MINIMAL-MASS BLOW-UP SOLUTIONS FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH GROWTH POTENTIALS

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Abstract. We consider the following nonlinear Schrödinger equation with growth potentials:

\[ i\frac{\partial u}{\partial t} + \Delta u + |u|^p u - V u - U u + \omega_1 |x|^{2\sigma_1} u - \omega_2 |x|^{2\sigma_2} u - \xi \cdot x |x|^\sigma u = 0 \]

in \( \mathbb{R}^N \). From the classical argument, the solution with subcritical mass \( \|u\|_2 < \|Q\|_2 \) is global and bounded in \( X_1 \), where \( X_1 \) is the domain of \( -\Delta + U + \omega_2 |x|^{2\sigma_2} \) in \( L^2(\mathbb{R}^N) \) and \( Q \) is the ground state of the mass-critical problem. Therefore, we are interested in the existence and behaviour of blow-up solutions for the threshold \( \|u\|_2 = \|Q\|_2 \).

Previous studies investigate the existence and behaviour of the critical-mass blow-up solution when the cases of algebraically tractable growth potential or no growth potential. In this paper, we construct a critical-mass blow-up solution for the equation with growth potentials which has no algebraic property.

1. Introduction

We consider the following nonlinear Schrödinger equation with potentials:

\[ i\frac{\partial u}{\partial t} + \Delta u + |u|^p u - V u - U u + \omega_1 |x|^{2\sigma_1} u - \omega_2 |x|^{2\sigma_2} u - \xi \cdot x |x|^\sigma u = 0 \]

in \( \mathbb{R}^N \).

Let be \( \omega_2 > 0 \) or \( (\omega_1, \omega_2, \xi) = (0, 0, 0) \). Let be \( \xi \in \mathbb{R}^N \). For \( \sigma_j \), we assume that

\[ 0 \leq \sigma_1 < \sigma_2 \leq 1, \quad -1 < \sigma_3 < 2\sigma_2 - 1. \]

Moreover, we assume that \( U \in C^\infty(\mathbb{R}^N), U \geq 0, \) and

\[ \left( \frac{\partial}{\partial x} \right)^\alpha U \in L^\infty(\mathbb{R}^N) \]

for any multi-index \( \alpha \) such that \( |\alpha| \geq 2 \). For the sake of clarity in notation, we define

\[ W(x) := V(x) + U(x) - \omega_1 |x|^{2\sigma_1} + \omega_2 |x|^{2\sigma_2} + \xi \cdot x |x|^\sigma. \]

Furthermore, we define Hilbert spaces \( X_k \) and \( \Sigma^k \) by

\[ X_k := \left\{ u \in H^k(\mathbb{R}^N) \mid (u + \omega_2 |x|^{2\sigma_2})^k u \in L^2(\mathbb{R}^N) \right\}, \quad (u, v)_{X_k} := (u, v)_{H^k} + \text{Re} \int_{\mathbb{R}^N} (U(x) + \omega_2 |x|^{2\sigma_2})^k u(x) \overline{v(x)} dx, \]

\[ \Sigma^k := \left\{ u \in H^k(\mathbb{R}^N) \mid |x|^k u \in L^2(\mathbb{R}^N) \right\}, \quad (u, v)_{\Sigma^k} := (u, v)_{H^k} + \text{Re} \int_{\mathbb{R}^N} |x|^{2k} u(x) \overline{v(x)} dx. \]

Then \( X_k \hookrightarrow \Sigma^k \) holds.

It is well known that if

\[ V \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \quad \left( p \geq 1 \text{ and } p > \frac{N}{2} \right), \]

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In addition, if $u_0 \in X_1$, there exists a unique maximal solution $u \in C((T_*, T^*), X_1) \cap C^1((T_*, T^*), X_1^*)$. Moreover, the mass (i.e., $L^2$-norm) and energy $E$ of the solution are conserved by the flow, where

$$E(u) := \frac{1}{2} \| \nabla u \|_2^2 - \frac{1}{2 + \frac{4}{N}} \| u \|_{2 + \frac{4}{N}}^{2 + \frac{4}{N}} + \frac{1}{2} \int_{\mathbb{R}^N} W(x)|u(x)|^2 \, dx.$$ 

Furthermore, there is a blow-up alternative

$$T^* < \infty \text{ implies } \lim_{t \nearrow T^*} \| u(t) \|_{X_1} = \infty.$$ 

In addition, if $u_0 \in \Sigma^1$, then the corresponding solution $u$ belongs to $C((T_*, T^*), \Sigma^1)$. Moreover, we consider

$$V \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \quad \left( p \geq 2 \text{ and } p > \frac{N}{2} \right).$$

If $u_0 \in X_2$, then the corresponding solution $u$ belongs to $u \in C((T_*, T^*), X_2) \cap C^1((T_*, T^*), L^2(\mathbb{R}^N))$. In addition, if $u_0 \in \Sigma^2$, then the solution $u$ belongs to $C((T_*, T^*), \Sigma^2)$. In this paper, we investigate the conditions for the potential related with the existence of minimal mass blow-up solution.

1.1. Critical problem. Firstly, we describe the results regarding the mass-critical problem:

$$\frac{\partial}{\partial t} u + \Delta u + |u|^{\frac{4}{N}} u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

In particular, (1) with $W = 0$ is reduced to (6).

