Perturbative aspects and conformal solutions of $F(R)$ gravity

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We investigate perturbative aspects of gravity with a general $F(R)$ Lagrangian, as well as non-perturbative dilatonic solutions. For the first part, we are interested in stability and the definition of asymptotic charges. The main result of this study is that, while generic $F(R)$ theories are stable under metric perturbations, they are expected to show instabilities against curvature perturbations when the Lagrangian includes $1/R$ terms. For the second part, one is interested on exact solutions, and we explicitly construct kink-like solutions of the Liouville type for the dilaton field for $F(R)$ having the form $R + \gamma R^n$, in two and in four dimensions.

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I. INTRODUCTION

In this work we investigate perturbative and nonperturbative effects in gravity, motivated by the cosmological constant problem [1]. As one knows, the cosmological constant engenders an important but very hard problem to solve due to both cosmological and phenomenological reasons. In the cosmic evolution of the Universe, particle physics introduces the main mechanisms to govern the several phases of the Universe, and cosmology drives the way the cosmological constant is accounted for in each phase. However, the presence of spontaneous symmetry breaking, which is the basic mechanism underlying unification of the fundamental interactions in nature, inevitably injects vacuum energy density even if no such term is present in the theory. For this reason, one realizes that not only the physics at the Planck scale, but also the low energy infrared dynamics of gravity itself is essential to the resolution of the cosmological constant problem. The reasoning establishes direct connection between nonperturbative effects in gravity and the cosmological constant problem.

Nowadays, the cosmological constant problem must include the important discovery that the Universe is currently evolving in an accelerating phase [2]. There are several distinct possibilities of taking acceleration into account: from one side, we can keep standard geometry, incorporating modifications on the matter contents of the model. Interesting investigations which deal with this possibility include, for instance, the cosmological constant [1], dynamical scalar fields [3], Chaplygin fluid [4], and phantom dynamics [5]. Another possibility deals with modifications of geometry, changing General Relativity: here an important class of theories includes models which depend on the Ricci scalar, usually named $F(R)$ theories [6]. In the present work, we will deal with the latter, that is, we will investigate extended gravity, changing $R \rightarrow F(R)$, for $F(R) = R + \gamma R^n$, including the cases $n = 2$ and $n = -1$. This modification brings interesting novelties, as we will show below. The motivation goes beyond the recent discovery of the accelerated expansion of the Universe [2], and it is very natural to expect that, due to the fact that the curvature of the present Universe is very small, the nonperturbative effects should be characteristic for some typical forms of $F(R)$. In particular, the interest in $n = 2$ also appear from semi-classical investigations, and the case of $F(R) = \exp(-\alpha R)$ suggests the presence of instanton effects.

The investigation of nonperturbative solutions is of direct interest to a diversity of applications, including black hole, integrability, noncommutative geometry and other problems [8, 10]. The nonperturbative issues are in general hard to solve, mainly in high energy physics in curved space-time. For this reason, we are planning to approach the problem with the use of the nonperturbative procedure referred to as the first-order formalism, in which one solves equations of motion with solutions of first-order differential equations. Some of us have recently done some progress in extending the formalism to curved space-time [11], and we now consider this possibility in connection with dilaton gravity.

In the present study we organize this Letter as follows. In the next Sec. III we derive the equations of motion...
for $F(R)$ gravity theories and the conserved charges, and then we study stability of solutions in these theories. In Sec. III we turn attention to gravity in the conformal sector, and there we describe the $F(R)$ model restricted to the conformal metric, investigating the corresponding equations of motion in two and in four dimensions. We end the work in Sec. IV where we include some final considerations.

II. EQUATIONS OF MOTION AND CONSERVED CHARGES

Let us lay down the notation, starting with the $F(R)$ action

$$S = - \int d^D x \sqrt{|g|} F(R)$$

where $F(R)$ is equal to $R + 2\Lambda$ for the usual Einstein-Hilbert action with a cosmological constant $\Lambda$. The equations of motion for this theory look like

$$\frac{1}{2} g_{ac} F(R) - F'(R) R_{ac} - F''(R) [\nabla_a \nabla_c R - \nabla_b \nabla^b R_{ac}] = - F'''(R) [\nabla_a R \nabla_c R - g_{ac} \nabla_a R \nabla^d R] = 0$$

Let us now compute the conserved charge associated with this Lagrangian, following the general procedure of [12]. From above, the boundary term one obtains from the Lagrangian is

$$\theta^b = - F'(R) [\nabla_a (\delta g^{ab}) - \nabla^b (\delta g)] - \nabla_a F'(R) \delta g^{ab} + \nabla^b F'(R) (\delta g)$$

