GLOBAL WELL-POSEDNESS AND EXPONENTIAL STABILITY FOR THE FERMION EQUATION IN WEIGHTED SOBOLEV SPACES

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ABSTRACT. This work deals with the Cauchy problem and the asymptotic behavior of the solution of the fermion equation in the Sobolev spaces with a polynomial weight in the torus. We first investigate the linearized equation and obtain the optimal exponential decay rate for the associated semigroup. Our strategy is taking advantage of quantitative spectral gap estimates in smaller reference Hilbert space, the factorization method and the enlargement of the functional space. We then turn to the nonlinear equation and prove the global existence and uniqueness of solutions in a close-to-equilibrium regime. Moreover, we prove an exponential stability for such a solution with the optimal decay rate given by the semigroup decay of the linearized equation.

1. Introduction and main results. The fermion equation is the simplest model describing a gas of fermions relaxing towards the thermodynamic equilibrium for a perfect Fermi gas which takes the form:

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\kappa} \int_{\mathbb{R}^d} \left[ \mu(v) (1 - f(v)) f(v_*) - \mu(v_*) (1 - f(v)) f(v) \right] dv_*, \quad (1)$$

$$f(0, x, v) = f_{in}(x, v) \geq 0, \quad (2)$$

where $x \in \mathbb{T}^d$ (the $d$-dimensional flat torus) and $v \in \mathbb{R}^d$ ($d \geq 1$). In (1), the unknown function $f = f(t, x, v) \geq 0$ describes the density of gas molecules with time $t \in \mathbb{R}^+$, the position $x \in \mathbb{T}^d$ and the velocity $v \in \mathbb{R}^d$. The constant $\kappa > 0$ denotes the Knudsen number which is essentially the ratio between the typical mean
free path and the typical length scale of the problem. \( \mu(v) \) denotes the normalized Maxwellian:

\[
\mu(v) = \frac{1}{(2\pi)^{d/2}} e^{-\frac{|v|^2}{2}}.
\]

The factors \((1 - f(v))\) and \((1 - f(v_*))\) take into account the quantum effects with Pauli exclusion principle. According to this, only one particle is allowed in each available quantum state. The values of the distribution function \( f \) have to respect the bounds \( 0 \leq f \leq 1 \). We assume without loss of generality that the torus has volume one. The equation (1) preserves the total mass of the distribution:

\[
\forall t \geq 0, \quad \int_{T^d \times \mathbb{R}^d} f(t, x, v) \, dx \, dv = \int_{T^d \times \mathbb{R}^d} f_{in}(x, v) \, dx \, dv = \rho
\]

and admits a unique global equilibrium (Fermi-Dirac statistics)

\[
f_\infty(v) = \frac{\kappa_\infty \mu}{1 + \kappa_\infty \mu},
\]

where \( \kappa_\infty \) is determined by the mass \( \rho \) of \( f_{in} \):

\[
\int_{\mathbb{R}^d} f_\infty(v) \, dv = \int_{T^d \times \mathbb{R}^d} f_{in}(x, v) \, dx \, dv.
\]

We consider the solution of the equation (1) around the global Maxwellian state \( f_\infty \), and rewrite the solution \( f \) as \( f = f_\infty + h \). Then the equation for \( h \) reads

\[
\partial_t h + v \cdot \nabla_x h = \frac{1}{\kappa} \left( \int_{\mathbb{R}^d} (1 + \kappa_\infty \mu(v_*)) h(v_*) \, dv_* \right) \frac{\mu}{1 + \kappa_\infty \mu} - \frac{\rho}{\kappa \kappa_\infty} (1 + \kappa_\infty \mu) h
\]

\[
+ \frac{1}{\kappa} \int_{\mathbb{R}^d} (\mu(v) - \mu(v_*)) \, h(v) h(v_*) \, dv_*.
\]

Dropping the nonlinear term, we have the linearized fermion equation

\[
\partial_t h = \mathcal{L}(h), \quad h_{in}(x, v) = f_{in} - f_\infty,
\]

where

\[
\mathcal{L}(h) := \frac{1}{\kappa} \left( \int_{\mathbb{R}^d} (1 + \kappa_\infty \mu(v_*)) h(v_*) \, dv_* \right) \frac{\mu}{1 + \kappa_\infty \mu} - \frac{\rho}{\kappa \kappa_\infty} (1 + \kappa_\infty \mu) h - v \cdot \nabla_x h.
\]

The asymptotic behavior of solutions to kinetic equations is a very important topic in statistical physics and there are many results on it. A fundamental question is to estimate the rate of relaxation of the solutions towards a global equilibrium. Since the literature on this subject is very huge, we only briefly review some works related to the study of this paper. Desvillettes and Villani proved convergence to the equilibrium in weighted Hilbert spaces for spatially inhomogeneous kinetic equations by a distinguished Lyapunov functional which is based on functional inequalities, time-derivative estimates and interpolation. It was used for the Fokker-Planck equation [7] and in its main achievement for the full Boltzmann equation [8]. The method of Desvillettes and Villani was also applied to equation (1) and (2) in [21]. They obtained the polynomial rates of convergence to the equilibrium. Let us also mention the problem that consists in identifying general structures, which interplay between a conservative part and a degenerate dissipative part. We know that the conservative part alone does not induce relaxation and the degenerate dissipative part is not sufficient, but the combination of the two parts leads to convergence to equilibrium. In the memoir [23], Villani gave a systematic study of hypocoercivity methods for a class of possibly unbounded operators which can be
written in “Hörmander form” on a given Hilbert space. This approaches are also used to study the linear kinetic equations without confinement such as the Fokker-Planck equation and the linear relaxation Boltzmann equation (i.e. equation (1) without quantum effects ) (see [2]). Mouhot and Neumann [20] established an abstract version of a powerful hypocoercivity theorem and obtained exponential decay to equilibrium for a general class of linear collisional kinetic models in the torus involving the equation (1).

Until recently, inspired from the methodological approach developed in [19] for the homogeneous Boltzmann equation for hard potentials with angular cut-off, Gualdani, Mischler, and Mouhot [13] established an abstract theory for enlarging the space where the spectral gap and the discrete part of the spectrum is known for a certain class of unbounded closed operators. The core of this method is a factorization argument for a class of operators written as a dissipative part plus a bounded part and the two parts satisfy a semigroup commutator condition of regularization. Then for this class of operators, the corresponding abstract theory for enlarging the space can be used to obtain the explicit decay semigroup estimates in the larger space. This can also be seen as a theory for obtaining quantitative spectral mapping theorem in the larger spaces. From the quantitative hypocoercivity theorem in a small Hilbert space setting in [20], they obtained the first constructive proof of exponential decay towards global equilibrium for the full nonlinear inhomogeneous Boltzmann equation for hard spheres in the torus. Later, this approach can be applied to the Landau kinetic equations. In [6], Carrapatoso, Tristani and Wu obtained the existence, uniqueness and exponential stability results for the Landau equation with hard, Maxwellian and moderately soft potentials in the closed to equilibrium setting, their results not only enlarged the function spaces, but also remove the velocity regularity assumption on the initial condition. In [5], Carrapatoso and Mischler extended the results to the Landau equation with very soft potential and Coulomb potential. This approach has also been applied recently for the kinetic Fokker-Planck equation [17], for the growth-fragmentation equation [18] and the inhomogeneous Boltzmann equation for hard potentials with a moderate angular singularity [15] (see also [1, 4, 16, 22, 24] for related works).

In spite of the above works, the asymptotic behavior of solutions to the fermion equation (1) only has been settled in the weighted Hilbert space \( H^{\ell} \) on a given Hilbert space. This approaches are also used to study the linear kinetic equations without confinement such as the Fokker-Planck equation and the linear relaxation Boltzmann equation (i.e. equation (1) without quantum effects ) (see [2]). Mouhot and Neumann [20] established an abstract version of a powerful hypocoercivity theorem and obtained exponential decay to equilibrium for a general class of linear collisional kinetic models in the torus involving the equation (1).

Before describing our main results, we need to give some notations and definitions. Let \( m = m(v) \) be a positive Borel weight function, we define the weighted Lebesgue space \( L^p_v L^q_x(m) \) as

\[
\left\| f \right\|_{L^p_v L^q_x(m)} := \left( \left( \int_{\mathbb{R}^d_v} \left( \int_{\mathbb{T}^d_x} |f(x,v)|^p \, dx \right)^{\frac{q}{p}} m(v)^q \, dv \right)^{\frac{1}{q}} \right)^{\frac{1}{p}}.
\]

We use the multi-index notation: given \( \alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d \), \( |\alpha| = \alpha_1 + \cdots + \alpha_d \) then \( \partial^\alpha x f(x) = \frac{\partial^{\alpha_1} f(x)}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d} f(x)}{\partial x_d^{\alpha_d}} = \partial^{\alpha_1} x_1 \cdots \partial^{\alpha_d} x_d f \). We also consider the higher-order Sobolev
space \( W_0^{s,q} W_2^{s,p}(m) \) for \( s, q \in \mathbb{N} \) defined by
\[
\|f\|_{W_0^{s,q} W_2^{s,p}(m)} := \sum_{\alpha, \beta \in \mathbb{N}^d, |\alpha| \leq s, |\beta| \leq q, |\alpha| + |\beta| \leq \max\{s, q\}} \|\partial^\alpha \partial^\beta f\|_{L^2(L^2(m))}. 
\]
This definition reduces to the usual weighted Sobolev space \( W_0^{s,q}(m) \) when \( q = p \) and \( s = q \), and we recall the shorthand notation \( H_0^{s,q} = W_2^{s,q} \).

For two given Banach spaces \( (E, \|\cdot\|_E) \) and \( (\mathcal{E}, \|\cdot\|_{\mathcal{E}}) \), we denote \( \mathcal{B}(E, \mathcal{E}) \) the space of bounded linear operators from \( E \) to \( \mathcal{E} \). We write \( \mathcal{B}(E) = \mathcal{B}(E, E) \) when \( E = \mathcal{E} \). We denote \( \mathcal{C}(E, \mathcal{E}) \) the space of closed unbounded linear operators from \( E \) to \( \mathcal{E} \) with dense domain, and \( \mathcal{C}(E) = \mathcal{C}(E, E) \) in the case \( E = \mathcal{E} \). For a Banach space \( X \) and an operator \( \mathcal{L} \in \mathcal{C}(X) \), we denote the semigroup generated by \( \mathcal{L} \) as \( S_\mathcal{L}(t) = e^{t\mathcal{L}}, t \geq 0 \), the domain as \( \mathcal{D}(\mathcal{L}) \), the null space as \( \mathcal{N}(\mathcal{L}) \), and the range as \( \mathcal{R}(\mathcal{L}) \).

