I. INTRODUCTION

The fundamental NLSE is one of the most investigated equations in describing the dynamics of multiple physical phenomena, in both discrete and continuous systems. It describes Bose-Einstein condensates (alias Gross-Pitaevskii equation) [1], the collapse of plasma waves, pulses in nonlinear optical fibers [2], the propagation of waves in nonlinear waveguides, and the interaction between solitons in nonlinear waveguides. One important admitted solution to this equation is soliton. It is a localized wave originating from the competition between dispersive and nonlinear effects [3]. In fact, it is the most essential phenomenon of the local NLSE. Solitons appear in many diverse systems such as plasmas, astrophysics, molecular biology, nonlinear optics, spin waves, superfluidity, and Bose-Einstein condensates (see for review Ref. [4]).

Many successful techniques have been developed to solve the NLSE. Among them are Painlevé analysis [5–7], Hirota method [8], similarity transformation method [9], Lax Pair (LP) and Darboux transformation (DT) method [10,11,12], Miura transformation [13,14], inverse scattering transform and Hamiltonian approach [15], homotopy analysis method [16], Exp-function method [17], the tanh-function method [18,19], the homogeneous balance method [20], and the F-expansion method [21].

Currently, there is a considerable interest in finding exact solutions to the nonlocal NLSE [22,23]. This non-Hermitian and PT-symmetric equation with the potential \( V(x,t) = u(x,t)u^*(-x,t) \), where \( u(x,t) \) is the mean-field wavefunction, satisfies the PT-symmetric condition, \( V(x,t) = V^*(-x,t) \). Several efforts were devoted in showing that this equation admits a soliton solution as well [22,24].

Briefly, the LP and DT method is based on searching for an appropriate pair of matrices that associates the nonlinear system to a linear system. The LP should be associated with the nonlinear model through what is called a compatibility condition. The obtained linear system is solved using a seed solution, \( u_0(x,t) \), which is a known exact solution of the nonlinear system. Each seed solution generates a family of exact solutions.

Prominent among the solutions of the NLSE is the two-soliton solution which can be obtained using the LP and DT method [27,28]. Employing the trivial seed solution, \( u_0(x,t) = 0 \), the DT generates a single soliton solution. Using the latter as seed, generates the two-soliton solution [27]. The binding energy, the force, and potential of interaction between solitons have been calculated and studied extensively [27,34].

The interest in two-soliton solution stems, not only from its importance on fundamental level, but also from its tremendous application as a data carrier in optical fibers [31]. It has been suggested that such a soliton-molecule may increase the data-carrying capacity [2,35,36]. The existence of a nonzero binding energy in terms of the width of the soliton is a signature on its stability.

Here, we follow the above-mentioned effort in the literature to find new exact solutions to the NLSE. Specifically, we use a seed solution of the form \( u_0(x,t) = 1/x \) to generate a two-soliton solution that is characterized by two diverging peaks known as singular solitons. It is found that such states are generated due to the strong self-repulsion [37,38]. Despite its divergency, the new solution describes a soliton molecule with a binding energy. We have calculated the potential of the interaction between the two solitons to show that it is of molecular type. To the best of our knowledge, this is a new kind of soliton molecule.

Furthermore, we have considered the nonlocal NLSE and used the same procedure to generate a new solution out of the \( u_0(x,t) = 1/x \) seed. It turned out that the new solution is much richer than the local case. Here, the new solution corresponds to the scattering of stationary soliton and two breathers on a finite background at half...
of space and inclined background at the other half.

The plan of this work is as follows. In Sec IV we derive a new two-soliton solution with two diverging peaks to the local NLSE using the LP and DT method with an algebraically-decaying seed. We then investigate the binding energy, the force, and potential of interaction for these new local NLSE solution, to check that it is indeed a soliton molecule. We also consider its scattering properties with other solitons to show that the integrity of individual solitons as well as the molecule is preserved after scattering. In Sec V we employ the same seed to obtain a new exact solution to the nonlocal NLSE, which demonstrates an elastic interaction between one bright soliton and two breather solutions, one on a finite flat background and the other on a ramp background. Section VI is devoted to the case with reverse-time nonlocal NLSE. We end with a summary of our main conclusions in Sec V.

