Level Set based Topology Optimization and its Application for Structures

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Abstract—Topology optimization has played a key role in structural optimization. It has been basically introduced by Bendsoe and Kikuchi in 1988. It is very important to use the material wisely to reduce the weight of the structure without compromising the strength and other desired properties. Now, the optimization concept has grown in many directions and can be done with material distribution methods, Heuristic (experience) based methods, and Boundary based method. This article presents an introduction to level set method (Boundary based method) and some of its application in optimization of 2D structures and shows the convergence behaviour of the objective function. Minimum compliance is taken as the objective function which is subjected to volume constraint for some static structures.

Keywords—Optimization, structural topology, Level set method, Matlab, Beam

INTRODUCTION

In today’s world, Optimality has become a driving a force for individuals(engineers,scientist, and mathematicians) and companies for research, better product and services for society. Optimization can be described as a best solution among the many available alternatives under a given set of predefined conditions. Better product, less overall cost, serviceability, maintenance has made optimization an essential part of every research and industry. Optimization not only application to a physical component, but can be broadly used in an assembly of components, products, plans as well as in management services. The advancement in computation has made this process very popular in recent years to minimize the time and to speed up the process of design and development.

Topology optimization has become an integral part of design and development of structural components in recent years. The main objective of topology optimization is the distribution of material with in the components to improve the performance of the structure. This is possible because of the material distribution method for generating optimal topologies of structural elements. In classical design where material played a very important role, with the development of topological optimization, the distribution of material becomes an essential part of designing. Initially this concept was used for mechanical design problems, but now it has spread its wing all around. The topology design is used broadly in structural problems (vibrations, buckling, stress constraints, pressure loads, compliant mechanisms, material design, support design, civil engineering applications and bio-mechanics, etc.). It can be used for linear and nonlinear kind of problems. Moreover, the new areas where it has been implemented successfully are electro-thermal actuators, MEMS, flow problems, transducers, electromagnetic, acoustics, optics etc.

The first paper on topology was published by an Australian named Michell in 1904 to optimize the weight of a truss. Later this theory was extended to optimize the beam system. The advancement in numerical techniques with the development of analytical and computational tools made the optimization process very easy and fast because of their capability to handle the large amount of optimization data. With the development of computational tools, new techniques for the optimization developed to best optimized structure.

The first general theory on topology optimization termed as optimal layout theory was presented by Prager and Rozvany in 1977 for grid type structures. In 1988 a paper by Bendsoe and Kikuchi [1] provided a numerical method for topology optimization. They have used homogenization method in their work. The major drawback of this method was creation of large amount of voids, and variable densities ranges from 0 to 1. Therefore it was difficult to interpret the exact shape of the optimized object.

The next development was to avoid the above said problem means to provide density to each element. This method is commonly known as a solid isotropic material with penalization method. In this method, the solution was in the form of check board with intermediate densities; i.e. the densities very between 0 to 1. Still there was no clear view of an optimized structure. To overcome the above problem, a new method known as evolutionary structural optimization approach (ESO) developed. The main motive of this method was to remove the weak elements of the object in a predefined design domain. By doing this exercise, all intermediate densities could be eliminated. The outcome of this method was an object having jagged voids in structure. There were no smooth boundaries around the voids. This kind of structure was not acceptable as a design point of view and as well as difficult to manufacture. This was the main limitation of ESO method.

Another method proposed by Querin et al. [2] was bi-directional evolutionary optimization approach (BESO). In this approach the efficient material to be added while the inefficient material is removed. Simultaneously many authors proposed very efficient method to eliminate the check board pattern in optimization techniques specially the sharp corners in check board pattern.
To overcome all above problems, a new method, implicit moving boundary method, also known as level set method used which was devised by Osher and Sethian[3] This method is capable of getting clear boundaries during all steps of optimization process. The major benefit of level set method is to track the motion of the boundaries. Additionally it automatically nucleates holes, merges hole with each other and with boundaries throughout the optimization process.

