Distinguishability Measures and Ensemble Orderings

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It is shown that different distinguishability measures impose different orderings on ensembles of \( N \) pure quantum states. This is demonstrated using ensembles of equally-probable, linearly independent, symmetrical pure states, with the maximum probabilities of correct hypothesis testing and unambiguous state discrimination being the distinguishability measures. This finding implies that there is no absolute scale for comparing the distinguishability of any two ensembles of \( N \) quantum states, and that distinguishability comparison is necessarily relative to a particular discrimination strategy.

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I. INTRODUCTION

The use of quantum channels to send classical information has many advantages over the use of classical channels. One of these is the fact that it enables one to establish cryptographic keys in a way that is provably secure, a feat which has never been achieved classically. Developments such as this have led to renewed attention being given to the problem of distinguishing between quantum states\(^1\). In state discrimination, we are faced with the following situation: a quantum system is prepared in one of \( N \) states. For the sake of simplicity, we will take these to be pure states \(|\psi_j\rangle\), where \( j = 1, \ldots, N \). The \( a \) priori probability of the state of the system being \(|\psi_j\rangle\) is \( p_j \). We would like to determine which state has been prepared. Unless the states are orthogonal we cannot determine the state perfectly. We are then faced with the problem of devising a strategy which discriminates between the \( N \) potential states as well as possible. This will involve some, possibly generalised, quantum measurement. This measurement should be optimised and the resulting figure of merit, which is typically a probability, can be regarded as a measure of the distinguishability of the states with these \( a \) priori probabilities. A set of quantum states \(|\psi_j\rangle\) considered together with their \( a \) priori probabilities \( p_j \) forms an ensemble \( \mathcal{E} = \{p_j, |\psi_j\rangle\} \). So, distinguishability measures will refer to ensembles.

We shall denote a generic distinguishability measure by \( D[\mathcal{E}] \). Several distinct measures are in common use for quantifying the distinguishability of states\(^2\). The question we address in this paper is: do all distinguishability measures impose the same ordering on ensembles of \( N \) quantum states? That is, that we have two ensembles, \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \), and two distinguishability measures, \( D_1 \) and \( D_2 \). Then, if \( D_1[\mathcal{E}_1] > D_1[\mathcal{E}_2] \), is it the case that \( D_2[\mathcal{E}_1] \geq D_2[\mathcal{E}_2] \)? We shall see that this is not necessarily so.

The distinguishability measures we choose are the maximum probabilities of correct hypothesis testing and unambiguous state discrimination. For the sake of simplicity, we restrict our attention to ensembles of states which have equal \( a \) priori probabilities. We show that for two equally-probable pure states, this effect cannot be demonstrated for these distinguishability measures, which leads us to consider ensembles of \( N > 2 \) states. We restrict our attention to linearly independent ensembles, since this is a requirement for unambiguous state discrimination\(^3\). The states in the chosen ensembles are taken to form symmetrical sets. This is done to take advantage of the fact that for such ensembles, our distinguishability measures can be calculated analytically for equal \( a \) priori probabilities. The effect we have described is shown to occur for \( N = 3 \). We conclude with a discussion of the connection between this effect and a related one recently discovered by Jozsa and Schlienz\(^2\).

II. DISTINGUISHABILITY MEASURES IMPOSING DIFFERENT ENSEMBLE ORDERINGS

The distinguishability measures we choose are the maximum probabilities of correct hypothesis testing and unambiguous state discrimination. In hypothesis testing among \( N \) states in the ensemble \( \mathcal{E}(p_j, |\psi_j\rangle) \), we consider an \( N \) outcome generalised measurement, in which the \( j \)th outcome is associated with the POVM element \( E_j \). Our hypothesis is that the outcome of the measurement corresponds exactly to the state. If the state is \(|\psi_j\rangle \) and outcome \( j \) is obtained, then the hypothesis is correct. If however, outcome \( j' \neq j \) is obtained, then the hypothesis will be incorrect. The maximum probability \( P_{HYP} \) that our hypothesis is correct is\(^3\)

