Remote state estimation problem: characterization of the data rate limits

C. Kawan*, A. Matveev†, A. Pogromsky‡

Abstract

In the context of control and estimation under information constraints, restoration entropy measures the minimal required data rate above which a system can be regularly observed. The remote observer here is assumed to receive its data through a communication channel of a finite bit-rate capacity. In this paper, we provide a new characterization of the restoration entropy which does not require to compute any temporal limit, i.e., an asymptotic quantity. Our new formula is based on the idea of finding a specific Riemannian metric on the state space which makes the metric-dependent upper estimate of the restoration entropy as tight as one wishes.

1 Introduction

Over the past half century, in accordance with Moore’s law, advances in constantly growing computing power have enabled us to put an artificial intelligence into tiny devices. At the same time, advances in communication technology have created the possibility of large-scale control systems, where the control tasks are distributed over many agents negotiating via a communication network. Empirically, this observation is formulated in a form of Metcalfe’s law, which claims that the network “effect” or “value” is proportional to the number of unique possible interconnections in the network and, hence, proportional to \( n^2 \) if \( n \) stands for the number of agents. The informational capacity of the network (the amount of traffic the network is able to handle) is also proportional to the number of interconnections and,

---

*Institute of Informatics, Ludwig-Maximilians-Universität, München, Germany (e-mail: christoph.kawan@lmu.de).

†Department of Mathematics and Mechanics, Saint Petersburg University, St. Petersburg, Russia (e-mail: almat1712@yahoo.com) and Department of Control Systems and Industrial Robotics, Saint-Petersburg National Research University of Information Technologies Mechanics and Optics (ITMO), Russia.

‡Department of Mechanical Engineering, Eindhoven University of Technology, Eindhoven, The Netherlands, (e-mail: A.Pogromsky@tue.nl) and Department of Control Systems and Industrial Robotics, Saint-Petersburg National Research University of Information Technologies Mechanics and Optics (ITMO), Russia.
hence, also must grow quadratically with respect to the number of agents in order to make the “value” of the network larger. With the development of the internet of things and intellectual vehicles, this capacity constraint bottlenecks the potential growth of the networks, particularly in communication challenging real-time environments. To overcome this issue, the agents should learn how to transfer more useful information while using less bits in their physical broadcast. This motivated a substantially growing attention to networked control systems [16, 15] and related problems of control and/or state estimation via communication channels with constrained bit-rates; for extended surveys of this area, we refer the reader to [29, 42, 1] and references therein.

One of the fundamental concerns in this context is to find a minimal data rate between the communication peers above which remote state estimation ([40], see also [11] for a related problem) is feasible. In other words, the receiver is supposed to reconstruct the current state of the remote system in the real time regime. The receiver is updated by means of a bit flow with limited bit-rate. A minimal threshold of this rate, which still ensures that the remote observer is able to keep track of the system, is the quantity one likes to evaluate in a constructive manner. Loosely speaking, the communication rate between the system and the observer has to exceed the rate at which the system “generates information”, while the latter concept is classically formalized in a form of entropy-like characteristic of the dynamical system at hands. The related mathematical results are usually referred to as Data Rate Theorems (see, e.g. [32, 29, 28] and references therein) - their various versions coexist to handle various kinds of observability and models of both the plant and the constrained communication channel.

Those results deliver a consistent message that the concept of the topological entropy (TE) of the system and its recent offshoots provide the figure-of-merit needed to evaluate the channel capacity for control applications; the mentioned modifications of TE are partly aimed to properly respond to miscellaneous phenomena crucial for control problems, like uncertainties in the observed system [36, 22, 23], implications of control actions [9, 13, 8], the decay rate of the estimation error [25], or Lipschitz-like relations between the exactness of estimation and the initial state uncertainty [28]. Keeping in mind relevance of communication constraints in modern control engineering, constructive methods to compute or finely estimate those entropy-like characteristics take on crucial not only theoretical but also practical importance. Several steps have been done in this direction in [34, 28, 12], where corresponding upper estimates were found by following up the ideas of the second Lyapunov method. Moreover, it was shown that for some particular prototypical chaotic systems of low dimensions, these upper estimates are exact in the sense that they coincide with the true value of the estimated quantity.

Whether these inspiring samples of precise calculations are mere incidents, or, conversely, are particular manifestations of a comprehensive capacity inherent in the employed approach? Confidence in the last option would constitute a rationale
for undertaking special efforts aimed to fully unleash the potential of this approach via its further elaboration.

The primary goal of the current paper is to answer the posed question; we show that among the above options, the last one is the true one. This is accomplished via a sort of a converse result, which is similar in spirit to the celebrated converse Lyapunov theorems. Among various descendants of TE, we pick the so-called restoration entropy (RE) \[28\] to deal with. In the previous work \[28\], it has been shown that an upper estimate of RE can be derived in terms of singular values of the derivative of the system flow, calculated with respect to some metric. Any metric involved in such calculations will result in a valid upper estimate. The main contribution of this paper is a result showing that one can find a metric which lets the corresponding upper estimate become arbitrarily close to the true value of RE.

The tractability of the developed approach is confirmed by a closed-form computation of the restoration entropy for the celebrated Landford system (see, e.g., \[2\]). Meanwhile, computation or even fine estimation of TE and the likes has earned the reputation of an extremely complicated matter \[10\].

The paper is organized as follows. Sec. 2 offers the remote state estimation problem statement and presents the main assumptions. Sec. 3 contains the main results, which are illustrated by an example in Sec. 4. The technical proofs of the main results are collected in the appendices.

The following notations are adopted in this paper: \(\log\) – logarithm base 2, \(Df\) – Jacobian matrix of function \(f\), \(P^t\) with \(t \in \mathbb{R}\) – \(t\)-th power of a symmetric positive definite matrix \(P\). Given a matrix \(A\), its singular values ordered in the nondecreasing order, are denoted by \(\alpha_i(A)\). If the matrix \(A\) is parameterized: \(A = A(x)\) or \(A = A(t,x)\), then for the sake of brevity, the corresponding singular values will be denoted as \(\alpha_i(x)\), or \(\alpha_i(t,x)\), provided the choice of the matrix \(A\) is clear from the context.

\[\text{2 State estimation via limited bit-rate communication and restoration entropy}\]

The objective of this section is to outline basic points of recent results that motivate the research reported in this paper.

We consider time-invariant dynamical systems of the form

\[x^\nabla(t) = \varphi[x(t)], \quad t \in \Omega_+, \quad x(0) \in K \subset \mathbb{R}^n \quad (1)\]

in the following two cases:

- **c-t** Either \(\Omega_+ = [0, \infty) \subset \mathbb{R}\), and the symbol \(x^\nabla\) stands for the derivative of the function: \(x^\nabla(t) := \dot{x}(t)\);
Or $\mathbb{T}_+$ is the set of integers $t \geq 0$, and $x^\triangledown$ denotes the forward time-shift by one step: $x^\triangledown(t) := x(t + 1)$.

In (1), $\varphi(\cdot)$ is of class $C^1$ and $K$ is a given compact set of initial states that are of our interest; in the case d-t), all considered time variables assume integer values by default.

We deal with a situation where at a remote site $S_{est}$, direct observation of the time-varying state $x(t)$ is impossible but a reliable estimate $\hat{x}(t)$ of $x(t)$ is needed at any time $t \in \mathbb{T}_+$. Meanwhile, data may be communicated to $S_{est}$ from another site, where $x(t)$ is fully measured at time $t$. The main issue of our interest arises from the fact that the channel of communication between these sites allows to transmit only finitely many bits per unit time. How large their number must be so that a good estimate can be generated?

We clarify this issue only in general terms by following [27, 28] and [29, Sec. 3.4], and refer the reader there for full details. The channel is a communication facility that transmits finite-bit data packets, transmission consumes time (during which the channel cannot process new packets). The time $t_\ast$ when transferring the packet is started, and its bit-size and contents are somewhat manipulable; the channel is used repeatedly, thus transmitting a flow of messages. Despite all variability in the ways of channel usage, the channel imposes an upper bound $b_\ast(r)$ on the total number of bits that can be transferred within any interval of duration $r$. On the positive side, there is a way to deliver no less that $b_\ast(r)$ bits. We consider the case where the two associated averaged per-unit-time amounts of the bits converge to a common value $c = \lim_{r \to \infty} b_\ast(r)/r$ as the duration $r$ grows without limits; this value $c$ is called the capacity of the channel.

In the situation at hands, the observer is composed of a coder and a decoder. The coder is located at the measurement site; its function is to generate the departure times $t_\ast$ and the messages to $S_{est}$ based on the preceding measurements, as well as on the initial estimate $\hat{x}(0)$ and its accuracy $\delta > 0$.

$$\|x(0) - \hat{x}(0)\| < \delta, \quad x(0), \hat{x}(0) \in K.$$  \hspace{1cm} (2)

The decoder is built at the site $S_{est}$ and has access to $\hat{x}(0)$ and $\delta > 0$; its duty is to generate an estimate $\hat{x}(t)$ of the current state $x(t)$ at the current time $t$ based on the messages fully received through the channel prior to this time. Both the coder and decoder are aware of $\varphi(\cdot)$ and $K$ from (1).

Given the plant (1), the possibility of building a reliable observer is dependent on the features of the employed channel. Among them, a comprehensive figure of merit is the channel capacity $c$ [27, 28]. Those values of $c$ that make reliable observation possible are in our focus. Their minimum is characterized by the system itself, is called the observability rate $R(\varphi, K)$, and is measured in bits per unit time. Depending on the concept of observation reliability, a whole variety of rates $R$’s thus arises.
The simplest concept requires that by choosing the initial error $\delta$ in (2) small enough, the overall error $\sup_{t \in \mathbb{T}_+} \| x(t) - \hat{x}(t) \|$ can be made as small as desired; the associated observability rate is denoted by $R_o(\varphi,K)$. The stronger regular observability demands that the overall error stays proportional to the initial one $\| x(t) - \hat{x}(t) \| \leq G\delta \forall t \in \mathbb{T}_+$ (with $G$ being independent of $\delta$ and $t$); the associated rate is denoted by $R_{\text{reg}}(\varphi,K)$ [27, 28]. The strongest fine observability additionally requests that the error exponentially decays $\| x(t) - \hat{x}(t) \| \leq G\delta e^{-\eta t} \forall t \in \mathbb{T}_+$ (where $\eta > 0$ does not depend on $\delta$ and $t$ and is not pre-specified); the related rate is denoted by $R_{\text{fine}}(\varphi,K)$ [27, 28].

These rates could be viewed as an answer to the key question posed at the start of the section, but only if a method to effectively compute them be available. Their definitions do not directly suggest such a method since they refer to complementing the system with an a priori uncertain object – an observer. So the first step is to get rid of this uncertainty and to fully express those rates via features of the system only.

