Switching time of spin-torque-driven magnetization in biaxial ferromagnets

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(Dated: 13 January 2020)

We analytically model the magnetization switching time of a biaxial ferromagnet driven by an antidamping-like spin torque. The macrospin magnetization dynamics is mapped to an energy-flow equation, wherein a rational-function approximation of the elliptic integrals for moderate spin current and small damping results in a closed-form expression of the switching time. Randomness in the initial angle of magnetization gives the distribution function of the switching time. The analytic model conforms to the results obtained from Monte Carlo simulation for a broad range of material parameters. Our results can ameliorate design and benchmarking of in-plane spin torque magnetic memory by obviating expensive numerical computation.

Keywords: Spin torque, switching time, biaxial anisotropy, constant-energy approximation

I. INTRODUCTION

Current-induced spin phenomena, such as spin-transfer torque (STT) and spin-orbit torque (SOT), allow for electrical manipulation of the magnetic order, and form the basis of emerging spintronic technologies such as non-volatile memory,1–3 magnetic interconnects,2 and radio-frequency oscillators.3 Spin current can transfer angular momentum to a magnetic layer and reorient its magnetization, similar to how electric current can transfer charge to a capacitor and modulate its voltage. As shown in Fig. 1(a), an electric current flowing orthogonal to the plane of the spin valve becomes spin polarized in a direction parallel to the magnetization of the fixed layer. This spin-polarized current affects the magnetization of the free layer due to STT.4,5 On the other hand, an in-plane electric current flowing through a nonmagnetic material with spin-orbit coupling is spin polarized in the plane of the nonmagnetic material, but transverse to the electric current due to the spin-Hall effect.6 This in-plane polarized spin current can exert SOT7 on the free-layer magnetization of the spin valve as shown in Fig. 1(b).

Thin-film magnets—the path toward miniaturized spintronics—are subject to epitaxial strain from substrate and finite-size effects, which can elicit an in-plane or a perpendicular spin orientation.8 The symmetries in the energy landscape of thin films up to quadratic (lowest-order) terms of magnetization components are characterized by a biaxial anisotropy, consisting of an axis of minimum energy that is ‘easy’ for spins to orient along and an orthogonal axis of maximum energy which is ‘hard’. The uniaxial anisotropy is a special case of the biaxial anisotropy where the hard axis is absent. For perpendicularly magnetized films, a perpendicular easy axis approximates the anisotropy by assuming symmetry in the plane. For in-plane magnetized films, an in-plane easy axis and a perpendicular hard axis offer the correct description.

In STT memory, the perpendicular free-layer configuration is superior to the in-plane configuration due to its lower switching current, faster speeds, and higher density.1 A key problem in writing STT memory is its vulnerability to dielectric breakdown of the tunnel barrier. This is addressed in SOT memory, where the writing occurs with an in-plane current that need not traverse the tunnel barrier. The three-terminal SOT memory separates the read and write paths, improving memory endurance at the cost of cell size. However, deterministic switching of the perpendicular free layer in SOT memory requires either a biasing magnetic field10 or additional layers in the device stack adding to its fabrication complexity.10 The in-plane free-layer configuration of the SOT memory is preferred due to its fabrication simplicity, magnetic-field-free switching, and lower switching currents,11 although its writing speed is inferior to that of the perpendicular SOT memory.

When the spin polarization of the injected spin current is antiparallel to the stable orientation of the free-layer magnetization, the spin torque is antidamping-like—it competes with the intrinsic damping to raise the macrospin energy—until halfway in the magnetization reversal process when it becomes damping-like—it contributes to damping to cause dissipation of macrospin energy. The switching process is characterized by the time to reverse the orientation of the free layer as a function of the input spin current. A closed-form expression of the switching time is useful to design and optimize the performance of spin-torque memory. Previous works7,12–14 have derived expressions of the switching time for the uniaxial anisotropy, but not for the more general biaxial anisotropy presented in this work.

There are three equivalent approaches to analyze the magnetization dynamics: perturbative approach,12 constructing a Fokker-Planck representation,15,16 or using a

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constant-energy stochastic equation. In this work, we adopt the constant-energy orbit averaging (CEOA) to study magnetization reversal in a biaxial magnetic system because this approach simplifies a coupled three-dimensional (3D) (two-dimensional if the magnetization has a constant magnitude) stochastic problem into a tractable one-dimensional (1D) problem at low temperature and for low-to-moderate applied spin current.

The switching dynamics is modeled as a slow perturbation of the rapid constant-energy gyration around the easy axis (Sec. II). In the deterministic limit, a closed-form expression of the switching time as a function of input spin current, initial magnetization energy, and material parameters is obtained. Average switching time and the probability distribution of the switching time follow for an initial Boltzmann distributed ensemble of spins (Secs. III and IV).

![Diagram](image)

**FIG. 1.** Injection of spin current $I_s$ (dashed red line) into a free magnetic layer produces (a) STT in a spin valve and (b) SOT in a spin-Hall structure. $\mathbf{m}$ represents the direction of free-layer magnetization, $\hat{n}_x$, $\hat{n}_y$, and $\hat{n}_z$ are the unit vectors along the easy, intermediate, and hard anisotropy axes, respectively. The flow of electron in the nonmagnetic (NM) layer is opposite to that of the injected charge current $I$.

