Differential-Dunkl Operators and Nonstandard Solutions to the Classical Yang-Baxter Equation

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Abstract. For every pair of positive coprime integers, $m$ and $n$, with $m < n$, there is an associated generalized Cremmer-Gervais $r$-matrix $r_{m,n} \in \mathfrak{sl}_n \wedge \mathfrak{sl}_n$ which provides a nonstandard quasitriangular solution to the classical Yang-Baxter equation. We give an interpretation of $r_{n-2,n}$ (for $n$ odd) in terms of differential-Dunkl operators related to the polynomial representation of dihedral-type rational Cherednik algebras. Finally, we use this interpretation to partially answer a conjecture of Gerstenhaber and Giaquinto concerning boundary solutions to the classical Yang-Baxter equation.

1. Introduction

In the early 1980’s, Belavin and Drinfeld [1] classified the quasitriangular solutions to the classical Yang-Baxter equation. The $r$-matrices $r_{m,n} \in \mathfrak{sl}_n \wedge \mathfrak{sl}_n$ of Cremmer and Gervais (see e.g. [7,10,14]), which are indexed by pairs of positive coprime integers $m$ and $n$ with $m < n$, provide an interesting family of such solutions.

In this paper, we give an interpretation of the Cremmer-Gervais $r$-matrices $r_{n-2,n}$ (for $n$ odd) in terms of differential-Dunkl operators related to the polynomial representation of certain dihedral-type rational Cherednik algebras. The rational Cherednik algebras are a family of algebras defined by Etingof and Ginzburg [11] in the context of symplectic reflection algebras. They are doubly degenerate versions of double affine Hecke algebras, which were first introduced by Cherednik [5] as a key tool to solving the Macdonald constant term conjectures, and have since been of importance in noncommutative algebras and quantum groups. For a more detailed exposition of these algebras, we refer the reader to [12].

The organization of this paper is as follows. Section 2 covers the preliminaries on classical $r$-matrices, the Belavin-Drinfeld classification theorem, and the conjecture of Gerstenhaber and Giaquinto [14, Conj. 5.7] concerning Cremmer-Gervais $r$-matrices and boundary solutions to the classical Yang-Baxter equation. In Section 3 we introduce an algebra of differential-Dunkl operators related to the polynomial representation of dihedral-type rational Cherednik algebras and give an interpretation of certain Cremmer-Gervais $r$-matrices in terms of these operators. In Section 4 we prove the Gerstenhaber-Giaquinto conjecture for the case when $m = n - 2$ (and $n$ is necessarily odd). Section 5 is dedicated to a proof of Thm. 2.3 which gives an explicit formula for the Cremmer-Gervais $r$-matrices.

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2. Classical $r$-matrices

2.1. Preliminaries and definitions. Throughout this paper, $m$ and $n$ denote fixed positive integers with $m < n$. Later, we make the further assumption that $m$ and $n$ are coprime. Let $k$ be an algebraically
closed field of characteristic zero. We mention here, however, that most of the statements and results of this paper hold over a field with \(2n \neq 0\) and containing an \(m\)-th root of unity. All tensor products are taken over \(k\). For a \(k\)-module \(V\), and vectors \(a, b \in V\), we use the notation \(a \wedge b\) as shorthand for \(\frac{1}{2}(a \otimes b - b \otimes a)\) and let \(V \wedge V\) be the vector subspace of \(V \otimes V\) spanned by the set \(\{a \wedge b : a, b \in V\}\).

Let \(g\) be a simple Lie algebra over \(k\) and let \(U(g)\) denote its universal enveloping algebra. For \(r = \sum_i a_i \otimes b_i \in g \otimes g\), define the following elements of \(U(g) \otimes U(g) \otimes U(g)\): \(r_{12} = \sum_i a_i \otimes b_i \otimes 1\), \(r_{23} = \sum_i 1 \otimes a_i \otimes b_i\), and \(r_{13} = \sum_i a_i \otimes 1 \otimes b_i\). We call \(r \in g \otimes g\) an \(r\)-matrix if
\[
\langle \langle r, r \rangle \rangle := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]
\]
is \(g\)-invariant. An \(r\)-matrix is called triangular if \(\langle \langle r, r \rangle \rangle = 0\) and quasitriangular if \(\langle \langle r, r \rangle \rangle \neq 0\). Although we don’t discuss Poisson-Lie groups or Lie bialgebras in this paper, we mention here that \(r\)-matrices arise in the context of determining Poisson structures on Lie groups which are compatible with the group multiplication. For further reference, see e.g. [3] Chapter 1.

In this paper we turn our attention to the case when \(g = \mathfrak{sl}_n\), the Lie algebra of traceless \(n \times n\) matrices. Let \(e_{ij}\) denote the elementary \(n \times n\) matrix having 1 in the \((i, j)\)-entry and zeros elsewhere. We identify \(e_{ij}\) with the \(k\)-linear map on \(k^n\) defined by \(e_{ij} : \delta_{e_{ij}}\), where \(e_1, ..., e_n\) denote the standard basis vectors of \(k^n\). Under this identification, we view an element \(r \in \mathfrak{sl}_n \otimes \mathfrak{sl}_n\) as a linear operator on \(k^n \otimes k^n\) that satisfies \(PrP = -r\), where \(P\) is the linear map that interchanges tensor components: \(u \otimes v \mapsto v \otimes u\). It turns out that the \(\mathfrak{sl}_n\)-invariants in \(U(\mathfrak{sl}_n) \otimes U(\mathfrak{sl}_n) \otimes U(\mathfrak{sl}_n)\) correspond to scalar multiples of the linear map given by \(u \otimes v \otimes w \mapsto w \otimes u \otimes v - v \otimes w \otimes u\) for all \(u, v, w \in k^n\). This prompts the following definitions.

**Definitions 2.1.** Let \(V\) be a \(k\)-module, and let \(\tau : V \otimes V \rightarrow V \otimes V\) be a linear map satisfying \(PrP = -\tau\).

1. For an arbitrary (but fixed) scalar \(\lambda \in k\), define the classical Yang-Baxter map \(CYB_\lambda : \text{End}_k(V^\otimes 3) \rightarrow \text{End}_k(V^\otimes 3)\) as follows,
\[
CYB_\lambda(\tau) := [\tau_{12}, \tau_{13}] + [\tau_{12}, \tau_{23}] + [\tau_{13}, \tau_{23}] - \lambda Z,
\]
where \(Z\) is the linear map defined by \(u \otimes v \otimes w \mapsto w \otimes u \otimes v - v \otimes w \otimes u\) for all \(u, v, w \in V\).

2. We call \(\tau\) an \(r\)-matrix if there exists \(\lambda \in k\) so that \(CYB_\lambda(\tau) = 0\). If \(CYB_0(\tau) = 0\) then \(\tau\) is called a triangular \(r\)-matrix. An \(r\)-matrix with \(CYB_0(\tau) \neq 0\) is called quasitriangular.

2.2. Quasitriangular \(r\)-matrices, Belavin-Drinfeld’s Classification. A classification of quasitriangular \(r\)-matrices for the case when \(g\) is a complex simple Lie algebra was completed in the early 1980’s by Belavin and Drinfeld’s [1].

In this section, we assume \(g\) is complex and simple. Let \(h \subseteq g\) denote a Cartan subalgebra, and let \(\langle , \rangle\) be a nondegenerate symmetric ad-invariant bilinear form on \(g\). The nondegeneracy of the bilinear form induces an inner product on the dual vector space \(g^*\), also denoted with \(\langle , \rangle\). For a root system \(\Phi = \Phi_+ \cup \Phi_-\) of \((g, h)\), let \(\Pi = \{\alpha_1, ..., \alpha_l\} \subseteq \Phi_+\) denote a basis of positive roots of \(g\), and let \(g \cong \mathfrak{sl}_n \otimes \mathfrak{h} \otimes \mathfrak{n}_+\) denote the corresponding triangular decomposition. For each positive root \(\alpha \in \Phi_+\), select root vectors \(e_\alpha \in \mathfrak{g}_\alpha, e_{-\alpha} \in \mathfrak{g}_{-\alpha}\) so that \(\langle e_\alpha, e_{-\alpha} \rangle = 1\). For each root \(\beta \in \Phi\), let \(h_\beta \in h\) denote the unique vector satisfying \(\langle h_\beta, H \rangle = \alpha(H)\) for all \(H \in h\). A BD-triple \((S_0, S_1, \zeta)\) is a bijection \(\zeta : S_0 \rightarrow S_1\) between subsets \(S_0 \subseteq \Pi\) and \(S_1 \subseteq \Pi\) and satisfies

1. the orthogonality condition: \(\langle \zeta(\alpha_i), \zeta(\alpha_j) \rangle = \langle \alpha_i, \alpha_j \rangle\) for every \(\alpha_i, \alpha_j \in S_0\), and
2. the nilpotency condition: for every \(\alpha \in S_0\), there exists \(N \in \mathbb{N}\) so that \(\zeta^N(\alpha) \in S_1 \setminus S_0\).

