NEARLY ALL k-SAT FUNCTIONS ARE UNATE

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Abstract. We prove that \(1 - o(1)\) fraction of all k-SAT functions on \(n\) Boolean variables are unate (i.e., monotone after first negating some variables), for any fixed positive integer \(k\) and as \(n \to \infty\). This resolves a conjecture by Bollobás, Brightwell, and Leader from 2003.

1. Introduction

1.1. Background. We study the following basic question on Boolean functions:

How many k-SAT functions on \(n\) Boolean variables are there? What does a typical such function look like?

This question was first studied by Bollobás, Brightwell, and Leader [8]. We focus on the regime where \(k\) is fixed and \(n \to \infty\). We count k-SAT functions in their disjunctive normal form (DNF). It would be an equivalent problem to enumerate k-SAT functions in their conjunctive normal form (CNF) since the negation of a DNF is a CNF. For our purpose, a k-SAT function on \(n\) Boolean variables is a function \(f: \{0,1\}^n \to \{0,1\}\) of the form

\[f(x_1, \ldots, x_n) = C_1 \lor C_2 \lor \cdots \lor C_m,\]

where each \(C_i\) has the form \(z_1 \land \cdots \land z_k\) with \(z_1, \ldots, z_k \in \{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n\}\). Here we call \(x_1, \ldots, x_n\) the variables. Each of \(x_i\) and \(\overline{x}_i\) is a literal (positive literal and negative literal, respectively). Each clause \(C_i\) is a conjunction (“and”) of \(k\) literals. We further restrict that every clause uses \(k\) distinct variables (e.g., both \(x_1 \land x_2\) and \(x_1 \land x_2 \land \overline{x}_2\) are invalid 3-SAT clauses). This restriction does not lose any generality (the first example can be replaced by \((x_1 \land x_2 \land x_3) \lor (x_1 \land x_2 \land \overline{x}_3)\) and the second example is a clause that is never satisfied and so can be deleted). To simplify notation, we will drop the “and” symbol \(\land\) when writing a clause. A formula is a set of clauses. For instance, the 2-SAT formula \((x_1\land x_2) \lor (\overline{x}_1 \land x_3) \lor (x_3 \land x_4)\) is written as \(\{x_1x_2, \overline{x}_1x_3, x_3x_4\}\). Every k-SAT function has a k-SAT formula, but different k-SAT formulae may correspond to the same k-SAT function.

Given the importance of k-SAT functions, we study the typical structure of such a function. While the total number of functions \(\{0,1\}^n \to \{0,1\}\) is \(2^{2n}\), the number of k-SAT functions is significantly smaller. As an easy upper bound, since there are \(2^k \binom{n}{k}\) possible clauses, the number of k-SAT formulae is \(2^k \binom{n}{k}\). Meanwhile, the number of k-SAT functions is at most \(2^{2^k \binom{n}{k}}\), which is significantly smaller than \(2^{2^n}\) for a fixed \(k\) and large \(n\). The actual number of k-SAT functions turns out to be considerably smaller than even this upper bound.

A k-SAT formula is monotone if it only uses positive literals. A k-SAT function is monotone if it has a monotone k-SAT formula. There are \(\binom{n}{k}\) possible monotone clauses, and every monotone k-SAT formula produces a unique monotone k-SAT function, hence there are \(2^{\binom{n}{k}}\) monotone k-SAT functions.

Balogh was supported in part by NSF grants DMS-1764123 and RTG DMS-1937241, FRG DMS-2152488, the Arnold O. Beckman Research Award (UIUC Campus Research Board RB 22000), the Langan Scholar Fund (UIUC).

Lidický was supported in part by NSF grant FRG DMS-2152490 and Scott Hanna professorship.

Mani was supported by the NSF Graduate Research Fellowship Program and a Hertz Graduate Fellowship.

Zhao was supported in part by NSF CAREER award DMS-2044606, a Sloan Research Fellowship, and the MIT Solomon Buchsbaum Fund.
A \(k\)-SAT function or formula is unate if it is monotone after replacing some variables with their negations (e.g., \(\overline{x_1} x_2, x_1 \overline{x_2}, x_2 x_3\) is unate but \(\overline{x_1} x_2, x_1 x_2, x_2 x_3\) is not). In other words, a formula is unate if each variable \(x_i\) shows up only as a positive literal \(x_i\) or only as a negative literal \(\overline{x_i}\). A \(k\)-SAT function is unate if it admits a unate formula. The number of unate \(k\)-SAT formulae that use all \(n\) variables is at least

\[
2^n \left( \binom{n}{k} - n2^{\binom{n-1}{k}} \right) = (1 + o(1))2^{n\binom{n}{k}},
\]

for fixed \(k\) as \(n \to \infty\). Indeed, for each variable \(x_i\), there are \(2^n\) choices as to whether to use it as a positive literal or a negative literal and there are at least \(2^{\binom{n}{k}} - n2^{\binom{n-1}{k}}\) monotone formulae that use all \(n\) variables. All unate \(k\)-SAT formulae represent distinct functions, and thus the number of unate \(k\)-SAT functions on \(n\) variables is at least \((1 + o(1))2^{n\binom{n}{k}}\).

Bollobás, Brightwell, and Leader [8] conjectured in 2003 that nearly all \(k\)-SAT functions are unate. We prove their conjecture.

**Theorem 1.1.** Fix \(k \geq 2\). The number of \(k\)-SAT functions on \(n\) Boolean variables is \((1 + o(1))2^{n\binom{n}{k}}\). Equivalently: a \(1 - o(1)\) fraction of all \(k\)-SAT functions on \(n\) variables are unate.

Bollobás, Brightwell, and Leader also proposed a weaker version of this conjecture, namely that the number of \(k\)-SAT functions on \(n\) Boolean variables is \(2^{(1+o(1))\binom{n}{k}}\), and established this weaker conjecture for \(k = 2\). Theorem 1.1 was proved for \(k = 2\) by Allen [1] and for \(k = 3\) by Ilinca and Kahn [18]. The proofs in [8, 1] for \(k = 2\) used the Szemerédi Regularity Lemma [26], ([17] gave an alternate regularity-free proof for \(k = 2\)), whereas the proof for \(k = 3\) [18] used the hypergraph regularity method.

To prove Theorem 1.1, we use hypergraph container and stability arguments to reduce Theorem 1.1 to a Turán density problem for partially directed hypergraphs. We then solve this Turán problem. The hypergraph container approach is simpler to carry out than the earlier regularity approaches. One of our insights is distilling the “right” Turán problem to be solved.

**Acknowledgments.** This paper integrates results presented at SODA ’22 by a subset of the authors (Dong, Mani, and Zhao), as well as STOC ’23 by all the authors. The reduction to the Turán problem, encompassing all sections other than Section 2, originates from the work presented at SODA ’22. Zhao presented these results at an Oberwolfach workshop in April 2022, sparking the collaboration that culminated in proving the main conjecture by solving the Turán problem (Section 2).

Zhao expresses gratitude to Jeff Kahn, who first introduced this problem during Zhao’s graduate studies, for his encouragement to pursue this challenge.

1.2. **Strategy.** Our setup is closest to that of Ilinca and Kahn [18] who solved the problem for 3-SAT functions. Instead of counting functions, we count minimal formulae. In [18], the term “non-redundant” is used for what we are calling “minimal.”

**Definition 1.2** (Minimal formula). A formula \(G = \{C_1, C_2, \ldots, C_l\}\) on variables \(\{x_1, \ldots, x_n\}\) is minimal if deleting any clause from \(G\) changes the resulting Boolean function. Equivalently, \(G\) is minimal if for each clause \(C_i \in G\), there is a witness assignment \(w \in \{0,1\}^n\) that satisfies \(\overline{C_i}\) but no other clause \(C_j \in G, j \neq i\).

**Example 1.3.** The 2-SAT formula \(\{wx, wy, x\overline{z}, \overline{yz}\}\) is not minimal since it is impossible to satisfy only \(wx\) and no other clause. Indeed, if we attempt to only satisfy \(wx\), we must assign \(w = 1\) and \(x = 1\); to avoid satisfying \(wy\) and \(x\overline{z}\), we must assign \(y = 0\) and \(z = 1\), which would then lead to the final clause \(\overline{yz}\) being satisfied.

Every \(k\)-SAT function can be expressed as a minimal formula, but possibly in more than one way. We upper bound the number of minimal \(k\)-SAT formulae, which in turn upper bounds the
number of $k$-SAT functions. Thus, to prove Theorem 1.1, we actually prove the following stronger statement.

**Theorem 1.4.** Let $k \geq 2$ be a fixed positive integer. The number of minimal $k$-SAT formulae on $n$ Boolean variables is $(1 + o(1))2^{n + \binom{n}{k}}$. Equivalently: a $1 - o(1)$ fraction of all minimal $k$-SAT formulae on $n$ variables are unate.

To upper bound the number of minimal $k$-SAT formulae, we identify a fixed finite set $B$ of “forbidden” non-minimal formulae. We then upper bound the number of $B$-free formulae, which are formulae (not necessarily minimal) not containing any subformula isomorphic to any element of $B$.

The problem of counting $B$-free formulae is analogous to counting $F$-free graphs on $n$ vertices. A classic result by Erdős, Kleitman, and Rothschild [12] shows that almost all triangle-free graphs are bipartite. Erdős, Frankl, and Rödl [11] generalized this result and showed that for a fixed graph $F$, the number of $n$-vertex $F$-free graphs is $2^{o(n^2)}$. The latter result was initially proved using the Szemerédi Regularity Lemma [26], i.e., the graph regularity method. It can also be proved as a quick application of the more recently developed hypergraph container method [3, 24] (see also the survey [4]).

Here is a quick sketch of how to use the container method to enumerate $n$-vertex triangle-free graphs. By an application of the hypergraph container theorem, there is a collection $G$ of $n$-vertex graphs (“containers”) each with at most $(1/4 + o(1))n^2$ edges, such that $|G| = 2^{o(n^2)}$ and every $n$-vertex triangle-free graph is a subgraph of some container $G \in G$. It then follows that the number of $n$-vertex triangle-free graphs is at most $\sum_{G \in G} 2^{e(G)} = 2^{(1/4+o(1))n^2}$. See Section 3.3 for more details.

As in the case of counting $F$-free graphs, the enumeration of $B$-free formulae reduces to a certain extremal problem. Here the situation diverges from the graph theory setting. For counting triangle-free graphs, the container method reduces the enumeration problem to Mantel’s theorem on the maximum number of edges in a triangle-free graph. It is more intricate to count $k$-SAT functions. We identify a structure that we call partially directed hypergraphs to cleanly capture the essence of the problem. Also, unlike counting $F$-free graphs, our extremal problem is not the direct analogue of the maximum size of an $F$-free graph.

We now begin to explain this reduction.

**Definition 1.5** (Simple formula). A $k$-SAT formula is simple if no two clauses use exactly the same set of $k$ variables.

**Example 1.6.** The 3-SAT formula $\{wxy, wxz, xyz\}$ is simple, whereas $\{xyz, x\overline{y}z\}$ is not.

Every minimal $k$-SAT formula on $n$ variables can be made simple by removing $o(n^k)$ clauses (Proposition 4.3). We then observe the following steps:

$$
\# \{k\text{-SAT functions}\} \leq \# \{\text{minimal } k\text{-SAT formulae}\}
$$

$$
\sim \# \{\text{simple minimal } k\text{-SAT formulae}\}
$$

$$
\leq \# \{\text{simple } B\text{-free } k\text{-SAT formulae}\}.
$$

Recall that $B$ is some collection of non-minimal formulae. Let us focus on enumerating simple $B$-free formulae. The container theorem gives us a collection $G$ of $n$-variable “container” formulae with $|G| = 2^{o(n^k)}$ such that every $n$-variable $B$-free formula is a subformula of some container $G \in G$. Furthermore, every $G \in G$ is “not too large” in a certain sense that we now describe. For a container $G \in G$, suppose $\alpha(n^k)$ $k$-subsets of variables support exactly one clause, and $\beta(n^k)$ $k$-subsets of variables support exactly two clauses. Let us ignore $k$-subsets of variables supporting more than two clauses since there are $o(n^k)$ of them (Lemma 4.11). For such a container $G$, how many simple subformulae of $G$ are there? For each $k$-subset of variables, if $G$ has exactly $m$ clauses supported on these $k$ variables, then to include at most one of the clauses in our choice of a simple subformula of
Given a pair of partially directed graphs $\vec{H}$ and $\vec{G}$, we say that $\vec{H}$ is a subgraph of $\vec{G}$ if one can obtain $\vec{H}$ from $\vec{G}$ by a combination of (1) removing vertices, (2) removing edges, and (3) removing the orientation of some edges.

The following partially directed graph plays a special role:

$\vec{T}_2 = \begin{array}{c}
\text{1} \\
\text{2}
\end{array}$

Below, the left graph contains $\vec{T}_2$ as a subgraph, and the right does not contain $\vec{T}_2$ as a subgraph.

The following statement implies Theorem 1.1 for $k = 2$. (See Theorem 1.14 for the full statement of the implication.)
Theorem 1.7. There is some constant $\theta > \log_2 3$ such that every $n$-vertex partially directed graph with $\alpha \binom{n}{2}$ undirected edges and $\beta \binom{n}{2}$ directed edges and not containing $\vec{T}_2$ as a subgraph satisfies

$$\alpha + \theta \beta \leq 1 + o_{n \to \infty}(1).$$

Now let us generalize the problem to hypergraphs. While there are many possible notions of a directed hypergraph, the relevant notion for us is the one where a directed edge is an edge with a pointed vertex on the edge. A partially directed hypergraph is formed from a hypergraph by directing some of its edges and leaving others intact.

**Definition 1.8 ($k$-PDG).** A partially directed $k$-graph, abbreviated $k$-PDG, is given by a set $V$ of $n$ vertices, and, for each unordered $k$-element subset $\{v_1, \ldots, v_k\}$, one of the following possibilities:

(a) No edge with vertices $\{v_1, \ldots, v_k\}$, or
(b) An undirected edge with vertices $\{v_1, \ldots, v_k\}$, or
(c) A directed edge using vertices $\{v_1, \ldots, v_k\}$ along with a choice of some $v_i \in \{v_1, \ldots, v_k\}$; we say that the edge is directed (or pointed) toward $v_i$. We notate such a directed edge by $v_1 \cdots \check{v}_i \cdots v_k$.

In particular, no two edges (whether directed or undirected) of a $k$-PDG can use the exact same set of $k$ vertices. We usually denote $k$-PDGs by simply writing their edge sets.

**Example 1.9.** This is the edge set of a 3-PDG: $\{123, 12\check{4}, 134, 235, 245\}$. Non-examples of edge sets of 3-PDGs include $\{123, 12\check{3}\}$ (two edges using the same triple of vertices), $\{1\check{2}\}$ (not a valid edge), and $\{12, 1\check{2}\}$ (not 3-uniform).

The following $k$-PDG plays a special role.

**Definition 1.10 ($\vec{T}_k$).** For each $k \geq 3$, define $\vec{T}_k$ to be the $k$-PDG obtained by starting with $\vec{T}_2$ defined earlier and then adding $k - 2$ common vertices to all three edges.

There are some examples:

$$\vec{T}_3 = \begin{array}{c}
\text{edges = \{123, 12\check{4}, 134\};}
\end{array}$$

$$\vec{T}_4 = \begin{array}{c}
\text{edges = \{1234, 123\check{5}, 1245\}.}
\end{array}$$

Given a $k$-PDG $\vec{H}$, we write

$$e_u(\vec{H}) = \text{the number of undirected edges in } \vec{H}, \text{ and}$$

$$e_d(\vec{H}) = \text{the number of directed edges in } \vec{H}.$$

Denote the edge densities by

$$\alpha(\vec{H}) = \frac{e_u(\vec{H})}{\binom{n}{k}} \text{ and } \beta(\vec{H}) = \frac{e_d(\vec{H})}{\binom{n}{k}}.$$

**Definition 1.11 (Subgraph of $k$-PDG).** We say that a $k$-PDG $\vec{F}$ is a subgraph of another $k$-PDG $\vec{H}$ if we can obtain a $k$-PDG isomorphic to $\vec{F}$ starting from $\vec{H}$ by some sequence of actions of the following types: (a) deleting an edge, (b) deleting a vertex, and (c) forgetting the direction of an edge.
**Definition 1.12** (*$\vec{F}$-free $k$-PDG*). Given $k$-PDG $\vec{F}$, we say that a $k$-PDG $\vec{H}$ is $\vec{F}$-free if $\vec{H}$ does not contain $\vec{F}$ as a subgraph.

**Definition 1.13** (Turán density for $k$-PDGs). Given a $k$-PDG $\vec{F}$, let $\pi(\vec{F}, \theta)$ be the smallest real number such that every $n$-vertex $\vec{F}$-free $k$-PDG $\vec{H}$ satisfies

$$\alpha(\vec{H}) + \theta \beta(\vec{H}) \leq \pi(\vec{F}, \theta) + o_{n \to \infty}(1).$$

The Turán problem for $\vec{T}_k$-free $k$-PDGs will play a central role, as $\vec{T}_k$ encodes non-minimal formulae that turn out to be critical for the $k$-SAT enumeration problem. Note that for every $k$ and $\theta$ we always have

$$\pi(\vec{T}_k, \theta) \geq 1,$$

since the complete undirected $k$-PDG is $\vec{T}_k$-free. (This is related to the fact that every monotone formula is minimal.) Also, note the trivial inequality $\pi(\vec{T}_k, \theta) \leq \pi(\vec{T}_k, \theta')$ for $\theta < \theta'$.

### 1.4. Results

We prove the following two results, which together imply Theorem 1.1.

**Theorem 1.14** (Reduction to Turán problem). Let $k \geq 2$. If $\pi(\vec{T}_k, \theta) = 1$ for some $\theta > \log_2 3$, then Theorem 1.1 and Theorem 1.4 are true for this $k$.

**Theorem 1.15** (Turán problem). For every $k \geq 2$, there is some constant $\theta > \log_2 3$ such that

$$\pi(\vec{T}_k, \theta) = 1.$$  

The proof of Theorem 1.15 is in Section 2. The proof of Theorem 1.14 occupies the rest of the paper starting in Section 3. On our way to proving Theorem 1.14, by the end of Section 4, we establish the following weaker conclusion.

**Theorem 1.16** (Weak reduction). For $k \geq 2$, the number of $n$-variable $k$-SAT functions is at most

$$2^{(\pi(\vec{T}_k, \log_2 3)+o(1))(\binom{n}{k})}.$$  

Theorems 1.15 and 1.16 already show that the number of $k$-SAT functions is $2^{(1+o(1))\binom{n}{k}}$ for all $k \geq 2$. It is easier to establish this weaker conclusion since it does not require the technical stability arguments that occupy the second half of the paper starting with Section 5. Most of the new ideas in our paper are contained in the proof of this weaker conclusion.

We prove Theorem 1.15 by reducing it to a simple though mysterious looking inequality. While this inequality is easy to check by hand, its discovery required a computer search, drawing inspiration from Razborov’s innovative flag algebras method [23]. Notably, our approach diverges from many other flag algebra applications commonly seen in extremal graph and hypergraph problems, as our proof certificate is remarkably concise and readily verifiable without computer assistance.

Despite the extensive study of hypergraph Turán density problems, definitive solutions have remained elusive, with only particular cases being resolved. For further insights into hypergraph Turán problems, we recommend Keevash’s comprehensive survey [21].

**Remark 1.17** (Earlier version). An earlier version of this work, available at arXiv:2107.09233v3, contains additional arguments and results that do not appear in this version since they are no longer necessary for proving Theorem 1.1. Here we summarize some these elements that might have potential future value.

First, we identify a family of forbidden $k$-PDGs that completely characterize the asymptotic enumeration of minimal $k$-SAT formulae. This means that the enumeration of $k$-SAT functions is equivalent to the Turán problem with this larger family of forbidden subgraphs. It turns out, as we now know from Theorem 1.15, that forbidding $\vec{T}_k$ alone is sufficient, although this was not clear a priori. It is possible that the additional forbidden configurations (that are no longer mentioned in this paper) may play a critical role in further extensions of this problem discussed next.
We also deleted a proof of Theorem 1.15 for \( k = 4 \) via an interesting argument that applied a recent result by Füredi and Maleki [14], as well as an appendix (by Nitya Mani and Edward Yu) that gives a computer verification of the \( k = 5 \) case of Theorem 1.15.

### 1.5. Further directions.

As suggested by Bollobás, Brightwell, and Leader [8], this investigation opens doors to a theory of random \( k \)-SAT functions. For example, Theorem 1.4 implies that a typical \( k \)-SAT function admits a unique minimal \( k \)-SAT formula, and furthermore this formula has \((1/2 + o(1))(\binom{n}{k})\) clauses. Note that our model is very different from that of random \( k \)-SAT formulae where clauses are added at random (e.g., the recent breakthrough on the satisfiability conjecture [10]). Rather, Theorem 1.4 concerns a random \( k \)-SAT formula conditioned on minimality. In this light, our results are analogous to the theory of dense Erdős–Rényi random graphs. In random graph theory, understanding the typical structure of a dense triangle-free graph has led to many fruitful research directions concerning typical structures and thresholds, and inspired important combinatorial techniques such as the hypergraph container method. One such direction concerns the typical structure of sparser triangle-free graphs. Another direction concerns the typical structure of a graph without a \( t \)-clique, when \( t \) increases with \( n \). We refer to [5, 6] and their references for discussions of the extensive work on these and related problems. It is natural to ask similar questions about \( k \)-SAT functions.

