Refined solvable presentations for polycyclic groups

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We describe a new type of polycyclic presentations, that we will call refined solvable presentations, for polycyclic groups. These presentations are obtained by refining a series of normal subgroups with abelian sections. These presentations can be described effectively by presentation maps which yield the basis data structure to define a polycyclic group in computer-algebra-systems like GAP or MAGMA. We study refined solvable presentations and, in particular, we obtain consistency criteria for them. This consistency implementation demonstrates that it is often faster than the existing methods for polycyclic groups.

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1 Introduction

A group $G$ is polycyclic if there exists a finite series of subnormal subgroups $G = G_1 \supseteq G_2 \supseteq \ldots \supseteq G_m \supseteq G_{m+1} = \{1\}$ so that each section $G_i/G_{i+1}$ is cyclic. Polycyclic groups play an important role in group theory as, for instance, each finite group with odd order is polycyclic. Moreover, polycyclic groups form a special class of finitely presented groups for which various algorithmic problems
are solvable. For instance, it is well-known that the word problem in a polycyclic group is solvable. More precisely, a polycyclic group $G$ can be described by a polycyclic presentation. This is a finite presentation with generators $\{a_1, \ldots, a_m\}$ and relations of the form

$$
a_i^{r_i} = a_{i+1}^{\alpha_i} \cdots a_m^{\alpha_m}, \quad i \in \mathcal{I}
$$

$$
a_i^{-1} a_j a_i = a_{i+1}^{\beta_{ij}} \cdots a_m^{\beta_{ij}}, \quad 1 \leq i < j \leq m
$$

$$
a_i^{-1} a_j^{-1} a_i = a_{i+1}^{\gamma_{ij}} \cdots a_m^{\gamma_{ij}}, \quad 1 \leq i < j \leq m, \quad j \notin \mathcal{I}
$$

$$
a_i a_j a_i^{-1} = a_{i+1}^{\delta_{ij}} \cdots a_m^{\delta_{ij}}, \quad 1 \leq i < j \leq m, \quad i \notin \mathcal{I}
$$

$$
a_i a_j^{-1} a_i^{-1} = a_{i+1}^{\varepsilon_{ij}} \cdots a_m^{\varepsilon_{ij}}, \quad 1 \leq i < j \leq m, \quad i, j \notin \mathcal{I}
$$

for a subset $\mathcal{I} \subseteq \{1, \ldots, m\}$ and integers $\alpha_{i,\ell}, \beta_{i,j,\ell}, \gamma_{i,j,\ell}, \delta_{i,j,\ell}, \varepsilon_{i,j,\ell} \in \mathbb{Z}$ that satisfy $0 \leq \alpha_{i,\ell}, \beta_{i,j,\ell}, \gamma_{i,j,\ell}, \delta_{i,j,\ell}, \varepsilon_{i,j,\ell} < r_i$ whenever $\ell \in \mathcal{I}$ holds. For further details on polycyclic presentations we refer to Section 9.4 of [14].

Given any finite presentation of a polycyclic group, the polycyclic quotient algorithm [11,12] allows one to compute a polycyclic presentation defining the same group. If, additionally, the polycyclic group is nilpotent, than any finite presentation can be transformed into a polycyclic presentation with the nilpotent quotient algorithm [13]. We further note that even certain infinite presentations (so-called finite L-presentations; see [2]) of a nilpotent and polycyclic group can be transformed into a polycyclic presentation [3]. We may therefore always assume that a polycyclic group is given by a polycyclic presentation.

In the group $G$, every element is represented by a word $a_1^{e_1} a_2^{e_2} \cdots a_m^{e_m}$ with $0 \leq e_i < r_i$ whenever $i \in \mathcal{I}$ holds. If this representation is unique, then the polycyclic presentation is consistent and it yields a normal form for elements in the group. This is a basis for symbolic computations within polycyclic groups. Various strategies for computing normal forms in a polycyclic group have been studied so far [10,16,6,1]. The current state of the art algorithm is collection from the left. But it is known that even ‘collection from the left’ is exponential in the number of generators [10]; see also [1].

In this paper, we concentrate on refined solvable presentations as a special class of polycyclic presentations that we describe in Section 2. We choose a finite series of normal subgroups so that the sections are abelian. A refined solvable presentation will be a certain polycyclic presentation that refines this series. Each weighted nilpotent presentation, as used extensively in the nilpotent quotient algorithms [13,3] and in [15], is of this type. A solvable presentation can be described effectively by presentation maps which we define in Section 2. Presentation maps can be considered as the basic data structure to define a polycyclic group in computer-algebra-systems like GAP or MAGMA. We obtain consistency criteria for refined solvable presentations in Section 3. This consistency check
has been implemented in the NQL-package [8]. Our implementation shows that the consistency checks for solvable presentations are often faster than the general methods for polycyclic groups. As an example, we consider nilpotent quotients of the Basilica group [7] and the BSV group [4].

