Influence of Lorentz- and CPT-violating terms on the Dirac equation

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The influence of Lorentz- and CPT-violating terms (in "vector" and "axial vector" couplings) on the Dirac equation is explicitly analyzed: plane wave solutions, dispersion relations and eigenenergies are explicitly obtained. The non-relativistic limit is worked out and the Lorentz-violating Hamiltonian identified in both cases, in full agreement with the results already established in the literature. Finally, the physical implications of this Hamiltonian on the spectrum of hydrogen are evaluated both in the absence and presence of a magnetic external field. It is observed that the fixed background, when considered in a vector coupling, yields no qualitative modification in the hydrogen spectrum, whereas it does provide an effective Zeeman-like splitting of the spectral lines whenever coupled in the axial vector form. It is also argued that the presence of an external fixed field does not imply new modifications on the spectrum.

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I. INTRODUCTION

Lorentz covariance, as is well-known, is a good symmetry of the fundamental interactions comprised in the traditional framework of a local Quantum Field Theory, from which the Standard Model is derived. However, since the beginning 90’s, Lorentz-violating theories have been proposed as a possible candidate of signature of a more fundamental physics defined in a higher scale of energy, not accessible to the present experiments. A pioneering work due to Carroll-Field-Jackiw [1] has proposed a CPT-odd Chern-Simons-like correction term \( \epsilon^{\mu
\nu\lambda\kappa\lambda} v_{\mu} A_{\nu} F_{\kappa\lambda} \) to the conventional Maxwell Electrodynamics, that preserves gauge invariance despite breaking Lorentz and parity symmetries. Some time later, Colladay & Kostelecky [2], [3] adopted a quantum field theoretical framework to address the issue of CPT- and Lorentz-breakdown as a spontaneous violation [4]-[6]. In this sense, they constructed the extended Standard Model (SME), an extension to the Standard Model which maintains unaffected the \( SU(3) \times SU(2) \times U(1) \) gauge structure of the usual theory and incorporates the CPT-violation as an active feature of the effective low-energy broken action. In the broken phase, the resulting effective action exhibits breakdown of CPT and Lorentz symmetries at the particle frame, but conservation of covariance under the perspective of the observer inertial frame. The parameters representing Lorentz violation are obtained as the vacuum expectation values of some tensor operators belonging to the underlying theory. The SME incorporates all the tensor terms that yield scalars (by contracting standard model operators with Lorentz breaking parameters) in the observer frame.

Timely, it is worthwhile to point out the existence of alternative mechanisms that bring about equivalent Lorentz-breaking effects. Indeed, noncommutative field theories [7]-[11] also generate Lorentz-violating terms of equal structure, able to imply similar effects to the ones of the SME phenomenology. Another mechanism is varying fundamental couplings [12]-[14] which amounts to the incorporation of Lorentz-violating terms in the action as well. In fact, varying couplings leads to the breaking of temporal and spatial translations, which may be seen as a particular case of Lorentz breakdown. In a cosmological environment, this issue may be used to investigate candidate fundamental theories containing a scalar field with a spacetime-varying expectation value, once the associated Lorentz-breaking effects may be taken as a signature for an underlying theory. Further, Lorentz violation still appears in other theoretical contexts, involving the consideration of loop gravity [15]-[16] and spacetime foam [17]-[18].

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The gauge sector of the SME model has been extensively studied in several works both in (1+3) and (1+2)-dimensions \cite{19-60}, with many interesting results. Concerning the fermion sector, in the context of the SME, Colladay & Kostelecky \cite{2, 3} have devised Lorentz-violating terms compatible with U(1) gauge symmetry and renormalizability. These terms are explicitly written as below:

\[
\mathcal{L} = -v_\mu \bar{\psi} \gamma^\mu \psi - b_\mu \bar{\psi} \gamma_5 \gamma^\mu \psi - \frac{1}{2} H_{\mu\nu} \bar{\psi} \sigma^{\mu\nu} \psi + \frac{i}{2} c_{\mu\nu} \bar{\psi} \gamma_5 \gamma^\mu \gamma^\nu \psi + \frac{i}{2} d_{\mu\nu} \bar{\psi} \gamma_5 \gamma^\mu \gamma^\nu \psi, \tag{1}
\]

where the Lorentz-breaking coefficients \(v_\mu, b_\mu, H_{\mu\nu}, c_{\mu\nu}, d_{\mu\nu}\) arise as vacuum expectation values of tensor quantities defined in an underlying theory. The first two terms are CPT-odd and the others are CPT-even. Firstly, the fermion sector of the SME model has been investigated in a general way (by discussing dispersion relations, plane-wave solutions, and energy eigenvalues). Later, it has been addressed in connection with CTP-violating probing experiments, which involve comparative studies of cyclotron frequencies of trapped-atoms \cite{61-62}, clock-comparison tests \cite{63}, spectroscopic comparison of hydrogen and antihydrogen \cite{64}, analysis of muon anomalous magnetic moment \cite{65}, study of macroscopic samples of spin-polarized solids \cite{66}, and so on.

The interest of the present work lays only on the two CPT-odd terms, linked to the fermion field by an assigned "vector" and "axial vector" coupling, respectively. The main objective is to examine the effects of the Lorentz-violating background on the Dirac equation and solutions, focusing on its nonrelativistic regime and possible implications on the hydrogen spectrum. Some results concerning this study were already discussed in the literature. Indeed, the nonrelativistic Hamiltonian associated with Lagrangian \(\mathcal{L}\) was already evaluated by means of a Foldy-Wouthuysen expansion in refs. \cite{67-68}. Moreover, the corresponding shifts of atomic levels were perturbatively carried out in a broad perspective in ref. \cite{64, 69}. In the present paper, however, the analysis of the hydrogen spectrum in the presence CPT-odd terms is done in a different way (more specific, direct and simpler), also including the action of an external constant magnetic field. The starting point is the Dirac Lagrangian supplemented by Lorentz and CPT-violating terms. The dispersion relations, plane-wave solutions and eigenenergies are carried out for each one of the considered couplings. In the sequel, the investigation of the nonrelativistic limit is performed. This is a point of interest due to its connection with real systems of Condensed Matter Physics, a true environment where the presence of a background may be naturally tested. The effect of the background on the spectrum of hydrogen atom is then evaluated, initially for the case of the vector coupling, for which it is reported no correction on the hydrogen spectrum. In the case of the axial vector coupling, the spinor solutions come out to be cumbersome and the nonrelativistic limit altered. The Pauli equation is supplemented by terms that effectively modify the spectrum of the hydrogen in a similar way as the usual Zeeman effect. This sort of theoretical modification may be combined with fine spectral analysis to set up precise bounds on the magnitude of the corresponding Lorentz-violating coefficient. It is still shown that the presence of an external fixed magnetic field does not lead to new Lorentz-violating effects.

