WEAK GREENBERG’S GENERALIZED CONJECTURE FOR
IMAGINARY QUADRATIC FIELDS

KAZUAKI MURAKAMI

Abstract. Let $p$ be an odd prime number and $k$ an imaginary quadratic field in which $p$ splits. In this paper, we consider a weak form of Greenberg’s generalized conjecture for $p$ and $k$, which states that the non-trivial Iwasawa module of the maximal multiple $\mathbb{Z}_p$-extension field over $k$ has a non-trivial pseudo-null submodule. We prove this conjecture for $p$ and $k$ under the assumption that the Iwasawa $\lambda$-invariant for a certain $\mathbb{Z}_p$-extension over a finite abelian extension of $k$ vanishes and that the characteristic ideal of the Iwasawa module associated to the cyclotomic $\mathbb{Z}_p$-extension over $k$ has a square-free generator.

1. Introduction

Let $p$ be a prime number and $k$ a number field. In Iwasawa theory, one studies the arithmetic of the multiple $\mathbb{Z}_p$-extensions over $k$, which are galois groups being topologically isomorphic to direct products of copies of the additive group $\mathbb{Z}_p$ of $p$-adic integers. In this paper, we consider the maximal multiple $\mathbb{Z}_p$-extension field $\tilde{k}$ of $k$. By class field theory, there exists a positive integer $r$ with $r \geq r_2(k)+1$ such that $\text{Gal}(\tilde{k}/k)$ is isomorphic to $\mathbb{Z}^\oplus r$, where $r_2(k)$ is the number of complex places of $k$. Let $L_{\tilde{k}}$ be the maximal unramified pro-$p$ abelian extension field of $\tilde{k}$. We denote the galois group $\text{Gal}(L_{\tilde{k}}/\tilde{k})$ by $X_{\tilde{k}}$ and call it the Iwasawa module of $\tilde{k}/k$. Iwasawa [15] and Greenberg [11] proved that Iwasawa modules are finitely generated torsion $\mathbb{Z}_p[[\text{Gal}(\tilde{k}/k)]]$-modules. This module has a very important role in Iwasawa theory. The usual form of Iwasawa’s main conjecture states that Iwasawa modules, regarded as algebraic objects, have a relation with analytic objects; roughly speaking, the characteristic ideal of a Iwasawa module should coincide with the ideal generated by a certain $p$-adic $L$-function. Under several assumptions, Mazur-Wiles [20] and Wiles [33] proved this conjecture for the cyclotomic $\mathbb{Z}_p$-extensions of real abelian fields and of totally real number fields, respectively.

Concerning the structure of Iwasawa modules, Greenberg formulated a conjecture which is called Greenberg’s generalized conjecture (or GGC, for short):

Conjecture ([13], Greenberg’s generalized conjecture (GGC)). For each prime number $p$ and number field $k$, the $\mathbb{Z}_p[[\text{Gal}(\tilde{k}/k)]]$-module $X_{\tilde{k}}$ is pseudo-null, in other words, the height of the annihilator ideal $\text{Ann}_{\mathbb{Z}_p[[\text{Gal}(\tilde{k}/k)]]}(X_{\tilde{k}})$ is greater than one.

Here a $\mathbb{Z}_p[[\text{Gal}(\tilde{k}/k)]]$-module is said to be pseudo-null if there exist two relatively prime annihilators. If $k$ is a totally real number field and if Leopoldt’s conjecture holds for $p$ and $k$, then this conjecture is equivalent to the usual form of Greenberg’s conjecture, which states that the Iwasawa module is finite ([12]). Furthermore, by the structure theorem (see Section 2), the usual form of Greenberg’s conjecture
holds if and only if the Iwasawa $\lambda$- and $\mu$-invariants of the cyclotomic $\mathbb{Z}_p$-extension over $k$ vanish.

In this paper, we consider a weaker form of this conjecture. We call the following conjecture weak Greenberg’s generalized conjecture (or weak GGC, for short):

**Conjecture (Weak Greenberg’s generalized conjecture (weak GGC)).** Let $p$ be a prime number and $k$ a number field. Assume that $X_k$ is not trivial. Then $X_k$ has a non-trivial pseudo-null $\mathbb{Z}_p[\text{Gal}(\bar{k}/k)]$-submodule.

We note that weak GGC is proposed by Nguyen Quang Do [24, 25] for totally real number fields and by Wingberg [34] for arbitrary number fields. In a very recent paper, Kataoka [18] proved that $X_k$ has no non-trivial finite $\mathbb{Z}_p[\text{Gal}(\bar{k}/k)]$-submodule for imaginary quadratic fields in which $p$ splits.

Throughout this paper, we suppose that $p$ is an odd prime number and that $k$ is an imaginary quadratic field. We assume that $p$ splits in $k$ into $p$ and $p^*$. Under this assumption, there exists a uniquely defined $\mathbb{Z}_p$-extension $N_{\infty}/k$ (respectively, $N_{\infty}^*$/k) such that the prime ideal $p^*$ (respectively, $p$) does not ramify. We also denote by $k_{\infty}^*$ the cyclotomic $\mathbb{Z}_p$-extension field of $k$.

To state our main theorem, we introduce the notion of $p$-split $p$-rational fields.

**Definition 1.** For an imaginary quadratic field $k$ in which $p$ splits into $p$ and $p^*$, $k$ is said to be $p$-split $p$-rational if $k$ satisfies that $L_k \subset \bar{k}$, $k^{\mathcal{O}_p} \neq k$, and that $\bar{k}^{\mathcal{O}_p} \subset L_k$, where $\mathcal{O}_p$ is the decomposition group of the prime $p$ in $\text{Gal}(\bar{k}/k)$, $\bar{k}^{\mathcal{O}_p}$ is the fixed field of $k$ by $\mathcal{O}_p$, and $L_k$ is the $p$-Hilbert class field of $k$.

If we suppose that $\bar{k}^{\mathcal{O}_p} \subset L_k$, then $\mathcal{O}_p$ is a normal subgroup of $\text{Gal}(\bar{k}/\mathbb{Q})$ (Lemma 3.8). Hence we have $\mathcal{O}_p = \mathcal{O}_{p^*}$. Therefore, the definition of a $p$-split $p$-rational field does not depend on the choice of $p$. If $k$ is $p$-split $p$-rational, the $p$-Sylow subgroup of the ideal class group of $k$ is cyclic from the assumption that $L_k \subset \bar{k}$. In Section 5, we will give a necessary and sufficient condition for $k$ to be $p$-split $p$-rational (Proposition 5.8).

In this paper, we prove the following

**Theorem 1.1.** Let $p$ be an odd prime number and $k$ an imaginary quadratic field which is not $p$-split $p$-rational. Assume the following conditions:

(i) The Iwasawa $\lambda$-invariant of $N_{\infty}/k$ is zero.

(ii) The characteristic ideal of $X_{k_{\infty}}$ has a generator which is square-free.

Then weak GGC holds for $p$ and $k$.

On the $\mathbb{Z}_p$-extension $N_{\infty}/k$, it is known that the Iwasawa $\mu$-invariant of $N_{\infty}/k$ is zero. This was proved by Gillard [10] and Schneps [30] for $p \geq 5$ and Oukhaba-Viguë [27] for $p = 2, 3$. By [21] Proposition 1.C], the Iwasawa $\lambda$-invariant of $N_{\infty}/k$ is zero if and only if every ideal class of the $p$-part of the ideal class group of $k$ becomes principal in $N_{\infty}$. We note that vanishing of the Iwasawa $\lambda$-invariant of $N_{\infty}/k$ here is similar to that in the usual form of Greenberg’s conjecture in the case where $p$ is totally ramified in $N_{\infty}/k$. No counter examples of the assumption (i) have been found yet. Fukuda-Komatsu [8, 9] checked that the Iwasawa $\lambda$-invariants of $N_{\infty}/k$ vanish for some imaginary quadratic fields with $p = 3$ under the assumption that $p$ is totally ramified in $N_{\infty}/k$. On the assumption (ii) in Theorem 1.1, we note that we do not need this condition in the case where the Iwasawa module $X_k$ is cyclic as a $\mathbb{Z}_p[[\text{Gal}(\bar{k}/k)]]$-module (Remark 5.14).
There are two key new ideas of the proof of this theorem. One is to find an annihilator of $X_k$, which is introduced in [22, Lemma 3.3]. We fix this annihilator and denote it by $f(S, T)$ (Lemma 3.3). Using this power series, we can prove that the Iwasawa module $X_k$ is pseudo-isomorphic to a certain $\mathbb{Z}_p[[\text{Gal}(\bar{k}/k)]]$-cyclic module, provided that weak GGC does not hold for $p$ and $k$ (Corollary 5.3). The other is to consider a certain subfield of $L_k^\lambda$, which is denoted by $M_p^\lambda(N_\infty)$. From the $\mathbb{Z}_p$-rank of $\text{Gal}(M_p^\lambda(N_\infty)/\bar{k})$, we consider two inequalities (A) and (B), which are stated in the end of Section 3. These inequalities contradict each other, which imply Theorem 1.1. We will prove them, assuming that weak GGC does not hold for $p$ and $k$.

In the case where $k$ is a $p$-split $p$-rational field, we know that the Iwasawa $\lambda$-invariant of $N_\infty/k$ is zero by genus formula (see for example [22, Remark 3.2]). Furthermore, we see that $X_k$ is cyclic as a $\mathbb{Z}_p[[\text{Gal}(\bar{k}/k)]]$-module ([22, Proposition 3.8]). To treat this case, we consider the $\mathbb{Z}_p$-extension $N_\infty N_{s+1}^*/N_{s+1}^*$, where $s$ is the positive integer satisfying $p^s = [L_k:k]$ and $N_{s+1}^*$ is the $(s+1)$-st layer of $N_\infty/k$. We put $H = N_{s+1}^*$ and $H_\infty = N_\infty N_{s+1}^*$. Then $H_\infty/H$ is a $\mathbb{Z}_p$-extension unramified outside all prime ideals of $H$ lying above $p$. We prove the following theorem under the assumption that the Iwasawa $\lambda$-invariant for $H_\infty/H$ vanishes, which is similar to that in the usual form of Greenberg's conjecture mentioned above.

**Theorem 1.2.** Let $p$ be an odd prime number and $k$ an imaginary quadratic field which is $p$-split $p$-rational. Suppose that the Iwasawa $\lambda$-invariant of $H_\infty/H$ is zero. Then weak GGC holds for $p$ and $k$.

There is a relation between weak GGC and GGC. Fujii verified that weak GGC for $p$ and $k$ implies GGC for $p$ and $k$ if the characteristic ideal of the $\text{Gal}(\bar{k}/k^c)$-coinvariant module of $X_k$ as a $\mathbb{Z}_p[[\text{Gal}(\bar{k}^c/k)]]$-module is a prime ideal ([4, Proposition 3.1]). Hence we obtain the following corollary from Theorems 1.1 and 1.2.

**Corollary 1.3.** Assume either the assumption of Theorem 1.1 or that of Theorem 1.2 holds. Assume also that the characteristic ideal of the $\text{Gal}(\bar{k}/k^c)$-coinvariant module of $X_k$ as a $\mathbb{Z}_p[[\text{Gal}(\bar{k}^c/k)]]$-module is a prime ideal. Then GGC holds for $p$ and $k$.

Furthermore, we apply Theorem 1.1 and Corollary 1.3 to imaginary quadratic fields $k = \mathbb{Q}(\sqrt{-d})$. Fujii and Shirakawa gave criteria of GGC for imaginary quadratic fields ([4, Theorem 5.1], [31, Theorem 5.2]). If $1 < d < 10^3$ and $d \equiv 2$ mod 3, they verified the conjecture using the first and the second layers of a certain $\mathbb{Z}_p$-extension except for $d = 971$. Combining Corollary 1.3 with Fujii’s criterion, we can verify the conjecture including the case of $d = 971$. We will give more precise calculation in Section 7. Therefore we obtain the following

**Corollary 1.4.** Let $d$ be a positive integer and $k = \mathbb{Q}(\sqrt{-d})$ an imaginary quadratic field in which $p$ splits. If $1 < d < 10^3$ and $d \equiv 2$ mod 3, then GGC holds for $p = 3$ and $k$.

Concerning GGC, the following is known. Let $p$ be an odd prime number. Minardi [21, Proposition 3.3] verified GGC for an imaginary quadratic field $k$ in which $p$ splits under the assumption that the class number of $k$ is not divisible by $p$. This result is generalized by Itoh [14] and Fujii [6]. They verified the conjecture for imaginary abelian quartic fields ([14]) and for CM-fields of degree greater than...
or equal to 4 \cite{12} under the assumption that \( p \) splits completely in the base field and that the class number of the base field is not divisible by \( p \), and that all Iwasawa invariants of the cyclotomic \( \mathbb{Z}_p \)-extension over the maximal totally real subfield of the base field are trivial. Using a method of Minardi in \cite{21}, Ozaki \cite{28} and Kataoka \cite{17} studied the behavior of Iwasawa \( \lambda \)- and \( \mu \)-invariants of infinitely many \( \mathbb{Z}_p \)-extensions over a fixed number field. They determined many Iwasawa \( \lambda \)- and \( \mu \)-invariants of \( \mathbb{Z}_p \)-extensions under the assumption that GGC holds for the base field.

The outline of this paper is as follows. In Section 2, we provide some basic definitions and notation. In Section 3, we study the decomposition of the prime number \( p \) in \( \bar{k}/\mathbb{Q} \). Furthermore, in the end of this section, we give two inequalities (A) and (B), which imply Theorem 1.1. These inequalities give an upper bound and a lower bound of the \( \mathbb{Z}_p \)-rank of a certain Galois group. In Sections 4 and 5 we give the proof of the inequalities (A) and (B), respectively, assuming that weak GGC does not hold for \( p \) and \( k \). In Section 6 we consider the case where \( k \) is a \( p \)-split \( p \)-rational and prove Theorem 1.2. In Section 7, we introduce some numerical examples including \( p = 3 \) and \( k = \mathbb{Q}(\sqrt{-97}) \).

2. Preliminary

2.1. Let \( p \) be an odd prime number and \( k \) an imaginary quadratic field in which \( p \) splits into \( p \) and \( p^* \). Let \( K \) be a \( \mathbb{Z}_p \)-extension or the \( \mathbb{Z}_p^{\otimes 2} \)-extension of \( k \). We denote by \( L_K/K \) the maximal unramified pro-\( p \) abelian extension and put \( X_K = \text{Gal}(L_K/K) \). Since the Galois group \( \text{Gal}(K/k) \) acts naturally on \( X_K \), it becomes a \( \mathbb{Z}_p[\text{Gal}(K/k)] \)-module. It is known that \( X_K \) is a finitely generated torsion \( \mathbb{Z}_p[\text{Gal}(K/k)] \)-module \cite{11, 15}.

