THE ANTI-SPERHALIC HECKE CATEGORIES
FOR HERMITIAN SYMMETRIC PAIRS

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Abstract. We calculate the \( p \)-Kazhdan–Lusztig polynomials for Hermitian symmetric pairs and prove that the corresponding anti-spherical Hecke categories are standard Koszul. We prove that the combinatorial invariance conjecture can be lifted to the level of graded Morita equivalences between subquotients of these Hecke categories.

Introduction

Anti-spherical Hecke categories first rose to mathematical celebrity as the centrepiece of the proof of the Kazhdan–Lusztig positivity conjecture [EW14, LW]. Understanding the \( p \)-Kazhdan–Lusztig polynomials of these categories subsumes the problem of determining prime divisors of Fibonacci numbers [Wil17]; this is a notoriously difficult problem in number theory, for which a combinatorial solution is highly unlikely. As \( p \to \infty \) the situation simplifies and we encounter the classical Kazhdan–Lusztig polynomials; these are important combinatorial objects which can be calculated via a recursive algorithm. We seek to understand this gulf between the combinatorial and non-combinatorial realms within \( p \)-Kazhdan–Lusztig theory.

Over fields of infinite characteristic, the families of Kazhdan–Lusztig polynomials which are best understood combinatorially are those for Hermitian symmetric pairs, \( P \leq W \). These polynomials admit inexplicably simple combinatorial formulae in terms of Dyck paths or Temperley–Lieb diagrams [Boe88, Bre07, Bre09, BS11a, ES16a]. Their importance derives from their universality: these polynomials control the structure of parabolic Verma modules for Lie algebras [ES87]; algebraic supergroups [Bru03]; Khovanov arc algebras [BS12b, ES17]; Brauer and walled Brauer algebras [BS12a, ES16b, Mar15, CD11]; categories \( \mathcal{O} \) for Grassmannians [LS81, BS11b]; topological and algebraic Springer fibres, Slodowy slices, and \( W \)-algebras [ES16a].

Koszulity and \( p \)-Kazhdan–Lusztig theory. The first main result of this paper extends our understanding of the Kazhdan–Lusztig theory for Hermitian symmetric pairs to all fields.

**Theorem A.** Let \( k \) be a field of characteristic \( p \geq 0 \) and \( (W, P) \) a Hermitian symmetric pair. The Hecke category, \( \mathcal{H}_{(W, P)} \), is standard Koszul and the \( p \)-Kazhdan–Lusztig polynomials are \( p \)-independent and admit closed combinatorial interpretations.

This provides the first family of \( p \)-Kazhdan–Lusztig polynomials to be calculated since Williamson’s famous torsion explosion examples to the Lusztig conjecture [Wil17]. It is very unusual that our infinite families of \( p \)-Kazhdan–Lusztig polynomials are independent of \( p \geq 0 \). We believe that the Hermitian symmetric pairs \( (W, P) \) are the only infinite families of parabolic Coxeter systems for which the polynomials \( p n_{\lambda, \mu} \) are entirely independent of \( p \).

Koszulity of Lie theoretic objects is usually difficult to prove and it is an incredibly rare attribute over fields of characteristic \( p > 0 \). Our proof of Koszulity explicitly constructs linear projective resolutions of standard modules using the following theorem, which recasts the results of Enright–Shelton’s monograph [ES87] in the setting of Hecke categories and generalises their results to fields of positive characteristic. We hence make headway on the difficult problem of constructing singular Soergel diagrammatics.

**Theorem B.** Let \( \tau \in W \) for \( (W, P) \) a Hermitian symmetric pair. We explicitly construct the singular Hecke category \( \mathcal{H}^\tau_{(W, P)} \) as a subcategory of \( \mathcal{H}_{(W, P)} \). Furthermore, we prove that \( \mathcal{H}^\tau_{(W, P)} \) is isomorphic to the Hecke category of a Hermitian symmetric pair \( (W, P)^\tau \) of smaller rank.

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The combinatorial shadow of Theorem B is a graded bijection between paths in the smaller Bruhat graph of \((W, P)\) and paths in a truncation of the larger Bruhat graph of \((W, P)\). This can be categorified to the level of “dilation” homomorphisms between the anti-spherical Hecke categories. Sections 5 and 6 are dedicated to constructing these dilation maps and proving that they are indeed homomorphisms. We depict an example of the embedding of Bruhat graphs and the effect of the homomorphism on the fork generator in Figure 1.

Figure 1. On the left we depict the embedding of the Bruhat graph of \((A_3, A_1 \times A_1)\) into \((A_5, A_2 \times A_2)\). On the right we depict the corresponding dilation map on the fork generator. Here \(i\) is a primitive 4th root of 1. The tri-colouring 243 of single edges in the Bruhat graph and Soergel diagram comes from a single tricoloured node in the truncated Coxeter diagram.

**Tetris presentations and combinatorial invariance.** The graph automorphisms of Coxeter graphs of type \(A\) and \(D\) gives rise to fixed point subgroups of types \(B\) and \(C\). A crucial step in the proof of Theorem A is to lift this to the level of the corresponding Hecke categories of Hermitian symmetric pairs — thus categorifying an observation of Boe, namely that the Kazhdan–Lusztig polynomials for these pairs coincide. This is an example of Lusztig–Dyer–Marietti’s combinatorial invariance conjecture: which states that anti-spherical Kazhdan–Lusztig polynomials depend only on local isomorphisms of the Bruhat graphs. For Hermitian symmetric pairs, this conjecture was proven by Brenti [Bre09]. We categorify this result and generalise it to fields of positive characteristic as follows:

**Theorem C.** Let \(\Pi\) and \(\Pi’\) be subquotients of the Bruhat graphs of Hermitian symmetric pairs \((W, P)\) and \((W’, P’)\). If \(\Pi\) and \(\Pi’\) are isomorphic as partially ordered sets, then the corresponding subquotients \(\mathcal{H}_{\Pi}^{(W, P)}\) and \(\mathcal{H}_{\Pi’}^{(W’, P’)}\) are highest-weight graded Morita equivalent.

In order to prove Theorem C we must provide new, non-monoidal, presentations of the \(\mathcal{H}_{(W, P)}\) for \((W, P)\) a Hermitian symmetric pair, see Theorem 4.21. While monoidal presentations have many advantages, they are ill-equipped for tackling the combinatorial invariance conjecture. This is because monoidal presentations are “too local” and therefore cannot possibly hope to reflect the wider structure of the Bruhat graph. Defining these new presentations requires the full power of Soergel diagrammatics and the development of new “Tetris style” closed combinatorial formulas for manipulation of diagrams in \(\mathcal{H}_{(W, P)}\). This provides an extremely thorough understanding of these Hecke categories, and we expect that it will serve as a springboard for further combinatorial analysis of more general Hecke categories.

When \((W, P)\) and \((W’, P’)\) are both simply laced, the subquotients of Theorem B are actually isomorphic and this isomorphism can be constructed without reference to the combinatorics of light leaves bases. This gives hope that the richer structure of the Hecke category offers new tools for tackling the classical combinatorial invariance conjecture. Remarkable new advances in our understanding of combinatorial invariance for parabolic Coxeter systems have come from both mathematicians [Mar18, BLP] and even Google’s artificial intelligence [BBD+, DVB+21] — we pose our categorification of this conjecture in full generality in Conjecture 4.4.
1. The Hecke categories for Hermitian symmetric pairs

Let \((W, S_W)\) be a Coxeter system: \(W\) is the group generated by the finite set \(S_W\) subject to the relations \((s \tau)^{m_{s \tau}} = 1\) for \(s, \tau \in S_W\), \(m_{s \tau} \in \mathbb{N} \cup \{\infty\}\) satisfying \(m_{s \tau} = m_{\tau s}\) and \(m_{\tau \tau} = 1\) if and only if \(s = \tau\). Let \(\ell : W \to \mathbb{N}\) be the corresponding length function. Consider \(S_P \subseteq S_W\) a subset and \((P, S_P)\) its corresponding Coxeter system. We say that \(P\) is the parabolic subgroup corresponding to \(S_P \subseteq S_W\). Let \(P W \subseteq W\) denote a set of minimal coset representatives in \(P \backslash W\). For \(w = \sigma_1 \sigma_2 \cdots \sigma_t\) an expression, we define a subword to be a sequence \(t = (t_1, t_2, \ldots, t_\ell) \in \{0, 1\}^\ell\) and we set \(w^t := \sigma_1^{t_1} \sigma_2^{t_2} \cdots \sigma_\ell^{t_\ell}\). We let \(w \leq y\) if for some reduced expression \(w\) there exists a subword \(t\) and a reduced expression \(y\) such that \(w^t = y\). The Hasse diagram of this partial ordering is called the Bruhat graph of \((W, P)\). For the remainder of this paper we will assume that \(W\) is a Weyl group and indeed that \((W, P)\) is a Hermitian symmetric pair, which are classified as follows:

**Definition 1.1.** Let \(W\) be a finite Coxeter group and \(P\) a parabolic subgroup. Then \((W, P)\) is a Hermitian symmetric pair if and only if it is one of the following: \((A_n, A_{k-1} \times A_{n-k})\) with \(1 \leq k \leq n\), \((D_n, A_{n-1})\), \((D_n, D_{n-1})\), \((B_n, B_{n-1})\), \((C_n, A_{n-k})\) for \(n \geq 2\), \((E_6, D_5)\), or \((E_7, E_6)\).

In Figure 2, we recall the Dynkin diagrams of Hermitian symmetric pairs explicitly. For type \(D\), we use a slightly unusual labelling of nodes, which will allow us to pass between types \(C\) and \(D\) more easily. The remainder of this section is dedicated to the combinatorics of Hermitian symmetric pairs. This has been lifted from [EHP14, Appendix: diagrams of Hermitian type], but has been translated into a more diagrammatic language.

**Figure 2.** Enumeration of nodes in the parabolic Dynkin diagram of types of type \((A_n, A_{k-1} \times A_{n-k})\), \((C_n, A_{n-1})\) and \((B_n, B_{n-1})\), \((D_n, A_{n-1})\) and \((D_n, D_{n-1})\) and \((E_6, D_5)\) and \((E_7, E_6)\) respectively. The single node not belonging to the parabolic is highlighted in pink in each case.

1.1. Tile partitions. The Bruhat graphs of Hermitian symmetric pairs can be encapsulated in terms of tilings of certain admissible regions of the plane, which we now define. In type \((A_n, A_{k-1} \times A_{n-k})\), the admissible region is simply a \((k \times (n-k+1))\)-rectangle, and the tilings governing the combinatorics are Young diagrams which fit in this rectangle. The general picture is as follows:

**Definition 1.2.** Let \((W, P)\) be a Hermitian symmetric pair of classical type. We call a point \([r, c] \in \mathbb{N}^2\) a tile. The admissible region \(A_{(W, P)}\) is a certain finite subset of tiles defined as follows:
- for type \((W, P) = (A_n, A_{k-1} \times A_{n-k})\), the admissible region is the subset of tiles
  \(\{[r, c] \mid r \leq n-k+1, c \leq k\}\).
Given \( \lambda \),

- **Definition 1.4.** Let \( \lambda \) say that \( \circ \) an angle of 45°.

  - for types \((W, P) = (C_n, A_{n-1}) \text{ and } (D_n, A_{n-1})\), the admissible region is the subset of tiles
    \[ \{ [r, c] \mid r, c \leq n \text{ and } r - c \geq 0 \}. \]
  - for type \((W, P) = (B_n, B_{n-1})\), the admissible region is the subset of tiles
    \[ \{ [r, c] \mid r = 1 \text{ and } c < n \} \cup \{ [r, c] \mid c = n \text{ and } r \leq n \}. \]
  - for type \((W, P) = (D_n, D_{n-1})\), the admissible region is the subset of tiles
    \[ \{ [r, c] \mid r = 1 \text{ and } c < n \} \cup \{ [r, c] \mid c = n \text{ and } r \leq n \} \cup \{ [2, n-1] \}. \]

We draw tiles and admissible regions in the “Russian” style, with rows (i.e. fixed values of \( r \)) pointing northwest and columns (i.e. fixed values of \( c \)) pointing northeast.

**Example 1.3.** We illustrate the admissible region for type \((A_8, A_4 \times A_3)\) in Figure 3, for types \((C_6, A_5)\) and \((D_6, A_5)\) in Figure 4, and for types \((B_6, B_5)\) and \((D_7, D_6)\) in Figure 5. For the two exceptional types \((E_6, D_5)\) and \((E_7, E_6)\), the admissible region consists of the subset of tiles pictured in Figure 6.

![Figure 3](image-url)

Figure 3. On the left we picture the admissible region for \((A_8, A_4 \times A_3)\). We then picture two tilings; the first of which is a tile partition, but the latter is not (the tile \([2, 4]\) is not supported).

Each tile \([r, c] \in A_{W,P}\) carries a coloured label, inherited from the Dynkin diagram of \( W \). This is explained in detail in [EHP14, Appendix], but can be deduced easily from Figures 3 to 6. Given \([r, c] \in A_{W,P}\), we let \( s_{[r,c]} \) denote the corresponding simple reflection in \( S \). For example, in types \((A_5, A_{k-1} \times A_{n-k})\) and \((C_n, A_{n-1})\), the reflection \( s_{[r,c]} \) is determined simply by the \( x \)-coordinate of the tile \([r,c] \in A_{W,P}\) (i.e. it is determined by \( c - r \)).

We say that a pair of tiles are neighbours if they meet at an edge (which necessarily has an angle of 45° or 135°, by construction). Given a pair of neighbouring tiles \( X \) and \( Y \), we say that \( Y \) supports \( X \) if \( X \) appears above \( Y \). We say that a tile, \( X \in A_{W,P}\), is supported if every admissible tile which can support \( X \) does support \( X \). We say that a collection of tiles, \( \lambda \subseteq A_{W,P}\), is a tile-partition if every tile in \( \lambda \) is supported. We let \( P_{A_{W,P}} \) denote the set of all tile-partitions. We depict a tile-partition \( \lambda \) by colouring the tiles of \( \lambda \). See Figure 3 for examples and non-examples of tile-partitions.

We define the length of a tile-partition \( \lambda \) to be the total number of tiles \([r, c] \in \lambda \). There is a natural bijection between \( \hat{P}W \) and \( P_{A_{W,P}} \) (see [EHP14, Appendix]) under which the length functions coincide. For \( \lambda, \mu \in P_{A_{W,P}} \), we define the Bruhat order on tile partitions by \( \lambda \leq \mu \) if

\[ \{ [r, c] \mid [r, c] \in \lambda \} \subseteq \{ [r, c] \mid [r, c] \in \mu \}. \]

Given \( \lambda \leq \nu \), we define the skew tile-partition \( \nu \setminus \lambda \) to be the set difference of \( \lambda \) and \( \nu \).

**Definition 1.4.** Let \([r, c] \in A_{W,P}\) denote any tile. We define \( \lambda_{[r,c]} \) to be the tile partition

\[ \lambda_{[r,c]} := \{ [x, y] \in A_{W,P} \mid x \leq r, y \leq c \} \]
Remark 1.5. In type $(A_n, A_{k-1} \times A_{n-k})$ a tile-partition $\lambda$ is the Young diagram (in Russian notation) of a classical partition with at most $k$ columns and $(n-k+1)$ rows. In this case $\lambda_{[r,c]}$ is the $(r \times c)$-rectangle. In other types, it is this rectangle intersected with the region $\mathcal{A}(W,P)$.

1.2. Tile tableaux. The combinatorics of reduced and non-reduced words for Hermitian symmetric pairs can be encapsulated in terms of tile-paths or tile-tableaux, which we now define. Given $\mu \in \mathcal{P}(W,P)$, we define the set of all addable and removable tiles to be $\text{Add}(\mu) = \{[r,c] \mid \mu \cup [r,c] \in \mathcal{P}(W,P)\}$ and $\text{Rem}(\mu) = \{[r,c] \mid \mu \setminus [r,c] \in \mathcal{P}(W,P)\}$ respectively. Abusing notation, we will write $\mu + [r,c]$ for $\mu \cup [r,c]$ and $\mu - [r,c]$ for $\mu \setminus [r,c]$. Any tile-partition $\mu$ has at most one addable or removable tile of any given colour $\tau \in S_W$. Thus given $[r,c] \in \mathcal{A}(W,P)$ with $\tau = s_{[r,c]}$, we often write $\tau \in \text{Add}(\mu)$ or $\tau \in \text{Rem}(\mu)$ and we write $\mu \tau := \mu + [r,c]$ or $\mu - \tau := \mu - [r,c]$.

Definition 1.6. For $\lambda \in \mathcal{P}(W,P)$ we define a tile-tableau of length $\ell$ and shape $\lambda$ to be a path

$$T : \emptyset = \lambda_0 \xrightarrow{[r_1,c_1]} \lambda_1 \xrightarrow{[r_2,c_2]} \lambda_2 \xrightarrow{[r_3,c_3]} \cdots \xrightarrow{[r_{\ell},c_{\ell}]} \lambda_\ell = \lambda$$

such that for each $k = 1, \ldots, \ell$, $\lambda_k \in \mathcal{P}(W,P)$ and $\lambda_k$ satisfies one of the following
Given the set of all paths obtained in this manner.

We say that a tile-tableau coincides with the usual notion of standard Young tableaux.

Remark 1.7. In type \((A_n, A_{k-1} \times A_{n-k})\), the notions of addable and removable tiles correspond to the familiar notions of addable and removable boxes for Young diagrams. The set of reduced tile tableaux coincides with the usual notion of standard Young tableaux.

For \(\lambda \in \mathcal{P}_{(W,P)}^\ell\) we identify a reduced tableau \(t \in \text{Std}(\lambda)\) with a bijective map \(t : \lambda \rightarrow \{1, \ldots, \ell\}\) and we record this by placing the entry \(t^{-1}(k)\) in the \([r,c]\)th tile, in the usual manner. In this fashion, we can identify \(\text{Std}(\lambda)\) with the set of all possible fillings of \(\lambda\) with the numbers \(\{1, \ldots, \ell\}\) in such a way that these numbers increase along the rows and columns of \(\lambda\). For \(\nu \setminus \lambda\) a skew tile partition, we can define \(\text{Std}(\nu \setminus \lambda)\) in the obvious fashion. Given \(1 \leq k \leq \ell\), we let \(t|_{\{1,\ldots,k\}}\) denote the restriction of the map to the pre-image of \(\{1, \ldots, k\}\). Examples are depicted in Figure 7.

Definition 1.8. Let \(\lambda, \mu \in \mathcal{P}_{(W,P)}\) and fix \(t \in \text{Std}(\mu)\) such that \(t([x_k,y_k]) = k\) for \(1 \leq k \leq \ell\). We say that a tile-tableau

\[ T : \emptyset = \lambda_0 \xrightarrow{[r_1,c_1]} \lambda_1 \xrightarrow{[r_2,c_2]} \lambda_2 \xrightarrow{[r_3,c_3]} \cdots \xrightarrow{[r_\ell,c_\ell]} \lambda_\ell = \lambda \]

is obtained by folding-up \(t \in \text{Std}(\mu)\) if \(s_{[r_k,c_k]}(t([x_k,y_k])) = t([x_k,y_k])\) for \(1 \leq k \leq \ell\). We let \(\text{Path}(\lambda,t)\) denote the set of all paths obtained in this manner.

Definition 1.9. Given \(\lambda \in \mathcal{P}_{(W,P)}^\ell\), \(t \in \text{Std}(\lambda)\) and \([r,c] \in \text{Add}(\lambda)\), we let \(t \otimes \tau \in \text{Std}(\lambda \tau)\) denote the tableau uniquely determined by \((t \otimes \tau)[x,y] = t[x,y]\) for \([x,y] \neq [r,c]\).
1.3. Parity conditions for non-simply laced tiles. Finally, we are now ready to explain the existence of ± signs in Figures 4 and 5. For \( W \) a non-simply-laced Weyl group, we refine the colouring of the graph by dividing certain tiles into ±-tiles, depending on the parity condition demonstrated in Figures 4 and 5. Given \( \lambda, \mu \in \mathcal{P}(W, P) \) and \( s \in \text{Std}(\mu) \), we let \( \text{Path}^\pm(\lambda, s) \subseteq \text{Path}(\lambda, s) \) denote the subset of paths which preserve this parity condition. We will see that these paths provide bases of \( \mathcal{H}(W, P) \) later on in Theorem 4.21.

1.4. The diagrammatic Hecke categories. Almost everything from this section is lifted from Elias–Williamson’s original paper [EW16] or is an extension of their results to the parabolic setting [LW]. Let \((W, S)\) denote a Coxeter system for \( W \) a Weyl group. Given \( \sigma \in S_W \) we define the monochrome Soergel generators to be the framed graphs

\[ 1_\emptyset = \quad 1_\sigma = \quad \text{spot}^\emptyset = \quad \text{fork}^{\sigma\sigma} = \]

and given any \( \sigma, \tau \in S_W \) with \( m_{\sigma\tau} = m = 2, 3 \) or 4 we have the bi-chrome generator \( \text{braid}^{\sigma\tau}_{\sigma\tau}(m) \) which is pictured as follows

for \( m \) equal to 2, 3 or 4 respectively. (We will also sometimes write \( \text{braid}^\sigma_{\sigma\tau}, \text{braid}^{\sigma\tau}_{\sigma\tau}, \) and \( \text{braid}^{\sigma\tau\sigma\tau}_{\sigma\tau\sigma\tau} \) for \( \text{braid}^{\sigma\tau}_{\sigma\tau}(m) \) with \( m = 2, m = 3, \) and \( m = 4 \) respectively.) Pictorially, we define the duals of these generators to be the graphs obtained by reflection through their horizontal axes. Non-pictorially, we simply swap the sub- and superscripts. We sometimes denote duality by \( * \). For example, the dual of the fork generator is pictured as follows

\[ \text{fork}^{\sigma\sigma} = \]

We define the northern/southern reading word of a Soergel generator (or its dual) to be word in the alphabet \( S \) obtained by reading the colours of the northern/southern edge of the frame respectively. Given two (dual) Soergel generators \( D \) and \( D' \) we define \( D \otimes D' \) to be the diagram obtained by horizontal concatenation (and we extend this linearly). The northern/southern colour sequence of \( D \otimes D' \) is the concatenation of those of \( D \) and \( D' \) ordered from left to right. Given any two (dual) Soergel generators, we define their product \( D \circ D' \) (or simply \( DD' \)) to be the vertical concatenation of \( D \) on top of \( D' \) if the southern reading word of \( D \) is equal to the northern reading word of \( D' \) and zero otherwise. We define a Soergel graph to be any graph obtained by repeated horizontal and vertical concatenation of (dual) Soergel generators.

For \( w = \sigma_1 \ldots \sigma_\ell \) an expression, we define \( 1_w = 1_{\sigma_1} \otimes 1_{\sigma_2} \otimes \cdots \otimes 1_{\sigma_\ell} \) and given \( k > 1 \) and \( \sigma, \tau \in S_W \) we set \( 1_{\sigma\tau}^k = 1_{\sigma} \otimes 1_{\tau} \otimes 1_{\sigma} \otimes 1_{\tau} \ldots \) to be the alternately coloured idempotent on

![Figure 7. Tableaux of shape (4, 3, 2^2) and (5^2, 4, 3) \setminus (4, 3, 2^2) in type (A_8, A_4 \times A_3).](image-url)
Given $\sigma, \tau \in S_W$ with $m_{\sigma \tau} = 2$, let $w = \rho_1 \cdots \rho_k (\sigma \tau) \rho_{k+3} \cdots \rho_t$ and $w = \rho_1 \cdots \rho_k (\tau \sigma) \rho_{k+3} \cdots \rho_t$ be two reduced expressions for $w \in W$. We say that $w$ and $w$ are adjacent and we set
\[
\text{braid}^\sigma_w = 1_{\rho_1} \cdots 1_{\rho_k} \otimes \text{braid}^{\sigma \tau}_{\tau \sigma}(2) \otimes 1_{\rho_{k+3}} \cdots 1_{\rho_t}.
\]
Now, given a sequence of adjacent reduced expressions, $w = w^{(1)}, w^{(2)}, \ldots, w^{(q)} = w$ and the value $q$ is minimal such that this sequence exists, then we set
\[
\text{braid}^\sigma_w = \prod_{1 \leq p < q} \text{braid}^{w^{(p)}}_{w^{(p+1)}}.
\]
Given $\sigma$, we define the corresponding “barbell” and “gap” diagrams to be the elements
\[
\text{bar}(\sigma) = \text{spot}^\sigma_\emptyset \text{spot}^\sigma_\emptyset, \quad \text{gap}(\sigma) = \text{spot}^\sigma_\emptyset \text{spot}^\sigma_\emptyset,
\]
respectively. Given $\mu \in \mathcal{P}^\ell(W, P)$, $t \in \text{Std}(\mu)$ and a $\sigma$-tile $[r, c] \in \mu$ such that $t([r, c]) = k$, we set
\[
\text{gap}(t - \sigma) = 1_{t_k(1 \ldots k-1)} \otimes \text{gap}(\sigma) \otimes 1_{t_k(k+1 \ldots k)}.
\]
We also define the corresponding “double fork” diagram to be the element
\[
\text{dork}^{\sigma \tau}_{\rho \sigma} = \text{fork}^\sigma_{\rho \sigma} \text{fork}^{\rho \sigma}_{\sigma \tau}.
\]
It is standard (in Soergel diagrammatics) to draw the element $\text{cap}^\sigma_{\tau \sigma} := \text{spot}^\emptyset_\sigma \text{for}^\tau_{\tau \sigma}$ simply as a strand which starts and ends on the southern edge of the frame. (We define $\text{cup}^\sigma_{\rho \sigma}$.)

For $\underline{w} = \sigma_1 \sigma_2 \cdots \sigma_t$ a word, we define $\underline{w}_{\text{rev}} = \sigma_t \cdots \sigma_2 \sigma_1$. Then we inductively define
\[
\text{cap}^\emptyset_{\underline{w}} := \text{cap}^\emptyset_{\underline{w}_{\text{rev}}} (1_{\underline{w}} \otimes \text{cap}^\emptyset_{\sigma_t \sigma_{t-1}} \otimes 1_{\underline{w}_{\text{rev}}}^{-1}),
\]
where $y = \sigma_1 \sigma_2 \cdots \sigma_{t-1}$. This diagram can be visualised as a rainbow of concentric arcs (with $\text{cap}^\emptyset_{\sigma_t \sigma_{t-1}}$ the innermost arc). Since we have $\underline{w}_{\text{rev}} = 1_W$ when evaluated in the group $W$, we will simply write $\text{cap}^\emptyset_{\underline{w}^{-1}}$ for $\text{cap}^\emptyset_{\underline{w}_{\text{rev}}}^{-1}$.

In order to make our notation less dense, we will often suppress mention of idempotents by including them in the sub- and super-scripts of other generators. This is made possible by recording where the edits to the underlying words are with emptysets. For example
\[
\text{spot}^\emptyset_{\alpha \beta \tau} := \text{spot}^\emptyset_\alpha \otimes \text{spot}^\emptyset_\beta \otimes \text{spot}^\emptyset_\tau, \quad \text{fork}^\gamma_{\gamma \beta \alpha \tau} := \text{fork}^\gamma_{\gamma \beta \alpha} \otimes \text{fork}^\alpha_{\alpha \tau},
\]
where $\gamma = \alpha \beta \gamma \beta \alpha \tau$. We make use of all of the above notational shorthands (even within the same equation).

