THE RANK STABLE TOPOLOGY OF INSTANTONS ON $\overline{\mathbb{CP}^2}$

JIM BRYAN
MARC SANDERS

Abstract. Let $\mathcal{M}_k^n$ be the moduli space of based (anti-self-dual) instantons on $\overline{\mathbb{CP}^2}$ of charge $k$ and rank $n$. There is a natural inclusion $\mathcal{M}_k^n \hookrightarrow \mathcal{M}_k^{n+1}$. We show that the direct limit space $\mathcal{M}_k^\infty$ is homotopy equivalent to $BU(k) \times BU(k)$. Let $\ell_\infty$ be a line in the complex projective plane and let $\overline{\mathbb{CP}^2}$ be the blow-up at a point away from $\ell_\infty$. $\mathcal{M}_k^n$ can be alternatively described as the moduli space of rank $n$ holomorphic bundles on $\overline{\mathbb{CP}^2}$ with $c_1 = 0$ and $c_2 = k$ and with a fixed holomorphic trivialization on $\ell_\infty$.

1. Introduction

In his 1989 paper [Ta], Taubes studied the stable topology of the based instanton moduli spaces. He showed that if $\mathcal{M}_k^n(X)$ denotes the moduli space of based $SU(n)$-instantons of charge $k$ on $X$, then there is a map $\mathcal{M}_k^n(X) \to \mathcal{M}_k^{n+1}(X)$ and, in the direct limit topology, $\mathcal{M}_k^\infty(X)$ has the homotopy type of $\text{Map}_0(X, BSU(n))$.

There is also a map $\mathcal{M}_k^n(X) \hookrightarrow \mathcal{M}_k^{n+1}(X)$ given by the direct sum of a connection with the trivial connection on a trivial line bundle and one can consider the direct limit $\mathcal{M}_k^\infty(X)$. For the case of $X = S^4$ with the round metric, it was shown by Kirwan and also by Sanders ([Ki], [Sa]) that the direct limit has the homotopy type of $BU(k)$.

In this note we consider the case of $X = \overline{\mathbb{CP}^2}$ where $\overline{\mathbb{CP}^2}$ denotes the complex projective plane with the Fubini-Study metric and the opposite orientation of the one induced by the complex structure. Our result is:

Theorem 1.1. $\mathcal{M}_k^\infty(\overline{\mathbb{CP}^2})$ has the homotopy type of $BU(k) \times BU(k)$.

The main tool in the proof of the theorem is a construction of the moduli spaces $\mathcal{M}_k^n(\overline{\mathbb{CP}^2})$ due to King [Ki]. In general, Buchdahl [Bu] has shown that, for appropriate metrics on the $N$-fold connected sum $\#_N \overline{\mathbb{CP}^2}$, the moduli spaces $\mathcal{M}_k^n(\#_N \overline{\mathbb{CP}^2})$ are diffeomorphic to certain spaces of equivalence classes of holomorphic bundles on $\overline{\mathbb{CP}^2}$ blown-up at $N$ points. The universal $U(k) \times U(k)$ bundle that appears giving the homotopy equivalence of theorem 1.1 can be constructed as higher direct image bundles (see section 3).

Remark 1.1. The cofibration $S^2 \to \overline{\mathbb{CP}^2} \to S^4$ gives rise to the fibration of mapping spaces $\Omega^4 BSU(n) \to \text{Map}_*(\overline{\mathbb{CP}^2}, BSU(n)) \to \Omega^2 BSU(n)$ which for K-theoretic reasons is a trivial fibration in the limit over $n$. The total space of this fibration is homotopy equivalent to the space of based gauge equivalence classes of all connections on $\overline{\mathbb{CP}^2}$. Thus, from Taubes’ result, $\mathcal{M}_k^\infty$ must have the property that taking the limit over $k$ gives $BU \times BU$. For $S^4$, similar remarks imply that $\lim_{k \to \infty} \mathcal{M}_k^\infty(S^4) \simeq BU$ and the inclusion of $\mathcal{M}_k^\infty(S^4)$ into this limit has been
shown to be (up to homotopy) the natural inclusion $BU(k) \hookrightarrow BU$ \[\text{(Sa)}\]. Theorem 1.1 and these results for $S^4$ suggest a general conjecture which is supported by the fact that the higher direct image bundle giving our homotopy equivalence generalizes in an appropriate way.

