NEW SPECIAL OPERATORS IN $W$–GRAVITY THEORIES

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ABSTRACT

We find new special physical operators of $W_3$–gravity having non trivial ghost sectors. Some of these operators may be viewed as the liouville dressings of the energy operator of the Ising model coupled to 2d gravity and this fills in a gap in the connection between pure $W_3$–gravity and Ising model coupled to 2d gravity found in our previous work. We formulate a selection rule required for the calculation of correlators in $W$–gravity theories. Using this rule, we construct the non ghost part of the new operators of $W_N$–gravity and find that they represent the $(N, N + 1)$ minimal model operators from both inside and outside the minimal table. Along the way we obtain the canonical spectrum of $W_N$–gravity for all $N$. 

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1. INTRODUCTION

$W_N$-algebras [1] [2], are non-linear extensions of Virasoro algebra by higher spin currents with spins $s$, $2 \leq s \leq N$, the spin – 2 current being the stress tensor, $T$. $W$-gravities are obtained by gauging $W$-algebras and their formal aspects have been studied in [2] [3] [4] while the physical properties and the spectrum of $W$-gravities and $W$-strings have been studied in detail in [5] for $W_3$ and $W_4$.

In [5], among other things we obtained the “canonical” physical operators of pure $W_N$-gravity, for $N = 3, 4$, and found an intriguing connection with unitary minimal models coupled to 2d gravity. We found that all the canonical physical operators of $W_N$-gravity can be regarded as the “liouville” dressings of the diagonal operators of the $(N, N + 1)$ minimal models coupled to 2d gravity, for $N = 3, 4$, and conjectured that this phenomenon is true for all $N$. However we were unable to represent the non diagonal operators in the context of $W$-gravity. Also, recent studies [6] [7] indicate that the minimal model operators from outside the minimal table do not decouple when coupled to gravity. But the representation of such operators in the $W$-gravity framework is not known either.

Connections such as above have appeared elsewhere too. The double scaling limit of the multimatrix models is believed to be $(p, q)$ minimal models coupled to gravity and the $W_k$-constraints appear [8] in the multimatrix models as Dyson – Schwinger equations. Also, in the topological field theoretic description of Ising model coupled to topological gravity, the diagonal operators are special and appear as topological matter primaries while the energy operator appears as “gravitational descendent”.

In view of such discoveries, it is important to know how to represent the non diagonal operators of the minimal model and also the operators from outside the minimal table in the framework of $W$-gravity theories. Knowing this is crucial for further understanding of $W$-gravity and $W_k$-constraints of multimatrix models.

In this paper, we describe the representation of such operators for Ising model
in the context of $W_3$-gravity. To be precise, we show how to represent the energy operator of the Ising model as new physical operators in the $W_3$-gravity spectrum. These operators have non trivial ghost sectors and, as an added bonus, they also represent another Ising model operator, from outside the minimal table.

We further formulate a selection rule, required for the calculation of correlators in $W$-gravity theories. Using this rule, we construct the non ghost part of the new special operators in $W_N$-gravity. They are found to represent the $(N, N+1)$ minimal model operators from both inside and outside the minimal table. (The construction of the non trivial ghost sectors for these new operators requires the BRST charge $Q_B$ for $W_N$-gravity, which is not known for $N > 3$).

Along the way we obtain the canonical physical operators of $W_N$-gravity, with the standard ghost sector, for all $N$ and find that they represent the diagonal operators of the $(N, N+1)$ minimal model coupled to 2d gravity. We also show that these canonical physical operators can be expressed as the non singular composites of the screening charges, a fact first discovered in [5] for $N = 3, 4$ and conjectured for all $N$.

2. $W$-GRAVITY SPECTRUM

In this section we briefly review the relevant aspects of [5] [2] and give the construction of the $W_N$-gravity spectrum for all $N$. A free field representation of $W$-algebra can be obtained in terms of $(N-1)$ free scalar fields $\Phi = (\phi_1, \phi_2, \ldots, \phi_{N-1})$ obeying the OPEs $\phi_a(z)\phi_b(0) = -2\delta_{ab} \log(z)$. (In what follows, all the vectors are $(N-1)$ dimensional vectors). We describe the construction of $W$-currents using the weights $\vec{h}_k$, the simple roots $\vec{e}_k = \vec{h}_k - \vec{h}_{k+1}$ and the fundamental weights $\vec{\omega}_k = \sum_{m=1}^k \vec{h}_m$ of the representation of SU(N); here $k = 1, 2, \ldots, N-1$. We further define a vector $\vec{h}_N$ by $\sum_{k=1}^N \vec{h}_k = 0$. Furthermore the Weyl vector $\vec{\rho}$ is given by $\vec{\rho} = \sum_{m=1}^{N-1} \vec{\omega}_m$. While the choice of $\vec{h}_m$ does not
matter, we often use the canonical choice:

\[ \vec{h}_1 = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \cdots, \frac{1}{\sqrt{N(N-1)}} \right), \quad \vec{h}_2 = \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \cdots, \frac{1}{\sqrt{N(N-1)}} \right), \]

\[ \vec{h}_3 = (0, -\frac{2}{\sqrt{6}}, \cdots, \frac{1}{\sqrt{N(N-1)}}) \quad \cdots \quad \vec{h}_N = (0, 0, \cdots, \frac{1}{\sqrt{N(N-1)}}) \]

With this choice the Weyl vector \( \vec{\rho} \) becomes \( \vec{\rho} = \frac{1}{2} (\sqrt{2}, \sqrt{6}, \cdots, \sqrt{N(N+1)}) \). Note also that \( \rho^2 = \frac{N(N^2-1)}{12} \).

Now we define the quantities \( U_k \) through the quantum Miura transformation

\[
\prod_{m=1}^{N} (\partial + \vec{h}_m \cdot \partial \vec{\phi}) = \sum_{k=0}^{N} U_k(z) \partial^{N-k} \]

where \( \partial = \frac{\partial}{\partial z} \) and \( \alpha_0 \) is a free parameter. Then, as explained in [5], the spin – k currents \( W_k(z) \) are given in terms of \( U_m(z), m \leq k \), and their derivatives up to an overall constant which is fixed by requiring the central terms to be in the canonical form. Thus for example, the stress tensor \( T( \equiv W_2 ) \) is given by

\[ T = (i\alpha_0 \sqrt{2})^2 U_2(z) = -\frac{1}{4}(\partial \vec{\phi})^2 + i\alpha_0 \vec{\rho} \cdot \partial^2 \vec{\phi}. \]

The modes of the \( W_k \) currents \( (k = 2, \cdots, N) \), defined by \( W_k(z) = \sum \frac{W_k(n)}{z^{n+k}}, L_n \equiv W_2(n) \), form the \( W_N \)-algebra with the central charge

\[ C(N) = (N-1)(1-2\alpha_0^2 N(N+1)). \]

In \( W \)-gravity, for each \( W_k \), one has the ghost fields \( (b_k, c_k) \) of dimensions \( (k, 1-k) \) respectively. Thus the central charge of the entire ghost system is given by

\[ C_{gh}(N) = \sum_{k=2}^{N} -2(1 + 6k^2 - 6k) = -(N-1)(4N^2 + 4N + 2). \]

The critical central charge is obtained by requiring \( C(N) + C_{gh}(N) = 0 \), which
gives
\[ \alpha_0^2 = \frac{(2N + 1)^2}{2N(N + 1)}. \] (6)

For future use, we further define the quantities \( \alpha_\pm \) by
\[ \alpha_\pm = \frac{1}{2}(\alpha_0 \pm \sqrt{\alpha_0^2 + 2}). \] (7)

In \( W \)-gravity the “canonical” physical states (BRST closed but not BRST exact) are of the form
\[ |\Psi_{phys} > = |\text{matter} > \otimes |0 >_{gh} \] (8)

upto BRST exact terms, where the ghost vacuum is given by
\[ |0 >_{gh} = \prod_{k=2}^{N} \prod_{n=1}^{k-1} c_k(n)|0 >. \] (9)

The “canonical” physical states \(|\Psi_{phys} >\) are assigned a ghost number zero. From above, one gets the \( L_0 \)-intercept for the matter sector as
\[ L_0|matter > = \frac{N(N^2 - 1)}{6}|matter > = (2\rho^2)|matter >. \] (10)

The other \( W_k(0), (k > 2) \), intercepts depend on the detailed structure of the BRST charge.

