CRITICAL VALUES OF RANKIN–SELBERG $L$-FUNCTIONS FOR $\text{GL}_n \times \text{GL}_{n-1}$ AND THE SYMMETRIC CUBE $L$-FUNCTIONS FOR $\text{GL}_2$

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With an appendix by Chandrasheela Bhagwat

1. Introduction and statements of results

In a previous article [31] an algebraicity result for the central critical value for $L$-functions for $\text{GL}_n \times \text{GL}_{n-1}$ over $\mathbb{Q}$ was proved assuming the validity of a nonvanishing hypothesis involving archimedean integrals. The purpose of this article is to generalize [31, Thm. 1.1] for all critical values for $L$-functions for $\text{GL}_n \times \text{GL}_{n-1}$ over any number field $F$ while using the period relations of [33] and some additional inputs as will be explained below. Thanks to some recent work of Binyong Sun [38], the nonvanishing hypothesis has now been proved. The results of this article are unconditional. Using such results for $\text{GL}_3 \times \text{GL}_2$, new unconditional algebraicity result for the special values of symmetric cube $L$-functions for $\text{GL}_2$ over $F$ have been proved. Previously, algebraicity results for the critical values of symmetric cube $L$-functions for $\text{GL}_2$ have been known only in special cases: see Garrett–Harris [9], Kim–Shahidi [21], Grobner–Raghuram [14], and Januszewski [19].

1.1. $L$-functions for $\text{GL}_n \times \text{GL}_{n-1}$. Suppose $\mathbb{A}_F$ is the ring of adèles of $F$. Let $\Pi$ be a regular algebraic cuspidal automorphic representation of $\text{GL}_n(\mathbb{A}_F)$. Such a representation contributes to the cuspidal cohomology of $G_n := \text{Res}_{F/\mathbb{Q}}(\text{GL}_n/F)$ with coefficients in a sheaf $\mathcal{M}_\mu$ attached to an algebraic irreducible representation $\mathcal{M}_\mu$ of $G_n$ with highest weight $\mu$. This information will be denoted as $\Pi \in \text{Coh}(G_n, \mu)$. Similarly, let $\Sigma$ be a regular algebraic cuspidal automorphic representation of $\text{GL}_{n-1}(\mathbb{A}_F)$, and let $\Sigma \in \text{Coh}(G_{n-1}, \lambda)$. Consider the Rankin–Selberg $L$-function $L(s, \Pi \times \Sigma)$ attached to such a pair of cohomological representations $(\Pi, \Sigma)$. Algebraicity results for the critical values of $L(s, \Pi \times \Sigma)$ are proved under a compatibility condition on the weights $\mu$ and $\lambda$.

Take a representation $\Pi \in \text{Coh}(G_n, \mu')$ as above; henceforth, as in [31], working with the dual weight $\mu'$ is only for convenience. Let $\Pi = \Pi_\infty \otimes \Pi_f$ be the usual decomposition of $\Pi$ into its archimedean part $\Pi_\infty$ and its finite part $\Pi_f$. The rationality field of $\Pi$ is denoted $\mathbb{Q}(\Pi)$; it is a number field. For a given weight $\mu$, cuspidal cohomology has a $\mathbb{Q}(\mu)$-structure (Clozel [5]) and hence the realization of $\Pi_f$ as a Hecke-summand in cuspidal cohomology in lowest possible degree has a $\mathbb{Q}(\Pi)$-structure. The choice of lowest possible degree—as will be explained below—is absolutely crucial for this paper. On the other hand, the Whittaker model $\mathcal{W}(\Pi_f)$ of the finite part of the representation admits a $\mathbb{Q}(\Pi)$-structure. By comparing these two $\mathbb{Q}(\Pi)$-structures, in a previous article with Shahidi [33], certain periods $p'(\Pi) \in \mathbb{C}^\times$ were defined and studied; here $\epsilon = (\epsilon_v)_{v \in S_r}$ is a collection of signs indexed by the set $S_r$ of real places of $F$. These signs can be arbitrary if $n$ is even, and are canonically determined by $\Pi_\infty$ if $n$ is odd; if the number of real places is $r_1$, then there are $2^{r_1}$ such periods of $\Pi$ if $n$ is even and only one period if $n$ is odd. For any $\sigma \in \text{Aut}(\mathbb{C})$,
one knows that $\sigma \Pi \in \text{Coh}(G_n, \rho')$; indeed, one defines periods $p^\sigma(\sigma \Pi)$ simultaneously for all the conjugates of $\Pi$. The collection of periods $\{p^\sigma(\sigma \Pi) : \sigma \in \text{Aut}(C)\}$ is well-defined as an element of $\left(\mathbb{Q}(\Pi) \otimes \mathbb{C}^*\right)/\mathbb{Q}(\Pi)^*$. When $F$ is totally real or a totally imaginary quadratic extension of a totally real field, then the constituents of the representation at infinity of $\sigma \Pi$ are, up to signs, a permutation of the constituents of $\Pi_\infty$ (see [8, Prop. 3.2]); however, in the case of a general number field this poses some additional difficulties involving a careful analysis at infinity. This issue is related to the set of possible weights that can support cuspidal cohomology for $\text{GL}_n/F$. We call them strongly pure weights. See the discussion involving purity in 2.3.4 and Defn. 2.5. Henceforth, we let $\mu \in X^+_0(T_n)$ to stand for a dominant integral strongly pure weight and we consider $\Pi \in \text{Coh}(G_n, \mu')$. The first main theorem of this article is the following:

**Theorem 1.1.** Let $\mu \in X^+_0(T_n)$ and $\Pi \in \text{Coh}(G_n, \mu')$ for $n \geq 2$. Similarly, let $\lambda \in X^+_0(T_{n-1})$ and $\Sigma \in \text{Coh}(G_{n-1}, \lambda')$. Assume that there is an integer $j$ such that $M_{\lambda+j} = M_\lambda \otimes \det^j$ appears in the restriction to $G_{n-1}$ of $M_\mu$, i.e.,

$$\{j \in \mathbb{Z} : \text{Hom}_{G_{n-1}}(M_\mu, M_{\lambda+j}) \neq (0)\}$$

is a non-empty set.

Let $s = \frac{1}{2} + \lambda \in \frac{1}{2} + \mathbb{Z}$ be critical for $L_f(s, \Pi \times \Sigma)$ which is the finite part of the Rankin–Selberg $L$-function attached to the pair $(\Pi, \Sigma)$. Then, there exist signs $\epsilon = (\epsilon_v)_{v \in S_\rho}$ and $\eta = (\eta_v)_{v \in S_\rho}$, with $\eta_v = (-1)^{n} \epsilon_v$, attached to the data $(\Pi_\infty, \Sigma_\infty)$ as in 2.3.3 such that:

1. If $n$ is even, then there exists $p^\sigma(\sigma(\mu + m, \lambda)) \in \mathbb{C}^*$, such that for any $\sigma \in \text{Aut}(\mathbb{C})$ we have

$$\sigma\left(\frac{L_f(\frac{1}{2} + m + \Pi \times \Sigma)}{p^\sigma(\sigma(\Pi))G_\Sigma(\omega_\Sigma)}p^{\sigma\eta}(\mu + m, \lambda)\right) = \frac{L_f(\frac{1}{2} + m, \sigma(\Pi) \times \sigma(\Sigma))}{p^\sigma(\sigma(\Pi))G_\Sigma(\omega_\Sigma)p^{\sigma\eta}(\mu + m, \sigma\lambda)},$$

where $\epsilon_m = (-1)^m$ and $G_\Sigma(\omega_\Sigma)$ is the Gauss sum attached to the central character of $\Sigma$. In particular, we have

$$L_f(\frac{1}{2} + m, \Pi \times \Sigma) \sim_{Q(\Pi, \Sigma)} p^\epsilon(\Pi)p^{\sigma(\Pi)}G_\Sigma(\omega_\Sigma)p^{\epsilon\eta}(\mu + m, \lambda),$$

where, by $\sim_{Q(\Pi, \Sigma)}$, we mean up to an element of the number field $Q(\Pi, \Sigma)$ which is the compositum of the rationality fields $Q(\Pi)$ and $Q(\Sigma)$ of $\Pi$ and $\Sigma$ respectively.

2. If $n$ is odd, then there exists $p^\sigma(\sigma(\mu, \lambda + m)) \in \mathbb{C}^*$, such that for any $\sigma \in \text{Aut}(\mathbb{C})$ we have

$$\sigma\left(\frac{L_f(\frac{1}{2} + m + \Pi \times \Sigma)}{p^\epsilon(\Pi)p^{\sigma(\Pi)}G_\Sigma(\omega_\Sigma)p^{\epsilon\eta}(\mu, \lambda + m)}\right) = \frac{L_f(\frac{1}{2} + m, \sigma(\Pi) \times \sigma(\Sigma))}{p^\epsilon(\sigma(\Pi)p^{\sigma(\Pi)}G_\Sigma(\omega_\Sigma)p^{\epsilon\eta}(\sigma\mu, \sigma\lambda + m)},$$

In particular,

$$L_f(\frac{1}{2} + m, \Pi \times \Sigma) \sim_{Q(\Pi, \Sigma)} p^\epsilon(\Pi)p^{\sigma(\Pi)}G_\Sigma(\omega_\Sigma)p^{\epsilon\eta}(\mu, \lambda + m).$$

For $F = \mathbb{Q}$, the above theorem or some variation of it has appeared in Kazhdan–Mazur–Schmidt [21], Mahnkopf [29], Kasten–Schmidt [23] and Raghuram [31]. Recently, Grobner–Harris [12] proved a similar result when $F$ is an imaginary quadratic field.

The basic idea of the proof of Theorem 1.1 involves interpreting the $L$-value as a Rankin–Selberg integral involving two carefully chosen cusp forms; see Prop. 2.1. This integral is then interpreted as a map in cohomology stemming from Poincaré duality; see the diagram in 2.5.1 and then see the main identity in Thm. 2.49. The reader is also referred to the introduction of [31] where this idea is explained in some detail.
We now address an important additional ingredient needed for the above generalization from $F = \mathbb{Q}$ to any number field. Kasten and Schmidt observed in [23, Thm. 2.3] that when $F = \mathbb{Q}$, if the weights $\mu$ and $\lambda$ satisfy (1.2), then for any $m \in \mathbb{Z}$ the point $\frac{1}{2} + m$ is critical for $L(s, \Pi \times \Sigma)$ if and only if $M^*_{\mu}$ contains $M_{\lambda + m}$. This is a purely local statement involving cohomological representations of $\text{GL}_n(\mathbb{R})$ and standard branching laws. What seems initially surprising, but rather natural after the fact, is that an identical statement (see Thm. [2.20]) holds true for cohomological representations of $\text{GL}_n(\mathbb{C})$ although the proof turns out to be combinatorially more challenging than for $\text{GL}_n(\mathbb{R})$; the main steps in the proof of Thm. [2.20] are contained in Lem. [2.23, Cor. 2.29 and Prop. 2.30]. (The reader should also look at Grobner–Harris [12, Lem. 4.7].) This particular observation, that we can deal with both real and complex cases on the same footing, was at the genesis of this article.

1.2. Relation with motivic periods and motivic $L$-functions. Let $M$ be a pure motive over $\mathbb{Q}$ with coefficients in a number field $\mathbb{Q}(M)$. Suppose $M$ is critical, then a celebrated conjecture of Deligne [6, Conj. 2.8] relates the critical values of its $L$-function $L(s, M)$ to certain periods that arise out of a comparison of the Betti and de Rham realizations of the motive. One expects a cohomological cuspidal automorphic representation $\Pi$ to correspond to a motive $M(\Pi)$; one of the properties of this correspondence is that the standard $L$-function $L(s, \Pi)$ is the motivic $L$-function $L(s, M(\Pi))$ up to a shift in the $s$-variable; see Clozel [5, Sect. 4]. With the current state of technology, it seems impossible to compare our periods $p'(\Pi)$ with Deligne’s periods $c^\pm(M(\Pi))$. However, one may ask if Thm. 1.1 is in any way compatible with Deligne’s conjecture.

One such compatibility is in terms of the internal structure of quantities in Thm. 1.1; one may ask if the periods of a tensor product motive decompose in a way suggested by our theorem. In the appendix, Chandrasheer Bhagwat gives a description of Deligne’s periods $c^\pm$ for the tensor product $M \otimes M'$, where $M$ and $M'$ are two pure motives over $\mathbb{Q}$ all of whose nonzero Hodge numbers are one, in terms of the periods $c^\pm$ and some other finer invariants attached to $M$ and $M'$ by Yoshida [12]. The main period relations in the appendix are in Thm. 4.6 Thm. 4.8 and Thm. 4.10. A comparison of our Thm. 1.1 with Bhagwat’s Thm. 4.6 makes it immediately clear that $p'(\Pi)$ and $c^\pm(M(\Pi))$ are two very different kind of periods. Turning this around, it is interesting to ask for a purely automorphic analogue of Bhagwat’s period relations; the problem is to describe Yoshida’s invariant $c_\Pi(M(\Pi))$ entirely in terms of $\Pi$. In general, this could be a hard problem, however, if the base field is an imaginary quadratic extension of $\mathbb{Q}$, and $\Pi$ is a base change from a unitary group, then the reader is referred to the period relation in Grobner–Harris [12, Thm. 6.7].

Another kind of compatibility of Thm. 1.1 with Deligne’s conjecture is to consider the behavior of $L$-values under twisting by characters. Blasius [2] and Panchishkin [30] have independently studied the behavior of Deligne’s periods upon twisting the motive by a Dirichlet character (more generally by Artin motives). Using Deligne’s conjecture, they predict the behavior of critical values of motivic $L$-functions upon twisting by Dirichlet characters. For a critical motive over $\mathbb{Q}$, assumed to be simple and of rank $2r$, this prediction looks like $L(m, M \otimes \chi_f) \sim \mathbb{Q}(M, \chi) L(m, M) G(\chi_f)^r$. Applying this to the tensor product motive $M(\Pi) \otimes M(\Sigma)$, which has rank $n(n-1)$, we have the following compatibility with Deligne’s conjecture:

**Corollary 1.3.** Let $\Pi \in \text{Coh}(G_n, \mu^n)$ and $\Sigma \in \text{Coh}(G_{n-1}, \lambda^n)$. Assume that the compatibility condition (1.2) holds. For any critical point $s = \frac{1}{2} + m$ of $L_f(s, \Pi \times \Sigma)$ and for any character $\chi : F^* \setminus \mathbb{A}_F^* \rightarrow \mathbb{C}^*$ of finite order, and any $\sigma \in \text{Aut}(\mathbb{C})$ we have

\[ \sigma \left( \frac{L_f(\frac{1}{2} + m, \Pi \times \Sigma \otimes \chi)}{G(\chi)^n(n-1)/2 L_f(\frac{1}{2} + m, \Pi \times \Sigma)} \right) = \frac{L_f(\frac{1}{2} + m, \Pi \times \Sigma \otimes \chi)}{G(\chi)^n(n-1)/2 L_f(\frac{1}{2} + m, \Pi \times \Sigma)} \cdot \]
It should be noted that Theorem 1.1 also gives some evidence toward a conjecture due to Gross [6] Conjecture 2.7(ii)] which says that the order of vanishing of a motivic $L$-function at a critical point is independent of which conjugate of the motive we are looking at, i.e., if $M$ is critical, then $\text{ord}_{s=M}L(s,\sigma,M)$ is independent of the embedding $\sigma: \mathbb{Q}(M) \to \mathbb{C}$. We are unable to say anything about the order of vanishing, however, it follows from our theorem that the property of vanishing is indeed independent of which particular conjugate of the representation we consider.

**Corollary 1.4.** Let $\Pi \in \text{Coh}(G_n,\mu^\nu)$ and $\Sigma \in \text{Coh}(G_{n-1},\lambda^\nu)$. Assume that the compatibility condition (1.2) holds. For any critical point $\frac{1}{2} + m$ of $L_f(s,\Pi \times \Sigma)$ and $\sigma \in \text{Aut}(\mathbb{C})$ we have
\[
L\left(\frac{1}{2} + m, \Pi \times \Sigma\right) = 0 \iff L\left(\frac{1}{2} + m, \sigma\Pi \times \sigma\Sigma\right) = 0.
\]

As in my paper with Wee Teck Gan [8, Thm. 1.1] implies an arithmeticity result for $GL_{n-1}$-periods of automorphic representations of $GL_n \times GL_{n-1}$. See [8, Thm. 7.1]. As explained in the introduction of that paper, this consequence is analogous to Gross’s conjecture but for automorphic periods. We state this consequence as the following

**Corollary 1.5.** Let $\Pi \in \text{Coh}(G_n,\mu^\nu)$ and $\Sigma \in \text{Coh}(G_{n-1},\lambda^\nu)$ be as in Thm. 1.1. Suppose $\mu$ and $\lambda$ satisfy (1.2). Consider the representation $\Pi \otimes \Sigma$ of $(G_n \times G_{n-1})(\mathbb{A})$. Let $\Delta G_{n-1}$ be the image of the diagonal embedding of $G_{n-1}$ in $G_n \times G_{n-1}$. Then
\[
\Pi \otimes \Sigma \text{ is } \Delta G_{n-1} \text{ distinguished } \implies \sigma \Pi \otimes \sigma \Sigma \text{ is } \Delta G_{n-1} \text{ distinguished } \forall \sigma \in \text{Aut}(\mathbb{C}).
\]

1.3. **Symmetric power $L$-functions for $GL_2$.** Let $\pi$ be a cohomological cuspidal automorphic representation of $GL_2$ over $F$. For any $r \geq 1$, let $\text{Sym}^r(\pi)$ denote the Langlands transfer of $\pi$ corresponding to the homomorphism $\text{Sym}^r: GL_2(\mathbb{C}) \rightarrow GL_{r+1}(\mathbb{C})$ of $L$-groups; $\text{Sym}^r(\pi)$ is conjecturally an isobaric automorphic representation of $GL_{r+1}$ over $F$. Such a transfer is known to exist for a general $\pi$ for $r \leq 4$ by the works of Gelbart–Jacquet, Kim, and Kim–Shahidi, and for all $r$ if $\pi$ is dihedral. We define the $r$-th symmetric power $L$-function as the standard $L$-function of the $r$-th symmetric transfer of $\pi$, i.e., $L(s,\text{Sym}^r(\pi)) = L(s,\text{Sym}^r(\pi))$; for more details and references, see [32 Sect. 3.1], [31 Sect. 5.1.1] and Sect. 3.1 below. If $\pi$ is dihedral then any symmetric power $L$-function is a product of $L$-functions of $GL_2$ and $GL_1$.

Suppose $\pi \in \text{Coh}(G_2,\mu^\nu)$ for $\mu \in X_{00}^*(T_2)$. Assume that $\pi$ is not of dihedral type; in particular, $\text{Sym}^2(\pi)$ is cuspidal. We define a weight $\text{Sym}^2(\mu) \in X_{00}^*(T_{r+1})$, and assuming automorphy and cuspidality of the $r$-th symmetric power transfer, we prove that $\text{Sym}^r(\pi) \in \text{Coh}(G_{r+1},\text{Sym}^r(\mu)^\nu)$; see Thm. 3.2 which generalizes [32 Thm. 5.5] and lends further evidence to the discussion in [32 Sect. 5.2] relating functoriality and the property of being cohomological. Similarly, we can define a weight $\det(\mu)$ so that $\omega_\pi \in \text{Coh}(G_1,\det(\mu)^\nu)$; see Sect. 3.1.2

In [31] Rankin–Selberg theory for $GL_n \times GL_{n-1}$ over $\mathbb{Q}$ was used to to get results on special values of odd symmetric power $L$-functions attached to modular forms. We completely work out this idea in the case of symmetric cube $L$-functions for cusp forms for $GL_2$ over any number field. The starting point is the factorization:
\[
L_f(s,\text{Sym}^2(\pi) \times \pi \otimes \xi) = L_f(s,\text{Sym}^3(\pi) \otimes \xi) \cdot L_f(s,\pi \otimes \omega_\pi \xi),
\]
where $\xi$ is a Hecke character of $F$ of finite order. We know the special values of the left hand side (resp., the second factor on the right hand side) by the $GL_3 \times GL_2$ case (resp., $GL_2 \times GL_1$ case) of Thm. 1.1 we deduce a result for the special values of symmetric cube $L$-functions. Take a
half-integer $\frac{1}{2} + m$ which is critical for $L_f(s, \Sym^3(\pi) \otimes \xi)$ and $L(s, \pi \otimes \omega_\pi \xi)$, then it is also critical for $L_f(s, \Sym^2(\pi) \times \pi \otimes \xi)$; see Sect. 3.2.2 for a description of the set of critical points. We write
\[
L_f\left(\frac{1}{2} + m, \Sym^3(\pi) \otimes \xi\right) = \frac{L_f\left(\frac{1}{2} + m, \Sym^2(\pi) \times \pi \otimes \xi\right)}{L_f\left(\frac{1}{2} + m, \pi \otimes \omega_\pi \xi\right)},
\]
provided $L\left(\frac{1}{2} + m, \pi \otimes \omega_\pi \xi\right) \neq 0$. For all critical points, except possibly for the central critical point, the denominator is indeed nonzero. If $w = w(\mu)$ is the purity weight of $\mu$, then $\pi = \pi^w \otimes |w/2|$ for a unitary cuspidal automorphic representation $\pi^w$, and $L\left(1 + m, \pi \otimes \omega_\pi \xi\right) = L\left(\frac{1}{2} + m + 3w/2, \pi^w \otimes \omega_\pi \xi\right)$.

Hence, $L\left(\frac{1}{2} + m, \pi \otimes \omega_\pi \xi\right) = 0$ can happen only when $m = -3w/2$. (This is the center of symmetry in the unitary case; note the necessary condition that $w$ should be even.) In this particular case, we use the main theorem of Rohrlich [35] and introduce another twisting character making the $L$-value nonzero, i.e., after relabeling if necessary, we may take $\xi$ to be such that $L\left((1 - 3w)/2, \pi \otimes \omega_\pi \xi\right) \neq 0$. Putting these together gives us the following

**Theorem 1.6.** Let $\pi \in \Coh(G_2, \mu')$ for $\mu \in \mathcal{X}_{00}(T_2)$. Assume that the pairs of weights $(\Sym^2(\mu), \mu)$ and $(\mu, \det(\mu))$ satisfy (1.2). Let $\xi : \mathbb{F}^\times \backslash \mathbb{A}_\mathbb{F} \to \mathbb{C}^\times$ be a character of finite order; if $w$ is even, then take $\xi$ such that $L\left((1 - 3w)/2, \pi \otimes \omega_\pi \xi\right) \neq 0$. Suppose $\frac{1}{2} + m$ is critical for $L_f(s, \Sym^3(\pi) \otimes \xi)$ and $L(s, \pi \otimes \omega_\pi \xi)$. Then we have:

\[
L_f\left(\frac{1}{2} + m, \Sym^3(\pi) \otimes \xi\right) \sim \epsilon_+ \left(\Sym^3(\pi)\right) \frac{p^{\epsilon_+ \epsilon_\xi(\pi)} - \epsilon_- \epsilon_\xi(\pi)}{p^{\epsilon_- \epsilon_\xi(\pi)}} \frac{\xi(2)}{\xi(\mu + m, \det(\mu))} g(\xi)^2 p^{\epsilon_+ \epsilon_\xi(\pi)} p^{\epsilon_- \epsilon_\xi(\mu + m, \det(\mu))},
\]

where, by $\sim$, we mean up to an element of $\mathbb{Q}(\pi, \xi)$; $\epsilon_+$ is the sign which is + everywhere; $\epsilon_- = -\epsilon_+$, and $\epsilon_\xi$ is the signature of $\xi$ as in Sect. 2.1.7. Furthermore, the ratio of the left hand side by the right hand side is equivariant under $\Aut(\mathbb{C})$.

