Off-shell invariant $D = N = 2$ twisted super Yang–Mills theory with a gauged central charge without constraints

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We formulate $N = 2$ twisted super Yang–Mills theory with a gauged central charge by superconnection formalism in two dimensions. We obtain off-shell invariant supermultiplets and actions with and without constraints, which is in contrast with the off-shell invariant $D = N = 4$ super Yang–Mills formulation with unavoidable constraints.

1. Introduction

Supersymmetry (SUSY) is one of the most important guiding principles in contemporary particle physics. In particular, it has been recognized that $D = N = 4$ super Yang–Mills (SYM) theory plays a crucial role in string-motivated gauge theory formulations [1]. It has also been recognized that $D = N = 2$ and $D = N = 4$ SYM formulation play a special role for the Dirac–Kähler twisting procedure [2,3], which gives links to quantization and supersymmetry [2–10], and to lattice supersymmetry [11–13]. There is a long-standing question as to whether one can find a superspace formulation for $D = N = 4$ SYM to obtain an off-shell invariant formulation. It is known that $D = N = 4$ SYM with SU(4) R-symmetry can be formulated only at the on-shell level [14], while one can find an off-shell invariant SYM formulation if one introduces a central charge and changes the R-symmetry from SU(4) to USp(4) [15–18]. In this case, however, we need a constraint equation which can be seen as a remnant of an equation of motion of higher dimensions. The corresponding superspace formulation has been developed in [19–23]. Harmonic superspace has also been developed in a similar context [24].

One may thus ask whether or not such a constraint is unavoidable for a gauged central charge SYM formulation. In this paper we investigate a $D = N = 2$ off-shell invariant twisted SYM with gauged central charge by superconnection formalism [25–29]. This kind of gauged central charge formulation is called a vector-tensor multiplet and has been investigated intensively [30–36]. It turns out that for the A model ansatz (see later) we obtain an off-shell invariant formulation with a constraint. Although this constraint can be solved in the case of an Abelian gauge group, it is not possible to solve the constraint, at least locally, in the case of a non-Abelian gauge group [37–39]. For the B model ansatz we obtain an off-shell invariant formulation without any constraints.
This paper is organized as follows: In Sect. 2 we first discuss twisted superalgebra with central charges. In Sect. 3 we consider three models for three different supercurvature ansatz. Then we obtain supermultiplets and actions in Sect. 4. We then summarize the results and discuss them in the last section.

2. $D = N = 2$ twisted superalgebra with central charge

In this section we introduce central charge to the $N = 2$ superalgebra in two dimensions and perform Dirac–Kähler twisting. We concentrate on Euclidean spacetime in this paper according to the twisting procedure [2,3].

2.1. Superalgebra with central charge

In two-dimensional Euclidean spacetime, $\gamma$-matrices satisfying

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta^\mu_\nu$$

and charge conjugation matrix $C$ can be chosen to satisfy [40]

$$C\gamma^\mu C^{-1} = \gamma^\mu, \quad C^T = C. \quad (2.1)$$

Thus we can choose the representation of these matrices as

$$\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = 1, \quad \gamma^5 \equiv i\gamma^1\gamma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \quad (2.2)$$

Note that $\gamma^\mu$ and $\gamma^5$ are symmetric and antisymmetric matrices, respectively.

The general extended superalgebra is given by

$$\{Q_{ai}, Q_{bj}\} = 2\delta_{ij}\gamma^\mu_{a\beta}P_\mu,$$

$$[Q_{ai}, R_1] = iS_{ij}Q_{aj},$$

$$[Q_{ai}, R_2] = S'_{ij}\gamma^5_{a\beta}Q_{bj},$$

$$[Q_{ai}, P_\mu] = [R_i, P_\mu] = [P_\mu, P_\nu] = [R_i, R_j] = 0, \quad (2.3)$$

where $Q_{ai}$ is supercharge and $R_1, R_2$ are the generators of the R-symmetry. Here we consider the $D = N = 2$ case. The Majorana condition is given by $Q_{ai} = Q_{ai}^\ast$. Jacobi identities w.r.t. $Q, Q, R_1$ and $Q, Q, R_2$ lead, respectively, to $S_{ij} = -S_{ji}$ and $S'_{ij} = -S'_{ji}$, which means $R_1$ and $R_2$ generate $U(1)$ symmetry.

We now introduce possible extra terms as follows:

$$\{Q_{ai}, Q_{bj}\} = 2\delta_{ij}\gamma^\mu_{a\beta}P_\mu + 2\delta_{a\beta}U_{ij} + 2\gamma^5_{a\beta}V_{ij}, \quad (2.4)$$

where $U_{ij} = U_{ji}, V_{ij} = -V_{ji}$ to be consistent with the simultaneous replacements of $\alpha \leftrightarrow \beta$ and $i \leftrightarrow j$. $U_{ij}$ and $V_{ij}$ get the following restrictions according to a Jacobi identity w.r.t. $Q, Q, R_1$:

$$U_{ik}S_{kj} - S_{ik}U_{kj} = 0, \quad V_{ik}S_{kj} + S_{ik}V_{kj} = 0. \quad (2.5)$$

We can then solve the constraints up to an overall constant as

$$U_{ij} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V_{ij} \sim \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (2.6)$$

On the other hand, a Jacobi identity w.r.t. $Q, Q, R_2$ leads to the relation:

$$U_{ik}S'_{kj} + S'_{ik}U_{kj} = 0, \quad V_{ik}S'_{kj} - S'_{ik}V_{kj} = 0, \quad (2.7)$$
which can be solved as
\[ U_{ij} \sim \begin{pmatrix} u' & u \\ u & -u' \end{pmatrix}, \quad V_{ij} = 0, \] (2.8)
where \( u, u' \) are real parameters. The solutions (2.6) and (2.8) are not compatible. In other words, we cannot keep both the \( R_1 \) and \( R_2 \) symmetries. In case where we choose the \( R_1 (\equiv R) \) symmetry we obtain the following algebra:
\begin{align*}
\{Q_{\alpha i}, Q_{\beta j}\} &= 2 \delta_{ij} \gamma^\mu P_\mu + 2 \delta_{\alpha \beta} \delta_{ij} U_0 + 2 \gamma^5 \gamma_{ij} V_5, \\
\{Q_{\alpha i}, R\} &= i S_{ij} Q_{\alpha j}, \\
[U_0, \text{any}] &= [V_5, \text{any}] = 0,
\end{align*}
(2.9) (2.10) (2.11)
where we identify \( U_0 \) and \( V_0 \) as central charges. On the other hand, if we choose the \( R_2 \) symmetry, we cannot carry out the Dirac–Kähler twisting procedure. We thus do not choose this case.

