NEW HYPERGEOMETRIC CONNECTION FORMULAE BETWEEN FIBONACCI AND CHEBYSHEV POLYNOMIALS

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ABSTRACT. We establish new connection formulae between Fibonacci polynomials and Chebyshev polynomials of the first and second kinds. These formulae are expressed in terms of certain values of hypergeometric functions of the type $_2F_1$. Consequently, we obtain some new expressions for the celebrated Fibonacci numbers and their derivatives sequences. Moreover, we evaluate some definite integrals involving products of Fibonacci and Chebyshev polynomials.

Keywords: Fibonacci polynomials; Fibonacci numbers; Chebyshev polynomials; connection coefficients; hypergeometric functions

1. Introduction

Fibonacci numbers appear in several disciplines of modern science. The wide spectrum of applications of these numbers in mathematics, computer science, physics, biology, graph theory and statistics justifies the growing interest of mathematicians in the properties enjoyed by these numbers. The beautiful book of Koshy, [14], exhibits some of the applications in which these numbers arise.

The family of Fibonacci polynomials $\{F_n(x)\}$ is defined by Fibonacci-like recurrence relations. In fact, the sequence of Fibonacci numbers can be obtained from the sequence of Fibonacci polynomials by setting $x = 1$. Therefore, the more knowledge we acquire on Fibonacci polynomials, the closer we get to understanding the qualities of Fibonacci numbers and other sequences of numbers. Yet, studying Fibonacci polynomials for their own sake provides us with a clearer idea concerning their combinatorial and analytic properties. A great deal of mathematical ingenuity has been invested in developing identities involving Fibonacci polynomials, Fibonacci numbers, and their generalizations, see [11–13, 19–21] and the references there for examples on such identities.

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In order to study Fibonacci polynomials, one may consider linking Fibonacci polynomials to other well-studied polynomials, such as Chebyshev polynomials. Chebyshev polynomials of the first and second kinds, $T_n(x)$ and $U_n(x)$ respectively, are subfamilies of the larger class of Jacobi polynomials. They are of crucial importance from both the theoretical and practical points of view. The interested reader in these polynomials may consult [16].

Given two sets of polynomials $\{P_i\}_{i\geq0}$ and $\{Q_j\}_{j\geq0}$, the so-called connection problem between these polynomials is to determine the coefficients $A_{i,j}$ in the expression

$$P_i(x) = \sum_{j=0}^{i} A_{i,j} Q_j(x).$$

The connection coefficients $A_{i,j}$ play an important role in many problems in pure and applied mathematics and in mathematical physics. The problem of finding connection coefficients between two sets of orthogonal polynomials has been investigated by many authors, see for instance [3,6,7,10,15,17,18]. In fact, most of the formulae for the connection coefficients between orthogonal polynomials are given in terms of terminating hypergeometric series of various types. For example, the connection formula of Jacobi-Jacobi polynomials is given in terms of a terminating hypergeometric series of the type $\,{}_{3}F_{2}(1)$, see [7].

In this article, we solve the connection problem between Fibonacci polynomials and the orthogonal polynomials $T_n(x)$ and $U_n(x)$. We exhibit the Chebyshev expression form of Fibonacci polynomials $F_n(x)$. In fact, the connection coefficients turn out to be terminating hypergeometric series of type $\,{}_{2}F_{1}(\lambda)$ where $\lambda$ is either $-4$ or $-1/4$. Furthermore, we tackle the inverse connection coefficients problem. In other words, we express $T_n(x)$ and $U_n(x)$ in terms of $F_n(x)$. Again, the latter coefficients involve the terminating hypergeometric series $\,{}_{2}F_{1}(\lambda)$.

Based on the new connection formulae that we derive, we display several identities satisfied by Fibonacci numbers, and we evaluate some finite sums. Moreover, we find relations between terminating hypergeometric series of type $\,{}_{2}F_{1}(\lambda)$ for certain values of $\lambda$ and specific parameters. Also, some identities involving the derivatives sequences of Fibonacci numbers are given. Finally, we express weighted definite integrals of products of Fibonacci and Chebyshev polynomials, and products of Fibonacci polynomials as sums involving hypergeometric series.
FIBONACCI AND CHEBYSHEV POLYNOMIALS

2. SOME RELEVANT PROPERTIES OF FIBONACCI AND CHEBYSHEV POLYNOMIALS

In this section, we recall the properties of Fibonacci and Chebyshev polynomials that we are going to use throughout the note.

