2-Categories, 4d state-sum models and gerbes

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In this article we focus on the third word in the title of our talk and on our motivation for getting involved with it.

The simplest state-sum model is the Dijkgraaf-Witten (DW) model [5]. In its most elementary form the DW-model associates to a smooth closed connected finite-dimensional manifold $M$ the following number:

$$\#\text{Hom}\{\pi_1(M), G\},$$

where $G$ is a finite group. If we want to understand the differential geometry behind the DW-model we have to give up the finiteness of $G$ of course. If $G$ is a Lie group, we can ask ourselves what geometric objects correspond to smooth homomorphisms from $\pi_1(M)$ to $G$ (we will not explain here what we mean by smoothness exactly). The answer is well known: principal $G$-bundles with flat connections. We explain this in some more detail. Let $\{U_i\}$ be a covering of $M$ by open sets such that all intersections $U_{i_1...i_p} = U_{i_1} \cap \cdots \cap U_{i_p}$ are contractible. We present a principal $G$-bundle, $P$, by its transition functions $g_{ij}: U_{ij} \to G$, which satisfy $g_{ji} = g_{ij}^{-1}$ and the cocycle condition

$$g_{ij}g_{jk}g_{ik}^{-1} = 1 \text{ on } U_{ijk}.$$ 

A connection, $A$, in $P$ can be defined in terms of local 1-forms, $A_i$ on $U_i$, with values in the Lie algebra of $G$, which satisfy

$$A_j - g_{ij}^{-1}A_i g_{ij} = g_{ij}^{-1}dg_{ij} \text{ on } U_{ij}.$$ 

Given a loop $\ell$ in $M$ one can define the holonomy, $\mathcal{H}(\ell) \in G$, of $A$ around $\ell$. In general the holonomies around two homotopic loops are different. However,

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if the two loops are thin homotopic, then the holonomies are equal, as was first remarked by Barrett [1]. There are several ways to define thin homotopy mathematically [1, 3]. For the purpose of this article it suffices to give the intuitive idea:

**Definition 1** (sketch) Two loops are thin homotopic if there exists a homotopy between them whose image has no area.

All homotopies involved in the standard proof that $\pi_1(M)$ is a group are thin as a matter of fact, so we can define the thin fundamental group of $M$, denoted $\pi_1^\text{thin}(M)$, by dividing out the set of loops only by thin homotopies. Note that $\pi_1(M)$ is a quotient of $\pi_1^\text{thin}(M)$. Thus $A$ gives rise to a holonomy homomorphism

$$H: \pi_1^\text{thin}(M) \to G,$$

which is smooth in a technical sense [1, 3]. Barrett [1] (see [3] for a proof of the analogous statement using a different definition of thin homotopy) proved that there is a converse statement:

**Theorem 2** Given a smooth homomorphism $H: \pi_1^\text{thin}(M) \to G$, there is a principal $G$-bundle with connection, unique up to equivalence, whose holonomy homomorphism is equal to $H$.

A connection is flat precisely when the corresponding $H$ factors through the ordinary $\pi_1(M)$.

Let us now assume that $\pi_1(M) = 0$. The next state-sum model that we consider associates to $M$ the number

$$\#\text{Hom}\{\pi_2(M), H\},$$

where $H$ is a finite Abelian group. This is a special case of the Yetter model [10], which involves the homotopy 2-type of $M$. The right algebraic framework for the Yetter model is that of 2-categories, which is how the first word in the title of our talk enters the picture. For more information about 2-categories and 4-dimensional state-sum models see the two papers [7, 8] by the first author and references therein. The question now arises whether we can understand the maps $\pi_2(M) \to H$ in an analogous differential geometric way. Let $H = U(1)$. We see immediately that the answer cannot be found in the framework of bundles and connections, because we need some geometric structure that gives rise to holonomy around surfaces rather than
loops. It is known (well-known would be an over-statement) that gerbes with gerbe-connections give rise to such holonomies \[2\]. A gerbe, \(G\), can be defined by functions on triple intersections, \(h_{ijk} : U_{ijk} \to U(1)\), which satisfy

\[ h_{\sigma(i)\sigma(j)\sigma(k)} = h_{ijk}^{(\sigma)} \]

for any \(\sigma \in S_3\), and the next order cocycle condition:

\[ h_{ijk} h_{ijl}^{-1} h_{ikl}^{-1} h_{jkl}^{-1} = 1 \text{ on } U_{ijkl}. \]

A gerbe-connection, \(B\), in \(G\) can be defined by 1-forms, \(A_{ij}\) on \(U_{ij}\), and 2-forms, \(F_i\) on \(U_i\), all with values in \(i\mathbb{R}\), such that \(A_{ji} = -A_{ij}\) and

\[ A_{ij} + A_{jk} - A_{ik} = h_{ijk}^{-1} dh_{ijk} \text{ on } U_{ijk}, \]

\[ F_j - F_i = dA_{ij} \text{ on } U_{ij}. \]

Gerbes were first defined by Giraud \[6\]. The standard reference nowadays is Brylinski’s book \[2\]. The properties of gerbes and gerbe-connections are analogous to those of line-bundles; e.g. the curvature of a gerbe-connection, \(\Omega|_{U_i} = dF_i\), is a closed integral 3-form, the cohomology class of which classifies the gerbe up to equivalence, and every closed integral 3-form is the curvature of a certain gerbe-connection in a certain gerbe. One can define the gerbe-holonomy \[2\], \(H(s)\), of \(B\) around any smooth map \(s : S^2 \to M\), henceforth referred to as a 2-loop. As for ordinary connections one can show \[9\] that the gerbe-holonomies around two thin homotopic 2-loops are equal.

**Definition 3** (sketch) *We say that two 2-loops are thin homotopic if there exists a homotopy between them whose image has no volume.*

The higher dimensional thin homotopy groups were first defined by Caetano and Picken in \[4\], where one can find the technical definition. Thus \(B\) gives rise to a smooth *gerbe-holonomy homomorphism*

\[ H : \pi_2^2(M) \to U(1), \]

where \(\pi_2^2(M)\) is the *thin second homotopy group*. In \[9\] we proved that there is a converse statement:

**Theorem 4** Assume that \(M\) is simply-connected. Given a smooth homomorphism \(H : \pi_2^2(M) \to U(1)\), there exists a gerbe with gerbe-connection, unique up to equivalence, whose holonomy map is equal to \(H\).
A gerbe-connection is flat precisely when its holonomy map factors through the ordinary $\pi_2(M)$, so we have achieved our goal of understanding the differential geometry of the Yetter model. The proof of Thm. 3 is fairly straightforward. If one does not assume that $M$ is simply-connected the analogous statement and its proof involve the less familiar mathematics of Lie (2-)groupoids. We believe that this case is very interesting because it fuses ideas from category theory and geometry into something that can best be called categorical geometry.

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