\[
P_{HYP}[\mathcal{E}] = \max \sum_j p_j(\psi_j|E_j|\psi_j) \tag{2.1}
\]

where the maximisation is carried out over all sets of \( N \) positive operators \( E_j \) such that \( \sum_j E_j = 1 \). We shall use the maximum probability of correct hypothesis testing, \( P_{HYP}(\mathcal{E}) \), as a measure of the distinguishability of the ensemble \( \mathcal{E} \).
When the a priori probabilities are equal, ∆ = 0 and an ensemble of just two equally-probable pure states. This is to such sets that we will restrict our attention. If $P_{\text{USD}}(|\psi_j\rangle)$ is the probability that, given the initial state was $|\psi_j\rangle$, we obtain a conclusive identification rather than an inconclusive result, then the maximum probability of unambiguous state discrimination is

$$P_{\text{USD}}(\mathcal{E}) = \max_{\{P_{\text{USD}}(|\psi_j\rangle)\}} \sum_j p_j P_{\text{USD}}(|\psi_j\rangle),$$

(2.2)

where the extremisation with respect to the $P_{\text{USD}}(|\psi_j\rangle)$ is discussed by Duan and Guo. The distinguishability measure we will use for this strategy will be the maximum probability $P_{\text{USD}}(\mathcal{E})$ of unambiguous determination of the state.

Here, we will show that there exist ensemble pairs for which

$$P_{\text{USD}}(\mathcal{E}_2) < P_{\text{USD}}(\mathcal{E}_1),$$

(2.3)

$$P_{\text{HYP}}(\mathcal{E}_2) > P_{\text{HYP}}(\mathcal{E}_1).$$

(2.4)

The effect we wish to demonstrate does not occur for an ensemble of just two equally-probable pure states. This can be seen by examining the values of $P_{\text{HYP}}$ and $P_{\text{USD}}$ for a pair of pure states, $|\psi_1\rangle$ and $|\psi_2\rangle$. For later convenience, we will give expressions for these with arbitrary a priori probabilities $p_1$ and $p_2$. The former is given by Helstrom’s bound:

$$P_{\text{HYP}} = \frac{1}{2} \left( 1 + \sqrt{1 - (1 - \Delta^2)|\langle \psi_1 | \psi_2 \rangle|^2} \right),$$

(2.5)

where $\Delta = |p_1 - p_2|$. Also, the maximum value of $P_{\text{USD}}$ for a pair of pure states is given by the Jaeger-Shimony bound:

$$P_{\text{USD}} = \left\{ \frac{1}{2} (1 + \Delta)(1 - |\langle \psi_1 | \psi_2 \rangle|^2) : \frac{1 - \Delta^2}{1 + \Delta} \leq |\langle \psi_1 | \psi_2 \rangle| \leq \frac{1 + \Delta^2}{1 + \Delta} \right\},$$

(2.6)

When the a priori probabilities are equal, $\Delta = 0$ and $P_{\text{HYP}} = (1/2)(1 + \sqrt{1 - (1 - (1 - P_{\text{USD}})^2)})$. From this, one can show that $P_{\text{HYP}}$ is an increasing function of $P_{\text{USD}}$, implying that inequalities (2.3) and (2.4) can never be simultaneously satisfied.

To find ensembles of equally-probable pure states for which both (2.3) and (2.4) are true, we have to consider at least three states. We focus on ensembles of equally-probable, linearly independent, symmetrical states, since the maximum probabilities for unambiguous discrimination and correct hypothesis testing can be calculated explicitly for these. We will demonstrate the existence of ensemble pairs $\mathcal{E}_1$ and $\mathcal{E}_2$ which satisfy inequalities (2.3) and (2.4), in the following way. Firstly, we will consider all sets of $N$ linearly independent, symmetric states with equal a priori probabilities which have the same, arbitrary but fixed, value of $P_{\text{USD}}$. Over this set, we will find the extremal values of $P_{\text{HYP}}$. Using this information, we choose ensembles pairs which satisfy inequality (2.3), but where $\mathcal{E}_1$ and $\mathcal{E}_2$ have respectively the minimum and maximum values of $P_{\text{HYP}}$ for their corresponding values of $P_{\text{USD}}$. We will find that, for $N = 3$, inequality (2.4) is satisfied for a large range of parameters.

