ON THE CLASSIFICATION PROBLEM FOR C*-ALGEBRAS

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Abstract

In the given article, we discuss the problem of the classification of general C*-algebras. There arises a question: whether can we generalize the definition of the notion of von Neumann algebra of type I so that we can apply this generalized definition to all C*-algebras? In the given article there was given an affirmative answer for this question. Namely, it was introduced a new notion of C*-algebra of von Neumann type I.

It is proved that any GCR-algebra is a C*-algebra of von Neumann type I, and any C*-algebra is a NGCR-algebra if and only if this C*-algebra does not have a nonzero abelian annihilator.

Also in the article there were proved that for a C*-algebra A there exist such unique C*-subalgebras A_I, A_{II}, A_{III} that A_I is a C*-algebra of von Neumann type I, there does not exist a nonzero abelian annihilator in the algebras A_{II} and A_{III}, the lattice \mathcal{P}_{A_{II}} of annihilators of A_{II} is locally modular, the lattice \mathcal{P}_{A_{III}} of annihilators of A_{III} is purely nonmodular. Moreover A_I \oplus A_{II} \oplus A_{III} is a C*-subalgebra of A and the annihilator of A_I \oplus A_{II} \oplus A_{III} is the set \{0\}, i.e. Ann_A(A_I \oplus A_{II} \oplus A_{III}) = \{0\}.

In the final part of the article there were introduced notions of C*-algebra of type I_n, C*-algebra of types II, II_1, II_\infty and III. Then we have proven that: any simple C*-algebra of von Neumann type I is a C*-algebra of type I_n for some cardinal number n, any C*-algebra of type II_1 is finite, any simple purely infinite C*-algebra is of type III and any W*-factor of type II_\infty has a proper ideal J such that J is a simple C*-algebra of type II_\infty. Finally it has been formulated a classification theorem for simple C*-algebras.

Introduction

In the theory of operator algebras the existence of projections in an algebra is important. It was developed a large theory using projections. In particular, it was investigated the problem of the classification of von Neumann algebras and AW*-algebras. A similar classification for general C*-algebras was not developed, because there does not exist necessary quantity of projections in these algebras.

There exist the definitions of C*-algebras of type I and GCR-algebras introduced by Dixmier and Kaplansky. It is known that these definitions are equivalent (see II).

In the given article, we discuss the classification problem of general C*-algebras. The situation around this problem is: there are notions of C*-algebras of type I, purely infinite C*-algebras, finite C*-algebras and properly infinite C*-algebras for general C*-algebras. But in many scientific papers all these notions are considered for simple C*-algebras. There is not a correspondence between C*-algebras of type I and von Neumann algebras of type I. Because, on the one hand, we can not apply the definition of von Neumann algebra of type I to C*-algebras. On the other hand
not any von Neumann algebra of type I is a C*-algebra of type I. There arises a question: whether can we generalize the definition of von Neumann algebra of type I so that we can apply this generalized definition to all C*-algebras? In the given article we give an affirmative answer for this question.

The next problem concerns the notions of von Neumann algebra of type II and von Neumann algebra of type III. In the situation with the notions of "finite" "infinite" "properly infinite" and "purely infinite" C*-algebras there exists some correspondence between C*-algebras and von Neumann algebras (in particular, these notions are discussed in [2]), that is in the case of von Neumann algebras the definitions of these notions are equivalent to the definitions of the same notions for von Neumann algebras. But there do not exist notions of algebra of type II and algebra of type III for general C*-algebras. In the given article we also discuss the generalization problem of the notions of von Neumann algebra of type II and von Neumann algebra of type III in the case of general C*-algebras.

In this paper we offer a new definition of the notion of C*-algebra of type I. For this propose we consider the set $\mathcal{P}$ of all annihilators of the positive elements' subsets of the given C*-algebra. We proved that this set is a complete orthomodular lattice. In the article an annihilator which is a two sided ideal is called a central annihilator and an annihilator, which is a commutative C*-algebra is called abelian annihilator. Note that in the case of AW*-algebras and von Neumann algebras this set $\mathcal{P}$ coincides with the set of all projections of the given algebra. Also, note that in the case of general C*-algebras the set $\mathcal{P}$ can play the role of the set of all projections. For the elements of the set $\mathcal{P}$ we introduced, also, such notions as orthogonality of two elements, equivalence of two elements, abelian annihilator and locally modularity of the lattice $\mathcal{P}$ and so on.

To generalize the notion of von Neumann algebra of type I we introduced a new notion of C*-algebra of type I as follows: a C*-algebra $A$ is called of von Neumann type I, if there exists such abelian annihilator $V \in \mathcal{P}$ that the largest lower bound $c(V)$ of the set of all central annihilators, containing $V$, coincides with $A$, i.e. $c(V) = A$. It is proved that any C*-algebra of type I in terms of known definition is a C*-algebra of type I in terms of this new definition, that is it is proven that any GCR-algebra is a C*-algebra of von Neumann type I, and any C*-algebra is a NGCR-algebra if and only if this C*-algebra does not have a nonzero abelian annihilator.

Also in this article the next statement have been proved: for any C*-algebra $A$ there exist such unique C*-subalgebras $A_I$, $A_{II}$, $A_{III}$ of $A$ that $A_I$ is a C*-algebra of von Neumann type I, there does not exist a nonzero abelian annihilator in the algebras $A_{II}$ and $A_{III}$, the lattice $\mathcal{P}_{A_{II}}$ of annihilators of $A_{II}$ is locally modular, the lattice $\mathcal{P}_{A_{III}}$ of annihilators of $A_{III}$ is purely nonmodular. Moreover $A_I \oplus A_{II} \oplus A_{III}$ is a C*-subalgebra of $A$ and the annihilator of $A_I \oplus A_{II} \oplus A_{III}$ is the set $\{0\}$, i.e. $\text{Ann}_A(A_I \oplus A_{II} \oplus A_{III}) = \{0\}$.

In the final part of the article there were introduced notions of C*-algebra of type $I_n$, C*-algebra of types $II$, $III$, $V_\infty$ and III. Then we have proven that: any simple C*-algebra of von Neumann type I is a C*-algebra of type $I_n$ for some cardinal number $n$, any C*-algebra of type $II_1$ is finite, any purely infinite C*-algebra is of type III and any W*-factor of type $V_\infty$ has a proper ideal $J$ such that $J$ is a simple C*-algebra of type $II_\infty$. 
1. Annihilators of a C*-algebra

Let $A$ be a unital C*-algebra. Recall that $A_{sa} = \{ a \in A : a^* = a \}$ and $A = A_{sa} + iA_{sa}$. $A_{sa} \cap iA_{sa} = \{ 0 \}$. Also $Ann_r(S) = \{ a \in A : sa = 0 \text{ for all } s \in S \}$, $Ann_l(S) = \{ a \in A : as = 0 \text{ for all } s \in S \}$ for $S \subseteq A$.

**Lemma 1.** Let $A$ be a unital C*-algebra and $a, b \in A$. Then

1) if $a \in A_+$, $b \in A_{sa}$ then the next conditions are equivalent
   (a) $ab + ba = 0$
   (b) $ab = 0$
   (c) $ba = 0$;

2) if $a \in A_+$, $b \in A$ then $ab + ba = 0$ if and only if $ab = ba = 0$.

**Proof.** 1) (a)$\Rightarrow$(b), (c): We have $ab = -ba$ and $aba = -ba^2$, $-a^2b = aba$, that is $a^2b = ba^2$. Then the elements $a^2$ and $b$ commute. There exists a maximal commutative C*-subalgebra $A_0$, containing the elements $a^2, b$. Then since $a = \sqrt{a^2}$ we have $a \in A_0$. Hence $ab = ba = 0$. 
   (b)$\Rightarrow$(a), (c): Now suppose that $ab = 0$. Then $ba = (ab)^* = 0$ and $ba = 0$. Hence $ab + ba = 0$. The implication (c)$\Rightarrow$(a) is also obvious.

2) Let $b = x + iy$, $x, y \in A_{sa}$. We have $ab + ba = ax + iay + xa + iya = 0$ and $b^*a + ab^* = xa - iya + ax - iay = 0$. Hence $ab + ba + b^*a + ab^* = 2(ax + xa) = 0$, that is $ax + xa = 0$. Analogously $ay + ya = 0$. By item 1) of lemma 1 $ax = xa = 0$, $ay = ya = 0$. Therefore $ba = ab = 0$.

The converse of the statement 2) is obvious. >

Let $A$ be a C*-algebra, $S \subseteq A$. Let $Ann(S) = Ann_A(S) = \{ a \in A : as + sa = 0, \text{ for all } s \in S \}$. The set $Ann(S)$ we will call annihilator of the set $S$.

Let $A$ be a unital C*-algebra and $a, b$ be elements of the set $A_{sa}$. Recall that, $A_+ = \{ a \in A_{sa} : \text{ there exists } b \in A \text{ such that } a = bb^* \}$. By lemma 1 for any set $S \subseteq A_+$ we have $Ann(S) = Ann_r(S) \cap Ann_l(S)$.

Let $A$ be a C*-algebra of bounded linear operators on a Hilbert space $H$. Then the weak closure in the algebra $B(H)$ of the set $B \subseteq A$ we denote by $w(B)$. Let $dV = \{ a \in A : xay + yax = 0, \text{ for any } x, y \in V \}$ for an arbitrary subset $V \subseteq A$. We will set an analog of decomposition on projections using annihilators for C*-algebras. First we prove the next useful lemma.

**Lemma 2.** Let $A$ be a C*-algebra. Then for any subset $S \subseteq A_+$, the sets $Ann(S)$, $Ann(Ann(S))$ are C*-subalgebras and $xAx \subseteq Ann(S)$, $yAy \subseteq Ann(Ann(S))$ for any elements $x \in Ann(S)$, $y \in Ann(Ann(S))$. The set $d(Ann(Ann(S))) \cap d(Ann(S))$ is a Banach space.

**Proof.** We prove that $Ann(S)$ is a C*-algebra. Let $a, b \in Ann(S)$. Then by lemma 1 $s(ab) + (ab)s = (sa)b + a(bs) = 0$ for any $s \in S$. Hence $ab \in Ann(S)$. Since the elements $a$ and $b$ are chosen arbitrarily then the set $Ann(S)$ is an associative algebra. Also by separately continuity of the associative multiplication on the norm $Ann(S)$ is a Banach algebra. Note that all conditions of the definition of C*-algebra hold for the algebra $Ann(S)$. Hence $Ann(S)$ is a C*-algebra.

By the previous part of the proof we have $Ann(S) = Ann(S)_+ + Ann(S)_+$. It is obvious that $Ann(Ann(S)) \subseteq Ann(Ann(S)_+)$. Let $a \in Ann(Ann(S)_+)$. In this case, if $s \in Ann(S)_+$, then $s = x + iy$, $x, y \in Ann(S)_+$ and $ax + xa = ay + ya = 0$. Hence $as + sa = 0$. Therefore $a \in Ann(Ann(S)_+)$. So $Ann(Ann(S)) = Ann(Ann(S)_+)$. Consequently, $Ann(Ann(S))$ is a C*-algebra.
It is clear that the sets $d(Ann(Ann(S)))$ and $d(Ann(S))$ are linear space. By the separately continuity of the associative multiplication on the norm they are Banach spaces. Then $d(Ann(Ann(S))) \cap d(Ann(S))$ is also a Banach space.

By lemma 1 and associativity of the multiplication of the algebra $A \times Ax \subseteq Ann(S)$, $yAy \subseteq Ann(Ann(S))$ for any elements $x \in Ann(S)$, $y \in Ann(Ann(S))$.

**Lemma 3.** Let $A$ be a C*-algebra of bounded linear operators on a Hilbert space $H$, $w(A)$ be the weak closure of the algebra $A$ in the algebra $B(H)$ of all bounded linear operators in $H$. Then for any subset $S \subseteq A_+$ the next conditions hold:

(a) There exist projections $f, e$ in $w(A)$ such that

- $w(Ann(Ann(S))) = ew(A)e$, $w(Ann(S)) = fw(A)f$ and $w(d(Ann(Ann(S)))) \cap d(Ann(S))) = (fw(A)f + ew(A)e)$,
- $Ann(S) = (fw(A)f \cap A$, $Ann(Ann(S)) = (ew(A)e \cap A$ and $d(Ann(Ann(S))) \cap d(Ann(S)) = [ew(A)e \circ f] \cap A$;

(b) $Ann[Ann(Ann(S)) \cap d(Ann(S))] \cap Ann(S)] = \{0\}$.

