Painlevé transcendent evaluation of the scaled distribution of the smallest eigenvalue in the Laguerre orthogonal and symplectic ensembles

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The scaled distribution of the smallest eigenvalue in the Laguerre orthogonal and symplectic ensembles is evaluated in terms of a Painlevé V transcendent. This same Painlevé V transcendent is known from the work of Tracy and Widom, where it has been shown to specify the scaled distribution of the smallest eigenvalue in the Laguerre unitary ensemble. The starting point for our calculation is the scaled $k$-point distribution of every odd labelled eigenvalue in two superimposed Laguerre orthogonal ensembles.

1 Introduction

A quite old result in mathematical statistics concerns the eigenvalue distribution of random matrices of the form $A = X^T X$ where $X$ is a $n \times N$ ($n \geq N$) rectangular matrix with real entries. First it was proved by Wishart [27] that with

$$ (dX) := \prod_{j=1}^{n} \prod_{k=1}^{N} dx_{jk}, \quad (dA) := \prod_{1 \leq j < k \leq N} da_{jk} $$

denoting the product of differentials of the independent elements, the change of variables from the elements of $X$ to the elements of $A$ takes place according to the formula

$$ (dX) = \left( \det A \right)^{(n-N-1)/2} (dA). \quad (1.1) $$

From this a description in terms of eigenvalues and eigenvectors can be undertaken by introducing the spectral decomposition

$$ A = O \Lambda O^T $$

where the columns of $O$ consist of the normalized eigenvectors of $A$, and $\Lambda$ is the diagonal matrix of eigenvalues. About a decade after the work of Wishart, it was proved by a number of authors (see e.g. [4]) that

$$ (dA) = \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j| \prod_{j=1}^{N} \lambda_j (O^T dO). \quad (1.2) $$

A significant qualitative feature of (1.2) is that the eigenvalue dependence factors from that of the eigenvectors.

Suppose now the elements of $X$ are identical, independently distributed standard Gaussian random variables so that the corresponding probability measure is proportional to

$$ \prod_{j=1}^{n} \prod_{k=1}^{N} e^{-x_{jk}^2/2} (dX) = e^{-\frac{1}{2} \text{Tr}(X^T X)} (dX) = e^{-\frac{1}{2} \sum_{j=1}^{N} \lambda_j} (dX). \quad (1.3) $$
Noting that \( \det A = \prod_{j=1}^{N} \lambda_j \), substituting (1.2) in (1.1), and then substituting the resulting formula for \((dX)\) in (1.3) gives the now standard result that the eigenvalue probability density function (p.d.f.) of the matrix \( A = X^T X \) is given by

\[
\frac{1}{C} \prod_{j=1}^{N} \lambda_j^{(n-N-1)/2} e^{-\lambda_j/2} \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|, \quad \text{(1.4)}
\]

where \( C \) denotes the normalization and \( \lambda_j > 0 \) \((j = 1, \ldots, N)\). This is referred to as the real Wishart distribution, or alternatively as the Laguerre orthogonal ensemble (LOE\(_N\)), the latter name originating from the occurrence of the classical Laguerre weight function \( \lambda^n e^{-\lambda} \) (up to scaling of \( \lambda \)) and the invariance of (1.4) under the mapping \( A \mapsto OA^T \).

Another random matrix structure which leads to the p.d.f. (1.4) is the block matrix

\[
\begin{pmatrix}
0_{n \times n} & X \\
X^T & 0_{N \times N}
\end{pmatrix}. \quad \text{(1.5)}
\]

It is straightforward to verify that in general (1.5) has \( n - N \) zero eigenvalues, while the remaining \( 2N \) eigenvalues come in \( \pm \) pairs. It is similarly easy to verify that with \( X \) distributed according to (1.3) the positive eigenvalues are distributed according to (1.4) but with \( \lambda_j \mapsto \lambda_j^2 \) and an additional factor of \( \prod_{j=1}^{N} |\lambda_j| \). Hence the square roots of the positive eigenvalues of (1.5) are distributed according to (1.4).

Over the past decade the p.d.f. (1.4) has found application in a number of physical problems. One example is in the theory of quantum transport through disordered wires, in which the matrix product \( X^T X \) for \( X \) a \( N \times N \) random matrix modelling the top right hand block of the transmission matrix occurs in the Landauer formula for the conductance (see e.g. [4]). Another is in quantum chromodynamics, where the structure (1.5) models a random Dirac operator in the chiral gauge, and the number of zero eigenvalues is prescribed [26].

Because \( A \) is positive definite and so has positive eigenvalues the eigenvalues near \( \lambda_j = 0 \) are said to be near the hard edge. For eigenvalues in this neighbourhood, it is known that with \( n - N \) fixed, the statistical properties tend to well defined limits in the \( N \to \infty \) scaled limits, where the scaling is

\[
\lambda_j \mapsto \lambda_j/4N \quad \text{(1.6)}
\]

In fact the scaled \( k \)-point distribution function is known exactly [12]. Thus with

\[
a := (n - N - 1)/2, \\
K^h(X, Y) := \frac{J_a(X^{1/2}Y^{1/2}J_a(Y^{1/2}) - X^{1/2}J_a(X^{1/2})J_a(Y^{1/2})}{2(X - Y)}, \quad \text{(1.7)}
\]

\[
D_1^h(x, y) := \frac{\partial}{\partial x} S_1(x, y), \\
P_1^h(x, y) := -\int_x^y S_1(x, z) \, dz - \frac{1}{2} \text{sgn}(x - y),
\]

\[
f_1(x, y) := \begin{bmatrix} S_1^h(x, y) & P_1^h(x, y) \\ D_1^h(x, y) & S_1^h(y, x) \end{bmatrix},
\]

\[
\rho_{\text{LOE}}^{\text{h}}(x_1, \ldots, x_k; a) := \lim_{N \to \infty} \left( \frac{1}{4N} \right)^k \rho_{\text{LOE}}^{\text{h}}(\frac{x_1}{4N}, \ldots, \frac{x_k}{4N}; a) \quad \text{(1.8)}
\]

we have

\[
\rho_{\text{LOE}}^{\text{h}}(x_1, \ldots, x_k; (a - 1)/2) = \text{qdet} [f_1(x_j, x_k)]_{j, l=1, \ldots, k} \quad \text{(1.9)}
\]
where qdet is the quaternion determinant introduced into random matrix theory by Dyson [6], and the superscript “h” denotes the hard edge scaling [1,6].

In this work we will compute the exact distribution of the smallest eigenvalue in the scaled LOE as specified by the \( k \)-point distribution (1.9) in terms of a certain Painlevé V transcendent. We will also compute the same distribution for the scaled Laguerre symplectic ensemble (LSE), which before scaling is specified by the eigenvalue p.d.f.

\[
\prod_{j=1}^{N} \lambda_j^a e^{-\lambda_j} \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^4.
\]

This p.d.f. results from positive definite matrices \( A = X^\dagger X \) when the matrix \( X \) has the 2 \( \times \) 2 block structure

\[
\begin{pmatrix}
z & w \\
-\bar{w} & \bar{z}
\end{pmatrix}
\]

of a real quaternion (each eigenvalue is then doubly degenerate as well as occuring in \( \pm \) pairs). The explict form of the corresponding scaled \( k \)-point distribution function is given in [12]; in taking the scaled limit with scaling (1.6) the ensemble LSE \( \frac{N}{2} \) is considered rather than \( \text{LSE}_N \).

