The classical prime number theorem asserts that
\[ \sum_{n \leq x} \Lambda(n) \sim x, \]
where \( \Lambda(n) \) is the von Mangoldt function. Depending on the quality of the error term, it is possible to deduce from this a prime number theorem for short intervals, in the form
\[ \sum_{x < n \leq x + h} \Lambda(n) \sim h, \tag{1.1} \]
provided that \( h \) is not too small; with the presently best known error terms, we may take \( h \) a bit smaller than \( x \) divided by any power of \( \log x \), but not as small as \( x^{1-\delta} \) for any \( \delta > 0 \). Improving the error bound in the prime number theorem to allow for \( h \) to be of size \( x^{1-\delta} \) is a monumentally hard task, known as the quasi-Riemann hypothesis, and amounts to showing that there are no zeros of the Riemann zeta function \( \zeta(s) \) in the region \( \Re(s) > 1-\delta \).

Nevertheless, in 1930, Hoheisel [17] made the remarkable observation that, with Littlewood’s improved zero-free region for \( \zeta(s) \), if there are simply not too many zeros in this region, then one can deduce (1.1) with \( h = x^{1-\delta} \). In particular, it turns out that
\[ N(\sigma, T) := \# \{ \rho = \beta + i\gamma : \zeta(\rho) = 0, \beta \geq \sigma, |\gamma| \leq T \} \ll T^{c(1-\sigma)} (\log T)^{c'}, \tag{1.2} \]
where \( c > 2 \) and \( c' > 0 \) are absolute constants; this is a so-called zero density estimate.

(In this section, \( c \) and \( c' \) will always denote positive absolute constants, though they may represent different values in each occurrence.) Recall that there are about \( \frac{T}{\pi} \log \frac{T}{2\pi e} \) zeros of
\( \zeta(s) \) with \(|\gamma| \leq T\), so that a vanishingly small proportion of zeros have real part close to 1. An explicit version of (1.2) enabled Hoheisel to prove the prime number theorem in short intervals (1.1) for \( h = x^{1-\delta} \) in the range \( 0 \leq \delta \leq \frac{1}{33000} \); it is now known that we may take \( 0 \leq \delta \leq \frac{512}{133000} \), due to Huxley [18] and Heath-Brown [14].

Another classical problem in analytic number theory is to determine the least prime in an arithmetic progression \( a (\mod q) \) with \((a, q) = 1\). Linnik [25] was able to show that the least such prime is no bigger than \( qA \), where \( A \) is an absolute constant; the best known value of \( A \) is 5, due to Xylouris [41] in his Ph.D. thesis. Modern treatments of Linnik’s theorem typically use a simplification due to Fogels [10], which involves proving a more general version of (1.2) for Dirichlet \( L \)-functions \( L(s, \chi) \). Specifically, if we define

\[
N_{\chi}(\sigma, T) := \# \{ \rho = \beta + i\gamma : L(\rho, \chi) = 0, \beta \geq \sigma, \text{ and } |\gamma| \leq T \},
\]

then Fogels showed that

\[
(1.3) \quad \sum_{\chi (\mod q)} N_{\chi}(\sigma, T) \ll T^{c(1-\sigma)}
\]

when \( T \geq q \). Due to the absence of a log \( T \) term as compared to (1.2), it is standard to call such a result a log-free zero density estimate. In this paper, we are interested in analogous log-free zero density estimates for automorphic \( L \)-functions and their arithmetic applications, specifically to analogues of Hoheisel’s and Linnik’s theorems.

We consider the following general setup. Let \( K/\mathbb{Q} \) be a number field with ring of adeles \( \mathbb{A}_K \), and let \( \pi \) be a cuspidal automorphic representation of \( \text{GL}_d(\mathbb{A}_K) \) with unitary central character. For simplicity, we simply refer to such a representation \( \pi \) as an automorphic representation. There is an \( L \)-function \( L(s, \pi, K) \) attached to \( \pi \) whose Dirichlet series and Euler product are given by

\[
L(s, \pi, K) = \sum_a \frac{\lambda_{\pi}(a)}{Na^s} = \prod_p \prod_{j=1}^{d} (1 - \alpha_{\pi}(j, p)Np^{-s})^{-1},
\]

where the sum runs over the non-zero integral ideals of \( K \), the product runs over the prime ideals, and \( Na = N_{K/\mathbb{Q}}a \) denotes the norm of the ideal \( a \).

Let \( \pi \) and \( \pi' \) be automorphic representations of \( \text{GL}_d(\mathbb{A}_K) \) and \( \text{GL}_{d'}(\mathbb{A}_K) \), respectively. The Rankin-Selberg convolution

\[
L(s, \pi \otimes \pi', K) = \sum_a \frac{\lambda_{\pi \otimes \pi'}(a)}{Na^s} = \prod_p \prod_{j_1=1}^{d} \prod_{j_2=1}^{d'} (1 - \alpha_{\pi}(j_1, p)\alpha_{\pi'}(j_2, p)(Np)^{-s})^{-1}
\]

is itself an \( L \)-function with an analytic continuation and a functional equation. Define \( \Lambda_{\pi \otimes \pi'}(a) \) by the Dirichlet series identity

\[
-\frac{L'}{L}(s, \pi \otimes \pi', K) = \sum_a \frac{\Lambda_{\pi \otimes \pi'}(a)}{Na^s}.
\]

If \( \tilde{\pi} \) is the representation which is contragredient to \( \pi \), then it follows from standard Rankin-Selberg theory and the Wiener-Ikehara Tauberian theorem that we have a prime number theorem for \( L(s, \pi \otimes \tilde{\pi}, K) \) in the form

\[
\sum_{Na \leq x} \Lambda_{\pi \otimes \tilde{\pi}}(a) \sim x.
\]
It is reasonable to expect (for example, it follows from the generalized Riemann hypothesis) that there is some small \( \delta > 0 \) such that for \( x \) sufficiently large and any \( h \geq x^{1-\delta} \), we have

\[
\sum_{x < Na \leq x + h} \Lambda_{\pi \otimes \tilde{\pi}}(a) \sim h.
\]

Unfortunately, a uniform analogue of Littlewood’s improved zero-free region does not yet exist for all automorphic \( L \)-functions, so it seems that (1.4) is currently inaccessible except in special situations. However, Moreno [31] proved that if \( K \) exists for all automorphic \( L \)-functions, so it seems that (1.4) is currently inaccessible except in special situations. However, Moreno [31] proved that if \( K = \mathbb{Q} \), \( L(s, \pi \otimes \tilde{\pi}, \mathbb{Q}) \) has a “standard” zero-free region (one of a quality similar to Hadamard’s and de la Vallée Poussin’s for \( \zeta(s) \)), and there is a log-free zero density estimate of the form

\[
N_{\pi \otimes \pi'}(\sigma, T) := \#\{\rho = \beta + i\gamma : L(\rho, \pi \otimes \pi', K) = 0, \beta \geq \sigma, |\gamma| \leq T\} \ll T^{c_{\pi,\pi'}(1-\sigma)}
\]

for \( L(s, \pi \otimes \tilde{\pi}, \mathbb{Q}) \), then for any \( 0 < \delta < \frac{1}{c_{\pi,\pi'}} \) and any \( h \geq x^{1-\delta} \), one has

\[
\sum_{x < Na \leq x + h} \Lambda_{\pi \otimes \tilde{\pi}}(a) \sim h,
\]

which Moreno called the Hoheisel phenomenon. However, at the time of Moreno’s work, such log-free zero density estimates only existed in special cases. Moreover, in general, it is only known that \( L(s, \pi \otimes \tilde{\pi}, K) \) has a standard zero-free region if \( \pi \) is self-dual.

Recall that \( \pi \) and \( \pi' \) are automorphic representations of \( GL_d(\mathbb{A}_K) \) and \( GL_{d'}(\mathbb{A}_K) \). Suppose that \( K = \mathbb{Q} \) and that either \( d \) and \( d' \) are both at most 2 or that one of \( \pi \) and \( \pi' \) is self-dual. Building on the work of Fogels, Akbary and Trudgian [1] proved in this case that if one has a certain amount of control over the Dirichlet coefficients of \( L(s, \pi, \mathbb{Q}) \) and \( L(s, \pi', \mathbb{Q}) \) in short intervals and \( T \) is sufficiently large in terms of \( \pi \) and \( \pi' \), then

\[
N_{\pi \otimes \pi'}(\sigma, T) \leq T^{c_{d,d'}(1-\sigma)},
\]

where \( c_{d,d'} > 2 \) is a constant depending on \( d \) and \( d' \). This allowed them to prove a variant of the Hoheisel phenomenon for \( L(s, \pi \otimes \tilde{\pi}, \mathbb{Q}) \) when \( \pi \) is self-dual. Unfortunately, \( c_{d,d'} \) was not made effective, whence also the length of the interval in their variant of the Hoheisel phenomenon. This makes their result difficult to use in situations where uniformity in parameters over several \( L \)-functions is required, especially when the \( L \)-functions in question vary in degree. Furthermore, the range of \( T \) for which their bound holds is also not made effective. This is necessary to obtain analogues of Linnik’s theorem.

Effective log-free zero density estimates have been proven for certain natural families of \( L \)-functions. Weiss [40] proved an effective analogue of (1.3) for the Hecke \( L \)-functions of ray class characters, which enabled him to access prime ideals of \( K \) satisfying splitting conditions in a finite extension \( M/K \). Additionally, Kowalski and Michel [24] obtained a log-free zero density estimate for \( L \)-functions associated to any family of automorphic representations of \( GL_d(\mathbb{A}_\mathbb{Q}) \) satisfying certain conditions, including the generalized Ramanujan conjecture (see Hypothesis 2.1). Their result works best when \( T \) is essentially constant, which is useful for variants of Linnik’s theorem but not for the Hoheisel phenomenon.

Our first result is a log-free zero density estimate for \( L(s, \pi \otimes \pi', K) \) which is effective in its dependence on \( \pi, \pi' \), and \( K \). This dependence is most naturally stated in terms of the analytic conductors \( q(\pi) \) and \( q(\pi') \) of \( \pi \) and \( \pi' \), respectively, whose definition we postpone to Section 2.1. We prove the following.

**Theorem 1.1.** Let \( K \) be a number field with absolute discriminant \( D_K \) and root discriminant \( \text{rd}_K := D_K^{1/[K:\mathbb{Q}]} \). Let \( \pi \) and \( \pi' \) be cuspidal automorphic representations of \( GL_d(\mathbb{A}_K) \) and
GL_d(𝔸_K), respectively. Suppose that either both \( d \leq 2 \) and \( d' \leq 2 \) or that at least one of \( \pi \) and \( \pi' \) is self-dual, and suppose that the generalized Ramanujan conjecture (GRC) holds for \( L(s, \pi, K) \). Let \( T \gg \max\{ (q(\pi)q(\pi'))^{1/[K:ℚ]}, \text{rd}_K[K:ℚ]\} \). There exists an absolute constant \( c_1 > 0 \) such that if \( \frac{1}{2} \leq \sigma \leq 1 \), then

\[
N_{\pi \otimes \pi'}(\sigma, T) \ll d^2 T^{c_1(d+d')^4[K:ℚ](1-\sigma)}.
\]

The following unconditional result follows immediately from Theorem 1.1 by taking \( \pi \) to be the trivial representation.

**Corollary 1.2.** Let \( K \) be a number field with absolute discriminant \( D_K \) and root discriminant \( \text{rd}_K \), and let \( \pi \) be a cuspidal automorphic representation of \( GL_d(𝔸_K) \). There exists an absolute constant \( c_2 > 0 \) such that if \( \frac{1}{2} \leq \sigma \leq 1 \), then

\[
N_\pi(\sigma, T) \ll T^{c_2d^4[K:ℚ](1-\sigma)},
\]

provided that \( T \gg \max\{ q(\pi)^{1/[K:ℚ]}, \text{rd}_K[K:ℚ]\} \).

**Remarks.**

1. We impose the self-duality condition in Theorem 1.1 in order to ensure that \( L(s, \pi \otimes \pi', K) \) has a standard zero-free region; see Lemma 2.1.

2. In the event that \( L(s, \pi \otimes \pi', K) \) has a Landau-Siegel zero \( \beta_1 \), one can improve the bounds in Theorem 1.1 by a factor of \((1 - \beta_1) \log T\). This leads to a generalization of the zero repulsion phenomenon of Deuring and Heilbronn. Such an improvement follows from a slightly more careful analysis as in the proof of Theorem 4.3 of [40]. Since the notation is unwieldy and the modifications are nearly identical to those in the proof of Theorem 4.3 of [40], we omit the proof.

3. Corollary 1.2 is the first unconditional log-free zero density estimate for all automorphic \( L \)-functions \( L(s, \pi, K) \). (Recall that Akbary and Trudgian’s result is conditional on a hypothesis on the Dirichlet coefficients of \( L(s, \pi, K) \) in short intervals.) In particular, Corollary 1.2 gives an unconditional log-free zero density estimate for \( L(s, \pi, ℚ) \) when \( \pi \) is an automorphic representation of \( GL_2(𝔸_ℚ) \) associated to a Hecke-Maass form, which was not previously known.

