New arrangement of common approach to calculating the QCD ground state

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The quark behaviour in the background of intensive stochastic gluon field is studied. An approximate procedure for calculating the effective Hamiltonian is developed and the corresponding ground state within the Hartree-Fock-Bogolyubov approach is found. The comparative analysis of various model Hamiltonian is given and transition to the chiral limit in the Keldysh model is discussed in detail.

We study the quark (anti-quark) behaviour while being influenced by intensive stochastic gluon field and work in the context of the Euclidean field theory. The corresponding Lagrangian density is the following

\[ \mathcal{L}_E = \bar{q} \left( i\gamma_\mu D_\mu + im \right) q , \]  

(1)

here \( q (\bar{q}) \) — are the quark (anti-quarks) fields with covariant derivative \( D_\mu = \partial_\mu - igA^a_\mu t^a \) where \( A^a_\mu \) is the gluon field, \( t^a = \lambda^a / 2 \) are the generators of colour gauge group \( SU(N_c) \) and \( m \) is the current quark mass. As the model of stochastic gluon field we refer to the example of (anti-)instantons considering an ensemble of these quasi-classical configurations. On our way to construct an effective theory (which usually encodes the predictions of a quantum field theory at low energies) the assumptions done are not of special importance. However, what is entirely restrictive to fix the effective action at really low energy (i.e. low cutoff) up to a few coupling constants is an idea to neglect all the contributions coming from gluon fields \( A_{ex} \) generated by the (anti-)quarks.

\[ A_{ex} \ll A . \]

Actually, it means the removal of corresponding cutoff(s) from consideration, but by the definition of an effective theory this operation does not pose itself. Then the corresponding Hamiltonian description results from

\[ \mathcal{H} = \pi \dot{q} - \mathcal{L}_E , \quad \pi = \frac{\partial \mathcal{L}_E}{\partial \dot{q}} = i q^+ , \]

(2)

and

\[ \mathcal{H}_0 = -\bar{q} \left( i\gamma_\nabla + im \right) q , \]

(3)

for noninteracting quarks. In Schrödinger representation the quark field evolution is determined by the equation for the quark probability amplitude \( \Psi \) as

\[ \dot{\Psi} = -\mathcal{H}\Psi , \]

(4)

with the density of interaction Hamiltonian

\[ \mathcal{V}_S = \bar{q}(x) t^a \gamma_\mu A^a_\mu(t, x) q(x) . \]

(5)

The explicit dependence on “time” is present at the gluon field only. The creation and annihilation operators of quarks and anti-quarks \( a^+, a, b^+, b \) have no “time” dependence and consequently

\[ q_{\alpha i}(x) = \int \frac{dp}{(2\pi)^3 (2|\mathbf{p}_4|)^{1/2}} \left[ a(\mathbf{p}, s, c) \ u_{\alpha i}(\mathbf{p}, s, c) e^{i\mathbf{p}\mathbf{x}} + b^+(\mathbf{p}, s, c) \ v_{\alpha i}(\mathbf{p}, s, c) e^{-i\mathbf{p}\mathbf{x}} \right] . \]

(6)
The stochastic character of gluon field (which we supposed) allows us to develop the approximate description of the state $\Psi$ if the following procedure of averaging

$$\Psi \rightarrow \langle \Psi \rangle = \int_0^t d\tau \frac{\Psi(\tau)}{t}$$

is introduced. With this procedure taken the further step is to turn to the approach of constructing a density matrix $\langle \hat{\Psi} \hat{\Psi} \rangle$. However, here we believe that at calculating the ground state (or more generally with quasi-stationary state) it might be sufficiently informative to operate with the averaged amplitude directly. Then in the interaction representation $\Psi = e^{H_{st}t}\Phi$ we have the equation for state $\Phi$ as

$$\dot{\Phi} = -V\Phi, \quad V = e^{H_{st}t}V_se^{-H_{st}}. \quad (7)$$

Now the "time" dependence appears in quark operators as well and after averaging over the short-wavelength component one may obtain the following equation

$$\langle \dot{\Phi}(t) \rangle = + \int_0^\infty d\tau \langle V(t)V(t-\tau) \rangle \langle \Phi(t) \rangle. \quad (8)$$

