [4], [5], [6], [7], [8] and cross references cited therein.

Heat and mass flux phenomena through variety of flow models available in literature, involving different variables has been discussed widely in the past. Numerous articles are reported analyzing the flow profiles for variation in various fluid parameters. However, the implementation of Jeffrey type nanofluid model in nanofluids through Darcy-Forchheimer relation has not been reported yet as per the knowledge of authors. The article is organized as follows. Firstly, an incompressible viscous Jeffrey type nanofluid is considered to flow via Darcy-Forchheimer relation. The porosity factor and local inertial force is involved. An externally induced magnetic effect parallel to the vertical coordinate of the plane is applied with small Reynolds. The results for Velocity field, temperature and concentration of nanoparticles are analyzed at various values of fluid parameters through graphs.

2. Problem formulation. We assume a two-dimensional viscous incompressible Jeffrey type nanofluid flow over stretching sheet using Darcy-Forchheimer relation that saturates the porous media. The $xy$ coordinates are incorporated such that horizontal coordinate extends along the surface of stretching surface/sheet and vertical coordinate is upwardly normal to it. Let $U_w = ax$ be the $x$-component of velocity, known as stretching velocity. The interfacing temperature distribution $T_w$ and concentration distribution of nanoparticles $C_w$ are taken as functions of $x, y$, respectively. Physical model can be seen in Fig. 1.

Therefore,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)$$

$$v \frac{\partial u}{\partial y} + u \frac{\partial u}{\partial x} = \frac{\nu}{1 + \lambda_1} \left[ \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) \right] + \lambda_2 \left( u \frac{\partial}{\partial x} \left( \frac{\partial^2 u}{\partial y^2} \right) + \frac{\partial u}{\partial y} \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) \right) - \frac{\sigma B_0^2}{\rho_f} \left( \frac{\partial T}{\partial x} \right)^2 - F \frac{u}{K} u - Fu^2, \quad (2)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{(\rho c)_p}{(pc)_f} \left( D_{Br} \left( \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} + \frac{D_{Th}}{T_\infty} \left( \frac{\partial T}{\partial y} \right)^2 \right) \right) + \alpha \frac{\partial^2 T}{\partial y^2}, \quad (3)$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = \frac{D_{Th}}{T_\infty} \left( \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} \right) + D_{Br} \left( \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} \right). \quad (4)$$

subject to following BCs,

$$v(x, 0) = 0, \quad u(x, 0) = ax, \quad T_w = T(x, 0), \quad C_w = C(x, 0), \quad (5)$$

$$v(x, \infty) \to 0, \quad u(x, \infty) \to 0, \quad T(x, \infty) \to T_\infty, \quad C(x, \infty) \to C_\infty, \quad (6)$$

where $\lambda_1$ stands for relaxing time, $\lambda_2$ stands for retarding time, $K$ stands for the porous media permeability, $C_b/K^{1/2} = F$ stands for coefficient of inertia with $C_b$ the drag coefficient, $\alpha$ stands for thermal diffusivity of the base fluid, $k$ denotes the thermal conductivity of the base fluid, $c_f$ involves heat capacity of the fluid in flow model, $(pc)_f$ involves the effective nanoparticles’ heat capacity in the flow
model, $D_{Br}$ represents Brownian diffusion, $D_{Th}$ represents thermophoretic diffusion, $a$ represents the positive stretching rate per second, $T_w$ denotes the constant temperature while $C_w$ stands for the concentration of nanoparticles at the surface, $T_\infty$ is used for the ambient liquid temperature and $C_\infty$ stands for the ambient nanoparticles concentration at infinite distance from surface. $B_o$ is used for applied magnetic field. Define,

$$u = ax \frac{\partial f}{\partial \eta}, \quad v = -\sqrt{a} \sqrt{\nu} f, \quad \theta(\eta)(T_w - T_\infty) = T - T_\infty, \quad \phi(\eta)(C_w - C_\infty) = C - C_\infty, \quad \eta = \sqrt{\frac{a}{\nu} y}. \quad (7)$$

