Abstract

We consider the (coupled) Davey-Stewartson (DS) system and its Bäcklund transformations (BT). Relations among the DS system, the double Kadomtsev-Petviashvili (KP) system and the Ablowitz-Ladik hierarchy (ALH) are established. The DS system and the double KP system are equivalent. The ALH is the BT of the DS system in a certain reduction. From the BT of coupled DS system we can obtain new coupled derivative nonlinear Schrödinger equations.

Key Words: Davey-Stewartson system, KP system, Ablowitz-Ladik hierarchy, coupled DNLS equations
1 Introduction

The Davey-Stewartson (DS) equation is one of a few integrable equations in multi-

dimensions which have significances, and has received considerable attention during the

last decade.\textsuperscript{[1]} The DS equation has a new type of solution which is called dromion.\textsuperscript{[2], [3]} The dromion has the remarkable property that it decays exponentially in two spatial

dimensions. It is well known that the DS equation is the simplest nontrivial equation in

the two-component Kadomtsev-Petviashvili (KP) equation.\textsuperscript{[1]}

Recently a new infinite set of commuting “ghost” symmetries was proposed for the

KP-type integrable hierarchy by Aratyn, Nissimov and Paceva.\textsuperscript{[5]} These symmetries allow

for a Lax representation in which the hierarchy is realized as standard isospectral flows. This

gives rise to a new double KP hierarchy embedding ghost and original KP type Lax

hierarchies connected to each other by a duality mapping.

The universality of the Ablowitz-Ladik hierarchy (ALH) was pointed out by V.E.Vekslerchik.\textsuperscript{[6]} The ALH “contains” the 2D Toda lattice, DS equation and KP equation.\textsuperscript{[1], [8]} (See [9].) In\textsuperscript{[7]} the several solutions were obtained using the fact that the solutions of the ALH satisfy DS equation.

In one component case there are two derivative nonlinear Schrödinger (DNLS) equa-
tions. One is Kaup-Newell (KN) equation and the other is Chen-Lee-Liu (CLL) equation.\textsuperscript{[10], [11]} It is well known that these two equations are gauge equivalent. In the previous paper we discuss the coupled DNLS equations which are the coupled version of KN equation.\textsuperscript{[12]}

In this paper we consider the (coupled) DS system and its Bäcklund transformations
(BT). Relations among the DS system, the double KP system and the ALH are estab-
lished. The DS system and the double KP system are equivalent. The ALH is the BT of
the DS system in the case $t_k = t_k^*$ and $u_k = v_k^*$ where the asterisk is conjugation. From
the BT of the coupled DS system we can obtain new coupled DNLS equations which are
the coupled type of CLL equation. This suggests that in the multi-component case there
are two types of coupled DNLS equations which are not gauge equivalent.

The paper is organized as follows. In section 2 we obtain the Lax pair of the coupled
DS equations. This is a new integrable system. In section 3 we construct the BT of the
coupled DS equations. In section 4 we establish a relation between the BT of the coupled
DS equation and the ALH. In section 5 we consider a relation between the double KP
system and the DS system. In section 6 we get new coupled DNLS equations using the BT
which is obtained in the section 2. The last section is devoted to the concluding remarks.
2 Coupled Davey-Stewartson system

We consider the Lax representation of spatially two-dimensional systems. We denote the time by \( T \) and the space coordinates by \( X \) and \( Y \). As the Lax representation let

\[
[L, \frac{\partial}{\partial T} - A] = 0, \tag{2.1}
\]

where

\[
L = a \frac{\partial}{\partial Y} + \begin{pmatrix} l + 1 & 0 & 0 \\ 0 & l & 0 \\ 0 & 0 & l \end{pmatrix} \frac{\partial}{\partial X} + \begin{pmatrix} 0 & \pm v^{(1)} & \pm v^{(2)} \\ u^{(1)} & 0 & 0 \\ u^{(2)} & 0 & 0 \end{pmatrix}, \tag{2.2}
\]

and

\[
A = \begin{pmatrix} a + 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \frac{\partial^2}{\partial X^2} + \begin{pmatrix} 0 & \pm v^{(1)} & \pm v^{(2)} \\ u^{(1)} & 0 & 0 \\ u^{(2)} & 0 & 0 \end{pmatrix} \frac{\partial}{\partial X} + \begin{pmatrix} r_{11} & \varphi_1 & \varphi_2 \\ \varphi_1 & r_{22} & r_{32} \\ \varphi_2 & r_{23} & r_{33} \end{pmatrix}. \tag{2.3}
\]

Here

\[
\varphi_i = -\alpha \frac{\partial u^{(i)}}{\partial Y} + (2a - l + 1) \frac{\partial u^{(i)}}{\partial X},
\]

\[
\tilde{\varphi}_i = \pm \left[ \alpha \frac{\partial v^{(i)}}{\partial Y} + (l - 2a) \frac{\partial v^{(i)}}{\partial X} \right], \quad i = 1, 2. \tag{2.4}
\]