This paper is outlined as follows. In Sec. II, it is considered the presence of the term \(v_\mu \bar{\psi} \gamma^\mu \psi\) in the Dirac Lagrangian. The modified Dirac equation, dispersion relations, plane-wave solutions and energy eigenvalues are evaluated. The nonrelativistic limit is analyzed and the corresponding Hamiltonian worked out. In a first order evaluation, it is shown that the Lorentz-violating terms do not modify the hydrogen spectrum. In Sec. III the presence of the axial vector term, \(b_\mu \bar{\psi} \gamma_5 \gamma^\mu \psi\), in the Dirac sector is considered. Again, the modified Dirac equation, dispersion relations, plane-wave solutions and eigenvalues are carried out. Finally, the low-energy limit is studied and the Hamiltonian evaluated. A first order computation shows that the Lorentz-violating terms contribute to the spectrum hydrogen, causing a Zeeman splitting of the spectral lines. In Sec. IV, one presents the Conclusion and final remarks.

II. LORENTZ-VIOLATING DIRAC LAGRANGIAN ("VECTOR" COUPLING)

The most natural and easy way to couple a fixed background \([\alpha^\mu = (v_0, \vec{v})]\) to a spinor field is defining a vector coupling, given as follows:

\[
\mathcal{L} = \mathcal{L}_{\text{Dirac}} - v_\mu \bar{\psi} \gamma^\mu \psi, \tag{2}
\]
where $\mathcal{L}_{\text{Dirac}}$ is the usual Dirac Lagrangian ($\mathcal{L}_{\text{Dirac}} = \frac{i}{2} \bar{\psi} \gamma^\mu \hat{\partial}_\mu \psi - m_e \bar{\psi} \psi$) and $v_\mu$ is one of the CPT-odd parameters that here represents the fixed background responsible for the violation of Lorentz symmetry in the frame of particles. In true, the term $v_\mu \bar{\psi} \gamma^\mu \psi$ behaves as a scalar just in the observer frame, in which $v_\mu$ is seen as a genuine 4-vector. The Euler-Lagrange equation applied on this Lagrangian provides the modified Dirac equation:

$$(i\gamma^\mu \partial_\mu - v_\mu \gamma^\mu - m_e) \psi = 0,$$

which corresponds to the usual Dirac equation supplemented by the Lorentz-violating term associated with the background. The initial task is to investigate the plane-wave solutions, which may be attained by writing the spinor in terms of a plane-wave decomposition, $\psi = N e^{-iE_p \cdot p} w(p)$, where $N$ is the normalization constant and $w(p)$ is the $(4 \times 1)$ spinor written in the momenta space. Taking it into account, eq. (3) is rewritten in momentum space:

$$(\gamma^\mu p_\mu - v_\mu \gamma^\mu - m_e) w(p) = 0. \quad (4)$$

It is possible to show that each component of the spinor $w$ satisfies a changed Klein-Gordon equation which represents the dispersion relation of this model. In fact, multiplying this equation on the left by $(\gamma^\mu p_\mu - v_\mu \gamma^\mu + m_e)$, it results:

$$((p \cdot p - 2p \cdot v + v \cdot v - m_e^2)) w(p) = 0, \quad (5)$$

whose energy solutions are: $E_\pm = v_0 \pm \sqrt{m_e^2 + (p - \vec{v})^2}$. Here, one has two different energy values, one positive ($E_+$), another negative ($E_-$). The negative solution should be reinterpreted as positive-energy anti-particles. Even after the reinterpretation, the eigenenergies remain different. This is an evidence of charge conjugation breakdown, as it will be properly discussed ahead.

Now, the spinors $w(p)$ compatible with such solution should be achieved. Adopt an explicit representation for the Dirac matrices\(^1\) and writing $w(p)$ in terms of two $2 \times 1$ spinors ($w_A$ and $w_B$), the following spinor equations are obtained:

$$w_A = \frac{1}{(E - v_0 - m_e)} \vec{\sigma} \cdot (\vec{p} - \vec{v}) w_B, \quad (6)$$

$$w_B = \frac{1}{(E - v_0 + m_e)} \vec{\sigma} \cdot (\vec{p} - \vec{v}) w_A. \quad (7)$$

In order to attain a simple solution, a usual procedure for construction of plane-wave spinors is followed: a starting form, $\left( \begin{array}{c} 1 \\ 0 \end{array} \right)$ or $\left( \begin{array}{c} 0 \\ 1 \end{array} \right)$, for one of them is proposed, so that the other is straightforwardly derived by means of eqs. (6), (7). These two $2 \times 1$ spinors must then be grouped in a single normalized $(4 \times 1)$ spinor. Following this procedure, after reinterpretation\(^2\), four independent $(4 \times 1)$ spinors, $u_i$ (particle solutions) and $v_i$ (anti-particle solutions), are attained:

$$u_1(p) = N \left( \begin{array}{c} 1 \\ 0 \\ \frac{p_x - v_x}{E + m_e - v_0} \\ \frac{(p_x - v_x)(p_y - v_y)}{E + m_e - v_0} \end{array} \right), \quad u_2(p) = N \left( \begin{array}{c} 0 \\ 1 \\ \frac{p_x - v_x}{E + m_e - v_0} \\ \frac{(p_x - v_x)(p_y - v_y)}{E + m_e - v_0} \end{array} \right), \quad (8)$$

1 Here, one adopts the Dirac representation for $\gamma$-matrices: $\gamma^0 = \left( \begin{array}{cc} I & 0 \\ 0 & -I \end{array} \right)$, $\gamma^i = \left( \begin{array}{cc} 0 & \sigma^i \\ -\sigma^i & 0 \end{array} \right)$, $\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \left( \begin{array}{cc} I & 0 \\ 0 & I \end{array} \right)$, with $\sigma^i = (\sigma_x, \sigma_y, \sigma_z)$ being the well-known Pauli matrices.

