TRAVELING WAVE SOLUTIONS IN A NONLOCAL REACTION-DIFFUSION POPULATION MODEL

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Abstract. This paper is concerned with a nonlocal reaction-diffusion equation with the form
\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u \left( 1 + \alpha u - \beta u^2 - (1 + \alpha - \beta)(\phi * u) \right), \quad (t, x) \in (0, \infty) \times \mathbb{R}, \]
where \( \alpha \) and \( \beta \) are positive constants, \( 0 < \beta < 1 + \alpha \). We prove that there exists a number \( c^* \geq 2 \) such that the equation admits a positive traveling wave solution connecting the zero equilibrium to an unknown positive steady state for each speed \( c > c^* \). At the same time, we show that there is no such traveling wave solutions for speed \( c < 2 \). For sufficiently large speed \( c > c^* \), we further show that the steady state is the unique positive equilibrium. Using the lower and upper solutions method, we also establish the existence of monotone traveling wave fronts connecting the zero equilibrium and the positive equilibrium. Finally, for a specific kernel function \( \phi(x) := \frac{1}{2\sigma} e^{-\frac{|x|}{\sigma}} \) (\( \sigma > 0 \)), by numerical simulations we show that the traveling wave solutions may connects the zero equilibrium to a periodic steady state as \( \sigma \) is increased.

1. Introduction. In this paper, we investigate the following nonlocal reaction-diffusion equation
\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u \left\{ 1 + \alpha u - \beta u^2 - (1 + \alpha - \beta)(\phi * u) \right\}, \quad (t, x) \in (0, \infty) \times \mathbb{R}, \quad (1) \]
where \( \alpha \) and \( \beta \) are positive constants, \( 0 < \beta < 1 + \alpha \), and
\[ (\phi * u)(x) := \int_{\mathbb{R}} \phi(x - y)u(t, y)dy, \quad x \in \mathbb{R}. \]
The kernel function \( \phi(x) \in L^1(\mathbb{R}) \) satisfies
(K1) \( \phi(x) \geq 0 \) for \( x \in \mathbb{R} \) and \( \int_{\mathbb{R}} \phi(x)dx = 1 \).
(K2) There exist positive constants \( \tilde{\lambda} \) and \( M \) such that \( \phi(x) \leq Me^{-\tilde{\lambda}|x|} \) for \( x \in \mathbb{R} \).

Equation (1) was initially introduced by Britton [10, 11] to model the behavior of a single species which is diffusing, aggregating, reproducing and competing for space and resources. The terms in (1) are interpreted as follows: \( \alpha u \) is a measure of the advantage to individuals in aggregating or grouping, \( -\beta u^2 \) represents competition...
for space (rather than resources), the integral (nonlocal) term denotes competition between the individuals for food resources (for more biological details, we refer to [13, 23]).

Gourley et al. [24] and Billingham [9] studied traveling wave solutions of equation (1) with the kernel function \( \phi(x) = \frac{1}{\sqrt{\pi}} e^{-x^2/\delta^2}, \delta > 0 \). It is clear that, as \( \delta \to 0 \), equation (1) reduces to the classical reaction-diffusion equation

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1 + \beta u)(1 - u), \quad (t, x) \in (0, \infty) \times \mathbb{R},
\]

which admits monotone traveling wave fronts connecting 0 to 1 for each speed \( c \geq 2 \). Using asymptotic methods, Gourley et al. [24] constructed traveling wave fronts of (1) connecting 0 to 1 for sufficiently large speed \( c > 2 \). In contrast to the well-known traveling wave fronts of Fisher’s equation, it was found that a hump may appears for traveling wave fronts of (1), which implies the non-monotonicity of traveling wave fronts of (1). Through stability analysis and numerical simulations, they confirmed that (1) indeed exists this wave when \( u \equiv 1 \) is stable. In addition, when \( u \equiv 1 \) is not stable, it was showed that the use of a suitably localized initial condition results in an invading wavefront moving out into the domain and leaving behind it a stable non-uniform steady state. Billingham [9] considered a strong situation (i.e. \( \delta \) is sufficiently large). Using numerical and asymptotic methods, he showed that the unsteady traveling waves, periodic traveling waves and steady traveling waves can develop from localized initial conditions in different and well-defined regions of parameter space.

Recently, there were some great progresses on traveling wave solutions of (1) with \( \alpha = \beta = 0 \), namely, the following equation

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1 - \phi * u), \quad (t, x) \in (0, \infty) \times \mathbb{R}. \tag{2}
\]

A earlier study on traveling wave fronts of (2) was done by Gourley [20] for both two cases that the nonlocality is sufficiently weak and strong by using numerical and perturbation methods. Berestycki et al. [8] proved that (2) admits the traveling wave solution connecting 0 to an unknown positive state for all \( c \geq c^* = 2 \) and there exists no such traveling wave solution with wave speed \( c < 2 \). After that, Nadin et al. [27] showed that the unknown steady state just is the equilibrium 1 for some traveling wave solutions. Fang and Zhao [15] further gave a sufficient and necessary condition for the existence of monotone traveling waves of (2) connecting two equilibria 0 and 1. More recently, Alfaro and Coville [2] rigorously proved that (2) admits the rapid traveling wave solutions connecting 0 to 1. In particular, this allows situations where 1 is unstable in the sense of Turing. Hamel and Ryzhik [25] proved that (2) exists a periodic steady state due to the instability of the equilibrium, that the solutions of the Cauchy problem has a uniform bound, and that the spreading rate of the solutions has upper and lower bounds with compactly supported initial data. Faye and Holzer[14] proved that (2) exists modulated traveling fronts of the form

\[
u(t, x) = U(x - ct, x), \quad \lim_{\xi \to -\infty} U(\xi, x) = 1 + P(x), \quad \lim_{\xi \to +\infty} U(\xi, x) = 1,
\]

where \( P(x) \) is a stationary periodic solution of

\[
0 = v_{xx} - \mu v - \mu v(\phi * v), \quad x \in \mathbb{R}.
\]
Besides, this model was also numerically investigated in [7, 9, 5, 24, 17, 20, 16], and those numerical results showed more behaviors than the theoretical ways. For more results on traveling wave solutions of (2) and other nonlocal reaction-diffusion equations, we refer to [1, 3, 4, 12, 13, 37, 21, 22, 26, 29, 31, 32, 33, 34] and the references therein.

It is clear that the study of Gourley et al. [24] and Billingham [9] on traveling wave solutions of (1) was mainly based on numerical and asymptotic methods, which result in the lack of rigorously mathematical proof on the existence of traveling wave solutions of (1) with general kernel function \( \phi \) and general admissible speed \( c \) (i.e. \( c \) is not sufficiently large). The purpose of this paper is to solve (at least partially) these questions by developing the methods of Berestycki et al. [8], Alfaro and Coville [2] and Fang and Zhao [15]. In contrast with (2), the main difficulty in the study of (1) is due to the positivity of the parameter \( \alpha \), which result in that the solutions of (1) can not be controlled by the solutions of the linearized equation of (1) at the zero equilibrium. To overcome such difficulty, we use the following nonlinear equation

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u \left( 1 + \alpha u - \beta u^2 \right), \quad (t, x) \in (0, \infty) \times \mathbb{R}
\]

(3) to construct a suitable upper solution in the proof of the existence of traveling wave solutions of (1).

The contents of this paper contain four parts. The first part is to establish the existence of traveling wave solutions of (1) with general kernel function \( \phi \) and general admissible speed \( c \), which is done in Section 2. The main method of this part is to use the comparison principle and construct the suitable lower and upper solutions. The following theorem is the main result of this part, which indicates that (1) admits traveling wave solutions connecting the equilibrium 0 to an unknown positive steady state.