### 2.2. Twisted superalgebra

The Dirac–Kähler twisting procedure includes two steps: expansion of the supercharge by the complete set of \( \gamma \)-matrices and redefinition of the Lorentz rotation generator [2,3,20,27–29].

We identify the representations of R-symmetry as that of spinor, and treat the extended SUSY suffix and spinor suffix of supercharge \( Q_{\alpha i} \) in the same manner. We thus expand the charge as
\[ Q_{\alpha i} = (1s + \gamma^\mu s_\mu - i \gamma^5 \tilde{s})_{\alpha i}, \] (2.12)
where \( s, s_\mu \), and \( \tilde{s} \) are called twisted supercharges. Note that these supercharges can be expressed by the original charge as
\[ s = \frac{1}{2} \text{tr} Q, \quad s_\mu = \frac{1}{2} \text{tr} \gamma^\mu Q, \quad \tilde{s} = -\frac{1}{2} \text{tr} \gamma^5 Q. \] (2.13)
The charges may look strange because \( s_\mu \) has, for example, a vector suffix even though it is a fermionic charge. This Dirac–Kähler mechanism can be understood in the following way. In two dimensions, the Lorentz generator is represented by one component generator \( J \) satisfying
\[ [P_\mu, J] = -i \epsilon_{\mu \nu} P_\nu, \] (2.14)
\[ [Q_{\alpha i}, J] = -\frac{1}{2} \gamma^5 \gamma_{\alpha i} Q_5. \] (2.15)
On the other hand, we can rewrite (2.10) in the same form as (2.15) because \( S_{ij} \) and \( \gamma^5_{ij} \) are both antisymmetric and thus can be chosen to be proportional to each other, then
\[ [Q_{\alpha i}, R] = -\frac{1}{2} \gamma^5_{ij} Q_{\alpha j}. \] (2.16)
Thus we can define \( J' = J + R \) and obtain
\[ [s, J'] = 0, \quad [s_\mu, J'] = -i \epsilon_{\mu \nu} s_\nu, \quad [\tilde{s}, J'] = 0, \] (2.17)
from (2.13), (2.15), and (2.16). These relations mean that twisted supercharges \( s, s_\mu \), and \( \tilde{s} \) transform as scalar, vector, and (pseudo-)scalar under \( J' \), respectively.

As can be seen above, the equivalence between the Lorentz group and the R-symmetry group is required to realize the Dirac–Kähler twist. The R-symmetry group is inevitably a compact group and
thus the Lorentz group also needs to be compact. There is a natural reason that Euclidean spacetime is chosen.

The algebra among the twisted supercharges is derived from (2.9) and (2.13) as follows:

\[ \{ s, s_\mu \} = p_\mu, \quad \{ \tilde{s}, s_\mu \} = -\epsilon_{\mu\nu} p_\nu, \quad \{ s, \tilde{s} \} = 0, \]
\[ s^2 = \tilde{s}^2 = \frac{1}{2}(U_0 - V_5), \quad \{ s_\mu, s_\nu \} = \delta_{\mu\nu}(U_0 + V_5). \tag{2.18} \]

This is the \( N = 2 \) twisted superalgebra with central charges in two dimensions.

### 3. Ansatz on supercurvature

In this section we consider the so-called superconnection formalism and find appropriate ansatz on supercurvatures based on the algebra derived in the previous section [20,27–29].

We introduce superfields in the superspace parametrized by \((x_\mu, \theta_A, z)\)

\[ \Phi(x_\mu, \theta_A, z) = \phi(x_\mu, z) + \theta_A \phi_A(x_\mu, z) + \frac{1}{2} \theta_A \theta_B \phi_{AB}(x_\mu, z) + \cdots, \tag{3.1} \]

where \( \theta_A \) represents \( \theta, \theta_\mu, \) and \( \tilde{\theta} \), which are Grassmann coordinates, and \( z \) is a real parameter associated with a central charge. Using the supercharge differential operator \( Q_A \) generating a parameter shift in the superspace, we define the supertransformations of the component fields \( \phi, \phi_A, \ldots \) as follows:

\[ \delta_\xi \Phi(x, \theta_A, z) = \delta_\xi \phi(x, z) + \theta_A \delta_\xi \phi_A(x, z) + \frac{1}{2} \theta_A \theta_B \delta_\xi \phi_{AB}(x, z) + \cdots \]
\[ \equiv (\xi Q + \xi^\mu Q_\mu + \tilde{\xi} \tilde{Q}) \Phi(x, \theta_A, z), \tag{3.2} \]

where \( \xi_A \) is a Grassmann parameter.

One can find the supercovariant derivative \( D_A \) which anticommutes with \( Q_A \) and then introduce a fermionic gauge supercovariant derivative as

\[ \nabla_A = D_A - i\Gamma_A, \tag{3.3} \]

where \( \Gamma_A \) is a fermionic superfield called a superconnection. Similarly, bosonic gauged supercovariant derivatives are introduced as

\[ \nabla_\mu = \partial_\mu - i\Gamma_\mu, \quad \nabla_z = \partial_z - i\Gamma_z, \tag{3.4} \]

where \( \Gamma_\mu \) and \( \Gamma_z \) are bosonic superfields. The gauge transformation of \( \nabla_I \equiv \{ \nabla_\mu, \nabla_A, \nabla_z \} \) is defined as

\[ \nabla_I' = e^K \nabla_I e^{-K}, \quad \text{or} \quad \delta_K \nabla_I = [\nabla_I, K]. \tag{3.5} \]

where \( K \) is a gauge parameter superfield. The zeroth-order terms of \( \Gamma_A \) and \( \Gamma_z \) w.r.t. \( \theta_A \) can be taken to be 0 by choosing the Wess–Zumino gauge, while that of \( \Gamma_\mu \) is defined as a usual gauge connection:

\[ \Gamma_A | = 0, \quad \Gamma_\mu | = A_\mu, \tag{3.6} \]

where \( | \) represents the zeroth-order term w.r.t. \( \theta_A \). We can thus define the standard gauge-covariant derivative as

\[ \nabla_\mu | \equiv D_\mu = \partial_\mu - iA_\mu. \tag{3.7} \]

We then define the supercurvatures by (anti-)commutation relations of all pairs of \( \nabla_I \). The supercurvatures transform gauge-covariantly under (3.5). Then some suitable ansatz on supercurvatures...
Table 1. Twisted version of supercurvature ansatz of (3.8). For example, \(\{\nabla, \nabla\} = X - X', [\nabla, \nabla_\mu] = -iF_\mu\). The positions of \(\nabla_\mu\) reflect those of \(P_\mu\) in (2.18).