Since we have \( \text{Gal}(\bar{k}/k) \cong \mathbb{Z}_p^{\otimes 2} \), \( k \) has two independent \( \mathbb{Z}_p \)-extensions. For example, the cyclotomic \( \mathbb{Z}_p \)-extension \( k^{\epsilon}_\infty \) and the anti-cyclotomic \( \mathbb{Z}_p \)-extension \( k^{\sigma}_\infty \) are disjoint over \( k \) and satisfy \( \bar{k} = k^{\epsilon}_\infty k^{\sigma}_\infty \). Furthermore, since we suppose that \( p \) splits in \( k \), there exist two \( \mathbb{Z}_p \)-extensions in which one of the prime ideals of \( k \) lying above \( p \) does not ramify. Let \( N_\infty/k \) (respectively, \( N^{\epsilon}_\infty/k \)) be the \( \mathbb{Z}_p \)-extension of \( k \) in which \( p^* \) (respectively, \( p \)) does not ramify. We note that \( N_\infty \) (respectively, \( N^{\epsilon}_\infty \)) coincides with the fixed field of \( k \) by the inertia subgroup of \( \text{Gal}(\bar{k}/k) \) for the prime ideal \( p^* \) (respectively, \( p \)) of \( k \) \cite{21 Lemma 3.2}.

Let \( \sigma \) and \( \tau \) be topological generators of \( \text{Gal}(\bar{k}/k^{\epsilon}_\infty) \) and \( \text{Gal}(\bar{k}/k^{\sigma}_\infty) \), respectively. By the isomorphism

\[
\text{Gal}(\bar{k}/k) \cong \text{Gal}(\bar{k}/k^{\epsilon}_\infty) \times \text{Gal}(\bar{k}/k^{\sigma}_\infty),
\]

we fix an isomorphism

(1) \( \mathbb{Z}_p[[\text{Gal}(\bar{k}/k)]] \cong \mathbb{Z}_p[[S, T]] \) \( (\sigma \leftrightarrow 1 + S, \ \tau \leftrightarrow 1 + T) \).

We put \( \Lambda = \mathbb{Z}_p[[S, T]] \). By this isomorphism, we regard \( X_{\bar{k}} \) as a \( \Lambda \)-module. We note that \( \Lambda \) is a unique factorization domain and a noetherian local integral domain with the maximal ideal \( (S, T, p) \).
2.2. For a ring $R$, we denote by $R^\times$ the unit group of $R$. We suppose that $R$ is the formal power series ring in one variable over a discrete valuation ring or suppose that $R$ is the formal power series ring $A$. For a finitely generated torsion $R$-module $M$, we define the characteristic ideal of $M$ as follows. By the structure theorem of $R$-modules ([23, Proposition 5.1.7]), there exists an $R$-homomorphism

$$M \rightarrow \bigoplus_{i=1}^l R/p_i^{m_i}$$

with pseudo-null kernel and pseudo-null cokernel, where $p_i$’s are prime ideals of height one and $l$ is a non-negative integer, and $m_i$’s are positive integers. Here, for an $R$-module $P$, $P$ is said to be pseudo-null if there are two relatively prime annihilators of $P$. We note that a pseudo-null $R$-module is an $R$-module of finite length if $R$ is the formal power series ring in one variable over a discrete valuation ring. Then, we define the characteristic ideal of $M$ by

$$\text{char}_R(M) = \left( \prod_{i=1}^l p_i^{m_i} \right),$$

which is an ideal in $R$.

Let $G$ be a profinite group. For any topological $G$-module $M$, we denote by $M^G$ the subset of elements of $M$ invariant under the action of $G$. We also denote by $M_G$ the largest quotient module of $M$ on which $G$ acts trivially, namely,

$$M_G = M/\langle \{ g-1 \}_m \mid g \in G, m \in M \rangle,$$

where $\langle \{ g-1 \}_m \mid g \in G, m \in M \rangle$ is the topological closure of $\{ g-1 \}_m \mid g \in G, m \in M \rangle$ in $M$.

3. The decomposition of the prime $p$

In this section, we roughly describe the decomposition of the prime $p$ in $\text{Gal}(\tilde{k}/\mathbb{Q})$. We first have the following

**Lemma 3.1** (See for example [23, Lemma 1]). Let $k$ be an imaginary quadratic field in which $p$ splits and $k_\infty$ a $\mathbb{Z}_p$-extension different from $N_\infty$ and $N^*_\infty$. Then there exists an exact sequence of $\mathbb{Z}_p[[\text{Gal}(k_\infty/k)]]$-modules:

$$0 \rightarrow (X_{\tilde{k}})_\text{Gal}(\tilde{k}/k_\infty) \rightarrow X_{k_\infty} \rightarrow \text{Gal}(\tilde{k}/k_\infty) \rightarrow 0.$$

For a $\mathbb{Z}_p$-extension $k_\infty/k$, we denote by $\lambda(k_\infty/k)$, $\mu(k_\infty/k)$ the Iwasawa $\lambda$-invariant, $\mu$-invariant of $k_\infty/k$, respectively. Since we suppose that $p$ splits in $k$, we have $\lambda(k_\infty^c/k) \geq 1$.

**Lemma 3.2.** The Iwasawa module $X_{\tilde{k}}$ is trivial if and only if $\lambda(k_\infty^c/k) = 1$.

**Proof.** In the case of $\lambda(k_\infty^c/k) = 1$, we have $(X_{\tilde{k}})_\text{Gal}(\tilde{k}/k_\infty) = 0$ using Lemma 3.1 and using the fact that $X_{k_\infty^c}$ has no non-trivial finite $\mathbb{Z}_p[[\text{Gal}(k_\infty^c/k)]]$-submodule ([32, Corollary 13.29]). By Nakayama’s lemma, we obtain $X_{\tilde{k}} = 0$. If we suppose that $X_{\tilde{k}} = 0$, then we have $X_{k_\infty^c} \cong \text{Gal}(\tilde{k}/k_\infty^c) \cong \mathbb{Z}_p$. This implies that $\lambda(k_\infty^c/k) = 1$. Thus we get the conclusion. □

We put $\lambda^* = \lambda(k_\infty^c/k) - 1$. We obtain an annihilator of $X_{\tilde{k}}$ from the following
Lemma 3.3 ([22] Lemma 3.3), Suppose that $\lambda^* \geq 1$, where $\lambda^*$ is the integer defined above. Then there exist power series $f(S, T) \in \text{Ann}_{\Lambda}(X_{\tilde{k}})$ and $g_i(S) \in \mathbb{Z}_p[[S]]$ $(i = 0, \ldots, \lambda^*-1)$ such that

$$f(S, T) = T^{\lambda^*} + g_{\lambda^*-1}(S)T^{\lambda^*} - 1 + \cdots + g_1(S)T + g_0(S).$$

Proof. We note that $X_{\kappa_\infty}$ has no non-trivial finite $\mathbb{Z}_p[[\text{Gal}(k_{\kappa_\infty}/k)]]$-submodule. Applying Lemma 3.1 to the case of $k_{\kappa_\infty} = k_{\kappa_\infty}$, we have

$$X_{\tilde{k}}/SX_{\tilde{k}} \cong (X_{\tilde{k}})_{\text{Gal}(\tilde{k}/k_{\kappa_\infty})} \cong \mathbb{Z}_p^{\oplus \Lambda^*}$$

by the isomorphism (1). Using Nakayama’s lemma, there exist $x_i \in X_{\tilde{k}}$ $(i = 1, \ldots, \lambda^*)$ such that $X_{\tilde{k}} = (x_1, \ldots, x_{\lambda^*})_{\mathbb{Z}_p[[S]]}$. Thus there exist $f_{ij}(S) \in \mathbb{Z}_p[[S]]$ $(i, j = 1, \ldots, \lambda^*)$ such that

$$TX_i = \sum_{j=1}^{\lambda^*} f_{ij}(S)x_j, \quad (i = 1, \ldots, \lambda^*).$$

By these relations, we have the following matrix

$$A = \begin{pmatrix} T - f_{11}(S) & -f_{12}(S) & \cdots & -f_{1\lambda^*}(S) \\ -f_{21}(S) & T - f_{22}(S) & \cdots & -f_{2\lambda^*}(S) \\ \vdots & \vdots & \ddots & \vdots \\ -f_{\lambda^*1}(S) & -f_{\lambda^*2}(S) & \cdots & T - f_{\lambda^*\lambda^*}(S) \end{pmatrix}.$$

We denote by $\det(A)$ the determinant of the matrix $A$. We put $f(S, T) = \det(A)$. Then we obtain

$$f(S, T) = T^{\lambda^*} + g_{\lambda^*-1}(S)T^{\lambda^*} - 1 + \cdots + g_1(S)T + g_0(S)$$

for some $g_i(S) \in \mathbb{Z}_p[[S]]$ $(i = 0, \ldots, \lambda^*-1)$. It is easy to see that $f(S, T)X_{\tilde{k}} = 0$. Thus we get the conclusion. □

Remark 3.4. The uniqueness of the power series $f(S, T)$ is not known. In this paper, we fix this power series in the case where $\lambda(k_{\kappa_\infty}/k) \geq 2$.

We fix an isomorphism

$$\mathbb{Z}_p[[\text{Gal}(k_{\kappa_\infty}/k)]] \cong \mathbb{Z}_p[[T]] \quad (r\text{Gal}(\tilde{k}/k_{\kappa_\infty}) \leftrightarrow 1 + T).$$

By this isomorphism, we identify these rings.

There exists a relation between the power series $f(S, T)$ and the characteristic ideal of $X_{k_{\kappa_\infty}}$ as follows.

Proposition 3.5. Assume that the characteristic ideal of $X_{k_{\kappa_\infty}}$ has a generator which is square-free or $X_{\tilde{k}}$ is cyclic as a $\Lambda$-module. Suppose that $\lambda(k_{\kappa_\infty}/k) \geq 2$. Then we have

$$\text{char}_{\mathbb{Z}_p[[T]]}(X_{k_{\kappa_\infty}}) = (f(0, T)),$$

where $f(S, T)$ is the same power series defined in Lemma 3.3.

Proof. Since $X_{\tilde{k}}$ is a finitely generated $\Lambda$-module, there exists a positive integer $r$ such that $(\Lambda/f(S, T)\Lambda)^{\oplus r} \to X_{\tilde{k}}$ is surjective. We can take $r = 1$ if $X_{\tilde{k}}$ is cyclic as a $\Lambda$-module. Hence we get a surjective homomorphism $(\mathbb{Z}_p[[T]]/(f(0, T)\mathbb{Z}_p[[T]]))^{\oplus r} \to X_{\tilde{k}}/SX_{\tilde{k}}$. This implies that

$$(Tf(0, T)^r) \subset \text{char}_{\mathbb{Z}_p[[T]]}(X_{k_{\kappa_\infty}}).$$
By Lemma 3.3, we note that \( f(0, T) \) is a polynomial with \( \deg(f(0, T)) = \lambda^* \), where \( \deg(f(0, T)) \) is the degree of the polynomial \( f(0, T) \). Since \( \text{char}_{\mathbb{Z}_p[[T]]}(X_{k_{\infty}}) \) has a generator which is square-free in the case of \( r \geq 2 \), we obtain

\[
(Tf(0, T)) \subset \text{char}_{\mathbb{Z}_p[[T]]}(X_{k_{\infty}}).
\]

By the definition of \( \lambda^* \), we have \( (Tf(0, T)) = \text{char}_{\mathbb{Z}_p[[T]]}(X_{k_{\infty}}) \). \( \square \)

Using the structure theorem, we can calculate the order of \( (X_{\bar{k}})_{\text{Gal} (\bar{k}/k)} \).

**Lemma 3.6.** Suppose that \( \lambda(k_{\infty}/k) \geq 2 \). With the same notation as above, we have the following:

\[
\#(X_{\bar{k}})_{\text{Gal} (\bar{k}/k)} = \#(\mathbb{Z}_p / g_0(0) \mathbb{Z}_p),
\]

where \( g_0(S) \) is the same power series defined in Lemma 3.3.

**Proof.** Using the structure theorem, we have an injective pseudo-isomorphism

\[
\Phi : (X_{\bar{k}})_{\text{Gal} (\bar{k}/k_{\infty})} \to \bigoplus_{j=1}^{l} \mathbb{Z}_p[[T]]/(f_j(T)^{n_j}),
\]

where \( l \) and \( n_j \)'s are positive integers and \( f_j(T) \)'s are irreducible elements of \( \mathbb{Z}_p[[T]] \). By Lemma 3.1 and Proposition 3.5, the characteristic ideal of \( (X_{\bar{k}})_{\text{Gal} (\bar{k}/k_{\infty})} \) is generated by \( f(0, T) \). Then we have a prime factorization \( f(0, T) = \prod_{j=1}^{l} f_j(T)^{n_j} \).

By [10] Proposition 6, \( f(0, T) \) is not divisible by \( T \). Hence we have a commutative diagram:

\[
\begin{array}{ccc}
0 & \longrightarrow & (X_{\bar{k}})_{\text{Gal} (\bar{k}/k_{\infty})} \\
& & \Phi \\
0 & \longrightarrow & \bigoplus_{j=1}^{l} \mathbb{Z}_p[[T]]/(f_j(T)^{n_j}) \\
& & \longrightarrow C \\
& & \longrightarrow 0
\end{array}
\]

where the vertical maps are multiplication by \( T \) and \( C \) is the cokernel of \( \Phi \). Since \( C \) is finite, we obtain

\[
\#(X_{\bar{k}})_{\text{Gal} (\bar{k}/k)} = \#(\mathbb{Z}_p / f(0, 0) \mathbb{Z}_p) = \#(\mathbb{Z}_p / g_0(0) \mathbb{Z}_p).
\]

\( \square \)

In the following lemmas and propositions, we consider the decomposition of the prime \( p \) in \( \bar{k}/\mathbb{Q} \). We first note that the prime number \( p \) is only finitely decomposed in \( \bar{k}/\mathbb{Q} \). This fact was verified by Minardi in his thesis using class field theory:

**Lemma 3.7.** [21] Lemma 3.1. Let \( \mathfrak{D}_p \) be the decomposition group of \( \mathfrak{p} \) in \( \text{Gal} (\bar{k}/k) \). Then \( \mathfrak{D}_p \) has finite index in \( \text{Gal} (\bar{k}/k) \).

We give a necessary and sufficient condition for the decomposition group \( \mathfrak{D}_p \) to be normal in \( \text{Gal} (\bar{k}/\mathbb{Q}) \).