2.1. Fibonacci polynomials. Fibonacci polynomials are generated via the following recurrence relation

\[ F_{n+2}(x) = x F_{n+1}(x) + F_n(x), \quad n \geq 0, \quad F_0(x) = 0, \quad F_1(x) = 1. \]

The \( n \)-th Fibonacci polynomial can be described explicitly as follows

\[ F_n(x) = \frac{(x + \sqrt{x^2 + 4})^n - (x - \sqrt{x^2 + 4})^n}{2^n \sqrt{x^2 + 4}}. \]

The latter expression has the following explicit power form representation

\[ F_n(x) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} x^{n-2j-1} \]

where \( \lfloor z \rfloor \) represents the largest integer less than or equal to \( z \). Now one can define the \( n \)-th Fibonacci number as follows

\[ F_n = F_n(1) = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}. \]

The corresponding derivatives sequences of the Fibonacci numbers are denoted by \( F^{(q)}_n \). They are defined via

\[ F^{(q)}_n = D^q F_n(x) \bigg|_{x=1}. \]

Some of the identities relating Fibonacci and Lucas numbers to complex values of Chebyshev polynomials of the first and second kinds are as follows, see \[1,22,\]

\[ F_{n+1} = \frac{1}{i^n} U_n \left( \frac{i}{2} \right), \]

\[ U_n (-2i) = \frac{(-i)^n}{2} F_{3(n+1)}. \]

There are several articles that study relations and identities satisfied by Fibonacci numbers and their derivatives sequences, see for example \[5,12,14,\].
2.2. Chebyshev polynomials of the first and second kinds. Chebyshev polynomials $T_n(x)$ and $U_n(x)$ of the first and second kinds, respectively, are polynomials in $x$, which can be defined by (see, Mason and Handscomb [16]):

$$T_n(x) = \cos(n \theta),$$

and

$$U_n(x) = \frac{\sin((n + 1)\theta)}{\sin \theta},$$

where $x = \cos \theta$. The polynomials $T_n(x)$ and $U_n(x)$ are orthogonal on $(-1, 1)$ with respect to the weight functions $\frac{1}{\sqrt{1-x^2}}$ and $\sqrt{1-x^2}$, that is

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_n(x) T_m(x) dx = \begin{cases} 0, & m \neq n, \\ \frac{\pi}{2} c_n, & m = n, \end{cases}$$

and

$$\int_{-1}^{1} \sqrt{1-x^2} U_n(x) U_m(x) dx = \begin{cases} 0, & m \neq n, \\ \frac{\pi}{2}, & m = n, \end{cases}$$

where

$$c_n = \begin{cases} 2, & n = 0, \\ 1, & n > 0. \end{cases}$$

The polynomials $T_n(x)$ and $U_n(x)$ may be generated, respectively, by means of the two recurrence relations:

$$T_n(x) = 2x T_{n-1}(x) - T_{n-2}(x), \quad n = 2, 3, \ldots,$$

with

$$T_0(x) = 1, \quad T_1(x) = x,$$

and

$$U_n(x) = 2x U_{n-1}(x) - U_{n-2}(x), \quad n = 2, 3, \ldots,$$

with

$$U_0(x) = 1, \quad U_1(x) = 2x.$$

They also have the following explicit power forms:

$$T_n(x) = \frac{n}{2} \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^r}{n-r} \binom{n-r}{r} (2x)^{n-2r},$$

and

$$U_n(x) = \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^r \binom{n-r}{r} (2x)^{n-2r},$$
The following special values are of important use later:

\[(5)\]
\[T_n(1) = 1, \quad U_n(1) = n + 1,\]

\[(6)\]
\[D^q T_n(1) = \prod_{i=0}^{q-1} \frac{(n-i)(n+i)}{2i+1}, \quad q \geq 1,\]

and

\[(7)\]
\[D^q U_n(1) = (n+1) \prod_{i=0}^{q-1} \frac{(n-i)(n+i+2)}{2i+3}, \quad q \geq 1.\]

3. New connection formulae between Chebyshev polynomials of the first and second kinds and Fibonacci polynomials

The following two theorems establish two new connection formulae between Fibonacci polynomials and Chebyshev polynomials of the first and second kinds. The formulae are given in terms of values of hypergeometric functions.

