AN INDEX THEORY FOR SINGULAR SOLUTIONS OF THE
NEWTONIAN $n$-BODY PROBLEM

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Abstract. In the Newtonian $n$-body problem, for solutions with arbitrary energy, that start and end either at infinity or total collision, we prove some basic results about their Morse and Maslov indices. Moreover for homothetic solutions with arbitrary energy, we give a simple and precise formula that relates the Morse indices of homothetic solutions to the spectra of the normalized potential at the corresponding central configurations. Potentially these results could be useful in the application of non-action minimization methods in the Newtonian $n$-body problem.

AMS Subject Classification: 70F16, 70F10, 37J45, 53D12

Key Words: celestial mechanics, index theory, hyperbolic solution.

1. Introduction

The Newtonian $n$-body problem studies the motion of $n$ point masses, $m_i > 0$, according to Newton’s law of universal gravitation. Let $M = \text{diag}(m_1I_d, \ldots, m_nI_d)$ be the mass matrix, where $I_d$ is the $d \times d$ identity matrix with $d \geq 1$. Then $q = (q_i)_{i=1}^n$ ($q_i \in \mathbb{R}^d$ represents the position of $m_i$) satisfies

$$M \ddot{q} = \nabla U(q),$$

The first author thanks the support of NSFC (No.11790271, 11425105). The second author thanks support of NSFC (No.11671215, 11801583). The last author thanks the support of MSRI in Berkeley (under the NSF Grant No. DMS-1440140) and Nankai Zhide Foundation.
where $U(q) = \sum_{1 \leq i < j \leq n} \frac{m_im_j}{|q_i - q_j|}$ is the potential function (the negative potential energy) and $V$ is the gradient with respect to the Euclidean metric.

The solutions of (1) are invariant under linear translations, so there is no loss of generality to restrict ourselves to the $n^* := d(n-1)$ dimension subspace

$$\mathcal{X} := \{ q \in \mathbb{R}^{dn} : \sum_{i=1}^{n} m_i q_i = 0 \},$$

where the center of mass is fixed at the origin.

Let $T\mathcal{X}$ be the tangent bundle of $\mathcal{X}$. The Lagrangian $L: T\mathcal{X} \to [0, +\infty) \cup \{ +\infty \}$

$$L(q,v) = K(v) + U(q), \quad \text{where } K(v) := \frac{1}{2} |v|_M^2 := \frac{1}{2} \langle Mv,v \rangle,$$

has singularities at the collision configurations

$$\Delta = \bigcup_{1 \leq i < j \leq n} \Delta_{ij}, \quad \text{where } \Delta_{ij} = \{ q \in \mathcal{X} : q_i = q_j \}.$$

It is well-known (see [2]) the Lagrangian action functional

$$\mathcal{A}(q; t_1, t_2) := \int_{t_1}^{t_2} L(q(t), \dot{q}(t)) dt,$$

is $C^2$ on $W^{1,2}([t_1, t_2], \mathcal{X})$, $\tilde{\mathcal{X}} := \mathcal{X} \setminus \Delta$ represents collision-free configurations, and any collision-free critical point of $\mathcal{A}$ is a classical solution of (1).

Because the Newtonian gravity is a weak force, the action value of a path with collisions could still be finite. This means the critical points we find using variational methods may contain a subset of collision moments with zero measure and only satisfies (1) in the complement of it. Such solutions were named generalized solutions by Bahri and Rabinowitz in [3] and [4].

In the last twenty years, a lot of progress has been made regarding how to rule out collisions in minimal critical points for the Newtonian $n$-body problem (or the general weak force $n$-body problem), details can be found in [15], [14], [20], [13], [34] and the references within.

However very few results are available on how to rule out collisions in non-minimal critical points. In [34], based on an idea of Tanaka [32], it was shown the Morse index can be used to given an upper bound of the number of possible binary collisions (in certain cases eliminate all the possible binary collisions). We believe similar results should be available for collisions with more than two bodies. However this demands an efficient way of computing the Morse indices of collision solutions, and this will be one of the main results of this paper.

Besides collision solutions we are also interested parabolic and hyperbolic solutions where the distance between each pairs of masses goes to infinity as time goes to negative or positive infinity. For simplicity, we treat them as different types of singularities alongside the collision singularity as defined below.

**Definition 1.1.** Let $q \in C^2((T^-, T^+), \tilde{\mathcal{X}})$ be a solution of (1), where $T^\pm \in \mathbb{R} \cup \{ \pm \infty \}$. We say

1. $q(T^\pm)$ is a collision singularity, if $T^\pm \in \mathbb{R}$ and $\lim_{t \to T^\pm} q(t) = 0$;
2. $q(T^\pm)$ is a parabolic singularity, if $T^\pm = \pm \infty$, $\lim_{t \to T^\pm} |q_i(t) - q_j(t)| = +\infty$, for all $i \neq j$, and $\lim_{t \to T^\pm} \dot{q}_i(t) = 0$, for all $i$;
3. $q(T^\pm)$ is a hyperbolic singularity, if $T^\pm = \pm \infty$, $\lim_{t \to T^\pm} |q_i(t) - q_j(t)| = +\infty$, for all $i \neq j$, and $\lim_{t \to T^\pm} \dot{q}_i(t)$ exists and does not equal to zero, for all $i$. 
Such a solution \( q(t) \) will be called a **singular solution**, if both \( q(T^-) \) and \( q(T^+) \) are one of the singularities defined above.

**Remark 1.1.** The readers should notice that in the above definition, collision means total collision, parabolic (hyperbolic) means complete parabolic (hyperbolic). There are singularities correspond to partial collision, elliptic-parabolic, elliptic-hyperbolic, parabolic-hyperbolic and so on, which will not be discussed in this paper.

Write a singular solution \( q(t) \) defined as above in polar coordinates:

\[
    r = \sqrt{I(q)}, \quad s = (s_i)_{i=1}^n = q/r = (q_i/r)_{i=1}^n,
\]

where \( I(q) = (Mq, q) \) is the moment of inertia. Then \( \mathcal{E} := \{ q \in \mathcal{X} : I(q) = 1 \} \) is the set of normalized configurations. The gradient of \( U \) restricted on \( \mathcal{E} \) is

\[
    \nabla U|_\mathcal{E}(s) = \nabla U(s) + U(s)Ms.
\]

It is well-known a collision-free configuration in \( \mathcal{E} \) is a **central configurations**, if and only if \( \nabla U|_\mathcal{E} \) vanishes.

When \( q(T^\pm) \) is a hyperbolic singularity, by Chazy [11], \( s(t) \) converges to some \( s^\pm \in \mathcal{E} \setminus \Delta \), as \( t \to T^\pm \). On the other hand, when \( q(T^\pm) \) is a collision or parabolic singularity, we only know \( s(t) \) converges to the set of central configurations, as \( t \to T^\pm \). It is not clear whether there is a definite limit, except when \( n \leq 3 \). In our results, we shall assume such a limit always exists.

The scattering theory concerns the possible pairs of normalized configurations that can be connected by singular solutions that come in and go out to infinity after normalization, for more details see [19]. Notice that when the energy is zero, the limit configurations must be central configurations. While it is possible to construct half of such a solution that either comes in from infinity or goes out to infinity using action minimization method, see [28] and [29], it is impossible to find the entire trajectory defined on \( \mathbb{R} \) that goes from infinity to infinity as a minimizer in the \( n \)-body problem, see [17]. Meanwhile for the spatial \( n \)-center problem, it is possible to construct entire solutions using some minimax approach [9]. To generalize the result to the \( n \)-body problem it also seems necessary to develop some Morse index theory for the singular solutions.

Since the domain of a singular solution is not compact, some care has to be taken when we try to define its Morse index. Notice that for any \( [t_1, t_2] \subset (T^-, T^+) \), \( q|_{[t_1, t_2]} \) is a collision-free critical point of the action functional \( \mathcal{A} \) in \( W^{1,2}([t_1, t_2], \mathcal{X}) \). Then the Morse index of \( q|_{[t_1, t_2]} \) in \( H^1_0([t_1, t_2], \mathcal{X}) \), denoted by \( m^-(q; t_1, t_2) \), is the dimension of the largest subspace in \( H^1_0([t_1, t_2], \mathcal{X}) \), where the second derivative \( d^2A(q; t_1, t_2) \) is negative.

**Definition 1.2.** Let \( \{t_k^\pm\}_{k \in \mathbb{Z}} \) be two sequences satisfying \( T^- < t_k^- < t_k^+ < T^+ \) and \( \lim_{k \to \infty} t_k^\pm = T^\pm \). We define the **Morse index** of the singular solution \( q \in C^2((T^-, T^+), \mathcal{X}) \) as

\[
    m^-(q; T^-, T^+) = \lim_{k \to \infty} m^-(q; t_k^-, t_k^+).
\]

**Remark 1.2.** Because of the following monotone property (see [10] or [25])

\[
    m^-(q; t_1, t_2) \leq m^-(q; t_1^*, t_2^*), \text{ if } t_1^* \leq t_1, t_2 \leq t_2^*,
\]

\( m^-(q) \) is well-defined and independent of the choice of the sequences \( \{t_k^\pm\} \). The above definition of Morse index is not a surprise. The main challenge is to compute it and this will be the main contribution of our paper.
For any \( s \in \mathcal{E} \setminus \Delta \), the Hessian of \( U \) at \( s \) restricted on \( \mathcal{E} \) with respect to the Euclidean inner product is
\[
D^2U|_{\mathcal{E}}(s) = D^2U(s) + U(s)M.
\]
Therefore with respect to the \( M \) inner product, the Hessian is
\[
M^{-1}D^2U|_{\mathcal{E}}(s) = M^{-1}D^2U(s) + U(s)I.
\]

**Definition 1.3.** Given a central configuration \( s_0 \in \mathcal{E} \). We denote the eigenvalues of \( M^{-1}D^2U|_{\mathcal{E}}(s_0) \) by
\[
\lambda_1(s_0) \leq \lambda_2(s_0) \leq \cdots \leq \lambda_{n^*-1}(s_0).
\]
We say \( s_0 \) satisfies
1. the **spiral** condition, if \( \lambda_1(s_0) < -\frac{1}{8}U(s_0) \),
2. the **non-spiral** condition, if \( \lambda_1(s_0) \geq -\frac{1}{8}U(s_0) \);
3. the **strict non-spiral** condition, if \( \lambda_1(s_0) > -\frac{1}{8}U(s_0) \).

**Remark 1.3.** In \([6]\), the strict non-spiral condition was called \([BS]\)-condition. We prefer the current name because of the following reason. In McGehee coordinates, see subsection, central configurations give rise to equilibria on the collision manifold. The above conditions determine whether some of the eigenvalues of the linearized systems at the equilibria have non-zero imaginary parts, see Remark, which then determine whether the corresponding stable and unstable manifolds spiral into and out of the equilibria. For details see \([15]\) and \([31]\).

**Theorem 1.1.** Given a singular solution \( q \in C^2((T^-, T^+), \mathbb{R}) \) with both limits exist \( \lim_{t \to T^\pm} s(t) = s^\pm \).

(a). If \( q(T^\pm) \) is a collision or parabolic singularity with the corresponding \( s^\pm \) satisfying the spiral condition, then for any \( t_1 \in (T^-, T^+) \),
\[
\lim_{t_2 \to T^\pm} m^-(q; t_1, t_2) = \frac{1}{3\sqrt{2\pi}} \sum_{i=1}^l \sqrt{-\frac{1}{8} \frac{\lambda_i(s^\pm)}{U(s^\pm)}}
\]
where \( l = \# \{ 1 \leq i \leq n^*-1 : \lambda_i < -\frac{U(s^*)}{8} \} \) and
\[
\beta(t) = \begin{cases} 
|t - T^\pm|, & \text{when } q(T^\pm) \text{ is a collision singularity}, \\
|t|, & \text{when } q(T^\pm) \text{ is a parabolic singularity}.
\end{cases}
\]
In particular this implies \( m^-(q; T^-, T^+) = +\infty \).

(b). If \( s^\pm \) satisfying the strict non-spiral condition, whenever \( q(T^\pm) \) is a collision or parabolic singularity, then \( m^-(q; T^-, T^+) \) is finite.

**Proof.** Property (a) and (b) follow from Theorem \([6, 3.1]\) and \([6, 3.2]\) in section \([6]\) respectively.

The result that \( m^-(q, T^-, T^+) = +\infty \) under the condition in property (a) of the above theorem were already obtained for solutions with collision singularities in \([7]\) and solutions with parabolic singularities in \([6]\). Compare with approaches used in these papers, the one used in this paper allows us to achieve two additional things: first we can given an estimate of how fast the Morse index grows when the solution approaches a collision or parabolic singularity as in \([3]\); second we are able
to give a simple and precise formula for the computation of the Morse index of homothetic solutions as stated in the next theorem.

Let \( q(t) \) be a homothetic solution, then \( s(t) \equiv s_0 \), for all \( t \), where \( s_0 \) is a normalized central configuration. When \( s_0 \) satisfies the spiral condition, then property (a) in Theorem 1.1 applies. Our next result tells what happens otherwise.

**Theorem 1.2.** Let \( H_0 \) be the energy of the homothetic solution \( q \in C^2((T^-, T^+), \hat{X}) \). If the corresponding normalized central configuration \( s_0 \) satisfies the non-spiral condition,

\[
m^-(q; T^-, T^+) = \begin{cases} m^-(M^{-1}D^2U|\mathcal{E}(s_0)), & \text{when } H_0 < 0; \\ 0, & \text{when } H_0 \geq 0,
\end{cases}
\]

where \( m^-(M^{-1}D^2U|\mathcal{E}(s_0)) \) is the number of negative eigenvalues of the matrix \( M^{-1}D^2U|\mathcal{E}(s_0) \).

**Remark 1.4.** We point out that in Theorem 1.2, the central configuration \( s_0 \) only needs to satisfy the non-spiral condition, while in property (b) of Theorem 1.1 the corresponding central configuration needs to satisfy the stronger strict non-spiral condition. The improvement is possible because we have precise expressions for the homothetic solutions.

**Proof.** The theorem follows from Proposition 4.6. \( \square \)

Our paper is organized as follows: in Section 2 the McGehee coordinate will be used to study the asymptotic behavior of the system near a collision/parabolic singularity and a variation of the McGehee coordinates (which will be called hyperbolic McGehee coordinates) is also introduced for understanding the asymptotic behavior of the system near a hyperbolic singularity; in Section 3 we establish an index theory for general singular solutions and prove Theorem 1.1 in Section 2 we show how to compute the Morse index of homothetic solutions with arbitrary energy and prove Theorem 1.2.

**Notations.** Following notations will be adopted throughout the paper.

- \( J \) is the identity matrix and \( J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \). The dimensions of these matrix will not always be the same, but can be easily found out through the context.
- Given a function \( f : \mathbb{R}^k \to \mathbb{R} \), \( \nabla f \) represents the gradient of \( f \) with respect to the column vector and \( D^2 \) the Hessian of \( f \).
- For any positive integer \( k \), \( V_D \) represents the momentum space \( \mathbb{R}^k \oplus \{0\} \), which corresponds to the Dirichlet boundary condition, and \( V_N \) represents the configuration space \( \{0\} \oplus \mathbb{R}^k \), which corresponds to the Neumann boundary condition.
- Given a vector \( \xi \in \mathbb{R}^k \), \( \langle \xi \rangle \) will represent the line subspace spanned by the vector.
- Given a finite set \( A \), \( \#A \) will represent the number of elements in the set.
2. McGehee coordinates and dynamics of the linear system along a singular solution

Throughout this section, let \( q \in C^2((T^-, T^+), \mathbb{R}^2) \) be a singular solution of the \( n \)-body problem with energy constant \( H_0 \) and satisfying
\[
\lim_{t \to T^\pm} s(t) = \lim_{t \to T^\pm} q(t)/r(t) := s^\pm.
\]

In order to compute the Morse index of such a singular solution, we need to understand its asymptotic behaviors as it approaches to the singularities. For a collision or parabolic singularity, this can be achieved by McGehee coordinates (see [30] and [31]). For a hyperbolic singularity, we introduce a new set of coordinates following the spirit of McGehee, which will be called hyperbolic McGehee coordinates.

