NON-UNIQUENESS RESULTS FOR ENTROPY TWO-PHASE
SOLUTIONS OF FORWARD-BACKWARD PARABOLIC
PROBLEMS WITH UNSTABLE PHASE

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Abstract. This paper study the well–posedness of the entropy formulat ion
given by Plotnikov in [Differential Equations, 30 (1994), pp. 614–622] for
forward–backward parabolic problem obtained as singular limit of a proper
pseudoparabolic approximation. It was proved in [C. Mascia, A. Terracina, and
A. Tesei, Arch. Ration. Mech. Anal., 194 (2009), pp. 887–925] that such for-
mulation gives uniqueness when the solution takes values in the stable phases.
Here we consider the situation in which unstable phase is taken in account,
proving that, in general, uniqueness does not hold.

phase transition, forward–backward equations, ill–posed problems

1. Introduction

In this paper we consider the following forward–backward parabolic problem:

\[
\begin{aligned}
  u_t &= \phi(u)_{xx} \quad \text{in } Q_T := \Omega \times (0, T) \\
  \phi(u)_{x}(0, t) &\equiv \phi(u)_{x}(L, t) \equiv 0 \quad \text{in } (0, T) \\
  u(x, 0) &= u_0(x) \quad \text{in } \Omega := (0, L)
\end{aligned}
\]

where \( \phi \) is a nonmonotone function. Obviously this kind of problem is ill–posed
whenever \( u \) takes values in the interval in which \( \phi \) decreases. In particular in this
paper we consider a piecewise linear function \( \phi \), namely:

\[
\phi(u) = \begin{cases} 
  \phi_1(u) & \text{for } u \leq b \\
  \phi_0(u) & \text{for } b < u < c \\
  \phi_2(u) & \text{for } u \geq d,
\end{cases}
\]

where

\[
\phi_i(u) := \alpha_i u + \gamma_i, \quad i = 1, 2, \quad \phi_0(u) := \frac{A(u - b) - B(u - c)}{c - b}.
\]

Here \(-\infty < b < c < \infty, \alpha_i > 0, \gamma_i \in \mathbb{R}, i = 1, 2, A := \phi_2(c) < \phi_1(b) =: B.\)

Let us denote with \( \beta_1 : (-\infty, B) \to \mathbb{R}, \beta_2 : (A, \infty) \to \mathbb{R}, \beta_0 : (A, B) \to \mathbb{R}, \)
respectively, the inverse function of \( \phi_1, \phi_2 \) and \( \phi_0 \) (see Fig.1).

The differential equation in (1) with the response function \( \phi \) of cubic type arises
in the theory of phase transition. The function \( u \) gives the phase fields and its values
characterize the different phases; the half-lines \(( -\infty, b ) \) and \(( c, \infty ) \) correspond to
stable phases and the interval \(( b, c ) \) to the unstable one (e.g., see [4])

In [13] (see also [14]) it was proved that the problem (1) with a piecewise function \( \phi \)
has an infinite number of solutions. It is interesting to underline that the solutions
given in [13] takes values only in the two stable phases when \( t > 0. \) In some sense
the presence of the unstable phase allow to pass from one phase to the other with too much freedom.

It is important to recall that for forward–backward parabolic problems it is possible to state uniqueness results assuming that the solution is quite regular (see [8] for the backward case and [16], [17] for the forward–backward case).

In order to give a good formulation for the problem (1) a natural approach is to introduce a proper regularization, obviously the choice of the regularization terms is related to the physical phenomenon that we want to describe. In the case of the model of phase transition the original problem is very complicated from a mathematical point of view since there are many terms to take in account. As a matter of fact, it is possible to choose different type of regularizations in which only some phenomena are considered (see e.g. [3], [4], [7], [12], [1], [2], [10], [32]). Using this point of view the choice of a particular regularization depends on the phenomena which we want to highlight.

Here we consider the following pseudoparabolic regularization

\begin{equation}
\begin{array}{ll}
    u_t = v_{xx} & \text{in } Q_T := (0, L) \times (0, T) \\
    v_x(0, t) = v_x(L, t) = 0 & \text{in } (0, T) \\
    u(x, 0) = u_0(x) & \text{in } (0, L),
\end{array}
\end{equation}

where $v = \phi(u) + \epsilon u_t$, $\epsilon > 0$.

The third order term in the right hand side of the differential equation in (3) is a viscosity term related to nonequilibrium effects (see e.g. [3], [9], [12]). Let us observe, that in the regularization (3) it is not considered the characteristic term of the Cahn–Hilliard equation that describes the cost of the inhomogeneities in phase transition models.

