\[ \mathcal{W}_{1+\infty} \text{ algebra, } \mathcal{W}_3 \text{ algebra, and} \\
\text{Friedan-Martinec-Shenker} \\
\text{bosonization} \]

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Abstract

We show that the vertex algebra \( \mathcal{W}_{1+\infty} \) with central charge \(-1\) is isomorphic to a tensor product of the simple \( \mathcal{W}_3 \) algebra with central charge \(-2\) and a Heisenberg vertex algebra generated by a free bosonic field. We construct a family of irreducible modules of the \( \mathcal{W}_3 \) algebra with central charge \(-2\) in terms of free fields and calculate the full character formulas of these modules with respect to the full Cartan subalgebra of the \( \mathcal{W}_3 \) algebra.

0 Introduction

In search of classification of conformal field theories, one is lead to study \( \mathcal{W} \) algebras which are extended chiral algebras (vertex algebras or vertex operator algebras in mathematical terminology) containing Virasoro algebra as a subalgebra. Since the first attempt was made by Zamolodchikov \([Z]\) there has been much further study of \( \mathcal{W} \) algebras (see the review paper \([BS]\) and references therein). A particularly interesting example of \( \mathcal{W} \) algebra, the so-called \( \mathcal{W}_{1+\infty} \) algebra \([PRS]\), appears to be a universal one among various \( \mathcal{W} \) infinite algebras in the \( N \rightarrow \infty \) limit of \( \mathcal{W}(sl_N) \) algebras, see e.g. \([Br, BK, PRS, Q]\). The \( \mathcal{W}(sl_N) \) algebras are often referred to as \( \mathcal{W}_N \) algebras in literature.

\footnote{We note a less-known fact that \( \mathcal{W}_N \) algebras were constructed in \([F2]\) for the particular central charge \( c = N - 1 \)}
In mathematics, $\mathcal{W}_{1+\infty}$ is known as the universal central extension $\hat{\mathcal{D}}$ of the Lie algebra $\mathcal{D}$ of differential operators on the circle. The first systematic study of the representation theory of the Lie algebra $\mathcal{D}$ was undertaken by Kac and Radul in their seminal paper [KR1] and there have been many further development [M, FKRW, AFMO, KR2, W1] since then, just to name some.

In [FKRW], Lie algebra $\hat{\mathcal{D}}$ and its representation theory are studied in the framework of vertex algebras [B, FLM, DL, K2, LZ2]. It turns out that the irreducible vacuum $\hat{\mathcal{D}}$-module with central charge $c$ admits a canonical vertex algebra structure, with infinitely many generating fields of conformal weights $2, 3, 4, \ldots$, which we will denote by $\mathcal{W}_{1+\infty,c}$. The case when the central charge is non-integral is not difficult to understand. The case when the central charge is a positive integer was studied in detail in [FKRW]. The vertex algebra $\mathcal{W}_{1+\infty,N}$ with a positive integral central charge $N$ has redundant symmetries, namely only the first $N$ generating fields are independent. More precisely $\mathcal{W}_{1+\infty,N}$ is shown to be isomorphic to a $\mathcal{W}$-algebra $\mathcal{W}(gl_N)$ with central charge $N$ and the irreducible modules of $\mathcal{W}_{1+\infty,N}$ are classified [FKRW].

In this paper we will take the first step to clarify the connection between vertex algebra $\mathcal{W}_{1+\infty,-N}$ and some other $\mathcal{W}$-algebra with finitely many generating fields. We prove that the vertex algebra $\mathcal{W}_{1+\infty,-1}$ is isomorphic to a $\mathcal{W}(gl_3)$ algebra, which is a tensor product of the simple $\mathcal{W}_3$ algebra with central charge $-2$ (denoted below by $\mathcal{W}_{3,-2}$) and a Heisenberg vertex algebra generated by a free bosonic field. We will construct explicitly a number of modules of the $\mathcal{W}_{3,-2}$ algebra parametrized by integers in terms of free fields. We prove the irreducibility of these modules. As a by-product, we obtain full character formulas for these representations. To our best knowledge, these seem to be the first known full character formula of any non-trivial module of the $\mathcal{W}_3$ algebra with any non-generic central charge. We mention a curious fact that a generating function of counting covers of an elliptic curve [Di] appears to be closely related to our character formulas and admits very interesting modular invariance properties [KZ].

The difficulties appearing in the negative integral central charge case in contrast to the positive integral central charge case are roughly the following: In both cases we have free field realizations. In the case of positive integral central charge case we need $bc$ fields which are free fermions while in the negative integral central charge case we need $\beta\gamma$ fields which are free bosonic ghosts. The structure of $\mathcal{W}_N$ algebra in the realization of $\mathcal{W}_{1+\infty,N}$ in terms of $bc$ fields can be identified relatively easily due to the very fact that the structure of the basic representation
of the affine Kac-Moody algebra $\hat{sl}_N$ is well understood \cite{K1}. However structures of representations of affine algebras with negative integral central charges are far from being clear.

One of the main technique we use in relating the $W_{1+\infty}$ algebra with central charge $-1$ to $W_3$ algebra with central charge $-2$ is the bosonization of $\beta\gamma$ fields \cite{FMS}. A similar construction was also given by Kac and van de Leur and used by them for a construction of a super KP hierarchy \cite{KV1, KV2}. More detailed structures in the bosonization of $\beta\gamma$ fields are further worked out in \cite{FF} and used for the computation of semi-infinite cohomology of the Virasoro algebra with coefficient in the module of its adjoint semi-infinite symmetric powers. It is well known that $\beta\gamma$ fields are fundamental ingredients in superstring theory \cite{FMS}, in realizations of level $-1$ representations of classical affine algebras \cite{Fer} and in the calculation of BRST cohomology of super-Virasoro algebras \cite{LZ1}. They are also closely related to the logarithmic conformal field theories which recently attract much attention from physicists, see e.g. \cite{F, Ka, GK}. We hope our results may shed some lights on these subjects.

