Abstract

Topological quantum field theories can be used as a powerful tool to probe geometry and topology in low dimensions. Chern-Simons theories, which are examples of such field theories, provide a field theoretic framework for the study of knots and links in three dimensions. These are rare examples of quantum field theories which can be exactly (non-perturbatively) and explicitly solved. Abelian Chern-Simons theory provides a field theoretic interpretation of the linking and self-linking numbers of a link. In non-Abelian theories, vacuum expectation values of Wilson link operators yield a class of polynomial link invariants; the simplest of them is the famous Jones polynomial. Other invariants obtained are more powerful than that of Jones. Powerful methods for completely analytical and non-perturbative computation of these knot and link invariants have been developed. In the process answers to some of the open problems in knot theory are obtained. From these invariants for unoriented and framed links in $S^3$, an invariant for any three-manifold can be easily constructed by exploiting the Lickorish-Wallace surgery presentation of three-manifolds. This invariant up to a normalization is the partition function of the Chern-Simons field theory. Even perturbative analysis of the Chern-Simons theories are rich in their mathematical structure; these provide a field theoretic interpretation of Vassiliev knot invariants. Not only in mathematics, Chern-Simons theories find important applications in three and four dimensional quantum gravity also.
1 Introduction

Many a time advances in mathematics and physics have occurred hand in hand. Newton’s theory of mechanics and the development of techniques of calculus are a classical example of this phenomenon. Another example is the developments in differential geometry inspired by Maxwell theory of electromagnetism and Einstein theory of general relativity. A recent glorious example is the developments of topological quantum field theories and their relevance to the study of geometry and topology of low dimensional manifolds.

The application of topological quantum field theories reflects the enormous interest generated both by mathematicians and field theoreticians in building a link between quantum physics through its path integral formulation on one hand and geometry and topology of low dimensional manifolds on the other. These are indeed deep links which are only now getting explored. It does appear that the properties of low dimensional manifolds can be very successfully unraveled by relating them to infinite dimensional manifolds of fields. This provides a powerful tool to study these manifolds notwithstanding the ‘lack of mathematical rigour’ in defining the functional integrals of quantum field theory. Indeed, an axiomatic formulation of topological quantum field theories has also been attempted.

Topological quantum field theories are independent of the metric of curved manifold on which these are defined; the expectation value of the energy-momentum tensor is zero, $\langle T_{\mu\nu} \rangle = 0$. These possess no local propagating degrees of freedom; only degrees of freedom are topological. Operators of interest in such a theory are also metric independent.

To illustrate how ideas of quantum field theory can be used to study topology, we shall focus our attention here on recent important developments in Chern-Simons gauge field theory as a topological quantum field theory on a three-manifold. This theory provides a field theoretic framework for the study of knots and links in a given three manifold[1] - [5]. It was A.S. Schwarz who first conjectured [3] that the now famous Jones polynomial [6] may be related to Chern-Simons theory. E. Witten in his pioneering paper about ten years ago demonstrated this connection[2]. In addition, he set up a general field theoretic framework to study knots and links. Since then enormous effort has gone into developing an exact and explicit non-perturbative solution of this field theory. Many of the standard techniques of field theory find applications in these developments. The interplay between quantum field theory and knot theory has paid rich dividends in both directions. Many of the open problems in knot theory have found answers in the process.

Wilson loop operators are the topological operators of the Chern-Simons gauge field theory. Their vacuum expectation values are the topological invariants for knots and links which do not depend on the exact shape, location or form of the knots and links but reflect only their topological properties. The power of this framework is so deep that it allows us to study these invariants not only on simple manifold such as three-sphere but also on any arbitrary three-manifold.
The knot and link invariants obtained from these field theories are also intimately related to the integrable vertex models in two dimensions [6, 5]. These invariants have also been approached in different mathematical frameworks. A quantum group approach to these polynomial invariants has been developed [8]. Last decade or so has seen enormous activity in these directions in algebraic topology.

A mathematically important development is that these link invariants provide a method of obtaining a specific topological invariant for three-manifolds [2, 9] in terms of invariants for framed unoriented links in $S^3$ [10, 5, 11]. In the following, we shall review these developments.

Not only in mathematics, Chern-Simons theory has also played a major role in quantum gravity. Three-dimensional gravity with a negative cosmological constant, itself a topological field theory, can be described by two copies of $SU(2)$ Chern-Simons theory. Even in four dimensional gravity, Chern-Simons theories find application. For example, the boundary degrees of freedom of a black hole in four dimensions, are described by an $SU(2)$ Chern-Simons field theory. This has allowed an exact calculation of quantum entropy of a non-rotating black hole. The formula so obtained for a Schwarzschild black hole, while agreeing with the Bekenstein-Hawking formula for large areas, goes beyond the semi-classical result.

Before explaining how a field theoretic framework for knots and links can be developed, let us start with a brief discussion of knots and links.

# 2 Knots and links: an elementary introduction

*What is knot?* A smooth non-intersecting closed curve in a three-manifold is a knot. Oriented closed curves are oriented knots. A string with its ends joined in the shape of a circle without any entanglements is a model for the simplest non-intersecting closed curve called *unknot*. With a given knot, we associate a *knot diagram* obtained by projecting the knot on to a plane with a minimum number of double points. In such a diagram over-crossings and under-crossing are to be clearly marked. The number of double points in a knot diagram is called its *crossing number*. A few simple knots with low crossing numbers are:

![Knots](image)

Clearly, for a given minimum number of crossings, there can be more than one type of topologically inequivalent knots. The number of knots increases rapidly with the crossing number. For crossing number 9, there are 49 knots (not distinguishing mirror reflections), for 10 there are 165 and for crossing number 11 we have 552 knots.
For 13 crossings, there are more than 10,000 different knots.

What is a link? A collection of a number of oriented non-intersecting loops (knots) is an oriented link. A knot then is single component link. Links like knots can be represented by their two dimensional projections, the link diagrams with minimum number of double points, but with the over-crossings and under-crossings clearly marked. Examples of a few two-component links are:

![Link Diagrams]

Links

To a topologist, length, thickness or precise shape of a knot are not of any interest. Two knots or links are to be identified if one can be made to go continuously into other by shrinking or stretching or wiggling without snapping the string. There is a minimal set of elementary rules which encode these qualitative notions more precisely. These are the three Reidemeister moves which do not change the topological type of a link:

![Reidemeister Moves]

Invariance under all these three moves is called invariance under ambient isotopy. If a quantity is invariant under type II and III moves only, but not under type I moves, it is said to be a regular isotopic invariant.

The Reidemeister move III is of particular interest. It represents a defining relation for the generators of braids. In addition, it is a graphical representation of the Yang-Baxter relation of statistical mechanical models. These facts are not accidental but reflect a deep connection that knots and links have with braids and exactly solvable two-dimensional vertex models. In fact this connection has been successfully exploited to obtain infinitely many new exactly solvable statistical mechanical models.

Though the Reidemeister rules are so simple, it is not an easy exercise in general
to tell whether given two knots or links are topologically distinct or not. For example, it took nearly eighty years, since the time of knot tables of C.N. Little from the end of last century to the work of K.A. Perko in 1972, to recognize that the knots in the figure below are isotopically equivalent: 

Finding mathematical methods for distinguishing knots and links is indeed an important problem in knot theory. To this end, some definite invariants, called link invariants, are associated with the links. These are mathematical expressions which depend only on the isotopic type of the link and not on any of its particular representations. Some such invariants are in the form of polynomials. First polynomial invariant was discovered in late twenties by J.W. Alexander. It took almost sixty more years before the next one was discovered by V.R.F. Jones. The new invariant proved to be topologically more powerful than that of Alexander. For example, unlike Alexander polynomial, Jones polynomial does distinguish many mirror reflected knots. Soon after, a two variable generalization of Jones invariant was found. Though two distinct Jones polynomials do represent two isotopically distinct knots, the converse is not always true. There are examples of distinct knots with same Jones polynomial. Still Jones’ work represents a leap forward in the developments of knot theory. What is impressive about the topological field theoretic description of knots is that it provides a whole variety of link invariants in a straightforward manner. Of these Jones one-variable polynomial and its two-variable generalization are the simplest examples.

