Entanglement properties and ground-state statistics of free bosons

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We calculate analytically the entanglement and Rényi entropies, the negativity and the mutual information together with all the density and many-particle correlation functions for free bosons on a lattice in the ground state, for both homogeneous and inhomogeneous systems. We show that all those quantities can be derived from a multinomial form of the reduced density matrix in the configuration space whose diagonal elements dictate the statistics of the particle distribution, while the off-diagonal coherence terms control the quantum fluctuations. We provide by this analysis a unified approach based on a reduced density matrix technique useful to calculate both the entanglement properties and an infinite number of correlation functions.

INTRODUCTION

In spite of its simplicity the system of non-interacting bosons placed on a $d$-dimensional lattice represents a very peculiar case having particle number fluctuations, within a subsystem $A$, which obey a volume law while the entanglement entropy scales logarithmically $[1,2,5,6]$, violating the so-called area law $[7]$, in the same way as in the ferromagnetic Heisenberg model $[8,11]$ and analogously to conformal field theories $[12,13]$, random $[14]$ or unconventional quantum spin chains $[15]$. Most of the studies devoted to the entanglement properties of bosonic systems are focused on relativistic free bosonic field theories, which can be mapped to coupled harmonic oscillators $[1,2,5,6]$ with an infinite number of modes. We will consider, instead, the entanglement properties and the ground-state statistics of systems made of a fixed number of free non-relativistic bosons which form a condensate on a lattice. In this case, the entanglement entropy and the particle number fluctuations can be explained by assuming a binomial distribution of the particles inside the subsystem under study $[3,4,16,17]$. However this distribution is verified also for a confined non-extended system like bosons in a double well $[18]$. In order to unveil the correct particle distribution in an extended region one has to consider moments of higher order than the variance of the number of particles finding, in this way, the full statistics in the ground state. This state is a Bose-Einstein condensate which can be described by a single-particle wavefunction. In the homogeneous case, all sites are equivalent, therefore geometry and dimension do not play any role in the bipartition of the system, only the number of sites in the subsystems really matter. We show that the reduced density matrix is once again the basic tool containing all the informations needed to explain either the entanglement properties of our quantum system and the many-particle correlation functions.

MODEL AND GROUND STATE

The model describing free bosons hopping on a generic $d$-dimensional lattice is the following

$$H = - \sum_{ij} t_{ij} b_i^\dagger b_j$$

(1)

where $t_{ij}$ is the positive-definite hopping amplitude for a particle to jump from a site $i$ to a site $j$ of the lattice with $V$ sites while $b_i, b_i^\dagger$ are the bosonic annihilation and creation operators. We will consider in particular a translationally invariant system with periodic boundary condition in $d$ dimensions, namely free bosons on a $d$-torus. At zero temperature, all particles are in the ground state, which, for a system made by $N$ particles
in $V$ sites ($V$ is the volume), is given by

$$|\Psi\rangle = \frac{1}{\sqrt{N!}} \left( \sum_j \xi_j b_j^N \right) |0\rangle$$

$$= \sum_{\{n_1, \ldots, n_V\} = 0}^N \frac{\sqrt{N!}}{\prod_{i=1}^V n_i!} \prod_{j=1}^V \xi_j^{n_j} b_j^{n_j} |0\rangle$$

(2)

where $\xi_j$ are complex numbers representing the single-particle wave-function fulfilling the normalization condition $\sum_j |\xi_j|^2 = 1$. For an homogeneous system, with periodic boundary conditions, one has $\xi_j = 1/\sqrt{V}$. The sum over $\{n_1, \ldots, n_V\}$, means the sum over all the configurations $(n_1, \ldots, n_V)$ of the occupation numbers with the constraint $\sum_i n_i = N$, namely the summation notation means

$$\sum_{\{n_1, \ldots, n_V\}} \equiv \sum_{n_1=0}^N \sum_{n_2=0}^{N-n_1} \cdots \sum_{n_V=0}^{N-V+1-n_{V-1}} \delta_{N-\sum_i n_i}$$

**REDUCED DENSITY MATRIX**

Let us now calculate the reduced density matrix $\rho$ after partitioning the full system in two blocks, $A \equiv [1, V_A]$, and $B \equiv [V_A+1, V]$, and tracing out the degrees of freedom belonging to the second block,

$$\rho = \text{Tr}_B(|\Psi\rangle\langle\Psi|) = \sum_{\{n_{V_A+1}, \ldots, n_V\} = 0}^{N_A} \frac{\langle 0_B | \prod_{i=V_A+1}^V b_i^{n_i} | \Psi \rangle}{\sqrt{\prod_{i=V_A+1}^V n_i!}} \times \frac{\langle \Psi | \prod_{i=V_A+1}^V b_i^{m_i} | 0_B \rangle}{\sqrt{\prod_{i=V_A+1}^V n_i!}}$$

(3)

where the sum over all the occupation number configurations $(n_{V_A+1}, \ldots, n_V)$ fulfilling $\sum_{i=V_A+1}^V n_i \leq N$ is denoted by $\sum_{\{n_{V_A+1}, \ldots, n_V\} = 0}^{N_A} \equiv \sum_{n_{V_A+1}=0}^{N_A} \sum_{n_{V_A+2}=0}^{N_A-n_{V_A+1}} \cdots \sum_{n_{V}=0}^{N_A-V+1-n_{V-1}}$ using the previous definition, or can be expressed as

$$\sum_{\{n_{V_A+1}, \ldots, n_V\} = 0}^{N_A} = \sum_{n_{V_A+1}=0}^{N_A} \sum_{n_{V_A+2}=0}^{N_A-n_{V_A+1}} \cdots \sum_{n_{V}=0}^{N_A-V+1-n_{V-1}}$$

and where the vacuum state is split as $|0\rangle = |0_A\rangle|0_B\rangle$. Performing the trace over $B$ we get

$$\rho = \sum_{N_A=0}^{N} \sum_{\{n_1, \ldots, n_{V_A}\} = 0}^{N_A} \frac{N! \left( 1 - \sum_{i=1}^{V_A} |\xi_i|^2 \right)^{N-N_A}}{(N-N_A)! \prod_{i=1}^{V_A} n_i! m_i!} \times \prod_{j=1}^{V_A} \xi_j^{n_j} b_j^{n_j} |0_A\rangle\langle 0_A| \prod_{j=1}^{V_A} \xi_j^{m_j} b_j^{m_j}$$

