ON THE NUMBER OF RATIONAL ITERATED PRE-IMAGES OF THE ORIGIN UNDER QUADRATIC DYNAMICAL SYSTEMS

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Abstract. For a quadratic endomorphism of the affine line defined over the rationals, we consider the problem of bounding the number of rational points that eventually land at the origin after iteration. In the article “Uniform Bounds on Pre-Images Under Quadratic Dynamical Systems,” by the present authors and five others, it was shown that the number of rational iterated pre-images of the origin is bounded as one varies the morphism in a certain one-dimensional family. Away from a finite collection of morphisms in the family, we provide a sharp bound on the number of rational iterated pre-images. Then we give a conditional bound over the whole family using a number of modern tools for locating rational points on high genus curves. We also provide further insight into the geometry of the “pre-image curves.”

1. Introduction

Fix a rational number \( c \in \mathbb{Q} \) and define an endomorphism of the affine line by

\[
f_c : \mathbb{A}^1_\mathbb{Q} \to \mathbb{A}^1_\mathbb{Q}, \quad f_c(x) = x^2 + c.
\]

If we define \( f_c^N \) to be the \( N \)-fold composition of the morphism \( f_c \), and \( f_c^{-N} \) to be the \( N \)-fold pre-image, then for \( a \in \mathbb{A}^1(\mathbb{Q}) \), the set of rational iterated pre-images of \( a \) is given by

\[
\bigcup_{N \geq 1} f_c^{-N}(a)(\mathbb{Q}) = \{ x_0 \in \mathbb{A}^1(\mathbb{Q}) : f_c^N(x_0) = a \text{ for some } N \geq 1 \}.
\]

Heuristically, finding iterated pre-images amounts to solving progressively more complicated polynomial equations, and so rational solutions should be a rarity. The situation becomes more interesting as we vary \( c \), which has the effect of varying the morphism \( f_c \). A special case of the main theorem in \cite{3} shows that, independent of \( c \), there is a bound on the size of the set of rational iterated pre-images:

**Theorem 1.1** (\cite{3} Thm. 1.2 for \( B = D = 1 \)). Fix a point \( a \in \mathbb{A}^1(\mathbb{Q}) \) and define the quantity

\[
\kappa(a) = \sup_{c \in \mathbb{Q}} \# \left\{ \bigcup_{N \geq 1} f_c^{-N}(a)(\mathbb{Q}) \right\}.
\]

Then \( \kappa(a) \) is finite.

In the present paper we study Theorem 1.1 in the special case \( a = 0 \) and obtain several refinements. The proof of Theorem 1.1 shows that when \( a \in \mathbb{Q} \setminus \{-1/4\} \) and \( N \geq 4 \), there are only finitely many values \( c \in \mathbb{Q} \) for which \( f_c^{-N}(a)(\mathbb{Q}) \) is nonempty. So if we define

\[
\overline{\kappa}(a) = \limsup_{c \in \mathbb{Q}} \# \left\{ \bigcup_{N \geq 1} f_c^{-N}(a)(\mathbb{Q}) \right\},
\]

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then one obtains a trivial upper bound \( π(a) \leq 14 \) for \( a \neq -1/4 \) by counting the degrees of \( f_c, f_c^2 \), and \( f_c^3 \). We can substantially improve this for \( a = 0 \):

**Theorem 1.2.** One has \( π(0) = 6 \).

The content of this result amounts to the following two statements: (1) For any \( c \in \mathbb{Q} \), the map \( f_c \) has at most two rational third pre-images of the origin, and (2) For all but finitely many \( c \in \mathbb{Q} \), if \( f_c \) has four rational second pre-images of the origin, then it has no rational third pre-images. In order to refine Theorem 1.2 further, one must understand the exceptional values of \( c \) arising in (2). These statements are proved in \( \S 5 \).

Note that the two values \( c = 0 \) and \( c = -1 \) correspond to morphisms \( f_c \) for which 0 is periodic of period 1 and 2, respectively. For these values of \( c \), the origin has at least one rational \( N^{th} \) pre-image for arbitrary \( N \). We can give the following conditional result.

**Theorem 1.3.** Suppose that for \( c \in \mathbb{Q} \setminus \{0, -1\} \) the morphism \( f_c \) admits no rational \( 4^{th} \) pre-images of the origin. Then \( κ(0) \leq 8 \).

Evidently \( π(0) \leq κ(0) \), but computational evidence suggests that \( κ(0) = 6 \) as well. We have verified that the number of rational pre-images of the origin for \( f_c \) is as expected for \( c \)-values up to absolute logarithmic height around \( 10^6 \) (see PARI/gp Code A.1). Consequently, we believe the hypothesis of Theorem 1.3 is superfluous. The proof of Theorem 1.3 appears in \( \S 6.1 \).

Define an algebraic set \( Y^{pre} (N, a) \) in the \((x,c)\)-plane by the equation \( f_c^N(x) = a \). The algebraic points \((x_0, c_0) \in Y^{pre} (N, a) (\mathbb{Q})\) are in bijection with the \( N^{th} \) pre-images \( x_0 = f_{c_0}^{\frac{N}{k}}(a) \). For generic \( a \), \( Y^{pre} (4, a) \) is a nonsingular curve of genus 5; the finiteness of its set of rational points played a key role in the proof of Theorem 1.1. We apply a refinement of the method of Chabauty and Coleman — due to Stoll — in order to address the size of this set.

**Theorem 1.4.** Let \( J^{pre} (4,0) \) denote the Jacobian of the complete nonsingular curve birational to \( Y^{pre} (4,0) \). Assume the rank of \( J^{pre} (4,0) (\mathbb{Q}) \) is 3. Then there are at most 3 values \( c \in \mathbb{Q} \setminus \{0, -1\} \) such that \( f_c \) has a rational \( 4^{th} \) pre-image of the origin.

We present these conditional results for two reasons. First, by working with the curves \( Y^{pre} (N, a) \), our problem can be reduced to the classical Diophantine pastime of finding rational points on a curve of high genus. Our setting provides a nice example on which to illustrate the use of several modern tools for finding rational points (or at least bounding their number): the Mordell-Weil group of the Jacobian, the method of Chabauty-Coleman, and the Weil conjectures \([4, 7, 10, 11, 13]\). Second, in order to carry out these calculations we produce an explicit quasi-projective embedding of the curve \( Y^{pre} (N, a) \). It is our hope that the simple nature of this embedding will be of assistance in future studies of the arithmetic of pre-images.

In the next section, we recall the analogy between one-dimensional dynamical systems and elliptic curves; it often provides inspiration for dynamical research. In the elliptic curve setting, the result analogous to Theorem 1.1 is Mazur’s theorem on torsion. In \( \S 3 \) we briefly summarize the necessary background on pre-image curves developed in \( \S 3 \). The entirety of \( \S 4 \) is devoted to exhibiting properties and consequences of a useful projective embedding of the curve \( Y^{pre} (N, a) \). In \( \S 5 \) we prove Theorem 1.2 and in \( \S 6 \) we prove Theorems 1.3 and 1.4 and turn to the arithmetic and geometric study of the Jacobian \( J^{pre} (4,0) \). This section makes heavy use of the algebra and number theory systems Magma \([1]\) and PARI/gp \([12]\). The code for all of our calculations is provided in an appendix.

**2. The elliptic curve analogy**

Many phenomena in the arithmetic theory of dynamical systems have strong analogues in the theory of abelian varieties. For a fixed positive integer \( m \) and an elliptic curve \( E/\mathbb{Q} \), denote the
multiplication-by-\( m \) map by \([m] : E \rightarrow E\). Let \( \mathcal{O} \) be the origin for the group law on \( E(\mathbb{Q}) \). The key to the analogy is the following fact: the set of rational iterated pre-images of \( \mathcal{O} \) by the morphism \([m]\) is precisely the \( m \)-power torsion subgroup of the Mordell-Weil group

\[
E[m^\infty](\mathbb{Q}) = \bigcup_{N \geq 1} [m]^{-N}(\mathcal{O})(\mathbb{Q}).
\]

To continue the analogy, we replace our family of quadratic dynamical systems with the family of all elliptic curves and ask if there exists a uniform bound on \( E[m^\infty](\mathbb{Q}) \) as we vary the elliptic curve \( E \). The answer is given by Mazur’s uniformity theorem for torsion in the Mordell-Weil group \( E_{\text{tors}}(\mathbb{Q}) \subset E(\mathbb{Q}) \). The following weak form of Mazur’s theorem parallels Theorem 1.1.

**Theorem 2.1** ([3]). Let \( E/\mathbb{Q} \) be an elliptic curve. Then \( \#E_{\text{tors}}(\mathbb{Q}) \leq 16 \). In particular, if we define the quantity

\[
\kappa' = \sup_{E/\mathbb{Q}} \# \left\{ \bigcup_{N \geq 1} [m]^{-N}(\mathcal{O})(\mathbb{Q}) \right\},
\]

then \( \kappa' \) is finite.

We have chosen not to give an explicit value for \( \kappa' \) in the statement above in order to draw out the analogy with Theorem 1.1. In fact, \( \kappa' = 16 \) is an optimal choice since there exist elliptic curves \( E/\mathbb{Q} \) with torsion subgroup isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \).

