Coherent states for a 2-sphere with a magnetic field

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Abstract
We consider a particle moving on a 2-sphere in the presence of a constant magnetic field. Building on our earlier work in the nonmagnetic case we construct coherent states for this system. The coherent states are labeled by points in the associated phase space, the (co)tangent bundle of $\mathbb{S}^2$. They are constructed as eigenvectors for certain annihilation operators and expressed in terms of a certain heat kernel. These coherent states are not of Perelomov type but rather are constructed according to the ‘complexifier’ approach of Thiemann. We describe the Segal–Bargmann representation associated with the coherent states which is equivalent to a resolution of the identity.

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1. Introduction

In [1], Hall introduces a unitary Segal–Bargmann transform for the group manifold of an arbitrary compact Lie group, mapping to an $L^2$-space of holomorphic functions on the associated complex group. (See also [2] for a survey of related results, [3, 4] for connections to the quantization of $(1+1)$-dimensional Yang–Mills theory, and [5] for connections to geometric quantization.) The transform consists of integrating the position wavefunction against certain coherent states, which are expressed in terms of the heat kernel on the group. In section 11 of [1], this transform is extended to compact symmetric spaces, such as the $d$-sphere $\mathbb{S}^d$. Stenzel [6] has given a particularly nice description of the transform for symmetric spaces, a description that brings out the role of the heat kernel for the dual noncompact symmetric space. The unitarity of the Segal–Bargmann transform can be expressed, in typical physics terminology, as a resolution of the identity for the associated coherent states. Work has also been done on noncompact symmetric spaces [7–12], but the situation there is much more complicated. (See also [13] for a Segal–Bargmann transform for the Heisenberg group.)
In a previous paper [14], we considered coherent states for a particle moving in a \(d\)-dimensional sphere. This means that we regard \(S^d\) as the configuration space of our system, with the associated phase space then being the cotangent bundle \(T^*S^d\), which may be identified with the tangent bundle \(TS^d\). (It is possible to regard the 2-sphere \(S^2\) as the phase space of a classical system, but that is a completely different problem.) Although the results about coherent states on \(S^d\) are in principle special cases of results of Hall and Stenzel, we gave a self-contained and substantially different treatment of the subject. In particular, we brought in the ‘complexifier’ method of Thiemann [15] and the ‘polar decomposition’ method of Kowalski and Rembieliński [16]. (The coherent states in [16] were constructed independently, without any knowledge of the work of Hall or Stenzel.) We gave an elementary proof of the resolution of the identity for the coherent states on \(S^d\) (compare [17] in the two-dimensional case), showing very concretely how the heat equation on the dual noncompact symmetric space, namely \(d\)-dimensional hyperbolic space, arises. We have also shown [18] that when \(d\) is odd, the coherent states we construct converge to the usual Gaussian coherent states on \(\mathbb{R}^d\) in the limit as the radius of the sphere tends to infinity. (The same result is expected to hold in the even-dimensional case.) In the case of \(S^3 = SU(2)\), many detailed properties of the coherent states were worked out in [19], with applications to quantum gravity.

In this paper, we consider a charged particle moving in \(S^2\) in the presence of a magnetic field of constant magnitude \(B\), pointing in the direction perpendicular to the sphere. (If we think of our particle as a three-dimensional particle that is constrained to move on \(S^2\), then the magnetic field may be thought of as coming from a magnetic monopole at the origin.) Since, as we will see, the quantum Hilbert space for such a particle is not the same as in the nonmagnetic case, the coherent states will necessarily have to be modified.

In section 3, we use the ‘complexifier’ method of Thiemann to construct a diffeomorphism \(a\) between the phase space \(TS^2\) and the complex sphere

\[ S^2_C = \{ \mathbf{a} \in \mathbb{C}^3 | a_1^2 + a_2^2 + a_3^2 = r^2 \} . \]

When the magnetic field strength \(B\) is set equal to zero, this diffeomorphism reduces to the one considered in [14]. In section 4, we then use the quantum version of the complexifier method to construct annihilation operators \(A_k\) satisfying \(A_k^2 + A_k^2 + A_k^2 = r^2\). In section 5, we construct our coherent states as simultaneous eigenvectors for the \(A_k\)'s. In the position representation, the coherent states can be expressed in terms of the heat kernel for a certain line bundle over \(S^2\).

We then turn, in section 6, to the construction of a (Segal–)Bargmann representation for the quantum Hilbert space, which is equivalent to a resolution of the identity. The density used in the definition of the Segal–Bargmann space is again a sort of bundle heat kernel, which may be constructed by the method of ‘reduction to the group case’. Once the Segal–Bargmann space has been constructed, we describe a unitary Segal–Bargmann transform between the position Hilbert space and the Segal–Bargmann space. This transform consists of applying the bundle heat operator to a section over the real sphere \(S^2\) and then analytically continuing to the complex sphere \(S^2_C\).

When the magnetic field is zero, the complex structure we get on \(TS^2\) by identifying it with \(S^2_C\) coincides with the ‘adapted complex structure’ on \(TS^2\), as introduced independently by Lempert–Szőke [20, 21] and Guillemin–Stenzel [22, 23]. Meanwhile, in [24], the construction of the adapted complex structure is interpreted in terms of the ‘imaginary-time geodesic flow’, following the complexifier approach of Thiemann. (‘Time’ here should not be understood as physical time but simply as the parameter in a flow.) More recently, Hall and Kirwin have introduced a ‘magnetic’ version of adapted complex structure [25]. In the case of a constant magnetic field on \(S^2\), the complex structure on \(TS^2\) given by the method of [25] (see section 5
of [25]) coincides with the complex structure obtained by identifying $TS^2$ with $S^2_C$ by means of the diffeomorphism $a$.

Finally, we note that if we apply the complexifier method for a particle moving in the plane in a constant magnetic field, we will obtain coherent states that are expressible in terms of the heat kernel (i.e. imaginary-time propagator) for the quantum Hamiltonian. Such coherent states do not agree with the coherent states introduced by Malkin and Man’ko in [26], nor do they agree with the coherent states introduced by Kowalski and Rembieliński in [27]. In particular, the ‘complexifier’ coherent states will not be stable under the time evolution of the system. On the other hand, the complexifier coherent states will pass over smoothly to the usual minimum-uncertainty Gaussian coherent states as the magnetic field strength tends to zero, something that seemingly cannot be true for any coherent states that are temporally stable. After all, when the magnetic field strength is zero, one should not expect temporally stable coherent states, because of the phenomenon of the spreading of the wave packet. The complexifier coherent states on the plane do have an associated Segal–Bargmann representation, which is described in section 4.2 of [13], with the parameter $\lambda$ in [13] is to be identified with the magnetic field strength. (See section 4.2 of [25] for an explicit connection between section 4.2 of [13] and the complexifier method.)

