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ALMOST PERIODIC MOTION OF A STRING VIBRATING AGAINST
A STRAIGHT FIXED OBSTACLE

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1. INTRODUCTION

In a plane with normalized coordinate system 0xu, we consider the small oscillations of a
vibrating string with fixed ends (±1/2, 0). The free oscillations of this string are perturbed by
the presence of a fixed obstacle \( u = -h \), against which it rebounds following a law which
insures conservation of the energy \( 0 \leq h < 1 \).

At time \( t = 0 \), the string is at rest in a position \( u(x, 0) = u_0(x) \) such that
\[
\begin{align*}
&u_0 \in H^1([-1/2, 1/2]) \\
&0 \leq u_0 \leq 1 \quad \text{on } [-1/2, 1/2].
\end{align*}
\]

There exists \( a, b \) in \([-1/2, 1/2[\), \( a \leq b \) such that
\[
\begin{align*}
&u_0 \text{ is non-decreasing and } u_0(x) < 1 \text{ on } [-1/2, a[ \\
&u_0(x) = 1 \quad \text{for } x \in [a, b] \\
&u_0 \text{ is non-increasing and } u_0(x) < 1 \text{ on } ]b, 1/2].
\end{align*}
\]

If there was no obstacle, the motion of the string would be described by the wave equation:
\[
\Box u = \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0,
\]
a classical approximation following d'Alembert [1] for the small (2-periodic) oscillations
around \( u = 0 \).

In case there is an obstacle, Amerio & Prouse [2, linear obstacle], then Schatzman [12, concave obstacle] studied a nonlinear version of (4), assuming that there is no longitudinal
perturbation and no energy loss during the shocks. Following [8] and setting \( \Omega = \]−1/2, 1/2[, one looks for solutions in the functional class
\[
u \in C(R^+, H^1_0(\Omega)) \cap W^{1, \infty}(R^+, L^2(\Omega))
\]
satisfying the system
\[
\begin{align*}
u \geq -h & \quad \text{in } R^+ \times \Omega \\
\text{Supp}(\Box u) \subset \{(x, t), u(x, t) > -h\}
\end{align*}
\]
\[
\frac{\partial}{\partial x} \left\{-2 \frac{\partial u}{\partial t} \cdot \frac{\partial u}{\partial x} \right\} + \frac{\partial}{\partial t} \left\{ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial t} \right)^2 \right\} = 0.
\] (8)

Condition (8), which is a strengthened version of energy conservation, must hold in the sense of \( \mathcal{D}' \{ 0, +\infty \times \Omega \} \). It implies that the energy integral
\[
\int_{\Omega} \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial t} \right)^2 \, dx = \int_{\Omega} \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial t} \right)^2 \, dx
\]
remains unchanged through the motion.

Notice that in case a shock takes place simultaneously along a set of non-zero measure in \( \Omega \), we cannot have
\[
\hat{\frac{\partial u}{\partial t}} \in C(R^+, L^2(\Omega)).
\]
Schatzman studied the existence and uniqueness of a solution of (5) such that
\[
u(\mathbb{R}, 0) = u_0(x), \quad \nu_\partial(x, 0) = \nu_\partial(x).
\]
Actually, the existence and uniqueness of a solution for (5)+(9) is obtained in the functional class (5). The solution constructed satisfies the additional property:
\[
\Box u \equiv 0 \quad \text{in} \; \mathcal{D}'(0, +\infty \times \Omega)
\]
which is used to get uniqueness, and is interpreted in [9] as equivalent to subsonic propagation of interactions.

In our case, since \( \partial u/\partial t (x, 0) = 0 \), we look for solutions (even as a function of \( t \)) defined on \( \mathbb{R} \times \Omega \). Our purpose is to give a precise meaning to the physical idea of ‘vibrations’ against the obstacle.

We prove that the solution is strongly almost-periodic as a function from \( \mathbb{R} \) to \( H^1(\Omega) \), generally not exactly periodic in \( t \). We also sketch out some simple results concerning the non-harmonic Fourier series of \( u(x, t) \) with respect to \( t \). Further computations in this direction are planned for the future.

2. STATEMENT OF THE RESULTS

Let us denote by (\( \Sigma \)) the system (5)+(10) with \( \nu_0 = 0 \) and \( \mathbb{R}^+ \) replaced by \( \mathbb{R} \) everywhere. We recall \( 0 \leq \eta < 1 \).

**Theorem 1.1.** If \( u_0 \) satisfies (1)-(3), the solution of (\( \Sigma \)) is such that \( u(t) - u(\cdot, t) \) is strongly almost-periodic as a function from \( \mathbb{R} \) to \( H^1(\Omega) \). Moreover, the map \( u_0 \mapsto u(t) \) is Lipschitz-continuous from \( H^1(\Omega) \) to \( L^2(\mathbb{R}, H^1(\Omega)) \).

