THE INDETERMINACY LOCUS OF THE VOISIN MAP

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Abstract. Beauville and Donagi proved that the variety of lines $F(Y)$ of a smooth cubic fourfold $Y$ is a hyperkähler variety. Recently, C. Lehn, M. Lehn, Sorger, and van Straten proved that one can naturally associate a hyperkähler variety $Z(Y)$ to the variety of twisted cubics on $Y$. Then, Voisin defined a degree 6 rational map $\psi: F(Y) \times F(Y) \to Z(Y)$. We will show that the indeterminacy locus of $\psi$ is the locus of intersecting lines.

1. Introduction

It is a classical result that a manifold with a Ricci flat metric has trivial first Chern class, and by Bogomolov’s decomposition \cite{Bog74, Bea83} such manifolds have a finite étale cover given by the product of a Torus, Calabi-Yau varieties and hyperkähler varieties. Hyperkähler manifolds are interesting in their own and they are the subject of an intensive research. The first examples are K3 surfaces, and Beauville proved in \cite[Théorèmes 3 and 4]{Bea83} that for any $n \geq 0$ the Hilbert schemes of points $X^[n]$ where $X$ is a K3 surface or the generalized Kummer varieties $K^nA$ associated to an Abelian surface $A$, are hyperkähler varieties. Any hyperkähler variety that is a deformation of $X^[n]$ where $X$ is a K3 surface (respectively, a generalized Kummer variety) is called $K3^[n]$-type (respectively, $K^nA$-type). Those examples are particularly interesting because they permit to construct hyperkähler varieties of any even complex dimension. Later O’Grady in \cite{O’G99, O’G03} constructed two new examples in dimension 6 and 10 of hyperkähler varieties that are not deformation of known types. There are few explicit complete families of hyperkähler manifolds of $K3^[n]$-type. Beauville and Donagi proved in \cite[Proposition 1]{BD85} that the variety of lines $F(Y)$ of a smooth cubic fourfold $Y \subseteq \mathbb{P}^5$ is an hyperkähler variety of $K3^[2]$-type. Another example was given much more recently by C. Lehn, M. Lehn, Sorger, and van Straten in \cite{LLSvS17}. They observed that if $F_3(Y)$ is the moduli space of generalized twisted cubic curves on $Y$, then $F_3(Y)$ is a $\mathbb{P}^2$-fibration over a smooth variety $Z(Y)$. Moreover, there is a divisor in $Z(Y)$ that can be contracted and this contraction produces a hyperkähler variety $Z(Y)$. This variety is of $K3^[4]$-type by \cite[Corollary]{AL17} or \cite[Corollary 6.2]{Leh18}.

On the other hand the study of $k$-cycles on smooth complex projective varieties is a classical subject and it is very interesting on hyperkähler manifolds with respect to several regards. For example, while it is a classical result that the cone of nef divisors is contained in the cone of pseudoeffective divisors, in general $\text{Nef}_k(X) \nsubseteq \text{Eff}_k(X)$ for $2 \leq k \leq \text{dim} X - 2$. The first example of such phenomenon was given in \cite{DELV11}.

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Later, Ottem proved that if the cubic \( Y \) is very general, then the second Chern class \( c_2(F(Y)) \) of the Fano variety of lines in \( Y \) is nef but it is not effective [Ott15, Theorem 1].

It is known, due to Mumford’s Theorem [Mum68, Theorem], that for a projective hyperkähler variety \( X \) of dimension \( 2n \) the kernel of the cycle map \( cl : A^{2n}(X) \to H^{4n}(X) \) is infinite-dimensional (see [Voi03, III.10] for more details). Nevertheless Beauville made the following

**Conjecture 1.1** ([Bea07]). Let \( X \) be a projective hyperkähler manifold. Then the cycle class map is injective on the subalgebra of \( A^*(X) \) generated by divisors.

See [Voi16] for an introduction to these topics. On the other hand Shen and Vial in [SV16] used a codimension 2 algebraic cycle to give evidence for the existence of a certain decomposition for the Chow ring of \( F(Y) \) for a very general cubic fourfold \( Y \).

Voisin constructed in [Voi16, Proposition 4.8] a degree 6 rational map

\[
\psi : F(Y) \times F(Y) \to Z(Y).
\]

Roughly speaking, the map \( \psi \) sends pairs of non-incident lines \((l, l') \in F(Y) \times F(Y)\) to the (class of the) degree 3 rational normal curve in the linear system \(|L - L' - K_{S_{l,l'}}|\) of the cubic surface \( S_{l,l'} := (L, L') \cap Y \). This article is devoted to the study of the indeterminacy locus of this map. In particular, we prove the following.

**Theorem 1.2.** The indeterminacy locus of the Voisin map \( \psi : F(Y) \times F(Y) \to Z(Y) \) is the variety \( I \) of intersecting lines in \( Y \).