2 It should be just remembered that the reinterpretation procedure consists in turning a negative-energy solution into a positive-energy anti-particle (for which the energy and momentum must be reverted: $E \rightarrow -E$, $\vec{p} \rightarrow -\vec{p}$).
\[
v_1(p) = N \begin{pmatrix} \frac{(p_x + v_3)}{E + m_\gamma + v_0} & \frac{(p_x + v_3) + (p_y + v_2)}{E + m_\gamma + v_0} \\ \frac{1}{E + m_\gamma + v_0} & 0 \end{pmatrix}, \quad v_2(p) = N \begin{pmatrix} \frac{(p_x + v_3) + (p_y + v_2)}{E + m_\gamma + v_0} & 0 \\ 0 & 1 \end{pmatrix},
\]

where \(N\) is the normalization constant. In the solutions [5], [9], one of the effects of the background is manifest: to shift the energy and momentum by a constant: \(E \rightarrow E - v_0, \mathbf{p} \rightarrow (\mathbf{p} - \mathbf{v})\). It is also instructive to exhibit the energy eigenvalues associated with the four solutions above. In this case, one can write two eigenvalue equations: \(H u_i = E_i^{(u)} u_i, H v_i = E_i^{(v)} v_i\), with \(i = 1, 2,\) and \(E_i^{(u)} = v_0 + \left[m_\epsilon^2 + (\mathbf{p} - \mathbf{v})^2\right]^{1/2}, E_i^{(v)} = \left[m_\epsilon^2 + (\mathbf{p} + \mathbf{v})^2\right]^{1/2} - v_0\). Here, \(E_i^{(u)}\) stands for the particle energy whereas \(E_i^{(v)}\) represents the anti-particle energy. In the reinterpretation procedure, it was obviously assumed that the magnitude background is minute near the electron mass \((v_0 << m_\epsilon)\), regarded as a correction effect. This must be so once many experiments demonstrate the validity of Lorentz covariance with high precision. It should still be pointed out that these energy values are in agreement with the similar ones obtained in refs. [2]-[3], [67]-[68].

In a well-known case, the nonrelativistic limit manifests: to shift the energy and momentum by a constant: \(E \rightarrow E - v_0, \mathbf{p} \rightarrow (\mathbf{p} - \mathbf{v})\). It is also instructive to exhibit the energy eigenvalues associated with the four solutions above. In this case, one can write two eigenvalue equations: \(H u_i = E_i^{(u)} u_i, H v_i = E_i^{(v)} v_i\), with \(i = 1, 2,\) and \(E_i^{(u)} = v_0 + \left[m_\epsilon^2 + (\mathbf{p} - \mathbf{v})^2\right]^{1/2}, E_i^{(v)} = \left[m_\epsilon^2 + (\mathbf{p} + \mathbf{v})^2\right]^{1/2} - v_0\). Here, \(E_i^{(u)}\) stands for the particle energy whereas \(E_i^{(v)}\) represents the anti-particle energy. In the reinterpretation procedure, it was obviously assumed that the magnitude background is minute near the electron mass \((v_0 << m_\epsilon)\), regarded as a correction effect. This must be so once many experiments demonstrate the validity of Lorentz covariance with high precision. It should still be pointed out that these energy values are in agreement with the similar ones obtained in refs. [2]-[3], [67]-[68].

The attainment of different energies for particle and anti-particle \(E_i^{(u)} \neq E_i^{(v)}\) is an evidence that the charge conjugation (\(C\)) symmetry has been broken. Indeed, the term \(v_\mu \bar{\psi} \gamma^\mu \psi\) is \(C\)-odd and PT-even, that is, it implies breakdown of charge conjugation, and conservation of combined PT operation. An ease way to demonstrate such a violation is to apply the charge conjugation operator \(C = i \gamma^0 \gamma^\mu\) on the modified Dirac equation, as given in eq. (11).

This procedure will lead to the corresponding Dirac equation for the charge conjugate spinor \((\Psi_c = C\Psi^*)\) with an opposite sign for the term \(v_\mu \bar{\psi} \gamma^\mu \psi\), which implies breaking of C-symmetry.

One should now enquire about the spin interpretation of these solutions. Obviously, such solutions will not present the same spin projection as the usual Dirac free-particle solutions. But in some particular cases, it is possible to show that such solutions exhibit the same spin projection. For instance, whenever the background and the momentum are aligned along the z-axis, the spinors take the form:

\[
\begin{align*}
\hat{u}_1 &= N \begin{pmatrix} 1 \\ 0 \\ \frac{(p_x - v_3)}{E + m_\gamma - v_0} \\ 0 \end{pmatrix}, \\
\hat{u}_2 &= N \begin{pmatrix} 1 \\ 0 \\ \frac{-(p_x - v_3)}{E + m_\gamma - v_0} \\ 0 \end{pmatrix}, \\
\hat{v}_1 &= N \begin{pmatrix} 0 \\ 0 \\ \frac{(p_x + v_3)}{E + m_\gamma + v_0} \\ 1 \end{pmatrix}, \\
\hat{v}_2 &= N \begin{pmatrix} 0 \\ 0 \\ \frac{- (p_x + v_3)}{E + m_\gamma + v_0} \\ 1 \end{pmatrix}.
\end{align*}
\] (10)

Such solutions are eigenstates of the helicity operator, \(\hat{S} \cdot \hat{p} = S_z = \frac{1}{2} \Sigma_z\), with: \(\Sigma_z = \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}\). Thus, the spinors \(\hat{u}_1\) and \(\hat{v}_1\) have eigenvalue \(+1\) (spin up) whereas the spinors \(\hat{u}_2\) and \(\hat{v}_2\) have eigenvalue \(-1\) (spin down).

Hence, the presence of the fixed background does not suffice in principle to change the spin polarization of the new states. A detailed study of the spin projections may only be obtained by constructing the spin projector operators. This point is addressed by Lehnert in ref. [67].