**Lemma 3.8.** The decomposition group \( \mathfrak{D}_p \) is a normal subgroup of \( \text{Gal} (\bar{k}/\mathbb{Q}) \) if and only if \( [\text{Gal} (\bar{k}/k) : \mathfrak{D}_p] \leq [L_k \cap \bar{k} : k] \).
Proof. First, we note that $k_{\infty}^c / \mathbb{Q}$ is a galois extension. Furthermore, the complex conjugation, namely, the generator of $\text{Gal}(k/\mathbb{Q})$ acts on $\text{Gal}(k_{\infty}^c / k)$ as inverse. Then we have $L_k \cap \bar{k} \subset k_{\infty}^c$. By the definition of $N_{\infty}$, $k_{\infty}^c \cap N_{\infty}$ coincides with $L_k \cap \bar{k}$. Since $p^*$ splits completely in $\bar{k}^{D_{\mathfrak{p}^*}}$, we have $\bar{k}^{D_{\mathfrak{p}^*}} \subset N_{\infty}$.

We suppose that $\mathfrak{D}_p$ is a normal subgroup of $\text{Gal}(k/\mathbb{Q})$. Then we have $\bar{k}^{D_{\mathfrak{p}^*}} = \bar{k}^{D_{\mathfrak{p}}}$. Hence $p$ splits completely in $\bar{k}^{D_{\mathfrak{p}^*}}$ and $\bar{k}^{D_{\mathfrak{p}}} = \bar{k}^{D_{\mathfrak{p}}}$ is a subfield of $k_{\infty}^c$. This implies that $[\text{Gal}(\bar{k}/k) : \mathfrak{D}_p] \leq [\bar{L}_k \cap \bar{k} : k]$. Conversely, we suppose that $[\text{Gal}(\bar{k}/k) : \mathfrak{D}_p] \leq [L_k \cap \bar{k} : k]$. Since $L_k \cap k$ and $\bar{k}^{D_{\mathfrak{p}}} = \bar{k}^{D_{\mathfrak{p}}} = \bar{k}^{D_{\mathfrak{p}}} \subset L_k \cap \bar{k}$. Then $\bar{k}^{D_{\mathfrak{p}}} / \mathbb{Q}$ is a galois extension. This implies that $\mathfrak{D}_p$ is normal. □

From the following proposition, we see that the $p$-adic valuation of the constant term of $f(S, T)$ gives an upper bound of the number of prime ideals of $\bar{k}$ lying above $p$.

**Proposition 3.9.** Suppose that $\lambda(k_{\infty}^c / k) \geq 2$. We have the following:

$$[\text{Gal}(\bar{k}/k) : \mathfrak{D}_p] \leq \min\{[L_k \cap \bar{k} : k], \#(\mathbb{Z}_p / g_0(0)\mathbb{Z}_p)\}$$

if $\mathfrak{D}_p$ is a normal subgroup of $\text{Gal}(k/\mathbb{Q})$,

$$[L_k \cap \bar{k} : k] < [\text{Gal}(\bar{k}/k) : \mathfrak{D}_p] \leq \#(\mathbb{Z}_p / g_0(0)\mathbb{Z}_p)$$

if $\mathfrak{D}_p$ is not a normal subgroup of $\text{Gal}(k/\mathbb{Q})$.

Proof. We will prove that $[\text{Gal}(\bar{k}/k) : \mathfrak{D}_p] \leq \#(\mathbb{Z}_p / g_0(0)\mathbb{Z}_p)$. By Lemma 3.1, we have an exact sequence

$$0 \to \langle X_k \rangle_{\text{Gal}(\bar{k}/k_{\infty}^c)} \to X_{k_{\infty}^c} \to \text{Gal}(\bar{k}/k_{\infty}^c) \to 0$$

as $\mathbb{Z}_p[[\text{Gal}(k_{\infty}^c / k)]]$-modules. The snake lemma gives the exact sequence

$$0 \to \left( \langle X_k \rangle_{\text{Gal}(\bar{k}/k_{\infty}^c)} \right)_{\text{Gal}(k_{\infty}^c / k)} \to X_{k_{\infty}^c} \to \left( \text{Gal}(\bar{k}/k_{\infty}^c) \right)_{\text{Gal}(k_{\infty}^c / k)} \to 0.$$

Then we have $\left( \langle X_k \rangle_{\text{Gal}(\bar{k}/k_{\infty}^c)} \right)_{\text{Gal}(k_{\infty}^c / k)} = 0$. Indeed, $T$ does not divide a generator of $\text{char}(\mathbb{Z}_p[[T]]) \left( \langle X_k \rangle_{\text{Gal}(\bar{k}/k_{\infty}^c)} \right)$. Furthermore, we have $\left( X_{k_{\infty}^c} \right)_{\text{Gal}(k_{\infty}^c / k)} = D_{k_{\infty}^c}$ by (26) Lemma 4.1, where $D_{k_{\infty}^c}$ is the decomposition group of a prime ideal of $k_{\infty}^c$ lying above $p$ in $X_{k_{\infty}^c} = \text{Gal}(L_{k_{\infty}^c} / k_{\infty}^c)$. Therefore we obtain an exact sequence

$$0 \to D_{k_{\infty}^c} \to \text{Gal}(\bar{k}/k_{\infty}^c) \to \left( X_{k_{\infty}^c} \right)_{\text{Gal}(k_{\infty}^c / k)} \to \text{Gal}(\bar{k}/k_{\infty}^c) \to 0.$$

This implies that

$$\text{Coker}(D_{k_{\infty}^c} \to \text{Gal}(\bar{k}/k_{\infty}^c)) \cong \frac{\text{Gal}(\bar{k}/k_{\infty}^c) / \ker(\text{Gal}(\bar{k}/k_{\infty}^c))}{\left( X_{k_{\infty}^c} \right)_{\text{Gal}(k_{\infty} / k)}} \cong \text{Image}(\text{Gal}(\bar{k}/k_{\infty}^c) \to \left( X_{k_{\infty}^c} \right)_{\text{Gal}(k_{\infty} / k)})$$

We note that

$$\text{Image}(D_{k_{\infty}^c} \to \text{Gal}(\bar{k}/k_{\infty}^c)) = \text{Gal}(\bar{k}/k_{\infty}^c) \cap \mathfrak{D}_p.$$
From the exact sequence \([\mathcal{D}_p : D_p] \subset [\mathcal{D}_p : D_p] \cap \mathcal{D}_p\), we obtain
\[
\begin{align*}
[\text{Gal}(\bar{k}/k) : \mathcal{D}_p] &= [\text{Gal}(\bar{k}/k_{\infty}) \mathcal{D}_p : \mathcal{D}_p] \\
&= [\text{Gal}(\bar{k}/k_{\infty}) : \text{Gal}(\bar{k}/k_{\infty}) \cap \mathcal{D}_p] \\
&= \#\text{Coker}(D_{k_{\infty}} \to \text{Gal}(\bar{k}/k_{\infty})) \\
&\leq \#(X_{k_{\infty}}(\text{Gal}(\bar{k}/k))).
\end{align*}
\]
By Lemma 3.6, we have completed the proof. \(\square\)

Let \(D_p\) (respectively, \(D_{p^*}\)) be the decomposition group of \(p\) (respectively, \(p^*\)) in \(\text{Gal}(N_{\infty}/k)\) (respectively, \(\text{Gal}(N_{\infty}^*/k)\)).

**Lemma 3.10.** With the same notation as above, we have the following:
\[
[\text{Gal}(\bar{k}/k) : \mathcal{D}_p] = [\text{Gal}(N_{\infty}/k) : D_p],
\]
\[
[\text{Gal}(\bar{k}/k) : \mathcal{D}_{p^*}] = [\text{Gal}(N_{\infty}/k) : D_{p^*}].
\]

**Proof.** From the natural exact sequence
\[
0 \to \text{Gal}(\bar{k}/N_{\infty}) \to \text{Gal}(\bar{k}/k) \to \text{Gal}(N_{\infty}/k) \to 0,
\]
we have
\[
0 \to \text{Gal}(\bar{k}/N_{\infty}) \cap \mathcal{D}_{p^*} \to \mathcal{D}_{p^*} \to D_{p^*} \to 0.
\]
Since all prime ideals of \(N_{\infty}\) lying above \(p^*\) ramify in \(\bar{k}/N_{\infty}\), we have \(\text{Gal}(\bar{k}/N_{\infty}) \cap \mathcal{D}_{p^*} = \text{Gal}(\bar{k}/N_{\infty})\). Thus we obtain \(\text{Gal}(\bar{k}/k)/\mathcal{D}_{p^*} \cong \text{Gal}(N_{\infty}/k)/D_{p^*}\). Hence we have proved the former part. We can prove that \([\text{Gal}(\bar{k}/k) : \mathcal{D}_p] = [\text{Gal}(N_{\infty}/k) : D_p]\) in the same way as above. Thus we get the conclusion. \(\square\)

**Proposition 3.11.** Suppose that \(\mathcal{D}_{p^*}\) is not a normal subgroup of \(\text{Gal}(\bar{k}/Q)\). Then, \(p\) splits completely in \(L_k \cap \bar{k}\).

**Proof.** We have \([\text{Gal}(\bar{k}/k) : \mathcal{D}_{p^*}] = [\text{Gal}(N_{\infty}/k) : D_{p^*}]\) by Lemma 3.10. Thus we have \(N_{\infty}^{D_{p^*}} = \bar{k}D_{p^*}\). Since \(D_{p^*}\) is not a normal subgroup of \(\text{Gal}(N_{\infty}/Q)\), we have
\[
N_{\infty}^{D_{p^*}} \supset L_k \cap \bar{k}, \quad N_{\infty}^{D_{p^*}} \neq L_k \cap \bar{k}.
\]
Indeed, if we suppose that \(N_{\infty}^{D_{p^*}} \subset L_k \cap \bar{k}\), then \(N_{\infty}^{D_{p^*}}\) is a subfield of \(k_{\infty}^a\). Hence \(N_{\infty}^{D_{p^*}}\) is galois over \(Q\). Therefore \(\mathcal{D}_{p^*}\) is a normal subgroup of \(\text{Gal}(\bar{k}/Q)\). This is a contradiction. By the same reason, the inclusion \(N_{\infty}^{D_p} \supset L_k \cap \bar{k}\) is proper. These imply that \(p\) and \(p^*\) split completely in \(L_k \cap \bar{k}\). Thus we get the conclusion. \(\square\)

For an algebraic extension \(K/k\), we denote by \(M_p(K)\) (respectively, \(M_{p^*}(K)\)) the maximal pro-\(p\) abelian extension of \(K\) unramified outside all prime ideals of \(K\) lying above \(p\) (respectively, \(p^*\)). To prove Theorem 1.1, we will show two inequalities below under the assumption that weak GGC does not hold for \(p\) and \(k\):
\[
\begin{align*}
(A) \quad &\text{rank}_{Z_p}(\text{Gal}(M_p/(N_{\infty})/\bar{k})) \leq \min\{|L_k \cap \bar{k} : k|, [\text{Gal}(\bar{k}/k) : \mathcal{D}_{p^*}]\} - 1, \\
(B) \quad &\text{rank}_{Z_p}(\text{Gal}(M_{p^*}(N_{\infty})/\bar{k})) \geq \min\{|L_k \cap \bar{k} : k|, [\text{Gal}(\bar{k}/k) : \mathcal{D}_{p^*}]\},
\end{align*}
\]
where we denote by \(\text{rank}_{Z_p}(*)\) the \(Z_p\)-rank of *. If both (A) and (B) hold, then this is a contradiction. Thus weak GGC holds for \(p\) and \(k\). In Sections 4 and 5, we will prove these inequalities.
Finally, we introduce the following proposition needed later. For convenience we include a proof.

**Proposition 3.12.** ([5 Theorem 2]) The following two conditions are equivalent:

(i) The Iwasawa module $X_{\bar{k}}$ has a non-trivial pseudo-null $\Lambda$-submodule.

(ii) We have $M_{p^*}(\bar{k}) \neq L_{\bar{k}}$.

**Proof.** We first suppose that $M_{p^*}(\bar{k}) = L_{\bar{k}}$. Then we have $X_{p^*}(\bar{k}) = X_{\bar{k}}$, where $X_{p^*}(\bar{k})$ is the Galois group of the extension $M_{p^*}(\bar{k})/\bar{k}$. By ([29]), $X_{p^*}(\bar{k})$ has no non-trivial pseudo-null $\Lambda$-submodule. Hence (i) implies that (ii) holds.

Next we suppose that (ii) holds. Using Lemma 2 in [5], we obtain $M_{p^*}(k_c^\infty) = L_{k_c^\infty}$. Hence we have a commutative diagram of $\Lambda$-modules:

$$
\begin{array}{c}
0 \longrightarrow X_{p^*}(\bar{k})/SX_{p^*}(\bar{k}) \longrightarrow X_{k_c^\infty} \longrightarrow \text{Gal}(\bar{k}/k_c^\infty) \longrightarrow 0 \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\end{array}
$$

This implies that

$$
X_{p^*}(\bar{k})/SX_{p^*}(\bar{k}) \cong X_{\bar{k}}/SX_{\bar{k}} \cong \mathbb{Z}_p^{\oplus \Lambda}.
$$

In the case of $\lambda(k_c^\infty/k) = 1$, we have proved that $X_{\bar{k}} = 0$ by Lemma 3.2. Then we have $X_{p^*}(\bar{k}) = 0$ by (4). This is a contradiction. Hence we have $\lambda(k_c^\infty/k) \geq 2$.

In this case, we see that $X_{\bar{k}}$ is not trivial. Hence, without loss of generality, we may assume that $f(S,T)$ annihilates $X_{p^*}(\bar{k})$, where $f(S,T)$ is the same power series defined in Lemma 3.3. We put $Y = \text{Gal}(M_{p^*}(\bar{k})/L_{\bar{k}})$. Then we have a commutative diagram of $\Lambda$-modules:

$$
\begin{array}{c}
0 \longrightarrow Y \longrightarrow X_{p^*}(\bar{k}) \longrightarrow X_{\bar{k}} \longrightarrow 0 \\
\downarrow s \times \downarrow s \times \downarrow s \times \\
0 \longrightarrow Y \longrightarrow X_{p^*}(\bar{k}) \longrightarrow X_{\bar{k}} \longrightarrow 0,
\end{array}
$$

where the vertical maps are multiplication by $S$. Using [1], we have a surjective homomorphism

$$
X_{\bar{k}}[S] \rightarrow Y/\text{SY},
$$

where $X_{\bar{k}}[S]$ is the $S$-torsion subgroup of $X_{\bar{k}}$. By Nakayama’s lemma, $X_{\bar{k}}[S]$ is not trivial because of the assumption that $Y \neq 0$. Furthermore, $X_{\bar{k}}[S]$ is pseudo-null since $S$ is coprime to $f(S,T)$. Therefore $X_{\bar{k}}$ has a non-trivial pseudo-null submodule. Thus we get the conclusion. □

4. Proof of the inequality (A)

In this section, we will prove the inequality (A):

$$
\text{rank}_{\mathbb{Z}_p}(\text{Gal}(M_{p^*}(N_{\infty})/\bar{k})) \leq \min\{[L_k \cap \bar{k} : k], [\text{Gal}(\bar{k}/k) : \mathcal{D}_{p^*}]\} - 1
$$

under the assumption of Theorem 1.1 and assuming that weak GGC does not hold for $p$ and $k$.