The continuity equations is satisfied. The remaining PDEs are transformed as follows,

$$\frac{\partial^3 f}{\partial \eta^3} + \beta \left[ \left( \frac{\partial^2 f}{\partial \eta^2} \right)^2 - f \frac{\partial^4 f}{\partial \eta^4} \right] + (1 + \lambda_1) \left[ f \frac{\partial^2 f}{\partial \eta^2} - \left( \frac{\partial f}{\partial \eta} \right)^2 - (M - \lambda) \frac{\partial f}{\partial \eta} - F_r \left( \frac{\partial f}{\partial \eta} \right)^2 \right] = 0, \quad (8)$$

$$\frac{1}{Pr} \frac{\partial^2 \theta}{\partial \eta^2} + Nt \left( \frac{\partial \theta}{\partial \eta} \right)^2 + f \frac{\partial \theta}{\partial \eta} + Nb \frac{\partial \phi}{\partial \eta} \frac{\partial \phi}{\partial \eta} = 0, \quad (9)$$

$$\frac{\partial^2 \phi}{\partial \eta^2} + Pr Le f \frac{\partial \phi}{\partial \eta} + Nt \frac{\partial^2 \theta}{\partial \eta^2} = 0, \quad (10)$$

$$f = 0, \quad \frac{\partial f}{\partial \eta} = 1, \quad \theta(\eta) = 1, \quad \phi(\eta) = 1 \text{ at } \eta = 0, \quad (11)$$

$$\frac{\partial f}{\partial \eta} \rightarrow 0, \quad \theta(\eta) \rightarrow 0, \quad \phi(\eta) \rightarrow 0 \text{ as } \eta \rightarrow \infty, \quad (12)$$

**Figure 1.** Physical model and coordinate system.
where $M$ (Hartmann number) denotes the magnetic parameter, $\lambda$ stands for the porosity parameter, $F_r$ represents the local inertial coefficient, $Pr$, $Nb$, $Nt$ and $Le$ are typical Prandtl, Brownian, Thermophoretic and Lewis numbers, respectively. Mathematically,

$$
M = \frac{\sigma B_0^2}{\mu a}, \quad \beta = \lambda_2 a, \quad \lambda = \frac{\nu}{K_f}, \quad F_r = \frac{C_w}{K_{1.2}}, \quad Pr = \frac{\nu}{\alpha}, \quad Le = \frac{\alpha}{D_{Br}}, \quad Nb = \frac{Du/(\rho c_p(C_w-C_\infty))}{\nu(p_c)_f}, \quad Nt = \frac{D_{Br}/(\rho c_p(T_w-T_\infty))}{\nu(p_c)_f T_\infty}.
$$

The mathematical expressions for the drag force and the Nusselt number are,

$$
C_{fx} = \frac{\tau_w}{\rho_f u_w^2}, \quad Nu_x = \frac{q_{wx}}{k(T_f-T_\infty)}, \quad Sh_x = \frac{xq_m}{D_{Br} (C_f-C_\infty)},
$$

where,

$$
\tau_w = \frac{\mu}{1+\lambda} \left[ \frac{\partial u}{\partial y} + \lambda_2 \left( u \frac{\partial^2 u}{\partial y \partial x} + v \frac{\partial^2 u}{\partial y^2} \right) \right],
$$

$$
q_w = -k \left( \frac{\partial T}{\partial y} \right),
$$

$$
q_m = -D_{Br} \left( \frac{\partial C}{\partial y} \right), \quad \text{at} \quad y = 0,
$$

are typical stress tensor, heat and mass flux at surfaces, respectively. Finally, the expressions for drag force, local Nusselt and local Sherwood are given below,

$$
Re_x^{1/2} C_{fx} = \frac{1}{x \chi^{N_t}} \left[ \Phi_{\beta}^2 \left( 0 \right) \right], \quad \text{where} \quad \Phi_{\beta}^2 \left( 0 \right) = \Phi_{\beta}^2 \left( 0 \right)
$$

$$
Re_x^{1/2} Nu_x = -\frac{\partial \theta}{\partial y}\left( 0 \right), \quad \text{and} \quad \textbf{h}_x = \frac{xq_m}{D_{Br} (C_f-C_\infty)},
$$

where $u_w x / \nu = Re_x$ is called local Reynolds number parallel to $x-$ direction.

