Some cute applications of Lagrangian cobordisms towards examples in quantitative symplectic geometry

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Abstract

We provide some constructions using Lagrangian cobordisms which improve known examples for some symplectic squeezing problems. Additionally, we prove a flexibility result that Lagrangian submanifolds which are Lagrangian isotopic are also Lagrangian cobordan.

1 Introduction

A general type of problem in symplectic geometry is the “squeezing problem”. A typical example of a squeezing problem asks: given subsets $A, B$ inside of a symplectic manifold $(X, \omega)$ does there exists a symplectic or Hamiltonian isotopy $\phi : X \to X$ so that $\phi(A) \subset B$. In the simplest cases one can use quantitative information from the symplectic form to solve a squeezing problem. For example, a first quantity that one can associate with a squeezing problem comes from the volume, which must satisfy the inequality $\int_A \omega^n \leq \int_B \omega^n$. It is natural to ask: can we find a solution to a squeezing problem given some quantitative information about the objects being squeezed. These questions have two components: constructing examples of squeezings and finding obstructions to the existence of squeezings.

In this paper, we give some improvements on known constructions of squeezings by using Lagrangian cobordisms to produce examples. In each problem, the quantitative nature of the construction exhibits itself by considering the shadow of the Lagrangian cobordisms used.

Problem I: Packing Lagrangian Tori

The integral Lagrangian packing problem asks how many pairwise disjoint integral Lagrangian\footnote{An integral Lagrangian is a Lagrangian submanifold such that the area homomorphism $H_2(X, L; \mathbb{Z}) \to \mathbb{R}$ takes only integer values.} one can find in a symplectic manifold $(X, \omega)$.

This problem has been studied by [HK21] for 4-dimensional polydisks. For $a, b > 0$, let the Lagrangian product torus be

$$L(a, b) := \{(z_1, z_2) \in \mathbb{C}^2 \mid \pi|z_1|^2 = a, \pi|z_2|^2 = b\}.$$  

Define the polydisk to be

$$P(a, b) := \{(z_1, z_2) \in \mathbb{C}^2 \mid \pi|z_1|^2 < a, \pi|z_2|^2 < b\}.$$  

Fix $a, b > 0$. For any $m, n \in \mathbb{N}$ with $m < a$ and $n < b$, $L(m, n)$ is an integral Lagrangian in $P(a, b)$. If every integral Lagrangian $L \subset P(a, b)$ intersects some $L(m, n)$, we say that the integral packing of $P(a, b)$ by $\{L(m, n)\}$ is maximal. The Lagrangian torus packing problem has the following obstruction and construction:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{packing_torus}
\caption{Can you pack an integral Lagrangian torus into the polydisk so that it avoids all other integral Lagrangian tori?}
\end{figure}
Obstruction [HK21, Theorem 1.1]: If $1 < a, b < 2$, the integral packing of $P(a, b)$ by $L(1, 1)$ is maximal.

Construction [HK21, Theorem 1.2]: If $\min(a, b) > 2$, then the integral packing of $P(a, b)$ by $\{L(m, n)\}$ is not maximal.

In Section 2, we provide the following construction.

**Construction 1.1.** If $1 < a$ and $2 < b$, then the integral packing of $P(a, b)$ by $\{L(m, n)\}$ is not maximal.

As a consequence, we answer [HK21, Question 1.5] negatively and [HK21, Question 1.6 and 1.7] affirmatively (see Corollary 2.1).

**Problem II: Displacing Lagrangian Tori**

The second problem we consider is displacing a Lagrangian link from itself in $S^2 \times S^2$, studied by the second author and Smith in [MS21].

Let $\omega$ be a symplectic form on $S^2$ with area 1. For any $A \in \mathbb{R}_{>0}$, we denote $(S^2, A\omega)$ by $S^2_A$. Let $L_1$ and $L_2$ be disjoint embedded circles in $S^2_{2B+C}$ such that the complement of $L_{B,C} := L_1 \cup L_2$ consists of two discs of area $B$ and one annulus of area $C$. Let $K_a$ be the equator in $S^2_a$. It is clear by area consideration that every connected component of the Lagrangian $L_{B,C}$ in $S^2_B \times S^2_C$ is Hamiltonian non-displaceable from $L_{B,C}$ if and only if $B - C \geq 0$. The following result concerns $L_{B,C} \times K_a$.

**Obstruction [MS21, Theorem 1.1]:** If $B - C \geq a > 0$, any connected component of the Lagrangian $L_{B,C} \times K_a$ in $S^2_{2B+C} \times S^2_2a$ is Hamiltonian non-displaceable from $L_{B,C} \times K_a$.

**Construction:** In contrast, $L_{B,C} \times K_a$ is Hamiltonian displaceable from itself when $a > B$.

Polterovich showed that $L_{B,C} \times K_a$ is Hamiltonian displaceable when $a$ is sufficiently large [Pol01, Example 6.3.C], [MS21, Lemma 1.11]. Dimitroglou Rizell explained to us the construction above which only requires $a > B$, and is a direct application of probe [McD11, Lemma 2.4] (see Figure 3).

While the obstruction has a dependence on $C$, the construction does not. We provide a construction with a dependence on $C$.

**Construction 1.2.** $L_{B,C} \times K_a$ is displaceable from one of its connected component when $a > 2(B - C)$

The particularly interesting case is when $B = C$. In this case, $L_{B,C}$ is a monotone link in the sense of [CG+21]. Recall that a union of pairwise disjoint circles $L = L_1 \cup \cdots \cup L_k$ in a closed symplectic surface $\Sigma$ is called a monotone link if the connected components of the complement of $L$ have the same area. Every monotone link is non-displaceable by area consideration, and it has a non-zero Heegaard Floer type invariant which plays an important role in [CG+21].

As a generalization of Construction 1.2, we show that

**Construction 1.3.** Let $L$ be a monotone link in a closed symplectic surface $\Sigma$. If it has more than 1 component and there is a component $L_1 \subset L$ which bounds a disk that is disjoint from the other components, then $L_1 \times K_a$ is displaceable from $L \times K_a$ in $\Sigma \times S^2_2a$ for any $a > 0$.

1.0.1 **Problem III: Constructing Lagrangian Submanifolds**
The third problem we examine constructs non-orientable Lagrangian submanifolds in $S^2 \times S^2$.