### II. THEORY

The dynamics of magnetization subject to an effective magnetic field, intrinsic damping, and spin torque is described by the Landau-Lifshitz-Gilbert (LLG) equation. Using dimensionless form of physical parameters listed in Table I, the LLG equation is

$$\frac{\partial \mathbf{m}}{\partial \tau} = - (\mathbf{m} \times \mathbf{h}_{\text{eff}}) - \alpha \mathbf{m} \times (\mathbf{m} \times \mathbf{h}_{\text{eff}}) - I_s \mathbf{m} \times (\mathbf{m} \times \hat{n}_p) + \alpha I_s (\mathbf{m} \times \hat{n}_p), \quad (1)$$

where $\mathbf{m}$ is the normalized free-layer magnetization, $\alpha$ is the Gilbert damping constant, $I_s$ is the input spin current, and $\hat{n}_p$ is the unit vector along spin polarization. The effective magnetic field, $\mathbf{h}_{\text{eff}}$, includes contributions from an internal field produced by the magnetic anisotropy, externally applied magnetic fields ($\mathbf{H}_a$), and a thermal field. The sum of the internal and external magnetic fields, normalized to the Stoner-Wohlfarth field, $H_k$, is given as

$$\mathbf{h} = -\frac{1}{2} \nabla_m [g_L(m, H_a)] \quad (2)$$

where $g_L$ is the free energy of the macrospin normalized to its uniaxial energy $K_u V$, where $K_u = \mu_0 M_s H_k/2$ ($\mu_0$ is the free-space permeability and $M_s$ is the saturation magnetization of the free layer). Neglecting higher order anisotropy terms,

$$g_L(m, H_a) = D_e (m \cdot \hat{n}_e)^2 + D_h (m \cdot \hat{n}_h)^2 - \frac{2}{H_k} m \cdot H_a, \quad (3)$$

where $D_e$ and $D_h$ are the effective anisotropy coefficients along the easy ($\hat{n}_e$) and hard axes ($\hat{n}_h$), respectively. For thin-film magnets, the energy landscape is characterized by a biaxial anisotropy with $D_e = -1$ and $D_h = M_s/H_k$.

Without loss of generality, $\hat{n}_e$ and $\hat{n}_h$ are assumed to coincide with $\hat{x}$ and $\hat{z}$ axes, respectively. Using Eq. (3) in Eq. (2), we obtain

$$\mathbf{h} = m, \hat{x} = R m, \hat{z} + H_a/H_k$$

where $R = D_h = M_s/H_k$.

The thermal field is a Langevin field that is spatially isotropic and uncorrelated in space and time,

$$\langle h_T(t) \rangle = 0, \quad \langle h_{T,r}(t) h_{T,r}(t_2) \rangle = D_{p,r} \delta(t_1 - t_2), \quad (4)$$

where $p$, $r$ represent Cartesian coordinates, $\delta_{p,r}$ is the Kronecker delta, and $\delta(t)$ is the Dirac delta. Assuming the macrospin to be in thermal equilibrium with the thermal bath and neglecting Joule heating, the diffusion coefficient

$$D = \frac{\alpha k_B T}{(1 + \alpha^2) K_u V} = \frac{\alpha}{(1 + \alpha^2) \Delta_0}, \quad (5)$$

where $k_B$ is the Boltzmann constant, $T$ is the temperature of the bath, and $\Delta_0$ measures the thermal stability of the macrospin and is referred to as the barrier height of a uniaxial anisotropy magnet.

### A. Constant-Energy Orbit Averaging (CEOA)

The energy of a macrospin is conserved when damping, thermal field (noise), external magnetic field, and spin
torque are absent. In this case, \( \mathbf{m} \) precesses around the easy axis on the unit magnetization sphere with a fixed macrospin energy \( g_L < 0 \). Trajectories of conserved motion, illustrated in Fig. 2(a), are obtained by solving Eqs. (1) and (3) with \( \alpha, \mathbf{H}_s \), and \( I_s \) set equal to zero, and \( |\mathbf{m}| = 1 \). However, with finite damping, \( \mathbf{m} \) loses energy, eventually relaxing to a stable equilibrium state \( (m_x = \pm 1) \). A non-zero spin torque can pump energy to the macrospin and act against its inherent damping, causing the magnetization to deviate from its equilibrium position.

In the case of a small to moderate input spin torque, small damping, and low temperature, two distinct time-scales of magnetization dynamics emerge: (i) a fast time-scale associated with constant-energy gyration around the easy axis and (ii) a slow time-scale corresponding to perpendicular diffusion of magnetization from one constant-energy orbit to another as a result of damping, spin torque, and thermal field. Figure 2(b) shows one such trajectory where the magnetization switches from an anti-parallel well to a parallel well under the simultaneous effects of damping, input spin current, and thermal field.

![FIG. 2. (a) Constant-energy curves for \( g_L < 0 \) in both anti-parallel and parallel wells for \( R = 15 \). (b) Switching trajectory of a magnetization initially in the anti-parallel well for \( R = 15 \) and a large input spin current. External field is assumed to be absent. The arrows shown in the figure mark the initial (magenta), intermediate (yellow) and final (green) positions of the magnetization, in the counter-clockwise sense. The dashed red curves in both figures correspond to a zero energy separatrix.](image)

The rate of change of macrospin energy due to the non-conservative torques is given as

\[
\frac{\partial g_L}{\partial \tau} = \nabla m g_L \cdot \frac{\partial \mathbf{m}}{\partial \tau} = 2 \left( R m_x \frac{\partial m_z}{\partial \tau} - m_z \frac{\partial m_x}{\partial \tau} \right). \tag{6}
\]

Averaging the above equation over one time period of the undamped motion reduces the coupled stochastic dynamics of Eq. (1) to a 1D stochastic dynamics as \(^{18,20}\) (see Appendix A)

\[
\left\langle \frac{\partial g_L}{\partial \tau} \right\rangle = \frac{\pi \alpha}{K(R,g_L)} \sqrt{\frac{R - g_L}{1 + R}} \left[ \frac{I_s}{\alpha} (1 + g_L) - \frac{2}{\pi} \sqrt{(1 + R)(R - g_L)} \left\{ E(R,g_L) + g_L K(R,g_L) \right\} \right] + 2 \sqrt{\frac{\alpha}{2 \Delta_0}} \sqrt{\frac{R - g_L}{K(R,g_L)}} E(R,g_L) + g_L K(R,g_L) \circ \hat{W}_{gL}. \tag{7}
\]

Here, \( \circ \) denotes multiplication of thermal noise in the Stratonovich sense\(^{18,22} \), while \( K(R,g_L) \) and \( E(R,g_L) \) are the complete elliptic integrals of the first and second kind, respectively. Time averaging \( \frac{\partial g_L}{\partial \tau} \) over a period of precessional motion enables us to study the dynamics due to slow diffusion of energy with respect to fast periodic oscillations\(^{18,20} \).

