Abstract. A fuzzy geometry is a certain type of spectral triple whose Dirac operator crucially turns out to be a finite matrix. This notion was introduced in [J. Barrett, J. Math. Phys. 56, 082301 (2015)] and accommodates familiar fuzzy spaces like spheres and tori. In the framework of random noncommutative geometry, we use Barrett’s characterization of Dirac operators of fuzzy geometries in order to systematically compute the spectral action $S(D) = \text{Tr} f(D)$ for $2n$-dimensional fuzzy geometries. In contrast to the original Chamseddine-Connes spectral action, we take a polynomial $f$ with $f(x) \to \infty$ as $|x| \to \infty$ in order to obtain a well-defined path integral that can be stated as a random matrix model with action of the type $S(D) = N \cdot \text{tr} F + \sum_i \text{tr} A_i \cdot \text{tr} B_i$, being $F, A_i$ and $B_i$ noncommutative polynomials in $2^{2n-1}$ complex $N \times N$ matrices that parametrize the Dirac operator $D$. For arbitrary signature—thus for any admissible KO-dimension—formulas for 2-dimensional fuzzy geometries are given up to a sextic polynomial, and up to a quartic polynomial for 4-dimensional ones, with focus on the octo-matrix models for Lorentzian and Riemannian signatures. The noncommutative polynomials $F, A_i$ and $B_i$ are obtained via chord diagrams and satisfy: independence of $N$; self-adjointness of the main polynomial $F$ (modulo cyclic reordering of each monomial); also up to cyclicity, either self-adjointness or anti-self-adjointness of $A_i$ and $B_i$ simultaneously, for fixed $i$. Collectively, this favors a free probabilistic perspective for the large-$N$ limit we elaborate on.

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1. Introduction

In some occasions, the core concept of a novel research avenue can be traced back to a defiant attitude towards a no-go theorem. However uncommon this is, some prolific theories that arose from a slight perturbation of the original assumptions, aiming at an escape from the no-go, have shaped the modern landscape of mathematical physics. Arguably, the best-known story fitting this description is supersymmetry.

Another illustration is found in noncommutative geometry (NCG) applications to particle physics: In an attempt to unify all fundamental interactions, the proposal of trading gravitation coupled to matter on a usual spacetime manifold $M$ by pure gravitation on an extended space $M \times F$ is bound to fail—as a well-known symmetry argument (amidst other objections) shows—as far as spacetime is extended by an ordinary manifold $F$. Rebelling against the no-go result, while not giving up a gravitational unification approach, shows the way out of the realm of commutative spaces (manifolds) after restating the symmetries in an algebraic fashion. For the precise argument we refer to [CM07, Sec. 9.9] and for the details on the obstruction to [Thu74, Mat74, Mat75, Eps70].

Following the path towards a noncommutative description of the ‘internal space’ $F$ (initially a two-point space), Connes was able to incorporate the Higgs field on a geometrically equal footing with gauge fields, simultaneously avoiding the Kaluza-Klein tower that an augmentation of spacetime by an ordinary space $F$ would cause. As a matter of fact, not only the Higgs sector but the whole classical action of the Standard Model of particle physics has been geometrically derived [CCM07, Bar07] from the Chamseddine-Connes spectral action $^1$ [CC97]. The three-decade-old history of the impact of Connes’ groundbreaking idea on the physics beyond the Standard Model is told in [CvS19] (to whose comprehensive references one

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$^1$To be precise, in this article ‘spectral action’ means ‘bosonic spectral action’. The derivation of the Standard Model requires also a fermionic spectral action $\langle J\hat{\psi}, D\hat{\psi} \rangle$ where $\hat{\psi}$ is a matrix (see [DDS18]) of classical fermions. See [Liz18] for a physics review and [vS15] and [CM07, Secs.9-18] for detailed mathematical exposition.
could add later works [Bes19, BF19, MS19, BS20]); see his own review [Con19] for the impact of the spectral formalism on mathematics.

On top of the very active quest for the noncommutative internal space $F$ that corresponds to a chosen field theory, it is pertinent to point out that such theory is classical and that quantum field theory tools (for instance, the renormalization group) are adapted to it. The proposals on presenting noncommutative geometries in an inherently quantum setting are diverse: A spin network approach led to the concept of gauge networks, along with a blueprint for spin foams in NCG, as a quanta of NCG [MvS14]; therein, from the spectral action (for Dirac operators) on gauge networks, the Wilson action for Higgs-gauge lattice theories and the Kogut-Susskind Hamiltonian (for a 3-dimensional lattice) were derived, as an interesting result of the interplay among lattice gauge theory, spin networks and NCG. Also, significant progress on the matter of fermionic second quantization of the spectral action, relating it to the von Neumann entropy, has been proposed in [CCvS18]; and a bosonic second quantization was undertaken more recently in [DK19]. The context of this paper is a different, random geometrical approach motivated by the path-integral quantization of noncommutative geometries

$$Z = \int_{\mathcal{M}} e^{-\text{Tr} f(D)} dD,$$

where $S(D) = \text{Tr} f(D)$ is the (bosonic) spectral action. The integration is over the space $\mathcal{M}$ of geometries encoded by Dirac operators $D$ on a Hilbert space that, in commutative geometry, corresponds to the square integrable spinors $\mathcal{H}_M$ (well-defining this $Z$ is a fairly simplified version of the actual open problem stated in [CM07, Ch. 18.4]). The meaning of this partition function $Z$ is not clear for Dirac operators corresponding to an ordinary spacetime $M$. In order to get a finite-rank Dirac operator one can, on the one hand, truncate the algebra $C^\infty(M)$ and the Hilbert space $\mathcal{H}_M$ in order to get a well-defined measure $dD$ on the space of geometries $\mathcal{M}$, now parametrized by finite, albeit large, matrices. On the other hand, one does not want to fall in the class of lattice geometries [HP03] or finite geometries [Kra98].

Fuzzy geometries are finite-dimensional geometries that escape the classification of finite geometries given in [Kra98], depicted in terms of the Krajewski diagrams, and [PS98]. In fact, fuzzy geometries retain also a (finite dimensional) model of the spinor space that is not present in a finite geometry. Moreover, in contradistinction to lattices, fuzzy geometries are genuinely—and not only in spirit—noncommutative. In particular, the path-integral quantization of fuzzy geometries differs also from the approach in [HP03] for lattice geometries.

Of course, fuzziness is not new [Mad92] and can be understood as limited spatial resolution on spaces. The prototype is the space spanned by finitely many spherical harmonics approximating the algebra of functions on the sphere $S^2$. This picture is in line with models of quantum gravity, since classical spacetime is expected to break down at scales below Planck length [DFR95].

Although the three components of a spectral triple have sometimes been evoked in the study of fuzzy spaces [DHO08] and their Dirac operators on some fuzzy spaces are well-studied (e.g. the Grosse-Prešnajder Dirac operator [GP95]), a novelty in [Bar15] is their systematic spectral triple formulation; for instance, fuzzy
tori, elsewhere addressed (e.g. [DO03, SchSt13]), acquire a spectral triple [BG19]. Spectral triples are data that algebraically generalize spin manifolds. More precisely, when the spectral triple is commutative (i.e. the algebraic structure that generalizes the algebra of coordinates is commutative, with additional assumptions we omit) a strong theorem is the ability to construct, out of it, an oriented, smooth manifold, with its metric and spin$^c$ structure. This has been proven by Connes [Con13] taking some elements from previous constructs by Rennie-Várilly [RV06].

This paper computes the spectral action for fuzzy geometries. Compared with the smooth case, our methods are simpler. For an ordinary manifold $M$ or an almost commutative space $M \times F$ (being $F$ a finite geometry [vS15, Sec. 8]), one commonly relies on a heat kernel expansion

$$\text{Tr}(e^{-tD^2}) \sim \sum_{n \geq 0} t^{n-\dim(M)/2} a_n(D^2) \quad (t^+ \to 0),$$

which allows, for $f$ of the Laplace-Stieltjes transform type $f(x) = \int_{\mathbb{R}_+} e^{-tx^2} d\nu(t)$, to determine the spectral action $\text{Tr}(f(D/\Lambda))$ in terms of the Seeley-DeWitt coefficients $a_{2n}(D^2)$ [Gil95], being $t = \Lambda^{-1}$ the inverse of the cutoff $\Lambda$; see also [EI19]. The elements of Gilkey’s theory are not used here. Crucially, $f$ is instead assumed to be a polynomial (with $f(x) \to \infty$ for $|x| \to \infty$), which enables one to directly compute traces of powers of the Dirac operator. This alteration of the Chamseddine-Connes spectral action —in which $f$ is typically a symmetric bump function around the origin— comes from a convergence requirement for the path-integral (1.1), as initiated in [BG16] (a polynomial spectral action itself is already considered in [MvS14], though, for gauge networks arising from embedded quivers in a spin manifold). The motivation of Barrett-Glaser is to access information about fuzzy geometries by looking at the statistics of the eigenvalues of $D$ using Markov chain Monte Carlo simulations. This and a posterior study [BDG19] deliver evidence for a phase transition to a 2-dimensional behavior (also of significance in quantum gravity [Car19]).

Finally, the paper is organized as follows: the next section, based on [Bar15], introduces spectral triples and fuzzy geometries in a self-contained way. The definition is slightly technical, but the essence of a fuzzy geometry can be understood from its matrix algebra, its Hilbert space $\mathcal{H}$ and Barrett’s characterization of Dirac operators (Secs. 2.3 and 2.4). In Section 3 we compute the spectral action in a general setting. A convenient graphical description of ‘trace identities’ for gamma matrices (due to the Clifford module structure of $\mathcal{H}$, Sec. 3.1) is provided in terms of chord diagrams, which later serve as organizational tool in the computation of $\text{Tr}(D^m)$, $m \in \mathbb{N}$. As the main results in Sections 4 and 5, we derive formulas for the spectral action for 2- and 4-dimensional fuzzy geometries, respectively. In the latter case, we elaborate on the Riemannian and Lorentzian cases, being these the first reported (analytic) derivations for the spectral action of $d$-dimensional fuzzy geometries with general Dirac operators in $d > 3$. Formulas for the spectral action for geometries of signature (0,3), which lead to a tetra-matrix model$^2$, were

$^2$Strictly seen, these lead to an octo-matrix model, but a simplification is allowed by the fact that the product of all gamma matrices is a scalar.
presented in [BG16, App. A.6]. Later, Glaser explored the phase transition of the fuzzy-sphere-like (1,3) case—that is of KO-dimension 2, as satisfied by the Grosse-Prešnajder operator—together with that of (1,1) and (2,0) geometries of KO-dimensions 0 and 6, respectively. For the (1,3) geometry, the spectral action used in the numerical simulations of [Gla17] was obtained inside MCMCv4, a computer code aimed at simulating random fuzzy geometries; the formula (in C++ language) for the spectral action can be found in the file Dirac.cpp of [Gla].

Our solely analytic approach to spectral action computations yields, out of a single general proof, a formula for any admissible KO-dimension, as it will become apparent in Proposition 5.4.

In Section 6, we restate our results, aiming at free probabilistic tools towards the large-$N$ limit (being $N$ the matrix size in Barrett’s parametrization of the Dirac operator). In order to define noncommutative (NC) distributions, one often departs from a self-adjoint NC polynomial. It turns out that only a weaker concept (‘cyclic self-adjointness’) defined here is satisfied by the main NC polynomial $P$ in

$$\text{Tr} f(D) = N \cdot \text{Tr}_N P + \sum_i \text{Tr}_N \Phi_i \text{Tr}_N \Psi_i;$$

the other NC polynomials $\Phi_i$ and $\Psi_i$ (for fixed $i$) either satisfy this very condition, or they are both cyclic anti-self-adjoint. The trace $\text{Tr}_N$ cannot tell apart these conditions from the actual self-adjointness of a NC polynomial.

The conclusions and the outlook are presented in the last two sections. The short Appendix A contains some useful information about (anti-)hermiticity of products of gamma matrices for general signature. To ease legibility, some steps in the proof of Proposition 4.1 (the sextic term in dimension 2) have been placed in Appendix B, which is also intended as a stand-alone example on how to gain NC polynomials from chord diagrams.

2. Fuzzy geometries as spectral triples

The formalism of spectral triples in noncommutative geometry can be very intricate and its full machinery will not be used here. We refer to [vS15] for more details on the usage of spectral triples in high energy physics.

The essential structure is the spectral triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is a unital, involutive algebra of bounded operators on a Hilbert space $\mathcal{H}$. The Dirac operator $D$ is a self-adjoint operator on $\mathcal{H}$ with compact resolvent and such that $[D,a]$ is bounded for all $a \in \mathcal{A}$. On $\mathcal{H}$, the algebraic behavior between of the Dirac operator and the algebra $\mathcal{A}$—and later also among $D$ and some additional operators on $\mathcal{H}$—encodes geometrical properties. For instance, the geodesic distance $d_g$ between two points $x$ and $y$ of a Riemannian (spin) manifold $(M,g)$, can be recovered from

$$d_g(x,y) = \sup_{a \in \mathcal{A}} \{ |a(x) - a(y)| : a \in \mathcal{A} \text{ and } ||[D,a]|| \leq 1 \},$$

being $\mathcal{A}$ the algebra of functions on $M$ and $D$ the canonical Dirac operator [Con94, Sect VI.1].

Precisely those additional operators lead to the concept of real, even spectral triple, which allows to build physical models. Next definition, taken from [Bar15], is given here by completeness, since fuzzy geometries are a specific type of real (in this paper all of them even) spectral triples.

**Definition 2.1.** A real, even spectral triple of KO-dimension $s \in \mathbb{Z}/8\mathbb{Z}$ consists in the following objects and relations:
(i) an algebra \( \mathcal{A} \) with involution \( * \)
(ii) a Hilbert space \( \mathcal{H} \) together with a faithful, \(*\)-algebra representation \( \rho : \mathcal{A} \to \mathcal{L}(\mathcal{H}) \)
(iii) an anti-linear unitarity (called real structure) \( J : \mathcal{H} \to \mathcal{H} \), \( \langle Jv, Jw \rangle = \langle w, u \rangle \), being \( \langle \cdot, \cdot \rangle \) the inner product of \( \mathcal{H} \)
(iv) a self-adjoint operator \( \gamma : \mathcal{H} \to \mathcal{H} \) commuting with the representation \( \rho \) and satisfying \( \gamma^2 = 1 \) (called chirality)
(v) for each \( a, b \in \mathcal{A} \), \( [\rho(a), J\rho(b)J^{-1}] = 0 \)
(vi) a self-adjoint operator \( D \) on \( \mathcal{H} \) that satisfies
\[
[ [D, \rho(a)], J\rho(b)J^{-1}] = 0, \quad a, b \in \mathcal{A}
\]
(vii) the relations
\[
\begin{align*}
J^2 &= \epsilon \\
JD &= \epsilon' DJ \\
J\gamma &= \epsilon'' \gamma J
\end{align*}
\]
with the signs \( \epsilon, \epsilon', \epsilon'' \) determined by \( s \) according to the following table:

| \( s \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-------|---|---|---|---|---|---|---|---|
| \( \epsilon \) | + | + | - | - | - | + | + | + |
| \( \epsilon' \) | + | - | + | + | - | + | + | + |
| \( \epsilon'' \) | + | + | - | + | + | - | + | + |

A fermion space of KO-dimension \( s \) is a collection of objects \( (\mathcal{A}, \mathcal{H}, J, \gamma) \) satisfying axioms (i) through (v) and (vii), except for eq. (2.1b).

2.1. Gamma matrices and Clifford modules. Given a signature \((p, q) \in \mathbb{Z}_{\geq 0}^2\), a spinor (vector) space \( V \) is a representation \( c \) of the Clifford algebra\(^3\) \( \mathcal{C}(p, q) \). Thus, elements of the basis \( e^a \) and \( \bar{e}^{\dot{a}} \) of \( \mathbb{R}^{p+q}, a = 1, \ldots, p \) and \( \dot{a} = 1, \ldots, q \), become endomorphisms \( c(e^a) = \gamma^a, c(e^{\dot{a}}) = \gamma^{\dot{a}} \) of \( V \). If \( d = q + p \) is even, \( V \) is assumed to be irreducible, whereas only the eigenspaces \( V^+, V^- \subset V \) of \( \gamma \) are, if \( q + p \) is odd. The size of these square matrices (the Dirac gamma matrices) is \( 2^{\lfloor (p+q) \rfloor} \).

It follows from the relations of the Clifford algebra that
\[
\gamma^\mu \gamma^\nu = \gamma^{[\mu \nu]} + \frac{1}{2} \{ \gamma^\mu, \gamma^\nu \} = \gamma^{|\mu \nu|} + g^\mu \nu 1_V,
\]
which can be used to iteratively compute products of gamma matrices in terms of \( g^\mu \nu \) and their anti-symmetrization. Taking their trace \( \text{Tr}_V \) (contained in the spectral action) gets rid of the latter, so we are left with \( \dim V \cdot g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} \cdots \). A product of an odd number of gamma matrices is traceless; the trace of a product of \( 2n \) gamma matrices can be expressed as a sum of over \( (2n - 1)! \) products of \( n \) bilinears \( g^\mu \nu \) that will be represented diagrammatically.

\(^3\)We recall that \( \mathcal{C}(p, q) \) is the tensor algebra of \( \mathbb{R}^{p+q} \) modulo the relation \( 2g(v, w) = v \otimes w + w \otimes v \) for each \( u, w \in \mathbb{R}^{p+q} \), being \( g = \text{diag}(+, \ldots, +, -, \ldots, -) \) the quadratic form with \( p \) positive and \( q \) negative signs.
2.2. Fuzzy geometries. Section 2.2 is based on [Bar15]. A fuzzy geometry can be thought of as a finite-dimensional approximation to a smooth geometry. A simple matrix algebra $M_N(\mathbb{C})$ conveys information about the resolution of a space (an inverse power of $N$, e.g. $\sim 1/\sqrt{N}$ for the fuzzy sphere $S^4_N$ [SpSt18]) where the noncommutativity effects are no longer negligible. To do geometry on a matrix algebra one needs additional information, which, in the case of fuzzy geometries, is in line with the spectral formalism of NCG.

**Definition 2.2** (Paraphrased from [Bar15]). A fuzzy geometry of signature $(p,q) \in \mathbb{Z}_{\geq 0}^2$ is given by

- a simple matrix algebra $\mathcal{A}$ with coefficients in $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$; in the latter case, $M_{N/2}(\mathbb{H}) \subset M_N(\mathbb{C})$, otherwise $\mathcal{A}$ is $M_N(\mathbb{R})$ or $M_N(\mathbb{C})$ — in this paper we take always $\mathcal{A} = M_N(\mathbb{C})$

- a Hermitian $\mathcal{C}(p,q)$-module $V$ with a chirality $\gamma$. That is a linear map $\gamma : V \rightarrow V$ satisfying $\gamma^\ast = \gamma$ and $\gamma^2 = 1$

- a Hilbert space $\mathcal{H} = V \otimes M_N(\mathbb{C})$ with inner product $\langle v \otimes R, w \otimes S \rangle = (v, w)\text{Tr}(R^\ast S)$ for each $R, S \in M_N(\mathbb{C})$, being $(\cdot, \cdot)$ the inner product of $V$

- a left-$\mathcal{A}$ representation $\rho(a)(v \otimes R) = v \otimes (aR)$ on $\mathcal{H}, a \in \mathcal{A}$ and $v \otimes R \in \mathcal{H}$

- three signs $\epsilon, \epsilon', \epsilon'' \in \{-1, +1\}$ determined through $s := q - p$ by the following table:

| $s \equiv q - p \pmod{8}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|--------------------------|---|---|---|---|---|---|---|---|
| $\epsilon$               | + | + | - | - | - | - | + | + |
| $\epsilon'$              | + | - | + | + | + | - | + | + |
| $\epsilon''$             | + | + | - | + | + | - | + | + |

- a real structure $J = C \otimes \ast$, where $\ast$ is complex conjugation and $C$ is an anti-unitarity on $V$ satisfying $C^2 = \epsilon$ and $C\gamma^\mu = \epsilon'\gamma^\mu C$ for all the gamma matrices $\mu = 1, \ldots, p + q$

- a self-adjoint operator $D$ on $\mathcal{H}$ satisfying the **order-one condition**

$$[[D, \rho(a)], J\rho(b)J^{-1}] = 0 \quad \text{for all } a, b \in \mathcal{A}$$

- a chirality $\Gamma = \gamma \otimes 1_\mathcal{A}$ for $\mathcal{H}$, where $\gamma$ is the chirality of $V$. These signs impose on the operators the following conditions:

$$J^2 = \epsilon, \quad JD = \epsilon' DJ, \quad J\Gamma = \epsilon'' \Gamma J.$$

For $s$ odd, $\Gamma$ can be thought of as the identity $1_\mathcal{H}$. The number $d = p + q$ is the **dimension** of the spectral triple and $s = q - p$ is its **KO-dimension**.

We pick gamma matrices that satisfy

$$\gamma^\mu = \text{Hermitian}, \quad (\gamma^\mu)^2 = +1, \quad \mu = 1, \ldots, p,$$

$$\gamma^\mu = \text{anti-Hermitian}, \quad (\gamma^\mu)^2 = -1, \quad \mu = p + 1, \ldots, p + q$$

in terms of which the chirality for $V$ is given by $\gamma = (-1)^{s(s-1)/2}\gamma_1 \cdots \gamma_{p+q}$. For mixed signatures it will be convenient to separate spatial from time like indices.
and denote by lowercase Roman the former \((a = 1, \ldots, p)\) and by dotted indices\(^4\) \((\dot{c} = p + 1, \ldots, p + q)\) the latter. The gamma matrices \(\gamma^\mu\) are Hermitian matrices squaring to \(+1\), and \(\gamma^\dot{c}\)'s denote here the anti-Hermitian matrices squaring to \(-1\). Greek indices are spacetime indices \(\alpha, \beta, \mu, \nu, \ldots \in \Delta_d := \{1, 2, \ldots, d\}\).

We let the gamma matrices generate \(\Omega := \langle \gamma^1, \ldots, \gamma^d \rangle_R\) as algebra; this splits as \(\Omega = \Omega^+ \oplus \Omega^-\), where \(\Omega^+\) contains products of an even number of gamma matrices and \(\Omega^-\) an odd number of them.

2.3. General Dirac operator. Using the spectral triple axioms for fuzzy geometries, their Dirac operators can be characterized as self-adjoint operators of the form [Bar15, Sec. 5.1]

\[
D(v \otimes R) = \sum_I \omega^I v \otimes (K_I R + \epsilon' R K_I), \quad v \in V, R \in \mathcal{M}_N(\mathbb{C}),
\]

(2.3)

with \(\{\omega^I\}_I\) a linearly independent set and \(I\) an abstract index to be clarified now. For \(r \in \mathbb{N}_{\leq d}\), let \(\Lambda^r_d\) be the set of \(r\)-tuples of increasingly ordered spacetime indices \(\mu_i \in \Delta_d\), i.e. \(\Lambda^r_d = \{(\mu_1, \ldots, \mu_r) \mid \mu_i < \mu_j \text{ if } i < j\}\). We let \(\Lambda_d = \cup_r \Lambda^r_d\), whose odd part is denoted by \(\Lambda^\pm_d\),

\[
\begin{align*}
\Lambda^d_+ &= \{(\mu_1, \ldots, \mu_r) \mid \text{for some odd } r, 1 \leq r \leq d \text{ & } \mu_i < \mu_j \text{ if } i < j\} \\
&= \{1, \ldots, d\} \cup \{(\mu_1, \mu_2, \mu_3) \mid 1 \leq \mu_1 < \mu_2 < \mu_3 \leq d\} \\
&\quad \cup \{(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5) \mid 1 \leq \mu_1 < \mu_2 \ldots < \mu_5 \leq d\} \cup \ldots
\end{align*}
\]

The most general Dirac operator in dimension \(d\) writes in terms of products \(\Gamma^I\) of gamma matrices that correspond to indices \(I\) in these sets, each bearing a matrix coefficient \(k_I\),

\[
D(p,q) = \begin{cases} 
\sum_{I \in \Lambda_d^+} \Gamma^I \otimes k_I & \text{for } d = p + q \text{ even}, \\
\sum_{I \in \Lambda_d^+} \Gamma^I \otimes k_I & \text{for } d = p + q \text{ odd}.
\end{cases}
\]

(2.4)

We elaborate on each of the tensor-factors, \(\Gamma^I\) and \(k_I\). First, \(\Gamma^I\) is the ordered product of gamma matrices with all single indices appearing in \(I\),

\[
\Gamma^I = \gamma^{\mu_1} \cdots \gamma^{\mu_r},
\]

for \(I = (\mu_1, \ldots, \mu_r) \in \Lambda_d^+\). This can be thought of as each gamma matrix \(\gamma^\mu\) corresponding to a one-form \(dx^\mu\) (in fact, via Clifford multiplication for canonical spectral triples) and \(\Lambda_d\) as the basis elements of the exterior algebra. The set \(\Lambda_d = \cup_r \Lambda^r_d\) can thus be seen as an abstract backbone of the de Rham algebra \(\Omega^r_{dR} = \bigoplus_r \Omega^r_{dR}\) and \(\Lambda^r_d\) of the \(r\)-forms \(\Omega^r_{dR}\). There are \(#(\Lambda^r_d) = \binom{d}{r}\) independent \(r\)-tuple products of gamma matrices. We now separate the cases according to the parity of \(s\) (or of \(d\)). Second, \(k_I(R) = (K_I R + \epsilon R K_I)\) is an operator on \(\mathcal{M}_N(\mathbb{C}) \ni R\), which needs a dimension-dependent characterization.

