Modelling meson clouds using coherent states

Manuel Fiolhais
Department of Physics and CFisUC, University of Coimbra, P-3004-516 Coimbra, Portugal
E-mail: tmanuel@uc.pt

Abstract. A simple model for the quark-pion interaction is used to test the validity of a variational approach which is often used in realistic models for baryons. That simple model is solvable in the weak coupling regime as well as in the strong one. As we show, the approach consisting of a meson description by coherent states supplemented by an angular momentum and isospin projection onto states of good quantum numbers (such as the nucleon or delta resonance) is flexible enough at least to reproduce the accurate results in those two limits. However, in order for these results to be achieved, the so-called variation after projection method has to be applied. This study gives confidence for using similar techniques in realist models of quarks and mesons in interaction.

1. Introduction
The research work presented in this communication has been done and reported long time ago. However, I felt it would be very interesting to refresh it in order to show it to a community whose general interest is the mathematical modelling in physical sciences, and not hadronic physics in particular. In the present case, we use a coherent state approach to describe a pion cloud around a bare core of three quarks, aiming at a description of baryons in the framework of effective models.

The work was done mostly in collaboration with people mentioned at the end, but here, in particular, I would like to warmly acknowledge Mitja Rosina (University of Ljubljana) with whom I learnt a lot on the physical and mathematical aspects of modelling the pion cloud by a projected coherent state.

The Quantum Chromodynamics (QCD) was established long ago as the theory for the strong interaction. At this most fundamental level, quarks interact among themselves by exchanging gluons which also experience self-interactions. In principle it is the theory for the description of baryons and mesons, as well as all atomic nuclei and all systems where that fundamental interaction plays a role. However, the equations to be solved for describing a simple isolated nucleon are still prohibitively complicated even using the most optimized algorithms and the most powerful supercomputers available for open science.

Nevertheless, progress has been done using the so-called lattice QCD approach. On the other hand, phenomenological models have been considered to address the description of the same physical strong interacting systems, such as the nucleon and its resonances.

The effective models try to replace the complicated quark-gluon fundamental interactions by more treatable ones from the mathematical viewpoint, such as meson-quark interactions, as depicted in figure 1.
Figure 1. A baryon, such as the nucleon, is a system of three quarks interacting through gluon exchange (also with self-interactions) — left. In effective models, these fundamental interactions are replaced by meson interactions — right.

Amongst the effective models, we should mention the cloudy bag models and the chiral soliton models, such as the linear sigma model, the Nambu-Jona Lasinio model, etc. In the former, quarks are absolutely confined, as it is observed in nature, whereas in the latter they are just bound.

The quarks are spin 1/2 particles and also isospin 1/2 particles, considering the non-strange sector only with the u and d quarks (the most relevant to describe the nucleon and its radial, spin and isospin resonances). In that case, three quarks may generate bare baryon states with angular momentum $J = \frac{1}{2}$ or $J = \frac{3}{2}$ and with isospin $T = \frac{1}{2}$ or $T = \frac{3}{2}$. In particular, we are interested in the nucleon (N) and delta ($\Delta$) sector, so our system is depicted in figure 2, with a pion cloud, $\pi(r)$, surrounding a bare baryon core.

Figure 2. A baryon modelled as a core, containing bare nucleons and deltas, surrounded by a pion cloud.

Either confined or just bound, the quark core creates a source with which the mesons interact. The pions are pseudo-scalar isovector mesons and, for them, the spin-isospin three quark source can be written as $\rho_t = \sum_k \sigma \cdot k \tau_t$, where $t = -1, 0, 1$ is the spherical isospin third component. In this expression, $\sigma$ and $\tau$ are the Pauli matrices acting on the spin and isospin, respectively, quark single particle space.
Using its creation an annihilation operators, the pion field operator, \( \hat{\pi}(r) \), can be schematically written as \( \hat{\pi}(r) = \sum_{k} [a_{\pi}(k) + (-1)^{l}a^{\dagger}_{\pi}(k)] \) and the classical solution, \( \pi(r) \), is to be interpreted as a coherent state of pions, i.e. \( \pi(r) = \langle \psi | \hat{\pi}(r) | \psi \rangle \), where \( | \psi \rangle \) is the coherent state

\[
| \psi \rangle = \exp \left[ \sum_{i,k} \xi_{i}(k)a_{i}^{\dagger}(k) \right] | B_{0} \rangle,
\]

with the functions (amplitudes) \( \xi_{i}(k) \) to be determined, usually variationally.