It is well known (2) (8) (13) that there exists a unique classical solution $Q$ for

$$-\Delta Q + Q - |Q|^\frac{4}{N} Q = 0, \quad Q \in H^1(\mathbb{R}^N), \quad Q > 0, \quad Q \text{ is radial},$$

which is called the ground state. If $\|u\|_2 = \|Q\|_2 (\|u\|_2 < \|Q\|_2, \|u\|_2 > \|Q\|_2)$, we say that $u$ has the critical mass (subcritical mass, supercritical mass, respectively).

We note that $E_{\text{crit}}(Q) = 0$, where $E_{\text{crit}}$ is the energy with respect to (6). Moreover, the ground state $Q$ attains the best constant in the Gagliardo-Nirenberg inequality

$$\|v\|_{2 + \frac{4}{N}}^{2 + \frac{4}{N}} \leq \left( 1 + \frac{2}{N} \right) \left( \frac{\|v\|_2}{\|Q\|_2} \right)^{\frac{4}{N}} \|\nabla v\|_2^2 \quad \text{for } v \in H^1(\mathbb{R}^N).$$

Therefore, for all $v \in H^1(\mathbb{R}^N)$,

$$E_{\text{crit}}(v) \geq \frac{1}{2} \|\nabla v\|_2^2 \left( 1 - \frac{\left( \frac{\|v\|_2}{\|Q\|_2} \right)^{\frac{4}{N}}}{\left( \frac{\|v\|_2}{\|Q\|_2} \right)^{\frac{4}{N}}} \right)$$

holds. This inequality and the mass and energy conservations imply that any subcritical mass solution for (6) is global and bounded in $H^1(\mathbb{R}^N)$.

Regarding the critical mass case, we apply the pseudo-conformal transformation

$$u(t, x) \mapsto \frac{1}{|t|^{\frac{4}{N}}} u \left( -\frac{1}{t}, \frac{x}{t} \right) e^{i |x|^2 \frac{4}{N}}$$

to the solitary wave solution $u(t, x) := Q(x)e^{it}$. Then we obtain

$$S(t, x) := \frac{1}{|t|^{\frac{4}{N}}} Q \left( \frac{x}{t} \right) e^{-\frac{t}{4} e^{i |x|^2 \frac{4}{N}}},$$

which is also a solution for (6) and satisfies

$$\|S(t)\|_2 = \|Q\|_2, \quad \|\nabla S(t)\|_2 \sim \frac{1}{|t|} \quad (t \nearrow 0).$$

Namely, $S$ is a minimal mass blow-up solution for (6). Moreover, $S$ is the only finite time blow-up solution for (6) with critical mass, up to the symmetries of the flow (see (10)).
Regarding the supercritical mass case, there exists a solution $u$ for (6) such that

$$
\|\nabla u(t)\|_2 \sim \sqrt{\frac{\log[\log|T^* - t|]}{T^* - t}} (t \searrow T^*)
$$

(see [12, 13]).

1.2. Previous results. We describe previous results regarding the following nonlinear Schrödinger equation with a real-valued potential:

$$
i \frac{\partial u}{\partial t} + \Delta u + |u|^{4 \nu} u + V(x)u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N.
$$

At first, [3, 4] give results for growth potentials.

**Theorem 1.1** (Carles and Nakamura [4]). If $V(x) = E \cdot x$ for some $E \in \mathbb{R}^N$, then (7) has a finite-time blow-up solution

$$
S(t, x) := \frac{1}{|t|^{\frac{4}{p}}} Q \left( \frac{x - t^2 E}{t} \right) \exp \left( i \left( \frac{1}{4t} \frac{|x - t^2 E|^2}{4} - \frac{1}{2} + tE \cdot x - \frac{t^3}{3} |E|^2 \right) \right).
$$

In particular, $\|S\|_2 = |Q|_2$.

**Theorem 1.2** (Carles [3]). If $V(x) = \omega^2 |x|^2$ for some $\omega \in \mathbb{R}^N$, then (7) has a finite-time blow-up solution

$$
S(t, x) := \left( \frac{2 \omega}{\sinh (2 \omega t)} \right)^{\frac{1}{3}} Q \left( \frac{2 \omega x}{\sinh (2 \omega t)} \right) \times \exp \left( i \left( \frac{\omega |x|^2}{2 \sinh (2 \omega t) \cosh (2 \omega t)} - \frac{\omega}{2 \tanh (2 \omega t)} + \frac{\omega}{2} |x|^2 \tanh (2 \omega t) \right) \right).
$$

In particular, $\|S\|_2 = |Q|_2$.

These results show that (7) may have a critical-mass blow-up solution with a blow-up rate of $t^{-1}$ when the potential $V$ is easy to handle algebraically and can be reduced to (6).

Theorems 1.1 and 1.2 construct blow-up solutions by applying the pseudo-conformal transformation to the ground states. In contrast to these, the seminal work Raphaël and Szeftel [14] constructs a minimal-mass blow-up solution for

$$
i \frac{\partial u}{\partial t} + \Delta u + k(x)|u|^{\frac{4}{p}} u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N
$$

without using the pseudo-conformal transformation. Le Coz, Martel, and Raphaël [7] based on the methodology of [14] obtains the following results for

$$
i \frac{\partial u}{\partial t} + \Delta u + |u|^{\frac{4}{p}} u \pm |u|^{p-1} u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N.
$$

**Theorem 1.3** (Le Coz, Martel, and Raphaël [7]). Let $N = 1, 2, 3, 1 < p < 1 + \frac{4}{N}$, and $\pm = +$. Then for any energy level $E_0 \in \mathbb{R}$, there exist $t_0 < 0$ and a radially symmetric initial value $u_0 \in H^1(\mathbb{R}^N)$ with

$$
\|u_0\|_2 = |Q|_2, \quad E(u_0) = E_0
$$

such that the corresponding solution $u$ for (8) with $u(t_0) = u_0$ blows up at $t = 0$ with a blow-up rate of

$$
\|\nabla u(t)\|_2 = \frac{C(p) + o_{t \nearrow 0}(t)}{|t|^{\sigma}},
$$

where $\sigma = \frac{4}{4 + N(p - 1)}$ and $C(p) > 0$.