2.1. Asymptotic estimates. Since \( E \) is an \( n^* := d(n-1) \)-dimension ellipse, we introduce a smooth coordinate chart \((\Omega, \psi)\) on \( E \),
\[
\psi : \Omega \to \psi(\Omega) \subset \mathbb{R}^{n^*}-1; \quad s \mapsto x.
\]
Set \( \hat{U}(x) := U(\psi^{-1}(x)) \) and
\[
\hat{M} := \left( \frac{\partial \psi^{-1}}{\partial x} \right)^T M \left( \frac{\partial \psi^{-1}}{\partial x} \right).
\]

Remark 2.1. Notice that \( \hat{M} \) depends on \( x \) and \( \hat{M}^T = \hat{M} \).

Under the new variables \((r, x)\), the Lagrangian can be written as
\[
L(r, x, \dot{r}, \dot{x}) = K(r, x, \dot{r}, \dot{x}) + r^{-1} \hat{U}(x) = \frac{1}{2} (r^2 + r^2 (\hat{M} \dot{x}, \dot{x})) + r^{-1} \hat{U}(x).
\]

Moreover
\[
\lim_{t \to T^\pm} x(t) = \lim_{t \to T^\pm} \psi(s(t)) = \psi(s^\pm) := x^\pm,
\]
and
\[
\hat{M}_\pm := \lim_{t \to T^\pm} \hat{M}(x(t)).
\]

Further introduce the new variables \((p_1, p_2)\) as
\[
p_1 = \dot{r}, \quad p_2 = r^2 \dot{M} \dot{x}.
\]

We obtain the corresponding Hamiltonian
\[
H(p_1, p_2, r, x) = \frac{1}{2} \left( p_1^2 + \frac{\langle \hat{M}^{-1} p_2, p_2 \rangle}{r^2} \right) - \frac{\hat{U}(x)}{r}.
\]

Let \( \zeta(t) = (p_1, p_2, r, x)(t) \) represent the singular solution \( q(t) \) in the new coordinates introduced as above. Then it satisfies the following Hamiltonian equation
\[
\dot{\zeta} = J \nabla H(\zeta),
\]
where
\[
\nabla H(\zeta) = \left( p_1, \frac{p_1^T \hat{M}^{-1} \hat{U}(x)}{r^2} - \frac{\langle \hat{M}^{-1} p_2, p_2 \rangle}{r^3}, D_x(\hat{M}^{-1} p_2, p_2) - \frac{D_x \hat{U}(x)}{r} \right)^T.
\]
Moreover the linearized system of (16) along $\zeta(t)$ is

\[(17)\quad \dot{\zeta}(t) = JB(t)\zeta(t) := JD^2H(\zeta(t))\zeta(t),\]

where

\[
D^2H(\zeta(t)) = \\
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{\hat{M}^{-1}}{r^2} & -\frac{2}{r} \hat{M}^{-1} p_2 & \frac{D_x(\hat{M}^{-1} p_2)}{r} \\
-\frac{2}{r} p_2 \hat{M}^{-1} & \frac{3(\hat{M}^{-1} p_2, p_2)}{r^3} - \frac{2\hat{U}(x)}{r^3} & \frac{D_x(\hat{M}^{-1} p_2)}{r^3} - \frac{D_x(\hat{M}^{-1} p_2, p_2)}{2r} \\
0 & \frac{\hat{U}^2}{r^3} - \frac{\nabla_x(\hat{M}^{-1} p_2)}{r^3} & \frac{3(\hat{M}^{-1} p_2, p_2)}{2r^3} - \frac{\hat{D}_x(\hat{M}^{-1} p_2)}{r} \\
\end{pmatrix}
\]

(18)

Lemma 2.1. (a) If $q(T^\pm)$ is a collision singularity or parabolic singularity,

\[
\lim_{t \to T^\pm} r(t)\beta^{-\frac{1}{2}}(t) = [18U(s^\pm)]^\frac{1}{4},
\]

\[
\lim_{t \to T^\pm} |\dot{r}(t)|\beta^{\frac{1}{2}}(t) = [2U(s^\pm)]^\frac{1}{4},
\]

\[
\lim_{t \to T^\pm} r^\frac{3}{2}(t)|\dot{s}(t)|_M = 0,
\]

where $\beta(t)$ is defined as in (9).

(b) If $q(T^\pm)$ is a hyperbolic singularity,

\[
\lim_{t \to T^\pm} r(t)|d|^{-1} = \sqrt{2H_0},
\]

\[
\lim_{t \to T^\pm} |\dot{r}(t)| = \sqrt{2H_0},
\]

\[
\lim_{t \to T^\pm} r(t)|\dot{s}(t)|_M = 0.
\]

Proof. (a). This is the well-known Sundman-Sperling estimates, detailed proofs can be found in [5] Theorem 4.18 or [8] Theorem 7.7.

(b). Since $q(T^\pm)$ is a hyperbolic singularity, $T^\pm = \pm \infty$. We show the details for $T^+ = +\infty$ ($T^- = -\infty$ can be proven similarly).

As $I(q) = \langle Mq, \dot{q} \rangle$,

\[
(19)\quad \ddot{I}(q) = 2\langle M\dot{q}, \dot{q} \rangle + 2\langle Mq, \ddot{q} \rangle.
\]

By the homogeneity of $U$, $\langle Mq, \dot{q} \rangle = -U(q)$. Then

\[
(20)\quad \ddot{I}(q) = 4K(\dot{q}) - 2U(q) = 4H_0 + 2U(q).
\]

Recall that $\lim_{t \to \pm \infty} |q_i - q_j| = 0$, for all $i \neq j$, $U(q(t)) \to 0$, as $t \to +\infty$. Therefore $\lim_{t \to +\infty} \dot{I}(q(t)) = 4H_0$. As a result,

\[
(21)\quad \lim_{t \to +\infty} \dot{I}(q(t))t^{-1} = 4H_0, \quad \lim_{t \to +\infty} \ddot{I}(q(t))t^{-2} = 2H_0.
\]

They immediately imply the first two identities in property (b), as $r(t) = \sqrt{I(q(t))}$.

Since $s = (s_i)_{i=1}^n$ and $s_i = I^{-\frac{1}{2}} q_i$,

\[
\dot{q}_i = I^{-\frac{1}{2}}(q)\dot{s}_i + \frac{1}{2} I^{-1}(q)\ddot{I}(q)q_i.
\]

Then

\[
2K(\dot{q}) = \langle M\dot{q}, \dot{q} \rangle = I(q)|\dot{s}|_M^2 + \frac{1}{4} I^{-1}(q)\ddot{I}(q).
\]
As a result,
$$I(q)|\dot{s}|_M^2 = 2K(q) - \frac{1}{4}I^{-1}(q)|\dot{I}^2(q)| = 2H_0 + 2U(q) - \frac{1}{4}I^{-1}(q)|\dot{I}^2(q)|.$$ 
Since \(21\) implies \(\lim_{t\to+\infty} \frac{1}{4}I^{-1}(q(t))|\dot{I}^2(q(t))| = 2H_0,\)

$$\lim_{t\to+\infty} I(q(t))|\dot{s}(t)|_M^2 = \lim_{t\to+\infty} U(q(t)) = 0.$$ 
This then implies the third identity in property (b). \(\square\)

2.2. **McGehee coordinates.** Following McGehee we define the new coordinates \(v\) and \(u\) as
\begin{equation}
    v = r^{\frac{1}{2}}p_1 = r^{\frac{1}{2}}\dot{r}, \quad u = r^{-\frac{1}{2}}p_2 = r^{-\frac{1}{2}}\dot{M}\dot{x}.
\end{equation}

Then equation \((22)\) becomes
\begin{equation}
\begin{aligned}
    \dot{v} &= -\frac{1}{2}(\dot{M}^{-1}v^2 + (\dot{M}^{-1}u, u) - \dot{U}(x)), \\
    \dot{u} &= -\frac{1}{2}uv + \dot{U}_x(x) - \frac{1}{2}((\dot{M}^{-1})_u u), \\
    \dot{r} &= -\frac{1}{2}v, \\
    \dot{x} &= -r^{-\frac{1}{2}}\dot{M}^{-1}u.
\end{aligned}
\end{equation}

By changing the time parameter from \(t\) to \(\tau\) with \(dt = r^{\frac{3}{2}}d\tau\), we have
\begin{equation}
\begin{aligned}
    v' &= \frac{1}{2}v^2 + (\dot{M}^{-1}u, u) - \dot{U}(x), \\
    u' &= -\frac{1}{2}uv + \dot{U}_x(x) - \frac{1}{2}((\dot{M}^{-1})_u u), \\
    r' &= rv, \\
    x' &= \dot{M}^{-1}u,
\end{aligned}
\end{equation}
where \(\dot{\cdot}\) means \(\frac{d}{d\tau}\) throughout the paper.

In these new coordinates, the energy identity becomes
\begin{equation}
\frac{1}{2}(\dot{M}^{-1}u, u) + v^2 - \dot{U}(x) = rH_0.
\end{equation}

**Remark 2.2.** Notice that \((\sqrt{2U(x_0)}, 0, 0, x_0)\) is an equilibrium of \((22)\), where \(x_0 = \psi(s_0)\) with \(s_0\) being a normalized central configuration. By a straightforward computation, one can see, if \(s_0\) satisfies the spiral condition, then at least one of the eigenvalues of the linearized system at the equilibrium will have non-zero imaginary part.

**Lemma 2.2.** If \(q(T^\pm)\) is a collision or parabolic singularity,
\begin{enumerate}
    \item \(\tau = \tau(t) \to \pm\infty\), as \(t \to T^\pm\); \\
    \item \(|v|, u(\tau) \to (\sqrt{2U(s^\pm)}), 0\), as \(\tau \to \pm\infty\).
\end{enumerate}

**Proof.** (a). By the definition of \(\tau\) and Lemma 2.1
\begin{equation}
\frac{d\tau}{dt} = r^{-\frac{3}{2}}(t) \to (3\sqrt{2U(s^\pm)}\beta(t))^{-1}, \quad \text{as } t \to T^\pm.
\end{equation}
This immediately implies property (a).

(b). Recall that \(v = r^{\frac{3}{2}}\dot{r}\) and \(u = r^{\frac{3}{2}}\dot{M}\dot{x}\). Lemma 2.1 implies
\begin{equation}
\lim_{\tau \to \pm\infty} |v(\tau)| = \lim_{t \to T^\pm} r^{\frac{3}{2}}(t)|\dot{r}(t)| = \sqrt{2U(s^\pm)},
\end{equation}
Remark 2.3. Since \( v = r^\frac{2}{3} \dot{r} \) with \( r > 0 \), the sign of \( v \) is the same as \( \dot{r} \). When \( \tau \to \pm \infty \), we can determine the sign of \( v(\tau) \) by simply checking whether the system is expending or shrinking. The same principle applies to Lemma 2.7 as well.

From an index point of view, it is difficult to work with (24), as it is not Hamiltonian. Hence we shall still work with the linear Hamiltonian system (17). However to use the information obtained through the McGehee coordinates, we need to make a couple of transformations of (17).

First we change the time parameter from \( t \) to \( \tau \). Then (17) becomes

\[
(26) \quad \xi'(\tau) = JB(\tau)\xi(\tau) := r^{\frac{2}{3}}(\tau)JD^2H(\xi(\tau))\xi(\tau),
\]

where

\[
(27) \quad B(\tau) =
\begin{pmatrix}
 r^{\frac{2}{3}} & 0 & 0 & 0 \\
 0 & -\frac{M^{-1}}{\langle \nu \rangle^2} & -\frac{2M^{-1}}{\langle \nu \rangle^2} & \frac{D_x(M^{-1}p_2)}{\langle \nu \rangle^2} \\
 0 & -\frac{2p_2^T M^{-1}}{\langle \nu \rangle^2} & \frac{3(M^{-1}p_2, p_2)}{\langle \nu \rangle^4} & \frac{2U(x)}{\langle \nu \rangle^2} & \frac{U_x(x) - (M^{-1}p_2, p_2)_x}{2\langle \nu \rangle^2} \\
 0 & \frac{2U(x)}{\langle \nu \rangle^2} & \frac{U_x(x)}{\langle \nu \rangle^2} - \frac{U_x(M^{-1}u, u)}{\langle \nu \rangle^2} & \frac{1}{2}(M^{-1}u, u)_xx - \dot{U}_x(x)
\end{pmatrix}.
\]

Next the variable \( r \) will be separated from the system. For this we introduce the following lemma (its proof is a straightforward computation).

Lemma 2.3. Let \( R(\tau) \) be a path of symplectic matrices depending on \( \tau \). If \( \xi'(\tau) = JB(\tau)\xi(\tau) \), then \( \eta(\tau) = R(\tau)\xi(\tau) \) satisfies \( \eta'(\tau) = J\Phi_R(\tau)\eta(\tau) \) with

\[
(28) \quad \Phi_R(B)(\tau) := -JR'(\tau)R^{-1}(\tau) + R^{-T}(\tau)B(\tau)R^{-1}(\tau).
\]

In particular, when \( R(\tau) \) is a constant matrix, \( \Phi_R(B)(\tau) = R^{-T}B(\tau)R^{-1} \).

In the rest of the subsection, set \( \tilde{B}(\tau) = \Phi_R(B)(\tau) \) with \( \Phi_R(B) \) given by (28) and

\[
(29) \quad R(\tau) = \text{diag}(r^{\frac{2}{3}}, r^{-\frac{1}{3}}I, r^{-\frac{1}{3}}I, r^\frac{2}{3}I)(\tau)
\]

By the above lemma, \( \eta(\tau) = R(\tau)\xi(\tau) \) satisfies

\[
(30) \quad \eta'(\tau) = J\tilde{B}(\tau)\eta(\tau),
\]

where

\[
(31) \quad \tilde{B}(\tau) =
\begin{pmatrix}
 1 & 0 & -\frac{3}{4}v & 0 \\
 0 & M^{-1} & -\frac{3}{4}v & -2M^{-1}u \\
 -\frac{3}{4}v & -2u^T M^{-1} & 3(M^{-1}u, u) - 2\dot{U}(x) & D_x(M^{-1}u) + \frac{u_x}{4} \\
 0 & \frac{2U(x)}{\langle \nu \rangle^2} & \frac{U_x(x) - \dot{U}_x(M^{-1}u, u)}{\langle \nu \rangle^2} & \frac{1}{2}(M^{-1}u, u)_{xx} - \ddot{U}_x(x)
\end{pmatrix}.
\]

Notice that by (22),

\[
v = r^\frac{2}{3}p_1 = r' r^{-1}, \quad u = r^{-\frac{1}{3}}p_2 = \dot{M}x'.
\]
To simplify notation, in the rest of this subsection we set

$$\tau^* = \pm \infty, \quad s^* = s^\pm, \quad x^* = x^\pm, \quad v^* = \pm \sqrt{2U(s^\pm)}, \quad \hat{M}_* = \hat{M}_\pm,$$

without specifying the $+$ or $-$. This is because as $\tau$ goes to $-\infty$ or $+\infty$, we need to make the corresponding choices between $-$ and $+$ for $s^*, x^*, v^*$ and $\hat{M}_*$. 