To simplify the expressions, we introduce

\[
\tilde{D}_1 = \alpha \frac{\partial}{\partial Y} + l \frac{\partial}{\partial X}, \quad \tilde{D}_2 = \alpha \frac{\partial}{\partial Y} + (l + 1) \frac{\partial}{\partial X},
\]

\[
\tilde{D}_3 = \alpha \frac{\partial}{\partial Y} + (l - 2a) \frac{\partial}{\partial X}, \quad \tilde{D}_4 = \alpha \frac{\partial}{\partial Y} + (l - 2a - 1) \frac{\partial}{\partial X},
\]

\[
2D_1 = \tilde{D}_1 \tilde{D}_4 + \tilde{D}_2 \tilde{D}_3, \quad D_2 = \tilde{D}_1 \tilde{D}_2, \quad D_3 = \tilde{D}_1 \tilde{D}_4. \tag{2.5}
\]

And the conditions (2.1) with (2.2)-(2.3) are reduced to a system of equations

\[
\frac{\partial u^{(1)}}{\partial T} = D_1 u^{(1)} + p^{(1)} u^{(2)} - r_{23} u^{(2)},
\]

\[
\frac{\partial u^{(2)}}{\partial T} = D_1 \varphi_1 + p^{(2)} u^{(2)} - r_{32} u^{(1)}, \tag{2.6}
\]

where \( p^{(1)} = r_{11} - r_{22}, \ p^{(2)} = r_{11} - r_{33} \) and

\[
D_2 p^{(1)} = \mp (2D_1 u^{(1)} v^{(1)} + D_3 u^{(1)} v^{(2)}),
\]

\[
D_2 p^{(2)} = \mp (2D_1 u^{(1)} v^{(2)} + D_3 u^{(1)} v^{(1)}),
\]

\[
\tilde{D}_1 r_{32} = \pm \tilde{D}_3 u^{(2)} v^{(1)}, \quad \tilde{D}_1 r_{23} = \pm \tilde{D}_3 u^{(1)} v^{(2)}. \tag{2.7}
\]
The particular cases come from the degenerations of the quadratic form of $D_1$. Let $a = 0$ then the system reduces to

$$\frac{\partial}{\partial t} u^{(1)} = \frac{\partial^2}{\partial x^2} u^{(1)} + p^{(1)} \phi_1 - r_{23} u^{(2)}, \quad \frac{\partial}{\partial t} u^{(2)} = \frac{\partial^2}{\partial x^2} u^{(2)} + p^{(2)} u^{(2)} - r_{32} u^{(1)},$$

$$\frac{\partial}{\partial y} (p^{(1)} \pm u^{(2)} v^{(2)}) = \mp 2 \frac{\partial}{\partial x} (u^{(1)} v^{(1)}), \quad \frac{\partial}{\partial y} (p^{(2)} \pm u^{(1)} v^{(1)}) = \mp 2 \frac{\partial}{\partial x} (u^{(2)} v^{(2)}),$$

$$\frac{\partial}{\partial y} r_{32} = \pm \frac{\partial}{\partial y} (u^{(2)} v^{(1)}), \quad \frac{\partial}{\partial y} r_{23} = \pm \frac{\partial}{\partial y} (u^{(1)} v^{(2)}),$$

(2.8)

where

$$\frac{\partial}{\partial y} = \alpha \frac{\partial}{\partial Y} + (l + 1) \frac{\partial}{\partial X}, \quad \frac{\partial}{\partial x} = \alpha \frac{\partial}{\partial Y} + l \frac{\partial}{\partial X},$$

(2.9)

and we rewrite $T = t$.

Let $a = -1$ then the system reduces to

$$\frac{\partial}{\partial \tau} u^{(1)} = \frac{\partial^2}{\partial y^2} u^{(1)} + q^{(1)} u^{(1)} - \tilde{r}_{23} u^{(2)}, \quad \frac{\partial}{\partial \tau} u^{(2)} = \frac{\partial^2}{\partial y^2} u^{(2)} + q^{(2)} u^{(2)} - \tilde{r}_{32} u^{(1)},$$

$$\frac{\partial}{\partial x} (q^{(1)} \pm u^{(2)} v^{(2)}) = \mp 2 \frac{\partial}{\partial y} (u^{(1)} v^{(1)}), \quad \frac{\partial}{\partial x} (q^{(2)} \pm u^{(1)} v^{(1)}) = \mp 2 \frac{\partial}{\partial y} (u^{(2)} v^{(2)}),$$

$$\frac{\partial}{\partial y} \tilde{r}_{32} = \pm (2 \frac{\partial}{\partial x} - \frac{\partial}{\partial y}) (u^{(2)} v^{(1)}), \quad \frac{\partial}{\partial y} \tilde{r}_{23} = \pm (2 \frac{\partial}{\partial x} - \frac{\partial}{\partial y}) (u^{(1)} v^{(2)}),$$

(2.10)

where $\tau = T$, $q_1 = p_1$ and $q_1 = p_1$.

