THE INHOMOGENEOUS HALL’S RAY

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1. Introduction

The expression
\[ M_+^{\alpha, \beta} = \liminf_{q \to \infty} q||q\alpha - \beta|| \]
measures how well multiples of a fixed irrational \( \alpha > 0 \) approximate a real number \( \beta \). A similar concept is defined by Rockett and Szüsz (26 Ch. 4, §9), where they consider the slight variant, \( M(\alpha, \beta) \), (the two-sided case) with the initial \( q \) replaced by \( |q| \). It is evident that (see, for example, (23, 20))
\[ M_+^{\alpha, \beta} = \min(M_+^{\alpha, \beta}, M_+^{\alpha, -\beta}). \]

We define
\[ S_+^{\alpha} = \{ M_+^{\alpha, \beta} : \beta \in \mathbb{R}^+ \}, \]
\[ S^{\alpha} = \{ M(\alpha, \beta) : \beta \in \mathbb{R}^+ \}. \]

We refer to the first set as the (one-sided) inhomogeneous approximation spectrum of \( \alpha \).

\( M_+^{\alpha, \beta} \) and the corresponding spectrum have been considered in precisely this form by various authors, (18 19 11 7), and the ideas relate to inhomogeneous minima of binary quadratic forms (2 7 11 6 1 3). In the celebrated paper (15), Hall showed that the Lagrange spectrum, \( \mathcal{L} = \{ M_+^{\alpha, 0} : \alpha \in \mathbb{R} \} \), contains an interval \([0, \mu_H]\) \((\mu_H > 0)\) subsequently called Hall’s Ray. The precise value of \( \mu_H \) has been determined by Freiman (13) in a heroic calculation; we refer the reader to (10), where this result is discussed in detail. Our aim here is to prove the existence of an interval \([0, \mu_{\alpha}]\) in the inhomogeneous spectrum for all irrationals \( \alpha \), though without a precise value for the maximum endpoint of the interval. It is clear that the result fails for rational \( \alpha \).

Since \( M_+^{\alpha, \beta} = M_+^{\alpha, \beta + 1} \), the values of \( \beta \) may be restricted to the unit interval \([0, 1]\). Similarly, we may assume without loss of generality that \( 0 \leq \alpha < 1 \). The key theorem of this paper is the following:

**Theorem 1.** For \( \alpha \) irrational, the set \( S_+^{\alpha} \) contains an interval of the form \([0, \mu_{\alpha}]\) for some \( \mu_{\alpha} > 0 \).

Once this is established, it is straightforward to extend to the two-sided case, and to binary quadratic forms.

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1.1. History. As far as we are aware, the first work on inhomogeneous minima dates back to Minkowski [21] who expressed his results in terms of binary quadratic forms. He showed that if \( a, b, c, d \) are real numbers with \( \Delta = ad - bc \neq 0 \) then, for any real numbers \( \lambda \) and \( \mu \), there are integers \( m, n \) such that

\[
|(am - bn - \lambda)(cm - dn - \mu)| \leq \frac{1}{4} \Delta.
\]

This implies that \( \inf_q ||q\alpha - \beta|| \leq \frac{1}{4} \) for all \( \alpha, \beta \). The same conclusion is true for \( \mathcal{M}(\alpha, \beta) \) but this requires more work. In fact Khintchine [17] proved that \( \mathcal{M}_+(\alpha, \beta) \leq \frac{1}{3} \), and the result with \( \frac{1}{4} \) replacing \( \frac{1}{3} \) is claimed by Cassels as derivable from his methods in [8].

Khintchine [16] showed that there exists \( \delta > 0 \) such that, for any \( \alpha \), there exists \( \beta \) for which

\[
\mathcal{M}(\alpha, \beta) \geq \delta.
\]

In fact, like Minkowski, he deals with the infimum rather than \( \lim \inf \). Fukusawa gave an explicit value for \( \delta \) of \( \frac{1}{457} \) and this was subsequently improved by Davenport (\( \delta = \frac{1}{73.9} \)) [12] and by Prasad (\( \delta = \frac{3}{32} \)) [25]. These papers are of special significance because they develop a methodology for handling calculations of values of \( \mathcal{M}_+(\alpha, \beta) \) that has been the cornerstone of much subsequent work, and underlies the techniques used in this paper.

Far too many authors have contributed to the understanding of \( \mathcal{M}(\alpha, \beta) \) and \( \mathcal{M}_+(\alpha, \beta) \) for us to reference all of the papers here. As far as we are aware, the first ray results occur in [14], Satz XIII, where it is shown that if, in a semi-regular continued fraction expansion of \( \alpha \), the partial quotients tend to \( \infty \) then \( \mathcal{S}(\alpha) \) contains the interval \( [0, \frac{1}{4}] \). Barnes obtains essentially the same result in [2] though he states a weaker one: that, for each \( t \in [0, \frac{1}{4}] \), there are uncountably many \( \alpha \)'s and \( \beta \)'s with \( \mathcal{M}(\alpha, \beta) = t \).

The predominant methodology for handling problems of this kind, originating with Davenport [12], invokes some form of continued fraction expansion of \( \alpha \) and a corresponding digit expansion of \( \beta \). We will use this methodology but choose to use the negative continued fraction because of the simple and “decimal”-like geometrical interpretation of the expansion of \( \beta \) associated with it (which we call the Davenport Expansion). Use of the regular continued fraction is possible, and was first done by Prasad [25], but makes the construction less intuitive and more complicated from our perspective, because divisions of subintervals alternate in direction. The general machinery for the regular continued fraction is well-exposed in Rockett and Szüsz [26]. Cassels also uses the Davenport expansion ideas in his paper [8], without attribution, where he shows that, except for special cases, \( \mathcal{M}_+(\alpha, \beta) \leq \frac{4}{11} \).

Several authors have contributed to refinement of the technique, including Sós [27], and Cusick, Rockett and Szüsz [9]. These authors ascribe the origin of the technique to Cassels in [8].

Almost all of the work for this paper, including a more complicated proof of the main theorem, was done in the early 1990’s, and versions of it have
been circulating privately since then. Its ideas and results have been used and cited in various places, in particular, in [23, 24].

2. The negative continued fraction expansion for $\alpha$

Here we briefly describe the features needed from the theory of the negative continued fraction. For a more complete discussion of the corresponding concepts for the regular continued fraction, see [26] or for the more general semi-regular continued fraction see Perron [22]. For $0 < \alpha < 1$, let $\alpha_1 = \alpha$, $a_1 = \left\lceil \frac{1}{\alpha_1} \right\rceil$, and define, recursively,

$$a_i = \left\lfloor \frac{1}{\alpha_i} \right\rfloor \quad \text{and} \quad \alpha_{i+1} = a_i - \frac{1}{\alpha_i}.$$ 

so that $a_i \geq 2$ and $0 < \alpha_{i+1} < 1$, for all $i$. Evidently, $\alpha$ has the continued fraction expansion

$$\alpha = \frac{1}{a_1 - \frac{1}{a_2 - \frac{1}{\ddots}}},$$

abbreviated as $\alpha = \langle a_1, a_2, a_3, \ldots \rangle$. The numbers $\alpha_i$ are called the $i$th complete quotients of $\alpha$ and satisfy

$$\alpha_i = \langle a_i, a_{i+1}, a_{i+2}, \ldots \rangle.$$ 

Since $\alpha$ is irrational, the partial quotients $a_i$ are greater than 2 for infinitely many indices $i$, and so there is a unique sequence $a'_1, a'_2, a'_3, \ldots$ of positive integers such that

$$a_1, a_2, a_3, \ldots = a'_1 + \frac{1}{a'_2 + \frac{1}{a'_3 + \frac{1}{\ddots}}},$$

It will be necessary occasionally to discuss the usual continued fraction expansion of $\alpha$, now expressible as

$$\alpha = \frac{1}{a'_1 + \frac{1}{a'_2 + \frac{1}{a'_3 + \frac{1}{\ddots}}}}.$$ 

Eventually, we will split the proof of Theorem 1 into two cases, corresponding to whether or not the sequence $(a'_n)$ is bounded.

We make use of the (negative continued fraction) convergents $p_i/q_i$ to $\alpha$:

$$\frac{p_i}{q_i} = \langle a_1, a_2, \ldots, a_i \rangle,$$
satisfying the recurrence relations

\[ p_{i+1} = a_{i+1}p_i - p_{i-1} \quad \text{and} \quad q_{i+1} = a_{i+1}q_i - q_{i-1} \]

where \( i \geq 1 \) and \( p_0, q_0 = 0, 1 \). Easily established are the following simple properties:

\[ 1 = p_iq_{i-1} - q_ip_{i-1} \]

\[ \alpha = (a_i - \alpha_{i+1})p_{i-1} - p_{i-2} = \frac{p_i - \alpha_{i+1}p_{i-1}}{q_i - \alpha_{i+1}q_{i-1}}. \]

Moreover, \( q_{i-1}/q_i = \overline{\alpha}_i \) where

\[ \overline{\alpha}_i = \langle a_i, a_{i-1}, \ldots, a_1 \rangle. \]

Since \( q_0 = 1 \), the identity

\[ q_i = \frac{1}{\overline{\alpha}_1 \overline{\alpha}_2 \cdots \overline{\alpha}_i} \]

follows.

This section concludes with a brief description of the Ostrowski expansion (see [26]) for positive integers. Any given integer \( q \geq 1 \) can be written as a sum of the form

\[ q = \sum_{k=1}^{n} c_kq_{k-1} \]

where

\[ c_n \geq 1 \quad \text{and} \quad 0 \leq c_k \leq a_k - 1 \quad \text{for} \quad 1 \leq k \leq n. \]

A greedy algorithm is used to determine the coefficients \( c_n \).

It is not hard to verify that

\[ q_k - 1 = (a_1 - 2)q_0 + (a_2 - 2)q_1 + \cdots + (a_{k-1} - 2)q_{k-2} + (a_k - 1)q_{k-1}. \]

This last identity yields that, for no pair of indices \( i \) and \( j \), is there a consecutive subsequence of coefficients of the form

\[ (c_i, c_{i+1}, \ldots, c_j) = (a_i - 1, a_{i+1} - 2, a_{i+2} - 2, \ldots, a_{j-1} - 2, a_j - 1). \]

The basic facts about the Ostrowski expansion are described in the following lemma.

**Lemma 2.1.** Each integer \( q \geq 1 \) has a unique expansion of the form [10] such that the constraint [11] holds and no consecutive sub-sequence of coefficients is of the form [13].
3. The Davenport Expansion of $\beta$

We now describe the Davenport Expansion for the elements $\beta$ of the interval $[0, 1)$. While the expansion is analogous to that used in [11], we remind the reader that it is based on a different continued fraction algorithm. This approach results in a “decimal”-like geometry of the Davenport expansion in the negative continued fraction case which makes more intuitive the invocation of Hall’s theorem on sums of Cantor sets [15] later. This is a key component of the proof in the bounded case.

For $0 \leq \beta < 1$, let $\beta_1 = \beta$ and define, inductively,

$$b_i = \left\lfloor \frac{\beta_i}{\alpha_i} \right\rfloor \quad \text{and} \quad \beta_{i+1} = \frac{\beta_i}{\alpha_i} - b_i,$$

so that $0 \leq b_i \leq a_i - 1$ and $0 \leq \beta_{i+1} < 1$. The convergent sum $\beta = \sum_{k=1}^{\infty} b_k D_k$ is called the Davenport expansion of $\beta$ or the Davenport sum of the sequence $(b_k)$ relative to $\alpha$. The integers $b_i$ are the Davenport coefficients.

In the same way as in the decimal expansion $0.999\ldots$ is identified with $1.000\ldots$, we identify (14) $b_1, b_2, \ldots, b_i, a_i - 1, a_{i+1} - 2, a_{i+2} - 2, \ldots$, with $b_1, b_2, \ldots, b_i + 1, 0, 0, \ldots$ for $b_i < a_i - 1$, since their Davenport sums are the same.