Crucial to our study is knowledge, in terms of a Painlevé V transcendent, of the distribution of the smallest eigenvalue in the scaled Laguerre unitary ensemble (LUE). Before scaling, the latter distribution is specified by the eigenvalue p.d.f.

\[
\prod_{j=1}^{N} \lambda_j^a e^{-\lambda_j} \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^2,
\]

and results from positive definite matrices \( A = X^\dagger X \) when the matrix \( X \) has complex elements. The corresponding scaled \( k \)-point distribution function

\[\rho_{(k)}^{\text{LUE h}}(x_1, \ldots, x_k) = \lim_{N \to \infty} \left( \frac{1}{4N} \right)^k \rho_{(k)}^{\text{LUE N}} \left( \frac{x_1}{4N}, \ldots, \frac{x_k}{4N} \right)\]

has the explicit form [9]

\[\rho_{(k)}^{\text{LUE h}}(x_1, \ldots, x_k) = \det[K^h(x_j, x_l)]_{j,l=1,\ldots,k},\] (1.11)

where \( K^h \) is given by (1.7).

In general the probability that there are no eigenvalues in an interval \( J \), \( E(0; J) \), can be written in terms of the corresponding \( k \)-point distribution by

\[E(0; J) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_J dx_1 \cdots \int_J dx_k \rho_{(k)}(x_1, \ldots, x_k).\] (1.12)

For \( \rho_{(k)} \) a \( k \times k \) determinant with entries \( g(x_j, x_k) \) the structure (1.12) is just the expansion of the Fredholm integral operator on \( J \) with kernel \( g(x, y) \). Thus it follows from (1.11) that

\[E^h_2(0; (0, s); a) = \det(1 - K^h)\]

where \( E^h_2(0; (0, s); a) \) denotes the probability there are no eigenvalues in the interval \( (0, s) \) for the scaled LUE (the subscript 2 characterizes the LUE via the exponent on the product of differences in (1.11)), while \( K^h \) denotes the integral operator on \( (0, s) \) with kernel \( K^h(x, y) \). With \( p^\text{min}_\beta(s; a) \)
denoting the distribution of the smallest eigenvalue in the scaled LOE ($\beta = 1$), LUE ($\beta = 2$) or LSE ($\beta = 4$) we have in general

$$p_{\beta}^{\min}(s; a) = -\frac{d}{ds} E_{\beta}^h(0; (0, s); a)$$

so to compute $p_{\beta}^{\min}(s; a)$ it suffices to compute $E_{\beta}^h(0; (0, s); a)$.

For general $a > -1$, $E_{\beta}^h(0; (0, s); a)$ has been computed in terms of a Painlevé transcendent by Tracy and Widom [22]. The Painlevé transcendent is denoted by $q_h$ (in [22] the subscript $h$ is not present), and specified as the solution of the nonlinear equation

$$s(q_h^2 - 1)(sq_h)' = q_h(sq_h)^2 + \frac{1}{4}(s - a^2)q_h + \frac{1}{4}sq_h^3(q_h^2 - 2)$$

subject to the boundary condition

$$q_h(s) \sim \frac{1}{2^{a/2} \Gamma(1 + a)} s^{a/2}.$$ 

That $q_h$ is a Painlevé transcendent follows from the transformation [22]

$$q_h(s) = \frac{1 + y(x)}{1 - y(x)}, \quad s = x^2$$

from which one can deduce that $y(x)$ satisfies the Painlevé V equation

$$y'' = \left(\frac{1}{2y} + \frac{1}{1 - y}\right)(y')^2 - \frac{1}{x} y' + \frac{(y - 1)^2}{x^2} (\alpha y + \beta) + \frac{\gamma y}{x} + \frac{\delta y(y + 1)}{y - 1}$$

with $\alpha = -\beta = a^2/8$, $\gamma = 0$ and $\delta = -2$. The result of [22] is that the Painlevé transcendent $q_h$ specifies $E_{\beta}^h$ via the formula

$$E_{\beta}^h(0; (0, s); a) = \exp \left( -\frac{1}{4} \int_0^s (\log s/t)(q_h(t))^2 \, dt \right).$$

In this work we will show that $E_{\beta}^h$ can also be evaluated in terms of $q_h(t)$. Specifically, we obtain the formula

$$\left( E_{\beta}^h(0; (0, s); (a - 1)/2) \right)^2 = E_{\beta}^h(0; (0, s); a) \exp \left( -\frac{1}{2} \int_0^s \frac{q_h(t)}{\sqrt{t}} \, dt \right).$$

With $E_{\beta}^h$ and $E_{\beta}^h(0; (0, s); a)$ known in terms of $q_h(t)$ the probability $E_{\beta}^h(0; (0, s); a)$ can also be expressed in terms of $q_h(t)$ by using the formula [13]

$$E_{\beta}^h(0; (0, s); a + 1) = \frac{1}{2} \left( E_{\beta}^h(0; (0, s); (a - 1)/2) + \frac{E_{\beta}^h(0; (0, s); a)}{E_{\beta}^h(0; (0, s); (a - 1)/2)} \right).$$

Thus

$$\left( E_{\beta}^h(0; (0, s); a + 1) \right)^2 = E_{\beta}^h(0; (0, s); a) \cosh^2 \left( \frac{1}{4} \int_0^s \frac{q_h(t)}{\sqrt{t}} \, dt \right).$$

Here $E_{\beta}^h$ is computed by scaling the ensemble LSE$_N/2$ according to [13]. Also, with $E_{\beta}^h(1; (0, s); a)$ denoting the probability that there is exactly one eigenvalue in the interval $(0, s)$ for the scaled LOE, we have the inter-relationship [13]

$$E_{\beta}^h(1; (0, s); (a - 1)/2) = E_{\beta}^h(0; (0, s); a + 1) - E_{\beta}^h(0; (0, s); (a - 1)/2).$$
Substituting (1.15) and (1.17) shows
\[
\left( E_1^h(1; (0, s); (a - 1)/2) \right)^2 = E_2^h(0; (0, s); a) \sinh^2 \left( \frac{1}{4} \int_0^s q_h(t) \, dt \right). \tag{1.18}
\]

Crucial to our derivation of (1.15) is a reworking of the derivation of Tracy and Widom \cite{23} giving the probability \( E_s^1(0; (s, \infty)) \) in terms of a Painlevé transcendent. Here \( E_1^s(0; (s, \infty)) \) denotes the probability that there are no eigenvalues in the interval \((s, \infty)\) for the scaled Gaussian orthogonal ensemble (GOE). As the density falls off rapidly as \( s \) increases, the region \((s, \infty)\) is said to be a soft edge, thus explaining the use of the superscript “s” in \( E_s^1 \).

The ensemble GOE\(_N\) refers to the eigenvalue p.d.f.
\[
\frac{1}{C} \prod_{j=1}^N e^{-x_j^2/2} \prod_{1 \leq j < k \leq N} |x_k - x_j|. \tag{1.19}
\]
This is realized by \( N \times N \) real symmetric matrices, with diagonal elements having the Gaussian distribution \( N[0, 1] \), and independent off diagonal elements having the distribution \( N[0, 1/\sqrt{2}] \). Tracy and Widom begin with the quaternion determinant expression \cite{16} for the \( k \)-point distribution function in the finite GOE (we also draw attention to the recent work \cite{2} in which Painlevé type recurrence equations are obtained for the analogue of the probabilities \( E_s^1 \) and \( E_1^h \) in the finite system). Instead, inspired by the observation of Baik and Rains \cite{3} that the square of the distribution of the largest eigenvalue in GOE is equal to the distribution of the largest eigenvalue in two independent, appropriately superimposed GOE’s, technically the ensemble
\[
\text{even}(\text{GOE}_N \cup \text{GOE}_N), \tag{1.20}
\]
we take as our starting point the \( k \)-point distribution of (1.20), scaled at the spectrum edge.

Now the \( k \)-point distribution of (1.20) is an ordinary determinant, whereas the \( k \)-point distribution of the GOE is a quaternion determinant. Furthermore the elements of the determinant contains terms familiar from the analysis of \( E^2 \) given in \cite{24}; this is also true of the quaternion determinant but the former involves only a subset of the latter. These facts together provide a simplified evaluation of \( E_s^1 \). The power of this derivation is demonstrated by its application to the evaluation of \( E_1^h \). We find that each step used in the derivation of \( E_s^1 \) has an analogous step in the case of \( E_1^h \) and this leads to (1.15).