3. Solution methodology. To implement HAM for numerical solutions, Define,

$$
f_0 = 2 - (1 + \exp(-\eta)), \quad \theta_0 = \exp(-\eta), \quad \phi_0 = \exp(-\eta),
$$

with

$$
\hat{L}_f = \frac{\partial^3 f}{\partial \eta^2} - \frac{\partial f}{\partial \eta}, \quad \hat{L}_\theta = \frac{\partial^2 \theta}{\partial \eta^2} - \theta, \quad \hat{L}_\phi = \frac{\partial^2 \phi}{\partial \eta^2} - \phi,
$$

such that

$$
\begin{align*}
\hat{L}_f \left[ E_1 + E_2 e^\eta + E_3 e^{-\eta} \right] &= 0, \\
\hat{L}_\theta \left[ E_4 e^\eta + E_5 e^{-\eta} \right] &= 0, \\
\hat{L}_\phi \left[ E_6 e^\eta + E_7 e^{-\eta} \right] &= 0,
\end{align*}
$$

where $E_j, \quad j = 1 - 7$ are temporary constants. Writing down the $0^{th}$ order transformations as follows,

$$
\begin{align*}
\hat{z}_f \left[ f(\eta,t) - \frac{\partial f}{\partial \eta} \right] - \frac{\partial^2 f}{\partial \eta^2} \left[ (1 + \lambda_2) \frac{\partial f}{\partial \eta} \right] &= \beta \left[ \Phi_{\beta}^2 \left( 0 \right) \right], \\
\hat{z}_\theta \left[ \theta(\eta,t), \dot{\theta}(\eta,t), \ddot{\theta}(\eta,t) \right] &= \frac{1}{Pr} \frac{\partial^2 \theta}{\partial \eta^2} = \frac{f \partial \dot{\theta}}{\partial \eta} + Nt \left( \frac{\partial \theta}{\partial \eta} \right), \\
\hat{z}_\phi \left[ \phi(\eta,t), \dot{\phi}(\eta,t), \ddot{\phi}(\eta,t) \right] &= \frac{\partial^2 \phi}{\partial \eta^2} = Pr Le \frac{\partial \phi}{\partial \eta} + Nt \left( \frac{\partial \phi}{\partial \eta} \right),
\end{align*}
$$

with boundary conditions as appearing in system (11) and (12) with an additional variable $t$ called the embedding parameter. Setting $t = 0$ we approach the initial
achieved under the following Taylor series expansions,

\[
\begin{align*}
\hat{f}(\eta,t) &= \sum_{n=1}^{\infty} f_n(\eta)t^n + f_0(\eta), \\
\hat{\theta}(\eta,t) &= \sum_{n=1}^{\infty} \theta_n(\eta)t^n + \theta_0(\eta), \\
\hat{\phi}(\eta,t) &= \sum_{n=1}^{\infty} \phi_n(\eta)t^n + \phi_0(\eta).
\end{align*}
\]

The convergence of above system is strongly dependent on choice of the auxiliary parameters. Such that, for \( t = 1 \),

\[
\begin{align*}
f &= f_0(\eta) + \sum_{n=1}^{\infty} f_n(\eta), \\
\theta &= \theta_0(\eta) + \sum_{n=1}^{\infty} \theta_n(\eta), \\
\phi &= \phi_0(\eta) + \sum_{n=1}^{\infty} \phi_n(\eta).
\end{align*}
\]

Consequently, the \( n^{th} \) order deformation problems with subsequent boundary conditions are stated below,

\[
\begin{align*}
\hat{Y}_f^n(\eta) &= f''_{n-1} + \beta \left( \sum_{i=0}^{n-1} f''_{n-1-i} f''_i \right) + (1 + \lambda_1) \\
&\quad \left( \sum_{i=0}^{n-1} f_{n-1-i} f''_i - \sum_{i=0}^{n-1} f'_{n-1-i} f'_i - (M - \lambda) \sum_{i=0}^{n-1} f'_{n-1-i} - F_i \sum_{i=0}^{n-1} f'_{n-1-i} \right), \\
\hat{Y}_\theta^n(\eta) &= \frac{1}{Pr} \theta''_{n-1} + \sum_{i=0}^{n-1} f_{n-1-i} \theta'_i + Nb \sum_{i=0}^{n-1} (\theta'_{n-1-i} \phi'_i) + Nt \sum_{i=0}^{n-1} (\theta'_{n-1-i} \phi'_i), \quad (29) \\
\hat{Y}_\phi^n(\eta) &= \phi''_{n-1}(\eta) + Pr Le \sum_{i=0}^{n-1} f_{n-1-i} \phi'_i - \frac{Nt}{Nb} \theta''_{n-1}, \\
f_n(0) &= 0, \quad f'_n(0) = 1, \quad f''_n(\infty) = 0, \\
\theta_n(0) &= 1 = \phi_n(0), \quad \theta_n(\infty) = 0 = \phi_n(\infty). \quad (30)
\end{align*}
\]