Figure 1 gives an illustration of the geometry of the situation for the case when $\alpha = (5, 3, 5, 3, \ldots)$. The interval $[0, 1)$ is subdivided by the numbers $n\alpha \pmod{1}$, $(n = 1, 2, 3, 4)$ into 5 intervals, the first four of which are “long” and the last “short” since $5\alpha > 1$. When we allow $n$ to range up to 13, each long interval is then subdivided into 3 intervals with the same pattern in each: 2 “long” intervals and 1 “short” interval, whereas the “short” interval is divided into just 1 “long” interval and 1 “short” interval. This pattern of “long” and “short” intervals is repeated at finer and finer resolutions as $n$ increases, reflecting, in this example, the periodic structure of the continued fraction. This structure corresponds to a “decimal” expansion with restrictions on digits, involving dependencies on the preceding digits. The general case is described below.

![Figure 1. The “Long-Short” Picture for $\alpha = (5, 3, 5, 3, \ldots)$](image)

From the inductive step in the Davenport expansion,

$$\beta_i = b_i \alpha_i + \beta_{i+1} \alpha_i,$$

and, as a result,

$$\beta_i = b_i \alpha_i + \beta_i b_{i+1} \alpha_{i+1} + \ldots + b_j (\alpha_i \alpha_{i+1} \ldots \alpha_j) + \beta_j (\alpha_i \alpha_{i+1} \ldots \alpha_j)$$
for all \( j \geq i \). Note that \( \beta_i \) is the location of \( \beta \) in the rescaled copy of the (long) interval in which it is contained. We define

\[
D_1 = 1, \quad D_i = \alpha_1 \alpha_2 \ldots \alpha_i
\]

and write

\[
\beta_i D_{i-1} = b_i D_i + b_{i+1} D_{i+1} + \cdots + b_j D_j + \beta_{j+1} D_j.
\]

\( D_i \) is the length of the long intervals at the \( i \)th level, and \( D_i - D_{i+1} \) is the length of the short intervals at that level.

The following result is straightforward.

**Theorem 2.** Let \( \beta = \sum_{k=1}^{\infty} b_k D_k \) where \((b_k)\) is a sequence of positive integers. Then \( 0 \leq \beta < 1 \) and \((b_k)\) are the Davenport coefficients of \( \beta \) if and only if \( b_i < a_i \) for all \( i \geq 1 \) and no block of the form

\[(16) \quad a_i - 1, a_{i+1} - 2, a_{i+2} - 2, \ldots, a_j - 1, a_j - 1\]

or of the form

\[(17) \quad a_i - 1, a_{i+1} - 2, a_{i+2} - 2, a_{i+3} - 2, \ldots\]

occurs in \((b_k)\).

The exceptional cases in this result; when \( b_i, b_{i+1}, \ldots, b_j \) is of the form \( a_i - 1, a_{i+1} - 2, a_{i+2} - 2, \ldots, a_j - 1, a_j - 1 \), correspond to the missing long intervals one level higher. As in the example in Figure 1, each short interval has one fewer long interval at the next level.

In the general geometric picture, \( a_1 - 1 \) multiples of \( \alpha \) subdivide the unit interval into \( a_1 \) intervals, the first \( a_1 - 1 \) of which have length \( \alpha \) and the last of length \( 1 - (a_1 - 1)\alpha \). The next multiple (modulo 1) is \( \alpha_1 \alpha_2 = a_1 \alpha - 1 \). This subdivides each of the long intervals at the previous level into \( a_2 - 2 \) intervals of the same length followed by a short interval. The final short interval of the initial subdivision is subdivided into \( a_2 - 2 \) long intervals followed by a short interval. This pattern is repeated at all finer resolutions with the appropriate partial quotients.

By means of the Davenport expansion, we can describe the integer pairs \((p, q)\) for which \( 0 < q \alpha - p < 1 \). It is straightforward to see that if \( q = \sum_{k=1}^{n} c_k q_{k-1} \) is the Ostrowski expansion of \( q \) then

\[
p = \sum_{k=1}^{n} c_k p_{k-1}, \quad i \geq 1.
\]

**Lemma 3.1.** (1) Let \( q \geq 1 \) be an integer with Ostrowski expansion as in \((10)\) and let \( p \) be defined by \((18)\). Then \( 0 < q \alpha - p < 1 \) and

\[
q \alpha - p = \sum_{k=1}^{\infty} b_k D_k
\]

is the Davenport expansion of \( q \alpha - p \), where \((b_k)\) is the sequence \( c_1, c_2, \ldots, c_n, 0, 0, 0, \ldots \).
Let $0 < \beta < 1$ and let $(b_i)$ be the Davenport coefficients of $\beta$. Then there are integers $q \geq 1$ and $p$ such that $\beta = qa - p$ if and only if there is $n \geq 1$ such that $b_i = 0$ for all $i > n$. Further, if that is so then $q = \sum_{k=1}^{n} b_k q_{k-1}$ and $p = \sum_{k=1}^{n} b_k p_{k-1}$.

4. Calculation of $\mathcal{M}_+ (\alpha, \beta)$ via the Davenport Expansion

The Davenport expansion will be used to calculate $\mathcal{M}_+ (\alpha, \beta)$. Again we stress that the underlying ideas are not really new, being essentially contained in the work of Davenport, Cassels, Sós, and others. Accordingly, we omit much of the justification and instead aim to provide geometrical insights.

To begin, let $0 \leq \beta < 1$ and let $(b_i)$ be the Davenport coefficients of $\beta$. We define

$$Q_n = \sum_{k=1}^{n} b_k q_{k-1}$$

$$Q'_n = \begin{cases} Q_n + q_{n-1} & \text{if } Q_n < q_n - q_{n-1} \\ Q_n + q_{n-1} - q_n & \text{if } Q_n \geq q_n - q_{n-1} \end{cases}$$

for all $n \geq 1$. The two cases here correspond to when $\beta$ lies in a long or a short interval, respectively, at the appropriate level of the decomposition of the interval. If $\beta$ is in a short interval, then the right endpoint of that interval occurred earlier in the decomposition; hence the $q_n - q_{n-1}$ term.

The next two lemmas are relatively straightforward consequences of these definitions and ideas.

\textbf{Lemma 4.1.}  
(1) $0 \leq Q_n < q_n$ for all $n \geq 1$ and $Q_n \geq q_n - q_{n-1}$ if and only if $b_n \neq 0$.  
(2) $Q_n \geq Q_{n-1}$ for all $n \geq 2$ and $Q_{n-1} = Q_n$ if and only if $b_n = 0$.  
(3) $0 \leq Q'_n < q_n$ for all $n \geq 1$ and $Q'_n \geq q_{n-1}$ if and only if $Q_n < q_n - q_{n-1}$.  
(4) $Q'_n \geq Q'_{n-1}$ for all $n \geq 2$ and $Q'_{n-1} = Q'_n$ if and only if $Q_n \geq q_n - q_{n-1}$.  
(5) The inequality $Q_n \geq q_n - q_{n-1}$ holds if and only if there is some index $m$ with $1 \leq m \leq n$ such that the sequence $b_m, b_{m+1}, \ldots, b_n$ is equal to $a_m - 1, a_{m+1} - 2, a_{m+2} - 2, \ldots, a_n - 2$.

The last condition, $Q_n \geq q_n - q_{n-1}$, occurs if the point $\beta$ is inside a short interval.

The integers $Q_n$ and $Q'_n$ are used to define quantities $\lambda_n(\beta)$ and $\rho_n(\beta)$, the significance of which will be evident from the following lemma.

\textbf{Definition 4.1.} Let $0 \leq \beta < 1$ and let $\beta_1, \beta_2, \beta_3, \ldots$ be the sequence of numbers generated by applying the Davenport expansion algorithm to $\beta$. We define

$$\lambda_n(\beta) = Q_n D_n \beta_{n+1}$$
Lemma 4.2. Let \( n < m, 0 < \beta < 1 \), and \( (b_i) \) be the Davenport coefficients of \( \beta \), with \( b_i \neq 0 \) for infinitely many \( i \). Then

\[
\begin{align*}
\lambda_n(\beta) &= Q_n||Q_n\alpha - \beta|| \quad \text{and} \quad \rho_n(\beta) = Q'_n||Q'_n\alpha - \beta||. \\
(1) & \\
(2) & \text{If } b_n = 0 \text{ then } \lambda_n(\beta) = \lambda_{n-1}(\beta). \text{ In other words if } \beta \text{ is in the first interval of the decomposition at level } n, \text{ then } Q_n\alpha \text{ is } Q_{n+1}\alpha \text{ modulo 1.} \\
(3) & \text{If } b_n \neq 0 \text{ and } b_m \neq 0 \text{ and } b_i = 0 \text{ for all } i \text{ which satisfy } n < i < m \text{ then } q_{n-1}D_m \leq \lambda_n(\beta) < q_nD_{m-1}. \\
(4) & \text{If } Q_n \geq q_n - q_{n-1} \text{ then } \rho_n(\beta) = \rho_{n-1}(\beta). \text{ In other words, if } \beta \text{ is a short interval (namely a rightmost) at level } n \text{ then } Q'_n\alpha \text{ is equal to } Q_n\alpha \text{ modulo 1.} \\
(5) & \text{If } Q_n < q_n - q_{n-1} \text{ and } Q_m < q_m - q_{m-1} \text{ and } Q_i \geq q_i - q_{i-1} \text{ for all } i \text{ which satisfy } n < i < m \text{ then } q_{n-1}D_m(1 - \alpha_{m+1}) \leq \rho_n(\beta) < q_nD_{m-1}(1 - \alpha_m) \text{ unless } m = n + 1 \text{ in which case } q_{n-1}D_{n+1}(1 - \alpha_{n+2}) \leq \rho_n(\beta) < q_nD_n. \\
\end{align*}
\]

The next lemma is a key step in calculating \( M_+(\alpha, \beta) \) in terms of \( \lambda_n(\beta) \) and \( \rho_n(\beta) \).

Lemma 4.3. For \( n \geq 1 \),

\[
\min\{\lambda_n(\beta), \rho_n(\beta), \lambda_{n+1}(\beta), \rho_{n+1}(\beta)\}
\]

is a lower bound for the infimum of the set \( \{q||q\alpha - \beta||: q_n \leq q < q_{n+1}\} \).

Proof. We sketch the proof of the result. The diagram showing the key ideas is given in Figure 2. Write \( I_n \) and \( I_{n+1} \) for the intervals prescribed by the Davenport expansion at level \( n \) and \( n+1 \) that contain \( \beta \): \( I_n = [Q_n\alpha, Q'_n\alpha], \ I_{n+1} = [Q_{n+1}\alpha, Q'_{n+1}\alpha] \). The obvious candidates for the smallest values of \( q||q\alpha - \beta|| \) for \( q_n \leq q \leq q_{n+1} \) are the cases \( q = Q_{n+1} \) or \( q = Q'_{n+1} \) — the left and right endpoints of the interval \( I_{n+1} \) at level \( n+1 \) containing \( \beta \). It is clear from fairly straightforward size considerations that they do better than
any \( q\alpha \in I_n (q_n \leq q \leq q_{n+1}) \). It is also clear that the candidates \( q = Q_n \) and \( q = Q'_n \) are better than any \( q\alpha \not\in I_n (q_n \leq q \leq q_{n+1}) \) since \( Q_n < q_n \). □

The key equation for calculation of \( M_+ (\alpha, \beta) \) is in the following theorem, which captures the important ingredient of the preceding lemma.

**Theorem 3.** If \( 0 < \beta < 1 \) and no integers \( q \geq 1 \) and \( p \) satisfy \( \beta = q\alpha - p \) then

\[
M_+ (\alpha, \beta) = \min \left\{ \liminf_{n \to \infty} \lambda_n (\beta), \liminf_{n \to \infty} \rho_n (\beta) \right\}.
\]

For completeness, we note that, in Theorem 3, we have not dealt with the possibility that \( \beta \) is of the form \( q\alpha - p \) where \( q \) and \( p \) are positive integers. In this case, we have

\[
M_+ (\alpha, \beta) = \liminf_{q' \to \infty} q'\|q'\alpha - q\alpha - p\| = \liminf_{q' \to \infty} q'\|(q' - q)\alpha\|
\]

and consequently

\[
M_+ (\alpha, \beta) = \liminf_{q' \to \infty} (q' - q)\|(q' - q)\alpha\| = M_+ (\alpha, 0).
\]

The quantity \( M_+ (\alpha, 0) \) it is, of course, the homogeneous approximation constant of \( \alpha \).