(4)

This reduced density matrix is a block diagonal matrix which can be written as

$$\rho = \left( \begin{array}{ccc} \rho_N & 0 & \ldots & 0 \\ 0 & \rho_{N-1} & \ldots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \rho_0 \end{array} \right)$$

(5)

where each block $\rho_{N_A}$, with $0 \leq N_A \leq N$, is a $[(N_A + V_A - 1)!/(N_A!V_A)!] \times [(N_A + V_A - 1)!/(N_A!V_A)!]$ fully sparse matrix whose elements, in the configuration space, are

$$\rho_{n_1, \ldots, n_{V_A} m_1, \ldots, m_{V_A}} = \frac{N! \left( 1 - \sum_{i=1}^{V_A} |\xi_i|^2 \right)^{N-N_A}}{(N-N_A)!} \prod_{i=1}^{V_A} \frac{n_i! m_i!}{n_i! m_i!}$$

(6)

with $N_A = \sum_{i=1}^{V_A} n_i = \sum_{i=1}^{V_A} m_i$. The null elements in Eq. (6) are those with $\sum_{i=1}^{V_A} n_i \neq \sum_{i=1}^{V_A} m_i$. It is convenient to write explicitly the diagonal elements of Eq. (6), denoted for simplicity $\rho_{n_1, \ldots, n_{V_A}} = \rho_{n_1, \ldots, n_{V_A}}^{\text{v}}$, and

$$\rho_{n_1, \ldots, n_{V_A}}^{\text{v}} = \frac{N! \left( 1 - \sum_{i=1}^{V_A} |\xi_i|^2 \right)^{N-N_A}}{(N-N_A)!} \prod_{i=1}^{V_A} \frac{n_i! m_i!}{n_i! m_i!}$$

(7)

which has the form of a multinomial distribution [23], and will be useful, as we will be seeing in the next sections, in deriving the density correlation functions.

We notice that Eq. (6) can be factorized so that, for any $0 \leq N_A \leq N$, we can rewrite it as the outer product of two vectors

$$\rho_{N_A} = C_{N_A} \tilde{W} \otimes \tilde{W}^* = C_{N_A} \tilde{W} \tilde{W}^\dagger$$

(8)

where

$$C_{N_A} = \frac{N!}{(N-N_A)!} \left( 1 - \sum_{i=1}^{V_A} |\xi_i|^2 \right)^{N-N_A}$$

$$\tilde{W}$$ a vector with dimension $(N_A + V_A - 1)!/(N_A!V_A)!$ in configuration space whose elements are

$$W_{n_1, \ldots, n_{V_A}} = \frac{V_A}{n_i!} \xi_i^{n_i}$$

(10)

The eigenvalues of Eq. (8) can now be easily calculated. They are all zeros except one which we call $\rho_{N_A}$ (without the hat symbol) given by

$$\rho_{N_A} = \text{Tr}(\rho_{N_A}) = C_{N_A} |\tilde{W}|^2 = C_{N_A} \sum_{n_1, \ldots, n_{V_A}} \prod_{i=1}^{V_A} |\xi_i|^{2n_i}$$

(11)

Performing the summation and using Eq. (9) we get the explicit eigenvalues of the reduced density matrix which have the form of a binomial distribution

$$\rho_{N_A} = \frac{N! \left( 1 - \sum_{i=1}^{V_A} |\xi_i|^2 \right)^{N-N_A}}{(N-N_A)!} \left( \sum_{i=1}^{V_A} |\xi_i|^2 \right)^{N_A}$$

(12)
This result is the same as that we could have obtained resorting to the Schmidt decomposition procedure \[3\]. The advantage of this approach is that we have the full density matrix in the configuration representation whose diagonal elements, fulfilling
\[
\sum_{\{n_1 \ldots n_{N_A}\}} N_A \rho_{n_1 \ldots n_{N_A}} = \rho_{N_A} \tag{13}
\]
will appear in the calculation of the density correlation functions while, as we will see, the off-diagonal elements, instead, are responsible for the many-particle quantum hopping processes.

**ENTANGLEMENT MEASURES**

After finding and diagonalizing the reduced density matrix we can calculate some entanglement properties.

**Entanglement entropy**

The Von Neumann entropy is given by

\[
S = -\text{Tr}(\hat{\rho} \ln \hat{\rho}) = - \sum_{N_A=0}^{N} \rho_{N_A} \ln \rho_{N_A} \tag{14}
\]

where \(\rho_{N_A}\) are the non-null eigenvalues of the reduced density matrix, Eq. \[3\], given in Eq. \[12\]. After performing the trace we get the general result for the entanglement entropy of a Bose-Einstein condensate, also in the presence of space modulations dictated by the single-particle wavefunction \(\xi_i\) inside the subsystem \(A\). For the homogeneous case we have that the single particle wavefunction is uniform everywhere and given by
\[
|\xi_i|^2 = \frac{1}{V} = \frac{\nu}{N} \tag{15}
\]

where we introduced the filling fraction \(\nu = N/V\). In the thermodynamic limit, \(N \to \infty, V \to \infty\), keeping \(\nu\) finite, since
\[
\lim_{N \to \infty} \left(\frac{N!}{(N-N_A)! N_A!}\right) = 1 \quad \text{(16)}
\]

\[
\lim_{N \to \infty} \left(1 - \frac{V_A \nu}{N}\right)^N = e^{-V_A \nu} \tag{17}
\]

Eq. \[12\] becomes simply
\[
\rho_{N_A} = \frac{1}{N_A!} (V_A \nu)^{N_A} e^{-V_A \nu} \tag{18}
\]

In the same limit the diagonal elements of \(\hat{\rho}\), Eq. \[7\], become
\[
\rho_{n_1 \ldots n_{N_A}} = \frac{1}{\prod_{i=1}^{N_A} n_i!} e^{-V_A \nu} \left(\sum_{i=1}^{N_A} n_i\right)^{N_A} \tag{19}
\]

which clearly fulfill Eq. \[13\]. From Eqs. \[14\] and \[18\] we get the following result for the entanglement entropy
\[
S = V_A \nu \left(1 - \ln(V_A \nu)\right) + e^{-V_A \nu} \sum_{n=0}^{\infty} \frac{(V_A \nu)^n}{n!} \ln n! \tag{20}
\]

