On the three-dimensional homogeneous
SO(2) - isotropic Riemannian manifolds

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Abstract
In this paper we consider some properties of the three-dimensional homogeneous SO(2)-isotropic Riemannian manifolds. In particular, we determine the geodesics, the totally geodesic surfaces, the totally umbilical surfaces and the geodesics of the rotational surfaces.

1 Introduction
We consider a two-parameter family of three-dimensional Riemannian manifolds \((M, ds^2_{\ell,m})\), where the metrics have the expression

\[
ds^2_{\ell,m} = \frac{dx^2 + dy^2}{1 + m(x^2 + y^2)} + \left(\frac{z + \frac{\ell}{2} \frac{ydx - xdy}{[1 + m(x^2 + y^2)]}}{1 + m(x^2 + y^2)}\right)^2,
\]

with \(\ell, m \in \mathbb{R}\). The underlying differentiable manifolds \(M\) are \(\mathbb{R}^3\) if \(m \geq 0\) and \(M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < -\frac{1}{m}\}\) otherwise. These metrics can be found in the classification of 3-dimensional homogeneous metrics given by L. Bianchi in 1897 (see [1]); later, they appeared in form (1.1) in É. Cartan ([2]) and G. Vranceanu (see [14]). For these reasons we call them the Cartan-Vranceanu metrics (C-V metrics). Their geometric interest lies in the following: the family of metrics (1.1) includes all 3-dimensional homogeneous metrics whose group of isometries has dimension 4 or 6, except for those of constant negative sectional curvature. We recall the spaces that correspond to the different values of \(\ell\) and \(m\).

- If \(\ell = 0\), then \(M\) is the product of a surface \(S\) with constant Gaussian curvature \(4m\) and the real line \(\mathbb{R}\).
- If \(4m - \ell^2 = 0\), then \(M\) has nonnegative constant sectional curvature.
- If \(\ell \neq 0\) and \(m > 0\), \(M\) is locally \(SU(2)\).
- Similarly, if \(\ell \neq 0\) and \(m < 0\), \(M\) is locally \(\tilde{SL}(2, \mathbb{R})\), while if
- \(m = 0\) and \(\ell \neq 0\) we get a left invariant metric on the Heisenberg Lie group \(\mathbb{H}_3\).
The isometry group of these spaces has a subgroup isomorphic to the group $SO(2)$, so there are rotational surfaces around $z$-axis. R. Caddeo, P. Piu, A. Ratto and P. Tomter (see [3], [4], [13]) have studied rotational surfaces in $\mathbb{H}^3$ with constant (mean or Gauss) curvature, while the CMC and CGC invariant surfaces of $\mathbb{H}^3$ and of the product space $\mathbb{H}^2 \times \mathbb{R}$ have been studied by C. Figueroa, F. Mercuri, R. Pedrosa, S. Montaldo and I. Onnis (see [5], [8], [7]). In this paper we obtain the Lie algebra of the Killing vector fields and thus the group of isometries for the C-V metrics. We explicitly determine the equations of the geodesics by using the Killing vector fields and obtain the equations of the surfaces which contain the geodesics. After having determined the totally geodesic surfaces isometrically immersed in the C-V spaces, we study the totally umbilical surfaces of these spaces, proving that the only totally umbilical surfaces are totally geodesic. We find the geodesics for the $SO(2)$-invariant surfaces of the Cartan-Vranceanu spaces, deduce the conditions that meridians and parallels must satisfy in order to be geodesics and show the analogies with the Euclidean case.