Assuming deterministic dynamics and using Eq. (7) to evaluate for zero energy flow at \( g_L = 0 \) and \( g_L = -1 \) leads to two different threshold currents\(^{12,16,18} \): the minimum current required to push the magnetization over the energy barrier into the adjoining basin, \( I_{s0}^{th} = \alpha \frac{2}{\pi} \sqrt{R(1 + R)} \) and the minimum current required to move the magnetization away from stable equilibrium, \( I_{s1}^{th} = \alpha [R/2 + 1] \). \( R_c \) denotes the critical value of \( R \) for which \( I_{s0}^{th} \) equals \( I_{s1}^{th} \). The threshold currents demarcate regions of deterministic switching from those that require thermal assistance as shown in Fig. 3(a). Figures 3(b) and 3(c) show the rate of change of energy for deterministic dynamics. It is positive for the complete range of macrospin energy only for \( I_s > \max (I^{th0}, I^{th1}) \) which is consistent with Fig. 3(a). For other values of current, thermal assistance is required for switching. The CEOA is valid when the variation in macrospin energy over one precessional cycle is small, i.e. \( |\frac{\partial g_L}{\partial \tau}| \ll \max |g_L| = 1 \).

To satisfy this constraint on the variation of macrospin energy, the maximum spin current is given as \( I_{sM}^{th} = I_{s0}^{th} 1 + 1 / \left( 8 \Delta_0 \sqrt{R} \right) \).

III. ANALYTIC SWITCHING TIME MODEL

The switching time due to spin torque is defined as the time required for the macrospin energy to change from an initial value \( g_{L1} \) to its final value \( g_{L2} \). Analytic solutions of the switching time are obtained by approximating the elliptic integrals in Eq. (7) as follows

\[
\frac{2}{\pi} |E(R,g_L) + g_L K(R,g_L)| = A(R)g_L^2 + B(R)g_L + C(R), \tag{8a}
\]

\[
K(x) = \frac{\pi}{2} \left[ \frac{x - 4}{2x - 4} \right], \tag{8b}
\]

where \( x = \frac{R^{1+g_L}}{R-g_L} \). Defining \( \tilde{I}_s = I_s/\alpha \) and using the approximate representation of the rate of change of magnetization energy, we obtain the magnetization switching...
integrated using partial fractions to arrive at the closed-form expression of 

$$
\tau_s = \frac{1}{4\alpha} \int_{g_L}^{g_L} \frac{3R-g_L(R+4)}{R-g_L(R+2)} \sqrt{\frac{1+R}{R-g_L}} \left[ \frac{1+R}{R-g_L} - \frac{\partial g_L}{\partial g_L} \right] \left[ g_L(R+2) - R - g_L \right] \sqrt{1+R} \sqrt{R+4} \left[ 1+R \right] \left[ \frac{\partial g_L}{\partial g_L} \right] \left[ g_L(R+2) - R - g_L \right].
$$

In the above equation, the parameters $A$, $B$, and $C$ are functions of $R$ given as $k_1 + k_2 R^k_3$, where the values of $k_1$, $k_2$ and $k_3$ are chosen for different intervals of $R$ to reduce the error in approximating the elliptic integrals. See Table II for details. Note that the values of $A$, $B$, and $C$ are independent of the device geometry and depend only on $R = M_s / H_k$. Equation (9) is simplified and integrated using partial fractions to arrive at the closed-form expression of switching time:

$$
\tau_s = \frac{1}{2\alpha (R+2)} \left[ \sum_{i=1}^{5} \frac{N}{D} \log \left( \frac{R-g_L}{R-g_L} - w_i \right) \right]
$$

$$
+ \sqrt{\frac{R+1}{R+2}} \left[ \log \left( \frac{\sqrt{R-g_L}(R+2)}{\sqrt{R-g_L}(R+2)} \right) + \frac{\sqrt{R-g_L}(R+2)}{\sqrt{R-g_L}(R+2)} \right] \left[ \frac{A \prod_{n=1}^{5} \left( \frac{R+1}{R+2} + w_i \right)}{A \prod_{n=1}^{5} \left( \frac{R+1}{R+2} - w_i \right)} \right],
$$

where $w_i$'s are the roots of a fifth-degree polynomial $x^5 - \left( \frac{B}{A} + 2R \right) x^3 + \frac{I_s}{A \sqrt{1+R}} x^2 + (R^2 + \frac{B}{A} R + C)x - \frac{I_s \sqrt{1+R}}{A}$.