We begin in Section 2 by providing the evaluation of the \( k \)-point distribution for the ensemble (1.20) in the scaled limit at the right hand soft edge, as well as that for the ensemble
\[
\text{odd}(\text{LOE}_N \cup \text{LOE}_N), \tag{1.21}
\]

in the scaled limit at the hard edge. In Section 3 we begin by using the evaluation of \( \rho(k) \) obtained in Section 2 as the starting point for the evaluation of \( E_1^h(0; (s, \infty)) \), and then proceed to mimic this calculation to evaluate \( E_1^h(0; (0, s); a) \). In Section 4 our evaluations (1.13) and (1.17) are related to previously known results.

### 2 The ensembles even/odd(\( \text{OE}_N(f) \cup \text{OE}_N(f) \)) for \( f \) classical

Let \( \text{OE}_N(f) \) denote the matrix ensemble with eigenvalue p.d.f.
\[
\prod_{j=1}^N f(x_j) \prod_{1 \leq j < k \leq N} |x_k - x_j|. \tag{2.1}
\]
We see from (1.4) and (1.19) that the LOE is of this form with
\[ f(x) = x^a e^{-x^2}, \quad (x > 0, \ a := (n - N - 1)/2) \] (2.2)
while the GOE is of this form with
\[ f(x) = e^{-x^2/2}. \]

In fact the four special choices of \( f \)
\[
f(x) = \begin{cases} 
  e^{-x^2/2}, & \text{Gaussian} \\
  x^{a-1/2}e^{-x^2/2} (x > 0), & \text{Laguerre} \\
  (1-x)^{(a-1)/2}(1+x)^{(b-1)/2} (-1 < x < 1), & \text{Jacobi} \\
  (1+x^2)^{-(a+1)/2}, & \text{Cauchy}
\end{cases}
\] (2.3)
(note that the exponent of \( x \) in the Laguerre case has been renormalized relative to (2.2)) have been shown in [13] to possess special properties in regards to the superimposed ensembles
\[
\text{even(OE}_N(f) \cup \text{OE}_N(f)) \quad \text{and} \quad \text{odd(OE}_N(f) \cup \text{OE}_N(f))
\] (2.4)
(amongst other superimposed ensembles). In particular the \( k \)-point distribution is given by a determinant formula with the same general structure in each case.

To present the formula for the \( k \)-point distributions, some additional theory from [13] must be recalled. In particular, it is found that each of the weight functions (2.3) is one member of a pair \((f, g)\) which naturally occur in the study of the superimposed ensembles. Explicitly the weight functions \( g \) are
\[
g = \begin{cases} 
  e^{-x^2}, & \text{Gaussian} \\
  x^a e^{-x} (x > 0), & \text{Laguerre} \\
  (1-x)^a(1+x)^b (-1 < x < 1), & \text{Jacobi} \\
  (1+x^2)^{-a}, & \text{Cauchy}
\end{cases}
\] (2.5)

Now, let \( \{p_n(x)\}_{n=0,1,...} \) denote the set of monic orthogonal polynomials associated with a particular weight function \( g \), and let \((p_n, p_n)\) denote the corresponding normalization. Then it is shown in [13] that for the ensemble even(OE\(_N(f) \cup \text{OE}_N(f))
\[
\rho(k)(x_1, \ldots, x_k) = \prod_{i=1}^{k} g(x_i) \det \left[ \sum_{l=0}^{N-2} \frac{p_l(x_i)p_l(x_j)}{(p_l, p_l)_2} + \frac{p_{N-1}(x_i)F_{N-1}(x_j)}{(p_{N-1}, F_{N-1})_2} \right]_{i,j=1,...,n},
\] (2.6)
where
\[
F_{N-1}(x) = \sum_{l=N-1}^{\infty} \frac{(p_l, I_-)_2}{(p_l, p_l)_2} p_l(x), \quad I_-(x) := \frac{f(x)}{g(x)} \int_{-\infty}^{x} f(t) \, dt.
\] (2.7)

Similarly for the ensemble odd(OE\(_N(f) \cup \text{OE}_N(f))\) the \( k \)-point distribution is again given by the formula (2.6) but with \( I_-\) in (2.7) replaced by
\[
I_+(x) := \frac{f(x)}{g(x)} \int_{x}^{\infty} f(t) \, dt.
\] (2.8)

The summation in (2.6) can be evaluated according to the Christoffel-Darboux formula, and the corresponding scaling limits are well known [1]. To compute the scaled limit of the quantity \( F_{N-1} \) in (2.6), we first make use of results from the work [1] to provide the explicit evaluation of the coefficients
\[
(p_l, I_-)_2 := \int_{-\infty}^{\infty} dx \, f(x)p_l(x) \int_{-\infty}^{x} dt \, f(t),
\]
applicable in all the classical cases (2.3). First we note
\[ \int_{-\infty}^{x} f(t) \, dt = \frac{1}{2} \int_{-\infty}^{\infty} \text{sgn}(x-t)f(t) \, dt + \frac{1}{2} \int_{-\infty}^{\infty} f(t) \, dt =: \tilde{\phi}_0(x) + \tilde{s}_0, \] (2.9)
which allows us to write
\[ (p_l, I_-)^2 = \int_{-\infty}^{\infty} f(x)p_l(x)\tilde{\phi}_0(x) \, dx + \tilde{s}_0 \int_{-\infty}^{\infty} f(x)p_l(x) \, dx. \] (2.10)
Now results in [1] give
\[ \tilde{\phi}_0(x) = \frac{1}{\gamma_0 f(x)} \sum_{\nu=0}^{\infty} \prod_{k=1}^{\nu} \left( \frac{\gamma_{2k-1}}{\gamma_{2k}} \right) \frac{P_{2\nu+1}(x)}{(2\nu+1, 2\nu+1)} \]
where
\[ \gamma_k(p_k, p_k)_2 = \begin{cases} 1, & \text{Gaussian} \\ \frac{1}{2}, & \text{Laguerre} \\ \frac{1}{2}(2k+2+a+b), & \text{Jacobi} \\ \alpha - 1 - k, & \text{Cauchy} \end{cases} \] (2.11)
This allows us to immediately evaluate the first term in (2.10).

It remains to compute the second term in (2.10). With the notation
\[ \tilde{s}_l := \frac{1}{2} \int_{-\infty}^{\infty} f(x)p_l(x) \, dx. \]
this term is given by \(2\tilde{s}_0\tilde{s}_l\). Consider first the case \(l\) even. With
\[ \tilde{\phi}_l(x) := \frac{1}{2} \int_{-\infty}^{\infty} \text{sgn}(x-t)p_l(t)f(t) \, dt \]
we know from [1] that
\[ \tilde{\phi}_{2k}(x) = \frac{1}{\gamma_{2k}} \prod_{l=1}^{k} \left( \frac{\gamma_{2l-1}}{\gamma_{2l}} \right) \frac{g(x)}{f(x)} \sum_{\nu=0}^{\infty} \prod_{k=1}^{\nu} \left( \frac{\gamma_{2l-1}}{\gamma_{2l}} \right) \frac{P_{2\nu+1}(x)}{(2\nu+1, 2\nu+1)} \]
Forming the ratio \(\tilde{\phi}_{2k}/\tilde{\phi}_{2k-2}(x)\) and taking the limit \(x \to \infty\) shows that \(\tilde{s}_{2k}/\tilde{s}_{2k-2} = \gamma_{2k-2}/\gamma_{2k-1}\) and thus we have the evaluation
\[ \tilde{s}_{2l} = \tilde{s}_0 \prod_{j=0}^{l-1} \frac{\gamma_{2j}}{\gamma_{2j+1}}. \]
To evaluate \(\tilde{s}_l\), \(l\) odd, we recall the formula [1]
\[ \tilde{\phi}_{2k+1}(x) - \frac{\gamma_{2k-1}}{\gamma_{2k}} \tilde{\phi}_{2k-1}(x) = -\frac{1}{\gamma_{2k}} \frac{g(x)}{f(x)} \frac{P_{2k}(x)}{(2k, 2k)}. \]
Taking the limit \(x \to \infty\) implies \(\tilde{s}_{2k+1} = (\gamma_{2k-1}/\gamma_{2k})\tilde{s}_{2k-1}\) and since \(\tilde{s}_{-1} := 0\) this gives
\[ \tilde{s}_{2l+1} = 0. \]
Thus the second term in (2.10) is fully determined.
Substituting the evaluation of \((p_l, I_-)_2\) obtained from the above working in (2.7) shows
\[
F_{N-1}(x) = \frac{1}{\gamma_0} \sum_{\nu=[(N-1)/2]}^\infty \frac{\prod_{l=1}^\nu (\gamma_{2\nu-1}/\gamma_{2\nu})}{(p_{2\nu+1}, p_{2\nu+1})^2} p_{2\nu+1}(x) + 2\tilde{s}_0^2 \sum_{l=[N/2]}^\infty \frac{\prod_{j=0}^{l-1} (\gamma_{2j}/\gamma_{2j+1})}{(p_{2l}, p_{2l})^2} p_{2l}(x). \tag{2.12}
\]
From this result we read off that
\[
(p_{N-1}, F_{N-1})_2 = \begin{cases} 
\frac{1}{\gamma_0} \prod_{l=1}^{(N-2)/2} (\gamma_{2l-1}/\gamma_{2l}) & N \text{ even} \\
2\tilde{s}_0^2 \prod_{j=1}^{(N-1)/2} (\gamma_{2j-2}/\gamma_{2j-1}) & N \text{ odd}
\end{cases} \tag{2.13}
\]
so all quantities in the expression (2.6) for \(\rho(k)\) are now known explicitly.