\(^4\)Dotted indices are here unrelated to their usual interpretation in the theory of spinors. Also for the Lorentzian signature, the 0,1,2,3 numeration (without any dots) is used.
2.4. Characterization of the Dirac operator in even dimensions. We constrain the discussion to even (KO-)dimension. In Definition 2.2 the table implies \( e' = 1 \); on top of this, the self-adjointness of \( D \) implies that for each \( I \), both \( \omega^I \) and \( R \mapsto (K_I R + R K_I^*) \) are either Hermitian or both anti-Hermitian. In terms of the matrices \( K_I \), this condition reads \( K_I^* = +K_I \) or \( K_I^* = -K_I \), respectively. In the first case we write \( K_I = H_I \), in the latter \( K_I = L_I \). One can thus split the sum in eq. (2.3) as

\[
D(v \otimes R) = \sum_I \omega^I v \otimes (H_I R + RH_I) + \sum_I \omega^I v \otimes (L_I R - RL_I). \tag{2.5}
\]

Additionally, since \( d = 2\rho - s \) is even, \( \gamma^a \gamma^a + \gamma^a \gamma = 0 \). Hence the same anti-commutation relation \( \gamma \omega + \omega \gamma = 0 \) holds for each \( \omega \in \Omega^- \). This leads to the splitting

\[
D = \sum_{l \in \Lambda_d^-} \tau^l \{ H_I, \cdot \} + \sum_{l \in \Lambda_d^+} \alpha^l \{ L_I, \cdot \}, \tag{2.6}
\]

where each \( \tau^l \) and \( \alpha^l \) is an odd product of gamma matrices, and \( \Lambda_d^- \) is the set of multi-indices of an odd number of indices \( \mu \in \{1, \ldots, d\} \). In summary,

\[
(\tau^l)^* = \tau^l \in \Omega^- \quad \text{and} \quad (\alpha^l)^* = -\alpha^l \in \Omega^- \quad \text{Hence } H_I^* = H_I, \quad L_I^* = -L_I.
\]

We generally treat commutators and anti-commutators as (noncommuting) let-

ers \( k_I = \{ K_I, \cdot \} \), for each \( I \in \Lambda_d^- \). The sign \( e_I = \pm \) determines the type of the letter for \( k_I \), being the latter defined by the rule

\[
e_I = \begin{cases} +1 & \text{then } K_I = H_I \quad \text{therefore } k_I = h_I, \\ -1 & \text{then } K_I = L_I \quad \text{therefore } k_I = l_I, \end{cases} \tag{2.7}
\]

so \( k_I(R) = K_I R + e_I R K_I \), for \( R \in M_N(\mathbb{C}) \). Explicitly, one has

\[
D^{(p,q)} = \sum_{\mu} \gamma^\mu \otimes k_{\mu} + \sum_{\mu,\nu,\rho} \gamma^\mu \gamma^\nu \gamma^\rho \otimes k_{\mu\nu\rho} + \cdots + \sum_{\mu,\nu,\rho} \Gamma_{\mu\nu\rho} \otimes k_{\mu\nu\rho} + \sum_{\bar{\mu}} \bar{\Gamma} \otimes k_{\bar{\mu}},
\]

which runs through \( \sum_{\mu} \gamma^{\mu_1} \gamma^{\mu_2} \cdots \gamma^{\mu_d/2} \otimes k_{\mu_1\mu_2\cdots\mu_d/2} \) if the 4 divides \( d \), or through

\[
\sum_{\mu} \Gamma^{\mu_1\ldots\mu_d/2-1} \otimes k_{\mu_1\mu_2\cdots\mu_d/2-1} + \sum_{\mu} \Gamma^{\mu_1\ldots\mu_d/2+1} \otimes k_{\mu_1\mu_2\cdots\mu_d/2+1}
\]

if \( d \) is even but not divisible by 4. Hatted indices are, as usual, those excluded from \( \{1, \ldots, d\} \),

\[
\mu \bar{\mu} \ldots \rho = (1, 2, \ldots, \mu - 1, \mu + 1, \ldots, \nu - 1, \nu + 1, \ldots, \rho - 1, \rho + 1, \ldots, d).
\tag{2.8}
\]

In order for a Dirac operator to be self-adjoint, \( k_I \) is constrained by the parity of \( r = r(I) \), being \( |I| = 2r - 1 \), and by the number \( u(I) \) of spatial gamma matrices in the product \( \Gamma^I \). In a mixed signature setting, \( p, q > 0 \), an arbitrary \( I \in \Lambda_d^- \) has the form \( I = (a_1, \ldots, a_t, \hat{e}_1, \ldots, \hat{e}_u) \) for \( 0 \leq t \leq p \), \( 0 \leq u \leq q \), and so the corresponding matrix satisfies

\[
(\Gamma^I)^* = (-1)^{u + \lfloor u + t/2 \rfloor} \Gamma^I = (-1)^{u + r - 1} \Gamma^I. \tag{2.9}
\]

The first equality is shown in detail in Appendix A. The second is just due to

\[
(-1)^{\lfloor u + t/2 \rfloor} = (-1)^{\lfloor (2r - 1)/2 \rfloor} = (-1)^{r - 1}. \]

This decides whether \( k_I \) should be an ‘\( h_I \)-operator’ or an ‘\( l_I \)-operator’ (see eq. 2.7), which is summarized in Table 1.
\begin{table}[h]
\centering
\begin{tabular}{ccc}
\hline
$u(I)$ & $r(I)$ & $k_I$ \\
\hline
even & odd & $h_I$ \\
odd & odd & $l_I$ \\
even & even & $l_I$ \\
odd & even & $h_I$ \\
\hline
\end{tabular}
\caption{For $I = (a_1, \ldots, a_u, \hat{c}_1, \ldots, \hat{c}_t) \in \Lambda_{d}^{2r-1}$ a Hermitian matrix $H_I$ or an anti-Hermitian matrix $L_I$ parametrizes $k_I$ according to the shown operators $h_I = \{H_I, \cdot \}$ or $l_I = [L_I, \cdot]$}
\end{table}

For indices running where the dimension bounds allow, one has

$$D^{(p,q)} = \sum_{a=1}^{p} \gamma^a \otimes h_a + \sum_{\hat{c}=p+1}^{p+q} \gamma^\hat{c} \otimes l_{\hat{c}}$$

$$+ \sum_{a,b,c} \gamma^a \gamma^b \gamma^c \otimes l_{abc} + \sum_{a,b,\hat{c}} \gamma^a \gamma^b \gamma^\hat{c} \otimes h_{ab\hat{c}}$$

$$+ \sum_{a,b,\hat{c}} \gamma^a \gamma^b \gamma^\hat{c} \otimes l_{ab\hat{c}} + \cdots$$

$$+ \begin{cases} 
\sum_a \Gamma^a \otimes h_a + \sum_{\hat{c}} \Gamma^{\hat{c}} \otimes l_{\hat{c}} & \text{if } q \text{ and } d/2 \text{ have same parity,} \\
\sum_a \Gamma^a \otimes l_a + \sum_{\hat{c}} \Gamma^{\hat{c}} \otimes h_{\hat{c}} & \text{if } q \text{ and } d/2 \text{ have opposite parity.} 
\end{cases}$$

The last term is a product of $d-1 = p+q-1$ matrices. This expression is again determined by observing that the operator $k_{\hat{\mu}}$ is self-adjoint if $(-1)^{u+d/2}$ equals $+1$ and otherwise anti-Hermitian, being $u$ the number of spatial gamma matrices in $\Gamma_{\hat{\mu}} = \gamma^1 \cdots \hat{\gamma}^{\hat{\mu}} \cdots \gamma^d$. We proceed to give some examples.

**Example 2.3 (Fuzzy $d = 2$ geometries).** The next operators appear in [Bar15]:

- **Type (0,2).** Then $s = d = 2$, so $\epsilon' = 1$. The gamma matrices are anti-Hermitian and satisfy $(\gamma^i)^2 = -1$. The Dirac operator is

$$D^{(0,2)} = \gamma^1 \otimes [L_1, \cdot] + \gamma^2 \otimes [L_2, \cdot]$$

- **Type (1,1).** Then $d = 2$, $s = 0$, so $\epsilon' = 1$.

$$D^{(1,1)} = \gamma^1 \otimes \{H_1, \cdot \} + \gamma^2 \otimes \{L_1, \cdot \}$$

- **Type (2,0).** Then $d = 2$, $s = 6$, so $\epsilon' = 1$. The gamma matrices are Hermitian and satisfy $(\gamma^i)^2 = +1$. The Dirac operator is

$$D^{(2,0)} = \gamma^1 \otimes \{H_1, \cdot \} + \gamma^2 \otimes \{H_2, \cdot \}$$

**Example 2.4 (Fuzzy $d = 4$ geometries).** For realistic models the most important 4-fuzzy geometries have signatures (0,4) and (1,3) corresponding to the Riemannian and Lorentzian cases. We derive the first one in detail in order to arrive at the result in [Bar15, Ex. 10]. The rest follows from considering eq. (2.9).

- **Type (0,4), $s = 4$, Riemannian.** Notice that the gamma matrices are all anti-Hermitian and square to $-1$ in this case. Therefore, products of three
gamma matrices are self-adjoint: $(\gamma^a \gamma^b \gamma^c)^* = (-)^3 \gamma^c \gamma^b \gamma^a = \gamma^a \gamma^b \gamma^c$. The accompanying operators have the form $\{ H_{abc}, \cdot \}$ for $H_{abc}$ self-adjoint:

$$D^{(0,4)} = \sum_a \gamma^a \otimes [L_\dot{a}, \cdot] + \sum_{\dot{a}<\dot{b}<\dot{c}} \gamma^{a} \gamma^{b} \gamma^{c} \otimes \{ H_{abc}, \cdot \}$$  \hspace{1cm} (2.10)

- **Type (1,3),** $s = 2$, Lorentzian. Call $\gamma^0$ the only gamma matrix that squares to +1, and denote the rest by $\gamma^\dot{a}$, $\dot{a} = 1, 2, 3$. Then

$$D^{(1,3)} = \gamma^0 \otimes \{ H, \cdot \} + \gamma^\dot{c} \otimes [L_{\dot{c}}, \cdot] + \sum_{\dot{a} < \dot{b} < \dot{c}} \gamma^{0} \gamma^{\dot{a}} \gamma^{\dot{b}} \gamma^{\dot{c}} \otimes \{ L_{\dot{a}\dot{b}\dot{c}}, \cdot \} + \gamma^{1} \gamma^{2} \gamma^{3} \otimes \{ \tilde{H}, \cdot \}$$  \hspace{1cm} (2.11)

(For the Lorentzian signature, the dotted-index convention is redundant with the usual 0, 1, 2, 3 spacetime indices; then we henceforward drop it).

- **Type (4,0),** $s = -4 = 4 \mod 8$. The opposite case to ‘Riemannian’: now all gamma matrices are Hermitian, square to +1, and triple products $\Gamma^a$ are skew-Hermitian:

$$D^{(4,0)} = \sum_a \gamma^a \otimes \{ H_a, \cdot \} + \sum_{a < b < c} \gamma^{a} \gamma^{b} \gamma^{c} \otimes [L_{abc}, \cdot]$$  \hspace{1cm} (2.12)

- **Type (2,2),** $s = 0$, two times. Choosing the first two gamma matrices such that they square to +1, and $\gamma^{\dot{3}}$ and $\gamma^{\dot{4}}$ to −1, one gets

$$D^{(2,2)} = \sum_{a=1,2} \gamma^a \otimes \{ H_a, \cdot \} + \gamma^{a} \gamma^{\dot{3}} \gamma^{\dot{4}} \otimes [L_a, \cdot] + \sum_{\dot{c}=\dot{3},\dot{4}} \gamma^{\dot{c}} \otimes [L_{\dot{c}}, \cdot] + \gamma^{1} \gamma^{2} \gamma^{\dot{c}} \otimes \{ \tilde{H}_{\dot{c}}, \cdot \}$$  \hspace{1cm} (2.13)

Here we made the notation lighter, writing $\tilde{L}_a = L_\dot{a} = L_{1\ldots\dot{a}\ldots 4}$ for the $L$-matrix with all indices but $a$. Similarly, $\tilde{H}_\dot{c} = H_{1\ldots\dot{c}\ldots 4}$.

- **Type (3,1),** $s = 6$.

$$D^{(3,1)} = \sum_{a=1,2,3} \gamma^a \otimes \{ H_a, \cdot \} + \gamma^{1} \gamma^{2} \gamma^{3} \otimes \{ \tilde{L}, \cdot \} + \gamma^{\dot{4}} \otimes \{ L, \cdot \} + \sum_{a < c} \gamma^{a} \gamma^{\dot{c}} \gamma^{\dot{4}} \otimes \{ H_{a\dot{c}4}, \cdot \}$$  \hspace{1cm} (2.14)

Fuzzy geometries with odd $s$ allow elements of $\Omega^+$ also to parametrize Dirac operators,

$$D^{(\mu,\nu)} = \sum_\mu \gamma^\mu \otimes k^\mu + \sum_{\mu_1,\mu_2} \gamma^{\mu_1} \gamma^{\mu_2} \otimes k_{\mu_1\mu_2} + \sum_{\mu,\nu,\rho} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \otimes k_{\mu\nu\rho} + \ldots$$

$$+ \sum_{\mu,\nu,\rho} \Gamma^{\mu\nu\rho} \otimes k_{\mu\nu\rho} + \sum_{\mu,\nu} \Gamma^{\mu\nu} \otimes k_{\mu\nu} + \sum_{\mu} \Gamma^{\mu} \otimes k_{\mu} \cdot$$  \hspace{1cm} (2.15)

Examples of $D$ for a $d = 3$ geometry are given in [Bar15] and are not treated here.

### 2.5. Random fuzzy geometries.

Given a fermion space of fixed signature $(p, q)$, that is to say, a list $(\mathcal{A}, \mathcal{H}, \cdot, J, \Gamma)$ satisfying the listed properties in Definition 2.1 ignoring those concerning $D$, we consider the space $\mathcal{M} \equiv \mathcal{M}(\mathcal{A}, \mathcal{H}, J, \Gamma, p, q)$ of all possible Dirac operators $D$ that make of $(\mathcal{A}, \mathcal{H}, D, J, \Gamma)$ a real even spectral triple of signature $(p, q) \in \mathbb{Z}^2_{>0}$.
The symmetries of a spectral triple are encoded in $\text{Aut}(\mathcal{A})$, $\text{Inn}(\mathcal{A})$ and $\text{Out}(\mathcal{A})$, none of which implies the Dirac operator. This can be compared with the classical situation, in which fixing the data $(\mathcal{A}, \mathcal{H}, J, \Gamma)$ can be interpreted as imposing symmetries on the system and subsequently finding compatible geometries, encoded in $D \in \mathcal{M}$, typically via the extremization $\delta S(D_0) = 0$ of an action functional $S(D)$ that eventually selects a unique classical solution $D_0 \in \mathcal{M}$. The random noncommutative setting that appears in [BG16], on the other hand, considers ‘off-shell’ geometries. These can be stated as the following matrix integral

$$Z^{(p,q)} = \int_{\mathcal{M}} e^{-S(D)} dD, \quad S(D) = \text{Tr} f(D),$$  \hspace{1cm} (2.16)

being $f(x)$ an ordinary polynomial of real coefficients and no constant term. We next compute the spectral action $\text{Tr} f(D)$.

3. Computing the spectral action

In the spectral action (2.16) the trace is taken on the Hilbert space $\mathcal{H}$. We do not label it but, to avoid confusion, we label traces on other spaces: the trace $\text{Tr}_V$ is that of the spinor space $V$, the trace of operators on the matrix space $M_N(\mathbb{C})$ is denoted by $\text{Tr}_{M_N(\mathbb{C})}$, and $\text{Tr}_N$ stands for the trace on $\mathbb{C}^N$.

A homogeneous element spanning the Dirac operator $D = \sum I \omega_I \otimes k_I$ contains a first factor $\omega_I$, consisting of products of gamma matrices, and a second factor $k_I$ determined by a matrix that is either Hermitian or anti-Hermitian [Bar15]. We describe each factor and then give a general formula to compute the spectral action.

3.1. Traces of gamma matrices. We now rewrite the quantity

$$\langle \mu_1 \ldots \mu_{2n} \rangle := \frac{1}{\dim V} \text{Tr}_V (\gamma^{\mu_1} \ldots \gamma^{\mu_{2n}})$$  \hspace{1cm} (3.1)

in terms of chord diagrams of $2n$ points, to wit $n$ (disjoint) pairings among $2n$ cyclically ordered points. These are typically placed on a circle in whose interior the pairings are represented by chords that might cross. One finds

$$\langle \mu_1 \ldots \mu_{2n} \rangle = \sum_{2n\text{-pt chord diagrams } \chi} (-1)^{\# \text{crossings of chords in } \chi} \prod_{i,j=1}^{2n} g^{\mu_i \mu_j},$$  \hspace{1cm} (3.2)

where $\sim_\chi$ means that the point $i$ is joined with $j$ in the chord diagram $\chi$. We denote the total number of crossings of chords by $\text{cr}(\chi)$. We count only simple crossings; for instance, the sign of the ‘pizza-cut’ 8-point chord diagram with longest chords in the upper left corner of Figure 1 is $(-1)^6$.

For a mixed signature, $q, p > 0$, any non-vanishing $\langle \mu_1 \ldots \mu_{2n} \rangle$ has the form (up to a reordering sign) $\langle a_1 \ldots a_r \hat{c}_1 \ldots \hat{c}_u \rangle$ with $r + u = n$. Since $g^{ae}$ vanishes, any chord diagram $\chi$ in the sum of eq. (3.2) splits into a pair $(\sigma, \rho)$ of smaller

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5In contrast to noncommutative polynomials mentioned below.

6In a more involved context, these are called ‘chord diagrams with one backbone’ [ACPRS13].
chord diagrams, of 2r and 2u points, whose chords do not cross (see Fig. 2), so 
cr(\chi) = cr(\sigma) + cr(\rho). Therefore
\[ \langle a_1 \ldots a_{2r} c_1 \ldots c_{2u} \rangle = \sum_{2n \text{-pt chord diagrams } \chi} (-1)^{cr(\chi)} \prod_{i,j} g^{a_i a_j} \prod_{u,v} g^{\ell_u \ell_v} \]
\[ = \sum_{(2r,2u) \text{-pt chord diagrams } (\rho,\sigma)} (-1)^{cr(\sigma)} \prod_{i,j} g^{a_i a_j} (-1)^{cr(\rho)} \prod_{u,v} g^{\ell_u \ell_v} \]
\[ = \langle a_1 \ldots a_{2r} \rangle \langle c_1 \ldots c_{2u} \rangle. \tag{3.3} \]

For the metric \( g^{\mu\nu} = \text{diag}(+,\ldots,+,\ldots,-) \) the two factors are
\[ \langle a_1 \ldots a_{2r} \rangle = \sum_{2r \text{-pt chord diagrams } \rho} (-1)^{cr(\rho)} \prod_{i,j} g^{a_i a_j}, \tag{3.4a} \]
\[ \langle c_1 \ldots c_{2u} \rangle = (-1)^{u} \sum_{2u \text{-pt chord diagrams } \sigma} (-1)^{cr(\sigma)} \prod_{u,v} g^{\ell_u \ell_v}. \tag{3.4b} \]

It will be convenient to denote by \( CD_{2n} \) the set of 2n-point chord diagrams and to associate a tensor \( \chi^{\mu_1 \ldots \mu_{2n}} \) with \( \chi \in CD_{2n} \) and an index set \( \mu_1, \ldots, \mu_{2n} \in \Delta_4 \):
\[ \chi^{\mu_1 \ldots \mu_{2n}} = (-1)^{\# \text{crossings of chords in } \chi} \prod_{i,j=1}^{2n} g^{\mu_i \mu_j}. \tag{3.5} \]

This \( \chi \)-tensor is a version of the chord diagram \( \chi \) whose \( i \)-th point is decorated with the spacetime index \( \mu_i \); thus \( \chi^{\mu_1 \ldots \mu_{2n}} \) depends on the dimension, although it is not explicitly so denoted. All known identities for traces of gamma matrices can be stated in terms of these tensors, for instance \( \text{Tr}_{\mathbb{C}^4} (\gamma^{\mu_1 \nu_1 \gamma^{\mu_2} \nu_2 \gamma^{\mu_3} \nu_3 \gamma^{\mu_4} \nu_4}) = 4(g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} - g^{\mu_1 \nu_3} g^{\mu_2 \nu_4} + g^{\mu_1 \mu_4} g^{\mu_2 \nu_3}) \) in four dimensions: If \( \theta, \xi, \zeta \) denote the three 4-point chord diagrams, one can rewrite in terms of their corresponding tensors
\[ \theta^{\mu_1 \mu_2 \mu_3 \mu_4} = \mu_1 \mu_2 \mu_3 \mu_4 \quad \xi^{\mu_1 \mu_2 \mu_3 \mu_4} = \mu_1 \mu_2 \mu_3 \mu_4 \quad \zeta^{\mu_1 \mu_2 \mu_3 \mu_4} = \mu_1 \mu_2 \mu_3 \mu_4 \quad \tag{3.6} \]
the aforementioned trace identity as
\[ \text{Tr}_{\mathbb{C}^4} (\gamma^{\mu_1 \nu_1 \gamma^{\mu_2} \nu_2 \gamma^{\mu_3} \nu_3 \gamma^{\mu_4} \nu_4}) = \dim \mathbb{C} (\theta^{\mu_1 \mu_2 \mu_3 \mu_4} + \xi^{\mu_1 \mu_2 \mu_3 \mu_4} + \zeta^{\mu_1 \mu_2 \mu_3 \mu_4}) \].

For small \( n \), this seems to be a heavy notation, which however will pay off for higher values (the double factorial growth \#CD_{2n} = (2n - 1)!! notwithstanding).

3.2. Traces of random matrices. The aim of this subsection is to compute traces of words of the form \( \text{Tr}_{M_N(\mathbb{C})} (k_{I_1} \cdots k_{I_{2n}}) \) using the isomorphism \( M_N(\mathbb{C}) = \mathbb{N} \otimes \tilde{\mathbb{N}} \) (being \( \mathbb{N} \) the fundamental representation) at the level of the operators. By \cite{BG16},
\[ k_I = K_I \otimes 1_N + e_I \cdot (1_N \otimes K_I^T), \quad e_I = \pm. \]
The sign \( e_I \) is determined by Table 1.
Figure 1. The $7!! = 105$ chord diagrams with eight points. These assist to compute $\text{Tr}_V(\gamma^a \gamma^b \cdots \gamma^\theta \gamma^\lambda)$ in any dimension with diagonal metric of any signature. The sign of a diagram $\chi$ is $(-1)^{\#\text{simple crossings of } \chi}$. Thus, the 'pizza-cut' diagram in the upper left corner that appears as a summand in the normalized trace $\langle\alpha \beta \mu \nu \xi \zeta \theta \lambda\rangle$ evaluates to $(-1)^{1+2+3} g^{\xi \zeta} g^{\theta \mu} g^{\lambda \nu}$, where $\langle \cdots \rangle = (1/\text{dim } V) \text{Tr}_V(\cdots)$.
Figure 2. Splitting of a chord diagram for indices of a mixed
signature. One of the diagrams appearing in the computation of
$(a_1 \cdots a_6 \hat{a}_1 \cdots \hat{a}_8)$ 14 points, all of which split into two (of 8 and 6
points). The equality of diagrams means equality of the product of
the bilinears $g^{a_1 a_3}$ and $g^{\hat{a}_1 \hat{a}_3}$ determined by the depicted chords
and the signs for simple crossing.

