ON MAGNETIC LEAF-WISE INTERSECTIONS

YOUNGJIN BAE

Abstract. In this article we introduce the notion of a magnetic leaf-wise intersection point which is a generalization of the leaf-wise intersection point with magnetic effects. We also prove the existence of magnetic leaf-wise intersection points under certain topological assumptions.

1. Introduction

Let \((N, g)\) be a closed connected orientable Riemannian manifold and \(\tau : T^*N \to N\) be its cotangent bundle. We consider an autonomous Hamiltonian system defined by a convex Hamiltonian

\[ F_U(q, p) = \frac{1}{2}|p|^2 + U(q) \]

with canonical symplectic form \(\omega_{\text{std}} = dp \wedge dq\). Here \((q, p)\) are the canonical coordinates on \(T^*N\), \(|p|\) denotes the dual norm of the Riemannian metric \(g\) on \(N\) and \(U : N \to \mathbb{R}\) is a smooth potential. This Hamiltonian system describes the motion of a particle on \(N\) subject to the conservative force \(-\nabla U\).

We consider a closed energy hypersurface \(F^{-1}_U(k) := \Sigma \subset T^*N\) in an exact symplectic manifold \((T^*N, \omega_{\text{std}}, F_U)\) such that \((\Sigma, \alpha := \lambda|_{\Sigma})\) is a contact manifold. \(\Sigma\) is foliated by the leaves of the characteristic line bundle which is spanned by the Reeb vector field \(R_{\Sigma}\) of \(\alpha\).

Let \(\phi_{\Sigma}^t : \Sigma \to \Sigma\) be the flow of \(R_{\Sigma}\). For \(x \in \Sigma\) we denote by \(L_x\) the leaf through \(x\) which can be parameterized as \(L_x = \{\phi_{\Sigma}^t(x) : t \in \mathbb{R}\}\). If \(L_x\) is closed, we call it a closed Reeb orbit.

If we take closed 2-form \(\sigma\) on \(N\) and consider the twisted symplectic form

\[ \omega_{\sigma} = \omega_{\text{std}} + \tau^*\sigma, \]

then \((T^*N, \omega_{\sigma}, F_U)\) is called a twisted cotangent bundle. The additionally chosen data \(\sigma\) could be interpreted as a magnetic field. A 2-form \(\sigma\) on \(N\) is called \(\tilde{d}\)-bounded if its pull-back \(\tilde{\sigma} \in \Omega^2(\tilde{N})\) is a differential of a bounded 1-form. In this article, we restrict ourselves to the case when \(\sigma\) is \(\tilde{d}\)-bounded. In order to introduce leaf-wise intersections with the above magnetic effect, we need the following definition.

Definition 1.1. A magnetic perturbation \(\mathfrak{m}\) is a triple \((\beta, \sigma, \theta)\) which consists of the following data:

- \(\beta \in \mathfrak{B} := \{\beta \in C^\infty(S^1) : \beta(t) = 0, \forall t \in [0, \frac{1}{2}]\}\);
- \(\sigma\) is \(\tilde{d}\)-bounded;
- \(\theta \in \mathcal{P}_\sigma := \{\theta \in \Omega^1(\tilde{N}) : d\theta = \sigma\}\).

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Let $\mathcal{M}$ be the set of such magnetic perturbations. We consider an $S^1$-parameterized symplectic form as follows
\[ \omega_m = \omega_{\beta \sigma} := \omega_{\text{std}} + \tau^* \beta \sigma. \]

**Definition 1.2.** Let us denote by
\[ \mathcal{H} := \{ H \in C^\infty_c(S^1 \times T^* N) : H(t, \cdot) = 0, \forall t \in [0, \frac{1}{2}] \}; \]
\[ \text{Diff}_c(T^* N, m) := \{ \phi_m = \phi^1_{X_m H} \in \text{Diff}(T^* N) : H \in \mathcal{H} \}. \]

The above diffeomorphism $\phi^1_{X_m H}$ is the time-1-map of the vector field $X_m H$ which is defined by
\[ \iota_{X_m H} \omega_m = dH. \]

When $m = 0$, $\text{Diff}_c(T^* N, m)$ coincides with the set of Hamiltonian diffeomorphisms, denoted by $\text{Ham}_c(T^* N)$.

**Definition 1.3.** A point $x \in \Sigma$ is called a magnetic leaf-wise intersection point, if $\phi_m(x) \in \mathcal{L}_x$ for a time-dependent diffeomorphism $\phi_m \in \text{Diff}_c(T^* N, m)$. In other words, there exists $\eta \in \mathbb{R}$ such that
\[ \phi^\Sigma_{\eta}(\phi_m(x)) = x. \]

Note that a leaf-wise intersection point is the $\sigma = 0$ case of a magnetic leaf-wise intersection point. The leaf-wise intersection problem asks whether a given diffeomorphism $\varphi$ has a leaf-wise intersection point in a given hypersurface $\Sigma$. If there exist leaf-wise intersections one can ask further a lower bound on the number of leaf-wise intersections. This problem was introduced by Moser in [36], and studied further in [11, 17, 25, 20, 16, 24, 5, 6, 8, 9, 26, 27, 28, 33]. See [4] for the brief history of these problems. In this article, we investigate the approaches in [3, 30] and generalize their results.

To state the existence results, we need the following preparation. Let $F : T^* N \to \mathbb{R}$ be a Hamiltonian function. The Rabinowitz action functional $A^F : \mathcal{L} \times \mathbb{R} \to \mathbb{R}$ is defined by
\[ A^F(u, \eta) := \int_0^1 u^* \lambda - \eta \int_0^1 F(t, u) dt. \]

Here $\mathcal{L} = \mathcal{L}_{T^* N} := C^\infty(S^1, T^* N)$. If $A^F$ is Morse, by the work of Abbondandolo-Schwarz [2] and Cieliebak-Frauenfelder-Oancea [14], we then have the following non-vanishing result when $* \neq 0, 1$.

\[ \text{FH}_*(A^F) = \begin{cases} H_*(\mathcal{L}_N), & \text{if } * > 1, \\ H^{-*+1}_-(\mathcal{L}_N), & \text{if } * < 0, \end{cases} \]

Here $\text{FH}_*(A^F)$ is the Floer homology for $A^F$ and $\mathcal{L}_N$ is the free loop space of $N$.

A leaf-wise intersection points $x \in \Sigma$ with respect to $\varphi \in \text{Ham}_c(T^* N)$ can be interpreted as a critical point of a perturbed Rabinowitz action functional
\[ A^F_H(u, \eta) := \int_0^1 u^* \lambda - \eta \int_0^1 F(t, u) dt - \int_0^1 H(t, u) dt. \]

Here the additional Hamiltonian $H : T^* N \to \mathbb{R}$ generates $\varphi$. For a generic Hamiltonian for which $A^F_H$ is Morse, Albers-Frauenfelder [4] constructed an isomorphism
\[ \text{FH}(A^F_H) \cong \text{FH}(A^F). \]
Now we construct the action functional whose critical points give rise to magnetic leaf-wise intersection points. On $A_F^H$, we add an additional decoration $B_m: L \to R$

$$u \mapsto \int_0^1 \beta(t)\tau(\vec{u}(t))|\partial_t\vec{u}(t)|dt.$$ 

Here $\tau: T^*\tilde{N} \to \tilde{N}$ and $\vec{u}: S^1 \to T^*\tilde{N}$ is a lifting of $u$. Now we define

$$A_m(u, \eta) = A_F^H(u, \eta) + B_m(u).$$

Then the critical point $(u, \eta)$ of $A_m$ gives a magnetic leaf-wise intersection $u(\frac{1}{T})$, see Proposition 2.2.

**Theorem 1.4.** Let $\Sigma \subset T^*N$ be a closed hypersurface with a defining Hamiltonian $F$. Let $(H, m)$ be a generic pair such that $A_m$ is Morse. Then $FH(A_m)$ is well-defined.

**Theorem 1.5.** Let $\Sigma \subset T^*N$ be a closed hypersurface with a defining Hamiltonian $F$. Let $(H, m)$ be a generic pair such that $A_F^H, A_m$ are Morse. Then

$$FH(A_F^H) \cong FH(A_m).$$

By these results, if $\dim H_*(L_N) = \infty$ then we have infinitely many critical points of $A_m$. This implies that there are infinitely many magnetic leaf-wise intersections or a periodic one which means that the leaf on which it lies forms a closed Reeb orbit. We exclude the latter case generically, as follows.

We call a hypersurface $\Sigma \subset T^*N$ non-degenerate if closed Reeb orbits on $\Sigma$ form a discrete set. A generic $\Sigma$ is non-degenerate, see [5 Theorem B.1]. If $\Sigma$ is non-degenerate, then periodic leaf-wise intersection points can be excluded by choosing a generic Hamiltonian function, see [3 Theorem 3.3]. With the above generic Hamiltonian, Albers-Frauenfelder conclude that there are infinitely many leaf-wise intersection points on $\Sigma$, under the topological assumption $\dim H_*(L_N) = \infty$.

By the above reason, we only consider non-periodic (magnetic) leaf-wise intersection points. In this article, a generic (Hamiltonian) diffeomorphism or a generic pair means that a certain action functional is Morse and there is no periodic leaf-wise intersections. Precise conditions are listed in Definition 2.3. Thus we conclude the following existence result for magnetic leaf-wise intersections.

**Corollary 1.6.** Let $N$ be a closed connected orientable manifold of dimension $n \geq 2$. Let $\Sigma$ be a non-degenerate hypersurface in $T^*N$. Suppose that $\dim H_*(L_N) = \infty$. If $\varphi_m \in \text{Diff}_c(T^*N, m)$ is generic then there exist infinitely many magnetic leaf-wise intersection points.

In order to state the further result, we need the following notion. Let $L_N$ be the free loop space of $(N, g)$. The energy functional $E_g: L_N \to \mathbb{R}$ is given by

$$E_g(q) := \int_0^1 \frac{1}{2}|\dot{q}|^2_g dt.$$

For given $0 < T < \infty$, denote by

$$L_N(T) := \left\{ q \in L_N : E_g(q) \leq \frac{1}{2}T^2 \right\}.$$
Let $\Sigma$ be a non-degenerate fiberwise starshaped hypersurface with a defining Hamiltonian $F$ and $\varphi = \phi^1_{X_H} \in \text{Ham}_c(T^* N)$ be a generic Hamiltonian diffeomorphism. Given $T > 0$ let us define

$$n_{\Sigma, \varphi}(T) := \# \{ x \in T^* N : \phi^\Sigma_\eta(\varphi(x)) = x, \ 0 < \eta < T \}.$$  

**Theorem 1.7.** Let $N$ be a closed connected oriented manifold of dimension $n \geq 2$. Let $\Sigma$ be a non-degenerate fiberwise starshaped hypersurface in $T^* N$. Let $g$ be a bumpy Riemannian metric on $N$ with $S^*_g N$ contained in the interior of the compact region bounded by $\Sigma$. Assume that $\varphi_m \in \text{Diff}_c(T^* N, m)$ is generic. Then there exists a constant $c = c(N, g, \Sigma, \varphi_m) > 0$ such that the following holds: For all sufficiently large $T > 0$,

$$n_{\Sigma, \varphi}(T) \geq c \cdot \text{rank} \{ \iota : H_* (\mathcal{L}_N(c(T - 1))) \to H_* (\mathcal{L}_N) \}.$$  

(1.5)

Under certain topological assumption on $N$, the right hand side of (1.5) grows exponentially with $T$. Denote by $\tilde{\pi}_1(N)$ the fundamental group of $N$ modulo conjugacy classes. Then the connected components of $\mathcal{L}_N$ corresponds to the elements of $\tilde{\pi}_1(N)$, hence the exponential growth rate of $\tilde{\pi}_1(N)$ implies that

$$\liminf_{T \to \infty} \text{rank} \{ \iota : H_0 (\mathcal{L}_N(T)) \to H_0 (\mathcal{L}_N) \}$$  

(1.6)

has also exponential growth with respect to $T$, see \[32\]. Then the following corollary comes from exponential growth of (1.6) and Theorem 1.7.

**Corollary 1.8.** Let $N$ be a closed connected oriented manifold of dimension $n \geq 2$. Let $\Sigma$ be a non-degenerate fiberwise starshaped hypersurface in $T^* N$. Suppose that $\tilde{\pi}_1(N)$ has exponential growth. If $\varphi_m \in \text{Diff}_c(T^* N, m)$ is generic then $n_{\Sigma, \varphi}(T)$ grows exponentially with $T$.

The main example of such $N$ is any surface of genus greater than one. In these case, the magnetic field $\sigma$ can be chosen by the volume form of that surface. Other candidates for $N$ are the symplectically hyperbolic manifolds which will be discussed in Definition 3.1, Proposition 3.2

1.1. **Overview.** We show Theorem 1.5 by constructing an explicit map between $\text{FC}(\mathcal{A}_H^f)$ and $\text{FC}(\mathcal{A}_m)$. In this case, the main issue is a construction of the continuation map between two different symplectic forms by counting gradient flow lines. These type of symplectic deformation problem is studied in \[10\]. The above construction is deduced from a certain type of isoperimetric inequality for a $d$-bounded magnetic 2-form.

In proving Theorem 1.7, we heavily need the following result in \[30\]. With the same assumption as in Theorem 1.7 for a generic $\varphi \in \text{Ham}_c(T^* N)$, the following holds for sufficiently large $T > 0$:

$$\dim \text{FH}^{(a,T)}(\mathcal{A}^f) \geq \text{rank} \{ \iota : H_* (\mathcal{L}_N(c(T - 1))) \to H_* (\mathcal{L}_N) \}.$$  

(1.7)

Here $\text{FH}^{(a,T)}(\mathcal{A}^f)$ is the filtered Floer homologies for $\mathcal{A}^f$ and $a = a(\varphi) > 0$ is a certain generic value which will not be explained here. In proving this result, Macarini-Merry-Paternain used the Abbondandolo-Schwarz isomorphism, the Morse homology theorem and a continuation map between a concentric family of fiberwise starshaped hypersurfaces, see \[30\] Section 4.2. Especially, the authors use a certain version of Rabinowitz action functional $\mathcal{A}^f$, see Section 3.2. The additional data $f$ in $\mathcal{A}^f$ is crucial in constructing the latter continuation map.

In this paper, we consider $\mathcal{A}_m$ a certain variant of $\mathcal{A}^f$ with respect to the $S^1$-parameterized symplectic form $\omega_m$ in (1.1). Magnetic leaf-wise intersections arise as critical points of $\mathcal{A}_m$. 


We construct again a continuation map between FC\(^{(s,T)}\)(\(A^I_m\)) and FC\(^{(s,T)}\)(\(A^f\)). With this continuation map, we can compare the growth rate of dim FH\(^{(s,T)}\)(\(A^I_m\)) and dim FH\(^{(s,T)}\)(\(A^f\)) when \(T \to \infty\). Finally we use the dimension estimate \((1.7)\) and conclude Theorem \((1.7)\).

1.2. Organization of the paper. This paper is organized as follows: In Section \((2)\) we show that FH\((A^I_m)\) is well-defined and construct a continuation map between FC\((A^I_f)\) and FC\((A^I_m)\). In Section \((3)\) we define \(A^I_m\) and check the well-definedness of the filtered Floer homology of \(A^I_m\). Then we construct a continuation map between the filtered Floer homologies of \(A^f\) and \(A^I_m\) in a certain action window. In Appendix \((A)\) and \((B)\) we study the generic properties of \(A^I_m\) with respect to \(\beta \in \mathfrak{B}\).

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2. A perturbation of the Rabinowitz action functional

Let us begin with the defining Hamiltonian \(F\) of \(\Sigma = F^{-1}_U(k) \subset T^*N\)

\[
F(t,x) = F_{U,k,\rho}(t,x) := \rho(t)(F_U(x) - k).
\]  

(2.1)

Here \(\rho : S^1 \to \mathbb{R}\) satisfies

\[
\int_0^1 \rho(t)dt = 1 \quad \text{and} \quad \text{supp}(\rho) \subset \left(0, \frac{1}{2}\right).
\]

Therefore, the Hamiltonian vector field satisfy

\[
X_F(t,x) = \rho(t)X_{F_{U-k}}(x) = \rho(t)X_{F_U}(x).
\]

(2.3)

**Definition 2.1.** Given a fiberwise starshaped hypersurface \(\Sigma \subset T^*N\),

\[\mathcal{D}(\Sigma) := \{F \in C^\infty(T^*N) : F^{-1}(0) = \Sigma, X_F|_{\Sigma} = R_{\Sigma}, X_F \text{ is compactly supported} \}\].

We call such Hamiltonians defining Hamiltonians for \(\Sigma\).

Now, let \(\Sigma \subset T^*N\) be a closed hypersurface with a defining Hamiltonian \(F \in \mathcal{D}(\Sigma)\). Let \(m = (\beta, \sigma, \theta)\) be a magnetic perturbation on \(T^*N\). Given \(\varphi_m = \phi^1_{X^m_H} \in \text{Diff}_c(T^*N, m)\), recall that the perturbed action functional \(A_m : C^0(S^1, T^*N) \times \mathbb{R} \to \mathbb{R}\) defined as follows

\[
A_m(u, \eta) = \int_0^1 u^\ast \lambda - \eta \int_0^1 F(t,u(t))dt - \int_0^1 H(t,u(t))dt + \int_0^1 \beta(t)\tilde{r}^\ast \theta(\tilde{u}(t))\partial_\tau \tilde{u}(t)dt.
\]

Critical points of \(A_m\) satisfy

\[
\begin{align*}
\partial_t u &= \eta X_F(t,u) + X^m_H(t,u) \\
\int_0^1 F(t,u)dt &= 0.
\end{align*}
\]

(2.4)

For convenience,

\[\text{Crit}(A_m) := \{w = (u, \eta) \in \mathcal{L} \times \mathbb{R} : (u, \eta) \text{ satisfies } (2.4)\};\]

\[\text{Crit}^{(a,b)}(A_m) := \{(u, \eta) \in \text{Crit}(A_m) : A_m(u, \eta) \in (a, b)\}.
\]

In the following proposition we interpret the critical point as a magnetic leaf-wise intersection point as in Definition \((1.3)\).
Proposition 2.2. Let \((u, \eta) \in \text{Crit}(A_m)\). Then \(x = u(\frac{1}{2})\) satisfies \(\varphi_m(x) \in L_x\), where \(\varphi_m = \phi_{X_{H}}^1\). Thus, \(x\) is a magnetic leaf-wise intersection point.