**Theorem 1.4** ([7]). Let $N = 1, 2, 3, 1 < p < 1 + \frac{4}{N}$, and $\pm = -$. If an initial value has critical mass, then the corresponding solution for (8) with $u(0) = u_0$ is global and bounded in $H^1(\mathbb{R}^N)$.

Based on the method of [7, 14, 8] shows the following results for the case where the potential $V$ is smooth and integrable but has no algebraic properties.
Theorem 1.5 \([8]\). We assume that \(V\) is locally Lipschitz function, \(V \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)\) for some \(p \in [2, \infty] \cap (\frac{N}{2}, \infty)\), and \(\nabla V \in L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)\) for some \(q \in [2, \infty] \cap (N, \infty)\). Then there exist \(t_0 < 0\) and a radial initial value \(u_0 \in \Sigma^1\) with \(\|u_0\|_2 = \|Q\|_2\) such that the corresponding solution \(u\) for (7) with \(u(t_0) = u_0\) blows up at \(t = 0\). Moreover,

\[
\left\| u(t) - \frac{1}{\lambda(t)^2} P\left( t, \frac{x + w(t)}{\lambda(t)} \right) e^{-\frac{\lambda(t)|x + w(t)|^2}{\lambda(t)^2} + \gamma(t)} \right\|_{\Sigma^1} \to 0 \quad (t \nearrow 0)
\]

holds for some \(C^1\) functions \(\lambda : (t_0, 0) \to (0, \infty), b, \gamma : (t_0, 0) \to \mathbb{R}\), and \(w : (t_0, 0) \to \mathbb{R}^N\) such that

\[
\lambda(t) = |t|^{1 + o(1)}, \quad b(t) = |t|^{1 + o(1)}, \quad \gamma(t) \sim |t|^{-1}, \quad |w(t)| = o(|t|)
\]
as \(t \nearrow 0\).

Theorem 1.6 \([9]\). Assume that \(V(x) := |x|^{-2\sigma}\), where \(0 < \sigma < \min\left\{ \frac{N}{2}, 1 \right\}\). Then for any energy level \(E_0 \in \mathbb{R}\), there exist \(t_0 < 0\) and a radially symmetric initial value \(u_0 \in H^1(\mathbb{R}^N)\) with

\[
\|u_0\|_2 = \|Q\|_2, \quad E(u_0) = E_0
\]
such that the corresponding solution \(u\) for (7) with \(u(t_0) = u_0\) blows up at \(t = 0\). Moreover,

\[
\left\| u(t) - \frac{1}{\lambda(t)^2} P\left( t, \frac{x}{\lambda(t)} \right) e^{-\frac{\lambda(t)|x|^2}{\lambda(t)^2} + \gamma(t)} \right\|_{\Sigma^1} \to 0 \quad (t \nearrow 0)
\]

holds for some blow-up profile \(P\) and \(C^1\) functions \(\lambda : (t_0, 0) \to (0, \infty)\) and \(b, \gamma : (t_0, 0) \to \mathbb{R}\) such that

\[
P(t) \to Q \quad \text{in} \quad H^1(\mathbb{R}^N),
\]

\[
\lambda(t) = C_1(\sigma)|t|^{\frac{\sigma}{1+\sigma}} (1 + o(1)), \quad b(t) = C_2(\sigma)|t|^{\frac{\sigma}{1+\sigma}} (1 + o(1)), \quad \gamma(t)^{-1} = O\left(|t|^{-\frac{\sigma}{1+\sigma}}\right)
\]
as \(t \nearrow 0\).

Theorem 1.7 \([9]\). Assume \(N \geq 2\) and \(V(x) := |x|^{-2\sigma}\), where \(0 < \sigma < \min\left\{ \frac{N}{2}, 1 \right\}\). If \(u_0 \in H^1_{\text{rad}}(\mathbb{R}^N)\) such that \(\|u_0\|_2 = \|Q\|_2\), the corresponding solution \(u\) for (7) with \(u(0) = u_0\) is global and bounded in \(H^1(\mathbb{R}^N)\).

The comparison of Theorem 1.3 with Theorem 1.6 and Theorem 1.7 with Theorem 1.8 suggests that a inverse power potential and a power-type nonlinearity have a similar effect on blow-up rate.

