Large $N$ quantum gravity

Alessandro Codello
SISSA via Bonomea 265, I-34136 Trieste, Italy
E-mail: codello@sissa.it

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Abstract. We obtain the effective action of four-dimensional quantum gravity, induced by $N$ massless matter fields, by integrating the renormalization group (RG) flow of the relative effective average action. By considering the leading approximation in the large $N$ limit, where one neglects the gravitational contributions with respect to the matter contributions, we show how different aspects of quantum gravity, such as asymptotic safety, quantum corrections to the Newtonian potential and the conformal anomaly-induced effective action, are all represented by different terms of the effective action when this is expanded in powers of the curvature.

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1. Introduction

Even though we are still lacking a quantum theory of gravitational phenomena, we are starting to accumulate interesting partial results which will probably be important bits of the final theory. These are, among others, the conformal anomaly-induced effective action, which was first written down in [1]; the low-energy corrections to the Newtonian gravitational potential analyzed in [2]; and the possible ultraviolet (UV) finite completion of the theory described in the asymptotic safety scenario [3]. In this paper, we want to show how these different aspects can all be seen as arising from different terms in the gravitational effective action when this is expanded in powers of the curvature. In particular, the UV properties of the theory are related to the zero- and first-order terms, i.e. to the renormalization of Newton's and cosmological constants, and to the renormalization of the coupling constants of all higher order local invariants; quantum gravitational corrections to the Newtonian potential are encoded in the finite part of the curvature square terms, while the conformal anomaly-induced effective action is just one of the four possible curvature structures.

We will use the effective average action formalism to obtain the gravitational effective action induced by $N_0$ massless scalar fields, $N_{\frac{1}{2}}$ massless Dirac spinors and $N_1$ Abelian gauge fields, interacting solely with the background geometry, as a result of the integration of the renormalization group (RG) flow. In the large $N$ expansion¹ one assumes that the number of matter fields $N$ grows large. In the leading approximation one simply neglects the gravitational contributions with respect to the matter contributions: this eliminates computational and conceptual issues related to the treatment of gravitational fluctuations. Within the effective average action formalism this point of view has been analyzed in [6], where the focus was on the local part of the effective average action and on the related renormalizability issues. Here we will extend the large $N$ analysis to a non-local truncation of effective average action where this is expanded in powers of the curvature.

2. Effective average action for matter fields on curved space

Following [6], we consider massless matter fields on a curved four-dimensional (4D) manifold equipped with a metric $g_{\mu \nu}$. The bare action we consider describes $N_0$ massless scalar fields, $N_{\frac{1}{2}}$ massless Dirac spinors and $N_1$ Abelian gauge fields interacting only with the background geometry:

$$S[\phi, \psi, A_{\mu}, \bar{c}, c; g] = \int d^4x \sqrt{g} \left\{ \sum_{i=1}^{N_0} \left[ \frac{1}{2} \partial_{\mu} \phi_i \partial^{\mu} \phi_i + \frac{\chi}{12} \phi_i^2 R \right] + \sum_{i=1}^{N_{\frac{1}{2}}} \bar{\psi}_i \nabla \psi_i + \sum_{i=1}^{N_1} \left[ \frac{1}{4} F_{\mu \nu, i} F_{\mu \nu}^i + \frac{1}{2 \alpha} \left( \partial_{\mu} A_{\mu}^i \right)^2 + \partial_{\mu} \bar{c}_i \partial^{\mu} c_i \right] \right\}. \quad (1)$$

Here $\chi$ is a parameter; only when $\chi = 1$ is the scalar action conformally invariant if the scalar has conformal weight one. The Dirac operator is defined using the covariant Dirac matrices

¹ $N$ is any one of $N_0$, $N_{\frac{1}{2}}$ and $N_1$. 

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\[ \gamma^\mu = e^\mu_\alpha \gamma^\alpha. \]

Note also that on an arbitrary curved manifold the Abelian ghosts do not decouple and cannot be discarded; we will choose the gauge \( \alpha = 1 \) from now on.

The effective average action \( \Gamma_k[\phi, \psi, A_\mu, \bar{c}, c; g] \) is a scale-dependent generalization of the standard effective action (depending on the infrared (IR) cutoff scale \( k \)) that interpolates smoothly between the bare action for \( k \rightarrow \infty \) and the full quantum effective action for \( k \rightarrow 0 \) [4]. It satisfies an exact RG flow equation describing its flow as the IR scale \( k \) is shifted. The flow equation relevant to the field content we are considering reads as follows:

\[
\partial_t \Gamma_k[\varphi; g] = \frac{1}{2} \text{Tr} \frac{\partial_t R_k[g]}{\Gamma^{(2,0)}_k[\varphi; g] + R_k[g]},
\]

where \( \Gamma^{(2,0)}_k[\varphi, g] \) is the Hessian of the effective average action taken with respect to the field multiplet \( \varphi = (\phi, \psi, A_\mu, \bar{c}, c) \) and the trace is a ‘super-trace’ on this space. Equation (2) is exact and is both UV and IR finite. The flow equation (2) can be seen as an RG improvement of the one-loop effective action which is derived from the modified bare action \( S[\varphi, g] \rightarrow S[\varphi, g] + \Delta S_k[\varphi, g] \), where \( \Delta S_k[\varphi, g] \) is a cutoff action quadratic in the fields. This cutoff action is constructed in such a way as to suppress the propagation of all field modes smaller than the RG scale \( k \).

As explained in [8], since we are considering matter fields interacting only with the background geometry, we can replace the Hessian of the effective average action in the rhs of (2) with the Hessian of the bare action (1). After performing the field multiplet trace, we find that\(^2\)

\[
\partial_t \Gamma_k[\phi, \psi, A_\mu, \bar{c}, c; g] = \frac{N_0}{2} \text{Tr}_0 h_k(\Delta_0) - \frac{N_1}{2} \text{Tr}_1 h_k(\Delta_1) + N_1[\frac{1}{2} \text{Tr}_1 h_k(\Delta_1)] - \text{Tr}_0 h_k(\Delta_{gh}),
\]

where we defined the function \( h_k(z) = \frac{\partial_t R_k(z)}{z + R_k(z)} \) and \( \text{Tr}_s \) indicates a trace over spin-\( s \) fields. The differential operators introduced in (3) are the following:

\[
\Delta_0 = \Delta + \frac{1}{6} R, \quad \Delta_1 = \Delta + \frac{1}{4} R, \quad (\Delta_1)^{\mu\nu} = \Delta g^{\mu\nu} + R^{\mu\nu}, \quad \Delta_{gh} = \Delta,
\]

where \( \Delta = -g^{\mu\nu} \nabla_\mu \nabla_\nu \) is the covariant Laplacian. In the next section, we will determine the RG flow of the effective average action by calculating the functional traces on the rhs of (3).