In the case of the ensemble odd\((\text{OE}_N(f) \cup \text{OE}_N(f))\) the definition (2.7) of \(F_{N-1}\) has \(I_-\) replaced by \(I_+\). Noting
\[
\int_{-\infty}^{\infty} f(t) \, dt = -\frac{1}{2} \int_{-\infty}^{\infty} \text{sgn}(x-t) f(t) \, dt + \frac{1}{2} \int_{-\infty}^{\infty} f(t) \, dt = -\tilde{\phi}_0(x) + \tilde{s}_0
\]
which differs from (2.9) only in the sign of the first term, we see by revising the working which led from (2.9) to (2.12) the only modification needed to the formula (2.12) is that a minus sign be placed in front of the first term (and similarly in (2.13)).

### 2.1 Guassian ensemble at the soft edge

In the Gaussian case
\[
f(x) = e^{-x^2/2}, \quad g(x) = e^{-x^2}, \quad p_l(x) = 2^{-l}H_l(x), \quad (p_l, p_l)_2 = \pi^{1/2}2^{-l}l!, \tag{2.14}
\]
where \(H_l(x)\) denotes the Hermite polynomial. The soft edge scaling is (2.15)
\[
x = (2N)^{1/2} + \frac{X}{2^{1/2}N^{1/6}} \tag{2.15}
\]
so we seek to compute
\[
\rho^{\text{GOE}}_{(k)}(X_1, \ldots, X_k) := \lim_{N \to \infty} \left( \frac{1}{2^{1/2}N^{1/6}} \right)^k \rho_{(k)}^E\left( (2N)^{1/2} + \frac{X_1}{2^{1/2}N^{1/6}}, \ldots, (2N)^{1/2} + \frac{X_k}{2^{1/2}N^{1/6}} \right)
\]
where \(E\) denotes the ensemble (1.20) and the r.h.s. is given by (2.6) with the substitutions (2.14).

Regarding the summation in (2.6), we know from the study of the GUE at the soft edge that (2.16)
\[
\lim_{N \to \infty} \frac{1}{2^{1/2}N^{1/6}} \sum_{l=0}^{N-2} \frac{p_l(x)p_l(y)}{(p_l, p_l)_2} \bigg|_{x=(2N)^{1/2}+X/2^{1/2}N^{1/6}}^{y=(2N)^{1/2}+Y/2^{1/2}N^{1/6}} = K^s(X, Y),
\]
where, with \(\text{Ai}(x)\) denoting the Airy function
\[
K^s(x, y) := \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x-y}. \tag{2.16}
\]
This is obtained using the Christoffel-Darboux summation formula and the asymptotic expansion (2.17)
\[
e^{-x^2/2}H_n(x) = \pi^{-3/4}2^n/2^{1/4}(n!)^{1/2}n^{-1/12}\left(\pi\text{Ai}(-u) + O(n^{-2/3})\right) \tag{2.17}
\]
where \(x = (2n)^{1/2} - u/2^{1/2}n^{1/6}.\)
It remains to compute the scaled limit of the term involving $F_{N-1}$ in (2.6). Now substituting the values of $\gamma_k$ and $(p_k, p_k)$ from (2.11) and (2.14), and noting $s_0^2 = \pi/2$, (2.12) reads

$$F_{N-1}(x) = \pi^{1/2} \sum_{\nu = [(N-1)/2]}^{\infty} \frac{\nu!}{(2\nu + 1)!} H_{2\nu+1}(x) + \pi \sum_{l=[N/2]}^{\infty} \frac{1}{2^{2l}l!} H_{2l}(x). \tag{2.18}$$

Next we want to combine this result with (2.13). For definiteness take $N$ to be even. Then we see that

$$\left(g(x)g(y)\right)^{1/2} \frac{p_{N-1}(x)F_{N-1}(y)}{(p_{N-1}, F_{N-1})} = e^{-x^2/2} 2^{-(N-1)} H_{N-1}(x) \left(A_1(y) + A_2(y)\right) \tag{2.19}$$

with

$$A_1(y) := \frac{e^{-y^2/2}}{(N/2 - 1)!} \sum_{\nu = 0}^{\infty} \frac{(N/2 - 1 + \nu)!}{(N - 1 + 2\nu)!} H_{N-1+2\nu}(y),$$

$$A_2(y) := \frac{\pi^{1/2} e^{-y^2/2}}{(N/2 - 1)!} \sum_{l=0}^{\infty} \frac{1}{2^{N+2l}(N/2 + l)!} H_{N+2l}(y). \tag{2.20}$$

We remark that the summation defining $A_1(y)$ (with the lower terminal $\nu = 0$ replaced by $\nu = 1$) occurs in the study of the soft edge distribution at $\beta = 1$ [12], and furthermore a procedure has been given to compute its asymptotic behaviour with the scaling (2.15), the key ingredient of which is the asymptotic expansion (2.17).

For the $x$-dependent terms in (2.19), (2.17) gives

$$e^{-x^2/2} 2^{-(N-1)} H_{N-1}(x) \bigg|_{x = (2N)^{1/2} + X/2^{1/2} N^{1/6}} \sim \pi^{1/4} 2^{-(N-1)/2 + 1/4} (N-1)!/2 N^{-1/12} Ai(X).$$

For $A_1(y)$, first note that for large $N$

$$\frac{(N/2 - 1 + \nu)!}{(N - 1 + 2\nu)!} \frac{(N - 1)!}{(N/2 - 1)!} \sim 2^{-\nu}.$$  

Then use of (2.17) with $n = N + 2\nu - 1$, $-u \sim Y - 2\nu N^{1/3}$ shows that the sum becomes the Riemann approximation to a definite integral, and thus

$$(N - 1)!/2 A_1((2N)^{1/2} + Y/2^{1/2} N^{1/6}) \sim 2^{(N-1)/2} 1/2 4^{-1/4} N^{-1/12} N^{1/3/2} \int_{0}^{\infty} Ai(Y - v) dv.$$  

Hence

$$\lim_{N \to \infty} \frac{1}{2^{1/2} N^{1/6}} e^{-x^2/2} 2^{-(N-1)} H_{N-1}(x) A_1(y) \bigg|_{y = (2N)^{1/2} + X/2^{1/2} N^{1/6}} = \frac{1}{2} Ai(X) \int_{0}^{\infty} Ai(Y - v) dv. \tag{2.21}$$