### A. Nonrelativistic limit

Every good relativistic theory must exhibit a sensible low-energy limit whose predictions may be compared with the results of other correlated nonrelativistic theories. Such a requirement sets up the correspondence between an intrinsically relativistic theory and a nonrelativistic one. In a well-known case, the nonrelativistic

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\(3\) One takes as starting point the Dirac equation \((i \gamma^\mu \partial_\mu - e \gamma^\mu A_\mu - v_\mu \gamma^\mu - m) \psi = 0\), which for an anti-particle must be rewritten with opposite charge sign: \((i \gamma^\mu \partial_\mu + e \gamma^\mu A_\mu - v_\mu \gamma^\mu - m) \psi_c = 0\), being \(\psi_c\) the anti-particle spinor. In the case the \(C\)-symmetry holds on, this exact equation might be also obtained by applying the charge conjugation operator \(C = i \gamma^0 \gamma^\mu\) on the initial Dirac equation. Making it, one attains: \((i \gamma^\mu \partial_\mu + e \gamma^\mu A_\mu + v_\mu \gamma^\mu - m) \psi_c = 0\), where one notes the opposite sign of the term \(v_\mu \gamma^\mu\). This puts in evidence the \(C\)-breakdown. A similar procedure may be employed to demonstrate the conservation of PT symmetry.
limit of the Dirac theory yields the Pauli equation, which consists of the Schrödinger equation supplemented with the spin-magnetic interaction. Hence, to work in the nonrelativistic limit allows to investigate quantum mechanical features of a system without losing relativistic effects (like spin) of the original theory. In the present case, where the Dirac theory is being corrected by a Lorentz-violating coupling term, one expects that the nonrelativistic regime be well described by the Pauli equation incorporating Lorentz-violating terms. It will be shown that this is exactly the case.

To correctly analyze the nonrelativistic limit of Lagrangian (2), this model is considered in the presence of an external electromagnetic field \((A_\mu)\), so that Lagrangian (2) is rewritten in the form:

\[
\mathcal{L} = \frac{1}{2} \bar{\psi} \gamma^\mu D_\mu \psi - m_e \bar{\psi} \psi - \nu_\mu \bar{\psi} \gamma^\mu \psi, 
\]

where \(D_\mu = \partial_\mu + ieA_\mu\). The external field is implemented into our previous equations by means of the direct substitution: \(p^\mu \to p^\mu - eA^\mu\). Replacing it into eqs. (6) and (7), there follows:

\[
w_A = \frac{1}{(E - eA^0 - m_e - v^0)} \vec{\sigma} \cdot (\vec{p} - e\vec{A} - \vec{v}) w_B, 
\]

\[
w_B = \frac{1}{(E - eA^0 + m_e + v^0)} \vec{\sigma} \cdot (\vec{p} - e\vec{A} - \vec{v}) w_A. 
\]

In the low-velocity limit, it obviously holds \((\vec{p})^2 \ll m_e^2, eA_0 \ll m_e\), conditions that impose the smallness of kinetic and potential energy before the relativistic rest energy \((m_e)\). With it, the energy of the system is written as \(E = m_e + H\), where \(H\) represents the nonrelativistic Hamiltonian. From eqs. (12) and (13), the spinors \(w_A, w_B\) are read as the large and the small components, once the magnitude of \(w_A\) is much larger than \(w_B\). By replacing eq. (13) into eq. (12) and implementing the low-energy conditions, one should retain only the equation for the strong component \((w_A)\),

\[
(H - eA^0 - v^0) w_A = \frac{1}{(2m_e + v^0)} \vec{\sigma} \cdot (\vec{p} - e\vec{A} - \vec{v}) \vec{\sigma} \cdot (\vec{p} - e\vec{A} - \vec{v}) w_A, 
\]

which describes the physics of the nonrelativistic limit. Using the identity, \((\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = \vec{a} \cdot \vec{b} + i \vec{\sigma} \cdot (\vec{a} \times \vec{b})\), eq. (14) is reduced to the form,

\[
H w_A = \left\{ \frac{(\vec{p} - e\vec{A} - \vec{v})^2}{2m_e} + \frac{1}{2m_e} \vec{\sigma} \cdot [\nabla \times (\vec{A} - \vec{v})] + (eA^0 + v^0) \right\} w_A, 
\]

where \(H\) is the nonrelativistic Hamiltonian. Specifically, concerning the spin-orbit interaction, one can see that such background does not yield any modification, once \(\nabla \times \vec{v} = 0\). Now, comparing Eq. (15) with the Pauli equation, the Hamiltonian takes a more familiar form:

\[
H = \left\{ \left[ \frac{(\vec{p} - e\vec{A})^2}{2m_e} - \frac{e\hbar}{2m} \vec{\sigma} \cdot \vec{B} + eA^0 \right] + \left[ -2(\vec{p} - e\vec{A}) \cdot \vec{v} \right] + v^0 + \frac{\vec{v}^2}{2m_e} \right\}. 
\]

The first term into brackets contains the well-known Pauli Hamiltonian, whereas the second one is the correction Hamiltonian arising from the Lorentz-violating background. This specific term, object of our attention, is rewritten below:

\[
H_{LV} = \frac{2i\vec{v} \cdot \vec{A}}{2m_e} + \frac{2e\vec{A} \cdot \vec{v}}{2m_e} + v_0 + \frac{\vec{v}^2}{2m_e}. 
\]

Here, note that the breakdown of charge conjugation is no more manifest, once the relativistic dispersion relation has degenerated in a single expression for particles and anti-particles. Looking at eq. (17), the last two terms change the nonrelativistic Hamiltonian only by a constant, which does not represent any physical change.
(it just shifts the levels as a whole, not modifying the transition energies). Thus, just the first and the second are able to induce modifications on a physical system. The purpose now is to investigate the contribution of these two terms on the 1-particle wave functions ($\Psi$) of the hydrogen. It should be taken into account only the first term, once the hydrogen atom is initially regarded as a free system ($\vec{A} = 0$). This contribution is expected to be null, once it represents an average of the linear momentum on an atomic bound state. Explicitly, this energy quantity is correctly worked out as a first order perturbation on the corresponding 1-particle wave functions, namely: $\Delta E = \frac{1}{m_e} (nlm) |\vec{r} \cdot \vec{A}/n|$, where $n, l, m$ are the usual quantum numbers that label the 1-particle wave function of the hydrogen atom, $\psi_{nlm}(r, \theta, \phi) = R_{nl}(\rho) \Theta_{lm}(\theta) \Phi_m(\phi)$. Replacing such a form in $\Delta E$, with the gradient operator written in spherical coordinates, it implies

$$
\Delta E = \frac{i}{m_e} \int \left\{ \frac{\partial R_{nl}(r)}{\partial r} |\Theta_{lm}(\theta)|^2 |\Phi_m(\phi)|^2 \vec{v} \cdot \vec{r} + \frac{\partial R_{nl}(r)}{r \sin \theta} |\Phi_m(\phi)|^2 \Theta_{lm}(\theta)^* \frac{\partial \Theta_{lm}(\theta)}{\partial \theta} \vec{v} \cdot \vec{\theta} \right\} d^3r.
$$