2. The classical mechanics of a particle in a magnetic field

2.1. The $\mathbb{R}^n$ case

We wish to give a Hamiltonian description of the motion of a charged particle in $\mathbb{R}^n$ in the presence of a time-independent magnetic field, described by a skew-symmetric matrix $B_{ijk}$. Since we are dealing with a single charged particle, we can incorporate the charge of the particle into the definition of the magnetic field. The condition $\nabla \cdot B = 0$ in $\mathbb{R}^3$ becomes the condition that the 2-form $(1/2)B_{ijkl} \, dx^i \wedge dx^j$ should be closed, or, equivalently, that

$$\frac{\partial B_{ik}}{\partial x_j} + \frac{\partial B_{jk}}{\partial x_i} + \frac{\partial B_{ij}}{\partial x_k} = 0$$

for all $i, j, k, l$. It is desirable to formulate the theory in $\mathbb{R}^n$ in a way that makes no reference to the vector potential, since in the sphere case there will be no globally defined vector potential. We consider, then, position variables $x_j$ and ‘kinetic’ momentum variables $p_j$ along with a Poisson bracket defined by

$$\{f, g\}_B = \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial x_i} + B_{ijk}(x) \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial p_k}$$

(sum convention). In particular, the relations among our position and momentum variables are

$$\{x_j, x_k\} = 0$$
$$\{x_j, p_k\} = \delta_{jk}$$
$$\{p_j, p_k\} = B_{jk}(x).$$

We then introduce a Hamiltonian $H$ by

$$H(x, p) = \frac{p^2}{2m}.$$  

(3)

The equations of motion are computed by using the general formula $df/dt = \{f, H\}$. Specializing to $f = x_j$ and to $f = p_j$ gives

$$\frac{dx_j}{dt} = \frac{p_j}{m},$$
$$\frac{dp_j}{dt} = \frac{1}{m} B_{jk}(x) p_k.$$  

(4)
Note that \( p_j = m \frac{dx_j}{dt} \); this relation accounts for the terminology ‘kinetic momentum’. Note also that in this approach, the magnetic field enters only into the Poisson bracket relations (2) and not into the Hamiltonian (3). In the case \( n = 3 \), the skew matrix \( B_{jk} \) can be encoded by a vector \( \mathbf{B} \), in which case the formula for the derivative of momentum becomes \( \frac{dp_j}{dt} = (\mathbf{p}/m) \times \mathbf{B} \). (Recall that we are absorbing the charge of the particle into the definition of the magnetic field.)

Although the approach we have just described is the best one for generalizing to manifolds, in the \( \mathbb{R}^n \) case, we may alternatively consider ‘canonical’ momentum variables \( \tilde{p}_j \) satisfying the usual Poisson bracket relations, that is, \( \{x_j, \tilde{p}_k\} = \delta_{jk} \) and all other brackets are zero. The Hamiltonian is then \( (\tilde{p} - A)^2/(2m) \), where \( A \) is the vector potential for \( B \). The two types of momentum variables are related by \( p_j = \tilde{p}_j - A_j \).

### 2.2. The manifold case

Let \( M \) be a Riemannian manifold with metric \( g \), thought of as the configuration space for our system. The phase space is then the cotangent bundle \( T^*M \). On \( T^*M \) we have the canonical 1-form \( \theta \), which is given in local coordinates as \( \theta = p_j \, dx_j \), along with the canonical 2-form \( \omega := -d\theta \), given in coordinates as \( \omega = dx_j \wedge dp_j \). We assume \( M \) is equipped with a ‘magnetic field’, which we model as a closed 2-form \( B \). If \( \pi : T^*M \to M \) is the projection onto the base, then the pulled-back form \( \pi^*(B) \) is a closed 2-form on \( T^*M \). In local coordinates, we have \( B = (1/2)B_{jk} \, dx_j \wedge dx_k \) for a unique skew-symmetric matrix \( B_{jk} \), in which case \( \pi^*(B) \) is given by the same formula, but with the \( x_j \)'s now viewed as functions on \( T^*M \).

We now consider the modified symplectic form \( \omega^B \) given by \( \omega^B = \omega - \pi^*(B) \). In the usual sort of cotangent bundle coordinates \( \{x_j, p_j\} \), we may represent \( \omega^B \) by the matrix

\[
\omega^B = \begin{pmatrix}
-B & I \\
-I & 0
\end{pmatrix}.
\]

Then, the Poisson bracket of any two functions \( f \) and \( g \) is defined by \( \{f, g\} = - (\omega^B)^{-1} (df \wedge dg) \). It is easily verified that the formula for \( \{f, g\} \) in coordinates is the same as in (1). In particular, the momentum variables do not in general Poisson commute, but rather satisfy \( \{p_j, p_k\}_B = B_{jk}(x) \). Thus, the \( p_j \)'s should be thought of as the kinetic momenta.

We introduce the Hamiltonian

\[
H(x, p) = \frac{1}{2m} g_{jk}(x) p_j p_k.
\]

The dynamics associated with the Hamiltonian \( H \) and the symplectic form \( \omega^B \) are the dynamics of a charged particle moving on \( M \) acted on by the magnetic field \( B \) (but no other forces). The equations of motion in coordinates are

\[
\frac{dx_j}{dt} = \{x_j, H\} = \frac{g_{jk}(x)}{m} p_k
\]

and

\[
\frac{dp_j}{dt} = \{p_j, H\} = -\frac{1}{2m} \frac{\partial g_{ij}}{\partial x_j} p_k p_l + B_{jk}(x) \frac{g_{kl}(x)}{m} p_l.
\]

The expression for \( dx_j/dt \) in terms of \( p_k \) is the same as for a free particle moving on \( M \), which justifies calling the \( p_j \)'s the kinetic momenta. Meanwhile, the expression for \( dp_j/dt \) differs from a free particle by the addition of the term involving \( B \). As in the \( \mathbb{R}^n \) case, none of the relevant formulas requires us to choose a vector potential for \( B \).
3. Complex coordinates on phase space

We now specialize to the case in which our configuration space is the 2-sphere $S^2$, consisting of points $x \in \mathbb{R}^3$ such that $x^2 = r^2$, for some positive constant $r$. On $S^2$, we consider a magnetic field equal to a constant $B$ times the area form:

$$\frac{1}{2}Be_{jkl} \frac{dx_j}{r} dx_k \wedge dx_k.$$  

(5)

Our goal in this section is to introduce on $T^*S^2$ certain complex valued functions $a_j$, $j = 1, 2, 3$, that will allow us to identify $T^*S^2$ with the complex sphere

$$S^2_C = \{ a \in \mathbb{C}^3 | a_1^2 + a_2^2 + a_3^2 = 1 \}.$$

Then, in the next section, we will quantize the functions $a_j$ to obtain operators $A_j$, which we think of as annihilation operators. Our coherent states will then be simultaneous eigenvectors for the annihilation operators.

3.1. Angular momentum

We permanently identify the cotangent bundle $T^*S^2$ with the tangent bundle $TS^2$, using the metric on $S^2$. Thus, we consider

$$TS^2 = \{(x, p)|x^2 = r^2, x \cdot p = 0\}.$$

The canonical 2-form $\omega$ is then given by

$$\omega_{(x,p)}(\langle a, b \rangle, \langle c, d \rangle) = a \cdot d - b \cdot c$$

for all $\langle a, b \rangle$ and $\langle c, d \rangle$ in $T_{(x,p)}(TS^2)$. We then subtract from $\omega$ the pull-back $\pi^*(B)$ of the ‘magnetic’ 2-form $B$ in (5) under the projection map $\pi$, where $\pi((x, p)) = x$. The resulting form $\omega^B := \omega - \pi^*(B)$ is closed and nondegenerate. The vector $p$ is to be thought of as the kinetic momentum of the system and not the canonical momentum.