Fast algorithms for polycyclic groups are of special interest as, for instance, the algorithm in [9] attempts to find periodicities in the Dwyer quotients of the Schur multiplier of a group. In order to observe these periodicities, the algorithm needs to compute with polycyclic presentations with some hundreds of generators and therefore fast algorithms for polycyclic groups are needed.

2 Refined solvable presentations

Let \( G \) be a polycyclic group with a strictly ascending chain of normal subgroups

\[
\{1\} = G_0 < G_1 < \cdots < G_r = G
\]

where \( G_i/G_{i-1} \) is abelian for \( i = 1, \ldots, r \). Since each subgroup of a polycyclic group is finitely generated, we can choose a finite generating set \( X \) for \( G \) which partitions as \( X = X_1 \cup X_2 \cup \cdots \cup X_r \) such that

\[
G_i/G_{i-1} = \bigoplus_{x \in X_i} \langle xG_{i-1} \rangle
\]

for \( i = 1, \ldots, r \) and where all the direct summands are non-trivial. We can furthermore make our choice so that for each \( x \in X_i \), either the order, \( o(xG_{i-1}) \), of \( xG_{i-1} \) is infinite or a power of a prime. Let \( \mathcal{P} \) denote the set of all primes. For each \( p \in \mathcal{P} \), let

\[
X_i(p) = \{ x \in X_i : o(xG_{i-1}) \text{ is a power of } p \}
\]

and let

\[
X_i(\infty) = \{ x \in X_i : o(xG_{i-1}) = \infty \}.
\]

Notice that the Sylow \( p \)-subgroup of \( G_i/G_{i-1} \) is

\[
(G_i/G_{i-1})_p = \bigoplus_{x \in X_i(p)} \langle xG_{i-1} \rangle.
\]

We order the generators in \( X \) such that the generators in \( X_i \) precede those in \( X_j \) whenever \( i < j \). Suppose that \( X = \{ x_1, \ldots, x_m \} \) with \( x_1 < x_2 < \ldots < x_m \). For each \( x \in X_i \) let \( n(x) = o(xG_{i-1}) \). If \( n(x) = \infty \), let \( \mathbb{Z}_x = \mathbb{Z} \) and otherwise let \( \mathbb{Z}_x = \{0, \ldots, n(x) - 1\} \). Each element \( g \in G \) has a unique normal form expression

\[
g = x_m^{r_m} x_{m-1}^{r_{m-1}} \cdots x_1^{r_1}
\]
where $r_i \in \mathbb{Z}_{x_i}$.

We next describe some relations that hold in the generators $x_1, \ldots, x_m$. If $x \in X_s(p)$ then we get a power relation of the form
\[
x^n(x) = x_m^{\alpha(x)(m)} \cdots x_1^{\alpha(x)(1)}
\]
with $\alpha_x(i) \in \mathbb{Z}_{x_i}$ and where $\alpha_x(i) = 0$ if $x_i \not\in X_1 \cup \cdots \cup X_{s-1}$.

For each pair of generators $x, y \in X$ with $x < y$ we also get a conjugacy relation
\[
x^y = x_m^{\beta(x,y)(m)} \cdots x_1^{\beta(x,y)(1)}
\]
where $\beta_{(x,y)}(i) \in \mathbb{Z}_{x_i}$.

**Remark.** There are three types of relations of the form (2).

**Type 1.** If $x, y \in X_s$ then $x$ and $y$ commute modulo $G_{s-1}$ and thus we get that $\beta_{(x,y)}(i) = 0$ if $x_i \not\in X_1 \cup \cdots \cup X_{s-1} \cup \{x\}$ and that $\beta_{(x,y)}(i) = 1$ if $x_i = x$.

Now suppose that $s < t$.

**Type 2.** If $x \in X_s(p)$ and $y \in X_t$ then $x^y G_{s-1} \in (G_s/G_{s-1})_p$ and thus we get a relation of the form (2) where $\beta_{(x,y)}(i) = 0$ if $x_i \not\in X_1 \cup \cdots \cup X_{s-1} \cup X_s(p)$.