We also denote the spectrum of \( \mathcal{L} \) as \( \Sigma(\mathcal{L}) \). For any \( z \in \mathbb{C} \) \( \setminus \Sigma(\mathcal{L}) \), the operator \( \mathcal{L} - z \) is invertible and the resolvent operator
\[
R_\mathcal{L}(z) := (\mathcal{L} - z)^{-1}
\]
is well-defined, belongs to \( \mathcal{B}(X) \) and has range equal to \( \mathcal{D}(\mathcal{L}) \). We recall that \( \xi \in \Sigma(\mathcal{L}) \) is said to be an eigenvalue if \( \mathcal{N}(\mathcal{L} - \xi) \neq \{0\} \). Moreover, an eigenvalue \( \xi \in \Sigma(\mathcal{L}) \) is said to be isolated if for some \( r > 0 \),
\[
\Sigma(\mathcal{L}) \cap \{z \in \mathbb{C} \mid |z - \xi| < r\} = \{\xi\}.
\]
In the case when \( \xi \) is an isolated eigenvalue, we may define \( \Pi_{\mathcal{L}, \xi} \in \mathcal{B}(E) \) the spectral projector by
\[
\Pi_{\mathcal{L}, \xi} := -\frac{1}{2\pi i} \int_{|z - \xi| = r'} R_\mathcal{L}(z) \, dz, \quad i = \sqrt{-1},
\]
with \( 0 < r' < r \). It is well-known that \( \Pi_{\mathcal{L}, \xi}^2 = \Pi_{\mathcal{L}, \xi} \), so that \( \Pi_{\mathcal{L}, \xi} \) is indeed a projector. Moreover the range of the spectral projector \( \mathcal{B}(\Pi_{\mathcal{L}, \xi}) \) has finite dimension. We say that \( \xi \) is a discrete eigenvalue, written as \( \xi \in \Sigma_d(\mathcal{L}) \). When \( \Sigma(\mathcal{L}) \subseteq \mathbb{R}^- \), we say that \( \mathcal{L} \) has a spectral gap when the distance between 0 and \( \Sigma(\mathcal{L}) \setminus \{0\} \) is positive, and use the "spectral gap" to denote this distance.

We shall define the convolution of semigroup. Consider three Banach spaces \( X_1, X_2, \) and \( X_3 \). For two given functions
\[
S_1 \in L^1(\mathbb{R}^+, \mathcal{B}(X_1, X_2)) \quad \text{and} \quad S_2 \in L^1(\mathbb{R}^+, \mathcal{B}(X_2, X_3)),
\]
we define the convolution \( S_2 * S_1 \in L^1(\mathbb{R}^+, \mathcal{B}(X_1, X_3)) \) by
\[
\forall t \geq 0, \quad (S_2 * S_1)(t) := \int_0^t S_2(t - \tau) S_1(\tau) \, d\tau.
\]
When \( S = S_1 = S_2 \) and \( X_1 = X_2 = X_3 \), we define inductively \( S^{(s)} = S \) and \( S^{(s+1)} = S * S^{(s)} \) for any \( s \geq 1 \).

Let us now introduce the notation of hypodissipative operators. If one consider a Banach space \( (X, \|\cdot\|_X) \) and an operator \( \Lambda \in \mathcal{B}(X), a \in \mathbb{R} \). \( \Lambda - a \) is said to be hypodissipative on \( X \) if there exists some norm \( |||\cdot|||_X \) on \( X \) equivalent to the initial norm \( \|\cdot\|_X \), such that
\[
\forall f \in \mathcal{B}(\Lambda), \quad \exists \varphi \in F(f), \quad \text{satus} \quad \Re \langle \varphi, (\Lambda - a) f \rangle \leq 0,
\]
where \( \langle \cdot, \cdot \rangle \) is the duality bracket for the duality in \( X \) and \( X^* \), \( F(f) \subseteq X^* \) is the dual set of \( f \) defined by
\[
F(f) = F(|||\cdot|||)(f) := \{ \varphi \in X^* \mid \langle \varphi, f \rangle = |||f|||_X = |||\varphi|||_X \}.
\]
Through this paper, we denote \( \langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}} \) and use the same notation \( C \) for positive constants which may be changed from line to line. Moreover, the notation \( a \approx b \) means that there exist two constants \( c_1, c_2 > 0 \) such that \( c_1 a \leq b \leq c_2 a \). We will assume the polynomial weight \( m(v) = (1 + \kappa_\infty \mu)(v) \) in the remaining part of the paper.

We first study the semigroup decay properties of the solution to the linearized problem \( (4) \). This results provide a constructive spectral gap estimate for the operator \( \mathcal{L} \) in space \( W_\sigma^p W^s_p(m) \) and the cornerstone of the proof of Theorem 1.2.

**Theorem 1.1.** For any initial datum \( h_{in} \in W_\sigma^p W^s_p(m) \) with \( \sigma, s \in \mathbb{N}, \sigma \leq s, 1 \leq p, q \leq +\infty, \frac{1}{q} + \frac{1}{q} = 1 \) and \( k > \frac{d}{q} \), there are constructive constants \( \lambda > 0 \) and \( C > 0 \), such that the associated solution \( h \) to the linearized equation \( (4) \) satisfies the following decay estimates

\[
\forall t \geq 0, \quad \|h\|_{W_\sigma^p W^s_p(m)} \leq C e^{-\lambda t} \|h_{in}\|_{W_\sigma^p W^s_p(m)},
\]

where \( \lambda \) is the spectral gap in \( H^s_x(v) \left( \frac{1 + \kappa_\infty \mu}{\sqrt{\kappa_\infty \mu}} \right) \), \( \ell \in \mathbb{N} \) defined in Lemma 2.6.

**Remark 1.**

(1) The associated solution \( f \) converges exponentially fast towards the equilibrium \( f_\infty \). Furthermore, the explicit rate \( \lambda \) is an optimal timescale that is equal to the spectral gap in the smaller reference space \( H^s_x(v) \left( \frac{1 + \kappa_\infty \mu}{\sqrt{\kappa_\infty \mu}} \right) \).

(2) The procedure of the proof of Theorem 1.1 can also be applied to the linear relaxation Boltzmann equation, i.e. equation \( (1) \) without quantum effects,

\[
\partial_t f + v \cdot \nabla_x f = \frac{1}{\kappa} \left[ \left( \int_{\mathbb{R}^d} f(t, x, v_s) dv_s \right) \mu(v) - f \right], \quad f(0, x, v) = f_{in}(x, v) \geq 0.
\]

(5)

In this case, other methods have been developed in [2, 3, 12, 20] to solve this question in Hilbert space with Gaussian weight. Furthermore, the exponential decay rate toward to the global equilibrium \( \mu \) of the equation \( (5) \) with an external confining potential is also obtained by [9, 10, 11, 14].

Now we state our main results on the Cauchy problem and exponential convergence to the equilibrium \( f_\infty \) for the solutions to the fully nonlinear problem \( (1)-(2) \). We consider the close-to-equilibrium regime.

**Theorem 1.2.** Consider the space \( L^1_{t\mu} W^s_p(m) \), \( k > 0, s \in \mathbb{N}, s \geq 0 \) when \( p = \infty \) and \( s \geq 0 \) when \( p = +\infty \). If there is some constructive constant \( \epsilon > 0 \) (small enough) such that for any initial datum \( f_{in} \in L^1_{t\mu} W^s_p(m) \) satisfying \( \|f_{in} - f_\infty\|_{L^1_{t\mu} W^s_p(m)} \leq \epsilon \), then there exists a unique global solution of problem \( (1)-(2) \) in space \( L^\infty_t \left( [0, +\infty), L^1_{t\mu} W^s_p(m) \right) \) and \( L^1_t \left( [0, +\infty), L^1_{t\mu} W^s_p(m) \right) \). This solution satisfies the decay estimates

\[
\forall t \geq 0, \quad \|f - f_\infty\|_{L^1_{t\mu} W^s_p(m)} \leq C e^{-\lambda t} \|f_{in} - f_\infty\|_{L^1_{t\mu} W^s_p(m)},
\]

where \( \lambda \) is given by Theorem 1.1, \( C > 0 \) is a constructive constant.

**Remark 2.**

(1) We obtain the exponential decay estimates in Sobolev spaces with a polynomial weight which is much slower than the usual inverse of the Gaussian weight. Moreover, the corresponding solutions decay exponentially fast in time with the same rate as the linearized flow.
(2) We prove an exponential in time convergence to equilibrium, partly improving results of Neumann and Schmeiser [21]. The paper [21] used the entropy dissipation method to prove algebraic decay to equilibrium under the assumption of uniform bound for $x$-derivatives of the solution.

(3) In the present paper, for simplicity, we can assume the Knudsen number to be equal 1. However, if one wants to study the hydrodynamic limit, then the explicit dependencies on the Knudsen number is needed.

The outline of this paper is as follows. We prove the linearized result (Theorem 1.1) in section 2 and the nonlinear result (Theorem 1.2) in section 3.

2. The linearized equation: Proof of Theorem 1.1. To prove Theorem 1.1, we first split the linearized operator $L := A + B$,

$$L := A + B,$$

where

$$A(h) := \frac{1}{\kappa} \left( \int_{\mathbb{R}^d} \chi_{(|v_*| \leq \delta^{-1})} (1 + \kappa_\infty \mu(v_*)) h(v_*) \, dv_* \right) \frac{\mu}{1 + \kappa_\infty \mu},$$

$$B(h) := \frac{1}{\kappa} \left( \int_{\mathbb{R}^d} \chi_{(|v_*| > \delta^{-1})} (1 + \kappa_\infty \mu(v_*)) h(v_*) \, dv_* \right) \frac{\mu}{1 + \kappa_\infty \mu} - \frac{\rho}{\kappa \kappa_\infty} (1 + \kappa_\infty \mu) h - v \cdot \nabla_x h,$$

for some constant $\delta > 0$ to be chosen later. Here $\chi(v)$ is the characteristic function.

Next, we will take advantage of the factorization method and the enlargement of the functional space developed by Gualdani, Mischler, and Mouhot in [13]. Now we can set $E := H_{\kappa_\infty \mu}^{\ell}(\sqrt{\kappa_\infty \mu})$, $E := W^{s, q}_v W^{s, p}_x (m)$ and $L|_E = L$. We prove that (i) the spectral gap property of $L$ in $E$ can be shown to hold for $L$ in the space $E$ and (ii) explicit estimates on the rate of decay of the semigroup $S_L(t)$ can be computed from the ones on $S_L(t)$. This is true for a class of operators $L$ which split as $L := A + B$, where $A$ is bounded, $B$ is dissipative, the spectrum of $B$ is well localized, and some appropriate combination of $A$ and $S_B(t)$ has some regularizing properties.

2.1. Regularization properties of the operator $A$. We first state the regularization estimate on the operator $A$.

**Lemma 2.1.** For any $\ell \in \mathbb{N}$, $q \in [1, +\infty]$, there exists a constant $C > 0$ such that for any $h \in L_{L^1}^1(1 + \kappa_\infty \mu)$,

$$\|A(h)\|_{W^{\ell, q}_v} \leq C \|h\|_{L^{L^1}_1(1 + \kappa_\infty \mu)}.$$

**Proof.** We only focus on the case $1 \leq q < \infty$ since the case $q = +\infty$ is obvious.
Considering first $\ell = 0$ and using the Minkowski’s integral inequality, we obtain that for $1 \leq q < \infty$,

$$\|A(h)\|_{L^q_v} = \left( \int_{\mathbb{R}^d} |A(h)|^q \, dv \right)^{\frac{1}{q}}$$

$$\leq \frac{1}{\kappa} \left( \int_{\mathbb{R}^d} \left( \int_{\{|v_*| \leq \delta^{-1}\}} (1 + \kappa_\infty \mu(v_*)) |h(v_*)| \, dv_* \frac{\mu}{1 + \kappa_\infty \mu} \right)^q \, dv \right)^{\frac{1}{q}}$$

$$\leq \frac{1}{\kappa} \int_{\{|v_*| \leq \delta^{-1}\}} \left( \int_{\mathbb{R}^d} (\frac{\mu}{1 + \kappa_\infty \mu})^q \, dv \right)^{\frac{1}{q}} (1 + \kappa_\infty \mu(v_*)) |h(v_*)| \, dv_*$$

$$\leq C \int_{\{|v_*| \leq \delta^{-1}\}} (1 + \kappa_\infty \mu(v_*)) |h(v_*)| \, dv_* \quad (6)$$

$$\leq C \|h\|_{L^1_{\mu}(1 + \kappa_\infty \mu)}.$$

Next we consider the high order derivatives, i.e., $\ell \geq 1$. One easily gets

$$\partial^\alpha_v A(h) = \frac{1}{\kappa} \left( \int_{\mathbb{R}^d} \chi_{\{|v_*| \leq \delta^{-1}\}} (1 + \kappa_\infty \mu(v_*)) h(v_*) \, dv_* \right) \partial^\alpha_v \left( \frac{\mu}{1 + \kappa_\infty \mu} \right), \forall |\alpha| \leq \ell.$$

By employing the fact that the integration

$$\int_{\mathbb{R}^d} \left| \partial^\alpha_v \left( \frac{\mu}{1 + \kappa_\infty \mu} \right) \right|^q \, dv < +\infty.$$

Likewise we can also obtain that

$$\|\partial^\alpha_v A(h)\|_{L^q_v} \leq C \|h\|_{L^1_{\mu}(1 + \kappa_\infty \mu)}, \quad \forall |\alpha| \leq \ell$$

for some constant $C > 0$.