### Table II

| $R$    | $k_1$      | $k_2$      | $k_3$      |
|--------|------------|------------|------------|
| 1, 3   | 0.35661    | -0.51244   | -0.38689   |
| 3, 50  | 1.05148    | -0.55504   | -0.28598   |
| 50, 100| 0.61670    | 0.03018    | -1.00153   |
| 50, 100| 0.20223    | -0.38439   | -0.68424   |
| 50, 100| 0.81746    | -0.34729   | -0.63039   |
| 50, 100| 0.61755    | 0.02625    | -1.01442   |

with $N = (R + 4) + \frac{2R(1+R)}{(R+2)w_i^2 - R(1+R)}$, and $D = 5A w_i^4 - 3(B + 2AR) w_i^2 + 2 \frac{I_s}{A \sqrt{1+R}} w_i + (AR^2 + BR + C)$. A major advantage of this analytic result is that the approximations of elliptic integrals are independent of the input spin current and Gilbert damping. Additionally, the results obtained in this work are valid for a broad range of $R$ as opposed to prior works$^{12,23}$ that are valid only for $R < R_c (= 5.09)$. Considering $R \to \infty$ in Eq. (7), the elliptic integrals are approximated as

$$
K(x) = \frac{\pi}{2} \left[ \frac{x - 4}{2x - 4} \right]; \quad E(x) = \frac{\pi}{2} \left[ 1 - \frac{x - 4}{4x^2 - 40x + 64} \right],
$$

where $x = R \frac{I_s}{A - g_L}$. These approximations of elliptic integrals simplify the energy flow equation and the switching
time is (see Appendix C for details)

\[
\tau_s = \frac{1}{2\alpha \left( \bar{I}_s - 0.5R \right)} \left[ \log \left[ \frac{1 + gL_f}{1 + gL_i} \right] + \frac{b(a - 4)(a - 8)}{32(a - b)} \log \left[ \frac{1 + gL_i - a}{1 + gL_i - a} \right] - \frac{a(b - 4)(b - 8)}{32(a - b)} \log \left[ \frac{1 + gL_f - b}{1 + gL_i - b} \right] \right],
\]

where \( a \) and \( b \) are the roots of the quadratic equation

\[
x^2 - \left( \frac{160L_s - 60R}{16L_s - 7R} \right)x + \left( \frac{256L_s - 128R}{16L_s - 7R} \right) = 0.
\]

In the limit \( R \to 0 \), the energy landscape becomes uniaxial and the switching time is \(12\) (see Appendix D for details)

\[
\tau_s = \frac{1}{2\alpha \left( \bar{I}_s - 1 \right)} \left[ \bar{I}_s \left\{ \log \left[ \frac{1 + \sqrt{-gL_i}}{1 + \sqrt{-gL_f}} \right] - \log \left[ \frac{1 + \sqrt{-gL_f}}{1 + \sqrt{-gL_i}} \right] \right\} - \log \left[ \frac{1 + gL_i}{1 + gL_f} \right] \right] + 2 \log \left[ \frac{\bar{I}_s - \sqrt{-gL_i}}{\bar{I}_s - \sqrt{-gL_f}} \right].
\]

A. Equilibrium Distribution

In the absence of input spin current, the magnetization is considered to be in thermal equilibrium in its stable energy well. An average switching time, \( \langle \tau_s \rangle \), is obtained by averaging \( \tau_s \) over the equilibrium energy distribution, which in the case of a large energy barrier is the Boltzmann distribution given as

\[
w_{eq}(m) = \frac{1}{Z(\Delta_0, R)} \exp \left( -\Delta_0 gL(m, 0) \right),
\]

where \( Z(\Delta_0, R) \) is the partition function. The Boltzmann distribution function is evaluated in terms of the random variable \( G_L = -M^2 \) (see Appendix E for details of the transformation). Accordingly the probability density function

\[
\rho_{GL}(g_L) = \frac{1}{Z(\Delta_0, R)} \exp \left( -\Delta_0 gL \right),
\]

the partition function

\[
Z(\Delta_0, R) = \int_{-1}^{0} \rho_{GL}(g_L) dx = 2 \exp (\Delta_0 R) F[\sqrt{\Delta_0}],
\]

and the cumulative distribution function

\[
P_{GL}(g_L) = \int_{-1}^{g_L} \rho_{GL}(g_L) dx
\]

\[= 1 - \exp (-\Delta_0 (1 + g_L)) \frac{F[\sqrt{-\Delta_0 gL}]}{F[\sqrt{\Delta_0}]},\]

where \( F[x] = \exp (-x^2) \int_{0}^{\infty} \exp (y^2) dy \) is the Dawson’s integral.

B. Average Switching Time and Model Validation

The magnetization is considered to have switched successfully when it crosses the separatrix \( (g_{L_f} = 0) \) and consequently moves into the adjoining energy well. Once the magnetization moves into the target energy well, the spin current could be switched off. The magnetization would eventually settle into its stable well due to its intrinsic damping. Therefore, \( \langle \tau_s \rangle \) is given as

\[
\langle \tau_s \rangle = \int_{-1}^{0} \rho_{GL}(g_{L_f}) \tau_s (g_{L_i}, g_{L_f} = 0) \, dg_{L_i}.
\]

To benchmark our analytic results, we solve Eq. (1) numerically using the Heun integration scheme implemented in CUDA and run in parallel on GPUs. Numerical simulations were calibrated against published results to ensure their accuracy.\(13,18,26\) For all simulations, the time step of integration was set as 0.3 ps. Other material parameters are \( \alpha = 0.03, K_u = 0.6 \times 10^6 \text{ J/m}^3 \), and \( V = \pi \times 15 \times 7.5 \times 2 \times 10^{-27} \text{ m}^{-3} \), while the value of \( M_s \) was varied. Simulations were also conducted for other parameter values but are not reported here. Generally, the key features and trends of switching dynamics remain the same as material parameters are varied. For each set of parameters, simulations were performed on an equilibrium ensemble of \( 10^4 \) independent macrospins. Figure 4 shows that Eqs. (14) and (16) describe the numerical distribution very well.

Figure 5 shows that Eqs. (14) and (16) describe the numerical data and closed-form solutions given in Eqs. (10) and (12) for moderate to large current levels. However, for current levels approaching the threshold switching currents, numerical results predict a lower average
FIG. 5. Average switching time \( \langle t_s \rangle \) as a function of injected spin current density, \( J_s \), for different values of \( R \). The lower horizontal dashed black line in each figure is the threshold current demarcating region of deterministic switching from that of thermal activation. The upper horizontal dashed cyan line is the maximum current for which CEOA is valid. The numerical data is obtained for an ensemble of \( 10^4 \) macrospins.

switching time as the presence of thermal noise aids the switching process.

