Abstract

In this paper the notion of Anti Fuzzy BRK-ideal of BRK-algebra is introduced. Several theorems are stated and proved. The epimorphic image and the into homomorphic inverse image of an anti fuzzy BRK-ideal is studied well. The Cartesian product of anti fuzzy BRK-ideal is introduced and studied.

Keywords: BRK-algebra; BRK-ideal; anti fuzzy BRK-ideal.

Mathematics subjects classification: 06F35, 03G25, 08A30.

1 Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [5]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [4], Q. P. Hu and X. Li introduced a wide class of abstract: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. In [10], J. Neggers, S. S. Ahn and H. S. Kim introduced Q-algebras which is a generalization of BCK / BCI-algebras and obtained several results. In 2002, Neggers and Kim [9] introduced a new notion, called a B-algebra, and obtained several results. In 2007, Walendziak [11] introduced a new notion, called a BF-algebra, which is a generalization of B-algebra. In [7], C. B. Kim and H.S. Kim introduced BG-algebra as a generalization of B-algebra. In 2012, R. K. Bandaru [1] introduces a new notion, called BRK-algebra which is a generalization of BCK / BCI / BCH / Q / QS-algebras and BH/BM-algebras [6,8]. In [3] I consider the fuzzification of BRK-ideal of BRK-algebra. I introduce the
notion of fuzzy BRK-ideal of BRK-algebra. In this paper I introduce a new notion which is the anti fuzzy BRK-ideal of BRK-algebras. I study the epimorphic image and the into homomorphic inverse image of an anti fuzzy BRK-ideal. I investigate some related properties. Also I introduce the Cartesian product of Anti fuzzy BRK-ideals and some related properties.

2 Preliminaries

Definition 2.1 [1]:

A BRK-algebra is a non-empty set \( X \) with a constant 0 and a binary operation “\(*\)” satisfying the following conditions:

\[
\begin{align*}
(BRK_1) & \quad x \ast 0 = x, \\
(BRK_2) & \quad (x \ast y) \ast x = 0 \ast y, \text{ for all } x, y \in X.
\end{align*}
\]

In a BRK-algebra \( X \), a partially ordered relation \( \leq \) can be defined by \( x \leq y \) if and only if \( x \ast y = 0 \).

Proposition 2.2 [1]:

If \( (X; \ast, 0) \) is a BRK-algebra, the following conditions hold:

\[
\begin{align*}
(1) & \quad x \ast x = 0, \\
(2) & \quad x \ast y = 0 \Rightarrow 0 \ast x = 0 \ast y, \\
(3) & \quad 0 \ast (x \ast y) = (0 \ast x) \ast (0 \ast y), \text{ for all } x, y \in X.
\end{align*}
\]

Definition 2.3 [1]:

A non empty subset \( S \) of a BRK-algebra \( X \) is said to be BRK-subalgebra of \( X \), if \( x, y \in S \), implies \( x \ast y \in S \).

Definition 2.4 (BRK-ideal of BRK-algebra):

A non empty subset \( I \) of a BRK-algebra \( X \) is said to be a BRK-ideal of \( X \) if it satisfies:

\[
\begin{align*}
(I_1) & \quad 0 \in I, \\
(I_2) & \quad 0 \ast (x \ast y) \in I \text{ and } 0 \ast y \in I \text{ imply } 0 \ast x \in I \text{ for all } x, y \in X.
\end{align*}
\]

Definition 2.5:

Let \( X \) be a set. A fuzzy set \( \mu \) in \( X \) is a function \( \mu : X \rightarrow [0,1] \).

