The longitudinal structure function of the proton for small x *

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Abstract
A comparison of the H1 data on the longitudinal structure function, $F_L$, at small $x$ with the predictions from the generalized vector dominance / color dipole picture (GVD/CDP) is presented. Using the set of parameters previously determined in the fits to the total cross section, $\sigma_{\gamma p}$, we find good agreement with the data for $F_L$. Scaling in $\eta = (Q^2 + m_0^2)/\Lambda^2(W^2)$ is discussed in detail for the longitudinal and transverse photoabsorption cross sections.

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A confrontation of predictions for polarisation phenomena with experimental data has frequently in the past provided crucial tests of a theoretical ansatz. This was also true for an analysis of the nucleon structure functions when discriminating between longitudinal and transverse virtual photons.

The H1 collaboration at HERA has recently presented data \[1, 2\] for the longitudinal structure function of the proton, \(F_L(x, Q^2)\), in the diffraction region of small values of \(x \approx Q^2/W^2 \ll 1\). In the present note, we compare the experimental data with the predictions from the generalized vector dominance/color dipole picture (GVD/CDP) \[3, 4, 5\]. We will present our analysis of the data in terms of the longitudinal structure function, \(F_L(x, Q^2)\), and in terms of the longitudinal part of the photoabsorption cross section, \(\sigma_L(W^2, Q^2)\). In the case of the longitudinal cross section, \(\sigma_L(W^2, Q^2)\), we will employ the scaling variable

\[
\eta(W^2, Q^2) = \frac{Q^2 + m_0^2}{\Lambda^2(W^2)}, \tag{1}
\]

recently introduced \[3, 4\] in our analysis of the total cross section, where in good approximation

\[
\sigma_{\gamma^* p}(W^2, Q^2) = \sigma_T(W^2, Q^2) + \sigma_L(W^2, Q^2) = \sigma_{\gamma^* p}(\eta(W^2, Q^2)). \tag{2}
\]

Here \(\Lambda^2(W^2)\) is a slowly increasing function of \(W^2\) proportional to the effective gluon momentum transfer squared, and \(m_0\) denotes a threshold mass. We will provide a detailed discussion of the behavior of \(\sigma_T\) and \(\sigma_L\) with respect to their dependence on \(\eta\), in particular in the \(Q^2 \to 0\) photoproduction limit, where \(\eta \to \eta_{\text{Min}} = m_0^2/\Lambda^2(W^2)\).

The GVD/CDP was described and compared with the experimental data for the total cross section, \(\sigma_{\gamma^* p}\), in refs.\[3, 4, 5\]. The GVD/CDP, equivalently, may be formulated in transverse position space or in momentum space.

The evaluation of the two-gluon exchange diagrams \[8\] in the low-\(x\) limit, upon transition to transverse position space, leads to the representation \[3\]

\[
\sigma_{\gamma^* p}(W^2, Q^2) = \int_0^1 dz \int d^2r_\perp |\psi|^2(r_\perp^2, z(1-z), Q^2)\sigma_{(qq)p}(r_\perp^2, z(1-z), W^2), \tag{3}
\]

\[1\]For somewhat related approaches compare refs.\[6, 7\]
where \( \psi \) denotes the so-called photon wave function, explicitly given in \([9]\). The color-dipole cross section, \( \sigma_{\langle q\bar{q}\rangle p} \), fulfills a Fourier representation of the form

\[
\sigma_{\langle q\bar{q}\rangle p}(\vec{r}_{\perp}^2, z(1-z), W^2) = \int d^2l_{\perp} \tilde{\sigma}_{\langle q\bar{q}\rangle p}(\vec{l}_{\perp}^2, z(1-z), W^2) \left(1 - e^{-i\vec{l}_{\perp} \cdot \vec{r}_{\perp}}\right).
\]

In (3) and (4), the variables \( \vec{r}_{\perp} \) and \( z \) denote the two-dimensional vector of the transverse interquark separation and the fraction of the photon momentum taken over by one of the incoming quarks. The representation (3), with (4), contains the underlying generic structure of two-gluon exchange, and, accordingly, it incorporates “color transparency” as well as hadronic unitarity provided appropriate convergence properties are readily fulfilled. The empirical scaling behavior (2) (compare \([3, 4]\)) is embodied in (3) by adopting the simple ansatz of

\[
\tilde{\sigma}_{\langle q\bar{q}\rangle p}(\vec{l}_{\perp}^2, z(1-z), W^2) = \sigma_{\infty} \frac{1}{\pi} \delta(\vec{l}_{\perp}^2 - \Lambda^2(W^2)z(1-z))
\]