For \( R > 0 \) (biaxial anisotropy), \( \langle t_s \rangle \) obtained using Eq. (11) predicts a larger average switching time compared to the numerical results for current levels comparable to the threshold value. This is expected since the analytic solutions neglect the effect of thermal noise during the switching process. As the input spin current density \( J_s \) increases beyond the threshold value, \( J_{th0} = (4eK_uV/\hbar)(I_{th0}/Ar) \), the agreement between analytic and numerical results improves. Near the threshold current level, the average switching time obtained from Eq. (10) is slightly lower than that obtained from numerical results. This slight deviation is due to the quadratic approximation of elliptic integrals in Eq. (7). Note that for \( J_s > J_{thM} \), the validity of CEOA is not fully justified, which could be the source of mismatch between analytic and numerical models. Here, \( J_{thM} = (4eK_uV/\hbar)(I_{thM}/Ar) \).

IV. SWITCHING TIME DISTRIBUTION AND ERROR RATE

A. Probability and Cumulative Distribution Functions

Defining a random variable switching time \( T_s \), the fraction of macrospins in an ensemble that have switched from an anti-parallel to a parallel state at time \( \tau_s \) is

\[
\rho_{T_s}(\tau_s) = \frac{dP_{T_s}[\tau_s]}{d\tau_s} = \frac{\sqrt{\Delta_0}}{2F[\sqrt{\Delta_0}]} \exp\left(-\Delta_0(1+g_L(\tau_s))\right) \left| \frac{dg_L(\tau_s)}{d\tau_s} \right|.
\]

where \( g_L(\tau_s) \) corresponds to the initial energy for switching time \( \tau_s \).

B. Write-Error Rate

The write-error rate (WER) quantifies the probability of unsuccessful spin torque switching of the magnet.
Using Eq. (18), the WER is
\[ WER = 1 - P_{T,\tau} \]
\[ = 1 - \exp \left( -\Delta_0 \left( 1 + g_{L,i} \tau \right) \right) \]
\[ \frac{F \left( \sqrt{-\Delta_0 g_{L,i} \tau} \right)}{F \left( \sqrt{\Delta_0} \right)}. \]

(20)

The probability distribution function (pdf) and WER obtained using analytic results of Eqs. (19) and (20) are compared against numerical solution in Figs. 6-9 for an ensemble of $10^6$ macrospins. Simulation parameters are noted in Table III. It is observed that for $R = 0$, the accuracy of the pdf and WER improves as the applied spin current increases. For $R \geq 15$, the accuracy of analytic solutions also increases as current increases from $J_{s,0}^{th}$ toward $J_{s,\beta}^{HM}$. However, for spin currents larger than $J_{s,\beta}^{HM}$, the accuracy of analytic results drops again as the validity of CEOA becomes questionable. To arrive at analytic results reported in Figs. 7-9, we numerically invert Eq. (11) due to its simplicity.

V. CONCLUSION

Analytic models of average switching time, probability distribution function of switching times, and write-error rate developed in this paper for thin-film magnets with biaxial anisotropy show good agreement against numerical results for moderate to large spin current densities. In the vicinity of the threshold spin current density, the error between analytic and numerical data is significant due to thermal noise. For very large spin current densities, the constant energy orbit averaging approach adopted in this work becomes inadequate, even though the error between numerical and analytic results is well under a tolerance limit. The models of this paper should complement experimental results and aid the analysis, design and development of non-volatile memory driven by both spin-transfer and spin-orbit torques.

Appendix A: Simplifying Eq. (6)

Without any loss of generality, we consider the easy and hard axes to coincide with the $\hat{x}$ and $\hat{z}$ axes, respectively. Next, we consider spin polarization of the fixed layer to be in the plane of the magnet at an angle $\phi$ with the easy axis, therefore, $\mathbf{h}_{n} = \cos \phi \hat{x} + \sin \phi \hat{y}$. Therefore, for zero external magnetic field ($H_e = 0$), the effective magnetic field $\mathbf{h}_{eff} = m_s \hat{x} - Rm_s \hat{z} + \mathbf{h}_T$, where $\mathbf{h}_T$ is the thermal field. As a result the three components of Eq. (1) become
\[ \frac{\partial m_y}{\partial t} = Rm_y m_z + \alpha m_x \left( 1 - m_x^2 + Rm_z^2 \right) \]
\[ + I_s (\cos \phi - m_x (\cos \phi m_x + \sin \phi m_y)) \]
\[ - \alpha I_s m_z \sin \phi \]
\[ + n_{S,x}, \]
\[ \frac{\partial m_z}{\partial t} = - (R + 1) m_x m_z - \alpha m_y \left( m_x^2 - Rm_z^2 \right) \]
\[ - I_s (\sin \phi - m_y (\cos \phi m_x + \sin \phi m_y)) \]
\[ + \alpha I_s m_z \cos \phi \]
\[ + n_{S,y}, \]
\[ \frac{\partial m_x}{\partial t} = m_x m_y - \alpha m_z \left( R + m_x^2 - Rm_z^2 \right) \]
\[ - I_s (\sin \phi m_x - \cos \phi m_y) \]
\[ + \alpha I_s (\sin \phi m_x - \cos \phi m_y) \]
\[ + n_{S,z}, \]
where each of $n_{S,p}$ denote thermal noise component in the Stratonovich sense. Assuming the thermal field $\mathbf{h}_T$ as
\[ \mathbf{h}_T(t) = \sqrt{\frac{\alpha}{(1 + \alpha^2) \Delta_0}} \left( \frac{\partial \mathbf{W}}{\partial t} \right) = \sqrt{D} \mathbf{W}, \]
we get
\[ \begin{pmatrix} n_{S,x} \\ n_{S,y} \\ n_{S,z} \end{pmatrix} = \mathcal{D} \circ \begin{pmatrix} h_{T,x} \\ h_{T,y} \\ h_{T,z} \end{pmatrix} = \sqrt{D} \mathcal{D} \circ \begin{pmatrix} \dot{W}_x \\ \dot{W}_y \\ \dot{W}_z \end{pmatrix}, \]
(44)

