A note on error analysis for a nonconforming discretisation of the tri-Helmholtz equation with singular data

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Abstract

We apply the nonconforming discretisation of [WX19] to the tri-Helmholtz equation on the plane where the source term is a functional evaluating the test function on a one-dimensional mesh-aligned embedded curve. We present error analysis for the convergence of the discretisation and linear convergence as a function of mesh size is recovered almost everywhere away from the embedded curve which aligns with classic regularity theory.

Elliptic PDE; Nonconforming method; Finite element method.

1 Introduction

We study the vector tri-Helmholtz equation on the unit square:

\[ Bu = f \quad \text{on} \quad \Omega := [0,1]^d, \]

where \( d = 2 \) subject to homogeneous Dirichlet conditions on \( \partial \Omega \), where \( B \) is the linear differential operator defined as follows by the \( L^2 \) inner product (denoted by \( \langle \cdot, \cdot \rangle_{0,\Omega} \)), for some \( b > 0 \) for smooth functions \( u \) and \( v \) and \( m = 3 \), where \( \text{id} \) represents the identity operator:

\[ \langle Bu, v \rangle_{0,\Omega} = \sum_{i=1}^{d} \int_{\Omega} \langle (\text{id} - b\Delta)^m u^i, v^i \rangle \, dx, \]
where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product. $f$ is a functional defined in the following way, where for the periodic domain $S^1 = [0, 2\pi]$, $g \in C^0(S^1, \Omega)$ is a continuous embedded planar curve:

$$v \mapsto f(v) := \int_{S^1} \tilde{f} \cdot v \circ g \, d\theta,$$

where $\tilde{f}$ is some function in $L^2(S^1)^d$. To simplify the analysis further we assert that the curve $g$ satisfies the following assumption:

$$\dist(\partial \Omega, \text{supp}(g)) =: \epsilon_g > 0.$$

In this paper we apply the nonconforming finite element discretisation of [WX19] to the components of $u$ in equation (1) and study the convergence properties of this scheme for source terms $f$ as defined in equation (3). We denote by $\Omega_h$ a shape-regular, quasi-uniform triangulation of the domain $\Omega$ [BS07, Definition 3.3.11] and we shall assume that the curve $g$ is aligned with its mesh skeleton.

The finite element method [BS07] is a ubiquitous method for discretising partial differential equations (PDEs) to which solutions are commonly sought after in piecewise polynomial spaces. Such finite element functions are identified by a set of basis coefficients and a polynomial basis. Determining what constitutes an adequate basis is a hugely important challenge. Generally speaking, the choice of the local degrees of freedom of a finite element affects the properties of the global approximation space. For example, when $X_h$ is constructed from linear polynomials i.e. for $K \in \Omega_h$, $X_h|_K = P^1(K)$ and in addition, functions in $X_h$ are linear interpolates of the nodal values at the vertices of $K$, then it is easy to see that $X_h \subset C^0(\Omega)$. A nonconforming linear space also exists where the local degrees of freedom are chosen at the midpoint of the edges of $\partial K$ see e.g. the construction in [BS07, Section 10.3]. This results in a locally continuous, globally discontinuous finite element and $X_h \not\subset X$. While the continuity of the former can be advantageous in light of Céa’s lemma [EG13, Lemma 2.28] for easy access to a convergence proof for the method, it is sometimes inconvenient or even computationally infeasible to design conforming finite element spaces e.g. when the continuous space $X$ is some higher-order Sobolev space. Generally speaking, for a $2m^{\text{th}}$-order PDE a conforming space must be a $C^{m-1}$-conforming subspace\footnote{The weak form equation would require $m$ weak derivatives which implies the existence of $m - 1$ continuous derivatives.}.
This poses a challenge for the design and implementation of finite element software. For instance, automation of code generation for $C^1(\Omega)$-conforming finite elements was only very recently solved \cite{Kir18, KM19}. Nonconforming FEMs are therefore attractive in developing numerical schemes for higher-order PDEs and - as we shall see - adequate convergence can be recovered in certain cases.

### 1.1 Nonconforming Finite Elements

For a triangulation $\Omega_h$ we define the mesh resolution, or mesh size, as $h = \max_{K \in \Omega_h} h_K$, where $h_K = \text{diam}(K)$, see \cite{BS07, Cia02}. As we will be studying convergence of finite element methods we introduce the notion of shape-regularity \cite[Definition 1.107]{EG13}. A family of meshes $\{\Omega_h\}_{h>0}$ is said to be shape-regular if there exists $c_0 > 0$ independent of $h$ such that

$$\Omega_h \frac{h_K}{\rho_K} \leq c_0, \quad \forall \Omega_h \in \{\Omega_h\}_{h>0}, \quad (5)$$

where $\rho_K$ is the diameter of the largest circle inscribed in $K$. We also assume that such families are quasi-uniform \cite[Definition 1.140]{EG13} i.e. there exists a constant $c > 0$ such that:

$$\forall h, \forall K \in \Omega_h, \quad h_K \geq ch.$$

This implies that $\epsilon_g$ from (4) satisfies $h < \epsilon_g$ i.e. the discretisation of the domain is so that the curve is at least $h$ away from the boundary. Given a triangulation $\Omega_h$ we define its interior facets (or edges, when $d=2$) $\mathcal{E}_h$ by $e \in \mathcal{E}_h$ if $\exists K, K' \in \Omega_h$ such that $e = K \cap K'$. We denote by $\partial \Omega_h$ the edges of $\partial K$, $K \in \Omega_h$ that trace $\partial \Omega$, and we say that $\mathcal{E}_h = \mathcal{E}_h \cup \partial \Omega_h$ is the mesh skeleton of $\Omega_h$. Sometimes we explicitly denote the dependence of the skeleton on its associated triangulation i.e. $\mathcal{E}_h(\Omega_h)$. We also use the notation $\int_{\mathcal{E}_h} = \sum_{e \in \mathcal{E}_h} \int_{e}$ with similar interpretations for $\int_{\partial \Omega_h}$ and $\int_{\mathcal{E}_h}$ and we have:

$$\int_{\mathcal{E}_h} = \int_{\mathcal{E}_h} + \int_{\partial \Omega_h}.$$

The space $C^k(O, \mathbb{R}^d)$ denotes the space of continuous functions over the domain $O$ with $k$ continuous derivatives taking value in $\mathbb{R}^d$. When the latter is omitted, the range is $\mathbb{R}$ e.g. $C^0(O)$ is the space of continuous functions over
\( O \) taking values in \( \mathbb{R} \). \( L^p(O) \), \( p \geq 1 \) and integer denotes the usual Lebesgue spaces over \( O \) with \( L^\infty(O) \) being the Banach space of essentially bounded functions on \( O \). For \( k \geq 0 \) (integer or fraction) we let \( W^{k,p}(O) \) define the usual Sobolev spaces, see [Eva10]. Note that \( W^{k,\infty}(O) \) is the Banach space of scalar \( k \)-Lipschitz continuous functions over the convex polygonal Lipschitz domain \( O \) taking values in \( \mathbb{R} \). For \( k \geq 0 \) (integer or fraction) we let \( W^{k,\infty}(O) \) define the usual Sobolev spaces, see [Eva10]. Note that \( W^{k,\infty}(O) \) is the Banach space of scalar \( k \)-Lipschitz continuous functions over the convex polygonal Lipschitz domain \( O \) taking values in \( \mathbb{R} \). We will be dealing with vector and tensor-valued functions as well, and therefore when necessary use superscripts e.g. \( L^\infty(O)^d \) to denote vector-valued functions \( \mathbb{R}^d \to \mathbb{R}^d \), or \( W^{k,\infty}(O)^{d \times d} \) denotes the space of tensor-valued \( k \)-Lipschitz functions on \( O \). This should be understood in the sense that each component considered as a scalar function satisfies the regularity of the given space. We use standard multi-index notation i.e. a \( d \)-dimensional multi-index \( \alpha \), with \( |\alpha| = \sum_{i=1}^{d} \alpha_i \), and we define \( \partial^\alpha \), the partial derivative with respect to \( \alpha \) as \( \partial^\alpha = \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \), where \( \partial^\alpha_i := \partial^{\alpha_i} / \partial x_i^{\alpha_i} \). In this paper, \( dx \) - sometimes also \( \hat{dx} \) - denotes the \( d \)-dimensional Lebesgue measure [BS07 Section 1.1], where \( d \) will be clear from context.

We make a clear distinction in our notation depending on the value of \( p \). This is reflected in the following norms when \( p = 2 \) or \( p = \infty \), respectively:

\[
\| f \|^2_{k,O} = \sum_{|\alpha| \leq k} \| \partial^\alpha f \|^2_{0,O},
\]

\[
\| f \|_{W^{k,\infty}(O)} = \sum_{|\alpha| \leq k} \| \partial^\alpha f \|_{\infty,O},
\]

where \( \| \partial^\alpha f \|^2_{0,O} = \int_O |\partial^\alpha f|^2 \, dx \) is the usual \( L^2 \) norm over \( O \) and \( \| \partial^\alpha f \|_{\infty,O} = \text{ess sup}_O |\partial^\alpha f| \) is the essential supremum norm. Note in particular that due to the presence of the Euclidean norm \( | \cdot | \) here, the norms in (6) are well-defined for scalar, vector and tensor-valued objects. When \( p = 2 \) we also use the notation \( H^k(O) := W^{k,2}(O) \) with \( H^0_0(O) \) consisting of such functions that vanish on \( \partial O \) in sense of traces. We also denote by \( | \cdot |_{k,O} \) the usual semi-norm over this space. The inner product on \( H^k(O) \) is denoted by \( \langle \cdot, \cdot \rangle_{k,O} \).

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2 We remark that the Lipschitz class of functions does not always coincide with \( W^{k,\infty}(O) \) depending on the nature of \( O \). Given that we are always dealing with convex domains we characterise the Lipschitz class with the Sobolev spaces \( W^{k,\infty}(O) \) (see [Hei05 Theorem 4.1]) and mention this no further.
Definition 1 (Local interpolant [BS07, Lemma 4.4.1 and Theorem 4.4.4]).