**Question 1.18.** Fix \( k \). For a given \( m = m(n) \), what is the typical structure of a \( k \)-SAT function on \( n \) Boolean variables that can be expressed as a formula with \( m \) clauses? What is the typical structure of a minimal \( k \)-SAT formula with \( n \) Boolean variables and \( m \) clauses? What is the threshold \( m_0 \) above which a typical such formula is unate?

Another interesting further direction concerns large \( k \). Bollobás and Brightwell [7] studied the \( k \)-SAT function enumeration problem for large \( k = k(n) \). They conjectured that as long as \( k \leq (1/2 - c)n \) for some constant \( c > 0 \), the number of \( k \)-SAT functions on \( n \)-variables is \( 2^{(1+o(1))(\binom{n}{k})} \). In that paper [7], they proved bounds on the number of \( k \)-SAT functions on \( n \) variables in the regime \( k \geq n/2 \), where a completely different asymptotic behavior arises.

**Question 1.19.** For a given \( k = k(n) \), how many \( k \)-SAT functions on \( n \) Boolean variables are there? What about minimal \( k \)-SAT formula? What is the threshold \( k_0(n) \) below which it is typically unate?

A significant bottleneck here is the quantitative bounds of the container lemma. Even with a more efficient container lemma [6], it seems difficult to analyze what happens when \( k \) grows faster than \( \log n \), let alone linear in \( n \).

### 1.6. Outline.

The rest of the paper can be divided into three parts.

**Part I. Turán problem.**

In Section 2, we prove Theorem 1.15 (the Turán density problem).

**Part II. Exponential asymptotics.**

In Section 3, we recall some tools for hypergraphs, including the hypergraph container theorem. In Section 4, we prove Theorem 1.16 giving an upper bound on the number of \( k \)-SAT functions.

**Part III. Stability.**

Finally, we prove Theorem 1.14 via a technical stability argument. In Section 5, by a more careful analysis of the containers, we reduce the problem to showing that there are negligibly many non-unate but nearly unate minimal formulae. This final claim is then established in Section 6 by extending the arguments in [18, Section 8].

### 2. A Turán density problem

In this section, we prove Theorem 1.15. We quote the following lemma.
Lemma 2.1 (Füredi [13, Lemma 2.1]). Given a graph $G = (V, E)$, let $G^2$ be the graph on vertex set $V$ with $xy \in E(G^2)$ if and only if there is some $z \in V$ with $xz, yz \in E$. Then, for any graph $G$,  
\[ e(G^2) \geq e(G) - \lfloor n/2 \rfloor. \]

Theorem 2.2. $\pi(T_2, 2) = 1$.

Proof. Let $\tilde{H}$ be $T_2$-free 2-PDG on $n$ vertices. Let $\alpha = \alpha(\tilde{H})$ and $\beta = \beta(\tilde{H})$. Let $G$ be the underlying simple graph of $\tilde{H}$ (the graph obtained from $\tilde{H}$ by replacing directed edges with simple edges). Then $e(G) = (\alpha + \beta)\binom{n}{2}$. We say that edge $xy \in E(G)$ is a triangular edge if there exists some $z \in V(G)$ so that $xz, yz \in E(G)$. Only non-triangular edges in $G$ can arise as directed edges in $\tilde{H}$, i.e., $|E(G) - E(G^2)| \geq \beta\binom{n}{2}$. We further note that all triangular edges in $G$ are in $G^2$, i.e., a pair of vertices either forms an edge in $G^2$, or in $G - G^2$, or in neither of them. Thus by Lemma 2.1,  
\[ \binom{n}{2} \geq e(G^2) + \beta\binom{n}{2} \geq e(G) + \beta\binom{n}{2} - O(n) = (\alpha + 2\beta)\binom{n}{2} - O(n), \]
which implies that $\alpha + 2\beta \leq 1 + o(1)$.

Theorem 2.3. $\pi(T_3, 5/3) = 1$.

Proof. Let $\tilde{H}$ be a $T_3$-free 3-PDG on vertex set $V$. Let $\alpha = \alpha(\tilde{H})$ and $\beta = \beta(\tilde{H})$. For each vertex $v \in V$, the link of $v$, denoted $\tilde{H}_v$, is the 2-PDG with vertex set $V \setminus \{v\}$ and with edges obtained by taking all edges of $\tilde{H}$ containing $v$ and then deleting the vertex $v$ from each edge. We preserve all the direction data, except that every edge directed toward $v$ in $\tilde{H}$ becomes an undirected edge in $\tilde{H}_v$. Write $\alpha_v = \alpha(\tilde{H}_v)$ and $\beta_v = \beta(\tilde{H}_v)$. By linearity of expectation over a uniform random vertex $v \in V$,  
\[ E_v\alpha_v = \alpha + \beta/3 \quad \text{and} \quad E_v\beta_v = 2\beta/3. \]
For every $v \in V$, $\tilde{H}_v$ is $T_2$-free (for otherwise combining with $v$ forms a $T_3$), so $\alpha_v + 2\beta_v \leq 1 + o(1)$ by Theorem 2.2. Hence $\alpha + 5\beta/3 = E_v[\alpha_v + 2\beta_v] \leq 1 + o(1)$.  

We only consider $k \geq 4$ from now on. Here is the key lemma.

Lemma 2.4. Let $k \geq 4$ be a positive integer. There exist $\theta > \log_2 3$ and $a, b \in \mathbb{R}$ (depending on $k$) such that the following holds. Suppose $\tilde{F}$ is a $T_k$-free $k$-PDG on $k+1$ vertices. Let $x_1, \ldots, x_{k-1}, y, z$ be a permutation of vertices of $\tilde{F}$ chosen uniformly at random.

Define $x := x_1 \ldots x_{k-1}$, let $xy$ denote the event that $x_1 \ldots x_{k-1}y$ forms an undirected edge in $\tilde{F}$, let $\tilde{xy}$ denote the event that $x_1 \ldots x_{k-1}\tilde{y}$ is a directed edge in $\tilde{F}$, and let $\tilde{yz}$ denote the event that there is no edge with vertices $x_1, \ldots, x_{k-1}, y, z$ in $\tilde{F}$. Then, we have that  
\[ \mathbb{P}(xy) + k\theta \mathbb{P}(\tilde{xy}) + a^2 \mathbb{P}(\tilde{xy} \land xz) - 2ab \mathbb{P}(\tilde{xy} \land x\tilde{z}) + b^2 \mathbb{P}(\tilde{xy} \land \tilde{z}x) \leq 1. \]

Remark 2.5. The proof gives $\theta = 1 + \frac{1}{\sqrt{2}} \geq 1.707 > 1.585 > \log_2 3$, with $a = \frac{1}{\sqrt{2}}$, $b = \frac{k(\theta - 1) - 1}{\sqrt{2}} = \frac{k}{2} - \frac{1}{\sqrt{2}}$.

Theorem 2.6. There exists $\theta > \log_2 3$ such that $\pi(T_k, \theta) = 1$ for all $k \geq 4$.

Proof. Let $\tilde{H}$ be an $n$-vertex $T_k$-free $k$-PDG with $\binom{n}{k}$ undirected edges and $\binom{n}{k-1}$ directed edges.

Let $x_1, \ldots, x_{k-1}, y, z$ be vertices of $\tilde{H}$ chosen without replacement uniformly at random. Applying Lemma 2.4 (by first conditioning on the set of $k + 1$ randomly selected vertices) and using the notation of the lemma,  
\[ \mathbb{P}(xy) + k\theta \mathbb{P}(\tilde{xy}) + a^2 \mathbb{P}(\tilde{xy} \land xz) - 2ab \mathbb{P}(\tilde{xy} \land x\tilde{z}) + b^2 \mathbb{P}(\tilde{xy} \land \tilde{z}x) \leq 1. \]

Note that $\mathbb{P}(xy) = \alpha$ and $k\mathbb{P}(\tilde{xy}) = \beta$. It remains to show that  
\[ a^2 \mathbb{P}(\tilde{xy} \land xz) - 2ab \mathbb{P}(\tilde{xy} \land x\tilde{z}) + b^2 \mathbb{P}(\tilde{xy} \land \tilde{z}x) \geq -o(1). \]
We will show that this inequality holds for every fixed choice of \( x \). Conditioned on \( x \), vertices \( y \) and \( z \) are uniformly chosen without replacement in \( V(\bar{F}) \setminus \{x_1, \ldots, x_k\} \). When \( n \) is large, this is not much different from choosing with replacement, in which case \( y \) and \( z \) would be conditionally independent, given \( x \). In particular, \( \mathbb{P}(xy \wedge x\bar{z} | x) = \mathbb{P}(xy | x)^2 + o(1) \), and \( \mathbb{P}(xy \wedge x\bar{z} | x) = \mathbb{P}(xy | x)\mathbb{P}(x\bar{y} | x) + o(1) \), and \( \mathbb{P}(x\bar{y} \wedge x\bar{z} | x) = \mathbb{P}(x\bar{y} | x)^2 + o(1) \). Then we can prove the above displayed inequality, conditioned on any \( x \), by observing that

\[
\begin{align*}
&= a^2\mathbb{P}(xy \wedge x\bar{z} | x) - 2ab\mathbb{P}(xy \wedge x\bar{z} | x) + b^2\mathbb{P}(x\bar{y} | x) \\
&= a^2\mathbb{P}(xy | x)^2 - 2ab\mathbb{P}(xy | x)\mathbb{P}(x\bar{y} | x) + b^2\mathbb{P}(x\bar{y} | x)^2 - o(1) \\
&= (a\mathbb{P}(xy | x) - b\mathbb{P}(x\bar{y} | x))^2 - o(1) \geq -o(1). \quad \square
\end{align*}
\]

Finally, it remains to prove Lemma 2.4.

**Proof of Lemma 2.4.** Let \( \bar{H} \) be a \( \bar{T}_k \)-free \( k \)-PDG on \( k+1 \) vertices. Construct the following associated digraph \( D \) on the same vertex set as \( \bar{H} \):

1. For every directed edge in \( \bar{H} \) missing vertex \( i \) and directed towards vertex \( j \), add the directed edge \( i \to j \) in \( D \).
2. For every undirected edge in \( \bar{H} \) missing vertex \( i \), add the loop \( i \to i \) in \( D \).

Notice that every vertex in \( D \) has out-degree at most 1. Moreover, since \( \bar{H} \) does not contain \( \bar{T}_k \) as a subgraph, \( D \) is free of the following forbidden pattern:

- **Forbidden pattern:** \( i_1 \to i_2 \to * \) and \( i_3 \to * \) for three distinct vertices \( i_1, i_2, i_3 \), where * can be any vertex (the two *’s do not have to be the same).

Indeed, if \( D \) contains the above pattern, then \( \bar{H} \) has an edge missing \( i_2 \), an edge missing \( i_3 \), and an edge containing both \( i_2 \) and \( i_3 \) that is directed at \( i_2 \). These three edges contain \( \bar{T}_k \) as a subgraph.

It is easy to deduce the following exhaustive classification of all digraphs \( D \) on \( k+1 \) vertices with out-degree at most 1 at every vertex and without the above forbidden pattern.

- **(A)** \( i \to j \to i \) for distinct \( i, j \). No other edges.
- **(B)** \( i \to j \to j \) for distinct \( i, j \). No other edges.
- **(C)** \( i_1 \to i_2 \to i_3 \) for distinct \( i_1, i_2, i_3 \). No other edges.
- **(D)** For every edge of the form \( i \to j \) with \( i, j \) distinct, \( j \) has zero out-degree. (In other words, \( D \) is vertex-disjoint union of self-loops and stars in which all edges are directed toward the central vertex.)

Write \( i \to \emptyset \) to denote that \( i \) has out-degree 0 in \( D \). The inequality in Lemma 2.4 then translates to

\[
\mathbb{P}(z \to z) + k\theta \cdot \mathbb{P}(z \to y) + a^2\mathbb{P}(y \to \emptyset \wedge z \to \emptyset) \\
- 2ab\mathbb{P}(y \to z \wedge z \to \emptyset) + b^2\mathbb{P}(y \to z \wedge z \to y) \leq 1. \quad (1)
\]
Let $u$ and $d$ denote the number of undirected and directed edges in $\vec{H}$, respectively. We calculate the terms in inequality (1):

\[
\begin{align*}
\mathbb{P}(z \rightarrow z) &= \frac{u}{k+1}, \\
\mathbb{P}(z \rightarrow y) &= \frac{d}{(k+1)k}, \\
\mathbb{P}(\varnothing \rightarrow z \land z \rightarrow \varnothing) &= \frac{(k+1-u-d)(k-u-d)}{(k+1)k}, \\
\mathbb{P}(y \rightarrow z \land z \rightarrow \varnothing) &= \begin{cases} 
0 & \text{in cases (A), (B),} \\
\frac{1}{(k+1)k} & \text{in case (C),} \\
\frac{d}{(k+1)k} & \text{in case (D),} 
\end{cases} \\
\mathbb{P}(y \rightarrow z \land z \rightarrow y) &= \begin{cases} 
2 & \text{in case (A),} \\
0 & \text{in cases (B), (C), (D).}
\end{cases}
\end{align*}
\]

Then (1) reduces to simpler inequalities in each of the four cases:

(A) When our digraph is of the form $i \rightarrow j \rightarrow i$ for distinct $i, j$ with no other edges, (1) simplifies to

\[
\frac{2\theta}{k+1} + \frac{(k-1)(k-2)}{(k+1)k} a^2 + \frac{2}{(k+1)k} b^2 \leq 1. 
\]  (2)

(B) Since $\theta \geq 1$, the associated inequality in this case is implied by (A); we note the inequality below, but do not need to consider this case separately:

\[
\frac{1}{k+1} + \frac{\theta}{k+1} + \frac{(k-1)(k-2)}{(k+1)k} a^2 \leq 1.
\]

(C) The inequality in this case is also implied by (A) (we will end up choosing $a, b > 0$) so we also need not consider it separately:

\[
\frac{2\theta}{k+1} + \frac{(k-1)(k-2)}{k(k+1)} a^2 - \frac{2}{k(k+1)} ab \leq 1.
\]

(D) In this case, if $d = 0$ then $u \leq k+1$, and if $d \geq 1$ then $u + d \leq k$. The inequality reduces to

\[
\frac{u}{k+1} + \frac{\theta d}{k+1} + \frac{(k+1-u-d)(k-u-d)}{(k+1)k} a^2 - \frac{2d}{(k+1)k} ab \leq 1.
\]  (3)

Thus it remains to show that we can choose parameters $\theta > \log_2 3$ and $a, b > 0$ to make the inequalities implied by (A) and (D) true. We choose

\[
\theta = 1 + \frac{1}{\sqrt{2}} > 1.707, \quad \text{satisfying} \quad 2\theta^2 - 4\theta + 1 = 0,
\]

\[
a = \frac{1}{\sqrt{2}}, \quad b = \frac{k(\theta - 1) - 1}{\sqrt{2}} = \frac{k}{2} - \frac{1}{\sqrt{2}}.
\]

We can verify (2), establishing the desired inequality for case (A) (and thus cases (B) and (C)) via a direct substitution:
vertex in \( A \) neighborhoods by Füredi, Pikhurko, and Simonovits [15].

Conjecture 2.8. 

The following sense.

For each \( k \geq 2 \), what is the largest \( \theta \) such that \( \pi(\vec{T}_k, \theta) = 1 \)?

Theorem 2.2 confirms this conjecture for \( k = 2 \). The conjectured optimal construction is reminiscent of the solution to the hypergraph Turán problem corresponding to 3-graphs with independent neighborhoods by Füredi, Pikhurko, and Simonovits [15].

Conjecture 2.8. \( \pi(\vec{T}_k, (1 - 1/k)^{-k+1}) = 1 \) for every \( k \geq 2 \).

Therefore, (1) always holds.



\[
\frac{2\theta}{k+1} + \frac{(k-1)(k-2)}{(k+1)k} a^2 + \frac{2}{(k+1)k} b^2 - 1
\]

\[
= \frac{2\theta}{k+1} + \frac{(k-1)(k-2)}{2(k+1)k} + \frac{(k(\theta - 1) - 1)^2}{(k+1)k} - 1
\]

\[
= \frac{4k\theta + (k-1)(k-2) + 2(k(\theta - 1) - 1)^2 - 2(k+1)k}{2(k+1)k}
\]

\[
= \frac{4 - k + k^2(2\theta^2 - 4\theta + 1)}{2(k+1)k}
\]

To verify (3) for (D), observe that the left-hand side of (3) is non-decreasing in \( u \). Indeed, replacing \( u \) by \( u+1 \) increases the expression by at least

\[
\frac{1}{k+1} - \frac{(k+1)k - k(k-1)}{(k+1)k} a^2 = \frac{1}{k+1} - \frac{2}{k+1} a^2 = 0.
\]

Consequently, it remains to verify case (D) when \( u \) is as large as possible, meaning \( u+d \in \{k, k+1\} \), which makes the third term in (3) zero. The left-hand side of (3) becomes

\[
\frac{u}{k+1} + \frac{\theta d}{k+1} - \frac{2d}{(k+1)k} ab.
\]

If \( d = 0 \), then \( u = k+1 \), and the inequality clearly holds. Otherwise, \( d+u = k \). Since the above expression is linear in \( u \) (or equivalently, linear in \( d \)), the maximum is attained at one of endpoints \((u,d) = (k,0)\) or \((0,k)\). The only nontrivial situation to check is \((u,d) = (0,k)\), in which case the above expression is

\[
\frac{k\theta}{k+1} - \frac{2}{k+1} ab = \frac{k\theta}{k+1} - \frac{k(\theta - 1) - 1}{k+1} = 1.
\]

Therefore, (1) always holds.

Our work also motivates the following natural Turán problem for \( k \)-PDGs.

Problem 2.7. For each \( k \geq 2 \), what is the largest \( \theta \) such that \( \pi(\vec{T}_k, \theta) = 1 \)?

3. Hypergraph Tools

Remarks on asymptotic notation. We view \( k \geq 2 \) as a fixed constant and do not explicitly mention dependence on \( k \) in the hidden constants. We use standard asymptotic notation including \( O(\cdot) \) and \( \Omega(\cdot) \). We use subscripts to denote that the hidden constant factors may depend on the the subscripted parameters (along with \( k \), which is omitted). For example, \( f(n) = \Omega_B(g(n)) \) for some \( g > 0 \) means that there is some constant \( c = c(B,k) > 0 \) such that \( f(n) \geq cg(n) \) for all \( n \).
3.1. Densities of blowups. Given a $k$-SAT formula $G$, its $b$-blowup, denoted $G[b]$, is obtained from $G$ by replacing each variable by $b$ identical duplicates. For example, for the 3-SAT formula $G = \{x_1 x_2 x_3, x_1 \overline{x}_2 x_4\}$, we have

$$G[2] = \{x_1 x_2 x_3, x_1 x_2 x_3', x_1 x_2 x_3', x_1 x_2 x_3, x_1 x_2 x_3, x_1 x_2 x_3, x_1 x_2 x_3', x_1 x_2 x_3', x_1 x_2 x_3, x_1 x_2 x_3'\}. $$

By a standard Cauchy–Schwarz argument, we have the following.

Lemma 3.1. For every $\varepsilon > 0$ and $k$-SAT formula $F$, there is some $\delta > 0$ and $n_0$ such that for all $n > n_0$, if a $k$-SAT formula on $n$ variables has at least $\varepsilon n^v(F)$ copies of $F$, then it has at least $\delta n^{v(F)}$ copies of $F[2]$.

Proof. Suppose $G$ is a $k$-SAT formula on variables $x_1, \ldots, x_n$ that has at least $\varepsilon n^v(F)$ copies of $F$. Let $\ell = v(F)$. Define $Z = \{x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_n\}$ to be the set of $2n$ literals, and $V$ to be the disjoint union of $\ell$ copies of $Z$, i.e., $V = Z_1 \cup \cdots \cup Z_\ell$ where every $Z_j = \{x_j^{(0)}, \ldots, x_j^{(j)}, \overline{x}_j^{(0)}, \ldots, \overline{x}_j^{(j)}\}$.

Let $W = \{w_1, \ldots, w_\ell\}$ be the set of variables that appeared in $F$. Construct the associated $\ell$-graph $A_F$ on $V$ as follows: for every subformula $F' \subseteq G$ isomorphic to $F$ with isomorphism $\phi : W \to Z$, add edge $\phi(w_1)(1) \cdots \phi(w_\ell)(\ell)$ to $A_F$. Then $A_F$ is an $\ell$-partite $\ell$-graph with at least $\varepsilon n^\ell$ edges.