Under certain technical assumptions, this feature is the topological entropy $H_{\text{top}}(\varphi,K)$ of the plant (1) in the case of $R_o(\varphi,K)$, which is simply equal to $H_{\text{top}}(\varphi,K)$; we refer the reader to [10] for the definition of $H_{\text{top}}(\varphi,K)$. In the case of both $R_{\text{reg}}(\varphi,K)$ and $R_{\text{fine}}(\varphi,K)$, this is another characteristic of the system. It is introduced in [28] for the continuous-time case and extended to the discrete-time one in [20], and is called the restoration entropy $H_{\text{res}}(\varphi,K)$.

To recall its definition, we adopt the following.

2.1 Assumption The map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is of class $C^1$ and the set $K \subset \mathbb{R}^n$ is compact and forward-invariant.

Further, $\varphi^t$ stands for the map $a \mapsto x(t,a)$.

Let a duration $\tau \in \mathbb{T}_+$, a state $a \in \mathbb{R}^n$, and an “error tolerance level” $\delta > 0$ be given. We denote by $N_{\varphi,K}(\tau,a,\delta)$ the minimal number of open $\delta$-balls needed to cover the image $\varphi^\tau[B_\delta(a) \cap K]$. Then

$$H_{\text{res}}(\varphi,K) := \lim_{\tau \to \infty} \frac{1}{\tau} \lim_{\delta \to 0} \sup_{a \in K} \log N_{\varphi,K}(\tau,a,\delta).$$

Here $\lim_{\tau \to \infty}$ exists by Fekete’s lemma and is due to the easily verifiable subadditivity of the concerned quantity in $\tau$. The interest in (3) is caused by the fact that $H_{\text{res}}(\varphi,K) = R_{\text{reg}}(\varphi,K) = R_{\text{fine}}(\varphi,K)$ [28, 20]. Also, $H_{\text{res}}(\varphi,K) \geq H_{\text{top}}(\varphi,K)$ [28, 20]. Meanwhile, the concerned two concepts are not identical: $H_{\text{top}} < H_{\text{res}}$ for the logistic map [33, Ex. 5.1], whereas [20] offers an exhaustive characterization of the systems with $H_{\text{top}} = H_{\text{res}}$ and suggests that $H_{\text{top}} = H_{\text{res}}$ is a relatively rare occurrence.
The restoration entropy can be also linked to the finite-time Lyapunov exponents $t^{-1} \ln \alpha_i(t, x)$, where $\alpha_1(t, x) \geq \ldots \geq \alpha_n(t, x)$ are the singular values of the Jacobian matrix $D\varphi^t(x)$. It is convenient for us to divide these exponents by $\ln 2$, which results in the change of the logarithm base:

$$\Lambda_i(t, x) := \frac{1}{t} \log \alpha_i(t, x).$$

As is shown in [28, 20],

$$H_{\text{res}}(\phi, K) = \max_{x \in K} \lim_{t \to \infty} \sum_{i=1}^{n} \max\{0, \Lambda_i(t, x)\},$$

$$= \lim_{t \to \infty} \max_{x \in K} \sum_{i=1}^{n} \max\{0, \Lambda_i(t, x)\}$$

(4)

provided that the following holds.

2.2 Assumption The set $K$ is the closure of its interior.

2.3 Remark (28) If Assumption 2.2 is dropped, the first relation from (4) remains true provided that $\leq$ is put in place of $=$ in it.

The topological entropy and Lyapunov exponents are classic and long-studied concepts. However, their use in assessing $H_{\text{res}}$ is highly impeded by the fact that constructive practical evaluation of both $H_{\text{top}}$ and the limit in (4) has earned the reputation of an intricate matter and is in fact a long-standing challenge that still remains unresolved in many respects.

We offer an alternative machinery for evaluation of $H_{\text{res}}$. By the foregoing, it can also be used to upper estimate $H_{\text{top}}$ and to quantify finite-time Lyapunov exponents for large times.

3 Main ideas and results

Due to practical needs, the definition (3) operates with balls in the Euclidean metric.

Meanwhile, balls and Lyapunov exponents are standard in studies of dynamics on Riemannian manifolds, where its own metric tensor is assigned to every point. A homogeneous, state-independent metric was used in Sec. The idea to modify the material of Sec. via assigning its own metric tensor to any point $x \in \mathbb{R}^n$, thus altering the neighboring balls and the finite-time Lyapunov exponents, may look as nothing but a complication with no reason. However, this hint is a keystone for a method that not only aids in practical estimation of $H_{\text{res}}$ but also carries a potential to compute $H_{\text{res}}$ with as high exactness as desired.
To specify this, we denote by $S$ the linear space of the real symmetric $n \times n$ matrices, and by $S^+ \subset S$ the subset of the positive definite ones. A continuous mapping $P : K \to S^+$ gives rise to a Riemannian metric on $K$ \[7\] by defining the state-dependent inner product

$$\langle v, w \rangle_{P,x} := \langle P(x) v, w \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product.

This Riemannian metric produces its own finite-time Lyapunov exponents and, on a more general level, singular values of $A(x) := D\phi(x)$, $x \in K$ for any map $\phi \in C^1(\mathbb{R}^n \to \mathbb{R}^n)$ such that $\phi(K) \subset K$. Indeed, then $A(x)$ should be viewed as an operator between two spaces, endowed with the inner products $\langle \cdot, \cdot \rangle_{P,x}$ and $\langle \cdot, \cdot \rangle_{P,\phi(x)}$, respectively. This obliges to treat the singular values of $A(x)$ as the square roots of the eigenvalues of the operator $D\phi(x)^* D\phi(x)$, where $D\phi(x)^*$ is the adjoint to $D\phi(x)$ with respect to that pair of inner products.

3.1 Lemma The singular values $\alpha_1^P(x|\phi) \geq \cdots \geq \alpha_n^P(x|\phi) \geq 0$ of the matrix $A(x) := D\phi(x)$ in the metric $\langle \cdot, \cdot \rangle_P$ are the square roots of the solutions $\lambda$ for the following algebraic equation:

$$\det \left[ A(x)^T P(\phi(x)) A(x) - \lambda P(x) \right] = 0. \quad (5)$$

These solutions are simultaneously the eigenvalues of the positive semi-definite matrix $B(x)^T B(x)$, where

$$B(x) := P[\phi(x)]^{1/2} A(x) P(x)^{-1/2}. \quad (6)$$

Proof To compute the adjoint to $A(x)$, we observe that

$$\langle (A(x)v, w) \rangle_{P,\phi(x)} = \langle P(\phi(x)) A(x)v, w \rangle$$

$$= \langle A(x)v, P(\phi(x))w \rangle = \langle v, A(x)^T P(\phi(x))w \rangle$$

$$= \langle v, P(x) P(x)^{-1} A(x)^T P(\phi(x))w \rangle$$

$$= \langle v, P(x)^{-1} A(x)^T P(\phi(x))w \rangle_{P,x}$$

implying $A(x)^* = P(x)^{-1} A(x)^T P(\phi(x))$. It remains to note that the associated singular value equation

$$\det \left[ P(x)^{-1} A(x)^T P(\phi(x)) A(x) - \lambda I_n \right] = 0 \quad (7)$$

is equivalent to both (5) and $\det[B(x)^T B(x) - \lambda I] = 0$, which can be seen via multiplying the matrix inside $[\ldots]$ in (7) by $P(x)[\ldots]$ and $P(x)^{1/2}[\ldots]P(x)^{-1/2}$ in the first and second case, respectively. \hfill \Box
3.1 Discrete time systems

Now we are in a position to state our first main result.

3.2 Theorem Let the case d-t) and Assumption[2.1] hold. Then the following statements are true (where log 0 := −∞):

(i) Any map \( P(\cdot) \in C^0(K, S^+) \) gives rise to the following bound on the restoration entropy of the system (1):

\[
H_{res}(\varphi, K) \leq \max_{x \in K} \Sigma^P(x|\varphi),
\]

where \( \Sigma^P(x|\varphi) := \sum_{i=1}^{n} \max \{0, \log \alpha^P_i (x|\varphi)\} \). (8)

(ii) Let the set \( K \) satisfy Assumption[2.2] and the Jacobian matrix \( D\varphi(x) \) be invertible for every \( x \in K \). Then for any \( \varepsilon > 0 \), there exists \( P \in C^0(K, S^+) \) such that

\[
H_{res}(\varphi, K) \geq \max_{x \in K} \Sigma^P(x|\varphi) - \varepsilon.
\]

The proof of this theorem is given in Appendix B.

In Thm. 3.2 \( \max_{x \in K} \) is attained since \( K \) is compact by Asm. 2.1 and \( \alpha^P_i (x|\varphi) \) depend continuously on \( x \). The latter holds since they are the singular values of the matrix (6), which is continuous in \( x \), whereas the singular values continuously depend on the matrix [19, Cor. 7.4.3.a].

The claim (i) converts any positive definite matrix function \( P \) into an upper estimate on \( H_{res}(\varphi, K) \), while taking no limits as \( t \to \infty \), unlike (3), (4). Meanwhile, (ii) shows that this estimate can be made as tight as one wishes by a proper choice of \( P \). So the presented method allows for computation of \( H_{res}(\varphi, K) \) with as high exactness as desired.

In the particular case of a constant matrix function \( P \), the claim (i) is covered by Thm. 12 in [27]. It is shown in [27] that the discussed method is largely constructive: intelligent choices of constant \( P \) and simple lower bounds on \( H_{res} \) result in closed-form expressions of \( H_{res} \) in terms of the parameters of some classic prototypical chaotic systems. The claim (ii) proves that these are not accidental successes but are manifestations of a fundamental trait of the approach.

3.3 Remark Under the assumptions of (ii) in Thm. 3.2 the claim (ii) yields the following new and exact formula:

\[
H_{res}(\varphi, K) = \inf_{P \in C^0(K, S^+)} \max_{x \in K} \Sigma^P(x|\varphi).
\]

In fact, \( \inf_P \) can be taken only over \( P \)'s that are extendible to an open vicinity \( V_P \) of \( K \) as \( C^\infty \)-smooth mappings to \( S^+ \). (Since this is not used in the paper, the proof is omitted.)
3.4 Corollary  Let $K$ meet Asm. 2.2 and $G := \{ \varphi \in C^1(\mathbb{R}^n \to \mathbb{R}^n) : \varphi(K) \subset K, \exists [D\varphi(x)]^{-1} \forall x \in K \}$ be endowed with $C^1$-topology. The map $\varphi \in G \mapsto H_{\text{res}}(\varphi, K)$ is upper semi-continuous: $H_{\text{res}}(\varphi, K) \geq \lim_{\varphi \to \varphi_0, \varphi_0 \in G} H_{\text{res}}(\varphi_0, K) \forall \varphi \in G$.