In addition to density estimates of the form (1.3), Jutila [20] proved a “hybrid” density estimate of the form

\[
\sum_{q \leq Q} \sum_{\chi \text{ mod } q}^* N_\chi(\sigma, T) \ll (Q^2 T)^c (\log QT)^{c'},
\]

where the \( ^* \) on the summation indicates it is to be taken over primitive characters; Montgomery [29] improved upon Jutila’s work to show that one may take \( c = \frac{5}{2} \). This simultaneously generalizes (1.3) and Bombieri’s large sieve density estimate [4]. As a consequence of (1.3), one sees that the average value of \( N_\chi(\sigma, T) \) is noticeably smaller that what is given by (1.3). Furthermore, (1.5) can be used to prove versions of the Bombieri-Vinogradov theorem in both long and short intervals.

Gallagher [11] proved that

\[
\sum_{q \leq T} \sum_{\chi \text{ mod } q}^* N_\chi(\sigma, T) \ll T^c (1-\sigma), \quad T \geq q,
\]

Unless mentioned otherwise, the implied constant in an asymptotic inequality is absolute and computable.
providing a mutual refinement of (1.3) and (1.5). Gallagher’s refinement can be used to prove both Linnik’s bound on the least prime in an arithmetic progression and variants of the Bombieri-Vinogradov theorem for short intervals. Our second result generalizes (1.6) to consider twists of Rankin-Selberg $L$-functions associated to automorphic representations over $Q$.

**Theorem 1.3.** Under the notation and hypotheses of Theorem 1.1 with $K = Q$, there exists an absolute constant $c_3 > 0$ such that

$$\sum_{q \leq T} \sum^* \chi \mod q N_{(\pi \otimes \pi') \otimes \chi}(\sigma, T) \ll d^2 T c_3 (d + d')^{1 - \sigma}.$$  

As with Theorem 1.1, by taking $\pi$ to be the trivial representation, we immediately obtain the following corollary.

**Corollary 1.4.** Under the notation and hypotheses of Corollary 1.1 with $K = Q$, there exists an absolute constant $c_4 > 0$ such that

$$\sum_{q \leq T} \sum^* \chi \mod q N_{\pi \otimes \chi}(\sigma, T) \ll T c_4 d^4 (1 - \sigma).$$

We now turn to the applications of Theorems 1.1 and 1.3 and their corollaries. We begin by considering a version of the Hoheisel phenomenon for $L$-functions satisfying the generalized Ramanujan conjecture. In some cases, it is desirable to incorporate an auxiliary splitting condition on the prime ideals. Thus, let $M/K$ be a Galois extension with Galois group $G$, let $C \subseteq G$ be a conjugacy class, and let $[M/K]_C$ denote the Artin symbol. For an ideal $a$, define $1_C(a)$ to be 1 if $a = p^k$ for some unramified prime $p$ with $[M/p]_C = C$ and to be 0 otherwise. It is then possible to prove an analogue of the Chebotarev density theorem for $L(s, \pi \otimes \tilde{\pi}, K)$ in the form

$$\sum \Lambda_{\pi \otimes \tilde{\pi}}(a) \sim |C|/|G|.$$  

Our first application is a short interval version of (1.7), with effective bounds on the size of the intervals.

**Theorem 1.5.** Assume the above notation. Let $\pi$ be a self-dual cuspidal automorphic representation of $GL_d(A_K)$ whose $L$-function $L(s, \pi, K)$ satisfies GRC. There exists a positive absolute constant $c_5 > 0$ such that if

$$\delta \leq \frac{c_5}{d^4 [M : Q] \log(3d[M : K])},$$

$x$ is sufficiently large, and $h \geq x^{1 - \delta}$, then

$$\sum_{x < Na \leq x + h} 1_C(a) \Lambda_{\pi \otimes \tilde{\pi}}(a) \asymp h,$$

where the implied constant depends on $\pi$ and the extension $M/K$. If $d = 2$, or if $\pi$ is the symmetric square of such a representation, then this is unconditional.
Remark. The key to removing the assumption of GRC in the cases mentioned is the factorization
\[(1.8) \quad L(s, \text{Sym}^n \pi \otimes \text{Sym}^n \tilde{\pi}, K) = L(s, \omega, K) \prod_{j=1}^n L(s, \text{Sym}^2 \pi, K),\]

where $\omega$ is the central character; an analogous factorization holds when the representations are twisted by Hecke characters. As with Corollaries 1.2 and 1.4, we see that the result is unconditional when the symmetric power $L$-functions are known to be automorphic and cuspidal. For $n = 1$, this follows from Kim and Shahidi [23], and for $n = 2$, this follows from Kim [21]. A case where this is interesting is when $\pi$ is associated to a Hecke-Maass form over $\mathbb{Q}$, where GRC is not known. However, in this case, Motohashi [34] recently established a log-free zero density estimate for $L(s, \text{Sym}^2 \pi, \mathbb{Q})$, thereby obtaining Theorem 1.5 in this case.

It is of course somewhat unsatisfying that we are not able to obtain an asymptotic formula in Theorem 1.5 to provide a true short interval analogue of (1.7), but, as remarked earlier, this is due to the lack of a strong zero-free region for general automorphic $L$-functions and seems unavoidable at present. Good zero-free regions of a quality better than Littlewood’s exist for Dedekind zeta functions (for example, due to Mitsui [28]), which enabled Balog and Ono [2] to prove a prime number theorem for primes in Chebotarev sets lying in short intervals.

Even though versions of Theorem 1.5 with asymptotic equality are only known in special cases, we can use Theorem 1.3 to show that the predicted asymptotics hold on average. We prove the following generalization of [11, Theorem 7]; to obtain unconditional results, we restrict ourselves to consider cuspidal automorphic representations of $\text{GL}_2(\mathbb{A}_\mathbb{Q})$.

**Theorem 1.6.** Let $\pi$ be either a self-dual cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_\mathbb{Q})$ with trivial central character or the symmetric square of such a representation. There exist absolute constants $c_6 > 0$ and $c_7 > 0$ such that if $\exp(\sqrt{\log x}) \leq Q \leq x^{c_6}$ and $x/Q \leq h \leq x$, then
\[
\sum_{q \leq Q} \sum_{\chi \bmod q}^* \left| \sum_{n \in [x, x+h]} \Lambda_{\pi \otimes \tilde{\pi}}(n) \chi(n) - \delta(\chi) h + \delta_{q, \ast}(\chi) h \xi^{\beta_1-1} \right| \ll h \exp \left( - \frac{c_7 \log x}{\log(Qq(\pi))} \right)
\]
for some $\xi \in [x, x+h]$. Here, $\delta(\chi) = 1$ if $\chi$ is the trivial character and is zero otherwise, and $\beta_1$ denotes the Landau-Siegel zero associated to an exceptional real Dirichlet character $\chi^*(\bmod q)$ if it exists. We set $\delta_{q, \ast}(\chi) = 1$ if $\chi = \chi^*$ and zero otherwise, including if the exceptional zero does not exist. The implied constant depends on at most $q(\pi)$.

Unlike the previous log-free zero density estimates for general automorphic $L$-functions discussed earlier, Theorem 1.6 allows us to handle questions where maintaining uniformity in parameters is crucial. One famous example of such an application is the Sato-Tate conjecture, which concerns the distribution of the quantities $\lambda_\pi(p)$ attached to a cuspidal automorphic representations $\pi$ of $\text{GL}_2(\mathbb{A}_K)$, where $K$ is a totally real field; for generalizations to higher degree representations, see, for example, Serre [37]. If $\pi$ has trivial central character and is genuine in the sense of Shahidi [38, Section 2] (in the case that $K = \mathbb{Q}$, this amounts to assuming that $\pi$ is associated with a holomorphic cuspidal Hecke newform), then, by work of Deligne [9], it satisfies the generalized Ramanujan conjecture that $|\lambda_\pi(p)| \leq 2$ at
all unramified $p$. We may thus write $\lambda_{\pi}(p) = 2\cos \theta_p$ for some angle $\theta_p \in [0, \pi]$, and the Sato-Tate conjecture predicts that if $I = [a, b] \subset [-1, 1]$ is a fixed subinterval, then
\[
\lim_{x \to \infty} \frac{1}{\pi_K(x)} \# \{Np \leq x : \cos \theta_p \in I \} = \frac{2}{\pi} \int_{I} \sqrt{1-t^2} \, dt =: \mu_{ST}(I),
\]
where $\pi_K(x) := \# \{p : Np \leq x \}$. The Sato-Tate conjecture is now a theorem, due to the remarkable work of Barnet-Lamb, Geraghty, Harris, and Taylor [3]. The proof relies upon showing that the symmetric power $L$-functions $L(s, \text{Sym}^n \pi, K)$ are all potentially automorphic, i.e., automorphic when restricted to some finite extension of $K$. It is expected that $L(s, \text{Sym}^n \pi, K)$ is automorphic over $K$ for each $n \geq 1$, but as of right now, this is known in general only for $n \leq 4$ (see [12, 21, 22, 23]). By recent work of Clozel and Thorne [6], if $\pi$ is associated to a classical modular form, then $L(s, \text{Sym}^n \pi, K)$ is automorphic for $n \leq 8$. Consequently, the number of symmetric powers needed to access the interval $I$ is incredibly important in this problem.

Recall that the Chebyshev polynomials $U_n(t)$, defined by
\[
\sum_{n=0}^{\infty} U_n(t)x^n = \frac{1}{1-2tx+x^2},
\]
form an orthonormal basis for $L^2([-1, 1], \mu_{ST})$. If $\pi_p$ is unramified, then $U_n(\cos \theta_p)$ is the Dirichlet coefficient of $L(s, \text{Sym}^n \pi, K)$ at the prime $p$. We say that a subset $I \subseteq [-1, 1]$ can be Sym$^n$-minorized if there exist $b_0, \ldots, b_n \in \mathbb{R}$ with $b_0 > 0$ such that
\[
(1.9) \quad \mathbf{1}_I(t) \geq \sum_{j=0}^{n} b_j U_i(t)
\]
for all $t \in [-1, 1]$, where $\mathbf{1}_I(\cdot)$ denotes the indicator function of $I$. Note that if $I$ can be Sym$^n$-minorized, then it is the union of intervals which individually need not be Sym$^n$-minorizable. We have the following.

**Theorem 1.7.** Assume the above notation. Let $K/\mathbb{Q}$ be a totally real field, and let $\pi$ be a genuine automorphic representation of $\text{GL}_2(\mathbb{A}_K)$ with trivial central character. Suppose that a fixed subset $I \subseteq [-1, 1]$ can be Sym$^n$-minorized and that the $L$-functions $L(s, \text{Sym}^j \pi, K)$ are automorphic for each $j \leq n$. Let $B = \max_{0 \leq j \leq n} |b_j|/b_0$, where $b_0, \ldots, b_n$ are as in (1.9). There exists an absolute constant $c_8 > 0$ such that if
\[
\delta \leq \frac{c_8}{n^4[K: \mathbb{Q}] \log(3Bn)},
\]
x is sufficiently large, and $h \geq x^{1-\delta}$, then
\[
\sum_{x < Np \leq x+h \atop \pi_p \text{ unramified}} \mathbf{1}_I(\cos \theta_p) \log Np \asymp h,
\]
where the implied constant depends on $B$, $I$, and $K$. In particular, if $I$ can be Sym$^4$-minorized, then this is unconditional.

**Remarks.** 1. For any fixed $n$, determining the subsets $I$ that can be Sym$^n$-minorized is an elementary combinatorial problem. We carry this out in Lemma A.1 to determine the intervals that can be Sym$^4$-minorized, which we consider to be the most interesting case; it turns out that the proportion of subintervals of $[-1, 1]$ which can be Sym$^4$-minorized...
is roughly 0.388. If one is not concerned with obtaining the optimal minorization or if \( n \) is large, it is likely more convenient to apply a standard minorant for \( I \) instead. For the Beurling-Selberg minorant (see Montgomery \cite[ Lecture 1]{montgomery2006}) , a tedious calculation shows that if \( n \geq 4(1 + \delta)/\mu_{ST}(I) - 1 \) for some \( \delta > 0 \), then \( I \) can be \( \text{Sym}^n \)-minorized with

\[
B \leq \frac{2 + 3/\delta}{\mu_{ST}(I)}.
\]

It follows that any interval can be \( \text{Sym}^n \)-minorized for \( n \) sufficiently large, and thus every interval is at least conditionally covered by Theorem \ref{thm:main}. Lemma \ref{lem:cond-cover} shows, however, that this minorant might be far from optimal. With the Beurling-Selberg minorant, we have unconditional results for intervals \( I \) satisfying \( \mu_{ST}(I) > \frac{4}{5} \). By contrast, Lemma \ref{lem:cond-cover} implies unconditional results for all intervals satisfying \( \mu_{ST}(I) \geq 0.534 \), and for some with measure as small as 0.139.

