Largest family without a pair of posets on consecutive levels of the Boolean lattice

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Abstract

Suppose $k \geq 2$ is an integer. Let $Y_k$ be the poset with elements $x_1, x_2, y_1, y_2, \ldots, y_{k-1}$ such that $y_1 < y_2 < \cdots < y_{k-1} < x_1, x_2$ and let $Y'_k$ be the same poset but all relations reversed. We say that a family of subsets of $[n]$ contains a copy of $Y_k$ on consecutive levels if it contains $k + 1$ subsets $F_1, F_2, G_1, G_2, \ldots, G_{k-1}$ such that $G_1 \subset G_2 \subset \cdots \subset G_{k-1} \subset F_1, F_2$ and $|F_1| = |F_2| = |G_{k-1}| + 1 = |G_{k-2}| + 2 = \cdots = |G_1| + k - 1.$ If both $Y_k$ and $Y'_k$ on consecutive levels are forbidden, the size of the largest such family is denoted by $La_{ac}(n, Y_k, Y'_k).$ In this paper, we will determine the exact value of $La_{ac}(n, Y_k, Y'_k).$

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1 Introduction

Given two partially ordered sets (posets) $P$ and $Q$, we say that $P$ is a subposet of $Q$ if there exists an injection $\phi : P \to Q$ such that $x \leq_P y$ implies $\phi(x) \leq_Q \phi(y).$ Viewing collections of sets as posets under the inclusion relation, we have the following extremal functions, first introduced by Katona and Tarján [9]. For any collection of finite posets $\mathcal{P}$, let $La(n, \mathcal{P})$ be the maximum size of a family of subsets of $[n] = \{1, 2, \ldots, n\}$ which does not contain any $P \in \mathcal{P}$ as a subposet.

This type of problems was first studied by Sperner [14].

Theorem 1.1 (Sperner [14]). Let $\mathcal{F}$ be a family of subsets of $[n]$ without inclusion relation between any two of the subsets. Then

$$|\mathcal{F}| \leq \binom{n}{\left\lfloor \frac{n}{2} \right\rfloor}.$$
A chain of length \( k \) is a poset with elements \( x_1, x_2, \ldots, x_k \) such that \( x_1 < x_2 < \cdots < x_k \). Let \( \sum(n, k) \) be the sum of the \( k \) largest binomial coefficients of the form \( \binom{n}{i} \). Then Sperner’s theorem can be extended as follows:

**Theorem 1.2** (Erdős [3]). Let \( \mathcal{F} \) be a family of subsets of \([n]\) without a chain of length \( k \). Then

\[
|\mathcal{F}| \leq \sum(n, k - 1).
\]

Before stating the next result, we need the following notation. Let \( 2 \leq k \leq n \) and \( 0 \leq r \leq n \) be two integers. The following lacunary sum of binomial coefficients was first introduced by Ramus [13] in 1834.

\[
S(n, k, r) = \sum_{i \equiv r \mod k} \binom{n}{i}.
\]

Clearly, if \( r_1 \equiv r_2 \pmod{k} \), then \( S(n, k, r_1) = S(n, k, r_2) \). So there are \( k \) distinct such sums \( S(n, k, 0), S(n, k, 1), \ldots, S(n, k, k - 1) \).

The first author [8] published the next two theorems which are analogs of the two theorems above.

**Theorem 1.3** (Katona [8]). Let \( \mathcal{F} \) be a family of subsets of \([n]\) such that no two of the subsets \( F_i, F_j \) satisfy \( F_i \subset F_j \) and \( |F_j| - |F_i| < k \). Then

\[
|\mathcal{F}| \leq \max\{S(n, k, r) \mid r \in \{0, \ldots, k - 1\}\}.
\]

**Theorem 1.4** (Katona [8]). Let \( h \leq k \) be an integer, and let \( \mathcal{F} \) be a family of subsets of \([n]\) such that at most \( h + 1 \) of the subsets \( F_{i_1}, \ldots, F_{i_{h+1}} \) satisfy \( F_{i_1} \subset \cdots \subset F_{i_{h+1}} \) and \( |F_{i_{h+1}}| - |F_{i_1}| < k \). Then

\[
|\mathcal{F}| \leq \max\{S(n, k, r_1) + S(n, k, r_2) + \cdots + S(n, k, r_h) \mid r_1 \neq r_2 \neq \cdots \neq r_h \in \{0, \ldots, k - 1\}\}.
\]

Another type of generalizations of Sperner Theorem is to determine the largest size of a family of subsets of \([n]\) without a copy of poset \( Y_k \) and \( Y'_k \) defined below. Let \( Y_k \) be the poset with elements \( x_1, x_2, y_1, y_2, \ldots, y_{k-1} \) such that \( y_1 < y_2 < \cdots < y_{k-1} < x_1, x_2 \), and let \( Y'_k \) be the same poset but all relations reversed. Katona and Tarján [9] gave the following result for \( k = 2 \). (In their paper, posets \( Y_2 \) and \( Y'_2 \) are denoted by \( V \) (the cherry poset) and \( \Lambda \) (the fork poset) respectively.)