**Remark.** In contrast with the case of equation (4), \( u(t) \) is generally not time-periodic. More precisely, we have:

**Theorem 1.2.** (a) \( \eta = p/q \) with \( p, q \) integers, the motion is periodic with \( p + q \) as a period if \( p + q \) is even, and \( 2(p + q) \) as a period if \( p + q \) is odd. In the special case when \( u_0(-x) = u_0(x) \), we have always the period \( p + q \).

In some cases the smallest period is smaller.
(b) If \( h \in Q \), the motion is never periodic, except in the single case \( u_0(x) = 1 - 2|x| \), where the motion has the period \( 1 + h \).

**Remark.** The aim of this paper is to give complete proofs of the results announced in [7], together with some more information.

Since \( u \) is almost-periodic with respect to \( t \), for every \( \lambda \in \mathbb{R} \), the limit

\[
\lim_{t \to \pm \infty} \frac{1}{2t} \int_{-t}^{t} u(x, \theta) \, d\theta = \hat{u}(x, \lambda)
\]

exists. The set \( \mathcal{E}(u) = \{ \lambda \in \mathbb{R}, \hat{u}(\cdot, \lambda) \neq 0 \} \) is called the set of exponents of \( u \), and in general (cf. [4]) is denumerable. Here, we have a more precise result.

**Theorem 1.3.** The exponents of \( u \) lie in the additive subgroup of \( \mathbb{R} \) generated by \( \pi \) and \( 2\pi/1 + h = \omega \).

More precisely:

(a) For general \( u_0 \) satisfying (1)-(3), we have

\[
\mathcal{E}(u) \subseteq \mathbb{Z} \pi + Z\omega.
\]  

(b) If \( u_0 \) is even, the motion of \( u(0, t) \) can be developed as a generalized Fourier series:

\[
u(0, t) = \lim_{N \to \infty} \left[ \frac{1-h}{2} + \sum_{p=1}^{N} \sum_{k=-\infty}^{\infty} \lambda_k \cos((2p+1)\omega + 2k\pi)t \right]
\]  

where the infinite sums on the right-hand side converge uniformly and the convergence with respect to \( N \) is uniform for \( t \in \mathbb{R} \).

(c) In the special case \( u_0(x) = 1 - 2|x + \lambda \sin 2\pi x| \), \( |\lambda| \leq \frac{1}{2\pi} \)

\[
\lambda_k = -\frac{8(1+h)}{(2p+1)^2\pi^2} J_k[(2p+1)\lambda\omega] \quad \text{for } k > 0
\]

\[
\lambda_{-k} = (-1)^k \lambda_k \quad \text{for } k > 0
\]

and

\[
\lambda_0 = -\frac{4(1+h)}{(2p+1)^2\pi^2} J_0[(2p+1)\lambda\omega].
\]

3. SOME EXPLICIT FORMULAS

a. **Case** \( u_0(x) = 1 - 2|x| \)

The solution of (Σ) is easy to describe. For \( 0 \leq t \leq t^* = \frac{1 + h}{2} \), the motion is given by

\[
\hat{u}(x, t) = \begin{cases} 
1 - 2 \sup\{|x|, t\} & \text{if } 0 \leq t \leq 1/2 \\
1 - 2 \inf\{1 - |x|, t\} & \text{if } 1/2 \leq t \leq \frac{1 + h}{2}.
\end{cases}
\]  

*†* Here \( J_k \) is Bessel's function of order \( k \).
At time $t^* - 1 + \frac{h}{2}$, the string hits the obstacle $\{u = -h\}$ along the line segment $\{u = -h, |x| \leq (1 - h)/2\}$, with a uniform velocity $v(t^*) = v^- = -2$.

Then the velocity is reversed into $v^+ = -v^- = +2$, and the motion proceeds backwards:

$$
\tilde{u}(x, t) = \tilde{u}(x, 1 + h - t), \forall t \in \left[\frac{1 + h}{2}, 1 + h\right].
$$

Thus $\partial \tilde{u}/\partial t(x, t)$ is discontinuous at $t = t^*$, while $\tilde{u}(x, t)$ remains continuous with respect to $t \in R$. The motion $u$ is periodic in $t$ with period $1 + h$.

Conditions (5) and (6) are easy consequences of (15), (16) as soon as (8) is checked (to insure strong continuity in $H^{1/2}(\Omega)$). It is obvious that $\tilde{u} \in W^{1,\infty}(R \times \Omega)$ and moreover, almost everywhere on $R \times \Omega$, we have

$$
\frac{\partial \tilde{u}}{\partial t} \cdot \frac{\partial \tilde{u}}{\partial x} = 0, \left(\frac{\partial \tilde{u}}{\partial t}\right)^2 + \left(\frac{\partial \tilde{u}}{\partial x}\right)^2 = 4.
$$

Then (8) follows as an immediate consequence.