3. The geometry of \( Y^{\text{pre}}(N, a) \)

Here we summarize the necessary geometric theory of pre-image curves developed in [3]. Most of the purely geometric results in [3] assume that the base field is algebraically closed. We will adapt the statement of Theorem 3.1 below to account for this disparity. It is an adjustment in viewpoint; no additional proof is necessary.

Let \( k \) be a field of characteristic different from 2. As in the introduction, we define a morphism \( f_c : \mathbb{A}^1_k \rightarrow \mathbb{A}^1_k \) for any \( c \in k \) by the formula

\[
f_c(x) = x^2 + c.
\]

We could view \( f_c \) as an endomorphism of \( \mathbb{P}^1_k \), but the point at infinity is totally invariant for this type of morphism, and hence dynamically uninteresting. Fix a basepoint \( a \in k \) and a positive integer \( N \). Define an algebraic set

\[
Y^{\text{pre}}(N, a) = V(f_c^N(x) - a) \subset \mathbb{A}^2_k = \text{Spec } k[x, c].
\]

If \( Y^{\text{pre}}(N, a) \) is geometrically irreducible, we define the \( N \)-th pre-image curve, denoted \( X^{\text{pre}}(N, a) \), to be the unique complete nonsingular curve birational to \( Y^{\text{pre}}(N, a) \). When we say a curve \( C/k \) is nonsingular, we mean that it is nonsingular after extending scalars to the algebraic closure. Recall also that the gonality of a curve \( C/k \) is the minimum degree of a nonconstant morphism \( C \rightarrow \mathbb{P}^1 \) (defined over \( k \)).

**Theorem 3.1** ([3] Cor. 2.4, Thm. 3.2, & Thm. 3.6)). Let \( a \in k \), and let \( N \geq 1 \) be an integer for which \( Y^{\text{pre}}(N, a) \) is nonsingular. Then \( Y^{\text{pre}}(N, a) \) is geometrically irreducible. Moreover, \( X^{\text{pre}}(N, a) \) is geometrically irreducible and has genus \( (N-3)2^{N-2} + 1 \). If \( N \geq 2 \), then \( X^{\text{pre}}(N, a) \) has gonality \( 2^{N-2} \).

We end this section with the result that ties the previous theorem into the special case \( a = 0 \).

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1It is a standard fact in algebraic geometry that such a curve exists and is unique when \( k \) is algebraically closed. Uniqueness follows in general from the fact that the curve is proper, but existence is a little trickier in the case of arbitrary \( k \). See Corollary 4.3 for a direct proof of existence in our case.
Proposition 3.2 ([3 Prop. 4.8]). Let \( k = \mathbb{Q} \). For every \( N \geq 1 \), the curve \( Y_{\text{pre}}(N,0) \) is nonsingular.

4. A Projective Embedding of \( X_{\text{pre}}(N,a) \)

The goal of this section is to describe a closed immersion of \( Y_{\text{pre}}(N,a) \) into affine \( N \)-space whose image has a projective closure that is a complete intersection of quadrics with no singularities on the hyperplane at infinity.\(^2\) It follows that if \( Y_{\text{pre}}(N,a) \) is nonsingular, then the projective closure is isomorphic to \( X_{\text{pre}}(N,a) \). We can use this embedding to gain an explicit description of the points at infinity — the points of \( X_{\text{pre}}(N,a) \setminus Y_{\text{pre}}(N,a) \).

The results in this section are geometric in nature (and not arithmetic), so we will work over an arbitrary field \( k \) of characteristic different from 2. For \( a \in k \), define a morphism \( \psi: Y_{\text{pre}}(N,a) \to \mathbb{A}^N \) by

\[
\psi(x,c) = (x, f_c(x), f_c^2(x), f_c^3(x), \ldots, f_c^{N-1}(x)).
\]

Lemma 4.1. The morphism \( \psi: Y_{\text{pre}}(N,a) \to \mathbb{A}^N \) is a closed immersion. If \( \mathbb{A}^N \) has coordinates \( z_0, \ldots, z_{N-1} \), then the ideal defining the image of \( \psi \) is \( I = (z_{N-1}^2 + z_i - z_{i-1}^2 - a : i = 1, 2, \ldots, N-1) \).

Proof. It suffices to prove that the induced homomorphism on rings of regular functions

\[
\psi^*: k[z_0, \ldots, z_{N-1}] \to k[x,c]/(f_c^N(x) - a)
\]

is surjective with kernel \( I \). Since \( \psi^*(z_0) = x \) and \( \psi^*(z_1 - z_0^2) = c \), we see that \( \psi^* \) is surjective. When \( N = 1 \) we have \( k[x,c]/(x^2 + c - a) \cong k[x] \), showing that \( \psi \) is an isomorphism and thus \( I \) is trivial. To exhibit the kernel of \( \psi^* \) for \( N \geq 2 \), we give an explicit elimination calculation. First, we choose equivalent generators of the ideal \( I \). For \( N \geq 2 \), we keep the first generator of \( I \) and replace the \( i \)th generator by its difference with the first:

\[
I = (z_{N-1}^2 + z_i - z_{i-1}^2 - a) + (z_i - z_{i-1}^2 - z_1 + z_0^2: i = 2, 3, \ldots, N-1).
\]

The key fact that allows us to simplify inductively is that \( f_c^{j-1}(x)^2 + c = f_c^j(x) \) for each \( j \geq 1 \). Now for each \( j = 1, 2, \ldots, N-1 \), define an ideal \( I_j \) of \( k[x,c,z_j, \ldots, z_{N-1}] \) by

\[
I_j = (z_{N-1}^2 + c - a, z_j - f_c^j(x)) + (z_i - z_{i-1}^2 - c: i = j + 1, \ldots, N-1)
\]

The map

\[
k[z_0, \ldots, z_{N-1}]/I \to k[x,c,z_1, \ldots, z_{N-1}]/I_1
\]

\[
z_0 \mapsto x \quad \text{and} \quad z_i \mapsto z_i, \quad i \geq 1
\]

is an isomorphism. Indeed, the inverse of the map is given by sending \( c \) to \( z_1 - z_0^2 \). Now we proceed by a chain of \( k \)-algebra isomorphisms given by eliminating one of the \( z_i \)'s at each step. We have

\[
k[z_0, \ldots, z_{N-1}]/I \cong k[x,c,z_1, \ldots, z_{N-1}]/I_1
\]

\[
\cong k[x,c,z_2, \ldots, z_{N-1}]/I_2
\]

\[
\vdots
\]

\[
\cong k[x,c,z_{N-1}]/I_{N-1}
\]

\[
\cong k[x,c]/(f_c^{N-1}(x)^2 + c - a)
\]

\[
\cong k[x,c]/(f_c^N(x) - a).
\]

\(^2\) This construction was inspired by a workshop talk given by Michael Stoll at the American Institute of Mathematics.
It follows from the definitions that the map that induces the isomorphism from the first algebra to the last algebra is precisely $\psi^*$.

The next proposition describes the projective closure of the image of $\psi$. If $\mathbb{P}^N$ has homogeneous coordinates $Z_0, \ldots, Z_N$, let us identify $\mathbb{A}^N$ with the subset of $\mathbb{P}^N$ where $Z_N \neq 0$.

**Proposition 4.2.**

(a) The projective closure of the image of $\psi$ is a complete intersection of quadrics with homogenous ideal

$$J = (Z_{N-1}^2 + Z_i Z_N - Z_{i-1}^2 - a Z_N^2 : i = 1, 2, 3, \ldots, N - 1).$$

(b) The points of $V(J)$ on the hyperplane $Z_N = 0$ have homogeneous coordinates

$$(\epsilon_0 : \cdots : \epsilon_{N-1} : 0), \quad \epsilon_i = \pm 1.$$  

In particular, there are $2^{N-1}$ of them. Moreover, they are all nonsingular points of $V(J)$.

(c) If $Y^{\text{pre}}(N,a)$ is nonsingular, then $X^{\text{pre}}(N,a) \cong V(J)$ and the complement of the affine part $X^{\text{pre}}(N,a) \setminus Y^{\text{pre}}(N,a)$ consists of $2^{N-1}$ points.

**Proof.** Let $J$ be the ideal defined in the statement of part (a). We will begin by working out various properties of the scheme $V(J)$, and then we will prove that it is actually the projective closure of the image of $\psi$. Let’s calculate the set-theoretic intersection of $V(J)$ with the hyperplane $Z_N = 0$. Killing $Z_N$ in all of the generators of $J$ gives the system

$$Z_0^2 = Z_1^2 = \cdots = Z_{N-1}^2.$$  

As one of the coordinates must be nonzero, we may assume that all of these squares are equal to 1, and consequently all of the coordinates must be $\pm 1$. Thus we obtain the set of $k$-valued points described in the statement of (b). After scaling the coordinates so that the first equals 1, we find the other $N - 1$ nonzero coordinates may be either of $\pm 1$. This proves there are $2^{N-1}$ points on $V(J)$ with $Z_N = 0$.