### 3. Flow equations and beta functions

The traces on the rhs of the flow equation (3) can be expanded in powers of the curvature by employing the non-local heat kernel expansion [7] as was done in [8, 9]. To second order we

\(^2\) We are using a Type II cutoff operator in the nomenclature of [5].
find the following result:\(^3\):

\[
(4\pi)^2 \partial_t \Gamma_k[g] = -\frac{N_0 - 4N_\frac{1}{2} + 2N_1}{2} Q_2[h_k] \int d^4x \sqrt{g} \\
+ \frac{(1 - \chi)N_0 + 2N_\frac{1}{2} - 4N_1}{12} Q_1[h_k] \int d^4x \sqrt{g} R \\
+ \int d^4x \sqrt{g} R \left[ \int_0^\infty ds \tilde{h}_k(s) s F_R(s\Delta) \right] R \\
+ \int d^4x \sqrt{g} C_{\mu\nu\alpha\beta} \left[ \int_0^\infty ds \tilde{h}_k(s) s F_C(s\Delta) \right] C^{\mu\nu\alpha\beta} \\
- \frac{N_0 + 11N_\frac{1}{2} + 62N_1}{720} Q_0[h_k] \int d^4x \sqrt{g} E \\
+ \frac{N_0 + N_\frac{1}{2} - 3N_1}{60} Q_0[h_k] \int d^4x \sqrt{g} \Delta R + O(\mathcal{R}^3),
\]

(5)

where \( \mathcal{R} \) stands for any curvature. The non-local heat kernel structure functions \( F_C(x) \) and \( F_R(x) \) in (5) are linear combinations of the basic non-local heat kernel structure function \( f(x) = \int_0^1 d\xi \ e^{-\xi(1-\xi)} \) and read as follows:

\[
F_R(x) = \frac{5N_0 - 20N_\frac{1}{2} + 10N_1}{48} \frac{f(x) - 1}{x^2} + \frac{(3 - 2\chi)N_0 - 2N_\frac{1}{2} - 2N_1}{48} \frac{f(x)}{x} \\
- \frac{(13 - 12\chi)N_0 + 8N_\frac{1}{2} - 22N_1}{288} \frac{1}{x} + \frac{(3 - 2\chi)^2 N_0 - 2N_1}{576} f(x),
\]

\[
F_C(x) = \frac{N_0 - 4N_\frac{1}{2} + 2N_1}{4} \frac{f(x) - 1}{x^2} - \frac{N_\frac{1}{2} - 2N_1}{4} \frac{f(x)}{x} \\
+ \frac{N_0 + 2N_\frac{1}{2} - 10N_1}{24} \frac{1}{x} + \frac{N_1}{8} f(x).
\]

(6)

In (5) \( \tilde{h}_k(s) \) is the (inverse) Laplace transform of the function \( h_k(x) \) and the \( Q \)-functionals are defined as \( Q_n[f] = \frac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} f(z) \) when \( n > 0 \) and as \( Q_n[f] = (-1)^n f^{(n)}(0) \) when \( n \leq 0 \).

Since the RG flow has generated all possible terms compatible with diffeomorphism invariance on the rhs of (5), we need to now consider the most general truncation for the gravitational effective average action \( \Gamma_k[g] \) to insert in the lhs of (5). As proposed in [8, 9], we consider an ansatz where the effective average action is expanded in powers of the curvature and where the scale dependence is carried both by running couplings and by running (possibly non-local) structure functions. To second order in the curvatures the expansion of the gravitational effective average reads as follows:\(^4\):

\[
\Gamma_k[g] = \int d^4x \sqrt{g} \left[ \frac{1}{16\pi G_k} (2\Lambda_k - R) + R f_{R,k}(\Delta) R \\
+ C_{\mu\nu\alpha\beta} f_{C,k}(\Delta) C^{\mu\nu\alpha\beta} + \frac{1}{\xi_k} E + \frac{1}{\tau_k} \Delta R \right] + O(\mathcal{R}^3).
\]

(7)

\(^3\) Here we define \( \Gamma_i[g] = \Gamma_i[0, 0, 0, 0; g] \).

\(^4\) This also implies that we are adding to the action (1) the non-dynamical term \( \Gamma_A[g] \).

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In (7) the couplings \( \Lambda_k \) and \( G_k \) are the running cosmological and Newton’s constants, \( f_{C,k} \) and \( f_{R,k} \) are the two independent curvature square running structure functions (which at this point are arbitrary functions of the covariant Laplacian \( \Delta \)), while \( \rho_k \) and \( \tau_k \) are the running couplings related to the Euler and total derivative invariants. The other two curvature square running couplings \( \lambda_k \) and \( \xi_k \) are related to the running structure functions by the following relations:

\[
 f_{C,k}(0) = \frac{1}{2\lambda_k}, \quad f_{R,k}(0) = \frac{1}{\xi_k}.
\]