that is to say, the full variation of the action can be written as $\delta \mathcal{L} = \sqrt{|g|} E_{ab} \delta g^{ab} + \sqrt{|g|} \nabla_a \theta^a$, where $E_{ab}$ are the equations of motion [2]. Now, the Noether procedure gives us a conserved current associated with a symmetry generated by a vector field $\xi^a$. Those types of diffeomorphisms change the metric as $\delta g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a$. The conserved current is $J^a[\xi] = \theta^a - \xi^a \mathcal{L}$. The existence of this current stems directly from the proof of the theorem. The hard step in any theory which is invariant under diffeomorphisms is to find the charge associated with $J^a$. Consider the Poincaré dual to $J^a$: $J_{a_1 \ldots a_n - 1} = \epsilon_{a_1 \ldots a_n - 1} J^a$, where $\epsilon_{a_1 \ldots a_n}$ is the volume form associated with the metric. We will exceptionally use $n$ for the dimension of space-time in this subsection to avoid cluttering the notation. Conservation of the current implies that, on shell, the $n - 1$ form $J$ is closed $dJ = 0$, so it can be written locally as a exterior derivative: $J_{a_1 \ldots a_n - 1} = \nabla_{[a_1} Q_{a_2 \ldots a_n - 1]} = \partial_{[a_1} Q_{a_2 \ldots a_n - 1]}$, or, analogously with the duals $J^b = \nabla_a Q^{[ab]}$. We can check directly that this equation above implies the conservation of the current

$$\nabla_b J^b = \nabla_b \nabla_a Q^{[ab]} = -\frac{1}{2} (\nabla_a \nabla_b Q^{[ab]} + \nabla_b \nabla_a Q^{[ab]} ) = \frac{1}{2} (R_{abc}^a Q^{[cb]} + R_{abc}^b Q^{[ac]} ) = \frac{1}{2} ( - R_{abc} Q^{[cb]} + R_{abc} Q^{[ac]} ) = R_{abc} Q^{[ac]} = 0$$

If we are able to find such $Q^{ab}$ we can write directly the conserved charge by integrating the corresponding $n - 2$ form $Q$ over a suitable $n - 2$ hypersurface. The procedure is then analogous to the definition of conserved charges in field theory which can be written under certain assumptions as the integral of a charge flux over a sphere (a $n - 2$ dimensional hypersurface in $\mathbb{R}^{3,1}$) “at infinity”.

As it turns out, one can indeed write a $Q^{ab}$ for the theory above; we suppose it has the form $Q^{ab}[\xi] = 2 W^a [\xi^b] + 2 X \nabla^a [\xi^b]$, for some metric-dependent $W^a$ and $X$. Taking its divergence, we will find several terms

$$\nabla_a Q^{ab} = \xi^b \nabla_a W^a - \xi^a \nabla_a W^b + W^a \nabla_a \xi^b - W^b \nabla_a \xi^a + \nabla_a X (\nabla^a \xi^b - \nabla^b \xi^a ) + X (\nabla_a \nabla^a \xi^b - \nabla_a \nabla^b \xi^a )$$

On the other hand, the direct calculation from (II) gives

$$J^b = F'(R) [\nabla_a (\nabla^a \xi^b + \nabla^b \xi^a ) - 2 \nabla^b \nabla_a \xi^a ] - F''(R) \nabla_a R (\nabla^a \xi^b + \nabla^b \xi^a - 2 g^{ab} \nabla_c \xi^c ) - 2 F'(R) \nabla^b R \nabla_a \xi^a + \xi^b F(R)$$

Comparing the terms involving two derivatives of $\xi^a$, we find that $X$ must be $X = F'(R)$, up to some term proportional to the equations of motion. The terms involving only one derivative of $\xi^a$

$$W^a \nabla_a \xi^b - W^b \nabla_a \xi^a + \nabla_a X (\nabla^a \xi^b - \nabla^b \xi^a ) \overset{?}{=} - F''(R) \nabla_a R [\nabla^a \xi^b + \nabla^b \xi^a - 2 g^{ab} \nabla_c \xi^c ]$$
which gives \( W_b = -2F''(R)\nabla_b R \). We now compare the term proportional to \( \xi^a \) in (5) and (6) to get

\[
\xi^b \nabla_a W^a - \xi^c \nabla_c W^b = -2F''(R)R^b_d \xi^d + F(R)\xi^d = (F(R)g^{ab} - 2F'(R)R^{ab})\xi_a
\]

With the choices above for \( X \) and \( W_b \) one has, for the left-hand side

\[
\xi^b \nabla_a W^a - \xi^c \nabla_c W^b = -2F''(R)\xi^b \nabla_a R^a - 2F''(R)\xi^c \nabla^2 R + 2F''(R)\xi^a \nabla_a R^b + 2F''(R)\xi^a \nabla_a \nabla^b R = 2F''(R)(\nabla^a R^b_d - g^{ab} \nabla_c R^c_d) + 2F''(R)(\nabla^a \nabla^b R - g^{ab} \nabla_c \nabla^c R)\xi_a
\]

which is the left-hand side of the equation (8) by the equations of motion (2). The conserved charge is thus given by \( Q^{ab} = 2F'(R)\nabla[a\xi^b + 4F''(R)\xi^a \nabla^b R \). In this expression the first term is analogous to the Komar mass term in usual Einstein-Hilbert whereas the second term is new.