The non ghost part of the operators which create the physical state are of the form
\[ V_{\tilde{\beta}}(z) = e^{i\tilde{\beta} \cdot \tilde{\phi}}. \] (11)

The vector \( \tilde{\beta} \) can be determined from the \( W_k(0), (2 \leq k \leq N) \) intercepts. However, since this involves a knowledge of the BRST charge \( Q_B \) which is not known for \( N > 3 \), we shall adopt a correspondence principle given in [5] to derive the \( W \)-gravity spectrum. This principle briefly means the following. When \( \tilde{\beta} = \tilde{\beta}_0 \equiv a\alpha_0\rho, a \) any

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constant, the exponent $\vec{\beta}_0 \cdot \vec{\phi}$ in $V_{\beta_0}$ is precisely in the combination which appears as the background charge term in the stress tensor $T$ in (3). For this and other reasons [5], these are referred to as “cosmological constant” operators. Moreover, this operator is a solution for $W_3$–gravity for any value of $\alpha_0$ and for $W_4$–gravity in the classical limit when $\alpha_0^2 \to -\infty$. We assume that a correspondence principle holds whereby the cosmological constant operator will continue to be a solution for $W_N$–gravity for any value of $N$ and $\alpha_0$. The value of $a$ in $\vec{\beta}_0 = a\alpha_0\bar{\rho}$ is determined by the $L_0$–intercept ($= 2\bar{\rho}^2$) in (10). One may use this solution to determine all the $W_k(0), (k > 2)$, intercepts and then solve for $\vec{\beta}$’s in (11) which have the same intercepts. They would constitute the canonical physical spectrum of $W$–gravity.

In solving for $\vec{\beta}$ we note, following [2], that by operating (2) on $V_\beta(z)$ in (11) and then operating the resulting equation on functions of $z$, say $z^j, j = 1, 2, \ldots, N - 2$, one can obtain the following equations

$$
\prod_{m=1}^{N} (N - m + j - \frac{\bar{h}_m \cdot \vec{\beta}}{\alpha_0}) = \sum_{k=0}^{j} \frac{j!}{(j-k)!} U_{N-k}(0)
$$

(12)

which determine the intercepts $U_k(0)$, and hence $W_k(0)$. Equation (12) has a permutation symmetry. Let $\vec{\beta}_0$ corresponding to the set $\{m_0\} = (1, 2, \ldots, N)$ be a solution to (12) and let $\vec{\beta}$ correspond to the set $\{m\} = (m_1, m_2, \ldots, m_N)$ obtained by a permutation of $\{m_0\}$. Then, the left hand side of (12) is invariant under the transformation $\vec{\beta}_0 \to \vec{\beta}$ and

$$
\bar{h}_{m_0} \cdot \vec{\beta}_0 + \alpha_0 m_0 = \bar{h}_m \cdot \vec{\beta} + \alpha_0 m.
$$

(13)

Hence, $\vec{\beta}$ satisfying (13) will be a new solution having the same $W_k(0)$ intercepts as $\beta_0$. Since, we know by the correspondence principle that the cosmological constant operator, $\vec{\beta}_0 = a\alpha_0\bar{\rho}$, is a solution of $W$–gravity constraints, we can obtain all the $N!$ solutions by solving (13) for $\vec{\beta}$ corresponding to $\{m\}$.
To simplify further calculations, we define new parameters $i_k$, $b_k$, $K_0$ and $K_\pm$ by

\[ m_{ik} = k, \quad m_{ik} \in \{m\}, \quad k = 1, 2, \ldots, N \]

\[ \alpha_0 = \frac{iK_0}{\sqrt{2N(N + 1)}}, \quad \alpha_\pm = \frac{iK_\pm}{\sqrt{2N(N + 1)}} \]

\[ \vec{\beta} = \frac{i}{\sqrt{2N(N + 1)}}(b_1\sqrt{2}, b_2\sqrt{6}, \ldots, b_{N-1}\sqrt{N(N - 1)}) \]

Note that from (6) and (7), $K_0 = 2N + 1, K_+ = N + 1, K_- = N$. However, we will not substitute these values in our expressions. Requiring the dimension of $V_{\vec{\beta}_0}$, $\vec{\beta}_0 = a\alpha_0\vec{\rho}$, to be $2\rho^2$, we get

\[ 1 - a = \pm \frac{1}{K_0} \]

where we have used the fact that the dimension of $V_\beta$ is $(\vec{\beta}^2 - 2a_0\vec{\rho} \cdot \vec{\beta})$. We choose the + sign in (15). (The other sign will correspond to one of the solutions obtained from (13)). Now solving (13) for $\vec{\beta}$ corresponding to a given $\{m\}$, parameterised as in (14), we obtain