By Lemma 3.7, the prime ideal $p^*$ finitely decomposes in $\bar{k}/k$. We put $p^{n_0} = [\text{Gal}(\bar{k}/k) : \mathcal{D}_{p^*}]$. Using Lemma 3.10, we have

$$
p^{n_0} = [\text{Gal}(\bar{k}/k) : \mathcal{D}_{p^*}] = [\text{Gal}(N_{\infty}/k) : D_{p^*}].
$$
Let \( k_\infty/k \) be a \( \mathbb{Z}_p \)-extension. For each \( n \geq 0 \), we denote by \( k_n \) the intermediate field of the \( \mathbb{Z}_p \)-extension \( k_\infty \) such that \( k_n \) is the unique cyclic extension over \( k \) of degree \( p^n \). Let \( K/k \) be an algebraic extension. For a prime ideal \( q \) of \( K \), let \( K_q \) be the completion of \( K \) at \( q \). We denote by \( U_q^{(1)} \) the principal local unit group with respect to \( q \) in \( K_q \).

### 4.1. Normal case.

In this subsection, we will prove the inequality (A) in the case where \( \mathcal{O}_p \) is a normal subgroup of \( \text{Gal}(\tilde{k}/\mathbb{Q}) \).

**Proposition 4.1.** Assume the same condition (i) as in Theorem 1. Assume also that weak GGC does not hold for \( p \) and \( k \) and that \( \mathcal{O}_p \) is a normal subgroup of \( \text{Gal}(\tilde{k}/\mathbb{Q}) \). Then the inequality (A) holds.

**Proof.** We assume that weak GGC does not hold for \( p \) and \( k \). By Proposition 3.12, we have \( M_p^*(\tilde{k}) = L_k^\prime \). By Lemma 3.8, \( p \) splits completely in \( \tilde{k}^{D_p} \). Since \( \mathcal{O}_p \) is a normal subgroup of \( \text{Gal}(\tilde{k}/\mathbb{Q}) \), we have \( \tilde{k}^{D_p} = \tilde{k}^{D_p} \). Let

\[
\{ p^*_{\infty,i} \mid 1 \leq i \leq p^*_{\infty} \}
\]

be the set of prime ideals of \( N_\infty \) lying above \( p^* \). We denote by \( I_{p^*_{\infty,i}} \) the inertia subgroup of \( \text{Gal}(M_p^*(N_\infty)/N_\infty) \) for the prime ideal \( p^*_{\infty,i} \). Then we have an exact sequence

\[
0 \to \sum_{i=1}^{p^*_{\infty}} I_{p^*_{\infty,i}} \to \text{Gal}(M_p^*(N_\infty)/N_\infty) \to \text{Gal}(L_{N_\infty}/N_\infty) \to 0.
\]

Since \( M_p^*(N_\infty)/\tilde{k} \) is an unramified extension, we have \( \text{Gal}(M_p^*(N_\infty)/\tilde{k}) \cap I_{p^*_{\infty,i}} = 1 \). Hence \( I_{p^*_{\infty,i}} \) is isomorphic to \( \mathbb{Z}_p \) for each \( i \). From [10, 30, 27], we obtain \( \mu(N_\infty/k) = 0 \). Furthermore, since we suppose that \( \lambda(N_\infty/k) = 0 \) and \( \text{Gal}(L_{N_\infty}/N_\infty) \) is finite. This implies that \( \text{Gal}(M_p^*(N_\infty)/N_\infty) \) is a finitely generated \( \mathbb{Z}_p \)-module. Therefore we obtain \( \text{rank}_{\mathbb{Z}_p}(\text{Gal}(M_p^*(N_\infty)/\tilde{k})) \leq p^*_{\infty} - 1 \). By Lemma 3.8, we have \( p^*_{\infty} \leq [L_k \cap \tilde{k} : k] \). Thus we get the conclusion. \( \square \)

### 4.2. Non-normal case.

In this subsection, we will prove the inequality (A) in the case where \( \mathcal{O}_p \) is not a normal subgroup of \( \text{Gal}(\tilde{k}/\mathbb{Q}) \).

**Proposition 4.2.** With the same notation as above, we have

\[
\text{rank}_{\mathbb{Z}_p}(\text{Gal}(M_p^*(N_{n_0})/N_{n_0})) = 1.
\]

**Proof.** Let \( \mathcal{O}_{N_{n_0}} \) be the ring of integers in \( N_{n_0} \) and

\[
\{ p^*_{n_0,i} \mid 1 \leq i \leq p^*_{n_0} \}
\]

be the set of prime ideals of \( N_{n_0} \) lying above \( p^* \). We put \( E_{n_0} = \{ u \in \mathcal{O}_{N_{n_0}}^\times \mid u \equiv 1 \mod p^*_{n_0,i} (1 \leq i \leq p^*_{n_0}) \} \). Since \( N_{n_0}/k \) is an abelian extension, Leopoldt's
We note that Proposition 4.3. Assume the same condition (i) as in Theorem 1.1. Assume also that weak GGC does not hold for \( p \) and \( k \) and that \( \mathfrak{D}_p \) is not a normal subgroup of \( \text{Gal}(\bar{k}/\mathbb{Q}) \). Then we have the inequality (A).

**Proof.** We note that \( p^* \) splits completely in \( N_{N_\infty}^{D_p} = N_{n_0} \) and that all prime ideals of \( N_{n_0} \) lying above \( p^* \) do not split in \( N_\infty \). Let

\[
\{ p^*_{\infty,i} \mid 1 \leq i \leq p^{n_0} \}
\]

be the set of prime ideals of \( N_\infty \) lying above \( p^* \) and \( I_{p^*_{\infty,i}} \) the inertia subgroup of \( \text{Gal}(M_{p^*}(N_{\infty})/N_\infty) \) for the prime ideal \( p^*_{\infty,i} \). By the same reason as in the proof of Proposition 1.1, we have \( I_{p^*_{\infty,i}} \cong \mathbb{Z}_p \) and \( \mu(N_\infty/k) = 0 \). Furthermore, the fixed field of \( M_{p^*}(N_\infty) \) by \( \sum_{i=1}^{p^{n_0}} I_{p^*_{\infty,i}} \) coincides with \( L_{N_\infty} \). Thus \( \text{Gal}(M_{p^*}(N_{\infty})/N_\infty) \) is a finitely generated \( \mathbb{Z}_p \)-module, since we suppose that \( \lambda(N_\infty/k) = 0 \). Let \( M' \) be the submodule of \( \mathbb{Z}_p \)-torsion in \( \text{Gal}(M_{p^*}(N_{\infty})/N_\infty) \). Hence we have an exact sequence

\[
0 \to M' \to \text{Gal}(M_{p^*}(N_{\infty})/N_\infty) \to \text{Gal}(M_{p^*}(N_{\infty})/N_\infty)/M' \to 0
\]

as \( \mathbb{Z}_p[[\text{Gal}(N_\infty/k)]] \)-modules. Since \( M' \) is a finite \( \mathbb{Z}_p \)-module, we have

\[
\text{char}_{\mathbb{Z}_p[[\text{Gal}(N_\infty/k)]]}(\text{Gal}(M_{p^*}(N_{\infty})/N_\infty)) = \text{char}_{\mathbb{Z}_p[[\text{Gal}(N_\infty/k)]]}(\sum_{i=1}^{p^{n_0}} I_{p^*_{\infty,i}}).
\]

For each \( i \) with \( 1 \leq i \leq p^{n_0} \), the decomposition group \( D_{p^*} \) acts on \( I_{p^*_{\infty,i}} \) because all prime ideals of \( N_{n_0} \) lying above \( p^* \) do not split in \( N_\infty \). Since we have \( \text{Gal}(M_{p^*}(N_{\infty})/k) \cap I_{p^*_{\infty,i}} = 1 \), there exists an injective homomorphism

\[
I_{p^*_{\infty,i}} \to \text{Gal}(\bar{k}/N_\infty)
\]
as $\mathbb{Z}_p[[\text{Gal}(N_\infty/N_{n_0})]]$-modules. Then $D_{p^*}$ acts on $I_{p^*_\infty,i}$ trivially because $\tilde{k}/N_{n_0}$ is an abelian extension. Hence $D_{p^*}$ acts on $\sum_{i=1}^{p^*_0} I_{p^*_\infty,i}$ trivially. Let $F$ be the fixed field of $M_{p^*}(N_\infty)$ by $M'$. By [5] and [6], $F/N_{n_0}$ is an abelian extension.

Let $s$ be a non-negative integer such that $p^s = [L_k \cap \tilde{k} : k]$. By Proposition 3.11 $p$ splits completely in $L_k \cap \tilde{k}$. We note that all prime ideals of $L_k \cap \tilde{k}$ lying above $p$ do not split in $N_\infty$. Let

$$\{p_{n_0,i} \mid 1 \leq i \leq p^s\}$$

be the set of prime ideals of $N_{n_0}$ lying above $p$ and $I_{p_{n_0,i}}$ the inertia subgroup of $\text{Gal}(F/N_{n_0})$ for the prime ideal $p_{n_0,i}$. Furthermore, by the same method as in the proof of Proposition 4.1 we have $I_{p_{n_0,i}} \cong \mathbb{Z}_p$ for each $i$. Since the fixed field in $F$ by $\sum_{i=1}^{p^s} I_{p_{n_0,i}}$ is contained in $M_{p^*}(N_{n_0})$, we obtain

$$\text{rank}_{\mathbb{Z}_p}(\text{Gal}(M_{p^*}(N_\infty)/N_{n_0})) = \text{rank}_{\mathbb{Z}_p}(\text{Gal}(F/N_{n_0})) \leq p^s + 1$$

by Proposition 4.2. Using Proposition 3.9 we have $p^s < p^{n_0}$. Thus we get the conclusion. \hfill \Box

5. Proof of the inequality (B)

In this section, we will prove the inequality (B):

$$\text{rank}_{\mathbb{Z}_p}(\text{Gal}(M_{p^*}(N_\infty)/\tilde{k})) \geq \min\{|L_k \cap \tilde{k} : k|, [\text{Gal}(\tilde{k}/k) : \mathcal{D}_{p^*}]\}$$

under the assumption of Theorem 1.1 and assuming that weak GGC does not hold for $p$ and $k$.

5.1. Module theory. In this subsection, we will prove some module theoretical properties needed later provided that weak GGC does not hold for $p$ and $k$. From Lemma 3.2 we have $\lambda(k_\infty/k) \geq 2$ if weak GGC does not hold for $p$ and $k$. Let $\text{Ass}_\Lambda(X_{\tilde{k}})$ be the set of associated prime ideals of $X_{\tilde{k}}$. In other words, we put

$$\text{Ass}_\Lambda(X_{\tilde{k}}) = \{p : \text{prime ideal} \mid p = \text{Ann}_\Lambda(x) \text{ for some element } x \text{ of } X_{\tilde{k}}\},$$

where we write $\text{Ann}_\Lambda(x) = \{a \in \Lambda \mid ax = 0\}$.

**Lemma 5.1.** Assume that weak GGC does not hold for $p$ and $k$. Let $Y$ be a $\Lambda$-submodule of $X_{\tilde{k}}$. If $Y$ is cyclic as a $\Lambda$-module, in other words, $Y$ is isomorphic to $\Lambda/\text{Ann}_\Lambda(Y)$, then $\text{Ann}_\Lambda(Y)$ is a principal ideal. In particular, an associated prime ideal of $X_{\tilde{k}}$ is principal.

**Proof.** We suppose that $Y$ is isomorphic to $\Lambda/\text{Ann}_\Lambda(Y)$. Let $g,h$ be elements of $\text{Ann}_\Lambda(Y)$. We denote by $G$ the greatest common divisor for $g$ and $h$. Then there exist $g', h' \in \Lambda$ such that $g = g'G, h = h'G$, and $g'$ is coprime to $h'$. Then $GY$ is a pseudo-null $\Lambda$-submodule of $X_{\tilde{k}}$. Since $X_{\tilde{k}}$ has no non-trivial pseudo-null $\Lambda$-submodule, we get $GY = 0$. Hence we have $G \in \text{Ann}_\Lambda(Y)$. This implies that $(g,h) \subset (G) \subset \text{Ann}_\Lambda(Y)$ as an ideal of $\Lambda$. Since $\text{Ann}_\Lambda(Y)$ is a finitely generated $\Lambda$-module, we can prove that $\text{Ann}_\Lambda(Y)$ is a principal ideal, inductively. \hfill \Box

If $X_{\tilde{k}}$ is cyclic as a $\Lambda$-module, we can determine the isomorphism class of $X_{\tilde{k}}$. 
**Proposition 5.2.** Assume that $X_{\tilde{k}}$ is cyclic as a $\Lambda$-module. Assume also that weak GGC does not hold for $p$ and $k$. Then we have

$$X_{\tilde{k}} \cong \Lambda/f(S,T)\Lambda$$

as $\Lambda$-modules, where $f(S,T)$ is the same annihilator of $X_{\tilde{k}}$ defined in Lemma 3.3.