Let $X$ be a 4-manifold with symplectic form $\omega$, and $\beta \in H_2(X, \mathbb{Z}/2\mathbb{Z})$ be a homology class. The minimal genus problem asks “what is the minimal (non-orientable) genus of a (possibly non-orientable) Lagrangian submanifold representing the class $\beta$?” We denote this quantity $\eta(X, \omega, \beta)$.

In [Eva21], Evans explores the behavior of $\eta(X, \omega, \beta)$ as $\omega$ is varied with fixed $X, \beta$. In the specific setting where $X = S^2_\lambda \times S^2_1$ with symplectic form $\omega_\lambda = \lambda \omega_{S^2} + \omega_{S^2}$ and $\beta = [S^2 \times \{\bullet\}]$, he conjectures that

$$\lim_{\lambda \to \infty} \eta(X, \omega_\lambda, \beta) = \infty.$$  \hfill (1)

As evidence towards the conjecture, Evans provides the following obstruction and construction.

**Obstruction** [Eva21, Theorem 3.1]: Consider the symplectic space $B^*_\lambda/2 \times S^1_1$, the product of a cylinder with area $\lambda$ and a sphere of area 1. For any $\epsilon > 0$, there does not exist a family of embedded Lagrangian Klein bottles $i_\lambda : L \to B^*_\lambda/2 \times S^1_1$ where $\lambda \in [1/2 - \epsilon, 1/2 + \epsilon]$, and $i_{1/2 - \epsilon}(L)$ is the visible Lagrangian Klein bottle drawn in Figure 4.

**Construction** [Eva21, Lemma 2.9] The minimal genus is bounded by

$$\eta(X, \omega_\lambda, \beta) < 20\ell + 2 \text{ when } \lambda < 10\ell + 2.$$  

The construction of [Eva21] Lemma 2.9 uses Lagrangian lifts of tropical curves; the jumps in the bound correspond to when the rectangle with side lengths of $(1, \lambda)$ can squeeze in another tropical curve. Using Lagrangian cobordisms, we provide a (slightly) improved bound whose jumps which occur at even integer values of $\lambda$.

**Construction 1.4.** $\eta(X, \omega_\lambda, \beta) < 4n + 1$, where $\lambda < 2n$.

The construction is based on the following lemma, which might be of independent interest.

**Lemma 1.5** (=Corollary 4.2). Suppose that $L_0, L_1$ are orientable and Lagrangian isotopic. Then $L_0$ and $L_1$ are Lagrangian cobordant.

We will only apply this lemma to the case that $L_0$ and $L_1$ are unions of circles. The non-orientable Lagrangians we construct will be a concatenation of Lagrangian cobordisms.
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1.1 Background: Lagrangian Cobordisms

Quantitative symplectic geometry looks at how the answers to the above problems change as we vary the quantities in the problem. In general, these problems are studied by producing constructions (providing an affirmative for some subset of the quantitative data) or developing obstructions (which can answer the question in the negative for some subsets of the quantitative datum). In this paper, we will use Lagrangian cobordisms to provide constructions for several examples of problems coming from quantitative symplectic geometry. A feature of these constructions is that the shadow metric of the Lagrangian cobordism provides a connection to the quantitative nature of the problem.

Definition 1.6 \([\text{Arn80}]\). Let \(L_0, \ldots, L_k\) be Lagrangian submanifolds of \(X\). A Lagrangian cobordism from \(\{L_j\}_{j=1}^k\) to \(L_0\) is a Lagrangian submanifold \(K \subset X \times \mathbb{C}\) which satisfies the following conditions:

- **Fibered over ends:** There exist constants \(t^- < t^+\) such that
  
  \[
  K \cap \{(x,z) : \text{Re}(z) \geq t^+\} = \bigcup_{j=1}^k L_j \times \{j \cdot \sqrt{-1} + \mathbb{R}_{\geq t^+}\}
  \]

  \[
  K \cap \{(x,z) : \text{Re}(z) \leq t^-\} = L_0 \times \mathbb{R}_{\leq t^-}.
  \]

- **Compactness:** There exists a constant \(s > 0\) so that the projection \(\pi_{\sqrt{-1}\mathbb{R}} : K \to \sqrt{-1}\mathbb{R} \subset \mathbb{C}\) is contained within an interval \([\sqrt{-1}s, \sqrt{-1}s]\).

We will write such a cobordism as \(K : (L_1, \ldots, L_k) \leadsto L_0\). The level sets of the real coordinate \(t = \text{Re}(z)\) in \(K\) are themselves Lagrangian submanifolds of \(X\).

Definition 1.7. Given \(K \subset X \times \mathbb{C}\) a Lagrangian cobordism, and \(t \in \mathbb{R}\) a value so that \(\pi_{\mathbb{R}}^{-1}(t) \subset X \times \mathbb{C}\) is transverse to \(K\), the slice of \(K\) at \(t\) is an immersed Lagrangian submanifold

\[
K|_t := \{(x,z) \in K : \text{Re}(z) = t\}.
\]

Given a Lagrangian cobordism \(K \subset X \times \mathbb{C}\), the shadow \(\text{Area}(K)\) is the infimum of the area of simply connected regions \(U \subset \mathbb{C}\) which contain \(\pi_{\mathbb{C}}(K)\).

Example 1.8. Let \(H = (H_t)_{t \in \mathbb{R}} : X \times \mathbb{R} \rightarrow \mathbb{R}\) be a time-dependent Hamiltonian with compact support, and \(i : L \rightarrow X\) be a Lagrangian submanifold. Let \(\phi_t : X \rightarrow X\) be the time \(t\) Hamiltonian flow. The suspension of \(H\) is the Lagrangian cobordism \(K_H\) which is parameterized by

\[
K_H := L \times \mathbb{R} \rightarrow X \times \mathbb{C}\]

\[
(q,t) \mapsto (\phi_t(i(q)), t + \sqrt{-1}H_t(i(q))\)
\]

It satisfies the property that \(\text{Area}(K_H) = \|H_t\|_L\), where \(\|H_t\|_L\) is the (relative) H"{o}fer norm

\[
\|H_t\|_L := \int_t^s \sup \{H_t(x) \mid x \in L\} - \inf \{H_t(x) \mid x \in L\} dt.
\]

We say that a Lagrangian homotopy \(i_t : L \times \mathbb{R} \rightarrow X\) is an exact homotopy if the flux class is exact, with primitive \(H_t : L \times \mathbb{R} \rightarrow \mathbb{R}\). In this setting, Eq. (2) also yields a Lagrangian cobordism between \(i(t \ll 0)\) and \(i(t \gg 0)\).
Proposition 1.10. Let \( f_0, f_1 : \mathbb{R} \to \mathbb{R} \) be smooth functions with compact support so that \( f_0 \leq f_1 \) and \( \int_{\mathbb{R}} (f_1 - f_0) dt = A \). Let \( K_{H_t} : L_0 \to L_1 \) be a suspension of an exact isotopy with \( \text{Area}(K_{H_t}) = A \). There exists \( K_{H_t'} : L_0 \to L_1 \) the suspension so that whenever \( z \in \pi_2(K_{H_t'}) \), \( f_0(Re(z)) < Im(z) < f_1(Re(z)) \).