For \(V_A \ll \nu^{-1}\) the entropy is \(S \approx V_A \nu \left(1 - \ln(V_A \nu)\right)\). For very large values of the filling fraction \(\nu (N \gg V)\), or for large subsystems, namely for \(V_A \nu \gg 1\), Eq. \[20\] becomes
\[
S \approx \frac{1}{2} \ln(2\pi e V_A \nu) \tag{21}
\]

which can be obtained approximating the last term in Eq. \[20\] as follows
\[
e^{-V_A \nu} \sum_{n=0}^{\infty} \frac{(V_A \nu)^n}{n!} \ln n! = \langle \ln N_A! \rangle \rho_{N_A} \sim \ln([V_A \nu]!) + \frac{1}{2} \ln(2\pi e V_A \nu)
\]

where \(\langle \ln N_A! \rangle \rho_{N_A}\) is a statistical average with weight Eq. \[18\], and using the Stirling series
\[
\ln([V_A \nu]!) \sim V_A \nu \ln(V_A \nu) - 1 + \frac{1}{2} \ln(2\pi e V_A \nu)
\]

Actually, in the latter situation, \(V_A \nu \gg 1\), the Poisson distribution Eq. \[12\] turns to a Gaussian one
\[
\rho_{N_A} \approx \frac{e^{-\frac{(N_A - V_A \nu)^2}{2 V_A \nu}}}{\sqrt{2\pi V_A \nu}} \tag{22}
\]

so that one can easily verify that in the thermodynamic limit the entropy is given by Eq. \[21\]. In the generic case, for an inhomogeneous system Eq. \[21\] can be generalized as follows
\[
S \approx \frac{1}{2} \ln \left[2\pi e N \left(\sum_{i=1}^{V_A} |\xi_i|^2 \right) \left(1 - \sum_{i=1}^{V_A} |\xi_i|^2\right)\right] \tag{23}
\]

obtained putting \(N \sum_{i=1}^{V_A} |\xi_i|^2\) instead of \(\nu V_A\) in Eq. \[22\]. Before we proceed a comment is in order. As already discussed in Ref. \[1\], the standard approach described in Refs. \[1, 2\] for calculating the entanglement entropy through the two-point correlation function matrix \(C_{ij} = \langle b_i \bar{b}_j \rangle\), truncated within the subsystem \(A\), does not lead to the correct result for a Bose-Einstein condensate. By that procedure the entanglement entropy reads \(S = \sum_i \left[1 + c_i \ln(1 + c_i) - c_i \ln c_i\right]\) with \(c_i\) the eigenvalues of \(C_{ij}\). For the homogeneous case there is only one non-zero eigenvalue of \(C_{ij}\), namely \(c_i = \nu V_A \delta_{ij}\), therefore, for large \(V_A\) the entropy is \(S \approx \ln(V_A \nu)\), which differs from Eq. \[21\] in the prefactor. This discrepancy originates from the fact that the approach used in Refs. \[1, 2\] relies on the conventional formulation of the Wick’s theorem for correlation functions, which cannot be applied for the ground-state in Eq. \[2\], as we will be seeing in what follows (see Eqs. \[34\] and \[53\]).
Rényi entropies

Let us calculate the Rényi entropies of generic order \( \alpha \) defined by

\[
S_\alpha = \frac{1}{1 - \alpha} \ln \text{Tr}(\hat{\rho}^\alpha)
\]

which, after diagonalizing the reduced density matrix \( \hat{\rho} \), becomes

\[
S_\alpha = \frac{1}{1 - \alpha} \ln \left[ \sum_{N_A=0}^N \rho_{N_A}^\alpha \right]
\]

where \( \rho_{N_A} \) is given in Eq. (23). In the Gaussian regime, namely for \( N \sum_{i=1}^{V_A} |\xi_i|^2 \gg 1 \), we can calculate \( S_\alpha \) analytically using the following result

\[
\text{Tr}(\hat{\rho}^\alpha) \approx \int_{-\infty}^{\infty} dx \left( \frac{e^{-\frac{(x-NP)}{2NP(1-P)}}}{\sqrt{2\pi NP(1-P)}} \right) ^\alpha = \left( \frac{2\pi NP(1-P)}{\alpha} \right)^{\frac{\alpha}{2}}
\]

where \( P = \sum_{i=1}^{V_A} |\xi_i|^2 \), getting

\[
S_\alpha \approx \frac{1}{2} \ln \left( 2\pi V_A \nu \right) + \frac{\ln \alpha}{\alpha - 1}
\]

Notice that in the limit \( \alpha \to 1 \) we recover the von Neumann entropy reported in Eq. (23). In the homogeneous case and in the thermodynamic limit \( NP(1-P) \to V_A \nu \) and we get

\[
S_\alpha \approx \frac{1}{2} \ln \left( 2\pi V_A \nu \right)
\]

which is the generalization of Eq. (21). We notice that the minimum Rényi entropy obtained for \( \alpha \to \infty \) in Eq. (28), known as min-entropy, is finite, \( \frac{1}{2} \ln (2\pi V_A \nu) \).

Negativity

Let us assume now to split the system \( \mathcal{A} \) in two parts \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) with number of sites \( V_1 \) and \( (V_A-V_1) \), respectively. We will calculate the quantum negativity measured after partial transposition of the second block, \( \mathcal{A}_2 \). It is convenient to rewrite Eq. (1) in the following form

\[
\hat{\rho} = \sum_{N_A=0}^N \sum_{\{m_{1\ldots m_{V_A}}\}} \sum_{\{n_{1\ldots n_{V_1}}\}} \frac{C_{N_A}}{V_1! V_A!} \frac{\prod_{i=1}^{V_1} \xi_i^{n_i} b_i^{n_i} |0_{A_1} \rangle \langle 0_{A_1}|}{V_1!} \frac{\prod_{i=1}^{V_A} \xi_i^{m_i} b_i^{m_i} |0_{A_2} \rangle \langle 0_{A_2}|}{V_A!} \\
\times \prod_{i=1}^{V_1} \xi_i^{n_i} b_i^{n_i} |0_{A_1} \rangle \langle 0_{A_1}| \frac{\prod_{i=1}^{V_A} \xi_i^{m_i} b_i^{m_i} |0_{A_2} \rangle \langle 0_{A_2}|}{V_A!}
\]

so that we can write the partial transpose reduced density matrix \( \hat{\rho}^{T_{A_2}} \) exchanging the partial configurations \( \{m_{V_1+1\ldots m_{V_A}}\} \) and \( \{n_{V_1+1\ldots n_{V_A}}\} \). The result matrix is not a block diagonal one anymore. It is much easier considering the homogeneous case in the thermodynamical limit, so that