In the one component case ($u^{(2)} = v^{(2)} = 0$) (2.8) and (2.10) become

$$\frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u + 2pu, \quad \frac{\partial}{\partial y} p = \mp \frac{\partial}{\partial x} (uv),$$

(2.11)

and

$$\frac{\partial}{\partial \tau} u = \frac{\partial^2}{\partial y^2} u + 2qu, \quad \frac{\partial}{\partial x} q = \mp \frac{\partial}{\partial y} (uv),$$

(2.12)

where $2p = p^{(1)}$ and $2q = q^{(1)}$. (2.11) and (2.12) are compatible and any linear combination of them is integrable. We call this system with higher flows the DS system.[1]

In the (1+1) dimensional case ($y = x$) (2.8) and (2.10) become the coupled nonlinear Schrödinger (NLS) equation[13]

$$\partial_t u^{(i)} = \partial_x^2 u^{(i)} \mp 2(u^{(1)} v^{(1)} + u^{(2)} v^{(2)}) u^{(i)}, \quad \partial_t v^{(i)} = \partial_x^2 v^{(i)} \mp 2(u^{(1)} v^{(1)} + u^{(2)} v^{(2)}) v^{(i)},$$

(2.13)

where $i = 1, 2$.

### 3 Bäcklund Transformations

We shall consider a sequence $u^{(i)}_n$ and $v^{(i)}_n$ generated by the auto-Bäcklund transformation (BT). In the operator form BT is defined by [14]

$$W_n L_n = L_{n+1} W_n.$$  

(3.14)
An implicit BT of (2.8) and (2.10) corresponds to

\[
W_n = \begin{pmatrix}
1 & -v_n^{(1)} & -v_n^{(2)} \\
n_n^{(1)} & \partial_y - \partial_x - w_n^{(11)} & -w_n^{(11)} \\
n_n^{(2)} & -w_n^{(12)} & \partial_y - \partial_x - w_n^{(22)}
\end{pmatrix},
\]

Note that we only consider the bright case, that is, the sign is “-” in (2.2) and (2.3).

From (2.2), (3.14) and (3.15) we can obtain

\[
-\partial_x u_n^{(1)} = u_n^{(1)} + w_n^{(11)} u_n^{(1)} + w_n^{(21)} u_n^{(2)} ,
-\partial_x u_n^{(2)} = u_n^{(2)} + w_n^{(22)} u_n^{(2)} + w_n^{(12)} u_n^{(1)} ,
\partial_x v_n^{(1)} = v_n^{(1)} + w_{n-1}^{(11)} v_n^{(1)} + w_{n-1}^{(21)} v_n^{(2)} ,
\partial_x v_n^{(2)} = v_n^{(2)} + w_{n-1}^{(22)} v_n^{(2)} + w_{n-1}^{(12)} v_n^{(1)} ,
\]

and

\[
\partial_y w_n^{(11)} = -w_{n+1}^{(11)} v_n^{(1)} + u_n^{(1)} v_n^{(1)} ,
\partial_y w_n^{(22)} = -w_{n+1}^{(22)} v_n^{(2)} + u_n^{(2)} v_n^{(2)} ,
\partial_y w_n^{(21)} = -w_{n+1}^{(21)} v_n^{(1)} + u_n^{(2)} v_n^{(1)} ,
\partial_y w_n^{(11)} = -w_{n+1}^{(11)} v_n^{(2)} + u_n^{(1)} v_n^{(2)} ,
\]

Here we introduce new nonlocal dynamical variables \(w_n^{(ij)}\).

### 4 Ablowitz-Ladik Hierarchy

We consider the one component case of (3.16) and (3.17)

\[
-\partial_x u_n = u_{n+1} + u_n w_n ,
\partial_x v_n = v_{n-1} + v_n w_{n-1} ,
\partial_y w_n = \partial_x (u_n v_{n+1}) = u_n v_n - u_{n+1} v_{n+1} .
\]

Here we set

\[
u_n = v_n^* , \quad x = y^* .
\]

The asterisk means conjugation:

\[
(f^*)^* = f ,
\]

where \(f\) is an arbitrary function. From this constraint we can obtain the following equations
\[
\begin{align*}
\partial_y u_n &= u_{n-1} + u_n w^*_n, \\
-\partial_y v_n &= v_{n+1} + v_n w^*_n, \\
\partial_x w^*_n &= \partial_y (u_{n+1} v_n) = u_n v_n - u_{n+1} v_{n+1}.
\end{align*}
\] (4.4)

We can obtain also (4.4) by an exchange of \(u_n \leftrightarrow v_n\) in (4.1).

From the last equations of (4.1) and (4.4), we have
\[
w_n = u_{n+1} v_n + C, \quad w^*_n = u_n v_{n+1} + C^*,
\] (4.5)

where \(C\) and \(C^*\) are constants. Hereafter for simplicity we set \(C = C^* = 0\). (4.1)-(4.5) are nothing but the Ablowitz-Ladik hierarchy (ALH). Thus, the ALH can be viewed as sequences of the BT of the DS system.