Definition 2.6:

Let \( (X, \ast, 0) \) be a BRK-algebra. A fuzzy set \( \mu \) in \( X \) is called a fuzzy BRK-ideal of \( X \) if it satisfies:

\[
\begin{align*}
(FI_1) & \quad \mu(0) \geq \mu(x), \\
(FI_2) & \quad \mu(0 \ast x) \geq \min\{\mu(0 \ast (x \ast y)), \mu(0 \ast y)\}, \text{ for all } x, y \in X.
\end{align*}
\]
3 Anti-Fuzzy BRK-Ideal of BRK-Algebra

Definition 3.1:
Let \((X,*;0)\) be a BRK-algebra. A fuzzy set \(\mu\) in \(X\) is called an anti fuzzy BRK-ideal of \(X\) if it satisfies:

\[(FT_1)\] \(\mu(0) \leq \mu(x)\),

\[(FT_2)\] \(\mu(0 \ast x) \leq \max \{\mu(0 \ast (x \ast y)), \mu(0 \ast y)\}\), for all \(x, y \in X\).

Example 3.2:
Let \(X = \{0, a, b, c\}\). Define \(*\) on \(X\) as the following table:

|     | 0  | a  | b  | c  |
|-----|----|----|----|----|
| 0   | 0  | b  | b  | 0  |
| A   | a  | 0  | 0  | b  |
| B   | b  | 0  | 0  | b  |
| C   | c  | a  | a  | 0  |

Then \((X, *;0)\) is a BRK-algebra. Define a fuzzy set \(\mu: X \rightarrow [0, 1]\) by \(\mu(0) = t_1\), \(\mu(a) = \mu(c) = \mu(b) = t_2\), where \(t_1, t_2 \in [0, 1]\) with \(t_1 < t_2\), routine calculation gives that \(\mu\) is an anti fuzzy BRK-ideal of BRK-algebra.

Proposition 3.3:
Let \(\mu\) be an anti fuzzy BRK-ideal of BRK-algebra \(X\) and if \(x \leq y\), then \(\mu(0 \ast x) \leq \mu(0 \ast y)\), for all \(x, y \in X\).

Proof. Let \(\mu\) be an anti fuzzy BRK-ideal of a BRK-algebra \(X\). For any \(x, y \in X\) such that \(x \leq y\).

Since \(x \leq y\), then \(x \ast y = 0\).

\[
\begin{align*}
\mu(0 \ast x) & \leq \max \{\mu(0 \ast (x \ast y)), \mu(0 \ast y)\} \\
& = \max \{\mu(0 \ast 0), \mu(0 \ast y)\} \\
& = \max \{\mu(0), \mu(0 \ast y)\} \\
& = \mu(0 \ast y).
\end{align*}
\]

Hence \(\mu(0 \ast x) \leq \mu(0 \ast y)\).

Theorem 3.4:
A fuzzy subset \(\mu\) of a BRK-algebra \(X\) is a fuzzy BRK-ideal of \(X\) if and only if its complement \(\mu^c\) is an anti fuzzy BRK-ideal of \(X\).

Proof. Let \(\mu\) be a fuzzy BRK-ideal of a BRK-algebra \(X\), and let \(x, y \in X\). Then
So, \( \mu(0) \geq \mu(x) \)
then 
\[ 1 - \mu(0) \leq 1 - \mu(x) \]
\[ \rho^c(0) \leq \rho^c(x). \]  \hspace{1cm} (FT_1)

And 
\[ \mu(0 \cdot x) \geq \min \{ \mu(0 \cdot (x \cdot y)), \mu(0 \cdot y) \} \]
\[ 1 - \mu(0 \cdot x) \leq 1 - \min \{ \mu(0 \cdot (x \cdot y)), \mu(0 \cdot y) \} \]
\[ \rho^c(0 \cdot x) \leq \max \{ 1 - \mu(0 \cdot (x \cdot y)), 1 - \mu(0 \cdot y) \} \]
\[ \rho^c(0 \cdot x) \leq \max \{ \rho^c(0 \cdot (x \cdot y)), \rho^c(0 \cdot y) \}. \] \hspace{1cm} (FT_2)

So, \( \rho^c \) is an anti fuzzy BRK-ideal of \( X \).