2. It is tempting to ask whether one can exploit existing results on potential automorphy for symmetric power \( L \)-functions and the explicit dependence on the base field in Theorem \ref{thm:main} to obtain unconditional, albeit weaker, results for all intervals. The proof of the Sato-Tate conjecture uses crucially work of Moret-Bailly \cite{moret-bailly1999} establishing the existence of number fields over which certain varieties have points. The proof of this result unfortunately only permits control over the ramification at finitely many places, so it is not possible to even obtain bounds on the discriminants of the fields over which the symmetric power \( L \)-functions are automorphic. Thus, the authors do not believe it is possible to obtain an unconditional analogue of Theorem \ref{thm:main} for all \( I \) at this time.

As mentioned earlier, Theorem \ref{thm:main} also allows us to access Linnik-type questions. As one such example, we consider an analogue of Linnik’s theorem in the context of the Sato-Tate conjecture.

**Theorem 1.8.** Assume the notation of Theorem \ref{thm:main}, and in particular that \( I \subset [-1, 1] \) can be \( \text{Sym}^n \)-minorized. There exists an absolute constant and \( c_0 \) such that if the \( L \)-functions \( L(s, \text{Sym}^j \pi, K) \) are automorphic for \( j \leq n \) and the minorant \ref{eq:minor} admits no Landau-Siegel zeros (see Remark 3 below), then there is a prime \( p \) satisfying both \( \cos \theta_p \in I \) and

\[
Np \ll q_0(\text{Sym}^n \pi)^{cn^2 \log(3Bn)},
\]

where the implied constant is absolute and where \( q_0(\text{Sym}^n \pi) = \max\{q(\text{Sym}^n \pi), D_K[K : \mathbb{Q}]^{[K : \mathbb{Q}]^3} \} \). In particular, if \( I \) can be \( \text{Sym}^n \)-minorized with \( b \leq 0 \), then this is unconditional.

**Remarks.** 1. When \( K = \mathbb{Q} \) and the arithmetic conductor of \( \pi \) is squarefree, Cogdell and Michel \cite{ cogdell2020} use the local Langlands correspondence to predict what \( q(\text{Sym}^n \pi) \) should be when \( \text{Sym}^n \pi \) is an automorphic representation satisfying Langlands functoriality. Under these assumptions, they prove that \( \log q(\text{Sym}^n \pi) \ll n \log q(\pi) \). In all other cases, Rouse \cite{rouse2016} proved that \( \log q(\text{Sym}^n \pi) \ll n^3[K : \mathbb{Q}] \log q(\pi) \) under the assumption of automorphy alone.

2. When \( I \) is fixed and \( \pi \) varies, the bound in Theorem \ref{thm:linnik} has the shape \( Np \leq q(\pi)^A \) for some absolute constant \( A \), and so is comparable to Linnik’s theorem. However, if \( \pi \) is fixed and \( I \) is varying, the dependence is much worse. This comes partially from the constants in the zero-free region for \( L(s, \text{Sym}^n \pi, K) \), where the \( n \) dependence in particular is of the form \( n^4 \). Without improving the quality of these constants, it seems likely that only minor improvements can be made to Theorem \ref{thm:linnik}.

3. One additional feature in the proof of Theorem \ref{thm:linnik} which is not seen in Theorem \ref{thm:main} is the role of a possible Landau-Siegel zero. To prove Linnik’s original result, one must obtain
lower bounds on \[1 - \frac{1}{\beta_1} \chi_1(a)x^{\beta_1-1},\] where \(\beta_1\) is the putative Landau-Siegel zero arising from the exceptional Dirichlet character \(\chi_1\). In doing so, one relies on the fact that \(|\chi_1(a)| \leq 1\), which makes it sufficient to bound \(1 - \frac{1}{\beta_1} x^{\beta_1-1}\) from below. In our case, however, the coefficients \(b_j\) may be arbitrary, and it is not clear how to obtain meaningful lower bounds in general. It is known for \(j = 1, 2,\) and 4 that \(L(s, \text{Sym}^j \pi, K)\) has no Landau-Siegel zero, and so there is no issue from these cases (see \([16, 15, 35]\) for \(j = 1, 2,\) and 4, respectively).

For the remaining symmetric powers, there are two courses. First, it might be the case that due to the Fourier decomposition of our \(\text{Sym}^n\)-minorant, the contribution from a putative Landau-Siegel zero of \(L(s, \text{Sym}^j \pi, K)\) may safely be discarded. This is the case when the associated Fourier coefficient \(b_j\) is non-positive. Otherwise, we use the work of Hoffstein and Ramakrishnan \([16]\), who prove that \(L\)-functions of degree \(n \geq 2\) satisfying what they call \(Hypothesis \ H(\pi)\), a consequence of Langlands functoriality, do not have exceptional zeros. Our hypothesis for Theorem 1.8 then states that some combination of these considerations holds for our \(\text{Sym}^n\)-minorant.

4. If \(K = \mathbb{Q}\), one may use Theorem 1.3 instead of Theorem 1.1 in the proof of Theorem 1.8. This would produce a bound on the least prime \(p \equiv a \pmod{q}\) such that \(\cos \theta_p \in I\).

This paper is organized as follows. In Section 2, we discuss the basic properties of automorphic \(L\)-functions that we will use in the proofs of the theorems and we prove a few useful lemmas. In Section 3, we prove Theorem 1.1. In Sections 4 and 6, we consider the arithmetic applications of Theorem 1.1. In particular, we prove Theorems 1.5 and 1.7 in Section 4 and Theorem 1.8 in Section 6.

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2. Preliminaries

2.1. Definitions and notation. We follow the account of Rankin-Selberg \(L\)-functions given by Brumley \([5, \text{Section 1}]\). Let \(K/\mathbb{Q}\) be a number field of absolute discriminant \(D_K\), and let \(n_K := [K : \mathbb{Q}]\). We also let \(\text{rd}_K := D_K^{1/n_K}\) denote the root discriminant of \(K\), a quantity in terms of which many of our estimates are more naturally stated. Let \(\mathbb{A}_K\) denote the ring of adeles over \(K\), and let \(\pi\) be an irreducible cuspidal automorphic representation of \(\text{GL}_d(\mathbb{A}_K)\) with unitary central character. For brevity, we will say that \(\pi\) is an automorphic representation.

We have the factorization \(\pi = \otimes_v \pi_v\) over the places of \(K\). For each nonarchimedean \(p\), we have the Euler factor

\[L_p(s, \pi, K) = \prod_{j=1}^d (1 - \alpha_{\pi}(j, p) Np^{-s})^{-1}\]

associated with \(\pi_p\). Let \(R_{\pi}\) be the set of primes ideals \(p\) for which \(\pi_p\) is ramified. We call \(\alpha_{\pi}(j, p)\) the local roots of \(L(s, \pi, K)\) at \(p\), and if \(p \notin R_{\pi}\), then \(\alpha_{\pi}(j, p) \neq 0\) for all \(1 \leq j \leq d\).
The representation $\pi$ has an associated automorphic $L$-function whose Euler product and Dirichlet series are given by

$$L(s, \pi, K) = \prod_p L_p(s, \pi, K) = \sum_a \frac{\lambda_\pi(a)}{Na^s},$$

where $p$ runs through the finite primes and $a$ runs through the non-zero integral ideals of $K$. This Euler product converges absolutely for $\text{Re}(s) > 1$, which implies that $|\alpha_\pi(j, p)| < Np$.

Luo, Rudnick, and Sarnak [27] showed that this may be improved to $|\alpha_\pi(j, p)| \leq Np^{\frac{1}{2} - \frac{1}{2d+1}}$, and the generalized Ramanujan conjecture asserts a further improvement.

**The generalized Ramanujan conjecture (GRC).** Assume the above notation. For each prime $p \notin R_\pi$, we have $|\alpha_\pi(j, p)| = 1$, and for each prime $p \in R_\pi$, we have $|\alpha_\pi(j, p)| \leq 1$.

**Remark.** It is expected that all automorphic $L$-functions $L(s, \pi, K)$ satisfy GRC. Indeed, it is already known for many of the most commonly used automorphic $L$-functions. Such $L$-functions include Hecke $L$-functions and the $L$-function of a cuspidal normalized Hecke eigenform of positive even integer weight $k$ on the congruence subgroup $\Gamma_0(N)$.

At each archimedean place $v$, we associate to $\pi_v$ a set of $n$ complex numbers $\{\mu_\pi(j, v)\}^d_{j=1}$, often called Langlands parameters, which are known to satisfy $\text{Re}(\mu_\pi(j, v)) > -1/2$ by the work of Luo, Rudnick, and Sarnak [27]. The local factor at $v$ is defined to be

$$L_v(s, \pi, K) = \prod_{j=1}^d \Gamma_{K_v}(s + \mu_\pi(j, v)),$$

where $\Gamma_\pi(s) = \pi^{-s/2}\Gamma_\pi(s)$ and $\Gamma_{\mathbb{C}}(s) = \Gamma_\pi(s)\Gamma_\pi(s + 1)$. Letting $S_\infty$ denote the set of archimedean places, we define the gamma factor of $L(s, \pi, K)$ by

$$\gamma(s, \pi, K) = \prod_{v \in S_\infty} L_v(s, \pi, K).$$

For notational convenience, we will define the complex numbers $\kappa_\pi(j)$ by

$$\gamma(s, \pi, K) = \prod_{j=1}^{dnK} \Gamma_\pi(s + \kappa_\pi(j)).$$

Any automorphic $L$-function $L(s, \pi, K)$ admits a meromorphic continuation to $\mathbb{C}$ with poles possible only at $s = 0$ and $1$. Let $r(\pi)$ denote the order of the pole at $s = 1$, and define the completed $L$-function

$$\Lambda(s, \pi, K) = (s(1-s))^{r(\pi)}q(\pi)^{s/2}\gamma(s, \pi, K)L(s, \pi, K),$$

where $q(\pi)$ is the conductor of $\pi$. It is well-known that $\Lambda(s, \pi, K)$ is an entire function of order $1$ and that there exists a complex number $\varepsilon(\pi)$ of modulus $1$ such that $\Lambda(s, \pi, K)$ satisfies the functional equation

$$\Lambda(s, \pi, K) = \varepsilon(\pi)\Lambda(1-s, \pi, K),$$

where $\pi$ is the representation contragredient to $\pi$. We have the relations

$$\alpha_\pi(j, p) = \overline{\alpha_\pi(j, p)}, \quad \gamma(s, \pi, K) = \gamma(s, \pi, K), \quad \text{and} \quad q(\pi) = q(\pi).$$
To maintain uniform estimates for the analytic quantities associated to \( L(s, \pi, K) \), we define the **analytic conductor** of \( L(s, \pi, K) \) by

\[
q(s, \pi) = q(\pi) \prod_{j=1}^{dn_K} \left( |s + \kappa_\pi(j)| + 3 \right).
\] (2.1)

We will frequently make use of the quantity \( q(0, \pi) \), which we denote by \( q(\pi) \).