**Theorem 1.5** (Katona and Tarján [9]).

\[
La(n, Y_2, Y'_2) = 2 \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}.
\]

De Bonis, Katona and Swanepoel [2] studied the case \( k = 3 \). We remark that they actually proved a result for the so-called butterfly poset, but their proof implies the following theorem.
Theorem 1.6 (De Bonis, Katona and Swanepoel [2]).

\[ \text{La}(n, Y_3, Y'_3) = \sum (n, 2). \]

For the case \( k \geq 4 \), Methuku and Tompkins [12] got the next theorem.

Theorem 1.7 (Methuku and Tompkins [12]). For \( n \geq k \geq 4 \),

\[ \text{La}(n, Y_k, Y'_k) = \sum (n, k - 1). \]

For other results related to the \( \text{La} \) function, see [1, 5, 6, 7, 11, 15]. Now, we consider the following problem, which is a combination of the two types of problems above. We say that a family of subsets of \([n]\) contains a copy of \( Y_k \) on consecutive levels if it contains \( k + 1 \) subsets \( F_1, F_2, G_1, G_2, \ldots, G_{k-1} \) such that \( G_1 \subset G_2 \subset \cdots \subset G_{k-1} \subset F_1, F_2 \) and \( |F_1| = |F_2| = |G_{k-1}| + 1 = |G_{k-2}| + 2 = \cdots = |G_1| + k - 1 \). We denote by \( \text{La}_c(n, Y_k, Y'_k) \) the largest size of a family of subsets of \([n]\) containing neither \( Y_k \) nor \( Y'_k \) on consecutive levels.

We note that this problem is mentioned in [4] in the language of the oriented hypercube. The authors in [4] gave an asymptotic formula of the size of largest family without tree posets in consecutive levels. For exact result, they proved

\[ \text{La}_c(n, Y_2, Y'_2) = 2^{n-1}. \]

Let \( m = \lfloor (n - k)/2 \rfloor \) in the rest of this paper. For \( k \geq 3 \), we have the following theorem.

Theorem 1.8. Let \( n \geq k \geq 3 \), then

\[ \text{La}_c(n, Y_k, Y'_k) = 2^n - S(n, k, m). \]

Remark 1.9. (I) Loehr and Michael [10] showed that

\[ S(n, k, m) = \min_{r: 0 \leq r \leq k-1} S(n, k, r). \]

So our result implies that the trivial construction is the best. Here, a trivial construction consists of all subsets except the ones with size \( s \equiv m \pmod{k} \).

(II) If \( n = k \geq 3 \), then \( m = 0 \) and \( S(n, k, m) = \binom{n}{0} + \binom{n}{n} = 2 \). In this case, Theorem 1.8 is trivial, since every family \( F \) with size \( 2^n - 1 \) contains a copy of \( Y_k \) or \( Y'_k \) on consecutive levels.

The rest of the paper is organized as follows. In the next section, we present some preliminary results. In Section 3, we will prove our main theorem (Theorem 2.1), which implies Theorem 1.8.

2 Preliminary results

A cyclic permutation \( \sigma \) of \([n]\) is a cyclic ordering \( a_1, a_2, \ldots, a_n, a_1 \), where \( a_i \in [n] \) for \( i = 1, 2, \ldots, n \). Let \( \mathcal{F} \) be a family of subsets of \([n]\) containing neither \( Y_k \) nor \( Y'_k \) on consecutive levels. We say a set \( F \in \sigma \) if \( F \) is an interval along the cyclic permutation \( \sigma \).
Figure 1: \(I(n)^\sigma\) for \(n = 6\) and \(\sigma = 1, 2, 3, 4, 5, 6, 1\) is shown by bold vertices, and two vertices from consecutive levels are connected by an edge if they have inclusion relation.

Now, we double count the sum \(S\) of \(\phi_F\) over all cyclic permutations \(\sigma\) and \(F \in \mathcal{F}\) such that \(F \in \sigma\), where

\[
\phi_F = \begin{cases} 
\binom{n}{|F|}, & \text{if } F \neq \emptyset \text{ and } F \neq [n]; \\
n, & \text{if } F = \emptyset \text{ or } F = [n].
\end{cases}
\]