For a convex simplex $K \subset \mathbb{R}^d$ and nonnegative integers $l$ and $m$ we define the usual local interpolant $I_K : \mathcal{C}^l(\bar{K}) \to H^m(K)$ satisfying the following estimate:

$$\|I_K u\|_{m,K} \lesssim \|u\|_{\mathcal{C}^l(\bar{K})}.$$  

When $m - l - d/2 > 0$ we have the following bound:

$$|u - I_K u|_{i,K} \lesssim h_{i,K}^{m-i}|u|_{m,K}, \quad 0 \leq i \leq m,$$

with a constant that depends on $m$, $d$ and $K$.

This extends trivially to vector or tensor-valued objects. We can therefore, under the right circumstances, approximate smooth functions on $K$ by a polynomial. For nonsmooth functions one must seek other interpolants since the nodal basis may not support point evaluation i.e. when the space $\mathcal{C}^l(\bar{K})$ in definition 1 is replaced by a weaker space like $L^2(\bar{K})$. Local averages are often taken as surrogates for the nodal basis in this case, see e.g. the Clément [Clé75], or for interpolation operator satisfying boundary conditions see Scott-Zhang [SZ90]. For more information on this see the discussion in [BS07, Section 4.8], and [DS80] for general polynomial approximation of Sobolev spaces. A central task in finite element theory is the construction of the nodal basis and the basis functions to cater for convergent approximations of some infinite-dimensional problem. We shall therefore leave these undefined here and return to their construction when necessary in the coming chapters.

From definition 1 we can define a global interpolant $I_h : X \to X_h$ by:

$$I_h v|_K = I_K v, \quad K \in O_h,$$

where $O_h$ is a triangulation of $O \subset \mathbb{R}^d$ over which the spaces $X$ and $X_h$ are defined. Global interpolation estimates require the notion of a broken Hilbert-Sobolev norm given by the following definitions:

$$\|u\|_{k,O_h}^2 := \sum_{K \in O_h} \|u\|_{k,K}^2,$$

with the associated broken inner product:

$$\langle u, v \rangle_{k,O_h} := \sum_{K \in O_h} \langle u, v \rangle_{k,K}.$$  

To study finite element methods we introduce the notion of approximability:
Lemma 2 (Approximability). Suppose $X = H^m(O)$ where $O$ is a Lipschitz domain in $\mathbb{R}^d$ and let $I_h$ be the global interpolant from (7). Then for $u \in H^{m+s}(O)$:

$$\|u - I_h u\|_{i, O_h} \leq h^{m+s-i} |u|_{m+s, O_h}, \quad 0 \leq i \leq m,$$

(9a)

$$\lim_{h \to 0} \inf_{v_h \in X_h} \|v - v_h\|_{X_h} = 0, \quad \forall v \in X,$$

(9b)

and we sometimes say $X_h$ approximates $X$.

Proof. (9a) follows from the definition of $I_h$ and definition 1. (9b) follows from continuity of the interpolant and a standard density argument; see e.g. the proof of [WX13, Theorem 2.1].

Convergence for nonconforming methods are usually established in norms such as (8) owing to their local nature. Approximation and consistency conditions for nonconforming finite elements can now be formulated using more suitable vocabulary, see [Stu79] or [WX13, Section 3.1] for the following definition.

Definition 3 (Consistent approximation). A finite element space $X_h$ over a bounded Lipschitz domain $\Omega$ is said to be a consistent approximation of $X = H^m(\Omega)$, $m \geq 1$, if and only if:

1. $\lim_{h \to 0} \inf_{v_h \in X_h} \|v - v_h\|_{X_h} = 0, \quad \forall v \in X$ (approximation).

2. For any sequence $\{v_h\}_{h>0}$ with $v_h \in X_h$ and $h \to 0$ such that $\{\partial^\alpha v_h\}_{h>0}$ is $L^2(\Omega)$-weakly convergent to some $v^\alpha \in X$ for all $|\alpha| \leq m$, it holds that $v_0 \in X$ and $v^\alpha = \partial^\alpha v_0$ for all $|\alpha| \leq m$ (consistency).

The first condition can be derived from (9a) of lemma 2 via a canonical interpolant. The second condition can be interpreted as a kind of weak compactness condition. In practice we employ so-called patch tests which are sufficient conditions for a consistency condition but motivated by the fact that they are much simpler to prove and sometimes even verify by numerical computation. These often amount to verifying whether integral moments of derivatives vanish on the mesh skeleton - a property called a weak continuity. Historically the first attempt at providing an automated solution to this issue was in engineering by Irons et al. [Baz65] in 1965 but proved to be neither necessary nor sufficient by Stummel [Stu80]. The generalised patch
test (GPT) of Stummel \cite{Stu79} states necessary and sufficient conditions leading to convergence of nonconforming elements. Generally speaking, the design of a finite element leads to some weak continuity of which the GPT does not make use in its original form. However, using the form of the GPT as stated in \cite[Equation 4.4]{Wan01}, \( X_h \) passes the GPT if and only if the following condition is satisfied:

\[
\lim_{h \to 0} \sup_{\|v_h\|_{X_h} \leq 1} |T_{\alpha,i} (\psi, v_h)| = 0, \quad |\alpha| < m, \quad 1 \leq i \leq d, \quad \forall \psi \in C^\infty(\bar{\Omega}),
\]

(10)

where:

\[
T_{\alpha,i} (\psi, v_h) = \sum_{K \in \Omega_h} \int_K \left[ \psi \frac{\partial}{\partial x_i} \partial^\alpha v_h + \frac{\partial \psi}{\partial x_i} \partial^\alpha v_h \right] \, dx.
\]

Equivalently, where \( \eta_K \) is the outward normal of \( K \in \Omega_h \):

\[
T_{\alpha,i} (\psi, v_h) = \sum_{K \in \Omega_h} \int_{\partial K} \psi \partial^\alpha v_h \eta_K \, ds.
\]

(11)

Alternatives were also proposed, see for instance F-E-M test \cite{Shi87} and the IPT test \cite{HM86}. A major step forward was taken in \cite{Wan01} where a conjecture was proved saying that given suitable approximation properties (in spirit of lemma 2) and weak continuity, a weak patch test (WPT) can be designed which is extremely simple and convenient. We refer the reader to \cite{Shi02} for an excellent overview of the nonconforming FEM literature for second and fourth order PDEs.

As mentioned, we shall use a finite element space from \cite{WX19} to discretise a weak formulation of (1) and we briefly summarise its construction. First we recall some notation. For a triangle \( K \), let \( P_i(K) \) denote the space of \( i \)th order polynomials on \( K \). Let \( q_K \) denote the bubble function on \( K \) which is a nonnegative cubic polynomial that is zero on \( \partial K \); see e.g. \cite{EG13} for more details. In the following, \( \frac{\partial}{\partial \nu_e} \) denotes a derivative in the direction of the outward normal \( \nu_e \) on an edge \( e \) in the mesh skeleton of a triangulation. We define the following shape function space on a triangle \( K \in \Omega_h \):

\[
\tilde{P}_K^{(3,2)} := P_3(K) + q_K P_1(K) + q_K^2 P_1(K).
\]
By [WX19, Lemma 4.1], the following degrees of freedom determine a function \( v \in \tilde{P}_K^{(3,2)} \):

\[
\begin{align*}
\frac{1}{|e|} \int_e \frac{\partial^2 v}{\partial y^2} \, ds & \quad \text{\( e \) is an edge of \( \partial K \),} \\
\nabla v(a) & \quad \text{\( a \) is a vertex of \( \partial K \),} \\
\frac{1}{|e|} \int_e \frac{\partial v}{\partial y_e} \, ds & \quad \text{\( e \) is an edge of \( \partial K \),} \\
v(a) & \quad \text{\( a \) is a vertex of \( \partial K \).}
\end{align*}
\]

We now define the scalar finite element space \( \tilde{V}_h \) defined over \( \Omega_h \) as consisting of all functions \( v_h|_K \in \tilde{P}_K^{(3,2)}, K \in \Omega_h \), with the degrees of freedom above being equal to zero if either the edge or vertex lies on the boundary of \( \Omega_h \) (see also [WX19, Equation 4.8] for the definition of \( \tilde{V}_h \)): This element is locally to \( K \) a seventh order polynomial. It is designed in such a way that makes it possible to describe convergent \( H^3 \)-nonconforming FEM. Our vector-valued finite element space \( \tilde{V}_d^h \) is then the space whose components occupy the scalar finite element space \( \tilde{V}_h \). As shown in the following corollary, this is a continuous finite element space.

**Corollary 4.** \( \tilde{V}_h \hookrightarrow C^0(\Omega) \).

**Proof.** Let \( K, K' \) be two elements of \( \Omega_h \) that share an edge \( e \). As shown in [WX19], there are 8 degrees of freedom on this edge. Let \( v \) and \( v' \) be the restriction of a function \( w \in \tilde{V}_h \) to \( \partial K \) and \( \partial K' \), where these functions occupy \( P^3(\partial K) \) and \( P^3(\partial K') \), respectively. Values and derivatives at the vertices are continuous in \( \tilde{V}_h \) in the sense that they are unique, so \( v \) and \( v' \) are order 3 polynomials that agree on 4 values which is only possible if \( v = v' \) everywhere on \( e \). Since \( e \) was arbitrary, \( \tilde{V}_h \hookrightarrow C^0(\Omega) \). \( \square \)

Although the finite element space above is continuous, this does not extend to higher-order derivatives. Nevertheless, the following central result highlights its weak continuity properties:

**Lemma 5** (Weak continuity [WX19, Lemma 4.2]). Let \( \Omega_h \) denote the mesh and \( e \in \mathcal{E}_h \) be an interior edge of a triangle \( K \in \Omega_h \). For any component \( u^i, i = 1, 2 \) of \( u \in \tilde{V}_d^h \) and \( K' \in \Omega_h \) such that \( e \cap K' \neq \emptyset \),

\[
\int_e \partial^\alpha u^i|_K \, ds = \int_e \partial^\alpha u^i|_{K'} \, ds, \quad |\alpha| = 1, 2.
\]
When $e \in \partial \Omega_h$,
\[ \int_e \partial^\alpha u^i|_K \, ds = 0. \]

**Remark 6.** This importance of this result cannot be understated. Weak continuity is the exact reason why we will be able to talk about convergence for nonconforming methods. A naive piecewise $H^3$ finite element space constructed from piecewise cubic polynomials functions $\mathcal{P}^3(K)$, $K \in \Omega_h$ does not lead to the same results.