**Proof.** Since $X_{\tilde{k}}$ is cyclic, we have $X_{\tilde{k}} \cong \Lambda/\text{Ann}_\Lambda(X_{\tilde{k}})$. By Lemma 5.1, $\text{Ann}_\Lambda(X_{\tilde{k}})$ is a principal ideal. Let $G(S,T) \in \Lambda$ be a generator of $\text{Ann}_\Lambda(X_{\tilde{k}})$. Using Lemma 3.1, we have $\text{char}_{\mathbb{Z}_p[[T]]}(X_{k_\infty}) = (TG(0,T))$. By Lemma 3.3, there exists a power series $H(S,T) \in \Lambda$ such that $f(S,T) = G(S,T)H(S,T)$. Using Proposition 3.5, we have $\text{char}_{\mathbb{Z}_p[[T]]}(X_{k_\infty}) = (Tf(0,T))$. This implies that $H(S,T) \in \Lambda^\times$. Therefore we have $\text{Ann}_\Lambda(X_{\tilde{k}}) = (f(S,T))$. Thus we get the conclusion. □

To consider the case where $X_{\tilde{k}}$ is not $\Lambda$-cyclic, we prove the following

**Lemma 5.3.** Suppose that $\lambda(k_\infty^c/k) \geq 2$ and that $X_{\tilde{k}}$ is not cyclic as a $\Lambda$-module. Let

$$f(S,T) = \prod_{i=1}^l f_i(S,T)^{n_i}$$

be a prime factorization, in other words, $l$ and $n_i$’s are positive integers and $f_i(S,T)$’s are irreducible elements of $\Lambda$. Assume that the characteristic ideal of $X_{k_\infty}$ has a generator which is square-free. Then $f_i(S,T)$ is a zero-divisor of $X_{\tilde{k}}$ for each $i$.

**Proof.** We note that $\text{char}_\Lambda(X_{\tilde{k}})$ has a square-free generator because the characteristic ideal of $X_{k_\infty}$ also has a square-free generator. Hence we have $n_i = 1$ for each $i$. In the case of $l = 1$, in other words, $f(S,T)$ is an irreducible element, $f(S,T)$ is a zero-divisor of $X_{\tilde{k}}$ by Lemma 3.3. Thus we get the conclusion. In the following, we suppose that $l \geq 2$. We put $F_i(S,T) = \prod_{j=1,j \neq i}^l f_j(S,T)$. We prove that $F_i(S,T)X_{\tilde{k}} \neq 0$. Suppose that $F_i(S,T)X_{\tilde{k}} = 0$. Since $X_{\tilde{k}}$ is a finitely generated $\Lambda$-module, there exists a positive integer $r$ and a surjective homomorphism

$$(\Lambda/F_i(S,T)\Lambda)^{\oplus r} \rightarrow X_{\tilde{k}}.$$ 

This homomorphism induces

$$(\mathbb{Z}_p[[T]]/F_i(0,T)\mathbb{Z}_p[[T]])^{\oplus r} \rightarrow X_{\tilde{k}}/SX_{\tilde{k}}.$$ 

Since the characteristic ideal of $X_{k_\infty}$ has a square-free generator, we have

$$(TF(0,T) \subset \text{char}_{\mathbb{Z}_p[[\text{Gal}(k_\infty^c/k)]]}(X_{k_\infty}) = (Tf(0,T))$$

by Proposition 3.5. Hence we have $\deg(F_i(0,T)) \geq \deg(f(0,T))$. This implies that $f_i(S,T)$ is a unit in $\Lambda$ because we have $f(0,T) = F_i(0,T)f_i(0,T)$. This is a contradiction. Thus we get the conclusion. □

**Proposition 5.4.** Assume that weak GGC does not hold for $p$ and $k$. Assume also that the characteristic ideal of $X_{k_\infty}$ has a generator which is square-free or $X_{\tilde{k}}$ is cyclic as a $\Lambda$-module. Then there exists a $\Lambda$-submodule $Y$ of $X_{\tilde{k}}$ such that $Y$ is isomorphic to $\Lambda/(f(S,T))$. 


Proof: If $X_{\tilde{k}}$ is cyclic as a $\Lambda$-module, then $X_{\tilde{k}}$ is isomorphic to $\Lambda/(f(S,T))$ by Proposition 5.2. Hence we can take $Y = X_{\tilde{k}}$. Thus we get the conclusion. We suppose that $X_{\tilde{k}}$ is not cyclic as a $\Lambda$-module. Since $\Lambda$ is a noetherian ring, $\bigcup_{p \in \text{Ass}_{\Lambda}(X_{\tilde{k}})} p$ is the set of zero-divisors of $X_{\tilde{k}}$ by [19, Theorem 6.1]. Let $p$ be an associated prime ideal of $X_{\tilde{k}}$. Let $f(S,T) = \prod_{i=1}^{l} f_{i}(S,T)$ be the same prime factorization as in Lemma 5.3. Since $f_{i}(S,T)$ is an irreducible element of $\Lambda$, $(f_{i}(S,T))$ is a prime ideal. We claim that $(f_{i}(S,T))$ is an associated prime ideal of $X_{\tilde{k}}$ for each $i$. Indeed, $f_{i}(S,T)$ is a zero-divisor of $X_{\tilde{k}}$ by Lemma 5.3. Hence we have $(f_{i}(S,T)) \subset \bigcup_{p \in \text{Ass}_{\Lambda}(X_{\tilde{k}})} p$. By the prime avoidance theorem, $(f_{i}(S,T)) \subset p$ for some associated prime ideal $p$. By Lemma 5.1, $p$ is a principal ideal. Therefore we obtain $p = (f_{i}(S,T))$. Furthermore, we can prove that

$$\text{Ass}_{\Lambda}(X_{\tilde{k}}) = \{(f_{i}(S,T)) \mid 1 \leq i \leq l\}.$$ 

Indeed, let $p$ be an associated prime ideal of $X_{\tilde{k}}$. There exists an element $x$ of $X_{\tilde{k}}$ such that $p = \text{Ann}_{\Lambda}(x)$. By Lemma 5.1, $p$ is a principal ideal of $\Lambda$. We put $p = (g)$ for some irreducible element $g$ in $\Lambda$. If $g$ is coprime to $f(S,T)$, then $\Lambda x$ is a non-trivial pseudo-null $\Lambda$-submodule of $X_{\tilde{k}}$. This is a contradiction. Therefore there exists an integer $i$ such that $(g) = (f_{i}(S,T))$.

By the structure theorem, there exists a pseudo-isomorphism

$$\varphi : \bigoplus_{i=1}^{l'} \Lambda/q_{i} \rightarrow X_{\tilde{k}},$$

where $l'$ is a positive integer and $q_{i}$'s are prime ideals of height one. We note that $\text{char}_{\Lambda}(X_{\tilde{k}})$ has a square-free generator by the same reason as in the proof of Lemma 5.3. Hence we obtain $q_{i} \neq q_{j}$ for $i \neq j$. Since $\bigoplus_{i=1}^{l'} \Lambda/q_{i}$ has no non-trivial pseudo-null $\Lambda$-submodule, this homomorphism is injective. In the following, we will prove that $q_{i}$ is an associated prime ideal for each $i$ and that $\text{Ass}_{\Lambda}(X_{\tilde{k}})$ coincides with the set $\{q_{i} \mid 1 \leq i \leq l'\}$. Indeed, for each $j$ with $1 \leq j \leq l'$, we have a homomorphism

$$\pi_{j} : \Lambda/q_{j} \rightarrow \bigoplus_{i=1}^{l'} \Lambda/q_{i} \rightarrow X_{\tilde{k}},$$

where $\Lambda/q_{j} \rightarrow \bigoplus_{i=1}^{l'} \Lambda/q_{i}$ is the natural injective homomorphism such that, for an element $a$ of $\Lambda/q_{j}$, we define $a \mapsto (a_{i})$, satisfying that $a_{i} = 0$ for $i \neq j$ and $a_{j} = a$. Hence $\pi_{j}$ is an injective homomorphism. Thus $q_{j}$ is an associated prime ideal. This implies that $\{q_{i} \mid 1 \leq i \leq l'\}$ is contained in $\text{Ass}_{\Lambda}(X_{\tilde{k}})$. Next we prove that $\text{Ass}_{\Lambda}(X_{\tilde{k}})$ is contained in $\{q_{i} \mid 1 \leq i \leq l'\}$. We assume that $\text{Ass}_{\Lambda}(X_{\tilde{k}}) \not\subset \{q_{i} \mid 1 \leq i \leq l'\}$. Then there exists an associated prime ideal $p$ such that $p \not\subset \{q_{i} \mid 1 \leq i \leq l'\}$. We note that $p$ is a prime ideal of height one by Lemma 5.1. Using localization with respect to
p, we obtain
\[(X_\tilde{k})_p \cong \left( \bigoplus_{i=1}^{l'} \Lambda/q_i \right)_p = 0\]
from (7). Hence \(\text{Ass}_\Lambda(X_\tilde{k})\) is not contained in the support of \(X_\tilde{k}\). This is a contradiction. Thus we get
\(\text{Ass}_\Lambda(X_\tilde{k}) = \{q_i \mid 1 \leq i \leq l'\} = \{(f_i(S,T)) \mid 1 \leq i \leq l\}\).
This implies that \(l = l'\). Therefore we have a natural injective homomorphism
\[\Lambda/(f(S,T)) \to \bigoplus_{i=1}^{l'} \Lambda/(f_i(S,T)) = \bigoplus_{i=1}^{l} \Lambda/q_i \to X_{\hat{k}}\]
since we have a prime factorization \(f(S,T) = \prod_{i=1}^{l} f_i(S,T)\). Thus we obtain the conclusion. \(\square\)

By the proposition above, we obtain the following

**Corollary 5.5.** Let \(M\) be a finitely generated torsion \(\Lambda\)-module. Assume that \(M/SM\) is a free \(\mathbb{Z}_p\)-module of finite rank and that \(\text{char} \mathbb{Z}_p[[T]](M/SM)\) has a generator which is square-free. Assume also that \(M\) has no non-trivial pseudo-null \(\Lambda\)-submodule. Then we have a pseudo-isomorphism
\[M \to \Lambda/\text{char}_\Lambda(M)\].

Let \(k_\infty/k\) be a \(\mathbb{Z}_p\)-extension. Then there exists a pair \((\alpha, \beta) \in \mathbb{Z}_p^* - p\mathbb{Z}_p^*\) such that \(k_\infty = k^{(\alpha \tau \beta)}\), where \(\sigma\) and \(\tau\) are the fixed topological generators defined in Section 2. It is easy to check the following

**Lemma 5.6.** Let \(k_\infty/k\) be a \(\mathbb{Z}_p\)-extension with \(k_\infty \cap k^a_\infty = k^a_m\), where \(k^a_m\) is the \(m\)-th layer of \(k_\infty^a\) and \(m\) is a non-negative integer. Then there exists a unit \(u \in \mathbb{Z}_p^*\) such that \(k_\infty = k^{(au^m \tau)}\).

We put \(k = \text{Gal}(\tilde{k}/N_\infty)\). Recall that \(p^* = [L_k \cap k : k]\). Hence we have \(N_\infty \cap k^a_\infty = L_k \cap k = k^a_\infty\). Then there exists a unit \(u\) such that
\[\text{Gal}(\tilde{k}/N_\infty) = \langle \sigma^{au^m \tau} \rangle\]
by the lemma above. We put \(T_\infty = (1 + S)^{au^m}(1 + T) - 1\).

The following proposition plays an important role to obtain a lower bound of the \(\mathbb{Z}_p\)-rank of \(\text{Gal}(M_p, (N_\infty)/k)\).

**Proposition 5.7.** Under the same assumption as in Proposition 5.4, we have \(\text{rank}_{\mathbb{Z}_p}((X_{\hat{k}})^\vee) \geq \text{rank}_{\mathbb{Z}_p}(Y^\vee)\), where \(Y\) is the same cyclic \(\Lambda\)-submodule of \(X_{\hat{k}}\) as in Proposition 5.4.

**Proof.** If \(X_{\hat{k}}\) is cyclic as a \(\Lambda\)-module, then we have \(Y = X_{\hat{k}}\). Thus we get the conclusion. We suppose that \(X_{\hat{k}}\) is not cyclic as a \(\Lambda\)-module. By Proposition 5.4 we have an exact sequence
\[0 \to Y \to X_{\hat{k}} \to \text{Coker} \to 0,\]
where we put \( \text{Coker} = X_k/Y \). Using the snake lemma, we get

\[
0 \to Y^\Gamma \to (X_k)^\Gamma \to \text{Coker}^\Gamma \to Y \to (X_k)_\Gamma \to \text{Coker}^\Gamma \to 0.
\]

Since \( X_k \) has no non-trivial pseudo-null \( \Lambda \)-submodule, we have \((X_k)^\Gamma = 0\). Indeed, we have \( T_s(X_k)^\Gamma = f(S,T)(X_k)^\Gamma = 0 \). Since the characteristic ideal of \( X_{k_\infty} \) is generated by a power series which is square-free, \( T_s \) is coprime to \( f(S,T) \). This implies that \((X_k)^\Gamma\) is pseudo-null. Hence we have \((X_k)^\Gamma = 0\). Then we obtain an exact sequence

\[
(8) \quad 0 \to \text{Coker}^\Gamma \to Y \to (X_k)_\Gamma \to \text{Coker}^\Gamma \to 0.
\]

We identify the following rings below, using isomorphisms

\[
(9) \quad \Lambda/T_s \Lambda \cong \mathbb{Z}_p[[S]] \cong \mathbb{Z}_p[\text{Gal}(N_\infty/k)],
\]

where we define

\[
G(S, T) \mapsto G(S, (1 + S)^{-up^s} - 1) \mapsto G(\sigma \text{Gal}(\bar{k}/N_\infty) - 1, \sigma^{-up^s} \text{Gal}(\bar{k}/N_\infty) - 1).
\]

Here we note that \( \mathbb{Z}_p[[S, T]] = \mathbb{Z}_p[[S, T]] \) because we have \( T = (1 + T_s)(1 + S)^{-up^s} - 1 \). We will prove that \( Y^\Gamma \) is a finitely generated \( \mathbb{Z}_p \)-module. Indeed, \( \text{Coker}^\Gamma \) is a finitely generated \( \mathbb{Z}_p \)-module since \((X_k)^\Gamma \), which is isomorphic to \( \text{Gal}(M_\infty/k) \), is a finitely generated \( \mathbb{Z}_p \)-module. Furthermore, we can show that \( \text{Coker}^\Gamma \) is a pseudo-null \( \Lambda \)-module by the same method as above. From (9), we have

\[
\begin{align*}
\text{char}_{\mathbb{Z}_p[[S]]}(\text{Coker}^\Gamma) &= \pi(\text{char}_\Lambda(\text{Coker})) \text{char}_{\mathbb{Z}_p[[S]]}(\text{Coker}^\Gamma),
\end{align*}
\]

where \( \pi \) is the natural projection from \( \Lambda \) to \( \Lambda/T_s \Lambda \). This implies that \( Y^\Gamma \) is a finitely generated \( \mathbb{Z}_p \)-module. Hence all terms in (8) have finite \( \mathbb{Z}_p \)-rank. Therefore we obtain

\[
\begin{align*}
\text{rank}_{\mathbb{Z}_p}(X_k)^\Gamma &= \text{rank}_{\mathbb{Z}_p}(Y^\Gamma) + \text{rank}_{\mathbb{Z}_p}(\text{Coker}^\Gamma) - \text{rank}_{\mathbb{Z}_p}(\text{Coker}^\Gamma) \\
&\geq \text{rank}_{\mathbb{Z}_p}(Y^\Gamma).
\end{align*}
\]

Thus we get the conclusion. \( \square \)

5.2. **Normal case.** In this subsection, we prove the inequality (B) in the case where \( k \) is not \( p \)-split \( p \)-rational assuming that weak GGC does not hold for \( p \) and \( k \). We first give a necessary and sufficient condition for \( k \) to be \( p \)-split \( p \)-rational.

