Strong lensing in the Einstein-Straus solution

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To the memory of Jürgen Ehlers

Abstract

We analyse strong lensing in the Einstein-Straus solution with positive cosmological constant. For concreteness we compare the theory to the light deflection of the lensed quasar SDSS J1004+4112.

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1 Introduction

In September last year Rindler & Ishak [1] corrected the general believe that the deflection angle of light passing near an isolated, static, spherically symmetric mass is independent of the cosmological constant. In their analysis the source emitting the light and the observer were supposed at rest with respect to the central mass and the masses of source and observer were neglected. Two subsequent papers [2, 3] confirmed Rindler & Ishak’s result. Khriplovich & Pomeransky [4] pointed out that, if the earth is taken comoving with respect to the exponentially expanding de Sitter space, then the effect of the cosmological constant on the deflection cancels. Park [5] re-did their analysis with McVittie’s solution and finds the same cancelation for the exponentially expanding de Sitter space.

The aim of this paper is to calculate the bending of light by a spherically symmetric mass, which is taken to be a cluster of galaxies, without the two mentioned simplifications: (i) the observer is allowed to move with respect to the cluster, (ii) the masses of the other clusters are included in the form of a homogeneous, isotropic dust. The observer is taken comoving with respect to the dust. This situation is described by the Einstein-Straus solution [6, 7] that matches the Kottler (or Schwarzschild-de Sitter solution) at the inside of the Schücking radius with a Friedmann solution at the outside. The first motivation of this solution was to explain why the cosmic expansion does not affect small length scales like in solar systems and atoms. Let us note that the Einstein-Straus solution is as unstable as Friedmann’s solutions [8]. This is the very instability that produces structure formation. Ishak et al. [9] have already used the Einstein-Straus solution in the context of light bending. They find that the dust partially screens the effect of the cosmological constant. Qualitatively this screening is easy to understand: The cosmological constant induces a repulsive force between the isolated cluster and the photon. This force increases with the distance between cluster and photon. Adding more clusters adds more repulsion. But the net force outside the Schücking radius vanishes due to the high symmetry of the dust. The present calculation will make this screening quantitative. It will show furthermore that the attractive force between cluster and photon, which is due to the central mass and which decreases with distance, is subject to sizeable anti-screening. An important part of this anti-screening will turn out to be of purely kinematical origin, coming from the velocity of the observer.

For numerical convenience, we use the following units: length is measured in astrometers (am), time in astroseconds (as) and mass in astrograms (ag),

\[
\begin{align*}
\text{am} & = 1.30 \cdot 10^{26} \text{ m} = 4221 \text{ Mpc}, \\
\text{as} & = 4.34 \cdot 10^{17} \text{ s}, \\
\text{ag} & = 6.99 \cdot 10^{51} \text{ kg} = 3.52 \cdot 10^{21} M_\odot.
\end{align*}
\]  

(1)

In these units, we have \( c = 1 \text{ am as}^{-1} \), \( 8\pi G = 1 \text{ am}^3\text{as}^{-2}\text{ag}^{-1} \), \( H_0 = 1 \text{ as}^{-1} \). For completeness we record Planck’s constant, which we do not use, \( \hbar = 3.86 \cdot 10^{-121} \text{ am}^2\text{as}^{-1}\text{ag} \). We will consider spatially flat universes where we may set the scale factor today \( a_0 = 1 \text{ am} \).
2 Bending of light in Kottler’s solution

Before we take up the Einstein-Straus solution, we review strong lensing in Kottler’s solution,

\[ \dd r^2 = B \dd T^2 - \frac{1}{B} \dd r^2 - r^2 (\sin^2 \theta \dd \varphi^2 + \dd \theta^2), \quad B := 1 - \frac{2GM}{r} - \frac{\Lambda}{3} r^2, \quad (2) \]

see figure 1, and include a radial velocity of the observer.

![Figure 1: Two geodesics](image)

In Kottler’s solution the geodesics can be integrated analytically to first order in the ratio peri-lens divided by Schwarzschild radius. We are interested in relating the physical observables of strong lensing, the two angles, \( \alpha \) and \( \alpha' \), between the images and the lens, the redshifts \( z_L \) of the lens and \( z_S \) of the source and the mass \( M \) of the lens. To be concrete we consider the lensed quasar SDSS J1004+4112 where [10, 11]

\[ \alpha = 10'' \pm 10\% , \quad z_L = 0.68 , \quad M = 5 \cdot 10^{13} M_\odot = 1.4 \cdot 10^{-8} M_\odot \pm 20\% \quad (3) \]
\[ \alpha' = 5'' \pm 10\% , \quad z_S = 1.734. \quad (4) \]

For this system, the above ratio is of order \( 10^{-5} \) and second order terms can indeed safely be neglected.