As in the introduction, define the von Mangoldt function \( \Lambda(\alpha) \) by

\[
- \frac{L'}{L}(s, \pi, K) = \sum_a \frac{\Lambda(\alpha)}{N\alpha^s},
\]

and let \( \Lambda_K(\alpha) \) be that associated to the Dedekind zeta function \( \zeta_K(s) \). Using the bounds for \( |\alpha_\pi(j, p)| \) from Luo, Rudnick, and Sarnak \[27\], we have that

\[
|\Lambda(\alpha)| \leq d\Lambda_K(\alpha)N^{\frac{1}{2} - \frac{1}{2d+1}},
\] (2.2)

and under GRC, we have

\[
|\Lambda(\alpha)| \leq d\Lambda_K(\alpha).
\]

Consider two cuspidal automorphic representations \( \pi \) and \( \pi' \) of \( \text{GL}_d(\mathbb{A}_K) \) and \( \text{GL}_{d'}(\mathbb{A}_K) \). We are interested in the Rankin-Selberg product \( \pi \otimes \pi' \) of \( \pi \) and \( \pi' \), which, at primes \( p \not\in R_\pi \cup R_{\pi'} \), has a local factor given by

\[
L_p(s, \pi \otimes \pi', K) = \prod_{j=1}^d \prod_{j_2=1}^{d'} \left( 1 - \alpha_\pi(j_1, p)\alpha_{\pi'}(j_2, p)Np^{-s} \right)^{-1}.
\]

For \( p \in R_\pi \cup R_{\pi'} \), there exist local roots \( \beta_{\pi \otimes \pi'}(j, p) \) which satisfy \( |\beta_{\pi \otimes \pi'}(j, p)| \leq 1 \) for all \( 1 \leq j \leq d'd \), and we define for such \( p \)

\[
L_p(s, \pi \otimes \pi', K) = \prod_{j=1}^{d'd} \left( 1 - \beta_{\pi \otimes \pi'}(j, p)Np^{-s} \right)^{-1}.
\]

This gives rise to the \( L \)-function \( L(s, \pi \otimes \pi', K) \), which we call the Rankin-Selberg convolution of \( \pi \) and \( \pi' \), whose Euler product and gamma factor are given by

\[
L(s, \pi \otimes \pi', K) = \prod_p L_p(s, \pi \otimes \pi', K)
\]

and

\[
\gamma(s, \pi \otimes \pi', K) = \prod_{\nu \in S_\infty} \prod_{j_1=1}^d \prod_{j_2=1}^{d'} \Gamma_{\kappa_\nu}(s + \mu_{\pi \otimes \pi'}(j_1, j_2, \nu)) = \prod_{j=1}^{d'dn_K} \Gamma_{\kappa_\nu}(s + \kappa_\pi \otimes \kappa_{\pi'}(j)),
\]

where \( \text{Re}(\mu_{\pi \otimes \pi'}(j_1, j_2, \nu)) > -1 \) and \( \text{Re}(\kappa_\pi \otimes \kappa_{\pi'}(j)) > -1 \). By Equation 8 of Brumley \[3\], we have

\[
q(s, \pi \otimes \pi') \leq q(\pi)^d q(\pi')^d d^d N^{dn_K}.
\]

Finally, we note that if \( \pi' = \tilde{\pi} \), then \( L(s, \pi \otimes \pi', K) \) has a pole at \( s = 1 \) of order 1, so that \( r(\pi \otimes \pi') = 1 \).
2.2. Preliminary lemmata. We begin with a zero-free region for $L(s,\pi \otimes \pi', K)$, obtained by adapting Theorem 5.10 of Iwaniec and Kowalski [19] to $L$-functions over arbitrary number fields.

Lemma 2.1. Suppose that either both $d$ and $d'$ are at most 2 or that at least one of $\pi$ and $\pi'$ is self-dual. Let $T \geq 3$, and let

$$\mathcal{L} = \mathcal{L}(T, \pi \otimes \pi', K) = (d' + d)^4 \log(q(\pi)q(\pi')T^{nK}).$$

There is a positive absolute constant $c_{10}$ such that the region

$$\{s = \sigma + it : \sigma \geq 1 - c_{10}L^{-1}, |t| \leq T\}$$

contains at most one zero of $L(s,\pi \otimes \pi', K)$. If such an exceptional zero $\beta_1$ exists, then it is real and simple, and $L(s,\pi \otimes \pi', K)$ must be self-dual. If $\pi'$ is trivial, then the same results hold for $L(s,\pi, K)$ with

$$\mathcal{L} = \mathcal{L}(T, \pi, K) = d'^4 \log(q(\pi)T^{nK}).$$

Proof. If $\pi'$ is isomorphic to neither $\pi$ nor $\tilde{\pi}$, this follows from Lemma 5.9 and Exercise 4 in Chapter 5 of [19] by considering the auxiliary $L$-function

$$L(s + it/2, \pi \otimes \pi', K)L(s,\pi \otimes \pi', K)L(s,\pi \otimes \pi, K)L(s - it/2,\pi \otimes \pi', K).$$

In the remaining cases, we have that $d = d'$ and both $\pi$ and $\pi'$ are self-dual. Thus one may use Moreno’s zero-free region [32, Theorem 3.3], which is stronger than the one presented here. The proof is essentially the same if $\pi'$ is trivial, but the auxiliary $L$-function is that which is used in the proof of [19, Theorem 5.10]. \hfill \Box

Lemma 2.2. Let $T \gg 1$, and let $\tau \in \mathbb{R}$ satisfy $|\tau| \leq T$.

(1) Uniformly on the disk $|s - (1 + i\tau)| \leq 1/4$, we have that

$$\frac{L'(s,\pi \otimes \pi', K)}{L(s,\pi \otimes \pi', K)} + \frac{r(\pi \otimes \pi')}{s} + \frac{r(\pi \otimes \pi')}{s - 1} - \sum_{|\rho - (1+i\tau)| \leq 1/2} \frac{1}{s - \rho} \ll \mathcal{L},$$

where the sum runs over zeros $\rho$ of $L(s,\pi \otimes \pi', K)$.

(2) For $1 \geq \eta \gg \mathcal{L}^{-1}$, we have that

$$\sum_{|\rho - (1+i\tau)| \leq \eta} 1 \ll \eta \mathcal{L}.$$

Proof. Part 1 is Lemma 2.4 of Akbary and Trudgian [1]. Part 2 follows from combining Theorem 5.6 of [19] and Proposition 5.8 of [19]. \hfill \Box

Lemma 2.3. If $0 < \eta \ll 1$ and $y \gg 1$, then

(1) $\sum_a |\Lambda_{\pi \otimes \pi'}(a)| Na^{1+\eta} \ll \frac{1}{\eta} + d'd \log(q(\pi)q(\pi'))$.

(2) $\sum_{Na \leq y} |\Lambda_{\pi \otimes \pi'}(a)| Na \ll \log y + d'd \log(q(\pi)q(\pi'))$.

\footnote{We denote by $c_1, c_2, \ldots$ a sequence of constants, each of which is absolute, positive, and effectively computable. We do not recall this convention in future statements, as we find it to be notationally cumbersome.}
Proof. By the Cauchy-Schwarz inequality, we have
\[
\sum_a \frac{|\Lambda_{\pi \otimes \pi'}(a)|}{(Na)^{1+\eta}} \ll \left( \sum_a \frac{\Lambda_{\pi \otimes \pi'}(a)}{(Na)^{1+\eta}} \right)^{1/2} \left( \sum_a \frac{\Lambda_{\pi' \otimes \pi''}(a)}{(Na)^{1+\eta}} \right)^{1/2} = \left( -\frac{L'}{L} (1 + \eta, \pi \otimes \tilde{\pi}, K) \right)^{1/2} \left( -\frac{L'}{L} (1 + \eta, \pi' \otimes \tilde{\pi}', K) \right)^{1/2}.
\]
We first estimate \(-\frac{L'}{L} (1 + \eta, \pi \otimes \tilde{\pi}, K)\), which is a positive quantity because \(\eta > 0\) is real and the Dirichlet coefficients of \(-\frac{L'}{L} (s, \pi \otimes \tilde{\pi}, K)\) are real and nonnegative. By Theorem 5.6 of [19] and part 3 of Proposition 5.7 of [19], we have that
\[
-\text{Re}\left(\frac{L'}{L} (1 + \eta, \pi \otimes \tilde{\pi}, K)\right) = \frac{1}{2} \log q(\pi \otimes \tilde{\pi}) + \text{Re}\left(\frac{\gamma'}{\gamma} (1 + \eta, \pi \otimes \tilde{\pi}, K)\right) + \frac{1}{1 + \eta} + \frac{1}{\eta} - \sum_{\rho \neq 0, 1} \text{Re}\left(\frac{1}{1 + \eta - \rho}\right).
\]
Since
\[
\text{Re}\left(\frac{1}{1 + \eta - \rho}\right) \geq \frac{\eta}{(1 + \eta)^2 + \gamma^2} > 0,
\]
we have that
\[
-\text{Re}\left(\frac{L'}{L} (1 + \eta, \pi \otimes \tilde{\pi}, K)\right) \leq \frac{1}{2} \log q(\pi \otimes \tilde{\pi}) + \text{Re}\left(\frac{\gamma'}{\gamma} (1 + \eta, \pi \otimes \tilde{\pi}, K)\right) + \frac{1}{1 + \eta} + \frac{1}{\eta}.
\]
By the proof of part 2 in Proposition 5.7 in [19], we have that
\[
\text{Re}\left(\frac{\gamma'}{\gamma} (s, \pi \otimes \tilde{\pi}, K)\right) = -\sum_{|s + \kappa_{\pi \otimes \pi'}(j)| < 1} \text{Re}\left(\frac{1}{s + \kappa_{\pi \otimes \pi'}(j)}\right) + O(\log q(\pi \otimes \tilde{\pi})).
\]
Since \(\text{Re}(\kappa_{\pi \otimes \pi'}(j)) > -1\) for all \(1 \leq j \leq d' n_K\), we find
\[
\text{Re}\left(\frac{1}{s + \kappa_{\pi \otimes \pi'}(j)}\right) \geq \frac{\eta}{(1 + \eta + \text{Re}(\kappa_{\pi \otimes \pi'}(j)))^2 + \text{Im}(\kappa_{\pi \otimes \pi'}(j))^2} > 0,
\]
so that
\[
-\frac{L'}{L} (1 + \eta, \pi \otimes \tilde{\pi}, K) \ll \frac{1}{\eta} + \log q(\pi \otimes \tilde{\pi}) \ll \frac{1}{\eta} + d \log q(\pi).
\]
Since the analogue must hold for \(\pi'\), part 1 follows. Part 2 follows by choosing \(\eta = \frac{1}{\log y}\). \(\square\)

We conclude this section with a bound on the mean value of a Dirichlet polynomial.

Lemma 2.4. Let \(T \gg r d_K n_K\) and \(u > y > T^{16 n_K}\). Define
\[
S_{y,u}(\tau, \pi \otimes \pi') := \sum_{y < N \leq u} \Lambda_{\pi \otimes \pi'}(p) \frac{Np^{1+\tau}}{Np^{1+\tau}}.
\]

1. If \(L(s, \pi, K)\) satisfies GRC, then
\[
\log y \int_{-T}^T |S_{y,u}(\tau, \pi \otimes \pi')|^2 d\tau \ll d^2((\log u)^2 + (d')^2(\log q(\pi'))(\log u)).
\]

2. If \(K = \mathbb{Q}\) and \(L(s, \pi, \mathbb{Q})\) satisfies GRC, then
\[
\sum_{q \leq T^2} \log y \sum_{\chi \mod q} \int_{-T}^T \left|S_{y,u}(\tau, (\pi \otimes \pi') \otimes \chi)\right|^2 dt \ll d^2((\log u)^2 + (d')^2(\log q(\pi'))(\log u)).
\]
**Proof.** 1. Let \( b(p) \) be a complex-valued function on the prime ideals of \( K \) such that \( \sum_p |b(p)| < \infty \) and \( b(p) = 0 \) whenever \( Np \leq y \). As a consequence of [30] Corollary 3.8, if \( T \gg \text{rd}_K n_K \) and \( y \geq T^{16n_K} \), then

\[
\int_{-T}^{T} \left| \sum_p b(p)Np^{-it} \right|^2 dt \ll \frac{1}{\log y} \sum_p Np|b(p)|^2.
\]

If we define \( b(p) \) by

\[
b(p) = \begin{cases} \frac{\Lambda_{\pi \otimes \pi'}(p)}{Np} & \text{if } y \leq Np \leq u, \\ 0 & \text{otherwise}, \end{cases}
\]

and recall the definition of \( S_{y,u}(\tau, \pi \otimes \pi') \), then it follows immediately that

\[
\int_{-T}^{T} \left| S_{y,u}(\tau, \pi \otimes \pi') \right|^2 dt \ll \frac{1}{\log y} \sum_{y<Np\leq u} \frac{|\Lambda_{\pi \otimes \pi'}(p)|^2}{Np}.
\]

By assuming GRC for \( L(s, \pi, K) \), we conclude that

\[
\sum_{y<Np\leq u} \frac{|\Lambda_{\pi \otimes \pi'}(p)|^2}{Np} \ll d^2 \sum_{y<Np\leq u} \frac{(\log Np)|\Lambda_{\pi \otimes \pi'}(p)|}{Np}.
\]

The claimed result now follows by partial summation using Part 2 of Lemma 2.3.

2. Let \( K = \mathbb{Q} \). Suppose that \( a(p) \) is a function on primes such that \( a(p) = 0 \) if \( p \leq Q \) and \( \sum_p |a(p)|^2 < \infty \). By [11] Theorem 4, we have that for \( T \geq 2 \),

\[
\sum_{q\leq Q} \log q \sum_{\chi \bmod q} \sum_{y<p} \left| \sum_p \frac{a(p)\chi(p)q^{-it}}{p} \right|^2 dt \ll \sum_{y<p} (Q^2 T + p)|a(p)|^2.
\]

If we define \( a(p) \) as we did \( b(p) \) above and let \( Q = T^2 \), then

\[
\sum_{q\leq T^2} \log \frac{T^2}{q} \sum_{\chi \bmod q} \sum_{y<p} \left| S_{y,u}(\tau, (\pi \otimes \pi') \otimes \chi) \right|^2 dt \ll \sum_{y<p} \frac{(T^5 + p)|\Lambda_{\pi \otimes \pi'}(p)|^2}{p^2}.
\]

By assuming GRC for \( L(s, \pi, \mathbb{Q}) \), it follows immediately that

\[
\sum_{y<p\leq u} \frac{(T^5 + p)|\Lambda_{\pi \otimes \pi'}(p)|^2}{p^2} \ll d^2 \sum_{y<p\leq u} \frac{|\Lambda_{\pi \otimes \pi'}(p)|}{p}.
\]

The claimed result now follows by partial summation using Part 2 of Lemma 2.3. \( \square \)

### 3. The zero density estimate

In this section, we prove Theorem 1.1 by generalizing Gallagher’s [11] and Weiss’s [40] treatment of Turán’s method for detecting zeros of \( L \)-functions, obtaining a result that is uniform in \( K, \pi, \) and \( \pi' \). The key result is the following technical proposition, whose proof we defer to the end of the section.

