Solving the Continuous Nonlinear Resource Allocation Problem with an Interior Point Method

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Abstract

The sparsity and algebraic structure of convex resource allocation problems have given rise to highly specialized methods. We show that the sparsity structure alone yields a closed-form Newton step for the generic primal-dual interior point method. Computational tests indicate that this makes the interior point method competitive with the leading special-purpose method on several important problem classes for which the latter can exploit additional algebraic structure. Moreover, the interior point method consistently outperforms the specialized method when no additional algebraic structure is used or available.

Keywords: convex programming; interior point methods; continuous knapsack

1 Introduction

The continuous nonlinear resource allocation problem (or continuous nonlinear knapsack problem) is a finite-dimensional convex optimization problem involving a separable objective, simple bounds on each variable, and a single constraint linking the variables by means of a separable function. The survey paper of Patriksson [25] shows that the resource allocation problem has a long history and diverse applications; see also the book by Ibaraki and Katoh [14]. The contexts in which the problem appears often demand that it be solved very quickly, even when the problem dimension is quite large. Consequently, researchers long ago moved beyond general-purpose nonlinear or convex programming procedures and focused on exploiting the special structure of the optimality conditions for the problem.

As noted by Patriksson [25], two frameworks have emerged as the most competitive for solving resource allocation problems: the pegging or variable-fixing methods and the breakpoint-search methods. Each involves solving a finite sequence of optimization subproblems. In a pegging method, the subproblems correspond to ignoring the bounds on some variables while holding other variables fixed ("pegged") at one of their bounds. Each iteration identifies at least one variable that can be pegged, and it stays pegged thereafter. The procedure ends when all unpegged variables (if any remain) in some subproblem satisfy their bounds at optimality when those bounds are not explicitly enforced. By contrast, a breakpoint search acts within the one-dimensional dual space corresponding to the Lagrange multiplier for the single linking constraint. The concave dual objective can be viewed as a piecewise-defined function whose breakpoints are easily listed, so a binary

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search identifies either a maximizer among those breakpoints or a pair of breakpoints that most closely bracket a maximizer. At each tested breakpoint, a separable optimization is performed in the primal variables; the final interpolating subproblem is essentially equivalent to a pegging subproblem.

The pegging and breakpoint-search frameworks have each been rediscovered many times, so original attribution is difficult. Patriksson [25] cites Theil and van de Panne [33] as a possible genesis for the pegging methods, with the first formal description apparently given by Sanathanan [29]. A dual approach seems to have been described first by Churchman et al. [8], with the earliest explicit search of breakpoints carried out by Charnes and Cooper [7]. In summarizing the state of the art, Patriksson [25] notes that computational studies in the literature have generally indicated that pegging is superior to breakpoint search when certain subproblems (see the next section) common to both methods are easily solved, whereas breakpoint search is faster otherwise. He also observes that numerical comparisons of either method with general-purpose solvers are essentially nonexistent in the literature.

In the current paper we present computational results showing that a primal-dual interior point method outperforms breakpoint search on problems where the latter is traditionally considered to be the best possible choice, namely, when its subproblems do not admit closed-form solutions and must be solved numerically. Each iteration of a primal-dual interior point method first calculates a search direction for Newton’s method applied to a perturbation of the Karush-Kuhn-Tucker conditions, and then chooses a step-size so that the primal and dual variables satisfy their respective bounds strictly [23]. The key to the method’s success in the present setting is the special form of the linear system that is solved when calculating the search direction. Aside from exploiting the linear-system structure, our implementation of the interior point method has been kept fairly simple and generic. By contrast, we have taken pains to make the breakpoint search as efficient and reliable as possible, so as to justify our conclusions about the superiority of interior point methods in this setting.

In the next section, we state the problem formally and review its optimality conditions. In §3 we describe the breakpoint search and interior point methods, along with details of their implementation. Section 4 lays out the problem instances used for the computational tests, and the results are discussed in §5. Some concluding remarks are provided in §6.

2 Notation, assumptions, and optimality conditions

We consider the resource allocation problem in the form

\[
\begin{align*}
\text{minimize} & \quad f(x) := \sum_{i=1}^{n} f_i(x_i) \quad \text{over all } x \\
\text{subject to} & \quad g(x) := \sum_{i=1}^{n} g_i(x_i) = b, \\
& \quad l \leq x \leq u.
\end{align*}
\]

Here \( x, l, \) and \( u \) are \( n \)-vectors of real numbers, whereas \( b \) is a real scalar. Inequalities of vectors are interpreted coordinate-wise. For the purpose of the present study, we make the following assumptions:

A1. The functions \( f_i \) and \( g_i \) are convex and twice differentiable on an open set containing the interval \([l_i, u_i]\).