We use the spatially flat \( \Lambda CDM \) model with \( \Lambda = 0.77 \cdot 3 \text{ am}^{-2} \pm 20\% \) to convert redshifts into angular distances with respect to the Earth, which we denote by \( d_L \) and \( d_S \) respectively. Then we obtain the coordinate distances [3],

\[ d_L = r_E, \quad d_S = \frac{r_E + r_S}{\sqrt{1 - \Lambda r_S^2/3}}, \quad (5) \]

and with the coordinate angle,

\[ \gamma := \arctan \left[ r_E \left| \frac{\dd \varphi}{\dd r} \right|_E \right], \quad (6) \]

we get the polar angle of the source,

\[ \varphi'_S \sim \gamma' \left( 1 + \frac{r_E}{r_S} \right) - \frac{4GM}{\gamma' r_E}. \quad (7) \]

Notice that this coordinate angle does not depend on the cosmological constant, which however re-enters through the relation between coordinate angles, \( \gamma, \gamma' \) and physical angles \( \alpha, \alpha' \):

\[ \tan \alpha \sim \sqrt{1 - \Lambda r_E^2/3} \tan \gamma, \quad (8) \]
for an observer at rest with respect to the lens. From \( \varphi_S = \varphi'_S \) we deduce:

\[
\frac{r_E}{r_S} \sim \frac{4GM}{\alpha\alpha'r_E}(1 - \Lambda r_E^2/3) - 1
\]

and the mass of the cluster \[12\], see table 1.

| \( \Lambda \pm 20\% \) | ±0 | + | - | + | - | + | - | + | - |
| \( \alpha \pm 10\% \) | ±0 | ±0 | ±0 | + | + | + | - | - | - |
| \( \alpha' \pm 10\% \) | ±0 | ±0 | ±0 | + | + | - | - | + | - |

\[
-\varphi_S \ [\pi] \\
M \ [10^{13} M_\odot]
\]

| 13.0 | 13.6 | 12.6 | 15.0 | 13.9 | 17.7 | 16.4 | 9.5 | 8.8 | 12.2 | 11.3 |
| 4.7 | 5.8 | 4.0 | 7.0 | 4.8 | 5.7 | 4.0 | 5.7 | 4.0 | 4.7 | 3.2 |

Table 1: Fitting the cluster mass in Kottler’s solution, earth at rest

We now want to take into account the velocity \( v_E \), that we suppose radially outward. Our task is to recalculate the relation between coordinate angle and physical angle, the latter being measured in nanoseconds over nanoseconds. Consider figure 2 in the \((r, \varphi)\) plane, \( \theta = \pi/2 \).

![Diagram](image)

The proper time \( d\tau_r \) it takes a photon to go from \((r_E, \pi)\) to \((r_E - dr, \pi)\) is computed from \( 0 = B dT^2_r - (1/B) dr^2 \), with \( dr^2 = B dT^2_r \). We get \( d\tau_r = B^{-1/2} dr \). During a lapse \( d\tau_\varphi \) the Earth has moved outwards by \( dv = v_E B^{1/2} d\tau_\varphi = v_E B dT_\varphi \). The proper time \( d\tau_\varphi \) it takes the photon to go from \((r_E - dr, \pi)\) to \((r_E - dr + dv, \pi - d\varphi)\) is computed from \( 0 = B dT^2_\varphi - (1/B)v_E^2 B^2 dT^2_\varphi - r_E^2 d\varphi^2 \). Therefore \( d\tau_\varphi = B^{1/2} dT_\varphi = (1 - v_E^2)^{1/2} r_E d\varphi \). Finally:

\[
\tan \alpha = \frac{d\tau_\varphi}{d\tau_r} = \frac{\sqrt{B}}{\sqrt{1 - v_E^2}} r_E \frac{d\varphi}{dr} = \frac{\sqrt{B}}{\sqrt{1 - v_E^2}} \tan \gamma.
\]

Imposing again \( \varphi_S = \varphi'_S \) we deduce:

\[
\frac{r_E}{r_S} \sim \frac{4GM}{\alpha\alpha'r_E} \frac{1 - \Lambda r_E^2/3}{1 - v_E^2} - 1.
\]
For an Earth comoving with the exponentially expanding de Sitter space, \( v_E = \sqrt{\Lambda / 3r_E} \), the cosmological constant indeed drops out \([4, 5]\). For the more realistic value \( v_E = H_0 r_E \) we obtain the values shown in Table 2.

Taking into account the Hubble velocity of the observer reduces the effect of the cosmological constant on the bending of light: a 20% increase of \( \Lambda \) decreases the cluster mass by 20% for the observer at rest, by only 10% for the comoving observer. Consequently the mass estimate of \( M = 4.7^{+2.3}_{-1.5} \cdot 10^{13} M_\odot \) for Kottler’s solution with the Earth at rest, which is nicely compatible with the observed value of \( M = 5.0^{+1.0}_{-1.0} \cdot 10^{13} M_\odot \) thanks to the positive cosmological constant, is brought down by the Hubble velocity of the observer to \( M = 3.0^{+1.4}_{-0.7} \cdot 10^{13} M_\odot \), now only marginally compatible with observation. Naturally we would like to include the effect of the other masses in the universe on the bending of light.

