CANONICAL DIMENSION OF PROJECTIVE
PGL₁(A)-HOMOGENEOUS VARIETIES

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Abstract. Let \( A \) be a central division algebra over a field \( F \) with \( \text{ind} A = n \). For integers \( 1 \leq d_1 < d_2 < \cdots < d_k \leq n - 1 \), let \( X_{d_1, d_2, \ldots, d_k}(A) \) be the variety of flags of right ideals \( I_1 \subset I_2 \subset \cdots \subset I_k \) of \( A \) with \( I_i \) of reduced dimension \( d_i \). In computing canonical \( p \)-dimension of such varieties, for \( p \) prime, we can reduce to the case of generalized Severi-Brauer varieties \( X_e(A) \) with \( \text{ind} A \) a power of \( p \) divisible by \( e \). We prove that canonical 2-dimension (and hence canonical dimension) equals dimension for all \( X_e(A) \) with \( \text{ind} A = 2 \) a power of 2.

1. Canonical \( p \)-dimension

We begin by recalling the definitions of canonical \( p \)-dimension, \( p \)-incompressibility, and equivalence.

Let \( X \) be a scheme over a field \( F \), and let \( p \) be a prime or zero. A field extension \( K \) of \( F \) is called a splitting field of \( X \) (or is said to split \( X \)) if \( X(K) \neq \emptyset \). A splitting field \( K \) is called \( p \)-generic if, for any splitting field \( L \) of \( X \), there is an \( F \)-place \( K \twoheadrightarrow L' \) for some finite extension \( L'/L \) of degree prime to \( p \). In particular, \( K \) is 0-generic if for any splitting field \( L \) there is an \( F \)-place \( K \twoheadrightarrow L \).

The canonical \( p \)-dimension of a scheme \( X \) over \( F \) was originally defined \([1, 7]\) as the minimal transcendence degree of a \( p \)-generic splitting field \( K \) of \( X \). When \( X \) is a smooth complete variety, the original algebraic definition is equivalent to the following geometric one \([7, 9]\).

Definition 1.1. Let \( X \) be a smooth complete variety over \( F \). The canonical \( p \)-dimension \( \text{cdim}_p(X) \) of \( X \) is the minimal dimension of the image of a morphism \( X' \to X \), where \( X' \) is a variety over \( F \) admitting a dominant morphism \( X' \to X \) with \( F(X')/F(X) \) finite of degree prime to \( p \). The canonical 0-dimension of \( X \) is thus the minimal dimension of the image of a rational morphism \( X \dashrightarrow X \).

In the case \( p = 0 \), we will drop the \( p \) and speak simply of generic splitting fields and canonical dimension \( \text{cdim}(X) \).

For a third definition of canonical \( p \)-dimension as the essential \( p \)-dimension of the detection functor of a scheme \( X \), we refer the reader to Merkurjev’s comprehensive exposition \([9]\) of essential dimension.

For a smooth complete variety \( X \), the inequalities

\[ \text{cdim}_p(X) \leq \text{cdim}(X) \leq \dim(X) \]

are clear from Definition 1.1. Note also that if \( X \) has a rational point, then \( \text{cdim}(X) = 0 \) (though the converse is not true).

Definition 1.2. When a smooth complete variety \( X \) has canonical \( p \)-dimension as large as possible, namely \( \text{cdim}_p(X) = \dim(X) \), we say that \( X \) is \( p \)-incompressible.
It follows immediately that if $X$ is $p$-incompressible, it is also incompressible (i.e. 0-incompressible).

When two schemes $X$ and $Y$ over a field $F$ have the same class of splitting fields, we call them equivalent and write $X \sim Y$. In this case

$$\text{cdim}_p(X) = \text{cdim}_p(Y)$$

for all $p$. If $X$ and $Y$ are smooth complete varieties, then they are equivalent if and only if there exist rational maps $X \dashrightarrow Y$ and $Y \dashrightarrow X$.

2. Reductions

Let $A$ be a central division algebra over a field $F$ with $\text{ind } A = n$. We consider the problem of computing the canonical $p$-dimension of the following varieties.

**Definition 2.1.** For integers $1 \leq d_1 < d_2 < \cdots < d_k \leq n - 1$, define $X_{d_1,d_2,\ldots,d_k}(A)$ to be the variety of flags of right ideals $I_1 \subset I_2 \subset \cdots \subset I_k$ of $A$ with $I_i$ of reduced dimension $d_i$. When the algebra $A$ is understood, we write simply $X_{d_1,d_2,\ldots,d_k}$.