Next we claim that $V(J)$ has pure dimension 1. Indeed, it is an intersection of $N-1$ hypersurfaces, so each irreducible component of $V(J)$ must have dimension at least 1. Let $W$ be the projective closure of $\text{im}(\psi)$. If we homogenize the generators of the ideal $I$ in the previous lemma, we get exactly the ideal $J$. This shows $W \subset V(J)$ and the two schemes agree on their intersection with $\mathbb{A}^N = \{Z_N \neq 0\}$. Since $W$ has dimension 1, and since $V(J)$ has only finitely many points outside $\mathbb{A}^N$, we find $V(J)$ has dimension at most 1. We are now forced to conclude that $V(J)$ has pure dimension 1. Moreover, we may infer that $W = V(J)$, as otherwise $V(J)$ would have a component of dimension zero. We have completed the proof of (b).

To see that all of the points of $V(J)$ on the hyperplane $Z_N = 0$ are nonsingular, we apply the Jacobian criterion. Dehomogenizing with respect to $Z_0$ and labeling the affine coordinates $(z_1, \ldots, z_N)$ shows the algebraic set $V(J) \cap \{Z_0 \neq 0\}$ is cut out by the polynomials

$$g_1 = z_{N-1}^2 + z_1 z_N - 1 - a z_N^2,$$

$$g_2 = z_{N-1}^2 + z_2 z_N - z_1^2 - a z_N^2,$$

$$g_3 = z_{N-1}^2 + z_3 z_N - z_2^2 - a z_N^2,$$

$$\vdots$$

$$g_{N-1} = z_{N-1}^2 + z_{N-1} z_N - z_{N-2}^2 - a z_N^2.$$
Notice that when $z_N = 0$, none of the other $z_j$’s vanish. The Jacobian matrix evaluated at $z_N = 0$ is given by

$$
\begin{bmatrix}
\frac{\partial g_i}{\partial z_j} \bigg|_{z_N=0}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 & 2z_{N-1} & z_1 \\
-2z_1 & 0 & 0 & \ldots & 0 & 2z_{N-1} & z_2 \\
0 & -2z_2 & 0 & \ldots & 0 & 2z_{N-1} & z_3 \\
0 & 0 & -2z_3 & \ldots & 0 & 2z_{N-1} & z_4 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & -2z_{N-2} & 2z_{N-1} & z_{N-1}
\end{bmatrix}.
$$

If we expand by minors across the first row, we can see that the determinant of the left $(N-1) \times (N-1)$ matrix is

$$\pm 2^{N-1} z_1 z_2 \cdots z_{N-1} \neq 0.$$ 

We conclude that the Jacobian has rank $N - 1$. The Jacobian criterion implies that $V(J)$ is nonsingular at each of these points. (Notice that we just used the fact that $V(J)$ has pure dimension 1.) This completes the proof of (b).

Finally, we assume that $Y^{\text{pre}}(N, a)$ is nonsingular. As $\psi$ is a closed immersion, its image in $\mathbb{A}^N$ is also nonsingular. Said another way, the part of $V(J)$ outside the hyperplane $Z_N \neq 0$ is nonsingular. By part (b), $V(J)$ is also nonsingular at the points lying on this same hyperplane, and so we find $V(J)$ is a nonsingular complete curve birational to $X^{\text{pre}}(N, a)$. Up to isomorphism there is only one such curve, and so $X^{\text{pre}}(N, a) \cong V(J)$. This proves (c). \(\square\)

Corollary 4.3. Fix $a \in k$. If $Y^{\text{pre}}(N, a)$ is nonsingular, then the complete curve $X^{\text{pre}}(N, a)$ can be defined over $k$, and it is both nonsingular and geometrically irreducible.

Proof. First, note that $Y^{\text{pre}}(N, a)$ is defined over $k$ because its defining equation has coefficients in $\mathbb{Z}[a]$. By Proposition 4.2, we see that $X^{\text{pre}}(N, a)$ is defined over $k$; indeed, it is cut out by a collection of polynomials with coefficients in $k$. Now $X^{\text{pre}}(N, a)$ is nonsingular by part (b) of the same proposition, and it is geometrically irreducible by Theorem 3.1. \(\square\)

5. The Arithmetic of Rational 2nd and 3rd Pre-Images

In this section we prove Theorem 1.2 which states that

$$\pi(0) = \limsup_{c \in \mathbb{Q}} \# \left\{ \bigcup_{N \geq 1} f_c^{-N}(0)(\mathbb{Q}) \right\} = 6.$$ 

Note that $f_c^2$ has degree 4, so that $f_c$ admits at most four rational second pre-images of the origin. We can explicitly determine the $c$-values for which this happens.

Proposition 5.1. The rational $c$-values for which $f_c$ admits four distinct rational second pre-images of the origin are parameterized by

$$c = -\frac{(t^2 + 1)^4}{16t^2(t^2 - 1)^2}, \quad t \in \mathbb{Q} \setminus \{0, \pm 1\}.$$ 

Proof. Let $c \in \mathbb{Q}$ be such that $f_c$ has four rational second pre-images of 0. Then in particular, it must have two rational first pre-images, which is to say that the equation $x^2 + c = 0$ has two distinct rational solutions in $x$. Evidently this can only happen if $c = -d^2$ for some nonzero rational number $d$.

The second pre-images are then the solutions to

$$(x^2 + c)^2 + c = (x^2 - d^2)^2 - d^2 = 0 \quad \iff \quad x = \pm \sqrt{d^2 \pm d}.$$
In order that all four of these solutions be rational, there must be \( r, s \in \mathbb{Q} \) such that

\[
(1) \quad r^2 = d^2 - a, \quad s^2 = d^2 + a.
\]

Adding these equations and dividing by \( d^2 \) gives

\[
\left( \frac{r}{d} \right)^2 + \left( \frac{s}{d} \right)^2 = 2.
\]

The rational solutions to the equation \( a^2 + b^2 = 2 \) can be parameterized by passing a line of slope \( t \) through the point \((a, b) = (-1, 1)\) and determining the other point of intersection with this circle. That is, set \( b = t(a + 1) + 1 \) and solve:

\[
a = \frac{r}{d} = -\frac{t^2 - 2t + 1}{t^2 + 1}, \quad b = \frac{s}{d} = -\frac{t^2 + 2t + 1}{t^2 + 1}.
\]

Divide both equations in (1) by \( d^2 \), substitute the parameterizations of \( r/d \) and \( s/d \) given above, and solve for \( d \) to arrive at

\[
d = -\frac{(t^2 + 1)^2}{4t(t^2 - 1)} \implies c = -d^2 = -\frac{(t^2 + 1)^4}{16t^2(t^2 - 1)^2}.
\]

Evidently \( t \neq 0, \pm 1 \).

Conversely, a direct computation shows that for any \( t \in \mathbb{Q} \setminus \{0, \pm 1\} \), the map \( f_c \) with \( c = -\frac{(t^2 + 1)^4}{16t^2(t^2 - 1)^2} \) has four distinct rational pre-images of the origin, namely

\[
x = \pm \frac{(t^2 + 1)(t^2 \pm 2t - 1)}{4t(t^2 - 1)}.
\]

□

Lemma 5.2. The complete curve \( X^{\text{pre}}(3, 0) \) is \( \mathbb{Q} \)-isomorphic to the elliptic curve \( E \) with affine equation \( v^2 = u^3 - u + 1 \). The birational transformation \( E \dashrightarrow Y^{\text{pre}}(3, 0) \) is given by

\[
(2) \quad x = \frac{v}{u^2 - 1} \quad c = \frac{-1}{(u^2 - 1)^2}.
\]

Moreover, this elliptic curve has Mordell-Weil group \( E(\mathbb{Q}) \cong \mathbb{Z} \).

We originally exhibited this birational map with the help of Maple. Thanks to one of the anonymous referees, we are able to provide the following self-contained derivation.

Proof. We begin by working with the nonsingular projective model \( X^{\text{pre}}(3, 0) \) for \( Y^{\text{pre}}(3, 0) \). Proposition 4.2(a) shows \( X^{\text{pre}}(3, 0) \) can be embedded in \( \mathbb{P}^3 \) as the intersection of two quadric hypersurfaces:

\[
Z_0^2 = Z_2^2 + Z_1Z_3 \quad Z_1^2 = Z_2^2 + Z_2Z_3.
\]

Setting \( Z_2 = 0 \) shows \((0 : 0 : 0 : 1)\) is the unique point on \( X^{\text{pre}}(3, 0) \) lying on the hyperplane \( Z_2 = 0 \). We now pass to affine coordinates that place this point at infinity by setting \( v = Z_0/Z_2 \), \( u = Z_1/Z_2 \), and \( w = Z_3/Z_2 \). The above equations then become

\[
v^2 = 1 + uw \quad u^2 = 1 + w.
\]

Eliminating the variable \( w \) yields \( v^2 = u^3 - u + 1 \), which proves that \( X^{\text{pre}}(3, 0) \) is isomorphic to the desired elliptic curve.