Similarly, for $A_2(y)$, noting that for large $N$

$$\frac{(N + 2l)!}{2^{N+2l}(N/2 + l)!} \frac{(N - 1)!}{(N/2 - 1)!} \sim \frac{1}{(2\pi)^{1/2}} 2^{-l},$$

and using (2.17) with $n = N + 2\nu$, $-u \sim Y - 2\nu N^{1/3}$ we find

$$(N - 1)!/2 A_2((2N)^{1/2} + Y/2^{1/2} N^{1/6}) \sim \frac{1}{\pi^{1/2}} A_1((2N)^{1/2} + Y/2^{1/2} N^{1/6})$$.
and hence
\[
\lim_{N \to \infty} \frac{1}{21/2^N N^{1/6}} e^{-x^2/2} 2^{-(N-1)} H_{N-1}(x) A_2(y) \bigg|_{x=(2N)^{1/2+X/2^{1/2}} N^{1/6}} = \frac{1}{2} \text{Ai}(X) \int_0^\infty \text{Ai}(Y-v) \, dv.
\]
(2.22)

The contributions (2.21) and (2.22) thus reinforce, so after adding to (2.16) we obtain
\[
\rho^{(\text{GOE})^2 h}(X_1, \ldots, X_k) = \det \left[ \left( K^h(X_j, X_l) + \text{Ai}(X_j) \int_0^\infty \text{Ai}(X_l - v) \, dv \right) \right]_{j, l = 1, \ldots, k}.
\]
(2.23)

### 2.2 Laguerre ensemble at hard edge

In the Laguerre case
\[
f(x) = x^{(a-1)/2} e^{-x/2}, \quad g(x) = x^a e^{-x}, \quad (x > 0)
\]
\[
p_l(x) = (-1)^l l! L_l^a(x), \quad (p_l, p_{l+1}) = \Gamma(l+1) \Gamma(a+l+1),
\]
\[
L_n^a = \text{the Laguerre polynomial}. \text{At the hard edge the appropriate scaling is specified by (1.6), so the task is to compute}
\]
\[
\rho^{(\text{LOE})^2 h}(X_1, \ldots, X_k; a) := \lim_{N \to \infty} \left( \frac{1}{4N} \right)^k \rho^{E}(X_1, \ldots, X_k)
\]
(2.25)

where E denotes the ensemble (1.21) and \(\rho(k)\) on the r.h.s. is specified by (2.4).

The scaled limit of the summation in (2.6) at the hard edge occurs in the study of the LUE and is known \([7]\). Thus making use of the Christoffel-Darboux formula and the large \(n\) asymptotic formula\([20]\)
\[
e^{-y/2} y^{a/2} L_n^a(y) \sim n^{a/2} J_a(2\sqrt{ny}).
\]
(2.26)

one finds
\[
\lim_{N \to \infty} \frac{1}{4N} \sum_{l=0}^{N-2} p_l(x) p_l(y) \bigg|_{x = X/4N, y = Y/4N} = K^h(X, Y)
\]
where \(K^h\) is specified by (1.7).

The first step in computing the term involving \(F_{N-1}\) in (2.4) is to substitute the formulas (2.24) in (2.12) modified so that there is a minus sign before the first term (recall the paragraph below (2.13)). This gives
\[
F_{N-1}(x) = \frac{2a!}{(a/2)!} \sum_{\nu=([N-1]/2)}^{\infty} \frac{2^{2\nu}(a/2 + \nu)\nu!}{(a + 2\nu + 1)!} L_{2\nu+1}^a(x)
\]
\[+ 2^a \frac{((a - 1)/2)^2(a/2)!}{a!} \sum_{l=[N/2]}^{\infty} \frac{(2l)!}{2^{2l} l!(a/2 + l)!} L_{2l}^a(x).
\]

We see from this formula and (2.13) (for definiteness in the latter we take \(N\) to be even; recall that in this case a minus sign must be inserted) that
\[
\left( g(x) g(y) \right)^{1/2} \frac{p_{N-1}(x) F_{N-1}(y)}{p_{N-1}, F_{N-1}} = (g(x))^{1/2} L_{N-1}^a(x) \left( \frac{(N-1)!}{2^{N-2}((N-2)/2)!(a/2 + (N-2)/2)!} \right)
\]
\[\times \left( \sum_{\nu=([N-2]/2)}^{\infty} \frac{2^{2\nu}(a/2 + \nu)\nu!}{(a + 2\nu + 1)!} (g(y))^{1/2} L_{2\nu+1}^a(y)
\]
\[+ 2^{a-1} \frac{((a - 1)/2)^2(a/2)!}{a!} \sum_{l=\lceil N/2 \rceil}^{\infty} \frac{(2l)!}{2^{2l} l!(a/2 + l)!} (g(y))^{1/2} L_{2l}^a(y) \right)
\]
\[:= (g(x))^{1/2} L_{N-1}^a(x) \frac{(N-1)!}{2^{N-2}((N-2)/2)!(a/2 + (N-2)/2)!} \left( B_1(y) + B_2(y) \right),
\]
(2.28)
where $B_1$ and $B_2$ denote the two terms in the final brackets of the line before.

Consider the function $B_1(y)$. From Stirling’s formula we have
\[
\frac{2^{2\nu} (a/2 + \nu)! \nu!}{(a + 2\nu + 1)!} \sim \pi^{1/2} 2^{-(a+1)\nu-a/2-1/2}.
\]

Using this and (2.26) we find that the sum defining $B_1(y)$ is the Riemann approximation to a definite integral and we find
\[
B_1 \left( \frac{Y}{4N} \right) \sim \frac{\pi^{1/2}}{2^{n/2+1}} \left( \frac{N}{2} \right)^{1/2} \int_1^\infty t^{-1/2} J_a(\sqrt{t}) \, dt = \frac{\pi^{1/2}}{2^{n/2}} \left( \frac{N}{2} \right)^{1/2} \int_1^\infty J_a(t) \, dt.
\]

Another application of Stirling’s formula shows
\[
\frac{(N-1)!}{2^{N-2}((N-2)/2)! (a/2 + (N-2)/2)!} \sim 2^{(a+1)/2} \pi^{-1/2} N^{1-a/2}
\]

so we see from further use of (2.26) together with (2.29) that
\[
\lim_{N \to \infty} (g(x))^{1/2} \frac{1}{4N} L_N^a - 1(x) 2^{-2} ((N-2)/2)! (a/2 + (N-2)/2)! B_1(y) \bigg|_{x = X/4N, y = Y/4N} = \frac{J_a(\sqrt{X})}{4\sqrt{X}} \int_{Y^{1/2}}^\infty J_a(t) \, dt.
\]

A similar analysis applied to $B_2(y)$ shows
\[
\lim_{N \to \infty} \frac{1}{4N} (g(x))^{1/2} L_N^a - 1(x) 2^{-2} ((N-2)/2)! (a/2 + (N-2)/2)! B_2(y) \bigg|_{x = X/4N, y = Y/4N} = \frac{J_a(\sqrt{X})}{4\sqrt{X}} \int_{Y^{1/2}}^\infty J_a(t) \, dt,
\]

which thus reinforces (2.30). Thus adding twice (2.30) to (2.27) we have
\[
\rho_{(k)}^{(LOE)q_h}(X_1, \ldots, X_k; (a - 1/2)) = \det \left[ (K^h(X_j, X_l) + \frac{J_a(\sqrt{X_j})}{2\sqrt{X_l}} \int_{\sqrt{X_l}}^\infty J_a(t) \, dt) \right]_{j,l=1,\ldots,k}
\]

### 3 Gap probabilities at the spectrum edge

#### 3.1 The probability $E_1^s(0; (s, \infty))$

It was remarked in the Introduction that the probability $E_1^s(0; (s, \infty))$ has been computed in terms of a Painlevé II transcendent by Tracy and Widom [23]. Explicitly let $q_s$ denote the solution of the particular Painlevé II differential equation
\[
(q_s)^{''} = sq_s + 2(q_s)^3
\]

subject to the boundary condition $q_s(s) \sim \text{Ai}(s)$ as $s \to \infty$. Then it is shown in [23] that
\[
\left( E_1^s(0; (s, \infty)) \right)^2 = E_2^s(0; (s, \infty)) \exp \left( - \int_s^\infty q_s(t) \, dt \right)
\]

where
\[
E_2^s(0; (s, \infty)) = \exp \left( - \int_s^\infty (t-s)q_s^2(t) \, dt \right).
\]
Here we will use the evaluation of the $k$-point distribution (2.23) to provide a simplified derivation of (3.2) while still following the essential strategy of [23].