For any \(F \in (\mathcal{F} \setminus \{\emptyset, [n]\})\), we have \(|F|!(n - |F|)!\) cyclic permutations \(\sigma\) satisfying \(F \in \sigma\). If \(\emptyset \in \mathcal{F}\) or \([n] \in \mathcal{F}\), all cyclic permutations satisfy the above condition, and the number of cyclic permutations is \((n - 1)!\). So

\[
S = |\{\emptyset, [n]\} \cap \mathcal{F}| \cdot n \cdot (n - 1)! + \sum_{F \in (\mathcal{F} \setminus \{\emptyset, [n]\})} |F|!(n - |F|)! \binom{n}{|F|}
\]

\[
= \left(|\{\emptyset, [n]\} \cap \mathcal{F}| + \sum_{F \in (\mathcal{F} \setminus \{\emptyset, [n]\})} 1\right) \cdot n! = |\mathcal{F}| \cdot n!.
\]

On the other hand, for any cyclic permutation \(\sigma\), let \(I(n)^\sigma\) be the family of intervals along \(\sigma\). The \(i^{th}\) level of \(I(n)^\sigma\) is the collection of its elements of size \(i\). (See Figure 1 for an example of \(I(6)^\sigma\), where \(\sigma = 1, 2, 3, 4, 5, 6, 1\).) For \(0 \leq i \leq n\), let \(x_i\) be the number of subsets in the \(i^{th}\) level of \(\mathcal{F} \cap I(n)^\sigma\). Then

\[
S = \sum_{\sigma} \left(|\{\emptyset, [n]\} \cap \mathcal{F}| \cdot n + \sum_{i=1}^{n-1} \binom{n}{i} x_i\right).
\]

Now, we need the following theorem to give an upper bound on \(|\{\emptyset, [n]\} \cap \mathcal{F}| \cdot n + \sum_{i=1}^{n-1} \binom{n}{i} x_i\) for every cyclic permutation \(\sigma\).

**Theorem 2.1.** Let \(n > k \geq 3\) and \(\mathcal{F}\) be a family of subsets of \([n]\) containing neither \(Y_k\) nor \(Y_k'\) on consecutive levels. Then for every cyclic permutation \(\sigma\), we have

\[
|\{\emptyset, [n]\} \cap \mathcal{F}| \cdot n + \sum_{i=1}^{n-1} \binom{n}{i} x_i \leq n \cdot (2^n - S(n, k, m)) + n - 1.
\]
Supposing that we know Theorem 2.1, then
\[ S = |\mathcal{F}| \cdot n! \leq \sum_{\sigma} \left( n \cdot (2^n - S(n, k, m)) + n - 1 \right) = (n - 1)! \cdot \left( n \cdot (2^n - S(n, k, m)) + n - 1 \right). \]
Since |\mathcal{F}| is an integer, we have
\[ |\mathcal{F}| \leq \left\lfloor \frac{n \cdot (2^n - S(n, k, m)) + n - 1}{n} \right\rfloor = 2^n - S(n, k, m), \]
as desired. So by (II) of Remark 1.9, it is sufficient to prove Theorem 2.1. The following two lemmas are needed to give constraints of \( x_0, x_1, \ldots, x_n \).

**Lemma 2.2.** If \( 0 \leq i \leq n - k + 1 \), then for all \( \sigma \),
\[ x_i + x_{i+1} + \cdots + x_{i+k-1} \leq (k - 1)n. \]

**Lemma 2.3.** For all \( \sigma \), if \( x_0 = 1 \) and \( x_1 + x_2 + \cdots + x_k = (k - 1)n \), then
\[ x_0 + x_1 + \cdots + x_{k-1} \leq (k - 1)n - \lfloor n/2 \rfloor. \]
For all \( \sigma \), if \( x_n = 1 \) and \( x_{n-k} + x_{n-k+1} + \cdots + x_{n-1} = (k - 1)n \), then
\[ x_{n-k+1} + x_{n-k+2} + \cdots + x_n \leq (k - 1)n - \lfloor n/2 \rfloor. \]

Before starting the proofs of the two lemmas above, we introduce some notations. Let \( \sigma = a_1, a_2, \ldots, a_n, a_1 \), where \( a_j \in [n] \) for \( 1 \leq j \leq n \). In the rest of this paper, we view \( \mathcal{I}(n)^\sigma \) (a subfamily of \( \mathcal{I}(n)^\sigma \)) as a graph. The vertex set of this graph is \( \mathcal{I}(n)^\sigma \) (the subfamily of \( \mathcal{I}(n)^\sigma \)), and two vertices are connected by an edge if they are from consecutive levels and they have inclusion relation (see Figure 2). We denote by \( I^*_s \) the vertex \( \{a_{t+1}, a_{t+2}, \ldots, a_{t+s}\} \) (with addition taken modulo \( n \)) of \( \mathcal{I}(n)^\sigma \). Clearly, \( |I^*_s| = s \), \( I^*_{s-1}, I^*_{s+1} \subseteq I^*_s \) for \( 2 \leq s \leq n - 1 \), and \( I^*_s \subseteq I^*_{s-1}, I^*_{s+1} \) for \( 1 \leq s \leq n - 2 \) (see Figure 2).