Now let \( \mu^c \) is an anti fuzzy BRK-ideal of BRK-algebra \( X \), and let \( x, y \in X \). Then

Since \[ \rho^c(0) \leq \rho^c(x) \]
then 
\[ 1 - \rho^c(0) \geq 1 - \rho^c(x) \]
\[ \mu(0) \geq \mu(x). \] \hspace{1cm} (FI_1)

And 
\[ \rho^c(0 \cdot x) \leq \max \{ \rho^c(0 \cdot (x \cdot y)), \rho^c(0 \cdot y) \} \]
\[ 1 - \rho^c(0 \cdot x) \geq 1 - \max \{ \rho^c(0 \cdot (x \cdot y)), \rho^c(0 \cdot y) \} \]
\[ \mu(0 \cdot x) \geq \min \{ 1 - \rho^c(0 \cdot (x \cdot y)), 1 - \rho^c(0 \cdot y) \} \]
\[ \mu(0 \cdot x) \geq \min \{ \mu(0 \cdot (x \cdot y)), \mu(0 \cdot y) \}. \] \hspace{1cm} (FI_2)

So, \( \mu \) is a fuzzy BRK-ideal of a BRK-algebra \( X \).

**Theorem 3.5:**

Let \( \mu \) be an anti fuzzy BRK-ideal of BRK-algebra \( X \). Then the set \( X_\mu = \{ x \in X \mid \mu(0 \cdot x) = \mu(0) \} \) is a BRK-ideal.

**Proof.** Clearly \( 0 \in X_\mu \). Let \( x, y \in X_\mu \) be such that \( (0 \cdot (x \cdot y)) \in X_\mu \) and \( 0 \cdot y \in X_\mu \).

Then \[ \mu(0 \cdot (x \cdot y)) = \mu(0 \cdot y) = \mu(0) . \]

It follows that
\[ \mu(0 \cdot x) \leq \max \{ \mu(0 \cdot (x \cdot y)), \mu(0 \cdot y) \} \]
\[ \mu(0 \cdot x) \leq \max \{ \mu(0), \mu(0) \} \]
\[ \mu(0 \cdot x) \leq \mu(0) \]

So, by combining with definition 3.1 FT_1, we get that \( \mu(0 \cdot x) = \mu(0) \), and hence \( 0 \cdot x \in X_\mu \).
Definition 3.6 [5]:

Let \( \mu \) be a fuzzy subset of a set \( X \), for \( t \in [0,1] \), the set \( \mu_t = \{ x \in X; \mu(x) \leq t \} \) is called a lower \( t \)-level subset of \( \mu \).

Definition 3.7:

Let \( \mu \) be a fuzzy BRK-ideal of BRK-algebra \( X \). The BRK-ideal \( \mu_t, t \in [0,1] \), is called a lower \( t \)-level BRK-ideal of \( \mu \).

Theorem 3.8:

Let \( \mu \) be a fuzzy subset of a BRK-algebra \( X \). If \( \mu \) is an anti fuzzy BRK-ideal of \( X \) then for each \( t \in [0,1] \), the lower \( t \)-level cut \( \mu_t \) is a BRK-ideal of \( X \).

Proof. Let \( \mu \) be an anti fuzzy BRK-ideal of \( X \) and let \( t \in [0,1] \) with \( t \geq \mu(0) \).

Clearly \( 0 \in \mu_t \). Let \( x, y \in X \) be such that \( (0 * (x * y)) \in \mu_t \) and \( (0 * y) \in \mu_t \). Then

\[
\mu(0 * (x * y)) \leq t, \text{ and } \mu(0 * y) \leq t
\]

Hence \( \mu(0 * x) \leq \max\{\mu(0 * (x * y)), \mu(0 * y)\} \leq t \).

So \( (0 * x) \in \mu_t \). Hence \( \mu_t \) is a BRK-ideal of \( X \).

