A HOLOMORPHIC VERTEX OPERATOR ALGEBRA OF CENTRAL CHARGE 24 WITH WEIGHT ONE LIE ALGEBRA $F_{4,6}A_{2,2}$

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Abstract. In this paper, a holomorphic vertex operator algebra $U$ of central charge 24 with the weight one Lie algebra $A_{8,3}A_{2,1}^2$ is proved to be unique. Moreover, a holomorphic vertex operator algebra of central charge 24 with weight one Lie algebra $F_{4,6}A_{2,2}$ is obtained by applying a $Z_2$-orbifold construction to $U$. The uniqueness of such a vertex operator algebra is also established. By a similar method, we also established the uniqueness of a holomorphic vertex operator algebra of central charge 24 with the weight one Lie algebra $E_{7,3}A_{5,1}$. As a consequence, we verify that all 71 Lie algebras in Schellekens’ list can be realized as the weight one Lie algebras of some holomorphic vertex operator algebras of central charge 24. In addition, we establish the uniqueness of three holomorphic vertex operator algebras of central charge 24 whose weight one Lie algebras have the type $A_{8,3}A_{2,1}^2$, $F_{4,6}A_{2,2}$, and $E_{7,3}A_{5,1}$.

1. Introduction

The classification of strongly regular holomorphic vertex operator algebras (VOA) of central charge 24 is an important problem in the theory of vertex operator algebras. In 1993, Schellekens [58] obtained a partial classification and determined the possible Lie algebra structures for the weight one subspaces of holomorphic VOAs of central charge 24 (see also [25]). There are 71 Lie algebras in his list but only 39 of the 71 cases in his list have been constructed explicitly at that time. In the recent years, many new holomorphic VOAs of central charge 24 have been constructed. In [42, 43], a class of holomorphic VOAs called framed VOAs were studied. In particular, 17 holomorphic VOAs were constructed. In addition, holomorphic VOAs with weight one Lie algebras $E_{6,3}G_{2,1}^2$, $A_{2,3}^6$ and $A_{5,3}D_{4,3}A_{4,1}^5$ have been constructed in [54, 57] using $Z_3$-orbifold constructions associated with lattice VOAs. Recently, van Ekeren, Möller and Scheithauer [25] have established the general $Z_n$-orbifold construction for elements of arbitrary orders and the constructions of holomorphic VOAs with the weight one Lie algebras $E_{6,4}C_{2,1}A_{2,1}$, $A_{4,5}^2$, $A_{2,6}D_{4,12}$, $A_{1,1}C_{5,3}G_{2,2}$ and $C_{4,10}$ were also discussed. In [44], the constructions of five other holomorphic VOAs have been obtained using an
orbifold construction associated with inner automorphisms. Moreover, a holomorphic VOA of central charge 24 whose weight one Lie algebra has the type $A_{6,7}$ has been constructed in [45]. Based on these results, 70 of 71 cases in Schellekens’ list have been constructed. There is only one remaining case and the corresponding Lie algebra has the type $F_{4,6}A_{2,2}$.

In this article, we shall construct a strongly regular holomorphic vertex operator algebra of central charge 24 whose weight one Lie algebra has the type $F_{4,6}A_{2,2}$. As a consequence, we verify that all 71 Lie algebras in Schellekens’ list can be realized as the weight one Lie algebras of some strongly regular holomorphic vertex operator algebras of central charge 24. Our method is basically a $\mathbb{Z}_2$-orbifold construction. We shall show that a strongly regular holomorphic VOA $\tilde{U}(g)$ of central charge 24 with $\tilde{U}(g)_1 = F_{4,6}A_{2,2}$ can be constructed by applying a $\mathbb{Z}_2$-orbifold construction to a holomorphic VOA $U$ with the weight one Lie algebra $A_{8,3}A_{2,1}^2$ and a suitable automorphism $g$ of order 2. However, there are some fundamental differences between our method and the previous constructions for the other cases. In our construction, the fixed point of the automorphism $g$ on $U_1 = A_{8,3}A_{2,1}^2$ should have the type $B_{4,6}A_{2,2}$. It means $g|U_1$ is an outer automorphism of the Lie algebra $U_1$. In general, it is very difficult to determine if an outer automorphism of Lie algebra $U_1$ can be lifted to an automorphism of the whole VOA. For $U_1 = A_{8,3}A_{2,1}^2$, there are at least two different constructions of a holomorphic VOA $U$ with $U_1 = A_{8,3}A_{2,1}^2$. One construction is based on orbifold method and is obtained in [44]. Another construction is based on mirror extensions of VOAs [11, 62]. The construction based on mirror extensions is first obtained by Xu [62] in terms of conformal nets and is proposed in [11] in VOA setting. In this article, we shall use the construction based on mirror extensions of VOAs. Using mirror extensions and the theory of modular invariants, we shall show that the VOA structure of a holomorphic VOA $U$ of central charge 24 with $U_1 = A_{8,3}A_{2,1}^2$ is unique, up to isomorphism (cf. Theorem 4.15). We also generalize a result of Shimakura [59, Proposition 3.2], which gives a sufficient condition for lifting of an automorphism of a subVOA to the whole VOA (cf. Theorem 3.7). In addition, we determine the subgroup of $\text{Aut}(U)$ which acts trivially on the weight one Lie algebra $U_1$ (see Section 4.4). By these facts, we are able to show if $U$ is a strongly regular holomorphic VOA of central charge 24 such that $U_1 = A_{8,3}A_{2,1}^2$, then there exists an involution $g \in \text{Aut}(U)$ such that $U_1^g$ is a Lie algebra of type $B_{4,6}A_{2,2}$. Moreover, we are able to determine the conformal weights of the unique irreducible $g$-twisted $U^T(g)$ of $U$ using the explicit action of $g$ on $U_1$. Finally, we shall establish the uniqueness for a holomorphic vertex operator algebra of central charge 24 with weight one Lie algebra $F_{4,6}A_{2,2}$ using the method of “Reverse orbifold”
proposed in [45]. The key idea is to study the automorphisms of $\text{Aut}(U)$ which act trivially on $U_1$ (see Sections 4.4 and 7.2). It turns out that the same method can be applied easily to the case when the weight one Lie algebra has the type $E_{7,3}A_{5,1}$ and we are able to establish the uniqueness for this case, also.

The organization of this article is as follows. In Section 2, we recall some basic facts about vertex operator algebras. In Section 3, we study some properties of the mirror extension $\widetilde{L}_{sl_9}(3,0)$ of affine vertex operator algebra $L_{sl_9}(3,0)$. The vertex operator algebra structure of $\widetilde{L}_{sl_9}(3,0)$ is proved to be unique. We also show that the automorphism $\theta$ of $L_{sl_9}(3,0)$ can be lifted to an automorphism of $\widetilde{L}_{sl_9}(3,0)$. In Section 4, we study the structure of holomorphic vertex operator algebra $U$ of central charge 24 with Lie algebra $A_{8,3}A_{2,1}$. It is proved that holomorphic vertex operator algebra of central charge 24 with Lie algebra $A_{8,3}A_{2,1}$ is unique. As an application, we obtain an involution $\widetilde{\theta} \otimes \sigma$ of $U$. In Section 5, we determine the conformal weights of $\widetilde{\theta} \otimes \sigma$-twisted irreducible $U$-modules. In Section 6, we construct a holomorphic vertex operator algebra of central charge 24 with Lie algebra $F_{4,6}A_{2,2}$ by orbifold construction. In Section 7, we prove that holomorphic VOAs of central charge 24 with weight one Lie algebras $F_{4,6}A_{2,2}$ and $E_{7,3}A_{5,1}$ are unique, up to isomorphisms.

2. Preliminaries

2.1. Basic definitions. In this subsection, we shall recall some notations about vertex operator algebras from [26, 27, 48, 63]. Let $(V, Y(\cdot, z), 1, \omega)$ be a vertex operator algebra as defined in [27]. The vacuum vector and the conformal element of $V$ are denoted by $1$ and $\omega$, respectively. The vertex operator $Y(v, z)$ corresponding to $v \in V$ is expanded as $Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$. We also use the standard notation $L(n)$ to denote the component operator of $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$. A linear automorphism $\sigma$ of $V$ is called an automorphism of $V$ if $\sigma(1) = 1$, $\sigma(\omega) = \omega$ and $\sigma(u_n v) = \sigma(u)_n \sigma(v)$ for any $u, v \in V$, $n \in \mathbb{Z}$. We denote the group of all automorphisms of $V$ by $\text{Aut}(V)$.

For a vertex operator algebra $V$, a weak $V$-module is a vector space $M$ equipped with a linear map

$$Y_M : V \rightarrow (\text{End}M)[[z, z^{-1}]],$$

$$v \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}, v_n \in \text{End}M$$

satisfying a number of conditions (cf. [16], [26]). A weak $V$-module $M$ is called an admissible $V$-module if $M$ has a $\mathbb{Z}_{\geq 0}$-gradation $M = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M(n)$ such that

$$a_m M(n) \subset M(wta + n - m - 1)$$
for any homogeneous \(a \in V\) and \(m, n \in \mathbb{Z}\), where \(\text{wt}(a) = s\) if \(a \in V_s\). If any admissible \(V\)-module is a direct sum of irreducible admissible modules, then \(V\) is called rational. It was proved in [16] that if \(V\) is rational then there are only finitely many irreducible admissible \(V\)-modules up to isomorphism.

A \(V\)-module is a weak \(V\)-module \(M\) which carries a \(\mathbb{C}\)-grading induced by the spectrum of \(L(0)\), that is, \(M = \bigoplus_{\lambda \in \mathbb{C}} M_{\lambda}\) where \(M_{\lambda} = \{w \in M|L(0)w = \lambda w\}\). Moreover, one requires that \(M_{\lambda}\) is finite dimensional and for fixed \(\lambda \in \mathbb{C}\), \(M_{\lambda + n} = 0\) for sufficiently small integer \(n\).

A rational vertex operator algebra is said to be holomorphic if it itself is the only irreducible module up to isomorphism. A vertex operator algebra \(V\) is said to be of CFT-type if \(V_0 = \mathbb{C}\) (note that \(V_n = 0\) for all \(n < 0\) if \(V_0 = \mathbb{C}\) [19]), and is said to be \(C_2\)-cofinite if the subspace \(C_2(V) = \langle u - 2v|u, v \in V\rangle\) has finite codimension in \(V\). A \(V\)-module \(M = \bigoplus_{\lambda \in \mathbb{C}} M_{\lambda}\) is said to be self-dual if \(M\) is isomorphic to \(M^\prime\), where \(M^\prime = \bigoplus_{\lambda \in \mathbb{C}} M^\prime_{\lambda}\) and the vertex operator \(Y_{M^\prime}\) is defined by the property

\[
\langle Y_{M^\prime}(a, z)u', v \rangle = \langle u', Y_{M}(e^{zL(1)}(-z^{-2})L(0)a, z^{-1})v \rangle,
\]

for \(a \in V, u' \in M^\prime\) and \(v \in M\). It is obvious that a holomorphic vertex operator algebra is simple and self-dual. A vertex operator algebra is said to be strongly regular if it is simple, rational, \(C_2\)-cofinite and of CFT-type. Note that a strongly regular vertex operator algebra is self-dual [49].

Let \(V\) be a CFT-type vertex operator algebra. It is well-known that \(V_1\) has a Lie algebra structure such that \([u, v] = u_0 v\) for any \(u, v \in V_1\) (cf. [6]). Moreover, it was proved in [20] that \(V_1\) is a reductive Lie algebra if \(V\) is a strongly regular vertex operator algebra.

We now recall the notions of intertwining operator and fusion rules from [26]. Let \(M^1, M^2, M^3\) be admissible \(V\)-modules. An intertwining operator \(\mathcal{Y}\) of type \(\begin{pmatrix} M^3 \\
M^1 & M^2 \end{pmatrix}\) is a linear map

\[
\mathcal{Y} : M^1 \to \text{Hom}(M^2, M^3)\{z\},
\]

\[
w^1 \mapsto \mathcal{Y}(w^1, z) = \sum_{n \in \mathbb{C}} w^1_n z^{-n-1}
\]

satisfying a number of conditions (cf. [26]). We use \(\mathcal{I}^M_{M^1, M^2}\) to denote the vector space of intertwining operators of type \(\begin{pmatrix} M^3 \\
M^1 & M^2 \end{pmatrix}\). If \(V\) is a rational vertex operator algebra and \(\{M^i|0 \leq i \leq p\}\) is the set of irreducible admissible \(V\)-modules, we define the fusion
rules to be the formal product rules

\[ M^i \times M^j = \sum_{0 \leq k \leq p} N_{M^i,M^j}^{M^k} M^k, \]

where \( N_{M^i,M^j}^{M^k} \) denotes the dimension of \( T_{M^i,M^j}^{M^k} \). The fusion ring of \( V \) is defined to be the ring with \( \{ M^i | 0 \leq i \leq p \} \) as a basis and with the fusion rules as the structural constants. Let \( V \) be a rational vertex operator algebra and \( M \) an irreducible admissible \( V \)-module. If for any irreducible admissible module \( M^2 \), there exists an irreducible admissible module \( M^3 \) such that \( M \times M^2 = M^3 \), then \( M \) is called a simple current module of \( V \).

2.2. Modular invariance of trace functions. We now recall the modular invariance property of vertex operator algebra from [63]. Let \( V \) be a rational vertex operator algebra and let \( M_0, ..., M_p \) be all the irreducible \( V \)-modules. Then \( M_i, 0 \leq i \leq p \), has the form

\[ M_i = \bigoplus_{n=0}^{\infty} M_{\lambda_i+n}, \]

with \( M_{\lambda_i} \neq 0 \) for some number \( \lambda_i \), which is called the conformal weight of \( M_i \). Let \( \mathcal{H} = \{ \tau \in \mathbb{C} | \text{Im} \tau > 0 \} \) be the upper half plane. The trace function associated with \( M_i \) is defined as follows: For any homogeneous element \( v \in V \) and \( \tau \in \mathcal{H} \),

\[ Z_{M_i}(v, \tau) := \text{tr}_{M_i} o(v) q^{L(0) - c/24} = q^{\lambda_i - c/24} \sum_{n \in \mathbb{Z}^+} \text{tr}_{M_{\lambda_i+n}} o(v) q^n, \]

where \( o(v) = v_{wt-1} \) and \( q = e^{2\pi \sqrt{-1} \tau} \). Assume further that \( V \) is a \( C_2 \)-cofinite vertex operator algebra, then \( Z_{M_i}(v, \tau) \) converges to a holomorphic function on the domain \( |q| < 1 \) [17, 63].

Recall that the full modular group \( SL(2, \mathbb{Z}) \) has generators \( S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), \( T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and acts on \( \mathcal{H} \) as follows:

\[ \gamma : \tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \]

Then the full modular group has an action on the trace functions. More precisely, we have the following result which was proved in [63] (also see [17]).

**Theorem 2.1.** Let \( V \) be a rational and \( C_2 \)-cofinite vertex operator algebra with the irreducible \( V \)-modules \( M_0, ..., M_p \). Then the vector space spanned by \( Z_{M_0}(v, \tau), ..., Z_{M_p}(v, \tau) \) is invariant under the action of \( SL(2, \mathbb{Z}) \) defined above, i.e., there is a representation
2.3. Quantum dimensions. In this subsection, we recall some facts about quantum dimensions of irreducible modules of vertex operator algebras from [12]. Let $V$ be a strongly regular vertex operator algebra and let $M^0 = V, M^1, ..., M^p$ be all the inequivalent irreducible $V$-modules. The quantum dimension of $M^i$ is defined to be

$$qdim_{V} M^i = \lim_{y \to 0^+} \frac{Z_{M^i}(\sqrt{-1}y)}{Z_{V}(\sqrt{-1}y)},$$

where $y$ is real and positive. The global dimension of the vertex operator algebra $V$ is defined to be

$$Glob V = \sum_{i=0}^{p} (qdim_{V} M^i)^2.$$

The following result was proved in [12].

**Theorem 2.2.** Let $V$ be a strongly regular vertex operator algebra and let $M^0 = V, M^1, ..., M^p$ be all the irreducible $V$-modules. Assume further that the conformal weights of $M^1, ..., M^p$ are greater than 0. Then

1. $qdim_{V} M^i \geq 1$ for any $0 \leq i \leq p$.
2. $M^i$ is a simple current $V$-module if and only if $qdim_{V} M^i = 1$.

Recall that a vertex operator algebra $U$ is called an extension vertex operator algebra of $V$ if $V$ is a vertex operator subalgebra of $U$ and $V$, $U$ have the same conformal element. The following result was proved in [2].

**Theorem 2.3.** Let $U$ and $V$ be strongly regular vertex operator algebras and let $M^0 = V, M^1, ..., M^p$ be all the irreducible $V$-modules. Assume further that $U$ is an extension vertex operator algebra of $V$ and that the conformal weights of $M^1, ..., M^p$ are greater than 0. Then, we have

$$Glob (U) = (qdim_{V} U)^2 Glob (V).$$

2.4. Modular invariants of vertex operator algebras. Next, we recall some facts about modular invariants of vertex operator algebras from [18]. We shall assume that $V$ is a strongly regular vertex operator algebra. Let $M^0 = V, M^1, ..., M^p$ be all the irreducible $V$-modules. A modular invariant of $V$ is a $(p + 1) \times (p + 1)$-matrix $X$ satisfying the following conditions:

1. The entries of $X$ are nonnegative integers.
2. $X_{0,0} = 1.$
(M3) $XS = SX$ and $XT = TX$, where we use $S, T$ to denote the modular transformation matrices $\rho(S)$ and $\rho(T)$ respectively.

In the following, we shall construct a modular invariant of $V$ from an extension vertex operator algebra of $V$. First, we have the following result (cf. [1, 34]).

**Theorem 2.4.** Let $V$ be a strongly regular vertex operator algebra and $U$ an extension vertex operator algebra of $V$. Assume further that $U$ is simple. Then, $U$ is rational and $C_2$-cofinite.

We now assume that $U, V$ are strongly regular vertex operator algebras and that $U$ is an extension vertex operator algebra of $V$. For $u, v \in V$, set

$$f_V(u, v, \tau_1, \tau_2) = \sum_{i=0}^{p} Z_{M_i}(u, \tau_1) \overline{Z_{M_i}(v, \tau_2)},$$

where $\tau_1, \tau_2 \in \mathcal{H}$.

Similarly, for $u, v \in U$, set

$$f_U(u, v, \tau_1, \tau_2) = \sum_{M} Z_M(u, \tau_1) \overline{Z_M(v, \tau_2)},$$

where $M$ ranges through the equivalent classes of irreducible $U$-modules. Since each irreducible $U$-module $M$ is a direct sum of irreducible $V$-modules, there exists a matrix $X = (X_{i,j})$ such that for all $i, j$ and for $u, v \in V$,

$$f_U(u, v, \tau_1, \tau_2) = \sum_{i,j=0}^{p} X_{i,j} Z_{M_i}(u, \tau_1) \overline{Z_{M_j}(v, \tau_2)}.$$

Moreover, the matrix $X = (X_{i,j})$ is uniquely determined by the following proposition.

**Proposition 2.5** ([18]). Let $M^0, ..., M^p$ be all the $V$-irreducible modules. Set

$$Z(u, \tau) = (Z_{M^0}(u, \tau), ..., Z_{M^p}(u, \tau))^T,$$

If $A = (a_{ij})$ is a matrix such that for any $u, v \in V$,

$$Z(u, \tau_1)^T A \overline{Z(v, \tau_2)} = 0,$$

then $A = 0$.

It was further shown in [18] that

**Theorem 2.6.** The matrix $X$ is a modular invariant of $V$.

Furthermore, by the discussion above, we have the following simple observation.

**Lemma 2.7.** Let $V, U, X$ be as above. Suppose that there exist irreducible $V$-modules $M^i$ and $M^j$ such that $X_{i,j} \neq 0$. Then we have $X_{i,i} \neq 0$ and $X_{j,j} \neq 0$. 

2.5. **Mirror extensions of vertex operator algebras.** In this subsection, we shall recall from [52] some facts about mirror extensions of vertex operator algebras. First, we have the following result which was proved in [33].

**Theorem 2.8.** Let $V$ be a strongly regular vertex operator algebra and let $C_V$ be the category of $V$-modules. Then $C_V$ is a modular tensor category such that $V$ is the unit object.

Recall from [4] that for any modular tensor category $(C, \otimes)$ and an object $M$ in $C$, there is a natural isomorphism

$$\theta_M : M \rightarrow M$$

such that

$$\theta_{M \otimes N} = c_{N,M}c_{M,N}(\theta_M \otimes \theta_N),$$

$$\theta_1 = \text{id},$$

$$\theta_{M^*} = (\theta_M)^*,$$

where 1 denotes the unit object of $C$, $M^*$ denotes the dual of $M$, $c_{-,-}$ denotes the braiding of $C$, and $(\theta_M)^* \in \text{Hom}(M^*,M^*)$ denotes the image of $\theta_M \in \text{Hom}(M,M)$ under the canonical map. It was shown in [34] that an extension vertex operator algebra $U$ of $V$ induces an etale algebra $A_U$ (cf. [10]) in $C_V$ such that $\theta_{A_U} = \text{id}$. Moreover, we have the following result which was proved in [34] (see also [52]).

**Theorem 2.9.** Let $V$ be a strongly regular vertex operator algebra and let $C_V$ be the category of $V$-modules. Then the following statements are equivalent

1. There exists an extension vertex operator algebra $U$ of $V$ such that $U$ viewed as a $U$-module is irreducible.
2. There exists an etale algebra $A_U$ in $C_V$ such that $A_U$ viewed as a $V$-module is isomorphic to $U$ and that $\theta_{A_U} = \text{id}$.

