Mixing and spectral-correlation properties of chaotic and stochastic systems: numerical and physical experiments

V S Anishchenko¹, G A Okrokvertskhov, T E Vadivasova and G I Strelkova

Institute of Nonlinear Dynamics, Physical Department, Saratov State University, 410126 Saratov, Astrakhanskaya str. 83, Russia
E-mail: wadim@chaos.ssu.runnet.ru, tanya@chaos.ssu.runnet.ru, galya@chaos.ssu.runnet.ru and george@chaos.ssu.runnet.ru

New Journal of Physics 7 (2005) 76
Received 6 October 2004
Published 8 March 2005
Online at http://www.njp.org/
doi:10.1088/1367-2630/7/1/076

Abstract. In the present paper, we analyse a mechanism of the onset of mixing and its interconnection with exponential instability of trajectories. We study statistical characteristics of chaotic oscillations that correspond to attractors of the spiral type and of the Lorenz type. It has been established that a random process of the ‘harmonic noise’ type can serve as a mathematical model of spiral chaos and a random telegraph signal as a model of the Lorenz attractor. It has been revealed that the instantaneous phase dynamics plays an important role in the case of spiral chaos and determines regularities of autocorrelation decay and power spectrum formation. The effect of external Gaussian noise sources on characteristics of chaotic oscillations is analysed in detail.

¹ Author to whom any correspondence should be addressed.
1. Introduction

Mixing is one of the fundamental properties of stochastic and chaotic systems. The presence of mixing enables one to use a statistical description for both classical stochastic systems and deterministic ones in the regime of dynamical chaos. What is the effect of mixing from a physical viewpoint? Let us consider a simple experiment. Imagine a glass of pure water and drop a small amount of ink into it. After some time the ink particles will spread over the whole volume of the water. This happens due to a random character of the thermal motion of water molecules—this is the reason for mixing. There is also another mechanism. Mixing can be caused by a dynamical instability. Let us mix the water with the ink drop by using the rotational motion of a spoon. The final effect will be the same as in the first case but the process will conclude much faster (see video1). Thus, for systems with mixing initially different parts of the phase volume are transformed (mixed) with time so that it is impossible to separate them. In ergodic theory this process can be illustrated by using the following representation. Let sets $A$ and $B$ have measures $\mu(A)$ and $\mu(B)$. The set $A$ evolves in time with the evolution operator $T^t$. If $T^t$ possesses the property of mixing, then as $t \to \infty$ we obtain

$$\mu(T^t(A) \cap B) = \mu(A)\mu(B).$$ (1)

Equation (1) means that all points starting from the set $A$ will spread over the whole phase volume, and after a sufficient time it is impossible to distinguish points from the sets $A$ and $B$. Mixing can also be introduced as a topological property. The transformation $T^t : X \to X$ is called the topological mixing if for any two sets (open and nonzero) $U, V \in X$ there is such $t_m > 0$ that $T^t(U) \cup V \neq 0$ for $t \geq t_m$ [1].

The presence of mixing in a system leads to the loss of memory about initial conditions, ergodicity, the origin of an invariant probability measure and decay of correlations in time. The
last statement means that if the transformation $T^t : X \to X$ has the property of mixing, then for two functions $f, g \in L^2(M, \mu)$ and for large enough $t$, the functions $f(T^tx)$ and $g(x)$ become statistically independent. This can be expressed as follows:

$$\lim_{t \to \infty} \int_M f(T^tx)g(x) \, d\mu(x) = \int_M f(x) \, d\mu(x) \int_M g(x) \, d\mu(x). \quad (2)$$

Random forces are inevitably present in any real system. On the other hand, deterministic chaos is a very typical regime for nonlinear systems with phase space of three and higher dimensions. In the most general way, we can treat the system under study as a Brownian particle driven by random and deterministic forces and can describe its motion by the following equation

$$\dot{\vec{x}} = \vec{f}(\vec{x}, t) + \vec{g}(\vec{x}, t) \xi(t). \quad (3)$$

Here, $\vec{f}(\vec{x}, t)$ and $\vec{g}(\vec{x}, t)$ are smooth deterministic functions of their arguments and $\xi(t)$ is a Gaussian white noise source. Over a sufficiently long time, the property of mixing was considered to be caused with a random force. However, as the theory of nonlinear dynamics and dynamical chaos was developing, the understanding came that in many cases mixing is mainly provided by a deterministic part of the evolution operator. In each particular case, one must examine which component plays the main role.

Systems with mixing must produce information. That is why the rate of mixing in deterministic chaotic systems is usually connected with the Kolmogorov–Sinai (KS) entropy $h$ [2]–[5]. If $\mu$ is an ergodic probability measure of a dynamical system, then one can introduce the KS entropy as the mean rate of information production. Let $A = (A_1, \ldots, A_a)$ be a finite partition of a compact support $\text{Supp} \, (\mu)$. For each part of the partition $A_i$ there exist points which are mapped into $A_i$ by a map $f^k$. These points can be defined as $f^{-1}A_i$. Then

$$A^{(n)} = A \lor f^{-1}A \lor \ldots \lor f^{-n+1}A,$$

where the elements of partition are

$$A_{i_1} \cap f^{-1}A_{i_2} \cap \ldots \cap f^{-n+1}A_{i_n}, \quad i_j \in [1, 2, \ldots, a].$$

The amount of information is defined as follows:

$$H(A) = -\sum_{i=1}^{a} \mu(A_i) \ln \mu(A_i).$$

Let us consider the limit

$$h(\mu, A) = \lim_{n \to \infty} \frac{1}{n} [H(A^{(n+1)}) - H(A^{(n)})] = \lim_{n \to \infty} \frac{1}{n} H(A^{(n)}).$$

Then the KS entropy reads

$$h(\mu) = \lim_{\text{diam} \, A \to 0} h(\mu, A).$$

This definition can be applied to continuous-time and discrete-time systems. The KS entropy is related with Lyapunov characteristic exponents of a system. For the differentiable map $f$ on

New Journal of Physics 7 (2005) 76 (http://www.njp.org/)
a finite-dimensional manifold with the ergodic measure $\mu$ the following inequality was proven [5]–[9]:

$$h(\mu) \leq \sum \lambda_i^+.$$  \hspace{1cm} (4)

If $\mu$ is an invariant of the diffeomorphism $f$ and has a smooth probability density, then the KS entropy reads [9, 10]

$$h(\mu) = \sum \lambda_i^+.$$  \hspace{1cm} (5)

Numerical results demonstrate that for typical chaotic attractors in dissipative systems such an evaluation of the KS entropy can also be acceptable. The Lyapunov exponents characterize stretching of a phase-space element in one direction along a trajectory and its compression in another one. Systems with mixing demonstrate stretching of a phase-space element with its consequent folding. The latter cannot be analysed in the framework of a linear theory of stability. Divergence of trajectories will undergo nonlinear restriction due to the finite size of an attractor. Thus, it is not enough to know the KS entropy or the Lyapunov exponents' spectrum to describe mixing [11]–[13]. A geometric structure of a chaotic attractor, the form of a probability distribution, and the behaviour of phase trajectories on an attractor can play a great role in the mixing character and rate. In the present paper, we review our recently obtained results on the study of mixing properties in chaotic and stochastic systems. Some of them have already been published. The aim of this paper is to summarize the already published results, to systemize them, and represent a qualitative picture of the mixing process as a general conception. In particular, we try to answer the following fundamental questions. What is the nature of deterministic mixing? Which peculiarities of the system dynamics are important from a viewpoint of the process of mixing? Which quantitative characteristics can describe mixing more properly? Are there common features for stochastic and deterministic mixing? All these questions must be solved in the framework of a general theory of dynamical mixing. However, till now such a theory has not yet been developed. Being experimentalists we cannot lay claim to the creation of this theory. However, we hope that the results of our numerical simulations and physical experiments that are presented in this paper can make an important contribution to the development of a general conception of dynamical mixing.