Finally, a lengthy but standard computation gives the expression of $\Box \tilde{u}$: for every $\varphi \in \mathcal{D}(R \times \Omega)$, we have

$$
\langle \Box \tilde{u}, \varphi \rangle = \frac{4}{2} \sum_{m \in \mathbb{Z}} \int_{\left\{|x| \leq \frac{1 - h}{2}\right\}} \varphi \left[x, (2m + 1) \frac{1 + h}{2}\right] dx.
$$

This formula implies (7) and (10).

b. The ‘regular’ case

In addition to (1)–(3), we assume now

$$
u_0 \in C^2([-1/2, 1/2]), \text{ and } u_0(-1/2) = u_0(1/2) = 0, \text{ and } u_0(a) = u_0(b) = u_0(a) = u_0(b) = 0.
$$

In order to compute $u(x, t)$, we generalize an idea used in [11] by Reder. We define a function $F: R \rightarrow R$ in several steps.

**Step 1.** If $x \in [-1/2, 1/2]$, we set

$$
F(x) = \begin{cases} 
\frac{u_0(x) - 1}{2} & \text{if } -1/2 \leq x \leq a \\
0 & \text{if } a \leq x \leq b \\
\frac{1 - u_0(x)}{2} & \text{if } b \leq x \leq 1/2.
\end{cases}
$$

It is clear from (19) that $F \in C^2([-1/2, 1/2])$.

**Step 2.** If $t \in [1/2, 3/2]$, we define

$$
F(t) = 1 - F(1 - t).
$$
It is clear that $F \in C^{2}([1/2, 3/2])$. Since $F''(1/2) = 0$, we have 

$$F \in C^{2}([-1/2, 3/2]).$$

**Step 3.** We extend $F$ on $R$ by the condition 

$$\forall t \in R, F(t + 2) = F(t) + 2.$$ 

We introduce $f(t) = F(t) - t$: then $f(t)$ is 2-periodic on $R$. 

We claim that $f$ and $F$ are in $C^{2}(R)$, because $F'(-1/2) = F'(3/2)$ and $F''(-1/2) = F''(3/2)$. 

The first condition is a consequence of (21), the second one follows from $F''(-1/2) = 0$.

Some other properties of $F$ are summarized below.

**Lemma 3.1.** $F$ is nondecreasing: $R \to R$. In addition (21) holds for $t \in R$, and we have also $F(t) = -1 - F(-1 - t), \forall t \in R$. 

For every $x \in \Omega$ and $t \in R$, we have 

$$-1 \leq F(x + t) + F(x - t) \leq 1.$$ 

**Proof.** Formula (21) means that for every $\theta \in [-1/2, 3/2]$, we have 

$$F(\theta) + F(1 - \theta) = 1.$$ 

Given $t \in R$, we choose $m \in \mathbb{Z}$ and $\theta \in [-1/2, 3/2]$ such that $t = 2m + \theta$. Then: 

$$F(t) + F(1 - t) = F(\theta) + F(1 - \theta) = 1.$$ 

Also, for $t \in R$, we have: 

$$F(t) + F(-1 - t) = F(t) + F(1 - t) - 2 = -1.$$ 

Let $x \leq 1/2$: by (21), we have 

$$F(x + t) + F(x - t) = F(x + t) - F(-x + t + 1) + 1.$$ 

But $x \leq 1/2 \Rightarrow -x + t + 1 \geq x + t$. 

Thus we obtain: $F(x + t) + F(x - t) \leq 1$. 

Also, if $x \geq -1/2$, we write 

$$F(x + t) + F(x - t) = F(x - t) - F(-1 - x - t) - 1 \geq -1,$n

since $x - t \geq -1 - x - t$. 

Since $-1/2 \leq F(x + t) + F(x - t)/2 \leq 1/2$, we introduce: 

$$u(x, t) = \bar{u}\left\{\frac{F(x + t) + F(x - t)}{2}, \frac{F(x + t) - F(x - t)}{2}\right\}$$

for $(x, t) \in \bar{\Omega} \times R$.