Furthermore, \( K_{H_0} \) and \( K_{H_1'} \) are Hamiltonian isotopic via Hamiltonian with compact support.

Proof. Let \( t_0 \) be sufficiently small so that \( f_0(t), f_1(t), H_t = 0 \) for all \( t < t_0 \). Without loss of generality, suppose that \( f_0 = 0 \) and \( \min_{q \in L} H_t(q) = 0 \). The proof of the proposition is the same as [Hic21, Proposition 3.1.4], where we modify the truncation profile \( \rho(t, s) \) so that at the upper edge of \( \text{Fig. 6} \),

\[
\int_{t_0}^{\rho(t_1, 1)} \left( \max_{q \in L} H_t(q) \right) dt = \int_{t_0}^{t_1} f_1(t) dt.
\]

Proposition 1.11. Let \( K_1 : L \to L_1 \) and \( K_2 : L \to L_2 \) be two suspension cobordisms of time-dependent Hamiltonian isotopies with support in \( (0, t_0) \). Let \( c > 0, \gamma_-, \gamma_+ \subset \mathbb{C} \) be the \( u \)-shaped curve with \( \gamma_0(0) = 0, \gamma_0(1) = -\sqrt{c} - c \) and \( \gamma_1 \subset \mathbb{C} \) the \( u \)-shaped curve with \( \gamma_+(0) = t_0, \gamma_+(1) = t_0 + \sqrt{c} - c \). Suppose that \( c, \gamma_0, \gamma_1 \) are chosen such that the following two sets are submanifolds of \( X \times \mathbb{C} \):

\[
(L_1 \times \gamma_-) \cup K_1|_{0 < Re(z) < t_0} \cup (K_1|_{0 < Re(z) < t_0} + \sqrt{-1} \cdot c) \cup (L \times \gamma_+)
\]

\[
(L_2 \times \gamma_-) \cup K_2|_{0 < Re(z) < t_0} \cup (K_2|_{0 < Re(z) < t_0} + \sqrt{-1} \cdot c) \cup (L \times \gamma_+)
\]

Then they are exactly isotopic.

The argument is the same as the construction of \( t^L_q \) from [Hic21, Page 53].
2 Construction One: Squeezing Lagrangian tori in \( P(a, b) \)

In this section, we present Construction \[1.1\]. The idea of the construction is to view \( P(a, b) \) as a symplectic disc bundle over a disc. Then we concatenate several Lagrangian suspension cobordisms of circles in the disc fibers to produce a Lagrangian in \( P(a, b) \) that helps us displace \( L(1, 1) \) from the integral packing.

**Proof of Construction** \[1.1\] Let \( X \) be the standard symplectic disk of area \( a > 2 \) and \( D \) be the standard symplectic disk of area \( b > 1 \) so that \( P(a, b) = X \times D \). Let \( \pi_X \) and \( \pi_D \) be the projection to the first and second factors respectively. Let \( C_{X,m} = \pi_X(L(m, n)) \) and \( C_{D,n} = \pi_D(L(m, n)) \) so that \( L(m, n) = C_{X,m} \times C_{D,n} \).

Let \( C'_{D,1} \subset D \) be a small Hamiltonian perturbation of \( C_{D,1} \) such that \( C'_D \cap C'_{D,1} \) consists of two points. Let \( L(1, 1)' = C_{X,1} \times C'_{D,1} \). To show that the integral packing of \( P(a, b) \) by \( \{L(m, n)\} \) is not maximal, it suffices to find a Lagrangian \( L \) in \( P(a, b) \) that is Hamiltonian isotopic to \( L(1, 1) \), disjoint from \( L(1, 1)' \) and disjoint from \( L(m, n) \) for all \((m, n) \in \mathbb{N}^2 \setminus \{(1, 1)\}\). We regard \( D \) as a subset of \( \mathbb{C} \) and \( P(a, b) \subset X \times \mathbb{C} \). A portion of \( L \) will be the suspension of a Hamiltonian function of \( X \) (see Eq. \[2\]).

Let \( U_0 \) and \( U_1 \) be the two small (closed) bigons in \( D \) with one side on \( C_{D,1} \) and the other side on \( C'_D \), and denote the area of \( U_i \) be \( \delta \). The \( L \) we construct will have shadow inside \( U_0 \).

For \( \epsilon > 0 \), let \( S_{X,\epsilon} \subset X \) be an embedded circle such that

- it is disjoint from \( C_{X,m} \) for all \( m \geq 2 \)
- it is Hamiltonian isotopic to \( C_{X,1} \) by a compactly supported Hamiltonian
- the intersection between the disk bounded by \( C_{X,1} \) and the disk bounded by \( S_{X,\epsilon} \) is a simply connected region with area less than \( \epsilon \)

(see Figure \[7a\]). With the last assumption, if \( a > 2 + \epsilon \), then we can find a compactly supported Hamiltonian \( H_1 \in C^\infty([0, 1] \times X) \) such that \( \|H_1\| = \epsilon \) and the time 1-flow \( \phi_{H_1}^1(S_{X,\epsilon}) \cap C_{X,1} = \emptyset \) (see Fig. \[7b\]). We choose \( \epsilon < \min\{a - 2, \delta/3\} \) and fix such a choice of \( H \).

Let \( S_D \subset D \) be a small Hamiltonian perturbation of \( C_{D,1} \) such that

- it is disjoint from \( C_{D,n} \) for all \( n \geq 2 \)
- \( S_D \cap C_{D,1} \cap C'_{D,1} = \emptyset \)
- \( S_D \cap C_{D,1} \) consists of two points \( p, q \in \partial U_0 \)
- \( S_D \cap C'_{D,1} \) consists of two points \( p', q' \in \partial U_0 \)

By possibly relabeling, we assume that \( S_D \cap U_0 = I_{p,p'} \cup I_{q,q'} \) where \( \partial I_{p,p'} = \{p, p'\} \) and \( \partial I_{q,q'} = \{q, q'\} \). We can also assume that \( S_D \setminus (I_{p,p'} \cup I_{q,q'}) = I_{p,q} \cup I_{p',q'} \) where \( \partial I_{p,q} = \{p, q\} \) and \( \partial I_{p',q'} = \{p', q'\} \) (see Figure \[7c\]).