When $k = 1$ we get the generalized Severi-Brauer varieties $X_d(A)$ of $A$. In particular, $X_1(A)$ is the Severi-Brauer variety of $A$.

It is known [8, Th. 1.17] that the generalized Severi-Brauer variety $X_d(A)$ has a rational point over an extension field $K/F$ if and only if the index $\text{ind } A$ divides $d_1$. As a consequence, $X_{d_1}(A) \sim X_d(A)$, where $d := \gcd(\text{ind } A, d_1)$. We record the easy generalization of this fact to varieties $X_{d_1,d_2,\ldots,d_k}(A)$.

**Proposition 2.2.** If $d := \gcd(\text{ind } A, d_1, d_2, \ldots, d_k)$, then

$$X_{d_1,d_2,\ldots,d_k}(A) \sim X_d(A)$$

and thus, for any $p$,

$$\text{cdim}_p(X_{d_1,d_2,\ldots,d_k}(A)) = \text{cdim}_p(X_d(A)).$$

**Proof.** If $X_{d_1,d_2,\ldots,d_k}(A)$ has a rational point over an extension field $K/F$, then by definition $A_K$ has right ideals of reduced dimensions $d_1, d_2, \ldots, d_k$. This is the case if and only if $\text{ind } A_K$ divides each of the $d_i$, or equivalently, $\text{ind } A_K$ divides $d$ (since $\text{ind } A_K$ always divides $\text{ind } A$).

Reading the argument backwards, $\text{ind } A_K$ dividing $d$ implies the existence of right ideals $I_1, I_2, \ldots, I_k$ in $A_K$ with reduced dimensions $d_1, d_2, \ldots, d_k$. In fact, the $I_1, \ldots, I_k$ can be chosen to form a flag. Suppose $d_j = m_j \text{ ind } A_K$ and $A_K \cong M_{m_j}(D)$ for some division algebra $D$. Then we take $I_i$ to be the set of matrices in $M_{m_j}(D)$ whose $t - m_i$ last rows are zero.

Hence it is enough to compute $\text{cdim}_p(X_d(A))$ for $d$ dividing $\text{ind } A$.

If the index of $A$ factors as $\text{ind } A = q_1 q_2 \cdots q_r$ with the $q_j$ powers of distinct primes $p_j$, then there exist central division algebras $A_j$ of index $q_j$ for $j = 1, \ldots, r$ such that

$$A \cong A_1 \otimes A_2 \otimes \cdots \otimes A_r.$$

**Proposition 2.3.** Given a positive integer $1 \leq d \leq \text{ind } A - 1$, with $q_j$ as above, define $e_j := \gcd(d,q_j)$ for $j = 1, \ldots, r$. Then

$$X_d(A) \sim X_{e_1}(A_1) \times X_{e_2}(A_2) \times \cdots \times X_{e_r}(A_r)$$
and thus, for any $p$,
\[
\text{cdim}_p(X_d(A)) = \text{cdim}_p(X_{e_1}(A_1) \times X_{e_2}(A_2) \times \cdots \times X_{e_r}(A_r)).
\]

Proof. The variety $X_d(A)$ has a rational point over an extension field $K/F$ if and only if $\text{ind} A_K$ divides $d$. Because
\[
\text{ind} A_K = (\text{ind}(A_1)_K) \cdots (\text{ind}(A_r)_K),
\]
this condition is equivalent to $\text{ind}(A_j)_K$ dividing $d$ for all $j$ (since $\text{ind}(A_j)_K$ always divides $\text{ind} A_j = q_j$). This holds if and only if each $X_{e_j}(A_j)$ has a rational point over $K$, which is equivalent to the product of the $X_{e_j}(A_j)$ having a rational point over $K$. □

The proposition gives the following upper bound on canonical $p$-dimension:

(1) \[
\text{cdim}_p(X_d(A)) \leq \dim \prod_{j=1}^r X_{e_j}(A_j) = \sum_{j=1}^r \dim X_{e_j}(A_j) = \sum_{j=1}^r e_j(q_j - e_j).
\]

If $p$ is prime, then there exists a finite, $p$-coprime extension $K$ of $F$ which splits the algebras $A_j$ for all $j$ with $p_j \neq p$. Since canonical $p$-dimension does not change under such an extension [9, Prop. 1.5 (2)], $\text{cdim}_p(X_d(A)) = 0$ unless some $p_s = p$, in which case
\[
\text{cdim}_p(X_d(A)) = \text{cdim}_p(X_{e_s}(A_s)).
\]

We see that it is enough, when $p$ is prime, to compute the canonical $p$-dimension of varieties of the form $X_d(A)$ with $\text{ind} A$ a prime power divisible by $e$. When $p = 0$, it is enough to compute the canonical dimension of products of such varieties.