(18)

For explicit calculation, the vector $\vec{v}$ can be placed along the $z$-axis, so that: $\vec{v} = v_z \hat{z}$, $\vec{v} \cdot \vec{\theta} = v_z \sin \theta$, $\vec{v} \cdot \vec{\theta} = 0$. Thus, one notes that the first two terms exhibit the presence of angular additional factors, $\cos \theta$ and $\sin \theta$, respectively. The first term is explicitly written as:

$$
\Delta E_1 = \frac{iv_z}{m_e} \int \left[ R_{nl}(r)^* \frac{\partial R_{nl}(r)}{\partial r} |\Theta_{lm}(\theta)|^2 \cos \theta \right] r^2 \sin \theta dr d\theta d\phi = 0.
$$

(19)

This null result is a consequence of $\int_0^\pi [\Theta_{lm}(\theta)]^2 \sin \theta d\theta = 0$, which holds for the associated Legendre functions. Following, the second term

$$
\Delta E_2 = -\frac{iv_z}{m_e} \int \left[ \frac{\partial R_{nl}(r)}{r} \Theta_{lm}(\theta)^* \frac{\partial \Theta_{lm}(\theta)}{\partial \theta} \sin \theta \right] r^2 \sin \theta dr d\theta d\phi,
$$

(20)

is now analyzed. The involved angular integration reads as $\int_0^\pi \Theta_{lm}(\theta) (\partial \Theta_{lm}/\partial \theta) \sin^2 \theta d\theta = \int_{-1}^1 |\Theta_{lm}(z)|^2 \frac{\partial \Theta_{lm}}{\partial z} (z^2 - 1) dz = 0$, which comes out null as consequence of the recurrence relation, $(z^2 - 1) \frac{\partial \Theta_{lm}}{\partial z} (z^2 - 1 + m) |\Theta_{l-1,m}(z)|$, and of the following orthogonality relation: $\int_{-1}^1 \Theta_{lm}(z) \Theta_{pm}(z) dz = 0$, for $l \neq p$. Therefore, the total energy correction is null, that is: $\Delta E = 0$. This means that the presence of the Lorentz-violating background does not imply any energy shift in the hydrogen spectrum. It is instructive to claim that this null correction is in full accordance with the role played by the term $v_\mu \psi \gamma^\mu \psi$: it only brings about a 4-momentum shift, $p^\mu \rightarrow p^\mu - v^\mu$, without any physical consequence on the spectrum of the system. It is also possible to understand it by reading the effect of the background as a gauge transformation. Indeed, making use of a field redefinition, $\psi \rightarrow \Psi(x) = \psi(x) e^{-iv^\mu x}$, it is possible to remove the background from the theory, so that Lagrangian (2) takes on the usual free form (written in terms of the field $\Psi$), namely: $\mathcal{L} = \mathcal{L}_{Dirac}$. This is true in any theory containing only one fermion field. For this result to remain valid in the case of a multifermion theory, the fermions families should be uncoupled with each other (no interacting fermions) and be coupled to the same Lorentz-violating parameter ($v_\mu$) [2-3].

The general result provided by the relativistic spectrum of the hydrogen may be attained by the exact solution of the modified Dirac equation (4), taken in the presence of the Coulombian potential. This solution, however, will yield nothing new, once it corresponds exactly to the conventional relativistic solution shifted according to $p^\mu \rightarrow p^\mu - v^\mu$. Finally, it should be noted that this null outcome is not due to the specific choice of the background spatial orientation, $v^\mu = (v_x, 0, 0, v_z)$; by adopting a background along an arbitrary direction, $\vec{v} = (v_x, v_y, v_z)$, identical calculations straightforwardly yield the same null result for $\Delta E$.

So far, the hydrogen spectrum has been investigated only in the absence of external field. In the presence of a fixed magnetic field, one notes that the term $e(\vec{A} \cdot \vec{v})/m_e$ of eq. (17) may contribute to a first order calculation by the quantity:

$$
\Delta E_{A} = \frac{e}{m_e} \int \Psi^* \left( \vec{A} \cdot \vec{v} \right) \Psi d^3r.
$$

(21)
Knowing that $\vec{A} = -\vec{\nabla} \times \vec{B}/2$, for a fixed magnetic field along the z-axis, $\vec{B} = B_0 \hat{z}$, it results: $\vec{A} = -B_0(y/2, -x/2, 0)$. This implies $\Delta E_{A\psi} = -(eB_0/2m_e) \int \Psi^* (y\vec{v}_z - x\vec{v}_y) \Psi d^3r$, whose explicit calculation leads to $\Delta E_{A\psi} = 0$. Therefore, one concludes that the presence of a fixed external magnetic field does not yield any Lorentz-violating contribution to hydrogen spectrum besides the usual Zeeman effect.

It is instructive to remark that these calculations hold equivalently for the case of a positron, for which the modified Pauli equation stems from (15), the positron nonrelativistic Hamiltonian exhibits opposite charge and opposite $\mu$ parameter, implying a Lorentz-violating Hamiltonian in the form $H_{LV} = -i\vec{\nabla} \cdot \vec{v}/m_e + e(\vec{A} \cdot \vec{v})/m_e - v_0 + \vec{v}^2/2m_e$. However, as in the electron case, this Hamiltonian yields no physically detectable energy shift. This issue is obviously related to the analysis of the hydrogen and antihydrogen spectroscopy, realized in wide sense in ref. [64]. In this work, it is also taken into account the effect of the Lorentz-violating background on the hyperfine structure (considering the proton spin).