The limitations to have such a factorization validated are well known in the theory of stochastic differential equations (see, for example, [1]). The integration interval in Eq.(8) may be extended to the infinite "time" because of the rapid decrease (supposed) of the corresponding correlation function. Now we are allowed to deal with amplitude $\langle \Phi(t) \rangle$ in the right hand side of Eq.(8) instead the amplitude with the shifted arguments in order to get an ordinary integro-differential equation. In the quantum field theory applications it is usually difficult to construct the correlation function in the most general form. However, if we are going to limit our interest by describing the long-wavelength quark component only then gluon field correlator $\langle A^a_i(x)A^b_j(y) \rangle$ may be factorized and as a result we have

$$\langle \dot{\Phi}(t) \rangle = \int d^4x \bar{q}(x,t) t^a\gamma_\mu q(x,t) \int_0^\infty d\tau \int d^4y \bar{q}(y,t-\tau) t^b\gamma_\nu q(y,t-\tau) g^2 \langle A^a_i(t,x)A^b_j(t-\tau,y) \rangle \langle \Phi(t) \rangle.$$

Having assumed the correlation function rapidly decreasing in "time" we could ignore all the retarding effects in the quark operators. Turning back to the Schrödinger representation we have for the state amplitude $\chi = e^{-H_{st}t}\langle \Phi \rangle$ the following equation

$$\dot{\chi} = -H_{ind} \chi,$$

$$H_{ind} = -\bar{q} (i\gamma \nabla + im) q - \bar{q} t^a\gamma_\mu q \int d^4y \bar{q}'(y) t^b\gamma_\nu q' \int_0^\infty d\tau g^2 \langle A^a_i A^b_j \rangle,$$

with $q = q(x)$, $\bar{q} = \bar{q}(x)$, $q' = q(y)$, $\bar{q}' = \bar{q}(y)$ and $A^a_i = A^a_i(t,x)$, $A^b_j = A^b_j(t-\tau,y)$. Now the correlation function might be presented as

$$\int_0^\infty d\tau g^2 \langle A^a_i A^b_j \rangle = \delta^{ab} F_{\mu\nu}(x-y),$$

with the corresponding formfactors $F_{\mu\nu}(x-y) = \delta_{\mu\nu} I(x-y) + J_{\mu\nu}(x-y)$. In our consideration we ignore the contribution of the second formfactor spanning on the components of the vector $x-y$. Thus, on output we receive the Hamiltonian of four-fermion interaction with the formfactor rooted in the presence of two quark currents in the points $x$ and $y$. With this form of the effective Hamiltonian we could apply the Hartree–Fock–Bogolyubov method [2] to find its ground state as one constructed by the quark–anti-quark pairs with the oppositely directed momenta

$$|\sigma\rangle = T \ |0\rangle,$$

$$T = \Pi_{p,s,c} \exp \left\{ \frac{\theta}{2} \left[ a^+(p, s, c) b^+(-p, s, c) + a(p, s, c) b(-p, s, c) \right] \right\}.$$

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where the parameter $\theta(p)$ characterizes the pairing strength. Introducing the creation and annihilation operators of quasi-particles $A = T a T^{-1}$, $A^+ = T a^{+T} T^{-1}$, $B = T b T^{-1}$, $B^+ = T b^{+T} T^{-1}$, we can rewrite the quark (anti-quark) operators as

\[ q(x) = \int \frac{dp}{(2\pi)^3} \frac{1}{|p_i|^{1/2}} \left[ A(p, s, c) U(p, s, c) e^{ipx} + B^+(p, s, c) V(p, s, c) e^{-ipx} \right], \]