It is convenient to calculate in terms of appropriately defined angular momentum functions. Since $B$ is invariant under rotations, $\omega^B$ is invariant under simultaneous rotations of $x$ and $p$. Let $E_1$ be the vector field denoting an infinitesimal rotation around the $e_1$-axis, in both $x$ and $p$, so that

$$E_1 = x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} + p_2 \frac{\partial}{\partial p_3} - p_3 \frac{\partial}{\partial p_2}.$$

We then define $E_2$ and $E_3$ by cyclic permutation of the indices in the definition of $E_1$. We then look for angular momentum functions $J_1, J_2, J_3$ such that

$$\omega^B(E_j, \cdot) = dJ_j.$$

These functions will have the property that $[J_j, f] = E_j f$, for $j = 1, 2, 3$.

Since $\omega^B$ is a sum of terms, one of which depends only on the position variables, we may look for $J_j$ of the same form. It is straightforward to check that

$$J(x, p) = x \times p - Brx.$$

(6)

The Poisson bracket relations involving $J$ and $x$ are

$$\{x_j, x_k\} = 0$$

$$\{J_j, x_k\} = \varepsilon_{jkl} x_l$$

$$\{J_j, J_k\} = \varepsilon_{jkl} J_l.$$

(7)

The first of these relations is true in general for magnetic symplectic forms (compare (1)) and the second and third relations hold because $[J_j, f]$ is an infinitesimal rotation of $f$. Although
one can work out the Poisson bracket relations involving the linear momentum by expressing \( p \) in terms of \( J \) as \( p = J \times x/r^2 \). We will not have need for these relations in this paper.

Although the relations (7) are identical to what we have in the \( B = 0 \) case, we should keep in mind that the \( J \) function is not the usual one. The ‘magnetic’ \( J \) is distinguished from the ordinary one by the algebraic relation

\[
J \cdot x = -r^3 B. \tag{8}
\]

Note also that

\[
J^2 = r^2 p^2 + r^4 B^2. \tag{9}
\]

The angular momentum is a constant of motion for the dynamics associated with the symplectic form \( \omega^B \) and the Hamiltonian \( H = p^2/(2m) \), thus, the particle’s position \( x \) will always lie in the circle obtained by intersecting \( S^2 \) with the plane \( J \cdot x = -r^3 B \), where \( J \) is the value of the angular momentum vector at the initial time.

### 3.2. The classical complexifier method

We now apply Thiemann’s complexifier method (section 2 of [15]), as we did in [14] in the nonmagnetic case. To do this, we take a constant \( \alpha \) (denoted \( \omega \) in [14]) with units of frequency, and we define our complexifier function by

\[
\text{complexifier} = \frac{\text{energy}}{\alpha} = \frac{p^2}{2m\alpha} = \frac{J^2}{2m\alpha r^2} + \text{const}.
\]

Since, as will be apparent shortly, adding a constant to the complexifier has no effect on the calculations, we will ignore the constant in the expression for the complexifier in terms of \( J^2 \).

Then, as in [14], we define complex-valued functions \( a_j \) on \( TS^2 \) by the formula

\[
a_j = e^{i \left\{ x_j, \frac{J^2}{2m\alpha r^2} \right\} (x)} = \sum_{n=0}^{\infty} \left( \frac{i}{2m\alpha r^2} \right)^n \frac{1}{n!} \left\{ \cdots \left\{ x_j, \frac{J^2}{2m\alpha r^2} \right\}, \cdots, \frac{J^2}{2m\alpha r^2} \right\}.
\]

(Note that replacing \( J^2 \) by \( J^2 + \text{const} \) plus a constant has no effect on the value of \( a_j \).) The ‘i’ in the exponent in the formula for \( a_j \) should not be understood as physical time, but merely as a parameter in our construction. That is to say, we are still going to consider quantum mechanics using ordinary (real) time.

Using (7) and the product rule, we calculate \( \left\{ x_j, \frac{J^2}{2m\alpha r^2} \right\} \) to be \( 2\varepsilon_{jkl} J_k x_l \). Thus, in vector notation,

\[
\left\{ x, \frac{J^2}{2m\alpha r^2} \right\} = \frac{1}{m\alpha r^2} J \times x.
\]

Since each \( J_j \) Poisson commutes with \( J^2 \), as is easily verified, we may treat \( J \) as a constant in computing subsequent commutators. Thus,

\[
a = \exp \left\{ \frac{i}{m\alpha r^2} J \times \cdot \right\} (x) = \sum_{n=0}^{\infty} \left( \frac{i}{m\alpha r^2} \right)^n \frac{1}{n!} J \times (\cdots J \times (J \times x))). \tag{10}
\]

Again, the dependence of (10) on the magnetic field strength \( B \) is through the dependence of \( J \) on \( B \).

**Theorem 1.** We have

\[
a(x, p) = (\cosh L) x + i \frac{\sinh L}{L} \frac{p}{m\alpha} = \frac{(\cosh L - 1)}{L^2} B \frac{J(x, p)}{m^2 \alpha^2 r}.
\]
where \( \mathbf{J}(\mathbf{x}, \mathbf{p}) \) is given by (6) and where \( L \) is a dimensionless version of the total angular momentum given by

\[
L = \frac{|\mathbf{J}(\mathbf{x}, \mathbf{p})|}{mar^2} = \sqrt{\frac{\mathbf{p}^2 + r^2B^2}{mar}}.
\]

When \( B = 0 \), the \( \mathbf{J} \) terms drop out, \( L \) becomes equal to \( \frac{\mathbf{p}}{mar} \), and we obtain the expression for \( a(\mathbf{x}, \mathbf{p}) \) in equation (18) of [14] (with \( \alpha \) being identified with \( \omega \) in [14]). We should mention that the \( B = 0 \) formula was already well known prior to [14], for example on p 410 of [21].

Proof. A simple computation shows that

\[
\mathbf{J} \times \mathbf{x} = r^2 \mathbf{p},
\]

\[
\mathbf{J} \times \mathbf{p} = -\frac{J^2}{r^2} \mathbf{x} - rBJ.
\]

Since also \( \mathbf{J} \times \mathbf{J} = 0 \), the action of ‘cross product with \( \mathbf{J} \)’ on the vectors \( \mathbf{x}, \mathbf{p} \) and \( \mathbf{J} \) may be represented by the matrix

\[
\begin{pmatrix}
0 & -\frac{J^2}{r^2} & 0 \\
\frac{J^2}{r^2} & 0 & 0 \\
0 & 0 & -rB
\end{pmatrix}.
\]

(11)

By (10), if we exponentiate \( i/(mar^2) \) times the matrix in (11), the first column of the resulting matrix will tell us the coefficients of \( a(\mathbf{x}, \mathbf{p}) \) in terms of the vectors \( \mathbf{x}, \mathbf{p} \) and \( \mathbf{J} \). The exponentiation can be done by hand or using a computer algebra program, with the result being the formula in the theorem.

\[\square\]

Theorem 2. Let \( S^2_C \) denote the set

\[
S^2_C = \{ \mathbf{a} \in \mathbb{C}^3 | a_1^2 + a_2^2 + a_3^2 = r^2 \}.
\]

Then, the map \( (\mathbf{x}, \mathbf{p}) \mapsto a(\mathbf{x}, \mathbf{p}) \) is diffeomorphism of \( TS^2 \) onto \( S^2_C \). Furthermore, we have

\[\{a_j, a_k\} = 0\]

for all \( j \) and \( k \).

