A comparison between Cole-Hopf Tranformation and Homotopy Perturbation Method for Viscous Burger Equation in Traffic Flow

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Abstract. This paper presents the comparison of Cole-Hopf transformation and Homotopy Perturbation Method for solving viscous Burgers Equation in traffic flow problems. A problem is presented to demonstrate the differences. The result shows that even though Cole-Hopf transformation brings exact solution for all cases, the formula is in complicated form. Compared to this, Homotopy Perturbation Method is more effective to solve burger equation with boundary condition in traffic flow problems.

1. Introduction
In 1915, Harry Bateman derived Burger Equation in physical content [1]. The initial value problem for one dimensional Burger Equation with boundary condition can be written as in equation (1).

\[ u_t + uu_x = \varepsilon u_{xx}, \quad x \in [x_1, x_2], \quad t > 0, \varepsilon > 0 \]
\[ u(x,0) = h(x), \quad x \in [x_1, x_2] \]
\[ u(x_1,t) = u_1(t), \quad t > 0 \]
\[ u(x_2,t) = u_2(t), \quad t > 0 \]

The exact solution of burger equation has been found by Hopf [2] dan Cole [3] by transforming this equation into heat equation. This transformation is wellknown as Cole-Hopf Transformation. Despite the simplicity of the Cole-Hopf Transformation, this method is need more full effort to obtain the solution since it is stated in very complex integral form. Therefore, the more effective method is needed to look for.

Some numerical methods has been applied to solve Burger Equation, for example the Adomian Decomposition Method. The comparison study of Cole-Hopf transformation and the Adomian Decomposition method has been done, [4]. The result showed that Adomian decomposition method is reliable, sufficient and simpler to solve Burger Equations. Nevertheless, this method found difficulties on calculate the Adomian polynomials. To overcome this problem, a new method named Homotopy Perturbation Method has been applied, [5]. Moreover, another study reveals that The Homotopy Perturbation Method is very effective and convenient to solve system of burger equation [6].

The Homotopy Perturbation Method (HPM) has been widely used by numerous researchers to solve nonlinear equations for both elementary or partial differential equations (ODE or PDE), since this
method able to deform the complex problem into simpler problem which is easier to solve, [6]. The aims of this paper is to study the exact solution of initial value problem using Cole-Hopf transformation and Homotopy Perturbation Method.

In its development, Burger Equations are not only applied to fluid flow problems but in wider area such as nonlinear acoustics, gas dynamics and traffic flow. In this paper, the comparison of both method are subjected to traffic flow problems.

2. Methods
In this section, it will be demonstrated the Cole-Hopf Transformation and the Homotopy Perturbation Method to solve Burger Equation.

2.1. Cole-Hopf Transformation. The Cole-Hopf transformation is defined by equation 2.

\[ u = -2\varepsilon \frac{\phi_x}{\phi} \]  

(2)

This transformation derive the strongly nonlinear viscous burger equation into heat equation. For better ways, first we transform using \[ u = \psi_x \] and then \[ \psi = -2\varepsilon \ln \phi \].

The solution of heat equation can be obtained by considering the domain. For infinite domain, the heat equation can be simply solved using fourier integral transformation, \[ \hat{\varphi}(\xi, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x, t) e^{i\xi x} dx \]. By inversing the fourier transformation next to Cole-Hopf transformation into the obtained solution, we obtain the analytic solution of viscous Burger Equation as given in equation 5.

\[ u(x, t) = \frac{1}{4\pi t} \int_{-\infty}^{\infty} \varphi_0(\xi, t) e^{\frac{-(x-\xi)^2}{4\alpha t}} d\xi \]

(3)

where \( \varphi_0(\xi, t) = e^{-\frac{1}{2\varepsilon t} \frac{n^\alpha}{e}} \). Meanwhile, for finite domain, the solution can be achieved by using Separation of Variables method, \( u(x, t) = X(x) \cdot T(t) \).

2.1.1. Homogeneous boundary conditions. For homogeneous boundary condition in both ends, the solution is simply

\[ \varphi(x, t) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{L} \right) e^{-\frac{n^\alpha}{t} \frac{1}{L}} \]

(4)

where \( B_n = \frac{2}{L} \int_{0}^{L} \varphi(x, 0) \sin \left( \frac{n\pi x}{L} \right) dx \). From this we obtain the solution for viscous burger equation as follows:

\[ u(x, t) = -2\varepsilon \frac{\sum_{n=1}^{\infty} B_n \frac{n\pi}{L} \cos \left( \frac{n\pi x}{L} \right) e^{-\frac{n\pi}{t} \frac{1}{L}}}{\sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{L} \right) e^{-\frac{n\pi}{t} \frac{1}{L}}} \]

