On the Casimir effect for parallel plates in the spacetime with one extra compactified dimension

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Abstract

In this paper, the Casimir effect for parallel plates in the presence of one compactified universal extra dimension is reexamined in detail. Having regularized the expressions of Casimir force, we show that the nature of Casimir force is repulsive if the distance between the plates is large enough, which is disagree with the experimental phenomena.

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The higher dimensional spacetime has become a powerful ingredient for unifying the interactions. Kaluza and Klein put forward the idea that our universe has more than four dimensions about 80 years ago [1, 2]. Their theory introduced an additional compactified dimension in order to unify gravity and classical electrodynamics. Now the string theory is developed to unify the quantum mechanics and gravity with the help of introducing seven extra spatial dimensions. It is interesting that the order of the compactification scale of the extra dimensions has not been confirmed. Some branches of string theory expect that the radii of the compactified universal extra dimensions should be Planck size which is beyond our experimental reach of today and near future [3, 4]. In some approaches large extra dimensions were also invoked for providing a breakthrough of hierarchy problem [5-10]. The gauge fields may be localized on a four-dimensional brane, our real universe, and only gravitons can propagate in the extra space transverse to the brane [8, 9]. It is possible to probe the large additional spatial dimensions.

The Casimir effect is a fundamental aspect of quantum field theory in confined geometries and the physical manifestation of zero-point energy [11-21]. The precision of the measurement has been greatly improved experimentally [22-25]. Therefore the Casimir effect can become a useful method for the study of a lot of topics. The magnitude of cosmological constant can be estimated with Casimir effect [26, 27]. The effect was also applied in the context of string theory [28-31]. Recently more progresses of the effect were made to investigate the properties of the spacetimes with extra dimensions [32-33]. As the first step of generalization, some topics in five-dimensional spacetimes were studied and the useful results were obtained [5-9]. Probing the possible existence and size of the extra dimension by means of Casimir effect attracts more attentions of the physical community [32-33]. The expressions of Casimir force between two parallel plates in the presence of one extra dimension differ from the force in the case without additional spatial dimensions. By comparison to experimental data the size of the universal extra dimension can be restricted to $L \leq 10nm$ for one extra dimension only when the distance between plates is very small [33].

The Casimir effect in the presence of a compactified universal extra dimension needs to be explored in detail. In this paper we reexamine the Casimir effect for parallel plates in the universe with only one additional dimension carefully. Having regularized the total energy, we obtain the Casimir energy, and then Casimir force. We find that the Casimir force is similar to the experimental data when the plates approach very close, and the upper limit of $L \leq 10nm$ on the extension of additional spatial dimension can also be obtained. According to our studies, the expression for Casimir force in the case with one extra dimension also shows that the plates will repulse each other for their large enough gap, but in the experiments the repulsive force has not appeared within the region of distance between plates [22-25]. The five-dimensional spacetime could not be feasible, and just be thought as a toy model. Here we rederive the Casimir energy and Casimir force with one extra dimension. We discuss these expressions for various ratio of plates distance and the radius of extra dimension. Finally the conclusions are emphasized.

In the Kaluza-Klein (KK) approach we study the scalar field in the system consisting of two
parallel plates in the spacetime with only one extra dimension. Along the extra dimension the wave vectors of the field have the form \( k_n = \frac{n}{L} \) with \( n \) and \( L \) being an integer and the radius of the extra dimension respectively. At the plates the fields satisfy the Dirichlet condition, leading the wave vector in the directions restricted by the plates to be \( k_l = \frac{\pi l}{R} \), \( l \) a positive integer and \( R \) the separation of the plates. Under the conditions mentioned above, the zero-point fluctuations of the fields can give rise to observable Casimir forces.

