1 Introduction

There are many natural forcing notions adding real numbers, most notably the precursors of this class of forcing notions, namely COHEN and Random forcing. Whereas Random forcing has been deliberately constructed in connection with the σ-algebra of LEBESGUE measurable sets and COHEN forcing is represented by the algebra of BOREL sets modulo the σ-ideal of meager sets, the other classical forcing notions, which add real numbers, e.g. SACKS forcing, MILLER forcing and LAVER forcing are not connected with a σ-algebra by virtue of their construction. But as one can see easily, each of these classical forcing notions corresponds naturally to a σ-algebra. MILLER alludes to this fact in the rather vague formulated problem (11.10) of his problem list. The main point of the problem is: What kinds of well-known results can be proved for these σ-algebras?

E.g. it is known that the analytic sets are LEBESGUE measurable and have the Baire property, and we have the first sets which do not have this property on the $\Delta^1_2$ level in $L$. So, given a forcing notion $P$ adding real numbers, the following two natural questions are connected with MILLER’s problem (11.10):

- Are the analytic sets $P$-measurable, i.e. are they elements of the σ-algebra naturally connected with $P$?
- Do we have $\Delta^1_2$ counterexamples in $L$? If yes, how can we characterize the axiom candidate “All $\Delta^1_2$ sets are $P$-measurable” (similarly for $\Sigma^1_2$)?

In this paper we will tackle the first of these questions. The second question is answered by BRENDLE and the present author in [BRENDLE–LÖWE 1997] for HECHLER and MILLER forcing. We will not only show that for all prominent forcing notions the analytic $P$-measurability is provable in ZFC, but provide a uniform proof technique for questions of analytic measurability based on the so called "SOLOVAY’s Unfolding Trick".

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1 The author expresses his deep gratitude towards Jörg Brendle who introduced him into the matter and offered countless pieces of advice. The paper was completed whilst the author held a grant by the Studienstiftung des Deutschen Volkes.

2 The early investigation of these algebras and ideals began with MARCZEWSKI 1933.

3 MILLER 1993.
2 Preliminaries

2.1 Notation and Definitions

A tree is a subset of $\omega^\omega$ closed under initial segments.

Let $T$ be a tree. We call a real $r \in \omega^\omega$ a branch of $T$ if $\forall n : r|_n \in T$. The set of all branches in $T$ is denoted by $[T]$. The (immediate) predecessor of a node $t \in T$ is uniquely determined and is denoted by $\operatorname{pred}(t)$. Obviously, if $\sup(s \in \omega^\omega : \forall x \in [T](s \subseteq x))$ does not exist, there is a real $y_T \in \omega^\omega$ with $[T] = \{y_T\}$. We then call the following object the stem of $T$:

$$\operatorname{stem}(T) := \begin{cases} \sup(s \in \omega^\omega : \forall x \in [T](s \subseteq x)) & \text{if the supremum exists} \\ y_T & \text{else} \end{cases}$$

To denote that a tree is a subtree with strictly longer stem we write

$$T \ll T' : \iff T \leq T' \text{ and } \operatorname{stem}(T) \neq \operatorname{stem}(T')$$

For a node $t \in T$ we denote by $\operatorname{Succ}(t) := \{s \in T : \exists n \in \omega(t^\frown n = s)\}$ the set of its (immediate) successors.

For a tree $T$ and a finite sequence $s \in \omega^\omega$ we define $T \uparrow s := \{t \in T : s \subseteq t \text{ or } t \subseteq s\}$. A node of a tree $T$ is called splitting node, if it has more than one successor, and $\omega$–splitting node, if it has infinitely many successors.

We call a notion of forcing $\mathbb{P}$ arboreal\(^4\) if there is a set $\mathcal{T}$ of trees partially ordered by inclusion, $\mathbb{P}$ is order–isomorphic to $\mathcal{T}$ and $\mathcal{T}$ has the following property:

$$\forall T \in \mathcal{T} \forall N > |\operatorname{stem}(T)| \exists T' \leq T : |\operatorname{stem}(T')| \geq N$$

If $\mathbb{P}$ is an arboreal forcing, we identify $p \in \mathbb{P}$ with the tree $T_p$. Without loss of generality all arboreal forceings have a largest element $1$ with $[1] = \omega^\omega$.

2.2 dramatis personae

1. A tree $L \subseteq \omega^\omega$ is called Laver tree, if all nodes above the stem are $\omega$–splitting nodes. We call the set of all Laver trees ordered by inclusion Laver forcing $\mathcal{L}$.

2. A tree $M \subseteq \omega^\omega$ is called superperfect, if every splitting node is an $\omega$–splitting node and every node has a successor which is a splitting node (and therefore an $\omega$–splitting node). Miller forcing $\mathcal{M}$ is the set of all superperfect trees ordered by inclusion.

\(^4\)This notion is far more general than the “perfect tree property” of [Groszek–Jech 1991], but in the applications we have to construct subtrees with specific properties and we need for these constructions many of the properties of “perfect tree forcing”. In particular all of the investigated forcing notions have the perfect tree property.
3. In analogy to the definition of $M$ we call a tree $P \subseteq 2^{<\omega}$ perfect, if below every node there is a splitting node and define Sacks forcing $S$ to be the set of all perfect trees ordered by inclusion.