Let us explain in more detail. It is well known \cite{M} that the Fock space $M_s$ of the $\beta\gamma$ fields as a module over $D$ can be decomposed into a direct sum of the modules $M^l_s$ parametrized by the $\beta\gamma$–charge number $l$. Recall \cite{FMS} that the $\beta\gamma$ fields are expressed in terms of two scalar fields $\psi(z)$ and $\phi(z)$. So the space $M_s$ can be identified with some subspace of the Fock space of the Heisenberg algebra of the two scalar fields $\psi(z)$ and $\phi(z)$. Indeed one can identify $M^l_s$ as $F^l \otimes H_{i(s+l)}$, $i = \sqrt{-1}$ (cf. e.g. \cite{FF}), where $F^l$ is a certain subspace of the Fock space of the Heisenberg algebra generated by the Fourier components of field $\psi(z)$ while $H_{i(s+l)}$ is the Fock space of the Heisenberg algebra of the field $\phi(z)$.

$W_{1+\infty,-1}$ acts on $M^l_s$ by means of fields

$$J^l(z) =: \gamma(z)\partial^l \beta(z) : = + \frac{1}{i + 1}s(s - 1) \cdots (s - i)z^{-i - 1}, i \in \mathbb{Z}_+.$$ 

For the sake of simplicity, the reader may understand the main results of this paper by taking $s = 0$ throughout this paper. By the celebrated boson-fermion correspondence, we have a pair of fermionic fields $b(z)$ and $c(z)$ expressed in terms of the scalar field $\psi(z)$. Furthermore we can construct two particular fields as some normally ordered polynomials of fields $b(z)$ and $c(z)$ and their derivative fields: a Virasoro field $T(z)$ of conformal weight 2 and a field $W(z)$ of conformal weight 3. These two fields $T(z)$ and $W(z)$ satisfy the operator product expansion of the $W_3$
algebra with central charge $-2$. The three fields $J^0(z), T(z)$ and $W(z)$ may be regarded as generating fields of a $\mathcal{W}(gl_3)$ algebra. We will show that all the $J^i(z) = : \gamma(z) \partial^i \beta(z) ;; i = 0, 1, \ldots$, can be expressed (See Lemmas 4.2 and 4.3) as some normally ordered polynomials in terms of $T(z), W(z)$ and $J^0(z)$ and their derivative fields. Since the space $\mathcal{F} \otimes \mathcal{H}_{i(s+l)}$, being isomorphic to $\mathcal{M}_{s}$, is an irreducible module over the vertex algebra $\mathcal{W}_{1+\infty,-1}$, it is also irreducible as a module over the $\mathcal{W}(gl_3)$ algebra.

One can show that $J^0(z) = i \partial \phi(z)$, by using Friedan-Martinec-Shenker bosonization. Note that when the $\mathcal{W}(gl_3)$ algebra acts on $\mathcal{F} \otimes \mathcal{H}_{i(s+l)}$, the Fourier components of fields $T(z)$ and $W(z)$ act only on the first factor $\mathcal{F}$, while $J^0(z)$ acts only on the second factor $\mathcal{H}_{i(s+l)}$. This implies that $\mathcal{F}$ is irreducible as a module over the $\mathcal{W}_{3,-2}$ algebra.

We obtain full character formulas of these irreducible modules $\mathcal{F}$ of the $\mathcal{W}_{3,-2}$ algebra as a consequence of our explicit free field realization. As a by-product of our free field realization of $\mathcal{F}$, we find that there exists non-split short exact sequences of modules over the $\mathcal{W}_3$ (resp. $\mathcal{W}_{1+\infty,-1}$) algebra with central charge $-2$ (resp. $-1$).

The plan of this paper goes as follows. In Section 1, we review the definition of $\hat{D}$ and the construction of vertex algebra $\mathcal{W}_{1+\infty,c}$. We present the free field realization of $\mathcal{W}_{1+\infty,-1}$ in terms of $\beta \gamma$ fields. In Section 2, we recall the bosonization of $\beta \gamma$ fields in detail. In Section 3 we review the $\mathcal{W}_3$ algebra in the framework of vertex algebras. In Section 4 we prove that vertex algebra $\mathcal{W}_{1+\infty,-1}$ is isomorphic to a tensor product of the simple $\mathcal{W}_{3,-2}$ algebra and a Heisenberg vertex algebra generated by a free bosonic field. We construct a number of irreducible modules of the $\mathcal{W}_{3,-2}$ algebra. In Section 5, we calculate the full character formula for representations of the $\mathcal{W}_{3,-2}$ algebra constructed in Section 4.

We take this opportunity to announce that we classify the irreducible modules of the $\mathcal{W}_{3,-2}$ algebra in our subsequent paper [W2]. It turns out that these irreducible modules are parametrized by points on a certain rational curve. We will also classify all the irreducible modules of $\mathcal{W}_{1+\infty,-1}$ algebra based on the relation between $\mathcal{W}_{3,-2}$ and $\mathcal{W}_{1+\infty,-1}$ algebras found in this paper.
1  Vertex algebra $\mathcal{W}_{1+\infty,c}$ and free fields realization of $\mathcal{W}_{1+\infty,-1}$

Let $\mathcal{D}$ be the Lie algebra of regular differential operators on the circle. The elements

$$J^l_k = -t^{l+k}(\partial_t)^l, \quad l \in \mathbb{Z}_+, k \in \mathbb{Z},$$

form a basis of $\mathcal{D}$. $\mathcal{D}$ has also another basis

$$L^l_k = -t^k D^l, \quad l \in \mathbb{Z}_+, k \in \mathbb{Z},$$

where $D = t\partial_t$. Denote by $\widehat{\mathcal{D}}$ the central extension of $\mathcal{D}$ by a one-dimensional center with a generator $C$, with commutation relation (cf. [KRI])

$$[t^r f(D), t^s g(D)] = t^{r+s} (f(D + s)g(D) - f(D)g(D + r)) + \Psi(t^r f(D), t^s g(D))C,$$

where

$$\Psi(t^r f(D), t^s g(D)) = \begin{cases} \sum_{-r \leq j \leq -1} f(j)g(j + r), & r = -s \geq 0 \\ 0, & r + s \neq 0. \end{cases}$$

Letting weight $J^l_k = k$ and weight $C = 0$ defines a principal graduation

$$\widehat{\mathcal{D}} = \bigoplus_{j \in \mathbb{Z}} \widehat{\mathcal{D}}_j.$$  \hfill (1.3)

Then we have the triangular decomposition

$$\widehat{\mathcal{D}} = \widehat{\mathcal{D}}_+ \bigoplus \widehat{\mathcal{D}}_0 \bigoplus \widehat{\mathcal{D}}_-,$$  \hfill (1.4)

where

$$\widehat{\mathcal{D}}_\pm = \bigoplus_{j \in \pm \mathbb{N}} \widehat{\mathcal{D}}_j, \quad \widehat{\mathcal{D}}_0 = \mathcal{D}_0 \bigoplus \mathbb{C}C.$$  

Let $\mathcal{P}$ be the distinguished parabolic subalgebra of $\mathcal{D}$, consisting of the differential operators that extends into the whole interior of the circle. $\mathcal{P}$ has a basis $\{J^l_k, l \geq 0, l+k \geq 0\}$. It is easy to check that the 2-cocycle $\Psi$ defining the central extension of $\widehat{\mathcal{D}}$ vanishes when restricted to the parabolic subalgebra $\mathcal{P}$. So $\mathcal{P}$ is also a subalgebra of $\widehat{\mathcal{D}}$. Denote $\widehat{\mathcal{P}} = \mathcal{P} \oplus \mathbb{C}C$. 