Finally, for distinct $\sigma, \tau \in S_W$ we recall the entries of the Cartan matrix corresponding to the Dynkin diagram of $(W, P)$:
\[
\langle \alpha_\sigma, \alpha_\tau \rangle=
\begin{cases} 
0 & \text{if } \sigma \not\rightarrow \tau, \\
-1 & \text{if } \sigma \not\leftarrow \tau, \\
-1 & \text{if } \sigma \not\Rightarrow \tau, \\
-2 & \text{if } \sigma \not\Leftarrow \tau.
\end{cases}
\]

**Definition 1.10.** Let $W$ be a Weyl group and $P$ be a parabolic subgroup. We define $\mathcal{H}_{(W, P)}$ to be the locally-unital associative $k$-algebra spanned by all Soergel-graphs with multiplication given by $\circ$-concatenation modulo the following local relations and their vertical and horizontal flips.

\[
\begin{align*}
1_\sigma 1_\tau &= \delta_{\sigma, \tau} 1_{\sigma} \quad & 1_\emptyset 1_\sigma &= 0 \quad & 1_\emptyset^2 &= 1_\emptyset \tag{1.1} \\
1_\emptyset \text{spot}^\emptyset_\emptyset 1_\sigma &= \text{spot}^\emptyset_\emptyset \quad & 1_\sigma \text{fork}^{\sigma \tau}_{\tau \sigma} 1_\sigma &= \text{fork}^{\sigma \tau}_{\sigma \tau} \quad & 1_\sigma^m \text{braid}^{\tau \sigma}_{\sigma \tau}(m) 1_\sigma^m &= \text{braid}^{\sigma \tau}_{\tau \sigma}(m) \tag{1.2}
\end{align*}
\]

For each $\sigma \in S$, we have the fork-spot, double-fork, circle-annihilation, and one-colour barbell relations
\[
\begin{align*}
(\text{spot}^\emptyset_\sigma \otimes 1_\sigma) \text{fork}^{\sigma \tau}_{\rho \sigma} &= 1_\sigma \quad & (1_\sigma \otimes \text{fork}^{\sigma \tau}_{\rho \sigma}) (\text{fork}^{\sigma \tau}_{\rho \sigma} \otimes 1_\sigma) &= \text{fork}^{\sigma \tau}_{\rho \sigma} \text{fork}^{\rho \tau}_{\sigma \rho} \tag{1.3} \\
\text{fork}^{\rho \sigma}_{\sigma \sigma} \text{fork}^{\sigma \tau}_{\rho \sigma} &= 0 \quad & \text{bar}(\sigma) \otimes 1_\sigma + 1_\sigma \otimes \text{bar}(\sigma) &= 2 \text{gap}(\sigma) \tag{1.4}
\end{align*}
\]
For every ordered pair \((\sigma, \tau) \in S^2_W\) with \(\sigma \neq \tau\), the bi-chrome relations: The \(\sigma \tau\)-barbell,
\[
\text{bar}(\tau) \otimes 1_\sigma - 1_\sigma \otimes \text{bar}(\tau) = \langle \alpha^\vee_\sigma, \alpha_\tau \rangle (\text{gap}(\sigma) - 1_\sigma \otimes \text{bar}(\sigma)).
\] (1.5)

For \(m = m_{\sigma \tau} \in \{2, 3, 4\}\) we also have the fork-braid relations
\[
\text{braid}_m^{\sigma \tau \tau \tau}(\text{fork}_m^{\sigma \tau} \otimes 1_m) (1_\sigma \otimes \text{braid}_m^{\tau \tau \tau \tau} (\text{fork}_m^{\sigma \tau} \otimes 1_{\tau} \otimes 1_\tau)) = (1_m \otimes \text{fork}_m^{\sigma \tau} \otimes 1_{\tau} \otimes 1_\tau) (\text{braid}_m^{\tau \tau \tau \tau} \otimes 1_m)
\]
for odd and even, respectively. We require the cyclicity relation,
\[
(1_m \otimes (\text{cap}^0_{\sigma \tau})) (1_\sigma \otimes \text{braid}_m^{\tau \tau \tau \tau} (\text{fork}_m^{\sigma \tau} \otimes 1_m)) (\text{cup}^0_{\tau} \otimes 1_m) = \text{braid}_m^{\tau \tau \tau \tau \tau \tau \tau \tau} \otimes 1_m
\]
\[
(1_m \otimes (\text{cap}^0_{\sigma \tau})) (1_\sigma \otimes \text{braid}_m^{\tau \tau \tau \tau} (\text{fork}_m^{\sigma \tau} \otimes 1_m)) (\text{cup}^0_{\tau} \otimes 1_m) = \text{braid}_m^{\tau \tau \tau \tau \tau \tau \tau \tau} \otimes 1_m
\]
for odd or even, respectively. For \(m = 2, 3, 4\) we have the double-braid relations\(^1\)
\[
1_{\sigma \tau} = \text{braid}_m^{\tau \tau \tau \tau} \text{braid}_m^{\tau \tau \tau \tau} - \text{spot}_m^{\sigma \tau} \text{fork}_m^{\sigma \tau} \text{fork}_m^{\tau \tau \tau \tau} \text{spot}_m^{\tau \tau \tau \tau} \text{spot}_m^{\sigma \tau}
\] (1.6)
\[
1_{\sigma \tau \tau \tau \tau} = \text{braid}_m^{\tau \tau \tau \tau \tau \tau \tau \tau} + (\alpha^\vee_\sigma, \alpha_\tau) \text{spot}_m^{\sigma \tau \tau \tau \tau \tau \tau \tau} \text{fork}_m^{\sigma \tau \tau \tau \tau \tau \tau \tau} \text{fork}_m^{\tau \tau \tau \tau \tau \tau \tau \tau} \text{spot}_m^{\tau \tau \tau \tau \tau \tau \tau \tau} \text{spot}_m^{\sigma \tau \tau \tau \tau \tau \tau \tau}
\]
\[
- \text{spot}_m^{\sigma \tau \tau \tau \tau \tau \tau \tau} \text{fork}_m^{\sigma \tau \tau \tau \tau \tau \tau \tau} \text{fork}_m^{\tau \tau \tau \tau \tau \tau \tau \tau} \text{spot}_m^{\tau \tau \tau \tau \tau \tau \tau \tau} \text{spot}_m^{\sigma \tau \tau \tau \tau \tau \tau \tau}
\] (1.7)
respectively. For \((\sigma, \tau, \rho) \in S_W \) with \(m_{\sigma \rho} = m_{\rho \tau} = 2\) and \(m_{\sigma \tau} = m\), we have
\[
(\text{braid}_m^{\sigma \tau \tau \tau \tau \tau \tau \tau \tau} \otimes 1_{\rho}) (\text{braid}_m^{\sigma \tau \tau \tau \tau \tau \tau \tau \tau} \otimes 1_{\rho}) = \text{braid}_m^{\sigma \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \tau \ta
2. Lifting full-commutativity to the Hecke categories of Hermitian symmetric pairs

Stembridge proved that the parabolic quotients for Hermitian symmetric pairs are fully commutative [Ste96, Theorem 6.1]. In other words, in $\mathcal{H}(W,P)$ the non-commuting braid relations are redundant. (This is discussed in more detail in terms of Temperley–Lieb diagrammatics in the companion paper [BDF+].) We now lift this idea to the 2-categorical level; in $\mathcal{H}(W,P)$ the non-commuting braid generators are redundant.

**Theorem 2.1.** Let $w \in \mathcal{H}(W,P)$ and let $w$ be any expression for $w$. In $\mathcal{H}(W,P)$ we have that

$$1_w \otimes \text{braid}_{\tau,\sigma}^P(m) = 0$$

for any $m = m(\sigma, \tau) > 2$.

Before proceeding, we first record a useful relation which we will use in what follows. For $\sigma \in S$, we have that

$$w = \begin{array}{c|c|c}
\begin{array}{c}
\begin{array}{c}
= \\
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
+ \\
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
- \\
\end{array}
\end{array}
\end{array} = \begin{array}{c|c|c}
\begin{array}{c}
\begin{array}{c}
= \\
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
+ \\
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
- \\
\end{array}
\end{array}
\end{array} - 2$$

which can be found (with a proof) in [EW16, Equation 5.15]. We also record the following useful observation.

**Proposition 2.2.** Let $w \in \mathcal{H}(W,P)$ and let $w, w'$ be a pair of expressions for $w$. We have that

$$1_w = \text{braid}_{\tau,\sigma}^w, \quad 1_w = \text{braid}_{\tau,\sigma}^w$$

**Proof.** For $(W,P)$ a Hermitian symmetric pair, [Ste96, Theorem 6.1] implies that any two words $w$ and $w'$ in $\mathcal{H}(W,P)$ differ only by application of the commuting braid relations of the Coxeter group. These lift to commuting braid generators in $\mathcal{H}(W,P)$ and the result follows. \hfill \square

We are now ready to prove the main result of this section.

**Proof of Theorem 2.1.** We proceed by induction on the length of the expression, $\ell = \ell(w)$. By [LW, Theorem 5.3], it is enough to consider reduced expressions $w$. If $\ell(w) = 0$, then $\text{braid}_{\tau,\sigma}^w(m) = 0$ for all $m \geq 2$ by relation 1.11. In what follows, we only explicitly consider the cases for which neither of $1_w \otimes 1_\sigma$ or $1_w \otimes 1_\tau$ is equal to zero by application of the commutativity and cyclotomic relations (as these cases are trivial). Now assume that $\ell(w) \geq 1$. We have two cases to consider.

**Case 1.** Either $w = x_\sigma$ or $w = x_\tau$ for some $x$ a reduced expression of some $x \in \mathcal{H}(W,P)$. We consider the former case, as the latter is identical. Now, we have that

$$1_x \otimes \text{braid}_{\tau,\sigma}^x(m)$$

$$= 1_x \otimes (1_\sigma \otimes 1_\tau \otimes 1_{\tau,\sigma}^{\sigma,\tau})$$

$$= 1_x \otimes (1_\sigma \otimes 1_\tau \otimes 1_{\tau,\sigma}^{\sigma,\tau})$$

The first term is zero by induction, since it factors through $1_x \otimes \text{braid}_{\tau,\sigma}^x(m)$ with $\ell(x) = \ell(w) - 1$. The other two terms factor through a diagram of the form

$$1_x \otimes (\text{fork}_{\tau,\sigma} \otimes 1_{\tau,\sigma}^{\sigma,\tau})$$

we apply the fork-braid relation to obtain

$$1_x \otimes \text{braid}_{\tau,\sigma}^x(1_{\tau,\sigma} \otimes \text{fork}_{\tau,\sigma}^x)(\text{braid}_{\tau,\sigma}^x \otimes 1_\tau)$$

for $m = 3$ or 4 respectively. In both cases, this is zero by induction.
Case 2. It remains to consider the case that \( w \neq x\sigma \) and \( w \neq x\tau \) for a reduced word \( x \) of some \( x \in P\mathcal{W} \). We first note that if \( w = x\rho \) with \( m(\sigma, \rho) = 2 = m(\tau, \rho) \) then we can choose a reduced word \( w \) such that \( w = x\rho \) and by induction \( 1_x \otimes \text{braid}^\rho_{\tau, \sigma}(m) = 0 \) and this implies (by the commutativity relations) that \( 1_w \otimes \text{braid}^\rho_{\tau, \sigma}(m) = 0 \), as required. Thus if \( w = x\rho \) for some reduced word \( x \) of some \( x \in P\mathcal{W} \), we can assume that \( \rho \neq \tau, \sigma \) and that either \( m(\sigma, \rho) > 2 \) or \( m(\rho, \tau) > 2 \). We can assume that \( w \) can be written in the form \( w = x\rho \) with \( m(\sigma, \rho) > 2 \), as the other case is identical.

**Subcase when \( \ell(w\sigma) < \ell(w) \).** We first consider the “generic” case in which \( \ell(w\sigma) < \ell(w) \). In which case, there exists a reduced word \( w \) of \( w \) such that

\[
   w = \begin{cases} 
      x\rho \sigma \rho \sigma & \text{if } m(\rho, \sigma) = 3 \\
      x\rho \sigma \rho \sigma \rho \sigma & \text{if } m(\rho, \sigma) = 4
   \end{cases}
\]

(2.2)

this covers all instances of types \( (A, A \times A) \), \( (C, A) \), and \( (D, A) \) cases and most other examples.

![Figure 8](image.png)

**Figure 8.** Some examples of the “exceptional” examples for which \( \ell(w\tau) < \ell(w) \) in Case 2 of the proof of Figure 8. The elements \( u \leq w \) depicted in grey and yellow, respectively. In the first case \( (\tau, \sigma) = (s_1, s_2) \), in the second case \( (\tau, \sigma) = (s_6, s_3) \), and in the third case \( (\tau, \sigma) = (s_5, s_4) \).

We consider the first generic case in which \( m(\rho, \sigma) = 3 \) (and \( m = m(\tau, \sigma) > 2 \)). We have

\[
   1_x \otimes 1_{\sigma, \rho} \otimes \text{braid}^\tau_{\sigma, \sigma}(m)
\]

\[
   = 1_x \otimes (\text{braid}^\rho_{\tau, \rho} \otimes 1^{m-1}_{\tau, \sigma} - \text{spot}_{\sigma, \rho} \text{spot}_{\rho, \sigma} \text{spot}_{\rho, \sigma})(1_{\sigma, \rho} \otimes \text{braid}^\tau_{\sigma, \sigma}(m)) \quad (2.3)
\]

using the double-braid relation. We now observe that

\[
   1_x \otimes \text{braid}^\rho_{\tau, \rho} \otimes 1^{m-1}_{\tau, \sigma} = 0 \\
   1_x \otimes 1_{\sigma} \otimes \text{spot}_{\rho} \otimes \text{braid}^\tau_{\sigma, \sigma}(m) = 0
\]

by induction (since \( \ell(x) \) and \( \ell(x\sigma) \) are both smaller than \( \ell(w) \)) and so both terms in equation (2.3) are zero, as required. We now consider the second generic case in which \( m(\rho, \sigma) = 4 \) (which implies that \( m = m(\tau, \sigma) = 3 \)). We have that

\[
   1_x \otimes 1_{\rho, \sigma} \otimes \text{braid}^\tau_{\sigma, \sigma}(m)
\]

\[
   = 1_x \otimes (\text{braid}^\rho_{\tau, \rho} \otimes \text{braid}^\rho_{\rho, \sigma} + (\alpha^\vee, \alpha) \text{spot}_{\rho, \sigma} \text{fork}_{\rho, \sigma} \text{fork}_{\rho, \sigma} \text{spot}_{\rho, \sigma})(1_{\rho, \sigma} \otimes \text{braid}^\tau_{\sigma, \sigma}(m))
\]

(2.3)

and all of these terms are zero by induction on length (similarly to the \( m(\rho, \sigma) = 3 \) case, above).

**Subcase when \( \ell(w\tau) < \ell(w) \).** If \( \ell(w\sigma) > \ell(w) \), this implies that \( \ell(w\tau) < \ell(w) \). It remains to consider this case. We remark that our assumptions on \( w \) and the fact that \( \ell(w\sigma) > \ell(w) \) implies that we must be in one of types \( (D_n, D_{n-1}) \), \( (B_n, B_{n-1}) \) and exceptional type and so we refer to this as the “exceptional” case. In this case, we can write \( w = uv \) where \( v \) is of maximal length such that \( v = s_i \ldots s_l \) and \( m(s_{ik}, \sigma) = 2 \) for all \( 1 \leq k \leq L \) and there exists a word \( u \) satisfying the conditions of equation (2.2). This case can be handled in an identical
fashion to the above, except that one must also apply the commutation relations in order to move strands past the idempotent $1_v$. Rather than go into the word combinatorics in detail (given that the only algebraic difference is the application of commutation relations) we simply provide illustrative examples in Figure 8 and leave this as an exercise for the reader. □

We now state the obvious corollary, for ease of reference.

**Definition 2.3.** We define a simple Soergel graph to be any Soergel graph which does not contain any barbells or braid $\sigma \tau^m$ for $m = m(\sigma, \tau) > 2$.

**Corollary 2.4.** Let $(W, P)$ be a Hermitian symmetric pair. We can define $H_{(W, P)}$ to be the locally-unital associative $k$-algebra spanned by all simple Soergel diagrams with multiplication given by vertical concatenation of diagrams modulo relations 1.1, 1.2, 1.3, 1.4, 1.5, 1.8, 1.9, 1.10, 1.11, for $(\sigma, \tau, \rho) \in S^3$ with $m(\sigma \rho) = m(\rho \tau) = m(\sigma \tau) = 2$, we have the commutation relations

$$\begin{align*}
\text{\begin{tikzpicture}[scale=0.3]
\draw[thick, dotted] (-1,0) -- (1,0);
\draw[thick, dotted] (-1,-2) -- (1,-2);
\draw (-1,0) to[out=90,in=180] (-1,-1);
\draw (-1,-2) to[out=90,in=0] (-1,-1);
\end{tikzpicture}} + \begin{tikzpicture}[scale=0.3]
\draw[thick, dotted] (-1,0) -- (1,0);
\draw[thick, dotted] (-1,-2) -- (1,-2);
\draw (-1,0) to[out=90,in=0] (-1,-1);
\draw (-1,-2) to[out=90,in=180] (-1,-1);
\end{tikzpicture} = 0
\end{align*}$$

and for $m(\sigma, \tau) = 3$, we have the null-braid relation

$$\begin{align*}
\text{\begin{tikzpicture}[scale=0.3]
\draw[thick, dotted] (-1,0) -- (1,0);
\draw[thick, dotted] (-1,-2) -- (1,-2);
\draw (-1,0) to[out=90,in=0] (-1,-1);
\draw (-1,-2) to[out=90,in=180] (-1,-1);
\end{tikzpicture}} + \begin{tikzpicture}[scale=0.3]
\draw[thick, dotted] (-1,0) -- (1,0);
\draw[thick, dotted] (-1,-2) -- (1,-2);
\draw (-1,0) to[out=90,in=180] (-1,-1);
\draw (-1,-2) to[out=90,in=0] (-1,-1);
\end{tikzpicture} + \begin{tikzpicture}[scale=0.3]
\draw[thick, dotted] (-1,0) -- (1,0);
\draw[thick, dotted] (-1,-2) -- (1,-2);
\draw (-1,0) to[out=90,in=0] (-1,-1);
\draw (-1,-2) to[out=90,in=180] (-1,-1);
\end{tikzpicture} + \begin{tikzpicture}[scale=0.3]
\draw[thick, dotted] (-1,0) -- (1,0);
\draw[thick, dotted] (-1,-2) -- (1,-2);
\draw (-1,0) to[out=90,in=180] (-1,-1);
\draw (-1,-2) to[out=90,in=0] (-1,-1);
\end{tikzpicture} = 0
\end{align*}$$

and their horizontal flips.

3. Light leaves for the Hecke categories of Hermitian symmetric pairs

In this section, we recall Libedinsky–Williamson’s construction of the light leaves basis in the case of Hermitian symmetric pairs. This could have been done in Subsection 1.4, however we delayed until now so that we could simplify the presentation of this material by virtue of Theorem 2.1. We regard $H_{(W, P)}$ as a locally unital associative algebra in the sense of [BS17, Section 2.2] via the following idempotent decomposition

$$H_{(W, P)} = \bigoplus_{x \in \text{exp}(x)} \bigoplus_{y \in \text{exp}(y)} x_1 H_{(W, P)} y_1.$$

**Remark 3.1.** Given $s, t \in \text{Std}(\lambda)$, by Proposition 2.2 we have that

$$\text{braid}_s^t \circ 1_s \circ \text{braid}_t^s = 1_s \quad \text{braid}_t^s \circ 1_s \circ \text{braid}_s^t = 1_t$$

Thus from now on, we may fix any preferred choice of $t_\lambda \in \text{Std}(\lambda)$, for each $\lambda \in \mathcal{P}_{(W, P)}$.

By Remark 3.1, we can truncate the set of weights to be “as small as possible”.

**Definition 3.2.** We set

$$1_{(W, P)} = \sum_{\mu \in \mathcal{P}_{(W, P)}} 1_{1_\mu} \quad \text{and} \quad h_{(W, P)} = 1_{(W, P)} H_{(W, P)} 1_{(W, P)}.$$

Let $\lambda, \mu \in \mathcal{P}_{(W, P)}$ and $T \in \text{Path}_\ell(\lambda, t_\mu)$ be a path of the form

$$T : \emptyset = \lambda_0 \to \lambda_1 \to \lambda_2 \to \cdots \to \lambda_\ell = \lambda$$
and we let $\tau \in \text{Add}(\mu)$. If $\tau \in \text{Add}(\lambda)$, we set $\lambda^+ = \lambda + \tau$ and $\lambda^- = \lambda$. If $\tau \in \text{Rem}(\lambda)$, we set $\lambda^+ = \lambda$ and $\lambda^- = \lambda - \tau$. We set $T^+$ and $T^-$ to be the paths

\[
T^+ : \emptyset \to \lambda_0 \to \lambda_1 \to \lambda_2 \to \cdots \to \lambda_\ell \to \lambda^+ \quad T^- : \emptyset \to \lambda_0 \to \lambda_1 \to \lambda_2 \to \cdots \to \lambda_\ell \to \lambda^-.
\]

We now inductively define the basis via certain “add” and “remove” operators. If $\tau \in \text{Add}(\lambda)$, then we define

\[
c_{T^+} = A^+_{\tau}(c_T) = \text{braid}^{\lambda+\tau}_{\lambda \otimes \tau}(c_T \otimes 1_{\tau}) \\
c_{T^-} = A^-_{\tau}(c_T) = c_T \otimes \text{spot}_{\tau}^0.
\]

If $\tau \in \text{Rem}(\lambda)$, then we define $\lambda = \lambda - \tau$ and we define

\[
c_{T^+} = R^+_{\tau}(c_T) = \text{braid}^{\lambda+\tau}_{\lambda \otimes \tau}(1_{\tau} \otimes \text{fork}^{\tau}_{\tau \otimes \tau})(\text{braid}^{\lambda+\tau}_{\lambda \otimes \tau} c_T \otimes 1_{\tau}) \\
c_{T^-} = R^-_{\tau}(c_T) = (1_{\tau} \otimes \text{cap}^{\tau}_{\tau})(\text{braid}^{\lambda+\tau}_{\lambda \otimes \tau} c_T \otimes 1_{\tau}).
\]

An example is given in the rightmost diagram in Figure 9.

**Theorem 3.3** ([EW16, Section 6.4] and [LW, Theorem 7.3]). The algebra $h_{(W,P)}$ is quasi-hereditary with graded cellular basis

\[
\{c_{ST}^\lambda : S \in \text{Path}_{(W,P)}(\lambda, t_\mu), T \in \text{Path}_{(W,P)}(\lambda, t_\nu), \lambda, \mu, \nu \in \mathcal{P}_{(W,P)}\}
\]

with respect to the Bruhat ordering on $\mathcal{P}_{(W,P)}$ and anti-involution $\ast$.

When it cannot result in confusion, we write $c_{ST}$ for $c_{ST}^\lambda$. For $\mu \in \mathcal{P}_{(W,P)}$, we define one-sided ideals

\[
h_{(W,P)}^{<\mu} = 1_{\leq \mu} h_{(W,P)} \\
h_{(W,P)}^{\leq \mu} = h_{(W,P)}^{<\mu} \cap \mathbb{k}\{c_{ST}^\lambda : S, T \in \text{Path}(\lambda), \lambda < \mu\}
\]

and we hence define the standard modules of

\[
\Delta(\mu) = \{c_{ST}^\mu + h_{(W,P)}^{<\mu} : S \in \text{Path}(\mu, t_\nu), \nu \in \mathcal{P}_{(W,P)}\}.
\]

We recall that the cellular structure allows us to define, for each $\mu \in \mathcal{P}_{(W,P)}$, a bilinear form $\langle \ , \ \rangle^\mu$ on $\Delta(\mu)$ which is determined by

\[
c_{ST}^\mu c_{UV}^\nu \equiv \langle c_T, c_U \rangle^\nu c_{SV}^\mu \pmod{h_{(W,P)}^{<\mu}}
\]

for any $S, T, U, V \in \text{Path}(\mu, -)$. When $\mathbb{k}$ is a field, we obtain a complete set of non-isomorphic simple modules for $h_{(W,P)}$ as follows

\[
L(\mu) = \Delta(\mu)/\text{rad}(\langle \ , \ \rangle^\mu)
\]

for $\mu \in \mathcal{P}_{(W,P)}$. The projective indecomposable $h_{(W,P)}$-modules are the direct summands

\[
1_{t_\mu} h_{(W,P)} = \bigoplus_{\lambda \leq \mu} \dim_{\mathbb{k}}(L(\lambda)1_{t_\mu})P(\lambda).
\]

The anti-spherical $p$-Kazhdan–Lusztig polynomials are defined as follows,

\[
p_{\lambda, \mu}(q) := \dim_{\mathbb{k}}(\text{Hom}_{h_{(W,P)}}(P(\lambda), \Delta(\mu))) = \sum_{k \in \mathbb{Z}} |\Delta(\mu) : L(\lambda)(k)| q^k
\]

for any $\lambda, \mu \in \mathcal{P}_{(W,P)}$. These polynomials were first defined via the diagrammatic character of [EW16, Definition 6.23] and [LW, Section 8] and rephrased as above in [Pla17, Theorem 4.8].
Remark 3.4. We note that $H_{(W,P)}$ and $h_{(W,P)}$ are graded Morita equivalent (as the latter is obtained from the former by a truncation which does not kill any simple module).

3.1. A “singular” horizontal concatenation. Singular Soergel bimodules were first considered in Williamson’s thesis, where it was proven that they categorify the Hecke algebroid [Wil11]. At present, we do not have a diagrammatic construction of the category of singular Soergel bimodules (although some progress has been made, see [EMTW20, Chapter 24]). In this paper, we will give a complete realisation of singular Soergel bimodules within the diagrammatic Hecke category for $(W, P)$ a Hermitian symmetric pair. In order to accomplish this goal, we first need to provide a Soergel-diagrammatic analogue of the tensor product (denoted $\otimes_R$) for singular Soergel bimodules “on a $\tau$-hyperplane” for $\tau \in SW$ (which we will denote by $\boxdot$).

Definition 3.5. We suppose that a diagram $D \in 1_{\mathcal{H}}(W,P)1_{\mathcal{Y}}$ is such that (i) the rightmost $\tau$-strand in the northern boundary is connected to the rightmost $\tau$ in the southern boundary by a strand and (ii) there are no barbells to the right of this strand. We say that such a diagram has a final exposed propagating $\tau$-strand. Similarly, we define a first exposed propagating $\tau$-strand by reflecting this definition through the vertical axis.

Definition 3.6. We let $D_1 \in 1_{\mathcal{H}}(W,P)1_{\mathcal{Y}}$ and $D_2 \in 1_{\mathcal{H}}(W,P)1_{\mathcal{Y}}$. We suppose that $D_1$ (respectively $D_2$) has a final (respectively first) $\tau$-propagating strand. We define

$$D_1 \boxdot D_2 = (D_1 \otimes 1_{\mathcal{Y}})(1_{\mathcal{Y}} \otimes D_2) = (1_{\mathcal{Y}} \otimes D_2)(D_1 \otimes 1_{\mathcal{Y}}) \quad (3.4)$$

Now suppose that $D_1' = \text{braid}_{\mathcal{Y}}^{-1}D_1\text{braid}_{\mathcal{Y}}^{-1} \neq 0$. We extend the above definition as follows

$$D_1' \boxdot D_2 = (\text{braid}_{\mathcal{Y}}^{-1} \otimes 1_{\mathcal{Y}})(D_1 \otimes D_2)(\text{braid}_{\mathcal{Y}}^{-1} \otimes 1_{\mathcal{Y}}). \quad (3.5)$$

![Figure 10. An example of $\boxdot$ merging the rightmost and leftmost $\tau$-strands.](image)

Remark 3.7. Diagrammatically, we can think of $\boxdot$ as identifying the rightmost $\tau$-strand of $D_1'$ with the leftmost $\tau$-strand of $D_2$. For examples, see Figures 9, 10 and 44.

Proposition 3.8. The operation $\boxdot$ satisfies the interchange law:

$$(D_1 \boxdot D_2) \circ (D_3 \boxdot D_4) = (D_1 \circ D_3) \boxdot (D_2 \circ D_4).$$

Proof. For equation (3.4), the result follows immediately by diagram chasing using the fact that

$$(D_1 \otimes 1_{\mathcal{Y}})(1_{\mathcal{Y}} \otimes D_2) = (1_{\mathcal{Y}} \otimes D_2)(D_1 \otimes 1_{\mathcal{Y}}) \quad (3.6)$$

The case of equation (3.5) follows by applying commuting braid generators.

We will abuse notation and use $\boxdot$ as a shorthand as follows.

Definition 3.9. We let $D_1 \in 1_{\mathcal{H}}(W,P)1_{\mathcal{Y}}$ and $D_2 \in 1_{\mathcal{H}}(W,P)1_{\mathcal{Y}}$. We suppose that $D_1$ (respectively $D_2$) has a final (respectively first) $\tau$-propagating strand. We define

$$D_1 \boxdot (\text{spot}_\tau \otimes 1_{\mathcal{Y}})D_2 = (1_{\mathcal{Y}} \otimes \text{spot}_\tau \otimes 1_{\mathcal{Y}})(D_1 \boxdot D_2). \quad (3.7)$$

We extend this via commuting braid generators in an analogous fashion to equation (3.5).

Remark 3.10. We note that the operation in equation (3.7) considers non-propagating $\tau$-strands and therefore does not satisfy the interchange law (as equation (3.6) no longer holds).
4. The categorical combinatorial invariance conjecture

We now propose a categorical lift of the famous combinatorial invariance conjecture and we prove this statement for Hermitian symmetric pairs (thus categorifying Brenti’s result for the Kazhdan–Lusztig polynomials [Bre09, Corollary 5.2]). In order to do this we must provide new, non-monoidal, presentations of the anti-spherical Hecke categories \( \mathcal{H}_{(W,P)} \) for \( (W,P) \) a simply laced Hermitian symmetric pair. While monoidal presentations have many advantages, they are ill-equipped for tackling the combinatorial invariance conjecture. This is because monoidal presentations are “too local” and they cannot reflect the structure of the Bruhat graph.

First, we let \( (W,P) \) be an arbitrary parabolic Coxeter system and we recall the classical notions of combinatorial invariance, so that we can categorify the question in full generality.

**Definition 4.1.** Let \( \Pi \) denote a subset of \( PW \). We say that \( \Pi \) is saturated if for any \( \alpha \in \Pi \) and \( \beta \in PW \) with \( \alpha < \beta \), we have that \( \beta \in \Pi \). We say that \( \Pi \) is co-saturated if its complement in \( PW \) is saturated. If a set is saturated, co-saturated, or the intersection of a saturated and a co-saturated set, we shall say that it is closed under the partial order.

Given any closed set, we can define a corresponding subquotient of \( \mathcal{H}_{(W,P)} \) as follows. (These subquotients also arise in the work of Achar–Riche–Vay in [ARV20].)

**Definition 4.2.** Let \( E \) and \( F \) denote subsets of \( PW \) which are saturated and co-saturated respectively and set \( \Pi = E \cap F \). We let
\[
e = \sum_{\mu \in E} 1_\mu \quad \text{and} \quad f = \sum_{\mu \in F} 1_\mu
\]
\( (4.1) \)
in \( \mathcal{H}_{(W,P)} \). We let \( \mathcal{H}_{(W,P)}^{\Pi} \) denote the subquotient of \( \mathcal{H}_{(W,P)} \) given by
\[
\mathcal{H}_{(W,P)}^{\Pi} = f(\mathcal{H}_{(W,P)}/(\mathcal{H}_{(W,P)}e\mathcal{H}_{(W,P)}))f.
\]

With this notation in place, we are now ready to pose the following question, which categorifies [Bre04, Problem 20] and extends this idea to fields of positive characteristic.

**Question 4.3.** Let \( (W,P) \) and \( (W',P') \) denote a pair of parabolic Coxeter systems. Let \( \Sigma \subseteq W \) and \( \Sigma' \subseteq W' \) be closed subsets and suppose that \( \Sigma \) and \( \Sigma' \) are isomorphic as partially ordered sets and moreover that this isomorphism restricts to an isomorphism between \( \Pi = \Sigma \cap PW \) and \( \Pi' = \Sigma' \cap PW' \). Under what conditions are \( \mathcal{H}_{(W,P)}^{\Pi} \) and \( \mathcal{H}_{(W',P')}^{\Pi'} \) highest-weight graded Morita equivalent?

