On the negative limit of viscosity solutions for discounted Hamilton-Jacobi equations

Ya-Nan Wang · Jun Yan · Jianlu Zhang

Abstract Suppose $M$ is a closed Riemannian manifold. For a $C^2$ generic (in the sense of Mañé) Tonelli Hamiltonian $H : T^*M \rightarrow \mathbb{R}$, the minimal viscosity solution $u_{-\lambda} : M \rightarrow \mathbb{R}$ of the negative discounted equation

$$-\lambda u + H(x, d_x u) = c(H), \quad x \in M, \quad \lambda > 0$$

with the Mañé’s critical value $c(H)$ converges to a uniquely established viscosity solution $u_0^-$ of the critical Hamilton-Jacobi equation

$$H(x, d_x u) = c(H), \quad x \in M$$

as $\lambda \rightarrow 0_+$. We also propose a dynamical interpretation of $u_0^-$. 

Keywords discounted Hamilton-Jacobi equation, viscosity solution, conjugated weak KAM solutions, Aubry Mather sets, Mañé’s genericity

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1 Introduction

Suppose $M$ is a closed manifold equipped with a smooth Riemannian metric $g$. $TM$ (resp. $T^*M$) denotes the tangent (resp. cotangent) bundles on $M$, on which a coordinate $(x,v) \in TM$ (resp. $(x,p) \in T^*M$) is involved. Since $M$ is compact, we can use $|\cdot|_x$ as the norm induced by $g$ on both the fiber $T_xM$ and $T^*_xM$, with a slight abuse of notations. A $C^2$ function $H: T^*M \times \mathbb{R} \to \mathbb{R}$ is called a Tonelli Hamiltonian, if

1. (Positive definiteness) $\forall (x,p) \in T^*M$, the Hessian matrix $\partial_{pp} H(x,p)$ is positive definite;
2. (Superlinearity) $H(x,p)$ is superlinear in the fibers, i.e.,

$$\lim_{|p|_x \to \infty} \frac{H(x,p)}{|p|_x} = +\infty, \quad \forall x \in M.$$ 

For such a Hamiltonian, we consider the negative discounted equation

$$-\lambda u + H(x, d_x u) = c(H), \quad x \in M, \quad \lambda > 0 \quad \text{(HJ}_\lambda^-\text{)}$$

with the Mañé critical value (see [3])

$$c(H) := \inf_{u \in C^\infty(M,\mathbb{R})} \sup_{x \in M} H(x, d_x u).$$

We can prove that the viscosity solution of $\text{(HJ}_\lambda^-\text{)}$ indeed exists but unnecessarily unique (see Sec. 2). Nonetheless, the ground state solution defined by

$$u^\lambda := \inf \{ u(x) | x \in M, u \text{ is a viscosity solution of } \text{(HJ}_\lambda^-\text{) } \}$$

is uniquely identified and satisfies a bunch of fine properties (see Sec. 2 for details). That urges us to explore the convergence of $u^\lambda$ as $\lambda \to 0^+$. 

**Theorem 11 (Main 1)** For a $C^2$ smooth Tonelli Hamiltonian $H(x,p)$ generic in the sense of Mañé [4], the ground state solution $u^\lambda$ of $\text{(HJ}_\lambda^-\text{)}$ converges to a specified viscosity solution $u^0$ of

$$H(x, d_x u) = c(H), \quad x \in M \quad \text{(HJ}_0\text{)}$$

as $\lambda \to 0^+$. 

\(^1\) See Definition A6 for the definitions of Mañé’s genericity.
Remark 12 Actually, we proved the convergence of $u^\lambda_n$ for any $C^2$ Tonelli Hamiltonian $H(x, p)$ satisfying the following:

For any sequence $\lambda_n \to 0_+$ as $n \to +\infty$,

$$\left( \bigcap_{N>0} \bigcup_{n \geq N} G_{\lambda_n} \right) \cap A_i \neq \emptyset, \quad \forall A_i \in A/d_c. \quad (\star)$$

see Sec. 2 for the definition of $G_\lambda$ and Appendix A for $A$, $A/d_c$. This is a generic condition for $C^2$ Tonelli Hamiltonian $H(x, p)$.

By using a variational analysis of contact Hamiltonian systems developed in [16,17,18,19], a breakthrough on the vanishing discount problem in the negative direction was recently made in [8]. Precisely, for any $C^3$ smooth Tonelli Hamiltonian $H(x, p)$ satisfying

constant functions are subsolutions of $(HJ^0)$, \( \quad (\diamondsuit) \)

they proved the convergence of the ground state solution $u^\lambda$ of $H_J^\lambda$ as $\lambda \to 0_+$. Actually, any $H(x, p)$ satisfying $(\diamondsuit)$ has to satisfy $(\star)$, since for any $\lambda > 0$ the associated $A \subset G_\lambda$ (proved in Proposition 3.2 of [8]). However, $(\diamondsuit)$ is not generic, that’s the reason we find a more general substitute $(\star)$ in this paper.

The significance of the convergence of viscosity solutions for discounted equations was firstly proposed by Lions, Papanicolaou and Varadhan in 1987 [11]. In [7], a rigorous proof for the convergence was finally given for the positive discounted systems. Also in recent works [3,20,21], the convergence of solutions for generalized 1st order PDE was discussed. Comparing to these works, the negative discount in $(HJ^\lambda)$ brings new difficulties to prove the existence of viscosity solutions, not to mention the convergence. By using a dual Lagrangian approach, we reveal that the negative discount limit of solutions actually conjugates to the positive discounted limit, see Sec. 3 for details.

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Organization of the article. In Sec. 2 we discuss the convergence of the Lax-Oleinik semigroup for $(HJ^\lambda)$ and the properties of the ground state solution $u^\lambda$. In Sec. 3 we finish the proof of Theorem 11. For the consistency and readability, a brief review of the Aubry-Mather theory and some properties of the Lax-Oleinik semigroup are moved to the Appendix.

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2 Generalities

Definition 21 (weak KAM solution) For $\lambda \geq 0$, a function $u : M \to \mathbb{R}$ is called a backward (resp. forward) $\lambda$–weak KAM solution if it satisfies:
for any \(x \in M\) there exists a curve \(\gamma_{x,\lambda}^-(t) : (-\infty, 0] \to M\) (resp. \(\gamma_{x,\lambda}^+ : [0, +\infty) \to M\)) ending with (resp. starting from) \(x\), such that for any \(s < t\) (resp. \(0 \leq s < t\)),

\[
e^{-\lambda t}u(\gamma_{x,\lambda}^-(t))-e^{-\lambda s}u(\gamma_{x,\lambda}^-(s)) = \int_s^t e^{-\lambda \tau} \left( L(\gamma_{x,\lambda}^-, \dot{\gamma}_{x,\lambda}^-) + c(H) \right) d\tau.
\]

( resp. \(e^{-\lambda t}u(\gamma_{x,\lambda}^+(t))-e^{-\lambda s}u(\gamma_{x,\lambda}^+(s)) = \int_s^t e^{-\lambda \tau} \left( L(\gamma_{x,\lambda}^+, \dot{\gamma}_{x,\lambda}^+) + c(H) \right) d\tau.\)

Such a \(\gamma_{x,\lambda}^-\) (resp. \(\gamma_{x,\lambda}^+\)) is called a backward (resp. forward) calibrated curve by \(u\).

Notice that

\[
u_{\lambda}^- := -\inf_{\gamma \in C^{ac}([0, +\infty), M)} \int_0^{+\infty} e^{-\lambda \tau} \left( L(\gamma(\tau), \dot{\gamma}(\tau)) + c(H) \right) d\tau
\]

is the unique forward \(\lambda\)-weak KAM solution of \((HJ_{\lambda})\) (see Proposition 22 for details). However, it’s not straightforward to get one backward \(\lambda\)-weak KAM solution for \((HJ_{\lambda})\). Usually, that relies on the backward Lax-Oleinik operator

\[
T_{t^-}^{\lambda} : C(M, \mathbb{R}) \to C(M, \mathbb{R})
\]

via

\[
T_{t^-}^{\lambda} \phi(x) = \inf_{\gamma(0) = x} \left\{ e^{\lambda t} \phi(-t) + \int_0^t e^{-\lambda \tau} \left( L(\gamma(\tau), \dot{\gamma}(\tau)) + c(H) \right) d\tau \right\}, \quad t \geq 0.
\]

If we can find a fixed point of \(T_{t^-}^{\lambda}\), then it automatically becomes a backward \(\lambda\)-weak KAM solution of \((HJ_{\lambda})\), then becomes a viscosity solution of \((HJ_{\lambda})\) (see (6) & (7) of Lemma C1).