In fact the second flow of the ALH are
\[
\begin{align*}
\partial_t u_n &= p_n p_{n+1} u_{n+2} + p_n u_{n+1} v_{n-1} u_n + p_n u^2_{n+1} v_n, \\
\partial_t v_n &= -p_n p_{n-1} v_{n-2} - p_n v^2_{n-1} u_n - p_n v_{n-1} v_n u_{n+1},
\end{align*}
\] (4.6) (4.7)

where \(p_n = 1 + u_n v_n\).

One can obtain straightforwardly [7]
\[
\partial_t u_n = \partial^2_x u_n + 2A_n u_n, \quad A_n = p_n u_{n+1} v_{n-1}.
\] (4.8)

On the other hand we can obtain
\[
\partial_y A_n = p_n (-v_n u_{n+1} + v_{n-1} u_n) = \partial_x (u_n v_n).
\] (4.9)

(4.8) and (4.9) is nothing but (2.11). In the same way we can get (2.12) using the other second flow. This proves that the ALH is the BT of the DS system with the constraints (4.2).

5 Double KP System

We first summarize some background information on the Kadomtsev-Petviashvili (KP) hierarchy and ghost symmetries. We use the Sato formalism of pseudo-differential operator calculus to describe KP type integrable hierarchies of integrable nonlinear evolution equations about the KP times \((t) = (t_1 \equiv x, t_2, \ldots)\):
\[
\mathcal{L} = D + \sum_{i=1}^{\infty} p_i \partial^{-i}, \quad \frac{\partial}{\partial t^l} \mathcal{L} = [\mathcal{L}^l, \mathcal{L}], \quad l = 1, 2, \ldots,
\] (5.1)
where $D$ stands for the differential operator $\partial/\partial x$, and the subscripts $+$ and $-$ of any pseudo-differential operator $A = \sum_j a_j D^j$ denote its purely differential part ($A_+ = \sum_{j\geq 0} a_j D^j$) and its purely pseudo-differential part ($A_- = \sum_{j<0} a_j D^j$) respectively.

In the present approach eigenfunction (EF) $\Phi(t)$ and adjoint eigenfunction (adj EF) $\Psi(t)$ which satisfy the following equations play crucial role,

$$\frac{\partial}{\partial t_k} \Phi = (L^k_+)^{\pm}(\Phi), \quad \frac{\partial}{\partial t_k} \Psi = (L'^k_+)^{\pm}(\Psi), \quad (5.2)$$

where $L^*$ is the adjoint operator of $L$. Note that the above eigenfunctions $\Phi$ and $\Psi$ do not need to be the Baker-Akhiezer eigenfunctions.

A ghost symmetry is defined through an action of a vector field $\hat{\alpha}$ on the KP Lax operator

$$\hat{\alpha} L = [M_\alpha, L], \quad M_\alpha = \sum_{a \in \{\alpha\}} \Phi_a D^{-1} \Psi_a, \quad (5.3)$$

where $(\Phi_a, \Psi_a)_{a \in \{\alpha\}}$ are the same set of functions indexed by $\{\alpha\}$. Commutativity of $\hat{\alpha}$ with $\partial_t$ implies that $(\Phi_a, \Psi_a)_{a \in \{\alpha\}}$ is a set of pairs of (adj-)EFs for $L$.

We now proceed to give an explicit construction of the ghost KP hierarchy. Consider an infinite system of independent (adj-)EFs $(\Phi_j, \Psi_j)_{j=1}^\infty$ of $L$ and define the following set of the ghost symmetry flows

$$\frac{\partial}{\partial t_s} L = [M_s, L], \quad M_s = \sum_{j=1}^s \Phi_{s-j+1} \partial^{-1} \Psi_j, \quad (5.4)$$

where $s = 1, 2, \cdots$. Note that the ghost symmetry flows $\frac{\partial}{\partial t_s}$ do commute.

To define the two Lax operators we introduce the Wronskian type determinant

$$W_k = W_k[\Phi_1, \cdots, \Phi_k] = \det ||\partial^{\alpha-1} \Phi_\beta||, \quad W_k = W_k[\Psi_1, \cdots, \Psi_k] = \det ||\partial^{\alpha-1} \Psi_\beta||, \quad (5.5)$$

where $\alpha, \beta = 1, 2, \cdots, k$.

We define the initial and the ghost Lax operators,

$$L = \partial + \sum_{k=1}^\infty a_k (D - \partial \ln \frac{W_{k+1}}{W_k})^{-1} \cdots (D - \partial \ln \frac{W_2}{W_1})^{-1}, \quad (5.6)$$

and

$$\bar{L} = \bar{\partial} + \sum_{k=1}^\infty b_k (\bar{D} + \bar{\partial} \ln \frac{W_{k+1}}{W_k})^{-1} \cdots (\bar{D} + \bar{\partial} \ln \frac{W_2}{W_1})^{-1}, \quad (5.7)$$

where $\bar{\partial} = \partial_{\bar{t}_1}$ and hereafter we define $y = -\bar{t}_1$.