**Lemma 3.2.** $u$ is the solution of $(\Sigma)$ with initial conditions $(u_0, 0)$. and we have 

$$\sup_{t \in K} |u(\cdot, t) - u'(\cdot, t)|_{H^0(\Omega)} \leq C|u_0 - u_0'\|_{H^0(\Omega)}$$

for $u_0, u_0'$ 'regular' satisfying (1-3).
Proof. Let us introduce for convenience
\[ X = \frac{F(x + t) + F(x - t)}{2}, \quad T = \frac{F(x + t) - F(x - t)}{2}, \]
Then, in the sense of \( \mathcal{D}'(\Omega \times R) \), we have
\[
\begin{align*}
\frac{\partial u}{\partial x} &= \frac{\delta u}{\delta X} \cdot \frac{F'(x + t) + F'(x - t)}{2} + \frac{\delta u}{\delta T} \cdot \frac{F'(x + t) - F'(x - t)}{2}, \\
\frac{\partial u}{\partial t} &= \frac{\delta u}{\delta X} \cdot \frac{F'(x + t) - F'(x - t)}{2} + \frac{\delta u}{\delta T} \cdot \frac{F'(x + t) + F'(x - t)}{2}.
\end{align*}
\]
\tag{25}
Using (18), we get immediately that \( \partial u/\partial t \) and \( \partial u/\partial x \) are in \( L^\infty(\Omega \times R) \) and moreover:
\[
\begin{align*}
\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial t} \right)^2 &= 2\left[ F'(x + t) F''(x + t) - F'(x - t) F''(x - t) \right], \\
\frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial t} &= F'(x + t) - F'(x - t)
\end{align*}
\]
\tag{26}
The properties (5) and (6) are immediately checked. On the other hand, in the sense of \( \mathcal{D}'(\Omega \times R) \), we obviously have
\[
\begin{align*}
\frac{\partial}{\partial t} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial t} \right)^2 \right] &= 4\left[ F'(x + t) F''(x + t) - F'(x - t) F''(x - t) \right], \\
\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) \cdot \frac{\partial u}{\partial t} &= 2\left[ F'(x + t) F''(x + t) - F'(x - t) F''(x - t) \right]
\end{align*}
\]
which implies (8).

Differentiating (25) in the distribution sense, we get after reduction:
\[
\Box u(x, t) = F'(x + t) F'(x - t) \Box \bar{u}[X(x, t), T(x, t)],
\tag{27}
\]
this formula making sense because \( \Box \bar{u} \) is a measure and \( F' \) is continuous. From (27), we deduce that (7) and (10) are satisfied.

As a consequence of lemma 3.1, we have for every \( t \in R \): \( X(1/2, t) = 1/2 \) and \( X(-1/2, t) = -1/2 \).
Thus \( u(1/2, t) = 0 \) and \( u(-1/2, t) = 0 \), and we have \( u(t) \in H^1_0(\Omega) \) for all \( t \).
To verify the initial condition, we remark that \( u(x, 0) = \bar{u}(F(x), 0) = \bar{u}_0(F(x)) = 1 - 2|F(x)| \) and (20) implies \( 1 - 2|F(x)| = u_0(x) \) in \( \Omega \).
It is easy to check that
\[
\frac{u(x, t) - u_0(x)}{t} \rightarrow 0 \quad \text{in} \quad C(\bar{\Omega}) \quad \text{as} \quad t \rightarrow 0, \quad \text{which implies} \quad \frac{\partial u}{\partial t}(x, 0) = 0.
\]
Finally, we can see that the mapping \( u_0 \rightarrow f \) is Lipschitzian from \( H^1_0(\Omega) \) to \( H^1([0, 2]) \). Since \( \bar{u} \in W^{1, \infty}(\Omega \times R) \), the mapping:
\[
f \mapsto u(\cdot, t) \quad \text{is Lipschitz-continuous from} \quad H^1([0, 2]) \quad \text{to} \quad L^\infty(R, H^1(\Omega)).
\]
Combining these two remarks, we get (24), which will appear a convenient tool to treat the more general case where \( u_0 \) is only in \( H^1_0(\Omega) \).
c. The general case

If \( u_0 \) satisfies conditions (1–3), we can still define \( F(t) \) as in the paragraph above. Then \( F(t) \) is non-decreasing, in \( H^1(R) \) and \( f(t) = F(t) - t \) is 2-periodic.

We compute \( u(x, t) \) by means of formula (23). To check that \( u(x, t) \) is the solution of (\( \Sigma \)) starting from \( (u_0, 0) \), we can choose a sequence of initials \( u_0 \) satisfying (19) with \( u_0 \to u_0 \) in \( H^1_0(\Omega) \). According to (24), \( u^0(x, t) \) will converge to \( u(x, t) \) in \( L^2(R, H^1_0(\Omega)) \). A similar calculation shows that \( \partial u^0/\partial t \) converges to \( \partial u/\partial t \) in \( L^2(R, L^2(\Omega)) \).

Thus all conditions (5) \to (10) are checked by density.