**Conjecture 1.1.** For appropriate metrics on $\#_N CP^2$, $M_k^\infty(\#_N CP^2)$ has the homotopy type of a product $BU(k) \times \cdots \times BU(k)$ with $N+1$ factors.

**Remark 1.2.** Combining theorem \[1.1\] with Taubes’ stabilization result leads to an alternate proof of Bott periodicity for the unitary group. There is a natural map from instantons on $S^4$ to those on $CP^2$ (given by pull-back) which in the limit as $n \to \infty$ is homotopy equivalent to the diagonal $BU(k) \to BU(k) \times BU(k)$. Taking the limit as $k \to \infty$ and applying Taubes’ result, the diagonal map appears as the inclusion of fibers in the fibration $BU \simeq \Omega_4^2 BU \to BU \times BU \to \Omega^2(BSU)$ (see remark 1.1). However, the map $BU \to BU \times BU$ has homotopy fiber $U \times U \cup U \simeq U$, and therefore, given the above fibration we must have $U \simeq \Omega^3 BSU \simeq \Omega^2 SU \simeq \Omega^2 U$.

Tian (\[13\]) noticed that results for limits of instantons on $S^4$ (see \[Kir\] or \[Sa\]) already imply the four-fold periodicity $\Omega^4 BU \simeq Z \times BU$. The $CP^2$ case thus gives the finer two-fold periodicity $\Omega^2 BU \simeq Z \times BU$, as one might expect due to the nontrivial $S^2 \hookrightarrow CP^2$.

In a future paper we will study limits of $Sp(n)$ and $SO(n)$ instantons on $CP^2$ and their relationships to those on $S^4$. As an amusing corollary, we will be able to rederive many of the Bott periodicity relationships among $Sp$, $U$, $SO$, and their homogeneous spaces.

2. The Construction of $M_k^\infty(\overline{CP^2})$

Let $x_0 \in \overline{CP^2}$ be the base point. Since $\overline{CP^2} \setminus \{x_0\}$ is conformally equivalent to $C^2$, the complex plane blown-up at the origin, $M_k^\infty(\overline{CP^2})$ can be regarded as instantons on $C^2$ based “at infinity”. Buchdahl \[Bu\] proved an analogue in this non-compact setting of Donaldson’s theorem relating instantons to holomorphic bundles: Let $C_N^2$ be the complex plane blown-up at $N$ points with a Kähler metric. Then $C_N^2$ has a “conformal compactification” to $\#_N CP^2$ and a “complex compactification” to $CP^2_N$ (the projective plane blown-up at $N$ points). We have added a point $x_0$ in the former case and a complex projective line $\ell_\infty$ in the latter.

Define $M_{alg,k}(\overline{CP^2}_N)$ to be the moduli space consisting of pairs $(\mathcal{E}, \tau)$ where $\mathcal{E}$ is a rank $n$ holomorphic bundle on $\overline{CP^2}_N$ with $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = k$, and where $\tau : \mathcal{E}|_{\ell_\infty} \to C^n \otimes \mathcal{O}_{\ell_\infty}$ is a holomorphic trivialization of $\mathcal{E}$ on $\ell_\infty$.