Mapping of an implicit and structural model of a continuously variable geometry is an important part of the level set method. The mapping can be done by two methods via density method and boundary method. In density method, Finite element inside the geometry is represented as solid and those outside or within a hole as empty. The limitation of this method is variation in density during the intersection of elements within the structural boundaries. This again gives an ambiguous design and cannot be predicted as desired result. However boundary-based method gives a clear prediction on an optimized structure.

Level-set-based methods (implicit moving boundary)

The interfaces propagate in variety of physical phenomenon. Ocean waves, combustion, and material boundaries are some of examples. In moving boundaries method, less obvious boundaries are also play a vital part. In boundary based method, the design variables directly control the exterior as well as interior of the design domain. The level set method was devised by Osher and Sethian (1988). It is much simpler and more versatile method for computing and analysing the motion of an interface in two or three dimensions. Initially the level set method was limited to the tracking of propagating interfaces. Later on it has been implemented for a wide range of problems, especially structural optimization. The level set models sometimes mentioned as implicit moving boundary (IMB) models. The level set models are very versatile as they can effortlessly represent intricate boundaries. It is having the ability of creating new holes, can split a bigger hole into multiple holes, or merge with other holes to form a single one.

Level set method is based on the concept of propagation of the level set surface. It is non-traditional method in which fixed grid mesh is used as a design space. It has gained the significant popularity as it works on the boundaries during whole optimization process. The design changes are carried out as a mathematical programming for the problem of optimization. The level set method can be categorized on the basis of level set function parameterization, geometry, and regularization techniques. The LSM can handle topological merging, nucleation and splitting of holes naturally by embedding the interface as the zero level set of a higher-dimensional function. The Implicit boundary method (The level set method) improves the accuracy near the boundaries and removes the ambiguity of intermediate densities as in density-based approach. Level-set-based topology optimization methods allow a convenient treatment of topological changes over explicit boundary descriptions. Additionally, the results of most-level-set-based topology optimization methods do not involve mesh-dependent which are often encountered in density-based topology optimization methods. LSMs for structural topology and shape optimization: In structural optimization the two-phase, material-void problem is the most simple and most frequently treated case. If we change level set function, the shape and topology of material domain changes.

Application: [3] has initially used LSM for tracking of the boundaries. LSM initially used for study of geometry, image enhancement and noise removal, combustion problems, crystal growth and shape detection. Later the application of LSM extended to structural optimization. 2D and 3D cases have been solved by many authors. Eigen value problem [4], contact problems [5], check board implementation of material [6], problems related to fluid, thermal electro-mechanical [7], electromagnetic [8], optical applications [9] shell structure (10) and many more are the examples of LSM implementation. Aircraft wing structure, bridges, medical instrument, Aircraft fuselage, bulkheads etc., dams, composite beam structure, wind excited tall buildings are few of them.

Mathematical formulation

Minimum compliance: method is a simplest method of formulation of a design problem. The general design problem can be formulated as: Maintaining the Integrity of the Specifications

Objective function \( c(x) \), subject to a volume constraint \( V(x) \leq 0 \) and \( N \) other constraints \( g_i \leq 0, i = 1 \ldots N \). The distribution of material is described by the density variable \( \rho(x) \) that will be in a 0-1 configuration. Mathematically, it can be written as [11]

Min: \( f(x) \)
Subject to: \( g_i(x) \leq 0 \ i = 1\ldots p \ (1) \)   \( h_j(x) = 0 \ j = 1\ldots q \)

In our case the objective function is minimization of compliance which is subjected to volume constraint.

\[
\min c(x) = F^T U = U^T K U = \sum_{k_e}^K k_{e} u_{e} \\
\text{Subject To: } V(x) = V(\text{req})
\]  