Proof. For \(t \in [0, \frac{1}{2}]\) we compute, using \(H(t, \cdot) = 0\) for all \(t \leq \frac{1}{2}\),

\[
\frac{d}{dt} (F_U(u(t))) = dF_U(u(t)) \cdot \partial_t u = dF_U(u(t)) \cdot [\eta X_F(t, u) + X_H^m(t, u)] = 0,
\]

since \(dF_U(X_{F_{u}}) = 0\). Hence \(F_U(u(t)) = c\) for some constant \(c\) when \(t \leq \frac{1}{2}\). Thus,

\[
0 = \int_0^1 F(t, u) dt = \int_0^1 \rho(t)(F_U(u(t)) - k) dt = c - k.
\]

Therefore \(F_U(u(t)) = k\) and since \(F_U^{-1}(0) = \Sigma\), we have \(u(t) \in \Sigma\) for \(t \in [0, \frac{1}{2}]\). In particular, \(u(\frac{1}{2}), u(0) = u(1) \in \Sigma\).

For \(t \in [\frac{1}{2}, 1]\) we have \(F(t, \cdot) = 0\). Thus, the loop \(u\) solves the equation \(\partial_t u = X_H^m(t, u)\) on \([\frac{1}{2}, 1]\), and therefore, \(u(1) = \varphi_m(u(\frac{1}{2}))\). We conclude that \(\varphi_m(u(\frac{1}{2})) \in \Sigma\). For \(t \in [0, \frac{1}{2}]\), \(\partial_t u = \eta X_F(t, u) + X_H^m(t, u) = \eta X_F(t, u) = \eta R_u\), since \(X_F|_\Sigma = R_u\). This means that \(\varphi_m(u(\frac{1}{2})) = u(1) = u(0) \in L_{u(\frac{1}{2})}\). Thus \(u(\frac{1}{2})\) is a magnetic leaf-wise intersection point. \(\square\)

2.1. Floer homology for \(A_m\). In this section, we show that \(\text{FH}(A_m)\) is well-defined. Throughout this section, we follow the strategy in [12] with minor modifications.

Definition 2.3. Let \(\Sigma\) be a non-degenerate hypersurface in \(T^*N\) with a defining Hamiltonian \(F\). A diffeomorphism \(\varphi_m \in \text{Diff}_c(T^*N, m)\) with \(\varphi_m = \phi_{X_{H}}^1\) or a pair \((H, m) \in \mathcal{H} \times \mathcal{M}\) is called regular with respect to \(F \in \mathcal{D}(\Sigma)\) if

- \(A_m = A_{H,m}^F\) is Morse.
- \(\varphi_m\) has no periodic leaf-wise intersection points.

For a given non-degenerate closed hypersurface \(\Sigma\), \(\varphi_m\) is regular for generic \(H \in \mathcal{H}\) and \(\beta \in \mathcal{B}\).

We discuss the generic property further in Appendix A and B.

Remark 2.4. In order to define gradient flow lines, we need an \(S^1\)-parameterized almost complex structure \(J(t)\) which is compatible with the \(S^1\)-parameterized symplectic form \(\omega_m\). This means that

\[
g_t(\cdot, \cdot) := \omega_m(\cdot, J(t) \cdot)
\]

defines a \(S^1\)-parameterized inner product on \(T^*N\). We denote the set of such almost complex structures as \(\mathcal{J}_m\).

Given \(J(t) \in \mathcal{J}_m\), we denote by \(\nabla_J A_m\) the gradient of \(A_m\) with respect to the inner product

\[
g_J((\tilde{u}_1, \tilde{\eta}_1), (\tilde{u}_2, \tilde{\eta}_2)) := \int_0^1 g_t(\tilde{u}_1, \tilde{u}_2) dt + \tilde{\eta}_1 \tilde{\eta}_2,
\]

where \((\tilde{u}_i, \tilde{\eta}_i) \in T_{(u, \eta)}(L \times \mathbb{R})\) for \(i = 1, 2\). One can check that

\[
\nabla_J A_m(u, \eta) = \left( -J(t, u)(\partial_t u - X_H^m(t, u) - \eta X_F(t, u)) \right).
\]
Definition 2.5. A positive gradient flow line of $A_m$ with respect to $J(t) \in J_m$ is a map $w = (u, \eta) \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R})$ solving the ODE
\[
\partial_s w(s) - \nabla J A_m(w(s)) = 0.
\]
According to Floer’s interpretation, this means that $u$ and $\eta$ are smooth maps $u : \mathbb{R} \times S^1 \to T^*N$ and $\eta : \mathbb{R} \to \mathbb{R}$ satisfying
\[
\begin{align*}
\partial_s u + J(t, u) \left( \partial_t u - X^H_m(t, u) - \eta X_F(t, u) \right) &= 0, \\
\partial_s \eta + \int_0^1 F(t, u) dt &= 0.
\end{align*}
\]
(2.6)

Proposition 2.6. If $\sigma$ is $\tilde{d}$-bounded then $\omega_m|\pi_2(T^*N) = 0$.

Proof. First choose a map $f : S^2 \to T^*N$, then it suffices to show that $\int_{f(S^2)} \omega_m = 0$.

\[
\int_{f(S^2)} \omega_m = \int_{f(S^2)} \omega_{\text{std}} + \tau^* \beta \sigma
= \int_{f(S^2)} \omega_{\text{std}} + \beta \int_{\tau f(S^2)} \sigma
= \int_{f(S^2)} \lambda + \beta \int_{\tau \tilde{\tau} f(S^2)} \tilde{\sigma}
= \int_{\partial f(S^2)} \lambda + \beta \int_{\tau \tilde{\tau} f(S^2)} d\theta
= \beta \int_{\partial \tilde{\tau} \tilde{\tau} f(S^2)} \theta
= 0.
\]
Here $\tilde{\tau} \circ f : S^2 \to \tilde{N}$ is a lifting of $\tau \circ f : S^2 \to \tilde{N}$.

Theorem 2.7. Let $w_n = (u_n, \eta_n)$ be a sequence of gradient flow lines for which there exists $a < b$ such that
\[
a \leq A_m(s)(w_n(s)) \leq b, \quad \forall s \in \mathbb{R}.
\]
Then for every reparameterization sequence $\mu_n$ the sequence $w_n(\cdot + \mu_n)$ has a subsequence which converges in $C^\infty_{\text{loc}}(\mathbb{R}, \mathcal{L} \times \mathbb{R})$.

Proof. If we show the following analytic properties then the proof follows from standard arguments in Floer theory:

1. a uniform $L^\infty$ bound on $u_n$;
2. a uniform $L^\infty$ bound on $\eta_n$;
3. a uniform $L^\infty$ bound on the derivatives of $u_n$.

The issue (1) and (3) are well-studied in usual Floer homology theory. The $L^\infty$ bound on $u_n$ follows from the convexity at infinity of $(T^*N, \omega_m)$. Suppose that $\eta_n$ is uniformly bounded then the $L^\infty$ bound on the derivatives of $u_n$ follows in the following way. If the derivatives would explode we then obtain a non-constant holomorphic sphere as limit. But by Proposition 2.6 there is no non-constant holomorphic sphere. So we concentrate on the problem (2) in the following arguments. □
Definition 2.8. Define a map $c : \mathcal{H} \times \mathfrak{M} \to [0, \infty)$ by

$$c(H, m) := \sup_{(t, u) \in S^1 \times \mathcal{L}} \left| \int_0^1 \bar{\lambda}_m(\bar{u}(t))[\bar{X}_H^m(t, u)] - H(t, u(t))dt \right|.$$ 

Here $\bar{\lambda}_m := \bar{\lambda} + \tau^* \beta \theta$ is a primitive of $\omega_m$ on the universal cover $T^*\bar{N}$.

We define

$$A_{H, m} := \int_0^1 \lambda(u(t))|\partial_t u| - \int_0^1 H(t, u(t))dt + \int_0^1 \beta(t)\theta(\bar{u}(t))[\partial_t \bar{u}(t)]dt$$

and denote the action spectrum of $A_{H, m}$ by

$$A(A_{H, m}) := \{A_{H, m}(u) : u \in \text{Crit}(A_{H, m})\}.$$

Then note that

$$c(H, m) = \sup\{|\eta| : \eta \in A(A_{H, m})\}.$$

Lemma 2.9. There exists $\epsilon > 0$ and $\tilde{c} > 0$ such that if $(u, \eta) \in C^\infty(S^1, T^*\bar{N}) \times \mathbb{R}$ satisfies $||\nabla_J A_m(u, \eta)||_J < \epsilon$ then

$$|\eta| \leq \tilde{c}(|A_m(u, \eta)| + 1).$$

(2.7)

Here $\| \cdot \|_J = \sqrt{g_J(\cdot, \cdot)}$.

Proof. The proof consists of 3 steps.

Step 1: There exist $\delta > 0$ and a constant $c_\delta < \infty$ such that if $u \in \mathcal{L}$ satisfies $u(t) \in U_\delta = F^{-1}(-\delta, \delta)$ for all $t \in [0, \frac{1}{2}]$, then

$$|\eta| \leq c_\delta (|A_m(u, \eta)| + \|\nabla_J A_m(u, \eta)\|_J + 1).$$

There exists $\delta > 0$ such that

$$\lambda(X_F(p)) \geq \frac{1}{2} + \delta, \quad \forall p \in U_\delta$$

We compute

$$|A_m(u, \eta)| = \left| \int_0^1 u^* \lambda - \int_0^1 H(t, u(t))dt - \eta \int_0^1 F(t, u(t))dt + B_m(u(t)) \right|$$

$$= \left| \int_0^1 \bar{u}^* \bar{\lambda}_m - \int_0^1 H(t, u(t))dt - \eta \int_0^1 F(t, u(t))dt \right|$$

$$= \left| \int_0^1 \bar{\lambda}_m(\bar{u}(t))|\partial_t \bar{u} - \eta \bar{X}_F(t, u) - \bar{X}_H^m(t, u)|dt \right|$$

$$+ \eta \int_0^1 \lambda(u(t))[X_F(t, u) - F(t, u(t))]dt + \int_0^1 \bar{\lambda}_m(\bar{u}(t))[\bar{X}_H^m(t, u)] - H(t, u(t))dt$$

$$\geq \frac{1}{2}|\eta| - c_\delta^m \|\partial_t \bar{u} - \bar{X}_H^m(t, u) - \eta X_F(t, u)\|_1 - c(H, m)$$

$$\geq \frac{1}{2}|\eta| - c_\delta^m \|\partial_t u - X_H^m(t, u) - \eta X_F(t, u)\|_2 - c(H, m)$$

$$\geq \frac{1}{2}|\eta| - c_\delta^m \|\nabla_J A_m(u, \eta)\|_J - c(H, m)$$
Here \( \tilde{\lambda}_m \) is the same as in Definition \( 2.3 \). \([-\] means the lifting of \([- \] to the universal cover and \( c'_{m,\delta} := \|\tilde{\lambda}_m\|_\infty \). Set \( c_\delta = \max\{2c'_{m,\delta}, 2c(H, m), 2\} \) then this inequality proves Step 1. Note that the finiteness of \( c'_{m,\delta} \) is guaranteed by the simple estimate as follows
\[
c'_{m,\delta} = \|\tilde{\lambda} + \tau^* \beta \theta\|_\infty \
\leq \|\tilde{\lambda}\|_\infty + \|\tau^* \beta \theta\|_\infty \
= \|\lambda\|_\infty + \|\beta \theta\|_\infty \
\leq \|\lambda\|_\infty + \|\beta \theta\|_\infty \
< \infty.
\]

**Step 2:** There exists \( \epsilon = \epsilon(\delta) \) with the following property. If there exists \( t \in [0, \frac{1}{2}] \) with \( F(u(t)) \geq \delta \) then \( \|\nabla_J A_m(u, \eta)\|_J \geq \epsilon \).

If in addition \( F(u(t)) \geq \frac{\delta}{2} \) holds for all \( t \in [0, \frac{1}{2}] \) then
\[
\|\nabla_J A_m(u, \eta)\|_J \geq \left| \int_0^1 F(t, u(t))dt \right| \geq \frac{\delta}{2} \int_0^1 \rho(t)dt = \frac{\delta}{2}.
\]
Otherwise there exists \( t' \in [0, \frac{1}{2}] \) with \( F(u(t')) \leq \frac{\delta}{2} \). Thus we may assume without loss of generality that \( 0 \leq a < b \leq \frac{1}{2} \) and \( \frac{\delta}{2} \leq |F(u(t))| \leq \delta \) for all \( t \in [a, b] \). Then we estimate
\[
\|\nabla_J A_m(u, \eta)\|_J \geq \|\partial_t u - X^m_H(t, u) - \eta X_F(t, u)\|_2 \\
\geq \left( \int_a^b \|\partial_t u - X^m_H(t, u) - \eta X_F(t, u)\|^2 dt \right)^{\frac{1}{2}} \\
\geq \int_a^b \|\partial_t u - \eta X_F(t, u)\|dt \\
\geq \frac{1}{\|\nabla F\|_\infty} \int_a^b \|\nabla F(u)\| \cdot \|\partial_t u - \eta X_F(t, u)\|dt \\
\geq \frac{1}{\|\nabla F\|_\infty} \int_a^b |g_t(\nabla F(u), \partial_t u - \eta X_F(t, u))|dt \\
= \frac{1}{\|\nabla F\|_\infty} \int_a^b \left| \frac{d}{dt} F(u(t)) \right| dt \\
\geq \frac{1}{\|\nabla F\|_\infty} \int_a^b \frac{d}{dt} F(u(t)) dt \\
= \frac{\delta}{2\|\nabla F\|_\infty}.
\]
Since \( \|\nabla F\|_\infty \) is bounded from above, we set \( \epsilon(\delta) := \min\{\frac{\delta}{2}, \frac{\delta}{2\|\nabla F\|_\infty}\} \). This proves Step 2.

**Step 3:** We prove the lemma.
Choose \( \delta \) as in Step 1, \( \epsilon = \epsilon(\delta) \) as in Step 2 and
\[
\bar{c} = c_3(\epsilon + 1).
\]
Assume that \( \| \nabla_J A_m(u, \eta) \|_J < \epsilon \) then
\[
|\eta| \leq c_3 (|A_m(u, \eta)| + \| \nabla_J A_m(u, \eta) \| J + 1) \leq \bar{c}(|A_m(u, \eta)| + 1).
\]
This proves the lemma.

**Definition 2.10.** The energy of a map \( w \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}) \) is defined as
\[
E(w) := \int_{-\infty}^{\infty} \| \partial_s w \|^2 ds.
\]

By a simple computation one can check that \( E(w) = A_m(w_+) - A_m(w_-) \).

**Proposition 2.11.** Let \( w_\pm \in \text{Crit}(A_m) \) and \( w = (u, \eta) \) be a gradient flow line of \( A_m \) with
\[
\lim_{s \to \pm \infty} w(s) = w_\pm.
\]
Then there exists a constant \( \kappa = \kappa(w_-, w_+) \) satisfying \( \| \eta \|_\infty \leq \kappa \).

**Proof.** Let \( \epsilon \) be as in Lemma 2.9. For \( l \in \mathbb{R} \), let \( \nu_w(l) \geq 0 \) be defined by
\[
\nu_w(l) := \inf\{ \nu_w \geq 0 : \| \nabla_J A_m[w(l + \nu_w)] \| J < \epsilon \}.
\]

Then \( \nu_w(l) \) is uniformly bounded as follows,
\[
A_m(w_+) - A_m(w_-) = \int_{-\infty}^{\infty} \| \partial_s w(s) \|^2 ds
= \int_{-\infty}^{\infty} \| \nabla_J A_m(w(s)) \|^2 J ds
\geq \int_{l}^{l+\nu_w(l)} \frac{\| \nabla_J A_m(w(s)) \|^2}{{\epsilon}^2} ds
\geq \epsilon^2 \nu_w(l).
\]

Now, we set
\[
\|F\|_\infty = \max_{(t,x) \in S^1 \times T^*N} |F(t,x)|, \quad K = \max\{|A_m(w_+)|, |A_m(w_-)|\}.
\]

By definition of \( \nu_w(l) \), we get \( \| \nabla_J A_m[w(l + \nu_w(l))] \| J < \epsilon \). Then we can use Proposition 2.9 to obtain the following estimate
\[
|\eta(l + \nu_w(l))| \leq \bar{c}(|A_m[w(l + \nu_w(l))]| + 1)
\leq \bar{c}(K + 1).
\]
By taking direct and inverse limit, we obtain

\[ M \] adding broken trajectories and this gives us

\[ \partial \] where the Floer homology group is defined by

\[ \text{For a generic almost complex structure } J(t) \in J \text{ and given } w_{\pm} \in \text{Crit}_{k_{\pm}}(A) \text{, we denote by} \]

\[ \text{Crit}_{k_{\pm}}(A) := \text{Crit}_{k_{\pm}}(A) \otimes \mathbb{Z}_2. \]

For a generic almost complex structure \( J(t) \in J \) and given \( w_{\pm} \in \text{Crit}_{k_{\pm}}(A) \), we denote by

\[ w_{\pm} := \{ w(s) : w \text{ satisfies (2.6), } \lim_{s \to \pm \infty} w(s) = w_{\pm} \}; \]

\[ \mathcal{M}(w_{-}, w_{+}) := \mathcal{M}(w_{-}, w_{+}) / \mathbb{R}. \]

The above \( \mathbb{R} \)-action is given by translating the \( s \)-coordinate. Suppose further that the almost complex structure \( J(t) \) is generic, so that \( \mathcal{M}(w_{-}, w_{+}) \) is a smooth manifold of dimension

\[ \dim \mathcal{M}(w_{-}, w_{+}) = \mu(w_{-}) - \mu(w_{+}) - 1. \]

The boundary operator \( \partial : FC_{k_{\pm}}(A) \to FC_{k-1}(A) \) is defined by

\[ \partial w_{-} := \sum_{\mu(w_{+}) = k-1} \#_{2} \mathcal{M}(w_{-}, w_{+}) w_{+}, \]

where \#_2 means \( \mathbb{Z}_2 \)-counting. When \( \dim \mathcal{M}(w_{-}, w_{+}) = 1 \) the moduli space is compactified by adding broken trajectories and this gives us \( \partial \circ \partial = 0 \). In the \( A \) case, the compactification of \( \mathcal{M}(w_{-}, w_{+}) \) is guaranteed by Theorem 2.7 with Proposition 2.11. Then the resulting filtered Floer homology group is defined by

\[ \text{FH}_{i}(A) := \text{H}_{i}(FC_{i}(A), \partial). \]

By taking direct and inverse limit, we obtain

\[ \text{FH}_{i}(A) := \lim_{a \to \infty} \lim_{b \to \infty} \text{FH}_{i}(A), \quad a, b \to \infty. \]
2.2. Continuation map between $\text{FH}(\mathcal{A}^F_H)$ and $\text{FH}(\mathcal{A}_m)$. In this section, we construct a continuation homomorphism

$$\tilde{\Psi}^m : \text{FH}(\mathcal{A}^F_H) \to \text{FH}(\mathcal{A}_m)$$

by counting gradient flow lines of the $s$-dependent perturbed Rabinowitz action functional. Here

$$\mathbf{m}(s) := (\gamma(s)\beta, \sigma, \theta)$$

are $s$-dependent magnetic perturbation data where $\gamma : \mathbb{R} \to [0, 1]$ is a cut-off function

$$\gamma(s) = \begin{cases} 0 & \text{for } s \leq 0 \\ 1 & \text{for } s \geq 1 \end{cases}$$

and $0 \leq \dot{\gamma}(s) \leq 2$ for all $s \in \mathbb{R}$. Then the corresponding action functional is

$$\mathcal{A}_{m(s)}(u, \eta) = \mathcal{A}^F_H(u, \eta) + \gamma(s)\mathcal{E}_m(u)$$

where $\mathcal{A}^F_H(u, \eta) = \int_0^1 u^*\lambda - \eta \int_0^1 F(t, u(t))dt - \int_0^1 H(t, u(t))dt$.