We view BD-triples graphically via two copies of the Dynkin diagram of \(g\) with arrows representing the bijection \(\zeta\) drawn between the nodes (see, e.g., Figures [1] and [2]).

**Figure 1.** The Cremmer-Gervais BD-triple \(T_{1,n}\).
For a BD-triple $T = (S_0, S_1, \zeta)$, we put
\[
\beta_T := \left\{ \beta \in h \wedge h : (1 \otimes (\zeta(\alpha) - \alpha))\beta = \frac{1}{2}(h_{\zeta(\alpha)} + h_\alpha) \text{ for all } \alpha \in S_0 \right\}.
\]
It turns out that $\beta_T$ is an affine algebraic subvariety of $h \wedge h$ of dimension $d(d-1)/2$ where $d = \#(\Pi - S_0)$ (see, e.g., Section 3.1). For $i \in \{0, 1\}$, put $\hat{S}_i := \mathbb{Z}S_i \cap \Phi_+$, and let $\hat{\zeta}$ denote the $\mathbb{Z}$-linear extension of $\zeta$ to $\hat{S}_0$. The BD-triple induces a partial ordering $\preceq$ on the positive roots as follows: for $\rho, \mu \in \Phi_+$, $\rho \prec \mu$ if and only if there exists $N \in \mathbb{N}$ so that $\hat{\zeta}^N(\rho) = \mu$.

It follows from the definition of quasitriangularity that if $r$ is quasitriangular, then $\lambda r$ is also quasitriangular for every $\lambda \in \mathbb{C}^\times$. Since the defining equations for an $r$-matrix are quadratic, we can view the set of all quasitriangular $r$-matrices as a projective subvariety $\mathcal{M}$ of $\mathbb{P}(g \wedge g)$. We also observe that the group $\text{Aut}_g(g) \times \mathbb{C}^\times$ acts on $\mathcal{M}$ by $(g, \lambda) r := \lambda(g \otimes g) r$. We call $r$ and $r'$ equivalent $r$-matrices if there exists a nonzero scalar $\lambda \in \mathbb{C}^\times$ and an inner automorphism $g \in \text{Int}_g(g)$ so that $(g, \lambda) r = r'$. Now we are ready to state the Belavin-Drinfel’d classification theorem.

**Theorem 2.2.** *(Belavin-Drinfel’d Classification Theorem)* For a BD-triple $T$ and $\beta \in \beta_T$,
\[
r_{T, \beta} = \alpha_T + \beta + \gamma \in g \wedge g
\]
is a quasitriangular $r$-matrix on $g$, where
\[
\alpha_T = 2 \sum_{\rho < \mu} e_{\rho} \wedge e_{-\mu}
\]
and
\[
\gamma = \sum_{\mu \in \Phi_+} e_{\mu} \wedge e_{-\mu}.
\]
Conversely, any quasitriangular $r$-matrix $r \in g \wedge g$ is equivalent to a unique $r$-matrix of the above form.

### 2.3. Generalized Cremmer-Gervais $r$-matrices.

The generalized Cremmer-Gervais $r$-matrices provide some interesting examples of quasitriangular $r$-matrices for the case when $g = sl_n$.

The linear span of $\{e_{ii} - e_{i+1,i+1} : 1 \leq i \leq n - 1\}$ is a Cartan subalgebra, which we denote as $h$. The bilinear form we use is the trace form $\langle X, Y \rangle := Tr(XY)$. The positive root vectors are $\{e_{ij} : i < j\}$ and the negative root vectors are $\{e_{ij} : i > j\}$. Let $\alpha_1, \ldots, \alpha_{n-1}$ denote the simple roots, where $\alpha_i$ is the linear functional on $h$ defined by the rule $\alpha_i(h) = \langle e_{ii} - e_{i+1,i+1}, h \rangle$ for all $h \in h$.

The generalized Cremmer-Gervais $r$-matrices are the quasi-triangular $r$-matrices associated to the maximal BD-triples with $\#(\Pi - S_0) = 1$. In [14], Gerstenhaber and Giaquinto show there are exactly $\phi(n)$ BD-triples of this type, where $\phi$ is the Euler-totient function. Additionally, for $m$ coprime to $n$ in $\{1, \ldots, n - 1\}$ they show the corresponding BD-triple $T_{m,n}$ is given by
\[
T_{m,n} := (\Pi - \{\alpha_{n-m}\}, \Pi - \{\alpha_m\}, \zeta : \alpha_s \mapsto \alpha_{s+m(\text{mod } n)}).
\]

**Figure 2.** The Cremmer-Gervais BD-triple $T_{n-2,n}$

The generalized Cremmer-Gervais $r$-matrices are particularly interesting because the variety $\beta_{T_{m,n}}$ is of minimal dimension. In fact, Gerstenhaber and Giaquinto [14] show that $\beta_{T_{m,n}}$ is a singleton set containing the point
\[
\beta_{m,n} := \sum_{1 \leq j < \ell \leq n} \left( -1 + \frac{2}{n} (j - \ell)m^{-1}(\text{mod } n) \right) e_{jj} \wedge e_{\ell\ell} \in h \wedge h.
\]
In order to describe the $\alpha_T$-part of the generalized Cremmer-Gervais $r$-matrices, which would consequently provide us with an explicit description of the entire $r$-matrix, we need to first introduce some notation. Here and elsewhere, when an integer is reduced modulo $\ell \in \mathbb{N}$, we always mean in $\{0, \ldots, \ell - 1\}$. Let $n$ and $m$ be relatively prime positive integers with $m < n$. We define a sequence of integers $i_0, i_1, i_2, \ldots$ recursively by setting $i_0 = n$, $i_1 = m$, and $i_t = -i_{t-2} \pmod{i_{t-1}}$ for $t > 1$. Eventually the sequence will reach $1$. Let $L$ be the smallest number so that $i_L = 1$. For all $0 \leq t \leq L$ and $j, \ell \in \mathbb{Z}$, define
\begin{equation}
\mathcal{J}_t(j, \ell) := 1 - i_t + \left( (n - \ell) \pmod{i_0} (\text{mod } i_1) \cdots (\text{mod } i_t) \right) + \left( (j - 1) \pmod{i_0} (\text{mod } i_1) \cdots (\text{mod } i_t) \right),
\end{equation}
and let $\psi_j$ be the unique integer in $\{1, \ldots, n\}$ satisfying $j = m\psi_j \pmod{n}$. Let $\text{sgn} : \mathbb{Z} \to \{-1, 0, 1\}$ denote the signum function: $\text{sgn}(0) = 0$ and for all nonzero $x$, $\text{sgn}(x) = \frac{|x|}{x}$. For $n \geq 3$ there is a non-trivial automorphism of the Dynkin diagram of $\mathfrak{sl}_n$. This corresponds to the involutive automorphism of $\mathfrak{sl}_n$ given by $\varphi : e_{jt} \mapsto -e_{n+1-\ell, n+1-j}$. One can readily compute that $(\varphi \otimes \varphi) r_{m,n} = r_{n-m, n}$ and we have the following

**Theorem 2.3.** The generalized Cremmer-Gervais $r$-matrix $r_{n-m,n}$ is given by the formula
\begin{equation}
r_{n-m,n}(e_j \otimes e_t) = \sum_{t=0}^{L-1} \sum_{N=0}^{\lfloor \mathcal{J}_t(j,\ell)/n \rfloor} e_j \mathcal{J}_t(j,\ell)+Nt+1 \otimes e_t \mathcal{J}_t(j,\ell)-Nt+1 - \sum_{t=0}^{L-1} \sum_{N=0}^{\lfloor \mathcal{J}_t(\ell,j)/n \rfloor} e_j \mathcal{J}_t(\ell,j)-Nt+1 \otimes e_t \mathcal{J}_t(\ell,j)+Nt+1 + \frac{1}{2} \text{sgn}(\psi_j - \psi_t) - \frac{1}{n}(\psi_j - \psi_t) e_j \otimes e_t - \frac{1}{2} \text{sgn}(j - \ell) e_t \otimes e_j.
\end{equation}