4. We call a perfect tree $P$ uniform, if it has the following property:
If $t_1, t_2 \in P$ with $|t_1| = |t_2|$, then $t_1^{\uparrow}(0) \in P \iff t_2^{\uparrow}(0) \in P$ and $t_1^{\uparrow}(1) \in P \iff t_2^{\uparrow}(1) \in P$.
The set of all uniform perfect trees ordered by inclusion is called Příkrý–Silver forcing $V$.
The uniformity of $V$ obliges us to introduce the combinatorial technique of amalgamation: Let $T$ be a uniform perfect tree and $t_1, t_2 \in T$ with $|t_1| = |t_2|$. Define $R_1 := T \uparrow t_1$ and $R_2 := T \uparrow t_2$. If we have $Q \leq R_2$, then we set
\[
\text{amal}(R_1, Q) := \{ t \in \omega^{<\omega} : t(i) = t_1(i) \text{ for } i < |t_1| \}
\]
and \[\exists r \in Q : t(i) = r(i) \text{ for } i \geq |t_1|\]
Obviously we construct a copy of $Q$ in $R_1$ so that amal($R_1, Q$) $\leq R_1$.

5. Take the following set $T := \{\langle s, A \rangle \in \omega^{<\omega} \times [\omega^{<\omega}]^\omega : \text{there is an enumeration of } s \cup A \text{ with } a_0 = s \text{ and } \min(\text{ran}(a_{i+1})) > \max(\text{ran}(a_i)) \text{ for all } i \in \omega \}$ and order it via
\[
\langle s, A \rangle \leq \langle t, B \rangle \iff s \supseteq t, \forall a \in A \exists \{b_1, \ldots, b_n\} \subseteq B(a = b_1^{+} \ldots b_n),
\exists \{\beta_1, \ldots, \beta_m\} \subseteq B(\text{ran}(s) \setminus \text{ran}(t) = \text{ran}(\beta_1^{+} \ldots \beta_m))
\]
We call this partial ordering Matet forcing\footnote{Matet 1988}. We define $x \in [\langle s, A \rangle] \iff s \subset x \land \exists A_0 = \{c_1, c_2, \ldots\} \in [A]^\omega$, so that $x = s \cdot c_1^{+} c_2^{+} \ldots$ and let $T_{\langle s, A \rangle}$ be the tree of all finite initial sequences of reals in $[s, A] := [\langle s, A \rangle]$.
As for Příkrý–Silver forcing we define for $T := \langle s, A \rangle$ and $S := \langle t, B \rangle$ with $\langle s, B \rangle \leq \langle s, A \rangle$:
\[
\text{amal}(T, S) := \langle s, B \rangle
\]
We define Willowtree forcing $W$\footnote{Brendle 1995} by
\[
\langle f, A \rangle \in W \iff A \in [\omega^{<\omega}]^\omega, \forall a, a' \in A :
\]
We call the elements of $W$ willowtrees. $W$ is arboreal via
\[
T_W := \{ s \in \omega^{<\omega} : \forall n < |s| (n \in \text{dom}(f) \rightarrow s(n) = f(n)) \}
\]
\[
\land \forall a \in A(\text{ran}(s[a]) = 1)\}
\]
where $W = \langle f, A \rangle$.
For every willowtree $W = \langle f, A \rangle$ we order the elements of $A$ as follows:

$$i \leq j \iff \min(a_i) \leq \min(a_j)$$

Since the elements of $A$ are disjoint sets, this is a linear ordering. Define the following relation on $W$:

$$W \geq_i W' : \iff \forall j : a_j \in A'$$

Proposition 2.1 When $W_0 \geq_0 W_1 \geq_1 W_2 \geq_2 \ldots$ is a sequence of willowtrees, then $\langle W' : i \in \omega \rangle$ is a fusion sequence.

Because of the affinity of Willowtree forcing with the uniform forcing notions $V$ and $T$ we need a form of amalgamation: For every willowtree $W$, $\sigma \in 2^{< \omega}$ and $i = |\sigma|$, set

$$W_\sigma^i := \langle f \cup \bigcup_{j=0}^{i-1} (a_j \times \{\sigma(j)\}), A \setminus \{a_0, \ldots, a_{i-1}\} \rangle =: \langle f_\sigma^i, A_\sigma^i \rangle$$

For amalgamation let now $W$ be an arbitrary willowtree, $|\sigma| = |\sigma'| = i, V \leq W_\sigma^i$ and $V = \langle g, B \rangle$.

In this case we define the function

$$h := \begin{cases} f_\sigma^i & \text{on } \bigcup_{j=0}^{i-1} a_j \\ g & \text{else} \end{cases}$$

and with this $\text{amal}(W_\sigma^i, V) := \langle h, B \rangle$.