Indeed, $\alpha_i^P(x|\varphi)$ continuously depend not only on $x$ but also on the map $\varphi \in G$, as can be easily seen by analysis of the arguments in the second paragraph following Thm. 3.2. Hence, $\Sigma^P(x|\varphi)$ and $\max_{x \in K} \Sigma^P(x|\varphi)$ also depend on $\varphi \in G$ continuously. It remains to note that the infimum (over $P$’s, in our case) of continuous functions is upper semi-continuous; see, e.g., [41, Ch. 3, Sec. 6].

In a practical setting, Cor. 3.4 asserts that any upper estimate of $H_{\text{res}}$ is robust: if an upper estimate is established for a nominal model (let the above $\varphi$ be associated with this model), this estimate remains true, possibly modulo small correction, under small uncertainties in the model and perturbations of its parameters. Here the “smallness” of the correction goes to zero as so do the uncertainties and perturbations.

3.2 Continuous time systems

For them, a placeholder to equation (5) looks as follows:

$$\det \left\{ 2[P(x)D\varphi(x)]^{\text{sym}} + \dot{P}(x) - \lambda P(x) \right\} = 0,$$  \hspace{1cm} (9)

where $B^{\text{sym}} := (B + B^T)/2$ is the symmetric part of the matrix $B \in \mathbb{R}^{n \times n}$ and $\dot{P}$ is the orbital derivative:

$$\dot{P}(a) := \lim_{\tau \to 0^+} \frac{P[x(\tau, a)] - P[a]}{\tau}. \hspace{1cm} (10)$$

This derivative is equal to $D P(x) \varphi(x)$ for continuously differentiable $P$’s. However, we shall consider maps $P(\cdot)$ with a limited differentiability, as is described in the following.

3.5 Assumption  The map $P(\cdot) \in C^0(K, S^+)$ is such that for any $a \in \mathbb{R}^n$, the limit in (10) exists and is orbitally continuous, i.e., the function $t \mapsto P[x(t, a)]$ is continuous.

3.6 Remark  The $x$-dependent roots of the algebraic equation (9) are the eigenvalues of the symmetric matrix

$$[P(x)]^{-1/2} \left\{ 2[P(x)D\varphi(x)]^{\text{sym}} + \dot{P}(x) \right\} [P(x)]^{-1/2}.$$  \hspace{1cm} (9)

So these roots are real. Being repeated in accordance with their multiplicity, they are denoted by $\varsigma_i^P(x)$, $i = 1, \ldots, n$.  \hspace{1cm} (9)
3.7 Theorem Suppose that the case c-t) and Assumption 2.1 hold. Then the following statements are true:

(i) For any map $P(\cdot)$ satisfying Assumption 3.5,

$$H_{\text{res}}(\varphi, K) \leq \frac{1}{2 \ln 2} \max_{x \in K} \sum_{i=1}^{n} \max \{0, \varsigma_{i}^{P}(x)\}. \tag{11}$$

(ii) Suppose that the set $K$ satisfies Assumption 2.2. Then for any $\varepsilon > 0$, there exists a map $P(\cdot)$ for which Assumption 3.5 and the following inequality are fulfilled:

$$H_{\text{res}}(\varphi, K) \geq \frac{1}{2 \ln 2} \max_{x \in K} \sum_{i=1}^{n} \max \{0, \varsigma_{i}^{P}(x)\} - \varepsilon.$$

The proof of this theorem is given in Appendix C. As in the case of discrete time, analogs of Remark 3.3 and Corollary 3.4 follow from Theorem 3.7.

4 Example: The Landford system

Consider the following system:

$$\begin{align*}
\dot{x} &= (a - 1)x - y + xz \\
\dot{y} &= x + (a - 1)y + yz, \quad x, y, z \in \mathbb{R}, \quad a > 0 \\
\dot{z} &= az - (x^2 + y^2 + z^2)
\end{align*} \tag{12}$$

This system is attributed to Landford and was studied in many publications, see, e.g. [2]. It is well-known that the system (12) has only two equilibrium points:

$$O_1 = [0, 0, 0]^T, \quad O_2 = [0, 0, a]^T.$$ 

The value $a = 2/3$ is of particular interest, since then there is a heteroclinic orbit connecting the equilibria [2].

Let $K$ be some compact forward-invariant set of (12).

4.1 Remark In $K$, we necessarily have $z \geq 0$.

Indeed, if $z(0) < 0$, then the third equation of (12) implies that the solution escapes to $-\infty$ in finite time ($\dot{z} < -z^2$).

The Jacobian matrix from (9) is now given as follows:

$$D\varphi(x, y, z) = \begin{bmatrix}
a - 1 + z & -1 & x \\
1 & a - 1 + z & y \\
-2x & -2y & a - 2z
\end{bmatrix}.$$
In (i) of Theorem 3.7, we take the following matrix function

\[ P(x, y, z) = P_0 e^{w(x, y, z)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \exp \left( \frac{2z}{a} \right), \]  

(13)

Straightforward calculations yield

\[ PD\varphi(x) = \begin{bmatrix} a - 1 + z & -1 & x \\ 1 & a - 1 + z & y \\ -x & -y & \frac{1}{2}(a - 2z) \end{bmatrix} e^w, \]

\[ D\varphi(x)^{\top} P = \begin{bmatrix} a - 1 + z & 1 & -x \\ 1 & a - 1 + z & -y \\ x & y & \frac{1}{2}(a - 2z) \end{bmatrix} e^w, \]

and therefore

\[ 2[P(x)D\varphi(x)]^{sym} = D\varphi(x)^{\top} P + PD\varphi(x) \]

\[ = e^w \begin{bmatrix} 2(a - 1 + z) & 0 & 0 \\ 0 & 2(a - 1 + z) & 0 \\ 0 & 0 & a - 2z \end{bmatrix}. \]

At the same time,

\[ \dot{P} - \lambda P = \left( \frac{2z}{a} - \lambda \right) e^w P_0 \]

(12)

\[ = e^w \left( \frac{2}{a} (az - x^2 - y^2 - z^2) - \lambda \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}. \]

Finally, the solutions of (12) can easily be found:

\[ \lambda_1 = 2(a - 2z) + \frac{2az - z^2 - x^2 - y^2}{a} \]

\[ \leq -\frac{2z^2}{a} - 2z + 2a \leq 2a, \]  

(14)

\[ \lambda_{2,3} = 2(a - 1 + z) + \frac{2az - z^2 - x^2 - y^2}{a} \]

\[ \leq -\frac{2z^2}{a} + 4z + 2(a - 1) \leq 2(2a - 1). \]  

(15)

By (i) of Theorem 3.7 the following upper estimate holds:

\[ H_{res}(K) \leq \frac{1}{2 \ln 2} \max_K \left[ \max \{0, \lambda_1\} + 2 \max \{0, \lambda_{2,3}\} \right] \]

\[ ^{\text{1}\text{For a more detailed treatment of the metric in this form for related problems of stability of forced oscillations, see [33].}} \]

11
\[ \frac{1}{2 \ln 2} \max\{ \max_k \lambda_1, 2 \max_k \lambda_{2,3}, \max_k (\lambda_1 + 2\lambda_{2,3}) \}. \]

Maximizing \( \lambda_1 + 2\lambda_{2,3} \) over \( z \in \mathbb{R} \) yields

\[ \max_{x,y,z} (\lambda_1 + 2\lambda_{2,3}) \leq \max_z \left[ 6a - 4 + 6z - \frac{6}{a}z^2 \right] = \frac{15}{2}a - 4. \]

By using (14) and (15), we thus arrive at the following.

4.2 Theorem Let \( K \) be a compact forward-invariant set of the system (12) with \( a \geq 2/3 \). Then

\[ H_{\text{res}}(\varphi, K) \leq \frac{2(2a - 1)}{\ln 2}. \]

Our next step is to derive a lower estimate for \( H_{\text{res}}(K) \) under an extra assumption imposed on the set \( K \). We start with the calculation of the proximate entropy \( H_L(O) \) around the system equilibria \( O \) (for the definition of the proximate entropy, see [28]). Calculating the eigenvalues of \( D\varphi(O_i) \), \( i = 1, 2 \) one can easily derive that

\[ H_L(O_1) = \begin{cases} 
\frac{1}{\ln 2} a & \text{if } 0 < a \leq 1, \\
\frac{1}{\ln 2} (3a - 2) & \text{if } a \geq 1
\end{cases}, \]

\[ H_L(O_2) = \begin{cases} 
\frac{1}{\ln 2} 0 & \text{if } 0 < a \leq 1/2, \\
\frac{1}{\ln 2} \frac{2(2a - 1)}{2a - 1} & \text{if } a \geq 1/2
\end{cases}, \]

\[ \max\{ H_L(O_1), H_L(O_2) \} \overset{a \geq 2/3}{=} H_L(O_2) = \frac{2(2a - 1)}{\ln 2}. \]

The last relation together with [28, Cor. 12] and Theorem 4.2 gives the following result.

4.3 Theorem Assume that \( a \geq 2/3 \). Let \( K \) be any compact forward-invariant set for system (12), which satisfies Assumption 2.2 and the inclusion \( O_2 \in \text{int}K \). Then

\[ H_{\text{res}}(\varphi, K) = \frac{2(2a - 1)}{\ln 2}. \]

At this point it is worth mentioning that the matrix \( P \) from (13) not only provides an upper estimate of the restoration entropy according to the statement (i) of Theorem 3.7 but also gives a Riemannian metric for which the lower estimate (see the statement (ii) of Theorem 3.7) holds true with \( \varepsilon = 0 \).

The remainder of the paper is devoted to proofs of Thm. 3.2 and 3.7.
A Technical preamble to proofs of Thm. 3.2 and 3.7

From now on, we use the following notations: \( \text{Gl}(n, \mathbb{R}) \) – group of real invertible \( n \times n \) matrices, \( I_n \) – \( n \times n \) identity matrix, \( \| \cdot \|_2 \) – standard Euclidean norm on \( \mathbb{R}^n \), \( x_i \) – the components of \( x \in \mathbb{R}^k \).