In order to prove the two kinds of constraints of \( k \) consecutive \( x_i \)'s above, we consider some typical structures in \( k \) consecutive levels of \( \mathcal{I}(n)^\sigma \). If the \( k \) levels are the levels from \( 0^{\text{th}} \) to \( (k - 1)^{\text{st}} \), then every \( Y_k \) on these levels must contain \( \emptyset \). In this case, we consider a special kind of \( Y_k \), which we denote by
\[ Y_k(j) = \{\emptyset, I^1_j, I^2_j, \ldots, I^{k-2}_j, I^{k-1}_j, I^{k-1}_{j-1}\} \]
where \( 1 \leq j \leq n \). (See Figure 3 for an example of \( Y_k(1) \).) If the \( k \) levels are the levels from \( (i + 1)^{\text{st}} \) to \( (i + k)^{\text{th}} \) (the middle part of Figure 2), then we introduce a new kind of structure on \( k \) consecutive levels. A family of \( k + 2 \) subsets is called \( X_k \) if it is
\[ \{I_{t+1}, I^1_t, I^2_t, \ldots, I^{k-2}_t, I^{k-1}_t, I^{k-1}_{t-1}\} \text{ or } \{I_{t-1}, I^1_{t-1}, I^2_{t-1}, \ldots, I^{k-2}_{t-1}, I^{k-1}_{t-1}\} \]
where \( 1 \leq t \leq n \). (See Figure 3 for two types of examples of \( t = 2 \) and \( t = k \) respectively.) In each \( X_k \), we call the 3 elements in the \( (i + k)^{\text{th}} \) level and the \( (i + k - 1)^{\text{st}} \) level a cherry, and the 3 elements in the \( (i + 1)^{\text{st}} \) level and the \( (i + 2)^{\text{nd}} \) level a fork.
Figure 2: Vertices $I^*_i$ of $I(n)^o$.

Figure 3: Examples of structures on $k$ consecutive levels.
Proof of Lemma 2.2: First, we show an element of the family \( \mathcal{I}(n)^{3} \) of \( I \) if \( I \) find a copy of the levels of \( x \) we have \( x \). Then if we sum over all \( 2^{i} \) function over its edges is at least \( F \). Remark 2.4, \( i \). On the other hand, we will prove below the claim that in each \( s \) level of the \( I \) and \( \psi \) is an element of the family \( F \).

**Proof of Lemma 2.2:** First, we show \( x_{0} + x_{1} + \cdots + x_{k-1} \leq (k-1)n \). Since \( x_{0} \leq 1 \) and \( x_{i} \leq n \) for \( 1 \leq i \leq k-1 \), we have \( x_{0} + x_{1} + \cdots + x_{k-1} \leq (k-1)n + 1 \). If \( x_{0} + x_{1} + \cdots + x_{k-1} = (k-1)n + 1 \), every subset of the levels from \( i^{th} \) to \( (k-1)^{st} \) in \( I \) is in \( F \), then one can easily find a copy of \( Y_k \) on consecutive levels. (See \( Y_k(1) \) for an example in Figure 3.) Similarly, we have \( x_{n-1+k} + x_{n-1+k-2} + \cdots + x_{n} \leq (k-1)n \).

Now, we prove that

\[
x_{i+1} + x_{i+2} + \cdots + x_{i+k} \leq (k-1)n
\]

for \( 0 \leq i \leq n - k - 1 \). Recall that there is an edge between two vertices \( I, I' \) if and only if \( I' \subset I \) and \( |I| = |I'| = 1 \). Then, we double count the sum \( T \) of the weight function \( \psi(e) \) over all edges \( e = \{ I, I' \} \) in the \( 2n \) \( X_k \)'s, where

\[
\psi(e) = \begin{cases} 0, & \text{if } I, I' \in F; \\ 0, & \text{if } I, I' \notin F; \\ 0, & \text{if } I \in F, I' \notin F \text{ and } |I| = i + k; \\ 0, & \text{if } I \notin F, I' \in F \text{ and } |I| = i + 2; \\ 1, & \text{otherwise.} \end{cases}
\]