Note that there are no absolute values in the definition of \( S^2_C \), which is a two-dimensional complex submanifold of \( \mathbb{C}^3 \). If \( C = J^2/(2mar^2) \) denotes the complexifier, then \( \{\cdot, C\} \) is a derivation, meaning that \( \{fg, C\} = f\{g, C\} + f\{g, C\} \). As a result, the exponential of \( i\{\cdot, C\} \) is multiplicative, by the usual power series argument for exponentials. Thus,

\[
a^2 = \sum_j (e^{i\{\cdot, C\} x_j})^2 = e^{i\{\cdot, C\}} \left( \sum_j x_j^2 \right) = e^{i\{\cdot, C\} (r^2)} = r^2.
\]

This shows that \( a(x, p) \) is contained in \( S^2_C \) for all \( x \) and \( p \). That \( a \) is a diffeomorphism of \( TS^2 \) onto \( S^2_C \) is shown in section 5 of [25]. Meanwhile, \( \{\cdot, C\} \) is also a derivation with respect to the Poisson bracket, so that the exponential of \( i\{\cdot, C\} \) preserves brackets. Thus, since \( x_j \) and \( x_k \) Poisson commute, \( a_j \) and \( a_k \) also Poisson commute.
4. The annihilation operators

4.1. Representations of the Euclidean group

We assume that the quantum Hilbert space carries an irreducible unitary representation of the uniquely simply connected Lie group $G$ whose Lie algebra is defined by the commutation relations in (7). This Lie algebra is easily identified as the Lie algebra $e(3)$ of the Euclidean group $E(3) = SO(3) \ltimes \mathbb{R}^3$. To determine the universal cover of $E(3)$, we first note that the universal cover of $SO(3)$ is $SU(2)$ and that the covering map $\Xi$ of $SU(2)$ onto $SO(3)$ is two-to-one and onto. For definiteness, let us choose one dimensional and of the form

$$\Xi \left( e^{i\phi/2} 0 0 \right) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (12)$$

The universal cover $\tilde{E}(3)$ of $E(3)$ is then given by

$$\tilde{E}(3) = SU(2) \ltimes \mathbb{R}^3.$$ 

Here, $SU(2)$ acts on $\mathbb{R}^3$ by first mapping to $SO(3)$ by the two-to-one covering map and then acting on $\mathbb{R}^3$ by rotations.

The irreducible representations of $\tilde{E}(3)$ are classified by the Wigner–Mackey method. (See, for example, [28].) To apply this method, we first choose an orbit of $SU(2)$ inside $\mathbb{R}^3$, which is a sphere of some radius $r$ that we assume is positive. (We identify this radius with the radius of the sphere whose cotangent bundle we are quantizing.) We then choose a point in $S^2$, which we take to be the north pole $n = (0, 0, r)$. The little group is then the subgroup of $SU(2)$ that maps $n$ to $n$. From (12), we can see that the little group is just the diagonal subgroup $D$ of $SU(2)$. The choice of an irreducible representation of the little group then completes the specification of an irreducible representation of $\tilde{E}(3)$. Every irreducible representation of $D$ is one dimensional and of the form

$$\left( e^{i\phi/2} 0 0 \right) \mapsto e^{i\theta},$$

for some integer or half-integer $l$. In the notation of Kowalski and Rembieliński [16], the parameter $l$ is the ‘twist’ of the system; it is analogous to the spin of a particle moving in $\mathbb{R}^3$.

We will use the standard basis $\{E_1, E_2, E_3\}$ of $SU(2)$, given by

$$E_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad E_2 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}; \quad E_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \quad (13)$$

These matrices satisfy $[E_j, E_k] = i\delta_{jk} E_l$. Let $\Sigma_{r,l}$ denote the representation of $\tilde{E}(3)$ corresponding to a choice of $r, l$. Then, the associated Lie algebra representation is described by ‘position’ operators $X_1, X_2, X_3$, whose joint spectrum is $S^2$, along with ‘angular momentum’ operators $\hat{J}_1, \hat{J}_2, \hat{J}_3$ given by

$$\hat{J}_l = i\hbar \frac{d}{dt} \Sigma_{r,l} (e^{iE_l}) \bigg|_{t=0}.$$ 

If $\psi$ is a (generalized) eigenvector for the position operators with $X_1 \psi = X_2 \psi = 0$ and $X_3 \psi = r \psi$, our choice of a representation of the little group means that

$$\hat{J}_3 \psi = \hbar l \psi.$$

The position and angular momentum operators satisfy relations analogous to (7):

$$\frac{1}{i\hbar} [X_j, X_k] = 0,$$
\[
\frac{1}{\hbar} [\hat{J}_j, X_k] = \varepsilon_{jkl} X_l,
\frac{1}{\hbar} [\hat{J}_j, \hat{J}_k] = \varepsilon_{jkl} \hat{J}_l.
\] (14)

The choice of a sphere of radius \(r\) in the Wigner–Mackey method gives us the additional algebraic relation
\[
X \cdot X = r^2.
\] (15)

Finally, the parameter \(l\), labeling the chosen representation of the little group, determines one additional relation
\[
\hat{J} \cdot X = r\hbar l.
\] (16)

To see that this relation is true, we can easily verify that \(\hat{J} \cdot X\) commutes with each \(\hat{J}_j\) and each \(X_j\), which means that this operator must act as a constant multiple of the identity in each irreducible representation. The value of this constant can be determined by evaluating \(\hat{J} \cdot X\psi\) on a (generalized) vector \(\psi\) such that \(X_1\psi = X_2\psi = 0\) and \(X_3\psi = r\), on which we have \(\hat{J} \cdot X\psi = r\hat{I}_3\psi = r\hbar l\psi\), by assumption.

Comparing (16) to (8) in the classical case, it is natural to make the following identification, which relates the value of \(l\) on the quantum side to the value of \(B\) on the classical side:
\[
-\frac{B}{r} = \frac{\hbar l}{r^2}.
\] (17)

That is to say, if we make the identification (17), then (16) becomes identical to the classical formula:
\[
\hat{J} \cdot X = -r^3 B.
\] (18)

Now, (17) is equivalent to the condition
\[
-\frac{(4\pi r^2)B}{2\pi \hbar} = 2l,
\] (19)

where \(2l\) is a non-negative integer. Equation (19) says that the area of the sphere, with respect to the magnetic 2-form—which is \(B\) times the area form—must be an integer multiple of \(2\pi \hbar\).

Since the restriction of the canonical 2-form \(\omega\) to \(S^2 \subset TS^2\) is zero, an equivalent formulation of the condition is that the symplectic area of the sphere \(S^2 \subset TS^2\) with respect to \(\omega^B\) has to be an integer multiple of \(2\pi \hbar\). This last condition is the usual integrality condition in the theory of quantization of symplectic manifolds. (See, for example, [29].)

Note that we have taken the position and angular momentum operators as the ‘basic’ operators of our theory. If we wish to introduce linear momentum operators, we must define them in terms of the angular momentum operators. Since we have, classically, \(p = J \times x/r^2\), it is reasonable to define the quantum version of \(p\) by the analogous relation:
\[
P := \frac{1}{r^2} \hat{J} \times X.
\] (20)

These linear momentum operators will come up in the computation of the annihilation operators in the next subsection.