By definition of (1.20) it follows that

$$E_{1}^{{\text{GOE}}_{N}}(0; (s, \infty))^{2} = E_{\text{even}(\text{GOE}_{N} \cup \text{GOE}_{N})}(0; (s, \infty)),$$

which with the scaling (2.15) implies

$$\left( E_{1}^{s}(0; (s, \infty)) \right)^{2} = E_{\text{even}(\text{GOE})}^{2}(0; (s, \infty)).$$

(3.4)

Now, recalling the determinant formula (2.23), we see from (1.12) and the text immediately below that $E_{\text{even}(\text{GOE})}^{2}(0; (s, \infty))$ can be written as the determinant of a Fredholm integral operator. Thus

$$\left( E_{1}^{s}(0; (s, \infty)) \right)^{2} = \det \left( 1 - (K^{s} + A \otimes B) \right)$$

(3.5)

where $K^{s}$ is the integral operator on $(s, \infty)$ with kernel (2.16) while $A$ is the operator which multiplies by $\text{Ai}(x)$, while $B$ is the integral operator with kernel $\int_{0}^{\infty} \text{Ai}(y - v) \, dv$.

Removing $(1 - K^{s})$ as a factor from (3.5) and recalling [21]

$$\det(1 - K^{s}) = E_{1}^{s}(0; (s, \infty))$$

we obtain

$$\left( E_{1}^{s}(0; (s, \infty)) \right)^{2} = E_{2}^{s}(0; (s, \infty)) \det \left( 1 - (K^{s})^{-1}A \otimes B \right)$$

$$= E_{2}^{s}(0; (s, \infty)) \left( 1 - \int_{s}^{\infty} (1 - K^{s})^{-1}A[y]B(y) \, dy \right)$$

(3.6)

where the second equality follows from the fact that $(1 - K^{s})^{-1}A[y]$ is the eigenfunction of the operator $(1 - K^{s})^{-1}A \otimes B$, so the eigenvalue is

$$\int_{s}^{\infty} (1 - K^{s})^{-1}A[y]B(y) \, dy$$

Analogous to the notation of [21] we put

$$\phi^{s}(x) := A(x) = \text{Ai}(x), \quad Q^{s}(x) := (1 - K^{s})^{-1}A[x]$$

so that

$$\int_{s}^{\infty} (1 - K^{s})^{-1}A[y]B(y) \, dy = \int_{s}^{\infty} dy Q^{s}(y) \int_{-\infty}^{y} \phi^{s}(v) \, dv =: u_{s}^{\epsilon}$$

(3.7)

(the notation $u_{s}^{\epsilon}$ – without the superscript $s$ – is used for an analogous quantity in [23]). Note from (3.6) that with the notation (3.7) we have

$$\left( E_{1}^{s}(0; (s, \infty)) \right)^{2} = E_{2}^{s}(0; (s, \infty))(1 - u_{s}^{\epsilon}).$$

(3.8)

Following [23], our objective is to derive coupled differential equations for $u_{s}^{\epsilon}$ and the quantity

$$q_{s}^{\epsilon} := \int_{s}^{\infty} dy \rho^{s}(s, y) \int_{-\infty}^{y} \phi^{s}(v) \, dv$$

(3.9)

where $\rho^{s}(x, y)$ is the kernel of the operator $(1 - K^{s})^{-1}$. These equations will involve

$$Q^{s}(s) := q_{s} = \int_{s}^{\infty} dy \rho^{s}(s, y) \phi^{s}(y),$$

(3.10)
which in [21] is shown to be the Painlevé II transcendent specified by the solution of (3.1), and their derivation relies on the formula [21]

$$\frac{\partial}{\partial s} Q^s(y) = -q_s \left( \delta^+(y - s) + \rho^s(s, y) \right),$$

(3.11)

where $\delta^+(y - s)$ is such that

$$\int_s^\infty \delta^+(y - s) f(y) \, dy = f(s),$$

as well as the formula [21]

$$\left( \frac{\partial}{\partial s} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \rho^s(x, y) = -Q^s(x)Q^s(y).$$

(3.12)

Now, differentiating (3.7) with respect to $s$ we have

$$(u^{s}_s)' = -q_s \int_{-\infty}^{s} \phi^s(v) \, dv + \int_{s}^{\infty} dy \left( \frac{\partial}{\partial s} Q^s(y) \right) \int_{-\infty}^{y} \phi^s(v) \, dv = -q_s u^{s}_s$$

(3.13)

where to obtain the second equality use has been made of (3.11) and the definition (3.9). We now seek a formula for $(q^{s}_s)'$. Making use of (3.12) in (3.9) shows

$$(q^{s}_s)' = -\int_{s}^{\infty} dy \frac{\partial}{\partial y} \rho^s(s, y) \int_{-\infty}^{y} \phi^s(v) \, dv - q_s \int_{s}^{\infty} dy Q^s(y) \int_{-\infty}^{y} \phi^s(v) \, dv$$

$$= \int_{s}^{\infty} \rho^s(s, y) \phi^s(y) \, dy - q_s u^{s}_s$$

$$= q_s (1 - u^{s}_s)$$

(3.14)

where the final equality follows from the definitions (3.7) and (3.10).

As $q_s$ is known, the system of equations (3.13) and (3.14) fully determines $u^{s}_s$ and $q^{s}_s$ once boundary conditions are specified. Now $Q^s(y)$ is smooth, so we see from (3.7) that

$$u^{s}_s \sim 0$$

(3.15)

as $s \to \infty$. On the other hand $\rho(s, y) = \delta^+(s - y) + R(s, y)$ where $R(s, y)$ is smooth, so for $s \to \infty$

$$q^{s}_s \sim \int_{-\infty}^{\infty} \phi^s(v) \, dv = \int_{-\infty}^{\infty} \text{Ai}(v) \, dv = 1.$$  

(3.16)

The unique solution of the coupled equations (3.13) and (3.14) satisfying (3.13) and (3.16) is easily shown to be

$$u^{s}_s = 1 - e^{-\mu_s}, \quad q^{s}_s = e^{-\mu_s}$$

(3.17)

where

$$\mu_s := \int_{s}^{\infty} q_s(x) \, dx.$$  

Substituting the evaluation of $u^{s}_s$ from (3.17) in (3.8) reclaims (3.2), as desired.

3.2 The probability $E^h_1(0; (0, s); a)$

All the steps leading to the rederivation of (3.2) given in the previous section have analogues for the probability $E^h_1(0; (0, s); a)$ which lead to the evaluation (3.13).