**Proposition 5.8.** The following two conditions are equivalent:
(i) The decomposition group \( D_p \) is a normal subgroup of \( \text{Gal}(k/\mathbb{Q}) \), \( [\text{Gal}(\bar{k}/k) : D_p] = \#(\mathbb{Z}_p/g_0(0)\mathbb{Z}_p) \), and \( \lambda(k_{\infty}/k) \geq 2 \), where \( g_0(S) \) is the same power series defined in Lemma 3.3.
(ii) The imaginary quadratic field \( k \) is \( p \)-split \( p \)-rational.
Proof. We suppose that (i) holds. Let $L$ be the fixed field of $L_{k_{\infty}}$ by $TX_{k_{\infty}}$. We note that $Gal(L/k_{\infty}) = (X_{k_{\infty}})_{Gal(k_{\infty}/k)}$. Using the exact sequence (3) in the proof of Proposition 3.9, we obtain

$$
\#\text{Image}(\text{Gal}(\tilde{k}/k_{\infty})) \to (X_{k})_{\text{Gal}(\tilde{k}/k)} = \#(X_{k})_{\text{Gal}(\tilde{k}/k)}
$$

from $[\text{Gal}(\tilde{k}/k) : D_p] = \#(\mathbb{Z}/g_0(0)\mathbb{Z})$ and from Lemma 3.6. Hence the homomorphism $\text{Gal}(\tilde{k}/k_{\infty}) \to (X_{k})_{\text{Gal}(\tilde{k}/k)}$ is surjective. This implies that $L = \tilde{k}$ from (3). Since $L_{k_{\infty}}/k_{\infty}$ is an abelian extension, $L_{k_{\infty}}$ is contained in $L$. Therefore, we obtain $L_k \subset \tilde{k}$. Then it is easy to check that $M_p(k) = \tilde{k}$, where $M_p(k)$ is the maximal pro-$p$ abelian extension field of $k$ unramified outside all prime ideals lying above $p$. By Lemma 3.8, we have $\tilde{k}^{D_p} \subset L_k$ since $D_p$ is a normal subgroup. Furthermore, we have $\tilde{k}^{D_p} \neq k$ from $\lambda(k_{\infty}/k) > 2$. Hence $k$ is $p$-split $p$-rational.

Conversely we suppose that $k$ is $p$-split $p$-rational. By Definition 1, we have $L_k \subset \tilde{k}$, $\tilde{k}^{D_p} \neq k$, $\tilde{k}^{D_p} \subset L_k$.

This implies that $M_p(k) = \tilde{k}$ and that $p$ splits in $\tilde{k}$. Hence the homomorphism $\text{Gal}(\tilde{k}/k_{\infty}) \to (X_{k})_{\text{Gal}(\tilde{k}/k)}$ above is surjective. We note that $\lambda(k_{\infty}/k) > 2$. By the exact sequence (3) in the proof of Proposition 3.9, we obtain

$$
[\text{Gal}(\tilde{k}/k) : D_p] = \#(X_{k})_{\text{Gal}(\tilde{k}/k)} = \#(\mathbb{Z}/g_0(0)\mathbb{Z})
$$

Since $\tilde{k}^{D_p}$ is contained in $L_k$, we have $[\text{Gal}(\tilde{k}/k) : D_p] \leq [L_k : k]$. By Lemma 3.8, $D_p$ is a normal subgroup of $Gal(\tilde{k}/\mathbb{Q})$. Thus we get the conclusion. □

For a power series $V(S) = \sum_{i=0}^{\infty} b_i S^i \in \mathbb{Z}[S]$, we put

$$
\mu(V(S)) = \sup \{ i \mid V(S) \equiv 0 \text{ mod } p^i \},
\lambda(V(S)) = \inf \{ i \mid b_i \neq 0 \text{ mod } p \}.
$$

Then we call $\mu(V(S)), \lambda(V(S))$ the $\mu$- and $\lambda$-invariant of $V(S)$, respectively.

Next, we get a lower bound of the $\lambda$-invariant of $g_0(S)$ in the case where the $\mu$-invariant of $g_0(S)$ is zero. Recall that $p^{n_0} = [\text{Gal}(\tilde{k}/k) : D_p]$.

**Proposition 5.9.** Assume that $D_p$ is a normal subgroup of $Gal(\tilde{k}/\mathbb{Q})$. Assume also that $k$ is not $p$-split $p$-rational and that $\lambda(k_{\infty}/k) > 2$. If $\mu(g_0(S))$ is zero, then we have $\lambda(g_0(S)) \geq p^{n_0}$.

Proof. Since $k_{a_n}/k$ is an abelian extension, Leopoldt’s conjecture holds for $p$ and $k_{a_n}$ (3). Hence we have $Gal(\overline{k_{a_n}}/k_{a_n}) \cong \mathbb{Z}/p^{n_0+1}$, where $\overline{k_{a_n}}$ is the composite of all $\mathbb{Z}/p$-extension fields over $k_{a_n}$. Since $p$ splits in $k_{a_n}$, $k_{a_n}/k_{\infty}$ is an unramified abelian extension. Thus we have a surjective homomorphism

$$
X_{k_{\infty}} \to Gal(\overline{k_{a_n}}/k_{a_n}).
$$

Furthermore, we see that

$$
Gal(\overline{k_{a_n}}/k_{\infty}) \cong \mathbb{Z}/[S]/((1 + S)^{p^{n_0}} - 1).
$$

Hence we have $\text{char}_{\mathbb{Z}[S]}(X_{k_{\infty}}) \subset ((1 + S)^{p^{n_0}} - 1)$. On the other hand, there exists a surjective homomorphism $(\Lambda/f(S,T)\Lambda)^{\mathbb{Z}/p^r} \to X_{\tilde{k}}$, where $r$ is a positive integer and
$f(S, T)$ is the same power series defined in Lemma \ref{lem:power_series}. Then we have a surjective homomorphism $(\mathbb{Z}_p[[S]]/g_0(S)\mathbb{Z}_p[[S]])^{\theta r} \to X_\kappa/TX_\kappa$. This implies that

$$(Sg_0(S)^r) \subset \text{char}_{\mathbb{Z}_p[[S]]}(X_{k_{\infty}}) \subset ((1 + S)^{p^{\mu_0}} - 1).$$

Since $(1 + S)^{p^{\mu_0}} - 1$ is square-free, we obtain $(Sg_0(S)^r) \subset ((1 + S)^{p^{\mu_0}} - 1)$. Then there exists a power series $\tilde{g}_0(S) \in \mathbb{Z}_p[[S]]$ such that $Sg_0(S) = \{(1 + S)^{p^{\mu_0}} - 1\}\tilde{g}_0(S)$. We have $p^{\mu_0} = [\text{Gal}(k/k) : \mathcal{O}_p] < \#(\mathbb{Z}_p/g_0(0)\mathbb{Z}_p)$ by Propositions \ref{prop:valuation} and \ref{prop:valuation2}.

Hence, we get $\text{ord}_p(\tilde{g}_0(0)) > 0$, where $\text{ord}_p$ is the normalized additive valuation on $\mathbb{Q}_p$ of $p$-adic numbers such that $\text{ord}_p(p) = 1$. Therefore, we obtain $\lambda(g_0(S)) \geq p^{\mu_0}$ because $\mu(g_0(S))$ is zero.

By the proposition above, we can prove the inequality (B).

**Proposition 5.10.** Assume the same condition as in Proposition \ref{prop:valuation}. Also assume that $\mathcal{O}_p$ is a normal subgroup of $\text{Gal}(\bar{k}/\mathbb{Q})$ and that $k$ is not $p$-split $p$-rational. Then the inequality (B) holds.

**Proof.** We have $\lambda(k_{\infty}/k) \geq 2$ since weak GGC does not hold. By Proposition \ref{prop:valuation} there exists a $\Lambda$-submodule $Y$ of $X_\kappa$ such that $Y$ is isomorphic to $\Lambda/(f(S, T))$. We note that $Y_\Gamma$ is a finitely generated $\mathbb{Z}_p$-module by the proof of Proposition \ref{prop:valuation2}. We claim that $\text{rank}_{\mathbb{Z}_p}(Y_\Gamma) \geq p^{\mu_0}$. By Lemma \ref{lem:valuation}, there exists a unit $u$ such that

$$\Gamma = \text{Gal}(\bar{k}/N_{\infty}) = \langle \sigma^{up^r} \rangle.$$

Then we have

$$Y_\Gamma \cong \Lambda/(f(S, T), (1 + S)^{up^r}(1 + T) - 1) \cong \mathbb{Z}_p[[S]]/(f(S, (1 + S)^{-up^r} - 1)).$$

By the definition of $f(S, T)$, we obtain

$$f(S, (1 + S)^{-up^r} - 1) = \{(1 + S)^{-up^r} - 1\}^{\lambda^*} + \sum_{i=0}^{\lambda^* - 1} g_i(S)(1 + S)^{-up^r} - 1)^i$$

$$\equiv \left\{ \sum_{j=1}^{\infty} \frac{(-u)}{j} S^{jp^r} \right\}^{\lambda^*} + \sum_{i=0}^{\lambda^* - 1} g_i(S) \left\{ \sum_{j=1}^{\infty} \frac{(-u)}{j} S^{jp^r} \right\}^i \mod p$$

$$\equiv \begin{cases} g_0(S) \mod (p, S^{p^r}) & \text{if } \mu(g_0(S)) = 0, \\ 0 & \text{if } \mu(g_0(S)) > 0. \end{cases}$$

By Proposition \ref{prop:valuation}, we obtain $\lambda(f(S, (1 + S)^{-up^r} - 1)) \geq p^{\mu_0}$. This implies that $\text{rank}_{\mathbb{Z}_p}(Y_\Gamma) \geq p^{\mu_0}$. Thus we obtain $\text{rank}_{\mathbb{Z}_p}(X_{k_{\infty}}^r) \geq p^{\mu_0}$ by Proposition \ref{prop:valuation2}. Using Proposition \ref{prop:valuation}, we get the conclusion.

### 5.3. Non-normal case.

In this subsection, we prove the inequality (B) in the case where $\mathcal{O}_p$ is not a normal subgroup of $\text{Gal}(\bar{k}/\mathbb{Q})$ under the assumption that weak GGC does not hold for $p$ and $k$. We first show the following two propositions.
Proposition 5.11. Assume the same condition as in Proposition 5.4. Assume also that \( D_p \) is not a normal subgroup of \( \text{Gal}(\bar{k}/\mathbb{Q}) \). Suppose that \( \mu(k^a_\infty/k) = 0 \), then we have \( \lambda(k^a_\infty/k) \geq [L_k \cap \bar{k} : k] + 1 \).

Proof. We suppose that \( \mu(k^a_\infty/k) = 0 \). By Proposition 5.4, we have a \( \Theta \)-submodule \( Y \). Assume the same condition as in Proposition 5.4. Assume also Proposition 5.11. Since we suppose that \( \lambda(k^a_\infty/k) \geq [L_k \cap \bar{k} : k] \), we have a surjective homomorphism \( X_{k^a_\infty} \to \text{Gal}(\bar{k}/k^a_\infty) \). Then we have \( \text{char}_{Z_p[[S]]}(X_{k^a_\infty}) = \text{char}_{Z_p[[S]]}(\text{Gal}(\bar{k}/k^a_\infty)) \) because we suppose that \( \lambda(k^a_\infty/k) = [L_k \cap \bar{k} : k] \). Using \( \text{char}_{Z_p[[S]]}(\text{Gal}(\bar{k}/k^a_\infty)) = ((1+S)^{p^*} - 1) \), we get \( \text{char}_{Z_p[[S]]}(X_{k^a_\infty}) = ((1+S)^{p^*} - 1) \). By Proposition 5.4, we have a \( \Theta \)-submodule \( Y \) of \( X_{\bar{k}} \) such that \( Y \) is isomorphic to \( \Lambda/(f(S,T)) \) and has an exact sequence

\[
0 \to Y \to X_{\bar{k}} \to \text{Coker} \to 0.
\]

The \( \Theta \)-modules \( (X_{\bar{k}})_{\text{Gal}(\bar{k}/k^a_\infty)} \) and \( \text{Coker}_{\text{Gal}(\bar{k}/k^a_\infty)} \) are pseudo-null as \( \Theta \)-modules because \( T \) is coprime to \( f(S,T) \). Hence we get the exact sequence

\[
0 \to \text{Coker}_{\text{Gal}(\bar{k}/k^a_\infty)} \to Y_{\text{Gal}(\bar{k}/k^a_\infty)} \to (X_{\bar{k}})_{\text{Gal}(\bar{k}/k^a_\infty)} \to \text{Coker}_{\text{Gal}(\bar{k}/k^a_\infty)} \to 0.
\]

Since we suppose that \( \mu(k^a_\infty/k) = 0 \), \( (X_{\bar{k}})_{\text{Gal}(\bar{k}/k^a_\infty)} \) is a finitely generated \( \mathbb{Z}_p \)-module. By the same method as in the proof of Proposition 5.7, we see that \( Y_{\text{Gal}(\bar{k}/k^a_\infty)} \) is finitely generated as a \( \mathbb{Z}_p \)-module. This implies that \( \mu(g_0(S)) = 0 \) and that

\[
\text{rank}_{Z_p}(X_{\bar{k}}/TX_{\bar{k}}) \geq \text{rank}_{Z_p}(Z_p[[S]]/Z_p[[S]]).
\]

By Lemma 3.3, there exists a surjective homomorphism \( (\Lambda/(f(S,T))\Lambda)^{\oplus r} \to X_{\bar{k}} \), where \( r \) is a positive integer. Hence this homomorphism induces a surjective homomorphism \( (Z_p[[S]]/Z_p[[S]])^{\oplus r} \to X_{\bar{k}}/TX_{\bar{k}} \). Then we have \( (g_0(S)^r) \subset \text{char}_{Z_p[[S]]}(X_{\bar{k}}/TX_{\bar{k}}) \). Since \( (1+S)^{p^*} - 1 \) is square-free, we obtain \( \text{char}_{Z_p[[S]]}(X_{\bar{k}}/TX_{\bar{k}}) \subset (g_0(S)) \). This implies that

\[
\text{char}_{Z_p[[S]]}(X_{k^a_\infty}) \subset (g_0(S)).
\]

Therefore there exists a power series \( \psi(S) \) such that \( (1+S)^{p^*} - 1 = Sg_0(S)\psi(S) \). Hence we have \( p^* = g_0(0)\psi(0) \) and \( s \geq \text{ord}_p(g_0(0)) \). This contradicts Proposition 3.9. Thus we get the conclusion. □

Proposition 5.12. Assume the same condition as in Proposition 5.4. Assume also that \( D_p \) is not a normal subgroup of \( \text{Gal}(\bar{k}/\mathbb{Q}) \). Suppose that \( \mu(g_0(S)) = 0 \), then we have \( \lambda(g_0(S)) \geq [L_k \cap k : k] \).