It is worth to note that the approximation equations (3) have an independent interest that is beyond the physical model. More precisely, when $\phi$ is a linear function these equations suggest a Yoshida approximation of the differential equation in (1). In fact, this kind of equations were introduced in quasi–reversibility methods to approximate backward parabolic problem, see [18], [27], [5].

Problem (3) with a general nonlinear function $\phi$ of cubic type was studied in [21], whereas the singular limit was analyzed by Plotnikov (see [23], [24], [25]), a similar analysis for other type of nonlinearity was considered in [28]. The idea, in
analog with conservation laws, is to give an entropy formulation of the ill-posed problem assuming that the physical solutions of it are that obtained when $\epsilon$ goes to $0^+$ as limit of solutions of problems.

In general, we point out that functions $u$ obtained as limit of the problems do not satisfy equation in the classical sense, more precisely we have a solution $(u, \lambda_i, v)$, $i = 0, 1, 2$, such that

$$u = \lambda_1 \beta_1(v) + \lambda_0 \beta_0(v) + \lambda_2 \beta_2(v),$$

where $\lambda_i(x, t) \geq 0$, $i = 0, 1, 2$, $\sum_{i=0}^{2} \lambda_i(x, t) = 1$ in $Q_T$ and the equation $u_t = v_{xx}$ is satisfied in the weak sense.

In this context the solution $u$ can be regarded as a superposition of different states and fulfills the differential equation in the sense of the Young measures (see e.g. [25], [19]).

We shall give the precise definition of entropy solution suggested by Plotnikov in Section 2. However, we anticipate that this definition provides a good formulation for solutions that takes values in the two stable phases. In particular, in [20] [30], it was introduced the “two–phase problem” for which the initial data and the solution takes values in the two stable phases and there is a regular interface that separates different phases. In this situation the entropy formulation of Plotnikov suggest an admissibility condition for the evolution of the interface. For this type of solutions local existence and uniqueness with the response function (2) was obtained in [20], global existence was proved in [30], while existence, uniqueness and the study of the singular limit for a general nonlinear cubic type function $\phi$ is established in [29]. It is easy to check that the admissibility condition along the interface does not allow to consider the solutions given in [13], we have a stricter condition for jumping from one phase to the other and this guarantees uniqueness.

Using these considerations, we can guess that the formulation of entropy solution given by Plotnikov could be satisfactory also in the general case where unstable phase is taken in account. It is necessary to observe, that the entropy formulation of Plotnikov, on the one hand introduce an admissibility condition that is crucial to have uniqueness at least when we consider stable phases and on the other hand allow the solution to satisfy the original forward–backward differential equation in a very weak way (see Definition 2.1). This is necessary since it is not possible to have existence for the classical backward parabolic equation with generic initial data.

The main result of this paper is to show that uniqueness of the entropy solution fails in the general case.

Examples of explicit entropy solutions of the forward–backward parabolic equation that takes values also in the unstable phase are given in [11], where it is considered the “Riemann problem” and a solution is obtained by self similar methods. More recently in [31], it was studied the “two–phase problem” where one of the two phases is the unstable one.

This paper is organized in two further sections. In Section 2 we shall state the precise definition of entropy solution, in particular we shall recall briefly the considerations that lead to this kind of formulation. Moreover, we shall give a characterization of the entropy solution, showing that this is related to the monotonicity of the coefficients $\lambda_i$, $i = 0, 1, 2$ respect to the variable $t$. More precisely, the coefficients $\lambda_1$, $\lambda_2$ corresponding to the stable phase
tend to increase and the coefficient \( \lambda_0 \), corresponding to the unstable phase, tends to decrease. We will be interested to the situation in which we have only unstable phase at initial time, supposing that a solution of the type (4) appears at positive time (see Definition 3.1).

In Section 3 we shall show that, choosing properly the initial data, we obtain infinite solutions that satisfy the entropy formulation of Plotnikov. This give a negative answer to the open question about the well–posedness of the entropy formulation for general initial data. It is interesting to note, that existence of entropy solutions that have the structure given in (4) are related to the following inverse parabolic problem

\[
\begin{align*}
\frac{u_t}{(5)} &= u_{xx} + f(x) & \text{in } Q_T := (0, L) \times (0, T) \\
u_x(0, t) &= u_x(L, t) = 0 & \text{in } (0, T) \\
u(x, 0) &= u_0(x) & \text{in } (0, L) \\
u(x, T) &= g(x) & \text{in } (0, L),
\end{align*}
\]

where \( u_0, g \) are given functions and \( f \) is the unknown to be determined. There is a wide literature about this kind of problem, here we just mention the classic book of Isakov [15] and we underline that since we have freedom in the choice of the data \( u_0 \) and \( g \) we can easily exhibit solutions using very classical methods.