In the case of one extra dimension, the total energy density of the fields in the interior of system is thus given by

\[
\varepsilon = \int \frac{d^2 k}{(2\pi)^2} \sum_{l=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{2} \sqrt{k^2 + \frac{l^2 \pi^2}{R^2} + \frac{n^2}{L^2}}
\]

(1)

where

\[
k^2 = k_1^2 + k_2^2
\]

(2)

\( k_1 \) and \( k_2 \) are the wave vectors in directions of the unbound space coordinates. Following [12-15, 19], Eq.(1) becomes,

\[
\varepsilon = -\frac{1}{8\pi^3} \Gamma(-\frac{3}{2}) E_2(\frac{\pi^2}{R^2}, \frac{1}{L^2}; -\frac{3}{2}) - \frac{\pi^3}{8} \Gamma(-\frac{3}{2}) \zeta(-3) \frac{1}{R^3}
\]

(3)

where Epstein zeta function \( E_p(a_1, a_2, \cdots, a_p; s) \) is defined as,

\[
E_p(a_1, a_2, \cdots, a_p; s) = \sum_{\{n\}=1}^{\infty} \left( \sum_{j=1}^{p} a_j n_j^2 \right)^{-s}
\]

(4)

here \( \{n\} \) represents a short notation of \( n_1, n_2, \cdots, n_p, n_\alpha \) a positive integer.

By regularizing Eq.(3), we obtain the Casimir energy density,

\[
\varepsilon_C = \left[ \frac{1}{16\pi^3} \Gamma(2) \zeta(4) - \frac{1}{16\pi^3} \mu \Gamma(\frac{5}{2}) \zeta(5) \right] - \frac{1}{4\pi^2 \mu} \sum_{n_1, n_2=1}^{\infty} \left( \frac{n_2}{n_1} \right)^2 K_2(2\mu n_1 n_2) - \frac{\pi^2}{720} \frac{1}{\mu^3} \frac{1}{L^5}
\]

(5)

where

\[
\mu = \frac{R}{L}
\]

(6)

and \( K_\nu(z) \) is the modified Bessel functions of the second kind and falls exponentially with \( z \). The terms with series converge very quickly and only the first several summands need to be taken into account for numerical calculation to further discussions. Having derived in detail, we are able to prove that the Eq. (5) can become the expression of Casimir energy for a massive scalar field with mass \( m = \frac{n}{L} \) [17]. We analyze the Casimir energy (5) in the limits. If the plates distance is much
larger or less than the radius of universal extra dimension, the expression for the Casimir energy becomes,

\[ \varepsilon_C(\mu \gg 1) = -\frac{1}{16\pi^2 \mu^5} \mu^5 \Gamma \left( \frac{5}{2} \right) \zeta(5) \frac{1}{L^3} \]  

or

\[ \varepsilon_C(\mu \ll 1) = -\frac{\pi^2}{720} \frac{1}{\mu^3} \frac{1}{L^3} \]  

respectively. In the case of \( \mu \ll 1 \), the first more summands in Eq. (5) need to be considered, but the value of the third term is much smaller than that of the last term because the term with series also converge very quickly according to the property of the modified Bessel functions of the second kind \( K_{\nu}(z) \). The numerical calculations of the Casimir energy (5) lead to the data presented in Figure 1. The figure shows that the sign of the Casimir energy depending on the ratio \( \mu \) keeps negative no matter what the value the ratio \( \mu \) is.

It is certainly fundamental to discuss the Casimir force between the plates in the background with one extra dimension in order to compare our results with the experimental phenomenon. The Casimir force is denoted as \( f_C = -\frac{1}{L} \frac{\partial \varepsilon_C}{\partial \mu} \), so its expression is obtained as follow,

\[ f_C = \left\{ \frac{1}{16\pi^2} \frac{\mu^5}{\pi^2} \frac{5}{2} \Gamma \left( \frac{5}{2} \right) \zeta(5) \frac{1}{L^3} - \frac{1}{4\pi^2 \mu^2} \sum_{n_1,n_2=1}^{\infty} \frac{(n_2)^2 K_2(2\mu n_1 n_2)}{n_1(n_1,n_2)} - \frac{1}{4\pi^2 \mu} \sum_{n_1,n_2=1}^{\infty} \frac{n_2^3}{n_1} \left[ K_1(2\mu n_1 n_2) + K_3(2\mu n_1 n_2) \right] - \frac{\pi^2}{240 \mu^3} \frac{1}{L^4} \right\} \]  