Thus we conclude the proof of Lemma 2.1. \qed

**Lemma 2.2.** The operator $A \in \mathcal{B}(H_{x,v}^\ell(\frac{1 + \kappa_\infty \mu}{\sqrt{\kappa_\infty \mu}}))$ and $A \in \mathcal{B}(W_{x,v}^\sigma W_{x,v}^p(m))$ with $\ell, \sigma, s \in \mathbb{N}, s \leq s, k \geq 0$.

**Proof.** Since $A$ is an operator acting only in $v$, we have

$$\int_{\mathbb{T}^d} |A(h)| \, dx \leq A \left( \int_{\mathbb{T}^d} |h| \, dx \right),$$

and

$$\sup_{x \in \mathbb{T}^d} (|A(h)|) \leq A \left( \sup_{x \in \mathbb{T}^d} (|h|) \right).$$

By using the interpolation inequality, we obtain that

$$\|A(h)\|_{L^p_v} \leq A(\|h\|_{L^p_v})$$

for any $p \in [1, +\infty]$.

More generally, we have

$$\|\partial^\alpha_v A(h)\|_{L^p_v} \leq |\partial^\alpha_v A(\|h\|_{L^p_v})|, \quad \forall p \in [1, +\infty], \quad \forall |\alpha| \leq \ell. \quad (7)$$
We then get that
\[
\|A(h)\|_{L^2_{x,v}(1+\frac{\|h\|}{\|A\|})} = \left\| A(h) \right\|_{L^2_{x,v}(1+\frac{\|h\|}{\|A\|})} \\
\leq \left\| A(h) \right\|_{L^2_{x,v}(1+\frac{\|h\|}{\|A\|})} \\
\leq C \left\| h \right\|_{L^2_{x,v}}.
\]

Since the $x$-derivatives commute with the operator $A$, we have the same manner
\[
\|\partial^\beta_x A(h)\|_{L^2_{x,v}(1+\frac{\|h\|}{\|A\|})} = \|A(\partial^\beta_x h)\|_{L^2_{x,v}(1+\frac{\|h\|}{\|A\|})} \leq C \|\partial^\beta_x h\|_{L^2_{x,v}(1+\frac{\|h\|}{\|A\|})}, \quad \forall |\beta| \leq \ell.
\]

Now we consider the case of $v$-derivatives, $\ell \geq 1$. From Lemma 2.1, we have
\[
\|\partial^\alpha_v A(h)\|_{L^2_{x,v}(1+\frac{\|h\|}{\|A\|})} \leq C \|h\|_{L^2_{x,v}(1+\frac{\|h\|}{\|A\|})}, \quad \forall |\alpha| \leq \ell.
\]

Similarly, we get
\[
\|\partial^\alpha_v A(h)\|_{L^2_{x,v}(1+\frac{\|h\|}{\|A\|})} \leq C \|h\|_{L^2_{x,v}(1+\frac{\|h\|}{\|A\|})}, \quad \forall |\alpha| \leq \ell.
\]

Consequently, we have
\[
\|A(h)\|_{H^\ell_{x,v}(1+\frac{\|h\|}{\|A\|})} \leq C \|h\|_{H^\ell_{x,v}(1+\frac{\|h\|}{\|A\|})}.
\]

Now by the estimate (6), it is straightforward to check that for $\sigma \geq 0$,
\[
\|\partial^\alpha_v A(h)\|_{L^q_{x,v}(m)} \leq C \|h\|_{L^q_{x,v}(m)}, \quad \forall |\alpha| \leq \sigma,
\]
then using (7) to find
\[
\|A(h)\|_{L^q_{x,v}(m)} = \left\| A(h) \right\|_{L^q_{x,v}(m)} \\
\leq \left\| A(h) \right\|_{L^q_{x,v}(m)} \\
\leq C \left\| h \right\|_{L^q_{x,v}(m)} \\
= C \|h\|_{L^q_{x,v}(m)}.
\]

For more general cases, similar to the above discussion, we have
\[
\|\partial^\beta_x A(h)\|_{L^q_{x,v}(m)} \leq C \|\partial^\beta_x h\|_{L^q_{x,v}(m)}, \quad \forall |\beta| \leq s,
\]
and
\[
\|\partial^\alpha_v A(h)\|_{L^q_{x,v}(m)} \leq C \|h\|_{L^q_{x,v}(m)}, \quad \forall |\alpha| \leq \sigma.
\]

Therefore
\[
\|A(h)\|_{W^{s,q}_{x,v}(m)} \leq C \|h\|_{W^{s,q}_{x,v}(m)}.
\]

This completes the proof of this lemma.
2.2. Dissipativity properties of the operator $\mathcal{B}$. We define the operator $\mathcal{B}_\delta$ by

$$
\mathcal{B}_\delta(h) := \frac{1}{\kappa} \left( \int_{\mathbb{R}^d} \chi_{\{|v| > \delta^{-1}\}} (1 + \kappa_\infty \mu(v_*)) h(v_*) dv_* \right) \frac{\mu}{1 + \kappa_\infty \mu}.
$$

Then, we have

**Lemma 2.3.** For any $q \in [1, +\infty]$ and $k > \frac{d}{q'}$, there exists a constructive constant $C > 0$, such that

$$
\forall h \in L^q (m), \quad \|\mathcal{B}_\delta(h)\|_{L^q(m)} \leq O(\delta) \|h\|_{L^q(m)},
$$

where $\frac{1}{q} + \frac{1}{q'} = 1$, $O(\delta)$ is a constructive constant depending on $k$ and approaches zero as $\delta$ goes to zero.

**Proof.** We first consider the case $1 < q < +\infty$. Applying the Minkowski’s integral inequality and Hölder’s inequality, we have

$$
\begin{align*}
\|\mathcal{B}_\delta(h)\|_{L^q(m)} &\leq \frac{1}{\kappa} \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \chi_{\{|v| > \delta^{-1}\}} (1 + \kappa_\infty \mu(v_*)) |h(v_*)| dv_* \right)^q \mu(v)^q \langle v \rangle^k dv \right)^{\frac{1}{q}} \\
&\leq \frac{1}{\kappa} \int_{\{|v| > \delta^{-1}\}} \left( \int_{\mathbb{R}^d} \mu(v)^q \langle v \rangle^k dv \right)^{\frac{1}{q}} (1 + \kappa_\infty \mu(v)) |h(v)| dv \\
&\leq C \int_{\{|v| > \delta^{-1}\}} (1 + \kappa_\infty \mu(v)) |h(v)| dv \\
&\leq C \left( \int_{\{|v| > \delta^{-1}\}} |h(v)|^q \left( (1 + \kappa_\infty \mu(v)) \langle v \rangle^k \right)^q dv \right)^{\frac{1}{q}} \left( \int_{\{|v| > \delta^{-1}\}} \frac{1}{\langle v \rangle^k} dv \right)^{\frac{1}{q'}} \\
&\leq C \left( \frac{\delta(q'k-d)}{q'k-d} \right)^{\frac{1}{q'}} \|h\|_{L^q(m)}.
\end{align*}
$$

The cases $q = 1$ and $q = +\infty$ are obvious and hence we omit the details. This completes the proof of lemma 2.3. $\square$

Let us now prove the dissipativity estimates for the operator $\mathcal{B}$.

**Lemma 2.4.** Consider the space $W^{\sigma,q}_v W^{s,p}_x (m)$ with $\sigma, s \in \mathbb{N}$, $\sigma \leq s$, $1 \leq p, q \leq +\infty$ and $k > \frac{d}{q'}$. Then there exist $\lambda_0 > 0$ and $\nu > 0$ such that $\lambda_0 = \lambda_0(k, \delta) \in (0, \nu]$ with $\lambda_0(k, \delta) \to \nu$ when $\delta \to 0^+$ and $\mathcal{B} + \lambda_0$ is hypodissipative in $W^{\sigma,q}_v W^{s,p}_x (m)$.

Namely

$$
\forall t \geq 0, \quad \|S_B(t)\|_{\mathcal{A}(W^{\sigma,q}_v W^{s,p}_x (m))} \leq e^{-\lambda_0 t}.
$$

**Proof.** Let $h$ be the solution of the linear equation

$$
\partial_t h = \mathcal{B}(h) = \mathcal{B}_\delta(h) - \nu(v) h - v \cdot \nabla_x h
$$

with given initial datum $h_0$, here

$$
\nu(v) := \frac{\rho}{\kappa \kappa_\infty} (1 + \kappa_\infty \mu),
$$

then we define $\underline{\nu}$ and $\overline{\nu}$ as

$$
\underline{\nu} := \inf_{v \in \mathbb{R}^d} (\nu(v)) \leq \nu(v) \leq \overline{\nu} := \sup_{v \in \mathbb{R}^d} (\nu(v)).
$$
We consider first that $\sigma = 0$, $s = 0$, $1 \leq p, q < +\infty$. Using Lemma 2.3, Hölder’s inequality and direct calculation, we obtain

$$\frac{d}{dt} \| h \|_{L^q_t L^p_x(m)} = \frac{d}{dt} \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{T}^d} |h|^p \, dx \right)^{\frac{q}{p}} \, dt \right)^{\frac{1}{q}}$$

$$= \| h \|_{L^q_t L^p_x(m)} \int_{\mathbb{R}^d} |h|^q \left( \int_{\mathbb{T}^d} |h|^{p-1} \text{sign}(h) \partial_t h \, dx \right) m^q \, dt$$

$$= \| h \|_{L^q_t L^p_x(m)} \int_{\mathbb{R}^d} |h|^q \left( \int_{\mathbb{T}^d} |h|^{p-1} \text{sign}(h) B_{\delta}(h) \, dx \right) m^q \, dt$$

$$- \| h \|_{L^q_t L^p_x(m)} \int_{\mathbb{R}^d} \nu(v) \| h \|_{L^q_x} m^q \, dt$$

$$- \| h \|_{L^q_t L^p_x(m)} \int_{\mathbb{R}^d} \left( \int_{\mathbb{T}^d} \frac{1}{p} \nu \cdot \nabla_x (|h|^p) \, dx \right) m^q \, dt$$

$$\leq \| h \|_{L^q_t L^p_x(m)} \int_{\mathbb{R}^d} \| B_{\delta}(h) \|_{L^q_x} \| h \|_{L^p_x}^{p-1} m^q \, dt - \nu \| h \|_{L^q_t L^p_x(m)}$$

$$\leq \| B_{\delta} \|_{L^q_x} \| h \|_{L^q_t L^p_x(m)}$$

$$\leq - \left( \nu - \nu - O(\delta) \right) \| h \|_{L^q_t L^p_x(m)}, \quad (8)$$

where we have utilized the estimates

$$\| B_{\delta}(h) \|_{L^q_x} \leq B_3(\| h \|_{L^q_x}),$$

for any $p \in [1, +\infty)$ and the term involving $v \cdot \nabla_x$ cancels from its divergence (in $x$) structure. Thus, the proof of dissipativity in this case is finished.