Tracing through the definitions of the maps \( Y^{\text{pre}}(3, 0) \dashrightarrow \mathbb{A}^3 \dashrightarrow \mathbb{P}^3 \) shows that the birational transformation \( \alpha : Y^{\text{pre}}(3, 0) \dashrightarrow E \) that we have just constructed is given by

\[
(4) \quad u = \frac{f_c(x)}{f_c^2(x)} \quad v = \frac{x}{f_c^2(x)}.
\]
If we define $\beta : E \to Y^\text{pre} (3,0)$ by the formulas in [3], then a straight-forward (albeit messy) computation shows $\alpha \circ \beta = \text{id}$ and $\beta \circ \alpha = \text{id}$.

Cremona’s elliptic curve database [2] tells us the curve $E$ has rank 1 and no torsion. □

The following corollary is an immediate consequence of Lemma 5.2

**Corollary 5.3.** The c-values for which $f_c$ admits a third rational pre-image of the origin are given by $c = 0$ and $c = -(u^2 - 1)^{-2}$, where $(u, v)$ is an affine rational point on the elliptic curve $v^2 = u^3 - u + 1$ and $u \neq \pm 1$.

**Proposition 5.4.** Given $c_0 \in \mathbb{Q}$, there are at most two rational third pre-images of the origin for the morphism $x \mapsto f_{c_0}(x)$.

**Proof.** For the proof, it will be useful to distinguish between the curve $X^\text{pre} (3,0)$ and the elliptic curve $E$ with affine Weierstrass model $v^2 = u^3 - u + 1$, even though the preceding lemma shows that they are isomorphic over $\mathbb{Q}$. For a point $P \in E(\mathbb{Q})$, we write $u(P)$ for its $u$-coordinate. (Here $u(P) = \infty$ if $P$ is the origin for the group law.)

Let $h$ be the naive logarithmic height on $\mathbb{P}^1$. The function $g(P) = -(u(P)^2 - 1)^{-2}$ is an even rational function on $E$, and so we may define a height function by $h_g(P) = h(g(P))$. The strategy of the proof is to show that if $(x_1, c_0)$ and $(x_2, c_0)$ are two points on $X^\text{pre} (3,0) (\mathbb{Q})$ corresponding to rational third pre-images of the origin, then they in turn correspond to two points $P_1$ and $P_2$ on the elliptic curve $E$. The function $g$ cannot distinguish between points with the same $c$-coordinate, and so $h_g(P_1) = h_g(P_2)$. On the other hand, since $E$ has rank 1, we will be able to show that if a point $P$ has sufficiently large height, then $-P$ is the only other point with the same height. This will reduce the problem to a finite amount of computation. We now make this strategy more explicit, although we omit many of the computational details.

Let $\hat{h}$ be the canonical height on $E$. Tracing through the arguments in [9], Proposition 2.13 and [8] VIII.4, 6, 9[3], we can bound the difference between the canonical height and the modified height $h_g$. As $g$ has degree 8, this difference is given by

$$|8\hat{h}(P) - h_g(P)| \leq \frac{1}{3} \log C \quad (P \in E(\mathbb{Q})),$$

where $C \approx 7.35 \times 10^{85}$ is an explicit constant we computed in PARI/gp. (See PARI/gp Code A.2.)

As $E(\mathbb{Q})$ has rank 1, we may choose a generator $P_0$. For any $n \geq 1$, the above estimate and properties of the canonical height show

$$h_g([n + 1]P_0) - h_g([n]P_0) > 8\hat{h} ([n + 1]P_0) - 8\hat{h} ([n]P_0) - \frac{2}{3} \log C$$

$$= 8(n + 1)^2 \hat{h}(P_0) - 8n^2 \hat{h}(P_0) - \frac{2}{3} \log C$$

$$= 8(2n + 1) \hat{h}(P_0) - \frac{2}{3} \log C.$$ (5)

The point $P_0 = (1,1)$ is a generator of $E(\mathbb{Q})$, and its canonical height is $\hat{h}(P_0) \approx 0.0249$ according to PARI/gp[4]. It follows that the final quantity in (5) is positive as soon as

$$n \geq \frac{1}{2} \left( \frac{\log C}{12\hat{h}(P_0) - 1} \right) \approx 330.3.$$

[3] There is a small error in the displayed estimate of the proof of Proposition VIII.9.1. The final inequality should read “$\leq C/(3 \cdot 4^{11})$. It has been corrected in the latest edition that appeared in 2009.

[4] Warning: PARI/gp computes the canonical height with respect to the divisor $2(\infty)$ on $E$. The canonical height of $P_0$ is given here with respect to the divisor $(\infty)$. 

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To conclude the proof, suppose that \( c_0 \in \mathbb{Q} \) and \( x_1, x_2 \) are two rational third pre-images of the origin for the map \( x \mapsto x^2 + c_0 \). We aim to show that \( x_1 = \pm x_2 \). To that end, we may assume that \( c_0 \neq 0 \) since there is exactly one third pre-image of the origin in this case. The pre-images \( x_1 \) and \( x_2 \) correspond to two points \( P_1 \) and \( P_2 \) in \( E(\mathbb{Q}) \) such that \( g(P_1) = g(P_2) = c_0 \). The fact that \( c_0 \neq 0 \) implies that neither \( P_i \) is the origin for the group law on \( E \). Recalling that \( P_0 \) is our fixed generator for \( E(\mathbb{Q}) \), there exist nonzero integers \( n_1, n_2 \) such that \( P_i = [n_i]P_0 \). Moreover, replacing \( P_i \) by \( -P_i \) has the effect of replacing \( x_i \) by \(-x_i\), as can readily be seen from \( (5) \). Hence we may assume that \( n_i > 0 \). After reordering if necessary, we may further suppose that \( n_1 \geq n_2 \). If they are equal, then we are finished, so assume \( n_1 > n_2 \).

If \( n_2 \geq 331 \), then the computation in \( (5) \) implies \( h_g(P_1) > h_g(P_2) \). But \( g(P_1) = g(P_2) \), so this is a contradiction. Hence \( 1 \leq n_2 \leq 330 \). For this range of \( n \), we now compute \( g([n]P_0) \) — the \( c \)-coordinate for the point corresponding to \([n]P_0\) — and verify that in all cases there are exactly two rational third pre-images for the map \( x \mapsto x^2 + c \). This computation was performed with \textit{PARI/gp}. (See \textit{PARI/gp} Code \( A.2 \)) \( \square \)

**Proposition 5.5.** There are at most finitely many \( c \in \mathbb{Q} \) such that the map \( f_c \) admits four distinct rational second pre-images and a rational third pre-image of the origin.

**Remark 5.6.** We expect that the set of \( c \in \mathbb{Q} \) described by the proposition is empty. If true, then the conditional bound in Theorem \( 1.3 \) can be reduced to 6. In the proof below we will construct two hyperelliptic curves \( C_\pm \) of genus 3. Understanding the rational points on these curves is equivalent to determining the exceptional values \( c \) appearing in the proposition. We have verified that these two curves have the expected number of rational points up to logarithmic height around \( 10^6 \). With the help of Adam Logan, we performed a 2-descent on these curves to determine that their Jacobians each have Mordell-Weil group isomorphic to \( \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \). One can apply the method of Chabauty and Coleman to bound the number of rational points on \( C_\pm \), but we were unable to obtain a sharp result.

**Proof of Proposition 5.5.** Let \( c_0 \in \mathbb{Q} \) be such that the map \( f_{c_0} \) admits four distinct rational second pre-images and a rational third pre-image of the origin. Proposition \( 5.1 \) shows that there exists \( t_0 \in \mathbb{Q} \setminus \{0, \pm 1\} \) such that

\[
c_0 = \frac{(t_0^2 + 1)^4}{16 t_0^2 (t_0^2 - 1)^2}.
\]

Let \( x_0 \) be a rational third pre-image of the origin for \( f_{c_0} \). Then \( f_{c_0}(x_0) \) is a rational second pre-image of the origin, and so it is given by \( (2) \). Also, \( f_{c_0}^2(x_0) = \pm \sqrt{c_0} \) is a rational first pre-image of the origin. Since \( f_{c_0} \) admits a rational third pre-image of the origin, Lemma \( 5.2 \) shows there is a rational pair \((u, v)\) satisfying \( v^2 = u^3 - u + 1 \), and equation \( (4) \) gives a formula for \( u \):

\[
u = \frac{f_{c_0}(x_0)}{f_{c_0}^2(x_0)} = \pm \frac{t_0^2 \pm 2 t - 1}{t_0^2 + 1}.
\]

Thus the equation \( v^2 = u^3 - u + 1 \) becomes

\[
v^2 = \pm \left( \frac{t_0^2 \pm 2 t - 1}{t_0^2 + 1} \right)^3 \mp \frac{t_0^2 \pm 2 t - 1}{t_0^2 + 1} + 1
\]

\[
= \frac{(t_0^2 + 1)(t_0^6 + 4 t_0^5 + (3 \pm 8)t_0^4 - 8t_0^3 + (3 \mp 8)t_0^2 + 4t + 1)}{(t_0^2 + 1)^4}.
\]

Define a pair of affine algebraic curves \( C_\pm \) over \( \mathbb{Q} \) by

\[
C_\pm : y^2 = (t_0^2 + 1)(t_0^6 + 4 t_0^5 + (3 \pm 8)t_0^4 - 8t_0^3 + (3 \mp 8)t_0^2 + 4t + 1).
\]

Then a rational point \((t, y)\) on either of \( C_\pm \) gives rise to parameter \( c_0 \) such that \( f_{c_0} \) has four rational second pre-images of the origin and a rational third pre-image of the origin. However, the curves \( C_\pm \)
have genus 3, and so by Faltings’ theorem we may conclude that they have finitely many rational points. Consequently, there are only finitely many parameters \( t \), and so finitely many parameters \( c_0 \), such that \( f_{c_0} \) has the desired number of rational pre-images.