where $\dot{\mathbf{W}}$ represents a 3D stochastic Wiener process whose each component is a Gaussian random variable with zero mean and unit standard deviation. $\mathcal{D}$ is referred to as the diffusion matrix and is given as
\[ \begin{pmatrix} \alpha (1 - m_x^2) m_z - \alpha m_x m_y - m_y - \alpha m_z m_x \\ -m_z - \alpha m_x m_y \alpha (1 - m_x^2) m_z - \alpha m_z m_y \\ m_y - \alpha m_x m_z - m_y - \alpha m_y m_z \alpha (1 - m_z^2) \end{pmatrix}. \]

Substituting Eqs. (A1) and (A3) into Eq. (6) leads to
\[ \begin{align*}
\frac{\partial g_{L}}{\partial t} &= 2Rm_z \left[ m_x m_y - \alpha m_x \right] \left( R - g_{L} \right) \\
&- I_s m_z \left( \cos \phi m_x + \sin \phi m_y \right) \\
&+ \alpha I_s \left( \sin \phi m_x - \cos \phi m_y \right) \bigg] \\
&- 2m_x \left[ Rm_y m_z + \alpha m_y \left( 1 + g_{L} \right) \right] \\
&+ I_s \left( \cos \phi - m_x \left( \cos \phi m_x + \sin \phi m_y \right) \right) \\
&- \alpha I_s \sin \phi m_y \bigg) \end{align*} \]
\[ + 2\sqrt{D} \begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix} \circ \begin{pmatrix} \dot{W}_x \\ \dot{W}_y \\ \dot{W}_z \end{pmatrix} \\
\text{where} \begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix} = \begin{pmatrix} Rm_z D_{11} - m_x D_{13} \\ Rm_z D_{12} - m_x D_{13} \\ Rm_z D_{33} - m_x D_{13} \end{pmatrix}. \]
FIG. 6. All results reported for $R = 0$ (uniaxial anisotropy) and $J_{\text{th}1} = 5.5 \times 10^6$ A/cm$^2$. The top panel shows the pdf, while the bottom panel shows the write-error rate (WER). The accuracy of analytical results improves as the input spin current density increases with respect to $J_{\text{th}1}$.

FIG. 7. All results are reported for $R = 15$ and $J_{\text{th}0} = 2.15 \times 10^8$ A/cm$^2$. The top panel shows the pdf, while the bottom panel shows the write-error rate (WER). The accuracy of analytical solutions improves for spin current densities larger than the threshold current density. However, CEOA is strictly valid for $J_s < J_{\text{th}M} = 2.1J_{\text{th}0}$.

It can be observed from Eq. (A5) that each of damping, spin current and thermal noise contribute to the rate of energy change. In addition, the last term in Eq. (A5) spin current and thermal noise contribute to the rate of energy change. In addition, the last term in Eq. (A5)
FIG. 8. All results are reported for $R = 50$ and $J_{th0} = 7.02 \times 10^8$ A/cm$^2$. The top panel shows the pdf, while the bottom panel shows the write-error rate (WER). The accuracy of analytic results improves for spin current density larger than the threshold current density but drops again for $J_s > J_{thM} = 1.6 J_{th0}$.

FIG. 9. All results are reported for $R = 100$ and $J_{th0} = 1.4 \times 10^9$ A/cm$^2$. The top panel shows the pdf, while the bottom panel shows the write-error rate (WER). The accuracy of analytical results improves for as $J_s$ increases above $J_{th0}$ but decreases when $J_s > J_{thM} = 1.42 J_{th0}$.

is simplified to $\sqrt{d_x^2 + d_y^2 + d_z^2} \sqrt{W_x^2 + W_y^2 + W_z^2}$ which leads to

$$\frac{\partial g_L}{\partial \tau} = -2\alpha [(1 + R) m_x^2 + g_L (R - g_L)$$
$$+ \frac{I_s}{\alpha} ((1 + g_L) \cos \phi m_x + g_L \sin \phi m_y)$$
\[ + I_s (\sin \phi m_x m_y - R (\sin \phi m_x m_z - \cos \phi m_y m_z)) \]
\[ + 2 \sqrt{\frac{\alpha}{\Delta_0}} (1 + R) m_z^2 + g_L (R - g_L) \circ \dot{W}_{gl}, \]
\[ (A6) \]

where \( \dot{W}_{gl} \) is 1D white Gaussian noise and it acts away from the constant-energy orbit along a normal.\(^{20}\)