## 5. The Unbounded Case

In this section we dispense quickly and relatively straightforwardly with the case where \( \alpha \) has unbounded partial quotients \( (a_n^\alpha) \) in its ordinary continued fraction, before turning to the much more difficult case of bounded partial quotients. We write

\[
M_+ (\alpha) = \sup_{\beta} M_+ (\alpha, \beta).
\]

The following theorem is the key result of this section.

**Theorem 4.** If \( \alpha \) has unbounded partial quotients in its ordinary continued fraction then

\[
\{ M_+ (\alpha, \beta) : \beta \in \mathbb{R} \} = [0, M_+ (\alpha)].
\]
Lemma 5.1. Let \( \lambda_n(\beta) \) be a sequence of indices on which the partial quotients are strictly monotonically increasing. Now let \( \lambda_n(\beta) \) be chosen sufficiently sparse for our purposes.

Set \( \alpha = [0; a_1, a_2, \ldots] \) and let \( (n_k) \) be a sequence of indices on which the partial quotients are strictly monotonically increasing. Now let \( 0 < c < \mathcal{M}_+(\alpha) \) and choose \( \beta \) for which \( c < \mathcal{M}_+(\alpha, \beta) \leq \mathcal{M}_+(\alpha) \). Let its Davenport coefficients be \( (b^*_{i,j}) \) in the ordinary continued fraction. We will construct a sequence \( (c^*_j) \) so that \( c^*_j = b^*_{i,j} \) except on a subsequence of the \( n_k \) which will be chosen sufficiently sparse for our purposes.

Since \( \beta = \sum_{k=1}^{\infty} b^*_{i,k} D_k^* \), where \( D_k^* = q_{k+1}^* \alpha - b^*_{i,k-1} \), we put

\[
\lambda_n^*(\beta) = Q_n^* ||Q_n^* \alpha - \beta||
\]

The ordinary case of (22) (see\[8\] or \[26\]) gives

\[
(25) \quad \lambda_n^*(\beta) = \left( \sum_{k=1}^{n} b^*_{i,k} q_{k-1}^* \right) \left| \sum_{k=n+1}^{\infty} b^*_{i,k} D_k^* \right|
\]

Note that \( q_n^* D_n^* = [a_n^*, a_{n+1}^*, \ldots] / [a_n^*, a_{n-1}^*, a_{n-2}^*, \ldots, a_2^*, a_1^*] \), and so is absolutely bounded above and away from zero. For this choice of \( \beta \), this product is always at least \( 1/30 \) and so the second two terms in the product are each at least \( 1/60 \). Changing the value of \( b^*_{i,k} \) by 1 will change the value of \( \lambda_n^* \) by at most \( 1/(a_n^* - 1) \), so by choosing \( n = n_k \) and adjusting the value of \( b^*_{i,k} \) to \( c_{n_k} \), we replace \( \beta \) by \( \tilde{\beta} \), so that

\[
c < \min(\lambda_n(\beta), \lambda_{n-1}(\tilde{\beta})) < c + 2/a_n^*.
\]

By making this change at the indices \( n_k \) (so that \( a_{n_k}^* \to \infty \)), and putting \( c_n^* = b^*_{i,k} \) elsewhere, we obtain a number \( \gamma = \sum_k c_k^* D_k^* \) for which

\[
\mathcal{M}_+(\alpha, \gamma) = c,
\]

since the effect of these changes for other \( \lambda_n^* \) is smaller than that at \( n = n_k \) or \( n = n_{k-1} \). In fact we have:

**Lemma 5.1.** Let \( \beta \) have Davenport coefficients \( (b^*_{i,j}) \) in the ordinary continued fraction. Given any \( \epsilon > 0 \) and \( k \) sufficiently large, there is an \( M = M(k) < k/2 \) and \( N = N(k) \) such that if \( m \not\in (k - M, k + N) \) then any change in \( b^*_{i,j} \) will not change \( \lambda_m^*(\beta) \) or \( \rho_m^*(\beta) \) by more than \( \epsilon \).
Proof. This follows quickly by (25), since \( \alpha \) must have infinitely many partial quotients in its continued fraction expansion which are larger than 2. If \( k \) is sufficiently large then there are at least \(- \log \epsilon / \log 2\) such terms \( a_n^* \) in \( n \in [k/2, k) \) and at least \(- \log \epsilon / \log 2\) such terms \( a_n^* \) in \((k, k + N)\). Consequently any change in \( b_k \) will make a variation in the value of \( \lambda_{m}^*(\beta) \) and \( \rho_{m}^*(\beta) \) less than \( \epsilon \).

We now choose a sequence of the \( n_k \) which are sufficiently sparse that these intervals do not overlap. Choose \( c_{n_k} \) so that

\[
c < \min_{j \in [n_k - M(n_k), n_k + N(n_k)]} \min(\lambda_j^*(\gamma), \rho_j^*(\gamma)) < c + 2/a_n^*.
\]

This is clearly possible using the fact that changing \( b_k \) by 1 increases or decreases the expression in (25) by no more than \( 1/(a_n^* - 1) \). This completes the proof of the fact that for such well approximable \( \alpha \) the spectrum consists of a single ray. \( \square \)

### 6. The Bounded Case

In the light of results of the previous section, we restrict attention from this point to the case where the ordinary continued has bounded partial quotients \((a_n^*)\). This translates in the case of the negative continued fraction to the sequence \(a_1, a_2, a_3, \ldots \) being bounded, with least upper bound \( M \), and the lengths of the blocks of consecutive 2’s also being bounded with least upper bound \( N - 1 \geq 0 \). Then it follows from equations (4) and (5) that

\[
\frac{1}{M} < \bar{\sigma}_i < \frac{N}{N + 1}
\]

hold for all \( i \geq 1 \). We choose \( L \) to be the smallest integer such that

\[
\left( \frac{N}{N + 1} \right)^L \leq \frac{1 - \frac{N}{N + 1}}{(N + 1)^2} \frac{1 - \frac{N^2}{(N + 1)^2}}{M^N(M^2 - 1)}.
\]

The numbers \( N \) and \( L \) will figure significantly in the proof in the bounded case.

#### 6.1. Computation of \( M_+(\alpha, \beta) \).

We will define a collection of \( \beta \)'s, in terms of their Davenport coefficients, which \( \beta \) have the property that for some subsequence \((k(i))\) of positive integers

\[
M_+(\alpha, \beta) = \lim \inf_{i \to \infty} \lambda_{k(i)}(\beta).
\]

This enables us to work with just the \( \lambda_{k(i)} \) rather than the \( \rho_n \) and simplifies the rest of the proof of our main theorem. We assume throughout the remainder of the proof of the bounded partial quotient case that \( \beta \neq n\alpha + m \) for some integers \( n \) and \( m \).

We record some simple results in the following lemma.
Lemma 6.1. (1) For $i < j$,
\begin{equation}
q_i D_j = \frac{\alpha_{i+1} \alpha_{i+2} \ldots \alpha_j}{1 - \alpha_i \alpha_{i+1}}.
\end{equation}

(2) Let $r$ and $s$ be positive integers satisfying $r \geq sL$. Then
\begin{equation}
q_u D_{v-1} < q_{n-1} D_m (1 - \alpha_{m+1}) < q_{n-1} D_m
\end{equation}
whenever $u, v, n$ and $m$ are positive integers with $u + r < v$ and $n < m \leq n + s + N$.

Proof. The first part is a simple calculation. For the second part, note that the right inequality is obviously true since $0 < \alpha_m + 1 < 1$. To prove the left inequality we observe that (29) implies
\begin{equation}
q_u D_{v-1} = \frac{\alpha_{u+1} \alpha_{u+2} \ldots \alpha_{v-1}}{1 - \alpha_u \alpha_{u+1}}.
\end{equation}

Using (26), (8) and $u + r < v$, we have
\begin{equation}
q_u D_{v-1} < \frac{R^{v-u-1}}{1 - R^2} \leq \frac{R^r}{1 - R^2},
\end{equation}
where $R = N/(N + 1)$. Similarly,
\begin{equation}
q_{n-1} D_m (1 - \alpha_{m+1}) = \frac{\alpha_n \alpha_{n+1} \ldots \alpha_m (1 - \alpha_{m+1})}{1 - \alpha_{n-1} \alpha_n} > \frac{M^{-(s+N+1)} (1 - R)}{1 - M^{-2}}.
\end{equation}

The lemma is, therefore, true if
\begin{equation}
\frac{R^r}{1 - R^2} \leq \frac{M^{-(s+N+1)} (1 - R)}{1 - M^{-2}}
\end{equation}

Since $r \geq sL$ and $R < 1$ and $s \geq 1$ and $R^L M < 1$ we have $R^r M^{s-1} < R^s L M^{s-1} < R^L$ and the result follows immediately from the definition of $L$. \qed

Theorem 5. Choose positive integers $r$ and $s$ with $r \geq sL$, and an increasing sequence of indices $(k(i))$ with $k(i+1) > k(i) + r$. Let $0 < \beta < 1$ with Davenport coefficients $(b_i)$ satisfy:

(1) for each $i \geq 1$ the sequence $b_{k(i)+1}, b_{k(i)+2}, \ldots, b_{k(i)+r}$ is a block of $r$ zeros;

(2) there is no block of $N + s$ consecutive zeros in $(b_n)$ between $k(i) + r$ and $k(i+1)$,

(3) $\beta$ is not in short intervals at level $n$ for $N + s$ consecutive values of $n$, in other words the Davenport coefficients of $\beta$ contain no sequence of the form $a_j - 1, a_{j+1} - 2, \ldots, a_{j+N+s-1} - 2$.

then
\begin{equation}
\mathcal{M}_+ (\alpha, \beta) = \liminf_{i \to \infty} \lambda_{k(i)} (\beta).
\end{equation}
Proof. By Theorem 3, it is enough to show that
\[ \lambda_n(\beta) \geq \lambda_{k(i)}(\beta) \quad \text{or} \quad \lambda_n(\beta) \geq \lambda_{k(i+1)}(\beta) \]
and
\[
\rho_n(\beta) \geq \lambda_{k(i)}(\beta)
\]
for all integers \( n \) with \( k(i) \leq n < k(i+1) \), for \( i \) sufficiently large. We choose \( i > i_0 \) to ensure that some \( b_j \neq 0 \) for some \( j < i_0 \) and that \( \beta \) has appeared in a long interval before that stage. If this were not possible \( \beta \) would be a multiple of \( \alpha \) modulo 1. Now fix \( n \) between \( k(i) \) and \( k(i+1) \). We will liberally use the fact stated in Lemma 4.2 that we can move back and forth between \( \lambda_n(\beta) \) and \( \lambda_m(\beta) \) provided the intervening \( b_k \) are all zero. Similarly, at the other extreme, we could move back and forward between \( \rho_n(\beta) \) and \( \rho_m(\beta) \) provided that at the intervening levels \( \beta \) is in short intervals.

Choose \( u \leq k(i) < v \) to be such that \( b_j = 0 \) if \( u < j < v \) and to be the extreme integers with that property. We observe that \( v - u > r \). It follows from Lemma 1.2 that
\[ \lambda_{k(i)} < q_u D_{v-1}. \]
If \( n < k(i) + r \) then \( \lambda_{k(i)} = \lambda_n \). If not, then \( b_n \) is followed by a block of at most \( N + s \) zeros unless \( b_m = 0 \) for all \( m \) with \( n < m < k(i+1) \), in which case \( \lambda_{k(i+1)} = \lambda_n \). If \( \lambda_{k(i)} \neq \lambda_n \neq \lambda_{k(i+1)} \) then
\[ q_{n-1} D_m \leq \lambda_n(\beta), \]
for some \( m \leq n + N + s \). That \( \lambda_{k(i)}(\beta) \leq \lambda_n(\beta) \) follows from
\[ q_u D_{v-1} \leq q_{n-1} D_m, \]
which follows immediately from \( m \leq n + s + N \) and \( u + r < v \).