Recalling $m = 3$, we equip $\tilde{V}_h^d$ with the broken $H^m$ norm $\| \cdot \|_{m, \Omega_h}$ dominating all derivatives up to third order. We shall also define a matrix-valued finite element space $\tilde{V}_h^{d \times d}$ whose components occupy $\tilde{V}_h$, see the aforementioned reference.

We observe the following standard result.

**Theorem 7.** Let $O$ be a convex bounded Lipschitz domain in $\mathbb{R}^d$ with polygonal boundary and $O_h$ a triangulation thereof satisfying the regularity requirements introduced above. Suppose further that $u \in C(\bar{O})^d$, $u|_K \in H^3(K)^d$ for $K \in O_h$. Then $u \in W^{1, \infty}(O)^d$.

**Proof.** The embedding theorem for homogeneous Sobolev spaces (i.e. with zero traces) into the space $C^j(\bar{O})$ is well-known. However, since the trace $u|_K$ of $u$ on $\partial K$, $K \in O_h$ may not be zero we appeal to a slightly different albeit standard result. By [Ada75, Theorem 5.4], however, $H^m(K) \hookrightarrow C^1_B(K)$, where:

\[ C^1_B(K) = \{ u \in C^1(K) \mid D^\alpha u \text{ is bounded on } K, |\alpha| \leq 1 \}. \]

This means any $H^m(K)$ function has a continuous representative with almost everywhere bounded first derivatives on $K$. Since $u \in C^0(\bar{O})$, $u$ is a continuous function with its first derivative a.e. bounded, implying a Lipschitz condition.

\[ \Box \]

## 2 Tri-Helmholtz equation with singular data

We define the bilinear form:

\[ a(u, v) = \sum_{i=1}^{d} \int_{\Omega} \sum_{j=0}^{m} b_j \left( \binom{m}{j} \langle D^j u^i, D^j v^i \rangle \right) dx, \quad (13) \]
where $D^0 = \text{id}$, and

$$D^j = \begin{cases} \nabla D^{j-1} & j \text{ is odd,} \\ \nabla \cdot D^{j-1} & j \text{ is even,} \end{cases}$$

and $b_j = b^j$. We then see that by integrating by parts in (1) we have the equivalence:

$$a(u, v) = \langle Bu, v \rangle_{0, \Omega},$$

for sufficiently smooth functions $u$ and $v$ with vanishing $j^{th}$ order derivatives on $\partial \Omega$, for $j = 0, \ldots, m - 1$. In the rest of this paper the embedding $g$ will be a continuous piecewise linear map from $S^1$ into the plane which is aligned with the mesh skeleton similar to what is shown in figure 1. Equipped with (13) we can pose the following weak version of (1) where we write the 6th order problem into a mixed system of two third order problems:
Problem 1 (Weak tri-Helmholtz)
Find \( u \in H^m_0(\Omega)^d \) such that:
\[
a(u, v) = \int_{S^1} \tilde{f} \cdot v \circ g \, d\theta, \quad \forall v \in H^3(\Omega)^d,
\]
(14)
where \( \tilde{f} \in L^2(S^1)^d \).

Remark 8 (On global regularity). Recall that \( d = 2 \) and that \( g \in C^0(S^1)^d \) forms an embedded curve \( G = g \circ S^1 \) in a sufficiently large bounded domain \( \Omega \) (away from the boundary \( \partial \Omega \)), and let \( \Omega^-, \Omega^+ \) denote the inside and outside of the curve. Suppose further that \( g \) is such that we can define trace operators
\[
\gamma_G^- : \Omega^- \to H^{1/2}(G) \quad \text{and} \quad \gamma_G^+ : \Omega^+ \to H^{1/2}(G).
\]
A functional of the form:
\[
H^1(\Omega) \ni w \mapsto \int_{S^1} \gamma_G^-[w] \circ g \, d\theta,
\]
occupies \( H^{-1}(\Omega) \), and, a fortiori, in \( H^{-3}(\Omega) \). It is of course equivalent to use the trace \( \gamma_G^+ \) in the example above. As such, these cannot be represented as inner products in \( L^2(\Omega) \). The \( H^{-1}(\Omega) \) regularity prohibits global higher-order elliptic regularity beyond \( H^3(\Omega) \) of the solution \( w \) to (14). We refer to [Gri92, Kon67] for a priori regularity estimates for general elliptic equations.

We write the broken analogue \( a_h \) of (13):
\[
a_h(u, v) = \sum_{i=1}^d \sum_{K \in \Omega_h} \int_K \sum_{j=0}^m b_j \binom{m}{j} \langle D^j u^i, D^j v^i \rangle \, dx,
\]
(15)
This defines the following energy norm as well:
\[
\| \cdot \|^2_{m, \Omega_h} = a_h(\cdot, \cdot),
\]
and we observe the following norm equivalences based on boundedness of \( b \):
\[
\sqrt{a_h(\cdot, \cdot)} \simeq \| \cdot \|_{m, \Omega_h}, \quad \sqrt{a(\cdot, \cdot)} \simeq \| \cdot \|_{m, \Omega}.
\]
Problem 2 (Discrete Weak tri-Helmholtz)

Find $u_h \in \tilde{V}_h^d$ such that:

$$a(u_h, v_h) = \int_{S^1} \tilde{f} \cdot v_h \circ g \, d\theta, \quad \forall v_h \in \tilde{V}_h^d \quad (16)$$

where $\tilde{f} \in L^2(S^1)^d$.

Corollary 9. Provided the right-hand sides of the problems (14) and (16) are well-defined, there exist unique solutions $u \in H^3(\Omega)^d$ and $u_h \in \tilde{V}_h^d$ to these equations, respectively.

Proof. The proof is trivial since the continuous bilinear forms $a$ and $a_h$ control all $j^{th}$ order terms for $j = 0, \ldots, 3$ as is therefore naturally coercive on their respective spaces.

We remark that a Poincaré-type lemma holds for functions in $\tilde{V}_h$:

Lemma 10 (Poincaré [WX19, Lemma 4.3]). It holds that:

$$\|v_h\|_{m,\Omega_h} \lesssim \|v_h\|_{0,\Omega_h} + |v_h|_{m,\Omega_h}, \quad \forall v_h \in \tilde{V}_h. \quad (17)$$

Using (17) for each component $i$ of some $v_h \in \tilde{V}_h^d$ we obtain:

$$\|v_{h, i}\|_{m,\Omega_h}^2 \lesssim \|v_{h, i}\|_{0,\Omega}^2 + |v_{h, i}|_{m,\Omega_h}^2,$$

so summing over the components $i$ we get:

$$\|v_h\|_{m,\Omega_h}^2 \lesssim \langle v_h, v_h \rangle_{0,\Omega_h} + \langle v_h, v_h \rangle_{m,\Omega_h} \quad \forall v_h \in \tilde{V}_h^d.$$ 

and as such the results and analysis we present here are equally valid had we selected e.g. $B = \text{id} - \Delta^m$.

We introduce some additional notation:

- $\Omega_h^+$ (resp. $\Omega_h^-$) denotes the collection of $K \in \Omega_h$ lying on the outside (resp. inside) of the curve $g \circ S^1$.

- We let $\mathcal{K}^+$ (resp. $\mathcal{K}^-$) denote the collection of $K \in \Omega_h^+$ (resp. $K \in \Omega_h^-$) such that $K \cap g \circ S^1 \neq \emptyset$. 

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• For $e \in \mathcal{E}_h$, $\mathcal{K}_e$ denotes the collection of $K \in \Omega_h$ such that $K \cap e \neq \emptyset$.

• For any piecewise affine function $g \in C^0(S^1)^d$ we define the finite index set $\mathcal{J}_g$ such that for each $i \in \mathcal{J}_g$, $g$ can be written as a linear function on some connected $S^1_i$ such that $S^1_i \subset S^1$ and $\bigcup_{i \in \mathcal{J}_g} S^1_i = S^1$ and for any $i, j \in \mathcal{J}_g$ we have $S^1_i \cap S^1_j = \emptyset$ unless $i = j$.

• For an edge $e$ on $\partial \Omega_h$ between two triangles $K^-$ and $K^+$ we let the jump $[\cdot]$ and average $\{\cdot\}$ of a function across $e$ be defined by:

$$[v] = \begin{cases} v^+\eta^+ + v^-\eta^- & \text{if } v \text{ is scalar,} \\ v^+ \cdot \eta^+ + v^- \cdot \eta^- & \text{if } v \text{ is vector-valued,} \end{cases}$$

$$\{v\} = \frac{1}{2}(v^+ + v^-),$$

where $v^+$ and $v^-$ are the restrictions of $v$ to $K^-$ and $K^+$, respectively. On an edge $e \in \partial \Omega_h$, $[v] = \eta v$ and $\{v\} = v$ where $\eta$ is the outward normal.

We recall the Strang lemma (Str72) lemma (see also [BBF+13, Section 2.2.3]). In the present context we can state it as follows:

**Lemma 11** (Strang). Let $u$ and $u_h$ be the solutions of (14) and (16), respectively. Then,

$$\|u_h - u\|_{m,\Omega_h} \lesssim \inf_{v_h \in \tilde{V}_h^d} \|u - v_h\|_{m,\Omega_h} + \sup_{v_h \in \tilde{V}_h^d} \left| \int_{S^1} \tilde{f} \cdot v_h \circ g \, d\theta - a_h(u, v_h) \right| \|v_h\|_{m,\Omega_h},$$

where the first term on the right-hand side is called the approximation term and the second a consistency term.

**Proof.** Let $v_h, w_h \in \tilde{V}_h^d$.

$$a_h(u_h - w_h, v_h) = a_h(u_h - u, v_h) + a_h(u - w_h, v_h) = \int_{S^1} \tilde{f} \cdot v_h \circ g \, d\theta - a_h(u, v_h) + a_h(u - w_h, v_h).$$
Dividing by $\|v_h\|_{m,\Omega_h}$ and using coercivity and continuity of $a_h$ we can therefore claim:

$$\|u_h - w_h\|_{m,\Omega_h} \lesssim \sup_{v_h \in \tilde{V}_h^d} \int_{S_1} \tilde{f} \cdot v_h \circ g \, d\theta - a_h(u, v_h) \|v_h\|_{m,\Omega_h} + \|u - w_h\|_{m,\Omega_h},$$

which together with the triangle inequality yields the result.