2.3 Measurability

We can connect every arboreal forcing naturally to a notion of measurability. In addition to the forcings defined there are the well-known forcing notions of COHEN forcing $\mathbb{C}$, HECHLER forcing $\mathbb{D}$ and MATHIAS forcing $\mathbb{R}$. These forcings form topology bases for the Baire topology $\mathbb{C}$, the dominating topology $\mathbb{D}$ and the ELLENTUCK topology $\mathbb{R}$ respectively. The forcings are therefore quite naturally connected to the $\sigma$–algebra of sets with the Baire property in these topologies. The $\mathbb{C}$–, $\mathbb{D}$– and $\mathbb{R}$–meager sets are also called $\mathbb{C}$–, $\mathbb{D}$– and $\mathbb{R}$–null sets.

In analogy to this situation we define in the case of non–topological forcings $\mathbb{P}$ a set of real numbers $A$ ($A \subseteq \omega^\omega$ or $A \subseteq 2^\omega$ according to the definition of $\mathbb{P}$) to be $\mathbb{P}$–measurable if

$$\forall p \in \mathbb{P} \exists p' : (\{p'\} \cap A = \emptyset \text{ or } \{p'\} \cap \omega^\omega \setminus A = \emptyset)$$

\footnote{For definitions cf. e.g. [Jech 1984], [Truss 1977]}
2.4 The Banach–Mazur games

The Polish school of set theorists and topologists commenced the study of analytic sets via infinite games. For this reason they defined the so-called Banach–Mazur games on a topological space.

Definition 2.2 Let $X$ be a topological space and $A \subseteq X$. Then we state the rules of the Banach–Mazur game $G_X(A)$:

- There are two players I and II, I begins.
- The players play open sets $U^I_i$ and $U^{II}_i$ respectively in turn.
- There are the following restrictions: $U^I_{i+1} \subseteq U^{II}_i$ and $U^{II}_i \subseteq U^I_i$.
- After playing countably many turns, I wins if $\emptyset \neq \bigcap_{i<\omega} U^I_i \subseteq A$. Else II wins.

Definition 2.3 Let $X$ be a topological space and $A \subseteq X \times \omega^\omega$. Then we state the rules of the Banach–Mazur game in the plane $G^2_X(A)$:

- There are two players I and II, I begins.
- The players play in turn; II plays open sets $U^{II}_i$, I plays pairs of open sets and natural numbers $(U^I_i, n_i)$.
- There are the following restrictions: $U^I_{i+1} \subseteq U^{II}_i$ and $U^{II}_i \subseteq U^I_i$.
- After playing countably many turns, I wins, if $\emptyset \neq (\bigcap_{i<\omega} U^I_i) \times \{(n_i)_{i<\omega}\} \subseteq A$. Else II wins.

We distinguish between the games $G_X(A)$ or $G^2_X(A)$ on the one hand, which is the set of all legal sequences according to the rules mentioned above, and a run of the game on the other, which is one particular sequence following the rules. A function on the initial segments of runs to the appropriate mathematical objects to play in the next turn is called a strategy. Obviously we call a strategy winning if it guarantees that its user will be the winner of the game. A set $A$ is called determined for a game, if either I or II has a winning strategy in this game and a topological space is called Choquet, if I has a winning strategy for $G_X(X)$.

We now state the fundamental theorems for Banach–Mazur games which decide our questions for the topological forcings $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{R}$:

Theorem 2.4 (Gale–Stewart 1953) Let $A$ be a closed set in $\omega^\omega$ or $(\omega^\omega)^2$. Then $A$ is determined for the games $G_X(A)$ and $G^2_X(A)$.

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8For proofs of all results in this section we refer to Kechris 1995, p. 51 sqq and 149 sqq.
Theorem 2.5 (Banach–Mazur) Let $X$ be a Choquet topological space which is a refinement of a metric space and $A \subseteq X$. Then for $G_X(A)$:

1. I has a winning strategy $\Rightarrow A$ is comeager in a non–empty open set
2. II has a winning strategy $\Rightarrow A$ is meager

Theorem 2.6 (Solovay’s Unfolding Trick) Let $X$ be a Choquet topological space, which is a refinement of a metric space, $C \subseteq X \times \omega^\omega$ and $A$ the projection of $C$ on $X$. Then for $G_2^X(C)$:

1. I has a winning strategy $\Rightarrow A$ is comeager in a non–empty open set
2. II has a winning strategy $\Rightarrow A$ is meager

As a consequence of all these results we get in the topological case:

Theorem 2.7 All analytic sets are $\mathbb{P}$–measurable for $\mathbb{P} \in \{C, D, R\}$.\footnote{9}

3 Generalized Banach–Mazur Games

3.1 Basic Notions and Generalized Gale–Stewart theorem

Our goal is now to prove a result similar to the Banach–Mazur theorem for the non–topological forcings. We define the generalized Banach–Mazur game and its variant in the plane:\footnote{10}

Definition 3.1 Let $\mathbb{P}$ be an arboreal forcing and $A \subseteq \omega^\omega$. Then the generalized Banach–Mazur game $G_{\mathbb{P}}(A)$ is played with the following rules:

I begins and I and II play in turn forcing conditions $p^I_i$ and $p^{II}_i$ with the following restricting property:

$$p^I_{i+1} \ll p^{II}_i \ll p^I_i$$

Then $f := \bigcap |p^I_i|$ is obviously a real number. I wins, if $f \in A$, otherwise II wins.