We now average \( \frac{\partial u}{\partial t} \) over a constant-energy orbit since the non-conservative effects act over long time-scales and we are interested in studying their effects on short time-scale rapid periodic motion. In the limit of zero damping, zero spin current at absolute zero temperature, the constant energy solutions to Eqs. (A1)-(A3) in the anti-parallel well are given as\(^{17,18,20}\)

\[
\begin{align*}
    m_x^c(t) &= -\frac{R - g_L}{R + 1} \sqrt{R - g_L} t, \\
    m_y^c(t) &= \sqrt{1 + g_L} \sin \left[ \sqrt{R - g_L} t, k^2 \right], \\
    m_z^c(t) &= \sqrt{1 + g_L} \cosh \left[ \sqrt{R - g_L} t, k^2 \right],
\end{align*}
\]

where \( k^2 = \frac{R + g_L}{R - g_L}, 0 < k^2 < 1 \) and \( \sin[\cdot], \cos[\cdot], \cosh[\cdot] \) are Jacobian elliptic functions. Magnetization oscillates around the easy axis with a time period, \( T(g_L) = \frac{1}{\sqrt{R - g_L}} K(k^2) \), where \( K(k^2) \) is the complete elliptic integral of first kind. In Eq. (A6) only the time averages of \( m_x \) and \( m_z^2 \) are non-zero while those of \( m_y, m_x m_y, m_x m_z \), and \( m_y m_z \) are zero due to their periodic nature. To evaluate these averages we geometrically parametrize\(^{18,20}\) the constant-energy orbit in the anti-parallel well as

\[
\cosh^2(w) - \sinh^2(w) = 1,
\]

\[
\frac{1}{g_L} m_x^2 - \frac{R}{-g_L} m_z^2 = 1
\]

which leads to the form as given below

\[
\begin{align*}
    m_x^c &= -\frac{g_L}{R} \cosh(w), \\
    m_y^c &= \pm \frac{g_L}{R} \sinh(w), \\
    m_z^c &= \pm \frac{R - g_L}{R} \sqrt{1 - \zeta^2} \cosh^2(w), \\
    \zeta^2 &= -\frac{g_L (R + 1)}{R - g_L},
\end{align*}
\]

where \( \zeta^2 = 1 - k^2 \), and \(- \cosh^{-1}(1/\zeta) < w < \cosh^{-1}(1/\zeta)\). Defining time average of any function \( p(t) \) as

\[
\langle p(t) \rangle = \frac{4}{T(g_L)} \int_0^{t=m_x^c \sqrt{1+g_L} \rightarrow 0} p(t) \, dt,
\]

the time averages of \( \langle m_x \rangle \) and \( \langle m_z^2 \rangle \) is evaluated as

\[
\begin{align*}
    \langle m_x \rangle &= \frac{4}{T(g_L)} \int_0^{\cosh^{-1}(1/\zeta)} m_x(w) \frac{\sqrt{R m_z}}{R m_y m_z} \, dw, \\
    \langle m_z^2 \rangle &= \frac{4}{T(g_L)} \int_0^{\cosh^{-1}(1/\zeta)} m_z^2(w) \frac{m_x \sqrt{R m_z}}{m_z \sqrt{R m_y m_z}} \, dw
\end{align*}
\]

and

\[
\begin{align*}
    \langle m_x \rangle &= -\frac{4}{T(g_L)} \int_0^{\cosh^{-1}(1/\zeta)} \cosh(w) \, dw, \\
    \langle m_z^2 \rangle &= -\frac{4}{T(g_L)} \int_0^{\cosh^{-1}(1/\zeta)} \cosh^2(w) \, dw
\end{align*}
\]

where we have used \( dt = \frac{dp(w)}{dp(t)} \, dw \). Substituting \( \langle m_x \rangle \) and \( \langle m_z^2 \rangle \) in Eq. (A6) results in the average rate of energy flow as

\[
\tilde{\langle \frac{\partial g_L}{\partial T} \rangle} = -\frac{\pi \alpha}{K(R, g_L)} \sqrt{R - g_L} \left[ \frac{I_s}{\alpha} \cos \phi (1 + g_L) \right. \\
&+ 2 \frac{\pi}{2} \sqrt{1 + R - g_L} \left( E(R, g_L) + g_L K(R, g_L) \right) \\
&\left. + 2 \sqrt{\frac{\alpha}{\Delta_0}} \frac{R - g_L}{K(R, g_L)} \sqrt{E(R, g_L) + g_L K(R, g_L) \circ \dot{W}_{gl}} \right]
\]

(A11)

In this paper, we have considered the spin polarization \( \hat{n}_p \) to be collinear to the easy axis \( \hat{n}_e \) so \( \phi = 0 \) which leads to Eq. (7).
Appendix B: Solving the integral in Eq. (9)

In Eq. (9), substitute $R - g_L = u^2$, so that we have

$$
\tau_s = \frac{1}{2A \alpha (R + 2)} \int R - g_L \left( \frac{u^2 - R(1 + R) R + 2}{u^2 - R(1 + R) R + 2} \right) \frac{du}{P(u)}, \quad (B1)
$$

where $P(u) = u^2 - (B/A + 2R)u^3 + \frac{L_1}{A\sqrt{1 + R}}u^2 + (R^2 + (B/A)R + C/A)u - \tilde{I}_s \sqrt{1 + R/A}$ and $w_i$ are the roots of the polynomial $P(u)$ which are evaluated numerically. Using partial fractions to resolve the denominator of the integral we finally have

$$
\tau_s = \frac{1}{2A \alpha (R + 2)} \left[ \sum_{i=1}^{5} \frac{N_i}{D} \log \left( \frac{(R - g_{L_i}) - w_i}{(R - g_{L_i}) - w_i} \right) + \frac{\log \left[ \frac{\sqrt{(R - g_{L_f})(R + 2)} - \sqrt{(R + 1 + R)}}{\sqrt{(R - g_{L_f})(R + 2)} + \sqrt{(R + 1 + R)}} \right]}{A \prod_{n=1}^{5} \left( \frac{R(1 + R)}{R + 2} + w_i \right)} \right]. \quad (B2)
$$

Appendix C: $R \to \infty$

For large values of $R$, Eq. (7) in the deterministic domain along with rational approximations for the elliptic integrals can be simplified as

$$
\frac{\partial g_L}{\partial \tau} = 4\alpha \left[ \frac{1 - g_L}{3 - g_L} \right] \left[ \tilde{I}_s (1 + g_L) - R \left\{ \frac{g_L 3 - g_L}{2 - 2g_L} - \frac{4}{1 + g_L} \left( \frac{1 + g_L}{2} \right)^2 - 28(1 + g_L) + 64 \right\} \right]. \quad (C1)
$$

