Abstract
In mechanism design, it is challenging to design the optimal auction with correlated values in general settings. Although value distribution can be further exploited to improve revenue, the complex correlation structure makes it hard to acquire in practice. Data-driven auction mechanisms, powered by machine learning, enable to design auctions directly from historical auction data, without relying on specific value distributions. In this work, we design a learning-based auction, which can encode the correlation of values into the rank score of each bidder, and further adjust the ranking rule to approach the optimal revenue. We strictly guarantee the property of strategy-proofness by encoding game theoretical conditions into the neural network structure. Furthermore, all operations in the designed auctions are differentiable to enable an end-to-end training paradigm. Experimental results demonstrate that the proposed auction mechanism can represent almost any strategy-proof auction mechanism, and outperforms the auction mechanisms wildly used in the correlated value settings.

Introduction
Optimal auction mechanism design is an important and active area in economics and computer science. Myerson first proposed an analytical optimal auction mechanism for single-item auctions with independent values (Myerson 1981), which is a milestone in the field of revenue-maximizing auctions. However, correlated values are more common in the industry, because the information every bidder used to evaluate items like the click-through rate of ad slot in online advertising, is estimated based on the common observation of user expressions. Exploiting the underlying information of correlated bidding can further improve the revenue, as the conditional distribution based on other bids provides information about the considered bid. Myerson discussed on a toy case how to draw more revenue in the auctions with correlated values, leveraging the side-bet phenomenon (Myerson 1981). One of the most known mechanisms for correlated values is the Cremer-McLean mechanism (Crémer and McLean 1985, 1988), which can extract the full social welfare in expectation. However, the mechanism only ensures interim-IR, which is unrealistic in practice because bidders may run out of budgets and have to withdraw from auctions with negative utilities. Ensuring ex-post IR is more desirable but also brings difficulties. Papadimitriou and Pierrakos showed that finding the optimal deterministic mechanism for more than two bidders with ex-post IC and ex-post IR is NP-hard (Papadimitriou and Pierrakos 2011), and designing the near-optimal auction mechanism is still an open problem in computer science.

Recently, with the rise of deep learning, auction mechanism design through deep learning has been applied in notorious settings where theoretical optimal mechanisms are unknown, and have produced impressive results (Dütting et al. 2019; Feng, Narasimhan, and Parkes 2018; Golowich, Narasimhan, and Parkes 2018). Although analytical mechanism design for correlated values has hit some limits, deep learning can implicitly model and exploit the correlation automatically from the historical bidding data. There are two potential paradigms to integrate deep learning into auction mechanism design. One way is to model the whole auction as a black-box neural network and satisfies strategy-proofness by introducing extra constraints in optimization. This approach is consistent with the design principle of deep learning, but also has defects in the perspective of mechanism design. On the one hand, it can not yield strictly strategy-proof mechanisms, which may leave potential advantages for participants to exploit. On the other hand, optimizing revenue and guaranteeing game theoretical properties at the same time reduced the training efficiency, because the optimization directions of them may conflict. A more desirable paradigm is to leverage the optimization ability of deep learning to maximize revenue, while ensuring strict strategy-proofness orthogonal to the optimization process by encoding game theoretical constraints into neural network architecture. A feasible approach following this paradigm is to build a rank score-based auction mechanism like GSP or Myerson auction, and ensure strategy-proofness by satisfying Myerson Lemma (Myerson 1981).

However, considering the correlated values brings challenges in building a strategy-proof learning auction mechanism. The first challenge is to ensure strong enough expressiveness to learn the optimal mechanism. Because of lacking the sufficient and necessary conditions to characterize the optimal auction mechanisms under the general value settings, we turn to construct a candidate space that
represents a sufficiently large number of single-item auction mechanisms, under which we can efficiently search for the (near-)optimal auction mechanism. Agnostic approaches like RegretNet (Dütting et al. 2019) can represent any auction mechanisms but do not rule out the not strategy-proof mechanisms. MyersonNet (Dütting et al. 2019), an imitation of Myerson Auction, cannot represent all the strategy-proof single-item auction mechanisms, and may miss the optimal mechanism when bidders’ values are not independent.

It is still an open problem of characterizing the conditions of neural networks to exactly encode all the strategy-proof auction mechanisms. The second challenge comes from the contradiction between auction mechanism design and machine learning principle. Auction mechanisms usually involve various discrete operations when calculating allocation and payments. For example, the top-k operation in the allocation may lead gradients too sparse to train the neural network. And the strategy-proof auction mechanism may need to calculate the critical bid as the payment following Myerson Lemma by binary search, when the allocation rule is complicated and cannot be inverted directly due to the correlated valuations. Machine learning follows an end-to-end continuous optimization pipeline, which requires the whole calculation process to be continuous and differentiable. It needs further efforts to make the allocation and pricing differentiable while still maintaining the characteristics of strategy-proofness.