Definition 3.2 Let $\mathbb{P}$ be an arboreal forcing and $A \in (\omega^\omega)^2$. Then the generalized Banach–Mazur game in the plane $G_2^{\mathbb{P}}(A)$ is played with the following rules:

\footnote{9} Of Moschovakis 1994, p. 299sq
\footnote{10} There are some precursors of these notions in Kechris 1978, but they focus on topological properties of forcing.\footnote{11}
I begins and plays pairs of forcing conditions $p^I_i$ and natural numbers $n_i$, and
II plays forcing conditions $p^{II}_i$ with the following property:

$$p^I_{i+1} \ll p^{II}_i \ll p^I_i$$

Then $f := \bigcap [p^I_i]$ is obviously a real number. I wins, if $\langle f, \langle n_i : i \in \omega \rangle \rangle \in A$ and
II wins, if $\langle f, \langle n_i : i \in \omega \rangle \rangle \not\in A$.

To get a result analogous to the topological case we have to generalize the Gale–Stewart theorem: For a set $A$ and a forcing condition $p$ define the relative game $G_P(A, p)$ or $G^2_P(A, p)$ simply by postulating that all played conditions lie below $p$.

**Proposition 3.3** If $A \subseteq \omega^\omega$ is a closed set, $\mathbb{P}$ an arboreal forcing and $p \in \mathbb{P}$, then all games $G_P(A, p)$, $G_P(A)$, $G^2_P(A, p)$ and $G^2_P(A)$ are determined.

**Proof:**
One can easily see: if II has no winning strategy for $G_P(A, p)$ (or $G^2_P(A, p)$) then there is a $q \leq p$ with the property that for all $r \leq q$ II has no winning strategy in $G_P(A, r)$.
Let $A$ be closed. Suppose that II has no winning strategy for $G_P(A, p)$ so there is such a $q \leq p$. Then I can choose a $q$ with strictly longer stem. Regardless of what $q'$ II answers, II will have no winning strategy in $G_P(A, q')$, so we can iteratively define a strategy for I.
We still have to show that this strategy is winning or, in other words, that

$$\bigcup_{n \in \omega} \text{stem}(p_n) = \bigcap_{n \in \omega} [p_n] =: f \in A.$$ 

Suppose not, then one can find a finite sequence $s \subseteq f$ such that $[s] \subseteq \omega^\omega \setminus A$, because the complement of $A$ is open. Without loss of generality we have for some $n_0$: $s = \text{stem}(p_{n_0})$. But then II would have a trivial winning strategy for $G_P(A, p_{n_0})$ in contradiction to our assumption.
For the games in the plane we regard in the proof simply the product ordering in $\mathbb{P} \times \omega^{<\omega}$ instead of $\mathbb{P}$.

q.e.d.

For the general context of uniform unfolding we need two fundamental notions:

**Definition 3.4** Let $\mathbb{P}$ be an arboreal forcing and $\langle \tau_i : i \in \omega \rangle$ a sequence of strategies for one fixed player in the game $G_{\mathbb{P}}$. A partial function $\alpha : \mathbb{P} \to \mathbb{P}$ will be called a $\mathbb{P}$–strategic fusion for $\langle \tau_i : i \in \omega \rangle$ if $\alpha(T)$ has the following properties:

(K1) $\alpha(T) \leq T$

(K2) There is a function $S : [\alpha(T)] \times \omega \to \{ P \in \mathbb{P} : P \leq T \} : \langle x, b \rangle \mapsto S_{x,b}$ with

$$S_{x,b+1} \ll \tau_b(S_{x,b}) \ll S_{x,b}$$
\( \text{dom}(\alpha) = \mathbb{P} \) if \( \tau_1 \) is a sequence of strategies for player II and \( \text{dom}(\alpha) = \{ \tau_0(1) \} \), if \( \tau_1 \) is a sequence of strategies for player I.

In most cases we can even construct a sequence \( \langle T_\sigma : \sigma \in \omega^{<\omega} \rangle \) with \( \alpha(T) = \bigcap_{i \in \omega} \bigcup_{|\sigma| = i} T_\sigma \) (hence the name fusion) with the properties:

(C1) For each \( x \in [\alpha(T)] \) there is an increasing sequence \( \langle \sigma_i : i \in \omega \rangle \) in \( \omega^{<\omega} \) with \( |\sigma_i| = i \), so that \( x \in [T_{\sigma_i}] \) for all \( i \in \omega \).

(C2) For each \( \sigma \in \omega^{<\omega} \) there is an \( S \in \mathbb{P} \) such that

\[
T_\sigma \ll \tau_{|\text{pred}(\sigma)|}(S) \ll S \leq T_{\text{pred}(\sigma)}
\]

In this case we call the function \( f \) constructive.

To see that such an \( f \) really is a strategic fusion, let \( x \in [f(T)] \). Because of (C1) there is a sequence \( \langle \sigma_i : i \in \omega \rangle \) so that \( x \in [T_{\sigma_i}] \) for all \( i \in \omega \). Take the tree \( S \) whose existence is postulated in (C2) to be \( S_{x,b} \). Obviously the so defined function \( S \) has property (K2).