Different from the approach in [8,18,19], here we proved the same conclusions as [8] in our own way:

**Proposition 22 (ground state)** For \(\lambda > 0\),

\[
u_{\lambda}^+ := \lim_{t \to +\infty} T_{t^-}^{\lambda} \nu_{\lambda}^-
\]

exists as a backward \(\lambda\)-weak KAM solution of \((HJ_{\lambda})\) and satisfies the following:
(a) \( u^\lambda_\gamma(x) \) is a viscosity solution of \((HJ^\lambda_\gamma)\);

(b) \( u^\lambda_\gamma(x) \geq u^\lambda_\gamma(x) \) for all \( x \in M \) and the equality holds on and only on the following set\(^2\)

\[
G^\lambda := \left\{ x \in M \mid \exists \gamma : \mathbb{R} \to M \text{ with } \gamma(0) = x, \ e^{-\lambda b} u^\lambda_\gamma(\gamma(b)) - e^{-\lambda a} u^\lambda_\gamma(\gamma(a)) = \int_a^b e^{-\lambda t} (L(\gamma(t), \dot{\gamma}(t)) + c(H)) \, dt, \ \forall \ a < b \in \mathbb{R} \right\}.
\]

(c) \( u^\lambda_\gamma \) is the minimal viscosity solution of \((HJ^\lambda_\gamma)\). Namely, any fixed point of

\[ T^\lambda_{-\gamma} : C(M, \mathbb{R}) \to C(M, \mathbb{R}) \]  

has to be greater than \( u^\lambda_\gamma \).

Proof As we can see,

\[
\tilde{u}^\lambda_\gamma(x) := -u^\lambda_\gamma(x)
\]

\[
= \inf_{\gamma \in C^{\infty}((-\infty, 0], M)} \int_0^\infty e^{-\lambda t} \left( L(\gamma(t), \dot{\gamma}(t)) + c(H) \right) \, dt
\]

\[
= \inf_{\gamma \in C^{\infty}((0, \infty), M)} \int_0^\infty e^{\lambda t} \left( \tilde{L}(\gamma(t), \dot{\gamma}(t)) + c(H) \right) \, dt,
\]

where \( \tilde{L}(x, v) := L(x, -v) \) for \((x, v) \in TM\). Due to Appendix 2 in \cite{[7]}, \( \tilde{u}^\lambda_\gamma \) is the unique viscosity solution of

\[
\lambda u + \tilde{H}(x, d_x u) = c(H), \quad x \in M, \lambda > 0
\]

with \( \tilde{H}(x, p) := H(x, -p) \) for \((x, p) \in T^*M\). Besides, for any \( x \in M \) the curve \( \tilde{\gamma}^\lambda_\gamma : t \in (-\infty, 0] \to M \) ending with \( x \) and attaining the infimum of \((\tilde{u}^\lambda_\gamma)\) always exists, and \( \| \tilde{\gamma}^\lambda_\gamma(t) \|_{L^\infty} \leq \alpha \) for some constant \( \alpha > 0 \) independent of \( x \) and \( \lambda \). Therefore, \( \gamma^\lambda_\gamma(t) := \tilde{\gamma}^\lambda_\gamma(-t) \) defines a curve for \( t \in [0, +\infty) \) starting from \( x \), which is forward calibrated by \( u^\lambda_\gamma \). On the other side, we can easily verify that \( u^\lambda_\gamma \) in \((S^1)\) is indeed a forward \( \lambda \)-weak KAM solution of \((HJ^\lambda_\gamma)\).

Since now \( \tilde{L}(x, v) \) is \( C^2 \)-smooth, due to the Tonelli Theorem and Weierstrass Theorem in \cite{[14]}, \( \gamma^\lambda_\gamma(\tau) \) is \( C^2 \)-smooth and \((\gamma^\lambda_\gamma(\tau), \dot{\gamma}^\lambda_\gamma(\tau)) \) solves the Euler-Lagragian equation:

\[
\begin{cases}
\dot{x} = v \\
\frac{d}{dt} \left( \frac{\partial L}{\partial v}(x, v) \right) - \lambda \frac{\partial L}{\partial x}(x, v) = \frac{\partial L}{\partial x}(x, v),
\end{cases}
\]

for \( \tau \in (0, +\infty) \). Hence, \( |\dot{\gamma}^\lambda_\gamma(\tau)| \leq \alpha \) for all \( \tau \in (0, +\infty) \). Moreover, due to Proposition 2.5 in \cite{[7]}, \( \{ \tilde{a}_\lambda(\tau) \}_{\lambda > 0} \) are equi-Lipschitz and equi-bounded, then consequently so are \( \{ u^\lambda_\gamma(x) \}_{\lambda > 0} \).

\(^2\) the following curve \( \gamma \) is called globally calibrated by \( u^\lambda_\gamma \)
Let $\Phi_{L,\lambda}^t : TM \to TM$ be the flow of (6). By an analogy of [13], we define
\[ \Sigma_\lambda = \{(x, v) \in TM | \gamma(s) = \pi \circ \Phi_{L,\lambda}^s(x, v) \text{ is a forward calibrated curve by } u_\lambda^+ \} \]
where $\pi : TM \to M$ is the standard projection. Apparently,
(i) $\Sigma_\lambda$ is compact.
(ii) $\Sigma_\lambda$ is forward invariant under $\Phi_{L,\lambda}^t$, i.e., $\Phi_{L,\lambda}^t(\Sigma_\lambda) \subset \Sigma_\lambda$, for all $t > 0$.
Then
\[ \tilde{G}_\lambda = \bigcap_{t \geq 0} \Phi_{L,\lambda}^t(\Sigma_\lambda) \] (7)
is nonempty, compact and $\Phi_{L,\lambda}$-invariant. Due to (a.3) of Sec. 4 in [13], $G_\lambda := \pi \tilde{G}_\lambda$ has an equivalent definition as in (3).

Next, we claim two conclusions:
(iii) $T_{-t}^\lambda u_\lambda^+(x) \geq u_\lambda^+(x)$, for all $x \in M$, $t > 0$ (proved in Lemma 23);
(iv) For any $\lambda > 0$ fixed, the set $T_{-t}^\lambda u_\lambda^+(x)$ are uniformly bounded from above for all $t \geq 1$ and $x \in M$ (proved in Lemma 24).

Benefiting from these conclusions and (5) of Lemmas C1, $T_{-t}^\lambda u_\lambda^+(x)$ converges uniformly to a function (denoted by $u_\lambda^-(x)$) on $M$ as $t$ tending to $+\infty$. Due to (1) of Lemma C1 for each $t_1 > 0$ we derive
\[ u_\lambda^-(x) = \lim_{t \to +\infty} T_{t_1+t}^\lambda u_\lambda^+(x) = \lim_{t \to +\infty} T_{t_1}^\lambda \circ T_{-t}^\lambda u_\lambda^+(x) \]
\[ = T_{t_1}^\lambda (\lim_{t \to +\infty} T_{-t}^\lambda u_\lambda^+(x)) = T_{t_1}^\lambda u_\lambda^-(x), \]
then by (6) of Lemma C1 $u_\lambda^-(x)$ is a backward $\lambda$-weak KAM solution of $\mathbf{HJ}_{-\lambda}$, and by (7) of Lemma C1 $u_\lambda^-(x)$ is a viscosity solution of $\mathbf{HJ}_{-\lambda}$. So (a) of Proposition 22 is proved.

Due to (iii), we get $u_\lambda^-(x) \geq u_\lambda^+(x)$. For any $x \in G_\lambda$, there exists a globally calibrated curve $\gamma_x : \mathbb{R} \to M$ passing it and achieving (1) on any interval $[a, b] \subset \mathbb{R}$. So for any $t > 0$,
\[ T_{-t}^\lambda u_\lambda^-(x) \leq e^{\lambda t} u_\lambda^-(\gamma_x(-t)) + \int_{-t}^0 e^{-\lambda \tau}(L(\gamma_x(\tau), \dot{\gamma}_x(\tau)) + c(H))d\tau \]
\[ = u_\lambda^+(x), \]
which implies $u_\lambda^-(x) = u_\lambda^+(x)$ on $G_\lambda$.