**Proof of Theorem 1.2.** Proposition 5.1 exhibits an infinite family of rational parameters \( c \) such that there are four distinct rational second pre-images of the origin. Applying \( f_c \) to these second pre-images gives two distinct rational first pre-images. So for each of these parameters \( c \), we have

\[
\overline{\pi}(0) \geq \# f_c^{-1}(0)(\mathbb{Q}) + \# f_c^{-2}(0)(\mathbb{Q}) = 6.
\]

Let \( S_1 \) be the set of \( c \in \mathbb{Q} \) such that \( f_c \) admits a rational 4th pre-image of the origin. The affine curve \( Y^{\text{pre}}(4,0) \) has genus 5 (Theorem 3.1), and so by Faltings’ theorem the set \( S_1 \) is finite.

Write \( S_2 \) for the set of \( c \in \mathbb{Q} \) such that \( f_c \) has four distinct rational second pre-images of the origin and a rational third pre-image. By Proposition 5.3, this set is also finite.

Now suppose that \( c \) is a rational parameter in the complement of the finite set \( S_1 \cup S_2 \). Then \( f_c^{-3}(0)(\mathbb{Q}) \) is empty. If \( f_c^{-3}(0)(\mathbb{Q}) \) is empty, then looking at the degrees of \( f_c \) and \( f_c^2 \) shows that \( f_c \) admits at most six rational iterated pre-images of the origin. On the other hand, if \( f_c^{-3}(0)(\mathbb{Q}) \) is nonempty, then Proposition 5.1 implies that it contains at most two elements. Since we have assumed \( c \notin S_2 \), it follows that \( f_c \) has at most two distinct rational second pre-images of the origin. There are never more than two first pre-images, so \( f_c \) has at most six rational iterated pre-images again. Hence \( \overline{\pi}(0) \leq 6 \).

**Remark 5.7.** Using techniques similar to the ones given here, the second author and several of his undergraduate students have performed a detailed analysis of \( \overline{\pi}(a) \) for \( a \in \mathbb{Q} \setminus \{-1/4\} \), and it appears that \( \overline{\pi}(a) = 6 \) in all cases. The main difficulty lies in the fact that \( X^{\text{pre}}(3,a) \) has generic rank 2, and so the method of Proposition 5.4 does not carry over. On the other hand, it suffices to show that the possible arrangements of more than six pre-images correspond to a finite collection of algebraic curves of genus at least two.

### 6. The Arithmetic of 4th Pre-images

6.1. **Conditional bounds for \( \kappa(0) \).** Recall from the introduction that we defined \( \kappa(0) \) to be the maximum number of rational iterated pre-images for a morphism \( f_c(x) = x^2 + c \):

\[
\kappa(0) = \sup_{c \in \mathbb{Q}} \# \left\{ \bigcup_{N \geq 1} f_c^{-N}(0)(\mathbb{Q}) \right\}.
\]

Controlling \( \kappa(0) \) hinges on the finiteness of the rational points of the curve \( Y^{\text{pre}}(4,0) \) or, equivalently, the pre-image curve \( X^{\text{pre}}(4,0) \). A height theoretic argument was necessary to connect this result to the higher pre-images \( f_c^{-N}(0)(\mathbb{Q}) \) for \( N > 4 \). It turns out that a strong bound for \( \# X^{\text{pre}}(4,0)(\mathbb{Q}) \) allows us to bound \( \kappa(0) \) without recourse to height machinery.

**Theorem 6.1.** If \( \# X^{\text{pre}}(4,0)(\mathbb{Q}) = 10 \), then \( \kappa(0) \leq 8 \).

Before proving Theorem 6.1, we need to collect a few facts about the complete curve \( X^{\text{pre}}(4,0) \). By Theorem 3.1 and Proposition 3.2, we see that \( X^{\text{pre}}(4,0) \) is a nonsingular, geometrically irreducible curve of genus 5. Using Proposition 4.2, we can embed the fourth pre-image curve in projective space as

\[
X^{\text{pre}}(4,0) \cong V(Z_3^2 + Z_1Z_4 - Z_0^2, Z_3^2 + Z_2Z_4 - Z_1^2, Z_3^2 + Z_3Z_4 - Z_2^2) \subset \mathbb{P}^4.
\]
It has ten rational points that one locates easily by inspection. There are the eight points at infinity given by Proposition 4.4.5:

\[ P_1 = (1 : 1 : 1 : 0) \quad P_2 = (1 : 1 : 1 : -1 : 0) \]
\[ P_3 = (1 : 1 : -1 : 1 : 0) \quad P_4 = (1 : 1 : -1 : -1 : 0) \]
\[ P_5 = (1 : -1 : 1 : 1 : 0) \quad P_6 = (1 : -1 : 1 : -1 : 0) \]
\[ P_7 = (1 : -1 : -1 : 1 : 0) \quad P_8 = (1 : -1 : -1 : -1 : 0) . \]

The origin in \( A^1(\mathbb{Q}) \) is periodic for the morphisms \( f_0 \) and \( f_{-1} \), with periods 1 and 2, respectively. Using the embedding in Lemma 4.1 these correspond to points on \( X_{\text{pre}}(4,0) \):

\[ P_9 = (0 : 0 : 0 : 0 : 1) \quad P_{10} = (0 : -1 : 0 : -1 : 1) . \]

Only \( P_9 \) and \( P_{10} \) correspond to parameters \( c \in \mathbb{Q} \), and hence Theorem 6.1 is a reformulation of Theorem 1.3 from the introduction.

**Proof of Theorem 6.1.** We begin by assuming that \( \# X_{\text{pre}}(4,0)(\mathbb{Q}) = 10 \). Our comments above show that \( X_{\text{pre}}(4,0)(\mathbb{Q}) \) has only two rational points corresponding to rational 4th pre-images of the origin, namely \( P_9 \) and \( P_{10} \). They are periodic points for the morphisms \( f_0 \) and \( f_{-1} \). A direct calculation shows

\[
\# \left\{ \bigcup_{N \geq 1} f_0^{-N}(0)(\mathbb{Q}) \right\} = 1 \quad \text{and} \quad \# \left\{ \bigcup_{N \geq 1} f_{-1}^{-N}(0)(\mathbb{Q}) \right\} = 3.
\]

For the remainder of the proof we assume that \( c_0 \in \mathbb{Q} \setminus \{0, -1\} \). Observe that if \( f_{c_0}^{-N}(0)(\mathbb{Q}) \) is nonempty for some \( N = M \), then it is nonempty for every \( N < M \):

\[ x_0 \in f_{c_0}^{-M}(0) \Rightarrow 0 = f_{c_0}^M(x_0) = f_{c_0}^N(f_{c_0}^{M-N}(x_0)) \Rightarrow f_{c_0}^{M-N}(x_0) \in f_{c_0}^{-N}(0) . \]

Our hypothesis on \( c_0 \) shows immediately that \( f_{c_0}^{-N}(0)(\mathbb{Q}) = \emptyset \) for \( N \geq 4 \).

Suppose that \( f_{c_0}^{-3}(0)(\mathbb{Q}) \) is empty. Then we obtain the result immediately from

\[
\# \left\{ \bigcup_{N \geq 1} f_{c_0}^{-N}(0)(\mathbb{Q}) \right\} \leq \# f_{c_0}^{-1}(0)(\mathbb{Q}) + \# f_{c_0}^{-2}(0)(\mathbb{Q}) \leq 2 + 4 = 6 .
\]

Finally, assume that \( f_{c_0}^{-3}(0)(\mathbb{Q}) \neq \emptyset \). By Proposition 5.4 there are at most two rational pairs \( (x_0, c_0) \) such that \( f_{c_0}^3(x_0) = 0 \). Therefore,

\[
\# \left\{ \bigcup_{N \geq 1} f_{c_0}^{-N}(0)(\mathbb{Q}) \right\} \leq \# f_{c_0}^{-1}(0)(\mathbb{Q}) + \# f_{c_0}^{-2}(0)(\mathbb{Q}) + \# f_{c_0}^{-3}(0)(\mathbb{Q}) \leq 2 + 4 + 2 = 8 .
\]

The proof is now complete. \( \square \)

6.2. **Conditional Bounds for Rational 4th Pre-images.** We now turn to the task of bounding the number of rational 4th pre-images of the origin — or what amounts to the same thing — bounding the size of \( X_{\text{pre}}(4,0)(\mathbb{Q}) \). Much of this section is based on [11]. For many of our calculations we used the computer algebra systems *Magma* [1] and *PARI/gp* [12]. For the reader who wishes to explore these matters more carefully, we have documented our code in the appendix.