There is a natural map $\Phi : M_k^\infty(\#_N CP^2) \to M_{alg,k}(\overline{CP^2}_N)$ defined as follows. Let $p : \overline{CP^2}_N \to \#_N CP^2$ be the map that collapses $\ell_\infty \mapsto x_0$. If $[A] \in M_k^\infty$ then the $\bar{\partial}$ operator that defines the holomorphic bundle $\mathcal{V} = \Phi(A)$ is taken to be $(d_{\bar{\partial}}(A))^{(0,1)}$, the anti-holomorphic part of the covariant derivative defined by the pullback of the connection. The anti-self-duality of $A$ implies that the curvature of $p^\ast(A)$ is a $(1, 1)$-form and so $\bar{\partial}^2 = 0$.

Buchdahl’s theorem is then
Theorem 2.1. The map $\Phi : M^0_k(\#_N \text{CP}^2) \to M^0_{alg,k}(\text{CP}^2)$ is a diffeomorphism.

The case $N = 1$ was first proved by King [Ki]. We now restrict ourselves to that case and simply write $M^0_k$ for $M^0_k(\text{CP}^2)$ and $M^0_{alg,k}(\text{CP}^2)$.

King constructed $M^0_k$ explicitly in terms of linear algebra data. We recall his construction. Consider configurations of linear maps:

$$
\begin{array}{c}
W_0 & \xrightarrow{a_1, a_2} & W_1 \\
b & \searrow & c \\
& V_\infty &
\end{array}
$$

where $W_0$, $W_1$ and $V_\infty$ are complex vector spaces of dimensions $k$, $k$, and $n$ respectively.

A configuration $(a_1, a_2, b, c, x)$ is called integrable if it satisfies the equation

$$a_1 x a_2 - a_2 x a_1 + bc = 0.$$

A configuration $(a_1, a_2, b, c, x)$ is non-degenerate if it satisfies the following conditions:

$$\forall (\lambda_1, \lambda_2), (\mu_1, \mu_2) \in \mathbb{C}^2 \text{ such that } \lambda_1 \mu_1 + \lambda_2 \mu_2 = 0 \text{ and } (\mu_1, \mu_2) \neq (0,0),$$

$$\hat{\Phi} v \in W_1 \text{ such that } \begin{cases} x a_1 v = \lambda_1 v \\ x a_2 v = \lambda_2 v \\ c v = 0 \end{cases}$$

and $\hat{\Phi} w \in W_0^*$ such that

$$\begin{cases} x^* a_1^* w = \lambda_1 w \\ x^* a_2^* w = \lambda_2 w \\ b^* w = 0 \end{cases}$$

Let $A^0_k$ be the space of all integrable non-degenerate configurations. $G = \text{Gl}(W_0) \times \text{Gl}(W_1)$ acts canonically on $A^0_k$. The action is explicitly given by

$$(g_0, g_1) \cdot (a_1, a_2, b, c, x) = (g_0 a_1 g_1^{-1}, g_0 a_2 g_1^{-1}, g_0 b, c g_1^{-1}, g_1 x g_0^{-1})$$

Theorem 2.2. The moduli space $M^0_k$ is isomorphic to $A^0_k/G$.

Proof. King uses such configurations to determine monads that in turn determine holomorphic bundles. Configurations in the same $G$ orbit determine the same bundle. For the sake of brevity we refer the reader to [Ki] or [Br] for details. The construction identifies the vector spaces $W_0$ and $W_1$ canonically as $H^1(\mathcal{E}(\ell_\infty))$ and $H^1(\mathcal{E}(\ell_\infty + E))$ respectively, where $E \subset \text{CP}_2$ is the exceptional divisor. The vector space $V_\infty$ is identified with the fiber over $\ell_\infty$.

3. Proof of theorem 1.1

We prove the theorem in two steps: We first show that the space of monad data $A^0_k$ forms a principal $G = \text{Gl}(k) \times \text{Gl}(k)$ bundle over $M^0_k$. We then show that the induced $G$-equivariant inclusion $A^0_k \hookrightarrow A^{n+2k}_k$ is null-homotopic so that we can conclude that $A^\infty_k$ is contractible.

Lemma 3.1. $G$ acts freely on the space of monad data $A^0_k$. 