As has been noted above, mixing in a system means the splitting of correlations. Therefore, the studies of mixing properties can be mainly reduced to the correlation analysis of system oscillations. Cross-correlation and autocorrelation functions (ACF) are very important characteristics of the processes in a system, especially from experimental and practical points of view, since they can be calculated directly from time series. Besides, the ACF is uniquely connected with a power spectrum of a signal through the Wiener–Khinchin relations. For dynamical systems with discrete time many theorems have been devoted to correlation properties. The most general result concerns a wide class of expanded maps with a smooth probability measure as well as Anosov’s diffeomorphisms. In this case, for any functions $f, g \in C^1$, the correlation function $\psi_{f,g}(k)$ approaches zero exponentially as $k \to \infty$ [4, 6]. This means that there are such constants $C$ and $0 < r < 1$ such that $\psi_{f,g}(k) \leq Cr^k$. Thus, only the rate of correlation decay can be estimated by the above method. Theoretical results cannot give answers on which law the correlation decay satisfies for a particular choice of the functions $f$ and $g$, and how the rate of correlation decay is connected with Lyapunov exponents. Except for several specific
cases, there are no theoretical results for continuous-time systems. The problem is complicated by the fact that the law of correlation decay is not an invariant and depends considerably on observed variables, i.e., on the choice of the functions $f$ and $g$. For example, the character and the rate of ACF decay can be essentially different for different dynamical variables of the same system. It seems reasonable to assume that in the general case the ACF for some observable $f(t)$ has the form of a linear combination of a number of exponential functions:

$$\psi_f(\tau) = \sum_i C_i \exp(-s_i \tau),$$

where $s_i$ are different decrements that characterize different rates of correlation decay in the system. The decrements must be connected in a certain way with the eigenvalues of the probability measure evolution operator, such as the Frobenius–Perron operator for discrete-time systems or the Fokker–Planck operator for continuous-time systems with Gaussian white noise [5, 12]. Contributions of these correlation components can be different and depend on an observable. Special research problems include the choice of variables which are most important for the dynamical system under study, estimation of the main correlation decay rates and finding of their connection with processes in dynamical systems and characteristics of a chaotic attractor. The most important characteristic is a decrement of the ‘slowest’ exponential function that characterizes the system relaxation time to a stationary probability distribution. We restrict our consideration to two types of chaotic attractors, namely, a spiral attractor [14] and the Lorenz attractor [15, 16], that cover a wide class of chaotic dynamical systems. Both types of attractors are well studied. The first one is a very common example of the non-hyperbolic chaotic attractor that can be realized in a wide class of nonlinear systems. The second type is a classical example of a quasi-hyperbolic attractor [16, 17]. The main point of our studies is to characterize the decay of autocorrelations. We would like to find how the rate of correlation decay is connected with different characteristics of a chaotic attractor. In the framework of correlation analysis, we compare chaotic oscillations of a dynamical system with classical models of stochastic processes, such as harmonic noise and a telegraph signal. Such analyses can bring a deeper understanding of the nature of dynamical mixing for a given type of chaotic attractors. We also consider the influence of noise on the behaviour of chaotic systems and their main characteristics.

2. Mechanism of mixing and its geometrical interpretation

More or less rigorous theoretical results on dynamical mixing have been obtained for one- and two-dimensional maps. They state that the process of mixing is connected with stretching of a phase-space element and the rate of mixing is defined by a positive Lyapunov exponent that characterizes a divergence of trajectories [9, 18, 19]. However, there is the simplest one-dimensional map that can exemplify that the connection between the rate of mixing and the positive Lyapunov exponent is not always obvious. Let us consider a map of the unit interval

$$x_{n+1} = Kx_n, \mod 1. \quad (6)$$

For $K > 1$, it is an expanding map with a mixing invariant measure [4], [19]–[21]. This map can be considered as the simplest model of a chaotic system. It is proven analytically that for integer $K \geq 1$, the ACF decays exponentially with a decrement that is equal to $\ln K$
Figure 1. Normalized ACF of a chaotic sequence \( x(n) \) generated by map (6) for \( K = 2 \) (a) and \( K = 1.1 \) (b). Dots correspond to calculation results and dashed lines give the estimation \( \exp(-m \ln K) \). In case (a), the presence of low-intensity noise enables one to avoid periodicity due to round-off errors at certain integer values of \( K \).

and corresponds to the Lyapunov exponent or the KS entropy\(^2\). Figure 1(a) shows a good fit of the numerical results to the theoretical data. However, for non-integer \( K \) and especially for \( K \) being very close to 1, the decay of correlations can be significantly different from the exponential behaviour (see figure 1(b)). The presented results indicate that in terms of ACF, the rate of mixing is not defined by the Lyapunov exponent for non-integer values of \( K \). Let us consider another statistical property. Figure 2 depicts the temporal behaviour of the square root of the variance \( \sigma_x^2(m) \) for the sequence \( x(m) \) generated by map (6). In order to make the results more illustrative, values of \( \sigma_x(m) \) are normalized as follows: \( \sigma_N(m) = (\sigma_x(m) - \sigma_{st})/\sigma_{st} \), where \( \sigma_{st} = \lim_{m \to \infty} \sigma_x(m) \). As seen from figure 2, for \( m \leq 25 \sigma_x(m) \) the variance grows exponentially with the decrement being equal to \( \ln K \). For the time interval \( m \geq 25 \), the growth of \( \sigma_x(m) \) is determined by the nonlinear effect of folding and is not directly connected with the value of \( K \). The ‘tail’ of its temporal dependence is similar to the ACF behaviour. The process of folding is ‘slower’ than exponential divergence. However, when \( K \) is near 1, the rates of divergence and folding are very close that leads to ‘slow asymptotics’. In a more general case and especially in the case of a flow system, ‘slower’ components of mixing and the slow asymptotics of the ACF can be produced by inhomogeneity of local instability properties in different regions of the phase space, by the existence of almost periodic oscillations or by the presence of switching-type effects.

\(^2\) Here and further on, the KS entropy is assumed to be defined by formula (5).
Figure 2. Normalized square root of the variance $\sigma_x(m)$ depending on time for map (6) at $K = 1.1$ (x) together with the normalized ACF (red - - - - -) and the exponential approximation $\exp(m \ln K)$ (blue - - - - -) of the initial part of the $\sigma_N(m)$.

All the above listed properties are most typical for chaotic attractors of a non-hyperbolic type [16, 22, 23] which are realized in different models of real dynamical systems. However, even for quasi-hyperbolic attractors (for example, the Lorenz attractor [16, 24]) the rate of correlation splitting is defined not only by the rate of exponential separation of trajectories.

The relaxation time to a stationary probability distribution is closely related with the rate of mixing. We have explored numerically the process of relaxation to a probability distribution for flow systems with different types of chaotic attractors in $R^3$ [25, 26]. Our results support the idea that the relaxation time, and hence the rate of mixing, is determined not only by the positive Lyapunov exponent but also depends on the instantaneous phase dynamics of chaotic oscillations. In the regime of spiral chaos, noise sources that cause the phase to change can essentially accelerate the process of relaxation. For chaotic attractors with a non-regular behaviour of the instantaneous phase, the rate of mixing is very high and cannot be significantly affected by weak noise. This statement is true for non-hyperbolic attractors of the funnel type and for attractors of the switching type, for example, for the quasi-hyperbolic Lorenz attractor.

2.1. Qualitative features of mixing for spiral chaos

Now we turn to a more complicated case—a three-dimensional differential system. Consider the well-known Rössler oscillator [27] that is governed by the following system of equations:

$$
\begin{align*}
\dot{x} &= -y - z, \\
\dot{y} &= x + ay, \\
\dot{z} &= \beta + z(x - \mu), \\
\alpha &= \beta = 0.2.
\end{align*}
$$

This system demonstrates a spiral (or phase-coherent) attractor in a wide region of parameter $\mu$ values. Such a type of attractors is formed through a sequence of period-doubling bifurcations and is referred to as a chaotic attractor of the non-hyperbolic type [14, 28]. The power spectrum of spiral chaos exhibits a well-pronounced peak at the basic (average) frequency and, consequently, the envelope of the ACF decays relatively slowly. From a physical viewpoint, chaotic attractors of the spiral type possess the properties of a noisy limit cycle [29, 30]. However, spiral attractors are realized in fully deterministic systems, i.e., without external fluctuations. Similar to the theory...
of the Van der Pol generator with external noise we can introduce the instantaneous phase and instantaneous amplitude and their ‘fluctuations’ (figure 3).