**Definition 3.5** We say that an arboreal forcing \( \mathbb{P} \) has the **linear dichotomy property** if for every \( A \subseteq \omega^{<\omega} \):

1. If I has a winning strategy in \( G_{\mathbb{P}}(A) \) then there is a \( q \in \mathbb{P} \), so that \( [q] \subseteq A \)
2. If II has a winning strategy in \( G_{\mathbb{P}}(A) \), then there is for every \( p \in \mathbb{P} \) a \( q \leq p \) with \( [q] \cap A = \emptyset \)

**Definition 3.6** We say an arboreal forcing has the **planar dichotomy property** if for every set \( C \subseteq (\omega^{\omega})^2 \) and its projection \( A \) the following hold:

1. If I has a winning strategy in \( G^2_{\mathbb{P}}(C) \), then there is a \( q \in \mathbb{P} \), so that \( [q] \subseteq A \)
2. If II has a winning strategy in \( G^2_{\mathbb{P}}(C) \), then there is for every \( p \in \mathbb{P} \) a \( q \leq p \) with \( [q] \cap A = \emptyset \)

### 3.2 The Unfolding Theorem

**Theorem 3.7** If there is for every strategy \( \tau \) of an arbitrary player a \( \mathbb{P} \)-strategic fusion \( \alpha \) for the constant sequence \( \langle \tau : i \in \omega \rangle \), then \( \mathbb{P} \) has the linear dichotomy property.
**Proof:**
Let $\tau$ be a winning strategy for II and $P \in \mathbb{P}$ arbitrary. Let $Q := \alpha(P) \leq P$ according to (K1). We have to show that $[Q] \cap A = \emptyset$. Take $f \in [Q]$ arbitrary. Then we have with (K2) for each $i \in \omega$:

$$S_{f,i+1} \leq \tau_i(S_{f,i}) = \tau(S_{f,i}) \leq S_{f,i}$$

Obviously the sequence

$$S_{f,0}, \tau(S_{f,0}), S_{f,1}, \tau(S_{f,1}), \ldots$$

is a run of the game $G_{\mathbb{P}}(A)$ according to $\tau$. Therefore $f \notin A$.
On the other hand if $\tau$ is a winning strategy for I, so we can construct the same sequence with one difference: (K3) says that $\alpha$ is not defined for arbitrary $p \in \mathbb{P}$, so the constructed run has to begin with the initial value of $\tau$, that is $\tau(1)$. This accounts for the asymmetry between I and II in the definition of the dichotomy property.

q.e.d.

**Theorem 3.8 (Unfolding Theorem)** If there is a $\mathbb{P}$–strategic fusion for any sequence of strategies for any of the two players then $\mathbb{P}$ has the planar dichotomy property.

**Proof:**
Let $\beta$ be a bijection between $\omega^{<\omega}$ and $\omega$ having the property: $s \subseteq t \Rightarrow \beta(s) \leq \beta(t)$.
Firstly, regard a winning strategy $\tau$ for II. We define for $P \in \mathbb{P}$ the following sequence of strategies: $\tau_i(P) := \tau((P, \beta^{-1}(i)))$. If $T \in \mathbb{P}$ is arbitrary, then there is according to our assumption the strategic fusion $\alpha$, so $T' := \alpha(T) \leq T$ via (K1).
We claim that $[T'] \cap A = \emptyset$, i.e., more precisely: If $x \in [T']$, then for all $y \in \omega^\omega$ it holds that $\langle x, y \rangle \notin C$.
Let now be $x \in [T']$ and $y \in \omega^\omega$. Then the sequence $(y|_k)_{k \in \omega} \subseteq \omega^{<\omega}$ is increasing in $\omega^{<\omega}$, and therefore $b_k := \beta(y|_k)$ is (because of the property postulated for $\beta$) an infinite increasing sequence in $\omega$. Via (K2) we get

$$S_{x,b_k+1} \leq \tau_b(S_{x,b_k}) \leq S_{x,b_k}$$

and so

$$S_{x,b_0} \gg \tau_{b_0}(S_{x,b_0}) \gg S_{x,b_1} \gg \tau_{b_1}(S_{x,b_1})$$

is a decreasing sequence of trees. So we can define a run

$$\langle S_{x,b_0}, y|_0 \rangle \geq \langle \tau((S_{x,b_0}, y|_0)), y|_0 \rangle \geq \langle S_{x,b_1}, y|_1 \rangle \geq \langle \tau((S_{x,b_1}, y|_1)), y|_1 \rangle \geq \ldots \geq \langle S_{x,b_k}, y|_k \rangle \geq \langle \tau((S_{x,b_k}, y|_k)), y|_k \rangle \geq \ldots$$
of the planar game $G_{P}^2(C)$ following the winning strategy $\tau$. Therefore $\langle x, y \rangle \not\in C$ and because $y$ was arbitrary, $x \not\in A$.

Secondly, let $\tau$ be a winning strategy for I. Now we have no function from $P \times \omega^{<\omega}$ to $P$, but a function from $P \times \omega^{<\omega}$ to $P \times \omega$. So $\tau(P, \beta^-(i))$ has two components. We denote the first of them by $\tau_i(P)$, and the second by $n_i(P)$.