This state is clearly a superposition of multi-pion states, i.e. it can be written as the sum \( | \psi \rangle = \alpha_{0}|B_{0} \rangle + \alpha_{1}|\pi B_{0} \rangle + \alpha_{2}|\pi\pi B_{0} \rangle + \alpha_{3}|\pi\pi\pi B_{0} \rangle + \ldots \), where the amplitudes \( \alpha_{n} \) result from the exponential expansion. Coherent states have been used to describe light beams, with a huge number of photons, but here they will be used to describe the pion cloud whose number of bosons is yet undetermined. As we shall see, the proper quantum mechanical treatment of the coherent state allows us to obtain a very good description even when the number of pions is small. This treatment is the so-called Peierls-Yoccoz projection method, which was developed in the framework of the traditional nuclear physics. Actually, the coherent state (1) is a superposition of states with good angular momentum and isospin, |\( JT \rangle = \sum_{JT} |JT \rangle \). In order to extract the states |\( JT \rangle \) with good angular momentum and isospin, those that can be assigned to the physical states, the Peierls-Yoccoz projector, \( P_{JT}^{\text{PY}} \), should be applied to \( | \psi \rangle \): \( |JT \rangle = P_{JT}^{\text{PY}} | \psi \rangle \), \( J = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \) and \( T = \frac{1}{2}, \frac{3}{2} \).

The projection technique on top of coherent states have been applied in realistic chiral-quark models. Here we are going to apply it to a simple model consisting of pions, with frozen radial function, interacting with quarks in the lowest s state, with also frozen radial wave-functions (section 2). Therefore, the effective dynamics takes place in the spin-isospin space and the quality of the projection techniques, briefly mentioned above, is going to be tested in the simple model described in the next section. The results for the nucleon and the delta will be presented in section 3 and the conclusions will be summarized in section 4.

2. The simple model

We use a simplified version of chiral-quark models, with pions coupled to a spin-isospin source. Once radial functions for the quarks and for the mesons have been obtained (typically by solving differential or integro-differential equations) we can solely consider the angular momentum and isospin spaces. Therefore, instead of realistic effective chiral models of mesons and quarks, we use a toy model whose Hamiltonian can be written as [1]

\[
\hat{H} = \sum_{tm} a_{tm}^{\dagger}a_{tm} + \mathcal{G} \sum_{tm} B_{tm} \left[ a_{tm} + (-1)^{t+m}a_{-t-m}^{\dagger} \right],
\]

where \( m \) is the spherical angular momentum third component and \( B_{tm} \) is the baryon isospin-spin operator (resulting from the quark operators)

\[
B_{tm} = \frac{3}{5} \sum_{i=1}^{3} \tau_{i}(i)\sigma_{m}(i) = \tau_{t}^{NN}\sigma_{m}^{NN} + \sqrt{\frac{72}{25}} \left( \tau_{t}^{N\Delta}\sigma_{m}^{N\Delta} + \tau_{t}^{\Delta N}\sigma_{m}^{\Delta N} \right) + \frac{4}{5} \tau_{t}^{\Delta\Delta}\sigma_{m}^{\Delta\Delta}
\]

which includes matrices that convert one baryon into another one.