The argument to show that (30) holds when \( k(i) \leq n < k(i+1) \) is similar but uses the fact that \( \beta \) is not in a long sequence of consecutive short intervals.

6.2. Elements of \( S_+(\alpha) \). Now we give a construction for certain elements of \( S_+(\alpha) \) using Theorem 5. First we impose additional constraints on the sequence \( (k(i)) \) so that the limits of the sequences \( \overline{\alpha}_{k(i)} \) and \( \alpha_{k(i)+1} \) both exist. Moreover, the limits lie strictly between 0 and 1, since (26) and (8) hold for all \( i \geq 1 \) and \( 0 < 1/M < N/(N+1) < 1 \). The collection of \( \beta \) to be described in terms of their Davenport expansions will be the ones for which \( \mathcal{M}(\alpha, \beta) \) are in the Hall’s Ray.

Definition 6.1. We choose \( (K(i)) \) be an increasing sequence of indices with gaps \( K(i+1) - K(i) \) tending to infinity such that the limits
\[
\begin{align*}
\frac{a_1}{2}, a_2, a_3, & \ldots = \lim_{i \to \infty} a_{K(i)}, a_{K(i)-1}, \ldots, a_2, a_1 \ldots \\
\frac{a_1}{2}, a_2, a_3, & \ldots = \lim_{i \to \infty} a_{K(i)+1}, a_{K(i)+2}, a_{K(i)+3}, \ldots,
\end{align*}
\]

THE INHOMOGENEOUS HALL’S RAY 13
exist; that is, that in each case the sequence of integers eventually becomes constant. The existence of such a sequence follows quickly by a diagonal argument from the finiteness of the alphabet from which the \( a_i \)'s are chosen.

We write
\[
\alpha^- = \langle a^-_1, a^-_2, a^-_3, \ldots \rangle \quad \text{and} \quad \alpha^+ = \langle a^+_1, a^+_2, a^+_3, \ldots \rangle.
\]

The following lemma is a straightforward consequence of the properties of the sequence \( a_1, a_2, a_3, \ldots \).

**Lemma 6.2.** Each of the sequences \( \alpha^- \) and \( \alpha^+ \) have all of their terms less than or equal to \( M \) and contain no block of \( N \) consecutive 2's.

Evidently, the numbers \( \alpha^- \) and \( \alpha^+ \) are irrational with \( 0 < \alpha^- < 1 \) and the partial quotients of their regular continued fraction expansions satisfy (26).

All of the theory in the preceding sections is applicable to \( \alpha^- \) or \( \alpha^+ \) in place of \( \alpha \). We introduce the following notation. For \( i \geq 1 \), define
\[
\alpha^-_i = \langle a^-_i, a^-_{i+1}, a^-_{i+2}, \ldots \rangle \quad \text{and} \quad \alpha^+_i = \langle a^+_i, a^+_{i+1}, a^+_{i+2}, \ldots \rangle
\]
and set
\[
D^-_i = \alpha^-_1 \alpha^-_2 \ldots \alpha^-_i \quad \text{and} \quad D^+_i = \alpha^+_1 \alpha^+_2 \ldots \alpha^+_i.
\]

It follows from (32), (33), and the discussion above that
\[
\alpha^-_k = \lim_{i \to \infty} \alpha_{K(i)-k+1} \quad \text{and} \quad \alpha^+_k = \lim_{i \to \infty} \alpha_{K(i)+k}.
\]
Hence
\[
D^-_k = \lim_{i \to \infty} \alpha_{K(i)} \alpha_{K(i)-1} \ldots \alpha_{K(i)-k+1} = \lim_{i \to \infty} \frac{q_{K(i)-k}}{q_{K(i)}} D^-_{K(i)}
\]
and
\[
D^+_k = \lim_{i \to \infty} \alpha_{K(i)+1} \alpha_{K(i)+2} \ldots \alpha_{K(i)+k} = \lim_{i \to \infty} \frac{D^+_{K(i)+k}}{D^+_{K(i)}}.
\]
The next lemma, a crucial one in the proof, makes use of these identities.

**Lemma 6.3.** Let \( (b_i) \) be the Davenport coefficients of a number \( \beta \in [0,1] \) for which both of the limits
\[
b^-_1, b^-_2, b^-_3, \ldots = \lim_{i \to \infty} b_{K(i)}, b_{K(i)-1}, \ldots, b_1, 0, 0, 0, \ldots
\]
\[
b^+_1, b^+_2, b^+_3, \ldots = \lim_{i \to \infty} b_{K(i)+1}, b_{K(i)+2}, b_{K(i)+3}, \ldots
\]
exist and let
\[
\beta^- = \sum_{k=1}^{\infty} b^-_k D^-_k \quad \text{and} \quad \beta^+ = \sum_{k=1}^{\infty} b^+_k D^+_k.
\]
Then
\[
\lim_{i \to \infty} \lambda_{K(i)}(\beta) = \frac{\beta^- \beta^+}{1 - \alpha^- \alpha^+}.
\]
Proof. By definition
\[ \lambda_{K(i)}(\beta) = Q_{K(i)}D_{K(i)}\beta_{K(i)+1} = \frac{Q_{K(i)}}{q_{K(i)}} q_{K(i)}D_{K(i)}\beta_{K(i)+1} \]
and, by (33),
\[ \lim_{i \to \infty} q_{K(i)}D_{K(i)} = \frac{1}{1 - \alpha^-\alpha^+}. \]
In consequence, it is sufficient to observe that
\[ \lim_{i \to \infty} \frac{Q_{K(i)}}{q_{K(i)}} = \beta^- \quad \text{and} \quad \lim_{i \to \infty} \beta_{K(i)+1} = \beta^+. \]
This is a straightforward consequence of the fact that
\[ D_{K(i)}\beta_{K(i)+1} = \sum_{k=1}^{\infty} b_{K(i)+k}D_{K(i)+k} \]
and a corresponding expression for the first limit. \(\square\)

Now we define two Cantor-like subsets of \([0,1)\) in terms of their Davenport expansions.

**Definition 6.2.**

(1) \(\beta \in E(\alpha, s)\) if and only if in its Davenport coefficients \((b_i)\) no block \(b_i, b_{i+1}, \ldots, b_{i+s}\) consists solely of zeros or is of the form
\[ a_i - 2, a_{i+1} - 2, \ldots, a_{i+s-1} - 2, a_{i+s} - 1. \]

Note that this does not preclude tail sequences of the form \(a_i - 1, a_{i+1} - 2, a_{i+2} - 2, \ldots.\)

(2) \(\beta \in F(\alpha, s)\) if and only if in the sequence \(b_1, b_2, b_3, \ldots\) no block \(b_i, b_{i+1}, \ldots, b_{i+s}\) consists solely of zeros or is of the form
\[ a_i - 1, a_{i+1} - 2, a_{i+2} - 2, \ldots, a_{i+s} - 2. \]

We note that both of \(F(\alpha, s)\) and \(E(\alpha, s)\) are closed subsets of \([0,1]\).

We now state and prove the main result of this section.

**Theorem 6.** Let \(r\) and \(s\) be positive integers which satisfy \(s \geq N\) and \(r \geq sL\) and let \(\alpha^-\) and \(\alpha^+\) be defined by (33) and \(\alpha^+_r\) by (33) and \(D^+_r\) by (34). For all \(e \in E(\alpha^-, s)\) and \(f \in F(\alpha^+_r, s)\) there is some \(\beta\) with \(0 < \beta < 1\) such that
\[ M_+(\alpha, \beta) = \frac{efD^+_r}{1 - \alpha^-\alpha^+}. \]

**Proof.** We will exhibit appropriate Davenport expansions of \(\beta\) to achieve this result for \(f \in F(\alpha^+_r, s)\) and \(e \in E(\alpha^-, s)\).
Let $e \in E(\alpha^-, s)$ and $f \in F(\alpha^+_{r+1}, s)$. We shall prove there is a $\beta$ with $0 < \beta < 1$ which satisfies (36) by constructing its Davenport coefficients $(b_i)$. Specifically, we shall construct $b_1, b_2, b_3, \ldots$ so that the limits

$$
\lim_{i \to \infty} b_{K(i)}^-, b_{K(i)-1}, \ldots, b_1, 0, 0, 0, \ldots, b_{K(i)}^+, b_{K(i)+1}, b_{K(i)+2}, b_{K(i)+3}, \ldots
$$

exist and

(37) \hspace{1cm} e = \sum_{k=1}^{\infty} b_k^- D_k^- \quad \text{and} \quad fD_r^+ = \sum_{k=1}^{\infty} b_k^+ D_k^+.

Lemma 6.3 then yields:

(38) \hspace{1cm} \lim_{i \to \infty} \lambda_{K(i)}(\beta) = \frac{e f D_r^+}{1 - \alpha^- \alpha^+}.

We describe sequences $b_1^+, b_2^+, b_3^+, \ldots$ and $b_1^-, b_2^-, b_3^-, \ldots$ for which (38) holds. Let $f_1, f_2, f_3, \ldots$ be the Davenport coefficients of $f$ with respect to $\alpha^+_{r+1}$ and observe that

$$
f = \sum_{k=1}^{\infty} f_k \alpha^+_{r+1} \alpha^+_{r+2} \ldots \alpha^+_{r+k}.
$$

Multiplication by $D_r^+$ gives

$$
fD_r^+ = \sum_{k=1}^{\infty} f_k D_{r+k}^+.
$$

and therefore the right hand formula in (37) holds if we define

(39) \hspace{1cm} b_1^+, b_2^+, b_3^+, \ldots = 0, \ldots, 0, f_1, f_2, f_3, \ldots.

It is easily seen that these satisfy the appropriate conditions for a Davenport expansion.

For a number $e \in E(\alpha^-, s)$, we let $e_1, e_2, e_3, \ldots$ be the $\alpha^-$-expansion of $e$ and as above we observe that

$$
e = \sum_{k=1}^{\infty} e_k D_k^-.
$$

The left hand formula in (37) then holds if we set

$$
b_1^-, b_2^-, b_3^-, \ldots = e_1, e_2, e_3, \ldots.
$$

Next, we specify enough of the sequence $b_1, b_2, b_3, \ldots$ to ensure that (39) and (37) hold. At this point Figure 3 illustrates definition of the various pieces of the sequence.
Such a choice is clearly possible. We choose the sequence

\[ u(i) = a_{i-2} \quad \text{for all } i \geq i_0 \]

Furthermore we can also assume that, for all \( i \geq i_0 \),

\[
\begin{align*}
K(i+1), a_{K(i+1)+1}, a_{K(i+1)+2}, \ldots, a_{u(i)} & = a_1^+, a_2^+, \ldots, a_1^+ - K(i) \\
K(i), a_{K(i)+1}, a_{K(i)+2}, \ldots, a_{v(i)} & = a_1^-, a_2^-, \ldots, a_1^- - K(i) + v(i) + 1
\end{align*}
\]

We ensure (39) and (37) hold by defining

\[
\begin{align*}
b_{K(i)+1}, b_{K(i)+2}, \ldots, b_{u(i)} & = b_1^+, b_2^+, \ldots, b_1^+ - K(i) \\
b_{K(i)+1}, b_{K(i+1)-1}, \ldots, b_{v(i)} & = b_1^-, b_2^-, \ldots, b_1^- - K(i) + v(i) + 1
\end{align*}
\]

for all \( i \geq i_0 \).