By the assumption, we have a strategic fusion $\alpha$ for the sequence $\langle \tau_i : i \in \omega \rangle$ so set $T' := \alpha(\tau_0(1))$. We now have to show that $[T'] \subseteq A$, i.e. $\forall x \in [T'] \exists y \in \omega^\omega : \langle x, y \rangle \in C$. Take now $x \in [T']$ and construct by recursion:

$$T_0 := \tau_0(1)$$
$$\sigma_0 := \langle n_0(1) \rangle$$
$$b_1 := \beta(\sigma_0)$$
$$T_1 := S_{x, b_1} \leq T_0$$

Then we have $S_{x, b_1+1} \leq \tau_{b_1}(T_1) \leq T_1$.

$$\sigma_{i+1} := \sigma_i \hat{\langle n_{b_i+1}(T_{i+1}) \rangle}$$
$$b_{i+2} := \beta(\sigma_{i+1})$$
$$T_{i+2} := S_{x, b_{i+2}}$$

By the assumption $b_{i+1} \geq b_i + 1$ and so $T_{i+1} \leq S_{x, b_i+1} \leq \tau_{b_i}(T_i) \leq T_i$. So we get the following run of the game $G_{P}^2$:

$$\langle T_0, \sigma_0 \rangle = \langle \tau_0(1), (n_0(1)) \rangle = \tau((1, \langle \rangle)) \geq \langle T_1, \sigma_0 \rangle \geq \langle \tau_{b_1}(T_1), \sigma_1 \rangle \geq \ldots$$

which is a run according to $\tau$ having $x$ in the first component as a result and we have for $y := \bigcup_{i \in \omega} \sigma_i$:

$$\langle x, y \rangle \in C$$

q.e.d.

### 3.3 Strategic Fusions

After having reduced the proof of uniform unfolding to the existence of strategic fusion by 3.8, we have to give explicitly a strategic fusion for all the forcings mentioned above.
Sacks forcing. For a perfect tree $P$ we denote by $h_P$ the first splitting node. Let $T$ be a fixed perfect tree and $(\tau_i : i \in \omega)$ a sequence of strategies. Regard the following sequence of perfect trees.

$$
P_{(0)} := T
$$

$$
P_{\sigma^*(0)} := \tilde{P}_{\sigma} \uparrow h_{\tilde{P}_{\sigma}}^-(0)
$$

$$
P_{\sigma^*(1)} := \tilde{P}_{\sigma} \uparrow h_{\tilde{P}_{\sigma}}^-(1)
$$

$$
\tilde{P}_\sigma := \tau_{|\sigma|}(P_\sigma)
$$

Because the $P^{(i)} := \bigcup_{|\sigma|=i} \tilde{P}_{\sigma}$ are a fusion sequence we can define $\alpha(P) := \bigcap_{i \in \omega} P^{(i)}$ and $\alpha$ is even constructive.

Miller forcing. The strategic fusion is exactly analogous to the Sacks run construction. Simply substitute 2 by $\omega$.

Laver forcing. If $\tau$ is a strategy for the Laver game and $T$ is a Laver tree, so we define a rank function:

$$
\text{rk}_T : T \to \text{Ord} \cup \{\infty\}
$$

$$
\text{rk}_T(t) = 0 \iff \exists S \ll T : \text{stem}(\tau(S)) = t
$$

$$
\text{rk}_T(t) \leq \alpha \iff \forall t' \in \text{Succ}(t) : \text{rk}_T(t') < \alpha
$$

$$
\text{rk}_T(t) = \infty \text{ otherwise}
$$

Then we can easily prove the following

**Lemma 3.9** If $\tau$ is a strategy, we have for every node $t \in T$: $\text{rk}_T(t) \in \text{Ord}$.

Assisted by this lemma we now define a strategic fusion: Let $T$ be an arbitrary Laver tree and $(\tau_i : i < \omega)$ a sequence of strategies. At first, we define recursively a sequence of Laver trees $\hat{H}_{\sigma}$. To begin the recursion we define $H_{(0)} := T$, $\hat{H}_{(0)} := T$, $R_{(0)} := T$ and $i_{(0)} := 0$.

Let $\hat{H}_{\sigma}$, $R_{\sigma}$ and $i_{\sigma}$ be already defined and suppose that we have $\text{rk}_{R_{\sigma}}^{\tau_{\sigma}}(\text{stem}(\hat{H}_{\sigma})) \neq 0$ (we will show after the construction that this is always the case). Then stem$(\hat{H}_{\sigma})$ has infinitely many successors with smaller rank. We enumerate these successors by $\zeta_k$ and define recursively:

$$
H_{\sigma^*(n)} := \hat{H}_{\sigma} \uparrow \zeta_n
$$

$$
\hat{H}_{\sigma^*(n)} := \begin{cases} 
\text{Case 1} : & H_{\sigma^*(n)} \text{ if } \text{rk}_{R_{\sigma}}^{\tau_{\sigma}}(\text{stem}(H_{\sigma^*(n)})) > 0 \\
\text{Case 2} : & \tau_{\sigma}(S) \text{ if } \text{rk}_{R_{\sigma}}^{\tau_{\sigma}}(\text{stem}(H_{\sigma^*(n)})) = 0
\end{cases}
$$
In this definition, $S$ is the condition below $R_{\sigma}$ with $\text{stem}(H_{\sigma}^{<\langle n \rangle}) = \text{stem}(\tau_{i_{\sigma}}(S))$ (which exists according to the definition of the rank).