Fix $c \in \mathbb{C}$. Denote by $C_c$ the 1–dimensional $\hat{P}$ module by letting $C$ acts as scalar $c$ and $P$ acts trivially. Fix a non-zero vector $v_0$ in $C_c$. The induced $\hat{D}$–module

$$M_c(\hat{D}) = \mathcal{U}(\hat{D}) \otimes_{\mathcal{U}(P)} C_c$$

is called the vacuum $\hat{D}$–module with central charge $c$. Here we denote by $\mathcal{U}(g)$ the universal enveloping algebra of a Lie algebra $g$. $M_c(\hat{D})$ admits a unique irreducible quotient, denoted by $\mathcal{W}_{1+\infty,c}$. Denote the highest weight vector $1 \otimes v_0$ in $M_c(\hat{D})$ by $|0\rangle$.

It is shown in [FKRW] that $\mathcal{W}_{1+\infty,c}$ carries a canonical vertex algebra structure, with vacuum vector $|0\rangle$ and generating fields

$$J^l(z) = \sum_{k \in \mathbb{Z}} J^l_k z^{-k-l-1},$$

of conformal weight $l+1, l = 0, 1, \cdots$. The fields $J^l(z)$ corresponds to the vector $J^l_{l-1}|0\rangle$ in $\mathcal{W}_{1+\infty,c}$. Below we will concentrate on the particular case $\mathcal{W}_{1+\infty,-1}$.

Recall that the bosonic $\beta \gamma$ fields are

$$\beta(z) = \sum_{n \in \mathbb{Z}} \beta(n) z^{-n+s}, \quad \gamma(z) = \sum_{n \in \mathbb{Z}} \gamma(n) z^{-n-s-1} \quad (s \in \mathbb{C}) \quad (1.5)$$

with the operator product expansions (OPEs)

$$\beta(z) \gamma(w) \sim -\frac{1}{z-w} (\frac{z}{w})^s, \quad \beta(z) \beta(w) \sim 0, \quad \gamma(z) \gamma(w) \sim 0. \quad (1.6)$$

In other words, we have the following commutation relations

$$[\gamma(m), \beta(n)] = \delta_{m,-n}, \quad [\beta(m), \beta(n)] = 0, \quad [\gamma(m), \gamma(n)] = 0. \quad (1.7)$$

Let us denote by $\mathcal{M}_s$ the Fock space of the $\beta \gamma$ fields, with the vacuum vector $|s\rangle$, and

$$\beta(n+1)|s\rangle = 0, \quad \gamma(n)|s\rangle = 0, \quad n \geq 0. \quad (1.7)$$

One can realize a representation of $\mathcal{W}_{1+\infty,-1}$ on $\mathcal{M}_s$ by letting (cf. [KR2, M], our convention here is a little different):

$$J^N(z) =: \gamma(z) \partial^N \beta(z) : + \frac{1}{N+1} s(s-1) \cdots (s-N) z^{-N-1}, N \in \mathbb{Z}_+.$$ 

(1.8)
The normal ordering $::$ is understood as moving the operators annihilating $|s\rangle$ to the right.

Note that $J^0(z) = \sum_{k \in \mathbb{Z}} J_k z^{-k-1}$ is a free bosonic field of conformal weight 1 with commutation relations

$$[J^0_m, J^0_n] = -m \delta_{m-n}, \quad m, n \in \mathbb{Z}.$$

We also have the following commutation relations:

$$[J^0_m, \beta(n)] = \beta(m+n), \quad [J^0_m, \gamma(n)] = -\gamma(m+n), \quad m, n \in \mathbb{Z}.$$ 

Then we have the $\beta \gamma$-charge decomposition of $\mathcal{M}_s$ according to the eigenvalues of the operator $-J^0_0$: $\mathcal{M}_s = \bigoplus_{l \in \mathbb{Z}} \mathcal{M}_s^l$. It is known \[\text{M, KR2}\] that $\mathcal{M}_0^0$ is isomorphic to $\mathcal{W}_{1+\infty,-1}$ as vertex algebras.

2 Bosonizations

In Section 2.1 we recall the well-known boson-fermion correspondence (cf. \[\text{F1}\]). In Section 2.2 we review the Friedan-Martinec-Shenker bosonization of the $\beta \gamma$ fields and some more detailed structures \[\text{FMS, FF}\].

2.1 Bosonization of fermions

Let $j(z)$ be a free bosonic field of conformal weight 1, namely

$$j(z)j(w) \sim \frac{1}{(z-w)^2},$$

or equivalently, by introducing $j(z) = \sum_{n \in \mathbb{Z}} j(n) z^{-n-1}$, we have

$$[j(m), j(n)] = m \delta_{m-n}.$$

Let us also introduce the free scalar field

$$\phi(z) = q + j(0) \ln z - \sum_{n \neq 0} \frac{j(n)}{n} z^{-n},$$

where the operator $q$ satisfies $[q, j(n)] = \delta_{n,0}$. Clearly $j(z) = \partial \phi(z)$.

Given $\alpha \in \mathbb{C}$, we denote by $\mathcal{H}_\alpha$ the Fock space of the free field $j(z)$ generated by the vacuum vector $|\alpha\rangle$ satisfying

$$j(n)|\alpha\rangle = \alpha \delta_{n,0} |\alpha\rangle, \quad n \geq 0.$$
It is well known that $\mathcal{H}_0$ is a vertex algebra, which we refer to as a Heisenberg vertex algebra. Easy to see that $\exp(\eta q)|\alpha\rangle = |\alpha + \eta\rangle$.