Now we consider the $(s, t)$-dependent almost complex structure $J(s, t)$ on $T^*N$ such that $J(s, t) \in \mathcal{J}_{m(s)}$ for all $s \in [0, 1]$ and $J(s, t)$ is independent of $s$ for $s \leq -1$ and $s \geq 1$. This almost complex structure induces the $(s, t)$-dependent inner product on $T^*N$

$$g_{s,t}(\cdot, \cdot) := \omega_{m(s)}(\cdot, J(s, t), \cdot),$$

and the following $s$-dependent inner product on $\mathcal{L} \times \mathbb{R}$

$$g_s((\hat{u}_1, \hat{\eta}_1), (\hat{u}_2, \hat{\eta}_2)) := \int_0^1 g_{s,t}(\hat{u}_1, \hat{u}_2)dt + \hat{\eta}_1\hat{\eta}_2,$$

where $(\hat{u}_i, \hat{\eta}_i) \in T_{(u, \eta)}(\mathcal{L} \times \mathbb{R})$ for $i = 1, 2$. With the above metric, we obtain

$$\nabla_s \mathcal{A}_{m(s)}(u, \eta) = \begin{pmatrix} -J(s, t, u)(\partial_tu - \eta X_F(t, u) - X_{m(s)}^H(t, u)) \\ -\int_0^1 F(t, u)dt \end{pmatrix}$$

Then the gradient flow line $w = (u, \eta) \in C^\infty(\mathbb{R} \times S^1, T^*N) \times C^\infty(\mathbb{R}, \mathbb{R})$ satisfies

$$\begin{cases} \partial_s u + J(s, t, u)(\partial_tu - \eta X_F(t, u) - X_{m(s)}^H(t, u)) = 0 \\ \partial_s \eta + \int_0^1 F(t, u)dt = 0 \end{cases}$$

with energy

$$E(w) = \int_{-\infty}^\infty \|\partial_sw\|^2ds.$$

In order to construct the continuation homomorphism $\tilde{\Psi}^m$, it suffices to show that the Lagrange multiplier $\eta$ and the energy of the time-dependent gradient flow line are uniformly bounded. For this purpose, we need the following preparations.

Lemma 2.12. There exists $\epsilon > 0$ and $\bar{c} > 0$ such that if $(u, \eta) \in C^\infty(S^1, T^*N) \times \mathbb{R}$ satisfies

$$\|\nabla_s \mathcal{A}_{m(s)}(u, \eta)\|_s < \epsilon$$

then

$$|\eta| \leq \bar{c}(\|\mathcal{A}_{m(s)}(u, \eta)\| + 1).$$

(2.11)

Here $\|\cdot\|_s := \sqrt{g_s(\cdot, \cdot)}$.

Proof. The proof is basically the same as in Lemma 2.9 by considering $m(s)$ instead of $m$. Here we omit the proof. □
**Definition 2.13.** For a given magnetic perturbation \( m = (\beta, \sigma, \theta) \), an isoperimetric constant \( C : \mathfrak{M} \rightarrow [0, \infty) \) is defined by

\[
C = C(m) := \|\beta\|_{\infty} \|\theta\|_{\infty}.
\]

Note that \( C \to 0 \) as \( \|\beta\|_{\infty} \to 0 \).

**Proposition 2.14.** Let \( w_- \in \text{Crit}(A^F_H), w_+ \in \text{Crit}(A_m) \) and \( w = (u, \eta) \) be a gradient flow line of \( A_{m(s)} \) with \( \lim_{s \to \pm \infty} w = w_{\pm} \). If \( \|\beta\|_{\infty} \) is sufficiently small then there exists a constant \( \kappa = \kappa(w_-, w_+) \) such that

\[
\|\eta\|_{\infty} \leq \kappa.
\]

**Proof.** We prove the proposition in 3 steps.

**Step 1:** Let us first bound the energy of \( w \) in terms of \( \|\eta\|_{\infty} \).

\[
E(w) = \int_{-\infty}^{\infty} \|\partial_s w\|^2 ds
= \int_{-\infty}^{\infty} \langle \partial_s w, \nabla_s A_{m(s)}(w(s)) \rangle_s ds
= \int_{-\infty}^{\infty} \frac{d}{ds} A_{m(s)}(w(s)) ds - \int_{-\infty}^{\infty} \dot{A}_{m(s)}(w(s)) ds
= A_{m(1)}(w_+) - A_{m(0)}(w_-) - \int_{-\infty}^{\infty} \dot{A}_{m(s)}(w(s)) ds.
\] (2.12)

We estimate the last term in (2.12) by using the isoperimetric constant \( C \)

\[
\left| \int_{-\infty}^{\infty} \dot{A}_{m(s)}(w(s)) ds \right| \leq \int_{-\infty}^{\infty} \left| \dot{A}_{m(s)}(w(s)) \right| ds
= \int_{-\infty}^{\infty} \dot{\gamma}(s) \left| \int_{S^1} \bar{u}^s \beta(t) \theta dt \right| ds
\leq \int_{-\infty}^{\infty} \dot{\gamma}(s) C \int_{S^1} |\partial_t u|_{s,t} dt ds.
\] (2.13)

Here \( | \cdot |_{s,t} := \sqrt{g_{s,t}(\cdot, \cdot)} \). From the gradient flow equation (2.10), we get

\[
\partial_t u = J(s, t, u) \partial_s u + \eta X_F(t, u) + X_{H}^{m(s)}(t, u).
\] (2.14)
By inserting (2.14) into the last term in (2.13), we then obtain
\[
\int_{-\infty}^{\infty} \left| \dot{A}_m(s)(w(s)) \right| ds \leq \int_{-\infty}^{\infty} \dot{\gamma}(s) C \int_{S^1} \left| \partial_s u \right|_{s,t} dt \, ds
\]
\[
= \int_{-\infty}^{\infty} \dot{\gamma}(s) C \int_{S^1} \left| J(s,t) \partial_s u + \eta X_F(t,u) + X_{m_H}^m(t,u) \right|_{s,t} dt \, ds
\]
\[
\leq 2C \int_{0}^{1} \int_{S^1} \left( \left| \partial_s u \right|_{s,t} + \left| \eta \right| X_F(t,u) + \left| X_{m_H}^m(t,u) \right|_{s,t} \right) dt \, ds
\]
\[
\leq 2C \int_{0}^{1} \int_{S^1} \left( \left| \partial_s u \right|_{s,t} + 1 + \left| \eta \right| \| X_F \|_{\infty} + \| X_{m_H}^m \|_{\infty} \right) dt \, ds
\]
\[
= 2CE(u) + 2C + 2d_m C + 2\| \eta \|_{\infty} d_F C
\]
\[
\leq 2CE(w) + 2C + 2d_m C + 2\| \eta \|_{\infty} d_F C
\]
where \( d_m = d_{H,m} := \sup_{s \in \mathbb{R}} \| X_{m_H}^m(s) \|_{\infty} \), \( d_F = \| X_F \|_{\infty} \).

Now by combining the above estimates (2.12) and (2.15), we deduce
\[
E(w) = A_{m(1)}(w_+) - A_{m(0)}(w_-) - \int_{-\infty}^{\infty} \dot{A}_m(s)(w(s)) ds
\]
\[
\leq A_{m(1)}(w_+) - A_{m(0)}(w_-) + 2CE(w) + 2C + 2d_m C + 2\| \eta \|_{\infty} d_F C.
\]
If \( \| \beta \|_{\infty} \) is sufficiently small, then we may assume that \( C \leq \frac{1}{7} \). Set \( \Delta := A_{m(1)}(w_+) - A_{m(0)}(w_-) \), then we get
\[
E(w) \leq 2A_{m(1)}(w_+) - 2A_{m(0)}(w_-) + 4C + 4d_m C + 4\| \eta \|_{\infty} d_F C
\]
\[
= 2\Delta + 4C + 4d_m C + 4\| \eta \|_{\infty} d_F C. \quad (2.16)
\]
This finishes Step 1.

**Step 2** : Let \( \epsilon \) be as in Lemma (2.10) For \( l \in \mathbb{R} \) let \( \nu_w(l) \geq 0 \) be defined by
\[
\nu_w(l) := \inf \left\{ \nu \geq 0 : \left\| \nabla_s A_{m(l+\nu)}(w(l+\nu)) \right\|_{s} < \epsilon \right\}.
\]
In this step we bound \( \nu_w(l) \) in terms of \( \| \eta \|_{\infty} \) for all \( l \in R \) as follows
\[
E(w) = \int_{-\infty}^{\infty} \left| \partial_s w \right|_{s}^2 ds
\]
\[
= \int_{-\infty}^{\infty} \left\| \nabla_s A_{m(s)} \right\|_{s}^2 ds
\]
\[
\geq \int_{l}^{l + \nu_w(l)} \left\| \nabla_s A_{m(s)} \right\|_{s}^2 ds \geq \epsilon^2 
\]
\[ \geq \epsilon^2 \nu_w(l). \]
Step 1 and the above estimate finish Step 2.

**Step 3** : We prove the proposition.

First set
\[
K = \max \{-A_{m(0)}(w_-), A_{m(1)}(w_+)\}
\]
By the definition of $\nu_w(l)$, we get $\|\nabla_s A_m[l+\nu_w(l)]w(l+\nu_w(l))\|_s < \epsilon$, which enables us to use Lemma 2.12. We obtain the following estimate by using (2.11), (2.15) and (2.16)

$$|\eta(l+\nu_w(l))| \leq \tilde{c} \left( |A_m[l+\nu_w(l)]w(l+\nu_w(l))| + 1 \right)$$

$$\leq \tilde{c} \left( K + \int_{-\infty}^{\infty} |A_m(s)| \, ds + 1 \right)$$

$$\leq \tilde{c} \left( K + 2C E(w) + 2C + 2d_m C + 2\|\eta\|_{\infty} d_F C + 1 \right)$$

$$\leq \tilde{c} \left[ K + 2C (2\Delta + 4C + 4d_m C + 4\|\eta\|_{\infty} d_F C) + 2C + 2d_m C + 2\|\eta\|_{\infty} d_F C + 1 \right].$$

By Step 2 and (2.16), we get the following inequalities

$$\left| \int_l^{l+\nu_w(l)} \hat{\eta}(s) \, ds \right| \leq \left| \int_l^{l+\nu_w(l)} \int_0^1 F(t, u(t)) \, dt \, ds \right|$$

$$\leq \|F\|_{\infty} \nu_w(l)$$

$$\leq \|F\|_{\infty} \frac{E(w)}{\epsilon^2}$$

$$\leq \frac{\|F\|_{\infty}}{\epsilon^2} (2\Delta + 4C + 4d_m C + 4\|\eta\|_{\infty} d_F C).$$

Combining the above two estimates (2.18) and (2.19), we conclude the following

$$|\eta(l)| \leq |\eta(l+\nu_w(l))| + \int_l^{l+\nu_w(l)} \hat{\eta}(s) \, ds$$

$$\leq \tilde{c} \left[ K + 2C (2\Delta + 4C + 4d_m C + 4\|\eta\|_{\infty} d_F C) + 2C + 2d_m C + 2\|\eta\|_{\infty} d_F C + 1 \right]$$

$$+ \frac{\|F\|_{\infty}}{\epsilon^2} (2\Delta + 4C + 4d_m C + 4\|\eta\|_{\infty} d_F C)$$

$$= \left( 8\tilde{c} d_F C + 2\tilde{c} d_F + 4 \frac{d_F}{\epsilon^2} \|F\|_{\infty} \right) C \|\eta\|_{\infty}$$

$$\leq \left\{ \frac{1}{2} \tilde{c} \right\} K$$

Since the above estimate is valid for all $l \in \mathbb{R}$,

$$\|\eta\|_{\infty} \leq \kappa_1 \|\eta\|_{\infty} + \frac{1}{2} \kappa.$$

If we choose $\beta \in \mathcal{B}$ such that the induced isoperimetric constant $C$ additionally satisfies

$$C \leq \frac{1}{4} \quad \text{and} \quad \kappa_1 \leq \frac{1}{2},$$

then finally we conclude

$$\|\eta\|_{\infty} \leq \kappa.$$

This proves the proposition. □
Lemma 2.15. With the same assumptions as in Proposition 2.14 let \( a = A_{m(0)}(w_{-}) \), \( b = A_{m(1)}(w_{+}) \). Then the following assertion meets:

1. If \( a \geq \frac{1}{9} \), then \( b \geq a^2 \);
2. If \( b \leq -\frac{1}{9} \), then \( a \leq b^2 \).

**Proof.** By Proposition 2.14, the Lagrange multiplier \( \eta \) of the gradient flow line is uniformly bounded as follows

\[
\|\eta\|_{\infty} \leq 2\tilde{c}[K + 2C(2\Delta + 4C + 4d_mC) + 2C + 2d_mC + 1] + \frac{2\|F\|_{\infty}}{\epsilon^2}(2\Delta + 4C + 4d_mC).
\]

Recall that \( K = \max\{-a, b\} \) and \( \Delta = b - a \). From the fact that \( E(w) \geq 0 \) and (2.16), we obtain the following inequality

\[
b \geq a - 2\tilde{c}d_FC + 4\tilde{c}d_FC(1 + d_m + 4C + 4d_mC)\]

\[
+ \frac{8d_F\|F\|_{\infty}C}{\epsilon^2(1 + d_m)} \leq \frac{1}{72}.
\]

Then (2.22) becomes

\[
b \geq a - 2C - 2d_mC - 2\|\eta\|_{\infty}d_FC
\]

\[
\geq a - 2C - 2d_mC - 4\left(\tilde{c}[K + 2C(2\Delta + 4C + 4d_mC) + 2C + 2d_mC + 1] + \frac{\|F\|_{\infty}}{\epsilon^2}(2\Delta + 4C + 4d_mC)\right)d_FC
\]

\[
= a - 4\tilde{c}d_FC K - 8\left(2\tilde{c}d_FC + \frac{d_F\|F\|_{\infty}}{\epsilon^2}\right)C \Delta - 2\left(\frac{8d_F\|F\|_{\infty}C}{\epsilon^2}(1 + d_m)\right)
\]

\[
+ 1 + d_m + 2\tilde{c}d_FC(1 + d_m + 4C + 4d_mC)C
\]

\[
\geq a - \frac{1}{8}K - \frac{1}{16}(b - a) - \frac{1}{36}.
\]

To prove the assertion (1), we first consider the case

\[|b| \leq a, \quad a \geq \frac{1}{9}.\]

In this assumption, we induce the following estimate from (2.24)

\[
b \geq a - \frac{1}{8}a - \frac{1}{8}a - \frac{1}{36} = 3a - \frac{1}{36} \geq \frac{a}{2}.
\]
Now we want to exclude the case
\[-b \geq a \geq \frac{1}{9}.\]
But in this case (2.24) implies the following contradiction:
\[b \geq \frac{1}{9} + \frac{1}{72} - \frac{1}{16}(b-a) - \frac{1}{36} \geq -\frac{1}{16}(b-a) > 0.\]
This proves the first assumption. To prove the assertion (2), we set
\[b' = -a, \quad a' = -b.\]
Then (2.24) also holds for \(b'\) and \(a'\). Thus we get the following assertion from (1)
\[-b \geq \frac{1}{9} \implies -a \geq \frac{b}{2}\]
which is equivalent to the assertion (2). This proves the lemma. 

\[\square\]

**Proof of Theorem 1.5.** Recall the perturbation data \(m\) is a triple which consists of \((\beta, \sigma, \theta)\). First we subdivide \(\beta \in \mathcal{B}\) into small pieces to have the following properties:
- \(m^i = (d^i \beta, \sigma, \theta)\), where \(0 = d^0 < d^1 < \ldots < d^N = 1;\)
- \(\mathcal{A}_{m^i} : \mathcal{L} \times \mathbb{R} \to \mathbb{R}\) is Morse for all \(i = 0, 1, \ldots, N;\)
- \(C^i = (d^{i+1} - d^i)\|\beta\|_{\infty}\|\sigma\|_{\infty}\) satisfies (2.20), (2.23) for all \(i = 0, 1, \ldots, N - 1.\)