**Lemma 2.4.** If $q(T^\pm)$ is a collision or parabolic singularity,

$$\dot{B}(\tau^*) = \lim_{\tau \to \tau^*} \dot{B}(\tau) = \begin{pmatrix} 1 & 0 & -\frac{3}{4}v^* & 0 \\ 0 & \hat{M}_*^{-1} & \frac{3}{4}v^* & 0 \\ -\frac{3}{4}v^* & 0 & -2\dot{U}(x^*) & 0 \\ 0 & \frac{3}{4}v^* & 0 & -\ddot{U}_{xx}(x^*) \end{pmatrix}.$$  

**Proof.** The result follows from Lemma 2.1 as it implies $\lim_{\tau \to \tau^*}(v, u)(\tau) = (v^*, 0)$. 

Rewrite $\dot{B}(\tau^*)$ as $\dot{B}(\tau^*) = \dot{B}_1(\tau^*) \circ \dot{B}_2(\tau^*)$, where

$$\dot{B}_1(\tau^*) = \begin{pmatrix} 1 & -\frac{3}{4}v^* \\ -\frac{3}{4}v^* & -2\dot{U}(x^*) \end{pmatrix}, \quad \dot{B}_2(\tau^*) = \begin{pmatrix} \hat{M}_*^{-1} & \frac{3}{4}v^* \\ \frac{3}{4}v^* & -\ddot{U}_{xx}(x^*) \end{pmatrix}.$$  

Here $\circ$ represents the symplectic sum introduced by Long (see [26]): for any two $2m_k \times 2m_k$ matrices, $O_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$, $k = 1, 2$,

$$O_1 \circ O_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$  

$J\dot{B}(\tau^*)$ being hyperbolic or not will play crucial roles in our proofs. We will show this only depends on $s^*$ (or equivalently $x^*$). To prove this, we introduce two lemmas first.

**Lemma 2.5.** Let $s_0$ be a normalized central configuration and $x_0 = \psi(s_0)$. Then $\lambda_i(s_0)$, $i = 1, \ldots, n^* - 1$ (see Definition [13]), are eigenvalues of $A^T \ddot{U}_{xx}(x_0)A$, where $A$ is a matrix satisfying $A^T M_0 A = I$ and

$$\dot{M}_0 = \left( \frac{\partial \psi^{-1}}{\partial x} \right)^T \bigg|_{x=x_0} M \left( \frac{\partial \psi^{-1}}{\partial x} \right) \bigg|_{x=x_0}.$$  

**Proof.** By [11] and the fact $\ddot{U}_{xx}(x_0) = (\frac{\partial \psi^{-1}}{\partial x})^T D^2 U|_{\mathcal{E}(s_0)} (\frac{\partial \psi^{-1}}{\partial x})$,

$$A^T \ddot{U}_{xx}(x_0)A = A^{-1} \dot{M}_0^{-1} \ddot{U}_{xx}(x_0)A$$

$$= A^{-1} \left( \frac{\partial \psi^{-1}}{\partial x} \right)^{-1} \dot{M}_0^{-1} \frac{\partial \psi^{-1}}{\partial x}^{-T} \ddot{U}_{xx}(x_0)A$$

$$= A^{-1} \frac{\partial \psi^{-1}}{\partial x}^{-1} \dot{M}_0^{-1} D^2 U|_{\mathcal{E}(s_0)} \frac{\partial \psi^{-1}}{\partial x} A.$$  

$\square$
Lemma 2.6. Given a matrix \( P = \begin{pmatrix} I & Q \\ Q^T & Q^T Q - R \end{pmatrix} \). If \( Q \) is symmetric, \( J P \) is symplectic similar to \( \hat{J} \hat{P} \) with \( \hat{P} = \begin{pmatrix} I & 0 \\ 0 & -R \end{pmatrix} \).

Proof. Since \( Q = Q^T \), \( O = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \) is symplectic. A direct computation shows \( O^T P O = \hat{P} \). As a result, \( J \hat{P} = J O^T P O = O^{-1}(J P) O \).

Proposition 2.1. If \( q(T^+) \) is a collision or parabolic singularity, \( J \hat{B}(\tau^*) \) is hyperbolic if and only if \( s^* \) satisfies the strict non-spiral condition.

Proof. Since \( J \hat{B}(\tau^*) = J \hat{B}_1(\tau^*) \circ J \hat{B}_2(\tau^*) \), it is hyperbolic if and only if \( J \hat{B}_1(\tau^*) \) and \( J \hat{B}_2(\tau^*) \) are hyperbolic. A direct computation shows the eigenvalues of \( J \hat{B}_1(\tau^*) \) are \( \pm \frac{\sqrt{v^*}}{16} \sqrt{U(x^*)}. \)

For \( J \hat{B}_2(\tau^*) \), we can always find an invertible matrix \( A \) satisfying \( A^T \hat{M} A = I \). Meanwhile \( A_d = \text{diag}(A, A^{-1}) \) is a symplectic matrix. Therefore

\[
A_d(J \hat{B}_2(\tau^*))A_d^{-1} = J A_d^{-T} \hat{B}_2(\tau^*) A_d^{-1} = J \Phi_{A_d}(\hat{B}_2(\tau^*))
\]

where \( \Phi_{A_d}(\hat{B}_2(\tau^*)) \) is defined as in (28). This means \( J \hat{B}_2(\tau^*) \) is hyperbolic if and only if \( J \Phi_{A_d}(\hat{B}_2(\tau^*)) \) is.

Now notice that

\[
\Phi_{A_d}(\hat{B}_2(\tau^*)) = \begin{pmatrix} I & \frac{1}{4}v^* I \\ \frac{1}{4}v^* I & -A^T \hat{U}_{xx}(x^*) A \end{pmatrix}.
\]

Let \( P = \Phi_{A_d}(\hat{B}_2(\tau^*)) \) and \( O = \begin{pmatrix} I & -\frac{1}{4}v^* I \\ 0 & I \end{pmatrix} \). By Lemma 2.6

\[
O^{-1} J \Phi_{A_d}(\hat{B}_2(\tau^*)) O = J \hat{P}, \quad \text{where} \quad \hat{P} = \begin{pmatrix} I & 0 \\ 0 & -\frac{(v^*)^2}{16} I + A^T \hat{U}_{xx}(x^*) A \end{pmatrix}.
\]

Recall that \( v^* = \pm \sqrt{2U(x^*)} = \pm \sqrt{2U(s^*)} \), by a simple computation,

\[
(J \hat{P})^2 = \begin{pmatrix} \frac{U(s^*)}{8} I + A^T \hat{U}_{xx}(x^*) A & 0 \\ 0 & \frac{U(s^*)}{8} I + A^T \hat{U}_{xx}(x^*) A \end{pmatrix}.
\]

By Lemma 2.5 the eigenvalues of \( J \hat{P} \) are \( \pm \sqrt{U(s^*)} \), \( i = 1, \ldots, n^* - 1 \).

This implies the desired result.

\[ \Box \]
2.3. **Hyperbolic McGehee coordinates.** To deal with the hyperbolic singularities, we introduce the following coordinates

\[ v = p_1 = \dot{r}, \quad u = r^{-1}p_2 = r\hat{M}\dot{x}. \]

Then equation (16) becomes

\[
\begin{aligned}
\dot{v} &= r^{-1}(\hat{M}^{-1}u - r^{-1}\hat{U}(x)), \\
\dot{u} &= r^{-1}(-uv + r^{-1}\hat{U}_x(x) - \frac{1}{2}(\hat{M}^{-1}u, u)), \\
\dot{r} &= v, \\
\dot{x} &= r^{-1}\hat{M}^{-1}u.
\end{aligned}
\]

Like the previous subsection, we change the time parameter from \( t \) to \( \tau \), but this time with \( dt = r\, d\tau \)

\[
\begin{aligned}
\dot{v} &= (\hat{M}^{-1}u) - r^{-1}\hat{U}(x), \\
\dot{u} &= -uv + r^{-1}\hat{U}_x(x) - \frac{1}{2}(\hat{M}^{-1}u, u), \\
\dot{r} &= rv, \\
\dot{x} &= \hat{M}^{-1}u.
\end{aligned}
\]

The energy identity now becomes

\[
\frac{1}{2}(\hat{M}^{-1}u, u) + v^2 - r^{-1}\hat{U}(x) = H_0.
\]

**Lemma 2.7.** If \( q(T^\pm) \) is a hyperbolic singularity,

(a). \( \tau = \tau(t) \to \pm\infty \), as \( t \to T^\pm \); \\
(b). \( |v|, u(\tau) \to (\sqrt{2H_0}, 0) \), as \( \tau \to \pm\infty \).

**Proof.** (a). Since \( q(T^\pm) \) is a hyperbolic singularity, the energy constant \( H_0 > 0 \). By Lemma 2.1

\[
\frac{d\tau}{dt} = r^{-1}(t) \to \frac{|t|}{\sqrt{2H_0}} \quad \text{as} \quad t \to T^\pm.
\]

Combining this with the fact that \( T^\pm = \pm\infty \) (see Definition 1.1), we get the desired property.

(b). By (35) and Lemma 2.1 we have

\[
\lim_{\tau \to \pm\infty} |v(\tau)| = \lim_{t \to T^\pm} |\dot{r}(t)| = \sqrt{2H_0};
\]

\[
\lim_{\tau \to \pm\infty} |u(\tau)|^2 = \lim_{t \to T^\pm} r^2(t)(\hat{M}\dot{x}, \hat{M}\dot{x}) = \lim_{t \to T^\pm} r^2(t)|\dot{s}(t)|^2 = 0.
\]

\( \square \)

Like in the previous subsection, we change the time parameter in the linear system (17) from \( t \) to \( \tau \). Now as \( dt = r\, d\tau \), we get

\[
\dot{\xi}(\tau) = JB(\tau)\xi(\tau) := r(\tau)J\nabla^2 H(\xi(\tau))\xi(\tau).
\]

For the rest of this subsection, set \( \hat{B}(\tau) = \Phi_R(B) \) with \( \Phi_R(B) \) given by (28) and

\[
R(\tau) = \text{diag}(r^{\frac{1}{2}}, r^{-\frac{1}{2}}I, r^{-\frac{1}{2}}, r^{\frac{1}{2}}I)(\tau).
\]
By Lemma 2.3

\[ (41) \]
\[ \hat{B}(\tau) = \begin{pmatrix} 1 & 0 & -\frac{1}{2}v^* & 0 \\ 0 & \hat{M}^{-1} & 0 & \frac{1}{2}v^* \\ -\frac{1}{2}v^* & 0 & 0 & 0 \\ 0 & \nabla_x(\hat{M}^{-1}u) + \frac{1}{2}vI & 0 & 0 \end{pmatrix} . \]

Make the same assumption in notations as in (42) for the rest of this subsection.

**Lemma 2.8.** If \( q(T^\pm) \) is a hyperbolic singularity,

\[ (42) \]
\[ \hat{B}(\tau^*) = \lim_{\tau \to \tau^*} \hat{B}(\tau) = \begin{pmatrix} 1 & 0 & -\frac{1}{2}v^* & 0 \\ 0 & \hat{M}^{-1} & 0 & \frac{1}{2}v^*I \\ -\frac{1}{2}v^* & 0 & 0 & 0 \\ 0 & \frac{1}{2}v^*I & 0 & 0 \end{pmatrix} . \]

**Proof.** Since \( r(\tau) \to +\infty \), when \( \tau \to \pm \infty \). The desired result follows from Lemma 2.1 as it implies \( \lim_{\tau \to \tau^*} (v, u)(\tau) = (v^*, 0) \).

As a result, we may write \( \hat{B}(\tau^*) = \hat{B}_1(\tau^*) \circ \hat{B}_2(\tau^*) \) with

\[ (43) \]
\[ \hat{B}_1(\tau^*) = \begin{pmatrix} 1 & -\frac{1}{2}v^* \\ -\frac{1}{2}v^* & 0 \end{pmatrix}, \quad \hat{B}_2(\tau^*) = \begin{pmatrix} \hat{M}^{-1} & \frac{1}{2}v^*I \\ \frac{1}{2}v^*I & 0 \end{pmatrix} . \]

By a similar argument as in the proof of Proposition 2.1 we find

**Proposition 2.2.** If \( q(T^\pm) \) is a hyperbolic singularity, \( \hat{J} \hat{B}(\tau^*) \) is always hyperbolic.

3. **A Morse index theory for singular solutions**

A proof of Theorem 1.1 will be given in this section. Since the Morse index of a singular solution is an integer associated to an unbounded Fredholm operator in an infinite dimension space, it is difficult to compute it directly. Meanwhile for a Hamiltonian system it is known the Morse index is related with the Maslov index, which is an intersection index between two paths of Lagrangian subspaces. In the following we give a brief introduction of the Maslov index that will be needed in our proofs.

Given a 2\( k \)-dimension symplectic space \((\mathbb{R}^{2k}, \omega)\), where \( \omega \) represents the standard symplectic form. Let \( \text{Lag}(2k) \) represents the Lagrangian Grassmanian, i.e. the space of all Lagrangian subspaces in \((\mathbb{R}^{2k}, \omega)\). For any two continuous paths \( L_1(t), L_2(t), t \in [a, b] \), in \( \text{Lag}(2k) \), the Maslov index \( \mu(L_1, L_2; [a, b]) \) is an integer invariant, uniquely determined by a set of axioms listed as Property I to VII below (for the details see [10]).

**Property I. (Reparametrization invariance)** Let \( \varphi : [c, d] \to [a, b] \) be a continuous and piecewise smooth function satisfying \( \varphi(c) = a, \varphi(d) = b \), then

\[ (44) \]
\[ \mu(L_1(t), L_2(t)) = \mu(L_1(\varphi(t)), L_2(\varphi(t))) . \]

**Property II. (Homotopy invariant with end points)** If two continuous families of Lagrangian paths \( L_1(s, t), L_2(s, t), 0 \leq s \leq 1, a \leq t \leq b \) satisfies
dim\( (L_1(s,a) \cap L_2(s,a)) = C_1 \), \( \dim(L_1(s,b) \cap L_2(s,b)) = C_2 \), for any \( 0 \leq s \leq 1 \), where \( C_1, C_2 \) are two constant integers, then
\[
(45) \quad \mu(L_1(0,t), L_2(0,t)) = \mu(L_1(1,t), L_2(1,t)).
\]

**Property III. (Path additivity)** If \( a < c < b \), then
\[
(46) \quad \mu(L_1(t), L_2(t)) = \mu(L_1(t), L_2(t); [a, c]) + \mu(L_1(t), L_2(t); [c, b]).
\]

**Property IV. (Symplectic invariance)** Let \( \gamma(t), t \in [a, b] \) be a continuous path of symplectic matrices in \( \text{Sp}(2n) \), then
\[
(47) \quad \mu(L_1(t), L_2(t)) = \mu(\gamma(t)L_1(t), \gamma(t)L_2(t)).
\]

**Property V. (Symplectic additivity)** Let \( W_i, i = 1, 2 \), be two symplectic spaces, if \( L_i \in C([a, b], \text{Lag}(W_i)) \) and \( \dot{L}_i \in C([a, b], \text{Lag}(W_i)), i = 1, 2 \), then
\[
(48) \quad \mu(L_1(t) \oplus \dot{L}_1(t), L_2(t) \oplus \dot{L}_2(t)) = \mu(L_1(t), L_2(t)) + \mu(\dot{L}_1(t), \dot{L}_2(t)).
\]

**Property VI. (Symmetry)** If \( L_i \in C([a, b], \text{Lag}(2n)), i = 1, 2 \), then
\[
(49) \quad \mu(L_1(t), L_2(t)) = \text{dim} L_1(a) \cap L_2(a) - \text{dim} L_1(b) \cap L_2(b) - \mu(L_2(t), L_1(t)).
\]

In the case \( L_1(t) \equiv V_0, L(t) = \gamma(t)V \), where \( \gamma \) is a path of symplectic matrix we have a monotonicity property (cfr. \[22\]).