We say that a discretisation for which the terms on the right-hand side of (18) vanish as $h \to 0$ is an approximate and consistent discretisation. Approximation is guaranteed via classic element-wise interpolation estimates - we briefly state the main results from [WX19] adapted to the current setting.

**Definition 12** (Canonical interpolation operator). For a given $K \in \Omega_h$ let us denote by $\Pi_K : H^m(K) \to \tilde{P}_K^{(3,2)}$, the canonical interpolation operator where $\tilde{P}_K^{(3,2)} \subset P^7(K)$ is a seventh order polynomial space defined in [WX19, Equation 4.1]. Section 4 of [WX19] describes this space in great detail. A technical result [WX19, Lemma 4.1] determines the local degrees of freedom that show unisolvency of $\tilde{P}_K^{(3,2)}$ i.e. those that uniquely determine any $v \in \tilde{P}_K^{(3,2)}$. This unisolvency property allows us to sketch the definition of $\Pi_K$:

$$\Pi_K v = \sum_{j=1}^{m} l_{K,j} n_{K,j}(v), \quad \forall v \in H^m(K),$$

where $\{l_{K,j}\}_{j=1}^{m}$ is a local basis and $\{n_{K,j}\}_{j=1}^{m}$ (for some positive integer $m$) is a nodal basis satisfying $n_{K,i}(l_{K,k}) = \delta_{ik}$, where $\delta_{ik}$ is the Kronecker delta. See section 2.3 of the aforementioned reference for an example of a constructing of such a local interpolant.

By standard interpolation theory ([BS07], [WX19, Lemma 2.4]) we have:

**Lemma 13.** For $s \in [0,1]$ and $k$ such that $0 \leq k \leq m$ and $K \in \Omega_h$:

$$|v - \Pi_K|_{k,K} \lesssim h_{K}^{m+s-k}|v|_{m+s,K}, \quad \forall v \in H^{m+s}(K).$$

The global operator $\pi_h$ on $H^m(\Omega)^d$ into $\tilde{V}_h^d$ is defined by $(\pi_h v)|_K = \Pi_h|_K(v)|_K$ for $K \in \Omega_h$ and satisfies the following standard result:

**Theorem 14.** For $v \in H^{m+s}(\Omega)$, $s \geq 0$ we have:

$$\|v - \pi_h v\|_{m,\Omega_h} \leq h^s |v|_{m+s,\Omega},$$

$$\lim_{h \to 0} \|v - \pi_h v\|_{m,\Omega_h} = 0.$$
Proof. The first property is obvious from Lemma 13. For the second, see [WX19, Theorem 2.5].

This states that globally $m+s$ weakly differentiable functions can be well-approximated in the norm $\| \cdot \|_{m,\Omega_h}$. However, by Remark 8, global higher-order regularity of solutions to (14) is not possible. Local regularity may, however, be recovered by the following lemma.

Lemma 15 (Hypoellipticity). Let $u$ be the solution to (14) and define by

$\Omega^-, \Omega^+$, (resp. $\Omega^-_h, \Omega^+_h$) the connected open sets bounded by $g \circ S^1$ and $\partial \Omega$ (resp. $\partial \Omega_h$ and $g \circ S^1$) on the inside and outside of the curve. Then the restriction of $u$ to the sets $\Omega^-, \Omega^+$ is $C^\infty$ in each component.

Proof. We recall that a differential operator $D$ is hypoelliptic [Bre10] if for any open set $\omega$, $C^\infty(\omega) \ni Dz \Rightarrow z \in C^\infty(\omega)$ [Fol95, Theorem 6.33]. Moreover, any constant coefficient elliptic operator is in fact hypoelliptic [Fol95, Corollary 6.34]. We observe that the operator $B$ associated to the bilinear form $a(\cdot, \cdot)$ is hypoelliptic on open sets, and so in fact, for any open set $\omega \subset \Omega$ such that $\bar{\omega} \subset \Omega$ is compact and such that $\omega \cap g \circ S^1 = \emptyset$, $Bu|_{\omega} \in C^\infty(\omega)$. In the case of the curve $g \circ S^1$ it is easy to see that for any $K \in \Omega_h$ we can solve an elliptic problem with smooth data on the interior and recover the local estimate. The key property here is convexity of $K$, see e.g. [Gri11, Chapter 4].

From Lemma 15 we can state the following nonconforming estimate using Theorem 14 applied in an element-wise manner.

Corollary 16. Let $v \in H^m(\Omega)$ be such that $v|_K \in H^{m+s}(K)^d$, $\forall K \in \Omega_h$, with $m = d + 1$, $s \geq 0$. Then:

$$\|v - \Pi_h v\|_{m,\Omega_h} \leq h^s \|v|_{m+s,\Omega_h},$$

$$\lim_{h \to 0} \|v - \Pi_h v\|_{m,\Omega_h} = 0.$$
estimating to what extent, as a function of $h$, the continuous solution $(14)$ satisfies the equation $(16)$. This property can be verified by e.g. the generalised patch test (GPT) of Stummel [Stu79] by means of the compactness condition given in definition 3 via the result in [WX13].

The proof of consistency of $\tilde{V}_h$ was omitted in [WX19] so for completeness we apply an argument similar to [WX13, Theorem 3.2] using the weak continuity lemma 5 to show this. This property states that integrals of derivatives of functions in $\tilde{V}_h^d$ are smooth across interior edges of a triangulation, rather than smoothness in a pointwise sense.

Remark 17 (Bibliographic note). Weak continuity is indeed sufficient (but not necessary) for the so-called “SPT” condition mentioned in Theorem 3.2 of [WX13]. The following theorems are not novel and are simply included to benefit any unfamiliar readers with the literature.

**Theorem 18.** $\tilde{V}_h^d$ is a consistent approximation of $H^3(\Omega)^d$.

**Proof.** We easily verify coercivity and continuity of $a$ and $a_h$ by corollary[9] required by the GPT [Stu79. Section 1.4, Equation 7]. Using the form of the test as stated in [Wan01, Equation 4.4], $\tilde{V}_h$ passes the GPT if and only if the following condition stated is satisfied:

\[
\lim_{h \to 0} \sup_{v_h \in \tilde{V}_h, \|v_h\|_{m, \Omega_h} \leq 1} |T_{\alpha, i} (\psi, v_h)| = 0, \quad |\alpha| < m, \quad 1 \leq i \leq d, \quad \forall \psi \in C^{\infty}(\bar{\Omega}),
\]

where:

\[
T_{\alpha, i} (\psi, v_h) = \sum_{K \in \Omega_h} \int_K \left[ \psi \frac{\partial^{\alpha} v_h}{\partial x_i} + \frac{\partial \psi}{\partial x_i} \partial^{\alpha} v_h \right] \, dx.
\]

Equivalently, where $\eta_K$ is the outward normal of $K$:

\[
T_{\alpha, i} (\psi, v_h) = \sum_{K \in \Omega_h} \int_{\partial K} \psi \partial^{\alpha} v_h \eta_K^i \, ds.
\]

For an edge $e \in \hat{E}_h$ between $K, K' \in \Omega_h$ and let $\eta_K$ (resp. $\eta_{K'}$) denote the
outward normal to $K$ (resp. $K'$). We can then write (20) as:

$$T_{\alpha,i}(\psi, v_h) = \sum_{e \in \hat{E}_h} \int_e \psi \partial^\alpha v_h|_K \eta^i_K + \psi \partial^\alpha v_h|_{K'} \eta^i_{K'} \, ds + \sum_{e \in \partial \Omega_h} \int_e \psi \partial^\alpha v_h \eta^i_{\partial \Omega_h} \, ds$$

$$= \sum_{e \in \hat{E}_h} \int_e \psi \left[ \partial^\alpha v_h|_K - \partial^\alpha v_h|_{K'} \right] \eta^i_{K'} \, ds + \sum_{e \in \partial \Omega_h} \int_e \psi \partial^\alpha v_h \eta^i_{\partial \Omega_h} \, ds$$

$$\leq \sum_{e \in \hat{E}_h} \|\psi\|_{L^\infty(e)} \int_e \left[ \partial^\alpha v_h|_K - \partial^\alpha v_h|_{K'} \right] \, ds$$

$$+ \sum_{e \in \partial \Omega_h} \|\psi\|_{L^\infty(e)} \int_e \partial^\alpha v_h \, ds$$

$$= 0.$$

Here we have used lemma \( \text{[5]} \) stating that facet integral moments of first and second order match between cells, so the consistency condition is satisfied so $\tilde{V}_h$ passes the GPT, which implies the same result for $\tilde{V}^d_h$ which is therefore a consistent approximation of $H^3(\Omega)^d$.

We use the consistent approximation properties of $\tilde{V}^d_h$ to show a convergence result for the discretisation in problem \( \text{[2]} \). First we need to prove a technical lemma:

**Lemma 19.** Let $v \in H^3(\Omega)^d$ and the sequence $\{v_h\}_{h>0}$ in $\tilde{V}^d_h$ be such that

$$\lim_{h \to 0} \|v - v_h\|_{m, \Omega_h} = 0.$$ 

Then, for any $g \in C^0(S^1)^d$ such that $g \circ S^1$ is comprised of edges $e \in \hat{E}_h$ of a shape-regular, quasi-uniform mesh $\Omega_h$, $(v_h - v) \circ g$ converges a.e. pointwise to zero on $S^1$ (with respect to the 1-dimensional Lebesgue measure).

**Proof.** We have:

$$\text{ess sup}_{\theta \in S^1} |(v_h - v) \circ g|(|\theta|) = \text{ess sup}_{x \in g \circ S^1} |v_h - v|(x)$$

$$= \max_{i \in J} \text{ess sup}_{x \in g \circ S^1_i} |v_h - v|(x).$$

Since $v_h - v$ is continuous everywhere, $v_h - v = \{v_h - v\}$ on edges of the interior mesh skeleton $\hat{E}_h$. Therefore, for any two elements $K^+_i$ and $K^-_i$
joining an edge \( g \circ S^1_i \) we can say:

\[
\max_{i \in J} \frac{1}{\text{ess sup}} \left\{ |v_h(x) - v(x)| \right\} \leq \max_{i \in J} \left[ \text{ess sup} \left\{ |v_h(x)|_{K^+} - |v(x)| \right\} \right],
\]

By the Sobolev embedding theorem in one dimension we have for any bounded open set \( U : f \in H^1(U) \Rightarrow f \in C^0(U) \), so:

\[
\frac{1}{\text{ess sup}} \left\{ |v_h(x)|_{K^+} - v \circ g(\theta) \right\} \leq \max_{i \in J} \|v_h(x)|_{K^+} - v\|_{1, g \circ S^1_i}.
\]

By the trace theorem we have for any \( i \in J \):

\[
\|v_h(x)|_{K^+} - v\|_{1, g \circ S^1_i} \lesssim \|v_h - v\|_{2, \Omega_h}.
\]

Since \( \|v_h - v\|_{2, \Omega_h} \) converges by assumption we conclude the proof by applying the same argument to the restriction of \( v_h \) to \( K^+ \).