The proof of the above theorem is given in Sect. 3.2.4. Furthermore, exactly as in [31, Sect. 5], we can get analogous results for higher odd symmetric power $L$-functions. For fifth and seventh symmetric power $L$-functions we would get partial results, and assuming Langlands’s functoriality, we would get conditional results for all odd symmetric power $L$-functions. We omit the details as any interested reader can proceed as in Sect. 3.2.4.

**Note to the reader:** Although we have tried to make this article self-contained, anyone who wishes to verify details, will need to keep copies of [31, 32] and [33] by his/her side.

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## 2. Arithmetic properties of Rankin–Selberg $L$-functions

### 2.1. Some notation and preliminaries

#### 2.1.1. The base field

Let $F$ be a number field of degree $d = [F : \mathbb{Q}]$ with ring of integers $\mathcal{O}$. For any place $v$ we write $F_v$ for the topological completion of $F$ at $v$. Let $S_\infty$ be the set of archimedean places of $F$. Let $S_\infty := S_r \cup S_c$, where $S_r$ (resp., $S_c$) is the set of real (resp., complex) places. Let $\mathcal{E}_F = \Hom(F, \mathbb{C})$ be the set of all embeddings of $F$ as a field into $\mathbb{C}$. There is a canonical surjective map $\mathcal{E}_F \to S_\infty$, which is a bijection on the real embeddings and real places, and identifies a pair of complex conjugate embeddings $\{\iota_v, \bar{\iota}_v\}$ with the complex place $v$. For each $v \in S_r$, we fix an isomorphism $F_v \cong \mathbb{R}$ which is canonical. Similarly for $v \in S_c$, we fix $F_v \cong \mathbb{C}$ given by (say) $\iota_v$; this choice is not canonical. Let $r_1 = |S_r|$ and $r_2 = |S_c|$; hence $d = r_1 + 2r_2$. If $v \notin S_\infty$, we let $\mathcal{O}_v$ be the
ring of integers of $F_v$, and $\mathfrak{o}_v$ it’s unique maximal ideal. Moreover, $\mathbb{A}_F$ denotes the ring of adeles of $F$ and $\mathbb{A}_{F,f}$ its finite part. The group of ideles of $F$ will be denoted $\mathbb{A}_F^\times$ and similarly, $\mathbb{A}_{F,f}^\times$ is the group of finite ideles. We will drop the subscript $F$ when talking about $\mathbb{Q}$. Hence, $\mathbb{A}$ is $\mathbb{A}_\mathbb{Q}$, etc. We use the local and global normalized absolute values and denote each of them by $| \cdot |$. Further, $\mathfrak{D}_F$ stands for the absolute different of $F$, i.e., $\mathfrak{D}_F^{-1} = \{ x \in F : Tr_{F/\mathbb{Q}}(x\mathcal{O}) \subset \mathbb{Z} \}$.

### 2.1.2. The groups $G_n \supset B_n \supset T_n \supset Z_n \supset S_n$. The algebraic group $GL_n / F$ will be denoted as $G_n$, and we put $G_n = R_{F/Q}(G_n)$. An $F$-group will be denoted by an underline and the corresponding $\mathbb{Q}$-group via Weil restriction of scalars will be denoted without the underline; hence for any $\mathbb{Q}$-algebra $A$ the group of $A$-points of $G_n$ is $G_n(A) = G_n(A \otimes_{\mathbb{Q}} F)$. Let $B_n = T_n U_n$ stand for the standard Borel subgroup of $G_n$ of all upper triangular matrices, where $U_n$ is the unipotent radical of $B_n$, and $T_n$ the diagonal torus. The center of $G_n$ will be denoted by $Z_n$. These groups define the corresponding $\mathbb{Q}$-groups $G_n \supset B_n = T_n U_n \supset Z_n$. Observe that $Z_n$ is not $\mathbb{Q}$-split, and we let $S_n$ be the maximal $\mathbb{Q}$-split torus in $Z_n$; we have $S_n \cong \mathbb{G}_m$ over $\mathbb{Q}$.

### 2.1.3. The groups at infinity. Note that

$$G_{n,\infty} := G_n(\mathbb{R}) = G_n(F \otimes_{\mathbb{Q}} \mathbb{R}) = \prod_{v \in S_{\infty}} GL_n(F_v) \cong \prod_{v \in S_r} GL_n(\mathbb{R}) \times \prod_{v \in S_c} GL_n(\mathbb{C}).$$

Let $K$ denote either $\mathbb{R}$ or $\mathbb{C}$. We have $Z_n(\mathbb{R}) = \prod_{v \in S_r} \mathbb{R}^\times \times \prod_{v \in S_c} \mathbb{C}^\times$, where each copy of $\mathbb{K}^\times$ consists of nonzero scalar matrices in the corresponding copy of $GL_n(\mathbb{K})$. The subgroup $S_n(\mathbb{R})$ of $Z_n(\mathbb{R})$ consists of $\mathbb{R}^\times$ diagonally embedded in $\prod_{v \in S_r} \mathbb{R}^\times \times \prod_{v \in S_c} \mathbb{C}^\times$. Let $C_{n,\infty} = \prod_{v \in S_r} O(n) \times \prod_{v \in S_c} U(n)$ be the maximal compact subgroup of $G_{n,\mathbb{R}}$, and let

$$K_{n,\infty} = S_n(\mathbb{R}) C_{n,\infty} \cong \mathbb{R}^\times \left( \prod_{v \in S_r} O(n) \times \prod_{v \in S_c} U(n) \right) \cong S_n(\mathbb{R})^0 C_{n,\infty}.$$

Let $K_{n,\infty}^0$ be the topological connected component of $K_{n,\infty}$. Hence

$$K_{n,\infty}^0 = S_n(\mathbb{R})^0 C_{n,\infty}^0 \cong \mathbb{R}^\times_+ \left( \prod_{v \in S_r} SO(n) \times \prod_{v \in S_c} U(n) \right).$$

For any topological group $\mathcal{G}$ we will let $\pi_0(\mathcal{G}) := \mathcal{G}/\mathcal{G}^0$ stand for the group of connected components. Inclusion of connected components induces the equality $\pi_0(K_{n,\infty}) = \pi_0(G_n(\mathbb{R}))$. Observe also that $\pi_0(K_{n,\infty}) \cong \prod_{v \in S_r} \{ \pm 1 \} \cong \prod_{v \in S_c} \{ \pm 1 \}$. The matrix $\delta_n = \text{diag}(-1, 1, \ldots, 1)$ represents the nontrivial element in $O(n)/SO(n)$, and if $n$ is odd, the scalar matrix $-1_n$ also represents this nontrivial element. We identify $\pi_0(G_n(\mathbb{R}))$ inside $G_n(\mathbb{R})$ via the $\delta_n$’s. The character group of $\pi_0(K_{n,\infty})$ is denoted $\pi_0(K_{n,\infty})$.

### 2.1.4. Lie algebras. The general notational principle we follow is that for a real Lie group $G$, we denote its Lie algebra by $\mathfrak{g}$ and the complexified Lie algebra by $\mathfrak{g}$, i.e., $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$. Hence, if $G$ is the Lie group $GL_n(\mathbb{R})$ then $\mathfrak{g}_0 = \mathfrak{gl}_n(\mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. On the other hand, if $G$ stands for the real Lie group $GL_n(\mathbb{C})$ then $\mathfrak{g}_0 = \mathfrak{gl}_n(\mathbb{C})$ as a $\mathbb{R}$-Lie algebra, and $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) \otimes \mathbb{C}$. With this notational scheme, we have $\mathfrak{g}_n$, $\mathfrak{b}_n$, $\mathfrak{t}_n$ and $\mathfrak{b}_n$ denoting the complexified Lie algebras of $G_n(\mathbb{R})$, $B_n(\mathbb{R})$, $T_n(\mathbb{R})$ and $K_{n,\infty}^0$ respectively. For example, $\mathfrak{g}_n = \prod_{v \in S_r} \mathfrak{gl}_n(\mathbb{C}) \times \prod_{v \in S_c} (\mathfrak{gl}_n(\mathbb{C}) \otimes \mathbb{R}) \mathbb{C}$.
2.1.5. **Finite-dimensional representations.** Consider $T_n(\mathbb{R}) = \mathcal{T}_n(F \otimes \mathbb{R}) \cong \prod_{v \in S_{\infty}} \mathcal{T}_n(F_v)$. We let $X^*(T_n)$ stand for the group of all algebraic characters of $T_n$, and let $X^+(T_n)$ stand for all those characters in $X^*(T_n)$ which are dominant with respect to $B_n$. A weight $\mu \in X^+(T_n)$ may be described as follows: $\mu = (\mu_v)_{v \in \mathbb{F}_p}$, where

- For $v \in S_p$, we have $\mu_v = (\mu_1^v, \ldots, \mu_n^v)$, $\mu_1^v \in \mathbb{Z}$, $\mu_1^v \geq \cdots \geq \mu_n^v$, and the character $\mu^v$ sends $t = \text{diag}(t_1, \ldots, t_n) \in \mathcal{T}_n(F_v)$ to $\prod_t t_i^{\mu_i^v}$.
- If $v \in S_{\mathbb{C}}$ then $\mu_v$ is the pair $(\mu_\infty^v, \mu_v^v)$, with $\mu_\infty^v = (\mu_1^v, \ldots, \mu_n^v)$, $\mu_1^v \geq \cdots \geq \mu_n^v$; likewise $\mu_\infty^v = (\mu_1^v, \ldots, \mu_n^v)$ and $\mu_1^v \geq \cdots \geq \mu_n^v$; the character $\mu^v$ is given by sending $t = \text{diag}(z_1, \ldots, z_n) \in \mathcal{T}_n(F_v)$ to $\prod_{i=1}^n z_i^{\mu_i^v}$, where $\bar{z}_i$ is the complex conjugate of $z_i$.

We also write $\mu = (\mu_v)_{v \in S_{\infty}}$, where $\mu_v = (\mu_\infty^v, \mu_v^v)$ for $v \in S_{\mathbb{C}}$.

For $\mu \in X^+(T_n)$, we define a finite-dimensional complex representation $(\rho_\mu, \mathcal{M}_\mu, \mathcal{C})$ of $G_n(\mathbb{R})$ as follows. For $v \in S_{\mathbb{C}}$, let $(\rho_\mu^v, \mathcal{M}_\mu^v, \mathcal{C})$ be the irreducible complex representation of $G_n(F_v) = \mathcal{G}_{n}\mathcal{L}_n(\mathbb{C})$ with highest weight $\mu_v$. For $v \in S_{\mathbb{C}}$, let $(\rho_\mu^v, \mathcal{M}_\mu^v, \mathcal{C})$ be the complex representation of the real algebraic group $G(F_v) = \mathcal{G}_{n}\mathcal{L}_n(\mathbb{C})$ defined as $\rho_\mu^v(g) = \rho_\mu^v(g) \otimes \rho_\mu^v(\overline{g})$; here $\rho_\mu^v$ (resp., $\rho_\mu^v$) is the irreducible representation of the complex group $\mathcal{G}_{n}\mathcal{L}_n(\mathbb{C})$ with highest weight $\mu^v$ (resp., $\bar{\mu}^v$). Now we let $\rho_\mu = \otimes_v \mathcal{G}_{n}\mathcal{L}_n(\mathbb{R})$.

Note that $\text{Aut}(\mathcal{C})$ acts on $X^*(T_n)$ as follows: if $\sigma \in \text{Aut}(\mathcal{C})$ and $\mu \in X^*(T_n)$ then $\sigma \cdot \mu \in X^*(T_n)$ is defined as: $\sigma \mu = (\sigma \mu_v)_{v \in \mathbb{F}_p}$, where $\sigma \mu_v := \mu_{\sigma^{-1} v}$. Define the rationality field $\mathbb{Q}(\mu)$ as the fixed field in $\mathbb{C}$ under all those automorphisms $\sigma$ which fix $\mu$. Consider the representation $(\rho_\mu, \mathcal{M}_\mu, \mathcal{C})$ of $G_n(\mathbb{R})$ of highest weight $\sigma \cdot \mu$. Consider $\mathcal{C}$ as a $(\mathbb{C}, \mathcal{C})$-bimodule, where the left module structure is via $\sigma$ and the right module structure is the usual multiplication in $\mathbb{C}$; denote this bimodule as $\mathcal{C}_\sigma$. Then the canonical map $t : \mathcal{M} \to \mathcal{M} \otimes \mathcal{C}_\sigma$ defined by $t(w) = w \otimes 1$ is a $\sigma$-linear isomorphism. Take $\mathcal{M} = \mathcal{M}_{\mu, \mathcal{C}}$ and denote by $(\sigma \rho_\mu, \sigma \mathcal{M}_{\mu, \mathcal{C}})$ the representation of $G_n(\mathbb{R})$ where a $g \in G_n(\mathbb{R})$ acts on $\sigma \mathcal{M}_{\mu, \mathcal{C}} = \mathcal{M}_{\mu, \mathcal{C}} \otimes \mathcal{C}_\sigma$ by $\rho_\mu(g) = t \circ \rho_\mu(g) \circ t^{-1}$. Then $\sigma \rho_\mu \simeq \rho_\mu$ as a representation of $G_n(F)$ (see [13] Lem. 7.1)) and this representation is defined over $\mathbb{Q}(\mu)$ which may be seen exactly as in Waldspurger [40] Prop.1.3. For any extension $E/\mathbb{Q}(\mu)$ we will let $\mathcal{M}_{\mu, E} = \mathcal{M}_{\mu, \mathbb{Q}(\mu)} \otimes \mathbb{Q}(\mu) E$ on which $G_n(F)$ acts via its action on the first factor.

2.1.6. **Automorphic representations.** Following Borel–Jacquet [4, §4.6], we say an irreducible representation of $G_n(A) = \mathcal{G}_{n}\mathcal{L}_n(A_F)$ is automorphic if it is isomorphic to an irreducible subquotient of the representation of $G_n(A)$ on its space of automorphic forms. We say an automorphic representation is cuspidal if it is a subrepresentation of the representation of $G_n(A)$ on the space of cusp forms $A_{\text{cusp}}(G_n(\mathbb{Q}) \backslash G_n(A)) = A_{\text{cusp}}(\mathcal{G}_{n}\mathcal{L}_n(F) \backslash \mathcal{G}_{n}\mathcal{L}_n(A_F))$. The subspace of cusp forms realizing a cuspidal automorphic representation $\pi$ will be denoted $V_\pi$. For an automorphic representation $\pi$ of $G_n(A)$, we have $\pi = \pi_{\infty} \otimes \pi_f$, where $\pi_{\infty}$ is a representation of $G_n(\mathbb{R})$, and $\pi_f = \otimes_{v \notin S_{\infty}} \pi_v$ is a representation of $G_n(A_f)$. The central character of $\pi$ will be denoted $\omega_{\pi}$.

2.1.7. **Algebraic Hecke characters.** (References: Deligne [7] §5], Schappacher [36] Chapter 0], Waldspurger [40] I.5] or Weil [11].) A continuous homomorphism $\omega : F^\times \backslash A_F^\times \to \mathbb{C}^\times$ is called a Hecke character of $F$. An element $\alpha = \sum_{v \in \mathbb{F}_p} a_v \omega_v$, with $a_v \in \mathbb{Z}$ is called an infinity type. A Hecke character $\omega$ is called an algebraic Hecke character of infinity type $\alpha$ if

- for $v \in S_p$, $\omega_v(x) = x^{a_v}$ for all $x \in \mathbb{R}_+^\times$
- for $v \in S_{\mathbb{C}}$, $\omega_v(z) = z^{a_v} \bar{z}^{\bar{a}_v}$ for all $z \in \mathbb{C}^\times$

Weil gave the appellation ‘characters of type $(A_0)$’ for such algebraic Hecke characters. The existence of an algebraic Hecke character $\omega$ with infinity type $\alpha$ implies the following purity constraint on $\alpha$:
(1) if \( S_r \) is not empty, i.e., if \( F \) has at least one real place, then the map from \( \mathcal{E}_F \to \mathbb{Z} \) given by \( \iota \mapsto a_{\iota} \) is constant; in this case, let \( w(\omega) := a_{\iota} \) for any \( \iota \).

(2) if \( S_r \) is empty, i.e., if \( F \) is a totally imaginary field, then the map from \( \mathcal{E}_F \times \text{Aut}(\mathbb{C}) \to \mathbb{Z} \) given by \( (\iota, \sigma) \mapsto a_{\sigma \iota} + a_{\iota} \) is constant; in this case, let \( w(\omega) := a_{\iota} + a_{\iota} \) for any \( \iota \).

In either case, we call \( w(\omega) \) the purity weight of \( \omega \).

Suppose that \( F \) has at least one real place, then we define the signature \( \epsilon_\omega \) of an algebraic Hecke character \( \omega \) as follows: By the purity constraint, the character \( \omega_0 := \omega |^{-w(\omega)} \) is a character of finite order. For \( v \in S_r \), define

\[
\epsilon_\omega = (-1)^{\epsilon(w(\omega))} \omega_0(-1).
\]

Now put \( \epsilon_\omega = (\epsilon_\omega_v)_{v \in S_r} \). The signature is an \( r_1 \)-tuple of signs indexed by real embeddings of \( F \).

For each finite place \( v \), and any smooth character \( \chi_v : F_v^\times \to \mathbb{C}^\times \), define the rationality field \( \mathbb{Q}(\chi_v) \) of \( \chi_v \) as the field obtained by adjoining the values of \( \omega_0 \) to \( \mathbb{Q} \). For an algebraic Hecke character \( \omega \), we define its rationality field \( \mathbb{Q}(\omega) \) as the compositum of the fields \( \mathbb{Q}(\omega_v) \) for all finite places \( v \) that are unramified for \( \omega \). It is a standard fact that \( \mathbb{Q}(\omega) \) is a number field, and as Weil notes in [31], the field \( \mathbb{Q}(\omega) \) need not contain the field \( F \).

### 2.1.8. Additive characters and Gauss sums

We fix an additive character \( \psi_\mathbb{Q} \) of \( \mathbb{Q}\backslash \mathbb{A} \), as in Tate’s thesis, namely, \( \psi_\mathbb{Q}(x) = e^{2\pi i \lambda(x)} \) with the \( \lambda \) as defined in [39, Sect. 2.2]. Next, we define a character \( \psi \) of \( F\backslash \mathbb{A}_F \) by composing \( \psi_\mathbb{Q} \) with the trace map from \( F \) to \( \mathbb{Q} \): \( \psi = \psi_\mathbb{Q} \circ \text{Tr}_{F/\mathbb{Q}} \). If \( \mathcal{O}_F = \prod_{v} \mathcal{O}_v \) with the product running over all prime ideals \( \mathfrak{p} \), and \( \psi = \otimes_v \psi_v \), then the conductor of \( \psi_v \) at a finite place \( v \) is \( \mathfrak{p}_v^{-\mathfrak{r}_v} \), i.e., \( \psi_v \) is trivial on \( \mathfrak{p}_v^{-\mathfrak{r}_v-1} \) and nontrivial on \( \mathfrak{r}_v^{-1} \). For any Hecke character \( \chi \) of \( F \), we define the Gauss sum \( G(\chi_f) \) of \( \chi_f \) exactly as in [31, Sect. 2]; this depends on choice of \( \psi \).

### 2.2. Rankin–Selberg \( L \)-functions: analytic aspects

This subsection is a brief summary of [31, Sect. 3.1]; see references therein for all the assertions made below.

#### 2.2.1. Rankin–Selberg zeta integrals for \( G_n \times G_{n-1} \)

Let \( \Pi \) (resp., \( \Sigma \)) be a cuspidal automorphic representation of \( G_n(\mathbb{A}) \) (resp., \( G_{n-1}(\mathbb{A}) \)). Let \( \phi \in V_\Pi \) and \( \phi' \in V_\Sigma \) be cusp forms. Consider

\[
I(s, \phi, \phi') = \int_{G_{n-1}(\mathbb{Q})\backslash G_{n-1}(\mathbb{A})} \phi(\iota(g))\phi'(g)\det(g)^{s-1/2} dg.
\]

The above integral converges for all \( s \in \mathbb{C} \). Suppose that \( w \in \mathcal{W}(\Pi, \psi) \) and \( w' \in \mathcal{W}(\Sigma, \psi^{-1}) \) are global Whittaker functions corresponding to \( \phi \) and \( \phi' \), respectively. We have

\[
I(s, \phi, \phi') = \Psi(s, w, w') := \int_{U_{n-1}(\mathbb{A})/G_{n-1}(\mathbb{A})} w(\iota(g))w'(g)\det(g)^{s-1/2} dg.
\]

The integral \( \Psi(s, w, w') \) converges for \( \text{Re}(s) \gg 0 \). Let \( w = \otimes w_v \) and \( w' = \otimes w'_v \), then \( \Psi(s, w, w') := \otimes \Psi_v(s, w_v, w'_v) \) for \( \text{Re}(s) \gg 0 \), where the local integral \( \Psi_v \) is given by a similar formula. Recall that the local integral \( \Psi_v(s, w_v, w'_v) \) converges for \( \text{Re}(s) \gg 0 \) and has a meromorphic continuation to all of \( \mathbb{C} \). We will choose the local Whittaker functions carefully so that the integral \( I(\frac{1}{2}, \phi, \phi') \) computes the special value \( L_f(\frac{1}{2}, \Pi \times \Sigma) \) up to quantities which are \( \text{Aut}(\mathbb{C}) \)-equivariant.

