The Atwood’s machine as a tool to introduce variable mass systems

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Abstract

This paper discusses an instructional strategy which explores eventual similarities and/or analogies between familiar problems and more sophisticated systems. In this context, the Atwood’s machine problem is used to introduce students to more complex problems involving ropes and chains. The methodology proposed helps students to develop the ability needed to apply relevant concepts in situations not previously encountered. The pedagogical advantages are relevant for both secondary and high school students, showing that, through adequate examples, the question of the validity of Newton’s second law may be introduced to even beginning students.
I. INTRODUCTION

The description of the motion of a uniform rope over a smooth pulley under the influence of gravity can be used to exemplify the behavior of systems with variable mass. The discussion of variable mass problems from the concept of momentum flux (e.g. Siegel 1972 and Sousa 2002) or from the generalization of the Newton’s second law (e.g. Sousa and Rodrigues 2004) is not adequate for introductory physics. However, the inter-disciplinarity of this important subject and the relevance of its applications in rocket theory (e.g. Meirovitch 1970, Tran and Eke 2005), astronomy (e.g. Kayuk and Denisenko 2004), biology (e.g. Canessa 2007, 2009), robotics (e.g. Djerassi 1998), mechanical and electrical machinery (e.g. Cveticanin 2010), etc, justify the introduction of this theme to students in the scientific areas as soon as possible.

We observe that sometimes students have relevant mathematical knowledge but fail to apply or interpret that knowledge in the context of physics. This produces a barrier and an additional difficulty to the robust use of concepts in complex problem solving. So, it is important to elaborate strategies that can help students employ the mathematical knowledge they already possess.

In a presentation to a group of pre-university students, we tried a new method to solve problems of ropes or chains; it starts by exploring eventual similarities and/or analogies with the most simple and familiar problem on the Atwood’s machine. The versatility of this system has been confirmed by many generations of teachers (Greenslade 1985). It provides a rich source of ideas for experiences and problems in the application of Newton’s second law to the motion of a compound system, being analysed by the generality of students of physics and engineering.

In the present paper, the Atwood’s machine problem is solved by the traditional method based on Newton’s second law for particles (Method 1). After that, we verify the results obtained by considering other methodologies, involving the concept of centre of mass (Method 2) and conservation of energy (Method 3), which are appropriate to the more complex problem of the rope.
II. THE ATWOOD’S MACHINE PROBLEM

Two blocks of masses $m_1$ and $m_2$ ($m_2 > m_1$) are connected by a massless string passing over a frictionless pulley of negligible mass. The mass $m_2$ is released from rest at $t = 0$ (figure 1. (a)). Find the acceleration of the blocks and the tension in the string.

Method 1.

The traditional analysis of this one-dimensional motion consists in the use of the Newton’s second law for translation

$$\vec{F} = m \vec{a},$$

where $\vec{F}$ is the sum of all forces acting on the particle with mass $m$ and acceleration $\vec{a}$.

The free-body diagram for each block contains the downward force of gravity and the upward tension force $\vec{T}$ exerted by the string. As the string is inextensible both masses have acceleration with equal magnitude $a$. As we assume $m_2 > m_1$, the object $m_1$ accelerates upward, and $m_2$ accelerates downward. As the motion of the blocks is one-dimensional in the vertical direction, there is no need to use vectors explicitly.
Applying Newton’s second law (1) to blocks 1 and 2 we obtain, respectively,

\[ T - m_1 g = m_1 a, \]  
(2)

and

\[ m_2 g - T = m_2 a. \]  
(3)

From these two equations it is easy to find

\[ a = g \frac{m_2 - m_1}{m_1 + m_2}, \]  
(4)

and

\[ T = 2g \frac{m_1 m_2}{m_1 + m_2}. \]  
(5)

The equations (5) and (4) satisfy two special cases: when \( m_1 = m_2 = m \), \( a = 0 \) and \( T = mg \); if \( m_2 \gg m_1 \), then \( a \simeq g \) and \( T \simeq 2m_1 g \).

At this point the teacher must remember that, in principle, equation (1) is not valid for variable mass systems.

**Method 2.**

The solution already obtained can be confirmed if we substitute one of the equations (2) or (3) by considering the system as being made up of both objects. The tension in the string is now an internal force, and the external forces acting on the system are the downward force of gravity and the upward force of reaction by the pulley, \( \vec{N} \), which magnitude is \( N = 2T \).

The Newton’s second law for the translation of the centre of mass of the system of particles is given by:

\[ \vec{F} = m \vec{a}_{cm}, \]  
(6)

where \( \vec{F} \) is the sum of all external forces acting on the total mass of the system, \( m = m_1 + m_2 \), and \( \vec{a}_{cm} \) is the acceleration of the centre of mass.