Definition 3.9 [Homomorphism of BRK-algebra]:

Let \( (X, \ast, 0) \) and \( (Y, \ast', 0') \) be BRK-algebras. A mapping \( f : X \rightarrow Y \) is said to be a homomorphism if

\[
f(x \ast y) = f(x) \ast' f(y), \text{ for all } x, y \in X.
\]

Definition 3.10:

Let \( f \) be a mapping from the set \( X \) to the set \( Y \). If \( \mu \) is a fuzzy subset of \( X \), then the fuzzy subset \( B \) of \( Y \) defined by

\[
\mu f^{-1}(y) = B(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x), & \text{if } f^{-1}(y) = \{ x \in X; f(x) = y \} \neq \phi \\ 0, & \text{otherwise} \end{cases}
\]

for all \( y \in Y \) is called the image of \( \mu \) under \( f \).

Similarly, if \( B \) is a fuzzy subset of \( Y \), then the fuzzy subset defined by \( B(f(x)) = \mu(x) \) for all \( x \in X \), is said to be the inverse image of \( B \) under \( f \).

Theorem 3.11:

The epimorphic image of an anti fuzzy BRK-ideal is also an anti fuzzy BRK-ideal.
A fuzzy relation on any set $S$ is a fuzzy subset $\mu: S \times S \rightarrow [0,1]$.

**Definition 5.1:**
Definition 5.2:

Let \( \mu \) and \( \beta \) be the fuzzy subsets of a set \( S \). The anti Cartesian product of \( \mu \times \beta : X \times X \rightarrow [0, 1] \) is defined by \( (\mu \times \beta)(x, y) = \max\{\mu(x), \beta(y)\} \), for all \( x, y \in S \).

Definition 5.3:

Let \( \mu \) and \( \beta \) be fuzzy subsets of a set \( S \). The Cartesian product of \( \mu \) and \( \beta \) is defined by \( (\mu \times \beta)(x, y) = \min\{\mu(x), \beta(y)\} \) for all \( x, y \in S \).

Corollary 5.4:

Let \( (X; \ast, 0) \) and \( (Y; \ast, 0') \) be BRK-algebras, we define \( \ast \) on \( X \times Y \) by: For every \((x, y), (u, v) \in X \times Y \) \((x, y) \ast (u, v) = (x \ast u, y \ast v) \) then \((X \times Y; \ast, (0, 0')) \) is a BRK-algebra.

Proof. Let \( (X; \ast, 0) \) and \( (Y; \ast, 0') \) be BRK-algebras (see definition 3.1).

For all \((x_1, x_2), (y_1, y_2) \in X \times Y \), then

1. \( (x_1, x_2) \ast (0, 0') = (x_1 \ast 0, x_2 \ast 0') = (x_1, x_2) \)
2. \( ((x_1, x_2) \ast (y_1, y_2)) \ast (x_1, x_2) = (x_1 \ast y_1, x_2 \ast y_2) \ast (x_1, x_2) \)
   \[= ((x_1 \ast y_1) \ast x_1, (x_2 \ast y_2) \ast x_2) \]
   \[= (0 \ast y_1, 0' \ast y_2). \]

So, \((X \times Y; \ast, (0, 0')) \) is a BRK-algebra.

Theorem 5.5:

If \( \mu \) and \( \beta \) are anti fuzzy BRK-Ideals of BRK-algebras \( X \), then \( \mu \times \beta \) is an anti fuzzy BRK-ideal of \((X \times X; \ast, (0, 0'))\).

Proof. Let \( x, x' \in X \times X \). Then

\[(\mu \times \beta)(0, 0') = \max\{\mu(0), \beta(0')\} \leq \max\{\mu(x), \beta(x')\} = (\mu \times \beta)(x, x') \]

For any \((x, x'), (y, y') \in X \times X \) we have

\[(\mu \times \beta)(0 \ast x, 0' \ast x') = \max\{\mu(0 \ast x), \beta(0' \ast x')\} \]
\[\leq \max\{\max\{\mu(0 \ast (x \ast y)), \mu(0 \ast y)\}, \max\{\beta(0' \ast (x' \ast y')), \beta(0' \ast y')\}\} \]
\[= \max\{\max\{\mu(0 \ast (x \ast y)), \beta(0' \ast (x' \ast y'))\}, \max\{\mu(0 \ast y), \beta(0' \ast y')\}\} \]
\[= \max\{(\mu \times \beta)((0, 0') \ast (x, x') \ast (y, y')) \}, (\mu \times \beta)(0, 0') \ast (y, y'))\} \]

Hence \( \mu \times \beta \) is an anti fuzzy BRK-ideal of \((X \times X; \ast, (0, 0'))\).
In our future work, we introduce the concept of anti fuzzy relation on $S$ that is an anti fuzzy relation on $\beta$, given by $\mu_{\beta}(x, y) = \max\{\beta(x), \beta(y)\}$ for all $x, y \in S$.