Since the Morse property is generic, we can guarantee the second property. Let \(m^i(s) = (\gamma(s)(d^{i+1} - d^i)\beta, \sigma, \theta)\) be a homotopy from \(m^i\) to \(m^{i+1}\). Now we consider a gradient flow line \(w = (u, \eta)\) of the time dependent action functional \(\mathcal{A}_{m^i(s)}\) which satisfies
\[
\begin{align*}
\partial_s u + J(s, t, u)(\partial_t u - X^m_{\mathcal{L}}(s, t, u) \eta X_{\mathcal{L}}(s, t, u)) = 0, \\
\partial_s \eta + \int_0^1 F(t, u) dt = 0,
\end{align*}
\]
and the limit condition
\[
\lim_{s \to -\infty} w(s) = w_- \in \text{Crit}(\mathcal{A}_{m^i}), \quad \lim_{s \to \infty} w(s) = w_+ \in \text{Crit}(\mathcal{A}_{m^{i+1}}).
\]

We then define a map
\[
\Psi_{m^{i+1}}^m : FC_s(\mathcal{A}_{m^i}) \to FC_s(\mathcal{A}_{m^{i+1}})
\]
given by
\[
\Psi_{m^{i+1}}^m(w_-) = \sum_{\mu(w_u) = \mu(w_-)} \#_2 \mathcal{M}_{m^i}^{m^{i+1}}(w_-, w_+). w_+.
\]
Here \(\#_2\) means the \(\mathbb{Z}_2\)-counting and 
\[
\mathcal{M}_{m^i}^{m^{i+1}}(w_-, w_+) = \{ w = (u, \eta) \mid w \text{ satisfies (2.25), (2.20)} \}.
\]

By Proposition 2.6 \((T^s N, \omega_{m^i(s)})\) is symplectically aspherical for all \(s \in \mathbb{R}\). So there is no bubbling. Thus it suffices to bound the energy \(E(w) = \int_{-\infty}^{\infty} \|w\|_{2}^2 ds\) and the Lagrange multiplier \(\eta\) in terms of \(w_-, w_+\) for the compactness of \(\mathcal{M}_{m^i}^{m^{i+1}}(w_-, w_+).\) By the above 3rd condition for \(\beta \in \mathcal{B}\), we can use the argument of Proposition 2.14. Especially (2.16), (2.21) implies that the energy of time-dependent gradient flow lines are uniformly bounded,
\[
E(w) \leq 2A_{m^i(1)}(w_+) - 2A_{m^i(0)}(w_-) + 4C + 4d_{m^i} C + 4\|\eta\|_{\infty} d_{\mathcal{L}} C
\]
\[
\leq 2A_{m^i(1)}(w_+) - 2A_{m^i(0)}(w_-) + 4C + 4d_{m^i} C + 4\kappa(w_-, w_+) d_{\mathcal{L}} C.
\]
Now, by virtue of Lemma 2.15, we obtain maps for $a \leq -\frac{1}{9}$ and $b \geq \frac{1}{9}$

$$\Psi_{m+1}^{m+1}(a, b) : FC^{(a, b)}(A_{m}) \rightarrow FC^{(a, b)}(A_{m+1})$$

defined by counting gradient flow lines of $A_{m(s)}$. Since the continuation maps $\Psi_{m}^{m+1}(a, b)$ commute with the boundary operators, this induces the following homomorphisms on homologies as follows

$$\bar{\Psi}_{m+1}^{m+1}(a, b) : FH^{(a, b)}(A_{m}) \rightarrow FH^{(a, b)}(A_{m+1}).$$

By taking the inverse and direct limit

$$FH^{*}(A_{m}) = \lim_{b \to \infty} \lim_{a \to -\infty} FH^{(a, b)}(A_{m}),$$

we deduce

$$\bar{\Psi}_{m+1}^{m+1} : FH(A_{m}) \rightarrow FH(A_{m+1}).$$

By juxtaposing $\{\bar{\Psi}_{m+1}^{m+1}\}_{i=0}^{N}$, we obtain

$$\bar{\Psi}^{m} : FH(A_{m}) \rightarrow FH(A_{m+1}).$$

Here $\bar{\Psi}^{m} = \bar{\Psi}^{m}_{N-1} \circ \cdots \circ \bar{\Psi}^{m}_{1} \circ \bar{\Psi}^{m}_{0}$. In a similar way, we construct

$$\bar{\Psi}^{m} : FH(A_{m}) \rightarrow FH(A_{m+1}),$$

by following the homotopies in opposite direction. By a homotopy-of-homotopies argument, we conclude $\bar{\Psi}^{m} \circ \bar{\Psi}^{m} = id_{FH(A_{m})}$ and $\bar{\Psi}^{m} \circ \bar{\Psi}^{m} = id_{FH(A_{m+1})}$. Therefore $\bar{\Psi}^{m}$ is an isomorphism with inverse $\bar{\Psi}^{m}$. \qed

**Proof of Corollary 1.6** In Theorem 1.5, we have the continuation isomorphism as follows

$$\bar{\Psi}^{m} : FH_{*}(A_{m}) \rightarrow FH_{*}(A_{m+1}).$$

Since we assume that $\dim H_{*}(\mathcal{L}_{N}) = \infty$, (1.3), (1.4) imply that $\dim FH_{*}(A_{m}) = \infty$ and consequently the Morse function $A_{m}$ has infinitely many critical points. Now Proposition 2.2 implies that there exist infinitely many magnetic leaf-wise intersections or a period leaf-wise intersection. But, by Theorem 3.1 the latter case can be excluded for a generic $\beta \in \mathcal{B}$. Hence there exist infinitely many magnetic leaf-wise intersections. \qed

3. **On the growth rate of magnetic leaf-wise intersections**

In [30], Macarini-Merry-Paternain prove the exponential growth rate of leaf-wise intersections with respect to the period when $\bar{\pi}_{1}(N)$ grows exponentially. Recall that $\bar{\pi}_{1}(N)$ is the fundamental group of $N$ modulo conjugacy classes.

**3.1. Symplectically hyperbolic manifolds.** In this section, we investigate the examples and the candidates for the above topological assumption.

**Definition 3.1.** Let $(N, \omega_{N})$ be a closed symplectic manifold of dimension $2n$. If the symplectic form $\omega_{N}$ is $d$-bounded, then $(N, \omega_{N})$ is called symplectically hyperbolic.

**Proposition 3.2** (Kędra [29]). Let $(N, \omega_{N})$ be a symplectically hyperbolic manifold then $\pi_{1}(N)$ grows exponentially.
Proof. The proof consists of 2 steps.

Step 1 : \( \pi_1(N) \) has exponential growth if and only if a ball in \( \tilde{N} \) grows exponentially with respect to the radius.

Let us choose a Riemannian metric \( g \) on \( N \) and a base point \( x_0 \in N \). Then we define \( l_g : \pi_1(N, x_0) \rightarrow \mathbb{R} \) by

\[
l_g(s) = \inf \left\{ \int_0^1 |\dot{q}(t)|_g dt \mid q : [0, 1] \rightarrow (N, x_0), [q] = s \in \pi_1(N, x_0) \right\}.
\]

Now we let \( (\tilde{N}, \tilde{\omega}_N) \) be the universal cover of \((N, \omega_N)\) and \( \tilde{g} \) be the Riemannian metric lifted from \( g \). Take a fundamental region \( \underline{N} \subset \tilde{N} \) with the base point \( \tilde{x}_0 \in \underline{N} \) which is a lift of \( x_0 \). If we consider the following set

\[
B_{\pi_1(N)}(T) = B_{\pi_1(N, g, x_0)}(T) := \{ s \in \pi_1(N, x_0) : l_g(s) \leq T \}.
\]

Then each \( s \in B_{\pi_1(N)}(T) \) corresponds to a deck transformation on \((\tilde{N}, \tilde{g})\). Especially, we translate the fundamental region \((\underline{N}, \tilde{x}_0)\) via \( s \in B_{\pi_1(N)}(T) \) and denote it as \( s \underline{N} \subset \tilde{N} \). Let us denote by \( B(\tilde{N}, \tilde{x}_0)(T) \) the ball of radius \( T \) which is centered at \( \tilde{x}_0 \in \tilde{N} \).

\[
\text{vol}(B(\tilde{N}, \tilde{x}_0)(T)) \leq \text{vol}(\bigcup_{l_g(s) \leq T} s \underline{N}) \leq \text{vol}(B(\tilde{N}, \tilde{x}_0)(T + \text{diam}\underline{N})).
\]

Here the volume form is given by \( \tilde{\omega}_N^n \). Obviously the above middle term has the same value with

\[
\#\{ s \in \pi_1(N, x_0) : l_g(s) \leq T \} \cdot \text{vol}(\underline{N}).
\]

This proves Step 1.

Step 2 : \( \text{vol}(B(\tilde{N}, \tilde{x}_0)(T)) \) grows exponentially with \( T \).

Let \( \theta \in \Omega^1(\tilde{N}) \) be a primitive of \( \tilde{\omega}_N \) and let \( X \) be the corresponding Liouville vector field on \( \tilde{N} \). That is \( \iota_X \tilde{\omega}_N = \theta \). We may assume that the given \( \tilde{g} \) satisfies \( \| \tilde{\omega}_N \|_{\tilde{g}} = 1 \), so the norm of \( X \) is uniformly bounded by \( C := \| \theta \|_{\tilde{g}} \). Since \( \text{vol} = \tilde{\omega}_N^n \), we have

\[
L_X \text{vol} = L_X \tilde{\omega}_N^n = n \tilde{\omega}_N^n = n \text{vol}.
\]

Let \( \psi : \mathbb{R} \rightarrow \text{Diff}(\tilde{N}) \) be the flow given by the vector field \( X \) and \( B := B_{(\tilde{N}, \tilde{x}_0)}(1) \). We compute that the volume of the image \( \psi_T(B) \) grows exponentially with \( T \) as follows

\[
\frac{d}{dt} \bigg|_{t=s} \text{vol}(\psi_t(B)) = \frac{d}{dt} \bigg|_{t=s} \int_{\psi_t(B)} \text{vol} = \frac{d}{dt} \bigg|_{t=s} \int_B \psi_t^* \text{vol} = \int_B \frac{d}{dt} \bigg|_{t=s} \psi_t^* \text{vol} = \int_B \psi_s^*(L_X \text{vol}) = \int_{\psi_s^*(B)} (n \text{vol}) = n \text{vol}(\psi_s(B)).
\]
Hence we get \( \text{vol}(\psi_T(B)) = e^{nT} \text{vol}(B) \). Since \( B_{(\tilde{N}, \tilde{x}_0)}(2CT + 1) \supset \psi_T(B) \), we conclude that
\[
\text{vol}(B_{(\tilde{N}, \tilde{x}_0)}(2CT + 1)) \geq \text{vol}(\psi_T(B)) = e^{nT} \text{vol}(B).
\]
This proves Step 2. \(\square\)

As mentioned in the introduction, we are interested in the growth rate of \( \tilde{\pi}_1(N) \). It is known that \( \tilde{\pi}_1(N) \) has exponential growth rate when \( N \) is a 2-dimensional symplectically hyperbolic manifold. But we don’t know the growth rate of \( \tilde{\pi}_1(N) \) for any higher dimensional symplectically hyperbolic manifold.

3.2. Perturbed \( \mathcal{F} \)-Rabinowitz action functional. In order to show the exponential growth rate of leaf-wise intersection points, Macarini-Merry-Paternain used the \( \mathcal{F} \)-Rabinowitz action functional as follows
\[
\mathcal{A}^f : \mathcal{L} \times \mathbb{R} \to \mathbb{R},
\]
\[
\mathcal{A}^f(u, \eta) = \mathcal{A}^{F,f}_{\mathcal{H}}(u, \eta) := \int_0^1 u^* \lambda - f(\eta) \int_0^1 F(t, u) dt - \int_0^1 H(t, u) dt.
\]
The above new ingredient \( f \in C^\infty(\mathbb{R}, \mathbb{R}) \) needs to satisfy the following properties:

(1) \( f \) is a smooth strictly positive, strictly increasing function.
(2) \( \lim_{\eta \to \pm \infty} f(\eta) = 0 \) and \( f' \) satisfies \( 0 < f'(\eta) \leq 1 \) for all \( \eta \in \mathbb{R} \).

The additional data \( f(\eta) \) is crucial to the construction of continuation maps between a concentric family of fiberwise starshaped hypersurfaces, see [30, Section 4.2]. We denote by \( \mathcal{F} \) the set of such \( f \in C^\infty(\mathbb{R}, \mathbb{R}) \) satisfying the above conditions.

If we additionally consider the magnetic perturbation, then the action functional becomes
\[
\mathcal{A}^{m,f}_{\mathcal{H}}(u, \eta) = \mathcal{A}^{F,f}_{\mathcal{H},m} := \mathcal{A}^f(u, \eta) + \mathcal{B}_m(u).
\]
One can check that a critical point of \( \mathcal{A}^{m,f}_{\mathcal{H}} \) satisfies
\[
\begin{align*}
\partial_t u &= f(\eta)X_{\mathcal{F}}(t, u) + X^m_{\mathcal{H}}(t, u) \\
f'(\eta) \int_0^1 F(t, u) dt &= 0.
\end{align*}
\]
Since \( f'(\eta) > 0 \) for all \( \eta \in \mathbb{R} \), it is equivalent to
\[
\begin{align*}
\partial_t u &= f(\eta)X_{\mathcal{F}}(t, u) + X^m_{\mathcal{H}}(t, u) \\
\int_0^1 F(t, u) dt &= 0.
\end{align*}
\]
Given \(-\infty \leq a \leq b \leq \infty \), we adopt the following notations:
\[
\begin{align*}
\text{Crit}(\mathcal{A}^{m,f}_{\mathcal{H}}) &:= \{ (u, \eta) \in \mathcal{L} \times \mathbb{R} : (u, \eta) \text{ satisfies (3.3)} \}; \\
\text{Crit}^{(a,b)}(\mathcal{A}^{m,f}_{\mathcal{H}}) &:= \{ (u, \eta) \in \text{Crit}(\mathcal{A}^{m,f}_{\mathcal{H}}) : \mathcal{A}^{m,f}_{\mathcal{H}}(u, \eta) \in (a, b) \}.
\end{align*}
\]
Since \( f \in \mathcal{F} \) is a positive function, we only consider positive (magnetic) leaf-wise intersection points.\(^2\) It would be convenient if \( f(\eta) = \eta \) on the action window \((a, b) \subset \mathbb{R}^+\) we work with.

**Definition 3.3.** Given \( a > 0 \),
\[
\mathcal{F}(a) := \{ f \in \mathcal{F} : f(\eta) = \eta, \forall \eta \in [a, \infty) \}.
\]
\(^2\)A (magnetic) leaf-wise intersection point is called positive or negative if \( \eta \) in (1.2) is positive or negative respectively.
Proposition 3.4. Let \( \varphi \) With this generic condition for \( \varphi \), there is a map

\[
c(H, m) = \sup_{(t,u) \in S^1 \times \mathcal{L}} \left| \int_{0}^{1} \tilde{\lambda}_m(\tilde{u}(t))[\tilde{X}_H^m(t, u)] - H(t, u(t))dt \right|
\]

and recall that

Moreover, if there is no periodic magnetic leaf-wise intersection points then \( \text{ev} \) is injective.

Define 3.5. A positive gradient flow line \( \mathcal{A}_m^f \) with respect to \( \varphi_m \) is a critical point of \( \mathcal{A}_m^f \) such that

\[
\mathcal{A}_m^f(u, \eta) = \int_{0}^{1} \tilde{\lambda}_m(f(\eta)) \tilde{X}_F(t, u) + \tilde{X}_H^m(t, u)) - \int_{0}^{1} H(t, u)dt
\]

thus we obtain

\[
|\mathcal{A}_m^f(u, \eta) - f(\eta)| \leq c(H, m).
\]

Suppose \( \mathcal{A}_m^f(u, \eta) \in (a + c(H, m), b - c(H, m)) \) then

\[
a < f(\eta) < b.
\]

Since \( f \in \mathcal{F}(a) \), we conclude that

\[
a < \eta < b.
\]

For a given almost complex structure \( J \in \mathcal{J}_1 \), let \( \nabla_J \mathcal{A}_m^f \) be the gradient of \( \mathcal{A}_m^f \) with respect to the metric \( g_J(\cdot, \cdot) \) in (2.5). One can check that

\[
\nabla_J \mathcal{A}_m^f(u, \eta) = \left( -J(t, u) (\partial_t u - f(\eta) X_F(t, u) - X_H^m(t, u)) \right)
\]

Definition 3.5. A positive gradient flow line of \( \mathcal{A}_m^f \) with respect to an \( S^1 \)-parameterized almost complex structure \( J(t) \in \mathcal{J}_1 \) is a map \( w : \mathbb{R} \to \mathcal{L} \times \mathbb{R} \) which solves

\[
\partial_s w - \nabla_J \mathcal{A}_m^f = 0.
\]

The above map is interpreted as \( w = (u, \eta) \) where \( u : \mathbb{R} \times S^1 \to T^*N \times \mathbb{R} \), \( \eta : \mathbb{R} \to \mathbb{R} \) such that

\[
\begin{align*}
\partial_s u + J(t, u) (\partial_t u - X_H^m(t, u) - f(\eta) X_F(t, u)) &= 0 \\
\partial_s \eta + f'(\eta) \int_{0}^{1} F(t, u)dt &= 0.
\end{align*}
\]
3.3. Floer homology for $A_m^f$. Let us first assume that the perturbed $F$-Rabinowitz action functional $A_m^f : \mathcal{L} \times \mathbb{R} \to \mathbb{R}$ is Morse in the sense of Corollary \ref{cor:morse}. In order to define the Floer homology for $A_m^f$, we need to show that the Lagrange multiplier $\eta$ is uniformly bounded. We follow the same strategy as in the $A_m$-case with minor modifications.

**Lemma 3.6.** There exist $\epsilon, c' > 0$ such that if $(u, \eta) \in \mathcal{L} \times \mathbb{R}$ satisfies $\| \nabla_J A_m^f (u, \eta) \|_J \leq \epsilon f'(\eta)$ then

\[
\frac{2}{3} \left( A_m^f (u, \eta) - c' \| \nabla_J A_m^f (u, \eta) \|_J - c \right) \leq f(\eta) \leq 2 \left( A_m^f (u, \eta) + c' \| \nabla_J A_m^f (u, \eta) \|_J + c \right). \tag{3.6}
\]

Here $c = c(H, m)$ as in Definition \ref{def:uniformly_bound}.