We now let $(U,Y,1,\omega)$ be a vertex operator algebra and let $(V,Y,1,\omega')$ be a vertex operator subalgebra of $U$ such that $\omega' \in U_2$ and $L(1)\omega' = 0$. It was proved in [28] that $(V^c,Y,1,\omega - \omega')$ is also a vertex operator subalgebra of $U$, where $V^c = C_U(V) = \{v \in U | \omega' v = 0\}$. Assume further that $(V,Y,1,\omega), (V^c,Y,1,\omega - \omega'), (U,Y,1,\omega')$ are strongly regular and $(V^c)^c = V$, and denote the tensor products of the module categories $C_V$, $C_{V^c}$ and $C_V \otimes V^c$ by $\boxtimes_V$, $\boxtimes_{V^c}$, $\boxtimes_{V \otimes V^c}$, respectively. Then we have the following results which were proved in [52].

**Theorem 2.10.** (1) As a $V \otimes V^c$-module, $U$ has the following decomposition

$$U = V \otimes V^c \oplus (\oplus_{i=1}^n M_i \otimes N_i),$$
where $M^0 = V, M^1, \ldots, M^n$ (resp. $N^0 = V^c, N^1, \ldots, N^n$) are mutually inequivalent irreducible $V$-modules (resp. $V^c$-modules).

(2) Let $K(C_V)$ and $K(C_{V^c})$ be the Grothendieck rings of $C_V$ and $C_{V^c}$, respectively. Then $ZM^0 \oplus \cdots \oplus ZM^n$ (resp. $ZN^0 \oplus \cdots \oplus ZN^n$) forms a subring of $K(C_V)$ (resp. $K(C_{V^c})$).

(3) For any $0 \leq i_1, i_2, i_3 \leq n$, $N_{M^1, M^2}^{i_1} = N_{N^1, N^2}^{i_1}$.

Let $C_V^0$ (resp. $C_{V^c}^0$) be the tensor subcategory of $C_V$ (resp. $C_{V^c}$) such that the Grothendieck ring of $C_V^0$ (resp. $C_{V^c}^0$) is isomorphic to the subring $ZM^0 \oplus \cdots \oplus ZM^n$ (resp. $ZN^0 \oplus \cdots \oplus ZN^n$) of $K(C_V)$ (resp. $K(C_{V^c})$). Then we have the following results which were also proved in [52].

**Theorem 2.11.** (1) There is a braid-reversing equivalence $T : C_V^0 \to C_{V^c}^0$ such that $T(M^i) \cong (N^i)'$.

(2) If there is a vertex operator algebra structure $Y_{V^c}(\cdot, z)$ on the $V$-module

$$V^c = V \oplus (\oplus_{i=1}^n m_i M^i),$$

where $m_i$'s are nonnegative integers, such that $(V^c, Y_{V^c}(\cdot, z))$ is an extension vertex operator algebra of $V$, then there exists a vertex operator algebra structure $Y_{(V^c)^c}(\cdot, z)$ on the $V^c$-module

$$(V^c)^c = V^c \oplus (\oplus_{i=1}^n m_i (N^i)'),$$

such that $((V^c)^c, Y_{(V^c)^c}(\cdot, z))$ is an extension vertex operator algebra of $V^c$. Moreover, $(V^c)^c$ is a simple vertex operator algebra if $V^c$ is a simple vertex operator algebra.

Furthermore, we have the following result about the uniqueness.

**Theorem 2.12.** Let $V^c$ and $(V^c)^c$ be as above. Suppose that for any two extension vertex operator algebras $(V^c, Y_{V^c}(\cdot, z))$, $(V^c, Y_{V^c}(\cdot, z))$ of $V$, there exists a linear isomorphism $\phi : V^c \to V^c$ satisfying the following conditions

$$\phi|_V = \text{id}, \quad \phi(Y^1_{V^c}(u^1, z)u^2) = Y^2_{V^c}(\phi(u^1), z)\phi(u^2), \text{ for any } u^1, u^2 \in V^c.$$

Then for any two extension vertex operator algebras $((V^c)^c, Y^1_{(V^c)^c}(\cdot, z))$, $((V^c)^c, Y^2_{(V^c)^c}(\cdot, z))$ of $V^c$, there exists a linear isomorphism $\phi^c : (V^c)^c \to (V^c)^c$ satisfying the following conditions

$$\phi^c|_{(V^c)^c} = \text{id}, \quad \phi^c(Y^1_{(V^c)^c}(v^1, z)v^2) = Y^2_{(V^c)^c}(\phi^c(v^1), z)\phi^c(v^2), \text{ for any } v^1, v^2 \in (V^c)^c.$$

**Proof:** Assume that there exist two vertex operator algebras $((V^c)^c, Y^1_{(V^c)^c}(\cdot, z))$ and $((V^c)^c, Y^2_{(V^c)^c}(\cdot, z))$ such that $((V^c)^c, Y^1_{(V^c)^c}(\cdot, z))$, $((V^c)^c, Y^2_{(V^c)^c}(\cdot, z))$ are extension vertex operator algebras of $V^c$. Let $A^1_{(V^c)^c}, A^2_{(V^c)^c}$ be the etale algebras in $C_{V^c}$ induced from
reversing equivalence, there exists a braid-reversing functor $G : C^0_{V^e} \to C^0_{V^e}$ such that $T \circ G$, $G \circ T$ are natural isomorphic to $id_{C^0_{V^e}}$, $id_{C^0_{V^e}}$, respectively, that is, there exist a family of isomorphisms $\eta^1(N) : T \circ G(N) \to N, N \in C^0_{V^e}$, and $\eta^2(M) : G \circ T(M) \to M, M \in C^0_{V^e}$ satisfying

$$\eta^1(N^2) \circ (T \circ G(g)) = g \circ \eta^1(N^1), \quad \eta^2(M^2) \circ (G \circ T(f)) = f \circ \eta^2(M^1),$$

for any $M^1, M^2 \in C^0_{V^e}, N^1, N^2 \in C^0_{V^e}$ and $f : M^1 \to M^2, g : N^1 \to N^2$ (see [39]).

Note that, by Theorem 2.11, $G(V^e)$ is isomorphic to $V$ and that $G(A^1_{(V^e)^e}), G(A^2_{(V^e)^e})$ are two etale algebras in $C^0_{V^e}$ such that $G(A^1_{(V^e)^e}), G(A^2_{(V^e)^e})$ viewed as $V$-modules are isomorphic to $V^e$. It follows from Theorem 2.9 that there exist two vertex operator algebra structures $(G((V^e)^e), Y^1(\cdot, z)), (G((V^e)^e), Y^2(\cdot, z))$ such that $(G((V^e)^e), Y^1(\cdot, z)), (G((V^e)^e), Y^2(\cdot, z))$ are extension vertex operator algebras of $G(V^e)$. By assumption, there exists a linear isomorphism $\phi : G((V^e)^e) \to G((V^e)^e)$ satisfying the following conditions

$$\phi|_{G(V^e)} = id, \quad \phi(Y^1(u^1, z)u^2) = Y^2(\phi(u^1), z)\phi(u^2), \text{ for any } u^1, u^2 \in G((V^e)^e).$$

Then we know that $\phi$ induces an etale algebra isomorphism $\tilde{\phi} : G(A^1_{(V^e)^e}) \to G(A^2_{(V^e)^e})$ such that $\tilde{\phi}|_{G(V^e)} = id$. Hence, $T(\tilde{\phi})$ is an etale algebra isomorphism from $T \circ G(A^1_{(V^e)^e})$ to $T \circ G(A^2_{(V^e)^e})$ such that $T(\tilde{\phi})|_{T(G(V^e))} = id$. Note that $\eta^1(A^1_{(V^e)^e}), \eta^1(A^2_{(V^e)^e})$ are algebra isomorphism (see Definition XI.4.1 of [39]). As a result, $\eta^1(A^2_{(V^e)^e}) \circ T(\tilde{\phi}) \circ \eta^1(A^1_{(V^e)^e})^{-1}$ is an etale algebra isomorphism from $A^1_{(V^e)^e}$ to $A^2_{(V^e)^e}$ such that $\eta^1(A^2_{(V^e)^e}) \circ T(\tilde{\phi}) \circ \eta^1(A^1_{(V^e)^e})^{-1}|_{V^e} = id$. Therefore, $\eta^1(A^2_{(V^e)^e}) \circ T(\tilde{\phi}) \circ \eta^1(A^1_{(V^e)^e})^{-1}$ will induce the desired isomorphism. The proof is complete. \qed

As a corollary, we have the following:

**Corollary 2.13.** Let $V^e$ and $(V^e)^e$ be as above. Suppose that there is a unique vertex operator algebra structure on $V^e$ as an extension vertex operator algebra of $V$. Then there is a unique vertex operator algebra structure on $(V^e)^e$ as an extension vertex operator algebra of $V^e$.

3. Mirror extensions of affine vertex operator algebra $L_{sl_2}(3\tilde{\Lambda}_0)$

### 3.1. Affine vertex operator algebras

In this subsection, we shall recall some facts about affine vertex operator algebras from [28] and [48]. Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra and $\langle \cdot, \cdot \rangle$ the normalized Killing form of $\mathfrak{g}$, i.e., $\langle \theta, \theta \rangle = 2$ for the highest root $\theta$ of $\mathfrak{g}$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and denote the corresponding root system by $\Delta_\mathfrak{h}$ and the root lattice by $\mathcal{Q}$. We further fix simple roots $\{\alpha_1, \cdots, \alpha_l\}$, and
denote the set of positive roots by $\Delta^+_g$. Then the weight lattice $P$ of $g$ is the set of $\lambda \in \mathfrak{h}$ such that $\frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha \in \Delta_g$. Note that $P$ is equal to $\bigoplus_{i=1}^l \mathbb{Z} \Lambda_i$, where $\Lambda_i$ are the fundamental weights defined by the equation $\frac{2(\Lambda_i,\alpha)}{(\alpha,\alpha)} = \delta_{i,j}$. We also use the standard notation $P_+$ to denote the set of dominant weights $\{\Lambda \in P \mid \frac{2(\Lambda,\alpha)}{(\alpha,\alpha)} \geq 0, \ 1 \leq j \leq l\}$. For any $\alpha \in \Delta^+_g$, we fix $x_\pm \in g_{\pm 1} \oplus g_{\pm 2}$ such that $[x_\pm, x_{-\pm}] = h_\pm$, $[h_\pm, x_\pm] = \pm 2 x_\pm$, where $h_\pm = \frac{2}{(\alpha,\alpha)} \alpha$.

Recall that the affine Lie algebra associated to $g$ is defined on $\hat{g} = g \otimes \mathbb{C}[t^{-1},t] \oplus \mathbb{C}K$ with Lie brackets

$$[x(m), y(n)] = [x, y](m+n) + \langle x, y \rangle m \delta_{m+n,0} K,$$

$$[K, \hat{g}] = 0,$$

for $x, y \in g$ and $m, n \in \mathbb{Z}$, where $x(n)$ denotes $x \otimes t^n$. In particular, $\hat{\mathfrak{h}} = h \otimes \mathbb{C}[t^{-1},t] \oplus \mathbb{C}K$ is a subalgebra of $\hat{g}$.

For a positive integer $k$ and a weight $\Lambda \in P$, let $L_g(\Lambda)$ be the irreducible highest weight module for $g$ with highest weight $\Lambda$ and define

$$V^g_k(\Lambda) = \text{Ind}_{g \otimes C[t] \oplus C K}^{\hat{g} \otimes C[t] \oplus C K} L_g(\Lambda),$$

where $L_g(\Lambda)$ is viewed as a module for $g \otimes \mathbb{C}[t] \oplus \mathbb{C}K$ such that $g \otimes t \mathbb{C}[t]$ acts as 0 and $K$ acts as $k$. It is well-known that $V^g_k(\Lambda)$ has a unique maximal proper submodule which is denoted by $J(k,\Lambda)$ (see [37]). Let $L_g(k,\Lambda)$ be the corresponding irreducible quotient module. It was proved in [28] that $L_g(k,0)$ has a vertex operator algebra structure such that the conformal element

$$\omega = \frac{1}{2(k + h)} \left( \sum_{i=1}^{\dim h} u_i(-1)u_i(-1)1 + \sum_{\alpha \in \Delta_g} \frac{\langle \alpha, \alpha \rangle}{2} x_\alpha(-1)x_{-\alpha}(-1)1 \right),$$

where $h^\vee$ denotes the dual Coxeter number of $g$ and $\{u_i|1 \leq i \leq \dim h\}$ is an orthonormal basis of $h$ with respect to $\langle , \rangle$.

**Theorem 3.1** ([28, 37]). Let $k$ be a positive integer. Then

1. $L_g(k,0)$ is a strongly regular vertex operator algebra;
2. $L_g(k,\Lambda)$ is a module for the vertex operator algebra $L_g(k,0)$ if and only if $\Lambda \in P^k_+$, where $P^k_+ = \{\Lambda \in P_+ | \langle \Lambda, \theta \rangle \leq k\}$;
3. If $L_g(k,\Lambda)$ is an $L_g(k,0)$-module such that $L_g(k,\Lambda) \not\cong L_g(k,0)$, then the conformal weight of $L_g(k,\Lambda)$ is positive.

We next recall some facts about simple current modules of $L_g(k,0)$. Let $\theta = \sum_{i=1}^l a_i \alpha_i$, $a_i \in \mathbb{Z}_+$, be the highest root. It is well-known that the irreducible $L_g(k,0)$-module
\( L_{\mathfrak{g}}(k, k\Lambda_i) \) is a simple current \( L_{\mathfrak{g}}(k, 0) \)-module if \( a_i = 1 \) (see \([15, 29, 30, 51]\)). In particular, for \( \mathfrak{g} = \mathfrak{sl}_{n+1}, \ L_{\mathfrak{g}}(k, k\Lambda_1), \ldots, L_{\mathfrak{g}}(k, k\Lambda_n) \) are simple current \( L_{\mathfrak{sl}_{n+1}}(k, 0) \)-modules. Moreover, these are all the simple current \( L_{\mathfrak{sl}_{n+1}}(k, 0) \)-modules (see \([15, 51]\)).

### 3.2. Mirror extensions of affine vertex operator algebra \( L_{\mathfrak{sl}_3}(3,0) \)

In this subsection, we shall construct some extension vertex operator algebra of \( L_{\mathfrak{sl}_3}(3,0) \). Consider the affine vertex operator algebra \( L_{\mathfrak{sl}_3}(1,0) \). It is well-known that \( L_{\mathfrak{sl}_3}(1,0) \) contains a vertex operator subalgebra isomorphic to \( L_{\mathfrak{sl}_3}(9,0) \otimes L_{\mathfrak{sl}_3}(3,0) \). To determine the decomposition of \( L_{\mathfrak{sl}_3}(1,0) \) viewed as an \( L_{\mathfrak{sl}_3}(9,0) \otimes L_{\mathfrak{sl}_3}(3,0) \)-module, we need to recall some notations from \([56]\). Let \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_k > 0 = \lambda_{k+1} = \cdots) \) be a partition of \( |\lambda| = \lambda_1 + \cdots + \lambda_k \). We define \( h(\lambda) = k \) and identify \( \lambda \) with its corresponding Young diagram; thus \( h(\lambda) \) is just the number of rows in this diagram. We write \( I_n \) for the set of all partitions with \( h(\lambda) \leq n \). Let \( I_{n, m} \) be the set of all \( \lambda \in I_n \) with \( \lambda_1 \leq m \). Hence \( \lambda \in I_{n,m} \) if and only if its Young diagram fits into an \( m \times n \) rectangle. Denote by \( \lambda^t \) the transposed partition of \( \lambda \). Clearly, \( \lambda \in I_{n,m} \) implies \( \lambda^t \in I_{m,n} \).

Let \( C_{n,m} = \{(a_0, a_1, \ldots, a_{n-1}) \in \mathbb{N}^n \mid a_0 + \cdots + a_{n-1} = m\} \). By identifying \( a = (a_0, a_1, \ldots, a_{n-1}) \) with \( \tilde{a} = a_1\Lambda_1 + \cdots + a_{n-1}\Lambda_{n-1} \), we know that \( C_{n,m} \) is exactly the set of dominant \( \mathfrak{sl}_n \)-weights of level \( m \). For any \( \lambda \in I_{n,m} \), we define

\[
w_{n,m}(\lambda) = (m - \lambda_1 + \lambda_n, \lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{n-1} - \lambda_n) \in C_{n,m}.
\]

Conversely, we define \( d_{n,m} : C_{n,m} \rightarrow I_{n,m} \) by sending \( (a_0, a_1, \ldots, a_{n-1}) \) to partition \((a_1 + \cdots + a_{n-1}, a_2 + \cdots + a_{n-1}, \ldots, a_{n-1}, 0, \cdots)\).

Note that \( |d_{n,m}(a_0, a_1, \ldots, a_{n-1})| = \sum i a_i \). For \( a = (a_0, a_1, \ldots, a_{n-1}) \in C_{n,m} \), we say that \( |d_{n,m}(a)| + n\mathbb{Z} \in \mathbb{Z}/n\mathbb{Z} \) is the degree of \( a \) and write it \( \deg(a) \). In particular, we will consider the subset \( C^0_{n,m} = \{a \in C_{n,m} \mid \deg(a) = 0 \mod n\} \). Let \( \rho_n : C_{n,m} \rightarrow C_{n,m} \) be the cyclic permutation \( \rho_n(a_0, a_1, \ldots, a_{n-1}) = (a_{n-1}, a_0, \ldots, a_{n-2}) \). We also define \( \tau : C^0_{n,m} \rightarrow C^0_{m,n} \) by

\[
\tau(a) = \rho_m^{-\frac{|d_{n,m}(a)|}{n}} (w_{m,n}(d_{n,m}(a)^t)).
\]

Then we have the following result which was proved in \([56]\).

**Theorem 3.2.** Let \( \Lambda_1, \Lambda_2 \) be the fundamental weights of \( \mathfrak{sl}_3 \) and let \( \Lambda_1, \ldots, \Lambda_8 \) be the fundamental weights of \( \mathfrak{sl}_9 \). Then the decomposition of \( L_{\mathfrak{sl}_3}(1,0) \) viewed as an \( L_{\mathfrak{sl}_3}(9,0) \otimes L_{\mathfrak{sl}_3}(3,0) \)-module is as follows:

\[
L_{\mathfrak{sl}_3}(1,0) = \bigoplus_{a \in C^0_{3,9}} L_{\mathfrak{sl}_3}(9, \tilde{a}) \otimes L_{\mathfrak{sl}_3}(3, \tau(a)),
\]

where \( \tilde{a} \) denotes \( a_1\Lambda_1 + a_2\Lambda_2 \) and \( \tau(a) \) denotes \( \tau(a)_1\Lambda_1 + \cdots + \tau(a)_8\Lambda_8 \).
On the other hand, it is well-known that $L_{\mathfrak{sl}_3}(9,0)$ has an extension vertex operator algebra $L_{E_6}(1,0)$ (see [3,8]). Moreover, it was shown in [38] that the vertex operator algebra $L_{E_6}(1,0)$ viewed as an $L_{\mathfrak{sl}_3}(9,0)$-module has the following decomposition

$$L_{E_6}(1,0) = L_{\mathfrak{sl}_3}(9,0) \oplus L_{\mathfrak{sl}_3}(9,9\Lambda_1) \oplus L_{\mathfrak{sl}_3}(9,9\Lambda_2) \oplus L_{\mathfrak{sl}_3}(9,\Lambda_1 + 4\Lambda_2) \oplus L_{\mathfrak{sl}_3}(9,4\Lambda_1 + \Lambda_2) \oplus L_{\mathfrak{sl}_3}(9,4\Lambda_1 + 4\Lambda_2).$$

Furthermore, we have the following result which is a slight generalization of Theorem 3.8 of [3].

**Theorem 3.3.** Let $(U^1, Y_1(\cdot, z)), (U^2, Y_2(\cdot, z))$ be extension vertex operator algebras of $L_{\mathfrak{sl}_3}(9,0)$ such that $(U^1, Y_1(\cdot, z))$, $(U^2, Y_2(\cdot, z))$ are strongly regular and that $U^1$, $U^2$ viewed as modules of $L_{\mathfrak{sl}_3}(9,0)$ have the following decomposition

$$U^1 \cong U^2 \cong L_{\mathfrak{sl}_3}(9,0) \oplus L_{\mathfrak{sl}_3}(9,9\Lambda_2) \oplus L_{\mathfrak{sl}_3}(9,9\Lambda_1) \oplus L_{\mathfrak{sl}_3}(9,\Lambda_1 + 4\Lambda_2) \oplus L_{\mathfrak{sl}_3}(9,4\Lambda_1 + \Lambda_2) \oplus L_{\mathfrak{sl}_3}(9,4\Lambda_1 + 4\Lambda_2).$$

Then there exists an isomorphism $\phi : U^1 \to U^2$ such that

$$\phi|_{L_{\mathfrak{sl}_3}(9,0)} = \text{id}, \quad \phi(Y_1(u^1, z)u^2) = Y_2(\phi(u^1), z)\phi(u^2), \text{ for any } u^1, u^2 \in U^1.$$

**Proof:** By assumption, $(U^1, Y_1(\cdot, z))$ is an extension vertex operator algebra of $L_{\mathfrak{sl}_3}(9,0)$ such that $(U^1, Y_1(\cdot, z))$ is strongly regular and that $U^1$ viewed as a module of $L_{\mathfrak{sl}_3}(9,0)$ has the following decomposition

$$U^1 \cong L_{\mathfrak{sl}_3}(9,0) \oplus L_{\mathfrak{sl}_3}(9,9\Lambda_2) \oplus L_{\mathfrak{sl}_3}(9,9\Lambda_1) \oplus L_{\mathfrak{sl}_3}(9,\Lambda_1 + 4\Lambda_2) \oplus L_{\mathfrak{sl}_3}(9,4\Lambda_1 + \Lambda_2) \oplus L_{\mathfrak{sl}_3}(9,4\Lambda_1 + 4\Lambda_2).$$

It follows from Theorem 3.8 of [3] that there exists an isomorphism $\psi_1 : U^1 \to L_{E_6}(1,0)$ such that

$$\psi_1(Y_1(u, z)v) = Y_{L_{E_6}(1,0)}(\psi_1(u), z)\psi_1(v)$$

for any $u, v \in U^1$, where $Y_{L_{E_6}(1,0)}(\cdot, z)$ denotes the vertex operator map of $L_{E_6}(1,0)$. In particular, $\psi_1|_{L_{\mathfrak{sl}_3}(9,0)}$ is an automorphism of $L_{\mathfrak{sl}_3}(9,0)$.