**Theorem 1.1.** Let \( c^* > 0 \) be the minimal wave speed of traveling wave fronts of (3) connecting the equilibria \( u_0 = 0 \) and \( u_+ = \frac{\alpha + \sqrt{\alpha^2 + 4\beta}}{2\beta} \). Then for any \( c > c^* \), there exists a traveling wave solution \((c, u)\) of (1) satisfying

\[
-cu' = u'' + u \left\{ 1 + \alpha u - \beta u^2 - (1 + \alpha - \beta)(\phi * u) \right\} \quad \text{in} \quad \mathbb{R}
\]

(4)

with the boundary conditions

\[
\lim_{x \to +\infty} u(x) = 0 \quad \text{and} \quad \liminf_{x \to -\infty} u(x) > 0.
\]

(5)

In particular, the wave profile \( u \) is decreasing on \([Z_0, +\infty)\) for some \( Z_0 > 0 \). Besides, there is no such traveling wave solution \((c, u)\) for speed \( c < 2 \).

**Remark 1.** Due to the assumption \( 1 + \alpha > \beta \), there is \( u_+ = \frac{\alpha + \sqrt{\alpha^2 + 4\beta}}{2\beta} > 1 \). In addition, one has \( c^* \in \left[ 2, 2\sqrt{1 + \frac{\alpha^2}{4\beta}} \right] \). In particular, \( c^* = 2 \) if \( \alpha \leq \sqrt{\frac{\beta}{2}} \), which implies that \( c = 2 \) is the minimal wave speed of positive traveling wave solutions of (1) when \( \alpha \leq \sqrt{\frac{\beta}{2}} \). When \( \alpha > \sqrt{\frac{\beta}{2}} \), we can not get from Theorem 1.1 that \( c = 2 \) is the minimal wave speed. However, we think that \( c = 2 \) is still the minimal wave speed even if \( \alpha > \sqrt{\frac{\beta}{2}} \). In fact, using the method of Wang et al. [33] we have that for any \( c > 2 \), there exists \( \rho(c) > 0 \) such that for any \( \rho < \rho(c) \), (1) admits a decreasing traveling wave front with speed \( c \) connecting 0 and 1 if we take the
kernel function $\phi$ with some specific functions, for example $\phi(x) = \frac{1}{2\rho} e^{-\frac{x^2}{2\rho}}$ and $\phi(x) = \frac{1}{\sqrt{4\pi\rho}} e^{-\frac{x^2}{4\rho}}$, $\rho > 0$.

For $c > c^*$, the traveling wave solution $u$ established by Theorem 1.1 converges to a steady state, but there is no information on the steady state. The second part of this paper is to confirm that the unknown steady state just is the positive equilibrium $u = 1$ for large $c$, which is done in Section 3. The main result of this part are as follows.

**Theorem 1.2.** Let

$$\tau := \frac{(1 + \alpha - \beta)(\alpha + \sqrt{\alpha^2 + 4\beta})}{2\beta} \left( \int_{\mathbb{R}} |z|^2 \phi(z) dz \right)^{\frac{1}{2}}.$$

Then the traveling wave solution $u$ constructed in Theorem 1.1 with speed $c > \tau$ actually satisfies $u(-\infty) = 1$.

As reported by Gourley et al. [24] and Billingham [9], the traveling wave solution $u$ of (1) may be not monotone even if $u$ satisfies $u(+\infty) = 0$ and $u(-\infty) = 1$. Therefore, the third part of this paper is concerned the existence of monotone traveling wave front of (1). In Section 4, we give a sufficient condition for the existence of monotone traveling wave front of (1) connecting 0 and 1 by constructing a pair of suitable lower and upper solutions for an appropriate monotone operator (see [15, 18]). To measures how localized the kernel is and weights the rate of nonlocal interactions (see [15]), we introduce a parameter $\sigma > 0$ into the kernel function, namely, let $\phi_\sigma(x) := \frac{1}{\sigma^2} \phi \left( \frac{x}{\sigma} \right)$ for $x \in \mathbb{R}$. We replace $\phi$ by $\phi_\sigma$ in (1) and rewrite equation (1) as follows:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u \left\{ 1 + \alpha u - \beta u^2 - (1 + \alpha - \beta)(\phi_\sigma \ast u) \right\}, \quad (t, x) \in (0, \infty) \times \mathbb{R}. \quad (6)$$

Then we have the following theorem.

**Theorem 1.3.** For any $c \geq 2\sqrt{1 + \frac{\alpha^2}{4\beta}}$, there exists $\sigma(c) \in (0, \infty)$, such that (6) admits a monotone traveling wave front connecting 0 to 1 if $\sigma \leq \sigma(c)$.

In Theorems 1.2 and 1.3, we know that the unknown steady state in Theorem 1.1 is the uniform steady state $u = 1$. Therefore, at the end of this paper (in Section 5), other possible forms of the unknown steady state are considered. By choosing a special kernel function $\phi(x) = \frac{1}{2\sigma^2} e^{-\frac{x^2}{2\sigma^2}}$ ($\sigma > 0$) and using numerical simulations, we show that the unknown steady state can be a periodic steady state. Furthermore, by linear stability analysis we show why and when this situation may happen, namely, the Turing bifurcation will occur.

2. **Existence of traveling wave solutions.** In this section, we prove Theorem 1.1. First, we consider traveling wave fronts of (3), namely, the following equation

$$u_t = u_{xx} + f(u), \quad (7)$$

where $f(u) = u(1 + \alpha u - \beta u^2)$. Setting $\xi = x - ct$ and looking for solutions of (7) with the form of $u(x, t) = U(\xi)$, we can get

$$-cU' - U'' = U(1 + \alpha U - \beta U^2). \quad (8)$$
Obviously, (8) has three equilibria
\[ u_{-} = \frac{\alpha - \sqrt{\alpha^2 + 4\beta}}{2\beta}, \quad u_0 = 0, \quad u_{+} = \frac{\alpha + \sqrt{\alpha^2 + 4\beta}}{2\beta}. \]
Due to \(1 + \alpha > \beta\), there is \(u_{+} > 1\). Since \(f(u) > 0\) for \(u \in (0, u_{+})\), \(f'(0) = 1 > 0\) and \(f'(u_{+}) < 0\), it follows from [6, 28, 30, 38] that we have the following lemma.

**Lemma 2.1.** There exists \(c^*\) satisfying
\[ 2 = 2\sqrt{f'(0)} \leq c^* \leq 2\sqrt{\frac{\sup_{(0, u_{+})} f(u)}{u}} = 2\sqrt{1 + \frac{\alpha^2}{4\beta}}, \]
such that (7) exists traveling front with the form of \(u(x, t) = U(x - ct)\) satisfying
\[ U(-\infty) = u_{+} \quad \text{and} \quad U(+\infty) = u_0, \]
if and only if \(c \geq c^*\).

**Lemma 2.2.** One has \(c^* = 2\) when \(\alpha \leq \sqrt{\frac{\beta}{2}}\).

*Proof.* Define \(S = \{g(s) \in C^1[0, u_{+}] \mid g(0) = g(u_{+}) = 0, \ g'(0) > 0, \ g'(u_{+}) < 0 \text{ and } g(s) > 0 \text{ for } s \in (0, u_{+})\}\). It follows from [30, Chapter 1, §2] that
\[ c^* = \inf_{g \in S} \sup_{u \in (0, u_{+})} \left\{ \frac{f(u)}{g(u)} + g'(u) \right\}. \]
Let
\[ g(u) = \frac{u(u_{+} - u)}{u_{+}}, \]
then we have
\[ c^* \leq \sup_{u \in (0, u_{+})} \left\{ \frac{f(u)}{g(u)} + g'(u) \right\}. \]
Substituting (10) into (9), we get \(c^* \leq 2\). Combining with Lemma 2.1, we know that \(c^* = 2\). This completes the proof. \(\Box\)

Let
\[ \lambda_1 = \frac{c - \sqrt{c^2 - 4}}{2} \quad \text{and} \quad \lambda_2 = \frac{c + \sqrt{c^2 - 4}}{2}, \]
be two real positive roots of the equation \(\lambda^2 - c\lambda + 1 = 0\), where \(c > c^*\). It follows from [6, 28, 30, 38] that the following lemma holds.