We hope that the explicit description of \( \text{Ind}(\psi) \) will contribute to the study of \( c_2(Z(Y)) \), the study of algebraic cycles on \( Z(Y) \) and to other aspects of the geometry of \( Z(Y) \). We hope to return to these topics in a future work.

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2. General facts

A variety \( X \) is an integral and separated algebraic scheme over \( \mathbb{C} \). A compact Kähler manifold is hyperkähler if it is simply connected and the space of its global holomorphic two-forms is spanned by a symplectic form.

Throughout this paper we will use the following.

**Notation.**

- Given a rational map of varieties \( f : A \to B \), we denote by \( \text{Ind}(f) \) the complement of the largest open subset of \( A \) on which \( f \) is represented by a regular function.
– $G(k, n)$ is the Grassmannian variety parametrizing $k$-dimensional linear subspaces of an $n$-dimensional vector space.
– $T(X)_x$ is the tangent space of a smooth variety $X$ at the point $x$.
– $Y$ is a smooth cubic fourfold that does not contain a plane.
– $F$ is the variety of lines on $Y$. By [BD85, Proposition 1] $F$ is a 4-dimensional hyperkähler subvariety of $G(2, 6)$.
– Given a point $l \in F$, the line in $Y$ that it represents will be indicated with the same letter in uppercase, i.e. $l = [L]$.
– We denote by (2.1)

\[ P := \{(l, x) \in F \times Y | x \in L\} \xrightarrow{q} Y \]

the universal family of lines in $Y$.
– $I$ is the closed subscheme, with reduced structure, of intersecting lines, i.e.

\[ I := \{(l, l') \in F \times F | L \cap L' \neq \emptyset\} . \]

– If $X_1, X_2$ are subvarieties in the same projective space, then $\langle X_1, X_2 \rangle$ is their linear span.
– If $(l, l') \notin I$, the cubic surface in $Y$ defined by $L$ and $L'$ is $S_{l, l'} := \langle L, L' \rangle \cap Y$.

Lemma 2.1. Let $A, B$ and $C$ be varieties sitting inside the following commutative diagram

\[ \begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{h} & & \downarrow{g} \\
C & \xrightarrow{} & \end{array} \]

where $f$ and $h$ are rational maps and $g$ is a morphism. Then $\text{Ind}(f) \subseteq \text{Ind}(h)$.

Proof. Since the composition $g \circ h$ is defined in the domain of $h$, then $f$ is defined in the domain of $h$ by commutativity. \hfill \Box

Lemma 2.2. Let $pr_1 : I \to F$ be the projection to the first component. Then each fibre of $pr_1$ is a surface. Furthermore, $pr_1^{-1}(l)$ is irreducible for general $l \in F$.

Proof. Let $x \in Y$ be a point. Let $C_x$ be the subvariety of $F$ parametrizing lines through $x$. Take a system of coordinates of $\mathbb{P}^5$ such that $x = [0 : 0 : 0 : 0 : 0 : 1]$. Then the equation of $Y$ is $x_5^2q_1 + x_5q_2 + q_3 = 0$, where the polynomial $q_i \in \mathbb{C}[x_0, ..., x_4]$ is homogeneous of degree $i$. Since $Y$ is smooth, $q_1$ is not the zero polynomial. The variety $C_x$ can now be seen in $\mathbb{P}^4 \cong \{x_5 = 0\}$ as given by $q_1 = q_2 = q_3 = 0$. Then $C_x$ can be seen as $q_2 = q_3 = 0$ in $\mathbb{P}^3 \cong \{x_5 = q_1 = 0\}$. Thus, it is connected by [Har77, Exercise II.8.4c]. Hence, a fibre of $pr_1$ is

\[ pr_1^{-1}(l) \cong \bigcup_{x \in L} C_x. \]

It is known that $C_x$ is a curve for $x \in Y$, except for finitely many $x \in Y$ such that $C_x$ is a surface [CS09, Proposition 2.4]. Hence $pr_1^{-1}(l)$ is a surface for all $l \in F$. Moreover,
pr_1^{-1}(l) is connected for all \( l \in F \). Indeed, \( pr_1^{-1}(l) \) is the union of connected subvarieties of all of them meeting at the point \((l, l)\). Furthermore, \( pr_1^{-1}(l) \) is smooth for general \( l \in F \) by [Voi86, Section 3, Lemme 1], hence irreducible.

I thank Mingmin Shen for suggesting the following proof to me.