We can extract the beta functions for the couplings \( \Lambda_k \), \( G_k \), \( \rho_k \), \( \tau_k \) and the flow equations for the running structure functions \( f_{C,k}, f_{R,k} \) by comparing (5) with (7). In particular, by matching the coefficients of the operators \( \int \sqrt{g} \) and \( \int \sqrt{g} R \), we find the following relations:

\[
 (4\pi)^2 \partial_t \left( \frac{\Lambda_k}{8\pi G_k} \right) = \frac{N_0 - 4N_1 + 2N_1}{2} Q_2[h_k],
\]

\[
 (4\pi)^2 \partial_t \left( -\frac{1}{16\pi G_k} \right) = \frac{(1 - \chi)N_0 + 2N_\frac{1}{2} - 4N_1}{12} Q_1[h_k],
\]

while by matching the non-local curvature square terms we find the following flow equation for both running structure functions:

\[
 \partial_t f_{i,k}(x) = \frac{1}{(4\pi)^2} \int_0^\infty ds \, \tilde{h}_k(s) \, s \, F_i(sx),
\]

where \( i = C, R \) and \( x \) stands for \( \Delta \). If we insert the explicit form of the functions \( F_i(x) \) from (6) we can rewrite equation (10) explicitly in terms of \( Q \)-functionals:

\[
 (4\pi)^2 \partial_t f_{R,k}(x) = \frac{(3 - 2\chi)^2 N_0 - 2N_1}{576} \int_0^1 d\xi \, Q_0[h_k(z + x\xi(1 - \xi))]
 + \frac{(3 - 2\chi) N_0 - 2N_\frac{1}{2} - 2N_1}{48} \int_0^1 d\xi \, Q_1[h_k(z + x\xi(1 - \xi))]
 - \frac{(13 - 12\chi) N_0 + 8N_\frac{1}{2} - 22N_1}{288} Q_1[h_k]
 + \frac{5N_0 - 20N_\frac{1}{2} + 10N_1}{48x^2} \left\{ \int_0^1 d\xi \, Q_2[h_k(z + x\xi(1 - \xi))] - Q_2[h_k] \right\}.\]

\[
 (4\pi)^2 \partial_t f_{C,k}(x) = \frac{N_1}{8} \int_0^1 d\xi \, Q_0[h_k(z + x\xi(1 - \xi))]
 - \frac{N_\frac{1}{2} - 2N_1}{4x} \int_0^1 d\xi \, Q_1[h_k(z + x\xi(1 - \xi))]
 + \frac{N_0 + 2N_\frac{1}{2} - 10N_1}{24x} Q_1[h_k]
 + \frac{N_0 - 4N_\frac{1}{2} + 2N_1}{4x^2} \left\{ \int_0^1 d\xi \, Q_2[h_k(z + x\xi(1 - \xi))] - Q_2[h_k] \right\}.\]

Finally, matching the remaining local curvature square terms gives

\[
 (4\pi)^2 \partial_t \left( \frac{1}{\rho_k} \right) = -\frac{N_0 + 11N_\frac{1}{2} + 62N_1}{360} Q_0[h_k],
\]

\[
 (4\pi)^2 \partial_t \left( \frac{1}{\tau_k} \right) = \frac{N_0 + N_\frac{1}{2} - 3N_1}{30} Q_0[h_k].
\]
The flow equations (9) and (11)–(13) completely describe the RG flow of the effective average action (7).

The \(Q\)-functionals in the flow equations just derived can be evaluated once a particular cutoff shape function has been chosen; it is possible to evaluate them analytically if we employ the so-called ‘optimized’ cutoff shape function \(R_k(z) = (k^2 - z)\theta(k^2 - z)\). We could easily find the cutoff shape-independent value \(Q_0[h_k] = 2\) and the results \(Q_n[h_k] = \frac{2}{n}k^{2n}\), while the evaluation of the parametric integrals of \(Q\)-functional present in (11) and (12) is more involved; see the appendix of [9] for further details. We can now write down the beta functions for the dimensionless cosmological constant, \(\Lambda_k = k^2\tilde{\Lambda}_k\), and for the dimensionless Newton’s constant, \(\tilde{G}_k = k^{2-d}\tilde{G}_k\), which follow from (9):

\[
\begin{align*}
\partial_t \tilde{\Lambda}_k &= -2\tilde{\Lambda}_k + \frac{N_0 - 4N_\frac{1}{2} + 2N_1}{4\pi} \tilde{G}_k + \frac{(1 - \chi)N_0 + 2N_\frac{1}{2} - 4N_1}{6\pi} \Lambda_k \tilde{G}_k, \\
\partial_t \tilde{G}_k &= 2\tilde{G}_k + \frac{(1 - \chi)N_0 + 2N_\frac{1}{2} - 4N_1}{6\pi} \tilde{G}_k^2.
\end{align*}
\tag{14}
\]

The first terms on the rhs of (14) represent the canonical scaling of the cosmological and Newton’s constants, while the other terms represent the running induced by the interaction with matter fields. Note also that conformal scalars, i.e. \(\chi = 1\), do not contribute to the running of Newton’s constant. Equations (11) and (12) can be compactly rewritten as

\[
\partial_t f_{i,k}(x) = \frac{1}{(4\pi)^2} g_i \left( \frac{x}{k^2} \right),
\tag{15}
\]

where we have defined the following cutoff shape-dependent functions:

\[
g_i(u) \equiv \int_0^\infty ds \tilde{h}_k(s) s F_i(sk^2u).
\tag{16}
\]

In (16) \(u = x/k^2\) stands for the covariant Laplacian in units of \(k\); note that the UV regime corresponds to small values of \(u\), while the IR regime corresponds to large values of \(u\). The integrals in (16) can be evaluated analytically [9]; we find the following forms:

\[
\begin{align*}
g_C(u) &= \frac{N_0 + 6N_\frac{1}{2} + 12N_1}{120} - \left[ \frac{N_0 + 6N_\frac{1}{2} + 12N_1}{120} - \frac{8N_0 + 8N_\frac{1}{2} - 64N_1}{120u} \right. \\
&\quad + \left. \frac{16N_0 - 64N_\frac{1}{2} + 32N_1}{120u^2} \right] \sqrt{1 - \frac{4}{u} \theta(u - 4)}, \\
g_R(u) &= \frac{(1 - \chi)^2N_0}{72} - \left[ \frac{(1 - \chi)^2N_0}{72} + \frac{(1 - \chi)N_0 + N_\frac{1}{2} - 2N_1}{18u} \right. \\
&\quad + \left. \frac{N_0 - 4N_\frac{1}{2} + 2N_1}{18u^2} \right] \sqrt{1 - \frac{4}{u} \theta(u - 4)}.
\end{align*}
\tag{17}
\]