A.1 Riemannian geometry on the set \( S^+ \)

Since the set \( S^+ \) is open in the vector space \( S \) of all \( n \times n \)-matrices, the first candidate for a Riemannian metric tensor on \( S^+ \) is the Euclidean metric inherited from \( S \). However, many applications motivate the use of the so-called trace metric \[3\]. It varies with the point \( p \in S^+ \) and defines the following inner product on the space \( T_p S^+ \) tangential to \( S^+ \) at \( p \), which is in fact a copy of \( S \):

\[
\langle v, w \rangle_p := \text{tr}(p^{-1}vp^{-1}w), \quad \forall v, w \in T_p S^+ \cong S.
\]

This metric makes \( S^+ \) a complete Riemannian manifold \[7\], assigns a special length to any smooth curve, and gives birth to the Riemannian distance \( d(p, q) \), defined as the minimal length of curves bridging the points \( p \) and \( q \) of \( S^+ \). The minimizer exists, is unique, is called a geodesic, and is given by \[3, \text{Thm. 6.1.6}\]:

\[
p\#_t q := p^{1/2}(p^{-1/2}q^{1/2}p^{-1/2})^tp^{1/2}, \quad t \in [0, 1]. \tag{16}
\]

Let \( \alpha_1(g) \geq \ldots \geq \alpha_n(g) > 0 \) be the singular values of \( g \in \text{Gl}(n, \mathbb{R}) \). We put

\[
\vec{\sigma}(g) := [\log \alpha_1(g), \ldots, \log \alpha_n(g)] \in a^+ := \{\xi \in \mathbb{R}^n : \xi_1 \geq \xi_2 \geq \ldots \geq \xi_n\} \tag{17}
\]

and endow \( a^+ \) with the following partial order:

\[
\xi \leq \eta \overset{\text{def}}{\iff} \begin{cases} \sum_{i=1}^k \xi_i \leq \sum_{i=1}^k \eta_i & \forall 1 \leq k \leq n-1, \\ \sum_{i=1}^k \xi_i = \sum_{i=1}^k \eta_i & \text{for } k = n. \end{cases} \tag{18}
\]

Any matrix \( g \in \text{Gl}(n, \mathbb{R}) \) defines an action (mapping) \( g^* \) on \( S^+ \), specifically, \( p \in S^+ \mapsto g^*p := gpg^{-1} \in S^+ \). It is easy to see that \( g_1^*(g_2^*p) = (g_1g_2)^*p \), \( I^*p = p, p_2 = g^*p_1 \iff p_1 = g^{-1} * p_2 \), and for any \( p, q \in S^+ \) there exists \( g \in \text{Gl}(n, \mathbb{R}) \) (e.g., \( g := q^{1/2}p^{-1/2} \)) such that \( q = g^*p \).

The following proposition is based on \[4\] and lists some properties of the so-called vectorial distance

\[
\vec{d}(p, q) := 2\vec{\sigma}(p^{-1/2}q^{1/2}) \in a^+, \quad p, q \in S^+. \tag{19}
\]

A.1 Proposition The following statements hold:

a) \( \vec{d}(I, p) = \vec{\sigma}(p) \) and \( \|d(p, q)\|_2 = d(p, q) \forall p, q \in S^+ \);
b) \( \vec{d}(p_1, q_1) = \vec{d}(p_2, q_2) \) if and only if \( g \ast p_1 = p_2 \) and \( g \ast q_1 = q_2 \) for some \( g \in \text{GL}(n, \mathbb{R}) \);

c) \( \vec{d}(p, p) = 0; \vec{d}(p, q) \leq \vec{d}(p, r) + \vec{d}(r, q) \);

d) \( \vec{d}(q, p) = i(\vec{d}(p, q)) \), where \( i(\xi) := - (\xi_n, \xi_{n-1}, \ldots, \xi_1) \);

e) The curve \((16)\) is a geodesic segment for the vectorial distance, i.e., there is \( \xi \in \mathbb{R}^{+} \) such that \( \vec{d}(p\#_t q, p\#_s q) = (s - t)\xi \) for all \( s \geq t \);

f) \( \vec{d}(r\#_{1/2} p, r\#_{1/2} q) \leq \frac{1}{2}\vec{d}(p, q) \) for all \( p, q, r \in S^+ \).

Here b) shows that the map \( g \ast \) is an isometry (preserves distances); c) and d) are akin to the axioms of the metric, and f) replicates formula (4-7) in [4].

A.2 Proposition The following holds for geodesics on \( S^+ \):

\[
\begin{align*}
g \ast (p\#_t q) &= (g \ast p)\#_t (g \ast q), \quad \text{(20)} \\
\vec{d}(p\#_t q, r\#_t o) &\leq (1 - t)\vec{d}(p, r) + t\vec{d}(q, o) \quad \forall t \in [0, 1]. \quad \text{(21)}
\end{align*}
\]

Proof Here (20) holds since the map \( g \ast \) is an isometry of the Riemannian space \( S^+ \). To prove (21), we use c) and f) in Prop. A.1 and the symmetry \( p\#_{1/2} q = q\#_{1/2} p \) [3, Sec. 4.1, 6.1.7] and see that

\[
\begin{align*}
\vec{d}(p\#_{1/2} q, r\#_{1/2} o)
&\leq \vec{d}(p\#_{1/2} q, p\#_{1/2} o) + \vec{d}(p\#_{1/2} o, r\#_{1/2} o) \\
&= \vec{d}(p\#_{1/2} q, p\#_{1/2} o) + \vec{d}(o\#_{1/2} p, o\#_{1/2} r) \\
&\leq \frac{1}{2}\vec{d}(q, o) + \frac{1}{2}\vec{d}(p, r),
\end{align*}
\]

(21) with \( t = 1/2 \). For all numbers \( t \) of the form \( t = k/2^n \) with integers \( k, n \) such that \( 0 \leq k \leq 2^n \), it follows by induction. Since these numbers are dense in \([0, 1]\) and (16) is continuous in \( t \), (21) extends to the entirety of \([0, 1]\) by continuity.

The point \( p\#_{1/2} q \) is in fact the barycenter of the two-point set \( \{ p, q \} \), i.e., a minimizer \( r \) of \( d(r, p)^2 + d(r, q)^2 \). This notion of barycenter has a far-reaching generalization. Specifically, let

\[
\Delta_m := \{ \omega \in \mathbb{R}^m : \omega_i \geq 0, \sum \omega_i = 1 \}
\]

be the standard simplex in \( \mathbb{R}^m \), let \( \omega \in \Delta_m \), and let \( p_1, \ldots, p_m \in S^+ \). Then the following point is well-defined (see, e.g., [25]):

\[
\bar{\text{bar}}(\omega; p_1, \ldots, p_m) := \arg \min_{q \in S^+} \sum_{i=1}^{m} \omega_i d(q, p_i)^2
\]

14
and is called a weighted barycenter of $p_1, \ldots, p_m$ (also is known under other names). The unweighted barycenter is given by

$$\text{bar}(p_1, \ldots, p_m) := \text{bar}(\omega^\top; p_1, \ldots, p_m), \quad \text{where } \omega_i := \frac{1}{m} \forall i.$$ 

There is no closed-form expression for $\text{bar}(\omega; p_1, \ldots, p_m)$, in general. The following theorem from [26] (see [18] for the unweighted case) characterizes the barycenter as the limit of a certain iterative process, which acts through a binary operation at each step $k = 1, 2, \ldots$. Specifically, let $i(mod m)$ stand for the remainder after dividing $i$ by $m$. We put $l(k) := \sum_{i=1}^{k} (\omega_i (mod m))$, $s_k := \omega_k(mod m)/l(k)$ and consider the following process:

$$\bar{p}_1 := p_1, \quad \bar{p}_k := \bar{p}_{k-1} \# s_k \ p_k \ (mod \ m).$$

A.3 Theorem ([18] [26]) $\text{bar}(\omega; p_1, \ldots, p_m) = \lim_{k \to \infty} \bar{p}_k$.

Now we list some properties of the weighted barycenter.

A.4 Proposition The following relations hold:

$$g \ast \text{bar}(\omega; p_1, \ldots, p_m) = \text{bar}(\omega; g \ast p_1, \ldots, g \ast p_m) \quad \forall g \in \text{Gl}(n, \mathbb{R}); \quad (22)$$

$$d(u, v) \leq \omega_m d(p_m, p_m') \quad \forall p_1, \ldots, p_m, p_m' \in S^+, \quad (23)$$

where

$$\begin{align*}
    u &= \text{bar}(\omega; p_1, \ldots, p_{m-1}, p_m), \\
    v &= \text{bar}(\omega; p_1, \ldots, p_{m-1}, p_m'); \\
    \text{bar}(\omega; p_1, \ldots, p_m) &= \text{bar}(\omega_{\sigma}; p_{\sigma(1)}, \ldots, p_{\sigma(m)}) \quad (24)
\end{align*}$$

for any permutation $\sigma$ of $\{1, \ldots, m\}$, where $\omega_{\sigma}$ is the result $(\omega_{\sigma(1)}, \ldots, \omega_{\sigma(m)})$ of its action on $\omega$.

Proof Here (22) and (24) are straightforward from the definition of the barycenter and the fact that $g \ast$ is an isometry. To prove (23), we consider the sequences $(\bar{p}_k)_{k \in \mathbb{N}}$ and $(\bar{q}_k)_{k \in \mathbb{N}}$ defined as in Thm. A.3 from the data $(\omega; p_1, \ldots, p_{m-1}, p_m)$ and $(\omega; p_1, \ldots, p_{m-1}, p'_m)$, respectively. By using (21), we obtain

$$\begin{align*}
    d(\bar{p}_{km}, \bar{q}_{km}) &= d(\bar{p}_{km-1} \# s_{km} \ p_m, \bar{q}_{km-1} \# s_{km} \ p'_m) \\
    &\leq (1 - s_{km}) d(\bar{p}_{km-1}, \bar{q}_{km-1}) + s_{km} d(p_m, p_m') \\
    &= (1 - s_{km}) d(\bar{p}_{km-2} \# s_{km-1} \ p_{m-1}, \bar{q}_{km-2} \# s_{km-1} \ p_{m-1}) \\
    &\quad + s_{km} d(p_m, p_m') \leq (1 - s_{km}) \left[ (1 - s_{km-1}) d(\bar{p}_{km-2}, \bar{q}_{km-2}) \\
    &\quad + s_{km-1} d(p_{m-1}, p_{m-1}) \right] + s_{km} d(p_m, p_m') \\
    &= 0
\end{align*}$$

15
\[
(1 - s_{km})(1 - s_{km-1})d(\bar{p}_{km-2}, \bar{q}_{km-2}) + s_{km}d(p_m, p'_m) \\
\leq \ldots \\
\leq \prod_{i=0}^{m-1} (1 - s_{km-i}) d(\bar{p}_{(k-1)m}, \bar{q}_{(k-1)m}) + s_{km}d(p_m, p'_m).
\]