4. PROOF OF THEOREM 1.1

Because of (24), it is enough to check the almost-periodicity when \( u_0 \) is ‘regular’, so that \( F \in C^1(R) \).

a. As a first step, we prove that \( t \to u(., t) \) is almost-periodic in \( C(\Omega) \), by using directly Bochner’s criterion.

Let \( \{t_n\} \) be any sequence of real numbers. We write:

\[
t_n = 2m_n + \rho_n \quad \rho_n \in [0, 2]
\]

\[
2m_n = k_n(1 + h) + \sigma_n \quad \sigma_n \in [0, 1 + h]
\]

where \( m_n \) and \( k_n \) are in \( \mathbb{Z} \). Thus

\[
u(x, t + t_n) = \hat{u} \left\{ \frac{F(x + t + \rho_n) + F(x - t - \rho_n)}{2}, \frac{F(x + t + \rho_n) - F(x - t - \rho_n)}{2} + \sigma_n \right\}
\]

We may assume \( \lim \rho_n = \rho \) and \( \lim \sigma_n = \sigma \), by extracting a subsequence of \( \{t_n\} \). Then, because of the uniform continuity of \( F \) and \( \hat{u} \) the sequence \( u(x, t + t_n) \) converges uniformly on \( \Omega \times R \) to

\[
u^*(x, t) = \hat{u} \left\{ \frac{F(x + t + \rho) + F(x - t - \rho)}{2}, \frac{F(x + t + \rho) - F(x - t - \rho)}{2} + \sigma \right\}.
\]

b. Precompactness of the range of \( u \) in \( H^1_0(\Omega) \)

Thanks to conservation of energy, \( u(t) \) is bounded in \( H^1_0(\Omega) \), thus weakly almost-periodic in \( H^1_0(\Omega) \). According to [10], Theorem 2.11, the strong almost-periodicity is equivalent to precompactness of the range in \( H^1_0(\Omega) \). Even in the ‘regular’ case, it is not so easy to check because we cannot approach \( u \) in \( L^\infty(R, H^1_0(\Omega)) \) by regular functions since \( \hat{u} \) has discontinuous derivatives. We need a technical lemma.

**Lemma 4.1.** With the notations of (a), there exists a denumerable set \( S \subset [-1/2, 1/2] \) such that

\[
x \in S \Rightarrow \lim_{k \to \infty} \frac{\partial u}{\partial x}(x, t_n_k) = \frac{\partial u^*}{\partial x}(x, 0).
\]

**Proof.** Let us use the notations

\[
X(x, t) = \frac{F(x + t) + F(x - t)}{2}, T(x, t) = \frac{F(x + t) - F(x - t)}{2}
\]

\[
X(x, t) = \frac{F(x + t) + F(x - t)}{2}, T(x, t) = \frac{F(x + t) - F(x - t)}{2}
\]
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Fig. 1.

\[ a(x, t) = \frac{\partial \hat{u}}{\partial x} (x, t) \quad \text{(for all } t \text{ and almost every } x) \]

\[ b(x, t) = \frac{\partial \hat{u}}{\partial t} (x, t) \quad \text{(for all } x \text{ and almost every } t). \]

The functions \( a, b \) take only the values 0, -2 and +2, their discontinuities lie on some curves of \( \hat{\Omega} \times R \).

The periodicity of \( \hat{u}/t \) allows us to draw a picture in \( \hat{\Omega} \times [0, 1 + h] \) (cf. Fig. 1). For \( x \in [-1/2, 1/2[ \) and \( 0 < \delta < 1/2 - x \), we define

\[ \alpha(x, t, \delta) = 1/\delta [\hat{u}[X(x + \delta, t) - T(x + \delta, t)] - \hat{u}[X(x, t), T(x + \delta, t)] \]

\[ \beta(x, t, \delta) = 1/\delta [\hat{u}[X(x, t), T(x + \delta, t)] - \hat{u}[X(x, t), T(x, t)] \]

Also, for \( t \) fixed, we introduce two open sets

\[ \omega = \left\{ x \in ]-1/2, +1/2[ , \frac{\partial X}{\partial x} (x, t) \neq 0 \right\} \]

\[ \omega' = \left\{ x \in ]-1/2, +1/2[ , \frac{\partial T}{\partial x} (x, t) \neq 0 \right\}. \]

First we notice that if \( x \not\in \omega \), then

\[ X(x + \delta, t) - X(x, t) = o(\delta) \quad \text{when } \delta \to 0 \]

Since \( \hat{u} \) is Lipschitzian in both variables, we conclude

\[ \lim_{\delta \to 0} \alpha(x, t, \delta) = 0 \quad \text{in this case.} \]
—Let \( \omega_n \) be the sequence of connected components of \( \omega \).