Following the classical theory of chaos, we assume that amplitude fluctuations are determined by an exponential divergence of trajectories in the transversal direction with a positive Lyapunov exponent. The Lyapunov exponent is zero in the tangent direction and an angular shift is called phase fluctuations. The latter cannot be described in the framework of a linear approach. It should be noted that in most cases the time scales of these two processes are very different. Since the attractor has a finite size, the exponential divergence undergoes a nonlinear restriction with folding. Theoretically, the phase fluctuations are not bounded due to zero Lyapunov exponent. Figure 4 (and the corresponding animation) allows us to understand these processes.

On figure 4 we choose three different regions on the attractor and colour them red, green and blue. These regions cover the transversal section of the attractor. Due to the size limitation, the process of folding ‘turns on’ very quickly. After some evolution it is clearly seen that at each time moment, closely located trajectories spread exponentially in transversal directions. The mechanism of folding works in a definite region of the attractor at each period of time. The process of folding is determined by a structure of manifolds. There are a number of works devoted to the topology of the Rössler attractor [31, 32]. Significant and bright descriptions and illustrations of the topology can be found in [32]. Particularly, it was shown that all manifolds in the Rössler system represent a Möbius sheet [32]. Since manifolds guide trajectories, then the structure of the manifolds can be visualized by ‘shadowing’ the trajectories (see figure 5).

We suppose that the mechanism of folding is realized by injecting some trajectories from one region at the ‘border’ to another one inside the attractor. This can be imagined as follows. While a trajectory is located near the centre of rotation, nonlinearity has no effect. But when the trajectory goes far from the centre, the nonlinearity ‘turns on’ and ‘throws’ the trajectory to the central region. As the angular velocity depends on the amplitude of oscillations and due to the presence of folding, the phase shift can be observed. This process is quite similar to the dynamics of Baker’s map. A qualitative description of the process of phase shift depends on the dynamics of a system under consideration.
2.2. Mixing for the switching-type attractor in the Lorenz system

Another widely known and frequently encountered type of a chaotic attractor is a switching-type attractor. In this case, the chaotic attractor demonstrates a rather complicated structure and can contain certain regions which are separated by manifolds of saddle points and cycles. Transitions (or switchings) between these regions can occur when certain conditions are fulfilled. Such oscillations can be observed, for example, in the Lorenz system [15]:

\[
\begin{align*}
\dot{x} &= -\sigma (x - y), \\
\dot{y} &= rx - y - xz, \\
\dot{z} &= -\beta z + xy,
\end{align*}
\]

\[(8)\]

\(\sigma = 10, \quad r = 28, \quad \beta = 8/3.\)

**Figure 4.** Illustration of mixing in the Rössler system. See animation video2.

**Figure 5.** Structure of manifolds in the Rössler system. Blue and red colours denote manifolds with different behaviour. Blue trajectories are injected between red ones.
As we have mentioned before, the existence of switching processes can lead to a slow asymptotics of the ACF. Even for the quasi-hyperbolic Lorenz attractor, the auto correlation function of the coordinate $x(t)$ can be approximated by an exponential law only from above. Let us discuss the mechanism of switching. The processes of switching in the Lorenz system are defined by the structure of its manifolds [17], [33]–[35]. In the phase space of the Lorenz system there are two saddle-foci that are symmetrical about $z$-axis and are separated by the stable manifold of a saddle point located at the origin. The stable manifold has a complex structure that allows trajectories to switch between the saddle-foci in a specific path [16, 35]. The structure of manifolds in the Lorenz system is qualitatively illustrated in figure 6.

Unwinding about one of the saddle-foci, the trajectory approaches the stable manifold and then can jump to the other saddle-focus with a certain probability. The rotation about the saddle-foci does not contribute considerably to the decay of the ACF, while the frequency of ‘random’ switchings essentially affects the rate of ACF decay. This fact can be illustrated as follows. Choose a small ball with a set of initial conditions near one of the saddle-foci. While evolving, the ball contracts to the attractor and spreads due to an exponential divergence. When the ball

Figure 6. Qualitative illustration of the structure of manifolds in the Lorenz system and the reconstruction of the local manifolds of real trajectories.
approaches the specific region on the attractor where the manifolds are split, a certain part of the ball is thrown to the other side and some part of the ball is left near the saddle-foci. The process of switching (throwing) causes the variance of the points to grow rapidly and at this moment a huge amount of information about initial conditions is lost (figure 7). This leads to a fast decay of the ACF.

3. Statistical properties of spiral chaos

3.1. Systems under study

In this section, we deal with two dynamical systems that can demonstrate a chaotic attractor of the spiral type. The first system is the Rössler oscillator (7), where we fix $\alpha = \beta = 0.2$ and $\mu = 6.5$ corresponding to the regime of a spiral attractor. We simulate the system equations numerically and explore their analogue model experimentally. The investigation of the analogue model is justified by the following reasons:

- a real physical process takes place in the electronic circuit.
- the electronic components have non-ideal characteristics and in this sense the ‘real’ equations for the circuit are more complicated than the source equations of the model.
• the electronic circuit is inevitably subject to the influence of fluctuations. Thus, one must typically write stochastic equations for the circuit, that may be practically impossible due to a nonlinear transformation of noise.

• each element of the circuit is a noise source. For example, according to the Nyquist theorem each resistor must be modelled as an ideal one with a noise source. Hence, the resulting stochastic equations will include both additive and multiplicative noise sources having the properties defined by the electronic scheme.

• the analogue model can be produced in several ways, for example, by means of integrators or differentiators. The reaction of different electronic realizations of the system on the noise influence can be quite different, and the way of introducing external noise source highly depends on an electronic scheme.

The second system we study is a generator with inertial nonlinearity (GIN) (Anishchenko–Astakhov’s generator) [23]. It represents a real radio-technical self-sustained generator with a Wien bridge and an inertial nonlinear feedback which controls the transmission coefficient of an amplifying cascade. In the limiting case when the parameter $g \to \infty$, the generator represents the classical Van der Pol generator. Thus, in this sense Anishchenko–Astakhov’s generator is the Van der Pol generator with an inertial feedback loop. With some simplifications, the generator is described by the following equations in normalized variables:

\[ \dot{x} = mx + y - xz - \delta x^3, \quad \dot{y} = -x, \quad \dot{z} = -gz + gF(x)x^2, \]

\[ F(x) = \begin{cases} 1 & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases} \]  

The third $z$ component in (9) describes the inertial feedback. We choose parameters $m$ and $g$ values that correspond to the regime of a spiral attractor and study system (9) experimentally. Principal schemes of the analog model for the Rössler system (7) and of the electronic circuit of Anishchenko–Astakhov’s generator are presented in figure 8.

The analogue model of system (7) is examined experimentally in the regime of spiral chaos for $\mu = 6.5$. Signals $x(t)$ and $y(t)$ are recorded using a fast analogue–digital converter (ADC) and then processed with a computer. The discretization frequency and the basic frequency of the analogue model are equal to 200 and 1 kHz, respectively. The instantaneous amplitude and phase are introduced according to formula (13). A statistical ensemble is obtained from one sufficiently long-time series that is divided into 9000–10 000 parts. The time is normalized in such a way that the mean frequency $\omega_0$ is close to its value for the same regime in the mathematical model. Anishchenko–Astakhov’s GIN is also investigated experimentally in the regime of spiral chaos. The basic frequency and the discretization frequency of the GIN are set to be 18.5 and 694.44 kHz, respectively. The signal $x(t)$ is only written with the ADC, and the instantaneous amplitude and phase are introduced according to expressions (11). The time is normalized as $t' = \omega_0 t/2\pi$ so that numerical values for the analog and numerical models can be compared.