We next show that there exists an automorphism $\psi_2$ of $L_{E_6}(1,0)$ such that $\psi_2 \circ \psi_1|_{L_{\mathfrak{sl}_3}(9,0)} = \text{id}$. It is good enough to show that any automorphism of $L_{\mathfrak{sl}_3}(9,0)$ can be lifted to an automorphism of $L_{E_6}(1,0)$. Note that any automorphism of $\mathfrak{sl}_3$ has the form $\varphi \exp(2\pi \sqrt{-1}h)$, where $\varphi$ denotes the diagram automorphism of $\mathfrak{sl}_3$ and $h$ is an element of a Cartan subalgebra of $\mathfrak{sl}_3$ (see [50]). Moreover, it was shown in subsection 2.5 of [38] that the diagram automorphism $\varphi$ of $\mathfrak{sl}_3$ can be lifted to an automorphism of $E_6$. Since for any $h$, $\exp(2\pi \sqrt{-1}h)$ is also an automorphism of $E_6$, we then know that
any automorphism of \( sl_3 \) can be lifted to an automorphism of \( E_6 \). It follows that any automorphism of \( L_{sl_3}(9,0) \) can be lifted to an automorphism of \( L_{E_6}(1,0) \).

Similarly, there exists a vertex operator algebra isomorphism \( \phi_2 : U^2 \to L_{E_6}(1,0) \) such that \( \phi_2|_{L_{sl_3}(9,0)} = \text{id} \). Hence, \( \phi_2^{-1} \circ \psi_2 \circ \psi_1 \) is the desired isomorphism. The proof is complete.

The next lemma can be proved by a direct calculation.

**Lemma 3.4.** Let \( \tau \) be the map defined as above. Then we have
\[
\begin{align*}
\tau((0,0,0)) &= (3,0,0,0,0,0,0,0), \\
\tau((0,9,0)) &= (0,0,0,3,0,0,0,0), \\
\tau((0,0,9)) &= (0,0,0,0,0,3,0,0), \\
\tau((4,4,1)) &= (0,0,0,1,0,0,0,1,1), \\
\tau((4,1,4)) &= (0,1,1,0,0,0,1,0,0), \\
\tau((1,4,4)) &= (1,0,0,0,1,1,0,0,0).
\end{align*}
\]

Since the affine vertex operator algebra \( L_{E_6}(1,0) \) is self-dual, we then have the following result by Theorems 2.11, 3.2 and Lemma 3.4.

**Theorem 3.5.** There is a vertex operator algebra structure on
\[
\tilde{L}_{sl_3}(3,0) = L_{sl_3}(3,0) \oplus L_{sl_3}(3,3\bar{\Lambda}_3) \oplus L_{sl_3}(3,3\bar{\Lambda}_6) \oplus L_{sl_3}(3,\bar{\Lambda}_1 + \bar{\Lambda}_2 + \bar{\Lambda}_6) \\
\oplus L_{sl_3}(3,\bar{\Lambda}_3 + \bar{\Lambda}_7 + \bar{\Lambda}_8) \oplus L_{sl_3}(3,\bar{\Lambda}_4 + \bar{\Lambda}_5)
\]
such that \( \tilde{L}_{sl_3}(3,0) \) is an extension vertex operator algebra of \( L_{sl_3}(3,0) \) and strongly regular.

Moreover, by Theorems 2.12 and 3.3 we have:

**Theorem 3.6.** Let \((U^1, Y_1(\cdot, z))\) and \((U^2, Y_2(\cdot, z))\) be two strongly regular vertex operator algebras satisfying the following conditions:
\[(1) \ U^1 \text{ and } U^2 \text{ are extension vertex operator algebras of } L_{sl_3}(3,0). \]
\[(2) \ U^1 \text{ and } U^2 \text{ viewed as } L_{sl_3}(3,0)-\text{modules are isomorphic to} \]
\[
L_{sl_3}(3,0) \oplus L_{sl_3}(3,3\bar{\Lambda}_3) \oplus L_{sl_3}(3,3\bar{\Lambda}_6) \oplus L_{sl_3}(3,\bar{\Lambda}_1 + \bar{\Lambda}_2 + \bar{\Lambda}_6) \\
\oplus L_{sl_3}(3,\bar{\Lambda}_3 + \bar{\Lambda}_7 + \bar{\Lambda}_8) \oplus L_{sl_3}(3,\bar{\Lambda}_4 + \bar{\Lambda}_5).
\]

Then there exists a vertex operator algebra isomorphism \( \phi^c : U^1 \to U^2 \) such that \( \phi^c|_{L_{sl_3}(3,0)} = \text{id} \).
3.3. **Lattice vertex operator algebras.** In this subsection, we recall some facts about lattice vertex operator algebras from [6], [27] and [48]. Let $L$ be a positive definite even lattice. We denote the $\mathbb{Z}$-bilinear form on $L$ by $\langle \cdot, \cdot \rangle$. There is a canonical $\mathbb{Z}$-bilinear form $c_0$ on $L$ defined as follows:

$$c_0 : L \times L \to \mathbb{Z}/2\mathbb{Z}$$

$$(\alpha, \beta) \mapsto \langle \alpha, \beta \rangle + 2\mathbb{Z}.$$ 

Since $L$ is an even lattice, the $\mathbb{Z}$-bilinear form $c_0$ is alternating. Thus there is a central extension $\widehat{L}$ of $L$ by the cyclic group $\langle \kappa \rangle$ of order 2 with generator $\kappa$, that is,

$$1 \to \langle \kappa \rangle \to \widehat{L} \to L \to 1,$$

such that the corresponding commutator map is $c_0$ (see [27]). We choose a section $e: L \to \widehat{L}$ such that $e_0 = 1$ and that the corresponding 2-cocycle $\epsilon_0: L \times L \to \mathbb{Z}/2\mathbb{Z}$, which is defined by $\epsilon_0(\alpha, \beta) = \kappa^{c_0(\alpha, \beta)} e_{\alpha + \beta}$ for $\alpha, \beta \in L$, is a $\mathbb{Z}$-bilinear form satisfying the following condition:

$$\epsilon_0(\alpha, \alpha) = \frac{1}{2} \langle \alpha, \alpha \rangle.$$

Hence, we have $\epsilon_0(\alpha, \beta) - \epsilon_0(\beta, \alpha) = c_0(\alpha, \beta)$ for $\alpha, \beta \in L$ (see [27]).

Set $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend the $\mathbb{Z}$-bilinear form on $L$ to $\mathfrak{h}$ by $\mathbb{C}$-linearity. The corresponding affine Lie algebra is $\widehat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ with Lie brackets

$$[x(m), y(n)] = \langle x, y \rangle m \delta_{m+n,0} c,$$

$$[c, \widehat{\mathfrak{h}}] = 0,$$

for $x, y \in \mathfrak{h}$ and $m, n \in \mathbb{Z}$, where $x(n)$ denotes $x \otimes t^n$. Set

$$\widehat{\mathfrak{h}}^- = \mathfrak{h} \otimes t^{-1} \mathbb{C}[t^{-1}].$$

Hence, $\widehat{\mathfrak{h}}^-$ is an abelian subalgebra of $\widehat{\mathfrak{h}}$. We then consider the induced $\widehat{\mathfrak{h}}$-module

$$M(1) = U(\widehat{\mathfrak{h}}) \otimes_{U(C[t, t^{-1}] \otimes \mathbb{C}c)} \mathbb{C} \cong S(\widehat{\mathfrak{h}}^-) \quad (\text{linearly}),$$

where $U(.)$ denotes the universal enveloping algebra and $C[t] \otimes \mathfrak{h}$ acts trivially on $\mathbb{C}$, $c$ acts on $\mathbb{C}$ as multiplication by 1.

Consider the $\widehat{L}$-module

$$\mathbb{C}\{L\} = C[\widehat{L}]/C[\widehat{L}](\kappa + 1),$$

where $\mathbb{C}\{\cdot\}$ denotes the group algebra. For $a \in \widehat{L}$, we use $\iota(a)$ to denote the image of $a$ in $\mathbb{C}\{L\}$. Then the action of $\widehat{L}$ on $\mathbb{C}\{L\}$ is given by

$$a \cdot \iota(b) = \iota(ab), \quad \kappa \cdot \iota(b) = -\iota(b).$$
for \(a, b \in \hat{L}\). For a formal variable \(z\) and an element \(h \in \mathfrak{h}\), we define an operator \(h(0)\) on \(\mathbb{C}\{L\}\) by \(h(0) \cdot \iota(a) = \langle h, \bar{a} \rangle \iota(a)\) and an action \(z^h\) on \(\mathbb{C}\{L\}\) by \(z^h \cdot \iota(a) = z^{(h, \bar{a})} \iota(a)\).

Set
\[
V_L = M(1) \otimes_{\mathbb{C}} \mathbb{C}\{L\}.
\]

Then \(\hat{L}, h(n)(n \neq 0), h(0)\) and \(z^h\) act naturally on \(V_L\) by acting on either \(M(1)\) or \(\mathbb{C}\{L\}\) as indicated above. Denote \(\iota(1)\) by \(1\) and set
\[
\omega = \frac{1}{2} \sum_{i=1}^{d} h_i (-1)^{i-1} 1,
\]
where \(h_1, ..., h_d\) is an orthonormal basis of \(\mathfrak{h}\). Then we know that \((V_L, Y(. , z), 1, \omega)\) has a vertex operator algebra structure (see [6, 27]), the vertex operator \(Y(. , z)\) is determined by
\[
Y(h(-1)1, z) = h(z) = \sum_{n \in \mathbb{Z}} h(n) z^{-n-1} (h \in \mathfrak{h}),
\]
\[
Y(a, z) = E^-(\bar{a}, z) E^+(\bar{a}, z) a z^\bar{a} \quad (a \in \hat{L}),
\]
where
\[
E^-(\bar{a}, z) = \exp\left(\sum_{n<0} \frac{\bar{a}(n)}{n} z^{-n}\right), \quad E^+(\bar{a}, z) = \exp\left(\sum_{n>0} \frac{\bar{a}(n)}{n} z^{-n}\right).
\]

3.4. Automorphisms of mirror extension \(\hat{L}_{sl_9}(3, 0)\). We now consider the root lattice of type \(A_8\) and let \(L\) be the positive definite even lattice isomorphic to \(A_8 \oplus A_8 \oplus A_8\).

Let \(\eta_i, i = 1, 2, 3\), be the natural inclusion of \(A_8\) into the \(i\)-th summand of \(L = A_8 \oplus A_8 \oplus A_8\).

Let \((A_8)_2 = \{\alpha \in A_8 \mid \langle \alpha, \alpha \rangle = 2\}\). Set \(\hat{h} = (\eta_1 + \eta_2 + \eta_3)(h)(-1) \cdot 1 \in V_L\) for any \(h \in A_8 \otimes_{\mathbb{Z}} \mathbb{C}\) and
\[
E_{\alpha} = \iota(e_{\eta_1(\alpha)}) + \iota(e_{\eta_2(\alpha)}) + \iota(e_{\eta_3(\alpha)}) \in V_L \quad \text{for} \quad \alpha \in (A_8)_2.
\]

Then the vertex operator subalgebra \(\langle S \rangle\) of \(V_L\) generated by
\[
S = \{\hat{h} | h \in A_8 \otimes_{\mathbb{Z}} \mathbb{C}\} \cup \{E_{\alpha} | \alpha \in A_8, \langle \alpha, \alpha \rangle = 2\}
\]
is isomorphic to the affine VOA \(L_{sl_9}(3, 0)\) (see [13, 27]). Moreover, it was proved in [27] that \(\{\hat{h} | h \in A_8 \otimes_{\mathbb{Z}} \mathbb{C}\}\) and \(\{E_{\alpha} | \alpha \in (A_8)_2\}\) satisfy the following relations
\[
[\hat{h}, E_{\alpha}] = \langle h, \alpha \rangle E_{\alpha},
\]
\[
[E_{\alpha}, E_{-\alpha}] = (-1)^{\langle \alpha, -\alpha \rangle} \hat{h},
\]
\[
[E_{\alpha}, E_{\beta}] = (-1)^{\langle \alpha, \beta \rangle} E_{\alpha + \beta}, \quad \text{if} \quad \alpha + \beta \in (A_8)_2,
\]
\[
[E_{\alpha}, E_{\beta}] = 0, \quad \text{otherwise}.
\]
Let $\Omega$ be the conformal element of $L_{\text{sl}_9}(3,0)$. By the Sugawara construction, it is given by

$$\Omega = \frac{1}{2(3+9)} \left[ \sum_{k=1}^{8} \hat{h}^k + \sum_{\alpha \in (A_8)_2} (E_\alpha)_{-1}(-E_\alpha) \right],$$

where $\{h^1,\ldots,h^8\}$ is an orthonormal basis of $A_8 \otimes \mathbb{C}$. Note that the dual vector of $E_\alpha$ is $-E_\alpha$ since $\epsilon_0(\alpha,\alpha) = 1$.

Let $E = \{ (\alpha, \alpha, \alpha) \mid \alpha \in A_8 \}$ and $P = \{ (\alpha, \beta, \gamma) \mid \alpha, \beta, \gamma \in A_8, \alpha + \beta + \gamma = 0 \}$. Then, by a direct calculation (see [9]), we have

$$\hat{\theta}$$

may be lifted to an automorphism of $V_M$. Moreover, by a direct calculation (see [9]), we have

$$\Omega = \omega_E + \frac{3}{4}\omega_P - \frac{1}{12} \sum_{\alpha \in (A_8)_2} E_{\eta_k(n_\alpha)} - \eta_j(n_\alpha),$$

where $\omega_S$ denotes the conformal element of the lattice VOA $V_S$.

Let $\theta : \hat{L} \to \hat{L}$ be the automorphism of $\hat{L}$ defined by $\theta(a) = \kappa^{e_0(\bar{\alpha},\bar{\alpha})}a^{-1}$. It was shown in [27] that $\theta$ induces an automorphism of $V_L$ which is also denoted by $\theta$. Explicitly, $\theta : V_L \to V_L$ is the linear map defined by

$$\theta(\alpha_1(-n_1) \cdots \alpha_k(-n_k) \otimes \iota(a)) = (-1)^k \alpha_1(-n_1) \cdots \alpha_k(-n_k) \otimes \iota(\theta(a)).$$

In particular, we have $\theta(\iota(e_{(\alpha,\alpha,0)})) = -\iota(e_{(-\alpha,\alpha,0)})$, $\theta(\iota(e_{(0,0,\alpha)})) = -\iota(e_{(0,0,-\alpha)})$ for any $\alpha \in (A_8)_2$. Therefore, we have $\theta(\langle S \rangle) = \langle S \rangle$, that is, $\theta|_{\langle S \rangle}$ induces an automorphism of $L_{\text{sl}_9}(3,0)$, which is also denoted by $\theta$.

As a key result of this section, we shall show that the automorphism $\theta$ of $L_{\text{sl}_9}(3,0)$ may be lifted to an automorphism of $\overline{L_{\text{sl}_9}(3,0)}$.

First, we have the following general result which is a slight generalization of Proposition 3.2 of [59].

**Theorem 3.7.** Let $V$ be a strongly regular vertex operator algebra. Let $g$ be an automorphism of $V$ and $U$ an extension vertex operator algebra of $V$. Assume further that $V$, $g$ and $U$ satisfy the following conditions:

1. $U$ viewed as a $V$-module has the decomposition
   $$U = M^0 \oplus M^1 \oplus \cdots \oplus M^k,$$
   such that $M^0 = V, M^1, \cdots, M^k$ are nonisomorphic irreducible $V$-modules. Moreover,
   $$\{M^0, M^1, \cdots, M^k\} = \{M^0 \circ g, M^1 \circ g, \cdots, M^k \circ g\},$$
   where $M^i \circ g$ denotes the $V$-module such that $M^i \circ g = M^i$ as vector space and the vertex operator $Y_{M^i \circ g}(v, z) = Y_{M^i}(g(v), z)$.

2. For any two strongly regular VOA structures $(U, Y_1(\cdot, z))$ and $(U, Y_2(\cdot, z))$ on $U =
Proof: The idea of the proof is similar to that in Theorem 2.1 of [60]. Let \( \Psi : \tilde{\Psi} \) be the permutation such that \( (U, Y_U(\cdot, z)) \) is also an extension vertex operator algebra of \( V \). Then by assumption (2), there exists a VOA isomorphism \( \Phi : (\mathcal{O}_V, 0) \rightarrow (\Phi(\mathcal{O}_V), \Phi(0)) \) such that \( \Phi|_{\mathcal{O}_V} = g^{-1} \). Hence, \( (U, Y_U(\cdot, z)) \) is also a strongly regular vertex operator algebra. Therefore, \( (\Phi(\mathcal{O}_V), \Phi(0)) \) is simple. Otherwise, assume that \( I \) is a proper ideal of \( (U, Y_U(\cdot, z)) \), it is clear that \( \Phi(I) \) is a proper ideal of \( (U, Y_U(\cdot, z)) \), this is a contradiction. By Theorem 2.4, we know that \( (U, Y_U(\cdot, z)) \) is also a strongly regular vertex operator algebra. Then by assumption (2), there exists a linear isomorphism \( \Psi : U \rightarrow U \) such that \( \Psi|_{\mathcal{O}_V} = \text{id} \) and that for any \( u, v \in \mathcal{O}_V \), it is easy to verify that \( (U, \tilde{Y}_U(\cdot, z)) \) is also a vertex operator algebra. Moreover, we have \( \Phi|_{\mathcal{O}_V} = g^{-1} \); it implies \( \tilde{Y}_U(u, z) = Y_U(u, z) \). Hence, \((U, \tilde{Y}_U(\cdot, z))\) is also an extension vertex operator algebra of \( V \). We next prove that \( (U, \tilde{Y}_U(\cdot, z)) \) is simple. Otherwise, assume that \( I \) is a proper ideal of \( (U, \tilde{Y}_U(\cdot, z)) \), it is clear that \( \Phi(I) \) is a proper ideal of \( (U, Y_U(\cdot, z)) \), this is a contradiction. By Theorem 2.4, we know that \( (U, Y_U(\cdot, z)) \) is also a strongly regular vertex operator algebra. Then by assumption (2), there exists a linear isomorphism \( \Psi : U \rightarrow U \) such that \( \Psi|_{\mathcal{O}_V} = \text{id} \) and that

\[
\Psi(\tilde{Y}_U(u^1, z)u^2) = Y_U(\Psi(u^1), z)\Psi(u^2),
\]

for any \( u^1, u^2 \in \mathcal{O}_V \). In particular, we have

\[
\Psi(\Phi^{-1}Y_U(\Phi(u^1), z)\Phi(u^2)) = Y_U(\Phi(u^1), z)\Psi(u^2),
\]

for any \( u^1, u^2 \in \mathcal{O}_V \). This implies that \( \Psi \circ \Phi \) is an automorphism of \( U \) such that \( \Psi \circ \Phi|_{\mathcal{O}_V} = g^{-1} \). Therefore, \( (\Psi \circ \Phi)^{-1} \) is the desired automorphism. The proof is complete. □

To prove that the automorphism \( \theta \) of \( L_{sl_2}(3, 0) \) can be lifted to an automorphism of \( \tilde{L}_{sl_2}(3, 0) \), we need the following:
Lemma 3.8. Let $L_{sl_9}(3, L(\tilde{\Lambda}))$ be an irreducible module $L_{sl_9}(3, 0)$. Then we have
\[ L_{sl_9}(3, L(\tilde{\Lambda})) \circ \theta \cong L_{sl_9}(3, L(\tilde{\Lambda})^\ast) \cong L_{sl_9}(3, L(\tilde{\Lambda}))', \]
where $L(\tilde{\Lambda})^\ast$ denotes the dual module of $L(\tilde{\Lambda})$.

**Proof:** For any $x \in sl_9$, let $Y_{L_{sl_9}(3, L(\tilde{\Lambda})) \circ \theta}(x, z) = \sum_{n \in \mathbb{Z}} x^\theta(n)z^{-n-1}$. By the definition of $L_{sl_9}(3, L(\tilde{\Lambda})) \circ \theta$, we have
\[ x^\theta(n)v = \theta(x)(n)v \]
for any $x \in sl_9$, $v \in L_{sl_9}(3, L(\tilde{\Lambda}))$ and $n \in \mathbb{Z}$. In particular, we have
\[ \tilde{h}^\theta(n)v = -\tilde{h}(n)v, \quad E_\alpha^\theta(n)v = -E_-\alpha(n)v, \] (3.1)
for any $\tilde{h} \in A_\tilde{h} \otimes \mathbb{C}$ and $\alpha \in (A_{\tilde{h}})_2$.

