On a conjecture related to integer-valued polynomials

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Abstract. Using the following $\binom{4}{3}$ transformation formula

$$\sum_{k=0}^{n} \left(-\frac{x-1}{k}\right)^{2} \binom{x}{n-k}^{2} = \sum_{k=0}^{n} \binom{n+k}{2k}^{2} \left(\frac{2k}{k}\right)^{2} \binom{2x+k}{2k},$$

which can be proved by Zeilberger’s algorithm, we confirm some special cases of a recent conjecture of Z.-W. Sun on integer-valued polynomials.

Keywords: Zeilberger’s algorithm; Chu-Vandermonde summation; integer-valued polynomials; multi-variable Schmidt polynomials

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1 Introduction

Recall that a polynomial $P(x) \in \mathbb{Q}[x]$ is called integer-valued, if $P(x) \in \mathbb{Z}$ for all $x \in \mathbb{Z}$. During the past few years, integer-valued polynomials have been investigated by several authors (see, for example, [4,9,14]). Recently, Z.-W. Sun [15, Conjectures 35(i)] proposed the following conjecture.

**Conjecture 1.1 (Z.-W. Sun).** Let $l, m, n$ be positive integers and $\varepsilon = \pm 1$. Then the polynomial

$$\frac{1}{n} \sum_{k=0}^{n-1} \varepsilon^{k}(2k+1)^{2l-1} \sum_{j=0}^{k} \left(-\frac{x-1}{j}\right)^{m} \binom{x}{k-j}^{m}$$

is integer-valued.

By the Chu-Vandermonde summation formula, we have

$$\sum_{j=0}^{k} \left(-\frac{x-1}{j}\right) \binom{x}{k-j} = \binom{-1}{k} = (-1)^{k}.$$ 

Thus, by [10, Lemmas 2.3 and 2.4], we see that Conjecture 1.1 is true for $m = 1$. In this note, we shall confirm Conjecture 1.1 for $m = 2$. 


Theorem 1.2. Let \( l \) and \( n \) be positive integers and \( \varepsilon = \pm 1 \). Then the polynomial
\[
\frac{1}{n} \sum_{k=0}^{n-1} \varepsilon^k (2k + 1)^{2l-1} \sum_{j=0}^{k} \binom{-x - 1}{j}^2 \binom{x}{k-j}^2
\]
(1.1)
is integer-valued.

We shall also prove the following result, which confirms the \( l = 1 \) cases of [15, Conjectures 35(ii)].

Theorem 1.3. Let \( n \) be a positive integer. Then the polynomial
\[
\frac{1}{n^2} \sum_{k=0}^{n-1} (2k + 1) \sum_{j=0}^{k} \binom{-x - 1}{j}^2 \binom{x}{k-j}^2
\]
(1.2)
is integer-valued.

2 Proof of Theorem 1.2

We first require the following \( 4F_3 \) transformation formula.

Lemma 2.1. Let \( n \) be a non-negative integer. Then
\[
\sum_{k=0}^{n} \binom{-x - 1}{k}^2 \binom{x}{n-k}^2 = \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k}^2 \binom{x+k}{2k}.
\]
(2.1)

Proof. Denote the left-hand side or the right-hand side of (2.1) by \( S_n(x) \). Applying Zeilberger’s algorithm (see [1, 11]), we obtain
\[
(n + 2)^3 S_{n+2}(x) - (2n + 3)(n^2 + 2x^2 + 3n + 2x + 3)S_{n+1}(x) + (3n^2 + 3n + 1)S_n(x) = 0.
\]
That is to say, both sides of (2.1) satisfy the same recurrence relation of order 2. Moreover, the two sides of (2.1) are equal for \( n = 0, 1 \). This completes the proof.

Remark. Using Zeilberger’s algorithm, Z.-W. Sun [12, Eq. (3.1)] found the following identity:
\[
16^n \sum_{k=0}^{n} \binom{-1/2}{k}^2 \binom{-1/2}{n-k}^2 = \sum_{k=0}^{n} \binom{2k}{k}^3 \binom{k}{n-k}(-16)^{n-k},
\]
(2.2)
and he [13, Eq. (3.1)] gave the following formula:
\[
64^n \sum_{k=0}^{n} \binom{-1/4}{k}^2 \binom{-3/4}{n-k}^2 = \sum_{k=0}^{n} \binom{2k}{k}^3 \binom{2n-2k}{n-k}16^{n-k}.
\]
(2.3)
Here we point out that, for \( x = -1/2 \) and \( -3/4 \), Eq. (2.1) gives identities different from (2.2) and (2.3).
In [2], Chen and the author introduced the multi-variable Schmidt polynomials
\[ S_n(x_0, \ldots, x_n) = \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k} x_k. \]

In order to prove Theorem 1.2, we also need the following result, which is a special case of the last congruence in [2, Section 4].