\[ b_1 + b_2 + \cdots + b_{N-1} + K_0 = \frac{aK_0(N + 1)}{2} + (1 - a)i_1K_0 \]

\[ -b_1 + b_2 + \cdots + b_{N-1} + 2K_0 = \frac{aK_0(N + 1)}{2} + (1 - a)i_2K_0 \]

\[ -2b_2 + \cdots + b_{N-1} + 3K_0 = \frac{aK_0(N + 1)}{2} + (1 - a)i_3K_0 \]

\[ \cdots \]

\[ -(N - 1)b_{N-1} + NK_0 = \frac{aK_0(N + 1)}{2} + (1 - a)i_NK_0 \]

The solution of the above equations is given by

\[ b_p = \left(\frac{K_0}{2} + \frac{l_p}{p(p + 1)}\right), \quad p = 1, 2, \ldots, N - 1 \]

where $l_p$ is defined by

\[ l_p = i_1 + i_2 + \cdots + i_p - pi_{p+1}. \]

Equation (17) describes the canonical physical spectrum of $W_N$--gravity for any
Denoting the dimension of $e^{i\beta_p \phi_p}$ by $\Delta_p (= \beta_p^2 - 2\alpha_0 \rho_p \beta_p)$, we get

$$\Delta_p = \frac{1}{(2\rho_p)^2 2N(N + 1)} (\frac{1}{2} K_0 p(p + 1) - l_p^2)$$ (19)

where $\rho_p = \frac{\sqrt{p(p+1)}}{2}$ is the $p$th component of the Weyl vector $\vec{\rho}$. In particular, the dimension of $e^{i\beta_1 \phi_1}$, where $\phi_1$ is the "liouville mode", is given by

$$\Delta_1 \equiv 1 - h_1(N) = 1 - \frac{l_1^2 - 1}{4N(N + 1)}.$$ (20)

From the definition of $l_1$, it follows that $1 \leq |l_1| \leq N - 1$. Comparing $h_1(N)$ in (20) with the dimensions $h_{p,q}$ of the $(p,q)$ minimal model operators [9] one sees that $e^{i\beta_1 \phi_1}$ can be interpreted as the "liouville" dressing of the diagonal operators of the $(N,N+1)$ minimal models, represented in $W_N$-gravity by the fields $(\phi_k; b_{k+1}, c_{k+1}), 2 \leq k < N$. Thus it follows that all the operators in the canonical spectrum of $W_N$-gravity, for any $N$, can be regarded as the dressings of the diagonal operators of the $(N,N+1)$ minimal models coupled to 2d gravity.

3. CONSTRUCTION OF ENERGY OPERATOR

In this section we describe the representation of the energy operator of the Ising model in the framework of $W_3$-gravity. Recently, Lian and Zuckerman have given in [10] a cohomological classification of physical states in $C \leq 1$ conformal matter coupled to 2d gravity and predicted new "discrete" states with non trivial ghost sectors. The discrete states have been the subject of several recent studies [11] [12] and the physical consequences of such states with non trivial ghost sectors have been examined by E. Witten [12] in detail. Some of these states with non trivial ghost sectors have been constructed explicitly in [12]. The method of construction is essentially as follows: consider the $(p,q)$ minimal models coupled to 2d gravity and the liouville operator $e^{i\beta_1 \phi_1}$ which dresses up a null state built over
a primary $\Phi_M$ of the minimal model. Then, reference [10] guarantees that there exist physical states with non trivial ghost number which can be built by acting upon $e^{i\beta L}\Phi_M c_1 |0>$ with matter, liouville and ghost oscillators. We adopt such a procedure for the case of $W_3$–gravity and construct explicitly new physical states with non trivial ghost number.

We note that for $W$–minimal models, no classification as in [10] exists. However, we give the following arguments to motivate our construction; the reasoning is admittedly not rigorous, particularly due to the lack of understanding of $W$–matter couplings to $W$–gravity. In the $W$–minimal models, there are null states built over the primaries. For example, in $W_3$–minimal models [3], each primary state is labelled by 4 integers $(n, n'; m, m')$ and it is known that null states over these primaries exist at levels $nm$ and $n'm'$ [13]. Now, if a $W$–minimal model is coupled to $W$–gravity, one can obtain the $W$–gravity dressings of such null states. Then analogous to the case of matter coupled to 2d gravity, one can try to construct states with non trivial ghost number by acting with oscillators on $W$–gravity dressings and the $W$–minimal model primaries. Specialising to the case of the identity operator of the $W$–minimal model with zero central charge, one should be able to obtain the physical states in $W$–gravity with non trivial ghost number.