2 Geodesics on C-V spaces

It is well known that a curve $\gamma : I \to M$ on a Riemannian manifold $(M, g, \nabla)$ with the Levi-Civita connection $\nabla$ is a geodesic if its velocity vector field is constant (parallel),

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0. \quad (2.2)$$

We also remember an important theorem of Levi-Civita:

**Theorem 2.1.** If $X$ is a Killing vector field for the Riemannian manifold $(M, g)$ then the equation of the geodesics $\gamma(t)$ admits the prime integral

$$g(\dot{\gamma}, X) = \text{const.}$$

**Proof.** The derivative with respect to $t$ of the scalar product $g(\dot{\gamma}, X) = \varphi(t)$ gives

$$\frac{d}{dt} g(\dot{\gamma}, X) = g(\nabla_{\dot{\gamma}} \dot{\gamma}, X) + g(\dot{\gamma}, \nabla_{\dot{\gamma}} X)$$

and this is zero, because $\gamma$ is a geodesic and $X$ a Killing vector field. Thus we have $\varphi = \text{const.}$

We want to obtain the geodesics for the simply connected homogeneous $SO(2)$-isotropic 3-dimensional Riemannian manifolds with isometry group of dimension 4, endowed with the C-V metrics. The Cartan-Vranceanu metric (1.1) can be written as:

$$ds_{C,V}^2 = \sum_{i=1}^{3} \omega^i \otimes \omega^i, \quad (2.3)$$
where, putting $D = 1 + m(x^2 + y^2)$,
\[
\omega^1 = \frac{dx}{D}, \quad \omega^2 = \frac{dy}{D}, \quad \omega^3 = dz + \frac{\ell y dx - x dy}{2D}.
\] (2.4)

The orthonormal basis of vector fields dual to the 1-forms (2.4) is
\[
E_1 = D \frac{\partial}{\partial x} - \frac{\ell y}{2} \frac{\partial}{\partial z}, \quad E_2 = D \frac{\partial}{\partial y} + \frac{\ell x}{2} \frac{\partial}{\partial z}, \quad E_3 = \frac{\partial}{\partial z}.
\] (2.5)

The Killing vector fields of the metric (1.1) are the vector fields $X = \xi^i E_i$ such that the Lie derivative with respect to $X$ of the metric is zero $L_X(ds^2) = 0$.

A basis for the Lie algebra of Killing vector fields has been computed (see [11]) and we found that it is given by
\[
X = \frac{2mxy}{D} E_1 + \left(1 - \frac{2m^2}{D}\right) E_2 - \frac{\ell x}{D} E_3,
\]
\[
Y = \left(1 - \frac{2my^2}{D}\right) E_1 + \frac{2mxy}{D} E_2 + \frac{\ell y}{D} E_3,
\]
\[
Z = E_3,
\]
\[
R = -\frac{y}{D} E_1 + \frac{x}{D} E_2 - \frac{\ell(x^2 + y^2)}{2D} E_3.
\]

Let $\gamma : I \rightarrow M$ be a geodesic on the manifold $(M, ds^2)$. The tangent vector field $\dot{\gamma}$ with respect to the orthonormal basis (2.5) is
\[
\dot{\gamma} = \frac{\dot{x}}{D} E_1 + \frac{\dot{y}}{D} E_2 + \left[\dot{z} - \frac{\ell x \dot{y} - y \dot{x}}{2D}\right] E_3.
\]

According to Theorem 2.1 we can write four prime integrals

\[
\begin{align*}
\frac{2mxy\dot{x}}{D^2} + \frac{(1 + m(y^2 - x^2))\dot{y}}{D^2} - \frac{\ell x}{D} &= a_1 \\
\frac{(1 + m(x^2 - y^2))\dot{x}}{D^2} + \frac{2mxy\dot{y}}{D^2} + \frac{\ell y}{D} &= a_2 \\
\dot{z} - \frac{\ell x \dot{y} - y \dot{x}}{2D} &= a_3 \\
\frac{\dot{x} \dot{y}}{D^2} - \frac{\dot{x} y}{D^2} - \frac{a_3(x^2 + y^2)}{2D} &= a_4
\end{align*}
\]  

$a_i \in \mathbb{R}$. 