A2. The relaxed problem, in which \(2\) is replaced by \( g(x) \leq b \), has no optimum with \( g(x) < b \).
A3. The function \( f_i \) is decreasing on \([l_i, u_i]\) and \( g_i \) is increasing on \([l_i, u_i]\) with \( g(l) < b < g(u) \).

The randomly generated test instances discussed in later sections all satisfy these assumptions.

In practice, we are actually interested in the relaxed problem of Assumption A2. However, we can cheaply determine whether that assumption holds provided we know the intervals of monotonicity for each \( f_i \). Indeed, most presentations of methods for resource allocation problems include Assumption A2 because it can be enforced through some combination of initialization, preprocessing, and data generation. The required computational time is typically quite small compared to the time needed to solve the problem; in many instances, the task is trivial. Likewise, many treatments also include some form of Assumption A3: it simplifies discussion and implementation, and it can be guaranteed inexpensively by first identifying the intervals of monotonicity for \( f_i \) and \( g_i \) and then, as needed, shrinking the intervals \([l_i, u_i]\) and reorienting the variables \( x_i \).

Assumptions A1–A3 imply that problem (1)–(3) and the relaxed problem are equivalent and admit an optimal solution, and they also guarantee that the Slater constraint qualification holds for the relaxed problem. By Lagrangian duality, necessary and sufficient optimality conditions for (1)–(3) can therefore be expressed as follows:

\[
\begin{align*}
\text{(4) } & \quad \text{minimize } f(x) + \rho g(x) \quad \text{over all } x \text{ subject to (3)}, \\
\end{align*}
\]

The dual objective is

\[
\rho \mapsto -bp + \sum_{i=1}^{n} \min_{x_i \in [l_i, u_i]} [f_i(x_i) + \rho g_i(x_i)],
\]

which attains its maximum; moreover, any maximizer \( \rho \) is necessarily nonnegative. The subproblem (4) has coordinate-wise optimality conditions given by

\[
\begin{align*}
& f_i'(x_i) + \rho g_i'(x_i) = 0, \quad \text{if } l_i < x_i < u_i, \\
& f_i'(x_i) + \rho g_i'(x_i) \geq 0, \quad \text{if } x_i = l_i, \\
& f_i'(x_i) + \rho g_i'(x_i) \leq 0, \quad \text{if } x_i = u_i.
\end{align*}
\]

The left-hand sides give the Karush-Kuhn-Tucker multipliers for the bounds \( l_i \leq x_i \) and \( x_i \leq u_i \), respectively, as

\[
\lambda_i := \max\{0, -[f_i'(x_i) + \rho g_i'(x_i)]\} \quad \text{and} \quad \mu_i := \max\{0, f_i'(x_i) + \rho g_i'(x_i)\}.
\]

Letting \( s := u - x \) denote the vector of slack variables for the upper bounds on \( x \), we can express the Karush-Kuhn-Tucker (KKT) conditions for (1)–(3) as

\[
\begin{align*}
\nabla f(x) + \rho \nabla g(x) - \lambda + \mu &= 0, \\
x + s &= u, \\
x \geq l, \quad &\lambda \geq 0, \quad s \geq 0, \quad \mu \geq 0, \\
\text{diag}(x - l)\lambda &= 0, \quad \text{diag}(s)\mu = 0, \\
g(x) &= b.
\end{align*}
\]

Here the notation \( \text{diag}(z) \) refers to the diagonal matrix whose diagonal entries are given by the entries in the vector \( z \). An equivalent version of the Lagrangian dual incorporates the multiplier vectors \( \lambda \) and \( \mu \), but that form is not used in the sequel.

The three solution frameworks discussed in the introduction exploit the above developments in different ways:
Pegging methods solve subproblems of the form \((1)-(2)\), but for which some variables are held fixed while the bounds \((3)\) for all remaining variables are omitted.

Breakpoint search methods maximize the dual objective \((5)\) by solving a sequence of subproblems of the form \((4)\) at various values of \(\rho\).

Primal-dual interior point methods apply Newton’s method to perturbations of the KKT system \((6)-(10)\).