For any irreducible $L_{sl_9}(3, 0)$-module $M$, set
\[ \Omega(M) = \{ v \in M \mid x(n)v = 0 \text{ for } x \in sl_9, n \geq 1 \}. \]
Then we know that $\Omega(M)$ is an irreducible $sl_9$-module (see [28]). Since $L_{sl_9}(3, L(\tilde{\Lambda}))$ is irreducible, we know that $L_{sl_9}(3, L(\tilde{\Lambda})) \circ \theta$ is also an irreducible $L_{sl_9}(3, 0)$-module. In particular, $\Omega(L_{sl_9}(3, L(\tilde{\Lambda})) \circ \theta)$ is an irreducible $sl_9$-module. By formula (3.1), we know that $\Omega(L_{sl_9}(3, L(\tilde{\Lambda})) \circ \theta)$ is a lowest weight $sl_9$-module with lowest weight $-\tilde{\Lambda}$. Hence, we have $\Omega(L_{sl_9}(3, L(\tilde{\Lambda})) \circ \theta) \cong L(\tilde{\Lambda})^\ast$. In particular, $L_{sl_9}(3, (\tilde{\Lambda})) \circ \theta \cong L_{sl_9}(3, L(\tilde{\Lambda}))'$.

On the other hand, by the definition of $L_{sl_9}(3, L(\tilde{\Lambda}))'$, we have
\[ \langle x(n)f, v \rangle = \langle f, -x(-n)v \rangle, \] (3.2)
for any $x \in sl_9$, $f \in L_{sl_9}(3, L(\tilde{\Lambda}))'$ and $n \in \mathbb{Z}$. Since $L_{sl_9}(3, L(\tilde{\Lambda}))'$ is an irreducible $L_{sl_9}(3, 0)$-module, we know that $\Omega(L_{sl_9}(3, L(\tilde{\Lambda}))')$ viewed as a vector space is equal to $L(\tilde{\Lambda})^\ast$ and is an irreducible $sl_9$-module. It follows from the formula (3.2) that $\Omega(L_{sl_9}(3, L(\tilde{\Lambda})))$ viewed as an irreducible $sl_9$-module is equal to $L(\tilde{\Lambda})^\ast$. Hence, we have $L_{sl_9}(3, L(\tilde{\Lambda})^\ast) \cong L_{sl_9}(3, L(\tilde{\Lambda}))'$. The proof is complete.

Combining Theorems 3.6, 3.7 and Lemma 3.8 we have:

Theorem 3.9. There exists an automorphism $\tilde{\theta}$ of $L_{sl_9}(3, 0)$ such that $\tilde{\theta}(L_{sl_9}(3, 0)) = L_{sl_9}(3, 0)$ and that $\tilde{\theta}|_{L_{sl_9}(3, 0)} = \theta$.

**Proof:** By Theorems 3.6, 3.7 it is sufficient to verify that
\[ \{ L_{sl_9}(3, 0), L_{sl_9}(3, 3\tilde{\Lambda}_3), L_{sl_9}(3, 3\tilde{\Lambda}_5), L_{sl_9}(3, \tilde{\Lambda}_1 + \tilde{\Lambda}_2 + \tilde{\Lambda}_6), L_{sl_9}(3, \tilde{\Lambda}_3 + \tilde{\Lambda}_7 + \tilde{\Lambda}_8), L_{sl_9}(3, \tilde{\Lambda}_4 + \tilde{\Lambda}_5) \} \]
\[ = \{ L_{sl_9}(3, 0) \circ \theta, L_{sl_9}(3, 3\tilde{\Lambda}_3) \circ \theta, L_{sl_9}(3, 3\tilde{\Lambda}_5) \circ \theta, L_{sl_9}(3, \tilde{\Lambda}_1 + \tilde{\Lambda}_2 + \tilde{\Lambda}_6) \circ \theta, \]
\[ L_{sl_9}(3, \tilde{\Lambda}_3 + \tilde{\Lambda}_7 + \tilde{\Lambda}_8) \circ \theta, L_{sl_9}(3, \tilde{\Lambda}_4 + \tilde{\Lambda}_5) \circ \theta \}. \] (3.3)

Note that $L_{sl_9}(3, 0)$ is self-dual, the formula (3.3) follows immediately from Lemma 3.8. The proof is complete.
4. Holomorphic VOA of central charge 24 with Lie algebra $A_{8,3}A_{2,1}^2$

In this section, we shall study the holomorphic vertex operator algebra of central charge 24 with Lie algebra $A_{8,3}A_{2,1}^2$. Recall from [41] that there exists a holomorphic vertex operator algebra $U$ such that the central charge of $U$ is 24 and the Lie algebra $U_1$ is isomorphic to $A_{8,3}A_{2,1}^2$. Moreover, it was also proved in [41] that $U$ is strongly regular. We shall study the structure of $U$ in this section.

4.1. Vertex operator subalgebras of $U$. In this subsection, we shall study some vertex operator subalgebras of $U$. Note that $U$ contains a vertex operator subalgebra isomorphic to $L_{sl_9}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$. Set $V^2 = C_U(L_{sl_9}(1,0) \otimes L_{sl_3}(1,0))$. Our goal in this subsection is to determine the structure of the vertex operator subalgebra $V^2$.

**Lemma 4.1.** The vertex operator algebra $V^2$ is strongly regular.

**Proof:** Since $U$ is of CFT-type, it follows that $V^2$ is of CFT-type. Note also that $V^2$ contains the vertex operator subalgebra $L_{sl_9}(3,0)$, hence $V^2$ is an extension vertex operator algebra of $L_{sl_9}(3,0)$. By Theorem 2.3 to prove that $V^2$ is strongly regular, it is sufficient to show that $V^2$ is simple. Note that $L_{sl_3}(1,0)$ is isomorphic to the lattice vertex operator algebra $V_{A_2}$, where $A_2$ denotes the root lattice of type $A_2$. Let $G$ be the dual group of $(A_2 \oplus A_2)^*/(A_2 \oplus A_2)$. Then we know that there is an action of $G$ on $U$ such that $U^G = V^2 \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$. It follows that $V^2 \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$ is simple (see [22]). This implies that $V^2$ is simple. The proof is complete. \qed

We next determine the global dimension of $V^2$. The following result was proved in [41].

**Theorem 4.2.** Let $V$ be a strongly regular vertex operator algebra and $W$ a strongly regular vertex operator subalgebra of $V$. Suppose also that the commutant $C_V(W)$ of $W$ is strongly regular and satisfies $C_V(C_V(W)) = W$. Then all the irreducible $W$-modules appear in some simple $V$-modules.

To apply Theorem 4.2 we also need to determine the commutant $C_U(V^2)$ of $V^2$. Let $\Lambda_1, \Lambda_2$ be the fundamental weights of $sl_3$. It is well-known that $L_{sl_3}(1,0)$ has three nonisomorphic irreducible modules $L_{sl_3}(1,0)$, $L_{sl_3}(1,\Lambda_1)$, and $L_{sl_3}(1,\Lambda_2)$. The conformal weights of $L_{sl_3}(1,0)$, $L_{sl_3}(1,\Lambda_1)$, $L_{sl_3}(1,\Lambda_2)$ are equal to 0, $1/3$, $1/3$, respectively. Moreover, $L_{sl_3}(1,\Lambda_1)$, $L_{sl_3}(1,\Lambda_2)$ are simple current $L_{sl_3}(1,0)$-modules such that $L_{sl_3}(1,\Lambda_1) \times L_{sl_3}(1,\Lambda_1) \cong L_{sl_3}(1,\Lambda_2)$ (see [13], [51]). In particular, the fusion ring of $L_{sl_3}(1,0)$ is isomorphic to $\mathbb{Z}_3$.\n


Lemma 4.3. The commutant $C_U(V^2)$ of $V^2$ is equal to $L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$.

Proof: Note that $L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$ is a vertex operator subalgebra of $C_U(V^2)$. Hence, $C_U(V^2)$ is an extension vertex operator algebra of $L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$. Comparing the conformal weights of irreducible $L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$-modules, we know that $C_U(V^2)$ is equal to $L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$. □

Hence, by Theorems 2.6, 2.10, 4.2 and Lemmas 4.1, 4.3 we have the following:

Theorem 4.4. (1) All the irreducible $V^2$-modules appear in $U$. Moreover, there are 9 nonisomorphic irreducible $V^2$-modules.

(2) All the irreducible $V^2$-modules are simple current modules. In particular, the fusion ring of $V^2$ is isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ and $\text{Glob} V^2 = 9$.

To determine the vertex operator algebra structure of $V^2$, note first that $V^2$ gives rise to a modular invariant of $L_{sl_9}(3,0)$ by Theorem 2.6. On the other hand, the modular invariants of the affine vertex operator algebra $L_{sl_9}(3,0)$ have been classified in [31]. To describe the result, we need to recall some notations from [31]. Recall that irreducible $L_{sl_9}(3,0)$-modules are parameterized by $C_{9,3}$. For any $a = (a_0, a_1, \ldots, a_8) \in C_{9,3}$, we use $Z_a$ to denote the trace function associated to the irreducible $L_{sl_9}(3,0)$-module $L_{sl_9}(3,a_1\Lambda_1 + \cdots + a_8\Lambda_8)$ and $(Z_a)$ to denote $\sum_{j=1}^{3} Z_{\rho_{9}^j(\cdot)}$, where $\rho_9 : C_{9,3} \to C_{9,3}$ is the map defined in subsection 2.4. Let $C : C_{9,3} \to C_{9,3}$ be the map defined by $C((a_0, a_1, a_2, \ldots, a_7, a_8)) = (a_0, a_8, a_7, \ldots, a_2, a_1).$ We then define a matrix $C$ by $C_{a,b} = \delta_{b,C(a)}$. For a modular invariant $\mathcal{M}$ of the affine vertex operator algebra $L_{sl_9}(3,0)$, we define the conjugate modular invariant of $\mathcal{M}$ to be the matrix $CM$ (cf. [31]).

Theorem 4.5. Any modular invariant of the affine vertex operator algebra $L_{sl_9}(3,0)$ is equal to one of the following modular invariants or their conjugate modular invariants:

| $A_{a,b} = \delta_{a,b}$ | $A$ |
|------------------------|-----|
| $D_{a,b}^{(1)} = \sum_{j=1}^{2} \delta^4(|d_{a,j}(a)| + 6j)\delta_{b,\rho_{9}^j(a)}$ | $D^{(1)}$ |
| $D_{a,b}^{(2)} = \sum_{j=1}^{2} \delta^3(|d_{a,j}(a)| + 18j)\delta_{b,\rho_{9}^{2j}(a)}$ | $D^{(2)}$ |
| $D_{a,b}^{(3)} = \delta^3(|d_{a,3}(a)| + 54)\delta_{b,\rho_{9}^{3}(a)}$ | $D^{(3)}$ |

where $a, b \in C_{9,3}$, $\delta^2(y) = 1$ if $y/z \in \mathbb{Z}$ and $\delta^2(y) = 0$ if $y/z \notin \mathbb{Z}$. Here, the modular invariants of types $\mathcal{E}, \mathcal{E}', \mathcal{E}''$ should be interpreted similarly as the modular invariants in Subsection 2.4.
To determine the modular invariant corresponding to $V^2$, we need the following:

**Lemma 4.6.** The modular invariant corresponding to $V^2$ is not equal to the modular invariant of type $E''$, $CE'$ or $CE''$.

**Proof:** By the definition of conjugate modular invariant, the modular invariant of type $CE'$ is equal to

\[
|\langle Z_{(0,0,0,0,0,0,0,0,0,0)} \rangle + \langle Z_{(1,0,0,0,0,0,0,0,0,0)} \rangle|\langle Z_{(1,0,1,0,0,0,0,0,0,0)} \rangle|^2 + |\langle Z_{(0,1,0,0,0,0,0,0,0,0)} \rangle|^2
\]

\[
+ |\langle Z_{(0,1,0,0,0,0,0,0,0,0)} \rangle + \langle Z_{(0,0,1,0,0,0,0,0,0,0)} \rangle |\langle Z_{(0,0,0,0,0,0,0,0,0,0)} \rangle + \langle Z_{(0,1,0,0,0,0,0,0,0,0)} \rangle|
\]

\[
+ |\langle Z_{(0,0,0,0,0,0,0,0,0,0)} \rangle + \langle Z_{(0,1,0,0,0,0,0,0,0,0)} \rangle |\langle Z_{(0,0,0,0,0,0,0,0,0,0)} \rangle + \langle Z_{(0,1,0,0,0,0,0,0,0,0)} \rangle|
\]

\[
+ |\langle Z_{(1,0,0,0,0,0,0,0,0,0)} \rangle + \langle Z_{(1,0,0,0,0,0,0,0,0,0)} \rangle |\langle Z_{(1,0,0,0,0,0,0,0,0,0)} \rangle + \langle Z_{(1,0,0,0,0,0,0,0,0,0)} \rangle|.
\]

Note that the modular invariant of type $CE'$ contains $Z_{(0,3,0,0,0,0,0,0,0,0,0)} Z_{(0,3,0,0,0,0,0,0,0,0,0)}$, but does not contain the term $Z_{(0,3,0,0,0,0,0,0,0,0,0)} Z_{(0,3,0,0,0,0,0,0,0,0,0)}$, it follows from Lemma 2.7 that the modular invariant of type $CE'$ cannot be realized by extension vertex operator algebra.

Note also that the map $C : C_{9,3} \to C_{9,3}$ maps \(\{\rho_j((1,0,1,0,1,0,0,0,0,0))|0 \leq i \leq 8\}\) to itself, and that

\[
\{\rho_j((1,0,1,0,1,0,0,0,0,0))|0 \leq i \leq 8\} \cap \{\rho_j((1,0,0,0,1,1,0,0,0,0))|0 \leq i \leq 8\} = \emptyset.
\]

It follows that the modular invariants of types $E''$ and $CE''$ contain

\[
Z_{\rho_j(0,1,0,1,0,1,0,0,0)} Z_{(0,1,0,0,0,1,1,0,0)},
\]

for some $j$, but does not contain the term $Z_{(0,1,0,0,0,1,1,0,0)} Z_{(0,1,0,0,0,1,1,0,0)}$. By Lemma 2.7, the modular invariants of types $E''$ and $CE''$ cannot be realized by extension vertex operator algebras. The proof is complete. \(\square\)

Combining Lemmas 4.1, 4.6 and Theorem 4.5 we have:

**Theorem 4.7.** The vertex operator algebra $V^2$ is isomorphic to $L_{sl_2}(3,0)$.

**Proof:** The idea is to show that the modular invariant associated to $V^2$ is equal to the modular invariant of type $E'$. Note first that the modular invariant associated to $V^2$ cannot be equal to the modular invariants of types $A$, $D^{(1)}$, $D^{(3)}$, $D^{(9)}$, $E$ or their conjugate modular invariants. Otherwise, $V^2$ must be a simple current extension of $L_{sl_2}(3,0)$. However, there are only three simple current $L_{sl_2}(3,0)$-modules $L_{sl_2}(3,0)$, $L_{sl_2}(3,3\bar{A}_3)$, $L_{sl_2}(3,3\bar{A}_5)$ that have integral conformal weights. This implies the global dimension of $L_{sl_2}(3,0)$ must be less than 81 by Theorem 2.3. Since $L_{sl_2}(3,0)$ has more than 81 nonisomorphic irreducible modules, this is a contradiction by Theorem 2.2. By Lemma 4.6 the modular invariant associated to $V^2$ must be equal to the modular
invariant of type $\mathcal{E}'$. It follows immediately that $V^2$ viewed as a module of $L_{sl_9}(3, 0)$ is isomorphic to

$$L_{sl_9}(3, 0) \oplus L_{sl_9}(3, 3\tilde{\Lambda}_3) \oplus L_{sl_9}(3, 3\tilde{\Lambda}_6) \oplus L_{sl_9}(3, \tilde{\Lambda}_1 + \tilde{\Lambda}_2 + \tilde{\Lambda}_6)$$

$$\oplus L_{sl_9}(3, \tilde{\Lambda}_3 + \tilde{\Lambda}_7 + \tilde{\Lambda}_8) \oplus L_{sl_9}(3, \tilde{\Lambda}_4 + \tilde{\Lambda}_5).$$

By Theorem 4.8, we know that $V^2$ is isomorphic to $L_{sl_9}(3, 0)$. The proof is complete. □

Since the modular invariant associated to $V^2$ is equal to the modular invariant of type $\mathcal{E}'$, we can obtain the following results immediately.

**Theorem 4.8.** (1) There exists a unique $\hat{L}_{sl_9}(3, 0)$-module structure on each of the following $\hat{L}_{sl_9}(3, 0)$-modules

$$\hat{L}_{sl_9}(3, 0) = L_{sl_9}(3, 0) \oplus L_{sl_9}(3, 3\tilde{\Lambda}_3) \oplus L_{sl_9}(3, 3\tilde{\Lambda}_6) \oplus L_{sl_9}(3, \tilde{\Lambda}_1 + \tilde{\Lambda}_2 + \tilde{\Lambda}_6)$$

$$\oplus L_{sl_9}(3, \tilde{\Lambda}_3 + \tilde{\Lambda}_7 + \tilde{\Lambda}_8) \oplus L_{sl_9}(3, \tilde{\Lambda}_4 + \tilde{\Lambda}_5),$$

$$\hat{L}_{sl_9}(3, 3\tilde{\Lambda}_1) = L_{sl_9}(3, 3\tilde{\Lambda}_1) \oplus L_{sl_9}(3, 3\tilde{\Lambda}_4) \oplus L_{sl_9}(3, 3\tilde{\Lambda}_7) \oplus L_{sl_9}(3, \tilde{\Lambda}_1 + \tilde{\Lambda}_5 + \tilde{\Lambda}_6)$$

$$\oplus L_{sl_9}(3, \tilde{\Lambda}_2 + \tilde{\Lambda}_3 + \tilde{\Lambda}_7) \oplus L_{sl_9}(3, \tilde{\Lambda}_4 + \tilde{\Lambda}_8),$$

$$\hat{L}_{sl_9}(3, 3\tilde{\Lambda}_2) = L_{sl_9}(3, 3\tilde{\Lambda}_2) \oplus L_{sl_9}(3, 3\tilde{\Lambda}_5) \oplus L_{sl_9}(3, 3\tilde{\Lambda}_8) \oplus L_{sl_9}(3, \tilde{\Lambda}_2 + \tilde{\Lambda}_6 + \tilde{\Lambda}_7)$$

$$\oplus L_{sl_9}(3, \tilde{\Lambda}_3 + \tilde{\Lambda}_4 + \tilde{\Lambda}_8) \oplus L_{sl_9}(3, \tilde{\Lambda}_1 + \tilde{\Lambda}_5).$$

(2) There exist two nonisomorphic $\hat{L}_{sl_9}(3, 0)$-module structures on each of the following $\hat{L}_{sl_9}(3, 0)$-modules

$$L_{sl_9}(3, \tilde{\Lambda}_2 + \tilde{\Lambda}_7) \oplus L_{sl_9}(3, \tilde{\Lambda}_1 + \tilde{\Lambda}_3 + \tilde{\Lambda}_5) \oplus L_{sl_9}(3, \tilde{\Lambda}_4 + \tilde{\Lambda}_6 + \tilde{\Lambda}_8),$$

$$L_{sl_9}(3, \tilde{\Lambda}_1 + \tilde{\Lambda}_3 + \tilde{\Lambda}_8) \oplus L_{sl_9}(3, \tilde{\Lambda}_2 + \tilde{\Lambda}_4 + \tilde{\Lambda}_6) \oplus L_{sl_9}(3, \tilde{\Lambda}_5 + \tilde{\Lambda}_7),$$

$$L_{sl_9}(3, \tilde{\Lambda}_2 + \tilde{\Lambda}_4) \oplus L_{sl_9}(3, \tilde{\Lambda}_3 + \tilde{\Lambda}_5 + \tilde{\Lambda}_7) \oplus L_{sl_9}(3, \tilde{\Lambda}_1 + \tilde{\Lambda}_6 + \tilde{\Lambda}_8).$$

4.2. **Fusion ring of mirror extension** $L_{sl_9}(3, 0)$. In Subsection 4.1, we already knew that the fusion ring of $L_{sl_9}(3, 0)$ is isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. In this subsection, we shall determine the fusion ring of $L_{sl_9}(3, 0)$ explicitly. First, we need to recall some facts about the Weyl group of $sl_{n+1}$. Consider the $(n+1)$-dimensional euclidean space $\mathbb{R}^{n+1}$. Let $\epsilon_1, \cdots, \epsilon_{n+1}$ be the standard basis of $\mathbb{R}^{n+1}$. It is well-known that

$$\{\epsilon_i - \epsilon_j | 1 \leq i \neq j \leq n + 1\}$$

forms a root system of type $A_n$ and that the fundamental weights are given by

$$\tilde{\Lambda}_i = \frac{1}{n+1}((n+1-i)(\epsilon_1 + \cdots + \epsilon_i) - i(\epsilon_i + \cdots + \epsilon_{n+1})).$$
1 ≤ i ≤ n (see [36]). It is also known that the Weyl group of the root system of type $A_n$ is isomorphic to the permutation group $S_{n+1}$. In particular, the reflection associated to the root $\epsilon_i - \epsilon_j$ corresponds to the permutation $(i, j)$ (see [36]). The longest element of the Weyl group of root system of type $A_n$ was also determined in [7], [36]. In particular, we have:

**Lemma 4.9.** The longest element $w_0$ of the Weyl group of root system of type $A_8$ is equal to the permutation

\[
1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \\
9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2 \ 1.
\]

In particular, we have $w_0(\Lambda_i) = -\Lambda_{9-i}$ for $1 ≤ i ≤ 8$.