\[
\hat{\rho} = e^{-V_A \nu} \sum_{n_{V_1+1\ldots m_{V_A}}} \sum_{m_{V_1+1\ldots m_{V_A}}} \rho_{n_{V_1+1\ldots m_{V_A}}}^{T_{A_2}} = e^{-V_A \nu} \sum_{n_{V_1+1\ldots m_{V_A}}} \sum_{m_{V_1+1\ldots m_{V_A}}} \rho_{n_{V_1+1\ldots m_{V_A}}}^{T_{A_2}}
\]

which is an infinite dimensional sparse matrix. We have, therefore, that, in the homogeneous case, \( \xi = \sqrt{\nu/N} \), and for \( N,V \to \infty \), the partial transpose of the reduced density matrix with respect to a block after a bipartition of the system \( \mathcal{A} \) becomes simply

\[
\hat{\rho}^{T_{A_2}} \to \hat{\rho}
\]

therefore the negativity, which counts the number of negative eigenvalues of \( \hat{\rho}^{T_{A_2}} \), vanishes. This result suggests that, in the thermodynamic limit, the state turns to be a so-called entangled PPT (positive partial transpose) state.

Mutual Information

The derivation of the mutual information between the two parts, \( \mathcal{A}_1 \), with size \( V_1 \), and \( \mathcal{A}_2 \), with size \( V_2 \), of the system \( A = A_1 \cup A_2 \) after tracing out all the rest, \( B \), is very simple. Tracing over \( B \cup \mathcal{A}_2 \) and \( B \cup \mathcal{A}_1 \) we get the entropies \( S_{A_1} \) and \( S_{A_2} \) respectively.

We have, therefore, \( S \) given by Eq. (20) and \( S_{A_1} \) and \( S_{A_2} \) with the same form as in Eq. (20), with \( V_1 \) and \( V_2 \), respectively, instead of \( V_A \). The mutual information, defined by \( I = S_{A_1} + S_{A_2} - S \), is then simply given by

\[
I = \nu [V_A \ln V_A - V_1 \ln V_1 - V_2 \ln V_2]
\]

and

\[
+ e^{-V_A \nu} \sum_{n=0}^{\infty} \frac{n^\nu}{n!} \left[ e^{V_2 \nu} V_1^n + e^{V_1 \nu} V_2^n - V_A^n \right] \ln n!
\]

For large filling fraction \( \nu \) or large subsystem sizes \( V_1, V_2 \) (and, therefore, large \( V_A = V_1 + V_2 \)) we can use Eq. (21) for the entropies getting the following simple form for the mutual information

\[
I = \frac{1}{2} \ln \left( \frac{V_1 V_2}{V_1+V_2} \right) + \frac{1}{2} \ln (2\pi \nu e)
\]

Notice that \( A \) can be formed by two disjoint parts, \( A_1 \) and \( A_2 \), which can be far apart from each other. This
means that the two subsystems are equally entangled no matter how distant they are. We notice that the same behavior occurs for some unconventional quantum spin chains [13].

**CORRELATION FUNCTIONS**

In what follows we show that the diagonal elements of the density matrix $\rho$ are useful to calculate the density correlation functions. We will see that looking only at the density-density correlations, the system of free bosons in the ground state at the thermodynamical limit behaves like a classical uncorrelated gas. On the other hand, the two-site single-particle correlations and, in general, the many-particle correlation functions unveil the off-diagonal long range order and the quantum coherence.

**Density correlation functions**

Let us first consider the expectation value of $b_i^\dagger b_j$, namely the single-particle correlation function, which, after some algebra can be found to be

$$\langle \Psi|b_i^\dagger b_j|\Psi\rangle = \frac{\xi_i^* \xi_j}{|\xi_j|^2} \sum_{n_j=0}^{N} n_j \rho_{n_j} = N \xi_i^* \xi_j$$

(34)

where

$$\rho_{n_j} = \frac{N!}{(N-n_j)! n_j!} (1-|\xi_j|^2)^{N-n_j} |\xi_j|^{2n_j}$$

(35)

is the binomial distribution. In particular, defining $\tilde{n}_i = b_i^\dagger b_i$, we have

$$\langle \Psi|\tilde{n}_i|\Psi\rangle = N|\xi_i|^2$$

(36)

The expectation value of the product of two single particle density operators, $\tilde{n}_i \tilde{n}_j$, namely the density-density correlation function $\langle \Psi|\tilde{n}_i \tilde{n}_j|\Psi\rangle = \langle \Psi|b_i^\dagger b_j^\dagger b_j b_i|\Psi\rangle$, reads

$$\langle \Psi|\tilde{n}_i \tilde{n}_j|\Psi\rangle = \sum_{n_i,n_j} n_i n_j \rho_{n_i,n_j} = (N^2-N)|\xi_i|^2 |\xi_j|^2$$

(37)

for any $i \neq j$ and where

$$\rho_{n_i,n_j} = \frac{N! (1-|\xi_i|^2 -|\xi_j|^2)^{N-n_i-n_j}|\xi_i|^{2n_i}|\xi_j|^{2n_j}}{(N-n_i-n_j)! n_i! n_j!}$$

(38)

is a trinomial distribution. Actually, since we can reduce the order of a multinomial distribution summing over some indices

$$\sum_{n_i=0}^{N-\sum_{j \neq i} n_j} \rho_{n_1 \ldots n_i \ldots n_f} = \rho_{n_1 \ldots \hat{n}_i \ldots n_f}$$

(39)

we can use always the reduced density matrix in Eq. (7) to write all the correlation functions choosing arbitrarily a subsystem $A$ which contains the sites involved. For example, for any $i \in [1,V_A]$, we can write

$$\langle \Psi|\hat{n}_i^2|\Psi\rangle = \sum_{\{n_1 \ldots n_{V_A}\}} n_i^2 \rho_{n_1 \ldots n_{V_A}} = \sum_{n_i=0}^{N} n_i^2 \rho_{n_i}$$

(40)

performing the summations of the multinomial distribution, Eq. (7), over all the indices $n_k$ with $k \neq i$ getting, as a result, a binomial distribution, Eq. (35).