**Proof.** (a) Since $Ann(S)$ is a C*-algebra (lemma 2) there exists an approximative units $(u_\lambda)$ in $Ann(S)$ such that $(\forall \lambda)\|u_\lambda\| \leq 1$, $(\forall \lambda \leq \mu)u_\lambda \leq u_\mu$ and $(\forall a \in A_\lambda)\|u_\lambda \circ a - a\| \to 0$. We calculate $sup u_\lambda$ in $w(A)$. By the definition of $(u_\lambda)$ $(\forall \lambda)\|u_\lambda \circ u_\mu - u_\mu\| \to 0$. Then the net $(u_\lambda \circ u_\mu)$ weakly converges to $u_\mu$ at $\lambda \to \infty$ for any $\mu$. At the same time, since $(u_\lambda)$ weakly converges to the element $sup u_\lambda$ (sup is took in $w(A)$), then the net $(u_\lambda \circ u_\mu)$ weakly converges to $sup u_\lambda \circ u_\mu$ at $\lambda \to \infty$ in fixed $\mu$. Hence $(\forall \lambda)sup u_\lambda \circ u_\mu = u_\mu$. Therefore the net $(sup u_\lambda \circ u_\mu)$ weakly converges to $sup u_\mu$. Also the net $(sup u_\lambda \circ u_\mu)$ weakly converges to $sup u_\lambda \circ sup u_\mu$. Hence $sup u_\mu = sup u_\lambda \circ sup u_\mu = [sup u_\mu]^2$. So, $sup u_\mu$ is a projection in $w(A)$. Let $g := sup u_\mu$.

By the definition of $(u_\lambda)$ the net $(sup u_\lambda)$ weakly converges to $s$ for any $s \in Ann(S)$, and, at the same time $(sup u_\lambda)$ weakly converges to $g\circ s$. Hence $(\forall s \in Ann(S))g\circ s = s$. Let $f = sup\{r(s) : s \in Ann(S)\}$ (in $w(A)$). Then $f \leq g$. Note that $Ann(S) \subseteq U_f(w(A))$. Hence, $(\forall \lambda)f \circ u_\lambda = u_\lambda$. Therefore $f \circ g = g$ and $f \geq g$. Consequently, $f = g$.

Now let $a$ be an arbitrary element in $U_f(w(A))$. Then there exists a net $(u_\alpha)$ in $A$ weakly converging to $a$. Note, then the net $(\{u_\lambda a_\alpha u_\mu\})$ weakly converges to $(u_\lambda a_\alpha u_\mu)$ in fixed $\lambda$ and $\mu$. It is easy to see, that $(u_\lambda a_\alpha u_\mu)$ weakly converges to $U_f a$, which belongs to $U_f(w(A))$. Since $a \in U_f(w(A))$, then $U_f a = a$. Hence, since the set $(\{u_\lambda a_\alpha u_\mu\})$ is a net in $Ann(S)$ relative to indexes $\alpha$, $\lambda$, and $\mu$, then $w(Ann(S)) = U_f(w(A))$.

Now, take the set $Ann(Ann(S))$. We have by lemma 3 $Ann(Ann(S))$ is a C*-subalgebra of the C*-algebra $A$. Also there exists approximative units $(v_\lambda)$ in $Ann(Ann(S))$. Let $g = sup v_\lambda$ and $e = sup\{r(s) : s \in Ann(Ann(S))\}$ (in $w(A)$)
Then the repeating of the above arguments give us that $g$ is a projection in $w(A)$ and $e = g$.

The proof of the second part of item a): Note that $r(a)r(b) = 0$ for any $a \in Ann(S)$ and $b \in Ann(Ann(S))$, where $r(c)$ is the range projection of an element $c \in w(A)$. Let $e = sup\{r(a) : a \in Ann(Ann(S))\}$. Then $f = sup\{r(b) : b \in Ann(S)\}$. By the definitions of the projections $e$, $f$ we have $ef = 0$.

Let $Ann_{w(A)}(S)$ be the annihilator of the set $S$ in the algebra $w(A)$. Then there exists a projection $p$ of $w(A)$ such that $Ann_{w(A)}(S) = pw(A)p$. At the same time
we have \( \text{Ann}(S) \subseteq \text{Ann}_{w(A)}(S) \) and \( \text{Ann}(S) = \text{Ann}_{w(A)}(S) \cap A \). Then \( \text{Ann}(S) = pw(A)p \cap A \). Hence \( f \leq p \), \( \text{Ann}(S) = f w(A)f \cap A \) and \( \text{Ann}(Ann(S)) = ew(A)e \cap A \).

It can be straightforwardly proved that \( d(\text{Ann}(Ann(S))) \cap d(Ann(S)) = [ew(A)f \oplus f w(A)e] \cap A \). Note that for any \( x \in \text{Ann}(S) \) and \( y \in \text{Ann}(Ann(S)) \) we have \( xAy + yAx \subseteq d(\text{Ann}(Ann(S))) \cap d(Ann(S)) \). Hence \( \text{Ann}(S) \neq \{0\} \) and \( \text{Ann}(Ann(S)) \neq \{0\} \) if \( d(\text{Ann}(Ann(S))) \cap d(Ann(S)) \neq \{0\} \), but not only if, because may hold case when \( A = \text{Ann}(S) \oplus \text{Ann}(Ann(S)) \).

Item (b) follows by the equality \( \text{Ann}[\text{Ann}(Ann(S)) \oplus \text{Ann}(S)] = \{0\} \). \( \triangleright \)

**Corollary 4.** Let \( A \) be a \( C^* \)-algebra of bounded linear operators on a Hilbert space \( H \), \( w(A) \) be the weak closure of the algebra \( A \) in the algebra \( B(H) \) of all bounded linear operators in \( H \). Then for any subset \( S \subseteq A_+ \) if \( d(\text{Ann}(Ann(S))) \cap d(Ann(S)) = \{0\} \), then the annihilators \( \text{Ann}(Ann(S)) \), \( \text{Ann}(Ann(S)) \) are two sided ideals. In this case there exist central projections \( f, e \) in \( w(A) \) such that \( w(Ann(Ann(S))) = \text{ew}(A)e \), \( w(Ann(S)) = \text{fw}(A)f \).

*Proof.* By lemma 3 there exist projections \( f, e \) in \( w(A) \) such that \( w(Ann(Ann(S))) = \text{ew}(A)e \) and \( w(Ann(S)) = \text{fw}(A)f \). By the condition and weakly continuity of the multiplication we have \( w(d(Ann(Ann(S))) \cap d(Ann(S))) = \{0\} \). Let \( p = e + f \). Then

\[
pw(A)p = \text{ew}(A)e \oplus \text{fw}(A)f,
\]

and \( e, f \) are central projections of the algebra \( pw(A)p \).

We assert that the map \( \phi : A \rightarrow pAp \), defined as \( \phi(a) = pap \), for all \( a \in A \), is a one-to-one correspondence between the space \( pAp \) and the algebra \( A \). Indeed, let \( a, b \) be two elements of \( A \). Suppose \( \phi(a) = \phi(b) \), i.e. \( pap = pbp \). Let \( x = a - b \), \( C^*(x) \) be a \( C^* \)-algebra, generated by \( x \). It is clear, that \( pC^*(x)p = 0 \) by the continuity of the associative multiplication. Let \( C^*(x) = \{ y \in C^*(x) : y^* = y \} \) and \( C^*(x)_+ = \{ y \in C^*(x) : y = zz^* \}, for some \( z \in C^*(x) \}. \) Then \( C^*(x) = C^*(x)_{sa} + iC^*(x)_{sa} \) and \( C^*(x)_{sa} = C^*(x)_+ - C^*(x)_- \) \( \). We have \( py = 0 \) for any \( y \in C^*(x)_+ \). Hence \( py + yp = 0 \) for any \( y \in C^*(x)_+ \). Therefore \( y \in \text{Ann}(Ann(S) \oplus \text{Ann}(Ann(S))) \) \( \). By the continuity of the associative multiplication \( py + yp = 0 \) for any \( y \in C^*(x)_+ \). Hence \( C^*(x) \subseteq \text{Ann}(Ann(S) \oplus \text{Ann}(Ann(S))) \). Since \( C^*(x) \subseteq A \) and \( \text{Ann}(Ann(S) \oplus Ann(Ann(S))) = \{0\} \), then this is a contradiction if \( x \neq 0 \). So, \( x = 0 \) and \( a = b \).

Hence, since the elements \( a, b \) are chosen arbitrarily, then the map \( \phi : A \rightarrow pAp \) is a one-to-one correspondence.

Now we prove that \( Ann(S) \) is a closed two sided ideal of \( A \). Let \( s \) be an arbitrary element of \( Ann(S) \) and \( a \) be an arbitrary element of \( Ann(Ann(S)) \). Then, let \( v \) be an arbitrary element of \( Ann(Ann(S)) \). Then, since \( p, e, f \) are central projections of the algebra \( pw(A)p \) and \( psap \in pw(A)p \) we have \( (psap)v = v(psap) = ef(psap)v = 0 \), \( v(psap) = v(pesap) = 0 \). Hence \( p(sav + vsa)p = psap + vpsap = psapv + vpsap = 0 \). Note that \( sav + vsa \in A \). At the same time by the previous part of the proof for any \( a \in A \) \( pap = 0 \) if and only if \( a = 0 \). Hence \( sav + vsa = 0 \). Therefore since the element \( v \) is chosen arbitrarily, then \( sa \in Ann(Ann(S)) \). But \( Ann(Ann(Ann(S))) = Ann(S) \). Hence \( sa \in Ann(S) \). Hence, since the elements \( s, a \) are chosen arbitrarily, then \( Ann(S)A \subseteq Ann(S) \)

Analogously we have \( AAnn(S) \subseteq Ann(S) \) and \( Ann(S) \) is a closed two sided ideal of \( A \). Analogously we have \( Ann(Ann(S)) \) is also a closed two sided ideal of \( A \). Then by weakly continuity of the multiplication we have \( w(Ann(S)) \) and \( w(Ann(Ann(S))) \) are closed two sided ideals of the algebra \( w(A) \). Hence the projections \( p, e, f \) are central projections of the algebra \( w(A) \). \( \triangleright \)
2. Lattice of annihilators of a $C^*$-algebra

Recall that a lattice $L$ with zero $0$, a unit $1$ and an one parameter operation (orthocomplementation) $(\cdot)^\perp : L \to L$ is called ortholattice if $L$ satisfies the next conditions

1. $x \wedge x^\perp = 0, \ x \vee x^\perp = 1$;
2. $x^{\perp \perp} := (x^\perp)^\perp = x$;
3. $(x \vee y)^\perp = x^\perp \wedge y^\perp, \ (x \wedge y)^\perp = x^\perp \vee y^\perp$.

An ortholattice $L$ is called orthomodular lattice, if in this lattice the orthomodular law holds: for any $x, y \in L, x \leq y$, follows $y = x \vee (y \wedge x^\perp)$.

Let $x$ and $y$ be elements of an ortholattice $L$. If $x = (x \wedge y) \vee (x \wedge y^\perp)$ then we say $x$ commutes with $y$ and write $x \triangleleft y$. It is clear that $x \triangleleft y$ if $x \leq y$. The relation $\triangleleft$ is not a symmetric relation.

Recall that a lattice is said to be modular, if it follows from $x, z \in L \ x \leq z$ that for any $y \in L, x \vee (y \wedge z) = (x \vee y) \wedge z$.

A subset $B$ of an orthomodular lattice $L$ is called boolean subalgebra, if $B$ is a boolean algebra with induced lattice operations and orthocomplement in the sense of boolean complement. Maximal elements of the set of all boolean subalgebras of $L$ ordered by inclusion we call maximal boolean subalgebras of $L$. By Kuratovskiy-Zorn's lemma for any boolean subalgebra there exist a maximal boolean subalgebra containing this boolean subalgebra. But the next improved result holds.

An orthomodular lattice is a boolean algebra if and only if any two elements of this lattice are compatible.

The intersection of all maximal boolean subalgebras of an orthomodular lattice $L$ is called center $Z(L)$ of the orthomodular lattice $L$. It is clear that the center $Z(L)$ consists of elements compatible with all elements of $L$. The center of an orthomodular lattice is a boolean algebra.

An orthomodular lattice $L$ is said to be order complete, if for every subset $M \subseteq L$ there exists a least upper bound $\bigvee M := \sup(M)$ in $L$. Of course, in this case also there exists $\bigwedge M := \inf(M)$ and $\bigwedge M = \left( \bigvee_{x \in M} x^\perp \right)^\perp$.

The center $Z(L)$ of a complete orthomodular lattice $L$ is a complete boolean algebra.