**Property VII (Monotone property)** Suppose for \( j = 1, 2, L_j(t) = \gamma_j(t)V \), where \( \dot{\gamma}_j(t) = J B_j(t) \gamma_j(t) \) with \( \gamma_j(t) = I_{2n} \). If \( B_j(t) \geq B_2(t) \) in the sense that \( B_1(t) - B_2(t) \) is non-negative matrix, then for any \( V_0, V_1 \in \text{Lag}(2n) \), we have
\[
(50) \quad \mu(V_0, \gamma_1 V_1) \geq \mu(V_0, \gamma_2 V_1).
\]

An efficient way to study the Maslov index is via crossing form introduced by \[32\]. For simplicity and since it is enough for our purpose, we only review the case of the Maslov index for a path of Lagrangian subspace with respect to a fixed Lagrangian subspace. Let \( \Lambda(t) \) be a \( C^1 \)-curve of Lagrangian subspaces with \( \Lambda(0) = \Lambda \), and let \( V \) be a fixed Lagrangian subspace which is transversal to \( \Lambda \). For \( v \in \Lambda \) and small \( t \), define \( w(t) \in V \) by \( v + w(t) \in \Lambda(t) \). Then the form
\[
(51) \quad Q(v) = \left. \frac{d}{dt} \right|_{t=0} \omega(v, w(t))
\]
is independent of the choice of \( V \) (cfr. \[32\]). A crossing for \( \Lambda(t) \) is some \( t \) for which \( \Lambda(t) \) intersects \( W \) nontrivially. At each crossing, the crossing form is defined to be
\[
(52) \quad \Gamma(\Lambda(t), W, t) = Q|_{\Lambda(t) \cap W}.
\]
A crossing is called regular if the crossing form is non-degenerate. If the path is given by \( \Lambda(t) = \gamma(t)\Lambda \) with \( \gamma(t) \in \text{Sp}(2n) \) and \( \Lambda \in \text{Lag}(2n) \), then the crossing form is equal to \( (-\gamma(t)^{T}J\gamma(t)v, v) \), for \( v \in \gamma(t)^{-1}(\Lambda(t) \cap W) \), where \( (\cdot, \cdot) \) is the standard inner product on \( \mathbb{R}^{2n} \).

For \( \Lambda(t) \) and \( W \) as before, if the path has only regular crossings, following \[27\], the Maslov index is equal to
\[
\mu(W; \Lambda(t)) = m^+(\Gamma(\Lambda(a), W, a)) + \sum_{a < t < b} \text{sign}(\Gamma(\Lambda(t), W, t)) - m^-(\Gamma(\Lambda(b), W, b)),
\]
where the sum runs all over the crossings \( t \in (a, b) \) and \( m^+, m^- \) are the dimensions of positive and negative definite subspaces, \( \text{sign} = m^+ - m^- \) is the signature. We note that for a \( C^1 \)-path \( \Lambda(t) \) with fixed end points, and we can make it only have regular crossings by a small perturbation.
When the Hamiltonian system is given by the Legendre transformation of a Sturm-Liouville system, since all the crossing are positive, we have
\[
\mu(V_0, \Lambda(t)) = \dim(\Lambda(a) \cap V_0) + \sum_{a < t < b} \dim(\Lambda(t) \cap V_0). 
\]

For more details see [32], [21].

Given a Lagrangian path \( t \mapsto \Lambda(t) \), the difference of the Maslov indices of it with respect to two Lagrangian subspaces \( V_0, V_1 \in \text{Lag}(2n) \), is given in terms of the Hörmander index (see [32, Theorem 3.5])
\[
(53) \quad s(V_0, V_1; \Lambda(0), \Lambda(1)) = \mu(V_0, \Lambda(t)) - \mu(V_1, \Lambda(t)).
\]

The Hörmander index is independent of the choice of the path connecting \( \Lambda(0) \) and \( \Lambda(1) \), and it satisfied
\[
(54) \quad |s(V_0, V_1; \Lambda(0), \Lambda(1))| \leq 2n.
\]

With the above preparation on index theory, now we are ready to study the index of singular solutions in the \( n \)-body problem. Let \( q \in C^2((T^-, T^+), \mathcal{X}) \) be a singular solution of the \( n \)-body problem. The linearized Hamiltonian system along such a solution was given in (17). We denote its the fundamental solution as \( \gamma(t, t_1), \) i.e.
\[
\gamma(t, t_1) = JB(t)\gamma(t, t_1), \quad \gamma(t, t_1) = I.
\]

Recall that for any \( [t^-, t^+] \subset (T^-, T^+) \), \( m^-(q; t^-, t^+) \) represents the Morse index of \( q|_{[t^-, t^+]} \). By the Morse index theorem (see [24]),
\[
(55) \quad m^-(q; t^-, t^+) + n^* = \mu(V_0, \gamma(t, t) V_0; [t^-, t^+]),
\]
where the right hand side is the Maslov index of the two paths \( V_0 \) and \( \gamma(t, t) V_0 \), \( t \in [t^-, t^+] \).

In order to compute the limit of the above Morse index as \( t^\pm \) converge to \( T^\pm \), it is crucial to use results obtained in the previous section. In particular we with use the hyperbolic McGehee coordinates, when the corresponding singularity is hyperbolic, and the usual McGehee coordinates, otherwise. Recall that the change of coordinates transfers (17) to the following system
\[
(56) \quad \eta'(\tau) = J\hat{B}(\tau)\eta(\tau), \quad \text{where} \quad \hat{B}(\tau) = \Phi_R(B)(\tau).
\]
Recall that \( \Phi_R(B)(\tau) \) is defined in [28] with \( R(\tau) \) given by [20] in McGehee coordinates and by [40] in hyperbolic McGehee coordinates. The fundamental solution of (56) will be denoted as \( \hat{\gamma}(\tau, \tau_1) \).

Next lemma shows the Maslov index is invariant under the change of the (hyperbolic) McGehee coordinates.

**Lemma 3.1.** For any \( T^- < t_1 < t_2 < T^+ \), in (hyperbolic) McGehee coordinates,
\[
\mu(V_0, \hat{\gamma}(\tau, \tau_1) V_0; [\tau_1, \tau_2]) = \mu(V_0, \gamma(t, t_1) V_0; [t_1, t_2]),
\]
where \( \tau_i = \tau(t_i), \) \( i = 1, 2 \).

**Proof.** Recall that \( \hat{\gamma}(\tau, \tau_1) = R(\tau)\gamma(\tau, \tau_1)R^{-1}(\tau_1) \), where \( R(\tau) \) is either [20] or [40]. Then \( R(\tau)V_0 = V_0, R^{-1}(\tau)V_0 = V_0 \) and
\[
\mu(V_0, \hat{\gamma}(\tau, \tau_1) V_0; [\tau_1, \tau_2]) = \mu(V_0, R(\tau)\gamma(\tau, \tau_1)R^{-1}(\tau_1) V_0; [\tau_1, \tau_2])

= \mu(R^{-1}(\tau)V_0, \gamma(\tau, \tau_1)R^{-1}(\tau_1) V_0; [\tau_1, \tau_2])

= \mu(V_0, \gamma(\tau, \tau_1) V_0; [\tau_1, \tau_2])
\]
Since the Maslov index is invariant under the change of time parameter, we have
\[ \mu(V_D, \gamma(t_1)V_D; [t_1, t_2]) = \mu(V_D, \gamma(t_2)V_D; [t_1, t_2]). \]
\[ \square \]

3.1. The spiral case. We will prove Property (a) in Theorem 1.1. For this we need the following lemma.

Lemma 3.2. For any \([t_1, t_2] \subset (T^-, T^+)\) and any \(\hat{t} \in [t_1, t_2],\)
\[ |m^-(q; t_1, t_2) - m^-(q; t_1, \hat{t}) - m^-(q; \hat{t}, t_2)| \leq 3n^* . \]

Proof. Let \(\Lambda_s, s \in [0, 1]\), be a continuous path of Lagrangian subspaces satisfying
\[ \Lambda_0 = \gamma(\hat{t}, t_1)V_D \] and \(\Lambda_1 = V_D\). By the homotopy invariant property of the Maslov index,
\[ \mu(V_D, \Lambda_s; [0, 1]) + \mu(V_D, \gamma(t, \hat{t})V_D; [\hat{t}, t_2]) = \mu(V_D, \gamma(t, \hat{t})\Lambda_0; [\hat{t}, t_2]) + \mu(V_D, \gamma(t_2, \hat{t})\Lambda_s; [0, 1]). \]

Then
\[ \mu(V_D, \gamma(t, \hat{t})V_D; [\hat{t}, t_2]) - \mu(V_D, \gamma(t, \hat{t})\Lambda_0; [\hat{t}, t_2]) = \mu(V_D, \gamma(t_2, \hat{t})\Lambda_s; [0, 1]) - \mu(V_D, \Lambda_s; [0, 1]) = s(\gamma(\hat{t}, t_2)V_D, V_D; \Lambda_0, V_D). \]

By the Morse index theorem (55),
\[ m^-(q; t_1, t_2) + n^* = \mu(V_D, \gamma(t, t_1)V_D; [t_1, t_2]) = \mu(V_D, \gamma(t_1)V_D; [t_1, \hat{t}]) + \mu(V_D, \gamma(t, \hat{t})\Lambda_0; [\hat{t}, t_2]). \]

Meanwhile
\[ m^-(q; t_1, \hat{t}) + m^-(q; \hat{t}, t_2) + 2n^* = \mu(V_D, \gamma(t_1)V_D; [t_1, \hat{t}]) + \mu(V_D, \gamma(\hat{t}, t_2)V_D; [\hat{t}, t_2]). \]

Combining this with (57), (58) and (54), we get
\[ |m^-(q; t_1, t_2) - m^-(q; t_1, \hat{t}) - m^-(q; \hat{t}, t_2)| \leq n^* + |s(\gamma(\hat{t}, t_2)V_D, V_D; \Lambda_0, V_D)| \leq 3n^*. \]

This completes the proof. \[ \square \]

Theorem 3.1. If \(q(T^\pm)\) is a collision or parabolic singularity with the corresponding \(s^\pm\) satisfying the spiral condition, then for any \(t_1 \in (T^-, T^+),\)
\[ \lim_{t_2 \to T^+} \frac{m^-(q; t_1, t_2)}{|\ln \beta(t_2)|} = \frac{1}{3\sqrt{2\pi}} \sum_{i=1}^{l} \sqrt{-1} \frac{\lambda_{i(s^+)} - \lambda_{i(s^-)}}{U(s^+)} \]
\[ \lim_{t_2 \to T^-} \frac{m^-(q; t_2, t_1)}{|\ln \beta(t_2)|} = \frac{1}{3\sqrt{2\pi}} \sum_{i=1}^{l} \sqrt{-1} \frac{\lambda_{i(s^-)} - \lambda_{i(s^+)}}{U(s^-)} , \]
where \(l = \# \{ i : \lambda_i < \frac{-U(s^+)}{8} \}\) and \(\beta(t)\) satisfies (10).
Proof. Let's assume \( q(T^+) \) is a collision or parabolic singularity with \( s^+ \) satisfying the spiral condition (the case for \( q(T^-) \) can be proven similarly).

To keep the notations consistent with those in Section 2, let

\[
\tau^* = +\infty, \quad s^* = s^+, \quad v^* = \sqrt{2U(s^+)} , \quad x^* = \psi(s^+), \quad \hat{M} = \hat{M}(x^*).
\]

Then

\[
\lim_{\tau \to \tau^*} \hat{B}(\tau^*) = \hat{B}_1(\tau^*) \circ \hat{B}_2(\tau^*),
\]

where \( \hat{B}_1(\tau^*), \hat{B}_2(\tau^*) \) are given by (64).

For \( \varepsilon > 0 \) small enough (its precise value will be given later), there exists a \( \tau_\varepsilon \) (depending on \( \varepsilon \)) large enough, such that \( \forall \tau > \tau_\varepsilon \),

\[
\hat{B}_1(\tau^*) \circ \hat{B}_2(\tau^*) \leq \hat{B}(\tau) \leq \hat{B}_1(\tau^*) \circ \hat{B}_2(\tau^*) + \varepsilon I_2 \circ E_{\hat{M}_*}.
\]

where \( E_{\hat{M}_*} = \text{diag}(\hat{M}_*^{-1}, \hat{M}_*) \).

Let \( \hat{\gamma}(\tau, \tau_\varepsilon) \) be the fundamental solution of (58), i.e.,

\[
\begin{align*}
\hat{\gamma}(\tau, \tau_\varepsilon) &= J \hat{B}(\tau) \hat{\gamma}(\tau, \tau_\varepsilon), \\
\hat{\gamma}(\tau_\varepsilon, \tau_\varepsilon) &= I.
\end{align*}
\]

By the monotone property of the Maslov index

\[
\begin{cases}
\mu(V_D, \hat{\gamma}(\tau, \tau_\varepsilon) \hat{V}_D; [\tau_\varepsilon, \tau_2]) \leq \mu(V_D, \hat{\gamma}^+(\tau, \tau_\varepsilon) \hat{V}_D; [\tau_\varepsilon, \tau_2]) \\
\mu(V_D, \hat{\gamma}(\tau, \tau_\varepsilon) \hat{V}_D; [\tau_\varepsilon, \tau_2]) \geq \mu(V_D, \hat{\gamma}^-(\tau, \tau_\varepsilon) \hat{V}_D; [\tau_\varepsilon, \tau_2])
\end{cases}
\]

where \( \hat{\gamma}^\pm(\tau, \tau_\varepsilon) \) satisfies

\[
\begin{align*}
\hat{\gamma}^\pm(\tau, \tau_\varepsilon) &= J(\hat{B}_1(\tau^*) \circ \hat{B}_2(\tau^*) \pm \varepsilon I_2 \circ E_{\hat{M}_*}) \hat{\gamma}^\pm(\tau, \tau_\varepsilon); \\
\hat{\gamma}^\pm(\tau_\varepsilon, \tau_\varepsilon) &= I.
\end{align*}
\]

Meanwhile by the decomposition property of the Maslov index,

\[
\mu(V_D, \hat{\gamma}^\pm(\tau, \tau_\varepsilon) \hat{V}_D; [\tau_\varepsilon, \tau_2]) = \sum_{i=1}^2 \mu(V_D, \hat{\gamma}^\pm_i(\tau, \tau_\varepsilon) \hat{V}_D; [\tau_\varepsilon, \tau_2]),
\]

where \( \hat{\gamma}^\pm_i(\tau, \tau_\varepsilon) \) satisfy

\[
\begin{align*}
\hat{\gamma}^\pm_i(\tau, \tau_\varepsilon) &= J(\hat{B}_1(\tau^*) \pm \varepsilon I) \hat{\gamma}^\pm_i(\tau, \tau_\varepsilon); \\
\hat{\gamma}^\pm_i(\tau_\varepsilon, \tau_\varepsilon) &= I,
\end{align*}
\]

Let \( \hat{\gamma}^\pm_1(\tau, \tau_\varepsilon) \hat{V}_D = (b^\pm(\tau), d^\pm(\tau))^T \), then

\[
\mu(V_D, \hat{\gamma}^\pm_i(\tau, \tau_\varepsilon) \hat{V}_D; [\tau_\varepsilon, \tau_2]) = \#\{ \tau : d^\pm(\tau) = 0, \tau \in [\tau_\varepsilon, \tau_2] \}.
\]