Another way to see this result is to recall that \( \tilde{V}_h^d|_K \subset L^\infty(K), K \in \Omega_h \). Since \( K \) is convex, \([\text{Ste70}, \text{Theorem 5}]\) tells us that an extension result exists and we can therefore ascribe meaning to the values of a function in \( \tilde{V}_h^d|_K \) on \( \partial K \).

**Corollary 20.** Let \( u \) and \( u_h \) solve (14) and (16). Then we have:

\[
\lim_{h \to 0} \|u - u_h\|_{m, \Omega_h} = 0.
\]

**Proof.** Essential to our proof is the fact that the curve \( J_g \) is independent of the mesh resolution. The reader should think about this corollary as a convergence result if we refine the mesh around the embedded curve \( g \).

We know that \( \|u_h\|_{m, \Omega_h} \lesssim \|f\|_{0, S^1} \) since \( \tilde{V}_h^d \) is Lipschitz-conforming cf. theorem \( \text{[7]} \) so \( \{u_h\}_{h \geq 0} \) is a bounded sequence. Using the definition of consistency in \( \text{[Stu79, WX13]} \) we say that \( \tilde{V}_h^d \) is a consistent approximation of \( H^3(\Omega)^d \) by theorem \( \text{[18]} \) and \( \text{[WX13, Section 3.1]} \) so we can extract a subsequence \( \{u_{h_k}\}_{k \geq 0} \) of \( \{u_h\}_{h \geq 0} \) and a function \( w' \in H^3(\Omega)^d \) such that
\[ \partial^\alpha u_h \to \partial^\alpha w' \text{ weakly in } L^2(\Omega) \text{ for all } |\alpha| \leq m. \]

We know by corollary \ref{corollary:weak_convergence} that given any \( v \in H^3(\Omega)^d \) there exists a sequence \( \{v_h\}_{h>0} \) in \( \tilde{V}_h^d \) such that
\[ \lim_{h \to 0} \|v - v_h\|_{m, \Omega_h} = 0. \]

Then we can say:
\[
|a_h(u_h, v_h) - a(w', v)| \leq |a_h(u_h, v_h - v)| + |a_h(u_h - w', v)|
\]
\[
\leq \|u_h\|_{m, \Omega_h} \|v_h - v\|_{m, \Omega_h} + |a_h(u_h - w', v)|
\]

which vanishes as \( k \to \infty \).

Next we show:
\[
\int_{S^1} \tilde{f} \cdot v_h \circ g \ d\theta \to \int_{S^1} \tilde{f} \cdot v \circ g \ d\theta \quad \text{as } \ h \to 0. \tag{21}
\]

By the dominated convergence theorem this requires at least pointwise convergence of \( (v_h - v) \circ g \) and pointwise boundedness by some integrable function. The latter is trivial since \( v_h \) and \( v \) are continuous over a bounded domain. The information we have is strong convergence of \( \|v_h - v\|_{m, \Omega_h} \) by the weak approximation property, so by lemma \ref{lemma:weakApproximation} \( (21) \) is verified. In light of this we must have \( a(w', v) = \int_{S^1} \tilde{f} \cdot v \circ g \ d\theta \) and so corollary \ref{corollary:weak_convergence} implies \( w' \equiv u \).

We now show strong convergence. Let \( \{w_h\}_{h>0} \) be a sequence in \( \tilde{V}_h^d \) such that \( \|u - w_h\|_{m, \Omega_h} \to 0 \) as \( h \to 0 \). Then,
\[
\|u - u_h\|_{m, \Omega_h}^2 \lesssim \|u - w_h\|_{m, \Omega_h}^2 + a_h(w_h - u_h, w_h - u_h)
\]
\[
\lesssim \|u - w_h\|_{m, \Omega_h}^2 + a(u, u) - 2a_h(u, u_h) + a_h(u_h, u_h)
\]
\[
\lesssim \|u - w_h\|_{m, \Omega_h}^2 + a(u, u) - 2a_h(u, u_h)
\]
\[
+ \int_{S^1} \tilde{f} \cdot u_h \circ g \ d\theta.
\]

Now using what we just derived above:
\[
a(u, u) - 2a_h(u, u_h) + \int_{S^1} \tilde{f} \cdot u_h \circ g \ d\theta
\]
\[
\to a(u, u) - 2a_h(u, u) + \int_{S^1} \tilde{f} \cdot u \circ g \ d\theta = 0
\]
as \( h \to 0 \).

\[ \square \]

**Remark 21** (Comparison with \( L^2(\Omega) \) source terms). When studying elliptic equations it is common for the source term to be defined in terms of some \( f \in \]
$L^2(\Omega)^d$ on the form $\langle f, v_h \rangle_{0,\Omega}$. Weak $L^2(\Omega)^d$ convergence therefore trivially completes the argument in this case since:

$$\langle f, v_h \rangle_{0,\Omega} \rightarrow \langle f, v \rangle_{0,\Omega}.$$ 

is by definition, weakly convergent. This argument is ill-suited to the setting above due to the singular nature of the $S^1$ integral as a source term. This is the reason for the use of lemma 19 in corollary 20.

The key thing to note in the result above is that the implicit conforming shape-regular family of triangulations $\{\Omega_h\}_{h_j}$ is such that for any $h_j \rightarrow 0$ as $j \rightarrow \infty$ there is a subset of $\mathcal{E}_{th_j}$ tracing out $g \circ S^1$.

We proceed to prove a convergence rate. To this end we introduce the notion of \textit{conforming relatives} first proposed in [Bre96].

**Proposition 22 (Conforming relatives).** There exists an $H^m$-conforming finite element space $V_{h}^{c,d} \subset H_0^m(\Omega)^d$ and an operator $\Pi_{c,h} : \bar{V}_h \rightarrow V_{h}^{c,d}$ such that $\forall v \in \bar{V}_h$:

$$\sum_{j=0}^{m-1} h^{2(j-m)} |v_h^i - \left[\Pi_{c,h} v_h\right]^i_{j,\Omega_h}|^2 \lesssim |v_h|^2_{m,\Omega_h}.$$ 

for each component $v_h^i$ of $v_h$.

This result can be seen in e.g. [HZ17] or [HMS14, Lemma 3.2]. The proof we provide next uses these operators as well as the hypoellipticity result in lemma 15.

**Corollary 23.** The convergence rate for the expression $\lim_{h \rightarrow 0} \| u - u_h \|_{m,\Omega_h} =$

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4The reference given here shows under which conditions the conforming relatives operator exists, for arbitrary $m \geq 1$. These are exactly the weak continuity properties provided by [WX19].
0 is almost linear in $h$ via the following bound:

\[
\|u - u_h\|_{m, \Omega_h} \lesssim h \left( \|\tilde{f}\|_{0, S^1} + \sum_{i=1}^{2} \|D^3 u_i\|_{0, \Omega_h} + \|D^5 u_i\|_{0, \Omega_h} + \|D^4 u_i\|_{0, \Omega_h} \right) \\
+ h \left( \sum_{e \in g \circ S^1} \|\|D^3 u_i\|_{0,e} + \|\|D^5 u_i\|_{0,e}\| + 2\|D^4 u_i\|_{0, \Omega_h} \right) \\
+ \sum_{e \in g \circ S^1} \|\|D^3 u_i - P_{K_e} [D^3 u_i]\|_{0,e} \\\n+ h^s |u|_{m+s, \Omega_h},
\]

for any integer $s \geq 0$, where $P_{K_e}$ is the projection operator on $L^2(K_e)$ defined by:

\[
P_{K_e} v = \left( \sum_{K \in K_e} |K| \right)^{-1} \sum_{K \in K_e} \int_K v \, dx, \quad v \in L^2(\Omega).
\]

**Proof.** For arbitrary $w_h \in \tilde{V}_h^d$ we seek to bound the following expression in terms of $h$:

\[
a_h(u - u_h, w_h) = a_h(u, w_h) - \int_{S^1} \tilde{f} \cdot w_h \circ g \, d\theta.
\]

We use the conforming relatives operator to say:

\[
a_h(u - u_h, w_h) = a_h(u, w_h) - \int_{S^1} \tilde{f} \cdot w_h \circ g \, d\theta \\
= a_h(u, w_h - \Pi_h^c w_h) - \int_{S^1} \tilde{f} \cdot (w_h - \Pi_h^c w_h) \circ g \, d\theta,
\]

since $a_h(u, \Pi_h^c w_h) - \int_{S^1} \tilde{f} \cdot \Pi_h^c w_h \circ g \, d\theta = 0$.