On the one hand, recall that every vertex in the \( s^{th} \) level of \( \mathcal{I}(n)^{3} \) have two neighbors in the \( (s+1)^{st} \) level and two neighbors in the \( (s-1)^{st} \) level, and note that only the case when one of the two vertices in the edge is not in \( F \) counts nonzero. Then by (II) of Remark 2.4,

\[
T = \left( 2 \cdot \sum_{I \in \mathcal{F}, |I| = i+k} \psi(e) + \sum_{I \notin \mathcal{F}, |I| \neq i+k} \psi(e) \right) + \left( 2 \cdot \sum_{I' \notin \mathcal{F}, |I'| = i+1} \psi(e) + \sum_{I' \in \mathcal{F}, |I'| \neq i+1} \psi(e) \right)
\leq \left( 2 \cdot 2(n - x_{i+k}) + \sum_{j=i+3}^{i+k-1} 2(n - x_{j}) \right) + \left( 2 \cdot 2(n - x_{i+1}) + \sum_{l=i+2}^{i+k-2} 2(n - x_{l}) \right)
= 4 \left( \sum_{h=i+1}^{i+k} (n - x_{h}) \right) - 2(n - x_{i+2}) - 2(n - x_{i+k-1}).
\]

On the other hand, we will prove below the claim that in each \( X_k \), the sum of the weight function over its edges is at least \( |\{ I \in \mathcal{F} \cap X_k \ | |I| = i+2 \}| + |\{ I \in \mathcal{F} \cap X_k \ | |I| = i+k-1 \}| \). Then if we sum over all \( 2n \) \( X_k \)'s, \( T \) is at least \( 2(x_{i+2} + x_{i+k-1}) \).
Now, we divide the cherries of all $2n \times X_k$’s into 4 types (see Figure 4), according to whether its 3 elements are in $\mathcal{F}$ or not. We call a cherry Type 1 (2 or 3) if the middle element and both (one or none) of its neighbors are in $\mathcal{F}$. The rest of the cases are Type 4, namely that the middle element is not in $\mathcal{F}$. Note that in Type 4, we do not distinguish the cases if two neighbors of the middle element are in $\mathcal{F}$ or not. Similarly, we divide all forks into Types 5, 6, 7 and 8 (see Figure 4).

![Diagram of Types 1-8](image)

Figure 4: Types 1 – 8: the vertices $I$ in $\mathcal{F}$ (not in $\mathcal{F}$ or not clear) are denoted by solid (hollow or dashed) vertices, respectively.

To prove the claim, we distinguish several cases by the types defined above. Note that in each $X_k$, in order to avoid copies of $Y_k$ and $Y'_k$ in $\mathcal{F}$, either at least 2 of the 4 vertices in the $(i + 1)^{st}$ level and the $(i + k)^{th}$ level are not in $\mathcal{F}$, or at least one of the $k - 2$ vertices in the levels from $(i + 2)^{nd}$ to $(i + k - 1)^{st}$ is not in $\mathcal{F}$. If the cherry and the fork in some $X_k$ are Types 1 and 5 respectively, then we must have some vertex $I$ not in $\mathcal{F}$ in the chain connecting the cherry and the fork in this $X_k$. Then along this chain in both directions from $I$, take the first vertices in $\mathcal{F}$ respectively. Thus, we can find at least two edges with weight 1. Note that such vertices exist since in Types 1 and 5, the vertices in the $(i + 2)^{nd}$ level and the $(i + k - 1)^{st}$ level are in $\mathcal{F}$. The same argument works for the pairs Types 1 and 6, and Types 2 and 5. If they are Types 1 and 7, then we already have two desired edges in the fork. If they are Types 1 and 8, then we find the first vertex in $\mathcal{F}$ along the chain from the fork to the cherry, this vertex and its neighbor below in the chain will give the edge we want. The same argument works for Types 4 and 5. In the rest cases, it is easy to find the desired edges in the cherry and the fork. (If $k = 3$, Types 1 or 2 or 3 and 8 (5 or 6 or 7 and 4) cannot appear in any $X_k$, since $i + 1 = i + k - 2$. If $k = 3$ or 4, Types 1 and 5 (1 and 6, or 2 and 5) cannot appear in any $X_k$, since they will form a $Y_k$ or a $Y'_k$ in $\mathcal{F}$.)

Therefore, we have

$$2(x_{i+2} + x_{i+k-1}) \leq 4 \left( \sum_{h=i+1}^{i+k} (n - x_h) \right) - 2(n - x_{i+2}) - 2(n - x_{i+k-1}).$$

That is,

$$x_{i+1} + x_{i+2} + \cdots + x_{i+k} \leq (k - 1)n,$$
as required. □

**Proof of Lemma 2.3:** By symmetry, it is enough to prove the first part of the lemma. That is, we need to prove

\[ x_0 + x_1 + \cdots + x_{k-1} \leq (k - 1)n - \lfloor n/2 \rfloor = (k - 2)n + \lceil n/2 \rceil. \]

We distinguish three cases to prove it.

**Case 1.** \( x_{k-1} \leq \lfloor n/2 \rfloor \) when \( n \) is odd or \( x_{k-1} \leq (n/2) - 1 \) when \( n \) is even.