As the centre of mass moves downward we can write the following equation of motion:

\[ (m_1 + m_2) g - 2T = (m_1 + m_2) a_{cm}, \]  
(7)
where the acceleration of the center of mass is given by

\[ a_{cm} = a \frac{m_2 - m_1}{m_1 + m_2}. \tag{8} \]

The insertion of equation (8) into (7), and using one of the equations of motion (2) or (3), the results already obtained by Method 1 are confirmed.

**Method 3.**

Let us now analyse the conservation of energy. Comparing the configuration of the system at the instant \( t \) with those at \( t = 0 \), we easily obtain an equation which determines the velocity as a function of \( x \) (see figure 1 (b)).

We consider that the potential energy, \( U \), is zero in the configuration of the system at \( t = 0 \). Assuming that the blocks are initially at rest, the kinetic energy, \( K \), is also zero at \( t = 0 \). The friction is negligible and the conservation of energy states

\[ K_0 + U_0 = K + U. \tag{9} \]

As \( K = \frac{1}{2} (m_1 + m_2) v^2 \) and \( U = (m_2 - m_1) g x \), by using equation (9) the squared velocity at position \( x \) of mass \( m_1 \) is found to be

\[ v^2 = 2 g x \frac{m_1 - m_2}{m_1 + m_2}. \tag{10} \]

This equation allows to apply a mathematical procedure that students already know from math classes, but that do not usually apply in classes of introductory physics. In fact, this equation allows directly to the acceleration \( a \) by using the identity

\[ a = \frac{1}{2} \frac{d v^2}{d x}. \tag{11} \]

This equation comes from the following mathematical procedure:

\[ a = \frac{d v}{d t} = \frac{d v}{d x} \frac{d x}{d t} = \frac{d v}{d x} v = \frac{1}{2} \frac{d v^2}{d x}. \tag{12} \]

Using equations (10) and (11) we easily recover the acceleration \( a \), as it must.

A more common procedure to obtain the acceleration from equation (10) consists in using the equations of motion for position and velocity as functions of time: \( x = a t^2 / 2 \).
and \( v = a t \). However, the procedure here adopted (less common for beginning students) is more convenient for cases where the acceleration is not constant as in the rope problem.

### III. THE FALLING ROPE PROBLEM

A uniform and flexible rope of length \( l \), and mass per unit length \( \lambda \) hangs almost symmetrically over a frictionless and small pulley. Due to a small perturbation, the rope begins to fall from rest at \( x = 0 \). Find the velocity of the rope when it leaves the pulley, as well as the acceleration of the rope. Figure 2 shows the configuration of the system at the instant \( t \).

In analogy with the Atwood’s machine problem, we can divide the whole system (constant mass) in two sub-systems I and II (variable mass). If we want to keep the problem suitable for beginning students, we should not consider Method 1 based on Newton’s second law for sub-systems I and II. However, some aspects of this subject can be discussed \textit{a posteriori}, following the methodology presented in the Appendix.

The radius of the pulley, quite big in the figure, is supposed to be very small compared with the length \( l \) of the chain. This means that the movement is, in good approximation,
one-dimensional in the vertical direction, and, also in this problem, there is no need to use vectors explicitly.

Let us start with the conservation of energy. To this purpose we define the position of the centre of mass by using the expression

\[ x_{cm} = \frac{x_I m_I + x_{II} m_{II}}{\lambda l}, \]  

where \( m_I = \lambda (l/2 - x) \), \( m_{II} = \lambda (l/2 + x) \), \( x_I = l/4 + x/2 \) and \( x_{II} = l/4 - x/2 \).

The variable \( x \) denotes the displacement of one end of the rope from its initial position as indicated in figure 2 (0 < \( x < l/2 \)).

The previous equations allow to obtain

\[ x_{cm} = \frac{l}{4} - \frac{x^2}{l}. \]  

The equation (14) satisfy two special cases: when \( x = 0 \), \( x_{cm} = l/4 \); if \( x = l/2 \), then \( x_{cm} = 0 \).

The mechanical energy initially (\( x = 0 \)) and at a generic configuration of the system (\( x \neq 0 \)) are given by

\[ K_0 + U_0 = \lambda g \frac{l^2}{4}, \]  

and

\[ K + U = \frac{1}{2} \lambda l v^2 + \lambda g \left( \frac{l^2}{4} - x^2 \right). \]

The velocity as a function of \( x \) follows explicitly from the conservation of energy (9)

\[ v = x \left( \frac{2 g}{l} \right)^{1/2}. \]  

Combining the expression of \( v^2 \) with equation (11) yields to the acceleration

\[ a = 2 g \frac{x}{l}. \]  

In order to test the validity of the expression (17) in the interval 0 < \( x < l/2 \), we must calculate the force of reaction by the pulley, \( \vec{N} \). To this purpose we consider Method 2 of the Atwood’s machine problem.