Introduce the vertex operator $X_\eta(z) = \sum_{n \in \mathbb{Z}} \exp(\eta q) z^n X_\eta(n) z^{-n}$ as follows. Let

$$X_\eta(z) = \exp(\eta \phi(z)) : = \exp(\eta \sum_{n > 0} j(-n) z^n / n) \exp(\eta \sum_{n < 0} j(-n) z^n / n).$$

The Fourier components of $X_\eta(z)$ act from $\mathcal{H}_\alpha$ to $\mathcal{H}_{\alpha + \eta}$. Furthermore we have the following OPE

$$j(z)X_\eta(w) \sim \frac{\eta X_\eta(w)}{z - w} + \frac{1}{\eta} \partial X_\eta(w),$$

or equivalently we have

$$[j(m), X_\eta(n)] = \eta X_\eta(m + n),$$

$$: j(z)X_\eta(z) : = \frac{1}{\eta} \partial X_\eta(z).$$

Also we have

$$X_\xi(z)X_\eta(w) \sim (z - w)^{\xi \eta} : X_\xi(z)X_\eta(w) :$$

In particular we have a pair of fermionic fields $X_\pm(z)$ with OPEs:

$$X_1(z)X_{-1}(w) \sim \frac{1}{z - w}, \quad X_{\pm 1}(z)X_{\pm 1}(w) \sim 0.$$  

This is the well-known boson-fermion correspondence.

### 2.2 Bosonization of bosons

First let us introduce the $bc$ fermionic fields

$$b(z) = \sum_{n \in \mathbb{Z}} b(n) z^{-n}, \quad c(z) = \sum_{n \in \mathbb{Z}} c(n) z^{-n-1}$$

with OPEs

$$b(z)c(w) \sim \frac{1}{z - w}, \quad b(z)b(w) \sim 0, \quad c(z)c(w) \sim 0.$$  

$$b(z)b(w) \sim 0, \quad c(z)c(w) \sim 0.$$  

$$2.12$$
In other words, we have
\[ [b(m), c(n)]_+ = \delta_{m,-n}, \quad [b(m), b(n)]_+ = 0, \quad [c(m), c(n)]_+ = 0. \]

We denote by \( \mathcal{F} \) the Fock space of the \( bc \) fields, generated by \( |bc\rangle \), satisfying
\[ b(n+1)|bc\rangle = 0, \quad c(n)|bc\rangle = 0, \quad n \geq 0. \]

Then
\[ j^{bc}(z) =: c(z)b(z) := \sum_{n \in \mathbb{Z}} j^{bc}_n z^{-n-1}, \]
is a free boson of conformal weight 1 with commutation relations
\[ [j^{bc}_m, j^{bc}_n] = m \delta_{m,-n}, \quad m, n \in \mathbb{Z}. \]

We further have the following commutation relations:
\[ [j^{bc}_m, b(n)] = -b(m+n), \quad [j^{bc}_m, c(n)] = c(m+n), \quad m, n \in \mathbb{Z}. \]

Then we have the \( bc \)-charge decomposition of \( \mathcal{F} \) according to the eigenvalues of \( j^{bc}_0 \):
\[ \mathcal{F} = \bigoplus_{l \in \mathbb{Z}} \mathcal{F}^l. \]

Following [FF], we consider the vector space
\[ N(s) = \sum_{l \in \mathbb{Z}} \mathcal{F}^l \otimes \mathcal{H}_{i(s+l)}, \]
and we define the actions of \( \beta(n), \gamma(n), n \in \mathbb{Z} \) on \( N(s) \) by letting [FMS]
\[ \beta(z) = \sum_{n \in \mathbb{Z}} \beta(n) z^{-n+s} = \partial b(z) X_{-i}(z), \quad (2.13) \]
\[ \gamma(z) = \sum_{n \in \mathbb{Z}} \gamma(n) z^{-n-s-1} = c(z) X_i(z). \quad (2.14) \]

It can be easily shown that the bosonic fields \( \beta(z), \gamma(z) \) defined above indeed satisfy the OPEs (1.6). The vector \( |bc\rangle \otimes |is\rangle \) satisfies the vacuum condition (1.7) by means of (2.13) and (2.14). Then we have a homomorphism \( \epsilon : \mathcal{M}_s \rightarrow N(s) \) as modules of the Heisenberg algebra spanned by \( \beta(n), \gamma(n), n \in \mathbb{Z} \), by letting
\[ \epsilon(|s\rangle) = |bc\rangle \otimes |is\rangle. \]

This homomorphism is obviously an embedding since \( \mathcal{M}_s \) is an irreducible module of the above Heisenberg algebra. The following proposition (cf. [FF]) tells us the precise image of this embedding, we reproduce the proof here since some crucial misprints in their proof in [FF] need to be corrected.
**Proposition 2.1** The image of the homomorphism $\epsilon$ coincides with the kernel of $c(0)$, acting from $N(s)$ to $N(s-1)$.

*Proof.* The operators $\beta(n), \gamma(n), n \in \mathbb{Z}$, given by (2.13) and (2.14) do not depend on $b(0)$ and therefore commute with $c(0)$. So we have $\text{Im} \epsilon \subset \ker c(0)$ since the operator $c(0)$ kills the vacuum vector $|bc\rangle$.

It is easy to see that the kernel of $c(0)$ is obtained by applying to $|bc\rangle \otimes |is\rangle$ the operators $j(n), c(n), n \in \mathbb{Z}$, and $b(m), m \in \mathbb{Z} - \{0\}$. So it remains to show that fields $j(z), c(z)$ and $\partial b(z)$ can be expressed in terms of fields $\beta(z), \gamma(z), X_{\pm i}(z)$ and their derivative fields. Indeed it is easy to show that

$$
\begin{align*}
\partial b(z) & = \partial \beta(z) X_i(z), \\
c(z) & = \partial \gamma(z) X_{-i}(z).
\end{align*}
$$

Recall that $J^0(z) =: \gamma(z) \beta(z) :$. Easy to check by (2.13) and (2.14) that $j(z) \equiv \partial_z \phi(z) = -i J^0(z)$. $\blacksquare$

Denote by $F^i$ the kernel of the operator $c(0)$ acting from $F^i$ to $F^{i+1}$. We now have a natural isomorphism:

$$
\mathcal{M}^i_k \cong F^i \otimes \mathcal{H}_{i(s+i)}.
$$

(2.15)

**Remark 2.1** $F^0$ is a vertex subalgebra of $F^0$. This is an example of the following well-known fact in the theory of vertex algebras: Given a vertex algebra $V$ and let $Y(a, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}$ be the field corresponding to some vector $a \in V$, then the kernel of the operator $a(0)$ acting on $V$ is always a vertex subalgebra of $V$.