By construction, the product Lagrangian \( L_S := S_{X,\epsilon} \times S_D \) does not intersect \( L(m, n) \) for all \((m, n) \neq (1, 1)\). It is also clear that \( L_S \) is Hamiltonian isotopic to \( L(1, 1) \). However, \( L_S \) intersects \( L(1, 1)' \) non-trivially in the fibers \( X \times \{p', q'\} \) so we need to modify \( L_S \) to get our \( L \).
(a) Lagrangians in $X$.

(b) A Hamiltonian isotopy of $S_{X,\epsilon}$

(c) Lagrangians in $D$

(d) A modification of $S_{D,\epsilon}$ at the insert area from Fig. 7c. The labels correspond to the $X$-component of the Lagrangian.

Figure 7
To do that, recall that $3\epsilon < \delta = \text{area}(U_0)$. We replace $S_{X,\epsilon} \times I_{p,p'} \subset L_S$ by a suspension $L_{H,\epsilon}$ of $H$ such that (cf. Example $[1.8]$ and Proposition $[1.10]$)

- $\pi_D(L_{H,\epsilon}) \subset U_0$ and it has area $\epsilon$,
- $L_{H,\epsilon} \cap X \times C_{D,1} = S_{X,\epsilon} \times \{p\}$
- $L_{H,\epsilon} \cap X \times C'_{D,1} = \phi_H(S_{X,\epsilon}) \times \{p'\}$

as drawn in Fig. [7.1]. By assumption, $\phi_H(S_{X,\epsilon}) \cap C_{X,1} = \emptyset$ so $L_{H,\epsilon} \cap L(1,1)' = \emptyset$. Similarly, we replace $S_{X,\epsilon} \times I_{a,q} \subset L_S$ by a suspension $L_{H,q}$ of $H$ such that the corresponding conditions are satisfied. Moreover, we can assume that $\pi_D(L_{H,\epsilon}) \cap \pi_D(L_{H,q}) = \emptyset$ because $3\epsilon < \delta$.

Our desired $L$ is the union of $S_{X,\epsilon} \times I_{p,q}$, $L_{H,\epsilon}$, $L_{H,q}$ and $\phi_H(S_{X,\epsilon}) \times I_{p',q'}$. It is clear that $L$ is an embedded Lagrangian which is disjoint from $L(1,1)'$ and $L(m,n)$ for all $(m,n) \neq (1,1)$. Moreover, there is an exact Lagrangian isotopy from $L$ to $L_S$ given by Proposition $[1.11]$.

\[ \square \]

**Corollary 2.1.** When $a > 2$ and $b > 1$, there are 3 pairwise disjoint integral Lagrangians in $P(a,b)$. Moreover, two of which are Hamiltonian isotopic to $L(1,1)$ and the remaining one is Hamiltonian isotopic to $L(2,1)$

When $a > 2$ and $b > 2$, there are 6 pairwise disjoint integral Lagrangians in $P(a,b)$. Moreover, two of which are Hamiltonian isotopic to $L(1,1)$, two of which are Hamiltonian isotopic to $L(1,2)$, one of which is Hamiltonian isotopic to $L(2,1)$ and the remaining one is Hamiltonian isotopic to $L(2,2)$.

**Proof.** The first statement is a direct consequence of the previous construction. To explain the second statement, we use the notation in the previous proof. Recall that the construction happens in a neighborhood of $X \times C_{D,1}$. Therefore, when $b > 2$ (i.e. the area of $D$ is greater than 2), we can apply similar construction in a neighborhood of $X \times C_{D,2}$ to get an extra integral Lagrangian. It implies the second statement.

\[ \square \]

### 3 Construction Two: On displacing Lagrangian links in $S^2 \times S^2$

In this section, we present Construction $[1.2]$. The nature of this problem is similar to the previous one. In fact, Construction $[1.2]$ can be viewed as a generalization of Construction $[1.1]$.

**Proof of Construction** $[1.2]$ Let $X = S^2_{2B+c}$ and $D = S^2_{2a}$. Let $B - C \geq 0$, otherwise, the construction is obvious. Let $K_a' \subset D$ be a circle that is Hamiltonian isotopic to $K_a$ such that $K_a \cap K_a'$ consists of two points. To complete the construction, it suffices to find a Lagrangian $L$ in $X \times D$ that is Hamiltonian isotopic to $L_1 \times K_a$ and disjoint from both $L_1 \times K_a'$ and $L_2 \times K_a$.

Let $U_0, U_1$ and $V_0, V_1$ be the four (closed) bigons in $D$ with one side on $K_a$ and the other side on $K_a'$ such that their interiors are pairwise disjoint. By possibly relabeling, we assume that $U_0$ and $U_1$ have the same area, denoted by $\delta_U$, and that $V_0$ and $V_1$ have the same area, denoted by $\delta_V$. Therefore, we have $2a = 2(\delta_U + \delta_V)$. Without loss of generality, we assume that $\delta_U \geq \delta_V$.

For $\epsilon > 0$, let $S_{X,\epsilon} \subset X$ be an embedded circle such that

- it is disjoint from $L_2$
- it is Hamiltonian isotopic to $L_1$
- the intersection between the disk with area $B$ bounded by $L_1$ and the disk with area $B$ bounded by $S_{X,\epsilon}$ is a simply connected region with area less than $B - C + \epsilon$

With the last assumption, we can find a compactly supported Hamiltonian $H \in C^\infty([0,1] \times X)$ such that $\|H\| = B - C + \epsilon$ and $\phi_H(S_{X,\epsilon}) \cap L_1 = \emptyset$. We fix such a choice of $H$.

Let $S_D \subset D$ be a circle such that

- it is Hamiltonian isotopic to $K_a$
\begin{itemize}
  \item $S_D \cap K_a$ consists of two points $p, q \in \partial U_0$
  \item $S_D \cap K'_a$ consists of two points $p', q' \in \partial U_0$
  \item it is disjoint from $U_1$
\end{itemize}

By possibly relabeling, we assume that $S_D \cap U_0 = I_{p,p'} \cup I_{q,q'}$, $S_D \cap V_0 = I_{p,q}$ and $S_D \cap V_1 = I_{p',q'}$, where the subscripts of $I$ are the two endpoints.