Of course the equation (9) needs to be analyzed in the limits. According to the property of function \( K_{\nu}(z) \) and discussion above, the expression for the Casimir force becomes,

\[ f_C(\mu \ll 1) = -\frac{\pi^2}{240 \mu^3} \frac{1}{L^4} \]  

for smaller separation. In the case of larger ones,

\[ f_C(\mu \gg 1) = \frac{1}{16\pi^2 \mu^5} \mu^5 \Gamma \left( \frac{5}{2} \right) \zeta(5) \frac{1}{L^3} > 0 \]  

which means that there exists a repulsive force in the system if the plates separation is large enough.

According to (9) the Casimir force for parallel plates in the spacetime with one extra dimension is depicted in Figure 2. After proceeding the numerical calculation, we obtain the special ratio \( \mu_f = 5.343 \) satisfying \( f_C(\mu = \mu_f) = 0 \). When \( \mu < \mu_f \), then \( f_C < 0 \), and in the opposite \( f_C > 0 \) for \( \mu > \mu_f \). In the case of \( \mu \ll \mu_f \), the part of curve of Casimir force for two parallel plates depicted in Figure 2 can be used to estimate the size of the extra dimension by comparison to the experimental data. K. Poppenhaeger et al have carried out the study for the case that the plates locate very close each other [33]. They compared the curves for different value of size of extra dimension \( L \) with real experimental data to discover that good agreement with the data can only
be obtained if the upper limit on the radius of the only one additional dimension is $L \leq 10\, nm$. In the spacetime without additional spatial dimensions, the standard Casimir force between parallel plates is $f_{C0} = -\frac{\pi^2}{240} \frac{1}{R^4}$. The ratio of the Casimir force (with one extra dimension) to the standard ones (without extra dimensions) is $q = \frac{f_C}{f_{C0}}$ and depends on $\mu$. The curve of ratio $q$ is shaped in Figure 3 and shows the extra-dimension correction to the standard Casimir force. The larger the plates separation is, the greater the correction is. However we should not neglect that the repulsive Casimir force denoted as $f_C > 0$ under $\mu > \mu_f$ is excluded in the practice [22-25]. We must point out that the experiment is always performed on electromagnetic fields that may obey more complicated boundary conditions than the scalar field we consider here, according to the 5-dimensional Kaluza-Klein theory the expressions of Casimir force for different kinds of fields with different boundary conditions will certainly be different, and the special ratio $\mu_f$ will also not be equal to what is obtained above, but the repulsive Casimir force must appear when the separation of two plates is sufficiently large. Our conclusion that there must exist the repulsive Casimir force in the Universe with only one extra dimension is kept.

In conclusion, the model that the spacetime with only one extra dimension can not be realistic. Having studied the Casimir effect for two parallel plates in the universe with one additional dimension, we find analytically that there must exist the repulsive Casimir force between the plates when the separation is large enough. The experimental data show that no repulsive Casimir force appeared. Therefore the results obtained form the Kaluza-Klein theory including only one compactified universal extra dimension disagree with the experimental results. The related topics need further research.

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Figure 1: The Casimir energy density versus $\frac{R}{L}$ for two parallel plates in the spacetime with one compactified extra dimension.
Figure 2: The Casimir force versus $\frac{R}{L}$ between two parallel plates in the spacetime with one compactified extra dimension.
Figure 3: The ratio of the Casimir force (with one extra dimension) to the standard ones (without extra dimension) versus $\frac{R}{L}$ between two parallel plates.