EXAMPLE OF A 6-BY-6 MATRIX WITH DIFFERENT TROPICAL AND KAPRANOV RANKS

YAROSLAV SHITOV

ABSTRACT. We provide an example of a 6-by-6 matrix $A$ such that $rk_t(A) = 4$, $rk_K(A) = 5$. This answers a question asked by M. Chan, A. Jensen, and E. Rubei.

KEYWORDS: matrix theory, tropical semiring, tropical rank, Kapranov rank.

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1 Introduction

We work over the tropical semiring $(\mathbb{R}, \oplus, \otimes)$ whose operations are

$$a \oplus b = \min\{a, b\}, \quad a \otimes b = a + b.$$ 

We consider tropical matrices, i.e. matrices over the tropical semiring. There exist many different ways to define the rank of a tropical matrix, see [1, 4]. We deal with the notions of tropical rank and Kapranov rank, see also [3, 5].

Definition 1.1. We define the permanent of a tropical matrix $S \in \mathbb{R}^{n \times n}$ as

$$\text{perm}(S) = \min_{\sigma \in S_n} \{s_{1, \sigma(1)} + \ldots + s_{n, \sigma(n)}\}. \quad (1.1)$$

Definition 1.2. The matrix $S$ is called tropically singular if the minimum in (1.1) is attained at least twice. Otherwise, $S$ is called tropically non-singular.

Definition 1.3. The tropical rank of a matrix $M \in \mathbb{R}^{p \times q}$ is the largest integer $r$ such that $M$ has a tropically non-singular $r$-by-$r$ submatrix. We denote the tropical rank of $M$ by $rk_t(M)$.

Let $K$ denote the field whose elements are formal sums

$$a(t) = \sum_{i=1}^{\infty} a_i t^{\alpha_i} \quad \text{such that } a_n \in \mathbb{C}, \alpha_n \in \mathbb{R}, \lim_{n \to \infty} \alpha_n = +\infty.$$
Let \( \deg : \mathbf{K}^* \to \mathbb{R} \) be a natural valuation sending \( a(t) \) to the least of the exponents \( \alpha_i \), i.e. \( \deg(a) = \min_{n:a_n \neq 0} \{\alpha_n\} \). By definition, assume \( \deg(0) = \infty \). We say that \( B \in \mathbf{K}^{m \times n} \) is a lift of \( T \in \mathbb{R}^{m \times n} \) if \( \deg(b_{ij}) = t_{ij} \) for any \( i, j \).

The notion of the Kapranov rank of a matrix can be defined in the following way, see [4, Corollary 3.4].

**Definition 1.4.** Let \( M \in \mathbb{R}^{m \times n} \). We define the Kapranov rank of \( M \) as

\[
\text{rk}_K(M) = \min_{\mathbf{K}_M} \{\text{rank}(\mathbf{K}_M)\},
\]

where the minimum is taken over all lifts of \( M \). The expression \( \text{rank}(\mathbf{K}_M) \) means the usual rank of a matrix \( \mathbf{K}_M \) over the field \( \mathbf{K} \).

The notion of Kapranov rank was deeply investigated in [3, 4, 5]. Develin, Santos, and Sturmfels in [4] show that \( \text{rk}_K(M) \geq \text{rk}_t(M) \) for every matrix \( M \). The following theorem points out the connection with matroids.

**Theorem 1.5.** [4, Corollary 7.4] Let \( \mathcal{M} \) be a matroid which is not representable over \( \mathbb{C} \). Then the Kapranov and tropical ranks of the cocircuit matrix \( \mathcal{C}(\mathcal{M}) \) are different.

Theorem 1.5 makes it possible to construct examples of matrices with different tropical and Kapranov ranks. The example of a 7-by-7 matrix with different ranks is provided in [4].

Kim and Roush in [5] mostly deal with algorithmical aspects of the Kapranov rank. They prove that determining Kapranov rank of tropical matrices is NP-hard. Also, in [5] it was shown that there exist matrices of tropical rank 3 and arbitrarily high Kapranov rank.

The following theorem was proven in [3].