**Proof.** The proof consists of 2 steps.

**Step 1:** There exist constants $\delta, c' > 0$ such that if $u \in \mathcal{L}$ satisfies

\[
u(t) \in U_\delta := F^{-1}(-\delta, \delta), \quad \forall t \in [0, \frac{1}{2}]
\]

then (3.6) holds.

There exist $\delta > 0$ such that

\[
\frac{1}{2} + \delta \leq \lambda(X_F(p)) \leq \frac{3}{2} - \delta, \quad \forall p \in U_\delta.
\]

Now we compute

\[
A_m^f (u, \eta) = \int_0^1 u^* \lambda - \int_0^1 H(t, u(t)) dt - f(\eta) \int_0^1 F(t, u(t)) dt + B_m(u(t))
\]

\[
= \int_0^1 \tilde{u}^* \tilde{\lambda}_m - \int_0^1 H(t, u(t)) dt - f(\eta) \int_0^1 F(t, u(t)) dt
\]

\[
= \int_0^1 \tilde{\lambda}_m(\tilde{u}(t)) \left[ \partial_t \tilde{u} - f(\eta) \tilde{X}_F(t, u) - \tilde{X}_H^m(t, u) \right] dt
\]

\[
+ f(\eta) \int_0^1 \lambda(u(t)) [X_F(t, u)] - F(t, u(t)) dt + \int_0^1 \tilde{\lambda}_m(\tilde{u}(t)) [\tilde{X}_H^m(t, u)] - H(t, u(t)) dt
\]

\[
\geq \left( \frac{1}{2} + \delta - \delta \right) f(\eta) - c' \| \partial_t u - X_H^{m(s)}(t, u) - f(\eta) X_F(t, u) \|_1 - c(H, m)
\]

\[
\geq \frac{1}{2} | f(\eta) | - c' \| \partial_t u - X_H^{m(s)}(t, u) - f(\eta) X_F(t, u) \|_2 - c(H, m)
\]

\[
\geq \frac{1}{2} | f(\eta) | - c' \| \nabla_J A_m(u, f(\eta)) \|_J - c(H, m),
\]
where $\tilde{\lambda}_m$ is the same as in Definition 3.3

$c' = c'(m, \delta) := \|\tilde{\lambda}_m|_{U_B}\|_\infty$. In a similar way, we get the following estimate

$$A^f_m(u, \eta) = \int_0^1 u^*\lambda - \int_0^1 H(t, u(t))dt - f(\eta)\int_0^1 F(t, u(t))dt + B_m(u(t))$$

$$= \int_0^1 \tilde{\lambda}_m - \int_0^1 H(t, u(t))dt - f(\eta)\int_0^1 F(t, u(t))dt$$

$$= \int_0^1 \lambda_m(\tilde{u}(t))|\partial_t\tilde{u} - m \tilde{X}_\eta(t, u) - \tilde{X}_\eta^m(t, u)dt$$

$$+ f(\eta)\int_0^1 \lambda(u(t))|\tilde{X}_\eta(t, u) - \tilde{X}_\eta^m(t, u)| - H(t, u(t))dt$$

$$\leq \left(\frac{3}{2} - \delta + \delta\right)f(\eta) + c'|\partial_tu - X_H^m(t, u) - f(\eta)X_F(t, u)|_1 + c(H, m)$$

$$\leq \frac{3}{2}|f(\eta)| + c'|\partial_tu - X_H^m(t, u) - f(\eta)X_F(t, u)|_2 + c(H, m)$$

$$\leq \frac{3}{2}|f(\eta)| + c'|\nabla J_m(u, f(\eta))|_J + c(H, m).$$

The above two estimates prove Step 1.

**Step 2:** For any $\delta > 0$ there exist $\epsilon > 0$ such that if $(u, \eta) \in L \times \mathbb{R}$

$$\|\nabla J A^f_m(u, \eta)\|_J \leq \epsilon f'(\eta)$$

then $u(t) \in U_\delta$ for all $t \in [0, \frac{1}{2}]$.

By a similar argument as in Lemma 2.9 Step 2, if $F(u(t)) \geq \frac{\delta}{2}$ for all $t \in [0, \frac{1}{2}]$ then

$$\|\nabla J A^f_m(u, \eta)\|_J \geq f'(\eta)\int_0^1 F(t, u(t))dt \geq f'(\eta)\frac{\delta}{2}.$$}

Now, if there exist $t_1, t_2$ in $[0, \frac{1}{2}]$ with $F(u(t_1)) \leq \frac{\delta}{2}$ and $F(u(t_2)) \geq \delta$ then

$$\|\nabla J A^f_m(u, \eta)\|_J \geq \frac{\delta}{2\|\nabla F\|_\infty}.$$}

If we set

$$\epsilon = \epsilon(\delta, F) := \min\left\{\frac{\delta}{2}, \frac{\delta}{2\|\nabla F\|_\infty}\right\}$$

and use the fact that $f'(\eta) \leq 1$ for all $\eta \in \mathbb{R}$ then this proves Step 2.

By combining Step 1 and Step 2, we immediately prove the lemma. \(\square\)

We need one more preparation. Now we consider a certain class of $f \in F(a)$ with the following condition.

**Definition 3.7.** Given $a, r > 0$,

$$F(a, r) := \{f \in F(a) : \exists A > 0 \text{ such that } Af'(-A) > r\}. \tag{3.7}$$

**Remark 3.8.** Given $a > 0$, the set $\bigcap_{r > 0} F(a, r)$ is non-empty and path-connected. An explicit construction of $f \in \bigcap_{r > 0} F(a, r)$ exists. There also exists a homotopy between two different $f_0, f_1 \in F(a, r)$. All these things are explained in [30] Remark 3.24, Lemma 3.25].
Proposition 3.9. Fix $F \in \mathcal{D}(\Sigma)$ and an action window $(a, b)$ such that $0 < a < b < \infty$. Let $c', \varepsilon > 0$ be the constants from Lemma 3.6. Choose $f \in \mathcal{F}(\frac{a}{b}, \frac{b-a}{\min\{c', a/4\}})$ and a generic pair $(H, m)$ such that $c(H, m) \leq \frac{a}{2}$. Let $w_{\pm} \in \text{Crit}^{(a,b)}(A_{m}^{f})$ and $w = (u, \eta)$ be a gradient flow line of $A_{m}^{f}$ with $\lim_{s \to \pm \infty} w(s) = w_{\pm}$. Then there exists a constant $\kappa = \kappa(a, b)$ satisfying $\|\eta\|_{\infty} \leq \kappa$.

Proof. For convenience, set
\[
\epsilon_1 := \min \left\{ \epsilon, \frac{a}{4c'} \right\}.
\]
First define a function $\nu_{w} : \mathbb{R} \to [0, \infty)$ for a given gradient flow line $w = (u, \eta)$ by
\[
\nu_{w}(l) := \inf \{ \nu \geq 0 : \|\nabla J A_{m}^{f}(w(l + \nu))\|_{J} \leq \epsilon_1 f'(\eta(l + \nu)) \}.
\]
Since $\lim_{s \to \infty} f'(\eta(s)) = 1$ and $\lim_{s \to \infty} \|\nabla J A_{m}^{f}(u, \eta(s))\|_{J} = 0$, $\nu_{w}$ is well-defined. We get the following estimate
\[
b - a \geq \lim_{s \to \infty} A_{m}^{f}(w(s)) - \lim_{s \to -\infty} A_{m}^{f}(w(s))
= \int_{-\infty}^{t_{l+\nu_{w}(l)}} \|\nabla J A_{m}^{f}(w(s))\|_{J}^{2} ds
\geq \int_{t}^{t_{l+\nu_{w}(l)}} \epsilon_1^2 f'(\eta(s))^2 ds
\geq \nu_{w}(l) \epsilon_1^2 i_{w}(l)^2,
\]
where $i_{w}(l) := \inf_{t \leq s \leq t_{l+\nu_{w}(s)}} f'(\eta(s))$. Hence we obtain
\[
\nu_{w}(l) \leq \frac{b - a}{\epsilon_1^2 i_{w}(l)^2}.
\]
Now observe that
\[
\left| \int_{t}^{t_{l+\nu_{w}(l)}} \eta(s) ds \right| \leq \int_{t}^{t_{l+\nu_{w}(l)}} |\eta(s)| ds
\leq \left( \nu_{w}(l) \int_{t}^{t_{l+\nu_{w}(l)}} |\eta(s)|^{2} ds \right)^{1/2}
\leq \left( \nu_{w}(l) \int_{t}^{t_{l+\nu_{w}(l)}} \|\nabla J A_{m}^{f}(w(s))\|_{J}^{2} ds \right)^{1/2}
\leq (\nu_{w}(l) E(w))^{1/2}
\leq \frac{b - a}{\epsilon_1 i_{w}(l)}.
\]
By Lemma 3.6 we get the following estimate for any \( l \in \mathbb{R} \)
\[
f[\eta(l + \nu_w(l))] \geq \frac{2}{3} \left( A_m^f[w(l + \nu_w(l))] - c' \| \nabla_J A_m^f(u, \eta) \|_J - c(H, m) \right) \leq \frac{a}{2}
\]
\[
\geq \frac{2}{3} \left( a - c' \epsilon_1 f'[\eta(l + \nu_w(l))] \right) - \frac{a}{2}
\]
\[
\geq \frac{a}{6}.
\]
Since \( f \in \mathcal{F}(\frac{a}{6}) \), we get
\[
\eta(l + \nu_w(l)) \geq \frac{a}{6},
\]
and hence
\[
\eta(l) \geq \eta(l + \nu_w(l)) - \int_l^{l + \nu_w(l)} \dot{\eta}(s) ds
\]
\[
\geq \frac{a}{6} - \frac{b - a}{\epsilon_1 i_w(l)}
\]
\[
> - \frac{b - a}{\epsilon_1 i_w(l)}.
\]
This implies
\[
f'(\eta(l)) \eta(l) \geq i_w(l) \eta(l) \geq - \frac{b - a}{\epsilon_1}.
\]
Now suppose that there exists \( l_0 \in \mathbb{R} \) such that \( \eta(l_0) < -A \) then there must be \( l_1 \in \mathbb{R} \) with \( \eta(l_1) = -A \). This induces the following contradiction by the choice of \( f \in \mathcal{F}(\frac{a}{6}, \frac{b-a}{\epsilon_1}) \) with \( f \in \mathcal{F}(\frac{a}{6}, \frac{b-a}{\epsilon_1}) \) with (3.7),
\[
- \frac{b - a}{\epsilon_1} > -f'(-A)A = f'(\eta(l_1)) \eta(l_1) > - \frac{b - a}{\epsilon_1}.
\]
So, we conclude that \( \eta(l) > -A \) for all \( l \in \mathbb{R} \).

Now consider the upper bound. Start with a new function \( \tilde{\nu}_w : \mathbb{R} \rightarrow [0, \infty) \) by
\[
\tilde{\nu}_w(l) := \inf \{ \nu \geq 0 : \| \nabla_J A_m^f[w(l + \nu)] \|_J \leq \epsilon_1 f'(-A) \}.
\]
By a similar argument as in (3.8) and (3.9), we see that
\[
\tilde{\nu}_w(l) \leq \frac{b - a}{\epsilon_1^2 f'(-A)^2}
\]
and
\[
|\eta(l) - \eta(l + \tilde{\nu}_w(l))| < \frac{b - a}{\epsilon_1 f'(-A)} < A
\]
where the last inequality comes from (3.7) again. By Lemma 3.6 we get
\[
f[\eta(l + \tilde{\nu}_w(l))] \leq 2 \left( A_m^f[w(l + \tilde{\nu}_w(l))] + c' \| \nabla_J A_m^f[w(l + \tilde{\nu}_w(l))] \|_J + c(H, m) \right)
\]
\[
\leq 2(b + c' \epsilon_1 f'(-A) + \frac{a}{2}) \leq \frac{a}{4} \leq 1
\]
\[
< 2a + 2b.
\]
This implies that \( \eta(l + v_w(l)) < 2a + 2b \) and by (3.10)
\[
\eta(l) < 2a + 2b + A.
\]
Thus we conclude that
\[
\|\eta\|_\infty < \kappa := 2a + 2b + A.
\]
\[\square\]

For simplicity, let us denote by
\[
A(A^f_m) := \{A^f_m(w) : w \in \text{Crit}(A^f_m)\}.
\]

**Theorem 3.10.** Fix \( F \in \mathcal{D}(\Sigma) \) and \( f \in \bigcap_{r>0} \mathcal{F}^b(\Sigma, r) \), see Definition 3.7. Choose a generic pair \((H, m)\). If \( \max\{1, 2c(H, m)\} < a < b \leq \infty \) and \( a, b \notin A(A^f_m) \), then \( \text{FH}^{(a,b)}(A^f_m) \) is well-defined.

The construction of \( \text{FH}^{(a,b)}(A^f_m) \) is the same as in the \( A_m \)-case. For \( w = (u, \eta) \in \text{Crit}^{(a,b)}(A^f_m) \), we define the index \( \mu(w) := \mu_{CZ}(u) \). Let us denote by
\[
\text{Crit}^{(a,b)}_k(A^f_m) := \{w \in \text{Crit}^{(a,b)}(A^f_m) : \mu(w) = k\};
\]
\[
\text{FC}^{(a,b)}_k(A^f_m) := \text{Crit}^{(a,b)}(A^f_m) \otimes \mathbb{Z}_2.
\]
For a generic almost complex structure \( J(t) \in \mathcal{J}_m \) and given \( w \in \text{Crit}^{(a,b)}(A^f_m) \), we define
\[
\widehat{\mathcal{M}}(w_-, w_+) := \{w(s) : w \text{ satisfies (3.5)} \}
\]
\[
\mathcal{M}(w_-, w_+) := \frac{\widehat{\mathcal{M}}(w_-, w_+)}{\mathbb{R}}.
\]
The above \( \mathbb{R} \)-action is given by translating the \( s \)-coordinate. Suppose further that the almost complex structure \( J(t) \) is generic, so that \( \mathcal{M}(w_-, w_+) \) is a smooth manifold of dimension
\[
\text{dim} \mathcal{M}(w_-, w_+) = \mu(w_-) - \mu(w_+) - 1.
\]
The boundary operator \( \partial : \text{FC}^{(a,b)}_k(A^f_m) \to \text{FC}^{(a,b)}_{k-1}(A^f_m) \) is defined by
\[
\partial w_- := \sum_{\mu(w_+) = k-1} \#_2 \mathcal{M}(w_-, w_+) w_+,
\]
where \( \#_2 \) means \( \mathbb{Z}_2 \)-counting. By virtue of Proposition 3.9 with Theorem 2.7, \( \partial \) satisfies \( \partial \circ \partial = 0 \). Then the resulting filtered Floer homology group is
\[
\text{FH}^{(a,b)}_*(A^f_m) = \text{H}_*(\text{FC}^{(a,b)}_*(A^f_m), \partial).
\]

### 3.4. Continuation map between \( \text{FH}(A^f) \) and \( \text{FH}(A^f_m) \)

In this section we construct a continuation homomorphism between \( \text{FC}(A^f) \) and \( \text{FC}(A^f_m) \) which induces a map on homologies on a suitable action window. The construction is given by counting gradient flow lines of the \( s \)-dependent action functional
\[
A^f_m(s)(u, \eta) := A^f(u, \eta) + \gamma(s) \mathcal{B}_m(u).
\]
Here $A^f(u, \eta) = \int_0^1 u^* \lambda - f(\eta) F(t, u(t)) dt - \int_0^1 H(t, u(t)) dt$ and $m(s)$ is defined in (2.8).

With the same metric as in (2.9), the gradient flow line $w = (u, \eta) \in C^\infty(\mathbb{R} \times S^1, T^*N) \times C^\infty(\mathbb{R}, \mathbb{R})$ satisfies

$$\begin{align*}
\partial_s u + J(s, t, u)(\partial_t u - f(\eta) X_F(t, u) - X_H^m(t, u)) &= 0 \\
\partial_s \eta + f'(\eta) \int_0^1 F(t, u) dt &= 0.
\end{align*}$$

(3.11)

In order to construct a continuation map, we need to check that the energy $\int_0^\infty \|\partial_s w\|_s^2 ds$ and the Lagrange multiplier $\eta$ of gradient flow lines $w$ are uniformly bounded. As in the $A_m$ case, we start with the fundamental lemma.

**Lemma 3.11.** There exist $\overline{c}, \overline{c}' > 0$ such that if $(u, \eta) \in \mathcal{L} \times \mathbb{R}$ satisfies

$$\|\nabla_s A^f_{m(s)}(u, \eta)\|_s \leq \overline{c} f'(\eta)$$

then

$$\frac{2}{3} \left( A^f_{m(s)}(u, \eta) - \overline{c}'\|\nabla_s A^f_{m(s)}(u, \eta)\|_s - \overline{c} \right) \leq f(\eta) \leq 2 \left( A^f_{m(s)}(u, \eta) + \overline{c}'\|\nabla_s A^f_{m(s)}(u, \eta)\|_s + \overline{c} \right).$$

Here

$$\overline{c} = \overline{c}(H, m) := \sup_{s \in \mathbb{R}} \sup_{(t, u) \in S^1 \times \mathcal{L}} \left| \int_0^1 \lambda_m(s)(\overline{u}(t)) |\dot{X}_H^m(s)(t, u)| - H(t, u(t)) dt \right|.$$

**Proof.** The proof is similar as in Lemma 3.10 with $m(s)$ instead of $m$. So we omit the proof.

With a simple computation, one checks that

$$\overline{c} = \overline{c}(\delta, F) := \min \left\{ \frac{\delta}{2}, \frac{\delta}{2\|\nabla F\|_\infty} \right\}$$

and

$$\overline{c}' = \overline{c}'(m, \delta) := \sup_{s \in \mathbb{R}} \|\overline{\lambda}_m(s)|\dot{u}_\delta\|_\infty.$$

Here $\delta$ is chosen satisfying

$$\frac{1}{2} + \delta \leq \lambda(X_F(p)) \leq \frac{3}{2} - \delta, \quad \forall p \in U_\delta.$$

\[\square\]

**Proposition 3.12.** Fix $F \in \mathcal{D}(\Sigma)$ and an action window $(a, 2a)$ such that $a \geq 2$. Let $\overline{c}'$, $\overline{c} > 0$ be the constants from Lemma 3.11. Choose $f \in \mathcal{F}(\frac{a}{2}, \min\{2a, \min\{2a, \frac{2a+1}{\min\{2a, 2a\} - 1}\}\})$ and a generic pair $(H, m)$ such that $c(H, m) \leq \frac{a}{2}$. Let $w$ be a gradient flow line of $A^f_{m(s)}$ with the following asymptotic conditions

$$\lim_{s \to -\infty} w(s) = w_- \in \text{Crit}^{(a, 2a)}(A^f_{m(0)}), \quad \lim_{s \to \infty} w(s) = w_+ \in \text{Crit}^{(a, 2a)}(A^f_{m(1)}).$$

If $\|\beta\|_\infty$ is sufficiently small then the $L^\infty$-norm of $\eta$ is uniformly bounded in terms of a constant which only depends on $w_-$, $w_+$.