2. Entropy formulation

In this section we go back quickly to the motivation that are beyond the definition of entropy solution of a forward–backward parabolic problem given in [23], [24], [25]. Then, we characterize the admissibility condition in order to built the counterexample of not uniqueness in Section 3.

As said in the Introduction, the idea is to consider the solution as that obtained as singular limit of the approximation problem (3). Problem (3) is analyzed in [21] also in the multidimensional case. Existence and uniqueness is proved by classical methods of ODE in Banach space (see also [19]). Moreover in [21] a viscous entropy inequality is obtained. More precisely, the solution \( u_\epsilon \) of problem (3) satisfies

\[
\int_{Q_T} \left\{ G(u_\epsilon) \psi_t - g(v_\epsilon) \nabla v_\epsilon \cdot \nabla \psi - g'(v_\epsilon) |\nabla v_\epsilon|^2 \psi \right\} \, dx \, dt \geq 0
\]

for any \( T > 0, \psi \in C^\infty_0(Q_T), \psi \geq 0, \) where for any function \( g \in C^1(\mathbb{R}), g' \geq 0, \)

\[
G(u) := \int_0^u g(\phi(s)) \, ds + K \quad (K \in \mathbb{R}).
\]

Using these inequalities and choosing properly the function \( g \) it is possible to obtain a priori estimates in \( L^\infty \) for \( u_\epsilon, v_\epsilon \) that do not depend on \( \epsilon \) (see [21] and [19] for the details).

Using this kind of estimates we can extract proper subsequences \( \{ u_{\epsilon_n} \}, \{ v_{\epsilon_n} \} \) that converge, respectively, in the \( L^\infty \) weak* topology, to function \( u \) and \( v \). Obviously, we can state that equation \( u_t = v_{xx} \) is satisfied in a weak sense but in general \( v \neq \phi(u) \) and we can not pass to the limit in the viscous entropy inequality (6).

In order to overcome such obstacle, Plotnikov in [24] studied the Young measure \( \nu_{(x,t)} \) associated to the converging sequence \( \{ u_{\epsilon_n} \} \), proving that this is is a superposition of Dirac measures concentrated on the three monotone branches of the graph of \( v = \phi(u) \); the functions \( \beta_1, \beta_2, \beta_0 \) defined in the previous section (see Figure 1).

More precisely
\( \nu(x,t)(\tau) = \sum_{i=0}^{2} \lambda_i(x,t)\delta(\tau - \beta_i(v(x,t))) \)

where \( \delta \) is the classical Dirac measure, \( \lambda_i(x,t) \in L^\infty(\Omega_T) \), \( \lambda_i \geq 0 \), \( (i = 0, 1, 2) \) and \( \sum_{i=0}^{2} \lambda_i(x,t) = 1 \) in \( \Omega_T \).

This implies, that, for every \( f \in C(\mathbb{R}) \)

\( f(\epsilon_n) \rightharpoonup f \) in \( L^\infty(\Omega_T) \);

where

\[
G^*(x,t) := \sum_{i=0}^{2} \lambda_i G(\beta_i(v(x,t)))
\]

for a.e. \((x,t) \in \Omega_T \). Then, choosing \( f(u) = \phi(u) \) we deduce the following relation between \( u \) and \( v \)

\[
u(x,t)(\tau) = \sum_{i=0}^{2} \lambda_i(x,t)\delta(\tau - \beta_i(v(x,t)))
\]

Moreover, choosing \( f(u) = \phi(u)^2 \), it easy to prove that \( \phi(\epsilon_n) \) converges in the strong topology \( L^2 \) to the function \( v \).

In fact it is possible to prove stronger convergences of the sequence \( \epsilon_n \) to the function \( v \) (see [19] for details), in particular this is true in the \( L^2((0,T), H^1((0,L)) \) topology. Using these considerations, we can pass to the limit along a proper subsequence in the viscous entropy inequality (6), proving that (see [24], [19]) the limit couple \( (u,v) \) satisfies

\[
\int \int_{\Omega_T} \left\{ G^* \psi_t - g(v) \nabla v \cdot \nabla \psi - g'(v) |\nabla v|^2 \psi \right\} \ dx \ dt \geq 0
\]

for any \( \psi \in C^\infty_0(\Omega_T) \), \( \psi \geq 0 \), where

\[
G^*(x,t) := \sum_{i=0}^{2} \lambda_i G(\beta_i(v(x,t)))
\]

Then, it is natural to choose (11) as the entropy inequality condition for the solution of the forward–backward problem (1) obtained as singular limit of the approximation problem (3). More precisely, we have the following “natural” definition.