Before completing the specification of \( (b_j) \) we further restrict \( i_0 \) and the sequences \( (u(i))_{i=i_0}^{\infty} \) and \( (v(i))_{i=i_0}^{\infty} \).

\[
\begin{align*}
b_{u(i)} & \neq 0 \\
b_{v(i)} & \neq 0
\end{align*}
\]

for all \( i \geq i_0 \). This is relatively easy to arrange from the properties of the \( K(i) \) in relation to the Davenport expansion, and of the sequences \( (u(i)) \) and \( (v(i)) \). To complete the specification of \( (b_j) \) we introduce one more sequence. We choose the sequence \( (w(i))_{i=i_0}^{\infty} \) so that

\[ u(i) < w(i) \leq u(i) + N \text{ and } a_{w(i)} \geq 3 \quad (i \geq i_0). \]

Such a choice is clearly possible.

We can now unambiguously define

\[
b_j = \begin{cases} 
0 & \text{if } 1 \leq j \leq K(i_0) \\
\ a_j - 2 & \text{if } u(i) < j < w(i) \text{ for some } i \geq i_0 \\
\ a_j - 3 & \text{if } j = w(i) \text{ for some } i \geq i_0 \\
\ a_j - 2 & \text{if } w(i) < j < v(i) \text{ for some } i \geq i_0.
\end{cases}
\]

It is not hard to verify that \( 0 \leq b_i < a_i \) for all \( i \geq 1 \) and since \( b_{w(i)} = a_{w(i)} - 3 \) for all \( i \geq i_0 \) it is also clear that no subsequence \( b_i, b_{i+1}, b_{i+2}, \ldots \) is
of the form $17$. It is easy to check that $(b_i)$ are Davenport coefficients by showing that no block $b_i, b_{i+1}, \ldots, b_j$ is of the form $16$.

Now we observe that the hypotheses of Theorem 5 holds with $k(i) = K(i)$ for all $i$ to complete the proof. □

6.3. Cantor dissections for $E(\alpha, s)$ and $F(\alpha, s)$. Our eventual aim is to show that if the integer $s$ is large enough then the product of the two sets $E(\alpha^-, s)$ and $F(\alpha^+, s)$, where $r \geq 1$, contains an interval. Towards that aim we describe each of these two sets in terms of Cantor dissections. We do this for a generic $\alpha$ rather than $\alpha^-$ and $\alpha^+$ at this stage. We collect together a few definitions.

**Definition 6.3.**

(1) $H(\alpha, s)$ and $G(\alpha, s)$ are the smallest closed intervals containing $F(\alpha, s)$ and $E(\alpha, s)$ respectively.

(2) For each sequence $c_n = c_1, c_2, \ldots, c_n$ of positive integers, define:

$$S(c_n) = \sum_{k=1}^{n} c_k D_k,$$

$$F(c_n) = \{ \gamma = \sum_{k=1}^{\infty} b_k D_k \in F(\alpha, s) : b_k = c_k, \ (k = 1, 2, \ldots, n) \}$$

where $\sum_{k=1}^{\infty} b_k D_k$ is the Davenport expansion of $\gamma$. Denote by $C(c_n)$ the smallest closed interval which contains $F(c_n)$. Observe that $C(c_n)$ may be the empty set.

(3) when $(c_n) \neq \emptyset$,

$$C(c_n) = [\underline{C}(c_n), \overline{C}(c_n)]$$

where

$$\underline{C}(c_n) = \inf C(c_n), \ \overline{C} = \sup C(c_n) \text{ and } |C(c_n)| = \overline{C}(c_n) - \underline{C}(c_n).$$

We allow the possibility that $n = 0$ in which case $C(\ ) = H(\alpha, s)$.

The dissection of $H(\alpha, s)$ to obtain $F(\alpha, s)$ begins by replacing $C(\ ) = H(\alpha, s)$ with the collection of intervals

$$\{C(0), C(1), \ldots, C(a_1 - 1)\}.$$

The $n$-th stage of the dissection replaces each non-empty interval $C(c_n)$ by the collection of intervals

$$(40) \quad \{C(c_{n+1}) : 0 \leq c_{n+1} < a_{n+1}\}.$$ 

From the definition of $C(c_n)$ it is clear that it is the smallest closed interval containing the collection of intervals $\{40\}$. Moreover the restrictions on the digits results in gaps between all of these. As an illustration, note that if $C(c_n)$ has the last $n - 1$ $c_k$ all equal to 0, then at the next level $C(c_n, 0) = \emptyset$. The same kind of phenomenon occurs at the opposite end because of the
restriction on the number of terms of the form $a_i - 2$. It is clear that this is a Cantor dissection that produces $F(\alpha, s)$, and we have

$$S(c_n) + D_{n+s+1} \leq C(c_n) \leq \overline{C}(c_n) \leq S(c_n) + D_n$$

and

$$C(c_{n+1}) \subset [S(c_n) + c_{n+1}D_{n+1} + D_{n+s+2}, S(c_n) + (c_{n+1} + 1)D_{n+1}]$$.

Clearly $|C(c_n)| \leq D_n$, and it is evident that

$$F(\alpha, s) = \bigcap_{n=1}^{\infty} \bigcup_{0 \leq c \leq a_i \leq a_i} \{C(c_n) \neq \emptyset : 0 \leq c_i < a_i\}.$$

Now we obtain more precise estimates of the values of the endpoints $C(c_n)$ and $\overline{C}(c_n)$.

**Lemma 6.4.** Let $s \geq N$, $C(c_n) \neq \emptyset$, $t$ the largest integer with $0 \leq t \leq n$ such that all of $c_{n-t+1}, c_{n-t+2}, \ldots, c_n$ are zero and $u$ the unique integer with $0 \leq u \leq n$ such that $c_{n-u+1}, c_{n-u+2}, \ldots, c_n$ is equal to

$$a_{n-u+1} - 1, a_{n-u+2} - 2, a_{n-u+3} - 2, \ldots, a_n - 2.$$

Then

$$C(c_n) \leq \begin{cases} S(c_n) + D_{n+s} & \text{if } t = 0 \\ S(c_n) + D_{n+1} + D_{n+s} & \text{if } t > 0 \end{cases}$$

and

$$\overline{C}(c_n) \geq \begin{cases} S(c_n) + D_n - D_{n+s-N} & \text{if } u = 0 \\ S(c_n) + D_{n+N+1} & \text{if } u > 0 \end{cases}$$

**Proof.** Write $C = C(c_n)$ and note that $C = C(c_n)$ is the number $\beta$ whose Davenport coefficients $(b_i)$ are of the form

$$c_1, c_2, \ldots, c_n, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, \ldots.$$

Note that $t \leq s$ else $c_1, c_2, \ldots, c_n$ ends with more than $s$ consecutive zeros and $C = \emptyset$. In other words,

$$C = \sum_{k=1}^{n} c_k D_k + D_{n+s-t+1} + D_{n+2s-t+2} + D_{n+3s-t+3} + \ldots,$$

and

$$C \leq \begin{cases} S(c_n) + D_{n+s+1} + D_{n+2s+2} + D_{n+3s+3} + \ldots & \text{if } t = 0 \\ S(c_n) + D_{n+1} + D_{n+s+2} + D_{n+2s+3} + \ldots & \text{if } t > 0 \end{cases}$$

Further, since $s \geq N$ we know

$$D_{n+s} > D_{n+s+1} + D_{n+2s+2} + D_{n+3s+3} + \ldots,$$

and

$$D_{n+s} > D_{n+s+2} + D_{n+2s+3} + D_{n+3s+4} + \ldots,$$

and the truth of the first statement of the lemma is evident.
We describe the Davenport expansion of $C = \overline{C}(c_n)$ next. Let $k(0) = n - u$ and inductively define the sequence $k(1), k(2), k(3), \ldots$ by choosing $k(i)$ to be the largest integer such that

$$k(i - 1) + 2 \leq k(i) \leq k(i - 1) + s + 1 \quad \text{and} \quad a_{k(i)} \geq 3.$$ 

This is possible by the properties of $a_n$ as enunciated in Lemma 6.2 for $\alpha = \alpha^-$ and $\alpha = \alpha^+$. Further,

$$k(i) \geq k(i - 1) + s - N + 2$$

because if not $a_k = 2$ for all $k$ between and including $k(i - 1) + s - N + 2$ and $k(i - 1) + s + 1$ contrary to the definition of $N$. Now $C$ is the number $\beta$ whose Davenport coefficients $(b_i)$ are defined by

$$b_i = \begin{cases} 
  c_i & \text{if } i \leq k(0) \\
  a_i - 1 & \text{if } i = k(j) + 1 \text{ for some } j \geq 0 \\
  a_i - 2 & \text{if } k(j) + 1 < i < k(j + 1) \text{ for some } j \geq 0 \\
  a_i - 3 & \text{if } i = k(j) \text{ for some } j \geq 1.
\end{cases}$$

These are clearly Davenport coefficients, and the sequence contains no block $b_i, b_{i+1}, \ldots, b_{i+s}$ of the form (35) nor does it contain a block of $b_i, b_{i+1}, \ldots, b_{i+s}$ consisting solely of zeros. We conclude that $\beta \in F(\alpha, s)$. It is also fairly clear that the sequence $b_1, b_2, b_3, \ldots$ begins with $c_1, c_2, \ldots, c_n$.

It remains to show that no other element of $C(c_n)$ is larger than $\beta$. If that were the case, and there were some $\beta' \in C(c_n)$ with Davenport coefficients $(b'_i)$ such that $\beta' > \beta$. However, the form of the definition of $\beta$ prohibits any possible increase in the values of the Davenport coefficients while staying a member of $F(\alpha, s)$ and starting with $c_1, c_2, \ldots, c_n$. Evidently $C = \sum_{k=1}^{\infty} b_k D_k$. By truncating this series at the term with index $k(1) + 1$ and making some minor rearrangements we find that

$$\overline{C} > \sum_{l=1}^{k(0)} c_l D_l + \sum_{l=k(0)+1}^{k(1)+1} (a_l - 2)D_l + D_{k(0)+1} - D_{k(1)} + D_{k(1)+1}.$$ 

We consider two cases. First we suppose $u = 0$ and hence $k(0) = n$. In this case, since

$$D_n < \sum_{l=n+1}^{k(1)+1} (a_l - 2)D_l + D_{n+1} + D_{k(1)+1}$$

we obtain $\overline{C} > S(c_n) + D_n - D_{k(1)}$. It is easy to deduce from (41) with $i = 1$ that $D_{k(1)} > D_{n+s-N}$ and the second statement of the lemma is proved. Now we suppose $u > 0$ and hence $k(0) < n$. Since $k(1) \geq n + 1$,

$$\overline{C} > \sum_{l=1}^{n} c_l D_l + \sum_{l=n+1}^{k(1)+1} (a_l - 2)D_l - D_{k(1)} + D_{k(1)+1}.$$
As \(a_{k(1)} \geq 3\) and \(a_{k(1)+1} \geq 2\) we have
\[
\bar{C} > \sum_{l=1}^{n} c_l D_l + \sum_{l=n+1}^{k(1)-1} (a_l - 2) D_l + D_{k(1)+1}.
\]
The definition of \(N\) implies there is some \(i\) with \(n + 1 \leq i \leq n + N + 1\) such that \(a_i \geq 3\) and so
\[
\sum_{l=n+1}^{n+N+1} (a_l - 2) D_l \geq D_{n+N+1}.
\]
It follows that if \(k(1) - 1 \geq n + N + 1\) then the second statement of the lemma is true. If on the other hand \(k(1) < n + N + 1\) then \(D_{k(1)+1} \geq D_{n+N+1}\) and again the truth of the second statement is clear. This completes the proof of the lemma.

A key remark about the \(F(\alpha,s)\) construction, that will not be true for the case if \(E(\alpha,s)\), is that, at least generically, the deleted intervals resulting from the “zeros” condition and the “\(a_n - 2\)” condition in this Cantor construction have the property that their left and right endpoints respectively are \(S(c_1,\ldots,c_n + 1)\).

Next we deal with the set \(E(\alpha,s)\). This is a little more complicated; the two restrictions on the Davenport coefficients of the elements of \(E(\alpha,s)\) no longer correspond to a single gap in the dissection of \(G(\alpha,s)\). We make the following definition.