**Proof.** The proof consists of 4 steps.

**Step 1:** The energy is bounded by $\|f(\eta)\|_\infty$.

By a similar argument as in Proposition 2.14 Step 1, we obtain

$$\int_{-\infty}^\infty \|A^f_{m(s)}(w(s))\| ds \leq 2CE(w) + 2C + 2d_m C + 2\|f(\eta)\|_\infty d_F C$$

(3.12)
and
\[ E(w) \leq 2\Delta + 4C + 4d_mC + 4\|f(\eta)\|_\infty d_FC, \]  
(3.13)
under the smallness condition on the isoperimetric constant
\[ C < \frac{1}{4}. \]  
(3.14)

For convenience, we summarize the notations as follows
\[ E(w) = \int_{-\infty}^{\infty} \|\partial_s w(s)\|^2 ds; \]
\[ C = \|\beta\|_\infty \|\theta\|_\infty; \]
\[ d_m = d_{H,m} = \sup_{s \in \mathbb{R}} \|X^m(s)\|_\infty; \]
\[ d_F = \|X_F\|_\infty; \]
\[ \Delta = A^f_{m(1)}(w_+) - A^f_{m(0)}(w_-). \]

If we choose \( \beta \in \mathfrak{B} \) with small norm \( \|\beta\|_\infty \) then we may assume that \( C \) is sufficiently small.

The smallness of \( C \) is important in the following steps.

**Step 2:** \( \eta(s) \) is uniformly bounded from above.

In this step, without loss of generality, we work on the region that \( \eta(s) \geq \frac{a}{6} \). Since \( f \in \mathcal{F}(\frac{a}{6}) \), \( f(\eta(s)) = \eta(s) \) and \( f'(\eta(s)) = 1 \). Then Lemma 3.11 implies the following:

There exist \( \epsilon, c, c' > 0 \) such that if \( (u, \eta) \in \mathcal{L} \times \mathbb{R} \) satisfies
\[ \|\nabla_s A^f_{m(s)}(u, \eta)\|_s \leq \epsilon \]
then
\[ f(\eta) \leq 2 \left( A^f_{m(s)}(u, \eta) + c' \|\nabla_s A^f_{m(s)}(u, \eta)\|_s + \epsilon \right), \]  
(3.15)
for all \( s \in \mathbb{R} \) satisfying \( \eta(s) \geq \frac{a}{6} \). Here \( \epsilon, \overline{c}, \overline{c}' > 0 \) come from Lemma 3.11.

Now define
\[ \overline{\nu}_w(l) := \inf \{ \overline{\nu} \geq 0 : \|\nabla_s A^f_{m(l+\overline{\nu})}(w(l + \overline{\nu}))\|_s < \overline{\nu} \}, \]
for \( l \in \mathbb{R} \) such that \( \eta(l) \geq \frac{a}{6} \). Then by a similar argument as in (2.17), we obtain the following estimate
\[ \overline{\nu}_w(l) \leq \frac{E(w)}{\overline{c}^2}. \]  
(3.16)
By the gradient flow equation \( (3.11) \) and \( (3.16) \), we have
\[
\left| \int_{l}^{l + \overline{w}(l)} \partial_{s} f(\eta(s)) ds \right| \leq \left| \int_{l}^{l + \overline{w}(l)} f'(\eta(s)) \partial_{s} \eta(s) ds \right| \leq 1 \leq \left| \int_{l}^{l + \overline{w}(l)} \partial_{s} \eta(s) ds \right| \leq \left| \int_{l}^{l + \overline{w}(l)} f'(\eta(s)) \int_{0}^{1} F(t, u) dt \ ds \right| \leq \left\| F \right\|_{\infty} \left\| \overline{w}(l) \right\| \leq \left| \int_{l}^{l + \overline{w}(l)} \partial_{s} \eta(s) ds \right| \leq \left| \int_{l}^{l + \overline{w}(l)} f'(\eta(s)) \int_{0}^{1} F(t, u) dt \ ds \right| \leq \left\| F \right\|_{\infty} \frac{E(w)}{\varepsilon^2}.
\]

Let us note that the following inequality holds for all \( s \in \mathbb{R} \)
\[
a - \int_{-\infty}^{\infty} |\hat{A}_{m}(w(s))| \ ds \leq \hat{A}_{m}(w(s)) \leq 2a + \int_{-\infty}^{\infty} \left| \hat{A}_{m}(w(s)) \right| \ ds. \tag{3.18}
\]

By the definition of \( \overline{w}(l) \) and the above estimates \( (3.15), (3.18) \) and \( (3.12) \) we get
\[
f[\eta(l + \overline{w}(l))] \leq 2 \left( \hat{A}_{m(l+\overline{w}(l))}[w(l + \overline{w}(l))] + \overline{\varepsilon} \left\| \nabla A_{m(l+\overline{w}(l))}[w(l + \overline{w}(l))] \right\| \right) \leq \overline{\varepsilon} \leq 2 \left( 2a + \int_{-\infty}^{\infty} |\hat{A}_{m}(w(s))| \ ds + \overline{\varepsilon} \overline{\varepsilon} + \overline{\varepsilon} \right) \tag{3.19}
\]

Now combine \( (3.17) \) and \( (3.19) \), we then obtain
\[
f(\eta(l)) \leq f[\eta(l + \overline{w}(l))] + \int_{l}^{l + \overline{w}(l)} \partial_{s} f(\eta(s)) ds \leq 2 \left( 2a + 2CE(w) + 2C + 2d_{m}C + 2\left\| f(\eta) \right\|_{\infty} d_{F} C + \overline{\varepsilon} \overline{\varepsilon} + \overline{\varepsilon} \right) \leq \left( 16Cd_{F} + 4d_{F} + \frac{4\left\| F \right\|_{\infty} d_{F}}{\varepsilon^2} \right) C \left\| f(\eta) \right\|_{\infty} \leq 16Cd_{F} + 4d_{F} + \frac{4\left\| F \right\|_{\infty} d_{F}}{\varepsilon^2} \right) C \left( 2C + 4 + 2d_{m}C \right) \leq \frac{1}{2} \tag{3.20}
\]

where for the last inequality we use \( (3.13) \). Note that the last line of the above estimate \( (3.20) \) does not depend on the choice of a gradient flow line \( w \) and \( l \in \mathbb{R} \). If we choose a sufficiently small \( C > 0 \) such that
\[
\left( 16Cd_{F} + 4d_{F} + \frac{4\left\| F \right\|_{\infty} d_{F}}{\varepsilon^2} \right) C \leq \frac{1}{2} \tag{3.21}
\]
If we choose a sufficiently small isoperimetric constant $C > \frac{a}{8}$, then \( \nu \) and define a function

\[
\nu = \min \left\{ \bar{r}, \frac{a}{8\bar{r}} \right\},
\]

and define a function \( \nu_w : \mathbb{R} \to \mathbb{R}^+ \) by

\[
\nu_w(l) := \inf \{ \nu \geq 0 : \| \nabla_s A^f_{m(l + \nu)}(w(l + \nu)) \|_s < \xi f'(\nu(l + \nu)) \}.
\]

**Step 3**: If \( \| \beta \|_{\infty} \) is sufficiently small then \( \int_{-\infty}^{\infty} |A^f_{m(s)}(w(s))| \, ds \leq \frac{a}{8} \).

The above estimates (3.12), (3.13), (3.22) and \( \Delta < a \) imply that

\[
\int_{-\infty}^{\infty} |A^f_{m(s)}| \, ds \leq CE(w) + 2C + 2d_m C + 2 \| f(\eta) \|_{\infty} \, df \cdot C
\]

\[
\leq (8d_F C^2 + 2d_F C) \| f(\eta) \|_{\infty} + 4aC + 8C^2 + 8d_m C^2 + 2C + 2d_m C
\]

\[
\leq 8(4d_F C^2 + d_F C) \left( 2a + 4aC + 8C^2 + 8C^2 d_m + 2C + 2d_m C + \bar{r}' \bar{r} + \bar{r} \right)
\]

\[
+ \| F \|_{\infty} \left( a + 2C + 2d_m C \right) \right) + 4aC + 8C^2 + 8d_m C^2 + 2C + 2d_m C
\]

\[
= \left( 8(4d_F C^2 + d_F C)(2 + 4C + \| F \|_{\infty} \) + 4C \right) a
\]

\[
+ 8(4d_F C^2 + d_F C) \left( 8C^2 + 8C^2 d_m + 2C + 2d_m C + \bar{r}' \bar{r} + \bar{r} \right)
\]

\[
+ \| F \|_{\infty} \left( 2C + 2d_m C \right) \right) + 8C^2 + 8d_m C^2 + 2C + 2d_m C.
\]

If we choose a sufficiently small isoperimetric constant $C > 0$ such that

\[
8(4d_F C^2 + d_F C)(2 + 4C + \| F \|_{\infty} \) + 4C \leq \frac{1}{16};
\]

\[
8(4d_F C^2 + d_F C) \left( 8C^2 + 8C^2 d_m + 2C + 2d_m C + \bar{r}' \bar{r} + \bar{r} \right)
\]

\[
+ \| F \|_{\infty} \left( 2C + 2d_m C \right) \right) + 8C^2 + 8d_m C^2 + 2C + 2d_m C \leq \frac{1}{8},
\]

then we get the following estimate

\[
\int_{-\infty}^{\infty} |A^f_{m(s)}| \, ds \leq \frac{a}{16} + \frac{1}{8} \leq \frac{a}{8},
\]

where for the last inequality we use $a \geq 2$. This proves Step 3.

**Step 4**: \( \eta(s) \) is uniformly bounded.

First set

\[
\xi := \min \left\{ \bar{r}, \frac{a}{8\bar{r}} \right\},
\]

and define a function \( \nu_w : \mathbb{R} \to \mathbb{R}^+ \) by

\[
\nu_w(l) := \inf \{ \nu \geq 0 : \| \nabla_s A^f_{m(l + \nu)}(w(l + \nu)) \|_s < \xi f'(\nu(l + \nu)) \}.
\]
Now set
\[ \nu_w(l) := \inf_{l \leq s \leq l + \nu_w(l)} f'(\eta(s)). \]
By similar arguments as in (3.8) and (3.9), we obtain the following estimates
\[ \nu_w(l) \leq \frac{E(w)}{\xi^2} \]
and
\[ |\eta(l) - \eta(l + \nu_w(l))| \leq \frac{E(w)}{\xi \nu_w(l)}. \]
By the definition of \( \nu_w \), Lemma 3.11 implies that for any \( l \in \mathbb{R} \)
\[
\begin{align*}
f[\eta(l + \nu_w(l))] &\geq \frac{2}{3} \left( A_{m[l+\nu_w(l)]}^f[w(l + \nu_w(l))] - \overline{c}' \| \nabla \eta A_{m[l+\nu_w(l)]}^f[w(l + \nu_w(l))] \|_s - \overline{c} \right) \\
&\geq \frac{2}{3} \left( A_{m[l+\nu_w(l)]}^f[w(l + \nu_w(l))] - \overline{c}' \frac{f'(\eta(l + \nu_w(l)) - \frac{a}{2})}{\leq 1} \right) \\
&\geq \frac{2}{3} \left( A_{m[l+\nu_w(l)]}^f[w(l + \nu_w(l))] - \frac{5}{8} a \right).
\end{align*}
\] (3.24)
The action estimate (3.18) and Step 3 give us the following estimate
\[
A_{m(s)}^f(w(s)) \geq a - \int_{-\infty}^{\infty} \left| A_{m(s)}^f(w(s)) \right| ds \geq \frac{7}{8} a.
\] (3.25)
Let us combine (3.24), (3.25) to obtain
\[
f[\eta(l + \nu_w(l))] \geq \frac{2}{3} \left( A_{m[l+\nu_w(l)]}^f[w(l + \nu_w(l))] - \frac{5}{8} a \right) \geq \frac{a}{6}.
\]
Since \( f \in \mathcal{F}(\frac{a}{6}) \),
\[
\eta(l + \nu_w(l)) \geq \frac{a}{6} > 0,
\]
and hence
\[
\eta(l) \geq \frac{a}{6} - \frac{E(w)}{\xi \nu_w(l)} \geq -\frac{E(w)}{\xi \nu_w(l)}.
\]
As a consequence,
\[
-f'(\eta(l))\eta(l) \leq -\nu_w(l)\eta(l) \leq \frac{E(w)}{\xi} \leq \frac{1}{\xi} (2\Delta + 4C + 4d_mC + 4\kappa d_F C),
\]
where the last inequality comes from (3.13) and (3.22). If we choose \( C \) sufficiently small such that
\[
(4 + 4d_m + 4\kappa d_F)C \leq 1,
\] (3.26)
then
\[
-f'(\eta(l))\eta(l) \leq -\frac{2a + 1}{\xi},
\]
here we use again \( \Delta < a \). Since \( f \in \mathcal{F}(\frac{a}{6}, \frac{2\Delta + 1}{\xi}) \), there exists \( A > 0 \) such that
\[
A f'(-A) > \frac{2a + 1}{\xi}.\]
Now suppose that there exists \( l_0 \in \mathbb{R} \) such that \( \eta(l_0) < -A \) then by continuity there exists \( l_1 \in \mathbb{R} \) such that \( \eta(l_1) = -A \) which leads to a contradiction via condition (3.7)

\[
\frac{2a + 1}{\xi} < f'(-A)A = -f'(\eta(l_1))\eta(l_1) < \frac{2a + 1}{\xi}.
\]

Thus we conclude that \( \eta(l) > -A \) for all \( l \in \mathbb{R} \), and hence

\[
\| \eta(l) \|_\infty \leq \kappa := \max\{ \bar{\kappa}, A \}.
\]

**Lemma 3.13.** Fix \( F \in D(\Sigma) \) and an action window \((a, 2a)\) such that \( a \geq 2 \). Let \( \bar{\kappa}' \), \( \bar{\kappa} > 0 \) be the constants from Lemma 3.11. Choose \( f \in F(\frac{a}{6}, \frac{2a + 1}{\min\{2\kappa/a, \sqrt{\kappa}\}}) \) and a generic pair \((H, m)\) such that \( c(H, m) \leq \frac{a}{2} \). Let \( w \) be a gradient flow line of \( A^f_{m(s)} \) with the following asymptotic conditions:

\[
\lim_{s \to -\infty} w(s) = w_- \in \text{Crit}^{(a,2a)}(A^f_{m(0)}), \quad \lim_{s \to \infty} w(s) = w_+ \in \text{Crit}^{(a,2a)}(A^f_{m(1)}).
\]

If \( \| \beta \|_\infty \) is sufficiently small, then

\[
A^f_{m(1)}(w_+) \geq \frac{9}{10} A^f_{m(0)}(w_-) - \frac{1}{10}.
\]

**Proof.** For notational simplicity, Let us denote by

\[
p = A^f_{m(0)}(w_-), \quad q = A^f_{m(1)}(w_+).
\]

By Step 2 in Proposition 3.12, \( f(\eta) \) is uniformly bounded as follows,

\[
\| f(\eta) \|_\infty \leq 2 \left( 2q + 8C(q - p) + 16C^2 + 16C^2d_m + 4C + 4d_mC + 2\bar{\kappa}'\bar{\kappa} + 2\bar{\kappa} \right.
\]

\[
+ \frac{\| F \|_\infty}{\varepsilon^2} (2(q - p) + 4C + 4d_mC) \bigg) .
\]

Since \( E(w) \geq 0 \), we obtain the following inequality from (3.13)

\[
q \geq p - 2C - 2d_mC - 2 \| f(\eta) \|_\infty d_FC.
\]

By taking a small isoperimetric constant \( C > 0 \) satisfying

\[
8d_F \left( \frac{\| F \|_\infty}{\varepsilon} + 4C + 1 \right) C \leq \frac{1}{9};
\]

\[
2 \left( 1 + d_m + 8d_F \frac{\| F \|_\infty}{\varepsilon^2} C + 8d_F d_mC \frac{\| F \|_\infty}{\varepsilon^2} C \right)
\]

\[
+ 32d_F C^2 + 32d_F d_mC^2 + 8d_FC + 8d_F d_mC + 4\bar{\kappa}'\bar{\kappa} d_F + 4\bar{\kappa} d_F \bigg) C \leq \frac{1}{9};
\]

(3.27)
we now get
\[ q \geq p - 2C - 2d_m C - 2\|f(\eta)\|_\infty d_F C \]
\[ \geq p - 2C - 2d_mC - 4d_F C \left( \frac{\|F\|_\infty}{\tau^2} (2q - p) + 4C + 4d_mC \right) \]
\[ + 2q + 8C(q - p) + 16C^2 + 16C^2 d_m + 4C + 4d_mC + 2\tau C \]
\[ = p + 8d_F \left( \frac{\|F\|_\infty}{\tau^2} + 4C \right) C \frac{p - 8d_F \left( \frac{\|F\|_\infty}{\tau} + 4C + 1 \right) C q - 2 \left( 1 + d_m + 8d_F \frac{\|F\|_\infty}{\tau^2} C \right) + 8d_F d_m C + 32d_F C^2 + 8d_F C + 8d_F d_mC C^2 + 4\tau C d_F + 4\tau d_F \right) C \]
\[ \geq p + 8d_F \left( \frac{\|F\|_\infty}{\tau^2} + 4C \right) C \frac{p - 1}{9} q - \frac{1}{9} \]
\[ \geq p - \frac{1}{9} q - \frac{1}{9}. \]

This proves the assertion. \(\square\)

For convenience, let us abbreviate
\[ h[p] := \frac{9}{10} p - \frac{1}{10}. \]