**Type 3.** Finally if $x \in X_s(\infty)$ and $y \in X_t$ then $x^y \in G_s$ and we get a relation of the type (2) where $\beta_{(x,y)}(i) = 0$ if $x_i \not\in X_1 \cup \cdots \cup X_s$.

**Remark.** By an easy induction on $m$, one can see that (1) and (2) also give us, for every pair of generators $x, y \in X$ such that $x < y$, a relation $x^y = \mu(x, y)$, where $\mu(x, y)$ is a normal form expression. Thus using only relations (1) and the three types of relations (2), we have a full information about $G$ and we can calculate inverses and products of elements of normal form and turn the result into a normal form expression using for example collection from the left.

The claim holds trivially for $m = 1$. Now suppose that $m \geq 2$ and that the claim holds for all smaller values of $m$. Consider the subgroup $H = \langle x_1, \ldots, x_{m-1} \rangle$. By the inductive hypothesis, every element in $H$ can be turned into a normal form expression using only relations (1) and (2). Now (2) gives us normal form expressions for $x_1^{x_m}, \ldots, x_{m-1}^{x_m}$ and this determines an automorphism $\phi \in \text{Aut}(H)$ induced by the conjugation of $x_m$. This then gives us $\phi^{-1}$ that gives us in turn normal form expressions for $x_1^{x_m^1}, \ldots, x_{m-1}^{x_m^1}$. This finishes the proof of the inductive step.
The point about this is that the relations \( x^y = \mu(x, y) \) are not defining relations but consequences of (1) and (2). So for a polycyclic group \( G \) we only need (1) and (2) to define it. For practical reasons we need however to determine the relations \( x^y = \mu(x, y) \) first to be able to perform calculations in \( G \). At the end of section 3, we describe an efficient method for doing this for the polycyclic presentations that we are about to introduce next, refined solvable presentations.

Suppose now conversely that we have a finite alphabet \( X = \{x_1, x_2, \ldots, x_m\} \) with an ordering \( x_1 < x_2 < \ldots < x_m \). Let \( F \) be the free group on \( X \). Partition \( X \) into some disjoint non-empty subsets \( X_1, \ldots, X_r \) such that the elements of \( X_i \) precede those in \( X_j \) whenever \( i < j \). Then partition further each \( X_i \) as a union of disjoint subsets (most empty of course)

\[
X_i = \left( \bigcup_{p \in P} X_i(p) \right) \cup X_i(\infty).
\]

Let \( Y = Z \setminus \{x \in X : n(x) = \infty\} \) and \( Z = \{(x, y) \in X \times X : x < y\} \). We introduce three maps that we will refer to as presentation maps. The first one is

\[
n : X \to \mathbb{N} \cup \{\infty\}
\]

such that \( n(x) = \infty \) if \( x \in X_i(\infty) \) and \( n(x) \) is a non-trivial power of \( p \) is \( x \in X_i(p) \).

The second presentation map is

\[
\pi : Y \to F
\]

where, if \( x \in X_s(p) \), \( \pi(x) = x_1^{\alpha_x(m)} \cdots x_1^{\alpha_x(1)} \) with \( \alpha_x(i) \in \mathbb{Z}_{x_i} \) and \( \alpha_x(i) = 0 \) whenever \( x_i \not\in X_1 \cup \cdots \cup X_{s-1} \). Notice that these are the conditions for the right hand side of the power relation (1). The final presentation map is

\[
\delta : Z \to F
\]

where \( \delta(x, y) = x_m^{\beta_{x,y}(m)} \cdots x_1^{\beta_{x,y}(1)} \) and the conditions for the right hand side of (2) above hold as indicated in the remark that follows it. So we have a data that consists of an alphabet \( X \) with a partition and three presentation maps. To this data we associate a presentation with generators \( x_1, \ldots, x_m \), power relations

\[
x^{n(x)} = \pi(x)
\]

for any \( x \in X \) such that \( n(x) \neq \infty \), and conjugacy relations

\[
x^y = \delta(x, y)
\]

for each pair \( (x, y) \in X \times X \) such that \( x < y \). We call such a presentation a refined solvable presentation. We have seen above that every polycyclic group has
a refined solvable presentation that is consistent. Conversely, we are interested in criteria for a given refined solvable presentation to be a consistent presentation for a polycyclic group $G$. In other words we want the group $G$ to be polycyclic and we want every element $g \in G$ to have a unique normal form expression

$$g = x_{r_1}^m \cdots x_1^r$$

with $r_i \in \mathbb{Z}_{x_i}$. In next section we describe such consistency criteria.