In this work, we aim to design an end-to-end neural network for auction mechanism design considering correlated values: learning the optimal auction mechanism under any joint valuation distributions, and ensuring strict strategy-proofness. We build a rank score function for each bidder to decide the allocation, and calculate the critical bid as the payment in auctions. In particular, we restrict the rank score function with single crossing condition to ensure the monotonic of allocation rules. We learn the rank score function by MIN-MAX network (Daniels and Végh 2010), a piece-wise linear function, which enables to calculate closed-form payments and a differentiable revenue. Our contributions can be summarized as follows:

• We make an in-depth discussion on the formulation of learning-based auction design with correlated values. We introduce a new rank score function, which allows the network to exploit the correlation for revenue optimization without violating the property of strategy-proofness.

• We propose Conditional Auction Network (CAN), a learning approach that enables us to design the optimal strategy-proof auction mechanisms for bidders with any joint value distributions. CAN encodes the game theoretical constraints into network architecture, making the training more efficient and the learned mechanism more understandable. The differentiable revenue expression makes the auction mechanism and the training process compatible.

• We provide a comparison of the expressiveness of CAN and other representative mechanisms for single-item auctions. The allocation figures intuitively demonstrate the advantage of CAN in fully exploiting the correlation to optimize the revenue. We evaluate CAN and some representative baselines, and the results show that CAN outperforms the widely used strategy-proof auctions mechanisms in terms of the expressiveness and the revenue.

Preliminaries

Single-Item Auction Model

We consider a single-item auction model in general settings. There are n bidders \( N = \{1, \ldots, n\} \) compete for one item. Each bidder \( i \) has some information to evaluate the item, which is denoted as signal \( s_i \). Signals are commonly correlated, because the evaluation process are based on some underlying unknown variables. That is, signals are drawn from a joint distribution \( F \) over the domain \([0, \omega_i]^n \) (\( \omega_i \) could be \( +\infty \)). We assume the the information to evaluate the item entirely comes from one’s own signal, which means the estimated value \( v_i = v_i(s_i) = s_i \) for every bidder \( i \). Bidders report their signals strategically, and seek to maximize their utilities. We can view the reported signals as the bid \( b \). The auctioneer collects bids from bidders, and determines allocation and payments. In this process, the auctioneer runs an auction mechanism \( M = (\mathcal{X}, \mathcal{P}) \), consisting of a collection of allocation rules \( \mathcal{X}(\mathcal{b}) : [0, \omega_i]^n \to \{0, 1\} \) and payment rules \( \mathcal{P}(\mathcal{b}) : [0, \omega_i]^n \to \mathbb{R}_{\geq 0} \). We use \( \mathcal{X}(\mathcal{b}) = 1 \) to denote the bidder \( i \) wins the ad slot and \( \mathcal{X}(\mathcal{b}) = 0 \) to denote the bidder \( i \) loses the auction. \( \mathcal{P}(\mathcal{b}) \) represents payment the bidder \( i \) needs to pay for the auction. The utility of bidder \( i \) is \( u_i(b, v) = \mathcal{X}(\mathcal{b})v_i - \mathcal{P}(\mathcal{b}) \). The auctioneer’s revenue in the auction is the sum of all bidders’ payments, denoted as \( u_0(b, M) = \sum_i \mathcal{P}(\mathcal{b}) \).

A desired auction mechanism need to ensure some game theoretical properties. Bidders may misreport their signals if they could increase the utilities. If the utilities of bidders are negative for any bid, bidders may refuse to participate in the auction. Thus, the auction mechanism needs to satisfy the following properties: bidders should maximize their utilities and obtain non-negative utilities when reporting signals truthfully, which we call incentive compatibility (IC) and individual rationality (IR), respectively.

Definition 1 (Ex-post IC and Ex-post IR Mechanism), (Roughgarden and Talmag-Cohen 2013) An auction mechanism is ex-post IC and ex-post IR if for every bidder \( i \), true value \( v_i \), reported bid \( t_i \) and other bidders’ bid \( b_{-i} \),

\[
\mathcal{X}(t_i, b_{-i})v_i - \mathcal{P}(t_i, b_{-i}) \geq \mathcal{X}(t_i, b_{-i})v_i - \mathcal{P}(t_i, b_{-i}), \\
\mathcal{X}(t_i, b_{-i})v_i - \mathcal{P}(t_i, b_{-i}) \geq 0.
\]

Similar to Myerson Lemma proposed in (Myerson 1981), under the setting of correlated values, the following lemma can still guarantee ex-post IC and ex-post IR.