| \(\nabla\) | \(\nabla\) | \(\nabla_v\) | \(\nabla_\mu\) | \(\nabla_z\) |
|---|---|---|---|---|
| \(\nabla\) | \(X - X'\) | 0 | \(-i(\nabla_\mu + iX_v)\) | \(-iF_\mu\) | \(iG\) |
| \(\nabla\) | \(X - X'\) | \(i\epsilon_{\nu\rho}(\nabla_\mu - iX_\rho)\) | \(-iF_\mu\) | \(iG\) |
| \(\nabla_\mu\) | \(\delta_\mu\nu(X + X')\) | \(-iF_\mu\) | \(iG_\mu\) |
| \(\nabla_\mu\) | \(-iF_\mu\) | \(iG_\mu\) |
| \(\nabla_z\) | \(-iW + \nabla_\nu\) | \(-i\nabla_\mu\) | \(-iF_\mu\) | \(iG\) |

Table 2. Supercurvature ansatz of the A model. Corresponding to the ± sign choice of algebra \(\{\nabla_\mu, \nabla_v\} = \pm\delta_\mu\nu(iW + \nabla_\nu)\), we take \(X = \nabla_\nu\) for + and \(X' = -\nabla_\nu\) for -.

| \(\nabla\) | \(\nabla\) | \(\nabla_v\) | \(\nabla_\mu\) | \(\nabla_z\) |
|---|---|---|---|---|
| \(\nabla\) | \(-iW + \nabla_\nu\) | 0 | \(-i\nabla_\mu\) | \(-iF_\mu\) | \(iG\) |
| \(\nabla\) | \(-iW + \nabla_\nu\) | \(i\epsilon_{\nu\rho}\nabla_\mu\) | \(-iF_\mu\) | \(iG\) |
| \(\nabla_\mu\) | \(\pm\delta_\mu\nu(iW + \nabla_\nu)\) | \(-iF_\mu\) | \(iG_\mu\) |
| \(\nabla_\mu\) | \(-iF_\mu\) | \(iG_\mu\) |
| \(\nabla_z\) | 0 |

leads to an irreducible supermultiplet. To find such an ansatz it is useful to introduce supercurvatures \(X, X', X_\mu\) defined as

\[
\{\nabla_\alpha i, \nabla_\beta j\} = -2i\delta_\alpha j \gamma^\mu_{\alpha\beta} \nabla_\mu + 2\delta_\alpha j \delta_\beta \delta_\mu X + 2\gamma^5_{\alpha\beta}\gamma^5_{ij} X' + 2\delta_\alpha j \gamma^\mu_{ij} X_\mu, \tag{3.8}
\]

where \(\nabla_\alpha i\) are gauged supercovariant derivatives corresponding to \(Q_\alpha i\) in (2.9). The right-hand side in (3.8) are the most general terms for consistency with simultaneous replacement of \(\alpha \leftrightarrow \beta\) and \(i \leftrightarrow j\). As can be seen from (2.9), \(X\) and \(X'\) can be identified as a gauged central charge of \(U_0\) and \(V_5\), respectively. It may be further possible to identify \(X\) or \(X'\) as \(\nabla_\nu\). Table 1 shows the relations between the gauged supercovariant derivative and supercurvatures given in (3.8) in the twisted space. It is, in principle, possible to find different types of supercurvature ansatz.

We eventually find three types of ansatz. The first ansatz is shown in Table 2. Here, one of \(X\) and \(X'\) is identified as \(\nabla_\nu\). We name it an A model ansatz when the bosonic scalar supercurvature \((W\) in the case) is placed in diagonal positions. It is also possible to include \(X_\mu\) as supercurvatures. In this case, the bosonic vector supercurvatures are placed in off-diagonal positions, which we call a B model ansatz. One naive candidate for a B model ansatz is given in Table 3. The Jacobi identities, however, lead to \(G_I = G_\mu = 0\), which coincides with a model without central charge. On the other hand, it is possible to formulate two kinds of B model ansatz for one central charge by imposing \(U_0 = \pm V_5\). In Tables 4 and 5 we show the two kinds of ansatz, which we name the B(0,0,Z) model ansatz and B(Z,Z,0) model ansatz, respectively.

Once suitable ansatz on the supercurvatures are obtained, a set of relations between the supercurvatures can be derived from the Jacobi identities w.r.t. \(\nabla_I\), by which the degrees of freedom of the component fields are reduced. We use the notation \(\nabla W \equiv [\nabla, W]\). For example, the component fields in an irreducible supermultiplet can be defined as

\[
W = A, \quad \nabla W = \rho, \ldots, \text{if } \nabla W \neq 0, \tag{3.9}
\]
Table 3. Naive candidate for B model ansatz.

| $\nabla$ | $\tilde{\nabla}$ | $\nabla_v$ | $\nabla_w$ | $\nabla_z$ |
|---|---|---|---|---|
| $\nabla$ | $\nabla_z$ | 0 | $-i(\nabla_w + F_v)$ | $-iF_w$ | $iG$ |
| $\tilde{\nabla}$ | $\nabla_z$ | $i\epsilon_{\nu\rho}(\nabla_{\rho} - F_{\rho})$ | $-i\tilde{F}_w$ | $i\tilde{G}$ |
| $\nabla_\mu$ | $\nabla_z$ | $\pm\delta_{\mu\nu}\nabla_z$ | $-iF_{\mu\nu}$ | $iG_{\mu}$ |
| $\nabla_\mu$ | $\nabla_z$ | $-iF_{\mu\nu}$ | $iG_{\mu}$ | 0 |

Table 4. Supercurvature ansatz of the B(0,0,Z) model.

| $\nabla$ | $\tilde{\nabla}$ | $\nabla_v$ | $\nabla_w$ | $\nabla_z$ |
|---|---|---|---|---|
| $\nabla$ | 0 | 0 | $-i(\nabla_w + F_v)$ | $-iF_w$ | $iG$ |
| $\tilde{\nabla}$ | 0 | $i\epsilon_{\nu\rho}(\nabla_{\rho} - F_{\rho})$ | $-i\tilde{F}_w$ | $i\tilde{G}$ |
| $\nabla_\mu$ | $\nabla_z$ | $\delta_{\mu\nu}\nabla_z$ | $-iF_{\mu\nu}$ | $iG_{\mu}$ |
| $\nabla_\mu$ | $\nabla_z$ | $-iF_{\mu\nu}$ | $iG_{\mu}$ | 0 |

