On the minimal rank in non-reflexive operator spaces over finite fields

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Abstract
Let $U$ and $V$ be vector spaces over a field $K$, and $S$ be an $n$-dimensional linear subspace of $\mathcal{L}(U,V)$. The space $S$ is called algebraically reflexive whenever it contains every linear map $g : U \to V$ such that, for all $x \in U$, there exists $f \in S$ with $g(x) = f(x)$. A theorem of Meshulam and Šemrl states that if $S$ is not algebraically reflexive then it contains a non-zero operator $f$ of rank at most $2n - 2$, provided that $K$ has more than $n + 2$ elements. In this article, we prove that the provision on the cardinality of the underlying field is unnecessary. To do so, we demonstrate that the above result holds for all finite fields.

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1 Introduction

Let $K$ be an arbitrary field and $U$ and $V$ be vector spaces over $K$. Given a linear subspace $S$ of the space $\mathcal{L}(U,V)$ of all linear maps from $U$ to $V$, its reflexive closure is defined as

$$\mathcal{R}(S) := \{g \in \mathcal{L}(U,V) : \forall x \in U, \exists f \in S : g(x) = f(x)\};$$

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it is obviously a linear subspace of $\mathcal{L}(U, V)$ that contains $\mathcal{S}$, and one checks that $\mathcal{R}(\mathcal{S}) = \mathcal{R}(\mathcal{R}(\mathcal{S}))$. One says that $\mathcal{S}$ is (algebraically) reflexive whenever $\mathcal{R}(\mathcal{S}) = \mathcal{S}$.

An active research topic consists in finding sufficient conditions for the reflexivity of an operator space in terms of the dimension of $\mathcal{S}$ and the rank of its elements. Denote by

$$\operatorname{mrk}(\mathcal{S}) := \min \{ \operatorname{rk}(f) \mid f \in \mathcal{S} \setminus \{0\} \}$$

the minimal rank among the non-zero operators in $\mathcal{S}$ (here we do not distinguish between infinite cardinals and simply write $\operatorname{rk}(f) = +\infty$ if $f$ is not a finite rank operator).

Assume now that $\mathcal{S}$ is non-reflexive and finite-dimensional. In [2], Larson showed that

$$\operatorname{mrk}(\mathcal{S}) < +\infty.$$  

Hence, a finite-dimensional operator space that contains no non-zero operator of finite rank is always reflexive. A natural improvement is to give an upper-bound for $\operatorname{mrk}(\mathcal{S})$ with respect to the dimension of $\mathcal{S}$. In [1], Ding showed that

$$\operatorname{mrk}(\mathcal{S}) \leq (\dim \mathcal{S})^2.$$  

Later, this upper-bound was substantially improved by Meshulam and Šemrl: in [5], they showed that

$$\# \mathbb{K} > \dim \mathcal{S} + 2 \Rightarrow \operatorname{mrk}(\mathcal{S}) \leq 2 \dim \mathcal{S} - 2.$$  

Earlier, this result had been obtained by Li and Pan for the field of complex numbers [3].

For 2-dimensional spaces, this upper bound is known to be optimal (see [5]). For algebraically closed fields, Meshulam and Šemrl further improved the upper-bound as follows in [6]:

$$\operatorname{mrk}(\mathcal{S}) \leq \dim \mathcal{S}.$$  

In [8], we examined whether the upper-bound $2 \dim \mathcal{S} - 2$ from Meshulam and Šemrl’s result was optimal or if one could improve it in the case when $\dim \mathcal{S} \geq 3$. First, it was proved that this upper-bound still held under the milder cardinality assumption $\# \mathbb{K} > \dim \mathcal{S}$, and then, under that provision, a classification of the non-reflexive $n$-dimensional operator spaces $\mathcal{S}$ such that $\operatorname{mrk} \mathcal{S} = 2n - 2$ was
achieved (see Theorem 6.1 of [8]): it was shown in particular that the existence of such spaces is connected to the existence of exotic division algebra structures over the field $\mathbb{K}$, called left-division-bilinearizable (LDB) division algebras. The existence of LDB division algebras over $\mathbb{K}$ is deeply connected to the quadratic structure of $\mathbb{K}$. LDB division algebras were entirely classified in [7], and as a consequence the following result was obtained:

**Theorem 1.1.** Let $\mathcal{S}$ be a non-reflexive $n$-dimensional subspace of $\mathcal{L}(U, V)$, with $\# \mathbb{K} > n \geq 3$. If $\mathbb{K}$ has characteristic not 2 and $n \notin \{3, 5, 9\}$, then

$$\text{mrk}(\mathcal{S}) \leq 2n - 3.$$ 

If $\mathbb{K}$ has characteristic 2 and $n - 1$ is not a power of 2, then

$$\text{mrk}(\mathcal{S}) \leq 2n - 3.$$ 

For finite fields, this can even be improved as follows:

**Theorem 1.2.** Let $\mathcal{S}$ be a non-reflexive $n$-dimensional subspace of $\mathcal{L}(U, V)$, with $\mathbb{K}$ a finite field such that $\# \mathbb{K} > n \geq 3$. Then,

$$\text{mrk}(\mathcal{S}) \leq 2n - 3.$$ 

This follows from Theorem 6.1 of [8] and from the fact, over a finite field, a quadratic form whose dimension is greater than 2 is always isotropic.

In this article, we consider the situation of small finite fields. Until now, the best known result over such fields was the following one:

**Proposition 1.3** (See Theorem 4.5 in [8]). Let $\mathcal{S}$ be an $n$-dimensional non-reflexive operator space. Then,

$$\text{mrk}(\mathcal{S}) \leq \frac{n(n + 1)}{2}.$$ 

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1When $U$ and $V$ are finite-dimensional, Theorem 1.1 is a straightforward consequence of Theorem 6.1 of [8] and of Corollary 1.3 of [7]. To obtain the general case, it suffices to extend the former to all vector spaces $U$ and $V$, which can be done by noticing that Meshulam and Šemrl's Corollary 2.5 of [5] states that if $\text{mrk}(\mathcal{S}) = 2 \dim \mathcal{S} - 2$ then all the operators in $\mathcal{S}$ have finite-rank (provided that $\# \mathbb{K} > n + 2$, but it has been shown in [8] that it suffices to assume that $\# \mathbb{K} > n$) and one can then simply apply the above results to the reduced space associated with $\mathcal{S}$, whose source and target spaces are finite-dimensional (see Section 3 for the definition of that reduced space).
Here, we shall improve this upper-bound as follows, thus generalizing Meshulam and Šemrl’s theorem to all fields:

**Theorem 1.4.** Let $S$ be an $n$-dimensional non-reflexive operator space. Then,

$$\text{mrk}(S) \leq 2n - 2.$$ 

To achieve this, we will prove that Theorem 1.4 holds for all finite fields. In the case when the source space of the operators in $S$ is finite-dimensional, we will use counting techniques together with very basic results from linear algebra to obtain the above result (Section 2). These methods were inspired by an article of Meshulam and Šemrl [4], in which a similar technique was used to study locally linearly dependent spaces of operators over finite fields. In the last section, the general result will be derived from this situation by using a theorem of Larson [2].

Before we proceed with the proof of Theorem 1.4 we would like to make a few observations. First of all, Theorem 1.2 shows that $2n - 2$ is not an optimal upper-bound for finite fields of large cardinality and $n \geq 3$. We do not know whether the upper bound $2n - 3$ holds for arbitrary finite fields when $n \geq 3$. In any case, it is known that the optimal upper-bound must be greater than or equal to $n$, owing to the existence of $n$-dimensional division algebras over any finite field (see [6]). Our last remark is that results on non-reflexive spaces are often obtained as special cases of results on locally linearly dependent operator spaces. Recall that the subspace $S \subset \mathcal{L}(U, V)$ is called **locally linearly dependent** (in short: **LLD**) whenever every vector $x \in U$ is annihilated by some operator $f \in S \setminus \{0\}$. If $S$ is non-reflexive and one chooses $g \in \mathcal{R}(S) \setminus S$, then $S \oplus \mathbb{K}g$ is an LLD space of which $S$ is a linear hyperplane. Most of the results we have cited are actually special cases of results on linear hyperplanes of LLD spaces (provided that $\dim S \geq 2$; on the other hand, every 1-dimensional operator space is reflexive). However, it is not true that a linear hyperplane of an LLD space is always non-reflexive. One special feature of the proof of Theorem 1.2 is that we shall use the full power of the non-reflexivity assumption instead of relying only upon local linear dependence. We do not know whether the upper bound $2n - 2$ in Theorem 1.4 holds for linear hyperplanes of locally linearly dependent spaces as well (with $n \geq 2$).
2 Proof of Theorem 1.4 in the finite-dimensional setting over a finite field