On the other side, if $u_\lambda^+(z) = u_\lambda^-(z)$ for some point $z \in M$, there exists a backward calibrated curve $\gamma_{z,\lambda} : (-\infty, 0] \to M$ ending with $z$ and a forward
calibrated curve \( \gamma^+_{z,\lambda} : [0, +\infty) \to M \) starting with \( z \). Note that \( u^+_\lambda(x) \leq u^-_\lambda(x) \) and \( u^+_\lambda \prec \lambda L + c(H) \), then for any \( t > 0 \),

\[
e^{-\lambda t} u^+_\lambda(\gamma^+_{z,\lambda}(t)) - u^-_\lambda(z) \leq e^{-\lambda t} u^-_\lambda(\gamma^+_{z,\lambda}(t)) - u^-_\lambda(z) \\
\leq \int_0^t e^{-\lambda \tau} (L(\gamma^+_{z,\lambda}(\tau)), \dot{\gamma}^+_{z,\lambda}(\tau)) + c(H) d\tau \\
= e^{-\lambda t} u^+_\lambda(\gamma^+_{z,\lambda}(t)) - u^-_\lambda(z),
\]

which implies \( u^+_\lambda(\gamma^+_{z,\lambda}(t)) = u^-_\lambda(\gamma^+_{z,\lambda}(t)) \) for all \( t > 0 \). Similarly \( u^+_\lambda(\gamma^-_{z,\lambda}(t)) = u^-_\lambda(\gamma^-_{z,\lambda}(-t)) \), \( t > 0 \). Hence, for any \( a < b \in \mathbb{R} \)

\[
e^{-\lambda b} u^+_\lambda(\gamma_{z,\lambda}(b)) - e^{-\lambda a} u^+_\lambda(\gamma_{z,\lambda}(a)) = \int_a^b e^{-\lambda \tau} (L(\gamma_{z,\lambda}(\tau)), \dot{\gamma}_{z,\lambda}(\tau)) + c(H) d\tau,
\]

where \( \gamma_{z,\lambda} : (-\infty, +\infty) \to M \) is defined by

\[
\gamma_{z,\lambda}(\tau) = \begin{cases} 
\gamma^+_{z,\lambda}(\tau), \tau \in (-\infty, 0], \\
\gamma^-_{z,\lambda}(\tau), \tau \in [0, +\infty).
\end{cases}
\]

So \( \gamma_{z,\lambda} : \mathbb{R} \to M \) is globally calibrated by \( u^-_\lambda \) (also by \( u^+_\lambda \)) and then \( z \in \mathcal{G}_\lambda \). Finally \( u^+_\lambda(x) = u^-_\lambda(x) \) if \( x \in \mathcal{G}_\lambda \), which implies item (b) of Proposition 22.

To prove (c) of Proposition 22, it suffices to show any backward \( \lambda \)-weak KAM solution \( v^- \) of \( (HJ^-) \) is greater than \( u^+_\lambda(x) \) (due to (6) and (7) of Lemma 31). If so,

\[
v^-(x) = T^\lambda_{t+1} v^- (x) \geq T^\lambda_{t+1} u^+_\lambda(x)
\]

for all \( t > 0 \), then \( v^- (x) \geq \lim_{t \to +\infty} T^\lambda_{t+1} u^+_\lambda(x) = u^-_\lambda(x) \).

For each \( \gamma : [0, t] \to M \) with \( \gamma(0) = x \) and \( t > 0 \), we define \( \tilde{\gamma} : [- (t+1), 0] \to M \) by

\[
\tilde{\gamma}(s) = \begin{cases} 
\gamma(s + t + 1), s \in [- (t+1), -1], \\
\beta(s), s \in [-1, 0],
\end{cases}
\]

where \( \beta \) is a geodesic satisfying \( \beta(-1) = \gamma(t) \), \( \beta(0) = x \), and \( |\dot{\beta}(s)| \leq \text{diam}(M) =: k_0 \). Then,

\[
v^-(x) = T^\lambda_{t+1} v^- (x) \leq e^{\lambda(t+1)} v^- (x) + \int_{- (t+1)}^0 e^{-\lambda \tau} (L(\gamma(\tau)), \dot{\gamma}(\tau)) + c(H) d\tau \\
\leq e^{\lambda(t+1)} v^- (x) + e^{\lambda(t+1)} \int_{- (t+1)}^0 e^{-\lambda \tau} (L(\gamma(\tau)), \dot{\gamma}(\tau)) + c(H) d\tau \\
+ \int_{- (t+1)}^0 e^{-\lambda \tau} (L(\beta(\tau)), \dot{\beta}(\tau)) + c(H) d\tau.
\]
Hence,
\[v^-(x) \geq e^{-\lambda(t+1)} v^-(x) - \int_0^t e^{-\lambda\tau} (L(\gamma(\tau), \hat{\gamma}(\tau)) + c(H)) d\tau - e^{-\lambda(t+1)} \int_{-1}^0 e^{-\lambda\tau} (L(\beta(\tau), \hat{\beta}(\tau)) + c(H)) d\tau.
\]

We derive
\[v^-(x) \geq e^{-\lambda(t+1)} v^-(x) + A_t(x) - e^{-\lambda(t+1)} \frac{(C_{k_0} + c(H))(e^\lambda - 1)}{\lambda},\]
where \(C_{k_0} = \max \{L(x,v)|(x,v) \in TM, |v| \leq k_0\}\) and
\[A_t(x) = \inf_{\gamma \in C^{\infty}([0,t],M)} \int_0^t e^{-\lambda\tau} (L(\gamma(\tau), \hat{\gamma}(\tau)) + c(H)) d\tau.
\]
To show \(v^-(x) \geq u^+_\lambda(x)\), it suffices to show \(A_t(x)\) converges uniformly to \(u^+_\lambda(x)\) on \(M\), as \(t \to +\infty\). For this purpose, we take \(\gamma : [0,t] \to M\) be a minimizer of \(A_t(x)\) and define \(\tilde{\gamma} : [0, +\infty) \to M\) by
\[\tilde{\gamma}(s) = \begin{cases} \gamma(s), & s \in [0,t], \\ \gamma(t), & s \in (t, +\infty). \end{cases}
\]
Then, there exists a constant \(C_0\) greater than \(\max_{x \in M} |L(x,0)|\) such that
\[A_t(x) - u^+_\lambda(x) \leq \int_0^{+\infty} e^{-\lambda\tau} (L(\tilde{\gamma}(\tau), \hat{\tilde{\gamma}}(\tau)) + c(H)) d\tau - \int_0^t e^{-\lambda\tau} (L(\gamma(\tau), \hat{\gamma}(\tau)) + c(H)) d\tau = \int_t^{+\infty} e^{-\lambda\tau} (L(\gamma(\tau), 0) + c(H)) d\tau \leq (C_0 + c(H)) \int_t^{+\infty} e^{-\lambda\tau} d\tau = \frac{(C_0 + c(H))e^{-\lambda t}}{\lambda}.
\]
On the other hand, let \(\gamma^+_\lambda : [0, +\infty) \to M\) be a minimizer of \(u^+_\lambda(x)\). Recall that \(\gamma^+_\lambda(t) = \gamma^-_{\lambda,\lambda}(-t)\), by a similar way,
\[u^+_\lambda(x) - A_t(x) \leq - \int_{-t}^{+\infty} e^{-\lambda\tau} (L(\gamma^+_{\lambda}(\tau), \hat{\gamma}^+_{\lambda}(\tau)) + c(H)) d\tau \leq (C(0) - c(H)) \int_{-t}^{+\infty} e^{-\lambda\tau} d\tau = \frac{(C(0) - c(H))e^{-\lambda t}}{\lambda},
\]
where \(C(0) = - \min \{L(x,v)|(x,v) \in TM\}\). Hence, we derive \(A_t(x)\) converges uniformly to \(u^+_\lambda(x)\) on \(M\), as \(t\) tending to \(+\infty\). \(\square\)
Lemma 23

\[ T^\lambda_t u^+_\lambda(x) \geq u^+_\lambda(x), \forall x \in M, t > 0. \]

Hence, \( T^\lambda_{t_2} u^+_\lambda(x) \geq T^\lambda_{t_1} u^+_\lambda(x) \), for \( t_2 \geq t_1 > 0 \).