II. NEW EXACT SOLUTION TO THE LOCAL NLSE

In this section, we apply the LP and DT method with the rational seed solution, \( u_0(x,t) = 1/x \), to the local NLSE given by

\[
iu_t + \frac{1}{2} u_{xx} - |u|^2 u = 0,
\]

The Lax pair of Eq. (1) is given by [12]

\[
\Phi_x = J \cdot \phi \cdot \Lambda + U \cdot \phi,
\]

and

\[
\Phi_t = i J \cdot \phi \cdot \Lambda^2 + i U \cdot \phi \cdot \Lambda + V \cdot \phi.
\]

The matrices \( U, V, \Lambda, \) and \( J \) are defined as

\[
U = \begin{pmatrix} 0 & u \\ -u^* & 0 \end{pmatrix},
\]

\[
V = \frac{i}{2} \begin{pmatrix} |u|^2 & u_x \\ u_x^* & -|u|^2 \end{pmatrix},
\]

\[
\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},
\]

\[
J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

where \( u^*(x,t) \) is the complex conjugate of \( u(x,t) \) and \( \lambda_{1,2} = \lambda_{1,2} r + i \lambda_{1,2} i \) are the spectral complex parameters with \( \lambda_{1,2} r \) and \( \lambda_{1,2} i \) are arbitrary real constants. The auxiliary field \( \Phi(x,t) \) is given by

\[
\Phi(x,t) = \begin{pmatrix} \psi_1(x,t) \\ \phi_1(x,t) \end{pmatrix},
\]

The compatibility condition \( \phi_{xt} = \phi_{tx} \) leads to

\[
U_t - V_x + \left[ U, V \right] = 0,
\]

where \([U, V]\) is the commutator between \( U \) and \( V \). The use of Eqs. (4), (5) and (10) yields the compatibility condition which establishes the link between the NLSE and the LP. The DT is defined by [12]

\[
\Phi[1] = \Phi \cdot \Lambda - \sigma \Lambda,
\]

where \( \Phi[1] \) is the transformed field, and \([J, \sigma]\) is the commutator between \( J \) and \( \sigma \), with \( \sigma \) given by

\[
\sigma = \Phi_0 \cdot \Lambda \cdot \Phi_0^{-1}.
\]

Here, \( \Phi_0 \) is a seed solution of the linear system for a given seed solution of the NLSE, \( u_0(x,t) \). The field \( \Phi \) represents any solution of the linear system and \( \Phi[1] \) is the new solution of this system which obeys

\[
\Phi[1] = J \cdot \Phi[1] \cdot \Lambda + U[1] \cdot \Phi[1],
\]

\[
\Phi[1] \ell = i J \cdot \Phi[1] \cdot \Lambda^2 + i U[1] \cdot \Phi[1] \cdot \Lambda + V[1] \cdot \Phi[1],
\]

where

\[
U[1] = U_0 + \left[ J, \sigma \right],
\]

\[
V[1] = V_0 + \left[ U_0, \sigma \right],
\]

and \( U_0 \) and \( V_0 \) are the LP in terms of the seed solution. The matrices \( J \) and \( \Lambda \) are constant and do not change under the DT. Equations (4), (5) and (10) give the new solution of the NLSE as follows

\[
U[1] = U_0 + Q (\psi_1, \phi_1),
\]

where the Darboux dressing is given by

\[
Q (\psi_1, \phi_1) = [2 (\lambda_1 - \lambda_2) \phi_1 \psi_1] / (\phi_1 \psi_2 - \phi_2 \psi_1).
\]

Using the following symmetry reductions

\[
\lambda_2^* = -\lambda_1, \quad \phi_2^* = \psi_1, \quad \psi_2^* = \phi_1,
\]

reduces Eqs. (2) and (3) for \( \psi_1, \phi_1, \psi_2, \) and \( \phi_2, \) to

\[
- \lambda_1 \psi_1 - iu \phi_1 + \psi_{1x} = 0,
\]

\[
\lambda_1 \phi_1 + iu^* \psi_1 + \phi_{1x} = 0,
\]

\[
i \psi_{1t} + \psi_1 \left( \lambda_1^* + \frac{|u|^2}{2} \right) + \phi_1 \left( \lambda_1 u + \frac{u_x}{2} \right) = 0,
\]

where \([U, V]\) is the commutator between \( U \) and \( V \). The use of Eqs. (4), (5) and (10) yields the compatibility condition which establishes the link between the NLSE and the LP. The DT is defined by [12]

\[
\Phi[1] = \Phi \cdot \Lambda - \sigma \Lambda,
\]

where \( \Phi[1] \) is the transformed field, and \([J, \sigma]\) is the commutator between \( J \) and \( \sigma \), with \( \sigma \) given by

\[
\sigma = \Phi_0 \cdot \Lambda \cdot \Phi_0^{-1}.
\]