#### 2.2.2. Action of \( \text{Aut}(\mathbb{C}) \) on Whittaker models

Any \( \sigma \in \text{Aut}(\mathbb{C}) \) gives an element \( t_\sigma \in \mathbb{A}_F^\times \). Given \( w \in \mathcal{W}(\Pi_f, \psi_f) \), define \( \sigma w \in \mathcal{W}(\Pi_f, \psi_f) \) by \( \sigma w(g_f) = \sigma(w(t_{\sigma,n}g_f)) \), \( g_f \in G(\mathbb{A}_f) \), where \( t_{\sigma,n} = \text{diag}(t_{\sigma,-1}^{-(n-1)}, t_{\sigma,-2}^{-(n-2)}, \ldots, t_{\sigma,-1}, 1) \). For more details, see [31, Sect. 3.1] and [33, Sect. 3.2].
2.2.3. Normalized new vectors. For this paragraph, let $F$ be a non-archimedean local field, $O_F$ the ring of integers of $F,$ and $\mathcal{P}_F$ the maximal ideal of $O_F.$ Let $(\pi, V)$ be an irreducible admissible generic representation of $GL_n(F).$ Let $K_n(m)$ be the ‘mirahoric subgroup’ of $GL_n(O_F)$ consisting of all matrices whose last row is congruent to $(0, \ldots, 0, \ast)$ modulo $\mathcal{P}_F^m.$ Let $V_m := \{v \in V \mid \pi(k)v = \omega_k(v), \forall k \in K_n(m)\}.$ Let $f(\pi)$ be the least non-negative integer $m$ for which $V_m \neq \{0\}.$ One knows that $f(\pi)$ is the conductor of $\pi$ and that $V_{f(\pi)}$ is one-dimensional. Any vector in $V_{f(\pi)}$ is called a new vector of $\pi.$ Fix a nontrivial additive character $\psi$ of $F,$ and assume that $V = W(\pi, \psi)$ is the Whittaker model for $\pi.$ If $\pi$ is unramified, i.e., $f(\pi) = 0,$ then we fix a specific new vector called the spherical vector, denoted $w^{sp}_\pi,$ and normalized as $w^{sp}_\pi(1_n) = 1.$ More generally, for any $\pi,$ amongst all new vectors, there is a distinguished vector, called the essential vector, denoted $w^{es}_\pi,$ and characterized by the property that for any irreducible unramified generic representation $\rho$ of $GL_{n-1}(F)$ one has

$$\Psi(s, w^{es}_\pi, w^{sp}_\pi) = \int_{U_{n-1}(F) \backslash G_{n-1}(F)} w^{es}_\pi(i(g))w^{sp}_\pi(g) |\det(g)|^{s-1/2} \, dg = L(s, \pi \times \rho).$$

If $\pi$ is unramified then $w^{es}_\pi = w^{sp}_\pi.$ In general, given $\pi$ there exists $t_{\pi} \in T_n(F)$ such that a new vector for $\pi$ is nonvanishing on $t_{\pi}.$ Note that necessarily $t_{\pi} \in T_n^+(F),$ i.e., if $t_{\pi} = \text{diag}(t_1, t_2, \ldots, t_n)$ then $t_i t_{i+1}^{-1} \in O_F$ for all $1 \leq i \leq n - 1.$ We let $w^0_\pi$ be the new vector normalized such that $w^0_\pi(t_{\pi}) = 1.$ If $\pi$ is unramified then we may and will take $t_{\pi} = 1_n,$ and so $w^0_\pi = w^{es}_\pi = w^{sp}_\pi.$ For any $\sigma \in \text{Aut}(\mathbb{C})$ we may and will take $t_{\pi^\sigma} = t_{\pi}.$ Then $\sigma w^0_\pi = w^0_{\pi^\sigma}.$ Define $c_{\pi} \in \mathbb{C}^\times$ by $w^0_\pi = c_{\pi} w^{es}_\pi,$ i.e., $c_{\pi} = w^{es}_\pi(t_{\pi})^{-1}.$ For more details, see [31] Sect. 3.1.3] and the references therein.

2.2.4. Choice of Whittaker vectors and cusp forms. We now go back to global notation and choose global Whittaker vectors $w_{\Pi} = \otimes_v w_{\Pi, v} \in \mathcal{W}(\Pi, \psi)$ and $w_{\Sigma} = \otimes_v w_{\Sigma, v} \in \mathcal{W}(\Sigma, \psi^{-1})$ as follows. Let $S_v$ be the set of finite places $v$ where $\Sigma_v$ is unramified.

(1) If $v \notin S_\Sigma \cup S_\infty,$ we let $w_{\Pi, v} = w^0_{\Pi, v},$ and $w_{\Sigma, v} = w^{sp}_{\Sigma, v}.$

(2) If $v \in S_\Sigma,$ we let $w_{\Sigma, v} = w^0_{\Sigma_v},$ and let $w_{\Pi, v}$ be the unique Whittaker function whose restriction to $GL_{n-1}(F_v)$ is supported on $U_{n-1}(F_v) \backslash G_{n-1}(f(\Sigma_v))$ and on this double coset it is given by $w_v(u) w^0_{\Sigma_v} = \psi(u) \omega_{\Sigma_v}^{-1}(k_{n-1, n, 1}),$ for all $u \in U_{n-1}(F_v)$ and for all $k \in K_{n-1}(f(\Sigma_v)).$

(3) If $v \in S_\infty,$ we let $w_{\Pi, v}$ and $w_{\Sigma, v}$ be arbitrary nonzero vectors. (Later, these will be cohomological vectors.)

Let $w_{\Pi_f} = \otimes_{v \notin S_\infty} w_{\Pi, v}, w_{\Pi_\infty} = \otimes_{v \in S_\infty} w_{\Pi, v},$ and $w_{\Pi} = w_{\Pi_\infty} \otimes w_{\Pi_f}.$ Similarly, let $w_{\Sigma_f}, w_{\Sigma_\infty} and w_{\Sigma}.$ Let $\phi_{\Pi}$ (resp., $\phi_{\Sigma}$) be the cusp form corresponding to $w_{\Pi}$ (resp., $w_{\Sigma}).$

2.2.5. Integral representation of the central $L$-value. For $\Re(s) \gg 0,$ define $\Psi_\infty(s, w_{\Pi_\infty}, w_{\Sigma_\infty})$ to be $\prod_{v \in S_\infty} \Psi_v(s, w_{\Pi, v}, w_{\Sigma, v});$ this admits a meromorphic continuation to all of $\mathbb{C}.$ One identifies local $L$-factors $L_v(s, \Pi_v \times \Sigma_v),$ and proves that $\Psi_v(s, w_{\Pi, v}, w_{\Sigma, v})/L_v(s, \Pi_v \times \Sigma_v)$ is entire. Later we will be taking $s = 1/2$ to be a critical point, which says that $L_v(1/2, \Pi_v \times \Sigma_v)$ is regular for all $v \in S_\infty;$ hence, criticality of $s = 1/2$ gives that $\Psi_\infty(1/2, w_{\Pi_\infty}, w_{\Sigma_\infty})$ is finite.

**Proposition 2.1.** We have

$$I(\frac{1}{2}, \phi_{\Pi}, \phi_{\Sigma}) = \frac{\Psi_\infty(\frac{1}{2}, w_{\Pi_{\infty}}, w_{\Sigma_{\infty}}) \text{vol}(\Sigma) \prod_{v \in S_\Sigma} \text{vol}(K_{n-1}(f(\Sigma_v)))}{\prod_{v \in S_\Sigma} L(\frac{1}{2}, \Pi_v \times \Sigma_v)} L_f(\frac{1}{2}, \Pi \times \Sigma),$$

where $\text{vol}(\Sigma) = \prod_{v \in S_\Sigma} \text{vol}(K_{n-1}(f(\Sigma_v))) \in \mathbb{Q}^*.$

**Proof.** See [31] Prop. 3.1].
The main theorem on critical values of Rankin–Selberg $L$-functions follows by interpreting the above proposition in cohomology.

2.3. Automorphic cohomology.

2.3.1. Locally symmetric spaces. (See Harder [15] 1.1.) Let $K_f$ be an open-compact subgroup of $G_n(\mathbb{A}) = \text{GL}_n(\mathbb{A}_F)$, and let us write $K_f = \prod_p K_p$ where each $K_p$ is an open compact subgroup of $G_n(\mathbb{Q}_p)$ and for almost all $p$ we have $K_p = \prod_{v|p} \text{GL}_n(\mathcal{O}_v)$. Define the double-coset space

$$S_{K_f}^G = G_n(\mathbb{Q}) \backslash G_n(\mathbb{A}) / K_{n,\infty}^0 K_f = \text{GL}_n(F) \backslash \text{GL}_n(\mathbb{A}_F) / K_{n,\infty}^0 K_f.$$ 

For brevity, let $K = K_{n,\infty}^0 K_f$, and define $X = G_n(\mathbb{A}) / K = G_n(\mathbb{R}) / K_{n,\infty}^0 \times G_n(\mathbb{A}_f) / K_f$, i.e., $X$ is the product of the symmetric space $G_n(\mathbb{R}) / K_{n,\infty}^0$ with a totally disconnected space; any connected component of $X$ is of the form $X_g = G_n(\mathbb{R})^0(g_\infty, g_f) K_f / K$ where $g = (g_\infty, g_f) \in G_n(\mathbb{A})$ with $g_\infty \in \pi_0(G_n(\mathbb{R})) \subset G_n(\mathbb{R})$; for the last inclusion see [2.4.3]. The stabilizer of $X_g$ inside $G_n(\mathbb{Q})$ is $\Gamma_g := \{ \gamma \in G_n(\mathbb{Q}) : \gamma \in G_n(\mathbb{R})^0 \cap g_f K_f g_f^{-1} \}$. Any connected component of $S_{K_f}^G$ is of the form $\Gamma_g \backslash X_g \cong \Gamma_g \backslash G_n(\mathbb{R})^0 / K_{n,\infty}^0$. However, $\Gamma_g$ does not act freely on $X_g$ since $S_{n,\infty} \subset K_{n,\infty}$. Indeed, the stabilizer of every point in $X_g$ contains a congruence subgroup $\Delta$ of $S_n(G_F)$; this $\Delta$ is independent of the point in $X_g$, but the congruence conditions on $\Delta$ depend on $K_f$. The group $\Gamma_g / \Delta$ acts freely on $X_g$ and the quotient $\Gamma_g / \Delta$ is a locally symmetric space. We will abuse terminology and sometimes refer to $S_{K_f}^G$ as a locally symmetric space of $G_n$ with level structure $K_f$.

2.3.2. Sheaves on locally symmetric spaces. Given a dominant-integral weight $\mu \in X^+(T_n)$ and the associated representation $\mathcal{M}_{\mu,E}$, where $E$ is an extension of $\mathbb{Q}(\mu)$, we get a sheaf $\widetilde{\mathcal{M}}_{\mu,E}$ of $E$-vector spaces on $S_{K_f}^G$ as follows: Let $\pi : G_n(\mathbb{A}) / K_{n,\infty}^0 K_f \rightarrow S_{K_f}^G$ be the canonical projection. For any open subset $U$ of $S_{K_f}^G$ define the sections over $U$ by:

$$\widetilde{\mathcal{M}}_{\mu}(U) := \left\{ s : \pi^{-1}(U) \rightarrow \mathcal{M}_{\mu,E} \mid s \text{ is locally constant, and } s(\gamma u) = \rho_\mu(\gamma) s(u), \text{ for all } \gamma \in G_n(\mathbb{Q}) \text{ and } u \in \pi^{-1}(U) \right\}. $$

This defines a sheaf of complex vector spaces on $S_{K_f}^G$. Note that even if $\mathcal{M}_{\mu,E} \neq 0$ it is possible that the sheaf $\widetilde{\mathcal{M}}_{\mu,E} = 0$. (See Harder [15] 1.1.3.) Indeed, $\widetilde{\mathcal{M}}_{\mu,E} = 0$ unless the central character of $\rho_\mu$ has the infinity type of an algebraic Hecke character of $F$. Suppose $\mu = (\mu^e)_{e \in E_F}$, then define $a_\mu(\mu) := \sum_{i=1}^n \mu_i^e$. The central character of $\rho_\mu$ has the infinity type of an algebraic Hecke character of $F$ if and only if the map $e \mapsto a_\mu(\mu)$ satisfies either (1) or (2) of [2.1.7]. Henceforth, we will assume that $\mu$ satisfies this condition.

2.3.3. Cuspidal cohomology. We are interested in the sheaf cohomology groups

$$H^\bullet(S_{K_f}^G, \widetilde{\mathcal{M}}_{\mu,E}).$$

Here $\widetilde{\mathcal{M}}_{\mu,E}$ is the sheaf attached to the contragredient representation $\mathcal{M}_{\mu}^\vee$ of $\mathcal{M}_{\mu}$. If $\mu^\vee = -w_0(\mu)$, where $w_0$ is the element of the Weyl group of longest length, then $\mathcal{M}_{\mu}^\vee = \mathcal{M}_{w_0^\vee}$. (This dualizing is only for convenience and is dictated by personal tastes. Dualizing here, avoids some negative signs elsewhere.) It is convenient to pass to the limit over all open-compact subgroups $K_f$ and let $H^\bullet(S_{K_f}^G, \widetilde{\mathcal{M}}_{\mu,E}^\vee) := \lim_{\rightarrow K_f} H^\bullet(S_{K_f}^G, \mathcal{M}_{\mu,E}^\vee)$. There is an action of $\pi_0(G_{n,\infty}) \times G_n(\mathbb{A}_f)$ on $H^\bullet(S_{K_f}^G, \widetilde{\mathcal{M}}_{\mu,E}^\vee)$, and the cohomology of $S_{K_f}^G$ is obtained by taking invariants under $K_f$, i.e., $H^\bullet(S_{K_f}^G, \widetilde{\mathcal{M}}_{\mu,E}^\vee) = H^\bullet(S_{K_f}^G, \widetilde{\mathcal{M}}_{\mu,E}^\vee)^{K_f}$. 


Working at a transcendental level, i.e., taking \( E = \mathbb{C} \), we can compute the above sheaf cohomology via the de Rham complex, and then reinterpreting the de Rham complex in terms of the complex computing relative Lie algebra cohomology, we get the isomorphism:

\[
H^\bullet(S^{G_n}, \widetilde{M}_{\mu, \mathbb{C}}^\vee) \simeq H^\bullet(\mathfrak{g}_n, K_{n, \infty}^0; C^\infty(G_n(\mathbb{Q}) \backslash G_n(\mathbb{A})) \otimes \mathcal{M}_{\mu, \mathbb{C}}^\vee).
\]

With level structure \( K_f \) we have: \( H^\bullet(S^{G_n}_{K_f}, \widetilde{M}_{\mu, \mathbb{C}}^\vee) \simeq H^\bullet(\mathfrak{g}_n, K_{n, \infty}^0; C^\infty(G_n(\mathbb{Q}) \backslash G_n(\mathbb{A}))^{K_f} \otimes \mathcal{M}_{\mu, \mathbb{C}}^\vee). \)

The inclusion \( C^\infty_{\text{cusp}}(G_n(\mathbb{Q}) \backslash G_n(\mathbb{A})) \hookrightarrow C^\infty(G_n(\mathbb{Q}) \backslash G_n(\mathbb{A})) \) of the space of all smooth functions induces, via results of Borel [3], an injection in cohomology; this defines cuspidal cohomology:

\[
\begin{array}{ccc}
H^\bullet(S^{G_n}, \widetilde{M}_{\mu, \mathbb{C}}^\vee) & \rightarrow & H^\bullet(\mathfrak{g}_n, K_{n, \infty}^0; C^\infty(G_n(\mathbb{Q}) \backslash G_n(\mathbb{A})) \otimes \mathcal{M}_{\mu, \mathbb{C}}^\vee) \\
\uparrow & & \uparrow \\
H^\bullet_{\text{cusp}}(S^{G_n}, \widetilde{M}_{\mu, \mathbb{C}}^\vee) & \rightarrow & H^\bullet(\mathfrak{g}_n, K_{n, \infty}^0; C^\infty_{\text{cusp}}(G_n(\mathbb{Q}) \backslash G_n(\mathbb{A})) \otimes \mathcal{M}_{\mu, \mathbb{C}}^\vee)
\end{array}
\]

Using the usual decomposition of the space of cusp forms into a direct sum of cuspidal automorphic representations, we get the following fundamental decomposition of \( \mathcal{M}_{\mu, \mathbb{C}}^\vee \):

\[
H^\bullet_{\text{cusp}}(S^{G_n}, \widetilde{M}_{\mu, \mathbb{C}}^\vee) = \bigoplus \Pi H^\bullet(\mathfrak{g}_n, K_{n, \infty}^0; \Pi_{\infty} \otimes \mathcal{M}_{\mu, \mathbb{C}}^\vee) \otimes \Pi_f
\]

We say that \( \Pi \) contributes to the cuspidal cohomology of \( G_n \) with coefficients in \( \mathcal{M}_{\mu, \mathbb{C}}^\vee \), and we write \( \Pi \in \text{Coh}(G_n, \mu^\vee) \), if \( \Pi \) has a nonzero contribution to the above decomposition. Equivalently, if \( \Pi \) is a cuspidal automorphic representation whose representation at infinity \( \Pi_{\infty} \) after twisting by \( \mathcal{M}_{\mu, \mathbb{C}}^\vee \) has nontrivial relative Lie algebra cohomology. With a level structure \( K_f \), \( \langle 2.3 \rangle \) takes the form:

\[
H^\bullet_{\text{cusp}}(S^{G_n}_{K_f}, \widetilde{M}_{\mu, \mathbb{C}}^\vee) = \bigoplus \Pi H^\bullet(\mathfrak{g}_n, K_{n, \infty}^0; \Pi_{\infty} \otimes \mathcal{M}_{\mu, \mathbb{C}}^\vee) \otimes \Pi_f^{K_f}
\]

We write \( \Pi \in \text{Coh}(G_n, \mu^\vee, K_f) \) if a cuspidal automorphic representation \( \Pi \) contributes nontrivially to \( \langle 2.3 \rangle \). Note that \( \text{Coh}(G_n, \mu^\vee, K_f) \) is a finite set, and \( \text{Coh}(G_n, \mu^\vee) = \cup_{K_f} \text{Coh}(G_n, \mu^\vee, K_f) \).

2.3.4. Purity. Let \( \mu \in X^+(T_n) \) be a dominant integral weight satisfying the condition in \( \langle 2.3 \rangle \). Suppose \( \Pi \in \text{Coh}(G_n, \mu^\vee) \). The fact that \( \mu^\vee \) supports cuspidal cohomology places some restrictions on \( \mu \). First of all, essential unitarity of \( \Pi \), and in particular \( \Pi_{\infty} \) gives, via Wigner’s Lemma, essential self-duality of \( \mu \): there is an integer \( w(\mu) \) such that

\begin{enumerate}
  \item For \( v \in S_r \) and \( 1 \leq i \leq n \) we have \( \mu_i^{\vee} + \mu_{n-i+1}^{\vee} = w(\mu) \);
  \item For \( v \in S_c \) and \( 1 \leq i \leq n \) we have \( \mu_i^{\vee} + \mu_{n-i+1}^{\vee} = w(\mu) \).
\end{enumerate}

We will call such a weight \( \mu \) a pure weight and call \( w(\mu) \) the purity weight of \( \mu \). Let \( X^+_0(T_n) \) denote the set of dominant integral pure weights.

Furthermore, any \( \Pi \in \text{Coh}(G_n, \mu^\vee) \) satisfies a purity condition (Clozel [3, Lem. 4.9]) which is a translation to the automorphic side of the phenomenon that the associated motive \( M(\Pi) \) is pure—a condition on the Hodge types of \( M(\Pi) \). For \( v \in S_\infty \), let \( r_L(\Pi_v) \) stand for the Langlands parameter of \( \Pi_v \); it is an \( n \)-dimensional semi-simple representation of the Weil group \( W_{F_v} \in F_v; \mathbb{C}^\times \subset W_{F_v} \) as a subgroup of index at most 2. The representation \( r_L(\Pi_v)|_{\mathbb{C}^\times} \) is a sum of \( n \) characters \( z \mapsto z^p \), \( p, q \in \mathbb{Z} \). Purity says that there is an integer \( w(\Pi) \) such that for any \( v \in S_\infty \) all the exponents in \( r_L(\Pi_v)|_{\mathbb{C}^\times} \) satisfy \( p + q = w(\Pi) \). We will call \( w(\Pi) \) the purity weight of \( \Pi \), and it is related to the purity weight of \( \mu \) via \( w(\mu) = n - 1 + w(\Pi) \).
Finally, by Clozel’s theorem that cuspidal cohomology has a rational structure, we get
\[
\Pi \in \text{Coh}(G_n, \mu^\vee) \implies \sigma \Pi \in \text{Coh}(G_n, \sigma \mu^\vee), \quad \forall \sigma \in \text{Aut}(\mathbb{C}).
\]
In particular, \( \sigma \mu \) also satisfies the purity conditions (1) and (2) above. Note that \( w(\mu) = w(\sigma \mu) \).
Since this purity weight is going to appear frequently, we will often denote: \( w := w(\mu) = w(\sigma \mu) \).

**Definition 2.5.** Let \( \mu \in X^+(T_n) \) be a dominant integral weight satisfying the condition in (2.3.2). We say \( \mu \) is strongly pure if \( \sigma \mu \) is pure for all \( \sigma \in \text{Aut}(\mathbb{C}) \). Let \( X^+_0(T_n) \) stand for the set of dominant integral strongly-pure weights. (Note that if a dominant integral weight \( \mu \) is such that for all \( \sigma \in \text{Aut}(\mathbb{C}) \), the weight \( \sigma \mu \) satisfies the purity conditions (1) and (2) above, then necessarily, \( \mu \) satisfies the the condition in (2.3.2).)

For any \( F \), we have the following inclusions \( X^+_0(T_n) \subset X^+_0(T_n) \subset X^+(T_n) \subset X^+(T_n) \) and in general they are all strict inclusions. If \( F \) is a totally real field or a CM field (totally imaginary quadratic extension of a totally real field) then \( \mu \) is purely if and only if \( \mu \) is strongly pure. However, this is not true in general; it is easy to give an example of a weight \( \mu \) which is pure but not strongly pure when the base field is \( F = \mathbb{Q}(2^{1/3}) \) or its Galois closure. For any number field, one may see that there are strongly pure weights \( \mu \). Take an integer \( b \) and integers \( a_1 \geq a_2 \geq \cdots \geq a_n \) such that \( a_j + a_{n-j+1} = b \); then for each \( \mu \) there is a strongly pure weight \( \mu \). Such a weight may be called a parallel weight.