Lemma 1. (Roughgarden and Talmag-Cohen 2013) For every correlated values setting, a single-item auction mechanism \( M(\mathcal{X}, \mathcal{P}) \) is ex-post IC and ex-post IR, if and only if
for any fixed $i$ and $b_{-i}$:

\begin{align}
\forall i \geq b_i, \quad & \mathcal{X}_i(t_i, b_{-i}) \geq \mathcal{X}_i(b_i, b_{-i}); \\
& u_i(b_i, b_{-i}, b) = u_i(0, b_{-i}, b) + \int_0^{b_i} \mathcal{X}_i(t_i, b_{-i}) \, dt_i; \\
& u_i(0, b_{-i}, b) \geq 0;
\end{align}

(1a) \hspace{1cm} (1b) \hspace{1cm} (1c)

Equation (1a) and equation (1b) together ensure the ex-post IC property, and the addition of (1c) can guarantee the ex-post IR property. Equation (1a) states that the allocation rule $\mathcal{X}_i(b)$ is monotonously non-decreasing in the bid $b_i$, for every $i$ and the bid profile $b_{-i}$. Equation (1b) states the relation between allocation rule and payment rule. We can rewrite (1b) as:

$$P_i(b) = \mathcal{X}_i(b)v_i - \int_0^{v_i} \mathcal{X}_i(t_i, b_{-i}) \, dt_i$$

(2)

In other words, if we obtain the allocation rule, We can directly derive the payment to satisfy the ex-post IC property.

**Problem Formulation**

We formulate the problem of revenue optimization for the single-item auction with correlated values, under the guarantee of above game theoretical properties. Given the bid vector $b$ and auctioneer’s revenue $u_0(b, \mathcal{M})$, we aim to design an auction mechanism $\mathcal{M} = (\mathcal{X}, \mathcal{P})$ to maximize the revenue in expectation:

$$\max_{\mathcal{M}} \mathbb{E}_{b \sim F}[u_0(s, \mathcal{M})]$$

(3)

s.t. Ex-post IC constraint, Ex-post IR constraint,

We need to restrict the feasible solution space to meet the constraints through the characterization results from Lemma 1. To be specific, we determine the monotonic allocation by ranking bidders according to their rank scores calculated by some algorithm, while the payments are determined by equation (2). Here, rather than learning allocation rules directly through a black-box model, we introduce an intermediate variable, rank score, for a clearer expression for Lemma 1. We use $r_i(b_i, b_{-i})$ to denote bidder $i$’s rank score, which can be represented by some explicit closed-form formula, or be represented by a neural network. Now we will show how to reduce problem (3), the design of the auction mechanism $\mathcal{M}$, to the search of the rank score functions $r_i(b_i, b_{-i})$, which is a learnable problem.

First, the auction mechanism should guarantee (1a), the monotonicity of allocation. We define the allocation rules based on the rank scores as follows:

- Allocation Rules $\mathcal{X}$: Bidders are sorted in a non-increasing order of rank score values $r_i(b_i, b_{-i})$, $r_1(b_1, b_{-1}) \geq r_2(b_2, b_{-2}) \geq \cdots \geq 0 \geq \cdots \geq r_n(b_n, b_{-n})$, then the bidder with the highest non-negative rank score would win the auction, with ties broken randomly. If all rank scores are negative, the item will be reserved.

- It is worth noting that each rank score function $r_i(b)$ is partially monotonic with respect to bidder $i$’s own bid $b_i$ but is not a sufficient condition for monotonic allocation. If the increase of one bidder’s bid causes another bidder’s rank score rises faster than her own rank score, obviously the allocation rules are not monotonic in this case. Here is a sufficient condition for the monotonicity of allocation rules used in our work.