Note that the form of (17) implies that conformally invariant matter contributes to \(g_R(u)\) and thus to the flow of \(f_{R,k}(x)\), showing that for \(k \neq 0\) the flow generates non-conformally invariant interactions. We will see later that the conformal invariance of the curvature square terms will be partially restored in the IR at \(k = 0\). The constant terms in (17) when matched with (8) give the beta functions for the couplings \(\lambda_k\) and \(\xi_k\). These, together with the explicit forms of the beta
functions (13), are

\[
\begin{align*}
\partial_t \lambda_k &= -\frac{1}{(4\pi)^2} \frac{N_0 + 6N_1}{60} \lambda_k^2, \\
\partial_t \xi_k &= -\frac{1}{(4\pi)^2} \frac{(1 - \chi)^2 N_0}{72} \xi_k^2, \\
\partial_t \rho_k &= \frac{1}{(4\pi)^2} \frac{N_0 + 11N_1}{180} \rho_k^2, \\
\partial_t \tau_k &= -\frac{1}{(4\pi)^2} \frac{N_0 + N_1 - 3N_1}{15} \tau_k^2.
\end{align*}
\] (18)

The numerical coefficients in (18) are scheme independent, i.e. they do not depend on \( R_k(z) \) and on other details of the cutoff choice. Note that the beta function of the coupling \( \xi_k \) vanishes when one is considering conformally invariant matter. The beta functions (14) and (18) agree with those obtained in [6].

4. Asymptotic safety

We proceed now to integrate the RG flow of the couplings of the local curvature invariants present in the truncation we are considering, i.e. the cosmological, Newton’s and the coupling constants of the curvature square terms. We will see that precisely the flow of these couplings is related to the UV renormalization of the theory, which in the effective average action formalism is encoded into the initial conditions of RG flow. These relations will tell us how to choose the bare couplings in order to achieve a finite continuum limit when UV regularization is removed.

Since (14) is a closed system for \( \tilde{\Lambda}_k \) and \( \tilde{G}_k \), we can start looking for non-Gaussian fixed points of the RG flow, which we need to find in order to construct a continuum limit, by first solving:

\[
\partial_t \tilde{\Lambda}_k = 0, \quad \partial_t \tilde{G}_k = 0.
\] (19)

It is easy to see that the system (19) admits both a Gaussian fixed point \( \tilde{\Lambda}_k = \tilde{G}_k = 0 \) and a non-Gaussian one [6]:

\[
\begin{align*}
\tilde{\Lambda}_* &= -\frac{3}{4} \frac{N_0 - 4N_1}{(1 - \chi)N_0 + 2N_1 - 4N_1}, \\
\tilde{G}_* &= -\frac{12\pi}{(1 - \chi)N_0 + 2N_1 - 4N_1}.
\end{align*}
\] (20)

Note that to have a fixed point with positive Newton’s constant, we need to satisfy the inequality \((1 - \chi)N_0 + 2N_1 - 4N_1 > 0\). The stability matrix around the non-Gaussian fixed point (20) is readily calculated and shows that it is attractive in the UV. This shows that both the cosmological and Newton’s constants are asymptotically safe couplings. This fact shows that, within the truncation we are considering and in the large \( N \) limit, 4D quantum gravity is asymptotically safe. The beta function system (14) can be easily integrated analytically.

\[^5\] It has eigenvalues \(-4, -2\).

\[^6\] See [5] for a discussion of asymptotic safety when gravitational fluctuations are also considered.
Integrating the second equation in (14) from the UV scale $\Lambda$ to the IR scale $k$ gives the following relation connecting the $k$-dependent, fixed point and bare Newton’s constants\(^7\):

$$
\frac{1}{\tilde{G}_k} = \frac{1}{\bar{G}_s} + \left( \frac{1}{\bar{G}_\Lambda} - \frac{1}{\bar{G}_s} \right) \left( \frac{\Lambda}{k_0} \right)^2 \left( \frac{k_0}{k} \right)^2.
$$

(21)

In (21) we introduced the reference scale $k_0$, which will play a role similar to the renormalization scale $\mu$ of standard perturbation theory. To make (21) finite in the continuum limit, i.e. in the limit $\Lambda \to \infty$, we need to renormalize Newton’s constant; this can be done by imposing the following condition:

$$
\left( \frac{1}{\bar{G}_{\Lambda}} - \frac{1}{\bar{G}_s} \right) \left( \frac{\Lambda}{k_0} \right)^2 = C_G,
$$

(22)

where $C_G$ is a constant that we will determine later. Solving (22) with respect to bare Newton’s constant $\tilde{G}_{\Lambda}$ gives

$$
\tilde{G}_\Lambda = \frac{\bar{G}_s}{1 + C_G \bar{G}_s \left( \frac{k_0}{\Lambda} \right)^2},
$$

(23)

which shows how we need to choose $\tilde{G}_\Lambda$ to approach $\tilde{G}_s$, as $\Lambda \to \infty$, in order that the lhs of (22) remains constant. Inserting now (22) in (21) gives the functional form of the $k$-dependent Newton’s constant:

$$
\tilde{G}_k = \frac{\bar{G}_s}{1 + C_G \bar{G}_s \left( \frac{k_0}{k} \right)^2}.
$$

(24)

Equations (23) and (24) are formally the same at the level of the approximation we are considering but have different physical meanings: the first equation tells us how the bare coupling changes as we vary the UV scale $\Lambda$, while the second equation shows how the $k$-dependent coupling flows as we integrate more and more degrees of freedom by lowering the RG scale $k$. From (24) we see that the dimensionful Newton’s constant, $G_k = k^{-2} \tilde{G}_k$, reaches the following renormalized value at $k = 0$:

$$
G_0 = \frac{1}{C_G k_0^2} = C_G = \frac{1}{\bar{G}_0 k_0^2},
$$

(25)

which fixes the value of the constant introduced (22). Similarly, we obtain the following relation between the $k$-dependent, fixed point and bare cosmological constants:

$$
\frac{\tilde{\Lambda}_k}{\bar{G}_s} = \frac{\tilde{\Lambda}_s}{\bar{G}_s} + \left( \frac{\tilde{\Lambda}_\Lambda}{\bar{G}_s} - \frac{\tilde{\Lambda}_s}{\bar{G}_s} \right) \left( \frac{\Lambda}{k_0} \right)^4 \left( \frac{k_0}{k} \right)^4.
$$