### 2.4. Archimedean considerations.

Let \( \mu \in X^+_0(T_n) \) be a strongly pure weight, and let \( \Pi \in \text{Coh}(G_n, \mu^\vee) \). The purpose of this section is to write down explicitly the representation \( \Pi_\infty \) at infinity in terms of \( \mu \); this is possible up to a sign; see Propositions 2.8, 2.10 and 2.13 below. We record a well-known conclusion—due to Clozel—on the possible degrees in which one has nontrivial cuspidal cohomology; see Prop. 2.14. This gives rise to a piquant numerology which ultimately permits us to give a cohomological interpretation to Rankin–Selberg \( L \)-values. Also, with local representations at hand, we compute the set of critical points of Rankin–Selberg \( L \)-functions. Since \( \Pi_\infty = \otimes_{v \in S_\infty} \Pi_v \), the problem of describing \( \Pi_\infty \) is a purely local one. We begin by taking up real and complex places separately.

#### 2.4.1. Cohomological representations of \( \text{GL}_n(\mathbb{R}) \).

(Fix a place \( v \in S_\rho \), and since \( v \) is fixed, we drop it from our notations just for this subsection. For example, \( \mu_v \) is just \( \mu = (\mu_1, \ldots, \mu_n) \), an \( n \)-tuple of integers with \( \mu_1 \geq \cdots \geq \mu_n \) and \( \mu_i + \mu_{n-i+1} = w \). We will now define the representation \( J_\mu \) if \( n \) is even; and two representations \( J_\mu^\pm \) if \( n \) is odd. For this we need to fix some notations for discrete series representations of \( \text{GL}_2(\mathbb{R}) \).

#### 2.4.1.1. Discrete series for \( \text{GL}_2(\mathbb{R}) \).

For any integer \( l \geq 1 \), let \( D_l \) stand for the discrete series representation with lowest non-negative \( K \)-type being the character \( (\cos \theta - \sin \theta) \mapsto e^{-i(l+1)\theta} \), and central character \( a \mapsto \text{sgn}(a)^{l+1} \). Note the shift from \( l \) to \( l + 1 \). The representation at infinity for a holomorphic elliptic modular cuspidal form of weight \( k \) is \( D_{k-1} \). It is well-known that discrete series representations of \( \text{GL}_2(\mathbb{R}) \), possibly twisted by a half-integral power of absolute value, have nontrivial cohomology. For brevity, let \( (g_2, K_2^0) := (g_2, \text{SO}(2) \mathbb{Z}_2(\mathbb{R})^0) \). For a dominant integral weight \( v = (a, b) \), with integers \( a \geq b \), the basic fact here is that there is a non-split exact sequence of \( (g_2, K_2^0) \)-modules:

\[
0 \to D_{a-b+1} \otimes |(a+b)/2 \to \text{Ind}_{B_2(\mathbb{R})}^{\text{GL}_2(\mathbb{R})} (X_{(a,b)}) |^{1/2} \otimes X_{(b,b)} |^{1/2} \to \mathcal{M}_{v, \mathbb{C}} \to 0,
\]
Moreover, $H^\bullet(g_2, K^0_2; (D_{a-b+1} \otimes \mathbb{R}^{(a+b)/2}) \otimes \mathcal{M}_{\nu, \mathbb{C}}^r) \neq 0$ if and only if $\bullet = 1$, and that dimension of $H^1(g_2, K^0_2; (D_{a-b+1} \otimes \mathbb{R}^{(a+b)/2}) \otimes \mathcal{M}_{\nu, \mathbb{C}}^r)$ is two, with both the characters of $O(2)/SO(2)$ appearing exactly once.  (For more details see RaghuRam-Tanabe [34] Sect. 3.1.)

2.4.1.2. Definition of $J_n$ when $n$ is even. Given $\mu = (\mu_1, \ldots, \mu_n)$ define an $n$-tuple $\ell = \ell(\mu) = (\ell_1, \ldots, \ell_n)$ by $\ell_i = 2\mu_i + 2\rho_n - w$, i.e., we have

$$\ell_i = 2\mu_i + n - 2i + 1 - w = \mu_i - \mu_{n-i+1} + n - 2i + 1, \quad 1 \leq i \leq n.$$  

Observe that $\ell_1 > \ell_2 > \cdots > \ell_{n/2} \geq 1$ and $\ell_i = -\ell_{n-i+1}$. Let $P$ be the $(2, \ldots, 2)$-parabolic subgroup of $GL_n(\mathbb{R})$, i.e., $P$ has the Levi quotient $L = \prod_{i=1}^{n/2} GL_2(\mathbb{R})$. Define the parabolically induced representation:

$$(2.7) \quad J_\mu := \text{Ind}_{P(\mathbb{R})}^{GL_n(\mathbb{R})} \left(D(\ell_1)|\det|^{w/2} \cdots D(\ell_{n/2})|\det|^{w/2}\right).$$

(This is a small change in notation from some of my earlier papers ([31], [32]) where $J_\mu$, following Mahnkopf [29], was denoted $J(w, \ell)$.) We will refer to the integers in $\ell$ as the cuspidal parameters of $J_\mu$. It is well-known that $J_\mu$ is irreducible, essentially tempered and generic (being fully induced from essentially discrete series), and $H^\bullet(g_n, SO(n); J_\mu \otimes \mathcal{M}_\mu^r) = H^\bullet(g_n, SO(n)\mathbb{R}_\nu^c; J_\mu \otimes \mathcal{M}_\mu^r) \neq 0$. The following proposition describes the local component for a real place of a global cohomological representation; it says that when $n$ is even, the highest weight $\mu_v$ determines the isomorphism class of $\Pi_v$.

**Proposition 2.8.** Let $\mu \in X_{00}^+(T_n)$ and $\Pi \in \text{Coh}(G_n, \mu^\vee)$. Suppose $n$ is even. Let $v \in S_r$ be a real place. Then $\Pi_v \cong J_{\mu_v}^\epsilon$.

2.4.1.3. Definition of $J_\mu^\epsilon$ when $n$ is odd. When $n$ is odd, for any sign $\epsilon$, we define a representation $J_\mu^\epsilon$ as follows. The cuspidal parameter $\ell$ is again defined by $\ell = 2\mu + 2\rho_n - w$. This time, let $P$ to be the $(2, \ldots, 2, 1)$-parabolic subgroup. Define

$$(2.9) \quad J_\mu^\epsilon := \text{Ind}_{P(\mathbb{R})}^{GL_n(\mathbb{R})} \left(D(\ell_1)|\det|^{w/2} \cdots D(\ell_{n/2})|\det|^{w/2} \epsilon|\det|^{w/2}\right).$$

It is well-known that $J_\mu^\epsilon$ is irreducible, essentially tempered, generic and that the relative cohomology group $H^\bullet(g_n, SO(n); J_\mu^\epsilon \otimes \mathcal{M}_\mu^r) = H^\bullet(g_n, SO(n)\mathbb{R}_\nu^c; J_\mu \otimes \mathcal{M}_\mu^r)$ is one-dimensional. Reverting to global notation, we have the following proposition which says that when $n$ is odd, the highest weight $\mu_v$ and the sign of the central character of $\Pi_v$ determine the isomorphism class of $\Pi_v$ as a representation of $GL_n(\mathbb{R})$.

**Proposition 2.10.** Let $\mu \in X_{00}^+(T_n)$ and $\Pi \in \text{Coh}(G_n, \mu^\vee)$. Suppose $n$ is odd. Let $v \in S_r$ be a real place. Then $\Pi_v \cong J_{\mu_v}^\epsilon$, where $\epsilon_v(-1) = \omega_{1_v}(-1) \cdot (-1)^{(n-1)/2}$.

The reader, who wishes to verify the above equality of signs, should note the following consequences of $n$ being odd: (1) The purity weight $w$ is even since $w = 2\mu(n+1)/2$, and (2) The cuspidal parameters are even since $\ell_i = 2\mu_i + n - 2i + 1 - w$.

2.4.2. Cohomological representations of $GL_n(\mathbb{C})$. (Reference: Clozel [5] §3.5.) Let $\mu \in X_{00}^+(T_n)$ and $\Pi \in \text{Coh}(G_n, \mu^\vee)$. For a complex place $v$, $\mu_v$ is a pair of $n$-tuples $(\mu^{ve}, \mu^{vw})$, where $\nu_v$ is a complex embedding corresponding to $v$ that has been noncanonically chosen and fixed; and $\ell_v$ is the conjugate embedding. Since $v \in S_c$ is fixed, we will drop it from our notations. Hence, $\mu = (\mu^e, \mu^f)$; we will further simplify our notation and write $\mu^e = (\mu_1, \ldots, \mu_n)$ and $\mu^f = (\mu_1^e, \ldots, \mu_n^e)$. Recall from Sect. 2.3.1 that the integers in the pair $(\mu^e, \mu^f)$ are related by: $\mu^e_1 + \mu^{ve}_{n-i+1} = w$. Hence, we have

$$\mu^e = (\mu_1, \ldots, \mu_n), \quad \text{and} \quad \mu^f = (w - \mu_n, \ldots, w - \mu_1).$$
Define the cuspidal parameters as:
(2.11) \[ a := \mu + \rho_n; \quad a = (a_1, \ldots, a_n) := \left( \mu_1 + \frac{n-1}{2}, \mu_2 + \frac{n-3}{2}, \ldots, \mu_n - \frac{(n-1)}{2} \right) \]
\[ b := w - \mu - \rho_n; \quad b = (b_1, \ldots, b_n) := \left( w - \mu_1 - \frac{n-1}{2}, w - \mu_2 - \frac{n-3}{2}, \ldots, w - \mu_n + \frac{(n-1)}{2} \right). \]

Now define the representation \( J_\mu \) to be induced from the Borel subgroup \( B(\mathbb{C}) \) of \( \text{GL}_n(\mathbb{C}) \) as:
(2.12) \[ J_\mu := \text{Ind}_{B(\mathbb{C})}^{\text{GL}_n(\mathbb{C})} \left( z^{a_1}z^{b_1} \cdots z^{a_n}z^{b_n} \right). \]

where, for any half-integers \( a, b \), by \( z^a z^b \) we mean the character of \( \mathbb{C}^\times \) which sends \( z \) to \( z^a z^b \). It is well-known that \( J_\mu \) is irreducible, essentially tempered, generic and \( H^\bullet(\mathfrak{gl}_n; U(n); J_\mu \otimes \mathcal{M}_\mu) \neq 0 \).

Reverting to global notation, we have:

**Proposition 2.13.** Let \( \mu \in X_{00}^+(T_n) \) and \( \Pi \in \text{Coh}(G_n, \mu^\vee) \). Let \( v \in S_e \), i.e., \( v \) is a complex place. Then \( \Pi_v \cong J_{\mu_v} \).

2.4.3. The cuspidal range. In this subsection we record well-known bounds for the possible degrees in which there can be nonzero cuspidal cohomology. These bounds depend only on the rank \( n \) of \( \text{GL}_n \), and the numbers \( r_1 \) and \( r_2 \) of real and complex embeddings of \( F \).

**Proposition 2.14.** Define the following numbers:

The bottom degrees:
\[ b_n^R := \left[ \frac{n^2}{4} \right], \quad b_n^C := n(n-1)/2. \]

The top degrees:
\[ t_n^R = b_n^R + \left\lfloor \frac{n-1}{2} \right\rfloor, \quad t_n^C = b_n^C + n - 1. \]

Now define the bottom degree and top degree for the group \( G_n = R_{F/\mathbb{Q}}(\text{GL}_n/F) \) as:
\[ b_n^F = r_1 t_n^R + r_2 t_n^C, \quad t_n^F = r_1 t_n^R + r_2 t_n^C, \quad t_n^F = t_n^F + |F : \mathbb{Q}| - 1. \]

Let \( \mu \in X_{00}^+(T_n) \). Then \( H^\bullet_{	ext{cuspid}}(S^\mu_{G_n}, \mathcal{M}_\mu) \neq 0 \iff b_n^F \leq \bullet \leq t_n^F \).

**Proof.** Given \( \mu \in X_{00}^+(T_n) \), and a \( \Pi \in \text{Coh}(G_n, \mu^\vee) \), we know that \( \Pi_\infty = \otimes_{v \in S_\infty} J_{\mu_v}^\vee \). (The symbol \( \epsilon_v \) is a nonempty condition only for odd \( v \) and \( v \in S_p \).) For \( v \in S_\infty \), let \( \tilde{G}_{n,v} = \{ g_v \in \text{GL}_n(F_v) : |\det(g_v)| = 1 \} \) and let \( \tilde{\mathfrak{g}}_{n,v} \) be the Lie algebra of \( \tilde{G}_{n,v} \). Note that \( \mathfrak{g}_{n,v} = \tilde{\mathfrak{g}}_{n,v} \oplus \mathfrak{g}_{n,v} \) and summing over \( v \in S_\infty \), we have \( \mathfrak{g}_n = \tilde{\mathfrak{g}}_n + \mathfrak{g}_n \). By Wigner’s Lemma we have:
\[ H^\bullet(\mathfrak{g}_n, K_{n,\infty}^0; \Pi_\infty \otimes \text{M}^\vee_{\mu,C}) = H^\bullet(\tilde{\mathfrak{g}}_n, C_{n,\infty}^0; \Pi_\infty \otimes \text{M}^\vee_{\mu,C}) \otimes \wedge^\bullet(\mathfrak{g}_n/\mathfrak{g}_n). \]

The term \( \wedge^\bullet(\mathfrak{g}_n/\mathfrak{g}_n) \) accounts for the difference between \( t_n^F \) and \( t_n^F \). By the Künneth formula we get
\[ H^q(\tilde{\mathfrak{g}}_n, C_{n,\infty}^0; \Pi_\infty \otimes \text{M}^\vee_{\mu,C}) = \bigotimes_{q \leq q} H^q(\tilde{\mathfrak{g}}_n, C_{n,\infty}^0; \Pi_\infty \otimes \text{M}^\vee_{\mu,C}). \]

From Clozel [5] Lemme 3.14], we get \( H^q(\tilde{\mathfrak{g}}_n, C_{n,\infty}^0; \Pi_\infty \otimes \text{M}^\vee_{\mu,C}) \neq 0 \) if and only if \( b_n^F \leq q \leq t_n^F \). \( \square \)

Motivated by the numerical coincidence to be discussed below, we will focus our attention exclusively on cohomology in degree \( \bullet = b_n^F \). (In contrast, see my paper with Grobner [14] where we considered top-degree cuspidal cohomology.)

**Corollary 2.15.** Let \( \Pi \in \text{Coh}(G_n, \mu^\vee) \) for \( \mu \in X_{00}^+(T_n) \).
for any open compact subgroup \( R \) of the connected component of the identity in the maximal compact subgroup where \( b \) component in cuspidal cohomology in degree \( \Pi \nabla \) where the right hand side is the \( \varpi \) interpretation to the Rankin–Selberg theory for \( G \) terms of the space \( G \) is an open compact subgroup of \( f \) with \( \Pi \nabla \) appears \( 2^{\rho_1} \) times in cuspidal cohomology in degree \( b_n^\varpi \).

(2) If \( n \) is odd then \( H^b_n(\mathfrak{g}_n, K_{n,\infty}^0; \Pi_\infty \otimes \mathcal{M}_{\mu, \mathcal{C}}) \) is one-dimensional and the character of \( \pi_0(K_{n,\infty}) \) which appears is denoted as \( \epsilon_{\Pi_\infty} \). In this case, cuspidal cohomology decomposes as

\[
H^b_n(S^G_n, \overline{\mathcal{M}}_{\mu}^\varpi) = \bigoplus_{\Pi} \epsilon_{\Pi_\infty} \otimes \Pi_f,
\]

where \( \Pi \) runs over \( \text{Coh}(G_n, \mu^\varpi) \); and each \( \Pi_f \) appears once.

As in \([33]\), we say that an \( r_1 \)-tuple of signs \( \epsilon = (\epsilon_v)_{v \in S_r} \) is permissible for \( \mu \) if \( \epsilon = \epsilon_{\Pi_\infty} \) for some \( \Pi \in \text{Coh}(G_n, \mu^\varpi) \) when \( n \) is odd, and is of any possible \( 2^{\rho_1} \) signatures when \( n \) is even. For any \( n \), given \( \mu \in X^+_0(T_n), \Pi \in \text{Coh}(G_n, \mu^\varpi) \), and given a permissible signature \( \epsilon \) for \( \mu \), the \( \epsilon \otimes \Pi_f \) isotypic component in cuspidal cohomology in degree \( b_n^\varpi \), via the isomorphism in \((2.2)\), can be expressed in terms of the space \( V_\Pi \) of cusp forms realizing \( \Pi \) as

\[
H^b_n(S^G_n, \overline{\mathcal{M}}_{\mu}^\varpi)(\epsilon \otimes \Pi_f) = H^b_n(\mathfrak{g}_n, K_{n,\infty}^0; V_\Pi \otimes \mathcal{M}_{\mu, \mathcal{C}})(\epsilon),
\]

where the right hand side is the \( \epsilon \)-isotypic component in \( H^b_n(\mathfrak{g}_n, K_{n,\infty}^0; V_\Pi \otimes \mathcal{M}_{\mu, \mathcal{C}}) \) for the action of \( K_{n,\infty}/K_{n,\infty}^0 \).

2.4.4. **A numerical coincidence.** We record a relation between the numbers \( b_n^\varpi \) and \( b_{n-1}^\varpi \) and the dimension of a locally symmetric space for \( G_{n-1} \) which is crucial for giving a cohomological interpretation to the Rankin–Selberg theory for \( \text{GL}_n \times \text{GL}_{n-1} \). Define

\[
\tilde{S}_{K_f}^{G_n} := G_n(\mathbb{Q}) \backslash G_n(\mathbb{A})/C_{n,\infty}^0 K_f = \text{GL}_n(F) \backslash \text{GL}_n(\mathbb{A})/C_{n,\infty}^0 K_f
\]

where \( K_f \) is an open compact subgroup of \( G_n(\mathbb{A}_f) \), and \( C_{n,\infty}^0 = \prod_{v \in S_r} \text{SO}(n) \times \prod_{v \in S_r} \text{U}(n) \) is the connected component of the identity in the maximal compact subgroup \( C_{n,\infty}^0 \) of \( G_n(\mathbb{R}) \). Since \( K_{n,\infty}^0 = S_n(\mathbb{R})^0 C_{n,\infty}^0 \), see Sect.2.1.3 for our notations, we get a canonical fibration \( \phi \) given by:

\[
\begin{align*}
S_{K_f}^{G_n} & = G_n(\mathbb{Q}) \backslash G_n(\mathbb{A})/C_{n,\infty}^0 K_f \\
\tilde{S}_{K_f}^{G_n} & = G_n(\mathbb{Q}) \backslash G_n(\mathbb{A})/K_{n,\infty}^0 K_f \\

\phi &
\end{align*}
\]

**Proposition 2.16.** Let \( n \geq 2 \). Let \( b_n^\varpi \) be the bottom degree for \( G_n \) as defined in Prop.2.14 Then

\[
b_n^\varpi + b_{n-1}^\varpi = \dim(\tilde{S}_{K_f}^{G_{n-1}}),
\]

for any open compact subgroup \( R_f \) of \( G_{n-1}(\mathbb{A}_f) \).

**Proof.** Note that

\[
\dim(\tilde{S}_{R_f}^{G_{n-1}}) = \dim(G_n(\mathbb{R})^0/C_{n,\infty}^0)
\]

\[
= r_1 \cdot \dim(\text{GL}_n(\mathbb{R})^0/\text{SO}(n-1)) + r_2 \cdot \dim(\text{GL}_n(\mathbb{C})/\text{U}(n-1)).
\]
The proof follows if we check that
\[ b_n^\mathbb{R} + b_{n-1}^\mathbb{R} = \dim(\text{GL}_{n-1}(\mathbb{R})^0/\text{SO}(n-1)), \]
\[ b_n^\mathbb{C} + b_{n-1}^\mathbb{C} = \dim(\text{GL}_{n-1}(\mathbb{C})/U(n-1)). \]
These are easy to verify using the definitions of \( b_n^\mathbb{R} \) and \( b_n^\mathbb{C} \); for example, \( b_n^\mathbb{R} + b_{n-1}^\mathbb{R} = n(n-1)/2 + (n-1)(n-2)/2 = (n-1)^2 = \dim(\text{GL}_{n-1}(\mathbb{C})/U(n-1)). \)

For \( F = \mathbb{Q} \) and \( n = 3 \), this kind of numerology is apparent in Schmidt [37] and Mahnkopf [28]. For \( F = \mathbb{Q} \) and general \( n \), see Kazhdan, Mazur and Schmidt [24, Table on p. 100]; in this situation, the numerology was cleverly exploited by Mahnkopf [29], and so was used in my paper [31]. For general \( n \) and \( F \), this was recently used by Januszewski [19] in his study of modular symbols.

2.4.5. Critical points and compatibility of coefficient systems.
2.4.5.1. Definition of the critical set. Consider the Rankin–Selberg \( L \)-function \( L(s, \Pi \times \Sigma) \) where \( \Pi \) (resp., \( \Sigma \)) is a cuspidal automorphic representation \( \text{GL}_n(\mathbb{A}_F) \) (resp., \( \text{GL}_{n-1}(\mathbb{A}_F) \)).

**Definition 2.17.** We say that \( s_0 = \frac{1}{2} + m \in \frac{1}{2} + \mathbb{Z} \) is critical for \( L(s, \Pi \times \Sigma) \) if both \( L_\infty(s, \Pi_\infty \times \Sigma_\infty) \) and \( L_\infty(1 - s, \Pi_\infty \times \Sigma_\infty) \) are regular at \( s = s_0 \), i.e., both the \( L \)-factors at infinity on either side of the functional equation are holomorphic at \( s = s_0 \).

By definition, we only look at half-integers, i.e, the critical set is a subset of \( \frac{1}{2} + \mathbb{Z} \). This has to do with the so-called motivic normalization: that if a cuspidal representation \( \Theta \) of \( \text{GL}_r \) corresponds to a motive \( M \) then under this correspondence, \( L(s + \frac{1-r}{2}, \Theta) = L(s, M) \). On the motivic side, one always looks for critical points amongst integers; see Deligne [4]. Transcribing to the automorphic side, one looks for critical points amongst \( \frac{1-r}{2} + \mathbb{Z} \); in particular, if \( r \) is even then we look for critical points in \( \frac{1}{2} + \mathbb{Z} \). In our situation, assuming Langlands’s functoriality, we have \( \Theta = \Pi \boxtimes \Sigma \), which is (usually) a cuspidal representation of \( \text{GL}_r(\mathbb{A}_F) \) with \( r = n(n-1) \); in particular \( r \) is even; hence the critical set for \( L(s, \Pi \times \Sigma) \) consists of only half-integers. Another easy point to note is that given a particular half-integer \( s_0 = \frac{1}{2} + m \), to check whether \( s_0 \) is critical or not is an entirely local calculation because \( L_\infty(s_0, \Pi_\infty \times \Sigma_\infty) = \prod_{v \in S_\infty} L_v(s_0, \Pi_v \times \Sigma_v) \) and local \( L \)-factors are nonvanishing everywhere.