We now let $L(\Lambda)$ be the irreducible highest weight module of $sl_9$ with highest weight $\Lambda$. It is well-known that the dual module $L(\Lambda)^* \text{ of } L(\Lambda)$ is isomorphic to $L(-w_0(\Lambda))$ (see [7], [35]). Hence, we have the following:

**Lemma 4.10.** Let $L(\Lambda)$ be the irreducible highest weight module of $sl_9$ with highest weight $\Lambda$. Then we have $L_{sl_9}(3, L(\Lambda))^* \cong L_{sl_9}(3, L(-w_0(\Lambda)))$.

We now turn back to determine the fusion ring of $L_{sl_9}(3, 0)$. Recall that there exist two nonisomorphic $L_{sl_9}(3, 0)$-module structures on

\[ L_{sl_9}(3, \Lambda_2 + \Lambda_7) \oplus L_{sl_9}(3, \Lambda_1 + \Lambda_3 + \Lambda_5) \oplus L_{sl_9}(3, \Lambda_4 + \Lambda_6 + \Lambda_9). \]

We denote them by $\tau_1$ and $\tau_2$, respectively. Then we have:

**Theorem 4.11.** Let $\tau_1$, $\tau_2$, $L_{sl_9}(3, 3\Lambda_1)$, $L_{sl_9}(3, 3\Lambda_2)$ be the irreducible $L_{sl_9}(3, 0)$-modules defined as before. Then we have

\[
\tau_1 \times \tau_1 \cong \tau_2, \\
\tau_1 \times \tau_2 \cong L_{sl_9}(3, 0), \\
L_{sl_9}(3, 3\Lambda_1) \times L_{sl_9}(3, 3\Lambda_1) \cong L_{sl_9}(3, 3\Lambda_2), \\
L_{sl_9}(3, 3\Lambda_1) \times L_{sl_9}(3, 3\Lambda_2) \cong L_{sl_9}(3, 0).
\]

Moreover, $L_{sl_9}(3, 3\Lambda_1) \times \tau_1$ and $L_{sl_9}(3, 3\Lambda_1) \times \tau_2$ viewed as $L_{sl_9}(3, 0)$-modules are isomorphic and $L_{sl_9}(3, 3\Lambda_2) \times \tau_1$ and $L_{sl_9}(3, 3\Lambda_2) \times \tau_2$ viewed as $L_{sl_9}(3, 0)$-modules are isomorphic.
Lemma 4.12. Modulo integers, the conformal weights of irreducible $\tilde{L}_{sl_2}(3, 3\Lambda_1)$-modules are as follows:

It follows immediately that

$$L_{sl_2}(3, 3\Lambda_1) \times L_{sl_2}(3, 3\Lambda_1) \cong L_{sl_2}(3, 3\Lambda_2), \quad L_{sl_2}(3, 3\Lambda_1) \times L_{sl_2}(3, 3\Lambda_2) \cong L_{sl_2}(3, 0).$$

We next show that $\tau_1 \times \tau_1 \cong \tau_2$. Recall that the fusion ring of $L_{sl_2}(3, 0)$ is isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, it is good enough to show that $\tau_1' \cong \tau_2$. Note that $\tau_1$ viewed as an $L_{sl_2}(3, 0)$-module is isomorphic to

$$L_{sl_2}(3, \Lambda_2 + \Lambda_7) \oplus L_{sl_2}(3, \Lambda_1 + \Lambda_3 + \Lambda_5) \oplus L_{sl_2}(3, \Lambda_4 + \Lambda_6 + \Lambda_8).$$

It follows immediately from Lemmas 4.9, 4.10 that $\tau_1' \cong \tau_1$ or $\tau_2$, which forces $\tau_1' \cong \tau_2$. Hence, we have

$$\tau_1 \times \tau_1 \cong \tau_2,$$

$$\tau_1 \times \tau_2 \cong L_{sl_2}(3, 0).$$

The last statement follows from the fact that $L_{sl_2}(3, 3\Lambda_1)$ and $L_{sl_2}(3, 3\Lambda_2)$ are simple current $L_{sl_2}(3, 0)$-modules. The proof is complete. 

4.3. Uniqueness of holomorphic VOA of central charge 24 with Lie algebra $A_{8,3}A_{2,1}$. In this subsection, we shall show that if $\tilde{U}$ is a holomorphic vertex operator algebra such that the central charge of $\tilde{U}$ is 24 and the Lie algebra $\tilde{U}_1$ is isomorphic to $A_{8,3}A_{2,1}$, then $\tilde{U}$ is isomorphic to $U$. As an application, we shall construct some special automorphism of $U$.

By the similar discussion as above, we know that $C_U(L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0))$ is also isomorphic to $L_{sl_2}(3, 0)$. Hence, $\tilde{U}$ and $U$ are extension vertex operator algebras of $L_{sl_2}(3, 0) \otimes L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)$. To determine the decompositions of $\tilde{U}$ and $U$ viewed as $L_{sl_2}(3, 0) \otimes L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)$-modules, we need the following:

Lemma 4.12. Modulo integers, the conformal weights of irreducible $\tilde{L}_{sl_2}(3, 0)$-modules are as follows:

| $L_{sl_2}(3, 3\Lambda_1)$ | $L_{sl_2}(3, 3\Lambda_2)$ | $\tau_1$ | $\tau_2$ | $L_{sl_2}(3, 3\Lambda_1) \times \tau_1$ | $L_{sl_2}(3, 3\Lambda_2) \times \tau_2$ | $L_{sl_2}(3, 3\Lambda_1) \times \tau_2$ | $L_{sl_2}(3, 3\Lambda_2) \times \tau_1$ | $\tau_1$ | $\tau_2$ | $L_{sl_2}(3, 3\Lambda_1) \times \tau_1$ | $L_{sl_2}(3, 3\Lambda_2) \times \tau_2$ |
|-------------------------|-------------------------|--------|--------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|--------|--------|--------------------------------|--------------------------------|
| $\uparrow$               | $\uparrow$              | $\uparrow$ | $\uparrow$ | $\uparrow$                  | $\uparrow$                  | $\uparrow$                  | $\uparrow$                  | $\downarrow$ | $\downarrow$ | $\downarrow$                  | $\downarrow$                  |

The following result follows immediately from Theorems 2.10, 4.11 and Lemma 4.12.
Lemma 4.13. The VOA $\tilde{U}$ viewed as an $L_{\text{sl}_3}(3, 0) \otimes L_{\text{sl}_3}(1, 0) \otimes L_{\text{sl}_3}(1, 0)$-module is isomorphic to one of the following:

\[ L_{\text{sl}_3}(3, 0) \otimes L_{\text{sl}_3}(1, 0) \otimes L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(3, 3\Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(3, 3\Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \]

\[ \oplus \tau_1 \otimes L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_1) \oplus L_{\text{sl}_3}(1, \Lambda_1) \oplus L_{\text{sl}_3}(1, \Lambda_1) \oplus L_{\text{sl}_3}(1, \Lambda_1) \oplus L_{\text{sl}_3}(1, \Lambda_1) \]

\[ \oplus (L_{\text{sl}_3}(3, 3\Lambda_2) \times \tau_1) \otimes L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(3, 3\Lambda_1) \times \tau_1 \]

\[ \oplus (L_{\text{sl}_3}(3, 3\Lambda_2) \times \tau_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, 0) = W^1, \]

\[ L_{\text{sl}_3}(3, 0) \oplus L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(3, 3\Lambda_1) \oplus L_{\text{sl}_3}(1, \Lambda_1) \oplus L_{\text{sl}_3}(3, 3\Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \]

\[ \oplus \tau_1 \oplus L_{\text{sl}_3}(1, \Lambda_1) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \]

\[ \oplus (L_{\text{sl}_3}(3, 3\Lambda_2) \times \tau_1) \otimes L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(3, 3\Lambda_1) \times \tau_1 \]

\[ \oplus (L_{\text{sl}_3}(3, 3\Lambda_2) \times \tau_2) \oplus L_{\text{sl}_3}(1, \Lambda_1) \oplus L_{\text{sl}_3}(1, 0) = W^2, \]

\[ L_{\text{sl}_3}(3, 0) \oplus L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(3, 3\Lambda_1) \oplus L_{\text{sl}_3}(1, \Lambda_1) \oplus L_{\text{sl}_3}(3, 3\Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \]

\[ \oplus \tau_1 \oplus L_{\text{sl}_3}(1, \Lambda_1) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \]

\[ \oplus (L_{\text{sl}_3}(3, 3\Lambda_2) \times \tau_1) \otimes L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(3, 3\Lambda_1) \times \tau_1 \]

\[ \oplus (L_{\text{sl}_3}(3, 3\Lambda_2) \times \tau_2) \oplus L_{\text{sl}_3}(1, \Lambda_1) \oplus L_{\text{sl}_3}(1, 0) = W^3, \]

\[ L_{\text{sl}_3}(3, 0) \oplus L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(3, 3\Lambda_1) \oplus L_{\text{sl}_3}(1, \Lambda_1) \oplus L_{\text{sl}_3}(3, 3\Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \]

\[ \oplus \tau_1 \oplus L_{\text{sl}_3}(1, \Lambda_1) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \]

\[ \oplus (L_{\text{sl}_3}(3, 3\Lambda_2) \times \tau_1) \otimes L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(3, 3\Lambda_1) \times \tau_1 \]

\[ \oplus (L_{\text{sl}_3}(3, 3\Lambda_2) \times \tau_2) \oplus L_{\text{sl}_3}(1, \Lambda_1) \oplus L_{\text{sl}_3}(1, 0) = W^4, \]

\[ L_{\text{sl}_3}(3, 0) \oplus L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(3, 3\Lambda_1) \oplus L_{\text{sl}_3}(1, \Lambda_1) \oplus L_{\text{sl}_3}(3, 3\Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \]

\[ \oplus \tau_1 \oplus L_{\text{sl}_3}(1, \Lambda_1) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \oplus L_{\text{sl}_3}(1, \Lambda_2) \]

\[ \oplus (L_{\text{sl}_3}(3, 3\Lambda_2) \times \tau_1) \otimes L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(3, 3\Lambda_1) \times \tau_1 \]

\[ \oplus (L_{\text{sl}_3}(3, 3\Lambda_2) \times \tau_2) \oplus L_{\text{sl}_3}(1, \Lambda_1) \oplus L_{\text{sl}_3}(1, 0) \oplus L_{\text{sl}_3}(1, 0) = W^5, \]
Proposition 4.14. Let \( V \) be a strongly regular vertex operator algebra and \( g \) an automorphism of \( V \). Let \( V^1, V^2 \) be extension vertex operator algebras of \( V \). Assume that \( V, g, V^1, V^2 \) satisfy the following conditions:

1. There exists a unique vertex operator algebra structure on \( V \) such that \( V^2 \) is an extension vertex operator algebra of \( V \).

2. \( V^1 \circ g \) viewed as a \( V \)-module is isomorphic to \( V^2 \). Moreover, there exists a \( V \)-module isomorphism \( \phi : V^1 \circ g \rightarrow V^2 \) such that \( \phi|_V = g^{-1} \).

Then the vertex operator algebra \( V^1 \) is isomorphic to \( V^2 \).

**Proof:** Let \( Y_1 \) and \( Y_2 \) be the vertex operator maps of \( V^1 \) and \( V^2 \), respectively. By assumption, there exists a \( V \)-module isomorphism \( \phi : V^1 \circ g \rightarrow V^2 \) such that \( \phi|_V = g^{-1} \). In particular, for any \( u, v \in V \), we have

\[
\phi(Y_1(\phi^{-1}(u), z)v) = Y_2(u, z)\phi(v).
\]

Define a linear map

\[
Y^g_2 : V^2 \rightarrow \text{End}(V^2)[[z^{-1}, z]]
\]

\[
v \mapsto \phi Y_1(\phi^{-1}(v), z)\phi^{-1}.
\]

It is easy to show that \( (V^2, Y^g_2) \) is a vertex operator algebra. Note also that \( Y^g_2(u, z)v = Y_2(u, z)v \) for any \( u, v \in V \). Hence, \( (V^2, Y^g_2) \) is also an extension vertex operator algebra of \( V \). By assumption, there exists a linear map \( \psi : V^2 \rightarrow V^2 \) such that \( \psi(Y^g_2(w^1, z)w^2) = Y_2(\psi(w^1), z)\psi(w^2) \) for any \( w^1, w^2 \in V^2 \). This implies

\[
\psi\phi(Y_1(v^1, z)v^2) = Y_2(\psi\phi(v^1), z)\psi\phi(v^2)
\]
for any $v^1, v^2 \in V^1$. Hence, the vertex operator algebra $V^1$ is isomorphic to $V^2$. The proof is complete. 

We are now ready to show the uniqueness of holomorphic VOA of central charge 24 with Lie algebra $A_{8,3}A_{2,1}^2$. The idea is to apply Proposition 4.14. We first recall some automorphisms of $L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)$. Let $\varphi$ be the diagram automorphism of $sl_3$. Then we know that $\varphi$ induces an automorphism of $L_{sl_3}(1, 0)$, which is also denoted by $\varphi$. By a direct calculation, we can show that $L_{sl_3}(1, \Lambda_1) \circ \varphi \cong L_{sl_3}(1, \Lambda_2)$ and $L_{sl_3}(1, \Lambda_2) \circ \varphi \cong L_{sl_3}(1, \Lambda_1)$. We also need an automorphism $\sigma$ of $L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)$, which is defined by

$$
\sigma(v \otimes w) = w \otimes v,
$$

for any $v, w \in L_{sl_3}(1, 0)$. By a direct calculation, for any $L_{sl_3}(1, 0)$-modules $M^1$ and $M^2$, we have $(M^1 \otimes M^2) \circ \sigma \cong M^2 \otimes M^1$.

**Theorem 4.15.** Let $\tilde{U}^1, \tilde{U}^2$ be holomorphic vertex operator algebras such that the central charges of $\tilde{U}^1, \tilde{U}^2$ are equal to 24 and the Lie algebras $\tilde{U}_1^1, \tilde{U}_1^2$ are isomorphic to $A_{8,3}A_{2,1}^2$. Then there exists a vertex operator algebra isomorphism $\Phi : \tilde{U}^1 \rightarrow \tilde{U}^2$.

**Proof:** We shall show that there exists a vertex operator algebra isomorphism from $\tilde{U}^1$ to $\tilde{U}^2$. Note that $\tilde{U}^1$ and $\tilde{U}^2$ are simple current extensions of $L_{sl_3}(3, 0) \otimes L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)$, it follows from Proposition 5.3 of [20] that $\tilde{U}^1$ and $\tilde{U}^2$ viewed as extension vertex operator algebras of $L_{sl_3}(3, 0) \otimes L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)$ are unique. In particular, if $\tilde{U}^1$ and $\tilde{U}^2$ viewed as $L_{sl_3}(3, 0) \otimes L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)$-modules have the same decomposition, then $\tilde{U}^1$ is isomorphic to $\tilde{U}^2$. We then assume that $\tilde{U}^1, \tilde{U}^2$ viewed as $L_{sl_3}(3, 0) \otimes L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)$-modules have different decompositions. Let $G$ be the subgroup of Aut $(L_{sl_3}(3, 0) \otimes L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0))$ generated by $\{1 \otimes \varphi \otimes 1, 1 \otimes 1 \otimes \varphi, 1 \otimes \varphi \otimes \varphi, 1 \otimes \sigma\}$. By Lemma 4.13, we can prove that there exists an automorphism $g \in G$ such that $\tilde{U}^1 \circ g$ viewed as an $L_{sl_3}(3, 0) \otimes L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)$-module is isomorphic to $\tilde{U}^2$. By Proposition 4.14, we know that the vertex operator algebra $\tilde{U}^1$ is isomorphic to $\tilde{U}^2$. The proof is complete.

Furthermore, we have the following:

**Theorem 4.16.** Let $(\tilde{U}^1, Y_1(\cdot, z)), (\tilde{U}^2, Y_2(\cdot, z))$ be holomorphic vertex operator algebras such that the central charges of $\tilde{U}^1, \tilde{U}^2$ are equal to 24 and the Lie algebras $\tilde{U}_1^1, \tilde{U}_1^2$ are isomorphic to $A_{8,3}A_{2,1}^2$. Assume further that $\tilde{U}^1, \tilde{U}^2$ viewed as $L_{sl_3}(3, 0) \otimes L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)$-modules have the same decomposition, then there exists a vertex operator algebra isomorphism $\Phi : \tilde{U}^1 \rightarrow \tilde{U}^2$ such that $\Phi|_{L_{sl_3}(3,0)\otimes L_{sl_3}(1,0)\otimes L_{sl_3}(1,0)} = id$. 


Proof: First, note that both $\hat{U}^1$ and $\hat{U}^2$ have a vertex operator algebra isomorphic to $L_{sl_3}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$. By Theorem 3.6, we know that there exists a vertex operator algebra isomorphism

$$f : L_{sl_3}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0) \rightarrow L_{sl_3}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$$

such that $f|_{L_{sl_3}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)} = \text{id}$ and that $f(Y_1(u,z)v) = Y_2(f(u),z)f(v)$ for any $u,v \in L_{sl_3}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$. Consider the linear isomorphism $\tilde{f} : \hat{U}^1 \rightarrow \hat{U}^2$ such that

$$\tilde{f}|_{L_{sl_3}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)} = f,$$

$$\tilde{f}|_{(L_{sl_3}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0))^\perp} = \text{id},$$

where $(L_{sl_3}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0))^\perp$ denotes the complement $L_{sl_3}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$-module of $L_{sl_3}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$ in $\hat{U}^1$. Define a new vertex operator by

$$Y_3(\cdot,z) : \hat{U}^1 \rightarrow \text{End}(\hat{U}^1)[[z^{-1},z]],$$

$$u \mapsto \tilde{f}^{-1}Y_3(\tilde{f}(u),z),$$

for any $u \in \hat{U}^1$. It is easy to show that $(\hat{U}^1,Y_3(\cdot,z))$ is a vertex operator algebra. Moreover, $(\hat{U}^1,Y_3(\cdot,z))$ is also an extension vertex operator algebra of $L_{sl_3}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$. Since $(\hat{U}^1,Y_1(\cdot,z))$ and $(\hat{U}^1,Y_3(\cdot,z))$ are simple current extensions of $L_{sl_3}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$, we know that there exists a linear isomorphism $g : \hat{U}^1 \rightarrow \hat{U}^1$ such that $g|_{L_{sl_3}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)} = \text{id}$ and that $g(Y_1(u,z)v) = Y_3(g(u),z)g(v)$ for any $u,v \in \hat{U}^1$ (see [20]). As a result, $\tilde{f} \circ g$ is the desired isomorphism. The proof is complete. □

4.4. Liftings of $\theta \otimes \sigma$. In this subsection, we shall show that there exists an automorphism $\bar{\theta} \otimes \bar{\sigma}$ of $U$ such that $\bar{\theta} \otimes \bar{\sigma}|_{L_{sl_3}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)} = \theta \otimes \sigma$ and has order two. Note that by Theorem 4.16 we may assume that $U$ viewed as an $L_{sl_3}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$-module is isomorphic to $W^1$. By Lemmas 3.8, 4.9, 4.10 and Theorems 3.7, 4.16 we immediately have:

**Theorem 4.17.** There exists an automorphism $\bar{\theta} \otimes \bar{\sigma}$ of $U$ such that

$$\bar{\theta} \otimes \bar{\sigma}(L_{sl_3}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)) = L_{sl_3}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$$

and $\bar{\theta} \otimes \bar{\sigma}|_{L_{sl_3}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)} = \theta \otimes \sigma$. 
Note that the order of $\widehat{\theta} \otimes \sigma$ may not be two; we next show that there exists a lifting $\widehat{\theta} \otimes \sigma$ of $\theta \otimes \sigma$ which has order two. First, we need to study automorphisms of $U$ such that the restrictions on $L_{sl_9}(3, 0) \otimes L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)$ are trivial.

**Lemma 4.18.** Let $f$ be an automorphism of $\widetilde{L_{sl_9}(3, 0) \otimes L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)}$ such that

$$f(L_{sl_9}(3, 0) \otimes L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)) = L_{sl_9}(3, 0) \otimes L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)$$

and $f|_{L_{sl_9}(3, 0) \otimes L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)} = \text{id}$. Then $f = \text{id}$.