It is clear that there are some (probably strong) restrictions required for the existence of such an equivalence. For instance, the \( p \)-Kazhdan–Lusztig polynomials of type \( B_n \) and \( C_n \) for \( n > 2 \) do not coincide [JW17, Section 5.4] for \( p = 2 \) and therefore one cannot have such an equivalence in this case (although the Bruhat graphs are obviously isomorphic). Thus one possible restriction is the following:

**Conjecture 4.4.** The restriction that \( k = \mathbb{C} \) is sufficient for the existence of an equivalence as in Question 4.3.

We remark that Conjecture 4.4, if true, implies the combinatorial invariance conjecture of Lusztig and Dyer and its antispherical analogue due Marietti (which state that the antispherical Kazhdan–Lusztig polynomials of these categories coincide) see [Bre04, Mar18] for more details. Our own contribution towards Conjecture 4.4 and Question 4.3 is the following categorification of [Bre09, Corollary 5.2]:

**Theorem 4.5.** Let \( k \) be a field and let \( (W,P) \) and \( (W',P') \) be Hermitian symmetric pairs. Let \( \Pi \subseteq PW \) and \( \Pi' \subseteq PW' \) be closed subsets and suppose that \( \Pi \) and \( \Pi' \) are isomorphic as partially ordered sets. Then \( \mathcal{H}_{(W,P)}^{\Pi} \) and \( \mathcal{H}_{(W',P')}^{\Pi'} \) are highest-weight graded Morita equivalent.

We remark that \( \Pi \cong \Pi' \) is a weakening of the condition in Question 4.3 and so this gives a positive answer to the question. This section is dedicated to the proof of Theorem 4.5. In order to do this, we must first provide “Tetris-style” presentations of these categories.
4.1. Tetris combinatorics for “gaps” in reduced words. In what follows we let \( \sigma, \tau \in S_W \) with \( m(\sigma, \tau) = 3 \) or 4. If \( m(\sigma, \tau) = 4 \) and \((W, P) = (C_n, A_{n-1})\), then suppose that \((\sigma, \tau) = (s_1, s_2)\) or if that \((W, P) = (B_n, B_{n-1})\), then \((\sigma, \tau) = (s_2, s_1)\). Then \( \langle \alpha_\sigma, \alpha_\tau \rangle = -1 \), and the two-colour \( \sigma \tau \)-barbell relation is

\[
\begin{align*}
\left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right) + \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right) &= \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right) + \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right)
\end{align*}
\]

\hspace{1cm} (4.2)

Using the one-colour barbell relation we then obtain

\[
\begin{align*}
\left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right) &= \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right) - \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right)
\end{align*}
\]

\hspace{1cm} (4.3)

Finally, for \( m(\sigma, \tau) = 3 = m(\sigma, \rho) \), by summing over the one and two colour barbell relations we obtain the following

\[
\begin{align*}
\left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right) + \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right) + \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right) &= \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right) + \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right) + \left( \begin{array}{c}
\cdot \\
\cdot
\end{array} \right)
\end{align*}
\]

\hspace{1cm} (4.4)

We now provide inductive versions of the equation (4.2) and (4.3). First, we define a type A string \( T \subseteq S_W \) to be an ordered set of reflections \( s_{i_1}, \ldots, s_{i_t} \) such that \( m(s_{i_j}, s_{i_k}) = 2 + \delta_{j,k+1} + \delta_{j,k-1} \) for \( 1 \leq j, k \leq t \). By induction on equation (4.3) we have the following:

**Lemma 4.6.** Let \( T \subseteq S_W \) be a type A string. We have that

\[
1_{s_{i_1}s_{i_2}\ldots s_{i_{t-1}}} \otimes \text{bar}(s_{i_t}) = \sum_{k=1}^{t} \text{bar}(s_{i_k}) \otimes 1_{s_{i_1}s_{i_2}\ldots s_{i_{k-1}}} - \sum_{k=1}^{t-1} 1_{s_{i_1}s_{i_2}\ldots s_{i_{k-1}}} \otimes \text{gap}(s_{i_k}) \otimes 1_{s_{i_{k+1}}\ldots s_{i_{t-1}}}
\]

By induction on equation (4.2) we have the following:

**Lemma 4.7.** Let \( T \subseteq S_W \) be a type A string. We have that

\[
\sum_{k=1}^{t} 1_{s_{i_2}s_{i_3}\ldots s_{i_{k}}} \otimes \text{bar}(s_{i_k}) = \text{bar}(s_{i_1}) \otimes 1_{s_{i_2}s_{i_3}\ldots s_{i_{t}}} + \sum_{k=2}^{t} 1_{s_{i_2}s_{i_3}\ldots s_{i_{k-1}}} \otimes \text{gap}(s_{i_k}) \otimes 1_{s_{i_{k+1}}\ldots s_{i_{t}}}
\]

**Figure 11.** The far left and right depict the two sides of the equation from equation (4.6). The middle diagram depicts how we rewrite these terms using the null-braid relation.

**Lemma 4.8.** Let \( \beta, \gamma \in S_W \) be such that \( m(\beta, \gamma) = 3 \). We have the following local relation

\[
1_\gamma \otimes (\text{bar}(\beta) + \text{bar}(\gamma)) \otimes 1_\beta = 0.
\]

**Proof.** We have that

\[
1_\gamma \otimes (\text{bar}(\beta) + \text{bar}(\gamma)) \otimes 1_\beta = 1_\gamma \otimes 1_\beta \otimes \text{bar}(\gamma) + 1_\gamma \otimes \text{gap}(\beta)
\]

\[
= \text{spot}_{\gamma}^{(\beta)} \left( 1_\gamma \beta \gamma + \text{spot}_{\gamma}^{(\beta)} \text{dork}_{\gamma}^{(\beta)} \text{spot}_{\gamma}^{(\beta)} \right) \text{spot}_{\gamma}^{(\beta)}
\]

and so the result follows from the \( \beta \gamma \)-null-braid relation. \( \Box \)

**Lemma 4.9.** Suppose that \( W \) is simply laced and that \( [r, c], [r + 1, c], [r, c - 1], [r + 1, c - 1] \in \mu \) for some \( \mu \in \mathcal{P}_{(W, P)} \). We have that

\[
\text{gap}(t_\mu - [r, c]) = \text{gap}(t_\mu - [r + 1, c - 1]).
\]

(4.6)
By inspecting Figure 3 to Figure 6, we realise that either for other cases. For concreteness we consider the type (\(\mu\)). The resulting diagram factors through the idempotent labelled by \(\gamma\). We apply the \(\gamma\)-null-braid relation to the left and righthand-sides of equation (4.6) respectively, as depicted in Figure 11. The result follows.

We now consider points “near the unique trivalent vertex” in types \(D\) and exceptional types.

**Lemma 4.10.** Let \(W\) be simply laced, \(\mu\) be a reduced word. If \([r,c]\) \(\in\) \(\text{Add}(\mu)\) and \(s_{[r-1,c-1]} \neq s_{[r,c]}\), then either \([r-1,c]\) or \([r,c-1]\) is not in \(\mathcal{A}(W,P)\). We have that \(\text{gap}(t_{\mu+[r,c]} - [r-1,1]) = 0\) \(\text{gap}(t_{\mu+[r,c]} - [r,c-1]) = 0\) for \([r,c-1] \notin \mathcal{A}(W,P)\) or \([r-1,c] \notin \mathcal{A}(W,P)\) respectively.

**Proof.** For concreteness, we consider the type \((D,A)\) case (the exceptional cases are similar and the \((D,D)\) case is trivial); this implies that \(r = c\) and we are in the first case, \([r-1,r] \notin \mathcal{A}(D,A)\). We set \(\tau = s_2 = s_{[r,r-1]}\); \(\alpha \in \{s_0,s_1\}\); \(\alpha \neq \gamma \in \{s_0,s_1\}\); and \(\beta = s_3 \in S_W\). By the commutativity relations, we can assume that \([r,r-1]\) is the unique element of Rem(\(\mu\)) so that \(\mu = \lambda_{[r,r-1]}\). We apply the commuting relations and then apply the \(\alpha\)-null-braid relation to the strands of \(\text{gap}(t_{\mu+[r,c]} - [r,r-1])\) labelled by the tiles \([r,r],[r-2,r-2],[r-1,r-2]\). The resulting diagram factors through the idempotent labelled by \(t_\nu\) for \(\nu = \mu - [r-1,r-2] - [r,r]-[r,r-1]\). We apply the \(\gamma\)-null-braid relation to the left and righthand-sides of equation (4.6) respectively, as depicted in Figure 11. The result follows.

**Lemma 4.11.** Let \(W\) be simply laced, \(\mu\) be a reduced word. If \([r,c] \in\) \(\text{Add}(\mu)\) and \([r-1,c-1] \notin \mathcal{A}(W,P)\), then either \([r-1,c]\) or \([r,c-1]\) is not in \(\mathcal{A}(W,P)\). We have that \(\text{gap}(t_{\mu+[r,c]} - [r-1,1]) = 0\) \(\text{gap}(t_{\mu+[r,c]} - [r,c-1]) = 0\) for \([r,c-1] \notin \mathcal{A}(W,P)\) or \([r-1,c] \notin \mathcal{A}(W,P)\) respectively.

**Proof.** By inspecting Figure 3 to Figure 6, we realise that either \(r = 1\) or \(c = 1\) or we are in a more interesting case for types \((D_n,D_{n-1})\), \((E_5,E_6)\), and \((E_6,E_7)\). The \(r = 1\) and \(c = 1\) cases follow immediately from the cyclotomic and commutativity relations, we now consider the other cases. For concreteness we consider the \((E_6,E_7)\) case, as the others are similar. Inspecting Figure 6, we have that \([r,c] \in \{[5,4],[5,5],[5,4],[9,8],[9,9]\}\). In the first two cases, we can apply the \(s_{2}s_{3}\)-null-braid to \(\text{gap}(t_{\mu+[5,4]} - [5,3])\) and the \(s_{1}s_{2}\)-null-braid to \(\text{gap}(t_{\mu+[5,3]} - [5,4])\) and the resulting diagram is zero by the commutation and cyclotomic relations. In the third case we can apply the \(s_{5}s_{8}\)-null braid followed by the \(s_{4}s_{5}\)-null braid to \(\text{gap}(t_{\mu+[8,4]} - [8,4])\) to obtain a diagram which is zero by the \([5,4]\)-case. In the penultimate case we apply the \(s_{3}s_{4}\)-null braid followed by the \(s_{4}s_{5}\)-null braid to \(\text{gap}(t_{\mu+[9,9]} - [9,7])\) and the result is zero by the \([9,4]\)-case. In the final case, we apply the \(s_{1}s_{2}\)-null braid to followed by the \(s_{4}s_{3}\)-null braid \(\text{gap}(t_{\mu+[9,9]} - [9,8])\) and the resulting diagram is zero by Lemma 4.10 (for \([r,c] = [8,4]\)).

**Proposition 4.12.** In type \((D_n,A_{n-1})\) set \(\sigma = s_0\) and \(\tau = s_1\). In types \((A_n,A_{n-1})\) and \((D_n,D_{n-1})\) let \((\sigma,\tau)\) be any pair such that \(m(\sigma,\tau) = 2\) except \((s_0,s_1)\) or \((s_1,s_0)\). We have \(1_{t_\mu} \otimes \text{braid}^{t_\sigma}_{t_\tau} = 0\) for all \(\mu \in \mathcal{P}(W,P)\). (4.7)

**Proof.** In type \((D_n,A_{n-1})\) the pair \((\sigma,\tau)\) satisfies the condition of Lemma 4.10. Therefore either \(1_{t_\mu} \otimes 1_{t_\tau} = 0\) or \(1_{t_\mu} \otimes 1_{t_\tau} = 0\) by Lemma 4.10. The same is true in types \((A_n,A_{n-1})\) and \((D_n,D_{n-1})\) by the commutativity relations, so in all cases the result follows.

**Remark 4.13.** We refer to pairs as in Proposition 4.12 as zero braid generators. The fact that they are zero will allow us to ignore them in Section 5.

Building on Lemma 4.10, we now show how one can rewrite the gap diagrams in terms of the cellular basis. We only explicitly consider type \((D_n,A_{n-1})\) as it will be needed later on) but similar combinatorics works in other types.
Figure 12. For \( \mu = (1, 2, 3, 4, 5, 6, 7, 8^2, 3, 1^4) \) and \( [r, c] = [6, 6] \) we depict on the left the decorations of Definition 4.14. Notice \([7 + 6, 7 - 6] = [13, 1] \in \mu\), but \([7 + 5, 7 - 5] = [12, 2] \not\in \mu\) and so \([13, 1]\) does not contain a label \(R_k^\pm\). On the right we depict \(\Omega_{[6,6]}(\mu)\), obtained by letting the tiles fall under gravity. Here \(\Sigma_{[6,6]}(\mu) = (1, 2, 3, 4, 5, 6^2, 2^2, 1^5)\) is shaded grey.

**Definition 4.14.** Let \([r, c] \in \mu \in \mathcal{P}(D_n, A_{n-1})\) and suppose that \([r, c + 1] \notin \mu\). Define tile-partitions \(\Sigma_{[r,c]}(\mu)\) and \(\Omega_{[r,c]}(\mu)\) as follows

(i) Place a black spot label in \([r, c]\). If \([r + 1 + k, c + 1 - k], [r + k, c - 1 - k] \in \mu\), all \(0 \leq k \leq t\), then place the labels \(R_k^+\) and \(A_k^+\) in the tiles \([r + 1 + t, c + 1 - t], [r + t, c - 1 - t]\) and the labels \(R_t^-\) and \(A_t^-\) in the tiles \([r + t + 1, c + t - 1], [r + 1 + t, c - 1 - t]\).

(ii) If \([r + 1 + k, c - k], [r + 1 + k, c - 1 - k] \in \mu\), all \(0 < k \leq s\), then place the labels \(R_s^+\) and \(A_s^-\) in the tiles \([r + 1 + s, c - s], [r + 1 + s, c - 1 - s]\).

(iii) Set \(\Sigma_{[r,c]}(\mu)\) to be the maximal \(\lambda \leq \mu\) consisting only of undecorated tiles. Set \(\Omega_{[r,c]}(\mu)\) to be the tile-partition obtained by deleting all decorated tiles and allowing the remaining tiles to drop into place as in Figure 12, if this is a tile-partition (and leave \(\Omega_{[r,c]}(\mu)\) undefined otherwise). We set \(\ell(\Omega_{[r,c]}(\mu))\) to be half the number of decorated tiles.

**Proposition 4.15.** Let \([r, c] \in \mu \in \mathcal{P}(D_n, A_{n-1}), [r, c + 1] \notin \mu\). We have

\[
\text{gap}(t_\mu - [r, c]) = (-1)^{\ell(\Omega_{[r,c]}(\mu))} c_{S}\n
\]

for \(S\) the unique element of \(\text{Path}(\Omega_{[r,c]}(\mu), t_\mu)\) if \(\Omega_{[r,c]}(\mu)\) is defined, and zero otherwise.

**Proof.** The proof proceeds in two steps: we first rewrite the lefthand-side using null-braid relations and then prove that the resulting diagram is the claimed cellular basis element. For each pair \((R_k^+, A_k^-)\) for \(k = 0, \ldots, s\) in Definition 4.14 we apply the corresponding null braid relation to \(1_{t_\mu}\) in turn. For each pair \((R_k^+, A_k^-)\) for \(k = 0, \ldots, t\) in Definition 4.14 we apply the corresponding null braid relation to the above, in turn. If \(\Omega_{[r,c]}(\mu)\) is a tile partition we are done; if not, then the diagram is zero by the commutativity and cyclotomic relations.

We claim that when \(\Omega_{[r,c]}(\mu)\) is defined, the diagram constructed above is an element of the cellular basis, \(c_{S}\). To see this, note that each tile-decoration is of the form \(R_k^+, A_k^+\) (if you delete the integer label we used in step one of the proof and simply attach each tile’s reflection label). Now, \(c_{S}\) is obtained by first doing \(A_k^{\pm}_{\tau[x,y]}\) for \([x, y] \in \Sigma_{[r,c]}(\lambda)\), then doing the operations \(X_{\tau[x,y]}^\pm\) corresponding to the decorations of all the decorated tiles \([x, y] \in \mu\), and then doing \(A_k^{\pm}_{\tau[x,y]}\) for the remaining undecorated tiles \([x, y] \in \mu\) (up to a choice of \(t_\mu\)). Uniqueness of \(S\) is not used in this paper (except for ease of exposition) but a proof that \(\text{Path}(W, P)(\lambda, t_\mu)\) has at most one element for any \(\lambda, \mu \in \mathcal{P}(W, P)\) for \((W, P)\) simply laced is given in [BDF+]. \(\Box\)
Remark 4.16. Step (i) in Definition 4.14 is empty unless \( r = c \). The assumption that \([r, c+1] \not\in \mu\) in Definition 4.14 is merely for ease of notation. Notice that all \( \text{gap}(t_\mu - [r, c]) \) in type \((D_n, A_{n-1})\) can be calculated using either Proposition 4.15 and Lemma 4.9 or Lemma 4.10.

4.2. Tetris combinatorics for barbells. We now provide closed combinatorial formulas for removing barbells from diagrams.

![Diagram](image)

Figure 13. Examples of trails, the first 4 of which are maximal length (the 5th is not). Here \([x, y] = [5, 4], [7, 4], [8, 3]\) and \((W, P) = (A_9, A_3 \times A_5), (D_8, A_7), (E_6, E_5)\) with \( \mu = (4, 3), (1, 2, 3, 4, 5, 2, 1^2)\) respectively (only the distinct cases are listed). The 5th trail is non-maximal length as it has length 6, whereas the 4th trail has length 8. The grey tiles \([a, b]\) are those such that \( \text{gap}(t_\mu - [a, b]) = 0 \) by the cyclotomic and commutativity relations. The second and third maximal length trails are the left-trail and the right-trail, respectively.

**Definition 4.17.** Let \( \mu \in \mathcal{P}_{(W, P)} \). Given \([x, y]\) a (possibly non-admissible) tile, we set \( SW[x, y] = [x - 1, y] \) and \( SE[x, y] = [x, y - 1] \). We define a trail from \([x, y]\) to the origin, denoted \( T_{[x, y] \to [1, 1]} \), to be a set of tiles

\[
[x, y] = T_1, T_2, \ldots, T_{x+y-1} = [1, 1]
\]

such that \( T_{i+1} = SW(T_i) \) or \( T_{i+1} = SE(T_i) \) for \( 1 < i < x + y - 1 \). We write

\[
T_{[x, y] \to [1, 1]}^\mu : = \mu \cap T_{[x, y] \to [1, 1]}
\]

and we define the \( \mu \)-length of the trail to be \( |T_{[x, y] \to [1, 1]}^\mu| \). Given \( \tau = [r, c] \in \text{Add}(\mu) \), we let \( \text{Hook}_\tau(\mu) \) denote any multiset of the form

\[
\text{Hook}_\tau(\mu) = \begin{cases} 
2 \cdot T_{[r-1, c-1] \to [1, 1]}^\mu & \text{if } [r - 1, c] \not\in \mathcal{A}(W, P) \text{ in type } (C_n, A_{n-1}) \\
T_{[r-1, c] \to [1, 1]}^\mu \cup T_{[1, n] \to [1, 1]}^\mu & \text{if } [r, c - 1] \not\in \mathcal{A}(W, P) \text{ in type } (B_n, B_{n-1}) \\
T_{[r-1, c-1] \to [1, 1]}^\mu \cup T_{[r, c-1] \to [1, 1]}^\mu & \text{otherwise}
\end{cases}
\]

for any preferred choices of maximal \( \mu \)-length trails on the right-hand side.

This allows us to provide a closed combinatorial formula for rewriting barbells in diagrams, as follows. This formula is essential to our proof of combinatorial invariance. Finding such formulas for general \((W, P)\) seems to be an impossible task in general.

**Proposition 4.18.** Let \((W, P)\) be a Hermitian symmetric pair. Let \( \mu \in \mathcal{P}_{(W, P)} \) and \( \tau = [r, c] \in \text{Add}(\mu) \). We have that

\[
1_{t_\mu} \otimes \text{bar}(\tau) = \sum_{[x, y] \in \text{Hook}_\tau(\mu)} \text{gap}(t_\mu - [x, y]). \tag{4.9}
\]

Before embarking on the proof, we emphasise that equation (4.9) has a lot of redundancy. Many of the terms on the righthand-side of this sum are zero, using the results of Subsection 4.1 (some of these are highlighted in Figure 13). We now pick preferred choices of the maximal length trails in the definition of \( \text{Hook}_\tau(\mu) \) and delete some of these redundant terms.
Definition 4.19. Let $\mu \in \mathcal{P}(W,P)$. For $[x,y]$ a (possibly non-admissible) tile, we define the left-trail $L_{[x,y] \to [1,1]}$ (respectively right-trail $R_{[x,y] \to [1,1]}$) to be the maximal length trails

$$[x,y] = T_1, T_2, \ldots, T_{x+y-1} = [1,1]$$

with $T_i = [x_i, y_i]$ such that $y_i$ is maximal (respectively $x_i$ is maximal) for $1 \leq i \leq x+y-1$. We define the left hook $LH_{[\tau],x,y}$ (respectively right hook, $RH_{[\tau],x,y}$) to be the intersection of $L_{[x,y] \to [1,1]}$ with the $y$th column of $\mu$ (respectively $R_{[x,y] \to [1,1]}$ with the $x$th row of $\mu$).

![Figure 14](image)

Figure 14. The multisets $\text{Hole}_r(\mu)$. The first case is the union $RH_{\mu[6,3]} \cup RH_{\mu[5,4]}$ for $\tau = [6,4]$ and $\mu = (5^2,4^3,3,1)$ in type $(A_{11}, A_4 \times A_6)$. The second case is $\tau = [6,4]$ and $\mu = (1,2,3,4^2,3,1)$ in type $(D_8, A_7)$. The third and fourth cases are for $\mu = (1,2,3,4,5^2)$ and $\tau = s_{[6,6]}$ in types $(D,A)$ and $(C,A)$ respectively. We highlight the tiles in $\text{Hole}_r(\mu)$ by placing a gap diagram in the tile and the multiplicity of that tile within the multiset.

Examples of left and right trails are given in Figure 13. Examples of left and right hooks are given in Figure 14. The following lemma simplifies the gap elements we need to consider.

Lemma 4.20. Let $(W,P)$ be a classical Hermitian symmetric pair, $\mu \in \mathcal{P}(W,P)$ and $\tau = [r,c] \in \text{Add}(\mu)$. We set

$$\text{Hole}_r[r,c] = \begin{cases} 2 \cdot RH_{\mu[r,r-1]} & \text{if } r = c \text{ in type } (C_n, A_{n-1}) \\ RH_{\mu[r-1,c]} \cup LH_{\mu[1,n]} & \text{if } [r,c-1] \notin \mathcal{P}(W,P) \text{ in type } (B_n, B_{n-1}) \\ RH_{\mu[r-1,c]} \cup RH_{\mu[r,c-1]} & \text{otherwise} \end{cases}$$

We have that

$$\sum_{[x,y] \in \text{Hole}_r(\mu)} \text{gap}(t_{\mu} - [x,y]) = \sum_{[x,y] \in \text{Hole}_r(\mu)} \text{gap}(t_{\mu} - [x,y])$$

and in particular the left-hand side is independent of the choice of maximal $\mu$-length trails.

Proof. We have that $LH_{\mu(x,y)} \subseteq L_{[x,y] \to [1,1]}$; thus we need only to show that $\text{gap}(t_{\mu} - [x,y]) = 0$ for $[x,y] \in L_{[x,y] \to [1,1]} \setminus LH_{\mu([x,y])}$ (and similarly for the right). For types $(B,B)$ and $(D,D)$ this follows from the cyclotomic and commutativity relations. For types $(A,A \times A), (D,A), (C,A)$ one further requires Lemmas 4.9 and 4.11.

Proof of Proposition 4.18. We consider the cases in which the parabolic is of type $A$. The other cases are left as an exercise for the reader. By the commutativity relations, it is enough to prove the result for $\tau$ and $\mu$ such that $\tau = s_{[r,c]}$ and $\mu = \lambda_{[r,c]}$ for some $r,c \geq 1$ (in the notation of Definition 1.4). If $r = c = 1$ then the result is immediate from the cyclotomic relation. If $r > 1$ and $c = 1$ or $r = 1$ and $c > 1$, then the result follows by Lemma 4.16 and the cyclotomic relation. Thus by Lemma 4.20 it is enough to show that $1_{t_{\mu}} \otimes \text{bar}(\tau) = \sum_{[x,y] \in \text{Hole}(\mu)} \text{gap}(t_{\mu} - [x,y])$ for $r,c > 1$. 


Case 1. We now assume that $r, c > 1$, and consider the case where $[r, c]$ is such that $[r, c - 1], [r - 1, c] \in \mu$ as this is uniform across all types. We set $\nu$ to be the partition obtained by removing the final box from each column of $\mu$, that is $\nu \sigma = \lambda_{[r-1,c]}$. We have that
\[
1_{t_\mu} \otimes \bar{\text{bar}}([r, c]) = 1_{t_\nu} \otimes \bar{\text{bar}}([r, c]) \otimes 1_{t_{\mu \setminus \nu}} + \sum_{y < c} 1_{t_\nu} \otimes \bar{\text{bar}}([r, y]) \otimes 1_{t_{\mu \setminus \nu}} - \sum_{1 \leq y < c} \text{gap}(t_\mu - [r, y])
\]
by Lemma 4.6 and applying equation (4.2) we obtain
\[
1_{t_\mu} \otimes \bar{\text{bar}}([r, c]) = 1_{t_\nu} \otimes \bar{\text{bar}}([r - 1, c]) \otimes 1_{t_{\mu \setminus \nu}} + \sum_{y < c} 1_{t_\nu} \otimes \bar{\text{bar}}([r, y]) \otimes 1_{t_{\mu \setminus \nu}} - \sum_{[x, y] \in \mu \setminus \nu} \text{gap}(t_\mu - [x, y]).
\]
We set $\pi$ to be the partition obtained by removing the final two rows of $\mu$, that is $\pi = \lambda_{[r-2,c]}$.
By Lemma 4.7, we have that
\[
\sum_{y < c} 1_{t_\nu} \otimes \bar{\text{bar}}([r, y]) \otimes 1_{t_{\mu \setminus \pi}} = 1_{t_\nu} \otimes \bar{\text{bar}}([r, 1]) \otimes 1_{t_{\mu \setminus \pi}} + \sum_{y < c} \text{gap}(t_\mu - [r - 1, y])
\]
and we note that the first term after the equality is zero by the commutativity and cyclotomic relations. Putting these two equations above together, we have that
\[
1_{t_\mu} \otimes \bar{\text{bar}}([r, c]) = 1_{t_\nu} \otimes \bar{\text{bar}}([r - 1, c]) \otimes 1_{t_{\mu \setminus \nu}} + \left( \sum_{y < c} \text{gap}(t_\mu - [r - 1, y]) - \sum_{[x, y] \in \mu \setminus \nu} \text{gap}(t_\mu - [x, y]) \right)
\]
and so the result follows by induction and Lemmas 4.9 and 4.10. This inductive step is visualised in Figure 15; we have bracketed the latter two terms above in order to facilitate comparison with the rightmost diagram in Figure 15.

![Figure 15](image-url)

**Figure 15.** The first term on the left-hand side depicts $1_{t_\nu} \otimes \bar{\text{bar}}([r - 1, c]) \otimes 1_{t_{\mu \setminus \nu}}$ (known by induction) the second term depicts the coefficients of the gap terms in the inductive step in the proof. The first equality records the cancellations; the second equality follows from Lemmas 4.9 and 4.11. The rightmost diagram depicts $1_{t_\nu} \otimes \bar{\text{bar}}([r, c])$ (for $r \neq c$ in types $C$ and $D$).

Case 2. Now consider the type $C$ and $D$ cases for $\tau = s_{[r,r]}$ with $r > 1$ and we let $\sigma = s_2 \in W$.
\[
1_{t_\mu} \otimes \bar{\text{bar}}([r, r]) = \begin{cases} 
1_{t_{\mu - \sigma}} \otimes (\bar{\text{bar}}(\tau) + 2\bar{\text{bar}}(\sigma)) \otimes 1_{\sigma} - 2\text{gap}(t_\mu - [r, r - 1]) & \text{in type } C \\
1_{t_{\mu - \sigma}} \otimes (\bar{\text{bar}}(\tau) + \bar{\text{bar}}(\sigma)) \otimes 1_{\sigma} - \text{gap}(t_\mu - [r, r - 1]) & \text{in type } D
\end{cases}
\]
and the $r = 2$ case now follows by the cyclotomic relation.

Now suppose $r > 2$. We first consider the type $C$ case. We set $\nu$ to be the partition such that $\nu \tau = \lambda_{[r-1,r-1]}$. By the commutativity and one-colour barbell relations, we have that
\[
1_{t_{\mu - \sigma}} \otimes \bar{\text{bar}}(\tau) \otimes 1_{\sigma} = -1_{t_\nu} \otimes \bar{\text{bar}}(\tau) \otimes 1_{t_{\mu \setminus \nu}} + 2\text{gap}(t_\mu - [r - 1, r - 1])
\]
and so
\[
1_{t_\nu} \otimes \bar{\text{bar}}([r, r]) = 2 \times 1_{t_{\mu - \sigma}} \otimes \bar{\text{bar}}([r, r - 1]) \otimes 1_{\sigma} - 1_{t_\nu} \otimes \bar{\text{bar}}([r - 1, r - 1]) \otimes 1_{t_{\mu \setminus \nu}}
\]
\[
+ 2(\text{gap}(t_\mu - [r - 1, r - 1]) - 2\text{gap}(t_\mu - [r, r - 1]))
\]
and so the result follows by induction. This inductive step is visualised in Figure 16; we have bracketed the latter two terms above in order to facilitate comparison with the rightmost diagram in Figure 16.
The left-hand side depicts $1_{t_\nu} \otimes \bar{\text{bar}}([r, r])$ in type $C$. The first term on the right-hand side depicts $2 \times 1_{t_{\mu - \nu}} \otimes \bar{\text{bar}}([r, r - 1]) \otimes 1_{\sigma}$; the second term depicts $-1_{t_\nu} \otimes \bar{\text{bar}}([r - 1, r - 1]) \otimes 1_{t_{\nu - \psi}}$; the third term depicts the coefficients of the gap terms in the inductive step.