3. Known results for Severi-Brauer varieties

We now recall what is already known about the canonical $p$-dimension of Severi-Brauer varieties $X_1(A)$, the $d = 1$ case.

For any $p$, if $d = 1$ in (1) above, then all of the $e_j = 1$, and the upper bound becomes

(2) \[
\text{cdim}_p(X_1(A)) \leq \sum_{j=1}^r (q_j - 1).
\]

In the special case $r=1$ and $p = p_1$, it is shown in [1, Th. 11.4], based on Karpenko’s [6, Th. 2.1], that the inequality (2) is actually an equality. Thus, for general $A$, we have
\[
\text{cdim}_{p_j}(X_1(A)) = \text{cdim}_{p_j}(X_1(A_j)) = q_j - 1
\]
for $j = 1, 2, \ldots, r$, while $\text{cdim}_p(X_1(A)) = 0$ for all other primes $p$ [7, Ex. 5.10].

Now let $p = 0$, $d = 1$. When $r = 1$, we again have equality in (2), since canonical dimension is bounded below by canonical $p$-dimension for every prime $p$. In [4, Th. 1.3], (2) is proven also to be an equality in the case $\text{ind} A = 6$ (i.e. $r = 2$, $q_1 = 2$, $q_2 = 3$) provided that $\text{char} F = 0$. The authors of [4] suggest that equality may indeed hold for any $A$ when $p = 0$, $d = 1$. 


4. 2-INCOMPRESSIBILITY OF $X_e(A)$ FOR $\text{ind} A = 2e$ A POWER OF 2

If $A$ is a central division algebra with $\text{ind} A = 4$, the variety $X_2(A)$ is known to be 2-incompressible. Indeed, if the exponent of $A$ is 2, then $X_2(A)$ is isomorphic to a 4-dimensional projective quadric hypersurface called the Albert quadric of $A$ [10, §5.2]. Such a quadric has first Witt index 1 [13, p. 93], hence is 2-incompressible by [5, Th. 90.2]. If the exponent of $A$ is 4, we can reduce to the exponent 2 case by extending to the function field of the Severi-Brauer variety of $A \otimes A$.

In what follows, we show 2-incompressibility for an infinite family of varieties which includes the varieties of the form $X_2(A)$ (with $\text{ind} A = 4$) mentioned above.

**Theorem 4.1.** Let $e = 2^a$, $a \geq 1$. For a central division algebra $A$ with $\text{ind} A = 2e$, the variety $X_e := X_e(A)$ is 2-incompressible. Thus

$$\text{cdim}_2(X_e) = \text{cdim}(X_e) = \text{dim}(X_e) = e(2e - e) = e^2 = 4^a.$$  

We briefly recall some terminology from [5, §62 and §75]. Let $X$ and $Y$ be schemes with $\text{dim} X = e$. A correspondence of degree zero $\delta : X \rightsquigarrow Y$ from $X$ to $Y$ is just a cycle $\delta \in \text{CH}_e(X \times Y)$. The multiplicity $\text{mult}(\delta)$ of such a $\delta$ is the integer satisfying $\text{mult}(\delta) \cdot [X] = p_*(\delta)$, where $p_*$ is the push-forward homomorphism

$$p_* : \text{CH}_e(X \times Y) \to \text{CH}_e(X) = \mathbb{Z} : [X].$$

The exchange isomorphism $X \times Y \to Y \times X$ induces an isomorphism

$$\text{CH}_e(X \times Y) \to \text{CH}_e(Y \times X)$$

sending a cycle $\delta$ to its transpose $\delta^t$.

To prove that a variety $X$ is 2-incompressible, it suffices to show that for any correspondence $\delta : X \rightsquigarrow X$ of degree zero,

$$\text{mult}(\delta) \equiv \text{mult}(\delta^t) \pmod{2}. \quad (3)$$

Indeed, suppose we have $f : X' \to X$ and a dominant $g : X' \to X$ with $F(X')/F(X)$ finite of odd degree. Let $\delta \in \text{CH}(X \times X)$ be the pushforward of the class $[X']$ along the induced morphism $(g, f) : X' \to X \times X$. By assumption, $\text{mult}(\delta)$ is odd, so by (3) we have that $\text{mult}(\delta^t)$ is odd. It follows that $f_*(\delta^t)$ is an odd multiple of $[X]$ and in particular is nonzero, so $f$ is dominant.