Proposition 3.1. For any $r \in \mathbb{N}$

$$\text{Tr}_{MN}(C)(k_{I_1} \cdots k_{I_r}) = \sum_{\Upsilon \in \mathcal{P}_r} \text{sgn}(I_{\Upsilon}) \cdot \text{Tr}_N(K_{I_{\Upsilon}}) \cdot \text{Tr}_N((K^T)_{I_{\Upsilon}}), \quad (3.7)$$

where

- $\text{Tr}_N$ and $\text{Tr}_{MN}(C)$ are the traces on $\text{End}(\mathbb{N})$ and $\text{End}(MN(\mathbb{C}))$, respectively
- $\mathcal{P}_r$ is the power set $2^{\{1, \ldots, r\}}$ of $\{1, \ldots, r\}$, and $\Upsilon^c = \{1, \ldots, r\} \setminus \Upsilon$
- $\text{sgn}(I_{\Upsilon})$ is $(-1)^\# \{\text{commutators appearing in all the } k_i, \text{ with } j \in \Upsilon\},$ that is

$$\text{sgn}(I_{\Upsilon}) = \left( \prod_{i \in \Upsilon} e_i \right) \in \{-1, 1\}$$

- and, finally, the cyclic order $(\ldots \rightarrow r \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow \ldots)$ on the set $\{1, \ldots, r\},$ which can be read off from the trace in the LHS of eq. (3.7), induces a cyclic order on a given subset $\Xi = \{b_1, \ldots, b_\xi\} \in \mathcal{P}_r.$ Respecting
this order, define

$$\diamond \ K_{I_{\Xi}} = K_{I_{b_1}} K_{I_{b_2}} \cdots K_{I_{b_\xi}} \quad \text{and} \quad \diamond \ (K^T)_{I_{\Xi}} = K^T_{I_{b_1}} K^T_{I_{b_2}} \cdots K^T_{I_{b_\xi}} = (K_{I_{b_1}} \cdots K_{I_{b_2}} K_{I_{b_1}})^T.$$

Proof. By induction on the number $r - 1$ of products, we prove first that

$$k_{I_1} \cdots k_{I_r} = \sum_{\Upsilon \in \mathcal{P}_r} \prod_{i \in \Upsilon} \text{sgn}(I_{\Upsilon}) K_{I_{\Upsilon}} \otimes (K^T)_{I_{\Upsilon}}.$$

The statement holds for $r = 2,$ by direct computation; we now prove that the
statement being true for $r$ implies its veracity for $r + 1.$ In the first line of the
RHS of the next equation we use the assumption and then directly compute:

$$(k_{I_1} \cdots k_{I_r}) k_{I_{r+1}} = \prod_{w=1}^r \left[ K_{I_w} \otimes 1_N + e_{I_w} \cdot (1_N \otimes K^T)_{I_w} \right]$$

$$= \left( \sum_{\Upsilon \in \mathcal{P}_r} \left( \prod_{i \in \Upsilon} e_i \right) K_{I_{\Upsilon}} \otimes (K^T)_{I_{\Upsilon}} \right)$$

$$\cdot \left( K_{I_{r+1}} \otimes 1_N + e_{I_{r+1}} \cdot (1_N \otimes K^T)_{I_{r+1}} \right)$$
Dirac operator satisfy total of indices, 2
Proposition 3.2. Given a collection of multi-indices, we now proceed to compute.

\[ \text{Tr}_{M_N(\mathbb{C})}(k_{I_1} \cdots k_{I_{2n}}) = \sum_{\mathcal{T} \in \mathcal{P}_r} \text{sgn}(I_{tT}) \cdot \text{Tr}_{N \otimes \mathbb{N}}(K_{I_{tT}} \otimes (K^T)_{I_T}) \]

\[ = \sum_{\mathcal{T} \in \mathcal{P}_{r+1}} \left( \prod_{i \in \mathcal{Y}} e_{I_i} \right) e_{I_{r+1}} K_{I_{r+1}} \otimes (K^T)_{I_{r+1}} \]

\[ = \sum_{\Theta \in \mathcal{P}_{r+1}} \left( \prod_{i \in \Theta} e_{I_i} \right) \cdot K_{I_{\Theta e}} \otimes (K^T)_{I_{\Theta e}}. \]

To the last equality one arrives by considering that any set \( \Theta \in \mathcal{P}_{r+1} \) either contains \( r+1 \) (thus \( \Theta = \mathcal{Y} \cup \{r+1\} \) for some \( \mathcal{Y} \in \mathcal{P}_r \)) or does not \( (\Theta = \mathcal{Y} \in \mathcal{P}_r) \). These two sets are listed in the sum after the third equal sign (concretely, the second term and the first one, respectively). Then, it only remains to take traces

\[ \text{Tr}_{M_N(\mathbb{C})}(k_{I_1} \cdots k_{I_{2n}}) = \sum_{\mathcal{T} \in \mathcal{P}_r} \text{sgn}(I_{tT}) \cdot \text{Tr}_{N \otimes \mathbb{N}}(K_{I_{tT}} \otimes (K^T)_{I_T}) \]

\[ = \sum_{\Theta \in \mathcal{P}_{r+1}} \left( \prod_{i \in \Theta} e_{I_i} \right) \cdot K_{I_{\Theta e}} \otimes (K^T)_{I_{\Theta e}}. \]

3.3. The general structure of \( \text{Tr} D^m \). From the analysis of the gamma matrices one infers that for a polynomial \( f(x) = \sum f_i x^i \) the spectral action selects only the even coefficients \( \text{Tr} f(D) = \sum f_{2i} \text{Tr} (D^{2i}) \). In order to compute the spectral action of any matrix geometry we only need to know the traces of the even powers, which we now proceed to compute.

Proposition 3.2. Given a collection of multi-indices \( I_i \in \Lambda_d \), let \( 2n \) denote the total of indices, \( 2n = 2n(I_1, \ldots, I_{2n}) := |I_1| + \ldots + |I_{2n}| \). The even powers of the Dirac operator satisfy

\[ \frac{1}{\dim V} \text{Tr}(D^{2i}) = \sum_{I_1, \ldots, I_{2n} \in \Lambda_d} \left\{ \sum_{\chi \in \mathcal{C}_{2s}} \chi^{I_1 \cdots I_{2n}} \right\} \times \left[ \sum_{\mathcal{T} \in \mathcal{P}_{2i}} \text{sgn}(I_{tT}) \cdot \text{Tr}_{N}(K_{I_{tT}}) \cdot \text{Tr}_{N}((K^T)_{I_T}) \right]. \]  

in whose terms the spectral action \( S(D) = \text{Tr} f(D) = \sum f_{2i} \text{Tr} (D^{2i}) \) can be completely evaluated.

Proof. For \( t \in \mathbb{N} \),

\[ \frac{1}{\dim V} \text{Tr}(D^{2i}) = \frac{1}{\dim V} \text{Tr} \left( \left( \sum_{I \in \Lambda_d} \Gamma^I \otimes k_I \right)^{2i} \right) \]

\[ = \sum_{I_1, \ldots, I_{2n} \in \Lambda_d} \frac{1}{\dim V} \text{Tr}_V(\Gamma^{I_1} \cdots \Gamma^{I_{2n}}) \text{Tr}_N(k_{I_1} \cdots k_{I_{2n}}) \]

\[ = \sum_{I_1, \ldots, I_{2n} \in \Lambda} (I_1 \ldots I_{2n}) \text{Tr}_{M_N(\mathbb{C})}(k_{I_1} \cdots k_{I_{2n}}). \]

One uses eq. (3.2) and Proposition 3.1 with the notation of eq. (3.5). □

Notice that since the indices \( \mu_i \) of a multi-index \( I = (\mu_1 \ldots \mu_I) \in \Lambda_d \) are pairwise different, the traces of the gamma matrices greatly simplify. This also
ensures that there are no contractions between indices of the same $k$-operator, say $g^\mu\nu k_{\mu\nu...}$ ($k$'s with repeated indices do not exist).

In even dimension $d$, the Dirac operator is spanned by the number $\kappa(d)$ of independent odd products of gamma matrices. This equals $\kappa(d) = \#(\Lambda_d^-) = (d^1_0 + d^1_1 + ... + d^d_d)$ which can be rearranged (using Pascal’s identity) as $\kappa(d) = (d^{-1}_0) + (d^{-1}_1) + ... + (d^{-1}_{d-2}) + (d^{-1}_{d-1}) = 2^{d-1}$. The Dirac operator has then as many ‘matrix coefficients’ and is therefore parametrized by (what will turn out to be a subspace of) $M_N(\mathbb{C})^{\otimes \kappa(d)}$. In this manner, the ‘random spectral action’ (2.16) becomes a $\kappa(d)$-tuple matrix model.

**Definition 3.3.** Given integers $t, n \in \mathbb{N}$ (interpreted as in Prop. 3.2) and a chord diagram $\chi \in \text{CD}_{2n}$, its action (functional) $a_n(\chi)$ is a $\mathbb{C}$-valued functional on the matrix space $M_N(\mathbb{C})^{\otimes \kappa(d)}$ defined by

$$a_n(\chi)[K] = \sum_{I_1, ..., I_{2t} \in \Lambda_d^-} \sum_{2n = \sum_i |I_i|} \chi^{I_1 \ldots I_{2t}} \left[ \sum_{\gamma \in \mathbb{P}_{2t}} \text{sgn}(I_{\gamma}) \cdot \text{Tr}_N(K_{I_{\gamma^c}}) \cdot \text{Tr}_N((K^T)_{I_{\gamma}}) \right]$$

(3.10)

for $K = \{ K_{I_i} \in M_N(\mathbb{C}) \mid I_i \in \Lambda_d^- \} \in M_N(\mathbb{C})^{\otimes \kappa(d)}$. We often shall omit the dependence on the matrices and write only $a_n(\chi)$. We define the bi-trace functional as a sum over the non-trivial subsets $\Upsilon$ in eq. (3.10)

$$b_n(\chi)[K] = \sum_{I_1, ..., I_{2t} \in \Lambda_d^-} \sum_{2n = \sum_i |I_i|} \chi^{I_1 \ldots I_{2t}} \left[ \sum_{\gamma, \gamma' \in \mathbb{P}_{2t}, \gamma \neq \emptyset} \text{sgn}(I_{\gamma}) \cdot \text{Tr}_N(K_{I_{\gamma^c}}) \cdot \text{Tr}_N((K^T)_{I_{\gamma^c}}) \right]$$

(3.11)

and the single trace functional $s_n(\chi)$ via $a_n(\chi) = N \cdot s_n(\chi) + b_n(\chi)$. The factor $N$ ensures that $s_n$ does not depend on $N$ (cf. Sec. 3.4).

The restriction $2n = \sum_i |I_i|$ allows one to exchange the sums over the multi-indices $I$ and the chord diagrams in eq. (3.8). Then one can restate Proposition 3.2 as $(1/\dim V)\text{Tr}(D^{2t}) = NS_{2t} + B_{2t}$, where

$$S_{2t} = \sum_{n=t}^{t}(d-1) \sum_{\chi \in \text{CD}_{2n}} s_n(\chi),$$

(3.12)

$$B_{2t} = \sum_{n=t}^{t}(d-1) \sum_{\chi \in \text{CD}_{2n}} b_n(\chi).$$

(3.13)

The parameter $n$ lists all the numbers of points $2n = 2t, 2(t+1), \ldots, 2t \cdot (d-1)$ that chord diagrams contributing to $\text{Tr}(D^{2t})$ can have in dimension $d$.

In view of eqs. (3.12) and (3.13), all boils down to computing the single trace $s_n(\chi)$ and multiple trace part $b_n(\chi)$ of chord diagrams. We begin with the former.

3.4. **Single trace matrix model in the spectral action—manifest $O(N)$.** For geometries with even KO-dimension $s = q - p$, the spectral action’s manifest leading order in $N$ can be found from last proposition (see Sec. 6). Aiming at their large-$N$ limit we state the following
Corollary 3.4. Let \( d = q + p \) (thus \( s \)) be even. The spectral action (2.16) for the fuzzy \((p, q)\)-geometry with Dirac operator \( D = D^{(p,q)} \) satisfies, for any polynomial \( f(x) = \sum_{1 \leq r \leq m} f_r x^r \), the following:

\[
\frac{1}{\dim V} \text{Tr} f(D) = \frac{1}{\dim V} \sum_{1 \leq 2t \leq m} f_{2t} \text{Tr} \left( (D^{(p,q)})^{2t} \right) = N \sum_{1 \leq 2t \leq m} f_{2t} S_{2t} + B
\]

where \( 2n(I_1, \ldots, I_2) = \sum_i |I_i| \). Here \( B \) stands for products of two traces, whose coefficients are all independent of \( N \), with \( S_{2t} \) given by eq. (3.12) and for \( \chi \in \text{CD}_{2n} \),

\[
s_n(\chi) = \sum_{I_1, \ldots, I_{2t} \in \Lambda_d^+} \chi^{I_1, \ldots, I_{2t}} \left( \text{Tr}_N(K_{I_1}K_{I_2} \cdots K_{I_{2t}}) + (e_{I_1} \cdots e_{I_{2t}}) \text{Tr}_N(K_{I_{2t}}K_{I_{2t-1}} \cdots K_{I_1}) \right).
\]

Proof. For any given collection of multi-indices \( I_1, \ldots, I_{2t} \) one selects in Proposition 3.2 the two sets \( \Upsilon = \emptyset \) and \( \Upsilon = \{1, \ldots, 2t\} \) which correspond with the first and second summands between curly brackets. The overall \( N \)-factor corresponds to \( \text{Tr}_N(1_N) \). Clearly any other subset \( \Upsilon \) has no factor of \( N \) since is of the form

\[
\text{Tr}_N \left( \prod_{i \in \{1, \ldots, 2t\} \setminus \Upsilon} K_{I_i} \right) \times \text{Tr}_N \left( \prod_{i \in \Upsilon} K_{I_i} \right),
\]

where none of the products is empty. The arrows indicate the order in which the product is performed \((\rightarrow \) preserves it and \( \leftarrow \) inverts it, but this irrelevant to the point of this corollary). Therefore no trace of \( 1_N \) appears. One easily arrives then to eq. (3.14) by excluding from \( s_n(\chi) \) all the non-trivial sets, that is \( \Upsilon, \Upsilon^c \neq \emptyset \). \( \square \)

3.5. Formula for \( \text{Tr} D^2 \) in any dimension and signature. We evaluate in this section \( \text{Tr}(D^2) \) for Dirac operators \( D \) of a fuzzy geometry in any signature \((p, q)\).

Proposition 3.5. The Dirac operator of a fuzzy geometry of signature \((p, q)\) satisfies for odd \( d = p + q \)

\[
\frac{1}{\dim V} \text{Tr} \left( (D^{(p,q)})^2 \right) = 2 \sum_{I \in \Lambda_d^-} (-1)^{u(I) + \binom{|I|}{2}} \left[ N \cdot \text{Tr}_N(K_I^2) + e_I(\text{Tr}_N K_I)^2 \right],
\]

being \( u(I) \) the number of spatial indices in \( I \). If \( d \) is even, then the sum is only over \( I \in \Lambda_d^- \), and the expression reads

\[
\frac{1}{\dim V} \text{Tr} \left( (D^{(p,q)})^2 \right) = 2 \sum_{I \in \Lambda_d^-} (-1)^{u(I) + r(I) - 1} \left[ N \cdot \text{Tr}_N(K_I^2) + e_I(\text{Tr}_N K_I)^2 \right],
\]

with \( |I| = 2r(I) - 1 \).

Proof. In order to use eq. (3.9), notice that \( \langle I_1 I_2 \rangle \neq 0 \) implies that \( I_1, I_2 \in \Lambda_d \) have the same cardinality. If that were not the case (wlog \( |I_1| > |I_2| \)), since any non-zero term from \( \langle I_1 I_2 \rangle \) arises from a contraction of indices (cf. eq. (3.2)), a different number of indices would imply that there is a chord connecting two indices \( \mu_i, \mu_j \) of \( I_1 = (\mu_1, \ldots, \mu_r) \). Since \( I_1 \in \Lambda_d \), those indices are different, so \( g^{\mu_i \mu_j} = 0 \). Thus only chord diagrams for pairings \( g^{\mu \nu} \) of indices with \( \mu \in I_1 \) and \( \nu \in I_2 \) survive. Since the indices of \( I_1 \) and \( I_2 \) are strictly increasing, both ordered
sets have to agree. This means that we only have to care about evaluating \( \langle I \rangle \), with \( I' = (\mu_1', \ldots, \mu_w') \) being a copy of \( I = (\mu_1, \ldots, \mu_w) \), i.e. \( \mu_i' = \mu_i \). Since this last equality is the only possible index repetition

\[
\langle \mu_1, \ldots, \mu_w, \mu_1', \ldots, \mu_w' \rangle = \sum_{\text{2-upt chord diagrams } \chi} (-1)^{\chi} \prod_{i,j} g^{\mu_i \mu_j} \delta_{ij} = (-1)^{\chi(\pi)} \prod_{\mu=1}^w g^{\mu \mu}
\]

where \( \pi \) is the (pizza-cut) diagram with longest chords, that is joining antipodal points. The number of crossings \( \chi(\pi) \) is \( \binom{w}{2} \). An additional sign \( (-1)^w \) comes from \( \prod_{i=1}^w g^{\mu_i \mu_i} \), being \( u \leq q \) the number of spatial indices in \( I \), yielding

\[
\langle I_1 I_2 \rangle = \delta_{I_1}^I ( -1 )^u(I) + \binom{w}{2} .
\]

From eq. (3.9) with \( t = 1 \) one has

\[
\frac{1}{\dim V} \text{Tr} \left[ (D^{(p,q)})^2 \right] = \sum_{I_1, I_2 \in \Lambda} \langle I_1 I_2 \rangle \text{Tr}_{MN(N)}(K_{I_1} K_{I_2})
\]

\[
= \sum_{I \in \Lambda} (-1)^{u(I)} + \binom{|I|}{2} \text{Tr}_{N \otimes N} \left[ (K_I \otimes 1_N + e_I \otimes K_I^T)^2 \right]
\]

\[
= \sum_{I \in \Lambda} (-1)^{u(I)} + \binom{|I|}{2} \left[ \text{Tr}_N(K_I) \text{Tr}_N(1_N) + \text{Tr}_N(1_N) \text{Tr}_N(K_I^T)
\right.
\]

\[
+ 2 e_I \text{Tr}_N(K_I) \text{Tr}_N(K_I^T) \right].
\]

In the second equality we used eq. (3.15). The third one follows from Proposition 3.1. For \( p + q \) even, the sum runs only over \( I \in \Lambda^d \). In the sign appearing in eq. (3.15), \( \binom{|I|}{2} \) could then be replaced by \( r(I) - 1 \) with \( 2r(I) - 1 = |I| \), for \( \binom{r}{2} \equiv r - 1 \) (mod 2). \( \square \)

4. TWO-DIMENSIONAL FUZZY GEOMETRIES IN GENERAL SIGNATURE

We compute traces of \( D^2, D^4, D^6 \) for 2-dimensional fuzzy geometries general signatures. Concretely, for \( d = 2 \) the spinor space is \( V = \mathbb{C}^2 \).

4.1. Quadratic term. For a metric \( g = \text{diag}(e_1, e_2) \) notice that \( e_\mu = (-1)^{u(\mu)} \).

Therefore, by Proposition 3.5 one gets

\[
\frac{1}{4} \text{Tr} \left[ (D^{(p,q)})^2 \right] = \sum_{\mu} (-1)^{u(\mu)} \text{Tr}_N(K_{\mu}^2) + \sum_{\mu} \left[ \text{Tr}_N(K_{\mu}) \right]^2
\]

\[
= N \sum_{\mu, \nu} g^{\mu \nu} \text{Tr}_N(K_{\mu} K_{\nu}) + \sum_{\mu} \left[ \text{Tr}_N(K_{\mu}) \right]^2 ,
\]

(4.1)

where \( u(\mu) = 0 \) if \( \mu \) is time-like and if its spatial, \( u(\mu) = 1 \). Case by case,

\[
\begin{cases}
\sum_{a=1}^2 \left( - N \cdot \text{Tr}_N(L_a^2) + \left[ \text{Tr}_N(L_a) \right]^2 \right) & \text{for } (p, q) = (0, 2) \\
N \cdot \text{Tr}_N(H^2 - L^2) + \left[ \text{Tr}_N(H) \right]^2 + \left[ \text{Tr}_N(L) \right]^2 & \text{for } (p, q) = (1, 1) \\
\sum_{a=1}^2 \left( + N \cdot \text{Tr}_N(H_a^2) + \left[ \text{Tr}_N(H_a) \right]^2 \right) & \text{for } (p, q) = (2, 0)
\end{cases}
\]

reproducing some of the formulae reported in [BG16, App. A].
4.2. **Quartic term.** In *op. cit.* also the quartic term for $d = 2$ was computed. We recompute for a general $d = 2$ geometry of arbitrary signature aiming at illustrating the chord diagrams at work. Since $d = 2$, multi-indices $\mu \in \Lambda_2^-$ are just spacetime indices $\mu = 1, 2$. Hence, after Proposition 3.2,

$$\frac{1}{2} \text{Tr}(D^4) = \sum_{\mu_1, \ldots, \mu_4 \in \Lambda_2^-} \left( \begin{array}{c} \mu_1 \\
\mu_2 \\
\mu_3 \\
\mu_4 \\
\end{array} \right) \times \left\{ \begin{array}{l}
N \cdot \text{Tr}_N(K_{\mu_1}K_{\mu_2}K_{\mu_3}K_{\mu_4}) \\
+ \sum_{i=1}^4 e_{\mu_i} \text{Tr}_N(K_{\mu_1} \cdots \hat{K}_{\mu_i} \cdots K_{\mu_4}) \text{Tr}_N(K_{\mu_i}) \\
+ \sum_{1 \leq i < j \leq 4} e_{\mu_i} e_{\mu_j} \left[ \text{Tr}_N(K_{\mu_i}K_{\mu_j}) \text{Tr}_N(K_{\mu_i}K_{\mu_j}) \right] \\
+ \sum_{i=1}^4 e \cdot e_{\mu_i} \cdot \text{Tr}_N(K_{\mu_i}) \text{Tr}_N(K_{\mu_4} \cdots \hat{K}_{\mu_i} \cdots K_{\mu_1}) \\
+ e \cdot N \cdot \text{Tr}_N(K_{\mu_4}K_{\mu_3}K_{\mu_2}K_{\mu_1}) \end{array} \right\}$$

with $e = e_{\mu_1}e_{\mu_2}e_{\mu_3}e_{\mu_4}$ and $\{i, j, v, w\} = \Delta_4$. In the first line, the value of the chord diagrams is $g_{\mu_1\mu_2}g_{\mu_3\mu_4} - g_{\mu_1\mu_3}g_{\mu_2\mu_4} + g_{\mu_1\mu_4}g_{\mu_2\mu_3}$, the signs $e_{\mu_i} = \pm$ appearing in $g = \text{diag}(e_1, e_2)$ being determined by $(p, q)$. Summing over all indices, one gets

$$\frac{1}{4} \text{Tr}([D^{(p,q)}]^4) = N \left[ \text{Tr}_N(K_1^4) + \text{Tr}_N(K_2^4) \right]$$

$$+ 4e_1e_2 \text{Tr}_N(K_1^2K_2^2) - 2e_1e_2 \text{Tr}_N(K_1^2K_2K_2)$$

$$+ 4 \left\{ \text{Tr}_N(K_1K_2) \right\}^2 + \sum_{\mu=1,2} \text{Tr}_N K_{\mu} \cdot \text{Tr}_N \left[ K_{\nu} (e_1K_1^2 + e_2K_2^2) \right]$$

$$+ 3 \sum_{\mu=1,2} \left\{ \text{Tr}_N(K_{\mu}^3) \right\}^2 + 2e_1e_2 \text{Tr}_N(K_1^2) \cdot \text{Tr}_N(K_2^2).$$

One gets directly the results of [BG16, App. A.3, A.4, A.5] by setting

$$K_1 = \begin{cases} 
H_1 & \text{for } (p, q) = (2, 0) \text{ and } (1, 1), \\
L_1 & \text{for } (p, q) = (0, 2),
\end{cases}$$

and

$$K_2 = \begin{cases} 
H_2 & \text{for } (p, q) = (2, 0), \\
L_2 & \text{for } (p, q) = (1, 1) \text{ and } (0, 2).
\end{cases}$$

The conventions for which these hold are $(\gamma^\nu)^* = e_{\nu}\gamma^\nu$ (no sum, $\nu = 1, 2$).