We now consider type $D$. We colour $s_{[r-1,r-1]}$ violet and set $\nu$, $\pi$, and $\rho$ to be the partitions $\nu + [r - 1, r - 1] = \lambda_{[r-1,r-1]}$, $\pi + [r - 2, r - 2] = \lambda_{[r-2,r-2]}$, and $\rho = \lambda_{[r-3,r-3]}$. We have that

$$1_{t_\mu} \otimes \bar{\text{bar}}([r, r]) = 1_{t_\mu} \otimes \bar{\text{bar}}([r - 1, r - 1]) \otimes 1_{t_{\nu - \psi}} + \sum_{y=1}^{r} 1_{t_y} \otimes \bar{\text{bar}}([r, y]) \otimes 1_{t_{\mu - \nu}} - \sum_{[x,y] \in \mu \setminus \nu} \text{gap}(t_\mu - [x, y])$$

by Lemma 4.6. The second term can be rewritten as follows

$$\sum_{y=1}^{r} 1_{t_y} \otimes \bar{\text{bar}}([r, y]) \otimes 1_{t_{\mu - \nu}} = \sum_{y=1}^{r} 1_{t_{y + r}} \otimes \bar{\text{bar}}([r, y]) \otimes 1_{t_{\nu - r}}$$

$$= \sum_{y=1}^{r-1} 1_{t_y} \otimes \bar{\text{bar}}([r, y]) \otimes 1_{t_{\nu - r}} + \text{gap}(t_\mu - [r - 2, r - 2])$$

$$= \sum_{y=1}^{r-2} \text{gap}(t_\mu - [r - 2, y]) + 1_{t_r} \otimes (\bar{\text{bar}}([r, 1]) + \bar{\text{bar}}([r, 2])) \otimes 1_{t_{\mu - \rho}}$$

$$= \sum_{y=1}^{r-2} \text{gap}(t_\mu - [r - 2, y])$$

where the first equality follows by repeated applications of equation (4.4); the second from equation (4.2); the third from Lemma 4.7 and the commutation relations; the fourth from the commutation and cyclotomic relations (notice that no tile in $\pi$ has colour label corresponding to the reflections $s_{[r,1]}$ or $s_{[r,2]}$). Substituting this into the above, we obtain

$$1_{t_\mu} \otimes \bar{\text{bar}}([r, r]) = 1_{t_\mu} \otimes \bar{\text{bar}}([r - 1, r - 1]) \otimes 1_{t_{\nu - \psi}} + \sum_{y=1}^{r-2} \text{gap}(t_\mu - [r - 2, y]) - \sum_{[x,y] \in \mu \setminus \nu} \text{gap}(t_\mu - [x, y])$$

The result follows by induction (see Figure 17 for a visualisation of this step).

The lefthand-side depicts $1_{t_\mu} \otimes \bar{\text{bar}}([r, r])$. The first term on the righthand-side depicts $1_{t_\mu} \otimes \bar{\text{bar}}([r - 1, r - 1]) \otimes 1_{t_{\nu - \psi}}$ (known by induction); the second and third terms depict $+ \sum_{y \leq r} \text{gap}(t_\mu - [r - 2, y])$ and $- \sum_{[x,y] \in \mu \setminus \nu} \text{gap}(t_\mu - [x, y])$ respectively, which provide the gap terms in the inductive step in the proof.
4.3. The Tetris-style presentation. We are now ready to provide a new presentation for the Hecke categories of simply laced Hermitian symmetric pairs. One should notice that this presentation is mainly given in terms of the tiling combinatorics and not the usual Dynkin diagram combinatorics (the exception to this being discussion of commuting relations which are “far apart” in the Dynkin diagram).

Theorem 4.21. Let \((W, P)\) denote a simply laced Hermitian symmetric pair. The algebra \(\mathcal{H}_{(W, P)}\) can be defined as the locally-unital associative \(k\)-algebra spanned by simple Soergel diagrams with multiplication given by vertical concatenation of diagrams modulo the following local relations and their horizontal and vertical flips. Firstly, for any \(\sigma \in S_W\) we have the relations

\[
1_{\sigma} 1_{\tau} = \delta_{\sigma \tau} 1_{\sigma} \quad \quad 1_{\emptyset} 1_{\sigma} = 0 \quad \quad 1_{\emptyset}^{2} = 1_{\emptyset}
\]

\(\emptyset\) spot \(1_{\sigma} = \text{spot}^{\emptyset}_{1_{\sigma}} \quad \quad 1_{\sigma} \text{fork}^{\sigma}_{1_{\sigma}} 1_{\sigma} = \text{fork}^{\sigma}_{1_{\sigma}} \quad \quad 1_{\tau} \text{braid}^{\sigma}_{\tau} 1_{\sigma} = \text{braid}^{\sigma}_{\tau}
\]

where the final relation holds for all ordered pairs \((\sigma, \tau) \in S_W^3\) with \(m(\sigma, \tau) = 2\). For each \(\sigma \in S_W\) we have fork-spot contraction, the double-fork, and circle-annihilation relations:

\[
\begin{array}{c}
\text{For } (\sigma, \tau, \rho) \in S^3 \text{ with } m_{\sigma \rho} = m_{\rho \tau} = m_{\tau \sigma} = 2, \text{ we have the commutation relations}
\end{array}
\]

\[
\begin{array}{c}
1_{\sigma} \otimes 1_{\tau} = 1_{\mu} \otimes \left( \text{spot}_{\sigma}^{\emptyset} \otimes \text{fork}^{\sigma}_{\sigma} + \text{spot}_{\emptyset}^{\tau} \otimes \text{fork}^{\tau}_{\sigma} \right) + \sum_{[r,c] \in \text{Hook}_{\sigma}(\mu)} \text{gap}(t_{\mu} - [r, c]) \otimes \text{dork}^{\sigma}_{\sigma}.
\end{array}
\]

For any \(\mu \in \mathcal{P}_{(W, P)}\) and \(\sigma \in \text{Add}(\mu)\), we have the monochrome Tetris relation

\[
1_{\mu} \otimes 1_{\sigma} = 1_{\mu} \otimes \left( \text{spot}_{\sigma}^{\emptyset} \otimes \text{fork}^{\sigma}_{\sigma} + \text{spot}_{\emptyset}^{\tau} \otimes \text{fork}^{\tau}_{\sigma} \right) + \sum_{[r,c] \in \text{Hook}_{\sigma}(\mu)} \text{gap}(t_{\mu} - [r, c]) \otimes \text{dork}^{\sigma}_{\sigma}.
\]

For any \(\mu \sigma \tau \in \mathcal{P}_{(W, P)}\), we have the null-braid relation

\[
1_{\mu} \otimes \left( 1_{\sigma \tau} + \left( 1_{\sigma} \otimes \text{spot}_{\emptyset}^{\tau} \otimes 1_{\sigma} \right) \text{dork}^{\sigma}_{\sigma} \left( 1_{\sigma} \otimes \text{spot}_{\emptyset}^{\tau} \otimes 1_{\sigma} \right) \right) = 0
\]

For \(\mu \in \mathcal{P}_{(W, P)}\) and \(\tau \in \text{Add}(\mu)\), we have the bi-chrome Tetris relation

\[
1_{\mu} \otimes \text{bar}(\tau) = - \sum_{[r,c] \in \text{Hook}_{\tau}(\mu)} \text{gap}(t_{\mu} - [r, c]).
\]

Further, we require the interchange law and the monoidal unit relation

\[
(D_1 \otimes D_2) \circ (D_3 \otimes D_4) = (D_1 \circ D_3) \otimes (D_2 \circ D_4) \quad \quad 1_{\emptyset} \otimes D_1 = D_1 = D_1 \otimes 1_{\emptyset}
\]

for all diagrams \(D_1, D_2, D_3, D_4\). Finally, we require the non-local cyclotomic relations

\[
\text{bar}(\sigma) \otimes D = 0 \quad \quad \text{for all } \sigma \in S_W \text{ and } D \text{ any diagram}
\]

\[
1_{\sigma} \otimes D = 0 \quad \quad \text{for all } \sigma \in S_F \subseteq S_W \text{ and } D \text{ any diagram}
\]

Proof. In light of Corollary 2.4, we need only show that the one and two colour barbell relations can be replaced by the monochrome and bi-chrome Tetris relations.

Given two simple Soergel diagrams \(D_1\) and \(D_2\), the one and two colour barbell relations allow us to inductively move leftwards any barbell anywhere in \(D_1D_2\); once all barbells are at the leftmost edge of the diagram these are zero by the cyclotomic relation. Thus the one and two colour barbell relations allow us to rewrite a product of simple Soergel diagrams as a linear combination of simple Soergel diagrams.

We now show that we can rewrite, using only the relations of Theorem 4.21, any diagram \(D_1D_2\) as a linear combination of simple Soergel diagrams. Any diagram can be rewritten in terms of the cellular basis and the cellular basis elements are all of the form \(c_{\lambda}^{\mu} 1_{\lambda} e_{\mu}\). Thus it suffices to have a list of rules which rewrites \(1_{\lambda} \otimes \text{bar}(\tau)\) as a linear combination of simple Soergel diagrams. This is precisely what the monochrome and bi-chrome Tetris relation do. The result follows. \(\square\)
4.4. Combinatorial invariance. Equipped with our Tetris style presentations, we are now ready to prove Theorem 4.5. We begin by restricting our explicit attention to simply laced types only, as this case follows easily from our Tetris-style presentation.

Proposition 4.22. Let \((W, P)\) and \((W', P')\) be simply laced Hermitian symmetric pairs. Let \(\Pi\) and \(\Pi'\) be closed subsets of \(\mathcal{P}(W, P)\) and \(\mathcal{P}(W', P')\) respectively. Given a map \(\varphi : \Pi \to \Pi'\), \(\varphi(\lambda) = \lambda'\), we have that \(\varphi\) is a poset isomorphism if and only if \(\varphi\) sends like-coloured tiles in \(\lambda\) to like-coloured tiles in \(\lambda'\).

**Proof.** We consider the classical types, as the exceptional cases can be verified by exhaustion. We restrict our attention to the case that \(\Pi\) has a unique maximal and minimal element (for ease of exposition) as any general \(\Pi\) can be obtained as a union of such posets. Any such \(\Pi\) consists of a rectangle intersected with the admissible region. If the intersection is merely a rectangle, then the colouring is simply given by the diagonals of the rectangle, as required.

We now assume that \(\Pi\) is non-rectangular. Then \(\Pi\) has a single diagonal in which the colouring alternates (either between \(s_0\) and \(s_1\) in type \((D, A)\) or between \(s_1\) and \(s_2\) in type \((D, D)\)). If such a \(\Pi\) fits within a \(3 \times 3\) rectangle in type \((D, A)\), then one can check that there exists an isomorphic \(\Pi'\) in type \((D, D)\) and that the colourings match-up in these small cases. Otherwise, if \(\Pi\) and \(\Pi'\) are two distinct isomorphic posets, then they are both of type \((D, A)\) and can be obtained by vertical translation, the result follows. \(\square\)

**Definition 4.23.** Let \(\Gamma\) (respectively \(\Gamma'\)) be the set of colours of the tiles in \(\Pi\) (respectively \(\Pi'\)). We let \(\varphi(\gamma) = \gamma'\) be a surjective map. We lift this to a recolouring map on Soergel diagram as follows. We set

\[
\varphi(1_{\gamma}) = 1_{\gamma'}, \quad \varphi(\text{spot}^0_{\gamma}) = \text{spot}^0_{\gamma'}, \quad \varphi(\text{fork}^1_{\gamma}) = \text{fork}^1_{\gamma'}, \quad \varphi(\text{braid}^{(1', 1'')}_{\gamma_1, \gamma_2}) = \text{braid}^{(1', 1'')}_{\gamma_1, \gamma_2}
\]

and we \(\varphi(D^*) = (\varphi(D))^*\). We then inductively define

\[
\varphi(D_1 \otimes D_2) = \varphi(D_1) \otimes \varphi(D_2), \quad \varphi(D_1 \circ D_2) = \varphi(D_1) \circ \varphi(D_2)
\]

and extend this map \(k\)-linearly.

**Proposition 4.24.** Suppose we have a poset isomorphism \(\varphi : \Pi \to \Pi'\) as in Proposition 4.22 inducing a recolouring map \(\varphi\). Then \(\varphi\) extends to a unique graded isomorphism of \(k\)-algebras \(\varphi : h_\Pi \to h_{\Pi'}\) satisfying

\[
\varphi(1_{\mu} \otimes D) = 1_{\mu'} \otimes \varphi(D)
\]

for \(\mu \in \Pi\) any minimal element and \(D\) any Soergel diagram, and

\[
\varphi(1_{\mu} \otimes_{\gamma} D) = 1_{\mu'} \otimes_{\gamma'} \varphi(D)
\]

for \(\mu \in \Pi\) any minimal element, \(\gamma \in \text{Rem}(\mu)\), and \(D\) any Soergel diagram for which the \(\otimes_{\gamma}\) and \(\otimes_{\gamma'}\)-products makes sense.

**Remark 4.25.** We can regard the diagram \(D\) as being “coloured by” the initial Bruhat graph \(P' \setminus W\) and the effect of the map is to “recolour” this diagram according to the new Bruhat graph \(P' \setminus W'\). The effect of changing \(t_\mu\) for \(t_{\mu'}\) is merely to identify the different regions \(\Pi\) and \(\Pi'\) within our Bruhat graphs. An example is given in Figure 18.

We are now almost ready to prove Proposition 4.24. We first observe that the Tetris relations are compatible with restriction to a closed subregion.

**Lemma 4.26.** Let \(\lambda \in \Pi\) with \([\tau, c] = \tau \in \text{Add}(\lambda)\). We let \(\mu \notin \Pi\) be a maximal element such that \(\mu \leq \lambda\). Given \([x, y] \in \text{Hook}_\tau(\lambda) \cap \mu\), we have that

\[
gap(t_\lambda - [x, y]) = 0 \text{ in } \mathcal{H}_\Pi.
\]
**Proof.** We suppose $W$ is classical and set $\lambda = \lambda_{[r,c]} - [r,c]$, the general case is similar. If $\delta_{[x,y]} = \sigma$ and $\ell_\sigma(\lambda - \mu) = 0$, then $t_{\lambda_{[x,y]}}$ is an expression for some $\nu \not\in \Pi$ and we are done. Otherwise, we have that $[x+1, y+1] \in \lambda - \mu$ and $\ell_\sigma(\lambda - \mu) = 1$ by inspection of Figures 3 to 6; moreover, $[x, y+1]$ or $[x+1, y]$ does not belong to $\text{Hook}_r(\lambda) \cap \mu$. In the latter case (the former is identical) we have that $\ell_\gamma(\lambda - \mu) = 0$; we apply the $\alpha\gamma$-bull-braid relation and the argument above and hence deduce the result. \hfill \Box

![Diagram](image)

**Figure 18.** The left diagram, $D$, is an element from $h_\Pi$ for $\Pi = \{(r,c) | 1 \leq r \leq 2, 2 \leq c \leq 3\} \subset \mathcal{S}(A_8, A_4 \times A_3)$ and the right diagram is the corresponding, $\iota(D)$, in $h_{\Pi'}$ for $\Pi' = \{(r,c) | 2 \leq r \leq 3, 2 \leq c \leq 3\} \subset \mathcal{S}(A_8, A_4 \times A_3)$. Compare the colouring with that of $(A_8, A_4 \times A_3)$ in Figure 3.

**Proof of Proposition 4.24.** Each cellular basis element $c_{ST}^\lambda$ in $h_\Pi$ can be written in the form $1_{\gamma_i} \otimes D$ or $1_{\gamma_i} \otimes \tau \otimes D$ for some simple Soergel graph $D$, so $j$ is well defined. The monochrome and idempotent relations are trivially preserved by the map $j$. Two commuting reflections, $\tau, \sigma \in S_W$ correspond to tiles $[x,y]$, $[r,c]$ from $\Pi$ if and only if $(x-y) - (r-c) \neq \pm 1$; this distance is preserved by the map $j : \Pi \rightarrow \Pi'$ (by Proposition 4.22) and so the commuting relations are preserved.

The Tetris relations for $\mathcal{H}_\Pi$ are written entirely in terms of the addable and removable nodes of tilings and the sets $\text{Hook}_r(\mu)$ and this is compatible with restriction to $\Pi$ (using Lemma 4.26). The sets $\text{Hook}_r(\mu)$ for $\mu \in \Pi$ depend only on information which is preserved under $j : \Pi \rightarrow \Pi'$ (using Lemma 4.20 to flip left versus right in the definition of $\text{Hook}_r(\mu)$, if necessary). Therefore the Tetris relations go through $j$. Finally, we note that the cyclotomic relations follow from the Tetris relations and Lemma 4.26. Thus the map $j$ is an algebra homomorphism.

One can similarly define $j^{-1}$ as the recolouring map in the opposite direction. We have that $j \circ j^{-1}$ and $j^{-1} \circ j$ are both identity maps (as they amount to recolouring and recolouring again) and so the map is indeed an algebra isomorphism. \hfill \Box

### 4.5. Fixed point subgroups and Morita equivalences

We now consider the group automorphisms, $\sharp$, for type $A_{2n-1}$ and $D_{2n+1}$ given by flipping the Coxeter diagrams through the horizontal and vertical axes, respectively. Explicitly, the map $\sharp$ is determined by $\sharp(s_i) = s_{2n-1-i}$ for the group of type $A_{2n-1}$. The map $\sharp$ is determined by $\sharp(s_0) = s_1$, $\sharp(s_1) = s_0$ and $\sharp(s_i) = s_i$ for the group of type $D_{2n+1}$. The fixed point groups of these automorphism are the groups $\langle s_i, s_{2n-1-i} \ | \ 1 \leq i < n \rangle$ of type $B_n$ and $\langle s_0 s_1, s_1 \ | \ 2 \leq i \leq n \rangle$ of type $C_n$. By restricting our attention from the group to its fixed point subgroup, we obtain a surjective map $\iota$ on the tile-colourings. These are depicted in Figures 19 and 20.

**Remark 4.27.** For the remainder of this section, we will fix $\sigma = s_2$ and $\tau = s_1$ for $(W,P) = (C_n, A_{n-1})$ and $\sigma = s_1$ and $\tau = s_2$ for $(W,P) = (B_n, B_{n-1})$.

We extend our colouring convention from Remark 4.27 by setting $\sigma = s_n$ in type $A_{2n-1}$ and $\sigma = s_2$ in type $D_{2n+1}$. We similarly set $\rho = s_{n-1}$, $\pi = s_{n+1}$ in type $A_{2n-1}$ and $\rho = s_0$, $\pi = s_1$ in type $D_{2n+1}$ so that green and purple map to blue in both cases. We can easily extend this colouring map to a map, $\iota$, on the level of paths and moreover we have the following:

**Lemma 4.28.** The colouring maps on paths map bijectively onto the subsets of parity preserving paths from Subsection 1.3. In other words,

\[
\iota : \text{Path}(D_{2n+1}, A_n)(\lambda, t_\mu) \rightarrow \text{Path}^+(C_n, A_{n-1})(\lambda, t_\mu),
\quad
\iota : \text{Path}(A_{2n-1}, A_{2n-2})(\lambda, t_\mu) \rightarrow \text{Path}^+(B_n, B_{n-1})(\lambda, t_\mu)
\]
are both grading-preserving bijections.

We now prove that $h(C_n, A_{n-1})$ and $h(B_n, B_{n-1})$ are graded Morita equivalent to $h(D_{n+1}, A_n)$ and $h(A_{2n-1}, A_{2n-2})$ respectively.

![Figure 19. An example of the colouring map $\tau$ from type $D_{n+1}$ to type $C_n$.](image)

![Figure 20. An example of the colouring map $\tau$ from type $A_{2n-1}$ to type $B_n$.](image)

**Lemma 4.29.** Let $m(\beta, \gamma) = 3$ or $m(\beta, \gamma) = 4$. If $m(\beta, \gamma) = 4$, then suppose that $(\beta, \gamma) = (\tau, \sigma)$ as in Remark 4.27. We have that

\begin{align*}
\text{fork}_{\beta \beta}(1_\beta \otimes \text{bar}(\gamma) \otimes 1_\beta) & = -1_\beta \\
\text{cap}^0_{\beta \beta}(1_\beta \otimes \text{bar}(\gamma) \otimes 1_\beta) & = -\text{bar}(\beta) \\
\text{fork}_{\beta \beta}(1_\beta \otimes \text{bar}(\gamma) \otimes \text{bar}(\beta) \otimes 1_\beta) & = -1_\beta \otimes (2\text{bar}(\gamma) + \text{bar}(\beta)) \quad (4.13)
\end{align*}

**Proof.** Equation (4.11) follows by applying the $\gamma\beta$-barbell relation, followed by the $\beta$-circle annihilation relation and $\beta$-fork-spot contraction relation. Equation (4.12) follows from equation (4.11) by applying the spot generator on top and bottom. Equation (4.13) follows by applying the $\gamma\beta$-barbell relation to the lefthand-side, followed by equation (4.11). $\square$

We will find the following shorthand useful,

\begin{align*}
\text{trid}_{\tau \sigma \tau}^T = \text{fork}_{\tau \tau}^T (1_\tau \otimes \text{spot}_\sigma \otimes 1_\tau) \\
\text{trid}^0_{\tau \sigma \tau} = \text{spot}^0_{\tau \sigma \tau} \text{trid}^T_{\tau \sigma \tau}
\end{align*}

the former of which can be pictured as a “trident”. We set

\begin{align*}
\text{trid}^0_{\tau \sigma \tau} = (\text{trid}^T_{\tau \sigma \tau})^* \\
\text{trid}^0_{\tau \sigma \tau} = \text{trid}^T_{\tau \sigma \tau} \text{trid}^T_{\tau \sigma \tau}
\end{align*}

By equation (4.11), we have that $-\text{trid}^0_{\tau \sigma \tau}$ is an idempotent and that

\begin{align*}
(1_{\tau \sigma \tau} + \text{trid}^0_{\tau \sigma \tau} \text{trid}^0_{\tau \sigma \tau}) = 0. \quad (4.14)
\end{align*}

**Definition 4.30.** Let $(W, P) = (C_n, A_{n-1})$ or $(B_n, B_{n-1})$ and suppose $\sigma, \tau \in S_W$ satisfy the assumptions of Remark 4.27. For $\mu \in \mathcal{P}(W, P)$ and $1 < k \leq \ell_\tau(\mu)$, we set

\begin{align*}
\rho_k = \begin{cases} 
(1, 2, 3, \ldots, k) & \text{if } W = C_n \\
(n, 1) & \text{if } W = B_n
\end{cases}
\end{align*}

and we set $\kappa = \rho_k - \tau - \sigma - \tau$ and define

\begin{align*}
e^K_\mu & = \text{braid}^{t_\nu}_{\kappa} \text{trid}^0_{\tau \sigma \tau} (1_\kappa \otimes \text{trid}^T_{\tau \sigma \tau} \otimes 1_{\mu - \rho_k}) \text{braid}^{t_\nu}_{\kappa} \text{trid}^0_{\tau \sigma \tau}. \quad (4.15)
\end{align*}
We define the idempotent

\[ e_\mu = \prod_{1 < k \in \ell_\tau(\mu)} (1_{t_\mu} + e_k^\mu) \]

and set \( e_\mu = 1_{t_\mu} \) if \( \ell_\tau(\mu) \leq 1 \). We define \( e = \sum_{\mu \in \mathcal{P}(w, p)} e_\mu \).

![Figure 19](image1)

Figure 19. The summands are \( 1_{t_\mu}, e_\mu^1, e_\mu^2, \) and \( e_\mu^3 e_\mu^3 \) respectively.

![Figure 20](image2)

Figure 21. The element \( e_\mu \) for \( \mu = (1, 2, 3) \) in type \( C \). The colouring is the same as that of Figure 19. The summands are \( 1_{t_\mu}, e_\mu^2, e_\mu^3, \) and \( e_\mu^3 e_\mu^3 \) respectively.

We can extend the maps \( \iota \) from Lemma 4.28 to injective \( k \)-linear maps \( \iota : h(D_{n+1}, A_n) \rightarrow h(C_n, A_{n-1}) \) and \( \iota : h(A_{2n-1}, A_{2n-2}) \rightarrow h(B_n, B_{n-1}) \) by setting \( \iota(c_{ST}) = c_{\iota(S)\iota(T)} \). We note that \( \iota \) is not a \( k \)-algebra homomorphism, but we will prove the following:

**Theorem 4.31.** The maps \( \Theta : h(D_n, A_n) \rightarrow eh(C_n, A_{n-1})e \) and \( \Theta : h(A_{2n-1}, A_{2n-2}) \rightarrow eh(B_n, B_{n-1})e \)
defined by \( \Theta(a) = e \circ \iota(a) \circ e \) are graded \( k \)-algebra isomorphisms. Moreover, as \( e \) is a full idempotent these maps give rise to graded Morita equivalences between \( h(D_{n+1}, A_n) \) and \( h(C_n, A_{n-1}) \) and between \( h(A_{2n-1}, A_{2n-2}) \) and \( h(B_n, B_{n-1}) \).

We note that the first isomorphism categorifies an observation of Boe in [Boe88]. This section is dedicated to the proof. We begin with the simpler result for orthogonal groups.

### 4.5.1. The orthogonal case

We first consider the case of the orthogonal group. We can simplify the proof by focussing on the cellular basis. We prove that if \( c_{STUV} = \sum a_{XY}c_{XY} \) for coefficients \( a_{XY} \in k \), then we have that

\[ \Theta(c_{ST})\Theta(c_{UV}) = \sum a_{XY}\Theta(c_{XY}) \]

for \( S, T, U, V, X, Y \in \text{Path}(A_{2n-1}, A_{2n-2}) \) and hence deduce that Theorem 4.31 holds for type \((B_n, B_{n-1})\). For \( \mu = (c-1) \) with \( [1, c] \in \text{Add}(\mu) \) with \( \gamma = s_{[1, c]} \) and \( 1 \leq c \leq n \) the elements \( e_\mu^c \)

are of the form

\[ 1_{t_\mu}, 1_{t_\mu} \otimes \text{spot}^\gamma_1, 1_{t_\mu} \otimes \text{spot}^\gamma_2, \text{ and } 1_{t_\mu} \otimes \text{gap}(\gamma). \]

Thus rewriting products in equation (4.16) requires only the idempotent, bi-chrome Tetris, commutativity and cyclotomic relations. We consider the bi-chrome Tetris relation as the others are trivial. By Proposition 4.18 and Lemma 4.20, we have that

\[ c_{CT}^\gamma = 1_{t_\mu} \otimes \text{bar}(\gamma) = \text{gap}(t_\mu - [1, c-1]) \]

for \( T \in \text{Path}(\mu, t_{\mu+n}) \). For \( c \leq n \), we have that \( e_{t_\mu(\gamma)} = 1_{t_{(c)}} = 1_c \) and

\[ \Theta(1_{t_{(c-1)}} \otimes \text{bar}(\gamma)) = 1_{(c-1)} \otimes \text{gap}(\iota(\gamma)) = \Theta(\text{gap}(t_\mu - [1, c-1])) \]

as required. For \( c = n + 1 \) we have that \( \iota(\gamma) = \tau \) and

\[ \Theta(1_{(n)} \otimes \text{bar}(\gamma)) = \iota(\tau) \circ (1_{(n, 1)} + 1_{(n-2)} \otimes \text{trid}_{\tau/(\alpha \tau)}) \circ \iota(c_\tau^2) = 1_{(n)} \otimes \text{bar}(\tau) + 1_{(n-1)} \otimes \text{gap}(\sigma) \]
where as required. Here the first equality follows from the definition of $e$; the second from the $\tau$-fork-spot contraction relation; the third equality from Proposition 4.18 and Lemma 4.20; and the fourth is trivial. For $c = n + 2$, we have that

$$
\Theta(1_{t(n+1)} \otimes \text{bar}(\gamma)) = 1_{t(n)} \otimes \text{bar}(\gamma) + e_{t(n-2)} \otimes \text{trid}_0^{\sigma} \otimes \text{bar}(\gamma)
$$

$$
= -\text{gap}(t(n,1) - [n,2]) - 2\text{gap}(t(n,1) - [n,1]) - e_{t(n-2)} \otimes \text{trid}_0^{\sigma} \otimes \text{trid}_\tau^{\sigma \tau} \otimes \text{bar}(\gamma)
$$

$$
= -\text{gap}(t(n,1) - [n,2]) - e_{t(n-2)} \otimes \text{trid}_0^{\sigma} \otimes \text{trid}_\tau^{\sigma \tau} \otimes \text{bar}(\gamma)
$$

where the first equality is trivial; the second follows by applying Proposition 4.18 and Lemma 4.20 to the first term and applying the $(\tau, \iota(\gamma))$-null-braid and $\iota(\gamma)$-fork-spot-contraction to the second term; the third follows by applying (2.1) to the $\tau$-strands in the second term, followed by Proposition 4.18 and the cyclotomic and commutativity relations. On the other hand,

$$
\Theta(1_{t(n)} \otimes \text{gap}(\gamma)) = - (1_{t(n,1)} + 1_{t(n,2)} \otimes \text{trid}_\tau^{\tau \tau})(1_{t(n)} \otimes \text{gap}(\tau))(1_{t(n,1)} + 1_{t(n,2)} \otimes \text{trid}_\tau^{\tau \tau})
$$

$$
= -1_{t(n-2)} \otimes (1_{\tau} \otimes \text{gap}(\tau) + \text{trid}_\tau^{\tau \tau} \otimes \text{spot}_0^{\tau}) + \text{spot}_0^{\tau} \otimes \text{trid}_\tau^{\tau \tau} + \text{trid}_\tau^{\tau \tau} \otimes \text{bar}(\sigma))
$$

$$
= -1_{t(n-2)} \otimes (1_{\tau} \otimes \text{gap}(\tau) - \text{trid}_0^{\tau \sigma} \text{trid}_0^{\tau \sigma} \otimes \text{bar}(\tau) \otimes \text{dork}_0^{\tau \tau} + 1_{\tau} \otimes \text{gap}(\sigma) \otimes 1_{\tau})
$$

$$
= -\text{gap}(t(n,1) - [n,2]) - e_{t(n-2)} \otimes \text{trid}_0^{\tau \sigma} \otimes \text{trid}_0^{\tau \sigma}
$$

as required. Here the penultimate equality follows by applying 2.1 to the middle two terms and applying Proposition 4.18 and Lemma 4.20 to the final term; the final equality follows from Proposition 4.18 and Lemma 4.20 and the commutativity and cyclotomic relations. Finally, we suppose that $c > n + 2$. We have that

$$
\Theta(1_{t(c-1)} \otimes \text{bar}(\gamma)) = (1_{t(n,1-c-1-n)} + e_{t(n,1-c-1-n)}) \otimes \text{bar}(\gamma)
$$

$$
= -2\text{gap}(t(n,1-c-1-n) - [1,n]) - \text{gap}(t(n,1-c-1-n) - [c-1-n,n])
$$

$$
= -\text{gap}(t(n,1-c-1-n) - [c-1-n,n])
$$

$$
= \Theta(\text{gap}(t(c+1) - [1,c+1]))
$$

as required. For the second equality, we apply Proposition 4.18 and Lemma 4.20 to the first term and observe that the second term is zero by applying the bull-braid relations followed by Proposition 4.18 and Lemma 4.20 and the commutativity and cyclotomic relations. The other equalities are trivial. Thus the bi-chrome Tetris relation holds in all cases and we are done.