(5)
2.1.2. Nonhomogeneous boundary conditions. For nonhomogeneous boundary condition in which the equilibrium solution is exist, we need to transform the problem into homogeneous boundary condition problem using

\[ w(x,t) = \varphi(x,t) - r(x,t) \]
\[ r(x,t) = \varphi_i(t) - \frac{x-x_i}{x_2-x_i}(\varphi_2(t) - \varphi_i(t)) \] (6)

By using this transformation, the solution of burger equation with nonhomogeneous boundary condition is given in equation (7).

\[
u(x,t) = -2v - \sum_{n=1}^{\infty} C_n \frac{n \pi}{L} \cos \left(\frac{n \pi}{L} x\right) e^{-\left(\frac{n \pi}{L} t\right)} - \frac{1}{2 \pi} \int_0 \frac{u(x,0) dx}{x = x_2 - \frac{1}{2 \pi} \int_0 \frac{u(x,0) dx}{x = x_1} - \frac{1}{2 \pi} \int_0 \frac{u(x,0) dx}{x = x_1} \sin \left(\frac{n \pi}{L} x\right) dx. \]

where

\[ C_n = \frac{2}{L} \int_0 \left[ e^{-\frac{1}{2 \pi} \int_0 \frac{u(x,0) dx}{x = x_1} - \frac{1}{2 \pi} \int_0 \frac{u(x,0) dx}{x = x_1} \sin \left(\frac{n \pi}{L} x\right) dx \right] \frac{n \pi}{L} dx. \]

2.2. Homotopy Perturbation Method

The Homotopy Perturbation Method (HPM) was firstly founded by Ji Huan He, [7]. The basic idea of HPM is by coupling the perturbation method and the homotopy method. In this paper, the HPM is used to find the exact solution of burger equation with perturbation as has been done by in [4] with initial value or boundary conditions. Construct a homotopy \( H(v, p) : \Omega \times [0,1] \rightarrow \mathbb{R} \) which satisfies

\[ H(v, p) = (1 - p) \left[ \frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} \right] + p \left[ \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} - \frac{\partial^2 v}{\partial x^2} \right] = 0. \] (8)

Next, assume that the solution of equation (5) can be expressed as a power series in \( p \),

\[ v = v_0 + p^2 v_2 + \cdots. \] (9)

By substituting equation (6) into equation (5) and equating the terms with the identical power of \( p \), we can obtain

\[ p^0 : \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0 \]
\[ p^1 : \frac{\partial v_1}{\partial t} = -\frac{\partial v_0}{\partial t} + V_0 \frac{\partial v_0}{\partial x} + \varepsilon \frac{\partial^2 v_0}{\partial x^2}, \quad v_1(x,0) = 0 \] (10)
\[ p^j : \frac{\partial v_j}{\partial t} = -\sum_{i=0}^{j-1} \frac{\partial v_{j-i}}{\partial x} + \varepsilon \frac{\partial^2 v_{j-1}}{\partial x^2}, \quad v_j(x,0) = 0 \quad \forall j \in Z, j > 1 \]

By this, we earn that the solution of the equation system are

\[ v_j = \int_{-\sum_{i=0}^{j-1} \frac{\partial v_{j-i}}{\partial x} + \varepsilon \frac{\partial^2 v_{j-1}}{\partial x^2} \partial t, \quad v_j(x,0) = 0 \quad \forall j \in Z, j \geq 1. \] (11)

The solution of the burger equation can be obtain by equation (12).

\[ u(x,t) = \lim_{p \rightarrow 1} v(x,t). \] (12)
3. Results and Discussion

3.1. Traffic flow model

To understand, illustrate and predict the traffic condition of a road, the traffic congestion models have been widely developed. The genealogy of traffic flow models has been presented, [8]. Consider the traffic flow on a highway with only one lane. Suppose $\rho$ is the traffic density and $f(\rho)$ is the flux or traffic flow, according to conservation law in which the studied highway has no exit-entrance and no cars are created or destroyed, then the boundary problem for this are given in equation (13).

\[
\begin{align*}
\rho_i + [f(\rho)]_i &= 0, \quad i \in [\hat{x}_1, \hat{x}_2], \hat{i} > 0 \\
p(\hat{x}, 0) &= g(\hat{x}), \quad \hat{x} \in [\hat{x}_1, \hat{x}_2] \\
p(\hat{x}_1, \hat{i}) &= \rho_1(\hat{i}), \quad \hat{i} > 0 \\
p(\hat{x}_2, \hat{i}) &= \rho_2(\hat{i}), \quad \hat{i} > 0
\end{align*}
\]

In many researchs, the traffic flux is determined using fundamental relationship, approximated with the multiplication of the density and velocity $f(\rho) = v_{\rho}$. In 1955, Lightill and Willian, which later supported by Richard, based on their research found the relation that velocity is a function of the density $v = v(\rho)$. Later, there are numerous model of this density-velocity relationship.