Proof. By the same argument in Proposition 5.11 we have

\[
\text{char}_{Z_p[[S]]}(X_{k^a_\infty}) \subset \text{char}_{Z_p[[S]]}(\text{Gal}(k^a_\infty/k^a_\infty)) = ((1+S)^{p^*} - 1)
\]

and have a surjective homomorphism

\[
(Z_p[[S]]/g_0(S)Z_p[[S]])^{\oplus r} \to X_{\bar{k}}/TX_{\bar{k}}
\]

for some positive integer \( r \). Thus we get

\[
(Sg_0(S)^r) \subset \text{char}_{Z_p[[S]]}(X_{k^a_\infty}) \subset ((1+S)^{p^*} - 1).
\]
Since \((1+S)^{p^r} - 1\) is square-free, we have \((Sg_0(S)) \subset ((1+S)^{p^r} - 1)\). By the \(p\)-adic Weierstrass preparation theorem (\([22\) Theorem 7.3]), there exist a distinguished polynomial \(\tilde{g}_0(S) \in \mathbb{Z}_p[[S]]\) and a unit \(u(S) \in \mathbb{Z}_p[[S]]^\times\) such that

\[
g_0(S) = \frac{(1+S)^{p^r} - 1}{S} \tilde{g}_0(S) u(S).
\]

Thus we obtain \(\text{ord}_p(\tilde{g}_0(0)) > 0\) by Proposition 3.9. This implies that \(\lambda(g_0(S)) \geq p^s\) if \(\mu(g_0(S)) = 0\).

By the following proposition, we can prove the inequality (B).

**Proposition 5.13.** Assume the same condition as in Proposition 5.4. Assume also that \(D_p\) is not a normal subgroup of \(\text{Gal}(\tilde{k}/\mathbb{Q})\). Then the inequality (B) holds.

**Proof.** We use the same method as in the proof of Proposition 5.10. By Proposition 5.4, there exists a \(\Lambda\)-submodule \(Y\) of \(\tilde{k}\) such that \(Y\) is isomorphic to \(\Lambda/(f(S,T))\).

We will prove that \(\text{rank}_{\mathbb{Z}_p}(Y_\Gamma) \geq p^s\). There exists a unit \(u\) such that

\[
\text{Gal}(\tilde{k}/N_{\infty}) = \langle \sigma u^p \tau \rangle.
\]

Then we have

\[
Y_\Gamma \cong \mathbb{Z}_p[[S]]/(f(S, (1+S)^{-u^p} - 1)).
\]

By the proof of Proposition 5.4, \(Y_\Gamma\) is a finitely generated \(\mathbb{Z}_p\)-module. Hence we have \(\mu(f(S, (1+S)^{-u^p} - 1)) = 0\). Furthermore, we have

\[
f(S, (1+S)^{-u^p} - 1) = \begin{cases} g_0(S) \mod (p, S^{p^r}) & \text{if } \mu(g_0(S)) = 0, \\ 0 \mod (p, S^{p^r}) & \text{if } \mu(g_0(S)) > 0. \end{cases}
\]

Using Proposition 5.12, we have \(\text{rank}_{\mathbb{Z}_p}(Y_\Gamma) \geq p^s\). By Proposition 5.7, we obtain \(\text{rank}_{\mathbb{Z}_p}(X_\tilde{k}) \geq p^s\). This implies the inequality (B) from Proposition 3.9. Thus we get the conclusion. \(\Box\)

Combining the results of Sections 4 and 5, we have completed the proof of Theorem 1.1.

**Remark 5.14.** Theorem 1.1 holds without the assumption (ii) if \(X_\tilde{k}\) is not cyclic as a \(\Lambda\)-module. Indeed, we have the inequality (A) under the assumption (i) in Theorem 1.1. Furthermore, we have verified the inequality (B) in Propositions 5.10 and 5.13. In these propositions, we assumed (ii) in the case where \(X_\tilde{k}\) is not cyclic as a \(\Lambda\)-module.

### 6. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. In the following, we prove some lemmas and propositions needed later. Let \(k\) be a \(p\)-split \(p\)-rational field. By Definition 1, we have \(k^{\mathbb{F}_p} = k^{\mathcal{D}_r}\) and \(|\text{Gal}(\tilde{k}/k) : \mathcal{D}_p| \geq p\). We put \(\nu_m(S) = ((1+S)^{p^m} - 1)/S\) for a non-negative integer \(m\). As we stated in Section 1, we see that \(X_\tilde{k}\) is \(\Lambda\)-cyclic by the following

**Proposition 6.1** (\([22\) Propositions 3.5 and 3.8]). Assume that \(L_k \subset \tilde{k}\) and that \(\lambda(k_{\infty}/k) \geq 2\). Then we have a surjective homomorphism

\[
\Lambda/f(S,T)\Lambda \rightarrow X_\tilde{k}
\]
as a $\Lambda$-module, where $f(S,T)$ is the same power series defined in Lemma 3.3. Furthermore, if $\mathcal{O}_p$ is a normal subgroup of $\text{Gal}(\bar{k}/\mathbb{Q})$, then there exists a power series $u(S) \in \mathbb{Z}_p[[S]]^\times$ such that

$$f(S,0) = \nu_{n_0}(S)u(S),$$

where $n_0$ is the non-negative integer satisfying $[\text{Gal}(\bar{k}/\mathbb{Q}) : \mathcal{O}_p] = p^{n_0}$.

We have $\lambda(N_{\infty}/k) = 0$ by genus formula (see for example [22, Remark 3.2]). By Proposition 6.1, we can determine the isomorphism class of $X_\bar{k}$ as a $\Lambda$-module provided that weak GGC does not hold for $p$ and $k$. In fact, by Proposition 5.2, we have an isomorphism

$$X_\bar{k} \cong \Lambda/f(S,T)\Lambda.$$  

Using the isomorphism above, we can determine the characteristic ideal of $X_{k_{\infty}}^a$.

**Proposition 6.2.** Let $k$ be a $p$-split $p$-rational field. Assume that weak GGC does not hold for $p$ and $k$. Then we have

$$\text{char}_{\mathbb{Z}_p[[S]]}(X_{k_{\infty}}^a) = ((1 + S)^{p^{n_0}} - 1).$$

**Proof.** By Proposition 6.2 we have

$$X_\bar{k}/TX_\bar{k} \cong \mathbb{Z}_p[[S]]/g_0(S)\mathbb{Z}_p[[S]].$$

We note that $f(S,0) = g_0(S)$. Hence we obtain $\text{char}_{\mathbb{Z}_p[[S]]}(X_{k_{\infty}}^a) = ((1 + S)^{p^{n_0}} - 1)$ by Lemma 3.1 and Proposition 6.1.

As in Section 5, we put $p^s = [L_k : k]$. Then there exists a unit $u \in \mathbb{Z}_p^\times$ such that

$$\text{Gal}(\bar{k}/N_{\infty}) = \langle \sigma^u \tau^s \rangle$$

by Lemma 5.6.

By the same methods as Sections 4 and 5, we have the following

**Proposition 6.3.** Under the same assumption as in Proposition 6.2 we have

$$(X_\bar{k})_{\text{Gal}(\bar{k}/N_{\infty})} \cong \mathbb{Z}_p[[S]]/\nu_{n_0}(S)\mathbb{Z}_p[[S]].$$

**Proof.** By the same reason as in the proof of Proposition 5.13 we have

$$(X_\bar{k})_{\text{Gal}(\bar{k}/N_{\infty})} \cong \mathbb{Z}_p[[S]]/f(S, (1 + S)^{−u^p} - 1)\mathbb{Z}_p[[S]].$$

We note that $\mu(f(S, (1 + S)^{−u^p} - 1)) = 0$. Using Proposition 4.1 we have $\text{rank}_{\mathbb{Z}_p}((X_\bar{k})_{\text{Gal}(\bar{k}/N_{\infty})}) \leq p^{n_0} - 1$. By the same method as in the proof of Propositions 5.10, 5.13 and 6.1, we obtain $\lambda(f(S, (1 + S)^{−u^p} - 1)) \geq p^{n_0} - 1$. Thus we obtain $\lambda(f(S, (1 + S)^{−u^p} - 1)) = p^{n_0} - 1$. In the same way as Proposition 4.3, we can verify that $M_\rho(N_{\infty})/N_{n_0}$ is an abelian extension. This implies that $M_\rho(N_{\infty}) = \hat{k}_{n_0}^a$. Furthermore, we have $k_{n_0}^a = L_{k_{n_0}}$, since $X_{k_{n_0}}^a$ is a free $\mathbb{Z}_p$-module from the proof of Proposition 6.2. Therefore we have

$$(X_\bar{k})_{\text{Gal}(\bar{k}/N_{\infty})} \cong \text{Gal}(L_{k_{n_0}}/\bar{k}) \cong \mathbb{Z}_p[[S]]/g_0(S)\mathbb{Z}_p[[S]].$$

Thus we get the conclusion.
We put \( \rho = \sigma^u p^r \tau \). We fix an isomorphism
\[
(10) \quad \mathbb{Z}_p[\text{Gal}(\bar{k}/N_{\infty})] \cong \mathbb{Z}_p[[T]] \quad (\rho \leftrightarrow 1 + T).
\]
Then we have \( T_s = (1 + S)^{up^r}(1 + T) - 1 \). We note that \( \Lambda = \mathbb{Z}_p[[S,T]] \) since we have \( T = (1 + T_s)(1 + S)^{-up^r} - 1 \). By the same method as Lemma 6.3, we can find an annihilator of \( X_{\bar{k}} \) as follows.

**Lemma 6.4.** Assume the same conditions as in Proposition 6.2. Then there exist
\[
q(S,T_s) \in \text{Ann}_\Lambda(X_{\bar{k}}) \quad \text{and} \quad r_1(T_s) \in \mathbb{Z}_p[[T_s]] \quad (i = 0, \ldots, p^{n_0} - 2)
\]
such that
\[
q(S,T_s) = S^{p^{n_0} - 1} + r_{p^{n_0} - 2}(T_s)S^{p^{n_0} - 2} + \cdots + r_1(T_s)S + r_0(T_s).
\]
**Proof.** Using the isomorphism (10) and Proposition 6.3, we have
\[
X_{\bar{k}}/T_sX_{\bar{k}} \cong \mathbb{Z}_p^{p^{n_0} - 1}.
\]
By Nakayama’s lemma, there exist \( y_i \in X_{\bar{k}} \) (\( i = 1, 2, \ldots, p^{n_0} - 1 \)) such that \( X_{\bar{k}} = \langle y_1, \ldots, y_{p^{n_0} - 1} \rangle \) \( \mathbb{Z}_p[[T_s]] \). By the same method as Lemma 3.3, we have relations
\[
Sy_i = \sum_{j=1}^{p^{n_0} - 1} h_{ij}(T_s)y_j \quad (i = 1, 2, \ldots, p^{n_0} - 1)
\]
for some \( h_{ij}(T_s) \in \mathbb{Z}_p[[T_s]] \). We put \( c = p^{n_0} - 1 \). From theses relations, we have a matrix
\[
A = \begin{pmatrix}
S - h_{11}(T_s) & -h_{12}(T_s) & \cdots & -h_{1c}(T_s) \\
-h_{21}(T_s) & S - h_{22}(T_s) & \cdots & -h_{2c}(T_s) \\
\vdots & \vdots & \ddots & \vdots \\
-h_{c1}(T_s) & -h_{c2}(T_s) & \cdots & S - h_{cc}(T_s)
\end{pmatrix}.
\]
We put \( q(S,T_s) = \det(A) \). As in Lemma 3.3, \( q(S,T_s) \) satisfies the desired result. Thus we get the conclusion. \( \square \)

We note that the uniqueness of the power series \( q(S,T_s) \) is not known. In this paper, we fix this power series.