**Lemma 2.3.** One has
\[ U(\xi) \sim e^{-\lambda_1 \xi}, \quad (\xi \to +\infty), \]
when \(c > c^*\).

**Proof of Theorem 1.1.** Now we begin to prove Theorem 1.1.

*Construction of lower solution.* It follows from Lemmas 2.1 and 2.3 that there exists \(A_1 > 0\) such that
\[ U(x) \leq A_1 e^{-\lambda_1 x}, \quad \forall x \in \mathbb{R}. \]
By the assumptions (K1) and (K2), there exists some constant \(A_2 > 0\) such that
\[ (\phi * U)(x) \leq A_2 Z_1 e^{-\theta_1 x}, \quad \forall x \in \mathbb{R}, \]
where \(\theta_1 = \min \left\{ \frac{\lambda_1}{2}, \frac{\lambda_2}{2} \right\}\) and \(Z_1 = \int_{\mathbb{R}} \phi(x) e^{\theta_1 x} \, dx\). Let
\[ p_c(x) = \frac{1}{B} e^{-\lambda_1 x} - e^{-(\lambda_1 + \epsilon)x}, \]
where \(\epsilon > 0\) and \(\epsilon \to 0^+\) as \(c \to c^*\). The construction of the lower solution follows.
where \( \varepsilon > 0 \) is small enough so that
\[
(2\lambda_1 - c)\varepsilon + \varepsilon^2 < 0, \quad \text{and} \quad 2\varepsilon < \theta_1,
\]
and \( B > 1 \) is large enough so that
\[
B > \left( \frac{\beta + (1 + \alpha - \beta)A_2Z_1}{c - 2\lambda_1} \right) \varepsilon - \varepsilon^2.
\]
Then for \( x \) with \( p_c(x) > 0 \), that is \( x > \frac{\ln B}{\varepsilon} > 0 \), we have
\[
- cp'(x) - p''(x) - p_c(x)(1 + \alpha p_c(x) - \beta p_c^2(x)) + (1 + \alpha - \beta)p_c(x)(\phi * U)(x)
\]
\[
= \left[ (2\lambda_1 - c)\varepsilon + \varepsilon^2 \right] e^{-(\lambda_1+\varepsilon)x} + (1 + \alpha - \beta)p_c(x)(\phi * U)(x)
\]
\[
- \left( \alpha - \beta \left( \frac{1}{B} e^{-\lambda_1 x} - e^{-(\lambda_1+\varepsilon)x} \right) \right) \left( \frac{1}{B} e^{-\lambda_1 x} - e^{-(\lambda_1+\varepsilon)x} \right)^2
\]
\[
\leq e^{-(\lambda_1+\varepsilon)x} \left[ (2\lambda_1 - c)\varepsilon + \varepsilon^2 + \frac{(1 + \alpha - \beta)A_2Z_1}{B} e^{-(\theta_1-\varepsilon)x} \right]
\]
\[
- \left( \alpha - \beta \left( \frac{1}{B} e^{-\lambda_1 x} + \beta e^{-(\lambda_1+\varepsilon)x} \right) \right) \left( \frac{1}{B} e^{-\varepsilon x} \right)^2 e^{-(\lambda_1-\varepsilon)x}
\]
\[
\leq e^{-(\lambda_1+\varepsilon)x} \left[ (2\lambda_1 - c)\varepsilon + \varepsilon^2 + \frac{\beta}{B^3} e^{-(2\lambda_1+\varepsilon)x} + \frac{(1 + \alpha - \beta)A_2Z_1}{B} e^{-(\theta_1-\varepsilon)x} \right]
\]
\[
ge^{-\lambda_1 x} \left[ (2\lambda_1 - c)\varepsilon + \varepsilon^2 + (\beta + (1 + \alpha - \beta)A_2Z_1) e^{-(\theta_1-\varepsilon)x} \right] < 0.
\]
Let
\[
\overline{p}_c(x) = \max(0, p_c(x)), \quad \forall x \in \mathbb{R},
\]
then for \( x \neq \frac{\ln B}{\varepsilon} \), there holds
\[
- cp'(x) - \overline{p}_c''(x) - \overline{p}_c(x)(1 + \alpha \overline{p}_c(x) - \beta \overline{p}_c^2(x)) + (1 + \alpha - \beta)\overline{p}_c(x)(\phi * U)(x) < 0.
\]

\textbf{A finite domain problem.} For \( c > c^* \), we consider the problem on \((-a, a)\):
\[
- cu' - u'' = u\{1 + \alpha u - \beta u^2 - (1 + \alpha - \beta)(\phi * u)\}, \quad u(\pm a) = \overline{p}_c(\pm a),
\]
(11)
where \( a > \frac{\ln B}{\varepsilon} \). Define a convex set
\[
\mathcal{M}_a = \{ u \in C(-a, a) : \overline{p}_c(x) \leq u(x) \leq U(x), \quad u(\pm a) = \overline{p}_c(\pm a) \}.
\]
Consider the following two-point boundary value problem
\[
- cu' - u'' + [(1 + \alpha - \beta)(\phi * u_0) + \gamma]u = \gamma u_0 + u_0(1 + \alpha u_0 - \beta u_0^2), \quad u(\pm a) = \overline{p}_c(\pm a),
\]
(12)
where \( u_0 \in \mathcal{M}_a \) and \( \gamma > 0 \) satisfies \( \min_{u \in [0, u_0]} \{1 + \gamma + 2\alpha u - 3\beta u^2\} \geq 0 \). Let \( \Psi_a \) be the solution mapping of (12), that is \( \Psi_a u_0 = u \). We show that the set \( \mathcal{M}_a \) is invariant for the mapping \( \Psi_a \). Given \( u_0 \in \mathcal{M}_a \). Since \( u \equiv 0 \) is a subsolution of (12), we have \( u(x) > 0 \) for any \( x \in (-a, a) \). Consequently,
\[
- cU' - U'' + [(1 + \alpha - \beta)(\phi * u_0) + \gamma]U
\]
\[
\geq - cU' - U'' + \gamma U
\]
\[
= \gamma U + U(1 + \alpha U - U^2) \geq \gamma u_0 + u_0(1 + \alpha u_0 - \beta u_0^2)
\]
and \( u(\pm a) = \overline{p}_c(\pm a) \leq U(\pm a) \). From the maximum principle, we know that \( u(x) \leq U(x) \) for all \( x \in (-a, a) \). On the other hand, since
\[
0 \leq \gamma u_0 + u_0(1 + \alpha u_0 - \beta u_0^2),
\]
we have that \( u = 0 \) is a subsolution, which implies \( u(x) \geq 0 \) for \( x \in (-a, a) \). Now, for \( x \in (-a, \frac{b}{2}) \), we have
\[
-c \overline{p}'_{c} - \overline{p}''_{c} + [(1 + \alpha - \beta)(\phi * u_{0}) + \gamma]\overline{p}_{c} \leq -c \overline{p}'_{c} - \overline{p}''_{c} + [(1 + \alpha - \beta)(\phi * U) + \gamma]\overline{p}_{c}
\]
\[
\leq \gamma\overline{p}_{c} + \overline{p}_{c}(1 + \alpha\overline{p}_{c} - \beta\overline{p}_{c}^{2})
\]
\[
\leq \gamma u_{0} + u_{0}(1 + \alpha u_{0} - \beta u_{0}^{2}),
\]
and \( u(\pm a) = \overline{p}_{c}(\pm a) \). The maximum principle implies that \( u(x) \geq \overline{p}_{c}(x) \) for all \( x \in (-a, a) \). Thus, we conclude that the set \( M_{a} \) is invariant. Using the \( L^{p} \) estimates of linear elliptic differential equations and the embedding theorem (see Gilbarg and Trudinger [19, Corollary 9.18 and Theorem 7.26], see also [35, 36]), we further have that \( \Psi_{a} \) mapping \( M_{a} \) to \( M_{a} \) is compact and continuous. Then it follows from the Schauder fixed point theorem that \( \Psi_{a} \) has a fixed point \( u_{a} \) in \( M_{a} \). Namely, there exists \( u_{a} \in M_{a} \) satisfying (11).