Where \( F \) represents global force, \( K \) represents global stiffness and \( U \) represents the global displacement. In general, the real life applications have many constraints, but in this case only a linear volume constraint has been taken for simplicity. The volume constraint improves the general convergence. If \( D \) is design domain, \( \psi \) is material part of the domain and \( (D/\psi) \) represents the void, then the level set function can be defined as [12][14][21-22]

\[
\phi(X) > r \iff X \in \psi(\text{material}) \\
\phi(X) = r \iff X \in \Gamma(\text{interface}) \\
\phi(X) < r \iff X \in (D/\psi)(\text{void})
\]

The level set function can be redefined on the basis of grid points in the design domain. If \( s_{e} \) is the center of element \( e \), then the modified level set function can be defined as [23]

\[
\phi(s_{e}) \left\{ \begin{array}{l}
< 0 \text{if } x_{e} = 1 \\
> 0 \text{if } x_{e} = 0
\end{array} \right.
\]

Hamilton Jacobi equation can be used to update the level set function.
\[ \frac{\partial \phi}{\partial \tau} + |\nabla \phi| v = 0 \] (5)

The above equation is applicable to a one dimensional problem. In this case the level set function changes along the surface which do not allow new void creation. 

\[ \frac{\partial \phi}{\partial \tau} + |\nabla \phi| v_n + \eta z = 0 \] (6)

For two dimensional problems, two new parameters added. Where \( \eta \) is the positive parameter and \( z \) is used for creating new voids in the structure. \( v_n \) is the normal velocity of the interface and represents the geometrical motion of the boundary of the structure. It is the derivative of the shape sensitivity of the Lagrangian. The modified Hamilton Jacobi equation can be used to update the modified level set function. Courant-Friedrichs-Lewy (CFL) condition is used to check the convergence.

\[ C = u * \Delta t / \Delta x \leq C_{\text{max}} \] (7)

Where \( \Delta x \) is the minimum grid points distance and \( \Delta t \) is the time step. Usually the value of \( C_{\text{max}} \) is taken as unity.

**Evaluation of velocity \( v \) and parameter \( z \)**

The velocity field and the term \( z \) can be calculated by using the shape and topology sensitivities of Lagrangian function. The Lagrangian can be a function of both compliance as well as volume with some constants in a quadratic form.

\[ L = c(x) + \alpha'(V^2) + \frac{1}{2} \beta' \sigma(V^2)^2 \] (8)

Where \( V^2 = V(x) - V(\text{req}) \).

The two parameters \( \alpha \) and \( \beta \) changes with each iteration.

In the next step, there is a need to find out the shape sensitivities of compliance and volume. Both the term will be used to find out the velocity of the front. The shape sensitivity \( \delta c/\delta \psi \) of the compliance objective is negative of the strain energy density function whereas the shape sensitivity \( \delta V/\delta \psi \) of volume is one.

Thus the normal velocity is given by

\[ v|_n = -\delta L/\delta \psi = \alpha c/\delta \psi + \alpha' \delta (V^2)/\delta \psi + 1/2 \beta' \delta (V^2)^2/\delta \psi \] (9)

This equation can be written in the form of

\[ v|_n = -u^T_k \mu_k - \alpha' - 1/2 \beta' * [V^2] \] (10)

The term \( z \) which is responsible for new voids can be taken as

\[ z = -\text{sgn}(\phi) \delta \phi L \]

\[ z = \begin{cases} \delta \phi L/|\phi| & \text{if } \phi < 0 \\ 0 & \text{if } \phi \geq 0 \end{cases} \] (11)

The next step is to find out the topological sensitivities of the Lagrangian, compliance and the volume \( V(x) \). Found that the topological sensitivity of volume is \(-\pi\) whereas the help of [24]

\[ \delta \phi L(x) = \pi (\lambda + 2\mu)/(|\mu(\lambda + \mu)| (4\mu ku + (\lambda - \mu)u^T ku) \] (12)

\( \delta \phi L \) is the topological sensitivity of Lagrangian \( L \) and \( \mu \) are the Lame's constant. By using these results we can get the value of the source term \( z \).