**Definition 6.4.** \(A(\ ) = [0,1 - D_1]\) and \(B(\ ) = [1 - D_1,1]\). For each sequence \(c_n = c_1,c_2,\ldots,c_n\) of positive integers, we define \(A(c_n)\) to be the smallest closed interval containing \(E(\alpha,s) \cap [S(c_n),S(c_n) + D_n - D_{n+1}]\), and \(B(c_n)\) to be the smallest closed interval containing \(E(\alpha,s) \cap [S(c_n) + D_n - D_{n+1}, S(c_n) + D_n]\).

The dissection of \(G(\alpha,s)\) begins by replacing \(G(\alpha,s)\) with the pair of intervals \(A(\ )\) and \(B(\ )\). The next step is the substitution
\[
A(\ ) \to \{A(0),B(0),A(1),B(1),\ldots,A(a_1 - 3),B(a_1 - 3),A(a_1 - 2)\}
\]
\[
B(\ ) \to \{B(a_1 - 2), A(a_1 - 1)\}.
\]
The \(n\)-th step of the dissection is
\[
\emptyset \neq A(c_n) \to \{A(c_{n+1}) : 0 \leq c_{n+1} \leq a_{n+1} - 2\}
\]
\[
\cup \{B(c_{n+1}) : 0 \leq c_{n+1} \leq a_{n+1} - 3\}
\]
\[
\emptyset \neq A(c_n) \to \{B(c_1,c_2,\ldots,c_n,a_{n+1} - 2), A(c_1,c_2,\ldots,c_n,a_{n+1} - 1)\}.
\]
where again we use the notation \(c_n = c_1,c_2,\ldots,c_n\).

We note that \(A(c_n)\) and \(B(c_n)\) are the smallest closed intervals containing the collections \(c_n\) at the previous level. This follows because
\[
S(c_n) + D_n - D_{n+1} = S(c_1,c_2,\ldots,c_n,a_{n+1} - 2) + D_{n+1} - D_{n+2}.
\]
For the moment, we write

\[ A = A(c_n) \quad B = B(c_n) \quad S = S(c_n). \]

Now let \( \beta \in E(\alpha, s) \) and suppose its sequence of Davenport coefficients \( (b_i) \) begins with \( c_1, c_2, \ldots, c_n \). Then the block \( b_{n+1}, b_{n+2}, \ldots, b_{n+s+1} \) does not consist entirely of zeros, and so \( b_i \geq 1 \) and \( \beta \geq S + D_i \) for some \( i \) with \( n + 1 \leq i \leq n + s + 1 \). Hence \( \beta \geq S + D_{n+s+1} \). Since \( A \) is the smallest closed interval containing all such numbers \( \beta \) which also satisfy \( \beta \leq S + D_n - D_{n+1} \) it follows that

\[ S + D_{n+s+1} \leq A \leq S + D_n - D_{n+1}. \]

In particular \( |A| \leq D_n \). If, on the other hand, \( \beta > S + D_n - D_{n+1} \), there is \( i \geq n + 1 \) such that the block \( b_{n+1}, b_{n+2}, \ldots, b_i \) is of the form

\[ a_{n+1} - 2, a_{n+2} - 2, \ldots, a_{i-1} - 2, a_i - 1. \]

We conclude that

\[ S + D_n - D_{n+1} + D_{n+s+1} \leq B \leq S + D_n. \]

In fact,

\[ A(c_{n+1}) \subset [S + c_{n+1}D_{n+1} + D_{n+s+2}, S + (c_{n+1} + 1)D_{n+1} - D_{n+2}] \]

\[ B(c_{n+1}) \subset [S + (c_{n+1} + 1)D_{n+1} - D_{n+2} + D_{n+s+2}, S + (c_{n+1} + 1)D_{n+1}]. \]

We note that all such intervals where \( 0 \leq c_{n+1} < a_{n+1} \) are disjoint, and since

\[ E(\alpha, s) = \bigcap_{n=1}^{\infty} \bigcup \{ A(c_n) \neq \emptyset, B(c_n) \neq \emptyset : 0 \leq c_i < a_i \}, \]

it is totally disconnected. Again the gaps arise because of the constraints on digits in the definition of \( E(\alpha, s) \).

Now we need to find estimates for the endpoints of the intervals \( A(c_n) \) and \( B(c_n) \), just as we have for \( C(c_n) \) in Lemma 6.4.

**Lemma 6.5.** Let \( s \geq N \) and \( A(c_n) \neq \emptyset \). Then

\[ \underline{A}(c_n) < S(c_n) + D_n - D_{n+1} - D_{n+3N} \]

and

\[ \overline{A}(c_n) = S(c_n) + D_n - D_{n+1}. \]

Further,

\[ A(c_n) < S(c_n) + D_{n+s} \]

whenever \( n = 0 \) or \( B(c_1, c_2, \ldots, c_{n-1}, c_n - 1) \neq \emptyset \).

**Proof.** Write \( A = A(c_n) \). We note first that \( A \) is the number \( \beta \) whose Davenport coefficients \( (b_i) \) are of the form

\[ c_1, c_2, \ldots, c_n, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, \ldots \]
where \( t \) is the largest integer with \( 0 \leq t \leq n \) for which \( c_{n-t+1}, c_{n-t+2}, \ldots, c_n \) are all zero, and observe that there is some \( j \) satisfying

\[
n + 1 \leq j \leq n + s - t + N + 1
\]

such that \( b_j \leq a_j - 3 \) and \( b_i = a_i - 2 \) for all \( i \) with \( n + 1 \leq i \leq j - 1 \). It follows that

\[
A \leq \sum_{k=1}^{n} c_k D_k + \sum_{k=n+1}^{j-1} (a_k - 2) D_k + (a_j - 3) D_j + D_j.
\]

Since \( j \leq n + 2N \),

\[
A \leq \sum_{k=1}^{n} c_k D_k + \sum_{k=n+1}^{n+2N} (a_k - 2) D_k.
\]

We know

\[
\sum_{k=n+1}^{n+2N} (a_k - 2) D_k = D_n - D_{n+1} - \sum_{k=n+2N+1}^{\infty} (a_k - 2) D_k
\]

and since the definition of \( N \) implies there is some \( k \) with \( n + 2N < k \leq n + 3N \) such that \( a_k \geq 3 \) we conclude that (46) does not exceed \( D_n - D_{n+1} - D_{n+3N} \). The truth of the first statement of the lemma is now evident.

Now we redefine \( \beta \) as

\[
\beta = S(c_n) + D_n - D_{n+1}
\]

and observe that it has Davenport coefficients

\[
c_1, c_2, \ldots, c_n, a_{n+1} - 2, a_{n+2} - 2, a_{n+3} - 2, \ldots.
\]

This contains no block \( b_i, b_{i+1}, \ldots, b_j \) of the form (16) nor a block \( b_i, b_{i+1}, \ldots, b_{i+s} \) of the form (35) and so \( \beta \in E(\alpha, s) \).

Now suppose \( n = 0 \) or \( B(c_1, c_2, \ldots, c_n - 1) \) is non-empty. If \( B(c_1, c_2, \ldots, c_n - 1) \neq \emptyset \) then \( c_n \geq 1 \) and so \( t = 0 \). Obviously \( t \) is also zero if \( n = 0 \). As a result,

\[
A = \sum_{k=1}^{n} c_k D_k + D_{n+s+1} + D_{n+2s+2} + D_{n+3s+3} + \ldots.
\]

Because \( s \geq N \) we know that

\[
D_{n+s} \geq D_{n+s+1} + D_{n+2s+2} + D_{n+3s+3} + \ldots,
\]

and this is enough to complete the proof. \( \square \)

**Lemma 6.6.** Let \( s \geq N \) and \( B(c_n) \neq \emptyset \). Then

\[
\overline{B}(c_n) < S(c_n) + D_n - D_{n+1} + D_{n+2} + D_{n+s},
\]

\[
\overline{B}(c_n) = S(c_n) + D_n.
\]

Further,

\[
\overline{B}(c_n) < S(c_n) + D_n - D_{n+1} + D_{n+s}
\]

whenever \( n = 0 \) or \( c_n \neq a_n - 2 \) and \( A(c_n) \) is non-empty.
Proof. As usual, we write \( B = B(c_n) \) and observe that \( B \) contains the number \( \beta \) whose Davenport coefficients \( b_1, b_2, b_3, \ldots \) are equal to
\[
c_1, c_2, \ldots, c_n, a_{n+1} - 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, \ldots
\]
Therefore
\[
B \leq \sum_{k=1}^{n} c_k D_k + (a_{n+1} - 1)D_{n+1} + D_{n+s+2} + D_{n+2s+3} + D_{n+3s+4} + \ldots
\]
We note that
\[
D_{n+s} \geq D_{n+s+2} + D_{n+2s+3} + D_{n+3s+4} + \ldots
\]
The first inequality of the lemma then follows since \( a_{n+1}D_{n+1} = D_n + D_{n+2} \).

For the second statement of the lemma we consider \( B = \beta \) where \( \beta \) is
\[
(47) \quad \beta = S(c_n) + D_n,
\]
so that \( \beta = \sum_{k=1}^{\infty} c_k D_k \) where \( b_1, b_2, b_3, \ldots \) is the sequence
\[
c_1, c_2, \ldots, c_n, a_{n+1} - 1, a_{n+2} - 2, a_{n+3} - 2, a_{n+4} - 2, \ldots,
\]
and the rest is clear.

Now suppose either \( n = 0 \) or \( A(c_n) \neq \emptyset \) and \( c_n \neq a_n - 2 \). In this case \( B \) is the number \( \beta \) whose Davenport coefficient sequence \( (b_i) \) begins with
\[
(48) \quad c_1, c_2, \ldots, c_n, a_{n+1} - 2, a_{n+2} - 2, \ldots, a_{n+s-1} - 2, a_{n+s} - 1
\]
and continues with
\[
(49) \quad \overbrace{0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, \ldots}^s
\]
Clearly \( (b_i) \) is a sequence of Davenport coefficients and \( \beta \in E(\alpha, s) \). Further, since \( (b_i) \) begins with \( (48) \),
\[
\beta \geq \sum_{k=1}^{n} c_k D_k + \sum_{k=n+1}^{n+s-1} (a_k - 2)D_k + (a_{n+s} - 1)D_{n+s}.
\]
As the sequence of Davenport coefficients of \( B \) begins with \( (48) \) and continues with \( (49) \),
\[
B = \sum_{k=1}^{n} c_k D_k + \sum_{k=n+1}^{n+s} (a_k - 2)D_k + D_{n+s} + D_{n+2s+1} + D_{n+3s+2} + \ldots
\]
By using the appropriate identities of Section 3 we obtain
\[
B = \sum_{k=1}^{n} c_k D_k + D_n - D_{n+1} + D_{n+s+1} + D_{n+2s+1} + D_{n+3s+2} + D_{n+4s+3} + \ldots
\]
The usual arguments yield
\[
D_{n+s} \geq D_{n+s+1} + D_{n+2s+1} + D_{n+3s+2} + D_{n+4s+3} + \ldots
\]
and the truth of the final statement of the lemma is clear. \( \square \)
6.4. Application of Hall’s Theorem. As mentioned in the introduction to the last section, we shall now use a theorem of Hall, namely Theorem 2.2 in [15], to show that if \( s \) is large enough then the product of the sets \( E(\alpha, s) \) and \( F(\alpha, s) \) contains an interval. This idea was used in the context of inhomogeneous diophantine approximation by Cusick, Moran and Pollington, see [11]. The actual statement of Hall’s theorem [15] concerns the sum of Cantor sets but, as Hall points out, his result can be applied to products by taking logarithms. Specifically, we have

\[
\log(E(\alpha, s) F(\alpha, s)) = \log E(\alpha, s) + \log F(\alpha, s)
\]

and since the logarithm function is continuous and strictly increasing, it maps the Cantor dissections of \( G(\alpha, s) \) and \( H(\alpha, s) \) to Cantor dissections of \( \log G(\alpha, s) \) and \( \log H(\alpha, s) \), respectively.