**Lemma 6.5.** Assume the same conditions as in Proposition 6.2. Then we have
\[
\text{Ann}_\Lambda(X_{\bar{k}}) = (q(S,T_s)),
\]
where \( q(S,T_s) \) is the same power series defined in Lemma 6.4.
**Proof.** By Proposition 5.2 and Lemma 6.4, we have \( q(S,T_s) \in \text{Ann}_\Lambda(X_{\bar{k}}) = (f(S,T)) \). Then, there exists a power series \( Q(S,T_s) \in \Lambda \) such that
\[
q(S,T_s) = f(S) + T_s f(S)(1 + S)^{-up^r} - 1)Q(S,T_s).
\]
Hence we have \( q(S,0) = f(S)(1 + S)^{-up^r} - 1)Q(S,0) \). By the definition of \( q(S,T_s) \), we have \( \mu(q(S,0)) = 0 \) and \( \lambda(q(S,0)) \leq p^{n_0} - 1 \). By Lemma 6.4 and Proposition 6.1, we have a surjective homomorphism \( \Lambda/\mu(q(S,T_s)) \Lambda \to X_{\bar{k}} \). This homomorphism induces a surjective homomorphism \( \mathbb{Z}_p[[S]]/\mu(q(S,0))\mathbb{Z}_p[[S]] \to \mathbb{Z}_p[[S]]/\mu(q(S,T_s)) \). Using Lemma 6.3, we obtain
\[
\text{rank}_{\mathbb{Z}_p}(q(S,0)) \geq p^{n_0} - 1.
\]
Thus the surjective homomorphism above is an isomorphism. Hence we obtain \( Q(S,0) \in \mathbb{Z}_p[[S]]^\times \). This implies that \( Q(S,T) \) is a unit in \( \Lambda \). Thus we get the conclusion. \( \square \)
We put $H = N_{k+1}^*$ and $H_\infty = N_\infty N_{k+1}^*$. Then $H_\infty/H$ is a $\mathbb{Z}_p$-extension of $H$ unramified outside all prime ideals of $H$ lying above $p$. We prove the following

**Proposition 6.6.** Under the same assumption as in Proposition 6.2, we have

$$\operatorname{rank}_{\mathbb{Z}_p}((X_{\overline{k}})_{\operatorname{Gal}(\overline{k}/H_\infty)}) = p(p^{n_0} - 1).$$

**Proof.** We note that $\operatorname{Gal}(\overline{k}/H_\infty) = \overline{(\mathbb{Z}/p)}$. By Lemma 6.5 we have

$$(X_{\overline{k}})_{\operatorname{Gal}(\overline{k}/H_\infty)} \cong \mathbb{Z}_p[[S,T]]/(q(S,T), (1 + T_s)^p - 1)$$

$$\cong ((Z_p[[T_s]]/((1 + T_s)^p - 1)\mathbb{Z}_p[[T_s]])^{p^{n_0} - 1}$$

Thus we get the conclusion. $\square$

We will prove Theorem 1.2

**Proof of Theorem 1.2.** We assume that weak GGC does not hold for $p$ and $k$. From Proposition 3.12, $M_p^*(k)$ coincides with $L_k$. We note that $p^*$ splits completely in $N_{\infty p}^* = N_{n_0}^*$. Hence all prime ideals of $H$ lying above $p^*$ do not split in $H_\infty$. Let

$$\{\mathcal{P}_{\infty,i}^* \mid 1 \leq i \leq p^{n_0}\}$$

be the set of prime ideals of $H_\infty$ lying above $p^*$. We denote by $I_{p^*_{\infty,i}}$ the inertia subgroup of $\operatorname{Gal}(M_p^*(H_\infty)/H_\infty)$ for the prime $\mathcal{P}_{\infty,i}^*$, where $M_p^*(H_\infty)$ is the maximal abelian pro-$p$ extension of $H_\infty$ unramified outside all prime ideals of $H_\infty$ lying above $p^*$. We have $\operatorname{Gal}(M_p^*(H_\infty)/k)\cap I_{p^*_{\infty,i}} = 1$ for each $i$ because $M_p^*(H_\infty)/k$ is unramified. Hence we have $I_{p^*_{\infty,i}} \cong \mathbb{Z}_p$. Using [10, 30, 27], we obtain $\mu(H_\infty/H) = 0$ since $H/k$ is an abelian extension. This implies that $\operatorname{Gal}(M_p^*(H_\infty)/H_\infty)$ is a finitely generated as a $\mathbb{Z}_p$-module. Furthermore, since we suppose that $\lambda(H_\infty/H) = 0$, $L_{H_{\infty}}/H_\infty$ is a finite extension. Thus we have $\operatorname{rank}_{\mathbb{Z}_p}(\operatorname{Gal}(M_p^*(H_\infty)/H_{\infty})) \leq p^{n_0}$. Therefore we obtain

$$\operatorname{rank}_{\mathbb{Z}_p}(\operatorname{Gal}(M_p^*(H_\infty)/\overline{k})) \leq p^{n_0} - 1.$$

On the other hand, we have

$$\operatorname{rank}_{\mathbb{Z}_p}(\operatorname{Gal}(M_p^*(H_\infty)/\overline{k})) = \operatorname{rank}_{\mathbb{Z}_p}((X_{\overline{k}})_{\operatorname{Gal}(\overline{k}/H_{\infty}))}$$

$$= p(p^{n_0} - 1)$$

$$> p^{n_0} - 1$$

by Proposition 6.6. This is a contradiction. Thus we have completed the proof. $\square$

7. **Numerical examples**

In this section, we introduce some numerical examples which were computed using PARI/GP.

Let $A_{k^c}$ be the $p$-Sylow subgroup of the ideal class group of $k^c$. Then, by class field theory, we have $X_{k^c} \cong \lim_{\bigwedge} A_{k^c}$, where the inverse limit is taken with respect to the relative norms. As in Sections 2 and 3, $X_{k^c}$ is a finitely generated torsion $\mathbb{Z}_p[[T]]$-module via an fixed isomorphism in Section 2. Let $h(T)$ be the distinguished polynomial which generates the characteristic ideal of $X_{k^c}$. We can calculate the polynomial $h(T) \mod p^n$ for small $n$ numerically using PARI/GP. By
this method, we can check whether the characteristic ideal of \( X_{k_{\infty}} \) has a square-free generator.

We first give a criterion whether weak GGC holds, using the characteristic ideal of \( X_{k_{\infty}} \):

**Proposition 7.1.** Assume that \( L_k \subset \bar{k}, \lambda(k_{\infty}^c/k) \geq 2 \), and that \( \text{ord}_p(g_0(0)) > [L_k:k] \), where \( g_0(S) \) is the same power series defined in Lemma 7.3. Assume also that the characteristic ideal of \( X_{k_{\infty}} \) has a square-free generator. Then weak GGC holds for \( p \) and \( k \).

**Proof.** We have \( \lambda(N_{\infty}/k) = 0 \) by \( L_k \subset \bar{k} \) from genus formula (see for example [22, Remark 3.2]). By Proposition 6.1 we have \( \text{ord}_p(g_0(0)) = [\text{Gal}(\bar{k}/k) : \mathcal{D}_p] \). From \( \text{ord}_p(g_0(0)) > [L_k:k] \), \( \mathcal{D}_p \) is not a normal subgroup of \( \text{Gal}(\bar{k}/\mathbb{Q}) \) by Lemma 3.8. Using Proposition 5.8, we see that \( k \) is not a \( p \)-split \( p \)-rational field. Furthermore, \( X_{\bar{k}} \) is cyclic as a \( \Lambda \)-module by Proposition 6.1. Therefore weak GGC holds for \( p \) and \( k \) by Theorem 1.1 and Remark 5.14.

We use the following criteria to check whether \( L_k \) is contained in \( \bar{k} \):

**Lemma 7.2 (21 Corollary of Proposition 6.B] and [1 Theorem 1]).** Let \( k = \mathbb{Q}(\sqrt{-d}) \) with a square-free positive integer \( d \).

(i) If \( p = 3 \) and \( d \neq 3 \) mod 9, then \( L_k \subset \bar{k} \) if and only if the class number of \( \mathbb{Q}(\sqrt{3d}) \) is not divisible by 3.

(ii) Assume that \( p \geq 5 \) and that \( k \) has the same \( p \)-class number as \( k(\zeta_p) \), then \( L_k \subset \bar{k} \), where \( \zeta_p \) is a primitive \( 3 \)-th root of unity.

As we stated in Section 1, we can prove GGC from weak GGC if the following condition is satisfied.

**Proposition 7.3 ([2 Proposition 3.1]).** Suppose that weak GGC holds for \( p \) and \( k \) and that \( \text{char}_{\mathbb{Z}_p[[\text{Gal}(k_{\infty}^c/k)]]}([X_{\bar{k}}]/\text{Gal}(\bar{k}/k_{\infty}^c)) \) is a prime ideal. Then GGC holds for \( p \) and \( k \).

Using Propositions 7.1, 7.3 and Lemma 7.2 above, we can prove GGC for \( (p,k) = (3, \mathbb{Q}(\sqrt{-971})), (3, \mathbb{Q}(\sqrt{-17291})), \) and \( (5, \mathbb{Q}(\sqrt{-2239})) \).

**Example 7.4.** Put \( p = 3 \) and put \( k = \mathbb{Q}(\sqrt{-971}) \). Then 3 splits completely in \( k \). By PARI/GP, we have \( A_k \cong \mathbb{Z}/3\mathbb{Z} \). We can check that \( L_k \subset \bar{k} \) by Lemma 7.2. Indeed, the class number of \( \mathbb{Q}(\sqrt{2913}) \) is 7. Hence, we have \( \lambda(N_{\infty}/k) = 0 \). Using PARI/GP, we have

\[
h(T) \equiv T^2 + 64638T \mod 3^{11}.
\]

By Hensel’s lemma, there exists an integer \( \alpha \in \mathbb{Z}_3 \) such that

\[
h(T) = T(T + \alpha),
\]

where \( \alpha \) satisfies that \( \alpha \equiv 486 \mod 3^6 \). Hence the characteristic ideal of \( X_{k_{\infty}} \) has a square-free generator. Let \( f(S,T) \) be the same power series defined in Lemma 3.3. By Proposition 3.3, we have \( \text{ord}_3(f(0,0)) = \text{ord}_3(\alpha) = 5 \). Hence \( k \) is not \( p \)-split \( p \)-rational and weak GGC holds by Proposition 7.1. Furthermore, by Proposition 7.3 GGC holds since we have \( \lambda(k_{\infty}^c/k) = 2 \).
Example 7.5. Put $p = 3$ and put $k = \mathbb{Q}(\sqrt{-17291})$. Then 3 splits completely in $k$. By PARI/GP, we have $A_k \cong \mathbb{Z}/3 \mathbb{Z}$. We can check that $L_3 \subset k$ by Lemma 7.2. Indeed, the class number of $\mathbb{Q}(\sqrt{51873})$ is 1. Hence we have $\lambda(N_\infty/k) = 0$. Furthermore we have

$$h(T) \equiv T^4 + 405T^3 + 72T^2 + 522T \mod 3^7.$$  

This implies that there exists an irreducible polynomial $g(T) \in \mathbb{Z}_p[[T]]$ such that $h(T) = Tg(T)$. Hence the characteristic ideal of $X_{k_\infty}$ has a square-free generator. Let $f(S,T)$ be the same power series defined in Lemma 3.3. By Proposition 3.5, we have $\operatorname{ord}_3(f(0,0)) = \operatorname{ord}_3(522) = 2$. Hence $k$ is not $p$-split $p$-rational and weak GGC holds by Proposition 7.1. Furthermore, by Proposition 7.3, GGC holds since $g(T)$ is irreducible.

Example 7.6. Put $p = 5$ and put $k = \mathbb{Q}(\sqrt{-2239})$. Then 5 splits completely in $k$. By PARI/GP, we have $A_k \cong \mathbb{Z}/5 \mathbb{Z}$. We can check that $L_5 \subset k$ by Lemma 7.2. Indeed, the class number of $k(\zeta_5)$ is 560. Hence we have $\lambda(N_\infty/k) = 0$. Furthermore we have

$$h(T) \equiv T^2 + 3100T \mod 5^5.$$  

By Hensel’s lemma, there exists an integer $\alpha \in \mathbb{Z}_5$ such that

$$h(T) = T(T + \alpha),$$

where $\alpha$ satisfies that $\alpha \equiv 100 \mod 5^3$. Hence the characteristic ideal of $X_{k_\infty}$ has a square-free generator. Let $f(S,T)$ be the same power series defined in Lemma 3.3. By Proposition 3.5, we have $\operatorname{ord}_5(f(0,0)) = \operatorname{ord}_5(\alpha) = 2$. Hence $k$ is not $p$-split $p$-rational. From Proposition 7.1, weak GGC for $p$ and $k$ holds. Furthermore, by Proposition 7.3, GGC holds for $k$ since $\lambda(k_{\infty}/k) = 2$.

Let $k_\infty/k$ be any $\mathbb{Z}_p$-extension. For a non-negative integer $n$, let $k_n$ be the $n$-th layer of $k_{\infty}/k$ and $\mathcal{O}_{k_n}$ the ring of integers of $k_n$. We denote by $A_{k_n}$ the $p$-Sylow subgroup of the ideal class group of $k_n$. We define a homomorphism $i_{k_n} : A_k \rightarrow A_{k_n}$ by sending the ideal class of $a$ to the class of $a\mathcal{O}_{k_n}$ for every ideal $a$ of $k$.

We use the following criteria to check whether $\lambda(N_\infty/k) = 0$.

Proposition 7.7 (21 Proposition 1.C]). The Iwasawa $\lambda$-invariant of $k_\infty/k$ is zero if and only if $A_k = \bigcup_{n \geq 0} \ker(i_n)$, in other words, every ideal class of $A_k$ capitulates in $k_\infty$.

Proposition 7.8 (27 Theorem 1]). Let $k_\infty/k$ be any $\mathbb{Z}_p$-extension. Let $c$ be a non-negative integer such that any prime ideal of $k$ which is ramified in $k_{\infty}/k$ totally ramified in $k_{\infty}/k_c$. If there exists an integer $n \geq c$ such that $\#A_{k_n} = \#A_{k_{n+1}}$, then $\#A_{k_m} = \#A_{k_n}$ for all $m \geq n$. In particular, we see that $\mu(k_{\infty}/k) = \lambda(k_{\infty}/k) = 0$.

We can prove GGC for $(p,k) = (3, \mathbb{Q}(\sqrt{-5069}))$ by Theorem 1.1 and propositions above.

Example 7.9. Put $p = 3$ and put $k = \mathbb{Q}(\sqrt{-5069})$. Then 3 splits completely in $k$. Using PARI/GP, we can check that $A_k \cong \mathbb{Z}/3 \mathbb{Z} \oplus \mathbb{Z}/3 \mathbb{Z}$ and that $\text{Cl}_k$ is generated...
by the classes of the prime ideals lying above 5 and 19. Using Fujii’s criteria ([4, Lemma 4.3]), we have $\text{Gal}(\tilde{k}/L \cap \tilde{k}) \cong \mathbb{Z}/3\mathbb{Z}$. Then we have

$$h(T) \equiv T(T + 1989) \mod 3.$$ 

There exists an integer $\alpha \in \mathbb{Z}_3$ such that $f(T) = T(T + \alpha)$ with $\alpha \equiv 45 \mod 3$. Hence the characteristic ideal of $X_{k^\alpha}$ is generated by a square-free generator.

Next we prove that $\lambda(N_\infty/k) = 0$. Using criteria in [2, Theorem 2], we obtain,

$$x^6 - 2x^5 - 33x^4 - 70x^3 + 5462x^2 - 38784x + 83808$$

as a defining polynomial of $k_1^\alpha$ over $\mathbb{Q}$. Then all prime ideal classes above 5 and 19 become principal in $N_2$. Hence $\lambda(N_\infty/k) = 0$. Using Proposition 7.7, we can also check this from Proposition 7.8. Indeed, using PARI/GP, we see that any prime ideal of $k$ which is ramified in $N_\infty/k$ totally ramified in $N_\infty/k_1^\alpha$. Furthermore, $k_1^\alpha$ and $N_2$ has the same $p$-class number since $\text{ord}_3(\#A_{k_1^\alpha}) = \text{ord}_3(\#A_{N_2}) = 3$. Hence we get $\lambda(N_\infty/k) = 0$ by Proposition 7.8. Therefore weak GGC holds by Theorem 1.1. Furthermore, by Proposition 7.3, GGC holds since $\lambda(k^\alpha_{\infty}/k) = 2$.

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