1.3. Main results. For \(\sigma_j\), we consider the following:

\[
\max\left\{ \frac{2 - N}{4}, 0 \right\} < \sigma_1 < \sigma_2 \leq 1, \quad \max\left\{ \frac{N}{2} - 1, 0 \right\} < \sigma_3 < 2\sigma_2 - 1.
\]

Moreover, for a potential \(V\), we consider the following:

\[
V \text{ is locally Lipschitz continuous on } \mathbb{R}^N,
\]

\[
\nabla V \in L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \quad (q \geq 2 \text{ and } q > N).
\]

Theorem 1.8 (Existence of a minimal mass blow-up solution). Let \(\sigma_j\) satisfy (3), \(\xi \in \mathbb{R}^N\), and the potential \(V\) satisfy (3), (10), and (11). Then there exist \(t_0 < 0\) and a radial initial value \(u_0 \in \Sigma^1\) with \(\|u_0\|_2 = \|Q\|_2\) such that the corresponding solution \(u\) for (11) with \(u(t_0) = u_0\) blows up at \(t = 0\). Moreover,

\[
\left\| u(t, x) - \frac{1}{\lambda(t)^2} Q\left( \frac{x + w(t)}{\lambda(t)} \right) e^{-\frac{\lambda(t)|x + w(t)|^2}{\lambda(t)^2} + \gamma(t)} \right\|_{\Sigma^1} \to 0 \quad (t \nearrow 0)
\]

holds for some \(C^1\) functions \(\lambda : (t_0, 0) \to (0, \infty), b, \gamma : (t_0, 0) \to \mathbb{R}\), and \(w : (t_0, 0) \to \mathbb{R}^N\) such that

\[
\lambda(t) = |t| (1 + o(1)), \quad b(t) = |t| (1 + o(1)), \quad \gamma(t) \sim |t|^{-1}, \quad |w(t)| = o(|t|)
\]
as \(t \nearrow 0\).
Remark 1.9. In contrast, if \( \sigma_j \) satisfy (2), \( \xi \in \mathbb{R}^N \), and \( V \) satisfies (1), then any subcritical mass solution for (1) exists globally in time and is bounded in \( X_1 \). This can be proved easily by the Gagliardo-Nirenberg inequality and the Sobolev embedding theorem. Therefore, the solution in Theorem 1.8 is a minimal mass blow-up solution.

1.4. Comments regarding the main result. We present some comments regarding Theorem 1.8.

The potential \( W \) is composed of several terms, due to the existence of several growth potentials: \( \pm |x|^{2\sigma} \) and \( x|x|^\sigma \). This is because we assume that \( V \) is locally Lipschitz continuous near the origin, and therefore we cannot successfully divide growth potentials into \( V \) and \( U \). On the other hand, let \( \chi \) be a non-negative decreasing cut-off function that \( \chi = 1 \) near the origin and we define

\[
V(x) := |x|^{2\sigma} \chi(x), \quad U(x) := |x|^{2\sigma}(1 - \chi(x)).
\]

Then \( V = |x|^{2\sigma} + U \) holds. From Theorem 1.8 there exists a minimal mass blow-up solution for

\[
i \frac{\partial u}{\partial t} + \Delta u + |u|^{4\sigma} u - Uu - |x|^{2\sigma^2} u = 0.
\]

In contrast, \( V \) is not Lipschitz continuous near the origin if \( \sigma \) is a sufficiently small. Therefore, it is suggested that the assumption that \( V \) is a locally Lipschitz continuous is not essential.

A comparison of Theorem 1.5 with Theorem 1.8 suggests that the behaviour of potentials at infinity, in particular its growth, does not affect blow-up rate. Moreover, in comparison with Theorem 1.1 and 1.2 we expect to be able to construct a minimal mass blow-up solution for (1) with a blow-up rate \( t^{-1} \) if \( W = \omega|x|^{2\sigma} \) where \( 0 < \sigma < 1 \) and \( \omega \in \mathbb{R}^N \). However, if \( \sigma \) is formally replaced by \( -\sigma \) in Theorem 1.6 the blow-up rate becomes \( t^{-\frac{4\sigma}{|\sigma|}} \), which does not achieve the uniqueness of blow-up rate may not hold for (1). However, the construction of such a solution is not easy, and the proof of Theorem 1.6 is difficult unless \( \sigma > 0 \). For example, it becomes non-trivial that [9, Lemma 5.2] holds.

2. Notation and preliminaries

We define

\[
(u, v)_2 := \text{Re} \int_{\mathbb{R}^N} u(x)\overline{v}(x)dx, \quad \|u\|_p := \left( \int_{\mathbb{R}^N} |u(x)|^pdx \right)^{\frac{1}{p}},
\]

\[
f(z) := |z|^{\frac{4}{N}} z, \quad F(z) := \frac{1}{2} + \frac{|z|^{2+\frac{4}{N}}}{N} \quad \text{for} \ z \in \mathbb{C}.
\]

By identifying \( \mathbb{C} \) with \( \mathbb{R}^2 \), we denote the differentials of \( f \) and \( F \) by \( df \) and \( dF \), respectively. We define

\[
\Lambda := \frac{N}{2} + x \cdot \nabla, \quad L_+ := -\Delta + 1 - \left(1 + \frac{4}{N}\right)Q^{\frac{4}{N}}, \quad L_- := -\Delta + 1 - Q^{\frac{4}{N}}.
\]

Namely, \( \Lambda \) is the generator of \( L^2 \)-scaling, and \( L_+ \) and \( L_- \) come from the linearised Schrödinger operator to close \( Q \). Then

\[
L_-Q = 0, \quad L_+\Lambda Q = -2Q, \quad L_-|x|^2Q = -4\Lambda Q, \quad L_+\rho = |x|^2Q, \quad L_-xQ = -\nabla Q
\]

hold, where \( \rho \in S(\mathbb{R}^N) \) is the unique radial solution for \( L_+\rho = |x|^2Q \). Furthermore, there exists \( \mu > 0 \) such that for any \( u \in H^1(\mathbb{R}^N) \),

\[
\langle L_+ \text{Re} u, \text{Re} u \rangle + \langle L_- \text{Im} u, \text{Im} u \rangle \geq \mu \|u\|_{H^1}^2 - \frac{1}{\mu} \left( (\text{Re} u, Q)^2 + |(\text{Re} u, xQ)|^2 + (\text{Re} u, |x|^2Q)^2 + (\text{Im} u, \rho)^2 \right)
\]

holds (see, e.g., [11, 12, 14, 16]). Finally, we use the notation \( \preceq \) and \( \succeq \) when the inequalities hold up to a positive constant. We also use the notation \( \approx \) when \( \preceq \) and \( \succeq \) hold.