**Lemma 2.2.** Let \( l \) and \( n \) positive integers and \( \varepsilon = \pm 1 \). Then all the coefficients in
\[ \sum_{k=0}^{n-1} \varepsilon^k (2k + 1) S_k(x_0, \ldots, x_k). \]

are multiples of \( n \).

**Proof of Theorem 1.2** For any non-negative integer \( k \), define
\[ x_k = \binom{2k}{k} \binom{x + k}{2k}. \]

Then the identity (2.1) may be rewritten as
\[ \sum_{k=0}^{n} \left( \frac{-x-1}{k} \right)^2 \binom{x}{n-k}^2 = \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k} x_k. \]  

(2.4)

It is easy to see that \( x_0, \ldots, x_n \) are all integers on condition that \( x \) is an integer. By Eq. (2.4) and Lemma 2.2 we see that the polynomial (1.1) is integer-valued. \( \square \)

### 3 Proof of Theorem 1.3

We need the following result, which can be easily proved by induction on \( n \). See also [2, Eq. (2.4)].

**Lemma 3.1.** Let \( n \) and \( k \) be non-negative integers with \( k \leq n-1 \). Then
\[ \sum_{m=k}^{n-1} (2m+1) \frac{m+k}{2k} \binom{2k}{k} = n \frac{n+k}{k+1} \binom{n+k}{k}. \]  

(3.1)

**Proof of Theorem 1.3** Using the identities (2.1) and (3.1), we have
\[ \sum_{m=0}^{n-1} (2m+1) \sum_{k=0}^{m} \left( \frac{-x-1}{k} \right)^2 \binom{x}{n-k}^2 = \sum_{m=0}^{n-1} (2m+1) \sum_{k=0}^{m} \binom{n+k}{2k} \binom{2k}{k}^2 \binom{x+k}{2k} = \sum_{k=0}^{n-1} n \frac{n+k}{k+1} \binom{n+k}{k} \binom{2k}{k} \binom{x+k}{2k}. \]  

3
It follows that the expression (1.2) can be written as
\[
\sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k+1} \binom{n+k}{k} \binom{2k}{k} \binom{x+k}{2k} = \sum_{k=0}^{n-1} \frac{1}{n} \binom{n-1}{k+1} \binom{n+k}{k} \binom{2k}{k} \binom{x+k}{2k}.
\]  
(3.2)

Since \(\frac{1}{k+1} \binom{2k}{k} = \binom{2k}{k} - \binom{2k}{k-1}\) is clearly an integer (the \(n\)-th Catalan number), we conclude that the right-hand side of (3.2) is also an integer whenever \(x\) is an integer. This proves the theorem.

4 Concluding remarks

Z.-W. Sun [15, Conjecture 35(ii)] conjectured that, for all positive integers \(l\) and \(n\), the polynomial
\[
\frac{(2l-1)!!}{n^2} \sum_{k=0}^{n-1} (2k+1)^{2l-1} \sum_{j=0}^{k} \binom{-x-1}{j} \binom{x}{k-j}^2
\]
is integer-valued. Here \((2l-1)!! = (2l-1)(2l-3)\cdots 1\).

We believe that the following (stronger) result is true.

**Conjecture 4.1.** Let \(l\) and \(n\) be positive integers and \(k\) a non-negative integer with \(k \leq n-1\). Then
\[
(2l-1)!! \sum_{m=k}^{n-1} (2m+1)^{2l-1} \binom{m+k}{2k}^2 \equiv 0 \pmod{n^2}.
\]  
(4.1)

Our proof of Theorem 1.3 implies that the above conjecture is true for \(l = 1\). In view of (2.1), Sun’s conjecture follows from (4.1) too.

Recently, \(q\)-analogues of congruences have been studied by many authors. See [3,5–8] and references therein. For \(l = 1\), we have a \(q\)-analogue of (4.1) as follows:
\[
\sum_{m=k}^{n-1} \left[\frac{2m+1}{2k}\right] \binom{m+k}{2k}^2 q^{(k+1)m} \equiv 0 \pmod{[n]^2},
\]  
(4.2)

where \([n] = 1 + q + \cdots + q^{n-1}\) is the \(q\)-integer and \([n]_k = \prod_{j=1}^{k} (1 - q^{n-k+j})/(1 - q^j)\) denotes the \(q\)-binomial coefficient. The proof of (4.2) is similar to that of Theorem 1.3. Nevertheless, we cannot found any \(q\)-analogue of (4.1) for \(l > 1\).

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