Note that \( x_i \leq n \) for \( i = 1, 2, \ldots, k - 2 \), then by the assumption \( x_0 = 1 \), we have

\[ x_0 + x_1 + \cdots + x_{k-1} \leq 1 + (k - 2)n + \lfloor n/2 \rfloor \leq (k - 2)n + \lceil n/2 \rceil, \]

if \( n \) is odd; and

\[ x_0 + x_1 + \cdots + x_{k-1} \leq 1 + (k - 2)n + (n/2) - 1 \leq (k - 2)n + (n/2), \]

if \( n \) is even.

**Case 2.** \( x_{k-1} = n/2 \) when \( n \) is even.

Suppose that the result is not true. That is,

\[ x_0 + x_1 + \cdots + x_{k-1} \geq (k - 2)n + (n/2) + 1. \]

Note that \( x_{k-1} = n/2 \) in this case and \( x_i \leq n \) for \( i = 1, 2, \ldots, k - 2 \). Then by the assumption \( x_0 = 1 \), we have \( x_1 = x_2 = \cdots = x_{k-2} = n \). Therefore, by the assumption \( x_1 + x_2 + \cdots + x_k = (k - 1)n \), we have \( x_k = n/2 \). Since \( x_{k-1} = n/2 \) and \( x_k = n/2 \), we can find 2 vertices such that they are in the \( k^{th} \) level and the \( (k - 1)^{st} \) level respectively, they are in \( \mathcal{F} \), and they have inclusion relation. Then a copy of \( Y'_k \) on the levels from \( 1^{st} \) to \( k^{th} \) can be found, a contradiction.

**Case 3.** \( x_{k-1} \geq \lfloor n/2 \rfloor + 1 \).

The number of pairs of vertices \( (I_{j-1}^k, I_j^k) \) in \( \mathcal{F} \) is at least \( x_{k-1} - (n - x_{k-1}) = 2x_{k-1} - n \) (see Figure 3). To avoid a copy of \( Y_k(j) \) in \( \mathcal{F} \), at least one vertex of \( Y_k(j) \) in the levels from \( 1^{st} \) to \( (k - 2)^{nd} \) is not in \( \mathcal{F} \). So by (I) of Remark 2.4, we have

\[ \sum_{i=0}^{k-1} x_i \leq 1 + (k - 2)n - (2x_{k-1} - n) + x_{k-1} = 1 + (k - 1)n - x_{k-1} \leq (k - 1)n - \lfloor n/2 \rfloor, \]

since \( x_{k-1} \geq \lfloor n/2 \rfloor + 1 \). This completes the proof of this case and the lemma. □

Before starting the proof of Theorem 2.1, we need the following notations and lemmas for helping us to use the constraints above. Recall the definition of \( S(n, k, r) \) and \( m = \lceil (n - k)/2 \rceil \). Now let \( z \) be an integer such that \( z \equiv r \pmod{k} \). Then we denote

\[ S(n, k, r \mid z) = \sum_{i=0}^{n} \binom{n}{i}; \]

\[ w_i = \begin{cases} S(n, k, i \mid i) - S(n, k, i - 1 \mid i - 1), & \text{if } i \leq m; \\ S(n, k, n - i - k + 1 \mid n - i - k + 1) - S(n, k, n - i - k \mid n - i - k), & \text{if } m + 1 \leq i. \end{cases} \]
Lemma 2.5. Let $t \leq n$ be an integer. We have the following equalities.
(I) If $t < 0$, $S(n,k,t|t) = 0$, and if $0 \leq t \leq k - 1$, $S(n,k,t|t) = \binom{n}{t}$.
(II) If $0 \leq t \leq n$, $S(n,k,t|t) = S(n,k,t - k|t - k) + \binom{n}{t}$.
(III) If $0 \leq t \leq n$, $S(n,k,t|t) + S(n,k,n - k - t|n - k - t) = S(n,k,t)$.
(IV) If $0 \leq t \leq n$, $S(n,k,t - k|t - k) + S(n,k,n - k - t|n - k - t) = S(n,k,t) - \binom{n}{t}$.

Proof: (I) follows from the definition of $S(n,k,t|t)$. (II) follows from (I) if $0 \leq t \leq k - 1$, and the definition of $S(n,k,t|t)$ if $k \leq t \leq n$. (IV) follows from (II) and (III). Now, we distinguish two cases to prove (III).

If $t > n - k$, then we have $t + k > n$ and $n - k - t < 0$. Hence, $S(n,k,t|t) = S(n,k,t)$ and $S(n,k,n - k - t|n - k - t) = 0$ by (I), and this completes the proof of this case.