for the gluon-gluon-proton-proton vertex function, \( \tilde{\sigma}_{\langle q\bar{q}\rangle p} \), in (4). Substitution of (5) into (4) yields

\[
\sigma_{\langle q\bar{q}\rangle p}(\vec{r}_{\perp}^2, z(1-z), W^2) = \sigma_{\infty}(1 - J_0(r_{\perp} \sqrt{z(1-z)} \Lambda(W^2))),
\]

where \( J_0 \) denotes a Bessel function. The asymptotic value of the dipole cross section for \( r_{\perp} \to \infty \) as well as \( W^2 \to \infty \) has been denoted by \( \sigma^{(\infty)} \). Actually, \( \sigma^{(\infty)} \) is constant in good approximation, and it is of typical hadronic magnitude. Equations (3) and (4) may be considered as the basic formulae of the GVD/CDP. They yield \( \sigma_{\gamma^*p} = \sigma_{\gamma^*p}(\eta) \), and accordingly, \( \sigma_{\gamma^*p} \) depends on the (threshold) mass \( m_0^2 \) and on the adjustable parameters describing the increase of of the average or effective gluon transverse momentum, \( \Lambda^2(W^2) \), with energy.

Actually, the evaluation of the GVD/CDP was carried out \([10, 3]\) in momentum space. Inserting (4) into (3) together with the Fourier representation of the photon wave function for longitudinal and transverse (virtual) photons takes us back to momentum space and leads to

\[
\sigma_{\gamma^*T,L}(W^2, Q^2) = \frac{\alpha R_{e^+e^-}}{3\pi} \sigma^{(\infty)} I_{T,L} \left(\eta, \frac{m_0^2}{\Lambda^2(W^2)}\right)
\]

with

\[
R_{e^+e^-} = 3 \sum Q_i^2,
\]

(7)
where \( R_{e^+e^-} = 2 \) is to be inserted, since upon specifying (7) to photoproduction, \( Q^2 = 0 \), only three flavors \((u,d,s)\) with charges \( Q_i \) contribute appreciably, while \( \sigma^{(\infty)} \approx 80 GeV^{-2} \approx 31 mb \). The (dimensionless) functions \( I_{T,L}(\eta, m_0^2/\Lambda^2(W^2)) \) in (7) denote integrals of the form of mass dispersion relations reminiscent of off-diagonal generalized vector dominance [11, 12]. They were represented [3] as a sum of two terms,

\[
I_{T,L} = I_{T,L}^{(1)} + I_{T,L}^{(2)}.
\]

The main transverse and longitudinal contributions \( I_{T,L}^{(1)} \), are given by

\[
I_{T}^{(1)} \left( \eta, \frac{m_0^2}{\Lambda^2(W^2)} \right)
= \frac{1}{\pi} \int_{m_0^2}^{\infty} dM^2 \int_{(M-\Lambda(W^2))^2}^{(M+\Lambda(W^2))^2} dM'^2 \omega(M^2, M'^2, \Lambda^2(W^2))
\times \left[ \frac{M^2}{(Q^2 + M^2)^2} - \frac{M'^2 + M^2 - \Lambda^2(W^2)}{2(Q^2 + M^2)(Q^2 + M'^2)} \right],
\]

and

\[
I_{L}^{(1)} \left( \eta, \frac{m_0^2}{\Lambda^2(W^2)} \right)
= \frac{1}{\pi} \int_{m_0^2}^{\infty} dM^2 \int_{(M-\Lambda(W^2))^2}^{(M+\Lambda(W^2))^2} dM'^2 \omega(M^2, M'^2, \Lambda^2(W^2))
\times \left[ \frac{Q^2}{(Q^2 + M^2)^2} - \frac{Q^2}{2(Q^2 + M^2)(Q^2 + M'^2)} \right].
\]

The terms \( I_{T,L}^{(2)} \) in (8) assure the correct threshold behavior of \( I_{T,L} \) in the off-diagonal contribution [3]. They are given by

\[
I_{T}^{(2)} \left( \eta, \frac{m_0^2}{\Lambda^2(W^2)} \right)
= \frac{1}{\pi} \int_{m_0^2}^{\infty} dM^2 \Theta(m_0^2 - (M - \Lambda(W^2))^2) \int_{(M-\Lambda(W^2))^2}^{m_0^2} dM'^2
\times \omega(M^2, M'^2, \Lambda^2(W^2)) \frac{M^2 + M'^2 - \Lambda^2(W^2)}{2(Q^2 + M^2)(Q^2 + M'^2)},
\]