Throughout this section, we assume that the field $\mathbb{K}$ is finite and $q$ denotes its cardinality. Let $U$ and $V$ be vector spaces over $\mathbb{K}$, and $S$ be a finite-dimensional non-reflexive subspace of $\mathcal{L}(U, V)$. Assume that $U$ is finite-dimensional, set

$$p := \dim U \quad \text{and} \quad n := \dim S,$$

and assume that

$$\mrk(S) > 2n - 2,$$

so that

$$p \geq 2n - 1.$$

We seek to find a contradiction. Classically, if $n \leq 1$ then $S$ would be reflexive and hence

$$n \geq 2.$$

Let us choose an operator $g \in R(S) \setminus S$. Then,

$$T := g + S$$

is an $n$-dimensional affine subspace of $\mathcal{L}(U, V)$ that does not contain $0$, and the assumption $g \in R(S)$ translates into:

$$\forall x \in U, \exists h \in T : h(x) = 0.$$

Let us consider the set

$$\mathcal{N} := \{(x, h) \in U \times T : h(x) = 0\}.$$

**Claim 1.** The affine space $T$ contains an operator $h$ such that $\rk h \leq n - 1$.

**Proof.** Assume on the contrary that no such operator exists. For all $x \in U$, either $x = 0$ and then all the operators $h \in T$ satisfy $h(x) = 0$, or $x \neq 0$ and then at least one operator $h \in T$ satisfies $h(x) = 0$. This leads to

$$q^n + q^p - 1 \leq \#\mathcal{N}.$$

On the other hand, for every $f \in T$, we have $\dim \ker f \leq p - n$ and hence at most $q^{p-n}$ vectors of $U$ are annihilated by $f$. This yields

$$\#\mathcal{N} \leq q^n q^{p-n} = q^p.$$

Combining the above two inequalities leads to $q^n - 1 \leq 0$, contradicting $n > 0$. \qed
Now, let us set 
\[ r := \min \{ \text{rk} \ h \ | \ h \in T \} . \]

Note that we have just proved that
\[ r \leq n - 1. \]

In the rest of the proof, we shall use the following simple remark: if there are distinct operators \( h_1 \) and \( h_2 \) in \( T \) such that \( \text{rk} h_1 = r \) and \( r k h_2 \leq 2n - 2 - r \), then \( h_1 - h_2 \) is a non-zero operator of \( S \) and
\[ \text{rk}(h_1 - h_2) \leq r + (2n - 2 - r) = 2n - 2, \]
which contradicts our assumption that \( \text{mrk}(S) > 2n - 2 \). Thus, we obtain:

**Claim 2.** The space \( T \) contains exactly one rank \( r \) operator, and all the other ones have their rank greater than \( 2n - 2 - r \).

Next, we prove:

**Claim 3.** One has \( r = n - 1 \).

**Proof.** Recall from the proof of Claim 1 that \( q^n + q^p - 1 \leq \#N \). On the other hand, the sole rank \( r \) operator of \( T \) annihilates exactly \( q^{p-r} \) vectors of \( U \), whereas every other operator in \( T \) annihilates at most \( q^{p-2n+1+r} \) vectors. This leads to
\[ \#N \leq q^{p-r} + (q^n - 1)q^{p-2n+1+r}, \]
and hence
\[ q^{p-r}(q^r - 1) \leq (q^n - 1)(q^{p-2n+1+r} - 1). \]

In particular, as \( r > 0 \) we find \( p - 2n + 1 + r > 0 \), and factoring yields
\[ q^{-n+1} \geq \frac{1 - q^{-r}}{(1 - q^{-n})(1 - q^{2n-p-1-r})}. \]

Obviously, as \( q \geq 2 \) and \( r > 0 \),
\[ \frac{1 - q^{-r}}{(1 - q^{-n})(1 - q^{2n-p-1-r})} > 1 - q^{-r} \geq \frac{1}{2}, \]
and hence \( r - n + 1 > -1 \), which leads to \( r \geq n - 1 \). \( \square \)
Now, we know that \( \mathcal{T} \) contains one rank \( n-1 \) operator, which we denote by \( h_0 \), and all the other ones have greater rank. Set

\[
m := \# \{ f \in \mathcal{T} : \operatorname{rk}(f) \leq n \},
\]

so that \( m \leq q^n \) and \( \mathcal{T} \) contains exactly \( m-1 \) rank \( n \) operators, and exactly \( q^n - m \) operators with rank greater than \( n \). This leads to

\[
\# \mathcal{N} \leq q^{n-1} + (m-1)q^{p-n} + (q^n - m)q^{p-n-1}.
\]