Let $A$ be a $C^*$-algebra. We introduce a set $\mathcal{P}$ of annihilators of $A$ as

$$\mathcal{P} = \{ V \subseteq A : \text{there exists such } S \subseteq A_+ \text{ that } V = \text{Ann}(\text{Ann}(S)) \}.$$ 

Note that, since $\text{Ann}(\text{Ann}(\text{Ann}(S))) = \text{Ann}(S)$, then

$$\mathcal{P} = \{ V \subseteq A : \text{there exists such } S \subseteq A_+ \text{ that } V = \text{Ann}(S) \}.$$ 

For any two elements $V, W$ of $\mathcal{P}$, if $V \subseteq W$ then we write $V \leq W$. So we define an order in $\mathcal{P}$.

**Lemma 5.** Let $A$ be a $C^*$-algebra, $\mathcal{P}$ be the set of annihilators, defined above. Then $(\mathcal{P}, \leq)$ is a complete lattice.

*Proof.* Let $V, W \in \mathcal{P}$. Then there exist such $S, P \subseteq A$ that $V = \text{Ann}(\text{Ann}(S)), \ W = \text{Ann}(\text{Ann}(P))$. It is clear that $V, W \subseteq \text{Ann}(\text{Ann}(P \cup S))$. Let $Z \in \mathcal{P}$ such that $V \subseteq Z, W \subseteq Z$. Then there exists such $Q \subseteq A_+$ that $\text{Ann}(\text{Ann}(Q)) = Z$. We note that $\text{Ann}(\text{Ann}(\text{Ann}(Q))) \subseteq \text{Ann}(\text{Ann}(\text{Ann}(S)))$. At the same time, $\text{Ann}(\text{Ann}(\text{Ann}(S))) = \text{Ann}(S)$ and $\text{Ann}(\text{Ann}(\text{Ann}(Q))) = \text{Ann}(Q)$. Hence, $\text{Ann}(Q) \subseteq$
Ann(S). Analogously, \( Ann(Q) \subseteq Ann(P) \). Hence by the definition of \( Ann \) \( Ann(Q) \subseteq Ann(P \cup S) \). Therefore \( Ann(Ann(P \cup S)) \subseteq Ann(Ann((Q)) \). Since \( Z \) is chosen arbitrarily then \( V \lor W = Ann(Ann(P \cup S)) \).

Note that \( Ann(Ann(P) \cup Ann(S)) \subseteq V \cap W \). Let \( Z \in P \) such that \( Z \subseteq V \), \( Z \subseteq W \). Then there exists such \( Q \subseteq \Lambda_{+} \) that \( Ann(Ann(Q)) = Z \). By the definition of \( 'Ann' \) we have that \( Ann(Ann(Q)) \subseteq Ann(Ann(S \cup Ann(P)) \). Since \( Z \) is chosen arbitrarily then \( V \land W = Ann(Ann(P) \cup Ann(S)) \).

Note, if \( S \subseteq \Lambda_{+} \) then \( Ann(Ann(S) = \{0\} \), \( Ann(Ann(S)) = \{A\} \), i.e. \( Ann(S) \land Ann(Ann(S)) = \{0\} \).

Hence, the set \( P \) equipped with the order \( \subseteq \) is a lattice.

Let \( \{V_{i}\} \) be an arbitrary subset of \( P \). Then there exist \( \{S_{i}\} \subseteq \Lambda_{+} \) such, that \( Ann(Ann(S_{i})) = V_{i} \) for all \( i \). We have \( V_{i} \subseteq Ann(Ann(\cup S_{i})) \) for any \( i \). Let \( Z \in P \) such that \( V_{i} \subseteq Z \), for any \( i \). Then there exists such \( Q \subseteq \Lambda_{+} \) that \( Ann(Ann(Q)) = Z \). We note that \( Ann(Ann(Ann(Q))) = Ann(Ann(Ann(S_{i}))) \) for any \( i \). At the same time, \( Ann(Ann(Ann(S_{i}))) = Ann(S_{i}) \) and \( Ann(Ann(Ann(Q))) = Ann(Q) \). Hence, \( Ann(Q) \subseteq Ann(S_{i}) \) for any \( i \). Hence by the definition of \( Ann \) \( Ann(Q) \subseteq Ann(\cup S_{i}) \).

Therefore \( Ann(\cup S_{i}) \subseteq Ann(Ann(Q)) \). Since \( Z \) is chosen arbitrarily then \( \bigvee_{i} V_{i} = Ann(\cup S_{i}) \).

Hence by the previous paragraph the lattice \( (P, \leq) \) is complete. \( \triangleright \)

**Lemma 6.** Let \( A \) be a C*-algebra and \( X, \ Y \in P \). Then

(a) \( X \land Ann(X) = \{0\} \), \( X \lor Ann(X) = A \);
(b) \( Ann(Ann(X)) = X \), and if \( X \neq A \) then \( Ann(X) \neq \{0\} \);
(c) \( Ann(X \lor Y) = Ann(X) \land Ann(Y) \),
\[ Ann(X \land Y) = Ann(X) \lor Ann(Y) \]

*Proof.* (a) Let \( S \subseteq \Lambda_{+} \) and \( X = Ann(Ann(S)) \). Then by the proof of lemma \( 9 \) \( Ann(S) \land Ann(Ann(S)) = \{0\} \). We have \( Ann(X) = Ann(Ann(Ann(S))) = Ann(S) \). Then \( Ann(X) \land X = \{0\} \). Analogously \( Ann(S) \lor Ann(Ann(S)) = A \) and \( Ann(X) \lor X = A \).

Item (b). Suppose that \( X \neq A \) and \( Ann(X) = \{0\} \). Then \( Ann(Ann(X)) = A \). But by the definition \( Ann(Ann(X)) = X \). This is a contradiction. Hence \( Ann(X) \neq \{0\} \).

(c) Let \( Q \subseteq \Lambda_{+} \) and \( Y = Ann(Ann(Q)) \). By the proof of lemma \( 9 \) \( Ann(X \lor Y) = Ann(Ann(Ann(S \cup Q))) = Ann(S \cup Q) \). At the same time \( Ann(X) \land Ann(Y) = Ann(Ann(Ann(S \cup Q))) \land Ann(Ann(Ann(Q))) = Ann(S \land Ann(Q)) \). We have \( Z \subseteq Ann(S) \land Ann(Q) \) for any \( Z \in P \) such that \( Z \subseteq Ann(S) \) and \( Z \subseteq Ann(Q) \). At the same time \( Ann(S \land Ann(Q)) = Ann(S \cup Q) \). Hence \( Ann(S \land Ann(Q)) = Ann(S \cup Q) \). Thus \( Ann(X) \land Ann(Y) = Ann(S \cup Q) \land Ann(X) \lor Y = Ann(X) \land Ann(Y) \).

Analogously we have \( Ann(X \lor Y) = Ann(X) \lor Ann(Y) \). \( \triangleright \)

*Example.* Let \( X \) be a compact, \( \tau_{X} \) be the topology of \( X \). We consider \( \tau_{X} \). Let \( \leq \) be an order in \( \tau_{X} \), defined as: if \( V, W \in \tau_{X} \) and \( V \subseteq W \) then \( V \leq W \).

The ordered set \( (\tau_{X}, \leq) \) is a lattice. Indeed, \( 1 = X, 0 = \{\emptyset\} \), \( V \lor W = V \cup W \), \( V \land W = V \cap W \) for \( V, W \in \tau_{X} \).

The ordered set \( (\tau_{X}, \leq) \) is a complete lattice. Indeed, let \( \{V_{i}\} \subseteq \tau_{X} \). Then \( \bigvee_{i} V_{i} = \cup_{i} V_{i} \) and \( \bigwedge_{i} V_{i} = \cup\{U \in \tau_{X} : \text{for any } i \ U \subseteq V_{i}\} \).


Moreover, \((\tau_X, \leq)\) is a complete boolean algebra. Indeed, for arbitrary \(V, W \in \tau_X\) we have \(V = V_1 \lor Z, W = W_1 \lor Z\), where \(V_1 = V \setminus (V \cap W), W_1 = W \setminus (V \cap W), Z = V \cap W\) and \(V_1, W_1, Z \in \tau_X\).

Let \(C^c(X)\) be the complex commutative algebra of continuous functions on the compact \(X\). Then the lattice \(\mathcal{P}_{C^c(X)}\) of annihilators of the algebra \(C^c(X)\) is a complete boolean algebra. Moreover, \(\mathcal{P}_{C^c(X)}\) is order isomorphic to the complete boolean algebra \((\tau_X, \leq)\), where the isomorphism is defined by the map

\[
\Phi(X) = \{x \in X : f(x) \neq 0\text{ for some function } f \in X\}, X \in \mathcal{P}_{C^c(X)}.
\]

Indeed, the set \(U_f = \{x \in X : f(x) \neq 0\}\), where \(f \in X\), is open in \(X\). Then the set \(U_X = \bigcup_{f \in X} U_f\) is also open in \(X\). We have \(\Phi(X) = U_X\). Therefore \(\Phi(X)\) is an open set in \(X\). We have the set \(C(\Phi(X))\) of all functions \(f \in C^c(X)\) such that \(\{x \in X : f(x) \neq 0\} \subseteq \Phi(X)\) forms a commutative subalgebra of \(C^c(X)\). Moreover \(C(\Phi(X)) \subseteq \mathcal{P}_{C^c(X)}\) and \(\text{Ann}(C(\Phi(X))) = \text{Ann}(X)\). Hence \(C(\Phi(X)) = X\). Let \(Y \in \mathcal{P}_{C^c(X)}\) and \(\Phi(X) = \Phi(Y)\). Then \(C(\Phi(Y)) = Y\). Since \(\Phi(X) = \Phi(Y)\) then \(X = Y\).

Let \(A\) be a \(C^*\)-algebra. An annihilator \(V \subseteq \mathcal{P}\) is called central, if

\[
\phi^{d}(\text{Ann}(\text{Ann}(S))) \cap \phi^{d}(\text{Ann}(S)) = 0,
\]

where \(S \subseteq A\) and \(V = \text{Ann}(\text{Ann}(S))\). The set of all central annihilators we denote by \(Z(\mathcal{P})\). Two annihilators \(V\) and \(W\) in \(\mathcal{P}\) are called orthogonal, if \(V \cdot W = 0\), where \(V \cdot W = \{vw : v \in V, w \in W\}\).

**Lemma 7.** Let \(A\) be a \(C^*\)-algebra of bounded linear operators on a Hilbert space \(H\), \(Z(\mathcal{P})\) be the set of all central annihilators of \(\mathcal{P}\). Then elements of \(Z(\mathcal{P})\) are pairwise commute, i.e. \(X = (X \land Y) \lor (X \land Y^\perp)\) for any \(X, Y \in Z(\mathcal{P})\).

**Proof.** Let \(X, Y \in Z(\mathcal{P})\). We have \(X \land Y = X \cap Y, X \land Y^\perp = X \cap Y^\perp\) and by lemma 3 \(X = A \cap ew(A)e, Y = A \cap f w(A)f, Y^\perp = A \cap (f) w(A)(\overline{f})\) for some projections \(e\) and \(f\) in \(w(A)\). Note that \(\text{Ann}(X) \subseteq \text{Ann}((X \cap Y) \cup (X \cap Y^\perp))\) and the projections \(e\) and \(f\) are central in \(w(A)\). Let \(X \cdot Y = \{xy : x \in X, y \in Y\}\). Then \(X \cdot Y \subseteq X \cap Y, X \cdot Y \subseteq ef w(A)ef, X \cap Y = A \cap ef w(A)ef\). Analogously \(X \cap Y^\perp = A \cap ef w(A)\).

Suppose that \(\text{Ann}(X) \neq \text{Ann}((X \cap Y) \cup (X \cap Y^\perp))\). Then there exists \(a \in \text{Ann}((X \cap Y) \cup (X \cap Y^\perp))\) such that \(a \neq \text{Ann}(X)\). Hence there exists \(x \in X\) such that \(ax \neq 0\). Since \(\text{Ann}(Y \cup Y^\perp) = \{0\}\) then there exists \(y \in Y \cup Y^\perp\) such that \((ax)y \neq 0\). We have \((ax)y = (a)(x)y\). Then \((ax)yx = (a)(x)yx\). But \(a(ef + fe) = 0\). Hence \(a \in \text{Ann}(X)\). Therefore \(\text{Ann}(X) = \text{Ann}((X \cap Y) \cup (X \cap Y^\perp))\) and since \((X \cap Y) \cup (X \cap Y^\perp) \in Z(\mathcal{P})\) then \(X = (X \land Y) \lor (X \land Y^\perp)\). □

**Lemma 8.** Let \(A\) be a \(C^*\)-algebra of bounded linear operators on a Hilbert space \(H\), \(Z(\mathcal{P})\) be the set of all central annihilators of \(\mathcal{P}\). Then \(Z(\mathcal{P})\) is a complete boolean algebra.

