On random intersections of two convex bodies

Appendix to: “Isoperimetry of waists and local versus global asymptotic convex geometries” by R. Vershynin

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In [V] it is proved that the existence of nicely bounded sections of two symmetric convex bodies $K$ and $L$ in $\mathbb{R}^n$ (of dimensions $k$ and $n - ck$) implies that the random intersection $K \cap UL$ is nicely bounded with high probability, where $U$ is a random unitary operator. Namely, the diameter of $K \cap UL$ is at most $C^{n/k}$ times the larger of the diameters of the two sections, with probability at least $1 - e^{-n}$.

In this appendix we show how to improve the exponential bound $C^{n/k}$ to a polynomial bound, say $C(n/k)^2$. The cost for this is decreasing the probability from $1 - e^{-n}$ to $1 - e^{-k}$.

**Theorem 1.** Let $0 < a < 0.03$. Assume that two symmetric convex bodies $K$ and $L$ in $\mathbb{R}^n$ have sections of dimensions at least $k$ and $n - ak$ respectively whose diameters are bounded by 1. Then for every $t > C(n/k)^a$ the random orthogonal operator $U \in O(n)$ satisfies

$$\mathbb{P}\{\text{diam}(K \cap UL) > tn/k\} < (ct)^{-k/16}.$$ 

**Remarks.** 1. Theorem 1.1 of [V] is a partial case of this theorem for $t = C_{1}^{n/k}$.

$$\mathbb{P}\{\text{diam}(K \cap UL) > C_{1}(n/k)^a\} < e^{-n}.$$ 

2. To obtain a polynomial bound on the diameter, one can choose $t = C_{1}(n/k)$ in Theorem 1 to get

$$\mathbb{P}\{\text{diam}(K \cap UL) > C_{1}(n/k)^2\} < (cC_{1}n/k)^{-k/16} < e^{-k}$$ 

for an appropriate absolute constant $C_{1}$. To summarize,

In Theorem 1.1 of [V], the body $K \cap UL$ has diameter bounded by $C_{1}(n/k)^2$ with probability at least $1 - e^{-k}$. 

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A new ingredient in the proof of Theorem 1 is the following covering lemma.

**Lemma 2.** Let $K$ be a convex body in $\mathbb{R}^n$ such that $K \supseteq \delta D$ for some $\delta > 0$. Assume that there exists an orthogonal projection $P$ with $\text{rank} P = n - k$ and such that $PK \supseteq PD$. Then

$$N(D, 4K) \leq \left(\frac{C}{\delta}\right)^{2k}.$$ 

**Proof.** Denote the range of $P$ by $E$. Let $f : PD \rightarrow \mathbb{R}^n$ be a lifting of $f$, i.e. a map such that

$$f(PD) \subseteq K \quad \text{and} \quad V := (\text{id} - f)(PD) \subseteq E^\perp.$$ (1)

Since $V \subset PD - f(PD) \subset D - f(PD)$, the assumptions on $K$ and (1) imply that $V \subset (\frac{1}{\delta} + 1)K \cap E^\perp$. Then by the standard volumetric argument we have

$$N(V, K) \leq \left(\frac{C}{\delta}\right)^k$$

as $\text{dim } E^\perp = k$. This will allow us to cover $PD$. Indeed, by (1),

$$PD \subset f(PD) + V \subseteq K + V.$$  

By the submultiplicative property $N(K_1 + K_2, D_1 + D_2) \leq N(K_1, D_1) N(K_2, D_2)$, which is valid for all sets $K_1, K_2, D_1, D_2$, we have

$$N(PD, 2K) \leq N(K + V, 2K) \leq N(V, K) \leq \left(\frac{C}{\delta}\right)^k.$$ (2)

Also by the assumption on $K$ and by the standard volumetric argument already used above,

$$N(D \cap E^\perp, K) \leq N(D \cap E^\perp, \delta D \cap E^\perp) \leq \left(\frac{C}{\delta}\right)^k.$$ (3)

Since $D \subseteq PD + D \cap E^\perp$, we have by the submultiplicative property that

$$N(D, 3K) \leq N(PD, 2K) N(P \cap E^\perp, K)$$

and we finish by applying (2) and (3). $\blacksquare$

**Proof of Theorem 1** We start the proof as in [V], by dualizing the statement and assuming that there exist orthogonal projections $P$ and $Q$ with $\text{rank} P = k$ and $\text{rank} Q = n - ak$, and such that

$$PK \supseteq PD, \quad QL \supseteq QD.$$ (4)

Then we must prove that for $t$ as in the theorem,

$$\mathbb{P}\{(k/tn)D \subseteq K + UL\} \geq 1 - (ct)^{-k/16}. $$ (5)
Let $\varepsilon > 0$ and let

$$\delta_K = \sqrt{1 - \frac{\varepsilon^2 k}{n}}.$$ 

By Proposition 3.1 and Corollary 2.6 (ii) of [V],

$$\sigma_{n-1}(K + \delta_K D) \geq \sigma_{n-1,k-1}^{\text{Lip}}(\sin^{-1} \delta_K) \geq 1 - (C \varepsilon K)^{k/4}.$$ 

Let $0 < \delta_L < 1$ be a parameter. By Lemma 2 applied to the body $L + \delta_L$,

$$N(D, 4(L + \delta_L D)) \leq (C/\delta_L)^{2ak}.$$ 

Writing this covering number as $N(2D, 8L + 8\delta_L D)$, we apply Lemma 4.1 of [V]. It states that if $\delta_K + 8\delta_L < 1$ then the inclusion

$$(1 - \delta_K - 8\delta_L)D \subseteq K + 8UL$$

holds with probability at least

$$1 - (C/\delta_L)^{2ak} (C\varepsilon)^{k/4}.$$  (6)

To finish the proof, we need to bound below the radius $1 - \delta_K - 8\delta_L$ in (6) and the probability (7). Since $\sqrt{1-x} \leq 1 - x/2$ for $0 < x < 1$, we set

$$\delta_L = \frac{\varepsilon^2 k}{32n}$$

to obtain

$$1 - \delta_K - 8\delta_L \geq \frac{\varepsilon^2 k}{4n}.$$ 

It remains to estimate the probability (7). If we require that

$$\varepsilon \leq c_0 (k/n)^{C_0 a}$$

for suitable absolute constants $c_0, C_0 > 0$, then $(C/\delta_L)^{2ak} < (C\varepsilon)^{-k/8}$, hence the probability

$$(7) \geq 1 - (C\varepsilon)^{k/8}.$$ 

We have thus proved that if $\varepsilon > 0$ satisfies (8) then

$$\mathbb{P}\{(\varepsilon^2 k/32n)D \subseteq K + UL\} \geq \mathbb{P}\{(\varepsilon^2 k/4n)D \subseteq K + 8UL\} \geq 1 - (C\varepsilon)^{k/8}.$$ 

It remains to set $\varepsilon^2/32 = 1/t$, and the proof is complete. \qed

References

[V] R. Vershynin, Isoperimetry of waists and local versus global asymptotic convex geometries, preprint