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Remark 2.1. As here we are considering homogeneous riemannian spaces, in order to obtain all the geodesics starting at a point \( p \) it is sufficient to translate the geodesics starting at the origin of the coordinate system by using the isometry that takes \( 0 \) to \( p \).

Considering thus the geodesics starting at origin such that \( \dot{\gamma}(0) = (u, v, w) \), the constants \( a_i \) take the values

\[
a_1 = v \quad a_2 = u \quad a_3 = w \quad a_4 = 0
\]

and the prime integrals become

\[
\begin{align*}
\frac{2mxy\ddot{x}}{D^2} + \frac{(1 + m(y^2 - x^2))\dot{y}}{D^2} - \frac{\ell xw}{D} &= v \\
\frac{(1 + m(x^2 - y^2))x}{D^2} + \frac{2mxy\dot{y}}{D^2} + \frac{\ell yw}{D} &= u \\
\dot{z} - \frac{\ell x\dot{y} - y\dot{x}}{2D} &= w \\
\frac{\dot{yx} - \dot{xy}}{D^2} - \frac{\ell(x^2 + y^2)w}{2D} &= 0.
\end{align*}
\]

(2.6)

We observe that the Killing vector field \( R \) that generates the rotations around the \( z \)-axis, may be written as a combination of the other Killing vector fields:

\[
R = \frac{-y}{m(x^2 + y^2) - 1} X + \frac{x}{m(x^2 + y^2) - 1} Y - \frac{\ell(x^2 + y^2)}{2(m(x^2 + y^2) - 1)} Z.
\]

Theorem 2.1 applied to \( R \), gives the following prime integral

\[
\frac{-y}{m(x^2 + y^2) - 1}a_1 + \frac{x}{m(x^2 + y^2) - 1}a_2 - \frac{\ell(x^2 + y^2)}{2(m(x^2 + y^2) - 1)}a_3 = 0,
\]

equivalent to the equation

\[
\ell(x^2 + y^2)w - 2xv + 2yu = 0.
\]

(2.7)

Then we have

Proposition 2.1. For the C-V metrics, the geodesics \( \gamma(t) \) starting at the origin and such that \( \dot{\gamma}(0) = (u, v, w) \) can be defined as the intersection of two surfaces:

- a circular cylinder with the generating line (ruling) parallel to the \( z \)-axis or a plane parallel to the \( z \)-axis;
• a surface of rotation around the z-axis.

Proof. According to equation (2.7), for \( \ell \neq 0 \) and \( w \neq 0 \) we find that the geodesics of the Heisenberg group \( \mathbb{H}_3 \), of the Lie group \( SU(2) \) and those of the universal covering of \( SL(2, \mathbb{R}) \) are contained in a cylinder at origin with generating lines parallel to the z-axis; for \( \ell \neq 0 \) and \( w = 0 \) the geodesics are contained in the plane \( vx - uy = 0 \) parallel to the z-axis.

For \( \ell = 0 \) we obtain that the geodesics of the product spaces \( S^2 \times \mathbb{R} \) and \( \mathbb{H}^2 \times \mathbb{R} \) are contained in the plane \( vx - uy = 0 \). The rotational surface is obtained by rotating a geodesic around the z-axis.

We shall describe briefly the method of finding the equations of geodesics taking into consideration the case \( m \neq 0 \) and \( \ell \neq 0 \), when the family of metrics (1.1) gives a metric of the spaces \( SU(2) \) and \( \tilde{SL}(2, \mathbb{R}) \). It is convenient to write the integrals in (2.6) in cylindrical coordinates. Considering the geodesic starting at origin and tangent at the vector \((u, v, w)\), the prime integrals of the equation of the geodesics and the unit norm of the vector \( \dot{\gamma} \) give the system

\[
\begin{align*}
\dot{\rho} \cos \theta & = u(1 + m\rho^2) \\
\dot{\rho} \sin \theta & = v(1 + m\rho^2) \\
\dot{z} & = 0 \\
\dot{\theta} & = 0 \\
\frac{\rho^2 \dot{\theta}}{(1 + m\rho^2)^2} & = \frac{l}{2} \frac{\rho^2 w}{1 + m\rho^2} = 0 \\
\frac{\rho^2 + \rho^2 \dot{\theta}^2}{(1 + m\rho^2)^2} & = u^2 + v^2 \\
\end{align*}
\]