**Definition 2 (Single Crossing Conditions).** The rank score functions $\{r_i(b)\}_{i \in N}$ satisfies the single crossing condition if for every $i, j \neq i$, and every $s$,

$$\frac{\partial r_i(b)}{\partial b_i} \geq 0 \quad \text{and} \quad \frac{\partial r_j(b)}{\partial b_i} \leq 0.$$  

(4)

Then means one bidder’s rank score should be partially non-decreasing with respect to her own bid and partially non-increasing with respect to others’ bids. Representing the allocations based on single crossing conditions will not affect the expressiveness of our network, which will be discussed later in the paper. Once rank score functions $r_i(b_i, b_{-i})$ are determined, according to Lemma 1, the payment rules could be formulated as follows:

- Payment Rules $\mathcal{P}$: The payment for each bidder is calculated by (2). To ensure that the auctioneer’s revenue from every auction is non-negative, we let $u_i(0, b_{-i}, b) = 0$ for every $i, b_{-i}$. Therefore, the payment is uniquely determined by the allocation rule:

$$P_i(b) = \mathcal{X}_i(b)v_i - \int_0^{v_i} \mathcal{X}_i(t_i, b_{-i}) \, dt_i.$$ 

(5)

More specifically, the payment is:

$$P_i(b) = \begin{cases} 0 & \mathcal{X}_i(b) = 0, \\
\inf \{ t \mid \mathcal{X}_i(t_i, b_{-i}) = 1 \} & \mathcal{X}_i(b) = 1,
\end{cases}$$ 

(6)

where $z_i(b) = \inf \{ t \mid \mathcal{X}_i(t_i, b_{-i}) = 1 \}$ is the critical bid.

From the above discussion, we reduce the design of the auction mechanism $\mathcal{M}$ to the design of allocation rules, or the search of the rank score functions $r_i(b_i, b_{-i})$. We also analyze the constraints that the rank score functions must satisfy. Any additional constraints may impair the expressiveness of the neural network in correlated value settings. We show the design of a neural network to learn such rank score functions and verify its expressiveness for optimal auction mechanism next.

**Conditional Auction Net**

In this section, we present a learning-based method, Conditional Auction Net (CAN), to design the single-item auction mechanisms that can exploit the correlation of values to maximize the revenue, while the strict ex-post IC and ex-post IR properties are always maintained, even in the training process.
Overall Architecture

As illustrated in Fig. 14, Conditional Auction Net consists of two modules: conditional rank score functions, in which the rank scores are conditional on bids from the others, and a differentiable revenue calculator. For rank score we build a MIN-MAX network [Daniels and Velikova 2010], which has the capability to approximate any function, for each bidder i to learn her rank score $r_i(b)$. MIN-MAX network provides convenience for satisfying single crossing conditions (4) and payment calculation (5). In particular, different from the classical results that reserve the item if all rank scores are negative, the case the item is reserved is viewed as that an additional virtual bidder participates in the auction with a bid 0 and wins. We use $r_0(b)$ to denote the additional rank score, which should also be learned. For revenue calculation, we convert the calculated rank scores into the output of the auction mechanism following the procedure in the previous section in a differentiable way, to fit in the framework of machine learning. On the one hand, we relax the output of allocation rules $X'$ from a spare (0,1) vector to a probability vector using softmax operator. On the other hand, we propose a closed-form formula to calculate critical bids based on piece-wise linear rank score functions that satisfy single crossing conditions. With all these components, CAN can yield relaxed negative revenue as the training loss and can be optimized in an end-to-end way, while strict ex-post IC and ex-post IR properties are always maintained in both model training and deployment.

Conditional Rank Score Function

We design a parameterized conditional rank score function $r_i(b)$ to transform each bidder’s bid to a rank score, which is implemented by the MIN-MAX neural network, a piece-wise linear function. As illustrated in Fig. 19, the inputs of bidder i’s rank score function are all bidders’ bids $b$. Given $|Z|$ groups of $|Q|$ linear functions, the MIN-MAX operation first akes the maximum value among $|Q|$ values in each group, and takes the minimum value among these $|Z|$ maximum values. As long as we ensure each linear function is partially monotonic, that is, associating positive weights $e^{a_{zq}^i}$ with $s_i$ and associate non-positive weights $ReLu(\gamma_{zq}^i)$ with $b_{-i}$ for bidder $i$, where $z = 1, \ldots, |Z|$ and $q = 1, \ldots, |Q|$, the final piece-wise linear function will also preserve the property of monotonicity. Here we assume the weights of $b_i$ are strictly positive, because zero weight makes no sense. Besides, the unconstrained intercepts are denoted as $\beta_{zq}^i$ for bidder $i$. We formalize the rank score function as:

$$r_i(b_i, b_{-i}) = \min_{z \in Z} \max_{q \in Q} \left( e^{a_{zq}^i} b_i - ReLu(\gamma_{zq}^i) \cdot b_{-i} + \beta_{zq}^i \right).$$

The parameters $a_{zq}^i, \beta_{zq}^i, \gamma_{zq}^i$ are learnable, and monotonic allocation can be maintained independent on the values of the parameters. In particular, the additional rank score to reserve the item is non-increasing with respect to all bidders’ bids.
bids the \( b \):

\[
r_0(b) = \min_{z \in Z} \max_{q \in Q} \left( -\text{ReLU}(\gamma_{eq}) \cdot b + \beta_{eq} \right).
\]