The cases $p = +\infty$ and $q = +\infty$ are then obtained by taking the corresponding limits in the above estimate (8).

Since the $x$-derivatives commute with the operator $B$, in the same manner, we have

$$\frac{d}{dt} \| \partial_x^\beta h \|_{L^q_t L^p_x(m)} \leq - \left( \nu - O(\delta) \right) \| \partial_x^\beta h \|_{L^q_t L^p_x(m)}, \quad \forall |\beta| \leq s. \quad (9)$$

Next, we consider the case of derivatives in $v$, say that $\sigma = 1$ and $s \geq 1$. Note that for simplicity, we can reduce to the case $s = 1$. We compute the evolution of the $v$-derivatives as follows, $\forall i \in \{1, \ldots, d\}$,

$$\partial_t \partial_v h = \partial_v B_{\delta}(h) - \partial_v \nu(v) h - \nu(v) \partial_v h - v \cdot \nabla_x (\partial_v h) - \partial_x h$$

$$\quad = B(\partial_v h) + \partial_v B_{\delta}(h) - B_{\delta}(\partial_v h) - \partial_v \nu(v) h - \partial_x h.$$
Therefore,
\[
\frac{d}{dt} \| \partial_v h \|_{L^2 L^2_t(m)} \leq - \left( \mu - O(\delta) \right) \| \partial_v h \|_{L^2 L^2_t(m)} + O(\delta) \| h \|_{L^2 L^2_t(m)} + C \| h \|_{L^2 L^2_t(m)} + \| \partial_x h \|_{L^2 L^2_t(m)},
\]
(10)

We introduce the new norm
\[
\| h \|_{W^{1,q}_v W^{1,p}_x(m)} := \| h \|_{L^2 L^2_t(m)} + \| \nabla_x h \|_{L^2 L^2_t(m)} + \varepsilon \| \nabla_v h \|_{L^2 L^2_t(m)},
\]
which is equivalent to the standard \( W^{1,q}_v W^{1,p}_x(m) \)-norm for some \( \varepsilon > 0 \) to be fixed later. We now gather (8), (9) and (10) together to deduce that
\[
\frac{d}{dt} \| h \|_{W^{1,q}_v W^{1,p}_x(m)} \leq - \left( \mu - O(\delta) \right) \| h \|_{L^2 L^2_t(m)} - \left( \mu - O(\delta) \right) \varepsilon \| \nabla_x h \|_{L^2 L^2_t(m)} + \varepsilon \| \nabla_v h \|_{L^2 L^2_t(m)} + C \varepsilon \| h \|_{L^2 L^2_t(m)} + \varepsilon \| \nabla_x h \|_{L^2 L^2_t(m)}
\]
which concludes the proof by taking \( \varepsilon \) small enough in terms of \( \delta \).

The higher-order estimates are performed with the norm
\[
\| h \|_{W^{s,q}_v W^{s,p}_x(m)} := \sum_{|\alpha| \leq s, |\beta| \leq s, |\alpha| + |\beta| \leq \max\{s,s\}} \varepsilon^{|\alpha|} \| \partial_v^\alpha \partial_x^\beta h \|_{L^2 L^2_t(m)}
\]
for some \( \varepsilon > 0 \) to be chosen small enough.

We denote \( \lambda_0 := \left( \mu - O(\delta) - O(\varepsilon) \right) > 0 \), for \( \delta, \varepsilon \) small enough. We can obtain that \( B + \lambda_0 \) is dissipative in \( W^{s,q}_v W^{s,p}_x(m) \) for the norm \( \| \cdot \|_{W^{s,q}_v W^{s,p}_x(m)} \) and thus is hypodissipative in \( W^{s,q}_v W^{s,p}_x(m) \). This completes the proof of Lemma 2.4. \( \square \)

2.3. Regularization properties of \( (AS_B(t))^{(\varepsilon,n)} \). We now define the time-indexed operator of \( T_n(t) := (AS_B(t))^{(\varepsilon,n)} \) and go into the analysis of regularization properties of \( T_n(t) \).

**Lemma 2.5.** For any \( \lambda_0 \in (0, \lambda_0) \) where \( \lambda_0 \) is provided by Lemma 2.4, \( p \in [1, +\infty) \), \( k > \frac{d}{p} \), then there is a constructive constant \( C > 0 \) such that for any \( t \geq 0 \)

1. if \( s \geq 1 \), \( \| T_1(t)(h) \|_{W^{s+1,p}_v} \leq C e^{-\lambda_0 t} \| h \|_{W^{s,q}_v(m)} \);
2. if \( s \geq 0 \), \( \| T_2(t)(h) \|_{W^{s+\frac{1}{p},p}_v} \leq C e^{-\lambda_0 t} \| h \|_{W^{s,q}_v(m)} \).

**Proof:** Since the \( x \)-derivatives commute with the operator \( T_1(t) \). It is enough to consider \( h \in W^{s,p}_v W^{1,p}_x(m), s \in \mathbb{N} \). We obtain from Lemma 2.1 that
\[
\| A(h) \|_{W^{s,p}_v} \leq C \| h \|_{L^p_t(m)}.
\]
It is easy to know that
\[
\| T_1(t)(h) \|_{W^{s+1,p}_v} \leq \| T_1(t)(h) \|_{W^{s+1,p}_L^2} + \| \nabla_x T_1(t)(h) \|_{W^{s+1,p}_L^2}.
\]
For the first term, we obtain from Lemma 2.2 and Lemma 2.4 that
\[
\| T_1(t)(h) \|_{W^{s+1,p}_L^2} = \| AS_B(t)(h) \|_{W^{s+1,p}_L^2} \leq C \| S_B(t)(h) \|_{L^p_t(m)} \leq C e^{-\lambda_0 t} \| h \|_{L^p_t(m)}.
\]
(11)
For the second term, we consider $g = S_B(t)h$. It is easy to check that $g$ satisfies

$$\partial_t g = B(g) = \mathcal{B}_\delta(g) - \nu(v)g - v \cdot \nabla_x g$$

with the initial datum $g_0 = h$.

As in [13], let us define the differential operator $D_t := t \nabla_x + \nabla_v$ which commutes with the free transport operator $\partial_t + v \cdot \nabla_x$. Namely,

$$(t \nabla_x + \nabla_v)(\partial_t + v \cdot \nabla_x)g = (\partial_t + v \cdot \nabla_x)(t \nabla_x + \nabla_v)g.$$  

We have

$$\partial_t(D_tg) + v \cdot \nabla_x(D_tg) = (t \nabla_x + \nabla_v)(\partial_t + v \cdot \nabla_x)g$$

$$= (t \nabla_x + \nabla_v)(\mathcal{B}_\delta(g) - \nu(v)g)$$

$$= \mathcal{B}_\delta(t \nabla_x g) + \nabla_v \mathcal{B}_\delta(g) - \nabla_v \nu(v)g - \nu(v) \nabla_v g - t \nu(v) \nabla_x g$$

$$= \mathcal{B}_\delta(D_tg) - \mathcal{B}_\delta(\nabla_x g) + \nabla_v \mathcal{B}_\delta(g) - \nabla_v \nu(v)g - \nu(v) D_tg.$$  

We rewrite the last term as

$$\partial_t(D_tg) = B(D_tg) + \Lambda(g),$$

where $\Lambda(g) := \nabla_v \mathcal{B}_\delta(g) - \mathcal{B}_\delta(\nabla_x g) - \nabla_v \nu(v)g$.

Now, direct calculation gives

$$\|\nabla_v \mathcal{B}_\delta(g)\|_{L^p_{x,v}(m)} \leq C \|g\|_{L^\infty_{x,v}(m)};$$

$$\|\mathcal{B}_\delta(\nabla_x g)\|_{L^p_{x,v}(m)} \leq C \|\nabla_x g\|_{L^\infty_{x,v}(m)},$$

and

$$\|\nabla_v \nu(v)g\|_{L^p_{x,v}(m)} \leq C \|g\|_{L^\infty_{x,v}(m)}.$$  

Gathering the above three estimates together, we get

$$\|\Lambda(g)\|_{L^p_{x,v}(m)} \leq C \left[\|g\|_{L^\infty_{x,v}(m)} + \|\nabla_v g\|_{L^\infty_{x,v}(m)}\right].$$

Now by taking the same arguments as that in Lemma 2.4, we have

$$\frac{d}{dt}\|D_tg\|_{L^p_{x,v}(m)} = \|D_tg\|_{L^p_{x,v}(m)}^{1-p} \int_{\mathcal{T}^1 \times \mathbb{R}^d} |D_tg|^{p-1} \partial_t(D_tg) \text{sign}(D_tg) m^p \, dx \, dv$$

$$\leq \|B(D_tg)\|_{L^p_{x,v}(m)} + \|\Lambda(g)\|_{L^p_{x,v}(m)}$$

$$\leq - \lambda_0 \|D_tg\|_{L^p_{x,v}(m)} + C \|g\|_{L^\infty_{x,v}(m)} + C \|\nabla_v g\|_{L^\infty_{x,v}(m)},$$  

(12)

and

$$\frac{d}{dt}\|g\|_{L^p_{x,v}(m)} \leq - \lambda_0 \|g\|_{L^p_{x,v}(m)}.$$  

(13)

Combining (12) with (13), we obtain, for any $\lambda' \in (0, \lambda_0)$ and for $\varepsilon$ small enough, that

$$\frac{d}{dt} \left(e^{\lambda't} \left(\varepsilon \|D_tg\|_{L^p_{x,v}(m)} + \|g\|_{L^\infty_{x,v}(m)}\right)\right)$$

$$\leq (\lambda' - \lambda_0) e^{\lambda't} \varepsilon \|D_tg\|_{L^p_{x,v}(m)} + C e^{\lambda't} \varepsilon \left[\|g\|_{L^\infty_{x,v}(m)} + \|\nabla_v g\|_{L^\infty_{x,v}(m)}\right]$$

$$+ (\lambda' - \lambda_0) e^{\lambda't} \|g\|_{L^p_{x,v}(m)}$$

$$\leq 0,$$

which implies that, for any $t \geq 0$,

$$e^{\lambda't} \left(\varepsilon \|D_tg\|_{L^p_{x,v}(m)} + \|g\|_{L^\infty_{x,v}(m)}\right) \leq \varepsilon \|\nabla_v h\|_{L^p_{x,v}(m)} + \|h\|_{L^\infty_{x,v}(m)}.$$
Therefore,
\[ \forall t \geq 0, \quad \| D_t g \|_{L^p_{\ell,v}(m)} + \| g \|_{L^p_{\ell,v}(m)} \leq \varepsilon^{-1} e^{-\lambda_0 t} \| h \|_{W^{1,p}_{\ell,x}(m)}. \] (14)

As in the previous proofs of estimate (10), it follows that, for \( \varepsilon \) small enough,
\[
\frac{d}{dt} \left( \varepsilon \| \nabla_v g \|_{L^p_{\ell,v}(m)} + \| g \|_{L^p_{\ell,v}(m)} \right)
\leq \varepsilon^{\lambda_0 t} \left[ O(\delta) \| g \|_{L^p_{\ell,v}(m)} + C \| g \|_{L^p_{\ell,v}(m)} + \| \nabla_v g \|_{L^p_{\ell,v}(m)} \right]
+ (\lambda_0 - \lambda_0) e^{\lambda_0 t} \varepsilon \| \nabla_v g \|_{L^p_{\ell,v}(m)} + (\lambda_0 - \lambda_0) e^{\lambda_0 t} \| g \|_{L^p_{\ell,v}(m)}
\leq 0,
\]
which implies
\[ \forall t \geq 0, \quad \| \nabla_v g \|_{L^p_{\ell,v}(m)} + \| g \|_{L^p_{\ell,v}(m)} \leq \varepsilon^{-1} e^{-\lambda_0 t} \| h \|_{W^{1,p}_{\ell,x}(m)}. \] (15)