### 3 The Einstein-Straus solution with a cosmological constant

In this section we streamline Schücking’s proof \([7]\) of the Einstein-Straus solution \([6]\) in its form generalized by Balbinot, Bergamini & Comastri \([13]\) to include a cosmological constant. We only consider the case of spatially flat universes. But we add to the results in the above references the Jacobian of the transformation passing between the Friedmann and the Schwarzschild coordinates, which we use in the next section to compute the geodesics of photons.

**Statement of the result:** We write the Kottler metric as

\[
\text{d}t^2 = B \text{d}T^2 - \frac{1}{B} \text{d}r^2 - r^2 \text{d}\Omega^2, \quad B := 1 - \frac{2GM}{r} - \frac{\Lambda}{3} r^2, \tag{12}
\]

and the Friedmann metric as

\[
\text{d}t^2 = \text{d}t^2 - a^2[\text{d}\chi^2 + \chi^2 \text{d}\Omega^2], \quad \frac{\text{d}a}{\text{d}t} = \sqrt{\frac{A}{a} + \frac{\Lambda}{3} a^2}, \tag{13}
\]

\[A := \frac{1}{3} \rho_{\text{dust}} a_0^3.\tag{14}\]

We suppose that the scale factor \( a(t) \) is strictly monotonic. Both solutions are glued together at the constant Schücking radius \( \chi_{\text{Schü}} \):

\[r_{\text{Schü}}(T) := a(t) \chi_{\text{Schü}}, \quad r \leq r_{\text{Schü}}, \quad \chi \geq \chi_{\text{Schü}}. \tag{15}\]
The central mass $M$ must be equal to the dust density times the volume of the ball with Schücking radius $r_{\text{Schü}}$,

$$A = \frac{2M}{8\pi \chi_{\text{Schü}}^3} = \frac{2GM}{\chi_{\text{Schü}}^3}. \quad (16)$$

Then at the Schücking radius,

$$B(r_{\text{Schü}}) = B_{\text{Schü}} = 1 - \frac{A}{a} \chi_{\text{Schü}}^2 - \frac{\Lambda}{3} a^2 \chi_{\text{Schü}}^2, \quad (17)$$

where we also define $C_{\text{Schü}} := \sqrt{1 - B_{\text{Schü}}}$. The coordinate transformation $(T, r) \rightarrow (t, \chi)$ at the Schücking radius is cumbersome to write down, not so its Jacobian,

$$\frac{\partial t}{\partial T} \bigg|_{\text{Schü}} = 1, \quad \frac{\partial t}{\partial r} \bigg|_{\text{Schü}} = -\frac{C_{\text{Schü}}}{B_{\text{Schü}}}, \quad (18)$$

$$\frac{\partial \chi}{\partial T} \bigg|_{\text{Schü}} = -\frac{C_{\text{Schü}}}{a}, \quad \frac{\partial \chi}{\partial r} \bigg|_{\text{Schü}} = \frac{1}{aB_{\text{Schü}}}. \quad (19)$$

The inverse of the Jacobian is,

$$\frac{\partial T}{\partial t} \bigg|_{\text{Schü}} = \frac{1}{B_{\text{Schü}}}, \quad \frac{\partial T}{\partial \chi} \bigg|_{\text{Schü}} = \frac{aC_{\text{Schü}}}{B_{\text{Schü}}}, \quad (20)$$

$$\frac{\partial r}{\partial t} \bigg|_{\text{Schü}} = C_{\text{Schü}}, \quad \frac{\partial r}{\partial \chi} \bigg|_{\text{Schü}} = a. \quad (21)$$

We will also need to compare coordinate times at the Schücking radius,

$$\frac{dt}{dT} \bigg|_{\text{Schü}} = B_{\text{Schü}}. \quad (22)$$

**Proof:** The scale factor $a(t)$ is supposed monotonic and may therefore serve as time coordinate, $(t, \chi) \rightarrow (a, \chi)$. Then the Friedmann metric reads,

$$d\tau^2 = \frac{da^2}{A/a + \frac{1}{3}\Lambda a^2} - a^2 d\chi^2 - a^2 \chi^2 d\Omega^2. \quad (23)$$