Before starting a discussion of knots and links in terms of a quantum field theory, let us make a few historical remarks about knots and links in physics.

A few historical remarks: Knots and links first captured the imagination of physicists when Lord Kelvin (William Thomas) introduced them as early as 1857 as fluid-mechanical models of atoms. Reluctant to accept the prevailing notion of an infinitely rigid point-like atom, he thought of atoms as vortex-lines in a perfect homogeneous fluid, the ether. Different sorts of atoms were then to differ in accordance with the number of intersections of these vortex rings. “Stability” of the atoms in this theory thus is a reflection of the fact that knots do preserve their essential knottedness during their movement. Indeed Lord Kelvin would have wanted to develop a new theory of gasses, theory of elastic solids and liquids based on the dynamics of these vortex atoms – a programme he did not complete nor was considered by
later day physicists worth while in this context. However, a new area, knot theory, of mathematics was born.

Two contemporary Scottish physicists, J.C. Maxwell and P.G. Tait did find Lord Kelvin’s hypothesis attractive enough. Tait had hoped to explain the position of lines in the spectrum of a chemical element from the knot type representing it. Thus, it was natural for him to attempt the formidable task of classifying knots in three-space. For this he needed some measure of complexity of a knot. Thus the concept of the degree of knottedness was introduced. This is what we nowadays call crossing number of a knot, a notion already defined above. Tait with this notion of crossing number, produced the first knot tables, listing knots in order of their increasing knottedness. If atoms had been really knots, we would have been studying these tables instead of the periodic table of chemical elements in our schools.

Since the pioneering work of these physicists, knot theory was solely investigated by mathematicians till about ten years ago when physicists came back to it through quantum field theories. This brings us to modern field theoretic interpretation of knots in three dimensions.

3 Abelian Chern-Simons field theory and knots and links

In a field theory, the properties of a system of infinitely many oscillators are represented collectively by a field, $\phi(x)$ defined over all the space though the space label $x$. An action functional is prescribed for these fields. For example, for a one-component scalar field $\phi(x)$, say in three dimensional flat Euclidean space $R^3$, the action functional may be taken to be:

$$S[\phi] = \frac{1}{2} \int d^3 x \, \delta^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) ,$$

where $\mu, \nu = 1, 2, 3$ are space indices and for $R^3$, the metric is flat $\delta^{\mu\nu} = dia (1,1,1)$. For a theory defined over a general curved three-manifold endowed with a metric $g_{\mu\nu}$ (and its inverse $g^{\mu\nu}$), this action generalizes to:

$$S[\phi] = \frac{1}{2} \int d^3 x \sqrt{g(x)} \, g^{\mu\nu}(x) \partial_\mu \phi(x) \partial_\nu \phi(x) ,$$

where $g(x) = det g_{\mu\nu}$.

Similarly for a vector field $A_\mu(x)$, the gauge field of Maxwell theory in three dimensions, we write the action functional as:

$$S[A_\mu] = \frac{1}{4} \int d^3 x \sqrt{g(x)} \, g^{\mu\alpha}(x)g^{\nu\beta}(x) \left[ \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \right] \left[ \partial_\alpha A_\beta(x) - \partial_\beta A_\alpha(x) \right]$$

Both these actions above are invariant under general coordinate transformations.
Quantum field theories normally studied, like the examples above, depend on the metric $g_{\mu\nu}$ of the three-manifold in which the theory is defined. The metric describes the geometric properties, such as distances, curvature etc. But, here we are interested in attempting a field theoretic description of knots and links in such a way that only their topological properties are represented. Their size, exact shape, location etc are not of our concern. The topological properties, unlike these, do not depend on the metric. Thus we are seeking a field theory which is independent of the metric. Such theories are called topological field theories. A simple example of metric independent field theory is the Chern-Simons gauge theory. Its action in the Abelian version (with convenient normalization) is given by:

$$k S[A_\mu] = -\frac{k}{8\pi} \int_{S^3} d^3x \ e^{\mu\nu\alpha} A_\mu(x) \partial_\nu A_\alpha(x)$$  \hspace{1cm} (1)$$

where $e^{\mu\nu\alpha}$ is a completely anti-symmetric contravariant three-tensor density whose only nonzero component is $e^{1\ 2\ 3} = 1$. For definiteness, we shall discuss this theory in a three-manifold $S^3$. Clearly this action is independent of the metric. Also it is invariant under general coordinate transformations. Like the Maxwell theory, this theory exhibits a gauge invariance.

The quantum version of this theory is described by the functional integral representing the partition function:

$$Z = \int [dA] \ e^{ikS}$$  \hspace{1cm} (2)$$

and for metric independent gauge invariant functionals $W[A_\mu]$ of the gauge field $A_\mu(x)$, we have the functional averages (vacuum expectation values of the associated operators):

$$\langle W \rangle = Z^{-1} \int [dA] \ W \ e^{ikS}$$  \hspace{1cm} (3)$$

Though the action and gauge invariant functionals $W$ do not depend on the metric, there are potential sources which can introduce metric dependence in these functional averages. The functional integration may be thought of to be done by discretizing the space into a mesh. Infinitely many ordinary integrals over $A_\mu(x)$ at every point $x$ of the mesh are to be done and finally the limit of mesh size going to zero is taken in some well defined manner. This is the usual way we understand these infinite dimensional integrals. Further, there is a gauge invariance in the theory, which like in other gauge theories needs to be fixed by a choice of gauge. Both the choice of mesh as well as gauge fixing condition are generically metric dependent. Thus the gauge fixed measure of integration $[dA_\mu(x)]$ in a field theory defined on a curved space, in general depends on the metric. However, despite these, it can be shown that various metric dependence so conspire in this topological theory that they cancel out without spoiling the metric independence of the functional averages [16].

Now let us give an explicit form of a topological operator $W$ in this Abelian Chern-Simons theory. Consider a link $L$ made up of knots $K_1, K_2, \ldots, K_s$. Wilson
knot operator for each these knots $K_\ell$ is given by $\exp[i\ n_\ell \oint_{K_\ell} dx^\mu A_\mu(x)]$ where $n_\ell$ is an integer measuring the charge on the loop. Clearly these are independent of the metric. Then the Wilson link operator is product of all such knot operators:

$$W[L] = \prod_{\ell=1}^{s} \exp \left[ i\ n_\ell \oint_{K_\ell} dx^\mu A_\mu(x) \right]$$  \hspace{1cm} (4)

If we expand the exponential here, the expectation value $\langle W[L] \rangle$ is given by the expectation values of the various terms in this expansion. This is a non-interacting theory; all these expectation values are given in terms of the “two-loop” expectation values only:

$$\langle \oint_{K_\ell} dx^\mu A_\mu(x) \oint_{K_m} dy^\nu A_\nu(y) \rangle, \ K_\ell \neq K_m; \quad \text{and} \quad \langle \oint_{K} dx^\mu A_\mu(x) \oint_{K} dy^\nu A_\nu(y) \rangle$$  \hspace{1cm} (5)

Here in the first expression the two loops are distinct in contrast to the second expression where both the loop integrals are along the same knot. Clearly, these expressions can be easily evaluated in terms of the two-point correlator $\langle A_\mu(x) A_\nu(y) \rangle$. To do this, we can locally identify the region containing our link with $R^3$ so that we can use the flat metric $g_{\mu\nu} = \delta_{\mu\nu}$ in this region. Then $x^\mu$ and $y^\nu$ are the Euclidean flat coordinates along the two knots $K_\ell$ and $K_m$ respectively. This allows us to do away with the complications connected with the curved nature of the three-manifold $S^3$; we can do all our calculations in flat Euclidean space without loss of generality. Elementary field theory allows us to read off the flat space two-point correlator from the action (subject to a gauge condition, which we choose to be the covariant Lorentz gauge $\delta^{\mu\nu} \partial_\mu A_\nu = 0$):

$$\langle A_\mu(x) A_\nu(y) \rangle = \frac{i}{k} \varepsilon_{\mu\nu\alpha} \frac{(x-y)^\alpha}{|x-y|^3}$$

so that

$$\langle \oint_{K_\ell} dx^\mu A_\mu(x) \oint_{K_m} dy^\nu A_\nu(y) \rangle = \frac{4\pi i}{k} \mathcal{L}(K_\ell, K_m)$$

where

$$\mathcal{L}(K_\ell, K_m) = \frac{1}{4\pi} \oint_{K_\ell} dx^\mu \oint_{K_m} dy^\nu \varepsilon_{\mu\nu\alpha} \frac{(x-y)^\alpha}{|x-y|^3}.$$  \hspace{1cm} (6)