Now we apply the expression for the charge found above to spherically symmetric spaces. As all of the solutions discussed here are static, the total mass will be given by the expression above when the vector field associated is the time translation vector \( \xi^a = (\partial_t)^a \), which is hypersurface orthogonal and its derivative is given by \( \nabla_a \xi_b = -\xi_a \nabla_b \log |\xi|, \)\] where in the first step we used the general expression found in equation (C.3.12) from [13], the second step uses the fact that \( g_{ab}\xi^b = f(r)\partial_a\xi \), and the third step defines the volume element of the “t” plane as \( \xi \partial_a = (dt)\partial_a \xi \). Last but not least, we have used the chain rule to expand \( \nabla_a f(r) = -(dx)_a f'(r) \). One can also derive this result by noting that the relevant part of the metric for spherically symmetric spaces is given by \( ds^2 = f(r)dt^2 - (dr)^2 / f(r) \). Notice that the expression for the derivative of \( \xi^a \) above allows us to write the charge as a term proportional to the spatial volume element \( \xi^a \epsilon_{a_1 \ldots a_{n-2}} : \)

\[
Q_{a_1 \ldots a_{n-2}} = -2F'(R)\nabla^b \log |\xi| \xi^c - 2F''(R)\nabla^b R \xi^c \epsilon_{a_1 \ldots a_{n-2}}.
\]

Applying the same procedure for \( \nabla_a R \), we can write the charge as

\[
Q_{ab} = 2(F'(R) f'(r) - 2F''(R) R')\epsilon_{ab}.
\]

We note that, since the space is spherically symmetric, the integral over a \( D = 2 \) dimensional sphere one has to perform to compute the mass is trivial. One should remark that this expression is really better suited for asymptotically flat space-times. The definition of mass on non-asymptotically flat spaces is murky, where one usually uses the trick of “shifting” the value of the mass by subtracting the “vacuum” value, i. e., the value of \( Q \) on a maximally symmetric solution with the same asymptotics. We would to conclude by commenting that flat-space-solutions like Schwarzschild-de Sitter will have its mass modified by the presence of the \( F'(R) \) term, although not by much. One does expect, however, that solutions with long-range modifications of the curvature, such as those with \( R \approx 1/r \) asymptotically, will show a dramatic effect, specially for \( F'(R) \) involving \( 1/R \) terms, for then \( F''(R) \) can be quite large.

Let us now focus attention on stability. We investigate small deviations of an Einsteinian space solution \( R_{ab} = \frac{R}{2}g_{ab} \) of the general equation of motion (2). In order to do this, let us take the variations of the Ricci tensor and the scalar curvature and write them in terms of the linearized metric components \( h_{ab} = \delta g_{ab} \):

\[
\delta R_{ab} = \frac{1}{2}\nabla^2 h_{ac} + \frac{1}{2}\nabla_a \nabla_c h - \nabla_b \nabla_{(a} h^b_{c)} = \frac{1}{2}\nabla^2 \tilde{h}_{ac} - \frac{1}{2(D-2)}\nabla^2 \tilde{h} h_{ac} - \nabla_b \nabla_{(a} \tilde{h}_{c)}
\]

and hence

\[
\delta R = -h^{ab} \delta R_{ab} + g^{ac} \delta R_{ac} = \frac{2R}{D(D-2)} \tilde{h} - \frac{1}{D-2} \nabla^2 \tilde{h}.
\]

In the expressions above, \( \tilde{h}_{ab} = h_{ab} - \frac{1}{2}h_{ab} \). Such a choice stems from the gauge invariance of linearized gravity: \( h_{ab} \) and \( h_{ab} + \nabla_a v_b + \nabla_b v_a \) correspond to the same perturbation [14]. By writing the variations in this form one can get rid of the first term in (11) by choosing \( v_a \) such that \( \nabla^2 v_a + R_a^{\ b} v_b = -\nabla_b \tilde{h}_{a} \). This fixes some of the gauge symmetry, but the equation above determines \( v_a \) up to a vector \( w_a \) which satisfies \( \nabla^2 w_a + R_a^{\ b} w_b = 0 \). We should point out that solving the two equations above amount to find a vector potential in Lorentz gauge for some distribution of current. With this choice, we can write the first term in (11) as

\[
\nabla_b \nabla_{(a} \tilde{h}_{c)} = \nabla_{(a} \nabla_b \tilde{h}_{c)} + [\nabla_b, \nabla_{(a} ] \tilde{h}_{c)} = -R_i^{d(ab)} \tilde{h}_{d} + R_{db(a} \tilde{h}_{c)} = -R_i^{cd(ab)} \tilde{h}_{d} + R_{d(a} \tilde{h}_{c)}
\]