For $U_1 \subseteq Z_1, \ldots, U_\ell \subseteq Z_\ell$, we say that $A_F[U_1, \ldots, U_\ell]$ is complete if $u_1 \cdots u_\ell \in E(A_F)$ for all $u_1 \in U_1, \ldots, u_\ell \in U_\ell$. For every $i \in [\ell]$, let $v_i, v_i'$ be two independent random literals in $Z_i$. For $j = 0, \ldots, \ell - 1$, define the probability

$$p_j = P[A_F[\{v_1, v_1'\}, \ldots, \{v_j, v_j'\}, \{v_{j+1}\}, \ldots, \{v_\ell\}] \text{ is complete}].$$

By the Cauchy–Schwarz inequality, for each $j = 0, \ldots, \ell - 1$,

$$p_{j+1} = E_{v_1, v_1', \ldots, v_{j+1}, v_{j+1}', \ldots, v_\ell, v_\ell'} \prod_{u_1 \in \{v_1, v_1'\}} \cdots \prod_{u_{j+1} \in \{v_{j+1}, v_{j+1}'\}} 1[u_1 \cdots u_{j+1} v_{j+2} \cdots v_\ell \in E(A_F)]$$

$$= E_{v_1, v_1', \ldots, v_\ell, v_\ell'} (E_{v_1, v_1'} \prod_{u_1 \in \{v_1, v_1'\}} \cdots \prod_{u_\ell \in \{v_\ell, v_\ell'\}} 1[u_1 \cdots u_{j+1} v_{j+2} \cdots v_\ell \in E(A_F)])^2$$

$$\geq (E_{v_1, v_1', \ldots, v_\ell, v_\ell'} \prod_{u_1 \in \{v_1, v_1'\}} \cdots \prod_{u_\ell \in \{v_\ell, v_\ell'\}} 1[u_1 \cdots u_{j+1} v_{j+2} \cdots v_\ell \in E(A_F)])^2 = p_j^2.$$ 

Since $p_0 = |E(A_F)|/n^\ell \geq \varepsilon$, we have $p_\ell \geq \varepsilon^2 \ell$. Therefore, the number of copies of $F[2]$ in $G$ is at least $p\ell n^{2\ell} - O(\ell n^{2\ell - 1})$ (the error term accounts for non-injective maps), which is at least $\varepsilon^2 n^{2\ell}/2$ for sufficiently large $n$.  

3.2. Kruskal–Katona theorem. We need the following special case of the Kruskal–Katona theorem [22, 20]. Here a simplex in a $k$-graph is a clique on $k + 1$ vertices.

Theorem 3.2. A $k$-uniform hypergraph on $n$-vertices with at most $\beta n^k / k!$ edges contains at most $\beta^{k+1} n^{k+1} / (k+1)!$ simplices.

Here is a quick proof of this statement using Shearer’s entropy inequality [9] (see also [19]).

Proof. Let $(X_1, \ldots, X_{k+1})$ be the vertices of a uniformly chosen simplex in the $k$-graph, with the $k + 1$ vertices permuted uniformly at random. By Shearer’s inequality, letting $X_{-i}$ denote
(\(X_1, \ldots, X_{k+1}\)) with the \(i\)-th coordinate removed, we have

\[
k \log((k + 1)! \# \text{simplices}) = kH(X_1, \ldots, X_{k+1}) \\
\leq H(\vec{X}_{-1}) + \cdots + H(\vec{X}_{-(k+1)}) \\
= (k + 1)H(X_1, \ldots, X_k) \leq (k + 1) \log(k! \# \text{edges}).
\]

Since \(k! \# \text{edges} \leq \beta n^k\), we have \((k + 1)! \# \text{simplices} \leq \beta^{k+1} n^{k+1}\). \qed

### 3.3. Hypergraph containers

For motivation, let us first recall a container theorem for triangle-free graphs [4, Theorem 2.1].

**Theorem 3.3 (Containers for triangle-free graphs).** For every \(\varepsilon > 0\), there exists \(C > 0\) such that for every \(n\), there exists a set \(\mathcal{G}\) of graphs on \(n\) labeled vertices, such that

1. every triangle-free graph on \(n\) labeled vertices is contained in some \(G \in \mathcal{G}\), and
2. each \(G \in \mathcal{G}\) contains at most \(\varepsilon n^3\) triangles, and
3. \(|\mathcal{G}| \leq n^{Cn^{3/2}}\).

This theorem outputs a collection of \(2^{o(n^2)}\) containers \(\mathcal{G}\), such that each triangle-free graph is contained within some container \(G \in \mathcal{G}\) and each container \(G\) is almost triangle-free.

We now state and prove an analogous result for formulae avoiding a given finite set of subformulae. For any formula \(B\), we let \(v(B)\) denote the number of variables that appear in some clause in \(B\). Recall that we view a formula as a set of clauses, so \(|B|\) denotes the number of clauses. We formally define the notions of a subformula and a formula being “free” of subformulae below.

**Definition 3.4 (Subformula).** A subformula of a formula \(G\) is a subset of clauses of \(G\). We say that two formulae are isomorphic if one can be obtained from the other by relabeling variables. Given another formula \(F\), a copy of \(F\) in \(G\) is a subformula of \(G\) that is isomorphic to \(F\). Given a set \(B\) of \(k\)-SAT formulae, we say that \(G\) is \(B\)-free if \(G\) has no copy of \(B\) for every \(B \in B\).

We write

\[
m(B) = \max_{B \in B} |B|.
\]

We shall apply the hypergraph container theorem, proved independently by Balogh, Morris, and Samotij [3] and Saxton and Thomason [24] to show the following.

**Theorem 3.5.** Let \(B\) be a finite set of simple \(k\)-SAT formulae. For every \(\delta > 0\), there exists \(C = C(\mathcal{B}, \delta) > 0\) such that for every \(n\), there exists a collection \(\mathcal{G} = \mathcal{G}(\mathcal{B}, \delta)\) of formulae on variable set \(X = \{x_1, \ldots, x_n\}\) such that:

1. every \(B\)-free \(k\)-SAT formula with variables in \(X\) is a subformula of some \(G \in \mathcal{G}\), and
2. for every \(G \in \mathcal{G}\) and \(B \in B\), \(G\) has at most \(\delta n^{v(B)}\) copies of \(B\), and
3. \(|\mathcal{G}| \leq n^{Cn^{k-1/(m(B))^{1-\delta}}/m(B)}\).

The rest of this section, we deduce the above claim from the more general hypergraph container theorem. To state the general result, we introduce some notation for an \(\ell\)-uniform hypergraph \(\mathcal{H}\). Let \(\mathcal{I}(\mathcal{H})\) denote the collection of independent vertex sets in \(\mathcal{H}\). For every \(j \in \{1, \ldots, \ell\}\), let \(\Delta_j(\mathcal{H})\) denote the maximum \(j\)-codegree of vertices in \(\mathcal{H}\), i.e., the maximum number of edges containing a given \(j\)-vertex subset in \(\mathcal{H}\). Let \(\mathcal{P}(V)\) denote the family of all subsets of \(V\). Let \((\mathcal{V}_s)\) denote the family of those subsets of \(V\) having size at most \(s\). For \(A \subseteq \mathcal{P}(V)\), write \(\overline{A} = \mathcal{P}(V) \setminus A\).

We apply the following version of the hypergraph container theorem as stated in [3, Theorem 2.2]. A similar result was proved in [24].

**Theorem 3.6 (Hypergraph container theorem).** For every \(\ell \in \mathbb{N}\) and \(c, \varepsilon > 0\), there exists \(C_0 = C_0(\ell, c, \varepsilon) > 0\) such that the following holds. Let \(\mathcal{H}\) be an \(\ell\)-uniform hypergraph and \(A \subseteq \mathcal{P}(V(\mathcal{H}))\)
be an increasing family of sets such that for all \( A \in \mathcal{A} \), we have \(|A| \geq \varepsilon v(\mathcal{H})\) and \( e(\mathcal{H}[A]) \geq \varepsilon e(\mathcal{H})\). Suppose there exists \( p \in (0,1) \) such that, for every \( j \in \{1, \ldots, \ell\} \), we have
\[
\Delta_j(\mathcal{H}) \leq c \cdot p^{j-1} \cdot \frac{e(\mathcal{H})}{v(\mathcal{H})}.
\]
Then there exists a family \( \mathcal{S} \subseteq \binom{V(\mathcal{H})}{\leq C_p v(\mathcal{H})} \) and functions \( f : \mathcal{S} \to \overline{\mathcal{A}}, g : \mathcal{I}(\mathcal{H}) \to \mathcal{S} \) such that for every \( I \in \mathcal{I}(\mathcal{H}) \), we have:
\[
\begin{align*}
(i) & \quad g(I) \subseteq I; \\
(ii) & \quad I \setminus g(I) \subseteq f(g(I)).
\end{align*}
\]
As motivation for our deduction of Theorem 3.5 using Theorem 3.6, we first review why Theorem 3.3 follows from Theorem 3.6. To count triangle-free graphs, we define a 3-uniform hypergraph \( \mathcal{H} \) encoding triangles, i.e., where \( V(\mathcal{H}) = \binom{[n]}{3} \) and \( E(\mathcal{H}) \) comprise those triples of edges that form triangles. The triangle-free graphs on \( n \) vertices then are in one-to-one correspondence to independent sets on \( \mathcal{H} \) and after verifying the codegree conditions with \( p = n^{-1/2} \), Theorem 3.6 implies Theorem 3.3.

Theorem 3.5 follows from Theorem 3.6 in a similar way. For a finite set \( \mathcal{B} \) of simple \( k \)-SAT formulae, we define a hypergraph \( \mathcal{H} \) encoding formulae in \( \mathcal{B} \), so that \( \mathcal{B} \)-free formulae become independent sets in \( \mathcal{H} \). We then use Theorem 3.6 to deduce Theorem 3.5 after verifying the required codegree conditions.

With this preparation, we now prove Theorem 3.5.

**Proof of Theorem 3.5 using Theorem 3.6.** Fix a finite set \( \mathcal{B} \) of simple \( k \)-SAT formulae. Let \( m = m(\mathcal{B}) = \max_{B \in \mathcal{B}} |B| \) and \( v = \max_{B \in \mathcal{B}} v(B) \).

Let \( \mathcal{H} \) be an \( m \)-uniform hypergraph, where \( V(\mathcal{H}) \) is the set of all \( k \)-SAT clauses on variable set \( \{x_1, \ldots, x_n\} \) (so \( v(\mathcal{H}) = 2^k \binom{n}{k} \)), and
\[
E(\mathcal{H}) = \left\{ G \in \binom{V(\mathcal{H})}{m} \mid G \text{ is simple; } v(G) \leq v; G \text{ contains a copy of some } B \in \mathcal{B} \right\}.
\]
Observe that, for any formula \( B \in \mathcal{B} \), there are at least \( \binom{n}{v(B)} \) different formulae supported on \( x_1, \ldots, x_n \) isomorphic to \( B \). Thus, with \( v = \max_{B \in \mathcal{B}} v(B) \), we have the lower bound \( e(\mathcal{H}) \geq \binom{n}{v} \). On the other hand, since \( e(\mathcal{H}) \) is at most the number of simple formulae on \( \leq v \) vertices, we have \( e(\mathcal{H}) = O(n^v) \).

Using the above bounds on \( e(\mathcal{H}) \) and \( v(\mathcal{H}) \), we obtain the following codegree bounds for \( \mathcal{H} \):

- Since every clause has \( k \) variables, we have
  \[
  \Delta_1(\mathcal{H}) = O_B(n^{v-k}) = O_B \left( e(\mathcal{H}) / v(\mathcal{H}) \right).
  \]
- Take \( j \in \{2, 3, \ldots, m\} \). Note that any \( G \in E(\mathcal{H}) \) arises from a simple formula, and thus any \( j \geq 2 \) clauses from such \( G \) must include at least \( k+1 \) variables. This implies that
  \[
  \Delta_j(\mathcal{H}) = O_B(n^{v-k-1}) = O_B \left( p^{j-1} e(\mathcal{H}) / v(\mathcal{H}) \right)
  \]
  with \( p = n^{-1/(m-1)} \).

Fix \( \delta > 0 \) and choose \( \varepsilon = \varepsilon(\mathcal{B}, \delta) > 0 \) sufficiently smaller than \( \delta \) to be determined later. Define the family
\[
\mathcal{A} = \{ A \subseteq V(\mathcal{H}) : e(\mathcal{H}[A]) \geq \varepsilon e(\mathcal{H}) \}.
\]
For every \( A \in \mathcal{A} \), since \( \varepsilon e(\mathcal{H}) \leq e(\mathcal{H}[A]) \leq \Delta_1(\mathcal{H})|A| \) with \( \Delta_1(\mathcal{H}) = O_B(n^{v-k}) \), we have \( |A| \geq \varepsilon e(\mathcal{H}) / \Delta_1(\mathcal{H}) = \Omega_B(\varepsilon) v(\mathcal{H}) \).

Applying Theorem 3.6, we know that there exists \( C = C(\mathcal{B}, \delta) \) and a family
\[
\mathcal{S} \subseteq \left( \binom{V(\mathcal{H})}{\leq C_p v(\mathcal{H})} \right)
\]
with functions \( f : \mathcal{S} \to \overline{\mathcal{A}}, g : \mathcal{I}(\mathcal{H}) \to \mathcal{S} \) such that for every \( I \in \mathcal{I}(\mathcal{H}) \), we have:
Lemma 4.1. The minimal formulae. We will select some fixed finite set \( I \) such that it does not contain a pair of clauses with the same associated variable set. We begin by observing that every simple minimal formula is \(-\)free. Then every simple minimal formula is a subformula of some container, and (ii) every container has few copies of elements in \( B \). Therefore, for every \( G \in \mathcal{G} \), we know that \( G \) has at most \( \delta n^v(B) \) copies of \( B \) for every \( B \in \mathcal{B} \).

(b) For every \( G \in \mathcal{G} \) and \( B \in \mathcal{B} \), \( G \) has at most \( \delta n^v(B) \) copies of \( B \).

(c) \( |G| \leq nC'B^{k-1/(m-1)} \) for some \( C' = C'(B, \delta) \).

Since \( |G| \leq |S| \), we have (recalling \( p = n^{-1/(m-1)} \))

\[
|G| \leq \left( \frac{v(H)}{Cpv(H)} \right) \leq n^{O_{B,\delta}(n^{k-1/(m-1)})}.
\]

\( \square \)

4. Weak upper bound on the number of \( k \)-SAT functions

In this section, we prove Theorem 1.16. Recall that a \( k \)-SAT formula is simple (Definition 1.5) if it does not contain a pair of clauses with the same associated variable set. We begin by observing that every minimal formula is nearly simple. It remains to upper bound the number of simple minimal formulae. We will select some fixed finite set \( B \) of non-minimal formulae (the forbidden subformulae). Then every simple minimal formula is \( B \)-free. To upper bound the number of simple \( B \)-formula, we proceed via the following steps.

(a) Container formulae. Theorem 3.5 gives a collection of \( 2^{o(n^k)} \) container formulae such that (i) each simple \( B \)-free formula is a subformula of some container, and (ii) every container has few copies of elements in \( B \).

(b) \( k \)-PDGs. Each container formula is nearly “semisimple” (Definition 4.4). We convert each container formula \( G \) to a \( k \)-PDG \( \widetilde{F} \). If \( \widetilde{F} \) had many copies of \( \widetilde{T}_k \), then \( G \) would have many copies of some element of \( B \), so \( G \) must be nearly \( \widetilde{T}_k \)-free. The Turán bound, Theorem 1.15, then implies an upper bound on the number of simple subformulae of \( G \).

4.1. Minimal formulae are nearly simple.

Lemma 4.1. The 2-blowup of a pair of clauses on the same set of \( k \) variables is non-minimal.

For example, the lemma tells us that the 2-blowup of \( \{xyz, xy\overline{z}\} \) is non-minimal.

Proof. Up to relabeling and/or negating some of the variables, we can write the pair of clauses as \( v_1v_2\cdots v_k \) and \( \overline{v}_1\cdots \overline{v}_jv_{j+1}\cdots v_k \) for some \( 1 \leq j \leq k \). Then its 2-blowup has clauses (among others)

\[
v_1v_2\cdots v_k, v_1'v_2v_3\cdots v_k, v_1v_2'v_3\cdots v_k, \ldots, v_1\cdots v_{j-1}v_j'v_{j+1}\cdots v_k, \overline{v}_1'\cdots \overline{v}_j'v_{j+1}\cdots v_k.
\]
It is impossible to satisfy the first clause only, since it would involve setting \( v_1 = \cdots = v_k = 1 \) and \( v'_1 = \cdots = v'_j = 0 \), which would then satisfy the last clause, so the 2-blowup is non-minimal. \( \Box \)

In the above proof, if \( j \geq 2 \), then the displayed sequence of clauses is simple. We record this fact here as it will be used later Section 4.4.

**Lemma 4.2.** Given a pair of clauses on the set of \( k \) variables such that the clauses differ by at least two variable negations, then its 2-blowup contains a simple non-minimal subformula. \( \Box \)

**Proposition 4.3.** Every minimal \( k \)-SAT formula on \( n \) variables can be made simple by deleting \( o(n^k) \) clauses.

**Proof.** Let \( G \) be a minimal \( k \)-SAT formula on \( n \) variables. By Lemma 4.1, \( G \) does not contain any 2-blowup of a pair of clauses on the same \( k \) variables. It follows from Lemma 3.1 that the number of \( k \)-subsets of variables that support at least two clauses in \( G \) is \( o(n^k) \). Removing all such clauses yields a simple subformula of \( G \). \( \Box \)

4.2. From \( k \text{-SAT} \) to \( k \text{-PDGs} \). As hinted by Lemma 4.2, for the container argument, we focus our attention on the following special type of formulae.

**Definition 4.4** (Semisimple formula). A \( k \)-SAT formula is *semisimple* if for every \( k \)-subset \( S \) of variables, one of the following is true:

(a) There is no clause on \( S \), or
(b) There is exactly one clause on \( S \), or
(c) There are exactly two clauses on \( S \) and they differ by exactly one variable negation.

**Example 4.5.** The 3-SAT formula \( \{x_1 \neg x_2 x_3, x_1 \neg x_2 \neg x_3, x_1 x_2 \neg x_4\} \) is semisimple, whereas the formula \( \{x_1 x_2 x_3, x_1 x_2 \neg x_3\} \) is not semisimple.

We define a forgetful map from semisimple formulae to \( k \)-PDGs.

**Definition 4.6** (Type map). Define

\[
\text{Type}: \{ \text{semisimple } k \text{-SAT formula on } n \text{ variables}\} \to \{k \text{-PDGs on } n \text{ vertices}\}
\]

as follows. Given a semisimple formula \( k \)-SAT formula, let \( \text{Type}(G) \) be the \( k \)-PDG \( \bar{F} \) formed as follows:

- the vertex set of \( \bar{F} \) are the variables of \( G \);
- for each \( k \)-subset of variables supporting exactly one clause in \( G \), we add an associated undirected edge to \( \bar{F} \);
- for every \( k \)-subset of variables supporting exactly two clauses in \( G \), we add a directed edge to \( \bar{F} \), directed towards the variable where these two clauses differ by negation.

**Example 4.7.** The semisimple formula

\[
G = \{abc, ab\bar{c}, a\bar{b}d, bd\bar{c}, \bar{b}d\bar{c}\}
\]

has \( \text{Type}(G) \) the 3-PDG with vertex set \( \{a, b, c, d, e\} \) and edge set \( \{abc, abd, \bar{b}de\} \).

4.3. Forbidden subgraphs and non-minimal formulae.

**Proposition 4.8.** Let \( G \) be a semisimple \( k \)-SAT formula with \( \text{Type}(G) = \bar{T}_k \). Then \( G[2] \) has a simple non-minimal subformula.

**Example 4.9.** We consider a 3-SAT example. An example of a semisimple formula with Type \( \bar{H} = \{\{1, 2, 3, 4\}, \{123, 134, 234\}\} \) is the collection of 4-clauses on 4-variables \( G = \{abc, a\bar{b}d, a\bar{b}d, b\bar{c}d\} \). The 2-blowup of \( G \) has the following simple non-minimal formula: \( B = \{abc, ab'd, a'b'd, \bar{b'}cd\} \). We can observe that \( B \) is non-minimal by noting that to only satisfy \( abc \), we must assign \( a, b, c \mapsto 1 \) which forces us to assign \( b' \mapsto 0 \) to avoid satisfying the second clause in \( B \). However then we must assign \( d \mapsto 0 \) to avoid satisfying the third clause in \( B \) and this combination of assignments satisfies the fourth clause in \( B \).
Proof of Proposition 4.8. Let $G$ be a semisimple formula with

$$\text{Type}(G) = \tilde{T}_k = (\{1, 2, \ldots, k + 1\}, \{123 \cdots k, 23 \cdots k(k + 1), 12 \cdots (k - 1)(k + 1)\}).$$

We suppose that variables $x_1, \ldots, x_{k+1}$ in $G$ corresponds to vertices $1, \ldots, k + 1$ in $\tilde{T}_k$ under the Type map. Without loss of generality, suppose that clause $C_a = x_1 x_2 \cdots x_k$ lies in $G$, which gets mapped to $e_a = 1 \cdots k$ under the Type map. We also suppose that the blowup $G[2]$ has variable set $\{x_1, \ldots, x_{k+1}, x_1', \ldots, x_{k+1}'\}$.

We will construct a simple non-minimal subformula of $G[2]$, that we term $H$. We begin by including the following clauses from $G[2]$ into $H$.

- $C_a = x_1 x_2 x_3 \cdots x_k$,
- $C_a^{(1)} = x_1' x_2 x_3 \cdots x_k$,
- $C_a^{(2)} = x_1 x_2' x_3 \cdots x_k$,
- $C_a^{(3)} = x_1 x_2 x_3' \cdots x_k$,
- $\vdots$
- $C_a^{(k)} = x_1 x_2 x_3' \cdots x_k'$.

If $H$ was minimal, then we should be able to find an assignment $\phi$ of variables $\{x_1, \ldots, x_{k+1}, x_1', \ldots, x_{k+1}'\}$ that only satisfies $C_a$ among all clauses in $H$. To satisfy $C_a$, this assignment $\phi$ must satisfy $\phi(x_i) = 1$ for all $i \in [k]$. Then, in order to avoid satisfying $C_a^{(i)}$ for any $i \in [k]$, we must assign $\phi(x_i') = 0$ for all $i \in [k]$.