**Lemma 2.3.** The scheme \( I \subseteq F \times F \) is irreducible of dimension 6.

**Proof.** Let \( J := (q \times q)^{-1}(\Delta_Y) \). Then \( J \) is locally defined by four equations in \( P \times P \), hence each component of \( J \) has dimension at least 6. The map

\[
p \times p : J \to I
\]

is surjective and only contracts \( \Delta_P \) to \( \Delta_F \). Hence \( p \times p \) is birational and each component of \( I \) has dimension at least 6. Since \( pr_1^{-1}(l) \) is a surface for all \( l \in F \) by Lemma 2.2, each component of \( I \) has dimension 6. Moreover, \( pr_1^{-1}(l) \) is irreducible for general \( l \in F \). It follows that only one component of \( I \) maps surjectively to \( F \). Indeed, let \( I_1 \) be any irreducible component of \( I \) such that \( pr_1(I_1) = F \), then we have a surjective map \( pr_{1|I_1} : I_1 \to F \) between two irreducibles varieties of dimension, respectively, 6 and 4. The fibres of this map are \( pr_{1|I_1}^{-1}(l) = pr_1^{-1}(l) \cap I_1 \). Thus, for dimensional reasons, the general fibre of \( pr_{1|I_1} \) is a surface, hence \( pr_1^{-1}(l) \cap I_1 \) is a component of \( pr_1^{-1}(l) \). As \( pr_1^{-1}(l) \) is irreducible for general \( l \in F \), it follows that \( pr_1^{-1}(l) \) has only one component, therefore

\[
pr_1^{-1}(l) = pr_1^{-1}(l) \cap I_1.
\]

We have proved that \( pr_1^{-1}(l) \subseteq I_1 \) for general \( l \in F \). If \( I_2 \) is another irreducible component of \( I \) such that \( pr_1(I_2) = F \), the same argument implies that \( pr_1^{-1}(l) \subseteq I_2 \) for general \( l \in F \). It follows that \( pr_1^{-1}(l) \subseteq I_1 \cap I_2 \) for general \( l \in F \), so \( I_1 = I_2 \) as they are both irreducible.

Any other component of \( I \) different from \( I_1 \) maps to a proper closed subset of \( F \). For dimensional reasons this component must have dimension at most 5, so that \( I = I_1 \). It follows that \( I \) is irreducible and \( \text{dim } I = 6 \).

We define

\[
\rho : G(2, 6) \times G(2, 6) \to G(4, 6)
\]

to be the rational map given by the linear span of two general lines inside a projective space of dimension 5. In the following proposition, we use an argument already used in [SV16, Proposition 20.7].

**Proposition 2.4.** The indeterminacy locus of the restricted rational map

\[
\rho_{|F \times F} : F \times F \to G(4, 6)
\]

is the variety \( I \).

**Proof.** This map is clearly defined in the open set of pairs \((l, l')\) such that \( L \cap L' = \emptyset \). In particular

\[
\text{Ind}(\rho_{|F \times F}) \subseteq I.
\]

To prove the other inclusion, let \((l, l') \in I \) be a general point of \( I \), and let \( x = L \cap L' \). By surjectivity of the first projection \( I \to F \), the point \( l \) is general in \( F \). This implies that \( l \) is of first type. That is, the normal bundle to \( L \) in \( Y \) is

\[
N_{L/Y} = O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(1),
\]
see, e.g., [SV16, Appendix A.3]. The bundle $N_{L/Y}$ is of rank 3, which implies that we have three linearly independent vectors that generate $N_{L/Y,x}$. Take $v_\ell$ to be the image of a generator of $T(L')_x$ in $N_{L/Y,x}$ and choose a basis $\{v_\ell, w_0, w_1\}$ of $N_{L/Y,x}$.

For every point $(a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, we have a vector $aw_0 + bw_1 \in N_{L/Y,x}$. Since $N_{L/Y}$ is globally generated, the evaluation at $x$

$$ev_x : H^0(L, N_{L/Y}) \to N_{L/Y,x}$$

is a surjective linear map of vector spaces. By choosing a section of $ev_x$, for each vector $aw_0 + bw_1$ we get a section $s_{a,b} \in H^0(L, N_{L/Y})$ such that $s_{a,b}(x) = aw_0 + bw_1$. By [EH16, Theorem 6.13], we have an isomorphism $H^0(L, N_{L/Y}) \cong T(F)_l$. In particular, there is a vector in $T(F)_l$ induced by $s_{a,b}$.

Let $C$ be a smooth curve in $F$ such that $l$ is in $C$ and the tangent direction of $C$ at $l$ is that induced by $s_{a,b}$. We may assume that for general $l_c \in C$, the lines $L_c$ and $L'$ are disjoint in $Y$. If we restrict $\rho_{|F \times F}$ to $C \times \{l'\}$, then $\rho_{|C \times \{l'\}}$ is defined at a general point (hence, at any point) of $C \times \{l'\}$. So we see that $\rho_{|C \times \{l'\}}$ is well defined at $(l, l')$. For every $l_c \in C$ we denote by $\mathbb{P}_{l_c}$ the linear subspace of $\mathbb{P}^5$ represented by $\rho_{|C \times \{l'\}}(l_c, l')$.