Time-averaged values like the ACF are computed in such a way that an increase the time-series length by 10% does not lead to the ACF change $> 0.1\%$. When $B_{\text{eff}}$ is computed as the spectral peak width, the FFT window length must be so big that the frequency step $\Delta \omega < B_{\text{eff}}$. In that case, the error of the FFT algorithm is less than the approximation error by a Lorenzian. The latter can be used as the calculation error for $B_{\text{eff}}$. 

New Journal of Physics 7 (2005) 76 (http://www.njp.org/)
Figure 8. Principal schemes of the analog model of the Rössler system (a) and of the electronic circuit of Anishchenko–Astakhov’s generator (b).
3.2. Definitions of the instantaneous phase and instantaneous frequency: features of amplitude and phase mixing

For a wide class of oscillating processes, both chaotic and stochastic, it is possible and useful to introduce in some way the instantaneous amplitude and phase. In many cases, the analysis of amplitude and phase behaviour is an important instrument for studying dynamical systems [30, 36]. Moreover, very often it is the instantaneous phase dynamics that plays the main role, and the instantaneous amplitude seems to be not so important in describing qualitatively observed phenomena. An amplitude–phase approach can be most successfully applied to analyse a phase-coherent (spiral) chaotic attractor [14, 29, 37]. The chaotic dynamics of spiral type can be described in terms of the instantaneous amplitude and phase. In this case self-sustained oscillations in the regime of spiral chaos can be compared with stochastic processes realized in a noisy quasi-harmonic self-sustained oscillator [29, 30, 38, 39]. The instantaneous phase behaviour of self-sustained oscillations in the regime of spiral chaos was suggested to be considered as a Wiener process [29]. If such a suggestion is valid, a complete analogy can be drawn between the spiral chaos mode and a quasi-harmonic regime in the presence of additive Gaussian white noise. Indeed, the recently obtained numerical and experimental results have verified that such an analogy really exists [38]–[40]. The definition of the instantaneous amplitude and phase of chaotic oscillations cannot be introduced uniquely [30, 39]. The most general definition is the one used in the theory of stochastic processes [30, 41, 42]. It is possible to construct the so-called analytical signal \( z(t) = x(t) + i x_h(t) = A(t) \exp(i\Phi(t)) \) for any process \( x(t) \) if the Hilbert transform \( x_h(t) \) exists

\[
x_h(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} \, d\tau. \tag{10}
\]

Here, the integral is calculated in the sense of the Cauchy principal value; \( i = \sqrt{-1} \). \( A(t) \) is the instantaneous amplitude (envelope), and \( \Phi(t) \) is the instantaneous phase of oscillations \( x(t) \). They are expressed as follows:

\[
A(t) = \sqrt{x^2(t) + x_h^2(t)},
\]

\[
\Phi(t) = \arctan \left( \frac{x_h(t)}{x(t)} \right) \pm \pi k \quad k = 0, 1, 2, \ldots. \tag{11}
\]

The choice of integer \( k \) is determined by the continuity condition for the function \( \Phi(t) \).

The theory of analytical signal does not require that the function \( x(t) \) to be a random one. Therefore, this approach can also be applied to chaotic processes, and the theory of analytical signal was firstly applied to chaotic time-series by Rosenblum, Pikovsky and Kurths [36]. In a general case, chaotic random functions \( A(t) \) and \( \Phi(t) \) can formally be introduced and used if the process under consideration is stationary and has the following form:

\[
x(t) = A(t) \cos(\omega_0 t + \phi(t)), \tag{12}
\]

where \( \omega_0 = \text{const} \) and functions \( A(t) \) and \( \phi(t) \) are slow-changing functions in comparison with \( \exp(i\omega_0 t) \). In this case, the process \( x(t) \) is often called as harmonic noise. The instantaneous amplitude \( A(t) \) can be visualized as the length of a radius-vector on the plane \( (x, x_h) \) and the instantaneous phase \( \Phi(t) \) represents the angle of rotation of an image point around the origin. \( \omega_0 \) denotes the mean rotation frequency.
In the case of spiral chaos, the instantaneous phase $\Phi(t)$ can be introduced by considering the rotation angle of an image point on a plane of two properly selected coordinates, for example, on the plane of dynamical variables $(x, y)$. In this case, the equilibrium state of the dynamical system is set to be the origin of the radius-vector around which the phase trajectory rotates. Then the instantaneous amplitude and phase can be defined as follows:

$$A(t) = \sqrt{x^2(t) + y^2(t)}, \quad \Phi(t) = \arctan \left( \frac{y(t)}{x(t)} \right) \pm \pi k, \quad k = 0, 1, 2, \ldots.$$  \hspace{1cm} (13)

For example, in many cases it seems to be convenient to choose the plane $(\dot{x}, \dot{y})$ that defines the corresponding components of the phase velocity [43]. Such a method may appear to be the most correct one as chaotic trajectories on this plane always rotate strictly around the origin of coordinates.

There are also methods of introducing the instantaneous amplitude and phase that use a sequence of time points $t_k$ corresponding to the instants when the trajectory crosses a given secant plane. In this case, for arbitrary time moments, a stepwise, piecewise-linear or any other phase approximation is used [23, 30]. In a piecewise-linear approximation of the instantaneous phase, $A(t)$ and $\Phi(t)$ can be determined as follows:

$$A(t) = x(t_k), \quad t_k < t \leq t_{k+1} \quad \Phi(t) = \pi \frac{t - t_k}{t_{k+1} - t_k} \pm \pi k, \quad k = 0, 1, 2, \ldots.$$ \hspace{1cm} (14)

When the instantaneous phase $\Phi(t)$ is defined in any way, we can introduce the instantaneous frequency of oscillations

$$\omega(t) = \frac{d\Phi(t)}{dt}$$ \hspace{1cm} (15)

and the mean frequency

$$\omega_0 = \left\langle \frac{d\Phi(t)}{dt} \right\rangle_t = \lim_{t \to \infty} \frac{\Phi(t) - \Phi(t_0)}{t - t_0},$$ \hspace{1cm} (16)

where the brackets $\langle \cdots \rangle_t$ denote time averaging and ‘\text{i.m}’ is the mean-squared limit.

What does the amplitude–phase approach give us to understand the process of mixing in a dynamical system? We can qualitatively imagine this process in a differential system as a composition of mixing across a flow and along a flow. The first term is connected with the instantaneous amplitude behaviour and the second one is defined by the instantaneous phase dynamics. The mixing across the flow assumes the mixing in a certain transversal section. It is directly connected with local instability characteristics of trajectories, i.e., with positive Lyapunov exponents and the Kolmogorov entropy. But stability along the flow is neutral. The corresponding Lyapunov exponent is equal to zero. Thus, the phase mixing is a principally nonlocal effect. It is not directly defined by characteristics of linear instability and is mostly connected with the geometry of an attractor and peculiarities of phase-trajectory rotations on it.