First, the analogue of (3.4) is

$$\left(E^h_1(0; (0, s); a)\right)^2 = E^{(LOE)^2}_h(0; (0, s); a),$$
while use of the determinant formula (2.32) in (1.12) then gives

\[
\left( E_1^h(0; (0, s); (a - 1)/2) \right)^2 = \det \left( 1 - (K^h + C \otimes D) \right) \cdot \quad (3.18)
\]

Here \(K^h\) is the integral operator on \((0, s)\) with kernel (1.7), while \(C\) is the operator which multiplies by \(J_a(\sqrt{y})\), while \(D\) is the integral operator with kernel

\[
\frac{1}{2\sqrt{y}} \int_\sqrt{y}^\infty J_a(t) \, dt. \quad (3.19)
\]

Recalling [22] \(\det(1 - K^h) = E_2^h(0; (0, s); a)\)

we see that analogous to (3.6), (3.18) can be rewritten

\[
\left( E_1^h(0; (0, s); (a - 1)/2) \right)^2 = E_2^h(0; (0, s); a) \left( 1 - \int_0^s (1 - K^h)^{-1}C[y]D(y) \, dy \right). \quad (3.20)
\]

Analogous to the notation of [22] we put

\[
\phi^h(x) := C(x) = J_a(\sqrt{x}), \quad Q^h(y) := (1 - K^h)^{-1}C[y].
\]

After changing variables \(t = \sqrt{u}\) in (3.19) we see that in terms of this notation

\[
\int_0^s (1 - K^h)^{-1}C[y]D(y) \, dy = \frac{1}{4} \int_0^s dy Q^h(y) \frac{1}{\sqrt{y}} \int_y^\infty du \frac{1}{\sqrt{u}} \phi^h(u) := u_c^h, \quad (3.21)
\]

and in turn this latter notation used in (3.20) gives

\[
\left( E_1^h(0; (0, s); (a - 1)/2) \right)^2 = E_1^h(0; (0, s); a)(1 - u_c^h). \quad (3.22)
\]

Now, with

\[
Q^h(s) := q_h = \int_0^s dy \rho^h(s, y) \phi^h(y),
\]

which in [22] is shown to be the Painlevé V transcendent specified by the nonlinear equation (1.13), analogous to (3.11) we have [22]

\[
\frac{\partial}{\partial s}Q^h(y) = q_h \left( \delta^+(y - s) + \rho^h(s, y) \right).
\]

Use of this formula in (3.21) then shows

\[
\left( u_c^h \right)' = \frac{1}{4} q_h q_c^h, \quad q_c^h := \int_0^s dy \rho^h(s, y) \frac{1}{\sqrt{y}} \int_y^\infty du \frac{1}{\sqrt{u}} \phi(u) \quad (3.23)
\]

which is the analogue of (3.13).

Next we seek a formula for the derivative with respect to \(s\) of \(q_c^h\). For this purpose we note from [22] that

\[
x \frac{\partial}{\partial x} \rho^h(x, y) + s \frac{\partial}{\partial s} \rho^h(x, y) = - \frac{\partial}{\partial y} \left( y \rho^h(x, y) \right) + \frac{1}{4} Q^h(x)Q^h(y)
\]
This formula applied to (3.23) shows
\[ s(q^h)' = -\int_0^s dy \left( \frac{d}{dy}(y\rho^h(s,y)) \right) \frac{1}{\sqrt{y}} \int_y^\infty du \frac{1}{\sqrt{u}} \phi^h(u) + q^h u^h \]
\[ = -\frac{1}{2} \int_0^s dy \rho^h(s,y) \frac{1}{\sqrt{y}} \int_y^\infty du \frac{1}{\sqrt{u}} \phi^h(u) - \int_0^s dy \rho^h(s,y) \phi^h(y) + q^h u^h \]
\[ = -\frac{1}{2} q^h - q^h (1 - u^h) \] (3.24)
The coupled equations (3.23) and (3.24) must be solved subject to the 
boundary conditions
\[ u^h \sim 0, \quad \sqrt{s} q^h \sim \int_0^\infty \frac{1}{\sqrt{u}} \phi(u) du = 2 \int_0^\infty J_0(v) dv = 2. \] (3.25)
The occurrence of \( \sqrt{s} q^h \) in (3.25) suggests we introduce
\[ \tilde{q}^h := \sqrt{s} q^h \]
in (3.23) and (3.24). Doing this gives the system of equations
\[ \sqrt{s}(u^h)' = \frac{1}{4} \tilde{q}^h, \quad \sqrt{s}(\tilde{q}^h)' = -q^h (1 - u^h). \] (3.26)
Introducing the new independent variable
\[ \mu^h := \int_0^s \frac{1}{x^{1/2}} \tilde{q}_h(x) dx \]
we see that (3.26) reduces to the system with constant coefficients
\[ \frac{d}{d\mu} u^h = \frac{1}{4} \tilde{q}^h, \quad \frac{d}{d\mu} \tilde{q}^h = -(1 - u^h). \] (3.27)
The solution satisfying (3.25) is
\[ u^h = 1 - e^{\frac{1}{2} \mu^h}, \quad \tilde{q}^h = 2e^{-\frac{1}{2} \mu^h}. \] (3.28)
The stated formula (1.15) for \( E_1^h(0; (0, s); (a - 1)/2) \) now follows by substituting this evaluation of \( u^h \) in (3.22).

4 Discussion

4.1 Special values of \( a \)

Edelman [7] was the first person to obtain the exact evaluation of \( E_1^h(0; (0, s); a) \), albeit for two special values of \( a \) only, namely \( a = -1/2 \) and \( a = 0 \). In terms of the scaling (1.6) the results of [7] are
\[ E_1^h(0; (0, s); -\frac{1}{2}) = e^{-(s/8+\sqrt{s}/2)} \] (4.1)
\[ E_1^h(0; (0, s); 0) = e^{-s/8}. \] (4.2)
Subsequently it was shown by the present author \cite{8} that $E^h_1(0; (0, s); a)$ for $a \in \mathbb{Z}_{\geq 0}$ can be expressed as a $2a$-dimensional integral. Explicitly

$$E^h_1(0; (0, s); a) = C e^{-s/8} \left( \frac{1}{2\pi s^{1/2}} \right)^{2a} \int_{[-\pi, \pi]^{2a}} \prod_{j=1}^{2a} e^{s^{1/2} \cos \theta_j} e^{i\theta_j} \prod_{1 \leq j < k \leq 2a} |e^{i\theta_k} - e^{i\theta_j}|^4 d\theta_1 \cdots d\theta_{2a}$$

(4.3)

where

$$C = \prod_{j=1}^{2a} \frac{\Gamma(3/2)\Gamma(3/2 + j/2)}{\Gamma(1 + j/2)}.$$  

We remark that well known integration procedures (see e.g. \cite{24}) allow this integral to be expressed as a Pfaffian. Such Pfaffian formulas, deduced in a different way, have been given in \cite{18}.

The formula (1.15) relates $E^h_1(0; (0, s); (a - 1)/2)$ to $E^h_2(0; (0, s); a)$, so it is appropriate to consider the evaluation of the latter for special values of $a$. Analogous to (4.3) we have that for $a \in \mathbb{Z}_{\geq 0}$ \cite{8}

$$E^h_2(0; (0, s); a) = e^{-s/4} \left( \frac{1}{2\pi} \right)^a \frac{1}{a!} \int_{[-\pi, \pi]^a} \prod_{j=1}^{a} e^{s^{1/2} \cos \theta_j} \prod_{1 \leq j < k \leq a} |e^{i\theta_k} - e^{i\theta_j}|^2 d\theta_1 \cdots d\theta_a$$

(4.4)

This integral can easily be written as a Toeplitz determinant, with the result \cite{11}

$$E^h_2(0; (0, s); a) = e^{-s/4} \det \left[ I_{j-k}(s^{1/2}) \right]_{j,k=1,\ldots,a}$$

(4.5)

where $I_n(x)$ denotes the Bessel function of purely imaginary argument. As an aside, it is interesting to note that the integral (4.4) is the generating function for the enumeration of various combinatorial objects, including quantities related to random permutations \cite{19}, random words \cite{25} and random walks \cite{10}.