Let $W$ be the restriction of the universal family $P \to F$ to $C$. The flat morphism $W \to C$ is a $\mathbb{P}^1$-bundle over a smooth curve. In particular, we can assume that there exists a section $\alpha : C \to W$ such that $\alpha(l) = x$. Consider the line from $x$ to $\alpha(l_c)$, which is clearly contained in $\mathbb{P}_{l_c}$. When $l_c$ approaches $l$, by continuity we get a line through $x$ in $\mathbb{P}_{l}$. Such a line is generated, in a natural way, by a lift of $s_{a,b}(x)$ to $T(\mathbb{P}^5)_x$. We can get such a lift by considering the following diagram of vector spaces with exact row and column

\[
\begin{array}{ccccccccc}
& & & & & & & & & \\
& & & & & & & & & 0 \\
& & & & & & & & & \mathbb{P}^5 \\
& & & & & & & & & \mathbb{P}^5 \\
& & & & & & & & & 0 \\
& & & & & & & & & N_{L/Y,x} \\
& & & & & & & & & \downarrow \\
& & & & & & & & & \downarrow \\
& & & & & & & & & T(L)_x \\
& & & & & & & & & \downarrow \\
& & & & & & & & & \downarrow \\
& & & & & & & & & 0 \\
& & & & & & & & & N_{L/Y,x} \\
& & & & & & & & & \downarrow \\
& & & & & & & & & T(L)_x \\
& & & & & & & & & \downarrow \\
& & & & & & & & & 0.
\end{array}
\]

We deduce that

$$\rho_{|C \times \{l'\}}(l, l') = \langle v_\ell, v_\ell', s_{a,b}(x) \rangle,$$

where $v_\ell$ spans the tangent direction of $L$ in $T(\mathbb{P}^5)_x$.

Notice that if we consider points $(a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ modulo a multiplicative constant, then for each $[a : b] \in \mathbb{P}^1$ we get a different space $\langle v_\ell, v_\ell', s_{a,b}(x) \rangle$. Indeed, the canonical map

$$T(L)_x \to N_{L/Y,x}$$

is zero, so $\{v_\ell, v_\ell', w_0, w_1\}$ is linearly independent in $T(\mathbb{P}^5)_x$. Note that we lifted the generators of $N_{L/Y,x}$ using (2.2). Obviously, the vector space generated by $\{v_\ell, v_\ell', s_{a,b}(x)\}$ depends only on $[a : b] \in \mathbb{P}^1$. In particular the image of $(l, l')$ depends on the curve $C$. This implies that $\rho_{|F \times F}$ cannot be extended to a general point of $I$ and we are done. □
3. General facts about the LLSS variety

**Definition 3.1.** A rational normal curve of degree 3, or twisted cubic for short, is a smooth curve \( C \subset \mathbb{P}^3 \) that is projectively equivalent to the image of \( \mathbb{P}^1 \) under the Veronese embedding \( \mathbb{P}^1 \to \mathbb{P}^3 \) of degree 3.

If \( X \subseteq \mathbb{P}^N \) is a projective variety, we denote by \( \text{Hilb}^{3z+1}(X) \) the Hilbert scheme of curves contained in \( X \) with Hilbert polynomial equal to \( 3z+1 \).

Piene and Schlessinger \[PS85\] showed that \( \text{Hilb}^{3z+1}(\mathbb{P}^3) = H_0 \cup H_1 \), where \( H_0 \) is a 12-dimensional smooth irreducible component such that the general point is a rational normal curve, and \( H_1 \) is a 15-dimensional smooth irreducible component such that the general point is a curve \( C \) such that \( C_{\text{red}} \) is a plane cubic. A generalized twisted cubic is a subscheme in \( \mathbb{P}^3 \) represented by a point in \( H_0 \).

We define \( \text{Hilb}^{gtc}(\mathbb{P}^3) \) as the subscheme \( \text{Hilb}^{gtc}(\mathbb{P}^3) := H_0 \subseteq \text{Hilb}^{3z+1}(\mathbb{P}^3) \). It is known that \( \text{Hilb}^{3z+1}(\mathbb{P}^5) \) contains a smooth irreducible component \( \text{Hilb}^{gtc}(\mathbb{P}^5) \) that parameterizes generalised twisted cubics.

The general cubic fourfold \( Y \) in \( \mathbb{P}^5 \) does not contain a plane, see \[Voi86\], Section 1, Lemme 1.

**Definition 3.2.** We set \( F_3(Y) := \text{Hilb}^{gtc}(\mathbb{P}^5) \cap \text{Hilb}^{3z+1}(Y) \).

By \[LLSvS17\], Theorem 4.7 \( F_3(Y) \) is a smooth variety of dimension 10.