Existence and nonexistence of traveling wave solutions. Since \( 0 \leq u_{a}(x) \leq U(x) \leq u_{+} = \frac{\alpha + \sqrt{\alpha^{2} + 4\beta}}{2\beta} \), we know that \( u_{a}(x) \) is uniformly bounded in \( C^{2, \alpha}(-\frac{b}{2}, \frac{b}{2}) \).

Let \( a \to +\infty \) (possibly along a subsequence), then there exists a function \( u(\cdot) \in C^{2}(\mathbb{R}) \) satisfying
\[
-cu' - u'' = u\{1 + \alpha u - \beta u^{2} - (1 + \alpha - \beta)(\phi * u)\}, \quad x \in \mathbb{R}.
\]

Besides, we know that \( \overline{p}_{c}(x) \leq u(x) \leq U(x), \forall x \in \mathbb{R} \). In particular, we have
\[
\lim_{x \to +\infty} u(x) = 0.
\]

Next, we prove the rest of Theorem 1.1 by three steps.

Step 1. We show there exists \( Z_{0} > 0 \) such that \( u(x) \) is monotonically decreasing for \( x > Z_{0} \).

By a contradiction argument we assume that \( u(x) \) is not eventually monotonic as \( x \to +\infty \), then there exists a sequence \( z_{n} \to +\infty \) such that \( u(x) \) achieves a local minimum at \( z_{n} \) and \( u(z_{n}) \to 0 \). It follows that
\[
(\phi * u)(z_{n}) \geq \frac{1 + \alpha u(z_{n}) - \beta u^{2}(z_{n})}{1 + \alpha - \beta}.
\]

Since \( u(z_{n}) \to 0 \) and \( u(x) \) is bounded in \( C^{2}(\mathbb{R}) \), by the Harnack inequality, we know that, for any \( Z > 0 \) and any \( \delta \in \left(0, \min\left\{\frac{\alpha}{2\beta}, \frac{1}{4}, \frac{1}{1+\alpha-\beta}\right\}\right) \), there exists \( N \), such that \( u(x) \leq \frac{\delta}{2} \) for all \( x \in (z_{n} - Z, z_{n} + Z) \), \( n \geq N \). Thus
\[
(\phi * u)(z_{n}) \geq \frac{1 + \alpha u(z_{n}) - \beta u^{2}(z_{n})}{1 + \alpha - \beta} \geq \frac{1}{1 + \alpha - \beta} > \delta. \quad (13)
\]

However, we have
\[
(\phi * u)(z_{n}) \leq \delta, \quad (14)
\]
when \( Z \) is sufficiently large and \( \delta \) is sufficiently small. From (13) and (14) we can get a contradiction.

Step 2. We show there exists no traveling wave solution of speeds \( c < 2 \).

Let \( v_{n}(x) = u(x + n)/u(x_{n}) \). Because \( (c, u) \) satisfies (4), then
\[
-cv'_{n} - v''_{n} = v_{n}\left\{1 + \alpha u(x_{n})v_{n} - \beta(u(x_{n})v_{n})^{2} - (1 + \alpha - \beta)(\phi * u_{n})\right\} \text{ in } \mathbb{R},
\]
where \( u_{n}(x) = u(x + x_{n}) \). From the Harnack inequality, we can also obtain that \( u_{n} \to 0 \) locally uniformly in \( x \) as \( n \to +\infty \) and \( v_{n}(x) \) is locally uniformly in \( x \). Thus
For convenience, we define $v_n \to v$ ($n \to +\infty$), where $v \in C^2(\mathbb{R})$ satisfies

$$-v'' - cv' = v$$

Because $v$ is non-negative and $v(0) = 1$, $v$ is positive in $\mathbb{R}$. In addition, we know that equation (15) has such a solution if and only if $c \geq 2$. Then we conclude that $c \geq 2$. Thus, there exists no traveling wave solution of speeds $c < 2$.

**Proof of Theorem 1.2**

Assume that $(c, u)$ satisfies (4) and (5), where $c > c^*$. In the following we show that $\lim_{x, y \to -\infty} u(x, y)$ is bounded. Since $u(x) \leq U(x), \forall x \in \mathbb{R}$, then $\|u\|_{L^\infty} \leq u_+ := \frac{\alpha + \sqrt{\alpha^2 + 4\beta}}{2\beta}$.

In addition, we know that $u(x)$ satisfies

$$u(x) = \frac{1}{\lambda_2 - \lambda_1} \int_x^{\infty} \left( e^{\lambda_1(x-y)} - e^{\lambda_2(x-y)} \right)$$

$$\times (-\alpha u^2(y) + \beta u^3(y) + (1 + \alpha - \beta)u(\phi * u)(y)) \, dy,$$

where $\lambda_1 < \lambda_2 < 0$ are two negative roots of the characteristic equation $\lambda^2 + c\lambda + 1 = 0$. Therefore,

$$u'(x) = \frac{1}{\lambda_2 - \lambda_1} \int_x^{\infty} \left( \lambda_1 e^{\lambda_1(x-y)} - \lambda_2 e^{\lambda_2(x-y)} \right)$$

$$\times (-\alpha u^2(y) + \beta u^3(y) + (1 + \alpha - \beta)u(\phi * u)(y)) \, dy,$$

which implies

$$|u'(x)| \leq \frac{2}{\sqrt{c^2 - 4}} \left((1 + 2\alpha - \beta)u^2_+ + \beta u^3_+ \right) =: M', \forall x \in \mathbb{R}.$$

**Step 2.** We show that $u' \in L^2(\mathbb{R})$ and $\lim_{x \to \pm\infty} u'(x) = 0$ if

$$c > (1 + \alpha - \beta)\sqrt{n_2} \|u\|_{L^\infty}.$$
Define $W'(x) = x(1 + \beta x)(1 - x)$. We rewrite the equation (4) as

$$cu' = -u'' - u(1 + \beta u)(1 - u) - (1 + \alpha - \beta)u(u - \phi * u).$$

Multiplying $u'$ for two sides and integrating from $-A < 0$ to $B > 0$, we can get

$$c\int_{-A}^{B} u'^2 \, dx = \left[ -\frac{1}{2}u'^2 - W(u) \right]_{-A}^{B} - (1 + \alpha - \beta)\int_{-A}^{B} u'(u - \phi * u).$$

The rest proof is similar to [2, p.2096-2097].

Step 3. We further show that $\lim_{x \to -\infty} u(x)$ exist and equal to 1 if $c > (1 + \alpha - \beta)\sqrt{\frac{1}{2}}\|u\|_{L^\infty}$.