(13)
We can now turn to the variation of the equations of motion \([2]\). The relevant terms are

\[
\frac{1}{2} h_{ac} F'(R) + \frac{1}{2} g_{ac} F'(R) \delta R - F''(R) \delta R R_{ac} - F'(R) \delta R R_{ac} - F''(R) [\nabla_n \nabla_c \delta R - \nabla^2 \delta R g_{ac}] = 0
\]

and we omitted terms which involve derivatives of \(R\), which are zero on shell since \(R\) is assumed constant. One can further simplify the equation above by noting that the variation of the curvature scalar only depends on \(\tilde{h}\), or, equivalently \(h\), the trace of the metric variation. By taking the trace of the equation above, on the absence of matter, one finds a fourth order equation for \(\tilde{h}\), which can yield the trivial solution \(h = 0\) if one picks the right initial values for the lower derivatives of \(h\). This can be accomplished by a suitable gauge transformation for \(w_a\). So, in short, we can, in a general \(F(R)\) theory, accomplish the transverse traceless gauge for metric perturbations: \(\nabla_a h_{ab} = 0, h = 0\). In this case \(h_{ab} = h_{ab}\) and then the only relevant terms in \([14]\) are those that depend on \(h_{ab}\). Picking those, we have a simple equation for the metric perturbations

\[
\frac{1}{2} F(R) h_{ac} - F'(R) \left( \frac{1}{2} \nabla^2 h_{ac} - R^b_{ac} d h_{bd} + R_{d(\alpha h_{c}) d} \right) = 0
\]

or, specializing to Einstein spaces,

\[
\nabla^2 h_{ac} + 2 C^b_{ac} d h_{bd} + \frac{2R}{D-1} h_{ac} - \frac{F(R)}{F'(R)} h_{ac} = 0,
\]

where \(C_{abcd}\) is the Weyl tensor. The causal structure of the equation above for Einstein spaces has been studied in \([15]\), and we will take the result of a particularly nice survey of \([16]\). As one can see from, for instance, the behavior of the equation above under conformal transformations, the causal structure of the equation above is governed by a “mass term” which depends on the spin of the particle as well as the scalar curvature. In this case, the effect is that the mass term \(M^2\) of spin 2 fields is modified by the presence of curvature to

\[
M^2 = - \frac{F(R)}{F'(R)} + \frac{2}{D} R
\]

Note that this means that the term proportional to the Riemann tensor in \([15]\) takes into account the causal structure of spin-2 particles and hence only the following terms contribute to the mass of the perturbations. This is essentially the spin-2 version of the Breitenlohner-Freedman found \([17]\) in the study of anti-de Sitter spaces. One can also check that constant curvature solutions of the generic Lagrangian (which includes Einstein-Hilbert with a cosmological constant as a particular case) always have massless perturbations (gravitons). Therefore, these solutions do not present us with new instabilities. These are essentially the same which arise in the study of Einstein-Hilbert Lagrangian.

The equation above \([17]\) gives a straightforward criterion for the stability of the solutions described below: if the effective squared mass of the gravitons is positive, small deviations of the metrics found will oscillate, otherwise they will grow exponentially and the space found will be unstable. For the special case \(D = 4\), the corresponding equation has been found by several authors – see \([18]\) and references therein.

Theories involving \(1/R\) terms, which we call “non-perturbative,” has attracted some attention recently, mainly because of the prospect of having a solution of positive curvature, \(i.e.,\) a asymptotically de Sitter space-time, without the need for a cosmological constant. We analyze one well-known solution of those theories which shows an instability against curvature perturbations which should plague those theories in general.

We remark that the mere existence of a term proportional to the inverse of the curvature scalar does not fit with what we know from renormalizable field theory, or even what little we can grasp of quantum gravity, as in string theories. Usual corrections from perturbation series in the interactions of quantum fields with classical gravity show the dynamical generation of terms up to \(R^{D/2}\) \([19]\). In string theory, general remarks given in \([20]\) can be applied here to constrain what corrections one can expect from string instantons such as those that arise from compactifications which respect \(N = 1\) SUGRA. Such compactifications have received a lot of attention in recent years since the work of \([21]\) introduced extra ingredients like confined fluxes, which allowed for meta-stable de Sitter vacua. At any rate, instantons and fluxes may break the Super-Weyl rescaling symmetry of SUGRA \([g_{ab} \rightarrow e^{2\lambda} g_{ab}\), and \(e^{2\lambda} = -3/(\Phi^+ \Phi + c_1 \Phi + c_2 \Phi^+ - 3)\), where \(\Phi\) is a chiral superfield] to a discrete set, as it happens with other...
superfields. The residual symmetry constrain the effective potential generated to be of the form $e^{-\alpha R}$. In this expression, $\alpha$ incorporates scalar condensates, but should not depend on the metric. One should note that even in this hypothetical scenario, the corrections one expects from quantum strings are perturbative in $R$.

All the preceding discussion may present an obstacle for the consideration of non-perturbative theories, and below we will present a concrete criticism, following the study of [7], while keeping the dimension arbitrary. We consider this case, the scalar curvature has the form $\gamma^2 R$. Taking the trace of (2) we find

$$\frac{D}{2} \left( R - \frac{\gamma^2}{R} \right) = \left( 1 + \frac{\gamma^2}{R^2} \right) R \implies R_0^2 = \frac{D + 2 \gamma^2}{D - 2 \gamma^2}$$

(18)

which gives a small curvature for $\gamma$ small, as desired. We will be interested on the positive curvature solutions. Now consider a metric perturbation such that the curvature scalar is no longer a constant. Taking the trace of (2) we find that

$$-(D - 1)(F'' \nabla^2 R + F'''(\nabla R)^2) = - (D - 1) \nabla^2 (F'(R)) = -\frac{D}{2} F(R) - F'(R)R.$$  