The curve \( X_{\text{pre}}(4,0) \) can be embedded in \( \mathbb{P}^3 \) as in [6], and the image is defined over \( \mathbb{Z} \). *Magma* has determined that \( X_{\text{pre}}(4,0) \) has good reduction outside the primes 2, 23, and 2551. (See *Magma* Code 4.3) We will use several primes of good reduction in the arguments below.
The following result strengthens Theorem 1.4. We thank one of the anonymous referees for suggesting a substantial improvement to our argument.

**Theorem 6.2.** If the rank of $J^{\text{pre}}(4, 0)(\mathbb{Q})$ is 3, then there are at most eight rational pairs $(x_0, c_0)$ such that $x_0$ is a 4th pre-image of 0 by the map $x \mapsto f_{c_0}(x)$. Among these eight pairs there are only five distinct values $c_0 \in \mathbb{Q}$.

**Proof.** We apply a version of the method of Chabauty-Coleman demonstrated by Stoll [10, Cor. 6.7]. Over $\mathbb{Q}$, it says the following. Let $C/\mathbb{Q}$ be a smooth curve of genus $g \geq 2$, let $r$ be the rank of the Mordell-Weil group of the Jacobian of $C$, and let $p$ be a prime of good reduction for $C$. If $r < g$, then

$$\#C(\mathbb{Q}) \leq \#C(\mathbb{F}_p) + f_{C/\mathbb{F}_p}(r) + \Delta_p(\#C(\mathbb{F}_p), f_{C/\mathbb{F}_p}(r))$$

where

$$f_{C/\mathbb{F}_p}(r) = \max\{\deg(D) : D \geq 0, h^0(K - D) \geq g - r\},$$

$K$ is a canonical divisor on $C$, and $\Delta_p(u, v)$ is a combinatorial quantity that we will not define here. We will only use the fact that $\Delta_p(u, v) \leq \lfloor v/(p - 2) \rfloor$ if $p > 2$ and $u, v$ are natural numbers, which is the content of [10, Lem. 6.2].

In our setting, the genus of $C = X^{\text{pre}}(4, 0)$ is 5, and by hypothesis the rank of $J^{\text{pre}}(4, 0)(\mathbb{Q})$ is 3. Hence $g - r = 2$. We claim that $f_{C/\mathbb{F}_p}(3) = 4$. Suppose $D$ is an effective divisor on $X^{\text{pre}}(4, 0)$ for which $h^0(K - D) \geq 2$. Then (perhaps after projection) the linear system $|K - D|$ gives rise to a morphism $X^{\text{pre}}(4, 0) \to \mathbb{P}^1$ of degree $\deg(K - D)$. The curve $X^{\text{pre}}(4, 0)/\mathbb{F}_p$ has gonality 4 whenever $p$ is a prime of good reduction by Theorem 3.3, so that $\deg(K - D) \geq 4$. Equivalently, $\deg(D) \leq 4$. On the other hand, since $X^{\text{pre}}(4, 0)$ has gonality 4, there exists a divisor $E$ of degree 4 with $h^0(E) \geq 2$. Set $D' = K - E$. Then $\deg(D') = 4$, so that $D'$ is linearly equivalent to an effective divisor $D$ with the property that $h^0(K - D) = h^0(K - D') = h^0(E) \geq 2$. Therefore, $f_{C/\mathbb{F}_p}(3) = 4$.

Combining the conclusions of the last two paragraphs, we see that if $p > 2$ is a prime of good reduction for $X^{\text{pre}}(4, 0)$, then

$$X^{\text{pre}}(4, 0)(\mathbb{Q}) \leq \#X^{\text{pre}}(4, 0)(\mathbb{F}_p) + 4 + \left\lfloor \frac{4}{p - 2} \right\rfloor.$$

For $p = 5$, a simple computer search shows that $\#X^{\text{pre}}(4, 0)(\mathbb{F}_5) = 12$. Thus

$$\#X^{\text{pre}}(4, 0)(\mathbb{Q}) \leq 17.$$

The eight points at infinity do not correspond to pairs $(x_0, c_0)$ on $Y^{\text{pre}}(4, 0)$. That is, there are at most nine rational pairs $(x_0, c_0)$ such that $x_0$ is a 4th pre-image of 0 by the map $x \mapsto f_{c_0}(x)$. Two of these pairs are $(0, 0)$ and $(0, -1)$. There are no other rational pairs $(x, c)$ with $x = 0$ since $f^4_c(0) = 0$ has precisely two rational solutions. The involution $(x, c) \mapsto (-x, c)$ acts without fixed points on the remaining pairs, and so they must be even in number. This immediately reduces us to six remaining pairs sharing only three distinct values $c_0 \in \mathbb{Q}$, which completes the proof. \qed

How plausible is our hypothesis on the rank of $J^{\text{pre}}(4, 0)(\mathbb{Q})$ in the previous theorem? We now provide a number of facts about this Jacobian in order to lend some support for it.

**Theorem 6.3.**

(a) The Jacobian $J^{\text{pre}}(4, 0)$ splits as a product of a simple abelian variety of dimension 4 and an elliptic curve with Weierstrass equation $y^2 = x^3 - u + 1$.

(b) The subgroup of $J^{\text{pre}}(4, 0)(\mathbb{Q})$ generated by the 10 known rational points on $X^{\text{pre}}(4, 0)$ is isomorphic to $\mathbb{Z}^3$. In fact, it is already generated by divisors supported on the points at infinity.
Proof. We can define a morphism \( \delta : X^\text{pre} (4,0) \to X^\text{pre} (3,0) \) as follows. For points of the affine piece \( Y^\text{pre} (4,0) \), the morphism is given by \((x,c) \mapsto (x^2 + c,c)\). In other words, it sends a 4\text{th} pre-image of the origin to a 3\text{rd} pre-image. As \( X^\text{pre} (4,0) \) and \( X^\text{pre} (3,0) \) are complete and nonsingular, the morphism extends over \( X^\text{pre} (4,0) \).

We saw in the proof of Theorem \[6.1\] that \( X^\text{pre} (3,0) \) is an elliptic curve isomorphic to one with Weierstrass equation \( v^2 = u^3 - u + 1 \). In particular, \( X^\text{pre} (3,0) \) is isomorphic to its Jacobian \( J^\text{pre} (3,0) \). Passing to the morphism on Jacobians induced by \( \delta \)

\[
J^\text{pre} (4,0) \to J^\text{pre} (3,0) \cong X^\text{pre} (3,0),
\]

we see \( J^\text{pre} (4,0) \) splits as a product of \( J^\text{pre} (3,0) \) and another abelian variety of dimension 4. We must check that the larger factor is simple.

The Weil conjectures allow us to compute the Euler factor of \( J^\text{pre} (4,0) \) at \( p = 3 \) by computing the cardinality \( \#X^\text{pre} (4,0) (\mathbb{F}_3^m) \) for \( m = 1, \ldots, 5 \), which we do using \textit{PARI/gp}. (See \textit{PARI/gp} Code \[A.4\]) We find the Euler factor to be

\[
(x^2 + 3x + 3)(x^8 + 3x^7 + 7x^6 + 16x^5 + 28x^4 + 48x^3 + 63x^2 + 81x + 81).
\]

Since the two factors are irreducible and since the splitting of the Jacobian is witnessed by a splitting of the pre-image \( X^\text{pre} (4,0) \), we see \( J^\text{pre} (4,0) \) splits into a product of at most two simple Abelian varieties. The first polynomial is exactly the Euler factor for the elliptic curve \( v^2 = u^3 - u + 1 \), completing the proof of \[21\]. (See \textit{PARI/gp} Code \[A.5\])

The prime-to-\( p \) torsion in \( J^\text{pre} (4,0) (\mathbb{Q}) \) injects into \( J^\text{pre} (4,0) (\mathbb{F}_p) \) for any prime \( p \) of good reduction \[5 \text{ Thm. C.4.} \] It suffices for our purposes to compute \( J^\text{pre} (4,0) (\mathbb{F}_p) \) for three odd primes (see \textit{Magma} Code \[A.6\]):

\[
\begin{align*}
\#J^\text{pre} (4,0) (\mathbb{F}_5) & = 2^3 \cdot 877 \\
\#J^\text{pre} (4,0) (\mathbb{F}_7) & = 2^4 \cdot 3 \cdot 7 \cdot 233 \\
\#J^\text{pre} (4,0) (\mathbb{F}_{11}) & = 2^2 \cdot 5 \cdot 7 \cdot 13759.
\end{align*}
\]

It follows that the torsion subgroup has order 1, 2, or 4.