### 3 $\mathcal{W}_3$ algebra

Denote by $U(\mathcal{W}_{3,c})$ ($c \in \mathbb{C}$ is the central charge) the quotient of the free associative algebra generated by $L_m, W_m, m \in \mathbb{Z}$ by the two-sided ideal generated by the following commutation relations (cf. e.g. [BMP]):

$$
\begin{align*}
[L_m, L_n] & = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}, \\
[L_m, W_n] & = (2m-n)W_{m+n}, \\
[W_m, W_n] & = (m-n)(\frac{1}{15}(m+n+3)(m+n+2) \\
& - \frac{1}{6}(m+2)(n+2))L_{m+n} \\
& + \beta(m-n)\Lambda_{m+n} + \frac{c}{360}m(m^2-1)(m^2-4)\delta_{m,-n},
\end{align*}
\quad (3.16)
$$

\[3\]
with $\beta = 16/(22 + 5c)$ and

$$\Lambda_m = \sum_{k \leq -2} L_k L_{m-k} + \sum_{k > -2} L_{m-k} L_k - \frac{3}{10} (m + 2)(m + 3)L_m.$$ 

Denote

$$\mathcal{W}_{3,+} = \{L_n, W_n, \pm n \geq 0\}, \quad \mathcal{W}_{3,0} = \{L_0, W_0\}.$$ 

A Verma module $\mathcal{M}_c(t,w)$ of $\mathcal{U}(\mathcal{W}_{3,c})$ is the induced module

$$\mathcal{M}_c(t,w) = \mathcal{U}(\mathcal{W}_{3,c}) \otimes_{\mathcal{U}(\mathcal{W}_{3,+} \oplus \mathcal{W}_{3,0})} \mathbb{C}_{t,w}$$

where $\mathbb{C}_{t,w}$ is the 1-dimensional module of $\mathcal{U}(\mathcal{W}_{3,+} \oplus \mathcal{W}_{3,0})$ such that

$$\mathcal{W}_{3,+} | t, w \rangle = 0, \quad L_0 | t, w \rangle = t| t, w \rangle, \quad W_0 | t, w \rangle = w| t, w \rangle. \quad (3.17)$$

$\mathcal{M}_c(t,w)$ has a unique irreducible quotient which is denoted by $\mathcal{L}_c(t,w)$. A singular vector in a $\mathcal{U}(\mathcal{W}_{3,c})$-module means a vector killed by $\mathcal{W}_{3,+}$. It is easy to see that $L_{-1}|0,0\rangle, W_{-1}|0,0\rangle$, and $W_{-2}|0,0\rangle$ are singular vectors in $\mathcal{M}(0,0)$. We denote by $\mathcal{VW}_{3,c}$ the vacuum module which is by definition the quotient of the Verma module $\mathcal{M}(0,0)$ by the $\mathcal{U}(\mathcal{W}_{3,c})$-submodule generated by the singular vectors $L_{-1}|0,0\rangle, W_{-1}|0,0\rangle$, and $W_{-2}|0,0\rangle$. We call $\mathcal{L}_c(0,0)$ the irreducible vacuum module. Let $I$ be the maximal proper submodule of the vacuum module $\mathcal{VW}_{3,c}$. Clearly $\mathcal{L}_c(0,0)$ is the irreducible quotient $\mathcal{VW}_{3,c}/I$ of $\mathcal{VW}_{3,c}$. It is easy to see that $\mathcal{VW}_{3,c}$ has a linear basis

$$L_{-i_1-2} \cdots L_{-i_m-2} W_{-j_1-3} \cdots W_{-j_n-3}|0,0\rangle,$$

$$0 \leq i_1 \leq \cdots \leq i_m, \quad 0 \leq j_1 \leq \cdots \leq j_n, \quad m, n \geq 0. \quad (3.18)$$

Introduce the following fields

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \quad W(z) = \sum_{n \in \mathbb{Z}} W_n z^{-n-3}. \quad (3.19)$$

It is well known that the vacuum module $\mathcal{VW}_{3,c}$ (resp. irreducible vacuum module $\mathcal{L}_c(0,0)$) carries a vertex algebra structure with generating fields $T(z)$ and $W(z)$. The $\mathcal{W}_3$ algebra with central charge $-2$ we have been referring to is the vertex algebra $\mathcal{L}_{-2}(0,0)$, which we denote by $\mathcal{W}_{3,-2}$ throughout our paper. Fields $T(z)$ and $W(z)$ correspond to the vectors $L_{-2}|0,0\rangle$ and $W_{-3}|0,0\rangle$ respectively. The field corresponding to the vector $L_{-i_1-2} \cdots L_{-i_m-2} W_{-j_1-3} \cdots W_{-j_n-3}|0,0\rangle$ is

$$\partial^{(i_1)} T(z) \cdots \partial^{(i_m)} T(z) \partial^{(j_1)} W(z) \cdots \partial^{(j_n)} W(z),$$
where $\partial^{(i)}$ denotes $\frac{1}{i!} \partial_z^i$. We can rewrite (3.16) in terms of the following OPEs in our central charge $-2$ case:

\begin{align}
T(z)T(w) &\sim \frac{-1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \\
T(z)W(w) &\sim \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w} \\
W(z)W(w) &\sim \frac{-2/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} \\
&\quad + \frac{1}{(z-w)^2} \left( \frac{8}{3} : T(w)T(w) : - \frac{1}{2} \partial^2 T(w) \right) \\
&\quad + \frac{1}{z-w} \left( \frac{4}{3} \partial : T(w)T(w) : - \frac{1}{3} \partial^3 T(w) \right). 
\end{align}

Represention theory of the vertex algebra $\mathcal{VW}_{3,c}$ is just the same as that of $\mathcal{U}(W_3)$. However note that $c = -2$ is not a generic central charge $[W2]$, namely the vacuum module $\mathcal{VW}_{3,c}$ with $c = -2$ is reducible, or in other word, the maximal proper submodule $I$ of $\mathcal{VW}_{3,c}$ is not zero. Thus representation theory of $W_{3,-2}$ becomes highly non-trivial due to the following constraints: a module $M$ of the vertex algebra $\mathcal{VW}_{3,c}$ can be a module of the vertex algebra $W_{3,-2}$ if and only if $M$ is annihilated by all the Fourier components of all fields corresponding to vectors in the maximal proper submodule $I \subset \mathcal{VW}_{3,c}$.