(19)

Taking a small curvature perturbation $R = R_0 + \delta R$, we will have

$$(D - 1)F''(R_0) \nabla^2 \delta R \approx - \left( \frac{D - 2}{2} F'(R_0) - F''(R_0) R_0 \right) \delta R$$

(20)

where we omitted a term involving one derivative of $\delta R$, which depends on the particular metric perturbation but will not change the conclusion. The equation above gives an effective mass for such perturbations

$$m^2 \approx - \frac{1}{D - 1} R_0 + \frac{D - 2}{2(D - 1)} \frac{F'(R_0)}{F''(R_0)}$$

(21)

Note that, if we have the Einstein-Hilbert term, $F'(R_0) \approx 1 + \mathcal{O}(R_0)$. Now, for perturbative theories, $F''(R_0) \approx 1$ is independent of $R$ and then the last term of the right-hand side dominates for small curvatures. However, for non-perturbative theories like $F(R) = R - \gamma^2 / R$ we will have problems; indeed, in this case we get

$$m^2 \approx - \frac{D + 2}{2(D - 1)} R_0,$$

(22)

which shows that such theories are generically unstable under curvature perturbations. We note that this result is below the negative mass bound since $R_0 > 0$. One may wonder, like in [8], whether higher order terms, perturbative in $R$, may help the situation. From the discussion above, one sees that if a term like $-\gamma^2 / R$ is present, then $F''(R) = -2\gamma^2 / R^3 + \mathcal{O}(R)$. For the (small) constant curvature solution, $R \propto \gamma$, and then $F''(R)$ is dominated by the non-perturbative term, while the first derivative of $F(R)$ is still of order $\gamma^0$, hence the expression (21) should not change considerably as higher order terms in $R$ are added to the Lagrangian. One should also keep in mind that the same arguments given above against the naturalness of a non-perturbative term to the Lagrangian also point to the fact that the numerical coefficients in from of higher order terms are too small to have any dramatic effect in the discussion above, as can be seen in [19].

III. THE DILATONIC SECTOR

Let us now study the conformally flat space [9], where the dynamics is restricted to the conformal sector and the metric tensor $g_{\mu\nu}(x)$ is given by $g_{\mu\nu}(x) = e^{\sigma(x)}g_{\mu\nu}$, with $\sigma(x)$ standing for the conformal factor, the dilaton field. In this case, the scalar curvature has the form

$$R = g^{\alpha\beta} R_{\alpha\beta} = e^{-\sigma} \left( \frac{D - 1}{2} \right) \left[ 2 \Box \sigma + \left( \frac{D - 2}{2} \right) \partial^\alpha \sigma \partial_\alpha \sigma \right].$$

(23)
The standard Einstein-Hilbert action which includes the cosmological constant is given by 
\[ S = - \int d^D x \sqrt{|g|} (R + 2\Lambda) \]
where we are using \( 8\pi G = 1 \). The action modified by the change \( R \rightarrow F(R) \) has now the form \([11]\). More explicitly, we can write
\[ S = - \int d^D x e^{\frac{2}{D} F} \left( e^{-\sigma} \left( \frac{D-1}{2} \right) \left[ 2\square \sigma + \left( \frac{D-2}{2} \right) \partial^\alpha \sigma \partial_\alpha \sigma \right] \right) \]  
(24)

This form shows clearly that for \( D = 2 \), and \( F(R) = R + 2\Lambda \), the action reduces to total derivative, whereas for \( D = 1 \) the action is reduced to the cosmological term alone.

The equation of motion looks like \( De^{\frac{2}{D} F} F(R) + 2e^{\frac{2}{D} F} F(R) \frac{\partial R}{\partial \sigma} = 0 \), where prime represents derivative, that is, \( F'(R) \equiv dF/dR \). In more manifest form, after substitution of the expression \([23]\) for the scalar curvature \( R \), we have
\[ De^{\frac{2}{D} F} F(R) - (D-1)F'(R)e^{\left( \frac{D-1}{D} \right) \sigma} \left[ 2\square \sigma + \left( \frac{D-2}{2} \right) \partial^\alpha \sigma \partial_\alpha \sigma \right] + 2(D-1)\square(e^{\left( \frac{D-1}{D} \right) \sigma} F'(R)) - (D-1)(D-2)\partial_\alpha(e^{\left( \frac{D-1}{D} \right) \sigma} F'(R) \partial^\alpha \sigma) = 0. \]  
(25)

This is the equation of motion we have to deal with to find nonperturbative solutions in dilaton gravity, modified to include the change \( R \rightarrow F(R) \). Due to intrinsic difficulties, in the following we first investigate the simpler case \( D = 2 \). The strategy used in the following will then be extended to the case \( D = 4 \), which is treated afterwards.