In general terms, we find that all the density correlation functions, for $k \leq V_A$, can be written as

$$\langle \Psi|\hat{n}_1 \hat{n}_2 \ldots \hat{n}_k|\Psi\rangle = \sum_{\{n_1 \ldots n_{V_A}\}} n_1 n_2 \ldots n_k \rho_{n_1 \ldots n_{V_A}}$$

(41)

The right-hand-side of Eq. (41) is a statistical average, weighted by the multinomial distribution given in Eq. (7), that we will denote by $\langle \ldots \rangle_\rho$, so we can write

$$\langle \Psi|\hat{n}_1 \hat{n}_2 \ldots \hat{n}_k|\Psi\rangle = \langle n_1 n_2 \ldots n_k \rangle_\rho$$

(42)

We find, therefore, that the reduced density matrix in Eq. (7) really plays the role of a statistical distribution so that all the quantum density correlation functions can be seen as related to the moments of such a distribution. In the thermodynamic limit and in the homogeneous case, we can use Eq. (19) so that from Eq. (42) we can easily calculate all possible density correlation functions exactly, also the expectation values of all possible products of powers of density operators

$$\langle \Psi|\hat{n}_1^{\alpha_1} \ldots \hat{n}_k^{\alpha_k}|\Psi\rangle = \langle \hat{n}_1^{\alpha_1} \ldots \hat{n}_k^{\alpha_k} \rangle_\rho = \prod_{i=1}^{k} B_{\alpha_i}(\nu)$$

(43)

where $B_{\alpha_i}(\nu)$ are Bell polynomials, which fulfill the following recurrence relation

$$B_{\alpha+1}(\nu) = \sum_{\beta=0}^{\alpha} \frac{\alpha!}{(\alpha-\beta)! \beta!} \nu B_{\alpha-\beta}(\nu)$$

(44)

with initial value $B_0(\nu) = 1$. For example, $B_1(\nu) = \nu$, $B_2(\nu) = (\nu^2 + \nu)$, $B_3(\nu) = (\nu^3 + 3\nu^2 + \nu)$, etc.

Before we proceed a comment is in order. We saw that, for the homogeneous case and in the thermodynamic limit, all the density correlation functions factorize, Eq. (43). This means, in particular, that the covariance matrix is diagonal, namely the two-point density-density connected correlation functions vanish when we go to the thermodynamic limit, $\forall i \neq j$,

$$\lim_{N,V \to \infty} \left( \langle \Psi|n_i n_j|\Psi\rangle - \langle \Psi|n_i|\Psi\rangle \langle \Psi|n_j|\Psi\rangle \right) = 0$$

(45)

which means that the particles become uncorrelated.
Particle number fluctuations. Using Eqs. (45) and (46), we can calculate the fluctuations of the total number of particles in the subsystem $A$. Calling $\hat{N}_A = \sum_{i=1}^{V_A} \hat{n}_i$ we have that the variance of this number is

$$\delta N_A = \langle \Psi | \hat{N}_A^2 | \Psi \rangle - \langle \Psi | \hat{N}_A | \Psi \rangle^2 = \sum_{i=1}^{V_A} \left( \langle \Psi | \hat{n}_i^2 | \Psi \rangle \right) - \left( \sum_{i=1}^{V_A} \langle \Psi | \hat{n}_i | \Psi \rangle \right)^2 = N \left[ \sum_{i=1}^{V_A} |\xi_i|^2 - \left( \sum_{i=1}^{V_A} |\xi_i|^2 \right)^2 \right]$$

(46)

For an homogeneous system on a torus we have $|\xi_i|^2 = \frac{1}{V}$ and Eq. (46) becomes simply

$$\delta N_A = V_A \nu \left( 1 - \frac{V_A}{V} \right)$$

(47)

where we used Eq. (11). In the thermodynamic limit, namely for $V \to \infty$, therefore, the fluctuations increase linearly with the volume $V_A$ of the subsystem $A$, consistently with the behavior of the entropy in Eq. (20) in the very dilute regime, $\nu \ll 1$. We clarified by this analysis how to reconcile the logarithmic behavior of the entanglement entropy with the volume law of the particle number fluctuations by means of a unified approach through the reduced density matrix.

Generating function

In order to calculate all the density correlation functions it is convenient to introduce the so called moment generating function $G$, introducing auxiliary source fields $\mu_i$ such that any density correlators can be obtained by deriving many times $G$ with respect to $\mu_i$. This quantity can be calculated as it follows

$$G(\{\mu_i\}) = \sum_{\{n_i\}} \rho_{n_1 \ldots n_{V_A}} e^{\sum_i \mu_i n_i}$$

(48)

which, explicitly, reads

$$G(\{\mu_i\}) = \frac{N! \left( 1 - \sum_{i=1}^{V_A} |\xi_i|^2 \right)^{N-N_A}}{(N-N_A)!} \times \prod_{i=1}^{V_A} \frac{|\xi_i|^{2n_i e^{\mu_i n_i}}}{n_i!}$$

(49)

Performing the summation we easily get

$$G(\{\mu_i\}) = \left[ \left( 1 - \sum_{i=1}^{V_A} |\xi_i|^2 \right) + \sum_{i=1}^{V_A} |\xi_i|^2 e^{\mu_i} \right]^N$$

(50)

Making the derivatives of this expression with respect to $\mu_i$ one can easily get all the density correlation functions. For instance, we have

$$\langle \Psi | n_1^{\alpha_1} \ldots n_k^{\alpha_k} | \Psi \rangle = \frac{\partial \sum_{i=1}^{V_A} \rho_{n_i} \frac{\partial}{\partial \mu_i}}{\partial |\mu_i|} \bigg|_{\mu_i=0}$$

(51)

which is the generalization of Eq. (43) to the inhomogeneous system. For the homogeneous case Eq. (49) reduces to

$$G(\{\mu_i\}) = \left( 1 - \frac{V_A \nu}{N} + \frac{V_A}{N} \sum_{i=1}^{V_A} e^{\mu_i} \right)^N$$

(52)

which, in the thermodynamic limit, namely for $N \to \infty$, becomes simply

$$G(\{\mu_i\}) = \exp \left[ \nu \sum_{i=1}^{V_A} e^{\mu_i} - V_A \right]$$

(53)

Taking the derivatives with respect to $\mu_i$ and the limit $\mu_i \to 0$ we generate the Bell polynomials, so that using Eq. (53) in Eq. (51) we obtain Eq. (43).