**Proof.** By lemma 7 elements of \(Z(\mathcal{P})\) are pairwise commute. Then as recalled above in this paragraph \(Z(\mathcal{P})\) is a boolean algebra.

Let \(\{V_i\} \subseteq Z(\mathcal{P})\). Then by corollary 4 for any \(i\) there exist such central projections \(e_i, f_i \in P(w(A))\), that \(w(V_i) = e_i(w(A))\), \(w(\text{Ann}(V_i)) = f_i(w(A))\), where \(w(S)\) is a weak closure of a set \(S \subseteq A\) in the algebra \(B(H)\). Then \(V_i, \text{Ann}(V_i)\) are closed two sided ideals of the algebra \(A\) for all indexes \(i\).
Let $a$, $v$ be arbitrary elements of $A$, $∩_i \text{Ann}(V_i)$, correspondingly. Then $v \in \text{Ann}(V_i)$ and the elements $av$, $va$ belong to $\text{Ann}(V_i)$ for all $i$. Hence the elements $av$, $va$ belong to $\cap_i \text{Ann}(V_i)$ to. Therefore, since $\cap_i \text{Ann}(V_i) = \text{Ann}(\cup_i V_i)$ then $\cap_i \text{Ann}(V_i)$ is a two sided closed ideal of the algebra $A$. We have $w(\cap_i \text{Ann}(V_i)) = f(w(A)f$ for some projection $f \in w(A)$. Then by the weak continuity of the multiplication the next equation hold

$$d(\text{Ann}(\text{Ann}(V_i))) \cap d(\text{Ann}(V_i)) = 0.$$ 

Therefore, by corollary 4 $\text{Ann}(\text{Ann}(\cup_i V_i)) \subseteq Z(\mathcal{P})$. At the same time by the proof of lemma 5 $\sup_i V_i = \text{Ann}(\text{Ann}(\cup_i V_i))$. Therefore $\sup_i V_i \subseteq Z(\mathcal{P})$.

Analogously $\inf_i V_i \subseteq Z(\mathcal{P})$. Then the lattice $Z(\mathcal{P})$ is complete. $\triangleright$

Let $V \in \mathcal{P}$. By lemma 5 the greatest lower bound $c(V)$ of such central annihilators $W \subseteq Z(\mathcal{P})$ that $V \subseteq W$, is also an annihilator. Moreover by lemma 8 $c(V)$ is central. The annihilator $c(V)$ we will call a central support of $V$.

**Lemma 9.** Let $A$ be a commutative $C^*$-algebra bounded linear operators on a Hilbert space $H$, $w(A)$ be an ultraweak closure of $A$ in the algebra $B(H)$. Let $X$ be the topological space of characters of the algebra $A$, $Y$ be the topological space of characters of the algebra $w(A)$. Let $\text{supp}(Y)$, $\text{supp}(X)$ be sets of all points of the spaces $Y$ and $X$ correspondingly. Then

(a) $\text{supp}(X) \subseteq \text{supp}(Y)$,

(b) the set $\text{supp}(X)$ of all points of the space $X$ is dense in the topological space $Y$,

(c) let $x \in X$ and there exists $U \in \tau_X$ such that $x \in U$. Then there exists $U_o \in \tau_Y$ such that $x \in U_o$, where $\tau_X$ and $\tau_Y$ are the topologies of the spaces $X$ and $Y$.

**Proof.** (a) Since any character on $A$ can be uniquely *-weakly extended to a character on the algebra $w(A)$, then $\text{supp}(X) \subseteq \text{supp}(Y)$.

(b) By contradiction. Suppose the set $\text{supp}(X)$ is not dense in $Y$. Let $C(X)$, $C(Y)$ be commutative algebras of complex-valued continuous functions on the topological spaces $X$, $Y$ respectively. Then $A \cong C(X)$, $w(A) \cong C(Y)$.

Note, that $a(x) = \bar{a}(x)$ for any $a \in C(X)$ and $x \in X$, where $\bar{a}$ is the image of the function $a$ in the algebra $C(Y)$ in point of $C(X) \subseteq C(Y)$. Let $Y_o$ is an open set of the compact $Y$ such, that $Y_o \cap X = \emptyset$. We have the set $C(Y_o)$ of all functions $f \in C(Y)$ such that $\{ x \in Y : f(x) \neq 0 \} \subseteq Y_o$ forms a commutative subalgebra of the algebra $C(Y)$. We have $C(X) \subseteq \text{Ann}_{C(Y)}(C(Y_o))$. Let $f$ be an arbitrary nonzero element of $C(Y_o)$. Then $f \cdot C(X) = 0$. By the separately weak continuity of the algebraic multiplication we have $f \cdot w(C(X)) = f \cdot C(Y) = 0$. Hence $f = 0$. This is a contradiction. Therefore $Y_o = \emptyset$.

(c) We have every open set in $\tau_X$ defines by elements in $A$ and every open set in $\tau_Y$ defines by elements in $w(A)$. Then the statement of item (c) of the lemma hold. $\triangleright$

**Lemma 10.** Let $A$ be a commutative $C^*$-algebra of bounded linear operators on a Hilbert space $H$, $\mathcal{P}$ be the set of annihilators, $w(A)$ be an ultraweak closure of $A$ in the algebra $B(H)$. Let $Y \in \mathcal{P}$, $w(Y)$ be a weak closure of $Y$ in the algebra $w(A)$, $X \in \mathcal{P}$ and $X$ be a subset of $Y$ such that the weak closure $w(X)$ of $X$ in $w(Y)$ coincides with $ew(Y)e$ for some projection $e \in w(Y)$ with the condition $e < 1$, i.e. $w(X) = ew(Y)e$. Then $\text{Ann}_Y(X) \neq \{ 0 \}$ and $\text{Ann}_Y(\text{Ann}_Y(X)) = X$. 

Proof. Let $Q$ be the topological space of characters of the algebra $A$, $\overline{Q}$ be the topological space of characters of the algebra $w(A)$. By item (a) of lemma 9 \( \text{supp}(Q) \subseteq \text{supp}(\overline{Q}) \). By item (b) of lemma 9 the set \( \text{supp}(Q) \) is dense in $\overline{Q}$.

Note that the sets $V = \{ x \in Q : e(x) \neq 0 \}$ and $W = \{ x \in Q : (1 - e)(x) \neq 0 \}$ are close-open subsets of the space $\overline{Q}$ and $Q = V \cup W$. Also we have the sets $Q_Y = \bigcup_{f \in Y} \{ x \in Q : f(x) \neq 0 \}$, $Q_X = \bigcup_{f \in X} \{ x \in Q : f(x) \neq 0 \}$ are open sets of the topological space $Q$. Let $\text{Cl}(Q_Y)$ be the closure of $Q_Y$ in $Q$ and $\text{Cl}(Q_X)$ be the closure of $Q_X$ in $Q$. If $Q_Y \neq Q_X$ then $\text{Cl}(Q_Y) \neq \text{Cl}(Q_X)$. Indeed, if $\text{Cl}(Q_Y) = \text{Cl}(Q_X)$ then $Q \setminus \text{Cl}(Q_Y) = Q \setminus \text{Cl}(Q_X)$. $Q \setminus \text{Cl}(Q_Y)$ is a nonempty open set in $Q$ and $\text{Ann}_A(Y) = \text{Ann}_A(X)$ (see the example above). Then $Y = X$. What is imposable. Hence $\text{Cl}(Q_Y) \neq \text{Cl}(Q_X)$, and, hence, $Q_Y \neq Q_X$. Otherwise also we get $\text{Cl}(Q_Y) = \text{Cl}(Q_X)$.

Then, since $Q_X \subset Q_Y$ then $Q_Y \setminus Q_X$ is an open set in $Q$. Therefore $\text{Ann}_Y(X) \neq \{ 0 \}$.

Since $Q_Y = (Q_Y \setminus Q_X) \cup Q_X$ we have $\text{Ann}_Y(\text{Ann}_Y(X)) = X$. \( \triangleright \)

**Lemma 11.** Let $A$ be a $C^*$-algebra of bounded linear operators on a Hilbert space $H$, $\mathcal{P}$ be the set of annihilators, $w(A)$ be an ultraweak closure of $A$ in the algebra $B(H)$. Let $Y \in \mathcal{P}$, $w(Y)$ be a weak closure of $Y$ in the algebra $w(A)$, $X \in \mathcal{P}$ and $X$ be a subset of $Y$ such that the weak closure $w(X)$ of $X$ in $w(Y)$ coincides with $ew(Y)e$ for some projection $e \in w(Y)$ with the condition $e < 1$, i.e. $w(X) = ew(Y)e$. Then $\text{Ann}_Y(X) \neq \{ 0 \}$ and $\text{Ann}_Y(\text{Ann}_Y(X)) = X$. \( \triangleright \)

**Theorem 12.** Let $A$ be a $C^*$-algebra, $\mathcal{P}$ be the set of annihilators, defined above. Then $(\mathcal{P}, \leq)$ is an orthomodular complete lattice.

**Proof.** Note that $X$ is a $C^*$-algebra. Hence for any maximal commutative subalgebra $X_o$ of $X$ we have $w(X_o) = ew(Y_o)e$, where $Y_o$ is a maximal commutative subalgebra of the algebra $Y$, containing $X_o$. Hence By lemma 10 $\text{Ann}_Y(X_o) \neq \{ 0 \}$ and $\text{Ann}_Y(\text{Ann}_Y(X_o)) = X_o$. Hence $\text{Ann}_Y(X) \neq \{ 0 \}$.

We have $\text{Ann}_Y(\text{Ann}_Y(X_o)) = X_o$ for any maximal commutative subalgebra $X_o$ of $X$ and for any maximal commutative subalgebra $Y_o$ of the algebra $Y$, containing $X_o$. Then $\text{Ann}_Y(\text{Ann}_Y(X)) = X$. \( \triangleright \)
$Ann_V(X) = Y$. Hence $Y = X \lor (Y \land X^\perp)$. Therefore $\mathcal{P}$ is an orthomodular lattice.

Let $A$ be a C*-algebra. An annihilator $V \in \mathcal{P}$ is called abelian, if $V$ is a commutative C*-subalgebra of the algebra $A$. Let $B$ be a C*-subalgebra of $A$ and $\mathcal{P}_B = \{V \subseteq B : \text{there exists such } S \subseteq B_+ \text{ that } V = Ann_B(Ann_B(S))\}$.

**Lemma 13.** Let $A$ be a C*-algebra. Then the next statements hold.

a) Let $V \in \mathcal{P}$. Then $\mathcal{P}_V = \{W \in \mathcal{P} : W \subseteq V\}$ and $\mathcal{P}_V$ is a complete sublattice of $\mathcal{P}$.

b) Let $V$ be an abelian annihilator. Then for any $W \in \mathcal{P}$, if $W \subseteq V$, then $W$ is an abelian annihilator to.

**Proof.** Item a): Let $Z \in \mathcal{P}_V$. Then $Ann(V) \subseteq Ann(Z)$ and $Ann(Ann(Z)) \subseteq Ann(Ann(Ann(V))) = V$. Since $Ann_V(Z) \subseteq Ann(Z)$ and $Ann(Ann(Z)) = Ann_V(Ann(Z)) = Ann_V(Ann_V(Z))$ then $Ann(Ann(Z)) = Z$. Hence $Z \in \mathcal{P}$. Now, let $Z \in \mathcal{P}$ and $Z \leq V$. Then by lemmas 3 and 11 we have $Z = Ann_V(Ann_V(Z))$. Hence $Z \in \mathcal{P}_V$.

Item b) is obvious. ▷

An annihilator $V$ is called modular, if $\mathcal{P}_V$ is a modular lattice. Also, the next lemma holds, by lemmas 8 and 13.

**Lemma 14.** Let $A$ be a C*-algebra. Then the next statements hold.

a) Let $V$ be an abelian annihilator of $A$. Then $\mathcal{P}_V$ is a Boolean algebra.

b) Any abelian annihilator is modular.

The results of the given paragraph can be summarized as the next theorem.