3.3. Statistical characteristics of amplitude and phase fluctuations (simulations and physical experiments)

Consider first the spiral chaos regime in the Rössler system (7) without noise. The instantaneous amplitude $A(t)$ and instantaneous phase $\Phi(t)$ are defined by expressions (13). A statistical
ensemble containing 15 000 trajectories is used to calculate characteristics of the instantaneous phase. Numerical results are shown in figure 9. The probability distribution of phase fluctuations $\phi(t) = \Phi(t) - \langle \Phi(t) \rangle$ differs from the Gaussian one. Moreover, this difference increases in time (figure 9(a)). In spite of this, it is possible to separate the time scale corresponding to a linear growth of the instantaneous phase variance $\sigma^2_{\phi} = \langle (\Phi(t))^2 \rangle - \langle \Phi(t) \rangle^2$. For the period of time $t$ from several periods $T_0$ to at least a thousand periods, the phase variance $\sigma^2_{\phi}(t)$ oscillates around a linearly increasing law (figure 9(b)). In this case one can introduce a diffusion coefficient of the instantaneous phase:

$$B(t) = \frac{1}{2} \frac{d \sigma^2_{\phi}(t)}{dt},$$

(17)

$B(t) = \text{const}$ for a Wiener process. But if the process slightly differs from the Wiener one, $B(t)$ depends on time. In this case, a linear growth dominates in the variance dependence. The linear part defines the effective phase-diffusion coefficient that can be computed by using, for example, the least-squares method (LSM). The advantage of this method is that it does not require any information about the distribution of experimental data. This method is applied when an experimental probability model is unknown. We use an approximation function $f(t) = at + b$, where values of $a$ and $b$ are defined by means of the LSM. The calculation error for $B_{\text{eff}} = 1/2a$ is estimated by using a maximal error of the LSM. In our numerical simulation, the
effective diffusion coefficient has the value of $B_{\text{eff}} = 0.00016 \pm 1 \times 10^{-5}$. This phase diffusion is connected only with the deterministic dynamics of the system (7). As can be seen from figure 9(c), the probability distribution for the instantaneous frequency $\omega(t)$ is not unimodal. Therefore, the system possesses several characteristic frequencies or time scales. Moreover, none of these frequencies coincides with the mean frequency $\omega_0 = \langle \dot{\Phi}(t) \rangle = 1.068 \pm 10^{-4}$. The ACF of the process $\omega(t)$ is calculated as follows

$$
\psi_\omega(\tau) = \langle \omega(t)\omega(t + \tau) \rangle_t - \langle \omega(t) \rangle_t \langle \omega(t + \tau) \rangle_t,
$$

(18)

where the angular brackets mean time averaging. The ACF of the instantaneous frequency, that is shown in figure 9(d), decays rapidly during the initial time interval $\tau < 10T_0$. However, the long ‘tail’ gives evidence to the existence of long-term correlations.

The characteristics of the instantaneous phase and frequency that are obtained numerically for the spiral chaos in system (7) by using definition (13) are essentially different from the characteristics of the harmonic noise model (12). However, the properties of the processes $\phi(t)$ and $\omega(t)$ can be sensitive to the instantaneous phase definition. In order to find out how the instantaneous phase definition can affect statistical characteristics of the phase fluctuations, we perform necessary calculations for the system (7) by using definition (14). Numerical results are presented in figure 10.
The comparison of the presented results shows that the usage of definition (13) leads to a more complicated distribution of $\phi(t)$, whereas the distributions obtained using (14) are nearly Gaussian (figure 10(a)). But in both cases the phase variance grows in time at the same rate. The effective diffusion coefficient has the value of $B_{eff} = 0.00018 \pm 2 \times 10^{-5}$ when definition (14) is applied. Within the calculation accuracy, this estimation coincides with the value of $B_{eff}$ obtained for definition (13). The instantaneous frequency probability distribution in the case of definition (14) (figure 10(d)) is considerably different from the one calculated through (13) (figure 9(d)). The shape of the distribution as well as its variance changes. However, for both definitions, the mean frequency value is the same, $\omega_0 = 1.0683 \pm 10^{-4}$. The fact that the mean frequency is invariant with respect to an instantaneous phase definition enables one to consider the mean frequency locking as a criterion of spiral chaos synchronization. Boundaries of the synchronization region do not depend on the phase definition [39]. Correlation properties of the instantaneous frequency $\omega(t)$ are also connected with a way of introducing the instantaneous phase. The correlation of the instantaneous frequency values in the case of definition (14) decays much more rapidly than in the case of (13). Generally speaking, for the time interval $T_0 \ll t < 1000T_0$ the phase fluctuations $\phi(t)$ are closer to a Wiener process in the case of definition (14) than (11) or (13).

Thus, a particular model of phase dynamics of chaotic oscillations depends on a method of defining the instantaneous phase. However, several important features are invariants, for example, the temporal dependence of the phase variance. Since there is a linear component in the variance growth, one can compare the instantaneous phase dynamics of deterministic chaos with the motion of a Brownian particle. However, some properties remain unclear. It is known that a linear increase of the variance is a property of stochastic processes with independent increments. For the process of phase fluctuations in a chaotic system the properties of phase differences as well as the properties of the instantaneous frequency essentially depend on a phase definition. It is unclear how in these circumstances the effective phase diffusion $B_{eff}$ can be independent of the phase definition. This question needs further investigation.

The numerical results described above are obtained for the chaotic system without noise. It has been shown in [38, 40, 44] that in the presence of noise the effective diffusion coefficient $B_{eff}$ of the instantaneous phase increases up to some value smoothly depending on the noise intensity. Noise sources are inevitably present in real systems. Their statistical properties cannot always be correctly described. That is why the regularities that are established for mathematical models can be violated in reality. In order to clear up the robustness of numerically obtained properties of the instantaneous phase dynamics of spiral chaos and their experimental observation we turn to results of physical experiments. Experimental results for the analog model of the Rössler are presented in figure 11 and demonstrate a good agreement with the numerical findings. A more smooth curve of the probability distribution for the instantaneous frequency (figure 11(c)) is connected with a relatively large discretization step that leads to errors in estimating instantaneous frequency values. The experimental value for the effective phase-diffusion coefficient $B_{eff} = 0.00016 \pm 6 \times 10^{-5}$ is sufficiently close to the calculation result. Experimental characteristics of the instantaneous phase dynamics of the GIN are shown in figure 12. The experimental results obtained for the GIN are qualitatively the same as those obtained for the Rössler analogue model. However, the temporal dependence of the variance for the GIN demonstrates a significant oscillating component that may be related with an insufficiently large statistical ensemble used for calculating the variance (only 5000 parts of a long-time series are used instead of 10000 parts in the case of the Rössler model).
Figure 11. Results of analogue modelling of the Rössler system (7) for $\mu = 6.5$. (a) Probability distributions of the instantaneous phase fluctuations $\phi(t)$ for time moments $t = 1000$ (curve 1) and $t = 5000$ (curve 2), (b) time dependence of the instantaneous phase variance, (c) probability distribution of the instantaneous frequency and (d) normalized ACF for the instantaneous frequency $\omega(t)$. Note that all time values are scaled to dimensionless ones as described in section 3.1.

3.4. ACF and power spectrum of chaotic oscillations in numerical simulation and physical experiments, and their approximations

The phase dynamics of chaos itself is not very interesting from the practical point of view. Moreover, it essentially depends on a way of phase definition. Characteristics that can be observed numerically and experimentally, such as the ACF and the spectral power density of oscillations, can give more important information on peculiarities of chaotic behaviour. Let us now examine correlation and spectral properties of chaotic oscillations of the spiral type and compare them with the corresponding properties of the harmonic noise model [38]–[40]. If the process $x(t)$ is described by the model of harmonic noise (12), its ACF can be written as follows [42], [45]–[47]:

$$\psi(\tau) = \frac{1}{2} K_A(\tau) \exp(-B|\tau|) \cos \omega_0 \tau,$$

where $K_A(\tau) = \langle A(t)A(t+\tau) \rangle_t$ is the covariation function of the instantaneous amplitude $A(t)$. Using the Wiener–Khinchin theorem, one can derive the corresponding expressions for the spectral power density. The ACF of the chaotic oscillations $x(t)$ is calculated as follows:

$$\psi_x(\tau) = \langle x(t)x(t+\tau) \rangle_t - \langle x(t) \rangle_t \langle x(t+\tau) \rangle_t,$$

where the angle brackets denote time averaging. Since the process is ergodic, this procedure can be replaced by ensemble averaging. For graphic presentation, the ACF is...
Figure 12. Experimental results for the GIN (9) in the regime of spiral chaos. (a) Probability distributions of instantaneous phase fluctuations $\phi(t)$ for the period of time $t = 5000$, (b) temporal dependence of the instantaneous phase variance, (c) probability distribution of the instantaneous frequency, and (d) normalized ACF for the instantaneous frequency $\omega(t)$. Note that all time values are scaled to dimensionless ones as described in section 3.1.