Define a set $\Gamma$ as limit points of $u$ at $-\infty$. Because $u$ is bounded, we know that $\Gamma$ is not empty. Let $\xi \in \Gamma$. There exists a sequence $x_n \to -\infty$ such that $u(x_n) \to \xi$. Thus $v_n(x) = u(x + x_n)$ satisfies

$$v''_n + cv'_n = -\left\{1 + \alpha v_n - \beta v_n^2 - (1 + \alpha - \beta)(\phi * v_n)\right\} \text{ in } \mathbb{R}.$$ 

From the interior elliptic estimates and the Sobolev embedding theorem, one can extract a subsequence of $v_n$, still denoted by $v_n$, satisfying $v_n \to v$ strongly in $C^{2,\beta}_{\text{loc}}(\mathbb{R})$ and weakly in $W^{2,p}_{\text{loc}}(\mathbb{R})$. Then by Step 2, we have

$$v'(x) = \lim_{n \to \infty} u'(x + x_n) = 0, \quad \forall x \in \mathbb{R}.$$ 

In addition, $v$ satisfies

$$v'' + cv' = -v \left\{1 + \alpha v - \beta v^2 - (1 + \alpha - \beta)(\phi * v)\right\} \text{ on } \mathbb{R},$$ 

which means $v \equiv 0$ or $v \equiv 1$. Since $v(0) = \lim_{n \to \infty} u(x_n) = \xi$, then $\xi \in \{0, 1\}$. Because $u$ is continuous and $\Gamma$ is connected, we know that $\Gamma = \{0\}$ or $\Gamma = \{1\}$. Combining with (5), we know that $\lim_{x \to -\infty} u(x) = 1$. This completes the proof of Theorem 1.2.

4. Existence of monotone traveling wave fronts. In Section 2, we know that equation (1) admits traveling wave solutions connecting 0 to a positive steady state for any $c > c^*$. In Section 3 we further show that the positive steady state is the equilibrium 1 for sufficiently large $c$. However, we still do not know whether equation (1) admits monotone traveling wave fronts. Therefore, in this section we will find a sufficient condition for the existence of monotone traveling waves fronts of equation (1), namely, we prove Theorem 1.3.

Setting $u(x, t) = U(x - ct)$ and substituting it into equation (6) gives

$$U''(\xi) + cU'(\xi) + U(\xi) \left\{1 + \alpha U(\xi) - \beta U^2(\xi) - (1 + \alpha - \beta)\int_{\mathbb{R}} U(\xi - s)\phi_\sigma(s)ds\right\} = 0.$$ 

Linearizing (16) at $u \equiv 1$, we can get the characteristic equation

$$\Psi(c, \sigma, \lambda) := \lambda^2 + c\lambda + (\alpha - 2\beta) - (1 + \alpha - \beta)\int_{\mathbb{R}} e^{-\lambda s}\phi_\sigma(s)ds$$

$$= \lambda^2 + c\lambda + (\alpha - 2\beta) - (1 + \alpha - \beta)\int_{\mathbb{R}} e^{-\lambda \sigma s}\phi(s)ds = 0,$$

which is equivalent to (by a change of variable $\lambda' = \sigma \lambda$)

$$\frac{\lambda^2}{\sigma^2} + \frac{c}{\sigma} + (\alpha - 2\beta) = (1 + \alpha - \beta)\int_{\mathbb{R}} e^{-\lambda \sigma s}\phi(s)ds = (1 + \alpha - \beta)L(\lambda),$$

where $L(\lambda) = \int_{\mathbb{R}} e^{-\lambda \sigma s}\phi(s)ds$. 

Next, we give the following result.

**Proposition 4.1.** For any $c \geq 0$, there exists $\sigma(c) \in (0, +\infty)$ such that

(i) if $\sigma < \sigma(c)$, then $\Psi(c, \sigma, \lambda) = 0$ has a smallest positive root $\lambda_1$, and there exists $\varepsilon = \varepsilon(c, \sigma) \in (0, \lambda_1)$ such that $\Psi(c, \sigma, \lambda_1 + \varepsilon) > 0$.

(ii) if $\sigma > \sigma(c)$, then $\Psi(c, \sigma, \lambda) = 0$ has no negative root.

(iii) $\sigma(c)$ is nondecreasing in $c \in [0, \infty)$.

The proof of the proposition is completely similar to that of Fang and Zhao [15, Proposition 2.1] and we omit it (in fact, the result is intuitional, see Fig 1).

To prove Theorem 1.3, we fix $c > 2\sqrt{1 + \alpha^2/4\beta}$ and $\sigma < \sigma(c)$. Define

$$ K[\psi] := \psi''(\xi) + c\psi'(\xi) + \psi(\xi) \left( 1 + \alpha\psi(\xi) - \beta\psi^2(\xi) - (1 + \alpha - \beta) \int_\mathbb{R} \psi(\xi - s)\phi(\xi) \phi(s) ds \right). $$

It is clear that finding a solution of (16) is equivalent to searching a function $u$ satisfying $K[u] = 0$. Next, we will find a solution $u$ by constructing lower and upper solutions and using the monotone iterative technique.

Let $\mu_1 < \mu_2 < 0$ be two negative roots of the following equation

$$ \Phi(c, \mu) := \mu^2 + c\mu + 1 + \frac{\alpha^2}{4\beta} = 0. $$

Define

$$ \psi_-(\xi) = \begin{cases} \frac{d\mu_1^\xi}{\mu_1}, & \xi > \xi_-, \\ 1 - e^{\lambda_1^\xi}, & \xi \leq \xi_-, \end{cases} $$

(17)
where
\[
\begin{align*}
\xi & = -\frac{1}{\lambda_1} \ln \left( 1 - \frac{\lambda_1}{\mu_1} \right), \\
d & = -\frac{\lambda_1}{\mu_1} \left( 1 - \frac{\lambda_1}{\mu_1} \right) \left( \frac{\xi}{d} - 1 \right),
\end{align*}
\]
(18)
which implies that
\[
\begin{align*}
d e^{\xi} - 1 & = e^{\lambda_1 \xi}, \\
d \mu_1 e^{\xi} & = -\lambda_1 e^{\lambda_1 \xi}.
\end{align*}
\]
In addition, a direct computation gives
\[
d e^{\xi} > 1 - e^{\lambda_1 \xi}, \forall \xi > \xi_-
\]
Set
\[
\tilde{\psi}(b, \xi) = 1 - e^{\lambda_1 \xi} + be^{(\lambda_1 + \varepsilon) \xi}, \forall \xi \in \mathbb{R},
\]
where \(\varepsilon > 0\) is determined in Proposition 4.1, \(b > 1\) is a constant which will be determined later. We know that \(\tilde{\psi}(b, \xi)\) attains its minimum at \(\xi_0 = \frac{1}{\varepsilon} \ln \left( \frac{\lambda_1}{\mu_1} \right)\)
and is monotone increasing for \(\xi > \xi_0\). Let \(\xi_0 = \frac{1}{\varepsilon} \ln \frac{1}{R}\), then \(\tilde{\psi}(b, \xi_0) = 1\) and
\[
\tilde{\psi}(b, \xi) \geq 1, \forall \xi > \xi_0.
\]
Choose \(b > 1\) large enough satisfying
\[
\frac{1}{\varepsilon} \ln \frac{1}{b} < \frac{1}{\lambda_1 - \varepsilon} \ln \left( \frac{b \Psi(c, \sigma, \lambda_1 + \varepsilon)}{3\beta + (1 + \alpha - \beta) \int_{\mathbb{R}} e^{-\lambda_1 \varepsilon \phi_0(s)} ds} \right),
\]
then for any \(\xi < \xi_0\), we have
\[
\begin{align*}
b \Psi(c, \sigma, \lambda_1 + \varepsilon) & + (\alpha - 3\beta)e^{(\lambda_1 - \varepsilon) \xi} (-1 + be^{\varepsilon})^2 - \beta e^{(2\lambda_1 - \varepsilon) \xi} (-1 + be^{\varepsilon})^3 \\
& + (1 + \alpha - \beta)e^{(\lambda_1 - \varepsilon) \xi} (-1 + be^{\varepsilon}) \left( \int_{\mathbb{R}} e^{-\lambda_1 \varepsilon \phi_0(s)} ds - be^{\varepsilon} \int_{\mathbb{R}} e^{-(\lambda_1 + \varepsilon) \phi_0(s)} ds \right) \\
& > b \Psi(c, \sigma, \lambda_1 + \varepsilon) - 3\beta e^{(\lambda_1 - \varepsilon) \xi} + (1 + \alpha - \beta)e^{(\lambda_1 - \varepsilon) \xi} (-1 + be^{\varepsilon}) \int_{\mathbb{R}} e^{-\lambda_1 \varepsilon \phi_0(s)} ds \\
& > b \Psi(c, \sigma, \lambda_1 + \varepsilon) - \left[ 3\beta + (1 + \alpha - \beta) \int_{\mathbb{R}} e^{-\lambda_1 \varepsilon \phi_0(s)} ds \right] e^{(\lambda_1 - \varepsilon) \xi} > 0.
\end{align*}
\]
Define
\[
\psi_+(\xi) = \begin{cases} 
\tilde{\psi}(b, \xi), & \xi < \xi_0, \\
1, & \xi \geq \xi_0.
\end{cases}
\]
**Lemma 4.2.** For large enough \(b > 1\), one has \(\psi_-(\xi) \leq \psi_+(\xi), \forall \xi \in \mathbb{R}\).

