COMPOSITIONAL THEORIES FOR HOST-CORE LANGUAGES

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Abstract. Linear type theories, of various types and kinds, are of fundamental importance in most programming language research nowadays. In this paper we describe an extension of Benton’s Linear-Non-Linear type theory and model for which we can prove some extra properties that make the system better behaved as far as its theory is concerned. We call this system the host-core type theory. The syntax of a host-core language is split into two parts, representing respectively a host language $H$ and a core language $C$, embedded in $H$. This idea, derived from Benton’s Linear-Non-Linear formulation of Linear Logic, allows a flexible management of data linearity, which is particularly useful in non-classical computational paradigms. The host-core style can be viewed as a simplified notion of multi-language programming, the process of software development in a heterogeneous programming language. In this paper, we present the typed calculus $HC$, a minimal and flexible host-core system that captures and standardizes common properties of an ideal class of host-core languages. We provide a denotational model in terms of enriched categories and we state a strong correspondence between syntax and semantics through the notion of internal language. The latter result provides some useful characterizations of host-core style, otherwise difficult to obtain. We also discuss some concrete instances, extensions and specializations of the system $HC$.

1. Introduction

The idea of dividing the syntax of a formal system into two communicating parts was developed both in logic and in computer science. In [Ben95] Benton introduced LNL (Linear-Non-Linear Logic), a presentation of Linear Logic where the bang (!) modality is decomposed into an adjunction between a symmetric monoidal closed category and a cartesian closed category. LNL models are a considerable simplification of those of Intuitionistic Linear Logic and, in the following years, LNL catalysed the attention of the categorical logic community. Maietti et al. in [MMdPR05] discussed models and morphisms for several versions of Linear Logic, including LNL. Benton’s idea enjoys an interesting interpretation also from a programming language perspective. In fact, the recent quantum computing literature shows that LNL provides a good foundation for paradigmatic calculi in which a “main language” controls the computation and delegates the management of quantum data to a linear embedded “core”. LNL has inspired interesting working quantum circuit definition languages such as

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QWire [PRZ17, RPZ17, Ran18, Pay18] and its generalization EWire, whose foundation and denotation have been partially studied by Staton et al. [RS20a].

A situation where a main language interacts with others is not a peculiarity of quantum languages. In practical programming, it is frequent to use a principal encoding language and to distribute some tasks to another (often domain specific) language, “called” or imported in terms of ready-to-use subroutines. Programming platforms and modern IDE systems provide efficient methods to manage and control the process of software development in a heterogeneous programming language environment and software developers need tools to combine different languages, sometimes called cross-language interoperability mechanisms [Chi13]. Multi-Language Programs, as named in a pioneering investigation by Matthews and Findler [MF07], became a flourishing research area in programming languages. Even if some interesting results have been achieved in the last decade [OSZ12, PPDA17, BM19, BCM20] this is still an open research field, in particular from the programming foundation perspective.

Following the intuitions described above, we propose a typed calculus called HC, built upon two communicating languages H and C. We design and study HC following two different traditions. On the one hand, we follow the categorial logic tradition started by Benton, and for that [MMdPR05] is our main technical reference. On the other hand, (quantum) circuit description languages such as QWire and EWire (and also their ancestor Quipper) are the kind of application we envisage: in particular, we adopt from [RS20a] the use of enriched categories as our denotational model. Beyond these technical references, we take some inspiration from practical multi-language programming. To better frame our work with respect to the state of art, we identify the notion of a host-core language as a simplification of the notion of a multi-language framework.

The syntax of a host-core language is split into two parts, one for the host, an arbitrarily powerful language H, and one for the core language C, which is embedded in H. Their interaction is controlled through mixed typing rules that specify how to “promote” well-formed terms from the core language to the host language and (possibly) conversely. With respect to a more general definition of multi-language framework (coming from [OSZ12, PPDA17, MF07, BM19, BCM20]):

1. we focus our attention on exactly two languages;
2. we model a restricted form of interoperability, in which the host can import all core terms (and types) but the core cannot import any terms from the host. This could be seen as a limitation, but we show how even this restricted case is challenging enough to model, and already provides interesting results. Notice that Benton’s LNL type theory, which represents our starting point, consists of the particular case of a host-core language where the core is strictly linear.

The ultimate goal of HC (and its future developments) is to define a completely compositional theory that allows one to reason about the host language, the core language, and their communication, pursuing full flexibility of the modelling framework. In this first investigation we propose a study of what we call a “kernel” type theory, common to an ideal class of host-core languages. We propose a semantics for these host-core languages in terms of enriched category theory, obtaining an internal language theorem (in the sense of categorical logic), which states a full correspondence between the syntax and semantics of HC type theories.
1.1. Main Results.

1) A “kernel” type theory for host-core languages: We describe a kernel type theory for host-core languages where the core language is linear and the host language includes the usual lambda-calculus: we define a systematised type theory, which formally defines this notion of host-core languages. We address the problem from a general algebraic perspective and we present a minimal system called HC, a simply-typed lambda calculus hosting a purely linear core language. Our starting point is a simple language but our goal is not only to achieve a Linear-Non-Linear system, but also to be able to easily extend both the host and the core fragments of the system. We choose to start from a linear core, since, as we will show, we can obtain a non-linear core as a particular instance. To this end we exploit the HC compositionality. The design of the HC type system “privileges” the host language H, establishing a hierarchical dependency of the core fragment on the control fragment: the embedded language C is fully described in the host H. We will provide a running example that explains the design choices we made, showing HC instances and extensions.

2) Denotation of host-core calculi: We provide a semantics for HC in terms of enriched categories, partially following the blueprint of [RS20b]. The compositionality we claim for the HC syntax is mirrored at the semantic level. The models of HC are pairs (H, C) where H is a cartesian closed category and C is a H-symmetric monoidal category. In particular, we show how the presentation of the calculus admits a natural correspondence between syntactic properties and semantics constraints: $\beta$ and $\eta$ rules become coherence conditions on the morphisms of the algebraic structures. Ideally, we want to pursue a full correspondence between syntax and HC models in order to obtain more refined type theories by simply adding syntactic rules and (equivalent) denotational properties, without changing the rules of the basic language HC.

3) Internal language for host-core calculi: We study the relationship between host-core languages and their models, using HC as a case study. We point out that the expected soundness and completeness theorems may be not enough to fully identify the most appropriate class of models for a type theory, as explained in the work [MMdPR05]. Moving beyond soundness and completeness, we relate host-core languages and their categorical models via the notion of internal language. One can find several examples and applications of the notion of internal language in the categorical logic literature, e.g. see [LS86, Pit95, Joh02]. By definition, HC provides an internal language of the category Model(HC) of its models if one proves, as we do in Section 3, an equivalence between Model(HC) and the category Th(HC) of the theories of the language. We remark that in this paper we are working in the context of enriched categories and the definition of a suitable notion of morphisms for the category of models Model(HC) is the interesting and challenging point of the proof. This requires the technical notion of “change of base” which formally describes how, changing the host language, one can induce a change of the embedding for a given core language. After we show the correspondence between syntax and semantics via an internal language theorem, we provide some concrete examples of models.

4) A bridge towards host-core programming theory: In the last part of the paper we show how the results we proved can be used to provide an initial but significant formal definition of host-core languages. In the last few years the foundations of multi-language
programming theory have been much discussed [OSZ12, PPDA17, BM19, BCM20], however a key question remains unexplored: is there a formal characterization of a programming theory, as a type theory and a denotational semantics, where different heterogeneous languages are used and allowed to communicate with each other?

Focusing on the “host-core” languages as a particular simplified case of the most general notion of multi-languages, we answer three instances of the generic questions below. Consider two programming languages $L_1$ and $L_2$, supplied with their own syntax, semantics and mathematical properties.

**Q1:** Is there a language $L_3$ built out from $L_1$ and $L_2$, that describes both $L_1$ and $L_2$ and their communication allowing for the import and export of terms from one syntax into the other? If so, what is $L_3$’s formal semantics?

**Q2:** If $L_1$ is a host-core language, can we compare $L_1$ with another standalone language $L_2$?

**Q3:** Does the internal language result represent a useful tool for designing new host-core languages?

The above questions are particularly relevant for host-core languages and are non-trivial, since $L_1$ and $L_2$ could be radically different. Moreover, we do not consider mere extensions of a host language. We try to understand when a given language is expressive enough to host another given language, to delegate specific computations to it, and to represent its programs at the host syntax level.

As shown in Section 4, we achieve this “semantically”. Thanks to the existence of the denotational model and the strong correspondence with the syntax, we can answer the question **Q1**, providing a definition of the host-core language $L_3$. As far as **Q2** is concerned, notice that the problem of formally comparing a standalone language with a host-core language is a very tricky task. Once again, the internal language theorem gives a preliminary solution. The correspondence provides an initial answer also to the question **Q3**.

**Synopsis.** This paper is organized as follows. In Section 2 we present the syntax, type system and evaluation rules of the calculus $HC$, as well as a running example. In Section 3 we describe the categorical models of $HC$ and we prove the correspondence between syntax and denotation via the notion of internal language. In Section 4 we use the notion of internal language to answer the questions above. In Section 5 we recall the state of the art on multi-language frameworks; discussions, conclusions and future work are in Section 5.2. Finally, in Appendix A we show the evaluation rules of the host languages $H$ and in Appendix B we recall some background notions about enriched categories.

## 2. A “kernel” host-core calculus

In this section we present the typed calculus $HC$, built upon two components: a *host language* $H$ designed as a simply typed lambda calculus, and linear *embedded core language* $C$.

The presentation of the typed calculus proposed here gives a *privileged* position to the host language, in the sense that the embedded language is fully described in the host one. This means that, differently from Benton’s logic, in $HC$ the communication between the languages $H$ and $C$ is not possible in both directions. Roughly speaking, we cannot carry all the terms of $H$ into the language $C$, modelling a hierarchical dependency of the core on the host language. From the point of view of the categorical semantics we will not have any comonad or adjunction between the categories that provide denotation to the host and the
compositional theories for host-core languages but, as we will see in Section 3, we have just an instance of an enriched category.

2.1. General presentation of the typed system. We introduce type constructors, well-formation and evaluation rules for $\mathcal{HC}$. Notice that we describe the rules in logical style, assuming that we have a universe of (base) types. Concrete instances of $\mathcal{HC}$ can be easily obtained setting base types to a set of interest and building other types by type constructors. A concrete instance of type and term syntax is in our running example (see Examples 2.2, 2.5 and 2.6 below). We strictly follow the notation used in [MMDPR05] and we consider two different kinds of types, terms-in-contexts and judgements: $\mathcal{H}$-types and $\mathcal{H}$-terms for the host language $\mathcal{H}$ and $\mathcal{C}$-types and $\mathcal{C}$-terms, for the pure linear language $\mathcal{C}$ which is embedded in $\mathcal{H}$.

Type and term judgements and type and term equalities for $\mathcal{H}$ are the following ones

$$
\vdash_{\mathcal{H}} X : \text{type} \quad \vdash_{\mathcal{H}} X = Y : \text{type} \quad \Gamma \vdash_{\mathcal{H}} t : X \quad \Gamma \vdash_{\mathcal{H}} t = s : X.
$$

The first judgement says that a syntactical object is of sort type (in the empty context), the second that two types are equal, the third states that a term $t$ in a context $\Gamma$ has type $X$ and the fourth that two terms are equal. Type and term judgements with corresponding equalities for the core language $\mathcal{C}$ are of the following form

$$
\vdash_{\mathcal{C}} A : \text{type} \quad \vdash_{\mathcal{C}} A = B : \text{type} \quad \Gamma \mid \Omega \vdash_{\mathcal{C}} f : A \quad \Gamma \mid \Omega \vdash_{\mathcal{C}} f = h : A
$$

where $\Gamma \mid \Omega$ denotes a mixed context. In a mixed context we split out the host part $\Gamma$ and the core part $\Omega$. Notice that, as a particular case, both $\Gamma$ and $\Omega$ can be empty.

Given a host context $\Gamma$ (resp. a core context $\Omega$) we denote as $\times_{\Gamma}$ (resp. $\otimes_{\Omega}$) the product of the elements in $\Gamma$ (resp. in $\Omega$).