(26)

After we impose the UV renormalization condition

$$
\left( \frac{\tilde{\Lambda}_\Lambda}{\bar{G}_s} - \frac{\tilde{\Lambda}_s}{\bar{G}_s} \right) \left( \frac{\Lambda}{k_0} \right)^4 = C_\Lambda,
$$

\(^7\) If we define the couplings $\tilde{\Lambda}_k = \frac{\Lambda}{k \bar{G}_s}$ and $B_k = \frac{1}{\bar{G}_s}$, the system (14) becomes $\partial_t \tilde{\Lambda}_k = 4(\tilde{\Lambda}_s - \tilde{\Lambda}_k)$ and $\partial_t \tilde{B}_k = 2(\tilde{B}_s - \tilde{B}_k)$ where $\Lambda_k = \tilde{\Lambda}_k k^4$ and $B_k = \tilde{B}_k k^2$. These relations immediately give $\tilde{\Lambda}_k = \tilde{\Lambda}_s + (\tilde{\Lambda}_\Lambda - \tilde{\Lambda}_s) \left( \frac{\Lambda}{k} \right)^4$ and $\tilde{B}_k = \tilde{B}_s + (\tilde{B}_\Lambda - \tilde{B}_s) \left( \frac{\Lambda}{k} \right)^2$ from which we read off (21) and (26).
which can be solved for the bare cosmological constant

\[ \bar{\Lambda}_k = \frac{\bar{\Lambda}_s + C_A \tilde{G}_s (\frac{k_0}{k})^4}{1 + C_G \tilde{G}_s (\frac{k_0}{k})^2}, \]  

(27)

we obtain the \( k \)-dependent cosmological constant

\[ \tilde{\Lambda}_k = \frac{\bar{\Lambda}_s + C_A \tilde{G}_s (\frac{k_0}{k})^4}{1 + C_G \tilde{G}_s (\frac{k_0}{k})^2}. \]  

(28)

As before, (27) and (28) are formally equivalent. The renormalized value of the dimensionful cosmological constant \( \Lambda_k = k^2 \tilde{\Lambda}_k \) is obtained from (28) and shows that \( C_A = \frac{\Lambda_0}{G_0(k_0)^2} \). Since both \( G_k \) and \( \Lambda_k \) are relevant parameters in the UV (they are both attracted by the non-Gaussian fixed point), we need to fix their value from experiments, i.e. we need to measure \( \Lambda_0/k_0^2 \) and \( G_0k_0^2 \). The phase diagram obtained by plotting equations (24) and (28) is shown in figure 1; we can clearly see that all flow trajectories reach the IR without any obstruction, a feature that is not yet present in the analogous RG flows that include gravitational fluctuations [5]. Note that the dimensionless product \( \Lambda_0G_0 \) is independent of the arbitrary scale \( k_0 \); thus \( \bar{\Lambda}_0 \bar{G}_0 \) is a true IR numerical prediction associated with a given RG trajectory. The Einstein–Hilbert terms in our ansatz (7) in the \( k \to 0 \) limit become

\[ \Gamma_0[g]_{\text{EIR}} = \frac{\Lambda_0}{8\pi G_0} \int d^4x \sqrt{g} - \frac{1}{16\pi G_0} \int d^4x \sqrt{g} R. \]  

(29)

Obviously this part of the gravitational effective action is not conformally invariant due to the presence of the scales \( \Lambda_0 \) and \( G_0 \). We see from (21) and (26) that the only way of obtaining conformal invariance in the IR is to set the UV couplings to their fixed point values: \( \bar{\Lambda}_s = \bar{\Lambda}_A \) and \( \bar{G}_A = \bar{G}_s \) so that \( \Lambda_0 = 0 \) and \( 1/G_0 = 0 \), obviously the only way of restoring conformal invariance at the quantum level.

Next we can integrate the beta functions (18) for the couplings \( \lambda_k, \xi_k, \rho_k, \tau_k \). These couplings have only the Gaussian fixed point, at least within the approximation we are
considering [10]. This can be easily done:
\[
\frac{1}{g_k} = \frac{1}{g_\Lambda} + C \log \frac{\Lambda}{k} = \frac{1}{g_\Lambda} + C \log \frac{\Lambda}{k_0} - C \log \frac{k}{k_0},
\]
where \( g_k \) stands for any one of the couplings \( \lambda_k, \xi_k, \rho_k \) and \( \tau_k \) and where the relative constants \( C \) can be read off from (18). We can now choose the following UV boundary conditions:
\[
g_\Lambda = -\frac{1}{C \log \frac{\Lambda}{k_0}},
\]
(30)
which imply the following \( k \)-dependence of the coupling constants:
\[
g_k = -\frac{1}{C \log \frac{k}{k_0}}.
\]
(31)
As we mentioned before, (30) and (31) are formally equivalent; they both show that these couplings are asymptotically free. Note that when we reinsert the \( g_k \) from (31) in the effective average action (7) and we take the limit \( k \to 0 \), the local curvature square contributions diverge. It seems that the effective action has an IR divergence. We will see in the next section that this is not the case: when we include the finite curvature square contributions in (7), the \( \log k \) in (31) combines with another such term coming from the running structure functions, so that their sum has a finite \( k \to 0 \) limit.

5. Curvature square terms and corrections to the Newtonian potential

In this section we start to look at the finite non-local part of the effective action and we show how this part of the action can be obtained as the result of the integration of the RG flow. We focus on the running structure functions in the curvature square terms, where we can perform all steps analytically.