**Lemma 3.14.** Fix \( F \in \mathcal{D}(\Sigma) \) and an action window \((a, 2a)\) such that \( a \geq 2 \). Let \( \tau', \tau > 0 \) be the constants from Lemma 3.11. Choose \( f \in \mathcal{F}(\frac{a}{\min(\tau', \tau)}, \frac{2a + 1}{8\tau'}) \) and a generic pair \((H, m)\) such that \( c(H, m) \leq \frac{a}{2} \). If \( \|\beta\|_\infty \) is sufficiently small then there exists a commutative diagram:

\[
\begin{array}{ccc}
\text{FH}^{(h^{-2}[a], 2a)}(A_{m(0)}^f) & \xrightarrow{i(h^{-2}[a], h^{2}[2a])} & \text{FH}^{(a, h^{2}[2a])}(A_{m(0)}^f) \\
\downarrow \Phi_{m} & & \downarrow \Phi_{m} \\
\text{FH}^{(h^{-1}[a], h^{2}[2a])}(A_{m(1)}^f) & \xrightarrow{i(h^{-1}[a], h^{2}[2a])} & \text{FH}^{(a, h^{2}[2a])}(A_{m(1)}^f)
\end{array}
\]

**Proof.** Let us first construct \( \tilde{\Phi}_{m}^f \). Let \( w \) be the gradient flow line of \( A_{m(s)}^f \) satisfying the limit conditions:

\[ \lim_{s \to -\infty} w(s) = w_- \in \text{Crit}^{(h^{-2}[a], 2a)}(A_{m(0)}^f), \quad \lim_{s \to \infty} w(s) = w_+ \in \text{Crit}^{(h^{-1}[a], h^{2}[2a])}(A_{m(1)}^f). \]

Let \( M^m(w_-, w_+) \) be the set of such gradient flow lines. If \( \mu(w_-) = \mu(w_+) \), then we may assume that \( M^m(w_-, w_+) \) is discrete for a generic almost complex structure \( J(s, t) \in \mathcal{J}_{m(s)} \). We now define a map

\[ \Phi_{m}^f : FC_*(A_{m(0)}^f) \to FC_*(A_{m(1)}^f) \]

given by

\[ \Phi_{m}^f(w_-) = \sum_{\mu(w_+)=\mu(w_-)} \#M^m(w_-, w_+) w_+. \]
Since \( \omega_{m(s)} \) is symplectically aspherical for all \( s \in \mathbb{R} \), there is no bubbling. In order to compactify the moduli space \( \mathcal{M}(w_-, w_+) \), it suffices to bound the energy \( E(w) = \int_{-\infty}^{\infty} \| w \|_s^2 ds \) and the Lagrange multiplier \( \eta \) in terms of \( w_-, w_+ \). Since \( \| \beta \|_\infty \) is small, we may choose a sufficiently small isoperimetric constant \( C \) satisfying (3.14), (3.21), (3.23), (3.26). Now we can use the argument of Proposition 3.12. Especially (3.13), (3.22) give us the following uniform energy bound

\[
E(w) \leq 2A_m(1)(w_+) - 2A_m(0)(w_-) + 4C + 4d_mC + 4\|f(\eta)\|_\infty d_FC
\]

and Proposition 3.12 enables us to conclude that the Lagrange multiplier \( \eta \) is also uniformly bounded. Let us choose a smaller \( \| \beta \|_\infty \) such that the isoperimetric constant \( C \) satisfy (3.27) additionally. By virtue of Lemma 3.13 we obtain the following map.

\[
\Phi_m : F\mathcal{C}(h^{-2}[a], 2a)(A_m(0)) \rightarrow F\mathcal{C}(h^{-1}[a], h[2a])(A_m(1)).
\]

Since the continuation map \( \Phi_m \) commutes with the boundary operators, this induces the following homomorphism on homologies as follows

\[
\tilde{\Phi}_m : F\mathcal{H}(h^{-2}[a], 2a)(A_m(0)) \rightarrow F\mathcal{H}(h^{-1}[a], h[2a])(A_m(1)).
\]

Now we consider the inverse homotopy of \( A_m(s) \). By modifying the above construction, we obtain

\[
\tilde{\Phi}_m : F\mathcal{H}(h^{-1}[a], h[2a])(A_m(1)) \rightarrow F\mathcal{H}(a, h^2[2a])(A_m(0)).
\]

By a homotopy-of-homotopies argument, we conclude that \( \tilde{\Phi}_m \circ \Phi_m \) is the identity map on \( F\mathcal{H}(h^{-2}[a], h^2[2a])(A_m(0)) \). This proves the lemma. \( \square \)

**Lemma 3.15.** Fix \( F \in \mathcal{D}(\Sigma) \) and \( f \in \bigcap_{r>0} \mathcal{F}(\frac{1}{r}, r) \), see Definition 3.7. If \((H, m)\) is a generic pair with sufficiently small \( \| \beta \|_\infty \), then

\[
\dim F\mathcal{H}(h^{-1}[a], h[T])(A_m) \geq \frac{1}{4} \dim F\mathcal{H}(h^{-2}[a], h^2[T])(A_m) \tag{3.29}
\]

also holds for generic \( a, T \) such that \( \max\{2, 2c(H, m)\} < a < T < \infty \).

**Proof.** By the commutative diagram in Lemma 3.14 we obtain the following dimension estimate

\[
\dim F\mathcal{H}(h^{-1}[a], h[2a])(A_m) \geq \dim F\mathcal{H}(h^{-2}[a], h^2[2a])(A_m) \geq \text{rank} \left( i(h^{-2}[a], h^2[2a]) \right) \geq \dim F\mathcal{H}(h^{-1}[a], h^2[2a])(A_m). \tag{3.30}
\]

Actually if we choose \( b \in \mathbb{R} \) such that \( a < b < 2a \) then

\[
\dim F\mathcal{H}(h^{-1}[a], h[b])(A_m) \geq \dim F\mathcal{H}(h^{-2}[a], h^2[b])(A_m)
\]

holds under the generic condition \( h[b] \notin A(A_m) \) and \( h^2[b] \notin A(A_m) \).

Now we construct a sequence \( \{ \alpha_i \}_{i=1}^\infty \) such that the following holds:

- \( \alpha_1 \geq \max\{2, 2c(H, m)\} \);
- \( \alpha_{i+1} = h^2[2a_i] \);
- \( h^{-1}[a_i] \notin A(A_m), \forall i \in \mathbb{N} \);
- \( h^{-2}[a_i], h^2[2a_i] \notin A(A_m), \forall i \in \mathbb{N} \).
Note that $a_1$ determines the sequence and obviously $\{a_i\}$ is strictly increasing. The 3rd and 4th conditions are guaranteed for a generic $a_1$. Let $a$ be the set of sequences satisfying the above conditions.

In order to compare $\dim FH^{[h^{-1}[a],h[T]}(A_m^f)$ and $\dim FH^{[h^{-2}[a],h^2[T]}(A^f)$, we use (3.30) inductively. Choose $\{a_i\} \in a$ then the following holds:

$$
\dim FH^{(h^{-1}(a_1),h(2a_k))}(A_m^f) = \sum_{i=1}^{k} \dim FH^{(h^{-1}[a_i],h[2a_i])}(A_m^f)
$$

$$
\geq \sum_{i=1}^{k} \dim FH^{(h^{-2}[a_i],h^2[2a_i])}(A^f).
$$

But there exist missing action intervals for $A^f$ in the last term of (3.31). To cover the missing intervals, we first observe that if $a \geq 2$ then the length of the action intervals for $A_m^f$ and $A^f$

$$
h[2a] - h^{-1}[a], \quad h^2[2a] - h^{-2}[a]
$$

are positive and increasing functions with respect to $a$. By a simple computation, one can check that its ratio satisfies

$$
\frac{h[2a] - h^{-1}[a]}{h^2[2a] - h^{-2}[a]} \leq 4
$$

for all $a \geq 2$. This implies that there exist 4 sequences $\{a_1^1\}, \{a_1^2\}, \{a_1^3\}, \{a_1^4\} \in a$ such that $a_1^1 < a_1^2 < a_1^3 < a_1^4 < a_2^1$ and

$$
(h^{-2}[a_1^1], h^2[2a_k^1]) \cup \bigcup_{i=1}^{k-1} \bigcup_{j=1}^{4} (h^{-2}[a_i^j], h^2[2a_i^j])
$$

covers $(h^{-2}[a_1^1], h^2[2a_k^1]) \subset \mathbb{R}^+$ for any $k \in \mathbb{N}$.

Now we obtain the following estimate

$$
4 \dim FH^{(h^{-1}[a_1^1],h[2a_k^1])}(A_m^f) \geq \sum_{i=1}^{k-1} \sum_{j=1}^{4} \dim FH^{(h^{-1}[a_i^j],h[2a_i^j])}(A_m^f) + \dim FH^{(h^{-1}[a_k^1],h[2a_k^1])}(A_m^f)
$$

$$
\geq \sum_{i=1}^{k-1} \sum_{j=1}^{4} \dim FH^{(h^{-2}[a_i^j],h^2[2a_i^j])}(A^f) + \dim FH^{(h^{-2}[a_k^1],h^2[2a_k^1])}(A^f)
$$

$$
\geq \dim FH^{(h^{-2}[a_1^1],h^2[2a_k^1])}(A^f).
$$

This proves the lemma.

**Proposition 3.16.** Fix $F \in D(\Sigma)$ and $f \in \bigcap_{r>0} \mathcal{F}(\frac{1}{6}, r)$. Let $\overline{c}, \overline{r} > 0$ be the constants from Lemma 3.11. If $(H, m)$ is generic then there exist

$$
n = n(N, g, F, \overline{c}, \overline{r}, H, m) \in \mathbb{N}
$$

such that

$$
\dim FH^{(a,T)}(A_m^f) \geq \frac{1}{4n} \dim FH^{(h^{-n}[a],h^n[T]}(A^f)
$$

holds for generic $a, T$ such that $\max\{h^{-1}[2], h^{-1}[2c(H, m)]\} < a < T < \infty$. 
Proof. In order to use Lemma 3.15, we first subdivide $m$ into small pieces to have the following properties:

- $m^i = (d^i \beta, \sigma, \theta)$, where $0 = d^0 < d^1 < \cdots < d^n = 1$;
- $A_{m_i}^f : L \times \mathbb{R} \to \mathbb{R}$ is Morse for all $i = 0, 1, \ldots, n$;
- $C_i = (d_i + 1 - d_i)\|\beta\|_\infty\|\sigma\|_\infty$ satisfies (3.14), (3.21), (3.23) and (3.27) for all $i = 0, 1, \ldots, n - 1$.

By the choice of $f$, it is not necessary for the isoperimetric constant $C$ to satisfy the condition [3.26]. This implies that the subdivision number $n$ for $m$ does not depend on the action window. Now choose $a, T$ such that the following conditions hold:

- $\max\{h^{-1}[2], h^{-1}[2c(H, m)]\} < a < T < \infty$;
- $h^{-n+i}[a], h^n-T \notin A(A_{m_i}^f) \forall i = 0, 1, \ldots, n$.

By the above second condition, $\text{FH}^{(h^{-n+i}[a], h^n-T)}(A_{m_i}^f)$ are well-defined for $0 \leq i \leq n$. Now we are ready to apply Lemma 3.15. If we use (3.29) inductively then we conclude that

$$\dim \text{FH}^{(a, T)}(A_{m_i}^f) = \dim \text{FH}^{(a, T)}(A_{m_i}^f)$$

$$\geq \frac{1}{4} \dim \text{FH}^{(h^{-1}[a], h[T])}(A_{m_i}^{f_{n+1}})$$

$$\geq \cdots$$

$$\geq \frac{1}{4^n} \dim \text{FH}^{(h^{-n}[a], h^n[T])}(A_{m_i}^f)$$

$$= \frac{1}{4^n} \dim \text{FH}^{(h^{-n}[a], h^n[T])}(A_{m_i}^f).$$

This proves the lemma. \hfill \Box

Remark 3.17. The argument in Proposition 3.16 holds for any $F \in D(\Sigma)$ and any generic $(H, m)$. Note $\overline{\tau} \in \tau$ depend on $F$ and a $\delta$–neighborhood of $F^{-1}(0)$. For a given diffeomorphism $\varphi_m \in \text{Diff}(\Sigma, m)$, consider all defining data $(H, m)$ for $\varphi_m$ such that $\varphi_m = \phi_1^X_{\mathcal{H}}$. Now we consider

$$n' := \inf_{(H, m) \in (F, \delta)} n(N, g, F, \overline{\tau} \in \tau, H, m)$$

then $n'$ depends only on $(N, g, \Sigma, \varphi_m)$. By abuse of notation, we write $n = n'$.

Proof of Theorem 1.7. We first fix a defining Hamiltonian $F$ for $\Sigma$ and a defining data $(H, m)$ for $\varphi_m$. Choose $f \in \bigcap_{r \geq 0} \mathcal{F}(\frac{1}{r}, r)$. By the generic assumption, $\varphi_m$ has no periodic leaf-wise intersection point and $A_{m_i}^f$ is Morse for the action window $(\frac{1}{r} + c(H, m), \infty]$, see Corollary A.4.

If we choose a generic action value $a, T$ such that $\max\{h^{-1}[2], h^{-1}[2c(H, m)]\} < a < T < \infty$ then Proposition 3.14 and Proposition 3.16 imply that

$$n_{\Sigma, \varphi_m}(T) \geq \# \text{Crit}(\frac{1}{r} + c(H, m), T-c(H, m))(A_{m_i}^f)$$

$$\geq \dim \text{FH}^{(a, T-c(H, m))}(A_{m_i}^f)$$

$$\geq \frac{1}{4^n} \dim \text{FH}^{(h^{-n}[a], h^n[T-c(H, m)])}(A_{m_i}^f).$$

(3.32)

Here $n = n(N, g, \Sigma, \varphi_m) \in \mathbb{N}$ is the constant from Proposition 3.16 with Remark 3.17.
Now we recall that \(\mathcal{L}_N\) is the free loop space of \((N, g)\). The energy functional \(\mathcal{E}_g : \mathcal{L}_N \rightarrow \mathbb{R}\) is given by

\[
\mathcal{E}_g(q) := \int_0^1 \frac{1}{2} \|q\|_g^2 dt.
\]

For given \(0 < T < \infty\), denote by

\[
\mathcal{L}_N(T) := \left\{ q \in \mathcal{L}_N : \mathcal{E}_g(q) \leq \frac{1}{2} T^2 \right\}.
\]

By the result of Macarini-Merry-Paternain [30, Proof of Theorem A, Remark 1.4], there exists a constant \(c' = c'(N, g, \Sigma, \varphi_m) > 0\) such that

\[
\dim \text{FH}^{[h^{-n}[a], h^n[T - c(H, m)]]}(A_f) \geq \text{rank}\{ t : H_* (\mathcal{L}_N(c'(T - 1))) \rightarrow H_* (\mathcal{L}_N) \}.
\]

If \(c := \min\{ \frac{1}{4n}, c' \} > 0\) then finally we obtain

\[
n_\Sigma, \varphi_m(T) \geq \frac{1}{4n} \text{rank}\{ t : H_* (\mathcal{L}_N(c'(T - 1))) \rightarrow H_* (\mathcal{L}_N) \}
\geq c \cdot \text{rank}\{ t : H_* (\mathcal{L}_N(c(T - 1))) \rightarrow H_* (\mathcal{L}_N) \}.
\]

This proves the theorem. \(\square\)

**Appendix A. The perturbed Rabinowitz action functional is generically Morse.**

In this section we study the Morse property of the perturbed Rabinowitz action functional. Note first that the action functional \(\mathcal{A}_m = \mathcal{A}_{H,m}^F\) is determined by the following data \(F \in D(\Sigma)\), \(H \in \mathcal{H}\) and \(m \in \mathcal{M}\). Especially \(m \in \mathcal{M}\) consist of \((\beta, \sigma, \theta)\), see Definition 1.1. We claim that \(\mathcal{A}_m\) is Morse for generic \(H \in \mathcal{H}\) and \(\beta \in \mathcal{B}\). The generic property for \(H \in \mathcal{H}\) is well-studied in [4, Appendix A]. So we concentrate on the Morse property of \(\mathcal{A}_m\) with respect to the case of \(\beta \in \mathcal{B}\). First recall that

\[
\mathcal{B} := \{ \beta \in C^\infty(S^1) : \beta(t) = 0, \forall t \in [0, 1/2] \}.
\]

**Theorem A.1.** For a generic \(\beta \in \mathcal{B}\) the perturbed Rabinowitz action functional \(\mathcal{A}_m\) is Morse.

**A.1. Preparations.** The proof of the genericity of the Morse property follows a standard method, that is, once it is shown that a certain linear operator is surjective then the theorem follows from Sard-Smale’s theorem. In this proof we follow the strategy of [4, Appendix A].