**Proposition 3.1.** Let \( T \gg \text{rd}_K n_K \), \( \mathcal{L} = (d + d')^4 \log(q(\pi)q(\pi')T^{n_K}) \), and \( y = e^{c_1 \mathcal{L}} \). Suppose that \( \eta \) satisfies \( \eta^{-1} \ll \eta \ll 1 \). Let

\[
S_{y,u}(\tau, \pi \otimes \pi') := \sum_{y<Np\leq u} \frac{\Lambda_{\pi \otimes \pi'}(p)}{Np^{1+it}}.
\]
If \( L(s, \pi \otimes \pi') \) has a non-exceptional zero \( \rho_0 \) satisfying \(|\rho_0 - (1 + i\tau)| \leq \eta\), then
\[
\frac{y^{c_{12}\eta}}{(\log y)^3} \int_y^{y^{c_{13}}}|S_{y,u}(\tau, \pi \otimes \pi')|^2 \frac{du}{u} \gg 1.
\]

We first deduce Theorem \([1.1]\) from Proposition \([3.1]\). The proof of Proposition \([3.1]\) relies on certain upper and lower bounds on the derivatives of \( \frac{L'}{L}(s, \pi \otimes \pi') \), which are proven and assembled subsequently.

### 3.1. Proof of Theorems \([1.1]\) and \([1.3]\).

By Theorem 5.8 of \([19]\), we have
\[
N_{\pi \otimes \pi'}(0, T) = \frac{T}{\pi} \log \left( \frac{q(\pi \otimes \pi') T^{d' \log q K}}{(2\pi e)^{d' \log K}} \right) + O(\log q(T, \pi \otimes \pi')).
\]
(3.1)
Thus it suffices to prove the theorem for \( 1 - \sigma \) sufficiently small. Since the left side of Theorem \([1.1]\) is a decreasing function of \( \sigma \) and the right side of Theorem \([1.1]\) is essentially constant for \( 1 - \sigma \ll L^{-1} \), it suffices to prove the theorem for \( 1 - \sigma \gg L^{-1} \). Therefore, we may assume that \( c_{14} \leq \sigma \leq 1 - c_{10}L^{-1} \), where \( \frac{1}{2} < c_{14} < 1 \) and \( c_{10} > 0 \) are chosen such that we may take \( \eta = \sqrt{2}(1 - \sigma) \) in Proposition \([3.1]\).

It follows from the same reasoning as in the proof of Theorem 4.3 in \([40]\) (with Proposition \([3.1]\) replacing Lemma 4.2 of \([40]\)) that
\[
N_{\pi \otimes \pi'}(\sigma, T) \ll L \frac{y^{c_{12}\eta}}{(\log y)^3} \int_y^{y^{c_{13}}} \left( \int_{-T}^{T} |S_{y,u}(\tau, \pi \otimes \pi')|^2 \frac{d\tau}{u} \right) \frac{du}{u}.
\]

Suppose that \( L(s, \pi', K) \) satisfies GRC. By Part 1 of Lemma \([2.4]\) the definition of \( y \), and the definition of \( S_{y,u}(\tau, \pi \otimes \pi') \),
\[
N_{\pi \otimes \pi'}(\sigma, T) \ll d^2 L \frac{y^{c_{12}\eta}}{(\log y)^4} \int_y^{y^{c_{13}}} (\log u)^2 + (d')^2(\log u) \log q(\pi') \frac{du}{u} \ll d^2 y^{c_{12}\eta}.
\]
Since we may take \( \eta = \sqrt{2}(1 - \sigma) \), recalling the definition of \( y \), we have
\[
N_{\pi \otimes \pi'}(\sigma, T) \ll d^2 y^{\sqrt{2}c_{12}(1-\sigma)} \ll d^2 (q(\pi)q(\pi') T^{n_K})^{(d+d')4\sqrt{2}c_{12}(1-\sigma)}.
\]

To conclude the proof of Theorem \([1.1]\) let \( c_{12} \) be sufficiently larger than \( c_{11} \) and set \( c_1 = \sqrt{2}c_{12} \).

Theorem \([1.3]\) is proven in almost exactly the same way as Theorem \([1.1]\) except that it requires Part 2 of Lemma \([2.4]\) we omit the proof.

### 3.2. Bounds on derivatives.

We begin by introducing notation which we will use throughout this section and the next. First, let \( r = r(\pi \otimes \pi') \) be the order of the possible pole of \( L(s, \pi \otimes \pi', K) \) at \( s = 1 \). We suppose that \( L(s, \pi \otimes \pi', K) \) has a non-exceptional zero \( \rho_0 \) satisfying
\[
|\rho_0 - (1 + i\tau)| \leq \eta,
\]
and we set
\[
F(s) = \frac{L'}{L}(s, \pi \otimes \pi', K).
\]
Suppose that \( |\tau| \leq T \), where \( T \geq 2 \), as in the statement of Proposition \([3.1]\). On the disk \( |s - (1 + i\tau)| < 1/4 \), by part 1 of Lemma \([2.2]\) we have
\[
F(s) + \frac{r}{s} + \frac{r}{s-1} = \sum_{|\rho - (1+i\tau)| \leq 1/2} \frac{1}{s-\rho} + G(s),
\]
where $G(s)$ is analytic and $|G(s)| \ll \mathcal{L}$. Setting $\xi = 1 + \eta + i\tau$, we have

\begin{equation}
(3.2) \quad \frac{(-1)^k}{k!} \frac{d^k F}{ds^k}(\xi) + r(\xi - 1)^{-(k+1)} = \sum_{|\rho-(1+i\tau)| \leq 1/2} (\xi - \rho)^{-(k+1)} + O(8^k \mathcal{L}),
\end{equation}

where the error term absorbs the contribution from integrating $G \ll \mathcal{B}$ by Lemma 2.2 (part 2), the sum over zeros has such that $s > 200$.

**Proof of Lemma 3.2.** Assume the notation above. For any $M \gg \eta \mathcal{L}$, there is some $k \in [M, 2M]$ such that

\[ \frac{\eta^{k+1}}{k!} \left| \frac{d^k F}{ds^k}(\xi) \right| \geq \frac{1}{2} (100)^{-(k+1)}, \]

where $\xi = 1 + \eta + i\tau$.

We prove Lemma 3.2 by using a version of Turán’s [39] power-sum estimate.

**Lemma 3.3 (Turán).** Let $z_1, \ldots, z_m \in \mathbb{C}$. If $M \geq m$, then there exists $k \in \mathbb{Z} \cap [M, 2M]$ such that $|z_1^k + \cdots + z_m^k| \geq (\frac{1}{100} |z_1|)^k$.

**Proof of Lemma 3.2.** We begin by considering the contribution to (3.2) from those zeros $\rho$ satisfying $200 \eta < |\rho - (1 + i\tau)| \leq 1/2$. In particular, by decomposing the sum dyadically and applying part 2 of Lemma 2.2, we find that

\[ \sum_{200 \eta < |\rho - (1+i\tau)| \leq 1/2} |\rho - \xi|^{-(k+1)} \ll \sum_{j=0}^{\infty} (2^j 200 \eta)^{-(k+1)} 2^{j+1} r \mathcal{L} \ll (200 \eta)^{-k} \mathcal{L}, \]

This shows that it suffices to consider the zeros $\rho$ whose distance from $1 + i\tau$ is less than $200 \eta$.

Since $\eta \ll 1$, we have

\begin{equation}
(3.3) \quad \frac{1}{k!} \frac{d^k F}{ds^k}(\xi) + r(\xi - 1)^{-(k+1)} \geq \left| \sum_{|\rho-(1+i\tau)| \leq 200 \eta} (\xi - \rho)^{-(k+1)} \right| - O((200 \eta)^{-k} \mathcal{L}).
\end{equation}

By Lemma 2.2 (part 2), the sum over zeros has $\ll \eta \mathcal{L}$ terms. Choosing $M \gg \eta \mathcal{L}$, Lemma 3.3 tells us that for some $k \in [M, 2M]$, the sum over zeros on the right side of (3.3) is bounded below by $(50 |\xi - \rho_0|)^{-(k+1)}$, where $\rho_0$ is the nontrivial zero which is being detected.

Since $|\xi - \rho_0| \leq 2\eta$, the right side of the above inequality is

\[ \geq (100 \eta)^{-(k+1)} (1 - O(2^{-k} \eta \mathcal{L})). \]

Since $k \geq M \gg \eta \mathcal{L}$ and $\mathcal{L}^{-1} \ll \eta \ll 1$, there is a constant $0 < \theta < 1$ so that

\[ O(2^{-k} \eta \mathcal{L}) = O(\theta^k \eta \mathcal{L}) \leq 1/4. \]

Therefore, for some $k \in [M, 2M]$ with $M \gg \eta \mathcal{L}$, we have

\[ \frac{\eta^{k+1}}{k!} \left| \frac{d^k F}{ds^k}(\xi) \right| + r\eta^{k+1}|(\xi - 1)^{-(k+1)}| \geq \frac{3}{4} (100)^{-(k+1)}. \]

During the proof of Theorem 4.2 in [40], Weiss proves that

\[ r\eta^{k+1}|(\xi - 1)^{-(k+1)}| \leq \frac{1}{4} (100)^{-(k+1)}. \]
The desired result now follows, that
\[ \frac{\eta^{k+1}}{k!} \left| \frac{d^k F}{ds^k} (\xi) \right| \geq \frac{1}{2} (100)^{-k-1}. \]

We now turn to obtaining an upper bound on the derivatives of \( F(s) \), for which we have the following.

**Lemma 3.4.** Assume the notation preceeding Lemma 3.2. Set \( M = 300 \eta \log y \), and let \( k \) be determined by Lemma 3.2. Then
\[ \frac{\eta^{k+1}}{k!} \left| \frac{d^k F}{ds^k} (\xi) \right| \leq \eta^2 \int_y^{y^{c_{13}}} |S_{y,u}(\tau, \pi \otimes \pi')| \frac{du}{u} + \frac{1}{4} (100)^{-k-1}, \]
where \( S_{y,u}(\tau, \pi \otimes \pi') \) is as in Proposition 3.1.

**Proof.** Let \( M = 300 \eta \log y \) and recall that \( y = e^{c_{11}L} \) for some \( c_{11} \), which we will take to be sufficiently large. For \( u > 0 \), define
\[ j_k(u) = \frac{u^k e^{-u}}{k!}, \]
which satisfies
\[ j_k(u) \leq \begin{cases} (100)^{-k} & \text{if } u \leq k/300, \\ (110)^{-k} e^{-u/2} & \text{if } u \geq 20k. \end{cases} \]
Letting \( c_{13} \geq 12000 \) be sufficiently large, we thus have
\[ j_k(\eta \log(Na)) \leq \begin{cases} (110)^{-k} & \text{if } Na \leq y, \\ (100)^{-k}(Na)^{-\eta/2} & \text{if } Na \geq y^{c_{13}}. \end{cases} \]
Differentiating the Dirichlet series for \( F(s) \) directly, we obtain
\[ \frac{(-1)^{k+1} \eta^{k+1}}{k!} \frac{d^k F}{ds^k} (\xi) = \eta \sum_a \frac{\Lambda_{\pi \otimes \pi'}(a)}{Na^{1+ir}} j_k(\eta \log(Na)) \]
Splitting the above sum \( \sum \) in concert with the inequality (3.4) and suppressing the summands, we write
\[ \sum = \sum_{Np \in (0,y] \cup (y^{c_{13}},\infty)} + \sum_{a \text{ prime}} + \sum_{y < Np \leq y^{c_{13}}} \]
We will estimate these three sums separately.