Pair correlation functions

We have seen already that the one-particle correlation functions, Eq. (34), are finite. Let us now consider the so-called pair correlation functions which describe the process for a couple of particles to make a quantum hopping from one site to another. From

$$b_j b_j | \Psi \rangle = \sum_{\{n_i\}_{i \neq j}}^{N} n_j (n_j - 1) \prod_{i=1}^{N} |\xi_i|^{n_i b_j^\dagger b_j^{n_j-2}}$$

(54)

after some algebraic steps we get the following expression for the pair correlation functions

$$\langle \Psi | n_1 b_1^\dagger b_j | \Psi \rangle = \xi_1^{2 \xi_j^2} \langle n_j (n_j - 1) \rangle$$

(55)

and using Eqs. (38) and (40) we find

$$\langle \Psi | b_1^\dagger b_1^\dagger b_j b_j | \Psi \rangle = \xi_1^{2 \xi_j^2} (N^2 - N)$$

(56)

Eq. (54) and Eq. (56), valid for any couple of points $(i,j)$ are the manifestation of the off-diagonal long-range order in the system. For the homogeneous case we have $|\xi_i|^2 = \nu / N$ so that, in the thermodynamic limit, Eq. (56) reduces to

$$\langle \Psi | b_1^\dagger b_1^\dagger b_j b_j | \Psi \rangle = \nu^2$$

(57)
This result can be generalized as shown in the next section. For completeness we can calculate analogously the four-point correlation functions, which turn out to be
\[
\langle \Psi | b_{i}^{\dagger} b_{k}^{\dagger} b_{j} b_{l} \rangle = \frac{\xi_{i}^{\alpha} \xi_{k}^{\alpha} \xi_{j}^{\alpha} \xi_{l}^{\alpha}}{\xi_{i}^{2} \xi_{k}^{2} \xi_{j}^{2} \xi_{l}^{2}} (n_{i} n_{k} n_{j} n_{l})_{\rho}. \quad (58)
\]
Using Eq. (57), we can write
\[
\langle \Psi | b_{i}^{\dagger} b_{k}^{\dagger} b_{j} b_{l} \rangle = \xi_{i}^{\alpha} \xi_{k}^{\alpha} \xi_{j}^{\alpha} \xi_{l}^{\alpha} (N^{2} - N) \quad (59)
\]
which, in the homogeneous case and in the thermodynamic limit, is equal to Eq. (57). Finally we notice that these correlation functions do not factorize according to the contraction prescriptions of Wick’s theorem.

Many-particle correlation functions

We generalize the single-particle and the pair correlation functions considering the many-particle correlation functions. For any integer \(\alpha\) we find that these two-point correlation functions are given by the following equation
\[
\langle \Psi | b_{i}^{\dagger} b_{j} | \Psi \rangle = \frac{\xi_{i}^{\alpha} \xi_{j}^{\alpha}}{|\xi_{i}|^{2} |\xi_{j}|^{2}} \left(\frac{n_{i} n_{j}^{1}}{(n_{j} - \alpha)!}\right)_{\rho} \quad (60)
\]
which, performing the summation, reads
\[
\langle \Psi | b_{i}^{\dagger} b_{j} | \Psi \rangle = \frac{2 F_{1}(1,1,\alpha;1)}{\Gamma(1-\alpha)} \times \left(1 - |\xi_{j}|^{2}\right)^{\alpha} \xi_{i}^{\alpha} \xi_{j}^{\alpha} \quad (61)
\]
where \(2 F_{1}(a,b,c;z)\) is a regularized hypergeometric function. For the homogeneous case and in the thermodynamic limit Eq. (61) simplifies as follows
\[
\langle \Psi | b_{i}^{\dagger} b_{j} | \Psi \rangle = \nu^{\alpha} \quad (62)
\]
This quantity describes the quantum hopping of \(\alpha\) particles from the site \(j\) to the site \(i\). The composite operator \(B^{(\alpha)} = b_{i}^{\dagger} b_{j}^{\dagger} b_{i} b_{j}\), therefore, acts within the same sector of fixed number of particles, exchanging two configurations of the occupation numbers,
\[
(n_{1}, \ldots n_{i}, \ldots n_{j}, \ldots) \rightarrow (n_{1}, \ldots (n_{i} + \alpha), \ldots (n_{j} - \alpha), \ldots)
\]
In other words, any correlation function of the type \(\langle \Psi | b_{i}^{\dagger} b_{j} \rangle | \Psi \rangle\) can be written as the trace of the product of a reduced density matrix \(\hat{\rho}\) times an operator \(B^{(\alpha)}\), off-diagonal in the configuration space, with elements
\[
B_{m_{1}, \ldots m_{VA}}^{(\alpha)} = \sqrt{(n_{i} + \alpha) n_{i}} \delta_{m_{i}, n_{i} + \alpha} \prod_{k \neq i,j} \delta_{m_{k}, n_{k}} \quad (63)
\]
The two-point many-particle correlation functions can be written, therefore, as follows
\[
\langle \Psi | b_{i}^{\dagger} b_{j} | \Psi \rangle = \text{Tr}(\hat{\rho} B^{(\alpha)}) \quad (64)
\]
which is a finite quantity only because of the off-diagonal coherence terms in \(\hat{\rho}\). Actually, putting Eqs. (6) and (63) in Eq. (64)
\[
\text{Tr}(\hat{\rho} B^{(\alpha)}) = \sum_{N_{A}=0}^{N} \sum_{N_{A}} \sum_{N_{A}} \rho_{n_{1}, \ldots n_{VA}} B_{m_{1}, \ldots m_{VA}}^{(\alpha)} \quad (65)
\]
we obtain Eq. (60) and, therefore, Eq. (61). Notice that the correlation functions written in Eq. (61) manifestly depend on the off-diagonal elements of the density matrix \(\hat{\rho}\), although in Eq. (60) they are written in terms of the diagonal elements.

Phase fluctuations

As a final remark, let us introduce a space-dependent phase, possibly caused by the presence of an inhomogeneity of the system, in the single particle wavefunction parametrized by a space-dependent modulus and angle
\[
\xi_{j} = z_{j} e^{i \phi_{j}} \quad (65)
\]
where \(z_{j}\) is a real number, so that, contrary to the density correlation functions which always depend on \(|\xi_{j}|^{2}\), the many-particle correlation functions in Eq. (61) will depend on the phase
\[
\langle \Psi | b_{i}^{\dagger} b_{j} | \Psi \rangle \propto e^{i \alpha \phi_{j} - \phi_{i}} \quad (66)
\]
At the same time, also the off-diagonal elements of the density matrix in Eq. (6) are phase dependent, providing that the phase is not uniform, in the following way
\[
\rho_{n_{i}, \ldots n_{VA}} \propto e^{i \sum_{j=1}^{VA} (n_{i} - n_{i}) \phi_{i}} \quad (67)
\]
with the constraint \(\sum_{i=1}^{VA} n_{i} = \sum_{i=1}^{VA} m_{i}\), therefore for constant \(\phi_{i} = \theta\) the phase dependence cancels out.