**Proof:** Note that for any irreducible $L_{sl_9}(3, 0) \otimes L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)$-submodule $M$ of $L_{sl_9}(3, 0) \otimes L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)$, we have $f(M) = M$. Recall that $L_{sl_9}(3, 0)$ has the decomposition

$$L_{sl_9}(3, 0) = L_{sl_9}(3) \oplus L_{sl_9}(3, 3\bar{\Lambda}_3) \oplus L_{sl_9}(3, 3\bar{\Lambda}_6) \oplus L_{sl_9}(3, \bar{\Lambda}_1 + \bar{\Lambda}_2 + \bar{\Lambda}_6)$$

$$\oplus L_{sl_9}(3, \bar{\Lambda}_3 + \bar{\Lambda}_7 + \bar{\Lambda}_8) \oplus L_{sl_9}(3, \bar{\Lambda}_4 + \bar{\Lambda}_5),$$

and that $L_{sl_9}(3, 3\bar{\Lambda}_3)$ and $L_{sl_9}(3, 3\bar{\Lambda}_6)$ are simple current modules of $L_{sl_9}(3, 0)$ such that

$$(L_{sl_9}(3, 3\bar{\Lambda}_3) \times L_{sl_9}(3, 3\bar{\Lambda}_6) = L_{sl_9}(3, 3\bar{\Lambda}_3) \otimes L_{sl_9}(3, 3\bar{\Lambda}_6) = L_{sl_9}(3, 0).$$

By the Schur’s Lemma, we know that

$$f|_{L_{sl_9}(3, 3\bar{\Lambda}_3) \otimes L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)} = \lambda \text{id}, \quad f|_{L_{sl_9}(3, 3\bar{\Lambda}_6) \otimes L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)} = \lambda^2 \text{id},$$

where $\lambda$ denotes a cube root of unity. Furthermore, by fusion rules between $L_{sl_9}(3, 0)$-modules, we know that there exists a complex number $\nu$ such that

$$f|_{L_{sl_9}(3, \bar{\Lambda}_4 + \bar{\Lambda}_5) \otimes L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)} = \nu \text{id},$$

and that

$$f|_{L_{sl_9}(3, \bar{\Lambda}_1 + \bar{\Lambda}_2 + \bar{\Lambda}_6) \otimes L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)} = \nu \lambda \text{id}, \quad f|_{L_{sl_9}(3, \bar{\Lambda}_3 + \bar{\Lambda}_7 + \bar{\Lambda}_8) \otimes L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)} = \nu \lambda^2 \text{id}.$$
of \( L_{\text{sl}_3}(9,0) \). This contradicts to Theorem 3.10 of [3]. By the similar argument, \( \nu \) cannot be \( \lambda \) or \( \lambda^2 \). Suppose that \( \nu \) is equal to \(-1\), \(-\lambda\), or \(-\lambda^2\). Then \( f^3 \) acts on 
\( (L_{\text{sl}_9}(3,0) \oplus L_{\text{sl}_3}(3,3\Lambda_3) \oplus L_{\text{sl}_9}(3,3\Lambda_6)) \otimes L_{\text{sl}_3}(1,0) \otimes L_{\text{sl}_3}(1,0) \) as id and on 
\( (L_{\text{sl}_9}(3,\Lambda_1 + \Lambda_2 + \Lambda_6) \oplus L_{\text{sl}_3}(3,\Lambda_3 + \Lambda_7 + \Lambda_8) \oplus L_{\text{sl}_9}(3,\Lambda_4 + \Lambda_5)) \otimes L_{\text{sl}_3}(1,0) \otimes L_{\text{sl}_3}(1,0) \) as \(-id\).

Therefore, \( L_{\text{sl}_9}(3,\Lambda_1 + \Lambda_2 + \Lambda_6) \oplus L_{\text{sl}_3}(3,\Lambda_3 + \Lambda_7 + \Lambda_8) \oplus L_{\text{sl}_9}(3,\Lambda_4 + \Lambda_5) \) must be a simple current \( L_{\text{sl}_9}(3,0) \oplus L_{\text{sl}_3}(3,3\Lambda_3) \oplus L_{\text{sl}_9}(3,3\Lambda_6) \)-module. This is a contradiction.

Hence, we have \( \lambda = 1 \). As a consequence, we have \( \nu^2 = 1 \). This further forces that \( \nu = 1 \).

The proof is complete. \( \square \)

**Theorem 4.19.** Let \( g \) be an automorphism of \( U \) such that \( g|_{L_{\text{sl}_9}(3,0) \otimes L_{\text{sl}_3}(1,0) \otimes L_{\text{sl}_3}(1,0)} = \text{id} \). Then the order of \( g \) is equal to 1 or 3.

**Proof:** Note that for any irreducible \( L_{\text{sl}_9}(3,0) \otimes L_{\text{sl}_3}(1,0) \otimes L_{\text{sl}_3}(1,0) \)-submodule \( M \) of \( U \), we have \( g(M) = M \). In particular, we have \( g|_{L_{\text{sl}_9}(3,0) \otimes L_{\text{sl}_3}(1,0) \otimes L_{\text{sl}_3}(1,0)} = L_{\text{sl}_9}(3,0) \otimes L_{\text{sl}_3}(1,0) \otimes L_{\text{sl}_3}(1,0) \). Thus, \( g \) induces an automorphism of \( L_{\text{sl}_9}(3,0) \otimes L_{\text{sl}_3}(1,0) \) such that

\[ g(L_{\text{sl}_9}(3,0) \otimes L_{\text{sl}_3}(1,0) \otimes L_{\text{sl}_3}(1,0)) = L_{\text{sl}_9}(3,0) \otimes L_{\text{sl}_3}(1,0) \otimes L_{\text{sl}_3}(1,0) \]

and \( g|_{L_{\text{sl}_9}(3,0) \otimes L_{\text{sl}_3}(1,0) \otimes L_{\text{sl}_3}(1,0)} = \text{id} \). By Lemma 4.18, we have \( g|_{L_{\text{sl}_9}(3,0) \otimes L_{\text{sl}_3}(1,0) \otimes L_{\text{sl}_3}(1,0)} = \text{id} \). Note that \( U \) is a simple current extension of \( L_{\text{sl}_9}(3,0) \otimes L_{\text{sl}_3}(1,0) \otimes L_{\text{sl}_3}(1,0) \), and the fusion ring of \( L_{\text{sl}_9}(3,0) \otimes L_{\text{sl}_3}(1,0) \otimes L_{\text{sl}_3}(1,0) \) is isomorphic to \( \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \).

By Proposition 4.2.9 of [61], we know that the order of \( g \) must be equal to 1 or 3. The proof is complete. \( \square \)

As a corollary, we have:

**Corollary 4.20.** There exists an order two automorphism \( \hat{\theta} \otimes \sigma \) of \( U \) such that

\[ \hat{\theta} \otimes \sigma(L_{\text{sl}_9}(3,0) \otimes L_{\text{sl}_3}(1,0) \otimes L_{\text{sl}_3}(1,0)) = L_{\text{sl}_9}(3,0) \otimes L_{\text{sl}_3}(1,0) \otimes L_{\text{sl}_3}(1,0) \]

and \( \hat{\theta} \otimes \sigma|_{L_{\text{sl}_9}(3,0) \otimes L_{\text{sl}_3}(1,0) \otimes L_{\text{sl}_3}(1,0)} = \theta \otimes \sigma \).

**Proof:** Let \( \overline{\theta} \otimes \sigma \) be a lift of \( \theta \otimes \sigma \) as in Theorem 4.17. Then

\[ (\overline{\theta} \otimes \sigma)^2|_{L_{\text{sl}_9}(3,0) \otimes L_{\text{sl}_3}(1,0) \otimes L_{\text{sl}_3}(1,0)} = \text{id} \]

By Theorem 4.19, we know that the order of \( \overline{\theta} \otimes \sigma \) is equal to \( 2k \), where \( k = 1 \) or 3. Then \( (\overline{\theta} \otimes \sigma)^k \) is the desired automorphism. \( \square \)

5. \( \hat{\theta} \otimes \sigma \)-twisted module

Let \( U \) be a strongly regular holomorphic VOA of central charge 24 and \( U_1 \cong A_{8,3}A_{2,1}^2 \).

By Theorem 4.15, the VOA structure of \( U \) is uniquely determined. Let \( g = \hat{\theta} \otimes \sigma \) be
the automorphism as given in Corollary 4.20. We shall study the unique irreducible $g$-twisted module $U^T$ of $U$ in this section. In particular, we show that the lowest (conformal) weight of $U^T$ is 1.

Recall that $U$ contains a full subVOA $\widetilde{L^{sl}_2}(3,0) \otimes L^{sl}_2(1,0) \otimes L^{sl}_2(1,0)$ and $g|_{\widetilde{L^{sl}_2}(3,0)} = \tilde{\theta}$ and $g|_{L^{sl}_2(1,0) \otimes L^{sl}_2(1,0)} = \sigma$. Therefore, $U^T$ is a direct sum of the tensor products of irreducible $\tilde{\theta}$-twisted $\widetilde{L^{sl}_2}(3,0)$-modules and irreducible $\sigma$-twisted $L^{sl}_2(1,0) \otimes L^{sl}_2(1,0)$-modules. Since $L^{sl}_2(3,0)$ is an extension of $L^{sl}_2(3,0)$, irreducible $\tilde{\theta}$-twisted modules of $\widetilde{L^{sl}_2}(3,0)$ are direct sum of irreducible $\theta$-twisted $L^{sl}_2(3,0)$-modules.

5.1. $\mathbb{Z}_2$-twisted module of lattice VOA. First, let us recall the construction of irreducible $\mathbb{Z}_2$-twisted modules of lattice VOA from [1,4] (see also [27]).

Let $L$ be an even lattice with a symmetric bilinear form $\langle \cdot, \cdot \rangle$ and let $\sigma$ be an isometry of $L$ of order 2. Let $\mathfrak{h} = L \otimes \mathbb{C}$ and extend the bilinear form $\langle \cdot, \cdot \rangle \mathbb{C}$-bilinearly to $\mathfrak{h}$. Set $\mathfrak{h}_{(0)} = \{ h \mid \sigma x = x \}$ and $\mathfrak{h}_{(1)} = \{ h \mid \sigma x = -x \}$. For $i = 0, 1$, let $P_i$ be the natural projection of $\mathfrak{h}$ to $\mathfrak{h}_{(i)}$. We also use $x_{(i)}$ to denote $P_i(x)$ for any $x \in \mathfrak{h}$.

The twisted affine algebra $\hat{\mathfrak{h}}[\sigma]$ is the Lie algebra

$$\hat{\mathfrak{h}}[\sigma] = \mathfrak{h}_{(0)} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathfrak{h}_{(1)} \otimes t^{1/2} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$$

with the bracket given by

$$[x \otimes t^m, y \otimes t^n] = m \langle x, y \rangle \delta_{m+n,0} c$$

and

$$[c, \hat{\mathfrak{h}}_{\mathbb{Z} + \frac{1}{2}}] = 0$$

where $x, y \in \mathfrak{h}_{(i)}$ and $m, n \in \mathbb{Z} + \frac{1}{2}, i = 0, 1$. The Lie algebra $\hat{\mathfrak{h}}[\sigma]$ has a triangular decomposition given by

$$\hat{\mathfrak{h}}[\sigma] = \hat{\mathfrak{h}}^+ [\sigma] \oplus \hat{\mathfrak{h}}^0 [\sigma] \oplus \hat{\mathfrak{h}}^- [\sigma]$$

where $\hat{\mathfrak{h}}^\pm [\sigma] = \bigoplus_{n=1}^{\infty} \left( \mathfrak{h}_{(0)} \otimes t^{\pm n} \oplus \mathfrak{h}_{(1)} \otimes t^{\pm(n-\frac{1}{2})} \right)$ and $\hat{\mathfrak{h}}^0 [\sigma] = \mathfrak{h}_{(0)} \oplus \mathbb{C}c$.

Let $N = (1 - P_0) \mathfrak{h} \cap L = \{ \alpha \in L \mid \langle \alpha, \mathfrak{h}_{(0)} \rangle = 0 \}$ and $M = (1 - \sigma)L$. Note that $M < N$ since $\langle M, \mathfrak{h}_{(0)} \rangle = 0$. Let $\langle \kappa \rangle$ be a cyclic group of order 2. On $N$, we define

$$C_N(\alpha, \beta) = \kappa^{\langle \alpha, \beta \rangle}, \quad \text{and} \quad R = \{ \alpha \in N \mid C_N(\alpha, N) = 1 \}.$$ 

Let

$$1 \longrightarrow \langle \kappa \rangle \longrightarrow \hat{N} \xrightarrow{\varphi} N \longrightarrow 1$$

be the central extension of $N$ associated with the commutator map $C_N$ and let $\hat{\sigma}$ be a lift of $\sigma$ in $Aut(\hat{N})$, i.e., $\varphi(\hat{\sigma}(a)) = \sigma(\varphi(a))$ for any $a \in \hat{N}$. Set $K = \{ a \hat{\sigma}(a)^{-1} \mid a \in \hat{N} \}$. Then $K$ is an index 2 subgroup of $\hat{M}$.

**Proposition 5.1 ([47]).** For any irreducible character $\chi : \hat{R}/K \rightarrow \mathbb{C}$ with $\chi(\kappa K) = -1$, there is a unique irreducible $\hat{N}/K$-module $T_\chi$ such that $\hat{R}$ acts according to $\chi$. Moreover, $\dim T_\chi = |N/R|^{1/2}$. 
Let \( L^* = \{ \alpha \in L \otimes \mathbb{Z} \mid \langle \alpha, L \rangle < \mathbb{Z} \} \) be the dual lattice of \( L \). For any coset \( \lambda + P_0(L) \in P_0(L^*)/P_0(L) \) and an irreducible character \( \chi \in \text{Irr}(\widehat{R/K}) \), denote
\[ V_{L,\chi}^{T,L} = S(\hat{\mathfrak{h}}[-\sigma]) \otimes \mathbb{C}[\lambda + P_0(L)] \otimes T_{\chi}. \]
It is shown in [14] (see also [47]) that \( V_{L,\chi}^{T,L} \) is an irreducible \( \widehat{\sigma} \)-twisted module of \( V_L \). Moreover, the lowest (conformal) weight of \( V_{L,\chi}^{T,L} \) is given by
\[ \frac{\text{rank}(N)}{16} + \frac{n_\lambda}{2}, \]
where \( n_\lambda = \min\{ \langle \alpha, \alpha \rangle \mid \alpha \in \lambda + P_0(L) \} \).

5.2. \( \sigma \)-twisted module of \( L_{sl_3}(1,0) \otimes L_{sl_3}(1,0) \). Next we study the \( \sigma \)-twisted modules for \( L_{sl_3}(1,0) \otimes L_{sl_3}(1,0) \). Recall that \( L_{sl_3}(1,0) \) is isomorphic to the lattice VOA \( V_{A_2} \) and hence, \( L_{sl_3}(1,0) \otimes L_{sl_3}(1,0) \cong V_{A_2 \oplus A_2} \).

Let \( \sigma : A_2 \oplus A_2 \to A_2 \oplus A_2 \) be an isometry defined by \( \sigma(\alpha, \beta) = (\beta, \alpha) \) and set \( A^+ = \{ (\alpha, \alpha) \mid \alpha \in A_2 \} \) and \( A^- = \{ (\alpha, -\alpha) \mid \alpha \in A_2 \} \). Then \( A^+ \) and \( A^- \) are the eigenlattices of \( \sigma \) of eigenvalues +1 and -1, respectively.

Let \( P_0 = \frac{1}{2}(1 + \sigma) \) be the natural projection from \( (A_2 \oplus A_2)^* \) to \( \mathfrak{h}(0) \). Then
\[ P_0(A_2 \oplus A_2) = \frac{1}{2} A^+ \quad \text{and} \quad (1 - P_0) \mathfrak{h} \cap (A_2^2) = A^- = (1 - \sigma)(A_2^2). \]

Moreover, we have \( P_0((A_2^2)^2) = \frac{1}{2} \{ (\alpha, \alpha) \mid \alpha \in A_2^* \} \) and \( |P_0((A_2^2)^2)/P_0(A_2^2)| = 3 \).

Let \( P_0((A_2^2)^2) = P_0(A_2^2) \cup (\lambda_1 + P_0(A_2^2)) \cup (\lambda_2 + P_0(A_2^2)) \) be the coset decomposition of \( P_0((A_2^2)^2) \) in \( P_0((A_2^2)^2) \). Then, by [14] (see also [13]), we have the following lemma.

**Lemma 5.2.** There are 3 inequivalent irreducible \( \sigma \)-twisted modules for \( V_{A_2 \oplus A_2} \) and they are given by
\[ W_0 = S(\hat{\mathfrak{h}}[-\sigma]) \otimes \mathbb{C}[P_0(A_2^2)] \otimes T, \]
\[ W_1 = S(\hat{\mathfrak{h}}[-\sigma]) \otimes \mathbb{C}[\lambda_1 + P_0(A_2^2)] \otimes T, \quad \text{and} \]
\[ W_2 = S(\hat{\mathfrak{h}}[-\sigma]) \otimes \mathbb{C}[\lambda_2 + P_0(A_2^2)] \otimes T, \]
where \( T \) is a one-dimensional irreducible module of \( \hat{A}^- \). The lowest weight of \( W_0 \) is \( 2/16 = 1/8 \) and the lowest weights of \( W_1 \) and \( W_2 \) are \( 1/8 + 1/6 = 7/24 \).

5.3. Irreducible \( \theta \)-twisted modules of \( L_{sl_3}(3,0) \). Next we consider the irreducible \( \theta \)-twisted modules of \( L_{sl_3}(3,0) \).

**Lemma 5.3.** There are exactly 5 irreducible \( \theta \)-invariant modules for \( L_{sl_3}(3,0) \), namely, \( L_{sl_3}(3), L_{sl_3}(\tilde{A}_1 + \tilde{A}_8), L_{sl_3}(3, \tilde{A}_2 + \tilde{A}_7), L_{sl_3}(3, \tilde{A}_3 + \tilde{A}_6) \), and \( L_{sl_3}(3, \tilde{A}_4 + \tilde{A}_5) \).

**Proof.** The lemma follows immediately from Lemma 5.8. \qed
By [17, Theorem 1.1], we also have the following result.

**Corollary 5.4.** There are exactly 5 inequivalent irreducible $\theta$-twisted modules for $L_{sl_9}(3, 0)$.

Now let $L = A_8^3$ be a root lattice of type $A_8^3$. For explicit calculations, we use the standard model for $A_8$, i.e.,

$$A_8 = \{(a_1, \ldots, a_9) \in \mathbb{Z}^9 \mid \sum_{i=1}^9 a_i = 0\}.$$

The following lemma can be obtained by a direct calculation.

**Lemma 5.5.** Every coset of $A_8/2A_8$ contains a vector of norm $\leq 8$ and the coset representatives (with minimal norm) are given as follows:

| Norm | Representatives | # of cosets |
|------|-----------------|-------------|
| 0    | 0               | 1           |
| 2    | $\pm(1, -1, 0^7)$ | 36          |
| 4    | $\pm(1^2, -1^2, 0^5)$ | 126        |
| 6    | $\pm(1^3, -1^3, 0^3)$ | 84          |
| 8    | $\pm(1^4, -1^4, 0)$ | 8           |

Let $X$ and $Y$ be sublattices of $A_8$ such that $X/2A_8$ and $Y/2A_8$ are maximal totally singular subspaces of $A_8/2A_8$ and $X + Y = A_8$. Recall that $A_8/2A_8$ forms a non-singular quadratic space associated with the standard quadratic form $q(\alpha + 2A_8) = \langle \alpha, \alpha \rangle/2 \mod 2$ since $\det(A_8) = 9$.

Set $\mu_{i,j} = \eta_i - \eta_j$ for $i \neq j$ and $\eta = \eta_1 + \eta_2 + \eta_3$ and let $\Phi$ be the sublattice of $L = A_8^3$ spanned by

$$\mu_{1,2}(A_8) \cup \eta(X) \cup 2L$$

and let $\Psi$ be the sublattice spanned by

$$\mu_{2,3}(A_8) \cup \eta(Y) \cup 2L.$$

Then $\Phi/2L$ and $\Psi/2L$ are maximal totally singular subspaces of $L/2L$ and $\Phi + \Psi = L$. Note that $\mu_{i,j}(A_8) \perp \eta(A_8)$.

Let $\chi_0$ be an irreducible character of $\hat{\Phi}/K$ such that $\chi_0(\iota(e_\alpha)) = 1$ and $\chi_0(\kappa K) = -1$. Then

$$T = \text{Ind}_{\hat{\Phi}/K}^{\hat{L}/K} F_{\chi_0},$$

where $F_{\chi_0}$ is the irreducible $\hat{\Phi}/K$-module affording the character $\chi_0$.

Take $0 \neq t_0 \in F_{\chi_0}$. Then $F_{\chi_0} = \mathbb{C}t_0$ and

$$T = \text{Span}_{\mathbb{C}} \{ e_{\mu_{2,3}(\beta) + \eta(\gamma)} \cdot t_0 \mid \beta \in A_8, \gamma \in Y \}.$$
For simplicity, we set $t_{(\beta, \gamma)} = e_{\mu_{2,3}(\beta)+\eta(\gamma)} \cdot t_0$.

Recall from Sec. 3.3 that the lattice VOA $V_{A_8}$ contains a subVOA isomorphic to the affine VOA $L_{sl_2}(3,0)$, which is generated by

$$
\tilde{h} = \eta(h)(-1) \cdot 1 \quad \text{for} \ h \in A_8 \otimes \mathbb{Z} \mathbb{C},
$$

$$
E_\alpha = t(e_{\eta_1}(\alpha)) + t(e_{\eta_2}(\alpha)) + t(e_{\eta_3}(\alpha)) \quad \text{for} \ \alpha \in (A_8)_2.
$$

The conformal element $\Omega$ of $L_{sl_2}(3,0)$ is given by

$$
\Omega = \omega_E + \frac{3}{4} \omega_P - \frac{1}{12} \sum_{\alpha \in (A_8)_2} e_{\mu_{i,j}(\alpha)},
$$

where $E = \eta(A_8), P = \{(\alpha, \beta, \gamma) \in A_8^3 \mid \alpha + \beta + \gamma = 0\}$ and $\omega_M$ denotes the conformal element of the lattice VOA $V_M$.

Next we shall construct some explicit eigenvectors of $\Omega_1$ on $T$.