By integrating the flow equation (15) for the running structure functions from the UV scale \( \Lambda \) to the IR scale \( k \), we find the relation
\[
f_{i,\Lambda}(x) - f_{i,k}(x) = \frac{1}{(4\pi)^2} \int_k^\Lambda \frac{dk'}{k'} g_i \left( \frac{x}{k'^2} \right),
\]
where \( f_{i,\Lambda}(x) \) are the bare structure functions, while \( f_{i,k}(x) \) are the \( k \)-dependent ones. Changing variables to \( u = x/k^2 \), with \( dk/k = -du/2u \), gives
\[
f_{i,k}(x) = f_{i,\Lambda}(x) - \frac{1}{(4\pi)^2} \int_{x/\Lambda^2}^{x/k^2} \frac{du}{2u} g_i(u).
\]
(32)
It should be noted that when we insert the explicit forms of the functions \( g_i(u) \) from (17), the constant terms will make the integrals in (32) logarithmically divergent at the lower limit when \( \Lambda \to \infty \). We can isolate these divergences by subtracting these constants from the integrals in (32) in the following way:
\[
f_{C,\Lambda}(x) = f_{C,\Lambda}(x) - \frac{1}{(4\pi)^2} \frac{N_0 + 6N_1 + 12N_{1/2}}{120} \log \frac{\Lambda}{k_0}
+ \frac{1}{(4\pi)^2} \frac{N_0 + 6N_1 + 12N_{1/2}}{120} \log \frac{k}{k_0} - \frac{1}{(4\pi)^2} \int_{x/\Lambda^2}^{x/k^2} \frac{du}{2u} g_C(u) - \frac{N_0 + 6N_{1/2} + 12N_1}{120}
\]
(33)
and

\[ f_{R,\Lambda}(x) = f_{R,\Lambda}(x) - \frac{1}{(4\pi)^2} \frac{(1 - \chi)^2 N_0}{72} \log \frac{\Lambda}{k_0} \]

\[ + \frac{1}{(4\pi)^2} \frac{(1 - \chi)^2 N_0}{72} \log \frac{k}{k_0} - \frac{1}{(4\pi)^2} \int_{\Lambda}^{x/k^2} \frac{du}{2u} \left[ g_R(u) - \frac{(1 - \chi)^2 N_0}{72} \right]. \]  

(34)

The log \( \frac{\Lambda}{k_0} \) terms in the first lines of (33) and (34) are the UV divergences that are also encountered in standard perturbation theory [11]. We can renormalize the running structure functions by imposing the following UV boundary conditions:

\[ f_{C,\Lambda}(x) = \frac{1}{(4\pi)^2} \frac{N_0 + 6N_0 + 12N_1}{120} \log \frac{\Lambda}{k_0} + c_C, \]

\[ f_{R,\Lambda}(x) = \frac{1}{(4\pi)^2} \frac{(1 - \chi)^2 N_0}{72} \log \frac{k}{k_0} + c_R, \]  

(35)

which make the first lines in (33) and (34) vanish. Here \( c_C \) and \( c_R \) are possible finite renormalization constants. In this way the second lines of (33) and (34) are finite in the UV \( \Lambda \to \infty \) and in the IR \( k \to 0 \) limits. Note also that the first terms of the second lines of (33) and (34), when combined with (8), correctly reproduce (31). Performing the integrals in (33) and (34) now gives the following finite contributions to the curvature square part of the renormalized effective action:

\[ \Gamma_0[g]_{R^2} = \frac{1}{2(4\pi)^2} \int d^4x \sqrt{g} \left[ \frac{N_0 + 6N_0 + 12N_1}{120} C_{\mu\nu\alpha\beta} \log \frac{\Delta}{k_0^2} C_{\mu\nu\alpha\beta} + \frac{(1 - \chi)^2 N_0}{72} R \log \frac{\Delta}{k_0^2} R \right], \]  

(36)

where we fixed the renormalization constants to the values

\[ c_C = \frac{-23N_0}{1800} - \frac{3N_1}{50} - \frac{4N_1}{75}, \quad c_R = \frac{1 + 10(1 - \chi) + 30(1 - \chi)^2 N_0}{2160} + \frac{N_1}{360} - \frac{N_1}{120}. \]

The effective action (36) is the same as that obtained using standard perturbation theory if we equate the arbitrary scale \( k_0 \) with the renormalization scale \( \mu \) [11]. In (36) we dropped the Euler and the total derivative terms; as we mentioned in the previous section, these terms are not finite in the IR limit \( k \to 0 \) since their running is of the form (31), but this does not cause any problem, since these terms are total derivatives and thus vanish in general. Note also that if we were employing a different cutoff shape function \( R_0(z) \) than that employed here, we would have found a different form for the flow for \( k \neq 0 \) but, as was shown in the 2D case [8], the \( k \to 0 \) limit would have always been equal to (36).

In (36) the only contribution to the renormalized Ricci structure function \( f_{R,0}(x) \) comes from the scalar fields and this contribution vanishes if we consider conformally invariant scalars by setting \( \chi = 1 \). Even when \( \chi = 1 \), the functional (36) is not conformally invariant. This happens because we are considering an ansatz of the effective average action structured as in (7), which by construction cannot accommodate a conformally invariant limit for \( k = 0 \) since the expansion we are performing is not so. As we will explain in the following section, we...

\footnote{The integral \( \int \sqrt{\mathcal{F}} = 32\pi^2\chi \) is the Euler characteristic of the 4D manifold and is thus non-zero in general. The renormalization of this term is probably related to topological fluctuations.}
expect that when $\chi = 1$ the only non-conformally invariant part of the effective action is the conformal anomaly-induced effective action $^{14}$. The form found in equation (36) must thus be a result of the expansion of a more general conformally invariant term, which to order curvature square reduces to (36). We can speculate such a term to be of the following form:

$$\int d^4x \sqrt{g} C_{\mu\nu\alpha\beta} \log \mathcal{O}_C C^{\mu\nu\alpha\beta},$$  

(37)

where the operator $\mathcal{O}_C$ is a fourth-order conformally invariant differential operator that acts on four-index tensors with the symmetries of the Weyl tensor. The explicit form of this differential operator is unknown $^{12}$, but it must have a form such that (37) has a smooth flat space limit. At this point it is important to note that a conformal variation of (36), when $\chi = 1$, will generate the anomaly term proportional to $C^2$ with the correct coefficient (see equation (44)). This fact has already been noted by several authors $^{11, 13}$; in particular, the second authors consider this a genuine anomaly term in contrast to the expectation that the effective action proposed in $^1$ is the only one that produces the conformal anomaly upon variation $^{14}$. Since these first authors obtained the form (36) by a flat space calculation, we can think of them to be in the same position as we are in our calculation: the non-conformally invariant form, and thus the anomalous term, is what remains of a truly conformally invariant term of the form (37) that reduces to a non-invariant term when, to be able to perform calculations, an expansion that breaks conformal symmetry is performed. This point of view can compose the different expectations expressed on the one hand in $^{13}$ and on the other in $^{14}$ and thus deserves further study.