By construction, the product Lagrangian $L_S := S_{X,\epsilon} \times S_D$ does not intersect $L_2 \times K_a$ but it intersects $L_1 \times K'_a$ non-trivially in the fibers $X \times \{p', q'\}$ so we need to modify $L_S$ to get our $L$.

To do that, we apply the same suspension trick as in Construction \ref{construction:3} to replace $S_{X,\epsilon} \times I_{p,p'}$, $S_{X,\epsilon} \times I_{q,q'}$ and $S_{X,\epsilon} \times I_{p',q'}$ by $L_{H,p}$, $L_{H,q}$ and $\phi_H(S_{X,\epsilon}) \times I_{p',q'}$, respectively. This is possible when the area of the shadow, $2(B - C + \epsilon)$, is less than $\delta_U$. By letting $\delta_V$ be sufficiently small, $\delta_U$ can be arbitrarily close to $a$ so the construction works as long as $2(B - C) < a$.

The proof of Construction \ref{construction:3} is entirely parallel.

4 Construction Three: Lagrangian submanifolds in $S^2 \times S^2$ of low genus

4.1 Isotopic Lagrangians are Lagrangian cobordant

Construction \ref{construction:3} is based on a special instance of the following general observation.

**Lemma 4.1.** Let $[\eta] \in H^1(L, \mathbb{R})$ be Poincaré dual to $[H] \in H_1(L, \mathbb{Z})$, where \( \eta \) is a closed 1-form and $H$ is an embedded hypersurface of $L$. Let $L_\eta \subseteq T^*L$ be the Lagrangian section given by the graph of $\eta$. There exists an embedded Lagrangian cobordism $K_{\eta,H} : L_0 \rightsquigarrow L_\eta$.

**Proof.** Let $H \times [-1, 1]$ be a normal neighborhood of $H$ in $L$. We can split $T^*(H \times [-1, 1]) = T^*H \times T^*[-1, 1]$.

Let $(p, q)$ be the coordinates on $T^*[-1, 1]$. Consider the curves $\gamma_1, \gamma_2 \subset T^*[-1, 1]$ drawn in Fig. \ref{fig:gamma1_gamma2}, which are chosen so that the area contained by the curve is at least 30. The Lagrangians $H \times \gamma_1$ and $H \times \gamma_2$ are Hamiltonian isotopic. Let $K_1 : H \times \gamma_1 \rightsquigarrow H \times \gamma_2$ be the suspension cobordism.

$(H \times \gamma_1) \cup (H \times \gamma_2)$ intersects $H \times [-1, 1] \subset L$ cleanly inside $T^*L$. Let $I_1 := [-1, 1] \cap \gamma_i$. Using [MW18, Corollary 2.22] we can perform a clean Lagrangian surgery, and let

$L' := (H \times \gamma_1)\#_{H \times I_1} L \#_{H \times I_2} (H \times \gamma_2)$.

We have a surgery trace cobordism

$K_2 : (H \times \gamma_1, L, H \times \gamma_2) \rightsquigarrow L'$.

The resulting surgery (drawn in Fig. \ref{fig:surgery_trace}) has two connected components, both of which are immersed. On the top connected component, perform an exact homotopy from $L'$ to $L''$ in the $T^*[-1, 1]$ factor as indicated in Fig. \ref{fig:surgery_trace}. The exact homotopy is constructed so that area swept over the blue region is 1 and so that each of the red regions has an area of 1/2. Let $K_3 : L' \rightsquigarrow L''$ be the suspension of this exact homotopy. Let $K_4 : L'' \rightsquigarrow (H \times \gamma_1', L'', H \times \gamma_2')$ be the antisurgery trace cobordism from Fig. \ref{fig:antisurgery_trace} to Fig. \ref{fig:surgery_trace}. The Lagrangians $H \times \gamma_1'$ and $H \times \gamma_2'$ are Hamiltonian isotopic; let $K_5$ be the suspension of this Hamiltonian isotopy. Note that

$L''' = (L \setminus (H \times [-1, 1])) \cup (H \times \gamma_3)$

where $\gamma_3$ is indicated in Fig. \ref{fig:gamma3}. This is the graph a section $\eta'$ of $T^*L$ with $[\eta'] = PD(H) = [\eta]$. We can therefore find a Hamiltonian isotopy between $L'''$ and $L_\eta$. Let $K_6 : L''' \rightsquigarrow L_\eta$ be the trace of this isotopy.

Finally, let $K'''' : L \rightsquigarrow L_\eta$ be the Lagrangian cobordism constructed by assembling $K_1, \ldots, K_6$ according to Fig. \ref{fig:assembled_cobordism}. This is an immersed Lagrangian cobordism; by perturbing and resolving the double points we obtain an embedded Lagrangian cobordism from $K_{\eta,H} : L \rightsquigarrow L_\eta$.

\[\square\]
(a) Projection of $H \times \gamma_1, H \times \gamma_2, H \times [-1, 1] \subset T^*H \times T^*[-1, 1]$ to the $T^*[-1, 1]$ component.

(b) Performing surgery at the clean self-intersections to obtain the Lagrangian $L'$.

(c) $L''$, which is exactly homotopic to $L'$. The red regions have an area of $1/2$, and the blue region an area of $1$.

(d) Performing antisurgery. The Lagrangians $\gamma_1'$ and $\gamma_2'$ are Hamiltonian isotopic.

(e) Using the cobordisms $K_1, \ldots, K_5$ to obtain a cobordism from $L$ to $L_\eta$. The labels on the diagram give the Lagrangian in $X$ above the corresponding portion of the Lagrangian cobordism.

Figure 8. Construction of a Lagrangian cobordism associated with a Lagrangian isotopy with integral flux class.
In a discussion with Álvaro Muñiz, it was observed that the construction of Lemma 4.1 gives a non-orientable Lagrangian cobordism.

**Corollary 4.2.** Let $X$ be a compact symplectic manifold. Whenever $L_0, L_1 \subset X$ are orientable and Lagrangian isotopic, there exists a Lagrangian cobordism $K : L_1 \rightsquigarrow L_0$.