In a next step we want to turn the $a^2 \chi^2$ factor in front of $d\Omega^2$ into $r^2$,

$$(a, \chi) \rightarrow (b, r), \quad a =: \Phi(b, r), \quad \chi =: r/\Phi(b, r), \quad (24)$$

with the boundary condition that at the Schücking radius $\chi_{\text{Schü}}$, old and new time coordinates coincide, $a = b = \Phi(b, b\chi_{\text{Schü}})$. Then with $C_1 := \sqrt{A/\Phi + \frac{1}{3}\Lambda \Phi^2}$ the metric tensor of the Friedmann solution becomes,

$$g^{\text{Frie}}_{bb} = \Phi_b^2 \left\{ \frac{1}{C_1^2} - \frac{r^2}{\Phi^2} \right\}, \quad g^{\text{Frie}}_{rr} = \left[ 1 - \frac{r}{\Phi} \Phi_r \right]^2 - \frac{\Phi_r^2}{C_1^2}, \quad (25)$$

$$g^{\text{Frie}}_{br} = \Phi_b \left\{ \frac{\Phi_r}{C_1^2} + \frac{r}{\Phi} \left[ 1 - \frac{r}{\Phi} \Phi_r \right] \right\}. \quad (26)$$
We do not want a mixed term, $g_{br}^{\text{Frie}} = 0$, which is equivalent to,

$$\Phi_r = - \frac{r}{\Phi} \frac{C_1^2}{B_1}, \quad B_1 := 1 - \frac{A r^2}{\Phi^3} - \frac{\Lambda}{3} r^2. \quad (27)$$

For every fixed $b$, this differential equation admits one local solution satisfying the boundary condition. We can simplify,

$$g_{bb}^{\text{Frie}} = \Phi_b^2 \frac{B_1}{C_1^2}, \quad g_{rr}^{\text{Frie}} = \frac{1}{B_1}. \quad (28)$$

Differentiating the boundary condition with respect to $b$, we have:

$$\Phi_b|_{\text{Sch}} := \Phi_b(b, b \chi_{\text{Sch}}) = 1 - \chi_{\text{Sch}} \Phi_r|_{\text{Sch}} = 1/B_{\text{Sch}}. \quad (29)$$

We now turn to the Kottler solution and change its coordinates:

$$(T, r) \rightarrow (b, r), \quad \frac{dT}{db} = \Psi(b). \quad (30)$$

This coordinate transformation still allows us the choice of one initial condition, which we will use later. In the new coordinates, the metric tensor of the Kottler solution is,

$$g_{bb}^{\text{Kott}} = \Psi^2 B, \quad g_{rr}^{\text{Kott}} = 1/B. \quad (31)$$

It is in these coordinates, $(b, r, \theta, \varphi)$, that we join together Friedmann’s and Kottler’s metric tensors continuously at the Schücking radius and for all times:

$$g_{bb}^{\text{Frie}}|_{\text{Sch}} = g_{bb}^{\text{Kott}}|_{\text{Sch}}; \quad g_{rr}^{\text{Frie}}|_{\text{Sch}} = g_{rr}^{\text{Kott}}|_{\text{Sch}}. \quad (32)$$

At this point we need the relation (16) between Friedmann’s dust density and the central mass $M$. This relation implies $B_1|_{\text{Sch}} = B_{\text{Sch}}$ and $C_1|_{\text{Sch}} = C_{\text{Sch}}/\chi_{\text{Sch}}$. For the gluing to be continuous we must therefore choose

$$\Psi(b) = \frac{\chi_{\text{Sch}}}{B_{\text{Sch}}(b) C_{\text{Sch}}(b)}. \quad (33)$$

Successive application of the chain rule then gives;

$$\frac{\partial t}{\partial T} = \frac{dt}{da} \frac{\partial \Phi}{\partial b} \frac{db}{dT}, \quad \frac{\partial t}{\partial T} = \frac{dt}{da} \frac{\partial \Phi}{\partial r}, \quad (34)$$

$$\frac{\partial \chi}{\partial T} = \frac{\partial \chi}{\partial b} \frac{db}{dT}, \quad \frac{\partial \chi}{\partial r} = \frac{1}{\Phi} - \frac{r}{\Phi^2} \frac{\partial \Phi}{\partial r}, \quad (35)$$

and restricting to the Schücking radius yields the desired Jacobian.

To compare the Friedmann and Schwarzschild coordinate times $t$ and $T$ at the Schücking radius, consider the parameterized curve, $T = p$, $r = \chi_{\text{Sch}}b$, $(\theta = \pi/2, \varphi = 0)$. Its 4-velocity is:

$$\frac{dT}{dp} = 1, \quad \frac{dr}{dp} = \chi_{\text{Sch}} \frac{db}{dT}|_{\text{Sch}} \frac{dT}{dp} = B_{\text{Sch}} C_{\text{Sch}}. \quad (36)$$
in Schwarzschild coordinates and in Friedmann coordinates:

\[
\frac{dt}{dp} = \frac{\partial t}{\partial T}_{\text{Schü}} \frac{dT}{dp} + \frac{\partial t}{\partial r}_{\text{Schü}} \frac{dr}{dp} = 1 \cdot 1 - \frac{C_{\text{Schü}}}{B_{\text{Schü}}} B_{\text{Schü}} = B_{\text{Schü}}, \tag{37}
\]

\[
\frac{d\chi}{dp} = \frac{\partial \chi}{\partial T}_{\text{Schü}} \frac{dT}{dp} + \frac{\partial \chi}{\partial r}_{\text{Schü}} \frac{dr}{dp} = -\frac{C_{\text{Schü}}}{a} \cdot 1 + \frac{1}{a B_{\text{Schü}}} B_{\text{Schü}} C_{\text{Schü}} = 0. \tag{38}
\]

Finally we obtain the desired relation: \( dt/dT = dt/dp \cdot dp/dT = B_{\text{Schü}}. \)

We conclude the proof by pointing out a few typeset errors in reference [13]: the cosmological constant has the wrong sign in equations (3.15), (4.5) and (4.7). In equation (3.19), \( \kappa \) should read \( \chi \). In the appendix, the definitions of \( S \) and \( T \) are missing.