4.3. **Sextic term.** We now compute the sixth-order term.
Proposition 4.1. Let \( g = \text{diag}(e_1, e_2) \) denote the quadratic form associated to the signature \((p, q)\) of a 2-dimensional fuzzy geometry with Dirac operator \(D\). Then
\[
\frac{1}{2} \text{Tr}(D^6) = N \cdot S_6[K_1, K_2] + B_6[K_1, K_2],
\]
where the single-trace part is given by
\[
S_6[K_1, K_2] = 2 \cdot \text{Tr}_N \left\{ e_1K_1^6 + 6e_2K_1^4K_2^2 - 6e_2K_1^2(K_1K_2)^2 + 3e_2(K_1^2K_2)^2 \right. \\
+ e_1K_2^6 + 6e_2K_2^4K_1^2 - 6e_1K_2^2(K_2K_1)^2 + 3e_1(K_2^2K_1)^2 \left. \right\}
\]
and the bi-trace part is
\[
B_6[K_1, K_2] = 6\text{Tr}_N(K_1) \left\{ 2\text{Tr}_N(K_1^5) + 2\text{Tr}_N(K_1K_2^4) \\
+ 6e_1e_2\text{Tr}_N(K_1^3K_2^2) - 2e_1e_2\text{Tr}_N(K_1^2K_2K_1K_2) \right\} \\
+ 6\text{Tr}_N(K_1^2) \cdot \left\{ e_1 \left[ 8\text{Tr}_N(K_1^2K_2^2) - 2\text{Tr}_N(K_2K_2K_1K_2) \right] \\
+ e_1 \left[ 5\text{Tr}_N(K_1^4) + \text{Tr}_N(K_1^2) \right] \right\} \\
+ 6\text{Tr}_N(K_1^2) \cdot \left\{ e_2 \left[ 8\text{Tr}_N(K_1^2K_2^2) - 2\text{Tr}_N(K_1K_2K_1K_2) \right] \\
+ e_2 \left[ 5\text{Tr}_N(K_2^4) + \text{Tr}_N(K_1^2) \right] \right\} \\
+ 4 \left( \frac{5}{2} \text{Tr}_N(K_1^3)^2 + 6e_1e_2\text{Tr}_N(K_1K_2^2)\text{Tr}_N(K_1^3) + 9\text{Tr}_N(K_1^2K_2)^2 \right) \\
+ 4 \left( \frac{5}{2} \text{Tr}_N(K_2^3)^2 + 6e_1e_2\text{Tr}_N(K_1^2K_2)^2\text{Tr}_N(K_2^3) + 9\text{Tr}_N(K_1K_2^2)^2 \right) .
\]

Proof. The part \( \langle \mu_1 \ldots \mu_6 \rangle \) concerning the chord diagrams evaluates to
\[
(-1)^0g^{\mu_1\mu_2}g^{\mu_3\mu_4}g^{\mu_5\mu_6} + (-1)^1g^{\mu_1\mu_2}g^{\mu_3\mu_5}g^{\mu_4\mu_6} + (-1)^0g^{\mu_1\mu_3}g^{\mu_2\mu_5}g^{\mu_4\mu_6}
+ (-1)^1g^{\mu_1\mu_3}g^{\mu_2\mu_4}g^{\mu_5\mu_6} + (-1)^1g^{\mu_1\mu_4}g^{\mu_2\mu_5}g^{\mu_3\mu_6} + (-1)^0g^{\mu_1\mu_4}g^{\mu_2\mu_6}g^{\mu_3\mu_5}
+ (-1)^0g^{\mu_1\mu_5}g^{\mu_2\mu_4}g^{\mu_3\mu_6} + (-1)^2g^{\mu_1\mu_5}g^{\mu_2\mu_6}g^{\mu_3\mu_5} + (-1)^1g^{\mu_1\mu_5}g^{\mu_2\mu_6}g^{\mu_4\mu_5}
+ (-1)^1g^{\mu_1\mu_5}g^{\mu_2\mu_4}g^{\mu_3\mu_6} + (-1)^2g^{\mu_1\mu_5}g^{\mu_2\mu_6}g^{\mu_3\mu_5} + (-1)^1g^{\mu_1\mu_5}g^{\mu_2\mu_6}g^{\mu_4\mu_5}
+ (-1)^0g^{\mu_1\mu_6}g^{\mu_2\mu_4}g^{\mu_3\mu_5} + (-1)^1g^{\mu_1\mu_6}g^{\mu_2\mu_4}g^{\mu_3\mu_5} + (-1)^0g^{\mu_1\mu_6}g^{\mu_2\mu_5}g^{\mu_3\mu_4}
\]
but it is actually useful to depict these terms as in Figure 3, for then, due to
the cyclicity of $\text{Tr}_N$, one can compute by classes (modulo $\pi \mathbb{Z}_6/3$-rotations) of diagrams. To each class, a Roman number is assigned:

One can relabel the $\mu_j$-indices to obtain

$$\frac{1}{\dim V} \text{Tr}(D^6) = \sum_{\chi \in CD_6} a(\chi) = 3a(I) + 6a(II) + 2a(III) + 3a(IV) + a(V),$$

the factors being the multiplicity of each diagram class. The single-trace part $S_6$ can be computed for each diagram directly (at the end of App. B one of these is shown). We simplified the notation: $a_3$ as $a$, and similarly we shall write $b_3$ for $b$, since therein only 6-point diagrams appear (a power $2t \geq 2$ of the Dirac operator determines the number of points of the chord diagrams only for dimensions $d \leq 2$).

We now compute the bi-trace term. Defining

$$O_{\mu \nu \rho} = e_{\nu} e_{\rho} \cdot \text{Tr}_N (K_\mu) \cdot \text{Tr}_N (K_\mu K_\rho K_\nu K_\rho K_\nu K_\nu),$$

$$P_{\mu \nu \rho} = e_{\nu} e_{\rho} \cdot \text{Tr}_N (K_\mu) \cdot \text{Tr}_N (K_\mu K_\rho K^2_\nu K_\nu),$$

$$Q_{\mu \nu \rho} = e_{\nu} e_{\rho} \cdot \text{Tr}_N (K_\mu) \cdot \text{Tr}_N (K_\mu K^2_\rho K^2_\nu),$$

$$R_{\mu \nu \rho} = e_{\rho} \cdot \text{Tr}_N (K_\mu K_\nu) \cdot \text{Tr}_N (K_\mu K_\nu K^2_\rho),$$

$$S_{\mu \nu \rho} = e_{\rho} \cdot \text{Tr}_N (K_\mu K_\nu) \cdot \text{Tr}_N (K_\mu K_\rho K_\nu K_\rho),$$

$$T_{\mu \nu \rho} = e_{\mu} e_{\nu} e_{\rho} \cdot \text{Tr}_N (K^2_\mu) \cdot \text{Tr}_N (K_\nu K_\rho K_\nu K_\rho).$$
\[ U_{\mu\nu\rho} = e_\mu e_\nu e_\rho \cdot \text{Tr}_N (K^2_{\mu}) \cdot \text{Tr}_N (K^2_{\nu} K^2_{\rho}) \] (4.8g)

\[ V_{\mu\nu\rho} = [\text{Tr}_N (K_\mu K_\nu K_\rho)]^2 \] (4.8h)

\[ W_{\mu\nu\rho} = e_\nu e_\rho \cdot \text{Tr}_N (K_{\mu} K^2_{\nu}) \cdot \text{Tr}_N (K_{\rho} K^2_{\nu}). \] (4.8i)

We can find by direct computation, that for any of the 6-point chord diagrams \( \chi \) there are integers \( p_\chi, q_\chi, \ldots, v_\chi, w_\chi \) such that

\[ b(\chi) = \sum_{\mu, \nu, \rho} o_\chi O_{\mu\nu\rho} + p_\chi P_{\mu\nu\rho} + q_\chi Q_{\mu\nu\rho} \]

\[ + r_\chi R_{\mu\nu\rho} + s_\chi S_{\mu\nu\rho} + t_\chi T_{\mu\nu\rho} + u_\chi U_{\mu\nu\rho} \] (4.9a)

\[ + v_\chi V_{\mu\nu\rho} + w_\chi W_{\mu\nu\rho} \] (4.9b)

The terms \( O, P, Q \) come from the \((1, 5)\) partition of 6, i.e. \( \text{Tr}_N (1 \text{ matrix}) \times \text{Tr}_N (5 \text{ matrices}) \); \( R, S, T, U \) terms come from the \((2, 4)\) partition and \( W, V \) from the \((3, 3)\) partition of 6. This claim is verified by direct computation; the proof for \( b(I) \) is presented in Appendix B and the rest is similarly obtained:

\[ b(I) = +2 \sum_{\mu, \nu, \rho} \left( 4O_{\mu\nu\rho} + 2P_{\mu\nu\rho} + 6R_{\mu\nu\rho} + 6S_{\mu\nu\rho} + 2T_{\mu\nu\rho} + U_{\mu\nu\rho} + 4V_{\mu\nu\rho} + 6W_{\mu\nu\rho} \right) \] (4.10a)

\[ b(II) = -2 \sum_{\mu, \nu, \rho} \left( 2O_{\mu\nu\rho} + 2P_{\mu\nu\rho} + 2Q_{\mu\nu\rho} + 8R_{\mu\nu\rho} + 4S_{\mu\nu\rho} + T_{\mu\nu\rho} + 2U_{\mu\nu\rho} + 4V_{\mu\nu\rho} + 6W_{\mu\nu\rho} \right) \] (4.10b)

\[ b(III) = +2 \sum_{\mu, \nu, \rho} \left( 6Q_{\mu\nu\rho} + 12R_{\mu\nu\rho} + 3U_{\mu\nu\rho} + 4V_{\mu\nu\rho} + 6W_{\mu\nu\rho} \right) \] (4.10c)

\[ b(IV) = +2 \sum_{\mu, \nu, \rho} \left( 2P_{\mu\nu\rho} + 4Q_{\mu\nu\rho} + 8R_{\mu\nu\rho} + 4S_{\mu\nu\rho} + 3U_{\mu\nu\rho} + 4V_{\mu\nu\rho} + 6W_{\mu\nu\rho} \right) \] (4.10d)

\[ b(V) = -2 \sum_{\mu, \nu, \rho} \left( 6O_{\mu\nu\rho} + 6R_{\mu\nu\rho} + 6S_{\mu\nu\rho} + 3T_{\mu\nu\rho} + 4V_{\mu\nu\rho} + 6W_{\mu\nu\rho} \right) \] (4.10e)

One then performs the sums explicitly and arrives to the claim for \( B = \sum_\chi b(\chi) \). □

5. Four-dimensional geometries in general signature

We compute now the spectral action for 4-dimensional fuzzy geometries.

5.1. The term \( \text{Tr} D^2 \). For any four-dimensional geometry \( p + q = 4 \) of signature \((p, q)\) there are eight matrices, \( K_1, K_2, K_3, K_4, X_1, X_2, X_3 \) and \( X_4 \in M_N(\mathbb{C}) \), parametrizing the Dirac operator

\[ D^{(p,q)} = \sum_{\mu=1}^{4} \gamma^\mu \otimes k_\mu + \Gamma^\hat{\mu} \otimes x_\mu. \] (5.1)
Here the lower case operators on \( M_N(\mathbb{C}) \) are related to said matrices by
\[
k_\mu = \{K_\mu, \cdot\}_{e_\mu} \quad \text{and} \quad x_\mu = \{X_\mu, \cdot\}_{e_\mu}
\]
where given a sign \( \varepsilon = \pm \), the braces \( \{A, B\}_\varepsilon = AB + \varepsilon BA \) represent a commutator or an anti-commutator. As before \( \Gamma^1 = \gamma^2\gamma^3\gamma^4, \Gamma^2 = \gamma^1\gamma^3\gamma^4 \), etc, but in favor of a lighter notation we have replaced \( K_\mu \) by \( X_\mu \). The metric here is \( g = \text{diag}(e_1, e_2, e_3, e_4) \) and the spinor space is \( V = \mathbb{C}^4 \).

The numbers \( u(\mu) \) and \( u(\hat{\nu}) \) of spatial subindices of each (multi-)index, \( \mu \) and \( \hat{\nu} \), can be written in terms of the signs \( e_\mu \) and \( e_{\hat{\nu}} \) that define the (anti-)hermiticity conditions—namely \( (\gamma^\mu)^* = e_\mu \gamma^\mu \) and \( (\gamma^\hat{\nu})^* = e_{\hat{\nu}} \gamma^\hat{\nu} \). First, trivially, \( e_\mu = (-1)^{u(\mu)} \).

On the other hand since \( u(\nu) + u(\hat{\nu}) \) is the total number \( q \) of spatial indices, one has, by Appendix A, \( e_{\hat{\nu}} = (-1)^{u(\nu)+[3/2]} = (-1)^{q+1+u(\nu)} = e_\nu(-1)^{q+1} \). Since the spinor space is four dimensional, by Proposition 3.5 one has
\[
\frac{1}{2 \cdot 4} \text{Tr} \left[ (D^{[\rho \sigma \nu]} \right)^2 \right] = \sum_{\mu=1}^{4} (-1)^{u(\mu)+[1/2]} \left[ N \cdot \text{Tr}_N (K_\mu^2) + e_\mu (\text{Tr}_N K_\mu)^2 \right] \\
+ \sum_{\nu=1}^{4} (-1)^{u(\nu)+[3/2]} \left[ N \cdot \text{Tr}_N (X_\nu^2) + e_{\hat{\nu}} (\text{Tr}_N X_\nu)^2 \right] \\
= \sum_{\mu=1}^{4} e_\mu N \cdot \text{Tr}_N \left[ K_\mu^2 + (-1)^{q+1} X_\mu^2 \right] + (\text{Tr}_N K_\mu)^2 + (\text{Tr}_N X_\mu)^2 .
\]

In Section 5.3 we specialize eq. (5.2) to fuzzy Riemannian and Lorentzian geometries. Before, it will be useful to obtain the quartic term in order to integrate it with the quadratic one.

5.2. The term \( \text{Tr} D^4 \). To access \( \text{Tr} (D^4) \) we now detect the non-vanishing chord diagrams.

5.2.1. Non-vanishing chord diagrams. In four dimensions chord diagrams of various number of points \( (2n = 4, 6, 8, 10, 12) \) have to be computed to access \( \text{Tr} (D^4) \). Next proposition helps to see the only non-trivial diagrams and requires some new notation. With each multi-index \( I_i \) running over eight values \( I_i = \mu, \hat{\nu} \) \((\mu, \nu = \Delta_4)\), the \( 8^4 \) decorations for the tensor \( \chi^{I_1I_2I_3I_4} \) fall into the following \( \tau \)-types:

\[
\chi^{I_1I_2I_3I_4} \in \left\{ \begin{array}{c}
\mu_1 \quad \mu_2 \quad \mu_3 \\
\mu_1 \quad \mu_2 \\
\mu_1 \quad \mu_2 \\
\mu_1 \quad \mu_2 \\
\nu_1 \quad \nu_2 \\
\nu_1 \quad \nu_2 \\
\nu_1 \quad \nu_2 \\
\nu_1 \quad \nu_2 \\
\end{array} \right\} 
\]

The leftmost diagram \( \chi \) is of generic type. On the other hand, not only do the diagrams in the list indicate the number of points (the total number of bars transversal to the circle), they also state how these are grouped: normal indices \( \mu_i = 1, \ldots, 4 \) being a single line and multiple \( \hat{\nu}_i \) a triple line. Although they are in
fact ordinary chord diagrams, they cannot have contractions between the grouped lines due to the strict increasing ordering of their indices.

If a diagram $\chi$ accepts a decoration of the type $\tau$ in the LHS of (5.3), up to rotation, we symbolically write $\chi \in \tau$. In the $\tau$-types of the RHS, however, $I_1$ corresponds strictly to the upper index of the respective diagram in the list, $I_2$ to the rightmost, and so on clockwise. One can sum over the $\tau_i$ classes — since we are interested in products of $\chi^1 \ldots I_k$ with traces (which are cyclic) and products of two traces (which are summed over all the subsets of $\{1, 2, 3, 4\}$, see eq. (3.10) for details) — and in order to do so, one has to include symmetry factors, namely $\{1, 4, 2, 4, 4, 4, 1\}$ in that order. All the chord diagrams contributing to $\text{Tr}(D^4)$ in $d = 4$ are then covered by

$$
\left( \sum_{\chi \in \tau_1} + 4 \sum_{\chi \in \tau_2} + 2 \sum_{\chi \in \tau_3} + 4 \sum_{\chi \in \tau_4} + 4 \sum_{\chi \in \tau_5} + \sum_{\chi \in \tau_6} \right) .
$$

A cross check is that the symmetry factors add up to 16 and, since each (multi)index in the list (5.3) can take four values, the number of all diagram index decorations is $4^4 \times 16 = 8^4$. Which of them survives is shown next:

**Proposition 5.1.** Let $g = \text{diag}(e_1, e_2, e_3, e_4)$ denote the quadratic form given by the signature $(p, q)$. For any $\mu, \nu, \mu_1, \mu_2, \mu_3, \mu_4, \nu_1, \nu_2, \nu_3, \nu_4 = 1, \ldots, 4$ the following holds for each one of the diagrams $\chi$ of the type $\tau_i$ —defined by eq. (5.3) — indicated to the right of each equation:

$$
\chi^{\mu_1 \mu_2 \mu_3 \mu_4} = e_{\mu_1} e_{\mu_2} \delta^{\mu_3}_{\mu_1} \delta^{\mu_4}_{\mu_2} - e_{\mu_1} e_{\mu_2} \delta^{\mu_3}_{\mu_2} \delta^{\mu_4}_{\mu_1} + e_{\mu_1} e_{\mu_3} \delta^{\mu_2}_{\mu_1} \delta^{\mu_4}_{\mu_3} ,
$$

$$
\chi^{\hat{\nu}_1 \hat{\nu}_2 \mu_3 \mu_4} = (-1)^{\sigma + 1} e_{\nu_1} e_{\nu_2} e_{\mu_3} \delta_{\nu_1 \mu_1 \mu_2} ,
$$

where $\sigma = \sigma(\mu, \nu) := (\alpha_1 \alpha_2 \alpha_3)_{\mu_1 \mu_2 \mu_3} \in \text{Sym}\{\alpha_1, \alpha_2, \alpha_3\}$, with $\hat{\nu} = (\alpha_1, \alpha_2, \alpha_3)$ ordered as $\alpha_1 < \alpha_2 < \alpha_3$. Also

$$
\delta_{\alpha \mu \nu \rho} = \begin{cases} 1 & \text{when } \{\alpha, \mu, \nu, \rho\} = \Delta_4 \\ 0 & \text{otherwise} \end{cases}
$$

(i.e. $\delta_{\alpha \mu \nu \rho}$ is the Levi-Civita symbol in absolute value). Whenever not all the four indices $\mu_1, \mu_2, \nu_1, \nu_2$ agree,

$$
\chi^{\mu_1 \nu_1 \mu_2 \nu_2} = -(-1)^{\mu_1 + \mu_2} e_1 e_2 e_3 e_4 (\delta^{\nu_1}_{\mu_1} \delta^{\nu_2}_{\mu_2} - \delta^{\nu_1}_{\mu_2} \delta^{\nu_2}_{\mu_1}) + e_\mu \left( \prod_{\alpha \neq \nu_1} e_\alpha \right) \delta^{\mu_2}_{\mu_1} \delta^{\nu_2}_{\nu_1} ,
$$

$$
\chi^{\mu_1 \mu_2 \nu_1 \nu_2} = +(-1)^{\mu_1 + \mu_2} e_1 e_2 e_3 e_4 (\delta^{\nu_1}_{\mu_1} \delta^{\nu_2}_{\mu_2} - \delta^{\nu_1}_{\mu_2} \delta^{\nu_2}_{\mu_1}) + e_\mu \left( \prod_{\alpha \neq \nu_1} e_\alpha \right) \delta^{\mu_2}_{\mu_1} \delta^{\nu_2}_{\nu_1} .
$$

(see below for the sign choice). Otherwise these two diagrams satisfy $\chi^{\mu \hat{\nu} \hat{\mu} \nu} = e_1 e_2 e_3 e_4$ and $\chi^{\mu \mu \hat{\nu} \hat{\mu}} = -e_1 e_2 e_3 e_4$. Moreover, letting $\sigma = \sigma(\nu, \mu) := (\vartheta_1, \vartheta_2, \vartheta_3)_{\nu_1 \nu_2 \nu_3} \in \text{Sym}\{\vartheta_1, \vartheta_2, \vartheta_3\}$, with $\hat{\nu} = (\vartheta_1, \vartheta_2, \vartheta_3)$ ordered as $\varsigma_1 < \vartheta_2 < \vartheta_3$, one has

$$
\chi^{\mu \hat{\nu} \hat{\nu} \nu} = (-1)^{\sigma + 1} e_{\nu_1} e_{\nu_2} \nu_{\nu_1 \nu_2 \nu_3} \delta_{\nu_1 \nu_2} ,
$$

and, finally, if $\chi \in \tau_6$

$$
\chi^{\nu_1 \nu_2 \nu_3 \nu_4} = \pm \left[ e_{\nu_1} e_{\nu_2} \delta^{\nu_3}_{\nu_1} \delta^{\nu_4}_{\nu_3} - e_{\nu_1} e_{\nu_2} \delta^{\nu_3}_{\nu_2} \delta^{\nu_4}_{\nu_3} + e_{\nu_1} e_{\nu_2} \delta^{\nu_3}_{\nu_3} \delta^{\nu_4}_{\nu_2} \right] ,
$$

The upper signs in equations (73), (74) and (76) are taken if $\chi$ has minimal crossings.
The minimality condition on the crossings, assumed for the $\tau_{3,4,6}$ classes, is meant to shorten the proof. Exactly for those classes, the spacetime indices do not necessarily determine a unique diagram by assuming that it does not vanish. This requirement can be left out, and in that case eq. (76) should have a global sign $\pm$ that depends on the diagram; in the $\tau_{3,4}$ cases (73) and (74) the term $e_{\mu_1} \left( \prod_{\alpha \neq \mu} e_{\alpha} \right) \delta_{\mu_1}^{\nu_1} \delta_{e_2}^{\nu_2}$ would undergo a diagram-dependent sign change. However, as we will see, these will be ‘effectively’ replaced by the minimal-crossing diagram, so the simplified claim suffices.

**Proof.** The proof is by direct, even if in cases lengthy, computation. One first seeks the conditions one has to impose on the indices for a diagram not to vanish, typically in terms of Kronecker deltas, and then one computes their coefficients in terms of the quadratic form $g = \text{diag}(e_1, e_2, e_3, e_4)$. We order the proof by similarity of the statements:

- **Type $\tau_1$.** The $\tau_1$-type diagram is well-known, for it is the only one here without any single multi-index (see the end of Sec. 3.1).

- **Type $\tau_6$.** Notice that at least two pairings of the $\nu_j$’s are needed for the diagram not to vanish: $\nu_1 = \nu_j =: \nu$ and $\nu_4 = \nu_r =: \mu$ with $\{i, j, t, r\} = \Delta_4$. Therefore, the Kronecker deltas are placed precisely as for the $\tau_1$ type. The computation of their $e$-factors is a matter of counting: for each chord joining two points labeled with, say, $\alpha$ there is an $e_{\alpha}$ factor. There are 6 such chords, labeled by $\{e_{\alpha}\}_{\alpha \neq \nu} \cup \{e_{\rho}\}_{\rho \neq \mu}$, for in $\hat{\nu}$ all the indices $\alpha \neq \nu$ appear and similarly for $\hat{\mu}$. Thus, if $\nu \neq \alpha \neq \mu$, $e_{\alpha}$ appears twice, so $e_{\alpha}^2 = 1$. The two remaining chord-labels are those appearing either in $\{e_{\alpha}\}_{\alpha \neq \nu}$ or in $\{e_{\rho}\}_{\rho \neq \mu}$. Thus the factor is $e_{\mu} e_{\nu}$ and we only have to compute the sign: the $\hat{\mu} \hat{\nu} \hat{\mu} \hat{\nu}$ configuration with minimal crossings has sign $(−1)^6$. For $\hat{\mu} \hat{\nu} \hat{\nu}$ and $\hat{\mu} \hat{\nu} \hat{\mu}$ the crossings yield a positive sign $(−1)^6$.