### 4.5.2. The symplectic case

We now consider the, more difficult, case of the symplectic group.

**Lemma 4.32.** For $e_{\tau \sigma \tau \sigma} = (1_{\tau \sigma \tau \sigma} + \text{trid}_0^{\tau \sigma} \otimes 1_{\sigma})(1_{\tau \sigma \tau \sigma} + 1_{\tau} \otimes \text{trid}_0^{\tau \sigma})$, we have that

$$
(1_{\tau} \otimes \text{trid}_0^{\tau \sigma} \otimes 1_{\tau})e_{\tau \sigma \tau \sigma} = -\text{trid}_0^{\tau \sigma}(1_{\tau} \otimes \text{trid}_0^{\tau \sigma} \otimes \tau) - \text{trid}_0^{\tau \sigma} \otimes \text{trid}_\tau^{\tau \sigma}
$$

$$
= -\text{trid}_0^{\tau \sigma}(1_{\tau} \otimes \text{trid}_0^{\tau \sigma} \otimes \tau) e_{\tau \sigma \tau \sigma}
$$

**Proof.** We prove the first equality, the second follows as $e_{\tau \sigma \tau \sigma}$ is an idempotent which kills the second term. Consider the $m(\sigma, \tau) = 4$ braid relation and tensor it on the left by $1_{\tau}$. Vertically concatenating $\text{trid}_0^{\tau \sigma} \otimes 1_{\sigma}$ on top of this combination of diagrams, we obtain

$$
1_{\tau \sigma} \otimes \text{trid}_0^{\tau \sigma} + \text{trid}_0^{\tau \sigma} \otimes 1_{\tau} + \text{trid}_0^{\tau \sigma}(1_{\tau} \otimes \text{trid}_0^{\tau \sigma} \otimes 1_{\tau}) + 2\text{trid}_0^{\tau \sigma} \otimes \text{trid}_\tau^{\tau \sigma} + 1_{\tau} \otimes \text{trid}_0^{\tau \sigma} \otimes 1_{\tau} = 0.
$$

Moving the third term and one copy (of the available two) of the fourth term to the right, the result follows. \(\square\)

We will split the proof of (the symplectic case of) Theorem 4.31 into two propositions. the first one, Proposition 4.33, shows that $\Theta$ is an isomorphism of graded vector spaces. The second one, Proposition 4.37, shows that $\Theta$ is an algebra homomorphism.
Proposition 4.33. We have that the map $\Theta : h_{(D_{n+1}, A_n)} \to e h_{(C_n, A_{n-1})} e$ given by $\Theta(c_{ST}) = e \circ \tau(c_{ST}) \circ e$ is an isomorphism of graded $k$-spaces.

Proof. We will show that the set

$$\{e \circ c_{ST} \circ e \mid S, T \in \text{Path}_{(C_n, A_{n-1})}^\pm(\lambda, t_\mu)\}$$

(4.17)

form a basis of $e h_{(C_n, A_{n-1})} e$ and thus deduce the result. We do this by considering $\Delta(\lambda)e$ for all $\lambda \in \mathcal{P}_{(C_n, A_{n-1})}$. We will prove the following claim:

$$(c_S + h^\gamma_{(C_n, A_{n-1})})e = \begin{cases} c_S + \sum_{T \in \text{Path}_{(C_n, A_{n-1})}^\pm(\lambda, t_\mu)} a_{TCT} + h^\gamma_{(C_n, A_{n-1})} & \text{if } S \in \text{Path}_{(C_n, A_{n-1})}^\pm(\lambda, t_\mu) \\ 0 & \text{otherwise} \end{cases}$$

from which we will immediately deduce the result. We first note that we can choose our $t_\mu$ for each $\mu \in \mathcal{P}_{(C_n, A_{n-1})}$ in such a way that $\sigma$ always occurs immediately prior to a $\tau$. We prove this for $t_{\mu+\gamma}$ assuming it holds for $t_\mu$ (with the $\ell(t_\mu) = 0$ case being trivial). Let $S \in \text{Path}(\lambda, t_\mu)$. For $\gamma \neq \tau$, we have that

$$(A^+_{\gamma}(c_S)e = A^+_{\gamma}(c_{SE}) \quad (R^+_{\gamma}(c_S))e = R^+_{\gamma}(c_{SE})$$

for $\gamma \in \text{Add}(\lambda)$ or $\gamma \in \text{Add}(\mu)$, respectively. (Whence $\ell_{\tau}(\mu + \gamma) = \ell_{\tau}(\mu)$ implies $e_{\mu+\gamma} = e_\mu \otimes 1_\gamma$.)

Thus we may now assume that $\gamma = \tau \in \text{Add}(\mu)$. In which case $\sigma = s_2 \in \text{Rem}(\mu)$ by our choice of $t_\mu$. We let $\mu' = \mu - \sigma$. We suppose $\ell_{\tau}(\mu)$ is odd (the even case is identical) so that $\pi \in \text{Add}(\iota^{-1}(\mu))$ and $\rho \in \text{Rem}(\iota^{-1}(\mu'))$. Given $\lambda' \subseteq \mu'$, we let $c_{S'} \in \text{Path}_{(C_n, A_{n-1})}(\lambda', t_{\mu'})$. We construct $c_S$ for $S \in \text{Path}_{(C_n, A_{n-1})}(\lambda, t_{\mu+\tau})$ by applying the inductive process twice: once for $\sigma$ and once for $\tau$ as follows,

$$c_S = X^+_{\tau} X^+_{\sigma}(c_{S'})$$

for $X \in \{A, R\}$. Note that

$$c_{SE} = (X^+_{\tau} X^+_{\sigma}(c_{SE})) (1_{\tau_{\sigma}} + e_{\mu+\tau}^{(\mu+1)}).$$

(4.18)

We assume, by induction, that the claim holds for $c_{S'}$. So we have

$$c_{SE} = c_{S'} + \sum_{T' \in \text{Path}_{(C_n, A_{n-1})}^\pm(\lambda', t_{\mu'})} a_{T'CT'} + h^\lambda_{(C_n, A_{n-1})}$$

(4.19)

since $T' \notin \text{Path}_{(C_n, A_{n-1})}^\pm(\lambda', t_{\mu'})$, this implies by definition $X^+_{\tau} X^+_{\sigma}(c_{T'}) \notin \text{Path}_{(C_n, A_{n-1})}^\pm(\lambda, t_{\mu})$. We will now consider

$$c_S = X^+_{\tau} X^+_{\sigma}(c_{S'})$$

for $S' \in \text{Path}_{(C_n, A_{n-1})}^\pm(\lambda', t_{\mu'})$. Before considering the above case-wise, we remark that either $\rho \in \text{Rem}(\iota^{-1}(\lambda'))$ or $\text{Add}(\iota^{-1}(\lambda'))$ (because it appears at the edge of the region).

Case 1. Suppose $\sigma \in \text{Add}(\lambda')$. This implies that $\rho \in \text{Rem}(\iota^{-1}(\lambda'))$. The first two subcases which we consider simultaneously are

$$A^+_{\tau} A^+_{\sigma}(c_{S'}) = c_{S'} \otimes 1_{\sigma} \otimes 1_{\tau} \quad A^+_{\tau} A^+_{\sigma}(c_{S'}) = c_{S'} \otimes 1_{\sigma} \otimes \text{spot}_{\tau}.$$  

Here we have that $c_S = A^+_{\tau} A^+_{\sigma}(c_{S'})$ satisfies $S \in \text{Path}_{(C_n, A_{n-1})}^\pm$. We have that

$$(A^+_{\tau} A^+_{\sigma}(c_{S'})) + h^\lambda_{(C, A)}(1 + e_{\mu+\tau}^{(\mu+1)}) = c_S + c_{S'} \otimes \text{trid}_{\sigma \tau}^{T_{\tau}^\sigma} + h^\lambda_{(C, A)} = c_S + h^\lambda_{(C, A)}$$

$$(A^+_{\tau} A^+_{\sigma}(c_{S'}) + h^\lambda_{(C, A)}(1 + e_{\mu+\tau}^{(\mu+1)}) = c_S + c_{S'} \otimes (\text{trid}_{\sigma \tau}^{T_{\tau}^\sigma} \otimes \text{spot}_{\tau}) + h^\lambda_{(C, A)} = c_S + h^\lambda_{(C, A)}$$

where in both cases the diagram $c_{S'} \otimes \ldots$ factors through the idempotent labelled by $t_{\lambda'}$ and so belongs to the ideal $h^\lambda_{(C, A)}$. The final two subcases which we will consider simultaneously are

$$R^+_{\tau} A^+_{\sigma}(c_{S'}) = c_{S'} \otimes \text{trid}_{\sigma \tau}^{T_{\tau}^\sigma} \quad R^+_{\tau} A^+_{\sigma}(c_{S'}) = c_{S'} \otimes \text{trid}_{\sigma \tau}^{T_{\tau}^\sigma}.$$  

Here $c_S = R^+_{\tau} A^+_{\sigma}(c_{S'})$ satisfies $S \notin \text{Path}_{(C_n, A_{n-1})}^\pm$. We note that $e_{\mu+\tau}^{(\mu+1)} = 1_{\tau_{\sigma}} \otimes (1_{\tau_{\sigma}} + \text{trid}_{\sigma \tau}^{T_{\tau}^\sigma})$ and hence applying equation (4.11) we obtain

$$(R^+_{\tau} A^+_{\sigma}(c_{S'}))(1 + e_{\mu+\tau}^{(\mu+1)}) = 0$$
Lemma 4.36. For $\sigma \in \text{Rem}(\lambda^\prime)$. This implies that $\rho \in \text{Add}(\tau^{-1}(\lambda^\prime))$. Since any two $\sigma$-relations in $\mu^\prime$ are separated by some $\tau$-tile (and $\tau \not\in \text{Rem}(\lambda^\prime)$ but $\sigma \in \text{Rem}(\lambda^\prime)$) we have that $c_{\sigma^\prime} = c_{\sigma^\prime} \circ \text{spot}_{\tau}^\theta$ for some $S'' \in \text{Path}^\pm_{(C_n,A_n-1)}(\lambda^\prime, \mu^\prime - \tau)$. Here we have that

$$R_{\alpha}^+(c_{\sigma^\prime}) = c_{\sigma^\prime} \oplus \text{fork}_{\alpha}^\tau = c_{\sigma^\prime} \circ \text{spot}_{\tau}^\theta \oplus \text{fork}_{\alpha}^\tau$$

We start with

$$(A_{+}^\tau R_{\sigma}^+(c_{\sigma^\prime}))((1_{\mu^\prime}, \rho_{\mu^\prime}^\tau + e_{\mu^\prime}^\tau(\rho))((1_{\mu^\prime}, \rho_{\mu^\prime}^\tau + e_{\mu^\prime}^\tau(\rho)) + 1),$$

and we first consider $A_{+}^\tau R_{\sigma}^+(c_{\sigma^\prime}).$ We note that $\tau \in \text{Add}(\lambda^\prime)$ but that $\rho \not\in \text{Add}(\tau^{-1}(\lambda^\prime))$ (rather, the “wrong” colour $\rho$ is). Therefore $S \not\in \text{Path}^\pm_{(C_n,A_n-1)}(\lambda, \mu^\prime)$ and using Lemma 4.32 we have

$$(A_{+}^\tau R_{\sigma}^+(c_{\sigma^\prime})) \circ (1_{\mu^\prime} + e_{\mu^\prime}(\rho)) \circ (1_{\mu^\prime} + e_{\mu^\prime}(\rho)) + 1) \in h_{(C,A)}^\lambda$$

since the terms in the sum over $\sigma^\prime$ through the idempotent $\lambda - \tau$. Therefore $c_{\sigma^\prime} = 0$ modulo $h_{(C,A)}^\lambda$, as required. Arguing in an identical manner, (or by simply “putting a blue spot on top of the above calculation”) we have that

$$(A_{+}^\tau R_{\sigma}^+(c_{\sigma^\prime}))((1_{\mu^\prime} + e_{\mu^\prime}(\rho))((1_{\mu^\prime} + e_{\mu^\prime}(\rho)) + 1) \in h_{(C,A)}^\lambda$$

as required. The final two subcases which we will consider simultaneously are

$$c_{\sigma} = R_{\sigma}^+ R_{\sigma}^+(c_{\sigma^\prime}) = c_{\sigma^\prime} \circ \text{spot}_{\tau}^\theta \oplus \text{fork}_{\sigma^\prime}^\tau$$

$$c_{\sigma} = R_{\sigma}^+ R_{\sigma}^+(c_{\sigma^\prime}) = c_{\sigma^\prime} \circ \text{spot}_{\tau}^\theta \oplus \text{fork}_{\sigma^\prime}^\tau$$

In both cases, $S \in \text{Path}^\pm_{(C_n,A_n-1)}$. We have that

$$(R_{+}^\tau R_{\sigma}^+(c_{\sigma^\prime}))((1_{\mu^\prime} + e_{\mu^\prime}(\rho))((1_{\mu^\prime} + e_{\mu^\prime}(\rho)) + 1) \in h_{(C,A)}^\lambda$$

In each case the former term on the right-hand-side of the equality is equal to $c_{\sigma}$ and the latter term is equal to $c_{\sigma^\prime}$ for $T \not\in \text{Path}^\pm_{(C_n,A_n-1)}(\lambda, \mu^\prime)$. The result follows.

Lemma 4.34. Let $e_{\tau \sigma \tau \sigma} = (1_{\tau \sigma \tau \sigma} + \text{trid}_{\tau \sigma \tau \sigma} \otimes 1_{\sigma}$), then we have

$$e_{\tau \sigma \tau \sigma} \cdot 1_{\tau \sigma \tau \sigma} \cdot e_{\tau \sigma \tau \sigma} = -e_{\tau \sigma \tau \sigma} \cdot (1_{\tau} \otimes \text{trid}_{\tau \sigma \sigma \tau}) e_{\tau \sigma \tau \sigma}$$

Proof. Applying $e_{\tau \sigma \tau \sigma}$ to both sides of the $m(\sigma, \tau) = 4$ null-braid relations and using equation (4.14) immediately gives the result.

Lemma 4.35. Let $\mu \in \mathcal{P}(C_{n+1} \setminus A_n)$. If $[r, c] \in \mu$ we have that $e \circ \text{gap}(t_\mu - [r, r - 1]) \circ e = 0$. Now, if $[r, c] \in \mu$ and $[r, c+1] \not\in \mu$, we have

$$e \circ \text{gap}(t_\mu - [r, c]) \circ e = (-1)^{\ell(\Omega_{[r,c]})} e \circ c_{SS} \circ e$$

for $S$ the unique element of $\text{Path}(\Omega_{[r,c]}(\mu), t_\mu)$, if $\Omega_{[r,c]}(\mu)$ is defined and zero otherwise. If $[r, c], [r+1, c], [r, c-1], [r+1, c-1] \in \mu$, we have that

$$\text{gap}(t_\mu - [r, c]) = \text{gap}(t_\mu - [r+1, c-1])$$

Proof. The first statement follows directly from equation (2.1) and (4.14). The latter two statements are symplectic analogues of Lemma 4.10 and Proposition 4.15; the proofs of which only make use of the simply laced null-braid relations (and commutativity and cyclotomic relations). Thus one can mimic the proofs and constructions verbatim in $e h_{(C_n,A_n-1)} e$ using Lemma 4.34 (and commutativity and cyclotomic relations).

Lemma 4.36. For $S \in \text{Path}(D_{n+1} \setminus A_n)(\lambda, t_\mu)$, we have that $\nu(c_S) \circ e = e \circ \nu(c_S) \circ e$. 

Proposition 4.37. The map $\Theta : h(D_{n+1}, A_n) \to eh(h_{C_n, A_{n-1}}) e$ given by $\Theta(c_{ST}) = e \circ i(c_{ST}) \circ e$ is a $k$-algebra homomorphism.

Proof. We check this on the cellular basis by showing that
$$e \circ i(c_{ST} c_{UV}) \circ e = \Theta(c_{ST} c_{UV}) = \Theta(c_{ST}) \Theta(c_{UV}) = e \circ i(c_{ST}) \circ e \circ i(c_{UV}) \circ e$$
for $S \in \text{Path}(D_{n+1}, A_n)(\nu, -)$, $T \in \text{Path}(D_{n+1}, A_n)(\nu, \mu)$, $U \in \text{Path}(D_{n+1}, A_n)(\eta, \mu)$, $V \in \text{Path}(D_{n+1}, A_n)(\eta, -)$. By Lemma 4.36, we have that
$$e \circ i(c_{ST}) \circ e \circ i(c_{UV}) \circ e = e \circ i(c_{ST}) \circ e \circ i(c_{UV}) \circ e \circ i(c_{U}) \circ e \circ i(c_{V}) \circ e.$$

We proceed by induction on $\ell(\mu)$, the base case $\ell(\mu) = 0$ is trivial. We can assume $\ell(\pi) = \ell(\rho) < \ell(\mu)$ as if $\ell(\pi) = \ell(\rho) = \ell(\mu)$ then $1_{\pi} = 1_{\rho} = 1_{\mu}$ and this product becomes $e \circ i(c_{SV}) \circ e$ as required. Similarly, if $\ell(\pi) = \ell(\mu)$ and $\ell(\rho) < \ell(\mu)$ (or vice versa) this product becomes
$$e \circ i(c_{ST}) \circ e \circ i(c_{U}) \circ e \circ i(c_{V}) \circ e = e \circ i(c_{ST}) \circ e \circ i(c_{UV}) \circ e \circ i(c_{V}) \circ e \circ i(c_{U}) \circ e$$
and so we can again appeal to our inductive assumption. We will focus on the middle of the product and prove that
$$e \circ i(c_{ST}) \circ e \circ i(c_{U}) \circ e = e \circ i(c_{TU}) \circ e. \quad (4.20)$$
As $\ell(\eta), \ell(\nu) \leq \ell(\mu)$ we can then apply induction to deal with the products with $e \circ i(c_{ST})$ and $i(c_{UV}) \circ e$. Now, the basis elements $c_{ST}$ and $c_{UV}$ are constructed inductively and we will consider cases depending on the last step in this inductive procedure.

Case 1. We first consider the case that $c_{ST} = A_{\pi}^+(c_{ST})$ and $c_{UV} = A_{\rho}^+(c_{UV})$. By induction, we can assume that
$$e \circ i(c_{ST}) \circ e \circ i(c_{UV}) \circ e = \sum X_{XY} a_{XY} e \circ i(c_{XY}) \circ e \quad \text{where} \quad c_{ST} c_{UV} = \sum X_{XY} a_{XY} c_{XY}$$
If $\alpha \neq \pi, \rho$, then $\ell_{\tau}(i(\mu)) = \ell_{\tau}(i(\mu - \alpha))$ and therefore
$$e \circ i(c_{ST}) \circ e \circ i(c_{UV}) \circ e = \sum X_{XY} a_{XY} (e \circ i(c_{XY}) \otimes 1_{\alpha}) \circ e.$$
If $\alpha = \pi$ (the $\rho$ case is identical) then $c_{ST} = A_{\pi}^+ A_{\sigma}^+(c_{ST})$, $c_{UV} = A_{\pi}^+ A_{\sigma}^+(c_{UV})$ and
$$e \circ i(c_{ST}) \circ e \circ i(c_{UV}) \circ e = e \circ i(\sum X_{XY} a_{XY} (e \circ i(c_{XY}) \otimes 1_{\alpha}) \circ e) \circ e \circ i(\sum X_{XY} a_{XY} (e \circ i(c_{XY}) \otimes 1_{\alpha}) \circ e)$$
the first equality follows from the definition of the idempotents; for the second equality, we note that the trident term in the sum is zero by equation (4.14); the third equality follows
by definition of the cellular basis elements and the idempotents; the final equality holds by induction. Thus in all cases, we have that
\[ e \circ i(c_T) \circ e \circ i(c_U^*) \circ e = \sum_{X,Y} a_{X,Y} e(i(c_{XY}) \otimes 1_{i(\alpha)}) e. \]  
\( (4.21) \)

It remains to show that every \( e(i(c_{XY}) \otimes 1_{i(\alpha)}) e = e(i(c_{XY} \otimes 1_{\alpha})) e \) for every \( X, Y \) appearing in the the above sum. We set \( \lambda := \text{Shape}(X) = \text{Shape}(Y) \).

\[ c_{XY} = c_{X'} \otimes \text{spot}^\emptyset_{\alpha} \otimes \text{fork}_{\sigma\sigma}^\emptyset \] 
\[ c_{Y} = c_{Y'} \otimes \text{spot}^\emptyset_{\alpha} \otimes \text{fork}_{\sigma\sigma}^\emptyset \] 
\( (4.22) \)

for some \( X', Y' \in \text{Path}_{D_n+1, A_n}(\lambda + \rho, t_{\nu} - \rho - \sigma) \). Now, as \( \pi \in \text{Rem}(\lambda - \sigma) \), using the \( \pi\sigma \)-bull-braid relations we get
\[ c_{XY} \otimes 1_\pi = -(c_{X} \otimes \text{spot}^\emptyset_{\alpha} \otimes \text{fork}_{\pi\pi}^\emptyset)(c_{Y} \otimes \text{spot}^\emptyset_{\alpha} \otimes \text{fork}_{\pi\pi}^\emptyset) \] 
\( (4.23) \)

On the other hand, using equation \( (4.22) \) and Lemma 4.32 we have
\[ e(i(c_{XY}) \otimes 1_\tau) e = -(1_{\tau_1} \otimes \text{spot}^\emptyset_{\alpha} \otimes \text{fork}_{\tau\tau}^\emptyset)(1_{\tau_1} \otimes \text{spot}^\emptyset_{\alpha} \otimes \text{fork}_{\tau\tau}^\emptyset) e \]
and similarly for \( (i(c_{Y}) \otimes 1_\tau) e \). Thus we get
\[ e(i(c_{XY}) \otimes 1_\tau) e = -(1_{\tau_1} \otimes \text{spot}^\emptyset_{\alpha} \otimes \text{fork}_{\tau\tau}^\emptyset)(1_{\tau_1} \otimes \text{spot}^\emptyset_{\alpha} \otimes \text{fork}_{\tau\tau}^\emptyset) e \]
\[ = -e(i(c_{X}) \otimes \text{spot}^\emptyset_{\alpha} \otimes \text{fork}_{\tau\tau}^\emptyset)(i(c_{Y}) \otimes \text{spot}^\emptyset_{\alpha} \otimes \text{fork}_{\tau\tau}^\emptyset) e \] 
\( (4.24) \)

comparing equation \( (4.23) \) and \( (4.24) \) we are done.

Figure 23. Case 1: a diagrammatic version of equation \( (4.21) \).

Subcase 1.1. If \( \alpha \in \text{Add}(\lambda) \) and \( i(\alpha) \in \text{Add}(i(\lambda)) \) then \( i(c_{XY}) \otimes 1_{i(\alpha)} \) and \( c_{XY} \otimes 1_{\alpha} \) are both cellular basis elements and we are done.

Subcase 1.2. If \( \alpha \notin \text{Add}(\lambda) \) and \( i(\alpha) \in \text{Add}(i(\lambda)) \) then we can assume that \( \alpha = \pi \) (the \( \alpha = \rho \) is identical). This implies that \( \rho \in \text{Add}(\lambda) \) (but by assumption \( \sigma \in \text{Rem}(\lambda) \) and \( \pi \in \text{Rem}(\lambda - \sigma) \)) this implies that we can write \( c_{X} \) and \( c_{Y} \) as
\[ c_{X} = c_{X'} \otimes \text{spot}^\emptyset_{\alpha} \otimes \text{fork}_{\sigma\sigma}^\emptyset \] 
\[ c_{Y} = c_{Y'} \otimes \text{spot}^\emptyset_{\alpha} \otimes \text{fork}_{\sigma\sigma}^\emptyset \] 
\( (4.22) \)

for some \( X', Y' \in \text{Path}_{D_n+1, A_n}(\lambda + \rho, t_{\nu} - \rho - \sigma) \). Now, as \( \pi \in \text{Rem}(\lambda - \sigma) \), using the \( \pi\sigma \)-bull-braid relations we get
\[ c_{XY} \otimes 1_\pi = -(c_{X} \otimes \text{spot}^\emptyset_{\alpha} \otimes \text{fork}_{\pi\pi}^\emptyset)(c_{Y} \otimes \text{spot}^\emptyset_{\alpha} \otimes \text{fork}_{\pi\pi}^\emptyset) \] 
\( (4.23) \)

On the other hand, using equation \( (4.22) \) and Lemma 4.32 we have
\[ e(i(c_{XY}) \otimes 1_\tau) e = -(1_{\tau_1} \otimes \text{spot}^\emptyset_{\alpha} \otimes \text{fork}_{\tau\tau}^\emptyset)(1_{\tau_1} \otimes \text{spot}^\emptyset_{\alpha} \otimes \text{fork}_{\tau\tau}^\emptyset) e \]
and similarly for \( (i(c_{Y}) \otimes 1_\tau) e \). Thus we get
\[ e(i(c_{XY}) \otimes 1_\tau) e = -(1_{\tau_1} \otimes \text{spot}^\emptyset_{\alpha} \otimes \text{fork}_{\tau\tau}^\emptyset)(1_{\tau_1} \otimes \text{spot}^\emptyset_{\alpha} \otimes \text{fork}_{\tau\tau}^\emptyset) e \]
\[ = -e(i(c_{X}) \otimes \text{spot}^\emptyset_{\alpha} \otimes \text{fork}_{\tau\tau}^\emptyset)(i(c_{Y}) \otimes \text{spot}^\emptyset_{\alpha} \otimes \text{fork}_{\tau\tau}^\emptyset) e \] 
\( (4.24) \)

comparing equation \( (4.23) \) and \( (4.24) \) we are done.