4.2. The quantum complexifier method

We work in a Hilbert space constituting an irreducible representation of \(\tilde{E}(3)\), with operators \(\hat{J}_j\) and \(X_j\) satisfying the commutation relations (14) along with the algebraic relations (15) and...
We assume that the quantum counterpart \( \hat{H} \) to the classical energy function is equal to \( \hat{J}^2/(2mr^2) \) plus a constant, in which case our complexifier operator is

\[
\text{complexifier} = \frac{\text{energy}}{\alpha} = \frac{\hat{J}^2}{2mr^2} + \text{const.},
\]

as in the classical case. Here, the parameter \( \alpha \), having units of frequency, is the same one used in section 3.2. As in the classical setting, the constant on the right-hand side of (21) has no effect on the complexifier method, as will be evident shortly.

Following the quantum version of Thiemann’s method [15], we define operators \( A_j \) by

\[
A_j = e^{i\left[\frac{\text{complexifier}}{\hbar}, X_j\right]} = \sum_{n=0}^{\infty} \left( \frac{1}{2mr^2\hbar} \right)^n \left[ \cdots \left[ X_j, \hat{J}_1^2 \right], \hat{J}_2^2, \ldots, \hat{J}_2 \left[ \right] \right].
\]

We will interpret these operators as the annihilation operators for our system. By a standard identity (see, for example, proposition 2.25 and exercise 2.19 in [30]), we have the alternative expression

\[
A_j = \exp \left\{ -\frac{\hat{J}^2}{2mr^2\hbar} \right\} X_j \exp \left\{ \frac{\hat{J}^2}{2mr^2\hbar} \right\}.
\]

For purposes of computing the coherent states, the expression (23) is the most useful formula for the annihilation operators. In particular, from (23), we can see that

\[
\frac{1}{\hbar} [A_j, A_k] = 0
\]

and

\[
A_j A_j = r^2.
\]

We now look for quantum counterparts to the expressions for \( a(x, p) \) in (10) and theorem 1. In computing the commutator of \( X_j \) with \( \hat{J}_2 \), we get products of \( X \)’s and \( \hat{J} \)’s in both orders. If we move, say, all the \( \hat{J} \)’s to the left we obtain a quantum correction as follows:

\[
\frac{1}{\hbar} \left[ X, \frac{\hat{J}^2}{2mr^2} \right] = \frac{1}{mar^2} \left( \hat{J} \times X - i\hbar X \right),
\]

as may easily be verified. Now, since \( \hat{J}^2 \) commutes with each \( \hat{J}_j \), we may treat \( \hat{J} \) as a constant in computing subsequent commutators. Thus,

\[
A = \exp \left\{ \frac{i(\hat{J} \times \cdot) + \hbar}{mar^2} \right\} (X).
\]

In the case \( \hbar/(mar^2) = 1 \), this expression is essentially equation (4.7) of [16], which should be expected, since we have thus far used only the commutation relations (14) and not (15) or (16). Equation (26) also coincides with the \( d = 2 \) case of equation (33) of [14]. (The notation \( JX \) in [14] corresponds, in the \( d = 2 \) case, to \( \hat{J} \times X \) in the notation of the current paper.)

We now compute the annihilation operators ‘explicitly’ in a form similar to the expressions for \( a(x, p) \) in theorem 1. Recalling the definition (20) of the linear momentum operators, a straightforward computation gives:

\[
\hat{J} \times X = r^2 P,
\]

\[
\hat{J} \times P = -\frac{\hat{J}^2}{r^2} X + i\hbar P - rB J.
\]
Thus, if we cross with \( \hat{J} \) repeatedly, we will obtain expressions involving \( \hat{J} \) in addition to \( X \) and \( P \). In the quantum case, \( \hat{J} \times \hat{J} \) is not zero:

\[
\hat{J} \times \hat{J} = i\hbar \hat{J}.
\]

The action of the operation of ‘crossing with \( \hat{J} \)’ on the vector operators \( X \), \( P \) and \( \hat{J} \) can thus be encoded in the following matrix:

\[
\hat{J} \times \cdot = \begin{pmatrix}
0 & -\frac{L}{\hbar} & 0 \\
r^2 & i\hbar & 0 \\
0 & -rB & i\hbar
\end{pmatrix}.
\] (27)

Since the entries of the matrix in (27) commute, we can think of it as an ordinary \( 3 \times 3 \) matrix. We then put in this matrix in place of the expression \( \hat{J} \times \cdot \) in (26) and exponentiate.

The matrix exponential can be computed explicitly by Mathematica, and the first column of the exponential gives us the result of applying the quantum complexifier to \( X \).

The matrix in (27) is block-upper triangular and the upper left \( 2 \times 2 \) block is the same as the matrix in [14]; as a result, the upper \( 2 \times 2 \) block in the exponential is the same as the exponential of the matrix in [14]. Thus, the expression for \( A \) will be the same as the \( d = 2 \) case of equation (38) of [14], except that there will be an extra term involving \( \hat{J} \). (Recall that what we call \( \alpha \) here corresponds to \( \omega \) in [14].) We record the answer in the following theorem.

**Theorem 3.** Let us introduce the shifted, dimensionless angular momentum operator

\[
\hat{L} = \sqrt{\hat{J}^2 + \frac{\hbar^2}{4m \alpha^2 r^2}}.
\]

Then, we obtain

\[
A = e^{\frac{\hbar}{(2ma^2 r^2)}} \left( \cosh \frac{\hbar}{2ma^2 r^2} \sinh \hat{L} \right) X + \frac{ie^{\hbar/(2ma^2 r^2) \sinh \hat{L}}}{\hat{L}} P - \Lambda B \frac{\hat{J}}{m^2 \alpha^2 r^2},
\] (28)

where

\[
\Lambda = \frac{1}{L^2 - (\hbar/(2ma^2 r^2))^2} \left( 1 + e^{\frac{\hbar}{(2ma^2 r^2)}} \left( \frac{\hbar}{2ma^2 r^2} \sinh \frac{\hat{L}}{L} - \cosh \frac{\hat{L}}{L} \right) \right).
\]

When \( B = 0 \), the expression for \( A \) agrees (upon setting \( \hbar/(ma^2 r^2) = 1 \)) with equation (4.16) of [17]. The \( B = 0 \) case of theorem 3 also agrees with the \( d = 2 \) case of equation (38) in [14]. On the other hand, taking the limit as \( \hbar \) tends to zero in (28)—and identifying \( \hat{L} \) with \( L \)—gives the expression for the classical function \( a(x, p) \) in theorem 1.

5. The coherent states

We define a state \( \psi \) to be a **coherent state** if \( \psi \) is a simultaneous eigenvector for the operators \( A_j \):

\[
A_j \psi = a_j \psi.
\]

Since the \( A_j \)'s commute (see (24)) it is reasonable to hope that there are many coherent states. Since also \( A_j A_j = r^2 \), we must have

\[
a_1^2 + a_2^2 + a_3^2 = r^2,
\]

meaning that the vector \( a := (a_1, a_2, a_3) \) must belong to \( S^2 \subset \mathbb{C}^3 \).
Theorem 4. For each \( a \in S^2_C \), there exists a nonzero, normalizable vector \( \chi_a \) in the quantum Hilbert space such that
\[
A_j \chi_a = a_j \chi_a.
\]
For \( a \) in the real sphere \( S^2 \), we may compute \( \chi_a \) as
\[
\chi_a = \exp \left\{ -\frac{\hat{J}^2}{2m^2\hbar^2} \right\} \delta_a,
\]
where \( \delta_a \) is a (non-normalizable) vector satisfying \( X_j \delta_a = a_j \delta_a \). For each \( a \) in the real sphere \( S^2 \), the space of (generalized) eigenvectors \( \psi \) for \( X \) satisfying \( X_j \psi = a_j \psi \) is one dimensional. When \( l \neq 0 \), there is no way to pick a nonzero element \( \delta_a \) of each eigenspace that depends continuously on \( a \). Thus, there is no continuous way to parameterize the coherent states as vectors, even for parameters in the real sphere. Physically, however, it is only the one-dimensional subspace spanned by the coherent state that is important, and these subspaces depend continuously (in fact, holomorphically) on \( a \in S^2_C \).