First, note that
\[ 1 \ll \eta L \ll \eta \log y \ll M \ll k. \]
We use Lemma 2.3 and (3.5) to obtain
\[
\left| \eta \sum_{\text{p not prime}} \right| \ll \eta(110)^{-k} \left( \sum_{a \leq y} \frac{\Lambda_{\pi \otimes \pi'}(a)}{Na} + \sum_{a} \frac{\Lambda_{\pi \otimes \pi'}(a)}{Na^{1+\eta/2}} \right) \\
\ll \eta(110)^{-k} \left( \frac{1}{\eta} + \log y + d'd \log(q(\pi)q(\pi')) \right) \\
\ll (110)^{-k} \left( 1 + \eta \log y + \eta L \right) \\
\ll k(110)^{-k}.
\]

If \( \eta \leq 1/55 \), which we may assume, then the identity \( \sum_{m \geq 0} j_m(u) = 1 \) implies that
\[
Na^{-1/2} j_k(\eta \log(Na)) = (2\eta)^k Na^{-\eta} j_k(\log(Na)/2) \leq (110)^{-k} Na^{-\eta}.
\]

Thus, as above,
\[
\left| \eta \sum_{a \text{ not prime}} \right| \ll \eta(110)^{-k} \sum_{a=p^m, m \geq 2} \frac{\Lambda_{\pi \otimes \pi'}(a)}{Na^{1/2+\eta}} \ll \eta(110)^{-k} \sum_{a} \frac{\Lambda_{\pi \otimes \pi'}(a)}{Na^{1+2\eta}} \ll k(110)^{-k},
\]

as well. Finally, recall that
\[
S_{y,u}(\tau, \pi \otimes \pi') = \sum_{y < Np \leq u} \frac{\Lambda_{\pi \otimes \pi'}(p)}{Np^{1+\tau}}.
\]

Summation by parts gives us
\[
\sum_{y < Np \leq y^{1/13}} S_{y,y^{13}}(\tau, \pi \otimes \pi') j_k(\eta \log(y^{13})) - \eta \int_{y}^{y^{13}} S_{y,u}(\tau, \pi \otimes \pi') j'_k(\eta \log u) \frac{du}{u}
\]
since \( S_{y,y}(\tau, \pi \otimes \pi') = 0 \). Much like above,
\[
|\eta S_{y,y^{13}}(\tau, \pi \otimes \pi') j_k(\eta \log(y^{13}))| \ll \eta(110)^{-k} y^{-c_{13}\eta/2} \sum_{Np \leq y^{13}} \frac{\Lambda_{\pi \otimes \pi'}(p)}{Np} \ll k(110)^{-k}.
\]

Therefore, since \( |j'_k(u)| = |j_{k-1}(u) - j_k(u)| \leq j_{k-1}(u) + j_k(u) \leq 1 \), we have
\[
\left| \eta \sum_{y^{13} < Np \leq y^{13}} \right| \leq \eta^2 \int_{y^{13}}^{y^{c_{12}\eta}} \left| S_{y,u}(\tau, \pi \otimes \pi') \right|^2 \frac{du}{u} + O(k(110)^{-k}).
\]

However, by (3.5) and \( \eta \gg L^{-1} \), we have that if \( k \) is sufficiently large, then each term of size \( O(k(110)^{-k}) \) is at most \( \frac{1}{16}(100)^{-(k+1)} \). The lemma follows. \( \square \)

### 3.3. Zero detection: The proof of Proposition 3.1

We now combine our upper and lower bounds on the derivatives of \( F \) to prove Proposition 3.1. Thus, we wish to show that if \( \rho_0 \) is a zero satisfying \( |\rho_0 - (1 + i\tau)| \leq \eta \), then
\[
\frac{y^{c_{12}\eta}}{(\log y)^3} \int_{y}^{y^{13}} \left| S_{y,u}(\tau, \pi \otimes \pi') \right|^2 \frac{du}{u} \gg 1.
\]

Combining Lemmas 3.2 and 3.4, we find that
\[
\eta^2 \int_{y}^{y^{13}} \left| S_{y,u}(\tau, \pi \otimes \pi') \right|^2 \frac{du}{u} \geq \frac{1}{4}(100)^{-(k+1)}.
\]
Using (3.5), we have
\[ \eta^2 \int_y^{y^{c_{13}}} |S_{y,u}(\tau, \pi \otimes \pi')| \frac{du}{u} \gg y^{-c_{12}\eta/4}, \]
where \( c_{12} \) is sufficiently large. Multiplying both sides by \( y^{-c_{12}\eta/4} \) yields
\[ y^{-c_{12}\eta/4}\eta^2 \int_y^{y^{c_{13}}} |S_{y,u}(\tau, \pi \otimes \pi')| \frac{du}{u} \gg y^{-c_{12}\eta/2}. \]

Since \( y^{-c_{12}\eta/4}\eta^2 \ll (\log y)^{-2} \), we have
\[ \frac{1}{(\log y)^2} \int_y^{y^{c_{13}}} |S_{y,u}(\tau, \pi \otimes \pi')| \frac{du}{u} \gg y^{-c_{12}\eta/2}. \]

Squaring both sides and applying the Cauchy-Schwarz inequality yields the proposition.

4. Proof of Theorems 1.5 and 1.7

In this section, we consider the arithmetic applications of the zero-density estimate provided in Theorem 1.1 to approximate versions of Hoheisel’s short interval prime number theorem. We begin in Section 4.1 by proving a useful proposition and using it to prove Theorem 1.5 concerning the Hoheisel phenomenon for a general automorphic \( L \)-function. We then specialize in Section 4.2 to the setup of the Sato-Tate conjecture, recalling the necessary notions, and prove Theorem 1.7.

4.1. The general setup and the proof of Theorem 1.5. As mentioned above, we begin by proving a general result from which our theorems will follow.

**Proposition 4.1.** Let \( \pi \) be a self-dual cuspidal automorphic representation of \( \text{GL}_d(\mathbb{A}_K) \) with unitary central character.

1. Suppose that \( L(s, \pi, K) \) satisfies GRC and \( T \gg \max\{q(\pi)^{2/n_K}, \text{rd}_K n_K\} \). For any fixed \( J \geq 2 \), set \( A = 32Jc_1d^4n_K \) and \( x = T^A \). If \( \rho = \beta + i\gamma \) runs through the nontrivial zeros of \( L(s, \pi \otimes \pi, K) \), then
   \[ \sum_{\rho=\beta+i\gamma, \theta=\gamma} x^{\beta-1} \leq c_{15}d^2e^{-Jc_1c_{10}}. \]

2. Let \( T \gg \max\{q(\pi)^{1/n_K}, \text{rd}_K n_K\} \). For any fixed \( J \geq 2 \), set \( A = 2Jc_2d^4n_K \) and \( x = T^A \). If \( \rho = \beta + i\gamma \) runs through the nontrivial zeros of \( L(s, \pi, K) \), then unconditionally, we have
   \[ \sum_{\rho=\beta+i\gamma, \theta=\gamma} x^{\beta-1} \leq c_{16}e^{-Jc_2c_{10}}. \]

**Proof.** We prove the first part; the second part is proven similarly. By the functional equation, it suffices to bound the summation over those zeros with \( \beta \geq 1/2 \). Using Lemma 2.1.
we have that
\[
\sum_{\rho = \beta + i\gamma \neq \beta_1 \atop |\gamma| \leq T} x^{\rho \beta - 1} = -\int_{1/2}^{1-\frac{c_1}{2}} x^{\sigma - 1} dN_{\pi \otimes \bar{\pi}}(\sigma, T)
\]
\[
= x^{-1/2}N_{\pi \otimes \bar{\pi}}(1/2, T) + \log x \int_{1/2}^{1-\frac{c_1}{2}} x^{\sigma - 1} N_{\pi \otimes \bar{\pi}}(\sigma, T) \, d\sigma.
\]

By (3.1) and our choice of \( x \), the first term is easily seen to be \( O(x^{-\frac{1}{2} + \frac{1}{4}} \log x) \) with an absolute implied constant. We employ Theorem 1.1 to bound the second term by
\[
d^2 \log x \int_{1/2}^{1-\frac{c_1}{2}} x^{\sigma - 1} T^{c_1(d + d')4nK(1 - \sigma)} \, d\sigma \ll d^2 \left( x^{1/4} + x^{-c_{10}/L} \right),
\]
where, again, the implied constant is absolute. The result now follows from the definition of \( L \) in Lemma 2.1. \( \square \)

**Proof of Theorem 1.5.** The upper bound is standard, so we prove only the lower bound. Choose \( g \in \mathcal{C} \), and let \( \mathcal{H} = \langle g \rangle \) be the cyclic group generated by \( g \). Regarding \( 1_c(\cdot) \) as a class function on \( \mathcal{G} \), we have the decomposition
\[
1_c = \frac{|C|}{|G|} \sum_{\chi \in \mathcal{H}} \bar{\chi}(g) \text{Ind}_{G}^{\mathcal{H}} \chi.
\]
Thus, if we let \( E \) be the fixed field of \( \mathcal{H} \) and set \( \psi = \pi \otimes \bar{\pi} \), by applying Frobenius reciprocity, we find that, as class functions of the absolute Galois group,
\[
\text{tr}(\psi) \cdot 1_c = \frac{|C|}{|G|} \sum_{\chi \in \mathcal{H}} \bar{\chi}(g) \cdot \text{tr}(\psi \otimes \text{Ind}_{G}^{\mathcal{H}} \chi) = \frac{|C|}{|G|} \sum_{\chi \in \mathcal{H}} \bar{\chi}(g) \cdot \text{tr}(\psi|_E \otimes \chi),
\]
where \( \psi|_E \) denotes the restriction of \( \psi \) to \( E \). At the level of primes, this translates to the equality
\[
\sum_{x < N \alpha \leq x + h} 1_c(a) \Lambda_{\pi \otimes \bar{\pi}}(a) = \frac{|C|}{|G|} \sum_{\chi \in \mathcal{H}} \bar{\chi}(g) \sum_{x < N \beta \leq x + h} \Lambda_{\psi|_E \otimes \chi}(b) + O(\log(q(\pi \otimes \pi')) \log x).
\]
Let \( \rho_\chi \) denote a nontrivial zero of \( L(s, \psi|_E \otimes \chi, E) \). Using the explicit formula in the form given by Equation 5.53 of [19] (which is valid under the assumption of GRC), we find that
\[
\frac{|G|}{|C|} \sum_{x < N \alpha \leq x + h} 1_c(a) \Lambda_{\pi \otimes \bar{\pi}}(a) = -1
\]
\[
= - \sum_{\chi \in \mathcal{H}} \bar{\chi}(g) \left( \sum_{\rho_\chi = \beta + i\gamma \neq \beta_1 \atop |\gamma| \leq T} (x + h)^{\rho_\chi} - x^{\rho_\chi} \right) h \rho_\chi + O\left( \frac{x \log x \log(q(\psi|_E \otimes \chi) x^{d_{2nM}})}{hT} \right)
\]
\[
\geq - x^{\beta_1 - 1} - \sum_{\chi \in \mathcal{H}} \left( \sum_{\rho_\chi = \beta + i\gamma \neq \beta_1 \atop |\gamma| \leq T} x^{\beta_1 - 1} + O\left( \frac{x \log x \log(q(\psi|_E \otimes \chi) x^{d_{2nM}})}{hT} \right) \right),
\]
Note that $|\mathcal{H}| \leq n_{M/K}$. Since we may assume that both $x$ and $T$ are $\gg q(\psi|_E \otimes \chi)$, we can apply Proposition 4.1 to obtain the lower bound

$$
\frac{|G|}{|C|h} \sum_{x < Nq \leq x+h} 1_{\mathcal{C}}(a) \Lambda_{\pi \otimes \hat{\pi}}(a) \geq 1 - c_{15} \frac{d^2 n_{M/K}}{eJc_{10}} - c_{17} \frac{d_n M^1 M x (\log x)^2}{hT} - o(1),
$$

where the $o(1)$ term comes from the possible Landau-Siegel zero. (The implied constant in the $o(1)$ term is not effectively computable.) Choosing

$$
J = \max \left\{ 2, \frac{1}{c_1 c_{10}} \log(4c_{15}d^2 n_{M/K}) \right\},
$$

and recalling that $A = 32 Jc_{1} d^4 n_K$ and $T = x^{1/2}$ as in Proposition 4.1, this yields

$$
\frac{|G|}{|C|h} \sum_{x < Nq \leq x+h} 1_{\mathcal{C}}(a) \Lambda_{\pi \otimes \hat{\pi}}(a) \geq \frac{3}{4} - o(1) - c_{17} \frac{d^2 n_{M/K} n_M x^{1 - \frac{1}{4}} (\log x)^2}{h}.
$$

Finally, taking

$$
h \geq 4c_{17} d^2 n_{M/K} n_M x^{1 - \frac{1}{4}} (\log x)^2,
$$

we obtain the desired lower bound when $x$ is sufficiently large. The unconditional part of the theorem is proven similarly using the arguments in the remarks following Theorem 4.5. \qed

4.2. The Sato-Tate conjecture. We now assume that we are in the situation where we may talk about the Sato-Tate conjecture. Thus, we assume that $K$ is a totally real field and that $\pi$ is a genuine cuspidal automorphic representation of $GL_2(\mathbb{A}_K)$ with trivial central character. Recall that the Sato-Tate conjecture concerns the distribution of the quantities $\lambda_{\pi}(p) = 2 \cos \theta_p$ as $p$ ranges over primes for which $\pi_p$ is unramified, where $\theta_p \in [0, \pi]$. At each such prime $p$, the local factor of the $n$-th symmetric power $L$-function is given by

$$
L_p(s, \text{Sym}^n \pi, K) = \prod_{j=0}^{n} (1 - e^{i\theta_p (n-2j)Np^{-s}})^{-1} = \sum_{k=0}^{\infty} \frac{U_n(\cos(k\theta_p))}{Np^s},
$$

where $U_n$ is the $n$-th Chebyshev polynomial of the second kind. At ramified primes $p$, there are real numbers $\alpha_n(p)$ and $\beta_n(p)$ of absolute value at most $Np^{\frac{1}{2} - \frac{1}{(n+1)^2 + 1}}$ for which the local factor is given by

$$
L_p(s, \text{Sym}^n \pi, K) = \prod_{j=0}^{n} (1 - \alpha_n(p)^j \beta_n(p)^{n-j}Np^{-s})^{-1}.
$$

We note that $L(s, \text{Sym}^1 \pi, K) = L(s, \pi, K)$ and $L(\text{Sym}^0 \pi, K) = \zeta_K(s)$.