**RESULT AND DISCUSSIONS**

Application in structures: This section shows the implementation of the level set method for partially loaded simply supported beam and rolling support, cantilever beam.

**Case 1**

Topology optimization of a simply supported 2D beam, fixed at both ends by using minimum compliance. The initial configuration of the beam is rectangular without any hole. Fig. 1 shows a simple structure with supports and partially distributed load.

Fig: 1 Simple structure with boundary conditions

The target value of reduction is set as 70%. During the optimization process, the material gradually moves out from the beam till it reaches to its target value.

\[ v|_n = -u^T_k \mu_k - \alpha' - 1/2 \beta' * [V^2] \]

Fig: 2 Optimization results
Fig. 2 shows the optimization of a simple beam. Initially the beam was in a solid (black) form. As the optimization process moves on, gradually the material comes out from the beam until the volume constraint met. In the above process, subtraction and addition of material both can takes place. This can be seen from iteration 90 and 110. Fig. 3 shows a variation in compliance with volume. In Fig. 4 the variation in volume can be seen with iteration number. Fig. 5 shows as the number of iteration increases, there is a random variation in the compliance. Later it stables.

**Case 2**

Topology optimization of a simply supported 2D beam, having one end rolling and other fixed by using minimum compliance. The static load applied at each node is unit. The initial configuration of the beam is a rectangular one without any hole. Fig. 6 shows geometry of Michell beam.

The target value of reduction is set as 70%. During the optimization process, the material gradually moves out from the beam till it reaches to its target value.
Fig. 7 shows the process of optimization of Michell beam for partially distributed load. Fig. 8, Fig. 9, Fig.10 shows the variation in compliance with volume, variation in volume with iteration number and variation in compliance with iteration number.

Case 3

The third case is a study of cantilever beam. The applied load is a partially distributed load. The target volume reduction in this case is taken as 60% of the total material.

Figure 12 shows the optimization of a cantilever beam. It also starts with a solid beam. As the optimization process progresses, new hole creation starts. The material gradually goes off from the beam to get the targeted value of the material.
density field. The physical application of this method is limited to few simple cases. For better and faster results, this method can be coupled with some other optimization method.

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CONCLUSION

The objective of this paper is to provide a brief introduction about the level set method and its application for simple structures. The major benefit of LSM is that, there is no need of pre-existing holes in the structure. It can create new voids, can merge small voids and can split bigger voids. The final solution will be 0/1 configuration without any intermediate densities. Secondly it works with moving boundaries. The level set method transforms the objective and the constraint in the form of speed of propagation of the boundary. The movements of the interfaces are governed by the Hamilton-Jacoby equation. LSM works only with the moving boundary, there is no need to pay any kind of attention inside or outside of the boundary.

Though ample amount of work has been done on application of Level Set method for structural topology optimization in recent years but still a lot of research work has to be done in this field. Till date the level set method is used for simple cases in structure optimization, which are based on
TABLE 1: Variation in compliance and volume with iteration number