Before applying Hall’s theorem we need to check that his Condition 1 holds. This condition states that if, in going from level \( n \) to \( n+1 \), an interval \( C \) is replaced by two disjoint intervals \( C_1 \) and \( C_2 \) with an open interval \( C_{12} \) between them, so that \( C_1 \cup C_{12} \cup C_2 = C \), then the length of \( C_{12} \) should not exceed the minimum of \( |C_1| \) and \( |C_2| \). We note, as Hall does in his discussion of bounded continued fractions, that the transition from the \( n \)th to the \( (n+1) \)th stage of the Cantor dissections leading to the sets \( F(\alpha, s) \) and \( E(\alpha, s) \) can be done by iteratively removing just one “middle” interval at a time. To verify Condition 1 of Hall, it is enough to show that for any pair of adjacent intervals formed at the \( n \)th stage of the Cantor dissection to produce either \( F(\alpha, s) \) or \( E(\alpha, s) \), the minimum of their lengths exceeds the length of the removed interval.

We can now verify this for the Cantor dissection for \( \log F(\alpha, s) \).

**Lemma 6.7.** There is an integer \( s_0 \geq N \) such that if \( s \geq s_0 \) and if \( C_1 \) and \( C_2 \) are non-empty neighbouring intervals arising at the same stage of the Cantor dissection for \( F(\alpha, s) \) then

\[
|\log C_{12}| \leq \min\{ |\log C_1|, |\log C_2| \}
\]

where \( C_{12} \) is the open interval lying between \( C_1 \) and \( C_2 \).

**Proof.** Let \( s \geq N \) and let \( C_1 \) and \( C_2 \) and \( C_{12} \) be as described. We assume without loss of generality that \( C_1 \) lies to the left of \( C_2 \). Our aim is to show that if \( s \) is large enough then the number

\[
|\log C_{12}| = \log C_2 - \log C_1
\]

is less than or equal to both

\[
|\log C_1| = \log C_1 - \log C_2 \text{ and } |\log C_2| = \log C_2 - \log C_1.
\]

By rearranging and using the properties of logarithms we reduce this statement to

\[
C_1 C_2 \leq C_1 C_2 \text{ and } C_2 C_2 \leq C_1 C_2.
\]
Note that, since
\[ 4C_1C_2 = (C_1 + C_2)^2 - (C_1 - C_2)^2, \]
to prove the first of the inequalities in (51) it is enough to show
\[ C_1 + C_2 < 2 \overline{C}_1, \]
and we concentrate on this.

Since \( C_1 \) and \( C_2 \) arise at the same stage of the dissection and \( C_1 \) lies to the left of \( C_2 \) we write
\[ C_1 = C(c_n) \quad \text{and} \quad C_2 = C(c_1, c_2, \ldots, c_{n-1}, c'_n) \]
where \( c'_n > c_n \). The key fact here is that \( C(c_1, c_2, \ldots, c_n, c) \) is empty only for the extreme values of \( c \), because of the conditions that describe \( F(\alpha, s) \). Hence \( c'_n = c_n + 1 \).

We write
\[ S_1 = S(c_n) \quad \text{and} \quad S_2 = S(c_1, c_2, \ldots, c_{n-1}, c'_n). \]
Note that \( S_2 = S_1 + D_n \). Assume \( t \) is the largest integer with \( 0 \leq t \leq n \) such that all of \( c_{n-t+1}, c_{n-t+2}, \ldots, c_n \) are zero and \( u \) the unique integer with \( 0 \leq u \leq n \) such that \( c_{n-u+1}, c_{n-u+2}, \ldots, c_n \) is equal to \( [45] \). We denote the corresponding integers for \( C_2 \) by \( t' \) and \( u' \), respectively. We know \( u' = 0 \) else \( c_1, c_2, \ldots, c_{n-1}, c'_n \) ends with
\[ a_{n-u+1} - 1, a_{n-u+2} - 2, a_{n-u+3} - 2, \ldots, a_{n-1} - 2, a_n - 1 \]
implying that \( C_2 = \emptyset \). Similarly, \( t' = 0 \) since \( c'_n \geq 1 \). Hence
\[ \overline{C}_1 > S_1 + D_n - D_{n+s-N} \quad \text{and} \quad \overline{C}_2 < S_2 + D_{n+s}. \]
and
\[ C_1 < S_1 + D_{n+1} + D_{n+s} \quad \text{and} \quad \overline{C}_2 > S_2 + D_{n+N+1} - D_{n+s-N}. \]
We are now ready to consider the inequalities in (51). The inequalities above imply that
\[ 2 \overline{C}_1 - (C_1 + C_2) > S_1 + 2D_n - 2D_{n+s-N} - S_2 - D_{n+1} - 2D_{n+s}. \]
Further, \( S_2 = S_1 + D_n \) and \( D_{n+s-N} \geq D_{n+s} \) and thus
\[ 2 \overline{C}_1 - (C_1 + C_2) > D_n - D_{n+1} - 4D_{n+s-N}. \]
Since (26) holds for all \( i \geq 1 \) we know there is some \( s_0 \geq N \) such that
\[ 1 - \alpha_{n+1} - 4 \alpha_{n+1} \alpha_{n+2} \ldots \alpha_{n+s-N} > 0 \]
and hence
\[ D_n - D_{n+1} - 4D_{n+s-N} > 0 \]
if \( s \geq s_0 \). Note that the size of \( s_0 \) is independent of \( n \). It follows that if \( s \geq s_0 \) then \( C_1 + C_2 < 2 \overline{C}_1 \) and we have the desired result.

For the second inequality in (51) we observe that
\[ \overline{C}_1 \overline{C}_2 - C_1 C_2 > (S_1 + D_n - D_{n+s-N}) \overline{C}_2 - (S_2 + D_{n+N+1} - D_{n+s-N}) - (S_2 + D_{n+s})^2. \]
Since $S_1 \geq 0$ and $S_2 \geq D_n$ and $D_{n+s-N} \geq D_{n+s}$ we have
\[ C_1 \overrightarrow{C_2} - C_2 \overrightarrow{C_2} > (D_n - D_{n+s-N})(D_n + D_{n+N+1} - D_{n+s-N}) - (D_n + D_{n+s-N})^2 \]
and hence
\[ C_1 \overrightarrow{C_2} - C_2 \overrightarrow{C_2} > D_n(D_{n+N+1} - 4D_{n+s-N}) - D_{n+N+1}D_{n+s-N}. \]
Clearly $D_n > D_{n+N+1}$ and therefore it suffices to show that if $s$ is large enough then
\[ D_{n+N+1} - 4D_{n+s-N} > D_{n+s-N} \]
or equivalently
\[ 1 > 5\alpha_n + 2\alpha_n + 3\ldots \alpha_n + s + n - N. \]
As above, this is an easy consequence of (51).

We can now verify that Hall’s Condition 1 holds for the dissection for $E(\alpha, s)$.

**Lemma 6.8.** There is an integer $s_0 \geq N$ such that if $s \geq s_0$ and if $C_1$ and $C_2$ are non-empty neighbouring intervals arising at the same stage of the Cantor dissection which produces $E(\alpha, s)$ then
\[ |\log C_{12}| \leq \min\{|\log C_1|, |\log C_2|\} \]
where $C_{12}$ is the open interval lying between $C_1$ and $C_2$.

**Proof.** Let $s \geq N$ and let $C_1$ and $C_2$ and $C_{12}$ be as described. We may assume without loss of generality that $C_1$ lies to the left of $C_2$. We know from proof of Lemma 6.7 that it is sufficient to prove the inequalities
(52)
\[ C_1 \overrightarrow{C_2} \leq \overrightarrow{C_1} \overrightarrow{C_1} \quad \text{and} \quad C_2 \overrightarrow{C_2} \leq \overrightarrow{C_1} \overrightarrow{C_2} \]
hold when $s$ is large enough. We can also make use of statement (51).

We consider two possibilities for $C_1$. We suppose first that
(53)
\[ C_1 = A(c_n). \]
In this case $B(c_1, c_2, \ldots, c_n) \neq \emptyset$ and therefore
\[ C_2 = B(c_n). \]
To see this, we produce a number $\beta$ that belongs to $B(c_n)$. To this end we note that in the Cantor dissection of $G(\alpha, s)$ the intervals
\[ A(c_1, c_2, \ldots, c_{n-1}, a_n - 2) \quad \text{and} \quad A(c_1, c_2, \ldots, c_{n-1}, a_n - 1) \]
have no right neighbours since they result from the dissection of $A(c_1, c_2, \ldots, c_{n-1})$ and $B(c_1, c_2, \ldots, c_{n-1})$, respectively. Therefore, either $n = 0$ or $c_n \leq a_n - 3$. It follows from the proof of Lemma 6.5 that $C_1$ lies in $E(\alpha, s)$ and has Davenport coefficients
\[ c_1, c_2, \ldots, c_n, a_n - 1, a_n + 2, a_n + 2, a_n + 3 - 2, \ldots. \]
Now let $\beta = \sum_{k=1}^{\infty} b_k D_k$ where $b_1, b_2, b_3, \ldots$ is the sequence
\[ c_1, c_2, \ldots, c_n, a_n + 1 - 1, a_n + 2 - 2, a_n + 3 - 2, a_n + 4 - 2, \ldots. \]
It is straightforward again to check that $\beta \in E(\alpha, s)$. It now follows from (44) that $\beta$ belongs to an interval of the form $A(c'_1, c'_2, \ldots, c'_n)$ or $B(c'_1, c'_2, \ldots, c'_n)$. By observing that $\beta = S(c_n) + D_n$

and applying the inequalities in (43), it can be seen that the only possibility is $\beta \in B(c_n) \neq \emptyset$.

We can now apply Lemmas 6.5 and 6.6 to $C_1$ and $C_2$. As usual, it is convenient to write $S = S(c_n)$. Lemma 6.5 implies

$$C_1 < S + D_n - D_{n+1} - D_{n+3N}$$

and $C_2 < S + D_n - D_{n+1} + D_{n+s}$

and Lemma 6.6 implies

$$C_1 = S + D_n - D_{n+1}$$

and $C_2 = S + D_n$.

It follows that

$$2 \overline{C_1} - (C_1 + C_2) > D_{n+3N} - D_{n+s}.$$

Since (26) holds for all $i \geq 1$ we know there is some $s_0 \geq 3N + 1$ such that

$$1 > \alpha_{n+3N+1} \alpha_{n+3N+2} \ldots \alpha_{n+s}.$$

We emphasis that the size of $s_0$ does not depend on $n$. For such a choice of $s_0$ we have $D_{n+3N} > D_{n+s}$ and hence $C_1 + C_2 < 2 \overline{C_1}$ for all $s \geq s_0$. An application of (51) gives first inequality in (52) for $s \geq s_0$.

For the second inequality in (52) we observe that

$$\overline{C_1} \overline{C_2} - C_1 C_2 > (S + D_n - D_{n+1})(S + D_n) - (S + D_n - D_{n+1} + D_{n+s})^2.$$

Since $S \geq 0$ it follows that

$$\overline{C_1} \overline{C_2} - C_1 C_2 > (D_n - D_{n+1})(D_{n+1} - 2D_{n+s}) - D_{n+s}^2.$$

Therefore it suffices to show there is some $s_0$ (which does not depend on $n$) such that

$$D_n - D_{n+1} > D_{n+s} \quad \text{and} \quad D_{n+1} - 2D_{n+s} > D_{n+s}$$

or equivalently

$$1 > \alpha_{n+1} + \alpha_{n+1} \alpha_{n+2} \ldots \alpha_{n+s} \quad \text{and} \quad 1 > 3 \alpha_{n+2} \alpha_{n+3} \ldots \alpha_{n+s}$$

for all $s \geq s_0$. This is easily done with the help of (26).