Let us introduce the non-standard orbit of successive Darboux-Bäcklund (DB) transformations for the initial KP system

$$L(n+1) = T_1(n) L(n) T_1^{-1}(n), \quad T_1(n) = \Phi_1 D \Phi_1^{-1} \equiv \Phi_1^{(n)} D(\Phi_1^{(n)})^{-1},$$

$$L(n-1) = \bar{T}_1(n) L(n) \bar{T}_1^{-1}(n), \quad \bar{T}_1(n) = \Psi_1 D \Psi_1^{-1} \equiv \Psi_1^{(n)} D(\Psi_1^{(n)})^{-1}. \quad (5.8)$$
The DB transformations for the ghost Lax operator is

$$\bar{\mathcal{L}}(n + 1) = \left( \frac{1}{\Phi_1^{(n+1)}} \bar{D}^{-1} \Phi_1^{(n+1)} \right) \bar{\mathcal{L}}(n) \left( \frac{1}{\Phi_1^{(n+1)}} \bar{D} \Phi_1^{(n+1)} \right).$$  (5.9)

We can construct an infinite set of (adj-)EFs \((\bar{\Phi}_j, \bar{\Psi}_j)_{j=1}^{\infty}\) for the ghost Lax operator \(\bar{\mathcal{L}}\). We can obtain the first relations

$$\bar{\Phi}_1^{(n)} = \Psi_1^{(n+1)}, \quad \bar{\Psi}_1^{(n)} = \Psi_1^{(n+1)}.$$  (5.10)

Both Lax operators \(\mathcal{L}\) and \(\bar{\mathcal{L}}\) define “double KP system”,

$$\frac{\partial}{\partial t_r} \mathcal{L} = [(\mathcal{L}^r)_+, \mathcal{L}], \quad \frac{\partial}{\partial t_s} \mathcal{L} = [\mathcal{M}_s, \mathcal{L}],$$
$$\frac{\partial}{\partial t_r} \bar{\mathcal{L}} = [(\bar{\mathcal{L}}^r)_+, \bar{\mathcal{L}}], \quad \frac{\partial}{\partial t_s} \bar{\mathcal{L}} = [ar{\mathcal{M}}_s, \bar{\mathcal{L}}],$$  (5.11)

where \(\bar{\mathcal{M}}_r\) is defined in terms of the \(\text{(adj)EFs : } \bar{\mathcal{M}}_r = \sum_{i=1}^{\infty} \Phi_{r-i+1} \bar{D}^{-1} \bar{\Psi}_i \).

For \(k = 2\) \((5.2)\) gives

$$\frac{\partial \Phi_1}{\partial t_2} = \frac{\partial^2 \Phi_1}{\partial x^2} + 2p_1 \Phi_1, \quad \frac{\partial \Psi_1}{\partial t_2} = - \frac{\partial^2 \Psi_1}{\partial x^2} - 2p_1 \Phi_1,$$  (5.12)

while for \(k = 3\) we obtain

$$\frac{\partial \Phi_1}{\partial t_3} = \frac{\partial^3 \Phi_1}{\partial x^3} + 3p_1 \frac{\partial \Phi_1}{\partial x} + 3p_2 \Phi_1 + 3 \frac{\partial p_1}{\partial x} \Phi_1,$$
$$\frac{\partial \Psi_1}{\partial t_3} = \frac{\partial^3 \Psi_1}{\partial x^3} + 3p_1 \frac{\partial \Psi_1}{\partial x} - 3p_2 \Psi_1.$$  (5.13)

On the other hand \((5.4)\) gives

$$\frac{\partial p_1}{\partial y} = \frac{\partial}{\partial x} (\Phi_1 \Psi_1),$$  (5.14)

If we set \(t_2 = t\), \((5.12)\) and \((5.14)\) are nothing but the tally of the DS system \((2.11)\).

From \((5.4)\) we can obtain

$$\frac{\partial p_2}{\partial y} = - \frac{\partial}{\partial x} (\Phi_1 \frac{\partial \Psi_1}{\partial x}).$$  (5.15)

Using \((5.13)\), \((5.14)\) gives

$$\frac{\partial \Phi_1}{\partial t_3} = \frac{\partial^3 \Phi_1}{\partial x^3} + 3(\frac{\partial}{\partial x} \int^y \Phi_1 \Psi_1 dy) \frac{\partial \Phi_1}{\partial x} + 3[I \int^y (\frac{\partial \Phi_1}{\partial x} \frac{\partial \Psi_1}{\partial x} + \frac{\partial^2 \Phi_1}{\partial x^2} \Psi_1) dy] \Phi_1,$$
$$\frac{\partial \Psi_1}{\partial t_3} = \frac{\partial^3 \Psi_1}{\partial x^3} + 3(\frac{\partial}{\partial x} \int^y \Phi_1 \Psi_1 dy) \frac{\partial \Psi_1}{\partial x} + 3[I \int^y (\frac{\partial \Phi_1}{\partial x} \frac{\partial \Psi_1}{\partial x} + \Phi_1 \frac{\partial^2 \Psi_1}{\partial x^2}) dy] \Psi_1,$$  (5.16)