The generalized solutions \( (f_n, \theta_n, \phi_n) \) take the following final form,

\[
\begin{align*}
f_n(\eta) &= f_n^*(\eta) + E_1 + E_2 \eta + E_3 \eta^2, \\
\theta_n(\eta) &= \theta_n^*(\eta) + E_4 \eta + E_5 \eta^2, \\
\phi_n(\eta) &= \phi_n^*(\eta) + E_6 \eta + E_7 \eta^2.
\end{align*}
\]

Where the constants \( E_j \) (\( j = 1 - 7 \)) such that,

\[
\begin{align*}
E_2 &= E_4 = E_6 = 0, \quad E_3 = \left. \frac{\partial f_n^*(\eta)}{\partial \eta} \right|_{\eta=0}, \quad E_1 = -E_3 - f_n^*(0), \\
E_5 &= -\theta_n^*(0), \quad E_7 = -\phi_n^*(0).
\end{align*}
\]
4. **Convergence.** The series solutions (26) – (28) involving auxiliary parameters $h_f, h_\theta, h_\phi$, directly depend on the suitable choice of values. The solution accelerates and converges fast if the choice is made correctly and appropriately. The so-called $h$–curves for flow profiles are sketched in Fig. 2. One can see that the intervals of convergence for three profiles are $-1.45 \leq h_f \leq -0.1, -1.45 \leq h_\theta \leq -0.40, -1.50 \leq h_\phi \leq -0.10$. Upto $25^{th}$ approximation order is required for convergence. Table 1 presents that $10^{th}$ order of deformations are necessary for velocity profile, and $25^{th}$ order deformations are necessary for the three profiles to converge.

![H-curves](image)

**Figure 2.** H-curves.

| Table 1: Convergence |
|----------------------|
| Order | $-\frac{\partial^2 f}{\partial \eta^2}(0)$ | $-\frac{\partial \theta}{\partial \eta}(0)$ | $-\frac{\partial \phi}{\partial \eta}(0)$ |
|-------|--------------------------------|--------------------------------|--------------------------------|
| 1     | 1.13462 | 0.60003 | 0.49222 |
| 5     | 1.15525 | 0.48633 | 0.43226 |
| 10    | 1.15526 | 0.47564 | 0.38262 |
| 15    | 1.15526 | 0.47404 | 0.37521 |
| 25    | 1.15526 | 0.46484 | 0.36552 |
| 35    | 1.15526 | 0.46484 | 0.36552 |
| 50    | 1.15526 | 0.46484 | 0.36552 |