\[ \bar{q}(x) = \int \frac{dp}{(2\pi)^3} \frac{1}{|p_i|^{1/2}} \left[ A^+(p, s, c) \bar{U}(p, s, c) e^{-ipx} + B(p, s, c) \bar{V}(p, s, c) e^{ipx} \right], \]

with the quasi-particle spinors

\[ U(p, s, c) = \cos \left( \frac{\theta}{2} \right) u(p, s, c) - \sin \left( \frac{\theta}{2} \right) v(-p, s, c), \]

\[ V(p, s, c) = \sin \left( \frac{\theta}{2} \right) u(-p, s, c) + \cos \left( \frac{\theta}{2} \right) v(p, s, c), \]

where $U(p, s, c) = U^+(p, s, c) \gamma_4$, $\bar{V}(p, s, c) = V^+(p, s, c) \gamma_4$. Minimizing the mean energy functional one is able to determine the angle $\theta$ magnitude

\[ \frac{d\langle \sigma | H_{\text{ind}} | \sigma \rangle}{d\theta} = 0. \]

Dropping the calculation details out we present here the following result for the mean energy as a function of the $\theta$ angle

\[ \langle \sigma | H_{\text{ind}} | \sigma \rangle = - \int \frac{dp}{(2\pi)^3} \frac{2N_c p_i^2}{|p_i|} (1 - \cos \theta) - \tilde{G} \int \frac{dp dq}{(2\pi)^6} \left\{ -(3 \bar{I} - \bar{J}) \frac{p_i q_i}{|p_i||q_i|} + (4 \bar{I} - \bar{J}) \frac{p_i q_i}{|p_i||q_i|} \left( \sin \theta - \frac{m}{p} \cos \theta \right) \left( \sin \theta' - \frac{m}{q} \cos \theta' \right) + (2 \bar{I} \delta_{ij} - 2 \bar{J}_{ij} + \bar{J} \delta_{ij}) \frac{p_i q_j}{|p_i||q_i|} \left( \cos \theta + \frac{m}{p} \sin \theta \right) \left( \cos \theta' + \frac{m}{q} \sin \theta' \right) \right\}, \]

where the following designations are used $p = |p|$, $q = |q|$, $\bar{I} = \bar{I}(p + q)$, $\bar{J}_{ij} = \bar{J}_{ij}(p + q)$, $\bar{J} = \sum_{i=1}^3 \bar{J}_{ii}$, $p^2 = q^2 = -m^2$, $\theta' = \theta(q)$ where $\tilde{G}$ is the constant of corresponding four-fermion interaction (the relevant details can be found in [3]). The first integral in Eq. (13) comes from free Hamiltonian, and we make a natural subtraction (adding the unit) in order to have zero mean free energy when the angle of pairing is trivial.

**Nambu–Jona-Lasinio model**

In order to get an idea of the parameter scales we continue with handling the model in which the formfactor behaves in the coordinate space as $I(x - y) = \delta(x - y)$, $J_{\mu\nu} = 0$, dropping contribution spanned on the $p_i q_j$ tensor also. Actually, it corresponds to the Nambu–Jona-Lasinio model [4]. As well known the model with such a formfactor requires the regularization and, hence, the cutoff parameter $\Lambda$ comes to the play

\[ W = \int^{\Lambda} \frac{dp}{(2\pi)^3} \left[ |p_i| (1 - \cos \theta) - \tilde{G} \frac{p}{|p_i|} \left( \sin \theta - \frac{m}{p} \cos \theta \right) \int^{\Lambda} \frac{dq}{(2\pi)^3} \frac{q}{|q_i|} \left( \sin \theta' - \frac{m}{q} \cos \theta' \right) \right]. \]

We adjust the NJL model with the parameter set given by Hatsuda and Kunihiro [4] in which $\Lambda = 631\text{MeV}$, $m = 5.5\text{MeV}$. One curious point of this model is that the solution for optimal angle $\theta$ in the whole interval $p \in [0, \Lambda]$ can be found by solving the simple trigonometrical equation

\[ (p^2 + m^2) \sin \theta - M_q (p \cos \theta + m \sin \theta) = 0, \]
Figure 1: Phase portrait of the Keldysh model, \( \sin \theta \) as a function of momentum \( p \) (MeV). The dotted curve corresponds to the solution with the negative values of angle in the chiral limit \( m = 0 \).