Table 5. Supercurvature ansatz of the B(Z,Z,0) model.

| $\nabla$ | $\tilde{\nabla}$ | $\nabla_v$ | $\nabla_w$ | $\nabla_z$ |
|---|---|---|---|---|
| $\nabla$ | $\nabla_z$ | 0 | $-i(\nabla_w + F_v)$ | $-iF_w$ | $iG$ |
| $\tilde{\nabla}$ | $\nabla_z$ | $i\epsilon_{\nu\rho}(\nabla_{\rho} - F_{\rho})$ | $-i\tilde{F}_w$ | $i\tilde{G}$ |
| $\nabla_\mu$ | $\nabla_z$ | 0 | $-iF_{\mu\nu}$ | $iG_{\mu}$ |
| $\nabla_\mu$ | $\nabla_z$ | $-iF_{\mu\nu}$ | $iG_{\mu}$ | 0 |

where $W$ represents a supercurvature while $A$ and $\rho$ are the bosonic and fermionic component fields, respectively. The supertransformations and central charge transformations are obtained by

$$sA = s(W|) = QW| = DW| = D W| - i[\Gamma, W]| = \nabla W| = \rho.$$  \hspace{1cm} (3.10)

The third equality holds at the zeroth order of $\theta_A$ while the fourth equality holds due to the first relation of (3.6). More-complicated supertransformations can be defined by more sophisticated Jacobi identities. One thus obtains all supertransformations of each component field in an irreducible supermultiplet.

4. Supermultiplets and actions

We derive the supermultiplets and the actions for each model with supercurvature ansatz found in the previous section. The A model, B(0,0,Z) model and B(Z,Z,0) model are considered in Sects. 4.1, 4.2, and 4.3, respectively. Note that $N = 2$ supersymmetric theory in two-dimensional Euclidean spacetime generally has four bosonic and four fermionic degrees of freedom at the off-shell level.
We now consider the following algebra:

\[
\{ s, s_\mu \} = P_\mu, \quad \{ \tilde{s}, s_\mu \} = -\epsilon_\mu\nu P_\nu, \quad \{ s, \tilde{s} \} = 0,
\]

\[
s^2 = \tilde{s}^2 = \frac{1}{2} Z, \quad \{ s_\mu, s_\nu \} = \pm \delta_\mu\nu Z, \quad [ s_I, Z ] = 0, \tag{4.1}
\]

where + represents the case \( Z = U_0 \) and - represents the case \( Z = -V_5 \) of (2.18) in the double sign.

The corresponding supercharge differential operators for the superspace are given by

\[
Q = \frac{\partial}{\partial \theta} + \frac{i}{2} \theta_\mu \partial_\mu + \frac{i}{2} \theta \partial_z,
\]

\[
Q_\mu = \frac{\partial}{\partial \theta_\mu} + \frac{i}{2} \theta \partial_\mu - \frac{i}{2} \epsilon_\mu\nu \partial_\nu \pm \frac{i}{2} \theta_\mu \partial_z,
\]

\[
\tilde{Q} = \frac{\partial}{\partial \tilde{\theta}} - \frac{i}{2} \theta_\mu \epsilon_\mu\nu \partial_\nu + \frac{i}{2} \tilde{\theta} \partial_z, \tag{4.2}
\]

where \( P_\mu \) and \( Z \) are represented by \(-i\partial_\mu \) and \(-i\partial_z \), respectively. The supercovariant derivatives are then found as

\[
D = \frac{\partial}{\partial \theta} - \frac{i}{2} \theta_\mu \partial_\mu - \frac{i}{2} \theta \partial_z,
\]

\[
D_\mu = \frac{\partial}{\partial \theta_\mu} - \frac{i}{2} \theta \partial_\mu + \frac{i}{2} \tilde{\theta} \epsilon_\mu\nu \partial_\nu \mp \frac{i}{2} \theta_\mu \partial_z,
\]

\[
\tilde{D} = \frac{\partial}{\partial \tilde{\theta}} + \frac{i}{2} \theta_\mu \epsilon_\mu\nu \partial_\nu - \frac{i}{2} \tilde{\theta} \partial_z, \tag{4.3}
\]

where \( \{ Q_A, D_B \} = 0 \). It should be noted that \( D_A \) satisfies the same algebraic relations as (4.1) with the identification of \( D_A \rightarrow s_A \), while \( Q_A \) satisfies similar relations with the replacements \( Q_A \rightarrow s_A \), \( P_\mu \rightarrow -P_\mu \), and \( Z \rightarrow -Z \) in (4.1).

We now consider the supercurvature ansatz in Table 2. The following relations can be derived by Jacobi identities:

\[
\nabla_\mu \nabla W = \epsilon_\mu\nu \nabla_\nu \tilde{\nabla} W, \quad F_\mu = -i \nabla_\mu W, \quad \tilde{F}_\mu = -i \epsilon_\mu\nu \nabla_\nu W,
\]

\[
F_{\mu\nu} = \pm i \delta_\mu\nu \nabla W \mp i \epsilon_\mu\nu \tilde{\nabla} W, \quad F_{\mu\nu} = \pm \epsilon_\mu\nu \tilde{\nabla} \nabla W + \frac{1}{2} \epsilon_\mu\nu \epsilon_{\rho\sigma} \nabla_\rho \nabla_\sigma W,
\]

\[
G = \nabla W, \quad \tilde{G} = \tilde{\nabla} W, \quad G_\mu = -\nabla_\mu W, \quad G_\mu = 2i \nabla_\mu \nabla W - \nabla_\mu W. \tag{4.4}
\]

In addition to these, we need to impose the following relation:

\[
\nabla \tilde{\nabla} W = \frac{1}{2} \epsilon_{\mu\nu} \nabla_\mu \nabla_\nu W. \tag{4.5}
\]

This relation is not derived by a Jacobi identity but interpreted as a constraint on supercurvatures to kill the reducibility of representation.

The component fields are then defined as

\[
W = \phi, \quad \nabla W = \rho, \quad \tilde{\nabla} W = \tilde{\rho}, \quad \nabla_\mu W = \lambda_\mu, \quad \nabla_z W = D, \quad G_\mu = g_\mu, \tag{4.6}
\]

where \( \phi, D, \) and \( g_\mu \) are bosonic fields, and \( \rho, \tilde{\rho}, \) and \( \lambda_\mu \) are fermionic fields. Table 6 shows the supertransformations of each component field. Note that \( F_{\mu\nu} = F_{\mu\nu} = i[D_\mu, D_\nu] \) is a curvature in
the usual gauge theory. Off-shell closure of the supertransformations up to gauge transformations is shown with the following constraint on the component fields:

\[ i D_\mu g_\mu \mp [\phi, D] - (\lambda_\mu, \lambda_\mu) \mp \{\rho, \rho\} \mp \{\tilde{\rho}, \tilde{\rho}\} = 0. \tag{4.7} \]

One can regard this constraint as the same type of constraint found for the \( D = N = 4 \) SYM case \[17,18\].