Figure 24. Subcase 1.2
Subcase 1.3. If $\alpha \in \text{Rem}(\lambda)$ and $\iota(\alpha) \in \text{Rem}(\iota(\lambda))$ then the monochrome Tetris relation implies that
\[
\begin{align*}
    c_{XY} \otimes 1_\alpha &= (c_X \oplus \text{fork}^\alpha_{\iota(\alpha)})^* (c_Y \oplus \text{spot}^\iota_{\iota(\alpha)} \otimes 1_\alpha) + (c_X \oplus \text{spot}^\iota_{\iota(\alpha)} \otimes 1_\alpha)^* (c_Y \oplus \text{fork}^\alpha_{\iota(\alpha)}) \\
    &\quad + (c_X \oplus \text{fork}^\alpha_{\iota(\alpha)})^* (\sum_{[x,y] \in \text{Hole}_\alpha(\lambda - \alpha)} \text{gap}(t_\lambda - [x,y])) (c_Y \oplus \text{fork}^\alpha_{\iota(\alpha)})
\end{align*}
\]
Similarly, we obtain
\[
e(\iota(c_{XY}) \otimes 1_{\iota(\alpha)}) = e(\iota(c_X) \oplus \text{fork}^\iota_{\iota(\alpha)}(\iota(\alpha)))^* (\iota(c_Y) \oplus \text{spot}^\alpha_{\iota(\alpha)} \otimes 1_{\iota(\alpha)}) e
\]
\[
+ e(\iota(c_X) \oplus \text{spot}^\alpha_{\iota(\alpha)} \otimes 1_{\iota(\alpha)})^* (\iota(c_Y) \oplus \text{fork}^\iota_{\iota(\alpha)}(\iota(\alpha))) e
\]
\[
+ e(\iota(c_X) \oplus \text{fork}^\iota_{\iota(\alpha)}(\iota(\alpha)))^* e(\sum_{[x,y] \in \text{Hole}_\iota(\iota(\alpha))} \text{gap}(t_\iota(\lambda) - [x,y])) e(\iota(c_Y) \oplus \text{fork}^\iota_{\iota(\alpha)}(\iota(\alpha))) e
\]
where we have inserted extra idempotents $e$ in the final summand using Lemma 4.36. Recall that we need to check that
\[
e(\iota(c_{XY} \otimes 1_\alpha)) e = e(\iota(c_{XY}) \otimes 1_{\iota(\alpha)}) e
\]
where the first two terms in each of the above equations obviously agree. For the final term, note that if $\alpha \neq \rho, \pi$ we have
\[
\text{Hole}_\alpha(\lambda - \alpha) = \text{Hole}_{\iota(\alpha)}(\iota(\lambda - \alpha)).
\]
If $\alpha = \rho$ then $\iota(\alpha) = \tau$ (the $\alpha - \pi$ case is identical). Say $s_{[r,r]} = \rho \in \text{Rem}(\lambda)$. Using Lemmas 4.9 and 4.20 we have that
\[
(\sum_{[x,y] \in \text{Hole}_\alpha(\lambda - \alpha)} \text{gap}(t_\lambda - [x,y])) = (\sum_{[x,y] \in \text{RH}_\rho[r,r-1]\sqcup \text{RH}_\rho[r,r-2]} \text{gap}(t_\lambda - [x,y]))
\]
and we note that
\[
\text{RH}_{\rho}[r,r-1]\sqcup \text{RH}_\rho[r,r-2] \sqcup \{[r,r-1]\} = \text{Hole}_\rho(\iota(\lambda) - \tau) \tag{4.25}
\]
where $s_{[r,r-1]} = \sigma$. Comparing Lemma 4.10 and Proposition 4.15 against Corollary 4.35, we deduce for any $[a,b] \in \lambda$ that $\text{gap}(t_\lambda - [a,b]) = \pm c_{SS}$ (respectively 0) if and only if $e \circ \text{gap}(t_\lambda - [a,b]) \circ e = \pm e \circ \iota(c_{SS}) \circ e$ (respectively zero) for $S$ the path described in the proof of Proposition 4.15. Now, for $\alpha = \rho = s_{[r,r]}$ we have $\text{gap}(t_\lambda - [r,r-1]) = 0$ using Lemma 4.10 and $e \circ \text{gap}(t_\lambda - [r,r-1]) \circ e = 0$ by Corollary 4.35. By equation (4.25) we can conclude that subcase 1.3 holds.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{subcase-1.3.png}
\caption{Subcase 1.3.}
\end{figure}

Subcase 1.4. If $\alpha \notin \text{Rem}(\lambda)$ and $\iota(\alpha) \in \text{Rem}(\iota(\lambda))$ then we can assume that $\alpha = \pi$ (the $\alpha = \rho$ is identical). Then we must have that $s_{[r,r]} = \rho \in \text{Rem}(\lambda)$ and $s_{[r+1,r]} = \sigma$ and $s_{[r+1,r+1]} = \pi$. We have that
\[
c_X = c_X' \oplus \text{spot}^\rho_{\sigma} \quad c_Y = c_Y' \oplus \text{spot}^\rho_{\sigma}
\]
for $X', Y' \in \text{Path}((D_{n+1}, A_n)(\lambda + \sigma, -))$. Therefore
\[
c_{XY} \otimes 1_\pi = (c_X' \otimes 1_\pi) \text{gap}(t_\lambda + \sigma + \pi - [r+1,r]) (c_Y' \otimes 1_\pi) = 0
\]
using Lemma 4.10. On the other hand
\[ e \circ (i(c_{XY}) \otimes 1_r) \circ e = e \circ (i(c_{XY}') \otimes 1_r) \circ e \circ \text{gap}(t_{(\lambda)+\sigma+r} - [r+1, r]) \circ e \circ (i(c_{U'}) \otimes 1_r) \circ e = 0 \]
where the first equality follows from Lemma 4.36 (inserting extra idempotents, e) and the second follows by Corollary 4.35.

\textbf{Subcase 1.5.} If \( i(\alpha) \not\in \text{Add}(i(\lambda)) \) nor \( \text{Rem}(i(\lambda)) \), then \( \alpha \not\in \text{Add}(\lambda) \) or \( \text{Rem}(\lambda) \). Take \([x, y]\) with \(x+y\) minimal such that \( s_{[x,y]} = \alpha\) and \([x, y] \not\in \lambda\). Then precisely one of \([x, y-1]\) or \([x-1, y]\) \(\in\lambda\). We assume \([x, y-1] \in \lambda\) and we set \( \gamma = s_{[x,y-1]} \). We have
\[ c_{XY} \otimes 1_\alpha = -(c_{XY} \otimes \text{spot}_\gamma \otimes \text{fork}_{\gamma\alpha}^\alpha)^* (c_{U} \otimes \text{spot}_\gamma \otimes \text{fork}_{\gamma\alpha}^\alpha) \]
by the \( \gamma\alpha\)-null-braid relation. This might not be a cellular basis diagram, but can be rewritten as such using Lemma 4.10 and Proposition 4.15. Similarly \( e(i(c_{XY}) \otimes 1_{i(\alpha)}) e \) can be rewritten in the same form using Corollary 4.35. Subcase 1.5 follows.

\textbf{Case 2.} We now consider the case that \( c_T = A_\alpha^+(c_{T'}) \) and \( c_U = A_{\alpha}^- (c_{U'}) \) (the dual case with \( T \) and \( U \) swapped is similar). If \( \alpha \neq \pi, \rho \), then
\[
eq e \circ i(c_{T'}) \circ 1_{i(\alpha)} \circ e \circ i(c_{U'}) \otimes \text{spot}_{\alpha}(i(\alpha)) \otimes \text{spot}_\rho \circ e = e \circ (i(c_{T'}) \circ e \circ i(c_{U'})) \otimes \text{spot}_\rho \circ e = e \circ (i(c_{T'}) \circ e \circ i(c_{U'})) \otimes \text{spot}_\rho \circ e \]
and so the result follows by induction on \( \ell(\mu) \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure26}
\caption{Case 2 for \( \sigma \neq \pi \)}
\end{figure}

If \( \alpha = \pi \), then \( c_T = A_\pi^+(c_{T'}) \) and \( c_U = A_{\pi}^-(c_{U'}) \) and we also set \( c_{T'} = A_{\pi}^+(c_{T''}) \) and \( c_{U'} = A_{\pi}^-(c_{U''}) \). Expanding out the final term of the middle idempotent e and applying equation (4.14), we obtain
\[
eq e \circ (i(c_{T'}) \circ 1_{\pi\sigma} \otimes 1_{\pi} \otimes \text{spot}_\rho) \circ e = e \circ (i(c_{T'}) \circ e \circ i(c_{U''})^* \otimes \text{spot}_\rho) \circ e = e \circ (i(c_{T'}) \circ e \circ i(c_{U''})^* \otimes \text{spot}_\rho) \circ e \]
and so the result again follows by induction on length, as above.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure27}
\caption{Case 2 for \( \pi \)}
\end{figure}
Case 3. We now consider the case that \( c_T = R_T^\circ(c_T) \) and \( c_U = A_U^\circ(c_U) \) (the dual case with \( T \) and \( U \) swapped is identical). By the same inductive argument as we used in Case 1 (in order to deduce equation \((4.21)\)), we have that
\[
e \circ i(c_T) \circ e \circ i(c_U^\circ) \circ e = \sum_{X,Y} a_{X,Y} e(i(c_X^\circ) \otimes \text{fork}^\circ_{\iota(\alpha)}) (i(\alpha) \otimes 1_{i(\alpha)}) e.
\]
We set \( \lambda = \text{Shape}(X) = \text{Shape}(Y) \). Observe that \( \alpha \in \text{Add}(\lambda) \) or \( \text{Rem}(\lambda) \). If \( \alpha \in \text{Add}(\lambda) \), then
\[
c_X^\circ \otimes \text{fork}^\circ_{\iota(\alpha)} = (X^\circ_\alpha(c_{X'})^\circ \otimes \text{fork}^\circ_{\iota(\alpha)} = (X^\circ_\alpha(c_{X'}))^\circ
\]
for \( X \in \{R, A\} \) and some \( X' \). We have that \( c_Y \otimes 1_\alpha = A_U^\circ(c_Y) \). In particular, both diagrams are light leaves basis elements. Exactly the same is true replacing \( c_X, c_Y, \) and \( \alpha \) with their images under \( \iota \). The result follows.

![Figure 28. Case 3: a diagrammatic version of equation \((4.26)\).](image)

Case 4. Now suppose \( c_T = R_T^\circ(c_T) \) and \( c_U = A_U^\circ(c_U) \). In this case, we are simply placing a dot on the diagrams from case 3 and so the result follows from case 3 and induction on \( \ell(\mu) \).

![Figure 29. Case 5 for \( \alpha = \sigma \).](image)

Case 5. Let \( c_T = A_T^\circ(c_T) \) and \( c_U = A_U^\circ(c_U) \). If \( i(\alpha) \neq \tau \) then \( \ell_T(i(\mu)) = \ell_T(i(\mu - \alpha)) \) and so
\[
e \circ i(c_T) \circ e \circ i(c_U)^\circ \circ e = (e \circ i(c_T) \circ e \circ i(c_U)^\circ \circ e) \otimes \text{bar}(i(\alpha))
= e \circ (1_{i(\tau)} \otimes \text{bar}(i(\alpha))) \circ e \circ i(c_T) \circ e \circ i(c_U)^\circ \circ e
\]
and if \( i(\alpha) = \tau \) (say \( \alpha = \rho \) as the \( \pi \) case is identical) we have
\[
e \circ i(c_T) \circ e \circ i(c_U)^\circ \circ e = e \circ (1_{i(\tau)} \otimes \text{bar}(i(\alpha))) \circ e \circ i(c_T) \circ e \circ i(c_U)^\circ \circ e
+ e \circ (1_{i(\tau)} \otimes \text{gap}(\sigma)) \circ e \circ i(c_T) \circ e \circ i(c_U)^\circ \circ e
\]
\[(4.27)\]
this is pictured in Figure 29. As observed in subcase 1.3, the rules for resolving \( 1_\tau \otimes \text{bar}(\alpha) \) in type \((D_{n+1}, A_n)\) and \( 1_{i(\tau)} \otimes \text{bar}(i(\alpha)) \) in type \((C_n, A_{n-1})\) are identical except when \( i(\alpha) = \tau \) in
which case we get an extra term; this term cancels with the second summand on the right of equation (4.27). Using Lemma 4.10 and Proposition 4.15 versus Corollary 4.35 we see that

\[ e \circ i(c_T) \circ e \circ i(c_U)^* \circ e = e \circ i(c_T) \circ e \]

by induction on \( \ell(\mu) \).

**Case 6.** Let \( c_T = R_{\alpha}^+(c_T) \) and \( c_U = A_{\alpha}(c_U) \) (the dual case is similar). If \( \iota(\alpha) \neq \tau \) then \( \ell_\tau(\iota(\mu)) = \ell_\tau(\iota(\mu - \alpha)) \) and using the fork-spot relation we have

\[ e \circ i(c_T) \circ e \circ i(c_U)^* \circ e = e \circ i(c_T) \circ e \circ i(c_U)^* \circ e \]

and so the result follows by induction. If \( \iota(\alpha) = \tau \) then \( \ell_\tau(\iota(\mu)) = \ell_\tau(\iota(\mu - \alpha) + 1 \) and we set \( \alpha = \rho \) (the \( \alpha = \pi \) case is identical) and we must have \( c_T = R_\rho^+ R_\sigma^- (c_T^\nu) \). We have that

\[ i(c_T)e = i(c_T)e_{i(\mu)} = i(c_T)(1_{i(\mu)} + \ell_\tau(i(\mu)))(c_{i(\mu)} - \tau \otimes 1_{\tau}) = i(c_T)(c_{i(\mu)} - \tau \otimes 1_{\tau}) \]

as illustrated in Figure 30. So we have

\[ e \circ i(c_T) \circ e_{i(\mu)} \circ i(c_U)^* \circ e = e \circ i(c_T) \circ (c_{i(\mu)} - \tau \otimes 1_{\tau}) \circ i(c_U)^* \circ e = e \circ i(c_T) \circ e \circ i(c_U)^* \circ e \]

using the fork-spot relation as above. Again we are done by induction.

![Figure 30. Case 6 for \( \alpha = \sigma \).](image)

**Case 7.** The case \( c_T = A_{\alpha}^- (c_T) \) and \( c_U = R_{\alpha}^- (c_U) \) follows from case 6 (in the manner that case 4 followed from case 3).

**Case 8.** We now consider the case that \( c_T = R_{\alpha}^+(c_T) \) and \( c_U = R_{\alpha}^-(c_U) \). By the same inductive argument as we used in Case 1 (in order to deduce equation (4.21)), we have that

\[ e \circ i(c_T) \circ e \circ i(c_U)^* \circ e = \sum_{X,Y} a_{X,Y} e(i(c_X)^* \otimes \text{fork}_{i(\alpha)\iota(\alpha)}(i(c_Y) \otimes \text{fork}_{i(\alpha)\iota(\alpha)})e. \]

We set \( \lambda = \text{Shape}(X) = \text{Shape}(Y) \). Note that either \( \alpha \in \text{Add}(\lambda) \) or in \( \text{Rem}(\lambda) \). If \( \alpha \in \text{Add}(\lambda) \), then arguing as in case 3 we get that

\[ c_X^* \otimes \text{fork}_{i(\alpha)\iota(\alpha)}(c_Y) \otimes \text{fork}_{i(\alpha)\iota(\alpha)}(c_Y) \otimes \text{fork}_{i(\alpha)\iota(\alpha)} = 0 \]

are cellular basis elements and so we are done. If \( \alpha \in \text{Rem}(\lambda) \), then we have

\[ (c_X^* \otimes \text{fork}_{i(\alpha)\iota(\alpha)}(c_Y) \otimes \text{fork}_{i(\alpha)\iota(\alpha)}) = c_{XY} + \text{fork}_{i(\alpha)\iota(\alpha)}(c_{XY}) \otimes \text{fork}_{i(\alpha)\iota(\alpha)} = 0 \]

and so we are done.

**Cases 9 and 10.** The case in which \( c_T = R^+_\alpha (c_T) \) and \( c_U = R^-_\alpha (c_U) \) and the case in which \( c_T = R^-_\alpha (c_T) \) and \( c_U = R^+_\alpha (c_U) \) both follow from case 8 (in the manner that case 4 followed from case 3).
5. Coxeter truncation

In this section we prove one of the main results of this paper: that singular Hecke categories for Hermitian symmetric pairs are Morita equivalent to (regular) Hecke categories for smaller rank Hermitian symmetric pairs. For the underlying Kazhdan–Lusztig polynomials, this was first observed by Enright–Shelton [ES87]. Our result lifts theirs to the 2-categorical level and to positive characteristic. By Theorem 4.31 we can focus on the simply laced case without loss of generality. Let $(W, P)$ a simply laced Hermitian symmetric pair of rank $n$ and fix $\tau \in S_W$. We define

$$\mathcal{P}_{(W,P)}^{\tau} = \{\mu \in \mathcal{P}_{(W,P)} \mid \tau \in \text{Rem}(\mu)\}.$$  

We will show that the subalgebra of $h_{(W,P)}$ spanned by

$$\{c_{ST} \mid S \in \text{Path}(\lambda, t_\mu), T \in \text{Path}(\lambda, t_\nu), \lambda, \mu, \nu \in \mathcal{P}_{(W,P)}^{\tau}\}$$

is isomorphic to $h_{(W,P)}^\tau$ for some Hermitian symmetric pair $(W, P)^\tau$ of strictly smaller rank.

5.1. The $\tau$-contraction tilings. In what follows, we let $(W, P)$ be a simply laced Hermitian symmetric pair and let $\tau \in S_W$. We now introduce a contraction map which will allow us to work by induction on the rank.

![Figure 31](image-url)

**Figure 31.** The contraction tilings for $(A_8, A_1 \times A_3)^\tau$ with $\tau = s_3$, and $s_1$ respectively. The tiling is discussed in Subsection 5.1 and the Dynkin diagrams are discussed in Subsection 5.2.

**Definition 5.1.** Given $(W, P)$ a Hermitian symmetric pair and $\tau \in S_W$, we let $[r_1, c_1] < [r_2, c_2] < \cdots < [r_k, c_k]$ denote the completely ordered set (according to the natural ordering on $r_i + c_i \in \mathbb{N}$) of all $\tau$-tiles in $\mathcal{A}_{(W,P)}$. Given two $\tau$-tiles, $[r_i, c_i]$ and $[r_{i+1}, c_{i+1}]$ that are adjacent in this ordering, we define

$$T_{i, i+1}^{\tau} = \{[x, y] \mid r_i \leq x \leq r_{i+1} \text{ and } c_i \leq y \leq c_{i+1} \text{ and } x, y \neq [r_i, c_i]\} \cap \mathcal{A}_{(W,P)}.$$  

Associated to the minimal tile $[r_1, c_1]$ we define a corresponding null-region

$$T_{0, 1}^{\tau} = \{[x, y] \mid x \leq r_1 \text{ and } y \leq c_1\} \cap \mathcal{A}_{(W,P)}$$

and for the maximal tile $[r_k, c_k]$, we define the maximal null-region

$$T_{\infty}^{\tau} = \{[x, y] \mid r_k \leq x \text{ and } c_k \leq y \text{ and } x, y \neq [r_k, c_k]\} \cap \mathcal{A}_{(W,P)}$$

and we set $N^{\tau} = T_{0, 1}^{\tau} \cup T_{\infty}^{\tau}$. We define the $\tau$-contraction tiling to be the disjoint union of the $T_{i, i+1}^{\tau}$-tiles and all remaining tiles in $\mathcal{A}_{(W,P)} \setminus N^{\tau}$. We refer to any tile in this overall tiling as a $\tau$-contraction tile. Given $\Upsilon$ a $\tau$-contraction tile. We define a reading word of $\Upsilon$ by recording the constituent tiles $[r, c]$ within $\Upsilon$ from bottom to top (that is, by the natural order on $r + c \in \mathbb{Z}_{\geq 0}$).

**Remark 5.2.** We note that any tile-partition $\lambda \in \mathcal{P}_{(W,P)}^{\tau}$ can be obtained by stacking $\tau$-contraction tiles on top of $T_{0, 1}^{\tau}$.
Figure 32. The contraction tilings \((D_6, A_5)^\tau\) for \(\tau = s_0, s_1, s_2 \in W\) respectively. In the first two cases, the grey node is labelled by \(21320\) and \(20321\).

Figure 33. The contraction tilings of \((D_7, D_6)^\tau\) for \(\tau = s_1, s_2, s_3, s_5\) in order.

**Example 5.3.** In the leftmost diagram in Figure 32 the large contraction tiles are all identical. The reading words of these tiles are \(21320\) and \(23120\). Notice that these words differ only by the commuting relations in the Coxeter groups.

There is, in essence, only one type of large contraction tile: this is the contraction tiles of type \((A, A \times A)\) and their augmentations pictured in Figure 37. We will see in Subsection 5.3 that these augmentations merely “bulk out” the corresponding Soergel diagram (using degree zero strands) without changing its substance. In more detail, the tricone is formed from three tiles in a formation \(T = \{[r, c], [r - 1, c], [r, c - 1]\}\). Here the tile \([r, c]\) is the only \(\tau\)-tile in \(T\). We augment this picture by adding \(k\) tiles symmetrically above and below the brim (thus displacing the \(\tau\)-tile) to obtain an augmented tile \(\text{aug}_k(T)\) as depicted in the rightmost diagram of Figure 37.

5.2. **The Dynkin types of \(\tau\)-contraction tilings.** We now identify the \(\tau\)-tilings of \(\mathcal{A}_{(W,P)}\) with the tilings of the admissible region of a Hermitian symmetric pair of smaller rank. The nodes of \((W, P)^\tau\) will be labelled by the reading words of the tiles in the \(\tau\)-contraction tiling.

**Proposition 5.4.** Let \((W, P)\) be a simply laced Hermitian symmetric pair and let \(\tau \in S_W\). There is an order preserving bijection \(\varphi: \mathcal{P}_{(W,P)}^\tau \to \mathcal{P}_{(W,P)}\) where \((W, P)^\tau = (W^\tau, P^\tau)\) is defined by
Figure 34. The contraction tilings of $(E_6, D_5)$ in order.

Figure 35. The first 4 of 7 contraction tilings for $(E_7, E_6)$, in order.

- $(A_n, A_k \times A_{n-k})^\tau = (A_{n-2}, A_{k-1} \times A_{n-k-1})$;
- $(D_n, A_{n-1})^\tau = (D_{n-2}, A_{n-3})$;
- $(D_n, D_{n-1})^\tau = (A_1, A_0)$;
- $(E_6, D_5)^\tau = (A_5, A_4)$;
- $(E_7, E_6)^\tau = (D_6, D_5)$.
Moreover, fixing a reading word for each $\tau$-contraction tile, this defines a reduced path $\varphi_\tau(t) \in \text{Std}(\varphi_\tau(\lambda))$ for each $t \in \text{Std}(\lambda)$ and $\lambda \in \mathcal{P}(W,P)^{\tau}$.

**Proof.** This following by inspection, comparing Figures 31 to 36 with Figures 3 to 6.

We label the nodes of the smaller rank Coxeter system with the reading words of the $\tau$-contraction tiles of the larger Coxeter system. The positions of these labels can easily be deduced from the $\tau$-tilings, see for example Figures 38 to 40.

**Figure 38.** An example of a graph and its contraction at the vertices 3 and 5 respectively.

In fact, it can be shown that the map $\varphi_\tau$ can be extended from reduced paths (where the observation is trivial) to non-reduced paths. The proof of this involves much more substantial combinatorics and so is postponed to the companion paper, [BDF+].
Figure 39. An example of a graph and its contractions at vertices 3 and 5, respectively.

Figure 40. The graphs obtained from the leftmost graph in Figure 39 by contraction at the type $D$ vertices 0, 1 and 2, respectively.

Theorem 5.5 ([BDF+]). We have a graded bijection
\[ \varphi_\tau : \text{Path}_{(W,P)_\tau} (\lambda, t_\mu) \rightarrow \text{Path}_{(W,P)} (\varphi_\tau (\lambda), t_{\varphi_\tau (\mu)}). \]

Example 5.6. In Figure 38 we depict the pair $(A_5, A_2 \times A_2)$ and the truncation at the node $\tau = s_3 \in S_W$. We have that $\varphi_\tau (243) = 2 \otimes 4 \otimes 3$. On the right of Figure 1 we depict the Bruhat graph for $(A_5 \setminus A_2 \times A_2)^\tau$. The bottommost (and topmost) edge of this graph is tricoloured by $2 \otimes 4 \otimes 3$ and maps to a concatenate of three distinct edges in the leftmost graph.

5.3. The dilation homomorphism. We now lift the map $\varphi_\tau$ of Theorem 5.5 to the level of a graded $k$-algebra isomorphism. We let $i$ denote a primitive 4th root of unity. We first define the dilation maps on the monoidal generators. Let $\sigma = s_1 s_2 \ldots s_\ell$ be a (composite) label of a node of the Coxeter graph $(W,P)^\tau$. (Note that $s_1, s_2, \ldots, s_\ell \in S_W$ belong to the dilated Coxeter group, whereas $\sigma \in S_{W^\tau}$.) We define the dilation map on the idempotent generators as follows
\[ \text{dil}_\tau (1_{\sigma}) = 1_{\tau} \otimes 1_{s_1} \otimes 1_{s_2} \otimes \cdots \otimes 1_{s_\ell}. \]

For a non-zero braid generator (see Proposition 4.12 and Remark 4.13 for the list of zero braid generators) we define the dilation map as follows
\[ \text{dil}_\tau (\text{braid}^{\sigma, \alpha}_{\sigma_\alpha}) = 1_{\tau} \otimes \text{braid}^{s_1, s_2, \ldots, s_\ell}_{s_1, s_2, \ldots, s_\ell}. \]

Examples are depicted in Figure 41.

Figure 41. Examples of the $\text{dil}_\tau$ map on idempotent and braid generators. The colouring corresponds to that of Figure 31. The leftmost $\tau$-strand is drawn as a dotted strand in order to remind the reader that we horizontally concatenate these diagrams using $\otimes$ (which identifies this blue strand with an earlier blue strand in the diagram).

We now define the dilation map on the fork and spot generators. For $\sigma = s_i$ a tile labelled by a singleton $s_i \in S_{W^\tau}$ (which necessarily commutes with $\tau \in S_W$) we define
\[ \text{dil}_\tau (\text{fork}^\sigma_{\sigma, \sigma}) = 1_{\tau} \otimes \text{fork}^{s_i}_{s_i} \quad \text{dil}_\tau (\text{spot}^\sigma) = 1_{\tau} \otimes \text{spot}^\sigma. \]

We set $\sigma = \alpha \gamma \tau$, the reading word of a tricorne tile. We define
\[ \text{dil}_\tau (\text{fork}^\sigma_{\sigma, \sigma}) = i \times 1_{\tau} \otimes (\text{fork}^{\alpha, \alpha}_{\gamma, \gamma} \otimes 1_{\tau}) (1_{\alpha} \otimes \text{braid}^{\alpha, \alpha}_{\gamma, \gamma} \otimes 1_{\gamma \tau}) (1_{\alpha} \otimes \text{spot}^\theta \otimes 1_{\alpha \gamma \tau}). \]
\[ \text{dil}_\tau(\text{spot}_0^\emptyset) = -i \times \text{fork}_\tau(1_\tau \otimes \text{spot}_0^\emptyset \otimes \text{spot}_\gamma^0 \otimes 1_\tau). \]

Examples are depicted in Figure 42.

We now describe how one can “augment” the diagrams of tricorners to obtain arbitrary diagrams. Let \( \sigma \) be a label of a vertex in the graph \((W,P)\) of the form \( \sigma = x\alpha\gamma x^{-1}_\tau \). (That is, \( \sigma \) is the reading word of an augmented tricorne.) We define \( \text{dil}_\tau(\text{fork}_\sigma^\emptyset) \) to be the element

\[ i^{\ell(x)+1}(1_{\tau_x} \otimes (\text{fork}_{\alpha\gamma\gamma} \text{braid}_{\alpha\alpha\gamma\gamma}(1_{\alpha\gamma} \otimes \text{cap}_{\tau_x}^\emptyset \otimes 1_{\alpha\gamma}(1_{\alpha\gamma\gamma^{-1}_x} \otimes \text{spot}_0^\emptyset \otimes 1_{\tau_x}))) \otimes 1_{\tau_x^{-1}}. \]

and we define \( \text{dil}_\tau(\text{spot}_0^\emptyset) \) to be the element

\[ (-i)^{\ell(x)+1} \text{fork}_\tau(1_{\tau_x} \otimes \text{cap}_{\tau_x}^\emptyset \otimes 1_\tau)(1_{\tau_x} \otimes \text{spot}_0^\emptyset \otimes \text{spot}_\gamma^0 \otimes 1_{\tau_x^{-1}}). \]

Examples are depicted in Figure 43.

Having defined \( \text{dil}_\tau \) on all Soergel generators, we set \( \text{dil}_\tau(D^*) = (\text{dil}_\tau(D))^* \). We now extend this definition to arbitrary Soergel diagrams and hence define our contraction homomorphisms.

**Definition 5.7.** Given diagrams \( D_1, D_2 \in \mathcal{H}(W,P) \), we inductively define

\[ \text{dil}_\tau(D_1 \otimes D_2) = \text{dil}_\tau(D_1) \otimes \text{dil}_\tau(D_2) \]

\[ \text{dil}_\tau(D_1 \circ D_2) = \text{dil}_\tau(D_1) \circ \text{dil}_\tau(D_2) \]

and we extend this map \( k \)-linearly. We hence define \( \varphi_\tau : \mathcal{H}(W,P) \to \mathcal{H}(W,P) \) as follows,

\[ \varphi_\tau(D) = 1_{T_{0 \to 1}} \otimes \text{dil}_\tau(D) \]

where we recall that \( T_{0 \to 1} \) is the null region at the bottom of the \( \tau \)-contraction tiling of \((W,P)\).

**Figure 44.** The map \( \varphi_\tau \) on a diagram, for the leftmost contraction tiling in Figure 31.
Remark 5.8. Each null-region tile $T_{0\to 1}$ has a unique reading word and so there is no ambiguity here. That the map $\varphi_\tau$ is well-defined on diagrams follows from the interchange law.

The map $\varphi_\tau$ preserves the light leaves basis (because our map is defined on monoidal generators) and thus lifts the map of Theorem 5.5 to an isomorphism of graded $k$-modules between

$$h_{(W,P)^\tau} = k\{c_{ST} \mid S \in \text{Path}(\lambda, t_\mu), T \in \text{Path}(\lambda, t_\nu), \lambda, \mu, \nu \in \mathcal{P}_{(W,P)^\tau} \}$$

and $h_{(W,P)}^\tau \subset h_{(W,P)}$ which we define to be the subspace with basis

$$k\{c_{ST} \mid S \in \text{Path}(\varphi_\tau(\lambda), t_{\varphi_\tau(\mu)}), T \in \text{Path}(\varphi_\tau(\lambda), t_{\varphi_\tau(\nu)}), \varphi_\tau(\lambda), \varphi_\tau(\mu), \varphi_\tau(\nu) \in \mathcal{P}_{(W,P)}^\tau \}.$$  

In fact we will now lift this to the level of graded $k$-algebras.

**Theorem 5.9.** Let $(W, P)$ be a Hermitian symmetric pair and $\tau \in S_W$. We have a graded $k$-algebra isomorphism $\varphi_\tau(h_{(W,P)^\tau}) \cong h_{(W,P)}^\tau$.

6. Proof of the Coxeter dilation homomorphism

This section is dedicated to the proof that the map of $\varphi_\tau$ is a homomorphism. This amounts to checking the relations for these algebras. By Definition 5.7, we have that

$$\varphi_\tau(D)\varphi_\tau(D') = (1_{T_{0\to 1}} \otimes \text{dil}_\tau(D))(1_{T_{0\to 1}} \otimes \text{dil}_\tau(D')) = 1_{T_{0\to 1}} \otimes \text{dil}_\tau(D \circ D')$$  \hspace{1cm} (6.1)

using the interchange law and moreover

$$\varphi_\tau(D \circ D') = 1_{T_{0\to 1}} \otimes \text{dil}_\tau(D \circ D').$$  \hspace{1cm} (6.2)

Therefore for the local relations, it suffices to show that

$$\text{dil}_\tau(D \circ D') = \text{dil}_\tau(D) \circ \text{dil}_\tau(D').$$  \hspace{1cm} (6.3)

Most of this section is dedicated to the proof that the relations of Corollary 2.4 are preserved under equation (6.3) (but replacing $\otimes$ with $\circ$). For the non-local relations, we check that equation (6.1) and (6.2) coincide at the end of the section. We now turn to the local relations. Relations 1.1 and 1.2, and 1.9 are all trivial. Relation 1.8 is satisfied by Proposition 3.8.