**Definition 2.1.** An entropy solution to problem (1) in \( \Omega_T \) is given by \( u, \lambda_0, \lambda_1, \lambda_2 \in L^\infty(\Omega_T) \), \( v \in L^\infty((0,T), H^1(\Omega)) \) such that:

(a) \( \sum_{i=0}^{2} \lambda_i = 1, \lambda_i \geq 0 \) and there holds:

\[
u(x,t)(\tau) = \sum_{i=0}^{2} \lambda_i(x,t)\delta(\tau - \beta_i(v(x,t)))
\]

with \( \lambda_1 = 1 \) if \( v < A \), \( \lambda_2 = 1 \) if \( v > B \);

(b) the couple \( (u,v) \) is a weak solution of the equation \( u_t = v_{xx} \) in \( \Omega_T \):

\[
\int \int_{\Omega_T} \left\{ u \psi_t - v \psi_x \ dx \ dt + \int_{\Omega} u_0(x)\psi(x,0)\ dx = 0
\]
for any $\psi \in C^1(\overline{Q_T})$, $\psi(\cdot, T) = 0$ in $\Omega$.

(c) Inequality (11) is satisfied for any $\psi \in C_0^\infty(Q_T)$, $\psi \geq 0$ and $g \in C^1(\mathbb{R})$, $g' \geq 0$.

As we anticipate in the Introduction, in general $\phi(u) \neq v$, then condition b) in the previous definition does not imply that the original forward–backward equation is satisfied in a weak way. This will be true if and only if for a.e. $(x, t) \in Q_T$, one of the coefficients $\lambda_i(x, t)$ is equal to 1 and consequently the others are equal to 0. Otherwise the equation will be satisfied in the sense of the measure–valued solution.

On the other hand, by construction, there always exists at least one entropy solution in the sense of Definition 2.1 for every initial data in $L^\infty$. This allows to give sense to the backward equation for a general class of initial data.

Obviously there are some natural questions related to this definition:

- Can we rewrite entropy condition (11) in a more explicit way? In particular which is the consequence of such condition on the coefficients $\lambda_i$?
- It is possible to state that, in the case in which the initial data takes values only in the two stable phases, the original forward–backward equation $u_t = \phi(u)_{xx}$ is satisfied at least in the distributional sense?
- Is there uniqueness for the forward–backward problem (1) in the class of entropy solutions introduced in Definition 2.1?

Regarding the first question there is the following result obtained in [24], [25].

Theorem 2.2. Let $(u, v, \lambda_0, \lambda_1, \lambda_2)$ be an entropy solution in the sense of Definition 2.1 to problem (1) in $Q_T$. Then $\lambda_i(x, \cdot) \in BV_{\text{loc}}(0, T)$ for almost every $x \in \Omega$ ($i = 0, 1, 2$). Moreover, if

$$\text{ess sup}_{t \in (t_1, t_2)} v(x, t) < B$$

for some interval $(t_1, t_2) \subseteq (0, T)$, then $\lambda_1(x, \cdot)$ is not decreasing in $(t_1, t_2)$. Similarly, if

$$\text{ess inf}_{t \in (t_1, t_2)} v(x, t) > A$$

for some interval $(t_1, t_2) \subseteq (0, T)$, then $\lambda_2(x, \cdot)$ is not decreasing in $(t_1, t_2)$.

This result suggests that coefficients $\lambda_1$ and $\lambda_2$ related to the stable phases tend to increase and therefore $\lambda_0$ decrease. In particular $\lambda_1$ and $\lambda_2$ do not decrease unless $v = B$ or $v = A$.

In order to give complete answers to the previous questions, at least for a subclass of initial data that takes values in the two stable phases, we introduce the “two–phase problem”. More precisely, let us consider an initial data $u_0 \in L^\infty((0, L))$ that satisfies

$$\begin{align*}
\text{u}_0 \leq b & \text{ in } (0, x_0), \\
\text{u}_0 \geq c & \text{ in } (x_0, L), \\
\phi(u_0) & \in H^1(\Omega).
\end{align*}$$

(15)

where $x_0 \in (0, L)$.

In view of the above assumptions (15), we look for a solution to problem (11) with a particular structure. More precisely, since the initial datum $u_0$ takes values only in the stable phases, we impose that solutions to problem (11) are again in these phases with a regular interface separating the rectangle $Q_T$ into two different regions. We require that the unstable phase $(b, c)$ does not influence the dynamics. Then,
in accordance with the general entropy formulation given in [24], the following definition of two-phase solution was done (see [20], [29]).