The $N$ pure states $|\psi_j\rangle$, where $j = 0, \ldots, N - 1$, are linearly independent and symmetric if and only if they can be written as

$$|\psi_j\rangle = \sum_{r=0}^{N-1} c_r e^{2\pi i j r / N} |x_r\rangle,$$

(2.7)

for some $N$ orthonormal states $|x_r\rangle$ and non-zero complex coefficients $c_r$, satisfying $\sum_r |c_r|^2 = 1$. Notice that the phase of $c_r$ may be absorbed by $|x_r\rangle$, which implies that we may, without loss of generality, take $c_r$ to be real and positive, which we shall. We will take all states to have equal a priori probabilities $p_j = 1/N$. The maximum unambiguous discrimination probability for these states is

$$P_{\text{USD}} = N \times \min_r c_r^2.$$

(2.8)

The optimum hypothesis testing strategy uses the so-called ‘square-root’ measurement. Define the operator

$$\Phi = \sum_j |\psi_j\rangle\langle \psi_j|,$$

(2.9)

and the states

$$|\omega_j\rangle = \Phi^{-1/2} |\psi_j\rangle.$$

(2.10)

One can quite easily show that the operators $E_r = |\omega_j\rangle\langle \omega_r|$ form a POVM (i.e., that $E_r \geq 0$ and $\sum_r E_r = 1$.) This POVM is the optimal hypothesis testing strategy, and we find that

$$P_{\text{HYP}} = \frac{1}{N} \left( \sum_r c_r \right)^2.$$

(2.11)

The square-root measurement optimally discriminates between any set of equally-probable, symmetric states, even if they are not linearly independent. This measurement has recently been carried out for symmetrical optical polarisation states. Applications of this measurement to quantum key distribution are discussed in [10].

We now calculate the global extrema of $P_{\text{HYP}}$ for a fixed value of $P_{\text{USD}}$. Our aim is to fix the smallest of the $c_r$, which is equivalent to fixing $P_{\text{USD}}$, and vary the remaining coefficients to obtain the extremal values of $P_{\text{HYP}}$. We may let $\min_r c_r = c_0$. This allows us to write

$$c_r = c_0 + \cos^2 \theta_r,$$

(2.12)
for \( r = 1, \ldots, N - 1 \) and some angles \( \theta_j \). We will now extremise \( P_{\text{HYP}} \) with respect to these angles using the method of Lagrange multipliers, in order to take into account the normalisation constraint. This method will yield the local extrema, over which we will subsequently optimise to find the global extrema. Let \( G = (\sum_{r=0}^{N-1} c_r^2) - 1 \). The constrained, local extrema of \( P_{\text{HYP}} \) occur where

\[
\frac{\partial P_{\text{HYP}}}{\partial \theta_r} = \lambda \frac{\partial G}{\partial \theta_r},
\]

and \( \lambda \) is our Lagrange multiplier. Inserting Eq. (2.12) into the definitions of \( P_{\text{HYP}} \) and \( G \), we find that this becomes

\[
\sin \theta_r \cos \theta_r \left[ c_0 + \frac{1}{N} \sum_{r'=1}^{N-1} \cos^2 \theta_{r'} - \lambda (c_0 + \cos^2 \theta_r) \right] = 0.
\]

It is a simple matter to show that the normalisation requirement will be violated if \( \sin \theta_r = 0 \) for any \( r \). So, for each \( r \), either \( \cos \theta_r = 0 \), in which case \( c_r = c_0 \), or the term in brackets is zero. To proceed, we will partition the \( c_r \) into two sets, where each set corresponds to one of these two possibilities. Let there be \( N_0 \) coefficients, including \( c_0 \), which are equal to \( c_0 \). For the remaining \( N - N_0 \) coefficients, the term in brackets is zero. All of the coefficients in this latter set must also be equal, as can be seen from the fact that the vanishing of the bracket term implies that the corresponding \( \cos^2 \theta_r \) are equal. The value that they have is easily deduced from the fact that the other \( N_0 \) coefficients are equal to \( c_0 \), and normalisation. We find they have the value

\[
c_r = \sqrt{\frac{1 - N_0 c_0^2}{N - N_0}}.
\]