(2.8)

For \( w = 0 \), the system (2.8) becomes

\[
\begin{align*}
\dot{\rho} \cos \theta & = u(1 + m\rho^2) \\
\dot{\rho} \sin \theta & = v(1 + m\rho^2) \\
\dot{z} & = 0 \\
\dot{\theta} & = 0 \\
\frac{\rho^2}{(1 + m\rho^2)^2} & = u^2 + v^2 \\
\end{align*}
\]

\[\Rightarrow\]

\[
\begin{align*}
\dot{\rho} \cos \theta & = u(1 + m\rho^2) \\
\dot{\rho} \sin \theta & = v(1 + m\rho^2) \\
z & = 0 \\
\theta & = a \\
\end{align*}
\]

\[\frac{\dot{\rho}}{(1 + m\rho^2)} = \pm \sqrt{u^2 + v^2}.
\]
The immediate integration of the last equation gives the equations of the geodesics for the spaces $SU(2)$ and $\tilde{S}L(2,\mathbb{R})$, respectively:

$$
\begin{align*}
\frac{x}{u^2+v^2} &= \tan(\sqrt{m(u^2+v^2)}t) \\
\frac{y}{u^2+v^2} &= \tan(\sqrt{m(u^2+v^2)}t) \\
z &= 0
\end{align*}
\begin{align*}
\frac{x}{\sqrt{u^2+v^2}} &= \tan(-m(u^2+v^2)t) \\
\frac{y}{\sqrt{u^2+v^2}} &= \tan(-m(u^2+v^2)t) \\
z &= 0
\end{align*}
$$

If $w \neq 0$, from the last two equations of system (2.8) we have

$$
d\theta = \frac{\ell w}{2(1 + m\rho^2)}dt, \quad d\rho = \frac{\ell w \tan \frac{\theta}{2}}{(1 + m\rho^2)\sqrt{(u^2 + v^2) - \ell^2 w^2\rho^2}} = \pm dt.
$$

Now we put $\ell^2 w^2 = a^2$, $u^2 + v^2 = b^2$ and $a^2 + b^2 m \neq 0$. Then by integrating we obtain

$$
\rho^2 = \frac{b^2 \tan At}{A^2 + a^2 \tan At}, \quad \theta = \arctan \frac{\ell w \tan At}{A},
$$

where $A = \sqrt{\ell^2 w^2 + 4m(u^2 + v^2)}$. We shall give a list of all geodesics obtained together with their graphical representation.

• For $\ell \neq 0$ and $\ell^2 w^2 + 4m(u^2 + v^2) > 0$, we have the following equations:

$$
\begin{align*}
x &= \frac{2 \tan(\frac{At}{2})}{\sqrt{A^2 + \ell^2 w^2 \tan^2(\frac{At}{2})}} (u \cos T - v \sin T) \\
y &= \frac{2 \tan(\frac{At}{2})}{\sqrt{A^2 + \ell^2 w^2 \tan^2(\frac{At}{2})}} (v \cos T + u \sin T) \\
z &= wt - \frac{\ell^2 w}{4m} t - \frac{\ell w}{2m} T,
\end{align*}
$$

with $T = \arctan \frac{\ell w \tan(\frac{At}{2})}{A}$.