In the next section we want to interpret the central mass as the mass of a cluster \( M \sim 10^{14} M_\odot \). For this interpretation to make sense we must have hierarchies of the following length scales, the Schwarzschild radius \( s \sim 10^{-9} \) am, the typical radius of a cluster \( r_{\text{cluster}} \sim 10^{-3} \) am, the Schücking radius \( r_{\text{Schü}} \sim 10^{-3} \) am, the typical distance between clusters \( D_{\text{cluster}} \sim 10^{-3} \) am and the de Sitter radius \( r_{\text{dS}} \sim 1 \) am:

\[
s < r_{\text{cluster}} < r_{\text{Schü}} < D_{\text{cluster}} \quad \text{and} \quad r_{\text{Schü}} < r_{\text{dS}}. \tag{39}
\]

4 Integrating the geodesics of light

The geodesics will be integrated piecewise, see figure 3: in spatially flat Friedmann’s solution with cosmological constant \( \Lambda = 0.77 \cdot 3 \) am\(^{-2}\) and dust \( \rho_{\text{dust}} = (3 - \Lambda) \alpha g/\text{am}^3 \) and in Kottler’s solution.

![Figure 3: Two piecewise differentiable geodesics](image)

The geodesics will be pasted together continuously at the Schücking radius with their first derivatives matched by using the Jacobian computed in the previous section. The scale factor \( a(t) \) is computed numerically with a Runge-Kutta method from Friedmann’s equation, which in our units reads

\[
2 \frac{1}{a} a_{tt} + \frac{1}{a^2} (a_t)^2 = \Lambda, \tag{40}
\]

with final condition, \( a_t(0) = a(0) = 1 \). On the other hand, the spatial part of the geodesics is easy to integrate: the photons follow ‘straight lines’ in the polar coordinates \((\chi, \varphi, \theta)\). In Kottler’s solution the geodesics are integrated manually to first order in the ratio perihel lens divided by Schwarzschild radius. To be concrete we use again the lensed quasar SDSS.
J1004+4112, for which the above ratio is of order $10^{-5}$ and second order terms can indeed safely be neglected.

As the physical angles are measured at the arrival, we will integrate backwards in time, i.e. negative $dt$, $dT$ and $dp$, $p$ being the affine parameter. We denote $d/dp$ by the over-dot $\dot{}$.

**Step 0** is the integration in Friedmann’s solution all the way back to the source without deflection.

We take the origin, $\chi = 0$, at the position of the lens and define the plane containing Earth, lens and source by $\theta = \pi/2$. Our final condition at $p = 0$ is $t = 0$, $\chi = \chi_E$, $\varphi = \pi$ and $\dot{t} = 1$, $\dot{\chi} = 1$, $\dot{\varphi} = 0$. Again we use a Runge-Kutta method to integrate $d\chi/dt = 1/a$ and we denote the solution by $\tilde{\chi}(t)$ and its inverse function by $\tilde{t}(\chi)$. With the definitions of redshift, $1 + z = 1/a$, and Schücking radius,

$$
\chi_{\text{Schü}} = \left( \frac{3M}{4\pi \rho_{\text{dust}} 0} \right)^{1/3},
$$

we get the values shown in table 3.

|        | Earth | Schücking radius | lens | Schücking radius | source |
|--------|-------|------------------|------|------------------|--------|
| $z$    | 0     | 0.68             | 0.68 | 1.734            |        |
| $t_0$  | 0     | -0.4566          | -0.4566 | -0.4576 | -0.7372 |
| $\chi$ | 0.5904 | 0.0017           | 0    | 0.0017           | 0.5942 |

Table 3: Passage times without deflection in astroseconds and comoving coordinate distances

In **step 1** we compute the trajectory of the photon between the Earth and the Schücking radius. Here we need the Christoffel symbols of the Friedmann metric in the plane $\theta = \pi/2$:

$$
\Gamma^t_{\chi\chi} = a_t a, \quad \Gamma^x_{\varphi\varphi} = -\chi, \quad \Gamma^\varphi_{t\varphi} = a_t/a, \quad \Gamma^t_{\varphi\varphi} = \chi^2 a_t a, \quad \Gamma^x_{tx} = a_t/a, \quad \Gamma^x_{\chi\varphi} = 1/\chi.
$$