**Remark.** Notice that there are groups with a refined solvable presentation that are not polycyclic. Take for example two variables $x_1 < x_2$ and let $X_1 = X_1(\infty) = \{x_1\}$, $X_2 = X_2(\infty) = \{x_2\}$. Here $Y = \emptyset$ and $Z = \{(x_1, x_2)\}$. For the presentation maps $n : X \to \mathbb{N} \cup \{\infty\}$ and $\pi : Y \to F$, we must have $n(x_1) = n(x_2) = \infty$ and $\pi$ must be empty. Suppose we choose $\delta : Z \to F$ such that $\delta(x_1, x_2) = x_1^2$. Then we get a presentation with two generators $x_1, x_2$ and one relation

$$x_1^2 = x_1^2.$$ 

The resulting group is not polycyclic. The criteria that we will describe in section 3 are thus not only consistency criteria but also criteria for the resulting group to be polycyclic.

### 3 The consistency criteria

Before establishing our consistency criteria, we first describe constructions that are central to what follows. Suppose we have a polycyclic group $G = \langle X \rangle$ that has a consistent refined solvable presentation as described above with a generating set $X = \{x_1, \ldots, x_m\}$ that is partitioned as described in section 2 and with presentation maps $n, \pi$ and $\delta$. Let $\phi \in \text{Aut}(G)$. We will consider two situations where we can use this data to get a consistent refined solvable presentation for a larger polycyclic group $\tilde{G}$. Add a new variable $x_{m+1}$ and extend our order on $\tilde{X} = X \cup \{x_{m+1}\}$ such that $x_{m+1}$ is larger than the elements in $X$. Let $\tilde{F}$ be the free group on $\tilde{X}$. Let $H$ be the semidirect product of $G$ with a infinite cyclic group $C_\infty = \langle x \rangle$ where the action from $C_\infty$ on $G$ is given by $g^x = g^\phi$.

For the first situation let $\tilde{G} = H$. We extend the presentation maps $n, \pi, \delta$ to $\tilde{n}, \tilde{\pi}, \tilde{\delta}$ so they involve $\tilde{X}$. We do this by letting $\tilde{n}(x_{m+1}) = \infty$ and

$$\tilde{\delta}(x_i, x_{m+1}) = x_i^\phi$$

(in a normal form expression in $x_1, \ldots, x_m$) for $i = 1, \ldots, x_m$. Notice that, since $n(x_{m+1}) = \infty$, $\tilde{\pi} = \pi$. The refined solvable presentation that we get using the extended presentation maps has all the relations for $G$ together with $m$ extra relations

$$x_i^{x_{m+1}} = \delta(x_i, x_{m+1}) = x_i^\phi.$$
for $i = 1, \ldots, m$. A moments glance should convince the reader that this is a refined solvable presentation for the polycyclic group $\tilde{G} = H$.

**Remark.** We haven’t said anything above about the partition of $\tilde{X} = \{x_1, \ldots, x_{m+1}\}$. The partition would be into $\tilde{X}_1 = X_1, \ldots, \tilde{X}_r = X_r, \tilde{X}_{r+1} = \{x_{m+1}\}$. If furthermore $x^{-1}x^\phi \in G_{r-1}$ for all $x \in X_r$ we could instead choose a partition with $\tilde{X}_1 = X_1, \ldots, \tilde{X}_{r-1} = X_{r-1}, \tilde{X}_r = X_r \cup \{x_{m+1}\}$.

The second situation is a variant of the first. Now suppose further that for some integer $e \geq 2$, that is a power of a prime $p$, and $g \in G$ we have that

\[ a^g = a^{g^e} \quad \text{(for all } a \in G) \]  
\[ g^\phi = g \]  

In this case $N = \langle g^{-1}x^e \rangle$ is a subgroup of the centre of $H$. Let $\tilde{G} = H/N$. $G$ embeds naturally into $\tilde{G}$ and we identify it with it’s image. We now extend the presentation maps $n, \pi, \delta$ to $\tilde{n}, \tilde{\pi}, \tilde{\delta}$ as follows. First we let $\tilde{n}(x_{m+1}) = e$ and $\tilde{\pi}(x_{m+1})$ be the normal form expression for $g$ in $x_1, \ldots, x_m$. Finally as before let $\tilde{\delta}(x_i, x_{m+1})$ be the normal form expression of $x_i^\phi$ in $x_1, \ldots, x_m$. The refined solvable presentation with respect to the presentation maps $\tilde{n}, \tilde{\pi}$ and $\tilde{\delta}$ is then a presentation with all the relations for $G$ and the extra relations

\[ x_{m+1}^n = \tilde{\pi}(x_{m+1}) = g \]

together with

\[ x_i^{x_{m+1}} = \tilde{\delta}(x_i, x_{m+1}) = x_i^\phi \quad \text{(in a normal form expression in } x_1, \ldots, x_m) \]

for $1 \leq i \leq m$. Again it is clear that this is a refined solvable presentation for the polycyclic group $\tilde{G} = H/N$. The remark above applies again for the partition in this case.