Because of the constraint, the degrees of freedom of \( g_\mu \) can be regarded as one. Thus, the bosonic degrees of freedom at the off-shell level is four (\( \phi, A_\mu, D, g_\mu \)). Note that the gauge field \( A_\mu \) has one bosonic degree of freedom at the off-shell level.

For an Abelian gauge group the constraint (4.7) becomes simply

\[ \partial_\mu g_\mu = 0, \tag{4.8} \]

which can be solved as

\[ g_\mu = \epsilon_{\mu \nu} \partial_\nu B. \tag{4.9} \]

The degrees of freedom is, in fact, one. The explicit form of an action which includes field \( B \) is

\[
S = \int d^2 x \text{Tr} \left( \pm \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{4} F^2_{\mu \nu} - \frac{1}{2} D^2 \pm \frac{1}{2} (\partial_\mu B)^2 \mp 2i \lambda_\mu (\partial_\mu \rho - \epsilon_{\mu \nu} \partial_\nu \tilde{\rho}) \\
+ e \left( \frac{1}{2} \phi \epsilon_{\mu \nu} F_{\mu \nu} + 2 \rho \tilde{\rho} + BD + \epsilon_{\mu \nu} \lambda_\mu \lambda_\nu \right) \right), \tag{4.10}
\]

where \( e \) is a parameter with mass dimension 1. In this case, invariance of the action and closure of the superalgebra are satisfied without constraints. It is interesting to note that the topological BF term is included in the action. For a non-Abelian gauge group the constraint cannot be solved locally \[37–39\].

Finally, one can find an action for a non-Abelian gauge group as

\[
S = \int d^2 x \text{Tr} \left( \pm \frac{1}{2} (D_\mu \phi)^2 - \frac{1}{4} F^2_{\mu \nu} - \frac{1}{2} D^2 \pm \frac{1}{2} g^2_{\mu \nu} \mp 2i \lambda_\mu (D_\mu \rho - \epsilon_{\mu \nu} D_\nu \tilde{\rho}) \\
- i \phi(\rho, \rho) - i \phi(\tilde{\rho}, \tilde{\rho}) \mp i \phi(\lambda_\mu, \lambda_\mu) \right). \tag{4.11}
\]

It is worth mentioning that this action cannot be derived by superspace.

In this subsection we found a SYM formulation with a constraint. In the next subsections we find SYM formulations without constraints.
4.2. $B(0,0,Z)$ model

The following algebra is considered:

\[
\{s, s_{\mu}\} = P_{\mu}, \quad \{\bar{s}, s_{\mu}\} = -\epsilon_{\mu\nu} P_{\nu}, \quad \{s, \bar{s}\} = 0, \\
s^2 = \bar{s}^2 = 0, \quad \{s_{\mu}, s_{\nu}\} = \delta_{\mu\nu} Z, \quad \{s_I, Z\} = 0, 
\]  \hspace{1cm} (4.12)

where $Z = 2U_0 = 2V_5$ in (2.18). The corresponding supercharge and supercovariant derivative differential operators are given by

\[
Q = \frac{\partial}{\partial \theta} + i \frac{1}{2} \theta_{\mu} \partial_{\mu}, \quad Q_{\mu} = \frac{\partial}{\partial \theta_{\mu}} - \frac{1}{2} \theta_{\mu} \partial_{\mu} + i \frac{1}{2} \theta_{\mu} \partial_{\mu} + \frac{1}{2} \theta_{\mu} \partial_{\mu}, \quad \tilde{Q} = \frac{\partial}{\partial \tilde{\theta}} - \frac{1}{2} \theta_{\mu} \partial_{\mu}, \quad D_{\mu} = \frac{\partial}{\partial \theta_{\mu}} - \frac{1}{2} \theta_{\mu} \partial_{\mu}. 
\]  \hspace{1cm} (4.13)

The $B(0,0,Z)$ model ansatz is shown in Table 4. The following relations are derived by Jacobi identities:

\[
G_{\mu} = 0, \quad \nabla F_{\mu} = \epsilon_{\mu\nu} \nabla \nabla F_{\nu}, \quad \nabla \nabla F_{\nu} + \nabla F_{\nu} \nabla F_{\mu} = \delta_{\mu\nu} \nabla \nabla F_{\rho}, \\
F_{\mu} = -i \nabla F_{\mu}, \quad \bar{F}_{\mu} = i \nabla \nabla F_{\mu}, \\
F_{\mu\nu} = -i \frac{1}{2} \delta_{\mu\nu} (\nabla \nabla F_{\rho} - G) + i \frac{1}{2} \epsilon_{\mu\nu} (\epsilon_{\rho\sigma} \nabla \nabla F_{\sigma} - \tilde{G}), \\
F_{\mu\nu} = \nabla_{\mu} \nabla F_{\nu} - \nabla_{\nu} \nabla F_{\mu} + i[F_{\mu}, F_{\nu}] + \frac{1}{2} \epsilon_{\mu\nu} \nabla \tilde{G}, \\
\nabla G = \nabla \tilde{G} = \nabla \tilde{G} + \nabla G = 0, \\
\nabla_{\nu} F_{\mu} = \frac{1}{2} (\nabla_{\mu} G - \epsilon_{\mu\nu} \nabla \tilde{G}), \quad G_{\mu} = \frac{1}{2} (\nabla_{\mu} G + \epsilon_{\mu\nu} \nabla \tilde{G}). 
\]  \hspace{1cm} (4.14-4.18)

The component fields are defined as

\[
F_{\mu} = \phi_{\mu}, \quad \nabla F_{\mu} = \lambda_{\mu}, \quad \nabla_{\nu} F_{\mu} = \frac{1}{2} (\delta_{\mu\nu} \rho + \epsilon_{\mu\nu} \tilde{\rho}), \quad \nabla_{\nu} \nabla F_{\mu} = D, 
\]  \hspace{1cm} (4.19)

where $\phi_{\mu}$ and $D$ are bosonic fields and $\rho$, $\tilde{\rho}$, and $\lambda_{\mu}$ are fermionic fields.