**Theorem 3.1.** For every \(j \geq 1\), the following connection formula holds:

\[T_j(x) = \left\lfloor \frac{j}{2} \right\rfloor \sum_{m=0}^{\left\lfloor \frac{j}{2} \right\rfloor} (-1)^m \binom{j-m}{j-2m} 2^{j-2m-1} \frac{2F1}{j-m+2} \left( \begin{array}{c} -m, j-m \\ j-2m+2 \end{array} \right) \right) F_{j-2m+1}(x).\]

**Theorem 3.2.** For every \(j \geq 1\), the following connection formula holds:

\[U_j(x) = 2^j \left\lfloor \frac{j}{2} \right\rfloor \sum_{m=0}^{\left\lfloor \frac{j}{2} \right\rfloor} (-1)^{m+1} \binom{j}{m} \frac{-j+2m-1}{j-m+1} 2F1 \left( \begin{array}{c} -m, -j+m-1 \\ j-2m+2 \end{array} \right) \right) F_{j-2m+1}(x).\]

We will prove Theorem 3.1. The proof of Theorem 3.2 is similar.

**Proof:** The connection coefficients are written using the following terminating hypergeometric series

\[2F1 \left( \begin{array}{c} -m, j-m \\ 2+j-2m \end{array} \right) \right) = \sum_{k=0}^{m} \frac{(-m)_k (j-m)_k (-4)^k}{(2+j-2m)_k k!},\]

Therefore it suffices to show that the following identity holds

\[T_j(x) = \phi_j(x) := \sum_{m=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \sum_{k=0}^{m} \frac{(-1)^m 2^{-1+j+2k-2m}}{k! (m-k)! (1+j+k-2m)! (m-k)!} F_{j-2m+1}(x).\]
In other words, we want to prove that the function \( \phi_j(x) \) satisfies the recurrence relation defining \( T_j(x) \). It is easy to see that \( \phi_1(x) = x \) and \( \phi_2(x) = 2x^2 - 1 \). We now make use of the recurrence relation

\[
x F_j(x) = F_{j+1}(x) - F_{j-1}(x)
\]
satisfied by Fibonacci polynomials

\[
2x \phi_j(x) = \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \sum_{k=0}^{m} \frac{(-1)^m 2^{j+2k-2m} j (1 + j - 2m) (j + k - m - 1)!}{k! (m-k)! (1 + j + k - 2m)!(m-k)!} F_{j-2m+2}(x) 
\]
\[
+ \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \sum_{k=0}^{m} \frac{(-1)^{m+1} 2^{j+2k-2m} j (1 + j - 2m) (j + k - m - 1)!}{k! (m-k)! (1 + j + k - 2m)!(m-k)!} F_{j-2m}(x).
\]

Some algebraic manipulations will yield

\[
2x \phi_j(x) = \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \sum_{k=0}^{m} \frac{(-1)^m 2^{j-2+j+2k-2m} (j-1) (j - 2m) (j + k - m - 2)!}{k! (m-k)! (1 + j + k - 2m)!(m-k)!} F_{j-2m}(x) 
\]
\[
+ \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \sum_{k=0}^{m} \frac{(-1)^{m+1} 2^{j+2k-2m} (j + 1) (2 + j - 2m) (j + k - m)!}{k! (m-k)! (2 + j + k - 2m)!(m-k)!} F_{j-2m+2}(x) 
\]
\[
= \phi_{j-1}(x) + \phi_{j+1}(x),
\]

Since Chebyshev polynomials \( T_j(x), j \geq 1 \), are uniquely determined by the recurrence relation

\[
2x T_j(x) = T_{j-1}(x) + T_{j+1}(x), T_1 = x, T_2(x) = 2x^2 - 1,
\]
it follows that \( \phi_j(x) \) is the \( j \)-th Chebyshev polynomial \( T_j(x) \) for \( j \geq 1 \). This completes the proof of Theorem 3.1. \( \square \)