Now we proceed as follows. Using the formulas of [3], one discovers that a null state can be built at level 2 over an identity operator of the $W$–minimal model with zero central charge. This would mean that the $W_3$–gravity dressing should have dimension 2 (instead of the usual 4) and a ghost number +1. (The canonical physical states of section 2 are assigned a ghost number zero). The knowledge of other $W$–intercepts can be obtained by the correspondence principle. Acting on such a state with required number of oscillators, we find a physical state which is BRST closed but not BRST exact.

We choose the ghost sector of the physical state to be

$$|0>_{gh} = c_2(1)c_3(0)c_3(1)c_3(2)|0> \quad (21)$$
which is equivalent to (9) and, hence, also assigned the same ghost number zero.

We start with the operators of the form $\Psi_\beta = e^{i\vec{\beta} \cdot \vec{\phi}} c_2(1)c_3(0)c_3(1)c_3(2)$ where the dimension of $e^{i\vec{\beta} \cdot \vec{\phi}}$ is 2. The BRST charge $Q_B$ is given in terms of modes in [14] and in terms of fields it is given by $Q_B = \oint dz j_B(z)$, where [15]

$$j_B(z) = c_2(T + T(b_3, c_3) + \frac{1}{2} T(b_2, c_2)) + c_3 W_3 + \frac{8}{261} T b_2 c_3 \partial c_3 - \frac{25}{522} b_2 \partial c_3 \partial^2 c_3 + \frac{25}{783} b_2 c_3 \partial^3 c_3.$$  

(22)

In the above expression $T(b_k, c_k)$ is the standard stress tensor [16] for the spin – k ghost fields. The operators $\Psi_\beta$ given above are not BRST closed. However, we find that the operators given by

$$\tilde{\Psi}_\beta = c_2(-2) \Psi_\beta,$$  

(23)

where $\tilde{\beta}$s are such that the dimension of $e^{i\tilde{\beta} \cdot \vec{\phi}}$ is 2, are BRST closed but not BRST exact and hence, are genuine physical states with ghost number +1.

Now using the methods of section 2, one can find all the solutions of $\tilde{\beta}$ that appear in (23). The dimension of the “liouville field” $e^{i\tilde{\beta}_1 \phi_1}$ turns out to be $\Delta_1 = 1 - h_1(3)$ where

$$h_1(3) = \frac{1}{2} , \quad \frac{33}{16}.$$  

(24)

Thus we see that all the physical operators $\tilde{\Psi}_\beta$, which have a non trivial ghost number +1, can be interpreted as the liouville dressings of either the energy operator, $\Phi_{2,1}$ or the operator $\Phi_{4,2}$ of the Ising model coupled to 2d gravity. Note that the second operator $\Phi_{4,2}$ is from outside the minimal table. However, it is not surprising that this operator appears in the theory, in fact it is rather desirable, in the light of the discovery of [6] [7] that operators from outside the minimal table fail to decouple when the minimal models are coupled to gravity. The decoupling in [6] [7] fails because of the poles arising from the liouville sector which precisely cancel the zeroes of the matter sector correlators involving operators from outside the minimal table.
Thus we have explicitly constructed in $W_3$–gravity new special physical operators with non trivial ghost sectors which can be interpreted as the liouville dressings of the operators of the Ising model from both inside and outside the minimal table. Moreover, the physical operators $\tilde{\Psi}_\beta$ also satisfy another selection rule described in section 4.

A similar analysis of $W_N$–gravity for $N > 3$, is not possible at present since the corresponding BRST charge, $Q_B$ is not known. However, it is quite likely that for any $N$, there exist special operators similar to the ones constructed here satisfying the selection rule mentioned above. These operators, as will be shown in the next section, correspond to the minimal model operators from both inside and outside the minimal table coupled to 2d gravity.