The geodesic reads:

$$
\ddot{t} + a_t a \, \dot{\chi}^2 + \chi^2 a_t a \, \dot{\varphi}^2 = 0, \quad (44)
$$
$$
\ddot{\chi} + 2a_t/a \, \dot{t} \dot{\chi} - \chi \, \dot{\varphi}^2 = 0, \quad (45)
$$
$$
\ddot{\varphi} + 2a_t/a \, \dot{t} \dot{\varphi} + 2/\chi \, \dot{\chi} \dot{\varphi} = 0. \quad (46)
$$

To define the final conditions at $p = 0$, we use the fact that the coordinate angle $\arctan(\chi \dot{\varphi}/\dot{\chi})$ coincides with the physical angle $\alpha'$ measured in nanoseconds/nanoseconds:

$$
t = 0, \quad \chi = \chi_E, \quad \varphi = \pi, \quad (47)
$$
$$
\dot{t} = 1, \quad \dot{\chi} = \cos \alpha', \quad \dot{\varphi} = \sin \alpha'/\chi_E. \quad (48)
$$
Then the solution of the geodesic equation is unique,

\[ \dot{t} = \frac{1}{a}, \quad \frac{\chi_P}{\chi} = \sin(\varphi - \alpha'), \quad \dot{\varphi} = \frac{\chi_P}{a^2 \chi^2}, \quad (49) \]

where \( \chi_P := \chi_E \sin \alpha' \) is the would-be peri-lens. Therefore the photon crosses the Schücking sphere in the half-space containing the Earth at the polar-angle

\[ \varphi_{\text{Sch"uck} E} = \pi - \arcsin \frac{\chi_P}{\chi_{\text{Sch"uck}}}, \quad (50) \]

and at the time

\[ t_{\text{Sch"uck} E} = \tilde{t} \left( \sqrt{\chi_E^2 + \chi_{\text{Sch"uck}}^2 + 2 \chi_E \chi_{\text{Sch"uck}} \cos \varphi_{\text{Sch"uck} E}} \right) \sim 0 t_{\text{Sch"uck} E}. \quad (51) \]

The difference between \( t_{\text{Sch"uck} E} \) and the non-deflected passage time \( 0 t_{\text{Sch"uck} E} \) is of second order in \( \pi - \varphi_{\text{Sch"uck} E} \). At this crossing, the 4-velocity of the photon is:

\[ \dot{t}_{\text{Sch"uck} E} = \frac{1}{a_{\text{Sch"uck} E}}, \quad \dot{\chi}_{\text{Sch"uck} E} = -\frac{\cos(\varphi_{\text{Sch"uck} E} - \alpha')}{a_{\text{Sch"uck} E}^2}, \quad \dot{\varphi}_{\text{Sch"uck} E} = \frac{\chi_P}{a_{\text{Sch"uck} E}^2 \chi_{\text{Sch"uck}}}, \quad (52) \]

with \( a_{\text{Sch"uck} E} := a(t_{\text{Sch"uck} E}) \). Let us call \( \gamma_F, F \) for Friedmann, the smaller physical angle between the (unoriented) direction of the photon and the direction towards the lens. We have

\[ \gamma_F = \arctan \left( \chi_{\text{Sch"uck} E} \frac{\dot{\varphi}_{\text{Sch"uck} E}}{\dot{\chi}_{\text{Sch"uck} E}} \right) = \pi - (\varphi_{\text{Sch"uck} E} - \alpha'). \quad (53) \]

In **step 2** we translate the 4-velocity into the coordinates \( T, r, \varphi \). We now use the free initial condition mentioned after equation (30) to set \( T_{\text{Sch"uck} E} = t_{\text{Sch"uck} E} \). Using the inverse Jacobian, equation (21), we have

\[ \dot{r}_{\text{Sch"uck} E} = \frac{C_{\text{Sch"uck} E} - \cos(\varphi_{\text{Sch"uck} E} - \alpha')}{a_{\text{Sch"uck} E}}. \quad (54) \]

Let us call \( \gamma_K, K \) for Kottler, the smaller **coordinate** angle between the (unoriented) direction of the photon and the direction towards the lens,

\[ \gamma_K := \arctan \left( r_{\text{Sch"uck} E} \frac{\dot{\varphi}_{\text{Sch"uck} E}}{\dot{r}_{\text{Sch"uck} E}} \right) = \arctan \frac{\sin \gamma_F}{C_{\text{Sch"uck} E} + \cos \gamma_F}. \quad (55) \]

These specify the final conditions for (the spatial part of) the geodesic equation inside the Schücking sphere.