4. Inversion Formulae between Fibonacci and Chebyshev Polynomials of the First and Second Kinds

In this section we are concerned with deriving the inversion formulae to those given in the previous section. We will express Fibonacci and Lucas polynomials in terms of Chebyshev polynomials. Again the connection coefficients turn out to be expressions involving values of hypergeometric functions of the type \( {}_2F_1 \). We will need the following lemma in order to proceed.
Lemma 4.1. We set $d_{j,m} = \binom{j}{m}^{-1} \left( \frac{-m, -j + m - 1}{j} \right)$. The following recurrence relation holds

\[
4 \left( \frac{j - 2}{m - 1} \right) (j - 2m + 1) (j - m + 1) d_{j-2,m-1} + \binom{j - 1}{m} (j - 2m + 2) (j - m) d_{j-1,m-1} + \frac{1}{m} (j - 2m) (j - m + 1) d_{j-1,m} - \frac{1}{m} (j - m) (j - 2m + 1) d_{j,m} = 0.
\]

Proof: Let

\[ e_{j,m} = \binom{j}{m} (j - 2m + 1) d_{j,m}. \]

In order to prove that the recurrence relation above is satisfied, it suffices to show that

\[ 4(j - m + 1) e_{j-2,m-1} + (j - m) e_{j-1,m-1} + (j - m + 1) e_{j-1,m} = (j - m) e_{j,m}. \]

Now, $e_{j,m}$ can be written in the form

\[ e_{j,m} = \binom{j}{m} (j - 2m + 1) \sum_{k=0}^{m} \frac{(-m)_k (m - j - 1)_k (-4)^k}{(-j)_k k!}. \]

Using the identity

\[ (-m)_k = \frac{(-1)^k k!}{(m - k)!}, \]

$e_{j,m}$ can be written equivalently as

\[ e_{j,m} = (j - m + 1)(j - 2m + 1) \sum_{k=0}^{m} \frac{(j - k)! 4^k}{k! (m - k)! (j - k - m + 1)!}. \]

It follows that

\[
4(j - m + 1) e_{j-2,m-1} + (j - m) e_{j-1,m-1} + (j - m + 1) e_{j-1,m} = (j - m)_2 \left[ 4(1 + j - 2m) \sum_{k=0}^{m-1} \frac{(j - k)! 4^k}{k! (m - k)! (j - k - m)!} + (j - 2m) \sum_{k=0}^{m} \frac{(j - k - 1)! 4^k}{k! (m - k)! (j - k - m)!} \right] + (2 + j - 2m) \sum_{k=0}^{m-1} \frac{(j - k - 1)! 4^k}{k! (m - k - 1)! (j - k - m + 1)!}.
\]

Taking a common denominator yields the following simplification

\[ 4(j - m + 1) e_{j-2,m-1} + (j - m) e_{j-1,m-1} + (j - m + 1) e_{j-1,m} = (j - m)_2 (1 + j - 2m) \sum_{k=0}^{m} \frac{(j - k)!}{k! (m - k - 1)! (j - k - m + 1)!} = (j - m) e_{j,m}. \]

Lemma 4.1 is now proved. \(\square\)
In the following two theorems, we exhibit the connection relation between Fibonacci polynomials and Chebyshev polynomials of the first and second kinds.

**Theorem 4.2.** For every nonnegative integer \( j \), the following connection formula holds

\[
F_{j+1}(x) = \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} c_{j-2m} \binom{j-m}{j-2m} 2^{-j+2m+1} \binom{-m, j-m+1}{j-2m+1} \frac{1}{4} T_{j-2m}(x),
\]

where

\[
c_j = \begin{cases} 
2, & j = 0, \\
1, & j > 0.
\end{cases}
\]

**Theorem 4.3.** For every nonnegative integer \( j \), the following connection formula holds

\[
F_{j+1}(x) = \frac{1}{2j} \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j}{m} \binom{j-2m+1}{j-m+1} 2F_1 \left( -m, -j+m-1 \middle| -4 \right) U_{j-2m}(x).
\]

The proofs of Theorems 4.2 and 4.3 are similar, so it suffices to prove Theorem 4.3.

**Proof:** We will proceed by induction. Assume that the above identity is valid for any \( k \leq j \). We know that

\[
F_{j+1}(x) = x F_j(x) + F_{j-1}(x),
\]

therefore using the induction hypothesis to write \( F_{j-1}(x) \) and \( F_j(x) \) in terms of Chebyshev polynomials, together with making use of the recurrence relation

\[
x U_j(x) = \frac{1}{2} \left[ U_{j-1}(x) + U_{j+1}(x) \right],
\]

yield

\[
F_{j+1}(x) = \frac{1}{2j} \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j-1}{m} \binom{j-2m}{j-m} 2F_1 \left( -m, -j+m \middle| 1-j \right) U_{j-2m-2}(x) + U_{j-2m}(x).
\]

\[
+ \frac{1}{2^{j-2}} \sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor} \binom{j-2m-1}{j-m-1} \binom{j-2}{m} 2F_1 \left( -m, -j+m+1 \middle| 2-j \right) U_{j-2m-2}(x).
\]