### III. LORENTZ-VIOLATING DIRAC LAGRANGIAN ("AXIAL VECTOR" COUPLING)

Amongst the possible coefficients involved with the breaking of Lorentz symmetry in the fermion sector of the SME, shown in eq. (1), our interest rest in one that is also CPT-odd, $b_\mu \bar{\psi} \gamma_5 \gamma^\mu \psi$. This torsion-like term [74] is linked with the fixed background by means of an assigned axial vector coupling. Taking it into account, one writes:

$$\mathcal{L} = \frac{1}{2} \bar{\psi} i\gamma^\mu \partial_\mu \psi - mc \bar{\psi} \psi - b_\mu \bar{\psi} \gamma_5 \gamma^\mu \psi.$$  \hfill (22)

The first step is to determine the new Dirac equation stemming from the above Lagrangian, namely:

$$(i\gamma^\mu \partial_\mu - b_\mu \gamma_5 \gamma^\mu - m_e) \psi = 0.$$  \hfill (23)

This modified equation is then rewritten in the momentum space,

$$(\gamma^\mu p_\mu - b_\mu \gamma_5 \gamma^\mu - m_e) w(p) = 0.$$  \hfill (24)

provided a plane-wave solution is proposed. In order to obtain the dispersion relation associated with such an equation, it should be multiplied by $(\gamma^\mu p_\mu - b_\mu \gamma_5 \gamma^\mu + m_e)$, so that one obtains: $[p^2 - m_e^2 - b^2 + \gamma_5(p\hat{\mathbf{p}} - \hat{\mathbf{p}}\mathbf{p})]w(p) = 0$. This expression presents contributions out of the main diagonal of the spinor space. In order to achieve an expression totally contained in the main diagonal, equally valid for each component of the spinor $w$, the preceding equation shall be multiplied by $(p^2 - m_e^2 - b^2 - \gamma_5(p\hat{\mathbf{p}} - \hat{\mathbf{p}}\mathbf{p}))$, which yields the following dispersion relation:

$$(p^2 - m_e^2 - b^2)^2 + 4p^2b^2 - 4(p\cdot\hat{\mathbf{p}})^2 = 0.$$  \hfill (25)

This is a fourth order relation for the energy that can be exactly solved only in special cases. In the case of a purely timelike background, $b^\mu = (b_0, 0)$, and a purely spacelike background, $b^\mu = (0, \hat{\mathbf{b}})$, one respectively achieves:

$$E = \pm \sqrt{\hat{\mathbf{p}}^2 + m_e^2 + b_0^2 \pm 2b_0|\hat{\mathbf{p}}|},$$  \hfill (26)

Notice that there is no breakdown of charge conjugation in this case. In fact, after usual reinterpretation both particle and anti-particle exhibit the same energy values, that is, the positive roots given in eqs. [25], [26].
Therefore, Lagrangian (22) does not imply C-violation. This may be explicitly demonstrated by means of the procedure employed in Footnote 2. Therefore, Lagrangian (22) does not imply C-violation. This may be explicitly demonstrated by means of the procedure employed in Footnote 2.

Taking into account the $\gamma$-matrices definition, given at footnote 1, eq. (23) gives rise to two coupled spinor equations for $w_A$ and $w_B$:

\[
\begin{align*}
(E - \vec{\sigma} \cdot \vec{b} - m_e)w_A + (b^0 - \vec{\sigma} \cdot \vec{p})w_B &= 0, \quad (27) \\
(\vec{\sigma} \cdot \vec{p} - b^0)w_A + (-E + \vec{\sigma} \cdot \vec{b} - m_e)w_B &= 0, \quad (28)
\end{align*}
\]

leading to the following spinor relations:

\[
\begin{align*}
w_A &= \frac{1}{E_2^2} \left\{ (E - m_e)(\vec{\sigma} \cdot \vec{p}) - (E - m_e)b^0 - b^0(\vec{\sigma} \cdot \vec{b}) + \vec{b} \cdot \vec{p} + i\vec{\sigma} \cdot \vec{c} \right\} w_B, \quad (29) \\
w_B &= \frac{1}{E_1^2} \left\{ (E + m_e)(\vec{\sigma} \cdot \vec{p}) - (E + m_e)b^0 - b^0(\vec{\sigma} \cdot \vec{b}) + \vec{b} \cdot \vec{p} + i\vec{\sigma} \cdot \vec{c} \right\} w_A, \quad (30)
\end{align*}
\]

where: $\vec{c} = \vec{b} \times \vec{\sigma}, E_2^2 = [(E + m_e)^2 - b \cdot b], E_1^2 = [(E - m_e)^2 - b \cdot b].$

To construct the plane-wave solutions, one follows the general procedure adopted in the preceding section. The resulting $4 \times 1$ spinor solutions are given below:

\[
\begin{align*}
u_1 &= N \begin{pmatrix}
1 \\
0 \\
\left[(E + m_e)(p_z - b^0) - b^0b_z + \vec{b} \cdot \vec{p} + ic_z \right]/E_1^2 \\
\left[(E + m_e)(p_x + ip_y) - b^0(b_x + ib_y) + i(c_x + ic_y) \right]/E_1^2
\end{pmatrix}, \quad (31) \\
\nu_2 &= N \begin{pmatrix}
0 \\
1 \\
\left[-(E + m_e)(p_x + b^0) + b^0b_z + \vec{b} \cdot \vec{p} - ic_z \right]/E_1^2 \\
\left[-(E + m_e)(p_z + b^0) - b^0b_z + \vec{b} \cdot \vec{p} - ic_z \right]/E_1^2
\end{pmatrix}, \quad (32)
\end{align*}
\]

\[
\begin{align*}
u_1 &= N \begin{pmatrix}
1 \\
0 \\
\left[(E + m_e)(p_z + b^0) + b^0b_z + \vec{b} \cdot \vec{p} - ic_z \right]/E_2^2 \\
\left[(E + m_e)(p_x + ip_y) - b^0(b_x + ib_y) + i(c_x + ic_y) \right]/E_2^2
\end{pmatrix}, \quad (33) \\
\nu_2 &= N \begin{pmatrix}
0 \\
1 \\
\left[-(E + m_e)(p_x - b^0) - b^0b_z + \vec{b} \cdot \vec{p} + ic_z \right]/E_2^2 \\
\left[-(E + m_e)(p_z - b^0) - b^0b_z + \vec{b} \cdot \vec{p} + ic_z \right]/E_2^2
\end{pmatrix}, \quad (34)
\end{align*}
\]

where $N$ is the normalization constant. The eigenvalues of energy are the ones evaluated in eqs. (25), (26) that are now exhibited in the following eigenenergy relations: $\hbar \nu_i = E_i^{(u)} u_i$, with $E_i^{(u)} = \left[\vec{\sigma}^2 + m_u^2 + b^2 + (1)^i 2b_0 \vec{p} \right]^{1/2}$, for $b^u = (b_0, 0, 0)$, and $E_i^{(v)} = \left[\vec{\sigma}^2 + m_v^2 + \vec{b}^2 + (1)^i 2b_0 \vec{p} \right]^{1/2}$, for $b^v = (0, \vec{b}, 0)$, and $i = 1, 2$. Here, $E_i^{(u)}$ stands for the particle and anti-particle energy. Despite the cumbersome form of these spinors, it is possible to show that in the case of $b^u = (b_0, 0, 0, b_z)$ and $p = (0, 0, p_z)$, such solutions are eigenstates of the spin operator $\Sigma_z$ with eigenvalues $\pm 1$, in much the same way as observed in the foregoing section.
A. Nonrelativistic limit