This double loop integral over two distinct knots ($K_\ell \neq K_m$) is a well known topological invariant, called Gauss linking number of the two closed curves. It measures the number of times one knot $K_\ell$ goes through the other knot $K_m$. Clearly, linking number of two knots is an integer. For example, for the right-handed Hopf link $H_+$,

![Right-handed Hopf link H+](image)
its value is $+1$. Its value for the mirror reflection of this link (left-handed Hopf) is $-1$. Linking number does not depend on the exact location of the two knots, nor on their size or shape. It depends only on their topological relationship with each other. This invariant has a physical interpretation due to Maxwell – in electrodynamics, it represents the work done to move a magnetic monopole around one knot in three-space while an electric current runs through the other knot.

The Abelian Chern-Simons theory also provides a representation for yet another simple topological quantity associated with an individual knot called its self-linking number and also sometimes framing number or simply framing. This is related to the second expectation value given in (5) where the two loop integrals are over the same knot. This expectation value is to be evaluated through a limiting procedure: To a knot $K$ parametrized by $x^\mu(s)$ $(0 \leq s \leq L)$ along the length of the knot by the parameter $s$, associate another closed curve $K_f$, called its frame, given by coordinates $y^\mu = x^\mu(s) + \epsilon n^\mu(s)$ where $\epsilon$ is a small parameter and $n^\mu(s)$ is a unit vector field normal (principal normal) to the curve at $s$. That is, $K_f$ is the curve $K$ displaced along the normal by a small amount. Then the linking number of the curve $K$ and its frame $K_f$ is called self-linking number $SL(K)$ of the knot:

$$\left\langle \oint_K dx^\mu A_\mu(x) \oint_K dy^\nu A_\nu(y) \right\rangle = \lim_{\epsilon \to 0} \left\langle \oint_K dx^\mu A_\mu(x) \oint_{K_f} dy^\nu A_\nu(y) \right\rangle$$

$$= \frac{4\pi i}{k} \mathcal{L}(K, K_f) = \frac{4\pi i}{k} SL(K)$$

This self-linking number is independent of the parameter $\epsilon$ and can easily be shown to obey the following important theorem, first proven by G. Calugareanu almost forty years ago [17]:

Calugareanu theorem: The self-linking number of a knot is the sum of its twist and writhe numbers:

$$SL(K) = T(K) + w(K)$$

$$T(K) = \frac{1}{2\pi} \int_K ds \epsilon_{\mu\nu\alpha} \frac{dx^\mu}{ds} n^\nu \frac{dn^\alpha}{ds}, \quad w(K) = \frac{1}{4\pi} \int_K dt \int_K ds \epsilon_{\mu\nu\alpha} e^\mu \frac{de^\nu}{ds} \frac{de^\alpha}{dt}$$

where the vector field $e^\mu$ is given by

$$e^\mu(s, t) = \frac{y^\mu(t) - x^\mu(s)}{|y(t) - x(s)|}$$

is a map $K \otimes K \mapsto S^2$ and $n^\mu(s)$ is the normal vector field along the length of the curve $K$ $(x^\mu(s), 0 \leq s \leq L)$. The quantities $T(K)$ and $w(K)$ represent well defined geometric properties of the knot. $T(K)$ represents the twist in the knot $K$ with reference to its frame $K_f$ and $w(K)$ is the amount of writhe or coiling of the knot. Clearly, the twist number and writhe number are not necessarily integers nor are they ambient isotopic invariants. But their sum, the self-linking number, is indeed an
integer and also an ambient isotopic invariant. This theorem can be easily appreciated if we recall that stretching a coiled up telephone cord reduces its coils but increases its twist and loosening of a twisted cord coils it up. The amount of coils lost (or gained) is exactly the same as the amount by which the twisting is gained (or lost) so that their sum is always unchanged. This theorem of Calugareanu when applied to circular ribbon (which can be thought of as a framed closed curve) has been put to good use in the study of the properties of circular polymers and circular DNA [18].

Notice that the self-linking number does carry dependence on the frame. The mathematical concept of framing of a knot is intimately connected to the concept of regularization in field theory. In order to avoid the coincidence singularity in the two-point correlator \( \lim_{x \to y} \langle A_\mu(x) A_\nu(y) \rangle \), we need to regularize it, say by point-splitting. Evaluating, ‘two-loop’ correlator of Eqn.(5), where the two loops are same, we face this same divergence, which, through framing, has been resolved by ‘loop-splitting’. Ordinarily, quantities in field theory do depend on the regularization. Like-wise the self-linking number here depends on the framing. But all those framing curves enveloping around the knot, which can be continuously deformed into each other without snapping the knot, form a topological class for which the self-linking number does not change. In field theory language, framing provides a topological regularization.

Now collecting all these pieces of information, the expectation value of the Wilson link operator for a link \( L = (K_1, K_2, \ldots K_s) \) in the Abelian Chern-Simons theory on \( S^3 \) can be written down in terms of the linking and self-linking (framing) numbers as:

\[
\langle W[L] \rangle = \exp \left\{ \frac{-2\pi i}{k} \sum_{\ell} n_\ell^2 S \mathcal{L}(K_\ell) + \sum_{\ell \neq m} n_\ell n_m \mathcal{L}(K_\ell, K_m) \right\} \tag{9}
\]

Thus, we have indicated here how this simple field theory does indeed, through expectation values of Wilson link operators, provide a field theoretic interpretation of some of the topological invariants, linking number and self-linking number of knots and links. Non-Abelian Chern-Simons theories are much richer in their structure; these capture even more complex topological properties of knots and links.

### 4 Non-Abelian Chern-Simons theory as a description of knots and links

A non-Abelian Chern-Simons theory, instead of being a gauge theory of one vector field, involves, say for gauge group \( SU(2) \), three such fields, \( A^a_\mu (a = 1, 2, 3) \). These three are collectively written as a matrix valued vector field \( A_\mu = A^a_\mu \sigma^a_\mu \), where anti-hermetian matrices \( \sigma^a_\mu \) are the generators of the group \( SU(2) \). Action functional defined in a three-manifold, say \( S^3 \), is given by:

\[
kS = \frac{k}{4\pi} \int_{S^3} d^3x \; e^{i\omega_\alpha} \; \text{tr} \left[ A_\mu(x) \partial_\nu A_\alpha(x) + \frac{2}{3} A_\mu(x) A_\nu(x) A_\alpha(x) \right] \tag{10}
\]
Like Abelian Chern-Simons theory, this action has no metric dependence. Besides a
gauge invariance, it is also invariant under general coordinate transformations.

The topological operators are the Wilson loop (knot) operators defined as

\[ W_j[K] = tr_j P exp \int_K dx^\mu A^a_\mu T^a_j \]  

(11)

for an oriented knot \( K \) carrying spin \( j \) representation reflected by the associated
representation matrices \( T^a_j \) \((a = 1, 2, 3)\). The symbol \( P \) stands for path ordering of
the exponential. This is done by breaking the length of the knot \( K \) into infinitesimal
intervals of size \( dx^\mu_m \) around the points labeled by the coordinates \( x^\mu_m \) along the knot.
Then path ordered exponential is:

\[ P exp \int_K dx^\mu A^a_\mu T^a_j = \prod_m [1 + dx^\mu_m A^a_\mu(x^\mu_m)T^a_j] \]

For a link \( L \) made up of oriented component knots \( K_1, K_2, \ldots K_s \) carrying spin
\( j_1, j_2, \ldots j_s \) representations respectively, we have the Wilson link operator defined as

\[ W_{j_1 j_2 \ldots j_s}[L] = \prod_{\ell=1}^s W_{j_\ell}[K_\ell] \]  

(12)

We are interested in the functional averages of these operators:

\[ V_{j_1 j_2 \ldots j_s}[L] = Z^{-1} \int [dA] W_{j_1 j_2 \ldots j_s}[L] e^{ikS}, \quad \text{where} \quad Z = \int [dA] e^{ikS} \]  

(13)

Here the integrands in the functional integrals are metric independent. So is the gauge
fixed measure \([13]\). Therefore, these expectation values depend only on the isotopy
type of the oriented link \( L \) and the set of representations \( j_1, j_2 \ldots j_s \) associated with
component knots.