2.4.5.2. Branching rule for the pair \((\text{GL}_n(\mathbb{C}), \text{GL}_{n-1}(\mathbb{C}))\). Working with local notations, let \( \mu = (\mu_1, \ldots, \mu_n) \) be a dominant integral weight for \( \text{GL}_n(\mathbb{C}) \) and \( M_\mu \) be the irreducible finite-dimensional representation of the algebraic Lie group \( \text{GL}_n(\mathbb{C}) \) with highest weight \( \mu \). Similarly, we have \( \lambda = (\lambda_1, \ldots, \lambda_{n-1}) \) and \( M_\lambda \) for \( \text{GL}_{n-1}(\mathbb{C}) \). The following branching rule is well-known (see, for example, Goodman-Wallach [11, Thm. 8.1.1]):

**Proposition 2.18.** The representation \( M_\lambda \) appears in the restriction to \( \text{GL}_{n-1}(\mathbb{C}) \) of the representation \( M_\mu \), i.e., \( \text{Hom}_{\text{GL}_{n-1}(\mathbb{C})}(M_\lambda, M_\mu) \neq 0 \) if and only if
\[ \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_n. \]
In this situation, \( M_\lambda \) appears with multiplicity one in \( M_\mu \). The conditions on the weights \( \mu \) and \( \lambda \) captured by the above inequalities will be denoted \( \mu \succ \lambda \), and we say \( \mu \) interlaces \( \lambda \).

Remember that we dualized the coefficient systems. Let us restate the above proposition in the form that we will need it later on. Given \( \mu = (\mu_1, \ldots, \mu_n) \) recall that \( \mu^\vee = (-\mu_n, \ldots, -\mu_1) \), and that the contragredient representation \( M^\vee_\mu \) of \( M_\mu \) is \( M_{\mu^\vee} \). Similarly, for \( \lambda \) and \( M_\lambda \).
These inequalities are then used to verify that suppose \( \mu \succ \lambda \), i.e.,
\[-\mu_n \geq \lambda_1 \geq -\mu_{n-1} \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq -\mu_1.\]
In this situation, \( \text{Hom}_{\text{GL}_{n-1}(\mathbb{C})}(\mathcal{M}_\mu^\vee \otimes \mathcal{M}_\lambda^\vee, \mathbb{C}) \neq 0 \) if and only if \( \mu^\vee \succ \lambda \), i.e.,
\[\text{verify that (2.21).}\]

**Proof.** \( \text{Hom}_{\text{GL}_{n-1}(\mathbb{C})}(\mathcal{M}_\mu^\vee \otimes \mathcal{M}_\lambda^\vee, \mathbb{C}) = \text{Hom}_{\text{GL}_{n-1}(\mathbb{C})}(\mathcal{M}_\mu^\vee, \mathcal{M}_\lambda). \)

We will also write \( \mathcal{M}_\lambda \hookrightarrow \mathcal{M}_\mu \) to say that \( \mathcal{M}_\lambda \) appears (with multiplicity one) in \( \mathcal{M}_\mu \). Given \( \lambda \) and an \( m \in \mathbb{Z} \), the representation \( \mathcal{M}_\lambda \otimes \text{det}^m \) corresponds to the weight \( \lambda + m \). In global notations; given a weight \( \lambda = (\lambda^i)_{i \in \mathcal{E}_F} \), the weight \( \lambda + m \) is simply \( (\lambda^i + m)_{i \in \mathcal{E}_F} \).

### 2.4.5.3. Critical set and compatibility

The main result of this section is the following theorem which relates the critical set to a condition on the coefficient systems.

**Theorem 2.20.** Let \( \mu \in X_0^+(T_n) \) and \( \Pi \in \text{Coh}(G_n, \mu^\vee) \). Similarly, let \( \lambda \in X_0^+(T_{n-1}) \) and \( \Sigma \in \text{Coh}(G_{n-1}, \lambda^\vee) \). Assume that there is an integer \( m_0 \) such that
\[M_{\lambda+m_0} \hookrightarrow M_\mu^\vee, \text{ that is } \mu^\vee \succ \lambda + m_0.\]
Then we have
\[\{ m \in \mathbb{Z} \mid \mu^\vee \succ \lambda + m \} = \{ m \in \mathbb{Z} \mid \frac{1}{2} + m \text{ is critical for } L(s, \Pi \times \Sigma) \}.\]
(In particular, the critical set is a finite set.)

**Proof.** Let us begin by noting that the assertion is of a purely local nature. If \( m \in \mathbb{Z} \) then \( \mu^\vee \succ \lambda + m \)
if and only if for every \( v \in S_\infty \) we have \( \mu^\vee \succ \lambda + m \). On the other hand, \( \frac{1}{2} + m \) is critical for
\( L(s, \Pi \times \Sigma) \) if and only if both \( L_v(s, \Pi \times \Sigma_v) \) and \( L_v(1-s, \Pi_v \times \Sigma_v^\vee) \) are holomorphic at \( s = \frac{1}{2} + m \)
for every \( v \in S_\infty \). We will consider the real and complex places separately.

For \( v \in S_r \), this has been observed by Kasten and Schmidt [23 Theorem 2.3]|. What is remarkable
is that their observation for a real place in fact goes through for a complex place also; see also
Grobner–Harris [12, Lem. 4.7]. We very briefly summarize the proof of [23 Theorem 2.3], so that
the reader may compare the similarities and the differences in the combinatorics in the real and
complex cases. Fix a real place \( v \in S_r \) and we drop it from the notations. The hypothesis \( \mu^\vee \succ \lambda \)
means
\[\mu_n \geq \lambda_1 \geq \mu_n \geq \cdots \geq \lambda_{n-1} \geq -\mu_1.\]
If \( \ell \) and \( \ell^\prime \) are the cuspidal parameters of \( \mu \) and \( \lambda \) (see 2.4.11), then define the cuspidal width between
\( \Pi \) and \( \Sigma \) as:
\[c_{\Pi, \Sigma} := \min\{|\ell_i - \ell^\prime_j| \mid 1 \leq i \leq n, 1 \leq j \leq n - 1\} \]
Verifies that \( s = 1/2 \) is critical if and only if
\[-(c_{\Pi, \Sigma} - 1) \leq (w + w^\prime) \leq c_{\Pi, \Sigma} - 1.\]
Verifies that (2.21) \( \implies \) (2.22), i.e., \( \mu^\vee \succ \lambda \) implies that \( s = 1/2 \) is critical. During this verification
one observes that the condition \( \mu^\vee \succ \lambda \) in fact implies
\[\ell_1 > \ell^\prime_1 > \ell_2 > \ell^\prime_2 > \cdots > \ell^\prime_{n-1} > \ell^\prime_n.\]
These inequalities are then used to verify that suppose \( \mu^\vee \succ \lambda + m_0 \) for some \( m_0 \) and \( \frac{1}{2} + m \) is
critical for \( L(s, \Pi \times \Sigma) \) then \( \mu^\vee \succ \lambda + m \). For more details the reader is referred to [23]. Now we
take the complex case, which is combinatorially more complicated.

For the rest of this proof, we assume \( v \in S_c \), i.e., that \( v \) is a complex place. Since we have now fixed
\( v \in S_c \), we drop it and use local notations as in 2.4.2. Recall: \( \mu = (\mu^\epsilon, \mu^\prime) \) with \( \mu^\epsilon = (\mu_1, \ldots, \mu_n) \)
and $\mu^i = (\mu^i_1, \ldots, \mu^i_n)$; the purity condition gives $\mu^i_1 + \mu_{n-i+1} = w$, where $w$ is the purity weight of the global $\mu$; the cuspidal parameters are:

$$(a_1, \ldots, a_n) = \left( \mu_1 + \frac{n-1}{2}, \ldots, \mu_n - \frac{(n-1)}{2} \right) \text{ and } (b_1, \ldots, b_n) = (w - a_1, \ldots, w - a_n);$$

the local representation $\Pi_v = J_\mu := \text{Ind} \left( z^{a_1} z^{b_1} \otimes \cdots \otimes z^{a_n} z^{b_n} \right)$. Note that $a_i + b_i = w$ and $a_i - b_i = 2\mu_i - w + n - 2i + 1$. Similarly, we have $\lambda = (\lambda^1, \lambda^2)$ with $\lambda^i = (\lambda_1, \ldots, \lambda_{n-1})$ and $\lambda^j = (\lambda^1_j, \ldots, \lambda^n_{n-1})$; purity gives $\lambda^j_1 + \lambda_{n-j} = w'$, where $w' = w(\lambda)$ is the purity weight of $\lambda$; the cuspidal parameters are:

$$(a'_1, \ldots, a'_{n-1}) = \left( \lambda_1 + \frac{n-2}{2}, \ldots, \lambda_{n-1} - \frac{(n-2)}{2} \right) \text{ and } (b'_1, \ldots, b'_{n-1}) = (w' - a'_1, \ldots, w' - a'_{n-1});$$

the local representation $\Sigma_v = J_\lambda := \text{Ind} \left( z^{a'_1} z^{b'_1} \otimes \cdots \otimes z^{a'_{n-1}} z^{b'_{n-1}} \right)$. Note that $a'_j + b'_j = w'$ and $a'_j - b'_j = 2\lambda^j_n - w' + n - 2j$.

It suffices to prove the theorem for a pair $(\Pi, \Sigma)$ such that $\mu^\nu \succ \lambda$. This is because, suppose we have a pair $(\Pi_1, \Sigma_1)$ such that $\mu^i_1 \succ \lambda_1 + m_0$ for an integer $m_0$, then we can consider $\Pi = \Pi_1$ and $\Sigma = \Sigma_1 \otimes |^\lambda_{m_0}$; for these representations the weights are $\mu = \mu_1$ and $\lambda = \lambda_1 + m_0$. It is easy to see that if the theorem holds for $(\Pi, \Sigma)$ then it holds for $(\Pi_1, \Sigma_1)$. Henceforth we assume therefore that $\mu^\nu \succ \lambda$. We have:

**Lemma 2.23.** If $\mu^\nu \succ \lambda$ then $s = \frac{1}{2}$ is critical for $L(s, \Pi \times \Sigma)$.

**Proof.** The hypothesis $\mu^\nu \succ \lambda$, means $\mu^\nu_1 \succ \lambda^1$ and $\mu^\nu_1 \succ \lambda^2$. By Cor. 2.19 and the formulae for $\mu^\nu = w - \mu_{n-i+1}$ and $\lambda^j = w' - \lambda_{n-j}$ we have:

\[
\begin{align*}
-\mu_n & \geq \lambda_1 \geq -\mu_{n-1} \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq -\mu_1, \\
w - \mu_n & \geq \lambda_1 - w' \geq w - \mu_{n-1} \geq \lambda_2 - w' \geq \cdots \geq \lambda_{n-1} - w' \geq w - \mu_1.
\end{align*}
\]

Using the formulae in Knapp [25, Sect. 4], for any complex place $v$ as above, we have

\[
L_v (s, \Pi_v \times \Sigma_v) = \prod_{i,j} L \left( s, z^{a_i + a'_j} z^{b_i + b'_j} \right) \sim \prod_{i,j} \Gamma \left( s + \frac{a_i + a'_j + b_i + b'_j}{2} + \left| \frac{a_i - b_i + a'_j - b'_j}{2} \right| \right),
\]

where, by $\sim$, we mean up to a nonzero exponential factor which is irrelevant in computing the critical set. In terms of the data going in to $\mu$ and $\lambda$, we have:

\[
\begin{align*}
L_v (s, \Pi_v \times \Sigma_v) &= \prod_{1 \leq i,j \leq n} \Gamma \left( s + \frac{w + w'}{2} + \frac{2\mu_i + 2\lambda_j + 2n - 2i - 2j + 1 - (w + w')}{2} \right), \\
L_v (s, \Pi'_v \times \Sigma'_v) &= \prod_{1 \leq i,j \leq n} \Gamma \left( s - \frac{w + w'}{2} + \frac{2\mu_i + 2\lambda_j + 2n - 2i - 2j + 1 - (w + w')}{2} \right).
\end{align*}
\]

It follows that $s = 1/2$ is critical if every factor on the right hand side above is regular at $s = 1/2$, i.e., if both the inequalities

\[
\begin{align*}
1 + (w + w') + \frac{2\mu_i + 2\lambda_j + 2n - 2i - 2j + 1 - (w + w')}{2} & \geq 1, \\
1 - (w + w') + \frac{2\mu_i + 2\lambda_j + 2n - 2i - 2j + 1 - (w + w')}{2} & \geq 1.
\end{align*}
\]

are satisfied. Consider two cases:
Case 1: $i + j \geq n + 1$. By (2.24), we have $\mu_i + \lambda_j \leq 0$ and $\mu_i + \lambda_j \leq w + w'$. Hence
\[
|2\mu_i + 2\lambda_j + 2n - 2i - 2j + 1 - (w + w')| = -2\mu_i - 2\lambda_j - 2n + 2i + 2j - 1 + (w + w').
\]
For a given $j$, this quantity is minimized if $i = n + 1 - j$. Hence (2.27) is satisfied since
\[
1 + (w + w') - 2\mu_{n+1-j} - 2\lambda_j + 1 + (w + w') = -\mu_{n+1-j} - \lambda_j + (w + w') + 1 \geq 1.
\]
Similarly, (2.28) is satisfied since
\[
1 - (w + w') - 2\mu_{n+1-j} - 2\lambda_j + 1 + (w + w') = 1 - \mu_{n+1-j} - \lambda_j \geq 1.
\]

Case 2: $i + j \leq n$. By (2.24), we have $\mu_i + \lambda_j \geq 0$ and $\mu_i + \lambda_j \geq w + w'$. Hence
\[
|2\mu_i + 2\lambda_j + 2n - 2i - 2j + 1 - (w + w')| = 2\mu_i + 2\lambda_j + 2n - 2i - 2j + 1 - (w + w').
\]
For a given $j$, this quantity is minimized if $i = n - j$. Hence (2.27) is satisfied since
\[
1 + (w + w') + 2\mu_{n-j} + 2\lambda_j + 1 - (w + w') = 1 + \mu_{n-j} + \lambda_j \geq 1.
\]
Similarly, (2.28) is satisfied since
\[
1 - (w + w') + 2\mu_{n-j} + 2\lambda_j + 1 - (w + w') = 1 + \mu_{n-j} + \lambda_j - (w + w') \geq 1.
\]

Corollary 2.29.
\[
\{ m \in \mathbb{Z} : \mu^s \succ \lambda + m \} \subset \{ m \in \mathbb{Z} : \frac{1}{2} + m \text{ is critical for } L(s, \Pi \times \Sigma) \}.
\]

Proof. If $m$ is such that $\mu^s \succ \lambda + m$ then by Lem, $m = 1/2$ is critical for $L(s, \Pi \times \Sigma(m))$ where $\Sigma(m) = \Sigma \otimes |m|$. But, $L(\frac{1}{2}, \Pi \times \Sigma(m)) = L(\frac{1}{2} + m, \Pi \times \Sigma)$.

Proposition 2.30. Assume $\mu^s \succ \lambda$. Define the numbers:
\[
m^+_{\mu, \lambda} := \min \{ -\mu_{n+1-j} - \lambda_j : 1 \leq j \leq n - 1 \} \cup \{ \mu_{n-j} + \lambda_j - (w + w') : 1 \leq j \leq n - 1 \}, \]
\[
m^-_{\mu, \lambda} := \max \{ -\mu_{n-j} - \lambda_j : 1 \leq j \leq n - 1 \} \cup \{ \mu_{n+1-j} + \lambda_j - (w + w') : 1 \leq j \leq n - 1 \}.
\]
Observe that $m^-_{\mu, \lambda} \leq 0 \leq m^+_{\mu, \lambda}$. We have
\[
1. \{ m \in \mathbb{Z} : \mu^s \succ \lambda + m \} = \{ m \in \mathbb{Z} : m^-_{\mu, \lambda} \leq m \leq m^+_{\mu, \lambda} \} \text{; and}
\]
\[
2. \{ m \in \mathbb{Z} : \frac{1}{2} + m \text{ is critical for } L(s, \Pi \times \Sigma) \} = \{ m \in \mathbb{Z} : m^-_{\mu, \lambda} \leq m \leq m^+_{\mu, \lambda} \}.
\]

Proof. That $m^-_{\mu, \lambda} \leq 0 \leq m^+_{\mu, \lambda}$ follows from (2.24). To prove (1), suppose $\mu^s \succ \lambda + m$, then applying (2.24) to the case of $\lambda + m$ we get:
\[
-\mu_n \geq \lambda_1 + m \geq -\mu_{n-1} \geq \lambda_2 + m \geq \cdots \geq \lambda_{n-1} + m \geq -\mu_1,
\]
\[
w - \mu_n \geq \lambda_1 - w' - m \geq w - \mu_{n-1} \geq \lambda_2 - w' - m \geq \cdots \geq \lambda_{n-1} - w' - m \geq w - \mu_1.
\]
From these inequalities, we deduce for all $1 \leq j \leq n - 1$:
\[
\mu_{n+1-j} + \lambda_j - w - w' \leq m \leq -\mu_{n+1-j} - \lambda_j,
\]
\[
-\mu_{n-j} - \lambda_j \leq m \leq \mu_{n-j} + \lambda_j - w - w'.
\]
Hence \( m_{\mu, \lambda}^- \leq m \leq m_{\mu, \lambda}^+ \). This entire chain of reasoning is reversible, which proves (1).

To prove (2), from Cor. [2.29] and (1), it suffices to show that if \( \frac{1}{2} + m \) is critical then the integer \( m \) satisfies: \( m_{\mu, \lambda}^- \leq m \leq m_{\mu, \lambda}^+ \). Suppose then that \( \frac{1}{2} + m \) is critical, then from (2.25) and (2.26) it follows that

\[
(2.32) \quad m + \frac{1 + (w + w') + |2 \mu_i + 2 \lambda_j + 2n - 2i - 2j + 1 - (w + w')|}{2} \geq 1,
\]

\[
(2.33) \quad -m + \frac{1 - (w + w') + |2 \mu_i + 2 \lambda_j + 2n - 2i - 2j + 1 - (w + w')|}{2} \geq 1.
\]

As in the proof of Lem. [2.23] consider two cases:

**Case 1:** \( i + j \geq n + 1 \). The hypothesis \( \mu^\nu \succ \lambda \) gives the inequalities (2.24) from which we get as before \( \mu_i + \lambda_j \leq 0 \) and \( \mu_i + \lambda_j \leq w + w' \). Hence

\[
|2 \mu_i + 2 \lambda_j + 2n - 2i - 2j + 1 - (w + w')| = -2 \mu_i - 2 \lambda_j - 2n + 2i + 2j - 1 + (w + w').
\]

In this case, (2.32) holds if and only if \( m \geq \mu_{n+1-j} + \lambda_j - (w + w') \), and (2.33) holds if and only if \( m \leq \mu_{n+1-j} - \lambda_j \).

**Case 2:** \( i + j \leq n \). In this case, exactly as above, (2.32) and (2.33) hold if and only if \( \mu_{n-j} - \lambda_j \leq m \leq \mu_{n-j} + \lambda_j - (w + w') \).

Putting both cases together, we see that \( \frac{1}{2} + m \) is critical implies that \( m_{\mu, \lambda}^- \leq m \leq m_{\mu, \lambda}^+ \), which concludes the proof of the proposition. \( \square \)

This also concludes the proof of Thm. [2.20] \( \square \)

Let’s revert to global notations, and for future reference, record the set of all critical points:

**Corollary 2.34.** Let \( \mu \in X_{00}^+(T_n) \) (resp., \( \lambda \in X_{00}^+(T_{n-1}) \)) and \( \Pi \in \text{Coh}(G_n, \mu^\nu) \) (resp., \( \Sigma \in \text{Coh}(G_{n-1}, \lambda^\nu) \)). Assume \( \mu^\nu \succ \lambda \). Define the numbers:

1. For \( v \in S_c \) which corresponds to \( t_v \in E_c \):
   - \( m_{\mu^v, \lambda^v}^+ = \min_{1 \leq j \leq n-1} \{-\mu_{n+1-j}^v - \lambda_j^v\} \);
   - \( m_{\mu^v, \lambda^v}^- = \max_{1 \leq j \leq n-1} \{-\mu_{n-j}^v - \lambda_j^v\} \).
2. For \( v \in S_c \) which corresponds to a pair of embeddings \( \{t_v, t_v\} \):
   - \( m_{\mu^v, \lambda^v}^+ = \min_{1 \leq j \leq n-1} \{-\mu_{n+1-j}^v + \lambda_j^v\} + \{\mu_{n-j}^v + \lambda_j^v - (w + w')\} \);
   - \( m_{\mu^v, \lambda^v}^- = \max_{1 \leq j \leq n-1} \{-\mu_{n-j}^v - \lambda_j^v\} + \{\mu_{n+1-j}^v + \lambda_j^v - (w + w')\} \).
3. Now define
   - \( m^+(\mu, \lambda) \): \( \min \{v \in S_\infty : m_{\mu^v, \lambda^v}^+\} \);
   - \( m^- (\mu, \lambda) \): \( \max \{v \in S_\infty : m_{\mu^v, \lambda^v}^-\} \).

Then, \( \frac{1}{2} + m \) is critical for \( L(s, \Pi \times \Sigma) \) if and only if \( m^- (\mu, \lambda) \leq m \leq m^+(\mu, \lambda) \).

**Remark 2.35.** Let us consider the hypothesis that there is an integer \( m_0 \) such that \( M_{\lambda+m_0} \hookrightarrow M_\mu' \), in the statement of the theorem. Suppose \( F = \mathbb{Q} \). Observe that this hypothesis always satisfied for \( GL_2 \times GL_1 \); take \( \lambda_2 = -\mu_2 \). But for \( n \geq 3 \) without that hypothesis, the conclusion of Theorem [2.20] need not hold. For example, consider \( GL_3 \times GL_2 \) over \( \mathbb{Q} \). Take \( \mu = (0, 0, 0) \); then \( w = 0 \); \( \ell = (2, 0, -2) \). Take \( \lambda = (1, -1) \); then \( w' = 0 \); \( \ell' = (3, -3) \). Then \( C_{H, \Sigma} = 1 \); hence \( \frac{1}{2} \) is critical since \( 1 - C_{H, \Sigma} \leq w + w' \leq C_{H, \Sigma} - 1 \). However, there is no integer \( m \) such that \( M_{\lambda+m} \hookrightarrow M_\mu' \). In other words, there are Rankin–Selberg \( L \)-funcitons \( L(s, \Pi \times \Sigma) \) which have critical values, but the underlying
sheaves are not compatible; in such situations we are unable to say anything about these critical values.