\[\text{As long as no detailed treatment of the influence of the charm-quark mass is included in (8), the dependence on the product } R_{e^+e^-} \cdot \sigma^{(\infty)} \text{ allows one to equally well insert } R_{e^+e^-} = 10/3 \text{ and } \sigma^{(\infty)} = 48 GeV^{-2} = 18.7 mb.\]

\[\text{Numerically, it turns out that } I_{T}^{(2)} \text{ is practically negligible in the HERA energy range, while } I_{L}^{(2)} \text{ contributes about 20\% at the lowest HERA energy and becomes negligible, when the energy reaches the highest HERA energy. Note that explicit analytic formulæ are available [3] for } I_{T,L}^{(1)}, \text{ and accordingly only the correction term } I_{L}^{(2)} \text{ must be evaluated by numerical integration.}\]
and

\[ I_L^{(2)} = \left( \eta, \frac{m_0^2}{\Lambda^2(W^2)} \right) = \frac{1}{\pi} \int_{m_0^2}^{\infty} dM^2 \Theta(m_0^2 - (M - \Lambda(W^2))^2) \int_{\Lambda(W^2)}^{m_0^2} dM^2 \]

\[ \times \omega(M^2, M'^2, \Lambda^2(W^2)) \frac{Q^2}{(Q^2 + M^2)(Q^2 + M'^2)}. \]

We note the relative minus sign between the diagonal \((M^2 = M'^2)\) and the off-diagonal propagator term in (10) and (11) that is characteristic for off-diagonal generalized vector dominance [11]. The two contributions with their relative minus sign are an outgrowth of the two-gluon exchange structure. Concerning \(\omega(M^2, M'^2, \Lambda^2(W^2))\), we only note the integration formulae,

\[ \frac{1}{\pi} \int_{(M+\Lambda(W^2))^2}^{(M-\Lambda(W^2))^2} dM'^2 \omega(M^2, M'^2, \Lambda^2(W^2)) = 1, \] (14)

and

\[ \frac{1}{\pi} \int_{(M-\Lambda(W^2))^2}^{(M+\Lambda(W^2))^2} dM'^2 \omega(M^2, M'^2, \Lambda^2(W^2))M'^2 = M^2 + \Lambda^2(W^2), \] (15)

and refer to ref. [3] for the explicit expressions.

In [3], we gave explicit analytical expressions for \(I_{T,L}^{(1)}\), and an approximation formula for the sum \(I_T^{(2)} + I_L^{(2)}\). These formulae allowed us to perform a fit to the data for the total cross section, \(\sigma_{\gamma p}(W^2, Q^2)\). The fit gave the parameters

\[ m_0^2 = 0.16 \pm 0.01 \text{ GeV}^2, \] (16)

as well as the parameters describing the increase of \(\Lambda^2(W^2)\) with energy, alternatively by a power law or a logarithm,

\[ \Lambda^2(W^2) = \begin{cases} C_1(W^2 + W_0^2)^{C_2}, \\
C'_1 \ln \left( \frac{W^2}{W_0^2} + C'_2 \right), \end{cases} \] (17)

where

\[ C_1 = 0.34 \pm 0.05 \text{ (GeV}^2)^{1-C_2}, \]
\[ C_2 = 0.27 \pm 0.01, \] (18)
\[ W_0^2 = 882 \pm 246 \text{ GeV}^2, \]
Figure 1: The H1 data for $F_L$ are compared with the predictions from the GVD/CDP.

and

$$C'_1 = 1.64 \pm 0.14 \text{(GeV}^2\text{)},$$

$$C'_2 = 4.1 \pm 0.4,$$

$$W^{'2}_0 = 1015 \pm 334 \text{GeV}^2.$$  \hspace{1cm} (19)$$

Note that $\Lambda^2(W^2)$ varies between $\Lambda^2 \simeq 2 \text{GeV}^2$ and $\Lambda^2 \simeq 7 \text{GeV}^2$ in the HERA energy range of $W^2 \simeq 1000 \text{GeV}^2$ to $90000 \text{GeV}^2$. Averaging over the configuration
variable $z$ yields $\langle \vec{l}^2 \rangle_{W^2} = (1/6)\Lambda^2$, i.e. a reasonable value of $\langle \vec{l}^2 \rangle \simeq 0.3\text{GeV}^2$ to $\langle \vec{l}^2 \rangle \simeq 1\text{GeV}^2$ for the average or effective gluon transverse momentum squared absorbed by one of the quarks.