Direct computations show that \( d(\tau) \) satisfy

\[
\begin{align*}
d^\pm(\tau) &= g^\pm(\varepsilon)d^\pm(\tau), \\
d^\pm(\tau_\varepsilon) &= 1 \pm \varepsilon, \\
d^\pm(\tau_\varepsilon) &= 0.
\end{align*}
\]

where

\[
g^\pm(\varepsilon) = \frac{9}{16} \varepsilon^2 + (1 \mp \varepsilon)(2U(s^+) \pm \varepsilon),
\]
and \( g(\varepsilon) > 0 \), when \( 0 < \varepsilon < \min \{ 1, 2U(s^*) \} \). Hence

\[
\mu(V_D, \hat{\gamma}^\pm_2(\tau, \tau_e) V_D; [\tau_e, \tau_2]) = 1.
\]

For \( \hat{\gamma}^\pm_2(\tau, \tau_e) \), there exist two matrices \( A \) and \( C \) satisfying

\[
A^T \dot{M}_* A = I, C^T C = I,
\]

and

\[
C^T (A^T \dot{U}_{xx}(x^*) A) = \text{diag}(\lambda_1(s^*), \ldots, \lambda_{n^* - 1}(s^*)) \).
\]

Set \( A_d = \text{diag}(A^T, A^{-1}) \) and \( C_d = \text{diag}(C^T, C^{-1}) \). Then

\[
\hat{\xi}^\pm(\tau, \tau_e) = C_d A_d \hat{\gamma}^\pm_2(\tau, \tau_e) A^{-1}_d C^{-1}_d
\]

satisfies equations,

\[
\begin{cases}
\hat{\xi}'^\pm(\tau, \tau_e) = J \Phi_{C_d A_d}(\hat{B}_2(x^*) \pm \varepsilon E_{M_*}) \hat{\xi}^\pm(\tau, \tau_e), \\
\hat{\xi}^\pm(\tau_e, \tau_e) = I.
\end{cases}
\]

where

\[
\Phi_{C_d A_d}(\hat{B}_2(x^*) \pm \varepsilon E_{M_*}) = \begin{pmatrix} 1 \pm \varepsilon & \frac{1}{4} v^* \\ \frac{1}{4} v^* & -\lambda_i(s^*) \pm \varepsilon \end{pmatrix} \cdots \begin{pmatrix} 1 \pm \varepsilon & \frac{1}{4} v^* \\ \frac{1}{4} v^* & -\lambda_{n^* - 1}(s^*) \pm \varepsilon \end{pmatrix}
\]

A similar argument as in the proof of Lemma 3.1 shows

\[
\mu(V_D, \hat{\gamma}^\pm_1(\tau, \tau_e) V_D; [\tau_e, \tau_2]) = \mu(V_D, \hat{\xi}^\pm(\tau, \tau_e) V_D; [\tau_e, \tau_2]).
\]

By the decomposition property of the Maslov index again,

\[
\mu(V_D, \hat{\xi}^\pm(\tau, \tau_e) V_D; [\tau_e, \tau_2]) = \sum_{i=1}^{n^* - 1} \mu(V_D, \hat{\xi}^\pm_i(\tau, \tau_e) V_D; [\tau_e, \tau_2]),
\]

where each \( \hat{\xi}^\pm_i(\tau, \tau_e) \) satisfies

\[
\begin{cases}
\hat{\xi}'^\pm_i(\tau, \tau_e) = J \begin{pmatrix} 1 \pm \varepsilon & \frac{1}{4} v^* \\ \frac{1}{4} v^* & -\lambda_i(s^*) \pm \varepsilon \end{pmatrix} \hat{\xi}_i^\pm(\tau, \tau_e), \\
\hat{\xi}^\pm_i(\tau_e, \tau_e) = I.
\end{cases}
\]

Let \( \hat{\xi}^\pm_i(\tau, \tau_e) V_D = (a^+_i(\tau), c^+_i(\tau))^T \). Then

\[
\mu(V_D, \hat{\xi}^\pm_i(\tau, \tau_e) V_D; [\tau_e, \tau_2]) = \# \{ \tau : c^+_i(\tau) = 0, \tau \in [\tau_e, \tau_2] \}.
\]

By a direct computation, \( c^+_i(\tau) \) satisfies

\[
\begin{cases}
\hat{c}''^+_i(\tau) = f^+_i(\varepsilon) c^+_i(\tau), \\
\hat{c}^+_i(\tau_e) = 1 \pm \varepsilon, \\
c_i(\tau_e) = 0,
\end{cases}
\]

where

\[
f^+_i(\varepsilon) = \frac{1}{16} v^2 + (1 \mp \varepsilon)(\lambda_i \pm \varepsilon).
\]

Recall that \( l = \# \{ 1 \leq i \leq n^* - 1 : \lambda_i(s^*) < \frac{-U(s^*)}{8} \} \). Further assume

\[
w = \# \{ 1 \leq i \leq n^* - 1 : \lambda_i(s^*) = \frac{-U(s^*)}{8} \}.\]
Let $\varepsilon$ be smaller than
\[
-\frac{\lambda_i(s^*)}{2} + \frac{1}{2} \sqrt{\frac{(\lambda_i(s^*) + 1)^2 + U(s^*)}{2}}
\]
and, if $l + w + 1 \leq n^* - 1$, also smaller than
\[
-\frac{\lambda_{l+w+1}(s^*)}{2} + \frac{1}{2} \sqrt{\frac{(\lambda_{l+w+1}(s^*) + 1)^2 + U(s^*)}{2}}.
\]
Then by (65), we get
\[
\varepsilon < \frac{1}{2} \sqrt{\frac{(\lambda_{l+w+1}(s^*) + 1)^2 + U(s^*)}{2}}.
\]

Then
\[
\begin{cases}
    f_i^+(\varepsilon) < 0, & \text{if } 1 \leq i \leq l + w,
    \\
    f_i^+(\varepsilon) > 0, & \text{if } l + w < i \leq n^* - 1.
\end{cases}
\]
and
\[
\begin{cases}
    f_i^-(\varepsilon) < 0, & \text{if } 1 \leq i \leq l,
    \\
    f_i^-(\varepsilon) > 0, & \text{if } l < i \leq n^* - 1.
\end{cases}
\]
Then by (65), we get
\[
\begin{aligned}
    c_i^+(\tau) &= \begin{cases} 
        \frac{1+\varepsilon}{\sqrt{-f_i^+(\varepsilon)}} \sin(\sqrt{-f_i^+(\varepsilon)}(\tau - \tau_\varepsilon)), & \text{if } 1 \leq i \leq l + w; \\
        1, & \text{if } l + w < i \leq n^* - 1,
    \end{cases} \\
    c_i^-(\tau) &= \begin{cases} 
        \frac{1-\varepsilon}{\sqrt{-f_i^-(\varepsilon)}} \sin(\sqrt{-f_i^-(\varepsilon)}(\tau - \tau_\varepsilon)), & \text{if } 1 \leq i \leq l; \\
        1, & \text{if } l < i \leq n^* - 1.
    \end{cases}
\end{aligned}
\]
These imply
\[
\# \{ \tau : c_i^+(\tau) = 0, \tau \in [\tau_\varepsilon, \tau_2] \} = \begin{cases} 
    \left\lfloor \frac{\sqrt{-f_i^+(\varepsilon)(\tau_2 - \tau_\varepsilon)}}{\pi} \right\rfloor + 1, & \text{if } 1 \leq i \leq l + w; \\
    1, & \text{if } l + w < i \leq n^* - 1,
\end{cases}
\]
and
\[
\# \{ \tau : c_i^-(\tau) = 0, \tau \in [\tau_\varepsilon, \tau_2] \} = \begin{cases} 
    \left\lfloor \frac{\sqrt{-f_i^-(\varepsilon)(\tau_2 - \tau_\varepsilon)}}{\pi} \right\rfloor + 1, & \text{if } 1 \leq i \leq l; \\
    1, & \text{if } l < i \leq n^* - 1.
\end{cases}
\]
Together with (59), (60), (61), (62), (63), we get
\[
\mu(V_\mathcal{D}, \hat{\gamma}(\tau_\varepsilon) \mathcal{V}_\mathcal{D}; [\tau_\varepsilon, \tau_2]) \leq \sum_{i=1}^{l+w} \left\lfloor \frac{\sqrt{-f_i^+(\varepsilon)(\tau_2 - \tau_\varepsilon)}}{\pi} \right\rfloor + n^*.
\]
\[
\mu(V_\mathcal{D}, \hat{\gamma}(\tau_\varepsilon) \mathcal{V}_\mathcal{D}; [\tau_\varepsilon, \tau_2]) \geq \sum_{i=1}^{l} \left\lfloor \frac{\sqrt{-f_i^-(\varepsilon)(\tau_2 - \tau_\varepsilon)}}{\pi} \right\rfloor + n^*.
\]
Let $t_\varepsilon = t(\tau_\varepsilon)$ and $t_2 = t(\tau_2)$ be the Newtonian times corresponding to $\tau_\varepsilon$ and $\tau_2$. By the Morse index theorem (55) and Lemma 3.1 we have
\[
\begin{aligned}
m^-(q; t_\varepsilon, t_2) &\leq \sum_{i=1}^{l+w} \left\lfloor \frac{\sqrt{-f_i^+(\varepsilon)(\tau_2 - \tau_\varepsilon)}}{\pi} \right\rfloor, \\
(67)
m^-(q; t_\varepsilon, t_2) &\geq \sum_{i=1}^{l} \left\lfloor \frac{\sqrt{-f_i^-(\varepsilon)(\tau_2 - \tau_\varepsilon)}}{\pi} \right\rfloor,
\end{aligned}
\]
From the proof of Lemma 3.2 we get
\[
\lim_{t \to T^+} \frac{\tau(t)}{|\ln \beta(t)|} = \frac{1}{3\sqrt{2U(s^+)}},
\]
Combining this with (67) gives us
\[
\lim_{t_2 \to T^+} \frac{m^-(q; t_x, t_2)}{|\ln \beta(t_2)|} \leq \frac{1}{3\sqrt{2\pi}} \sum_{i=1}^{l+w} \sqrt{-f^+_i(\varepsilon)} U(s^+),
\]
\[
\lim_{t_2 \to T^+} \frac{m^-(q; t_x, t_2)}{|\ln \beta(t_2)|} \geq \frac{1}{3\sqrt{2\pi}} \sum_{i=1}^{l} \sqrt{-f^-_i(\varepsilon)} U(s^+).
\]
Fix an arbitrary \( t_1 < t_x \), combining the above inequalities with Lemma 3.2 and the fact that \( \lim_{t_2 \to T^+} |\beta(t_2)| = +\infty \) gives us
\[
\lim_{t_2 \to T^+} \frac{m^-(q; t_1, t_2)}{|\ln \beta(t_2)|} \leq \lim_{t_2 \to T^+} \frac{m^-(q; t_x, t_2)}{|\ln \beta(t_2)|} \leq \frac{1}{3\sqrt{2\pi}} \sum_{i=1}^{l+w} \sqrt{-f^+_i(\varepsilon)} U(s^+),
\]
\[
\lim_{t_2 \to T^+} \frac{m^-(q; t_1, t_2)}{|\ln \beta(t_2)|} \geq \lim_{t_2 \to T^+} \frac{m^-(q; t_x, t_2)}{|\ln \beta(t_2)|} \geq \frac{1}{3\sqrt{2\pi}} \sum_{i=1}^{l} \sqrt{-f^-_i(\varepsilon)} U(s^+).
\]
Since the above equalities hold for any \( \varepsilon > 0 \) small enough, the desired result follows from the fact that
\[
\lim_{\varepsilon \to 0} f^\pm_i(\varepsilon) = \begin{cases} 
\frac{U(s^+)}{8} + \lambda_i, & \text{when } 1 \leq i \leq l; \\
0, & \text{when } l + 1 \leq i \leq l + w.
\end{cases}
\]
\[\square\]

### 3.2. The strict non-spiral case
In this subsection, we assume \( q(T^\pm) \) is either a hyperbolic singularity or a collision/parabolic singularity with the corresponding limit \( s^\pm = \lim_{t \to T^\pm} s(t) \) exists and satisfies the strict non-spiral condition. A proof of Property (b) in Theorem 1.1 will be given in this subsection.

In this case, it is difficult to compute the Maslov index in (55) directly. Instead we introduce and compute another Maslov index. Then estimate the difference between these two different Maslov indices using the Hörmander index. This new Maslov index was first introduced in [12] to study the index of homoclinic solutions (further works can be found in [21] and [23]).

**Definition 3.1.** At each moment \( \tau \in \mathbb{R} \), we define the **stable subspace** \( V^s(\tau) \) and **unstable subspace** \( V^u(\tau) \) associated with the linear system (56) as
\[
V^s(\tau) = \{ v \in \mathbb{R}^{2n} | \lim_{\sigma \to +\infty} \tilde{\gamma}(\sigma, \tau)v = 0 \},
\]
\[
V^u(\tau) = \{ v \in \mathbb{R}^{2n} | \lim_{\sigma \to -\infty} \tilde{\gamma}(\sigma, \tau)v = 0 \}.
\]

To study the limiting behaviors of the stable and unstable subspaces. Some topology of linear subspaces will be needed. Let \( \mathcal{G}(\mathbb{R}^{2k}) \) be the Grassmannian of \( \mathbb{R}^{2k} \), i.e. the set of all closed linear subspaces of \( \mathbb{R}^{2k} \). For any \( W \in \mathcal{G}(\mathbb{R}^{2k}) \), let \( P_W \) be the orthogonal projection of \( \mathbb{R}^{2k} \) to \( W \). Then
\[
\text{dist}(W, W^*) := \| P_W - P_{W^*} \|, \text{ for any } W, W^* \in \mathcal{G}(\mathbb{R}^{2k}),
\]
defines a complete metric on \( \mathcal{G}(\mathbb{R}^{2k}) \). Here \( \| \cdot \| \) represents the metric on the space of bounded linear operators from \( \mathbb{R}^{2k} \) to itself.
Lemma 3.3. Given an arbitrary hyperbolic matrix $A$, we define $V^+(A)$ and $V^-(A)$ as the invariant linear subspaces corresponding to the eigenvalues with positive and negative real parts.

Set $\tau^* = \pm \infty$. By Proposition 2.1 and 2.2, $\hat{B}(\tau^*) = \lim_{\tau \to \tau^*} \hat{B}(\tau)$ exist and $J\hat{B}(\tau^*)$ always is a hyperbolic matrix under our assumption. Next result gives the limits of the linear subspaces introduced above, for a proof see [1, Theorem 2.1].

Proposition 3.1. (a) If $J\hat{B}(+\infty)$ is a hyperbolic matrix and $W$ is a linear subspace satisfying $W \cap V^+(\tau_1)$ in $\mathbb{R}^{2n'}$, for some $\tau_1$, then
\[
\lim_{\tau \to +\infty} V^+(\tau) = V^+(J\hat{B}(+\infty)), \quad \lim_{\tau \to +\infty} \hat{\gamma}(\tau, \tau_1)W = V^+(J\hat{B}(+\infty)).
\]
(b) If $J\hat{B}(-\infty)$ is a hyperbolic matrix and $W$ is a linear subspace satisfying $W \cap V^-(\tau_1)$ in $\mathbb{R}^{2n'}$, for some $\tau_1$, then
\[
\lim_{\tau \to -\infty} V^-(\tau) = V^-(J\hat{B}(-\infty)), \quad \lim_{\tau \to -\infty} \hat{\gamma}(\tau, \tau_1)W = V^-(J\hat{B}(-\infty)).
\]

With the above proposition, we define the following Maslov index for heteroclinics.