The case \( D = 2 \) is motivated by the well-known fact that gravity in \((1,1)\) dimensions has the direct advantage of eliminating intricate structures which are present in higher dimensions, easing the investigation due to technical simplifications specific of the \((1,1)\) dimensions. From the equation of motion for \( D \) arbitrary, we see that in the \( D = 2 \) case the equation simplifies to give
\[ e^\sigma F'(R) - F'(R) \Box \sigma + \Box F'(R) = 0. \]  
(26)

We follow the first-order formalism of Ref. \([11]\), so we concentrate on the presence of static solutions, searching for a first-order equation connecting the derivative of the dilaton with some specific function of it. The static dilaton only depends on \( x \equiv x^1 \). Using the signature \((+,-)\) for the two-dimensional case, we can write the scalar curvature as
\[ R = -e^{-\sigma} \sigma'' , \]  
(27)

where we are using the notation \( \sigma' = d\sigma/dx \), \( \sigma'' = d^2 \sigma/dx^2 \), etc. In this case, the equation of motion \([20]\) takes the form
\[ e^\sigma F'(R) + F'(R) \sigma'' - d^2 F'(R)/dx^2 = 0. \]  

Direct calculation leads to \( d^2 F'(R)/dx^2 = F''(R) \cdot (\sigma'' - \sigma''')^2 e^{-2\sigma} + F''(R) \cdot e^{-\sigma} (\sigma'' + 2\sigma' \sigma'' - \sigma'^2 \sigma'' - \sigma'''') \), and so the equation of motion gets to the new form
\[ e^\sigma F'(R) + F'(R) \sigma'' - F''(R) \cdot (\sigma'' - \sigma''')^2 e^{-2\sigma} - F''(R) \cdot e^{-\sigma} (\sigma'' + 2\sigma' \sigma'' - \sigma'^2 \sigma'' - \sigma''') = 0. \]  
(28)

We now proceed with solving this equation. Our first strategy is to choose \( F(R) \) in the form \( F(R) = R + \gamma R'' + 2\Lambda \), with \( \gamma \) being a constant. Our second strategy for solving the Eq. \([25]\) is to choose the ansatz
\[ \sigma' = -2\Lambda e^{\sigma/2} , \]  
(29)

This is the first-order equation we need: it is inspired in \([11]\) and the specific form follows from the particular profile of the equation of motion \([25]\). The first-order equation allows writing \( \sigma'' = 2a A^2 e^{(a-1)\sigma} \), \( \sigma''' = -4a A^3 e^{3a\sigma/2} \), \( \sigma'''' = 12a^3 A^4 e^{2a\sigma} \). Therefore, we find the following expressions: \( R = -2a A^2 e^{(a-1)\sigma} \), \( \sigma'' - \sigma'''' = 4A^3 a(a - 1) e^{3a\sigma/2} \), and \( \sigma'' + 2\sigma' \sigma'' - \sigma'^2 \sigma'' - \sigma''''' = -4a A^4 (a - 1)(3a - 2) e^{2a\sigma} \). Substituting these expressions into Eq. \([25]\) leads to the simpler equation
\[ e^{n(a-1)\sigma} (-2a A^2)^n (n - 1) \left( 1 - \frac{2}{a} n(n - 2)(a - 1)^2 - \frac{1}{a} n(a - 1)(3a - 2) \right) - \frac{2\Lambda}{\gamma} = 0. \]  
(30)

which can be examined in the two distinct cases \( \Lambda \neq 0 \) and \( \Lambda = 0 \). The cases \( n = 0 \) and \( n = 1 \) are trivial and will not be considered.
If $\Lambda \neq 0$, we have to make $a = 1$. Thus, for $a = 1$ the equation (30) reduces to the following equation for the value of $A$: $(-2A^2)^n(n-1) - 2A/\gamma = 0$. For $n$ odd we get $A = \pm \sqrt[2]{\gamma/(2\gamma (n-1))}$, and for $n$ even, $A = \pm \sqrt[n]{\gamma/2\gamma (n-1)}$. These values of $A$ should be substituted to the ansatz equation (29) which yields the solution

$$\sigma(x) = -2 \ln(Ax + C),$$

(31)

where $C$ is an integration constant. This is similar to the Liouville-like kink solution found in Ref. [22]. We note that, for $a = 1$ the curvature given above is constant and negative, equal to $R = -A^2$; thus, we have just found an anti de Sitter (adS) solution.

Note also that the condition of reality of the dilaton field $\sigma(x)$ restricts possible signs of $\Lambda$ and $\gamma$, and the situation can differ for different values of $n$. For example, in the case $n = -1$ we get

$$\sigma(x) = -2 \ln \left( \pm \frac{1}{2} \sqrt[2]{\gamma} x + C \right).$$

(32)

As a result, we find that both the constants $\Lambda$ and $\gamma$ should be either positive or negative. In the other interesting case, $n = 2$, one finds that $A = \pm \sqrt[2]{\Lambda/\gamma}$.

If $\Lambda = 0$, we do not need to choose $a = 1$ anymore; instead, the equation (30) is now reduced to the following equation relating $a$ and $n$: $1 - (2n/a)(n-2)(a-1)^2 - (3/a)(a-1)(a-2/3) = 0$. The solutions of this equation are $a_1 = 2n/(2n-1)$, and $a_2 = (n-1)/n$. The ansatz equation in this case yields

$$\sigma(x) = -\frac{2}{a} \ln(Ax + C).$$

(33)

Again, we get to similar Liouville-like kink solution. However, in this case the value of $A$ cannot be fixed. We notice that the $n = 1/2$ case is very specific, yielding the single root $a = -1$. Also, we note that this solution does not depend on $\gamma$.