These expectation values can be obtained non-perturbatively. For example, for
knots and links carrying only the spin 1/2 representations, Witten has shown that
the link invariants (expectation values of the associated Wilson link operators) satisfy
a simple relation. This relation is given for three link diagrams which are identical
everywhere except for one crossing where they differ in that it is an over-crossing
\((L_+)\), or no-crossing \((L_0)\) or an under-crossing \((L_-)\) as shown in the figure below:

```
      L_+
     / \    \     \n    Over-crossing No-crossing Under-crossing L_-
```

Then the invariant for such links are related as:

\[ q V_{1/2}[L_+] - q^{-1} V_{1/2}[L_-] = (q^{1/2} - q^{-1/2}) V_{1/2}[L_0] \]  

(14)

where \( q \) is a root of unity related to the Chern-Simons coupling \( k \) through the relation
\( q = exp[2\pi i/(k+2)] \). This is precisely the well known generating skein relation for the
Jones polynomials. Indeed $V_{1/2}[L]$, which is the expectation value of the Wilson link operator where every component knot carries the doublet spin 1/2 representation, is the one-variable Jones polynomial.

The above skein relation is powerful enough that it recursively yields Jones polynomial for any arbitrary link. For example consider following three link diagrams:

We use an important factorization property of these invariant: the link invariant of two distant (disjoint) links (that is, with no mutual entanglement) is simply the product of invariants for the individual links. That is, for the link $L_0$ above, $V_{1/2}[U \cup U] = (V_{1/2}[U])^2$, where symbol $U$ represents the unknot. Then use of the skein relation yields:

$$q V_{1/2}[U] - q^{-1}V_{1/2}[U] = (q^{1/2} - q^{-1/2}) (V_{1/2}[U])^2$$

so that spin 1/2 invariant for an unknot is given by: $V_{1/2}[U] = q^{1/2} + q^{-1/2}$.

Next apply the skein relation to three links, where the $L_+$ is the right-handed Hopf link, $L_-$ is simply the union of two (unlinked) unknots and $L_0$ is an unknot:

This yields, the invariant for the right-handed Hopf link $H_+$ as: $V_{1/2}[H_+] = 1 + q^{-1} + q^{-2} + q^{-3}$. Now use recursion relation for the three links:

where $L_+$ is a right-handed trefoil ($T_+$), $L_-$ is an unknot and $L_0$ is a right-handed Hopf $H_+$. This gives us the invariant for the trefoil knot as $V_{1/2}[T_+] = q^{-1/2} + q^{-3/2} + q^{-5/2} - q^{-9/2}$. This way invariant for any arbitrary link can be recursively obtained.

Jones polynomial is in fact the simplest of the examples of a whole host of new link invariants that emerge naturally from this field theory. More general invariants are the
expectation values of Wilson link operators with arbitrary spin representations placed on the knots. The formalism does also allow for placing different representations on each of the component knots. This leads to so-called coloured polynomial invariants. Besides, instead of the gauge group \( SU(2) \), Chern-Simons theory based on any other semi-simple group can be used. These then yield even richer spectrum of the new invariants.

While Jones polynomial can be obtained by recursive use of the skein relation, other more general invariants (for spin representations \( j = 1, 3/2, \ldots \)) can not be obtained in this manner. Of course there are generalizations of the skein relations for an arbitrary spin invariants. But these do not possess recursively complete solutions (except for spin 1/2 case above). Therefore methods had to be developed to obtain expectation values of Wilson operators with arbitrary representations living on the component knots of a link. One such method in its complete manifestations has been presented in ref [4]. This allows us to present a complete and explicit solution of the Chern-Simons theory. This is a non-perturbative method which, generalizing the formalism set up by Witten, makes use of two ingredients, one from quantum field theory and other from mathematics of braids:

(i) **Field theoretic input:** Chern-Simons theory on a three-manifold with boundary is essentially characterized by a corresponding two dimensional Wess-Zumino conformal field theory on that boundary\([2]\):

\[
\text{SU}(2) \text{ CS theory with coupling } k \text{ on } M \quad \quad \begin{array}{c}
\text{SU}(2) \text{ WZ theory on } \Sigma
\end{array}
\]

And Chern-Simons functional average for Wilson lines ending at \( n \) points in the boundary is described by the associated Wess-Zumino theory on the boundary with \( n \) punctures carrying the representations of the free Wilson lines:

\[
\text{SU}(2) \text{ CS theory with coupling } k \text{ on the manifold } M \quad \quad \begin{array}{c}
\text{SU}(2) \text{ WZ theory on } \Sigma \text{ with } n \text{ punctures carrying primary fields in representations } j_1, j_2, \ldots, j_n
\end{array}
\]

The Chern-Simons functional integral can be represented \([2]\) by a vector in the Hilbert space \( \mathcal{H} \) associated with the space of \( n \)-point correlator of the Wess-Zumino conformal field theory on the boundary \( \Sigma \). In fact, these correlators provide a basis for this

\[13\]
boundary Hilbert space. There are more than one possible basis. These different bases are related by duality of the correlators of the conformal field theory\[1\].

(ii) Mathematical input: The second ingredient used is the close connection knots and links have with braids. An \(n\)-braid is a collection of non-intersecting strands connecting \(n\) points on a horizontal plane to \(n\) points on another horizontal plane directly below the first set of \(n\) points. The strands are not allowed to go back upwards at any point in their travel. The braid may be projected onto a plane with the two horizontal planes collapsing to two parallel rigid rods. The over-crossings and under-crossings of the strands are to be clearly marked. When all the strands are identical, we have ordinary braids. The theory of such braids, first developed by Artin, is well studied. These braids form a group. However, we may wish to orient the individual strands and further distinguish them by putting different colours on them. These different colours are represented by different \(SU(2)\) spins. These braids, unlike braids made from unoriented identical strands, have a more general structure than a group. These instead form a groupoid. The necessary aspects of the theory of such braids have been presented in ref.[4]

One way of relating the braids to knots and links is through closure of braids. We obtain the closure of a braid by connecting the ends of the first, second, third, ..... strands from above to the ends of respective first, second, third, ..... strands from below as shown in (A):

There is a theorem by Alexander\[19\] which states that \textit{any knot or link can be obtained as closure of a braid}. This construction of a knot or link is not unique.

There is another construction associated with braids which relates them to knots and links. This is called platting. Consider a \(2m\)-braid, with pairwise adjacent strands carrying the same colour and opposite orientations. Then connect the \((2i−1)\)th strand with \((2i)\)th from above as well as from below. This yields the plat of the given braid as shown in (B) above. There is a theorem due to Birman\[20\] which relates plats to links. This states that \textit{a coloured-oriented link can be represented (though not uniquely) by the plat of an oriented-coloured 2m-braid.}

Use of these two inputs, namely relation of Chern-Simons theory to the boundary Wess-Zumino conformal field theory and presentation of knots and links as closures or plats of braids leads to an explicit, complete and non-perturbative solution of the Chern-Simons theory. Conformal field theory on associated boundary gives matrix representations for braids and plating or closing of a braid corresponds to taking a specific matrix element of these braid representations. This then yields the expec-
tation value of the Wilson link operator associated with that link. For example this invariant for an unknot $U$ carrying spin $j$ representation turns out to be:

$$V_j[U] = [2j + 1] \quad \text{where} \quad [x] = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}}$$

The square bracket indicates a $q$-number. Jones polynomial above corresponds to spin $j = 1/2$. And for a right-handed trefoil $T_+$, the invariant turns out to be:

$$V_j[T_+] = \sum_{m=0,1,2,\ldots,\min(2j,k-2j)} [2m + 1] \ (-)^{2j+m} q^{-6C_j + \frac{2}{q}C_m}$$

where $C_j = j(j+1)$ is the quadratic Casimir of the spin $j$ representation. For $j = 1/2$, this expression agrees with the polynomial obtained above by using the skein relation.