The pegging and breakpoint search methods both benefit considerably when minimization of \(x_i \mapsto f_i(x_i) + \rho g_i(x_i)\) can be handled efficiently. As noted by Patriksson [25], previous computational studies have indicated that pegging is generally the fastest method for resource allocation problems in situations where the pegging subproblem can be solved in closed form, and that breakpoint search is the best method otherwise. The present article shows that, in fact, interior point methods are competitive with breakpoint search in the latter situation. Because we focus on problems for which breakpoint search dominates pegging, we do not include pegging methods in this study. Indeed, the pegging approach is not even well-defined for some of the problems we consider, because the pegging subproblems do not admit optimal solutions.

3 The methods and their implementation

In this section, we describe the two main approaches tested in our computational study. In addition to the details indicated below, our MATLAB implementations of both methods employ vectorized operations with logical indexing wherever they provide an efficiency gain over explicit loops or conditionals.

3.1 Breakpoint search

Breakpoint search (see Figure 1) is based on the observation that the dual objective \((5)\) is concave and defined piecewise with a finite number of easily calculated breakpoints. The derivative, or subdifferential, of this objective is nonincreasing. A binary search of the breakpoints therefore identifies either one that is itself a root or a pair of breakpoints that most closely bracket a root.

There are at most \(2n\) breakpoints, occurring at \(\rho\)-values where some \(x_i \mapsto f_i(x_i) + \rho g_i(x_i)\) attains its minimum over \([l_i, u_i]\) at an endpoint \(l_i\) or \(u_i\). Equivalently, a breakpoint makes the derivative \(x_i \mapsto f'_i(x_i) + \rho g'_i(x_i)\) nonnegative at \(l_i\) or nonpositive at \(u_i\). Consequently, all breakpoints have the form

\[
\rho_i^+ := -f'_i(l_i)/g'_i(l_i) \quad \text{or} \quad \rho_i^- := -f'_i(u_i)/g'_i(u_i).
\]

The monotonicity of \(f_i\) and \(g_i\) allow us to define \(\rho_i^+ = \infty\) when \(g'_i(l_i) = 0\) and to guarantee that \(g'_i(u_i) > 0\) in the definition of \(\rho_i^-\). The convexity and monotonicity of \(f_i\) and \(g_i\) also guarantee that \(0 \leq \rho_i^- \leq \rho_i^+\).

The binary search sequentially refines a bracketing \(\rho^- < \rho^* < \rho^+\) until the true root \(\rho^*\) lies between two consecutive breakpoints. The bracket is adjusted inward by extracting a breakpoint \(\rho\) within it and testing the sign of the derivative of the dual objective \((5)\). To evaluate that derivative at \(\rho\), we first fix

\[
(11) \quad x_i := \begin{cases} l_i, & \text{if } \rho \geq \rho_i^+, \\ u_i, & \text{if } \rho \leq \rho_i^- . \end{cases}
\]
Input: data $f, g, b, l, u$ defining an instance of problem (1)–(3) satisfying assumptions A1–A3 of §2.

Output: approximate solution $x$ for problem (1)–(3).

1. [Initialization]
   Set $I := \{1, \ldots, n\}$, $\rho^- := 0, \rho^+ := \infty$ and $\hat{b} := b$.
   For all $i \in I$, set $\rho_i^+ := -\frac{f_i'(l_i)}{g_i'(l_i)}$ and $\rho_i^- := -\frac{f_i'(u_i)}{g_i'(u_i)}$.
   Set $R := \{\rho_i^-\}_{i \in I} \cup \{\rho_i^+\}_{i \in I}$, stored as an unsorted list of possibly replicated values.

2. [Breakpoint selection]
   Select $\rho \in R$ as a median value from the list representing $R$.

3. [Subproblem solution]
   Set $I^- := \{i \in I : \rho \leq \rho_i^-\}$ and $\beta^- := \sum_{i \in I^-} g_i(u_i)$.
   Set $I^+ := \{i \in I : \rho \geq \rho_i^+\}$ and $\beta^+ := \sum_{i \in I^+} g_i(l_i)$.
   For each $i \in I \setminus (I^- \cup I^+)$, find $x_i$ to minimize $f_i(x_i) + \rho g_i(x_i)$ subject to $l_i \leq x_i \leq u_i$.
   Set $\beta := \beta^- + \beta^+ + \sum_{i \in I \setminus (I^- \cup I^+)} g_i(x_i)$.