Now we observe that \( s_{km} = \omega_m / k \) and

\[
\prod_{i=0}^{m-1} (1 - s_{km-i}) = \prod_{i=0}^{m-1} \left(1 - \frac{\omega_{m-i}}{k - 1 + \sum_{j=1}^{m-i} \omega_j}\right) = \prod_{i=0}^{m-1} \frac{k - 1 + \sum_{j=1}^{m-i} \omega_j - \omega_{m-i}}{k - 1 + \sum_{j=1}^{m-i} \omega_j} = \frac{k - 1}{k}.
\]

Hence, we end up with

\[
d(\bar{p}_{km}, \bar{q}_{km}) \leq \frac{k - 1}{k} d(\bar{p}_{(k-1)m}, \bar{q}_{(k-1)m}) + \frac{\omega_m}{k} d(p_m, p'_m).
\]

Iterating this estimate all the way down to \( k = 1 \), we find that

\[
d(\bar{p}_{km}, \bar{q}_{km}) \leq \left(\frac{\omega_m}{k} + \frac{k - 1}{k} \frac{\omega_m}{k - 1} + \frac{k - 1}{k} \frac{k - 2}{k - 1} \frac{\omega_m}{k - 2} + \ldots \right)
\]

\[
+ \prod_{i=1}^{k} \frac{k - i}{k - i + 1} \omega_m d(p_m, p'_m) = \omega_m d(p_m, p'_m).
\]

Letting \( k \to \infty \) with regard to Thm. [A.3] completes the proof. □

The set of \( m \) points \( p_i \in S^+ \) and the tuple of weights \( \omega \in \Delta_m \) give birth to a probability measure on \( S^+ \): this is the convex combination \( \mu \) of \( m \) Dirac measures \( \delta_{p_i} \) with the coefficients \( \omega_i \). The weighted barycenter is the minimizer \( p \) of the function

\[
p \mapsto \int_{S^+} d(p, q)^2 d\mu(q).
\]

This observation permits to extend the notion of the barycenter to any probability measure \( \mu \) on the Borel \( \sigma \)-algebra of \( S^+ \).

More rigorously, let \( \mathcal{P}^1(S^+) \) be the set of all Borel probability measures on \( S^+ \) with finite first moment, i.e.,

\[
\mathcal{P}^1(S^+) := \left\{ \mu : \int_{S^+} d(I, p) \, d\mu(p) < \infty \right\}.
\]

This set can be equipped with the 1-Wasserstein metric [39]:

\[
W_1(\mu, \nu) := \inf_{P \in \mathcal{P}^1(S^\times S^+)} \int_{S^+ \times S^+} d(p, q) \, dP(p, q),
\]
where \((\mu|\nu)\) is the set of all probability measures \(P\) on \(S^+ \times S^+\) whose projection along the first and second coordinate coincides with \(\mu\) and \(\nu\), respectively. Simultaneously [39, Rem. 6.5],

\[
W_1(\mu, \nu) = \sup_{\|\psi\|_{\text{Lip}} \leq 1} \left[ \int_{S^+} \psi \, d\mu - \int_{S^+} \psi \, d\nu \right],
\]

where \(\|\psi\|_{\text{Lip}} := \sup_{p_1 \neq p_2, p_i \in S^+} |\psi(p_1) - \psi(p_2)|/d(p_1, p_2).

Remark 4.2 and Lemma 5.1 in [37] give rise to the following.

A.5 Lemma (i) The sums of Dirac measures with identical weights \(\frac{1}{m}(\delta_{p_1} + \ldots + \delta_{p_m})\) form, in total, a set that is dense in \(P^1(S^+)\). (ii) For any isometry \(I : S^+ \rightarrow S^+\), the associated push-forward transformation of probability measures \(\mu \mapsto \nu, \nu(E) := \mu(I^{-1}E)\) isometrically maps \(P^1(S^+)\) into itself.

The next proposition asserts existence of the measure barycenter and reports some of its properties.

A.6 Proposition (Lem. 5.1, Thm. 6.3, Prop. 4.3 in [37]) There exists a map \(\text{bar} : P^1(S^+) \rightarrow S^+\) with the following properties:

1. Whenever an isometry \(I : S^+ \rightarrow S^+\) pushes a probability measure \(\mu\) forward into the measure \(\nu\), we have \(\text{bar}[\nu] = I(\text{bar}[\mu])\);

2. The map \(\text{bar}[\cdot]\) is 1-Lipschitz:

\[
d(\text{bar}[\mu], \text{bar}[\nu]) \leq W_1(\mu, \nu) \quad \forall \mu, \nu \in P^1(S^+);
\]

3. The minimum of (25) is attained at \(\text{bar}[\mu]\) provided that the r.h.s. is well-defined, i.e., \(\int_{S^+} d(I, q)^2 \, d\mu(q) < \infty\).

The last inequality implies that in (25), \(\int_{S^+} d(p, q)^2 \, d\mu(q) < \infty\) for any \(p \in S^+\) since \(d(p, q) \leq d(p, I) + d(I, q)\) and the constant \(d(p, I)\) is of class \(L_2\) with respect to the probability measure.

A.7 Corollary The unweighed barycenter \(\text{bar}(p_1, \ldots, p_m)\) depends continuously on \(p_i \in S^+\).

This is immediate from (2) in Proposition A.6 since (26) implies that \(W_1(\mu, \nu) \leq \max_i d(p_i, p_i')\) if \(\mu\) and \(\nu\) are the sums of equi-weighted Dirac measures at points \(p_i\) and \(p_i', \ldots, p_m', \) respectively.
A.2 Derivatives of singular values and other matrix functionals

The \textit{m-th largest eigenvalue} of a symmetric matrix \( A \in S \) is the \( m \)-th term \( \lambda_m(A) \) of the decreasing sequence \( \lambda_1(A) \geq \ldots \geq \lambda_n(A) \) of its eigenvalues, each repeated with regard to its algebraic multiplicity \( r_m \). We put \( i_m := m - m_{\text{first}} + 1 \), where \( m_{\text{first}} \) is the first position in the sequence that accommodates the number \( \lambda_m(A) \), whereas \( m_{\text{last}} \) is the last position. In particular, \( i_m = 1 \) if \( r_m = 1 \). Also let \( U \) be an orthogonal matrix that reduces \( A \) to a diagonal matrix \( U^T A U \) with decreasing diagonal entries. We denote by \( \tilde{U}_m \) the matrix that is obtained by depriving \( A \) of all columns, except for those in the range from \( m_{\text{first}} \) to \( m_{\text{last}} \).

A.8 Theorem ([17, Thm. 4.5]) Let \( O \subset \mathbb{R}^p \) be an open set and \( A : O \to S \) be a map of class \( C^1 \). For any \( m = 1, \ldots, n, \xi_0 \in O \) and unit-length vector \( d \in \mathbb{R}^p \), the derivative \( \lim_{\varepsilon \to 0^+} \frac{f(\xi_0 + \varepsilon d) - f(\xi_0)}{\varepsilon} \) of the function \( \xi \in O \mapsto f(\xi) := \lambda_m[A(\xi)] \) in direction \( d \) exists. It equals the \( i_m \)-largest eigenvalue of the following \( r_m \times r_m \) matrix built from \( A(x_0) \) (along with \( \tilde{U}_m, r_m, i_m \))

\[
\tilde{U}_m \left[ \sum_{j=1}^p d_j \frac{\partial A}{\partial \xi_j}(\xi_0) \right] \tilde{U}_m.
\]

A.9 Corollary For any \( C^1 \)-smooth function \( g : \mathbb{R} \to \text{Gl}(n, \mathbb{R}) \) with \( g(0) = I \) and \( \dot{g}(0) = H \in \mathbb{R}^{n \times n} \), the l.h.s. of (17) with \( g = g(t) \) has the right derivative at \( t = 0 \) and

\[
\frac{d}{dt} [g(t)]_{t=0^+} = \frac{1}{2 \ln(2)} [\lambda_1(H + H^T), \ldots, \lambda_n(H + H^T)].
\]

\textbf{Proof} The map \( t \mapsto A(t) := g(t)g(t)^T \in S \) is of class \( C^1 \) and

\[
\bar{g}[g(t)] = \frac{1}{2} [\log \lambda_1(A(t)), \ldots, \log \lambda_n(A(t))], \quad (28)
\]

\[
A(0) = I \Rightarrow r_m = n, i_m = m, \tilde{U}_m = I, \quad \tilde{A}(0) = H + H^T.
\]

By applying Thm. A.8 to \( p = 1, O := \mathbb{R}, \xi_0 := 0 \), we see that the right derivative of \( \lambda_m[A(t)] \) exists and is equal to \( \lambda_m[\tilde{A}(0)] = \lambda_m(H + H^T) \). This and (28) complete the proof. \( \square \)

A.10 Lemma The function \((p,q) \in S^+ \times S^+ \mapsto \Upsilon(p,q) := p^{-1/2}q^{1/2} \in \text{Gl}(n, \mathbb{R}) \) is of class \( C^1 \) and its derivative at any point with \( p = q \) is given by

\[
\text{D}\Upsilon(p,p) \begin{bmatrix} v_p \\ v_q \end{bmatrix} = p^{-1/2}h(v_q - v_p) \quad \forall v_p, v_q \in S,
\]

where \( h = h(v) \in S \) is the unique solution of the equation

\[
h p^{1/2} + p^{1/2} h = v \in S. \quad (29)
\]
Proof. Existence and uniqueness of the solution for the Lyapunov equation (29) is a well known fact; see e.g., [24, Ch. 3]. We put \( \Xi(p) := p^{-1}, \Omega_{\pm}(p) := p^{\pm 1/2} \forall p \in \mathcal{S}^+ \). It is immediate from [33, Thm. D2] and [30, Thm. 1.1] that \( \Xi, \Omega, \Upsilon \) are \( C^1 \)-smooth and

\[
\begin{align*}
D\Xi(p)v &= -p^{-1}vp^{-1}, \quad D\Omega_{\pm}(p)v = h(v) \quad \forall v \in \mathcal{S}.
\end{align*}
\]

Since \( \Omega_{-} = \Xi \circ \Omega_{+} \), we have \( D\Omega_{-}(p) = D\Xi [\Omega_{+}(p)] \circ D\Omega_{+}(p) \leftrightarrow D\Omega_{-}(p)v = -p^{-1/2}h(v)p^{-1/2} \). Finally, \( \Upsilon(p, q) = \Omega_{-}(p) \times \Omega_{+}(q) \), hence

\[
\begin{align*}
D\Upsilon(p, q) \left[ \begin{array}{c} v_p \\ v_q \end{array} \right] &= D\Omega_{-}(p)v_p \times \Omega_{+}(p) + \Omega_{-}(p) \times D\Omega_{+}(p)v_q \\
&= -p^{-1/2}h(v_p)p^{-1/2} \times p^{1/2} + p^{-1/2} \times h(v_q) \\
&= p^{-1/2}[h(v_q) - h(v_p)] = p^{-1/2}h(v_q - v_p).
\end{align*}
\]

\[\square\]