Proof Let \( \gamma \in C^{\infty}([-t, 0], M) \) with \( \gamma(0) = x \) and \( \eta \in C^{\infty}([0, +\infty), M) \) with \( \eta(0) = x \). Define \( \tilde{\gamma} : [-t, +\infty) \to M \) by

\[
\tilde{\gamma}(\tau) = \begin{cases} 
\gamma(\tau), & \tau \in [-t, 0], \\
\eta(\tau), & \tau \in [0, +\infty).
\end{cases}
\]

Then,

\[
e^{\lambda t}u^+_\lambda(\gamma(-t)) + \int_{-t}^{0} e^{-\lambda \tau} (L(\gamma(\tau), \dot{\gamma}(\tau)) + c(H))d\tau \\
\geq -\int_{-t}^{+\infty} e^{-\lambda \tau} (L(\tilde{\gamma}(\tau), \dot{\tilde{\gamma}}(\tau))) + c(H))d\tau + \int_{-t}^{0} e^{-\lambda \tau} (L(\gamma(\tau), \dot{\gamma}(\tau))) + c(H))d\tau \\
= -\int_{0}^{+\infty} e^{-\lambda \tau} (L(\eta(\tau), \dot{\eta}(\tau)) + c(H))d\tau.
\]

By the arbitrariness of \( \gamma \) and \( \eta \), we derive that \( T^\lambda_t u^+_\lambda(x) \geq u^+_\lambda(x) \). \( \square \)

Lemma 24 Given \( \lambda > 0 \), the set \( \{ T^\lambda_t u^+_\lambda(x)|t \geq 1, x \in M \} \) is bounded from above.

Proof Let \( z \in G_\lambda \), \( \gamma_{z, \lambda}(s) \) be the globally calibrated curve by \( u^+_\lambda \). Let \( \beta : [-1, 0] \to M \) be a geodesic satisfying \( \beta(-1) = \gamma_{z, \lambda}(-1), \beta(0) = x \), and \( |\dot{\beta}(\tau)| \leq \text{diam}(M) = k_0 \). Then,

\[
T^\lambda_t u^+_\lambda(x) \leq u^+_\lambda(\gamma_{z, \lambda}(-t))e^{\lambda t} + \int_{-t}^{-1} e^{-\lambda \tau} (L(\gamma_{z, \lambda}(\tau), \dot{\gamma}_{z, \lambda}(\tau)) + c(H))d\tau \\
+ \int_{-1}^{0} e^{-\lambda \tau} (L(\beta(\tau), \dot{\beta}(\tau)) + c(H))d\tau \\
= e^{\lambda t}u^+_\lambda(\gamma_{z, \lambda}(-1)) + \int_{-1}^{0} e^{-\lambda \tau} (L(\beta(\tau), \dot{\beta}(\tau)) + c(H))d\tau \\
\leq e^{\lambda t}u^+_\lambda(\gamma_{z, \lambda}(-1)) + (C_{k_0} + c(H))\int_{-1}^{0} e^{-\lambda \tau}d\tau \\
\leq (K + \frac{C_{k_0} + c(H)}{\lambda})e^{\lambda t},
\]

where \( K = \|u^+_\lambda\| \). \( \square \)

Proposition 25 \( \{u^-_\lambda\}_{\lambda \in (0, 1)} \) is equi-Lipschitz and equi-bounded.
Proof By (2) of Proposition 22 and item (5) of Lemma C1, we derive \( \{u^\lambda(x)\}_{\lambda \in (0,1]} \) is Lipschitz continuous with the Lipschitz constant \( \kappa \) independent of \( \lambda \). Recall that \( \{u^\lambda(x)\}_{\lambda > 0} \) is equi-bounded, namely

\[
-K_0 \leq u^\lambda(x) \leq K_0, \quad \forall x \in M, \forall \lambda > 0
\]

is Lipschitz continuous with the Lipschitz constant \( \kappa \) independent of \( \lambda \). Recall that \( \{u^\lambda(x)\}_{\lambda > 0} \) is equi-bounded, namely

\[
-K_0 \leq u^\lambda(x) \leq K_0, \quad \forall x \in M, \forall \lambda > 0
\]

for some constant \( K_0 \). Then (2) of Proposition 22 and Lemma 23 indicate

\[
u^\lambda(x) = \lim_{t \to +\infty} T_t^\lambda u^\lambda(x) \geq u^\lambda(x) \geq -K_0, \quad \forall x \in M, \lambda \in (0,1].
\]

To show \( \{u^\lambda(x)\}_{\lambda \in (0,1]} \) is bounded from above, it suffices to show \( \{u^\lambda(x)\}_{x \in \mathcal{G}_\lambda, \lambda \in (0,1]} \) is bounded, since \( \{u^\lambda\}_{\lambda \in (0,1]} \) is Lipschitzian with Lipschitz constant \( \kappa \) independent of \( \lambda \). Indeed, for each \( x \in M, z \in \mathcal{G}_\lambda \),

\[
u^\lambda(z) \leq u^\lambda(z) + \kappa \cdot d(x, z).
\]

Let \( \gamma_{z,\lambda} : (-\infty, +\infty) \to M \) be a globally calibrated curve by \( u^+ \) with \( \gamma_{z,\lambda}(0) = z \). Then, for \( t > 0 \),

\[
T_t^\lambda u^\lambda(z) \leq u^\lambda(\gamma_{z,\lambda}(-t)) e^{\lambda t} + \int_{-t}^0 e^{-\lambda \tau}(L(\gamma_{z,\lambda}(\tau), \dot{\gamma}_{z,\lambda}(\tau)) + c(H)) d\tau
\]

We derive \( u^\lambda(z) \leq K_0 \) on \( \mathcal{G}_\lambda \) for all \( \lambda \in (0,1] \).

\( \Box \)

3 Convergence of ground state solutions

Before we prove Theorem 11, we first propose the following definition:

Definition 31 (Conjugate weak KAM solutions [9]) A backward (resp. forward) 0−weak KAM solution \( u^- \) (resp. \( u^+ \)) of \( (HJ_0) \) is said to be conjugated to \( u^+ \) (resp. \( u^- \)), if

\[
- u^+ \leq u^- ;
- u^- = u^+ \text{ on the projected Mather set } \Lambda^\lambda.
\]

Such a pair \( (u^-, u^+) \) is called conjugate.

Lemma 32 (Corollary 5.1.3. of [9]) Any backward 0−weak KAM solution \( u^- \) of \( (HJ_0) \) conjugates to one and only one forward 0−weak KAM solution \( u^+ \), vice versa.

On the other side, due to an analogue skill as in [7], we get:

Lemma 33 The function \( u^+_\lambda \) in \( \text{S}^+ \) converges to a uniquely identified forward 0−weak KAM solution \( u_0^+ \) of \( (HJ_0) \).

\(^a\) see Definition A4
Proof From the proof of Proposition (22), \( \hat{u}_\lambda = -u^+_\lambda \) is the unique viscosity solution of (5). Due to Theorem 1.1 of [7], \( \hat{u}_\lambda \) converges to a unique backward 0-weak KAM solution \( \hat{u}_0^- \) of
\[
\hat{H}(x, d_x u) = c(H), \quad x \in M.
\]
Due to Theorem 4.9.3 of [9], \( u^+_0 := -\hat{u}_0^- \) is a forward 0−weak KAM solution of
\[
H(x, d_x u) = c(H), \quad x \in M
\]
and \( u^+_0 = -\lim_{\lambda \to 0} \hat{u}_\lambda = \lim_{\lambda \to 0} u^+_\lambda. \)

\( \Box \)

Proof of Theorem 11: Suppose \( u^-_0 \) is the uniform limit of \( u^-_{\lambda_n} \) as \( \lambda_n \to 0_+ \), then \( u^-_0 \geq u^+_0 \). On the other side, \( u^+_\lambda = u^+_\lambda \) on \( G_{\lambda} \), if (\( \ast \)) holds, then for any Aubry class \( \Lambda_i \), there exists at least one point \( z \in \Lambda_i \) which can be accumulated by a sequence \( \{z_n \in G_{\lambda_n}\}_{n \in \mathbb{N}} \). Consequently, \( u^-_0 = u^+_0 \) on \( A \). Recall that \( M \subset A \) due to Proposition A5, then \( u^-_0 \) is conjugated \( u^+_0 \), which is unique due to Lemma 32. So we get the uniqueness of \( u^-_\lambda \) as \( \lambda \to 0_+ \).

For a generic \( C^1 \) Tonelli Hamiltonian \( H(x, p) \), there exists a unique ergodic Mather measure (see Proposition A7), then \( \mathcal{A}/d_c \) is a singleton due to Proposition A8 (namely \( \mathcal{A} \) consists of a unique Aubry class). Furthermore, by Proposition D1 and Proposition A9, the condition (\( \ast \)) always holds.