**Differentiable Revenue Expression**

Next, we will calculate the payments and the revenue in a differentiable way based on the above designed rank score functions. As mentioned in [9], the minimum bid to maintain winning is the payment for the winner. Assuming bidder \( i \) is the winner, when decreasing her bid \( b_i \) to \( b_i' \), her rank score \( r_i \) decreases and the others’ \( r_{-i} \) increases. At some \( b_i' \), bidder \( i \) tied with some other bidder \( j \) for the first time with respect to rank scores. The threshold \( b_i' \) is the critical bid and is the abscissa of the intersection of the two rank scores \( r_i(b_i') \) and \( r_j(b_j') \). Therefore, the problem of calculating the critical bid can be abstracted as solving the intersection of piece-wise linear functions. Given two piece-wise linear functions:

\[
y_i = \min_{z \in Z} \max_{q \in Q} \left( \alpha_{i}^1 b_i + \beta_{i}^1 \right),
\]

and

\[
y_j = \min_{z \in Z} \max_{q \in Q} \left( \alpha_{j}^2 b_j + \beta_{j}^2 \right),
\]

we can assume \( y_i \) and \( y_j \) is increasing and non-increasing with respect to \( b_i \), respectively. Then the abscissa of the intersection \( p_{ij}^c \) is

\[
p_{ij}^c = \min_{z \in Z} \min_{q \in Q} \left( \frac{\beta_{i}^1 - \beta_{j}^2}{\alpha_{i}^1 - \alpha_{j}^2} \right),
\]

for which we provide proof in the appendix. In practice, \( y_i \) and \( y_j \) represent the winner’s own rank score \( r_i(b_i) \) and other bidders’ rank scores \( r_{-i}(b_i) \) (including the additional rank score function \( r_0(b_i) \)) with respect to \( s_i \), respectively. We need to calculate \( |b_{-i}| + 1 \) times the abscissa of the intersection and take the largest one. It is exactly the payment \( P_i(b_i, b_{-i}) \) for the winner \( i \):

\[
P_i(b_i, b_{-i}) = \mathcal{X}_i(b_i, b_{-i}) \max \left\{ \left( \max_{j \in N, j \neq i} p_{ij}^c \right), p_{i0}^c \right\},
\]

where \( \mathcal{X}_i(b_i, b_{-i}) \) is the allocation result, a \((0,1)\) vector. The revenue is the sum of payments.

However, the actual revenue is not suitable to be the training loss due to sparse gradients. We relaxed the allocation \( \mathcal{X}_i(b_i, b_{-i}) \) by a softmax operator, and get a probability vector \( \tilde{\mathcal{X}}_i(b_i, b_{-i}) \). As the critical bid of losing bidders is higher than their bids, to avoid IR violation misleading the training direction, we truncate \([9]\) by the bidder’s own bid. Thus, the relaxed payment for each bidder is

\[
\tilde{P}_i(b_i, b_{-i}) = \tilde{\mathcal{X}}_i(b_i, b_{-i}) \cdot \min \left\{ \max \left\{ \left( \max_{j \in N, j \neq i} p_{ij}^c \right), p_{i0}^c \right\}, b_i \right\},
\]

and the training loss \( \mathcal{L} \) is the negative relaxed revenue

\[
\mathcal{L} = -\sum_i (\tilde{P}_i(b_i, b_{-i})).
\]

**Expressiveness of Mechanisms**

In this section, we compare the expressiveness of CAN and other strategy-proof mechanisms under general single-item auction settings, to show the special design of CAN for more strong expressiveness. [Papadimitriou and Pierrakos] provided a geometric characterization of strategy-proof single-item auctions with general joint value distributions [Papadimitriou and Pierrakos 2011], with which we can show the expressiveness of the mechanism intuitively.

We assume there are two bidders in an auction without loss of generality. An arbitrary strategy-proof mechanism can be expressed by its allocation rules, or two minimum bid curves, \( f_A \) and \( f_B \), which partition the bidding space into several regions as shown in Fig. 2a. If the bid falls above (the direction of the axis of \( b_A \)) \( f_A \), bidder A wins the item and pays \( p_A \). The same for the other bidder. Otherwise, the item is reserved. In Fig. 2b, there are two regions the item should be reserved. The minimum bid curves only need to satisfy two conditions: (1) the minimum bid curve of bidder \( i \) is a multivariate function of other bidders’ bids, i.e., \( f_i = f(b_{-i}) \). (2) Curves are non-crossing, i.e., \( b_i > f_i(b_{-i}) \implies \forall j \neq i, b_j < f_j(b_{-i}) \). Condition (1) is the requirement of the monotonic allocation and condition (2) guarantees that the item is not over-allocated.