To stress the distinction between the host and the core parts of the syntax, we write $X, Y, Z \ldots$ for the $\mathcal{H}$-types, $\Gamma, \Gamma'$ for $\mathcal{H}$-contexts, $x, y, z \ldots$ for $\mathcal{H}$-variables, and $s, t, v \ldots$ for $\mathcal{H}$-terms. We write $A, B, C \ldots$ for the $\mathcal{C}$-types, $\Omega, \Omega'$ for $\mathcal{C}$-contexts, $a, b, c \ldots$ for $\mathcal{C}$-variables, and $h, f, g \ldots$ for $\mathcal{C}$-terms.

We consider two fixed sets of base types: a set $\Sigma_{\mathcal{C}}$ of base $\mathcal{C}$-types, ranged over by $\alpha$ (possibly indexed) and a set $\Sigma_{\mathcal{H}}$ of base $\mathcal{H}$-types, ranged over by $\iota$ (possibly indexed).

Similarly to [PRZ17, RS20a], we have a host-type of the form $\text{Proof}(A, B)$ (where $A$ and $B$ are core-types). The proof-theoretical meaning of a type $\text{Proof}(A, B)$ is that it represents the type of proofs from $A$ to $B$ in the host language. Given a core term, we can read it as a proof and in this way we can speak about $\mathcal{C}$-proofs at the host level.

The type constructor rules are presented in Figure 1. Notice that in (t2c) we obtain a $\mathcal{H}$-type starting from two $\mathcal{C}$-types.
\[(t_{01}c)\quad \vdash_{C} \alpha : \text{type} \quad \text{for any base C-type } \alpha\]

\[(t_{02}c)\quad \vdash_{C} I : \text{type}\]

\[(t_{1c})\quad \vdash_{C} A : \text{type} \quad \vdash_{C} B : \text{type} \quad \vdash_{C} A \otimes B : \text{type}\]

\[(t_{2c})\quad \vdash_{C} A : \text{type} \quad \vdash_{C} B : \text{type} \quad \vdash_{H} \text{Proof}(A, B) : \text{type}\]

\[(t_{01}h)\quad \vdash_{H} \iota : \text{type} \quad \text{for any base H-type } \iota\]

\[(t_{02}h)\quad \vdash_{H} 1 : \text{type}\]

\[(t_{1h})\quad \vdash_{H} X : \text{type} \quad \vdash_{H} Y : \text{type} \quad \vdash_{H} X \times Y : \text{type}\]

\[(t_{2h})\quad \vdash_{H} X : \text{type} \quad \vdash_{H} Y : \text{type} \quad \vdash_{H} X \rightarrow Y : \text{type}\]

Figure 1: Type constructors

We require that all the type constructors preserve the equality of types as usual. For example, we require \(\text{Proof}(-,-)\) to preserve the equality of types, i.e. we have to add the rule

\[(eqT)\quad \vdash_{C} A = B : \text{type} \quad \vdash_{C} C = D : \text{type} \quad \vdash_{H} \text{Proof}(A, C) = \text{Proof}(B, D) : \text{type}\]

to the system.

Let \(\mathcal{J}_C\) be the set of core types. The grammar of core types is defined as:

\[A, B, C ::= I \mid \alpha \mid A \otimes B\]

where \(I\) is the unit type and \(\alpha \in \Sigma_C\).

Let \(\mathcal{J}_H\) the set of host types. The grammar of host types is defined as

\[X, Y, Z ::= 1 \mid \iota \mid X \times Y \mid X \rightarrow Y \mid \text{Proof}(A, B)\]

where \(1\) is the unit type, \(\iota \in \Sigma_H\) and \(A\) and \(B\) in \(\text{Proof}\) are \(C\)-types. \(\Sigma_H\) could be instantiated, for example, to the set \{Nat, Bool\}. The type \(\text{Proof}(A, B)\) represents and models the meeting point between the two syntaxes.

The grammar of \(C\) terms is

\[f, g ::= \bullet \mid a \mid f \otimes g \mid \text{let } f \text{ be } a \otimes b \text{ in } h \mid \text{let } f \text{ be } \bullet \text{ in } h \mid \text{derelict}(s, f)\]

where \(s\) is a host term, i.e. it ranges over the following \(H\) grammar:

\[s, t ::= * \mid x \mid (s, t) \mid \pi_1(s) \mid \pi_2(s) \mid \lambda x : X.t \mid st \mid \text{promote}(a_1, \ldots, a_n, f)\]

with \(f\) a core term.

The grammars of terms are those of a simply-typed lambda calculus and a linear calculus with \text{let} constructors for the host \(H\) and the core \(C\), respectively. We also introduce the host unit term \(*\) of host unit type 1, and the core unit term \(\bullet\) of core unit type \(I\). Moreover, we add to the host and the core grammars respectively the functions \text{promote}(\cdot, \cdot)\) and \text{derelict}(\cdot, \cdot)\) that model the communication between the host and the core syntaxes.
The typing rules for $\mathcal{HC}$ are in Figure 2.

\begin{align*}
\text{(av)} & \quad \Gamma_1, x : X, \Gamma_2 \vdash_{\mathcal{H}} x : X \\
\text{(pv)} & \quad \Gamma \vdash_{\mathcal{H}} s : X \quad \Gamma \vdash_{\mathcal{H}} t : Y \\
\text{(π1v)} & \quad \Gamma \vdash_{\mathcal{H}} \pi_1(v) : X \quad \Gamma \vdash_{\mathcal{H}} \pi_2(v) : Y \\
\text{(π2v)} & \quad \Gamma \vdash_{\mathcal{H}} v : X \times Y \\
\text{(aiv)} & \quad \Gamma_1 \vdash_{\mathcal{H}} t : \Lambda \quad \Gamma \vdash_{\mathcal{H}} s : X \quad \Gamma \vdash_{\mathcal{H}} (s, t) : X \times Y \\
\text{(ac)} & \quad \Gamma \vdash_{\mathcal{H}} s \cdot t : \Lambda \\
\text{(uc)} & \quad \Gamma \vdash_{\mathcal{H}} (−) \cdot I \\
\text{(tc)} & \quad \Gamma \vdash_{\mathcal{H}} promote(a) : \Lambda \\
\text{(let1c)} & \quad \Gamma \vdash_{\mathcal{H}} let f \ be \ a \ in \ h : C \\
\text{(let2c)} & \quad \Gamma \vdash_{\mathcal{H}} let f \ be \ \bullet \ in \ h : C \\
\text{(prom)} & \quad \Gamma \vdash_{\mathcal{H}} promote(a) : \Lambda \\
\text{(der)} & \quad \Gamma \vdash_{\mathcal{H}} derelict(t) : \Lambda 
\end{align*}

Figure 2: Typing rules

Notice that in the dereliction rule (der) a host-term is “derelicted” to the core when applied to a (linear) core-term of the correct type. This style is consistent with the standard way of presenting this kind of rule (e.g., see [RS20a]). Similarly, the promotion rule (prom) “promotes” a term from the core to the host, by “preserving” the linear context. In the following, we denote by $\Gamma \vdash_{\mathcal{H}} \text{promote}(−, f) : \text{Proof}(I, A)$ the result of the promotion of a core term of the shape $\Gamma \vdash_{\mathcal{H}} f : A$. The duality between (der) and (prom) is formalized in Figure 4.

2.2. Evaluation. In this section, we introduce an equational theory for $\mathcal{HC}$. First we present a substitution lemma. It is straightforward to prove (by induction on the derivations) that all substitutions are admissible.

Notice also that our substitutions are quite similar to the substitutions presented in the enriched effect calculus EEC, see [EMS14, Prop. 2.3]

**Lemma 1 (Substitution).** The following hold:
• (sub1) if $\Gamma \vdash_H s : X$ and $\Gamma, x : X \mid \Omega \vdash_C e : A$ then $\Gamma \mid \Omega \vdash_C e[s/x] : A$;
• (sub2) if $\Gamma \mid \Omega_1 \vdash_C g : A$ and $\Gamma \mid \Omega_2, a : A \vdash_C f : B$ then $\Gamma \mid \Omega_1, \Omega_2 \vdash_C f[g/a] : B$;
• (sub3) if $\Gamma \vdash_H s : X$ and $\Gamma, x : X \vdash_H t : Y$ then $\Gamma \vdash_H t[s/x] : Y$.

Notice that (sub1) allows us to substitute a host term for a variable in a core term. In designing (sub2), we follow the line adopted in languages like QWire and EWire, which have some primitive notions of substitution for composing functions of Proof$(\cdot, \cdot)$ type. Finally, (sub3) models the replacement of a term for a variable in the host language as well as in simply typed lambda calculus.

The $\beta$ and $\eta$ rules for the $H$-terms are those of simply typed lambda calculus with the connectives $\times$ and $\to$, see Appendix A.

The $\beta$ and $\eta$ rules for the $C$-terms are those of the modality-free fragment of Linear Logic (without exponents) and are in Figure 3.

\begin{align*}
(\text{el1}) & \quad \frac{\Gamma \mid \Omega_1, a : A, b : B \vdash_C f : C \quad \Gamma \mid \Omega_2 \vdash_C g : A \quad \Gamma \mid \Omega_3 \vdash_C h : B}{\Gamma \mid \Omega_1, \Omega_2, \Omega_3 \vdash_C \text{let } g \otimes h \text{ be } a \otimes b \text{ in } f = f[g/a, h/b] : C} \\
(\text{el2}) & \quad \frac{\Gamma \mid \Omega_1, c : A \otimes B \vdash_C f : C \quad \Gamma \mid \Omega_2 \vdash_C g : A \otimes B}{\Gamma \mid \Omega_1, \Omega_2 \vdash_C \text{let } g \otimes be a \otimes b \text{ in } f[a \otimes b/c] = f[g/c] : C} \\
(\text{el3}) & \quad \frac{\Gamma \mid \Omega \vdash_C f : A}{\Gamma \mid \Omega \vdash_C \text{let } \bullet \text{ be } \bullet \text{ in } f = f : A} \\
(\text{el4}) & \quad \frac{\Gamma \mid \Omega_1, a : I \vdash_C f : A \quad \Gamma \mid \Omega_2 \vdash_C g : I}{\Gamma \mid \Omega_1, \Omega_2 \vdash_C \text{let } g \otimes be \bullet \text{ in } f[\bullet/a] = f[g/a] : A}
\end{align*}

Figure 3: Let-Evaluation rules for the core language $C$

We also add the following rules in Figure 4, modelling the duality between promotion and dereliction operations, where $\Omega := [a_1 : A_1, \ldots, a_n : A_n]$:

\begin{align*}
(\text{der-prom}) & \quad \frac{\Gamma \mid \Omega \vdash_C f : A \quad \Gamma \mid \Omega_1 \vdash_C g_1 : A_1 \ldots \Gamma \mid \Omega_n \vdash_C g_n : A_n}{\Gamma \mid \Omega \vdash_C \text{derelict.promote}(a_1, \ldots, a_n.f), g_1 \otimes \ldots \otimes g_n = f[g_1/a_1 \ldots g_n/a_n] : A} \\
(\text{prom-der}) & \quad \frac{\Gamma \vdash_H f : \text{Proof}(A, B)}{\Gamma \vdash_H \text{promote}(a.\text{derelict}(f, a)) = f : \text{Proof}(A, B)}
\end{align*}

Figure 4: Promote and Derelict duality rules

With respect to Benton’s formulation of the LNL system, here we add the $\eta$ rules to the calculus following the presentation of [MMdPR05]. Observe that this is necessary to get a completeness result.

Moreover, we highlight again that a crucial difference between the LNL original presentation and our calculus $HC$ is that the interaction between the $H$-language and $C$ is not symmetric, since we are giving a privileged position to the host language $H$. 
Observe that if we consider the pure $C$-judgements, which are those of the form

$$- | \Omega \vdash_c f : A$$

we obtain exactly a pure linear language, with tensor products.

Since we are programmatically defining $HC$ as a minimal system, having a linear core is a natural starting choice. This could appear very limiting and difficult to overcome. However, it is not the case, as shown in the following example.

**Example 2.1** (Breaking the linearity of the core $C$, Part I). To extend $C$ to a non-linear core, we have to force the tensor product to be a cartesian product. This can be done by adding to the core type system the rules $(uv)$, $(pv)$, $(\pi 1)$, $(\pi 2)$ (with mixed contexts) and (eventually) new base types. Notice that old rules for tensor product are derivable from the new ones.