**Proof.** From (18), we know that
\[
\xi_- = -\frac{1}{\lambda_1} \ln \left( 1 - \frac{\lambda_1}{\mu_1} \right),
\]
so \(\xi_- > \xi_0\), when \(b\) is sufficiently large. Then for \(\xi < \xi_0\), we have
\[
\psi_-(\xi) = 1 - e^{\lambda_1 \xi} < \psi_+(\xi) = 1 - e^{\lambda_1 \xi} + be^{(\lambda_1 + \varepsilon) \xi},
\]
and for \(\xi \geq \xi_0\), we have \(\psi_-(\xi) = de^{\lambda_1 \xi} < 1 = \psi_+(\xi)\). This completes the proof. \(\square\)

The next lemma shows that \(\psi_-\) and \(\psi_+\) are lower and upper solutions to \(K[\psi] = 0\), respectively.
Lemma 4.3. (i) $K[\psi_-](\xi) \leq 0, \forall \xi \in \mathbb{R}\setminus\{\xi_-\}$. 

(ii) For sufficiently large $b > 1$, there holds $K[\psi_+](\xi) \geq 0$ for $\xi \in \mathbb{R}\setminus\{\xi_b\}$.

The proof of Lemma 4.3 is long and complex, for reader’s convenience, we prove it in Appendix.

Proof of Theorem 1.3. We first consider $c > 2\sqrt{1 + \frac{\alpha^2}{3\beta}}$ and $\sigma < \sigma(c)$. We know that $K[\psi] = 0$ is equivalent to

$$\psi(\xi) = T[\psi](\xi),$$

where

$$T[\psi](\xi) = \frac{1}{\mu_1 - \mu_2} \int_{-\infty}^{\xi} e^{\mu_2(x-y)} e^{\mu_1(y-x)} \left[ \beta \psi^3 - \alpha \psi^2 + \frac{\alpha^2}{3\beta} \psi + (1 + \alpha - \beta) \psi \times (\psi \ast \phi)(y) \right] dy$$

and $\mu_1 < \mu_2 < 0$ is the root of $\mu^2 + \mu + 1 + \frac{\alpha^2}{3\beta}$ = 0. Since $3\beta x^2 - 2\alpha x + \frac{\alpha^2}{3\beta} \geq 0$ for all $x \in \mathbb{R}$, the operator $T[\psi]$ is monotone. Let $\psi_-$ and $\psi_+$ are respectively defined in (17) and (20). From Lemma 4.3, combining with [18, Corollary 16], we see that

$$\psi_- \leq T[\psi_-] \quad \text{and} \quad \psi_+ \geq T[\psi_+].$$

Define an iteration sequence: $v_0 = \psi_+, \ v_{n+1} = T[v_n], \forall \ n \geq 0$, then

$$\psi_- \leq \ldots \leq v_n \leq v_{n-1} \leq \ldots \leq v_1 \leq v_0 = \psi_+.$$

Let

$$u(\xi) := \lim_{n \to \infty} v_n(\xi), \quad \forall \xi \in \mathbb{R}.$$

Then $u(\xi)$ is a continuous and nonincreasing function on $\mathbb{R}$, and both $u(\pm \infty)$ exist. Obviously, $u = T[u]$ and $\psi_- \leq u \leq \psi_+$. Moreover, it follows from (21) that $u(\pm \infty)$ satisfies

$$\left(1 + \frac{\alpha^2}{3\beta}\right) x = \beta x^3 - \alpha x^2 + \frac{\alpha^2}{3\beta} x + (1 + \alpha - \beta) x^2.$$

Combining $\psi_-(-\infty) \leq u(-\infty) \leq \psi_+(-\infty)$ and $\psi_-(+\infty) \leq u(+\infty) \leq \psi_+(+\infty)$, we know that

$$u(-\infty) = 1 \quad \text{and} \quad u(+\infty) = 0.$$

For $c = 2\sqrt{1 + \frac{\alpha^2}{3\beta}}$ or $\sigma = \sigma(c)$, we only consider the case $c = 2\sqrt{1 + \frac{\alpha^2}{3\beta}}$ and $\sigma = \sigma(c)$ without loss of generality. Choose sequences $c_n > 2\sqrt{1 + \frac{\alpha^2}{3\beta}}$ and $\sigma_n < \sigma(c_n, \phi)$ with $c_n \to 2\sqrt{1 + \frac{\alpha^2}{3\beta}}$ and $\sigma_n \to \sigma(c)$. Obviously, $u_n$ is a monotone traveling wave with speed $c_n$ for each $n$. Fix $u_n(0) = \frac{1}{2}$. It follows from Helly’s theorem that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k}$ converges to a non-increasing function $u$ in $\mathbb{R}$ pointwise as $k \to \infty$. Furthermore, by Lebesgue’s dominated convergence theorem, we can get $u = T[u]$ and hence $u \in C^2$. In addition, we have $u(-\infty) = 1$ and $u(+\infty) = 0$ since $u(0) = \frac{1}{2}$. This completes the proof of Theorem 1.3.
5. Numerical simulations and stability analysis. In Sections 3 and 4, we know that the unknown steady state is $u = 1$ under suitable conditions. In this section we show that the unknown steady state may be a period steady state. Specifically, we first provide some numerical simulations and then, by linear stability analysis we give the explanations to these numerical results. Here we note that in this section we only consider the specific kernel function $\phi(x) := \frac{1}{2\sigma}e^{-\frac{|x|}{\sigma}}, \sigma > 0$.

Let $w(t, x) := (\phi * u)(t, x) = \int_{\mathbb{R}} \frac{1}{2\sigma}e^{-\frac{|x-y|}{\sigma}} u(t, y) dy,$

then equation (1) can be written as

\[
\begin{cases}
  u_t = u_{xx} + u\{1 + \alpha u - \beta u^2 - (1 + \alpha - \beta)w\}, \\
  0 = w_{xx} + \frac{1}{\sigma^2}(u - w).
\end{cases}
\]  

(22)

Before our numerical simulation, the initial value problem needs to be developed first. We define the initial value of $u(t, x)$ as

\[
  u(0, x) = \begin{cases}
    1, & \text{for } x \leq L_0, \\
    0, & \text{for } x > L_0.
  \end{cases}
\]  

(23)

Since $w(x, 0)$ is determined by

\[
w(0, x) = \int_{\mathbb{R}} \frac{1}{2\sigma}e^{-\frac{|x-y|}{\sigma}} u(0, y) dy,
\]

then

\[
w(0, x) = \begin{cases}
  1 - \frac{1}{2}e^{-(x-L_0)} & \text{for } x \leq L_0, \\
  \frac{1}{2} & \text{for } x > L_0.
\end{cases}
\]  

(24)

**Figure 2.** The time and space evolution for nonlocal equation (1). Our computational domain is $x \in [0, 40]$ and $t \in [0, 15]$. The corresponding parameter values are: $L_0 = 10$, $\alpha = 0.2$, $\beta = 0$, $\sigma = 4$. 
In addition, the zero-flux boundary condition is considered here. Along with (23) and (24), the system (22) can be simulated through the pdepe package in Matlab (see Fig 2 and Fig 3).