4 Relations between $W_3$ algebra and vertex algebra $W_{1+\infty,-1}$

Define

\begin{equation}
T(z) \equiv \sum_{n \in \mathbb{Z}} L_n z^{-n-2} = : \partial b(z)c(z) :. 
\end{equation}

Easy to check that $T(z)$ is a Virasoro field with central charge $-2$. We also define another field of conformal weight 3:

\begin{equation}
W(z) \equiv \sum_{n \in \mathbb{Z}} W_n z^{-n-3} = \frac{1}{\sqrt{6}} \left( : \partial^2 b(z)c(z) : - : \partial b(z)\partial c(z) : \right). 
\end{equation}

We have the following proposition whose proof is straightforward however tedious by using Wick’s theorem.

**Proposition 4.1** Fields $T(z)$ and $W(z)$ satisfy the OPEs (3.20) of $W_3$ algebra with central charge $-2$. 

We note that this $\mathcal{W}_3$ algebra structure in $bc$ fields was also observed in [BCMN]. We rescale $W(z)$ to be $\tilde{W}(z) = \frac{1}{2} \sqrt{6} W(z)$, namely

$$\tilde{W}(z) = \sum_{n \in \mathbb{Z}} \tilde{W}_n z^{-n-3} = \frac{1}{2} \left( \partial^2 b(z)c(z) : - : \partial b(z) \partial c(z) : \right). \quad (4.23)$$

We will see later that it is more convenient to work with the rescaled field $\tilde{W}(z)$. Now we can state our first main results of this paper.

**Theorem 4.1**

1) The vertex algebra $\mathcal{F}_0$ is isomorphic to the simple vertex algebra $\mathcal{W}_{3,-2}$, with generating fields $T(z)$ and $W(z)$. $\mathcal{F}_l$ ($l \in \mathbb{Z}$) are irreducible modules of the $\mathcal{W}_{3,-2}$ algebra.

2) The vertex algebra $\mathcal{W}_{1+\infty,-1}$ is isomorphic to a tensor product of the vertex algebra $\mathcal{W}_{3,-2}$, and the Heisenberg vertex algebra $\mathcal{H}_0$ with $J^0(z)$ as a generating field.

Proof of the above theorem reply on the following three lemmas:

**Lemma 4.1** The vector space $\mathcal{M}_s$ is irreducible regarded as a module of the vertex algebra $\mathcal{W}_{1+\infty,-1}$ via the free field realization [L3].

**Lemma 4.2** The fields $J^a(w) =: \gamma(w) \partial^a \beta(w) :$, $n \geq 0$ acting on the Fock space $\mathcal{M}_0$ can be expressed as a normally ordered polynomial in terms of fields $\partial^i b(w) \partial^j c(w) :$, $i + j \leq n$, $i > 0$, and $\partial^k j(w)$, $k = 0, 1, \ldots, n$.

More precisely, we have

$$\gamma(w) \partial^n \beta(w) : = \sum_{1 \leq k \leq n} \left[ k \binom{n}{k} \partial^{n-k+1} b(w)c(w) : P_{k-1}(j) \right] + C_n P_{n+1}(j), \quad (4.24)$$

where $C_n = (n + 2) \sum_{m=0}^{n} (-1)^{m+1} \frac{1}{m+2}$ is some constant depending on $n$, and the normally ordered polynomial $P_m(j)$ (or denoted by $P_m(j(w))$ when it is necessary to specify the variable in $j(w)$) in terms of the field $j(w)$ and its derivative fields is defined as (recall that $j(w) = \partial \phi(w)$)

$$P_m(j) = \frac{\partial^m_w : e^{-i\phi(w)} :}{: e^{-i\phi(w)} :}, \quad m \geq 0. \quad (4.25)$$
Proof of Lemma 4.2.
We will calculate the normally ordered product : $\partial^n \beta(w)\gamma(w) :$ instead of : $\gamma(w)\partial^n \beta(w) :$. These two normally ordered products coincide since both $\beta(w)$ and $\gamma(w)$ are free fields.

By formulas (2.13) and (2.14), we have

\[
: \partial^n \beta(w)\gamma(w) : = : \partial^n (\partial b(w)X_{-i}(w)) (c(w)X_i(w)) : = \sum_{0 \leq k \leq n} \binom{n}{k} [ : \partial^{n-k+1}b(w)\partial^k X_{-i}(w)c(w)X_i(w) : ].
\] (4.26)

It follows from the OPEs (2.12) that

\[
\partial_z^{n-k+1}b(z)c(w) = \frac{(-1)^{n-k+1}(n-k+1)!}{(z-w)^{n-k+2}} + : \partial^{n-k+1}b(w)c(w) : + \text{higher terms},
\] (4.27)

It follows from the OPEs (2.11) that

\[
\partial_z^{k}X_{-i}(z)X_i(w) = \partial_z^k \left( \sum_{m \geq 0} \frac{(z-w)^{m+1}}{m!} P_m(j(w)) \right).
\] (4.28)

Since : $\partial^n \beta(w)\gamma(w) :$ is the constant term in the expansion of power series of $z - w$ in the operator product expansion of $\partial^n \beta(z)\gamma(w)$, we see from equations (4.26), (4.27) and (4.28) that the only terms in equation (4.28) which will contribute to : $\partial^n \beta(w)\gamma(w) :$ non-trivially is the two terms $m = k - 1$ and $m = n + 1$. Namely we have

\[
: \partial^n \beta(w)\gamma(w) : = \sum_{0 \leq k \leq n} \binom{n}{k} \left[ k : \partial^{n-k+1}b(w)c(w) : P_{k-1}(j) \right] + \sum_{1 \leq k \leq n} \binom{n}{k} \left[ : \partial^{n-k+1}b(w)c(w) : P_{k-1}(j) : \right] + C_n P_{n+1}(j),
\] (4.29)

where

\[
C_n = (n+2) \sum_{m=0}^{n} (-1)^{m+1} \frac{1}{m+2}.
\]
Remark 4.1 1) $P_m(j)$ defined in equation (4.25) reads as follows for small $m$:

\[
\begin{align*}
P_1(j) &= -ij(w), \\
P_0(j) &= 1, \\
P_2(j) &= -i\partial j(w) - j(w)^2, \\
P_3(j) &= -i\partial^2 j(w) - 3 : j(w)\partial j(w) : + i : j(w)^3 :.
\end{align*}
\]