Returning to the cases $n = 2$ and $n = -1$, we find that in the $\Lambda = 0$ situation both these case are possible yielding for $n = -1$ $a = 2/3$ and $a = 2$ (for instance, for $a = 2$ the solution looks like $\sigma(x) = -\ln(2Ax + C)$) whereas for $n = 2$ one finds $a_1 = 4/3$ and $a_2 = 1/2$. We note that in this case, the constant curvature is not a solution anymore. The curvature is $R = -2A^2 e^{(a-1)\sigma} = -2A^2(Ax + C)^{\frac{1}{a}-2}$, so one can see that for $a > 1$ the curvature decays to zero as $|x|$ increases to large values. We find that for $n = 2$ the localized solution gives $a = 4/3$, whereas for $n = -1$ one obtains $a = 2$.

Perhaps the most interesting result is that the value $\Lambda = 0$ somehow induces a phase transition: although the solutions do not change appreciably, the scalar curvature cannot be constant in the limit of a vanishing cosmological constant.

Before ending this section, let us examine the possibility of finding adS solutions for $F(R)$ general. Here we keep considering the first-order ansatz equation (29) for the dilaton, thus the curvature (27) yields

$$R = -2aA^2 e^{(a-1)\sigma}$$

(34)

We get to constant curvature with $a = 1$, leading to adS geometry with $R = -2A^2$. Indeed, we have found that the constant curvature $R$ is related to the dilaton field by $\sigma'' = -Re^{\sigma}$. Thus, with $a = 1$ the equation of motion (28) is reduced to the much simpler equation

$$F(R) - RF'(R) = 0.$$  

(35)

Let us consider two more possibilities for the function $F(R)$: (i) $F(R) = R + aR^m + bR^m + 2A$, and (ii) $F(R) = e^{-\alpha R} + 2A$. In the first case, we find that the curvature is related to the constant parameters $a$ and $b$ via equation $(1 - n)aR^n + (1 - m)bR^m + 2A = 0$ which shows that for $\Lambda \neq 0$ the flat space solution $R = 0$ is impossible. In the second case, the equation is $e^{-\alpha R}(1 + \alpha R) + 2A = 0$, which admits the flat solution $R = 0$, for $2\Lambda = -1$. 

We now move to four dimensional space-time. We use the same methodology, but here the equation of motion \( \Box \) takes the form
\[
2e^{2\sigma}F(R) - F'(R)e^\sigma \left( \frac{3}{2} \Box \sigma + \partial^\alpha \sigma \partial_\alpha \sigma \right) + 3e^\sigma F'(R) - 3\partial_\alpha (e^\sigma F'(R) \partial^\alpha \sigma) = 0. \tag{36}
\]
The scalar curvature in this case is equal to
\[
R = \frac{3}{2}e^{-\sigma} \left[ 2\Box \sigma + \partial^\alpha \sigma \partial_\alpha \sigma \right]. \tag{37}
\]
Let us choose the function \( F(R) \) again in the form \( F(R) = R + \gamma R^n + 2\Lambda \). Thus, the equation of motion takes the form
\[
Re^{2\sigma} + \gamma (2 - n) R^n e^{2\sigma} + 3n\gamma (e^\sigma \Box R^{n-1} + \partial_\alpha R^{n-1} \partial^\alpha e^\sigma) + 4\Lambda e^{2\sigma} = 0. \tag{38}
\]
We substitute the value of scalar curvature (37) to get
\[
\frac{3}{2} \Box \sigma + \partial^\alpha \sigma \partial_\alpha \sigma \equiv -2\sigma'' - \frac{4}{r} \sigma' - (\sigma')^2 \equiv T. \tag{40}
\]
And now the equation takes the form
\[
\frac{3}{2} e^{\sigma} T + \left( \frac{3}{2} \right)^n \gamma (2 - n) e^{(2-n)\sigma} T^n + 2 \left( \frac{3}{2} \right)^n n\gamma \left[ e^\sigma \left( - \frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} \right) (e^{- (n-1)\sigma} T^{n-1}) - \frac{d}{dr} \left( e^{- (n-1)\sigma} T^{n-1} \right) \frac{d}{dr} e^\sigma \right] + 4\Lambda e^{2\sigma} = 0. \tag{41}
\]
To solve this equation, we suppose the geometry to be \( \text{adS} \), taking \( R = -(3/2)C_0 = \text{const} \). This leads us to
\[
-T = 2\sigma'' + \frac{4}{r} \sigma' + (\sigma')^2 = C_0 e^\sigma. \tag{42}
\]
We substitute the above equation into the equation of motion (41) to get to the simpler equation relating \( C_0 \) and \( \Lambda \):
\[-3C_0 + 2\gamma (2 - n) (-3C_0/2)^n + 8\Lambda = 0. \]
The dilaton field being the solution of Eq. (42) looks like \( \sigma(r) = -2 \ln r \), which makes \( C_0 \) to be zero. In this case, both \( R \) and \( \Lambda \) vanish. In other words, an exact spherically symmetric constant curvature solution is possible for \( \Lambda = 0 \) and corresponds to the flat space. We can go further and try another solution of (42), for \( C_0 \) very small. We change \( C_0 \to \epsilon \), and consider \( \sigma(r) = -2 \ln(r) + \epsilon g(r) \). We get \( \sigma(r) = -2[1 + (\epsilon/4)] \ln(r) \), which makes the scalar curvature very small, but now the cosmological constant does not vanish, being \( \Lambda = (3/8)\epsilon \).