We will check that the condition (3) holds for the variety $X_e$. A correspondence of degree zero $\delta : X_e \rightsquigarrow X_e$ is just an element of $\text{CH}_e^2(X_e \times X_e)$. Using the method of Chernousov and Merkurjev described in [2], we can decompose the Chow motive of $X_e \times X_e$ as follows. See also [3] for examples of similar computations.

We first realize $X_e$ as a projective homogeneous variety. Let $n := \text{ind} A = 2e = 2^{n+1}$. Let $G$ denote the group $PGL_1(A)$, and let $\Pi$ be a set of simple roots for the root system $\Sigma$ of $G$. If $\varepsilon_1, \ldots, \varepsilon_n$ are the standard basis vectors of $\mathbb{R}^n$, we may take

$$\Pi = \{\alpha_1 := \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{n-1} := \varepsilon_{n-1} - \varepsilon_n\}.$$  

Then $X_e$ is a projective $G$-homogeneous variety, namely the variety of all parabolic subgroups of $G$ of type $S_i$ for the subset $S = \Pi \setminus \{\alpha_e\}$ of the set of simple roots.

Let $W$ denote the Weyl group of the root system $\Sigma$. There are $e+1$ double cosets $D \in W_P \setminus W/W_P$ with representatives $w$ as follows, where $w_{\alpha_k}$ denotes the reflection induced by the root $\alpha_k$.  

The subset of $\Pi$ associated to $w = 1$ is of course $S = \Pi \setminus \{ \alpha_e \}$. The general nontrivial representative

$$w = w^{-1} = (w_{\alpha_e} \cdots w_{\alpha_{i-1}}) \cdots (w_{\alpha_{e+i}} \cdots w_{\alpha_e}),$$

for $i \in \{ 0, \ldots, e-1 \}$, has the effect on $\mathbb{R}^n$ of switching the tuple of standard basis vectors $(\varepsilon_{e-i}, \ldots, \varepsilon_e)$ with the tuple $(\varepsilon_{e+1}, \ldots, \varepsilon_{e+1+i})$. The resulting subset associated to $w$ is therefore

$$\Pi \setminus \{ \alpha_{e-(i+1)}, \alpha_e, \alpha_{e+(i+1)} \}$$

for $i = 0, \ldots, e-2$ and $\Pi \setminus \{ \alpha_e \}$ for $i = e-1$.

From Theorem 6.3 of [2], we deduce the following decomposition of the Chow motive of $X_e \times X_e$, where the relation between the indices $i$ above and $l$ below is $l = i + 1$.

$$M(X_e \times X_e) \simeq M(X_e) \oplus \bigoplus_{l=1}^{e-1} M(X_{e-l,e+e+l}) (l^2) \oplus M(X_e)(e^2)$$

This in turn yields a decomposition of the middle-dimensional component of the Chow group of $X_e \times X_e$.

$$\text{CH}_{e^2}(X_e \times X_e) \simeq \text{CH}_{e^2}(X_e) \oplus \bigoplus_{l=1}^{e-1} \text{CH}_{(e-l)(e+l)}(X_{e-l,e+e+l}) \oplus \text{CH}_0(X_e)$$

It now suffices to check the congruence $\text{mult}(\delta) \equiv \text{mult}(\delta^l) \pmod{2}$ for $\delta$ in the image of any of these summands. We treat the first and last summands separately from the rest.

The embedding of the first summand $\text{CH}_{e^2}(X_e)$ is induced by the diagonal morphism $X_e \to X_e \times X_e$, so the multiplicities of $\delta$ and $\delta^l$ are equal by symmetry.

For the last summand $\text{CH}_0(X_e)$ we need the following fact.