This is the tally of the higher DS system. If we set \(x = y\) we can obtain the complex modified Korteweg-de Vries (cmKdV) equation which is the first higher order equations in the nonlinear Schrödinger hierarchy.
According to (5.11) and (5.3) there exists a duality mapping between the double KP systems of (5.11) defined by \( L \) and \( \bar{L} \), respectively under the change \((t) \leftrightarrow (\bar{t}), \Phi \leftrightarrow \bar{\Phi} \) and \( \Psi \leftrightarrow \bar{\Psi} \). Then for example we can get the equations by the duality mapping of (5.12) and (5.14)

\[
\frac{\partial \bar{\Phi}}{\partial \bar{t}} = \frac{\partial^2 \Phi}{\partial y^2} + 2q_1 \Phi, \quad \frac{\partial \bar{\Psi}}{\partial \bar{t}} = -\frac{\partial^2 \Psi}{\partial y^2} - 2q_1 \Psi,
\]

and

\[
\frac{\partial q_1}{\partial x} = \frac{\partial}{\partial y} (\Phi \bar{\Phi}).
\]

Using (5.10) for any site \( n \) on the DB-orbit we can obtain

\[
\frac{\partial \Phi}{\partial \bar{t}} = -\frac{\partial^2 \Phi}{\partial y^2} - 2q_1 \Phi, \quad \frac{\partial \Psi}{\partial \bar{t}} = \frac{\partial^2 \Psi}{\partial y^2} + 2q_1 \Psi,
\]

and

\[
\frac{\partial q_1}{\partial x} = \frac{\partial}{\partial y} (\Phi \Psi),
\]

where we rewrite \( \bar{p}_1 = q_1 \). If we set \( \tau = -\bar{t}_2 \), these equations are nothing but the tally of the DS system (2.12). In the same way we can obtain the tally of the higher DS system. If we set \( y = \bar{t}_1 \), we can obtain the other type (the sign is “\(-\)” in (2.11) and (2.12)) system. Note that these equations have dark type solutions.

From these results we can obtain one hierarchy from the flows with respect to \((t)\) and \(y\) of double KP system. On the other hand we can get the other hierarchy from the flows with respect to \((-\bar{t})\) and \(x\). These two hierarchies are compatible and any linear combination of them is integrable. The couple of these hierarchies are the DS system. Note that dependent variables of DS system is (adj-)EF of double KP system.

6 Coupled Derivative Nonlinear Schrödinger equations

If we set \( x = y \) in (3.16) and (3.17), we can obtain

\[
\begin{align*}
-\partial_x u_n^{(i)} &= u_{n+1}^{(i)} + (u_n^{(1)} v_{n+1}^{(1)} + u_n^{(2)} v_{n+1}^{(2)}) v_n^{(i)}, \\
\partial_x v_n^{(i)} &= v_{n-1}^{(i)} + (u_{n-1}^{(1)} v_{n}^{(1)} + u_{n-1}^{(2)} v_{n}^{(2)}) v_n^{(i)},
\end{align*}
\]

where \( i = 1, 2 \). (6.21) are the BT of the coupled NLS equations (2.13).

Let us consider the following transformation,

\[
U_n^{(i)} = u_n^{(i)}, \quad V_n^{(i)} = v_{n+1}^{(i)}, \quad i = 1, 2.
\]
This transformation is obviously invertible. In the new variables chain equations (6.21) take the form

\[- \partial_x U_n^{(i)} = U_{n+1}^{(i)} + (U_{n}^{(1)} V_n^{(1)} + U_{n}^{(2)} V_n^{(2)}) U_n^{(i)}; \]
\[\partial_x V_n^{(i)} = V_{n-1}^{(i)} + (U_{n}^{(1)} V_n^{(1)} + U_{n}^{(2)} V_n^{(2)}) V_n^{(i)},\]  

(6.23)

where \(i = 1, 2\).