**Proof.** See Section 5. 

**2.4. Frobenius and Quasi-Frobenius Lie algebras.** There is not an analogue of the Belavin-Drinfeld classification theorem for triangular $r$-matrices. Instead, they are characterized by a homological condition \[ [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g} \] Assume $r \in \mathfrak{g} \wedge \mathfrak{g}$ is a triangular $r$-matrix over the base field $k$. The *carrier* of $r$ is the Lie subalgebra $\mathfrak{f} \subseteq \mathfrak{g}$ spanned by $\{(\xi \otimes 1)r : \xi \in \mathfrak{g}^*\}$. The map $\tilde{r} : \mathfrak{f}^* \to \mathfrak{f}$ defined by $\xi \mapsto (\xi \otimes 1)r$ is a linear isomorphism of $\mathfrak{f}$ with its dual $\mathfrak{f}^*$ and induces a nondegenerate skew-symmetric bilinear form
\begin{equation}
F : \mathfrak{f} \times \mathfrak{f} \to k
\end{equation}
\begin{equation}
(X,Y) \mapsto \langle \tilde{r}^{-1}X,Y \rangle.
\end{equation}
Since $r$ is a triangular $r$-matrix, it follows that $F$ is Lie algebra 2-cocycle on $\mathfrak{f}$ with coefficients in the trivial representation:
\begin{equation}
F([X,Y], Z) + F([Z,X], Y) + F([Y,Z], X) = 0
\end{equation}
for all $X, Y, Z \in \mathfrak{f}$. Here, we call $(\mathfrak{f}, F)$ a *quasi-Frobenius Lie algebra*. The above process can be inverted to give a one-to-one correspondence between quasi-Frobenius Lie algebras $(\mathfrak{f}, F)$ and *nondegenerate* triangular $r$-matrices on $\mathfrak{f}$. Thus, a classification of triangular $r$-matrices for $\mathfrak{g}$ would entail classifying all quasi-Frobenius subalgebras $\mathfrak{f} \subseteq \mathfrak{g}$. One way to obtain a skew-symmetric bilinear form $F$ satisfying Eqn. (2.12) is to choose an arbitrary functional $\eta \in \mathfrak{f}^*$ and set $F(X,Y) = \eta(\langle X,Y \rangle) \langle X,Y \rangle$. In this setting $(\mathfrak{f}, \eta)$ is called a Frobenius Lie algebra provided $F$ is nondegenerate. The functional $\eta \in \mathfrak{f}^*$ is called the Frobenius functional of $\mathfrak{f}$.

**2.5. Maximal Parabolic Subalgebras of $\mathfrak{sl}_n$.** Certain maximal parabolic subalgebras of $\mathfrak{sl}_n$ provide interesting examples of Frobenius Lie algebras. For $1 \leq m < n$, let $\mathfrak{p}_{m,n}$ denote the maximal parabolic subalgebra of $\mathfrak{sl}_n$ obtained by deleting the $m$-th simple negative root $e_{m+1,m}$: i.e.
\[ \mathfrak{p}_{m,n} = \text{span}_k \{e_{jt} : j \leq m \text{ or } m < \ell \} \subseteq \mathfrak{sl}_n. \]
The following theorem can be found in [14] Theorem 5.3.

**Theorem 2.4.** The second cohomology group $H^2(\mathfrak{p}_{m,n}, k)$ is trivial for all $1 \leq m < n$. 


This implies the following.

**Corollary 2.5.** A maximal parabolic subalgebra of $\mathfrak{sl}_n$ is quasi-Frobenius if and only if it is Frobenius.

The following theorem was proved by Elashvili [9].

**Theorem 2.6.** The maximal parabolic subalgebra $p_{m,n}$ of $\mathfrak{sl}_n$ is Frobenius if and only if $m$ and $n$ are relatively prime.

### 2.6. Boundary $r$-matrices

In [14], it was shown that points lying in the Zariski boundary of the variety $M$ are triangular $r$-matrices. Due to the close relationship that boundary $r$-matrices have with quasitriangular $r$-matrices, one would expect a classification result for boundary $r$-matrices that closely parallels the Belavin-Drinfeld classification. However very little is currently known about boundary $r$-matrices. The most general class of known examples include the generalized Jordanian $r$-matrices, which are triangular $r$-matrices for the Lie algebra $\mathfrak{sl}_n$ having carrier $p_{1,n} \subseteq \mathfrak{sl}_n$. In [14], it was shown that the generalized Jordanian $r$-matrices lie on the boundary of the component of $M$ corresponding to the Cremmer-Gervais $r$-matrices $r_{1,n}$.

This raises an interesting question and prompted Gerstenhaber and Giaquinto to make the following conjecture [14, Conj. 5.7].

**Conjecture 2.7. (The Gerstenhaber-Giaquinto Conjecture)** Suppose $m$ and $n$ are relatively prime positive integers with $m < n$. Then the triangular $r$-matrix with carrier $p_{m,n}$ is a boundary $r$-matrix and lies in the closure of the $SL_n$-orbit of $r_{m,n}$.

In [14, Thm. 5.9], they prove their conjecture in the case when $m = 1$. The following theorem provides a way to construct boundary $r$-matrices.

**Theorem 2.8.** [14, Prop. 5.1] Let $t \in k$ be nonzero. Suppose $r \in \mathfrak{g} \wedge \mathfrak{g}$ and $r_t = r + tr' + \cdots + t^d r'' \in \mathfrak{g} \wedge \mathfrak{g}$ are $r$-matrices with $\langle(r, r)\rangle = \langle(r_t, r_t)\rangle$. Then $r''$ is a boundary $r$-matrix.

In particular, if $r$ is an $r$-matrix and $X \in \mathfrak{g}$ is nilpotent. Then the highest degree term on $\exp(tX).r$ is a boundary $r$-matrix. For instance, let

$$X = \frac{1}{2}[(n - 1)e_{12} + (n - 2)e_{23} + \cdots + 1 \cdot e_{n-1,n}] \in \mathfrak{sl}_n.$$  

Then $\exp(tX).r_{1,n} = r_{1,n} + t[X, r_{1,n}]$. Thus, $[X, r_{1,n}]$ is a boundary $r$-matrix. We have the following.

**Theorem 2.9.** [14, Thm. 5.9] The boundary $r$-matrix $[X, r_{1,n}]$ lies in the closure of the $SL_n$-orbit of $r_{1,n}$. The carrier of $[X, r_{1,n}]$ is the maximal parabolic subalgebra $p_{1,n} \subseteq \mathfrak{sl}_n$.

The $r$-matrices $[X, r_{1,n}]$ are referred to as the generalized Jordanian $r$-matrices of Cremmer-Gervais type (or the Jordanian $r$-matrices for short) [10]. This is in reference to the fact that $[X, r_{1,2}]$ is the Jordanian $r$-matrix for $\mathfrak{sl}_2$.

### 3. The Polynomial Representation of the Rational Cherednik Algebras, the algebra $\tilde{H}$ of Differential-Dunkl Operators, and the Generalized Cremmer-Gervais $r$-matrices

The aim of this section is to demonstrate how Eqn. 2.10 above, which describes the action of the generalized Cremmer-Gervais $r$-matrices on $k^n \otimes k^n$, is related to the action of certain elements in the polynomial representation of the rational Cherednik algebras. To illustrate, when $m = 1$, we have

$$r_{n-1,n}(e_j \otimes e_{\ell}) = \text{sgn}(j - \ell)
\left(\frac{1}{2} (e_j \otimes e_{\ell} + e_{\ell} \otimes e_j) + \sum_{s \text{ strictly between } j \text{ and } \ell} e_s \otimes e_j + e_{\ell} - s\right) - \frac{j - \ell}{n} e_j \otimes e_{\ell}$$

for all $j, \ell \in \{1, \ldots, n\}$. We let $\text{TruncPol}_n(k[x_1, x_2])$ denote the $k$-linear subspace of $k[x_1, x_2]$ spanned by the monomials $x_1^j x_2^\ell$ with $0 \leq j, \ell < n$. Identifying the vector spaces $\text{TruncPol}_n(k[x_1, x_2])$ and $k^n \otimes k^n$ via $x_1^j x_2^\ell \leftrightarrow e_j \otimes e_{\ell}$ yields the formula

$$r_{n-1,n} = -\frac{1}{n} (x_1 \partial_1 - x_2 \partial_2) + \frac{\Delta}{2},$$

5
where $\Delta$ is the divided difference operator

$$
\Delta := \frac{x_1 + x_2}{x_1 - x_2}(1 - \sigma),
$$

$\sigma$ is the operator that interchanges the variables $\sigma.f(x_1, x_2) = f(x_2, x_1)$ for all $f \in \text{TruncPol}_n(k[x_1, x_2])$, and $\partial_1$ and $\partial_2$ are the partial derivative operators with respect to $x_1$ and $x_2$. In [20], it was shown that Eqn. (3.2) above can be expressed in terms of the so-called Dunkl operators, which arise in the polynomial representations of rational Cherednik algebras.

### 3.1. Rational Cherednik Algebras of Type $G(m,1,2)$

First we review the construction of the rational Cherednik algebra $H_{\kappa,c}(W)$ associated to a reflection group $W$ (and reflection representation $W \to GL(V)$, where $V$ is equipped with a nondegenerate inner product). While most of the statements and results from this section are true over a base field $k$ containing a primitive $m$-th root of one and with $2n \neq 0$ (for a fixed choice of positive integers $m$ and $n$), we assume the base field $k$ is algebraically closed of characteristic zero. For our purposes, the only reflection group we are concerned with is $W = G(m,1,2)$, the group of $2 \times 2$ monomial matrices having entries in $\{1, \omega, \omega^2, ..., \omega^{m-1}\}$, where $\omega \in k$ is a primitive $m$-th root of unity.

We may assume $V$ is the natural 2-dimensional representation of $G(m,1,2)$. For a more detailed treatment on the rational Cherednik algebras, especially for the type we consider, we refer the reader to the work of Chmutova [6] and Griffeth [16][17], where they discuss the representation theory of such algebras.