Definition 3.3. For any $\tau_0 \in \mathbb{R}$, we define the Maslov index as
\[
\mu(\hat{B}; \tau_0) := \mu(V_\mathbb{D}, V^+(\tau); (-\infty, \tau_0]).
\]
and its limit as (if the limit exists)
\[
\mu(\hat{B}; \mathbb{R}) := \lim_{\tau_0 \to -\infty} \mu(\hat{B}; \tau_0).
\]
We define the degenerate index as
\[
\nu(\hat{B}) := \dim(V^+(0) \cap V^*(0)).
\]
We say $\hat{B}$ is degenerate, if $\nu(\hat{B}) \neq 0$.

The index defined above was introduced in the study of heteroclinic orbits (see [21, 23]). Since we have assumed $B(t)|_{V_\mathbb{D}} > 0$, then all crossing are positive [32].

(69) \[
\mu(V_\mathbb{D}, V^+(\tau); \mathbb{R}) = \sum_{\tau \in \mathbb{R}} \dim(V^+(\tau) \cap V_\mathbb{D}).
\]
Under the condition $J\hat{B}(\pm \infty)$ is hyperbolic, $-J \frac{d}{dt} - \hat{B}$ is a Fredholm operator and
\[
\nu(\hat{B}) = \dim \ker(-J \frac{d}{dt} - \hat{B}),
\]
For the details, see [23].

Now we use Hörmander index to estimates the difference between the two Maslov indices.

Lemma 3.3. (a) For any $\tau_1 < \tau_2$,
\[
\mu(V_\mathbb{D}, \hat{\gamma}(\tau, \tau_1) V_\mathbb{D}; [\tau_1, \tau_2]) - \mu(V_\mathbb{D}, V^+(\tau); [\tau_1, \tau_2]) = s(\hat{\gamma}(\tau, \tau_2) V_\mathbb{D}, V_\mathbb{D}; V^+(\tau_1), V_\mathbb{D}).
\]
(b) For any $\tau_0 \in \mathbb{R}$ and $T_0 = t(\tau_0)$. If $V_\mathbb{D} \cap V^+(\tau_0)$ and $V_\mathbb{D} \cap V^+(J\hat{B}(-\infty))$,
\[
\mu(V_\mathbb{D}, \hat{\gamma}(\tau, \tau_0) V_\mathbb{D}; [\tau_1, \tau_2]) = s(\hat{\gamma}(\tau, \tau_0) V_\mathbb{D}, V_\mathbb{D}; V^-(J\hat{B}(-\infty)), V_\mathbb{D}).
\]


(c). If $V^s(0) \ni V^u(0)$, $V_D \ni V^+(J \hat{B}(\pm \infty))$ and $V_D \ni V^+(J \hat{B}(-\infty))$, then

\[ m^-(q; T^-, T^+) + n^* - \mu(\hat{B}; \mathbb{R}) = s(V^+(J \hat{B}(-\infty)), V_D; V^+(J \hat{B}(-\infty)), V_D). \]

**Proof.**

(a). Let $\Lambda_s, s \in [0, 1]$, be a continuous path of Lagrangian subspaces satisfying $\Lambda_0 = V^u(\tau_1)$ and $\Lambda_1 = V_D$. By the homotopy invariant property of the Maslov index,

\[ \mu(V_D, \Lambda_s; [0, 1]) + \mu(V_D, \hat{\gamma}(\tau, \tau_1)V_D; [\tau_1, \tau_2]) = \mu(V_D, V^u(\tau); [\tau_1, \tau_2]) + \mu(V_D, \hat{\gamma}(\tau_2, \tau_1)V_D; [0, 1]). \]

Then

\[ \mu(V_D, \hat{\gamma}(\tau, \tau_1)V_D; [\tau_1, \tau_2]) - \mu(V_D, V^u(\tau); [\tau_1, \tau_2]) = \mu(V_D, \hat{\gamma}(\tau_2, \tau_1)V_D; [0, 1]) - \mu(V_D, \Lambda_s; [0, 1]) \]

\[ = \mu(\hat{\gamma}(\tau_1, \tau_2)V_D; V^u(\tau_1), V_D). \]

(b). For any $\tau_\tau < \tau_0$. Let $t_\tau = t(\tau)$. By (55) and Lemma 3.1

\[ m^-(q; t_\tau, T_0) + n^* = \lim_{\tau_\tau \to -\infty} \mu(V_D, \hat{\gamma}(\tau, \tau_\tau)V_D; [\tau_\tau, \tau_0]). \]

Then Proposition 3.1 and the monotone property of Morse index implies

\[ m^-(q; t_\tau, T_0) + n^* = \lim_{\tau_\tau \to -\infty} \mu(V_D, \hat{\gamma}(\tau, \tau_\tau)V_D; [\tau_\tau, \tau_0]). \]

Since $V_D \ni V^u(\tau_0)$, then $\lim_{\tau_\tau \to -\infty} \hat{\gamma}(\tau, \tau_0)V_D = V^-(J \hat{B}(-\infty))$. Then

\[ \lim_{\tau_\tau \to -\infty} s(\hat{\gamma}(\tau, \tau_0)V_D, V_D; V^u(\tau), V_D) \]

\[ = s(V^-(J \hat{B}(-\infty)), V_D; V^+(J \hat{B}(-\infty)), V_D). \]

$V_D \ni V^+(J \hat{B}(-\infty))$ implies that

\[ \mu(\hat{B}; \tau_0) = \lim_{\tau_\tau \to -\infty} \mu(V_D, V^u(\tau); [\tau_\tau, \tau_0]). \]

(70) now follows from (73), (74) and (75).

(c). Since $V^s(0) \ni V^u(0)$, by Proposition 3.1

\[ \lim_{\tau \to -\infty} V^u(\tau) = V^+(J \hat{B}(-\infty)), \lim_{\tau \to +\infty} V^u(\tau) = V^+(J \hat{B}(+\infty)). \]

This means there exist $\tau_1 < 0$ small enough and $\tau_2 > 0$ large enough, such that

$V_D \ni V^u(\tau), \forall \tau \in (-\infty, \tau_1] \cup [\tau_2, +\infty)$. Hence for any $\tau_1' \leq \tau_1$ and $\tau_2' \geq \tau_2$, $\mu(V_D, V^u(\tau); [\tau_1', \tau_2'])$ is a finite constant and

\[ \mu(\hat{B}; \mathbb{R}) = \mu(V_D, V^u(\tau); \mathbb{R}) = \mu(V_D, V^u(\tau); [\tau_1', \tau_2 ]). \]

This implies $\mu(\hat{B}; \mathbb{R})$ must be finite.

Let $t_i = t(\tau_i)$, $i = 1, 2$. By (55) and Lemma 3.1

\[ m^-(q; t_1, t_2) + n^* = \lim_{\tau_2 \to +\infty} \lim_{\tau_1 \to -\infty} \mu(V_D, \hat{\gamma}(\tau, \tau_1)V_D; [\tau_1, \tau_2]). \]

Then Proposition 3.1 and the monotone property of Morse index implies

\[ m^-(q; T^-, T^+) + n^* = \lim_{\tau_2 \to +\infty} \lim_{\tau_1 \to -\infty} \mu(V_D, \hat{\gamma}(\tau, \tau_1)V_D; [\tau_1, \tau_2]). \]
Since \( V_D \cap V^+(J\hat{B}(+\infty)) \), the second identity in \((76)\) implies \( V_D \cap V^u(\tau_2^l), \forall \tau_2^l \geq \tau_2 \), when \( \tau_2 \) is large enough. Then by Proposition \( 3.1 \)
\[
\lim_{\tau_1 \to -\infty} \hat{\gamma}(\tau_1, \tau_2^l)V_D = V^- (J\hat{B}(-\infty)), \forall \tau_2^l \geq \tau_2.
\]
As a result, for \( \tau_2 \) large enough,
\[
\lim_{\tau_2 \to +\infty} \lim_{\tau_1 \to -\infty} s(\hat{\gamma}(\tau_1, \tau_2)V_D, V_D; V^u(\tau_1), V_D) = s(V^- (J\hat{B}(-\infty)), V_D; V^+(J\hat{B}(-\infty)), V_D)
\]
Combining \((72), (77), (78)\) and \((79)\), we get the desired identity. \( \square \)

The identity \((74)\) express the difference between the two different Maslov indices as the Hörmander index. The next lemma will be useful in computing the Hörmander index.

**Lemma 3.4.** Given a matrix \( B = \text{diag}(1, b) \) with \( b < 0 \). Then
\[
V^\pm(JB) = \left\{ \left( \pm \sqrt{-b} \right) \right\} \cap V_D
\]
and
\[
s \left( V^- (JB), V_D; V^+(JB), V_D \right) = 1.
\]

**Proof.** Let \( \Lambda_t = \left\{ \left( \frac{\sqrt{-b}}{1 - t} \right) \right\}, t \in [0, 1] \), then \( \Lambda_0 = V^+(JB), \Lambda_1 = V_D \). The result follows from the facts that \( \mu(V^- (JB), \Lambda_t) = 0 \) and \( \mu(\Lambda_1, \Lambda_t) = -1 \).

\( \square \)

Now we are ready to prove property \((b)\) in Theorem \( 1.1 \)

**Theorem 3.2.** Let \( q \in C^2(T^-, T^+), \hat{X} \) be a singular solution satisfying the conditions given in property \((b)\) of Theorem \( 1.1 \)

(a). For any \( \tau_0 \in \mathbb{R} \), let \( T_0 = t(\tau_0) \) be the corresponding Newtonian time. If \( V_D \cap V^u(\tau_0) \), then
\[
m^- (q; T^-, T_0) = \mu(\hat{B}; \tau_0).
\]
Moreover if \( V^+(0) \cap V^u(0) \), then
\[
m^- (q; T^-, T^+) = \mu(\hat{B}; \mathbb{R}).
\]
(b). \( m^- (q; T^-, T^+) \) is finite.

**Proof.** \((a)\). We will prove \((30)\) first. Depending on the type of singularity of \( q(T^-) \), different McGehee coordinates need to be used. First let’s assume \( q(T^-) \) is a collision/parabolic singularity. Set \( \tau^* = -\infty \). Following the notations from Section \( 2 \), \( s^* = \lim_{\tau \to -\tau^*} s(t) \) is a central configuration, \( x^* = \psi(s^*) \), and \( \hat{B}(\tau^*) = \hat{B}_1(\tau^*) \circ \hat{B}_2(\tau^*) \) with \( \hat{B}_1(\tau^*), \hat{B}_2(\tau^*) \) given as in \( (34) \). Recall that Proposition \( 2.1 \) shows \( J\hat{B}(\tau^*) \) is a hyperbolic matrix.

By Lemma \( 2.2 \) there exists a matrix \( A \), such that \( A^T \hat{M}, A = I \) and the eigenvalues of \( A^T \hat{U}_{xx}(x^*)A \) are \( \lambda_i(s^*), i = 1, \cdots, n^* - 1 \). We can further find an orthogonal matrix \( C \), such that
\[
C^T (A^T \hat{U}_{xx}(x^*)A)C = \text{diag}(\lambda_1(s^*), \cdots, \lambda_{n^* - 1}(s^*)).
\]
This finishes our proof of property (a).

where

\[ R_2 = \begin{pmatrix} -\frac{3}{2}v^* & 0 \\ 1 & 1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & \frac{3}{2}v^* \\ 0 & 1 \end{pmatrix}, \]

are \( 2(n^* - 1) \times 2(n^* - 1) \) matrices. According to (28),

\[ \Phi_R(\hat{B})(\tau^*) = \begin{pmatrix} 1 & -\frac{25}{8}U(s^*) \\ 0 & 1 - \frac{U(s^*)}{8} - \lambda_1(s^*) \end{pmatrix} \circ \cdot \cdot \cdot \circ \begin{pmatrix} 1 & \frac{3}{2}v^* \\ 0 & 1 \end{pmatrix}. \]

Recall that the strict non-spiral condition implies

\[-\frac{U(s^*)}{8} - \lambda_i(s^*) < 0, \quad 1 \leq i \leq n^* - 1.\]

Combining these with Lemma 3.4 we get

\[ s(V^-(J\Phi_R(\hat{B})(\tau^*)), V_{\mathcal{D}}, V^+(J\Phi_R(\hat{B})(\tau^*)), V_{\mathcal{D}}) = n^*. \]

Direct computations show

\[ RV_{\mathcal{D}} = V_{\mathcal{D}}, \quad RV^\pm(J\hat{B}(\tau^*)) = V^\pm(J\Phi_R(\hat{B})(\tau^*)). \]

Therefore

\[ s(V^-(J\hat{B}(\tau^*)), V_{\mathcal{D}}; V^+(J\hat{B}(\tau^*)), V_{\mathcal{D}}) = s(RV^-(J\hat{B}(\tau^*)), RV_{\mathcal{D}}; RV^+(J\hat{B}(\tau^*)), RV_{\mathcal{D}}) \]

\[ = s(V^-(J\Phi_R(\hat{B})(\tau^*)), V_{\mathcal{D}}; V^+(J\Phi_R(\hat{B})(\tau^*)), V_{\mathcal{D}}) = n^*. \]

Together with (70), they imply (71).

To prove (71), let’s assume \( V^*(0) \cap V^u(0) \), then together with (71) implies the result.

The proof is exact the same, when \( q(T^-) \) is a hyperbolic singularity. The only difference is in this case \( B_1(\tau^*), B_2(\tau^*) \) are given by (23), and

\[ R_1 = \begin{pmatrix} 1 & -\frac{1}{2}v^* \\ 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} A^T & 0 \\ 0 & A^{-1} \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & \frac{3}{2}v^* \\ 0 & 1 \end{pmatrix}. \]

This finishes our proof of property (a).

(b) Now let’s drop the assumption that \( V^*(0) \cap V^u(0) \). In this case, we can always find another Lagrange subspace \( V \) satisfying \( V \cap V^*(0) \) and \( V \cap V^u(0) \). By Proposition 3.1 this implies

\[ \lim_{\tau \to +\infty} \hat{\gamma}(\tau, 0)V = V^+(J\hat{B}(+\infty)), \quad \lim_{\tau \to -\infty} \hat{\gamma}(\tau, 0)V = V^-(-\hat{\gamma}(J\hat{B}(-\infty))). \]

After replacing \( V^u(\tau) \) by \( \hat{\gamma}(\tau, 0)V \), a similar argument as in the proof of property (a) of Lemma 3.3 shows, for any \( \tau_1 < \tau_2 \),

\[ \mu(V_{\mathcal{D}}, \hat{\gamma}(\tau, \tau_1)V_{\mathcal{D}}; \tau_1, \tau_2]) \]

\[ = \mu(V_{\mathcal{D}}, \hat{\gamma}(\tau, 0)V; [\tau_1, \tau_2]) + s(\hat{\gamma}(\tau_1, \tau_2)V_{\mathcal{D}}; \hat{\gamma}(\tau_1, 0)V, V_{\mathcal{D}}). \]

Meanwhile \( V_{\mathcal{D}} \cap V^+(J\hat{B}(+\infty)) \), \( V_{\mathcal{D}} \cap V^-(J\hat{B}(-\infty)) \). With (83), they imply

\[ \lim_{\tau_2 \to +\infty} \lim_{\tau_1 \to +\infty} \mu(V_{\mathcal{D}}, \hat{\gamma}(\tau, 0)V; [\tau_1, \tau_2]) < +\infty. \]
Since the Hörmander index \( s(\gamma(t_1, t_2)V_D, V_D; \gamma(t_1, 0)V, V_D) \) is also bounded (54), together with the monotonicity of \( \mu(V_D, \gamma(t, \tau_1)V_D; [\tau_1, \tau_2]) \), they imply
\[
\lim_{\tau_2 \to +\infty} \lim_{\tau_1 \to -\infty} \mu(V_D, \gamma(t, \tau_1)V_D; [\tau_1, \tau_2]) < +\infty.
\]
Then the finiteness of \( m^-(q; T^-, T^+) \) follows from (78).