By direct computations, we write
\[ t \nabla_x T_1(t)(h) = t \nabla_x A(g) = A(D_t g) - A(\nabla_v g). \]
Applying (14) and (15), we hence obtain that
\[
t \| \nabla_x T_1(t)h \|_{W^{1+1,p}_{\ell,x}} \leq \| A(D_t g) \|_{W^{1+1,p}_{\ell,x}} + \| A(\nabla_v g) \|_{W^{1+1,p}_{\ell,x}}
\leq C \left[ \| D_t g \|_{L^p_{\ell,v}(m)} + \| \nabla_v g \|_{L^p_{\ell,v}(m)} \right]
\leq C \varepsilon^{-1} e^{-\lambda_0 t} \| h \|_{W^{1,p}_{\ell,x}(m)}.
\]

Therefore,
\[ \| \nabla_x T_1(t)h \|_{W^{1+1,p}_{\ell,x}} \leq C \varepsilon^{-1} e^{-\lambda_0 t} \| h \|_{W^{1,p}_{\ell,x}(m)}. \] (16)

If \( s \geq 1 \), combining (11) with (16), we can get
\[
\| T_1(t)(h) \|_{W^{1+1,p}_{\ell,x} W^{s,p}_{\ell,v}} \leq \| T_1(t)(h) \|_{W^{1+1,p}_{\ell,x}} + \| \nabla_x T_1(t)(h) \|_{W^{1+1,p}_{\ell,x}}
\leq C e^{-\lambda_0 t} \| h \|_{L^p_{\ell,v}(m)} + C \varepsilon^{-1} e^{-\lambda_0 t} \| h \|_{W^{1,p}_{\ell,x}(m)}
\leq C \varepsilon^{-1} e^{-\lambda_0 t} \| h \|_{W^{1,p}_{\ell,x}(m)}
\leq C \varepsilon^{-1} e^{-\lambda_0 t} \| h \|_{W^{1,p}_{\ell,x}(m)},
\]

which implies the expected result (1) because \( T_1(t) \) commutes with \( x \)-derivatives, that is
\[ \forall s \geq 1, \quad \| T_1(t)(h) \|_{W^{1+1,p}_{\ell,x} W^{s,p}_{\ell,v}} \leq C \varepsilon^{-1} e^{-\lambda_0 t} \| h \|_{W^{1,p}_{\ell,x}(m)}. \]

To prove (2), for a given \( s \geq 0 \), we can interpolate between the above estimate and (11), which gives
\[
\| T_1(t)(h) \|_{W^{1+1,p}_{\ell,x} W^{s,p}_{\ell,v}} \leq C e^{-\lambda_0 t} \left( \frac{e^{-\lambda_0 t}}{t} \right)^{\frac{1}{2}} \| h \|_{W^{1,p}_{\ell,x}(m)}
\leq C \frac{e^{-\lambda_0 t}}{\sqrt{t}} \| h \|_{W^{1,p}_{\ell,x}(m)}.
\]
Thus, for $s \geq 0$, we get
\[
\|T_2(t)(h)\|_{W^{s+\frac{1}{2},p}} \leq \int_0^t \|T_1(t-\tau)T_1(\tau)h\|_{W^{s+\frac{1}{2},p}} d\tau \leq C \int_0^t e^{-\lambda_0(t-\tau)} \|T_1(\tau)h\|_{W^{s,p}_x L^p_t(m)} d\tau \leq C \left( \int_0^t \frac{e^{-\lambda_0(t-\tau)}}{\sqrt{t-\tau}} d\tau \right) \|h\|_{L^p_x L^p_t(m)} \leq C e^{-\lambda_0 t} \|h\|_{W^{s,p}_x L^p_t(m)}
\]
for some constant $C > 0$. This proves the assertion (2). \qed

2.4. Proof of Theorem 1.1. We know from [20] (see also [3]) that the linearized operator $\mathcal{L}$ admits a spectral gap in the Hilbert space $H^\ell_x \left( \frac{1 + \kappa \infty}{\sqrt{\kappa \infty \mu}} \right)$, $\ell \in \mathbb{N}$. Let us summarize it as follows

**Lemma 2.6 ([20], Theorem 3.1).** There is a constructive constant $\lambda > 0$, such that for any $h \in H^\ell_x \left( \frac{1 + \kappa \infty}{\sqrt{\kappa \infty \mu}} \right)$, $\ell \in \mathbb{N}$,

$$\forall t \geq 0, \quad \|S_{\mathcal{L}}(t)h - \Pi h\|_{H^\ell_x \left( \frac{1 + \kappa \infty}{\sqrt{\kappa \infty \mu}} \right)} \leq e^{-\lambda t} \|h - \Pi h\|_{H^\ell_x \left( \frac{1 + \kappa \infty}{\sqrt{\kappa \infty \mu}} \right)},$$

where $\Pi$ stands for the projection onto $\mathcal{N}(\mathcal{L}) = \text{Span} \left\{ \frac{\mu}{(1 + \kappa \infty \mu)^2} \right\}$, more explicitly,

$$\Pi h := \left( \int_{\mathbb{T}^d \times \mathbb{R}^d} h \, dx \, dv \right) \frac{\mu}{(1 + \kappa \infty \mu)^2}.$$

Moreover, $\lambda \in (0, \nu)$ is the spectral gap of $\mathcal{L}$ in $H^\ell_x \left( \frac{1 + \kappa \infty}{\sqrt{\kappa \infty \mu}} \right)$.

We now present an enlargement of the functional space of a quantitative spectral mapping theorem (in the sense of semigroup decay estimate) in [13] to prove Theorem 1.1.

**Theorem 2.7 ([13], Theorem 2.13).** Let $E, \mathcal{E}$ be two Banach spaces with $E \subseteq \mathcal{E}$ dense with continuous embedding, and consider $L \in \mathcal{C}(E)$, $\mathcal{L} \in \mathcal{C}(\mathcal{E})$ with $\mathcal{L}_{|E} = L$ and $a > 0$. We assume the following:

1. $L$ generates a semigroup $e^{tL}$ on $E$, $L + a$ is dissipative on $\mathcal{R}(\text{Id} - \Pi_{L,a})$ and $\Sigma(L) \cap \{ z \in \mathbb{C} | \Re z > -a \} = \{ \xi \} \subseteq \Sigma_d(L)$.

2. There exist $A, B \in \mathcal{C}(\mathcal{E})$ such that $\mathcal{L} = A + B$ (with corresponding restrictions $A$ and $B$ on $E$) and a constant $C_a > 0$ so that

   (i): $B + a$ is hypodissipative on $\mathcal{E}$;
   (ii): $A \in \mathcal{B}(\mathcal{E})$ and $A \in \mathcal{B}(E)$;
   (iii): $\exists n \geq 1$, the operator $T_n(t) := (AS_B(t))^{(s_n)}$ satisfies

   $$\forall t \geq 0, \quad \|T_n(t)\|_{\mathcal{B}(E,E)} \leq C_a e^{-at}.$$

Then $\mathcal{L}$ is hypodissipative in $\mathcal{E}$ with

$$\forall t \geq 0, \quad \|S_{\mathcal{L}}(t) - S_t \Pi_{\mathcal{L},a}\|_{\mathcal{B}(\mathcal{E})} \leq C'_a \max\{1, t^{n-1}\} e^{-at}$$

for some explicit constant $C'_a > 0$ depending on the constants in the assumptions.
Next, we also give a lemma which provides a practical criterion for proving assumption (2)-(iii) in the above abstract theorem.

**Lemma 2.8** ([13], Lemma 2.17). Let $E, \mathcal{E}$ be two Banach spaces with $E \subseteq \mathcal{E}$ dense with continuous embedding, and consider $L \in \mathcal{C}(E), \mathcal{L} \in \mathcal{C}(\mathcal{E})$ with $\mathcal{L}|_E = L$ and $a > 0$. We assume that there exist some “intermediate spaces”

$$E = \mathcal{E}_1 \subseteq \mathcal{E}_{j-1} \subset \ldots \subset \mathcal{E}_2 \subseteq \mathcal{E}_1 = \mathcal{E}, \quad J \geq 2$$

such that, denoting $A_j := A|_{\mathcal{E}_j}$ and $B_j := B|_{\mathcal{E}_j}$

(i): $B_j + a$ is hypodissipative and $A_j$ is bounded on $\mathcal{E}_j$ for $1 \leq j \leq J$;

(ii): there are some constants $\gamma_0 \in \mathbb{N}^+, C \geq 1, K \in \mathbb{R}, \varepsilon \in [0,1)$ such that

$$\forall t \geq 0, \quad ||T_{\gamma_0}(t)||_{\mathcal{B}(\mathcal{E}_j, \mathcal{E}_{j+1})} \leq C e^{\frac{Kt}{t}}$$

for $1 \leq j \leq J - 1$, with the notation $T_{\gamma}(t) := (\mathcal{AS}_\mathcal{E}(t))^{(\gamma)}$.

Then for any $a' < a$, there exist some constructive constants $n \in \mathbb{N}^+, C_{a'} \geq 1$ such that

$$\forall t \geq 0, \quad ||T_{\gamma}(t)||_{\mathcal{B}(\mathcal{E}, \mathcal{E})} \leq C_{a'} e^{-a't}.$$  

Proof of Theorem 1.1. Now we can set $E := H_{x,v}^{\ell,\nu}(\frac{1+\kappa}{\sqrt{\kappa}})$, $\mathcal{E} := W_\nu^{s,q} W_{x,v}^{s,p}(m)$, $m(v) = (1 + \kappa_\mu)\kappa(v)^k$ with $\ell, \sigma, s \in \mathbb{N}, \sigma \leq s, p, q \in [1, +\infty], k > \frac{d}{2}$. Then for $\ell$ large enough, $E \subseteq \mathcal{E}$ is dense with continuous embedding, and $L \in \mathcal{C}(E), \mathcal{L} \in \mathcal{C}(\mathcal{E})$ with $\mathcal{L}|_E = L$.

The conclusion of Theorem 1.1 is a direct consequence of the abstract Theorem 2.7. First, if the spectral gap $\lambda \in (0, \nu)$ from Lemma 2.6 is larger than the $\lambda_0$ provided by Lemma 2.4. In this case, because $\lambda_0(\delta) \rightarrow \nu$ as $\delta \rightarrow 0^+$, we can choose $\delta$ small enough so that $\lambda < \lambda_0$. Hence, we can always assume that $\lambda < \lambda_0$.

Next, the assumption (1) is nothing but Lemma 2.6, assumption (2)-(i) comes from Lemma 2.4, (2)-(ii) comes from Lemma 2.2 and (2)-(iii) comes from Lemma 2.5 and Lemma 2.8 (we refer the reader to [4] for more details and references therein).

We remark that $\mathcal{B}$ generates a strongly continuous semigroup $S_{\mathcal{B}}(t)$ on $E$. Because of the continuous and dense embedding $E \subseteq \mathcal{E}$ and the hypodissipativity of $\mathcal{B}$, we can extend this semigroup from $E$ to $\mathcal{E}$. So we can obtain that $\mathcal{B}$ generates a semigroup $S_{\mathcal{B}}(t)$ on $E$. Since $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and $\mathcal{A} \in \mathcal{B}(\mathcal{E})$, we deduce that $\mathcal{L}$ generates a semigroup from the classical linear operator perturbation theory. This proves the exponential decay on the semigroup $\mathcal{E}$:

$$\forall h_{in} \in \mathcal{E}, \quad \forall t \geq 0, \quad ||S_{\mathcal{L}}(t) h_{in} - S_L h_{in}||_{\mathcal{E}} \leq C e^{-\lambda t} ||h_{in} - \Pi h_{in}||_{\mathcal{E}}.$$  

Let us notice that $f_{in}$ has the same mass as $f_\infty$, it implies that $\Pi h_{in} = 0$. This quantity is conserved by the linearized equation (4). By $S_L \Pi = \Pi S_L$, we deduce that

$$\forall t \geq 0, \quad ||h||_{\mathcal{E}} \leq C e^{-\lambda t} ||h_{in}||_{\mathcal{E}}.$$  

Hence the proof of Theorem 1.1 is completed.