For every \( n \), we have either \( \partial X / \partial x(\mathbf{x}, t) > 0 \) in \( \omega_n \), or \( \partial X / \partial x(\mathbf{x}, t) < 0 \) in \( \omega_n \). Thus, in each \( \omega_n \), there are at most 2 values \( (x_{n,k})_{k \in \{1,2\}} \) such that \( (a) \) is discontinuous at \( [X(x_{n,k}, t), T(x_{n,k}, t)] \). As a consequence, if \( x \in \bigcup_{n,k} \omega_n \) we find

\[
\lim_{\delta \to 0} a(x, t, \delta) = a(X, T) \frac{\partial X}{\partial x}.
\]

In the same way, we obtain that except for denumerable values of \( x \), we have

\[
\lim_{\delta \to 0} \beta(x, t, \delta) = b(X, T) \frac{\partial T}{\partial x}.
\]

As a consequence, there exists \( S \), denumerable such that, for every \( x \in S \):

\[
\frac{\partial}{\partial x} [u(x, t)] = a(X, T) \frac{\partial X}{\partial x} + b(X, T) \frac{\partial T}{\partial x}.
\]

Let \( S_{\text{in}} = \bigcup_{k \in \mathbb{N}} S_k \). Then, for \( x \in \bigcup_{k \in \mathbb{N}} S_k \), we have

\[
\forall k \in \mathbb{N}, \frac{\partial}{\partial x} [u(x, t_k)] = a(X_k(x), T_k(x)) \frac{\partial X_k}{\partial x} + b(X_k(x), T_k(x)) \frac{\partial T_k}{\partial x}
\]

where \( X_k(x) = X(x, t_k) \) and \( T_k(x) = T(x, t_k) \). Finally, let

\[
\begin{align*}
X_{\rho}(x) &= X(x, \rho) \\
T_{\rho, \sigma}(x) &= T(x, \rho) + \sigma.
\end{align*}
\]

By a very similar argument as above, we find \( S_{\text{in}} \) denumerable in \( [-1/2, 1/2] \) such that

\[
x \in \bigcup_{k \in \mathbb{N}} S_k, x \in S \Rightarrow \lim_{k \to \infty} \frac{\partial}{\partial x} [u(x, t_k)] = a(X_{\rho}, T_{\rho, \sigma}) \frac{\partial X_{\rho}}{\partial x} + b(X_{\rho}, T_{\rho, \sigma}) \frac{\partial T_{\rho, \sigma}}{\partial x}.
\]

But, except for \( x \in S^* \) denumerable, this is precisely equal to \( \partial / \partial x [u^*(x, 0)] \). Finally, (29) holds with \( S = S^* \cup \bigcup_{k \in \mathbb{N}} S_k \).

**Proposition 4.1.** The range of \( u(t) \) is precompact in \( H^i_0(\Omega) \) (‘regular’ case).

**Proof.** Let \( (t_n) \) be an arbitrary sequence of reals. There exists a subsequence \( (t_{n_k}) \), \( \rho \) and \( \sigma \) such that

\[
u(x, t_{n_k}) \to u^*(x, 0) \quad \text{in } C(\Omega).
\]

By lemma: \( \partial u / \partial x (x, t_{n_k}) \to \partial u^* / \partial x (x, 0) \) a.e. in \( \Omega \).

Since \( \partial u / \partial x (x, t_{n_k}) \) remains bounded in \( L^\infty (\Omega) \) by (30), Lebesgue’s theorem implies

\[
\int_{\Omega} \left| \frac{\partial u}{\partial x} (x, t_{n_k}) - \frac{\partial u^*}{\partial x} (x, 0) \right|^2 \, dx \to 0.
\]

Hence, \( u(x, t_{n_k}) \to u^*(x, 0) \) in \( H^i_0(\Omega) \).
End of the proof of Theorem 1.1. In the regular case, theorem 1.1 is now an obvious consequence of (a), proposition 4.1 and [10, theorem 2.11, p. 48], applied with $Y = C = C(\Omega)$ and $Z = H^1(\Omega)$.

In the general case, we can use (24) which shows that $u(x, t)$ is the limit in $L^\infty(R, H^1(\Omega))$ of solutions associated to 'regular' initials $u_0^\alpha$. This finishes the proof of theorem 1.1, since the statement on $u_0 \to u(t)$ is actually a consequence of (24).

5. PROOF OF THEOREM 1.2
(a) Since $F(t) - t$ is 2-periodic and $\tilde{u}(x, t)$ is $1 + h$-periodic in $t$, the first assertion is an obvious consequence of formula (23).

If $u_0(-x) = u_0(x)$, then $b = -a$ and $F(t)$ is odd. Then $F(t) - t$ is 1-periodic, and the second assertion follows.