Here, \( \Phi_0 \) is a seed solution of the linear system for a given seed solution of the NLSE, \( u_0(x,t) \). The field \( \Phi \) represents any solution of the linear system and \( \Phi[1] \) is the new solution of this system which obeys

\[
\Phi[1] = J \cdot \Phi[1] \cdot \Lambda + U[1] \cdot \Phi[1],
\]

\[
\Phi[1] \ell = i J \cdot \Phi[1] \cdot \Lambda^2 + i U[1] \cdot \Phi[1] \cdot \Lambda + V[1] \cdot \Phi[1],
\]

where

\[
U[1] = U_0 + \left[ J, \sigma \right],
\]

\[
V[1] = V_0 + \left[ U_0, \sigma \right],
\]

and \( U_0 \) and \( V_0 \) are the LP in terms of the seed solution. The matrices \( J \) and \( \Lambda \) are constant and do not change under the DT. Equations (4), (5) and (10) give the new solution of the NLSE as follows

\[
U[1] = U_0 + Q (\psi_1, \phi_1),
\]

where the Darboux dressing is given by

\[
Q (\psi_1, \phi_1) = [2 (\lambda_1 - \lambda_2) \phi_1 \psi_1] / (\phi_1 \psi_2 - \phi_2 \psi_1).
\]

Using the following symmetry reductions

\[
\lambda_2^* = -\lambda_1, \quad \phi_2^* = \psi_1, \quad \psi_2^* = \phi_1,
\]

reduces Eqs. (2) and (3) for \( \psi_1, \phi_1, \psi_2, \) and \( \phi_2, \) to

\[
- \lambda_1 \psi_1 - iu \phi_1 + \psi_{1x} = 0,
\]

\[
\lambda_1 \phi_1 + iu^* \psi_1 + \phi_{1x} = 0,
\]

\[
i \psi_{1t} + \psi_1 \left( \lambda_1^* + \frac{|u|^2}{2} \right) + \phi_1 \left( \lambda_1 u + \frac{u_x}{2} \right) = 0,
\]
Substituting the seed $u_0 = 1/x$ in (19)-(22) and solving for $\psi_1$ and $\phi_1$ reads

$$\psi_1(x, t) = e^{-x\lambda_1}/4x \left[ 4c_1 e^{-i\lambda_1 t} + c_2(2\lambda_1 x - 1)e^{2\lambda_1 x + i\lambda_1 t} \right]/\lambda_1^2,$$

(23)

$$u_1(x, t) = \frac{16A_1 x_1^2 + 2A_1 [2A_1 x_0 (\lambda_1 - \lambda_1^* + 2x_1 x)] - A_0 \lambda_1^2 e^{2\lambda_1 x - i\lambda_1 t} \rho_2^{-2} \lambda_1^2 \lambda_1 e^{2\lambda_1 x + i\lambda_1 t}}{4A_0 \lambda_1^2 x_0 e^{2\lambda_1 (x - i\lambda_1 t)} + A_1 [2A_1 x_0 + A_0 (2x_1 x - x_0)] e^{2\lambda_1 (x - i\lambda_1 t)} e^{2\lambda_1 (x + i\lambda_1 t)} - 8A_1 \lambda_1^2 (x_0 + 2x_1 x)},$$

(25)

where $x_0 = \lambda_1 + \lambda_1^*$, $x_1 = \lambda_1 \lambda_1^*$, $A_0 = c_2 c_2^*/2\lambda_1^2 c_1 c_1^*$, and $A_1 = c_2/c_1$.

Depending on the values of these parameters, one can distinguish three different regimes, namely:

(i) **Symmetric coalescing solitons:**

For $\lambda_1 = \lambda_1^* = 1/2$, $c_1 = 10 + 10i$ and $c_2 = 10 - 10i$, the solution (25) reduces to

$$u_1(x, t) = \frac{e^{\lambda_1 x} (2e^{x^2} + i - ie^{2x} + i)}{i(x^2 - 2e^{x^2} + x^2 - 2e^{2x})},$$

(26)

which is displayed in Fig. 1(a). Here, we observe that the two solitons collide periodically.