3. **The nonlinear equation: Proof of Theorem 1.2.** This section is devoted to the proof of Theorem 1.2. Our strategy is based on the decay of the semigroup of the linearized equation (4) obtained in previous section. We can see that the linear part is dominated under a suitable norm, then we go back to the fully nonlinear problem. Assume the initial datum is close enough to the equilibrium $f_\infty$, one can
constructed solutions to the nonlinear equation (3) and prove exponential stability with optimal decay rates. We recall that the weight \( m(v) = (1 + \kappa_{\infty} \mu)(v)^k \).

### 3.1. Linearized estimate in a new norm.

**Lemma 3.1.** Consider the space \( L^k_x W^{x,p}_\varepsilon(m) \) with \( k > 0, \ s \in \mathbb{N}, \ p \in [1, +\infty] \). We introduce a dissipative Banach norm

\[
\| h \|_{L^k_x W^{x,p}_\varepsilon(m)} := \eta \| h \|_{L^k_x W^{x,p}_\varepsilon(m)} + \int_0^{+\infty} \| S_\varepsilon(\tau) h \|_{L^k_x W^{x,p}_\varepsilon(m)} d\tau, \quad \eta > 0,
\]

which is equivalent with the original norm \( \| \cdot \|_{L^k_x W^{x,p}_\varepsilon(m)} \).

Then \( \exists \eta > 0, \ \exists \lambda' \in (0, \lambda) \) such that for any \( h_{in} \in L^k_x W^{x,p}_\varepsilon(m) \), the solution \( h(t) := S_\varepsilon(t) h_{in} \) to the linearized equation (4) satisfies:

\[
\forall t \geq 0, \quad \frac{d}{dt} \| h \|_{L^k_x W^{x,p}_\varepsilon(m)} \leq -\lambda' \| h \|_{L^k_x W^{x,p}_\varepsilon(m)}.
\]

**Proof.** From the decay property of the semigroup \( S_\varepsilon(t) \) provided by Theorem 1.1, we have

\[
\forall h \in L^k_x W^{x,p}_\varepsilon(m), \quad \forall t \geq 0, \quad \| S_\varepsilon(t) h \|_{L^k_x W^{x,p}_\varepsilon(m)} \leq C e^{-\lambda t} \| h \|_{L^k_x W^{x,p}_\varepsilon(m)},
\]

here we use that \( \Pi h = \Pi h_{in} = 0 \) due to the mass conservation of the linearized equation (4). Therefore the new norm \( \| \cdot \|_{L^k_x W^{x,p}_\varepsilon(m)} \) is well-defined and we deduce that

\[
\eta \| h \|_{L^k_x W^{x,p}_\varepsilon(m)} \leq \| h \|_{L^k_x W^{x,p}_\varepsilon(m)}
\]

\[
\leq \eta \| h \|_{L^k_x W^{x,p}_\varepsilon(m)} + C \left( \int_0^{+\infty} e^{-\lambda \tau} d\tau \right) \| h \|_{L^k_x W^{x,p}_\varepsilon(m)}
\]

\[
= \left( \eta + \frac{C}{\lambda} \right) \| h \|_{L^k_x W^{x,p}_\varepsilon(m)}.
\]

Thus the norms \( \| \cdot \|_{L^k_x W^{x,p}_\varepsilon(m)} \) and \( \| \cdot \|_{L^k_x W^{x,p}_\varepsilon(m)} \) are equivalent for any \( \eta > 0 \).

Since the \( x \)-derivatives commute with the linearized operator \( \mathcal{L} \), without loss of generality, we can set \( s = 0 \). We consider first \( p \in [1, +\infty) \) and compute the time derivative of the norm

\[
\frac{d}{dt} \| h \|_{L^k_x L^p_x(\varepsilon)} = \frac{d}{dt} \left( \eta \| h \|_{L^k_x L^p_x(\varepsilon)} + \int_0^{+\infty} \| S_\varepsilon(\tau) h \|_{L^k_x L^p_x(\varepsilon)} d\tau \right)
\]

\[
= \eta \int_{\mathbb{R}^d} \| h \|_{L^p_x}^{-1-p} \left( \int_{\mathbb{R}^d} |h|^{p-1} \text{sign}(h) \partial_t h \, dx \right) m \, dv
\]

\[
+ \int_0^{+\infty} \frac{\partial}{\partial t} \| S_\varepsilon(\tau) h \|_{L^k_x L^p_x(\varepsilon)} d\tau
\]

\[
= \eta \int_{\mathbb{R}^d} \| h \|_{L^p_x}^{-1-p} \left( \int_{\mathbb{R}^d} |h|^{p-1} \text{sign}(h) \mathcal{L}(h) \, dx \right) m \, dv
\]

\[
+ \int_0^{+\infty} \frac{\partial}{\partial t} \| S_\varepsilon(\tau) h \|_{L^k_x L^p_x(\varepsilon)} d\tau
\]

\[
= : I_1 + I_2.
\]
Concerning the first term $I_1$ and arguing as in the proof of Lemma 2.2 and Lemma 2.4, we have

$$I_1 = \eta \int_{\mathbb{R}^d} ||h||_{L^p_x} \left( \int_{\mathbb{R}^d} |h|^{p-1} \text{sign}(h) (A(h) + B(h)) \, dx \right) \, dv$$

$$\leq \eta \left( \int_{\mathbb{R}^d} ||h||_{L^p_x} \left( \int_{\mathbb{R}^d} |h|^{p-1} \text{sign}(h) \, dx \right) \, dv + ||A(h)||_{L^1_x L^p_v(m)} \right)$$

$$\leq \eta \left( -\lambda_0 ||h||_{L^1_x L^p_v(m)} + C ||h||_{L^1_x L^p_v(m)} \right) \quad (17)$$

for some constants $\lambda_0$, $C > 0$.

The second term $I_2$ is computed directly as

$$I_2 = \int_0^{+\infty} \frac{\partial}{\partial t} ||S_{\mathcal{L}}(\tau) h||_{L^1_x L^p_v(m)} \, d\tau = \int_0^{+\infty} \frac{\partial}{\partial \tau} ||S_{\mathcal{L}}(\tau) h||_{L^1_x L^p_v(m)} \, d\tau$$

$$= -||h||_{L^1_x L^p_v(m)}. \quad (18)$$

The combination of the two inequalities (17) and (18) yields the desired result

$$\frac{d}{dt} ||h||_{L^1_x L^p_v(m)} \leq \eta \left( -\lambda_0 ||h||_{L^1_x L^p_v(m)} + C ||h||_{L^1_x L^p_v(m)} \right) - ||h||_{L^1_x L^p_v(m)}$$

$$\leq -K ||h||_{L^1_x L^p_v(m)}$$

$$\leq -\frac{\lambda K}{\lambda \eta + C} ||h||_{L^1_x L^p_v(m)}$$

with $K > 0$, by choosing $\eta$ small enough. We define $\lambda' := \frac{\lambda K}{\lambda \eta + C}$, by taking $C > 0$ big enough, then $\lambda' \in (0, \lambda)$.

Then the case $p = +\infty$ is obtained by passing to the limit.

Next, thanks to the $x$-derivatives commute with the linearized equation (4). Arguing as before, we can also obtain that

$$\frac{d}{dt} ||\partial_x^2 h||_{L^1_x L^p_v(m)} \leq -\lambda' ||\partial_x^2 h||_{L^1_x L^p_v(m)} \quad \forall |\beta| \leq s.$$ 

Finally, we deduced that

$$\frac{d}{dt} ||h||_{L^1_x W^{s,p}_x(m)} \leq -\lambda' ||h||_{L^1_x W^{s,p}_x(m)}.$$ 

$\square$

### 3.2. Bilinear estimates.

We establish bilinear estimates on the nonlinear term in equation (3).

**Lemma 3.2.** Consider the space $L^q_x W^{s,p}_x(m)$, $k > 0$, $s \in \mathbb{N}$, $q \in [1, +\infty]$, $s > \frac{2d}{p}$ when $p \in [1, +\infty)$ or $s \geq 0$ when $p = +\infty$. We define the symmetric form as

$$Q(f, g)(v) := \frac{1}{2K} \int_{\mathbb{R}^d} (\mu(v) - \mu(v_*) \left[ g(v)f(v_*) + g(v_*)f(v) \right] \, dv_*$$

then we have

$$||Q(f, g)||_{L^q_x W^{s,p}_x(m)} \leq C \left[ \|g\|_{L^q_x W^{s,p}_x(m)} \|f\|_{L^q_x W^{s,p}_x(m)} + \|g\|_{L^1_x W^{s+1,p}_x(m)} \|f\|_{L^q_x W^{s,p}_x(m)} \right].$$
Proof. If $q$ is finite, using the Minkowski’s integral inequality, we first easily compute that

$$\|Q(f, g)\|_{L^q_t(m)} = \frac{1}{2K} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\mu(v) - \mu(v_\star)) [g(v)f(v_\star) + g(v_\star)f(v)] \, dv \|s^q m^q \, dv \right)^{1\over q}$$

$$\leq C \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |g(v)|^q |f(v_\star)|^q m^q \, dv \right)^{1\over q} \, dv_\star$$

$$+ C \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |g(v_\star)|^q |f(v)|^q m^q \, dv \right)^{1\over q} \, dv$$

$$\leq C \left[ \|g\|_{L^q_t(m)} \|f\|_{L^q_t} + \|g\|_{L^q_t} \|f\|_{L^q_t(m)} \right].$$

The above result is obvious if $q = +\infty$.

Next, we consider the position variable of the norm, thanks to the distributive property

$$\partial_x Q(f, g) = Q(\partial_x f, g) + Q(f, \partial_x g).$$

The case $p \in [1, +\infty)$, we use the Sobolev embedding $W^{2,p}_x(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$ with continuous embedding since $s > \frac{2d}{p}$. We can also check that the case $p = +\infty$ is obvious.

3.3. A priori stability estimate. We obtain the following uniform in time a priori estimate.

**Lemma 3.3.** Consider the space $L^1_t W^{s,p}_x(m)$, $k > 0$, $s \in \mathbb{N}$, $s > \frac{2d}{p}$ when $p \in [1, +\infty)$ or $s \geq 0$ when $p = +\infty$. Consider a solution $h \in L^1_t W^{s,p}_x(m)$ to the nonlinear equation (3),

$$\partial_t h = \mathcal{L}(h) + Q(h, h), \quad h_{in} = f_{in} - f_\infty,$$

then $h$ satisfies the estimate

$$\forall t \geq 0, \quad \frac{d}{dt} \|h\|_{L^1_t W^{s,p}_x(m)} \leq \left( C \|h\|_{L^1_t W^{s,p}_x(m)} - \lambda' \right) \|h\|_{L^1_t W^{s,p}_x(m)}$$

for some constants $C > 0$ and $\lambda' \in (0, \lambda)$ given by Lemma 3.1.