Proof. This is essentially proved in [K] where it is implicitly shown that the non-degeneracy conditions are precisely the conditions that guarantee freeness. We point out that this also follows more conceptually from the existence of a universal family $\mathbb{E} \to \mathcal{M}_k^n \times \mathbb{C}\mathbb{P}^2$ and the cohomological interpretation of $W_0$ and $W_1$:

First, the existence of a universal family can be shown via the gauge theoretic construction: Let $V$ be a smooth hermitian vector bundle on $\mathbb{C}\mathbb{P}^2$ with $c_1(V) = 0$ and $c_2(V) = k$. Let $A_{1,1}^n$ denote unitary connections on $V$ with curvature of pure type $(1,1)$ and that restrict to the trivial connection on $E_\infty$, then $\mathcal{M}_k^n = A_{1,1}^n/\mathbb{G}_0^C$. The quotient

$$(A_{1,1}^n \times V)/\mathbb{G}_0^C \to \mathcal{M}_k^n \times \mathbb{C}\mathbb{P}^2$$

will form a universal bundle if the moduli space is smooth and no $E \in \mathcal{M}_k^n$ has non-trivial automorphisms (c.f. [Fr-Mo] Chapt. IV):

**Lemma 3.2.** $\mathcal{M}_k^n$ is smooth and any $E \in \mathcal{M}_k^n$ has no non-trivial automorphisms preserving $\tau: E|_{E_\infty} \to \mathbb{C}^n \otimes \mathcal{O}_{E_\infty}$.

By Serre duality $H^2(E \otimes E^*) = H^0(E \otimes E^* \otimes K)^*$. Since $E \otimes E^*$ is trivial on $E_\infty$, it is also trivial on nearby lines. Any section of $E \otimes E^* \otimes K$ restricts to a section of $\mathbb{C}^n \otimes \mathcal{O}_{E_\infty}(-3)$ and so must vanish on $E_\infty$. Likewise, it must vanish on nearby lines and so it is 0 on an open set and must be identically 0. Thus $H^2(E \otimes E^*) = 0$ and smoothness follows once we show there are no automorphisms.

Suppose that there exists an automorphism $\phi \in H^0(E \otimes E^*)$ such that $\phi \neq \mathbb{1}$ and $\phi$ preserves $\tau$ so that $\phi|_{E_\infty} = \mathbb{1}|_{E_\infty}$. Then $\phi - \mathbb{1}$ is a non-zero section of $E \otimes E^*$ vanishing on $E_\infty$. We then get an injection $0 \to \mathcal{O}(E_\infty) \to E \otimes E^*$. Restricting this sequence to $E_\infty$ we get an injection $0 \to \mathcal{O}(E_\infty) \to E \otimes E^*$. Using Serre duality and the same argument, one gets the vanishing for $H^2$.

Thus the vector spaces $W_0$ and $W_1$ are the fibers of the vector bundles $R^1 \pi_* (E(-E_\infty))$ and $R^1 \pi_* (E(-E_\infty + E))$. The $G$-orbit of a configuration giving a bundle $E$ can be identified with the group of isomorphisms $g_0 : H^1(E(-E_\infty)) \to \mathbb{C}^k$ and $g_1 : H^1(E(-E_\infty + E)) \to \mathbb{C}^k$. Thus $A_k^n$ is realized precisely as the total space of the principal $GL(k) \times GL(k)$ bundle associated to $R^1 \pi_* (E(-E_\infty)) \oplus R^1 \pi_* (E(-E_\infty + E))$. 

Recall that the map $\mathcal{M}_k^n \to \mathcal{M}_k^{n+1}$ is defined by the direct sum with the trivial connection: $[A] \mapsto [A \oplus \theta]$. In terms of holomorphic bundles this is $E \mapsto E \oplus \mathcal{O}$. Tracing through the monad construction, it is easy to see that the inclusion induces the $G$-equivariant map $A_k^n \to A_k^{n+1}$ given by $(a_1, a_2, x, b, c) \mapsto (a_1, a_2, x, b', c')$ where $b'$ is $b$ with an extra first column of zeroes and $c'$ is $c$ with an extra first row of zeroes. Define $A_k^\infty$ to be the direct limit $\lim_{n \to \infty} A_k^n$ so that there is a homeomorphism between $\mathcal{M}_k^\infty$ and $A_k^\infty / G$.