In **step 3** we integrate this geodesic equation. To this end we need the Christoffel symbols of the Kottler solution with \( \theta = \pi/2 \) and denoting \( ' := \frac{d}{dr} \),

\[ \Gamma^T_{rr} = B'/2B, \quad \Gamma^r_{TT} = BB'/2, \quad \Gamma^r_{rr} = -B'/2B, \quad (56) \]

\[ \Gamma^r_{\varphi\varphi} = -rB, \quad \Gamma^r_{r\varphi} = 1/r. \quad (57) \]
The geodesic equations read:

\[\ddot{T} + \frac{B'}{B} \dot{T} \dot{r} = 0,\]  
\[\ddot{r} + \frac{1}{2} B B' \dot{T}^2 - \frac{1}{2} B B' \dot{r}^2 - r B \dot{\varphi}^2 = 0,\]  
\[\ddot{\varphi} + 2 r^{-1} \dot{r} \dot{\varphi} = 0,\]

We immediately get three first integrals:

\[\dot{T} = \frac{1}{B}, \quad r^2 \dot{\varphi} = J, \quad \dot{r}^2 / B + J^2 / r^2 - 1 / B = -E.\]

The last two come from invariance of the metric under rotations and time translations and the integration constants \(J\) and \(E\) have the meaning of angular momentum and energy per unit of mass. For the photon, \(E = 0\). Eliminating affine parameter and coordinate time we get:

\[\frac{d\varphi}{d\varphi} = \pm r \sqrt{r^2 / J^2 - B}.\]

At the peri-lens \(r_P\), \(d\varphi / d\varphi(r_P) = 0\) and therefore \(J = r_P B(r_P)^{-1/2}\). Substituting \(J\) into equation (62), the cosmological constant drops out and we have:

\[\frac{d\varphi}{dr} = \pm \frac{1}{r \sqrt{r^2 / r_P^2 - 1}} \left[1 - \frac{s}{r} - \frac{s}{r_P} - \frac{r}{r + r_P}\right]^{-1/2},\]

where we have written \(s := 2GM\) for the Schwarzschild radius. From now on we will omit terms of order \((s/r_P)^2\), which in our case are of order \(10^{-10}\), and write equalities up to this order with a \(\sim\) sign. In this approximation the peri-lens is

\[r_P \sim r_{\text{Sch}} E \sin \gamma_K - \frac{1}{2} s.\]

Note that for the upper trajectory, \(d\varphi / dr\) is positive for \(r\) between \(r_{\text{Sch}} E\) and \(r_P\), negative between \(r_P\) and \(r_{\text{Sch}} S\). Therefore

\[\varphi_{\text{Sch}} E - \varphi_{\text{Sch}} S = \int_{r_P}^{r_{\text{Sch}} E} \left| \frac{d\varphi}{dr} \right| dr + \int_{r_P}^{r_{\text{Sch}} S} \left| \frac{d\varphi}{dr} \right| dr.\]

Using \(\int x^{-1}(x^2 - 1)^{-1/2} dx = -\arcsin 1 / x\), \(\int x^{-2}(x^2 - 1)^{-1/2} dx = (x^2 - 1)^{1/2} / x\), \(\int (x + 1)^{-1}(x^2 - 1)^{-1/2} dx = [(x - 1) / (x + 1)]^{1/2}\), we get to linear order:

\[\varphi_{\text{Sch}} S \sim \varphi_{\text{Sch}} E - \pi + \arcsin \frac{r_P}{r_{\text{Sch}} E} + \arcsin \frac{r_P}{r_{\text{Sch}} S}\]

\[-\frac{1}{2} \frac{s}{r_{\text{Sch}} E} \sqrt{\frac{r_{\text{Sch}} E^2}{r_P^2} - 1} - 1 - \frac{1}{2} \frac{s}{r_{\text{Sch}} S} \sqrt{\frac{r_{\text{Sch}} S^2}{r_P^2} - 1} - 1\]

\[-\frac{1}{2} \frac{s}{r_P} \sqrt{\frac{r_{\text{Sch}} E - r_P}{r_{\text{Sch}} E + r_P} - 1} - \frac{1}{2} \frac{s}{r_P} \sqrt{\frac{r_{\text{Sch}} S - r_P}{r_{\text{Sch}} S + r_P}}.\]
To proceed we need the coordinate time $T_{\text{Schü}}$ at which the photon crosses the Schücking sphere at the source side and its corresponding time $t_{\text{Schü}}$ in Friedmann’s coordinates. Recall that we have defined the time coordinates such that $T_{\text{Schü}} = t_{\text{Schü}}$. The time the undeflected photon takes to cross the Schücking sphere is $0_t^{t} = 2 \times 10^{-3}$ as. Its time delay due to bending is of the order of 10 years \[14\] or $10^{-9}$ as. The difference $t_{\text{Schü}} - T_{\text{Schü}}$ can be estimated with equation (22) and an intermediate value theorem:

$$t_{\text{Schü}} - T_{\text{Schü}} = C_{\text{Schü}} (t_{\text{Schü}} - t_{\text{Schü}}), \quad C_{\text{Schü}}^2 := \frac{A}{a(t)} \chi_{\text{Schü}}^2 + \frac{\Lambda}{3} a(t)^2 \chi_{\text{Schü}}, \quad (67)$$

with an intermediate value $t_i \in [t_{\text{Schü}}, t_{\text{Schü}}]$. The function $C_{\text{Schü}}$ varies slowly, in our example by less than half a per mil, and is small, of the order of $10^{-5}$. We will therefore put $t_{\text{Schü}} = 0_t^{t}$ and $r_{\text{Schü}} = a(0_t^{t}) \chi_{\text{Schü}}$.