Suppose clause $C_b \in G$ gets mapped to $e_b$, and clauses $C_c, C_c' \in G$ get mapped to $e_c$ under the Type map. Write

- $C_b = z_2 \cdots z_k z_{k+1}$,
- $C_c = w_1 \cdots w_{k-1} z_{k+1}$,
- $C_c' = w_1 \cdots w_{k-1} z_{k+1}$,

with literals $z_i, w_i \in \{x_i, \overline{x_i}\}$ for every $i$. If $\phi(x_1) = \cdots = \phi(x_k) = 1$ and $\phi(x_1') = \cdots = \phi(x_k') = 0$, then no matter which literals the $z_i, w_i$ are equal to, we can always pick clauses $D_b \in C_b[2]$, $D_c \in C_c[2] \cup C_c'[2]$ such that $\phi(D_b) = \phi(z_{k+1})$ and $\phi(D_c) = \phi(z_{k+1})$. Here is an illustrating example: if

- $C_b = \overline{x_2} x_3 x_4$,
- $C_c = x_1 \overline{x_2} x_4$,
- $C_c' = x_1 \overline{x_2} x_4$,

then we can pick

- $D_b = \overline{x_2} x_3 x_4 \in C_b[2]$ and $D_c = x_1 \overline{x_2} x_4$ such that $\phi(D_b) = \phi(x_3) = 1$, which gives $\phi(D_b) = \phi(x_4)$ and $\phi(D_c) = \phi(x_4)$. Therefore, $\phi$ must satisfy one of $D_b$ and $D_c$.

Let $H = \{C_a, C_a^{(1)}, \ldots, C_a^{(k)}, D_b, D_c\} \subseteq G[2]$. Per above, we know that there is no assignment $\phi$ of variables $\{x_1, \ldots, x_{k+1}, x_1', \ldots, x_{k+1}'\}$ that only satisfies $C_a$ among all clauses in $H$, so $H$ is non-minimal. \hfill \square

4.4. Supersaturation. For a $k$-SAT formula $G$ on $n$ variables, define $\alpha_i(G)$ by

$$\alpha_i(G) \binom{n}{k} = \left| \left\{ S \subseteq \binom{V}{k} \mid G \text{ has exactly } i \text{ clauses with variables } S \right\} \right|.$$

For example, $\alpha_2(G) \binom{n}{k}$ is the number of $k$-subsets of variables $S$ such that $G$ has exactly 2 clauses supported on $S$.

Furthermore, we let $\alpha'_2(G) \binom{n}{k}$ be the number of $k$ subsets of variables $S$ where $G$ has exactly 2 clauses supported on $S$ and this pair of clauses supported on $S$ differs in exactly one variable (which is negated in one clause and positive in the other). We also let $\alpha'_2(G) := \alpha_2(G) - \alpha'_2(G)$.
Lemma 4.10. For every $\varepsilon, \theta > 0$ there exists $\delta > 0$ such that for all sufficiently large $n$, if an $n$-variable $k$-SAT formula $G$ satisfies $\alpha_1(G) + \theta \alpha'_2(G) \geq n \pi(\bar{T}_k, \theta) + \varepsilon$, then there is some simple non-minimal subformula $B$ with at most $2(k+1)$ variables, such that $G$ contains at least $\delta n^{v(B)}$ copies of $B$ as subformulae.

Proof. Fix $\varepsilon > 0$ and $\theta > 0$. Consider an $n$-variable $k$-SAT formula $G$ such that $\alpha_1(G) + \theta \alpha'_2(G) \geq n \pi(\bar{T}_k, \theta) + \varepsilon$. After deleting all clauses of $G$ not accounted for by $\alpha_1(G)$ and $\alpha'_2(G)$, we arrive at a semisimple subformula. We restrict our attention to this subformula and thus assume hereafter without loss of generality that $G$ is semisimple.

Let $\bar{H} = \text{Type}(G)$. Then $\bar{H}$ has $\alpha_1(G) \binom{n}{k}$ undirected edges and $\alpha'_2(G) \binom{n}{k}$ directed edges. It follows by the definition of Turán density $\pi(\bar{T}_k, \theta)$ and a standard supersaturation argument that $\bar{H}$ contains at least $\Omega_{\varepsilon, \theta}(n^{k+1})$ copies of $\bar{T}_k$. So there is some $k$-SAT semisimple formula $G'$, with $\text{Type}(G') = \bar{T}_k$, such that $G$ contains $\Omega_{\varepsilon, \theta}(n^{k+1})$ copies of $G'$ as subformulae. Then, by Lemma 3.1, $G$ contains $\Omega_{\varepsilon, \theta}(n^{2(k+1)})$ copies of $G'[2]$ as subformulae. By Proposition 4.8, each copy of $G'[2]$ contains some simple non-minimal subformula. Thus there is some simple non-minimal subformula $B$ on at most $2(k+1)$ variables that appears $\Omega_{\varepsilon, \theta}(n^{v(B)})$ times.

Lemma 4.11. Let $\varepsilon > 0$. There exists $\delta = \delta(k, \varepsilon) > 0$ such that for all sufficiently large $n$, if an $n$-variable $k$-SAT formula has at least $\alpha_i(G) \geq \varepsilon$ for some $i \geq 3$ or $\alpha''_2(G) \geq \varepsilon$, then there is some simple non-minimal subformula $B$ on at most $2k$ vertices such that $G$ contains at least $\delta n^{v(B)}$ copies of $B$ as subformulae.

Proof. Consider an $n$-variable $k$-SAT formula $G$ such that $\alpha_i(G) \geq \varepsilon$ for some $i \geq 3$, or $\alpha''_2(G) \geq \varepsilon$. By the pigeonhole principle, there is some $2 \leq j \leq k$ such that $G$ has $\Omega_{\varepsilon}(n^k)$ subformulae of the form $G' = \{v_1v_2 \cdots v_k, \bar{v}_1 \cdots \bar{v}_j, v_{j+1} \cdots v_k\}$ where each $v_i$ is some literal in $\{x_i, \bar{x}_i\}$. In other words, $G$ has $\Omega_{\varepsilon}(n^j)$ pairs of clauses that are supported on the same variable set but differ by the negation on at least $k$ literals. By Lemma 3.1, $G$ contains $\Omega_{\varepsilon}(n^{2k})$ copies of the 2-blowups of these clause pairs as subformulae. By Lemma 4.2, each copy of $G'[2]$ has a non-minimal simple subformula. The conclusion then follows as earlier.

4.5. Applying the container theorem. Recall the definition of the $\alpha_i$'s from the beginning of Section 4.4.

Definition 4.12. Define the weight of a $k$-SAT formula $G$ by

$$\text{wt}(G) = \alpha_1(G) + \log_2 3 \cdot \alpha_2(G) + \log_2 4 \cdot \alpha_3(G) + \cdots + \log_2 (2^{k+1} - 1) \cdot \alpha_{2k}(G).$$

Given a $k$-SAT formula $G$ on $n$ variables, the number of simple $k$-SAT subformulae of $G$ is

$$\prod_{j=1}^{2^k} (j+1)^{\beta_j(G)} \binom{n}{k} = 2^{\text{wt}(G)} \binom{n}{k},$$

since there are $j+1$ ways to choose at most one clause from $j$ clauses. This motivates our definition of weight.

We are now ready to prove Theorem 1.16 that the number of minimal $k$-SAT formulae on $n$ variables is at most $2^{(\pi(\bar{T}_k, \log_2 3) + o(1)) \binom{n}{k}}$.

Proof of Theorem 1.16. Let $\mathcal{B}$ be the set of all non-minimal simple formulae on at most $2(k+1)$ variables, one for each isomorphism class (so that $\mathcal{B}$ is finite). Fix any $\varepsilon > 0$. Pick $\delta > 0$ to be smaller than the minimum of the $\delta$'s chosen in Lemma 4.10 (with $\theta = \log_2 3$) and Lemma 4.11. Applying the container result, Theorem 3.5, we find a collection $\mathcal{G}$ of $n$-variable formulae on $X = \{x_1, \ldots, x_n\}$ such that

(a) Every $\mathcal{B}$-free formula on $X$ is a subformula of some $G \in \mathcal{G}$, so in particular, every simple minimal formula on $X$ is a subformula of some $G \in \mathcal{G}$; and
(b) For every $G \in \mathcal{G}$ and $B \in \mathcal{B}$, $G$ contains at most $\delta n^{o(B)}$ copies of $B$, which implies, by Lemmas 4.10 and 4.11, that $\alpha_1(G) + \log_2 3 \cdot \alpha_2'(G) < \pi(\tilde{T}_k, \log_2 3) + \varepsilon$, $\alpha_2'(G) < \varepsilon$ and $\alpha_i(G) < \varepsilon$ for all $i \geq 3$; and

c] |G| \leq 2^{o(n^k)}.

The number of simple minimal $k$-SAT subformula of each $G \in \mathcal{G}$ is thus at most

$$2^{\omega(G)}(\mu) \leq 2^{(\pi(\tilde{T}_k, \log_2 3) + O(\varepsilon))(\mu)},$$

By taking a union bound over all $G \in \mathcal{G}$, of which there are at most $2^{o(n^k)}$, and noting that $\varepsilon$ can be taken to be arbitrarily small, the number of simple minimal $k$-SAT formula on $n$ variables is $2^{(\pi(\tilde{T}_k, \log_2 3) + o(1))(\mu)}$.

Finally, by Proposition 4.3, we can obtain any minimal $k$-SAT formulae by adding $o(n^k)$ clauses to a simple minimal formula. This adds a negligible factor $2^{o(n^k)}$. So the total number of minimal $k$-SAT formulae on $n$ variables is $2^{(\pi(\tilde{T}_k, \log_2 3) + o(1))(\mu)}$. $\Box$

5. Stability I: near vs. far from unate

We prove Theorem 1.14 in the following two sections. Let us recall some definitions. The variables are $x_1, \ldots, x_n$. Each variable can appear as either a positive literal $x_i$ or a negative literal $\overline{x}_i$. We say that a clause uses a variable $x$ if it contains one of the literals $x$ and $\overline{x}$.

We usually omit mentioning dependencies on $k$ as we consider it fixed throughout. Recall that a clause is monotone if it only uses positive literals. An assignment $w$ of variables is a witness for a clause $C$ in a formula if $w$ satisfies $C$ and no other clauses. A formula is minimal if every clause has a witness.

We next define several useful properties a $k$-SAT formula might have.

Definition 5.1. Let $\Phi$ be an $n$-variable $k$-SAT formula and let $\zeta, \zeta' > 0$.

- $\Phi$ is $\zeta$-nearly monotone if it has at most $\zeta n^k$ non-monotone clauses.
- $\Phi$ is $\zeta$-sparsely minimal if it is minimal and every clause has a witness assignment with $< \zeta n$ variables assigned to 1.
- $\Phi$ is positive-dominant if for all variables $x$, $m(x) \geq m(\overline{x})$, where for a literal $v$, $m(v)$ is the number of occurrences of $v$ in $\Phi$.
- $\Phi$ is $\zeta'$-dense if every variable is used by at least $\zeta' \binom{n-1}{k-1}$ clauses.

Let $\mathcal{I}(n)$ be the set of minimal $k$-SAT formulae on $n$ variables, and let $\mathcal{I}'(n, \zeta, \zeta')$ be the set of minimal $k$-SAT formula that enjoy the above properties with parameters $\zeta, \zeta'$.

Here is the main result of this section.

Proposition 5.2. Suppose $\theta > \log_2 3$ satisfies $\pi(\tilde{T}_k, \theta) = 1$. For every $\zeta > 0$, there exists $\varepsilon > 0$ (depending only on $k, \theta, \zeta$) such that for all sufficiently large $n$,

$$|\mathcal{I}(n)| \leq 2^{(1-\varepsilon)(\mu)} + 2^{\frac{3}{2}}(\mu) |\mathcal{I}(n-1)| + 2^n \mathcal{I}'(n, \zeta, \frac{1}{10k}).$$

Remark 5.3 (Notation for setting sufficiently small constants). Throughout the next two sections, our arguments often involve picking a sequence of constants. We use $\varepsilon \ll \delta$ to mean that $\varepsilon > 0$ is a sufficiently small constant depending on $\delta > 0$. Here we omit from the notation the dependencies on $k$ and $\theta \gg \log_2 3$, which we view as fixed throughout.

To prove Proposition 5.2, we refine the proof of Theorem 1.16 from Section 4 by separating containers that are nearly unate from those that far from unate.

Definition 5.4. For every $\rho > 0$, we say that a $k$-SAT formula $G$ is $\rho$-nearly unate if it can be made unate by removing up to $\rho n^k$ clauses. Otherwise we say that $G$ is $\rho$-far from unate.
5.1. Nearly complete far from unate simple formulae have many forbidden subformulae.

We call a simple \( k \)-SAT formula with \( k + 1 \) variables and \( k + 1 \) clauses a simplex. An example of a simplex with \( k = 3 \) is \( \{wxy, w\overline{xz}, w\overline{yz}, xy\overline{z}\} \).

**Lemma 5.5.** The 2-blowup of every non-unate simplex is non-minimal.

**Proof.** The proof is analogous to that of Proposition 4.8. Let \( x_1, \ldots, x_{k+1} \) be the variables of the simplex. Denote the vertices of the 2-blowup by \( x_1, x_1', \ldots, x_{k+1}, x_{k+1}' \). Without loss of generality (by negating or relabeling variables), the non-unate simplex contains the following three clauses:

\[
\begin{align*}
C_1 &= x_1x_2x_3 \cdots x_{k-1}x_k, \\
C_2 &= y_2y_3 \cdots y_{k-1}y_kx_{k+1}, \\
C_3 &= z_1z_2z_3 \cdots z_{k-1}z_kx_{k+1},
\end{align*}
\]

where \( y_i, z_i \in \{x_i, \overline{x_i}\} \) for each \( i \). Let us attempt to find a witness assignment \( w \) of the variables that satisfies only the clause \( x_1 \cdots x_k \) in the 2-blowup. For each \( i = 1, \ldots, k \), we have \( w(x_i) = 1 \). Furthermore, since \( w \) does not satisfy the clause \( x_1 \cdots x_k' \cdots x_k \) from the 2-blowup of \( C_1 \), we have \( w(x_k') = 0 \). If \( w(x_{k+1}) = 1 \), then there is a clause in the 2-blowup of \( C_2 \) that is satisfied under \( w \), whereas if \( w(x_{k+1}) = 0 \), then there is clause in the 2-blowup of \( C_3 \) that is satisfied. So \( w \) cannot be a witness. Thus the 2-blowup is non-minimal. \( \square \)

**Lemma 5.6.** For every \( \rho > 0 \), there exists \( \varepsilon > 0 \) such that for all sufficiently large \( n \), every simple \( \rho \)-far from unate \( n \)-variable \( k \)-SAT formula with at least \( (1 - \varepsilon) \binom{n}{k} \) clauses contains \( \Omega(\rho n k^{-1}) \) non-unate simplices.

**Proof.** Let \( G \) be a simple \( \rho \)-far from unate \( n \)-variable \( k \)-SAT formula with at least \( (1 - \varepsilon) \binom{n}{k} \) clauses. For each variable \( x \), let \( G_x \) denote the subformula consisting of clauses containing \( x \) (as a positive literal), and let \( G_{\overline{x}} \) the subformula consisting of clauses containing \( \overline{x} \) (as a negative literal).

Since \( G \) is \( \rho \)-far from unate, there are at least \( \rho n/2 \) variables \( x \) with

\[
\min \{|G_x|, |G_{\overline{x}}|\} \geq \rho n k^{-1}/2. \tag{5}
\]

Since \( |G| \geq (1 - \varepsilon) \binom{n}{k} \), for all but at most \( k\sqrt{n} \) variables \( x \),

\[
|G_x \cup G_{\overline{x}}| \geq (1 - \sqrt{\varepsilon}) \binom{n - 1}{k - 1}. \tag{6}
\]

By choosing \( \varepsilon \) small enough so that \( k\sqrt{n} \leq \rho/4 \), there is some set \( S \) of variables with \( |S| \geq \rho n/4 \) so that (5) and (6) both hold for every \( x \in S \).

For every \( x \in S \), by the Kruskal–Katona theorem (Theorem 3.2),

\[
\# \{\text{simplices in } G_x\} \leq \frac{(k - 1)! |G_x|^{k/(k-1)}}{k!},
\]

and

\[
\# \{\text{simplices in } G_{\overline{x}}\} \leq \frac{(k - 1)! |G_{\overline{x}}|^{k/(k-1)}}{k!}.
\]

Adding the two inequalities, and noting that \( |G_x| + |G_{\overline{x}}| \leq \binom{n-1}{k-1} \), \( |G_x| = \Omega(\rho n k^{-1}) \), \( |G_{\overline{x}}| = \Omega(\rho n k^{-1}) \), and the convexity of \( t \mapsto t^{k/(k-1)} \), we have

\[
\# \{\text{simplices in } G_x\} + \# \{\text{simplices in } G_{\overline{x}}\} \leq (1 - \Omega(\rho)) \frac{n^k}{k!}. \tag{7}
\]

For each \( x \in S \), due to (6), there are \( (1 - O(\sqrt{\varepsilon})) \binom{n-1}{k-1} \) simplices in \( G_x \cup G_{\overline{x}} \). Combining with (7), we see that there are \( \Omega(\rho n k) \) simplices that simultaneously contain both \( x \) and \( \overline{x} \) as literals. Such simplices are non-unate. Since \( |S| = \Omega(\rho n) \), there are \( \Omega(\rho^2 n k^{-1}) \) non-unate simplices. \( \square \)
Lemma 5.7. For every $\rho > 0$, there exists $\varepsilon > 0$ such that for all sufficiently large $n$, every simple $\rho$-far from unate $n$-variable $k$-SAT formula with at least $(1-\varepsilon)\binom{n}{k}$ clauses contains at least $\Omega_\rho(n^{v(B)})$ copies of some non-minimal simple subformula $B$ on at most $2(k+1)$ variables.

Proof. Let $G$ be such a formula. By Lemma 5.6, $G$ contains $\Omega_\rho(n^{k+1})$ non-unate simplices. By Lemma 3.1, $G$ contains $\Omega_\rho(n^{2k+2})$ copies of $2$-blowups of non-unate simplices, and each such copy, by Lemma 5.5, contains a non-minimal subformula on $2k+2$ variables. The result then follows. □

5.2. Small and large containers. Recall from Section 4.4 that the definition of $\alpha_i(G)$, namely that $\alpha_i(G)\binom{n}{k}$ is the number of $k$-tuples of variables that support exactly $i$ clauses. Also the weight (Definition 4.12) is defined by $\text{wt}(G) = \sum_{i=1}^{2^k} (\log_2 i)\alpha_i(G)$. We consider a container “large” if its weight is close to 1.

Lemma 5.8 (Large containers are nearly unate). Suppose $\theta > \log_2 3$ satisfies $\pi(T_k, \theta) = 1$. For every $\rho > 0$, there exists $\delta > 0$ such that if $G$ is an $n$-variable $k$-SAT formulae with $\text{wt}(G) \geq 1 - \delta$ and at most $\delta n^{v(B)}$ copies of each non-minimal simple formula $B$ on at most $2(k+1)$ variables, then $G$ is $\rho$-nearly unate.

Proof. Select constants $0 < \delta \ll \varepsilon \ll \rho$. Since $G$ has at most $\delta n^{v(B)}$ copies of each non-minimal simple formula $B$ on at most $2(k+1)$ variables, we have $\alpha_1(G) + \theta \alpha_2(G) \leq 1 + \varepsilon$ by Lemma 4.10, and $\alpha_2(G) \leq \varepsilon$ and $\alpha_i(G) \leq \varepsilon$ for each $i \geq 3$ by Lemma 4.11. Thus

$$1 - \delta < \text{wt}(G) = \alpha_1(G) + (\log_2 3)\alpha_2(G) + O(\varepsilon)$$

$$= \alpha_1(G) + \theta \alpha_2(G) - (\theta - \log_2 3)\alpha_2(G) + O(\varepsilon)$$

$$\leq 1 - (\theta - \log_2 3)\alpha_2(G) + O(\varepsilon).$$

Thus $\alpha_2(G) = O(\varepsilon)$ (recall that we treat $\theta > \log_2 3$ as a fixed constant whose dependence is omitted), and $\alpha_1(G) = 1 - O(\varepsilon)$. Hence, by Lemma 5.7, $G$ is $\rho$-nearly unate. □

Lemma 5.9. For every $\zeta > 0$, there exists $\rho > 0$ such that the following holds. Suppose $(G, H)$ is a pair of formulae such that $G$ is $\rho$-nearly unate, $H$ is minimal and $H \subseteq G$. Then, at least one of the following holds for $(G, H)$:

(1) $H$ is not $\frac{1}{10k}$-dense.

(2) There exists a variable negation of $G$ into $G'$ such that $G'$ has $\leq \rho n^k$ non-monotone clauses, but the resulting $H' \subseteq G'$ is not $\zeta/2$-sparingly minimal.

(3) We can negate a subset of variables to obtain $H^+ \in T^*(n, \zeta, \frac{1}{10k})$ from $H$.

Proof. Suppose $(G, H)$ does not satisfy conditions (1)(2) in lemma. Since $G$ is $\rho$-nearly unate, we can negate a subset of variables to obtain some formula $G'$ that has $\leq \rho n^k$ non-monotone clauses. Consequently, the formula $H' \subseteq G'$ also has $\leq \rho n^k$ non-monotone clauses.