The nonrelativistic limit of the model described by Lagrangian \(\mathcal{L}\) is now worked out in much the same way of the previous section. The objective is to identify the corrected Hamiltonian and possible energy shifts induced on the spectrum of hydrogen in the presence and absence of an external magnetic field. Considering the presence of an external electromagnetic field minimally coupled to the spinor field:

\[
\mathcal{L} = \frac{1}{2} \bar{\psi} \gamma^\mu \gamma^5 D_\mu \psi - m_e \bar{\psi} \psi - b_\mu \bar{\psi} \gamma^\mu \psi,
\]

where \(D_\mu = \partial_\mu + ieA_\mu\). Taking into account the external field, eq. (27) and (28) take on the form:

\[
\begin{align*}
[E - \vec{\sigma} \cdot \vec{b} - m_e - eA_\mu] w_A + \left[b^0 - \vec{\sigma} \cdot (\vec{b} - eA)\right] w_B &= 0, \\
\left[\vec{\sigma} \cdot (\vec{b} - eA) - b^0\right] w_A - \left[E - \vec{\sigma} \cdot \vec{b} + m_e - eA_\mu\right] w_B &= 0.
\end{align*}
\]

The low-energy limit is implemented by the following conditions: \((\vec{p})^2 \ll m_e^2\), \(eA_0 \ll m_e\), \(E = m_e + H\). Furthermore, one still assumes that the factor \(\vec{\sigma} \cdot \vec{b}\) must be neglected in eq. (37), once the background is supposed to be small whenever compared with the electron mass. Implementing all these conditions, it holds for the strong component:

\[
H w_A = \left\{ \left[\vec{\sigma} \cdot (\vec{p} - eA) - b^0\right] w_A - \left[E - \vec{\sigma} \cdot \vec{b} + m_e - eA_0\right] w_B \right\}.
\]

After some algebraic calculations, one achieves:

\[
H = H_{\text{Pauli}} + \frac{\vec{\sigma} \cdot \vec{b} - 2b_0 \vec{\sigma} \cdot (\vec{p} - eA) - m_e + b_0^2}{2m_e + eA_0 + \vec{\sigma} \cdot \vec{b}}.
\]

This is the modified full Hamiltonian, composed by the Pauli and a Lorentz-violating part \((H_{LV})\), where in lies our interest. Provided that \(H_{LV}\) has two interesting new terms (the third one is constant), one should try to figure out whether these terms imply real corrections to the spectrum of hydrogen. Taking into account these informations, the effective Lorentz-violating Hamiltonian assumes the form: \(H_{LV} = \vec{\sigma} \cdot \vec{b} - 2b_0 (\vec{\sigma} \cdot \vec{p}) / 2m_e\), where it was taken \(A = 0\). One then starts analyzing the term \(\vec{\sigma} \cdot \vec{b}\), whose first order contribution is:

\[
\Delta E_{\sigma \cdot b} = \langle nlmjm_s | \vec{\sigma} \cdot \vec{b} | nlmjm_s \rangle.
\]

Here, \(n, l, j, m_j\) are the quantum numbers suitable to address a situation where occurs addition of angular momenta \((L \text{ and } S)\). To solve this calculation, it is necessary to write the \(|jm_j\rangle\) kets in terms of the spin eigenstates \(|mm_s\rangle\), which is done by means of the general expression: \(|jm_j\rangle = \sum_{m, m_s} \langle mm_s | jm_j \rangle |mm_s\rangle\), where \(\langle mm_s | jm_j \rangle\) are the Clebsch-Gordon coefficients. Evaluating such coefficients for the case \(j = l + 1/2, m_j = m + 1/2\), one has: \(\langle jm_j \rangle = \alpha_1 |m \uparrow\rangle + \alpha_2 |m + 1 \downarrow\rangle\); one the other hand, for \(j = l - 1/2, m_j = m + 1/2\), it results: \(\langle jm_j \rangle = \alpha_2 |m \uparrow\rangle - \alpha_1 |m + 1 \downarrow\rangle\), with: \(\alpha_1 = \sqrt{(l + m + 1)/(2l + 1)}\), \(\alpha_2 = \sqrt{(l - m)/(2l + 1)}\). Now, taking into account the orthonormalization relation \(\langle mm'_s | mm''_s \rangle = \delta_{m_m} \delta_{m'_m} \delta_{m''_m}\), it is possible to show that eq. (40) reduces simply to \(\Delta E_{\sigma \cdot b} = \langle jm_j | \sigma_z b_z | jm_j \rangle\), whose explicit calculation leads to:

\[
\Delta E_{\sigma \cdot b} = \pm \frac{b_z m_j}{2l + 1}.
\]
where the positive and negative signs correspond to \( j = l + 1/2 \) and \( j = l - 1/2 \), respectively. Thus, in this first order evaluation the energy turns out corrected by a quantity depending on \( \pm m_j \), in a very similar way to the well-known Zeeman effect. Indeed, each line of the spectrum is split into \((2j + 1)\) lines, with a \( b_z/(2l + 1) \) linear separation. This correction was also obtained in ref. [70]. Once the magnitude of such splitting depends directly on the modulus of the background, this theoretical outcome may be used to set up an upper bound on the breaking parameter \((b^0)\).