A.3 Linear cocycles

A dynamical system on a metric space \( X \) is given by its evolution function (dynamic semiflow) \( \Phi : \mathcal{T}_+ \times X \rightarrow X \), where \( \mathcal{T}_+ \) is like in (1) and \( \Phi(0, x) = x \forall x, \Phi[s, \Phi(t, x)] = \Phi[s + t, x] \forall s, t \in \mathcal{T}_+, x \in X \). For this system, a linear cocycle is defined as a mapping \( (t, x) \mapsto A^{(t)}(x) \in \text{Gl}(k, \mathbb{R}) \) such that

\[
A^{(0)}(x) = I \quad \text{and} \quad A^{(s+t)}(x) = A^{(s)}[\Phi(t, x)]A^{(t)}(x)
\] (30)

for all \( x \in X, s, t \in \mathcal{T}_+ \). In the discrete-time case, the semiflow is determined by \( \Psi(\cdot) := \Phi(1, \cdot) \) (specifically, \( \Phi(t, \cdot) \) is the \( t \)-th iterate of \( \Psi(\cdot) \) for \( t > 0 \) and the identity map for \( t = 0 \)), and the cocycle by its generator \( A(\cdot) := A^{(1)}(\cdot) \) since (30) shapes into

\[
A^{(0)}(x) = I, \quad A^{(t)}(x) = A[\Psi^{t-1}(x)] \cdots A[\Psi^{1}(x)]A[x].
\]

Given two linear cocycles \( A, B : \mathcal{T}_+ \times X \rightarrow \text{Gl}(k, \mathbb{R}) \) for a common dynamical system, they are said to be conjugate if there exists a continuous map \( V : X \rightarrow \text{Gl}(k, \mathbb{R}) \) (the conjugacy) such that \( B^{(t)}(x) = V[\Phi(t, x)]^{-1}A^{(t)}(x)V(x) \forall x \in X, t \in \mathcal{T}_+ \). In the discrete-time case, it suffices to test only \( t = 1 \), i.e., the relation \( \mathcal{B}(x) = V[\Psi(x)]^{-1}A(x)V(x) \forall x \in X \), where \( \mathcal{B}(\cdot) := B^{(1)}(\cdot) \).

B Proof of Theorem 3.2

In this section, the assumptions of this theorem are adopted; in particular, we consider the discrete-time system (1) (the case d-t)). The first step towards proving (i) in Thm. 3.2 is the following.
B.1 Lemma There are $C_+ \in \mathbb{R}$ such that for all $t \in \Sigma_+, x \in K,$

$$-\infty < C_- \leq \Sigma^P(t, x) - \Sigma^I(t, x) \leq C_+ < \infty.$$  

(31)

Here $\Sigma^P(t, x) := \Sigma[x|\varphi^t]$ is defined by using (8), and $\Sigma^I(t, x)$ employs the ordinary singular values $\alpha_i(t, x)$ in (8).

Proof Let $\omega_k(C)$ be the product of $k$ largest singular values of the square matrix $C$ if $k \geq 1$; and $\omega_0(C) := 1$. We apply Lemma 3.1 to the $t$-th iterate $\phi := \varphi^t$ of $\varphi$, denote by $A(t)(x)$ and $B(t)(x)$ the respective $A(x)$ and $B(x)$ from that lemma, and see that thanks to the last claim of that lemma,

$$\Sigma^P(t, x) - \Sigma^I(t, x) = \log \max_{0 \leq k \leq n} \omega_k(B(t)(x)) \max_{0 \leq k \leq n} \omega_k(A(t)(x)).$$

By Horn’s inequality [6, Prop. 2.3.1],

$$\omega_k(B(t)(x)) \leq \omega_k(P(\varphi^t(x))^{1/2}) \omega_k(A(t)(x)) \omega_k(P(x)^{-1/2}).$$

Here $x \in K \Rightarrow y := \varphi^t(x) \in K \forall t$ by Asm. 2.1. Since the singular values continuously depend on the matrix [19, Thm. 2.6.4], so does $\omega_k$. Hence the following functions are continuous as well:

$$y \in K \mapsto \omega_k(P(y)^{1/2}) \quad \text{and} \quad y \in K \mapsto \omega_k(P(y)^{-1/2})$$

So the maximum over the compact set $K$ is attained and finite for the both. This observation yields the upper estimate in (31). The lower estimate is obtained likewise by applying Horn’s inequality to $\omega_k(A(t)(x))$. Thus we see that (31) does hold.

□

Proof (of (i) in Thm. 3.2) By combining (31) with Remark 2.3, we get

$$H_{\text{res}}(\varphi, K) \leq \max_{x \in K} \limsup_{t \to \infty} \frac{1}{t} \Sigma^P(t, x).$$

(32)

The generalized Horn’s inequality [6, Prop. 7.4.3]) implies that $\Sigma^P[t + s, x] \leq \Sigma^P[s, \varphi^t(x)] + \Sigma^P[t, x]$. Then by [31] Thm. A.3 and the definition of $\Sigma^P(t, x)$, the r.h.s. of (32) equals

$$\inf_{t > 0} \max_{x \in K} \frac{1}{t} \sum_{i=1}^{n} \max \{0, \log \alpha_i^P(t, x)\} \leq \max_{x \in K} \sum_{i=1}^{n} \max \{0, \log \alpha_i^P(x)\}.$$ 

□

A key to the proof of (ii) in Theorem 3.2 is the following lemma, which uses the concepts and notations from Subsection A.3.
B.2 Lemma  Let \( A \) be a continuous linear cocycle over a continuous discrete-time dynamical system \( \Psi \) on a compact metric space \( X \). For any natural \( N \), there exists a continuous cocycle \( B \), conjugate to \( A \), such that for its generator \( \mathcal{B} \), the following holds:

\[
\tilde{\sigma}[\mathcal{B}(x)] \leq N^{-1} \tilde{\sigma}[A^{(N)}(x)] \quad \text{for all } x \in X. \tag{33}
\]

The involved conjugacy \( V \) can be chosen so that \( V(x) \in S^+ \) for all \( x \).

**Proof**  For \( x \in X \), let \( Q(x) \in S^+ \) be the unweighted barycenter:

\[
Q(x) := \text{bar}(I, A^{(1)}(x)^{-1} \ast I, \ldots, A^{(N-1)}(x)^{-1} \ast I).
\]

By (22) and (24), the matrix \( A^{(1)}(x) \ast Q(x) \) is equal to

\[
\text{bar}(A^{(1)}(x) \ast I, A^{(1)}(\Psi(x))^{-1} \ast I, \ldots, A^{(N-2)}(\Psi(x))^{-1} \ast I) = \text{bar}(I, A^{(1)}(\Psi(x))^{-1} \ast I, \ldots, A^{(N-2)}(\Psi(x))^{-1} \ast I, A^{(1)}(x) \ast I).
\]

Meanwhile, \( Q[\Psi(x)] \) is the following matrix:

\[
\text{bar}(I, A^{(1)}(\Psi(x))^{-1} \ast I, \ldots, A^{(N-1)}(\Psi(x))^{-1} \ast I).
\]

For the generator \( A(x) = A^{(1)}(x) \), we thus have

\[
d[Q(\Psi(x)), A(x) \ast Q(x)] \leq \frac{d[A^{(N-1)}(\Psi(x))^{-1} \ast I, A(x) \ast I]}{N} \tag{34}
\]

Thanks to Corollary A.7, the maps \( Q(\cdot) \) and \( x \in X \mapsto V(x) := Q(x)^{-1/2} \in S^+ \) are continuous; also, \( V(x) \ast Q(x) = Q(x)^{-1/2}Q(x)V(x)^{-1/2} = I \). For the conjugate linear cocycle generated by \( \mathcal{B}(x) := V(\Psi(x))A(x)V(x)^{-1} \), we have

\[
\tilde{\sigma}[\mathcal{B}(x) \ast I] = d[I, \mathcal{B}(x) \ast I] \tag{40}
\]

\[
= d\left\{ Q[\Psi(x)]^{1/2} \ast I, Q[\Psi(x)]^{1/2} \mathcal{B}(x) \ast I \right\} = d\left\{ Q[\Psi(x)], V[\Psi(x)]^{-1} \mathcal{B}(x) \ast I \right\} = d\left\{ Q[\Psi(x)], A(x) V(x)^{-1} \ast I \right\} = d\left\{ Q[\Psi(x)], A(x) \ast Q(x) \right\}
\]

\[
\leq N^{-1} d[I, A^{(N)}(x) \ast I] \tag{10}
\]

To get (33), it remains to note that \( \tilde{\sigma}(g \ast I) = 2\tilde{\sigma}(g) \) by (17).

**Proof (of (ii) in Thm. 3.2)** By (4), there exists \( N \) such that

\[
H_{\text{res}}(\varphi, K) \geq \frac{1}{N} \max_{x \in K} \sum_{i=1}^{n} \max \left\{ 0, \log \alpha_i(N, x) \right\} - \varepsilon
\]

\[
= \max_{x \in K} \frac{1}{N} \sum_{k=0}^{n} \sum_{i=1}^{k} \log \alpha_i(N, x) - \varepsilon,
\]

\[
= \max_{x \in K} \frac{1}{N} \sum_{k=0}^{n} \sum_{i=1}^{k} \log \alpha_i(N, x) - \varepsilon, \tag{35}
\]
where \( \sum_{i=1}^{k} \log \alpha_i[B(x)] \leq \frac{1}{N} \sum_{i=0}^{k} \log \alpha_i(N, x) \quad \forall k = 0, \ldots, n \)

\[ \Rightarrow H_{\text{res}}(\varphi, K) \geq \max_{x \in K} \max_{k=0, \ldots, n} \sum_{i=1}^{N} \log \alpha_i[B(x)] - \varepsilon \]

It remains to note that \( B(x) = V[\varphi(x)]^{-1}A(x)V(x) \forall x \in X \) by the last sentence in Subsection A.3 and so \( \alpha_i[B(x)] = \alpha_i^P(x|\varphi) \) for \( P(x) := V(x)^{-2} \) by the last claim of Lem. 3.1.

### C Proof of Theorem 3.7

In this section, the assumptions of this theorem are adopted; in particular, the system (1) is described by ODE \( \dot{x} = \varphi(x) \). We use the dynamical flow \( \varphi^t \) introduced just after Asm. 2.1 and the associated linear cocycle \((t, x) \mapsto A(t)(x) := D\varphi^t(x) \in GL(n, \mathbb{R})\).