Now we explain our numerical results. In Figure 2, it is shown that a hump occurs, which agrees well with Figure 1 of Gourley et al. [24]. In Figure 3, we see that the wave is monotonic and connects 0 to 1 for the parameters $\alpha = 2$, $\beta = 0.4$ and $\sigma = 1$. Fix $\alpha = 2$ and $\beta = 0.4$. As $\sigma$ is increased, namely, $\sigma$ takes $\frac{5}{3}$ and 2 respectively, the wave loses its monotonicity and humps appear, which is similar to that in Figure 2. Finally, when $\sigma$ takes $\frac{10}{3}$, we find that a periodic steady state appears and the wave connects the zero equilibrium and the periodic steady state.

In the following, by the linear stability analysis we discuss why and when the wave will connect the zero equilibrium and a periodic steady state (in other words, a periodic steady state appears) as $\sigma > 0$ is increased. Here we would like to point out that similar question has been discussed by Gourley et al. [24] for the parameter $\alpha$.

It is easy to see that system (22) has three equilibria $(0, 0)$, $(1, 1)$ and $(-\frac{1}{\beta}, -\frac{1}{\beta})$. But from the biological point of view, the point $(1, 1)$ is just our main focus.
Setting \( u = 1 + \tilde{u} \) and \( w = 1 + \tilde{w} \) into (22), we get a linearized system
\[
\begin{cases}
\tilde{u}_t = \tilde{u}_{xx} + (\alpha - 2\beta) \tilde{u} - (1 + \alpha - \beta) \tilde{w} \\
0 = \tilde{w}_{xx} + \frac{1}{\sigma^2} (\tilde{u} - \tilde{w}).
\end{cases}
\]  
(25)

Take the test function with the form
\[
\left( \frac{\tilde{u}}{\tilde{w}} \right) = \sum_{k=1}^{\infty} \left( \frac{C^1_k}{C^2_k} \right) e^{i \lambda t + ikx},
\]  
(26)

where \( k \in \mathbb{R} \) is the wave number. Substituting (26) into (25) yields
\[-(\alpha - 2\beta - k^2 - \lambda) \left( \frac{1}{\sigma^2} + k^2 \right) + (1 + \alpha - \beta) \frac{1}{\sigma^2} = 0,
\]
then
\[\lambda = (\alpha - 2\beta - k^2) - \frac{1 + \alpha - \beta}{1 + k^2 \sigma^2}.\]  
(27)

Figure 4. The time and space evolution for equation (1). Our computational domain is \( x \in [0, 80] \) and \( t \in [0, 15] \). The corresponding parameter values are: \( L_0 = 40, \alpha = 0.9, \beta = 0.5 \) and \( \sigma \) follows by 1, \( \frac{10}{3} \), 5, 10.
Let $\alpha > 2\beta$. Obviously, $\lambda \in \mathbb{R}$ for all $\sigma > 0$, so the Hopf bifurcations from (1, 1) of equation (22) are impossible. Moreover, because $\lambda$ given by (27) is negative for sufficiently small $\sigma$, $(u, w) = (1, 1)$ is linear stable (which implies that the discussion in Section 4 is meaningful). However, $\lambda$ might pass through 0 as $\sigma$ is increased, which implies $(1, 1)$ will loss the stability. For a fixed wave number $k$ with $\alpha - 2\beta - k^2 > 0$, this will happen when

$$\sigma^2 = \frac{1 + \beta + k^2}{k^2(\alpha - 2\beta - k^2)},$$

(28)

$\sigma$, which is given in (28), has a minimum $\sigma_c$ at $k = k_c$, where

$$k_c^2 = -(1 + \beta) + \sqrt{(1 + \beta)^2 + (1 + \beta)(\alpha - 2\beta)} := -(1 + \beta) + B$$

and

$$\sigma_c = \left[ \frac{B}{(-1 + \beta + B)(1 + \alpha - \beta - B)} \right]^{\frac{1}{2}}.$$

Thus, the uniform steady state (1, 1) will lose the stability and a non-uniform steady state similar to $e^{ik_c x}$ will appear near (1, 1) as $\sigma$ is increased through $\sigma_c$ (a full steady state bifurcation analysis can be done similar to that in section 4 of Gourley et al. [24]). A direct computation gives $\sigma_c = 2.37261965$ for $\alpha = 2$ and $\beta = 0.4$, which coincide the numerical results of Figure 3.

If $\alpha \leq 2\beta$, then $\lambda < 0$ for any $\sigma > 0$, which implies the equilibrium (1, 1) is always stable for any $\sigma > 0$. In this case the non-uniform steady states around (1, 1) do not appear, and hence, the traveling wave solutions always connect equilibria $u = 0$ and $u = 1$ although they may be not monotonic (see Figure 4). In particular, for such a kernel function $\phi(x) := \frac{1}{\sigma^2} e^{-|x|/\sigma}$, the traveling wave solutions of (2) always connect equilibria $u = 0$ and $u = 1$

**Appendix.** In the following we give the proof of Lemma 4.3.

**Proof of Lemma 4.3.** (i). When $\xi \geq \xi_-$, we have

$$K[\psi_-](\xi) = -\beta \psi_- \left( \psi_- - \frac{\alpha}{2\beta} \right)^2 - (1 + \alpha - \beta) \psi_- \int_{-\infty}^{\infty} \psi_- (\xi - s) \phi_\sigma(s) ds < 0.$$ 

When $\xi < \xi_-$, since

\[
1 - \int_{-\infty}^{\xi_-} e^{\lambda_1 s} \phi_\sigma(s) ds = \int_{-\infty}^{\xi_-} (1 - e^{\lambda_1 s}) \phi_\sigma(s) ds - \int_{\xi_-}^{\infty} e^{\mu_1 s} \phi_\sigma(s) ds \\
= \int_{-\infty}^{\xi_-} e^{\lambda_1 s} \phi_\sigma(s) ds + \int_{\xi_-}^{\infty} (1 - e^{\lambda_1 s} - e^{\mu_1 s}) \phi_\sigma(s) ds \\
\leq \int_{\xi_-}^{\xi_+} e^{\lambda_1 s} \phi_\sigma(s) ds = e^{\lambda_1 \xi} \int_{-\infty}^{\xi} e^{-\lambda s} \phi_\sigma(s) ds,
\]
we have

\[ K[\psi_-](\xi) = (-\lambda_1^2 - c\lambda_1)e^{\lambda_1\xi} + (1 - e^{\lambda_1\xi}) \left\{ 1 + \alpha(1 - e^{\lambda_1\xi}) - \beta(1 - e^{\lambda_1\xi})^2 \right\} - (1 + \alpha - \beta) \int_{\mathbb{R}} \phi_\sigma(s) \psi_-(\xi - s) ds \]

\[ = (-\lambda_1^2 - c\lambda_1)e^{\lambda_1\xi} + (1 - e^{\lambda_1\xi}) \left\{ 1 + \alpha - \beta + (2\beta - \alpha)e^{\lambda_1\xi} - \beta e^{2\lambda_1\xi} \right\} - (1 + \alpha - \beta) \int_{\mathbb{R}} \phi_\sigma(s) \psi_-(\xi - s) ds \]

\[ \leq (-\lambda_1^2 - c\lambda_1)e^{\lambda_1\xi} + (1 + \alpha - \beta)(1 - e^{\lambda_1\xi})e^{\lambda_1\xi} \int_{\mathbb{R}} e^{-\lambda_1^x} \phi_\sigma(s) ds \]