In what follows, we let $\sigma$ be a reading word of some $\tau$-contraction tile. In the case that $\sigma$ is an augmented tricorne $\sigma = x\alpha\gamma x^{-1} \tau$, we set and $x = \sigma_k \ldots \sigma_1$. In diagrams we put a gradient which reflects the ordering of the purple strands (with $k$ the lightest and 1 the darkest). We label strands in a diagram simply by $1 \leq j \leq k$ (rather than by $\sigma_j$) for brevity.

6.1. The dilated fork-spot relation. We first consider the leftmost relation in 1.3, namely the fork-spot relation

$$(\text{dil}_\tau(1_\sigma) \otimes \text{dil}_\tau(\text{spot}_\sigma^0)) \circ \text{dil}_\tau(\text{fork}_\sigma^0) = \text{dil}_\tau((1_\sigma))$$  \hspace{1cm} (6.4)

For $\sigma = s_i \in S_W$, it follows trivially. For $\sigma = x\alpha\gamma x^{-1} \tau$, the equality follows by one application of each of the $\alpha$- $\gamma$- and $\tau$-fork-spot contractions and the $\alpha\gamma$-commutativity relation (and monoidal unit relation); and by “straightening out” the $x$-strands via application of the fork-spot and double-fork relations (this is sometimes referred to simply as “isotopy” in the literature). For a tricorne, this relation is depicted in Figure 45. For an augmented tricorne, one side of this relation is depicted in Figure 46 and it is easy to see that the argument goes through unchanged. The argument for the horizontal and vertical flips of equation (6.4) is similar.

6.2. The dilated double-fork relation. We now consider the rightmost relation in 1.3, namely, the double-fork relation

$$(\text{dil}_\tau(1_\sigma) \otimes \text{dil}_\tau(\text{fork}_\sigma^0)) \circ (\text{dil}_\tau(\text{fork}_\sigma^0) \otimes \text{dil}_\tau((1_\sigma))) = \text{dil}_\tau(\text{fork}_\sigma^0) \circ \text{dil}_\tau(\text{fork}_\sigma^0).$$

We apply the double-fork relation to every constituent doubly-forked strand in the diagram in turn, and the result follows. See Figure 47 for the corresponding picture for tricornes, the augmented tricorne picture can be obtained in a similar fashion to Figure 46.
τ = τ = τ.

Figure 45. The dilation of the fork-spot relation and its flip through the vertical axis, for a tricorne \( \sigma = \alpha \gamma \tau \). (We note that the the scalar coefficient for both these products is \( i \times -i = 1 \).)

\( \tau k \alpha \gamma \tau \).

Figure 46. The lefthand-side of the dilated fork-spot relation for an augmented tricorne \( \sigma = x \alpha \gamma x^{-1} \tau \). (We note that the the scalar coefficient for both these products is \( i \times -i = 1 \) or \( 1 \times 1 = 1 \) depending on the parity of \( k \geq 1 \)).

\( \tau k \alpha \gamma \tau \).

Figure 47. The dilated double-fork relation for \( \sigma = \alpha \gamma \tau \) the reading word of a tricorne. The equality follows by applying the double-fork relation to the \( \alpha \)- and \( \gamma \)-strands.

6.3. The dilated circle annihilation relation. We now verify the leftmost relation in 1.4, namely, the circle-annihilation relation

\[
\text{dil}_\tau (\text{fork}_{\sigma \sigma}^\sigma) \text{dil}_\tau (\text{fork}_{\sigma \sigma}^\sigma) = 0. \tag{6.5}
\]

For a tricorne \( \sigma = \alpha \gamma \tau \) we have that \( \text{dil}_\tau (\text{fork}_{\sigma \sigma}^\sigma) \text{dil}_\tau (\text{fork}_{\sigma \sigma}^\sigma) \) is equal to

\[
-1 \tau \otimes \text{fork}_{\alpha \gamma \alpha \gamma}^{\alpha \gamma \alpha \gamma} (1 \otimes \text{bar}(\tau) \otimes 1) \text{braid}_{\alpha \gamma \alpha \gamma}^{\alpha \gamma \alpha \gamma} \text{fork}_{\alpha \gamma \alpha \gamma}^{\alpha \gamma \alpha \gamma} \otimes 1 \tau \tag{6.6}
\]

by definition (note \((-i)^2 = -1 \)). Applying equation (4.3) to the \( \tau \gamma \)-strands in equation (6.6) we obtain

\[
-1 \tau \otimes \text{fork}_{\alpha \gamma \alpha \gamma}^{\alpha \gamma \alpha \gamma} (1 \otimes \text{bar}(\gamma) \otimes 1) \text{braid}_{\alpha \gamma \alpha \gamma}^{\alpha \gamma \alpha \gamma} \text{fork}_{\alpha \gamma \alpha \gamma}^{\alpha \gamma \alpha \gamma} \otimes 1 \tau \\
-1 \tau \otimes \text{fork}_{\alpha \gamma \alpha \gamma}^{\alpha \gamma \alpha \gamma} (1 \otimes \text{gap}(\gamma) \otimes 1) \text{braid}_{\alpha \gamma \alpha \gamma}^{\alpha \gamma \alpha \gamma} \text{fork}_{\alpha \gamma \alpha \gamma}^{\alpha \gamma \alpha \gamma} \otimes 1 \tau \\
+ 1 \tau \otimes \text{fork}_{\alpha \gamma \alpha \gamma}^{\alpha \gamma \alpha \gamma} (1 \otimes \text{bar}(\gamma) \otimes 1) \text{braid}_{\alpha \gamma \alpha \gamma}^{\alpha \gamma \alpha \gamma} \text{fork}_{\alpha \gamma \alpha \gamma}^{\alpha \gamma \alpha \gamma} \otimes 1 \tau
\]

(these three terms are depicted in Figure 48). Now, the first term is zero by the \( \alpha \gamma \)-commutativity relation and the \( \gamma \)-circle annihilation relation. The second and third terms are zero by the \( \alpha \gamma \)-commutativity relation and the \( \alpha \)-circle annihilation relation.

We now consider the case of an augmented tricorne \( \sigma = x \alpha \gamma x^{-1} \tau \), with \( x = \sigma_k \ldots \sigma_1 \). The diagram \( \text{dil}_\tau (\text{fork}_{\sigma \sigma}^\sigma) \text{dil}_\tau (\text{fork}_{\sigma \sigma}^\sigma) \) has a \( \tau \)-barbell in the centre of \( k \) concentric circles with the innermost circle labelled by \( \sigma_k \) and the outermost labelled by \( \sigma_1 \) (as pictured in the diagram on the lefthand-side of Figure 49). We pull this barbell through these \( k \) circles using \( k \) applications
Figure 48. The circle annihilation relation for $\sigma = \alpha \gamma \tau$. We apply the $\gamma \tau$-barbell relation to the lefthand-side. The first term (respectively latter two terms) on the righthand-side is zero by the circle annihilation relation for $\gamma$ (respectively $\alpha$) and the $\alpha \gamma$-commutativity relations.

of equation (4.12) and hence obtain

$$\text{cap}_{\text{circ}^{-1}}^\beta(1_{\Sigma} \otimes \text{bar}(\tau) \otimes 1_{\Sigma}^{-1}) \cup_{\emptyset}^{\text{circ}^{-1}} = (-1)^k \text{bar}(\sigma_1).$$

We therefore have that $\text{dil}_x((\text{fork}_{\delta \sigma}) \text{dil}_x((\text{fork}_{\beta \gamma})$ is equal to

$$1_{\tau} \otimes 1_{\Sigma} \otimes (\text{fork}_{\alpha \alpha \gamma \gamma}) (1_{\alpha} \otimes \text{bar}(\sigma_1) \otimes 1_{\alpha}) \text{braid}_{\alpha \alpha \gamma \gamma} (\text{fork}_{\alpha \alpha \gamma \gamma}) \otimes 1_{\Sigma}^{-1} \otimes 1_{\tau}.$$  

We can now apply the $\gamma \sigma_1$-barbell relation and show that the three resulting terms are zero exactly as in the case of the tricorn, above. (See also Figure 49.)

Figure 49. Simplifying the lefthand-side of the circle annihilation relation (equation (6.5)) using equation (4.12). Compare the righthand-side of the equation above with the lefthand-side of the equation pictured in Figure 48.

6.4. The dilated null-braid relations. Let $\beta \in S_{\nu \tau}$ and $m(\sigma, \beta) = 3$. By inspection of Figures 31 to 36, we see that $\beta$ must be a singleton label and either $(i)$ $m(\beta, \alpha) = 3$, $m(\beta, \gamma) = m(\beta, \tau) = m(\beta, \sigma_1) = 2$ for all $1 \leq i \leq k$ $(ii)$ $m(\beta, \gamma) = 3$, $m(\beta, \alpha) = m(\beta, \tau) = m(\beta, \sigma_1) = 2$ for all $1 \leq i \leq k$. We assume without loss of generality that $m(\beta, \alpha) = 3$. We must prove that

$$\text{dil}_x(1_{\beta \sigma \beta}) = -\text{dil}_x((\text{spot}_{\beta \sigma \beta}) \text{dil}_x(\text{dork}_{\beta \beta}) \text{dil}_x((\text{spot}_{\beta \sigma \beta}),$$  

$$\text{dil}_x(1_{\sigma \beta \sigma}) = -\text{dil}_x((\text{spot}_{\sigma \beta \sigma}) \text{dil}_x(\text{dork}_{\sigma \sigma}) \text{dil}_x((\text{spot}_{\sigma \beta \sigma}).$$  

We first prove equation (6.7). We first apply the commutativity relations to the two $\beta$-strands in $\text{dil}_x(1_{\beta \sigma \beta})$ in order to bring them as close to the $\alpha$-strand as possible (to obtain the top-right diagram of Figure 50) and we then apply the $\alpha \beta$-null-braid (to obtain $-1$ times the middle-left diagram of Figure 50). We then apply the $\gamma \sigma_1$-null-braid (to obtain the middle-right diagram of Figure 50) followed by the $\sigma_1 \sigma_{i+1}$-null-braids for $1 \leq i < k$ in turn (to obtain $(-1)^{k+1}$ times the bottom-left diagram of Figure 50). Finally, we apply the $\tau \sigma_i$-null-braid (to obtain $(-1)^{k+2}$ times the bottom-right diagram of Figure 50) and hence obtain $-\text{dil}_x((\text{spot}_{\beta \sigma \beta}) \text{dil}_x(\text{dork}_{\beta \beta}) \text{dil}_x((\text{spot}_{\beta \sigma \beta})$ as required.

We now prove equation (6.8) in a similar fashion. We first apply the $\tau \sigma_i$-null-braid relation to $\text{dil}_x(1_{\beta \sigma \beta})$ followed by the $\sigma_1 \sigma_{i+1}$-null-braid relations for $k > i \geq 1$ (to obtain $(-1)^k$ times the second diagram of Figure 51). We then apply the $\gamma \beta$-null-braid and $\alpha \sigma_1$-null-braid relations (to obtain $(-1)^k$ times the third diagram of Figure 51) and hence obtain $-\text{dil}_x((\text{spot}_{\beta \sigma \sigma}) \text{dil}_x(\text{dork}_{\sigma \sigma}) \text{dil}_x((\text{spot}_{\beta \sigma \sigma})$ as required.
6.5. The dilated barbell relations. We now consider the one and two colour barbell relations.

Lemma 6.1. For $\alpha, \gamma, \tau \in S^3_W$ with $m(\alpha, \tau) = 3 = m(\gamma, \tau)$ and $m(\alpha, \gamma) = 2$ we have that

$$\text{fork}_{\tau} \left( 1_{\tau} \otimes \text{bar}(\alpha) \otimes \text{bar}(\gamma) \otimes 1_{\tau} \right) \text{fork}_{\tau} = - \left( \text{bar}(\alpha) + \text{bar}(\tau) + \text{bar}(\gamma) \right) \otimes 1_{\tau} = -1_{\tau} \otimes \left( \text{bar}(\alpha) + \text{bar}(\tau) + \text{bar}(\gamma) \right).$$

Proof. We prove the first equality; the second is given by equation (4.4) and recorded here only for reference. We first move the $\alpha$ barbell to the left through the $\tau$ strand using 4.3 (and hence obtain 3 terms); for the first two of these terms (in which the $\tau$-strand remains in tact) we then again use 4.3 to move the $\gamma$ barbell to the left through the $\tau$-strand. We hence obtain a sum...
involving 7 terms, 4 of which are zero by the \( \tau \)-circle-annihilation relation; this leaves us with the required 3 terms.

We now “augment” the previous lemma so that it applies to augmented tricorneres.

**Lemma 6.2.** Let \( \sigma \) be an augmented tricorner, \( \sigma = \mathcal{Z} \alpha \mathcal{Z}^{-1} \tau \) and \( \mathcal{Z} = \sigma_k \ldots \sigma_1 \). We have that

\[
\forall k \geq 0, (1_{\tau} \otimes \text{cap}^\emptyset_{\mathcal{Z}^{-1}} \otimes 1_{\tau})(1_{\tau} \otimes \text{bar}(\alpha) \otimes \text{bar}(\gamma) \otimes 1_{\mathcal{Z}^{-1} \tau})(1_{\tau} \otimes \text{cup}^\emptyset_{\mathcal{Z}^{-1}} \otimes 1_{\tau}) \forall k \geq 0
\]

(6.9)

\[
= (-1)^{k+1} 1_{\tau} \otimes (\sum_{i=1}^{k} 2\text{bar}(\sigma_i) + \text{bar}(\alpha) + \text{bar}(\gamma) + \text{bar}(\tau))
\]

(6.10)

**Proof.** We proceed by induction on \( k \geq 0 \), with the \( k = 0 \) base case taken care of in Lemma 6.1. By induction, we can rewrite the left-hand side of 6.9 as follows

\[
(-1)^k\text{fork}_{\tau}^\sigma \text{spot}_{\sigma_k}^\sigma \tau (1_{\tau} \otimes (-1)^k 1_{\tau} \otimes (\sum_{i=1}^{k-1} 2\text{bar}(\sigma_i) + \text{bar}(\alpha) + \text{bar}(\gamma) + \text{bar}(\sigma_k)) \otimes 1_{\tau}) \forall k \geq 0
\]

which is equal to

\[
(-1)^k \text{fork}_{\tau}^\sigma \text{spot}_{\sigma_k}^\sigma \tau (1_{\tau} \otimes (\sum_{i=1}^{k} 2\text{bar}(\sigma_i) + \text{bar}(\alpha) + \text{bar}(\gamma) + \text{bar}(\sigma_k)) \otimes 1_{\tau}) \forall k \geq 0
\]

The term involving a tensor product \( \text{bar}(\sigma_k) \otimes \text{bar}(\sigma_k) \) can be rewritten using equation (4.13). The remaining terms involve a tensor product of two distinctly coloured barbells, one of which commutes with the \( \tau \)-strand; thus we can apply equation (4.11) to these terms. Rewriting all the terms in the above manner and summing over the resulting elements, we obtain 6.9. Equation (6.10) follows by equation (4.4). □

We are now ready to construct the dilated barbell diagrams.

**Lemma 6.3.** Let \( \sigma \) be an augmented tricorner, \( \sigma = \mathcal{Z} \alpha \mathcal{Z}^{-1} \tau \) and \( \mathcal{Z} = \sigma_k \ldots \sigma_1 \). We have that

\[
\text{dil}_{\tau}(\text{bar}(\sigma)) = 1_{\tau} \otimes (\sum_{i=1}^{k} 2\text{bar}(\sigma_i) + \text{bar}(\alpha) + \text{bar}(\gamma) + \text{bar}(\tau))
\]

(6.11)

\[
= (\sum_{i=1}^{k} 2\text{bar}(\sigma_i) + \text{bar}(\alpha) + \text{bar}(\gamma) + \text{bar}(\tau)) \otimes 1_{\tau}
\]

(6.12)

\[
\text{dil}_{\tau}(1_{\sigma}) \otimes \text{dil}_{\tau}(\text{bar}(\sigma)) = 1_{\tau} \text{dil}_{\tau}(\text{bar}(\sigma)) = 1_{\tau} \text{dil}_{\tau}(\text{bar}(\sigma))
\]

(6.13)

\[
\text{dil}_{\tau}(\text{bar}(\sigma)) \otimes \text{dil}_{\tau}(1_{\sigma}) = 1_{\tau} \text{dil}_{\tau}(\text{bar}(\sigma)) = 1_{\tau} \text{dil}_{\tau}(\text{bar}(\sigma))
\]

(6.14)

\[
\text{dil}_{\tau}(\text{gap}(\sigma)) = 1_{\tau} \text{dil}_{\tau}(\text{gap}(\sigma)) = 1_{\tau} \text{dil}_{\tau}(\text{gap}(\sigma))
\]

(6.15)

**Proof.** Equation (6.11) follows directly from Lemma 6.2. We now consider equation (6.13) and equation (6.14). We have that

\[
\text{dil}_{\tau}(1_{\sigma}) \otimes \text{dil}_{\tau}(\text{bar}(\sigma)) = 1_{\tau} \text{dil}_{\tau}(\text{bar}(\sigma)) = 1_{\tau} \text{dil}_{\tau}(\text{bar}(\sigma))
\]

where the first equality follows equation (6.11); the second from summing over relations 1.4 and 1.5; the third from Lemma 4.8. We repeat the final two steps above a further \( k - 2 \) times and hence obtain

\[
\text{dil}_{\tau}(1_{\sigma}) \otimes \text{dil}_{\tau}(\text{bar}(\sigma)) = 1_{\tau} \text{dil}_{\tau}(\text{bar}(\sigma)) = 1_{\tau} \text{dil}_{\tau}(\text{bar}(\sigma))
\]

where the second and fourth equalities follow from Lemma 4.8; the third from equation (4.4); the fifth from the \( \gamma \alpha \)-commutativity relations. We now consider equation (6.14). We have that

\[
\text{dil}_{\tau}(\text{bar}(\sigma)) \otimes \text{dil}_{\tau}(1_{\sigma}) = 1_{\tau} \text{dil}_{\tau}(\text{bar}(\sigma)) = 1_{\tau} \text{dil}_{\tau}(\text{bar}(\sigma))
\]
where the first equality follows from equation (6.11) and the second follows by the exact same argument as for the case of equation (6.13). Finally, we consider equation (6.15). We have that
\[
\dil_r(\gap(\sigma)) = \dil_r(\spot^\sigma)\dil_r(\spot^\emptyset)
\]
\[
= (-1)^{k+1}\spot^{\tau_\emptyset\alpha\gamma\sigma^\emptyset \gamma_1 \gamma_2 \tau_2 \bar{s}} \dil(\spot^\emptyset)(1_r \otimes \cap^\emptyset_{\bar{s}-1} \otimes 1_r) \dil(\spot^\emptyset)(1_r \otimes \cap^\emptyset_{\bar{s}-1} \otimes 1_r) \spot^\emptyset \alpha \gamma_1 \gamma_2 \tau_2 \bar{s}^{-1}
\]
\[
= 1_{\tau_2} \otimes \gap(\alpha) \otimes 1_{\gamma_2^{-1} \tau_2}
\]
where the first and second equalities are by definition; the third follows by applying the \(\tau \sigma_i\)-null-braid relation followed by the \(\sigma_i \sigma_{i+1}\)-null-braid relations for \(k > i \geq 1\) followed by the \(\gamma \sigma_i\)-null-braid relation.

**6.5.1. The dilated one colour barbell relation.** Let \(\sigma\) be a reading word of some \(\tau\)-contraction tile. We now verify the rightmost relation in 1.4. We have that
\[
\dil_r(\bar{1}_\sigma) + \dil_r(\bar{1}_\sigma) + \dil_r(\bar{1}_\sigma)
\]
\[
= 1_{\tau_2} \otimes \bar{1}_\sigma \otimes 1_{\alpha \gamma_1 \gamma_2 \tau_2} + 1 \dots + 1 \dots
\]
\[
= 2 \cdot 1_{\tau_2} \otimes \gap(\alpha) \otimes 1_{\gamma_2^{-1} \tau_2}
\]
as required. Here the first equality follows from equation (6.13) and (6.14); the second follows from the one-colour-barbell relation; and the third from equation (6.15).

**6.5.2. The dilated two colour barbell relations.** Let \(\beta \in S_{W_\tau}\), as noted in Subsection 6.4, we can assume that \(\beta\) is a singleton which commutes every label in \(\sigma\) except \(\alpha\). We have that
\[
\dil_r(\bar{1}_\beta) - \dil_r(\bar{1}_\beta) + \dil_r(\bar{1}_\beta)
\]
\[
= 1_{\tau_2} \otimes 1_{\alpha \gamma_1 \gamma_2 \tau_2} - 1_{\tau_2} \otimes 1_{\alpha \gamma_1 \gamma_2 \tau_2} + 1_{\tau_2} \otimes 1_{\alpha \gamma_1 \gamma_2 \tau_2}
\]
as required. Here, the first equality follows from equation (6.11); the second from the commutativity relations; the third from the \(\alpha \beta\)-barbell relation; the fourth follows by definition.

We now turn to the other two-colour barbell relation (in which the roles of \(\beta\) and \(\sigma\) are swapped). We have that
\[
\dil_r(\bar{1}_\sigma) - \dil_r(\bar{1}_\sigma) + \dil_r(\bar{1}_\sigma)
\]
\[
= 1_{\tau_2} \otimes 1_{\alpha \gamma_1 \gamma_2 \tau_2} - 1_{\tau_2} \otimes 1_{\alpha \gamma_1 \gamma_2 \tau_2} + 1_{\tau_2} \otimes 1_{\alpha \gamma_1 \gamma_2 \tau_2}
\]
as required. Here the first equality follows by definition; the second by the commutativity relations; the third by the \(\alpha \beta\)-barbell; the fourth by equation (6.13) and (6.15).

**6.6. The dilated \(m = 2\) relations.** For \(\sigma, \beta, \pi \in S_{W_\tau}\) with \(m(\sigma, \beta) = m(\pi, \beta) = m(\sigma, \pi) = 2\) we need to check the dilated versions of the relations
\[
\text{braid}^{\beta \sigma} \text{braid}^{\sigma \beta} = \text{braid}^{\beta \sigma} \text{braid}^{\sigma \beta}
\]
\[
\text{braid}^{\beta \sigma} \text{fork}^{\sigma \beta} = \text{fork}^{\beta \sigma} \text{braid}^{\sigma \beta}
\]
and their horizontal and vertical flips, along with the diagrams obtained by swapping the roles of \(\beta\) and \(\sigma\). Note that by Proposition 4.12 and Remark 4.13, both sides of all of these equations

\[
\text{braid}^{\beta \sigma} \text{braid}^{\sigma \beta} = 1_{\beta \sigma}
\]
\[
\text{braid}^{\beta \sigma} \text{braid}^{\sigma \beta} = 1_{\beta \sigma}
\]
\[
\text{braid}^{\beta \sigma} \text{braid}^{\sigma \beta} = 1_{\beta \sigma}
\]
\[
\text{braid}^{\beta \sigma} \text{braid}^{\sigma \beta} = 1_{\beta \sigma}
\]
vanish when \((W,P)^\tau = (A_n, A_{n-1})\), or \((W,P)^\tau = (D_n, D_{n-1})\), or \((W,P)^\tau = (D_n, A_{n-1})\) with \(\{\beta, \sigma\} = \{s_0, s_1\}\). In all other cases, we have that \(\sigma\) is a (possibly) composite label and \(\beta\) (and \(\pi\)) are singleton labels which commute with every constituent label of \(\sigma\). Thus all these relations are trivially satisfied.

### 6.7. The cyclotomic relations.

We finish by showing that the dilations of the non-local relation 1.10 and 1.11 are also preserved by \(\varphi_\tau\). It is easy to see that

\[
\varphi_\tau(1_\sigma \otimes 1_w) = 1_{T_{0\to 1}} \odot \text{dil}_\tau(1_\sigma) \otimes \text{dil}_\tau(1_w) = 0
\]

whenever \(\sigma \in S_{P^\tau}\) using (possibly) the null braid relations, the commutativity relations, and the cyclotomic relation in \(H_{(W,P)}\). It remains to show that

\[
\varphi_\tau(\text{bar}(\sigma) \otimes 1_w) = 0
\]

for \(\sigma\) the unique element of \(S_{W^\tau} \setminus S_{P^\tau}\). We will show that

\[
1_{T_{0\to 1}} \odot \text{dil}_\tau(\text{bar}(\sigma)) = 0
\]

for such \(\sigma\) and hence deduce the result. For the remainder of this section, we set \(T_{0\to 1} = \rho_1 \rho_2 \ldots \rho_r\) and we note that \(\rho_r = \tau\).

**Case 1.** Suppose that \(\sigma\) is a singleton. Then there exists \(1 \leq j \leq r\) such that \(m(\sigma, \rho_i) = 2\) for all \(i \neq j\) and \(m(\sigma, \rho_j) = 3\); this can be seen by inspection of Figures 31 to 36. We have that

\[
1_{T_{0\to 1}} \odot \text{dil}_\tau(\text{bar}(\sigma)) = 1_{\rho_1 \rho_2 \ldots \rho_r} \otimes \text{bar}(\sigma)
\]

as required. Here the first equality is the definition; the second follows by the commuting relations; the third by the two-colour barbell relation; the fourth by the commuting and cyclotomic relations; the fifth follows by repeating the arguments above.

**Case 2.** We now suppose that \(\sigma = \alpha \gamma \tau\), a tricone. By inspecting Figures 31 to 36, we deduce that \(m(\alpha, \rho_i) = 2 = m(\gamma, \rho_i)\) for all \(1 \leq i \leq r\). We have that

\[
1_{T_{0\to 1}} \odot \text{dil}_\tau(\text{bar}(\sigma)) = 1_{\rho_1 \rho_2 \ldots \rho_r} \otimes 1_\tau \otimes (\text{bar}(\alpha) + \text{bar}(\gamma) + \text{bar}(\tau))
\]

as required. Here the first equation follows from equation (6.11); the second by equation (4.4); the third by the commuting and cyclotomic relations; the fourth follows as in Case 1.

**Case 3.** We now suppose that \(\sigma = \alpha_1 \alpha_2 \ldots \alpha_k \gamma \tau\), an augmented tricone. By inspecting Figures 32 to 36, we deduce that \(m(\alpha, \rho_i) = m(\gamma, \rho_i) = m(\sigma_j, \rho_i) = 2\) for \(j \neq k, 1 \leq i \leq r\); and \(m(\sigma_k, \rho_r) = 3\) (recall \(\tau = \rho_r\)). We have that

\[
1_{T_{0\to 1}} \odot \text{dil}_\tau(\text{bar}(\sigma)) = 1_{\rho_1 \rho_2 \ldots \rho_r} \otimes 1_\tau \otimes (\sum_{i=1}^{k} 2 \text{bar}(\sigma_i) + \text{bar}(\alpha) + \text{bar}(\gamma) + \text{bar}(\tau))
\]

as required. The first equality follows from equation (6.11); the second from the commutativity relations; the third from equation (4.4); the fourth by commutativity relations; the fifth equality follows as in Case 1.
7. Graded decomposition numbers and Koszul resolutions

We are now ready to determine the main structural results concerning the Hecke categories of Hermitian symmetric pairs. Specifically, we will calculate the graded composition multiplicities and radical filtrations of standard modules and prove that the \( h_{(W,P)} \) are standard Koszul. Our treatment of this material is inspired by similar ideas in [BS10].

**Theorem 7.1** ([BDF+]). Let \((W,P)\) be a simply laced Hermitian symmetric pair. For any \( \lambda \neq \mu \), we have that

\[
\sum_{S \in \text{Path}(\lambda, t_{\lambda})} q^{|\text{deg}(S)|} \in q\mathbb{Z}_{\geq 0}[q].
\]

**Theorem 7.2.** Let \((W,P)\) be an arbitrary Hermitian symmetric pair and \( k \) be a field of characteristic \( p \geq 0 \). The graded decomposition numbers

\[
[\Delta(\lambda) : L(\mu)]_q = \sum_{k \in \mathbb{Z}} [\Delta(\lambda) : L(\mu)(k)] q^k
\]

of \( h_{(W,P)} \) are independent of the prime \( p \geq 0 \). For \((W,P)\) of simply laced type, the algebra \( h_{(W,P)} \) is basic and the modules \( 1_{t_{\lambda}} h_{(W,P)} \) for \( \lambda \in P_{(W,P)} \) provide a complete set of non-isomorphic projective indecomposable right \( h_{(W,P)} \)-modules.

**Proof.** By Theorem 4.31, it is enough to restrict our attention to simply laced type. In this case the result follows from Theorem 7.1 and the following pair of facts: for our algebras \( \dim_q(L(\lambda)) = 1 \) (by definition) and for general graded cellular algebras \( \dim_q(L(\lambda)) \in \mathbb{Z}_{\geq 0}[q + q^{-1}] \) (by [HM10, Proposition 2.18]).