The other possibility for $C_1$ is that

$$C_1 = B(c_n).$$

It is easy to see that $\beta = \sum_{k=1}^{\infty} b_k D_k \in A(c_1, c_2, \ldots, c_{n-1}, c_n + 1)$ where $b_1, b_2, b_3, \ldots$ is the sequence

$$c_1, c_2, \ldots, c_{n-1}, c_n + 1, a_{n+1} - 2, a_{n+2} - 2, a_{n+3} - 2, \ldots,$$

and so $A(c_1, c_2, \ldots, c_{n-1}, c_n + 1) \neq \emptyset$. Therefore

$$C_2 = A(c_1, c_2, \ldots, c_{n-1}, c_n + 1).$$
Again we apply Lemmas 6.5 and 6.6 to \( C_1 \) and \( C_2 \). This time we write
\[
S_1 = S(c_n) \quad \text{and} \quad S_2 = S(c_1, c_2, \ldots, c_{n-1}, c_n + 1).
\]
Note that \( S_2 = S_1 + D_n \). Lemma 6.6 implies
\[
C_1 < S_1 + D_n - D_{n+1} + D_{n+2} + D_{n+s} \quad \text{and} \quad \overline{C}_1 = S_1 + D_n
\]
and Lemma 6.5 implies
\[
C_2 < S_2 + D_{n+s} \quad \text{and} \quad \overline{C}_2 = S_2 + D_n - D_{n+1}.
\]
These combine to yield
\[
2 \overline{C}_1 - (C_1 + C_2) > D_{n+1} - D_{n+2} - 2D_{n+s}.
\]
Using (26) we know there is some \( s_0 \geq 1 \) (which does not depend on \( n \)) such that
\[
1 > \alpha_{n+2} + 2 \alpha_{n+2} \alpha_{n+3} \ldots \alpha_{n+s}
\]
and hence \( D_{n+1} > D_{n+2} + 2D_{n+s} \) for all \( s \geq s_0 \). As a result \( C_1 + C_2 < 2 \overline{C}_1 \)
if \( s \geq s_0 \) and using (51) we conclude that the first inequality in (52) holds if \( s \) is large enough.

To see that the second inequality in (52) is true we note that
\[
\overline{C}_1 \overline{C}_2 - C_1 C_2 > (S_1 + D_n)(S_2 + D_n - D_{n+1}) - (S_2 + D_{n+s})^2.
\]
Since \( S_2 = S_1 + D_n \) and \( S \geq 0 \) it follows that
\[
\overline{C}_1 \overline{C}_2 - C_1 C_2 > D_n(D_n - D_{n+1} - 2D_{n+s}) - D_{n+s}^2.
\]
Therefore it suffices to show there is some \( s_0 \) (which does not depend on \( n \)) such that
\[
D_n - D_{n+1} - 2D_{n+s} > D_{n+s}
\]
or equivalently
\[
1 > \alpha_{n+1} + 3 \alpha_{n+1} \alpha_{n+2} \ldots \alpha_{n+s}
\]
for all \( s \geq s_0 \). Again this is easily done with the help of (26). \( \square \)

These sequence of lemmas leads to the following key precursor to the main result.

**Theorem 7.** There is an integer \( s_0 \geq N \) such that if \( s \geq s_0 \) and \( R = N/(N + 1) \) and
\[
P_1 = \frac{R^{2s}}{(1 - R(s-N))^2} \quad \text{and} \quad P_2 = 1 - R^{(s-N)}
\]
then \( P_2 \geq P_1 \) and the interval \([P_1, P_2]\) lies in the product of the sets \( E(\alpha, s) \) and \( F(\alpha, s) \).

**Proof.** The proof applies Theorem 2.2 in Hall’s paper [15] to the sum
(54)
\[
\log E(\alpha, s) + \log F(\alpha, s).
\]
It is appropriate to outline why this is possible. In the last section we showed that the sets \( E(\alpha, s) \) and \( F(\alpha, s) \) are the result of Cantor dissections of the intervals \( G(\alpha, s) \) and \( H(\alpha, s) \). By applying the logarithm function it follows
that the sets $\log E(\alpha, s)$ and $\log F(\alpha, s)$ are the result of Cantor dissections of the intervals $\log G(\alpha, s)$ and $\log H(\alpha, s)$. We know from Lemmas 6.7 and 6.8 that these dissections satisfy Condition 1 in Hall’s paper, if $s$ is large enough. In other words, there is some $s_0 \geq N$ such that for all $s \geq s_0$ Hall’s theorem applies to the sum (54). Note that since $R < 1$ we can choose $s_0$ so that we also have $P_2 > P_1$.

Hall’s theorem implies that the sum (54) contains the interval 
\[ [\log x_2 + \log y_2 - 2 \min\{\log x_2 - \log x_1, \log y_2 - \log y_1\}, \log x_2 + \log y_2] \]

where 
\[ x_1 = G(\alpha, s) \quad x_2 = G(\alpha, s) \quad y_1 = H(\alpha, s) \quad y_2 = H(\alpha, s). \]

It follows immediately that the product of $E(\alpha, s)$ and $F(\alpha, s)$ contains the interval 
\[ [x_2y_2(\max\{x_1/x_2, y_1/y_2\})^2, x_2y_2]. \]

To prove the lemma it suffices to show 
\[ (55) \quad x_2y_2(\max\{x_1/x_2, y_1/y_2\})^2 \leq P_1 \quad \text{and} \quad x_2y_2 \geq P_2. \]

To this end we observe that $G(\alpha, s) = B(\alpha, s)$ and $H(\alpha, s) = C(\alpha, s)$ and hence Lemma 6.4 and 6.6 imply $x_2 = 1$ and $y_2 > 1 - D_{s-N}$. Therefore $x_2y_2 > 1 - D_{s-N}$. We know $D_{s-N} = \alpha_1 \alpha_2 \ldots \alpha_{s-N}$ and since (26) holds for all $i \geq 1$ it is easy to see that the second inequality in (55) is true.

For the first inequality in (55) we observe that $G(\alpha, s) = A(\alpha, s)$ and $H(\alpha, s) = C(\alpha, s)$ and hence Lemmas 6.4 and 6.5 imply $x_1 < D_s$ and $y_1 < D_s$. Thus 
\[ x_1/x_2 < D_s \quad \text{and} \quad y_1/y_2 < D_s/(1 - D_{s-N}). \]

Clearly $x_2y_2 < 1$ and it follows that 
\[ x_2y_2(\max\{x_1/x_2, y_1/y_2\})^2 < \frac{D_s^2}{(1 - D_{s-N})^2}. \]

The truth of the first inequality in (55) can now be seen by expressing $D_{s-N}$ and $D_s$ in terms of the numbers $\alpha_i$ and applying (26). \qed

Finally, we return to the sets $E(\alpha^-, s)$ and $F(\alpha^+_{r+1}, s)$, where $r \geq 1$. Recall that $\alpha^-$ and $\alpha^+$ are defined by (33) and $\alpha^+_{r+1}$ by (33). We know from Lemma 6.2 that $\alpha^-$ and $\alpha^+$ satisfy all the constraints we have placed on $\alpha$. Clearly the same is true of $\alpha^+_{r+1}$. We can, therefore, replace $E(\alpha, s)$ and $F(\alpha, s)$ in Theorem 6.7 with $E(\alpha^-, s)$ and $F(\alpha^+_{r+1}, s)$, respectively. In this manner we obtain the following corollary.

**Corollary 1.** There is an integer $s_0 \geq N$ such that if $s \geq s_0$ and $R = N/(N + 1)$ and 
\[ P_1 = \frac{R^{2s}}{(1 - R^{s-N})^2} \quad \text{and} \quad P_2 = 1 - R^{s-N}. \]
then $P_2 \geq P_1$ and the product of the sets $E(\alpha^-, s)$ and $F(\alpha_{r+1}^+, s)$, where $r \geq 1$, contains the interval $[P_1, P_2]$.

6.5. The existence of Hall’s ray. In this section, we prove the existence of a Hall’s ray in the set $S_+(\alpha)$ in (11); that is we prove Theorem 1.

Proof of Theorem 1. The proof of this theorem consists of showing that the set $S_+(\alpha)$ contains a chain of intersecting intervals whose endpoints converge to zero. We shall construct the chain with the help of Theorem 6 and the Corollary to Theorem 7.

Let $s_0 \geq N$ be the integer mentioned in the Corollary to Theorem 7 and define $r_0$ to be the smallest integer which is greater than or equal to $s_0$. Note that since $L \geq 1$ we have $s_0 = \lfloor r_0/L \rfloor$ where as usual $\lfloor x \rfloor$ denotes the largest integer which is less than or equal to $x$. Now let $r$ be an integer with $r \geq r_0$ and put $s = \lfloor r/L \rfloor$. Since $r/s \geq L$ we can apply Theorem 6. Thus for every number $x$ in the product of the sets $E(\alpha^-, s)$ and $F(\alpha_{r+1}^+, s)$ there is some $\beta$ with $0 < \beta < 1$ such that

$$M^+(\alpha, \beta) = \frac{xD_r^+}{1 - \alpha^- \alpha^+}.$$

Because $r \geq r_0$ we know that $s \geq s_0$. Therefore Theorem 7 implies $P_1 \leq P_2$ and the product of the sets $E(\alpha^-, s)$ and $F(\alpha_{r+1}^+, s)$ contains the interval $[P_1, P_2]$ where

$$P_1 = \frac{R^{2s}}{(1 - R(s-N))^2} \quad \text{and} \quad P_2 = 1 - R^{(s-N)}$$

and $R = N/(N + 1)$. It follows that for every number $\mu$ in the interval

$$\left[ \frac{P_1 D_r^+}{1 - \alpha^- \alpha^+}, \frac{P_2 D_r^+}{1 - \alpha^- \alpha^+} \right]$$

there is some $\beta$ with $0 < \beta < 1$ such that $M(\alpha, \beta) = \mu$. In other words the interval (56) lies in the set $S_+(\alpha)$. Since $r$ was any integer with $r \geq r_0$ we conclude that $S_+(\alpha)$ contains a chain of intervals.

By choosing $s_0$ large enough we can ensure that the intervals just mentioned intersect. To this end let $s' = \lfloor (r + 1)/L \rfloor$ and set

$$P_1' = \frac{R^{2s'}}{(1 - R(s'-N))^2} \quad \text{and} \quad P_2' = 1 - R^{(s'-N)}.$$

Note that $s' \geq s$. According to the argument above, the interval for the integer $r + 1$ is

$$\left[ \frac{P_1 D_{r+1}^+}{1 - \alpha^- \alpha^+}, \frac{P_2 D_{r+1}^+}{1 - \alpha^- \alpha^+} \right].$$

It will overlap the interval (56) if both the inequalities

$$\frac{P_1 D_{r+1}^+}{1 - \alpha^- \alpha^+} \leq \frac{P_2 D_r^+}{1 - \alpha^- \alpha^+} \quad \text{and} \quad \frac{P_1 D_r^+}{1 - \alpha^- \alpha^+} \leq \frac{P_2 D_{r+1}^+}{1 - \alpha^- \alpha^+}$$

are satisfied.
hold. These inequalities become

\[ P_1' \alpha + r + 1 \leq P_2 \quad \text{and} \quad P_1 \leq P_2' \alpha + r + 1 \]

and, substituting for \( P_1, P_2, P_1' \) and \( P_2' \) and rearranging, we have

\[ R^{2s'} \alpha + r + 1 \leq (1 - R^{(s'-N)})^2 (1 - R^{(s-N)}) \]

and

\[ R^{2s} \leq (1 - R^{(s-N)})^2 (1 - R^{(s'-N)}) \alpha + r + 1 \]

Now we observe that \( R < 1 \) and hence the quantities \( R^{2s} \) and \( R^{(s-N)} \) and \( R^{2s'} \) and \( R^{(s'-N)} \) all converge to zero as \( s \) and \( s' \) increase to infinity. Since \( s' \geq s \geq s_0 \) and the term \( \alpha_{r+1}^+ \) satisfies \( 1/M < \alpha_{r+1}^+ < N/(N+1) \), it is clear that by choosing \( s_0 \) sufficiently large we can ensure that (57) always holds. We conclude as indicated that \( s_0 \) can be chosen so that successive members in the chain of intervals in \( S^+(\alpha) \) intersect one another. Evidently the endpoints of the interval (56) converge to zero as \( r \) increases to infinity.  

\[ \square \]

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THE INHOMOGENEOUS HALL’S RAY

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