It follows from the above system of equations that variables \(u_n^{(i)}, v_n^{(i)}\), can be expressed in terms of \(U_n^{(i)}, V_n^{(i)}\),

\[u_n^{(i)} = U_n^{(i)}, \quad v_n^{(i)} = V_n^{(i)} = \partial_x V_n^{(i)} - (U_n^{(1)} V_n^{(1)} + U_n^{(2)} V_n^{(2)}) V_n^{(i)}, \quad i = 1, 2.\]  

(6.24)

This defines a Miura transformation (MT)

\[u^{(i)} = U^{(i)}, \quad v^{(i)} = \partial_x V^{(i)} - (U^{(1)} V^{(1)} + U^{(2)} V^{(2)}) V^{(i)}, \quad i = 1, 2.\]  

(6.25)

To find the transformed equations let us rewrite the system of (2.13) in the new variables \(U_n^{(i)}, V_n^{(i)}\). Hereafter we only consider the bright case, that is, the sign is “+” in (2.13). It follows from (6.22) that

\[\partial_t U_n^{(i)} = \partial_x^2 U_n^{(i)} + 2(U_n^{(1)} V_{n-1}^{(1)} + U_n^{(2)} V_{n-1}^{(2)}) U_n^{(i)}; \]
\[-\partial_t V_n^{(i)} = \partial_x^2 V_n^{(i)} + 2(U_{n+1}^{(1)} V_n^{(1)} + U_{n+1}^{(2)} V_n^{(2)}) V_n^{(i)}, \quad i = 1, 2.\]  

(6.26)

The variables \(V_n^{(i)}\) is already expressed in terms of \(U_n^{(i)}, V_n^{(i)}\) (6.24). It follows from (6.23) that

\[U_{n+1}^{(i)} = -\partial_x U_n^{(i)} - (U_n^{(1)} V_n^{(1)} + U_n^{(2)} V_n^{(2)}) U_n^{(i)}.\]  

(6.27)

Thus the transformed equations by the MT (6.23) of the system (2.13) is of the form

\[\partial_t U^{(i)} = \partial_x^2 U^{(i)} + 2(U^{(1)} \partial_x V^{(1)} + U^{(2)} \partial_x V^{(2)}) U^{(i)} - 2(U^{(1)} V^{(1)} + U^{(1)} V^{(1)})^2 U^{(i)}, \]
\[-\partial_t V^{(i)} = \partial_x^2 V^{(i)} - 2(V^{(1)} \partial_x U^{(1)} + V^{(2)} \partial_x U^{(2)}) U^{(i)} - 2(U^{(1)} V^{(1)} + U^{(1)} V^{(1)})^2 V^{(i)},\]  

(6.28)

where \(i = 1, 2\). (6.28) are coupled derivative nonlinear Schrödinger (DNLS) equations. There are two MT between (2.13) and (6.28). To construct the other MT we expressed variables \(u^{(i)} = u_{n+1}^{(i)}, v^{(i)} = v_{n+1}^{(i)}\) in terms of \(U^{(i)} = U_n^{(i)}, V^{(i)} = V_n^{(i)}\). It follows from (6.22) \(u_{n+1}^{(i)} = U_{n+1}^{(i)}, v_{n+1}^{(i)} = V_{n}^{(i)}\) and from (6.27) that

\[u^{(i)} = -U_x^{(i)} - (U^{(1)} V^{(1)} + U^{(1)} V^{(1)}) U^{(i)}, \quad v^{(i)} = V^{(i)}.\]  

(6.29)

This is the second MT linking (2.13) and (6.28).

From (6.22), \(u_{n+1}^{(i)} = U_{n+1}^{(i)}\) and \(v_{n+1}^{(i)} = V_{n}^{(i)}\) we can conclude that the difference between the coupled NLS equation and the coupled DNLS equations are the complex conjugation between \(\{u_n^{(i)}\}\) and \(\{v_n^{(i)}\}\). For the coupled NLS equations \(u_n^{(i)}\) and \(v_n^{(i)}\) are complex
conjugate. On the other hand for the coupled DNLS equations $u_n^{(i)}$ and $v_{n+1}^{(i)}$ are complex conjugate.

Here we consider the coupled version of the Chen-Lee-Liu (CLL) type equations [11], [16],

$$i\hat{Q}_{T}^{(i)} = \hat{Q}_{XX}^{(i)} + i\alpha(\sum_{k}^{N} |\hat{Q}^{(k)}|^{2})\hat{Q}_{X}^{(i)},$$  \hspace{1cm} (6.30)

If we set

$$Q^{(i)} = \hat{Q}^{(i)} \exp \left( -2i\delta \sum_{k}^{N} \int_{X}^{X} |\hat{Q}^{(k)}|^{2}dX \right),$$  \hspace{1cm} (6.31)

then (6.30) is gauge-equivalent [18] to

$$iQ_{T}^{(i)} = Q_{XX}^{(i)} - 2i\delta A Q^{(i)} + 2i\delta B Q^{(i)} + i(4\delta + \alpha)\rho_{Q}^{2} Q_{X}^{(i)} + \delta(4\delta - \alpha)\rho_{Q}^{4} Q^{(i)},$$  \hspace{1cm} (6.32)

where

$$A = \sum_{k}^{N} (Q_{X}^{(k)} Q_{X}^{(k)*} + Q_{X}^{(k)*} Q_{X}^{(k)}), \quad B = \sum_{k}^{N} (Q_{X}^{(k)} Q_{X}^{(k)*} + Q_{X}^{(k)*} Q_{X}^{(k)}),$$

$$\rho_{Q}^{2} = \sum_{(k)}^{N} |Q^{(k)}|^{2}.$$  \hspace{1cm} (6.33)

(6.32) are the new coupled version of the generalized coupled DNLS equations. (6.32) is different from the the generalized coupled DNLS equations which we obtain in the previous works. These equations are the coupled version of the Kaup-Newell equation, [10]

$$i\hat{Q}_{T}^{(i)} = \hat{Q}_{XX}^{(i)} + i\alpha(\sum_{k}^{N} |\hat{Q}^{(k)}|^{2})\hat{Q}_{X}^{(i)},$$  \hspace{1cm} (6.34)

They do not have MT to CNLS equations.