Let $X_{H'}$ be the set of variables with $m(x) < m(\overline{x})$ in $H'$. Observe that $|X_{H'}| \leq 100k^{k+1}\rho n$, as otherwise, since $H'$ is $\frac{1}{10k}$-dense, there would be more than

$$100k^{k+1}\rho n \cdot \frac{1}{20k} \binom{n-1}{k-1} \cdot \frac{1}{k} > \rho n^k$$

non-monotone clauses in $H'$. Therefore, by negating those variables in $X_{H'}$, we can obtain a positive-dominant formula $H^+$ from $H'$ that has at most (suppose $\rho \ll \zeta$)

$$100k^{k+1}\rho n \binom{n-1}{k-1} + \rho n^k \leq \zeta n^k$$

non-monotone clauses. It is clear that $H^+$ is $\frac{1}{10k}$-dense. Since every clause in $H'$ has a witness assignment with fewer than $\zeta n/2$ variables assigned to 1, every clause in $H^+$ has a witness assignment
with fewer than \( \zeta n/2 + 100k^{k-1}pn \leq \zeta n \) variables assigned to 1. Thus, \( H^\uparrow \) is \( \zeta \)-sparsely minimal. Combining the above gives that \( H^\uparrow \in I'(n, \zeta, 1_{10k}). \)

**Lemma 5.10.** For every \( \zeta > 0 \), there exists \( \rho > 0 \) such that the following holds for all sufficiently large \( n \): if \( G \) is a collection of \( \rho \)-nearly unate \( n \)-variable \( k \)-SAT formulae with \( |G| \leq 2^\rho n^k \), then

\[
|\{(G, H) : G \in G, H \subseteq G, H \text{ minimal and } (G, H) \text{ meets condition (2) in Lemma 5.9}\}| \leq 2^{(1-\rho)(n^k)}.
\]

**Proof.** Let \( G \) be a collection of \( \rho \)-nearly unate \( n \)-variable \( k \)-SAT formulae, with \( |G| \leq 2^\rho n^k \). Per Lemma 5.9, we will count pairs \((G, H)\) such that

1. \( G \in G, H \subseteq G, H \) is minimal;
2. there exists a variable negation of \( G \) into \( G' \) such that \( G' \) has \( \leq \rho n^k \) non-monotone clauses, but the resulting \( H' \subseteq G' \) is not \( \zeta/2 \)-sparsely minimal.

Every such \((G, H)\) is uniquely determined by the following data:

- \((a)\) \( G \in G \leq 2^\rho n^k \) possibilities;
- \((b)\) The variable negation of \( G \) into \( G' \leq 2^n \) possibilities;
- \((c)\) The non-monotone clauses of \( H' \subseteq G' \leq 2^\rho n^k \) possibilities since \( G' \) is \( \rho \)-nearly monotone;
- \((d)\) The \( \lfloor \zeta n/2 \rfloor \) lexicographically smallest variables \( x_i \) where \( w_{H'}(x_i) = 1 \) for some witness \( w_{H'} \) to clause \( C_{H'} \leq \binom{\lfloor \zeta n/2 \rfloor}{k} \) possibilities;
- \((e)\) All monotone clause(s) supported on the above \( \lfloor \zeta n/2 \rfloor \) variables (which must be either uniquely \( C_{H'} \) or none due to \( w \) being a witness for \( C_{H'} \leq \binom{\lfloor \zeta n/2 \rfloor}{k} + 1 \) possibilities);
- \((f)\) The remaining monotone clauses of \( H' \) not covered by \((e)\) \( \leq 2^\rho n^k \) \( \binom{\lfloor \zeta n/2 \rfloor}{k} \) possibilities.

Combining the bounds, we see that with \( \rho \ll \zeta \), the number of such pairs is at most

\[
\leq 2^{\rho n^k} \cdot 2^n \cdot 2^{\rho n^k} \cdot \binom{n}{\lfloor \zeta n/2 \rfloor} \cdot \left( \binom{\lfloor \zeta n/2 \rfloor}{k} + 1 \right) \cdot 2^\rho n^k \binom{\lfloor \zeta n/2 \rfloor}{k} \leq 2^{(1-\rho)(n^k)}.
\]

We will use \( H(\cdot) \) to denote the binary entropy function \( H(p) = -p \log_2 p - (1-p) \log_2 (1-p) \), which satisfies the standard inequality

\[
\left( \leq \lfloor pm \rfloor \right) \leq 2^{H(p)n} \quad \text{for all } 0 < p \leq 1/2 \text{ and } n \in \mathbb{N}.
\]

**Proof of Proposition 5.2.** We first count the number of formulae in \( I(n) \) that are not \( 1/10k \)-dense. Every formula \( G \in I(n) \) that is not \( 1/10k \)-dense is uniquely determined by the following information:

1. The lexicographically smallest variable \( x \in X \) that is used by less than \( 1/10k \binom{n-1}{k-1} \) clauses in \( G \);
2. The clauses in \( G \) that use the variable \( x \);
3. The clauses in \( G \) that do not use the variable \( x \).

We get the following bound by upper bounding the number of choices for each item:

\[
|\{G \in I(n) : G \text{ is not } 1/10k \text{-dense}\}| \leq \frac{n}{10k} \binom{2^k \binom{n-1}{k-1}}{\binom{n}{10k}} |I(n - 1)| < 2^k \binom{n-1}{k-1} |I(n - 1)|,
\]

where the last inequality uses (8), since \( H \left( \frac{1}{10k} \right) 2^k < \frac{1}{3} \) for \( k \geq 3 \) and \( n \) is sufficiently large.
As in Section 4, let $B$ vertices. Applying the container theorem, Theorem 3.5, we obtain a collection $G$ of formulae satisfying:

- Every $B$-free $n$-variable $k$-SAT formula is a subformula of some $G \in G$;
- For every $G \in G$ and $B \in B$, $G$ has at most $\delta n^v(B)$ copies of $B$;
- $|G| = 2^{o(n^k)}$.

We partition $G = G^{sm} \cup G^{lg}$ where

$G^{sm} = \{ G \in G \mid \text{wt}(G) \leq 1 - \delta \}$ and $G^{lg} = \{ G \in G \mid \text{wt}(G) > 1 - \delta \}$.

We know from Proposition 4.3 that every minimal formulae can be made simple by deleting constants satisfying:

By Lemma 5.8, every $G \in G^{sm}$ of simple clauses. The number of simple subformulae of each $G \in G^{sm}$ is $2^{\text{wt}(G)\binom{n}{k}} \leq 2^{(1-\delta)(\binom{n}{k})}$, and so the number of minimal subformulae of $G$ is at most $2^{(1-\delta+o(1))(\binom{n}{k})}$. Taking a union bound over all $G \in G^{sm}$, as $|G^{sm}| = 2^{o(n^k)}$,

$$\left| \bigcup_{G \in G^{sm}} \{ \text{minimal } H \subseteq G \} \right| \leq 2^{(1-\delta+o(1))(\binom{n}{k})}.$$ 

By Lemma 5.8, every $G \in G^{lg}$ is $\rho$-nearly unate. By Lemma 5.9 and Lemma 5.10, we have

$$\left| \bigcup_{G \in G^{lg}} \left\{ \text{minimal, } \frac{1}{10k}\text{-dense } H \subseteq G \right\} \right| \leq 2^{(1-\rho)(\binom{n}{k})} + 2^n \left| I^*(n, \zeta, \frac{1}{10k}) \right|.$$ 

Since every minimal formula is a subformula of some element of $G = G^{sm} \cup G^{lg}$, combining these two bounds, we get that

$$\left| \left\{ G \in I(n) : G \text{ is } \frac{1}{10k}\text{-dense} \right\} \right| \leq 2^{(1-\epsilon)(\binom{n}{k})} + 2^n \left| I^*(n, \zeta, \frac{1}{10k}) \right|. \quad \square$$

6. Stability II: Nearly Monotone and Sparingly Minimal Formulae

Recall that in the previous section, we proved Proposition 5.2, which reduced the problem of bounding the number of minimal $k$-SAT formula to counting formulae in

$$I_1^*(n, \zeta) := I^*\left(n, \zeta, \frac{1}{10k}\right)$$

(see Definition 5.1). In this section, we complete the proof of Theorem 1.14 by establishing a recursive inequality for $|I_1^*(n, \zeta)|$. The inequality is analogous to equation (20) in Ilinca–Kahn [18].

**Theorem 6.1.** For $k \geq 3$, there exist $\zeta > 0$ and $c > 0$ such that for all $n$,

$$|I_1^*(n, \zeta)| \leq 2^{\binom{n}{k}} + \exp_2 \left[ (1-c)\binom{n}{k} \right] + \sum_{i=1}^{k-2} \exp_2 \left[ i(1-c)\binom{n}{k-1} \right] |I(n-i)|$$

$$+ \exp_2 \left[ \frac{k}{2}(1-c)\binom{n}{k-1} \right] |I(n-k)| + \exp_2 \left[ \frac{k}{2}(1-c)\binom{n}{k-1} - cn \right].$$

When $k = 2$, we have

$$|I_1^*(n, \zeta)| \leq 2^{\binom{n}{2}} + 2 \exp_2 \left[ (1-c)\binom{n}{2} \right] + \exp_2 \left[ \frac{k}{2}(1-c)\binom{n}{k-1} \right] |I(n-k)|.$$
We first prove Theorem 1.14 assuming Theorem 6.1.

**Proof of Theorem 1.14.** Choose \( \theta > \log_2 3 \) with \( \pi(\tilde{T}_1, \theta) = 1 \) and \( \zeta > 0 \) as in Theorem 6.1. We then choose \( \varepsilon > 0 \) as in Proposition 5.2, so that for \( n \) sufficiently large, we have

\[
|I(n)| \leq 2^{(1-\varepsilon)(\binom{n}{k})} + 2^{\frac{1}{2} \binom{n-1}{k-1}} |I(n-1)| + 2^n |I_1^*(n, \zeta)|.
\]

We show by induction that there exist \( c_1, c_0 > 0 \) such that for all \( n \),

\[
|I(n)| \leq (1 + 2^{c_1-c_0 n}) 2^{\binom{n}{k}+n}.
\]  \hspace{1cm} (9)

We first choose \( 0 < c < 1 \) sufficiently small so that Theorem 6.1 holds. We then choose \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \), the following conditions hold:

(i) \( \binom{n}{k} - \binom{n-i}{k} > i \left(1 - \frac{d}{2}\right) \binom{n}{i} \) for all \( i \in [k] \);
(ii) if \( k \geq 3 \), we require that \( \frac{d}{2} \binom{n}{k-1} \geq 2n \);
(iii) \( 2^{-\frac{1}{2} \binom{n}{k}+1} + k 2^{-n+1} + 2^{-c_0 n+1} \leq 2^{-\frac{1}{2} \varepsilon n} \).

Choose \( c_0 > 0 \) such that for all \( n > n_0 \), we have

\[
2^{-n} + 2^{-\frac{1}{2} \binom{n-1}{k-1}} + 2^{-\frac{1}{2} \varepsilon n} \leq 2^{-c_0 n}.
\]  \hspace{1cm} (10)

Then, take \( c_1 > 0 \) such that (9) holds for all \( n \leq n_0 \). It remains to show that (9) holds for all \( n > n_0 \). Fix one such \( n \) and assume by induction that (9) holds for all smaller values of \( n \). By the inductive hypothesis and condition (i) on \( c \), for every \( i \in [k] \), we have

\[
\exp_2 \left[ (1-c) \binom{n}{k-1} \right] |I(n-i)| \leq (1 + 2^{c_1-c_0(n-i)} \binom{n}{k-i}+n-i \cdot 2^i (1-c) \binom{n}{k-i}) \leq (1 + 2^{c_1-c_0(n-i)} \binom{n}{k}+n-i \cdot 2^i (1-c) \binom{n}{k-i}).
\]

When \( k \geq 3 \), substituting the above into Theorem 6.1 gives that

\[
|I_1^*(n, \zeta)| \leq 2^{\binom{n}{k}} + \exp_2 \left[ (1-c) \binom{n}{k} \right] + \sum_{i=1}^{k} \exp_2 \left[ (1-c) \binom{n}{k-1} \right] |I(n-i)| + \exp_2 \left[ \binom{n}{k} - c n \right] \leq 2^{\binom{n}{k}} + \exp_2 \left[ (1-c) \binom{n}{k} \right] + \sum_{i=1}^{k} (1 + 2^{c_1-c_0(n-i)} \binom{n}{k-i}+n-i \cdot 2^i (1-c) \binom{n}{k-i}+n-i \cdot 2^i (1-c) \binom{n}{k-i}).
\]

Let

\[
J_{n,k} := \sum_{i=1}^{k} (1 + 2^{c_1-c_0(n-i)} \binom{n}{k-i}+n-i \cdot 2^i (1-c) \binom{n}{k-i}+n-i \cdot 2^i (1-c) \binom{n}{k-i}).
\]

We know from condition (ii) that \( \frac{d}{2} \binom{n}{k-1} \geq 2n \). Thus, we have

\[
J_{n,k} \leq k (1 + 2^{c_1}) 2^{\binom{n}{k}+n} - n \cdot \frac{d}{2} \binom{n}{k-1} - i.
\]

which gives

\[
|I_1^*(n, \zeta)| \leq 2^{\binom{n}{k}} + 2^{(1-c) \binom{n}{k}} + k (1 + 2^{c_1}) 2^{\binom{n}{k}+n} + 2^{\binom{n}{k}+n} - c n \leq (1 + 2^{c_1} (2^{-c_1} \binom{n}{k} + k 2^{-n+1} + 2^{-c_0 n})) 2^{\binom{n}{k}} \leq (1 + 2^{c_1} - \frac{1}{2} \varepsilon n) 2^{\binom{n}{k}},
\]

where the final inequality follows from condition (iii).
When $k = 2$, we apply the simpler recurrence of Theorem 6.1 and obtain:
\[
\mathcal{I}^*_1(n, \zeta) \leq 2^{\binom{n}{2}} + 2 \exp_2 \left[ (1 - c) \binom{n}{2} \right] + 2^{(1 - c)n} (1 + 2c_1) 2^{\binom{n-1}{2} + n - 1} \\
\leq 2^{\binom{n}{2}} + 2 \exp_2 \left[ (1 - c) \binom{n}{2} \right] + (1 + 2c_1) 2^{\binom{n}{2} + cn + 1} \\
\leq (1 + 2c_1 (2^{-c(\binom{n}{2} + 1) + 2^{-cn + 1}})) 2^{\binom{n}{2}} \leq (1 + 2^{c_1 - \frac{c}{2}} n) 2^{\binom{n}{2}},
\]
where the final inequality follows from condition (iii).

Proposition 5.2 then implies
\[
|\mathcal{I}(n)| \leq 2^{(1 - c) \binom{n}{k}} + 2^{\frac{1}{2} \binom{n}{k-1}} |\mathcal{I}(n - 1)| + 2^n |\mathcal{I}^*_1(n, \zeta)| \\
\leq (2^{-n} + 2^{\frac{1}{2} \binom{n-1}{k-1} + c_1 + 1 + 2^{c_1 - \frac{c}{2} n}}) 2^{\binom{n}{2} + n} \\
\leq (1 + 2^{c_1 - c_0 n}) 2^{n + \binom{n}{2}},
\]
where the final inequality is a consequence of our choice of $c_0$ satisfying (10). This proves the inductive hypothesis. \qed

Our proof of Theorem 6.1 extends the arguments in [18, Section 8]. The following steps Section 6.1 to Section 6.4 roughly correspond to Steps 1–5 in [18, Section 8].

6.1. **Basic properties of** $\mathcal{I}^*_1$. It remains for us to count formulae in $\mathcal{I}^*_1(n, \zeta)$. We recall that such formulae enjoy the following properties:

- $m(x) \geq m(\overline{\tau})$ for every variable $x \in X$;
- every such formula has at most $\zeta n^k$ non-monotone clauses;
- every clause has a witness $w$ supported on fewer than $< \zeta n$ variables;
- every variable $x \in X$ is used by at least $\frac{1}{10k} \binom{n-1}{k-1}$ clauses.

We will show that in addition, most clauses are monotone in every formula in $\mathcal{I}^*_1(n, \zeta)$. The following observations will be useful in our subsequent steps.

**Lemma 6.2.** For every $G \in \mathcal{I}^*_1(n, \zeta)$, the following are true.

(a) For every $k - 1$ literals $z_1, \ldots, z_{k-1}$, there are at most $\zeta n$ variables $w$ such that $z_1 \cdots z_{k-1} \overline{w} \in G$.

(b) For every variable $v$ and $j \in \{1, \ldots, k - 1\}$, there are at most $\zeta n^{k - 1}$ clauses in $G$ containing the positive literal $v$ and exactly $j$ negative literals.

(c) For every variable $v$ and $j \in \{1, \ldots, k - 1\}$, there are at most $\zeta n^{k - 1}$ clauses in $G$ containing the negative literal $\overline{v}$ and exactly $j$ negative literals other than $\overline{v}$.

**Proof.** Consider any formula $G \in \mathcal{I}^*_1(n, \zeta)$. We verify properties (a)-(c):

(a) Fix $k - 1$ literals $z_1, \ldots, z_{k-1}$. Suppose there are more than $\zeta n$ variables $w$ such that $z_1 \cdots z_{k-1} \overline{w} \in G$. Let $w_0$ be one of these variables, and $w \in \{0, 1\}^n$ be the witness that satisfies $z_1 \cdots z_{k-1} \overline{w_0}$ but no other clauses in $G$. Then for any other $w \neq w_0$ such that $z_1 \cdots z_{k-1} \overline{w} \in G$, $w$ must assign $w$ to 1. But then $w$ assigns at least $\zeta n$ variables to 1, contradicting the definition of $\mathcal{I}^*_1(n, \zeta, \zeta')$ (Definition 5.1).

(b) Fix a variable $v$. By (a), for any $j - 1$ negative literals $\overline{v}_1, \ldots, \overline{v}_{j-1}$ and $k - j$ positive literals $v_j, \ldots, v_{k-2}$, there are at most $\zeta n$ variables $w$ such that $G$ contain $v_1 \cdots \overline{v}_{j-1} v_j \cdots v_{k-2} \overline{w}$ as a clause. Summing over all possible choices $\overline{v}_1, \ldots, \overline{v}_{j-1}$ and $v_j, \ldots, v_{k-2}$ gives the result.

(c) Fix a variable $v$. By (a), for any $j - 1$ negative literals $\overline{v}_1, \ldots, \overline{v}_{j-1}$ and $k - j$ positive literals $v_j, \ldots, v_{k-2}$, there are at most $\zeta n$ variables $w$ such that $G$ contain $\overline{v}_1 \cdots \overline{v}_{j-1} v_j \cdots v_{k-2} \overline{w}$ as a clause. Summing over all possible choices $\overline{v}_1, \ldots, \overline{v}_{j-1}$ and $v_j, \ldots, v_{k-2}$ gives the result. \qed
6.2. Most minimal formulae have few clauses with negated literals. We will eventually show that most formulae in \( \mathcal{I}_1(n, \zeta) \) have few non-monotone clauses. In this subsection, we give an upper bound on the number of formulae in \( \mathcal{I}_1^*(n, \zeta) \) that have a large numbers of clauses containing \( i \) negative literals, for any \( i \in [k-2] \).

**Definition 6.3.** For every \( \beta_1 > 0 \), define

\[
\mathcal{I}_{2,1}^*(n, \zeta, \beta_1) = \left\{ G \in \mathcal{I}_1^*(\zeta) : \text{for all } u \in X, G \text{ contains at most } \beta_1 n^{k-1} \text{ clauses of the form } \overline{u}v_1 \cdots v_{k-1} \right\}.
\]

For \( k \geq 4 \), for every \( i \in \{2, \ldots, k-2\} \) and \( \beta_i > 0 \), define

\[
\mathcal{I}_{2,i}^*(n, \zeta, \beta_1, \beta_i) = \left\{ G \in \mathcal{I}_{2,1}^*(n, \zeta, \beta_1) : \text{for all } \{u_1, \ldots, u_i\} \in \binom{X}{i}, G \text{ contains at most } \beta_i n^{k-i} \text{ clauses of the form } \overline{u_1} \cdots \overline{u_i}v_1 \cdots v_{k-i} \right\}.
\]

For \( k \geq 3 \) and every \( \beta > 0 \), define

\[
\mathcal{I}_{2}^*(n, \zeta, \beta) = \bigcap_{i=2}^{k-2} \mathcal{I}_{2,i}^*(n, \zeta, \beta_1, \beta_i)
= \left\{ G \in \mathcal{I}_1^*(\zeta) : \text{for all distinct variables } u_1, \ldots, u_i \in X \text{ where } i \in [k-2], G \text{ contains at most } \beta n^{k-i} \text{ clauses of the form } \overline{u_1} \cdots \overline{u_i}v_1 \cdots v_{k-i} \right\}.
\]

For \( k = 2 \), we define \( \mathcal{I}_{2}^*(n, \zeta, \beta) = \mathcal{I}_{2,1}^*(n, \zeta, \beta_1) \).

We will first show that most formulae in \( \mathcal{I}_1^*(n, \zeta) \) lie in \( \mathcal{I}_{2,1}^*(n, \zeta, \beta_1) \).