In fact the latter identity can be simplified and rewritten as follows

\[
F_{j+1}(x) = \sum_{m=0}^{\lfloor \frac{j-1}{2} \rfloor} g_{j,m} U_{j-2m}(x) + \frac{1}{2} g_{j-1, \mu_j} U_{j-2, \mu_j-2}(x) + g_{j-2, \nu_j-1} U_{j-2, \nu_j}(x) \theta_j,
\]
where

\[ g_{j,m} = \frac{1}{2^j} \left\{ \binom{j-1}{m-1} \binom{j-2m+2}{j-m+1} 2F_1 \left( \begin{array}{c} 1-m, -j+m-1 \\ j-1 \end{array} \mid -4 \right) + \right. \\
\left. 4 \frac{(j-2)}{(m-1)} \frac{(j-2m+1)}{j-m} 2F_1 \left( \begin{array}{c} 1-m, m-j \\ 2-j \end{array} \mid -4 \right) + \right. \\
\left. \binom{j-1}{m} \frac{(j-2m)}{j-m} 2F_1 \left( \begin{array}{c} -m, m-j \\ 1-j \end{array} \mid -4 \right) \right\}, \]

and

\[ \mu_j = \left\lfloor \frac{j-1}{2} \right\rfloor, \quad \nu_j = \left\lfloor \frac{j}{2} \right\rfloor, \quad \theta_j = \begin{cases} 1, & \text{if } j \text{ even,} \\ 0, & \text{if } j \text{ odd}. \end{cases} \]

Making use of Lemma \[4.1\] together with some manipulations imply that \( g_{j,m} \) can be reduced to take the form

\[ g_{j,m} = \frac{j}{2^j (j-m+1)} 2F_1 \left( \begin{array}{c} -m, -1+m-j \\ -j \end{array} \mid -4 \right). \]

This yields that

\[ F_{j+1}(x) = \frac{1}{2^j} \sum_{m=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \frac{j}{m} \frac{(j-2m+1)}{j-m+1} 2F_1 \left( \begin{array}{c} -m, -j+m-1 \\ -j \end{array} \mid -4 \right) U_{j-2m}(x), \]

which completes the proof of Theorem \[4.3\] \( \square \)

5. SOME APPLICATIONS

In this section, we introduce three applications to the new derived connection formulae and their inversion ones: (i) We display some new expressions involving Fibonacci and Lucas numbers. (ii) Some new expressions for the derivatives sequences of Fibonacci numbers are given. (iii) We evaluate some definite integrals involving certain products of Fibonacci and Chebyshev polynomials.

5.1. New expressions involving Fibonacci and Lucas numbers. In this section we use the results we developed in \$3\$ and \$4\$ to evaluate finite sums involving certain values of hypergeometric functions and Fibonacci numbers.
Corollary 5.1. For every nonnegative integer \( j \), the following two identities hold:

\[
\sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^m \binom{j-m}{j-2m} \frac{2^{j-2m-1}}{j-m} \binom{-m-j-m}{j-2m+2} \binom{4}{j} F_{j-2m+1} = 1,
\]

and

\[
2^j \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor+1} (-1)^{m+1} \binom{j}{m} \frac{(-j+2m-1)}{j-m+1} \binom{-m-j+m-1}{j-2m+1} \binom{-4}{j} F_{j-2m+1} = j+1.
\]

Corollary 5.2. For every nonnegative integer \( j \), the following two expressions for Fibonacci numbers are valid

\[
F_{j+1} = \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \frac{1}{c_{j-2m}} \binom{j-m}{j-2m} 2^{-j+2m+1} \binom{-m-j+m+1}{j-2m+1} \binom{-4}{j},
\]

\[
F_{j+1} = \frac{1}{2^j} \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j}{m} \frac{(j-2m+1)^2}{j-m+1} \binom{-m-j+m+1}{j-2m+1} \binom{-4}{j}.
\]

Proof: The proof of Corollaries 5.1 and 5.2 are immediately obtained from Theorems 3.1, 3.2, 4.2 and 4.3, respectively, by setting \( x = 1 \).