- **Type $\tau_2$.** Since $\hat{\nu} \in \Lambda_{d=4}$, all the three indices $\alpha_i$ in $\hat{\nu} = (\alpha_1, \alpha_2, \alpha_3)$ different. For this diagram not to vanish, the set equality $\{\alpha_1, \alpha_2, \alpha_3\} = \{\mu_1, \mu_2, \mu_3\}$ should hold, i.e. a permutation $\sigma \in \{\mu_1, \mu_2, \mu_3\}$ with $\alpha_{\sigma(i)} = \mu_i$ is needed. This says first, that $\nu$ cannot be any of $\mu_i$ (whence the $\delta_{\mu_12\mu_3}$) and second, that each of the three chords yields a factor $e_{\mu_i}$ with a sign $(-1)^{|\sigma|+1}$. The extra minus is due to the convention to place the indices, e.g. for 4123, the numbers 123123 are put cyclicly; this permutation $\sigma$ is the identity, which nevertheless looks like the ‘V diagram’ in eq. (4.7).

- **Type $\tau_5$.** Suppose that two indices of a non-vanishing diagram agree. Then either $\mu = \nu_i$ or (wlog) $\nu_1 = \nu_2$. In the first case notice that in $\mu$ and $\hat{\nu}_i$ the indices 1, 2, 3, 4 all appear listed. This implies for the remaining two multi-indices have to be of the form $\hat{\nu}_i = (***)$ and $\hat{\nu}_m = (1*4)$ or $\hat{\nu}_l = (1**) and $\hat{\nu}_m = (**4)$ where $\{i, l, m\} = \{1, 2, 3\}$ and $\ast \in \Delta_4$.

- **Type $\tau_7$.** Suppose that two indices of a non-vanishing diagram agree. Then either $\mu = \nu_i$ or (wlog) $\nu_1 = \nu_2$. In the first case notice that in $\mu$ and $\hat{\nu}_i$ the indices 1, 2, 3, 4 all appear listed. This implies for the remaining two multi-indices have to be of the form $\hat{\nu}_i = (***)$ and $\hat{\nu}_m = (1*4)$ or $\hat{\nu}_l = (1**) and $\hat{\nu}_m = (**4)$ where $\{i, l, m\} = \{1, 2, 3\}$ and $\ast \in \Delta_4$. In the first case, $\hat{\nu}_i = (***) and $\hat{\nu}_m = (1*4)$ the numbers $\rho, \rho, 2, 3$ (for some $\rho \in \Delta_4$) should fill the placeholders *. Then $\rho$ has to appear in both $\hat{\nu}_i$ and $\hat{\nu}_m$, but no value of $\rho$ fulfills this if the increasing ordering is to be preserved, hence we are only left with next case.
\[ \hat{n}_1 = (1 \ast \ast) \text{ and } \hat{n}_m = (\ast \ast 4), \text{ say } \hat{n}_1 = (1wx) \text{ and } \hat{n}_m = (yz4), \text{ then } \{ w, x \} \text{ and } \{ y, z \} \text{ are the sets } \{ 2, \rho \} \text{ and } \{ 3, \rho \} \text{ (not necessarily in this order), for some } \rho. \text{ Clearly, } \rho \text{ cannot be either 1 or 4 since each appears once in one multi-index. But } \rho = 2, 3 \text{ would force also a repetition of indices in at least one multi-index, which contradicts } \hat{n}_m, \hat{n}_1 \in \Lambda_{\Delta 4}^- . \]

This contradiction implies that if } \mu \text{ equals some } \nu \text{ then the diagram vanishes. By a similar analysis one sees that a repetition } \nu_1 = \nu_j \text{ implies also that the diagram is zero. Hence the diagram is a multiple of } \delta_{\mu_1 \nu_1 \nu_2 \nu_3}. \text{ Thereafter it is easy to compute the } e \text{-coefficients following a similar argument to the given for the type } \tau_3 \text{ diagram to arrive at the sign } (-1)^{\chi+1} \text{ for } \lambda \text{ a permutation of } \{ \nu_1, \nu_2, \nu_3 \}. \]

- Type } \tau_3. \text{ If no indices coincide then one gets two different numbers } i, j \in \Delta 4 \text{ appearing exactly once in the list } \mu_1, \nu_1, \nu_2, \nu_2. \text{ Since these cannot be matched by a chord, a non-zero diagram requires repetitions.}

- If } \mu_1 = \mu_2 \text{ then } \nu_1 = \nu_2. \text{ Since by hypothesis the four cannot agree the minimal crossings for this configuration is seen to be one, so the sign is } (-1). \text{ The } e \text{-factors are: } e_{\mu_1} \text{ for the chord between } \mu_1 \text{ and } \mu_2, \text{ the product of three } \prod_{\alpha \neq \nu_1} e_\alpha, \text{ for the three chords between } \hat{n}_1 \text{ and } \hat{n}_2. \text{ This account for } -e_{\mu_1} \left( \prod_{\alpha \neq \nu_1} e_\alpha \right) \delta_{\mu_1} \delta_{\nu_1}. \]

- If } \mu_1 = \nu_1, \text{ then again } \mu_2 = \nu_2 \text{ in order for the indices listed in } \mu_1, \hat{n}_1, \nu_2, \nu_2 \text{ to appear precisely twice. Since } \mu_1 = \nu_1 \text{ implies that } \mu_1 \text{ does not appear in } \hat{n}_1, \text{ there is one chord (thus a factor } e_\alpha) \text{ for each } \alpha \in \Delta 4. \text{ After straightforward (albeit neither brief nor very illuminating) computation one finds the sign } (-1)^{\mu_1 + \mu_2 + 1}. \text{ All in all, one gets } (-1)^{\mu_1 + \mu_2 + 1} e_1 e_2 e_3 e_4 \delta_{\mu_1} \delta_{\mu_2}. \]

- If } \mu_1 = \nu_2, \text{ then again } \mu_2 = \nu_1. \text{ But this is the same as the last point with } \nu_1 \leftrightarrow \nu_2. \text{ This account for } (-1)^{\mu_1 + \mu_2} e_1 e_2 e_3 e_4 \delta_{\mu_1} \delta_{\mu_2}. \]

- Type } \tau_4. \text{ Mutatis mutandis from the type } \tau_3. \]

\[ \textbf{Remark 5.2.} \text{ We just used the ‘minimal’ number of crossings for diagrams with a more than two-fold index repetition. For instance, for the four point diagram evaluated in } \chi^{1111} = 1 \text{ there might be one crossing or no crossings, but crucially two diagrams have no crossing so } \sum_{\chi} \chi^{1111} \times \text{traces} = (1 - 1 + 1) \times \text{traces}. \text{ This reappears in the computation of twelve-point diagrams in a nested fashion, as shown in Figure 4. If we pick } 2424 \text{ as configuration of the indices, then imposing } \chi^{2424} \neq 0 \text{ does not determine } \chi \in C\bar{D}_6: \text{ the lines joining the two 2-indices and the two 4-indices diagonally are mandatory, but for the four 1-indices and four 3-indices one can choose at the blobs tagged with } \phi, \psi \text{ one of three possibilities (shown in the diagrams of eq. (3.6) as } \theta, \xi, \zeta). \text{ Then there are 9 possible sign values. Again, it is essential that there are 5 positive and 4 negative global signs in } \chi^{2424}(\phi, \psi)_{\psi, \phi} \text{ and the sum } \sum_{\chi} \chi^{2424}(\text{traces}) \text{ can be replaced by the diagram with minimal crossings (of global positive sign).} \]

As a last piece of preparation, we need to determine the signs } e_{13} e_{13} e_{13} e_{13} \text{ for each } \tau_i \text{-type. These turn out to be constant and fully determined by the } \tau_i \text{-type:}
Claim 5.3. Assuming that for $I_1, I_2, I_3, I_4 \in \Lambda_{d=4}^-$ the tensor $\chi^{I_1 I_2 I_3 I_4}$ does not vanish, then $e_{\mu_1} e_{\mu_2} e_{\mu_3} e_{\mu_4}$ reads in each case

\begin{align*}
e_{\mu_1} e_{\mu_2} e_{\mu_3} e_{\mu_4} &\equiv +1 \quad (5.6) \\
e_{\nu} e_{\mu_1} e_{\mu_2} e_{\mu_3} &\equiv -1 \quad (5.7) \\
e_{\mu_1} e_{\mu_2} e_{\nu_1} e_{\nu_2} &\equiv +1 \quad (5.8) \\
e_{\nu_1} e_{\nu_2} e_{\nu_3} e_{\nu_4} &\equiv +1 \quad (5.10)
\end{align*}

Proof. To obtain these relations one needs Proposition 5.1. The first and last cases are obvious, since $\chi^{I_1 I_2 I_3 I_4} \neq 0$ requires in each case a repetition $e_{\mu_1}^2 e_{\nu_2}^2 = 1$ or $e_{\nu_1}^2 e_{\nu_3}^2 = 1$.

For the second, $e_{\nu} e_{\mu_1} e_{\mu_2} e_{\mu_3} = (-1)^{u(\nu)+[3/2]+\sum u(\mu)}$, by Appendix A. The non-vanishing of $\chi^{\mu_1 \mu_2 \mu_3 \mu_4}$ implies that $\nu$ is the multi-index containing $\mu_1, \mu_2, \mu_3$, so $u(\mu_1) + u(\mu_2) + u(\mu_3) = u(\nu)$ and eq. (5.7) follows.

For the third identity, if $\chi^{\mu_1 \mu_2 \nu_1 \nu_2}$ (and thus $\chi^{\mu_1 \nu_1 \mu_2 \nu_2}$) does not vanish, then it is either of the form $\chi^{\mu \nu \bar{\mu} \bar{\nu}}$, $\chi^{\mu \nu \bar{\rho} \bar{\nu}}$ or $\chi^{\mu \nu \bar{\mu} \bar{\rho}}$ ($\mu \neq \nu$). Only for the latter one needs a non-trivial check: $e_{\mu} e_{\nu} e_{\bar{\rho}} e_{\bar{\nu}} = e_{\mu} \cdot (-1)^{1+u(\Delta_1-\{\mu\})} e_{\nu} \cdot (-1)^{1+u(\Delta_1-\{\nu\})} = e_{\mu} e_{\nu} (-1)^{2u(\Delta_1-\{u,v\})} (-1)^{u(\mu)+u(\nu)}$, and since $e_{\mu} = (-1)^{u(\mu)}$, $e_{\nu} e_{\bar{\rho}} e_{\bar{\nu}} = 1$. In either case, eq. (5.8) follows.

We are left with the fourth identity. By assumption all the indices $\nu_j \neq \nu_i \neq \mu$ if $i \neq j$. Then by eq. (2.9)

\[
e_{\mu} e_{\nu_1} e_{\nu_2} e_{\nu_3} = e_{\mu} \cdot (-1)^{3 \times [3/2]+u(\Delta_1-\{\nu_1\})+u(\Delta_1-\{\nu_2\})+u(\Delta_1-\{\nu_3\})} = -e_{\mu} (-1)^{3u(\mu)} (-1)^{2u(\nu_1)} (-1)^{2u(\nu_2)} (-1)^{2u(\nu_3)} = -1
\]

From the first to the second line we used $\Delta_4 - \{\nu_1\} = \{\mu, \nu_2, \nu_3\}$, and similar relations. \qed
5.2.2. Main claim. With help of these two results, we state the main one. We recall that the definition of the permutation $\sigma(\nu, \mu)$, appearing next, is given in eq. (5.5).

**Proposition 5.4.** For a 4-dimensional fuzzy geometry of signature $(p, q)$, the purely quartic spectral action $\frac{1}{4} \text{Tr}(D^4) = NS_4 + B_4$ is given by

$$S_4 = \text{Tr}_N \left\{ 2 \sum_{\mu} K_\mu^4 + X_\mu^4 \right\}$$

and

$$B_4 = 8 \sum_{\mu, \nu} (-1)^{q+1} e_\nu \text{Tr}_N X_\mu \cdot \text{Tr}_N (X_\mu X_\nu^2) + e_\nu \text{Tr}_N (K_\mu) \cdot \text{Tr}_N (K_\mu K_\nu^2)$$

$$+ \sum_{\mu < \nu} 4 e_\mu e_\nu (2K_\mu^2 K_\nu^2 + 2X_\mu^2 X_\nu^2 - K_\mu K_\nu K_\mu K_\nu - X_\mu X_\nu X_\mu X_\nu)$$

$$- \sum_{\alpha, \beta, \gamma, \mu, \nu} \delta_{\alpha \beta \mu \nu} e_\alpha e_\beta (K_\mu X_\nu^2 + 2K_\mu^2 X_\nu^2) + 2(-1)^{q} \sum_{\mu} (K_\mu X_\mu^2 - 2K_\mu^2 X_\mu^2)$$

$$+ 8(-1)^{q+1} \sum_{\mu, \nu} (-1)^{\sigma(\nu, \mu)} \delta_{\mu \nu 1 \nu 2 \nu 3} e_\mu (X_\mu K_{\nu 1} K_{\nu 2} K_{\nu 3} + K_\mu X_{\nu 1} X_{\nu 2} X_{\nu 3}) \right\},$$

and

$$+ 4 \sum_{\mu, \nu=1} 2 \text{Tr}_N (X_\mu^2) \cdot \text{Tr}_N (X_\nu^2) + 4 e_\mu e_\nu \left[ \text{Tr}_N (X_\mu X_\nu) \right]^2 \right\}$$

$$+ \sum_{\mu, \nu=1} 2 \text{Tr}_N (K_\mu^2) \cdot \text{Tr}_N (K_\nu^2) + 4 e_\mu e_\nu \left[ \text{Tr}_N (K_\mu K_\nu) \right]^2 \right\}$$

$$+ \sum_{\mu=1} 2(-1)^{1+q} e_\mu \text{Tr}_N (K_\mu) \cdot \text{Tr}_N (K_\mu X_\mu^2) + 2 e_\mu \text{Tr}_N (X_\mu) \cdot \text{Tr}_N (X_\mu K_\mu^2)$$

$$+ (-1)^{1+q} \text{Tr}_N (X_\mu^2) \cdot \text{Tr}_N (K_\mu^2) + 2 \left[ \text{Tr}_N (K_\mu X_\mu^2) \right]^2 \right\}$$

$$- 8 \sum_{\nu, \mu=1} (-1)^{\sigma(\nu, \mu)} \delta_{\mu \nu 1 \nu 2 \nu 3} \cdot \left\{ - \text{Tr}_N (X_\nu) \cdot \text{Tr}_N (K_{\mu 1} K_{\mu 2} K_{\mu 3})$$

$$+ e_{\mu 2} e_{\mu 3} \left( \text{Tr}_N K_{\mu 1} \cdot \text{Tr}_N (X_\nu K_{\mu 2} K_{\mu 3}) \right)$$

$$+ \text{Tr}_N X_{\mu 1} \cdot \text{Tr}_N (K_{\nu 2} X_{\mu 2} X_{\mu 3}) + (-1)^{q} \text{Tr}_N K_{\nu} \cdot \text{Tr}_N (X_{\mu 1} X_{\mu 2} X_{\mu 3}) \right\}$$

$$+ 24 \sum_{\mu \neq \nu=1} (-1)^{1+q} e_\nu \text{Tr}_N (K_\mu) \cdot \text{Tr}_N (K_\nu X_\mu^2) + e_\mu \text{Tr}_N (X_\nu) \cdot \text{Tr}_N (K_\mu^2 X_\nu)$$

$$+ 12 \sum_{\mu \neq \nu} \left\{ 2 \left[ \text{Tr}_N (K_\mu X_\nu^2) \right]^2 + e_\mu e_\nu (-1)^{q+1} \text{Tr}_N (K_\mu^2) \cdot \text{Tr}_N (X_\nu^2) \right\}.$$  

The eight matrices $K_\mu, X_\mu$ satisfy the following (anti-)hermiticity conditions:

$$K_\mu^* = e_\mu K_\mu \quad \text{and} \quad X_\mu^* = e_\mu (-1)^{q+1} X_\mu \quad \text{for any} \quad \mu \in \Delta_4,$$

where each $e_\mu \in \{+1, -1\}$ is determined by $g = \text{diag}(e_1, e_2, e_3, e_4).$
Proof. We first find $S_4 = \sum_{n=2}^{6} s_n(\chi)$ using eqs. (5.3) and (5.4). By direct computation
\[
\sum_{\chi \in CD_2} s_2(\chi) = 2 \sum_{\mu} \text{Tr}_N(K^4_\mu) + 8 \sum_{\mu < \nu} e_\mu e_\nu \text{Tr}_N(K^2_\mu K^2_\nu) - 4 \sum_{\mu < \nu} e_\mu e_\nu \text{Tr}_N(K_\mu K_\nu K_\mu K_\nu).
\] (5.14)

In view of (5.6) and (5.10) and the similarity of the type diagrams, one gets the same result by replacing $K_\mu$ by $K_\mu = X_\mu$, namely
\[
\sum_{\chi \in CD_6} s_6(\chi) = 2 \sum_{\mu} \text{Tr}_N(X^4_\mu) + 8 \sum_{\mu < \nu} e_\mu e_\nu \text{Tr}_N(X^2_\mu X^2_\nu) - 4 \sum_{\mu < \nu} e_\mu e_\nu \text{Tr}_N(X_\mu X_\nu X_\mu X_\nu).
\] (5.15)

Next, using eq. (5.7), the 6-point diagrams are evaluated:
\[
\sum_{\chi \in CD_3} s_3(\chi) = 4 \sum_{\nu, \mu} \text{Tr}_N \left[ \delta_{\nu_1 \mu_1 \mu_2} e_\nu e_\mu e_\nu (-1)^{1+|\sigma_\mu|} \right. \\
\left. \cdot (X_\nu K_{\mu_1} K_{\mu_2} K_{\mu_3} - K_{\mu_1} K_{\mu_2} K_{\mu_3} X_\nu) \right] \\
= -8 \sum_{\nu, \mu} \text{Tr}_N \left[ \delta_{\nu_1 \mu_1 \mu_2 \mu_3} e_\nu e_\mu e_\nu (-1)^{|\sigma_\mu|} X_\nu K_{\mu_1} K_{\mu_2} K_{\mu_3} \right] \\
= 8 (-1)^{1+q} \sum_{\nu, \mu} \text{Tr}_N \left[ \delta_{\nu_1 \mu_1 \mu_2 \mu_3} e_\nu e_\mu e_\nu (-1)^{|\sigma_\mu|} X_\nu K_{\mu_1} K_{\mu_2} K_{\mu_3} \right].
\] (5.16)

Here, again using the duality between $\tau_2$ and $\tau_5$ evident in Proposition 5.1 and Claim 5.3, the $s_5$ term can be computed by swapping each $K_\mu$ matrix with the $K_\mu$ matrix:
\[
\sum_{\chi \in CD_5} s_5(\chi) = \sum_{\chi \in CD_3} s_3(\chi) \bigg|_{K_\nu \leftrightarrow X_\nu \text{ for all } \nu = 1, 2, 3, 4} \tag{5.17}
\]

(but there is no the sign swap $e_\mu \leftrightarrow e_\bar{\mu}$).

Finally, we split the sum $\sum_{\chi \in CD_4} = 2 \sum_{\chi \in \tau_3} + 4 \sum_{\chi \in \tau_4}$ in order to compute the term $s_4$. The calculation simplifies using eq. (5.8) and noticing that (for $\mu_1 = \mu_2 = \nu_1 = \nu_2$ being false), one has
\[
\sum_{\nu, \mu} (-1)^{\mu_1 + \mu_2} \left[ \delta_{\mu_1 \mu_2} \delta_{\nu_1 \nu_2} - \delta_{\mu_1 \nu_1} \delta_{\nu_2 \nu_2} \right] \text{Tr}_N \left( K_{\mu_1} X_{\nu_1} K_{\mu_2} X_{\nu_2} \right) \\
+ e_{\mu_1} e_{\nu_1} e_{\nu_2} e_{\nu_2} \text{Tr}_N \left( X_{\nu_2} K_{\mu_2} X_{\nu_1} K_{\mu_1} \right) \\
= \sum_{\mu \neq \nu} (-1)^{\mu + \nu} \text{Tr}_N \left\{ K_{\mu} X_{\mu} K_{\nu} X_{\nu} + X_{\mu} K_{\nu} X_{\mu} K_{\nu} \right. \\
\left. - K_{\mu} X_{\nu} K_{\nu} X_{\mu} - X_{\mu} K_{\nu} X_{\mu} K_{\nu} \right\} = 0,
\] (5.18)

using the cyclicity of the trace. Therefore both equations $(\tau_3)$ and $(\tau_4)$ the only contribution to $s_4$ comes from the term $e_{\mu_1} \left( \prod_{\alpha \neq \nu} e_{\alpha} \right) \delta_{\mu_1 \nu_1} \delta_{\nu_2 \nu_2}$ (which require $\mu_1 \neq \nu_2$) and from the terms $\chi^{\mu_1 \mu_2 \nu}$ and $\chi^{\mu_2 \nu \mu}$. These terms appear, respectively, in the
first and second lines of
\[
\sum_{\chi \in \mathcal{CD}_4} \mathfrak{s}_2(\chi) = - \sum_{\alpha, \beta, \mu, \nu} \delta_{\alpha \beta \mu \nu} \epsilon_\alpha \epsilon_\beta \Tr_N \left[ (K_\mu X_\nu)^2 + 2K_\mu^2 X_\nu^2 \right] + 2\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 \sum_{\mu} \Tr_N \left[ (K_\mu X_\mu)^2 - 2K_\mu^2 X_\mu^2 \right]
\] (5.19)

Expressing this via the delta \( \delta_{\alpha \beta \mu \nu} \) is motivated by \( \epsilon_\mu \left( \prod_{\rho \neq \nu} \epsilon_\rho \right) = \prod_{\rho \neq \nu} \epsilon_\rho \).

We now compute in steps the bi-tracial functional
\[
\mathcal{B}_4 = \sum_{I \in (\Lambda_4^+) \times 4} \left\{ \sum_{\chi \in \mathcal{CD}_{4n}(I)} \chi^{I_1 I_2 I_3 I_4} \times \left[ \sum_{i=1}^{4} \epsilon_{I_i} \Tr_N (K_{I_i} \cdots \overline{K}_{I_i} \cdots K_{I_i}) \cdot \Tr_N K_{I_i} \right. \right.
\]
\[
+ \sum_{1 \leq i < j \leq 4} \sum_{v, w \neq i, j} e_{I_i} e_{I_j} \left( \Tr_N (K_{I_i} K_{I_w}) \Tr_N (K_{I_i} K_{I_i}) \right) \right. \]
\[
+ \sum_{i=1}^{4} \left( \prod_{j \neq i} \epsilon_{I_j} \right) \Tr_N (K_{I_i}) \cdot \Tr_N (K_{I_i} \cdots \overline{K}_{I_i} \cdots K_{I_i}) \right\}.
\] (5.20)

The contribution to \( \mathcal{B}_4 \) arising from the term in the square brackets in the first, second and third lines are referred to as the (1,3), (2,2) and (3,1) partitions, respectively. For a fixed number 2r of points, these are denoted by \( \sum_{\chi} b_r^\pi(\chi) \), for \( \pi \in \{ (1,3), (2,2), (3,1) \} \). In view of the partial duality established in Proposition 5.1, we obtain the contributions to \( \mathcal{B}_4 \) by similarity; thus we first compute 12-pt and 4-pt diagrams together and later 6-pt and 10-pt diagrams. This duality would be perfect if both replacements \( K_\mu \leftrightarrow K_\nu = X_\nu \) and \( e_\nu \leftrightarrow e_\rho \) would swap the eqs. \( (\tau_1 \leftrightarrow \tau_6) \) and \( (\tau_2 \leftrightarrow \tau_3) \) in Proposition 5.1. However, \( e_\nu \leftrightarrow e_\rho \) is not needed for the swapping to hold.