First we estimate $\int_{S^1} \tilde{f} \cdot (w_h - \Pi_h^c w_h) \circ g \, d\theta$ and show linear convergence in $h$. The key tools are the Sobolev embedding theorem and (22):

\[
\int_{S^1} \tilde{f} \cdot (w_h - \Pi_h^c w_h) \circ g \, d\theta \leq \|\tilde{f}\|_{0, S^1} \|(w_h - \Pi_h^c w_h) \circ g\|_{0, S^1} \\
\lesssim \|\tilde{f}\|_{0, S^1} \|(w_h - \Pi_h^c w_h) \circ g\|_{\infty, S^1} \\
= \|\tilde{f}\|_{0, S^1} \|w_h - \Pi_h^c w_h\|_{\infty, g \circ S^1} \\
\lesssim \|\tilde{f}\|_{0, S^1} \|w_h - \Pi_h^c w_h\|_{1, g \circ S^1} \text{ (embedding)} \\
\lesssim \|\tilde{f}\|_{0, S^1} \|w_h - \Pi_h^c w_h\|_{2, \Omega_h}
\]
Now using the bound in (22):
\[
\int_{S^1} \tilde{f} \cdot (w_h - \Pi_h^i w_h) \circ g \, d\theta \lesssim h \| \tilde{f} \|_{0,S^1} \| w_h \|_{m, \Omega_h}.
\] (23)

Second we estimate \( a_h(u, w_h - \Pi^h c h w_h) \) by writing the variational problem for \( w \) in a strong form. For the sake of exposition we recall the standard integration by parts identities on elements \( K \) of \( \Omega_h \) in our notation:
\[
\int_K \langle Du^i, Dw_h^i \rangle \, dx = - \int_K \langle D^2 u^i, w_h^i \rangle \, dx + \int_{\partial K} \langle \eta_K Dw_h^i, \eta \rangle \, ds,
\]
\[
\int_K \langle D^2 u^i, Dw_h^i \rangle \, dx = \int_K \langle D^4 u^i, w_h^i \rangle \, dx + \int_{\partial K} \langle \eta_K D^2 u^i, Dw_h^i \rangle \, ds
\]
\[
- \int_{\partial K} \langle D^3 u^i, \eta_K w_h^i \rangle \, ds,
\]
\[
\int_K \langle D^3 u^i, Dw_h^i \rangle \, dx = - \int_K \langle D^5 u^i, w_h^i \rangle \, dx + \int_{\partial K} \langle \eta_K D^4 u^i, Dw_h^i \rangle \, ds
\]
\[
- \int_{\partial K} \langle \eta_K D^4 u^i, Dw_h^i \rangle \, ds + \int_{\partial K} \langle D^3 u^i, \eta_K D^2 w_h^i \rangle \, ds,
\]

Recall that \( \Omega_h^{-} \) (resp. \( \Omega_h^{+} \)) denotes the parts of domain \( \Omega_h \) restricted to the inside (resp. outside) of the curve traced by \( g \circ S^1 \).

**Remark 24** (Strong form equation). The strong form version of (14) can be written by using the identities above:

\[
Bu = 0, \quad \text{on} \quad \Omega_h^{-} \text{ and } \Omega_h^{+},
\]
\[
\int_{S^1} \tilde{f} \cdot v \circ g \, d\theta = \sum_{i=1}^{d} \sum_{e \in g \circ S^1} \int_e b_2 \left( \frac{m}{2} \right) \langle \| D^3 u^i \|, v^i \rangle - b_3 \left( \frac{m}{3} \right) \langle \| D^5 u^i \|, v^i \rangle
\]
\[
+ b_3 \left( \frac{m}{3} \right) \langle \| D^4 u^i \|, Dv^i \rangle - b_3 \left( \frac{m}{3} \right) \langle \| D^3 u^i \|, D^2 v^i \rangle \, ds,
\]
\[
\forall v \in H^3(\Omega).
\]

Indeed, for a test function \( v \in H^3(\Omega)^d \) we have:
\[
\sum_{i=1}^{d} \int_{\Omega_h^{-}} \sum_{j=0}^{m} b_j \left( \frac{m}{j} \right) \langle D^j u^i, D^j v^i \rangle \, dx
\]
\[
= \sum_{i=1}^{d} \int_{\Omega_h^{-}} \sum_{j=0}^{m} b_j \left( \frac{m}{j} \right) \langle D^j u^i, D^j v^i \rangle \, dx + \int_{\Omega_h^{+}} \sum_{j=0}^{m} b_j \left( \frac{m}{j} \right) \langle D^j u^i, D^j v^i \rangle \, dx.
\]
So integrating by parts:

\[
\sum_{i=1}^{d} \int_{\Omega_h} \sum_{j=0}^{m} b_j \left( \frac{m}{j} \right) \langle D^j u^i, D^j v^i \rangle \, dx
= \sum_{i=1}^{d} \int_{\Omega_h^-} B u^i \cdot v^i \, dx + \sum_{i=1}^{d} \int_{\Omega_h^+} B u^i \cdot v^i \, dx
+ \int_{\partial \Omega_1} b_2 \left( \frac{m}{2} \right) \langle [D^3 u^i], v^i \rangle - b_3 \left( \frac{m}{3} \right) \langle [D^5 u^i], v^i \rangle
+ b_3 \left( \frac{m}{3} \right) [D^4 u^i] D v^i - b_3 \left( \frac{m}{3} \right) [D^3 u^i D^2 v^i] \, ds.
\]

Since the integral over \( \Omega_h^- \) and \( \Omega_h^+ \) of \( B u^i \cdot v^i \) vanishes we can relate the singular source term to a jump condition on \( g \circ S^1 \) via the following variational equality:

\[
\int_{\partial \Omega_1} \tilde{f} \cdot v \circ d\theta = \sum_{i=1}^{d} \sum_{e \in g \circ S^1} \int_{e} b_2 \left( \frac{m}{2} \right) \langle [D^3 u^i], v^i \rangle - b_3 \left( \frac{m}{3} \right) \langle [D^5 u^i], v^i \rangle
+ b_3 \left( \frac{m}{3} \right) \langle [D^4 u^i], D v^i \rangle
- b_3 \left( \frac{m}{3} \right) \langle [D^3 u^i], D^2 v^i \rangle \, ds, \quad \forall v \in H^3(\Omega).
\]

Then:

(24)
\begin{align*}
a_h(u, w_h - \Pi_h^c w_h) &= \sum_{i=1}^{2} \sum_{K \in \Omega_h} \int_K B u^i \cdot (w_h^i - [\Pi_h^c w_h]^i) \, dx \\
&\quad + \int_{\partial K} b_1 \left( \begin{smallmatrix} m \\ 1 \end{smallmatrix} \right) \langle D u^i, \eta_K (w_h^i - [\Pi_h^c w_h]^i) \rangle \, ds \\
&\quad + \int_{\partial K} b_2 \left( \begin{smallmatrix} m \\ 2 \end{smallmatrix} \right) \langle \eta_K D^2 u^i, D (w_h^i - [\Pi_h^c w_h]^i) \rangle \, ds \\
&\quad - \int_{\partial K} b_2 \left( \begin{smallmatrix} m \\ 2 \end{smallmatrix} \right) \langle D^3 u^i, \eta_K (w_h^i - [\Pi_h^c w_h]^i) \rangle \, ds \\
&\quad - \int_{\partial K} b_3 \left( \begin{smallmatrix} m \\ 3 \end{smallmatrix} \right) \langle D^5 u^i, \eta_K (w_h^i - [\Pi_h^c w_h]^i) \rangle \, ds \\
&\quad + \int_{\partial K} b_3 \left( \begin{smallmatrix} m \\ 3 \end{smallmatrix} \right) \langle \eta_K D^4 u^i, D (w_h^i - [\Pi_h^c w_h]^i) \rangle \, ds \\
&\quad - \int_{\partial K} b_3 \left( \begin{smallmatrix} m \\ 3 \end{smallmatrix} \right) \langle D^3 u^i, \eta_K D^2 (w_h^i - [\Pi_h^c w_h]^i) \rangle \, ds.
\end{align*}

We now write the boundary terms as jumps on the mesh skeleton. The expressions \( w_h^i - [\Pi_h^c w_h]^i \) and \( D^2 u^i \) have unique traces everywhere, while the expressions \( D^j u^i, j = 3, 4, 5 \) only have unique traces on \( \hat{E}_h \setminus g \circ S^1 \). Lastly, the expressions \( D^j (w_h^i - [\Pi_h^c w_h]^i), j = 1, 2 \) are not uniquely defined anywhere on \( \hat{E}_h \). We can therefore write:
\[
\begin{align*}
a_h(u, w_h - \Pi_h^c w_h) \\
= & \sum_{i=1}^{2} \int_{\hat{e}_h} b_2 \left( \begin{array}{c} m \\ 2 \end{array} \right) \langle D^2 u^i, [D(w_h^i - [\Pi_h^c w_h]^i)] \rangle \ ds \\
+ & \int_{\hat{e}_h \setminus g_0 S^1} b_3 \left( \begin{array}{c} m \\ 3 \end{array} \right) \langle D^4 u^i, [D(w_h^i - [\Pi_h^c w_h]^i)] \rangle \\
- & b_3 \left( \begin{array}{c} m \\ 3 \end{array} \right) \langle D^3 u^i, [D^2(w_h^i - [\Pi_h^c w_h]^i)] \rangle \ ds \\
+ & \int_{\partial S^1} -b_2 \left( \begin{array}{c} m \\ 2 \end{array} \right) \langle [D^3 u^i], w_h - [\Pi_h^c w_h]^i \rangle \\
- & b_3 \left( \begin{array}{c} m \\ 3 \end{array} \right) \langle [D^5 u^i], w_h - [\Pi_h^c w_h]^i \rangle \\
+ & b_3 \left( \begin{array}{c} m \\ 3 \end{array} \right) \langle [D^4 u^i D(w_h^i - [\Pi_h^c w_h]^i)] \rangle \\
- & b_3 \left( \begin{array}{c} m \\ 3 \end{array} \right) \langle [D^3 u^i D^2(w_h - [\Pi_h^c w_h]^i)] \rangle, \\
\end{align*}
\]

where we recall that \([D^4 u^i D(w_h^i - [\Pi_h^c w_h]^i)]\) and \([D^3 u^i D^2(w_h - [\Pi_h^c w_h]^i)]\) on \(e = K_e^+ \cap K_e^-\) are defined as:

\[
\begin{align*}
\langle [D^4 u^i D(w_h^i - [\Pi_h^c w_h]^i)] \rangle &= \langle \eta_{K_e^+} D^4 u^i |_{K_e^+}, D(w_h^i - [\Pi_h^c w_h]^i) |_{K_e^+} \rangle \\
&+ \langle \eta_{K_e^-} D^4 u^i |_{K_e^-}, D(w_h^i - [\Pi_h^c w_h]^i) |_{K_e^-} \rangle, \\
\langle [D^3 u^i D^2(w_h - [\Pi_h^c w_h]^i)] \rangle &= \langle \eta_{K_e^+} \cdot D^3 u^i |_{K_e^+}, D^2(w_h^i - [\Pi_h^c w_h]^i) |_{K_e^+} \rangle \\
&+ \langle \eta_{K_e^-} \cdot D^3 u^i |_{K_e^-}, D^2(w_h^i - [\Pi_h^c w_h]^i) |_{K_e^-} \rangle.
\end{align*}
\]

Many of the terms in (25) are estimated using the same technique and rely on the weak continuity result for \(V_h^c\) in lemma [5]. First we highlight the main inequalities on which we rely. Let \(e\) be an edge of an arbitrary \(K \in \Omega_h\) with length \(|e|\). Further, let \(K_e\) denote the set of triangles in \(\Omega_h\) having \(e\) as an edge. We recall [BSO7] Equations 10.3.8, 10.3.9 and the trace theorem:

\[
\begin{align*}
|e|^{-1} \|f\|^2_{0,e} \lesssim & \|f\|^2_{0,K} + |f|^2_{1,K}, \quad f \in H^1(K), \\
|e| \|w\|^2_{0,e} \lesssim & \sum_{K \in K_e} \|w\|^2_{1,K}, \quad w|_K \in H^1(K), \\
\|f\|^2_{1,K} \lesssim & \|f\|^2_{1,K}, \quad f|_K \in H^1(K).
\end{align*}
\]

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We now estimate each term in (25) starting with (25a). We now drop the binomial coefficients to ease the notation.