**Remark 2.36.** The main results of this paper are proved using the theory of cohomology of arithmetic groups to study the arithmetic nature of critical values for $L$-functions for $\text{GL}_n \times \text{GL}_{n-1}$. This is possible only when the sheaves $\tilde{\mathcal{M}}_\mu$ and $\tilde{\mathcal{M}}_\lambda$ are compatible. But if they are compatible, then the methods of this paper says that we can prove a theorem for every critical value, and only for critical values. In other words, even if the sheaves are compatible, we are unable to say anything about noncritical values.

### 2.5. The proof of Theorem 1.1

#### 2.5.1. The main idea behind a cohomological interpretation of $L(\frac{1}{2}, \Pi \times \Sigma)$

We interpret the Rankin–Selberg integral $I(\frac{1}{2}, \phi_\Pi, \phi_\Sigma)$ in terms of Poincaré duality. More precisely, the vector $w_{\Pi_f}$ will correspond to a cohomology class $\vartheta_{\Pi}$ in degree $b_1^F$, and similarly $w_{\Sigma_f}$ will correspond to a class $\vartheta_{\Sigma}$ in degree $b_{n-1}^F$. These classes, after dividing by certain periods, have good rationality properties. Pull back $\iota^*\vartheta_{\Pi}$ along the proper map $\iota : \tilde{S}^{G_{n-1}} \to S^{G_n}$, and wedge (or cup) with $\vartheta_{\Sigma}$, to give a top degree class (by Prop. 2.16) on $\tilde{S}^{G_{n-1}}$ with coefficients in a tensor product sheaf. Now if the constituent sheaves are compatible ($\mu^\vee > \lambda$), which by [2.4.5] is the same as saying $s = 1/2$ is critical, then we get a top-degree class on $\tilde{S}^{G_{n-1}}$ with constant coefficients. Apply Poincaré duality, i.e., fix an orientation on $\tilde{S}^{G_{n-1}}$ and integrate. One realizes then that this is essentially the Rankin–Selberg integral in Prop. 2.1, interpreting that integral, and hence the critical $L$-value $L_f(\frac{1}{2}, \Pi \times \Sigma)$ in cohomology, permits us to study it’s arithmetic properties, since Poincaré duality is Galois equivariant. All this may be summarized in the diagram:

\[
\begin{array}{c}
\mathcal{W}(\Pi_f) \times \mathcal{W}(\Sigma_f) \xrightarrow{f} H^{b_1^F}(g_n, K_{n,\infty}; V_{\Pi} \otimes \mathcal{M}_{\mu,\mathbb{C}}(\epsilon)) \times H^{b_{n-1}^F}(g_{n-1}, K_{n-1,\infty}; V_{\Sigma} \otimes \mathcal{M}_{\lambda,\mathbb{C}}(\eta))
\end{array}
\]

#### 2.5.2. Review of certain periods and period relations

We briefly review the definition of certain periods attached to $\Pi \in \text{Coh}(G_n, \mu)$, and their behavior upon twisting by $\Pi$ by algebraic Hecke characters. The reader is referred to my paper with Shahidi [33] for all the details.
2.5.2.1. Comparing Whittaker models and cohomological representations. Given any $\Pi \in \text{Coh}(G_n, \mu^\vee)$ and a permissible signature $\epsilon$ for $\Pi$, fix a generator $[\Pi, \cdot]_\epsilon$ of the one-dimensional space $H^{b_{\epsilon}}(\mathfrak{g}_n, K_{n, \infty}^0; W(\Pi_{\infty}) \otimes \mathcal{M}_{\mu, C}^\epsilon(\epsilon))$. We have the following comparison isomorphism:

$$
F_{\Pi} : W(\Pi_f) \rightarrow H^{b_{\epsilon}}(\mathfrak{g}_n, K_{n, \infty}^0; V_{\Pi} \otimes \mathcal{M}_{\mu, C}^\epsilon(\epsilon)).
$$

See [33, Sect. 3.3].

2.5.2.2. Rationality fields and rational structures. Given $\mu \in X_{00}^+(T_n)$ and $\Pi \in \text{Coh}(G_n, \mu^\vee)$, define the number field $\mathbb{Q}(\mu)$ as in [2.5.1] and the rationality field $\mathbb{Q}(\Pi_f)$ as in [13, Sect. 7.1]. Now define $\mathbb{Q}(\Pi)$ as the compositum of $\mathbb{Q}(\mu)$ and $\mathbb{Q}(\Pi_f)$. The Whittaker model $W(\Pi_f)$ of the finite part of $\Pi$ admits a $\mathbb{Q}(\Pi_f)$-structure, and hence a $\mathbb{Q}(\Pi)$-structure; see [33, Sect. 3.2]. This rational structure is generated by normalized new vectors. The cohomological model $H^{b_{\epsilon}}(\mathfrak{g}_n, K_{n, \infty}^0; V_{\Pi} \otimes \mathcal{M}_{\mu, C}^\epsilon(\epsilon))$ admits a $\mathbb{Q}(\Pi)$-structure arising from purely geometric considerations, since the sheaf $\mathcal{M}_{\mu, C}$ has a $\mathbb{Q}(\mu)$-structure $\mathcal{M}_{\mu, C}(\mu)$; see [33, Sect. 3.3].

2.5.2.3. Definition of the periods. The isomorphism $F_{\Pi}$ need not preserve rational structures on either side. Each side is an irreducible representation space for the action of $\pi_0(G_n(\mathbb{R})) \times G_n(\mathbb{A}_f)$ and rational structures being unique up to homotheties, we can adjust the isomorphism $F_{\Pi}$ by a scalar—which is the period—so as to preserve rational structures. There is a nonzero complex number $p^\epsilon(\Pi)$ attached to the datum $(\Pi_f, \epsilon, [\Pi, \cdot]_\epsilon)$ such that the normalized map $F_{\Pi, 0} := p^\epsilon(\Pi)^{-1}F_{\Pi}$ is $\text{Aut}(\mathbb{C})$-equivariant, i.e., the following diagram commutes:

$$
\begin{array}{ccc}
W(\Pi_f) & \xrightarrow{F_{\Pi, 0}} & H^{b_{\epsilon}}(\mathfrak{g}_n, K_{n, \infty}^0; V_{\Pi} \otimes \mathcal{M}_{\mu, C}^\epsilon(\epsilon)) \\
\downarrow \sigma & & \downarrow \sigma \\
W(\sigma \Pi_f) & \xrightarrow{F_{\sigma, 0}} & H^{b_{\epsilon}}(\mathfrak{g}_n, K_{n, \infty}^0; V_{\sigma \Pi} \otimes \mathcal{M}_{\mu, C}^{\epsilon(\sigma \epsilon)}(\sigma \epsilon))
\end{array}
$$

The complex number $p^\epsilon(\Pi)$ is well-defined up to multiplication by elements of $\mathbb{Q}(\Pi)^\times$. If we change $\Pi$ to $\alpha \Pi$ with $\alpha \in \mathbb{Q}(\Pi)^\times$, then the period $p^\epsilon(\alpha \Pi)$ changes to $\sigma(\alpha)p^\epsilon(\Pi)$. In other words the collection $\{p^\epsilon(\alpha \Pi) : \sigma \in \text{Aut}(\mathbb{C})\}$ is well-defined in $\mathbb{Q}(\Pi) \otimes \mathbb{C})^\times / \mathbb{Q}(\Pi)^\times$. In terms of the un-normalized maps, we can write the above commutative diagram as

$$
(2.37) \quad \sigma \circ F_{\Pi} = \left( \frac{\sigma(p^\epsilon(\Pi))}{p^\epsilon(\alpha \Pi)} \right) F_{\sigma \Pi} \circ \sigma.
$$

See [33, Defn./Prop. 3.3].

2.5.2.4. Behaviour of the periods under twisting. Let $\xi$ be an algebraic Hecke character with signature $\epsilon_\xi$ as in [2.1.7] and Gauß sum $G(\xi_f)$ as in [2.1.8]. Then [33, Thm. 4.1] says that

$$
(2.38) \quad \sigma \left( \frac{p^{\epsilon_\xi}(\Pi_f \otimes \xi_f)}{G(\xi_f)^{n(n-1)/2} p^\epsilon(\Pi_f)} \right) = \left( \frac{p^{\epsilon_\xi(\sigma \Pi_f \otimes \xi_f)}}{G(\sigma \xi_f)^{n(n-1)/2} p^\epsilon(\sigma \Pi_f)} \right).
$$

2.5.3. The cohomology classes and the global pairing.
2.5.3.1. The cohomology classes. Recall our choice of Whittaker vectors \( w_\Pi = w_{\Pi_f} \otimes w_{\Pi_\infty} \in \mathcal{W}(\Pi, \psi) \) and \( w_\Sigma = w_{\Sigma_f} \otimes w_{\Sigma_\infty} \in \mathcal{W}(\Sigma, \psi^{-1}) \) from \([2.24]\). Define
\[
(2.39) \quad \vartheta^\Pi_\Pi := \mathcal{F}_\Pi(w_{\Pi_f}) \quad \text{and} \quad \vartheta^n_\Sigma := \mathcal{F}^n_\Sigma(w_{\Sigma_f}).
\]
These classes are transcendental, whereas the normalized classes
\[
(2.40) \quad \vartheta^\Pi_{\Pi,0} := \mathcal{F}^c_{\Pi,0}(w_{\Pi_f}) = p^c(\Pi_f)^{-1} \vartheta^\Pi_\Pi \quad \text{and} \quad \vartheta^n_{\Sigma,0} := \mathcal{F}^n_{\Sigma,0}(w_{\Sigma_f}) = p^n(\Sigma_f)^{-1} \vartheta^n_\Sigma.
\]
are rational, i.e., \( \vartheta^\Pi_{\Pi,0} \in H^1_{k_1}(S^{G_n}, \mathcal{M}^\nu_\mu) \) and \( \vartheta^n_{\Sigma,0} \in H^1_{r_{n-1}}(S^{G_{n-1}}, \mathcal{M}^\lambda_\lambda) \).

2.5.3.2. Compatibility of the sheaves and the map \( \mathcal{T} \). The hypothesis \([1.22]\) on the weights \( \mu \) and \( \lambda \), implies via Thm \([2.20]\) that the critical set is non-empty. In particular, \( s = 1/2 \) is critical, and hence \( \mu^\vee > \lambda \). Apply the branching rule Cor. \([2.19]\) at every archimedean place; for \( v \in S_\infty \), fix a nonzero \( \mathcal{T}_v \in \text{Hom}_{GL_n(F_v)}(\mathcal{M}^\nu_\mu \otimes \mathcal{M}^\lambda_\lambda, \mathbb{I}) \), where \( \mathbb{I} \) is the trivial representation. Define
\[
(2.41) \quad \otimes_{v \in S_\infty} \mathcal{T}_v =: \mathcal{T} \in \text{Hom}_{G_{n-1}}(\mathcal{M}^\nu_\mu \otimes \mathcal{M}^\lambda_\lambda, \mathbb{I}).
\]

2.5.3.3. The orientation class. Consider \( S_{R_f}^{G_{n-1}} \), for an open-compact subgroup \( R_f \in G_{n-1}(\mathbb{A}_f) \), as defined in \([2.44]\). From the description in \([2.3.1]\), the connected components of \( S_{R_f}^{G_{n-1}} \) are of the form \( \Gamma_i \backslash \mathcal{G}_{n-1}(\mathbb{R})^0 / C_0^{\mathbb{R},n-1,\infty} \). We fix an orientation on \( \mathcal{G}_{n-1}(\mathbb{R})^0 / C_0^{\mathbb{R},n-1,\infty} \), push it down to all such connected components, and then take the sum over the index \( i \) (as in \( \Gamma_i \)) to get the orientation class on \( S_{R_f}^{G_{n-1}} \), which we denote as \([S_{R_f}^{G_{n-1}}]\). For each \( v \in S_r \), define \( \delta_v \in \pi_0(K_{n-1,\infty}) \) to be the element which is trivial at all places other than \( v \), and at \( v \) it is \( \delta_v = \text{diag}(-1,1,1,\ldots,1) \); see \([21.3]\). Then \( \pi_0(K_{n-1,\infty}) \) is generated by \( \{ \delta_v : v \in S_r \} \).

**Lemma 2.42.** The group \( \pi_0(K_{n-1,\infty}) \) acts on the orientation class \([S_{R_f}^{G_{n-1}}]\) via:
\[
\delta_v \cdot [S_{R_f}^{G_{n-1}}] = (-1)^n [S_{R_f}^{G_{n-1}}], \quad \forall v \in S_r.
\]

**Brief sketch of proof.** Let \( \{ X_1, \ldots, X_m \} \) be an ordered basis for \( \mathfrak{g}_{n-1} / \mathfrak{so}(n-1) \) consisting of:
\[
\{ H_i = E_{ii} \mid 1 \leq i \leq n-1 \} \cup \{ X_{ij} = E_{ij} + E_{ji} \mid 1 \leq i < j \leq n-1 \},
\]
where \( E_{ij} \) is the matrix with 1 in the \((i,j)\)-th entry and 0 elsewhere. Then \( \delta = \text{diag}(-1,1,1,\ldots,1) \) fixes each \( H_i \) and each \( X_{ij} \) with \( i \neq 1 \), and takes \( X_{ij} \) to \(-X_{ij} \). Hence \( \delta^* (X_1 \wedge \cdots \wedge X_m) = (-1)^{n-2}(X_1 \wedge \cdots \wedge X_m) \). The proof consists in applying this remark to \( \delta \) as a given \( \delta_v \), which has no effect on the basis vectors coming from places other than \( v \). (See also \([31]\) Lem. 3.4.) \( \square \)

2.5.3.4. The global pairing. Referring to the diagram in \([2.5.1]\) we wish to compute the pairing
\[
(2.43) \quad \langle \vartheta^\Pi_{\Pi,0}, \vartheta^n_{\Sigma,0} \rangle := \mathcal{I}_{[S_{R_f}^{G_{n-1}}]}(\mathcal{T}^* (\epsilon^* \vartheta^\Pi_{\Pi} \wedge \phi^* \vartheta^n_{\Sigma})) \quad \text{or} \quad \langle \vartheta^\Pi_{\Pi,0}, \vartheta^n_{\Sigma,0} \rangle = \frac{\langle \vartheta^\Pi_{\Pi,0}, \vartheta^n_{\Sigma,0} \rangle}{p^c(\Pi) p^n(\Sigma)}.
\]
2.5.3.5. The choice of signs $\epsilon$ and $\eta$. Recall that $\epsilon = (\epsilon_v)_{v \in S_r}$, and $\eta = (\eta_v)_{v \in S_r}$. It is clear from Lem. 2.42 that we can expect to have a nonzero pairing in (2.43) only when

$$\epsilon_v = (-1)^n \eta_v, \quad \forall v \in S_r.$$

Further, by parity constraints as in 2.43, we know that $\epsilon$ or $\eta$ is uniquely determined by $\Pi$ or $\Sigma$, depending on whether $n$ is odd or even. To summarize, for every $v \in S_r$, we have

1. $\epsilon_v = (-1)^n \eta_v.$
2. If $n$ is odd then $\epsilon_v = \omega_{\Pi_v}(-1) \cdot (\epsilon^w(\mu)/2$;
3. if $n$ is even then $\eta_v = \omega_{\Sigma_v}(-1) \cdot (\epsilon^w(\lambda)/2$.

2.5.3.6. The pairing at infinity. As in [31. Sect. 3.2.5], the computation of the pairing in (2.43) involves a certain archimedean contribution. Recall from 2.5.2.1 the generator $[\Pi_\infty]^\epsilon$ of the one-dimensional space $H^F_{n, 0} (g_n, K_{n, \infty}; \mathcal{W}(\Pi_\infty) \otimes M_{\mu, \mathbb{C}})(\epsilon)$, and similarly, we have $[\Sigma_\infty]^\eta$ generating the one-dimensional $H^F_{n-1, 0} (g_{n-1}, K_{n-1, \infty}; \mathcal{W}(\Sigma_\infty) \otimes M_{\mu, \mathbb{C}})(\eta)$. To compute the pairing at infinity, fix a basis $\{y_j: 1 \leq i \leq d_n^F\}$ for $(g_{n-1, \infty}/t_{n-1, \infty})^\epsilon$ such that $y_j : 1 \leq j \leq d_n^F - 1$ is a basis for $(g_{n-1, \infty}/t_{n-1, \infty})^\epsilon$. Next, fix a basis $x_i : 1 \leq i \leq d_n^F - 1$ for $(g_{n, \infty}/t_{n, \infty})^\epsilon$, such that $v^* x_j = y_j$ for all $1 \leq j \leq d_n^F - 1$ and $v^* x_i = 0$ if $i > d_n^F - 1$. We further note that $y_1 \wedge y_2 \wedge \cdots \wedge y_{d_n^F - 1}$ is a $\mathbb{Q}$-basis for $M_{\mu}$ (resp., $M_{\lambda}$). The class $[\Pi_\infty]^\epsilon$ is represented by a $K^0_{n, \infty}$-invariant element in $\wedge^b_n (g_{n, \infty}/t_{n, \infty})^\epsilon \otimes \mathcal{W}(\Pi_\infty) \otimes M_{\mu, \mathbb{C}}$ which we write as

$$[\Pi_\infty]^\epsilon = \sum_{i_1 \cdots i_n} \sum_{\alpha} x_i \otimes w_{\infty, i, \alpha} \otimes m_{\alpha},$$

where $w_{\infty, i, \alpha} \in \mathcal{W}(\Pi_\infty, \psi_\infty)$. Thinking in terms of the Künneth theorem for relative Lie algebra cohomology, the choice of basis $\{x_i\}$ and $\{m_{\alpha}\}$ can be made so that we have:

$$[\Pi_\infty]^\epsilon = \otimes_{v \in S_r} [\Pi_v]^\epsilon.$$

(It is understood that for $v \in S_c$ there is no sign $\epsilon_v$.)

Similarly, $[\Sigma_\infty]^\eta$ is represented by a $K^0_{n-1, \infty}$-invariant element in $\wedge^b_{n-1} (g_{n-1, \infty}/t_{n-1, \infty})^\epsilon \otimes \mathcal{W}(\Sigma_\infty) \otimes M_\lambda^\epsilon$ which we write as:

$$[\Sigma_\infty]^\eta = \sum_{j_1 < \cdots < j_{d_{n-1}^F}} \sum_{\beta} y_j \otimes w_{\infty, j, \beta} \otimes m_{\beta},$$

with $w_{\infty, j, \beta} \in \mathcal{W}(\Sigma_\infty, \psi^{-1})$. Again, we have

$$[\Sigma_\infty]^\eta = \otimes_{v \in S_r} [\Sigma_v]^\eta.$$

We now define a pairing at infinity by

$$\langle [\Pi_\infty]^\epsilon, [\Sigma_\infty]^\eta \rangle = \sum_{i, j} s(i, j) \sum_{\alpha, \beta} T(m_{\beta}^\mu \otimes m_{\alpha}^\epsilon) \Psi_{\infty} (1/2, w_{\infty, i, \alpha}, w_{\infty, j, \beta}),$$

where $s(i, j) = \{0, -1, 1\}$ is defined by $v^* x_i \wedge y_j = s(i, j) y_1 \wedge y_2 \wedge \cdots \wedge y_{d_n^F - 1}$. Recall that $\Psi_{\infty} (1/2, w_{\infty, i, \alpha}, w_{\infty, j, \beta})$ is defined only after meromorphic continuation. However, the assumption on $\mu$ and $\lambda$ in (1.2) guarantees that $s = 1/2$ is critical which ensures that the integrals
\( \Psi_\infty(1/2, w_{\infty,i,1}, w_{\infty,j,2}) \) are all finite, hence \( \langle [\Pi_\infty],[\Sigma_\infty] \rangle \) is finite. Furthermore, computing this pairing is a purely local problem since

\[
\langle [\Pi_\infty]^\epsilon,[\Sigma_\infty]^\eta \rangle = \prod_{v \in \mathcal{S}_\infty} \langle [\Pi_v]^\epsilon_v,[\Sigma_v]^\eta_v \rangle.
\]

Binyong Sun \cite{38} has recently proved that the local pairings are all nonzero giving us the following

**Theorem 2.47.** \( \langle [\Pi_\infty],[\Sigma_\infty] \rangle \neq 0. \)

The quantity \( \langle [\Pi_\infty],[\Sigma_\infty] \rangle \) depends only on the weights \( \mu \) and \( \lambda \), and the signs \( \epsilon \) and \( \eta \) which are determined as in \( 2.5.3.5 \) Now define:

\[
p^\epsilon^\eta_{\infty}(\mu,\lambda) := \frac{1}{\langle [\Pi_\infty],[\Sigma_\infty] \rangle}.
\]

Ultimately, one should expect \( p^\epsilon^\eta_{\infty}(\mu,\lambda) \) to be a power of \( (2\pi i) \).

### 2.5.4. The main identity.

The following theorem is a generalization of \cite{31} Thm. 3.12.

**Theorem 2.49** (Main Identity). Let \( \mu \in \mathcal{X}_0^s(T_0) \) and \( \Pi \in \text{Coh}(G_n,\mu) \). Let \( \lambda \in \mathcal{X}_0^s(T_{n-1}) \) and \( \Sigma \in \text{Coh}(G_{n-1},\lambda) \). Assume that \( \mu \) and \( \lambda \) satisfy \( \mu \bowtie \lambda \); in particular, \( s = 1/2 \) is critical for \( L_f(s,\Pi \times \Sigma) \). Take the signs \( \epsilon,\eta \) as in \( 2.5.3.5 \). Let \( \vartheta_{\Pi_0}^1 \) and \( \vartheta_{\Sigma_0}^1 \) be the normalized classes defined in \( 2.4(4) \). Then

\[
\frac{L_f(\frac{1}{2},\Pi \times \Sigma)}{p^s(\Pi)p^\epsilon(\Sigma)p^\eta_{\infty}(\mu,\lambda)} = \prod_{v \in \mathcal{S}_\infty} \frac{L(\frac{1}{2},\Pi_v \times \Sigma_v)}{\text{vol}(\Sigma)} \prod_{v \in \mathcal{S}_\infty} \frac{\vartheta_{\Pi_0,v}^1}{\vartheta_{\Sigma_0,v}^1},
\]

where the pairing on the right hand side is defined in \( 2.4(9) \), the nonzero rational number \( \text{vol}(\Sigma) \) is as in Prop. \( 2.4(2) \), and \( c_{\Pi_v} \) is defined in \( 2.2.3 \).

Since \( c_{\Pi_v} = 1 \) for an unramified places \( v \), the infinite product in the denominator of the right hand side is in fact a finite product.