A physical reason for expecting a term like (36) in the effective action, at second order in the curvatures, is that it generates the quantum gravitational corrections to Newton’s gravitational potential$^9$. These corrections can be obtained along the lines of [9] and are the following:

$$V(r) = -\frac{MG_0}{r} \left\{ 1 + \left( \frac{43}{30\pi} + \frac{[4 + 5(1 - \chi)^2] N_0 + 24N_\frac{7}{2} + 48N_1}{180\pi} \right) \frac{G_0}{r^2} \right\}. \quad (38)$$

The first part of the quantum gravitational correction in (38) is the purely gravitational one found in [9]. As explained in the last reference, the correction (38) is equivalent to that obtained using effective field theory arguments. Note that every field species (even conformally invariant matter) gives a positive contribution, of the same form as the purely gravitational one. From (38), we can see that when the number of massless matter fields $N$ grows larger, their contributions soon dominate the purely gravitational contribution.

6. Higher order terms and the conformally induced effective action

In the last two sections, we explicitly derived and integrated the RG flow of the gravitational effective average action to second order in the curvature expansion. In this section we now look at the third- and fourth-order terms.

One can consider the curvature cube terms in the effective average action ansatz (7) and can project their flow by employing in (5) the non-local heat kernel expansion to third order $^{15}$. There are now ten independent running structure functions and the curvature expansion of the

$^9$ This is only the ‘graviton polarization’ part of the correction.
effective average action has the following form:

$$\Gamma_k[g]|_{\mathcal{R}^3} = \left(\frac{1}{4\pi}\right)^2 \int d^d x \sqrt{g} f_{i,k}(\Delta_1, \Delta_2, \Delta_3) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(i),$$

(39)

where the curvature structures $\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(i)$ are given in [15]. For example, $\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(1) = R_1 R_2 R_3$, $\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(2) = R_{1\mu} R_{2\nu} R_{3\nu}$ and so on; following the conventions of [15], the Laplacians $\Delta_i$ act only on the curvatures $\mathcal{R}_i$ for $i = 1, 2, 3$. The third-order terms on the rhs of the flow equation (5) can now be written as

$$\partial_t \Gamma_k[g]|_{\mathcal{R}^3} = \left(\frac{1}{4\pi}\right)^2 \int d^d x \tilde{\mathcal{H}}_k(s) s^{\alpha_i-1} \int d^d x \sqrt{g} F_i(s \Delta_1, s \Delta_2, s \Delta_3) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(i),$$

(40)

where the explicit form of the non-local heat kernel structure functions $F_i(x_1, x_2, x_3)$ can be obtained from those given in [15]. Since the curvature structures $\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(i)$ may contain covariant derivatives, third-order terms are proportional to different powers of $s$. In fact, we have $\alpha_1 = \alpha_2 = \alpha_3 = 3$, $\alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = 4$, $\alpha_8 = \alpha_9 = 5$ and $\alpha_{10} = 6$. By comparing (39) to (40), we obtain the flow equations for the third-order running structure functions:

$$\partial_t f_{i,k}(x_1, x_2, x_3) = \left(\frac{1}{4\pi}\right)^2 g_i \left(\frac{x_1}{k^2}, \frac{x_3}{k^2}, \frac{x_3}{k^2}\right),$$

(41)

where we introduced the scheme-dependent functions:

$$g_i(u_1, u_2, u_3) = \left(\frac{1}{4\pi}\right)^2 \int_0^\infty ds \tilde{\mathcal{H}}_k(s) s^{\alpha_i-1} F_i(s k^2 u_1, s k^2 u_2, s k^2 u_3),$$

(42)

where $u_i = x_i/k^2$ for $i = 1, 2, 3$. At this order, computations become increasingly hard and it seems rather difficult to straightforwardly evaluate the $s$ integrals on the rhs of (42), even if we employ the optimized cutoff shape function $R_k(z) = (k^2 - z)\theta(k^2 - z)$. It was shown in [6] that if one employs this cutoff shape function, the couplings of the cubic and higher invariants, i.e. $g_{1,k} = g_{i,k}(0, 0, 0)$, have zero beta functions. With this cutoff choice, we need to UV renormalize only the couplings we have already considered at lower order of the curvature expansion. In spite of this, the structure functions $g_i(x_1, x_2, x_3)$ are non-zero even if they have a vanishing Taylor expansion around the origin (since $Q_{-n}[h_k] = 0$ for all $n \geq 1$); probably they are proportional to theta functions as are the second-order functions $g_i(u)$ (17). For this reason, no UV divergences arise when we integrate the flow equation (42) and we find a finite result for the renormalized third-order structure functions:

$$f_{i,0}(x_1, x_2, x_3) = \left(\frac{1}{4\pi}\right)^2 \int_0^\infty \frac{k^2}{k} g_i \left(\frac{x_1}{k^2}, \frac{x_3}{k^2}, \frac{x_3}{k^2}\right).$$

(43)

In (43) we imposed the UV boundary conditions $f_{i,\Lambda}(x_1, x_2, x_3) = 0$. The absence of UV divergences can also be understood by looking at the local expansion of the gravitational effective average action where, as we just noted, the dimensionful beta functions turn out to be zero. $\partial_t g_{i,k} = 0$, thus implying $\tilde{g}_{i,k} = \left(\frac{1}{k}\right)^{\alpha_i} g_{i,\Lambda}$. where $g_{i,k} = \tilde{g}_{i,k}$ are the dimensionless couplings with $d_i < 0$. Thus the limit $\Lambda \to \infty$ is finite and we have $g_{i,k} \to 0$. We finally obtain the cubic curvature part of the gravitational effective action $\Gamma_0[g]|_{\mathcal{R}^3}$ when we re-insert (43) into (39).