The fact that Fibonacci numbers themselves can be expressed as values of terminating hypergeometric series can be exploited in order to find a linear relation between hypergeometric series of type \( \binom{a}{c} \) with different parameters at a specific value.

The \( n \)-th Fibonacci number can be written as a hypergeometric series itself. In fact one knows that

\[
F_n = \frac{n}{2^{n-1}} \binom{\frac{1-n}{2}}{\frac{2-n}{2}} \binom{5}{5} = \binom{\frac{1-n}{2}}{\frac{2-n}{2}} \binom{-4}{1-n},
\]

see [5] for example. One of the linear transformations of hypergeometric series is given by the following identity

\[
\binom{a}{c} z = (1-z)^{-a} \binom{a-c+b}{c} \binom{z}{z-1}.
\]

Putting these together, one can rewrite Corollary 5.2 as follows.
Corollary 5.3. For every nonnegative integer $j$, the following two expressions identities hold true

$$2F_1\left(\frac{-j, 1-j}{-j} \middle| -4\right) = \sum_{m=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \frac{1}{c_{j-2m}} \left(\frac{j-m}{j-2m}\right) 2^{-j+2m+1} 2F_1\left(-m, j-m+1 \middle| -1 \right| \frac{-4}{j-2m+1}\right),$$

$$2F_1\left(\frac{-j, 1-j}{\frac{3}{2}} \middle| 5\right) = \frac{2}{j+1} \sum_{m=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \frac{5^m}{c_{j-2m}} \left(\frac{j-m}{j-m+1}\right) 2F_1\left(-m, j-m+1 \middle| -4 \right| \frac{1}{j-2m+1}\right),$$

Furthermore, one may obtain similar identities to the ones above relating values of the hypergeometric series $2F_1$ evaluated at $1/5$ or $4/5$ and several other values, see §4 in [5], using different linear transformations.

Now, and based on the identities [(1)] and [(2)], along with the connection formulae obtained in §4, the following identities follow.

Corollary 5.4. For every nonnegative integer $j$, and $i^2 = -1$, the following identities hold:

$$i^j F_{j+1} = 2^j \sum_{m=0}^{\left\lfloor \frac{j}{2} \right\rfloor} (-1)^{m+1} \left(\frac{j-m}{j-m+1}\right) F_{j-2m+1} \left(\frac{\frac{j}{2}}{\frac{j-2m+1}{2}} \middle| -1 \right| \frac{1}{4}\right),$$

$$(-i)^j F_{j+3} = 2^{j+1} \sum_{m=0}^{\left\lfloor \frac{j}{2} \right\rfloor} (-1)^{m+1} \left(\frac{j-m}{j-m+1}\right) F_{j-2m+1} \left(\frac{-2i}{\frac{j-2m+1}{2}} \middle| -1 \right| \frac{1}{4}\right).$$

One may also obtain some new trigonometric identities making use of the fact that $T_n(\cos \theta) = \cos(n \theta)$. In fact one has

$$F_{j+1}(\cos \theta) = \sum_{m=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \frac{1}{c_{j-2m}} \left(\frac{j-m}{j-2m}\right) 2^{-j+2m+1} 2F_1\left(-m, j-m+1 \middle| -1 \right| \frac{-4}{j-2m+1}\right) \cos ((j-2m)\theta),$$

where

$$c_j = \begin{cases} 2, & j = 0, \\ 1, & j > 0. \end{cases}$$
Another interesting identity is obtained by observing that $T_n \left( \frac{x + x^{-1}}{2} \right) = \frac{x^n + x^{-n}}{2}$, whence
\[
F_{j+1} \left( \frac{x + x^{-1}}{2} \right) = \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \frac{1}{c_{j-2m}} \left( j - m \right) 2^{-j+2m} \binom{m, j-m+1, j-2m+1}{m, j-2m+2} \binom{-1}{-4} \left( x^{j-2m} + x^{2m-j} \right).
\]

### 5.2. New derivatives sequences identities

Based on the connection formulae introduced in Theorems 3.1, 3.2, 4.2, and 4.3, we may obtain new formulae for the derivatives sequences identities.