We begin with the 12-point diagrams for the (1,3) and (3,1) partitions. As consequence of Claim 5.3,
\[
\sum_{\chi \in \tau_6} b_6^{(1,3)}(\chi) + b_6^{(3,1)}(\chi) = \sum_{\nu_1, \ldots, \nu_4=1}^{4} \sum_{\chi \in \tau_6} \chi^{\nu_1 \nu_2 \nu_3 \nu_4} \left[ e_{\mu_1} \Tr_N X_{\nu_1} \cdot \Tr_N (X_{\nu_2} \{X_{\nu_3}, X_{\nu_4}\}) \right] + \text{cyclic}
\]
\[
= 4 \sum_{\mu, \nu} \epsilon_\mu \epsilon_\nu \epsilon_\mu \epsilon_\nu \Tr_N X_\mu \cdot \Tr_N (X_\nu \{X_\mu, X_\nu\})
\]
\[
= 8 \sum_{\mu, \nu} (-1)^{q+1} e_\mu \Tr_N X_\mu \cdot \Tr_N (X_\nu X_\nu^2)
\] (5.21)

after some simplification; the last equality follows from eq. (A.2). The (2,2)-partition evaluates similarly to
\[
\sum_{\chi \in \tau_6} b_6^{(2,2)}(\chi) = 2 \sum_{\mu, \nu} \left\{ \epsilon_\mu \epsilon_\nu \epsilon_\mu \epsilon_\nu \Tr (X_\mu^2) \cdot \Tr (X_\nu^2) + 2\epsilon_\mu \epsilon_\nu \epsilon_\mu \epsilon_\nu \Tr (X_\mu X_\nu)^2 \right\}
\]
\[
= 4 \sum_{\mu, \nu=1}^{4} \left\{ 2\Tr_N (X_\mu^2) \cdot \Tr_N (X_\nu^2) + 4\epsilon_\mu \epsilon_\nu \left[ \Tr_N (X_\mu X_\nu)^2 \right] \right\},
\] (5.22)
since \( \epsilon_\mu \epsilon_\nu \epsilon_\mu \epsilon_\nu = (-1)^{2(1+q)} \) by eq. (A.2). One then computes \( \sum_{\chi \in \tau_6} b_6(\chi) \) by summing eqs. (5.22) and (5.21).

The four-point diagrams contain ordinary indices and their computation is not illuminating. Since it moreover resembles that for the 12-point diagrams we omit.
it and present the result:

\[
\sum_{\chi \in \tau_1} b_2(\chi) = \sum_{\mu,\nu=1}^{4} 8e_\nu \text{Tr}_N(K_\mu) \cdot \text{Tr}_N(K_\mu K_\nu^2) \\
+ \sum_{\mu,\nu=1}^{4} \left\{ 2\text{Tr}_N(K_\mu^2) \cdot \text{Tr}_N(K_\nu^2) + 4e_\mu e_\nu \left[ \text{Tr}_N(K_\mu K_\nu) \right]^2 \right\}.
\]

We now present the computation of 6-point and 10-point diagrams. Again, we remark that the terms corresponding to the (1,3) and (3,1) partitions agree, \( \sum_\chi b_3^{(1,3)}(\chi) = \sum_\chi b_3^{(3,1)}(\chi) \). In order to see this, first we notice that \( \sum_\chi b_3^{(1,3)}(\chi) \) equals

\[
4 \sum_{\nu,\mu}(\chi) \delta_{\nu \mu_1 \mu_2 \mu_3} \cdot \left\{ -\text{Tr}_N X_\nu \cdot \text{Tr}_N(K_{\mu_1} K_{\mu_2} K_{\mu_3}) \\
+ e_{\mu_2} e_{\mu_3} \text{Tr}_N(K_{\mu_1} \cdot \text{Tr}_N(X_\nu K_{\mu_2} K_{\mu_3}) \\
+ e_{\mu_1} e_{\mu_3} \text{Tr}_N(K_{\mu_2} \cdot \text{Tr}_N(X_\nu K_{\mu_1} K_{\mu_3}) \\
+ e_{\mu_1} e_{\mu_2} \text{Tr}_N(K_{\mu_3} \cdot \text{Tr}_N(X_\nu K_{\mu_1} K_{\mu_2})) \right\},
\]

due to eq. (72) and \( e_\nu e_{\mu_1} e_{\mu_2} e_{\mu_3} = -1 \) (Claim 5.3). But also departing from \( \sum_\chi b_3^{(3,1)}(\chi) \), using (5.7) to convert the triple signs to a single one, (e.g. \( e_\nu e_{\mu_1} e_{\mu_3} = -e_{\mu_2} \)), renaming indices (which gets rid of the minus sign via the skew-symmetric factor \( (1)^{1+|\sigma|} \)) one arrives again to the same expression. Thus \( \sum_\chi b_3^{(1,3)}(\chi) + \sum_\chi b_3^{(3,1)}(\chi) \) equals

\[
8 \sum_{\nu,\mu}(\chi) \left\{ -\text{Tr}_N X_\nu \cdot \text{Tr}_N(K_{\mu_1} K_{\mu_2} K_{\mu_3}) + e_{\mu_2} e_{\mu_3} \text{Tr}_N(K_{\mu_1} \cdot \text{Tr}_N(X_\nu K_{\mu_2} K_{\mu_3})) \right\}.
\]

Using the skew-symmetry of \( (1)^{1+|\sigma|} \) and the cyclicity of the trace, one proves easily that the \((2,2)\)-partition \( b_3^{(2,2)} \) vanishes, and so does in fact \( b_3^{(2,2)} \). Thus the only contributions from 10-point diagrams are the partitions \((1,3)\) and \((3,1)\) which can be computed similarly as for the 6-point contributions, by a similar token. Thus

\[
\sum_{\chi \in \text{CD}_{10}} b_5(\chi) = 2 \sum_{\chi \in \tau_5} b_5^{(1,3)}(\chi) = -8 \sum_{\mu,\nu_1,\nu_2,\nu_3} \delta_{\nu_1 \nu_2 \nu_3} (\chi) (-1)^{|\lambda_{\nu}|} e_{\nu_1} e_{\nu_2} e_{\nu_3} \\
\times \left[ e_{\mu} \text{Tr}_N K_\mu \cdot \text{Tr}_N(X_{\nu_1} X_{\nu_2} X_{\nu_3}) + e_{\nu_1} \text{Tr}_N X_{\nu_1} \cdot \text{Tr}_N(K_\mu X_{\nu_2} X_{\nu_3}) \\
+ e_{\nu_2} \text{Tr}_N X_{\nu_2} \cdot \text{Tr}_N(K_\mu X_{\nu_1} X_{\nu_3}) + e_{\nu_3} \text{Tr}_N X_{\nu_3} \cdot \text{Tr}_N(K_\mu X_{\nu_1} X_{\nu_2}) \right]
\]

By performing the sum of the terms in the last line one sees that they cancel out due to the skew-symmetry of \( (1)^{|\lambda_{\nu}|} \). The only contribution come therefore from the two first terms in the square brackets, which are directly seen to yield

\[
\sum_{\chi \in \text{CD}_{10}} b_5(\chi) = -8 \sum_{\mu,\nu_1,\nu_2,\nu_3} \delta_{\nu_1 \nu_2 \nu_3} (\chi) (-1)^{|\lambda_{\nu}|} \times \left[ (-1)^{\mu} \text{Tr}_N K_\mu \cdot \text{Tr}_N(X_{\nu_1} X_{\nu_2} X_{\nu_3}) \\
+ e_{\nu_2} e_{\nu_3} \text{Tr}_N X_{\nu_1} \cdot \text{Tr}_N(K_\mu X_{\nu_2} X_{\nu_3}) \right].
\]
Concerning the 8-point diagrams,

\[
\sum_{\chi \in \text{CD}_8} b_8(\chi) = \sum_{\mu_1, \mu_2, \nu_1, \nu_2 \text{ not all equal}} \left( 2\chi^{\mu_1 \hat{\nu}_1 \mu_2 \hat{\nu}_2} + 4\chi^{\mu_1 \mu_2 \hat{\nu}_1 \hat{\nu}_2} \right) \{\text{non-trivial partitions}\}
\]

\[
+ \sum_{\mu} (2\chi^{\mu \mu \mu \mu} + 4\chi^{\mu \mu \mu \mu}) \{\text{non-trivial partitions}\}. \tag{5.24}
\]

The sum over the 8-point chord diagrams is spitted in the \(\tau_3\) and \(\tau_4\) types with their symmetry factors; in each line these are, respectively, the two summands in parenthesis. Here ‘non-trivial partitions’ in curly brackets refers to (1,3), (2,2) and (3,1). We call the second line \(\Delta\), for which straightforward computation yields

\[
\Delta = 4(-1)^{1+q} \sum_{\mu=1}^4 \left\{ 2e_\mu \text{Tr}_N (K_\mu) \cdot \text{Tr}_N (K_\mu X_\mu^2) + 2e_\mu \text{Tr}_N (X_\mu) \cdot \text{Tr}_N (X_\mu K_\mu^2)
\]

\[
+ \text{Tr}_N (X_\mu^2) \cdot \text{Tr}_N (K_\mu^2) + 2(-1)^{1+q} \left[ \text{Tr}_N (K_\mu X_\mu)^2 \right] \right\} \tag{5.25}
\]

by rewriting \(e_1 e_2 e_3 e_4 = (-1)^q\). We now compute the first line of eq. (5.24) considering first only the \(\tau_3\) diagrams (the \(\tau_4\)-type is addressed later). The sum (1,3) + (3,1) of partitions can be straightforwardly obtained:

\[
2 \sum_{\mu_1, \mu_2, \nu_1, \nu_2 \text{ not all equal}} \chi^{\mu_1 \hat{\nu}_1 \mu_2 \hat{\nu}_2} \{\text{(1,3) + (3,1 partitions)}\} \tag{5.26}
\]

\[
= 2 \sum_{\mu_1, \mu_2, \nu_1, \nu_2 \text{ not all equal}} \left[ -(-1)^{\mu_1 + \mu_2} e_1 e_2 e_3 e_4 (\delta^{\mu_1}_{\mu_2} \delta^{\nu_2}_{\nu_1} - \delta^{\mu_2}_{\mu_1} \delta^{\nu_1}_{\nu_2}) - e_{\mu_1} \left( \prod_{\alpha \neq \nu_1} e_\alpha \right) \delta^{\mu_2 \nu_2}_{\mu_1 \nu_1} \right]
\]

\[
\times \left[ e_{\mu_1} \text{Tr}_N (K_{\mu_1}) \cdot \text{Tr}_N \left( X_{\nu_1} \{K_{\mu_2}, X_{\nu_2}\} \right) + e_{\mu_2} \text{Tr}_N (K_{\mu_2}) \cdot \text{Tr}_N \left( K_{\mu_1} \{X_{\nu_1}, X_{\nu_2}\} \right)
\]

\[
+ e_{\nu_1} \text{Tr}_N (X_{\nu_1}) \cdot \text{Tr}_N \left( K_{\mu_1} \{K_{\mu_2}, X_{\nu_2}\} \right) + e_{\nu_2} \text{Tr}_N (X_{\nu_2}) \cdot \text{Tr}_N \left( K_{\mu_1} \{K_{\mu_2}, X_{\nu_1}\} \right) \right]
\]

\[
= -8 \sum_{\mu \neq \nu} \left( \prod_{\alpha \neq \nu} e_\alpha \right) \left( \text{Tr}_N (K_{\mu}) \cdot \text{Tr}_N \left( K_{\mu} X_\nu^2 \right) + e_\mu e_\nu \text{Tr}_N (X_\nu) \cdot \text{Tr}_N \left( K_{\mu}^2 X_\nu \right) \right)
\]

\[
= 8 \sum_{\mu \neq \nu} (-1)^{1+q} e_\mu \text{Tr}_N (K_{\mu}) \cdot \text{Tr}_N \left( K_{\mu} X_\nu^2 \right) + e_\mu \text{Tr}_N (X_\nu) \cdot \text{Tr}_N \left( K_{\mu}^2 X_\nu \right).
\]

In the first equality we just used the expression for \(\chi^{\mu_1 \hat{\nu}_1 \mu_2 \hat{\nu}_2}\). In order to obtain the second one, it can be shown that the terms proportional to \((\delta^{\mu_1}_{\mu_2} \delta^{\nu_2}_{\nu_1} - \delta^{\mu_2}_{\mu_1} \delta^{\nu_1}_{\nu_2})\) cancel out. Using eq. (A.2) one simplifies the signs to obtain the last equality. The condition of \(\mu \neq \nu\) in the sum of the last equations reflects only the fact that the four indices cannot coincide (cf. assumptions in Prop. 5.1). The remaining partition reads
2 \sum_{\mu_1, \mu_2, \nu_1, \nu_2} \chi^{\mu_1 \nu_1 \mu_2 \nu_2} \times \{(2, 2) \text{ partition}\} \quad (5.27)

= -4 \sum_{\mu \neq \nu} \left( \prod_{\alpha \neq \nu} e_\alpha \right) \left[ 2e_\nu \text{Tr}_N (K_\mu X_\nu)^2 + e_\mu \text{Tr}_N (K_\mu^2) \cdot \text{Tr}_N (X_\nu^2) \right]

= +4 \sum_{\mu \neq \nu} \left[ 2 \left( \text{Tr}_N (K_\mu X_\nu)^2 \right)^2 + e_\mu e_\nu (-1)^{\sigma + 1} \text{Tr}_N (K_\mu^2) \cdot \text{Tr}_N (X_\nu^2) \right].

Using a similar approach (which would be redundant here), one can similarly show that the contribution of the \( \tau_q \)-diagrams is precisely twice that of \( \tau_3 \), obtaining in total \( 3 \times \{\text{eqs.}(5.26) + (5.27)\} \) for the 8-point diagrams. The claim follows from

\[ S_4 = \sum_{r=2}^{6} \sum_{\chi \in \text{CD}_2} s_\pi (\chi) \quad \text{and} \quad B_4 = \sum_{\pi} \sum_{r=2}^{6} \sum_{\chi \in \text{CD}_2} b_\pi^\tau (\chi) \quad (5.28) \]

where \( \pi \) runs over the non-trivial partitions \( \pi \in \{(1, 3), (2, 2), (3, 1)\} \). \( \square \)

5.3. Riemannian and Lorentzian geometries. Before writing down the action functionals for Riemannian and Lorentzian geometries, it will be helpful to restate eqs. (5.11) and (5.12) via

\[- \sum_{\alpha, \beta, \mu, \nu} \delta_{\alpha \beta \nu \mu} e_\alpha e_\beta \text{Tr}_N \left[ (K_\mu X_\nu)^2 + 2K_\mu^2 X_\nu^2 \right] \]

\[= (-1)^{\sigma + 1} \sum_{\mu \neq \nu} 2e_\mu e_\nu \text{Tr}_N \left[ (K_\mu X_\nu)^2 + 2K_\mu^2 X_\nu^2 \right] \]

and by writing out \( \text{‘cycl.’ next means equality after cyclic reordering) \)

\[8(-1)^{\sigma + 1} \sum_{\mu, \nu} (-1)^{\sigma (\nu, \mu)} \delta_{\mu_1 \nu_1 \nu_2 \nu_3} e_\mu (X_\mu K_{\nu_1} K_{\nu_2} K_{\nu_3} + K_\mu X_{\nu_1} X_{\nu_2} X_{\nu_3}) \]

\[\equiv -8(-1)^{\sigma} \left[ c_1 X_1 \left( K_2[K_3, K_4] + K_3[K_4, K_2] + K_4[K_2, K_3] \right) \right. \quad (5.29)\]

\[+ e_2 X_2 \left( K_1[K_3, K_4] + K_3[K_4, K_1] + K_4[K_1, K_3] \right) \]

\[+ e_3 X_3 \left( K_1[K_2, K_4] + K_2[K_4, K_1] + K_4[K_1, K_2] \right) \]

\[+ e_4 X_4 \left( K_1[K_2, K_3] + K_2[K_3, K_1] + K_3[K_1, K_2] \right) \]

\[- 8(-1)^{\sigma} \left[ c_1 K_1 \left( X_2[X_3, X_4] + X_3[X_4, X_2] + X_4[X_2, X_3] \right) \right. \]

\[+ e_2 K_2 \left( X_1[X_3, X_4] + X_3[X_4, X_1] + X_4[X_1, X_3] \right) \]

\[+ e_3 K_3 \left( X_1[X_2, X_4] + X_2[X_4, X_1] + X_4[X_1, X_2] \right) \]

\[+ e_4 K_4 \left( X_1[X_2, X_3] + X_2[X_3, X_1] + X_3[X_1, X_2] \right) \],
as well as
\[
-8 \sum_{\mu, \nu} (-1)^{\sigma(\mu, \nu)} \delta_{\mu_1 \mu_2 \mu_3} \cdot \left\{ -\text{Tr}_N K_{\mu_1} \cdot \text{Tr}_N (K_{\mu_2} K_{\mu_3}) \\
+ e_{\mu_2} e_{\mu_3} \left( \text{Tr}_N K_{\mu_1} \cdot \text{Tr}_N (X_\nu K_{\mu_2} K_{\mu_3}) + \text{Tr}_N X_{\mu_1} \cdot \text{Tr}_N (K_{\nu} X_{\mu_2} X_{\mu_3}) \right) \\
+ (-1)^q \text{Tr}_N K_\nu \cdot \text{Tr}_N (X_{\mu_1} X_{\mu_2} X_{\mu_3}) \right\}
\] 
(5.30)

\[= + 24 \text{Tr}_N X_1 \cdot \text{Tr}_N \left( K_2 [K_3, K_4] \right) + 24 \text{Tr}_N X_2 \cdot \text{Tr}_N \left( K_1 [K_3, K_4] \right) \]

\[+ 24 \text{Tr}_N X_3 \cdot \text{Tr}_N \left( K_1 [K_2, K_4] \right) + 24 \text{Tr}_N X_4 \cdot \text{Tr}_N \left( K_1 [K_2, K_3] \right) \]

\[-8 \text{Tr}_N K_1 \cdot \text{Tr}_N \left( e_3 e_4 [K_3, K_4] X_2 + e_2 e_4 [K_2, K_4] X_3 + e_2 e_3 [K_2, K_3] X_4 \right) \]

\[-8 \text{Tr}_N K_2 \cdot \text{Tr}_N \left( e_3 e_4 [K_3, K_4] X_1 + e_1 e_4 [K_4, K_1] X_3 + e_1 e_3 [K_3, K_1] X_4 \right) \]

\[-8 \text{Tr}_N K_3 \cdot \text{Tr}_N \left( e_2 e_4 [K_2, K_4] X_1 + e_1 e_4 [K_4, K_1] X_2 + e_1 e_2 [K_1, K_2] X_4 \right) \]

\[-8 \text{Tr}_N K_4 \cdot \text{Tr}_N \left( e_2 e_3 [K_2, K_3] X_1 + e_1 e_3 [K_1, K_3] X_2 + e_1 e_2 [K_1, K_2] X_3 \right) \]

\[+ (-1)^{1+q} \left\{ 24 \text{Tr}_N K_1 \cdot \text{Tr}_N \left( X_2 [X_3, X_4] \right) + 24 \text{Tr}_N K_2 \cdot \text{Tr}_N \left( X_1 [X_3, X_4] \right) \right. \]

\[+ 24 \text{Tr}_N K_3 \cdot \text{Tr}_N \left( X_1 [X_2, X_4] \right) + 24 \text{Tr}_N K_4 \cdot \text{Tr}_N \left( X_1 [X_2, X_3] \right) \right\} \]

\[-8 \text{Tr}_N X_1 \cdot \text{Tr}_N \left( e_3 e_4 [X_3, X_4] K_2 + e_2 e_4 [X_2, X_4] K_3 + e_2 e_3 [X_2, X_3] K_4 \right) \]

\[-8 \text{Tr}_N X_2 \cdot \text{Tr}_N \left( e_3 e_4 [X_3, X_4] K_1 + e_1 e_4 [X_4, X_1] K_3 + e_1 e_3 [X_3, X_1] K_4 \right) \]

\[-8 \text{Tr}_N X_3 \cdot \text{Tr}_N \left( e_2 e_4 [X_2, X_4] K_1 + e_1 e_4 [X_4, X_1] K_2 + e_1 e_2 [X_1, X_2] K_4 \right) \]

\[-8 \text{Tr}_N X_4 \cdot \text{Tr}_N \left( e_2 e_3 [X_2, X_3] K_1 + e_1 e_3 [X_1, X_3] K_2 + e_1 e_2 [X_1, X_2] K_3 \right) \].

From these expressions the Riemannian and Lorentzian cases can be readily derived.

5.3.1. Riemannian fuzzy geometries. The metric \( g = \text{diag}(-1, -1, -1, -1) \) implies \( e_\mu = -1 \) for each \( \mu \in \Delta_4 \) and \( q = 4 \). The Dirac operator \( D = D(L, \hat{H}) \) (Ex. 2.4) is parametrized by four anti-Hermitian matrices \( K_\mu = L_\mu \) (where \( [L_\mu, \cdot] \) corresponds to the derivatives \( \partial_\mu \) in the smooth case) and four Hermitian matrices \( X_\mu = \hat{H}_\mu \) (corresponding to the spin connection \( \omega_\mu \) in the smooth spin geometry case represented by \( [\hat{H}_\mu, \cdot] \) here). In Example 2.4 above these have been called
\(\hat{H}_1 = H_{234}, \ldots, \hat{H}_4 = H_{123}\). The bi-tracial octo-matrix model has the following quadratic part

\[
\frac{1}{8} \text{Tr} \left( |D|^2 \right) = \frac{1}{8} \sum_{\mu=1}^{4} \text{Tr} \left[ \hat{H}_\mu^2 - L_\mu^2 \right] + (\text{Tr}_N \hat{H}_\mu)^2 + (\text{Tr}_N L_\mu)^2,
\]

which directly follows from eq. (5.2). The quartic part is more complicated:

\[
\frac{1}{4} \text{Tr} \left( |D|^4 \right) = NS_4^{\text{Riemann}} + B_4^{\text{Riemann}}
\]

having single-trace action

\[
S_4^{\text{Riemann}} = \text{Tr}_N \left\{ 2 \sum_{\mu} (L_\mu^4 + \hat{H}_\mu^4) + 4 \sum_{\mu < \nu} (2L_\mu^2 L_\nu^2 + 2\hat{H}_\mu^2 \hat{H}_\nu^2 - L_\mu L_\nu L_\mu L_\nu - \hat{H}_\mu \hat{H}_\nu \hat{H}_\mu \hat{H}_\nu) 
- \sum_{\mu \neq \nu} \left[ 2(L_\mu \hat{H}_\nu)^2 + 4L_\mu^2 \hat{H}_\nu^2 \right] + \sum_{\mu} \left[ 2(L_\mu \hat{H}_\mu)^2 - 4L_\mu^2 \hat{H}_\mu^2 \right] + 8 \bar{H}_1 \left[ L_2[L_3, L_4] + L_3[L_4, L_2] + L_4[L_2, L_3] \right]
+ \bar{H}_2 \left[ L_1[L_3, L_4] + L_3[L_4, L_1] + L_4[L_1, L_3] \right]
+ \bar{H}_3 \left[ L_1[L_2, L_4] + L_2[L_4, L_1] + L_4[L_1, L_2] \right]
+ \bar{H}_4 \left[ L_1[L_2, L_3] + L_2[L_3, L_1] + L_3[L_1, L_2] \right] \right\},
\]

and bi-tracial action

\[
B_4^{\text{Riemann}} = 8 \sum_{\mu, \nu} \text{Tr}_N \hat{H}_\mu \cdot \text{Tr}_N (\hat{H}_\mu \hat{H}_\nu^2) - \text{Tr}_N (L_\mu) \cdot \text{Tr}_N (L_\mu L_\nu^2) \]

\[
+ \sum_{\mu, \nu \neq 1} \left\{ 2 \text{Tr}_N \left( \hat{H}_\mu^2 \right) \cdot \text{Tr}_N \left( \hat{H}_\nu^2 \right) + 4 \left[ \text{Tr}_N \left( \hat{H}_\mu \hat{H}_\nu \right) \right]^2 \right\}
+ \sum_{\mu, \nu \neq 1} \left\{ 2 \text{Tr}_N \left( L_\mu^2 \right) \cdot \text{Tr}_N \left( L_\nu^2 \right) + 4 \left[ \text{Tr}_N (L_\mu L_\nu) \right]^2 \right\}.
\]
5.3.2. Lorentzian fuzzy geometries. Here we keep the usual conventions: the index 0 for time and (undotted) Latin spatial indices $a = 1, 2, 3$. In the Lorentzian setting $g = \text{diag}(+1, -1, -1, -1)$, so $q = 3$, $e_0 = +1$ and $e_a = -1$ for each spatial $a$. A parametrization of the Dirac operator of the form $D = D(H, L_a, Q, R_a)$ by six anti-Hermitian matrices $K_a = L_a$, $X_a = R_a$ and two Hermitian matrices $K_0 = H$ and $X_0 = Q$ follows. As before, we give first the quadratic part and then

\[ + 4 \sum_{\mu=1}^{4} \left\{ 2 \text{Tr}_N L_\mu \cdot \text{Tr}_N (L_\mu \tilde{H}_\mu^2) - 2 \text{Tr}_N \tilde{H}_\mu \cdot \text{Tr}_N (\tilde{H}_\mu L_\mu^2) \right\} \]

\[ - \text{Tr}_N (\tilde{H}_\mu^2) \cdot \text{Tr}_N (L_\mu^2) + 2 \left[ \text{Tr}_N (L_\mu \tilde{H}_\mu) \right]^2 \}