The mechanisms CAN can represent are shown in Fig. 2b. The difference is that CAN can only reserve the item in a connected region. The minimum bid curves of two bidders are no longer separated after they overlap. We call such an allocation has the property of reserve monotonicity. It is the most intuitive allocation rule as it reserves the item if bids are all low and sells it if every bidder bid high. Only when the joint value distribution is very abnormal, such a allocation rule will not be optimal. Under the constraint of reserve monotonicity, CAN can reach full expressiveness of strategy-proof single-item auction mechanisms.

Formally, we provide proof to show the expressiveness of CAN. Assume functions \( h_*(\cdot) \) with any index like \( h_1(\cdot), h_2(\cdot), \ldots \) are arbitrary monotonic non-decreasing functions. The minimum bid curve of bidder A can be viewed as two parts: compete with the reserve price and other bidders’ bids, i.e., \( f_A = f(b_{-1}) \). Curves are non-crossing, i.e., \( b_i > f_i(b_{-i}) \implies \forall j \neq i, b_j < f_j(b_{-i}) \). Condition (1) is the requirement of the monotonic allocation and condition (2) guarantees that the item is not over-allocated.

We notice that the left side of the equation is strict monotonic increasing with \( b_A \), while the right side is not a monotonic function with respect to \( b_B \). It means only \( b_A \) can be inverted and it is an arbitrary function of \( b_B \). For the latter part, it can be reduced by \( r_A(b_A, b_B) = r_B(b_A, b_B) \), or

\[
\begin{align*}
h_1(b_A) - h_2(b_B) &= -h_3(b_A) - h_4(b_B), \\
h_1(b_A) + h_3(b_A) &= h_2(b_B) - h_4(b_B).
\end{align*}
\]

We notice that the left side of the equation is strict monotonic increasing with \( b_B \). For the former part, it can be reduced by \( r_A(b_A, b_B) = r_B(b_A, b_B) \), or

\[
\begin{align*}
h_1(b_A) - h_2(b_B) &= -h_3(b_A) + h_6(b_B), \\
h_1(b_A) + h_3(b_A) &= h_2(b_B) + h_6(b_B).
\end{align*}
\]

This time both sides of the equation are strict monotonic increasing, which means \( b_A \) is the function of \( b_B \) and \( b_B \) is the

\[4\]For \( n \) bidders, there are \( n \) surfaces that divide an \( n \)-dimensional bidding space into \( n \) winning regions for each bidder.
function of $b_A$. In other words, $b_A$ is a monotonic function of $b_B$. Thus the total minimum bid curve of bidder A is still an arbitrary function of $b_B$ and mechanisms represented by CAN satisfy condition (1) if the minimum bid curves are not separate again. And condition(2) is naturally satisfied because mechanisms only pick at most one winner with the largest rank score.

A consequent question is that what will happen if the reserve price is zero rather than be viewed as an additional bidder. We denote it as CANw/oA and mechanisms it can represent are shown in Fig. 2c. CANw/oA cannot represent a mechanism that may changes from reserving the item to allocating it to one bidder by increasing others’ bids. Thus, in addition to the condition (1) of Fig. 2b, here the minimum bid curves should be monotonic multivariate functions with respect to others’ bids.

Finally, we show how taking all bidders’ bids as the parameters of rank score functions affects the expressiveness. Dütting et al. propose a single-item auction mechanism namely MyersonNet (Dütting et al. 2019) to deal with auctions with independent values. The rank score function of MyersonNet only takes bidder’s own bid as the parameter, i.e., $r_i = r_i(s_i)$. The auction mechanisms MyersonNet can represent are shown in Fig. 2d. Bidder $i$ can affect the allocation result of bidder $j$ only when they are directly competing with each other. Besides lacking expressiveness on the reserve price shown in the figure, it also cannot encode the influence among bids. Here is a toy case. There are three bidders $A$, $B$, and $C$ participating in auctions. The first time their all bid 1 and bidder $A$ won the auction. The second time only bidder $C$ raises his bid to 2. If rank score $r_1$ is calculated only by $s_i$, bidder $B$ may never win. But if rank score $r_1$ is calculated by $s$, bidder $B$ may win as long as the increase of $C$’s bid makes the rank score of $B$ exceed that of $A$. This shows the mechanism exploits the correlation among signals, and the estimated expected bid of $A$ rises faster than $B$ when $C$’s bid increases. While MyersonNet lacks the ability to represent such mechanisms.