Proof. As explained in detail in section 6, there is a ‘position representation’ in which our Hilbert space is the space of square-integrable sections of complex line bundle over \( S^2 \). Then for \( a \in S^2_C \), the coherent state \( \chi_a \) is nothing but the ‘bundle heat kernel’, evaluated at a point in the fiber over \( a \). It is well known that such a bundle heat kernel always exists and is smooth, so that \( \psi_a \) is a normalizable (finite-norm) vector. For general \( a \in S^2_C \), we need to show that the bundle heat kernel can be analytically continued with respect to the parameter \( a \) from \( S^2 \) to \( S^2_C \). It suffices to show that any solution of the bundle heat equation can be analytically continued from \( S^2 \) to \( S^2_C \), which we will show in section 6 by the method of reduction to the group case. (In the group case, the existence of the analytic continuation of the heat kernel was shown in detail in section 4.2 of [1].)

Computations of the heat kernel for the ‘spinor’ case \( (l = 1/2) \) can be found in [31].

6. The Segal–Bargmann representation

In this section, we construct a Segal–Bargmann representation associated with the coherent states, and an associated unitary Segal–Bargmann transform. That is to say, the transform consists of taking the inner product of a state \( \psi \) with each coherent state \( \chi_a \), resulting in a function of \( a \). Because the coherent states depend holomorphically on \( a \), we get a holomorphic function on \( S^2_C \), or rather, a holomorphic section of a certain line bundle over \( S^2_C \). The unitarity of the Segal–Bargmann transform is equivalent to resolution of the identity for the coherent states, as we explain in section 7 of [14].

6.1. The Schrödinger Hilbert space

In section 4.1, we considered irreducible representations of the double cover of the Euclidean group ‘in the abstract’. That is, we never give a concrete realization of the Hilbert space, but rather perform all calculations using only the commutation relations of the Lie algebra together with the two algebraic relations (15) and (16) that characterize the particular irreducible representation. We now wish to give a concrete realization of a given irreducible representation, as a space of square-integrable ‘sections’ over the real sphere.

If \( (\Pi, V) \) is a finite-dimensional representation of \( SU(2) \), define operators \( \sigma_j \) by
\[
\sigma_j = i\hbar \frac{d}{dt} \Pi (e^{itE_j}) \bigg|_{t=0}.
\]
where \( F_j \) is defined in (13). For each non-negative integer or half-integer \( l \), let \((\Pi_l, V_l)\) be the irreducible unitary representation of \( SU(2) \) in which the largest eigenvalue of \( \sigma_3 \) is \( il \). Now let \( L^2(S^2; V_l) \) denote the space of square-integrable functions on \( S^2 \) with values in \( V_l \). Define angular momentum operators \( \hat{J}_j \) on this space by
\[
\hat{J}_j = L_j + \sigma_j, \tag{30}
\]
where the \( L_j \)’s are the usual orbital angular momentum operators given by
\[
L_1 = -i\hbar \left( x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} \right)
\]
and relations obtained from this by cyclic permutations of the indices.

For any integer or half-integer \( l \) (positive or negative), the Schrödinger ‘realization’ of the associated representation of \( \hat{E}(3) \) corresponding to that value of \( l \) will be a certain subspace of the Hilbert space \( L^2(S^2; V_{|l|}) \). (Here \( l \) is the parameter in (16) and (17) in our analysis of the representations of \( \hat{E}(3) \).)

**Definition 5.** For any integer or half-integer \( l \), the **Schrödinger Hilbert space**, denoted \( \Gamma^2(S^2; l) \), is the subspace of \( L^2(S^2; V_{|l|}) \) consisting of functions \( \psi : S^2 \to V_{|l|} \) with the property that for all \( \mathbf{x} \in S^2 \),
\[
(\sigma \cdot \mathbf{x}) \psi(\mathbf{x}) = r\hbar l \psi(\mathbf{x}). \tag{31}
\]
Here, \( \sigma \) is defined by (29). The norm of such a function \( \psi \) is computed as
\[
\|\psi\|^2 = \int_{S^2} |\psi(\mathbf{x})|^2 \, d\mathbf{x},
\]
where \( |\cdot| \) is the \( SU(2) \)-invariant norm on \( V_l \) and \( d\mathbf{x} \) is the surface-area measure on \( S^2 \).

The notation \( \Gamma^2 \) is commonly used to denote sections of a vector bundle over some manifold. The notation \( \Gamma^2(S^2; l) \) then denotes the space of square-integrable sections of the complex line bundle over \( S^2 \) labeled by \( l \).

At each point \( \mathbf{x} \in S^2 \), the space of possible values for \( \psi(\mathbf{x}) \) is one dimensional. If, for example, we take \( \mathbf{x} = \mathbf{n} \), then \( \psi(\mathbf{n}) \) must lie in the eigenspace for \( \sigma_3 \) with eigenvalue \( il \).

If \( l = 1 \), the matrices \( F_j := \frac{d\Pi_l(e^{iE_j})}{dt}|_{t=0} \) form the standard basis for \( SO(3) \), and we may calculate that
\[
\mathbf{x} \cdot \sigma = i\hbar \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}\]
and thus that
\[
(\mathbf{x} \cdot \sigma)(\mathbf{v}) = i\hbar \mathbf{x} \times \mathbf{v}.
\]

The Schrödinger Hilbert space may then be described as the space of square-integrable functions \( \psi : S^2 \to \mathbb{C}^3 \) such that
\[
\mathbf{x} \times \psi(\mathbf{x}) = i\hbar \psi(\mathbf{x}). \tag{32}
\]

We have described the Schrödinger Hilbert space as a Hilbert space; it remains to describe the action of the Euclidean Lie algebra on it. The action of the position operators is simple enough: we put \( X_j \psi(\mathbf{x}) \) equal to \( x_j \psi(\mathbf{x}) \). For the action of the angular momentum operators, we wish to continue to use the formula in (30). For this to make sense, we must show that the space \( \Gamma^2(S^2; l) \) of functions satisfying (31) is invariant under the operators \( \hat{J}_j \) in (30). To
verify this invariance, we may easily verify that $\Gamma^2(S^2; l)$ is invariant under the action $SU(2)$ given by

$$ (U \cdot \psi)(x) = \Pi_I(U) \psi(R_I x), \tag{33} $$

where as in definition 5, $R_I$ is the element of $SO(3)$ corresponding to the element $U$ of $SU(2)$. The operators $\hat{J}_j$ are obtained by differentiating the action of $\exp(tF_j)$ at $t = 0$. Since $\Gamma^2(S^2; l)$ is invariant under the group action in (33), it is also invariant under the associated Lie algebra action in (30).