We notice that for \( \epsilon \) positive, the solution leads to \( \text{AdS} \) geometry with \( \Lambda \) positive, and for \( \epsilon \) negative, we get to \( \text{dS} \) geometry with \( \Lambda \) negative.

Another possibility in \( D = 4 \) is to use the domain wall ansatz in which the dilaton depends only on a single spatial coordinate, for instance \( \sigma = \sigma(x) \). For a review on domain walls in supergravity, see Ref. [24]. In this case we replace \( \Box \) by \(-d^2/dx^2, 2\Box \sigma + \partial^\alpha \sigma \partial_\alpha \sigma \) by \(-2\sigma'' - (\sigma')^2 \) etc. The equation of motion becomes
\[
\frac{3}{2} e^{\sigma} (2\sigma'' - (\sigma')^2) + \left( \frac{3}{2} \right)^n \gamma (2 - n) e^{(2-n)\sigma} (2\sigma'' - (\sigma')^2)^n + 2 \left( \frac{3}{2} \right)^n n\gamma \left[ e^\sigma \frac{d^2}{dx^2} (e^{-(n-1)\sigma} (2\sigma'' - (\sigma')^2)^{n-1}) - \frac{d}{dx} (e^{-(n-1)\sigma} (2\sigma'' - (\sigma')^2)^{n-1}) \frac{d}{dx} e^\sigma \right] + 4\Lambda e^{2\sigma} = 0. \tag{43}
\]
where prime now stands for derivative with respect to $x$. Here we used the fact that $\frac{d}{dx}e^{\sigma} = e^{\sigma}\sigma'$. To solve this equation, as before we consider the ansatz $\sigma' = -2Be^{\sigma/2}$. It allows writing
\[2\sigma'' + (\sigma')^2 = 8B^2e^{\sigma}.\]
This leads to $(-12B^2)^n\gamma(2-n) - 12B^2 + 4\Lambda = 0$, and for $\Lambda \neq 0$ we can obtain the constant $B$. We solve the equation (44) to get
\[\sigma(x) = -2\ln(Bx + C),\]
with $C$ being an integration constant. The scalar curvature in this case is $R = -12B^2$.

If we impose the condition that the scalar curvature is constant, we arrive at the following reduction of the Eq. (43): $2F(R) - RF'(R) = 0$. Thus, if we follow the steps done in two space-time dimensions, we can consider other possibilities for the function $F(R)$: (i) $F(R) = R + aR^n + bR^m + 2\Lambda$, and (ii) $F(R) = e^{-\alpha R} + 2\Lambda$. In the first case, the curvature turns out to be related to the constant parameters $a$ and $b$ via equation $R+(2-n)aR^n+(2-m)bR^m+4\Lambda = 0$ which shows that for $\Lambda \neq 0$ the flat space solution $R = 0$ is impossible. In the second case, the equation is $e^{-\alpha R}(2 + aR) + 4\Lambda = 0$, which leads to vanishing curvature for $2\Lambda = -1$. Note that, by the criterion (17), such solutions are always stable for $n, m > 0$.

IV. ENDING COMMENTS

In this work we have studied general features of $F(R)$ gravity models and derived a formula for the asymptotic mass of such theories, showing that for perturbative theories in which $F(R)$ allow a perturbative expansion the value of the mass do not change significantly from that of usual Einstein-Hilbert solutions. We have analyzed the behavior of these theories under perturbations of the solutions and concluded that, while metric perturbations are essentially the same as in Einstein-Hilbert models, the scalar curvature perturbations are unstable for positive curvature solutions of models which modify the usual Lagrangian by $R^n$ terms with $n < 0$. It would seem that, on top of providing an effective action mechanism by which such term would arise, one is riddled by the stability of the solution found. Static solutions of the equations of motion are found for the $F(R)$ gravity models restricted to the conformal sector. We see that these solutions have the Liouville-like kink form in two space-time dimensions. Also, very similar effects are achieved in four-dimensional space-time, both for spherically symmetric or domain-wall-like solutions. We also have found that the limit $\Lambda \to 0$ takes these solutions with $R \neq 0$, then still maintaining the desired behaviour of $F(R)$ theories to present curved backgrounds without the need for a cosmological constant. The presence of first-order equation which solves the equation of motion with adS geometry indicates that the system may admit supersymmetric extensions, at least for $n$ positive. If this is indeed the case, the supersymmetric realization will be new, given the non-standard form of the graviton Lagrangian. This is an issue in which we will report elsewhere.

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