If $0 \leq t \leq n - k$, let $0 \leq r \leq k - 1$ be the remainder of $n - k - t$ divided by $k$. We have

$$
S(n,k,t|t) + S(n,k,n - k - t|n - k - t) = S(n,k,t|t) + \binom{n}{r} + \binom{n}{r + k} + \cdots + \binom{n}{n - k - t} = S(n,k,t|t) + \binom{n}{n - r} + \binom{n}{n - r - k} + \cdots + \binom{n}{k + t} = S(n,k,t|t) + \binom{n}{t + k} + \cdots + \binom{n}{n - r - k} + \binom{n}{n - r} = S(n,k,t),
$$

since $r < k$. \qed

Lemma 2.6. Let $s$ be an integer.
(I) $w_s = 0$ if $s < 0$ or $s > n - k + 1$, and $w_s > 0$ if $0 \leq s \leq n - k + 1$.
(II) If $0 \leq s \leq m$ or $m + k \leq s \leq n$, then

$$
\sum_{i=s-k+1}^{s} w_i = \binom{n}{s}.
$$

(III) If $m < s < m + k$,

$$
\sum_{i=s-k+1}^{s} w_i = S(n,k,m) - S(n,k,s) + \binom{n}{s}.
$$

(IV) $\sum_{i=0}^{n-k+1} w_i = S(n,k,m)$. 

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Proof: (I) It follows from the definition of $w_s$ and (I) of Lemma 2.5.

(II) By (II) of Lemma 2.5, if $0 \leq s \leq m$, it follows from (I) that

$$
\sum_{i=s-k+1}^{s} w_i = \sum_{i=s-k+1}^{s} (S(n,k,i|i) - S(n,k,i-1|i-1)) = S(n,k,s) - S(n,k,s-k) = \binom{n}{s},
$$

and if $m + k \leq s \leq n$, we have $s - k + 1 \geq m + 1$, and so

$$
\sum_{i=s-k+1}^{s} w_i = \sum_{i=s-k+1}^{s} (S(n,k,n-k-i+1|n-k-i+1) - S(n,k,n-k-i|n-k-i))
$$

$$
= S(n,k,n-s|n-s) - S(n,k,n-s-k|n-s-k) = \binom{n}{n-s} = \binom{n}{s}.
$$

(III) If $m < s < m+k$, then $s-k+1 \leq m$ and $s \geq m+1$. So by (III) and (IV) of Lemma 2.5, we have

$$
\sum_{i=s-k+1}^{s} w_i = \sum_{i=s-k+1}^{m} (S(n,k,i|i) - S(n,k,i-1|i-1))
$$

$$
+ \sum_{j=m+1}^{s} (S(n,k,n-k-j+1|n-k-j+1) - S(n,k,n-k-j|n-k-j))
$$

$$
= S(n,k,m|m) - S(n,k,s-k|s-k)
$$

$$
+ S(n,k,n-k-m|n-k-m) - S(n,k,n-k-s|n-k-s)
$$

$$
= (S(n,k,m|m) + S(n,k,n-k-m|n-k-m))
$$

$$
- (S(n,k,s-k|s-k) + S(n,k,n-k-s|n-k-s))
$$

$$
= S(n,k,m) - \binom{n}{n-k} = S(n,k,m) - S(n,k,s) + \binom{n}{s}.
$$

(IV)

$$
\sum_{i=0}^{n-k+1} w_i = \sum_{i=0}^{n} \left( \sum_{j=0}^{i} w_j \right) = S(n,k,m),
$$

by (I) and (II). \hfill \square

\section{3 Proof of Theorem 2.1}

First, we consider a linear programming problem: maximize $\sum_{i=0}^{n} \binom{n}{i} x_i$ subject to the constraints $x_i + x_{i+1} + \cdots + x_{i+k-1} \leq (k-1)n$ for $i = 0, 1, \ldots, n-k+1$ (Lemma 2.2), $0 \leq x_0, x_n \leq 1$, and $0 \leq x_i \leq n$ for all $i = 1, \ldots, n-1$.

We assign the weight $w_i$ (see the definition and properties of $w_i$ in Section 2) to the constraint $x_i + x_{i+1} + \cdots + x_{i+k-1} \leq (k-1)n$ for every $0 \leq i \leq n-k+1$. By (I) of Lemma 2.6, $w_i > 0$ for $0 \leq i \leq n-k+1$, so we have

$$
\sum_{i=0}^{n-k+1} w_i(x_i + x_{i+1} + \cdots + x_{i+k-1}) \leq (k-1)n \cdot \sum_{i=0}^{n-k+1} w_i.
$$

(1)
Then by (I), (II) and (III) of Lemma 2.6, the LHS of (1) is equal to
\[
\sum_{i=0}^{n} \left( \sum_{j=i-k+1}^{i} w_j \right) x_i = \sum_{i=0}^{n} \binom{n}{i} x_i + \sum_{j=m+1}^{m+k-1} (S(n, k, m) - S(n, k, j)) x_j.
\]