The link invariants calculated from the field theory depend on the regularization used to define the coincident loop correlators, that is, the framing of the knots. The invariants above have been obtained in a specific framing called *standard framing*. In particular, the skein relation for spin 1/2 invariants given above is in this framing. In this framing, the self-linking (framing) number of every knot is zero. The invariants so obtained are unchanged under all the three Reidemeister moves. That is, this yields ambient isotopic invariants. There is another framing choice which has been of special interest. In this case, the frame is thought to be just vertically above the two dimensional projection of the knot. In this framing, known as *vertical framing*, Reidemeister moves II and III do leave the link invariants unchanged, but Reidemeister move I changes them.

The general framework developed provides a powerful method of calculating knot and link invariants. This has in the process also provided answers to some of the open problems of knot theory. For example, one such problem is to find polynomial invariants which would discriminate between two chiralities of a given knot. The invariants for the mirror reflected knots are give by simple complex conjugation. Up to ten crossing number, there are six chiral knots, $9_{42}$, $10_{48}$, $10_{71}$, $10_{91}$, $10_{104}$ and $10_{125}$ (as listed in the knot tables in Rolfsen’s book [21]) which are not distinguished from their mirror images by spin 1/2 (Jones) polynomials. Spin one (Kauffman/Akutsu-Wadati) polynomials do detect the chirality of four of them, namely $10_{48}$, $10_{91}$, $10_{104}$ and $10_{125}$. But for $9_{42}$ and $10_{71}$ both Jones and Kauffman polynomials are not changed under chirality transformation ($q \rightarrow q^{-1}$). However, the new spin 3/2 invariants are powerful enough to distinguish these knots from their mirror images[22].

Another problem of knot theory that has been provided with an answer is to do with so called *mutant* knots. A mutant of a knot or link is obtained in the following way: isolate a portion of the knot in such a way that it has two strands going into and two strands leaving from it. Scoop it out and rotate it through $\pi$ about any of three orthogonal axes (rotations about only two of these are really independent). Glue it back after, if necessary, changing the orientations on the strands to match the free ends of strands of rest of the knot to which the free ends of the rotated portion are glued. This yields a mutant of the original knot. It has been possible to
prove that polynomial invariants obtained from a Chern-Simons theory based on *any arbitrary non-Abelian gauge group* do not distinguish isotopically inequivalent mutant knots\[23\]. As an example consider the following sixteen crossing mutant knots:

![A 16 crossing mutant pair](image)

The two knots are related by a mutation of the portion indicated by dashed enclosure. Like all other mutants, the invariants obtained from any non-Abelian Chern-Simons theory for them are identical. What is of particular interest about this pair is that one of them is chiral, other is not. This then yields an example of a chiral knot whose chirality can not be detected by any of these invariants.

The general framework developed to study knots and links is also applicable to another set of gauge invariant operators called graphs. For $SU(2)$ Chern-Simons theory, these are the graphs containing vertices with three legs. The edges of the graph between vertices carry Wilson line operators. More general gauge invariant operators which include links attached to the edges of graphs can also be evaluated in this framework.

## 5 Three-manifold invariants

The invariants of knots and links in $S^3$ obtained from the Chern-Simons theory can be used to construct a special three-manifold invariant\[2, 9, 10, 5\]. This provides an important tool to study topological properties of three-manifolds. Starting step in this construction is a theorem due to Lickorish and Wallace\[24, 21\]:

**Fundamental theorem of Lickorish and Wallace:** Every closed, orientable, connected three-manifold, $M^3$ can be obtained by surgery on an unoriented framed knot or link $[L, f]$ in $S^3$.

As described earlier, the framing $f$ of a link $L$ is defined by associating with every component knot $K_s$ of the link an accompanying closed curve $K_{sf}$ parallel to the knot and winding $n(s)$ times in the right-handed direction. That is, the linking number $lk(K_s, K_{sf})$ of the component knot and its frame (self-linking number of the knot $K_s$) is $n(s)$. For the construction of three-manifold invariants, we use vertical framing where the frame is thought to be just vertically above the two dimensional projection of the knot as shown below. This is some times indicated by putting $n(s)$ writhes in the strand making the knot or even by just simply writing the integer $n(s)$ next to the knot as shown below:
Next the surgery on a framed link \([L, f]\) made of component knots \(K_1, K_2, ..., K_r\) with framing \(f = (n(1), n(2), ..., n(r))\) in \(S^3\) is performed in the following manner. Remove a small open solid torus neighbourhood \(N_s\) of each component knot \(K_s\), disjoint from all other such open tubular neighbourhoods associated with other component knots. In the manifold left behind \(S^3 - (N_1 \cup N_2 \cup ... \cup N_r)\), there are \(r\) toral boundaries. On each such boundary, consider a simple closed curve (the frame) going \(n(s)\) times along the meridian and once along the longitude of the associated knot \(K_s\). Now do a modular transformation on such a toral boundary such that the framing curve bounds a disc. Glue back the solid tori into the gaps. This yields a new manifold \(M^3\). The theorem of Lickorish and Wallace assures us that every closed, orientable, connected three-manifold can be constructed in this way.

This construction of three-manifolds by surgery is not unique: surgery on more than one framed link can yield homeomorphic manifolds. But the rules of equivalence of framed links in \(S^3\) which yield the same three-manifold on surgery are known. These rules are known as Kirby moves.\(^\text{25}\)

**Kirby calculus on framed links in \(S^3\):** Following two elementary moves (and their inverses) generate Kirby calculus:

**Move I.** For a number of unlinked strands belonging to the component knots \(K_s\) with framing \(n(s)\) going through an unknotted circle \(C\) with framing \(+1\), the unknotted circle can be removed after making a complete clockwise twist from below in the disc enclosed by the circle \(C\):

\[
\begin{align*}
C \quad \begin{array}{c}
\uparrow \\
\downarrow \\
\uparrow \\
\downarrow \\
\uparrow \\
\downarrow \\
\uparrow \\
\downarrow \\
\uparrow \\
\downarrow \\
\end{array} +1 \\
n(s)
\end{align*}
\]

\[
L \quad \begin{array}{c}
\uparrow \\
\downarrow \\
\uparrow \\
\downarrow \\
\uparrow \\
\downarrow \\
\uparrow \\
\downarrow \\
\uparrow \\
\downarrow \\
\end{array} \equiv \begin{array}{c}
\uparrow \\
\downarrow \\
\uparrow \\
\downarrow \\
\uparrow \\
\downarrow \\
\uparrow \\
\downarrow \\
\uparrow \\
\downarrow \\
\end{array} \\
n'(s) = n(s) - (\text{lk}(K_s, C))^2
\]

In the process, in addition to introducing new crossings, the framing of the various resultant component knots, \(K'_s\) to which the affected strands belong, change from \(n(s)\) to \(n'(s) = n(s) - (\text{lk}(K_s, C))^2\).