6.2. The Segal–Bargmann Hilbert space

We now define the Segal–Bargmann space over $S_C^2$ associated with a given value of $l$ and an arbitrary positive, matrix-valued density $\nu$ on $S_C^2$.

Definition 6. For any integer or half-integer $l$, the space of holomorphic sections over $S_C^2$, denoted $\mathcal{H}(S_C^2; l)$, is the space of holomorphic functions $\Psi : S_C^2 \to V[l]$ with the property that

$$ (\sigma \cdot \mathbf{a})(\Psi(\mathbf{a})) = \hbar l \Psi(\mathbf{a}) \tag{34} $$

for all $\mathbf{a} \in S_C^2$. Let $\nu : S_C^2 \to \text{Pos}(V[l])$ be a continuous map into the space of positive, self-adjoint operators on $V[l]$. Then, the Segal–Bargmann space, denoted $\mathcal{H}(S_C^2; l, \nu)$, associated with $\nu$ is the space of $\Psi$ in $\mathcal{H}(S_C^2; l)$ such that

$$ \| \Psi \|_l^2 := \int_{S_l} \langle \Psi(z) | M(z) | \Psi(z) \rangle dz < \infty. $$

Here, $dz$ is the $SO(3; \mathbb{C})$-invariant measure on $S_C^2$ described in section 6 in [14], and $\langle \cdot | \cdot \rangle_l$ is the $SU(2)$-invariant inner product on $V_l$.

We now wish to describe a map from $\Gamma^2(S^2; l)$ to $\mathcal{H}(S_C^2; l)$. This map will consist of applying a smoothing operator to $\psi \in \Gamma^2(S^2; l)$ and then analytically continuing from $S^2$ to $S_C^2$.

Proposition 7. For all $\psi \in \Gamma^2(S^2; l)$, consider the section $\Psi$ given by

$$ \Psi = \exp \left\{ -\frac{\mathbf{J}^2}{2m \alpha^2 \hbar} \right\} \psi. \tag{35} $$

Then, $\Psi$ admits a unique extension from $S^2$ to a holomorphic map of $S_C^2$ into $V_l$, and this extension is an element of $\mathcal{H}(S_C^2; l)$. We refer to $\Psi$ as the Segal–Bargmann transform of $\psi$.

Note the similarity between the definition (35) of the Segal–Bargmann transform and the formula for the coherent states in theorem 4. This similarity indicates that the Segal–Bargmann transform $\Psi$ at a point $\mathbf{a}$ is simply the inner product of $\psi$ with the coherent state $\chi_{\mathbf{a}}$. The Segal–Bargmann transform is the more convenient description in this case, simply because there is no continuous way of parameterizing the coherent states, due to the nontriviality of the bundle we are working with.

The Segal–Bargmann space and transform should be compared to the spaces in [32] and [33] in the $\mathbb{C}^n$ case. (See [34] for more information.) The proof of this result is deferred to section 6.4, where it will be proved by reduction to the group case. We will also see that there is a certain natural choice for $\nu$ such that the Segal–Bargmann transform is unitary.
6.3. Unitarity of the Segal–Bargmann transform

For convenience of computation, let us write the operator occurring in the exponent in (35) as
\[ \frac{\hat{J}^2}{2m^2\hbar} = \frac{1}{2} \tau \tilde{J}^2, \]
where \( \tilde{J} \) is a dimensionless version of \( \hat{J} \) given by
\[ \tilde{J}_k = \frac{\hat{J}_k}{\hbar}, \]
and where \( \tau \) is the dimensionless parameter given by
\[ \tau = \frac{\hbar}{m^2r^2}. \]
(The reader should not confuse \( \tilde{J} \) with the dimensionless quantities \( L \) and \( \hat{L} \) occurring in the formulas for the map \( a(x, p) \) and the annihilation operators. In computing \( L \), we divide by \( m^2r^2 \) rather than by \( \hbar \).)

In [14], we argued that \( \tau \) controls the ratio of the spatial width of the coherent states to the radius of the sphere. Specifically, if \( \Delta X \) denotes the spatial width of a coherent state (measured in some reasonable way), then we expect that \( \Delta X \approx \sqrt{\frac{\tau}{2}} \), at least when \( \tau \ll 1 \).

**Theorem 8.** For each integer or half-integer \( l \) and each \( \tau > 0 \), there exists a function \( \nu(l, \tau) \) with values in positive operators on \( V_l \) such that the Segal–Bargmann transform is a unitary map of \( \Gamma^1(S^2; l) \) onto \( \mathcal{H}^2(S^2; l, \nu(l, \tau)) \).

This result is proved in the following subsection. See theorem 10 for a formula for \( \nu(l, \tau) \).

6.4. Reduction to the group case

In this subsection, we begin with the Segal–Bargmann transform for the compact Lie group \( SU(2) \), as described in [1]. We then ‘twist’ this transform with the space \( V_l \) carrying an irreducible representation of \( SU(2) \). Next, we allow the resulting transform to descend from \( SU(2) \) to \( SU(2)/D = S^2 \), obtaining a Segal–Bargmann transform for \( V_l \)-valued functions on \( S^2 \), with respect to a ‘covariant’ Laplacian that can be computed as the sum of squares of the operators \( \hat{J}_j \) in (30). Finally, we restrict the Segal–Bargmann transform for \( V_l \)-valued functions on \( S^2 \) to the subspace of functions satisfying the condition (31).

Recall from (13) the basis \( \{E_1, E_2, E_3\} \) for \( SU(2) \), satisfying \( [E_j, E_k] = \varepsilon_{jkl}E_l \). We then take the inner product on \( SU(2) \) for which these elements are orthonormal. For each \( j \), we form the self-adjoint operator \( \Sigma_j \) on \( L^2(SU(2)) \) (with respect to the Haar measure), given by
\[ (\Sigma_j \phi)(x) = i \frac{d}{dt} \phi(e^{-iE_j x}) \bigg|_{t=0}. \]
We then form the Laplacian \( \Delta \) (here taken to be a positive operator) given by
\[ \Delta = \Sigma_j \Sigma_j. \]
Using \( \Delta \), we form the heat operator \( e^{-\tau \Delta/2} \).
Theorem 9. Fix a positive number \( \tau \). Then, for each \( \phi \in L^2(SU(2)) \), the function \( \Phi := e^{-\tau \Delta/2} \phi \) admits a holomorphic extension from \( SU(2) \) to \( SL(2; \mathbb{C}) \). Furthermore, there is a smooth positive density \( \nu_\tau \) on \( SL(2; \mathbb{C}) \) such that
\[
\| \phi \|_{L^2(SU(2))}^2 = \int_{SU(2)} |\Phi(g)|^2 \nu_\tau(g) \, dg. \tag{37}
\]

Finally, if \( \Phi \) is any holomorphic function on \( SL(2; \mathbb{C}) \) for which the integral on the right-hand side of (37) is finite, there is a unique \( \phi \in L^2(SU(2)) \) for which \( \Phi \mid_{SU(2)} = e^{-\tau \Delta/2} \phi \).