An element \( [\Gamma] \in F_3(Y) \) is the class of a one dimensional subscheme \( \Gamma \) of \( Y \) with Hilbert polynomial equal to \( 3z+1 \). The linear span of \( \Gamma \) is a 3-dimensional space \( \langle \Gamma \rangle \cong \mathbb{P}^3 \), and \( [\Gamma] \in \text{Hilb}^{gtc}(\langle \Gamma \rangle) \). So the span induces a morphism \( F_3(Y) \to G(4,6) \) and, as stated by \[LLSvS17\], p.113 and Theorem 4.8, there is a commutative diagram

\[
\begin{array}{ccc}
F_3(Y) & \longrightarrow & G(4,6) \\
\phi \downarrow & & \downarrow g \\
Z' \nearrow & & \nearrow \\
\end{array}
\]

where \( Z' \) is a smooth irreducible projective variety. The diagram (3.1) has the following remarkable properties:

- The morphism \( \phi \) is a \( \mathbb{P}^2 \)-fibration.
- The morphism \( g \) is finite on the open subset \( g^{-1}(W_{\text{ADE}}) =: V_{\text{ADE}} \subseteq Z' \) where \( W_{\text{ADE}} := \{ P \in G(4,6) | P \cap Y \text{ has ADE singularities or is smooth} \} \).
- The degree of \( g \) on \( V_{\text{ADE}} \) is 72 by \[LLSvS17\], Theorem 2.1 and Table 1.

Moreover, by \[LLSvS17\], Theorem 4.11 and Proposition 4.5 there exists a divisorial contraction

\[
\begin{array}{ccc}
Z' & \rightarrow & \mathbb{P}(T_Y) \\
\sigma \downarrow & & \downarrow \\
Z & & Y \\
\end{array}
\]
making $Z'$ the blow up of a variety $Z$ over a subvariety canonically isomorphic to $Y$. By [LLSvS17, Theorem 4.19], $Z$ is an hyperkähler variety.

Obviously, both $Z'$ and $Z$ depend on $Y$, so they should be denoted by $Z'(Y)$ and $Z(Y)$. We choose to keep the same notation as [LLSvS17]. So, when no confusion is possible, we will simply write $Z'$ and $Z$.

4. The Voisin map

In [Vois16, Proposition 4.8] Voisin defined a rational map $\psi : F \times F \rightarrow Z'$ using the following nice geometric argument. Let $(l, l') \in F \times F$ be a general point, that is, $l$ and $l'$ are the classes of two disjoint lines $L$ and $L'$ such that the following surface

$$S_{l,l'} := \langle L, L' \rangle \cap Y$$

is smooth. The point $(l, l')$ defines a linear system in $S_{l,l'}$ given by the divisor

$$D_{l,l'} = L - L' - K_{S_{l,l'}}.$$ 

Since $\mathcal{O}_{S_{l,l'}}(1) = \mathcal{O}_{S_{l,l'}}(-K_{S_{l,l'}})$, in $|D_{l,l'}|$ there is a curve of the form $L \cup C'_x$, where $x$ is any point of $L$ and $C'_x$ is the unique conic such that $\langle x, L' \rangle \cap Y = L' \cup C'_x$. Then any member of this linear system is a generalized twisted cubic contained in $Y$. Voisin defines the map $\psi$ by setting $\psi(l, l')$ to be the class in $Z$ of any member of $|D_{l,l'}|$. The degree of the map is obtained as follows. It can be seen that $D_{l,l'}$ defines a morphism $\varphi_{D_{l,l'}} : S_{l,l'} \rightarrow \mathbb{P}^2$ that contracts exactly 6 lines. The members of $|D_{l,l'}|$ are pull-backs of lines in $\mathbb{P}^2$. The line $L$ is the inverse image of a blown up point, thus it is a component of the pull-back of any line through that point. We can see, by intersection theory in $S_{l,l'}$, that $L'$ is the strict transform of a conic through the other five points. Then we have 6 lines that are components of some rational normal curve in $|D_{l,l'}|$, so we have 6 possible choices of pairs of lines $R, R' \subseteq S_{l,l'}$ such that $|D_{l,l'}| = |D_{R,R'}|$.

We will describe Voisin’s construction in full detail. Note that there exists a rational map

$$\psi' : F \times F \rightarrow Z',$$

which differs from $\psi$ by a birational map, i.e., $\psi = \sigma \circ \psi'$. In particular, by Voisin’s construction of $\psi$ we already know that $\psi'$ is dominant and has degree 6. In Proposition 4.1 we will check that $\text{Ind}(\psi') = I$ and give a different argument for the degree of $\psi'$.