**Theorem 15.** Let $A$ be a C*-algebra and $\mathcal{P}$ be the set of all annihilators of subsets of $A_+$. Then

a) $\mathcal{P}$ is a lattice with the order $\subseteq$,

b) the annihilator $\{0\}$ is zero $0$ and $A$ is the unit $1$ of the lattice $\mathcal{P}$,

c) $\mathcal{P}$ is an orthomodular lattice with an orthocomplementation defined as $(\cdot)^\perp : \mathcal{P} \to (V)^\perp = Ann(V), V \in \mathcal{P}$,

d) elements $V, W \in \mathcal{P}$ are orthogonal as elements of the ortholattice $\mathcal{P}$ if $V \circ W = \{0\}$,

e) the center of the orthomodular lattice $\mathcal{P}$ coincides with the set $Z(\mathcal{P})$ of all central annihilators of $\mathcal{P}$,

f) $\mathcal{P}$ is order complete under the order $\subseteq$.

g) the center $Z(\mathcal{P})$ of $\mathcal{P}$ is a complete boolean algebra.

**Remark.** The lattice $\mathcal{P}$ of annihilators of a von Neumann algebra $A$ is a sublattice of the lattice $J(A)$ of weak* closed inner ideals of the algebra $A$. The lattice $J(A)$ is not orthomodular, but, since it possesses a complementation, such concepts as orthogonality and center remain meaningful nevertheless (see [3], [4]). At the same time, since $\mathcal{P}$ can be identified with the lattice $P(A)$ of all projections of $A$, $\mathcal{P}$ is orthomodular. Note that, in the case of anisotropic Jordan *-triples annihilators in $\mathcal{P}$ are also inner ideals. In this case elements of $\mathcal{P}$ are defined by Jordan multiplication. Therefore the results in [3] also hold for annihilators.

3. C*-algebras of type I

Now recall the definition of C*-algebra of type I. Let $A$ be a C*-algebra and $\pi$ be a representation of $A$. We say the representation $\pi$ of the algebra $A$ is of type I,
if the von Neumann algebra, generated by $\pi(A)$ is of type I. The $C^*$-algebra $A$ is called of type I, if all representations of this algebra are of type I.

Let $H$ be a complex Hilbert space and $\xi \in H$. The closure $\pi(A)\xi$ is a closed on norm vector subspace of the space $H$, invariant in relation to $\pi(A)$. If this subspace is $H$, then $\xi$ is called totalitarianizing vector for the representation $\pi$.

Let $A$ be a $C^*$-algebra bounded linear operators on the Hilbert space $H$, $\pi$ be a representation of the algebra $A$ in the Hilbert space $H$. We say that $\pi$ is irreducible, if $H \neq 0$ and the commutant of $\pi(A)$ in $B(H)$ is $C1$.

A $C^*$-algebra $A$ is called CCR-algebra, if for any irreducible representation $\pi$ of the $C^*$-algebra $A$ and for any $x \in A$ the operator $\pi(x)$ is compact. A $C^*$-algebra $A$ is called GCR-algebra, if any nonzero factor-$C^*$-algebra of $A$ has a nonzero closed two sided CCR-ideal. It is known that a $C^*$-algebra $A$ is a GCR-algebra if and only if $A$ is a $C^*$-algebra of type I by Dixmier. (see [1])

The theory of lattices of annihilators developed above allows us to introduce the next definitions.

Definition. A $C^*$-algebra $A$ is called $C^*$-algebra of von Neumann type I, if there exists such abelian annihilator $V$ in $P$ that $c(V) = A$.

Lemma 16. Let $A$ be a $C^*$-algebra of bounded linear operators on a Hilbert space $H$, $Y$ be a $C^*$-subalgebra of $A$ such that $w(Y) = f w(A)f$ and $Y = A \cap f w(A)f$ for some projection $f$ of $w(A)$, $X \in P$, $Y \subseteq X$. Suppose that $Ann_X(Y) = \{0\}$. Then $X = Y$.

Proof. First, suppose that the algebra $A$ is commutative. Let $Q$ be the topological space of characters of the algebra $A$, $\overline{Q}$ be the topological space of characters of the algebra $w(A)$. By item (a) of lemma 9 $supp(Q) \subseteq supp(\overline{Q})$. By item (b) of lemma 9 the set $supp(Q)$ is dense in $\overline{Q}$.

We have the sets $Q_Y = \bigcup_{f \in Y} \{x \in Q : f(x) \neq 0\}$, $Q_X = \bigcup_{f \in X} \{x \in Q : f(x) \neq 0\}$ are open sets of the topological space $Q$. Let $Z = \{f \in A : \{x \in Q : f(x) \neq 0\} \subset Q_Y\}$. Then by the condition $Y = Z$. We have $Ann_A(Ann_A(Z)) = Z$, i.e. $Z \in P$. Hence $Y \in \mathcal{P}$ and $Ann_A(Ann_A(Y)) = Y$.

Now, suppose that $A$ is an arbitrary $C^*$-algebra. Note that $Y$ is a $C^*$-algebra. By the condition $w(Y) = f w(A)f$ and $Y = A \cap f w(A)f$ for some projection $f$ of $w(A)$. Hence for any maximal commutative subalgebra $Y_0$ of $Y$ we have $w(Y_0) = f w(A_0)f$, where $A_0$ is a maximal commutative subalgebra of the algebra $A$, containing $Y_0$. Hence by the previous part of the proof $Ann_A(Ann_A(Y_0)) = Y_0$. We have $Ann_A_0(Ann_A(Y_0)) = Y_0$, $w(Y_0) = f w(A_0)f$, $f \in w(A_0)$ for any maximal commutative subalgebra $Y_0$ of $Y$ and for any maximal commutative subalgebra $A_0$ of the algebra $A$, containing $Y_0$. Then $Ann_A(Ann_A(Y)) = Y$. Hence $Y \in \mathcal{P}$. Now we can apply lemma 11 and write $Ann_X(Ann_X(Y)) = Y$. Therefore, since $Ann_X(Y) = \{0\}$ then $X = Y$. $\triangleright$

Theorem 17. Let $A$ be a GCR-algebra of bounded linear operators on a Hilbert space $H$. Then $A$ is a $C^*$-algebra of von Neumann type I.

Proof. By lemma 4.4.4 in [1] there exists a nonzero element $x$ in $A$ such that $\pi(x) = 0$ or $\pi(x)$ has rank 1 for any representation $\pi$ of the algebra $A$ in a Hilbert space. Then $\pi(xAx) = \pi(x)\pi(A)\pi(x)$ and $\pi(xAx)$ is commutative for any representation $\pi$ of the algebra $A$ in a Hilbert space. Therefore $xAx$ is commutative. We note, that $xAx$ is a commutative $C^*$-algebra.
Let $A_o$ be a maximal commutative subalgebra of the algebra $xAx$, then for some maximal commutative subalgebra $A_o$ of the weak closure $w(xAx)$ of the algebra $xAx$ in the algebra $w(A)$ we have $A_o \subseteq A_w$. There exists a hyperstonean compact $Q$ such that $A_o \cong C(Q)$. If we suppose that $x \in A_o$, then there exists a monotone increased sequence $(x_n)$ (for example approximative unit of the algebra $A_o$) such that $\sup x_n = e$. Therefore the weak limit of the sequence $(x_n)$ is a projection $e$, where $e$ is the unit of the algebra $w(xAx)$. Then we have $w(xAx) = ew(A)e$. Therefore by the separately weak continuity of the operation of multiplication we have the algebra $ew(A)e$ is commutative.

We have $A \cap ew(A)e$ is an abelian $C^*$-algebra. Let $X = Ann(Ann(A \cap ew(A)e))$. Then $Ann_X(A \cap ew(A)e) = \{0\}$. Indeed, otherwise the set $Ann_X(A \cap ew(A)e)$ does not belong to $X$. This is a contradiction. Hence $Ann_X(A \cap ew(A)e) = \{0\}$. Then by lemma 16 $X = A \cap ew(A)e$ and $X$ is an abelian $C^*$-algebra. Thus the algebra $A$ contains a nonzero abelian annihilator $X \in \mathcal{P}$.

Let $\{E_i\}$ be a maximal set of abelian annihilators with pairwise orthogonal central supports. We should prove that the central support of $\bigvee_i E_i$ is $A$ that is if $c(E_i)$ is a central support annihilator of $E_i$ for any $i$, then $\bigvee_i c(E_i) = A$. If it is not true then $\bigvee_i c(E_i) < A$ and $Ann(\bigvee_i c(E_i)) \neq \{0\}$. Note that $Ann(\bigvee_i c(E_i))$ is a central annihilator and a $C^*$-algebra. By theorem 4.3.5 in [1] the annihilator $Ann(\bigvee_i c(E_i))$ is a GCR-algebra. Hence there exists an abelian annihilator $F$ in the algebra $Ann(\bigvee_i c(E_i))$ with a central support $Z \subseteq Ann(\bigvee_i c(E_i))$. This is contradicts the maximality of the set $\{E_i\}$. Thus $\bigvee_i E_i = A$. Hence $A$ is a $C^*$-algebra of type $I$ by von Neumann. $>$

**Remark.** The converse of the statement of theorem 27 is not true. For example, let $H_1$, $H_2$, ... be Hilbert spaces of dimensions 1, 2, ... Then the $C^*$-algebra

$$\sum_{n=1,2,...} \bigoplus B(H_1)$$

is not a GCR-algebra, but this algebra is a von Neumann algebra of type $I$. Hence this algebra is a $C^*$-algebra of von Neumann type $I$. Hence the new class of $C^*$-algebras of von Neumann type $I$ is wider than the class of $C^*$-algebras of type $I$ (that is the class of GCR-algebras, see [1]).

4. **Functional representation of a von Neumann algebra of type I**

Let $n$ be an infinite cardinal number, $E$ a set of indexes of cardinality $n$. Let $\{e_{ij}\}$ be a set of matrix units such that $e_{ij}$ is a matrix, whose $(i,j)$-s component is 1 and the rest components are zero. Let $\{m_\xi\}_{\xi \in E}$ be a set of $n \times n$ matrices. By $\sum_{\xi \in E} m_\xi$ we denote the matrix whose components are sums of the corresponding components of the matrices of the set $\{m_\xi\}_{\xi \in E}$. Let

$$M_n(C) = \{ \sum_{ij \in E} \lambda_{ij} e_{ij} : \text{for any indexes } ij \lambda_{ij} \in C, \text{there exists such number } K \in R \text{ that for any } n \in N \text{ and for any } \{e_{kl}\}_{k,l=1}^n \subseteq \{e_{ij}\},\| \sum_{k,l=1}^n \lambda_{kl} e_{kl} \| \leq K \},$$
where \( \| \cdot \| \) is a matrix norm. Then \( \mathcal{M}_n(\mathbb{C}) \) is a von Neumann algebra of infinite \( n \times n \) matrices on \( \mathbb{C} \) who is defined by its own infinite order decomposition (see [5]). Note that \( \mathcal{M}_n(\mathbb{C}) \) is a factor of type \( I_n \).

Let \( X \) be a hyperstonean compact, \( C(X) \) the commutative algebra of all complex-valued continuous functions on the compact \( X \) and
\[
C(X) \otimes \mathcal{M}_n(\mathbb{C}) = \{ \sum_{ij \in \Xi} \lambda_{ij}(x)e_{ij} : (\forall i,j) \lambda_{ij}(x) \in C(X) \}
\]
\[
(\exists K \in R)(\forall m \in N)(\forall \{e_{kl}\}_{kl=1}^n \subseteq \{e_{ij}\}) \| \sum_{kl=1}^m \lambda_{kl}(x)e_{kl} \| \leq K,
\]
where \( \| \sum_{kl=1}^m \lambda_{kl}(x)e_{kl} \| \leq K \) means \((\forall x_0 \in X)\| \sum_{kl=1}^m \lambda_{kl}(x_0)e_{kl} \| \leq K\). The set \( C(X) \otimes \mathcal{M}_n(\mathbb{C}) \) is an algebra with the order, agreed with pointwise algebraic operations. If we define a norm as
\[
\| a \| = \sup_{\{e_{kl}\}_{kl=1}^n \subseteq \{e_{ij}\}} \| \sum_{kl=1}^n \lambda_{kl}(x)e_{kl} \|,
\]
where \( a \in C(X) \otimes \mathcal{M}_n(\mathbb{C}) \) and \( a = \sum_{ij \in \Xi} \lambda_{ij}(x)e_{ij} \), then \( C(X) \otimes \mathcal{M}_n(\mathbb{C}) \) is a norm closed associative algebra. It can be straightforwardly checked that all axioms of the definition of \( C^* \)-algebra hold. Hence, \( C(X) \otimes \mathcal{M}_n(\mathbb{C}) \) is a \( C^* \)-algebra.

**Theorem 18.** The \( C^* \)-algebra \( C(X) \otimes \mathcal{M}_n(\mathbb{C}) \) is a von Neumann algebra of type \( I_n \).