It would follow from Langlands functoriality that $\text{Sym}^n \pi$ is a cuspidal automorphic representation of $GL_{n+1}(\mathbb{A}_K)$ with trivial central character for all $n \geq 1$, in which case $L(s, \text{Sym}^n \pi, K)$ would have an analytic continuation to the entire complex plane and satisfy a functional equation of the type described in Section 2. Unfortunately, this is only known for $n \leq 4$, which poses problems if one wants finer distributional information about the sequence $\{\cos \theta_p\}$ than the ineffective equidistribution result of Barnet-Lamb, Geraghty, Harris, and Taylor [3].

In Theorem 4.7 our goal is to estimate for $I \subseteq [-1, 1]$ the summation

$$
\sum_{x < Np \leq x+h} 1_{I(\cos \theta_p)} \log Np,
$$

where the $o(1)$ term comes from the possible Landau-Siegel zero. (The implied constant in the $o(1)$ term is not effectively computable.) Choosing
where \( h \geq x^{1-\delta} \) for some \( \delta > 0 \). We recall from the introduction that \( I \) can be \( \text{Sym}^n \)-minorized if there exist \( b_0, \ldots, b_n \) with \( b_0 > 0 \) such that

\[
1 + \frac{1}{b_j} \geq \sum_{j=0}^{n} \frac{b_j U_j(t)}{I(t)}
\]

for all \( t \in [-1, 1] \). Thus, if \( I \) can be \( \text{Sym}^n \)-minorized, we can obtain a non-trivial lower bound for \( (4.1) \) by considering an appropriate linear combination of the logarithmic derivatives of \( L(s, \text{Sym}^j \pi, K) \) for \( j \leq n \). We are now able to use Proposition 4.1 to prove Theorem 1.7.

**Proof of Theorem 1.7.** As in Theorem 1.5, the upper bound is standard and follows from the prime ideal theorem in short intervals, say, or even Hoheisel’s original result over \( \mathbb{Q} \). Thus, we prove only the lower bound. The proof will be mostly similar to that of Theorem 1.5, but we need a slightly different version of the explicit formula as GRC is only known at the unramified primes. Specifically, modeled on the proof of Theorem 2.1 of Liu and Ye [26], we prove that

\[
\sum_{\text{Sym}^j \pi(a) \leq x} \Lambda_{\text{Sym}^j \pi}(a) = rx - \sum_{\rho_j = \beta_j + i\gamma_j \atop |\gamma_j| \leq T} \int_{\rho_j}^{\sigma_0 + iT} \frac{L'(s, \text{Sym}^j \pi)}{L(s, \text{Sym}^j \pi)} x^s ds + O\left( x \sum_{a} \frac{|\Lambda_{\text{Sym}^j \pi}(a)|}{Na^{\sigma_0}} \min\left\{ 1, \frac{1}{T|\log x Na|} \right\} \right).
\]

We begin with the standard Perron integral (see [8, Chapter 17]), finding that if \( \sigma_0 = 1 + \frac{1}{\log x} \), then

\[
\sum_{\text{Sym}^j \pi(a) \leq x} \Lambda_{\text{Sym}^j \pi}(a) = -\frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{L'(s, \text{Sym}^j \pi)}{L(s, \text{Sym}^j \pi)} x^s ds + O\left( x \sum_{a} \frac{|\Lambda_{\text{Sym}^j \pi}(a)|}{Na^{\sigma_0}} \min\left\{ 1, \frac{1}{T|\log x Na|} \right\} \right).
\]

Let \( H \geq 2 \). When \( Na < x(1 - \frac{1}{H}) \) or \( Na > x(1 + \frac{1}{H}) \), we have that

\[
|\log \frac{x}{Na}| \gg \frac{1}{H}.
\]

By reasoning similar to the proof of Lemma 2.3, we thus have

\[
\sum_{|Na-x| \leq \frac{x}{H}} \frac{|\Lambda_{\text{Sym}^j \pi}(a)|}{Na^{\sigma_0}} \min\left\{ 1, \frac{1}{T|\log x Na|} \right\} \ll \frac{H}{T} \sum_{a} \frac{|\Lambda_{\text{Sym}^j \pi}(a)|}{Na^{\sigma_0}} \ll \frac{H}{T} (\log x + \log q(\text{Sym}^j \pi)).
\]

It remains to estimate

\[
\sum_{|Na-x| \leq \frac{x}{H}} \frac{|\Lambda_{\text{Sym}^j \pi}(a)|}{Na^{\sigma_0}} = \sum_{|Np^m-x| \leq \frac{x}{H} \atop \pi \text{ unram. at } p} \frac{|\Lambda_{\text{Sym}^j \pi}(p^m)|}{(Np^m)^{\sigma_0}} + \sum_{|Np^m-x| \leq \frac{x}{H} \atop \pi \text{ ram. at } p} \frac{|\Lambda_{\text{Sym}^j \pi}(p^m)|}{(Np^m)^{\sigma_0}}.
\]

There are \( \ll n_K x/H \) powers of unramified primes with norm between \( x(1 - \frac{1}{H}) \) and \( x(1 + \frac{1}{H}) \), so Deligne’s bound yields

\[
\sum_{|Np^m-x| \leq \frac{x}{H} \atop \pi \text{ unram. at } p} \frac{|\Lambda_{\text{Sym}^j \pi}(p^m)|}{(Np^m)^{\sigma_0}} \ll n_K(j + 1) \frac{\log x}{H}.
\]
There are $\ll (\log x)(\log q(\text{Sym}^j \pi))$ powers of ramified primes with norm between $x(1-\frac{1}{H})$ and $x(1+\frac{1}{H})$, so using (2.2) yields
\[
\sum_{\substack{|Np^m-x| \leq \frac{x}{n} \text{ ram. at } p}} \frac{|A_{\text{Sym}^j \pi}(p^m)|}{(Np^m)^{\sigma_0}} \ll n_K(j+1)(\log x)^2(\log q(\text{Sym}^j \pi))x^{-\frac{1}{2}}x^{(j+1)^2+1}
\]
Collecting the above estimates, we choose $x > q(\text{Sym}^j \pi)$, $T \leq x$, and $H = T^{1/2}$ to obtain
\[
\sum_{N \leq x} A_{\text{Sym}^j \pi}(a) = -\frac{1}{2\pi i} \int_{C_x} L'(s, \text{Sym}^j \pi) \frac{x^s}{s} ds + O\left(n_K(j+1)\frac{x(\log x)^3}{\sqrt{T}}\right).
\]
By a standard computation using the residue theorem, we then find (4.3).

Suppose that $I \subset [-1, 1]$ can be Sym$^n$-minorized and that $L(s, \text{Sym}^j \pi, K)$ is automorphic for each $0 \leq j \leq n$. Let $b_0, \ldots, b_n$ be as in (1.2) and set $B = \max_{0 \leq j \leq n} |b_j|/b_0$. We use Proposition 4.1 with $\pi'$ trivial, $A = 2\log(n+1)^4n_K$, and $x = TA$. Letting $\rho_j$ run through the nontrivial zeros of $L(s, \text{Sym}^j \pi, K)$, and letting both $T$ and $x$ be $\gg q(\text{Sym}^n \pi)\sqrt{\log K}$, we have that
\[
\frac{1}{h} \sum_{x < Np \leq x+h} 1_I(\cos \theta_p) \log Np
\]
is bounded below by
\[
\sum_{j=0}^{n} b_j \left( \frac{1}{h} \sum_{x < Np \leq x+h} U_j(\cos \theta_p) \log Np \right) \geq b_0 - \sum_{j=0}^{n} b_j \left( \sum_{\rho = \beta_j + i\gamma_j, \gamma_j \leq T} (x+h)^{\rho_j} - x^{\rho_j} \right) \frac{c_{17}n_K(j+1)x(\log x)^3}{h\sqrt{T}} \geq b_0 \left(1 - c_{15}(n+2)B e^{-Jc_{210}} - \frac{c_{17}(n+2)^2Bn_Kx^{1-\frac{3}{4}}(\log x)^3}{h} - o(1)\right),
\]
where the $o(1)$ term arises from the contributions of the possible exceptional zeros. We now choose $J = \max\{2, \frac{1}{c_{210}}\log(4c_{15}B(n+1))\}$, so that
\[
\frac{1}{h} \sum_{x < Np \leq x+h} 1_I(\cos \theta_p) \log Np \geq b_0 \left(\frac{3}{4} - o(1) - \frac{c_{17}Bn_K(n+1)^2x^{1-\frac{3}{4}}(\log x)^3}{h}\right).
\]
Choosing $h \geq 4c_{17}Bn_K(n+1)^2x^{1-\frac{3}{4}}(\log x)^3$, we obtain the desired lower bound. \qed

5. Proof of Theorem 1.6

In this section, all implied constants depend at most on $q(\pi)$.

Proof of Theorem 1.6. Our proof will handle the case where $\pi$ is a self-dual cuspidal automorphic representation of $GL_2(\mathbb{A}_Q)$ with trivial central character; the case where it is the symmetric square of such a form is proven similarly.

Let $Q^5 = T \leq x^{\frac{3}{10\log q}}$, and suppose that $x \leq hQ$ and $\log x \leq (\log Q)^2$. Let $\chi$ be a primitive Dirichlet character modulo $q \leq Q$. By (1.8) and the assumption that $\pi$ has trivial central
character, we have that
\[(5.1) \quad L(s, (\pi \otimes \pi) \otimes \chi, \mathbb{Q}) = L(s, \chi, \mathbb{Q})L(s, \text{Sym}^2 \pi \otimes \chi, \mathbb{Q}).\]
Furthermore, \(L(s, \text{Sym}^2 \pi \otimes \chi, \mathbb{Q})\) has no Siegel zero (see, e.g., Theorem A of Ramakrishnan and Wang [33]), so \(L(s, (\pi \otimes \pi) \otimes \chi, \mathbb{Q})\) has a Siegel zero if and only if it is inherited from \(L(s, \chi, \mathbb{Q})\).

By arguments similar to those in the above proofs, we have that
\[
\sum_{x < n \leq x + h} \Lambda_{\pi \otimes \pi}(n) \chi(n) - \delta(\chi) h + h \xi_{\beta_1 - 1} \ll h \left( \sum_{\gamma \leq T} x^{\beta_1 - 1} + Q^2 / T \right),
\]
where the summation on the right-hand side is over the nontrivial zeros of \(L(s, (\pi \otimes \pi) \otimes \chi, \mathbb{Q})\) which are not \(\beta_1\). Thus
\[
\sum_{q \leq Q \chi \mod q} \sum_{x < n \leq x + h} \Lambda_{\pi \otimes \pi}(n) \chi(n) - \delta(\chi) h + \delta_{\beta_1}(\chi) h \xi_{\beta_1 - 1} \ll h \left( \sum_{q \leq Q \chi \mod q} \sum_{\gamma \leq T} x^{\beta_1 - 1} + Q^4 / T \right).
\]

Using the factorization (5.1), the triple sum in (5.2) is bounded by
\[
\log x \int_{1/2}^{1} x^{(\sigma-1)} \sum_{q \leq Q \chi \mod q} \sum_{\chi} N_{\chi}(\sigma, T) d\sigma + x^{-1/2} \sum_{q \leq Q \chi \mod q} \sum_{\chi} N_{\chi}(\sigma, T)
+ \log x \int_{1/2}^{1} x^{(\sigma-1)} \sum_{q \leq Q \chi \mod q} \sum_{\chi} N_{\text{Sym}^2 \pi \otimes \chi}(\sigma, T) d\sigma + x^{-1/2} \sum_{q \leq Q \chi \mod q} \sum_{\chi} N_{\text{Sym}^2 \pi \otimes \chi}(\sigma, T),
\]
Using Corollary 1.4 and recalling our choice of \(T\), the triple sum is now bounded by
\[
\log x \int_{1/2}^{1-\frac{\log 2}{2}} x^{(\sigma-1)} d\sigma + x^{-\frac{1}{4}} \ll x^{-\frac{\log 2}{2}} + x^{-\frac{1}{4}},
\]
where \(\mathcal{L}' = 256 \log(q(\text{Sym}^2 \pi) QT)\). Since \(T = Q^5\) and \(\log q(\text{Sym}^2 \pi) \asymp \log q(\pi)\) (with an absolute implied constant), the right-hand side of (5.2) is bounded by the quantity claimed in the statement of the theorem.