normalized: $\Psi_x(\tau) = \psi_x(\tau)/\psi_x(0)$. Figure 13 shows results for $\Psi_x(\tau)$ (grey dots 1) of the Rössler system both without noise and in the presence of noise. We use the white Gaussian noise added in the first equation of (7) as the term $\sqrt{2D}\xi(t)$, where $\langle \xi(t) \rangle \equiv 0$, $\langle \xi(t)\xi(t + \tau) \rangle = \delta(\tau)$ and $D$ is the noise intensity. The ACF decays almost exponentially both without noise (figure 13(a)) and in the presence of noise (figure 13(b)). Additionally, as seen from figure 13(c), for $\tau < 20$ there is an interval on which the correlations decrease much faster. The covariance function of the amplitude $K_A(t)$ is also calculated. Using expression (19) and the values of $B_{eff}$ instead of $B$, we can approximate envelopes of the calculated ACF $\Psi_x(\tau)$. To do this, we substitute the numerically computed characteristics $K_A(\tau)$ and $B_{eff}$ into an expression for the normalized envelope $\Gamma(\tau)$

$$\Gamma(\tau) = \frac{K_A(\tau)}{K_A(0)} \exp (-B_{eff} |\tau|).$$

Calculation results for $\Gamma(\tau)$ are shown in figures 13(a) and (b) by curve 2. Equation (21) describes quite well the behaviour of the envelope of $\Psi_x(\tau)$. Note that taking into account the multiplier $K_A(\tau)/K_A(0)$ enables us to obtain a good approximation for all times $\tau$. This means that the amplitude fluctuations play a significant role on short-time intervals, whereas the slow process of the correlation decay is mainly determined by the phase diffusion. Thus, we can observe a surprisingly good agreement between the numerical results for the spiral chaos and the data for the classical model of harmonic noise. At the same time, it is quite difficult to explain rigorously the reason for such a good agreement. Firstly, the relationship (19) was obtained by
assuming the amplitude and phase values to be statistically independent. However, this approach cannot be applied to a chaotic regime. Secondly, when deriving (19), we used the fact that the phase fluctuations are described by a Wiener process. But our results obtained above show that \( \phi(t) \) for the spiral chaos represents a more complicated process. It is especially important to note that the findings presented in figure 13(a) were obtained in the regime of purely deterministic chaos, that is, without noise in the system.

We have shown that for \( \tau \) being larger than the correlation time the envelope of the ACF for chaotic oscillations can be approximated by the exponential law \( \exp(-B_{\text{eff}}|\tau|) \). Then according to the Wiener–Khichin theorem, the spectral peak at the average frequency \( \omega_0 \) must have a Lorenzian shape and its width is defined by the effective phase-diffusion coefficient \( B_{\text{eff}} \):

\[
S(\omega) = C \frac{B_{\text{eff}}}{B_{\text{eff}}^2 + (\omega - \omega_0)^2}, \quad C = \text{const.} \quad (22)
\]

The calculation results presented in figure 14 justify this statement. The basic spectral peak is approximated by using (22) and this fits quite well with the numerical results for the power spectrum of the \( x(t) \) oscillations. We note that the data shown in figures 13 and 14 for the noise intensity \( D = 10^{-3} \) have also been verified for different values of \( D \), \( 0 < D < 10^{-2} \), as well as for the range of parameter values \( m \) that correspond to the regime of spiral chaos. Our findings for the approximation of the ACF and the shape of the basic spectral peak are completely confirmed by our investigations of the spiral attractor in Anishchenko–Astakhov’s generator [38].
Now let us turn to experimental studies of the correlations and spectra in the regime of spiral chaos. Experimental results of the ACF for the Rössler analog model are shown in figure 15. The ACF behaviour was also analysed experimentally in the GIN model in the presence of external noise. With this purpose, system (9) was subjected to a broadband noise from an external noise generator whose intensity can be varied. Figure 16 illustrates logarithmic plots of normalized ACF envelopes obtained experimentally for different values of the external noise intensity. The obtained dependences were approximated according to the exponential law \[ \Psi_{\text{app}}(\tau) = \exp(-B_{\text{eff}}\tau), \] where \( B_{\text{eff}} \) is the experimentally found effective diffusion coefficient of the instantaneous phase. Approximation plots are shown in figure 16 by symbols.

The power spectrum of a diffusive process looks like a Lorenzian having the width that is defined by the effective phase-diffusion coefficient. For the normalized spectrum, the Lorenzian is given by the expression (22). In experiments, the effective diffusion coefficient can be
Figure 16. ACF envelopes (——) obtained experimentally for the GIN model for different root-mean-square values of the external noise intensity: \(1 - D = 0\), \(2 - D = 0.0005\) mV(RMS), and \(3 - D = 0.001\) mV(RMS); and their exponential approximations (----) with the decrement of decay \(B_{\text{eff}} = 0.00024 \pm 3.4 \times 10^{-5}\), \(B_{\text{eff}} = 0.00033 \pm 2.1 \times 10^{-5}\) and \(B_{\text{eff}} = 0.000439 \pm 2.8 \times 10^{-5}\), respectively. Note that all time values are scaled to dimensionless values as described in section 3.1.

Figure 17. (a) Experimental power spectrum of the \(x(t)\) oscillations in system (9) and its theoretical approximation by (22) with \(B_{\text{eff}} = 0.00033\) in the presence of noise with \(D = 0.0005\) and (b) power spectra for \(D = 0.001\) (curve 1) and \(D = 0\) (curve 2). Note that all time values are scaled to dimensionless values as described in section 3.1.

Independently defined by measuring the spectral peak width. To obtain a more precise value of the diffusion coefficient, we approximate the spectral peak with formula (22) by varying \(B_{\text{eff}}\). The resulting value of the coefficient is the one at which the approximation error is minimal (see figure 17(a)). Figures 17(a) and 17(b) illustrate parts of the experimental power spectra of the GIN without and in the presence of external noise sources. The spectrum was calculated by means of a standard FFT method with averaging. The window length was \(2^{18}\) points, and the total...
Table 1. Comparison of phase-diffusion coefficient values obtained by different methods without and in the presence of noise with different intensities.

| $D (mV)$ | $B_{\text{eff}}$ (Hilbert) | $\text{max} \Delta B_{\text{eff}}$ | $B_{\text{eff}}$ (spectrum) | $\text{max} \Delta B_{\text{eff}}$ |
|---------|--------------------------|----------------|--------------------------|----------------|
| 0       | $2.44 \times 10^{-4}$    | $2.66 \times 10^{-4}$ |
| $5.0 \times 10^{-4}$ | $3.30 \times 10^{-4}$ | $3.4 \times 10^{-5}$ |
| $1.0 \times 10^{-3}$ | $4.39 \times 10^{-4}$ | $4.43 \times 10^{-4}$ |

The number of windows was 50. The main result is that the effective phase-diffusion coefficient values estimated from the spectra correlate with the values obtained from the linear approximation of the growth of the instantaneous phase variance. The corresponding phase-diffusion coefficient values are given in Table 1 for three different levels of the external noise. The values $\text{max} \Delta B_{\text{eff}}$ and $\text{max} \Delta B_{\text{eff}}^0$ represent the maximal calculation error of $B_{\text{eff}}$ defined via the Hilbert transformation and the spectrum, respectively. Thus, it has been experimentally established that the ACF of the spiral chaos decays in time according to the nearly exponential law $\exp(-B_{\text{eff}} \tau)$. The spectral linewidth of oscillations at the basic frequency $\omega_0$ is defined by the effective phase-diffusion coefficient from expression (22).