**Move II.** Drop a disjoint unknotted circle with framing \(-1\) without any change
Thus Lickorish-Wallace theorem and equivalence of surgery under Kirby moves reduces the theory of closed, orientable, connected three-manifolds to the theory of framed unoriented links via a one-to-one correspondence:

\[
\left( \text{Framed links in } S^3 \text{ modulo } \text{equivalence under Kirby moves} \right) \leftrightarrow \left( \text{Closed, orientable, connected three-manifolds modulo homeomorphisms} \right)
\]

This consequently allows us to characterize three-manifolds by the invariants of associated unoriented framed knots and links obtained from the Chern-Simons theory in \( S^3 \). This can be done by constructing an appropriate combination of the invariants of the framed links which is unchanged under Kirby moves:

\[
\left( \text{Invariants of a framed unoriented link which do not change under Kirby moves} \right) = \left( \text{Invariants of associated three-manifold} \right)
\]

One such invariant has been constructed in ref \[5\]. It is given in terms of invariants for unoriented links obtained from \( SU(2) \) Chern-Simons theory. The link invariants discussed in Sec.4 above are obtained in standard framing. These are sensitive to the relative orientations of the component knots. Here we shall use invariants for unoriented links in vertical framing. But, unlike the invariants in standard framing which exhibit ambient isotopic invariance, those obtained in vertical framing have only regular isotopic invariance. That is, in standard framing, a writhe can be stretched (a Reidemeister move I) without affecting the link invariant, in vertical framing this is not so. A link invariant in vertical framing gets changed by a phase when a writhe is smoothed out as:

\[
\begin{align*}
\text{right-handed crossing} & : q^{C_j \cap^j} \\
\text{left-handed crossing} & : q^{-C_j \cap^j}
\end{align*}
\]

where \( C_j = j(j + 1) \). Here we have represented the link invariant by the affected portion of the link. Thus, in vertical framing, invariant for an unknot with self-linking (framing) number \( +1 \) or \( -1 \) is related to the invariant for an unknot with zero self-linking number as:

\[
\begin{align*}
V_j \left[ \begin{array}{c}
\text{right-handed unknot with } +1 \\
\text{framing}
\end{array} \right] & = q^{C_j} V_j \left[ \begin{array}{c}
\text{unknot with } 0 \\
\text{framing}
\end{array} \right] = q^{C_j}[2j + 1], \\
\text{and} \\
V_j \left[ \begin{array}{c}
\text{left-handed unknot with } -1 \\
\text{framing}
\end{array} \right] & = q^{-C_j} V_j \left[ \begin{array}{c}
\text{unknot with } 0 \\
\text{framing}
\end{array} \right] = q^{-C_j}[2j + 1].
\end{align*}
\]

In this framing, each right-(left-) handed crossing in a knot introduces a self-linking number \( +1 \) \( (-1) \). For a right-handed trefoil (self-linking number \( = 3 \)), the invariant
in this framing turns out to be:

\[ V_j[T_+] = \sum_{m=0,1,...\min(2j,k-2j)} [2m+1] (-)^m q^{-3C_j+\frac{3}{2}C_m} \]

A three-manifold invariant is constructed from these link invariants in vertical framing. It has been shown that\[5\]: For a framed link \([L, f]\) with component knots, \(K_1, K_2, ..., K_r\) and their framings respectively as \(n(1), n(2), ..., n(r)\), the quantity

\[ \hat{F}[L, f] = \alpha^{-\sigma[L, f]} \sum_{\{j_i\}} \mu_{j_1} \mu_{j_2} ... \mu_{j_r} V[L; n(1), n(2), ..., n(r); j_1, j_2, ..., j_r] \quad (15) \]

constructed from invariants \(V\) of the unoriented framed link in vertical framing, is an invariant of the associated three-manifold obtained by surgery on that link. Here the coefficients \(\mu_\ell\) are given by

\[ \mu_\ell = S_{0\ell} \quad \text{where} \quad S_{j\ell} = \sqrt{\frac{2}{k+2}} \sin \frac{\pi(2j+1)(2\ell+1)}{k+2} \]

and \(\alpha = \exp 3\pi ik/[2(k+2)]\), and \(\sigma[L, f]\) is the signature of linking matrix \(W[L, f]\):

\[ \sigma[L, f] = (\text{no. of +ve eigenvalues of } W) - (\text{no. of -ve eigenvalues of } W) \]

The off diagonal elements of the linking matrix \((W[L, f])_{ij}\) are given by linking number \(lk(K_i, K_j)\) for the distinct knots \((i \neq j)\) and diagonal elements \((i = j)\) are the self-linking number (frame number) of the knot \(K_i\): \((W[L, f])_{ii} = SL(K_i) = n_i\).

It can be directly verified that this three-manifold invariant \((15)\) is unchanged under Kirby moves I and II.

**Explicit examples:** Now computation of this invariant for various three-manifolds is rather straightforward. We present its value for a few three-manifolds. The surgery description of manifolds \(S^3\), \(S^2 \times S^1\) and \(RP^3\) are given by an unknot with framing +1, 0 and +2 respectively. As indicated above the invariant for an unknot with zero framing carrying spin \(j\) representation is \([2j+1] = S_{0j}/S_{00}\), where the square bracket represents the \(q\)-number. Thus the invariant for \(S^3\) is:

\[ \hat{F}[S^3] = \hat{F} \left[ +1 \bigotimes^j \right] = \alpha^{-1} \sum_{\ell=0,1/2,1,...,k/2} \mu_\ell q^{C_\ell} \frac{S_{0\ell}}{S_{00}} \]

where \(\mu_\ell = S_{0\ell}\) and the factor \(q^{C_\ell}\) is the effect from framing +1 (one right-handed writhe). We make use of an identity: \(\sum_\ell S_{j\ell} q^{C_\ell} S_{\ell m} = \alpha q^{-C_j-C_m} S_{jm}\) which is closely related to the modular transformations of a torus. Thus this invariant for \(S^3\) is simply:

\[ \hat{F}[S^3] = 1 \]

For the three-manifold \(S^2 \times S^1\) (with surgery representation as an unknot with zero framing):
\[ \hat{F}[S^2 \times S^1] = \hat{F} \left[ \begin{array}{c} 0 \\ \circ \circ \end{array} \right] = \sum_{\ell} \mu_{\ell} \frac{S_{0\ell}}{S_{00}} = \sum_{\ell} \frac{S_{0\ell}S_{\ell0}}{S_{00}} = \frac{1}{S_{00}} \]

where orthogonality property of the \( S \) matrix, \( \sum_{\ell} S_{j\ell} S_{\ell m} = \delta_{jm} \), has been used.

Next for the three-dimensional real projective space \( RP^3 \) (this is an \( S^3 \) with antipodal points identified), the invariant is:

\[ \hat{F}[RP^3] = \hat{F} \left[ \begin{array}{c} +2 \\ \circ \circ \end{array} \right] = \alpha^{-1} \sum_{j=0, \frac{1}{2}, 1, ... \frac{k}{2}} \frac{S_{0j} q^{2c_j} S_{j0}}{S_{00}}. \]

A slightly more complex example we take up is the Poincare manifold \( P^3 \) (also known as dodecahedral space or Dehn’s homology sphere). It is a homology three-sphere given by the set of points \((u, v, w)\) in complex 3-space such that \( u^2 + v^3 + w^5 = 0 \) and \(|u|^2 + |v|^2 + |w|^2 = 1 \). Its surgery presentation is given \[21\] by a right-handed trefoil knot with framing +1:

\[
\begin{array}{c}
+1 \\
R \\
\end{array}
\]

\[
\begin{array}{c}
R \\
L \\
\end{array}
\]

\[
\begin{array}{c}
L \\
\end{array}
\]

Notice, each of three right-handed crossings introduces +1 linking number between the trefoil knot and its vertical framing, and each of two left-handed writhes contributes −1 so that the total framing number of this knot is +1. Now using the knot invariant for trefoil in vertical framing given above, the invariant for this three-manifold can easily be written down:

\[ \hat{F}[P^3] = \alpha^{-1} \sum_{j=0, \frac{1}{2}, 1, ... \frac{k}{2}} S_{0j} q^{-2c_j} \sum_{m=0, 1, 1, ... \min(2j, k-2j)} (-)^m [2m + 1] q^{-3c_j + \frac{2c_m}{2}} \]

The two left-handed writhes introduce a factor of \( q^{-2c_j} \).

The invariant \( \hat{F} \) for a manifold \( M^3 \) constructed above is same, up to a normalization, as the partition function of an \( SU(2) \) Chern-Simons theory on that manifold\[11\]:

\[ Z[M^3] = \hat{F}[M^3] S_{00}. \tag{16} \]

Generally, it is rather difficult to obtain the Chern-Simons partition function for a given three-manifold \( M^3 \) directly. But, the formulae above, make its computation through \( \hat{F} \) rather easy.