**Definition 2.3.** Let us suppose that \( u_0 \in L^\infty((0, L)) \) satisfies (15) and \( \phi(u_0) \in C((0, L)) \). By a two-phase solution to problem (1) we mean a triple \((u, v, \xi)\) such that:

1. \( u \in L^\infty(Q_T), \ v \in C(\overline{Q_T}) \cap L^2((0, T); H^1(\Omega)), \) and \( \xi : [0, T] \to \overline{\Omega}, \ \xi \in C^1([0, T]), \) \( \xi(0) = x_0; \)

2. we have:

   \[ u = \beta_i(v) \quad in \quad V_i \quad (i = 1, 2), \]

   where

   \[ V_1 := \{(x, t) \in \overline{Q_T} \mid 0 < x < \xi(t), \ t \in [0, T]\}, \]

3. and

   \[ \gamma := \partial V_1 \cap \partial V_2 = \{\{(\xi(t), t) \mid t \in [0, T]\}; \]

   (iii) \( u \) satisfies conditions b) and c) of Definition [2.1]

   Obviously equations (16) imply that \( v = \phi(u) \) and the coefficients \( \lambda_1, \lambda_2 \) are, respectively, equal to \( I_{V_1}, I_{V_2} \), where \( I_E \) denotes the characteristic function of the set \( E \).

   In this class of solutions it is possible to give a characterization of the entropy inequality (11) in terms of admissibility condition for the evolution of the interface \( \xi(t) \). More precisely we have the following result (see [29] for a proof in the more general case)

**Proposition 2.4.** Let \((u, v, \xi)\) be a two-phase solution of problem (1). Then

\[
\xi'(t) \begin{cases} 
\leq 0 & \text{if } v(\xi(t), t) = B \\
= 0 & \text{if } v(\xi(t), t) \in (A, B) \\
\geq 0 & \text{if } v(\xi(t), t) = A.
\end{cases}
\]

This means that interface moves only at the critical value \( A, B \). This is in accordance with the results in Theorem [2.2] for any fixed value \( \overline{a}, \lambda_1(\overline{a}, \cdot) \) can pass from the value 1 to the value 0 \( (\lambda_1 \text{ decrease}) \) only at time \( \overline{t} \) when \( v(\overline{a}, \overline{t}) = B \), analogously \( \lambda_2(\overline{a}, \cdot) \) can pass from the value 1 to the value 0 \( (\lambda_2 \text{ decrease}) \) only when \( v(\overline{a}, \overline{t}) = A \).

For this class of problems, we can give the answers to all the previous questions. In fact Proposition [2.3] gives the characterization requested in the first question.

In order to obtain an answer to the second question in the class of data satisfying condition (15), we have to prove that there is existence of two-phase entropy solutions introduced in Definition [2.3]. This problem was studied in [20], [30] in the piecewise linear case and in [29] for the nonlinear cubic case. Then local existence was proved in [20], [29], and global existence in [30]. Regarding uniqueness in the class of two phase entropy solutions, this is proved in [20] in the piecewise linear case and in [29] in the general nonlinear case.

The purpose of this paper is to prove that uniqueness fails in the general contest of Definition [2.1]. Then the last question has a negative answer. The counterexample that we shall give in Section 3 is for initial data that take values in the unstable
phase. Therefore the question for initial data that takes values only in the stable phases is still open. However, it is worth to note, that, the results obtained for the “two–phase problem” suggest a different answer for this restricted class of initial data.

In the last part of this section we analyze a new characterization of the entropy condition \([\Pi]\) for a class of solutions which will be introduced in Section 3. First of all, observe that, if we impose that one coefficient \(\lambda_i\) is equal to 0, e.g. \(\lambda_1 \equiv 0\), we can choose one of the other coefficient in function of the third one, e.g. \(\lambda_0 = 1 - \lambda_2\). In some sense, we have again a two phase solution but with a more general structure.

We have the following

**Proposition 2.5.** Let \(v \in C^2(Q_T)\) be such that:

i) there exists \(\lambda \in C(1)(Q_T)\) such that \(\lambda_i \geq 0\), \(0 \leq \lambda \leq 1\) in \(Q_T\);

ii) \(u := (1 - \lambda)\beta_0(v) + \lambda\beta_2(v)\) and \(u_t = v_{xx}\) in \(Q_T\).

Then \(v\) satisfies the entropy inequality \([\Pi]\).

**Proof.** We have to prove that for any \(g \in C^1\), \(g' \geq 0\)

\[
(21) \quad \iint_{Q_T} \left\{ G^* \psi_t - g(v) \nabla v \cdot \nabla \psi - g'(v) |\nabla v|^2 \psi \right\} \, dx \, dt \geq 0
\]

for any \(\psi \in C^\infty_0(Q_T), \psi \geq 0\), where

\[
G^*(x, t) := (1 - \lambda)G(\beta_0(u(x,t))) + \lambda G(\beta_2(u(x,t)))
\]

for a.e. \((x, t) \in Q_T\)

with \(G\) given in \((7)\).