The main tool for proving the remaining portion of the assertions is the homomorphism

\[
\Phi_S : \bigoplus_{i=1}^{10} \mathbb{Z} P_i \to \text{Pic}_{X^\text{pre}(4,0)}(\mathbb{Q}) \to \prod_{p \in S} \text{Pic}_{X^\text{pre}(4,0)/\mathbb{F}_p}(\mathbb{F}_p)
\]

where \( S \) is a set of primes of good reduction. Here \( \Phi_S \) maps a divisor to its class in \( \text{Pic}_{X^\text{pre}(4,0)}(\mathbb{Q}) \), and then sends it to the product of its reductions for primes in \( S \). Let \( G \) be the subgroup of \( J^\text{pre} (4,0) (\mathbb{Q}) \) generated by the 10 points \( P_i \). Then \( G \) is given by the image of the degree-zero part of \( \bigoplus_{i=1}^{10} \mathbb{Z} P_i \) inside \( \text{Pic}_{X^\text{pre}(4,0)}(\mathbb{Q}) \). \( A \text{ priori} \) we see that the rank of \( G \) is at most 9.

Take \( S = \{3,5,7,11,13,17\} \) and consider the kernel of \( \Phi_S \). Some elements of the kernel correspond to linear equivalence relations among the \( P_i \), and others are artifacts of the reductions modulo primes of \( S \). Using \textit{Magma} we can exhibit rational functions on \( X^\text{pre} (4,0) \) that show the following (independent) linear equivalence relations (see \textit{Magma} Code \[A.7\]):

\[
\begin{align*}
P_3 + P_4 + P_7 + P_8 & \sim P_1 + P_6 + P_9 + P_{10} \\
P_2 + P_3 + P_6 + P_7 & \sim P_1 + P_4 + P_5 + P_8 \\
P_3 + P_8 + P_9 + P_{10} & \sim P_1 + P_2 + P_5 + P_6 \\
P_5 + P_6 + P_7 + P_8 & \sim 2P_1 + P_2 + P_9 \\
P_4 + P_5 + 2P_9 & \sim P_1 + P_2 + P_4 + P_8 \\
P_1 + P_2 + P_7 + P_8 & \sim P_3 + P_6 + 2P_{10}.
\end{align*}
\]
These six relations show that $G$ has rank at most three.

From the definitions, we can see that the map $\Phi_S$ induces a homomorphism

$$G \rightarrow \prod_{p \in S} \text{Pic}^0_{X_{p^\infty}(4,0)/\mathbb{F}_p}(\mathbb{F}_p)$$

whose image is equal to the degree-zero part of the image of $\Phi_S$, denoted $\text{im}(\Phi_S)^0$. We can compute the image of $\Phi_S$ in Magma (see Magma Code A.7) by taking the quotient of $\bigoplus Z P_i$ by the kernel of $\Phi_S$; passing to the degree-zero part, we find $\text{im}(\Phi_S)^0$ is isomorphic (as an abelian group) to $\mathbb{Z}/28\mathbb{Z} \times \mathbb{Z}/1680\mathbb{Z} \times \mathbb{Z}/392857929108811200\mathbb{Z}$.

As $G$ contains no 7-torsion points while its quotient $\text{im}(\Phi_S)^0$ contains a factor $(\mathbb{Z}/7\mathbb{Z})^3$, we conclude that $G$ has rank at least three. Hence its rank is exactly three.

Now let $G'$ be the degree-zero part of $\bigoplus_{i=1}^{10} Z P_i$ modulo the relations in (7). We may use these relations to eliminate the generators $P_9$, $P_{10}$, $P_1$, $P_4$, $P_7$, and then $P_6$. (It is of course possible to do the elimination in other ways.) One now sees that, as abelian groups, $G' \cong \mathbb{Z}(P_2 - P_3) \oplus \mathbb{Z}(P_2 - P_5) \oplus \mathbb{Z}(P_2 - P_8)$.

As $G'$ surjects onto $G$ and as they have the same rank, we find $G$ is also free abelian. Moreover, this calculation shows that $G$ is generated by divisors supported on the points at infinity. The proof of (b) is now complete. $\square$

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**Appendix A. Magma and PARI/gp Code**

**PARI/gp Code A.1.**

```plaintext
\begin{verbatim}
e(x) = x^3-x+1;
f(x,c)=x^2+c;
X(u,v) = -v/(u^2-1);
Y(u,v) = -1/(u^4-2*u^2+1);
E=ellinit([0,0,0,-1,1]);
P0=[1,1];
numit=10000;

{P = elladd(E,P0,P0);
 newP=P;
 for(i=2,numit,
   if(i%10==0,print("***",i,:,",",round(log(denominator(P))/log(10))));
   newP=elladd(E,P0,P);
   PP=[X(newP[1],newP[2]),Y(newP[1],newP[2])];
}
\end{verbatim}
```

14
\texttt{P=newP; C=PP[2];}

\texttt{\textbackslash \textbackslash see if there are more than 2 rational second pre-images}
\texttt{Cfac=factor(f(f(x,C),C)); Total=0;}
\texttt{for(j=1,matsize(Cfac)[1],}
\texttt{\hspace{1cm}if(poldegree(Cfac[j,1])==1,}
\texttt{\hspace{2cm}Total=Total + Cfac[j,2];}
\texttt{\hspace{1cm});}
\texttt{if(Total>2, print("4 2nd pre-images for c= ",C,":\",Total);;)}

\texttt{\textbackslash \textbackslash see if there are any rational 4th pre-images}
\texttt{Cfac2=factor(f(f(f(f(x,C),C),C),C)); Total=0;}
\texttt{for(j=1,matsize(Cfac2)[1],}
\texttt{\hspace{1cm}if(poldegree(Cfac2[j,1])==1,}
\texttt{\hspace{2cm}Total=Total + Cfac2[j,2];}
\texttt{\hspace{1cm});}
\texttt{if(Total>1, print("4th pre-images for c= ",C,":\",Total);;)}
\texttt{)}

\textbf{PARI/gp Code A.2.}

\texttt{\textbackslash \textbackslash x coordinate of 2P}
\texttt{f(p,q) = (p^4 + 2*q^2*p^2 - 8*q^3*p + q^4)/(4*q*p^3 - 4*q^3*p + 4*q^4)}
\texttt{F(x,y) = numerator(1/((f(x,y)^2-1)^2)) \textbackslash \textbackslash numerator of the c value}
\texttt{G(x,y) = denominator(1/((f(x,y)^2-1)^2)) \textbackslash \textbackslash denominator of the c value}

\texttt{\textbackslash \textbackslash To determine the upper bound on the height function we take maximum of}
\texttt{\textbackslash \textbackslash the sum of the coefficients of F and G.}
\texttt{\{}
\texttt{EF=vector(17);}
\texttt{EG=vector(17);}
\texttt{SF=0;SG=0;}
\texttt{for(i=1,17,}
\texttt{\hspace{1cm}EF[i]=subst(polcoeff(F(x,y),i-1,y),x,1);}
\texttt{\hspace{1cm}SF=SF+abs(EF[i]);}
\texttt{\hspace{1cm}EG[i]=subst(polcoeff(G(x,y),i-1,y),x,1);}
\texttt{\hspace{1cm}SG=SG+abs(EG[i]);}
\texttt{)}
\texttt{\}}
\texttt{\textbackslash \textbackslash SF = 16128  SG = 29496}

\texttt{\textbackslash \textbackslash To determine the lower bound we apply the theory of resultants}
\texttt{\textbackslash \textbackslash to find f_1,g_1,f_2,g_2 such that}
\texttt{\textbackslash \textbackslash f_1(x,y)F(x,y) + g_1(x,y)G(x,y) = Res(F,G)*x^{-31}}
We can then use these auxiliary polynomials to construct a lower bound on the height function.

\[ M = \text{matrix}(32,32); \]
\[ \{ \text{for}(i=1,16, \text{for}(j=1,32, \text{if}(j>=i&&j-(i))<=16, M[i,j]=EF[j-(i-1)], M[i,j]=0) ) \} \]
\[ \{ \text{for}(i=17,32, \text{for}(j=1,32, \text{if}(j>=i-16&&j-(i-16))<=16, M[i,j]=EG[j-(i-17)], M[i,j]=0) ) \} \]
\[ \text{matdet}(M) = \text{resultant} \]
\[ \text{Madj=matadjoint}(M); \]
\[ X = \text{matrix}(32,1); \]
\[ \{ \text{for}(i=1,16, X[i,1] = x^{(16-i)} y^{(i-1)} A ); \]
\[ \{ \text{for}(i=17,32, X[i,1] = x^{(16-(i-16))} y^{(i-17)} B ); \]
\[ T = \text{Madj} \times X; \]
\[ \{ \text{T[1,1] with A->F(x,y) B -> G(x,y) gives the relation} \]
\[ \text{f}_1(x,y)F(x,y) + g_1(x,y)G(x,y) = \text{Res} x^{31} \]
\[ \text{T[32,1] with A->F(x,y) B -> G(x,y) gives the relation} \]
\[ \text{f}_2(x,y)F(x,y) + g_2(x,y)G(x,y) = \text{Res} y^{31} \]
\[ \{ \]
\[ \text{Sf1=0;Sf2=0;Sg1=0;Sg2=0;} \]
\[ \{ \]
\[ \text{for}(i=1,16, \text{Sf1=Sf1+abs(subst(polcoeff(subst(subst(T[1,1],B,0),A,1),i-1,y),x,1))}; \]
\[ \text{Sg1=Sg1+abs(subst(polcoeff(subst(subst(T[1,1],A,0),B,1),i-1,y),x,1))}; \]
\[ \text{Sf2=Sf2+abs(subst(polcoeff(subst(subst(T[32,1],B,0),A,1),i-1,y),x,1))}; \]
\[ \text{Sg2=Sg2+abs(subst(polcoeff(subst(subst(T[32,1],A,0),B,1),i-1,y),x,1))}; \]
\[ \} \]
\[ \}
\[ \text{C=2*max(max(max(Sf1,Sf2),Sg1),Sg2)}; \]
\[ \}
\[ \text{We now determine the lower bound on n for } [n]P \]
\[ \text{to guarantee the resulting c values are distinct.} \]
\[ E = \text{ellinit}([0,0,0,-1,1]); \]
\[ C2 = \text{log}(C)/(6*\text{ellheight}(E,[1,1])); \]
\[ n = (C2-1)/2; \]
\[ n \approx 330.3 \]
\[ \text{Checking the first 330 points for the number of third preimages.} \]
\[ \text{Note that we exclude the first three as } x = \pm 1 \text{ does not correspond to a } c \text{ value and } u=0 \text{ corresponds to } c=-1 \text{ which has 2 3rd preimages } \{1,-1\}. \]
\[ e(x) = x^3-x+1; \]
\[ F(x,c)=x^2+c; \]
\[ XX(u,v) = -v/(u^2-1); \]
\[ YY(u,v) = -1/(u^4-2*u^2+1); \]
\[ \{ \]
\[ \text{for}(i=4,330, \]
P=[1,1];
Q=ellpow(E,[1,1],i);
c=YY(Q[1],Q[2]);