\( \Box \)

A Aubry Mather Theory of Tonelli Hamiltonians

As is known, the \( C^2 \) Tonelli Hamiltonian \( H(x, p) \) has a dual Tonelli Lagrangian
\[
L(x, v) := \max_{p \in T^*_x M} (p, v) - H(x, p), \quad (x, v) \in TM
\]
which is also \( C^2 \) and strictly convex, superlinear in \( v \). Consequently, for any \( x, y \in M \) and \( t > 0 \), the action function
\[
h^t(x, y) := \inf_{\gamma \in C^{ac}([0, t], M)} \int_0^t L(\gamma, \dot{\gamma}) + c(H) ds
\]
always attains its infimum at a \( C^2 \)-smooth minimizing curve \( \gamma_{min} : [0, t] \to M \), satisfying the Euler-Lagrange equation
\[
\begin{cases}
\frac{d}{dt} x = v, \\
\frac{d}{dt} L_v(x, v) = L_x(x, v),
\end{cases}
\]
due to the Tonelli Theorem and the Weierstrass Theorem, see [14]. A curve \( \gamma : \mathbb{R} \to M \) is called critical, if \( (\gamma, \dot{\gamma}) \) solves (\( \ast \)). Denote the Lagrangian flow by \( \Phi^t_L : TM \to TM \), then \( \Phi^t_L \) is well defined for \( t \in \mathbb{R} \) since \( H(x, L_v(x, v)) \) is invariant w.r.t. it.
Definition A1  In [75], the Peierls barrier function

\[ h^\infty(x, y) := \lim_{t \to \infty} h^1(x, y) \]  

(10)

is proved to be well-defined and continuous on \( M \times M \). Consequently, the projected Aubry set is defined by

\[ A := \{ x \in M : h^\infty(x, x) = 0 \}. \]

With respect to the pseudo metric

\[ d_c(x, y) := h^\infty(x, y) + h^\infty(y, x), \quad \forall x, y \in A, \]

we can decompose \( A \) into a bunch of connected subsets (static classes in [72]), such that any two point in the same class has a trivial \( d_c \)-distance. Without loss of generality, let's denote by \( A/d_c \) the set of all the static classes.

Consider \( TM \) (resp. \( M \)) as a measurable space and \( \mathcal{P}(TM) \) (resp. \( \mathcal{P}(M) \)) by the set of all Borel probability measures on it. A measure on \( TM \) is denoted by \( \tilde{\mu} \), and we remove the tilde if we project it to \( M \). We say that a sequence \( \{ \mu_n \}_n \) of probability measures weakly converges to a probability measure \( \tilde{\mu} \) if

\[ \lim_{n \to +\infty} \int_{TM} f(x, v) d\tilde{\mu}_n(x, v) = \int_{TM} f(x, v) d\tilde{\mu}(x, v) \]

for any \( f \in C_c(TM, \mathbb{R}) \). Accordingly, the deduced probability measure \( \mu_n \) weakly converges to \( \mu \), i.e.

\[ \lim_{n \to +\infty} \int_M f(x) d\mu_n(x) := \lim_{n \to +\infty} \int_{TM} f \circ \pi(x, v) d\tilde{\mu}_n(x, v) \]

\[ = \int_{TM} f \circ \pi(x, v) d\tilde{\mu}(x, v) =: \int_M f(x) d\mu(x) \]  

(11)

for any \( f \in C(M, \mathbb{R}) \).

Definition A2  A probability measure \( \tilde{\mu} \) on \( TM \) is closed if it satisfies:

- \( \int_{TM} |v| d\tilde{\mu}(x, v) < +\infty \);
- \( \int_{TM} (\nabla \phi(x), v) d\tilde{\mu}(x, v) = 0 \) for every \( \phi \in C^1(M, \mathbb{R}) \).

Let's denote by \( \mathcal{F}_c(TM) \) the set of all closed measures on \( TM \), then the following conclusion is proved in [32].

Theorem A3  \( \min_{\tilde{\mu} \in \mathcal{F}_c(TM)} \int_{TM} L(x, v) d\tilde{\mu}(x, v) = -c(H) \). Moreover, the minimizer is \( \Phi_L^* \)-invariant and called a Mather measure.

Definition A4  Define by \( \mathfrak{M} \) the set of Mather measures, which can be projected to \( 2\mathfrak{M} \subset \mathcal{P}(M) \) consisting of all the projected Mather measures due to [11]. The projected Mather set is defined by

\[ \mathcal{M} := \bigcup_{\mu \in \mathfrak{M}} \text{supp}(\mu) \subset M. \]

Proposition A5  [32] \( \mathfrak{M} \subset A \).

Definition A6 (Mañé's genericity[12])  A property is called \( (C^2) \)-generic (in the sense of Mañé) for \( H(x, p) \), if there exists a residual set \( \mathcal{O} \subset C^2(M, \mathbb{R}) \) such that for any \( \psi \in \mathcal{O} \), the property holds for \( H + \psi \). Accordingly, a Tonelli Hamiltonian \( H(x, p) \) is called generic, if we can find another Tonelli \( H_0(x, p) \) and a residue set \( \mathcal{O} \subset C^2(M, \mathbb{R}) \), such that \( H - H_0 \in \mathcal{O} \).

Proposition A7 (Theorem C of [12])  For a generic \( C^2 \) Tonelli Hamiltonian, the associated Mather measure is uniquely ergodic.
Proposition A8 (Lemma 5.3 of [6]) If the Mather measure is uniquely ergodic, the Aubry class has to be unique.

Proposition A9 (Proposition 5.3 of [2]) For autonomous Tonelli Lagrangian, if $\mathcal{A}$ is of a unique Aubry class, then $\mathcal{A} = \mathcal{G}$ with

$$\mathcal{G} = \{ x \in M \mid \text{there exists } \gamma : \mathbb{R} \rightarrow M \text{ with } \gamma(0) = x, \text{ such that } \forall a < b \in \mathbb{R}, \gamma|_{[0,1]} \text{ realizes } h^{b-a}(\gamma(a), \gamma(b)) \}.$$ 

B Viscosity solutions of \( (HJ_0) \)

Lemma B1 (Theorem 7.6.2. of [9]) The backward 0–weak KAM solution has to be a viscosity solution, vice versa.

Lemma B2 (item 3 of Remark 4.9.3 in [5]) For any $y \in M$ fixed, $h^\infty(y, \cdot)$ is a backward 0–weak KAM solution.

– for any $y \in M$ fixed, $-h^\infty(\cdot, y)$ is a forward 0–weak KAM solution;

Proposition B3 (Theorem 8.6.1 of [9]) Any viscosity solution $u$ of \( (HJ_0) \) can be formally expressed by

$$u(x) := \inf_{x_0 \in A} \{ u(x_0) + h^\infty(x_0, x) \}, \quad \forall x \in M.$$ 

Lemma B4 (10) For any $\Lambda_i \in A/dc$, any two viscosity solutions of \( (HJ_0) \) differs by a constant on $\Lambda_i$.

Proof This conclusion is a direct corollary of previous Proposition [53]. Precisely, for any viscosity solution $u$, we have

$$u(x) = \inf_{x_0 \in A} \{ u(x_0) + h^\infty(x_0, x) \} = \inf_{x_0 \in A} \{ u(x_0) + h^\infty(x_0, y) + h^\infty(y, x) \} = \inf_{x_0 \in A} \{ u(x_0) + h^\infty(x_0, y) \} + h^\infty(y, x) = u(y) + h^\infty(y, x)$$

as long as $x, y$ belonging to the same static class. Therefore,

$$\omega(y) - u(y) = \omega(x) - u(x), \quad \forall x, y \in \Lambda_i$$

for any two viscosity solutions $u$ and $\omega$.

C Properties of $T_\lambda^{\lambda^-}$

Lemma C1 (1). For $s, t > 0$ and $\phi(x) \in C(M, \mathbb{R})$,

$$T_\lambda^{\lambda^-} \phi(x) = T_{t+}^{\lambda^-} \circ T_{t-}^{\lambda^-} \phi(x).$$

(2). Let $\phi_1(x), \phi_2(x) \in C(M, \mathbb{R})$. Then,

$$||T_\lambda^{\lambda^-} \phi_1 - T_\lambda^{\lambda^-} \phi_2|| \leq e^M ||\phi_1 - \phi_2||, \quad \forall t > 0.$$ 

(3). For $\lambda \in (0, 1)$, each minimizer $\gamma$ of $T_\lambda^{\lambda^-} \phi(x)$ is $C^2$ and $|\gamma(t)| \leq \alpha_0$, where $\alpha_0$ is independent of $\lambda$. 