Since for the hypothesis \(\lambda\) and \(v\) are regular functions, we can integrate by parts in the first member of \((21)\), obtaining

\[
\iint_{Q_T} \left\{ - G^*_t \psi + (g(v))_x \nabla \psi - g'(v) |v_x|^2 \psi \right\} \, dx \, dt = \iint_{Q_T} \left( - G^*_t + (g(v))_x \right) \psi \, dx \, dt.
\]

Then, it is enough to prove that \(-G^*_t + (g(v))_x \geq 0\) in \(Q_T\).

Observe that

\[
-G^*_t = \lambda_t G(\beta_0(u(x,t))) - \lambda_t G(\beta_2(u(x,t))) - (1 - \lambda)_t [G(\beta_0(u(x,t)))\b] \geq 0
\]

Here we use the definition of \(G\), that gives

\[
[G(\beta_i(v))]_t = g(\phi(\beta_i(v))) \b = g(v) \b
\]

for i=0,1,2.

For condition ii) we have

\[
-G^*_t = \lambda_t G(\beta_0(u(x,t))) - \lambda_t G(\beta_2(u(x,t))) - u_t g(v) + [-\lambda_t \beta_0(v) + \lambda_t \beta_2(v)] g(v).
\]

Since \(u_t = v_{xx}\), we obtain

\[
g(v)v_{xx} - G^*_t = \lambda_t [G(\beta_0(v)) - G(\beta_2(v))] + (\beta_2(v) - \beta_0(v)) g(v) .
\]

Therefore, since \(\lambda_t \geq 0\), it remains to prove that \(G(\beta_0(v)) - G(\beta_2(v)) + (\beta_2(v) - \beta_0(v)) g(v) \geq 0\).
Using again the definition of $G$, we get
\[ G(\beta_0(v)) - G(\beta_2(v)) + (\beta_2(v) - \beta_0(v)) g(v) = \int_{\beta_0(v)}^{\beta_2(v)} [g(v) - g(\phi(s))] \, ds \]
and we obtain the thesis since $g' \geq 0$, $\beta_0(v) \leq \beta_2(v)$ and $v \geq \phi(s)$ for every $s \in (\beta_0(v), \beta_2(v))$.

Proposition 2.5 suggest a way to obtain an entropy solution with unstable phase. Here we have only one coefficient $\lambda$ that correspond to $\lambda_2$. Again the entropy condition is equivalent to the request that the coefficient corresponding to the stable phase does not decrease.

3. Non–uniqueness results

In order to produce the non existence counterexample, we consider an entropy solution that satisfies the hypothesis of Proposition 2.5. We call this kind of solution a “two phase, measure–valued, regular solution” of problem (1). More precisely

Definition 3.1. We say that the triple $u \in C^2(Q_T) \cap C((0, L) \times [0, T))$, $\lambda \in C^1(Q_T) \cap C((0, L) \times [0, T))$, $v \in C^2(Q_T)$ is a “two phase, measure–valued, regular solution” of problem (1) if and only if:

i) $u(x, 0) = u_0(x)$ for every $x \in (0, L)$;

ii) there exist $\lim_{x \to 0^+} v_+(x, t) = \lim_{x \to L^-} v_-(x, t) = 0$ for every $t \in (0, T)$;

iii) $v \geq A$ and $\lambda = 1$ if $v > B$;

iv) the functions $u$, $\lambda$, $v$ are related by following equations:

\[ u = (1 - \lambda)\beta_0(v) + \lambda \beta_2(v) \quad \text{in} \quad Q_T, \]
\[ u_t = u_{xx} \quad \text{in} \quad Q_T; \]
\[ v) \lambda_t \geq 0, \quad 0 \leq \lambda \leq 1 \quad \text{in} \quad Q_T. \]

Then we have the following

Proposition 3.2. A “two phase, measure–valued, regular solution” of problem (1) is also a solution in the sense of Definition 2.1 of problem (1).

Proof. This is consequence of Proposition 2.5 that assures that the entropy inequality (11) is satisfied. The other conditions requested in Definition 2.1 are immediate. □

It is natural to consider solutions in which at least at initial time $t = 0$ there is not superposition of phases. This means $v(x, 0) = \phi(u_0(x))$. Then $\lambda(x, 0) \equiv 1$ in $(0, L)$ or $\lambda(x, 0) \equiv 0$ in $(0, L)$. The former case consists of initial data $u_0$ that are in the stable phase, in this situation $\lambda \equiv 1$ in $Q_T$ and we obtain a classical solution solving a forward parabolic problem. In the second case the initial data is in the unstable phase, then we have a purely backward parabolic problem that we can solve by using the auxiliary function $\lambda$. Observe that condition $v)$ of Definition 3.1 suggests that at positive time a superposition of phases appears and in particular the stable phase becomes dominant respect to the unstable one.