Next, one evaluates the first order contribution of the second term of \( H_{LV} \) to the hydrogen spectrum, namely:

\[
\Delta E_{\sigma,p} = \frac{i b_0}{m_e} \langle nljm_jm_s| \vec{\sigma} \cdot \vec{v} |nljm_jm_s\rangle,
\]

(42)

The 1-particle wave function, \( \Psi_{nljm_jm_s} = \psi_{nljm_j}(r, \theta, \phi)\chi_{sm_s} \), now contains a spin function, \( \chi_{sm_s} \). In order to solve eq. (42), one should note that the gradient operator acts on the spatial function \( \psi_{nljm_j} \), whereas \( \vec{\sigma} \) operates on the spin function, so that it reads:

\[
\Delta E_{\sigma,p} = \frac{i b_0\pm m_j}{(2l+1)m_e} \int \left\{ R_{nl}^* (r) \frac{\partial R_{nl} (r)}{\partial r} |\Theta_{lm} (\theta)|^2 |\Phi_m (\phi)|^2 \langle jm_j | \vec{\sigma} \cdot \hat{r} | jm_j \rangle + \frac{|R_{nl} (r)|^2 |\Phi_m (\phi)|^2}{r} \Theta_{lm}^* (\theta) \frac{\partial \Theta_{lm} (\theta)}{\partial \theta} \langle jm_j | \vec{\sigma} \cdot \hat{\phi} | jm_j \rangle \right\} d^3 r.
\]

(43)

Writing the spherical versors in terms of the Cartesian ones, one obtains:

\[
\vec{\sigma} \cdot \hat{r} = \sin \theta \cos \phi \sigma_x + \sin \theta \sin \phi \sigma_y + \cos \theta \sigma_z, \quad \vec{\sigma} \cdot \hat{\phi} = -\sin \phi \sigma_x + \cos \phi \sigma_y.
\]

It is clear that only the terms proportional to \( \sigma_z \) yield non-null expectation values on the kets \( |jm_j\rangle \), which implies:

\[
\Delta E_{\sigma,p} = \frac{\pm i b_0 m_j}{(2l+1)m_e} \int \left\{ R_{nl}^* (r) \frac{\partial R_{nl} (r)}{\partial r} |\Theta_{lm} (\theta)|^2 \cos \theta - \frac{|R_{nl} (r)|^2 |\Phi_m (\phi)|^2}{r} \Theta_{lm}^* (\theta) \frac{\partial \Theta_{lm} (\theta)}{\partial \theta} \sin \theta \right\} d^3 r.
\]

(44)

These are exactly the same integrals involved in the expressions of \( \Delta E_1 \) and \( \Delta E_2 \), already evaluated in the previous section. So, it is obvious that: \( \Delta E_{\sigma,p} = 0 \). Hence, the sole non-null first order effect on the hydrogen spectrum is a Zeeman-like splitting stemming from the correction term \( \vec{\sigma} \cdot \vec{b} \).

Another point that deserves attention is related to the correction term \( 2eb_0 \vec{\sigma} \cdot \hat{A} \), present in eq. (39). This term is obviously null for the "free" hydrogen atom (once \( \hat{A} = 0 \)). For the case the atom is subjected to the influence of an external magnetic field, however, this term must be taken into account. For a fixed magnetic field along the z-axis, \( \vec{B} = B_0 \hat{z} \), one has \( \hat{A} = -B_0(y/2, -x/2, 0) \), so that the correction may be written as:

\[
\Delta E_{\sigma,A} = \frac{b_0 e}{m_e} \langle nljm_jm_s | \vec{\sigma} \cdot \hat{A} | nljm_jm_s \rangle = -\frac{B_0b_0 e}{2m_e} \langle nljm_jm_s | y\sigma_x - x\sigma_y | nljm_jm_s \rangle.
\]

(45)

Considering the effect of the spin operators on the kets \( |jm_j\rangle \), a null correction \((\Delta E_{\sigma,A} = 0)\) turns out. One should remark that this result remains null even for an arbitrary orientation of the magnetic field. Therefore, the conclusion is that an external fixed field does not imply any additional correction to the well-known Zeeman effect. In this case, the Lorentz-violating effect of eq. (44) corrects the usual Zeeman splitting just by a small quantity proportional to \( |\vec{b}| \). The general result provided by the relativistic spectrum of the hydrogen may be examined by the exact solution of the modified Dirac equation (3) in the presence of the Coulombian potential. This case implies qualitative modifications on the usual relativistic hydrogen spectrum, both in the case of a purely timelike or purely spacelike background. It is now under development.

IV. CONCLUSION

In this work, the effects of CPT- and Lorentz-violating background terms (stemming from a more fundamental theory) on the Dirac equation have been studied. This analysis has considered two different ways of coupling the fermion field to the background. One has started with the vector coupling, for which the modified Dirac
equation with corresponding solutions and eigenenergies have been determined. The results agree with those already known in the literature [2]-[3], [67]-[68]. The nonrelativistic regime has been assessed. It was verified that the background implies modifications on the Pauli equation, but they are such that do not yield any energy shift for the hydrogen spectrum. This is an expected result, once the vector coupling might be seen simply as a momentum shift ($p^\mu \rightarrow p^\mu - v^\mu$) unable to bring about physical modifications, or as a gauge transformation that absorbs entirely the background. In the sequel, one has analyzed the case in which the background is coupled to spinor field in an axial vector way. Again, the free-particle, dispersion relation and eigenenergies have been calculated and the nonrelativistic limit has been discussed. It was argued that the Lorentz-violating corrections to the Pauli equation are able to provide new effects on the spectrum of hydrogen. Indeed, it has been shown that the background may induce a Zeeman-like splitting of the spectral lines arising from a spin interaction. This effect may be used to set up bounds on the magnitude of the Lorentz-violation coefficient, $b^\mu$, according to precise observations of hydrogen spectrum. The presence of an external homogenous magnetic field has been also considered, but it has been shown that it does not add new corrections on the usual Zeeman effect beyond the ones already associated with the coefficient $b_\mu$.

Further comments refer to the possibility of inducing topological phases in the electron wave function by the Lorentz-violating terms considered. In a recent work [71], it has been argued that the fixed background, whenever non-minimally coupled to the gauge and spinor fields by means of a Carroll-Field-Jackiw-like term, $\epsilon_{\mu\nu\alpha\beta}^{\gamma\delta} F^{\alpha\beta}$, is able to induce an Aharonov-Casher phase in the wave function of an electron. This occurs whenever the canonical momentum is changed by a term whose curl is non null. In the case of the CPT- and Lorentz-violating coupling terms investigated in this work, however, no topological phase is generated. In effect, in both cases the canonical momentum is changed by a constant quantity ($\vec{p} \rightarrow \vec{p} - \vec{v}$) or remains invariant. In this paper, however, it was not addressed the possible effects induced by the non-minimally coupled background on a low-energy atomic spectrum. This issue has been just recently addressed [72], with new interesting results. Another continuation of the present line of investigation consists in examining the solution of the full Lorentz-violating relativistic Dirac equation for an interacting configuration, such as the Coulombian potential. In this case, only the axial vector coupling should be considered, once the vector coupling just implies a momentum shift unable to modify the solutions. One then expects that the relativistic spectrum solution may reveal new effects, at the same time it recovers the results here evaluated in the nonrelativistic regime.

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