We can now calculate the local extrema of \( P_{\text{HYP}} \). Evaluating Eq. (2.11) where \( N_0 \) coefficients are equal to \( c_0 \), with the remaining \( N - N_0 \) coefficients being given by Eq. (2.15), and making use of the fact that \( c_0 = \sqrt{P_{\text{USD}}/N} \), we find that the local extrema of \( P_{\text{HYP}} \) for fixed \( P_{\text{USD}} \) are

\[
P_{\text{HYP}} = \frac{1}{N^2} \left( N_0 \sqrt{P_{\text{USD}}} + \sqrt{(N - N_0)(N - N_0 P_{\text{USD}})} \right)^2.
\]

Finding the global extrema of \( P_{\text{HYP}} \) amounts to extremisation of this expression with respect to \( N_0 \). One method of doing this is to treat \( N_0 \) as a continuous parameter in the interval \([1, N - 1]\) and differentiate Eq. (2.16) with respect to it. After some algebra, we find that \( \partial P_{\text{HYP}}/\partial N_0 \leq 0 \), for \( P_{\text{HYP}} \) given by Eq. (2.16). This implies that the global maximum and minimum of \( P_{\text{HYP}} \) occur at the minimum and maximum values of \( N_0 \) respectively. Thus, the maximum value of \( P_{\text{HYP}} \) occurs at \( N_0 = 1 \), which gives us the tight upper bound

\[
P_{\text{HYP}} \leq \frac{1}{N^2} \left( \sqrt{P_{\text{USD}}} + \sqrt{(N - 1)(N - P_{\text{USD}})} \right)^2.
\]

We also see that the minimum value of \( P_{\text{HYP}} \) occurs at \( N_0 = N - 1 \), which gives us the tight lower bound

\[
P_{\text{HYP}} \geq \frac{1}{N^2} \left( (N - 1) \sqrt{P_{\text{USD}}} + \sqrt{N - (N - 1)P_{\text{USD}}} \right)^2.
\]

When \( P_{\text{USD}} = 1 \), the minimum of \( P_{\text{HYP}} \) is also 1, as we would expect. However, if \( P_{\text{USD}} = 0 \), then the minimum of \( P_{\text{HYP}} \) is \( 1/N \), which corresponds to a random guess of which of the \( N \) possible states the system has been prepared in. In fact, it is quite easily shown that in this case, all \( |\psi_j\rangle \) are, up to a phase, equal to \( |x_0\rangle \). Here, the states are linearly dependent and this case is of no interest to us in the present context.

Let us now use the bounds in (2.17) and (2.18) to demonstrate the existence of ensembles for which inequalities (2.3) and (2.4) hold. For the sake of simplicity,
III. DISTINGUISHABILITY AND INFORMATION

We have seen that different distinguishability measures impose different orderings on ensembles of pure quantum states. We considered linearly independent states and demonstrated this effect using the maximum probabilities of correct hypothesis testing and unambiguous state discrimination as distinguishability measures.

This result has important implications for any situation where we wish to transmit nonorthogonal states to send classical information. For example, in quantum key distribution, we may have a choice of sending states prepared in either ensemble $E_1$ or ensemble $E_2$. The distinguishability of these ensembles may determine the potential information available to an eavesdropper and also the rate at which key bits can be generated. It would therefore be desirable to be able to compare the distinguishability of both ensembles. What the results in the preceding show is that this cannot be done in any absolute sense; we must know in advance which distinguishability measure is being used. However, we may not know which measure would apply to a potential eavesdropper since we may not know in advance which detection strategy they would employ.