6.2.1. Formulae with lots of clauses \( \overline{w}v_1 \cdots v_{k-1} \). We begin by studying the \( k = 2 \) case.

**Lemma 6.4.** Suppose \( k = 2 \). For every \( \beta_1 < 1/2 \), if \( \zeta \) is sufficiently small relative to \( \beta_1 \), there exists \( c > 0 \) such that

\[
|\mathcal{I}_1^*(n, \zeta) \setminus \mathcal{I}_{2,1}^*(n, \zeta, \beta_1)| \leq \exp_2 \left[ (1 - c) \binom{n}{k} \right].
\]

**Proof.** For every formula \( G \in \mathcal{I}_1^*(n, \zeta) \setminus \mathcal{I}_{2,1}^*(n, \zeta, \beta_1) \), there exists some variable \( u \in X \) such that \( G \) contains more than \( \beta_1 n \) clauses of the form \( \overline{w}v \). From the fact that (1) \( u \) is used by at least \( n/\beta_1 \) clauses in \( G \), (2) \( m(u) \geq m(\overline{w}) \) and (3) \( G \) contains at most \( \zeta n \) clauses of the form \( \overline{w}v \), we know that \( G \) contains at least \((1/40 - \zeta)n \) clauses of the form \( uv \). Define

\[
N_u = \{ v \in X_{-u} : uv \in G \}, \\
N_{\overline{w}} = \{ v \in X_{-u} : \overline{w}v \in G \}.
\]

Observe that, to avoid the non-minimal formula \( \{ \overline{w}v_2, uv_1, v_1v_2 \} \), all monotone clauses \( v_1v_2 \) with \( v_1 \in N_u, v_2 \in N_{\overline{w}} \) cannot lie in \( G \). Thus, given \( N_u \) and \( N_{\overline{w}} \), we know at least \(|N_u||N_{\overline{w}}| - n \) monotone clauses that cannot lie in \( G \).

Each formula \( G \in \mathcal{I}_1^*(n, \zeta) \setminus \mathcal{I}_{2,1}^*(n, \zeta, \beta_1) \) is then uniquely determined by specifying the following in order:

1. \( u \in X \);
2. \( N_u, N_{\overline{w}} \subseteq X \);
3. \( \{ v \in X : \overline{w}v \in G \} \) and \( \{ v \in X : \overline{w}v \in G \} \);
4. the non-monotone clauses in \( G \) that do not use the variable \( u \);
5. the monotone clauses in \( G \) that do not use the variable \( u \).
We then have the following upper bound on $|\mathcal{I}^*_1(n, \zeta) \setminus \mathcal{I}^*_{2,1}(n, \zeta, \beta_1)|$, in terms of the number of choices for each item.

$$|\mathcal{I}^*_1(n, \zeta) \setminus \mathcal{I}^*_{2,1}(n, \zeta, \beta_1)| \leq \frac{n}{k} \left( \frac{2^k - 1}{2} \right)^{\left( \frac{n}{k} \right)^2} \left( \frac{2^k}{2} \right)^{\beta_1(1/40-\zeta)2^{-n}} \leq \exp \left[ (1 - c) \left( \frac{n}{2} \right)^k \right]. \quad \square$$

We now turn to the $k \geq 3$ case.

**Lemma 6.5.** For $k \geq 3$ and any positive $\beta_1 < 1/k$, if $\zeta$ is sufficiently small relative to $\beta_1$, there exists $c > 0$ such that

$$|\mathcal{I}^*_1(n, \zeta) \setminus \mathcal{I}^*_{2,1}(n, \zeta, \beta_1)| \leq \exp \left[ (1 - c) \left( \frac{n}{k} \right)^k \right] \exp \left[ (1 - c) \left( \frac{n}{k} \right)^k \right] |\mathcal{I}(n-1)|.$$

**Proof.** Choose constants $0 < c \ll \zeta \ll \theta_1 \ll \beta_1 \ll 1/k$.

For every formula $G \in \mathcal{I}^*_1(n, \zeta) \setminus \mathcal{I}^*_{2,1}(n, \zeta, \beta_1)$, by definition, there is some variable $u \in X$ such that $G$ contains more than $\beta_1 n^{k-1}$ clauses of the form $\tau v_1 \cdots v_{k-1}$. Fix this variable $u$ and let $X_{-u} = X \setminus \{u\}$. Define the *monotone links* of literals $u$ and $\bar{u}$ to be the $(k-1)$-graphs with edge sets given by

$$N_u = \{(v_1, \ldots, v_{k-1}) \subseteq X_{-u} : uv_1 \cdots v_{k-1} \in G\},$$

$$N_{\bar{u}} = \{(v_1, \ldots, v_{k-1}) \subseteq X_{-u} : \bar{u}v_1 \cdots v_{k-1} \in G\}.$$

Now consider the collections of $(k-2)$-subsets of $X_{-u}$ that have small co-degree (less than $\theta_1 n$) in the hypergraphs $N_u, N_{\bar{u}}$. Precisely,

$$S_u = \{(v_1, \ldots, v_{k-2}) \subseteq X_{-u} : |\{w : \{w, v_1, \ldots, v_{k-2}\} \in N_u\}| < \theta_1 n\},$$

$$S_{\bar{u}} = \{(v_1, \ldots, v_{k-2}) \subseteq X_{-u} : |\{w : \{w, v_1, \ldots, v_{k-2}\} \in N_{\bar{u}}\}| < \theta_1 n\}.$$

Finally, let $T_{-u}$ be the $(k-2)$-subsets of $X_{-u}$ that fall in neither $S_u$ nor $S_{\bar{u}}$, i.e., that have large co-degree into both $N_u$ and $N_{\bar{u}}$, so that $T_{-u} = \left( X_{-u} \right)^{\left( k-2 \right)} \setminus (S_u \cup S_{\bar{u}})$. Since $|S_u|, |S_{\bar{u}}| < \left( \frac{n}{k} \right)^{k-2}$, we see that

$$|N_u| \leq \theta_1 n |S_u| + n (|S_u| + |T_{-u}|) \leq n (|S_u| + |T_{-u}|) + \frac{\theta_1 n^{k-1}}{(k-2)!}.$$

We assumed that $G \notin \mathcal{I}^*_{2,1}(n, \zeta, \beta_1)$, so $|N_{\bar{u}}| \geq \beta_1 n^{k-1}$.

Combining this lower bound with the above upper bound on $|N_{\bar{u}}|$ means that at least one of the following must hold:

1. $|T_{-u}| > \theta_1 n^{k-2}$;
2. $|T_{-u}| \leq \theta_1 n^{k-2}$; in this case, from the above inequalities, we can get a lower bound for $|S_u|$:

$$|S_u| \geq \beta_1 n^{k-2} - \frac{\theta_1 n^{k-2}}{(k-2)!} - |T_{-u}| \geq \beta_1 n^{k-2} - \frac{\theta_1 n^{k-2}}{(k-2)!} - \frac{\theta_1 n^{k-2}}{(k-2)!} > \theta_1 n^{k-2},$$

as $\theta_1 \ll \beta_1$.

We upper bound the number of formulae in $\mathcal{I}^*_1(n, \zeta) \setminus \mathcal{I}^*_{2,1}(n, \zeta, \beta_1)$ in each of these two cases.

**Claim 6.6.** There are at most $\exp \left[ (1 - c) \left( \frac{n}{k} \right)^k \right]$ formulae $G \in \mathcal{I}^*_1(\zeta) \setminus \mathcal{I}^*_{2,1}(n, \zeta, \beta_1)$ such that for some variable $u \in X$, we have $|T_{-u}| > \theta_1 n^{k-2}$.

**Proof.** Each such formula $G$ is uniquely determined by specifying the following information in order:

1. $u \in X$;
2. $T_{-u} \subseteq \left( \frac{X}{k-2} \right)$;
3. $\{(v_1, \ldots, v_{k-1}) \in N_u : \text{there exists } i \in [k-1] \text{ such that } \{v_1, \ldots, v_{k-1}\} \setminus \{v_i\} \in T_{-u}\}$;
(4) \( \{v_1, \ldots, v_{k-1}\} \in N_{\overline{u}} \): there exists \( i \in [k-1] \) such that \( \{v_1, \ldots, v_{k-1}\} \setminus \{v_i\} \in T_{-u} \);
(5) The non-monotone clauses in \( G \);
(6) The monotone clauses in \( G \).

We prove the claim by giving an upper bound on the number of choices for each piece of information:

\[
\frac{n}{(1)} \cdot 2^{\binom{\nu_2}{2}} \cdot \frac{2}{|T_{-u}|} |T_{-u}| \cdot n \cdot \left( \binom{2k-1}{n} - 1 \right) \exp_2 \left[ \frac{n}{k} \right] < \exp_2 \left[ (1 - c) \right] = \exp_2 \left[ (1 - c) \right]
\]

(recall \( c \ll \zeta \ll \theta_1 \)). In particular, bounds (5) and (6) come from the following observations:

(5) Since \( G \in I_1^\circ(n, \zeta) \) has at most \( \zeta n^k \) non-monotone clauses.
(6) Suppose we have specified (1)–(4). Recall from the definition of \( T_{-u} \) that for every \( \{v_1, \ldots, v_{k-2}\} \in T_{-u} \), there are at least \( \theta_1 n \) variables \( w \in X \) such that \( v_1 \cdots v_{k-2} w u \in G \), and at least \( \theta_1 n \) variables \( w' \in X \) such that \( v_1 \cdots v_{k-2} w' \overline{u} \in G \). For every such pair \( w, w' \), the monotone clause \( v_1 \cdots v_{k-2} w u \) is forbidden from \( G \), else \( G \) contains the non-minimal formula

\[
\{ v_1 \cdots v_{k-2} w u, \ v_1 \cdots v_{k-2} w u, \ v_1 \cdots v_{k-2} w \overline{u} \}
\]

(as it is impossible to satisfy only the first clause). Therefore, given (1)–(4) and the hypothesis that \( |T_{-u}| > \theta_1 n^k - 2 \), there are at least \( |T_{-u}| \cdot (\theta_1 n)^2/\binom{k}{2} = \Omega(n^k) \) monotone clauses that cannot lie in \( G \).

**Claim 6.7.** There are at most \( \exp_2 \left[ (1 - c) \binom{n}{k} \right] |I(n-1)| \) formulae \( G \in I_1^\circ(\zeta) \setminus I_{\overline{2}}(n, \zeta, \beta_1) \) such that for some variable \( u \in X \), we have \( |T_{-u}| \leq \theta_1 n^k - 2 \) and \( |S_u| > \theta_1 n^k - 2 \).

**Proof.** We follow a similarly inspired approach to the previous claim; each such formula \( G \) is uniquely determined by specifying the following information in order:

1. \( u \in X \);
2. \( S_u, S_{\overline{u}}, T_{-u} \subseteq \binom{X-u}{k-2} \);
3. \( \{v_1, \ldots, v_{k-1}\} \in N_u \): there exists \( i \in [k-1] \) such that \( \{v_1, \ldots, v_{k-1}\} \setminus \{v_i\} \in S_u \cup T_{-u} \);
4. \( \{v_i, \ldots, v_{k-1}\} \in N_{\overline{u}} \): there exists \( i \in [k-1] \) such that \( \{v_1, v_i, \ldots, v_{k-1}\} \setminus \{v_i\} \in S_{\overline{u}} \cup T_{-u} \);
5. \( \{v_1, \ldots, v_{k-1}\} \in N_u \): for all \( i \in [k-1] \), \( \{v_1, \ldots, v_{k-1}\} \setminus \{v_i\} \in S_{\overline{u}} \setminus S_u \);
6. \( \{v_1, \ldots, v_k\} \in N_{\overline{u}} \): for all \( i \in [k-1] \), \( \{v_1, \ldots, v_k\} \setminus \{v_i\} \in S_u \setminus S_{\overline{u}} \);
7. The monotone clauses in \( G \) containing \( u \);
8. The clauses in \( G \) containing \( \overline{u} \) and \( k-1 \) positive literals;
9. The clauses in \( G \) that use the variable \( u \) and contain some negative literal other than \( \overline{u} \);
10. The clauses in \( G \) that do not use the variable \( u \).

Note that (7) and (8) are uniquely determined by (3)–(6). We prove the claim by giving an upper bound on the number of choices for each item:

\[
\frac{n}{(1)} \cdot 2^{\binom{\nu_2}{2}} \cdot \frac{2}{|T_{-u}|} |T_{-u}| \cdot n \cdot \left( \binom{2k-1}{n} - 1 \right) \exp_2 \left[ \frac{n}{k} \right] < \exp_2 \left[ (1 - c) \right] \]

(recall \( c \ll \zeta \ll \theta_1 \)). These bounds come from the following observations:

(2) Since \( T_{-u} \) is uniquely determined by \( S_u, S_{\overline{u}}, \) there are at most \( 4^{\binom{n}{k-2}} \) possible triples \( S_u, S_{\overline{u}}, T_{-u} \).
(3)(4) For every \( (k-2) \)-set in \( T_{-u} \), there are at most \( 2^n \) ways to choose the \( (k-1) \)-sets in \( N_u \) containing it. For every \( (k-2) \)-set in \( S_u \), the number of \( (k-1) \)-sets in \( N_u \) containing it
The candidate elements for (5) are those \((k-1)\) subsets of \(X_u\) whose \((k-2)\)-subsets all lie in \(S_{\pi} \setminus S_u\). By the Kruskal–Katona theorem (Theorem 3.2), the number of such candidates is at most \(((k-2)!|S_u \setminus S_{\pi}|)^{\frac{k-1}{k-2}}/(k-1)!\). By a similar reasoning, the number of candidates for (6) is at most \(((k-2)!|S_{\pi} \setminus S_u|)^{\frac{k-1}{k-2}}/(k-1)!\). We know from the claim hypothesis that \(|S_u| \geq \theta_1 n^{k-2}\). Moreover, since \(G\) is \(\frac{1}{10k}\)-dense and positive-dominant, there are at least \(\frac{1}{20k} \binom{n}{k-1}\) clauses in \(G\) that contains the positive literal \(u\). By Lemma 6.5(b), at most \((k-1)\zeta n^{k-1}\) of these clauses are non-monotone, so we have \(|N_u| \geq \frac{1}{20k} \binom{n}{k-1} - (k-1)\zeta n^{k-1}\).

Since

\[
|N_u| \leq \frac{1}{k-1} \left( \theta_1 n |S_u| + n \left( \binom{n-1}{k-2} - |S_u| \right) \right)
\]

with \(\zeta \ll \theta_1 \ll 1/k\), we also have the upper bound \(|S_u| \leq \binom{n}{k-2} - \theta_1 n^{k-2}\). Thus, we have the upper bound

\[
\frac{((k-2)!|S_u \setminus S_{\pi}|)^{\frac{k-1}{k-2}}}{(k-1)!} + \frac{((k-2)!|S_{\pi} \setminus S_u|)^{\frac{k-1}{k-2}}}{(k-1)!}
\]

\[
= \frac{((k-2)!)^{\frac{k-1}{k-2}}}{(k-1)!} \left( |S_u \setminus S_{\pi}|^{\frac{k-1}{k-2}} + |S_{\pi} \setminus S_u|^{\frac{k-1}{k-2}} \right)
\]

\[
\leq \frac{((k-2)!)^{\frac{k-1}{k-2}}}{(k-1)!} \left( \theta_1^{\frac{k-1}{k-2}} + \left( \frac{1}{(k-2)!} - \theta_1 \right)^{\frac{k-1}{k-2}} \right) n^{k-1},
\]

where (*) follows from the convexity of \(x \mapsto x^{(k-1)/(k-2)}\) and the fact that \(\theta_1 n^{k-2} \leq |S_u| \leq \binom{n}{k-2} - \theta_1 n^{k-2}\):

\[
|S_u \setminus S_{\pi}|^{\frac{k-1}{k-2}} + |S_{\pi} \setminus S_u|^{\frac{k-1}{k-2}} \leq |S_u|^{\frac{k-1}{k-2}} + \left( \binom{n}{k-2} - |S_u| \right)^{\frac{k-1}{k-2}}
\]

\[
\leq \left( \theta_1^{\frac{k-1}{k-2}} + \left( \frac{1}{(k-2)!} - \theta_1 \right)^{\frac{k-1}{k-2}} \right) n^{k-1}.
\]

Thus the number of choices for (5)(6) is at most

\[
\exp_2 \left[ \frac{((k-2)!)^{\frac{k-1}{k-2}}}{(k-1)!} \left( \theta_1^{\frac{k-1}{k-2}} + \left( \frac{1}{(k-2)!} - \theta_1 \right)^{\frac{k-1}{k-2}} \right) n^{k-1} \right] \leq \exp_2 \left[ \left( \theta_1^{\frac{k-1}{k-2}} + \frac{1}{(k-1)!} \right) n^{k-1} \right],
\]

since \(\frac{((k-2)!)^{\frac{k-1}{k-2}}}{(k-1)!} \leq \frac{1}{2}\) for \(k \geq 3\).

(9) By Lemma 6.2(b), for every \(j \in \{1, \ldots, k-1\}\), there are at most \(\zeta n^{k-1}\) clauses in \(G\) containing the positive literal \(u\) and exactly \(j\) negative literals; hence there are at most \(\binom{\binom{n}{k-2}}{k-1}^{\frac{k-1}{k-1}}\) ways to determine the non-monotone clauses in \(G\) that contain the positive
For every Claim 6.9, there are at most
\[
\left(\binom{2n-2}{k-1}\right)^{k-1} \leq \exp_2 \left[ \binom{2n-2}{k-1} H \left( \frac{\zeta^{n-1}}{\binom{2n-2}{k-1}} \right) (k-1) \right]
\]  
(Equation 8)

ways to determine the clauses in $G$ that contain $\overline{u}$ and some other negative literal. $\blacksquare$

Combining Claim 6.6 and Claim 6.7, we get that
\[
|\mathcal{I}_1^*(n, \zeta) \setminus \mathcal{I}_{2,1}^*(n, \zeta, \beta_1)| \leq \exp_2 \left[ (1 - c) \left( \frac{n}{k} \right) \right] + \exp_2 \left[ (1 - c) \left( \frac{n}{k-1} \right) \right] |\mathcal{I}(n-1)|.
\]

6.2.2. Formulae with lots of clauses $\overline{u} \cdots \overline{v}_{i-1}$.

Throughout this subsection, we assume $k \geq 4$. Next, we upper bound the size of $\mathcal{I}_{2,1}^*(n, \zeta, \beta_1) \setminus \mathcal{I}_2^*(n, \zeta, \beta_1)$. Recall from Definition 6.3 that for every $i \in \{2, \ldots, k-2\}$ and $\beta_i > 0$, we have
\[
\mathcal{I}_{2,i}^*(n, \zeta, \beta_1, \beta_i) = \{ G \in \mathcal{I}_{2,1}^*(n, \zeta, \beta_1) : \text{for all distinct variables } u_1, \ldots, u_i \in X, G \text{ contains at most } \beta_i n^{k-i} \text{ clauses of the form } \overline{u}_1 \cdots \overline{u}_i \cdots \overline{v}_{k-i}, \}
\]
and $\mathcal{I}_2^*(n, \zeta, \beta_1) = \bigcap_{k=2}^{k-2} \mathcal{I}_{2,i}^*(n, \zeta, \beta_1, \beta_i)$.

We give upper bound on the size of $\mathcal{I}_{2,1}^*(n, \zeta, \beta_1) \setminus \mathcal{I}_2^*(n, \zeta, \beta_1, \beta_i)$ for every $i \in \{2, \ldots, k-2\}$.

**Lemma 6.8.** For $k \geq 4$ and every $i \in \{2, \ldots, k-2\}$ and $\beta_i > 0$, if positive $\zeta$ and $\beta_1$ are chosen sufficiently small relative to $\beta_i$, there exists $c > 0$ such that
\[
|\mathcal{I}_{2,1}^*(n, \zeta, \beta_1) \setminus \mathcal{I}_{2,i}^*(n, \zeta, \beta_1, \beta_i)| \leq \exp_2 \left[ (1 - c) \left( \frac{n}{k} \right) \right] + \exp_2 \left[ (1 - c) \left( \frac{n}{k-1} \right) \right] |\mathcal{I}(n-i)|.
\]

**Proof.** For every $i \in \{2, \ldots, k-2\}$ and $\beta_i > 0$, choose constants $0 < c \ll \zeta \ll \theta_i \ll \beta_i \ll 1/k$ and $\beta_1 \ll \beta_i$. We will follow a similar proof strategy to that used in Lemma 6.5.

For every $G \in \mathcal{I}_{2,1}^*(n, \zeta, \beta_1) \setminus \mathcal{I}_{2,i}^*(n, \zeta, \beta_1, \beta_i)$, by definition, there exist some $u_1, \ldots, u_i \in X$ such that there are more than $\beta_i n^{k-i}$ clauses of the form $\overline{u}_1 \cdots \overline{u}_i \cdots \overline{v}_{k-i}$. Let $X_u = X \setminus \{u_1, \ldots, u_i\}$ and define the associated $(k-i)$-graph
\[
N_{\overline{u}} = \{ \{v_1, \ldots, v_{k-i}\} \subseteq X_u : \overline{u}_1 \cdots \overline{u}_i \cdots \overline{v}_{k-i} \in G \},
\]
so that by our assumption on $G$, $|N_{\overline{u}}| > \beta_i n^{k-i}$. For every $v = \{v_1, \ldots, v_{k-i}\} \subseteq X_u$, let $N_{u,v}$ be the $(i-1)$-graph comprising the common co-neighborhood of sets $\{u_j, v_1, \ldots, v_{k-i}\}$ over all $j \in [i]$. Formally, let
\[
N_{u,v} = \{ \{w_1, \ldots, w_{i-j}\} \subseteq X_u : u_j v_1 \cdots v_{k-i} w_{i-1} \cdots w_1 \in G \text{ for all } j \in [i]\},
\]
\[
A_u = \{ v \in N_{\overline{u}} : |N_{u,v}| \geq \theta_i n^{i-1} \},
\]
\[
B_u = N_{\overline{u}} \setminus A_u.
\]

Since $|N_{\overline{u}}| > \beta_i n^{k-i}$, at least one of the following holds:

1. $|A_u| > \theta_i n^{k-i}$;
2. $|B_u| > (\beta_i - \theta_i) n^{k-i}$.