First of all, we notice that if $L$ is a line and $C$ is a conic in a projective space such that $L$ is not contained in the plane defined by $C$ and if $L \cap C = \{x\}$, then $L \cup C$ is a limit of rational normal curves [Har82, Section 1.b p. 39]. If both $L$ and $C$ are contained in $Y$, we have $[L \cup C] \in F_3(Y)$. As already pointed out in [Vois16, Proposition 4.8], there is a rational map

$$\psi_1 : P \times F \rightarrow F_3(Y)$$

defined as follows. Let $(l, l')$ be not in $I$, let $x \in L$ be a point and let $C'_x$ be the unique conic such that

$$\langle x, L' \rangle \cap Y = L' \cup C'_x.$$

Then

$$\psi_1(l, x, l') := [L \cup C'_x].$$
Consider
\[ U_{ADE} := \{(l, l') \in F \times F | (l, l') \notin I, S_{l, l'} := \langle L, L' \rangle \cap Y \text{ has } ADE \text{ singularities or is smooth} \} . \]

Pick \((l, l') \in U_{ADE} \). Since the linear span of \( L \cup C'_x \) is \( \langle L, L' \rangle \) for each \( x \in L \), then the image of the curve
\[ \Gamma_{l, l'} = \{ [L \cup C'_x] | x \in L \cong \mathbb{P}^1 \} \]
under the span map
\[ F_3(Y) \to G(4, 6) \]
is the point \( \langle L, L' \rangle \). In other words the curve \( \Gamma_{l, l'} \) is contracted by \( g \circ \phi \). By construction, \( g \circ \phi(\Gamma_{l, l'}) \subset W_{ADE} \) so that
\[ \phi(\Gamma_{l, l'}) \subset V_{ADE} . \]
The curve \( \Gamma_{l, l'} \) must be contracted by \( \phi \), since \( \phi(\Gamma_{l, l'}) \) is in the set where \( g \) is finite. Let
\[ p_1 : P \times F \to F \times F \]
be the canonical \( \mathbb{P}^1 \)-bundle. Then the restricted map
\[ \phi \circ \psi_1 : p_1^{-1}(U_{ADE}) \to Z' \]
contracts all the fibres of
\[ p_1 : p_1^{-1}(U_{ADE}) \to U_{ADE} . \]
Since \( p_1 \) is a linear \( \mathbb{P}^1 \)-bundle, we can consider an open set \( U \subseteq U_{ADE} \) trivializing \( p_1 \) [BS95, Section 3.2]. Then the diagram
\[ \begin{array}{ccc}
U & \xrightarrow{p_1} & Z' \\
\downarrow{p_1, U} & & \\
\mathbb{P}^1 \times U & \xrightarrow{\phi \circ \psi_1} & Z' \\
\end{array} \]
(4.1)
satisfies the hypothesis of the Rigidity Lemma [GW10, Proposition 16.54]. Indeed: \( U \) is reduced, \( Z' \) is separated and \( \mathbb{P}^1 \) is reduced, connected and proper. It follows that there exists a unique morphism \( U \to Z' \) making (4.1) commutative. We can cover \( U_{ADE} \) by trivializing open subsets, repeat that argument and get a map
\[ U_{ADE} \to Z' \]
which point-wise is
\[ (l, l') \mapsto \phi([L \cup C'_x]) . \]
Because we know that this map does not depend on the choice of \( x \in L \). We have therefore defined a rational map
\[ \psi' : F \times F \dashrightarrow Z' \]
\[ (l, l') \mapsto \phi([L \cup C'_x]) , \]
for all \((l, l')\) in the open subset \( U_{ADE} \) of \( F \times F \). As we said before, the map \( \psi' \) is dominant of degree 6 because these properties hold for \( \psi \), as Voisin proved. In the following proposition, we will see how the degree of \( \psi' \) can be obtained also from the map \( g : Z' \to G(4, 6) \).

**Proposition 4.1.** The rational map \( \psi' : F \times F \dashrightarrow Z' \) defined above is dominant, has degree 6 and \( \text{Ind}(\psi') = I \).
Proof. The composition of $\psi'$ with $g : Z' \to G(4, 6)$ gives rise to a commutative diagram

\[
\begin{array}{c}
F \times F \\ \downarrow \psi' \\
\downarrow \\
Z'
\end{array} \xrightarrow{\rho|_{F \times F}} G(4, 6)
\]

where $\rho$ is the span map. The inclusion $I \subseteq \text{Ind}(\psi')$ is an application of Lemma 2.1 and Proposition 2.4 to the Diagram (4.2). To prove the other inclusion, let

\[ p_1 : P \times F \to F \times F \]

be the canonical $\mathbb{P}^1$-bundle and consider the diagram

\[
\begin{array}{c}
P \times F \\ \downarrow \psi_1 \\
\downarrow \phi \\
F \times F \\ \psi' \\
\end{array} \xrightarrow{\rho|_{F \times F}} Z'.
\]

All maps in this diagram are defined on $p_1^{-1}(U_{ADE})$ and the diagram commutes there. Indeed, if $(l, x, l') \in p_1^{-1}(U_{ADE})$, then we know that $\psi_1$ is defined in $(l, x, l')$ as $(l, l') \in I$.