For any constant $c_e$ and using Cauchy-Schwartz:

$$
\sum_{i=1}^{2} \int_{\hat{\mathcal{E}_h}} b_2 \langle D^2 u^i, [D(\mathbf{w}_h^i - [\Pi_h^c \mathbf{w}_h^i])] \rangle \, ds
$$

$$
= \sum_{i=1}^{2} \int_{\hat{\mathcal{E}_h}} b_2 \langle D^2 u^i - c_e, [D(\mathbf{w}_h^i - [\Pi_h^c \mathbf{w}_h^i])] \rangle \, ds
$$

$$
\leq b_2 \sum_{i=1}^{2} \sum_{e \in \hat{\mathcal{E}_h}} \|D^2 u^i - c_e\|_{0,e} \|D(\mathbf{w}_h^i - [\Pi_h^c \mathbf{w}_h^i])\|_{0,e}.
$$

Now we multiply by $1 = |e|^{-1/2}|e|^{1/2}$:

$$
\sum_{i=1}^{2} \int_{\hat{\mathcal{E}_h}} b_2 \langle D^2 u^i, [D(\mathbf{w}_h^i - [\Pi_h^c \mathbf{w}_h^i])] \rangle \, ds
$$

$$
= b_2 \sum_{i=1}^{2} \sum_{e \in \hat{\mathcal{E}_h}} |e|^{-1/2} \|D^2 u^i - c_e\|_{0,e} |e|^{1/2} \|D(\mathbf{w}_h^i - [\Pi_h^c \mathbf{w}_h^i])\|_{0,e}
$$

$$
\leq b_2 \left( \sum_{i=1}^{2} \sum_{e \in \hat{\mathcal{E}_h}} |e|^{-1} \|D^2 u^i - c_e\|_{2,e}^2 \right)^{1/2} \left( |e| \|D(\mathbf{w}_h^i - [\Pi_h^c \mathbf{w}_h^i])\|_{0,e} \right)^{1/2}.
$$

Now using (26):

$$
\sum_{i=1}^{2} \int_{\hat{\mathcal{E}_h}} b_2 \langle D^2 u^i, [D(\mathbf{w}_h^i - [\Pi_h^c \mathbf{w}_h^i])] \rangle \, ds
$$

$$
\lesssim b_2 \left( \sum_{i=1}^{2} \sum_{e \in \hat{\mathcal{E}_h}} \min_{K \in K_e} \left[ h_K^{-2} \|D^2 u^i - c_e\|_{0,K}^2 + |D^2 u^i|_{1,K}^2 \right] \right)^{1/2}
$$

$$
\times (h_K^2 |D(\mathbf{w}_h^i - [\Pi_h^c \mathbf{w}_h^i])|_{1,K}^2)^{1/2}.
$$

Now since $c_e$ was arbitrary we can take it to be the constant in $\mathbb{R}$ that is the closest approximation of $D^2 u^i$. Projection of Sobolev functions onto constants is well-understood; by e.g. [EG13, Proposition 1.135] we know
that:
\[ \| D^2 u^i - |K|^{-1} \int_K D^2 u^i \, dx \|_{0,K} \lesssim h_K |D^2 u^i|_{1,K}, \]
with a constant independent of \( h_K \). Squaring on both sides cancels the dependence on \( h_K \) in the previous estimate so we get:
\[
\sum_{i=1}^{2} \int_{E_h} b_2 \min_{K \in K_e} \langle D^2 u^i, [D(w_h^i - [\Pi_h^c w_h]^i)] \rangle \, ds \\
\lesssim b_2 \sum_{i=1}^{2} \sum_{e \in E_h} \min_{K \in K_e} |D^2 u^i|^2_{1,K}^{1/2} (h_K |D(w_h^i - [\Pi_h^c w_h]^i)]|^2_{1,K}^{1/2} \\
\lesssim b_2 h \| D^3 u^i \|_{0, \Omega_h} \| w_h^i - [\Pi_h^c w_h]^i \|_{m, \Omega_h} \\
\lesssim b_2 h \| D^3 u^i \|_{0, \Omega_h} \| w_h \|_{m, \Omega_h},
\]
using (22) in the last step.

Next we estimate (25b). It is clear that the only difference from (25a) here is the operator \( D^4 \) instead of \( D^2 \) applied to \( u^i \), so by the same steps as previously:
\[
\sum_{i=1}^{2} \int_{E_h \setminus \partial S^1} b_3 \langle D^4 u^i, [D(w_h^i - [\Pi_h^c w_h]^i)] \rangle \, ds \\
\lesssim b_3 h \| D^5 u^i \|_{0, \Omega_h} \| w_h \|_{m, \Omega_h}.
\]
We now treat (25c). For an arbitrary constant \( c_e \):
\[
\sum_{i=1}^{2} \int_{E_h \setminus \partial S^1} b_3 \langle D^3 u^i - c_e, [D^2(w_h^i - [\Pi_h^c w_h]^i)] \rangle \, ds \\
= \sum_{i=1}^{2} \int_{E_h \setminus \partial S^1} b_3 \langle D^3 u^i - c_e, [D^2(w_h^i - [\Pi_h^c w_h]^i)] \rangle \, ds \\
\leq b_3 \left( \sum_{i=1}^{2} \sum_{e \in E_h \setminus \partial S^1} \| D^3 u^i - c_e \|_{0, e}^2 \right)^{1/2} \left( \| D^2(w_h^i - [\Pi_h^c w_h]^i) \|_{0, e}^2 \right)^{1/2}.
\]
Now using (26) again and Cauchy-Schwartz:

\[
\sum_{i=1}^{2} \int_{\hat{\Omega}} b_3 \langle D^3 u^i, [D^2 (w^i_h - [\Pi^i_h w_h])] \rangle \, ds
\]

\[
\leq b_3 \left( \sum_{i=1}^{2} \sum_{e \in \hat{\Omega}} \min_{K \in \mathcal{K}_e} h_K^{-2} \| D^3 u^i \|^2 - c_e \| D^3 u^i \|^2_{1,K} \right)^{1/2}
\]

\[
\times \left( h_K^2 \| D^2 (w^i - [\Pi^i_h w_h]) \|^2_{1,K} \right)^{1/2}
\]

\[
\lesssim b_3 \left( \sum_{i=1}^{2} \sum_{e \in \hat{\Omega}} \| D^3 u^i \|^2_{1,K} \right)^{1/2} \left( h_K^2 \| D^2 (w^i - [\Pi^i_h w_h]) \|^2_{1,K} \right)^{1/2}.
\]

Using the trace theorem we get:

\[
\left| \sum_{i=1}^{2} \int_{\hat{\Omega}} b_3 \langle D^3 u^i, [D^2 (w^i_h - [\Pi^i_h w_h])] \rangle \, ds \right|
\]

\[
\lesssim b_3 h \left( \sum_{i=1}^{2} \sum_{K \in \Omega_h} \| D^3 u^i \|^2_{1,K} \right)^{1/2} \| [\Pi^i_h w_h] \|^1_{m, \Omega_h}
\]

\[
= b_3 h \| D^4 u^i \|^1_{0, \Omega_h} \| w^i_h - [\Pi^i_h w_h] \|^1_{m, \Omega_h}
\]

\[
\lesssim b_3 h \| D^4 u^i \|^1_{0, \Omega_h} \| w^i_h \|^1_{m, \Omega_h}.
\]

Estimating (25d), (25e) and (25f) relies on the trace theorem and a property of the conforming relatives operator in (22). For (25d) we have:

\[
\sum_{i=1}^{2} \int_{\hat{\Omega}} -b_2 \langle [D^3 u^i], w^i_h - [\Pi^i_h w_h] \rangle \, ds
\]

\[
\leq b_2 \sum_{i=1}^{2} \sum_{e \in \hat{\Omega}} \| [D^3 u^i] \|^1_{0,e} \| w^i_h - [\Pi^i_h w_h] \|^1_{0,e}.
\]

By the trace theorem we have \( \| w^i_h - [\Pi^i_h w_h] \|^1_{0,e} \lesssim \sum_{K \in \mathcal{K}_e} \| w^i_h - [\Pi^i_h w_h] \|^1_{1,K} \).
Using this and (22) in the above estimate gives us:

\[ | \sum_{i=1}^{2} \int_{g \circ S^1} -b_2 \langle \| D^3 u^i \|, w_h^i - [\Pi_h^c w_h]^i \rangle \, ds | \]
\[ \lesssim b_2 \sum_{i=1}^{2} \sum_{e \in g \circ S^1} \| D^3 u^i \|_0, e \| w_h^i - [\Pi_h^c w_h]^i \|_{1, \Omega_h} \]
\[ \lesssim b_2 h^2 \sum_{i=1}^{2} \sum_{e \in g \circ S^1} \| D^3 u^i \|_0, e \| w_h \|_{m, \Omega_h}. \]

By a similar argument for (25e):

\[ | \sum_{i=1}^{2} \int_{g \circ S^1} -b_3 \langle \| D^5 u^i \|, w_h^i - [\Pi_h^c w_h]^i \rangle \, ds | \]
\[ \lesssim b_3 \sum_{i=1}^{2} \sum_{e \in g \circ S^1} \| D^5 u^i \|_0, e \| w_h^i - [\Pi_h^c w_h]^i \|_{1, \Omega_h} \]
\[ \lesssim b_3 h^2 \sum_{i=1}^{2} \sum_{e \in g \circ S^1} \| D^5 u^i \|_0, e \| w_h \|_{m, \Omega_h}. \]