The supertransformations of each component field are derived straightforwardly. In contrast with the previous model where $G_A$ is related to $W$, $G$ and $\tilde{G}$ should satisfy (4.17) and seem to be independent of $F_{\mu}$ as far as Jacobi identities are concerned. The superalgebra is closed up to gauge transformations without constraints at the off-shell level.

To obtain the action, we introduce a linear combination of $s_{\mu}$,

\[
s_{\pm} \equiv s_1 \pm is_2, 
\]  \hspace{1cm} (4.20)

which satisfies

\[
s_{\pm}^2 = 0, \quad \{s_{+}, s_{-}\} = 2Z. 
\]  \hspace{1cm} (4.21)

We define $\lambda_{\pm} \equiv \lambda_1 \pm i\lambda_2$ similarly, and introduce the notation $\nabla_{\mu}^{\mp} \equiv \nabla_{\mu} \pm F_{\mu}$ and $D_{\mu}^{\pm} \equiv D_{\mu} \pm \phi_{\mu}$ for convenience. Then one can derive the action by using the nilpotency of $\phi$, $\tilde{\phi}$, and $s_{\pm}$. In fact, we can find $S_1$, $S_2$, and $S_3$ satisfying $sS_1 = \bar{s}S_1 = s_{+}S_2 = s_{-}S_3 = 0$, where $S_1$, $S_2$, and $S_3$ are not
generally identical. However, in the case of $\nabla \tilde{G} = \tilde{V} G = 0$ together with (4.17), we find

$$S_1 = \frac{1}{2} \int d^2 x \text{Tr} \tilde{s} \tilde{\rho} \tilde{\rho} = S_0 \quad (4.22)$$

$$S_2 = \frac{1}{2} \int d^2 x \text{Tr} s_- \lambda_- \lambda_+ = S_0 + \int d^2 x \text{Tr} \left\{ -\frac{1}{2}(\epsilon_{\mu \nu} D^-_{\mu} \lambda_+ + i(-D^+_{\mu} \lambda_\mu))(G) + i\tilde{G} \right\}, \quad (4.23)$$

$$S_3 = \frac{1}{2} \int d^2 x \text{Tr} s_+ \lambda_+ \lambda_- = S_0 + \int d^2 x \text{Tr} \left\{ \frac{1}{2}(\epsilon_{\mu \nu} D^+_{\mu} \lambda_\nu + i(-D^-_{\mu} \lambda_\mu))(G) - i\tilde{G} \right\}, \quad (4.24)$$

where

$$S_0 = \int d^2 x \text{Tr} \left\{ \frac{1}{2}(D_{\mu} \phi_v)^2 + \frac{1}{4} F^2_{\mu \nu} + \frac{1}{2} D^2 - i \rho D^+_{\mu} \lambda_\mu - i \tilde{\rho} \epsilon_{\mu \nu} D^-_{\mu} \lambda_\nu - \frac{1}{4}[\lambda_\mu, \phi_v]^2 \right\}, \quad (4.25)$$

which corresponds to the action without a central charge and to the twisted versions of the action in [41,42]. Here it is important to recognize that we can find solutions satisfying $\nabla \tilde{G} = \tilde{V} G = 0$ and (4.17),

$$G = a \epsilon_{\mu \nu} \nabla^-_{\mu} \nabla F_{\nu}, \quad \tilde{G} = -a \nabla^+_{\mu} \nabla F_{\mu}, \quad (4.26)$$

where $a$ is a parameter with mass dimension $-1$. Moreover, the above choice of $G$ and $\tilde{G}$ makes $S_2$ and $S_3$ identical:

$$S_2 = S_3 = S_0 - ia^{-1} \int d^2 x \text{Tr} G|\tilde{G}|. \quad (4.27)$$

We can then find the following action satisfying $sS = \tilde{s}S = s_\mu S = 0$, where the supertransformations are given in Table 7,

$$S = S_0 - ia^{-1} \int d^2 x \text{Tr} G|\tilde{G}|. \quad (4.28)$$

---

Table 7. Supertransformations of the B(0,0,Z) model. $A_{\mu v} = \frac{1}{2} \delta_{\mu v} D - \frac{1}{2} (D_{\mu} \phi_v + D_{\nu} \phi_\mu - \delta_{\mu \nu} D_{\rho} \phi_\rho) + \frac{1}{2} (F_{\mu \nu} - i[\phi_\mu, \phi_\nu])$.