Our numerical and experimental studies confirm our idea that the instantaneous phase dynamics plays a principal role in the process of mixing and correlation decay in the regime of spiral chaos. The instantaneous phase behaviour both in a pure deterministic case and in the presence of noise possesses the peculiarities of a Brownian motion. The growing temporal dependence of the instantaneous phase variance demonstrates a linear component. The effective diffusion coefficient $B_{\text{eff}}$ characterizing the rate of this linear growth mainly defines the rate of correlation decay. Hence, mixing is connected directly not only with a local exponential divergence of trajectories on a chaotic attractor but also with a complicated nonlocal process of the phase behaviour. The ACF of spiral chaos decays in time according to the law (19) with the effective diffusion coefficient $B_{\text{eff}}$ instead of the coefficient $B$. If an external noise source is added to a system, the phase variance also grows linearly but the value of $B_{\text{eff}}$ increases. In spite of the linear increase of the phase variance and the validity of expression (19), the instantaneous phase of spiral chaos cannot be described by a Wiener process. Probability distributions of the instantaneous phase and frequency are not exactly Gaussian laws, and instantaneous frequency values are characterized by long-term correlations. Thus, it is experimentally proven that relation (19) can be applied in more general conditions than those for which the theoretical relation was obtained. As follows from (19), taking into account small values of $B_{\text{eff}}$ enables us to determine the phase dynamics that defines the rate of mixing for large periods of time. With this, the mixing rate is independent of the Kolmogorov entropy value (in the sense of Ruelle and Pesin).

4. Correlation and spectral properties of the Lorenz system

In the previous section, we have used the effective phase-diffusion coefficient to describe the correlation properties of the Rössler system and Anishchenko–Astakhov’s generator. However, such an approach cannot be applied to approximate ACFs of chaotic oscillations of a switching type. Consider the time series of the $x$ coordinate of the Lorenz system, that is shown in figure 18.
By taking into consideration the symbol dynamics methods, i.e., to exclude the rotation about saddle-foci, one can get a telegraph-like signal. Figure 19 shows the ACF of the $x(t)$ oscillations for the Lorenz attractor and of the corresponding telegraph signal.

Comparing these two dependences we can state that the time of correlation splitting and the ACF behaviour on this period of time are determined by switchings only, whereas the rotation about the saddle-foci makes a minor contribution to the ACF decay on large time intervals. It is worth noting that the ACF decreases linearly on short times. This fact is remarkable as the linear decaying of the ACF corresponds to a discrete equidistant residence time probability distribution in the form of delta-peaks. Additionally, the probability of switchings between the two states is equal to $1/2$ [48, 49].

Figure 20 shows the residence time distribution calculated for the telegraph signal resulting from switchings in the Lorenz system. As can be seen from figure 20(a), the residence time distribution in two regions of the attractor really has a structure that is quite similar to an equidistant discrete distribution. At the same time, the peaks are characterized by a finite width. Figure 20(b) represents the probability distribution of switchings which occur at time moments that multiples of $\xi_0$, where $\xi_0$ is a minimal residence time in one of the states. This dependence...
shows that the transition probability at the time moment $\xi_0$ is close to $1/2$. The discrete character of switchings can be explained by peculiarities in the structure of manifolds of the Lorenz system (see figure 6). In the vicinity of the origin $x = 0$, $y = 0$ the manifolds are split into two leaves. This leads to the fact that the probability of switchings between the two states in one revolution about the fixed point is approximately equal to $1/2$. Such peculiarities of the trajectory behaviour ensure that the ACF of the $x(t)$ and $y(t)$ oscillations on the Lorenz attractor must be defined by the expression for the ACF of a generalized telegraph signal [38]. However, the finite width of the peaks in the distribution and inaccurate equality of the probability to $1/2$ can lead to the fact that the ACF decays to a certain finite value only but not to zero. Long-term correlations can be estimated by applying a change of variables that uses properties of $z$-symmetry. This change is continuous and is onto the plane transformation. However, it is irreversible without an additional construction. This transformation maps each pair of symmetrical points in the $X-Y$ plane onto one point in the $U-V$ plane [50]:

$$t = \sqrt{(r-1)\sigma} t_L, \quad x = \frac{1}{\sqrt{(r-1)b}} X,$$

$$y = \frac{1}{r-1} \sqrt{\frac{\sigma}{b}} (X - Y), \quad z = \frac{1}{r-1} \left( Z - \frac{1}{2\sigma} X^2 \right),$$

$$u = \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}, \quad v = \frac{2xy}{\sqrt{x^2 + y^2}}. \quad (23)$$

The first four equations in system (23) rescale the variables to normalized coordinates and the last two map them onto the $U-V$ plane. The normalized coordinates are very suitable to carry out natural experiments with an analogue model of the Lorenz system. The size of the attractor changes when the control parameters are varied but it remains unchanged in the normalized coordinates. The form of the attractor in the $U-V$ coordinates is similar to the spiral attractor in the Rössler system (see figure 21). For such types of attractors there is an analytical approach for describing the ACF of one of the coordinates [38, 40, 44]. The ACF decay for such attractors is mostly connected with the effective diffusion coefficient of the instantaneous phase. To prove the validity of the obtained numerical results, a series of natural experiments were conducted on an analog model of the Rössler system. Experimental data were recorded using a computer with an ADC board with 12-bit accuracy and sampling frequency 300 kHz. Using the obtained
Figure 21. $U - V$ projection of the Lorenz attractor in natural experiments (a) and the ACF of the $U(t)$ signal approximated with the phase-diffusion coefficient (b).

Figure 22. ACF of the $x(t)$ signal and its approximation (solid line).

data for the $x(t)$ and $U(t)$ signals the ACF of a telegraph signal and of the rotation motion was computed. The combination of these two ACF can successfully approximate the full ACF of the $x(t)$ signal (figure 22).

5. Conclusion

Our numerical and experimental studies presented in this paper enable us to answer the questions formulated in section 1. We have examined two widely known types of chaotic attractors that are realized in flow dynamical systems with phase-space dimension $N \geq 3$. The first one is a spiral attractor being a particular case of a chaotic attractor of the saddle-focus type. The second type represents the Lorenz attractor that is an attractor of the switching type. For these types of chaotic attractors we have shown that the instability of trajectories on a chaotic attractor is a necessary but insufficient condition for mixing. Lyapunov exponents do not determine the rate of
mixing on the attractor and do not provide any necessary information on the process of mixing. The mechanism of mixing in deterministic chaotic systems and quantitative characteristics of mixing depend on the topology of an unstable manifold of trajectories on the attractor. Thus, for the types of chaotic attractors considered, we have qualitatively described the mechanism of mixing that is connected with the attractor structure.

Mixing in systems with chaotic attractors of the spiral type (and, apparently, of the funnel type) takes place due to a complicated multi-sheet structure of an unstable manifold. Such a structure ensures transition of a part of initially close trajectories to the inner part of an attractor that is accompanied by the change in their instantaneous phase. Thus, the process of mixing occurs both in the transverse section of the flow and ‘along the flow’. In this case the behaviour of the instantaneous phase of chaotic oscillations is an important characteristic of this process. It has been established numerically and experimentally that the instantaneous phase diffusion in the regime of spiral chaos is close to the normal one and can quantitavely be characterized by some effective diffusion coefficient $B_{\text{eff}}$. It has also been shown that the spectral and correlation properties of spiral chaos are in good agreement with a classical model of ‘harmonic noise’. With this, the effective diffusion coefficient $B_{\text{eff}}$ is a major factor that defines the rate of correlation splitting in the system and, correspondingly, the basic spectral line width.

Mixing in systems with chaotic attractors of the switching type is mainly related with the circumstance that one part of initially close trajectories can remain in the neighbourhood of one unstable equilibrium state and the remaining part switches to the vicinity of another one. In a deterministic chaotic system, switchings of trajectories are ensured by the structure of an unstable manifold, and the rate of mixing is mainly defined by the switching statistics. A quasi-hyperbolic attractor in the Lorenz system represents a classical example of switching chaos. The structure of the unstable manifold of the Lorenz attractor gives the possibility for a trajectory to switch in definite time moments only. Accordingly, such a chaotic process can be modelled by means of a quasi-random telegraph signal. Our studies have demonstrated that the presence of a characteristic interval $T_0$ between possible switchings determines the switching probability close to $1/2$. Therefore, the ACF of switching events in the Lorenz attractor are characterized by a quite long part of the linear decay, and the rate of decay is defined by a minimal switching time.