For the ground state \( Q \), the following property holds:

Proposition 2.1 (E.g., [7]). For any multi-index \( \alpha \), there exist \( C_\alpha, \kappa_\alpha > 0 \) such that

\[
\left| \left( \frac{\partial}{\partial x} \right)^\alpha Q(x) \right| \leq C_\alpha Q(x), \quad \left| \left( \frac{\partial}{\partial x} \right)^\alpha \rho(x) \right| \leq C_\alpha (1 + |x|)^{\kappa_\alpha} Q(x).
\]
We estimate the error terms $\Psi$ that is defined by
\[ \Psi(y) := \lambda^2 W(\lambda y - w)Q(y). \]

Moreover, we define $\kappa$ by
\[ \kappa := \min \left\{ 1 - \frac{N}{q}, 2\sigma_1, 2\sigma_2, \sigma_3 + 1 \right\} \in (0, 1]. \]

Without loss of generality, we may assume that $V(0) = U(0) = 0$.

**Proposition 2.2** (Estimate of $\Psi$). There exists a sufficiently small constant $\epsilon' > 0$ such that
\[ \left\| \epsilon' |y| \Psi \right\|_2 + \left\| \epsilon' |y| \nabla \Psi \right\|_2 \lesssim \lambda^{1+\kappa}(\lambda + |w|) \]
for $0 < \lambda \ll 1$ and $w \in \mathbb{R}^N$ such that $|w| \leq 1$.

**Proof.** As in [8], we have the estimate. We show only for $|x|^{2\sigma}$.

Firstly, we obtain
\[ \left| \nabla \left( |\lambda y - w|^{2\sigma} Q(y) \right) \right| \lesssim |\lambda| |\lambda y - w|^{2\sigma - 1} Q(y) + |\lambda y - w|^{2\sigma} |\nabla Q(y)|. \]

If $2\sigma \geq 1$, then we obtain
\[ \lambda^2 \left\| |\lambda y - w|^{2\sigma - 1} Q e^{\epsilon' |y|^2} \right\|_2 \lesssim \lambda^3 (\lambda^{2\sigma - 1} + w^{2\sigma - 1}) \lesssim \lambda^{1+2\sigma}(\lambda + w). \]

On the other hand, if $2\sigma \leq 1$, then we obtain
\[
\begin{aligned}
\left\| |\lambda y - w|^{2\sigma - 1} Q e^{\epsilon' |y|^2} \right\|_2 &\leq \lambda^{2\sigma} \left\| |y|^{2\sigma - 1} Q \left( y + \frac{w}{\lambda} \right) e^{\epsilon' |y + \frac{w}{\lambda}|^2} \right\|_2 \\
&\lesssim \lambda^{2\sigma} \left( \left\| |y|^{2\sigma - 1} Q \left( y + \frac{w}{\lambda} \right) e^{\epsilon' |y + \frac{w}{\lambda}|^2} \right\|_{L^2(|y| \leq 1)} + \left\| |y|^{2\sigma - 1} Q \left( y + \frac{w}{\lambda} \right) e^{\epsilon' |y + \frac{w}{\lambda}|^2} \right\|_{L^2(|y| \geq 1)} \right)
\end{aligned}
\]

Therefore, from (9),
\[ \lambda^2 \left\| |\lambda y - w|^{2\sigma - 1} Q e^{\epsilon' |y|^2} \right\|_2 \lesssim \lambda^{2+2\sigma} \]
holds.

Accordingly, we obtain Proposition 2.2. \hfill \Box

At the end of this section, we state the following standard result. For the proof, see [12, 9].