Then

\[
\phi(\psi_1(l, x, l')) = \phi([L \cup C_x]) = \phi(\Gamma_{l,l'}) = \psi'(l, l') = \psi'(p_1(l, x, l')).
\]

Consider a local section $s : F \times F \to P \times F$ of $p_1$, and let

\[
\psi'' := \phi \circ \psi_1 \circ s
\]

be the rational map defined on $F \times F \setminus I$. By commutativity of (4.3), $\psi''$ coincides with $\psi'$ on $U_{ADE}$. This implies that $\psi'$ can be extended to every point of $F \times F \setminus I$. We have then proved that

\[
\text{Ind}(\psi') \subseteq I.
\]

Let $M \in G(4, 6)$, then

\[
\rho^{-1}(M) = \left\{ (l, l') \in F \times F \setminus I \mid \langle L, L' \rangle = M \right\}.
\]

In particular, the pairs $(l, l') \in \rho^{-1}(M)$ represent pairs of disjoint lines contained in the cubic surface $M \cap Y$. If $M$ is sufficiently general, then $M \cap Y$ is smooth and contains 27 lines, each of them meeting exactly 10 other lines [GH78, pag 485]. It can easily be seen that there are $27 \cdot (27 - 11) = 432$ pairs of lines contained in $\rho^{-1}(M)$, and therefore $\rho$ is generically finite of degree 432. All the manifolds appearing in the commutative diagram (4.2) are 8-dimensional. Hence, as $\rho$ is generically finite, it is also dominant. This implies that also $g$ and $\psi'$ are dominant and generically finite. Since $\deg \rho = \deg \psi' \deg g$ by commutativity of (4.2), the degree of $\psi'$ is

\[
\deg \psi' = \frac{432}{72} = 6.
\]

$\square$
**Definition 4.2.** Let \( \psi \) be the Voisin map. A resolution of the indeterminacy of the map \( \psi \) is a commutative diagram
\[
\begin{array}{c}\xymatrix{ \widetilde{F \times F} \ar[rr]^\pi \ar[dr]_\psi & & \widetilde{F \times F} \ar[dr]_\psi \ar[r] & Z } \\
& F \times F \ar[r]^\psi & & \end{array}
\]
where \( \widetilde{F \times F} \) is a non-singular variety and \( \pi \) is a birational morphism that is an isomorphism outside \( \text{Ind}(\psi) \).

We will denote simply by \( \tilde{\psi} : \widetilde{F \times F} \to Z \) a fixed resolution of the indeterminacy of \( \psi \). Moreover, we denote by \( E \) the support of the exceptional divisor of \( \pi \). The existence of such a resolution follows by [Hir64, I. Question (E) p.140]. The map \( \pi \) may be obtained as a sequence of blow-ups along smooth subvarieties.

**Remark 4.3.** Notice that we have the following commutative diagram.
\[
\begin{array}{c}\xymatrix{ P \times F \ar[r]_{\psi_1} & F_3(Y) \ar[dl]_{\psi'} \ar[dr]^{\sigma} \\
F \times F \ar[r]_\psi & Z } \\
\end{array}
\]
Hence, the Voisin map defined in [Voi16, Proposition 4.8] is the composition \( \sigma \circ \psi' \).

For the reader’s convenience, we collect the following.

**Lemma 4.4** ([Voi16, Remark 4.10]). The map \( \psi : F \times F \to Z \) is étale of degree 6 where it is defined. Furthermore, the image of the exceptional divisor of the resolution \( \pi \) of \( \psi \) contains a divisor.

**Proof.** The map \( \psi \) is dominant of degree 6 because it is the composition of a dominant degree 6 rational map and of a blow up. Let \( R_{\tilde{\psi}} \) be the ramification divisor of \( \tilde{\psi} \), that is the divisor supported in the subset of points of \( \widetilde{F \times F} \) where the induced map \( d\tilde{\psi} : T_{\widetilde{F \times F}} \to \tilde{\psi}^*T_Z \) is not an isomorphism. The scheme structure is given locally by the vanishing of the Jacobian determinant \( \det d\tilde{\psi} \), see [Ful98, Example 3.2.20]. Thus we have the formula
\[
K_{\widetilde{F \times F}} = \pi^*K_{F \times F} + E' = \tilde{\psi}^*K_Z + R_{\tilde{\psi}}
\]
and since the first Chern class of \( F \) and \( Z \) is trivial, \( E' = R_{\tilde{\psi}} \). This implies that the ramification locus of \( \tilde{\psi} \) is \( E = \text{Supp} E' \), so that the Jacobian matrix is of maximal rank outside \( E \), in other words, \( \psi \) is étale where it is defined. Let \( D = \tilde{\psi}(E) \) be the image of the exceptional divisor. If we denote \( G := \tilde{\psi}^{-1}(D) \supset E \), then \( \tilde{\psi}_{|F \times F \setminus G} : F \times F \setminus G \to Z \setminus D \) is a nontrivial topological cover of degree 6. Since \( Z \) is simply connected [LLSvS17,
Theorem 4.19], if $D$ does not contain a divisor then by [God71, Chap. X Théorème 2.3] also $Z \setminus D$ is simply connected, so it cannot have nontrivial topological cover. Hence, an irreducible component of $D$ must be a divisor and we are done. \hfill \Box

We are now ready for the following.