\[ + (2\beta - \alpha)e^{\lambda_1\xi} - (3\beta - \alpha)e^{2\lambda_1\xi} + \beta e^{3\lambda_1\xi} \]

\[ = \left[ -\lambda_1^2 - c\lambda_1 + (2\beta - \alpha) + (1 + \alpha - \beta) \int_{\mathbb{R}} e^{-\lambda_1^x} \phi_\sigma(s) ds \right] e^{\lambda_1\xi} \]

\[ - (1 + \alpha - \beta)e^{2\lambda_1\xi} \int_{\mathbb{R}} e^{-\lambda_1^x} \phi_\sigma(s) ds - (3\beta - \alpha)e^{2\lambda_1\xi} + \beta e^{3\lambda_1\xi} \]

\[ = \left[ -(1 + \alpha - \beta) \int_{\mathbb{R}} e^{-\lambda_1^x} \phi_\sigma(s) ds - (3\beta - \alpha) \right] e^{2\lambda_1\xi} + \beta e^{3\lambda_1\xi} \]

\[ = \left[ -\lambda_1^2 + (\lambda_1^2 + c\lambda_1) - (\alpha - 2\beta) + (\alpha - 3\beta) \right] e^{2\lambda_1\xi} + \beta e^{3\lambda_1\xi} \]

\[ = \left[ -\lambda_1^2 + c\lambda_1 - \beta + \beta e^{\lambda_1\xi} \right] e^{2\lambda_1\xi} < 0. \]

(ii). When \( \xi \geq \xi_b \), we have

\[ K[\psi_+] = 1 + \alpha - \beta - (1 + \alpha - \beta)(\phi_\sigma * \psi_+) \geq 0. \]

When \( \xi < \xi_b \), note that

\[ 1 - \int_{\mathbb{R}} \psi_+(\xi - s) \phi_\sigma(s) ds \]

\[ = 1 - \int_{\mathbb{R}} \psi_+(s) \phi_\sigma(\xi - s) ds \]

\[ = 1 - \int_{-\infty}^{\xi_b} \phi_\sigma(\xi - s) \psi_+(b, \xi) ds - \int_{\xi_b}^{+\infty} \phi_\sigma(\xi - s) ds \]

\[ = 1 - \int_{\mathbb{R}} \phi_\sigma(\xi - s) \psi_+(b, \xi) ds + \int_{\xi_b}^{+\infty} \left( \psi_+(b, \xi) - 1 \right) \phi_\sigma(\xi - s) ds \]

\[ \geq 1 - \int_{\mathbb{R}} \phi_\sigma(\xi - s) \psi_+(b, \xi) ds \]

\[ = e^{\lambda_1\xi} \int_{\mathbb{R}} e^{-\lambda_1^x} \phi_\sigma(s) ds - be^{(\lambda_1 + \epsilon)\xi} \int_{\mathbb{R}} e^{-(\lambda_1 + \epsilon)^x} \phi_\sigma(s) ds. \]
It follows from (19) and (29) that
\[ K[\psi_+] = -\lambda_1^2 - c\lambda_1 e^{\lambda_1 \xi} + b \left[ (\lambda_1 + \varepsilon)^2 + c(\lambda_1 + \varepsilon) \right] e^{(\lambda_1 + \varepsilon)\xi} \]
\[ + \left( 1 - e^{\lambda_1 \xi} + be^{(\lambda_1 + \varepsilon)\xi} \right) \left\{ 1 + \alpha \left( 1 - e^{\lambda_1 \xi} + be^{(\lambda_1 + \varepsilon)\xi} \right) \right. \]
\[ - \beta \left( 1 - e^{\lambda_1 \xi} + be^{(\lambda_1 + \varepsilon)\xi} \right)^2 - (1 + \alpha - \beta) \int_{\mathbb{R}} \phi_\sigma(s)\psi_+(\xi - s) ds \} \]
\[ = -\lambda_1^2 - c\lambda_1 - (\alpha - 2\beta) e^{\lambda_1 \xi} \]
\[ + b \left[ (\lambda_1 + \varepsilon)^2 + c(\lambda_1 + \varepsilon) - (\alpha - 2\beta) \right] e^{(\lambda_1 + \varepsilon)\xi} \]
\[ + (1 + \alpha - \beta) \left( 1 - e^{\lambda_1 \xi} + be^{(\lambda_1 + \varepsilon)\xi} \right) \left( 1 - \int_{\mathbb{R}} \phi_\sigma(s)\psi_+(\xi - s) ds \right) \]
\[ + (\alpha - 3\beta) \left( -e^{\lambda_1 \xi} + be^{(\lambda_1 + \varepsilon)\xi} \right)^2 - \beta \left( -e^{\lambda_1 \xi} + be^{(\lambda_1 + \varepsilon)\xi} \right)^3 \]
\[ \geq -\lambda_1^2 - c\lambda_1 - (\alpha - 2\beta) e^{\lambda_1 \xi} \]
\[ + b \left[ (\lambda_1 + \varepsilon)^2 + c(\lambda_1 + \varepsilon) + (\alpha - 2\beta) \right] e^{(\lambda_1 + \varepsilon)\xi} \]
\[ + (1 + \alpha - \beta) \left( 1 - e^{\lambda_1 \xi} + be^{(\lambda_1 + \varepsilon)\xi} \right) \left( e^{\lambda_1 \xi} \int_{\mathbb{R}} e^{-\lambda_1 s} \phi_\sigma(s) ds \right) \]
\[ - be^{(\lambda_1 + \varepsilon)\xi} \int_{\mathbb{R}} e^{-(\lambda_1 + \varepsilon)s} \phi_\sigma(s) ds + (\alpha - 3\beta) \left( -e^{\lambda_1 \xi} + be^{(\lambda_1 + \varepsilon)\xi} \right)^2 \]
\[ - \beta \left( -e^{\lambda_1 \xi} + be^{(\lambda_1 + \varepsilon)\xi} \right)^3 \]
\[ = -e^{\lambda_1 \xi} \Psi(c, \sigma, \lambda_1) + be^{(\lambda_1 + \varepsilon)\xi} \Psi(c, \sigma, \lambda_1 + \varepsilon) \]
\[ + (\alpha - 3\beta) \left( -e^{\lambda_1 \xi} + be^{(\lambda_1 + \varepsilon)\xi} \right)^2 \]
\[ - \beta \left( -e^{\lambda_1 \xi} + be^{(\lambda_1 + \varepsilon)\xi} \right)^3 + (1 + \alpha - \beta) \left( -e^{\lambda_1 \xi} + be^{(\lambda_1 + \varepsilon)\xi} \right) \]
\[ \times \left( e^{\lambda_1 \xi} \int_{\mathbb{R}} e^{-\lambda_1 s} \phi_\sigma(s) ds - be^{(\lambda_1 + \varepsilon)\xi} \int_{\mathbb{R}} e^{-(\lambda_1 + \varepsilon)s} \phi_\sigma(s) ds \right) \]
\[ = e^{(\lambda_1 + \varepsilon)\xi} \left[ b\Psi(c, \sigma, \lambda_1 + \varepsilon) + (\alpha - 3\beta) e^{(\lambda_1 - \varepsilon)\xi} (-1 + be^{\varepsilon}) \right]^2 \]
\[ - \beta e^{(2\lambda_1 - \varepsilon)\xi} (-1 + be^{\varepsilon})^3 + (1 + \alpha - \beta) e^{(\lambda_1 - \varepsilon)\xi} (-1 + be^{\varepsilon}) \]
\[ \left( \int_{\mathbb{R}} e^{-\lambda_1 s} \phi_\sigma(s) ds - be^{\varepsilon} \int_{\mathbb{R}} e^{-(\lambda_1 + \varepsilon)s} \phi_\sigma(s) ds \right) \]
\[ > 0. \]

This completes the proof.
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