7.1. Induction. We wish to use the dilation homomorphism in order to prove results by induction on rank. In order to do this, we define

\[
e_\tau = \sum_{\mu \in P_{(W,P)}} 1_{t_{\mu}} \otimes 1_{-\tau}.
\]

We have that \( e_\tau h_{(W,P)} \) carries the structure of a \((h^e_{(W,P)}, h_{(W,P)})\)-bimodule. The action on the right is by concatenation of diagrams. The action on the left is given by first conjugating \( h^e_{(W,P)} \) by a (commuting) braid so that the colour sequences match-up, and then concatenating diagrams. (Recall from Remark 3.1 that this simply amounts to changing our choice of tableaux.) With this isomorphism in place (and the isomorphism of Theorem 5.9) we are now able to define an induction functor

\[
G^\tau : h_{(W,P)}^- \mod \longrightarrow h_{(W,P)}^\tau \mod
\]

\[
M \mapsto M \otimes h_{(W,P)}^\tau e_\tau h_{(W,P)}^{-\langle -1 \rangle}
\]

using the identification \( h_{(W,P)}^- \cong h^e_{(W,P)} \subseteq h_{(W,P)}^\tau \). The degree shift in this definition ensures that the functor \( G^\tau \) commutes with duality (see Theorem 7.5 below). We have that

\[
P^\tau_{(W,P)} : = \{ \lambda \in P_{(W,P)} \mid \tau \in \text{Rem}(\lambda) \} \leftrightarrow P_{(W,P)}^\tau
\]

and for \( \lambda \in P^\tau_{(W,P)} \), we write \( \lambda_{\downarrow \tau} \) for the image on the righthand-side (so that \( \varphi^\tau(\lambda_{\downarrow \tau}) = \lambda \)). We say that \( \lambda_{\downarrow \tau} \) is the contraction of \( \lambda \) at \( \tau \). In what follows, we will write \( 1_{\mu} \) instead of \( 1_{t_{\mu}} \) to simplify notations.

**Theorem 7.3.** The functor \( G^\tau \) is exact.

**Proof.** We need to show that \( e_\tau h_{(W,P)} \) is projective as both a right \( h_{(W,P)}^- \)-module and as a left \( h_{(W,P)} \)-module. As a right \( h_{(W,P)}^- \)-module, \( e_\tau h_{(W,P)} \) is a direct summand of \( h_{(W,P)} \) (as \( e_\tau \) is an idempotent) and so it is clearly projective. It remains to show that \( e_\tau h_{(W,P)} \) is projective as a left \( h_{(W,P)}^- \)-module. We can decompose this module as follows

\[
e_\tau h_{(W,P)} = \oplus_\mu e_\tau h_{(W,P)} 1_{\mu}.
\]
We will show that each of these summands is projective as a left \( h_{(W,P)} \)-module. For the remainder of the proof, all statements concerning modules or homomorphisms will be taken implicitly to be of left \( h_{(W,P)} \)-modules. In all of the following cases, we will use the fact that \( c^\lambda_{ST} \in e_\tau h_{(W,P)} \) implies \( S \in \text{Path}(\lambda, \tau) \) such that \( \tau \in \text{Rem}(\nu) \). This, in turn, implies that \( \tau \in \text{Rem}(\lambda) \) or in \( \text{Add}(\lambda) \).

**Case 1.** We first assume that \( \tau \in \text{Rem}(\mu) \). We claim that in this case
\[
e_{\tau} h_{(W,P)} 1_\mu \cong h_{(W,P)} 1_{\mu^-_\tau} \oplus h_{(W,P)} 1_{\mu^+_\tau} \quad (2).
\]
The module \( e_{\tau} h_{(W,P)} 1_\mu \) has a basis
\[
B = \{ c^\lambda_{ST} | S \in \text{Path}(\lambda, \tau), T \in \text{Path}(\lambda, \mu), \text{ with } \lambda \in \mathcal{P}_{(W,P)} \text{ and } \nu \in \mathcal{P}_{(W,P)} \}
\]
which decomposes as a disjoint union \( \{ c^\lambda_{ST} \in B \mid \tau \in \text{Rem}(\lambda) \} \sqcup \{ c^\lambda_{ST} \in B \mid \tau \in \text{Add}(\lambda) \} \). Now we have
\[
\langle c^\lambda_{ST} \in B \mid \tau \in \text{Rem}(\lambda) \rangle = h^T_{(W,P)} 1_\mu \cong h_{(W,P)} 1_{\mu^-_\tau}.
\]

Now consider the quotient \( e_{\tau} h_{(W,P)} 1_\mu / h^T_{(W,P)} 1_\mu \). It has a basis given by the elements \( c^\lambda_{ST} + h^T_{(W,P)} 1_\mu \) with \( \tau \in \text{Add}(\lambda) \). These satisfy \( S = X^\tau_\tau(S') \) and \( T = X^\tau_\tau(T') \) for a (possibly different) choice of \( X = A \) or \( R \) for each one. If we take \( U = X^\tau_\tau(S') \) and \( V = X^\tau_\tau(T') \) then we can write
\[
c^\lambda_{ST} = c^\lambda_{\bar{U}} c_T = c^\lambda_{\bar{U}} (1_\lambda \otimes \text{gap}(\tau)) c_V.
\]
If \( T = A^\tau_\tau(T') \) and so \( V = A^\tau_\tau(T') \) then it becomes
\[
c^\lambda_{ST} = c^\lambda_{UV} (1_{\mu^-_\tau} \otimes \text{gap}(\tau)).
\]
If \( T = R^\tau_\tau(T') \) and so \( V = R^\tau_\tau(T') \) then we can factorise \( c^\lambda_{ST} \) as
\[
c^\lambda_{ST} = c^\lambda_{\bar{U}} (1_\lambda \otimes (\text{spot}_T^\varnothing \text{cap}_T^\varnothing)) (c_T \otimes 1_\tau).
\]
Now applying \( 1_\tau \otimes \text{spot}_\tau^\varnothing \) to equation equation \( (2.1) \) we get
\[
1_\tau \otimes \text{spot}_\tau^\varnothing = \text{spot}_\tau^\varnothing \otimes 1_\tau + \text{spot}_\tau^\varnothing \text{cap}_\tau^\varnothing - \text{bar}(\tau) \otimes \text{fork}_\tau^T.
\]
Thus we can rewrite equation \( (7.1) \) as
\[
c^\lambda_{ST} = c^\lambda_{\bar{U}} (1_\lambda \otimes (1_\tau \otimes \text{spot}_\tau^\varnothing)) (c_T \otimes 1_\tau) - c^\lambda_{\bar{U}} (1_\lambda \otimes (\text{spot}_\tau^\varnothing \otimes 1_\tau)) (c_T \otimes 1_\tau) + c^\lambda_{\bar{U}} (1_\lambda \otimes (\text{bar}(\tau) \otimes \text{fork}_\tau^T)) (c_T \otimes 1_\tau)
\]
Now note that the last two terms belong to \( h^T_{(W,P)} 1_\mu \) and the first one can be rewritten as
\[
c^\lambda_{UV} (1_{\mu^-_\tau} \otimes \text{gap}(\tau)).
\]
where \( c^\lambda_{UV} \in h^T_{(W,P)} 1_\mu \). This shows that the quotient is isomorphic to \( h^T_{(W,P)} 1_\mu \). As it is projective, it splits and we have
\[
e_{\tau} h_{(W,P)} 1_\mu \cong h^T_{(W,P)} 1_\mu \oplus h_{(W,P)} 1_\mu (1_{\mu^-_\tau} \otimes \text{gap}(\tau)),
\]
thus proving the claim.

**Case 2.** We now assume that \( \tau \in \text{Add}(\mu) \). We claim that in this case
\[
e_{\tau} h_{(W,P)} 1_\mu \cong h_{(W,P)} 1_{\mu^-_\tau} \quad (1).
\]
To see this, we will show that
\[
e_{\tau} h_{(W,P)} 1_\mu \cong h^T_{(W,P)} 1_{\mu^-_\tau} (1_\mu \otimes \text{spot}_\tau^T).
\]
Indeed for any \( c^\lambda_{ST} \in e_{\tau} h_{(W,P)} 1_\mu \) we have that \( S = X^\tau_\tau(S') \). If \( S = X^\tau_\tau(S') \) then we must have \( \tau \in \text{Add}(\lambda) \) and we define \( U = X^\tau_\tau(S') \) and \( V = A^\tau_\tau(T') \). If \( S = X^\tau_\tau(S') \) then we must have \( \tau \in \text{Rem}(\lambda) \) and we define \( U = S \) and \( V = R^\tau_\tau(T') \). Then in both cases we can write
\[
c^\lambda_{ST} = c^\lambda_{UV} (1_\mu \otimes \text{spot}_\tau^T).
\]
Note that \( c_{UV} \in h^T_{(W,P)} 1_{\mu^-_\tau} \) so we’re done.

**Case 3.** It remains to consider the case that \( \tau \not\in \text{Rem}(\mu) \) or \( \text{Add}(\mu) \). We now consider the case that \( \tau \not\in \text{Rem}(\mu) \) of \( \text{Add}(\mu) \), but there exists \( \sigma \in \text{Rem}(\mu) \) with \( m(\sigma, \tau) = 3 \). Note that we can
assume that \( \tau \in \text{Rem}(\mu - \sigma) \) as otherwise we would be in Case 2. This will serve as the base case for the inductive step in Case 4. We claim that in this case
\[
e_{\tau} h_{(W,P)} 1_\mu \cong h_{(W,P)^\tau} 1_{(\mu-\sigma) \downarrow_{\tau}}(1).
\]
To see this, we will show that
\[
e_{\tau} h_{(W,P)} 1_\mu = h_{(W,P)^\tau} 1_{(\mu-\sigma) \otimes \text{spot}_{\sigma}}
\]
Our assumptions that \( \sigma \in \text{Rem}(\mu) \) and \( \tau \in \text{Rem}(\nu) \) imply that there are two cases to consider:
\( \sigma \in \text{Rem}(\lambda) \) and \( \tau \in \text{Add}(\lambda) \) versus \( \sigma \in \text{Add}(\lambda) \) and \( \tau \in \text{Rem}(\lambda) \). In the latter case, we have that \( T = A_{\tau}(T') \) and so
\[
c_{ST} = c^\sigma_{ST}(c_{VT} \otimes \text{spot}_{\sigma}) = c^\tau_{ST} \otimes \text{spot}_{\tau}
\]
with \( c_{ST} \in h_{(W,P)} 1_{\mu-\sigma} \) as required. In the first case, we have \( T = R_{\sigma} X^-_{\tau}(T') \) and \( S = X^-_{\tau}(S') \).
Setting \( U = X^+_T(S') \) and \( V = X^+_T(T') \), we can write
\[
c_{ST} = c_{ST}^\sigma(1_{\lambda-\sigma} \otimes \text{trid}_{\tau \sigma} \otimes \text{spot}_{\sigma})(c_{VT} \otimes 1_{\sigma}) = -c_{UV} \otimes \text{spot}_{\tau}
\]
where the last equality follows by applying \( 1_{\sigma} \otimes \text{spot}_{\sigma} \) to the \( \sigma \tau \)-nullbraid relations. Again we have that \( c_{UV} \in h_{(W,P)} 1_{\mu-\sigma} \) so we are done.

**Case 4.** If \( \mu \) is not as in cases 1 to 3, then we must have \( \sigma \in \text{Rem}(\mu) \) with \( \sigma \) and \( \tau \) commuting. We will show that \( e_{\tau} h_{(W,P)} 1_{\mu} \) is either 0 or projective-indecomposable as a left \( h_{(W,P)^\tau} \)-module.
We proceed by induction on the rank of \( W \).
Note that as \( \sigma \) and \( \tau \) commute, \( \sigma \) labels a node in the Dynkin diagram for \( (W,P)^\tau \) and so it makes sense to consider \( e_{\sigma}^{(W,P)^\tau} \in h_{(W,P)^\tau} \) and \( (W,P)^\tau \). We claim that
\[
e_{\tau} h_{(W,P)} 1_\mu \cong h_{(W,P)^\tau} e_{\sigma}^{(W,P)^\tau} \otimes h_{(W,P)^{\sigma \tau}} e_{\tau}^{(W,P)^{\sigma \tau}} h_{(W,P)^{\sigma \tau}} 1_{(\mu-\sigma) \downarrow_{\tau}}
\]
(7.2)
as a left \( h_{(W,P)^\tau} \)-module. Note that any basis element in \( e_{\tau} h_{(W,P)} 1_{\mu} \) has the form \( c_{ST}^\sigma \) for \( S \in \text{Path}(\lambda, t_\mu) \), \( T \in \text{Path}(\lambda, t_\nu) \) with \( \tau \in \text{Rem}(\nu) \) and \( \sigma \in \text{Rem}(\mu) \). So either \( \sigma \in \text{Rem}(\lambda) \) or \( \sigma \in \text{Add}(\lambda) \) and similarly either \( \tau \in \text{Rem}(\mu) \) or \( \tau \in \text{Add}(\lambda) \). To prove the claim, it is enough to show that any such \( c_{ST}^\sigma \) can be written as a product
\[
c_{ST}^\sigma = c_{PQ}^\sigma c_{UV}^\sigma
\]
where \( c_{PQ}^\sigma \in h_{(W,P)}^\tau \) and \( c_{UV}^\sigma \in h_{(W,P)}^{\sigma \tau} \). There are four distinct cases to consider. If \( \sigma, \tau \in \text{Rem}(\lambda) \) then \( S = X^+_\sigma(S) \), \( T = X^+_\sigma(T') \) and we pick \( P = S \), \( Q = U = t_\lambda \) and \( V = T \). If \( \sigma \in \text{Rem}(\lambda) \) and \( \tau \in \text{Add}(\lambda) \) then \( S = X^-_\sigma(S') \), \( T = X^-_\sigma(T') \) and we pick \( P = X^+_\sigma(S) \), \( Q = A^+_\sigma(t_\lambda) \), \( U = A^+_\sigma(t_\lambda) \) and \( V = T \). If \( \sigma \in \text{Add}(\lambda) \) and \( \tau \in \text{Rem}(\lambda) \) then \( S = X^-_\tau(S') \), \( T = X^-_\tau(T') \) and we pick \( P = S \), \( Q = A^-_\sigma(t_\lambda) \), \( U = A^-_\sigma(t_\lambda) \) and \( V = X^-_{\tau}(T') \). Hence we have proven equation (7.2).

By induction, \( e_{\sigma}^{(W,P)^\tau} h_{(W,P)^\tau} 1_{(\mu-\sigma) \downarrow_{\tau}} \) is either 0, or it is a projective indecomposable \( h_{(W,P)^{\sigma \tau}} \)-module, say \( h_{(W,P)^{\sigma \tau}} 1_{\eta} \). Substituting into equation (7.2), we obtain that \( e_{\tau} h_{(W,P)} 1_{\mu} \) is either 0, or
\[
e_{\tau} h_{(W,P)} 1_\mu \cong h_{(W,P)^\tau} e_{\sigma}^{(W,P)^\tau} \otimes h_{(W,P)^{\sigma \tau}} 1_{\eta} \cong h_{(W,P)^\tau} 1_{\tau \downarrow_{\eta}}
\]
which is projective indecomposable.

\( \square \)

**Lemma 7.4.** There is a graded \((h_{(W,P)^\tau}, h_{(W,P)^\tau})\)-bimodule homomorphism
\[
\psi : e_{\tau} h_{(W,P)} e_{\tau} \to h_{(W,P)^\tau}(2).
\]

**Proof.** The module \( e_{\tau} h_{(W,P)} e_{\tau} \) has basis given by
\[
B = \{ c_{ST}^\lambda | S \in \text{Path}(\lambda, t_\mu), T \in \text{Path}(\lambda, t_\nu), \text{ with } \lambda \in \mathcal{P}_{(W,P)} \text{ and } \mu, \nu \in \mathcal{P}_{(W,P)} \}.
\]
which decomposes as a disjoint union \( \{ c^\lambda_{ST} \in B \mid \tau \in \text{Rem}(\lambda) \} \cup \{ c^\lambda_{ST} \in B \mid \tau \in \text{Add}(\lambda) \} \). By Theorem 5.9, we have a \((h_{(W,P)^\tau}, h_{(W,P)^\tau})\)-bimodule isomorphism
\[
h_{(W,P)^\tau} \cong h^\tau_{(W,P)} = (c^\lambda_{ST} \in B \mid \tau \in \text{Rem}(\lambda)) \subseteq e_\tau h_{(W,P)} e_\tau.
\]
Following the proof of case 1 of Theorem 7.3, we see that
\[
e_\tau h_{(W,P)} e_\tau / k \{ c^\lambda_{ST} \in B \mid \tau \in \text{Rem}(\lambda) \} \cong h^\tau_{(W,P)} (\sum_{\mu \in \mathcal{P}_{(W,P)}} 1_{\mu \rightarrow \tau} \otimes \text{gap}(\tau))
\]
as left \(h_{(W,P)^\tau}\)-modules and similarly, flipping diagrams across the horizontal axis we get that
\[
e_\tau h_{(W,P)} e_\tau / k \{ c^\lambda_{ST} \in B \mid \tau \in \text{Rem}(\lambda) \} \cong (\sum_{\mu \in \mathcal{P}_{(W,P)}} 1_{\mu \rightarrow \tau} \otimes \text{gap}(\tau)) h^\tau_{(W,P)}
\]
as right \(h_{(W,P)^\tau}\)-modules. This shows that
\[
e_\tau h_{(W,P)} e_\tau / k \{ c^\lambda_{ST} \in B \mid \tau \in \text{Rem}(\lambda) \} \cong h^\tau_{(W,P)} (2) \cong h_{(W,P)^\tau} (2)
\]
as \((h_{(W,P)^\tau}, h_{(W,P)^\tau})\)-bimodules as required.

Let \( M \) be a right \( h_{(W,P)} \)-module. Define the right \( h_{(W,P)} \)-module \( M^* \) by \( M^* = \text{Hom}_k(M, k) \) as a vector space and for \( f \in M^*, a \in h_{(W,P)} \) we define \( fa \in M^* \) by \( (fa)(m) = f(ma^*) \) where \( a^* \) is the dual element in \( h_{(W,P)} \) (given by flipping a diagram across the horizontal axis).

**Theorem 7.5.** For an \( h_{(W,P)} \)-module we have that \( G_\tau(M^*) \cong (G^\tau(M))^* \).

**Proof.** We have that
\[
G^\tau(M^*) = M^* \otimes_{h_{(W,P)^\tau}} e_\tau h_{(W,P)}(-1) \quad \text{and} \quad (G^\tau(M))^* = \text{Hom}_k(M \otimes_{h_{(W,P)^\tau}} e_\tau h_{(W,P)}(-1), k)
\]
We define \( \vartheta : G^\tau(M^*) \rightarrow (G^\tau(M))^* \) by setting \( f \otimes a \mapsto \vartheta f \otimes a \) for \( f \in M^* \) and \( a \in e_\tau h_{(W,P)}(-1) \) where
\[
\vartheta f \otimes a(m \otimes b) = f(m \psi(ba^*))
\]
for \( m \in M \) and \( b \in e_\tau h_{(W,P)}(-1) \). Note that this makes sense because \( ba^* \in e_\tau h_{(W,P)} e_\tau(-2) \) and so \( \psi(ba^*) \in h_{(W,P)^\tau} \). Also \( \vartheta \) is well-defined as \( \psi \) is a bimodule homomorphism.

We now show that \( \vartheta \) is a \( h_{(W,P)^\tau} \)-homomorphism. On one hand, we have
\[
\vartheta(f \otimes a)(m \otimes b) = \vartheta f \otimes a(m \otimes b) = f(m \psi(ba^*))
\]
on the other hand, we have
\[
(\vartheta(f \otimes a)x)(m \otimes b) = \vartheta f \otimes a((m \otimes b)x^*) = \vartheta f \otimes a(m \otimes bx^*) = f(m \psi(bx^*a^*))
\]
as required. We now show that \( \vartheta \) is a vector space isomorphism. It is enough to check that \( \vartheta : G_\tau(M^*)1_{\mu} \rightarrow (G^\tau(M))^*1_{\mu} \) is a vector space isomorphism for each \( \mu \in \mathcal{P}_{(W,P)} \). We have
\[
G_\tau(M^*)1_{\mu} = M^* \otimes_{h_{(W,P)^\tau}} e_\tau h_{(W,P)}(-1)1_{\mu} \\
(G^\tau(M))^*1_{\mu} = \text{Hom}_k(M \otimes_{h_{(W,P)^\tau}} e_\tau h_{(W,P)}(-1), k)1_{\mu} \\
= \text{Hom}_k(M \otimes_{h_{(W,P)^\tau}} e_\tau h_{(W,P)}(-1)1_{\mu}, k) \\
= (M \otimes_{h_{(W,P)^\tau}} e_\tau h_{(W,P)}(-1)1_{\mu})^*
\]
We have seen in the proof of Theorem 7.3 that \( e_\tau h_{(W,P)} 1_{\mu} \) is either zero or isomorphic to (possibly two shifted copies of) \( h_{(W,P)^\tau} 1_{\nu} \) for some \( \nu \in \mathcal{P}_{(W,P)^\tau} \). So it is enough to note that
\[
M^*1_{\nu} = M^* \otimes_{h_{(W,P)^\tau}} h_{(W,P)} 1_{\nu} \cong (M \otimes_{h_{(W,P)^\tau}} h_{(W,P)} 1_{\nu})^* = (M1_{\nu})^*
\]
as required.

Using our induction functor, we will relate (sequences of) \( h_{(W,P)^\tau} \)-modules labelled by \( \lambda \in \mathcal{P}_{(W,P)^\tau} \) with (sequences of) \( h_{(W,P)} \)-modules labelled by
\[
\lambda^+ := \varphi(\lambda) \quad \text{and} \quad \lambda^- := \varphi(\lambda) - \tau.
\]
We note that this is the typical Kazhdan–Lusztig “doubling-up” that we expect.

**Proposition 7.6.** For each \( \lambda \in \mathcal{P}_{(W,P)^\tau} \), we have \( G^\tau(P(\lambda)) = P(\lambda^+)(-1) \).
Proof. Recall that \((W, P)\) is a simply laced Hermitian symmetric pair. By Theorem 7.2, the projective indecomposable modules are \(P(\lambda) = 1_{\lambda} h_{(W, P)}\) for \(\lambda \in \mathcal{P}_{(W, P)}\). Therefore

\[ G^\tau(P(\lambda)) = 1_{\lambda} h_{(W, P)} \otimes_{h_{(W, P)}} \mathcal{c}_\tau h_{(W, P)}(-1) = 1_{\varphi(\lambda)} h_{(W, P)}(-1) = P(\lambda^+)(-1) \]

as required. \(\square\)

**Proposition 7.7.** For each \(\mu \in \mathcal{P}_{(W, P)}\), we have

\[ 0 \to \Delta(\mu^-) \to G^\tau(\Delta(\mu)) \to \Delta(\mu^+)(-1) \to 0 \]

**Proof.** We have an exact sequence

\[ 0 \to h_{(W, P)}^\mu \to P(\mu) \to \Delta(\mu) \to 0 \]

where \(h_{(W, P)}^\mu = \sum_{\nu < \mu} 1_{\mu} h_{(W, P)} 1_{\nu} h_{(W, P)}.\) The modules \(P(\mu)\) and \(h_{(W, P)}^\mu\) have bases

\[ \{1_{\mu} c_{ST}^\mu \mid S, T \in \text{Path}(\nu, -), \nu < \mu\} \quad \{1_{\mu} c_{ST}^\mu \mid S, T \in \text{Path}(\nu, -), \nu < \mu\} \]

respectively. Since \(G^\tau\) is exact, we obtain an exact sequence

\[ 0 \to G^\tau(h_{(W, P)}^\mu) \to G^\tau(P(\mu)) \to G^\tau(\Delta(\mu)) \to 0 \]

where \(G^\tau(P(\mu)) \cong P(\mu^+)(-1).\) Therefore \(G^\tau(\Delta(\mu)) = P(\mu^+)(-1)/G^\tau(h_{(W, P)}^\mu)\) has basis given by

\[ \{c_U(-1), c_V \otimes \text{spot}_V^{\mu^+}(-1) \mid U \in \text{Path}(\mu^+, -), V \in \text{Path}(\mu^-, -)\}. \]

It is clear that, as a right \(h_{(W, P)}\)-module

\[ \{c_V \otimes \text{spot}_V^{\mu^+}(-1) \mid V \in \text{Path}(\mu^-, -)\} \]

is a submodule of \(G^\tau(\Delta(\mu))\) isomorphic to \(\Delta(\mu^-)\) and the quotient is isomorphic to \(\Delta(\mu^+)(-1).\) \(\square\)

7.2. **Koszulity.** We are now able to use the ideas of the previous section in order to prove that \(h_{(W, P)}\) is standard Koszul.

**Definition 7.8.** For \(\lambda, \mu \in \mathcal{P}_{(W, P)}\), we define polynomials \(p_{\lambda, \mu}(q)\) inductively on the rank and Bruhat order as follows. We set \(p_{\lambda, \lambda}(q) = 1\) and for \(\lambda \nsubseteq \mu\) we set \(p_{\lambda, \mu}(q) = 0.\) If \(\lambda \subset \mu,\) pick \(\tau\) such that \(\tau \in \text{Rem}(\lambda).\) We set

\[ p_{\lambda, \mu}(q) = \begin{cases} p_{\lambda_{\tau^{-1}, \mu_{\tau^{-1}}}}(q) + q \times p_{\lambda_{\tau}, \mu}(q) & \text{if } \tau \in \text{Rem}(\mu); \\ q \times p_{\lambda_{\tau}, \mu}(q) & \text{if } \tau \notin \text{Rem}(\mu). \end{cases} \]

We write \(p_{\lambda, \mu}(q) = \sum_{n=0} P_{\lambda, \mu}(n) g^n.\)

**Theorem 7.9.** For \(\lambda \in \mathcal{P}_{(W, P)},\) we have an exact sequence

\[ \cdots \to P_2(\lambda) \to P_1(\lambda) \to P_0(\lambda) \to \Delta(\lambda) \to 0 \]

where \(P_0(\lambda) = P(\lambda)\) and for \(n \geq 1\) we have \(P_n(\lambda) = \oplus_{\mu \in \mathcal{P}_{(W, P)}} P_{\lambda, \mu}(n) P(\mu)/n.\)

**Proof.** We proceed by induction on the rank of \(W\) and the Bruhat order on \(\mathcal{P}_{(W, P)}\). If \(\lambda = \emptyset\) is the minimal element in the Bruhat order, then \(\Delta(\emptyset) = P(\emptyset)\) and we are done. Assume \(\emptyset \neq \lambda \in \mathcal{P}_{(W, P)},\) then there exists some \(\tau \in \text{Rem}(\lambda)\) and we have that \(\lambda - \tau \in \mathcal{P}_{(W, P)}\) and \(\lambda_{\tau} \in \mathcal{P}_{(W, P)}.\) By induction we have exact sequences,

\[ \cdots \to P_2(\lambda - \tau) \to P_1(\lambda - \tau) \to P_0(\lambda - \tau) \to \Delta(\lambda - \tau) \to 0 \]
\[ \cdots \to P_2(\lambda_{\tau}) \to P_1(\lambda_{\tau}) \to P_0(\lambda_{\tau}) \to \Delta(\lambda_{\tau}) \to 0 \]
in $h_{(W,P)}$-mod and $h_{(W,P)^\tau}$-mod respectively. Applying the induction functor $G^\tau$ to the latter sequence, and lifting the injective homomorphism from Proposition 7.7 we obtain a commutative diagram with exact rows.

\[
\cdots \longrightarrow P_2(\lambda - \tau) \longrightarrow P_1(\lambda - \tau) \longrightarrow P_0(\lambda - \tau) \longrightarrow \Delta(\lambda - \tau) \longrightarrow 0
\]

\[
\cdots \longrightarrow G^\tau(P_2(\lambda_{\downarrow \tau})) \longrightarrow G^\tau(P_1(\lambda_{\downarrow \tau})) \longrightarrow G^\tau(P_0(\lambda_{\downarrow \tau})) \longrightarrow G^\tau(\Delta(\lambda_{\downarrow \tau})) \longrightarrow 0
\]

Taking the total complex of this double complex (that is, summing over the dotted lines) and then taking the quotient by the complex

\[
\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \Delta(\lambda - \tau) \longrightarrow \Delta(\lambda - \tau) \longrightarrow 0
\]

we obtain

\[
\cdots \longrightarrow G^\tau(P_2(\lambda_{\downarrow \tau})) \oplus P_1(\lambda - \tau) \longrightarrow G^\tau(P_1(\lambda_{\downarrow \tau})) \oplus P_0(\lambda - \tau) \longrightarrow G^\tau(P_0(\lambda_{\downarrow \tau})) \longrightarrow \Delta(\lambda)(-1) \longrightarrow 0.
\]

We have $G^\tau(P_0(\lambda_{\downarrow \tau})) = P(\lambda)(-1)$. By induction, for $n \geq 1$ we have that

\[
G^\tau(P_n(\lambda_{\downarrow \tau})) = P_{n-1}(\lambda - \tau)
\]

where the last equality follows by the definition of $p_{\lambda,\mu}(q)$. Thus we obtain an exact sequence

\[
\cdots \longrightarrow P_2(\lambda)(-1) \longrightarrow P_1(\lambda)(-1) \longrightarrow P_0(\lambda)(-1) \longrightarrow \Delta(\lambda)(-1) \longrightarrow 0.
\]

Applying a degree shift $(1)$ gives the required linear projective resolution for $\Delta(\lambda)$.

\[\square\]

**Corollary 7.10.** The algebra $h_{(W,P)}$ is Koszul.

**Proof.** The algebra $h_{(W,P)}$ is graded quasi-hereditary algebra with (right) standard modules $\Delta(\lambda)$; the linear projective resolutions of these modules are given in Theorem 7.9. Twisting with the anti-automorphism $*$ we also get that its left standard modules have linear projective resolutions. Therefore $h_{(W,P)}$ is Koszul by [ADL03, Theorem 1].

\[\square\]

**Corollary 7.11.** For $\mu \in \mathcal{P}_{(W,P)}$, we have that the radical filtration of $\Delta(\mu)$ coincides with the grading filtration

\[
\Delta(\mu) = \Delta_{\geq 0}(\mu) \supset \Delta_{\geq 1}(\mu) \supset \Delta_{\geq 2}(\mu) \supset \ldots
\]

where we define $\Delta_{\geq k}(\mu) = \{S \mid S \in \text{Path}(\lambda, t_\mu), \deg(S) \geq k\}$.

**Proof.** We have that $h_{(W,P)}$ is Koszul by Corollary 7.10. That the radical and grading series coincide follows from [BGS96, Proposition 2.4.1].

\[\square\]

**Acknowledgements.** The first and third authors are grateful for funding from EPSRC grant EP/V00090X/1 and the Royal Commission for the Exhibition of 1851, respectively.
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