**Proof of the Theorem 1.2.** From the commutative diagram

\begin{equation}
F \times F \xrightarrow{\psi} Z
\end{equation}

we have the inclusion

\begin{equation}
\text{Ind}(\psi) \subseteq I,
\end{equation}

by Lemma 2.1 and Proposition 4.1. Since $\sigma : Z' \to Z$ is a blow up along $Y \subseteq Z$, then on $Z \setminus Y$ we can compose its inverse with $g : Z' \to G(4,6)$ and get a map

\[ g_{|Z} : Z \setminus Y \to G(4,6). \]

Let $\tilde{\psi} : F \times F \to Z$ be as in Definition 4.2, and set

\begin{equation}
W := \pi(\tilde{\psi}^{-1}(Y)).
\end{equation}

We point out that outside the exceptional divisor $E$, the map $\tilde{\psi}$ is quasi-finite by Lemma 4.4. It follows that there is a commutative diagram of rational maps

\begin{equation}
F \times F \setminus W \xrightarrow{\rho_{F \times F \setminus W}} G(4,6)
\end{equation}

We argue by contradiction. Suppose $\text{Ind}(\psi) \subsetneq I$, and set

\[ T := \pi(\tilde{\psi}^{-1}(Y) \cap \pi^{-1}(I)). \]

**Claim.** The set $T$ is dense in $I$.

**Proof of the Claim.** By Lemma 2.3 and by our assumption $\text{Ind}(\psi) \subsetneq I$, the set $I \setminus \text{Ind}(\psi)$ is dense open in $I$. It follows that also

\[ \tilde{I} = \pi^{-1}(I \setminus \text{Ind}(\psi)) \]

is dense in $\pi^{-1}(I)$, since $\pi$ is an isomorphism outside $\text{Ind}(\psi)$. Now $\tilde{I} \setminus (\tilde{\psi}^{-1}(Y) \cap \tilde{I})$ is dense in $\tilde{I}$ since it is open and not empty. Indeed, if it were empty then $\tilde{I} \subseteq \tilde{\psi}^{-1}(Y)$, so

\[ \tilde{\psi}(\tilde{I}) \subseteq Y. \]
That is impossible since $Y$ has dimension 4 and $\tilde{\psi}(\tilde{I})$ has dimension 6 (because $\tilde{I}$ is contained in the open set where $\tilde{\psi}$ is quasi-finite). Since the map $\pi_{|\tilde{I}}: \tilde{I} \to I$ is dominant, the image of the dense open set $\tilde{I}\setminus(\tilde{\psi}^{-1}(Y) \cap \tilde{I})$ is dense. But

$$\pi(\tilde{I}\setminus(\tilde{\psi}^{-1}(Y) \cap \tilde{I})) = \pi(\pi^{-1}(I\setminus\text{Ind}(\psi))\setminus(\tilde{\psi}^{-1}(Y) \cap \pi^{-1}(I\setminus\text{Ind}(\psi)))) \subseteq \pi(\pi^{-1}(I)\setminus(\tilde{\psi}^{-1}(Y) \cap \pi^{-1}(I))) = T,$$

so that $T$ is dense. □

Thus there exists a point

$$(l, l') \in T \cap (I\setminus\text{Ind}(\psi)), \tag{4.10}$$

and since $\pi$ is an isomorphism outside $\text{Ind}(\psi)$, there exists a unique $u \in \tilde{F} \times \tilde{F}$ such that

$$\pi(u) = (l, l'). \tag{4.11}$$

By definition of $T$, the point $u$ is not in $\tilde{\psi}^{-1}(Y)$. If we apply Lemma 2.1 to Diagram (4.9) we get

$$\text{Ind}(\rho_{F \times F\setminus W}) \subseteq \text{Ind}(\psi|_{F \times F\setminus W})$$

and thus

$$I\setminus(W \cap I) \subseteq \text{Ind}(\psi\setminus(W \cap \text{Ind}(\psi))).$$

In particular, we have a dense subset of $I$ contained in $\text{Ind}(\psi)$, so that $I \subseteq \text{Ind}(\psi)$. Hence by (4.7) we get $I = \text{Ind}(\psi)$. This contradicts the assumption $\text{Ind}(\psi) \subsetneq I$ and we are done. □

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