In the following we always assume that $\lambda(x, 0) \equiv 0$ in $(0, L)$.

Let us fix $T > 0$, $b', c' \in (b, c)$ such that $b' < c'$. We choose $g(x) \in C^1([0, L])$ such that $g'(0) = g'(L) = 0$, $g(x) \in (b', c')$ for any $x \in [0, L]$.
Let us consider the following problem

\begin{equation}
\begin{aligned}
& u_t = \phi(u)_{xx} & \text{in } Q_T = \Omega \times (0, T) \\
& u_x(0, t) = u_x(L, t) \equiv 0 & \text{in } (0, T) \\
& u(x, T) = g(x) & \text{in } \Omega = (0, L).
\end{aligned}
\end{equation}

This is a well-posed parabolic problem, since it is a backward problem with condition at final time $T$.

Denote with $\mathfrak{F}$ the unique solution of problem (22). Therefore for the maximum principle, we have $\mathfrak{F}(x, t) \in (\mathfrak{U}, \mathfrak{U}')$ for any $(x, t) \in Q_T$. Let $u_0(x) := \mathfrak{F}(x, 0)$. Obviously $\mathfrak{F}$ is a “two phase, measure–valued, regular solution” of problem (1). We put $\mathfrak{F} = \phi(\mathfrak{F})$, observing that it satisfies the following problem

\begin{equation}
\begin{aligned}
& (\beta_0(v))_t = v_{xx} & \text{in } Q_T = \Omega \times (0, T) \\
& v_x(0, t) = v_x(L, t) \equiv 0 & \text{in } (0, T) \\
& v(x, T) = \phi(g(x)) & \text{in } \Omega = (0, L).
\end{aligned}
\end{equation}

Moreover, since $\phi$ is piecewise linear, we have $\mathfrak{F}(x, 0) = \phi(u_0)$.

Now we want to obtain a different “two phase, measure–valued, regular solution” that has the same initial data $u_0$.

Let us impose that the triple of function $u$, $\lambda$, $v$ satisfies condition iv) in Definition 3.1. Then we get

\begin{equation}
[(1 - \lambda)\beta_0(v) + \lambda\beta_2(v)]_t = v_{xx}.
\end{equation}

Let us integrate (24) in $(0, t)$, for any fixed $(x, t) \in Q_T$. Since $\lambda(\cdot, 0) \equiv 0$, we obtain

\begin{equation}
[(1 - \lambda)\beta_0(v) + \lambda\beta_2(v)](x, t) - \beta_0(v(x, 0)) = \int_0^t v_{xx}(x, s) \, ds.
\end{equation}

Observe that $\beta_2(v(x, t)) = \beta_0(v(x, t))$ if and only if $v(x, t) = A$. In these points (25) becomes

\begin{equation}
\int_0^t v_{xx}(x, s) - [\beta_0(v(x, s))]_s \, ds = 0.
\end{equation}

On the other hand, in any point $(x, t) \in Q_T$ such that equation (26) is satisfied, we have $\lambda(x, t) = 0$ or $v(x, t) = A$.

In the following we assume $v > A$ in $Q_T$ and we introduce the function $m(x, t) = v_{xx}(x, t) - [\beta_0(v(x, t))]_t$.

Therefore, we get

\begin{equation}
\lambda(x, t) = \frac{\beta_0(v(x, 0)) - \beta_0(v(x, t)) + \int_0^t v_{xx}(x, s) \, ds}{\beta_2(v(x, t)) - \beta_0(v(x, t))} = \frac{\int_0^t v_{xx}(x, s) - [\beta_0(v(x, s))]_s \, ds}{\beta_2(v(x, t)) - \beta_0(v(x, t))}.
\end{equation}

In order to impose the monotonicity condition for the coefficient $\lambda$ we derive equation (27) respect to the variable $t$.

Then, we obtain

\begin{equation}
\lambda_t(x, t) = \frac{m(x, t)}{\beta_2(v(x, t)) - \beta_0(v(x, t))} - \frac{[\beta_2(v(x, t)) - \beta_0(v(x, t))]_t \int_0^t m(x, s) \, ds}{(\beta_2(v(x, t)) - \beta_0(v(x, t)))^2}.
\end{equation}
It is clear that the sign of \( m \) it is strictly related to the monotonicity condition of the coefficient \( \lambda \).

We have the following result that gives sufficient conditions.