**EXPERIMENTAL EVALUATIONS**

**Experiment Setup**

We implement CAN using the TensorFlow deep learning library. We use 4 groups of 4 linear functions in the MIN-MAX neural network. For all settings, we train networks for 100,000 iterations with a minibatch size of 64. The training set has 100,000 samples and the evaluation set has 10,000 samples. We use Adam optimizer with learning rate 0.001.

**Two Bidders Auctions**

We first evaluate CAN on a two-bidder single-item auction case. To create a general joint value distribution, we take bidding samples randomly from two multivariate normal distributions. As shown in Fig. 3a-Fig. 3c, on the first joint value distribution, monotonic minimum bid curves can represent the optimal allocation rules, and thus CANw/oA can get almost the same mechanism and revenue as CAN. While MyersonNet can only learn a constant reserve price, which leads to a loss of revenue when values are correlated. Fig. 3d-Fig. 3f illustrate allocation rules learned on another joint value distribution. The optimal minimum bid curves are not monotonic and CANw/oA can only learn monotonic minimum bid curves. Therefore, CAN can draw more revenue than CANw/oA and MyersonNet.

**Three Bidders Auctions**

We also evaluate CAN and MyersonNet on a three-bidder single-item auction case to show the interaction among bidders’ bids. As we mentioned, besides the form of reserve prices, another difference in expressiveness between CAN

| Setting | CAN  | CANw/oA | MyersonNet | SP   |
|---------|------|---------|------------|------|
| (a)     | 0.445| 0.443   | 0.424      | 0.417|
| (b)     | 0.636| 0.623   | 0.622      | 0.414|
| (c)     | 0.648| 0.637   | 0.630      | 0.442|
| (d)     | 0.747| 0.747   | 0.724      | 0.704|
| (e)     | 0.831| 0.831   | 0.814      | 0.785|
| (f)     | 0.890| 0.890   | 0.869      | 0.858|
| (g)     | 0.899| 0.899   | 0.869      | 0.858|

Figure 2: (a): Allocation rules of an arbitrary auction mechanism satisfying IC and IR. (b): Allocation rules CAN can represent. (c): Allocation rules CANw/oA can represent. (d): Allocation rules MyersonNet can represent.
and MyersonNet is that one’s bid can affect other bidders’ allocations not through direct competition. As shown in Fig.3, the mechanism learned by CAN always maximizes the revenue based on the conditional joint value distribution as bidder C rise her bid. While in Fig.4, because the bid of bidder C is too small to directly compete with the other two bidders A and B, the mechanism learned by MyersonNet cannot change the allocation rule for the two bidders. This lack of expressiveness leads to a reduction in revenue.

Evaluation on More Bidders

We further evaluate our approach with more bidders. All the results of the evaluated revenue are shown in Table.2. With the rise of the number of bidders, the fierce competition makes CAN without a can draw almost the same revenue as CAN because the item is more likely to be allocated. But due to the defects in the expressiveness of MyersonNet, CAN can always outperform it even in large-scale auctions.

Related Work

Dütting et al. first introduced deep learning for auction mechanism design in an agnostic approach (Dütting et al. 2019). The proposed RegretNet shows great generality and has been adopted in various settings by enforcing additional desirable constraints (Feng, Narasimhan, and Parkes 2018; Kuo et al. 2020; Peri et al. 2021; Sakurai et al. 2019; Golowich, Narasimhan, and Parkes 2018; Luong et al. 2018). However, they learn mechanisms in an agnostic way and cannot ensure strict strategy-proof. Thus, some researchers focus on designing strict truthful mechanisms through deep learning (Shen, Tang, and Zuo 2018; Liu et al. 2021; Curry, Sandholm, and Dickerson 2022). In particular, Curry, Sandholm, and Dickerson analyzed the existing methods of mechanism design through learning, which shed light on the research of ideal mechanism design.

Regarding auctions with correlated valuations, Papadimitriou and Pierrakos shows finding the deterministic, ex-post IC and ex-post IR, optimal mechanism for more than two bidders is NP-hard (Papadimitriou and Pierrakos 2011). Roughgarden and Talgam-Cohen proposed an optimal auction mechanism under specific conditions on correlated distributions inspired by Myerson auction (Roughgarden and Talgam-Cohen 2013). Many researchers try to find an approximate algorithm to design near-optimal auction mechanisms in general setting (Ronen 2001; Dobzinski, Fu, and Kleinberg 2011), but it is still an open problem.