We now turn back to our task of finding a consistency criteria for power-conjugate presentations of poly-cyclic groups. Suppose $G = \langle x_1, \ldots, x_m \rangle$ is a poly-cyclic group with a refined solvable presentation as described above. So we have some partition of $X = \{x_1, \ldots, x_m\}$ and presentation maps $n, \pi, \delta$ giving us relations

\[ x^n(x) = x_1^{x_m(1)} \cdots x_1^{x_m(m)} \underbrace{\pi(x)} \]

for $x_1 \leq x \leq x_m$ with $n(x) < \infty$ and

\[ x^y = x_1^{x_m(y)(1)} \cdots x_1^{x_m(y)(m)} \underbrace{\delta(x, y)} \]
for $x_1 \leq x < y \leq x_m$. For $k = 0, 1, \ldots, m$, let $H_k$ be the group satisfying the sub-presentation with generators $x_1, \ldots, x_k$ and those of the relations involving only $x_1 \leq x < y \leq x_k$. The idea is to establish inductively criteria for the refined solvable presentation for $H_k$ to be a consistent presentation of a polycyclic group. The induction basis $k = 0$ doesn't need any work. Now suppose that we have already obtained criteria for the refined solvable presentation for $H_k$, where $0 \leq k \leq m - 1$, to be a consistent presentation of a polycyclic group. Using the presentation map $\delta$ we define a function $\delta(x_{k+1}) : H_k \to H_k$ by first defining the values of the generators as $x_i^{\delta(x_{k+1})} = \delta(x_i, x_{k+1})$ for $i = 1, \ldots, k$. We then extend this to the whole of $H_k$ by letting $\delta(x_{k+1})$ act on normal form expressions as follows

$$(x_k^r \ldots x_1^r)_{\delta(x_{k+1})} = (x_k^{\delta(x_{k+1})})^r \ldots (x_1^{\delta(x_{k+1})})^r.$$ 

Suppose the resulting map $\delta(x_{k+1})$ is an automorphism. If $n(x_{k+1}) = \infty$, we have that the presentation for $H_{k+1}$ is a consistent presentation for the semidirect product of $H_k$ with the infinite cyclic group $C_{\infty} = \langle x \rangle$ where $g^x = g^{\delta(x_{k+1})}$. Now suppose that $n(x_{k+1}) \neq \infty$. Using the second construction above and taking into account conditions (3) and (4), we get a presentation for $H_{k+1}$ that is a consistent presentation of a polycyclic group, provided that

$$\pi(x_{k+1})^{\delta(x_{k+1})} = \pi(x_{k+1})$$

$$x_i^{\delta(x_{k+1}) n(x_{k+1})} = x_i^{\pi(x_{k+1})}$$

for $i = 1, \ldots, k$. It remains to find criteria for $\delta(x_{k+1})$ to be an automorphism. This problem we turn to next.

Let $G = \langle X \rangle$ be a poly-cyclic group with a consistent refined solvable presentation as described above. For $s = 1, \ldots, r$ let $G_s = \langle X_1 \cup \cdots \cup X_s \rangle$, $G_s(p) = \langle X_1 \cup \cdots \cup X_{s-1} \cup X_s(p) \rangle$ and let $\tau(G_s) = \langle X_1 \cup \cdots X_{s-1} \cup (\bigcup_{p \in \mathcal{P}} X_s(p)) \rangle$. For each $x \in X$ choose an element $x^\phi$ subject to the following conditions:

$$x^\phi \in G_i \text{ if } x \in X_i$$

$$x^\phi \in G_i(p) \text{ if } x \in X_i(p).$$

(5)