4. [Bracket update]
   Set $\Delta b := 0$.
   If $\beta \geq \hat{b}$, then:
     Set $\rho^- := \rho, \Delta b := \beta^-, I := I \setminus I^- \setminus I^+$ and $R := \{r \in R : r > \rho\}$.
     For each $i \in I^-$, set $x_i := u_i$.
   If $\beta \leq \hat{b}$, then:
     Set $\rho^+ := \rho, \Delta b := \beta^-, I := I \setminus I^- \setminus I^+$ and $R := \{r \in R : r < \rho\}$.
     For each $i \in I^+$, set $x_i := l_i$.
   Set $\hat{b} := \hat{b} - \Delta b$.

5. [Stopping criterion]
   If $R \neq \emptyset$, then go to step 2. Otherwise, continue to step 6.

6. [Interpolation within final bracket]
   Find $\rho \in [\rho^-, \rho^+]$ and $x_i$ for each $i \in I$ so that $f_i'(x_i) + \rho g_i'(x_i) = 0$ and $\sum_{i \in I} g_i(x_i) = b$.

Figure 1: Breakpoint search

The remaining minimizers are critical points: $f_i'(x_i) + \rho g_i'(x_i) = 0$ and $l_i < x_i < u_i$. Depending on the problem data (see §4), these critical points might be found (a) in closed form, (b) by using a
problem-specific implementation of Newton’s method, or (c) by means of a general-purpose New-
ton’s method with Armijo linesearch for sufficient decrease and damping (as needed) to maintain
\( l_i < x_i < u_i \). The derivative value at \( \rho \) is then given by \(-b + \sum g_i(x_i)\), the sign of which determines
whether \( \rho \) becomes the new \( \rho^- \) or \( \rho^+ \). This in turn determines, through (11), that some values of
\( x_i \) shall remain fixed and can therefore be removed from further consideration.

The final bracket, if nontrivial, consists of two closest breakpoints with the optimal value of \( \rho \)
lying somewhere between them. To interpolate between them, our implementation finds \( \rho \)
and the unfixed \( x_i \)-coordinates (denoted by \( i \in I \)) simultaneously by applying a multi-dimensional Newton’s
method with Armijo linesearch to the corresponding Lagrange multiplier conditions \( \sum g_i(x_i) = \hat{b} \) and
\( f'_i(x_i) + \rho g'_i(x_i) = 0 \) for \( i \in I \).

Throughout the procedure, the subproblem optimizations are initialized using the corresponding
solutions from prior iterations. Also, we extract the required median values without sorting the list
of breakpoints in advance, which can yield significant computational savings if each subproblem
solution requires only a few operations per index \( i \) [6, 24, 28, 16].

3.2 Interior point method

The primal-dual interior point method solves the KKT optimality conditions (6)–(10) for the vari-
ables \((x, \lambda, s, \mu, \rho)\). Its operation preserves strict inequality for the simple bounds (8), only allowing
them to become active in the limit. The method is based on the perturbed KKT system

\[
\nabla f(x) + \rho \nabla g(x) - \lambda + \mu = 0, \\
x + s = u, \\
\text{diag}(x - l)\lambda = \tau e, \quad \text{diag}(s)\mu = \tau e, \\
g(x) = b,
\]

where \( e \) denotes the \( n \)-vector of all ones and the inequalities \( x > l, \lambda > 0, s > 0, \mu > 0 \) are enforced
separately. The system (12)–(15) is algebraically equivalent to the Lagrange multiplier equations
for a related optimization problem involving barrier functions for the bounds (3):

\[
\text{minimize} \quad f(x) - \tau \sum_{i=1}^n [\ln(x_i - l_i) + \ln(s_i)] \quad \text{over all} \quad x > l, \ s > 0 \\
\text{subject to} \quad g(x) = b, \ x + s = u.
\]

As the barrier parameter \( \tau > 0 \) is driven to zero, we expect the (unique) solution \((x, \lambda, s, \mu, \rho)\) of
(12)–(15) to tend toward the solution set of the original KKT system (6)–(10).

In each iteration of the interior point method, we calculate a Newton search direction for the
perturbed system (12)–(15) and then take a step along that direction, damped so as to preserve
\( x > l, \lambda > 0, s > 0, \mu > 0 \). Next, the value of \( \tau \) is adjusted and the iteration repeats. The method
is formally stated in Figure 2 using notation defined below. It is essentially a general-purpose
interior point method for a nonlinear programming problem formulated with simple bounds and
equality constraints. The notation emphasizes the single equality constraint of the present setting
and the initialization exploits that aspect as well. Procedure parameters are given specific values
\((10^{-10}, 0.25, \text{and} \ 0.8 \text{in steps 2, 3, and 5, respectively})\) that gave reliable performance in preliminary
testing.