On the other hand, by (IV) of Lemma 2.6, the RHS of (1) is equal to \((k - 1)n \cdot S(n, k, m)\). Hence, by (1), we have
\[
\sum_{i=0}^{n} \binom{n}{i} x_i + \sum_{j=m+1}^{m+k-1} (S(n, k, m) - S(n, k, j)) x_j \leq (k - 1)n \cdot S(n, k, m). \tag{2}
\]

Note that \(S(n, k, m) = \min_{0 \leq r \leq k-1} S(n, k, r)\) by Remark 1.9, and \(0 \leq x_j \leq n\) for \(m + 1 \leq j \leq m + k - 1\). So we have
\[
\sum_{j=m+1}^{m+k-1} (S(n, k, j) - S(n, k, m)) x_j \leq \sum_{j=m+1}^{m+k-1} (S(n, k, j) - S(n, k, m)) n. \tag{3}
\]

Therefore, combining (2) and (3), we have
\[
\sum_{i=0}^{n} \binom{n}{i} x_i \leq (k - 1)n \cdot S(n, k, m) + \sum_{j=m+1}^{m+k-1} (S(n, k, j) - S(n, k, m)) n
\]
\[
= \sum_{j=m+1}^{m+k-1} (S(n, k, j) - S(n, k, m) + S(n, k, m)) n
\]
\[
= n \cdot \sum_{j=m+1}^{m+k-1} S(n, k, j) = n \cdot (2^n - S(n, k, m)). \tag{4}
\]

Now, we use (4) to prove Theorem 2.1. If \(|\mathcal{F} \cap \{\emptyset, [n]\}| = 0\), then the statement of the theorem follows from (4). Then if \(|\mathcal{F} \cap \{\emptyset, [n]\}| = 1\), we may suppose that \(\emptyset \in \mathcal{F}\) and \([n] \notin \mathcal{F}\), and so \(x_0 = 1\) and \(x_n = 0\). Thus, it follows that
\[
n \cdot x_0 + \sum_{i=1}^{n-1} \binom{n}{i} x_i + n \cdot x_n = \sum_{i=0}^{n} \binom{n}{i} x_i + n - 1 \leq n \cdot (2^n - S(n, k, m)) + n - 1,
\]
as required.

If \(\emptyset \in \mathcal{F}\) and \([n] \in \mathcal{F}\), we have \(x_0 = 1\) and \(x_n = 1\). By Lemma 2.2, \(x_1 + x_2 + \cdots + x_k \leq (k - 1)n\) and \(x_{n-k} + x_{n-k+1} + \cdots + x_{n-1} \leq (k - 1)n\). According to these two constraints, we distinguish two subcases.

If \(x_1 + x_2 + \cdots + x_k \leq (k - 1)n - 1\) or \(x_{n-k} + x_{n-k+1} + \cdots + x_{n-1} \leq (k - 1)n - 1\), we may assume \(x_1 + x_2 + \cdots + x_k \leq (k - 1)n - 1\). Then in this case, (1) should be
\[
\sum_{i=1}^{n-k+1} w_i (x_i + x_{i+1} + \cdots + x_{i+k-1}) \leq (k - 1)n \cdot \sum_{i=0}^{n-k+1} w_i - w_1.
\]

By the assumption of Theorem 2.1, \(n > k\), so \(m \geq 1\) and \(w_1 = n - 1\). Then (4) should be
\[
\sum_{i=0}^{n} \binom{n}{i} x_i \leq n \cdot (2^n - S(n,k,m)) - (n - 1), \text{ and so}
\]
\[
n(x_0 + x_n) + \sum_{i=1}^{n-1} \binom{n}{i} x_i = (n - 1)(x_0 + x_n) + \sum_{i=0}^{n} \binom{n}{i} x_i
\]
\[
\leq 2(n - 1) + n \cdot (2^n - S(n,k,m)) - (n - 1)
\]
\[
= n \cdot (2^n - S(n,k,m)) + (n - 1).
\]

If \(x_1 + x_2 + \cdots + x_k = (k-1)n\) and \(x_{n-k} + x_{n-k+1} + \cdots + x_{n-1} = (k-1)n\), then by Lemma 2.3, \(x_0 + x_1 + \cdots + x_{k-1} \leq (k-1)n - \lfloor \frac{n}{2} \rfloor\) and \(x_{n-k+1} + x_{n-k+2} + \cdots + x_n \leq (k-1)n - \lfloor \frac{n}{2} \rfloor\). Similarly, one can modify (1) and (4) to show that
\[
n \cdot (x_0 + x_n) + \sum_{i=1}^{n-1} \binom{n}{i} x_i \leq 2(n - 1) + n \cdot (2^n - S(n,k,m)) - 2 \lfloor \frac{n}{2} \rfloor
\]
\[
\leq (n - 1) + n \cdot (2^n - S(n,k,m))
\]
by \(w_0 = w_{n-k+1} = 1\). This completes the proofs of Theorems 2.1 and 1.8.

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