with the dynamical quark mass

\[
M_q = 2G \int^\Lambda \frac{dp}{(2\pi)^3} \frac{p}{|p_4|} \left( \sin \theta - \frac{m}{p} \cos \theta \right).
\]

Eventually the results obtained look like \( M_q = -335 \) MeV for dynamical quark mass and \( \langle \sigma|\bar{q}q|\sigma \rangle = -i (245 \text{ MeV})^3 \) for the quark condensate with the following definition of the quark condensate

\[
\langle \sigma|\bar{q}q|\sigma \rangle = i \frac{N_c}{\pi^2} \int_0^\infty dp \frac{p^2}{|p_4|} \left( p \sin \theta - m \cos \theta \right).
\]

**The Keldysh model**

Now we are going to analyse the limit, in some extent, opposite to the NJL model, i.e. we are dealing with the formfactor behaving as a delta function but in the momentum space (analogously the Keldysh model, well known in the physics of condensed matter [5]), \( I(p) = (2\pi)^3 \delta(p) \). Here the mean energy functional has the following form

\[
W(m) = \int \frac{dp}{(2\pi)^3} \left| p_4 \right| \left( 1 - \cos \theta \right) - G \frac{p^2}{|p_4|^2} \left( \sin \theta - \frac{m}{p} \cos \theta \right)^2.
\]

contrary to the NJL model there is no need to introduce any cut off. The equation for calculating the optimal angle \( \theta \) becomes the transcendental one

\[
\left| p_4 \right|^3 \sin \theta - 2G \left( p \cos \theta + m \sin \theta \right) \left( p \sin \theta - m \cos \theta \right) = 0,
\]

and, clearly, it is rather difficult to get its solution in a general form. Fortunately, it is much easier and quite informative to analyse the model in the chiral limit \( m = 0 \). There exist one trivial solution \( \theta = 0 \) and two nontrivial ones (for the positive and negative angles) which obey the equation

\[
\cos \theta = \frac{p}{2G}.
\]

Obviously, these solutions are reasonable if the momentum is limited by \( p < 2G \). Then for the mean energy we have \( W_\pm(0) = -\frac{G^4}{15\pi^2} \) if the quark condensate defined as \( \langle \sigma|\bar{q}q|\sigma \rangle(0) = \frac{i N_c G^3}{2\pi} \). For
the trivial solution the mean energy equals to zero together with the quark condensate \( W_0(0) = 0, \langle \sigma | \bar{q} q | \sigma \rangle_0(0) = 0 \). Introducing the practical designation \( \sin \theta = \frac{M_\theta}{(p^2 + M_\theta^2)^{1/2}} \) which characterizes the pairing strength by the parameter \( M_\theta \) we have, for example, for the nontrivial solution \( M_\theta = (4G^2 - p^2)^{1/2} \). In order to compare the results with the NJL model we fixed the value of four-fermion interaction constant as \( M_\theta(0) = 2G = 335 \text{ MeV} \). It is interesting to notice that the respective energy becomes constant \( E(p) = \sqrt{p^2 + M^2_\theta}, E(p) = 2G \).

Figure 2: The optimal angle \( \theta \) as a function of momentum \( p \)(MeV). The solid line corresponds to the NJL model and the dashed one to the Keldysh model. The current quark mass is \( m = 5.5 \text{ MeV} \) and \( p_\theta \sim 40 \text{ MeV} \).