**Theorem 1.6.** [3, Corollary 1.5] Let \( M \in \mathbb{R}^{m \times n}, \min\{m, n\} \leq 5 \). Then \( \text{rk}_K(M) = \text{rk}_t(M) \).

Chan, Jensen, and Rubei in [3] point out the connection with the notion of tropical basis. They ask the following question.

**Question 1.7.** [3, Question 1.1] For which numbers \( d, n, r \) do the \((r+1) \times (r+1)\)-minors of a \( d \)-by-\( n \) matrix form a tropical basis? Equivalently, for which \( d, n, r \) does every \( d \)-by-\( n \) matrix of tropical rank at most \( r \) have Kapranov rank at most \( r \)?
In [3] the following conjecture was also made.

**Conjecture 1.8.** [3, Conjecture 1.6] The \((r+1) \times (r+1)\) minors of a \(d\)-by-\(n\) matrix are a tropical basis if and only if either \(r \leq 2\) or \(r \geq \min\{d, n\} - 2\).

Also, in [3] it was asked whether there exists a 6-by-6 matrix with different tropical and Kapranov ranks. We answer this question by providing an example of a 6-by-6 matrix with tropical rank 4 and Kapranov rank 5.

Now let us take into account the equivalence given in Question 1.7. Our example shows that the 5-by-5 minors of a 6-by-6 matrix are not a tropical basis. Thus we disprove Conjecture 1.8.

Additionally, we note that the difference between the tropical and Kapranov ranks of our matrix does not have a matroidal nature. Indeed, matroids with at most 6 elements are all representable over \(\mathbb{C}\), see [2].

### 2 The Example

**Example 2.1.** Let

\[
A = \begin{pmatrix}
0 & 0 & 4 & 4 & 4 & 4 \\
0 & 0 & 2 & 4 & 1 & 4 \\
4 & 4 & 0 & 0 & 4 & 4 \\
2 & 4 & 0 & 0 & 2 & 4 \\
4 & 4 & 4 & 0 & 0 & 0 \\
2 & 4 & 1 & 4 & 0 & 0
\end{pmatrix}.
\]

Then \(\text{rk}_t(A) = 4\), \(\text{rk}_K(A) = 5\).

**Proof.** 1. Note that every 5-by-5 submatrix of \(A\) can be written in some of the following forms (up to permutations of rows and columns):

\[
S' = \begin{pmatrix}
0 & s_{12}' & s_{13}' & s_{14}' & s_{15}' \\
s_{21}' & 0 & 0 & s_{24}' & s_{25}' \\
s_{31}' & 0 & 0 & s_{34}' & s_{35}' \\
s_{41}' & s_{12}' & s_{43}' & 0 & 0 \\
s_{51}' & s_{52}' & s_{53}' & 0 & 0
\end{pmatrix}, \quad
S'' = \begin{pmatrix}
0 & 4 & 4 & 4 & 4 \\
0 & x & 4 & y & 4 \\
0 & 0 & 4 & 4 \\
0 & 0 & 0 & z & 4 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

where \(x, y, z \in \{1, 2\}, s_{ij}', s_{ij}'' \in \{1, 2, 4\}\). By Definition 1.1, \(\text{perm}(S') = 0\). The minimum in (1.1) for \(S'\) is given by \(\text{id}, (23) \in S_5\). Analogously, \(\text{perm}(S'') = y\), the minimum is given by \((24), (243) \in S_5\). Thus by Definition 1.2, every
5 × 5-submatrix of $A$ is tropically singular. From Definition 1.3 it follows that $rk_t(A) \leq 4$.

Now consider the 4-by-4 submatrix which is formed by the 1st, 2nd, 4th, and 6th rows and the 1st, 4th, 5th, and 6th columns of $A$:

\[
\begin{pmatrix}
0 & 4 & 4 & 4 \\
0 & 4 & 1 & 4 \\
2 & 0 & 2 & 4 \\
2 & 4 & 0 & 0
\end{pmatrix}.
\]

The minimum in the expression for its permanent is given by the only permutation $(23) \in S_4$. Thus by Definition 1.3, $rk_t(A) = 4$.