If $m > 0$ and $4m \neq \ell^2$ these equations determine the geodesics of $SU(2)$, while if $4m = \ell^2$ we have the geodesics of the sphere $S^3$. If $m < 0$ these equations determine the geodesics of $\tilde{S}L(2,\mathbb{R})$. 
For \( \ell \neq 0 \) and \( \ell^2 w^2 + 4m(u^2 + v^2) < 0, (m < 0) \), the parametric equations of the geodesics starting at the origin of \( \tilde{SL}(2, \mathbb{R}) \) in the case \( w \neq 0 \) are:

\[
\begin{align*}
    x &= \frac{2 \tanh(Ct)}{\sqrt{C^2 + \ell^2 w^2 \tanh^2(Ct)}} (u \cos T' - v \sin T') \\
    y &= \frac{2 \tanh(Ct)}{\sqrt{C^2 + \ell^2 w^2 \tanh^2(Ct)}} (v \cos T' + u \sin T') \\
    z &= wt - \frac{\ell^2 w}{4m} t - \frac{\ell w}{2m} T',
\end{align*}
\]

where \( C = \sqrt{-\ell^2 w^2 - 4m(u^2 + v^2)} \) and \( T' = \arctan \frac{\ell w \tanh(Ct)}{C} \).

For \( \ell \neq 0 \) and \( \ell^2 w^2 + 4m(u^2 + v^2) = 0, (m < 0) \), the parametric equations of the geodesics starting at the origin of \( \tilde{SL}(2, \mathbb{R}) \) are, for \( w \neq 0 \),

\[
\begin{align*}
    x &= \frac{2t}{\sqrt{4 + \ell^2 w^2 t^2}} (u \cos T - v \sin T) \\
    y &= \frac{2t}{\sqrt{4 + \ell^2 w^2 t^2}} (v \cos T + u \sin T) \quad T = \arctan \frac{\ell wt}{2}.
\end{align*}
\]

If \( m = 0 \) and \( \ell \neq 0 \), the parametric equations of the geodesics arising from the origin of the Heisenberg group \( \mathbb{H}_3 \) in the cases \( w \neq 0 \) and \( w = 0 \), are, respectively,

\[
\begin{align*}
    x(t) &= \frac{v}{\ell w} \cos(\ell wt) + \frac{u}{\ell w} \sin(\ell wt) - \frac{v}{\ell w} \\
    y(t) &= \frac{v}{\ell w} \sin(\ell wt) - \frac{u}{\ell w} \cos(\ell wt) + \frac{v}{\ell w} \\
    z(t) &= wt + \frac{u^2 + v^2}{2w} t - \frac{u^2 + v^2}{2w} \sin(\ell wt) \quad \begin{align*}
        x &= ut \\
        y &= vt \\
        z &= 0
    \end{align*}
\]

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• If \( m > 0, \ell = 0 \), we have the following cartesian equations of the geodesics starting at the origin of \( S^2 \times \mathbb{R} \), in the cases \( w \neq 0 \) and \( w = 0 \):

\[
\begin{align*}
&vx - uy = 0 \\
&x^2 + y^2 = \frac{1}{m} \tan^2 \left( \frac{\sqrt{m(u^2 + v^2)}}{w} z \right) \\
\end{align*}
\]

\[
\begin{align*}
&vx - uy = 0 \\
&z = 0
\end{align*}
\]

• If \( m < 0, \ell = 0 \), we have the following cartesian equations of the geodesics starting at the origin of \( H^2 \times \mathbb{R} \), respectively in the case \( w \neq 0 \) and \( w = 0 \):

\[
\begin{align*}
&vx - uy = 0 \\
&x^2 + y^2 = -\frac{1}{m} \tanh^2 \left( \frac{\sqrt{-m(u^2 + v^2)}}{w} z \right) \\
\end{align*}
\]

\[
\begin{align*}
&vx - uy = 0 \\
&z = 0
\end{align*}
\]

3 Totally geodesic submanifolds

By definition, a submanifold \( V \subset M \) is \textbf{totally geodesic} if each of its geodesics is also a geodesic of \( M \). We recall the following