(4). Let $\phi_1, \phi_2 \in C(M, \mathbb{R})$ and $\phi_1(x) \leq \phi_2(x), x \in M$. Then,
\[
T_t^{\lambda} \phi_1(x) \leq T_t^{\lambda} \phi_2(x), \quad \forall t \geq 0, x \in M.
\]
(5). For $\lambda \in (0,1]$ and $t > \text{diam}(M)$, the map $x \mapsto T_t^{\lambda} \phi(x)$ is equi-Lipschitz, i.e.,
\[
|T_t^{\lambda} \phi(x) - T_t^{\lambda} \phi(y)| \leq \kappa \text{diam}(x,y), \quad t > \text{diam}(M),
\]
where $\kappa$ is independent of $\lambda, \phi$ and $t$.
(6). $u \in C(M, \mathbb{R})$ is a backward $\lambda$-weak KAM solution if and only if
\[
T_t^{\lambda} - u(x) = u(x), \forall x \in M, t > 0.
\]
(7). Each fixed point of $T_t^{\lambda}$ is a viscosity solution of $\mathcal{H}_{1,2}$, vice versa.

Proof (1). Note that
\[
h^{-(s+1),0}(y, x) = \inf_{s \in M}\{h^{-(s+1),-t}(y, z) + h^{t,0}(z, x)\}
\]
and
\[
h^{-(s+1),-t}(y, z) = e^{\lambda t} h^{-(s+1),0}(y, z).
\]
We derive
\[
T_{s+t}^{\lambda} \phi(x) = \inf_{y \in M}\{\phi(y) e^{\lambda(s+t)} + h^{-(s+1),0}(y, x)\}
\]
\[
= \inf_{y \in M}\{\phi(y) e^{\lambda(s+t)} + \inf_{z \in M}\{h^{-(s+1),-t}(y, z) + h^{t,0}(z, x)\}\}
\]
\[
= \inf_{z \in M}\{\inf_{y \in M}\{\phi(y) e^{\lambda(s+1)} + e^{\lambda t} h^{-(s+1),0}(y, z) + h^{t,0}(z, x)\}\}
\]
\[
= \inf_{z \in M}\{e^{\lambda t} T_s^{\lambda} - \phi(z) + h^{t,0}(z, x)\}
\]
\[
= T_t^{\lambda}\circ T_s^{\lambda} - \phi(x).
\]
(2). For every $x \in M$, let $\gamma_1 : [-t, 0] \to M$ be a minimizer of $T_t^{\lambda} - \phi_1(x)$. Then,
\[
T_t^{\lambda} - \phi_2(x) - T_t^{\lambda} - \phi_1(x) \leq \phi(\gamma_1(-t)) e^{\lambda t} - \phi(\gamma_2(-t)) e^{\lambda t}
\]
\[
\leq e^{\lambda t}\|\phi_2 - \phi_1\|.
\]
Similarly, we can obtain
\[
T_t^{\lambda} - \phi_1(x) - T_t^{\lambda} - \phi_2(x) \leq e^{\lambda t}\|\phi_2 - \phi_1\|.
\]
Hence,
\[
\|T_t^{\lambda} - \phi_1 - T_t^{\lambda} - \phi_2\| \leq e^{\lambda t}\|\phi_2 - \phi_1\|.
\]
(3). By the Weierstrass Theorem, we derive that $\gamma$ solves Euler-Lagrangian equation $\mathcal{H}_{1,2}$. Hence, $\gamma$ is $C^2$. By the definition of $T_t^{\lambda} - \phi(x)$, for $\frac{1}{2} < s_2 - s_1 < 1$
\[
e^{\lambda s_1 - \tau} T_{s_2}^{\lambda} - \phi(\gamma(-s_1)) - e^{\lambda s_2} T_{s_2}^{\lambda} - \phi(\gamma(-s_2)) = \int_{s_2}^{s_1} e^{-\lambda s} (L(\gamma(\tau), \dot{\gamma}(\tau)) + c(H)) d\tau.
\]
On the other hand, let $\beta : [-s_2, -s_1] \to M$ be a geodesic satisfying $\beta(-s_1) = \gamma(-s_1), \beta(-s_2) = \gamma(-s_2)$ and $|\dot{\gamma}(\tau)| \leq 2 \text{diam}(M) =: k_0$,
\[
e^{\lambda s_1 - \tau} T_{s_2}^{\lambda} - \phi(\gamma(-s_1)) - e^{\lambda s_2 - \tau} T_{s_2}^{\lambda} - \phi(\gamma(-s_2)) \leq \int_{s_2}^{s_1} e^{-\lambda s} (L(\beta(\tau), \dot{\beta}(\tau)) + c(H)) d\tau
\]
\[
\leq (C_{k_0} + c(H)) \int_{s_2}^{s_1} e^{-\lambda s} d\tau.
\]
Then,
\[
\int_{-s_2}^{-s_1} e^{-\lambda\tau} (|\dot{\gamma}(\tau)| - C(1) + c(H)) d\tau \leq \int_{-s_2}^{-s_1} e^{-\lambda\tau} (L(\gamma(\tau), \dot{\gamma}(\tau)) + c(H)) d\tau
\]
\[
\leq (C_{h_0} + c(H)) \int_{-s_2}^{-s_1} e^{-\lambda\tau} d\tau.
\]

There exists \( t_0 \in (-s_2, -s_1) \) such that \( |\dot{\gamma}(t_0)| \leq C_{h_0} + C(1) \). Note that \( \gamma(\tau) \) solves Euler-Lagrangian equation (6). We derive \( |\dot{\gamma}(\tau)| \leq \alpha_0 \).

(4). For each \( \gamma \in C^{\alpha,\beta}([-t, 0], M) \) with \( \gamma(0) = x \), it holds
\[
\phi_1(\gamma(-t))e^{\lambda t} + \int_{-t}^{0} e^{-\lambda\tau} (L(\gamma(\tau), \dot{\gamma}(\tau)) + c(H)) d\tau
\]
\[
\leq \phi_2(\gamma(-t))e^{\lambda t} + \int_{-t}^{0} e^{-\lambda\tau} (L(\gamma(\tau), \dot{\gamma}(\tau)) + c(H)) d\tau.
\]

Then, \( T_{\lambda}^{\gamma} - \phi_1(x) \leq T_{\lambda}^{\gamma} - \phi_2(x) \).

(5). Let \( \gamma_{x,\lambda} : [-t, 0] \to M \) be a minimizer of \( T_{\lambda}^{\gamma} - \phi(x) \) and \( \Delta t = d(x, y) \) and let \( \beta : [-\Delta t, 0] \to M \) be a geodesic satisfying \( \beta(-\Delta t) = \gamma_{x}(-\Delta t), \beta(0) = y, \) and
\[
|\beta(\tau)| \equiv \frac{d(\gamma_{x}(-\Delta t), y)}{\Delta t} \leq \frac{d(\gamma_{x}(-\Delta t), x)}{\Delta t} + 1 \leq \alpha_0 + 1,
\]
where \( |\gamma_{x}(\tau)| \leq \alpha_0 \).

\[
T_{\lambda}^{\gamma} \phi(y) - T_{\lambda}^{\gamma} \phi(x) \leq \int_{-\Delta t}^{0} e^{-\lambda\tau} (L(\beta(\tau), \dot{\beta}(\tau)) - L(\gamma(\tau), \dot{\gamma}(\tau))) d\tau
\]
\[
\leq (C_{\alpha_0 + 1} + C(0)) \int_{-\Delta t}^{0} e^{-\lambda\tau} d\tau
\]
\[
= (C_{\alpha_0 + 1} + C(0)) \cdot d(x, y).
\]

Similarly, we have
\[
T_{\lambda}^{\gamma} \phi(x) - T_{\lambda}^{\gamma} \phi(y) \leq (C_{\alpha_0 + 1} + C(0)) \cdot d(x, y).
\]

Let \( \kappa = C_{\alpha_0 + 1} + C(0) \). We have
\[
|T_{\lambda}^{\gamma} \phi(y) - T_{\lambda}^{\gamma} \phi(x)| \leq \kappa \cdot d(x, y).
\]

(6). Let \( u(x) \) be a backward \( \lambda \)-weak KAM solution and \( \gamma_{x,\lambda} : (-\infty, 0] \to M \) be a backward calibrated curve satisfying \( \gamma_{x,\lambda}(0) = x \). For each \( t > 0 \),
\[
u(x) - e^{\lambda t} u(\gamma_{x,\lambda}^{-}(t)) = \int_{-t}^{0} e^{-\lambda\tau} (L(\gamma_{x,\lambda}^{-}(\tau), \dot{\gamma}_{x,\lambda}^{-}(\tau)) + c(H)) d\tau.
\]