For large phase fluctuations, the many-particle hopping described by Eq. (63) can be suppressed within the stationary phase approximation, as well as the off-diagonal coherence terms of the reduced density matrix, Eq. (67).

This observation suggests that, after a bipartition, if one could introduce a wide space modulation of the phase or a random phase field, for instance, by breaking time reversal symmetry so that \(t_{ij} \rightarrow t_{ij} e^{i (\theta_{i} - \theta_{j})}\) with \(\theta_{i}\) random variables, one could spoil the coherence and the quantum fluctuations, getting, at the same time, a diagonal reduced density matrix. As a result, we would get an extensive entropy fulfilling a volume law, but with the drawback of a vanishing mutual information. The Von Neumann entropy, therefore, would resemble a “thermal” entropy of a gas of classical particles in a grand canonical ensemble.
CONCLUSIONS

We calculate analytically the reduced density matrix in the configuration space, the entanglement entropy, the Rényi entropies, the quantum negativity and the mutual information between two separated regions for an extended system of free bosons. We show that, after a bipartition, the mixed state described by the reduced density matrix in the thermodynamic limit is an entangled PPT state, with vanishing negativity but finite space-independent mutual information.

Moreover we show that the reduced density matrix written in terms of occupation numbers can be seen as a statistical distribution useful to derive all the particle correlation functions. We find that in the thermodynamic limit, looking only at the density-density correlation functions, the system of free bosons behaves like an uncoupled gas since all the density operators are diagonal in the configuration space and can be calculated exactly at any order from the moment generating function of a multinomial distribution. The coherence among the particles, instead, generates off-diagonal terms in the reduced density matrix and off-diagonal long-range order described by always finite many-particle correlation functions. Finally we discussed how, within this description, a random field might spoil the quantum fluctuations.

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APPENDIX

Here we present some details of the calculations reported in the main text.

Ground state. Assuming $t_{ij}$ a hermitian matrix which can be diagonalized by a unitary transformation $U$,

$$ - \sum_{ij} U_{ki} t_{ij} U_{jq}^\dagger = \epsilon_k \delta_{kq} \tag{68} $$

we can define the canonically transformed fields

$$ \tilde{b}_k = \sum_i U_{ki} b_i \tag{69} $$

since $U$ preserves the commutation relations. The Hamiltonian in Eq. (1) becomes

$$ H = \sum_k \epsilon_k \tilde{b}_k^\dagger \tilde{b}_k. \tag{70} $$

At zero temperature the lowest energy state denoted by $k = k_0$ hosts all the $N$ particles, forming a condensate, so that, calling $\xi_i \equiv U_{ki}^{\dagger}$, the ground state is given by

$$ |\Psi\rangle = \frac{1}{\sqrt{N!}} \left( \tilde{b}_k^\dagger \right)^N |0\rangle = \frac{1}{\sqrt{N!}} \left( \sum_i \xi_i b_i^\dagger \right)^N |0\rangle \tag{71} $$

reported in Eq. (2). For a translationally invariant system $U_{kj} = \frac{1}{\sqrt{V}} e^{-ik \cdot j}$, namely it is the Fourier transformation which diagonalizes the Hamiltonian. The lowest energy state is at $k = 0$, so that we have $\xi_i = U_{ki}^{\dagger} = 1/\sqrt{V}$.

Reduced density matrix. For the reduced density matrix, Eq. (3), we need to evaluate

$$ \langle 0_B | B \prod_{i=V_A+1}^V b_i^{n_i} | \Psi \rangle \tag{72} $$

where $| \Psi \rangle$ is given in Eq. (2), therefore, after splitting the two subsystems $A$ and $B$, it can be written as

$$ \langle 0_B | B \prod_{i=V_A+1}^V b_i^{n_i} | \Psi \rangle = \frac{\sqrt{N!}}{\prod_{i=1}^N n_i!} \prod_{j=1}^N V_A \xi_j^{n_j} b_j^{n_j} |0_A\rangle $$

$$ \times \langle 0_B | B \prod_{i=V_A+1}^V b_i^{n_i} | \Psi \rangle = \sqrt{N!} \prod_{j=V_A+1}^V \xi_j^{n_j} \sum_{\{n_j\}_{j=1}^N} \prod_{i=1}^{V_A} n_i! b_i^{n_i} |0_A\rangle \tag{73} $$

where we used

$$ \langle 0 | b_i^{n_i} b_j^{n_j} | 0 \rangle = \delta_{n_i} \delta_{n_j} \delta_{i} \delta_{j} n! \tag{74} $$

For the trace over $B$ we used the following relation (let us call $V_A = V_A + 1$ for brevity)

$$ \sum_{\{n_j\}_{j=1}^{N_A}} \prod_{j=V_A}^V |\xi_j^{2n_j}| \prod_{j=V_A}^V n_j! = \frac{\prod_{j=V_A}^V |\xi_j^{2n_j}|^2 N - N_A}{(N - N_A)!} \tag{75} $$

getting the final result in Eq. (4).

Correlation functions. From Eq. (2), applying an annihilation operator we get

$$ b_j |\Psi\rangle = \sum_{\{n_i\}_{i=1}^N} n_j \frac{\sqrt{N!}}{\prod_{i=1}^N n_i!} \xi_j^{n_j - 1} \prod_{k \neq j} b_k^{n_k} |0\rangle \tag{76} $$
and using again the identity Eq. (74), after some algebraic steps we get

\[
\langle \Psi | b_j^\dagger b_j | \Psi \rangle = N! \sum_{\{n_1, \ldots, n_N\}} n_j \prod_{k=1}^{N} \frac{\xi_k^{2n_k}}{n_k!} n_j^{\xi_j} \frac{\xi_j!}{\xi_j!}
\]

\[
= N! \sum_{n_j=0}^{N} \sum_{\{n_1, \ldots, n_N\} \neq \emptyset} \prod_{k \neq j} \frac{\xi_k^{2n_k}}{n_k!} n_j^{\xi_j} \frac{\xi_j!}{\xi_j!}
\]

\[
= N! \sum_{n_j=0}^{N} \frac{\xi_j^{n_j}}{n_j!} n_j^{\xi_j} \frac{\xi_j!}{\xi_j!}
\]

\[
= \sum_{n_j=0}^{N} n_j \rho_{n_j} \frac{\xi_j^{n_j}}{n_j!} = N \xi_i \xi_j
\]  \hspace{1cm} (77)

where \(\rho_{n_j}\) is the binomial distribution defined in Eq. (59).