The model describes a system of non self-interacting pions linearly coupled to a bare baryon core, \( \mathcal{G} \) being the effective coupling constant. This model is simple enough for its exact solutions to be worked out in the strong and weak regimes. These accurate solutions are then compared with variational approximate solutions based on projected coherent states.
2.1. Mean-field approach

The classical approximation consists in assuming the pion field as a c-number (and not an operator obeying canonical commutation relations). The simplest way to keep the quantum mechanical description, though in an approximate way, is to interpret this c-number as an expectation value of the corresponding operator in coherent states. It is known that the coherent states are the “most classical” ones since they minimize the Heisenberg uncertainty relation. For the Hamiltonian (2), the coherent states to be considered are

\[ |\psi\rangle = N(\xi) \exp \left( \sum_{tm} \xi_{tm} a_{tm}^\dagger \right) |B\rangle \]

(4)

where \( N(\xi) \) is a normalization constant, \( \xi_{tm} \) are the amplitudes and \( |B\rangle \) is a general bare baryon state. The coherent states are eigenstates of the annihilation operator, \( a_{tm}|\psi\rangle = \xi_{tm}|\psi\rangle \), a very useful property in the development of the formalism.

In the mean-field approximation (MFA) the energy, \( E_{\text{MFA}} \), is the expectation value of (2) in the state (4). The calculation of the energy is straightforward and leads to

\[ E_{\text{MFA}} = \langle \psi | \hat{H} | \psi \rangle = \sum_{tm} \{ \xi_{tm}^* \xi_{tm} - G \left[ \xi_{tm}^* - (-1)^{t+m} \xi_{-t-m} \right] v_{tm} \} \]

(5)

where

\[ v_{tm} = \langle B | B_{tm} | B \rangle \]

(6)

refers to the baryon sector. The variational principle consists in minimizing the energy (5) in order to find the amplitudes. The variational equation \( \partial E_{\text{MFA}} / \partial \xi_{tm}^* = 0 \) allows us to find the best bare baryon state, which is the so-called hedgehog state

\[ |B\rangle_h = \frac{1}{\sqrt{2}} (|N\rangle_h + |\Delta\rangle_h) \]

(7)

where the nucleon and the delta hedgehogs are

\[ |N\rangle_h = \frac{1}{\sqrt{2}} \left( |N^+\rangle_\uparrow + |N^0\rangle_\uparrow \right) , \quad |\Delta\rangle_h = \frac{1}{2} \left( |\Delta^+\rangle_\uparrow - |\Delta^+\rangle_\downarrow + |\Delta^0\rangle_\downarrow - |\Delta^0\rangle_\uparrow \right) \]

(8)

It is worth noticing that the state (7) results from the configuration \( \langle q \rangle_h^3 \), with \( |q\rangle_h = \frac{1}{\sqrt{2}} (|u \downarrow\rangle - |d \uparrow\rangle) \) at the most fundamental quark level.

The baryon expectation value (6) is \( v_{tm} = \frac{9}{5} \left( \delta_{t1} \delta m - 1 + \delta_{t-1} \delta m1 - \delta_0 \delta m0 \right) \) and the amplitudes turn out to be \( \xi_{tm} = G v_{tm} \). Going back to the full space, the expectation value of the pion field is \( \vec{\pi}(r) = \hat{r} \phi(r) \) (here, the arrow denotes isovector), where \( \phi \) is a radial function. It is this peculiar correlation between isospin and angular momentum that led to the name “hedgehog”.

The hedgehog solution is the best mean-field approximation state or, in other words, it is the exact solution in the limit of large number of pions (very strong coupling regime). Numerically, one obtains \( E_{\text{MFA}} = -9.72 \, G^2 \).

2.2. Projection method

As already mentioned in the Introduction, the hedgehog state cannot directly describe a physical state since it is neither an isospin nor an angular momentum eigenstate. Nevertheless, using projection techniques we can extract from it such eigenstates, in particular to be identified with the nucleon and with the delta physical states. The projection method is similar to the one invented long ago in the context of nuclear physics.
Altogether, we are using variational methods and it turns out to be convenient to slightly enlarge the variational space, namely by generalizing the state (7), allowing for different amounts of nucleon hedgehog and delta hedgehog [2]. This more general hedgehog state (we call it generalized hedgehog or $\delta$-hedgehog) can be written as

\[ |B(\delta)\rangle_h = \cos \delta |N\rangle_h + \sin \delta |\Delta\rangle_h , \]

where the mixing angle, $\delta$, is to be treated as a variational parameter. Obviously, at mean-field level one has $\delta = 45^\circ$ as mentioned in the previous subsection.