**Proof.** Assume first $s = 0$ and $p \in [1, +\infty)$. We calculate

$$\frac{d}{dt} \|h\|_{L^1_t L^q_x(m)} = \eta \int_{\mathbb{R}^d} \|h\|^{1-p}_{L^p_x} \left( \int_{\mathbb{T}^d} |h|^{p-1} \text{sign}(h) \partial_t h \, dx \right) \, m \, dv$$

$$+ \int_0^{+\infty} \int_{\mathbb{R}^d} \|S_{\mathcal{L}}(\tau) h\|^{1-p}_{L^p_x} \left( \int_{\mathbb{T}^d} |S_{\mathcal{L}}(\tau) h|^{p-1} \text{sign}(S_{\mathcal{L}}(\tau) h) \right.$$ 

$$\times \partial_t (S_{\mathcal{L}}(\tau) h) \, dx \right) \, m \, dv \, d\tau$$

$$= \eta \int_{\mathbb{R}^d} \|h\|^{1-p}_{L^p_x} \left( \int_{\mathbb{T}^d} |h|^{p-1} \text{sign}(h) (\mathcal{L}(h) + Q(h, h)) \, dx \right) \, m \, dv$$

$$+ \int_0^{+\infty} \int_{\mathbb{R}^d} \|S_{\mathcal{L}}(\tau) h\|^{1-p}_{L^p_x} \left( \int_{\mathbb{T}^d} |S_{\mathcal{L}}(\tau) h|^{p-1} \text{sign}(S_{\mathcal{L}}(\tau) h) \times S_{\mathcal{L}}(\tau) (\mathcal{L}(h) + Q(h, h)) \, dx \right) \, m \, dv \, d\tau$$

$$=: I_1 + I_2,$$
Now, we take the limit and we assume

\[ |h|_{L^p}^{1-p} \left( \int_{\mathbb{R}^d} |h|^{p-1} \text{sign}(h) \mathcal{L}(h) \, dx \right) \]

with

\[ I_1 = \eta \int_{\mathbb{R}^d} |h|_{L^p}^{1-p} \left( \int_{\mathbb{R}^d} |h|^{p-1} \text{sign}(h) \mathcal{L}(h) \, dx \right) \, m \, dv \]

and

\[ + \int_0^{+\infty} \int_{\mathbb{R}^d} \left| S_{\mathcal{L}}(\tau) h \right|_{L^p}^{1-p} \left( \int_{\mathbb{R}^d} \left| S_{\mathcal{L}}(\tau) h \right|^{p-1} \text{sign}(S_{\mathcal{L}}(\tau) h) \times S_{\mathcal{L}}(\tau) (\mathcal{L}(h)) \, dx \right) \, m \, dv \, d\tau, \]

\[ I_2 = \eta \int_{\mathbb{R}^d} |h|_{L^p}^{1-p} \left( \int_{\mathbb{R}^d} |h|^{p-1} \text{sign}(h) Q(h, h) \, dx \right) \, m \, dv \]

and

\[ + \int_0^{+\infty} \int_{\mathbb{R}^d} \left| S_{\mathcal{L}}(\tau) h \right|_{L^p}^{1-p} \left( \int_{\mathbb{R}^d} \left| S_{\mathcal{L}}(\tau) h \right|^{p-1} \text{sign}(S_{\mathcal{L}}(\tau) h) \times S_{\mathcal{L}}(\tau) (Q(h, h)) \, dx \right) \, m \, dv \, d\tau. \]

In Lemma 3.1, by choosing \( \eta > 0 \) small enough, it holds

\[ I_1 \leq -\lambda' |||h|||_{L^1_t L^p_x(m)}, \]

where \( \lambda' \in (0, \lambda) \).

For the second term, using the Hölder’s inequality and Lemma 3.2, we obtain

\[ I_2 \leq \eta \|Q(h, h)\|_{L^1_t L^p_x(m)} + \int_0^{+\infty} \|S_{\mathcal{L}}(\tau) (Q(h, h))\|_{L^1_t L^p_x(m)} \, d\tau \]

\[ = \|Q(h, h)\|_{L^1_t L^p_x(m)} \]

\[ \leq C \left( \eta + \frac{C}{\lambda} \right) \frac{2}{\eta^2} |||h|||_{L^1_t L^p_x(m)} |||h|||_{L^1_t L^p_x(m)}. \]

Therefore, we deduce that

\[ \frac{d}{dt} |||h|||_{L^1_t L^p_x(m)} \leq \left( C |||h|||_{L^1_t L^p_x(m)} - \lambda' \right) |||h|||_{L^1_t L^p_x(m)}. \]

Now, we take the limit \( p \to +\infty \). This concludes the proof in the case \( s = 0 \) when \( p = +\infty \). In the case \( s > \frac{2d}{p} \) when \( p \in [1, +\infty) \), one uses the distributive property of the \( x \)-derivative and Sobolev embeddings.

3.4. Proof of Theorem 1.2.

Proof of Theorem 1.2. We shall construct solutions by considering the following iterative scheme

\[ \partial_t h^n = \mathcal{L}(h^n) + Q(h^n, h^{n-1}), \quad n \geq 1 \]

with the initialization

\[ \partial_t h^0 = \mathcal{L}(h^0), \quad h^0(0) = h_{in}, \]

and we assume \( |||h_{in}|||_{L^1_t W^{s,p}_x(m)} \leq \frac{\varepsilon}{2} \).

The sequence of functions \( \{h^n\}, n \geq 0 \), is well-defined in \( L^1_t W^{s,p}_x(m) \) for all times \( t \geq 0 \) thanks to the study of the semigroup in Theorem 1.1 and the stability estimates proven below.

We split the proof of Theorem 1.2 into four steps. The first two steps of the proof establish the stability and convergence of the iterative scheme, and they are mainly an elaboration upon the key a priori estimate of the previous results. The third step proves the uniqueness of the solution. The fourth step consists of a
bootstrap argument in order to recover the optimal decay rate of the semigroup for the linearized equation.

**Step 1: Stability of the iterative scheme**

We can prove by induction the following control

\[ \forall n \geq 0, \quad \sup_{t \geq 0} \left( \| h^n \|_{L^p \to L^p} + \lambda' \int_0^t \| h^n \|_{L^p \to L^p} \, d\tau \right) \leq \epsilon \]  

(19)

under a smallness condition on \( \epsilon \).

The case \( n = 0 \) follows from Lemma 3.1, that is

\[ \forall t \geq 0, \quad \frac{d}{dt} ||| h^0 |||_{L^p \to L^p} \leq -\lambda' ||| h^0 |||_{L^p \to L^p}. \]

An integration of the above differential inequality from 0 to \( t \) and the fact that \( ||| h_{in} |||_{L^p \to L^p} \leq \frac{\epsilon}{\lambda'} \) imply that

\[ \forall t \geq 0, \quad ||| h^0 |||_{L^p \to L^p} + \lambda' \int_0^t ||| h^0 |||_{L^p \to L^p} \, d\tau \leq ||| h_m |||_{L^p \to L^p} \leq \frac{\epsilon}{\lambda'}. \]

Let us now assume that (19) is satisfied at rank \( N \) and let us prove it for \( n = N + 1 \).

A similar computation as in the proof of Lemma 3.3 implies

\[ \frac{d}{dt} ||| h^{N+1} |||_{L^p \to L^p} \leq -\lambda' ||| h^{N+1} |||_{L^p \to L^p} + C ||| Q(h^{N+1}, h^N) |||_{L^p \to L^p} \]

\[ \leq -\lambda' ||| h^{N+1} |||_{L^p \to L^p} + C ||| h^{N+1} |||_{L^p \to L^p} \]

\[ + C ||| h^N |||_{L^p \to L^p} ||| h^N |||_{L^p \to L^p} \]

for some constant \( C > 0 \).

Hence we get

\[ ||| h^{N+1} |||_{L^p \to L^p} + \lambda' \int_0^t ||| h^{N+1} |||_{L^p \to L^p} \, d\tau \leq ||| h_{in} |||_{L^p \to L^p} + C \int_0^t ||| h^{N+1} |||_{L^p \to L^p} \, d\tau \]

\[ \leq \epsilon + C \sup_{t \geq 0} (||| h^N |||_{L^p \to L^p}) \int_0^t \| h^{N+1} \|_{L^p \to L^p} \, d\tau \]

\[ \leq \frac{\epsilon}{2} + C \epsilon \int_0^t ||| h^{N+1} |||_{L^p \to L^p} \, d\tau, \]

from which it follows

\[ \forall t \geq 0, \quad \sup_{t \geq 0} \left( \| h^n \|_{L^p \to L^p} + \lambda' \int_0^t \| h^n \|_{L^p \to L^p} \, d\tau \right) \leq \epsilon, \]

as soon as \( \epsilon < \frac{\lambda'}{2C}. \)

The induction is proven.

**Step 2: Convergence of the scheme**

Let us now denote by \( d^n := h^{n+1} - h^n \). It satisfies for all \( n \geq 1 \)

\[ \partial_t d^n = \mathcal{L}(d^n) + Q(d^n, h^n) + Q(h^n, d^{n-1}), \]

and

\[ \partial_t d^0 = \mathcal{L}(d^0) + Q(h^1, h^0). \]

So, \( d^n(0) = 0 \) for any \( n \geq 0 \).
Let us define by

\[ A_n(t) := \sup_{0 \leq \tau \leq t} \left( \|d^\rho\|_{L^1_t W_x^s p(m)} + \lambda \int_0^\tau \|d^\rho\|_{L^1_t W_x^s p(m)} \, ds \right), \]

and let us prove by induction that

\[ \forall t \geq 0, \quad \forall n \geq 0, \quad A_n(t) \leq (C \epsilon)^n \]

for some constant \( C > 0 \) uniformly as \( \epsilon \) goes to zero.

The case \( n = 0 \) is obtained by using the estimate

\[
\frac{d}{dt} \|d^0\|_{L^1_t W_x^s p(m)} \leq -\lambda' \|d^0\|_{L^1_t W_x^s p(m)} + C \|Q(h^0)\|_{L^1_t W_x^s p(m)}
\]

\[
\leq -\lambda' \|d^0\|_{L^1_t W_x^s p(m)} + C \|h^0\|_{L^1_t W_x^s p(m)} \|d^0\|_{L^1_t W_x^s p(m)}.
\]

Then

\[
\|d^0\|_{L^1_t W_x^s p(m)} + \lambda' \int_0^t \|d^0\|_{L^1_t W_x^s p(m)} \, d\tau \leq C \int_0^t \|h^1\|_{L^1_t W_x^s p(m)} \|h^0\|_{L^1_t W_x^s p(m)} \, d\tau \leq C \sup_{t \geq 0} (\|h^0\|_{L^1_t W_x^s p(m)}) \int_0^t \|h^1\|_{L^1_t W_x^s p(m)} \, d\tau \leq C \epsilon^2 \leq \epsilon
\]

for \( \epsilon \) small enough.