(1)

For every \( h \in \mathcal{T} \) such that \( \operatorname{rk} h = n \), we have

\[
\dim(Ker h \cap Ker h_0) \geq \dim Ker h + \dim Ker h_0 - \dim U = p - 2n + 1.
\]

Thus, at least \( q^{p-2n+1} \) vectors of \( Ker h_0 \) belong to \( Ker h \). Considering the subset

\[
\mathcal{N}' := \mathcal{N} \cap ((Ker h_0 \setminus \{0\}) \times (\mathcal{T} \setminus \{h_0\})�
\]

this leads to

\[
\# \mathcal{N}' \geq (q^{p-2n+1} - 1)(m-1),
\]

and hence

\[
q^n + q^p - 1 + (q^{p-2n+1} - 1)(m-1) \leq \# \mathcal{N}.
\]

(2)

Combining (1) with (2) leads to

\[
q^n + q^p - q^{p-2n+1} - q^{p-n+1} + q^{p-n} - q^{p-1} \leq m(q^{p-n} - q^{p-n-1} - q^{p-2n+1} + 1). \]

(3)

As \( q \geq 2 \) we have on the other hand

\[
q^{p-n} - q^{p-n-1} - q^{p-2n+1} + 1 \geq q^{p-n-1}(q - 1) - q^{p-2n+1} \geq q^{p-n-1} - q^{p-2n+1} \geq 0,
\]

where the last inequality comes from \( n \geq 2 \). As \( m \leq q^n \) we deduce that

\[
q^n + q^p - q^{p-2n+1} - q^{p-n+1} + q^{p-n} - q^{p-1} \leq q^n(q^{p-n} - q^{p-n-1} - q^{p-2n+1} + 1).
\]

Expanding and simplifying leads to

\[
q^{p-n} \leq q^{p-2n+1}.
\]

Yet, \( q^{p-2n+1} < q^{p-n} \) since \( n \geq 2 \).

This final contradiction shows that our initial assumption was wrong. This yields

\[
\operatorname{mrk}(\mathcal{S}) \leq 2n - 2,
\]

thereby completing the proof of Theorem 1.4 in the special case when \( K \) is finite and the source space of \( \mathcal{S} \) is finite-dimensional.
3 The generalization to operator spaces between infinite-dimensional spaces

Now, we complete the proof of Theorem 1.4 for finite fields. Assume that \( \mathbb{K} \) is finite.

We lose no generality in assuming that \( S \) is a minimal non-reflexive space. Then, by a theorem of Larson [2, Corollary 2.8], all the operators in \( S \) have finite rank. It follows that

\[
U_0 := \bigcap_{f \in S} \ker f
\]

has finite codimension in \( U \). Then, every \( f \in \mathcal{R}(S) \) naturally induces a linear operator

\[
\overrightarrow{f} : U/U_0 \to V
\]

with the same rank as \( f \), to the effect that the reduced space

\[
\overline{S} := \{ \overrightarrow{f} \mid f \in S \}
\]

has dimension \( n \) and the vector space \( \overline{\mathcal{R}(S)} \) is isomorphic to \( \mathcal{R}(S) \), whose dimension is greater than \( n \). One checks that \( \overline{\mathcal{R}(S)} \subset \overline{\mathcal{R}(S)} \) (actually, those spaces are equal), and hence \( \overline{S} \) is non-reflexive. Then, as \( U/U_0 \) is finite-dimensional, we deduce from Section 2 that

\[
\text{mrk}(S) = \text{mrk}(\overline{S}) \leq 2n - 2,
\]

which completes the proof.

Remark 3.1. If \( S \) contains an operator with infinite rank, then applying the above result to the subspace \( S_F \) of all finite rank operators in \( S \) - which, by a theorem of Larson [2], is non-reflexive - yields \( \text{mrk} S = \text{mrk} S_F \leq 2n - 4 \). Therefore, if \( \text{mrk} S \geq 2n - 3 \) then \( S \) contains only finite rank operators.

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