Nondegeneracy of the inner product $(\ ,\ )$ on $V$ induces an inner product on the dual vector space $V^*$, also denoted with $(\ ,\ )$. A reflection is an element $s \in W$ so that $\text{codim}[\text{Ker}(Id - s)] = 1$. Let $R$ denote the set of reflections in $W$. To each reflection $s$, let $h_s \subseteq V$ be the hyperplane fixed by $s$, and let $\alpha_s \in V^*$ be a nonzero functional that vanishes on $h_s$. Put $\alpha_s^\vee = \frac{2\alpha_s}{(\alpha_s, \alpha_s)} \in V^{**} \cong V$ and let $\{e_s\}_{s \in R}$ be a collection of scalars in $k$ indexed by the reflections and satisfying $e_{wsw^{-1}} = e_s$ for all $s \in R, w \in W$. For us, the set of reflections is $\{\sigma\} \cup \{\xi_1^j, \xi_2^j: j = 1, ..., m - 1\}$, where

$$
(3.4) \quad \xi_1 = \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$

There are $m$ conjugacy classes of reflections. For $j \in \{1, ..., m - 1\}$, let $c_j$ denote the scalar corresponding to the conjugacy class $\{\xi_1^j, \xi_2^j\}$, and let $c_0$ denote the scalar associated to the conjugacy class $\{\sigma, \xi_2^{-1}\sigma, ..., \xi_2^{m-1}\sigma, \xi_1\sigma\}$. Let $\kappa \in k$ be a fixed scalar. Finally, let $F$ be the $k$-algebra freely generated by the group algebra $kW$ and the symmetric algebra $k[V]$ and $k[V^*]$, where the natural inclusion maps $k[W] \to F$, $k[V] \to F$, $k[V^*] \to F$ are algebra homomorphisms. The rational Cherednik algebra $H_{\kappa,c}(W)$ is the quotient of $F$ by the relations

$$
wxw^{-1} = x^w, \quad wyw^{-1} = y^w, \quad yx - xy = \kappa (x,y) - \sum_{s \in R} c_s (\alpha_s, y) (x, \alpha_s^\vee) s,
$$

for all $w \in W, x \in V^*$, and $y \in V$. The algebra $H_{\kappa,c}(W)$ is $\mathbb{Z}$-graded with $\text{deg}(x) = 1, \text{deg}(y) = -1, \text{deg}(g) = 0$ for every $x \in V^*$, $y \in V$, and $g \in kW$. For a $kW$-module $\Lambda$, define the Verma module $M(\Lambda)$ by

$$
M(\Lambda) := \text{Ind}_{k[V^*] \times kW}^W \Lambda,
$$

where $y \in k[V^*]$ acts via multiplication by $y(0)$ on $\Lambda$. When $\Lambda$ is the trivial representation $1$, we obtain the polynomial representation $M(1) \cong k[V]$ and the elements of $V$ act via Dunkl operators,

$$
y.f = \kappa \partial_y f - \sum_{s \in R} c_s (\alpha_s, y) \frac{f - s f}{\alpha_s}
$$

for every $y \in V$ and $f \in k[V]$. 

6
Definition/Proposition 3.1. The rational Cherednik algebra $H_{\kappa,e}(G(m,1,2))$ is the $k$-algebra generated by $\sigma$, $\xi_1$, $\xi_2$, $x_1$, $x_2$, $y_1$, $y_2$ and has the following defining relations:

\begin{align}
\sigma^2 &= \xi_1^m = \xi_2^m = 1, \\
\xi_i \xi_j &= \xi_j \xi_i, \quad x_i x_j = x_j x_i, \quad y_i y_j = y_j y_i, \quad (i, j \in \{1, 2\}), \\
\xi_i y_i &= \omega y_i \xi_i, \quad \xi_i x_i = \omega^{-1} x_i \xi_i, \\
\sigma x_i &= x_{\sigma^{-1} i}, \quad \sigma y_i = y_{\sigma^{-1} i}, \quad \sigma \xi_i = \xi_{\sigma^{-1} i}, \\
\xi_i x_j &= x_j \xi_i, \quad \xi_i y_j = y_j \xi_i, \quad (i \neq j), \\
[y_1, x_1] &= \kappa - c_0 \sum_{r=0}^{m-1} \xi_1^r \xi_2^{r-1} \sigma - \sum_{r=1}^{m-1} c_r (1 - \omega^{-r}) \xi_1^r, \\
[y_1, x_2] &= c_0 \sum_{r=0}^{m-1} \omega^{-r} \xi_1^r \xi_2^{r-1} \sigma, \\
[y_2, x_1] &= c_0 \sum_{r=0}^{m-1} \omega^r \xi_1^r \xi_2^{r-1} \sigma.
\end{align}

In the polynomial representation of $H_{\kappa,e}(G(m,1,2))$, the Dunkl operators act via the following formulas (c.f. [2] Prop 2.2):

\begin{align}
y_1 x_1^j x_2^\ell &\mapsto \kappa j x_1^{j-1} x_2^\ell - m c_0 \sum_{N=0}^{m-1} x_1^{j-1-Nm} x_2^{\ell + Nm} - \sum_{N=1}^{m} x_1^{j-1 + Nm} x_2^{\ell - Nm} - \sum_{N=1}^{m} c_N (1 - \omega^{-Nj}) x_1^{j-1} x_2^\ell, \\
y_2 x_1^j x_2^\ell &\mapsto \kappa \ell x_1^{j-1} x_2^\ell + m c_0 \sum_{N=1}^{m} x_1^{j-Nm} x_2^{\ell + Nm} - \sum_{N=0}^{m-1} x_1^{j+Nm} x_2^{\ell - Nm} - \sum_{N=1}^{m} c_N (1 - \omega^{-N\ell}) x_1^{j} x_2^{\ell-1},
\end{align}

for all $j, \ell \geq 0$.

3.2. The Algebra $\tilde{H}$ and the $r$-matrices $r_{n-1,n}$ and $r_{n-2,n}$. Given the similarity between the formulas for the actions of $r_{n-m,n}$ on $TruncPol_n(k[x_1, x_2])$ and the Dunkl operators for the algebra $H_{\kappa,e}(G(m,1,2))$ (compare Eqns. 3.10 with Eqns. 3.11 and 3.12), this suggests that one may reinterpret the Cremmer-Gervais $r$-matrices in terms of the algebras $H_{\kappa,e}(G(m,1,2))$, or at least modified versions of $H_{\kappa,e}(G(m,1,2))$. The following result verifies this statement for the case when $m = 1$.

Theorem 3.2. [20] Section 4] The action of the Cremmer-Gervais $r$-matrix $r_{n-1,n}$ on $k^n \otimes k^n$ coincides with the action of $\frac{1}{n} (x_1 y_1 - x_2 y_2) \in H_{1,n/2}(G(1,1,2))$ on the truncated polynomial ring $TruncPol_n(k[x_1, x_2])$.

Proof. In the rational Cherednik algebra $H_{\kappa,e}(G(1,1,2))$, the Dunkl operators are $y_1 = \kappa \partial_1 - c_0 \frac{1-\sigma}{x_1-x_2}$ and $y_2 = \kappa \partial_2 + c_0 \frac{1-\sigma}{x_1-x_2}$. The result follows from substituting these expressions into Eqn. 3.2.

Throughout the remainder of this section, we focus on the case when $m = 2$ and $n$ is odd. The main algebra we consider is the rational Cherednik algebra associated to the reflection group $G(2,1,2)$. For brevity, put

$H = H_{\kappa,e}(G(2,1,2))$.

A straightforward yet lengthy computation yields the following proposition, which we provide to give an example of an $r$-matrix arising from the polynomial representation of $H$.

Proposition 3.3. Let $\mathcal{E} = x_1 y_1 - x_2 y_2 + c_0 (\xi_1 - \xi_2) \sigma \in H$. As an operator on the polynomial representation $k[x_1, x_2] \cong k[x] \otimes k[x]$, we have CYB$_{4e_0}(\mathcal{E}) = 0$.

Converting $\mathcal{E}$ into an element of $gl_n \wedge gl_n$ yields

$$\mathcal{E} = \sum_{1 \leq j < \ell \leq n} (a_{j,j} e_{jj} \wedge e_{\ell\ell} + b_{j,j} e_{jj} \wedge e_{\ell\ell}) + \sum_{1 \leq p < j < \ell \leq n} c_{j,j} e_{jj} \wedge e_{\ell-p,j-p}$$
where \( a_{j\ell}, b_{j\ell}, c_{j\ell} \) are the constants
\[
\begin{align*}
a_{j\ell} &= 4c_0 - 2\kappa(\ell - j) + 2c_1 \left( (-1)^\ell - (-1)^j \right) \\
b_{j\ell} &= 2c_0 \left( -1 - (-1)^j + (-1)^\ell - (-1)j \right) \\
c_{j\ell} &= 4c_0 \left( 1 + (-1)^{j+\ell} \right).
\end{align*}
\]
Setting the deformation parameters to \( c_0 = \frac{3}{4} \) and \( c_1 = 0 \) is the only instance when \( E \in sl_n \cap sl_n \).