2) The formula (4.24) reads as follows for small $n$:

\[
\begin{align*}
: \gamma(w)\beta(w) &\equiv J^0(w) = ij(w), \\
: \gamma(w)\partial\beta(w) &= : \partial b(w)c(w) : - \frac{1}{2} : J^0(w)^2 : + \frac{1}{2} \partial J^0(w), \\
: \gamma(w)\partial^2\beta(w) &= 2 : \partial^2 b(w)c(w) : - 2 : \partial b(w)c(w) : J^0(w) \\
&\quad+ \frac{5}{3} \partial^2 J^0(w) + \frac{5}{3} : J^0(w)^3 : \\
&\quad- 5 : J^0(w)\partial J^0(w) :.
\end{align*}
\]

Lemma 4.3 Each field $: \partial^i b(z)\partial^j c(z) :$, $i > 0, j \geq 0$ can be expressed as a normally ordered polynomial in terms of $T(z)$ and $W(z)$ defined in (4.21) and (4.22) and their derivative fields.

Proof of Lemma 4.3. We first prove the following statement:

Claim $A_n$: Any field $: \partial^i b(z)\partial^{n-i+1} c(z) :$, $1 \leq i \leq n + 1$ can be written as a linear combination of the following $n + 1$ fields

\[
\partial \left( : \partial^i b(z)\partial^{n-k} c(z) : \right), 1 \leq k \leq n \text{ and } : T(z)\partial^{n-1} b(z)c(z) :.
\]

Indeed one can calculate directly by using (4.21) and Wick’s Theorem that

\[
: T(z) \left( \partial^{n-1} b(z)c(z) \right) := \frac{1}{2} : \partial^{n-1} b(z)\partial^2 c(z) : + \frac{1}{n} : \partial^{n+1} b(z)c(z) :.
\]

(4.30)

Also since the derivation of a normally ordered product satisfies the Leibniz rule we have

\[
\partial \left( : \partial^i b(z)\partial^{n-k} c(z) : \right) = : \partial^{k+1} b(z)\partial^{n-k} c(z) : + : \partial^k b(z)\partial^{n-k+1} c(z) :.
\]

(4.31)

The $n + 1$ fields

\[
\partial \left( : \partial^i b(z)\partial^{n-k} c(z) : \right), 1 \leq k \leq n \text{ and } : T(z)\partial^{n-1} b(z)c(z) :
\]
can be obtained from the \( n + 1 \) fields \( \partial^i b(z) \partial^{n-i+1} c(z) \), \( 1 \leq i \leq n + 1 \) through a linear transformation given by the following \((n+1) \times (n+1)\) matrix

\[
\begin{array}{cccccc}
1 & 1 & & & & \\
0 & 1 & & & & \\
& \ddots & \ddots & \ddots & & \\
& & \ddots & 1 & 1 & 0 \\
& & & 0 & 1 & 1 \\
& & & & \frac{1}{2} & 0 & \frac{1}{n+1}
\end{array}
\]

It is easy to see this matrix has determinant \( \frac{n+3}{2(n+1)} \), so it is invertible. By inverting the matrix we prove the Claim \( A_n \).

Now we are ready to prove the following claim by induction on \( n \) which is a reformulation of Lemma \( \ref{L3} \). Claim \( B_n \): Any field \( \partial^i b(z) \partial^{n-i} c(z), 1 \leq i \leq n \) can be written as a normally ordered polynomial in terms of \( T(z), W(z) \) and their derivative fields.

When \( n = 1 \), \( \partial b(z)c(z) \) is just \( T(z) \) itself.

When \( n = 2 \), \( \partial^2 b(z)c(z) \) and \( \partial b(z)\partial c(z) \) are clearly linear combinations of the fields

\[
\partial T(z) = \partial (\partial (b(z)c(z) :)) = : \partial^2 b(z)c(z) : + : \partial b(z)\partial c(z) : \\
W(z) = \frac{1}{\sqrt{6}} \left( : \partial^2 b(z)c(z) : - : \partial b(z)\partial c(z) : \right).
\]

So Claim \( B_2 \) is true.

Assume that the statement \( B_n \) is true. Then particularly the field \( \partial^{n-1} b(z)c(z) : \) can be written as a normally ordered polynomial of \( T(z) \) and \( W(z) \). And so is \( T(z) \partial^{n-1} b(z)c(z) : \). Then the Claim \( B_{n+1} \) follows from Claim \( A_n \) (cf. equation \( \ref{4.30} \)). \( \square \)

Proof of Theorem \( \ref{4.1} \). Lemmas \( \ref{1.1} \) and \( \ref{1.2} \) imply immediately that \( \mathcal{F} \) is irreducible under the actions of the Fourier components of fields \( \partial^i b(z) \partial^j c(z) :, i > 0, j \geq 0 \). Together with Lemma \( \ref{L3} \), this implies that \( \mathcal{F} \) is irreducible under the actions of \( L_n, W_n, n \in \mathbb{Z} \). So the vertex algebra \( \mathcal{F}^0 \) is isomorphic to \( \mathcal{W}_{3,-2} \) by Proposition \( \ref{1.1} \). The free field construction of \( \mathcal{F} \) guarantees that \( \mathcal{F} \) is a module of the vertex algebra \( \mathcal{W}_{3,-2} \). The second statement of Theorem \( \ref{4.1} \) now follows from the isomorphism of vertex algebras \( M^0_0 \cong \mathcal{F}^0 \otimes \mathcal{H}_0 \) given by \( \ref{2.13} \). \( \square \)
We have the following proposition from the explicit free field realization of modules $F^l$ of the $W_{3,-2}$ algebra. Also see Remark 4.3 in [W1] for some further implication.

**Proposition 4.2**

1) There exists a non-split short exact sequence of modules over the vertex algebra $W_{3,-2}$:

$$0 \rightarrow F^l \rightarrow F^l \rightarrow F^l/ F^l \rightarrow 0. \quad (4.32)$$

2) There exists a non-split short exact sequence of modules over the vertex algebra $W_{1+\infty,-1}$:

$$0 \rightarrow M^l_{s-l} \rightarrow M \rightarrow M^l_{s-l+1} \rightarrow 0. \quad (4.33)$$

Here $M$ is isomorphic to $F^l \otimes H_{is}$ as vector spaces.