**Corollary 5.5.** For all $q \geq 1$, the following two formulae are valid
\[
\sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^m \binom{j - m}{j - 2m} \frac{2^{j-2m-1}}{j-m} \frac{\binom{2F_1}{-m, j-m \mid j-2m+2 \mid -4}}{F_{j-2m+1}^{(q)}} = \frac{(-1)^{q+1} \sqrt{\pi} j (1-j)_{q-1} (j+1)_{q-1}}{2^q \Gamma(q + \frac{1}{2})},
\]
and
\[
\sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^{m+1} \binom{j}{m} \frac{(-j + 2m - 1)}{j-m+1} \frac{\binom{2F_1}{-m, -j + m - 1 \mid -j \mid -1}}{F_{j-2m+1}^{(q)}} = \frac{(-1)^{q+1} \sqrt{\pi} j(j+1)_{q-1} (j+3)_{q-1}}{2^{j+q+1} \Gamma(q + \frac{1}{2})}.
\]

**Corollary 5.6.** For all $q \geq 1$, the following two formulae are valid
\[
F_{j+1}^{(q)} = \frac{(-1)^{q+1} \sqrt{\pi}}{\Gamma(q + \frac{1}{2})} \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \frac{1}{c_{j-2m}} \left( j - m \right) 2^{-j+2m-q+1} \frac{(j-2m)^2 (j-2m+1)_{q-1}}{(j-2m+1)_{q-1}} \times (j + 2m + 1)_{q-1} \frac{2F_1}{-m, j - m + 1 \mid j - 2m + 1 \mid -1},
\]
and
\[
F_{j+1}^{(q)} = \frac{(-1)^{q+1} \sqrt{\pi}}{2^{j+q+1} \Gamma(q + \frac{3}{2})} \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \frac{1}{c_{j-2m}} \binom{j}{m} \frac{(j-2m)(j-2m+1)^2 (j-2m+2)}{j-m+1} \times (j-2m+3)_{q-1} \frac{2F_1}{-m, -j + m - 1 \mid -j \mid -4}.
\]
Proof: In order to prove Corollaries 5.5 and 5.6 one needs to differentiate the connection formulae in Theorems 3.1, 3.2, 4.2, and 4.3, then one sets $x = 1$. Now the identities follow from (6) and (7). □

5.3. Some integrals formulae involving Chebyshev and Fibonacci polynomials. The following two integrals formulae are direct consequences of Theorems 4.2 and 4.3.

Corollary 5.7. For all $j \geq k$, the following two integrals formulae hold:

$$\int_{-1}^{1} \frac{F_{j+1}(x)T_{k}(x)}{\sqrt{1-x^2}} \, dx = \begin{cases} \pi \binom{j}{k} \left( \frac{k-j}{2} \right)^{2j+k-2} \binom{j}{k} \binom{2j-k+2}{j} \left( -\frac{k-j}{2} \right) \binom{k-j}{k+1}, & \text{if } (k+j) \text{ even,} \\ 0, & \text{otherwise} \end{cases}$$

and

$$\int_{-1}^{1} \frac{F_{j+1}(x)}{\sqrt{1-x^2}} \, dx = \begin{cases} \pi \binom{j}{k} \left( \frac{k-j}{2} \right)^{2j+k-2} \binom{j}{k} \binom{2j-k+2}{j} \left( -\frac{k-j}{2} \right) \binom{k-j}{k+1}, & \text{if } (k+j) \text{ even,} \\ 0, & \text{otherwise} \end{cases}$$

Now one can find an explicit description for weighted definite integrals of products of Fibonacci polynomials in terms of hypergeometric series.

Corollary 5.8. For all $j \geq k$, the following two integrals formulae hold:

$$\int_{-1}^{1} \frac{F_{j+1}(x)F_{k+1}(x)}{\sqrt{1-x^2}} \, dx = \pi \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \frac{2^{4m} \binom{j-m}{k} \binom{k-m}{j-2m} \binom{j-m}{k-2m} \binom{k-m}{j-2m}}{c_{k-2m}c_{j-2m}} \binom{-m, k-m+1}{k-2m+1} \binom{-1}{4} \times \binom{-m, j-m+1}{j-2m+1} \binom{-1}{4}.$$  

and

$$\int_{-1}^{1} \frac{F_{j+1}(x)}{\sqrt{1-x^2}} \, dx = \pi \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j}{m} \binom{k}{m} \binom{k-2m+1}{k-m+1} \binom{j-2m+1}{j-m+1} \binom{-m, -k+m-1}{-k} \binom{-4}{-4} \times \binom{-m, -j+m-1}{-j} \binom{-4}{-4}.$$
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