In what follows we express \( r_{n-2,n} \) in terms of operators in a slightly modified version of \( H \). Let \( \tilde{H} \) denote the \( k \)-algebra
\[
(3.17) \quad \tilde{H} := H \langle x_{1}^{-1}, x_{2}^{-1}, \partial_{1}, \partial_{2} \rangle.
\]
This is the algebra obtained from \( H \) by adjoining the inverses of \( x_1 \) and \( x_2 \) as well as the partial derivative operators, \( \partial_1 \) and \( \partial_2 \). The commuting relations involving \( x_1^{-1} \) and \( x_2^{-1} \) can be obtained from the relations in \( H \) involving \( x_1 \) and \( x_2 \). The partial derivatives \( \partial_1 \) and \( \partial_2 \) commute with each other and satisfy the following relations in \( \tilde{H} \):
\[
[\partial_j, x_\ell] = \delta_{j\ell}, \quad \partial_j \sigma = \sigma \partial_{j-1}, \quad \partial_j \xi_\ell = (-1)^{j+\ell} \xi_\ell \partial_j,
\]
for \( j, \ell \in \{1, 2\} \). The commuting relations involving the partial derivatives with the Dunkl operators \( y_1 \) and \( y_2 \) are less obvious, but we will not make use of them. In Theorem 3.4 below, we will demonstrate how the \( r \)-matrix \( r_{n-2,n} \) can be interpreted in terms of the algebra \( \tilde{H} \). First we define the following elements of \( kW \subset H \):
\[
g_1 = -\frac{\sigma}{4c_0}, \quad g_2 = \frac{\kappa \sigma}{4c_0} - \frac{1}{2n}, \quad g_3 = \frac{1}{4}(1 - \xi_1)(1 - \xi_2), \quad g_4 = -\frac{c_1}{4c_0}(\xi_1 - \xi_2) \sigma.
\]
We have the following

**THEOREM 3.4.** Identifying \( k^n \otimes k^n \) with \( \text{TruncPol}_n(k[x_1, x_2]) \) yields
\[
(3.18) \quad r_{n-2,n} = g_1(x_1 y_1 - x_2 y_2) + g_2(x_1 \partial_1 - x_2 \partial_2) + \left( \frac{x_2}{x_1} - \frac{x_1}{x_2} \right) g_3 + g_4 \in \tilde{H}.
\]

**PROOF.** When \( m = 2 \), Eqn. (2.10) becomes
\[
r_{n-2,n}(e_j \otimes e_\ell) = \sum_{N=0}^{\lfloor \ell-j \rfloor} e_{\ell+2N} \otimes e_{j-2N} - \sum_{N=0}^{\lfloor \ell-j \rfloor} e_{\ell-2N} \otimes e_{j+2N}
\]
\[
+ \frac{1}{4}(1 - (-1)^j)(1 - (-1)^\ell) \left( (j - 1) \otimes e_{\ell+1} - e_{\ell+1} \otimes e_{j-1} \right)
\]
\[
+ \left( \frac{1}{2} - \frac{1}{n} \left( (j - \ell) \frac{n+1}{2} \right) \right) e_j \otimes e_\ell - \frac{1}{2} \text{sgn}(j - \ell) e_\ell \otimes e_j.
\]
The \( \alpha, \beta, \) and \( \gamma \) parts of \( r_{n-2,n} \) (refer to Eqns. 5.17, 5.18) act via
\[
\alpha \mapsto -\frac{M}{4}(1 + \xi_1 \xi_2) - \frac{\Delta}{4} + \frac{1}{4} \xi_1 \Delta \xi_2 + \left( \frac{x_2}{x_1} - \frac{x_1}{x_2} \right) g_3,
\]
\[
\beta \mapsto \frac{M}{4}(1 + \xi_1 \xi_2) - \frac{1}{2n}(x_1 \partial_1 - x_2 \partial_2),
\]
\[
\gamma \mapsto \frac{1}{2} \sigma M,
\]
where \( M \) is the linear operator on \( k[x_1^{\pm 1}, x_2^{\pm 1}] \) defined by \( M : x_1^j x_2^\ell \mapsto \text{sgn}(j - \ell)x_1^j x_2^\ell \). Summing \( \alpha, \beta, \) and \( \gamma \) gives us
\[
r_{n-2,n} = -\frac{1}{2n}(x_1 \partial_1 - x_2 \partial_2) + \frac{1}{4} \left( \Delta + \xi_1 \Delta \xi_2 + 4 \left( \frac{x_2}{x_1} - \frac{x_1}{x_2} \right) g_3 \right).
\]
In the algebra \( H \), the Dunkl operators are \( y_1 = \kappa \partial_1 - c_0 \frac{1-\sigma}{x_1-x_2} - c_0 \frac{1-\xi_1 \xi_2}{x_1+x_2} - c_1 \frac{1-\xi_1}{x_1} \) and \( y_2 = \kappa \partial_2 + c_0 \frac{1-\sigma}{x_1-x_2} - c_0 \frac{1-\xi_1 \xi_2}{x_1+x_2} - c_1 \frac{1-\xi_2}{x_2} \). Substituting these into the above formula yields Eqn. (3.18). \( \Box \)
To conclude this section, we verify that \( r_{n-2,n} \), expressed in terms of the algebra \( \tilde{H} \), is indeed an \( r \)-matrix. In fact, we have the following lemma, which is a more general result.

**Lemma 3.5.** Let \( \Delta = \frac{r_1 + r_2}{x_1 - x_2} (1 - \sigma) \). Then

\[
CYB_4 \left( \Delta + \xi_1 \Delta \xi_2 + a_1 \left( \frac{x_2}{x_1} - \frac{x_1}{x_2} \right) g_3 + a_2 (x_1 \partial_1 - x_2 \partial_2) \right) = 0
\]

for all \( a_1, a_2 \in k \).

**Proof.** Compute.

**Corollary 3.6.** We have \( CYB_4(r_{n-2,n}) = 0 \). □

**Proof.** Setting \( a_1 = 1 \) and \( a_2 = \frac{2}{n} \) in Eqn. 3.19 coincides with \( 4r_{n-2,n} \). Thus, \( CYB_4(4r_{n-2,n}) = 0 \) (equivalently \( CYB_4(r_{n-2,n}) = 0 \)). □

4. A Note on the Gerstenhaber-Giaquinto Conjecture: the case when \( m = n - 2 \)

In this section, we prove the Gerstenhaber-Giaquinto conjecture for the case when \( m = n - 2 \) (and \( n \) is necessarily odd). First, we define the following:

\[
E_1 := \frac{1}{2} (x_1^{-1} \partial_1 + x_2^{-1} \partial_2) - \frac{1}{4} (x_1^{-2} (1 - \xi_1) + x_2^{-2} (1 - \xi_2)) \in \tilde{H},
\]

\[
E_2 := \frac{1}{2} (x_1 + x_2 - x_1 \xi_1 - x_2 \xi_2) \in \tilde{H}.
\]

The truncated polynomial ring \( TruncPol_n(k[x_1, x_2]) \) is stable under the actions of \( E_1 \) and \( E_2 \). Under the vector space isomorphism \( TruncPol_n(k[x_1, x_2]) \cong k^n \otimes k^n, E_1 \) and \( E_2 \) are identified with the matrices \( \sum_{j=1}^{n-2} \frac{i+j}{4} e_{i,j+2} \in \mathfrak{sl}_n \) and \( e_{n,n-1} + e_{n-2,n-3} + \cdots + e_{3,2} \in \mathfrak{sl}_n \), respectively. Let \( \mathfrak{n} \) denote the Lie algebra generated by \( E_1 \) and \( E_2 \). One may readily check that the commutator \([E_1, E_2]\) is nonzero and commutes with \( E_1 \) and \( E_2 \). Hence, \( \mathfrak{n} \) is isomorphic to the Heisenberg Lie algebra. The defining relations of the universal enveloping algebra \( U(\mathfrak{n}) \) are \( E_1^2 E_2 - 2E_1 E_2 E_1 + E_1 E_2^2 = 0 \) and \( E_2^2 E_1 - 2E_2 E_1 E_2 + E_2 E_1^2 = 0 \). Since \( r_{n-2,n} \in \mathfrak{sl}_n \land \mathfrak{sl}_n \), the adjoint action of \( \mathfrak{sl}_n \) on \( \mathfrak{sl}_n \land \mathfrak{sl}_n \) gives us a natural way to generate a \( U(\mathfrak{n}) \)-module. A straightforward computation will verify the following lemma.