**Proof.** As a vector space we have a direct sum $F^l = F^l \oplus b(0)\bar{F}^{-1}$. Then it is not hard to see that as a $W_{3,-2}$-module, $F^l/ F^l$ is isomorphic to the irreducible module $F^l/ F^l$. So the following non-split short exact sequence of modules over the $W_{3,-2}$ algebra

$$0 \rightarrow \bar{F} \rightarrow F^l \rightarrow F^l/ F^l \rightarrow 0$$

is isomorphic to the one in (4.32).

Note that $M^l_j$ is isomorphic to $F^l \otimes H_{is+j}$ as modules over the vertex algebra $W_{1+\infty,-1}$ by Theorem 4.1. Then the non-split short exact sequence (4.33) can be obtained by tensoring the one in (4.32) with $H_{is}$.  

\[\square\]

5 Character formulas of modules over $W_3$ algebra with central charge $-2$

Denote by

$$\Psi(z, q, p) \equiv \sum_{l \in \mathbb{Z}} z^l \psi_l(q, p) = \text{Tr} |_{\bigoplus_{l \in \mathbb{Z}} F^l} z^{-j_0} q^{L_0} p \tilde{W}_0$$

the full character of $\bigoplus_{l \in \mathbb{Z}} F^l$, a direct sum of irreducible modules $F^l$ over the $W_{3,-2}$ algebra. Here $\psi_l(q, p)$ is the full character of $F^l$, $l \in \mathbb{Z}$. Then the full character formula $\psi_l(q, p)$ of the irreducible $W_{3,-2}$-module $F^l$ can be recovered from $\Psi(z, q, p)$ by taking residue

$$\psi_l(q, p) = \text{Res}_{z=0} z^{l+1} \Psi(z, q, p).$$

We will need the following lemma.
Lemma 5.1 We have the following OPEs:

\[
T(z)b(w) \sim \frac{\partial b(w)}{z - w}, \\
T(z)c(w) \sim \frac{c(w)}{(z - w)^2} + \frac{\partial c(w)}{z - w}, \\
\tilde{W}(z)b(w) \sim \frac{\frac{1}{2}\partial b(w)}{(z - w)^2} + \frac{\partial^2 b(w)}{z - w}, \\
\tilde{W}(z)c(w) \sim \frac{-c(w)}{(z - w)^3} + \frac{-\frac{3}{2}\partial c(w)}{(z - w)^2} + \frac{-\partial^2 c(w)}{z - w}. \tag{5.34}
\]

Proof. We will prove the OPE (5.34) only and the other OPEs can be proved similarly by using Wick’s Theorem.

Since

\[
b(z)c(w) \sim \frac{1}{z - w},
\]

we have

\[
\left(\partial^2_z b(z)\right)c(w) \sim \frac{2}{(z - w)^3}.
\]

Since \(c(z)c(w) \sim 0\) and \(b(z), c(z)\) are fermionic fields, we have by Wick’s Theorem

\[
\left(\partial^2_z b(z)c(z)\right)c(w) \sim -\frac{2c(z)}{(z - w)^3}.
\]

We also have by Wick’s Theorem

\[
\left(\partial_z b(z)\partial_z c(z)\right)c(w) \sim \frac{\partial_z c(z)}{(z - w)^2} \sim \frac{\partial_w c(w)}{(z - w)^2} + \frac{\partial^2_w c(w)}{z - w}. \tag{5.37}
\]

Now the OPE (5.34) follows from (5.36), (5.37) and the definition of \(\tilde{W}(z)\) in (4.23).

In particular Lemma 5.1 implies

Corollary 5.1 We have the following commutation relations \((n \in \mathbb{Z})\):

\[
[L_0, b(n)] = -nb(n), \quad [L_0, c(n)] = -nc(n), \\
[W_0, b(n)] = n^2 b(n), \quad [W_0, c(n)] = -n^2 c(n).
\]
Proof. By comparing the coefficients of the $z^{-3}$ terms in both sides of the OPE (5.34), we get

$$[W_0, b(w)] = w \partial b(w) + w^2 \partial^2 b(w).$$

(5.38)

Comparing the coefficients of the $w^n$ terms in both sides of (5.38), we get $[W_0, b(n)] = n^2 b(n)$. Proofs of the other commutation relations in Corollary 5.1 are similar.

The following full character formula follows now from Corollary 5.1 and the characterization of $\mathcal{F}$ as the subspace of $\mathcal{F}^I$ consisting of vectors which do not involve $b(0)$, the zero-th Fourier component of the field $b(z)$.

**Theorem 5.1** The full character formula $\Psi(z, q, p)$ is given by

$$\Psi(z, q, p) = \prod_{n \geq 1} \left( 1 + zq^np^n \right) \left( 1 + z^{-1}q^np^{-n} \right).$$

**Remark 5.1**

1) By using the Jacobi triple identity, one can easily show that

$$\psi_1(q, 0) = \frac{1}{\Pi_{n \geq 1}(1 - q^n)} \sum_{k \geq |l|} (-1)^{k+l} q^{k(k+1)/2}. $$

This is consistent with the explicit decomposition of $\mathcal{F}$ with respect to the Virasoro algebra generated by the Fourier components of the field $T(z) [FF]$.  

2) If we consider instead

$$\tilde{\Psi}(z, q, p) \equiv \text{Tr} \left| \bigoplus_{l \in \mathbb{Z}} \mathcal{F} \right| z^{-j_0} q^{-L_0} p^{W_0},$$

then we can show similarly that

$$\tilde{\Psi}(z, q, p) = (1 + z) \prod_{n \geq 1} \left( 1 + zq^np^n \right) \left( 1 + z^{-1}q^np^{-n} \right).$$

(5.39)

Essentially the same formula as in (5.37) up to some simple changes of variables appears in [Di] as some generating function of counting covers of an elliptic curve. Modular invariance and some other interesting properties of the function $\tilde{\Psi}(z, q, p)$ were discussed in detail in [KZ]. It is suggested that $\tilde{\Psi}(z, q, p)$ may
be an indication of the existence of generalized Jacobi forms involving several (possibly infinitely many) variables. We hope that full character formulas of representations of \( \mathcal{W} \)-algebras in general may provide further natural examples of generalized Jacobi forms.

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