5. **Results and discussion.** Under this heading we have managed to explore the variations in numerical values of porosity factor ($\lambda$), local inertial impact ($F_r$), Prandtl, Brownian and Thermophoretic factors ($Pr, Nb, (Nt)$), relaxation and retardation parameters to analyze their effect on velocity field $\frac{\partial f}{\partial \eta}$, temperature field $\theta(\eta)$ and nanoparticles concentration in base fluid $\phi(\eta)$ in Figs. 3 – 16. Fig. 3 explores the impact of dimensionless quantity $\beta$ on the dimensionless velocity $\frac{\partial f}{\partial \eta}$. For
elevated values of $\beta$, the profile shows strong reduction and the respective boundary layer becomes thinner. The reduction in velocity profile occurs because $\beta$ is product of stretching coefficient $a$ and retardation time $\lambda_2$. For elevation in $\beta$ the stretching as well as retardation time increases that results in the reduction of velocity. Fig. 4 depicts the impact of local inertial coefficient on flow velocity $\frac{\partial u}{\partial \eta}$. Here the respective profile is a declining function of $F_r$. For augmented values of $F_r$, the resistance offered to the fluid enhances that certainly reduces the flow motion. The behavior of flow motion with variation in relaxation time $\lambda_1$ is depicted in Fig. 5. Augmented values of $\lambda_1$ correspond to enhancement in relaxation time that means a fluid particle requires more than expected time to reach back from perturbed state to equilibrium state resulting a drop down in liquid velocity. Fig 6. is the presentation of porosity factor impacting on the momentum profile. The velocity shows decreasing behavior with elevated values of $\lambda$ for enhancement in the resistance offered by the porous media to the fluid motion. The relaxation time parameter $\lambda_1$ results in enhancement of temperature profile as portrayed in Fig. 7. The enhancement of relaxation time allows more fluid and nanoparticles to penetrate near the surface. This penetration results in more enhanced collision between the particles resulting a rise in the thermal profile and associated boundary layer. The impact of Brownian diffusion factor and Thermophoretic factor on the temperature field is predicted in Fig. 8 and Fig. 9. In both cases, a rise in temperature profile is noticed. The in-predictive motion of nanoparticles together with a stronger Thermophoretical force compels the fluid to leave the position more abruptly. This sudden and fast transition concludes in enhancement of the temperature field and respective thermal layer for both the cases. The resistance offered by porosity factor to the fluid flow drops down the flow motion and the fluid penetrates near the surface. This penetration results in temperature projection towards higher profile, as depicted in Fig. 10 for variation of temperature profile against $\lambda$. The impact of $Pr$ is noticed in Fig. 11. The enhancement in Prandtl number results in lower thermal diffusivity of the fluid because of inverse relation in kinematic viscosity and thermal diffusivity ($Pr = \nu/\alpha$). Therefore, augmented behavior of $Pr$ results in low temperature field and respective thermal layer thickness reduces as well. The retardation time, the inertial force and the relaxation time result in enhancement of the concentration of nanoparticles near the surface as plotted in Fig. 12, 13 and 14, respectively. The more is the relaxation time more enhanced penetration of particles is observed near the surface. The inertial force offers more resistance to the fluid flow that results in enhancement of the concentration distribution. However, the profile shows decreasing behavior with enhancement in Lewis number as shown in Fig. 15. Since, Lewis factor is the ratio of Schmidt number to $Pr$ therefore, stronger Lewis number means a weaker Prandtl factor that result in low concentration of nanoparticles and corresponding boundary layer drops down. Fig. 16 is the display of behavior of concentration profile with variation in Thermophoretic parameter. Stronger Thermophoretic force results in more concentration of nanoparticles near the sheet surface that rises the respective boundary layer. Table 2 explores the behavior of drag force $C_{fx}$ for pertinent fluid parameters. It is significant that frictional force (drag force) coefficient $C_{fx}$ enhances when local inertial coefficient $F_r$ and porosity parameter $\lambda$. The reason behind this fact is that porous medium gives rise to skin-friction coefficient $C_{fx}$. Porosity parameter $\lambda$ is directly related to square of inertial coefficient $F_r$. This fact leads to the dominating drag force. Table 3 elaborates the behavior of heat and mass flux rates for dimensionless quantities.
Nusselt number $\text{Nu}_x$ is reduced when porosity parameter $\lambda$ and inertial coefficient $F_T$ are enhanced. This leads to the fact that conduction dominates over convection. Similar behavior is noticed for Sherwood number $Sh_x$, the ratio of mass transfer rate and diffusion rate. The increasing value of $Le$ causes to increase the $Sh_x$ which leads to the enhancing mass flux. So, this is the fact that thickness of concentration boundary layer increases.

**Figure 3.** Graph of $f'(\eta)$ against $\beta$.

**Figure 4.** Graph of $f'(\eta)$ against $F_T$. 

Figure 5. Graph of $f' (\eta)$ against $\lambda_1$.

Figure 6. Graph of $f' (\eta)$ against $\lambda$. 
Figure 7. Graph of $\theta(\eta)$ against $\lambda_1$.

Figure 8. Graph of $\theta(\eta)$ against $N_b$. 
Figure 9. Graph of $\theta(\eta)$ against $N_t$.

Figure 10. Graph of $\theta(\eta)$ against $\lambda$. 
Figure 11. Graph of $\theta(\eta)$ against $Pr$.

Figure 12. Graph of $\phi(\eta)$ against $\beta$. 

\[ \beta = 0.20, \lambda = \lambda_1 = 0.30, Fr = 0.20, \eta = 0.10, \eta = 0.20, Le = 1.00 \]

\[ \text{Pr} = 0.40, \quad \text{Pr} = 0.70, \quad \text{Pr} = 1.0, \quad \text{Pr} = 1.30 \]

\[ \beta = 0.00, \quad \beta = 0.30, \quad \beta = 0.60, \quad \beta = 1.00 \]
Figure 13. Graph of $\phi(\eta)$ against $F_r$.

Figure 14. Graph of $\phi(\eta)$ against $\lambda_1$. 
6. Concluding remarks. An incompressible viscous Jeffery type nanofluid is considered to flow via Darcy-Forchheimer relation. The porosity factor and local inertial force is involved. An external induced magnetic effect parallel to the vertical
coordinate of the plane is applied with small magnetic Reynolds. Key findings are itemized as follows.