Since the above state only contains states with $J = T$ it is enough to project only either in angular momentum or in isospin. The states to be identified with the nucleon or delta are

\[ |JM; T = JM_T = -M\rangle = \hat{P}_{JM}|\psi(\delta)\rangle \]

where $|\psi(\delta)\rangle$ contains the bare baryon state given by (9):

\[ |\psi(\delta)\rangle = N(\xi) \exp \left[ \frac{\xi}{\sqrt{3}} \left( a_{11}^\dagger + a_{-11}^\dagger - a_{00}^\dagger \right) \right] |B(\delta)\rangle_h \]

In the state (10), the projector operator is [3] $\hat{P}_{JM} = \frac{2J+1}{8\pi^2} \int d\Omega D_{JM}^\ast(\Omega)\hat{R}(\Omega)$, containing the Wigner $D$-functions and the rotation operator ($\Omega$ stands for the three Euler angles).

The calculation of the projected energy, for both the nucleon and the delta, can be carried on analytically, and one obtains after a lengthy but straightforward calculation [1, 2]:

\[ E_{1/2} = \frac{S}{3} \left[ 1 + \frac{f_2(\delta,S)}{f_2(\delta,S)} \right] - 2\sqrt{3} S \left[ 1 + \frac{2}{3} \sin 2\delta \frac{f_N(\delta) + f_{\Delta}(\delta)}{f_2(\delta,S)} \right] \]

where $S = \xi^2$ and

\[ f_J(\delta,S) = \langle \psi(\delta)|\hat{P}_{JM}|\psi(\delta)\rangle = \cos^2 \delta \ f_N(\delta) + \sin^2 \delta \ f_{\Delta}(\delta), \]

with

\[ f_N(\delta) = \frac{1}{2} e^{z/2} (k + 1) I_{k+1}(z), \quad f_{\Delta}(\delta) = \frac{1}{2z} e^{z/2} [k I_k(z) + (k + 2) I_{k+2}(z)] . \]

Here, we have introduced $z = \frac{2S}{3}$, $k = J - \frac{1}{2}$ and the modified Bessel functions, $I_v(z)$. Similarly as for the nucleon energy, one also obtains a closed expression for the delta energy:

\[ E_{3/2} = \frac{S}{3} \left[ 1 + \frac{f_2(\delta,S) + f_3(\delta,S)}{f_2(\delta,S)} \right] - 2\sqrt{3} S \left[ 1 + \frac{2}{3} \sin 2\delta \frac{f_N(\delta) + f_{\Delta}(\delta)}{f_2(\delta,S)} \right] . \]

The numerical results for the nucleon and the delta will be presented and discussed in Sec. 3.

2.3. Weak coupling limit

Using up to one pion admixtures, the perturbative state can be constructed for the nucleon and the delta, and the perturbative energy can be readily obtained using standard perturbation theory techniques. In the limit $\mathcal{G} \to 0$ these results are exact ones.

For the nucleon and for the delta the calculations lead, respectively, to $E_{1/2}^{\text{pert.}} = -20.52G^2$ and $E_{3/2}^{\text{pert.}} = -11.88G^2$. These energies are below the mean-field energy mentioned in subsection 2.1, as expected, and the nucleon energy is well below the delta one.
3. More results and discussion
In the previous section we already present the exact results of the model in the strong and in the weak coupling regimes. Regarding the variational approach, we just presented the final expressions, since the numerical results will be presented here.

The angular momentum (or isospin) projection method described above can be implemented in two different ways: the so called “Variation before projection” (VBP) method and the “Variation after projection” (VAP) method. It is known from nuclear physics that the former does not lead to the correct mass parameter of the system, a drawback corrected by the latter method.

In VBP, one first finds the best “mean-field” state, and then project it onto good spin-isospin states, according to the sequence:

$$E_{MFA} = \langle \psi | \hat{H} | \psi \rangle \rightarrow \delta E_{MFA} = 0 \rightarrow |\psi\rangle_0 \rightarrow |JT\rangle = P_{PY}^{JT} |\psi\rangle_0 ,$$

with $|\psi\rangle_0$ the best variational mean-field state. This means that the best variational parameters are obtained at the mean-field level. For the mixing angle, this leads to $\delta = 45^\circ$.