**Proof.** The proof of \cite{31} Thm. 3.12] goes through mutatis mutandis, and so we omit the details. \( \square \)

### 2.5.5. The proof of Theorem 1.1.

**2.5.5.1. Central critical value.** Suppose \( \mu \bowtie \lambda \) then \( s = 1/2 \) is critical. The proof of \cite{31} Thm. 1.1] goes through mutatis mutandis giving an algebraicity result for the central critical value; the proof entails verifying that the right hand side of the main identity in Thm. \( 2.4(9) \) above is Galois equivariant, i.e., well-behaved under the action of \( \sigma \in \text{Aut}(\mathbb{C}) \). This is the essence of \cite{31} Sect. 3.3], all of which appropriately generalizes to the situation of this paper and so we omit the details.

**2.5.5.2. All critical values.** Now suppose \( \mu \) and \( \lambda \) satisfy \( 1.2 \], and suppose \( \frac{1}{2} + m \) is critical. Then we take a suitable Tate twist and consider a situation when \( \frac{1}{2} \) is critical, and then apply the period relations in \( 2.5.2.4 \). The reader should bear in mind that Thm. \( 2.20 \] says that the set of possible Tate-twists we can take subject to the restriction imposed by the compatibility condition \( 1.2 \] exactly covers all the critical points for \( L(s,\Pi \times \Sigma) \). However, the parity of \( \eta \) will play a role, because this will affect the recipe for signs \( \epsilon \) and \( \eta \) as in \( 2.5.3.5 \). We argue as follows:

Suppose \( \eta \) is even, then we absorb the \( m \) into the representation of \( \text{GL}_n \) as:

\[
L_f(\frac{1}{2} + m,\Pi \times \Sigma) = L_f(\frac{1}{2},\Pi(m) \times \Sigma) \sim p^s(\Pi(m))p^\epsilon(\Sigma)G(\omega_{\Sigma_f})p^\eta_{\infty}(\mu + m,\lambda).
\]

Using \( 2.38 \) we can write this as \( p^{m+\epsilon}(\Pi)p^\epsilon(\Sigma)G(\omega_{\Sigma_f})p^\eta_{\infty}(\mu + m,\lambda) \). Suppose \( \eta \) is odd, then we absorb the \( m \) into the representation of \( \text{GL}_{n-1} \), and argue similarly.
3. Symmetric power $L$-functions

3.1. Symmetric power transfers are cohomological.

3.1.1. Definition of $\text{Sym}^r(\pi)$ and $L(s, \text{Sym}^r(\pi) \otimes \chi)$. Let $\pi$ be a cohomological cuspidal automorphic representation of $GL_2$ over $F$. Let $\text{Sym}^r : GL_2(\mathbb{C}) \to GL_{r+1}(\mathbb{C})$ be the $r$-th symmetric power of the standard representation of $GL_2(\mathbb{C})$. By the local Langlands correspondence, see Harris–Taylor [16] and Henniart [17] for the non-archimedean places and Langlands [27] for the archimedean places, the local transfer $\text{Sym}^r(\pi_v)$ is defined as an irreducible admissible representation of $GL_{r+1}(F_v)$. Now define $\text{Sym}^r(\pi) := \bigotimes_v \text{Sym}^r(\pi_v)$ which is a well-defined irreducible admissible representation of $GL_{r+1}(A_F)$. Langlands’s functoriality predicts that $\text{Sym}^r(\pi)$ is an isobaric automorphic representation of $GL_{r+1}(A_F)$; this is known for $r \leq 4$ by Gelbart–Jacquet [10], Kim–Shahidi [22], and Kim [20] for a general $\pi$, and it is known for all $r$ if $\pi$ is dihedral. Define the $r$-th symmetric power $L$-function of $\pi$ as the standard $L$-function of the $r$-th symmetric power transfer of $\pi$; we may also introduce a twisting Hecke character $\chi$. We are interested in the special values of such twisted symmetric power $L$-functions of $\pi$: $L(s, \text{Sym}^r(\pi) \otimes \chi)$. As in [31] we approach odd symmetric power $L$-functions inductively. To get started, consider the case when $r = 1$.

3.1.2. $L$-functions for $GL_2 \times GL_1$. Let’s explicate the $n = 2$ case of Thm [17]. The reader should note that this particular case is not new; see, for example, Hida [18], although our notations are rather different.

Let $\pi \in \text{Coh}(G_2, \mu^w)$ with $\mu \in X^+_0(T_2)$. Let $\mu = (\mu^i)_{i \in \mathcal{E}_F}$, and $\mu^i = (a^i, b^i) \in \mathbb{Z}^2$ with $a^i \geq b^i$. Let $w = w(\mu)$ be the purity weight of $\mu$. Recall: for any $v \in S_r$, $w = a^i + b^i$, and for any $v \in S_c$, if $\mu^i = (a^i, b^i)$ then $\mu^i = (w - b^i, w - a^i)$.

Let $\lambda \in \mathbb{Z}$ be thought of as an element $\lambda = (\lambda^i)_{i \in \mathcal{E}_F} \in X^+_0(T_1)$, where each $\lambda^i = \lambda$. The corresponding representation $M_\lambda$ of $G_1$ is $\text{det}^\lambda$. The purity weight is $w(\lambda) = 2\lambda$. Let $\chi \in \text{Coh}(G_1, \lambda^w)$, then $\chi = \chi^0 \otimes | \cdot |^{\lambda}$, where $\chi^0 : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ is a character of finite order. For any $v \in S_r$, define $\epsilon_{\chi,v} = (-1)^{\lambda} \cdot \chi^0(-1)$; the signature of $\chi$ is $\epsilon_{\chi} = (\epsilon_{\chi,v})_{v \in S_r}$.

The compatibility condition (1.2) for the pair $(\mu, \lambda)$ is satisfied if there is an integer $j$ such that $-b^i \geq \lambda + j \geq -a^i$ for all $i$; this is best considered in two cases:

1. $v \in S_c$: here we have $a^i - w \geq \lambda + j \geq -a^i$. Note a consequence of these inequalities is that $a^i - w \geq -[w/2] \geq -a^i$.

2. $v \in S_r$: here we have $-b^i \geq \lambda + j \geq -a^i$ and $a^i - w \geq \lambda + j \geq b^i - w$. A consequence of both these inequalities are: $-b^i \geq -[w/2] \geq -a^i$ and $a^i - w \geq -[w/2] \geq b^i - w$.

It is important to appreciate the possibility that the compatibility condition (1.2) need not always be satisfied. For example, take $F$ to be imaginary quadratic extension of $\mathbb{Q}$, take $a^i = b^i = 0$ and $w \neq 0$; then clearly the necessary condition in (2) is not satisfied. Furthermore, the reader can check in this situation that $L(s, \pi)$ does have any critical points. However, at the other extreme, if $F$ is totally real, then (1.2) is satisfied for $j = -\lambda - [w/2] = -(w(\mu) + w(\mu))/2$; and by Thm 2.20 there are critical points for $L(s, \pi \otimes \chi)$. Finally, let’s note that for any number field with at least one real place, if $\mu$ is a parallel weight then (1.2) is satisfied and $L(s, \pi \otimes \chi)$ has critical points. Assume henceforth that (1.2) holds for $(\mu, \lambda)$.

Let $\frac{1}{2} + m \in \frac{1}{2} + \mathbb{Z}$ be any critical point for $L(s, \pi \otimes \chi)$. By Thm 2.20 this is the same as $m \in \mathbb{Z}$ such that $\mu^r > \lambda + m$. The sign $\eta = \epsilon_{\chi}$ as given above, and since $n = 2$, $\epsilon = \eta$. Further, $\chi$ being on $GL_1$, we have $p^{\theta}(\chi) \sim 1$. The statement of Thm 1.1 gives:

$$L_f(\frac{1}{2} + m, \pi \otimes \chi) \sim p^{\epsilon_m \epsilon_{\chi}}(\pi) G(\chi)p_{\infty}(\mu + m, \lambda),$$
and more generally, the ratio of the left hand side by the right hand side is Aut(ℂ)-equivariant. When F is totally real, the quantity $P^\chi_p(\mu + m, \lambda)$ has been explicitly calculated in [34] Prop. 3.24. In general, one expects, as in the totally real case, that this quantity is a rational multiple of an integral power of $2\pi i$.

We will apply the above discussion to the situation $L(s, \pi \times \omega_\pi \xi)$ where $\omega_\pi$ is the central character of $\pi$ and $\xi$ is a finite order Hecke character of $F$. Given $\mu$ as above, define $\det(\mu)$ by $\det(\mu)^t = a^t + b^t$; then $w(\det(\mu)) = 2w(\mu)$. We have:

$$\pi \in \text{Coh}(G_2, \mu^\nu) \implies \omega_\pi \in \text{Coh}(G_1, \det(\mu)^\nu).$$

The latter also implies that $\omega_\pi \xi \in \text{Coh}(G_1, \det(\mu)\nu)$. As above, the compatibility condition (1.2) for the pair $(\mu, \det(\mu))$ need not be satisfied in general.

3.1.3. Transferring the weights $\mu$. Let $\mu \in X_{00}^+(T_2)$. As above, suppose $\mu = (\mu^t)_{t \in E_F}$, with $\mu^t = (a^t, b^t) \in \mathbb{Z}^2$ and $a^t \geq b^t$. Define a weight $\text{Sym}^r(\mu) = (\text{Sym}^r(\mu^t))_{t \in E_F} \in X^+(T_{r+1})$, where for each $t$ we have:

$$\text{Sym}^r(\mu^t) := (ra^t, (r-1)a^t + b^t, \ldots, a^t + (r-1)b^t, rb^t).$$

If $\nu = w(\mu)$ be the purity weight of $\mu$, then it is easy to check that $\text{Sym}^r(\mu)$ is also pure and it’s purity weight is $w(\text{Sym}^r(\mu)) = rw$. Furthermore, one checks that $\text{Sym}^r(\mu)$ is strongly pure, i.e., $\text{Sym}^r(\mu) \in X_{00}^+(T_{r+1})$. Also, if $\mu$ is a parallel weight, then so is $\text{Sym}^r(\mu)$.

3.1.4. Symmetric power transfer preserves the property of being cohomological. The following theorem is a generalization of our result with Shahidi [32] Thm. 5.5 and is a variation of Labesse–Schwermer [26] Prop. 5.4.

**Theorem 3.2.** Let $\mu \in X_{00}^+(T_2)$ and $\pi \in \text{Coh}(G_2, \mu^\nu)$. Suppose $\text{Sym}^r(\pi)$ is a cuspidal automorphic representation of $G_{r+1}$, then $\text{Sym}^r(\pi) \in \text{Coh}(G_{r+1}, \text{Sym}^r(\mu)^\nu)$.

**Proof.** The proof is purely local, as it suffices to check that for $v \in S_{\infty}$, the representation $\text{Sym}^r(\pi)_v$ has nontrivial relative Lie algebra cohomology after twisting by $M_{\text{Sym}^r(\mu)_v}$. Consider two cases:

$v \in S_r$: Let’s suppress the $\ell_v = \ell$ and write $\mu^t = (a, b)$. Then $\pi_v = D(a - b + 1) | |^w2$, whose Langlands parameter we denote for brevity as $I(\xi_l)(w/2)$ with $l = a - b + 1$; it is, up to twisting by $| |^w2$, the two-dimensional representation of the Weil group $W_R$ of $\mathbb{R}$ which is induced from the Weil group $W_\mathbb{C} = \mathbb{C}^\times$ of $\mathbb{C}$ by the character $\xi_l$ given by $z = re^{i\theta} \mapsto e^{i\theta}$. A pleasant exercise gives:

$$\text{Sym}^r(I(\xi_l)) = \begin{cases} I(\xi_l) \oplus I(\xi_{2l}) \oplus \cdots \oplus I(\xi_{rl}), & \text{if } r \text{ is odd,} \\ \text{sgn}^{rl/2} \oplus I(\xi_{2l}) \oplus I(\xi_{4l}) \oplus \cdots \oplus I(\xi_{rl}), & \text{if } r \text{ is even.} \end{cases}$$

Also, $\text{Sym}^r(I(\xi_l)(w/2)) = \text{Sym}^r(I(\xi_l))(rw/2)$. By the local Langlands correspondence for $\text{GL}_{r+1}(\mathbb{R})$ (see Knapp [25]), we see that $\text{Sym}^r(\pi_v)$ is the representation in [27] or (2.4) exactly when the corresponding highest weight is taken as $\text{Sym}^r(\mu)$ when defined as in Sect. 3.1.3.

$v \in S_\mathbb{C}$: Suppressing again the notations for $\{\xi_v, \ell_v\}$, we write $\mu_v = \{(a, b), (w - b, w - a)\}$. Then, the Langlands parameter of $\pi_v = J_{\mu_v}$, see (2.12), is $\xi(a + \frac{1}{2}, w - a - \frac{1}{2}) \oplus \xi(b - \frac{1}{2}, w - b + \frac{1}{2})$, where for half-integers $p, q$ by $\xi(p, q)$ is meant the character of $W_{\mathbb{C}}$ given by $z \mapsto z^p\overline{z}^q$. Then the Langlands parameter of $\text{Sym}^r(\pi_v)$ is given by:

$$\xi(ra + \frac{r}{2}, rw - ra - \frac{r}{2}) \oplus \xi((r-1)a + b + \frac{r-2}{2}, rw - (r-1)a - b - \frac{r-2}{2}) \oplus \cdots \oplus \xi(rb - \frac{r}{2}, rw - rb + \frac{r}{2}).$$

Hence, $\text{Sym}^r(\pi_v)$, via (2.11) and (2.12), corresponds to the highest weight $\text{Sym}^r(\mu_v)$. □
Thm.3.2 and 3.1 lend further evidence to the discussion in [32 Sect. 5.2] relating functoriality and the property of being cohomological.

3.2. Special values of Symmetric power $L$-functions.

3.2.1. A factorization of $L$-functions. Let $\pi$ be a cuspidal automorphic representation of $G_2(\mathbb{A})$, and suppose $\text{Sym}^r(\pi)$ is automorphic for all $r$, then our main idea behind special values of symmetric power $L$-functions is:

$$L_f \left( s, \text{Sym}^r(\pi) \times \text{Sym}^{r-1}(\pi) \right) = \prod_{a=1}^{r} L_f \left( s, \text{Sym}^{2a-1}(\pi) \otimes \chi_{r-a} \right);$$

see [31 Cor. 5.2]. Now suppose that $\pi \in \text{Coh}(G_2, \mu^r)$ and suppose that the symmetric power transfers are all cuspidal, then we apply Thm. 3.1 to get algebraicity results for the values of $L_f \left( s, \text{Sym}^r(\pi) \times \text{Sym}^{r-1}(\pi) \right)$, and inductively, we get special value results for odd symmetric power $L$-functions on the right hand side. As can be seen in [31 Prop. 5.4], where the base field $F$ was $\mathbb{Q}$, carrying out this exercise can be combinatorially tedious.

In the rest of this article, we carry through the above idea for symmetric cube $L$-functions of $\pi$. Algebraicity results for the critical values of symmetric cube $L$-functions are available in the literature in various special cases; see Garrett–Harris [9 Thm. 6.2], Kim–Shahidi [21 Prop. 4.1], Grobner–Raghuram [14 Cor. 8.1.2], and Januszewski [19 Sect. 6]. The following results are new when $F$ is a general number field and the representation $\pi$ is cohomological with respect to a general strongly pure coefficient system $\mu$.

3.2.2. Critical points for symmetric cube $L$-functions.

Proposition 3.3. Let $\pi \in \text{Coh}(G_2, \mu^r)$ with $\mu \in X^+_0(T_2)$. Let $\mu = (\mu^i)_{i \in \mathcal{E}_F}$, and $\mu' = (a^i, b^i) \in \mathbb{Z}^2$ with $a^i \geq b^i$. Let $w = w(\mu)$ be the purity weight of $\mu$. Let $\chi$ be any Hecke character of $F$ of finite order. Then, $\frac{1}{2} + m \in \frac{1}{2} + \mathbb{Z}$ is critical for $L \left( s, \text{Sym}^2(\pi) \otimes \chi \right)$ if and only if $m$ satisfies the inequalities in (3.4) and (3.5) below. For $\nu \in S_r$, and $\iota_{\nu} \in \mathcal{E}_F$ the corresponding embedding, we have

$$-2a^{i_v} - b^{i_v} \leq m \leq -2b^{i_v} - a^{i_v}. \tag{3.4}$$

For $\nu \in S_c$, and $\{\iota_{\nu}, \iota_{\nu} \}$ the corresponding pair of embeddings, suppose $\alpha_{\nu}$ stands for the minimum of $\{|6a^{i_v} - 3w + 3|, |4a^{i_v} + 2b^{i_v} - 3w + 1|, |2a^{i_v} + 4b^{i_v} - 3w - 1|, |6b^{i_v} - 3w - 3| \}$, then

$$\frac{1 - 3w - \alpha_{\nu}}{2} \leq m \leq \frac{-1 - 3w + \alpha_{\nu}}{2}. \tag{3.5}$$

Proof. The proof is a somewhat tedious exercise, and we merely sketch it leaving all the details to the reader. Consider two cases:

$v \in S_r$: suppressing the superscript $i_v$, let $\mu_v = (a, b)$. Then up to nonzero constants and exponential functions, we have

$$L(s, \text{Sym}^3(\pi_v)) = \Gamma \left( s + \frac{3w}{2} + \frac{a - b + 1}{2} \right) \Gamma \left( s + \frac{3w}{2} + \frac{3(a - b + 1)}{2} \right).$$

We leave it to the reader to check that $L(s, \text{Sym}^3(\pi_v))$ and $L(1 - s, \text{Sym}^3(\pi_v)^\vee)$ are regular at $s = \frac{1}{2} + m$ if and only if $m$ satisfies (3.4).
\(v \in S_c\): suppressing the superscripts \(\iota_v\) and \(\iota_v^\prime\), up to nonzero constants and exponential functions, we have
\[
L(s, \text{Sym}^3(\pi_v)) = \Gamma \left( \frac{3w}{2} + \frac{6a - 3w + 3}{2} \right) \Gamma \left( \frac{3w}{2} + \frac{4a + 2b - 3w + 1}{2} \right) \cdot \Gamma \left( \frac{3w}{2} + \frac{2a + 4b - 3w - 1}{2} \right) \Gamma \left( \frac{3w}{2} + \frac{6b - 3w - 3}{2} \right).
\]
We leave it to the reader to check that \(L(s, \text{Sym}^3(\pi_v))\) and \(L(1-s, \text{Sym}^3(\pi_v)^\vee)\) are regular at \(s = \frac{1}{2} + m\) if and only if \(m\) satisfies (3.5).

The following special case of a parallel weight when \(F\) has at least one real place is interesting, and the reader can just as well consider only this case:

**Corollary 3.6.** Let \(\pi \in \text{Coh}(G_2, \mu^\vee)\) with \(\mu \in X^+_0(T_2)\) being a parallel weight \(\mu^\vee = (a, b) \in \mathbb{Z}^2\) with \(a \geq b\). Suppose \(S_f \neq \emptyset\), i.e., \(F\) is not totally imaginary then the purity weight is \(w = w(\mu) = a + b\) and furthermore, the set
\[
\left\{ \frac{1}{2} + m \in \frac{1}{2} + \mathbb{Z} : -2a - b \leq m \leq -a - 2b \right\}
\]

is the critical set for \(L(s, \text{Sym}^3(\pi) \otimes \chi)\), \(L(s, \pi \otimes \omega_\pi \chi)\) and \(L(s, \text{Sym}^3(\pi) \times \pi \otimes \chi)\).

**Proof.** The inequalities in (3.4) and (3.5) boil down to \(-2a - b \leq m \leq -a - 2b\) which takes care of \(L(s, \text{Sym}^3(\pi) \otimes \chi)\). The critical sets for \(L(s, \pi \otimes \omega_\pi \chi)\) and \(L(s, \text{Sym}^3(\pi) \times \pi \otimes \chi)\) follow from \(n = 3\) and \(n = 2\) cases respectively of Thm [2.20] we leave the details to the reader. \(\square\)

### 3.2.3. The compatibility condition (1.2)

Let \(\pi \in \text{Coh}(G_2, \mu^\vee)\) with \(\mu \in X^+_0(T_2)\). Consider two successive symmetric power transfers of \(\pi\). We address the question of whether Thm (1.1) is applicable to \(L(s, \text{Sym}^r(\pi) \times \text{Sym}^{r-1}(\pi))\), i.e., after applying Thm (3.2) we ask whether the transferred weights \(\text{Sym}^r(\mu)\) and \(\text{Sym}^{r-1}(\mu)\) satisfy the compatibility condition (1.2). For this we are seeking an integer \(j\) such that for all \(\iota \in \mathcal{E}_F\), we should have \(\text{Sym}^r(\mu^\vee) \neq \text{Sym}^{r-1}(\mu^\vee) + j\), i.e.,
\[
-rb^\vee \geq (r - 1)a^\vee + j \geq -(r - 1)b^\vee - a^\vee \geq (r - 2)a^\vee + b^\vee + j \geq \cdots \geq (r - 1)b^\vee + j \geq -ra^\vee.
\]
Consider two cases: if \(\iota\) corresponds to a real place then we want
\[
-ra^\vee - (r - 1)b^\vee \leq j \leq -(r - 1)a^\vee - rb^\vee,
\]
and if \(\iota\) corresponds to a complex place, then not only do we want the above (at \(i\)) but also at the conjugate embedding, since \((a^\vee, b^\vee) = (w - b^\vee, w - a^\vee)\), we would also want
\[
-(2r - 1)w + (r - 1)a^\vee + rb^\vee \leq j \leq -(2r - 1)w + ra^\vee + (r - 1)b^\vee.
\]
Such an integer \(j\) may not exist, because, if \(j\) exists, then both the above inequalities give the necessary condition:
\[
2ra^\vee + 2(r - 1)b^\vee \geq (2r - 1)w \geq 2(r - 1)a^\vee + 2rb^\vee,
\]
which need not be satisfied. (For example, take \(F\) to be an imaginary quadratic extension of \(\mathbb{Q}\), and take \(a^\vee = b^\vee, w \neq 2a^\vee\), and any \(r \geq 1.1\))

On the other hand, suppose \(F\) is not totally imaginary, i.e., \(S_f \neq \emptyset\), then \(w = a^\vee + b^\vee\) and all the above conditions are equivalent to \(-rw + b^\vee \leq j \leq -rw + a^\vee\). For brevity, let \(j' = j + rw\). Then we are seeking \(j'\) such that \(b^\vee \leq j' \leq a^\vee\) for all \(\iota\); this is possible because \(b^\vee \leq w/2 \leq a^\vee\); take \(j' = \lfloor w/2 \rfloor\).
3.2.4. **Proof of Theorem 1.6.** The proof, as explained in the introduction, is to explicate Thm. 1.1 for $L_f(\frac{1}{2} + m, \pi \otimes \omega_\pi \xi)$ and $L_f(\frac{1}{2} + m, \text{Sym}^2(\pi) \times \pi \otimes \xi)$ which we take up in the following two paragraphs:

3.2.4.1. **Thm. 1.1** for $L_f(\frac{1}{2} + m, \pi \otimes \omega_\pi \xi)$. If $\pi \in \text{Coh}(G_2, \mu^n)$ then $\omega_\pi \xi \in \text{Coh}(G_1, \text{det}(\mu)^n)$. We are in the situation when $n = 2$, hence $\epsilon = \eta$ and

$$\eta_v = (1 - (1 - (-1)^{w(\text{det}(\mu))/2}) = \xi_v(-1).$$

Hence $\eta = \epsilon_\xi$. Thm. 1.1 takes the form:

$$L_f(\frac{1}{2} + m, \pi \otimes \omega_\pi \xi) \sim p^\epsilon_m \xi(\pi) \text{G}(\omega_\pi \xi) p^\epsilon_m (\mu + m, \text{det}(\mu))$$

$$\sim p^\epsilon_m \xi(\pi) \text{G}(\omega_\pi) \xi(\xi) p^\epsilon_m (\mu + m, \text{det}(\mu)).$$

(3.7)

3.2.4.2. **Thm. 1.1** for $L_f(\frac{1}{2} + m, \text{Sym}^2(\pi) \times \pi \otimes \xi)$. Here, $n = 3$, hence $\eta = -\epsilon$ and

$$\epsilon_v = \omega_\text{Sym}^2(\pi)v(1 - (1 - (-1)^{w(\text{Sym}^2(\mu))/2}) = \omega_\pi^2(-1) \cdot (-1)^{w} = 1.$$ 

Hence $\epsilon = \epsilon_+ = -\epsilon_-$ and $\eta = -\epsilon_+ = -\epsilon_+ = \epsilon_-$.

$$L_f(\frac{1}{2} + m, \text{Sym}^2(\pi) \times \pi \otimes \xi) \sim p^{\epsilon_+} \text{Sym}^2(\pi) \cdot p^{-\epsilon_+} (\pi \otimes \xi) \cdot \text{G}(\omega_\pi \otimes \xi) \cdot p^{\epsilon_+} (\text{Sym}^2(\mu), \mu + m)$$

$$\sim p^{\epsilon_+} \text{Sym}^2(\pi) \cdot p^{-\epsilon_+} (\pi) \text{G}(\xi) \cdot \text{G}(\omega_\pi) \text{G}(\xi)^2 \cdot p^{\epsilon_+} (\text{Sym}^2(\mu), \mu + m).$$

(3.8)

Proof of Theorem 1.6 follows from (3.7) and (3.8).