**Theorem 3.1.** A submanifold \((V, g)\) of a Riemannian manifold \((M, \overline{g})\) is totally geodesic if and only if its second fundamental form is identically zero.
In [6], Hadamard took into consideration the problem of determining, locally, the Riemannian manifolds \((M,\bar{g})\), of dimension 3, endowed with a foliation \(\mathcal{F}\) so that each geodesic of \(M\) tangent to a leaf at a point is completely contained in the leaf. Then the leaves of the foliation \(\mathcal{F}\) are totally geodesic submanifolds and that is the reason why such a foliation is called \textbf{totally geodesic}.

Cesare Rimini in [12] studied the problem proposed by Hadamard. He looked for the three-dimensional manifolds with a complete group of isometries of dimension 4 that admit totally geodesic foliations. He gave some important properties of these manifolds, found the conditions so that such manifolds admit totally geodesic foliations and, finally, determined these foliations.

The results obtained by Rimini are resumed in the following theorems.

\textbf{Theorem 3.2.} \textit{(Rimini)} A Riemannian manifold \((M,\bar{g})\) admits a foliation \(\mathcal{F}\) of totally geodesic hypersurfaces of equation \(x^n = \text{const.}\) if and only if they are isometric and the isometries are determined by their orthogonal trajectories.

\textbf{Theorem 3.3.} \textit{(Ricci - Rimini)} If a space \((M, g)\), \(\dim(M) = 3\), contains a family of totally geodesic surfaces, then this consists on surfaces orthogonal to a principal Ricci direction \(\xi\).

The principal Ricci curvature of the space, relative to \(\xi\), calculated at \(p\), is equal to the Gauss curvature of the surface of the family mentioned above passing through \(p\), and thus it is constant along every principal curve.

\textbf{Theorem 3.4.} \textit{(Rimini)} In a Riemannian manifold of dimension 3, with isometry group of dimension 4, there are not totally geodesic surfaces, except for the product spaces. In these spaces there are two types of totally geodesic surfaces:

- the surfaces orthogonal to the systatic geodesics, that form a family of totally geodesic surfaces \(\mathcal{F}\) with non zero constant gaussian curvature;

- a cylinder having as generating lines the systatic lines and as generating curve a geodesic of one of the totally geodesic surfaces of the foliation \(\mathcal{F}\). Each of these totally geodesic surfaces has the Gaussian curvature equal to zero.

We want to determine the totally geodesic surfaces isometrically immersed in the Cartan-Vranceanu manifolds. According to Theorem 3.3 we have that a foliation \(\mathcal{F}\) is totally geodesic if at each point the orthogonal trajectory to \(\mathcal{F}\) is an isometry generated by a principal Ricci direction. The principal Ricci directions of \((M, ds^2_{\nu_m})\) are determined by the vector fields \(E_1, E_2, E_3\) in (2.5) and the only isometry generated by a principal Ricci direction is that generated by \(E_3\). Hence,
supposing that the foliation $\mathcal{F}$ is totally geodesic, the leaves must be orthogonal to the principal Ricci direction $E_3 = Z = \frac{\partial}{\partial z}$ and we have that

$$\mathcal{F} = \text{Ker}(\omega),$$

where

$$\omega = dz + \ell \frac{ydx - xdy}{2[1 + m(x^2 + y^2)].}$$

But

$$\omega \wedge d\omega = \ell dx \wedge dy \wedge dz,$$

and therefore the form $\omega$ will be integrable if and only if $\ell = 0$. So we have proved the following

**Theorem 3.5.** In a Cartan-Vranceanu space $(M, ds^2_{lm})$ there are not totally geodesic surfaces, with the exception of the product spaces

$$S^2(c) \times \mathbb{R} \quad \mathbb{H}^2(-c) \times \mathbb{R}.$$

In such spaces there are two types of totally geodesic surfaces:

- the surfaces $S^2(c) \times \{a\}$ and $\mathbb{H}^2(-c) \times \{a\}$;

- a cylinder having as generating lines the curves tangent to $E_3$ and as the generating curve a geodesic of $S^2(c) \times \{a\}$ or $\mathbb{H}^2(-c) \times \{a\}$. Each of these totally geodesic surfaces has the Gaussian curvature zero.