We extend this to a map $\phi : G \to G$ by letting $\phi$ act on normal form expressions as:

$$(x_m^r \ldots x_1^r)^\phi = (x_m^\phi)^r \ldots (x_1^\phi)^r.$$ 

Notice that the condition (5) implies that $\phi$ induces maps $\phi_s : G_s \to G_s$, $s = 1, \ldots, r$, where $\phi_s = \phi|_{G_s}$. It also induces maps $\phi_{(s,p)} : G_s(p)/G_{s-1} \to G_s(p)/G_{s-1}$ and maps $\phi_{(s,\infty)} : G_s/\tau(G_s) \to G_s/\tau(G_s)$. 
Lemma 1 \ The map $\phi : G \to G$ is a homomorphism if and only if
\[
\pi(x)^\phi = (x^\phi)^{n(x)} \quad (x_1 \leq x \leq x_m)
\]
and
\[
x^{y^\phi} = x^{y^\phi} \quad (x_1 \leq x < y \leq x_m).
\]
$\phi$ is furthermore an automorphism if for $s = 1, \ldots, r$ we have
\[
\text{det}(\phi_{(s,p)}) \not\equiv 0 \pmod{p} \quad (3)
\]
\[
\text{det}(\phi_{(s,\infty)}) = \pm 1. \quad (4)
\]

Proof. Consider the homomorphism $\psi : F \to F$ on the free group $F = \langle x_1, \ldots, x_m \rangle$ induced by the values $x^\psi = x^\phi$ for $x_1 \leq x \leq x_m$. Let $R$ be the normal subgroup generated by the defining polycyclic relations for $G$. This means that $G = F/R$. Then conditions (1) and (2) imply that $R^\psi \leq R$ and thus $\psi$ induces a homomorphism on $G = F/R$. This homomorphism is clearly the map $\phi$.

The homomorphism $\phi$ is bijective if and only if the induced linear maps $\phi_{(s,p)}$ and $\phi_{(s,\infty)}$ are bijective and this happens if and only if condition (3) holds. \(\square\)

Remark. The condition (1) in the lemma above is of course only relevant when $n(x) < \infty$. To avoid making the statement more complicated we can decide that $\pi(x) = 1$ and $u^{n(x)} = 1$ for all $u \in G$ in the case when $n(x) = \infty$.

We now turn back again to the problem of establishing criteria for refined solvable presentations to be a consistent presentation of a polycyclic group. Let $G = \langle x_1, \ldots, x_m \rangle$ be a group satisfying a refined solvable presentation as described above with relations
\[
x^{n(x)} = x_m^{\alpha_x(m)} \cdots x_1^{\alpha_x(1)} \quad (x_1 \leq x \leq x_m)
\]
\[
x^y = x_m^{\beta_{(x,y)}(m)} \cdots x_1^{\beta_{(x,y)}(1)} \quad (x_1 \leq x < y \leq x_m).
\]

We let $H_k$ be the group satisfying the sub-presentation with generators $x_1, \ldots, x_k$ and those of the relations where $x_1 \leq x < y \leq x_k$. We establish inductively criteria for the presentation for $H_k$ to be a consistent presentation of a polycyclic group. Suppose this has been achieved for some $k$. We want to add criteria so that the presentation for $H_{k+1}$ is a consistent presentation for a polycyclic group. We let $\delta(x_{k+1}) : H_k \to H_k$ be the map induced by the values $x_k^{\beta(x_{k+1})}$ as described above. As we pointed out, the presentation for $H_{k+1}$ is a consistent presentation
of a polycyclic group if and only if the map $\delta(x_{k+1})$ is an automorphism and that we have the extra criteria that

$$\pi(x_{k+1})^\delta(x_{k+1}) = \pi(x_{k+1})$$

$$x_i^{\delta(x_{k+1})^{n(x_{k+1})}} = x_i^{\pi(x_{k+1})}.$$ 

From Lemma 1 we have criteria for $\delta(x_{k+1})$ to be an automorphism. Suppose that $x_{k+1} \in X_s$. Then $\delta(x_{k+1})$ acts trivially on $G_s/G_{s-1}$ and so to establish that $\delta(x_{k+1})$ is bijective we only need to show that $\delta(x_{k+1})_{(t,p)}$ and $\delta(x_{k+1})_{(t,\infty)}$ are bijective for $1 \leq t < s$.

For $z \in X$ let $r(z)$ be the integer such that $z \in X_{r(z)}$. Adding up for $k = 0, \ldots, m-1$, we obtain the following consistency criteria.