Similarly as before, the propagation of the induction is obtained by using the induction assumption on \( A_N(t) \):

\[
\frac{d}{dt} \|d^{N+1}\|_{L^1_t W_x^s p(m)} \leq -\lambda' \|d^{N+1}\|_{L^1_t W_x^s p(m)} + C \|Q(d^{N+1}, h^{N+1})\|_{L^1_t W_x^s p(m)}
\]

\[
+ C \|Q(h^{N+1}, d^N)\|_{L^1_t W_x^s p(m)} \leq -\lambda' \|d^{N+1}\|_{L^1_t W_x^s p(m)} + C \|d^{N+1}\|_{L^1_t W_x^s p(m)} \|h^{N+1}\|_{L^1_t W_x^s p(m)}
\]

\[
+ C \|h^{N+1}\|_{L^1_t W_x^s p(m)} \|d^N\|_{L^1_t W_x^s p(m)}.
\]
Thus,
\[
\|d^{N+1}\|_{L^1_tW^{s,p}_x(m)} + \lambda' \int_0^t \|d^{N+1}\|_{L^1_tW^{s,p}_x(m)} \, dt \\
\leq C \int_0^t \|h^{N+1}\|_{L^1_tW^{s,p}_x(m)} \|h^{N+1}\|_{L^1_tW^{s,p}_x(m)} \, dt \\
+ C \int_0^t \|h^{N+1}\|_{L^1_tW^{s,p}_x(m)} \|d^N\|_{L^1_tW^{s,p}_x(m)} \, dt \\
\leq C \sup_{t \geq 0} (\|h^{N+1}\|_{L^1_tW^{s,p}_x(m)} \int_0^t \|d^N\|_{L^1_tW^{s,p}_x(m)} \, dt) \\
+ C \sup_{t \geq 0} (\|h^{N+1}\|_{L^1_tW^{s,p}_x(m)} \int_0^t \|d^N\|_{L^1_tW^{s,p}_x(m)} \, dt) \\
\leq C \epsilon \int_0^t \|d^{N+1}\|_{L^1_tW^{s,p}_x(m)} \, dt + C \epsilon \cdot \epsilon^N.
\]

If \( \epsilon \) is small enough so that \( \epsilon < \frac{\lambda'}{2C} \), we deduce that
\[
A_{N+1}(t) = \|d^{N+1}\|_{L^1_tW^{s,p}_x(m)} + \lambda' \int_0^t \|d^{N+1}\|_{L^1_tW^{s,p}_x(m)} \, dt \leq (C \epsilon)^{N+1}
\]
for some constant \( C > 0 \), which concludes the induction.

Hence for \( \epsilon \) small enough, the series \( \sum_{n \geq 0} A_n(t) \) is summable for any \( t \geq 0 \). It suffices to show that the sequence of functions \( \{h^n\} \) has the Cauchy property in \( L^\infty_t \left( [0, +\infty), L^1_tW^{s,p}_x(m) \right) \) and \( L^1_t \left( [0, +\infty), L^1_tW^{s,p}_x(m) \right) \), which proves the convergence of the iterative scheme. The limit \( h \) as \( n \) goes to infinity satisfies the equation in the strong sense when the norm \( L^1_tW^{s,p}_x(m) \) involves enough derivatives, or else in the weak sense.

**Step 3: Uniqueness**

We now consider the solution \( h \) constructed so far. From Step 1 we take the limit \( n \rightarrow +\infty \) in the stability estimates and get
\[
\sup_{t \geq 0} \left( \|h\|_{L^1_tW^{s,p}_x(m)} + \lambda' \int_0^t \|h\|_{L^1_tW^{s,p}_x(m)} \, dt \right) \leq \epsilon.
\]

Let \( g \) is another solution to the problem (3) with the same initial data \( g(0, x, v) = h_{in} \), which then satisfies
\[
\sup_{t \geq 0} \left( \|g\|_{L^1_tW^{s,p}_x(m)} + \lambda' \int_0^t \|g\|_{L^1_tW^{s,p}_x(m)} \, dt \right) \leq \epsilon.
\]

The difference \( h - g \) satisfies
\[
\partial_t (h - g) = \mathcal{L}(h - g) + Q(h - g, h) + Q(g, h - g)
\]
with \( g(0) = h(0) = h_{in} \).

We then compute
\[
\frac{d}{dt} \|h - g\|_{L^1_tW^{s,p}_x(m)} \\
\leq - \lambda' \|h - g\|_{L^1_tW^{s,p}_x(m)} + C \|h - g\|_{L^1_tW^{s,p}_x(m)} \|h\|_{L^1_tW^{s,p}_x(m)} \\
+ C \|h - g\|_{L^1_tW^{s,p}_x(m)} \|g\|_{L^1_tW^{s,p}_x(m)}.
\]
Whence, integrating in time implies
\[
\|h - g\|_{L^p_t W^s_x(m)} + \lambda' \int_0^t \|h - g\|_{L^p_t W^s_x(m)} \, d\tau \\
\leq C \int_0^t \left( \|h - g\|_{L^p_t W^s_x(m)} + \|h - g\|_{L^p_t W^s_x(m)} \|g\|_{L^p_t W^s_x(m)} \right) \, d\tau \\
\leq C \left( \sup_{t \geq 0} \|h\|_{L^p_t W^s_x(m)} + \sup_{t \geq 0} \|g\|_{L^p_t W^s_x(m)} \right) \int_0^t \|h - g\|_{L^p_t W^s_x(m)} \, d\tau \\
\leq C \epsilon^2.
\]

We conclude the proof of uniqueness for \( \epsilon > 0 \) small enough.

**Step 4: Optimal rate of decay**

We can apply Lemma 3.3 to the solution \( h \):
\[
\frac{d}{dt} \|h\|_{L^p_t W^s_x(m)} \leq \left( C \|h\|_{L^p_t W^s_x(m)} - \lambda' \right) \|h\|_{L^p_t W^s_x(m)} \\
\leq (C \epsilon - \lambda') \|h\|_{L^p_t W^s_x(m)} \\
\leq - \frac{\lambda'}{2} \|h\|_{L^p_t W^s_x(m)},
\]
under the smallness condition \( \epsilon < \frac{\lambda'}{2C} \).

Employing the Gronwall’s inequality, we get that
\[
\|h\|_{L^p_t W^s_x(m)} \leq e^{-\frac{\lambda'}{2} t} \|h_{in}\|_{L^p_t W^s_x(m)}.
\]

Since \( \|h\|_{L^p_t W^s_x(m)} \to 0 \) as \( t \to +\infty \), we integrate the previous a priori estimate (20) from \( t \) to \( +\infty \) to get
\[
\frac{\lambda'}{2} \int_t^{+\infty} \|h\|_{L^p_t W^s_x(m)} \, d\tau \leq e^{-\frac{\lambda'}{2} t} \|h_{in}\|_{L^p_t W^s_x(m)} \leq e^{-\frac{\lambda'}{2} t} \|h_{in}\|_{L^p_t W^s_x(m)},
\]
which implies
\[
\int_t^{+\infty} \|h\|_{L^p_t W^s_x(m)} \, d\tau \leq \frac{2}{\lambda'} \|h_{in}\|_{L^p_t W^s_x(m)} \leq e^{-\frac{\lambda'}{2} t} \|h_{in}\|_{L^p_t W^s_x(m)}.
\]

Due to the norms \( \| \cdot \|_{L^p_t W^s_x(m)} \) and \( \| \cdot \|_{L^p_t W^s_x(m)} \) are equivalent, then we have
\[
\int_t^{+\infty} \|h\|_{L^p_t W^s_x(m)} \, d\tau \leq C e^{-\frac{\lambda'}{2} t} \|h_{in}\|_{L^p_t W^s_x(m)}
\]
for some constant \( C > 0 \).

Inspired by [13], we shall use a bootstrap argument in order to ensure that the solution \( h \) enjoys to the same optimal decay rate as the linearized semigroup in Theorem 1.1. Assume that the solution is known to decay as
\[
\|h\|_{L^p_t W^s_x(m)} \leq C e^{-\lambda_0 t} \|h_{in}\|_{L^p_t W^s_x(m)},
\]
where \( \lambda_0 = \frac{\lambda'}{2} \in (0, \lambda) \), \( C > 0 \).

Now, we assert that it indeed decays faster like
\[
\|h\|_{L^p_t W^s_x(m)} \leq C e^{-\lambda_1 t} \|h_{in}\|_{L^p_t W^s_x(m)}
\]
with \( \lambda_1 = \min\{\lambda_0 + \frac{\lambda'}{4}, \lambda\} \).

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We use Duhamel’s formula for the solution of the problem (3):

\[ h = S_L(t)h_{in} + \int_0^t S_L(t - \tau)Q(h, h) \, d\tau. \]

We go back to the original norm and we deduce from Theorem 1.1 and Lemma 3.2 (noticing that \( IIQ(h, h) = 0 \)) that

\[
\|h\|_{L_t^1 W_x^{s, p}(m)} \leq C e^{-\lambda t} \|h_{in}\|_{L_t^1 W_x^{s, p}(m)} + C \int_0^t e^{-\lambda(t - \tau)} \|Q(h, h)\|_{L_t^1 W_x^{s, p}(m)} \, d\tau \\
\leq C e^{-\lambda t} \|h_{in}\|_{L_t^1 W_x^{s, p}(m)} + C \int_0^t e^{-\lambda(t - \tau)} \|h\|^2_{L_t^1 W_x^{s, p}(m)} \, d\tau. \tag{21}
\]

We simply estimate

\[
\int_0^t e^{-\lambda(t - \tau)} \|h\|^2_{L_t^1 W_x^{s, p}(m)} \, d\tau \\
\leq C \int_0^t e^{-\lambda_1(t - \tau)} \|h\|^2_{L_t^1 W_x^{s, p}(m)} \, d\tau \\
\leq C e^{-\lambda_1 t} \left( \int_0^t e^{(\lambda_1 - \lambda_0)\tau} \|h\|_{L_t^1 W_x^{s, p}(m)} \, d\tau \right) \|h_{in}\|_{L_t^1 W_x^{s, p}(m)} \tag{22}
\]

and then by integration by parts, we have

\[
\int_0^t e^{(\lambda_1 - \lambda_0)\tau} \|h\|_{L_t^1 W_x^{s, p}(m)} \, d\tau \\
= \int_0^t e^{(\lambda_1 - \lambda_0)\tau} \left( - \int_\tau^t \|h\|_{L_t^1 W_x^{s, p}(m)} \, d\tau \right) \\
= \int_0^t \|h\|_{L_t^1 W_x^{s, p}(m)} \, d\tau + (\lambda_1 - \lambda_0) \int_0^t e^{(\lambda_1 - \lambda_0)\tau} \left( \int_\tau^t \|h\|_{L_t^1 W_x^{s, p}(m)} \, d\tau \right) \, d\tau \\
\leq C \left( \int_0^t e^{-\lambda_0 t} \, d\tau \right) \|h_{in}\|_{L_t^1 W_x^{s, p}(m)} \\
+ (\lambda_1 - \lambda_0) \int_0^t e^{(\lambda_1 - \lambda_0)\tau} \left( \int_\tau^t \|h\|_{L_t^1 W_x^{s, p}(m)} \, d\tau \right) \, d\tau \\
\leq C \|h_{in}\|_{L_t^1 W_x^{s, p}(m)} + C (\lambda_1 - \lambda_0) \left( \int_0^t e^{(\lambda_1 - \lambda_0 - \lambda\|)\tau} \, d\tau \right) \|h_{in}\|_{L_t^1 W_x^{s, p}(m)} \\
\leq C \|h_{in}\|_{L_t^1 W_x^{s, p}(m)} \tag{23}
\]

for some constant \( C > 0 \).

Substituting (22) and (23) into (21), we deduce

\[
\|h\|_{L_t^1 W_x^{s, p}(m)} \leq C e^{-\lambda t} \|h_{in}\|_{L_t^1 W_x^{s, p}(m)} + C e^{-\lambda_1 t} \|h_{in}\|^2_{L_t^1 W_x^{s, p}(m)} \\
\leq C e^{-\lambda t} \|h_{in}\|_{L_t^1 W_x^{s, p}(m)}. 
\]

This proves the claim.

Next we can apply the same argument on \( \lambda_2 = \min\{\lambda_1 + \frac{\lambda}{4}, \lambda\} \), etc. Hence in a finite number of steps, it proves the desired optimal decay rate:

\[
\|h\|_{L_t^1 W_x^{s, p}(m)} \leq C e^{-\lambda t} \|h_{in}\|_{L_t^1 W_x^{s, p}(m)}. 
\]

Thus we finish the proof of Theorem 1.2. \( \square \)
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