Before we proceed to the presentation of $HC$, we recall the “roadmap” that motivates our investigation. Linearity is an important property when designing languages for quantum computing. At the same time, when designing multi-language frameworks, some of the languages involved might have linear features. Both of these lines of enquiry require us to know how to deal with a minimal linear system suitably embedded into a host language. We designed $HC$ following these two intuitions. On the one hand, we follow the categorical logic tradition started by Benton. On the other hand, we also want to model some specific languages for quantum computing and in general for circuit manipulation. At the same time, we take inspiration from the practice of embedded programming. Our goal is not to provide a full description of host-core programming theory, but to show how a principled minimal system such as $HC$ works and can be extended to provide a useful basis for future systems.

Next, we develop our running example built upon the concrete syntax of $HC$ (see Examples 2.2, 2.5 and 2.6). We sketch how to equip $HC$ with constants and functions to support basic circuit definition and manipulation, suggesting a specialization of $HC$ as a system hosting a (toy) hardware definition language.

**Remark 1.** We would like to emphasize that the complete definition of concrete instances of syntax for the definition of paradigmatic languages, as well as the study of operational semantics, are beyond the scope of this work. The exploration of these intriguing developments is left to future research. The examples are intended to be understood as support for the reader and as a tool to suggest possible connections with the literature on languages for circuit manipulation, which are particularly relevant for both classical and probabilistic, reversible, and quantum computations.

**Example 2.2** (A circuit core language). Let $C^*$ be an extension of $C$ for basic circuits definition and manipulation, and let $H^*$ be the corresponding extension of $H$. The new terms and types of $C^*$ are defined as follows: we denote by $\text{Bit}$ a new type representing bits, and by $0$ and $1$ two constants typed as follows:

$$\Gamma \mid - \vdash_c 0 : \text{Bit} \quad \Gamma \mid - \vdash_c 1 : \text{Bit}$$

Moreover, we consider some new functions $\text{not}$, $\text{and}$ and $\text{cnot}$ (the controlled-not), representing boolean gates, together with their typing judgements:
\[
\Gamma \vdash f : \text{Bit} \\
\text{and (not and (f, g)) : Bit}
\]

Combining these new operators with the substitution rule (\text{sub2}) one can easily build, for example, the “not-and” \text{nand} operator as follows:

\[
\Gamma \vdash f : \text{Bit} \quad \Gamma \vdash g : \text{Bit} \\
\Gamma \vdash f, g : \text{Bit}
\]

Similarly, one can derive, for example, \(\Gamma \vdash \text{and}(\text{not}(h), \text{and}(f, g)) : \text{Bit}\):

\[
\Gamma \vdash h : \text{Bit} \quad \Gamma \vdash f : \text{Bit} \quad \Gamma \vdash g : \text{Bit} \\
\Gamma \vdash \text{and}(\text{not}(h), \text{and}(f, g)) : \text{Bit}
\]

Notice that the linear constraints on \text{Bit} forbid the duplication and discarding of the variables of type \text{Bit}. This can be crucial if we want strictly control linearity in contexts like quantum circuits manipulation, where data duplication violates crucial physical properties such as the no-cloning property.

In the following remarks, we discuss how \(\mathcal{HC}\) can manage the composition of core functions once a function has been promoted in the host part of the syntax.

**Remark 2.3** (Composition). Given the following terms

\[
\Gamma \vdash s : \text{Proof}(A, B) \quad \Gamma \vdash t : \text{Proof}(B, C) \quad \Gamma \vdash a : A \vdash A : A
\]

we can construct a new host-term denoted by

\[
\Gamma \vdash \text{comp}(a, s, t) : \text{Proof}(A, C)
\]

which represents the composition in \(\mathcal{H}\), given by the following derivation:

\[
\Gamma \vdash s : \text{Proof}(A, B) \quad \Gamma \vdash a : A \vdash A \vdash A : A \\
\Gamma \vdash \text{derelict}(s, a) : B \\
\Gamma \vdash t : \text{Proof}(B, C) \\
\Gamma \vdash \text{derelict}(t, \text{derelict}(s, a)) : C
\]

**Remark 2.4** (Associativity of the composition). Combining the axioms with the substitutions rules, we can derive the following equalities describing the composition rule
where the H-terms \( \Gamma \vdash_H \text{id}_A : \text{Proof}(A, A) \) and \( \Gamma \vdash_H \text{id}_B : \text{Proof}(B, B) \) denote the promotion of \( \Gamma \mid a : A \vdash_C a : A \) and \( \Gamma \mid b : B \vdash_C b : B \) respectively, i.e. \( \text{id}_A := \text{promote}(a.a) \) and \( \text{id}_B := \text{promote}(b.b) \).

We start by checking the associativity law: by definition, the term \( \Gamma \vdash_H \text{comp}(a.\text{comp}(a.s, t), u) : \text{Proof}(A, D) \) is equal to

\[
\Gamma \vdash_H \text{promote}(a.\text{derelict}(u, \text{derelict}(\text{promote}(a.\text{derelict}(t, \text{derelict}(s, a))), a))) : \text{Proof}(A, D)
\]

Similarly, we have that the term \( \Gamma \vdash_H \text{comp}(a.s, \text{comp}(b.t, u)) : \text{Proof}(A, D) \) is given by

\[
\Gamma \vdash_H \text{promote}(a.\text{derelict}(\text{promote}(b.\text{derelict}(u, \text{derelict}(t, b))), \text{derelict}(s, a))) : \text{Proof}(A, D)
\]

Now, since \( \text{derelict}(s, a) \) is of type \( B \), we can soundly apply the duality rule \( \text{der-prom} \) obtaining

\[
\Gamma \vdash_H \text{promote}(a.\text{derelict}(u, \text{derelict}(t, \text{derelict}(s, a)))) : \text{Proof}(A, D).
\]

Therefore, this shows that the composition is associative.

Similarly, we show that \( \Gamma \vdash_H \text{comp}(a.s, \text{id}_B) = s : \text{Proof}(A, B) \) can be derived. By definition, the term \( \Gamma \vdash_H \text{comp}(a.s, \text{id}_B) : \text{Proof}(A, B) \) is

\[
\Gamma \vdash_H \text{promote}(a.\text{derelict}(\text{promote}(b.b), \text{derelict}(s, a))) : \text{Proof}(A, B)
\]

Notice that \( \text{derelict}(s, a) \) has type \( B \), so we can apply the duality rule \( \text{der-prom} \), obtaining

\[
\Gamma \vdash_H \text{promote}(a.\text{derelict}(s, a)) : \text{Proof}(A, B)
\]

Finally, we can conclude that this is exactly

\[
\Gamma \vdash_H s : \text{Proof}(A, B)
\]

by means of the duality rule \( \text{prom-der} \).

Finally, the equality \( \Gamma \vdash_H \text{comp}(a.\text{id}_A, s) = s : \text{Proof}(A, B) \) can be derived as follows. Consider the explicit definition of \( \text{comp}(a.\text{id}_A, s) \):

\[
\Gamma \vdash_H \text{promote}(a.\text{derelict}(s, \text{derelict}(\text{promote}(a.a), a))) : \text{Proof}(A, B)
\]

By applying the \( \text{der-prom} \) rule to the (core) subterm \( \text{derelict}(\text{promote}(a.a), a) \) we obtain

\[
\Gamma \vdash_H \text{promote}(a.\text{derelict}(s, a)) : \text{Proof}(A, B)
\]

and then by means of the \( \text{prom-der} \) rule we can conclude

\[
\Gamma \vdash_H s : \text{Proof}(A, B)
\]
In Examples 2.5 and 2.6 we suggest how to accommodate the syntax of the host language for the extension defined in Example 2.2. In particular, we focus on the circuits manipulation at host level.

**Example 2.5** (Hosted Circuits). Let us consider the extension of $\mathcal{HC}$ defined in Example 2.2. Employing the promotion rule $(\text{prom})$, we can represent the new core constants $0$ and $1$ at the host level:

$$Γ \vdash_\mathcal{H} \text{prom}_{\bot} \cdot 0 : \text{Proof}(I, \text{Bit}) \quad Γ \vdash_\mathcal{H} \text{prom}_{\bot} \cdot 1 : \text{Proof}(I, \text{Bit})$$

Similarly, one can formally reason at the host-level about circuits and gates (see Example 2.2), by promoting the new core terms, e.g,

$$Γ \vdash_\mathcal{H} \text{prom}_{\bot}(a_1 \ldots a_n, \text{not}(f)) : \text{Proof}(\otimes \Omega, \text{Bit})$$

and

$$Γ \vdash_\mathcal{H} \text{prom}_{\bot}(b_1 \ldots b_n, c_1 \ldots c_k, \text{not}(g, h)) : \text{Proof}(\otimes \Omega_1, \Omega_2, \text{Bit})$$

where $Ω = [a_1 : A_1 \ldots a_n : A_n], Ω_1 = [b_1 : B_1 \ldots b_n : B_n]$ and $Ω_2 = [c_1 \ldots c_k]$ are the linear contexts of $f, g,$ and $h$ respectively and we denote $A_1 \otimes \ldots \otimes A_n \otimes B_1 \otimes \ldots \otimes B_n$ as $\otimes Ω_1, Ω_2$.

The language $\mathcal{H}^*$ can also manage a flow-control on (core) circuits: for example, it can erase and duplicate circuits and it can compose circuits in parallel and sequence.

Given the host terms $Γ \vdash_\mathcal{H} f : \text{Proof}(A, B), Γ \vdash_\mathcal{H} g : \text{Proof}(C, D), Γ \vdash_\mathcal{H} h : \text{Proof}(B, D)$, and the core terms $Γ \vdash a : A \vdash_\mathcal{C} a : A, Γ \vdash c : C \vdash_\mathcal{C} c : C$, we can infer derivations

$$\text{parallel}(a.f, c.g) := \text{prom}_{\bot}(a, c.(\text{derelict}(f, a) \otimes \text{derelict}(g, c)))$$

$$\text{comp}(a.f, h) := \text{prom}_{\bot}(a, (\text{derelict}(f, \text{derelict}(f, a))))$$

which encode parallelization and sequentialization of core circuits respectively (for suitable given inputs). The term $\text{parallel}(a.f, c.g)$ has type $\text{Proof}(A \otimes C, B \otimes D)$ and the term $\text{comp}(a.f, h)$ has type $\text{Proof}(A, D)$ . Notice that we encode circuits sequentialization exactly as the $\text{comp}(-, f, g)$ in Remark 2.3.

In Figure 5 we show the derivation for the closed host term

$$\lambda x_0 : \text{Proof}(A_0, B_0). \lambda x_1 : \text{Proof}(A_1, B_1). \text{parallel}(a_0, x_0, a_1, x_1)$$

with core inputs $a_0 : A_0$ and $a_1 : A_1$. This term has type $\text{Proof}(A_0, B_0) \rightarrow \text{Proof}(A_1, B_1) \rightarrow \text{Proof}(A_0 \otimes A_1, B_0 \otimes B_1)$. Notice that the core variables $a_0$ and $a_1$ do not freely occur in the host. For the sake of space, we write $Γ_{01}$ to denote $x_0 : \text{Proof}(A_0, B_0), x_1 : \text{Proof}(A_1, B_1)$.

$$\begin{array}{ll}
Γ_{01} \vdash_\mathcal{H} x_0 : \text{Proof}(A_0, B_0) & Γ_{01} \vdash a_0 : A_0 \vdash_\mathcal{C} a_0 \vdash A_0 \\
Γ_{01} \vdash a_0 : A_0 \vdash \text{derelict}(x_0, a_0) : B_0 & Γ_{01} \vdash a_1 : A_1 \vdash \text{derelict}(x_1, a_1) : B_1 \\
Γ_{01} \vdash x_0 : \text{Proof}(A_0, B_0) \vdash_\mathcal{H} x_1 : \text{Proof}(A_1, B_1) & Γ_{01} \vdash a_1 : A_1 \vdash_\mathcal{C} a_1 : A_1 \\
Γ_{01} \vdash \text{parallel}(a_0, x_0, a_1, x_1) : \text{Proof}(A_0 \otimes A_1, B_0 \otimes B_1) & \vdash_\mathcal{H} \lambda x_0 : \text{Proof}(A_0, B_0). \lambda x_1 : \text{Proof}(A_1, B_1). \text{parallel}(a_0, x_0, a_1, x_1)
\end{array}$$

Figure 5: Host type derivation for parametric parallelleli zation of circuits.
Similarly, we can also derive the closed term

\[ \lambda x : \text{Proof}(A, B). \lambda z : \text{Proof}(B, C). \text{comp}(a.x, z) \]

which has type \( \text{Proof}(A, B) \rightarrow \text{Proof}(B, C) \rightarrow \text{Proof}(A, C) \).