The formula (1.14) gives

$$-4 \frac{d}{ds} \left( \frac{s}{d} \log E^h_2(0; (0, s); a) \right) = \left( q_h(s) \right)^2,$$

(4.6)

so (4.5) implies that for $a \in \mathbb{Z}_{\geq 0}$, $q_h^2$ can be expressed in terms of Bessel functions. The simplest case is $a = 0$, when we have

$$E^h_2(0; (0, s); a) = e^{-s/4}.$$  

(4.7)

Substituting in (4.6) gives \cite{22}

$$q_h(s) = 1$$

(4.8)

and substituting this in (1.15) we reclaim (4.1). The next simplest case is $a = 1$ when we have

$$E^h_2(0; (0, s); 1) = e^{-s/4} I_0(s^{1/2})$$

(4.9)

and so

$$\left( q_h(s) \right)^2 = 1 - 4 \frac{d}{ds} \left( \frac{s}{d} \log I_0(s^{1/2}) \right).$$

For this to be consistent with (1.15) we see that the identity

$$I_0(s^{1/2}) = \exp \left( \frac{1}{2} \int_0^s \frac{1}{\sqrt{t}} \left( 1 - 4 \frac{d}{dt} \left( \frac{t}{d} \log I_0(t^{1/2}) \right) \right)^{1/2} dt \right)$$
must hold. This in turn is equivalent to the statement that
\( y := \log I_0(s^{1/2}) \) satisfies the nonlinear equation
\[
4sy'' + 4s(y')^2 + 4y' - 1 = 0,
\]
a fact which is readily verified using Bessel function identities.

Special evaluations are also known for \( E^h_{2\beta}(0; (0, s); a) \) in the case \( a \in \mathbb{Z}_{\geq 0} \) [8]. These evaluations are in terms of a certain generalized hypergeometric function based on Jack polynomials, while in the case \( a \) even Pfaffian formulas are also known [18]. In quoting from the results, one must be aware that \( E^h_{2\beta} \) is defined starting with the ensemble \( \text{LSE}_{N/2} \), and scaling according to (1.6). This means that the results of [8] require some rescaling of \( s \). Doing this, we note from [8] that the simplest cases are \( a = 0 \) and \( a = 1 \), when we have
\[
E^h_2(0; (0, s); 0) = e^{-s/8}, \quad (4.10)
\]
\[
E^h_2(0; (0, s); 1) = e^{-s/8} F_1\left(\frac{1}{2}; -\frac{s}{16}\right) = e^{-s/8} \cosh \frac{\sqrt{s}}{2}. \quad (4.11)
\]

The result (4.10) can only be related to (1.17) in the limit \( a \to -1^- \), since for \( a \leq -1 \) \( E^h_2(0; (0, s); a) = 0 \). However, as the limiting forms of the quantities on the r.h.s. are not known we cannot readily check the consistency with (4.10). In contrast the consistency between (1.17) and (4.11) is immediate upon recalling (4.7) and (4.8).

### 4.2 Connection between \( E^h_{2\beta} \) and \( E^s_{\beta} \)

In previous articles [9, 12] it has been noted that for \( a \to \infty \), after appropriate rescaling of the coordinates, the scaled \( k \)-point distribution function for the infinite Laguerre ensemble at the hard edge becomes equal to the scaled \( k \)-point distribution function for the infinite Gaussian ensemble at the soft edge. Note that in the symplectic case the scalings are done starting with the ensembles \( \text{GSE}_{N/2} \) and \( \text{LSE}_{N/2} \). Let
\[
a(\beta) = \begin{cases} 
(a - 1)/2, & \beta = 1 \\
a, & \beta = 2 \\
a + 1, & \beta = 4
\end{cases}
\]
Explicitly, it was checked that the scaled \( k \)-point distribution function for the infinite Laguerre ensemble at the hard edge, with parameter \( a \mapsto a(\beta) \) and after the rescaling of coordinates
\[
x \mapsto a^2 - 2a(a/2)^{1/3}x,
\]
equals the soft edge distribution functions for the corresponding Gaussian ensemble results in the \( a \to \infty \) limit. We must therefore have
\[
\lim_{a \to \infty} E^h_{\beta}(0; (0, a^2 - 2a(a/2)^{1/3} s); a(\beta)) = E^s_{\beta}(0; (0, \infty)). \quad (4.12)
\]
To verify (4.12), we first recall some additional results from [22]. Write
\[
\frac{1}{2} q_h = \left((sR_h)'\right)^{1/2}, \quad (4.13)
\]
so that, after integrating by parts, (1.14) reads
\[
E^h_2(0; (0, s); a) = \exp \left(- \int_0^s R_h(t) \, dt \right). \quad (4.14)
\]
Then \( \sigma := sR_h(s) \) is shown to satisfy the particular Painlevé III equation in \( \sigma \) form (for an account of the latter see \([15, 5]\))

\[
(s\sigma'')^2 + \sigma'(\sigma - s\sigma')(4\sigma' - 1) - a^2(\sigma')^2 = 0. \tag{4.15}
\]

Similarly let \( q_\sigma(s) = (-R'_\sigma(s))^{1/2} \) so that \((3.3)\) reads

\[
E^s_2(0; (s, \infty)) = \exp \left( - \int_{\infty}^{s} R_\sigma(t) \right). \tag{4.17}
\]

It is shown in \([21]\) that \( R_\sigma \) satisfies the particular Painlevé II equation in \( \sigma \) form

\[
(R''_\sigma)^2 + 4R'_\sigma((R'_\sigma)^2 - sR'_\sigma + \sigma) = 0. \tag{4.18}
\]

Substituting \((4.14)\) and \((4.17)\) in \((4.12)\) with \( \beta = 2 \) we deduce that the validity of the latter is equivalent to the statement

\[
2a(a/2)^{1/3}R_h(a^2 - 2a(a/2)^{1/3}t) \sim a \rightarrow \infty R_\sigma(t). \tag{4.19}
\]

This can be verified by introducing the function

\[
\bar{\sigma}(s) = \frac{2a(a/2)^{1/3}}{a^2} \sigma \left( a^2 - 2a(a/2)^{1/3}s \right) \sim 2a(a/2)^{1/3}R_h \left( a^2 - 2a(a/2)^{1/3}s \right)
\]

into \((4.15)\) and taking the limit \( a \rightarrow \infty \). One finds the differential equation \((4.18)\) results with \( R_\sigma = \bar{\sigma}(s) \).

In terms of \((4.13)\), the evaluations \((4.14)\) and \((4.17)\) read

\[
\begin{align*}
\left( E^h_1(0; (0, s); (a - 1)/2) \right)^2 &= E^h_2(0; (0, s); a) \exp \left( - \int_{0}^{s} \frac{((tR_h(t)))^{1/2}}{\sqrt{t}} dt \right) \\
\left( E^h_4(0; (0, s); a + 1) \right)^2 &= E^h_2(0; (0, s); a) \cosh^2 \left( \frac{1}{2} \int_{0}^{s} \frac{(tR_h(t)))^{1/2}}{\sqrt{t}} dt \right).
\end{align*}
\]

Making use of \((4.12)\) in the case \( \beta = 2 \) (which has just been verified), and \((4.19)\) together with \((4.16)\), we see that

\[
\begin{align*}
\lim_{a \rightarrow \infty} \left( E^h_1(0; (0, a^2 - 2a(a/2)^{1/3}s); (a - 1)/2) \right)^2 &= E^s_2(0; (s, \infty)) \exp \left( - \int_{s}^{\infty} q_\sigma(t) dt \right) \\
\lim_{a \rightarrow \infty} \left( E^h_4(0; (0, a^2 - 2a(a/2)^{1/3}s); a + 1) \right)^2 &= E^s_2(0; (s, \infty)) \cosh^2 \left( \frac{1}{2} \int_{s}^{\infty} q_\sigma(t) dt \right).
\end{align*}
\]

We recognize the right hand sides in these expressions as \( E^s_\beta(0; (s, \infty)) \) for \( \beta = 1 \) and 4 respectively \([23]\) (recall eq. \((3.2)\); as noted in \([13]\) \( E^s_1(0; (s, \infty)) \) can be deduced from \( E^s_1(0; (s, \infty)) \) and \( E^s_2(0; (s, \infty)) \) because of the validity of an inter-relationship analogous to \((1.16)\)).

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