**Theorem 2** The refined solvable presentation for $G$ is a consistent presentation for a polycyclic group if and only if the following criteria hold. Firstly we must have for all $x_2 \leq z \leq x_m$ that

$$\pi(z)^{\delta(z)} = \pi(z)$$

$$\pi(x)^{\delta(z)} = (x^{\delta(z)})^{\pi(x)} \quad (x_1 \leq x < z)$$

$$x^{\delta(z)^{n(x)}} = x^{\pi(z)} \quad (x_1 \leq x < z)$$

$$x^{y^{\delta(z)}} = x^{y^{\delta(z)}} \quad (x_1 \leq x < y < z).$$

We also need for $1 \leq s < r(z)$ that

$$\det(\delta(z)_{(s,p)}) \neq 0 \pmod{p}$$

$$\det(\delta(z)_{(s,\infty)}) = \pm 1.$$

**Remarks.** (1) Recall that we established the consistency of the polycyclic group $H_k$ recursively for $k = 0, 1, \ldots, m$. So according to the proof we should check (1)-(5) for $z = x_2, \ldots, x_m$ in ascending order. If $z = x_{k+1}$ then the consistency of $H_{k+1}$ follows from the consistency of $H_k$ together with relations (1)-(5) of Theorem 2 where $z = x_{k+1}$. So when doing the check for $z = x_{k+1}$ we can assume that the presentation for $H_k$ is consistent. Using the definition of $\delta(z)$ we first transform all the expressions in (1)-(4) into expressions in $H_k$. Then we turn each side of the equations into normal form in $H_k$ and compare. It is interesting to note that (provided the check has been positive so far) $H_k$ has a consistent presentation and so the normal form in each case is independent of how we calculate. We can however do the check in any order we like (and still sticking to the assumption that $H_k$ has a consistent presentation). The reason for this is that we will at some point reach the smallest $z$ where the check fails (provided that we haven’t got a negative result in the mean time). Hence if the presentation is
not a consistent presentation of a polycyclic group, this will be recognised.

(2) How does this approach compare to the existing ones. Our approach is to consider functions $\delta(z)$ defined on a group $G_z$ with a subpresentation (involving only the generators less than $z$). Modulo consistency of $G_z$ the conditions (1)-(5) in Theorem 2 are conditions for the map $\delta(z)$ to be an automorphism ((2), (4) and (5)) and for the resulting cyclic extension to have a consistent presentation ((1) and (3)). The emphasis is thus on the function $\delta(z)$ rather than the group operation (as in [14]). It is our belief that this viewpoint makes things look a bit clearer.

(3) It should be noted however that our conditions (1)-(4) have equivalent criteria in the standard approach. See the list (*) in [14], page 424. The 'overlaps' (1),(2),(3) and (5) in that list correspond to (4),(2),(3) and (1) in Theorem 2. The condition (5) is however new and is a biproduct of working with an ascending normal solvable series. In the standard approach one works with a ascending subnormal series with cyclic factors. It should also be noted that the idea of obtaining consistency recursively for $H_k, k = 0,\ldots,m$, through working with $\delta(z)$, is also implicit in [14] but is kept in the background within the proof. Our conditions (1)-(5) bring this to the surface.

A method for obtaining inverse conjugation relations. For practical checks using these consistency criteria one needs to determine first normal form expressions $xz^{-1}$ for $x < z < x_m$ (in order to be able to transform any expression in $H_k$ to an normal form expression). Note however that this is of course only needed when $z$ is of infinite order. Another advantage of our approach is that it becomes quite simple and effective to determine these after having produced all the linear maps $\delta(z)_{(s,p)}$ and $\delta(z)_{(s,\infty)}$, $2 \leq s \leq r$. Suppose that $z \in X_s$ for some $2 \leq s \leq r$. We now describe how to obtain normal form expressions for $x^{-1}$ recursively for $x < z$.

We can suppose that we already know that the sub-presentation for the group $G^*$ generated by the generators $\{x \in X : x < z\}$ (using only the relations involving these generators) is consistent. The presentation for $G^*$ is built around an ascending normal $z$-invariant series with each factor either a finite abelian $p$-group or a finitely generated torsion-free abelian group.

Now suppose that we are looking at one such factor $K/H$ and that the extra generators needed to generate $K$ are $y_1,\ldots,y_e$. We can suppose inductively that we have obtained normal form expressions for all $x^{-1}$ when $x$ is a generator of $H$. We want to extend this to $y_i^{-1}$ for $i = 1,\ldots,e$. 
Let $v_1 = y_1H, \ldots, v_e = y_eH$ be the generators of $K/H$. Let $\phi$ be the automorphism on $K/H$ induced by the conjugation action by $z$ and let $\psi$ be the inverse of $\phi$. Suppose $\psi$ is represented by the matrix $B = (b_{ij})$. Since $\phi(\psi(v_i)) = v_i$, we have
\[ b_{ei}\phi(v_e) + \cdots + b_{2i}\phi(v_2) + b_{1i}\phi(v_1) = v_i. \]
It follows that (using the presentation and calculating in $K$) we get
\[ (y_e^z)^{b_{ei}} \cdots (y_2^z)^{b_{2i}} (y_1^z)^{b_{1i}} = y_i u. \]
Where $u$ is a normal form expression in the generators of $H$ (and we already know how $z^{-1}$ acts on $u$. It follows that
\[ y_i^{-1} = y_e^{b_{ei}} \cdots y_2^{b_{2i}} y_1^{b_{1i}} u^{-1}. \]