**Lemma 5.6.** For $\alpha \in (A_8)_2$ and $(\beta, \gamma) \in A_8 \times Y$, we have

$$
\begin{cases}
\frac{1}{16} t_{(\beta, \gamma)} & \text{if } \langle \alpha, \beta \rangle = 0 \mod 2, \\
\frac{1}{16} (2t_{(\alpha+\beta, \gamma)} - t_{(\beta, \gamma)}) & \text{if } \langle \alpha, \beta \rangle = 1 \mod 2.
\end{cases}
$$

**Proof.** Recall from Sec. 3.3 that

$$
e_{\mu_{1,2}(\alpha)} e_{\mu_{2,3}(\beta)} = (-1)^{\langle \alpha, \beta \rangle} e_{\mu_{2,3}(\beta)} e_{\mu_{1,2}(\alpha)};
$$

$$
e_{\mu_{2,3}(\alpha)} e_{\mu_{2,3}(\beta)} = e_{\mu_{2,3}(\alpha+\beta)};
$$

$$
e_{\mu_{1,3}(\alpha)} = (-1)^{\alpha(\alpha, \alpha)} e_{\mu_{2,3}(\alpha)} e_{\mu_{1,2}(\alpha)} = -e_{\mu_{2,3}(\alpha)} e_{\mu_{1,2}(\alpha)}.
$$

Since $e_{\mu_{1,2}(\alpha)} \cdot t_0 = t_0$ for all $\alpha \in (A_8)_2$, we have

$$
\begin{align*}
(\mu_{1,2}(\alpha) + e_{\mu_{2,3}(\alpha)} + e_{\mu_{1,3}(\alpha)}) t_{(\beta, \gamma)} \\
= \frac{1}{24} (e_{\mu_{1,2}(\alpha)} + e_{\mu_{2,3}(\alpha)} + e_{\mu_{1,3}(\alpha)}) \cdot e_{\mu_{2,3}(\beta)} e_{\eta(\gamma)} \cdot t_0 \\
= \frac{1}{16} \left( (-1)^{\langle \alpha, \beta \rangle} t_{(\beta, \gamma)} + t_{(\alpha+\beta, \gamma)} - (-1)^{\langle \alpha, \beta \rangle} t_{(\alpha+\beta, \gamma)} \right).
\end{align*}
$$

Thus we have the desired conclusion. \(\square\)

**Notation 5.7.** For $n = 0, 1, 2, 3, 4$, let $C_{2n}$ be the set of cosets of $2A_8$ in $A_8$ with minimal norm $2n$, i.e., $C_{2n} = \{\alpha + 2A_8 \mid \min\{\langle a, a \rangle \mid a \in \alpha + 2A_8\} = 2n\}$. Note that a coset $\alpha + 2A_8 \in A_8/2A_8$ is in $C_{2n}$ if it contains a vector of the shape $(1^n, -1^n, 0^{9-2n})$. 
Lemma 5.8. Let \( \mu = (1, -1, 0^T) \) and denote \( v_0 = t_0 \) and
\[
v_1 = 21t_{(\mu, 0)} + \sum_{(\beta, \mu) = 0 \mod 2, \beta + 2A_8 \in C_2 \setminus \{\mu + 2A_8\}} t_{(\beta, 0)} - 3 \sum_{(\beta, \mu) = 1 \mod 2, \beta + 2A_8 \in C_2} t_{(\beta, 0)}. \]

Then we have
\[
\Omega_1 v_0 = \frac{7}{8} v_0, \quad \text{and} \quad \Omega_1 v_1 = \frac{29}{24} v_1.
\]

Proof. First we note that
\[
(\omega_E)_t = \frac{8}{16} t \quad \text{and} \quad (\omega_P)_t = \frac{16}{16} t
\]
for any \( t \in T \). Note that \( \text{rank}(E) = 8 \) and \( \text{rank}(P) = 16 \).

For \( v_0 = t_0 \), we have
\[
\left( \sum_{\alpha \in (A_8)_2 \atop 1 \leq i < j \leq 3} e_{\mu, j}(\alpha) \right) v_0 = \frac{72}{16} v_0
\]
by Lemma 5.6. Notice that there are 72 elements in \((A_8)_2\). Hence,
\[
\Omega_1 v_0 = \frac{1}{2} v_0 + \frac{3}{4} v_0 - \frac{1}{12} \cdot \frac{72}{16} v_0 = \frac{7}{8} v_0.
\]

For any \( \beta + 2A_8 \in C_2 \), we have
\[
|\{\alpha \in (A_8)_2 \mid \langle \beta, \alpha \rangle = 0 \mod 2\}| = 44;
\]
\[
|\{\alpha \in (A_8)_2 \mid \langle \beta, \alpha \rangle = 1 \mod 2\}| = 28.
\]

Let \( \mu = (1, -1, 0^T) \). If \( \beta + 2A_8 = \mu + 2A_8 \), then \( \langle \alpha, \beta \rangle = 1 \mod 2 \) implies
\[
\langle \alpha + \beta, \mu \rangle = \langle \alpha + \beta, \beta \rangle = 1 \mod 2
\]
and there are 28 such \( \alpha \in (A_8)_2 \).

If \( \beta + 2A_8 \neq \mu + 2A_8 \) and \( \langle \beta, \mu \rangle = 0 \mod 2 \), then
\[
|\{\alpha \in (A_8)_2 \mid \langle \beta, \alpha \rangle = 1, \langle \alpha + \beta, \mu \rangle = 0 \mod 2\}| = 20;
\]
\[
|\{\alpha \in (A_8)_2 \mid \langle \beta, \alpha \rangle = 1, \langle \alpha + \beta, \mu \rangle = 1 \mod 2\}| = 8.
\]

If \( \langle \beta, \mu \rangle = 1 \mod 2 \), then
\[
|\{\alpha \in (A_8)_2 \mid \langle \beta, \alpha \rangle = 1, \langle \alpha + \beta, \mu \rangle = 0 \mod 2, \alpha + \beta + 2A_8 \neq \mu + 2A_8\}| = 12;
\]
\[
|\{\alpha \in (A_8)_2 \mid \alpha + \beta + 2A_8 = \mu + 2A_8\}| = 2;
\]
\[
|\{\alpha \in (A_8)_2 \mid \langle \beta, \alpha \rangle = 1, \langle \alpha + \beta, \mu \rangle = 1 \mod 2\}| = 14.
\]
Then by Lemma 5.6, we have

\[
16 \left( \sum_{\alpha \in (A_8)_{<3}} e_{\mu_i,j}(\alpha) \right) v_1 = (21 \cdot (44 - 28) - 3 \cdot 2 \cdot 28)t_{(\mu,0)} + (((44 - 28) + 2 \cdot 20 - 3 \cdot 2 \cdot 8) \sum_{\langle \beta,\mu \rangle=0 \mod 2 \beta+2A_8 \in C_{2n} \setminus \{\mu+2A_8\}} t_{(\beta,0)} - (3 \cdot (44 - 28) + 3 \cdot 2 \cdot 14 - 2 \cdot 12 - 21 \cdot 2 \cdot 2) \sum_{\langle \beta,\mu \rangle=1 \mod 2 \beta+2A_8 \in C_{2n}} t_{(\beta,0)}
\]

\[
= 8 \left( 21t_{(\mu,0)} + \sum_{\langle \beta,\mu \rangle=0 \mod 2 \beta+2A_8 \in C_{2n}} t_{(\beta,0)} - 3 \sum_{\langle \beta,\mu \rangle=1 \mod 2 \beta+2A_8 \in C_{2n}} t_{(\beta,0)} \right) = 8v_1.
\]

Hence we have

\[
\Omega_1 v_1 = \frac{1}{2} v_1 + \frac{3}{4} v_1 - \frac{1}{12} \cdot \frac{8}{16} v_1 = \frac{29}{24} v_1.
\]

\[\square\]

**Notation 5.9.** For any \(\mu \in A_8\), we denote

\[
P_\mu^{2n} = \sum_{\langle \beta,\mu \rangle=0 \mod 2 \beta+2A_8 \in C_{2n}} t_{(\beta,0)}, \quad N_\mu^{2n} = \sum_{\langle \beta,\mu \rangle=1 \mod 2 \beta+2A_8 \in C_{2n}} t_{(\beta,0)}.
\]

**Lemma 5.10.** Let \(\mu_4 = (1^2, -1^2, 0^5), \mu_6 = (1^3, -1^3, 0^3)\) and \(\mu_8 = (1^4, -1^4, 0)\) and denote \(v_2 = 10P_2^{\mu_4} - 7N_2^{\mu_4}, v_3 = 6P_2^{\mu_6} - 7N_2^{\mu_6}, \) and \(v_4 = 5P_8^{\mu_8} - N_8^{\mu_8}\). Then we have

\[
\Omega_1 v_2 = \frac{59}{48} v_2, \quad \Omega_1 v_3 = \frac{55}{48} v_3, \quad \text{and} \quad \Omega_1 v_4 = \frac{17}{16} v_4.
\]

**Proof.** The proof is similar to the previous lemma. First we have

\[
(\omega_E + \frac{3}{4} \omega_P) t = \frac{5}{4} t
\]

for any \(t \in T\).

Let \(\mu = \mu_4 = (1^2, -1^2, 0^5)\). For any \(\beta + 2A_8 \in C_2\), we have

\[
|\{\alpha \in (A_8)_{<3} | \langle \beta, \alpha \rangle = 0 \mod 2\}| = 44;
|\{\alpha \in (A_8)_{<3} | \langle \beta, \alpha \rangle = 1 \mod 2\}| = 28.
\]

If \(\langle \beta, \mu \rangle = 0 \mod 2\), then

\[
|\{\alpha \in (A_8)_{<3} | \langle \beta, \alpha \rangle = 1, \langle \alpha + \beta, \mu \rangle = 0 \mod 2\}| = 8;
|\{\alpha \in (A_8)_{<3} | \langle \beta, \alpha \rangle = 1, \langle \alpha + \beta, \mu \rangle = 1 \mod 2\}| = 20.
\]
If \((\beta, \mu) = 1 \mod 2\), then

\[
|\{\alpha \in (A_8)_2 \mid \langle \beta, \alpha \rangle = 1, \langle \alpha + \beta, \mu \rangle = 0 \mod 2\}| = 14;
\]

\[
|\{\alpha \in (A_8)_2 \mid \langle \beta, \alpha \rangle = 1, \langle \alpha + \beta, \mu \rangle = 1 \mod 2\}| = 14.
\]

Then by Lemma 5.6, we have

\[
16 \left( \sum_{\substack{\alpha \in (A_8)_2 \\backslash \\{0\} \\backslash 1 \leq i < j \leq 3}} e_{\mu, j}(\alpha) \right) v_2
= (10 \cdot (16 + 16) - 40 \cdot 7) P_2^{\mu_4} - (7 \cdot (16 + 28) - 28 \cdot 10) N_2^{\mu_4}
= 40 P_2^{\mu_4} - 28 N_2^{\mu_4} = 4v_2.
\]

Hence we have

\[
\Omega_1 v_2 = \frac{5}{4} v_1 - \frac{1}{12} \cdot \frac{4}{16} v_2 = \frac{59}{48} v_2.
\]

The other cases can be proved by the similar method.

By Corollary 5.4 and the lemmas above, we have the following result.

**Lemma 5.11.** There are 5 inequivalent irreducible \(\theta\)-twisted modules for \(L_{sl_9}(3, 0)\) and their lowest conformal weights are \(7/8, 29/24, 59/48, 55/48\) and \(17/16\).

**Lemma 5.12.** Let \(\alpha\) be a root of \(A_8\) and \((\beta, \gamma) \in A_8 \times Y\). Then

\[
E_{\alpha} \cdot t_{(\beta, \gamma)} = \begin{cases} 
1 \cdot t_{(\beta, \gamma + \alpha)} & \text{if } \langle \alpha, \beta \rangle = 0 \mod 2, \\
2t_{(\beta + \alpha, \gamma + \alpha)} - t_{(\beta, \gamma + \alpha)} & \text{if } \langle \alpha, \beta \rangle = 1 \mod 2.
\end{cases}
\]

**Proof.** First we note that

\[
e_{\eta_1}(\alpha) = e_{\eta_1}(\alpha) e_{\mu_2, 3}(\alpha) e^{-2\eta_2(\alpha)},
\]

\[
e_{\eta_2}(\alpha) = -e_{\eta_1}(\alpha) e_{\mu_2, 3}(\alpha) e^{-\mu_1, 2}(\alpha) e^{-2\eta_2(\alpha)},
\]

\[
e_{\eta_3}(\alpha) = e_{\eta_1}(\alpha) e_{\mu_1, 2}(\alpha) e^{-2\eta_2(\alpha)}.
\]

Thus,

\[
E_{\alpha} = (e_{\eta_1}(\alpha) + e_{\eta_2}(\alpha) + e_{\eta_3}(\alpha)) \cdot t_{(\beta, \gamma)}
= t_{(\beta + \alpha, \gamma + \alpha)} - (-1)^{\langle \alpha, \beta \rangle} t_{(\beta + \alpha, \gamma + \alpha)} + (-1)^{\langle \alpha, \beta \rangle} t_{(\beta, \gamma + \alpha)}
\]

and we have the desired result.

**Lemma 5.13.** Let \(M\) be the \(\theta\)-twisted \(L_{sl_9}(3, 0)\)-module generated by \(t_0\). Let \(M(0)\) be the top module of an irreducible (twisted or untwisted) module \(M\). Then we have

\[
M(0) = \text{Span}_\mathbb{C}\{t_{(0, \gamma)} \mid \gamma \in Y\}.
\]

In particular, \(M(0)\) has dimension 16.
Proposition 5.14. Let $U$ be a holomorphic VOA of central charge 24 and $U_1 \cong A_{8,3}A_{2,1}^2$. Let $g = \theta \otimes \sigma$ be an involution of $U$ as given in Theorem 4.17. Let $U^T$ be the unique irreducible $g$-twisted module of $U$. Then the lowest conformal weight of $U^T$ is 1 and $\dim(U^T_1) \geq 16$.

Proof. We first note that $U^T$ is a direct sum of the tensor products of irreducible $\tilde{\theta}$-twisted $L_{sl_9}(3,0)$-modules and irreducible $\tilde{\sigma}$-twisted $L_{sl_3}(1,0)$-modules. Moreover, every $\tilde{\theta}$-twisted $L_{sl_9}(3,0)$-module is a direct sum of irreducible $\theta$-twisted $L_{sl_9}(3,0)$-modules.

By [17, Theorem 1.6], the conformal weights of $U^T$ are in $\frac{1}{2}\mathbb{Z}$. Then by Lemmas 5.2 and 5.11, the lowest conformal weights for irreducible $\tilde{\theta} \otimes \tilde{\sigma}$-twisted $L_{sl_9}(3,0) \otimes L_{sl_3}(1,0)$-submodules of $U^T$ are $1 = 7/8 + 1/8$ and $3/2 = 29/24 + 7/24$. Therefore, the lowest conformal weight of $U^T$ is 1. That $\dim(U^T_1) \geq 16$ follows from Lemma 5.13. □

6. $\tilde{\theta} \otimes \tilde{\sigma}$-Orbifold Construction

Let $U$, $U^T$ and $g$ be defined as in Proposition 5.14. Since the conformal weights of $U^T$ are in $\frac{1}{2}\mathbb{Z}$, we can apply an orbifold construction to $U$ using $g$ (see [25, Theorem 5.15] and [8]) and obtain a strongly regular holomorphic VOA

$$\tilde{U}(g) = U^g \oplus (U^T)_2$$

of central charge 24. The following lemma is a generalization of [44, Theorem 4.3] (see also [55]).

Lemma 6.1. Let $U$ and $g$ be as above. Then

$$\dim U_1 + \dim \tilde{U}(g)_1 = 3 \dim(U^g)_1 + 24(1 - \dim(U^T)_{1/2}).$$

Proof. Let $Z_U(g, \tau) = q^{-c/24} \sum_{n=0}^{\infty} \text{Tr} g|_{U_n} g^n$ be the trace function of $g$ on $U$ and let $Z_{U^T}(\tau) = q^{-c/24} \sum_{n=1}^{\infty} \dim(U^T)_n q^{n/2}$ be the character of $U^T$, where $q = e^{2\pi \sqrt{-1} \tau}$ and $\tau$ is in the upper half plane $\mathfrak{H}$.

It was proved in [17] that $Z_U(g, \tau)$ and $Z_{U^T}(\tau)$ both converge to holomorphic functions in $\mathfrak{H}$ and

$$Z_U(g, S\tau) = Z_U(g, -\frac{1}{\tau}) = \lambda Z_{U^T}(\tau)$$

for some $\lambda \in \mathbb{C}$. Moreover, it was proved in [25, Proposition 5.5] that $\lambda = 1$. Therefore, $U$ and $U^T$ satisfy Assumptions (A1) and (A2) of [44, Section 4.2]. Hence the proof of Theorem 4.3 of [44] still holds for $U$ and $g$ and we have

$$\dim U_1 + \dim \tilde{U}(g)_1 = 3 \dim(U^g)_1 + 24(1 - \dim(U^T)_{1/2}),$$
as desired. □

By a direct calculation, we also have the following.

**Lemma 6.2.** Let $U$, $U^T$ and $g$ be defined as above.
1. The weight one Lie algebra of $U^g$ has the type $B_4A_2$ and has dimension 44.
2. $\dim(\tilde{U}(g)_1) = 60$.

**Proof.** Recall that $g|_{\mathfrak{sl}_3(3,0) \otimes \mathfrak{sl}_3(1,0) \otimes \mathfrak{sl}_3(1,0)} = \theta \otimes \sigma$. Since the fixed point Lie algebra of $\theta$ on $\mathfrak{sl}_3$ has type $B_4$ and the fixed point of $\sigma$ on $\mathfrak{sl}_3 \oplus \mathfrak{sl}_3$ has type $A_2$, we have (1).

For (2), we have $\dim(\tilde{U}(g)_1) = 3 \times 44 + 24 - 96 = 60$ by Lemma 5.1 □

**Theorem 6.3.** Let $U$, $U^T$ and $g$ be defined as above. Then $\tilde{U}(g) = U^g \oplus (U^T)_Z$ is a strongly regular holomorphic VOA of central charge 24 and $\tilde{U}(g)_1$ has the type $F_{4,6}A_{2,2}$.

**Proof.** Since $\dim(\tilde{U}(g)_1) = 60$, the ratio $\frac{h^V_k}{h^U_k} = \frac{60 - 24}{24} = 3/2$ (cf. [20]). Therefore, the dual Coxeter number of any simple ideal of $\tilde{U}(g)_1$ must be divisible by 3 and hence a simple ideal must have the type $A_2$, $C_2$, $A_5$, $C_5$, $D_4$, or $F_4$. That $\dim(\tilde{U}(g)_1) = 60$ implies that $\tilde{U}(g)_1$ has the type $C_{2,2}^6$, $D_{4,4}A_{2,2}^4$ or $F_{4,6}A_{2,2}^2$. Since $\tilde{U}(g)_1$ contains a Lie subalgebra of type $B_4A_2$, $\tilde{U}(g)_1 = F_{4,6}A_{2,2}$ is the only possibility. □

7. **Uniqueness of holomorphic VOAs of central charge 24 with weight one Lie algebras $F_{4,6}A_{2,2}$ and $E_{7,3}A_{5,1}$**

In this section, we shall prove that holomorphic VOAs of central charge 24 with weight one Lie algebras $F_{4,6}A_{2,2}$ and $E_{7,3}A_{5,1}$ are unique.

7.1. **Reverse orbifold construction of holomorphic VOAs.** First, we recall the reverse orbifold method from [40] (see also [46]). Let $V$ be a strongly regular holomorphic VOA of central charge 24, $g$ be an automorphism of $V$ of prime order $p$. We then know that there is a unique $g^r$-twisted $V$-module $V^T(g^r)$ for each $1 \leq r \leq p - 1$ ([17, Theorem 10.3]). Moreover, the fixed point subspace $V^g$ of $V$ with respect to $g$ is a sub VOA of $V$. We say that the pair $(V, g)$ satisfies the orbifold condition if there exists a unique simple VOA $\hat{V}$ such that $V^g$ is embedded in $\hat{V}$ and $\hat{V} \cong V^g \oplus \bigoplus_{r=1}^{p-1} V^T(g^r)_Z$ as a $V^g$-module, where $V^T(g^r)_Z$ is the subspace of $V^T(g^r)$ of integral conformal weights (cf. [25]). If $(V, g)$ satisfies the orbifold condition, the VOA $\hat{V}$ which satisfies the above assumptions is strongly regular and holomorphic. We refer to $\hat{V}$ as the VOA obtained by applying the $\mathbb{Z}_p$-orbifold construction to $V$ and $g$, and denote the VOA $\hat{V}$ by $\hat{V}(g)$. If we further define an automorphism $a = a_{V,g}$ of $\hat{V}(g)$ by $a|_{V^g} = 1$ and $a|_{V^T(g^r)_Z} = e^{2\pi \sqrt{-1} r/p}$ $(1 \leq r \leq p - 1)$, we then know that the pair $(\hat{V}(g), a)$ satisfies the orbifold condition and $\hat{V}(g)(a) \cong V$ (see [25]).
We are now ready to state the results about reverse orbifold construction from \[46\].

**Theorem 7.1** \([46\), Theorem 5.2\]. Let \(\mathfrak{g}\) be a Lie algebra and \(\mathfrak{p}\) a subalgebra of \(\mathfrak{g}\). Let \(n \in \mathbb{Z} > 0\) and let \(U\) be a strongly regular holomorphic VOA of central charge \(c\). Assume that for any strongly regular holomorphic VOA \(W\) of central charge \(c\) whose weight one Lie algebra is \(\mathfrak{g}\), there exists an order \(n\) automorphism \(\psi\) of \(W\) such that the following conditions hold:

1. \(\mathfrak{g}^\psi \cong \mathfrak{p}\);
2. For \(1 \leq i \leq n - 1\), the lowest \(L(0)\)-weight of \(W^T(\psi^i)\) belongs to \((1/n)\mathbb{Z} > 0\);
3. \(\hat{\mathfrak{W}}(\psi)\) is isomorphic to \(U\).