Example 2.5 shows how a simply typed language hosting a minimalistic circuit design calculus allows some interesting manipulation of core programs and, in some sense, a limited and oriented form of interoperability.

Example 2.6 (Controlling Circuits, some reflections). Consider the instance \( H^* \) from Example 2.5 and assume to add boolean constants \( \text{true} \) and \( \text{false} \), both typed \( \text{bool} \) (we omit the obvious typing judgements), and also the \text{if then else} construct, typed by the rule

\[
\frac{\Gamma \vdash t : \text{bool} \quad \Gamma \vdash s_0 : X \quad \Gamma \vdash s_1 : X}{\Gamma \vdash \text{if} t \text{ then } s_0 \text{ else } s_1 : X}
\]

Notice that we can use the \text{if then else} for the control of core programs. For example, given two circuits and according to the guard evaluation, we can decide which circuit one should modify and use. Let \( \Gamma \) be \( x : \text{bool}, f : \text{Proof}(\text{Bit} \otimes \text{Bit}, \text{Bit} \otimes \text{Bit}), g : \text{Proof}(\text{Bit} \otimes \text{Bit}, \text{Bit} \otimes \text{Bit}) \).

It is easy to verify that the following judgments, which enable the representation of both input and core programs at the host level, are well-typed:

\[
\begin{align*}
\Gamma & \vdash_H \text{comp}(a.\text{promote}(-.0 \otimes 0), f) : \text{Proof}(I, \text{Bit} \otimes \text{Bit}) \\
\Gamma & \vdash_H \text{comp}(a.\text{promote}(-.1 \otimes 1), g) : \text{Proof}(I, \text{Bit} \otimes \text{Bit}) \\
\Gamma & \vdash_H \text{comp}(b.\text{promote}(-.1 \otimes 1), \text{comp}(c.f, g)) : \text{Proof}(I, \text{Bit} \otimes \text{Bit})
\end{align*}
\]

where \( a : I, b : I \) and \( c : \text{Bit} \otimes \text{Bit} \). The first term, call it \( s \), represents the application of the program \( f \) to the (promoted) input \( \text{promote}(-.0 \otimes 0) \). The second, call it \( t \), represents the application of the program \( g \) to the (promoted) input \( \text{promote}(-.1 \otimes 1) \). The third, call it \( w \), models the application to a promoted input \( \text{promote}(-.1 \otimes 1) \) of the sequentialization of the program (circuits) \( f \) and \( g \).

Now, let us endeavor to compose a conditional construct that assesses two distinct programs, effectively corresponding to two disparate circuit configurations.

For example, we can derive the term \text{IfCirc}, defined as

\[
\lambda x. \lambda f. \lambda g. \text{if } x \text{ then } \text{comp}(a.\text{promote}(-.0 \otimes 0), f) \text{ else } \text{comp}(b.\text{promote}(-.1 \otimes 1), \text{comp}(c.f, g))
\]

where, for the sake of readability, we omitted the type of lambda-abstracted variables.

The term \text{IfCirc} performs a simple computation: evaluates a guard (the parameter passed to the bound variable \( x \)) and, according to the result, evaluates the subprogram \( \lambda f.s \) or \( \lambda g.w \). It is direct to check that this term (from the empty context) has the following type:

\[
\text{Bit} \rightarrow \text{Proof}(\text{Bit} \otimes \text{Bit}, \text{Bit} \otimes \text{Bit}) \rightarrow \text{Proof}(\text{Bit} \otimes \text{Bit}, \text{Bit} \otimes \text{Bit}) \rightarrow \text{Proof}(I, \text{Bit} \otimes \text{Bit})
\]

Example 2.6 shows a very basic form of control by the host on the core. According to further extensions of the core \( C \), one can define more significant control operations. For example, if we move to a Turing complete host, or at least to a typed system as expressive as G"odel System \( T \), we can define interesting circuit construction operators such as the parametric circuit sequentialization and parallelization (which take a numeral \( n \) as input and iterate the operation \( n \) times). These operations are quite useful in circuit construction: for instance, in quantum programming languages, they allow us to encode parametric versions.
of well-known quantum algorithms [PPZ19]. We postpone this intriguing topic to future endeavors (refer to Section 5.2).

2.3. The category $\text{Th}(\mathcal{HC})$. We conclude this section with a key definition that allows us to formulate our main result, the internal language Theorem 3.11. To prove the strong correspondence between syntax and models, we first define the category $\text{Th}(\mathcal{HC})$, whose objects are theories (Definition 2.7) and whose morphisms are translations (Definition 2.8). In this section, we strictly follow the notation of [MMdPR05].

**Definition 2.7 (HC-theory).** A typed system $T$ is a **HC-theory** if it is an extension of $\mathcal{HC}$ with **proper-T-axioms**, namely with new base type symbols, new type equality rules, new term symbols and new equality rules of terms.

**Definition 2.8 (HC-translation).** Given two $\mathcal{HC}$-theories $T_1$ and $T_2$, a **HC-translation** $M$ is a function from types and terms of $T_1$ to types and terms of $T_2$ preserving type and term judgements, which means that it sends a type $X$ for which $\vdash_{\mathcal{H}} M(X) : \text{type}$ is derivable in $T_1$ to a type $M(X)$ such that $\vdash_{\mathcal{H}} M(X) : \text{type}$ is derivable in $T_2$, it sends a type $A$ for which $\vdash_{\mathcal{C}} A : \text{type}$ is derivable in $T_1$ to a type $M(A)$ such that $\vdash_{\mathcal{C}} M(A) : \text{type}$ is derivable in $T_2$, it sends an $\mathcal{H}$-term $t$ such that $\Gamma \vdash_{\mathcal{H}} t : X$ is derivable in $T_1$ to an $\mathcal{H}$-term $M(t)$ such that

$$M(\Gamma) \vdash_{\mathcal{H}} M(t) : M(X)$$

is derivable in $T_2$, where $M(\Gamma) \equiv [x_1 : M(X_1), \ldots, x_n : M(X_n)]$ if $\Gamma \equiv [x_1 : X_1, \ldots, x_n : X_n]$, and sending a $\mathcal{C}$-term $f$ for which $\Gamma \vdash_{\mathcal{C}} f : A$ is derivable in $T_1$ to a typed term $M(f)$ such that

$$M(\Gamma) \vdash_{\mathcal{C}} M(f) : M(A)$$

is derivable in $T_2$, where $M(\Delta) \equiv [a_1 : M(A_1), \ldots, a_m : M(A_m)]$ if $\Delta \equiv [a_1 : A_1, \ldots, a_m : A_m]$, satisfying in particular

$$M(x) = x \text{ and } M(a) = a$$

and preserving the $\mathcal{HC}$-types and terms constructors, and their equalities.

**Definition 2.9 (Category of HC-theories).** The category $\text{Th}(\mathcal{HC})$ has $\mathcal{HC}$-theories as objects and $\mathcal{HC}$-translations as morphisms.

Observe that a translation between the two host parts of theories induces a **change of base** for the core language, which means the core part of the first host language can be embedded in the second one using the translation between the hosts. In particular a translation can be formally split into two components: one acting on the host part, and the other one acting on the core language induced by the change of base. We will provide a formal definition of change of base in the next section where we introduce the $\mathcal{HC}$-semantics.

3. Categorical semantics

In this section we show that a model for $\mathcal{HC}$ is given by a pair $(\mathcal{H}, \mathcal{C})$, where $\mathcal{H}$ is a cartesian closed category, and $\mathcal{C}$ is a $\mathcal{H}$-enriched symmetric monoidal category, see [Kel05]. Moreover we prove a correspondence between the typed calculus $\mathcal{HC}$ and its categorical models via the notion of **internal language**.

Recall that $\mathcal{HC}$ provides an internal language of the category $\text{Model}(\mathcal{HC})$ of its models if one proves an equivalence between $\text{Model}(\mathcal{HC})$ and the category $\text{Th}(\mathcal{HC})$ of the theories
of the language. In the categorical logic literature several examples and applications of the notion of internal language can be found, e.g. see [LS86, Pit95, Joh02]. In this paper we mainly follow [MMdPR05, MdPR00], where the authors discuss models and morphisms for Intuitionistic Linear Logic (ILL), Dual Intuitionistic Linear Logic (DILL) and Linear-Non-Linear Logic (LNL). The leading idea of [MMdPR05, MdPR00] is that soundness and completeness theorems are not generally sufficient to identify the most appropriate class of denotational models for a typed calculus, unless, as pointed out in [MMdPR05], the same typed calculus provides the internal language of the models we are considering, as anticipated in the introduction.

In Section 4 we also use the internal language theorem to answer some questions about host-core languages, showing how this notion is useful also out from its usual range of applications.

Before we proceed to state and prove technical definitions and results, we sketch the key steps necessary to achieve the goal. Given the typed calculus $\mathcal{HC}$, consider its category of theories $\text{Th}(\mathcal{HC})$ (as in Definition 2.9). Our purpose is to define a category of models $\text{Model}(\mathcal{HC})$ such that $\mathcal{HC}$ provides an internal language for these models. This is the more interesting and challenging part of the proof and concerns the definition of the notion of morphism of models. The objects of $\text{Model}(\mathcal{HC})$ are models of $\mathcal{HC}$ (see Definition 3.1), and the suitable definition of morphisms of $\text{Model}(\mathcal{HC})$ requires introducing the formal notion of change of base (see Definition 3.8).

Once $\text{Model}(\mathcal{HC})$ has been defined, we known from [MMdPR05] that the typed calculus $\mathcal{HC}$ provides an internal language of $\text{Model}(\mathcal{HC})$ if we can establish an equivalence $\text{Th}(\mathcal{HC}) \cong \text{Model}(\mathcal{HC})$ between the category of models and the category of theories for $\mathcal{HC}$.

To show this equivalence, we need to define a pair of functors: the first functor

$$S: \text{Th}(\mathcal{HC}) \longrightarrow \text{Model}(\mathcal{HC})$$

assigns to a theory its syntactic category, and the second functor

$$L: \text{Model}(\mathcal{HC}) \longrightarrow \text{Th}(\mathcal{HC})$$

assigns to a pair $(\mathcal{H}, C)$ (a model of $\mathcal{HC}$, see Definition 3.1) its internal language, as a $\mathcal{HC}$-theory. Following the schema sketched above, we can proceed with technical results.

Since we will work in the context of enriched categories, we first recall some basic notions about them. We refer to [Kel05, Lac10] for a detailed introduction to the theory of enriched categories, and we refer to [RS17, LMZ18] for recent applications and connections with theoretical computer science.

Let $\mathcal{H}$ be a fixed monoidal category $\mathcal{H} = (\mathcal{H}_0, \otimes, I, a, l, r)$, where $\mathcal{H}_0$ is a (locally small) category, $\otimes$ is the tensor product, $I$ is the unit object of $\mathcal{H}_0$, $a$ defines the associativity isomorphism and $l$ and $r$ define the left and right unit isomorphism, respectively.

An enriched category $\mathcal{H}$-category $\mathcal{C}$ consists of a class $\text{ob}(\mathcal{C})$ of objects, a hom-object $\mathcal{C}(A, B)$ of $\mathcal{H}_0$ for each pair of objects of $\mathcal{C}$, a composition law

$$c_{ABC}: \mathcal{C}(B, C) \otimes \mathcal{C}(A, B) \longrightarrow \mathcal{C}(A, C)$$

for each triple of objects, and an identity element

$$j_A: I \longrightarrow \mathcal{C}(A, A)$$

for each object, subject to associativity and identity laws, see [Kel05, Lac10].
Much of category theory can be reformulated in the enriched setting, such as the notion of monoidal category we will consider here. Taking for example $\mathcal{H} = \text{Set}, \text{Cat}, 2, \text{Ab}$ one can re-find the classical notions of (locally small) ordinary category, 2-category, pre-ordered set, and additive category. The class of $\mathcal{H}$-categories forms a category denoted by $\mathcal{H}$-$\text{Cat}$, whose morphisms are $\mathcal{H}$-functors.