4. **Appendix: Periods of tensor product motives.**

*by Chandrasheel Bhagwat*

We give a description of Deligne’s periods for the tensor product of two pure motives over $\mathbb{Q}$ all of whose nonzero Hodge numbers are one. The main period relations in the appendix are in Thm. 4.6 Thm. 4.8 and Thm. 4.10 which describe $c^\pm(M \otimes M')$ in terms of periods $c^\pm$ and some other finer invariants attached to $M$ and $M'$ by Yoshida [42].

4.1. **Preliminaries.**

4.1.1. **Critical motives.** Let $M$ be a motive defined over $\mathbb{Q}$ with coefficients in a number field $\mathbb{E}$. Let $H_B(M)$ be the Betti realization of $M$. It is a finite-dimensional vector space over $\mathbb{E}$. The rank $d(M)$ of $M$ is defined to be $\dim_{\mathbb{E}} H_B(M)$. Write $H_B(M) = H_{B,0}(M) \oplus H_{B,1}(M)$, where $H_{B,1}(M)$ are the $\pm$-eigenspaces for the action of complex conjugation $\rho$ on $H_B(M)$. Let $d_\pm(M)$ be the $\mathbb{E}$-dimensions of $H_B^\pm(M)$. The Betti realization has a Hodge decomposition:

$$H_B(M) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p, q \in \mathbb{Z}} H^{p, q}(M),$$

where $H^{p, q}(M)$ is a free $\mathbb{E} \otimes \mathbb{C}$-module of rank $h^{p, q}_M$. The numbers $h^{p, q}_M$ are called the Hodge numbers of $M$. We say that $M$ is pure if there is an integer $w$ (which is called the purity weight of $M$) such that $H^{p, q}(M) = \{0\}$ if $p + q \neq w$. Henceforth, we assume that all the motives we consider are pure. We also have $\rho(H^{p, q}(M)) = H^{q, p}(M)$; and hence $\rho$ acts on the (possibly zero) middle Hodge type $H^{w/2, w/2}(M)$.

Let $H_{DR}(M)$ be the de Rham realization of $M$; it is a $d(M)$-dimensional vector space over $\mathbb{E}$. There is a comparison isomorphism of $\mathbb{E} \otimes \mathbb{Q}$-$\mathbb{C}$-modules:

$$I : H_B(M) \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow H_{DR}(M) \otimes_{\mathbb{Q}} \mathbb{C}.$$
The de Rham realization has a Hodge filtration $F^p(M)$ which is a decreasing filtration of $\mathbb{E}$-subspaces of $H_{DR}(M)$ such that $I \left( \bigoplus_{p' \geq p} H^{p'} \right) = F^p(M) \otimes_{\mathbb{Q}} \mathbb{C}$. Write the Hodge filtration as

\begin{equation}
H_{DR}(M) = F^{p_1}(M) \supseteq F^{p_2}(M) \supseteq \cdots \supseteq F^{p_n}(M) \supseteq F^{p_{n+1}}(M) = \{0\};
\end{equation}

all the inclusions are proper and there are no other filtration-pieces between two successive members. We assume that the numbers $p_\mu$ are maximal among all the choices. Let $s_\mu = h_{DR}^{p_\mu - p_{\mu - 1}}$ for $1 \leq \mu \leq m$. Purity plus the action of complex conjugation on Betti realizations of $H_{DR}$ says that the numbers $p_j$ and $s_\mu$ satisfy $p_j + p_{m+1-j} = w, \forall 1 \leq j \leq m$, and $s_\mu = s_{m+1-\mu}, \forall 1 \leq \mu \leq m$.

We say that the motive $M$ is critical if there exist $p^+, p^- \in \mathbb{Z}$ such that $\sum_{i=1}^{p^+} s_i = d^+(M)$ and $\sum_{i=1}^{p^-} s_i = d^-(M)$. In this case one says that $F^\pm(M)$ exists and equals $F^{p^\pm}(M)$.

4.1.2. Tensor product of motives. Let $M$ and $M'$ be pure motives defined over $\mathbb{Q}$ and with coefficients in a number field $\mathbb{K}$. Suppose that their ranks are $n$ and $n'$ and purity weights are $w$ and $w'$ resp. We further assume that all the non-zero Hodge numbers of $M$ and $M'$ equal to 1.

Suppose $H_B(M) \otimes \mathbb{C} = \bigoplus_{j=1}^{n} H^{p_j, w-p_j}(M)$, where $p_j$ are integers such that $p_1 < p_2 < \ldots p_n$. Similarly, suppose $H_B(M') \otimes \mathbb{C} = \bigoplus_{j=1}^{n'} H^{q_j, w'-q_j}(M')$, with $q_1 < q_2 < \ldots q_{n'}$.

Since all the non-zero Hodge numbers of $M$ and $M'$ equal to 1, it follows that the Hodge filtrations of the de Rham realizations of $M$, $M'$ and $M \otimes M'$ are given by

$$
H_{DR}(M) = F^{p_1}(M) \supseteq F^{p_2}(M) \supseteq \cdots \supseteq F^{p_n}(M) \supseteq (0),
$$

$$
H_{DR}(M') = F^{q_1}(M) \supseteq F^{q_2}(M) \supseteq \cdots \supseteq F^{q_{n'}}(M') \supseteq (0),
$$

$$
H_{DR}(M \otimes M') = F^{r_1}(M \otimes M') \supseteq F^{r_2}(M \otimes M') \supseteq \cdots \supseteq F^{r_m}(M \otimes M') \supseteq (0).
$$

Let $u_t$ denote the dimension of $F^{r_t}(M \otimes M')/F^{r_{t+1}}(M \otimes M')$ for $1 \leq t \leq m$. Let us further assume that $M \otimes M'$ is critical. Consider the complex conjugation action on Betti realizations for the motives $M$ and $M'$.

If the dimension $nn'$ is an even integer, it follows that $d^\pm(M \otimes M')$ are equal to $\frac{nn'}{2}$. From the criticality of $M \otimes M'$, it follows that there is $k^+ = k^- = k_0 \geq 1$ such that

$$
u_1 + \nu_2 + \ldots + \nu_{k_0} = d^+(M \otimes M') = \frac{nn'}{2}.
$$

Let $1 \leq i \leq n$ and $1 \leq j \leq n'$. Following Yoshida [42], we define:

$$a_i = |\{j : 1 \leq j \leq n' : p_i + q_j < r_{k_0}\}|, \quad a_j = |\{i : 1 \leq i \leq n : p_i + q_j \leq r_{k_0}\}|.
$$

If $nn'$ is odd, there exist $k^+, k^-$ such that

$$u_1 + u_2 + \ldots + u_{k^+} = d^+(M \otimes M') = \frac{nn' + 1}{2},
$$

$$u_1 + u_2 + \ldots + u_{k^-} = d^-(M \otimes M') = \frac{nn' - 1}{2}.
$$

It follows that $k^+ - k^- = d^+(M \otimes M') - d^-(M \otimes M') = \pm 1$. Let $1 \leq i \leq n$ and $1 \leq j \leq n'$. In this case we define:

$$a_i^+ = |\{j : 1 \leq j \leq n' : p_i + q_j \leq r_{k^+}\}|, \quad a_j^{+,+} = |\{i : 1 \leq i \leq n : p_i + q_j \leq r_{k^+}\}|.$$

Let us further assume that $M \otimes M'$ is critical. Consider the complex conjugation action on Betti realizations for the motives $M$ and $M'$.
4.1.3. Invariant polynomials and periods. The period matrix of \( M \) is defined in terms of \( \mathbb{E} \)-bases for the spaces \( H^\pm_B(M) \) and \( H_{DR}(M) \). Let \( \{ v_1, v_2, \ldots, v_{d^+(M)} \} \) be an \( \mathbb{E} \)-basis of \( H^+_B(M) \), and similarly, \( \{ v_{d^+(M)+1}, v_{d^+(M)+2}, \ldots, v_{d(M)} \} \) be an \( \mathbb{E} \)-basis of \( H^-_B(M) \). Let \( \{ w_1, w_2, \ldots, w_{d(M)} \} \) be a basis of \( H_{DR}(M) \) over \( \mathbb{E} \) such that \( \{ w_{s_1+s_2+\ldots+s_{\mu-1}+1}, \ldots, w_{d(M)} \} \) is a basis of \( F^\mu(M) \) for \( 1 \leq \mu \leq m \). The period matrix \( X \) of \( M \) is the matrix which represents the comparison isomorphism between the two realizations of \( M \) with respect to the bases chosen above. The fundamental periods \( c^\pm(M) \) and \( \delta(M) \) are related to the matrix \( X \) through certain invariant polynomials.

Let \( \mathbb{F} \) be a number field. Suppose \( d \) is a positive integer. Fix a partition \( s_1 + s_2 + \ldots + s_m = d \). Let \( P_m \) be the corresponding lower parabolic subgroup of \( \text{GL}(d, F) \). Given an \( m \)-tuple of integers \( (a_i)_{1 \leq i \leq m} \), define an algebraic character \( \lambda_1 \) of \( P_m \) by

\[
\lambda_1 \left( \begin{pmatrix} p_{11} & 0 & \ldots & 0 \\ * & p_{22} & \ldots & 0 \\ * & * & \ddots & \vdots \\ * & * & \ddots & p_{mm} \end{pmatrix} \right) = \prod_{1 \leq i \leq m} (\det p_{ii})^{a_i}; \quad p_{ii} \in \text{GL}(s_i).
\]

Let \( d = d^+ + d^- \). Given \( k^+, k^- \in \mathbb{Z} \), define a character \( \lambda_2 \) of \( \text{GL}(d^+) \times \text{GL}(d^-) \) by

\[
\lambda_2 \left( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right) = (\det a)^{k^+}(\det b)^{k^-}, \quad a \in \text{GL}(d^+), \quad b \in \text{GL}(d^-).
\]

Let \( f(x) \) be a polynomial with rational coefficients which satisfies the following equivariance condition with respect to the left action of \( P_m \) and the right action of \( \text{GL}(d^+) \times \text{GL}(d^-) \) on the matrix ring \( M_d(F) \):

\[
(4.3) \quad f(px\gamma) = \lambda_1(p)f(x)\lambda_2(\gamma), \quad \forall \ p \in P_m, \ \forall \ \gamma \in \text{GL}(d^+) \times \text{GL}(d^-).
\]

A polynomial satisfying (4.3) is said to have admissibility type \( \{(a_1, a_2, \ldots, a_m), (k^+, k^-)\} \). Yoshida [24, Theorem 1] proves that the space of polynomials of a given admissibility type is at most one.

**Lemma 4.4.** If the polynomial \( f(x) \) has admissibility type \( \{(a_1, a_2, \ldots, a_m), (k^+_1, k^-_1)\} \), and \( g(x) \) has admissibility type \( \{(b_1, b_2, \ldots, b_m), (k^+_2, k^-_2)\} \), then the polynomial \( h(x) = f(x)g(x) \) has admissible type is given by

\[
\{(a_1 + b_1, a_2 + b_2, \ldots, a_m + b_m), (k^+_1 + k^+_2, k^+_2 + k^-_2)\}.
\]

**Proof.** Follows from (4.3). \( \square \)

The admissibility type of \( f(x) = \det(x) \) for \( x \in M_d(F) \), is \( \{(1,1,1,\ldots,1), (1,1)\} \). Let \( f^\pm(x) \) be the upper left (resp., upper right) \( d^\pm \times d^\pm \) determinant of \( x \). Then it can be seen that the admissibility types of \( f^+(x) \) and \( f^-(x) \) are respectively given by

\[
\{(1,1,1,\ldots,1,0,0,\ldots,0), (1,0)\}, \quad \{(1,1,1,\ldots,1,0,0,\ldots,0), (0,1)\}.
\]

Yoshida interprets the period invariants to the period matrix \( X \) via invariant polynomials as \( \delta(M) = f(X) \) and \( c^\pm(M) = f^\pm(X) \). The determinant of the period matrix is an element of \( \mathbb{E} \times \mathbb{C} \), and the making a choice of basis says that it is well-defined modulo \( \mathbb{E}^\times \).
4.2 Calculation of motivic periods \(c^\pm(M \otimes M')\). One knows from the results of Yoshida [42] that the motivic periods \(c^\pm(M \otimes M')\) can be expressed as monomials in the other period invariants \(\delta(M), c^\pm(M), c_p(M), c_p(M'), (p \text{ runs over finite set})\). For the definitions of these period invariants, see [42]. In this section we calculate these monomials explicitly.

First we consider the case where the ranks of motives have opposite parities. Let \(M\) and \(M'\) be motives defined over \(\mathbb{Q}\) with coefficients in \(\mathbb{E}\) as in the section 4.1 with ranks \(n = 2k\) and \(n' = 2k' + 1\) resp. We set \(\epsilon(M') := d^+(M') - d^-(M') = \pm 1\). Thus \(d^\pm(M) = k, d^+(M') = k' + \epsilon(M')\) and \(d^-(M') = k' - \epsilon(M')\). Let’s define two finite sets by,

\[
\mathcal{P} = \{1, 2, \ldots, k - 1 = \min\{d^\pm(M)\} - 1\},
\]
\[
\mathcal{P}' = \{1, 2, \ldots, k' - 1 = \min\{d^\pm(M')\} - 1\}.
\]

Consider the expression for \(c^+(M \otimes M')\) as a monomial in other period invariants with integer exponents as follows:

\[
(4.5) \quad c^+(M \otimes M') = \delta(M)^\alpha \delta(M')^\beta \ c^+(M)^{\alpha^+} \ c^-(M)^{\alpha^-} \ c^+(M')^{\beta^+} \ c^-(M')^{\beta^-} \prod_{p \in \mathcal{P}} c_p(M)^{\alpha_p} \prod_{p \in \mathcal{P}'} c_p(M')^{\beta_p}.
\]

We have a similar expression for \(c^-(M \otimes M')\). From Yoshida [42], we know that admissibility types for the period invariants \(\delta(M), \delta(M'), c^+(M), c^+(M'), c_p(M), c_p(M')\) are given by:

\[
\delta(M) : \underbrace{(1, 1, \ldots, 1)}_{n \text{ times}}, \quad \delta(M') : \underbrace{(1, 1, \ldots, 1)}_{n' \text{ times}},
\]
\[
c^+(M) : \underbrace{(1, 1, \ldots, 1, 0, \ldots, 0)}_{d^+(M) \text{ times}}, \quad c^-(M) : \underbrace{(1, 1, \ldots, 1, 0, \ldots, 0)}_{d^-(M) \text{ times}},
\]
\[
c^+(M') : \underbrace{(1, 1, \ldots, 1, 0, \ldots, 0)}_{d^+(M') \text{ times}}, \quad c^-(M') : \underbrace{(1, 1, \ldots, 1, 0, \ldots, 0)}_{d^-(M') \text{ times}},
\]
\[
c_p(M) : \underbrace{(1, 1, \ldots, 1, 0, \ldots, 0)}_{p \text{ times}} \cup \underbrace{(1, 1, \ldots, 1, 0, \ldots, 0)}_{\text{min}(\{1, 1, \ldots, 1, 0, \ldots, 0\}) \forall \ p \in \mathcal{P}, \text{ and}}
\]
\[
c_p(M') : \underbrace{(1, 1, \ldots, 1, 0, \ldots, 0)}_{p \text{ times}} \cup \underbrace{(1, 1, \ldots, 1, 0, \ldots, 0)}_{\text{min}(\{1, 1, \ldots, 1, 0, \ldots, 0\}) \forall \ p \in \mathcal{P}').
\]

Let \(X\) and \(Y\) be the period matrices corresponding to the comparison isomorphism between Betti and de Rham realizations. Then \(c^\pm(M \otimes M')\) equals the product of \(\phi^\pm(X) \psi^\pm(Y)\) and their admissibility types are:

\[
\phi^\pm(X) : (a_1, a_2, \ldots, a_n), \quad (d^\pm(M'), d^\mp(M')) \quad \text{and} \quad \psi^\pm(Y) : (a_1^*, a_2^*, \ldots, a_n^*), \quad (d^\pm(M), d^\mp(M)).
\]

Using the admissibility types and (4.5) we get:

\[
\alpha = a_n, \quad \alpha^+ = a_k - n'/2 + \epsilon(M')/2, \quad \alpha^- = a_k - n'/2 - \epsilon(M')/2, \quad \alpha_p = a_p - a_{p+1} \forall \ p \in \mathcal{P},
\]
\[
\beta = a_n^*, \quad \beta^+ = \beta^- = a_k^* - n/2, \quad \beta_p = a_p^* - a_{p+1}^* \forall \ p \in \mathcal{P}'.
\]
Theorem 4.6. If the ranks of $M$ and $M'$ are even and odd respectively and $M \otimes M'$ is critical, then the periods $c^\pm (M \otimes M')$ are given by:

$$c^+(M \otimes M') = Tc^{(M')}(M), \quad c^-(M \otimes M') = Tc^{-\epsilon(M')}(M),$$

where $T$ is defined as:

$$T = \delta(M)^{\alpha_n} \delta(M')^{a^+_n} \left[ c^+(M)c^-(M) \right]^{a_k-k'} \left[ c^+(M')c^-(M') \right]^{a^+_k-k} \cdot \prod_{p \in \mathcal{P}} c_p(M)^{a_p-a_{p+1}} \prod_{p \in \mathcal{P}'} c_p(M')^{a^+_p-a^+_{p+1}}.$$

The ± sign in the exponent of $c^\pm (M)$ period in the above expression is determined by the sign of $\epsilon(M')$. This in particular is consistent with the following result of [1].

(4.7) $$\frac{c^+(M \otimes M')}{c^-(M \otimes M')} = \left( \frac{c^+(M)}{c^-(M)} \right)^{\epsilon(M')}.$$

The case when both $M$ and $M'$ have same parity can be handled in an exactly analogous manner. We consider the two cases.

**Case 1:** Both $M$ and $M'$ have odd ranks $n = 2k + 1$, $n' = 2k' + 1$, respectively. From the definition of the integers $a^\pm_i$ and $a^\pm_*$ it follows that

$$a^+_i = a^-_i \quad \forall \ 1 \leq i \leq \min \{d^\pm(M)\}, \quad \text{and} \quad a^\pm_* = a^\pm_* \quad \forall \ 1 \leq j \leq \min \{d^\pm(M')\}.$$

Theorem 4.8. If both $M$ and $M'$ are of odd rank and $M \otimes M'$ is critical, then

$$c^+(M \otimes M') = Tc^{\epsilon(M')}(M)c^{\epsilon(M)}(M'), \quad c^-(M \otimes M') = Tc^{-\epsilon(M')}(M)c^{-\epsilon(M)}(M'),$$

where the period $T$ is defined by the same formula as in Thm. 4.6, but note that $k$ and $\mathcal{P}$ (resp., $k'$ and $\mathcal{P}'$) depend on $n$ (resp., $n'$).

The following period relation from [1] is an easy consequence:

(4.9) $$\frac{c^+(M \otimes M')}{c^-(M \otimes M')} = \left( \frac{c^+(M)}{c^-(M)} \right)^{\epsilon(M')} \left( \frac{c^+(M')}{c^-(M')} \right)^{\epsilon(M)}.$$

**Case 2:** If both $M$ and $M'$ have even ranks then it turns out that $c^+(M \otimes M') = c^-(M \otimes M')$. Set $k = n/2$ and $k' = n/2$, then we have:

Theorem 4.10. If both $M$ and $M'$ are of even rank and $M \otimes M'$ is critical, then

$$c^\pm(M \otimes M') = \delta(M)^{\alpha_n} \delta(M')^{a^+_n} \left( c^+(M)c^-(M) \right)^{a_k-k'} \left( c^+(M')c^-(M') \right)^{a^+_k-k} \cdot \prod_{p \in \mathcal{P}} c_p(M)^{a_p-a_{p+1}} \prod_{p \in \mathcal{P}'} c_p(M')^{a^+_p-a^+_{p+1}}.$$

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