Now substituting $1 + g_L = x$ in the previous equation leads us to

$$
\frac{dx}{d\tau} = 4\alpha \left[ \frac{x - 2}{x - 4} \right] \left[ \tilde{I}_s x - R \left\{ (x - 1) \frac{x - 4}{2x - 4} + \frac{1}{4} \frac{x^2 - 28x + 64}{4x^2 - 40x + 64} \right\} \right], \quad (C2)
$$

which can then be simplified and rearranged to an integral of the form

$$
\int_{1 + g_L}^{1 + g_L} \frac{(x - 4)(x - 8)dx}{x(x^2 - Ex + F)} = \frac{\alpha}{4} \left( 16\tilde{I}_s - 7R \right) \tau_s, \quad (C3)
$$

where $E = \frac{160\tilde{I}_s - 60R}{16\tilde{I}_s - 7R}$ and $F = \frac{256\tilde{I}_s - 128R}{16\tilde{I}_s - 7R}$. If the roots of the quadratic equation is $x^2 - Ex + F = 0$ are $a$ and $b$ then the switching time $\tau_s$ can be evaluated as

$$
\tau_s = \frac{1}{2\alpha \left( \tilde{I}_s - 0.5R \right)} \left\{ \log \left[ \frac{1 + g_L}{1 + g_L_i} \right] + \frac{b(a - 4)(a - 8)}{32(a - b)} \log \left[ \frac{1 + g_L_a - a}{1 + g_L_i - a} \right] - \frac{a(b - 4)(b - 8)}{32(a - b)} \log \left[ \frac{1 + g_L_i - b}{1 + g_L_i - b} \right] \right\}. \quad (C4)
$$

Appendix D: Uniaxial Limit

In the uniaxial limit Eq. (7) in the deterministic domain reduces to

$$
\left\langle \frac{\partial g_L}{\partial \tau} \right\rangle = 2\alpha \sqrt{-g_L} \left( 1 + g_L \right) \left( \tilde{I}_s - \sqrt{-g_L} \right), \quad (D1)
$$

which can then be integrated as

$$
\int_{g_L}^{g_L_f} \frac{dg_L}{\sqrt{-g_L} \left( 1 + g_L \right) \left( \tilde{I}_s - \sqrt{-g_L} \right)} = 2\alpha \tau_s.
$$

Substituting $g_L = -u^2$, and using partial fractions to separate the terms in the denominator and integrating with proper limits leads us to

$$
\tau_s = \frac{1}{2\alpha \left( \tilde{I}_s^2 - 1 \right)} \left\{ \tilde{I}_s \left\{ \log \left[ \frac{1 + \sqrt{-g_L_i}}{1 + \sqrt{-g_L_f}} \right] - \log \left[ \frac{1 + g_L_i}{1 + g_L_f} \right] \right\} - 2\log \left[ \frac{\tilde{I}_s - \sqrt{-g_L_i}}{\tilde{I}_s - \sqrt{-g_L_f}} \right] \right\}. \quad (D2)
$$

Appendix E: Eqs. (14) and (16)

Using $R = 0$ in Eq. (13), the probability for a random variable $G_L = -M_x^2$ is obtained as

$$
P_{G_L} [g_L] = P [G_L \leq g_L] = P [-M_x^2 \leq g_L] = 1 - P [M_x \leq -\sqrt{-g_L}] = 1 - P [-\sqrt{-g_L} \leq M_x \leq -\sqrt{-g_L}] = 1 - \frac{2}{Z(\Delta_0, 0)} \int_0^{\sqrt{-g_L}} dm_x \exp \left( \Delta_0 m_x^2 \right)
$$

$$
= 1 - \frac{2}{Z(\Delta_0, 0)} \int_0^{\sqrt{-g_L}} dm_x \exp \left( \Delta_0 m_x^2 \right)
$$
\[ 1 - \frac{2}{Z(\Delta_0, 0) \sqrt{\Delta_0}} \int_0^\infty \exp (\Delta_0 m_z^2) \]
\[ = 1 - \frac{2 \exp (\Delta_0 g_L)}{Z(\Delta_0, 0) \sqrt{\Delta_0} \exp (\Delta_0 g_L)} \int_0^\infty du \exp (u^2) \]
\[ = 1 - \frac{2 \exp (\Delta_0 g_L)}{Z(\Delta_0, 0) \sqrt{\Delta_0}} F[\sqrt{\Delta_0} g_L]. \quad (E1) \]

In step 6 above we have substituted \( \sqrt{\Delta_0 m_z} = u \), and used \( F[x] = \exp (-x^2) \int_0^\infty dy \exp (y^2) \), the Dawson’s integral in step 7. Also, \( \rho_{GL}[-1] = 0 \), therefore,
\[ Z(\Delta_0, 0) = \frac{2 \exp (\Delta_0)}{\sqrt{\Delta_0}} F[\sqrt{\Delta_0}]. \quad (E2) \]

Eqs. (E1) and (E2) together lead to Eq. (16).

In order to arrive at the probability distribution function (pdf) we now differentiate Eq. (E1) with respect to \( g_L \) at step 4 to get
\[ \rho_{GL}[g_L] = \frac{1}{Z(\Delta_0, 0)} \frac{\exp (-\Delta_0 g_L)}{\sqrt{-gL}} \]
\[ = \frac{\sqrt{\Delta_0}}{2F[\sqrt{\Delta_0}]} \frac{\exp (-\Delta_0 (1 + g_L))}{\sqrt{-gL}}. \quad (E3) \]

We have used \( Z(\Delta_0, R) = Z(\Delta_0, 0) \) and have found that all the results match well.

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