We upper bound the number of formulae in $\mathcal{I}_{2,1}^*(n, \zeta, \beta_1) \setminus \mathcal{I}_{2,i}^*(n, \zeta, \beta_1, \beta_i)$ under each of these two cases.

**Claim 6.9.** There are at most $\exp_2 \left[ (1 - c) \left( \frac{n}{k} \right) \right]$ formulae $G \in \mathcal{I}_{2,1}^*(n, \zeta, \beta_1) \setminus \mathcal{I}_{2,i}^*(n, \zeta, \beta_1, \beta_i)$ such that the following holds for some $u_1, \ldots, u_k \in X$: (1) $G$ contains more than $\beta_i n^{k-i}$ clauses of the form $\overline{u}_1 \cdots \overline{u}_i \cdots \overline{v}_{k-i}$; (2) $|A_u| > \theta_i n^{k-i}$.
Proof. Every such $G$ is uniquely determined by sequentially specifying the following information:

1. $u_1, \ldots, u_i \in X$;
2. $A_u \in \binom{X_i}{k-i}$;
3. $N_{u,v}$ for every $v \in A_u$;
4. The non-monotone clauses in $G$;
5. The monotone clauses in $G$.

We prove the claim by giving an upper bound on the number of choices for each item:

\[
\left( \frac{n}{i} \right) \binom{n}{k} \binom{n}{k} \leq \left( \frac{2^k(n)}{k} \right) \exp \left[ \left( \frac{n}{k} \right) - \Omega(n^k) \right] \leq \exp \left[ (1 - c) \left( \frac{n}{k} \right) \right]
\]

(recall that $c \ll \zeta \ll \theta_i$). The bounds come from the following observations:

3. For every fixed $v \in A_u$, there are $2^{(n-1)}$ possible choices for $N_{u,v}$. Since $|A_u| \leq \binom{n}{k-i}$, the number of possible choices for (3) is at most $2^{n-1}(\binom{n}{k-i}) \leq 2^{n-k-1}$.
4. Since $G \in \mathcal{I}_4(n, \zeta)$, it has at most $\zeta n^k$ non-monotone clauses.
5. For every $v = \{v_1, \ldots, v_k\} \in A_u$ and $\{w_1, \ldots, w_{i-1}\} \in N_{u,v}$, to avoid the non-minimal subformula

\[
\{v_1 \cdots v_k w_1 \cdots w_{i-1} z,
\begin{align*}
&v_1 \cdots v_k | w_1 \cdots w_{i-1} u_1, \ldots, v_1 \cdots v_k | w_1 \cdots w_{i-1} u_i, \\
&\bar{\pi}_1 \cdots \bar{\pi}_i v_1 \cdots v_k \}
\end{align*}
\]

(as it is impossible to satisfy only the first clause), we cannot have $v_1 \cdots v_k w_1 \cdots w_{i-1} z \in G$ for any $z \in X$. Therefore, (1)-(3) have determined at least

\[
\frac{1}{i^{(k)}} |A_u| \theta_i n^{i-1}(n - (k - 1)) \geq \frac{1}{i^{(k)}} \theta_i n^{k-i} \theta_i n^{i-1}(n - (k - 1)) \geq \frac{\theta_i^2 n^k}{i^{(k)}} = \Omega(n^k)
\]

monotone clauses that cannot belong to $G$.

\[\square\]

Claim 6.10. There are at most $\exp \left[i(1-c)\binom{n}{k-i}\right] |\mathcal{I}(n-i)|$ formulae $G \in \mathcal{I}_{2,1}(n, \zeta, \beta_1) \setminus \mathcal{I}_{2,1}(n, \zeta, \beta_1, \beta_i)$ such that the following holds for some $u_1, \ldots, u_k \in X$: (1) $G$ contains more than $\beta_i n^{k-i}$ clauses of the form $\bar{\pi}_1 \cdots \bar{\pi}_i v_1 \cdots v_k$, (2) $|B_u| > (\beta_i - \theta_i)n^{k-i}$.

Proof. Every such $G$ is uniquely determined by sequentially specifying the following information:

1. $u_1, \ldots, u_i \in X$;
2. $B_u \in \binom{X_i}{k-i}$;
3. $N_{u,v}$ for every $v \in B_u$;
4. The monotone clauses in $G$ that use at least one of the variables $u_1, \ldots, u_i$;
5. The clauses in $G$ that contain one of the negative literals $\bar{\pi}_1, \ldots, \bar{\pi}_i$ and $k - 1$ other positive literals;
6. The clauses in $G$ that (i) use at least one of the variables $u_1, \ldots, u_i$, and (ii) contain a negative literal that is different from the negation of this variable;
7. The clauses in $G$ that do not use any of the variables $u_1, \ldots, u_i$. 

We prove the claim by giving an upper bound on the number of choices for each item:

\[
\sum_{j=1}^{n} \frac{\binom{n}{j}}{2^{k-1} \binom{n}{j}} \leq \beta_1 \exp \left[ i \left( \frac{n - 1}{k - 1} - (i - \log(2^i - 1)) \Omega(\beta_i) \right) \right]
\]

\[
\leq \beta_1 \exp \left[ i \left( \frac{2^{k-1} \binom{n}{j}}{k - 1} \right) \right] \leq k \zeta \exp \left[ i \left( \frac{n - 1}{k - 1} \right) \right] \Omega(\beta_i)
\]

\[
< \exp \left[ i \left( 1 - c \right) \left( \frac{n}{k - 1} \right) \right] \Omega(\beta_i)
\]

(recall that \( c \ll \zeta \ll \theta_i \ll \beta_i \) and \( \beta_1 \ll \beta_i \)). The bounds come from the following observations:

3. For every fixed \( v \in B_u \), since \(|N_{u,v}| < \theta_i n^{i-1}\), there are at most \( \binom{n}{i-1} \exp \left[ i \left( \frac{n - 1}{k - 1} \right) \right] \Omega(\beta_i)
\] possible choices for \( N_{u,v} \).

4. Let \( Q_u = \{ v_1 \cdots v_{k-i} w_1 \cdots w_{i-1} \} \subset \binom{X_u}{k-1} \) such that \( v = \{v_1, \ldots, v_{k-i} \} \in B_u \) and \( \{w_1, \ldots, w_{i-1} \} \notin N_{u,v} \).

   By definition of \( N_{u,v} \), for all \( \{v_1 \cdots v_{k-i} w_1 \cdots w_{i-1} \} \in Q_u \), there exists some \( j \in [i] \) such that \( (v_1 \cdots v_{k-i} w_1 \cdots w_{i-1}) \in G \). Since \( |B_u| > (\beta_i - \theta_i)n^{k-i} \) and \(|N_{u,v}| < \theta_i n^{i-1}\) for every \( v \in B_u \), (1)-(3) have specified at least

\[
\frac{1}{i-1} \binom{k-1}{i-1} (\beta_i - \theta_i)n^{k-i} \left( \binom{n - 1}{i - 1} - \theta_i n^{i-1} \right) \geq \Omega(\beta_i)
\]

\( (k-1) \)-subsets of \( X \) that lie in \( Q_u \) (recall that \( \theta_i \ll \beta_i \)). For every \( \{v_1 \cdots v_{k-i} w_1 \cdots w_{i-1} \} \in Q_u \), there are at most \( 2^i - 1 \) ways to choose which of \( u_1 v_1 \cdots v_{k-i} w_1 \cdots w_{i-1} \), \( u_i v_i \cdots v_{k-i} w_i \cdots w_{i-1} \) lie in \( G \). Therefore, there are at most

\[
\exp \left[ i \left( \frac{n - 1}{k - 1} \right) \right] \Omega(\beta_i)
\]

choices for the monotone clauses in \( G \) that contain some of \( u_1, \ldots, u_i \).

5. For every \( j \in [i] \), since \( G \in \mathcal{I}^*_2(n, \zeta, \beta_1) \), there are at most \( \beta_1 n^{k-1} \) clauses in \( G \) that contain \( \overline{w_j} \) and \( k - 1 \) positive literals.

6. For every variable \( u_j \), by Lemma 6.2(b)(c), there are at most \( k \zeta n^{k-1} \) clauses in \( G \) that contain the positive literal \( u_j \) and some negative literal; there are at most \( k \zeta n^{k-1} \) clauses in \( G \) that contain the negative literal \( \overline{u_j} \) and some other negative literal. Hence there are at most

\[
\left( \frac{2^{k-1} \binom{n}{j}}{k - 1} \right) \Omega(\beta_i)
\]

choices for (6).

Combining Claim 6.9 and Claim 6.10, we get that

\[
|\mathcal{I}^*_2(n, \zeta, \beta_1) \setminus \mathcal{I}^*_2(n, \zeta, \beta_1, \beta_i)| \leq \exp \left[ (1 - c) \left( \frac{n}{k} \right) \right] + \exp \left[ i(1 - c) \left( \frac{n}{k} \right) \right] \Omega(\beta_i). \]

6.3. \textit{k-wise boolean combinations of monotone neighborhoods are typically not too small}. It remains to upper bound the number of formulae in \( \mathcal{I}^*_2(n, \zeta, \beta) \). In this section, we make some preparations for giving this bound.
For every $k$ variables $u_1, \ldots, u_k \in X$, define
\[
X_{-u} = X \setminus \{u_1, \ldots, u_k\},
\]
\[
N_{u_j}^u = \{v_1, \ldots, v_{k-1} \} \subseteq X_{-u} : u_jv_1 \cdots v_{k-1} \in G,
\]
\[
\frac{N_{u_j}^u}{N_{u_j}^u} = \left(\frac{X_{-u}}{k-1}\right) \setminus \{v_1, \ldots, v_{k-1} \} \subseteq X_{-u} : u_jv_1 \cdots v_{k-1} \in G.
\]
We say that vectors $\vec{a}, \vec{b}$ have $\vec{a} \succ \vec{b}$, if $a$’s entries are component-wise larger than $b$’s entries.

**Definition 6.11.** For $\vec{\beta} \succ 0$, let
\[
\mathcal{I}_3^*(n, \zeta, \vec{\beta}) = \left\{ G \in \mathcal{I}_2(n, \zeta, \vec{\beta}) : \text{ for all } u_1, u_2, \ldots, u_k \in X,
M_{u_1} \in \{N_{u_1}^{-u}, \overline{N_{u_1}^{-u}}\}, \ldots, M_{u_k} \in \{N_{u_1}^{-u}, \overline{N_{u_k}^{-u}}\},
\right.
\left.
\text{ we have } |M_{u_1} \cap \cdots \cap M_{u_k}| \geq \frac{1}{2^k} \binom{n-k}{k-1}\right\}.
\]

**Lemma 6.12.** For all $k \geq 2$ and all sufficiently small $\vec{\beta} \succ 0$, if $\zeta$ is sufficiently small relative to $\beta_1$, there exists $c > 0$ such that
\[
|\mathcal{I}_2(n, \zeta, \vec{\beta}) \setminus \mathcal{I}_3^*(n, \zeta, \vec{\beta})| \leq \exp \left[ \frac{k}{2} (1-c) \left( \frac{n}{k-1} \right) \right] |\mathcal{I}(n-k)|.
\]

**Proof.** Fix $\vec{\beta} \succ 0$ sufficiently small and choose $0 < c \ll \zeta \ll \beta_1$. For every formula $G \in \mathcal{I}_2(n, \zeta, \vec{\beta}) \setminus \mathcal{I}_3^*(n, \zeta, \vec{\beta})$, there exist variables $u_1, \ldots, u_k \in X$ and $M_{u_1} \in \{N_{u_1}^{-u}, \overline{N_{u_1}^{-u}}\}, \ldots, M_{u_k} \in \{N_{u_1}^{-u}, \overline{N_{u_k}^{-u}}\}$ such that $|M_{u_1} \cap \cdots \cap M_{u_k}| < \frac{1}{2^k} \binom{n-k}{k-1}$.

Every formula $G \in \mathcal{I}_2(n, \zeta, \vec{\beta}) \setminus \mathcal{I}_3^*(n, \zeta, \vec{\beta})$ is uniquely determined by sequentially specifying the following information.

1. $u_1, \ldots, u_k$;
2. The non-monotone clauses in $G$ that use at least one of the variable $u_1, \ldots, u_k$.
3. The monotone clauses in $G$ that use at least one of the variable $u_1, \ldots, u_k$.
4. The clauses in $G$ that do not use any of the variables $u_1, \ldots, u_k$.

We upper bound $|\mathcal{I}_2(n, \zeta, \vec{\beta}) \setminus \mathcal{I}_3^*(n, \zeta, \vec{\beta})|$ by upper bounding the number of choices for each item:

\[
|\mathcal{I}_2(n, \zeta, \vec{\beta}) \setminus \mathcal{I}_3^*(n, \zeta, \vec{\beta})| \leq \left(\frac{n}{k}\right)^k \left(\frac{(2^k - 1)(n-1)}{k-1}\right)^k \exp \left[ H \left( \frac{1}{2^k} \right) \left( \frac{n-k}{k-1} \right) \right] |\mathcal{I}(n-k)|
\]
\[
\leq \exp \left[ \frac{k}{2} (1-c) \left( \frac{n}{k-1} \right) \right] |\mathcal{I}(n-k)|
\]
(recall that $c \ll \zeta \ll \beta_1$). In particular, upper bounds for (2) and (3) arise from the following observations.

1. For every variable $u_j$, by Lemma 6.2(b)(c), $G$ has at most $2(k-1)\zeta n^{k-1}$ non-monotone clauses that use the variable $u_j$ and contain some negative literal other than $\overline{u_j}$. Moreover, since $G \in \mathcal{I}_2(n, \zeta, \beta_1)$, $G$ has at most $\beta_1 n^{k-1}$ clauses of the form $\overline{u_j}v_1 \cdots v_{k-1}$.
2. There are $2^O(n^{k-2})$ possibilities to choose the monotone clauses in $G$ that use at least two of the variables $u_1, \ldots, u_k$. The monotone clauses in $G$ that use exactly one of the variables $u_1, \ldots, u_k$ is completely determined by
\[
\{M_{u_1} \cap \cdots \cap M_{u_k} : M_{u_1} \in \{N_{u_1}^{-u}, \overline{N_{u_1}^{-u}}\}, \ldots, M_{u_k} \in \{N_{u_k}^{-u}, \overline{N_{u_k}^{-u}}\}\},
\]
Thus, it suffices to show that $B$ show that $|W| \leq |N_{B_H}(W)|$ for every $W \subseteq E(H)$.
For every $W \subseteq E(H)$, $N_{B_H}(W)$ consists of all $S \subseteq \binom{V(H)}{\ell-1}$ (each repeated $L$ times) that are contained in some $e \in W$. By the Kruskal–Katona theorem (Theorem 3.2), every collection $S \subseteq \binom{V(H)}{\ell-1}$ covers at most $((\ell-1)!|S|)^{\frac{L}{\ell}} / \ell!$ edges in $H$, so every $W \subseteq E(H)$ satisfies

$$|W| \leq ((\ell-1)!|N_{B_H}(W)|/L)^{\frac{L}{\ell}} / \ell!.$$ 

Rearranging gives

$$|N_{B_H}(W)| \geq \frac{(\ell)!^{\frac{L}{\ell}} |W|^{\frac{L}{\ell}} L}{(\ell-1)!} \geq \frac{(\ell)!^{\frac{L}{\ell}} |W|^{\frac{L}{\ell}} L}{(\ell-1)!} \cdot (\ell-1)! \cdot e(H)^{\frac{L}{\ell}} = |W|^{\frac{L}{\ell}} \cdot e(H)^{\frac{L}{\ell}} \geq |W|. \quad \Box$$

We will also use the following version of Shearer’s inequality [9] (see also [18, Lemma 6.5]).

**Lemma 6.16.** Let $W$ be a set and $\mathcal{F}$ be a family of subsets of $W$. Let $H$ be a hypergraph with $V(H) = W$ and $\deg_H(v) \geq k$ for every $v \in W$. Then

$$\log_2 |\mathcal{F}| \leq \frac{1}{k} \sum_{e \in E(H)} \log_2 |\text{Tr}(\mathcal{F}, e)|,$$

where $\text{Tr}(\mathcal{F}, e) = \{F \cap e : F \in \mathcal{F}\}$.

Next, we verify the following technical result.

**Lemma 6.17.** Suppose $\bar{\beta}, \zeta$ are all sufficiently small constants (relative to $k$). Then, there exists $c > 0$ such that for all $0 \leq t_1, \ldots, t_{k-2} \leq \zeta n^k$, we have

$$|\left\{ G \in \mathcal{I}_3^*(n, \zeta, \bar{\beta}) \setminus \mathcal{I}_4^*(n, \zeta, \bar{\beta}) : G \text{ contains no clause with exactly } k \text{ negative literals},
|G_1| = t_1, \ldots, |G_{k-2}| = t_{k-2}\right\}|$$

$$< \exp_2 \left[ \binom{n}{k} - 2cn \right],$$

where $G_i$ is the set of clauses in $G$ with exactly $i$ negative literals (so $G$ is the disjoint union of $G_0, G_1, \ldots, G_{k-2}$).

The proof of this lemma is rather involved, so below we give a brief informal outline of the general argument:

- We consider some $k$-SAT formula $G$ satisfying the lemma conditions for a particular choice of parameters $(t_1, \ldots, t_{k-2})$.
- We partition the clauses of $G$ based on the number of negative literals $i$, and for each $i$ we construct an auxiliary oriented hypergraph $H_i$.
- Each edge of our hypergraph $H_i$ encodes a constraint on which clauses can and cannot appear in $G$.
- Thus, our primary proof strategy is to enumerate the number of such $G$ by counting (a) the number of possible hypergraph sequences $\{H_i\}_{i \in [k-2]}$ and (b) the number of possible formulae that correspond to a given hypergraph sequence.

**Proof of Lemma 6.17.** Fix $0 \leq t_1, \ldots, t_{k-2} \leq \zeta n^k$. Consider some $G \in \mathcal{I}_3^*(n, \zeta, \bar{\beta}) \setminus \mathcal{I}_4^*(n, \zeta, \bar{\beta})$ such that $G$ does not contain clauses with exactly $k$ negative literals, and $|G_1| = t_1, \ldots, |G_{k-2}| = t_{k-2}$.

For every $i \in [k-2]$, we construct an auxiliary collection $G'_i \subseteq \binom{X}{3} \times \binom{X}{k-3} \times X$ based on $G_i$, by doing the following: for every clause $\overline{u_1} \cdots \overline{u_i} v_1 \cdots v_{k-i} \in G_i$, we pick one $j \in \{1, \ldots, k-i\}$, and put the element $(\{u_1, \ldots, u_i\}, \{v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{k-i}\}, v_j)$ in $G'_i$. In other words, we obtain $G'_i$ from $G_i$ by making the set of positive literals in every clause “directed” at some $v_j$. 
There are clearly many ways to build $G'_i$. Using Lemma 6.15, we show that there exists some $G'_i$ such that for every $u_1, \ldots, u_i \in X$ and $v_1, \ldots, v_{k-i-1} \in X \setminus \{u_1, \ldots, u_i\}$, there are at most

$$\left\lfloor \frac{(k-i-1)!}{(k-i)!} \cdot \frac{1}{\beta_i^{k-i}} \cdot n \right\rfloor$$

elements of the form $\{(u_1, \ldots, u_i), (v_1, \ldots, v_{k-i-1}), w\}$. For every $u_1, \ldots, u_i \in X$, define the $(k-i)$-hypergraph $H_u$ by

$$V(H_u) = X \setminus \{u_1, \ldots, u_i\},$$

$$E(H_u) = \{\{v_1, \ldots, v_{k-i}\} : \overline{\pi}_1 \cdots \overline{\pi}_i \cdots v_{k-i} \in G_i\}.$$

Since $G$ contains at most $\beta_i n^{k-i}$ clauses of the form $\overline{\pi}_1 \cdots \overline{\pi}_i \cdots v_{k-i}$, we have $e(H_u) \leq \beta_i n^{k-i}$.

By Lemma 6.15, $H_u$ can be directed such that every $k-i-1$ vertices in $X \setminus \{u_1, \ldots, u_i\}$ are contained in at most

$$\left\lfloor \frac{(k-i-1)!}{(k-i)!} \cdot \frac{1}{\beta_i^{k-i}} \cdot n \right\rfloor$$

edges as the undirected part. Combining orientations over all $X_u$ gives us $G'_i$ as wanted.

Now fix the above $G'_i$. For every $i \in \{1, \ldots, k-2\}$, define the $(i+1)$-multigraph $H_i$ by

$$V(H_i) = \binom{X}{k-i},$$

$$E(H_i) = \left\{\{u_1, v_1, \ldots, v_{k-i-1}\}, \ldots, \{u_i, v_1, \ldots, v_{k-i-1}\}, \{w, v_1, \ldots, v_{k-i-1}\} : (\{u_1, \ldots, u_i\}, \{v_1, \ldots, v_{k-i-1}\}, w) \in G'_i\right\},$$

so that $|E(H_i)| = t_i$. We emphasize that $G_i$ is uniquely determined by $H_i$. Moreover, due to the orientation of $G'_i$, $H_i$ has small maximum degree.