The very simple density-velocity relationship but widely used was found by Greenshield, which later called as Greenshield model. This model used assumption that the velocity is linearly decreasing of the traffic density [8]. If the density is zero, then the car could ride in maximum velocity. As inversed, if the traffic is bumper-to-bumper, the car would stop. This model, later was generalized by including the driver behavior. If the upcoming traffic is heavier, the driver tend to reduce the velocity. As result, the traffic flux density corrected into the equation (14).

\[
f(\rho) = v_f\left[\rho - \frac{\rho^2}{\rho_{\text{max}}}\right] - D\rho_x
\]

Suppose $v_f = \frac{L}{T}$, by scaling $x = \hat{x}\frac{L}{\lambda}$ and $t = \hat{t}\frac{T}{\lambda}$, then $u = 1 - \frac{2\rho}{\rho_{\text{max}}}$ transformation leads to viscous Burger Equation as given in equation (15).

\[
\begin{align*}
u_i + uu_x &= \varepsilon u_{xx}, \quad x \in \Omega, t > 0, \varepsilon > 0 \\
u(x, 0) &= h(x), \quad x \in \Omega \\
u(x_1, t) &= u_1(t), t > 0 \\
u(x_2, t) &= u_2(t), t > 0
\end{align*}
\]

where $\varepsilon = \frac{D}{v_f}$, $x_1 = \hat{x}_1\frac{L}{\lambda}$, $x_2 = \hat{x}_2\frac{L}{\lambda}$ dan $\Omega = [x_1, x_2]$.

3.2. The comparison of the solutions

The burger equation will be implemented to study the traffic condition as given in table 1 which is presented in [9].

| TABLE 1. The parameter values | Parameter | Value |
|-------------------------------|----------|-------|
| $v_f$                         | 60 km/hour (kmh) |
| $\rho_{\text{max}}$          | 10.0 cars/km |
| $L$                           | 4 km |
| $T$                           | 4 second (s) |
\[\hat{g}(x) = \sin\left(\frac{\pi x}{4}\right), 0 < \hat{x} < 4\]

\[\rho_1(\hat{t}) = 0, \quad \hat{t} > 0\]

\[\rho_2(\hat{t}) = 0, \quad \hat{t} > 0\]

The initial and boundary condition after transform it into burger problem for this case were

\[u(x,0) = \sin(2\pi x), \quad x \in [0,1]\]

\[u(0,t) = 0, t > 0\]

\[u(1,t) = 0, t > 0\]

Using Cole-Hopf transformation as given in equation (5), the coefficient \(B_n\) solution is in form of equation (16).

\[B_n = \frac{2}{L} \int_0^L \varphi(x,0) \sin\left(\frac{n\pi x}{L}\right) dx.\]  

(16)

where \(\varphi_0(\xi, t) = e^{\frac{x}{2\varepsilon}}.\) This shows the solution in complex integral form. Next, we used Homotopy Perturbation Method to solve the problem.

Let choose, \(v_0(x,t) = u_0(x,t) = \sin(2\pi x).\) By solving equation 10, we obtain that

\[v_1(x,t) = -\pi(4\sin(2\pi x)\varepsilon + \sin(4\pi x))\]

\[v_2(x,t) = \frac{1}{2} \pi^2 t^2 \left(16\sin(2\pi x)\varepsilon^2 + 24\sin(4\pi x)\varepsilon - \sin(2\pi x) + 3\sin(6\pi x)\right)\]

Note that, \(v(x,t) = v_0(x,t) + pv_1(x,t) + p^2v_2(x,t)\) for \(p \to 1\) has already satisfy the burger equation and the initial boundary condition. Therefore, the solution is

\[u(x,t) = \sin(2\pi x) - \pi(4\sin(2\pi x)\varepsilon + \sin(4\pi x))\]

\[+ \frac{1}{2} \pi^2 t^2 \left(16\sin(2\pi x)\varepsilon^2 + 24\sin(4\pi x)\varepsilon - \sin(2\pi x) + 3\sin(6\pi x)\right).\]  

(17)

Eventhough the solution can be obtained using HPM, but we need to find the homotopy function and \(v_0(x,t).\)

4. Conclusion

In this work, CHT is very powerfull but it brings difficulty since the integral form of the solution was generally presented in very complex form. Meanwhile, since the exact way to determine the homotopy function and \(v_0(x,t)\) has not been found for initial and boundary burger problems, we need to guess those function. In general, this study show that comparing both methods, Homotopy Perturbation Method is more effective to solve burger equation with boundary condition in traffic flow problems.

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