**Lemma 2.3** (Decomposition). There exists $\overline{C} > 0$ such that the following statement holds. Let $I$ be an interval and $\delta > 0$ be sufficiently small. We assume that $u \in C(I, H^1(\mathbb{R}^N)) \cap C^1(I, X_1)$ satisfies
\[ \forall \ t \in I, \ \left\| \lambda(t) \frac{\chi}{\lambda(t)} u(t, \lambda(t) y - w(t)) e^{i\tilde{\gamma}(t)} - Q \right\|_{H^1} < \delta \]
for some functions $\lambda : I \to (0, \infty)$, $\gamma : I \to \mathbb{R}$, and $w : I \to \mathbb{R}^N$. Then there exist unique functions $\tilde{\lambda} : I \to (0, \infty)$, $\tilde{b} : I \to \mathbb{R}$, $\tilde{\gamma} : I \to \mathbb{R}/2\pi \mathbb{Z}$, and $\tilde{w} : I \to \mathbb{R}^N$ such that
\[
\begin{aligned}
\tilde{\lambda}(t) - 1 &+ |\tilde{b}(t)| + |\tilde{\gamma}(t) - \gamma(t)|_{\mathbb{R}/2\pi \mathbb{Z}} + \left| \frac{\tilde{w}(t) - w(t)}{\lambda(t)} \right| < \overline{C}
\end{aligned}
\]
hold, where $|c|_{\mathbb{R}/2\pi \mathbb{Z}}$ is defined by
\[ |c|_{\mathbb{R}/2\pi \mathbb{Z}} := \inf_{m \in \mathbb{Z}} |c + 2\pi m|, \]
and that $\tilde{\epsilon}$ satisfies the orthogonal conditions
\[
\begin{aligned}
(\tilde{\epsilon}, i\Lambda Q)_2 &= (\tilde{\epsilon}, |y|^2 Q)_2 = (\tilde{\epsilon}, i\rho)_2 = 0, \quad (\tilde{\epsilon}, yQ)_2 = 0
\end{aligned}
\]
on $I$. In particular, $\tilde{\lambda}, \tilde{b}, \tilde{\gamma}$, and $\tilde{w}$ are $C^1$ functions and independent of $\lambda, \gamma$, and $w$. 
3. Uniformity estimates for modulation terms

From this section to Section 6 we prepare lemmas for the proof of Theorem 1.8.

Given $t_1 < 0$ which is sufficiently close to 0, we define $s_1 := -t_1^{-1}$ and $\lambda_1 = b_1 = s_1^{-1}$. Let $u(t)$ be the solution for (11) with an initial value

$$u(t_1, x) := \frac{1}{\lambda_1} Q \left( \frac{x}{\lambda_1} \right) e^{-i \frac{|x|^2}{2 \lambda_1^2}}. \quad (15)$$

Note that $u \in C((T_s, T^*), \Sigma^2(\mathbb{R}^N))$ and $|x| \nabla u \in C((T_s, T^*), L^2(\mathbb{R}^N))$. Moreover,

$$\text{Im} \int_{\mathbb{R}^N} u(t_1, x) \nabla \psi(t_1, x) dx = 0$$

holds.

Since $u$ satisfies the assumption in Lemma 2.3 in a neighbourhood of $t_1$, there exist decomposition parameters $\tilde{\lambda}_1$, $\tilde{b}_1$, $\tilde{\gamma}_1$, $\tilde{w}_1$, and $\tilde{e}_1$ such that (13) and (14) hold in the neighbourhood. We define the rescaled time $s_{t_1}$ by

$$s_{t_1}(t) := s_1 - \int_t^{t_1} \frac{1}{\lambda_1(s)} \frac{d\tau}{\lambda_1(s)}.$$

Moreover, we define

$$t_{t_1} := s_{t_1}^{-1}, \quad \lambda_{t_1} := \tilde{\lambda}_{t_1}(t_{t_1}(s)), \quad b_{t_1} := \tilde{b}_{t_1}(t_{t_1}(s)), \quad \gamma_{t_1}(s) := \tilde{\gamma}_{t_1}(t_{t_1}(s)), \quad w_{t_1}(s) := \tilde{w}_{t_1}(t_{t_1}(s)), \quad \varepsilon_{t_1}(s, y) := \tilde{\varepsilon}_{t_1}(t_{t_1}(s), y).$$

For the sake of clarity in notation, we often omit the subscript $t_1$. Furthermore, let $I_{t_1}$ be the maximal interval of the existence of the decomposition such that (13) and (14) hold and we define

$$J_{s_1} := s_{t_1}(I_{t_1}).$$

Additionally, let $s_0$ ($\leq s_1$) be sufficiently large and

$$s' := \max \{s_0, \inf J_{s_1}\}.$$

Let $L$, $M$, and $M'$ satisfy

$$1 < M < L \leq 1 + \frac{\kappa}{2}, \quad 0 < M' < \min\{2(L-1), M\}.$$

Moreover, we define $s_*$ by

$$s_* := \inf \{s \in (s', s_1] \mid (13) \text{ holds on } [\sigma, s_1]\},$$

where

$$\text{Mod}(s) := \left\{ \begin{array}{l}
\|\varepsilon(s)\|_{L^2}^2 + b(s)^2 \|\varepsilon(s)\|_{L^2}^2 < s^{-2L}, \\
|s\lambda(s) - 1| < s^{-M}, \quad |sb(s) - 1| < s^{-M'}, \quad |w(s)| < s^{-1}.
\end{array} \right. \quad (16)$$

Note that for all $s \in (s_*, s_1]$, we have

$$s^{-1}(1 - s^{-M}) < \lambda(s) < s^{-1}(1 + s^{-M}), \quad s^{-1}(1 - s^{-M'}) < b(s) < s^{-1}(1 + s^{-M'}).$$

Finally, we define

$$\text{Mod}(s) := \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b, \frac{\partial b}{\partial s} + b^2, 1 - \frac{\partial \gamma}{\partial s} \cdot \frac{\partial w}{\partial s} \right).$$

The goal of this section is to estimate of $\text{Mod}(s)$.