Let us now calculate the density-density correlation functions making the scalar product of two vectors of the form

\[
b_j^\dagger b_j = \sum_{\{n_1, \ldots, n_N\}} n_j \prod_{k=1}^{N} \frac{\xi_k^{2n_k}}{n_k!} n_j^{\xi_j} (b_j)^{n_j} |0\rangle
\]  \hspace{1cm} (78)

so that, after analogous algebraic steps, we have

\[
\langle \Psi | \hat{n}_i \hat{n}_j | \Psi \rangle = N! \sum_{\{n_1, \ldots, n_N\}} \prod_{k=1}^{N} \frac{\xi_k^{2n_k}}{n_k!} n_i n_j
\]

\[
= N! \sum_{\{n_1, \ldots, n_N\}} (\sum_{k \neq i} \xi_k^{2N-n_i-n_j} |\xi_i|^{2n_i} |\xi_j|^{2n_j}) (N-n_i-n_j)! \frac{n_i! n_j!}{n_i! n_j!}
\]

\[
= \sum_{\{n_1, \ldots, n_N\}} n_i n_j \rho_{n_i n_j}
\]  \hspace{1cm} (79)

where \(\rho_{n_i n_j}\) is the trinomial distribution in Eq. (59), and the sum means \(\sum_{\{n_1, \ldots, n_N\}} = \sum_{n_1=0}^{N} \sum_{n_j=0}^{N-n_1} \ldots \sum_{n_N}=0\).

One can easily verify that, in general

\[
\langle \Psi | \hat{n}_1 \hat{n}_2 \ldots \hat{n}_k | \Psi \rangle = N! \sum_{\{n_1, \ldots, n_N\}} \prod_{i=1}^{N} \frac{\xi_i^{2n_i}}{n_i!} n_1 n_2 \ldots n_k
\]

\[
= \sum_{\{n_1, \ldots, n_N\}} n_1 n_2 \ldots n_k \rho_{n_1 \ldots n_k}
\]  \hspace{1cm} (80)

where \(\rho_{n_1 \ldots n_k}\) is a multinomial distribution of order \(k\). We can show now that we can reduce the order of a multinomial distribution summing over some indices, for instance

\[
N - \sum_{j \neq i} n_j \sum_{n_i = 0}^{N} \rho_{n_1 \ldots n_i \ldots n_k} = N! \prod_{j \neq i} \frac{\xi_j^{2n_j}}{n_j!}
\]

\[
\times \sum_{n_i = 0}^{N} \xi_i |\xi_i|^{2n_i} (N - \sum_{j=1}^{\ell} n_j) n_i !
\]

\[
= N! (1 - \sum_{j \neq i} |\xi_j|^{2n_j})(N - \sum_{j=1}^{\ell} n_j) \prod_{j \neq i} |\xi_j|^{2n_j}
\]

\[
\times (N - \sum_{j \neq i} n_j) !
\]  \hspace{1cm} (81)

where in the second equation we recognize the sum of a binomial distribution with \((N - \sum_{j \neq i} n_j)\) total number of particles. In this way we can use just a single multinomial distribution, Eq. (7), to write all the density correlation functions in terms of its moments, providing that the subset \(A\) contains all the sites involved, as reported in Eq. (111) and Eq. (112), where we defined

\[
\langle \hat{n}_{i1} \hat{n}_{i2} \ldots \hat{n}_{ik} \rangle = \sum_{\{n_1, \ldots, n_N\}} n_1 n_2 \ldots n_k \rho_{n_1 \ldots n_N}
\]  \hspace{1cm} (82)

For example the variance and the covariance (for \(i \neq j\)) of the multinomial distribution Eq. (7) are

\[
\text{Var}(n_i) = \langle n_i^2 \rangle - \langle n_i \rangle^2 = N|\xi_i|^2 (1 - |\xi_i|^2)
\]

\[
\text{Cov}(n_i, n_j) = \langle n_i n_j \rangle - \langle n_i \rangle \langle n_j \rangle = -N|\xi_i|^2 |\xi_j|^2
\]  \hspace{1cm} (83)

For the homogeneous case \(|\xi_i|^2 = 1/V = \nu/N\) and in the thermodynamic limit, \(V, N \to \infty\), but finite \(\nu = N/V\), Eq. (7) simplifies to Eq. (110) so that, for instance, \(\text{Var}(n_i) = \nu, \text{Cov}(n_i, n_j) = 0\) and in general terms, since

\[
\sum_{n=0}^{\infty} \frac{\nu^n}{n!} n^\alpha = e^\nu B_\alpha(\nu)
\]  \hspace{1cm} (85)

where \(B_\alpha(\nu)\) are Bell polynomials, we can easily verify the result for any kind of density correlation functions reported in Eq. (43).

For the many-particle correlation functions we have to apply many times \(b_j\) on the ground state

\[
b_j^\dagger b_j^\alpha |\Psi\rangle = \sum_{\{n_1, \ldots, n_N\}} \frac{n_j!}{(n_j - \alpha)!} \prod_{k=1}^{V} |\xi_j|^{2n_k} |b_j^{n_j-\alpha} \prod_{k \neq j} \xi_k^{2n_k} b_j^{n_k} |0\rangle
\]  \hspace{1cm} (86)

so that, making the scalar product with another vector of the same kind, and using Eq. (74), we get

\[
\langle \Psi | b_i^\dagger b_j^\alpha |\Psi\rangle = \frac{\xi_i^\alpha}{\xi_j^\alpha} N! \sum_{\{n_1, \ldots, n_N\}} \frac{n_j!}{(n_j - \alpha)!} \prod_{k=1}^{V} \xi_k^{2n_k} n_k!
\]  \hspace{1cm} (87)
which can be recast in terms of the multinomial distribution as in Eq. (60).

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