**Proof.** The algebra \( C(X) \otimes \mathcal{M}_n(\mathbb{C}) \) is monotone complete. Indeed, let \( (x_\alpha) \) is a bounded, monotone increased net of elements of the algebra \( C(X) \otimes \mathcal{M}_n(\mathbb{C}) \), and
\[
x_\alpha = \sum_{ij \in \Xi} \lambda_{ij}(x)e_{ij}, \text{ for any } \alpha.
\]
We assert that for all \( i, j \)
\[
\sup_\alpha (\lambda_{ii}(x)e_{ii} + \lambda_{ij}(x)e_{ij} + \lambda_{ji}(x)e_{ji}) = \lambda_{ii}(x)e_{ii} + \lambda_{ij}(x)e_{ij} + \lambda_{ji}(x)e_{ji} + \lambda_{jj}(x)e_{jj},
\]
for some functions \( \lambda_{ii}(x), \lambda_{ij}(x), \lambda_{ji}(x), \lambda_{jj}(x) \) in \( C(X) \). Since
\[
\sum_{\xi, \eta \in \{i, j, l\}} C(X)e_{\xi \eta} \cong C(X) \otimes M_3(\mathbb{C}),
\]
then for all \( i, j \) and \( l \) in \( \Xi \)
\[
\lambda_{ij}(x) = \lambda_{ii}(x), \lambda_{jj}(x) = \lambda_{jj}(x).
\]
Hence, there exists a set \( \{\lambda_{ij}(x)\} \subseteq C(X) \) such that
\[
\sup_\alpha (\lambda_{ii}(x)e_{ii} + \lambda_{ij}(x)e_{ij} + \lambda_{ji}(x)e_{ji}) = \lambda_{ii}(x)e_{ii} + \lambda_{ij}(x)e_{ij} + \lambda_{ji}(x)e_{ji},
\]
for any \( i \) and \( j \). Thus \( \sum_{ij \in \Xi} \lambda_{ij}(x)e_{ij} \in C(X) \otimes \mathcal{M}_n(\mathbb{C}) \). By the properties of least upper bound there exists a set \( Y \) dense in \( X \) such that
\[
\sup_\alpha \sum_{kl=1}^m \lambda_{kl}(x)e_{kl} = \sum_{kl=1}^m \lambda_{kl}(x)e_{kl}, (\forall x \in Y).
\]
By the definition of the set \( Y \) and since \( \lambda_{ij}(x) \in C(X) \) for any \( i, j \) we have \( \sum_{ij \in \Xi} \lambda_{ij}(x)e_{ij} \) is an element of the algebra \( C(X) \otimes \mathcal{M}_n(\mathbb{C}) \). It can be straightforwardly checked that \( \sup_\alpha x_\alpha = \sum_{ij \in \Xi} \lambda_{ij}(x)e_{ij} \). Hence the \( C^* \)-algebra \( C(X) \otimes \mathcal{M}_n(\mathbb{C}) \) is monotone complete.
Now we show a separate set of normal functionals for the algebra $C(X) \otimes M_n(\mathbb{C})$. Let $a$ be an arbitrary element of the algebra $C(X) \otimes M_n(\mathbb{C})$, and $H_o$ be such finite dimensional Hilbert subspace of the Hilbert space $l_2(\Xi)$, that the inducing $a|_{H_o}$ of the element $a$ on the Hilbert space $H_o$ is nonzero. Then $B(H_o)$ is finite dimensional and $C(X) \otimes B(H_o)$ is a von Neumann algebra. Let $\rho$ be a normal functional on the algebra $C(Q) \otimes B(H_o)$ such that $\rho(a|_{H_o}) \neq 0$. Then the functional $\rho$ can be extended on the algebra $C(X) \otimes M_n(\mathbb{C})$ as follows: let $e$ is the unit of the algebra $C(Q) \otimes B(H_o)$. Then $e(C(Q) \otimes M_n(\mathbb{C}))e = C(Q) \otimes B(H_o)$ and for any $a \in C(Q) \otimes M_n(\mathbb{C})$, $\rho(a) = \rho(eae) + 0$, that is $\rho((1 - e)(C(Q) \otimes M_n(\mathbb{C}))(1 - e) \oplus e(C(Q) \otimes M_n(\mathbb{C}))(1 - e) \oplus (1 - e)(C(Q) \otimes M_n(\mathbb{C}))e) = 0$. Then $\rho$ is a normal functional on the algebra $C(X) \otimes M_n(\mathbb{C})$ and $\rho(a) \neq 0$. Since the element $a$ is arbitrarily chosen the algebra $C(X) \otimes M_n(\mathbb{C})$ has a separate set of normal functionals. Thus $C(X) \otimes M_n(\mathbb{C})$ is a von Neumann algebra.

We have $M_n(\mathbb{C})$ is a von Neumann algebra of type $I_n$. If $1$ is the unit of the algebra $C(X)$ and $\{e_i\}$ is a maximal orthogonal set of abelian projections with the central support $1 \in M_n(\mathbb{C})$, then the set $\{1 \otimes e_i\}$ is a maximal orthogonal set of abelian projections of $C(X) \otimes M_n(\mathbb{C})$ with central support $1$. Hence $C(X) \otimes M_n(\mathbb{C})$ is a von Neumann algebra of type $I_n$. $\dagger$

By the theory of von Neumann algebras any von Neumann algebra of type $I_n$, where $n$ is a cardinal number, is isomorphic to an algebra $C(X) \otimes M_n(\mathbb{C})$ for some hyperstonean compact $X$ and some cardinal number $n$.

5. $C^*$-ALGEBRAS WITHOUT NONZERO ABELIAN ANNIHILATORS

A $C^*$-algebra $A$ is called NGCR-algebra, if this algebra does not have nonzero two sided CCR-ideals.

**Theorem 19.** Let $A$ be a $C^*$-algebra of bounded linear operators on a Hilbert space $H$. Then the algebra $A$ is a NGCR-algebra if and only if the algebra $A$ does not have a nonzero abelian annihilator.

**Proof.** Suppose that the algebra $A$ is a NGCR-algebra. Suppose that $A$ has a nonzero abelian annihilator $X$. By lemma 3 there exists a projection $p \in A$ such that $w(X) = pw(A)p$. By separately weak continuity of the multiplication $w(X)$ is commutative. Hence the projection $p$ is abelian. Let a central projection $e \in w(A)$ is a central support of the projection $p$. Then the algebra $ew(A)e$ is a von Neumann algebra of type I. Then $ew(A)e$ is a direct sum of homogenous von Neumann algebras (that is von Neumann algebra of type $I_m$, where $m$ is a cardinal number). Then by paragraph 7 there exist a set $\{Q_\xi\}$ of hyperstonean compacts and a set $\{H_\xi\}$ of Hilbert spaces such that

$$ew(A)e \cong \bigoplus_\xi (C(Q_\xi) \otimes B(H_\xi)),$$

where $\sum_\xi (C(Q_\xi) \otimes B(H_\xi))$ is a direct sum of the algebras $C(Q_\xi) \otimes B(H_\xi)$. Let

$$ew(A)e = \bigoplus_\xi (C(Q_\xi) \otimes B(H_\xi)),$$

$$J = \{a \in ew(A)e : \text{for any } Q_\xi, \text{ for any } y \in Q_\xi, a(y) \in KB(H_\xi)\},$$
where $KB(H_\xi)$ is an algebra of all bounded linear compact operators on the Hilbert space $H_\xi$ for any $\xi$. Now, let

$$I = \{ a \in J : \text{for any } y \in CL(dom(a)) \setminus U(dom(a)), \quad a(y) = 0 \},$$

where $dom(a) = \{ y \in \cup_\xi Q_\xi : a(y) \neq 0 \}$ and $CL(dom(a))$ is the closure of the set $dom(a)$ in $\cup_\xi Q_\xi$ and $U(dom(a))$ is the interior of the set $dom(a)$. Note, that

$$I = \{ a \in J : U(dom(a)) = dom(a) \}.$$

Then $I$ is an CCR-ideal of the algebra $ew(A)c$. If we define $L = I \cap A$ then $L \neq \{0\}$. Indeed, by the definition $X \subseteq pw(A)p \subseteq ew(A)c$. Since $pw(A)p = \sum_\xi (C(Q_\xi) \otimes e_\xi)$, where $e_\xi$ is an abelian projection of the algebra $B(H_\xi)$ with the central support $1_\xi$, who is the unit of the algebra $B(H_\xi)$, then $X = \sum_\xi (C(Q_\xi^{weak}) \otimes e_\xi)$, where $Q_\xi^{weak}$ is a set of all points of the compact $Q_\xi$ with some topology, which is weaker then the topology of the compact $Q_\xi$. If

$$X_\alpha = \{ a \in X : \text{for any } y \in CL(dom(a)) \setminus U(dom(a)), \quad a(y) = 0 \},$$

then $X_\alpha \subseteq I$. Hence $X_\alpha \subseteq L$ and $L \neq \{0\}$. By the definition $L$ is a two sided CCR-ideal of the algebra $A$. This contradicts the assumption that $A$ is a NGCR-algebra.

Suppose that $A$ does not have a nonzero abelian annihilator and there exists a two sided CCR-ideal $I$ in $A$. Then by the arguments of the proof of theorem 17 there exists an element $x \in I$ such that $xIx$ is a commutative $^{*}$-algebra. We have $xAx \subseteq I$. Therefore $xIx = xAx$ and as in the proof of theorem 17 the $^{*}$-algebra $Ann(Ann(xAx))$ is commutative. This is a contradiction of the supposition that $A$ does not have a nonzero abelian annihilator. Hence $A$ does not have a two sided CCR-ideal. Therefore $A$ is a NGCR-algebra. \>

Let $A$ be a $^{*}$-algebra, $P$ be the corresponding lattice of annihilators. $P$ is called locally modular, if there exists such modular annihilator $V \in P$ that $c(V) = A$. It is clear that in this case if $V = A$ then the lattice $P$ is modular. The lattice $P$ is called purely nonmodular, if there does not exist a nonzero modular annihilator in $P$. Recall that two annihilators $V$ and $W$ in $P$ are said to be orthogonial, if $V \cdot W = 0$, where $V \cdot W = \{ vw : v \in V, w \in W \}$.

Let $\{ Z_\xi \}$ be an orthogonal set of central annihilators of $P$. Let $\sum_\alpha w(Z_\xi)$ be a set of consequences $(a_\xi)$, where $a_\xi \in w(Z_\xi)$ with the bounded set $\{ \|a_\xi\| \}$ of the norms of the elements $a_\xi$. $\sum_\alpha w(Z_\xi)$ is a von Neumann algebra with the componentwise algebraic operations and the norm defined by the least upper bound of the norms of components $a_\xi$.

**Theorem 20.** Let $A$ be a $^{*}$-algebra of bounded linear operators on a Hilbert space $H$, $w(B)$ be a weak closure of a subset $B \subseteq B(H)$. Then there exist such unique $^{*}$-subalgebras $A_1$, $A_{11}$, $A_{111}$ of $A$ that

1. $A_I$ is a $^{*}$-algebra of von Neumann type $I$, there does not exist a nonzero abelian annihilator in the algebras $A_{11}$ and $A_{111}$, the lattice $P_{A_{11}}$ is locally modular, the lattice $P_{A_{111}}$ is purely nonmodular,
2. the $^{*}$-subalgebras $A_1$, $A_{11}$, $A_{111}$ belong to $Z(P)$,
3. $A_1 \oplus A_{11} \oplus A_{111}$ is a $^{*}$-subalgebra of $A$ and

$$Ann(A_1 \oplus A_{11} \oplus A_{111}) = \{0\}.$$

**Proof.** Let $\{ V_\xi \}$ be a maximal set of annihilators with pairwise orthogonal central supports $\{ Z_\xi \}$, i.e. for any $\xi$ the annihilator $Z_\xi$ is a central support of $V_\xi$ and
that $w(\xi) \circ Z_{\eta} = 0$ for such $\xi$ and $\eta$ that $\xi \neq \eta$. Let $\sum^\oplus V_\xi$ be a set of consequences $(a_\xi)$, where $a_\xi \in V_\xi$ with bounded set $\{||a_\xi||\}$ of the norms of the elements $a_\xi$. The set $\sum^\oplus V_\xi$ is a C*-algebra with componentwise algebraic operations and the norm, defined as the least upper bound $\sup\{||a_\xi||\}$ of the norms $||a_\xi||$ of the components $a_\xi$.