Actually, $F(t) = t + \alpha \sin 2\pi x$ is non-decreasing as soon as $2\pi |\alpha| < 1$, and corresponds to the initial datum $u_0(x) = 1 - 2|x + \alpha \sin 2\pi x|$. In this case, $F(t) - t$ is $1/n$-periodic.

If for instance $h = 1/n$, then $(n + 1)/n = 1 + h$ is a period for $u(., t)$, which is strictly (for $n \geq 2$) smaller than $n + 1$. For $n = 2p$, we get a period smaller than $n + 1 = p + q$.

(b) If $u(., t)$ is periodic with a period $\tau$, then the functions

$$g_{\tau, \alpha, \beta}(t) = \int_0^\tau \int_0^\beta \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial \theta} \right)^2 + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial \theta} \right] dx \, d\theta$$

are also $\tau$-periodic, independently of $\varepsilon, \alpha, \beta$ for $-\frac{1}{2} \leq \alpha < \beta \leq \frac{1}{2}$. As a consequence of (26),

$$g_{\tau, \alpha, \beta}(t) = 4 \int_0^\tau \int_0^\beta \int dx \, d\sigma$$

Now it is obvious that

$$\lim_{\varepsilon \to 0} \frac{1}{4\varepsilon} g_{\tau, \alpha, \beta}(t) = \int_\alpha^\beta \int_0^\tau F'^2(x + \theta) \, dx \, d\theta = \int_\alpha^\beta F'^2(\sigma) \, d\sigma$$

Thus for every $l \in [0, 1]$, the function

$$\varphi(l, t) = \int_l^{l+1} F'^2(\sigma) \, d\sigma$$

is $\tau$-periodic.

Now we distinguish two possible cases:

1st case: $\tau \notin Q$.

Since $F'$ is 2-periodic, we must have $\varphi(l, t) = \varphi(l)$ for $l \in [0, 1]$. Since $\varphi$ is a continuous function of $l$, it is immediate to deduce

$$\int_l^{l+1} F'^2(\sigma) \, d\sigma = k^2 l, \quad k \geq 0$$

Then at every Lebesgue's point of $F'^2$, we must have $F'^2(\sigma) = k^2$. Since $F' \equiv 0$ a.e., this allows us to conclude that $F' = k$ a.e. As a consequence of (22), we must have $k = 1$. Also by (21), $F(1/2) = 1/2$.

Thus $F(t) = t$, and $u_0(x) = 1 - 2|F(x)|$ implies $u_0(x) = 1 - 2|x|$. 

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2nd case: \( \tau = p/q \in Q \setminus \{0\} \)

Then \( 2q\tau = 2p \) is also a period for \( u(. , t) \).

On the other hand, by (23), we have:

\[
\begin{align*}
    u(x, t + 2p) &= \ddot{u} \left( \frac{F(x + t) + F(x - t)}{2}, \frac{F(x + t) - F(x - t)}{2} + 2p \right) \\
    &= \ddot{u} (x, t) + \frac{F(x + t) + F(x - t)}{2} + 2p.
\end{align*}
\]

Choosing \( t = 0 \) and \( x = a \) for instance, we get \( u(a, 2p) = \ddot{u}(0, 2p) = \ddot{u}(0, 0) = 1 \). The only possibility is \( 2p = m(1 + h), \) \( m \in Z \setminus \{0\} \).

Thus \( h = 2p/m - 1 \in Q \).

6. THE EXPONENTS OF \( u(t) \). (PROOF OF THEOREM 1.3)

(a) Let \( \omega = 2\pi/1 + h \), we can write \( \ddot{u} \) as a Fourier series in \( t \):

\[
\ddot{u}(x, t) = \sum_{p=0}^{\infty} a_p(x) \cos(p\omega t)
\]

Since \( \ddot{u} \) is uniformly convergent on \( \Omega \times R \), this series is uniformly convergent on \( \Omega \times R \). The coefficients \( a_p \) are \( C^1 \) functions on \( \Omega = [-1/2, +1/2] \) and can be computed easily. Thus we have:

\[
u(x, t) = \lim_{N \to +\infty} \sum_{p=0}^{N} a_p[X(x, t)] \cos[p\omega T(x, t)]
\]

uniformly on \( \Omega \times R \) and we just need to examine the exponents of each term \( a_p[\dot{X}(x, t)] \cos[p\omega T(x, t)] \).