After having done the analysis in the chiral limit which is shown by the dotted line in Fig.1 we would like to comment the situation beyond this limit. The evolution of corresponding branches is available on the same plot \( \overset{\text{1}}{\text{1}} \). One solution denoted by A is developing in the local vicinity of coordinate origin and for small values of quark mass this domain is practically indistinguishable. In order to make it noticeable (to have a reasonable resolution on the plot) the quark mass was put as \( m = 50 \text{ MeV} \). Besides, there are two solutions \( a \) and \( b \) in the domain denoted by \( I \), three solutions denoted by 1, 2, 3 in the domain \( II \) and one solution \( B \) in the domain \( III \). The minimum of mean energy functional can be realized with the piecewise continuous functions. At the local vicinity of coordinate origin we start with the solution branch A, then relevant solution passes to the branch \( a \) or \( b \) interchanging its position from \( a \) to \( b \) in any subinterval. But in any case there is only one way to continue the solution at streaming to the infinite limit and it is related with the branch \( B \) where the angle is going to the zero value. As to the functional \( \overset{\text{1}}{\text{5}} \) the contribution of the term proportional to the cosine in the second parenthesis is divergent even if the angle \( \theta \) is zero. It means the mean energy out of chiral limit goes to an infinity at any nonzero value of quark mass. The same conclusion is valid for the chiral condensate. In principle this functional could be regularized and corresponding continuation might be done but it is out of this presentation scope. It is not difficult to demonstrate the similar discontinuities of functional are present, for example, for Gaussian \( I(x) = G \exp(-a^2 x^2) \), and exponential \( I(x) = G \exp(-a|x|) \), formfactors and they are present even in the NJL model but this fact is masked by the cut off parameter.

Comparing the optimal angles in the NJL and Keldysh models (see Fig. 2) it is interesting to notice that the formation of quasiparticles becomes significant at some momentum value close to the origin \( p_\theta \sim 40 \text{ MeV} \) but not directly at the zero value. It is clear the inverse value of this parameter determines the characteristic size of quasiparticle. Parameter \( M_\theta \) as a function of
The parameter $M_\theta$(MeV) as a function of momentum $p$(GeV) corresponding to the best fit of the NJL data $M_q = 335$ MeV, $\langle \sigma|\bar{q}q|\sigma \rangle = -i (245$ MeV)$^3$. The solid line corresponds to the Gaussian formfactor in chiral limit and the dashed line corresponds to the magnitude of current quark mass $m = 5.5$ MeV. The exponentially behaving formfactor is represented by the dotted lines and $p_\theta \sim 150$ MeV.

momentum $p$ corresponding to the best fit to the NJL data $M_q = 335$ MeV, $\langle \sigma|\bar{q}q|\sigma \rangle = -i (245$ MeV)$^3$ is shown in Fig.3. The solid line corresponds to the Gaussian formfactor in the chiral limit and the dashed one shows the same dependence for the current quark mass $m = 5.5$ MeV. This dependence for exponential behaviour of formfactor is presented by the dotted lines on the same plot (the characteristic angle is $p_\theta \sim 150$ MeV in this case). Analysing the discontinuity of mean energy functional and quark condensate we face some troubles at fitting the quark condensate, for example. However, the dynamical quark mass and quark condensate are nonobservable quantities and it is curious to remark here that although the mean energy of the quark system is minus infinity the meson observables are finite and even in Keldysh model the mesons are recognizable with reasonable scale and we can in principle make a fit for this observables.

References

[1] N. G. Van Kampen, Phys. Rep. 24 (1976) 171.

[2] N. N. Bogolyubov, Izv. AN. USSR. Sect. phys., 11 (1947) 77.

[3] S. V. Molodtsov, G. M. Zinovjev, Theor. Math. Phys., to be published (2009).

[4] M. K. Volkov, A. E. Radzhabov, Phys. Usp. 176 (2006) 569;
   M. K. Volkov, Sov. J. Part. and Nucl., 17 (1986) 433;
   T. Hatsuda and T. Kunihiro, Phys. Rep. 247 (1994) 221.

[5] L. V. Keldysh, Doctor. Thesis (FIAN, 1965);
   E. V. Kane, Phys. Rev. 131 (1963) 79;
   V. L. Bonch-Bruevich, in "Physics of solid states", M., VINITI, 1965.