**Proof.** First, suppose that $L_0$ is the zero section of $T^*L_0$, and that $L_1$ is some other section given by the graph of a 1-form $\eta$. Write $[\eta] = \sum_j \alpha_j PD([H_j])$ for some embedded hypersurfaces $H_j$. By performing the construction from Lemma 4.1 (and making the area of the blue region in Fig. 8c equal to $\alpha_j$) for each $H_j$, we obtain a Lagrangian cobordism $K_\eta$ from $L_0$ and $L_\eta$.

For $L_0, L_1 \subset X$, take a subdivision $0 = t_0 < \cdots < t_k = 1$, so that:

- There exists a Weinstein neighborhood $U$ of $L_{t_0}$ for which $L_{t_{i+1}}$ is the graph of a section $L_{\eta_i} \subset U \subset T^*L_{t_i}$.
- The $\eta_i$ are small enough so that the Lagrangian cobordisms $K_\eta : L_{t_i} \rightsquigarrow L_{\eta_i}$ constructed above fit entirely within $U \times \mathbb{C} \subset T^*L_{t_i} \times \mathbb{C}$.

Because $X$ is compact, such a subdivision can be found. Then by taking $K$ to be the concatenation of the $K_\eta$, we obtain a Lagrangian cobordism from $L_1$ to $L_0$.

We say that a Lagrangian submanifold $K : L_1 \rightsquigarrow L_0$ satisfying the properties of Lemma 4.1 is “topologically efficient” if

$$\mathbb{b}_4(K_{\eta,H}) := \dim(H^1(K_{\eta,H}, L_0))$$

is close to 0. Observe that when $K$ is topologically $L_0 \times \mathbb{R}$ then $\mathbb{b}_4(K) = 0$. This can also be expressed as $\dim(H^1_{\text{Morse}}(K, \pi_2[K]))$.

The construction given in Lemma 4.1 is not very topologically efficient. A more efficient (although more lengthy to explicitly describe) cobordism from $L \rightsquigarrow L''$ can be built by generalizing Lagrangian antisurgery disks from [Han20]. Let $\gamma$ be the upper semicircle belonging to the curve $\gamma_1$ from Fig. 8a so that $H \times \gamma$ is a Lagrangian with boundary cleanly meeting $H \times [-1, 1]$. This Lagrangian gives an anti-surgery data generalizing the data of a Lagrangian antisurgery disk considered by [Han20]. By performing antisurgery along $H \times \gamma$, we obtain a Lagrangian $\mathcal{T}$, which is the upper connected component of $L'$ from Fig. 8b. We then perform the same exact isotopy as in Fig. 8c and then a clean surgery to obtain $L''$.

A different measure of the “efficiency” of a Lagrangian cobordism is the shadow of the cobordism. Yet another measure of distance between isotopic Lagrangians can be computed using the flux symplectic isotopy, which measures the symplectic area swept by 1-cycles of $L$ over the isotopy. There is a trade-off between topologically efficient and shadow efficient Lagrangian cobordisms. In the simplest example that we consider (where $\eta = \alpha PD[H]$) the cobordism constructed $K_{\eta,H}$ has shadow $\text{Area}(K_{\eta,H}) > \alpha$. Let us additionally assume that the Lagrangian cobordism $K_\eta$ in the construction of $K_{\eta,H}$ is trivial, so we can assume that $\text{Area}(K_{\eta,H}) \approx A \cdot \alpha$ for some fixed constant of proportionality $A$.

An easy way to lower the shadow of the cobordism we construct is to take $n$-disjoint copies of $H$, so that $\eta = \alpha \cdot \frac{n}{n} \cdot \text{PD}[H]$. The construction we give has a shadow of approximately

$$\text{Area}(K_{\eta,nH}) \approx \frac{A \cdot \alpha}{n} \mathbb{b}_4(K_{\eta,nH})$$

If additionally we assume that $H$ is connected, this is proportional to $\frac{\alpha}{\mathbb{b}_4(K_{\eta,nH})}$. So, the shadow can be traded for the topological complexity of a cobordism.

More generally, an argument due to Emmy Murphy [CS10, Section 4.4] shows that given $K : L_1 \rightsquigarrow L_0$, for all $C > 0$ there exists a Lagrangian cobordism $K' : L_1 \rightsquigarrow L_0$ with $\text{Area}(K') < C$ (but $\mathbb{b}_4(K') \gg \mathbb{b}_4(K)$).

Given $L_0, L_1$ which are Lagrangian homotopic, let $\mathcal{T}(L_0, L_1) := \{ i : L \times I \to X \mid i_0(L) = L_0, i_1(L) = L_1, \omega|_{L_0} = 0 \}$ denote the set of Lagrangian homotopies between $L_0$ and $L_1$. Let $\mathcal{B}$ denote the set of basis for $H_1(L, \mathbb{Z})$. We look at the minimal flux

$$d(L_0, L_1) := \inf_{i \in \mathcal{T}(L_0, L_1)} \sum_{B \in \mathcal{B}} |\text{Flux}_{i(B)}|.$$
Figure 9. Some Lagrangian submanifolds in $X = T^*S^1$. (a) Two zig-zag Lagrangians when $n = 3$. The intersection points $x_i$ are marked with $\times$, and the $y_i$ are marked with circles. (b) The slice $K_{b,0}$ obtained from surgering the two zig-zags at the $x_i$. The green region has area $C + \epsilon/8$. (c) A Lagrangian obtained after applying an exact homotopy which removes the green region. (d) The slice $K_{d,-t_0}$, which is obtained from the previous Lagrangian by surgering at the $y_i$.

**Conjecture 4.3.** There exists a constant $A$ so that for any pair of Lagrangian isotopic submanifolds $L_0, L_1$ and Lagrangian cobordism $K : L_1 \leadsto L_0$, we have

$$\text{Area}(K) > \frac{A \cdot d(L_0, L_1)}{b_1(K)}. \tag{4}$$

The two terms in Eq. (4) cannot be bounded in the other direction, as whenever $L_0$ and $L_1$ are non-trivially Hamiltonian isotopic, the left term is non-zero while the right term is zero. We now provide a specific example motivating this bound.

**Example 4.4.** For this example we let $X = T^*S^1$. Let $S^1_1 \subset X$ be the Lagrangian submanifold given by the graph of $c \cdot d\theta$, so that $d(S^1_{\lambda/2}, S^1_{-\lambda/2}) = \lambda$ as in Fig. 9d. Pick $\epsilon > 0$ and $n \in \mathbb{N}$. We now describe a Lagrangian null cobordism of $K_{\lambda,n,\epsilon} : (S^1_{\lambda/2} \cup S^1_{-\lambda/2}) \leadsto \emptyset$, with the property that

$$\frac{\lambda}{2n} < \text{Area}(K_{\lambda,n,\epsilon}) < \frac{\lambda}{2n} + \epsilon$$

and $b_1(K_{\lambda,n,\epsilon}) = 4n + 1$.