**Proof (of (i) in Thm. 3.7)** The proof is accomplished by following the arguments from the proof [28, Thm. 14], where \( v_d(x) := 0 \). The only difference is that the claim \( P(\cdot) \in C^1 \) from [28] is relaxed to Asm. 3.5. But this does not destroy these arguments, which can be seen by inspection that encompasses the proof of the underlying [34, Prop. 8.6].

In proving (ii), the role of Lemma B.2 is played by the following.

**C.1 Proposition** For every \( T > 0 \), there exists a mapping \( P : K \to S^+ \) that obeys Asm. 3.5 and the following inequality

\[
\frac{1}{2 \ln(2)}[\varsigma_1^P(x), \ldots, \varsigma_n^P(x)] \preceq \frac{1}{T} \sigma[A(T)(x)] \forall x \in K,
\]

where \( \varsigma_1^P(x) \geq \varsigma_2^P(x) \geq \ldots \geq \varsigma_n^P(x) \) are the solutions of (3).
\[ Q(x) := \text{bar} [\mu_{0 \to T|x}] \in S^+, \ x \in K, \] 
where \( g \ast p = g p^\top \in S^+ \ \forall g \in \text{Gl}(n, \mathbb{R}), p \in S^+. \) \hfill (38)

C.2 Lemma The mapping \((38)\) is continuous.

Proof For two continuous \( \gamma_i : [a, b] \to S^+ \) and \( \psi \in C^0(S^+, \mathbb{R}) \) with \( \| \psi \|_{\text{Lip}} \leq 1 \), the following holds for the probability measures \( \mu_i = \frac{1}{b-a} \int_a^b \delta_{\gamma_i(t)} \, dt \) by the definition of “pushing-forward”

\[
\left| \int \psi \, d\mu_1 - \int \psi \, d\mu_2 \right| = \frac{1}{b-a} \left| \int_a^b \{\psi[\gamma_1(t)] - \psi[\gamma_2(t)]\} \, dt \right| \\
\leq \frac{1}{b-a} \int_a^b |\psi[\gamma_1(t)] - \psi[\gamma_2(t)]| \, dt \leq \frac{1}{b-a} \int_a^b d[\gamma_1(t), \gamma_2(t)] \, dt.
\]

So \((26)\) yields that \( W_1(\mu_1, \mu_2) \leq \int_a^b d[\gamma_1(t), \gamma_2(t)] \, dt \to 0 \) as \( \gamma_2(t) \) goes to \( \gamma_1(t) \) uniformly in \( t \in [a, b] \). It remains to note that this uniform convergence does hold for \( \gamma_i(t) : = A^i(t) (x_i)^{-1} \ast I \) as \( x_2 \to x_1 \) and to employ \((2)\) in Prop. A.6. \hfill \( \square \)

C.3 Lemma For any continuous curve \( \gamma : [a, b] \to S^+ \) and map \( J : S^+ \to S^+ \), pushing the measure \( \mu := \int_a^b \delta_{\gamma(t)} \, dt \) forward by \( J \) results in the measure \( \nu \) that is equal to \( \int_a^b \delta_{J[\gamma(t)]} \, dt \). In particular, pushing \( \mu_{0 \to T|x} \) forward by the map \( P \in S^+ \mapsto A^i(t) (x)^{-1} \ast P \) results in \( \mu_{t \to t+T|x} \).

Proof For any \( \psi \in C^0(S^+, \mathbb{R}) \), the change-of-variable formula for the pushforward measure yields

\[
\int_{S^+} \psi \, d\nu = \int_{S^+} \psi \circ J \, d\mu = \int_a^b \psi[J[\gamma(t)]] \, dt.
\]

It remains to note that by the same argument, the last expression is the integral of \( \psi \) with respect to \( \int_a^b \delta_{J[\gamma(t)]} \, dt \). Now, for \( \mu_{0 \to T|x} \) and \( P \mapsto A^i(t) (x)^{-1} \ast P \), we obtain the measure \( \frac{1}{T^1} \int_0^T \delta_{A^i(t) (x)^{-1} \ast [A^i(t) (x)^{-1} \ast s]} \, ds \) and it remains to invoke \((30)\) (with \( \Phi(t, x) : = \varphi^i(x) \)) and \((39)\). \hfill \( \square \)

Now we recall that a certain \( T > 0 \) was fixed just after Proposition C.1.

C.4 Lemma For any \( \varepsilon > 0 \) and the vector \( \mathbb{1} \in \mathbb{R}^n \) composed of 1’s, the following relation holds for all small enough \( t > 0 \):

\[
\frac{d}{dt} \{ Q[\varphi^i(x)], A^i(t) (x)^{-1} \ast Q(x) \} \leq 2 \bar{\sigma} \left[ A^i(T|x) \right] + \varepsilon \mathbb{1}. \hfill (40)
\]

Proof Since \( A^i(x)^{-1} \ast I \in S^+ \) is continuous in \( s \), there exists \( \theta \in (0, T) \) such that whenever \( |s - \tau| \leq \theta \) and \( \tau \in [0, T] \), we have

\[
d[\varphi^i(t) (x)^{-1} \ast I, A^i(s) (x)^{-1} \ast I] \leq T \varepsilon /2. \hfill (41)
\]
For $t \in (0, \theta)$ and the map (38), we have
\[
A^{(t)}(x)^{-1} * Q[\varphi'(x)] \overset{\text{(38)}}{=} A^{(t)}(x)^{-1} * \text{bar}[\mu_{0\to T}|\varphi'(x)|]
\]
and
\[
d\left\{ Q[\varphi'(x)], A^{(t)}(x) * Q(x) \right\} \\
\overset{\text{b) in Prop. A.1 and (1) in Prop. A.6}}{=} d\left\{ A^{(t)}(x)^{-1} * Q[\varphi'(x)], Q(x) \right\} \\
\overset{\text{by (i) in Lem. A.5}}{=} d\left( \text{bar}[\mu_{t\to t+T}|x], \text{bar}[\mu_{0\to T}|x] \right).
\]
\[
\mu_{t\to t+T}|x \overset{\text{(38)}}{=} (1 - \frac{t}{T}) \mu_{t\to T}|x + \frac{t}{T} \mu_{T\to T+t}|x;
\]
\[
\mu_{0\to T}|x \overset{\text{(38)}}{=} \frac{t}{T} \mu_{0\to T}|x + \left(1 - \frac{t}{T}\right) \mu_{t\to T}|x.
\]
\[
d\left( \text{bar}[\mu_{t\to t+T}|x], \text{bar}[\mu_{0\to T}|x] \right)
\overset{\text{c) in Prop. A.1}}{\leq} d\left( \text{bar}[\mu_{t\to t+T}|x], \text{bar}[a_{t}\mu + b_{t}\delta_{A(T)}(x-1|x)] \right)
+ d\left( \text{bar}[a_{t}\mu + b_{t}\delta_{A(T)}(x-1|x)], \text{bar}[a_{t}\mu + b_{t}\delta_{T}] \right)
+ d\left( \text{bar}[a_{t}\mu(t) + b_{t}\delta_{T}], \text{bar}[\mu_{0\to T}|x] \right).
\]

The addends to the right of $\leq$ are sequentially denoted by $A_1, A_2, A_3$. Thanks to a) in Prop. A.1 and (27), we have
\[
\|A_1\|_2 \leq W_1 \left( a_{t}\mu + b_{t}\delta_{T}, a_{t}\mu + b_{t}\mu_{0\to T}|x| \right)
\]
\[
\overset{\text{(38) and (42)}}{=} b_{t} \sup_{\|\psi\|_{\text{Lip}} \leq 1} \left( \int_{S^+} \psi \, d\delta_{T} - \frac{1}{t} \int_{S^+} \psi(A(s)(x)^{-1} * I) \, ds \right)
\]
\[
= \frac{1}{T} \sup_{\|\psi\|_{\text{Lip}} \leq 1} \int_{0}^{t} \left[ \psi(I) - \psi(A(s)(x)^{-1} * I) \right] \, ds
\]
\[
\overset{\text{(38)}}{\leq} \frac{1}{T} \int_{0}^{t} d[I, A(s)(x)^{-1} * I] \, ds \overset{\text{(41)}}{\leq} t\varepsilon/2.
\]

By arguing likewise, we establish that $\|A_1\|_2 \leq t\varepsilon/2$. Bringing the pieces together and invoking (A.1) results in
\[
t^{-1}d\left\{ Q[\varphi'(x)], A^{(t)}(x) * Q(x) \right\} \leq \varepsilon + t^{-1}A_2.
\]

By (i) in Lem. A.3, $\mu$ is the $W_1$-limit of a sequence whose terms are measures of the form $\mu' = \frac{1}{s}(\delta_{p_1} + \ldots + \delta_{p_s})$, where $s \in \mathbb{N}$ and $p_1, \ldots, p_s \in S^+$ are individual for
any term. Relation (23) with \( m := s + 1, \omega := (a_t/s, \ldots, a_t/s, \beta_t = t/T) \in \Delta_m \) yields
\[
\bar{d} \left( \operatorname{bar}[a_t \mu + b_t \delta A(t)] \right) \\
= \bar{d} \left( \operatorname{bar}[\omega; p_1, \ldots, p_m, A(t)^{-1} I \right), \operatorname{bar}(\omega; p_1, \ldots, p_m, I)] \\
\leq t/T \cdot \bar{d}(A(t)^{-1} I, I).
\]

Letting \( \mu' \to \mu \) results in
\[
t^{-1} A_2 \leq \bar{d}(A(t)^{-1} I, I)/T
\]

where (a) holds since the involved matrices have the same singular values. The proof is completed by injecting (43) into (42).