Recall that given $A$ and $B$ two $\mathcal{H}$-categories, a $\mathcal{H}$-functor $F : A \longrightarrow B$ consists of a function

$$F : \text{ob}(A) \longrightarrow \text{ob}(B)$$

together with, for every pair $A, B \in \text{ob}(A)$, a morphism of $\mathcal{H}$

$$F_{AB} : A(A, B) \longrightarrow B(FA, FB)$$

subject to the compatibility with the composition and with the identities. Again we refer to [Kel05] and Appendix B for the details.

As a first step, we can provide a model for $\mathcal{H}$-$\text{Cat}$ in terms of an instance of an enriched category, by interpreting the host part into a category $\mathcal{H}$ and the core part into a suitable category $\mathcal{C}$ enriched in $\mathcal{H}$.

**Definition 3.1 (Models of $\mathcal{H}$-$\text{Cat}$).** A model of $\mathcal{H}$-$\text{Cat}$ is a pair $(\mathcal{H}, \mathcal{C})$ where $\mathcal{H}$ is a cartesian closed category and $\mathcal{C}$ is a $\mathcal{H}$-symmetric monoidal category.

**Notation:** For the rest of this section let $\mathcal{H}$ be a cartesian closed category and let $\mathcal{C}$ be a $\mathcal{H}$-symmetric monoidal category.

3.1. Structures and Interpretation. A structure for the language $\mathcal{H}$-$\text{Cat}$ in $(\mathcal{H}, \mathcal{C})$ is specified by giving an object $\lfloor X \rfloor \in \text{ob}(\mathcal{H})$ for every base $\mathcal{H}$-type $X$, and an object $\lfloor A \rfloor \in \text{ob}(\mathcal{C})$ for every base $\mathcal{C}$-type $A$. Given these assignments, the other types are interpreted recursively as presented in Figure 6:

$$\lfloor 1 \rfloor = 1 \quad \lfloor X \times Y \rfloor = \lfloor X \rfloor \times \lfloor Y \rfloor \quad \lfloor X \to Y \rfloor = \lfloor X \rfloor \to \lfloor Y \rfloor$$

$$\lfloor \text{Proof}(A, B) \rfloor = \mathcal{C}(\lfloor A \rfloor, \lfloor B \rfloor) \quad \lfloor A \otimes B \rfloor = \lfloor A \rfloor \otimes \lfloor B \rfloor \quad \lfloor I \rfloor = I$$

Figure 6: Type interpretation

**Notation:** given a $\mathcal{H}$-context $\Gamma = \lfloor x_1 : X_1, \ldots, x_n : X_n \rfloor$, we define $\lfloor \Gamma \rfloor := \lfloor X_1 \rfloor \times \cdots \times \lfloor X_n \rfloor$. Similarly given a $\mathcal{C}$-context $\Omega = \lfloor a_1 : A_1, \ldots, a_m : A_m \rfloor$ we define $\lfloor \Omega \rfloor := \lfloor A_1 \rfloor \otimes \cdots \otimes \lfloor A_m \rfloor$.

The interpretation of an $\mathcal{H}$-terms in context $\Gamma \vdash_{\mathcal{H}} s : X$ is given by a morphism of $\mathcal{H}$

$$\lfloor \Gamma \vdash_{\mathcal{H}} s : X \rfloor : \lfloor \Gamma \rfloor \longrightarrow \lfloor X \rfloor$$

defined by induction on the structure of terms as usual. This means, for example, that we define

$$\lfloor x_1 : X_1, \ldots, x_n : X_n \vdash_{\mathcal{H}} x_i : X_i \rfloor = \text{pr}_i$$

where $\text{pr}_i : \lfloor X_1 \rfloor \times \cdots \times \lfloor X_n \rfloor \longrightarrow \lfloor X_i \rfloor$ is the $i$-th projection.

The interpretation of a $\mathcal{C}$-term in context $\Gamma \mid \Omega \vdash_{\mathcal{C}} f : A$ is given by a morphism

$$\lfloor \Gamma \mid \Omega \vdash_{\mathcal{C}} f : A \rfloor : \lfloor \Gamma \rfloor \longrightarrow \mathcal{C}(\lfloor \Omega \rfloor, \lfloor A \rfloor)$$
Then the interpretation of the composition defined in Remark 2.3 is
\[
\Gamma \vdash_C a : A \vdash_C a : A] := [\Gamma \vdash 1 \rightarrow 1^{|A|} C([A], [A])].
\]
Moreover the terms constructed using promotion and dereliction are interpreted as follows:
\[
\Gamma \vdash_H \text{promote}(a_1, \ldots, a_n, f) : \text{Proof}(\otimes_{\Omega}, A)] := [\Gamma \vdash_C f : A]
\]
\[
[\Gamma \vdash_C \text{derelict}(t, f) : B] = [\Gamma \vdash (|t|, |f|) C([|A|], [|B|]) \times C([|\Omega|, [A]]) = C([|\Omega|, [B]])].
\]

**Definition 3.2.** A structure on \( (H, C) \) satisfies a \( H \)-equation in context \( \Gamma \vdash_H s : t : X \) if \( \Gamma \vdash_H s : X = [\Gamma \vdash_H t : X] \). Similarly, a structure satisfies a \( C \)-equation in context \( \Gamma \vdash_C g : f : A \) if \( \Gamma \vdash_C g : A = [\Gamma \vdash_C f : A] \).

**Remark 3.3.** As a direct consequence of the previous definitions, we have that
\[
[\Gamma \vdash a : A \vdash_C \text{derelict}(t, a) : B] = [\Gamma \vdash_H t : \text{Proof}(A, B)]
\]
because the second component of the arrow
\[
[\Gamma \vdash_C a : A \vdash_C a : A] := [\Gamma \vdash 1 \rightarrow 1^{|A|} C([A], [A])].
\]
acts as the identity with respect to the enriched composition. Therefore, we can conclude that
\[
[\Gamma \vdash_H \text{promote}(a, \text{derelict}(t, a)) : \text{Proof}(A, B)] = [\Gamma \vdash_H t : \text{Proof}(A, B)].
\]
Similarly, one can check that the other duality equation of promotion and dereliction is satisfied by our enriched models.

**Remark 3.4.** Consider two \( H \)-judgements \( \Gamma \vdash_H s : \text{Proof}(A, B) \) and \( \Gamma \vdash_H t : \text{Proof}(B, C) \). Then the interpretation of the composition defined in Remark 2.3
\[
\Gamma \vdash_H \text{comp}(a, s, t) : \text{Proof}(A, C)
\]
is given by the arrow of \( H \) obtained by the following composition
\[
[\Gamma \vdash_H \text{comp}(a, s, t) : \text{Proof}(A, C)] = [\Gamma \vdash \{\text{comp}(a, s, t)\} C([B], [C]) \times C([A], [B]) = C([A], [C])].
\]
Notice that in the interpretation the dependency on the term \( \Gamma \vdash_C a : A \) disappears because, by definition, the second component of the arrow
\[
[\Gamma \vdash_C a : A \vdash_C a : A] := [\Gamma \vdash 1 \rightarrow 1^{|A|} C([A], [A])].
\]
acts as the identity with respect to the enriched composition.

**Example 3.5** (Breaking the linearity of the core, Part II). If we want to extend \( C \) to a non-linear core, as described in Example 2.1, we have to force the tensor product of the enriched category \( C \) to be a \( H \)-cartesian product. Notice that the new product is a particular case of the \( H \)-tensor product.
3.2. Relating Syntax and Semantics (I): Soundness and Completeness. We prove the usual relationship between syntax and semantics of $\mathcal{HC}$ by means of soundness and completeness theorems.

**Theorem 3.6** (Soundness and Completeness). Enriched symmetric, monoidal categories on a cartesian closed category are sound and complete with respect to the calculus $\mathcal{HC}$.

*Proof.* The soundness for the pure $C$-judgements and pure $H$-judgements is standard, because $H$ is cartesian closed, and $C$ is symmetric monoidal with hom-sets $C_0(A, B) = H(I, C(A, B))$. What remains to be proven is that the equations in context satisfied by the structure on $(H, C)$ are closed under the mixed rules, but this follows directly by Remark 3.3 and by the coherence of the $H$-functors $- \otimes -$ and $I \otimes -$. It is a direct generalization of the non-enriched case.

To get completeness, starting from a $\mathcal{HC}$-theory $T$, we build a cartesian closed category $\mathcal{H}_T$ whose objects are $H$-types and whose morphisms are $y : Y \vdash_H s : X$, both modulo the corresponding equalities. The $\mathcal{H}_T$-category $C_T$ is given by the $C$-objects and the enrichment is given by setting $C_T(A, B) := \mathcal{HC}(A, B)$ and by the mixed judgements together with the corresponding equalities. For example the composition morphism

\[ e_{ABC} : C_T(B, C) \times C_T(A, B) \to C_T(A, C) \]

is given by

\[
\begin{array}{c}
\Gamma \vdash_H x : \text{Proof}(A, B) \\
\Gamma \vdash_A a : A \\
\Gamma \vdash_C \text{derelict}(x, a) : B \\
\Gamma \vdash_H y : \text{Proof}(B, C) \\
\Gamma \vdash_A a : A \\
\Gamma \vdash_C \text{derelict}(y, \text{derelict}(x, a)) : C \\
\end{array}
\]

(where $\Gamma := [y : \text{Proof}(B, C), x : \text{Proof}(A, B)]$) which is exactly what we denote by $\text{comp}(a.x, y)$ in Remark 2.3, and it is associative by Remark 2.4. Similarly the identity element

\[ j_A : 1 \to C_T(A, A) \]

is given by

\[
\begin{array}{c}
- \vdash_A a : A \\
- \vdash_H \text{promote}(a.a) : \text{Proof}(A, A) \\
\end{array}
\]

3.3. Relating Syntax and Semantics (II): $\mathcal{HC}$ is an Internal Language.

In order to prove the stronger result about the correspondence between the category of theories and the category of models, we need to define a suitable notion of morphisms for the latter.

Recall that a *cartesian closed functor* is a functor preserving finite products and exponents up to isomorphisms, and it is said *strict* cartesian closed functor if it preserves such structures on the nose.

A key notion which will play a central role in the definition of the category of models is the notion of *change of base*. This notion explains formally how by changing the host language one can induce a change of the embedded language. All the definitions and remarks...
we are going to state work in a more general context as well, see [Kel05], but we state them just for our case of interest.

Given two cartesian closed categories $\mathcal{H}_1$ and $\mathcal{H}_2$, a cartesian closed functor between them

$$F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$$

induces a functor

$$F_* : \mathcal{H}_1\text{-Cat} \longrightarrow \mathcal{H}_2\text{-Cat}$$

sending a $\mathcal{H}_1$-category $\mathcal{C}$ to the $\mathcal{H}_2$-category $F_*(\mathcal{C})$ such that

- $F_*(\mathcal{C})$ has the same objects as $\mathcal{C}$;
- for every $A$ and $B$ of $\mathcal{C}$, we define

$$F_*(\mathcal{C})(A, B) = F(\mathcal{C}(A, B))$$

- the composition, unit, associator and unitor morphisms in $F_*(\mathcal{C})$ are the images of those of $\mathcal{C}$ composed with the structure morphisms of the cartesian closed structure on $F$.

See [Rie14, Lemma 3.4.3] for all the details.