In the following, positive constants $C$ and $\epsilon$ are sufficiently large and small, respectively. If necessary, we retake $s_0$ and $s_1$ sufficiently large in response to $\epsilon$. 
Lemma 3.1 (The equation for $\varepsilon$). In $J_{s_1}$,
\begin{equation}
\Psi = i \frac{\partial \varepsilon}{\partial s} + \Delta \varepsilon - \varepsilon + f(Q + \varepsilon) - f(Q) - \lambda^2 W(\lambda y - w) \varepsilon
- i \left( \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right) \Lambda(Q + \varepsilon) + \left( 1 - \frac{\partial \gamma}{\partial s} \right) (Q + \varepsilon) + \frac{1}{\lambda} \frac{\partial w}{\partial s} \cdot \nabla(Q + \varepsilon) + \frac{1}{\lambda} \frac{\partial w}{\partial s} \cdot y(Q + \varepsilon)
\end{equation}
holds.
\textbf{proof.} This result is proven via direct calculation. \hfill \Box

Lemma 3.2. For $s \in (s_*, s_1]$,
\begin{align}
|\text{Im}(\varepsilon(s), \nabla Q)| & \lesssim s^{-(2L-1)}, \\
(\varepsilon(s), Q)_2 & = -\frac{1}{2} \|\varepsilon(s)\|_2^2, \\
|\text{Mod}(s)| & \lesssim s^{-2L}
\end{align}
\textbf{proof.} As in \cite{8}, we have the estimate. \hfill \Box

4. Modified energy function

In this section, we proceed with a modified version of the technique presented in Le Coz, Martel, and Raphaël \cite{7} and Raphaël and Szeftel \cite{14}. Let $m$ satisfy
\[2 < m \leq 2L.\]
Moreover, we define
\[H(s, \varepsilon) := \frac{1}{2} \|\varepsilon\|^2_{H^1} + \epsilon_1 b^2 \|y|\varepsilon\|_2^2 - \int_{\mathbb{R}^N} \left( F(Q(y) + \varepsilon(y)) - F(Q(y)) - \frac{dF(Q(y))}{dy}(\varepsilon(y)) \right) dy
+ \frac{1}{2} \lambda^2 \int_{\mathbb{R}^N} W(y)|\varepsilon(y)|^2 dy,
\]
\[S(s, \varepsilon) := \frac{1}{\lambda^m} H(s, \varepsilon),\]
where $\epsilon_1$ is a sufficiently small.

Lemma 4.1. For $s \in (s_*, s_1]$,
\[\frac{C_1}{\lambda^m} \left( \|\varepsilon\|^2_{H^1} + \epsilon_1 b^2 \|y|\varepsilon\|_2^2 \right) \leq S(s, \varepsilon) \leq \frac{C_2}{\lambda^m} \left( \|\varepsilon\|^2_{H^1} + b^2 \|y|\varepsilon\|_2^2 \right),
\]
\[\frac{b}{\lambda^m} \left( \|\varepsilon\|^2_{H^1} + b^2 \|y|\varepsilon\|_2^2 - \epsilon s^{-(2L+\kappa')} \right) \lesssim \frac{d}{ds} S(s, \varepsilon(s))
\]
hold.
\textbf{proof.} As in \cite{8}, we have the estimates. \hfill \Box

5. Bootstrap

In this section, we establish the estimates of the decomposition parameters by using a bootstrap argument and the estimates obtained in Section \cite{4}.
Lemma 5.1. There exists a sufficiently small \( \epsilon_2 > 0 \) such that for all \( s \in (s_*, s_1] \),
\[
\| \varepsilon(s) \|_{H^1}^2 + b(s) \| y \varepsilon(s) \|_{H^2}^2 \lesssim s^{-(2L+\kappa')},
\]
\[
| s_1 \lambda(s) - 1 | < (1 - \epsilon_2)s^{-M},
\]
\[
| s_1 b(s) - 1 | \lesssim s^{-2(L-1) + s^{-M}},
\]
\[
| w'(s) | \lesssim s^{-(2L-1)}.
\]
Moreover, \( s_* = s' = s_0 \) if \( s_0 \) is sufficiently large.

**proof.** See [8] for details of the proof.

\[ \square \]

6. Conversion of estimates

In this section, we rewrite the estimates obtained for the time variable \( s \) in Lemma 5.1 into an estimates for the time variable \( t \).

**Lemma 6.1 (Interval).** If \( s_0 \) is sufficiently large, then there exists \( t_0 < 0 \) such that
\[
[t_0, t_1] \subset s_1^{-1}([s_0, s_1]), \quad |s_1(t) - |t|| \lesssim |t|^{M+1} \quad (t \in [t_0, t_1])
\]
hold for \( t_1 \in (t_0, 0) \).

**Lemma 6.2 (Conversion of estimates).** For \( t \in [t_0, t_1] \),
\[
\bar{\lambda}_{t_1}(t) = |t| \left( 1 + \epsilon_{\bar{\lambda}, t_1}(t) \right), \quad \bar{b}_{t_1}(t) = |t| \left( 1 + \epsilon_{\bar{b}, t_1}(t) \right), \quad |\bar{w}_{t_1}(t)| \lesssim |t|^{2L},
\]
\[
\| \bar{\varepsilon}_{t_1} \|_{H^1} \lesssim |t|^{L + \frac{\kappa'}{2}}, \quad \| y \bar{\varepsilon}_{t_1} \|_{H^1} \lesssim |t|^{L + \frac{\kappa'}{2} - 1}
\]
holds. Furthermore,
\[
\sup_{t \in [t_0, t_1]} \left| \epsilon_{\bar{\lambda}, t_1}(t) \right| \lesssim |t|^{M}, \quad \sup_{t \in [t_0, t_1]} \left| \epsilon_{\bar{b}, t_1}(t) \right| \lesssim |t|^{M'}.
\]

See [8] for details of the proofs.