Since the unstable manifold structure is preserved in the presence of noise in a system, the established regularities must possess the property of robustness with respect to weak noise perturbations. Indeed, the robustness of mixing characteristics is confirmed by the correspondence between numerical and experimental results.

Nevertheless, many questions concerning mechanisms of mixing remain unsolved. We would like to describe some of them that have arised during our studies. For example, in our work, the mixing process on spiral and switching-type attractors is analysed in terms of the topological structure of these attractors. However, we have presented only a qualitative picture of mixing and the problem of a mathematical description of quantitative characteristics of mixing (for instance, the rate of ACF decay) on the basis of topological properties of an attractor is still open. Some approaches to solving this problem can be based on [32] and some other works.

Another unsolved problem concerns the choice of the most universal and physically justified characteristics of mixing. In our opinion, the most frequently used characteristics, such as the rate of ACF decay, the relaxation time to stationary values of some moments of the probability distribution, are, unfortunately, not universal, i.e., they depend on the choice of observable variables.
The reasons why chaotic oscillations in the regime of spiral attractor are so similar to harmonic noise require further studies. We suggest that such a similarity can be explained as follows. The complex structure of an unstable manifold resembles the instantaneous phase difference on an ensemble of trajectories, but the independence of the phase increments is apparently connected with an exponential instability. However, these suggestions need to be explored further.

We have examined only one example of a switching-type attractor—the Lorenz attractor. We assume that the method of separating characteristic motions (switchings, rotations, etc) can also be applied to other attractors, including attractors corresponding to ‘chaos–chaos’ intermittency. However, in each particular case the problem of switching statistics and its relation with mixing characteristics remains open. One can assume that other types of chaotic attractors realized in flow systems with \( N \geq 3 \) will demonstrate their own specific mechanisms of mixing at different and still unexplored regularities concerning quantitative characteristics of mixing. Both qualitative description of the mixing process and its quantitative characteristics in continuous-time chaotic systems are a wide and very little studied part of nonlinear dynamics. We hope that the numerical and physical experiments described in this paper as well as those planned to realize further will promote the creation of a general conception of dynamical mixing.

Acknowledgments

This work was supported by the Program BRHE (grant No SR-006-X1) and the Russian Foundation for Basic Research (grant No 04-02-16283).

References

[1] Broer H W, Dumortier F, van Strien S J and Takens F 1991 Structures in Dynamics. Finite Dimensional Deterministic Studies (Amsterdam: North-Holland)
[2] Komogorov A N 1959 Dokl. Acad. Nauk. USSR 124 754–55 (in Russian)
[3] Sinai Y 1959 Dokl. Acad. Nauk. USSR 124 768–71 (in Russian)
[4] Bowen R 1975 Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms (Lecture Notes in Mathematics vol 470) (Berlin: Springer)
[5] Ruelle D 1989 Math. Phys. 125 239
[6] Ruelle D 1976 Am. J. Math. 98 619
[7] Ruelle D 1978 Bol. Soc. Bras. Math. 9 83
[8] Bowen R and Ruelle D 1975 Invent. Math. 29 181
[9] Eckmann J P and Ruelle D 1985 Rev. Mod. Phys. 57 617
[10] Pesin Y 1977 Russian Math. Surveys 32 55
[11] Christiansen F, Paladin G and Rugh H H 1990 Phys. Rev. Lett. 65 2087
[12] Liverani C 1995 Ann. Math. 142 239
[13] Froyland G 1997 Commun. Math. Phys. 189 237
[14] Arneodo A, Collet P and Tresser C 1981 Commun. Math. Phys. 79 573
[15] Lorenz E N 1963 J. Atmos. Sci. 20 130–41
[16] Shilnikov L P 1997 Int. J. Bif. Chaos 7 1953
[17] Afraimovich V S, Bikov V V and Shilnikov L P 1977 Dokl. Acad. Nauk. USSR 234 336 (in Russian)
[18] Ruelle D and Takens F 1971 Commun. Math. Phys. 20 167
[19] Zaslavsky G M 1985 Chaos in Dynamical Systems (New York: Harwood Academic)
[20] Renyi A 1957 Acta Math. Acad. Sci. Hungary 8 477

New Journal of Physics 7 (2005) 76 (http://www.njp.org/)
[21] Rokhlin V A 1961 Izv. Acad. Nauk. USSR 25 499 (in Russian)
[22] Afraimovich V S 1989 Nonlinear Waves—I (Berlin: Springer) pp 14–28
[23] Anishchenko V S, Astakhov V V, Neiman A B, Vadivasova T E and Schimansky-Geier L 2002 Nonlinear Dynamics of Chaotic and Stochastic Systems (Berlin: Springer)
[24] Afraimovich V S and Shilnikov L P 1983 Nonlinear Dynamics and Turbulence (Boston, MA: Pitman) pp 1–43
[25] Anishchenko V S, Vadivasova T E, Kopeikin A S, Kurths J and Strelkova G I 2001 Phys. Rev. Lett. 87 054101
[26] Anishchenko V S, Vadivasova T E, Kopeikin A S, Kurths J and Strelkova G I 2002 Phys. Rev. E 65 036206
[27] Rössler O E 1976 Phys. Lett. A 57 397
[28] Crutchfield J, Framer D, Packard N, Shaw R, Jones G and Donnelly R J 1980 Phys. Lett. A 76 1–4
[29] Farmer J D 1981 Phys. Rev. Lett 47 179–82
[30] Pikovsky A, Rosenblum M and Kurths J 2001 Synchronization. A Universal Concept in Nonlinear Sciences (Cambridge: Cambridge University Press)
[31] Letellier G, Duterte P and Maheu B 1995 Chaos 5 271–82
[32] Gilmore R and Lefranc M 2002 The Topology of Chaos: Alice in Stretch and Squeezeland (New York: Wiley)
[33] Guckenheimer J and Holmes P 1983 Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields (Berlin: Springer)
[34] Jackson E A 1989 Perspectives of Nonlinear Dynamics vol 1 (Cambridge: Cambridge University Press)
[35] Jackson E A 1991 Perspectives of Nonlinear Dynamics vol 2 (Cambridge: Cambridge University Press)
[36] Rosenblum M, Pikovsky A and Kurths J 1996 Phys. Rev. Lett. 76 1804
[37] Anishchenko V S 1995 Dynamical Chaos—Models and Experiments (Singapore: World Scientific)
[38] Anishchenko V S, Vadivasova T E, Okrokvertskhov G A and Strelkova G I 2003 Physica A 325 199–212
[39] Anishchenko V S, Vadivasova T E, Strelkova G I, Kopeikin A S and Kurths J 2003 Fluctuation and Noise Lett. 3 213
[40] Anishchenko V S, Vadivasova T E, Okrokvertskhov G A and Strelkova G I 2004 Phys. Rev. E 69 036215
[41] Gabor D 1946 J. Inst. Electr. Eng. 93 429
[42] Rytov S M, Kravtsov Y A and Tatarskii V I 1987 Principles of Statistical Radiophysics (Berlin: Springer)
[43] Osipov G V, Hu B, Zhou C, Ivanchenko M V and Kurths J 2003 Phys. Rev. Lett. 91 024101
[44] Anishchenko V S, Vadivasova T E, Strelkova G I and Okrokvertskhov G A 2004 Math. Biosci. Eng. 1 161–84
[45] Stratonovich R L 1963 Topics in the Theory of Random Noise vol 1 (New York: Gordon and Breach)
[46] Stratonovich R L 1967 Topics in the Theory of Random Noise vol 2 (New York: Gordon and Breach)
[47] Malakhov A N 1968 Fluctuations in Auto-Oscillating Systems (Moscow: Nauka)
[48] Rytov S M 1966 Introduction in Statistical Radiophysics (Moscow: Nauka) (in Russian)
[49] Tikhonov V I and Mironov M A 1977 Markovian Processes (Moscow: Sov. Radio) (in Russian)
[50] Yamrom B, Kunin I and Chernykh G 2003 Int. J. Eng. Sci. 41 475–82