Cfac2=factor(F(F(F(x,c),c),c));

Total=0;
for(j=1,matsize(Cfac2)[1],
   if(poldegree(Cfac2[j,1])==1,
      Total=Total + Cfac2[j,2];
   );
);
if(Total !=2, print(i,"\n","Total"););
}

\textit{Magma} Code A.3.

\begin{verbatim}
A<x0,x1,x2,x3,x4> := ProjectiveSpace(Integers(),4);
C:=Curve(A,[x3^2 + x1*x4 -x0^2,x3^2+x2*x4-x1^2,x3^2+x3*x4-x2^2]);
Is := [Ideal(SingularSubscheme(AffinePatch(C, i))) : i in [1..5]];
[Factorization(Integers()!Basis(EliminationIdeal(I, {}))\[1\]) : I in Is];
\end{verbatim}

\textit{PARI/gp} Code A.4.

\begin{verbatim}
S(x,p) = (x + sqrt(x^2 - 4*p))/2;
T(x,p) = (x - sqrt(x^2 - 4*p))/2;
f(x,c) = x^2+c;
C4(x,c) = f(f(f(f(x,c),c),c),c);

{ 
N=vector(5);
minn=1;
maxn=5;
p=3;
for(n=minn,maxn,
   t=ffinit(p,n,x);
   res=vector(n,x,x=0);
   points=0;
   for(i=1,p^n,
      temp=i;
      for(k=1, n,
         res[k]=(temp%((p^k))/(p^(k-1));
         temp=temp-((temp%((p^k));
      );
      X=Mod(Pol(Mod(res,p),x),t);
   for(j=1,p^n,
      temp=j;
   for(k=1, n,
      res[k]=(temp%((p^k))/(p^(k-1));
      temp=temp-((temp%((p^k));
   );
}\}
\end{verbatim}
Y=Mod(Pol(Mod(res,p),x),t);
if(C4(X,Y)==0,points++);
);
N[n]=points+8;
);

Pi is the ith power sum of the eigenvalues. These formulas were determined by examining the coefficients of the Zeta function
P1=p+1-N[1];
P2=p^2+10*p+1-N[2];
P3=p^3 + 1 + 3*p*P1-N[3];
P4=p^4 -10*p^2+1+4*p*P2-N[4];
P5=p^5 +1+ 5*p*P3 - 5*p^2*P1 -N[5];

ci is the ith symmetric polynomial of the eigenvalues. These formulas are the Newton-Girard Formulas.
c1=P1;
c2=(c1^2-P2)/2;
c3=(P3+3*c1*c2-c1^3)/3;
c4=(c1^4 - 4*c2*c1^2 + 4*c3*c1 + 2*c2^2-P4)/4;
c5=(-c1^5 + 5*c2*c1^3 - 5*c3*c1^2 - 5*c2^2*c1 + 5*c4*c1 +5*c3*c2 +P5)/5;
F=x^5-c1*x^4+c2*x^3-c3*x^2+c4*x-c5;
R=polroots(F);
alpha=listcreate(10);
for(i=1,5,
    listput(alpha,S(R[i],p));
    listput(alpha,T(R[i],p));
);
G=1;
for(i=1,10,
    G=G*(x-alpha[i]);
);
G=round(G);
print(factor(G));
}

_PARI/gp_ Code A.5.
E(u,v)=u^3-u+1-v^2;
{
N=vector(2);
minn=1;
maxn=2;
p=3;
for(n=minn,maxn,
t=ffinit(p,n,x);
res = vector(n, x, x=0);
points = 0;
for (i=1, p^n,
    temp = i;
    for (k=1, n,
        res[k] = (temp % (p^k)) / (p^(k-1));
        temp = temp - (temp % (p^k));
    );
    X = Mod(Pol(Mod(res, p), x), t);
    for (j=1, p^n,
        temp = j;
        for (k=1, n,
            res[k] = (temp % (p^k)) / (p^(k-1));
            temp = temp - (temp % (p^k));
        );
        Y = Mod(Pol(Mod(res, p), x), t);
        if (E(X, Y) == 0, points++);
    );
); N[n] = points + 1;

%\Pi is the ith power sum of the eigenvalues. These formulas were
%determined by examining the coefficients of the Zeta function
P1 = 4 - N[1];
P2 = 10 - N[2];

%ci is the ith symmetric polynomial of the eigenvalues.
%These formulas are the Newton-Girard Formulas.
c1 = P1;
c2 = (c1^2 - P2) / 2;
F = x^2 - c1*x + c2;
print(factor(F));

\textit{Magma Code A.6.}
\begin{verbatim}
  A<x0,x1,x2,x3,x4> := ProjectiveSpace(Integers(),4);
  C:=Curve(A,[x3^2 + x1*x4 -x0^2,x3^2+x2*x4-x1^2,x3^2+x3*x4-x2^2]);
  C5 := BaseChange(C, GF(5));
  Invariants(ClassGroup(C5));
  C7 := BaseChange(C, GF(7));
  Invariants(ClassGroup(C7));
  C11 := BaseChange(C, GF(11));
  Invariants(ClassGroup(C11));
\end{verbatim}

\textit{Magma Code A.7.}
\begin{verbatim}
  A<x0,x1,x2,x3,x4> := ProjectiveSpace(Integers(),4);
  C:=Curve(A,[x3^2 + x1*x4 -x0^2,x3^2+x2*x4-x1^2,x3^2+x3*x4-x2^2]);
  Z10 := FreeAbelianGroup(10);
\end{verbatim}
ptsC := [C![1,e2,e3,e4,0] : e2,e3,e4 in [1,-1]] cat [C![0,0,0,0,1], C![0,-1,0,-1,1]];

C3 := BaseChange(C, GF(3));
Cl3, mCl3 := ClassGroup(C3);
h3 := hom<Z10 -> Cl3 | [Divisor(C3!Eltseq(pt)) @@ mCl3 : pt in ptsC]>
C5 := BaseChange(C, GF(5));
Cl5, mCl5 := ClassGroup(C5);
h5 := hom<Z10 -> Cl5 | [Divisor(C5!Eltseq(pt)) @@ mCl5 : pt in ptsC]>
C7 := BaseChange(C, GF(7));
Cl7, mCl7 := ClassGroup(C7);
h7 := hom<Z10 -> Cl7 | [Divisor(C7!Eltseq(pt)) @@ mCl7 : pt in ptsC]>
C11 := BaseChange(C, GF(11));
Cl11, mCl11 := ClassGroup(C11);
h11 := hom<Z10 -> Cl11 | [Divisor(C11!Eltseq(pt)) @@ mCl11 : pt in ptsC]>
C13 := BaseChange(C, GF(13));
Cl13, mCl13 := ClassGroup(C13);
h13 := hom<Z10 -> Cl13 | [Divisor(C13!Eltseq(pt)) @@ mCl13 : pt in ptsC]>
C17 := BaseChange(C, GF(17));
Cl17, mCl17 := ClassGroup(C17);
h17 := hom<Z10 -> Cl17 | [Divisor(C17!Eltseq(pt)) @@ mCl17 : pt in ptsC]>

K := Kernel(h3) meet Kernel(h5) meet Kernel(h7) meet Kernel(h11)
  meet Kernel(h13) meet Kernel(h17);
B := Basis(LLL(Lattice(Matrix([Eltseq(Z10!g): g in OrderedGenerators(K)]))));
CQ := BaseChange(C, Rationals());
for j := 1 to 6 do
  IsPrincipal(Divisor([<Place(CQ!Eltseq(ptsC[i])), B[j][i] : i in [1..10]])
end for;

QG, qmap := quo<Z10 | K>;
Invariants(QG);

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