\[ + 24 \sum_{\mu \neq \nu=1}^{4} \text{Tr}_N L_\mu \cdot \text{Tr}_N (L_\mu X_\nu^2) - \text{Tr}_N \tilde{H}_\nu \cdot \text{Tr}_N (L_\mu^2 \tilde{H}_\nu) \]

\[ + 12 \sum_{\mu \neq \nu}^{4} \left\{ 2 \left[ \text{Tr}_N (L_\mu \tilde{H}_\nu) \right]^2 - \text{Tr}_N (L_\mu^2) \cdot \text{Tr}_N (\tilde{H}_\nu^2) \right\} \]

\[ + 24 \text{Tr}_N \tilde{H}_1 \cdot \text{Tr}_N (L_2 [L_3, L_4]) + 24 \text{Tr}_N \tilde{H}_2 \cdot \text{Tr}_N (L_1 [L_3, L_4]) \]

\[ + 24 \text{Tr}_N \tilde{H}_3 \cdot \text{Tr}_N (L_1 [L_2, L_4]) + 24 \text{Tr}_N \tilde{H}_4 \cdot \text{Tr}_N (L_1 [L_2, L_3]) \]

\[ - 8 \text{Tr}_N L_1 \cdot \text{Tr}_N \left( [L_3, L_4] \tilde{H}_2 + [L_2, L_4] \tilde{H}_3 + [L_2, L_3] \tilde{H}_4 \right) \]

\[ - 8 \text{Tr}_N L_2 \cdot \text{Tr}_N \left( [L_3, L_4] \tilde{H}_1 + [L_4, L_1] \tilde{H}_3 + [L_3, L_1] \tilde{H}_4 \right) \]

\[ - 8 \text{Tr}_N L_3 \cdot \text{Tr}_N \left( [L_2, L_4] \tilde{H}_1 + [L_4, L_1] \tilde{H}_2 + [L_1, L_2] \tilde{H}_4 \right) \]

\[ - 8 \text{Tr}_N L_4 \cdot \text{Tr}_N \left( [L_2, L_3] \tilde{H}_1 + [L_1, L_3] \tilde{H}_2 + [L_1, L_2] \tilde{H}_3 \right) \]

\[ - 24 \text{Tr}_N L_1 \cdot \text{Tr}_N \left( \tilde{H}_2 [\tilde{H}_3, \tilde{H}_4] \right) - 24 \text{Tr}_N L_2 \cdot \text{Tr}_N \left( \tilde{H}_1 [\tilde{H}_3, \tilde{H}_4] \right) \]

\[ - 24 \text{Tr}_N L_3 \cdot \text{Tr}_N \left( \tilde{H}_1 [\tilde{H}_2, \tilde{H}_4] \right) - 24 \text{Tr}_N L_4 \cdot \text{Tr}_N \left( \tilde{H}_1 [\tilde{H}_2, \tilde{H}_3] \right) \]

\[ - 8 \text{Tr}_N \tilde{H}_1 \cdot \text{Tr}_N \left( [\tilde{H}_3, \tilde{H}_4] L_2 + [\tilde{H}_2, \tilde{H}_4] L_3 + [\tilde{H}_2, \tilde{H}_3] L_4 \right) \]

\[ - 8 \text{Tr}_N \tilde{H}_2 \cdot \text{Tr}_N \left( [\tilde{H}_3, \tilde{H}_4] L_1 + [\tilde{H}_4, \tilde{H}_1] L_3 + [\tilde{H}_3, \tilde{H}_1] L_4 \right) \]

\[ - 8 \text{Tr}_N \tilde{H}_3 \cdot \text{Tr}_N \left( [\tilde{H}_2, \tilde{H}_4] L_1 + [\tilde{H}_4, \tilde{H}_1] L_2 + [\tilde{H}_1, \tilde{H}_2] L_4 \right) \]

\[ - 8 \text{Tr}_N \tilde{H}_4 \cdot \text{Tr}_N \left( [\tilde{H}_2, \tilde{H}_3] L_1 + [\tilde{H}_1, \tilde{H}_3] L_2 + [\tilde{H}_1, \tilde{H}_2] L_3 \right) . \]
the quartic. The former follows from eq. (5.2),
\[
\frac{1}{8} \text{Tr} D^2 = N \text{Tr}_N \left\{ H^2 + Q^2 - \sum_a (L_a^2 + R_a^2) \right\}
+ \left( \text{Tr}_N H \right)^2 + \left( \text{Tr}_N Q \right)^2 + \sum_a \left( \text{Tr}_N L_a \right)^2 + \left( \text{Tr}_N R_a \right)^2. \tag{5.34}
\]
Using eqs. (5.29) and (5.30) to rewrite Proposition 5.4, one gets
\[
S_4^{\text{Lorentz}} = \text{Tr}_N \left\{ 2H^4 + 2Q^4 + \sum_a (L_a^4 + R_a^4) - \sum_a \left[ 2(L_a R_a)^2 + 4L_a^2 \right]
+ \sum_{a < c} \left[ -8H^2 L_a^2 - 8Q^2 R_a^2 + 4(HL_a)^2 + 4(QR_a)^2 \right]
+ \sum_a \left[ 8L_a^2 L_c^2 + 8R_a^2 R_c^2 - 4(L_a L_c)^2 - 4(R_a R_c)^2 \right]
- \sum_a \left[ 2(H R_a)^2 + 4H^2 R_a^2 + 2(L_a Q)^2 + 4L_a^2 Q^2 \right]
+ \sum_{a \neq c} 2(L_a R_c)^2 + 4L_a^2 R_c^2 - 2(HQ)^2 + 4H^2 Q^2
+ 8 \left[ Q \left( L_1[L_2, L_3] + L_2[L_3, L_1] + L_3[L_1, L_2] \right)
- R_1 \left( H[L_2, L_3] + L_2[L_3, H] + L_3[H, L_2] \right)
- R_2 \left( H[L_1, L_3] + L_1[L_3, H] + L_3[H, L_1] \right)
- R_3 \left( H[L_1, L_2] + L_1[L_2, H] + L_2[H, L_1] \right) \right]
+ 8 \left[ H \left( R_1[R_2, R_3] + R_2[R_3, R_1] + R_3[R_1, R_2] \right)
- L_1 \left( Q[R_2, R_3] + R_2[R_3, Q] + R_3[Q, R_2] \right)
- L_2 \left( Q[R_1, R_3] + R_1[R_3, Q] + R_3[Q, R_1] \right)
- L_3 \left( Q[R_1, R_2] + R_1[R_2, Q] + R_2[Q, R_1] \right) \right] \right\}, \tag{5.35}
\]
and
\[
B_4^{\text{Lorentz}} = 8 \text{Tr}_N Q \cdot \text{Tr}_N \left\{ Q^3 - \sum_a (QR_a^2 + 3L_a Q^2) + QH^2 \right.
+ 3L_1[L_2, L_3] + [R_3, R_2]L_1 + [R_3, R_1]L_2 + [R_2, R_2]L_3 \right.
+ 8 \text{Tr}_N H \cdot \text{Tr}_N \left\{ H^3 - \sum_a (HL_a^2 - 3HR_a^2) + HQ^2 \right. \tag{5.36}
\]
\[ + 3R_1[R_2, R_3] + [L_3, L_2]R_1 + [L_2, L_1]R_2 + [L_2, L_1]R_3 \] 

+ 8 \sum_a \text{Tr}_N R_a \cdot \text{Tr}_N \left\{ R_a H^2 - R_a \sum_c R_c^2 - L_a R_a^2 \right. 

\quad \left. + 3H^2 R_a - 3 \sum_{c(a\neq a)} R_c^2 \right\} 

- 8 \text{Tr}_N R_1 \cdot \text{Tr}_N \left( [R_2, R_3]H - [R_3, Q]L_2 - [R_2, Q]L_3 + 3H[L_3, L_2] \right) 

- 8 \text{Tr}_N R_2 \cdot \text{Tr}_N \left( [R_1, R_3]H - [R_3, Q]L_1 - [Q, R_1]L_3 + 3H[L_3, L_1] \right) 

- 8 \text{Tr}_N R_3 \cdot \text{Tr}_N \left( [R_1, R_2]H - [Q, R_2]L_1 - [Q, R_1]L_2 + 3H[L_2, L_1] \right) 

+ 8 \sum_a \text{Tr}_N L_a \cdot \text{Tr}_N \left\{ L_a H^2 - L_a \sum_c L_c^2 - R_a L_a^2 \right. 

\quad \left. + 3L_a Q^2 - 3 \sum_{c(a\neq a)} L_c R_c^2 \right\} 

- 8 \text{Tr}_N L_1 \cdot \text{Tr}_N \left( [L_2, L_3]Q - [L_3, H]R_2 - [L_2, H]R_3 + 3Q[R_3, R_2] \right) 

- 8 \text{Tr}_N L_2 \cdot \text{Tr}_N \left( [L_1, L_3]Q - [L_3, H]R_1 - [H, L_1]R_3 + 3Q[R_3, R_1] \right) 

- 8 \text{Tr}_N L_3 \cdot \text{Tr}_N \left( [L_1, L_2]Q - [H, L_2]R_3 - [H, L_1]R_2 + 3Q[R_2, R_1] \right) 

+ 6 \left[ \text{Tr}_N Q^2 \right]^2 + \sum_a \left\{ 4 \text{Tr}_N \left( R_a^2 \right) \cdot \text{Tr}_N \left( Q^2 \right) - 8 \left[ \text{Tr}_N \left( QR_a \right) \right]^2 \right\} 

+ \sum_{a,c} \left\{ 2 \text{Tr}_N \left( R_a^2 \right) \cdot \text{Tr}_N \left( R_c^2 \right) + 4 \left[ \text{Tr} \left( R_a R_c \right) \right]^2 \right\} 

+ 6 \left[ \text{Tr}_N H^2 \right]^2 + \sum_a \left\{ 4 \text{Tr}_N \left( L_a^2 \right) \cdot \text{Tr}_N \left( H^2 \right) - 8 \left[ \text{Tr}_N \left( HL_a \right) \right]^2 \right\} 

+ \sum_{a,c} \left\{ 2 \text{Tr}_N \left( L_a^2 \right) \cdot \text{Tr}_N \left( L_c^2 \right) + 4 \left[ \text{Tr} \left( L_a L_c \right) \right]^2 \right\} 

+ 4 \text{Tr}_N \left( Q^2 \right) \cdot \text{Tr}_N \left( H^2 \right) + 8 \left[ \text{Tr}_N \left( HQ \right) \right]^2 

+ 4 \sum_a \text{Tr}_N \left( R_a^2 \right) \cdot \text{Tr}_N \left( L_a^2 \right) + 8 \left[ \text{Tr}_N \left( L_a R_a \right) \right]^2 

+ 24 \sum_a \left\{ \left[ \text{Tr}_N \left( H R_a \right) \right]^2 + \left[ \text{Tr}_N \left( Q L_a \right) \right]^2 \right\} 

+ 12 \sum_{a \neq c} \left\{ \left[ \text{Tr}_N \left( L_a R_c \right) \right]^2 + \text{Tr}_N \left( L_a^2 \right) \cdot \text{Tr}_N \left( R_c^2 \right) \right\} 

- 12 \sum_a \left\{ \text{Tr}_N \left( L_a^2 \right) \cdot \text{Tr}_N \left( Q^2 \right) + \text{Tr}_N \left( R_a^2 \right) \cdot \text{Tr}_N \left( H^2 \right) \right\} .

Remark 5.5. Each anti-hermitian parametrizing matrix \( L \) can be replaced by a traceless one \( L' = L - (\text{Tr}_N L/N) \cdot 1_N \), since \( L \) appears in \( D \) only via anti-commutators; since \( \text{Tr}_N L \) is purely imaginary, \( L' \) is also anti-hermitian.
6. Large-\(N\) limit via free probability?

Some evidence for viability of a free probabilistic approach to the large-\(N\) limit is the numerical analysis \([BG16]\) of

\[
F = \frac{\sum_{\mu} [\text{Tr}_N H_{\mu}]^2}{N \sum_{\mu} \text{Tr}_N (H_{\mu}^2)}. 
\]

The observable \(F\) does not trivially vanish at \(N \to \infty\), since the double trace \(\text{Tr}_N \cdot \text{Tr}_N\) in the numerator competes with the denominator’s \(N \times \text{Tr}_N\). In \([BG16, \text{Fig. 14}]\), for the model \(\text{Tr}(D^4 - |\lambda|D^2)\) in some low-dimensional geometries of signatures \((1,0), (1,1), (2,0)\) and \((0,3)\), \(F\) is plotted in the region \(\lambda \in (1, 5)\). For each signature, a vanishing \(F \approx 0\) is found in the regions \(1 \lessapprox \lambda \lessapprox 2\). Preliminarily, \(F\) is related to the quotient \(B_2/S_2\) of the bi-trace and single-trace functionals introduced here. This motivates free probabilistic tools.

It is well-known that free probability and (multi)matrix models are related \([NS06, GJS07, GN14, NT18]\). One could start with noncommutative self-adjoint polynomials \(P \in \mathbb{R}(x_1, \ldots, x_\kappa)\) \([Spe19]\). To wit, \(P(x_1, \ldots, x_\kappa) = P(x_1, \ldots, x_\kappa)^*\), if each of the noncommutative variables \(x_i\) satisfies formal self-adjointness \(x_i^* = x_i\).

For instance, the next polynomials are self-adjoint:

\[
P_2(x_1, \ldots, x_\kappa) = x_1^2 + \ldots + x_\kappa^2 + \frac{\lambda}{2} \sum_{i \neq j} x_i x_j, \quad (6.1a)
\]

\[
P_4(x_1, \ldots, x_\kappa) = x_1^4 + \ldots + x_\kappa^4 + \frac{1}{2} \sum_{i \neq j} (\lambda_1 x_i x_j x_i x_j + \lambda_2 x_i^2 x_j^2), \quad (6.1b)
\]

being \(\lambda_i, \lambda\) real coupling constants. One can instead evaluate \(P\) in square matrices of size, say, \(N\) and define

\[
d\nu_N(X^{(N)}) = d\nu_N(X_1^{(N)}, \ldots, X_\kappa^{(N)}) = C_N \cdot e^{-N^2 \text{Tr}_N [P(X_1^{(N)}, \ldots, X_\kappa^{(N)})]} \cdot d\Lambda(X_1^{(N)} \ldots d\Lambda(X_\kappa^{(N)}) \quad (6.2)
\]

being \(C_N\) a normalization constant and \(\Lambda\) the Lebesgue measure

\[
d\Lambda(Y) = \prod_i dY_{ii} \prod_{i < j} \mathbb{R}(dY_{ij}) \Im(dY_{ij}), \quad Y \in \mathcal{M}_N(\mathbb{C}).
\]

The distributions \(\varphi_N\) defined by

\[
\varphi_N(X_{j_1}^{(N)}, \ldots, X_{j_k}^{(N)}) = \int \text{Tr}_N(X_{j_1}^{(N)} \cdots X_{j_k}^{(N)}) d\nu_N(X^{(N)}) \quad (6.3)
\]

and their eventual convergence to distributions \(\varphi(X_{j_1}, \ldots, X_{j_k})\) for large \(N\) is of interest in free probability. As we have shown in Sections 3, 4 and 5, we have developed a geometrically interesting way to produce noncommutative polynomials. Although these are not directly self-adjoint, self-adjointness is not essential in order for one to ponder the possible convergence of the measures they define. As far as the trace does not detect it, a weaker notion suffices:

**Definition 6.1.** Given variables \(z_1, \ldots, z_\kappa\), each of which satisfies either formal self-adjointness (i.e. for an involution \(\ast\), \(z_i^\ast = +z_i\) holds, in whose case we let \(z_i =: h_i\)) or formal anti-self-adjointness (\(z_i^\ast = -z_i\); and if so write \(z_i =: l_i\),

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a noncommutative (NC) polynomial $P \in \mathbb{R}(z_1, \ldots, z_\kappa)$ is said to be cyclic self-adjoint if the following conditions hold:

- for each word $w$ (or monomial) of $P$ there exists a word $w'$ in $P$ such that
  \[ [w(z_1, \ldots, z_\kappa)]^* = + (\sigma \cdot w')(z_1, \ldots, z_\kappa) \text{ holds for some } \sigma \in \mathbb{Z}/|w'|\mathbb{Z}, \quad (6.4) \]
  being
  \[ \circ \ |w| \text{ the length of the word } w \text{ (or order of the monomial } w) \text{ and} \]
  \[ \circ \sigma \cdot w' \text{ the action of } \mathbb{Z}/|w'|\mathbb{Z} \text{ on the word } w' \text{ by cyclic permutation of its letters.} \]
- The map defined by $w \mapsto w'$ is a bijection in the set of the words of $P$.

Similarly, a polynomial $G \in \mathbb{R}(z_1, \ldots, z_\kappa)$ is cyclic anti-self-adjoint if for each of its words $w$ if there exist a $\sigma \in \mathbb{Z}/|w'|\mathbb{Z}$ for which the condition

\[ [w(z_1, \ldots, z_\kappa)]^* = -(\sigma \cdot w')(z_1, \ldots, z_\kappa) \quad (6.5) \]

holds, and if, additionally, the map that results from this condition, $w \mapsto w'$, is a bijection in the set of words of $G$.

**Example 6.2.** Consider the formal adjoint $P^*$ of the NC polynomial $P$ given by $P(h, l_1, l_2, l_3) = l_1 \{h(l_2l_3 - l_3l_2) + l_2(l_3h - hl_3) + l_3(hl_2 - l_2h)\}$. One obtains

\[
[P(h, l_1, l_2, l_3)]^* = (h[l_2, l_3] + l_2[h, l_3] + l_3[h, l_2])^* l_1^* \\
= \{(l_2l_3 - l_3l_2)h + (l_3h - hl_3)l_2 + (hl_2 - l_2h)l_3\} l_1^* \quad (6.6)
\]

where $[\cdot, \cdot, \cdot]$ is the commutator in $\mathbb{R}(h, l_1, l_2, l_3)$. Clearly $P^* \neq P$, but up to the cyclic permutation $\sigma \in \mathbb{Z}/4\mathbb{Z}$ defined by bringing the letter $l_1$ from the last to the first position of each word, a bijection of words in $P$ is established. Hence $P$ is cyclic self-adjoint. On the other hand, take $\Psi(h, l_2, l_3) = (l_2l_3 - l_3l_2)h$. By a similar token, one sees that $\Psi$ is cyclic anti-self-adjoint.

It would not be surprising that the spectral action $\text{Tr} f(D)$ for fuzzy geometries (since it has to be real) in any dimension and allowed KO-dimension leads for any ordinary polynomial $f$ to the type of NC polynomials we just introduced. Preliminarily, we verify this statement only for the explicit computations we performed in this article:

**Corollary 6.3.** For the following cases

- $d = 2$, in arbitrary signature and being $f$ a sextic polynomial; and
- Riemannian and Lorentzian signatures ($d = 4$) being $f$ quartic polynomial,

the spectral action $\text{Tr} f(D) = \dim V(N \cdot S_f + B_f)$ for fuzzy geometries has the form

\[ S_f = \text{Tr}_N P \quad \text{and} \quad B_f = \sum_i \text{Tr}_N \Phi_i \cdot \text{Tr}_N \Psi_i, \quad (6.7) \]

where $P, \Phi_i, \Psi_i \in \mathbb{R}(x_1, \ldots, z_{\kappa(d)})$ with $\kappa(d) = 2^{d-1}$ are NC polynomials such that

- $P$ is cyclic self-adjoint
- and $\Phi_i$ and $\Psi_i$ are both either cyclic self-adjoint or both cyclic anti-self-adjoint.
Proof. Up to an irrelevant (as to assess cyclic self-adjointness) ambiguity in a global factor for Φi and Ψi, all the NC polynomials can be read off from eqs. (4.1), (4.3) and Proposition 4.1 for the d = 2 case. For d = 4, the result follows by inspection of each term, which is immediate since formulae (5.36), (5.33), (5.31) and (5.34) are given in terms of commutators. Then one uses that [h, l]∗ = [h, l], [l1, h2]∗ = −[h1, h2] and [l1, l2]∗ = −[l1, l2].

The only non-obvious part is dealing with expressions like
\[ P = Q\{L1[L2, L3] + L2[L3, L1] + L3[L1, L2]\}, \]
which appears in \( S^{\text{Lorentz}}_4 \) according to eq. (5.35). However, if P is the NC polynomial given in eq. (6.6) then \( P(Q, L1, L2, L3) \) equals \( P(Q) \), hence it is cyclic self-adjoint by Example 6.2. Also for the NC polynomial \( \Psi \) defined there, \( -8\text{Tr}_N L1 \cdot \text{Tr}_N (\Psi(Q, L2, L3)) \) appears in the expression given by eq. (5.36) for \( B^{\text{Lorentz}}_4 \), being both \( \Phi(L1) = L1 \) and \( \Psi \) cyclic anti-self-adjoint. \( \square \)

7. Conclusions

We computed the spectral action for fuzzy geometries of even dimension \( d \), whose ‘quantization’ was stated as a 2\( d-1 \)-matrix model with action \( \text{Tr} f(D) = \dim V(N : S_f + B_f) \) being the single trace \( S_f \) and bi-tracial parts \( B_f \) of the form
\[ S_f = \text{Tr}_N F, \quad B_f = \sum_i (\text{Tr}_N \otimes \text{Tr}_N)\{\Phi_i \otimes \Psi_i\}. \]

With the aid of chord diagrams that encode non-vanishing traces on the spinor space \( V \), we organized the obtention of (finitely many) noncommutative polynomials \( F, \Phi_i \) and \( \Psi_i \) in \( \mathbb{L}^{d-1} \) Hermitian or anti-Hermitian matrices in \( M_N(\mathbb{C}) \). These polynomials are defined up to cyclic permutation of their words and have integer coefficients that are independent of \( N \). We commented on a free probabilistic perspective towards the large-\( N \) limit of the spectral action and adapted the concept of self-adjoint noncommutative polynomial to a more relaxed one (cyclic self-adjointness) that is satisfied by \( F \). Furthermore, for fixed \( i \), either both \( \Phi_i \) and \( \Psi_i \) are cyclic self-adjoint or both are cyclic anti-self-adjoint.

On the one hand, we elaborated on 2-dimensional fuzzy geometries in arbitrary signature \((p, q)\). When quantized (or randomized), the corresponding partition function is
\[ Z^{(p,q)} = \int_{\mathcal{M}^{p,q}} e^{-S(D)} dK_1 dK_2 \quad D = D(K_1, K_2), \quad p + q = 2. \quad (7.1) \]
The space \( \mathcal{M}^{p,q} \) of Dirac operators is \( \mathcal{M}^{p,q} = (\mathbb{H}_N)^{\times p} \times \mathfrak{su}(N)^{\times q} \), but this simple parametrization does not generally hold for \( d > 2 \). Here \( \mathfrak{su}(N) = \text{Lie}(\text{SU}(N)) \) stands for the (Lie algebra of) traceless \( N \times N \) skew-Hermitian matrices and \( \mathbb{H}_N \) for Hermitian matrices. Concretely, in Section 4 formulas for \( S(D) = \text{Tr}(D^2 + \lambda_4 D^4 + \lambda_6 D^6) \) are deduced, but the present method enables to obtain \( S(D) = \text{Tr} f(D) \) for polynomial \( f \). This first result is an extension (by the sextic term) of the spectral action presented by Barrett-Glaser [BG16] up to quartic polynomials in \( d = 2 \).

On the other hand, the novelties (to the best of our knowledge) are the analytic derivations we provided for Riemannian and Lorentzian fuzzy geometries—and in fact, in arbitrary signature in 4 dimensions—as well as a systematic approach that maps random fuzzy geometries to multi-matrix bi-tracial models. For one thing,
this sheds some light on arbitrary-dimensional geometries and, for other thing, on extensions to (quantum) models including bosonic fields. For the quadratic-quartic spectral action $S(D) = \text{Tr}(D^2 + \lambda_4 D^4)$ computed in Section 5 one could study the octo-matrix model

$$Z^{(p,q)} = \int_{\mathcal{M}^{p,q}} e^{-S(D)} \prod_{\mu=1}^4 dK_\mu dX_\mu \quad D = D(K_\mu, X_\mu), \quad p + q = 4 \quad (7.2)$$

being, in particular,

$$\mathcal{M}^{p,q} = \begin{cases} \mathbb{H}^{4 \times 4}_N \times \mathfrak{su}(N)^{\times 4} & p = 0, q = 4 \text{ (Riemannian)}, \\ \mathbb{H}^{2 \times 4}_N \times \mathfrak{su}(N)^{\times 6} & p = 1, q = 3 \text{ (Lorentzian)}. \end{cases} \quad (7.3)$$

For rest of the signatures, $\mathcal{M}^{p,q}$ can be readily obtained with the aid of the eq. (A.2) as described for the Lorentzian and Riemannian cases. As a closing point, it is pertinent to remark that determining whether the Dirac operator of a fuzzy geometry is a truncation of a spin$^c$ geometry is a subtle problem addressed in [GS19] from the viewpoint of the Heisenberg uncertainty principle [CCM14].