First, let us recall the definition of the perturbed Rabinowitz action functional

\[
\mathcal{A}_m : \mathcal{L} \times \mathbb{R} \rightarrow \mathbb{R}
\]

\[
(u, \eta) \mapsto \int_0^1 u^* \lambda - \eta \int_0^1 F(t, u(t)) dt - \int_0^1 H(t, u(t)) dt + \int_0^1 \tau^* \beta(t) \theta(\tilde{u}(t)) [\partial_t \tilde{u}(t)] dt
\]

where in this section, from now on \(\mathcal{L} \equiv W^{(1,2)}(S^1, T^* N)\) is the completed loop space of \(T^* N\). For convenience we abbreviate

\[
\mathcal{F} : \mathcal{L} \rightarrow \mathbb{R}
\]

\[
u \mapsto \int_0^1 F(t, u) dt,
\]

\[
\mathcal{A}_m = \mathcal{A}_{H, m} := \int_0^1 \lambda(u(t)) [\partial_t u] - \int_0^1 H(t, u(t)) dt + \int_0^1 \beta(t) \theta(\tilde{u}(t)) [\partial_t \tilde{u}(t)] dt
\]
and
\[ A^{\eta_0,F}_m := A^{\eta_0,F + H,m}_m. \]
Thus, \( A_m(u, \eta) = A_m(u) - \eta F(u). \) We note that \( A_m(u, \eta) = A^{\eta_0,F}_m(u) + (\eta_0 - \eta) F(u), \) and therefore
\[ dA_m(u, \eta)[\dot{u}, \dot{\eta}] = dA^{\eta_0,F}_m(u)[\dot{u}] - \dot{\eta} F(u) + (\eta_0 - \eta) dF(u)[\dot{u}] \]
where \( \dot{u} \in \Gamma^{1,2}(u^* \mathcal{T}\mathcal{T}(T^* N)), \) the space of \( W^{1,2} \) vector fields along \( u \) and \( \eta \in \mathbb{R}. \) Hence at a critical point \( w_0 = (u_0, \eta_0) \in \text{Crit}(A_m) \) the Hessian equals
\[ \mathcal{H}_{A_m}(w_0)[(\dot{u}_1, \dot{\eta}_1), (\dot{u}_2, \dot{\eta}_2)] = \mathcal{H}_{A^{\eta_0,F}_m}(u_0)[\dot{u}_1, \dot{u}_2] - \dot{\eta}_1 dF(u_0)[\dot{u}_2] - \dot{\eta}_2 dF(u_0)[\dot{u}_1]. \]

For a function \( P : [0,1] \times T^* N \to \mathbb{R}, \) an \( S^1 \)-parameterized symplectic form \( \omega_m \) and the corresponding \( \phi_{X_P}^1 \in \text{Diff}(T^* N, m), \) we define
\[ \mathcal{L}_{P,m} := \{ v \in W^{1,2}([0,1], T^* N) : v(0) = \phi_{X_P}^1(v(1)) \}, \tag{A.1} \]
the twisted loop space, and introduce the diffeomorphism \( \Phi_{P,m} : \mathcal{L}_{P,m} \to \mathcal{L} \)
\[ \Phi_{P,m}(v)(t) = \phi_{X_P}^1(v(t)). \]
For a fixed critical point \( w_0 = (u_0, \eta_0) \) of \( A_m \) we use this diffeomorphism to pull back \( A_m \)
\[ \overline{A}^{\eta_0,F}_m := (\Phi_{\eta_0,F + H,m} \times \text{id}_{\mathbb{R}})^* A_m : \mathcal{L}_{\eta_0,F + H,m} \times \mathbb{R} \to \mathbb{R}. \]
We set \( v_0 := \Phi_{\eta_0,F + H,m}^{-1} \circ u_0, \) thus \( v_0 = \text{const}. \) Then using
\[ (\Phi_{H,m}^* dA_m)(v)[\dot{v}] = \int_0^1 \omega(\partial_t v(t), \dot{v}(t)) dt \]
we obtain
\[ \mathcal{H}_{\overline{A}^{\eta_0,F}_m}(v_0, \eta_0)[(\dot{v}_1, \dot{\eta}_1), (\dot{v}_2, \dot{\eta}_2)] = \int_0^1 \omega(\partial_t \dot{v}_1, \dot{v}_2) dt - \dot{\eta}_1 dF(v_0)[\dot{v}_2] - \dot{\eta}_2 dF(v_0)[\dot{v}_1], \]
where \( \mathcal{F} := F \circ \Phi_{\eta_0,F + H,\alpha \beta}. \) Since \( F(t, x) = 0 \) for \( t \in [\frac{1}{2}, 1] \) and \( H(t, x), \beta(t) \) vanish for \( t \in [0, \frac{1}{2}], \) we compute
\[ \mathcal{F}(v) = \int_0^1 F(t, \varphi_{\eta_0,F + H,m}^t(v)) dt = \int_0^{\frac{1}{2}} F(t, \varphi_{\eta_0,F + H,m}^t(v)) dt \]
\[ = \int_0^{\frac{1}{2}} F(t, \varphi_{\eta_0,F}^t(v)) dt = \int_0^1 F(t, v) dt \]
\[ = \int_0^1 F(t, v) dt. \]
Thus, the Hessian of \( \overline{A}^{\eta_0,F}_m \) simplifies as follows
\[ \mathcal{H}_{\overline{A}^{\eta_0,F}_m}(v_0, \eta_0)[(\dot{v}_1, \dot{\eta}_1), (\dot{v}_2, \dot{\eta}_2)] \]
\[ = \int_0^1 \omega(\partial_t \dot{v}_1, \dot{v}_2) dt - \dot{\eta}_1 \int_0^1 dF(t, v_0)[\dot{v}_2] dt - \dot{\eta}_2 \int_0^1 dF(t, v_0)[\dot{v}_1] dt \tag{A.2} \]
A.2. The linearized operator. We denote by \( \mathfrak{B}^k = \{ \beta \in C^k(S^1) : \beta(t) = 0, \forall t \in [0, 1/2] \} \). For \( v \in \mathcal{L}_{H,m} \) (see equation (A.1) for the definition) we define the bundle \( \mathcal{E}_{H,m} \rightarrow \mathcal{L}_{H,m} \) by

\[
(\mathcal{E}_{H,m})_v := L^2([0, 1], v^* T(T^* N)).
\]

**Definition A.2.** Let \((u_0, \eta_0)\) be a critical point of \( \mathcal{A}_m \) and \((v_0, \eta_0)\) the corresponding critical point of \( \mathcal{A}_m^{0,F} \), that is the point defined by the equation \( u_0 = \Phi_{\eta_0 F + H,m}(v_0) \). Then we define the linear operator

\[
L_{(v_0, \eta_0, \beta)} : (T_{v_0} \mathcal{L}_{\eta_0 F + H,m}) \times \mathbb{R} \times \mathfrak{B}^k \rightarrow (\mathcal{E}_{\eta_0 F + H,m})^* \times \mathbb{R}
\]

via the pairing with \((\hat{v}_2, \hat{\eta}_2)\) ∈ \((\mathcal{E}_{\eta_0 F + H,m})^* \times \mathbb{R}

\[
\langle L_{(v_0, \eta_0, \beta)} \hat{v}_1, \hat{\eta}_1, \hat{\beta} \rangle, (\hat{v}_2, \hat{\eta}_2) \rangle := \mathcal{H}_{\mathcal{A}_m^{0,F}}(v_0, \eta_0) \langle (\hat{v}_1, \hat{\eta}_1), (\hat{v}_2, \hat{\eta}_2) \rangle + \int_0^1 \hat{\beta}(t) \sigma(\tau_s \partial_t \hat{v}_1(t), \tau_s \hat{v}_2(t)) dt
\]

**Proposition A.3.** The operator \( L_{(v_0, \eta_0, \beta)} \) is surjective. In fact, \( L_{(v_0, \eta_0, \beta)} \) is surjective when restricted to the space

\[
\mathcal{V} := \{ (\hat{v}, \hat{\eta}, \hat{\beta}) \in (T_{v_0} \mathcal{L}_{\eta_0 F + H,m}) \times \mathbb{R} \times \mathfrak{B}^k : \hat{v}(1/2) = 0 \}.
\]

**Proof.** The \( L^2 \)-Hessian is a self-adjoint operator. Thus, the operator \( L_{(v_0, \eta_0, \beta)} \) has closed image. Therefore, it suffices to prove that the annihilator of the image of \( L_{(v_0, \eta_0, \beta)} \) vanishes. Let \((\hat{v}_2, \hat{\eta}_2)\) be in the annihilator of the image of \( L_{(v_0, \eta_0, \beta)} \); that is

\[
\langle L_{(v_0, \eta_0, \beta)} \hat{v}_1, \hat{\eta}_1, \hat{\beta} \rangle, (\hat{v}_2, \hat{\eta}_2) \rangle = 0
\]

for all \((\hat{v}_1, \hat{\eta}_1, \hat{\beta}) \in (T_{v_0} \mathcal{L}_{\eta_0 F + H,m}) \times \mathbb{R} \times \mathfrak{B}^k \). This is equivalent to the following two equations:

\[
\mathcal{H}_{\mathcal{A}_m^{0,F}}(v_0, \eta_0) \langle (\hat{v}_1, \hat{\eta}_1), (\hat{v}_2, \hat{\eta}_2) \rangle = 0, \quad \forall (\hat{v}_1, \hat{\eta}_1) \in (T_{v_0} \mathcal{L}_{\eta_0 F + H,m}) \times \mathbb{R}
\]

and

\[
\int_0^1 \hat{\beta}(t) \sigma(\tau_s \partial_t \hat{v}_1(t), \tau_s \hat{v}_2(t)) dt = 0, \quad \forall \hat{\beta} \in \mathfrak{B}^k.
\]

Since the Hessian \( \mathcal{H}_{\mathcal{A}_m^{0,F}} \) is a self-adjoint operator, equation (A.2) and (A.3) imply by elliptic regularity that \( \hat{v}_2 \in C^{k+1}([0, 1], T_{v_0} T^* N) \) and satisfies the equation

\[
\partial_t \hat{v}_2 - \hat{\eta}_2 X_F(t, v_0) = 0
\]

and the linearized boundary condition

\[
\hat{v}_2(0) = d\Phi_{\eta_0 F + H,m}(v_0)[\hat{v}_2(1)].
\]

In fact, if we restrict the Hessian to \( \mathcal{V} \) then equation (A.5) holds except at \( t = 1/2 \), since the Hessian is a local operator. Thus, by continuity, equation (A.5) is still valid for all \( t \in [0, 1] \).

From equation (A.3) we deduce that

\[
\hat{v}_2(t) = 0, \quad \forall t \in \left[ \frac{1}{2}, 1 \right].
\]

By (2.3), (A.5) becomes

\[
\partial_t \hat{v}_2 - \hat{\eta}_2 \rho(t) X_{F_2}(v_0) = 0.
\]
This is a linear ODE in the vector space $T_{v_0} T^* N$ as follows
\[ \dot{v}_2(t) = \dot{v}_2(0) + \eta_2 \left( \int_0^t \rho(\tau)d\tau \right) X_{F_U}(v_0). \] (A.8)
Recall (2.2) that $\int_0^t \rho(\tau)d\tau = 1$ for all $t \in [\frac{1}{2}, 1]$. Combining this with equation (A.7) we conclude for $t \geq \frac{1}{2}$
\[ 0 = \dot{v}_2(t) = \dot{v}_2(0) + \eta_2 X_{F_U}(v_0) \] (A.9)
By using equations (A.6) and (A.7) at $t = 1$, we deduce $\dot{v}_2(0) = 0$. Now, put this into (A.9)
we have
\[ \eta_2 X_{F_U}(v_0) = 0 \]
Since $(v_0, \eta_0)$ comes from a critical point $(u_0, \eta_0)$ of $A_m$, we know $F_U(v_0) = F_U(u(0)) = k$, and we already assume that $k$ is a regular value of $F_U$. In particular,
\[ \eta_2 = 0 \] (A.10)
Equation (A.9), (A.10) and $\dot{v}_2(0) = 0$ imply
\[ \dot{v}_2(t) = 0, \quad \forall t \in [0, 1]. \]
Therefore, the annihilator of the image of $L_{(v_0, \eta_0, \beta)}$ vanishes and thus $L_{(v_0, \eta_0, \beta)}$ is surjective.

Proof of Theorem A.1. First recall that $L = W^{1,2}(S^1, T^* N)$ and $\mathfrak{B}^k = \{ \beta \in C^k(S^1) : \beta(t) = 0, \quad \forall t \in [0, 1/2] \}$. We define the Banach space bundle $\mathcal{E} \to L$ by $\mathcal{E}_u = L^2(S^1, u^* T(T^* N))$.
Now consider the section $S : L \times \mathbb{R} \times \mathfrak{B}^k \to \mathcal{E}' \times \mathbb{R}$ given by the differential of the Rabinowitsch action functional $A_m$
\[ S(u, \eta, \beta) := dA_m(u, \eta). \] (A.11)
where the perturbation $\beta \in \mathfrak{B}^k$ is considered as an additional variable. Its vertical differential $DS : T_{(u_0, \eta_0, \beta)}L \times \mathbb{R} \times \mathfrak{B}^k \to \mathcal{E}'_{(u_0, \eta_0, \beta)}$ at $(u_0, \eta_0, \beta) \in S^{-1}(0)$ is
\[ DS_{(u_0, \eta_0, \beta)}[\dot{u}, \dot{\eta}, \dot{\beta}] = \mathcal{H}_{A_m} f(u_0, \eta_0)[(\dot{u}, \dot{\eta}), \cdot ] + \int_0^1 \dot{\beta}(t) \sigma(\tau_* \partial_t \dot{u}(t), \cdot )dt \] (A.12)
Since the pull-back of $DS$ under the diffeomorphism $\Phi_{R, F^H, F^F, m} \times \text{id}_{\mathbb{R}} \times \text{id}_{\mathfrak{B}^k}$ is the operator $L_{(v_0, \eta_0, \beta)}$ in Proposition A.3, the operator $DS$ is surjective. Thus, by the implicit function theorem the universal moduli space
\[ M := S^{-1}(0) \]
is a smooth Banach manifold. We consider the projection $\Pi : M \to \mathfrak{B}^k$. Then the $A_m$ is Morse if and only if $\beta$ is a regular value of $\Pi$. By the Sard-Smale theorem this forms a generic set for $k$ large enough. Moreover, the Morse condition is $C^k$-open. Thus, for function in an open and dense subset of $\mathfrak{B}^k$ the Rabinowitsch action functional is Morse. Taking the intersection of all $k$ concludes the proof of Theorem A.1.

Now we discuss the Morse property of $A_m$. Since we are interested in critical points of $A_m$ with positive action value, it suffices to check the Morse property for the positive critical points.

Corollary A.4. Given $a > 0$ and choose $f \in \mathcal{F}(a)$ (see, Definition 3.3). For a generic $\beta \in \mathfrak{B}$ the perturbed $F$-Rabinowitsch action functional $A_m = A_{F, m}^f$ is Morse on the action window $(a + c(H, m), \infty)$. 

40 YOUNGJIN BAE
Proof. Let $w_0 = (u_0, \eta_0)$ be a critical point of $A_m$ with $A_m(u_0, \eta_0) > a - c(H, m)$ then by the argument in Proposition 3.2 we obtain $f(\eta_0) > a$. Since $f \in F(a)$, we conclude $f'(\eta_0) = 1$. Hence the argument in the proof of Theorem A.1 definitely holds. This proves the corollary. \hfill \Box

**Appendix B. No periodic magnetic leaf-wise intersection points**

In this section, we study the second regularity property of $\varphi_m \in \text{Diff}_c(T^*N, m)$, see Definition 2.3. The claim is that $\varphi_m$ has no periodic leaf-wise intersection points for generic $H \in \mathcal{H}$ and $\beta \in \mathcal{B}$. In [3] Albers-Frauenfelder already studied the above property with respect to $H \in \mathcal{H}$. So as in Appendix A we still work with $\beta \in \mathcal{B}$ and modify the strategy of [3].

Recall that the hypersurface $\Sigma \subset T^*N$ is called non-degenerate if closed Reeb orbits on $\Sigma$ form a discrete set. A generic $\Sigma$ is non-degenerate, see [12] Theorem B.1. If the critical points of $A_m$ does not meet any closed Reeb orbit then there are no periodic leaf-wise intersection points. Thus it suffices to prove the following theorem.

**Theorem B.1.** Let $\Sigma \subset T^*N$ be a non-degenerate starshaped hypersurface and $R$ be a set of closed Reeb orbit on $\Sigma$ which form a discrete set. If $\dim N \geq 2$ then the set

$$\mathcal{B}_\Sigma := \{ \beta \in \mathcal{B} : A_m \text{ is Morse and } \text{im}(x) \cap \text{im}(y) = \emptyset, \forall x \in \text{Crit}(A_m), y \in R \} \quad (B.1)$$

is generic in $\mathcal{B}$, see Definition 1.1.

Proof. We first define the evaluation map

$$\text{ev} : \mathcal{M} \to \Sigma \quad (B.2)$$

where $\mathcal{M}$ is the same as in the proof of Theorem A.1. Combining Proposition A.3 with Lemma B.2 below it follows that the evaluation map $\text{ev}_\beta := \text{ev}(\cdot, \beta) : \text{Crit}(A_m) \to \Sigma$ is a submersion for a generic choice of $\beta$. Since $\dim T^*N \geq 4$, the preimage of the one dimensional set $\mathcal{R}^\tau := \{ \text{closed Reeb orbits with period } \leq \tau \}$ under $\text{ev}_\beta$ does not intersect. Therefore, the set

$$\mathcal{B}_\Sigma^\beta := \{ \beta \in \mathcal{B}^n : A_m \text{ is Morse and } \text{im}(x) \cap \text{im}(y) = \emptyset, \forall x \in \text{Crit}(A_m), y \in R^n \} \quad (B.3)$$

is generic in $\mathcal{B}$ for all $n \in \mathbb{N}$. Now, the set $\mathcal{B}_\Sigma^n$ is a countable intersection of the set $\mathcal{B}_\Sigma^\beta$, $n \in \mathbb{N}$. This proves the Theorem B.1. \hfill \Box

The following lemma is contained in [3]. The proof is included for the reader’s convenience.

**Lemma B.2.** Let $E \to S$ be a Banach bundle and $s : S \to E$ a smooth section. Moreover, let $\psi : \mathcal{B} \to N$ be a smooth map into the Banach manifold $N$. We fix a point $x \in s^{-1}(0) \subset S$ and set $K := \ker d\psi(x) \subset T_x \mathcal{B}$ and assume the following two conditions.

1. The vertical differential $Ds|_K : K \to E_x$ is surjective.
2. $d\psi(x) : T_x \mathcal{B} \to T_{\psi(x)}N$ is surjective.

Then $d\psi(x)|_{\ker Ds(x)} : \ker Ds(x) \to T_{\psi(x)}N$ is surjective.

Proof. We fix $\xi \in T_{\psi(x)}N$. Condition (2) implies that there exists $\eta \in T_xS$ satisfying $d\psi(x)\eta = \xi$. Condition (1) implies that there exists $\zeta \in K \subset T_xS$ satisfying $Ds(x)\zeta = Ds(x)\eta$. We set $\tau := \eta - \zeta$ and compute

$$Ds(x)\tau = Ds(x)\eta - Ds(x)\zeta = 0.$$
thus, $\tau \in \ker Ds(x)$. Moreover,

$$d\psi(x)\tau = d\psi(x)\eta - d\psi(x)\zeta = d\psi(x)\eta = \zeta.$$ 

This proves the lemma. \qed

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Youngjin Bae, Department of Mathematics and Research Institute of Mathematics, Seoul National University
E-mail address: jini0919@snu.ac.kr