4 Implementation and some applications of our consistency checks

We have implemented our consistency check in the NQL package [8] of the computer-algebra-system GAP; see [5]. In this section, we demonstrate how this method yields a significant speed-up in checking consistency of large polycyclic presentations (with some hundreds of generators). For this purpose, we consider nilpotent quotients of the Basilica group $\Delta$ from [7] and the Brunner-Sidki-Vieira-Group BSV from [4]. Both groups are two-generated but infinitely presented. The Basilica group admits the following infinite presentation
\[
\Delta \cong \langle \{a, b\} \mid [a, a^b]^\sigma^i, i \in \mathbb{N}_0 \rangle
\]
where $\delta$ is the endomorphism of the free group over $a$ and $b$ induced by the mapping $a \mapsto b^2$ and $b \mapsto a$; see [7]. The BSV group admits the infinite presentation
\[
\text{BSV} \cong \langle \{a, b\} \mid [b, b^a]^\varepsilon^i, [b, b^{a^3}]^\varepsilon^i, i \in \mathbb{N}_0 \rangle,
\]
where $\varepsilon$ is the endomorphism of the free group over $a$ and $b$ induced by the mapping $a \mapsto a^2$ and $b \mapsto a^2b^{-1}a^2$. The nilpotent quotient algorithm in [3] computes a weighted nilpotent presentation for the lower central series quotient $G/\gamma_c(G)$ for a group $G$ given by an infinite presentation as above (a so-called finite $L$-presentation; see [2]). A weighted nilpotent presentation is a polycyclic presentation which refines the lower central series of the group. We note that the weighted nilpotent presentations for the quotients $\Delta/\gamma_c\Delta$ and BSV/\gamma_cBSV are refined solvable presentations.

In order to verify consistency of a given polycyclic presentation, the algorithm in [14, p. 424] rewrites the overlaps of the rewriting rules and compares the result;
that is, the algorithm checks the underlying rewriting system for local confluence. As even the state of art algorithm ‘collection from the left’ is exponential [10], the number of overlaps is a central bottleneck here. There are improvements known which make use of the structure of a polycyclic presentation in order to reduce the number of overlaps. For instance, for weighted nilpotent presentations, a weight function allows one to reduce the number of overlaps significantly; see [14, p. 431].

Our method replaces some overlaps by the computation of determinants of integral matrices and it can easily be combined with the method for weighted nilpotent presentations. This promising approach yields a significant speed-up as the following table shows. The timings were obtained on an Intel Pentium 4 processor with a clock-speed of 2.4 GHz.

| Quotient | #gens | Usual   | Solv   | Weight | Solv+Weight |
|----------|-------|---------|--------|--------|-------------|
| BSV, class 25 | 106   | 0:00:05 | 0:00:04 | 0:00:01 | 0:00:01     |
| BSV, class 35 | 179   | 0:01:35 | 0:01:06 | 0:01:06 | 0:00:48     |
| BSV, class 40 | 219   | 0:04:26 | 0:03:00 | 0:03:22 | 0:02:25     |
| BSV, class 45 | 259   | 0:10:27 | 0:06:54 | 0:08:28 | 0:06:05     |
| BSV, class 50 | 301   | 6:31:17 | 3:52:36 | 6:30:13 | 4:43:52     |
| Δ, class 35   | 185   | 0:00:31 | 0:00:31 | 0:00:02 | 0:00:02     |
| Δ, class 80   | 609   | 1:19:22 | 1:15:03 | 0:29:48 | 0:27:36     |
| Δ, class 100  | 821   | 8:25:37 | 7:39:54 | 5:45:40 | 5:18:08     |

The method Usual denotes the algorithm in [14, p. 424] for polycyclic presentations, the method Solv denotes our new method, the method Weight denotes the method for weighted nilpotent presentation as in [14, p. 431], and the method Solv+Weight denotes the combination of both of the latter methods. The number #gens denotes the number of generators of the considered polycyclic presentation. In summary, our method always yields here a significant speed-up compared with the standard method for polycyclic groups.

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