For fixed $n \in \mathbb{N}$, consider the Lagrangian $L_1$ which is the zig-zag curve drawn in Fig. 9a. $L_1$ crosses the zero section $2n$-times. Let $L_2$ be the reflection of $L_1$ across the zero section. The Lagrangians $L_i$ are constructed so that the regions indicated in Fig. 9a satisfy $2A + B + C = \lambda/(2n)$. The value of $C$ can be taken as small as desired. The Lagrangian $L_1$ is Hamiltonian isotopic to the zero section, and the Höfer norm of the Hamiltonian isotopy is $A$. Let $K_1 : L_1 \to S^1_2$ be the suspension of this Hamiltonian isotopy. Similarly, let $K_2 : L_2 \to S^1_2$ be the suspension of the Hamiltonian isotopy from $L_2$ to the zero section. Then $\text{Area}(K_1) = \text{Area}(K_2) = A$. The Lagrangians $L_1$ and $L_2$ intersect transversely at $2n$ points. Label the intersection points $x_i = (i/n, 0)$ and $y_i = (i/n + 1/2n, 0)$ for $i \in \{0, \ldots, n - 1\}$. Consider the Lagrangian submanifold $L_{1x, \#y}, L_2$ which is obtained from $L_1$ and $L_2$ by performing surgery

- with $L_1$ as the first factor and $L_2$ as the second factor for surgeries performed at $\{x_i\}$ with surgery size $B$,
- with $L_2$ as the first factor, $L_1$ as the second factor at surgeries performed at $\{y_i\}$ with surgery size $C$

For these choices of Lagrangian surgeries, $L_{1x, \#y} L_2 = (S^1_{\lambda/2} \cup S^1_{-\lambda/2})$.

We now describe an embedded Lagrangian cobordism $K_3$ whose ends are $(L_1, L_2) \leadsto L_{1x, \#y}, L_2$. The construction is similar to the description of the surgery trace (Example 1.9), but requires an additional
argument due to the differing order of branches employed in the Polterovich surgery. First, consider the Lagrangian \( K_a := (L_1 \times \ell_1) \cup (L_2 \times \ell_2) \) as drawn in Fig. 10a. All of the self-intersections of \( K_a \) lie in the \( t = 0 \) slice, where \( K_a |_{s=0} = L_1 \cup L_2 \). Resolve the self-intersections of the form \( \{(x_1, 0)\} \) using Polterovich surgery with surgery neck size \( B + C + \epsilon/8 \) to obtain \( K_b \), drawn in Fig. 10b. The self intersections of \( K_b \) still live in the slice \( K_b |_{t=0} \), and are in bijection with \( \{(y_1) \times (\ell_1 \cap \ell_2)\} \). We draw the slice (Definition 1.7) \( K_b |_{t=0} \) in Fig. 8b.

Pick a value \(-t_0\) so that the area bounded by the lines \( \text{Re}(z) = -t_0 \), \( \text{Re}(z) = 0 \) and the shadow of \( K_b \) is \( C + \epsilon/8 \) (each green region from Fig. 10b). Using the discussion following [Hic 21, Example 4.2.2], reparameterize the Lagrangian cobordism \( K_b \) between \(-t_0, -t_1\) to create a bottleneck at \(-t_0\). Call the resulting Lagrangian \( K_c \) (Fig. 11c). This Lagrangian is immersed with transverse intersections, and corresponds to the suspension of an exact homotopy between the slices \(-t_0\) and 0. We have that \( K_b |_{t=0} = K_c |_{t=0} \), and the slice \( K_c |_{t=0} \) is drawn in Fig. 8c. Observe that Figs. 9b and 9c differ by the light green regions (also of area \( C + \epsilon/8 \)). We now apply Lagrangian surgery to the self-intersections of \( K_c \) at \( t = 0 \) and \( t = -t_0 \).

- At the self-intersections in the slice \( t = 0 \), perform surgery of all the points with a surgery neck width of \( \epsilon/8 \). This corresponds to the small red region in Fig. 10d.

- At the self-intersections in the slice \( t = -t_0 \), perform surgery with neck width of \( C \). This corresponds to the small yellow region drawn in Fig. 10d.

Call the subsequent Lagrangian submanifold \( K_d \). The dark gray region belonging to \( K_d \) has area \((C + \epsilon/8) - C = \epsilon/8 \). The slice \( K_d |_{t=0} = L_{1,x}, \# y, L_2 \) as desired. The Lagrangian submanifold \( K_3 \) is obtained by truncating ( [Hic21, Definition 3.1.6]) \( K_d \) at time \(-t_0\). We have from inspection that

\[
\text{Area}(K_3) = C + \epsilon/8 + 2\epsilon/8 + (C + \epsilon/8) + A = A + 2C + \epsilon/2
\]

Finally, let \( K_4 : (S_0^1, S_0^1) \leadsto \emptyset \) be a u-shaped null cobordism. The concatenation \( \tilde{K}_{\lambda,n,\epsilon} := K_4 \circ (K_1 \cup K_2) \circ (K_3)^{-1} : (S_{\lambda/2}^1 \cup S_{\lambda/2}^1) \leadsto \emptyset \) drawn in Fig. 11 is a null cobordism. The concatenation is performed so the “gap” in the middle of the cobordism has area \( \epsilon/2 \). In this way,

\[
\frac{\lambda}{2n} = 2B + A + C < \text{Area}(\tilde{K}_{\lambda,n,\epsilon}) < (A + 2C + \epsilon/2) + 2B + \epsilon/2 = \frac{\lambda}{2n} + \epsilon + C
\]

We recall that \( \epsilon, C \) were independently chosen as small as desired. For convenience, we merge these two constants into one, so that \( \text{Area}(\tilde{K}_{\lambda,n,\epsilon}) < \lambda/(2n) + \epsilon \). By rearranging the ends of \( \tilde{K}_{\lambda,n,\epsilon} \), we obtain a Lagrangian cobordism \( K_{\lambda,n,\epsilon} : S_{\lambda/2}^1 \leadsto S_{\lambda/2}^1 \). \( K_d \) is constructed by employing \( 3n \) Lagrangian surgeries to \((L_1 \times \ell_1) \cup (L_2 \times \ell_2) \). We have \( b_1(L_1 \times \ell_1) = 0 \). Each Lagrangian surgery (after the first, which only contributes 1) contributes 2 to \( b_1 \), so \( b_1(K_d) = 2 \cdot (3n - 1) + 1 \). When we truncate \( K_d \) to obtain \( K_3 \), we

---

**Figure 10.** Some shadows of Lagrangian cobordisms. (a) Two trivial suspensions. (b) Surgering the Lagrangians at their intersection. The green region has area \( C + \epsilon/8 \), while the purple region has area \( B \). (c) Flipping the left end to create a bottleneck. Areas are preserved. (d) Applying a surgery at the intersections in the slices \(-t_0\) and 0. The yellow region has area \( C \).
Figure 11. Concatenation to form a null-cobordism of $S^1_\lambda \cup S^1_{-\lambda}$. The center region has area $\epsilon/2$.