**Lemma 4.1.**

1. The \( U(\mathfrak{n}) \)-module generated by \( r_{n-2,n} \) has a \( k \)-basis \( \{ r_{n-2,n}, v_1, v_2, v_3, v_4 \} \), where the vectors \( v_1, \ldots, v_4 \) are given by the formulas

\[
v_1 = \frac{1}{4} \left( \Delta - \xi_1 \Delta \xi_2 + \xi_1 - \xi_2 \right) - \frac{1}{4n} \left( 2(x_1^{-1} \partial_1 - x_2^{-1} \partial_2) - x_1^{-2} (1 - \xi_1) + x_2^{-2} (1 - \xi_2) \right),
\]

\[
v_2 = \frac{1}{4} \left( x_2 \xi_1 - x_1 \xi_2 + (x_1 - x_2) \xi_1 \xi_2 + \frac{1}{n} (x_1 (1 - \xi_1) - x_2 (1 - \xi_2)) \right),
\]

\[
v_3 = \frac{1}{2} (x_1 x_2)^{-1} \left( x_1 \xi_1 - x_2 \xi_2 - (x_1 - x_2) \xi_1 \xi_2 + \frac{1}{n} (x_2 (1 - \xi_1) - x_1 (1 - \xi_2)) \right),
\]

\[
v_4 = \frac{1}{2} \left( \frac{x_2 - x_1}{x_1} \right) (1 - \xi_1 - \xi_2 + \xi_1 \xi_2).
\]

2. The \( U(\mathfrak{n}) \)-module structure for \( U(\mathfrak{n}) r_{n-2,n} \) is given by \( E_1 r_{n-2,n} = v_1, E_1 v_2 = \frac{1}{2} v_3, E_2 r_{n-2,n} = v_2, E_2 v_1 = v_3, E_2 v_3 = v_4, \) and \( E_1 v_1 = E_1 v_3 = E_1 v_4 = E_2 v_2 = E_2 v_4 = 0 \).

**Proof.** Parts (1) and (2) can be obtained by using the commutation relations among the generators of \( \tilde{H} \). Once (2) is established, it follows that \( \{ r_{n-2,n}, v_1, v_2, v_3, v_4 \} \) is a spanning set for \( U(\mathfrak{n}) r_{n-2,n} \). To obtain linearly independence, observe that \( \tilde{H} \) is \( \mathbb{Z} \)-graded with \( \text{deg}(x_i) = 1, \text{deg}(y_i) = \text{deg}(\partial_k) = -1, \text{deg}(\xi_i) = 0 \) for \( i = 1, 2 \). Linear independence follows from the fact that \( \text{deg}(v_1) = -2, \text{deg}(v_2) = 1, \text{deg}(v_3) = -1 \) and \( \text{deg}(r_{n-2,n}) = \text{deg}(v_4) = 0 \) (but \( r_{n-2,n} \) is not a \( k \)-multiple of \( v_4 \)). □

Furthermore, we have the following.
PROPOSITION 4.2. Any $k$-linear combination of $v_1, v_2, v_3, v_4$ is a triangular $r$-matrix.

PROOF. For a pair of linear operators $L', L''$ on the vector space $k[x_1^{\pm 1}, x_2^{\pm 1}] \cong k[x_1^{\pm 1}] \otimes k[x_2^{\pm 1}]$, we introduce the notation $\langle \langle L', L'' \rangle \rangle$ to mean

$$\langle \langle L', L'' \rangle \rangle := [L'_{12}, L''_{13}] + [L'_{12}, L''_{23}] + [L'_{13}, L''_{23}] \in \text{End}_k(k[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]) \cong \text{End}_k(k[x_1^{\pm 1}]^{\otimes 3})$$

(in particular, $\langle \langle L, L \rangle \rangle = CYB_0(L)$ for all $L \in \text{End}_k(k[x_1^{\pm 1}, x_2^{\pm 1}])$). Let $r' = e^{uE_2}e^{tE_1}r_{n-2,n}$ and $r'' = e^{uE_3}e^{tE_2}r_{n-2,n}$. From Lemma 4.1 it follows that

$$r' = r_{n-2,n} + uv_1 + tv_2 + tuv_3 + \frac{1}{2}t^2uv_4$$

and

$$r'' = r_{n-2,n} + uv_1 + tv_2 + \frac{1}{2}tuv_3.$$ 

Since $r'$, $r''$, and $r_{n-2,n}$ are equivalent $r$-matrices, we have $\langle \langle r', r' \rangle \rangle - \langle \langle r_{n-2,n}, r_{n-2,n} \rangle \rangle = 0$ and $\langle \langle r'', r'' \rangle \rangle - \langle \langle r_{n-2,n}, r_{n-2,n} \rangle \rangle = 0$. Expanding $\langle \langle r', r' \rangle \rangle - \langle \langle r_{n-2,n}, r_{n-2,n} \rangle \rangle$ and $\langle \langle r'', r'' \rangle \rangle - \langle \langle r_{n-2,n}, r_{n-2,n} \rangle \rangle$ yields expressions of the form

$$0 = uA_1 + tA_2 + tuA_3 + t^2uA_4 + u^2A_5 + tu^2A_6 + t^2u^2A_7 + t^2A_8 + t^3A_9 + t^4A_{10} + t^3A_{11}$$

and

$$0 = uB_1 + tB_2 + tuB_3 + t^2uB_4 + u^2B_5 + tu^2B_6 + t^2u^2B_7 + t^2B_8$$

respectively, where $A_1, ..., A_1$ and $B_1, ..., B_9$ are certain cross-terms, e.g. $A_1 = \langle \langle r_{n-2,n}, v_1 \rangle \rangle + \langle \langle v_1, r_{n-2,n} \rangle \rangle$, $A_2 = \langle \langle r_{n-2,n}, v_2 \rangle \rangle + \langle \langle v_2, r_{n-2,n} \rangle \rangle$, etc. It follows that $A_1 = \cdots = A_{11} = 0$ and $B_1 = \cdots = B_9 = 0$. For $a, b, c, d \in k$, let $r_{abcd} = av_1 + bv_2 + cv_3 + dv_4$. We have

$$\langle \langle r_{abcd}, v_{abcd} \rangle \rangle = a^2 \langle \langle v_1, v_1 \rangle \rangle + ab \langle \langle v_1, v_2 \rangle \rangle + ab \langle \langle v_2, v_1 \rangle \rangle + \cdots + d^2 \langle \langle v_4, v_4 \rangle \rangle$$

$$= a^2A_5 + ab(2B_3 - A_3) + acA_6 + ad(2A_7 - 8B_7) + b^2A_8$$

$$+ 2bcB_6 + 2bdA_9 + 4c^2B_7 + 2cdA_{11} + 4d^2A_{10}$$

$$= 0.$$

We conclude this section by showing that the Gerstenhaber-Giaquinto conjecture holds for the case when $m = n - 2$. For parameters $u, t \in k$, define first the following:

$$b_{CG}(u,t) := uv_1 + tv_2 + tuv_3 + \frac{1}{2}t^2uv_4 \in \tilde{H}. $$

The vector space $\text{TruncPol}_n(k[x_1, x_2])$ is stable under the action of $b_{CG}(u,t)$, hence we can identify $b_{CG}(u,t)$ with an element of $sl_n \wedge sl_n$.

The following theorem proves the Gerstenhaber-Giaquinto conjecture for the case when $m = n - 2$. The proof of part 1 of Thm. 1.3 below was communicated to us by Anthony Giaquinto [15].

THEOREM 4.3.

(1) Let $u, t \in k$. Then

$$e^{uE_2}e^{tE_1}r_{n-2,n} = r_{n-2,n} + b_{CG}(u,t).$$

Hence $b_{CG}(u,t)$ is a triangular $r$-matrix. For $u$ and $t$ not both zero, $b_{CG}(u,t)$ lies in the boundary component of the Cremmer-Gervais $r$-matrix $r_{n-2,n}$.

(2) For $u, t \neq 0$, the carrier of $b_{CG}(u,t)$ (viewed as an element of $sl_n \wedge sl_n$) is the maximal parabolic subalgebra $p_{n-2,n} \subseteq sl_n$.

(3) The Frobenius functional associated to $b_{CG}(u,t)$ is

$$u^{-1}(e_{13}^* + e_{24}^* + \cdots + e_{n-2,n}^*) - te_{n-1,n} + 2(e_{n-1,n-1} - e_{nn})^* \in p_{n-2,n}^\ast.$$
PROOF. Proposition 1.2 implies $b_{CG}(u, t)$ is a triangular $r$-matrix. For part (2), we compute. As operators on $k[x_1^\pm, x_2^\pm]|_{\mathcal{P}}$, where $\mathcal{P}$ is true and $\mathcal{P} = 0$ if $\mathcal{P}$ is false. Restricting these operators to the truncated polynomial ring $\text{TruncPol}_n(k[x_1, x_2])$ and then using the isomorphism $\text{TruncPol}_n(k[x_1, x_2]) \cong k^n \otimes k^n$ translates the above formulas for $v_1, \ldots, v_4$ into the following elements of $\mathfrak{sl}_n \otimes \mathfrak{sl}_n$:

$$v_1 = 2 \left( \sum_{1 \leq \ell < j \leq n} e_{\ell+2N-2,j} \otimes e_{j-2N,\ell} + \sum_{1 \leq \ell < j \leq n} e_{j-1,j} \otimes e_{\ell-1,\ell} + \sum_{j=1}^{n-2} e_{j,j+2} \otimes h_j \right),$$