The three-manifold invariant presented here is given in terms of link invariants from \( SU(2) \) Chern-Simons theory. It is clear that a similar construction can be done with link invariants from Chern-Simons gauge theories based on other semi-simple groups. This would yield a new method of obtaining the partition function of such Chern-Simons theories.
Next question we may ask is: Is this three-manifold invariant complete? Two
manifolds \( M \) and \( M' \) for which the invariants \( \hat{F}[M] \) and \( \hat{F}[M'] \) are different can not
be homeomorphic to each other. But the converse is not always true; for two arbitrary
manifolds, the invariants need not be always different. Recall the invariants obtained
from Chern-Simons theory for mutant knots are not distinct. Hence, manifold ob-
tained by surgery on topologically inequivalent mutant knots can not be distinguished
by this three-manifold invariant.

6 Perturbative non-Abelian Chern-Simons theory

Though Chern-Simons theories have been solved exactly and non-perturbatively as
discussed above, perturbative analysis of these theories are also rich in their math-
ematical structure. If we expand the expectation value of the Wilson loop operator
associated with a knot as a perturbative power series in the coupling constant, the
coefficients of such an expansion have a deep mathematical meaning. These on their
own are topological invariants characteristic of the knot.

Last decade has also witnessed enormous research activity in direct perturbative
calculations in Chern-Simons gauge field theory [26]. By simple power counting this
theory is superrenormalizable. There are divergences, which need to be regularized.
The effective coupling constant \( k \) does in general depend on the regularization. In
a class of regularizations, a shift in the coupling constant takes place: \( k \rightarrow k + 2 \)
for \( SU(2) \) theory. This shift is consistent with the effective coupling in the non-
perturbative studies of the theory.

It is very easy to see that the first order contribution to the vacuum expectation
value of the Wilson loop operator for a knot is the self-linking number of knot up
to some group theoretic factors. This is so because at this order, the theory reduces
essentially to Abelian Chern-Simons theories. Topological regularization of the coin-
cident loop integrals through framing as discussed in Sec.3 earlier, leads to this result.
Higher order contributions to the expectation value of a Wilson loop operator in an
\( SU(2) \) Chern-Simons theory yield the famous Vassiliev invariants. These were first
introduced by V.A. Vassiliev in 1990 from a totally different mathematical framework
involving a study of the space of all smooth maps of \( S^1 \) into \( S^3 \). These maps have
different types of singularities. According to the type of singularities, this space of
maps divides into classes, each of which corresponds to a knot type. These classes
are characterized by the families of invariants characterizing the knot [27].

Perturbative studies of Chern-Simons theory have provided new insights into the
theory of Vassiliev invariants. In a gauge theory, perturbative calculations are to
be performed in a definite gauge. Calculations in the Landau gauge [28] lead to
covariant integral representations of Vassiliev invariants, also known as configuration
space integrals first developed by Bott and Taubes in 1994 [29]. Another integral
representation of the Vassiliev invariants was introduced by M. Kontsevich in 1993
[30]. This corresponds to perturbative calculation of the Chern-Simons theory in
light-cone gauge \[31\]. It is rather very difficult to realize that these two integrals represent the same invariant. However, from a field theoretic point of view, this is simply a consequence of gauge invariance. Calculations in the temporal gauge have yielded yet another formulation of these invariants, leading to combinatorial formulae for them \[32\].

7 Gravity and Chern-Simons theory

While Chern-Simons theories have provided a powerful framework for theory of knots, these field theories are also of direct relevance in physics. For example there is an intimate relationship between these field theories and three dimensional gravity which is also a topological field theory. In fact two copies of SU(2) Chern-Simons theories represent gravity in Euclidean three-space with a negative cosmological constant\[33\]. To see this, just consider the partition function of two SU(2) Chern-Simons theories recast in terms of an SL(2, C) Chern-Simons theory as:

\[
Z = \int [dA, d\bar{A}] \exp \left\{ \frac{ik}{8\pi} \int_{M^3} d^3x \epsilon^{\mu\nu\alpha} \left[ tr (A_\mu \partial_\nu A_\alpha + \frac{2}{3} A_\mu A_\nu A_\alpha) - tr (\bar{A}_\mu \partial_\nu \bar{A}_\alpha + \frac{2}{3} \bar{A}_\mu \bar{A}_\nu \bar{A}_\alpha) \right] \right\}
\]

where \(A\) is an the SL(2, C) gauge field and \(\bar{A}\) its conjugate. This partition function is square of two SU(2) partition functions: \(Z_{SL(2, C)} = |Z_{SU(2)}|^2\). Make a change of variables \(A = \omega + ie/\ell\) and \(\bar{A} = \omega - ie/\ell\), where \(\omega\) and \(e\) are the gravitational spin connection and triad respectively. Writing \(kS[A] = \frac{k}{8\pi} \int d^3x \epsilon^{\mu\nu\alpha} tr [A_\mu \partial_\nu A_\alpha + \frac{2}{3} A_\mu A_\nu A_\alpha]\), this then relates the action of these two Chern-Simons theories to Einstein-Hilbert action for three dimensional gravity:

\[
ik(S[A] - S[\bar{A}]) = \frac{1}{16\pi G} \int_{M^3} d^3x \sqrt{g} \left( R + \frac{2}{\ell^2} \right)
\]

where the cosmological constant = \(-1/\ell^2\) is negative and the Chern-Simons coupling is related to the gravitational coupling as \(k = \ell/(4G)\).

This is closely related to another development in gravity. Three-dimensional gravity has a lattice formulation, first introduced by G. Ponzano and T. Regge in 1968 \[34\]. Here the three-manifold is decomposed into simplices. Each three-simplex is a tetrahedron. To each edge of the tetrahedron, a half-integral spin \(j\), called its colour, is assigned so that its length is given by \(\sqrt{j(j+1)}\). The spins on the three edges of each triangular face satisfy the triangular angular momentum inequality relations. The gravitational partition function is constructed in terms of Racah-Wigner six-\(j\) symbols for each tetrahedron in the simplicial decomposition of the manifold. For large spins, the six-\(j\) symbols reproduce the ordinary gravitational action. Ponzano-Regge partition function suffers from a problem: it diverges as all possible spin values are allowed to live on the edges. This, therefore requires a
regularization. A slightly more complex generalization of this lattice gravity model, which also provides a regularization, is related to a model first introduced by V.G. Turaev and O.Y. Viro [35]. It replaces the ordinary 6-\(j\) symbols by their \(q\)-deformed analogues (with \(q\) as a root of unity). For large spin values, the \(q\)-six-\(j\) symbol can be shown to give Regge action for a tetrahedron and represents Euclidean gravity action with a negative cosmological constant. The Turaev-Viro model would then be a quantum description of this three dimensional gravity.

For a triangulation of the three-manifold in terms of tetrahedra labeled by \(t\) and coloring \(j_e\) of its edges labeled by \(e\), Turaev-Viro partition function for a manifold without boundary is given by the formula:

\[
Z_{TV} = \sum_{\text{colourings } j_e \leq k/2} \prod_{\text{vertices}} \frac{1}{\Lambda} \prod_{\text{edges } e} (-1)^{2j_e} [2j_e + 1] \times \prod_{\text{tetrahedra } t} \exp\left(-i\pi \sum_i j_i(t)\right) \left\{ \begin{array}{ccc} j_1(t) & j_2(t) & j_3(t) \\ j_4(t) & j_5(t) & j_6(t) \end{array} \right\}_q
\]

The the square brackets indicate a \(q\)-numbers, and curly brackets represent a \(q\)-6\(j\) symbol. The deformation parameter \(q\) is related to the Chern-Simons coupling by \(q = \exp[2\pi i/(k + 2)]\) and \(\Lambda = -2(k + 2)/(q^{1/2} - q^{-1/2})^2 = (S_{00})^{-2}\). This partition function is naturally regularized and finite due to the restriction on the spins living on the edges \((j_e \leq k/2)\) introduced by the fact that the deformation parameter is a root of unity. Further this partition function can be shown to be exactly square of an SU(2) Chern-Simons partition function, \(Z_{TV} = |Z_{SU(2)}|^2\). This provides yet another representation for the Chern-Simons partition function.