2. Let us consider the matrix

\[
M_0 = \begin{pmatrix}
1 & 1 & t^4 & t^4 & t^4 & t^4 \\
-1 & -1 & t^2 & t^4 & t & t^4 \\
t^4 & t^4 & 1 - t^2 & 1 & -t^4 & -t^4 \\
t^2 & t^4 & -1 - t & -1 & t^2 & -t^4 \\
-t^4 & -t^4 & -t^4 & -1 - t^2 & 1 \\
-t^2 & -t^4 & t & -t^4 & 1 - t & -1
\end{pmatrix} \in K^{6 \times 6},
\]

which is a lift of $A$. The sum of the rows of $M_0$ is the zero row, so that the rank of $M_0$ is at most 5. Thus by Definition 1.4, $rk_K(A) \leq 5$.

Now let $H \in K^{6 \times 6}$ be an arbitrary lift of $A$. It follows directly from definitions that $deg(ab) = deg(a) + deg(b)$, $deg(a + b) \geq \min\{deg(a), deg(b)\}$ for any $a, b \in K$. Since $deg(h_{pq}) = a_{pq}$ for any $p, q$, we obtain the following expression for the minor $H_{25}$:

\[
H_{25} = h_{12}h_{34}h_{41}h_{56}h_{63} + h_{12}h_{33}h_{44}h_{56}h_{61} - h_{12}h_{34}h_{43}h_{56}h_{61} + g_1,
\]

where $deg(g_1) \geq 4$. Analogously, the minor $H_{61}$ can be expressed as

\[
H_{61} = h_{12}h_{25}h_{33}h_{44}h_{56} - h_{12}h_{25}h_{34}h_{43}h_{56} + g_2, \ deg(g_2) \geq 4.
\]

We denote $\Delta = h_{33}h_{44} - h_{34}h_{43}$, $\delta = deg(\Delta)$. We obtain

\[
H_{25} = h_{12}h_{34}h_{41}h_{56}h_{63} + h_{12}\Delta h_{56}h_{61} + g_1, \ deg(h_{12}h_{34}h_{41}h_{56}h_{63}) = 3, \quad
\]

\[
\begin{aligned}
&deg(h_{12}\Delta h_{56}h_{61}) = 2 + \delta; \\
&\text{(2.1)}
\end{aligned}
\]
\[ H_{61} = h_{12} h_{25} \Delta h_{56} + g_2, \quad \text{deg}(h_{12} h_{25} \Delta h_{56}) = 1 + \delta. \]  

(2.2)

It follows from definitions that \( \text{deg}(v_1 + v_2) = \min\{\text{deg}(v_1), \text{deg}(v_2)\} \) for any \( v_1, v_2 \in \mathbb{K} \) such that \( \text{deg}(v_1) \neq \text{deg}(v_2) \). Thus if \( \delta > 1 \), then from (2.1) it follows that \( \text{deg}(H_{25}) = 3 \), i.e. \( H_{25} \neq 0 \). Analogously, if \( \delta < 1 \), then \( \text{deg}(H_{25}) = 2 + \delta \), i.e. \( H_{25} \neq 0 \). Finally, if \( \delta = 1 \), then from (2.2) it follows that \( \text{deg}(H_{61}) = 2 \), i.e. \( H_{61} \neq 0 \). We see that some of the minors \( H_{25} \) and \( H_{61} \) differs from 0. This shows that the rank of \( H \) is at least 5. By Definition 1.4, \( \text{rk}_K(A) \geq 5 \). The proof is complete. \( \square \)

**Theorem 2.2.** The matrix \( A \) from Example 2.1 contains the least number of rows and the least number of columns among tropical matrices \( M \) such that \( \text{rk}_K(M) \neq \text{rk}_t(M) \).

**Proof.** Follows from Theorem 1.6 and Example 2.1. \( \square \)

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Faculty of Algebra, Department of Mathematics and Mechanics, Moscow State University, GSP-1, 119991 Moscow, Russia.

E-mail: yaroslav-shitov@yandex.ru