By \( u <_\lambda L + c(H) \), we derive
\[
u(x) \in C^{\alpha,\beta}([-t, 0], M) \left\{ e^{\lambda t} u(\gamma_{x,\lambda}^{-}(t)) + \int_{-t}^{0} e^{-\lambda\tau} (L(\gamma_{x,\lambda}^{-}(\tau), \dot{\gamma}_{x,\lambda}^{-}(\tau)) + c(H)) d\tau \right\} = T_{\lambda}^{\gamma} \nu(x).
\]

On the other hand, we assume \( T_{\lambda}^{\gamma} \nu(x) = \nu(x) \). Let \( \gamma \in C^{\alpha,\beta}([t_1, t_2], M) \). Then,
\[
T_{t_2-t_1}^{\lambda \gamma} \nu(\gamma(0)) \leq e^{\lambda (t_2-t_1)} u(\gamma(t_1 - t_2)) + \int_{t_1-t_2}^{0} e^{-\lambda\tau} (L(\gamma(\tau), \dot{\gamma}(\tau)) + c(H)) d\tau.
\]
where $\hat{\gamma} \in C^{0,\alpha}([t_1 - \Delta t, 0]; M)$ is defined by $\hat{\gamma}(s) = \gamma(s + t_2)$. From $T_{t_2-t_1} u(x) = u(x)$, it follows that

$$u(\gamma(t_2)) \leq e^{\Lambda(t_2-t_1)} u(\gamma(t_1)) + e^{\Lambda t_2} \int_{t_2}^{t_2+\Delta t} e^{-\lambda t} (L(\gamma(\tau), \dot{\gamma}(\tau)) + c(H)) d\tau,$$

which implies

$$e^{-\Lambda t_2} u(\gamma(t_2)) \leq e^{-\Lambda t_1} u(\gamma(t_1)) + e^{\Lambda t_2} \int_{t_2}^{t_2+\Delta t} e^{-\lambda t} (L(\gamma(\tau), \dot{\gamma}(\tau)) + c(H)) d\tau.$$

Hence, $u \prec L + c(H)$.

For each $n \in N$, we assume $\gamma_n : [-n, 0] \to M$ is a minimizer of $T_n^{1,\alpha} u(x)$. Then, for each $t \in [0, n]$, we derive

$$u(x) - e^{\Lambda t} u(\gamma_n(t)) = T_n^{1,\alpha} u(x) - e^{\Lambda t} T_n^{-1} u(\gamma_n(t)) = \int_{-t}^{0} e^{-\lambda s} (L(\gamma_n(s), \dot{\gamma}_n(s))) + c(H)) ds.$$

Note that $|\dot{\gamma}_n(s)| \leq n$ for all $n \in M$. By the Ascoli Theorem, there exists a subsequence $\{\gamma_{nk}\}$, such that $\gamma_{nk}$ converges uniformly to $\gamma_n \in C^{0,\alpha}((\infty, n]; M)$ on any finite interval of $(-\infty, 0]$. Then, for each $t > 0$, we derive

$$u(x) - e^{\Lambda t} u(\gamma_n(t)) \geq \int_{-t}^{0} e^{-\lambda s} (L(\gamma_n(s), \dot{\gamma}_n(s))) + c(H)) ds,$$

which implies

$$u(x) - e^{\Lambda t} u(\gamma_n(t)) = \int_{-t}^{0} e^{-\lambda s} (L(\gamma_n(s), \dot{\gamma}_n(s))) + c(H)) ds,$$

since $u \prec L + c(H)$. This means $\gamma_n : (\infty, 0] \to M$ is a backward calibrated curve. Hence, $u$ is a $\lambda$-weak KAM solution.

(7). Assume $u(x) = T_{t_2-t_1} u(x)$ for $t > 0$. Due to (6), $u(x)$ is a backward $\lambda$-weak KAM solution of $(HJ)$. Let $x_0 \in M$, $v_0 \in T_{x_0} M$ and let $\phi(x) \in C^1(M, \mathbb{R})$ and $u - \phi$ attains maximum at $x_0$. For $\Delta t < 0$, we assume $\gamma : [t + \Delta t, t] \to M$ is an absolutely continuous curve with $\gamma(t) = x_0$ and $\dot{\gamma}(t) = v_0$. Then,

$$e^{-\lambda t} (\phi(\gamma(t)) - \phi(\gamma(t + \Delta t))) \leq e^{-\lambda t} u(\gamma(t)) - e^{-\lambda t} u(\gamma(t + \Delta t))$$

$$\leq e^{-\lambda t} u(\gamma(t)) - e^{-\lambda t} u(\gamma(t + \Delta t)) + e^{-\lambda (t + \Delta t)} u(\gamma(t + \Delta t)) - e^{-\lambda t} u(\gamma(t + \Delta t)).$$

Then,

$$e^{-\lambda t} \frac{\phi(\gamma(t + \Delta t)) - \phi(\gamma(t))}{\Delta t} \leq \frac{1}{\Delta t} \int_t^{t+\Delta t} e^{-\lambda \tau} (L(\gamma(\tau), \dot{\gamma}(\tau))) + c(H) d\tau$$

$$- e^{-\lambda (t + \Delta t)} \cdot u(\gamma(t + \Delta t)).$$

Taking $\Delta t \to 0^+$, we derive

$$e^{-\lambda t} d_x \phi(x_0) \cdot v_0 \leq e^{-\lambda t} (L(x_0, v_0) + c(H)) + \lambda e^{-\lambda t} \cdot u(x_0),$$

which implies

$$-\lambda u(x_0) + d_x \phi(x_0) \cdot v_0 - L(x_0, v_0) \leq c(H).$$
Hence, 
\[-\lambda u(x_0) + H(x_0, d_x \phi(x_0)) \leq c(H)\].
On the other hand, let \(\psi(x) \in C^1(M, \mathbb{R})\) and \(u - \psi\) attains the minimum at \(x_0\) and let \(\gamma_x : [t + \Delta t, t] \to M\) be a calibrated curve by \(u\) with \(\gamma_x(t) = x_0\). Then, 
\[\psi(\gamma_x(t)) - \psi(\gamma_x(t + \Delta t)) \geq u(\gamma_x(t)) - u(\gamma_x(t + \Delta t))\].

Note that 
\[e^{-\lambda t} (u(\gamma_x(t)) - u(\gamma_x(t + \Delta t)))\] 
\[= e^{-\lambda t} u(\gamma_x(t)) - e^{-\lambda(t+\Delta t)} u(\gamma_x(t + \Delta t))\] 
\[+ e^{-\lambda(t+\Delta t)} u(\gamma_x(t + \Delta t)) - e^{-\lambda t} u(\gamma_x(t + \Delta t))\] 
\[= \int_t^{t+\Delta t} e^{-\lambda \tau} (L(\gamma_x(\tau), \dot{\gamma}_x(\tau)) + c(H))d\tau + (e^{-\lambda(t+\Delta t)} - e^{-\lambda t}) u(\gamma_x(t + \Delta t)).\]

We derive 
\[e^{-\lambda t} \frac{\psi(\gamma_x(t + \Delta t)) - \psi(\gamma_x(t))}{\Delta t} \geq \frac{1}{\Delta t} \int_t^{t+\Delta t} e^{-\lambda \tau} (L(\gamma_x(\tau), \dot{\gamma}_x(\tau)) + c(H))d\tau\] 
\[- e^{-\lambda(t+\Delta t)} + e^{-\lambda t} \cdot u(\gamma_x(t + \Delta t)).\]

Taking \(\Delta t \to 0^+\), we derive that 
\[-\lambda u(x_0) + d_x \psi(x_0) \cdot v_0 - L(x_0, v_0) \geq c(H),\]
which implies 
\[-\lambda u(x_0) + H(x_0, d_x \psi(x_0)) \geq c(H),\]
so \(u\) is a viscosity solution of \(\mathcal{H}^\lambda\).