In fig. 1, we compare our results for

$$F_L(x, Q^2) = \frac{Q^2}{4\pi^2\alpha_L(x, Q^2)}$$

with the H1 data using the parameters (16) as well as (18) or (19) previously deter-

mined in the fits to $\sigma_{\gamma^*p}$. There is reasonable agreement with the experimental data.

Figure 2: The H1 data, expressed in terms of $\sigma_L$, are compared with the predictions from the GVD/CDP.

From the point of view of the GVD/CDP it is more appropriate to plot the data in terms of the longitudinal part of the total cross section. In fig. 2, we show the data for $\sigma_L$, calculated from the data for $F_L$ according to (20), plotted against the scaling variable $\eta$. As expected from fig. 1, there is reasonable agreement, with a tendency for the data to lie somewhat lower than the theoretical predictions, as also seen in fig. 1.
The lower limit of $\eta$ determined by the photoproduction limit of $Q^2 = 0$,

$$\eta \geq \eta_{\text{Min}}(Q^2 = 0, W^2) = \frac{m_0^2}{\Lambda^2(W^2)}, \quad (21)$$
decreases with increasing energy. The vanishing of $\sigma_L(\eta, \eta_{\text{Min}})$ with $\eta - \eta_{\text{Min}} = Q^2/\Lambda^2(W^2) \to 0$ as a function of $\eta$, accordingly, occurs at values of $\eta_{\text{Min}}$ that decrease with increasing energy. Compare fig. 2. In the low $Q^2$ regime of $\eta \approx \eta_{\text{Min}}$, the longitudinal cross section, $\sigma_L$, strongly depends on $\eta_{\text{Min}}$ at fixed $\eta$. Scaling in $\eta$ is strongly violated. This is in contrast, as previously discussed [3], to the total cross section, where $\sigma_{\gamma^*p} = \sigma_{\gamma^*p}(\eta)$.

![Figure 3: The theoretical predictions of the GVD/CDP for $\sigma_L$, $\sigma_T$ and $\sigma_{\gamma^*p} = \sigma_L + \sigma_T$. Note that the validity of the theoretical predictions is restricted by $x \lesssim 0.01$. This restriction has been imposed on the curves in the figure.](image)

The behavior of $\sigma_T$, $\sigma_L$ as well as $\sigma_{\gamma^*p} = \sigma_T + \sigma_L$ as a function of $\eta$ is shown in fig. 3. In order to illuminate the behavior of the different contributions to the total cross section, we note that $I_T^{(1)}$ and $I_L^{(1)}$ from (10) and (11) depend on $\eta$ and $\eta_{\text{Min}}$;

$$I_{T,L} = I_{T,L}(\eta, \eta_{\text{Min}}). \quad (22)$$
Explicit formulae were given in ref. [3]. Numerically, $\eta_{\text{Min}} < 0.1$. Expressing $I_{T,L}$ in (22) in terms of $\eta$ and

$$\epsilon = \eta - \eta_{\text{Min}},$$

(23)

by substitution of $\eta_{\text{Min}} = \eta - \epsilon$, and noting that $\epsilon \cong \eta$ as soon as $\eta > 1$, we immediately see that $I_{T,L} = I_{T,L}(\eta)$ for $\eta > 1$; we have scaling in $\eta$ for $\sigma_T$ as well as $\sigma_L$ for $\eta > 1$. We turn to $\eta \cong \eta_{\text{Min}} < 1$. From the analytic expressions given in Appendix B of ref. [3], one immediately notes that $I_T$ contains an additive contribution, opposite in sign, but equal in magnitude to $I_L$. The violent increase of $I_L$ at fixed $\eta$ with decreasing $\eta_{\text{Min}}$ for $\eta \cong \eta_{\text{Min}} < 1$ is indeed seen to be entirely cancelled, once the sum of $I_T$ and $I_L$ is taken. An expansion of the sum of $I_T$ and $I_L$ at fixed $\eta$ as a function of $\eta_{\text{Min}}$ shows that any additional dependence on $\eta_{\text{Min}}$ is negligible [3].

In summary, we have shown that the GVD/CDP with the parameters previously determined in a fit to $\sigma_{\gamma^* p}$ describes the longitudinal structure function, $F_L$, or, equivalently, the longitudinal photoabsorption cross section, $\sigma_L$, at low $x$. We have given a detailed discussion on how scaling in $\eta$ for $\sigma_{\gamma^* p}$, i.e. $\sigma_{\gamma^* p} = \sigma_{\gamma^* p}(\eta)$, arises despite the fact that scaling is strongly violated for the longitudinal cross section in the region of $\eta \cong \eta_{\text{Min}}$.

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