If we have Case 1 for $n \in \omega$, then we define:

$$R_{\sigma}^{<\langle n \rangle} := R_{\sigma}$$
$$i_{\sigma}^{<\langle n \rangle} := i_{\sigma}$$

If we have Case 2, we define instead of that:

$$R_{\sigma}^{<\langle n \rangle} := \hat{H}_{\sigma}^{<\langle n \rangle}$$
$$i_{\sigma}^{<\langle n \rangle} := i_{\sigma} + 1$$

If we have Case 2 at a tree, we call this tree a switching point. If we have $k$ switching points among the predecessors of another switching point, we call it a switching point of order $k + 1$. As we remarked above, $\text{rk}_{\hat{H}_{\sigma}}^{\langle n \rangle}(\text{stem}(H_{\sigma}))$ can never be zero, because either $H_{\sigma}$ was a switching point, then we have $H_{\sigma} = R_{\sigma}$ and (since strategies always strictly prolongate the stem) the image of a strategy cannot have the same stem, or $H_{\sigma}$ was no switching point and the rank is larger than zero.

As the ranks are descending chains of ordinals, we know that after each tree there is a switching point. So the set of all switching points is order isomorphic to $\omega^{<\omega}$ and also the set of the corresponding trees $S$. Now take this set of all these trees $S$ and denote them by $(T_{\sigma} : \sigma \in \omega^{<\omega})$. Then the map $\alpha(T) := \bigcap_{i \in \omega} T^{(i)}$ with $T^{(i)} := \bigcup_{|\sigma| = i} T_{\sigma}$ is a constructive strategic fusion.

Příkrý–Silver forcing  As in the case of Sacks forcing we denote the first splitting node of $P$ by $h_{P}$. We identify finite sequences from the set $2^n$ with the corresponding binary numbers. When we refer to the binary numbers, we write an upper index $[n]$ to indicate the length of the original sequence. So we have $P_{\langle 1,0,0 \rangle} = P_{1}^{[3]}$ and $P_{\langle 1,0 \rangle} = P_{1}^{[2]}$.

Take the following sequence of uniform perfect trees for an arbitrary tree $T$:

$$P_{\langle \rangle} := T$$
$$P_{\sigma}^{<\langle 0 \rangle} := P_{\sigma} \uparrow h_{P_{\sigma}}^{<\langle 0 \rangle}$$
$$P_{\sigma}^{<\langle 1 \rangle} := P_{\sigma} \uparrow h_{P_{\sigma}}^{<\langle 1 \rangle}$$

For $|\sigma| = n$ we have $2^n$ trees $P_{\sigma}$, which are ordered via the natural ordering of the binary numbers corresponding to the $\sigma$s. We define iteratively:

$$U_{0}^{[n]} := P_{0}^{[n]}$$
$$T_{i}^{[n]} := \tau_{n}(U_{i}^{[n]})$$
\[ U_i^{[n]} := \text{amal}(P_i^{[n]}, T_i^{[n]}) \]
\[ \tilde{P}_i^{[n]} := \text{amal}(P_i^{[n]}, T_{2^n-1}^{[n]}) \]

With the same arguments as for Sacks forcing \( \alpha(T) := \bigcap_{i \in \omega} \bigcup_{|\sigma| = i} \tilde{P}_\sigma \) is a constructive strategic fusion.

**Willowtree forcing**  Let \( W := (f, A) \) be an arbitrary willowtree, where \( A = \{ a_i : i \in \omega \} \). We define the following sequence of willowtrees:

\[
T_{\langle \rangle} := W
\]

Suppose that \( T_\sigma \) is defined for \( \sigma \in 2^{<\omega} \) with \( |\sigma| = n - 1 \) and write \( T_\sigma = (f_\sigma, A_\sigma) \).

Then:

\[
H_{\sigma^{(0)}} := (f_\sigma \cup (a_{|\sigma|} \times \{0\}), A_\sigma \setminus \{ a_{|\sigma|} \})
\]
\[
H_{\sigma^{(1)}} := (f_\sigma \cup (a_{|\sigma|} \times \{1\}), A_\sigma \setminus \{ a_{|\sigma|} \})
\]
\[
\tilde{H}_{0}^{[n]} := \tau_n(H_{0}^{[n]})
\]
\[
\tilde{H}_{i+1}^{[n]} := \tau_n(\text{amal}(H_{i+1}^{[n]}, \tilde{H}_i^{[n]}))
\]
\[
T_{2^n-1}^{[n]} := \tilde{H}_{2^n-1}
\]
\[
T_{i-1}^{[n]} := \text{amal}(\tilde{H}_{i-1}^{[n]}, T_{2^n-1}^{[n]})
\]

As one can see easily, \( T^{(n)} \) is a fusion sequence and therefore \( W' := \bigcap_{n \in \omega} T^{(n)} \) is a willowtree. So \( \alpha(W) := W' \) has all properties of a constructive strategic fusion.