In step 4 we translate the four-velocity at $t = 0_t^{t}$, $\chi = \chi_{\text{Schü}}$, $\varphi = \varphi_{\text{Schü}}$,

$$\begin{align*}
\dot{T}_{\text{Schü}} &= \frac{1}{B_{\text{Schü}}} , \\
\dot{r}_{\text{Schü}} &= -\sqrt{1 - \frac{r_p^2}{r_{\text{Schü}}^2}} \, \frac{B_{\text{Schü}}}{B_p} , \\
\dot{\varphi}_{\text{Schü}} &= \frac{r_p}{r_{\text{Schü}}^2} \, \sqrt{B_p} ,
\end{align*} \quad (68)$$

back into Friedmann’s solution:

$$\dot{\chi}_{\text{Schü}} = -\frac{1}{a_{\text{Schü}} B_{\text{Schü}}} \left( C_{\text{Schü}} + \sqrt{1 - \frac{r_p^2}{r_{\text{Schü}}^2}} \, \frac{B_{\text{Schü}}}{B_p} \right) , \quad (69)$$

Using the same geometry as in step 1 we get the initial polar-angle of the emitted photon:

$$\varphi_s = \varphi_{\text{Schü}} - \gamma_{FS} + \arcsin \frac{\chi_{\text{Schü}} \sin \gamma_{FS}}{\chi_s} , \quad \gamma_{FS} := \arctan \frac{-\chi_{\text{Schü}} \dot{\varphi}_{\text{Schü}} S}{\chi_{\text{Schü}}} . \quad (70)$$

5 Results and conclusion

First we must point out that the peri-cluster is of the order $r_P \sim 10^{-5}$ am which is very small with respect to the typical radius of a cluster $r_{\text{cluster}} \sim 10^{-3}$ am.

| $\Lambda \pm 20\%$ | $\pm 0$ | $+ \pm - + - + - + - + -$ |
| $\alpha \pm 10\%$ | $\pm 0$ | $\pm 0 \pm 0 + + - + - + - + -$ |
| $\alpha' \pm 10\%$ | $\pm 0$ | $\pm 0 \pm 0 + + - + - + - + -$ |
| $-\varphi_s [\mu]$ | $10.0$ | $9.0 \ 10.6 \ 9.9 \ 11.6 \ 11.7 \ 13.7 \ 6.3 \ 7.4 \ 8.1 \ 9.5$ |
| $M \ [10^{15} M_\odot]$ | $1.8$ | $1.7 \ 1.8 \ 2.2 \ 2.2 \ 1.8 \ 1.8 \ 1.8 \ 1.8 \ 1.5 \ 1.5$ |

Table 4: Fitting the cluster mass in Einstein-Straus’ solution

We compute the two angles $\varphi_s$, one with $\alpha$ and one with $\alpha'$. For the chosen values of the cluster mass $M$ and $\Lambda$, the two angles do not coincide. Even within the error bars for $M$, $\alpha$ and $\alpha'$, there is no value of $\Lambda$ with positive dust-density making the $\varphi_s$ coincide. We
therefore keep the experimentally favoured cosmological constant $\Lambda = 0.77\cdot 3 \text{ am}^{-2} \pm 20\%$ and fit the mass $M$ in order to achieve coincidence. The results are displayed in table 4.

Taking into account the Hubble velocity of the observer had already reduced the effect of the cosmological constant on the bending of light in Kottler’s solution: a 20 % increase of $\Lambda$ decreases the cluster mass by 20 % for the observer at rest, by only 10 % for the comoving observer. Now, with realistic velocity and masses in the universe, an increase of $\Lambda$ by 20 % only decreases the cluster mass by 5 %. The dependence on $\Lambda$ comes in step 0 through passage times and comoving distances, in step 2 through the inverse Jacobian and in step 4 through the Jacobian. But at the same time, the central value of the cluster mass has decreased even further, see table 5 and is now incompatible with observation.

|        | $M = 5.0^{+1.0}_{-1.0} \cdot 10^{13} M_\odot$ |
|--------|-----------------------------------------------|
| observation | |
| Kottler, static observer | $M = 4.7^{+2.3}_{-1.5} \cdot 10^{13} M_\odot$ |
| Kottler, comoving observer | $M = 3.0^{+1.1}_{-0.7} \cdot 10^{13} M_\odot$ |
| Einstein-Straus | $M = 1.8^{+0.4}_{-0.3} \cdot 10^{13} M_\odot$ |

Table 5: Mass estimates for the lensing cluster SDSS J1004+4112

There is quite a number of systems where the central mass computed from lensing is up to two times too large compared to the mass inferred from x-rays and it should be interesting to redo the present analysis for those systems. Also the computation of the time delay should be worth to be reconsidered in the Einstein-Straus solution.

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