$$v_2 = 2E^- \otimes h_{n-1},$$

$$v_3 = 4E^+ \otimes h_{n-1},$$

$$v_4 = 4E^+ \otimes E^-, $$

where $E^+ = e_{12} + e_{34} + \cdots + e_{n-2,n-1}$, $E^- = e_{n,n-1} + e_{n-2,n-3} + \cdots + e_{32}$, and

$$h_j := \sum_{N=0}^{\lfloor j/2 \rfloor} e_{j-2N,j-2N} - \frac{1}{n} \left[ \frac{j+1}{2} - 1 \right] e_{1,1} + \cdots + e_{n,n}$$

for $j = 1, \ldots, n-1$. Following the notation from Section 2.4, one can verify that the linear maps

$$\tilde{r}: p_{n-2,n}^* \rightarrow p_{n-2,n},$$

and

$$\tilde{r}^{-1}: p_{n-2,n} \rightarrow p_{n-2,n}^*,$$
are given by
\[
\tilde{r} : e^*_{j\ell} \mapsto \begin{cases}
  u(e_{\ell - 2, j} + e_{\ell - 4, j - 2} + \cdots), & (\ell > j + 2) \text{ or } (j \text{ even and } \ell = j + 1), \\
  -u(e_{\ell + 2, j} + e_{\ell + 4, j + 4} + \cdots), & (\ell < j - 1) \text{ or } (j \text{ even and } \ell = j - 1), \\
  -u(e_{\ell, j+2} + e_{\ell+2, j+4} + \cdots) + 2tu_{h_{n-1}} + t^2u_{E^-}, & (j \text{ odd and } \ell = j + 1), \\
  -u(e_{\ell, j+2} + e_{\ell+2, j+4} + \cdots) + th_{n-1} - t^2u_{E^+}, & (j \text{ odd and } \ell = j - 1), \\
  uh_j, & (\ell = j + 2),
\end{cases}
\]
\[
\tilde{r} : h^*_{j} \mapsto \begin{cases}
  -ue_{j,j+2}, & (j \neq n - 1), \\
  -tE^- - 2tu_{E^+}, & (j = n - 1), \\
  -2e^*_{n-1,n} - t^{-1}h^*_{n-1} - u^{-1}e^*_{n-3,n}, & (j, \ell) = (n, n - 1), \\
  -u^{-1}h^*_j, & (\ell = j + 2), \\
  u^{-1}e^*_{3,3}, & (j, \ell) = (1, 2), \\
  2e^*_{n,n-1} - th^*_{n-1} - u^{-1}e^*_{n-2,n-1}, & (j, \ell) = (n - 1, n), \\
  -u^{-1}(e^*_{j,j+2} - e^*_{e_{j,j+2}}), & \text{otherwise},
\end{cases}
\]
\[
\tilde{r}^{-1} : e_{j\ell} \mapsto \begin{cases}
  -u^{-1}(e^*_{1,3} + e^*_{2,4} \cdots e^*_{n-2,n} - te^*_{n-1,n} + 2(e_{n-1,n-1} - e_{mn})^* \in p^*_{n-2,n}. \text{ One can check that } \varphi[X,Y] = \langle X, \tilde{r}^{-1}(Y) \rangle \text{ for all } X, Y \in p_{n-2,n}. \]
\]

5. Proof of Theorem 2.3

As before, let \( m \) and \( n \) be relatively prime positive integers with \( m < n \) and let \( T_{m,n} = (S_0, S_1, \zeta) \) denote the unique BD-triple for \( sl_n \) with \( S_1 = \Pi - \{ \alpha_m \} \). Let \( \varphi : sl_n \to sl_n \) be the involutive Lie algebra automorphism defined by \( e_{j\ell} \mapsto -e_{t'j',t} \) (where \( t' := n + 1 - t \) for all \( t \in \{1, ..., n\} \)). Furthermore, recall that the corresponding generalized Cremmer-Gervais r-matrix associated to \( T_{m,n} \) is \( r_{m,n} = \alpha_{m,n} + \beta_{m,n} + \gamma_n \) where
\[
(5.1) \quad \alpha_{m,n} := 2 \sum_{e_{j_1,j_2} \prec e_{j_3,j_4}} e_{j_1,j_2} \wedge e_{j_3,j_4},
\]
\[
(5.2) \quad \beta_{m,n} := \sum_{1 \leq j < \ell \leq n} \left[ -1 + \frac{2}{n}[(j - \ell)m^{-1}(\text{mod } n)] \right] e_{jj} \wedge e_{\ell\ell},
\]
\[
(5.3) \quad \gamma_n := \sum_{1 \leq j < \ell \leq n} e_{j\ell} \wedge e_{\ell j}.
\]

Here the sum on \( \alpha_{m,n} \) is over all appropriate quadruples of numbers \( j_1, j_2, j_3, j_4 \) with \( e_{j_1,j_2} \prec e_{j_3,j_4} \). The remainder of this section is devoted to giving a more precise description of such quadruples. Define first the sets
\[
(5.4) \quad S_{m,n}(j_1, j_2) := \{ s \in \{1, ..., n\} \mid e_{j_1,s} \prec e_{j_1+j_2-s,j_2} \text{ for the BD-triple } T_{m,n} \},
\]
\[
(5.5) \quad \overline{S}_{m,n}(j_1, j_2) := \{ s \in \{1, ..., n\} \mid e_{j_1,s} \leq e_{j_1+j_2-s,j_2} \text{ for the BD-triple } T_{m,n} \}.
\]

We readily compute \( \alpha_{n-m,n} = (\varphi \otimes \varphi)\alpha_{m,n} \) and the action of \( \alpha_{n-m,n} \) on \( e_j \otimes e_\ell \in k^n \otimes k^n \) is given by
\[
(5.6) \quad \alpha_{n-m,n}(e_j \otimes e_\ell) = \text{sgn}(\ell - j)e_\ell \otimes e_j + \sum_{s \in S_{m,n}(j',t')} e_{j'} \otimes e_{j'+\ell-s'} - \sum_{s \in \overline{S}_{m,n}(t',j')} e_{j'+\ell-s'} \otimes e_{j'}
\]
\[
= \sum_{s \in \overline{S}_{m,n}(j',t')} e_{j'} \otimes e_{j'+\ell-s'} - \sum_{s \in \overline{S}_{m,n}(t',j')} e_{j'+\ell-s'} \otimes e_{j'}
\]
\]

Our goal now is to explicitly determine the elements of the sets \( \overline{S}_{m,n}(1,1), \overline{S}_{m,n}(1,2), ..., \overline{S}_{m,n}(n,n) \). The running example we consider is the case when \( n = 31 \) and \( m = 12 \). In Figure 3 we’ve arranged the integers 1 through 31 in two concentric wheels which we view as having the ability to rotate independently around a
central axis. In each wheel the numbers are arranged so that as one traverses counterclockwise, the numbers increase by 12 modulo 31. Spokes are placed after certain numbers, indicating when a reduction modulo 31 occurs. The spokes partition the integers into strings. Here, the strings are

\{1, 13, 25\}, \{6, 18, 30\}, \{11, 23\}, \{4, 16, 28\}, \{9, 21\}, \{2, 14, 26\},
\{7, 19, 31\}, \{12, 24\}, \{5, 17, 29\}, \{10, 22\}, \{3, 15, 27\}, \{8, 20\}.

Each string has a smallest integer, called the minimal element. Here, the minimal elements of each string listed above are

1, 6, 11, 4, 9, 2, 7, 12, 5, 10, 3, 8

respectively. For example, suppose \(j = 17\), \(\ell = 10\), and we wish to determine \(\mathfrak{S}_{12,31}(17', 10')\). Figure 3 indicates how 19 \(\in \mathfrak{S}_{12,31}(17', 10')\), for instance. First, we locate \(j' = 15\) on the inner wheel and \(\ell' = 22\) on the outer wheel. As we traverse counterclockwise 8 units from the aligned pair of numbers \((j', 19)\), we see for instance that

\(e_{j', 19} = e_{15, 19} \prec e_{27, 31} \prec \cdots \prec e_{6, 10} \prec e_{18, 22} = e_{18, \ell'}\).

Hence, \(e_{j', 19} \prec e_{18, \ell'}\). In terms of the BD-triple \(\tilde{T}_{12,31}\) (recall Eqn. 227), this translates to

\[\hat{\zeta}^s(e_{j', 19} - e_{19}) = e_{18} - e_{\ell'}\].

Notice however that \(e_{j', 19} \not\prec e_{11, j'}\) because there does not exist \(N \in \mathbb{Z}_{\geq 0}\) so that \(\hat{\zeta}^N(e_{j', 19}) = e_{11} - e_{j'}\). This is indicated by the spokes on the two wheels not lining up correctly after traversing 9 units counterclockwise from the pair \((j', 19)\). However, there are other legitimate choices for \(s\) with \(e_{j', s} \not\leq e_{\ell' + j' - s, \ell'}\). We can check all possibilities and see that \(s = 16, 17, 19, 22\) are the only allowed values. However, we seek a more efficient algorithm and explicit formulae for finding all appropriate values of \(s\) in \(\mathfrak{S}_{m,n}(j', \ell')\).