Suppose \(\omega(x)\) is a viscosity solution of \(\mathcal{H}^\lambda\), then \(\omega(x)\) is Lipschitz due to the superlinearity of \(H(x, p)\), see [3]. For the reduced Lipschitz Lagrangian \(L^\lambda(x, v) := L(x, v) + \lambda \omega(x)\), \(\omega(x)\) is also the viscosity solution of \(\mathcal{H}^\lambda(x, d_x \omega(x)) = c(H)\), where 
\[\mathcal{H}^\lambda(x, p) = H(x, p) - \lambda \omega(x)\]
is the corresponding Hamiltonian. Then, \(U(x, t) \in M \times [0, +\infty) \to M\) defined by 
\[U(x, t) := \inf_{\gamma \in C^0([t, t+\Delta t], M)} \{\omega(\gamma(t)) + \int_{t}^{t+\Delta t} L^\lambda(\gamma(\tau), \dot{\gamma}(\tau)) + c(H) d\tau\}, \quad \forall t \geq 0,\]
is a viscosity solution of the Cauchy problem 
\[\begin{aligned}
\partial_t u + \mathcal{H}^\lambda(x, d_x \omega) &= c(H), \\
u(x, 0) &= \omega(x), \quad t \geq 0.
\end{aligned}\] (13)

Note that \(\omega(x)\) is also a viscosity of \(\mathcal{H}^\lambda(x, d_x \omega(x)) = c(H)\). We derive \(\omega(x)\) is a solution to the Cauchy problem [13]. From the uniqueness of viscosity solution, it follows that \(U(x, t) = \omega(x)\) for \(x \in M, t \geq 0\). Hence, for each absolutely continuous curve \(\gamma : [s, t] \to M\), 
\[\omega(\gamma(t)) - \omega(\gamma(s)) \leq \int_s^t L^\lambda(\gamma(\tau), \dot{\gamma}(\tau)) + c(H) d\tau.\]

Fix a sequence \(\{t_n\}_{n \in \mathbb{N}}\) tending to \(+\infty\) as \(n \to \infty\). Due to \(U(x, t_n) = \omega(x)\), for each \(n \in \mathbb{N}\), there exists an absolutely continuous curve \(\gamma_n : [-t_n, 0] \to M\) such that \(\gamma_n(0) = x\) and 
\[\omega(x) = \omega(\gamma_n(-t_n)) + \int_{-t_n}^0 L^\lambda(\gamma_n(\tau), \dot{\gamma}_n(\tau)) d\tau + c(H) d\tau.\]
From the superlinearity of $L^\lambda$ in $v$ and Lipschitz continuity of $\omega$, we derive $\{\|\gamma_n\|_{L^\infty}\}_{n}$ is equi-bounded. By Ascoli Theorem, there exists a subsequence of $\{\gamma_n\}$ (denoted still by $\gamma_n$) uniformly converging to an absolutely continuous curve $\gamma: (-\infty, 0] \to M$ on each finite interval of $(-\infty, 0]$, such that $\gamma_n(0) = x$ and

$$
\omega(x) - \omega(\gamma_n(-t)) = \int_{-t}^{0} L^\lambda(\gamma_n(\tau), \dot{\gamma}_n(\tau)) + c(H) \, d\tau, \ t > 0. \quad (14)
$$

Let $\gamma \in C^{ac}([a, b], M)$. Then, $\omega(\gamma(\tau))$ and

$$
s \mapsto \int_{s}^{0} L(\gamma(\tau), \dot{\gamma}(\tau)) + c(H) + \lambda \omega(\gamma(\tau)) \, d\tau
$$

are differentiable a.e. on $[a, b]$. For $t \in [a, b]$ and $\Delta t \neq 0$ with $t > \Delta t \in [a, b]$,

$$
\frac{\omega(\gamma(t + \Delta t)) - \omega(\gamma(t))}{\Delta t} \leq \frac{1}{\Delta t} \int_{t}^{t + \Delta t} L(\gamma(\tau), \dot{\gamma}(\tau)) + c(H) + \lambda \omega(\gamma(\tau)) \, d\tau.
$$

Taking $\Delta t$ tending to 0, we derive that

$$
d\omega(\gamma(t)) \leq L(\gamma(t), \dot{\gamma}(t)) + c(H) + \lambda \omega(\gamma(t)), \ a.e. \ t \in [a, b].
$$

Then,

$$
\frac{d}{dt} \left( e^{-\lambda t} \omega(\gamma(t)) \right) \leq e^{-\lambda t} (L(\gamma(t), \dot{\gamma}(t)) + c(H)), \ a.e. \ t \in [a, b].
$$

Integrating on $[a, b]$, we derive

$$
e^{-\lambda b} \omega(\gamma(b)) - e^{-\lambda a} \omega(\gamma(a)) \leq \int_{a}^{b} e^{-\lambda \tau} (L(\gamma(\tau), \dot{\gamma}(\tau)) + c(H)) \, d\tau,
$$

which implies $\omega \prec_{\lambda} L + c(H)$. By a similar discussion, we derive from (13) that $\gamma_n$ is a calibrated curve by $\omega$, i.e.,

$$
\omega(x) - e^{\lambda t} \omega(\gamma_n(-t)) = \int_{-t}^{0} e^{-\lambda \tau} (L(\gamma_n(\tau), \dot{\gamma}_n(\tau)) + c(H)) \, d\tau, \ \forall t > 0,
$$

which implies $\omega$ is a backward $\lambda$-weak KAM solution of $\{HJ^\lambda\}$. \hfill \square

**D Upper semi-continuity of $G_\lambda$**

**Proposition D1 (Upper semicontinuity)** As a set-valued function,

$$
\lim_{\lambda \to \lambda_+} G_\lambda \subset \mathcal{G} \subset M.
$$

**Proof** Let $\gamma_n$ be a globally calibrated curve by $u^+_{\lambda_n}$ with the parameter $\lambda_n \to 0_+$, as $n \to \infty$. To show the proposition, it suffices to show any accumulating curve $\gamma^*$ of $\{\gamma_n\}$ realizes $h^{b-a}(\gamma^*(a), \gamma^*(b))$, $a < b \in \mathbb{R}$.

Otherwise, there exists an interval $[a, b]$ and a curve $\eta^* \in C^{ac}([a, b], M)$ such that $\eta^*(a) = \gamma^*(a)$, $\eta^*(b) = \gamma^*(b)$, and

$$
h^{b-a}(\gamma^*(a), \gamma^*(b)) = \int_{a}^{b} L(\eta^*, \dot{\eta}^*) + c(H) \, d\tau < \int_{a}^{b} L(\gamma^*, \dot{\gamma}^*) + c(H) \, d\tau. \quad (15)
$$
By Weierstrass Theorem, one can easily check \( \eta^* \) is \( C^2 \) and \( |\eta^*| \leq \kappa_0 \). For sufficiently large \( n \in \mathbb{N} \), we define \( \eta_n \in C^\infty([a,b],M) \) by

\[
\eta_n(s) = \begin{cases} 
\beta_{1,n}(s), & s \in [a,a+d_{1,n}], \\
\eta^*(s), & s \in [a+d_{1,n},b-d_{2,n}], \\
\beta_{2,n}(s), & s \in [b-d_{2,n},b],
\end{cases}
\]

where \( d_{1,n} = d(\gamma_n(a),\gamma^*(a)) \), \( d_{2,n} = d(\gamma_n(b),\gamma^*(b)) \), \( \beta_{1,n} \) is the geodesic connecting \( \gamma_n(a) \) and \( \eta^*(a+d_{1,n}) \) with \( |\beta_{1,n}| \leq \kappa_0 + 1 \), \( \beta_{2,n} \) is the geodesic connecting \( \eta^*(b-d_{2,n}) \) and \( \gamma_n(b) \) with \( |\beta_{2,n}| \leq \kappa_0 + 1 \). Then, \( \eta_n \) converges uniformly to \( \eta^* \) on \([a,b]\) and \( \int_a^b |e^{-\lambda_n \tau}(L(\eta_n,\dot{\eta}_n) + c(\tau))|d\tau \) is bounded.

By Dominated Convergence Theorem, we derive

\[
\lim_{n \to +\infty} \int_a^b e^{-\lambda_n \tau}(L(\eta_n,\dot{\eta}_n) + c(\tau))d\tau = \int_a^b L(\eta^*,\dot{\eta}^*) + c(\tau)d\tau
\]

Hence,

\[
h^{b-a}(\gamma^*(a),\gamma^*(b)) = \int_a^b L(\eta^*,\dot{\eta}^*) + c(\tau)d\tau = \lim_{n \to +\infty} \int_a^b e^{-\lambda_n \tau}(L(\eta_n,\dot{\eta}_n) + c(\tau))d\tau \\
\geq \liminf_{n \to +\infty} \int_a^b e^{-\lambda_n \tau}(L(\gamma_n,\dot{\gamma}_n) + c(\tau))d\tau \\
\geq \int_a^b L(\gamma^*,\dot{\gamma}^*) + c(\tau)d\tau.
\]

Combining (15), we derive a contradiction. Hence, for each \( a < b \in \mathbb{R} \),

\[
h^{b-a}(\gamma^*(a),\gamma^*(b)) = \int_a^b L(\gamma^*,\dot{\gamma}^*) + c(\tau)d\tau
\]

Then the assertion follows. \( \square \)

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