4. SCREENING CHARGES AND SELECTION RULE

There has been some progress recently [6] [7] [17] in calculating the three point functions of $c \leq 1$ matter coupled to 2d gravity. The main ingredient in these works is the idea of analytically continuing $s$, the number of screening charges that appear in the correlation functions after integrating out the zero modes of the liouville field, from positive integer values to fractional [7] [17] or negative integer [6] values. In the approach of [7] [17] a semiclassical correspondence principle is used to select the screening charge term in the liouville action and hence after integrating out the liouville zero mode, only one type of screening charge appears. This necessitates the analytic continuation of $s$ into fractional values. In the more natural approach of [6] no semiclassical correspondence principle is used and both types of screening charges appear after doing the liouville sector functional integral perturbatively. Because of this it is sufficient to analytically continue $s$ into negative integers only.

The screening charges in $W_N$–gravity are $S_k^\pm = e^{i\alpha_\pm \vec{e}_k \cdot \vec{\phi}}$, $k = 1, 2, \ldots, N - 1$. However, no semiclassical correspondence principle is known for $W_N$–gravity and hence it is natural to include all the screening charge terms in the action. Thus in calculating the correlation functions $< \prod_{i=1}^L V_{\beta_i} >$ in $W$–gravity, one would
obtain the following condition a la [6]:

\[
\sum_{i=1}^{L} \beta_i + \sum_{m=1}^{N-1} \left( \alpha_+ E_m^+ + \alpha_- E_m^- \right) \epsilon_m = 2\alpha_0 \rho.
\]  

(25)

Since we have included all the screening charge terms we expect to analytically continue \( E_m^\pm \) into negative integers only and not into fractional values. In what follows we impose it as a condition and require \( E_m^\pm \) to take integer values only. Since this requirement has to hold good for any arbitrary \( L \) and for any \( \vec{\beta} \) in the \( W^- \) gravity spectrum and since \( 2\alpha_0 \rho = \sum_{m=1}^{N-1} m(N-m)(\alpha_+ + \alpha_-) \epsilon_m \), (25) implies that the \( \vec{\beta} \)'s must be expressible as

\[
\vec{\beta} = \sum_{m=1}^{N-1} \left( \alpha_+ \epsilon_m^+ + \alpha_- \epsilon_m^- \right) \epsilon_m
\]

(26)

where \( \epsilon^\pm \) are positive or negative integers only. This imposes a selection rule on \( \vec{\beta} \) and hence on the physical operators of \( W^- \) gravity. Note that the canonical physical operators in section 2 and the new physical operators \( \tilde{\Psi}_\beta \) in section 3 all satisfy this rule.

We now proceed to construct other new operators in the \( W^- \) gravity theory. Since we have neither the BRST charge \( Q_B \), nor the analog of Lian-Zuckerman theorem we proceed as follows. In general, the physical states \( \tilde{\Psi}_\beta \) with non trivial ghost numbers have total dimension zero and are produced by acting with oscillators on operators of the type \( \Psi_\beta = e^{i\vec{\beta} \cdot \vec{\phi}} \prod_{k=2}^{N} \prod_{n=1}^{k-1} c_k(n) \) (note that the states obtained by acting with \( c_k(0), 2 < k \leq N \) on \( \Psi_\beta \) are equivalent to \( \Psi_\beta \) and, hence, can be taken as a starting point instead of \( \Psi_\beta \) and assigned the same ghost number zero ). Hence, we consider \( \vec{\beta} \) such that the dimension \( \Delta \) of \( e^{i\vec{\beta} \cdot \vec{\phi}} \) is given by

\[
\Delta = 2\rho^2 - G
\]

(27)

where \( G \) is a non negative integer. We then impose (26) and obtain the special operators of \( W^- \) gravity by using the methods of section 2.
We have the cosmological constant operator $\vec{\beta}_0 = a a_0 \vec{\rho}$ as a solution by the correspondence principle. From (27), we get the value of $a$ as

$$1 - a = \pm \frac{X}{K_0}$$

(28)

where, using $K_0 = 2N + 1$ and $\rho^2 = \frac{N(N^2-1)}{12}$, we get

$$X = (1 + \frac{24}{N-1}G)^{1/2}. $$

(29)

The equation (16) for $b_p, p = 1, 2, \ldots, N-1$ which parametrise $\vec{\beta}$ as in (14), remain the same and the solution is given by

$$b_p = \frac{K_0}{2} + \frac{X l_p}{p(p+1)}$$

(30)

with $l_p$ defined as in (18). This gives the complete solution of the spectrum of $W-$gravity satisfying the condition (27). Now we further impose the requirement (26) where $\epsilon_m^\pm$ are positive or negative integers only. Using (14) the equation (26) becomes

$$K_+ \epsilon_m^+ + K_- \epsilon_m^- = m \sum_{p=m}^{N-1} b_p$$

(31)