| Sr. No. | Iteration Number | (case 1) Compliance (m/N) | V/V | (case 2) Compliance (m/N) | V/V | (case 3) Compliance (m/N) | V/V |
|---------|------------------|--------------------------|-----|--------------------------|-----|--------------------------|-----|
| 1       | 1                | 3325.8855                | 1   | 3009.1718                | 1   | 2430.4078                | 1   |
| 2       | 27               | 3325.8855                | 1   | 3009.1718                | 1   | 2430.4078                | 1   |
| 3       | 28               | 3325.8855                | 1   | 3009.1718                | 1   | 2430.4178                | 0.994|
| 4       | 45               | 3325.8855                | 1   | 3009.1718                | 1   | 2456.8571                | 0.924|
| 5       | 46               | 3325.8855                | 1   | 3009.1718                | 1   | 2462.9463                | 0.918|
| 6       | 47               | 3325.8855                | 1   | 3009.9064                | 0.987| 2473.2483                | 0.909|
| 7       | 48               | 3325.8855                | 1   | 3009.9064                | 0.987| 2483.0267                | 0.9   |
| 8       | 49               | 3326.0004                | 0.987| 3012.928                 | 0.97 | 2513.5596                | 0.892|
| 9       | 50               | 3326.0004                | 0.987| 3012.928                 | 0.97 | 2531.9765                | 0.869|
| 10      | 51               | 3329.2236                | 0.963| 3020.3874                | 0.95 | 2553.1156                | 0.857|
| 11      | 52               | 3329.2236                | 0.963| 3020.3962                | 0.946| 2553.1156                | 0.857|
| 12      | 53               | 3339.1675                | 0.934| 3036.5112                | 0.927| 2588.6334                | 0.841|
| 13      | 54               | 3339.1675                | 0.934| 3042.1663                | 0.912| 2626.5872                | 0.826|
| 14      | 62               | 3578.6611                | 0.764| 3368.1505                | 0.741| 3501.7507                | 0.632|
| 15      | 71               | 4514.0167                | 0.59 | 4743.0792                | 0.524| 5163.7482                | 0.426|
| 16      | 72               | 4788.7108                | 0.568| 5019.1833                | 0.501| 5297.072                 | 0.415|
| 17      | 73               | 161193.7156              | 0.539| 5264.63                  | 0.487| 5386.5637                | 0.406|
| 18      | 74               | 4958.8547                | 0.547| 5585.8191                | 0.47 | 5430.9251                | 0.4   |
| 19      | 75               | 327469.571               | 0.511| 138247.0879              | 0.451| 5538.9944                | 0.393|
| 20      | 85               | 6958.4508                | 0.414| 10319.8081               | 0.331| 5748.725                 | 0.373|
| 21      | 86               | 876048.1131              | 0.339| 11003.985                | 0.299| 5748.725                 | 0.373|
| 22      | 87               | 136011.861               | 0.362| 14141.8649               | 0.237| 5738.56                  | 0.373|
| 23      | 88               | 7904.7421                | 0.368| 13403.3681               | 0.212| 5726.5346                | 0.374|
| 24      | 89               | 1247969.147              | 0.282| 18945.2337               | 0.196| 5726.5346                | 0.374|
| 25      | 95               | 10024.7732               | 0.265| 9247.7606                | 0.238| 5111.9533                | 0.412|
| 26      | 96               | 79392.0043               | 0.253| 9097.9355                | 0.24 | 5111.9533                | 0.412|
| 27      | 97               | 9744.5134                | 0.258| 7731.5676                | 0.279| 5111.9533                | 0.412|
| 28      | 98               | 68535.2299               | 0.254| 7718.2944                | 0.279| 5111.9533                | 0.412|
| 29      | 99               | 9661.0897                | 0.257| 6858.764                 | 0.312| 5111.9533                | 0.412|
| 30      | 100              | 9612.3744                | 0.258| 6858.764                 | 0.312| 5126.6501                | 0.411|
| 31      | 101              | 9306.8337                | 0.262| 6774.7996                | 0.315| 5145.1525                | 0.409|
| 32      | 110              | 7906.3072                | 0.302| 6909.6188                | 0.301| 5300.0327                | 0.397|
| 33      | 111              | 7906.3072                | 0.302| 6935.178                 | 0.299| 5300.0327                | 0.397|
| 34      | 112              | 7906.3072                | 0.302| 6935.178                 | 0.299| 5300.0327                | 0.397|
| 35      | 113              | 7906.3072                | 0.302| 6935.178                 | 0.299| 5300.0327                | 0.397|
| 36      | 114              | 7906.3072                | 0.302| 6935.178                 | 0.299| 5300.0327                | 0.397|