Remark 3.7. Notice that if $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ and $G : \mathcal{H}_2 \longrightarrow \mathcal{H}_3$ are cartesian closed functors, then the functor $(G \circ F)_* : \mathcal{H}_1\text{-Cat} \longrightarrow \mathcal{H}_3\text{-Cat}$ coincides with the functor $G_* \circ F_*$. In fact, it is direct to check that for every $\mathcal{H}_1$-category $\mathcal{C}$, the categories $(G \circ F)_*(\mathcal{C})$ and $G_*(F_*(\mathcal{C}))$ have the same objects and, for every $A$ and $B$ of $\mathcal{C}$, we have that

$$(G \circ F)_*(\mathcal{C})(A, B) = (G \circ F)(\mathcal{C}(A, B)) = G(F(\mathcal{C}(A, B))) = (G_* \circ F_*)(\mathcal{C})(A, B).$$

Similarly, it is direct to check that the functor $(\text{Id}_{\mathcal{H}})_* : \mathcal{H}\text{-Cat} \longrightarrow \mathcal{H}\text{-Cat}$ induced by the identity functor on $\mathcal{H}$ coincides with the identity functor on $\mathcal{H}\text{-Cat}$.

Definition 3.8 (Change of base). Given a cartesian closed functor $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$, the induced functor $F_* : \mathcal{H}_1\text{-Cat} \longrightarrow \mathcal{H}_2\text{-Cat}$ is called change of base.

A standard example is the underlying set functor $\mathcal{H}(I, -) : \mathcal{H} \longrightarrow \text{Set}$, sending an object $A$ of $\mathcal{H}$ to the underlying set of elements $\mathcal{H}(I, A)$. This lax-monoidal functor induces a functor $\mathcal{H}(I, -)_* : \mathcal{H}\text{-Cat} \longrightarrow \text{Cat}$, see [Rie14, Proposition 3.5.10] or [Kel05].

Remark 3.9. Observe that if $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a strict cartesian closed functor and $\mathcal{C}$ is a $\mathcal{H}_1$-symmetric monoidal category, then $F_*(\mathcal{C})$ is a $\mathcal{H}_2$-symmetric monoidal category.

We exploit the equivalence between the category of theories and the category of models to show, from a type theoretical point of view, that the change of base has the following meaning: if one has a translation between two host languages $H_1$ and $H_2$ and $C$ is an embedded language in $H_1$, then one can use this translation of host languages to change the host language for $C$, obtaining an embedding in $H_2$.

Now we can define the category of models Model($\mathcal{H}$).

Definition 3.10 (Category of $\mathcal{H}$ models). The objects of Model($\mathcal{H}$) are models in the sense of Definition 3.1, and an arrow in Model($\mathcal{H}$) between two models $(\mathcal{H}_1, \mathcal{C}_1)$ and $(\mathcal{H}_2, \mathcal{C}_2)$ is given by a pair $(F, f)$ where $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a strict cartesian closed functor, and
The typed calculus $\mathcal{H}$ provides an internal language for $\text{Model}(\mathcal{H})$, i.e. $\text{Th}(\mathcal{H})$ is equivalent to $\text{Model}(\mathcal{H})$.

Proof. We define a functor $S: \text{Th}(\mathcal{H}) \longrightarrow \text{Model}(\mathcal{H})$ by mapping a $\mathcal{H}$-theory to its syntactic category defined as in Theorem 3.6. On morphisms, the functor $S$ takes a translation between theories to the pair of functors induced on the syntactic categories by mapping terms and types to their respective translations.

Conversely, the functor $L: \text{Model}(\mathcal{H}) \longrightarrow \text{Th}(\mathcal{H})$ is defined by mapping an object $(\mathcal{H}, \mathcal{C})$ of $\text{Model}(\mathcal{H})$ to the $\mathcal{H}$-theory obtained by extending $\mathcal{H}$ with:

- new $\mathcal{C}$-types $A$ with the corresponding axiom $\vdash_{\mathcal{C}} A : \text{type}$ for each element of $\text{ob}(\mathcal{C})$, which means that we are naming the objects of $\mathcal{C}$ in the calculus and we extend the interpretation of the $\mathcal{C}$-types of $\mathcal{H}$ by interpreting the new names with the corresponding objects;
- new $\mathcal{H}$-types $X$ with the corresponding axiom $\vdash_{\mathcal{H}} X : \text{type}$ for each element of $\text{ob}(\mathcal{H})$ by naming the objects of $\mathcal{H}$ as we did for $\mathcal{C}$, and by renaming the types and the axioms induced by objects of the form $\mathcal{C}(A, B)$ as $\vdash_{\mathcal{H}} \text{Proof}(A, B) : \text{type};$
- new $\mathcal{C}$-terms $f$ and $\Gamma \vdash_{\mathcal{C}} f : A$ for each morphism $f: \Gamma \longrightarrow \mathcal{C}(\Omega, A)$ of $\mathcal{H}$ having the interpretation of $\Gamma$ as domain and of $\mathcal{C}(\Omega, A)$ as codomain;
- new $\mathcal{H}$-terms $s$ and $\Gamma \vdash_{\mathcal{H}} s : X$ for each morphism $s: \Gamma \longrightarrow X$ of $\mathcal{H}$ having the interpretation of $\Gamma$ as domain and of $X$ has codomain;
- new equality axioms $\vdash_{\mathcal{C}} A = B : \text{type}$ if the interpretation of $A$ is equal to that of $B$ in $\mathcal{C};$
- new equality axioms $\vdash_{\mathcal{H}} X = Y : \text{type}$ if the interpretation of $X$ is equal to that of $Y$ in $\mathcal{H};$
- new equality axioms between $\mathcal{C}$-terms $\Gamma \vdash_{\mathcal{C}} f = g : A$ if the interpretation of $\Gamma \vdash_{\mathcal{H}} s = t : X$ if the interpretation of $\Gamma \vdash_{\mathcal{H}} s : X$ is equal to that of $\Gamma \vdash_{\mathcal{H}} t : X$ by interpreting the new term symbols in the morphisms they name;
- new equality axioms between $\mathcal{H}$-terms $\Gamma \vdash_{\mathcal{H}} s = t : X$ if the interpretation of $\Gamma \vdash_{\mathcal{H}} s : X$ is equal to that of $\Gamma \vdash_{\mathcal{H}} t : X$ by interpreting the new term symbols as the morphisms they name.

The morphisms of $\text{Model}(\mathcal{H})$ give rise to translations, because both the components of morphisms are strict. It is direct to check that the previous functors define an equivalence of categories.

Remark 3.12. Observe that functors defined in Theorem 3.11 define the isomorphisms $(\mathcal{H}, \mathcal{C}) \cong S(L(\mathcal{H}, \mathcal{C}))$ and $T \cong L(S(T))$. 

$f: F_*(\mathcal{C}_1) \longrightarrow C_2$ is a $\mathcal{H}_2$-functor, preserving strictly all the structures. Given two arrows $(F, f): (\mathcal{H}_1, \mathcal{C}_1) \longrightarrow (\mathcal{H}_2, \mathcal{C}_2)$ and $(G, g): (\mathcal{H}_2, \mathcal{C}_2) \longrightarrow (\mathcal{H}_3, \mathcal{C}_3)$ the composition of these is given by the pair $(G, g)(F, F) := (GF, gf)$ where $gf := gG_*(f)$.

Observe that the requirement that all the functors and enriched functors must be strict reflects the definition of translation of theories.

We can conclude with the theorem showing the equivalence between the categories of models and that of theories for $\mathcal{H}$.
Some other authors consider only the equivalence $(\mathcal{H}, \mathcal{C}) \equiv S(L(\mathcal{H}, \mathcal{C}))$, see for example [BW90], as the characterizing property of the internal language, however in this case one does not obtain the equivalence between the categories of models and that of theories, but just a sort of bi-equivalence.

We conclude this section with some concrete examples of $\mathcal{HC}$-models.

**Example 3.13.**

1. Every locally small symmetric monoidal category $\mathcal{M}$ is by definition $\textbf{Set}$-enriched, and therefore gives rise to a $\mathcal{HC}$-model given by the pair $(\textbf{Set}, \mathcal{M})$.

2. The category of $\text{C}^\ast$-algebras provides a concrete example of model in the form $(\textbf{Set}, \mathcal{M})$. Recall that a (unital) $\text{C}^\ast$-algebra is a vector space over the field of complex numbers with multiplication, unit and involution. Moreover, it satisfies associativity and unit laws for multiplication, involution laws and the spectral radius provides a norm making it a Banach space [Sak12, RS18, RS17]. Finite dimensional $\text{C}^\ast$-algebras and completely positive unital linear maps form a symmetric monoidal category denoted by $\text{FdC}^\ast\text{-Alg}$. See [RS18, RS17]. Therefore the pair $(\textbf{Set}, \text{FdC}^\ast\text{-Alg})$ is a model of $\mathcal{HC}$.

3. The pair $(\text{Dcpo}_\perp, \text{Rel})$ provides a model for $\mathcal{HC}$, where $\text{Rel}$ is the category of sets and relations. Recall that $\text{Dcpo}_\perp$ is cartesian closed and $\text{Rel}$ is enriched in $\text{Dcpo}_\perp$ by the usual subsets ordering, and the least element of each hom-set is given by the empty relation. This pair is particularly interesting when one plans to move toward quantum languages (with classical control [Zor16]). In Section 4 we develop this example to show some possible applications of Theorem 3.11.

4. **The host-core “internal language” in action**

The equivalence $\text{Th}(\mathcal{HC}) \equiv \text{Model}(\mathcal{HC})$ provided in Theorem 3.11 can be read as follow: we can reason equivalently either starting from the syntax or from the semantics. Thanks to this strong correspondence, we are able to answer the main questions raised in the introduction.

4.1. **Answering Q1: A formal definition of Host-Core Languages.**

**Definition 4.1.** Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two standalone languages with their syntactic denotations $D_1$ and $D_2$. We say that $\mathcal{L}_2$ is embeddable in $\mathcal{L}_1$ if the pair $(D_1, D_2)$ belongs to $\text{Model}(\mathcal{HC})$.

Employing the equivalence $\text{Th}(\mathcal{HC}) \equiv \text{Model}(\mathcal{HC})$ of Theorem 3.11, and the functor $L: \text{Model}(\mathcal{HC}) \longrightarrow \text{Th}(\mathcal{HC})$ defined in the proof of such a theorem, we can conclude that if $\mathcal{L}_2$ is embeddable in $\mathcal{L}_1$ in the sense of the previous definition, then there exists a host-core language $\mathcal{L}_3 := L(D_1, D_2)$ such that $\mathcal{L}_1$ hosts $\mathcal{L}_2$. The requirement that the denotation of $\mathcal{L}_2$ is enriched in the denotation of $\mathcal{L}_1$ from a syntactic point of view means that there exists a mixed language whose “host” part is $\mathcal{L}_1$ and whose “pure” core part is $\mathcal{L}_2$. This coincides with the intuitive notion one could have of embedded style languages.

Notice that this semantic definition of the host-core languages confirms what we explain in the introduction. We are not defining a mere extension of $\mathcal{L}_1$, but we want to describe the fact that $\mathcal{L}_1$ is able to host $\mathcal{L}_2$ and provide the interface between the two languages.
This is syntactically characterized through the mixed syntax, and semantically captured by the enrichment.

4.2. Answering Q2: Relating the standalone and the host-core style, a first step. The equivalence \( \text{Th}(\mathcal{H}C) \equiv \text{Model}(\mathcal{H}C) \) provides a useful tool for comparing languages presented with different features and styles, as standalone and host-core languages. Indeed, without these categorical results, it would be hard to find the right criteria to compare such different languages, while lifting this problem in the categorical settings and using the correspondence between theories and models, a comparison can be easily done just by studying their categorical models.

Consider Benton’s presentation LNL of Intuitionistic Linear Logic (ILL). In [MMdPR05] the authors proved that Model(LNL) is a full subcategory of Model(ILL). This means that a “mixed grammar” in the style of LNL is equivalent to another one written in ILL in a “standalone” style, in the sense that they have the same models. Similarly, we are able to compare the pure linear (exponential-free) fragment of ILL, called here RLL as in [MMdPR05], with \( \mathcal{H}C \). In fact, we can show that Model(RLL) is a subcategory of Model(\( \mathcal{H}C \)). The objects of Model(RLL) are small symmetric monoidal categories and then, by definition, they are Set-enriched. Each morphism of Model(RLL), which by definition preserves all appropriate structure, can be extended to a morphism of Model(\( \mathcal{H}C \)). This morphism is of the form \((\text{Id}, m)\).