The first candidates we check are those in the same string as \(\ell'\). Thus, we test the integers of the form \(\ell' - Nm\) for \(N = 0, 1, 2, \ldots\). The only condition that needs to be satisfied is \(\ell' - Nm > j'\), or equivalently \(N \leq \left\lfloor \frac{\ell' - j'}{m} \right\rfloor \). So we imagine first aligning \(j'\) on the inner wheel with \(\ell'\) on the outer wheel and then rotating the inner wheel clockwise so that \(j'\) aligns with \(\ell' - m\), then with \(\ell' - 2m\), etc. This will give us \(1 + \left\lfloor \frac{\ell' - j'}{m} \right\rfloor \) valid candidates (provided \(\ell < j\)). In our example, we check \(s = 22\) and \(s = 10\), but since \(10 < j\), this implies 10 is not valid. Thus, so far this only gives us \(s = 22 \in \mathfrak{S}_{12,31}(15, 22)\).
To find the other $s \in \mathcal{S}_{12,31}(15, 22)$, we continue rotating the inner wheel clockwise so that the minimal element in the string directly counterclockwise to the string containing $j'$ lines up with the minimal element of the string containing $\ell'$. We denote these minimal elements by $A(j')$ and $B(\ell')$ respectively. They can be calculated according to the formulas

$$A(j') := m - [(n - j')(\mod m)],$$

$$B(\ell') := \ell' - m \left\lfloor \frac{\ell' - 1}{m} \right\rfloor = 1 + [(\ell' - 1)(\mod m)].$$

In our example, the string adjacent to the one containing $j'$ is $\{8, 20\}$, so we align 8 with 10 (see Figure 4).

This will reduce the problem to a smaller pair a relatively prime positive integers, $i_1$ and $i_2$, with $i_2 < i_1$. Notice that we can naturally arrange the minimal elements on a pair of smaller rotating wheels, as in Figure 4. As we circumnavigate the larger wheels counterclockwise, the minimal elements of each string are 1, 6, 11, 4, 9, 2, 7, 12, 5, 10, 3 and 8. The minimal elements are increasing by $-n(\mod m)$. The spokes in the smaller wheel of minimal elements also tell us which minimal elements belong to large or small strings: the minimal elements of small strings have a spoke next to them in the counterclockwise direction. In particular, the spokes in the wheel of minimal elements tell us how the spokes align in the larger wheel.

Next, we define integers

$$C_1(\ell') := m + 1 - B(\ell') = m - [(\ell' - 1)(\mod m)],$$

$$D_1(j') := m + 1 - A(j') = 1 + [(n - j')(\mod m)]$$

and determine all values $\bar{s} \in \mathcal{S}_{n(\mod m), m}(C_1(\ell'), D_1(j'))$, i.e. reduce to the smaller pair of wheels. Continuing with our example, we seek all $\bar{s} \in \mathcal{S}_{5,12}(3, 5)$. We get $\mathcal{S}_{5,12}(3, 5) = \{4, 5, 7\}$. However, since $e_{3,4} \leq e_{4,5}$, $e_{3,5} \leq e_{3,5}$, and $e_{3,7} \leq e_{1,5}$ hold in the $T_{5,12}$-case, this will imply $e_{15,16} \leq e_{21,22}$, $e_{15,17} \leq e_{20,22}$, and $e_{15,19} \leq e_{18,22}$ for the $T_{12,31}$-case.
In general, the approach we take is to check all candidates in the same string containing \( \ell' \), then reduce the problem to a smaller case. Thus,

\[
\sum_{s \in \mathcal{S}_{m,n}(\ell', \ell)} e_s = \sum_{m=0}^{\ell' - N_m} e_{\ell' - N_m} + \sum_{s \in \mathcal{S}_{n \mod m, m}(C_{\ell'}(\ell'), D_{\ell'}(j'))} e_{j'} + (s - C_{\ell'}(1)) 
\]

We define a sequence of integers \( i_0, i_1, i_2, \ldots \) recursively by \( i_0 := n, i_1 := m, \) and \( i_t := -i_{t-2} \mod i_{t-1} \) for \( t > 1 \). Eventually the sequence will reach 1 (e.g. 31, 12, 5, 3, 1). Let \( L \) denote the smallest number so that \( i_L = 1 \). With this setup, we recursively get

\[
\sum_{s \in \mathcal{S}_{m,n}(\ell', \ell)} e_s = \sum_{t=0}^{L-1} \sum_{N=0}^{\ell' - N_m} e_{j'} + D_{t}(j') - C_{\ell'}(j') - N_{i_t} + \sum_{s \in \mathcal{S}_{n \mod m, m}(C_{\ell'}(\ell'), D_{\ell'}(j'))} e_{j'} + (s - C_{\ell'}(1)) \cdot
\]

where

\[
C_{t}(\ell) := \begin{cases} 
    i_t - ((i_0 - \ell) \mod i_0)(\mod i_1) \cdots (\mod i_t), & (t \text{ is even}), \\
    i_t - ((\ell - 1) \mod i_0)(\mod i_1) \cdots (\mod i_t), & (t \text{ is odd}), 
\end{cases}
\]

\[
D_{t}(\ell) := \begin{cases} 
    1 + ((\ell - 1) \mod i_0)(\mod i_1) \cdots (\mod i_t), & (t \text{ is even}), \\
    1 + ((i_0 - \ell) \mod i_0)(\mod i_1) \cdots (\mod i_t), & (t \text{ is odd}), 
\end{cases}
\]

for all \( 0 \leq t \leq L - 1 \) and \( 1 \leq \ell \leq n \). After simplifying Eqn. \[5.12\] we get

\[
\sum_{s \in \mathcal{S}_{m,n}(\ell', \ell)} e_s = \sum_{t=0}^{L-1} \sum_{N=0}^{\ell' - N_m} e_{j'} + J_{t}(j, \ell) - N_{i_t + 1}
\]

where

\[
J_{t}(j, \ell) := \begin{cases} 
    D_{t}(\ell') - C_{t}(j'), & (t \text{ is even}), \\
    D_{t}(j') - C_{t}(\ell'), & (t \text{ is odd}), 
\end{cases}
\]

\[
= 1 - i_t + [(n - \ell) \mod i_0)(\mod i_1) \cdots (\mod i_t)] + [(j - 1) \mod i_0)(\mod i_1) \cdots (\mod i_t)].
\]

Therefore,

\[
\alpha_{n-m,n}(e_j \otimes e_\ell) = \text{sgn}(\ell - j) e_\ell \otimes e_j + \sum_{t=0}^{L-1} \sum_{N=0}^{\ell' - N_m} e_{j'} + J_{t}(j, \ell) + N_{i_t + 1} \otimes e_\ell + J_{t}(j, \ell) - N_{i_t + 1}
\]

\[
- \sum_{t=0}^{L-1} \sum_{N=0}^{\ell' - N_m} e_{j'} + J_{t}(j, \ell) - N_{i_t + 1} \otimes e_\ell - J_{t}(j, \ell) + N_{i_t + 1}.
\]

Recall that \( \beta_{m,n} = \sum_{1 \leq j < \ell \leq n} \left[ -1 + \frac{2}{n}[(j - \ell) m^{-1}(\mod n)] \right] e_{jj} \wedge e_{\ell \ell} \) and \( \gamma_n = \sum_{1 \leq j < \ell \leq n} e_{jj} \wedge e_{\ell \ell} \). Applying the automorphism \( \phi \otimes \varphi \) yields \( (\varphi \otimes \phi) \beta_{m,n} = -\beta_{m,n} = \beta_{n-m,n} \) and \( (\varphi \otimes \phi) \gamma_n = \gamma_n \). Thus,

\[
\beta_{n-m,n}(e_j \otimes e_\ell) = \left( \frac{1}{2} - \frac{1}{n}[(j - \ell) m^{-1}(\mod n)] - \frac{1}{2} \delta_{j,\ell} \right) e_j \otimes e_\ell,
\]

\[
\gamma_n(e_j \otimes e_\ell) = \frac{1}{2} \text{sgn}(j - \ell) e_\ell \otimes e_j.
\]
For $j \in \{1, \ldots, n\}$, let $\psi_j$ denote the unique integer in $\{1, \ldots, n\}$ satisfying $j = m\psi_j \pmod{n}$. With this notation, the action of $r_{n-m,n}$ is

\begin{equation}
\tag{5.20}
r_{n-m,n}(e_j \otimes e_\ell) = \sum_{t=0}^{L-1} \sum_{N=0}^{J_t(j,\ell)-1} e_j - J_t(j,\ell) + Nt + 1 \otimes e_\ell - J_t(\ell,j) - Nt + 1 \otimes N^t = 0 e_j - J_t(j,\ell) + Nt + 1 \otimes e_\ell - J_t(\ell,j) - Nt + 1 \otimes e_j + \frac{1}{2} \text{sgn}(\psi_j - \psi_\ell) - \frac{1}{2} \text{sgn}(j - \ell) - \frac{1}{n} (\psi_j - \psi_\ell) e_j \otimes e_\ell.
\end{equation}

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