Indeed, the last assertion follows by $\sum^\oplus V_\xi \subset \sum^\oplus w(V_\xi)$, where $\sum^\oplus w(V_\xi)$ is a von Neumann algebra. By the separately weak continuity of the multiplicity for any $w(V_\xi)$ is a commutative von Neumann algebra and there exists such $p_\xi \in P(w(A))$ that $w(V_\xi) = p_\xi(w(A))p_\xi$. Let $p = \sup_\xi p_\xi$. Then $\sum^\oplus w(V_\xi) = \sum^\oplus p_\xi(w(A))p_\xi \cong p(w(A))p$ and $p_\xi w(V_\xi) p_\xi$ is commutative. Note that $\sum^\oplus V_\xi$ is commutative. It is easy to set $Ann(Ann(\cup_\xi V_\xi)) = \sum^\oplus V_\xi$. Hence $\sum^\oplus V_\xi \in P$. Also, $\sum^\oplus V_\xi$ is abelian with the central support $A_I = \bigvee_\xi Z_\xi$ and $A_I$ is a C*-algebra of von Neumann type I. Let $Z = Ann(A_I)$. Then $Z$ is a central annihilator. Analogously we can find such central annihilator $A_{II}$ in the subalgebra $Z$ that $P_{A_{II}}$ is locally modular and $w(A_{II}) \circ w(A_{II}) \subseteq w(Z)$, where $A_{II} = Ann_Z(A_{II})$, $Ann_Z(A_{II})$ is an annihilator of $A_{II}$ in $Z$. By the definition of $A_{II}$ we have $P_{A_{II}}$ is purely nonmodular. By item a) of lemma 13 we have $A_{II}, A_{II} \in P$. It is clear that $A_I \oplus A_{II} \oplus A_{II}$ is a C*-subalgebra of $A$. We have $Ann(A_I \oplus A_{II} \oplus A_{II}) = \{0\}$. The uniqueness of the subalgebras $A_I, A_{II}, A_{II}$ hold by their definitions. $\triangleright$

**Corollary 21.** Let $A$ be a C*-algebra of bounded linear operators on a Hilbert space $H$, $w(B)$ be a weak closure of a subset $B \subseteq B(H)$. Then there exist such unique C*-subalgebras $A_I, A_{NGCR}$ of $A$ that

(a) the C*-subalgebra $A_I$ is a C*-algebra of von Neumann type I, the C*-subalgebra $A_{NGCR}$ is a NGCR-algebra,

(b) $A_I \oplus A_{NGCR}$ is a C*-subalgebra of $A$ and

$$Ann(A_I \oplus A_{NGCR}) = \{0\}.$$  

**Proof.** The corollary follows by theorems 19 and 20.

**Remark.** By the theory developed in this article we can also introduce the notions of C*-algebra of type II, III as follows: A C*-algebra $A$ is called of type II, if there exists such modular annihilator $V \in P$ that $c(V) = A$ and there does not exist a nonzero abelian annihilator (that is the lattice $P_A$ is locally modular). A C*-algebra $A$ is called of type III, if there does not exist a nonzero modular annihilator (that is the lattice $P_A$ is purely nonmodular) (see paragraph 8).

In the book of Dixmier *C*-algebras and their representations* the notion C*-algebra of type $I$ was introduced and considered together with other equivalent notions as GCR-algebras of Kaplansky and the notion of Makey. Then such notions as representations of types $I, III$ have also been introduced. However the notions of C*-algebras of type II and III haven’t been introduced and investigated yet. The reason for it is that, if C*-algebra has a representations of type II (of type III) then this algebra necessarily has a representation of type III (accordingly of type II). Therefore it is impossible to introduce the notions of C*-algebras of types II and III using representations of types II and III. A NGCR-algebra has representations of types II and III, but does not have representations of type I. As for the new notions, introduced in the given article, if a C*-algebra is of type II, then in this algebra does not exist a nonzero central annihilators, being a C*-algebra of type III or of type I. Analogously, if a C*-algebra is of type III, then in this algebra does not exist a nonzero central annihilator, being a C*-algebra of type II or of type I.
Also, in this case, theorem 20 will be some analog of the classification for $C^*$-algebras on types I, II and III.

6. Classification of simple $C^*$-algebras of type I

**Theorem 22.** Let $A$ be a simple $C^*$-algebra of type I of bounded linear operators on a Hilbert space $H$. Then the algebra $w(A)$ is a $W^*$-factor of type I.

**Proof.** Let $X$ be an abelian annihilator in $\mathcal{P}$ such that $c(X) = A$. Then by lemma 3 there exists an abelian projection $e$ such that $w(X) = ew(A)e$. Let $z$ be a central projection in the algebra $w(A)$ such that $c(e) = z$, i.e. $z$ is a central support of $e$. Then $X \subseteq zw(A)w$. Let $I = zw(A) \cap A$. Since $zw(A)$ is an ideal of $w(A)$, i.e. $zw(A)w(A) \subseteq zw(A)$, then $IA \subseteq I$. Hence $I$ is an ideal of the algebra $A$. Since $A$ is simple then $I = A$. Hence $w(A) = zw(A)$ and $w(A)$ is of type I.

Let $z$ be a central projection of the algebra $w(A)$ and $z < 1$. Then $zX$ or $(1-z)X$ is not equal to $\{0\}$. We note that $(1-z)e \neq 0$, $ze \neq 0$ and $(1-z)e, ez \in w(X)$.

Let $Q$ be the topological space of characters of the algebra $X$, $\bar{Q}$ be the topological space of characters of the algebra $w(X)$. By lemma 9 $supp(Q) \subseteq supp(\bar{Q})$ and the set $supp(Q)$ is dense in $\bar{Q}$.

We have the sets $V = \{ q \in \bar{Q} : ze(q) \neq 0 \}$ and $W = \{ q \in \bar{Q} : (1-z)e(q) \neq 0 \}$ are close-open sets of $Q$ and $supp(Q) = V \cup W$. Note that $V \cap supp(Q)$ is dense in $V$ and the set $W \cap supp(Q)$ is dense in $W$. Suppose that $W$ does not contain an open set of the space $Q$. Then the set $W \cap supp(Q)$ does not contain an open set of the space $Q$. In this case $V \cap supp(Q)$ is dense in $Q$. Indeed, if $V \cap supp(Q)$ is not dense in $Q$ then the closure $Cl(V \cap supp(Q))$ of the set $V \cap supp(Q)$ in $Q$ is not equal to $Q$, i.e. $Cl(V \cap supp(Q)) \neq supp(Q)$ and $supp(Q) \setminus Cl(V \cap supp(Q))$ is an open set in $Q$, that contains in the set $W \cap supp(Q)$. This is a contradiction.

Thus $V \cap supp(Q)$ is dense in $Q$. Then every function $f$ in the algebra $C(C)$ of all real-valued continuous functions on the locally compact space $Q$ is an unique extension of the function $f_{V \cap supp(Q)}$, defined on the set $V \cap supp(Q)$. Therefore the algebra $C(Q)$ can be embedded in $C(V) = \{ f \in C(\bar{Q}) : \{ x \in \mathbb{Q} : f(x) \neq 0 \} \subseteq V \}$. Then since $supp(Q)$ is dense in $\bar{Q}$, then every function $f$ in the algebra $C(V)$ has an unique continuous extension on $\bar{Q}$, that is the algebras $C(V)$ and $C(Q)$ can be identified in the sense of $V \subseteq Q$. Hence $V$ is dense in $\bar{Q}$. Otherwise the set $supp(Q) \setminus Cl(V)$ is open and nonempty in $\bar{Q}$. In this case since $V$ is close-open in $\bar{Q}$ then we have $Cl(V) = V$, $W = supp(Q) \setminus Cl(V)$ and $supp(Q) \setminus Cl(V)$ is also close-open in $\bar{Q}$. Then $C(\bar{Q}) = C(V) \oplus C(W)$ and $C(W) \neq \{0\}$, what contradicts the identifiability of the algebras $C(V)$ and $C(Q)$. Thus $V$ is dense in $\bar{Q}$. Then since $V$ is a close-open set in $\bar{Q}$ we have $V = \bar{Q}$. Hence $ze$ a unitary element of the algebra $w(X)$, i.e. $ze = e$. Then by the previous part of the proof $z = 1$.

Now, suppose that $W$ contains an open set $U$ of the space $Q$. We have the set $C(U)$ of all functions $f \in C(Q)$ such, that the set $x \in \mathbb{Q} : f(x) \neq 0$ belongs to $U$, is a subalgebra of the algebra $C(Q)$. Since $U \neq \emptyset$ and $U$ is a close-open set of $Q$, then the algebra $C(U)$ has a nonzero function. Therefore the set $(1-z)ew(X) \cap X$ and, hence, the set $(1-z)w(A) \cap A$ are not empty. Then the set $I_o = (1-z)w(A) \cap A$ is an ideal of the algebra $A$ that is $I_oA \subseteq I_o$. Hence, since $A$ is simple, then $I_o = A$ and $1 - z = 1$. Hence $z = 0$.

Then, since the central projection $z$ is chosen arbitrarily, we have $w(A)$ is a factor. ▷
Theorem 23. Let $A$ be a simple $C^*$-algebra of bounded linear operators on a Hilbert space $H$. Then $A$ is a CCR-algebra if and only if $A$ is of von Neumann type I.

Proof. By theorem 17, if $A$ is a CCR-algebra, then $A$ is of von Neumann type I. Now, suppose $A$ is of von Neumann type I. Let $\pi$ be a representation of the algebra $A$ in the Hilbert space $H$. Then $\pi(A)$ is also a simple $C^*$-algebra. We assert that for an abelian annihilator $X \in \mathcal{P}$ such that $c(X) = A$ we have $\pi(X) = \{0\}$. Indeed, by theorem 22 there exists a minimal projection $p \in A$ such that $X = \mathbb{C}p$. We have $\pi(p) \neq 0$. By theorem 22 since $A$ is simple then $A$ is a CCR-algebra.

Let $A$ be a $C^*$-algebra, $\mathcal{P}$ a lattice of annihilators of $A$, $n$ be a cardinal number and $\Xi$ be a set of indexes such that $|\Xi| = n$. We say $A$ is a $C^*$-algebra of type $I_n$, if there is a set $\{P_i\}_{i \in \Xi}$ of pairwise orthogonal abelian annihilators with central support $A \in \mathcal{P}$ and $\sup_i \{P_i\}_{i \in \Xi} = A$.

Theorem 24. Let $A$ be a simple $C^*$-algebra of von Neumann type I. Then there exists a cardinal number $n$ such that $A$ is a $C^*$-algebra of type $I_n$.

Proof. Let $\{P_i\}$ be a maximal set of orthogonal abelian annihilators with a set of indexes $\Xi$. It is clear that the lattice $\mathcal{P}$ has only central elements $\{0\}$ and $A$. By theorem 22 $w(A)$ is a $W^*$-factor of type I and $\{P_i\}$ is an orthogonal set of minimal projections of the algebra $w(A)$ (hence of the algebra $A$).

We suppose that $\sup_i \{P_i\}_{i \in \Xi} < A$. Then $Ann(\sup_i \{P_i\}_{i \in \Xi}) \neq \{0\}$. By theorem 23 $A$ is a GCR-algebra. Hence the C*-subalgebra $Ann(\sup_i \{P_i\}_{i \in \Xi})$ is also a GCR-algebra. Hence, by the proof of theorem 17 there exists a nonzero abelian annihilator $X$ in the subalgebra $Ann(\sup_i \{P_i\}_{i \in \Xi})$. The last statement contradicts maximality of the set $\{P_i\}$. Therefore $\sup_i \{P_i\}_{i \in \Xi} = A$. Hence $A$ is a $C^*$-algebra of type $I_n$, where $n = |\Xi|$.

Examples. 1. Let $A$ be the closure on the norm of the inductive limit $A_0$ of the inductive system

$$C \xrightarrow{M_2(C)} M_3(C) \xrightarrow{M_4(C)} \ldots,$$

where $M_n(C)$ is mapped into the upper left corner of $M_{n+1}(C)$. Then $A$ is a $C^*$-algebra. The algebra $A$ contains the minimal projections of the form $e_{ii}$, where $e_{ij}$ is a matrix, whose $(i, i)$-component is 1 and the rest components are zero. These projections form the countable orthogonal set $\{e_{ii}\}_{i \geq 1}$ of minimal projections. We have $\{Ce_{ii}\}_{i \geq 1} \subseteq \mathcal{P}_A$ and $c(Ce_{ii}) = A$ for any $i$. Hence $A$ is a $C^*$-algebra of type $I_{\infty}$, where $\mathcal{N}_0 = \{|\{Ce_{ii}\}_{i \geq 1}\}$.