In addition, we assume that any automorphism \(\phi \in \text{Aut}\ (U)\) of order \(n\) satisfying the conditions (A), (B) and (C) below belongs to a unique conjugacy class in \(\text{Aut}\ (U)\):

- (A) \((U^\phi)_1\) is isomorphic to \(\mathfrak{p}\);
- (B) For \(1 \leq i \leq n - 1\), the lowest \(L(0)\)-weight of \(U^T(\phi^i)\) belongs to \((1/n)\mathbb{Z} > 0\);
- (C) \((\hat{U}(\phi))_1\) is isomorphic to \(\mathfrak{g}\).

Then any strongly regular holomorphic VOA of central charge \(c\) with weight one Lie algebra \(\mathfrak{g}\) is isomorphic to \(\hat{U}(\phi)\). In particular, such a holomorphic VOA is unique up to isomorphism.

**7.2. Reverse orbifold constructions by inner automorphisms.** We now begin to prove the uniqueness of the holomorphic VOA of central charge 24 with weight one Lie algebra \(F_{4,6}A_{2,2}\). To apply Theorem 7.1 we let \(\mathfrak{g} = F_{4,2}, \mathfrak{p} = B_{4,2}\) and \(W, U\) be holomorphic VOAs of central charge 24 with weight one Lie algebras \(F_{4,6}A_{2,2}, A_{8,3}A_{2,1}^2\), respectively. We then need to choose an appropriate automorphism \(\psi\) of \(W\). Take \(h = (\Lambda_4, 0) \in F_{4,6}A_{2,2} = W_1\), and define \(\sigma_h = \exp(2\pi \sqrt{-1}h_0)\), where \(\Lambda_i\) denotes the fundamental weight of \(F_i\). Then we know that \(\sigma_h\) is an inner automorphism of \(W\). We shall show that \(\sigma_h\) is the desired automorphism. To verify that \(\sigma_h\) satisfies the condition (b), we need the following result which was proved in \[44\].

**Proposition 7.2.** Let \(V\) be a strongly regular holomorphic VOA. Assume that the Lie algebra \(\mathfrak{g} = V_1\) is semisimple. Let \(\mathfrak{g} = \bigoplus_{i=1}^t \mathfrak{g}_i\) be the decomposition into the direct sum of \(t\) simple ideals \(\mathfrak{g}_i\). Let \(\langle V_1 \rangle\) be the subVOA of \(V\) generated by \(V_1\). Let \(\tilde{h}\) be an element in a (fixed) Cartan subalgebra \(H\) of \(\mathfrak{g}\) such that \(\text{Spec} \tilde{h}_0 \subseteq (1/T)\mathbb{Z}\) on \(V\) for some \(T \in \mathbb{Z} > 0\). Let \(\tilde{h}_{(i)}\) be the image of \(\tilde{h}\) under the canonical projection from \(H\) to \(H \cap \mathfrak{g}_i\). We further assume that

1. the conformal vectors of \(V\) and \(\langle V_1 \rangle\) are the same, i.e., \(\langle V_1 \rangle\) is a full subVOA of \(V\);
2. \(\langle \tilde{h}|\alpha \rangle \geq -1\) for all roots \(\alpha \in H\) of \(\mathfrak{g}\), where \(\langle \cdot | \cdot \rangle\) is the normalized Killing form on \(\mathfrak{g}\) so that \(\langle \beta | \beta \rangle = 2\) for any long root \(\beta\);
(3) for some $i$, $-\tilde{h}_{(i)}$ is not a fundamental weight.  
Then the lowest $L(0)$-weight of $V^{(\tilde{h})}$ is positive, where $V^{(\tilde{h})}$ denotes the unique $\sigma_{\tilde{h}}$-twisted $V$-module.

As a consequence, we have the following.

**Lemma 7.3.** Let $\sigma_{\tilde{h}}$ be the automorphism of $W$ defined above. Then $\sigma_{\tilde{h}}$ is an order two automorphism and satisfies the conditions (a), (b).

**Proof:** First, by Theorem 8.6 and Proposition 8.6 in [44], we know that $\sigma_{\tilde{h}}|_{F_4A_2}$ is an order two automorphism of $F_4A_2$ and that $(F_4A_2)^\sigma_{\tilde{h}} \cong B_4A_2$. Moreover, since for any fundamental weight $\Lambda_i$ of $F_4$ we have $(\Lambda_4|\Lambda_i) \in (1/2)\mathbb{Z}$, we then know that $\sigma_{\tilde{h}}$ is an order two automorphism of $W$.

We next verify the condition (b). By Proposition 4.1 of [24], we know that the vertex operator subalgebra of $W$ generated by $W_1$ has the same conformal vector as that of $W$. Moreover, it is straightforward to verify that $(\Lambda_4|\alpha) \geq -1$ for any root of $F_4$. Thus, by Proposition 7.2, $\sigma_{\tilde{h}}$ is an automorphism of $W$ satisfying the condition (b). $\square$

To verify the condition (c), we need the following results which were proved in [44].

**Theorem 7.4.** Let $V$ be a strongly regular holomorphic VOA with central charge $24$, $\tilde{h}$ be a semisimple element in a Cartan subalgebra $V_1$ such that: (i) Spec $\tilde{h}_0 \subseteq (1/2)\mathbb{Z}$ and Spec $\tilde{h}_0 \not\subseteq \mathbb{Z}$; (ii) $\langle \tilde{h}, \tilde{h} \rangle \in \mathbb{Z}$, where $\langle \cdot, \cdot \rangle$ denotes the unique symmetric invariant bilinear form on $V$ such that $\langle 1, 1 \rangle = -1$; (iii) The lowest weight of $V^{(\tilde{h})}$ is positive; (iv) $(V, \sigma_{\tilde{h}})$ satisfies the orbifold condition. Then we have $\dim V_1 + \dim \tilde{V}(\sigma_{\tilde{h}})_1 = 3 \dim V_1^{\sigma_{\tilde{h}}} + 24(1 - \dim(V^{(\tilde{h})})_{1/2})$.

As a consequence, we have the following.

**Lemma 7.5.** Let $W$ and $\tilde{h} \in W_1$ be as before. Then the automorphism $\sigma_{\tilde{h}}$ satisfies the condition (c).

**Proof:** By Proposition 5.3 of [44], we know that $(W, \sigma_{\tilde{h}})$ satisfies the orbifold condition. Thus, $\tilde{h}$ satisfies the conditions in Theorem 7.4 and we have $\dim \tilde{W}(\sigma_{\tilde{h}})_1 = 3 \dim W_1^{\sigma_{\tilde{h}}} + 24(1 - \dim(W^{(\tilde{h})})_{1/2}) - \dim W_1$. We next determine the dimension of $(W^{(\tilde{h})})_{1/2}$. By Lemma 2.7 of [44], the lowest conformal weight of $(L_{F_4}(6, \lambda_1) \otimes L_{sl_3}(2, \lambda_2))^{(\tilde{h})}$ is equal to $l(\lambda_1, \lambda_2) + \sum_{i=1}^{2} \min\{ \langle h_{(i)}|\mu \rangle | \mu \in \Pi(\lambda_i) \} + \langle \tilde{h}|\tilde{h} \rangle /2$, where $l(\lambda_1, \lambda_2)$ is the lowest conformal weight of $L_{F_4}(6, \lambda_1) \otimes L_{sl_3}(2, \lambda_2)$ and $\Pi(\lambda_i)$ is the set of all weights of the irreducible module $L(\lambda_i)$ with the highest weight $\lambda_i$. By a direct computation, the possible pairs $(\lambda_1, \lambda_2)$ such that $L_{F_4}(6, \lambda_1) \otimes L_{sl_3}(2, \lambda_2)$ has integral conformal weights are $(0, 0)$, $(\Lambda_4, \Lambda_1 + \Lambda_2)$, $(3\Lambda_4, \Lambda_1 + \Lambda_2)$, $(4\Lambda_4, 0)$, $(6\Lambda_4, \Lambda_1 + \Lambda_2)$, $(2\Lambda_3 + \Lambda_4, \Lambda_1 + \Lambda_2)$,
(3Λ3,0), (Λ2 + Λ4, 1), (Λ2 + Λ4, 2), (Λ2 + 2Λ4, 2), (Λ2 + 2Λ4, 4), (Λ1 + 3Λ4, 0),
(Λ1 + Λ3 + 2Λ4, 1), (Λ1 + Λ3 + 2Λ4, 2), (Λ1 + Λ3 + 2Λ4, 4), (Λ1 + 2Λ4, 0), (2Λ1, 2Λ1), (2Λ1, 2Λ1), (2Λ1 +
Λ3, 1, 1), the lowest conformal weights are 0, 1, 2, 4, 3, 2, 2, 3, 3, 3, 3, 2, 2, 3, respectively, where Λi denotes the fundamental weights of sl3. Since (Λ1|A1) = 1,
(Λ1|A2) = 2, (Λ1|A3) = 3/2 and (Λ1|A4) = 1, we have dim(W(h))1/2 = 0 by Lemma 4.1 in [40]. It follows that W(σh)1 is a semisimple Lie algebra of dimension 96.

We next determine the Lie algebra structure of W(σh)1. Assume that W(σh)1 \cong \frak{g}_{1,k_1} \oplus \cdots \oplus \frak{g}_{t,k_t}, where \frak{g}_{i,k_i} means the level of \frak{g}_i is equal to k_i. By Theorem 3 in [24], we have h_\psi/k_i = 3, where h_\psi denotes the dual Coxeter number of \frak{g}_i. It follows that the possible Lie subalgebras of W(σh)1 are A2, B2, A5, C5, D4, A8, B5, F4, E6. Moreover, we know that there exists an automorphism a_{W,σh} of W(σh)1 such that the fixed point subalgebra W(σh)1^{a_{W,σh}} is isomorphic to B4A2 (see Subsection 7.4). By Propositions 3.1, 3.3 of [40], W(σh)1 should be isomorphic to A8,3A2,1A2,1. The proof is complete.

7.3. Conjugacy classes of the automorphism group Aut (U). In this subsection, we shall prove that automorphisms of U satisfying the conditions (A), (B) and (C) belong to a unique conjugacy class in Aut (U). Let \Phi be an order two automorphism of U such that U_1^\Phi is isomorphic to B4A2. Then we know that \Phi|_{U_1} is conjugate to \theta \otimes \sigma under Aut (U_1) [32, Chapter X, Theorem 6.1]. In the following, we will further prove that \Phi is conjugate to one of liftings of \theta \otimes \sigma under Aut (U). First, by the similar argument in Lemma 4.2.8 of [61], we have:

Theorem 7.6. Let V be a VOA, \tilde{V} an extension of V and \psi an automorphism of V. Assume that \tilde{V} viewed as a V-module has the decomposition \tilde{V} = V \oplus M^1 \oplus \cdots \oplus M^k, where M^0 = V, M^1, \cdots , M^k are nonisomorphic irreducible V-modules, and that there exists an automorphism \tilde{\psi} of \tilde{V} such that \tilde{\psi}(V) = V and \tilde{\psi}|_V = \psi. Then \{M^0, M^1, \cdots , M^k\} = \{M^0 \circ \psi, M^1 \circ \psi, \cdots , M^k \circ \psi\}.

As a consequence, we have the following.

Theorem 7.7. Let \Phi be as above. Then \Phi is conjugate to one of liftings of \theta \otimes \sigma under Aut (U).

Proof: Since \Phi|_{U_1} is conjugate to \theta \otimes \sigma under Aut (U_1), there exists an automorphism f of U_1 such that \Phi|_{U_1} = f(\theta \otimes \sigma)f^{-1}. It is well-known that f has the form \exp(2\pi \sqrt{-1}u)\mu o \exp(2\pi \sqrt{-1}u)\mu o (1 \otimes \sigma), where u is an element of a Cartan subalgebra of U_1 and \mu is one of the following automorphisms

\theta \otimes 1 \otimes 1, \theta \otimes \varphi \otimes 1, \theta \otimes 1 \otimes \varphi, \theta \otimes \varphi \otimes \varphi, 1 \otimes \varphi \otimes 1, 1 \otimes 1 \otimes \varphi, 1 \otimes \varphi \otimes \varphi, 1 \otimes 1 \otimes 1.
Here, \( \varphi \) denotes the diagram automorphism of \( sl_3 \). Note that \((1 \otimes \sigma)(\theta \otimes \sigma)(1 \otimes \sigma)^{-1} = \theta \otimes \sigma\), we then may assume that \( f \) has the form \( \exp(2\pi\sqrt{-1}u)\mu \).

By assumption, \( U \) viewed as an \( L_{sl_3}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0) \)-module has the following decomposition

\[
L_{sl_3}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0) \oplus L_{sl_3}(3,3\Lambda_1) \oplus L_{sl_3}(1,\Lambda_1) \otimes L_{sl_3}(1,\Lambda_2) \oplus L_{sl_3}(3,3\Lambda_2) \oplus L_{sl_3}(1,\Lambda_2) \oplus L_{sl_3}(1,\Lambda_1)
\]

\[
\oplus \tau_1 \otimes L_{sl_3}(1,\Lambda_1) \otimes L_{sl_3}(1,\Lambda_1) \otimes \tau_2 \otimes L_{sl_3}(1,\Lambda_2) \otimes L_{sl_3}(1,\Lambda_2) \oplus (L_{sl_3}(3,3\Lambda_1) \times \tau_1) \otimes L_{sl_3}(1,\Lambda_2) \otimes L_{sl_3}(1,0)
\]

\[
\oplus (L_{sl_3}(3,3\Lambda_2) \times \tau_2) \otimes L_{sl_3}(1,\Lambda_1) \otimes L_{sl_3}(1,0).
\]

Note that the inner automorphism \( \exp(2\pi\sqrt{-1}u) \) of \( U_1 \) can be lifted to an automorphism of \( U \) and that

\[
L_{sl_3}(3,3\Lambda_1) \otimes L_{sl_3}(1,\Lambda_1) \otimes L_{sl_3}(1,\Lambda_2) \circ (\theta \otimes \varphi \otimes 1)(\theta \otimes \sigma)(\theta \otimes \varphi \otimes 1)^{-1} \cong L_{sl_3}(3,3\Lambda_2) \otimes L_{sl_3}(1,\Lambda_1) \otimes L_{sl_3}(1,\Lambda_2).
\]

It follows from Theorem 7.6 that \( \mu \) cannot be equal to \( \theta \otimes \varphi \otimes 1 \). Note also that

\[
L_{sl_3}(3,3\Lambda_1) \otimes L_{sl_3}(1,\Lambda_1) \otimes L_{sl_3}(1,\Lambda_2) \circ (\theta \otimes \varphi \otimes 1)(\theta \otimes \sigma)(\theta \otimes \varphi \otimes 1)^{-1} \cong L_{sl_3}(3,3\Lambda_2) \otimes L_{sl_3}(1,\Lambda_1) \otimes L_{sl_3}(1,\Lambda_2).
\]

\[
L_{sl_3}(3,3\Lambda_1) \otimes L_{sl_3}(1,\Lambda_1) \otimes L_{sl_3}(1,\Lambda_2) \circ (\theta \otimes \varphi \otimes 1)(\theta \otimes \sigma)(\theta \otimes \varphi \otimes 1)^{-1} \cong L_{sl_3}(3,3\Lambda_2) \otimes L_{sl_3}(1,\Lambda_1) \otimes L_{sl_3}(1,\Lambda_2).
\]

It follows that \( \mu \) cannot be equal to \( 1 \otimes \varphi \otimes 1, \theta \otimes \varphi \otimes 1 \otimes \varphi \) or \( 1 \otimes \varphi \otimes \varphi \). As a result, \( \mu \) is equal to \( \theta \otimes 1 \otimes 1, \theta \otimes \varphi \otimes \varphi, 1 \otimes 1 \otimes 1 \) or \( \varphi \otimes \varphi \). Note that in these cases we have \( \mu(\theta \otimes \sigma)\mu^{-1} = \theta \otimes \sigma \). Thus, we have \( \Phi|_{U_1} = \exp(2\pi\sqrt{-1}u)(\theta \otimes \sigma)\exp(2\pi\sqrt{-1}u)^{-1} \). It follows that \( \exp(2\pi\sqrt{-1}u)^{-1}\Phi\exp(2\pi\sqrt{-1}u) \) is a lifting of \( \theta \otimes \sigma \). The proof is complete.

We next prove that any liftings \( g_1, g_2 \) of \( \theta \otimes \sigma \) of order two are conjugate under \( \text{Aut}(U) \).

**Theorem 7.8.** Let \( g_1, g_2 \) be liftings of \( \theta \otimes \sigma \) of order two. Then \( g_1, g_2 \) are conjugate under \( \text{Aut}(U) \).

**Proof:** By Theorem 4.19, we know that the order of \( g_1g_2^{-1} \) is equal to 1 or 3. If the order of \( g_1g_2^{-1} \) is equal to 1, then \( g_1 = g_2 \). If the order of \( g_1g_2^{-1} \) is equal to 3, then we have \( g_2g_1g_2g_1g_2 = g_1 \). This implies that \( (g_1g_2)^{-1}g_2(g_1g_2) = g_1 \). Therefore, we always have \( g_1, g_2 \) are conjugate under \( \text{Aut}(U) \). \( \square \)

We now in a position to prove our main result in this section.

**Theorem 7.9.** Let \( W^1, W^2 \) be holomorphic VOAs of central charge 24 with weight one Lie algebras \( F_{4,6}A_{2,2} \). Then \( W^1, W^2 \) are isomorphic.

**Proof:** This follows immediately from Theorems 7.1, 7.8 and Lemmas 7.3, 7.5. \( \square \)

As another application of Theorem 4.19, we can also establish the uniqueness of the holomorphic VOA of central charge 24 with one Lie algebra \( E_{7,3}A_{5,1} \).
**Theorem 7.10.** Let $W^1, W^2$ be holomorphic VOAs of central charge 24 with weight one Lie algebras $E_{7,3}A_{5,1}$. Then $W^1, W^2$ are isomorphic.

**Proof:** To apply the reverse orbifold construction method, we take $W$ and $U$ to be holomorphic VOAs of central charge 24 with weight one Lie algebras $E_{7,3}A_{5,1}, A_{8,3}A_{2,1}^2$, respectively, and take $g, p$ to be Lie algebras $E_7A_5, A_7A_2^2U(1)$, respectively. Let $h = \frac{1}{2}(\tilde{A}_2, \tilde{A}_3) \in W$ and take $\psi$ to be the inner automorphism $\sigma_h$ of $W$, where $\tilde{A}_i, \tilde{A}_i^*$ denote the fundamental weights of $E_7$ and $A_5$, respectively. It was proved in [44] that $\sigma_h$ is an involution of $W$ and satisfies the conditions (a), (b), and (c).

We next prove that any involutions $\Phi_1, \Phi_2$ of $U$ satisfying the conditions (A), (B), (C) are conjugate under Aut$(U)$. By assumption, we have $U^{\Phi_1}$ is isomorphic to $A_7A_2^2U(1)$. Note that the fixed point subalgebra of $A_{8,3}A_{2,1}^2$ under the action of the inner automorphism of $\exp(\pi \sqrt{-1}(\tilde{A}_1 + \tilde{A}_2))$ is also $A_7A_2^2U(1)$, where $\tilde{A}_i$ denotes the fundamental weight of $A_8$. Since $\Phi_1$ has order 2, it follows that $\Phi_1|_{U_1}$ is conjugate to $\exp(\pi \sqrt{-1}(\tilde{A}_1 + \tilde{A}_2))$ under Aut$(U_1)$. In particular, there exists an automorphism $f$ of $U_1$ such that $\Phi_1|_{U_1} = f \exp(\pi \sqrt{-1}(\tilde{A}_1 + \tilde{A}_2))f^{-1}$. It is well-known that $f$ has the form $\exp(2\pi \sqrt{-1}u)\mu$ or $\exp(2\pi \sqrt{-1}u)\mu \circ (1 \otimes \sigma)$, where $u$ is an element of a Cartan subalgebra of $U_1$ and $\mu$ is one of the following automorphisms

$$\theta \otimes 1 \otimes 1, \theta \otimes \varphi \otimes 1, \theta \otimes 1 \otimes \varphi, \theta \otimes \varphi \otimes \varphi, 1 \otimes \varphi \otimes 1, 1 \otimes 1 \otimes \varphi, 1 \otimes \varphi \otimes \varphi, 1 \otimes 1 \otimes 1.$$  

Here, $\varphi$ denotes the diagram automorphism of $sl_3$. By the same arguments as in Theorem 7.7, we can take $f$ to be $\exp(2\pi \sqrt{-1}u) \theta \otimes \sigma$. Since $\exp(2\pi \sqrt{-1}u)$ and $\exp(2\pi \sqrt{-1}u)\theta \otimes \sigma$ can be lifted to automorphisms of $U$, it follows that $\Phi_1$ is conjugate to one of liftings of $\exp(\pi \sqrt{-1}(\tilde{A}_1 + \tilde{A}_2))$. Similarly, $\Phi_2$ is conjugate to one of liftings of $\exp(\pi \sqrt{-1}(\tilde{A}_1 + \tilde{A}_2))$. Thus, by the same argument in the proof of Theorem 7.8, we can prove that $\Phi_1, \Phi_2$ are conjugate under Aut$(U)$. Hence, by Theorem 7.1 any two holomorphic VOAs of central charge 24 with the weight one Lie algebra $E_{7,3}A_{5,1}$ are isomorphic. The proof is complete. 

\[\square\]

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