Note that we are considering only the purely gravitational curvature structures of [15].
It will be interesting to extend the curvature expansion to the next (quartic) order in the curvatures, since the conformal anomaly-induced effective action is of this order \[1\]. This effective action is usually obtained by integrating the conformal anomaly; for the field content we are considering and for \(\chi = 1\), this is the following:

\[
\langle T_\mu^\mu \rangle = \frac{1}{(4\pi)^2} \left[ \frac{N_0 + 6N_1}{120} C^2 - \frac{N_0 + 11N_1 + 62N_1}{360} E + \frac{N_0 + N_1 - 3N_1}{60} \Delta R \right]. \tag{44}
\]

To derive (44), one uses the relation \(\langle T_\mu^\mu \rangle = (4\pi)^{-2} \int \sqrt{g} \text{tr} b_4 (S^{(2)})\), where \(S^{(2)}\) is the Hessian of (1) and the \(b_4\) are the relative local heat kernel coefficients. It was shown in \([1]\) that the following effective action has an energy–momentum tensor whose trace reproduces (44):

\[
\Gamma_{CA}[g] = \frac{1}{8(4\pi)^2} \int d^4 x \sqrt{g} \left[ \frac{N_0 + 6N_1}{60} C^2 \right. \\
- \frac{N_0 + 11N_1 + 62N_1}{360} \left( E + \frac{2}{3} \Delta R \right) \left( E + \frac{2}{3} \Delta R \right), \tag{45}
\]

where \(\Delta_{4,0} = \Delta^2 + 2R^{\mu\nu\rho\sigma} \nabla_\mu \nabla_\nu + \frac{2}{3} R \Delta + \frac{1}{3} \nabla^\mu R \nabla_\mu\) is the conformal covariant fourth-order operator acting on conformal invariant scalars. The conformal anomaly-induced effective action (44) will constitute, together with other conformally invariant terms, the four curvature terms in the curvature expansion of the effective action. But since to quartic order in the curvatures we do not even know the form of the non-local heat kernel expansion, any calculation along the lines of the previous sections is, for the moment, a very hard task to perform. In spite of this, we believe that the integration of the flow of the effective average action will generate the non-local effective action (44) as one of the many fourth-order terms. In \([8]\), it was shown that this is indeed what happens in the analogous 2D case.

### 7. Conclusions

As a first step toward understanding the gravitational effective action, we studied the effective action induced by massless matter fields interacting solely with the background geometry. We proposed an expansion of the effective action in powers of the curvatures and we showed how various known results about quantum gravity could be seen as arising from different terms of it. We showed that asymptotic safety and UV renormalization are related to the running of the couplings of local curvature invariants, the first of these being the cosmological and Newton’s constants. Then we derived the quantum gravitational corrections to the Newtonian potential from the quadratic curvature terms and we explained that all other quantum gravitational effects, including those induced by the conformal anomaly, are contained in the higher order contributions.

Once matter contributions are well understood, one can infer something about the gravitational effective action when matter contributions become dominant in the large \(N\) limit. When instead gravitational fluctuations become large, we must consider the fact that gravity may actively react to matter inducing corrections to the effective action that may be of the same order as those induced by matter. The exact solution of 2D quantum gravity shows that this is indeed what happens in the 2D case \([16]\). It is thus important to extend this work and the program started in \([9]\) to the treatment of full gravitational dynamics.
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References

[1] Riegert R J 1984 *Phys. Lett. B* **134** 56
Fradkin E S and Tseytlin A A 1984 *Phys. Lett. B* **134** 187
[2] Donoghue J F 1994 *Phys. Rev. D* **50** 3874 (arXiv:gr-qc/9405057)
[3] Percacci R 2009 *Approaches to Quantum Gravity* ed D Oriti (Cambridge: Cambridge University Press) arXiv:0709.3851 [hep-th]
[4] Wetterich C 1993 *Phys. Lett. B* **301** 90
Berges J, Tetradis N and Wetterich C 2002 *Phys. Rep.* **363** 223
[5] Codello A, Percacci R and Rahmede C 2009 *Ann. Phys.* **324** 414 (arXiv:0805.2909 [hep-th])
[6] Percacci R 2006 *Phys. Rev. D* **73** 041501 (arXiv:hep-th/0511177)
[7] Barvinsky A O and Vilkovisky G A 1990 *Nucl. Phys. B* **333** 471
Avramidi I G 1991 *Nucl. Phys. B* **355** 712
Avramidi I G 1998 *Nucl. Phys. B* **509** 557 (erratum)
Vilkovisky G A 1992 Heat kernel: rencontre entre physiciens et mathematiciens CERN-TH-6392-92
[8] Codello A 2010 *Ann. Phys.* **325** 1727 (arXiv:1004.2171 [hep-th])
[9] Satz A, Codello A and Mazzitelli F D 2010 *Phys. Rev. D* **82** 084011 (arXiv:1006.3808 [hep-th])
[10] Benedetti D, Machado P and Saueressig F 2009 *Mod. Phys. Lett. A* **24** 2233
Benedetti D, Machado P and Saueressig F 2010 *Nucl. Phys. B* **824** 168
[11] Shapiro I L 2008 *Class. Quantum Gravity* **25** 103001 (arXiv:0801.0216 [gr-qc])
[12] Hamada K-j 2001 *Prog. Theor. Phys.* **105** 673 (arXiv:hep-th/0012053)
[13] Deser S and Schwimmer A 1993 *Phys. Lett. B* **309** 279 (arXiv:hep-th/9302047)
Deser S 1996 *Helv. Phys. Acta* **69** 570 (arXiv:hep-th/9609138)
[14] Mazur P O and Mottola E 2001 *Phys. Rev. D* **64** 104022 (arXiv:hep-th/0106151)
[15] Barvinsky A O, Gusev Yu V, Vilkovisky G A and Zhytnikov V V 1994 *J. Math. Phys.* **35** 3525 (arXiv:gr-qc/9404061)
Barvinsky A O, Gusev Yu V, Zhytnikov V V and Vilkovisky G A 2009 arXiv:0911.1168 [hep-th]
[16] Ambjorn J, Durhuus B and Jonsson T 1997 *Quantum Geometry: A Statistical Field Theory Approach* (Cambridge: Cambridge University Press)

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