**Proof (of Proposition C.1)** The remaining step of this proof largely comes to the computing the limit of l.h.s. in (40) as \( t \to 0^+ \). In this l.h.s., the orbital derivative \( \dot{Q}(x) := \frac{d}{dt} Q(\varphi(t)) \big|_{t=0^+} \) exists due to [5] Lem. 4.5. As for the other argument of \( \bar{d}[\cdot, \cdot] \), we have
\[
\frac{d}{dt} A(t)^{-1} Q(x) \big|_{t=0^+} = \frac{d}{dt} \left( D\varphi(t) Q(x) \right) \left( D\varphi(t) \right)^\top \big|_{t=0^+}
\]

\[
\frac{d}{dt} D\varphi(t) \big|_{t=0^+} = D\varphi(x) \text{ by [13] Thm. 3.1, Ch. V]}
\]

\[
D\varphi(x) Q(x) + Q(x) D\varphi(x)^\top.
\]

Formula (19) is our incentive to use Lemma A.10 and study
\[
\frac{d}{dt} \mathbf{Y} \{ Q(\varphi(t)), A(t)^{-1} * Q(t) \} \big|_{t=0^+}
\]

\[
= D\mathbf{Y} \left( Q(x), Q(x) \right) \left( D\varphi(x) Q(x) + Q(x) D\varphi(x)^\top \right) \text{ Lem. A.10]}
\]

\[
Q(x)^{-1/2} h,
\]

where \( h \in \mathcal{S} \) is the unique solution of the Lyapunov equation
\[
hQ(x)^{1/2} + Q(x)^{1/2} = D\varphi(x) Q(x) + Q(x) D\varphi(x)^\top - \dot{Q}(x).
\]

In the l.h.s. of (40), we replace \( \bar{d} \) by \( 2\bar{\sigma} \) as stated in (19). Then by Corollary A.9 the l.h.s. converges to the following limit as \( t \to 0^+ \):
\[
\frac{1}{\ln(2)} \left\{ \lambda_1 [Q(x)^{-1/2} h + hQ(x)^{-1/2}], \ldots, \lambda_n [Q(x)^{-1/2} h + hQ(x)^{-1/2}] \right\}.
\]

The function \( P(x) := Q(x)^{-1} \in \mathcal{S}^+ \) is continuous by Lemma C.2; the foregoing and [38] Thm. D2 imply that its orbital derivative \( \dot{P}(x) \) exists and \( Q(x) = \)
\[-P(x)^{-1}\dot{P}(x)P(x)^{-1} .\] Multiplying the Lyapunov equation by \(Q(x)^{-1/2}\) from the left and right, we get
\[
Q(x)^{-1/2}h + hQ(x)^{-1/2} = 
P(x)^{1/2}\left\{D\varphi(x)P(x)^{-1} + P(x)^{-1}D\varphi(x)^\top
+P(x)^{-1}\dot{P}(x)P(x)^{-1}\right\}P(x)^{1/2}
= P(x)^{-1/2}\left\{P(x)D\varphi(x) + D\varphi(x)^\top P(x) + \dot{P}(x)\right\}P(x)^{-1/2}.
\]
Due to Remark 3.6, this means that the l.h.s. of (40) converges to twice the l.h.s. of (36) as \(t \to 0^+\). By letting \(t \to 0^+\) and then \(\varepsilon \to 0^+\) in (40), we arrive at (36).

\[\square\]

Proof (of (ii) in Thm. 3.7) By (4), (35) is still valid with some real \(N\), now denoted by \(T\). Hence,
\[
H_{\text{res}}(\varphi, K) \geq \max_{x \in K} \frac{1}{T} \max_{k=0,\ldots,n} \sum_{i=1}^{k} \log \alpha_i(T, x) - \varepsilon
\geq \frac{1}{2 \ln 2} \max_{x \in K} \max_{k=0,\ldots,n} \sum_{i=1}^{k} \log \varsigma^P_i(x) - \varepsilon
= \frac{1}{2 \ln 2} \max_{x \in K} \sum_{i=1}^{n} \max\{0, \varsigma^P_i(x)\} - \varepsilon.
\]

\[\square\]

Acknowledgements

A. Pogromsky acknowledges his partial support by the UCoCoS project which has received funding from the European Unions Horizon 2020 research and innovation programme under the Marie Skodowska-Curie grant agreement No 675080 (Secs. 1, 7, 8). A. Matveev acknowledges his support by the Russian Science Foundation grant 19-19-00403. C. Kawan is supported by the German Research Foundation (DFG) through the grant ZA 873/4-1. A preliminary version of this paper was presented at the 2020 IFAC World Congress [21].

References

[1] B. R. Andrievsky, A. S. Matveev, and A. L. Fradkov. Control and estimation under information constraints: Toward a unified theory of control, computation and communications. *Automation and Remote Control*, 71(4):572–633, 2010.
[2] V.E. Belozyorov. Exponential-algebraic maps and chaos in 3D autonomous quadratic systems. *International J. Bifurcation and Chaos*, 25(4):1550048, 2015.

[3] R. Bhatia. *Positive Definite Matrices*. Princeton University Press, Princeton and Oxford, 2007.

[4] J. Bochi. Ergodic optimization of Birkhoff averages and Lyapunov exponents. In *Proceedings of the International Congress of Mathematicians*, Rio de Janeiro, 2018.

[5] J. Bochi and A. Navas. A geometric path from zero lyapunov exponents to rotation cocycles. *Ergodic Theory & Dynamical Systems*, 35:374–402, 2015.

[6] V. A. Boichenko, G. A. Leonov, and V. Reitman. *Dimension Theory for Ordinary Differential Equations*. Teubner Verlag, Wiesbaden, Germany, 2005.

[7] I. Chavel. *Riemannian Geometry: A Modern Introduction*. Cambridge University Press, NY, 2006.

[8] F. Colonius and C. Kawan. Invariance entropy for control systems. *SIAM Journal on Control and Optimization*, 48(3):1701–1721, 2009.

[9] F. Colonius, C. Kawan, and G. Nair. A note on topological feedback entropy and invariance entropy. *Systems & Control Letters*, 62(5):377–381, 2013.

[10] T. Downarowicz. *Entropy in Dynamical Systems*. Cambridge University Press, NY, 2011.

[11] A. L. Fradkov, B. Andrievsky, and M. S. Ananyevskiy. Passification based synchronization of nonlinear systems under communication constraints and bounded disturbances. *Automatica*, 55:287–293, 2015.

[12] S. Hafstein and C. Kawan. Numerical approximation of the data-rate limit for state estimation under communication constraints. *J. Math. Anal. Appl.*, 473(2):1280–1304, 2019.

[13] R. Hagihara and G. N. Nair. Two extensions of topological feedback entropy. *Math. of Cont., Signals, and Systems*, 25(4):473–490, 2013.

[14] P. Hartman. *Ordinary Differential Equations*. Birkhäuser, Boston, second edition, 1982.

[15] W. P. M. H. Heemels, A. R. Teel, N. Van De Wouw, and D. Nešić. Networked control systems with communication constraints: Tradeoffs between transmission intervals, delays and performance. *IEEE Trans. Aut. Control*, 55(8):1781–1796, 2010.
[16] J. P. Hespanha, P. Naghshtabrizi, and X. Yonggang. A survey of recent results in networked control systems. *Proceedings of IEEE*, 95(1):138–162, 2007.

[17] J.-B. Hiriart-Urruty and D. Ye. Sensitivity analysis of all eigenvalues of a symmetric matrix. *Numerische Mathematik*, 70:45–72, 1995.

[18] J. Holbrook. No dice: a deterministic approach to the Cartan centroid. *J. Ramanujan Math. Soc.*, 27:509–521, 2012.

[19] R. Horn and C. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, UK, 2013.

[20] C. Kawan. On the relation between topological entropy and restoration entropy. *Entropy*, 21(1):7, 2019.

[21] C. Kawan, A. Matveev, and A. Pogromsky. Data rate limits for the remote state estimation problem. In *Proceedings of the 2020 IFAC World Congress*, Berlin, 2020.

[22] C. Kawan and S. Yüksel. Metric and topological entropy bounds on state estimation for stochastic non-linear systems. In *2017 IEEE International Symposium on Information Theory (ISIT)*, pages 1455–1459. IEEE, 2017.

[23] C. Kawan and S. Yüksel. On optimal coding of non-linear dynamical systems. *IEEE Trans. Inform. Theory*, 64(10):6816–6829, 2018.

[24] H. K. Khalil. *Nonlinear Systems*. Prentice-Hall, Upper Saddle River, NJ, third edition, 2002.

[25] D. Liberzon and S. Mitra. Entropy and minimal bit rates for state estimation and model detection. *IEEE Trans. Automat. Control*, 63(10):3330–3344, 2018.

[26] Y. Lim and M. Palfia. Weighted deterministic walks for the least squares mean on Hadamard spaces. *Bull. London Math. Soc.*, 46:561–570, 2014.

[27] A. S. Matveev and A. Y. Pogromsky. Observation of nonlinear systems via finite capacity channels: constructive data rate limits. *Automatica*, 70:217–229, 2016.

[28] A. S. Matveev and A. Y. Pogromsky. Observation of nonlinear systems via finite capacity channels, part II: Restoration entropy and its estimates. *Automatica*, 103:189–199, 2019.

[29] A. S. Matveev and A. V. Savkin. *Estimation and Control over Communication Networks*. Birkhäuser, Boston, 2009.

[30] P. Del Moral and A. Niclas. A Taylor expansion of the square root matrix functional. *Journal of Mathematical Analysis and Applications*, 465(1):259–266, 2018.
[31] I. Morris. Mather sets for sequences of matrices and applications to the study of joint spectral radii. *Proc. Lond. Math. Soc.*, 107(1):121–150, 2013.

[32] G. N. Nair, F. Fagnani, S. Zampieri, and R. J. Evans. Feedback control under data rate constraints: an overview. *Proceedings of the IEEE*, 95(1):108–137, 2007.

[33] A. Pogromsky and A. Matveev. Data rate limitations for observability of nonlinear systems. *IFAC-PapersOnLine*, 49(14):119–124, 2016.

[34] A. Y. Pogromsky and A. S. Matveev. Estimation of topological entropy via the direct Lyapunov method. *Nonlinearity*, 24:1937–1959, 2011.

[35] A. Y. Pogromsky and A. S. Matveev. Stability analysis via averaging functions. *IEEE Transactions on Automatic Control*, 61(4):1081–1086, 2016.

[36] A. V. Savkin. Analysis and synthesis of networked control systems: topological entropy, observability, robustness, and optimal control. *Automatica*, 42(1):51–62, 2006.

[37] K.-Th. Sturm. Probability measures on metric spaces of nonpositive curvature. In P. Auscher, T. Coulhon, and A. Grigor’yan, editors, *Heat Kernels and Analysis on Manifolds, Graphs, and Metric Spaces (Contemporary Mathematics, 338)*, pages 357–390. American Mathematical Society, Providence, RI, 2003.

[38] A. van den Bos. *Parameter Estimation for Scientists and Engineers*. John Wiley & Sons, Upper Saddle River, NJ, third edition, 2007.

[39] C. Villani. *Optimal transport: old and new*. Springer, Berlin, 2009.

[40] W. S. Wong and R. G. Brockett. Systems with finite communication bandwidth constraints - Part I: State estimation problems. *IEEE Trans. Automat. Control*, 42(9):1294–1299, 1997.

[41] W. H. Young. *The Fundamental Theorems of the Differential Calculus*. Cambridge University Press, Cambridge, UK, 1910.

[42] S. Yüksel and T. Basar. *Stochastic Networked Control Systems: Stabilization and Optimization under Information Constraints*. Birkhäuser, Boston, 2013.