8. Outlook

We present a miscellanea of short outlook topics after elaborating on the next:

8.1. An auxiliary model for the $d = 1$ case. One could reformulate the partition functions of $d = 1$-models in terms of auxiliary models that do not contain multi-traces. We pick for concreteness the signature $(p, q) = (1, 0)$ and the polynomial $f(x) = (x^2 + \lambda x^4)/2$ for the spectral action $\text{Tr} f(D)$. We explain why the ordinary matrix model given by $Z_{\text{aux}}^{(1,0)} = \int_{\mathbb{H}^d_N} e^{\eta \text{Tr} H - \alpha \text{Tr}(H^2) + \beta \text{Tr}(H^3)} dH$ over the Hermitian $N \times N$ matrices $\mathbb{H}_N^d$, allows to restate the quartic-quadratic $(1,0)$-type Barrett-Glaser model with partition function

$$Z_{\text{BG}}^{(1,0)} = \int_{\mathcal{M}} e^{-\frac{1}{2} \text{Tr}(D^2 + \lambda D^4)} dD \quad (8.1)$$

as formally equivalent to the functional

$$\langle \exp\{-3\lambda \text{Tr} H^2 + 4\lambda \text{Tr} H \cdot \text{Tr} H^3 + (\text{Tr} H)^2\}\rangle_{\text{aux},0}, \quad (8.2)$$

where the expectation value of an observable $\Phi$ is taken with respect to the auxiliary model

$$\langle \Phi \rangle_{\text{aux}} = \frac{1}{Z_{\text{aux}}^{(1,0)}} \int_{\mathbb{H}^d_N} \Phi(H) e^{-\mathcal{S}(H)} dH,$$

being $\mathcal{S}(H) = \alpha \text{Tr}(H^2) + N \lambda \text{Tr}(H^4) + \eta \text{Tr} H + \beta \text{Tr}(H^3)$. The zero subindex ‘aux,0’ means evaluation in the parameters

$$\alpha = N, \gamma = \beta = 0. \quad (8.3)$$

Indeed, one can use the explicit form of the Dirac operator $D = \{H, \cdot \}$ to rewrite the integral in terms of the matrix $H$. One gets

$$\frac{1}{2} \text{Tr}(D^2 + \lambda D^4) = N[\text{Tr}(H^2) + \lambda \text{Tr}(H^4)]$$

$$+ 3\lambda[\text{Tr}(H^2)]^2 + 4\lambda \text{Tr} H \cdot \text{Tr} H^3 + [\text{Tr}(H)]^2$$

$$= N[\text{Tr}(H^2) + \lambda \text{Tr}(H^4)] + b |H|, \quad (8.4)$$
The second line of eq. (8.4) contains the bi-tracial terms; this term will be denoted by $b(H)$. Inserting last equations into (8.1)

$$
Z_{BG}^{(1,0)} = e^{-N \text{Tr}(H^2 + \lambda H^4)} e^{-b[H]} dH
$$

Since $\mathcal{S}(H)|_{\alpha=\gamma=\beta=0} = N[\text{Tr}(H^2) + \lambda \text{Tr}(H^4)]$, one can replace the first exponential by $e^{-\mathcal{S}(H)}$ and evaluate the parameters as in eq. (8.3):

$$
Z_{BG}^{(1,0)} = e^{-\mathcal{S}(H)} e^{-b[H]} dH.
$$

If one knows the partition function $Z_{aux}^{(1,0)}$, one can compute the model in question by taking out $e^{-b(H)}$ from the integral and accordingly substituting the traces by the appropriate derivatives:

$$
Z_{BG}^{(1,0)} = e^{-b_\beta} \left[ e^{\mathcal{S}(H)} dH \right]_{0} \quad \text{where} \quad b_\beta = 3\lambda \delta_{\alpha}^2 + 4\lambda \partial_{\beta} \partial_{\gamma} + \partial_{\alpha}.
$$

That is $Z_{BG}^{(1,0)} = [e^{-b_\beta} Z_{aux}^{(1,0)}]_0$, which also proves eq. (8.2). This motivates to look for similar methods in order restate, for $d \geq 2$, the bi-tracial part of the models addressed here as single-trace auxiliary multi-matrix models.

8.2. Miscellaneous.

- **Gauge theory.** The NCG-framework pays off in high energy physics precisely for gauge-Higgs theories. A natural step would be to come back to this initial motivation and to define almost commutative fuzzy geometries (ongoing project) in order to derive from them the Yang-Mills–Higgs theory on a fuzzy base.

- **Analytic approach.** A non-perturbative approach to matrix models, which led to the solvability of all quartic matrix models [GHW19] (after key progress in [PW18]) consists in exploiting the $U(N)$-Ward-Takahashi identities in order to descend the tower of the Schwinger-Dyson (or loop) equations (SDE). This was initially formulated for a quartic analogue of Kontsevich’s model [Kon92], but the Grosse-Wulkenhaar approach (SDE + Ward Identity [GW14]) showed also applicability to tensor field theory [Pér18, PPW17], and seems to be flexible.

- **Topological Recursion.** Probably the analytic approach would lead to a (or multiple) Topological Recursion (TR), as it appeared in [GHW19]. Alternatively, one could build upon the direct TR-approach [AK19]. Namely, the blobbed [Bor15] Topological Recursion [EO07, Eyn14, CE06, CEO06] has been lately applied [AK19] to general multi-trace models that encompass the 1-dimensional version of the models derived here. An extension of their TR to dimension $d \geq 2$ would be interesting.

- **Combinatorics.** Finally, chord diagrams are combinatorially interesting by themselves. For instance, together with decorated versions known as Jacobi and Gauß diagrams, they are used in algebraic knot theory [CDM12, Secs 3.4 and 4] in order to describe Vassiliev invariants. Those appearing here are related to the Penner matrix model [Pen88]. One can still explore
their generating function [ACPRS13, AFMPS17] in relation to the matrix model with action
\[ S(X) = \text{Tr}_N[X^2/2 - st \cdot (1-tX)^{-1}], \]
for \( X \in \mathbb{H}_N \). The free energies \( F_g \) of this Andersen-Chekhov-Penner-Reidys-Sulkowski (ACPRS) model \( Z_{\text{ACPRS}} = \sum g^{N^2-2g} F_g \) generate numbers that are moreover important in computational biology, as they encode topologically non-trivial complexes of interacting RNA molecules. These numbers are related to the isomorphism classes of chord diagrams with certain number cuts in the circle, leaving segments ('backbones', cf. [ACPRS13]) but also a connected diagram. For the ACPRS-model there is also a Topological Recursion (op. cit.).

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Appendix A. Some properties of gamma matrices

In order to deal with \( d \)-dimensional matrix geometries we prove some of the properties of the corresponding gamma matrices.

First, notice that in any signature for each multi-index \( I = (\mu_1, \ldots, \mu_r) \in \Lambda_d \) one has
\[ \gamma^{\mu_r} \cdots \gamma^{\mu_1} = (-1)^{r(r-1)/2} \gamma^{\mu_1} \cdots \gamma^{\mu_r} = (-1)^{|r/2|} \gamma^{\mu_1} \cdots \gamma^{\mu_r}. \] (A.1)
This can be proven by induction on the number \( r-1 \) of products. For \( r = 2 \), this is just \( \{\gamma^{\mu_1}, \gamma^{\mu_2}\} = 0 \), which holds since the indices are different. Suppose that eq. (A.1) holds for an \( r \in \mathbb{N} \). Then if \( (\mu_1, \ldots, \mu_{r+1}) \in \Lambda_d \), one has
\[ \gamma^{\mu_{r+1}} \gamma^{\mu_r} \cdots \gamma^{\mu_2} \gamma^{\mu_1} = (-1)^r (\gamma^{\mu_r} \cdots \gamma^{\mu_2} \gamma^{\mu_1}) \gamma^{\mu_{r+1}} \]
\[ = (-1)^{r^{2(r-1)/2}} \gamma^{\mu_1} \cdots \gamma^{\mu_{r+1}} \]
\[ = (-1)^{(r+1)/2} \gamma^{\mu_1} \cdots \gamma^{\mu_{r+1}} \]
\[ = (-1)^{|(r+1)/2|} \gamma^{\mu_1} \cdots \gamma^{\mu_{r+1}}. \]

Now let us fix a signature \((p, q)\). We dot the spacial indices \( \dot{c} = 1, \ldots, q \) and the temporal in usual Roman lowercase, \( a = 1, \ldots, p \). Given a multi-index \( I = (a_{t}, \ldots, a_{t}, \dot{c}_1, \ldots, \dot{c}_u) \in \Lambda_d \) (so \( t \leq p \) and \( u \leq q \)) it will be useful to know whether \( \Gamma^I \) is Hermitian or anti-Hermitian. We compute its Hermitian conjugate (\( \Gamma^I \))*:
\[
(\gamma^{a_1} \cdots \gamma^{a_i} \gamma^{\dot{c}_1} \cdots \gamma^{\dot{c}_u})^* = (\gamma^{\dot{c}_u})^* \cdots (\gamma^{\dot{c}_1})^* (\gamma^{a_1})^* \cdots (\gamma^{a_i})^*
\]
\[ = (-1)^u \gamma^{\dot{c}_u} \cdots \gamma^{\dot{c}_1} \gamma^{a_1} \cdots \gamma^{a_i}
\]
\[ = (-1)^u + |(u+t)/2| \gamma^{a_1} \cdots \gamma^{a_i} \gamma^{\dot{c}_1} \cdots \gamma^{\dot{c}_u}. \]

With the conventions set in eq. (2.2), have the following
- In Riemannian signature \((0, d)\) a product of \( u \) gamma matrices associated to \( I \in \Lambda_d \) is (anti-)Hermitian if \( u(u+1)/2 \) is even (odd).
In \((d,0)\)-signature a product of \(u\) gamma matrices associated to \(I \in \Lambda_d\) is Hermitian if \(t(t - 1)/2\) is even, and anti-Hermitian if it is odd. In the main text, it will be useful to know that
\[
e_{\mu}e_{\bar{\mu}} = (-1)^{q+1} \quad d = 4, \text{ with signature } (p, q).
\] (A.2)
This follows from \((\Gamma^\mu)^* = e_{\bar{\mu}} \Gamma^\mu\), from \(e_1e_2e_3e_4 = (-1)^q\) and from \((\Gamma^\mu)^* = (\gamma^4)^* \cdots (\gamma^{\mu})^* \cdots (\gamma^1)^* = -e_1e_2e_3e_4\gamma_1 \cdots \gamma^{\bar{\mu}} \cdots \gamma^4 = -e_1e_2e_3e_4e_{\mu} \Gamma^\bar{\mu}\).

**APPENDIX B. FULL COMPUTATION OF ONE CHORD DIAGRAM**

Since \(t = 2n = 6\) are constant in this section, we drop the subindices \(n\) in \(a_n, b_n\) and \(s_n\). Exclusively in this appendix, we abbreviate the traces as follows:
\[
|\mu_i \mu_j \ldots \mu_m| := \text{Tr}_V (K_{\mu_i}K_{\mu_j} \cdots K_{\mu_m}) \quad \mu_i, \mu_j, \ldots, \mu_m \in \{1, 2\}.
\]
Then the action functional \(a(\chi)\) of a chord diagram \(\chi\) of six points is given by
\[
a(\chi) = \sum_{\mu_1 \mu_2 \ldots \mu_6} (-1)^{cr(\chi)} \left( \prod_{w \prec v} g^{\mu_w \mu_v} \right) \left( \sum_{\gamma \in \Psi_6} \left[ \prod_{i \in \gamma} e_{\mu_i} \right] \cdot |\mu(\gamma^c)| \cdot |\mu(\gamma)| \right),
\]
that is,
\[
\sum_{\mu_1 \mu_2 \ldots \mu_6} (-1)^{cr(\chi)} \prod_{w \prec v} g^{\mu_w \mu_v} \left[ \sum_{\gamma \in \Psi_6} \left[ \prod_{i \in \gamma} e_{\mu_i} \right] \cdot |\mu(\gamma^c)| \cdot |\mu(\gamma)| \right] + \sum_i e_i \left( |\mu_1 \mu_2 \ldots \mu_i \ldots \mu_6| + e |\mu_6 \mu_5 \ldots \mu_i \ldots \mu_1| \right) \cdot |\mu_i| + \sum_{i<j} e_i e_j \left( |\mu_1 \mu_2 \ldots \mu_i \ldots \mu_j \ldots \mu_6| + e |\mu_6 \mu_5 \ldots \mu_j \ldots \mu_i \ldots \mu_1| \right) \cdot |\mu_i \mu_j| + \sum_{i<j<k} e_i e_j e_k \left( |\mu_1 \mu_2 \ldots \mu_i \ldots \mu_j \ldots \mu_k \ldots \mu_6| \cdot |\mu_i \mu_j \mu_k| \right),
\]
where \(e = e_{\mu_1} \ldots e_{\mu_6} = e_{\mu_1} \ldots e_{\mu_6}.\) We just conveniently listed the terms corresponding to \(\gamma\) and \(\gamma^c\) together, in the first line displaying those with \(#\gamma = 0\) and \(#\gamma^c = 6\) (‘trivial partitions’); in the second \(#\gamma = 1\) or \(5\); on the third line \(#\gamma = 2\) or \(4\); the fourth line corresponds to the \(#\gamma = 3\) cases. We also used the fact that \(e_{\mu}\) is a sign \(\pm\), and that \(e \cdot e_{\mu} e_{\mu_j} \cdots e_{\mu_k}\) equals the product of the \(e_{\mu}\)’s with \(r \neq i, j, \ldots, v\), i.e. precisely those not appearing in \(e_{\mu_i} e_{\mu_j} \cdots e_{\mu_k}\). But since in the non-vanishing terms \(e\) implies a repetition of indices by pairs, \(e \equiv 1\) for non-vanishing terms. Then we gain a factor \(2\) for those terms (i.e. except for traces of three matrices) and \(a(\chi)\) is therefore given by
\[
\sum_{\mu_1 \mu_2 \ldots \mu_6} \left( (-1)^{cr(\chi)} \prod_{w \prec v} g^{\mu_w \mu_v} \right) : \left\{ \sum_{\mu_1 \mu_2 \ldots \mu_6} 2N|\mu_1 \mu_2 \ldots \mu_6| \right\} \quad (B.1a)
\]
\[
+ \sum_i 2e_i |\mu_1 \mu_2 \ldots \mu_i \ldots \mu_6| \cdot |\mu_i| \quad (B.1b)
\]
\[
+ \sum_{i<j} 2e_i e_j |\mu_1 \mu_2 \ldots \mu_i \ldots \mu_j \ldots \mu_6| \cdot |\mu_i \mu_j| \quad (B.1c)
\]
\[
+ \sum_{i<j<k} e_i e_j e_k |\mu_1 \mu_2 \ldots \mu_i \ldots \mu_j \ldots \mu_k \ldots \mu_6| \cdot |\mu_i \mu_j \mu_k| \right\}. \quad (B.1d)
\]
We thus compute the first diagram of 6-points by giving line by line last expression. We perform first the computation for the third line (B.1c) (since this is the longest) explicitly, which can be expanded as

\[
2 \sum_{\mu} \left[ e_{\mu_1} e_{\mu_2} [\mu_1 \mu_2] \cdot [\mu_3 \mu_4 \mu_5 \mu_6] + e_{\mu_1} e_{\mu_3} [\mu_1 \mu_3] \cdot [\mu_2 \mu_4 \mu_5 \mu_6] + e_{\mu_1} e_{\mu_5} [\mu_1 \mu_5] \cdot [\mu_2 \mu_3 \mu_4 \mu_6] + e_{\mu_2} e_{\mu_3} [\mu_2 \mu_3] \cdot [\mu_1 \mu_4 \mu_5 \mu_6] + e_{\mu_2} e_{\mu_4} [\mu_2 \mu_4] \cdot [\mu_1 \mu_3 \mu_5 \mu_6] + e_{\mu_2} e_{\mu_5} [\mu_2 \mu_5] \cdot [\mu_1 \mu_3 \mu_4 \mu_6] + e_{\mu_3} e_{\mu_4} [\mu_3 \mu_4] \cdot [\mu_1 \mu_2 \mu_5 \mu_6] + e_{\mu_3} e_{\mu_5} [\mu_3 \mu_5] \cdot [\mu_1 \mu_4 \mu_4 \mu_6] + e_{\mu_4} e_{\mu_5} [\mu_4 \mu_5] \cdot [\mu_1 \mu_2 \mu_3 \mu_6] + e_{\mu_5} e_{\mu_6} [\mu_5 \mu_6] \cdot [\mu_1 \mu_2 \mu_3 \mu_4] \right].
\]  

(B.2)

The diagram’s meaning is the sign and product of \( g^{e_{\nu} e_{\rho}} \)'s before the braces in eq. (B.1). After contraction with the term in square brackets in (B.2) one gets

\[
2 \sum_{\mu, \nu, \rho} (e_{\mu} e_{\nu} e_{\rho}) \left\{ e_{\rho} [\mu \nu] \cdot [\nu \rho \mu \rho] + e_{\rho} [\mu \nu] \cdot [\rho \mu \nu \rho] + e_{\rho} [\mu \nu] \cdot [\rho \nu \mu \rho] + e_{\rho} [\mu \nu] \cdot [\nu \rho \mu \rho] \right\}
\]  

(B.3)

where the signs \( e_{\mu} e_{\nu} e_{\rho} \) are due to \( g^{\alpha \beta} = e_{\lambda} \delta^\lambda_{\alpha \beta} \) (no sum). Following the notation of eq. (4.8), using the cyclicity of the trace and renaming indices, this expression can be written as

\[
2 \sum_{\mu, \nu, \rho} (6R_{\mu \nu \rho} + 6S_{\mu \nu \rho} + 2T_{\mu \nu \rho} + U_{\mu \nu \rho}) .
\]  

Similarly, for the terms obeying \( #\Upsilon \) or \( #\Upsilon^c = 1 \), i.e. line (B.1b), one has

\[
2 \sum_{\mu} \left[ e_{\mu_1} [\mu_1] \cdot [\mu_2 \mu_3 \mu_4 \mu_5 \mu_6] + e_{\mu_2} [\mu_2] \cdot [\mu_1 \mu_3 \mu_4 \mu_5 \mu_6] + e_{\mu_3} [\mu_3] \cdot [\mu_1 \mu_2 \mu_4 \mu_5 \mu_6] + e_{\mu_4} [\mu_4] \cdot [\mu_1 \mu_2 \mu_3 \mu_5 \mu_6] + e_{\mu_5} [\mu_5] \cdot [\mu_1 \mu_2 \mu_3 \mu_4 \mu_6] + e_{\mu_6} [\mu_6] \cdot [\mu_1 \mu_2 \mu_3 \mu_4 \mu_5] \right],
\]  

which amounts to

\[
2 \sum_{\mu, \nu, \rho} e_{\nu} e_{\rho} [\mu] \left\{ [\nu \rho \mu \rho] + [\nu \rho \mu \rho] + [\nu \rho \mu \rho] + [\nu \rho \mu \rho] + [\nu \rho \mu \rho] \right\},
\]  

or, relabeling, to

\[
2 \sum_{\mu, \nu, \rho} (4O_{\mu \nu \rho} + 2P_{\mu \nu \rho}) .
\]  

(B.1b')
The terms with \( \# \Upsilon = 3 \) remain to be computed:

\[
\sum_{\mu} \mu_1 \mu_2 \mu_3 \times \left\{ (e_{\mu_1} e_{\mu_2} e_{\mu_3} + e_{\mu_4} e_{\mu_5} e_{\mu_6}) |\mu_1 \mu_2 \mu_3| \cdot |\mu_4 \mu_5 \mu_6| \\
\quad + (e_{\mu_1} e_{\mu_2} e_{\mu_4} + e_{\mu_3} e_{\mu_5} e_{\mu_6}) |\mu_1 \mu_2 \mu_4| \cdot |\mu_3 \mu_5 \mu_6| \\
\quad + (e_{\mu_1} e_{\mu_2} e_{\mu_5} + e_{\mu_3} e_{\mu_4} e_{\mu_6}) |\mu_1 \mu_2 \mu_5| \cdot |\mu_3 \mu_4 \mu_6| \\
\quad + (e_{\mu_1} e_{\mu_3} e_{\mu_6} + e_{\mu_2} e_{\mu_5} e_{\mu_4}) |\mu_1 \mu_3 \mu_6| \cdot |\mu_2 \mu_5 \mu_4| \\
\quad + (e_{\mu_1} e_{\mu_2} e_{\mu_6} + e_{\mu_3} e_{\mu_4} e_{\mu_5}) |\mu_1 \mu_2 \mu_6| \cdot |\mu_3 \mu_4 \mu_5| \\
\quad + (e_{\mu_1} e_{\mu_3} e_{\mu_5} + e_{\mu_2} e_{\mu_4} e_{\mu_6}) |\mu_1 \mu_3 \mu_5| \cdot |\mu_2 \mu_4 \mu_6| \\
\quad + (e_{\mu_1} e_{\mu_4} e_{\mu_5} + e_{\mu_2} e_{\mu_3} e_{\mu_6}) |\mu_1 \mu_4 \mu_5| \cdot |\mu_2 \mu_3 \mu_6| \\
\quad + (e_{\mu_1} e_{\mu_4} e_{\mu_6} + e_{\mu_2} e_{\mu_3} e_{\mu_5}) |\mu_1 \mu_4 \mu_6| \cdot |\mu_2 \mu_3 \mu_5| \\
\quad + (e_{\mu_1} e_{\mu_5} e_{\mu_6} + e_{\mu_2} e_{\mu_3} e_{\mu_4}) |\mu_1 \mu_5 \mu_6| \cdot |\mu_2 \mu_3 \mu_4| \right\} .
\]

Although \( \text{Tr}_N(M_1 M_2 M_3) = \text{Tr}_N(M_3 M_2 M_1) \) is false for general matrices \( M_1, M_2 \)
and \( M_3 \) (e.g. for \( M_j = \sigma_j \), the Pauli matrices), having at our disposal only two
matrices, \( K_1 \) and \( K_2 \), the relation \( \text{Tr}_N(K_1 K_2 K_3) = \text{Tr}_N(K_2 K_1 K_3) \) does hold.
This fact was used to obtain the last equation. Contracting with the diagram, as
we already did for other partitions, one gets

\[
\sum_{\mu, \nu, \rho} 8V_{\mu \nu \rho} + 12W_{\mu \nu \rho} .
\]

By collecting the terms from the three equations with primed tags, the bi-trace
term for the I-diagrams one obtains

\[
b(I) = +2 \sum_{\mu, \nu, \rho} \left( 4O_{\mu \nu \rho} + 2P_{\mu \nu \rho} + 6R_{\mu \nu \rho} + 6S_{\mu \nu \rho} \\
+ 2T_{\mu \nu \rho} + U_{\mu \nu \rho} + 4W_{\mu \nu \rho} \right) ,
\]

which is a claim amid the proof of Proposition 4.1.

Notice that these integer coefficients add up to 62, and so will these (denoted
\( p_\chi, q_\chi, \ldots, w_\chi \) in the main text) in absolute value for a general diagram \( \chi \). There
are two missing terms to get the needed \( 2^6 = \# \mathcal{P}_6 \) terms. These are the trivial
cases \( \Upsilon, \Upsilon^c = \emptyset \), which can be readily computed.

For the I-diagram,

\[
s(I) = 2N \cdot \sum_{\mu} \mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \times |\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6|
\]

\[
= 2N \cdot \sum_{\mu, \nu, \rho} e_{\mu} e_{\nu} e_{\rho} |\mu \nu \rho| \\
= 2N \cdot \text{Tr}_N \left\{ e_1 K_1^6 + 2e_2 (K_1 K_2)^2 K_1^2 + e_1 (K_2 K_1)^2 \\
+ e_2 K_2^6 + 2e_1 (K_2 K_1)^2 K_2^2 + e_2 (K_1 K_2)^2 \right\} .
\]
The single-trace action $S_6$ in Proposition 4.1 is then obtained by summing over all 6-point chord diagrams $\sum_\chi s(\chi)$, whose values are found by similar computations.

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