(ii) **Asymmetric noncoalescing solitons:**

For $\lambda_1 = 1/4$, $c_1 = 2i$ and $c_2 = -4i$, we get

$$u_1(x, t) = \frac{e^{-2i\lambda_1 x^2} + 64e^{x^2} - 1}{-64(x^2 - 4)e^{x^2} + 32e^{2x^2} + e^{x^2}(x^2 + 4) + 32e^{x^2}/2},$$

(27)

Here the two solitons still form a molecule but they do not coalesce, as is seen in Fig. 1(b).

(iii) **Asymmetric coalescing solitons:**

For small intersoliton distance, i.e., $\lambda_1 = 1$, $c_1 = 1 + i$ and $c_2 = 1 + i$, Eq. (24) simplifies to

$$u_1(x, t) = \frac{e^{2it} (16x e^{2x^2} + 2e^{4x} - e^{4x})}{(x-1)e^{4x} + 4xe^{4x} - 16e^{2it}(x^2 + 1) + 4e^{2x}},$$

(28)

In this case the solitons coalesce, as in the first case, but the inter-soliton oscillation is performed mainly by one soliton, as in the second case. This is shown in Fig. 1(c).

\[ \phi_1(x, t) = \frac{i e^{-\lambda_1 (x + \lambda_1 t)}}{4\lambda_1^2} \left[ 4c_1 \lambda_1^2 (2\lambda_1 x + 1) - c_2 e^{2\lambda_1 (x + \lambda_1 t)} \right], \]

(24)

where $c_1$ and $c_2$ are arbitrary complex constants. Substituting (23) and (24) into (16), we obtain a new exact solution to the local NLSE, Eq. (1), namely:

\[ \phi_1(x, t) = \frac{i e^{-\lambda_1 (x + \lambda_1 t)}}{4\lambda_1^2} \left[ 4c_1 \lambda_1^2 (2\lambda_1 x + 1) - c_2 e^{2\lambda_1 (x + \lambda_1 t)} \right], \]

\[ V = m(a_r - a_l), \]

FIG. 1. Singular solitons molecules with zero relative velocity. (a) Symmetric coalescing solitons (26). (b) Asymmetric noncoalescing solitons (24). (c) Asymmetric coalescing solitons (28). Blue lines show the soliton’s trajectory.

A. Interactions of singular solitons

Let us now discuss the interaction of the two local solitons given in Eq. (25). Figures 2(a) and 2(b) show the scattering of two solitons where they preserve their integrity after collision.

To determine the force of interaction between the two solitons, we calculate first the acceleration, $a_r$ of the right soliton and $a_l$ of the left soliton. This is performed by extracting the center of mass of each soliton, $x_r$ for the right soliton and $x_l$ for the left soliton, and then taking the second derivative, $a_r = \ddot{x}_r$ and $a_l = \ddot{x}_l$. The force of interaction between the two solitons is proportional to the second derivative of their separation $F = m(a_r - a_l)$, where $m = \int_{-\infty}^{\infty} |u_1(x, t)|^2 dx$. Then, the integration of the force gives the potential $V = -\int F dx$.

When the two solitons are well-separated from each other, analytic expressions for their positions can be extracted from the exact solution, Eq. (25), which lead to the following expressions for their accelerations:

\[ a_r = \frac{-x^3 + 4x^2 - 13x + 64e^{x}(4 - x) - 4}{8e^x(x^3 - 6x^2 + 12x - 8) + 1536(x^3 - 2x^2 - 4x + 8)}, \]

(29)
and

\[ a_t = \frac{64 \left( x^3 - 4x^2 + 13x + 4 \right) + e^x(x - 4)}{512e^x(x^3 - 6x^2 + 12x - 8) + 48(x^3 - 2x^2 - 4x + 8)}. \]  

(30)

Similarly, when the solitons are close to each other, the accelerations read

\[ a_r = \frac{64 \left( -16x^3 - 16x^2 - 13x + 1 \right) + 4e^{-x}(-x - 1)}{-384x^3 - 192x^2 + e^{-4x}(-8x^3 - 12x^2 - 6x - 1) + 96x + 48}, \]  