Notice that the integration measure in the partition function of two Chern-Simons theories above is \([dA, d\bar{A}]\), whereas for the gravity partition function, it is \([de, d\omega]\). Since \(A = \omega + ie/\ell\) and \(\bar{A} = \omega - ie/\ell\), the relation between the two involves \(1/\ell\) factors as the Jacobian. In fact in more exact treatment, it becomes clear that the Jacobian for this change of variables introduces exactly a factor of \(\Lambda\) for every vertex of the triangulation, so that the gravity partition function is just the Turaev-Viro partition function without the \(1/\Lambda\) factors:

\[
Z_{\text{grav}} = \sum_{\text{colourings } j_e \leq k/2} \prod_{\text{edges } e} (-1)^{2j_e} [2j_e + 1] \times \prod_{\text{tetrahedra } t} \exp\left(-i\pi \sum_i j_i(t)\right) \left\{ \begin{array}{ccc} j_1(t) & j_2(t) & j_3(t) \\ j_4(t) & j_5(t) & j_6(t) \end{array} \right\}_q
\]

For a manifold with boundary, this expression has additional factors of \(\exp(i\pi j_b) \sqrt{[2j_b + 1]}\) for every boundary edge with a spin \(j_b\). This partition function then is a functional of the boundary triangulation and spins of edges on the boundary.

There are many interesting questions which can be addressed in this framework for three-dimensional gravity. Some of these are: how does a black hole look in this formulation? What is its entropy? Analysis shows that a black hole (Banados-Teitelboim-Zanelli black hole) is given by a solid torus. Its horizon is given by the
longitudinal circle at the core of this solid torus. The possible states associated with
this black hole are the states associated with different triangulations of the black
hole manifold, with the restriction that the longitudes have same circumference. It
can be shown that correct semi-classical behaviour of entropy is reproduced by states
-cororresponding to all possible triangulations of such an Euclidean black hole [36]. The
dominant contribution comes from the states at the horizon.

Chern-Simons theories have also played an important role in non-perturbative
formulation of canonical quantum gravity in four dimensions [37]. In this approach,
the physical states are given by spin-networks with associated graphs in three-space,
where edges are labeled by $SU(2)$ spins (colours) and vertices are given by interwin-
ning operators. Quantum mechanical operators corresponding to lengths, areas and
-volumes all have discrete spectrum. It can be argued that the boundary degrees of
freedom of a black hole, say Schwarzschild black hole, in this four dimensional theory
can be described by a Chern-Simons theory [38, 39]. The action embodying the appro-
riate boundary conditions on the black hole horizon consists of, in addition to the
Einstein-Hilbert action (in suitable variables), an $SU(2)$ Chern-Simons gauge theory
living on a coordinate chart of a constant finite cross-sectional area on the horizon.
The Chern-Simons coupling $k$ is proportional to this constant cross-sectional area.
As the fundamental quantum excitations are polymer like, the horizon area is gen-
erated by the punctures where these spin-polymers pierce it. A bulk polymer state
that gives the horizon its area in this manner has to be compatible with the sur-
face states on the horizon itself. These boundary states are described by a quantum
$SU(2)$ Chern-Simons theory on the horizon. That is, the space of these boundary
degrees of freedom is given by the space of states of Chern-Simons theory on a three-
manifold with an $S^2$ boundary with finitely many punctures on which spins live.
The entropy of the black hole emerges from these boundary states. For large areas,
where essentially $U(1)$ subgroup of $SU(2)$ contributes, the entropy is calculated by
- counting these states. Their number grows exponentially with horizon area yielding
the semi-classical Bekenstein-Hawking expression for black hole entropy [39]. For fi-
nite areas, full $SU(2)$ counting has to be done. This has been done by exploiting
the relation between the boundary states of Chern-Simons theory and the space of
conformal blocks of associated Wess-Zumino conformal field theory on the boundary
two-sphere, a relationship which played a crucial role in obtaining the link invariants
in Sec.4. This yields an _exact_ formula for entropy of a non-rotating black hole which
for large areas reproduces the semi-classical formula, but for finite areas goes beyond
the Bekenstein-Hawking result [40].

8 Summary and Concluding remarks

We have made an attempt here to indicate how quantum field theories, which have
been successfully used to describe physics of fundamental interactions of Nature, can
also be used to study geometry and topology of low dimensional manifolds. These
developments not only provide new insights into old problems of topology of these manifolds but also have been responsible for profoundly interesting new mathematical results. These developments make use of many of the recent developments in quantum field theories. The interaction between quantum physics and mathematics has enriched both.

Chern-Simons gauge field theory, a topological quantum field theory, provides a powerful framework for modern theory of knots and links in any three-manifold. This is one of the rare quantum field theories which can be explicitly and non-perturbatively solved. While Abelian Chern-Simons theory provides a simple description of linking and self-linking numbers of a link, non-Abelian theories are even richer. For every representation of any non-Abelian gauge group, there is a new link invariant. Jones polynomial associated with spin $1/2$ representation in an $SU(2)$ Chern-Simons theory, is the simplest example of such link invariants. Even more general invariants (coloured invariants) are obtained if we place different representations on the component knots. The framework is rich enough to discuss the knots and links not only in simple manifold like $R^3$ or $S^3$, but any arbitrary three-manifold. Chern-Simons partition function is a particularly interesting three-manifold invariant for which a simple and efficient computational method is available now. Perturbative studies of Chern-Simons theory have given a new framework for describing Vassiliev invariants.

In the process of developing this framework for knot theory, new representations of braids also have been obtained. The close connection that braids have with Yang-Baxter equation, has provided methods of obtaining a variety of new exactly solvable two-dimensional statistical mechanical models in physics. These models are the higher vertex generalizations of the six-vertex model of Lieb and Wu and 19-vertex model of Zamolodchikov and Fateev.

Chern-Simons field theories are also of direct interest in other areas of physics. One area where these have found profound application is quantum gravity. Three-dimensional gravity with a negative cosmological constant, itself a topological field theory, is essentially described by two $SU(2)$ Chern-Simons theories. Micro-states of a black hole in the four dimensional spin-polymer gravity can also be modeled by a Chern-Simons theory. This allows an exact computation of black hole entropy going beyond the semi-classical result. These calculations so far have been done for non-rotating black holes only. These need to be extended for charged and rotating black holes, which requires certain amount of technical work. Further, while an exact formula for quantum entropy of a non-rotating black hole has been derived, a similar exact formula for the expectation value of the area operator in the Chern-Simons approach is not known. Also, a satisfactory understanding of Hawking radiation in this picture is yet to be developed.

String theory is another interesting framework in which black hole entropy has been analyzed in recent times. Though it provides a fundamental quantum description, unfortunately, calculations in this theory can be done for extremal or near extremal black holes only. These despite their mathematical interest are not astro-
physically realistic. In particular, black holes of interest such as a Schwarzchild black
hole are not generally amenable to analysis in this approach. Also supersymmetry
plays an important role in the string picture. In contrast, modeling of micro-states
of a black hole by an effective Chern-Simons theory is not limited by the constraint
of extremality or near extremality. This framework handles the curved geometry of
the black hole directly without invoking supersymmetry.

There are other topological quantum field theories also. One particularly inter-

esting class is so called cohomological field theories. These are the field theoretical
interpretations of four-manifold invariants obtained by S. Donaldson in 1983. His
work is an example of developments in mathematics which have made critical use of
some of the notions of physics [11]. His theory provides an understanding of the ge-
ometry in four dimensions through self-dual and anti-self-dual Yang-Mills gauge fields
known to physicists as ‘instantons and anti-instantons’. Five years later, E. Witten
provided a quantum field theoretical framework for Donaldsons’s work in terms of
a four dimensional topological Yang-Mills gauge field theory [12]. This field theory
has certain kind of twisted supersymmetry. Donaldson invariants are given as the
correlation functions in this field theory. In recent years, this area has registered even
further boost through the work of Seiberg and Witten [13]. These developments use
the powerful electric-magnetic duality to relate the cohomological field theory based
on gauge group $SU(2)$ to that based on $U(1)$. This brings in completely new insights
into this area and makes calculation of Donaldson four manifold invariants rather
easy.

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