The VAP method, more difficult to implement, is superior. It explores a larger variational space, since it is the “projected energy” that is minimized (this leads e.g. to a $\delta$-hedgehog mixing angle in general different from $45^\circ$). The procedure flow is now

$$|JT\rangle = P_{PY}^{JT} |\psi\rangle \rightarrow E_{JT} = \langle JT | \hat{H} | JT \rangle \rightarrow \delta E_{JT} = 0 \rightarrow |JT\rangle_0 ,$$

the last state being the one representing the physical states.

In both limits, $G \rightarrow 0$ and $G \rightarrow \infty$ it is possible to work out analytically the energies for both nucleon and delta and, in table 1, we summarize these results.

| Physical | $G \rightarrow 0$ | $G \rightarrow \infty$ |
|----------|-----------------|-----------------|
|         | $\delta$ | $E_{VBP}$ | $\delta$ | $E_{VAP}$ | $E_{\text{pert.}}$ | $\delta$ | $\text{Energies}$ |
| Nucleon | 45°     | $-19.44 G^2$ | 58.0°     | $-20.52 G^2$ | $-20.52 G^2$ | 45°     | $-9.72 G^2$ |
| Delta   | 45°     | $-9.72 G^2$  | 68.2°     | $-11.88 G^2$ | $-11.88 G^2$ | 45°     | $-9.72 G^2$ |

Table 1. Energies and angles $\delta$ obtained in VBP and VAP calculations. For $G \rightarrow 0$ the perturbative energies are also shown. For $G \rightarrow \infty$ all methods (MFA, VBP, VAP) lead to the same results.

The most interesting point is that the variational method VAP is able to reach both the accurate perturbative results and the mean-field ones which are accurate in the strong coupling regime. It is also worth noticing that the generalization of the hedgehog is crucial in the small coupling regime. In this limit, the exact results and the prediction of the variational calculation match perfectly, a remarkable result indeed.

Figure 3 shows the numerical results in the full range of the coupling constant. The projected energies are always below the mean-field energy, though the difference gets smaller for large $G$. The VBP method, only displayed for the nucleon, in clearly inferior with respect to the VAP method, since the energy is higher. The improvement resulting from the new $\delta$ degree of freedom in the ansatz is mainly observed for small $G$ values.
4. Conclusions
Based on a projected coherent state description of the pion cloud, we used a simple model to
test its validity in the strong and in the weak coupling limits. As the results show, the coherent
state approach, supplemented with the Peierls-Yoccoz projection method, works rather well if
the variation-after-projection method is implemented. The result is not surprising in the strong
coupling limit since coherent states are indeed a good description for a large number of bosons
(mesons, photons,...). The surprise appears in the other limit: provided a proper quantum
mechanical approach is considered, one observes that the perturbative exact result is achieved.

Long ago I had the privilege to show these results to Sir Rudolf Peierls in one of his visits to the
University of Coimbra (late eighties or early nineties, I cannot precise). I was at the beginning of
my career and it was a unique opportunity for me to interact with such a great personality, the
author of several books, including *Surprises in Theoretical Physics* (1979) and *More Surprises
in Theoretical Physics* (1991). Our interesting discussion finished with him looking at me and,
with a smile, commenting my work: “another surprise in theoretical physics”.

Acknowledgments
My expertise on projected coherent states resulted a lot from the interaction with many
colleagues and friends, whom I would like to thank: Pedro Alberto, Bojan Golli, Mitja Rosina,
José Urbano, among others. In this regard, it is my duty also to recall the memory of the late
colleague and friend Klaus Goeke.

References
[1] Fiolhais M, Rosina M 1986 *Portugaliae Physica* 17 49
[2] Čibej M, Fiolhais M, Golli B, Rosina M 1992 *Journal of Physics G: Nuclear and Particle Physics* 18 49
[3] Peierls R Yoccoz J 1957 *Proc. Phys. Soc.* A70 381