**Lemma 3.3.** $A_k^\infty$ is a contractible space.
Proof. Since the $A^n_k$’s are algebraic varieties and the maps $A^n_k \to A^{n+1}_k$ are algebraic, they admit triangulations compatible with the maps. Thus $A^\infty_k$ inherits the structure of a CW-complex and so it is sufficient to show that all of its homotopy groups are zero. To this end we prove that for any $k$ and $l$ there is an $r > l$ such that the natural inclusion from $A^n_k \hookrightarrow A^r_k$ is homotopically trivial.

Consider the homotopy $H_t : A^n_k \to A^{2k+n}_k$ defined as follows:

$$H_t((a_1, a_2, x, b, c)) = ((1-t)a_1, (1-t)a_2, (1-t)x, b_t, c_t)$$

where

$$c_t = \begin{pmatrix} tI_k \\ 0_{k,k} \\ (1-t)c \end{pmatrix}, \quad b_t = (0_{k,k}, tI_k, (1-t)^2b),$$

$I_k$ is the $k \times k$ identity matrix and $0_{k,k}$ is the $k \times k$ zero matrix. To see that $H_t(v) \in A^{n+2k}_k$ for any $v \in A^n_k$, we check that the integrability and non-degeneracy conditions are satisfied for all $0 \leq t \leq 1$. Integrability holds because $b_tc_t = (1-t)^3bc$. Non-degeneracy is satisfied for all $t \neq 0$ because there is a full rank $k \times k$ block, $tI_k$, in both $c_t$ and $b_t$. Furthermore, $H_0$ is just the inclusion $A^n_k \hookrightarrow A^{n+2k}_k$, so non-degeneracy also holds when $t = 0$. Finally, note that $H_1$ is a constant map.

These lemmas show that $A^\infty_k$ is a contractible space acted on freely by $G = Gl(k) \times Gl(k)$ and $A^\infty_k/G = M^\infty_k$. Thus $M^\infty_k$ is homotopic to $BG$ which in turn has the homotopy type of $BU(k) \times BU(k)$. We end by remarking that the proof shows that the universal $U(k) \times U(k)$ bundle is the bundle that restricts to any of the finite $M^n_k$’s as $R^1\pi_*(E(-\ell_\infty)) \oplus R^1\pi_*(E(-\ell_\infty + E))$.

References

[Br] Bryan, J. Symplectic Geometry and the Relative Donaldson Invariants of $\mathbb{C}P^2$, to appear in Forum Math.

[Bu] Buchdahl, N. Instantons on nCP$^2$. J. Diff. Geo. 37(1993).

[Fr-Mo] Robert Friedman and John Morgan. The Differential Topology of Complex Surfaces. Springer-Verlag, 1994.

[Ki] King, A. Instantons and Holomorphic Bundles on the Blown Up Plane, Ph.D. Thesis, Oxford, 1989.

[Kir] Kirwan, F. Geometric invariant theory and the Atiyah-Jones conjecture, Proceedings of the Sophus Lie Memorial Conference (Oslo, 1992), editors O. A. Laudal and B. Jahren, Scandinavian University Press, 1994, 161-188.

[Sa] Sanders, M., Classifying spaces and dirac operators coupled to instantons, Trans. of the A.M.S. Vol. 347, No. 10 1995.

[Ta] Taubes, C. The stable topology of self-dual moduli spaces J. Diff. Geo. 29 (1989).

[Ti] Tian, Y. The Atiyah-Jones Conjecture for classical groups and Bott periodicity, to appear in J. Diff. Geo.