### 4 Totally umbilical surfaces

A **principal curve** on a surface is a curve whose tangent vectors are all contained in a principal Ricci direction and an **umbilical point** on a surface is a point where the principal Ricci curvatures are equal.

The hypersurfaces whose first and second fundamental form differ by a constant factor are called **totally umbilical**.

Let $\mathcal{F}$ be a foliation of codimension 1 defined on a Riemannian manifold of dimension 3. Let $\xi$ be a vector field normal to the leaves of $\mathcal{F}$. Then the Codazzi equation is

$$X \langle B(Y, Z), \xi \rangle - Y \langle B(X, Z), \xi \rangle - \langle B([X, Y], Z), \xi \rangle - \langle B(Y, \nabla_X Z), \xi \rangle + \langle B(X, \nabla_Y Z), \xi \rangle = \mathcal{R}(X, Y, \xi, Z).$$

If $\mathcal{F}$ is totally umbilical ($B = \lambda g$) then the first member of the equation is zero and thus

$$\mathcal{R}(X, Y, \xi, Z) = 0.$$

This relation implies that the integral curves of $\xi$ form one of the principal congruences of the considered space. So we have
Theorem 4.1. If in a Riemannian manifold of dimension 3 there is a totally umbilical foliation $\mathcal{F}$ of codimension 1, then this is orthogonal to a principal congruence.

Let $(M, \mathcal{F})$ be a 3-dimensional Riemannian manifold with isometry group of dimension 4. Then we have:

Theorem 4.2. If $(N, g) \subset (M, \mathcal{F})$ is a totally umbilical surface isometrically immersed in $M$, then $N$ is totally geodesic and $M$ is a product manifold.

Proof. Taking into consideration Theorem 4.1, the totally umbilical surfaces $(N, g) \subset (M, \mathcal{F})$, if there are any, must be orthogonal to one of the principal congruences. From Theorem 3.4 we have that the congruence of the systatic geodesics of $M$ admits orthogonal surfaces only if $M$ is $S^2(c) \times \mathbb{R}$ or $H^2(-c) \times \mathbb{R}$ and we have seen that these surfaces are totally geodesic. Hence each totally umbilical surface $(N, g) \subset (M, \mathcal{F})$, must contain the systatic geodesics passing through its points. If $B = \lambda g$, with respect to a basis $\{X, Y\}$ of orthonormal vector fields on $N$, the Gauss equation becomes

\[ R(X, Y, X, Y) - \overline{R}(X, Y, X, Y) = \lambda^2. \]

It follows that the wanted surface must satisfy

\[ R(X, Y, X, Y) - \overline{R}(X, Y, X, Y) \geq 0. \]

We have seen ([12], [11]) that any surface that contains the systatic geodesics has the Gaussian curvature $G = R(X, Y, X, Y)$ equal to zero. Therefore the sectional curvature $\overline{R}(X, Y, X, Y)$ must also be zero and thus

\[ \lambda = 0. \]

Hence we have that the second fundamental form is identically zero and thus $N$ is totally geodesic. We find then that $M$ is a product space $S^2(c) \times \mathbb{R}$ or $H^2(-c) \times \mathbb{R}$ and that the totally umbilical surfaces $N$ are totally geodesic. \qed