**MATET forcing**  Let \( T := (s, A) \) be a MATET condition where we have \( A := \{ a_i : 1 \leq i < \omega \} \). Let \( (\tau_i : i \in \omega) \) be a sequence of strategies. Define the mapping \( \Theta_i := \tau_i \circ \ldots \circ \tau_0 \). Then regard the following sequence of MATET conditions:

\[
T_0 := T
\]
\[
s_{(0)} := s \cdot a_1
\]
\[
s_{(1)} := s
\]
\[
H_{(0)} := T \uparrow s_{(0)}
\]
\[
H_{(1)} := T \uparrow s_{(1)}
\]
\[
\tilde{H}_{(0)} := \Theta_0(H_{(0)})
\]
\[
\tilde{H}_{(1)} := \text{amal}(H_{(1)}, \tilde{H}_{(0)})
\]
\[
T_1 := \tilde{H}_{(0)} \cup \tilde{H}_{(1)}
\]
Now write $T_1$ as $\langle s, A^{(1)} \rangle$ where $A^{(1)} = \{a^{(1)}_i : i \in \omega \}$. Now we suppose that in the $i$th step the sequences $s_\sigma$ with $|\sigma| = i - 1$ are already constructed and that we have the tree $T_{i-1} = \langle s, A^{(i-1)} \rangle$. Define

$$s_\sigma(0) := s_\sigma \cdot a^{(i)}_1$$

$$s_\sigma(1) := s_\sigma$$

$$H_\sigma := T \uparrow s_\sigma$$

Again we identify binary numbers and 0–1 sequences and set

$$\tilde{H}_0[i] := \Theta_i(H_0[i])$$

$$\tilde{H}_k[i] := \begin{cases} \text{amal}(H_k[i], \tilde{H}_{k-1}[i]) & \text{if } k \text{ is odd} \\ \Theta_i(\text{amal}(H_k[i], \tilde{H}_{k-1}[i])) & \text{if } k \text{ is even} \end{cases}$$

$$\tilde{H}_{2^i-1}[i] := \tilde{H}_{2^i-1}$$

$$\tilde{H}_k[i] := \text{amal}(\tilde{H}_k[i], \tilde{H}_{k+1}[i])$$

$$T_i := \bigcup_{|\sigma| = i} \tilde{H}_\sigma$$

Now we claim that $\alpha(T) := \bigcap_{i \in \omega} T_i$ is a strategic fusion. Let $x \in \bigcap_{i \in \omega}[T_i]$ and $b \in \omega$. Then there is an increasing sequence $\langle k_i : i \in \omega \rangle$ with

$x = s^\cdot a^{(k_0)}_1 \cdot a^{(k_1)}_2 \cdot \ldots$

Define $S_{x,0} := \text{amal}(\tilde{H}^{[k_0]}_{l-1}, H_{k_0})$ where $\ell < 2^{k_0}$ is the appropriate integer such that $x \in [\tilde{H}^{[k_0]}_{\ell}]$. For $b > 0$ take

$k(b) := \min\{k_i : b \leq k_i \text{ and } \forall c < b(S_{x,c} \text{ was not constructed on level } k_i)\}$

Again take the corresponding $\ell < 2^{k(b)}$ and then define

$S_{x,b} := \Theta_{b-1}(\text{amal}(\tilde{H}^{[k(b)]}_{\ell-1}, H^{[k(b)]}_b))$

This function verifies that $\alpha$ is a strategic fusion.

**3.4 Results**

Concluding we get:

**Theorem 3.10** For forcing notions $\mathbb{P} \in \{S, M, T, V, W, L\}$ all analytic sets are $\mathbb{P}$–measurable.
Proof:

Together with the strategic fusions from the preceding section and 3.8 this proof is similar to 2.7.

Let $P$ be one of the mentioned forcings and $A$ be analytic in $\omega$. Then define $P_A := \{ P \in P : \exists R \leq P \text{ with } [R] \subseteq A \}$. We have to show that for all $Q \notin P_A$ we have: $\exists P' \leq Q : [P'] \cap A = \emptyset$. Take $Q \notin P_A$, so we have no $R \leq Q$ with $[R] \subseteq A$. But $A \cap [Q]$ is analytic, too, so we have a closed set $C$ in the plane with the projection $A \cap [Q]$. With Proposition 3.3 $C$ is determined, according to Section 3.3 all of the forcings have strategic fusions and with 3.8 we get: Either $\exists P : [P] \cap \omega \setminus (A \cap [Q]) = \emptyset$ or $\forall P \exists P' \leq P : [P'] \cap (A \cap [Q]) = \emptyset$. Since $Q \notin P_A$ the first case is impossible. Choose now $P := Q$, then we have a $P' \leq Q$ with $[P'] \cap (A \cap [Q]) = [P'] \cap A = \emptyset$.

q.e.d.

What we have won are previously unknown results for $\forall$, $\forall$ and $T$ (the results for $\forall$, $\forall$ and $\forall$ were already known implicitly because of results connected with the asymmetric games $\forall$) and a uniform method for proving the same result in case a new arboreal forcing should appear.

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