(31)

and

\[ a_l = \frac{4 \left( -16x^3 - 16x^2 - 13x + 1 \right) + 64e^{-4x}(-x - 1)}{16e^{-4x}(-8x^3 - 12x^2 - 6x - 1) + 3(-8x^3 - 4x^2 + 2x + 1)}. \]  

(32)

It is clearly apparent from Fig. 3(b) that for large separation, the interaction potential is negative showing the stability and the robustness of the bond between the solitons. It is noticed that while each of the two solitons experiences the same force of interaction, they have different acceleration due to their different masses, as is shown in Figs. 3(a) and 3(b). The situation is quite different when the two solitons are close to each other; the acceleration and the interaction potential well become narrower and deeper a fact that influences the formation of the bound state as is seen in Figs. 4(a) and 4(b), respectively. Figures 3 and 4 clearly show molecular type of potential between the two singular solitons.

III. NEW EXACT SOLUTION TO THE NONLOCAL NLSE

In this section, we apply the LP and DT method with the same rational seed solution, \( u_0(x, t) = 1/x \), to the nonlocal NLSE which can be written as:

\[ iu_t + \frac{1}{2}u_{xx} + u^2u = 0, \]  

(33)

where \( \bar{u} = u^*(-x, t) \). The Lax pair of Eq. (33) can be found via Eqs. (29) and (30), which is proportional to the mutual force of interaction, \( F \). The situation is quite different when the two solitons are close to each other; the acceleration and the interaction potential well become narrower and deeper a fact that influences the formation of the bound state as is seen in Figs. 3(a) and 3(b). The situation is quite different when the two solitons are close to each other; the acceleration and the interaction potential well become narrower and deeper a fact that influences the formation of the bound state as is seen in Figs. 4(a) and 4(b), respectively. Figures 3 and 4 clearly show molecular type of potential between the two singular solitons.
In Fig. 5, the interaction between a stationary soliton and two breathers is shown: one on an inclined background, where the breather soliton on the right side is in an inclined background and the left breather is in flat background. Parameters are: \( \lambda_{1i} = 0.08 \) and \( \lambda_{1r} = 1 \).

The Jacobian matrix is given by:

\[
J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]  

(37)

With the use of the symmetry reductions in (18), the linear system (2) and (4), is reduced to

\[
u \phi_1 - \lambda_1 \psi_1 + \psi_1 x = 0,
\]

(38)

\[
\bar{u} \psi_1 - \lambda_1 \phi_1 - \phi_1 x = 0,
\]

(39)

Applying the seed solution, \( u_0 = 1/x \), in (38) and solving for \( \psi_1 \) and \( \phi_1 \), one obtains:

\[
\psi_1(x, t) = \frac{1}{x} \left[ c_1 - c_2 (2 \lambda_1 x - 1) e^{2 \lambda_1 (x + i \lambda_1 t)} \right] e^{-\lambda_1 (x + i \lambda_1 t)},
\]

(42)

\[
\phi_1(x, t) = \frac{1}{x} \left[ c_1 + c_2 e^{2 \lambda_1 (x + i \lambda_1 t) + 2 c_1 \lambda_1 x} e^{-\lambda_1 (x + i \lambda_1 t)} \right].
\]

(43)

Then, the new exact solution to the nonlocal NLSE is obtained by substituting (42) and (43) into (10):

\[
u_1(x, t) = \frac{z_1(x, t)}{z_2(x, t)},
\]

(44)

where

\[
z_1(x, t) = c_1^2 (-2 \lambda_2 x + e^{-2 \lambda_2 (x + i \lambda_2 t)}) + c_1 c_2 ((4 \lambda_2^2 x^2 - 1) e^{2 \lambda_2 (x + i \lambda_2 t)} + (1 - 2 \lambda_2 x) \times e^{2 \lambda_1 (x + i \lambda_1 t) + 2 \lambda_2 x + c_2^2 (2 \lambda_2 x (1 - 2 \lambda_1 x)) e^{2 \lambda_2 (x + i \lambda_2 t) + (2 \lambda_2 x - 1)(4 \lambda_1 x^2 - 2 \lambda - 2 \lambda_2) + 1}) + 1,
\]