**Claim 6.18.** For every $i \in \{1, \ldots, k-2\}$, we have

$$\Delta_1(H_i) \leq (k-i) \left( \binom{n}{i-1} \left\lfloor \frac{(k-i-1)!}{(k-i)!} \cdot \frac{1}{\beta_i^{k-i}} \cdot n \right\rfloor + \zeta n \right).$$

**Proof.** Fix some $v \in \binom{X}{k-i}$. Every edge $e \in E(H_i)$ in which $v$ takes the role of $\{u_1, v_1, \ldots, v_{k-i-1}\}$ (resp. $\{u_2, v_1, \ldots, v_{k-i-1}\}, \ldots, \{u_i, v_1, \ldots, v_{k-i-1}\}$) is uniquely determined by the following:

(i) $u_1 \in v$;
(ii) $u_2, \ldots, u_i \in X$;
(iii) $w$.

Unlike in some earlier arguments in this section that look superficially similar, here, the underlying hypergraph $H_i$ is fixed, as is $G$. Thus, the number of such edges is just at most

$$\left( \binom{k-i}{i-1} \binom{n}{i-1} \left\lfloor \frac{(k-i-1)!}{(k-i)!} \cdot \frac{1}{\beta_i^{k-i}} \cdot n \right\rfloor \right),$$

where (iii) arises from the fact that given $u_1, \ldots, u_i, v_1, \ldots, v_{k-1-i}$, there are

$$\left\lfloor \frac{(k-i-1)!}{(k-i)!} \cdot \frac{1}{\beta_i^{k-i}} \cdot n \right\rfloor$$

elements of the form $\{(u_1, \ldots, u_i), (v_1, \ldots, v_{k-1-i}), w\}$ in $G'_i$. Meanwhile, every edge $e \in E(H_i)$ in which $v$ takes the role of $\{w, v_1, \ldots, v_{k-1-i}\}$ is uniquely determined by the following:

(i) $w \in v$;
(ii) $u_1, \ldots, u_{i-1} \in X$;
(iii) $u_i$.  
The number of such edges is at most
\[
\binom{k-i}{i} \binom{n}{i-1} \zeta n,
\]
where (iii) follows from the fact that given \(u_1, \ldots, u_{i-1}, v_1 \ldots v_{k-i-1}, w\), by Lemma 6.2(a), there are at most \(\zeta n\) variables \(u_i\) in \(G\) such that \(\overline{u}_1 \cdots \overline{u}_i v_1 \cdots v_{k-i-1} w \in G\).

Let \(\nu_i\) and \(\tau_i\) denote the matching and vertex cover numbers of \(H_i\). Suppose \(0 \ll \beta_i \ll \rho_i \ll 1\) and \(0 \ll \zeta \ll 1\), so that we have the degree bound
\[
\Delta_i(H_i) \leq (k-i) \binom{n}{i-1} \left(\frac{(k-i-1)!}{(k-i)!} \cdot \beta_i^{k-i} \cdot n\right) + \zeta n \leq \rho_i n^i.
\]

We thus have the inequality (recall that \(|E(H_i)| = t_i\))
\[
(i+1)\nu_i \geq \tau_i \geq \frac{t_i}{\rho_i n^i}.
\]

We now give an upper bound on the number of choices for \(H_i\) given \(t_i, \nu_i\) and \(\tau_i\).

**Claim 6.19.** For every \(i \in \{1, \ldots, k-2\}\), given \(t_i, \nu_i\) and \(\tau_i\), the number of choices for \(H_i\) is at most
\[
\binom{n}{k-i} \left(\binom{\tau_i}{t_i} \binom{k}{i+1}\right).
\]

**Proof.** Recall that \(|V(H_i)| = \binom{n}{k-i}| = \binom{n}{k-i}|\). Observe that \(H_i\) is uniquely determined by the following:

1. A vertex cover \(\mathcal{T}_i\) of \(H_i\);
2. The edge set \(E(H_i)\) of size \(t_i\), such that every edge contains at least one vertex in \(\mathcal{T}_i\).

Given \(t_i, \nu_i\) and \(\tau_i\), there are \(\binom{|V(H_i)|}{\tau_i} = \binom{n}{\tau_i}\) possible vertex covers of \(H_i\). Moreover, any vertex cover of \(H_i\) gives at most \(\tau_i \binom{k}{i+1}\) edges that could possibly lie in \(H_i\); thus, it gives at most \(\frac{\tau_i \binom{k}{i+1}}{t_i}\) possible choices for \(H_i\). Hence given \(t_i, \nu_i\) and \(\tau_i\), the number of possible choices for \(H_i\) is at most
\[
\binom{n}{k-i} \left(\binom{\tau_i}{t_i} \binom{k}{i+1}\right).
\]

We now upper bound the number of choices for \(G_0\) given \(H_1, \ldots, H_{k-2}\). Observe that for every \(e = \{\{u_1, v_1, \ldots, v_{k-i-1}\}, \ldots, \{u_i, v_1, \ldots, v_{k-i-1}\}, \{w, v_1, \ldots, v_{k-i-1}\}\} \in E(H_i)\) and any other \(i\) variables \(a_1, \ldots, a_i \notin \{u_1, u_i, v_1, \ldots, v_{k-i-1}, w\}\), since \(\overline{u}_1 \cdots \overline{u}_i v_1 \cdots v_{k-i-1} w \in G_i\), to avoid the non-minimal subformula
\[
\{v_1 \cdots v_{k-i-1} a_1 \cdots a_i w, v_1 \cdots v_{k-i-1} a_1 \cdots a_i u_1, \ldots, v_1 \cdots v_{k-i-1} a_1 \cdots a_i u_i, \overline{u}_1 \cdots \overline{u}_i v_1 \cdots v_{k-i-1} w\}
\]
(as it is impossible to satisfy only the first clause), we have
\[
|G_0 \cap F_{e,a_1,\ldots,a_i}| < i + 1,
\]
with \( F_{e,a_1,\ldots,a_i} \) defined by
\[
F_{e,a_1,\ldots,a_i} = \{ v_1 \cdots v_{k-i-1} a_i \cdots a_i w, \, v_1 \cdots v_{k-i-1} a_i u_1, \ldots, \, v_1 \cdots v_{k-i-1} a_i u_i \}.
\]

**Claim 6.20.** Suppose \( H_1, \ldots, H_{k-2} \) have been determined. Then the number of possible choices for \( G_0 \) is at most
\[
\exp_2 \left[ \binom{n}{k} - \sum_{i=1}^{k-2} \delta_i \binom{n-k}{i} \right],
\]
where
\[
\delta_i = \frac{i + 1 - \log_2(2i+1 - 1)}{(2^k - k - 2)(i+1)}\]

**Proof.** Let \( \mathcal{M}_i \) be a maximum matching of \( H_i \), so that \(|\mathcal{M}_i| = \nu_i\). Define the family of monotone formulae \( \mathcal{J}_i \) by
\[
\mathcal{J}_i = \bigcup_{e \in \mathcal{M}_i} \{ F_{e,a_1,\ldots,a_i} : a_1, \ldots, a_i \not\in \{ u_1, \ldots, u_i, v_1, \ldots, v_{k-i-1}, w \} \}
\]
with \( e = \{ \{u_1, v_1, \ldots, v_{k-i-1}\}, \ldots, \{a_i, v_1, \ldots, v_{k-i-1}\}, \{w, v_1, \ldots, v_{k-i-1}\} \} \). For every possible \( G_0 \) and every \( F \in \mathcal{J}_i \), by the above argument, we have \(|G_0 \cap F| < i + 1\). Moreover, since every \( e \in \mathcal{M}_i \) gives \( \binom{n-k}{i} \) different edges in \( \mathcal{J}_i \), and every \( F \in \mathcal{J}_i \) can arise from at most \( \binom{k-1}{i} \) different \( e \in \mathcal{M}_i \), we have
\[
|\mathcal{J}_i| \geq \frac{\nu_i(n-k)}{(k-1)}.
\]

For every monotone clause \( C \in \binom{X}{k} \), let \( d_i(C) \) denote the number of elements \( F \in \mathcal{J}_i \) that contain \( C \). Note that we always have \( d_i(C) \leq \binom{k}{i} \), or there would be two edges \( e \in \mathcal{M}_i \) sharing the same vertex, contradicting the fact that \( \mathcal{M}_i \) is a matching. Let \( \mathcal{H} \) be a (non-uniform) multigraph on the set of monotone clauses, defined by
\[
V(\mathcal{H}) = \binom{X}{k}, \quad E(\mathcal{H}) = \left( \bigcup_{i=1}^{k-2} \mathcal{J}_i \right) \cup \left( \bigcup_{C \in \binom{X}{k}} \frac{\{C\} \cdots \{C\}}{\sum_{i=1}^{k-2} \binom{k}{i} - d_i(C)} \right),
\]
so that every vertex in \( \mathcal{H} \) has degree \( \sum_{i=1}^{k-2} \binom{k}{i} = 2^k - k - 2 \). Let \( \mathcal{G}_0 \) be the collection of all possible choices for \( G_0 \). By Lemma 6.16, we have
\[
(2^k - k - 2) \log_2 |\mathcal{G}_0| \leq \sum_{F \in E(\mathcal{H})} \log_2 |\text{Tr}(\mathcal{G}_0, F)|.
\]

For all \( i \in [k-2] \) and \( F \in \mathcal{J}_i \), since \(|G_0 \cap F| < i + 1\) for all \( G_0 \in \mathcal{G}_0 \), we have \(|\text{Tr}(\mathcal{G}_0, F)| \leq 2^{i+1} - 1\). Meanwhile, since all \( \text{Tr}(\mathcal{G}_0, \{C\}) \leq 2 \), it follows that
\[
(2^k - k - 2) \log_2 |\mathcal{G}_0| \leq \sum_{C \in \binom{X}{k}} (2^k - k - 2 - d_i(C)) + \sum_{i=1}^{k-2} |\mathcal{J}_i| \log_2 (2^{i+1} - 1).
\]
Since $\sum_{C \in \binom{V}{i}} d_i(C) = (i + 1)|J_i|$, we then have

\[(2^k - k - 2) \log_2 |G_0| \leq (2^k - k - 2) \binom{n}{k} - \sum_{i=1}^{k-2} (i + 1 - \log_2(2^i + 1))|J_i|\]

\[\leq (2^k - k - 2) \binom{n}{k} - \sum_{i=1}^{k-2} \frac{(i + 1 - \log_2(2^i + 1))(n-k)_i}{i!} \nu_i \quad (|J_i| \geq \frac{\nu_i}{(k-i)_i})\]

\[\leq (2^k - k - 2) \binom{n}{k} - \sum_{i=1}^{k-2} \frac{(i + 1 - \log_2(2^i + 1))(n-k)_i}{(i+1)(k-i)_i} \tau_i \quad (\frac{\tau_1}{i+1} \leq \nu_i)\]

Therefore, given $H_1, \ldots, H_{k-2}$, we have

\[|G_0| \leq \exp_2 \left[ \binom{n}{k} - \sum_{i=1}^{k-2} \delta_i \binom{n-k}{i} \tau_i \right]. \]

To summarize, every $G$ in Lemma 6.17 is uniquely determined by sequentially specifying the following information:

1. $0 \leq \nu_i \leq \binom{n}{i+1}$ for $i = 1, \ldots, k-2$;
2. $0 \leq \tau_i \leq \binom{n}{k-i}$ for $i = 1, \ldots, k-2$;
3. $H_i$ for $i = 1, \ldots, k-2$;
4. $G_0$.

There are at most

\[\prod_{i=1}^{k-2} \left( \frac{\binom{n}{i} - 1}{i+1} + 1 \right) \left( \binom{n}{k-i} + 1 \right) \leq n^{2k^2}\]

choices for (1)(2). For every fixed sequence $\tau_1, \ldots, \tau_{k-2}$, by Claim 6.19 and Claim 6.20, the number of choices for (3)(4) is at most

\[\exp_2 \left[ \binom{n}{k} - \sum_{i=1}^{k-2} \delta_i \binom{n-k}{i} \tau_i \prod_{i=1}^{k-2} \left( \binom{n-k-i}{i} \right) \left( \binom{k}{i} \tau_i \binom{n}{i} \right) \right] \leq \exp_2 \left[ \binom{n}{k} + \sum_{i=1}^{k-2} \left( \frac{O(\log n) + H \left( \frac{i\rho_i}{n} \right) \binom{k}{i} n^i - \delta_i \binom{n-k}{i} \right) \tau_i \right] \quad (\tau_i \geq \frac{i}{\rho_i n^i})\]

(recall that $\rho_i \ll \delta_i$). Hence given $t_1, \ldots, t_{k-2}$, the number of choices for $G \in I_4^*(n, \zeta, \bar{\beta})$ in Lemma 6.17 is at most

\[n^{2k^2} \exp_2 \left[ \binom{n}{k} - 3cn \right] \leq \exp_2 \left[ \binom{n}{k} - 2cn \right]. \]

We now can show that most formulae in $I_3^*(n, \zeta, \bar{\beta})$ also lie in $I_4^*(n, \zeta, \bar{\beta})$.

**Lemma 6.21.** Suppose $\bar{\beta}, \zeta$ are all sufficiently small constants (relative to $k$). Then, there exists $c > 0$ such that

\[|I_3^*(n, \zeta, \bar{\beta}) \setminus I_4^*(n, \zeta, \bar{\beta})| < \exp_2 \left[ \binom{n}{k} - cn \right]. \]
Proof. Let each of the entries of $\vec{\beta}$ and $\zeta$ be sufficiently small positive constants (as a function of $k$), and let $c > 0$ be sufficiently small relative to all of $\vec{\beta}, \zeta$.

Observe first that every $G \in \mathcal{I}_3^*(n, \zeta, \vec{\beta})$ cannot contain any clause with exactly $k - 1$ negative literals. Indeed, if $\overline{w}_1 \cdots \overline{w}_{k-1} v \in G$, then since $N_{u_1} \cap \cdots \cap N_{u_{k-1}} \cap N_v \neq \emptyset$ (Definition 6.11), $G$ would have a non-minimal suformula of the form

$$\{w_1 \cdots w_{k-1} v, w_1 \cdots w_{k-1} u_1, \ldots, w_1 \cdots w_{k-1} u_{k-1}, \overline{w}_1 \cdots \overline{w}_{k-1} v\}.$$ 

Next, we upper bound the number formulae $G \in \mathcal{I}_3^*(n, \zeta)$ that contain at least one clause with exactly $k$ negative literals. Every such $G$ is uniquely determined by sequentially specifying the following information:

1. The lexicographically smallest variable set $\{u_1, \ldots, u_k\} \subseteq X$ such that $\overline{w}_1 \cdots \overline{w}_k \in G$;
2. $\bigcap_{j=1}^k N_{u_j} \subseteq (X \setminus X - \{u_j\})$;
3. The non-monotone clauses in $G$;
4. The monotone clauses in $G$.

The number of such formulae is at most

$$\binom{n}{k} 2^{k \binom{n}{k}} \leq \exp_2 \left[ \binom{n}{k} - \frac{n - 2k + 1}{k \cdot 2^{k+1}} \frac{n - k}{k - 1} \right] \leq \exp_2 \left[ (1 - c) \binom{n}{k} \right]$$

(recall that $c \ll \zeta$). In particular, bounds (3) and (4) arise from the following observations:

3. $G \in \mathcal{I}_1^*(n, \zeta)$ has at most $\zeta n^k$ non-monotone clauses.
4. Since $\overline{w}_1 \cdots \overline{w}_k \in G$, for all $\{w_1, \ldots, w_{k-1}\} \subseteq \bigcap_{j=1}^k N_{u_j}$, to avoid the non-minimal suformula

$$\{w_1 \cdots w_{k-1} v, w_1 \cdots w_{k-1} u_1, \ldots, w_1 \cdots w_{k-1} u_{k}, \overline{w}_1 \cdots \overline{w}_{k-1} v\},$$

there cannot be any $v \in X$ such that $w_1 \cdots w_{k-1} v \in G$. Thus (1)(2) have determined there are at least

$$\frac{1}{k} \cdot (n - 2k + 1) \left| \bigcap_{i=1}^k N_{u_i} \right| \geq \frac{n - 2k + 1}{k \cdot 2^{k+1}} \frac{n - k}{k - 1}$$

monotone clauses that cannot belong to $G$.

Finally, we upper bound the number of formulae $G \in \mathcal{I}_3^*(n, \zeta, \vec{\beta}) \setminus \mathcal{I}_1^*(n, \zeta, \vec{\beta})$ that do not contain clauses with exactly $k$ negative literals. For every such $G$, we write

$$G = G_0 \cup G_1 \cup \cdots \cup G_{k-2},$$

where $G_i$ is the set of clauses in $G$ with exactly $i$ negative literals. Since $G$ has at most $\zeta n^k$ non-monotone clauses total, we have $|G_1|, \ldots, |G_{k-2}| \leq \zeta n^k$.

Since the number of choices of $t_1, \ldots, t_{k-2}$ is at most $(\zeta n^k + 1)^{k-2}$, by Lemma 6.17, there are at most $(\zeta n^k + 1)^{k-2} \exp_2 \left[ \binom{k}{k} - 2^{cn} \right]$ formulae in $\mathcal{I}_3^*(n, \zeta, \vec{\beta}) \setminus \mathcal{I}_1^*(n, \zeta, \vec{\beta})$ that do not contain clauses with exactly $k$ negative literals. Combining with the previous two observations, we get the bound (since $c \ll \zeta$)

$$|\mathcal{I}_3^*(n, \zeta, \vec{\beta}) \setminus \mathcal{I}_1^*(n, \zeta, \vec{\beta})| \leq (\zeta n^k + 1)^{k-2} \exp_2 \left[ \binom{n}{k} - 2^{cn} \right] + \exp_2 \left[ (1 - c) \binom{n}{k} \right]$$

$$\leq \exp_2 \left[ \binom{n}{k} - cn \right].$$
6.5. Bound on $|I_1^*(n, \zeta)|$.

Proof of Theorem 6.1. For $k \geq 4$, suppose $0 < c \ll \zeta \ll \beta_1 \ll \beta_i \ll 1/k$ for all $i \in \{2, \ldots, k-2\}$. We have

$$|I_1^*(n, \zeta, \beta_1) \setminus I_2^*(n, \zeta, \beta)| \leq (k-2) \exp_2 \left[ (1 - c) \binom{n}{k} \right] + \exp_2 \left[ (1 - c) \binom{n}{k-1} \right] |I(n-1)|$$

$$+ \sum_{i=2}^{k-2} \exp_2 \left[ i(1 - c) \binom{n}{k-1} \right] |I(n-i)|, \quad \text{(Lemmas 6.5 and 6.8)}$$

$$|I_2^*(n, \zeta, \beta) \setminus I_3^*(n, \zeta, \beta)| \leq \exp_2 \left[ \frac{k}{2} (1 - c) \binom{n}{k-1} \right] |I(n-k)|, \quad \text{(Lemma 6.12)}$$

$$|I_3^*(n, \zeta, \beta) \setminus I_4^*(n, \zeta, \beta)| \leq \exp_2 \left[ \binom{n}{k} - cn \right], \quad \text{(Lemma 6.21)}$$

$$|I_4^*(n, \zeta, \beta)| \leq 2^n.$$

Combining the above inequalities gives the upper bound

$$|I_1^*(n, \zeta)| \leq 2^n + (k-2) \exp_2 \left[ (1 - c) \binom{n}{k} \right] + \exp_2 \left[ (1 - c) \binom{n}{k-1} \right] |I(n-1)|$$

$$+ \exp_2 \left[ \frac{k}{2} (1 - c) \binom{n}{k-1} \right] |I(n-k)| + \exp_2 \left[ \binom{n}{k} - cn \right],$$

which yields the desired bound by replacing $c$ by a constant multiple.

For $k = 3$, suppose $0 < c \ll \zeta \ll \beta_1 \ll 1/k$. Since

$$|I_1^*(n, \zeta, \beta_1) \setminus I_2^*(n, \zeta, \beta)| \leq \exp_2 \left[ (1 - c) \binom{n}{k} \right] + \exp_2 \left[ (1 - c) \binom{n}{k-1} \right] |I(n-1)|, \quad \text{(Lemma 6.5)}$$

we get that

$$|I_1^*(n, \zeta)| \leq 2^n + \exp_2 \left[ (1 - c) \binom{n}{k} \right] + \exp_2 \left[ (1 - c) \binom{n}{k-1} \right] |I(n-1)|$$

$$+ \exp_2 \left[ \frac{k}{2} (1 - c) \binom{n}{k-1} \right] |I(n-k)| + \exp_2 \left[ \binom{n}{k} - cn \right].$$

For $k = 2$, suppose $0 < c \ll \zeta \ll \beta_1 \ll 1/k$. Since

$$|I_1^*(n, \zeta, \beta_1) \setminus I_2^*(n, \zeta, \beta)| \leq 2 \exp_2 \left[ (1 - c) \binom{n}{k} \right], \quad \text{(Lemma 6.4)}$$

$$|I_3^*(n, \zeta, \beta) \setminus I_4^*(n, \zeta, \beta)| = 0, \quad \text{(Lemma 6.14)}$$

we get that

$$|I_1^*(n, \zeta)| \leq 2^n + 2 \exp_2 \left[ (1 - c) \binom{n}{k} \right] + \exp_2 \left[ \frac{k}{2} (1 - c) \binom{n}{k-1} \right] |I(n-k)|.$$

□