If we set in (6.28)

1) $x = iX$, $t = -iT$, $Q^{(i)} = U^{(i)}$, $Q^{(i)*} = V^{(i)}$, $\alpha = -4\delta = 2$,
2) $x = iX$, $t = iT$, $Q^{i} = V^{(i)}$, $Q^{(i)*} = U^{(i)}$, $-\alpha = 4\delta = 2$,  \hspace{1cm} (6.35)

then (6.28) is gauge equivalent to (6.32).

7 Concluding Remarks

We have considered the (coupled) Davey-Stewartson (DS) system and its Bäcklund transformations (BT). Relations among the DS system, the double KP system and the Ablowitz-Ladik hierarchy (ALH) have been established.

The double KP system has two sets of times, the original $(t) \equiv (t_1 \equiv x, t_2, \cdots)$ and the ghost ones $(\bar{t}) \equiv (\bar{t}_1 \equiv -y, \bar{t}_2, \cdots)$. Then we can obtain one hierarchy about the flows to
and \((t)\). We can get the other one from the flows to \(x\) and \((-\hat{t})\). The first equations in each hierarchy are DS equations \((2.11)\) and \((2.12)\). These two hierarchies are compatible and any linear combination of them is integrable. The couple of these hierarchies are the DS system. Note that dependent variables of DS system is \((\text{adj})\)-EF of double KP system.

The ALH has been found to be the BT of the DS system in the case \(t_k = \hat{t}_k\) and \(u_n = v_n^*\). \(u\) and \(v\) are dependent variables of DS system. The asterisk is conjugation. The flows of ALH are also those of double KP and DS systems. For this constraints the first equations which contain no discrete variables \(n\) are the complex sine-Golden equations instead of the DS equations.[19]

From the BT of the coupled DS equations we can obtain the new coupled derivative nonlinear Schrödinger (DLNS) equations. We can obtain the Miura transformations (MT) between the coupled NLS and the coupled DNLS equations. It is the new type of the coupled DNLS equations. The other coupled DNLS equations do not have the MT to the CNLS equations. We will report on the relations between these two coupled DNLS equations and investigations about these equations in next paper.

References

[1] V.E.Zakharov, S.V.Manakov, S.P.Novikov and L.P.Pitaevsky Theory of Solitons (Consultant Bureau, New York, 1984).

[2] M.Boiti, J.Leon, L.Martina and F.Pempinelli, Phys.Lett.A.132(1988)432.

[3] A.S.Fokas and P.M.Santini, Phys.Rev.Lett.63(1989)1379.

[4] L.V.Bogdanov and B.G.Konopelchenko, Analytic-bilinear approach to integrable hierarchies. II. Multicomponent KP and 2D Toda lattice hierarchies [solv-int/9705009].

[5] H.Aratyn, E.Nissimov and S.Pacheva, A New “Dual” Symmetry Structure of the KP Hierarchy solve-int/9712012.

[6] V.E.Vekslerchik, Inverse Problems.12(1996)1057.

[7] V.E.Vekslerchik, Functional representation of the Ablowitz-Ladik hierarchy [solv-int/9707008].

[8] V.E.Vekslerchik, Inverse Problems.11(1995)463.

[9] M.Hisakado, J.Phys.Soc.Jpn.66(1997)1939.

[10] D.J. Kaup and A. C. Newell, J. Math. Phys.19(1978)798.

[11] H. H. Chen, Y. C. Lee and C. S. Liu, Phys. Scr.20(1979)490.
[12] M. Hisakado and M. Wadati, J. Phys. Soc. Jpn. 64 (1995) 408.

[13] S. V. Manakov, Sov. Phys. JETP. 40 (1974) 269.

[14] A. V. Mikhailov and R. I. Yamilov, Phys. Lett. A. 230 (1997) 295.

[15] M. Hisakado, T Iizuka and M. Wadati, J. Phys. Soc. Jpn. 63 (1994) 2887.

[16] S. Kakei, N. Sasa and J. Satsuma, J. Phys. Soc. Jpn. 64 (1995) 1519.

[17] M. Hisakado and M. Wadati, J. Phys. Soc. Jpn. 63 (1994) 3962.

[18] M. Wadati and K. Sogo, J. Phys. Soc. Jpn. 52 (1983) 394.

[19] V. E. Vekslerchik, J. Phys. A. 27 (1996) 6299.