**Proposition 3.3.** Suppose that the function \( v \) fulfills the following conditions

\[
(29) \quad \text{there exist } T' \in (0, T), c_1 > 0, \ s.t. \ \beta_2(v(x,t)) - \beta_0(v(x,t)) \geq c_1 \text{ in } Q_{T'},
\]

\[
(30) \quad \text{there exist } T'' \in (0, T), c_2 > 0, \ s.t. \ m \geq c_2 \text{ in } Q_{T''},
\]

then there exists \( T \in (0, T) \) such that \( \lambda \) is not negative in \( Q_T \).

The proof is an immediate consequence of (28).

Let us choose \( v(x, t) = T + t \), where \( T \) is the solution of problem (23), then function \( v \) satisfies the following

\[
(31) \quad \left\{ \begin{array}{l}
v_{xx} - (\beta_0(v))_t = |\sigma| & \text{in } Q_T = \Omega \times (0, T) \\
v_x(0,t) \equiv v_x(L,t) \equiv 0 \quad & \text{in } (0, T) \\
v(x,0) = \phi(u_0) \quad & \text{in } \Omega = (0, L).
\end{array} \right.
\]

Here \( \sigma = \frac{-b}{A - B} \) is obtained by the definition of the piecewise function \( \phi \) given in (2). In particular the hypothesis (30) is satisfied by the function \( v \).

Moreover, we can choose \( T' \) small enough, such that \( v \in (b', c'') \) in \( Q_{T'} \) with \( c'' \in (c', c) \), then we obtain (29) with a proper constant \( c_1 \).

Finally, since \( \lambda(\cdot, 0) \equiv 0 \), using (28) we can choose time \( T \) such that \( \lambda \in [0, 1) \) for any \( (x, t) \in Q_T \). Then we can exhibit two different “two phase, measure–valued, regular solution” of problem (1) with initial data \( u_0 \). The first one is given by the triple \( \pi, \lambda \equiv 0 \) and \( \pi \), and the second one is given by \( u, \lambda, v \) where \( v \) is previously defined, \( \lambda \) is given in (27) and \( u = (1 - \lambda)\beta_0(v) + \lambda\beta_2(v) \).

Let us observe that we can obtain an infinite family of “two–phase, measure–valued, regular solution” with \( u_0 \) as initial condition. Actually we only need to check that conditions (29), (30) are satisfied. For example we can find solutions of \( |\sigma| v_t + v_{xx} = f(x) \geq c_2 > 0 \) that fulfill initial and boundary condition by standard method of eigenfunction expansion. By straightforward calculation we obtain a solution for every source function \( f \geq c_2 > 0 \), such that

\[
f(x) = \sum_{k=0}^{N} a_k \cos \left( \frac{k\pi x}{L} \right)
\]

with \( N \in \mathbb{N} \) and \( a_k \in \mathbb{R}, k = 0 \ldots, N \).

Analogous techniques could be used to prove existence of a “two–phase, measure–valued, regular solution” with general initial data \( u_0 \) that takes values in the unstable phase. We are not interested to consider in detail existence problems. We limit ourself to highlight that, in order to obtain existence using Proposition 3.3, it is useful to consider the following parabolic backward inverse problem, where the unknown data \( f \) has to be strictly positive.

\[
(32) \quad \left\{ \begin{array}{l}
|\sigma| v_t + v_{xx} = f(x, t) & \text{in } Q_T = \Omega \times (0, T) \\
v_x(0,t) \equiv v_x(L,t) \equiv 0 & \text{in } (0, T) \\
v(x,0) = \phi(u_0) & \text{in } \Omega = (0, L).
\end{array} \right.
\]

In general problem (32) is underdetermined unless we fix a final data \( v(x, T) = v_T(x) \). On the other hand using the methods of eigenfunction expansion, we see that it is necessary to impose some restriction on the initial data. In order to
give the idea, we consider the simple case in which we suppose that the solution $f$ depends only on the variable $x$. Then, if

$$v_0(x) = \phi(u_0) = \sum_{k=0}^{\infty} a_k \cos \left( \frac{k\pi x}{L} \right)$$

and

$$v_T(x) = \sum_{k=0}^{\infty} b_k \cos \left( \frac{k\pi x}{L} \right),$$

we obtain

$$f(x) = \sum_{k=0}^{\infty} f_k \cos \left( \frac{k\pi x}{L} \right)$$

such that

$$f_0 = \frac{(b_0 - a_0)|\sigma|}{T};$$

$$f_k = \frac{\pi^2 k^2 \left( b_k - a_k e^{\frac{2\pi^2}{L^2}|\sigma|} \right)}{L^2 \left( e^{\frac{2\pi^2}{L^2}|\sigma|} - 1 \right)} \quad k \geq 1.$$  

This suggest that we can choose properly $v_T$ in order to have a solution $f(x) \geq c_2$ but it is necessary to impose a summability condition to the coefficients $a_k$.

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