- Augmented values of inertial force parameter $F_r$ result in decrement of fluid velocity but enhances the concentration of nanoparticles.
- Velocity field $\frac{\partial f}{\partial \eta}$ and connected momentum layer show declination with augmented values of porosity factor $\lambda$ but opposite behavior is witnessed in temperature profile $\theta(\eta)$.
- Velocity field $\frac{\partial f}{\partial \beta}$ is proved as a declining function of both the relaxation time parameter and $\beta = \lambda^2 \alpha$.

Table 2: Variation in Skin - friction (Drag force)

| $M$ | $F_r$ | $\lambda$ | $-\text{Re}_x^{1/2} C_{fx}$ |
|-----|-------|-----------|-----------------------------|
| 0.1 | 0.1   | 0.2       | 1.2551                      |
| 0.5 | 1.3559|
| 1.0 | 1.8553|
| 0.2 | 0.1   | 0.3       | 1.1532                      |
| 0.5 | 1.3452|
| 1.0 | 1.5028|
| 0.2 | 0.1   | 0.1       | 1.0732                      |
| 0.5 | 1.3132|
| 1.0 | 1.5532|

Table 3: Variation in Nusslt and Sherwood numbers

| $M$ | $F_r$ | $\lambda$ | $Pr$ | $Nb$ | $Nt$ | $Le$ | $-\text{Re}_x^{-1/2} N_{ux}$ | $-\text{Re}_x^{-1/2} S_{hx}$ |
|-----|-------|-----------|------|------|------|------|-----------------------------|-----------------------------|
| 0.1 | 0.1   | 0.2       | 1.0  | 0.2  | 0.1  | 1.0  | 0.4998                      | 0.3992                      |
| 0.5 |       |           |      |      |      |      | 0.4772                      | 0.3658                      |
| 1.0 |       |           |      |      |      |      | 0.4320                      | 0.3324                      |
| 0.2 | 0.1   | 0.2       | 1.0  | 0.2  | 0.1  | 1.0  | 0.5014                      | 0.4158                      |
| 0.5 |       |           |      |      |      |      | 0.4882                      | 0.3852                      |
| 1.0 |       |           |      |      |      |      | 0.4724                      | 0.3682                      |
| 0.2 | 0.1   | 0.1       | 1.0  | 0.2  | 0.1  | 1.0  | 0.5656                      | 0.4442                      |
| 0.5 |       |           |      |      |      |      | 0.4965                      | 0.4012                      |
| 1.0 |       |           |      |      |      |      | 0.4647                      | 0.3687                      |
| 0.2 | 0.1   | 0.2       | 0.1  | 0.2  | 0.1  | 1.0  | 0.4254                      | 0.3220                      |
| 0.5 |       |           |      |      |      |      | 0.5185                      | 0.4132                      |
| 1.3 |       |           |      |      |      |      | 0.6067                      | 0.5568                      |
| 0.2 | 0.1   | 0.2       | 1.0  | 0.4  | 0.1  | 1.0  | 0.4232                      | 0.5158                      |
| 0.8 |       |           |      |      |      |      | 0.3637                      | 0.5370                      |
| 1.2 |       |           |      |      |      |      | 0.3175                      | 0.5701                      |
| 0.2 | 0.1   | 0.2       | 1.0  | 0.2  | 0.1  | 1.0  | 0.4344                      | 0.5598                      |
| 0.5 |       |           |      |      |      |      | 0.3988                      | 0.4007                      |
| 1.0 |       |           |      |      |      |      | 0.3655                      | 0.3202                      |
| 0.2 | 0.1   | 0.2       | 1.0  | 0.2  | 0.1  | 0.4 | 0.4656                      | 0.2331                      |
| 0.8 |       |           |      |      |      |      | 0.5112                      | 0.4125                      |
| 1.2 |       |           |      |      |      |      | 0.5551                      | 0.6365                      |
• Elevation in Prandtl factor results in decreasing behavior of temperature distribution.
• Thermophoresis results in enhancement of the concentration field near the surface.
• Augmented values of Lewis factor, the ratio of Schmidt and Prandtl numbers result in reduction of concentration profile.
• Both the inertial force and the porosity result in stronger drag force appearing in the fluid motion.
• Both the inertial force and the porosity result in reduction of heat and mass flux.
• Augmented Lewis number results in more mass flux.

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Received July 2019; revised September 2019.

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