Note that the left hand side of (31) is an integer since all the quantities appearing there are integers. The right hand side can be written, taking the + sign in (28), as

$$m \sum_{p=m}^{N-1} b_p = m \frac{2}{2}(K_0 - X)(N + 1) - \frac{m(m+1)}{2}K_0 + X \sum_{p=1}^{m} i_p.$$ 

(32)

Consider the right hand side of the above equation. The second term is always an integer. For the remaining sum to be an integer for all $i_p$'s and $m$'s, $X$ must necessarily be an integer. However, this is also a sufficient condition since when $X$ is odd $(K_0 - X)$ is even and when $X$ is even it can be seen from (29) that $N$ must be odd. Hence the right hand side of (32) is always an integer. Thus imposing (26) is equivalent to requiring $X$ to be an integer.
The case $X = 1$ corresponds to the canonical $W_N$–gravity spectrum with zero ghost number. For this case, equations (31) and (32) imply a very interesting phenomenon. Noting that $\sum_{p=1}^{m} i_p$ takes the values $\frac{1}{2} m (m + 1) + p, p = 0, 1, 2, \ldots, m(N - m)$, the solutions for $\epsilon_m^\pm$ can be written as

$$\epsilon_m^+ = p, \quad \epsilon_m^- = m(N - m) - p, \quad p = 0, 1, 2, \ldots, m(N - m).$$

(33)

In particular $\sum_{m=1}^{N-1} (\epsilon_m^+ + \epsilon_m^-) = (2\rho^2)$, the $L_0$–intercept of the physical operators. Hence it follows that all the physical operators in $W_N$–gravity with zero ghost number can be expressed as the non singular composites of the screening charges. See [5] for more details where this fact was first discovered for $N = 3, 4$ and conjectured for all $N$.

Now, from (30) the dimension $\Delta_p$ of $e^{i\beta_p \phi_p}$ can be computed. It is given by

$$\Delta_p = \frac{1}{(2\rho_p)^2 2N(N + 1)} \left( \frac{K_0 p(p + 1)}{2} - X^2 l_p^2 \right).$$

(34)

In particular, the dimension $\Delta_1$ of $e^{i\beta_1 \phi_1}$, where $\phi_1$ is the “liouville mode”, is given by

$$\Delta_1 \equiv 1 - h_1(N) = 1 - \frac{X^2 l_1^2 - 1}{4N(N + 1)}.$$ 

(35)

As before, the range of $l_1$ is $1 \leq |l_1| \leq N - 1$. Comparing $h_1(N)$ above with $h_{p,q}$ of $(N, N + 1)$ minimal models we see that the special operators of $W$–gravity constructed above can be interpreted as the liouville dressings of the $(N, N + 1)$ minimal model operators, from both inside and outside the minimal table. The fact that the new operators represent also the minimal model operators from outside the minimal table is a welcome feature in the light of the discovery [6] [7] that such minimal model operators fail to decouple when coupled to 2d gravity.

Since the expression for the BRST charge $Q_B$ for $W_N$–gravity, $N > 3$, is not known we are not able to construct the full physical operators that include the
ghost sector with non trivial ghost number. However it is quite likely that the
special operators, with their $W$–gravity sector constructed as above, exist and
are part of the full physical spectrum as we have seen explicitly in the case of
$W_3$–gravity, indicating a deep relation between $W$–gravity theories and minimal
models coupled to 2d gravity.

We conclude by noting that there does not yet exist a complete classification
of $W$–gravity states that include those with non trivial ghost numbers, analogous
to that of [10]. Though this is a very interesting problem in its own right, its
solution may have to await a thorough understanding of $W$–matter couplings to
$W$–gravity. However, our explicit construction of new states in section 3 appears
to indicate that the Ising model coupled to 2d gravity possesses a higher symmetry
- $W_3$ symmetry - that can be seen by representing the Ising model in a special
way by the fields $(\phi_2; b_3, c_3)$. We also need to understand better the meaning and
the origin of the selection rule described in section 4. These aspects may shed
light on the special states of $W$–gravity theories constructed in this paper and
the intriguing connection between $W$–gravity and minimal models coupled to 2d
gravity. This may also help us understand better $W$–gravity itself and the origin
of $W_k$–constraints in the matrix models.

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