7. Proof of Theorem 1.8

In this section, we prove Theorem 1.8. See [7, 8] for details of the proof.

**Proof of Theorem 1.8.** Let \( \{t_n\}_{n \in \mathbb{N}} \subset (0, 0) \) be a monotonically increasing sequence such that \( \lim_{n \to \infty} t_n = 0 \). For each \( n \in \mathbb{N} \), let \( u_n \) be the solution for (NLS) with an initial value
\[
u_n(t_n, x) := \frac{1}{\lambda_{1,n}^{\frac{1}{2}}} Q \left( \frac{x}{\lambda_{1,n}} \right) e^{-\frac{b_{1,n}}{2} \left| x \right|^2} \lambda_{1,n}^{-\frac{1}{2}}
\]
at \( t_n \), where \( b_{1,n} = \lambda_{1,n} = s_n^{-1} = -t_n \). According to Lemma 2.3 with an initial value \( \bar{\gamma}_n(t_n) = 0 \), there exists a decomposition
\[
u_n(t, x) = \frac{1}{\lambda_n(t)^{\frac{3}{2}}} (Q + \tilde{\varepsilon}_n) \left( t, \frac{x + \tilde{\gamma}_n(t)}{\lambda(t)} \right) e^{-\frac{b}{2} \left| x + \tilde{\gamma}_n(t) \right|^2} e^{i\tilde{\gamma}_n(t)}
\]
on \([t_0, t_n]\). Then, \( \{u_n(t_n)\}_{n \in \mathbb{N}} \) is bounded in \( \Sigma^1 \). Therefore, up to a subsequence, there exists \( u_{\infty}(t_0) \in \Sigma^1 \) such that
\[
u_n(t_n) \to u_{\infty}(t_0) \quad (n \to \infty)
\]
and
\[
u_n(t_0) \to u_{\infty}(t_0) \quad (n \to \infty).
\]
Let \( u_{\infty} \) be the solution for (NLS) with an initial value \( u_{\infty}(t_0) \) and \( T^* \) be the supremum of the maximal existence interval of \( u_{\infty} \). Moreover, we define \( T := \min \{0, T^*\} \). Then, for any \( T' \in [t_0, T] \), \( [t_0, T'] \subset [t_0, t_n] \) if \( n \) is sufficiently large. Then, there exists \( n_0 \) such that
\[
sup_{n \geq n_0} \| u_n \|_{L^\infty([t_0, T'], \Sigma^1)} \lesssim (1 + |T'|^{-1}) (1 + |t_0|^{L'})
\]
holds. Therefore, we obtain

\[ u_n \to u_\infty \text{ in } C([t_0, T'), L^2(\mathbb{R}^N)) \quad (n \to \infty). \]

In particular, \( u_n(t) \to u_\infty(t) \) in \( \Sigma^1 \) for any \( t \in [t_0, T) \). Furthermore, we have

\[ \|u_\infty(t)\|_2 = \|u_\infty(t_0)\|_2 = \lim_{n \to \infty} \|u_n(t_0)\|_2 = \lim_{n \to \infty} \|u_n(t_n)\|_2 = \|Q\|_2. \]

Based on weak convergence in \( H^1(\mathbb{R}^N) \) and Lemma \ref{lem:weak_convergence}, we decompose \( u_\infty \) to

\[ u_\infty(t, x) = \frac{1}{\hat{\lambda}(t)} (Q + \hat{\epsilon}) \left( t, x + \hat{\omega}(t) \right) e^{-i \frac{k_\infty(t)}{\hat{\lambda}(t)} (t + \hat{\omega}(t))^2 + i \gamma(t)} \]

on \( [t_0, T) \) for some initial value of \( \hat{\gamma} \) is \( \gamma_\infty(t_0) \in \left( |0|^{-1} - \pi, |0|^{-1} + \pi \right] \). Furthermore, as \( n \to \infty \),

\[ \hat{\lambda}_n(t) \to \hat{\lambda}_\infty(t), \quad \hat{b}_n(t) \to \hat{b}_\infty(t), \quad \hat{\omega}_n(t) \to \hat{\omega}_\infty(t), \quad e^{i \gamma_n(t)} \to e^{i \gamma_\infty(t)}, \quad \hat{\epsilon}_n(t) \to \hat{\epsilon}_\infty(t) \]

in \( \Sigma^1 \) holds for any \( t \in [t_0, T) \). Therefore, we have

\[ \hat{\lambda}_\infty(t) = |t| (1 + \epsilon_{\hat{\lambda},0}(t)), \quad \hat{b}_\infty(t) = |t| (1 + \epsilon_{\hat{b},0}(t)), \quad \|\hat{\omega}_\infty(t)\|_2 \lesssim |t|^{2L}, \]

\[ \|\hat{\epsilon}_\infty(t)\|_H^2 \lesssim |t|^{L + \epsilon_{\gamma,0}'}, \quad \|\|\hat{\epsilon}_\infty(t)\|_2 \| \lesssim |t|^{L + \epsilon_{\gamma,0}' - 1}, \]

\[ |\epsilon_{\hat{\lambda},0}(t)| \lesssim |t|^M, \quad |\epsilon_{\hat{b},0}(t)| \lesssim |t|^M \]

from a uniform estimate of Lemma \ref{lem:uniform_estimate}. Consequently, we obtain Theorem \ref{thm:main_theorem}. \( \square \)

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