2. Let $H$ be an infinite dimensional complex Hilbert space. It is known that the space $K(H)$ of all compact bounded linear operators on $H$ is a $C^*$-algebra. Moreover it is a CCR-algebra. The algebra $K(H)$ has a maximal set of minimal projections. Every of these minimal projections generates an abelian annihilator, which is isomorphic to $C$. These annihilators form a maximal orthogonal set of abelian annihilators in $\mathcal{P}_{K(H)}$ with central support $K(H)$. Hence $K(H)$ is a $C^*$-algebra of type $I_n$, where $n = \dim(H)$.

7. Equivalence relation in C*-algebras

Let $A$ be a $C^*$-algebra of bounded linear operators on a Hilbert space $H$. Let $V, W$ be annihilators of $\mathcal{P}$. We will write $V \approx W$ in the algebra $A$ (in the algebra $w(A)$), if there exists such Banach subspace $B \subseteq A$ (accordingly $B \subseteq w(A)$) that $V_+ = \{bb^* : b \in B\}$ and $W_+ = \{b^*b : b \in B\}$.
Lemma 25. Let \( p, q \) be projections of a C*-algebra \( A \) of bounded linear operators on a Hilbert space \( H \). Then \( p \sim q \) in \( w(A) \) if and only if \( pAq \cong qAq \) in the algebra \( A \).

Proof. It is obvious that \( pAq, qAq \in \mathcal{P} \). Suppose \( p \sim q \). Then there exists such \( x \in A \) that \( xx^* = p, x^*x = q \). Also we have \( px = x, x^*p = x^*, qx = x, qx^* = x^* \), \( pAq = xAq \). Let \( B = \{ xbx : b \in A \} \). Then \( B \) is a Banach space, \( \{ bb^* : b \in B \} = xA, \{ b^*b : b \in B \} = x^*A, x \). Indeed, \( xA \) is a C*-subalgebra and \( \{ xx^*xx^* : b \in \{ A \} \subseteq xA, x \). Therefore \( pAq \cong qAq \) in the algebra \( A \).

Conversely, let \( pAq \cong qAq \) in \( A \) by some Banach space \( B \subseteq A \). Then there exists such \( b \in B \) that \( b^*b = p \). Then \( b^*b \) is a projection and \( bb^* \in qAq \). Hence \( p \sim q \).

Analogously, there exists such \( d \in B \) that \( dd^* = q \). Then \( d^*d \) is a projection and \( d^*d \in pAq \). Hence \( p \sim q \) and \( p \sim q \) in the algebra \( w(A) \).

Lemma 26. Let \( A \) be a C*-algebra of bounded linear operators in a Hilbert space \( H, V, W, U \in \mathcal{P}, S = pAq \cap A, p, q \) be the units of the subalgebras \( w(V) \) and \( w(W) \) accordingly. Then

(a) \( V \cong W \) in \( A \) if and only if \( \{ ss^* : s \in S \} = V_+ \) and \( \{ s^*s : s \in S \} = W_+ \),
(b) if \( V \cong W \) in \( A \) then \( q \sim p \) in \( w(A) \). Conversely, if \( q \sim p \) in \( w(A) \) then \( V \cong W \) in \( w(A) \),
(c) if \( V \cong W \) and \( W \cong U \) in \( w(A) \), then \( V \cong U \) in \( w(A) \).

Proof. We prove (a). Let \( V \cong W \) in \( A \). Then there exists such Banach space \( B \) that \( \{ bb^* : b \in B \} = V_+ \) and \( \{ b^*b : b \in B \} = W_+ \). Let’s notice, that \( B \subseteq pAq \cap A \). Hence \( \{ bb^* : b \in B \} \subseteq \{ ss^* : s \in S \}, \{ b^*b : b \in B \} \subseteq \{ s^*s : s \in S \} \). Since \( \{ ss^* : s \in S \} \subseteq pAq \cap A_+ \), \( \{ s^*s : s \in S \} \subseteq qAq \cap A_+ \), then \( \{ ss^* : s \in S \} = V_+ \) and \( \{ s^*s : s \in S \} = W_+ \). The converse is obvious.

(b) Suppose \( V \cong W \) in \( A \). We have there exists such Banach subspace \( B \subseteq A \) that \( V_+ = \{ bb^* : b \in B \} \) and \( W_+ = \{ b^*b : b \in B \} \). By item (a) we can regard that \( B \subseteq pAq \cap A \). We prove that \( w(B) = pw(A)q \). Let \( (v_n) \) be an increasing approximate identity of the subalgebra \( V \), then a least upper bound of the net \( (v_n) \) in \( w(V) \) is the unit of \( w(V) \). Let \( (w_n) \) be an increasing approximate identity of the subalgebra \( W \), then a least upper bound of the net \( (w_n) \) in \( w(W) \) is the unit of \( w(W) \). Let \( a \in w(A) \). Then \( paq \in pw(A)q \). There exists a net \( (a_n) \subseteq A \) such that \( (a_n) \) weak converge to \( a \). We have the net \( (v_n, a_n, w_n) \) is a subset of the set \( B \) and weak converge to the element \( paq \). Hence \( paq \in w(B) \) and since the element \( a \) was chosen arbitrarily then \( w(B) = pw(A)q \). Then we have \( w(V)_+ = \{ ss^* : s \in pw(A)q \} \) and \( w(W)_+ = \{ s^*s : s \in pw(A)q \} \), that is \( w(V) \cong w(W) \) in \( w(A) \). Then by lemma 25 \( q \sim p \) in \( w(A) \).

The second part of the statement of item (b) is proved as in the proof of lemma 25.

(c) Let \( g \) be the unitary element of \( w(U) \). Then by item (b) of lemma 26 \( p \sim q \sim g \). Hence \( p \sim g \) and by the second part of item (b) of lemma 26 \( V \cong U \) in \( w(A) \).

It can be noted that by lemma 26 the relation \( V \cong W \) in \( w(A) \), for any \( V, W \in \mathcal{P} \) for a C*-algebra \( A \), is an equivalent relation of elements of the orthomodular lattice \( \mathcal{P} \).
8. Simple $C^*$-algebras without nonzero abelian annihilators

A projection $p$ of a $C^*$-algebra is said to be finite, if there does not exist subprojection $q$ of $p$, which equivalent to $p$. A simple $C^*$-algebra is said to be finite, if every projection of this algebra is finite.

It is known that there exists a faithful dimension function on any modular lattice. Therefore, if for a given lattice does not exist a faithful dimension function, then this lattice is not modular (see [7]).

Definition. Let $A$ be a $C^*$-algebra, $\mathcal{P}$ be the corresponding lattice of annihilators. $A$ is called $C^*$-algebra of type II, if $\mathcal{P}$ is locally modular. $A$ is called $C^*$-algebra of type II$_1$, if $\mathcal{P}$ is modular. $A$ is called $C^*$-algebra of type III, if $\mathcal{P}$ is purely nonmodular.

Theorem 27. A $C^*$-algebra of type II$_1$ is finite.

Proof. Let $A$ be a $C^*$-algebra of type II$_1$. Then there exists a faithful dimension function $D$ on the modular lattice $\mathcal{P}$. By lemma 26 values of the dimension function $D$ on equivalent annihilators coincide. By lemma 25 and the additivity of $D$ all projections of the algebra $A$ are finite. Hence $A$ is finite. $\triangleright$

A simple $C^*$-algebra $A$ is said to be purely infinite if every nonzero hereditary subalgebra of $A$ contains an infinite projection.

Theorem 28. A simple purely infinite $C^*$-algebra is of type III.

Proof. Let $A$ be a purely infinite $C^*$-algebra. We note that, any subalgebra of the form $xAx^*$ is hereditary. Hence, by lemmas 2, 25 and 26 for any $X \in \mathcal{P}$ there does not exist a nonzero faithful dimension function on $X$. Hence any annihilator in $\mathcal{P}$ is nonmodular. Hence, $A$ is of type III. $\triangleright$

Definition. Let $A$ be a $C^*$-algebra, $\mathcal{P}$ be the corresponding lattice of annihilators. $A$ is called $C^*$-algebra of type II$_\infty$, if $\mathcal{P}$ is locally modular and the annihilator $A$ is nonmodular.

Theorem 29. Let $A$ be a $W^*$-factor of type II$_\infty$. Then $A$ has a proper ideal $J$ such that $J$ is a simple $C^*$-algebra of type II$_\infty$.

Proof. We take a maximal chain $J_0 \supseteq J_1 \supseteq \cdots \supseteq J_n \supseteq J_{n+1} \supseteq \cdots$ of proper ideals of the algebra $A$, where $n$ is a cardinal number, i.e. if for an arbitrary ideal $I$ for any index $n J_n \supseteq I$ then $I \in \{J_n\}$. Then $\bigcap_n J_n$ is an ideal and a simple $C^*$-algebra. Indeed, from the definition $\bigcap_n J_n$ is a $C^*$-subalgebra. Moreover $\bigcap_n J_n$ is a proper ideal of the algebra $A$. If there exists a proper ideal $I$ of the $C^*$-algebra $\bigcap_n J_n$, then $I$ is also an ideal of the algebra $A$. Hence $I \in \{J_n\}$. Therefore $I = \bigcap_n J_n$ and the $C^*$-algebra $\bigcap_n J_n$ is simple.

We assert that for any projection $p$ of the algebra $A$ there exists an element $a \in J$ such that $pa = a$. Indeed, by the definition $pJ \subset J$. Hence, $pb \in J$ for any $b \in J$. Then, if $pJ \neq 0$ then there exists an element $b \in J$ such that $pb \neq 0$ and $pb \in J$. In this case as the element $a$ we take $pb$, i.e. $a = pb$. Otherwise, if $pJ = 0$, then $pA(1 - p)J$ is not a subset of $J$. This is a contradiction. Thus for any projection $p$ of the algebra $A$ there exists an element $a \in J$ such that $pa = a$.

Therefore for a finite projection $e \in A$ we have $J \cap eAe \neq \{0\}$. Hence, $J \cap eAe$ is a nonzero annihilator of $J$.

Now we assert that $J \cap eAe$ is modular. Note that for any projection $q \in A$ there exists a positive element $a \in J$ such that $qa \neq 0$. Hence, $qa \in J$ and $qaq \in J$, that is $qaq \cap J \neq \{0\}$. Moreover the weak closure $w(qAq \cap J)$ of the set $qAq \cap J$ contains
the projection $q$. Hence, the map $\Phi : P(A) \to P_J$, defined as

$$\Phi(q) = qAq \cap J,$$

is an order isomorphism. Indeed, let $p$ and $q$ be various projections of the algebra $A$. We have $\sup\{r(a) : a \in pAp \cap J\} = p$ and $\sup\{r(a) : a \in qAq \cap J\} = q$. If $pAp \cap J = qAq \cap J$ then we have $p = q$. What is a contradiction. Therefore $pAp \cap J \neq qAq \cap J$. Hence the map $\Phi$ is a one-to-one map. Hence $\Phi$ is an order isomorphism.

Theorem 30. For any simple C*-algebra $A$ one of the next conditions holds:

(a) $A$ is of type $I_n$, where $n$ is a natural number;

(b) $A$ is of type $I_n$, where $n$ is an infinite cardinal number;

(c) $A$ is of type $II_1$;

(d) $A$ is of type $II_\infty$;

(e) $A$ is of type III.

Proof. The theorem follows by theorems 20, 24 and the definitions of C*-algebras of types $II_1, II_\infty$. ▷

Remark. Note that in the case of von Neumann algebras the definitions of C*-algebras of types $I_n$, where $n$ is a cardinal number, $II_1, II_\infty$ and III are equivalent to the definitions of von Neumann algebras of types $I_n, II_1, II_\infty$ and III. By the theory, developed above, there exist simple C*-algebras of types $I_n, II_1, II_\infty$ and III. At the same time there exist only simple von Neumann algebras of types $I_n$, with $n$ finite, $II_1$ and III in the case of von Neumann algebras. Therefore in view of the theory, developed in the given article, we cannot similarly divide the (infinite dimensional) simple C*-algebras into two types; a finite type resembling the type $II_1$-factors and an infinite type resembling the type III-factors (see [19]).

Note that, in the work [17] M.Rordam have given an example of a simple C*-algebra with a finite and an infinite projection. The question "What type is this C*-algebra with a finite and an infinite projection of?" is open.

The approach to the classification problem for C*-algebras described in the given article may closely be connected to the Elliott classification conjecture. Indeed, on the one hand, theorem 30 is a completion of the theory, developed on the base of the Elliott classification conjecture and other methods ([2], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20]). On the other hand, the further developing the theory based on the notions introduced and studied in the given article may allow to add new type invariants to the list of the invariants of the Elliott classification conjecture and form new classification conjecture based on the Elliott classification conjecture.

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