5 Geodesics for the rotational surfaces

The metrics ([11]) are invariant with respect to the rotations around the $z$-axis and this leads to the study of the rotational surfaces, of the form

\[ X(u, v) = (f(u) \cos v, f(u) \sin v, g(u)), \]

where $0 \leq v < 2\pi$ and $f, g$ are real functions with $f > 0$. In order to obtain the geodesics of the rotational surfaces we use the Euler-Lagrange equations

\[
\begin{aligned}
\frac{d}{dt} \left[ 2 \left( \frac{f'(u)^2}{[1 + mf(u)]^2} + g'(u)^2 \right) \dot{u} - \frac{ff'(u)^2g'(u)}{1 + mf(u)} \dot{v} \right] &= E'(u)\dot{u}^2 + 2F'(u)\dot{u}\dot{v} + G'(u)\dot{v}^2 \\
\frac{d}{dt} \left[ 4(f(u)^2 + \ell^2 f(u)^4) \right] \dot{u} - \frac{\ell f(u)^2 g'(u)}{1 + mf(u)} \dot{u} \end{aligned}
\]

\[ = 0. \]
We obtain that (see [10]):

**I. The parallels** \( u = u_0 \) will be geodesics if

\[
\frac{f'(u_0)[2 + \ell^2 f(u_0)^2 - 2mf(u_0)^2]}{[1 + mf(u_0)^2]^3} = 0,
\]

(5.9)

and thus we have:

- the only parallels which are geodesics of the rotational surfaces for \( \mathbb{H}_3, \mathbb{SL}(2, \mathbb{R}), \mathbb{H}^2 \times \mathbb{R} \) are, just as in the Euclidean case, those generated by the rotation of a point of the generating curve where the tangent is parallel to the axis of rotation \( (f' = 0) \). [For these spaces we have \( \ell^2 \geq 2m \).]

- For the rotational surfaces of the product manifold \( \mathbb{S}^2 \times \mathbb{R} \) the parallels which are geodesics have \( f' = 0 \) or \( f(u_0) = \frac{\sqrt{m}}{m} \). In this case we have \( \ell^2 < 2m \).

- For the rotational surfaces of \( SU(2) \), besides the parallels with \( f' = 0 \), there are the parallels for which \( f(u_0) = \sqrt{\frac{2}{2m+\tau^2}} \).

**II. The meridians** \( v = v_0 \) are geodesics if

\[
\frac{\ell f(u)^2 \sqrt{[1 + mf(u)^2]^2 - f'(u)^2}}{[1 + mf(u)^2]^2} = \text{const}.
\]

It follows that

- if \( \ell = 0 \) then for the rotational surfaces of the product manifold \( \mathbb{S}^2 \times \mathbb{R} \) and \( \mathbb{H}^2 \times \mathbb{R} \), as in the euclidian case, all the meridians are geodesics;

- all the meridians of the cylinders \( f(u) = \text{const.} \) are geodesics;

- if \( \ell \neq 0 \) the meridians are geodesics for \( m \geq 0 \) if the function \( f \) is

\[
f(u) = \tan(\frac{\sqrt{m}u + c}{\sqrt{m}}), \quad f(u) = u, \quad f(u) = \frac{\tanh(\sqrt{-m}u + c)}{\sqrt{-m}}
\]

or if \( f \) is a solution of the equation

\[
2f'(u) + 4mf(u)^2 f'(u) + 2m^2 f(u)^4 f'(u) - 2f'(u)^3
\]

\[
+ 2m f(u)^2 f'(u)^3 - f(u)f'(u)f''(u) - mf(u)^3 f'(u)f''(u) = 0.
\]

In the particular case of the **cylinder** of equation

\[
S(u, v) = (a \cos v, a \sin v, u), \quad a \in \mathbb{R},
\]

(5.10)

we obtain the following
Proposition 5.1. The geodesics of the cylinder are the curves of equation
\[ \gamma(s) = (a \cos(As + B), a \sin(As + B), Cs + D). \]
that include:
- the meridians;
- the parallels,
- the helices, that is curves with constant geodesic curvature and geodesic torsion, analogous of the helices of \( \mathbb{R}^3 \).

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