4.3. Answering Q3: From semantics back to syntax, an empirical example. The functors \( L : \text{Model}(\mathcal{H}C) \to \text{Th}(\mathcal{H}C) \) and \( S : \text{Th}(\mathcal{H}C) \to \text{Model}(\mathcal{H}C) \) defined in the proof of Theorem 3.11 provide a useful tool to define a \( \mathcal{H}C \)-theory \( L(\mathcal{H}, \mathcal{C}) \) starting from a pair \((\mathcal{H}, \mathcal{C})\), where \( \mathcal{H} \) is a cartesian closed category and \( \mathcal{C} \) is a \( \mathcal{H} \)-symmetric monoidal category, and vice versa. Here we focus on the former direction. In particular, the explicit way in which the \( \mathcal{H}C \)-theory \( L(\mathcal{H}, \mathcal{C}) \) is defined from a given model \((\mathcal{H}, \mathcal{C})\) suggests that the functor \( L : \text{Model}(\mathcal{H}C) \to \text{Th}(\mathcal{H}C) \) can be used to reason backwards, from semantics to syntax. In Example 4.2 we sketch how to extract a basis kernel type theory for a quantum language with classical control starting from a model (two categories of interest). What we obtain can be used as a starting point to fully define the syntax and the well-typing rules. Starting from the semantics one gets a kernel set of syntactical constructors mirroring the denotation. This seems to be useful in non-classical computations, and in particular in quantum languages, where the definition of suitable semantics is not always trivial at all (see e.g., [CdVW19]).

Example 4.2. We apply the functor \( L : \text{Model}(\mathcal{H}C) \to \text{Th}(\mathcal{H}C) \) to the pair of models \((\text{Dcpo}_\perp, \text{Rel})\), where \text{Rel} is enriched in \text{Dcpo}_\perp. The pair \((\text{Dcpo}_\perp, \text{Rel})\) is of interest since it captures a communication with (the denotation of) a “classical” computational system represented by the \text{Dcpo}_\perp with (the denotation of) another system having a “quantum flavour”, since \text{Rel} is a dagger category in the sense of Selinger (see e.g., [Sel07]).

The \( \mathcal{H}C \)-theory \( L(\text{Dcpo}_\perp, \text{Rel}) \) is obtained by extending \( \mathcal{H}C \) with:

- a new C-type \( A \) for every object \( A \) of \text{Rel};
- a new H-type \( X \) for every object \( X \) of \text{Dcpo}_\perp; moreover, for each \( A, B \in \text{Rel} \), we add a new H-type \( \text{Proof}(A, B) \) which is the set of binary relations between \( A \) and \( B \);
- a new C-term \( f \) for each morphism \( f : \Gamma \to \text{Rel}(\Omega, A) \) of \text{Dcpo}_\perp;
a new $H$-term $s$ for each morphism $s : \Gamma \to X$ of $\text{Dcpo}_\bot$;

new equality axioms $\vdash_C A = B : \text{type}$ if the sets $A$ and $B$ are equal sets;

new equality axioms $\vdash_H X = Y : \text{type}$ if $X$ and $Y$ are equal objects of $\text{Dcpo}_\bot$;

new equality axioms between $C$-terms $\vdash_C f = g : A$ if $f : \Gamma \to \text{Rel}(\Omega, A)$ and $g : \Gamma \to \text{Rel}(\Omega, A)$ are equal morphisms of $\text{Dcpo}_\bot$;

new equality axioms between $H$-terms $\vdash_H s = t : X$ if $s : \Gamma \to X$ is equal to $t : \Gamma \to X$ as morphisms of $\text{Dcpo}_\bot$.

This example highlights the intuition behind the notion of enrichment. In few words, we can speak about relations between sets in the context of $\text{Dcpo}_\bot$. This is particularly evident if we consider pure core judgements (i.e., defined in a context whose host part is empty). For each object in $\text{Rel}(A, B)$ we have in $\text{Dcpo}_\bot$ a function defined from the initial object 1 to $\text{Rel}(A, B)$, i.e. $\vdash_H \text{promote}(a.f) : \text{Proof}(A, B)$ can be viewed as a constant and we can plainly use constants, that are “silent programs” in the host syntax.

Since $\text{Rel}$ is a dagger category, in this syntax we can explicitly specify the term constructor $\dagger$. For example:

\[
\frac{\cdot | a : A \vdash f : B}{\cdot | b : B \vdash f^\dagger : A}
\]

We can require that the operator satisfies all the expected equations [Sel07] and so model this through syntactical rules. The real advantage of such a $HC$-theory is that one can equivalently reason about its term judgements both syntactically or semantically, viewing them as a concrete functions. We plan to develop the example above and use the set of types, terms and judgements we extracted as a basis for quantum language design (see our future work, Section 5.2).

5. Related work, Discussions and Future Work

5.1. Related Work. Benton’s Linear-Non-Linear Logic provides an elegant presentation of Linear Logic [GL87], and in the last years its models have been studied both from a categorical logic and a computer science perspective.

An important contribution in the first direction has been given by Maietti et al. in [MMdPR05], where the authors discuss models and morphisms for Intuitionistic Linear Logic (ILL), Dual Intuitionistic Linear Logic (DILL) and Linear-Non-Linear Logic (LNL). The crucial point is that soundness and completeness theorems are not generally sufficient to identify the most appropriate class of denotational models for a typed calculus, unless the same typed calculus provides an internal language of the category of models we are considering.

In the context of the foundation of programming theory, the language $\text{EWire}$ is studied in [RS18] as the basis of a denotational semantics based on enriched categories. It is built from a simple first-order linear language for circuits embedded in a more powerful host language. The circuit language is interpreted in a category that is enriched in the category denoting the host part. Moreover, some interesting extensions of the host language are proposed. In particular, the authors use the enrichment of the category of $W^*$-algebras in the $\text{Dcpo}$-category to accommodate recursive types. This allows them to model arbitrary types and is directly connected with the possibility of easily encoding parametric quantum
algorithms. In [RS17] the authors also show a relation with Benton’s Linear-Non-Linear models.

An inspiring work both for HC and EWire is the enriched effect calculus (EEC) [EMS14], whose models are given in terms of enriched categories. The idea behind the semantics in [RS18] and in [EMS14] is quite similar. Also in [EMS14] a deep comparison with LNL models is provided, showing that every LNL model with additives determines an (EEC) model. The authors also prove soundness and completeness of the equational theories, with respect to the interpretation, but they do not consider the notion of internal language.

Looking for non-classical computation, EWire has been defined as a generalization of a version of QWire ("choir") [PRZ17, RPZ17, RPLZ19], which is one of the most advanced programming platforms for the encoding and the verification of quantum circuits. The circuit language of QWire can be treated as the quantum plugin for the host classical language, currently in the Coq proof assistant [RPZ17, Ran18]. The type system is inspired by Benton’s (LNL) Logic and supports both linear and dependent types. The circuit language essentially controls the well-formed expressions concerning wires, i.e. circuit inputs/outputs, whereas the host language controls higher-order circuit manipulation and evaluation. The host language also controls the boxing mechanism (a circuit can be “boxed” and then promoted as a classical resource/code. See [Ran18] for a complete account about the use of advanced operations and techniques designed for QWire. Further developments on this line are provided in [LMZ18], where the authors introduce the lambda calculus ECLNL for string diagrams, whose primary purpose is to generate complicated diagrams from simpler components. In particular, the language ECLNL adopts the syntax (and operational semantics) of Proto-Quipper-M, while the categorical model is again given by a LNL model, but endowed with an additional enrichment structure. The abstract model of ECLNL satisfies the soundness, while completeness and internal language are not discussed in [LMZ18]. However, notice that ECLNL can be seen as a particular specialization of HC, and also its categorical model is a particular case of the model we introduced. This is not surprising, since one of the main motivation for the design of HC claimed at the beginning of our work is that HC has to embody the principal properties of those languages dealing with host-core situations.

A notion of multi-language related to HC is defined and studied in [BM19, BCM20] on the basis of the pioneering investigation [MF07]. The authors address the problem of providing a formal semantics to the combination of programming languages by introducing an algebraic framework based on order-sorted algebras. The framework provides an abstract syntax (induced by the algebraic structure), that works regardless of the inherent nature of the combined languages. While the notion of multi-language they introduced in [BM19, BCM20] seems similar to the one we propose in this paper, motivations, syntactic features and semantics are deeply different. First, the authors consider two languages combined in a unique standalone language essentially given by the union of the two syntaxes (without a “mixed zone”). Thus, two languages can be considered at the same level, i.e. there is not a language that has a privileged position with respect to the other. This is in contrast with our formulation, which models situations in which the host can delegate some computations to the core. The absence of hierarchy between syntaxes composing the multi-language framework defined in [BM19, BCM20] is reflected also semantically: given two signatures $S_1$ and $S_2$, the categorical model of the multi-language obtained from them is given essentially by a pair of algebras $A_1$ and $A_2$, where the first is an algebra over $S_1$, and the second one
is an algebra over $S_2$. No enrichment is required, since, as just said, in the resulting multilanguage the set of sorts is obtained by the union of the sets of sorts of $S_1$ and $S_2$. Finally, with respect to [BM19, BCM20] we provide a type theory and a denotational semantics together with the notion of internal language, which is to the best of our knowledge, new in the context of host-core languages.

5.2. Discussions and Future Work. In this paper, we design the host-core calculus $\mathcal{HC}$ and provide its denotational semantics in terms of enriched categories.

We remark that the point of view we developed in this paper is different from the programming language design one, and we are aware that $\mathcal{HC}$ “as is” can not provide a full theoretical account of host-core programming. Notwithstanding, we believe our investigation provides the first steps towards a foundation for a particular case of multi-language interaction systems. We tried to show this throughout the paper, through examples and applications of the notion of internal language. We claim that a principled theory, with a pre-existing, extensible library of traditional type-theoretical results, relating the syntax and semantics of the underlying main notions of our host-core system, is the appropriate basis on which to build more complicated, less homogeneous systems, more adapted to the applications at hand.

A crucial aspect that will be central in any future refinement of $\mathcal{HC}$ is the communication between host and core. This fact becomes central when one wants to model a notion of interoperability. The issue is out of the scope of our investigation, but we quickly comment on the point from a type theory perspective. The way host $\mathcal{H}$ and core $\mathcal{C}$ communicate with each other is strongly related to the choice one makes to design the host and to the expressive power of $\mathcal{H}$ and $\mathcal{C}$.

More expressive languages can arise from more complex forms of interaction. At the level of the mixed type theory, a “realistic” version of communication able to reflect real language interoperability necessarily requires a notion of casting between host and core types. Some host types $X$ can be “isomorphic” to types of the shape $\text{Proof}(I, A)$ of promoted core terms. For example, one could have a boolean type $\text{bool}$ and state it is equivalent to a type $\text{Proof}(I, \text{Bit})$, where $\text{Bit}$ is the core type of bits. It is natural to think of adding a constant $\text{cast} : \text{Proof}(I, A) \mapsto X$ that allows $\mathcal{H}$ to explicitly read results of the evaluation in $\mathcal{C}$. One could consider also a reverse casting implementing a bi-directional notion of interoperability.

5.2.1. Future Work. Our investigation leads into (at least) three directions: the improvement of the expressive power of $\mathcal{HC}$, a complete operational study of $\mathcal{HC}$, and finally its quantum specialization.

• We design $\mathcal{HC}$ pursuing a notion of compositionality. We consider the system as a kernel calculus both for extensions and specializations.

We aim to use the direct correspondence between syntax and semantics to obtain more refined type theories by adding syntactical rules and (equivalent) denotational properties, without changing the rules of the basic language.

Each extension requires the addition of syntactical primitives and, mirroring the syntax, a suitable definition of models $(\mathcal{H}, \mathcal{C})$, where the core category $\mathcal{C}$ denotes the peculiar features of the paradigm and is enriched in $\mathcal{H}$. 
We plan to study the operational semantics of \( HC \) and its related safety properties such as Subject Reduction and Progress Theorems, as well as a notion of normal form for the \( HC \) computations. The topic is already interesting for the current formulation of the syntax and seems to become challenging if one considers more expressive languages. We expect that our operational semantics will not be similar to that of \( QWire \) and \( EWire \), since the communication between \( H \) and \( C \) and the evaluation style we designed are different. We are also interested in the study of reduction strategies.