This result is the \( K = SU(2) \) case of theorem 2 of [1]. There is a trivial extension of this theorem in which \( \phi \) takes values in \( V_l \) instead of in \( \mathbb{C} \). This extended Segal–Bargmann transform maps \( L^2(SU(2); V_l) \) (square-integrable, \( V_l \)-valued functions on \( SU(2) \)) to holomorphic, \( V_l \)-valued functions on \( SL(2; \mathbb{C}) \). To get something slightly less trivial, we are going to ‘twist’ our functions by the action of \( SU(2) \) on \( V_l \). When we apply the associated ‘twisted Laplacian’ with functions that are invariant under the right action of the diagonal subgroup \( D \) of \( SU(2) \), we obtain precisely the operator (36). The result of applying theorem 9 to functions of this type is the following.

Theorem 10. Let \( \psi \) be any function in \( L^2(S^2; V_l) \) and let \( \psi \) be the holomorphic function on \( S^2_E \) whose restriction to \( S^2 \) is given by
\[
\psi = \exp(-\tau \tilde{P}/2) \psi.
\]

Then,
\[
\| \psi \|_{L^2(S^2; V_l)}^2 = \int_{S^2_E} \langle \Psi(z), v_l(R) \Psi(z) \rangle \, dz,
\]
where \( v_l \) is the function with values in positive operators on \( V_l \) given by
\[
v_l(R) = \int_{D_E} \Pi_l ( ((gh)^{-1})^* (gh)^{-1} ) \nu_\tau(gh) \, dh
\]
for each \( g \in SL(2; \mathbb{C}) \). Here, \( R \) is the element of \( SO(3; \mathbb{C}) \) associated with \( g \in SL(2; \mathbb{C}) \). Furthermore, if \( \psi \) has the property (31), the \( \Psi \) has the property (34).

Using a slight variant of the method of Flensted-Jensen [35], one can show that the function \( v_l \) satisfies a bundle heat equation over hyperbolic 2-space, which is the noncompact symmetric space dual (in the usual duality between compact and noncompact symmetric spaces) to \( S^2 \).

Proof. To each function \( \phi \in L^2(SU(2); V_l) \), let us associate another function \( \tilde{\phi} \) given by
\[
\tilde{\phi}(x) = \Pi_l(x) \phi(x).
\]
Since \( \Pi_l \) is unitary, \( \phi \) and \( \tilde{\phi} \) have the same norm.

For \( y \in SU(2) \), let \( L_y \) denote the ‘ordinary’ left action of \( y \) on some \( \psi \in L^2(SU(2); V_l) \), namely
\[
(L_y \phi)(x) = \phi(y^{-1}x).
\]
We may also introduce the twisted left action \( \tilde{L}_y \) given by
\[
(\tilde{L}_y \psi)(x) = \Pi(y) \psi(y^{-1}x).
\]
It is easily verified that
\[
(\tilde{L}_y \phi) = \tilde{L}_y \tilde{\phi}.
\]
Differentiating this relation, we find that
\[ (\tilde{\Sigma}_j \phi) = \tilde{\Sigma}_j \phi, \]
where
\[ (\tilde{\Sigma}_j \psi)(x) = \frac{d}{dr} \psi(e^{-rE_j}x) + \tau(E_j) \psi(x). \]
It is then easy to see that
\[ (e^{-r\tilde{\Delta}/2} \phi) = e^{-r\tilde{\Delta}/2} \tilde{\phi}, \]
where
\[ \tilde{\Delta} = \tilde{\Sigma}_j \tilde{\Sigma}_j. \]
Thus, if \( \Phi \) is the holomorphic extension of \( e^{-r\tilde{\Delta}/2} \phi \) and \( \tilde{\Phi} \) is the holomorphic extension of \( e^{-r\tilde{\Delta}/2} \tilde{\phi} \), we have
\[ \Phi(g) = \Pi(g^{-1}) \tilde{\Phi}(g). \quad (38) \]

Let us now apply the Segal–Bargmann transform for \( SU(2) \) (trivially extended to \( V_l \)-valued functions), to \( \psi \), and express the result in terms of \( \Psi \) by means of (38):
\[
\|\tilde{\phi}\|^2_{L^2(SU(2); V_l)} = \|\phi\|^2_{L^2(SU(2); V_l)} = \int_{SL(2; \mathbb{C})} \langle \Phi(g), \Phi(g) \rangle_{V_l} (g) \, dg.
\]
\[
= \int_{SL(2; \mathbb{C})} \langle \Pi(g^{-1}) \tilde{\Phi}(g), \Pi(g^{-1}) \tilde{\Phi}(g) \rangle_{V_l} (g) \, dg.
\]
\[
= \int_{SL(2; \mathbb{C})} \langle \tilde{\Phi}(g), \Pi((g^{-1})^* (g^{-1})^{-1}) \tilde{\Phi}(g) \rangle_{V_l} (g) \, dg.
\]
\[
= \int_{SL(2; \mathbb{C})/D_C} \langle \tilde{\Phi}(g), \Pi((g^{-1})^* (g^{-1})^{-1}) \tilde{\Phi}(g) \rangle_{V_l} (g) \, dg. \quad (39)
\]

Now, let \( D \) be the diagonal subgroup of \( SU(2) \), so that \( SU(2)/D = S^2 \), and let \( D_C \) be the complexification of \( D \), which is just the diagonal subgroup of \( SL(2; \mathbb{C}) \). It is easy to see that the twisted left action of \( SU(2) \) commutes with the ordinary right action of \( SU(2) \). It follows that the space of functions on \( SU(2) \) that are invariant under the ordinary right action of \( D \) is invariant under \( \tilde{\Delta} \) and thus under the heat operator \( e^{-r\tilde{\Delta}/2} \). Thus, if we apply (39) in the case that \( \phi \) is invariant under the ordinary right action of \( D \), \( \tilde{\Phi} \) will be invariant under the ordinary right action of \( D \) and thus also (because \( \tilde{\Phi} \) is holomorphic) under the ordinary right action of \( D_C \). Meanwhile, we can break up the integration over \( SL(2; \mathbb{C}) \) into an integral over \( D_C \) followed by an integration over \( SL(2; \mathbb{C})/D_C \). Thus, (39) becomes, when \( \phi \) is right-\( D \)-invariant,
\[
\int_{SL(2; \mathbb{C})/D_C} \int_{D_C} \langle \Phi(g), \Pi(((gh)^{-1})^* (gh)^{-1}) \Phi(g) \rangle_{V_l} (gh) \, dh \, d[g].
\]
where \( d[g] \) is the \( SL(2; \mathbb{C}) \)-invariant volume measure on \( SL(2; \mathbb{C})/D_C \). After identifying \( SL(2; \mathbb{C})/D_C \) with \( S^2_C \) and \( d[g] \) with the invariant volume measure on \( S^2_C \), we obtain the first claimed result in the theorem.

If \( \psi = \tilde{\phi} \) is a right-\( D \)-invariant function on \( SU(2) \), it descends to a function on \( SU(2)/D = S^2 \). Furthermore, the action \( \tilde{\Sigma}_j \) on this function corresponds to the action of the \( \tilde{J}_j \) on the associated function on the sphere. Thus, the twisted Segal–Bargmann transform for \( \psi \) is just the transform associated with the operator \( \exp(-r\tilde{F}/2) \) on \( S^2 \). Meanwhile, it is easily seen that operators \( \tilde{J}_j \) preserve the condition (31). Thus, we can specialize our transform on \( L^2(S^2; V_l) \) to the subspace \( \Gamma^2(S^2; I) \).

\( \square \)
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