The interior point method presented here is rudimentary and could likely be refined to provide
theoretical guarantees of global convergence, superlinear convergence, or polynomiality in the num-
er of calls to \( f, \) \( g \) and their derivatives; see [27, 34, 20, 21, 19, 10, 32, 31, 26, 35] for treatments
Input: data \( f, g, b, l, u \) defining an instance of problem (1)–(3) satisfying assumptions A1–A3 of §2.
Output: approximate solution \( x \) for problem (1)–(3).

1. [Initialization]
   Choose \( t \in (0, 1) \) with \( g(tl + (1 - t)u) = b \) and set \( x := tl + (1 - t)u \) and \( s := u - x \).
   Set \( \rho := 1, \lambda_i := \max \{0, -[f_i'(x_i) + \rho g_i'(x_i)]\} \) and \( \mu_i := \max \{0, f_i'(x_i) + \rho g_i'(x_i)\} \).

2. [Termination criterion]
   Set \( r_d := \nabla f(x) + \rho \nabla g(x) - \lambda + \mu, r_l := \text{diag}(x - l)\lambda, r_u := \text{diag}(s)\mu, \) and \( r_g := g(x) - b. \)
   If none of the relative errors
   \[
   \frac{\|r_d\|_1}{1 + \|\nabla f(x)\|_1 + |\rho| + \|\nabla g(x)\|_1 + \|\lambda\|_1 + \|\mu\|_1},
   \frac{\|r_l\|_1}{1 + \|x - l\|_1 + \|\lambda\|_1},
   \frac{\|r_u\|_1}{1 + \|s\|_1 + \|\mu\|_1},
   \frac{|r_g|}{1 + \|g(x)\|_1 + |b|}
   \]
   exceeds \( 10^{-10} \), then we are done: \( x \) is considered optimal to within numerical tolerance.

3. [Barrier parameter selection]
   Set \( \sigma := \frac{\lambda \cdot x + \mu \cdot s}{2n} \) and \( \tau := 0.25\sigma \).

4. [Search direction]
   Set \( r_l := r_l - \tau e \) and \( r_u := r_u - \tau e \).
   Calculate \((\Delta x, \Delta \lambda, \Delta s, \Delta \mu, \Delta \rho)\) to solve the linear system (16).

5. [Step size]
   Set \( \hat{t} > 0 \) to be the largest value for which
   \[
   (x, \lambda, s, \mu) - \hat{t}(\Delta x, \Delta \lambda, \Delta s, \Delta \mu) \geq (l, 0, 0, 0). \]
   Set \( t := \min \{1, 0.8\hat{t}\} \).

6. [Update]
   Set \( (x, \lambda, s, \mu, \rho) := (x, \lambda, s, \mu, \rho) - t(\Delta x, \Delta \lambda, \Delta s, \Delta \mu, \Delta \rho) \).
   Go to step 2.

Figure 2: Interior point method

of such matters in similar contexts. However, the version presented here correctly solves all the instances described in §4 and handily outperforms breakpoint search on challenging problems of moderate to very large dimension. It therefore meets the needs of the present study. The key to making it competitive is the fact that the linear system in step 4 can be solved in \( Cn \) arithmetic operations for a small fixed value of \( C \), as we show next.

To simplify the notation, we introduce a vector \( h \) with entries \( h_i := f_i''(x_i) + \rho g_i''(x_i) \) and let
ξ denote $x - l$. We also use uppercase letters to denote these diagonal matrices: $\Xi := \text{diag}(\xi)$, $\Lambda := \text{diag}(\lambda)$, $S := \text{diag}(s)$, $M := \text{diag}(\mu)$, $H := \text{diag}(h)$. The linear system satisfied by the search direction is then

$$
\begin{bmatrix}
H & -I & 0 & I & \nabla g \\
\Lambda & \Xi & 0 & 0 & 0 \\
0 & 0 & M & S & 0 \\
I & 0 & I & 0 & 0 \\
\nabla g^T & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\Delta x \\
\Delta \lambda \\
\Delta s \\
\Delta \mu \\
\Delta \rho
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\eta
\end{bmatrix}.
$$

(16)