One can build a new quantum specialization of \( HC \). We aim to start from a sufficiently expressive host language \( H \) and a quantum “tuning” of a circuit description core in the style of \( C^* \) (Example 2.2), possibly improving its expressiveness. Once we have defined the quantum specialization of \( HC \), we plan to compare it with other established and influential languages, such as \( QWire \) and \( EWire \). As a parallel task in this quantum setting, we are interested in further developments of Example 4.2, focusing on pairs of enriched categories such as \( (\text{Dcpo}, \text{Rel}) \). Thanks to the internal language theorem, we hope to reverse the perspective in language design, by extracting new type systems backward from the semantics.

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Appendix A. Equational theory of the host languages $H$

The equations of the host language are the standard one of simply typed lambda calculus:

\[
\begin{align*}
(\beta) & \quad \frac{\Gamma \vdash_H \lambda x : X. s : X \to Y \quad \Gamma \vdash_H t : X}{\Gamma \vdash_H (\lambda x : X.s)t = s[t/x] : Y} \\
(\pi_1) & \quad \frac{\Gamma \vdash_H \langle s, t \rangle : X \times Y}{\Gamma \vdash_H \pi_1(s, t) = s : X} \\
(\pi_2) & \quad \frac{\Gamma \vdash_H \langle s, t \rangle : X \times Y}{\Gamma \vdash_H \pi_2(s, t) = t : Y} \\
(\eta) & \quad \frac{\Gamma \vdash_H \lambda x : X. s : X \to Y}{\Gamma \vdash_H \lambda x : X. s(x) = s : Y} \\
(l.a) & \quad \frac{\Gamma \vdash_H s = t : X \quad \Gamma \vdash_H u : X \to Y}{\Gamma \vdash_H us = ut : Y} \\
(r.a) & \quad \frac{\Gamma \vdash_H s = t : X \to Y \quad \Gamma \vdash_H u : X}{\Gamma \vdash_H su = tu : Y} \\
(in.\lambda) & \quad \frac{\Gamma, x : X \vdash_H s = t : Y}{\Gamma \vdash_H \lambda x.s = \lambda x.t : X \to Y} \\
(refl) & \quad \frac{\Gamma \vdash_H s : X}{\Gamma \vdash_H s = s : X} \\
(sym) & \quad \frac{\Gamma \vdash_H s = t : X}{\Gamma \vdash_H t = s : X} \\
(trans) & \quad \frac{\Gamma \vdash_H s = t : X \quad \Gamma \vdash_H t = u : X}{\Gamma \vdash_H s = u : X}
\end{align*}
\]

Figure 7: Evaluation rules for the host language $H$

For the $\eta$ rule one has the usual constraints that the variable $x$ does not appear free in the term $s$.

Appendix B. Enriched categories

We recall here some basic background about monoidal categories and enriched categories. See [Kel05, Lac10] for a complete account.
B.1. **Monoidal categories.** It is often useful to reason in a very abstract sense about processes and how they compose. Category theory provides the tool to do this.

A monoidal category is a category equipped with extra data, describing how objects and morphisms can be combined in parallel. The main idea is that we can interpret objects of categories as systems, and morphisms as processes.

One could interpret this for example, as running computer algorithms in parallel, or from a proof-theoretical point of view, as using separate proofs of $P$ and $Q$ to construct a proof of the conjunction $(P \land Q)$.

**Definition B.1.** A *monoidal category* $\mathcal{H} = (\mathcal{H}_0, \otimes, I, a, l, r)$ consists in giving:

- a category $\mathcal{H}_0$;
- an object $I$ of $\mathcal{H}_0$, called the *unit*;
- a bifunctor $\otimes : \mathcal{H}_0 \times \mathcal{H}_0 \to \mathcal{H}_0$, called *tensor product*, and we write $A \otimes B$ for the image under $\otimes$ of the pair $(A, B)$;
- for every $A, B, C$ objects of $\mathcal{H}_0$, an *associativity isomorphism*:
  $$a_{ABC} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$
  such that $a : ((- \otimes -) \otimes -) \cong (- \otimes (- \otimes -))$ is a natural isomorphism.
- for every object $A$, a *left unit* isomorphism
  $$l_A : I \otimes A \cong A$$
  such that $l : (I \otimes -) \cong \text{id}_{\mathcal{H}_0}$ is a natural isomorphism;
- for every object $A$, a *right unit* isomorphism
  $$r_A : A \otimes I \cong A$$
  such that $r : (- \otimes I) \cong \text{id}_{\mathcal{H}_0}$ is a natural isomorphism.

This data must satisfy the *pentagon* and *triangle* equations, for all objects $A, B, C$ and $D$:

\[
\begin{align*}
((A \otimes B) \otimes C) \otimes D & \quad \cong \quad (A \otimes (B \otimes C)) \otimes D \\
\cong & \quad (A \otimes B) \otimes (C \otimes D) \\
\end{align*}
\] (B.1)

\[
\begin{align*}
A \otimes ((B \otimes C) \otimes D) & \quad \cong \quad (A \otimes (B \otimes C)) \otimes D \\
\cong & \quad (A \otimes B) \otimes (C \otimes D) \\
\end{align*}
\] (B.2)

A special kind of example, called a cartesian monoidal category, is given by taking for $\mathcal{H}_0$ any category with finite products, by taking for $\otimes$ and $I$ the product $\times$ and the terminal object $1$, and by taking for $a, l, r$ the canonical isomorphisms.
Important particular cases of this are the categories Set, Cat, Grp, Ord, Top of sets, (small) categories, groupoids, ordered sets, topological spaces.

A collection of non-cartesian examples are Ab, Hilb, Rel of abelian groups, Hilbert spaces, sets and relations.

**Definition B.2.** A monoidal category \( H \) is said to be symmetric when for every \( A, B \) there is an isomorphism

\[
s_{AB} : A \otimes B \longrightarrow B \otimes A
\]

such that

- the morphisms \( s_{AB} \) are natural in \( A, B \);
- **associativity coherence:** for every \( A, B, C \) the following diagram commutes:

\[
\begin{array}{ccc}
(A \otimes B) \otimes C & \longrightarrow & (B \otimes A) \otimes C \\
\downarrow s_{AB} \otimes \text{id}_C & & \downarrow s_{BAC} \\
A \otimes (B \otimes C) & \longrightarrow & B \otimes (A \otimes C)
\end{array}
\]

\[
\begin{array}{ccc}
(B \otimes C) \otimes A & \longrightarrow & B \otimes (C \otimes A) \\
\downarrow \text{id}_B \otimes s_{AC} & & \\
A \otimes B & \longrightarrow & B \otimes A
\end{array}
\]

- **unit coherence:** for every \( A \) the following diagram commutes:

\[
\begin{array}{ccc}
A \otimes I & \longrightarrow & I \otimes A \\
\downarrow s_{AI} & & \downarrow \text{id}_A \\
A & \longrightarrow & I
\end{array}
\]

- **symmetric axiom:** for every \( A, B \) the following diagram commutes:

\[
\begin{array}{ccc}
A \otimes B & \longrightarrow & B \otimes A \\
\downarrow \text{id}_A \otimes B & & \downarrow s_{BA} \\
A \otimes B & \longrightarrow & B \otimes A
\end{array}
\]

**B.2. Enriched categories.** In this section we provide some basic notions about enriched categories. For details see e.g. [Sel09, Kel05, Js91].

An enriched category is a category in which the hom-functors take their values not in Set, but in some other category \( H \). The theory of enriched categories is now very well developed in category theory, see [Kel05] and [Lac10], and recently it finds interesting applications in theoretical computer science, see [RS17] and [LMZ18]. For the rest of this section, let \( H \) be a fixed monoidal category \( H = (H_0, \otimes, I, a, l, r) \), where \( H_0 \) is a category, \( \otimes \) is the tensor product, \( I \) is the unit object of \( H_0 \), \( a \) defines the associativity isomorphism and \( l \) and \( r \) define the left and right unit isomorphism respectively.

**Definition B.3** (Enriched Category). A \( H \)-category \( A \) consists of a class \( \text{ob}(A) \) of objects, a hom-object \( A(A, B) \) of \( H_0 \) for each pair of objects of \( A \), and

- **composition law** \( c_{ABC} : A(B, C) \otimes A(A, B) \longrightarrow A(A, C) \) for each triple of objects;
• identity element \( j_A : I \longrightarrow \mathcal{A}(A, A) \) for each object subject to the associativity and unit axioms expressed by the commutativity of the following diagrams

\[
\begin{align*}
&\xymatrix{
(A(C, D) \otimes A(C, B)) \otimes A(A, B) \ar[r]^\alpha \ar[d]_{\varepsilon_{BCD} \otimes \text{id}} & A(C, D) \otimes (A(C, B) \otimes A(A, B)) \ar[d]_{\text{id} \otimes \varepsilon_{ABC}} \\
A(D, B) \otimes A(A, B) \ar[r]^{c_{ABD}} & A(A, D) \ar[r]^{c_{ACD}} & A(C, D) \otimes A(A, C)
}
\end{align*}
\]

\[
\begin{align*}
&\xymatrix{
A(B, B) \otimes A(A, B) \ar[r]^{c_{ABB}} \ar[d]_{j_B \otimes \text{id}} & A(A, B) \ar[r]^{c_{AAB}} & A(A, B) \otimes A(A, A) \\
I \otimes A(A, B) \ar[rr]^-{\text{id}} & & A(A, B) \otimes I
}
\end{align*}
\]

Taking \( \mathcal{H} = \text{Set, Cat, 2, Ab} \) one can re-find the classical notions of (locally small ) ordinary category, 2-category, pre-ordered set, additive category.

**Definition B.4 (\( \mathcal{H} \)-functor).** Let \( \mathcal{A} \) and \( \mathcal{B} \) be \( \mathcal{H} \)-categories. A \( \mathcal{H} \)-\textbf{functor} \( F : \mathcal{A} \longrightarrow \mathcal{B} \) consists of a function

\[ F : \text{ob}(\mathcal{A}) \longrightarrow \text{ob}(\mathcal{B}) \]

together with, for every pair \( A, B \in \text{ob}(\mathcal{A}) \), a morphism of \( \mathcal{H} \)

\[ F_{AB} : \mathcal{A}(A, B) \longrightarrow \mathcal{B}(FA, FB) \]

subject to the compatibility with the composition and with the identities expressed by the commutativity of

\[
\begin{align*}
&\xymatrix{
\mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \ar[r]^c \ar[d]_{F_{BC} \otimes F_{AB}} & \mathcal{A}(A, C) \ar[d]_{F_{AC}} \\
\mathcal{B}(FB, FC) \otimes \mathcal{B}(FA, FB) \ar[r]^c & \mathcal{B}(FA, FC)
}
\end{align*}
\]

\[
\begin{align*}
&\xymatrix{
\mathcal{A}(A, A) \ar[d]_{j_A} \\
\mathcal{B}(FA, FA)
}
\end{align*}
\]

**Definition B.5 (\( \mathcal{H} \)-Natural Transformations).** Let \( F, G : \mathcal{A} \longrightarrow \mathcal{B} \) be \( \mathcal{H} \)-functors. A \( \mathcal{H} \)-\textbf{natural transformation} \( \alpha : F \longrightarrow G \) is an \( \text{ob}(\mathcal{A}) \)-indexed family of \textbf{components}

\[ \alpha_A : I \longrightarrow \mathcal{B}(FA, GA) \]

satisfying the \( \mathcal{H} \)-naturality condition expressed by the commutativity of the following diagram

\[
\begin{align*}
&\xymatrix{
I \otimes \mathcal{A}(A, B) \ar[r]^{\alpha_B \otimes F_{AB}} \ar[d]_{I^{-1}} & \mathcal{B}(FB, GB) \otimes \mathcal{B}(FA, FB) \\
\mathcal{A}(A, B) \ar[r]_{\varepsilon} \ar[d]_{I^{-1}} & \mathcal{B}(FA, GB)
}
\end{align*}
\]

\[
\begin{align*}
&\xymatrix{
\mathcal{A}(A, B) \otimes I \ar[r]^{G_{AB} \otimes \alpha_A} \ar[d]_{\varepsilon} & \mathcal{B}(GA, GB) \otimes \mathcal{B}(FA, GA) \\
\mathcal{A}(A, B) \ar[r]_{\varepsilon} & \mathcal{B}(FA, GB)
}
\end{align*}
\]