The velocity and the acceleration of the centre of mass, which moves downward, must be calculated. We obtain successively,
\[ v_{cm} = \frac{v m_{II} - v m_I}{\lambda l} = 2 v \frac{x}{l}, \quad (19) \]

and

\[ a_{cm} = \frac{d v_{cm}}{dt} = 2 a \frac{x}{l} + 2 \frac{v^2}{l} = 8 g \frac{x^2}{l^2}, \quad (20) \]

where equations (17) and (18) have also been used to obtain the last equation.

The Newton’s second law for the system of particles (6), applied to the whole rope, allows to write

\[ \lambda l g - N = \lambda l a_{cm} = 8 \lambda g \frac{x^2}{l}. \quad (21) \]

Therefore we may obtain the normal force

\[ N = N(x) = \lambda l g \left( 1 - 8 \frac{x^2}{l^2} \right), \quad (22) \]

allowing to conclude that the solution given by equation (17) is valid only till the value \( x = l/(2 \sqrt{2}) \), where the normal force attains the value \( N = 0 \) (Calkin 1989). This fact can be demonstrated to students in the classroom, by observing some whip-lashing behavior of the rope before its whole length goes over the pulley.

It is interesting to notice that the expression for the acceleration (18), although dependent on the coordinate \( x \), is consistent with the expression for the acceleration in the Atwood’s machine (4), i.e., satisfies to the relation \( a = g (m_{II} - m_I)/(m_I + m_{II}) \) in the referred interval of \( x \). However, as the mass of the rope is different from zero, an analogous equation for the tension in the string is not verified, i.e., \( 2 T - N \neq 0 \) (see the Appendix).

So, the teacher can also guide students to focus on important differences between both problems. We point out:

- Sub-systems I and II of variable mass can be described by the equation of motion \( F = m a \), but \( F = dp/dt \) is not satisfied (see the Appendix). The blocks of the Atwood’s machine can be described by both forms of the Newton’s second law.

- The forces acting on the small piece of the rope over the pulley satisfies the condition \( 2 T - N = 0 \), in the Atwood’s machine problem, whereas \( 2 T - N \neq 0 \), in the
rope problem (see the Appendix). This fact also gives the opportunity to enlarge the discussion to the concept of tension when the mass of the rope is nonzero.

In our opinion, the teacher must leave the complete analysis of the problem, from the point of view of a variable mass system, for students in intermediate courses of mechanics.

IV. CONCLUSIONS

The Atwood’s machine problem solved by the traditional methodology does not require a sophisticated level of mathematics. However, the enlargement suggested in the present paper indicates that students’ understanding can be probed more deeply. In this way they will be more prepared to re-create the physical situation under study, or make a model to better visualize the problem, or even think on analogous problems they have solved before. This strategy can guide the design of instruction to match the needs and performances of students in introductory physics.

In conclusion, the present strategy provides an elegant way to solving certain aspects of the rope problem. However, at the same time, leaves aside other equally important aspects which may be discussed in intermediate courses of mechanics. Finally, we remark that similar problems, such as a massive rope sliding freely over a smooth nail, can be treated along the lines indicated in the present work (Sousa and Rodrigues 2004). Another example would be the motion of a chain, part of which is hanging off the edge of a smooth table.

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Appendix
This Appendix applies the Newton’s second law for variable mass systems. As the motion is one-dimensional, this equation of motion can be written in the form (Sousa and Rodrigues 2004)

\[
\frac{dp}{dt} = F + u \frac{dm}{dt},
\]

where \(m\) is the instantaneous mass and \(p = mv\) its linear momentum, \(F\) is the net external force acting upon the variable mass system, and \(udm/dt\) is the rate at which momentum is carried into or away from the system of mass \(m\).
As in this case the velocity of the mass being transferred between the sub-systems is \( u = v \), and both sub-systems have the velocity \( v \), equation (23) can be re-arranged, confirming that a general equation of type \( F = ma \) applies to both sub-systems.

Denoting by \( T \) the tension in the rope at the pulley, the equations of motion of sub-systems I and II read,

\[
T - \lambda g (\frac{l}{2} - x) = \lambda g (\frac{l}{2} - x) a, \tag{24}
\]

and

\[
\lambda g (\frac{l}{2} + x) - T = \lambda g (\frac{l}{2} + x) a. \tag{25}
\]

Using the expression of the acceleration (18) we easily obtain the tension

\[
T = \frac{1}{2} \lambda gl (1 - 4 \frac{x^2}{l^2}). \tag{26}
\]

The resultant force on the small piece of rope over the pulley \( 2T - N \) can then be calculated using (22) together with the last equation, giving

\[
2T - N = 4\lambda g \frac{x^2}{l} = 2\lambda v^2. \tag{27}
\]

This result shows that in the interval \( \Delta t \) the momentum of this massive small piece of rope \( (\lambda \Delta x) \) increases by \( 2\lambda v^2 \Delta t \) downward.

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