**Proposition 5.7:**
For a given anti fuzzy subset $\beta$ of a BRK-algebra $X$, let $\mu_\beta$ be the strongest anti fuzzy relation on $X$. If $\mu_\beta$ is an anti fuzzy BRK-ideal of $(X \times X; \ast, (0,0))$, then $\beta(0) \leq \beta(x)$ for all $x \in X$.

**Proof.** Since $\mu_\beta$ is an anti fuzzy BRK-ideal of $X \times X$, it follows from (FT$_1$) that $\mu_\beta(0,0) \leq \mu_\beta(x,x)$.

So that $\mu_\beta(0,0) = \max\{\beta(0), \beta(0)\} \leq \min\{\beta(x), \beta(x)\} = \mu_\beta(x,x)$.

This implies that $\beta(0) \leq \beta(x)$.

**Theorem 5.8:**
Let $\beta$ be an anti fuzzy subset of BRK-algebras $X$, and $\mu_\beta$ be the strongest anti fuzzy relation on $X$. If $\beta$ is an anti fuzzy BRK-ideal of $X$ then $\mu_\beta$ is an anti fuzzy BRK-ideal of $(X \times X; \ast, (0,0'))$.

**Proof.** Suppose that $\beta$ is an anti fuzzy subset of a BRK-ideal $X$, and $\mu_\beta$ is the strongest anti fuzzy relation on $X$. Then $\mu_\beta(0,0') = \max\{\beta(0), \beta(0')\} \leq \max\{\beta(x), \beta(y)\} = \mu_\beta(x,y)$, for all $(x, y) \in X \times X$.

For all $(x_1, x_2), (y_1, y_2) \in X \times X$, we get that

$$
\mu_\beta((0,0') \ast (x_1, x_2)) = \mu_\beta(0 \ast x_1, 0' \ast x_2) = \max\{\beta(0 \ast x_1), \beta(0' \ast x_2)\} \\
\leq \max\{\max\{\beta(0 \ast (x_1 \ast y_1)), \beta(0 \ast y_1)\}, \max\{\beta(0' \ast (x_2 \ast y_2)), \beta(0' \ast y_2)\}\} \\
= \max\{\max\{\beta(0 \ast (x_1 \ast y_1)), \beta(0 \ast y_1)\}, \max\{\beta(0' \ast (x_2 \ast y_2)), \beta(0' \ast y_2)\}\} \\
= \max\{\mu_\beta(0 \ast (x_1 \ast y_1), 0' \ast (x_2 \ast y_2)) \pm \mu_\beta(0 \ast y_1, 0' \ast y_2)\} \\
= \max\{\mu_\beta(((0,0') \ast ((x_1, x_2) \ast (y_1, y_2))), \mu_\beta((0,0') \ast (y_1, y_2)) \}
$$

Hence $\mu_\beta$ is an anti fuzzy BRK-ideal of $X \times X$.

**5 Conclusion**

In this paper, we have introduced the concept of anti fuzzy BRK-ideal of BRK-algebra and studied their properties.

In our future work, we introduce the concept of Cubic fuzzy BRK-ideal of BRK-algebra, interval-valued Fuzzy BRK-ideal of BRK-algebra, intuitionistic fuzzy structure of BRK-ideal of BRK-algebra, intuitionistic fuzzy BRK-ideals of BRK-algebra, and intuitionistic $L$-fuzzy BRK-ideals of BRK-algebra. I hope this work would serve as a foundation for further studies on the structure of BRK-algebras.
Competing Interests

Author has declared that no competing interests exist.

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