4. The Morse indices of homothetic solutions

In this section, we give a proof of property (b) in Theorem 1.2. Throughout the entire section, we assume \( q \in C^2((T^-, T^+), \mathcal{X}) \) is a homothetic solution with energy \( H_0 \). As a result, there is a normalized central configuration \( s_0 \), such that
\[
s(t) = q(t)/\sqrt{I(q(t))} = s_0, \forall t \in (T^-, T^+).
\]

When \( H_0 < 0 \), the homothetic solution starts from a total collision at a finite time and comes back to the total collision at another finite time. When \( H_0 \geq 0 \), the homothetic solution either starts from a total collision at a finite time and goes to infinity, as time goes to positive infinite, or it ends at the total collision at a finite time and goes to infinity, as time goes to negative infinity. However a simple time reversal changes one to the other. Without loss of generality, in the rest of the section we assume \( q(T^-) \) is a total collision with a finite \( T^- \in \mathbb{R} \). Then \( T^+ \in \mathbb{R} \) is finite, when \( H_0 < 0 \) and \( T^+ = +\infty \), when \( H_0 \geq 0 \).

To compute the index of \( q(t) \), we rewrite it in McGehee coordinates \((v, u, r, x)(\tau)\) with time parameter \( \tau \) as defined in subsection 2.2. Although when the energy \( H_0 \) is positive, we have a hyperbolic singularity, as \( t \) goes to \( T^+ = +\infty \). However we will not switch to the hyperbolic McGehee coordinates, but continue to use the McGehee coordinate. Because of this, when \( H_0 > 0 \), the corresponding time \( \tau = \tau(t) \) goes to some finite \( \tau^+ \), when \( t \) goes to \( T^+ = +\infty \).

Since \( q(t) \) is a homothetic solution, \( x(\tau) \equiv x_0 = \psi(s_0) \) and \( u(\tau) \equiv 0 \), for all \( \tau \). Then the second and fourth equations in (24) are always zero on both sides, and the first and third equations become
\[
\begin{align*}
v' &= \frac{1}{2}v^2 - b, \\
r' &= rv
\end{align*}
\]
where \( b = \dot{U}(x_0) \) through out this section. The energy identity (24) now reads
\[
\frac{1}{2}v^2 - b = rH_0.
\]

Moreover (31) now becomes \( \dot{B}(\tau) = \dot{B}_1(\tau) \circ \dot{B}_2(\tau) \), where
\[
\dot{B}_1(\tau) = \begin{pmatrix} 1 & -\frac{3}{4}v \\ -\frac{1}{2}v & -2b \end{pmatrix}, \quad \dot{B}_2(\tau) = \begin{pmatrix} M^{-1} & \frac{1}{2}v I \\ \frac{1}{2}v I & -\dot{U}_{xx}(x_0) \end{pmatrix}.
\]

Here and in the rest of the section, we set
\[
\dot{M} = \dot{M}(x_0) = \left( \frac{\partial \psi^{-1}}{\partial x} \right)^T_{x=x_0} M \left( \frac{\partial \psi^{-1}}{\partial x} \right)_{x=x_0}.
\]
Let \( V_1^u, V_2^u \) is unstable subspaces of \( \dot{B}_1 \) and \( \dot{B}_2 \) respectively according to Definition 31. Then \( V^u = V_1^u \oplus V_2^u \). By the Symplectic additivity of Maslov index, we have
\[
\mu(\dot{B}; \tau_0) = \mu(\dot{B}_1; \tau_0) + \mu(\dot{B}_2; \tau_0),
\]
When both limits $\lim_{\tau \to +\infty} V_i^u(\tau)$, $i = 1, 2$, exist. After passing $\tau_0$ to $+\infty$, we get
\begin{equation}
(88) \quad \mu(\tilde{B}_i; \mathbb{R}) = \mu(\tilde{B}_1; \mathbb{R}) + \mu(\tilde{B}_2; \mathbb{R}).
\end{equation}

Let $R_1(\tau) = \begin{pmatrix} 1 & -\frac{2}{3}v(\tau) \\ 0 & 1 \end{pmatrix}$. By Lemma 2.3 $\xi_1(\tau)$ satisfies $\xi_1^\prime(\tau) = J\dot{B}_1(\tau)\xi_1(\tau)$ if and only if $\eta_1(\tau) = R_1(\tau)\xi_1(\tau)$ satisfies
\begin{equation}
(89) \quad \eta_1^\prime(\tau) = J\Phi_{R_1}(\tilde{B}_1(\tau))\eta_1(\tau),
\end{equation}
where according to (28),
\begin{equation}
(90) \quad \Phi_{R_1}(\tilde{B}_1) = \begin{pmatrix} 1 & \frac{2}{3}v' - \frac{9}{17}v^2 - 2b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{2}{3}v - \frac{14}{17}b \\ 0 & 1 \end{pmatrix}.
\end{equation}

Next lemma will be useful in computing $\mu(V_2, V^u(\tau), (-\infty, \tau_0])$.

**Lemma 4.1.** [21] Lemma 3.10 If $y(\tau), \tau \in \mathbb{R}$, satisfies
\begin{equation}
(91) \quad y''(\tau) = f(\tau)y(\tau),
\end{equation}
where $f \in C^\infty(\mathbb{R}, [0, +\infty))$ satisfies $\int_{\tau_1}^{\tau_2} f(\tau)d\tau > 0$ for any $\tau_1 < \tau_2$, and the $\lim_{\tau \to \pm \infty} f(\tau) > 0$. Then
(a). for any $\tau_1 < \tau_2$, there is no non-trivial solution of (91), which satisfies $y(\tau_1)y''(\tau_1) \geq y(\tau_2)y''(\tau_2)$;
(b). there is no non-trivial solution of (91) satisfying $y(\tau_0)y''(\tau_0) = 0$ for some $\tau_0 \in \mathbb{R}$ and $y(\tau) \to 0, y''(\tau) \to 0$ as $\tau \to -\infty$ (or $\tau \to +\infty$);
(c). there is no non-trivial solution of (91), which is bounded on $\mathbb{R}$.

**Proof.** The key idea of the proof is to multiply both sides of (91) $y$, and then use integration by parts. For details see [21] Lemma 3.10. \hfill $\square$

**Proposition 4.1.** Given a homothetic orbit with arbitrary energy. For any $\tau_0 \in \mathbb{R}$,
\begin{equation}
\mu(\tilde{B}_1; \tau_0) = 0.
\end{equation}

**Proof.** Let $V^u(\tau), \tilde{V}^u(\tau)$ be the unstable subspaces of the linear Hamiltonian system associated to $\tilde{B}_1$ and $\Phi_{R_1}(\tilde{B}_1)$ respectively. By Lemma 2.28 $\tilde{V}^u(\tau) = R_1V^u(\tau)$. Since $R_1V_2 = V_2$, the symplectic invariant property of Maslov index implies
\begin{equation}
(92) \quad \mu(\tilde{B}_1; \tau_0) = \mu(V_2, V^u(\tau); (\infty, \tau_0]) = \mu(V_2, \tilde{V}^u(\tau); (\infty, \tau_0]).
\end{equation}
Hence it is enough to show $\mu(V_2, \tilde{V}^u(\tau); (\infty, \tau_0]) = 0$. Notice that $\tilde{V}^u(\tau)$ is a 1-dim space. There exist $\xi(\tau) = (y(\tau), x(\tau))^T$, such that $\tilde{V}^u = (\xi(\tau))$. Then $\xi(\tau)$ satisfies (89) and $\lim_{\tau \to -\infty} \xi(\tau) = 0$, which implies
\begin{equation}
x''(\tau) = \frac{3}{11}v^2(\tau) + \frac{11}{4}b)\xi(\tau) \quad \text{and} \quad \lim_{\tau \to -\infty} x(\tau) = 0.
\end{equation}
Since $\frac{3}{11}v^2 + \frac{11}{4}b > \frac{11}{4}b$, from property (b) of Lemma 4.1, we have $x(\tau) \neq 0, \forall \tau \in \mathbb{R}$. This implies that there is no crossing occur for $V_2$, then we have the result. \hfill $\square$

Now we show how to compute $\mu(\tilde{B}_2, \mathbb{R})$. Recall that we can find a matrix $A$ such that $A^TMA = I$ and
\begin{equation}
(93) \quad \Phi_{A_i}(\tilde{B}_2(\tau)) = \begin{pmatrix} I & \frac{1}{4}vI \\ \frac{1}{4}vI & -A^T U_{xx}(x_0)A \end{pmatrix},
\end{equation}
where $A_d = \text{diag}(A^{-T}, A)$. Let $\lambda_i$, $i = 1, \ldots, n^* - 1$, represent the eigenvalues of $M^{-1}D^2U|_e(s_0)$. By Lemma 2.10, they are the eigenvalues of $A^T\hat{U}_{xx}(x_0)A$ as well. Then after a change of basis, we have

$$
\left( \begin{array}{c|c}
1 & \frac{1}{4}vI \\
\frac{1}{4}vI & -A^T\hat{U}_{xx}(x_0)A \\
\end{array} \right) = \left( \begin{array}{c|c}
1 & \frac{1}{4}v \\
\frac{1}{4}v & -\lambda_1 \\
\end{array} \right) \circ \cdots \circ \left( \begin{array}{c|c}
1 & \frac{1}{4}v \\
\frac{1}{4}v & -\lambda_{n^*-1} \\
\end{array} \right).
$$

To simplify notation, let $\lambda$ represent any $\lambda_i$, $i = 1, \ldots, n^* - 1$. Let $\tilde{V}_\lambda^u(\tau)$ be the unstable subspaces of the linear Hamiltonian system

$$
(93) \quad \eta'(\tau) = J\beta_\lambda(\tau)\eta(\tau), \quad \text{where} \quad \beta_\lambda = \left( \begin{array}{c|c}
1 & \frac{1}{4}v \\
\frac{1}{4}v & -\lambda \\
\end{array} \right).
$$

By the symplectic additivity of Maslov index, we have

$$
(94) \quad \mu(\hat{B}_2; \tau_0) = \sum_{i=1}^{n^*-1} \mu(\beta_{\lambda_i}; \tau_0).
$$

Let $R_2(\tau) = \left( \begin{array}{cc} 1 & \frac{1}{4}v(\tau) \\
0 & 1 \end{array} \right)$, then

$$
(95) \quad \Phi_{R_2}(\beta_\lambda) = \left( \begin{array}{cc}
1 & -\frac{3}{4}v^2 + \frac{b}{4} - \lambda \\
0 & 0 \\
\end{array} \right) = \left( \begin{array}{cc}
1 & 0 \\
0 & -\frac{3}{4}rH_0 - \frac{b}{8} - \lambda \\
\end{array} \right),
$$

where we have use the energy relation (85). Let $\hat{V}_\lambda^u(\tau)$ be the unstable subspaces of the linear Hamiltonian system $\eta'(\tau) = J\Phi_R(\beta_\lambda)\eta(\tau)$. Then $\hat{V}_\lambda^u(\tau) = R_2(\tau)V_\hat{\lambda}^u(\tau)$. Since $R_2(\tau)V_\hat{\beta} = V_\hat{\beta}$, we have

$$
(96) \quad \mu(V_\hat{\beta}, \hat{V}_\lambda^u(\tau); (-\infty, \tau_0]) = \mu(V_\hat{\beta}, \hat{V}_\lambda^u(\tau); (-\infty, \tau_0]),
$$

which is equivariant to $\mu(\beta_{\lambda}; \tau_0) = \mu(\Phi_{R_2}(\beta_\lambda); \tau_0)$.

If $H_0 \geq 0, \frac{3}{8}rH_0 + \frac{b}{8} + \lambda > 0$. Then a similar argument as in the proof of Proposition 4.1 shows that for any $\tau_0 \in \mathbb{R}$,

$$
\mu(V_\hat{\beta}, \hat{V}_\lambda^u(\tau); (\infty, \tau_0]) = 0.
$$

From (94), we get the following proposition

**Proposition 4.2.** For $H_0 \geq 0$, $\mu(\hat{B}_2; \tau_0) = 0$.

Now we compute the Morse index of $q(t)$. Since the homothetic solution behaviors quite differently for different energies, we first consider the case that the energy is non-negative.

**Proposition 4.3.** If the energy $H_0 \geq 0$, then $m^-(q; T^-, T^+) = 0$.

**Proof.** First let’s assume $H_0 = 0$. Then the energy identity (83) implies $v(\tau) \equiv \sqrt{2b}$, for all $\tau \in \mathbb{R}$. Notice that $v(\tau)$ is positive, as we assume $q(t)$ goes from the total collision to infinity. From Proposition 4.1 and Proposition 4.2, we have

$$
\mu(\hat{B}; \tau_0) = 0, \forall \tau_0 \in \mathbb{R}.
$$

Since the system is non-degenerate, let $\tau_0 \rightarrow +\infty$, we have

$$
(97) \quad \mu(\hat{B}; \mathbb{R}) = 0.
$$

From (83), we have

$$
(98) \quad m^-(q; T^-, T^+) = \mu(\hat{B}; \mathbb{R}) = 0.
$$
Now let’s assume \( H_0 > 0 \). Recall that by our assumption \( q(T^-) \) is a total collision with a finite \( T^- \in \mathbb{R} \) and \( q(T^+) \) is a hyperbolic singularity with \( T^+ = +\infty \). Let’s fix an arbitrary finite \( T_0 > T^- \), and rewrite \( q(t), t \in (T^-, T_0) \) in McGehee coordinates. Then by the energy identity \( S(t) \),

\[
v(\tau) = \sqrt{2(H_0 r(\tau) + b)}, \quad \forall \tau \in (-\infty, \tau_0).
\]

Here \( \tau_0 = \tau(T_0) \) is the time moment in McGehee coordinates corresponding to \( T_0 \). Notice that since \( q(T^+) \) is a hyperbolic singularity and we in McGehee coordinates instead of the hyperbolic McGehee coordinates, there is a finite \( \tau^+ \), such that \( \lim_{T_0 \to T^+} \tau(T_0) = \tau^+ \).