6. Proof of Theorem 1.8

We now turn to the problem of determining the least prime produced in the Sato-Tate conjecture, recalling the setup in Section 4.2. We begin with some notation. We let
\[
T = c_{18} \max\{q(\text{Sym}^n \pi)^{1/nK}, rd_{K^{nK}}\}, \quad x = T^{c_{2nK}},
B = \max_{0 \leq j \leq n} \left| b_j \right| / b_0, \quad A = 8c_{19} \left( 1 + \frac{(n + 1)^4 \log(32n)}{\min\{c_2 c_{10}, 1\}} \right),
\]
where \(c_{19} \geq 2\). Let
\[
K(s) = x^{\frac{s}{2}} \left( x^s - 1 \right) / s \log x
\]
and
\[
R(y) = \frac{1}{2\pi i} \int_{2 - i\infty}^{2 + i\infty} K(s)^2 y^{-s} ds.
\]
As shown by Graham [13] (though his definition of $K(s)$ has a typo – it is missing the factor of $s$ in the denominator), the support of $R(y)$ is the interval $[x^{A-2}, x^A]$, and on that interval we have that $R(y) \ll (\log x)^{-1}$. Furthermore, Graham shows that for all $\sigma < 0$, we have that

$$|K(s)|^2 \leq x^{(A-2)\sigma} \min\left\{1, \frac{4}{|s \log x|^2}\right\}.$$

Our ultimate goal is to show that the sum

$$S = \sum_{p: \pi p \text{ unram.}} \frac{(\log Np) R(Np)}{Np} \Lambda_f(\cos \theta_p),$$

is positive. We first prove the following lower bound for $S$.

**Lemma 6.1.** If $I$ can be $\text{Sym}^n$-minorized as in (4.2) without admitting any Landau-Siegel zeros (as in Remark 3 following Theorem 1.8), then

$$S \geq b_0 - \sum_{j=0}^n b_j \left( \sum_{\rho=\beta_j+i \gamma_j} K(\rho - 1)^2 + O(x^{-3/2}) \right),$$

where the second sum is over all nontrivial zeros $\rho_j$ of $L(s, \text{Sym}^j \pi, K)$.

**Proof.** If $I$ can be $\text{Sym}^n$-minorized as in (4.2), then

$$S \geq \sum_{j=0}^n b_j \sum_{p \text{ unramified}} \frac{(\log Np) R(Np)}{Np} U_j(\cos \theta_p).$$

Consider the individual summands

$$S_j = \sum_{p \text{ unramified}} \frac{(\log Np) R(Np)}{Np} U_j(\cos \theta_p)$$

and the related contour integrals

$$I_j = \frac{1}{2\pi i} \int_{2-\infty}^{2+i\infty} K(s)^2 \left( - \frac{L'}{L}(s+1, \text{Sym}^j \pi, K) \right) ds.$$

Since

$$- \frac{L'}{L}(s+1, \text{Sym}^j \pi, K) = \sum_a \frac{\Lambda_{\text{Sym}^j \pi}(a)}{Na^{s+1}},$$

integrating term by term yields the equality

$$I_j = \sum_a \frac{\Lambda_{\text{Sym}^j \pi}(a) R(Na)}{Na}.$$

Thus, the difference between $S_j$ and $I_j$ satisfies

$$|S_j - I_j| = \left| \sum_{p \text{ ramified}} \frac{\Lambda_{\text{Sym}^j \pi}(p) R(Np)}{Np} + \sum_{a=p^m} \frac{\Lambda_{\text{Sym}^j \pi}(a) R(Na)}{Na} \right|,$$
but $x$ is larger than any ramified prime. Thus the first sum is empty, and we can use the bounds $R(y) \ll 1/\log x$ and $|\Lambda_{\text{Sym}^3\pi}(a)| \leq (j+1)\Lambda_K(a)N^{1/2} - \frac{1}{(j+1)^2+1}$ to obtain

$$|S_j - I_j| = \left| \sum_{a=p^n \atop m \geq 2} \frac{\Lambda_{\text{Sym}^3\pi}(a)R(Na)}{Na} \right| \ll \frac{j}{\log x} \sum_{m \geq 2} \frac{\Lambda_K(p^n)}{(Np^m)^{1/2} (j+1)^2+1}.$$ 

There are at most $n_K x^{A/2}$ terms in this summation, each of which is bounded by

$$\frac{\log(x^A)}{x^{(A-2)\left(1 + \frac{1}{(j+1)^2+1}\right)}} \leq Ax^{2-\frac{4}{2} - \frac{A}{(j+1)^2+1}} \log x,$$

so that $|S_j - I_j| \ll n_K j Ax^{2-\frac{A}{(j+1)^2+1}}$. Since $x > \max\{3, n_K, n\}$ and $A > 8((n+1)^2 + 1)$, we find that

$$|S_j - I_j| \ll Ax^{4 - \frac{A}{(j+1)^2+1}} \ll x^{-2},$$

where, as always, the implied constant is absolute. Thus, we now estimate the integrals $I_j$.

By shifting the contour, it is straightforward to show that for our choice of $A$,

$$I_j = \delta_{0,j} - \sum_{\rho_j = \beta_j + i\gamma_j \atop 0 \leq \beta_j < 1} K(\rho_j - 1)^2 + o(x^{-3/2}).$$

Thus

$$S \geq \sum_{j=0}^n b_j S_j = \sum_{j=0}^n b_j (I_j + O(x^{-2})) = b_0 - \sum_{j=0}^n b_j \left( \sum_{\rho_j = \beta_j + i\gamma_j \atop 0 \leq \beta_j < 1} K(\rho_j - 1)^2 + o(x^{-3/2}) \right).$$

Note that if $\rho$ is an exceptional real zero of some $L(s, \text{Sym}^3\pi, K)$ and $b_j < 0$, then its contribution to the sum is real and positive. It may thus be safely discarded in the lower bound, and this is precisely what we have assumed to be able to do for $\pi$ and its minorant. □

**Proof of Theorem 1.8.** We now estimate the sum over nontrivial zeros in Lemma 6.1, as described above, we may assume that each nontrivial zero is unexceptional. We write

(6.2) $\sum_{\rho_j = \beta_j + i\gamma_j \atop 0 \leq \beta_j < 1} K(\rho_j - 1)^2 = \sum_{\rho_j = \beta_j + i\gamma_j \atop \frac{1}{2} \leq \beta_j \leq 1 - \epsilon_0 \mathcal{L}_j^{-1}} K(\rho_j - 1)^2 + \sum_{\rho = \beta_j + i\gamma_j \atop 0 \leq \beta_j < \frac{1}{2}} K(\rho_j - 1)^2 + \sum_{\rho_j = \beta_j + i\gamma_j \atop |\gamma_j| > T} K(\rho_j - 1)^2,$

where we have $\mathcal{L}_j = \mathcal{L}$ to emphasize the dependence of $\mathcal{L}_j$ on the degree of $L(s, \text{Sym}^3\pi, K)$. (For the exact dependence of $\mathcal{L}_j$ on $j$, see Lemma 2.1.) Writing the three sums on the right
hand side of (6.2) as $S_1 + S_2 + S_3$, we use Corollary 1.2 and that $x = T^{c_{10}nK}$ to obtain

$$S_1 \ll \int_{\frac{1}{2}}^{1-c_{10}L_j^{-1}} x^{(A-2)(\sigma - 1)} dN_{\text{Sym}^\prime}(\sigma, T)$$

$$\ll x^{(A-2)(\sigma - 1)} N_{\text{Sym}^\prime}(\sigma, T) $$

$$\ll x^{(A-2)(\sigma - 1)} N_{\text{Sym}^\prime}(\sigma, T) \left| 1 - \frac{c_{10}}{L_j} \right| - (A - 2) \log x \int_{\frac{1}{2}}^{1 - \frac{c_{10}}{L_j}} x^{(A-2)(\sigma - 1)} N_{\text{Sym}^\prime}(\sigma, T) d\sigma$$

Finally, for the third sum, we have

$$S_3 \ll \sum_{\substack{\rho_j = \beta_j + i\gamma_j \\mid \gamma_j > T \\mid}} \frac{x^{-(A-2)(1-\beta)}}{|(\rho - 1)| \log x} \ll \frac{1}{(\log x)^2} \sum_{\substack{\rho_j = \beta_j + i\gamma_j \\mid |\gamma_j| > T \\mid}} \frac{1}{|\rho - 1|^2} \ll \frac{\mathcal{L}_j \log T}{T(\log x)^2}.$$

Collecting all of our estimates, it follows from our choice of $T$ that

$$S \gg b_0 \left( 1 - O\left( B \sum_{j=0}^{n} x^{-\frac{c_{10}(\beta_j)}{L_j}} \right) \right) \gg b_0 \left( 1 - O\left( B(n + 1)x^{-\frac{c_{10}(\beta_j)}{L_n}} \right) \right).$$

By our choice of $A$, if $c_{19}$ is made sufficiently large, then $S \geq \frac{b_0}{2}$, as desired. \hfill \Box

### Appendix A. Sym$^n$-minorants

We close with two easy lemmas on Sym$^n$-minorants. The first explicitly classifies the intervals which can be Sym$^4$-minorized, i.e. those intervals we can access unconditionally for any $L(s, \pi, K)$. The second concerns the asymptotics of the smallest $n$ needed to access the set of primes with $|\lambda_{\pi}(p)| > 2(1 - \delta)$ as $\delta \to 0$ and obtains an improvement over the naïve Fourier bound.

**Lemma A.1.** Let $\beta_0 = \frac{1+\sqrt{7}}{6} = 0.6076 \ldots$ and $\beta_1 = \frac{-1+\sqrt{7}}{6} = 0.2742 \ldots$. The interval $[a, b] \subseteq [-1, 1]$ can be Sym$^4$-minorized if and only if it satisfies one of the following conditions:

1. $a = -1$ and $b > -\beta_0$,
2. $-1 < a \leq -\beta_0$ and $b > \frac{a + \sqrt{16a^2 - 11a^2 + 2}}{2(1 - 4a^2)}$,
3. $-\beta_0 \leq a \leq -\beta_1$ and $b > \frac{1}{6a}$,
4. $-\beta_1 \leq a \leq \beta_1$ and $b > \frac{a + \sqrt{16a^2 - 11a^2 + 2}}{2(1 - 4a^2)}$,
5. $\beta_1 \leq a \leq \beta_0$ and $b = 1$. 


Proof. We begin with sufficiency. For each case, we list a polynomial $F(x)$ which, for $x \in [-1, 1]$, is positive only if $x \in [a, b]$. We then compute

$$b_0(F) := \int_{-1}^{1} F d\mu_{ST}$$

and verify that it is positive. This is sufficient, since any such $F(x)$ can be scaled to minorize the indicator function.

1. $F(x) = (x - 1)(x - b)(x - \beta_1)^2$ and $b_0(F) = (b + \beta_0)((14 + \sqrt{7})/36)$.
2. $F(x) = -(x - a)(x - b)(x + a + b)/(4ab + 1)/(4ab + 1)$ and $b_0(F) = -(3/(4ab + 1))$.
3. $F(x) = (x^2 - a^2)(x + \beta_1)^2$ and $b_0(F) = (\beta_0 - a)((14 + \sqrt{7})/36)$.

The proof of necessity necessarily involves tedious casework, which we omit. Let us say only that we consider polynomials $F(x)$, ordered by degree, the number of real roots, and the placement of those roots relative to $a, b, 1,$ and $-1$, and in each case we determine conditions under which $b_0(F) > 0$. □

**Lemma A.2.** For any $n \geq 1$, the set $[-1, -a] \cup [a, 1]$ can be $\text{Sym}^{2n}$-minorized if $a < \sqrt{1 - 3/2(n+1)}$.

Proof. We recall the well-known fact that

$$\int_{-1}^{1} x^{2m} d\mu_{ST} = \frac{1}{m+1} \binom{2m}{m} =: C_m.$$

Given $n$ and $a$ satisfying the conditions of the lemma, we use the minorant $f_{n,a}(x) = (x^2 - a^2)x^{2n-2}$, and we find that

$$\int_{-1}^{1} f_{n,a} d\mu_{ST} = \frac{C_{n-1}}{4^{n-1}} \left(1 - a^2 - 3/2 \frac{3/2}{n+1}\right).$$

□

**Remark.** The Sato-Tate measures of the sets considered in Lemma A.2 satisfy $\mu^{-1} \gg n^{3/2}$, so the minorants in the proof provide a significant improvement over those arising from a naive Fourier approximation.

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