**Proposition 4.2.** Any element of $\text{CH}_0(X_e)$ has even degree.

**Proof.** If $\text{CH}_0(X_e)$ has an element of odd degree, then there exists a field extension $K/F$ of odd degree over which $X_e$ has a rational point. By [8, Prop. 1.17], $\text{ind } A_K$ divides $e$. Since the degree of $K$ over $F$ is relatively prime to $\text{ind } A = 2e = 2^{n+1}$, extension by $K$ does not reduce the index of $A$ [11, Th. 3.15a]. Thus $\text{ind } A = \text{ind } A_K$ divides $e$, a contradiction. \hfill $\Box$

Let the element $\gamma \in \text{CH}_0(X_e)$ have image $\delta \in \text{CH}_{e^2}(X_e \times X_e)$. By the proposition, $\deg(\gamma)$ is even. For some field $E/F$ over which $X_e$ has a rational point, we set $X_e := (X_e)_E$. Since $\text{CH}_0(X_e)$ is generated by a single element of degree 1, the image of $\gamma$ in $\text{CH}_0(X_e)$ is divisible by 2. It follows that $\delta \in \text{CH}_{e^2}(X_e \times X_e)$ is also divisible
by 2 and, since multiplicity does not change under field extension, \( \text{mult}(\delta) \) is even. The same argument can be applied to \( \delta' \), so \( \text{mult}(\delta) \equiv 0 \equiv \text{mult}(\delta') \pmod{2} \).

The remaining summands are dealt with by the following proposition.

**Proposition 4.3.** Let \( Fl := X_{d_1,d_2,\ldots,d_k}(A) \) with \( d := \gcd(e,d_1,d_2,\ldots,d_k) < e \), and let the correspondence \( \alpha : Fl \sim X_e \times X_e \) induce an embedding \( \alpha_* : \text{CH}_e(Fl) \hookrightarrow \text{CH}_{c,2}(X_e \times X_e) \).

Then for any \( \delta \) in the image of \( \alpha_* \), \( \text{mult}(\delta) \equiv 0 \equiv \text{mult}(\delta') \pmod{2} \).

**Proof.** Consider the diagram below of fiber products, where we select either of the projections \( p_i \) and choose the other morphisms accordingly.

\[
\begin{array}{c}
(Fl)_F(X_e) \\
(Fl \times X_e)_F(X_e) \quad (X_e)_F(X_e) \quad \text{Spec } F(X_e) \\
Fl \times X_e \times X_e \quad X_e \times X_e \quad X_e \\
Fl
\end{array}
\]

Taking push-forwards and pull-backs, we get the following diagram which commutes except for the triangle at the bottom. The push-forward by \( p_i \) takes a cycle \( \delta \in \text{CH}_{c,2}(X_e \times X_e) \) to \( \text{mult}(\delta) \) if we chose the first projection \( p_1 \) or to \( \text{mult}(\delta') \) if we chose the second projection \( p_2 \).

\[
\begin{array}{c}
\text{CH}_0 ((Fl)_F(X_e)) \\
\text{CH}_0 ((Fl \times X_e)_F(X_e)) \quad \text{deg} \quad \text{deg} \quad \mathbb{Z} \\
\text{CH}_{c,2} (Fl \times X_e \times X_e) \quad \text{CH}_{c,2} (X_e \times X_e) \quad \text{mult} \quad \text{(mult)} \circ \text{(transpose)} \quad \mathbb{Z} \\
\text{CH}_e(Fl) \quad \cdots \quad \alpha_* \quad \cdots \quad \cdots
\end{array}
\]

Any \( \delta \in \text{im}(\alpha_*) \) also lies in the image of \( \text{CH}_{c,2} (Fl \times X_e \times X_e) \), by the definition of the push-forward. Chasing through the diagram, one sees that \( \text{mult}(\delta) \) (and similarly \( \text{mult}(\delta') \)) must lie in \( \text{deg} \text{CH}_0 ((Fl)_F(X_e)) \). We will be done if we can show that no element of \( \text{CH}_0 ((Fl)_F(X_e)) \) has odd degree.

Note that \( Fl_{F(X_e)} = X_{d_1,d_2,\ldots,d_k}(A)^{F(X_e)} \cong X_{d_1,d_2,\ldots,d_k} (A_{F(X_e)}) \), where \( A_{F(X_e)} \) has index equal to \( \gcd(2e,e) = e \) [12, Th. 2.5]. If some element of \( \text{CH}_0 ((Fl)_F(X_e)) \) has odd degree, then there exists a field extension \( K/F(X_e) \) of odd
degree over which \((F^l)_e\) has a rational point. By Proposition 2.2, \(X_d(A^{F(X_e)})\) also has a rational point over \(K\). Thus \(\text{ind}_A\) divides \(d < e\), which contradicts \(\text{ind}_A = e\), since an odd degree extension cannot reduce the index of \(A^{F(X_e)}\) [11, Th. 3.15a].

This completes the proof of the theorem.

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