To put the results of the preceding section in context, it is helpful to consider a related finding recently made by Jozsa and Schlienz. They considered two ensembles of pure states, $E_1$ and $E_2$, with all states in each ensemble having equal $a$ priori probabilities. They used the pairwise overlaps of pairs of states as a means of quantifying the distinguishability of each ensemble, and the von Neumann entropy $S_1$ of the density operator of ensemble $E_1$ as a means of quantifying information content. They showed that there exist ensemble pairs in which the pairwise overlaps, and hence the distinguishability of $E_2$, are less than those of $E_1$, yet where $S_2 > S_1$, implying that the information content of $E_2$ is greater than that of $E_1$. This is somewhat counterintuitive, as information and distinguishability are often regarded as interchangeable concepts.

This finding suggests that there are distinctions to be made between information and distinguishability. That they are not entirely interchangeable is perhaps suggested by the fact that they arise in different contexts. In attempting to distinguish between a set of states, we are given only one copy, and must make optimum use of it. In information transmission however, we perform a collective measurement on a large number of states drawn from the same ensemble. The length of the strings and the subset chosen from the set of all possible strings is such that these strings are highly distinguishable. Indeed, they are chosen in such a way that the probability of failing to distinguish between the states perfectly can be made arbitrarily small. The information content is the number of bits of classical information that can be transmitted per signal in each string, where the signals obey the statistical constraints of the ensemble. In information transmission, we select strings which are sufficiently long so that we can neglect the distinguishability issue in one way. However, it is possible that this issue does return in some form to dictate the information content of a single signal in ways that are not currently well-understood.

A conclusive distinction, which does not apply to situations in which all states have the same $a$ priori probabilities, is that information content and distinguishability can vary in opposite ways when the $a$ priori probabilities of the states are altered. To illustrate this, consider two ensembles, $E_1$ and $E_2$, where both consist of the pure states $|\psi_1\rangle$ and $|\psi_2\rangle$. In $E_1$, both states have equal $a$ priori probabilities equal to $1/2$. However, in $E_2$, they have unequal $a$ priori probabilities $p_1$ and $p_2$. In state discrimination, our goal is to determine the state. If one state is more probable than the other, then we can take advantage of this fact to tailor our measurement to weight it in favour of the state which has higher $a$ priori probability, so that, on average, we will be able to improve our ability to determine the state. It follows that $E_2$ ought to have higher distinguishability than $E_1$. This is made explicit if we look back at either the Helstrom bound (2.5) or the Jaeger-Shimony bound (2.6). These both increase as $\Delta = |p_1 - p_2|$ increases, confirming this expectation.

However, if we wish to use states in ensemble $E_2$ to send messages, then the constraint that one state must always be more probable than the other implies that, to regain our freedom in the message we might choose to send, we must send more signal states per message as $\Delta$ increases. Quantitatively, for $E_2$, the information content is given by the von Neumann entropy of the ensemble density operator, which is equal to the binary entropy function

$$S_2 = -x \log_2 x - (1 - x) \log_2 (1 - x)$$

where

$$x = \frac{1}{2} \left(1 + \sqrt{1 - (1 - \Delta^2)(1 - |\langle \psi_1 | \psi_2 \rangle|^2)} \right).$$

As $\Delta$ increases, $x$ also increases which implies that, as
can easily be shown, $S_2$ decreases. The extreme situa-
tion is where one of the \textit{a priori} probabilities, say $p_1$, is
equal to 1, in which case $p_2 = 0$. Here, we know in ad-
vance what the state is, and so we can always determine
it perfectly. We may say that this ensemble has perfect
distinguishability. However, the sender has no freedom
in which state to send and the state is always entirely
predictable. For this reason, it is impossible to trans-
mit any information using this ensemble, for which $S_2$
is easily seen to be zero.

As the above argument demonstrates, a clear distinc-
tion between information content and distinguishability
emerges if variation of the \textit{a priori} probabilities of the
states is permitted. However, it is not entirely clear why
they should be distinct concepts when the \textit{a priori}
probabilities of all states are equal. Consequently, for en-
sembles with equal \textit{a priori} probabilities, the observation
that information and distinguishability impose different
orderings remains to be properly understood.