| $s$ | $s_\mu$ | $\tilde{s}$ | $Z$ |
|-----|---------|-------------|-----|
| $\phi_v$ | $\lambda_v$ | $\frac{1}{2} (\delta_{\mu v} \rho + \epsilon_{\mu v} \tilde{\rho})$ | $-\epsilon_{\nu \rho} \lambda_\rho$ | $\frac{1}{2} (\nabla_\nu G| - \epsilon_{\nu \rho} \nabla_\rho \tilde{G})$ |
| $A_v$ | $-i\lambda_v$ | $-\frac{1}{2} \delta_{\mu v} \rho + \frac{1}{2} \epsilon_{\mu v} \tilde{\rho}$ | $-i \epsilon_{\nu \rho} \lambda_\rho$ | $\frac{1}{2} (\nabla_\nu G| + \epsilon_{\nu \rho} \nabla_\rho \tilde{G})$ |
| $\lambda_v$ | $0$ | $A_{\mu v}$ | $0$ | $-\frac{1}{2} (D^-_{\rho} \rho - \epsilon_{\rho \sigma} D^+_{\rho} \tilde{\rho})$ |
| $\rho$ | $\frac{1}{2} [D^-_{\rho}, D^+_{\rho}] - D_{\rho}$ | $\frac{1}{2} (\nabla_\mu G| - \epsilon_{\mu \nu} \nabla_\nu \tilde{G})$ | $\frac{1}{2} \epsilon_{\rho \sigma} [D^-_{\rho}, D^+_{\rho}]$ | $\frac{1}{2} (\nabla_\nu \tilde{G} - \epsilon_{\rho \sigma} \nabla_\nu \nabla_\rho \tilde{G})$ |
| $\tilde{\rho}$ | $-\frac{1}{2} \epsilon_{\rho \sigma} [D^-_{\rho}, D^+_{\rho}]$ | $\frac{1}{2} (\nabla_\mu \tilde{G} + \epsilon_{\mu \nu} G)$ | $-\frac{1}{2} [D^-_{\rho}, D^+_{\rho}] - D_{\rho}$ | $\frac{1}{2} (\nabla_\nu \tilde{G} + \epsilon_{\rho \sigma} \nabla_\nu \nabla_\rho \tilde{G})$ |
| $D$ | $-iD^+_{\rho} \lambda_\rho$ | $-\frac{1}{2} (D^-_{\rho} G| - \epsilon_{\rho \sigma} D^+_{\rho} \tilde{G})$ | $-i \epsilon_{\rho \sigma} D^+_{\rho} \tilde{\rho}$ | $-\frac{1}{2} (D^-_{\rho} \nabla_\rho G| + \epsilon_{\rho \sigma} D^+_{\rho} \nabla_\sigma \tilde{G})$ |
| $\lambda_\rho$ | $0$ | $A_{\mu v}$ | $0$ | $i[\rho, G]| + i[\tilde{\rho}, \tilde{G}]$ |
| $\rho$ | $\frac{1}{2} [D^-_{\rho}, D^+_{\rho}] - D_{\rho}$ | $\frac{1}{2} (\nabla_\mu G| - \epsilon_{\mu \nu} \nabla_\nu \tilde{G})$ | $\frac{1}{2} \epsilon_{\rho \sigma} [D^-_{\rho}, D^+_{\rho}]$ | $\frac{1}{2} (\nabla_\nu \tilde{G} - \epsilon_{\rho \sigma} \nabla_\nu \nabla_\rho \tilde{G})$ |
| $\tilde{\rho}$ | $-\frac{1}{2} \epsilon_{\rho \sigma} [D^-_{\rho}, D^+_{\rho}]$ | $\frac{1}{2} (\nabla_\mu \tilde{G} + \epsilon_{\mu \nu} G)$ | $-\frac{1}{2} [D^-_{\rho}, D^+_{\rho}] - D_{\rho}$ | $\frac{1}{2} (\nabla_\nu \tilde{G} + \epsilon_{\rho \sigma} \nabla_\nu \nabla_\rho \tilde{G})$ |
| $D$ | $-iD^+_{\rho} \lambda_\rho$ | $-\frac{1}{2} (D^-_{\rho} G| - \epsilon_{\rho \sigma} D^+_{\rho} \tilde{G})$ | $-i \epsilon_{\rho \sigma} D^+_{\rho} \tilde{\rho}$ | $-\frac{1}{2} (D^-_{\rho} \nabla_\rho G| + \epsilon_{\rho \sigma} D^+_{\rho} \nabla_\sigma \tilde{G})$ |
In Table 7 the following expressions are used:

\[ G| = a\epsilon_{\mu\nu} D_\mu^- \lambda_\nu, \quad \tilde{G}| = -a D_\mu^+ \lambda_\mu, \]

\[ \nabla_\mu G| = a \left( -\epsilon_{\mu\nu} \{ \lambda_\nu, \tilde{\rho} \} + \frac{1}{2} \{ \lambda_\mu, \tilde{G}| \} + \epsilon_{\rho\sigma} D_\rho^- A_{\mu\sigma} \right), \]

\[ \nabla_\mu \tilde{G}| = a \left( -\epsilon_{\mu\nu} \{ \lambda_\nu, \tilde{\rho} \} - \frac{1}{2} \{ \lambda_\mu, G| \} - D_\nu^+ A_{\mu\nu} \right), \]

\[ Z_\rho = \frac{a}{2} \left( [\nabla_\mu \tilde{G}|, \lambda_\mu] + \epsilon_{\mu\nu} [\nabla_\nu G|, \lambda_\nu] - 2[D, \tilde{\rho}] + \epsilon_{\mu\nu} [A_{\mu\nu}, G|] \right. \]

\[ - i\epsilon_{\mu\nu} D_\mu^+ D_\nu^- \tilde{\rho} - iD_\mu^+ D_\mu^- \tilde{\rho} - \frac{i}{2} \epsilon_{\mu\nu} D_\mu^- D_\mu^+ G| - \frac{i}{2} D_\mu^- D_\mu^+ \tilde{G}| \left. \right), \]

\[ Z_\tilde{\rho} = \frac{a}{2} \left( -[\nabla_\mu G|, \lambda_\mu] + \epsilon_{\mu\nu} [\nabla_\nu \tilde{G}|, \lambda_\nu] + 2[D, \tilde{\rho}] + \epsilon_{\mu\nu} [A_{\mu\nu}, \tilde{G}|] \right. \]

\[ + iD_\mu^- D_\mu^+ \tilde{\rho} - i\epsilon_{\mu\nu} D_\mu^- D_\mu^+ \tilde{\rho} - \frac{i}{2} D_\mu^- D_\mu^+ G| - \frac{i}{2} \epsilon_{\mu\nu} D_\mu^+ D_\mu^- \tilde{G}| \right), \]

(4.29)

where \( \tilde{\rho} \equiv \rho - \frac{1}{2} G| \) and \( \tilde{\tilde{\rho}} \equiv \tilde{\rho} - \frac{1}{2} \tilde{G}|. \)

### 4.3. B(Z,Z,0) model

We consider the following algebra:

\[ \{ s, s_\mu \} = P_\mu, \quad \{ \tilde{s}, s_\mu \} = -\epsilon_{\mu\nu} P_\nu, \quad \{ s, \tilde{s} \} = 0, \]

\[ s^2 = \tilde{s}^2 = Z, \quad \{ s_\mu, s_\nu \} = 0, \quad \{ s_\mu, Z \} = 0, \]

(4.30)

where \( Z = 2U_0 = -2V_3. \) The model is completely similar in construction to the previous model, and we thus show only the main results. The B-(Z,Z,0) supercurvature ansatz is shown in Table 5. The component fields are defined as in (4.19).

The key relation derived by Jacobi identity is

\[ \nabla_\mu G_\nu + \nabla_\nu G_\mu = 0. \]

(4.31)

Similarly to the previous model, \( G_\mu \) is not directly related to \( F_\mu \) by Jacobi identity; it is then necessary to solve (4.31). As far as the above relation holds, one can derive the supertransformations of each component field and show off-shell closure up to gauge transformations without constraints.