Figure 12. Dividing the $S^2_\lambda \times S^2_1$ into two portions. The left-hand side $K_1$ is a 2-punctured sphere. The right hand side $K_2$ is described in Example 4.4.

remove $2n$ index-1 critical points of $\text{Re} : K_d \to \mathbb{R}$, so $b_1(K_3) = 6n - 1 - 2n = 4n - 1$. Since $b_1(K_4) = 1$, we obtain $b_1(K_{\lambda,n,\epsilon}) = 4n$, from which it follows that

$$b_1(K_{\lambda,n,\epsilon}) = 4n + 1.$$

From this construction, we obtain Lagrangians satisfying

$$2 \frac{d(S^1_{\lambda/2}, S^1_{-\lambda/2})}{b_1(K_{\lambda,n,\epsilon})} = \frac{\lambda}{2n + 1/2} < \frac{\lambda}{2n} < \text{Area}(K_{\lambda,n,\epsilon}) < \frac{\lambda}{2n} + \epsilon < 2 \frac{d(S^1_{\lambda/2}, S^1_{-\lambda/2})}{b_1(K_{\lambda,n,\epsilon})} + \frac{\lambda}{8n^2} + \epsilon.$$

4.2 Application:

We now use the previous discussion to explain Construction 4.4.

Proof. Let $n$ be the smallest integer greater than $\lambda/2$, and choose $\epsilon$ small enough so that

$$\frac{\lambda - \epsilon}{2n} + 2\epsilon < 1.$$

Consider the decomposition

$$S^2_\lambda \times S^2_1 = (S^2_\lambda \times D^2_\epsilon) \cup (S^2_\lambda \times D^2_{1-\epsilon}),$$

where $D^2_A$ is the symplectic disk with area $A$. In the first component, use [Mik19] to construct $K_1$, the Lagrangian lift of the tropical curve drawn on the left-hand side of Fig. 12. The topology of $K_1$ is a two-punctured sphere whose boundary is $((\bullet) \times (S^1_{-\lambda/2-\epsilon} \cup S^1_{\lambda/2-\epsilon})) \subset \{\bullet\} \times S^2_\lambda$. Let $B^1_{\lambda} S^1$ be the portion of the cotangent bundle of $S^1$ between $S^1_{-\lambda/2}$ and $S^1_{\lambda/2}$. On the right hand side of Fig. 12 consider the Lagrangian null cobordism:

$$\tilde{K}_{\lambda-2\epsilon,n,\epsilon} : S^1_{-\lambda/2-\epsilon} \cup S^1_{\lambda/2-\epsilon} \to \emptyset$$
from Example 4.4. Observe that

$$\text{Area}(K_{\lambda - 2\epsilon, n}) < \frac{\lambda - \epsilon}{2n} + \epsilon < 1 - \epsilon.$$ 

Therefore, this Lagrangian cobordism fits inside $B^*_\lambda S^1 \times D^2_{1-\epsilon} \subset S^2_\lambda \times D^2_{1-\epsilon}$. We call this Lagrangian chart $K_2$. $K_1$ and $K_2$ share the same boundary, so we can define $K = K_1 \cup K_2$. This is in the $\mathbb{Z}/2\mathbb{Z}$ homology class of $[S^2_\lambda \times \{\bullet\}]$ and has a non-orientable genus $4n + 1$.

\[\square\]

References

[Arn80] Vladimir Igorevich Arnol’d. “Lagrange and Legendre cobordisms. I.” In: Funktsional’nyi Analiz i ego Prilozheniya 14.3 (1980), pp. 1–13.

[CG+21] Daniel Cristofaro-Gardiner et al. “Quantitative Heegaard Floer cohomology and the Calabi invariant.” In: arXiv:2105.11026 (2021). doi:10.48550/ARXIV.2105.11026

[CS19] Octav Cornea and Egor Shelukhin. “Lagrangian cobordism and metric invariants.” In: Journal of Differential Geometry 112.1 (2019), pp. 1–45.

[Eva21] Jonny Evans. “A Lagrangian Klein bottle you can’t squeeze.” In: Journal of Fixed Point Theory and Applications (2021).

[Hau20] Luis Haug. “Lagrangian antisurgery.” In: Mathematical Research Letters 27.5 (2020), pp. 1423–1464.

[Hic21] Jeff Hicks. Lagrangian cobordisms and Lagrangian surgery. 2021. arXiv:2102.10197 [math.SG]

[HK21] Richard K. Hind and Ely Kerman. Packing Lagrangian tori. 2021. arXiv:2109.01772 [math.SG]

[McD11] Dusa McDuff. “Displacing Lagrangian toric fibers via probes.” In: Low-dimensional and symplectic topology. Vol. 82. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 2011, pp. 131–160. doi:10.1090/pspum/082/2768658

[Mik19] Grigory Mikhalkin. “Examples of tropical-to-Lagrangian correspondence.” In: European Journal of Mathematics 5.3 (2019), pp. 1033–1066.

[MS21] Cheuk Yu Mak and Ivan Smith. “Non-displaceable Lagrangian links in four-manifolds.” In: Geometric and Functional Analysis 31.2 (2021), pp. 438–481.

[MW18] Cheuk Yu Mak and Weiwei Wu. “Dehn twist exact sequences through Lagrangian cobordism.” In: Compositio Mathematica 154.12 (2018), pp. 2485–2533.

[Pol01] Leonid Polterovich. The geometry of the group of symplectic diffeomorphisms. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2001, pp. xii+132. ISBN: 3-7643-6432-7. doi:10.1007/978-3-0348-8299-6