By a density argument, it is sufficient to do this in the ‘regular’ case. Then \( F(t) - t = f(t) \) is a \( C^1 \) function, periodic with period 2, and we get a first development:

\[
a_p[X(x, t)] = \sum_{k=0}^{\infty} a_k^p(x) \cos k \pi t, \text{ with (for instance)} \]

\[
\sum_{k=0}^{\infty} |a_k^p|_\infty < +\infty.
\]

As a second step, we may write:

\[
\cos[p\omega T(x, t)] = \cos[p\omega [t + 1/2(f(x + t) - f(x - t))] = \cos[p\omega \beta(x, t)] - \sin[p\omega \gamma(x, t)].
\]

The functions \( \beta \) and \( \gamma \) are in \( C^1(\Omega \times R) \) and thus:

\[
\beta(x, t) = \sum_{k=0}^{\infty} [\beta_k^p(x) \cos k \pi t + \delta_k^p(x) \sin k \pi t]
\]

\[
\gamma(x, t) = \sum_{k=0}^{\infty} [\gamma_k^p(x) \cos k \pi t + \epsilon_k^p(x) \sin k \pi t]
\]

with

\[
\sum_{k=0}^{\infty} (|\beta_k^p|_\infty + |\gamma_k^p|_\infty + |\delta_k^p|_\infty + |\epsilon_k^p|_\infty) < +\infty.
\]

Thus the Cauchy product of the two series defining \( a_p[X(x, t)] \) and \( \cos[p\omega T(x, t)] \) is absolutely
convergent in $C(\Omega)$, and after reduction, we are left with a development of the type:

$$a_p[X] \cos(p \omega t) = \sum_{k \in Z} [\lambda_k(x) \cos(p \omega t + k \pi t) + \mu_k(x) \sin(p \omega t + k \pi t)]$$

which is strongly almost-periodic in $C(\Omega)$ with exponents contained in $\{p \omega\} + Z\pi$ in the $L^2(\Omega)$ sense. Thus the exponents of $u(\cdot, t)$ in the sense of $L^2(\Omega)$ are in $Z\omega + Z\pi$, and of course it is the same set of exponents than in $H_0^1(\Omega)$.

(b) In case $u_0$ is even and we take $x = 0$, some simplifications occur in the previous computations. We have

$$\begin{align*}
a_0(0) &= \frac{1 - h}{2} \\
a_p(0) &= -4(1 + h) \frac{\sin^2(p/2)\pi}{p^2\pi^2}.
\end{align*}$$

We also obtain

$$\begin{align*}
cos(p \omega f(t)) &= \sum_{k > 0} \beta_k^p \cos(2k\pi t) \\
sin(p \omega f(t)) &= \sum_{k > 0} \alpha_k^p \sin(2k\pi t)
\end{align*}$$

with

$$\begin{align*}
\beta_k^p &= 2 \int_0^1 \cos(p \omega f(t)) \cos(2k\pi t) \, dt, \\
\beta_0^p &= \int_0^1 \cos(p \omega f(t)) \, dt
\end{align*}$$

and

$$\begin{align*}
\alpha_k^p &= 2 \int_0^1 \sin(p \omega f(t)) \sin(2k\pi t) \, dt,
\end{align*}$$

and we deduce easily

$$u(0, t) = \lim_{N \to \infty} \left[ \frac{1 - h}{2} - 4(1 + h) \Phi_N(t) \right],$$

where

$$\Phi_N(t) = \sum_{p=1}^N \frac{1}{(2p + 1)^2\pi^2} \sum_{k \in Z} A_k^p \cos\{(2p + 1)\omega + 2k\pi|t|\}$$

and

$$A_k^p = \begin{cases} \frac{1}{2} (\beta_k^{p+1} + \alpha_k^{p+1}) & \text{if } k > 0 \\
\beta_k^{p+1} & \text{if } k = 0 \\
\frac{1}{2} (\beta_k^{p+1} - \alpha_k^{p+1}) & \text{if } k < 0. \end{cases}$$

(c) Case $u_0(x) = 1 - 2|x + \lambda \sin 2\pi x|$.

Then $f(t) = \lambda \sin 2\pi t$, and the condition $F' \geq 0$ is equivalent to $|\lambda| \leq 1/2\pi$. 

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Setting $c_p = \frac{1}{2}(2p + 1)\lambda \omega$, we obtain

$$A_k^p = 2J_k[(2p + 1)\lambda \omega] = 2 \sum_{r=0}^{\infty} \frac{(-1)^r(c_p)^{2r+k}}{r!(r+k)!}$$

if $k > 0$

$$A_0^p = J_0[(2p + 1)\lambda \omega] = \sum_{r=0}^{\infty} \frac{(-1)^r(c_p)^{2r}}{(r!)^2}$$

if $k = 0$

$$A_k^p = 2(-1)^k J_k[(2p + 1)\lambda \omega] = 2(-1)^k \sum_{r=0}^{\infty} \frac{(-1)^r(c_p)^{2r+|k|}}{r!(r+|k|)!}$$

if $k < 0$.

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