Eventually, one can find that the following \( G_\mu \) satisfy \( \nabla_\mu G_\nu = 0: \)

\[ G_\mu = a\epsilon_{\rho\sigma} (\nabla_\rho \nabla_\sigma F_\mu + [F_\rho, \nabla_\sigma F_\mu]). \]

(4.32)

The following action has full supersymmetry for the supertransformations given in Table 8:

\[ S = S_0 - i a^{-1} \int d^2 x \text{Tr} \frac{1}{2} \epsilon_{\mu\nu} G_\mu |G_\nu|. \]

(4.33)

In Table 8 the following expressions are used:

\[ G_\mu| = a(D_\mu^- \tilde{\rho} + \epsilon_{\mu\nu} D_\nu^+ \rho), \]

\[ \nabla G_\mu| = a \left( -\frac{i}{2} \epsilon_{\rho\sigma} D_\mu^- D_\rho^- D_\sigma^+ + i \epsilon_{\mu\nu} D_\nu^- D_\rho^+ \phi_\rho - \epsilon_{\mu\nu} D_\nu^- D_\rho^+ D - 2 \{ \tilde{\rho}, \tilde{\lambda}_\mu \} + \frac{1}{2} \epsilon_{\mu\nu} \{ \rho, G_\nu | \} \right). \]

\[ \tilde{\nabla} G_\mu| = a \left( i D_\mu^- D_\mu^+ \phi_\nu - D^- D + 2 \{ \rho, \tilde{\lambda}_\mu \} + \frac{1}{2} \epsilon_{\mu\nu} \{ \tilde{\rho}, G_\nu | \} \right). \]
Table 8. Supertransformations of the B(Z,Z,0) model. \( A_{\mu \nu} \) is defined in the same way as in Table 7.

| \( s \) | \( s_{\mu} \) | \( \tilde{s} \) | \( Z \) |
|---|---|---|---|
| \( \phi_{\nu} \) | \( \lambda_{\nu} \) | \( \frac{1}{2}(\delta_{\mu \nu} \rho + \epsilon_{\mu \nu} \tilde{\rho}) \) | \( -\epsilon_{\nu \rho} \lambda_{\rho} \) | \( \frac{i}{2}(\nabla G_{\nu} - \epsilon_{\nu \rho} \tilde{G}_{\rho}) \) |
| \( A_{\nu} \) | \( -i \lambda_{\nu} + \frac{1}{2}G_{\nu} \) | \( \frac{1}{2}(\delta_{\mu \nu} \rho + \epsilon_{\mu \nu} \tilde{\rho}) \) | \( -i \epsilon_{\nu \rho} \lambda_{\rho} + \frac{1}{2}i \epsilon_{\nu \rho} G_{\rho} \) | \( \frac{i}{2}(\nabla G_{\nu} + \epsilon_{\nu \rho} \tilde{G}_{\rho}) \) |
| \( \lambda_{\nu} \) | \( \frac{i}{2}(\nabla G_{\nu} - \epsilon_{\nu \rho} \tilde{G}_{\rho}) \) | \( A_{\mu \nu} \) | \( \frac{i}{2}(\tilde{G}_{G_{\mu} + \epsilon_{\mu \nu} G_{\nu}}) \) | \( \frac{i}{2} \bar{\nabla}_{G_{\nu}} - \frac{i}{2} \epsilon_{\nu \rho} \tilde{G}_{\rho} \) |
| \( \rho \) | \( -iD_{\rho} \phi_{\rho} - D \) | 0 | \( iD_{\rho} D_{\rho} - D \) | \( -iD_{\rho} G_{\rho} \) |
| \( \tilde{\rho} \) | \( -i \epsilon_{\rho \sigma} [D_{\rho}^{+}, D_{\sigma}^{+}] \) | 0 | \( iD_{\rho} \phi_{\rho} - D \) | \( -i \epsilon_{\rho \sigma} G_{\rho} \) |
| \( D \) | \( -iD_{\rho} \lambda_{\rho} + \frac{1}{2}D_{\rho} G_{\rho} \) | \( \frac{i}{2}(D_{\mu}^{+} - \epsilon_{\mu \nu} D_{\nu}^{+} \tilde{\rho}) \) | \( -i \epsilon_{\rho \sigma} D_{\rho}^{+} \lambda_{\sigma} + \frac{1}{2}i \epsilon_{\rho \sigma} D_{\rho} G_{\sigma} \) | \( \frac{i}{2}(D_{\rho} \nabla G_{\rho} + \epsilon_{\rho \sigma} D_{\rho}^{+} \tilde{G}_{\sigma}) \) | \( -2i[\lambda_{\rho}, G_{\rho}] + \frac{i}{2} \{ G_{\rho}, G_{\rho} \} \) |

\[
Z_{\lambda_{\mu}} = \frac{a}{2} \left\{ \frac{1}{2} \nabla G_{\mu} + \epsilon_{\mu \nu} \nabla G_{\nu}, \tilde{\rho} \right\} - \frac{1}{2} \left\{ \nabla G_{\mu} - \epsilon_{\mu \nu} \nabla G_{\nu}, \rho \right\} + 4 \epsilon_{\mu \nu} [K, \tilde{\lambda}_{\nu}] \\
- 2i \epsilon_{\mu \nu} D_{\nu}^{+} D_{\rho}^{+} \tilde{\lambda}_{\rho} + 2i \epsilon_{\mu \nu} D_{\rho}^{+} D_{\nu}^{+} \tilde{\lambda}_{\rho} + \frac{i}{2} \epsilon_{\mu \nu} D_{\nu}^{+} D_{\rho}^{+} G_{\rho} \right\} - \frac{i}{2} \epsilon_{\rho \sigma} D_{\mu}^{+} D_{\rho}^{+} G_{\sigma} \right\} \\
- \frac{i}{2} \epsilon_{\mu \nu} D_{\nu}^{+} D_{\rho}^{+} G_{\rho} \right\} - \frac{i}{2} \epsilon_{\rho \sigma} D_{\mu}^{+} D_{\rho}^{+} G_{\mu} \right\}, \tag{4.34}
\]

where \([,]\) denotes the antisymmetrization of suffixes and \( \tilde{\lambda}_{\mu} \equiv \lambda_{\mu} - \frac{1}{3} G_{\mu} \).

5. Conclusion and discussion

We have constructed an off-shell invariant \( N = 2 \) twisted SYM theory with a gauged central charge in two dimensions. Depending on the supercurvature ansatz we have introduced A and B models.

In the A model, the superalgebra is closed at the off-shell level with an extra constraint (4.7). This model has a similarity with \( D = N = 4 \) SYM theory with gauged central charge, with an unavoidable extra constraint. For an Abelian gauge group we can explicitly solve the constraint (4.7), and we can thus construct the off-shell supertransformations and action without any other constraints. It is interesting to note that the action has a two-dimensional topological BF term. We cannot, however, solve the constraint for the non-Abelian case.

On the other hand, we have found two types of B model whose superalgebra is closed at the off-shell level without any constraints. This gives us the hope that we may use an ansatz similar to the B model in four dimensions to get an off-shell invariant \( N = 4 \) formulation without constraints.\(^1\)

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