Term (25f) is estimated similarly:

\[ \sum_{i=1}^{2} \int_{g \circ S^1} b_3 \| D^4 u^i D(w_h^i - [\Pi_h^c w_h]^i) \| \, ds \]
\[ = b_3 \sum_{i=1}^{2} \int_{g \circ S^1} \langle \eta_{K^+}^* D^4 u^i |_{K^+}, D(w_h^i - [\Pi_h^c w_h]^i) |_{K^+} \rangle \]
\[ + \langle \eta_{K^-}^* D^4 u^i |_{K^-}, D(w_h^i - [\Pi_h^c w_h]^i) |_{K^-} \rangle \, ds. \]

Using Cauchy-Schwartz twice, the trace theorem and (22) we proceed as
previously:

\[
\sum_{i=1}^{2} \int_{g \circ S^1} b_3 \| D^4 u^i D (w^i_h - \left[ \Pi^c_h w_h \right]^i) \| \, ds \\
\leq b_3 \sum_{i=1}^{2} \sum_{e \in g \circ S^1} \| \eta_{K^+_e} D^4 u^i \|_{K^+_e} \| D (w^i_h - \left[ \Pi^c_h w_h \right]^i) \|_{K^+_e} \\
+ \| \eta_{K^-_e} D^4 u^i \|_{K^-_e} \| D (w^i_h - \left[ \Pi^c_h w_h \right]^i) \|_{K^-_e} \\
\lesssim b_3 \sum_{i=1}^{2} \sum_{e \in g \circ S^1} \left( \| \eta_{K^+_e} D^4 u^i \|_{K^+_e} \|_{0,e} + \| \eta_{K^-_e} D^4 u^i \|_{K^-_e} \|_{0,e} \right) \| D (w^i_h - \left[ \Pi^c_h w_h \right]^i) \|_{1, \Omega_h} \\
\lesssim b_3 h \left( \sum_{i=1}^{2} \sum_{e \in g \circ S^1} \left( \| \eta_{K^+_e} D^4 u^i \|_{K^+_e} \|_{0,e} + \| \eta_{K^-_e} D^4 u^i \|_{K^-_e} \|_{0,e} \right) \right) \| w_h \|_{m, \Omega_h} \\
\leq b_3 h \| D^4 u \|_{0, \Omega_h} \| w_h \|_{m, \Omega_h}.
\]

Finally we estimate (25g):

\[
\left| \sum_{i=1}^{2} \int_{g \circ S^1} -b_3 \| D^3 u^i D^2 (w^i_h - \left[ \Pi^c_h w_h \right]^i) \| \, ds \right| \\
= \left| -b_3 \sum_{i=1}^{2} \int_{g \circ S^1} \{ D^3 u^i \} \cdot \{ D^2 (w^i_h - \left[ \Pi^c_h w_h \right]^i) \} + \{ D^3 u^i \} \{ D^2 (w^i_h - \left[ \Pi^c_h w_h \right]^i) \} \, ds \right| \\
= T_1 + T_2.
\]

We estimate \( T_1 \) using the same method as for the term (25c). The only difference is the presence of the averaging operator, but we can write \( \{ D^3 u^i \} = \frac{1}{2} D^3 u^i |_{K^+_e} + \frac{1}{2} D^3 u^i |_{K^-_e} \), and since the restrictions of \( D^3 u^i \) to \( K^+_e \) and \( K^-_e \) are smooth there are no changes to the procedure for the term (25c).

\[
T_1 \lesssim b_3 h \| D^4 u \|_{0, \Omega_h} \| w_h \|_{m, \Omega_h}.
\]

Obtaining a convergence rate for \( T_2 \) is more difficult owing to the presence of the \( \{ D^2 (w^i_h - \left[ \Pi^c_h w_h \right]^i) \} \) term. It is easy to see why, since for an arbitrary edge \( e \in \mathcal{E}_h \), \( \| D^2 (w^i_h - \left[ \Pi^c_h w_h \right]^i) \|_{0,e} \lesssim \| D^2 (w^i_h - \left[ \Pi^c_h w_h \right]^i) \|_{1,K} \) for some \( K \in \Omega_h \) with \( e \) as an edge. Then by (22) we know:

\[
\left( \sum_{i=1}^{2} \sum_{K \in \Omega_h} \| D^2 (w^i_h - \left[ \Pi^c_h w_h \right]^i) \|_{1,K} \right)^{1/2} \lesssim \| w_h \|_{m, \Omega_h},
\]

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but we do not recover a factor of $h$ in this bound. We can however bound \([25g]\) as follows using an averaging operator over a neighbourhood near an edge (see also \([WX19, \text{Equation 3.12}]\) for an example). We define $P_{K_e}$ as the projection operator on $L^2(K_e)$ defined by:

$$P_{K_e}v = \left( \sum_{K \in K_e} |K| \right)^{-1} \sum_{K \in K_e} \int_K v \, dx, \quad v \in L^2(\Omega).$$

Then,

$$T_2 = |b_3 \sum_{i=1}^2 \int_{g \circ S^1} \left[ D^3 u^i \right] \{ D^2 (w_h^i - [\Pi_h^i w_h])^i \} \, ds |$$

$$= |b_3 \sum_{i=1}^2 \int_{g \circ S^1} \left[ D^3 u^i - P_{K_e} [D^3 u^i] \right] \{ D^2 (w_h^i - [\Pi_h^i w_h])^i \} \, ds |$$

$$\leq b_3 \sum_{i=1}^2 \sum_{e \in g \circ S^1} \| \left[ D^3 u^i - P_{K_e} [D^3 u^i] \right] \|_{0,e} \| \{ D^2 (w_h^i - [\Pi_h^i w_h])^i \} \|_{0,e}.$$  

Now using the trace theorem and \([22]\):

$$T_2 \lesssim b_3 \sum_{i=1}^2 \sum_{e \in g \circ S^1} \| \left[ D^3 u^i - P_{K_e} [D^3 u^i] \right] \|_{0,e} \| \{ D^2 (w_h^i - [\Pi_h^i w_h])^i \} \|_{0,m,\omega_h}$$

$$\lesssim b_3 \sum_{i=1}^2 \sum_{e \in g \circ S^1} \| \left[ D^3 u^i - P_{K_e} [D^3 u^i] \right] \|_{0,e} \| w_h^i \|_{m,\omega_h}.$$  

Collecting the bounds we have for \([25]\) so far and dividing by $\| w_h \|_{m,\omega_h}$ yields:

$$\frac{a_h(u - u_h, w_h)}{\| w_h \|_{m,\omega_h}} \lesssim h \left( \sum_{i=1}^2 \| D^3 u^i \|_{0,\omega_h} + \| D^5 u^i \|_{0,\omega_h} + \| D^4 u^i \|_{0,\omega_h} \right)$$

$$+ h \sum_{e \in g \circ S^1} \| \{ D^3 u^i \} \|_{0,e}$$

$$+ h \sum_{e \in g \circ S^1} \| \{ D^5 u^i \} \|_{0,e} + 2 \| D^4 u^i \|_{0,\omega_h}$$

$$+ \sum_{e \in g \circ S^1} \| \left[ D^3 u^i - P_{K_e} [D^3 u^i] \right] \|_{0,e}$$
Using Strang’s lemma \[11\] and corollary \[16\] we can therefore say:

\[
\|u - u_h\|_{m, \Omega_h} \lesssim h\left(\|\tilde{f}\|_{0,S^1} + \sum_{i=1}^{2} \|D^3 u^i\|_{0, \Omega_h} + \|D^5 u^i\|_{0, \Omega_h} + \|D^4 u^i\|_{0, \Omega_h} \right) \\
+ h \sum_{e \in g \cap S^1} \|\|D^3 u^i\|\|_{0,e} \\
+ h \sum_{e \in g \cap S^1} \|\|D^5 u^i\|\|_{0,e} + 2\|D^4 u^i\|_{0, \Omega_h} \\
+ \sum_{e \in g \cap S^1} \|\|D^3 u^i - P_{K_e}[D^3 u^i]\|\|_{0,e} \\
+ h^s|u|_{m+s, \Omega_h},
\]

for any \(s \geq 0\), so the result follows. \(\square\)

**Remark 25** (Conforming reconstruction operator). In the analysis above we make heavy use of the projection operator \(P_{K_e}\) as it is constructed in such a way as to be continuous over the edge \(e\) (trivially so as the range of this operator is the space of constants). \[GP17\] proposes a \(C^1\) reconstruction operator with attractive properties (see e.g. equation (16) in the reference) that could be explored instead. These properties concern error estimation of the projection error on elements \(K\) of the triangulation. However, convergence in terms of \(h\) is not readily obtained using such operators as the preceding analysis deals with terms of like \(\sum_{e \in g \cap S^1} \|\|D^3 u^i - P_{K_e}[D^3 u^i]\|\|_{0,e}\) concerning the jump over edges. By using a trace inequality to pull the estimate to \(K\) we lose an order of differentiation which is reflected in the estimation properties of the aforementioned operator (roughly speaking: the \(L^2\) norm on \(\partial K\) can be bounded by the \(H^1\) norm on \(K\)). It is therefore not certain that such a conforming reconstruction operator yields estimates that scale well with \(h\).

In summary, the finite element discretisation in \[16\] is convergent for \(\tilde{f} \in L^2(S^1)^d\) and mesh-aligned curve \(g \in C^0(S^1)^d\). We saw that the presence of the singular curve integral as a source term prohibits global regularity, and that despite recovering almost everywhere \(C^\infty\) regularity we cannot attain linear convergence as a function of \(h\) using the conforming relatives used in corollary \[23\].
3 Summary

We have presented error analysis for a higher-order nonconforming finite element method with singular data. In doing so we have confirmed a standard result: nonconforming finite elements are well-suited, in terms of proving convergence results for the discretisation, to problems with smooth data. The lack of at least global $L^2$ regularity of the source terms considered here prohibits the discovery of convergence rate across the singularity. We therefore emphasise the need for higher-order conforming finite element methods as this would greatly simplify the analysis carried out in section 2.

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