The results in this paper suggest a potential explana-
tion of this phenomenon. Given that it is not clear why
information content should not be regarded as a distin-
guishability measure when we restrict our attention to
ensembles of states with equal \textit{a priori} probabilities, let
us assume that it is in fact a suitable distinguishability
measure under these circumstances. The Jozsa-Schlienz
effect could then be explained as a demonstration of the
fact that, even for ensembles of equally-probable states,
different distinguishability measures impose different en-
samble orderings, which is what we have shown in this
paper, using quantities which are indisputably suitable
distinguishability measures.

It remains to be understood why different distinguisha-
bility measures impose different orderings. Perhaps, for
two distinguishability measures to be non-trivially dis-
tinct from one another, they must be sensitive to different
aspects of ensembles of states, which can vary, with some
degree of independence, from one ensemble to another.
This seems reminiscent of the discovery recently made by
Virmani and Plenio[1] that different, good measures of
the entanglement of mixed states must order these states
differently.

IV. DISCUSSION

We have shown in this paper that two of the most
common measures of the distinguishability of states, the
maximum probabilities of correct hypothesis testing and
unambiguous state discrimination, are essentially incompat-
ible with each other. By this, we mean that they
do not impose the same ordering on ensembles of pure
states. It is possible to have one ensemble, $E_1$, which is
more distinguishable than another ensemble $E_2$ when the
states must be distinguished unambiguously, yet where
$E_2$ is more distinguishable than $E_1$ if we wish to identify
the state by minimum error hypothesis testing. In gen-
eral, we cannot, in any absolute sense, characterise one
ensemble of states as being more distinguishable than
another. Distinguishability comparison must necessarily
refer to a particular discrimination strategy.

As with any interesting phenomenon, it is important
to determine the conditions under which this effect can
be demonstrated optimally. Here, we are faced with the
fact that, for generic ensembles of states, the optimisation
problems which must be solved to obtain values of
distinguishability measures are difficult to tackle analyt-
ically. At the time of writing, the only ensembles for
$N > 2$ for which both of our chosen distinguishability
measures can be calculated analytically are ensembles of
equally-probable, linearly independent, symmetric pure
states. For such ensembles, inequalities (2.16) and (2.17)
give the extremal values of the correct hypothesis testing
probability for an fixed value of the maximum probability
of unambiguous state discrimination, and so the optimal
conditions for demonstrating this effect for these ensem-
bles are given by these relations. We also explored in
some detail the case of $N = 3$.

The effect we have demonstrated is somewhat reminis-
cent of a related one recently discovered by Jozsa and
Schlienz. These authors showed that one can construct
an ensemble pair, $E_1$, $E_2$, where the distinguishability of
$E_1$, as measured by the pairwise overlaps of states, is
greater than that of $E_2$, yet where the information content
of $E_2$, which is quantified by the von Neumann entropy,
is greater than that of $E_1$.

This is an important finding, not least because the con-
cepts of distinguishability and information content are
sometimes used interchangeably. We described how es-
tential differences between these concepts do arise when,
in moving from one ensemble to another, the \textit{a priori}
probabilities of the states are changed. However, if the \textit{a}
priori probabilities of all states in both ensembles are the
same, it is by no means clear that, despite the contrast
between the asymptotic nature of information and the
‘one shot’ nature of distinguishability, information mea-
sures cannot also serve as measures of distinguishability.

Jozsa and Schlienz demonstrated their effect using
such ensembles. What the results in this paper suggest
is a possible interpretation of the Jozsa-Schlienz effect.
They showed that, for ensembles of equally-probable
pure states, information and a particular distinguishabil-
ity measure suffer an ordering incompatibility problem.
Does this imply that for such ensembles, information con-
tent is not a suitable distinguishability measure? The
results in this paper suggest that this is not necessarily
the case, since different distinguishability measures suffer
analogous ordering problems. It could then be the case
that, for the ensembles considered by Jozsa and Schlienz,
information content is in fact a suitable distinguishability
measure and that the ordering incompatibility they dis-
covered is a consequence of the fact that such problems
arise with regard to generic distinguishability measures.
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