From Proposition \( \text{1.1} \) and \( \text{1.2} \) we have

\[
\mu(\hat{\beta}; \tau_0) = 0, \quad \forall \tau_0 \in (-\infty, \tau^+).
\]

The proofs of Proposition \( \text{1.1} \) and \( \text{1.2} \) show that \( V_D \cap V^u(\tau_0) \). From \( \text{S(1)} \), we have

\[
m^-(q; T^-, T_0) = \mu(\hat{\beta}; \tau_0) = 0
\]

for any \( T_0 < T^+ \). Since this is true for any finite \( T_0 > T^- \), we have

\[
m^-(q; T^-, T^+) = \lim_{\tau_0 \to T^+} m^-(q; T^-, T_0) = 0.
\]

\[\square\]

From now let’s assume the energy \( H_0 < 0 \). Notice that in this case \( v(\tau_0) = 0 \) for some finite \( \tau_0 \), where \( r(\tau) \) reaches its maximum. Without loss of generality, let’s assume \( \tau_0 = 0 \). Solving the first equation in \( \text{S(1)} \) directly, we get

\[
v(\tau) = -\sqrt{2b} \tanh \frac{\sqrt{2b}\tau}{2}.
\]

As a result,

\[
B_\lambda(\pm \infty) = \lim_{\tau \to \pm \infty} B_\lambda(\tau) = \left( \begin{array}{cc} 1 & \mp \sqrt{2b}/4 \\ \mp \sqrt{2b}/4 & -\lambda \end{array} \right)
\]

**Lemma 4.2.** For any \( \lambda > -b/8 \), the eigenvalues of \( JB_\lambda(\pm \infty) \) are \( \pm \sqrt{b/8} + \lambda \), and the corresponding eigenspaces are

\[
V^{\pm}(JB_\lambda(-\infty)) = \left\{ \begin{pmatrix} \pm \sqrt{b/8} + \lambda - \sqrt{2b}/4 \\ 1 \end{pmatrix} \right\},
\]

\[
V^{\pm}(JB_\lambda(+\infty)) = \left\{ \begin{pmatrix} \pm \sqrt{b/8} + \lambda + \sqrt{2b}/4 \\ 1 \end{pmatrix} \right\}.
\]

Moreover,

\[
s(V_D, V_H; V^+(JB_\lambda(-\infty)), V^+(JB_\lambda(\infty))) = \begin{cases} 1, & \text{if } \lambda \in (-b/8, 0), \\ 0, & \text{if } \lambda \in (0, +\infty). \end{cases}
\]

**Proof.** The eigenvalues and eigenspaces follow from direct computations. To compute the Hörmander index. Let

\[
\Lambda_\lambda(t) = \left\{ \begin{pmatrix} \sqrt{b/8} + \lambda + t\sqrt{2b}/4 \\ 1 \end{pmatrix} \right\}, \quad \text{for } t \in [-1, 1].
\]

Then \( \Lambda_\lambda(-1) = V^+(JB_\lambda(-\infty)) \) and \( \Lambda_\lambda(1) = V^+(JB_\lambda(\infty)) \). When \( \lambda > 0 \),

\[
\sqrt{b/8} + \lambda + t\sqrt{2b}/4 > 0, \quad \forall t \in [-1, 1].
\]
Therefore $\Lambda_\tau(t)$ is transversal to $V_D$ and $V_R$, for any $t \in [-1, 1]$. Then $\mu(V_D, \Lambda_\tau) = \mu(V_D, \Lambda_\lambda) = 0$. This implies

$$s(V_D, V_R, V^+(JB_\lambda(-\infty)), V^+(JB_\lambda(\infty))) = 0.$$  

When $-b/8 < \lambda < 0$, $\Lambda_\tau(t) \cap V_D, \forall t \in [-1, 1]$, which means $\mu(V_D, \Lambda_\tau) = 0$. Meanwhile notice that there is only one negative crossing for $V_R$, so $\mu(V_D, \Lambda_\lambda) = -1$. As a result,

$$s(V_D, V_R, V^+(JB_\lambda(-\infty)), V^+(JB_\lambda(\infty))) = 1.$$  

This completes our proof. \qed

In the following, let $V^s_\lambda(\tau)$ and $V^u_\lambda(\tau)$ be the stable and unstable subspaces of the linear Hamiltonian system (103) according to Definition 3.1. For $\lambda = 0$, $B_\lambda$ is degenerate according to Definition 3.3 as shown in the next lemma.

**Lemma 4.3.** $\mu(B_0; \mathbb{R}) = 0$ and $\nu(B_0) = 1$.

**Proof.** Notice that $\xi_1(\tau) = \tau \frac{r}{r}(\tau)(0, 1)^T$, $\tau \in \mathbb{R}$, is a solution of (103) with $\lambda = 0$, where $r(\tau)$ is a homothetic solution with negative energy, $r(\tau)$ as well as $\xi_1(\tau)$ goes to 0, as $\tau \to \pm \infty$. Therefore $V^s_0(\tau) = V^u_0(\tau) = \langle \xi_1(\tau) \rangle$. By Definition 3.3, this implies $\nu(B_0) = 1$. Meanwhile since $V^u_0(\tau)$ has no crossing with $V_D$, we have $\mu(B_0; \mathbb{R}) = 0$. \qed

Next lemma shows $B_\lambda$ is non-degenerate, for $\lambda \neq 0$.

**Lemma 4.4.** $\nu(B_\lambda) = 0$, for any $\lambda \in (-b/8, 0) \cup (0, +\infty)$.

**Proof.** By contradiction, suppose that there is a $\lambda_0 \in (-b/8, 0) \cup (0, +\infty)$, such that $\nu(B_{\lambda_0}) = 1$. This means $V^s_{\lambda_0}(0) = V^u_{\lambda_0}(0)$. This further implies $V^s_{\lambda_0}(\tau) = V^u_{\lambda_0}(\tau)$. Then there exists a solution $\xi_1(\tau) = (y(\tau), x(\tau))^T$ of (103), such that

$$\langle \xi_0(\tau) \rangle = V^s_{\lambda_0}(\tau) = V^u_{\lambda_0}(\tau), \quad \forall \tau \in \mathbb{R}.$$  

Then $\lim_{\tau \to \pm \infty} \xi_0(\tau) = \lim_{\tau \to \pm \infty} (y(\tau), x(\tau))^T = (0, 0)^T$. Meanwhile $\langle \xi_0(\tau) \rangle \to V^+(JB_{\lambda_0}(\infty))$, when $\tau \to -\infty$. Since $\lambda_0 \neq 0$, by (100), $y(\tau)$ can not be zero, when $\tau$ is close enough to $-\infty$.

On the other hand, as $\xi_0(\tau)$ satisfies (103), we have

$$\begin{align*}
y' &= -\frac{v}{4}y + \lambda_0 x, \\
x' &= y + \frac{v}{4}x
\end{align*}$$  

Take derivative with respect to $\tau$ on both sides of the first equation in (103). After simplification we get

$$y'' = f(\tau)y,$$

where

$$f(\tau) := \frac{b}{8}(1 - \tanh^2(\frac{\sqrt{2b\tau}}{2})) + \lambda_0 + \frac{b}{8} > 0.$$  

From property (c) of Lemma 4.1, there is no non-trivial bounded solution of (104), which is a contradiction to what I showed above. \qed
By the previous two lemmas and the decomposition property, we get
\[ \nu(\hat{B}_2) = \# \{ \lambda_i : \lambda_i = 0, 1 \leq i \leq n^* - 1 \} = \dim(\ker(M^{-1}D^2U|_{\mathcal{C}}(s_0))). \]
This immediately implies.

**Corollary 4.1.** \( \nu(\hat{B}) = \nu(\hat{B}_1) + \nu(\hat{B}_2) = \dim(\ker(M^{-1}D^2U|_{\mathcal{C}}(s_0))). \)

**Lemma 4.5.** For \( \lambda > -b/8 \) and \( \lambda \neq 0 \), \( \mu(V_{\mathcal{R}}, V^u_{\lambda}; \mathbb{R}) = 0. \)

**Proof.** The proof is similar to Lemma [4.4]. Assume \( V^u_{\lambda}(\tau) = (\xi(\tau) = (y(\tau), x(\tau))^T) \), for all \( \tau \). Then \( y(\tau) \) satisfies [104] and \( \lim_{\tau \to -\infty} y(\tau) = 0 \). Since \( y(\tau) \) can not be zero for all \( \tau \), from (ii) of Lemma [4.1] we have \( y(\tau) \neq 0 \) for \( \tau \in \mathbb{R} \). This implies that \( V^u(\tau) \cap V_{\mathcal{R}} \) for any \( \tau \in \mathbb{R} \), so we get the results.

**Proposition 4.4.**
\[
\mu(V_{\mathcal{D}}, V^u_{\lambda}(\tau); \mathbb{R}) = \begin{cases} 
1, & \text{if } \lambda \in (-b/8, 0), \\
0, & \text{if } \lambda \in [0, +\infty). 
\end{cases}
\]

**Proof.** For \( \lambda = 0 \), the desired result follows from Lemma [4.4]. For \( \lambda \neq 0 \), the system is non-degenerate, so
\[
\lim_{\tau \to -\infty} V^u_{\lambda}(\tau) = V^+(JB_{\lambda}(-\infty)), \quad \lim_{\tau \to +\infty} V^u_{\lambda}(\tau) = V^+(JB_{\lambda}(\infty)).
\]
Then
\[
\mu(V_{\mathcal{D}}, V^u_{\lambda}; \mathbb{R}) - \mu(V_{\mathcal{R}}, V^u_{\lambda}; \mathbb{R}) = s(V_{\mathcal{D}}, V_{\mathcal{R}}; V^+(JB_{\lambda}(-\infty)), V^+(JB_{\lambda}(\infty))).
\]
Now the desired result follows from [102] and Lemma [4.5].

With the above result, we have the following proposition which implies property (b) in Theorem [4.2] when \( H_0 < 0 \).

**Proposition 4.5.** If the energy \( H_0 \) of \( q(t) \) is negative with the associated the normalized central configuration \( s_0 \) satisfies the strict non-spiral condition, then
\[
m^-(q; T^-, T^+) = m^-(M^{-1}D^2U|_{\mathcal{C}}(s_0)).
\]

**Proof.** From [88] and [104],
\[
\mu(\hat{B}; \mathbb{R}) = \mu(\hat{B}_1; \mathbb{R}) + \mu(\hat{B}_2; \mathbb{R}) = \mu(\hat{B}_1; \mathbb{R}) + \sum_{i=1}^{n^*} \mu(B_{\lambda_i}; \mathbb{R}).
\]
Then Proposition [4.1] and [4.3] implies
\[
\mu(\hat{B}; \mathbb{R}) = m^-(M^{-1}D^2U|_{\mathcal{C}}(s_0)).
\]
This actually means \( \mu(\hat{B}; (-\infty, \tau_0]) = m^-(M^{-1}D^2U|_{\mathcal{C}}(s_0)) \) for any \( \tau_0 \) large enough. Let \( T_0 = t(\tau_0) \) be the corresponding Newtonian time. Meanwhile for any \( \tau_0 \) large enough, \( V_{\mathcal{D}} \cap V^u(\tau) \). By Property (a) in Theorem [3.2]
\[
m^-(q; T^-, T_0) = \mu(\hat{B}; (-\infty, \tau_0]) = m^-(M^{-1}D^2U|_{\mathcal{C}}(s_0)).
\]
Since \( T_0 = t(\tau_0) \) goes to \( T^+ \), when \( \tau_0 \) goes to \( +\infty \). We get
\[
m^-(q; T^-, T^+) = \mu(\hat{B}; \mathbb{R}) = m^-(M^{-1}D^2U|_{\mathcal{C}}(s_0)).
\]
\[\square\]
Proposition 4.6. If $s_0$ satisfies the non-spiral condition, then
\[
m^-(q; T^-, T^+) = \begin{cases} \frac{m^-}{m^+} (M^{-1} D^2 U|_s(s_0)), & \text{when } H_0 < 0; \\
0, & \text{when } H_0 \geq 0. \end{cases}
\]

Proof. We shall use a perturbation argument. For $\varepsilon > 0$ small enough, set
\[
\hat{B}(\tau, \varepsilon) = \hat{B}_1(\tau) \circ \hat{B}_2(\tau, \varepsilon), \quad \text{where } \hat{B}_2(\tau, \varepsilon) = 
\begin{pmatrix}
\hat{M}^{-1} & \frac{1}{2} v I \\
\frac{1}{2} v I & -\hat{U}_{xx}(x_0) - \varepsilon \hat{M}
\end{pmatrix},
\]

Then $\hat{B}(\tau) - \hat{B}(\tau, \varepsilon)$ is a non-negative matrix. By the monotone property of Maslov index (see Property VII in page 14),
\[
\mu(V_{\mathcal{D}}, \hat{\gamma}(\tau, \tau_1) V_{\mathcal{D}} ; [\tau_1, \tau_2]) \geq \mu(V_{\mathcal{D}}, \hat{\gamma}_\varepsilon(\tau, \tau_1) V_{\mathcal{D}} ; [\tau_1, \tau_2]),
\]
where $\hat{\gamma}_\varepsilon(\tau, \tau_1)$ satisfies
\[
\begin{cases}
\hat{\gamma}_\varepsilon(\tau, \tau_1) = J \hat{B}(\tau, \varepsilon) \hat{\gamma}(\tau, \tau_1) ; \\
\hat{\gamma}(\tau_1, \tau_1) = I.
\end{cases}
\]

This system satisfies the strict non-spiral condition.

First let’s assume $H_0 < 0$. By previous results,
\[
\mu(V_{\mathcal{D}}, \hat{\gamma}(\tau, \tau_1) V_{\mathcal{D}} ; [\tau_1, \tau_2]) \geq \mu(V_{\mathcal{D}}, \hat{\gamma}_\varepsilon(\tau, \tau_1) V_{\mathcal{D}} ; [\tau_1, \tau_2]) - n^* = m^- (\hat{U}_{xx}(x_0) + \varepsilon \hat{M}).
\]

Meanwhile for $\varepsilon$ small enough,
\[
m^-(\hat{U}_{xx}(x_0) + \varepsilon \hat{M}) = m^- (\hat{U}_{xx}(x_0)) = m^- (M^{-1} D^2 U|_s(s_0)).
\]

By the monotone property (105) and the Morse index theorem (55),
\[
m^-(q; T^-, T^+) \geq m^- (M^{-1} D^2 U|_s(s_0)).
\]

Meanwhile (105) implies that for any $\tau_1 < \tau_2$,
\[
\mu(V_{\mathcal{D}}, \hat{\gamma}_\varepsilon(\tau, \tau_1) V_{\mathcal{D}} ; [\tau_1, \tau_2]) - n^* \leq m^- (M^{-1} D^2 U|_s(s_0))
\]

By contradiction, let’s assume the desired identity does not hold. Then
\[
m^-(q; T^-, T^+) \geq m^- (M^{-1} D^2 U|_s(s_0)) + 1.
\]

According to the Morse index theorem (55) and Lemma 5.1, there exist $\tau_1 < \tau_2$, such that
\[
\mu(V_{\mathcal{D}}, \hat{\gamma}(\tau, \tau_1) V_{\mathcal{D}} ; [\tau_1, \tau_2]) - n^* \geq m^- (M^{-1} D^2 U|_s(s_0)) + 1.
\]

From the index theorem (see [24]), the index $\mu(V_{\mathcal{D}}, \hat{\gamma}(\tau, \tau_1) V_{\mathcal{D}} ; [\tau_1, \tau_2])$ is equivalent to the Morse index of the Sturm-Liouville operator of this system. This implies such an index will not decrease by a small perturbation. Hence for $\varepsilon$ small enough, we still have
\[
\mu(V_{\mathcal{D}}, \hat{\gamma}_\varepsilon(\tau, \tau_1) V_{\mathcal{D}} ; [\tau_1, \tau_2]) - n^* \geq m^- (M^{-1} D^2 U|_s(s_0)) + 1.
\]

This contradicts (107). Hence under the non-spiral condition, we still have
\[
m^-(q; T^-, T^+) = m^- (M^{-1} D^2 U|_s(s_0)).
\]

The case for $H_0 \geq 0$ is similar and even simpler. For any $\tau_1 < \tau_2$, we have
\[
m^-(q; \tau_1, \tau_2) = \mu(V_{\mathcal{D}}, \hat{\gamma}_\varepsilon(\tau, \tau_1) V_{\mathcal{D}} ; [\tau_1, \tau_2]) - n^* = 0,
\]
where \( m^-(q; t(\tau_1), t(\tau_2)) \) is the Morse index of the \( \epsilon \) perturbation system. Since the Morse index is non-decrease under small perturbations, we have
\[
m^-(q; t(\tau_1), t(\tau_2)) = 0,
\]
for any \( \tau_1 < \tau_2 \). Let \( \tau_1 \to -\infty, \tau_2 \to +\infty \) (\( \tau_2 \to \tau^+ \) in the case of \( H_0 > 0 \)), we get the desired result.

\[\Box\]

Acknowledgments. The first author thanks useful discussions with Vivina Barutello, Alessandro Portaluri and Susanna Terracini. The last two authors wishes to thank School of Mathematics, Shandong University for its hospitality, where part of the work was done when they were visitors there.

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