Conclusion

In this work, we have studied the problem of single-item auction mechanism design with correlated values through the power of machine learning, where there are not many positive theoretical results. We make an in-depth discussion on the feasible solution space for the strategy-proof auction mechanisms without losing the expressiveness ability.
We also make the allocation and pricing differentiable by the MIN-MAX network and the closed-form revenue expression, which is compatible with the machine learning pipeline. Experimental results demonstrate that CAN outperforms the auction mechanisms wildly used in terms of the expressiveness and the revenue in the correlated value settings. It is of great significance to clarify the expressiveness and game theoretical property guarantee of the neural network to ensure the interpretability of the learned auction mechanisms. It still needs further efforts to achieve full expressiveness of learning based auction mechanism design approaches.

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Missing Proofs

Proof in Calculating the Payment

We first prove one lemma to simplify the proof.

**Lemma 2.** If \( \forall z \in Z \) and \( \forall q \in Q \), \( \alpha_{zq} > 0 \), then the abscissa \( z_0 \) of the intersection of a piece-wise linear function

\[
y_i = \min_{z \in Z} \max_{q \in Q} \left( \alpha_{zq} z + \beta_{zq}^1 \right)
\]

and a monotonic non-increasing function \( y_3 = f(z) \) is

\[
z_0 = \max_{z \in Z} \min_{q \in Q} (g^{-1}_{zq}(0)),
\]

where

\[
g_{zq}(t) = \alpha_{zq}^1 t + \beta_{zq}^1 - f(t).
\]

**Proof.** Assume the intersection is \((y_0, z_0)\). Substitute intersection coordinates, then

\[
y_0 = \min_{z \in Z} \max_{q \in Q} \left( \alpha_{zq}^1 z_0 + \beta_{zq}^1 \right)
\]

and \( y_0 = f(z_0) \). Thus we can get

\[
f(z_0) = \min_{z \in Z} \max_{q \in Q} (\alpha_{zq}^1 z_0 + \beta_{zq}^1),
\]

that is

\[
\min_{z \in Z} \max_{q \in Q} (g_{zq}(z_0)) = 0.
\]

Due to \( g_{zq}(z_0) \) is monotonic increasing, the inverse operation need to change \( \max \) to \( \min \) and change \( \min \) to \( \max \). Finally we get

\[
z_0 = \max_{z \in Z} \min_{q \in Q} (g^{-1}_{zq}(0)).
\]
Then we can come back to the problem. Given two piecewise linear functions

\[ y_i = \min_{z \in Z} \max_{q \in Q} \left( \alpha_{zq}^1 z + \beta_{zq}^1 \right) \]

and

\[ y_j = \min_{z \in Z} \max_{q \in Q} \left( \alpha_{zq}^2 z + \beta_{zq}^2 \right), \]

where \( y_i \) is monotonic increasing with respect to \( z \) and \( y_j \) is monotonic non-increasing with respect to \( z \). We can view \( y_j \) as the non-increasing function \( f(z) \) in Lemma 2 and \( y_i \) is the same. According to Lemma 2, the abscissa \( z_0 \) of the intersection is

\[ z_0 = \max_{z \in Z} \min_{q \in Q} (g_{zq}^{-1}(0)), \]

where

\[ g_{zq}(t) = \alpha_{zq}^1 t + \beta_{zq}^1 - \min_{z \in Z} \max_{q \in Q} \left( \alpha_{zq}^2 t + \beta_{zq}^2 \right). \]

The problem goes to get the abscissa \( z_{0,q} \) of the intersection of a monotonic increasing function \( y_i = \alpha_{zq}^1 t + \beta_{zq}^1 \) and a monotonic non-decreasing function

\[ y_j = \min_{z \in Z} \max_{q \in Q} \left( \alpha_{zq}^2 t + \beta_{zq}^2 \right). \]

The monotonicity is contrary to that in Lemma 2 but the proof is similar. The only difference is this time \( g_{zq}(z_{0,q}) \) is monotonic decreasing and we don’t need to change \( \min \) and \( \max \) this time. The value of \( z_{0,q} \) is

\[ z_{0,q} = \min_{z \in Z} \max_{q \in Q} \left( \frac{\beta_{zq}^1 - \beta_{zq}^2}{\alpha_{zq}^1 - \alpha_{zq}^2} \right). \]

And the final value of \( z_0 \) is

\[ z_0 = \min_{z \in Z} \max_{\tilde{q} \in Q} \min_{q \in Q} \max_{z \in Z} \left( \frac{\beta_{zq}^1 - \beta_{zq}^2}{\alpha_{zq}^2 - \alpha_{zq}^1} \right). \]