and

\[
z_2(x, t) = x (-2 c_2^2 x (\lambda_1 - c_2 \lambda_2 e^{2 \lambda_2 (x + i \lambda_2 t)} + c_2^2 (1 + 2 \lambda_2 x) e^{2 \lambda_1 (x + i \lambda_1 t) + 2 \lambda_2 x + c_2^2 (2 \lambda_2 x (1 - 2 \lambda_1 x)) c_1 c_2 \times (2 x \lambda_1 e^{2 \lambda_1 (x + i \lambda_1 t)} + 2 c_2 x \lambda_2 (-1 + 2 \lambda_1 x) e^{2 \lambda_2 (x + i \lambda_2 t) + 2 \lambda_2 x + c_2^2 (2 \lambda_2 x (1 - 2 \lambda_2 x)) - (1 + 2 x \lambda_1) x (1 - 1 + 2 x \lambda_2) e^{2 \lambda_2 (x + i \lambda_2 t))})),
\]

where \( c_1 \) and \( c_2 \) are arbitrary real constants. For simplicity, we take \( c_1 = c_2 = 1 \) thus, the solution \( u_1(x, t) \) takes the following form:

\[
u_1(x, t) = \frac{-4 i \lambda_1 x (|\lambda_1|^2 - \lambda_1^2) e^{2 q_1(x, t) - \lambda_1^2 e^{2 q_1(x, t)}} + \lambda_1^2 e^{2 q_1(x, t)} + \lambda_1^2 e^{2 q_2(x, t)}}{2 i \lambda_1 \cos[q_1(x, t)] + \lambda_1 \cosh[2 \lambda_1 x - 2 i \lambda_1 (x + 2 i \lambda_1 t)]} e^{q_2(x, t)},
\]

(45)

IV. NEW EXACT SOLUTIONS TO THE REVERSE-TIME NLSE

Another interesting possibility is an NLSE which is nonlocal in time rather than space, as for example

\[
u \bar{u}_t + \frac{1}{2} u_{xx} - u^2 \bar{u} = 0,
\]

(46)

where \( \bar{u} = u^*(x, -t) \). It was found that this equation admits a Lax pair only when the coefficient of the time derivative term is real \([22, 23]\). Nevertheless, in the following discussion, we present three exact solutions to (46) using the traditional separation-of-variables method.

(i) \( t \)-independent solution:
Let us write
\[
u(x,t) = F(x),
\]
and substitute into (46) to get
\[
F''(x) - 2F^3(x) = 0,
\]
with the solution
\[
F(x) = \frac{c}{cx - 1},
\]
where \(c\) is an arbitrary real constant.

(ii) \(x\)-independent solution:

We express the solution as
\[
u(x,t) = Z(T),
\]
where \(T = it\). Inserting in Eq. (46) leads to
\[
Z'(T) + Z^3(T) = 0,
\]
with the solution
\[
u(x,t) = \pm \frac{1}{\sqrt{2it - 2c}},
\]
where \(c\) is an arbitrary real constant.

(iii) \(t\)- and \(x\)-dependent solution:

Finally, we express the solution as
\[
u(x,t) = Z(T) e^{i x}.
\]
Substituting this into Eq. (46) leads to
\[
Z'(T) + Z^3(T) + Z(T) = 0,
\]
which yields to the following exact solution to (46)
\[
u(x,t) = \pm \frac{c e^{i x}}{\sqrt{e^{it} - 2c^2}},
\]
where \(c\) is an arbitrary real constant. Using any of these solutions as a seed for the DT will lead to higher order solutions corresponding to either a multi-singular soliton solution or breather.

V. CONCLUSION

In this paper, we derived a new two-soliton solution of the local NLSE using the LP and DT method with an algebraically-decaying seed solution. Within this seed, the DT method leads to the so-called singular molecule soliton which is a higher-order solution composed of two diverging peaks. The constructed solution allowed us to control practically all the characteristics of the molecule such as the binding energy, the force, and potential of interaction between the two solitons. We discussed in addition the scattering properties of such solitons. The time evolution of the soliton width has been also addressed to reveal the survival of these new structures. Furthermore, employing the LP in DT with the same seed solution as in the local NLSE case, we obtained an exact solution presented an elastic interaction between one soliton and two breather solitons, on a flat and ramp backgrounds. The case with reverse-time nonlocal NLSE has been also highlighted.

While the solitons found here are singular and hence have a diverging norm, they may describe realistic situations such as the collapsing dynamics of a Bose-Einstein condensate when the repulsive interatomic interactions are switched to attractive \[39\]. In addition, a recent preprint by Sackakouji et al. \[38\] has argued that such singular solitons may have some realistic relevance under certain circumstances.

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