To solve (16), first calculate the vectors

$$
w := h + \Xi^{-1}\lambda + S^{-1}\mu,$$

$$y := r_d + \Xi^{-1}r_l - S^{-1}r_u,$$

$$z := W^{-1}\nabla g$$

and let $\eta := -1/\nabla g^T z$. Notice that multiplying (16) from the left by

$$
\begin{bmatrix}
W^{-1} & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & I & 0 \\
-\eta z^T & 0 & 0 & 0 & \eta
\end{bmatrix}
\begin{bmatrix}
I & \Xi^{-1} & S^{-1}M & 0 \\
0 & \Xi^{-1} & 0 & 0 & 0 \\
0 & 0 & S^{-1} & 0 & 0 \\
0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\Delta x \\
\Delta \lambda \\
\Delta s \\
\Delta \mu \\
\Delta \rho
\end{bmatrix} =
\begin{bmatrix}
W^{-1}y \\
\Xi^{-1}r_l \\
S^{-1}r_u \\
\eta (r_g - z^T y)
\end{bmatrix}.
$$

From this we can read off the solution to (16) as follows:

$$
\Delta \rho = \eta (r_g - z^T y),
$$

$$
\Delta x = W^{-1}y - (\Delta \rho)z,
$$

$$
\Delta s = -\Delta x,
$$

$$
\Delta \lambda = \Xi^{-1}r_l - \Xi^{-1}\Lambda \Delta x,
$$

$$
\Delta \mu = S^{-1}r_u - S^{-1}M \Delta s.
$$

The values $\Xi^{-1}r_l$, $\Xi^{-1}\Lambda$, $S^{-1}r_u$, $S^{-1}M$ can be saved during the prior calculation of $w$ and $y$ and reused here. The solution of (16) therefore requires $9n - 1$ additions/subtractions, $5n + 1$ multiplications, and $6n + 1$ divisions.

4 Test problems

We classify the test problems considered in this work according to the availability of problem-specific subproblem solvers for breakpoint search. Namely, we place the problems into three categories as follows:
(a) Closed-form solutions are available for the separable breakpoint subproblem, but not for the breakpoint-interpolating (or pegging) subproblems.

(b) No closed-form subproblem solutions are available, but each separable breakpoint subproblem has a readily identified initial guess that guarantees convergence of the standard Newton iteration. Typically, this guarantee is based on the knowledge of the convexity and inflection points of the nondecreasing function \( x_i \mapsto f_i'(x_i) + \rho g_i'(x_i) \).

(c) Neither of the above seem to apply.

As described in the preceding section, we implemented the breakpoint search with a generic iterative solver for the subproblems. This was applied to all problem instances, regardless of category. However, we also tested implementations that exploit the problem-specific opportunities above when available. Specifically, for the first two categories above, breakpoint search was tested both with and without the problem-specific subproblem solvers; we therefore considered three approaches (including interior point methods) for those instances. For the third category above, only the two more general approaches were considered.

Problems within each category are described in the subsections below. Instances were generated so that assumptions A1–A3 of §2 were satisfied, after possible reorientation of intervals. The notation \( z \sim U(a, b) \) indicates that the value \( z \) was selected according to a continuous uniform distribution on the open interval \((a, b)\), whereas \( z \sim N(\mu, \sigma) \) indicates that \( z \) was selected according to a normal distribution with mean \( \mu \) and standard deviation \( \sigma \).

Some of the problem classes considered here have already been studied within the resource allocation literature, while others represent mathematical forms that commonly appear in operations research and its applications. We also draw attention to several important classes that have been left out: the quadratic knapsack \([13, 12, 6, 24, 28, 17]\), stratified sampling \([22, 30, 9, 3]\), and manufacturing capacity planning \([8, 36, 5, 2, 14]\) problems. These admit closed-form solutions to the pegging subproblems described in §2 and so the pegging method handily outperforms any of the methods investigated here. However, we mention in passing that the relative performance of our non-pegging methods on these three classes is very similar to what we report in §5 for the nonlinear lot-sizing problem of §4.1.1.

### 4.1 Instances with closed-form solutions for breakpoint subproblems

The problems in this subsection admit closed-form solutions to the breakpoint subproblem \([4]\), but not to the interpolation or pegging subproblem.

#### 4.1.1 Nonlinear lot-sizing

Problem in this class have \( f_i(x_i) = a_i x_i + c_i / x_i \) and \( g_i(x_i) = d_i / x_i \) with domains \( x_i > 0 \), discussed in the book of Churchman, Ackoff and Arnoff \([8]\). Instances for the present study were generated as follows:

- \( a_i, c_i \sim U(1, 5) \) and \( d_i \sim U(1, 11) \);
- \( l_i = \sqrt{c_i / a_i} \) and \( u_i \sim U(l_i, l_i + 4) \);
- \( b \sim U(g(l), g(u)) \).
4.1.2 Sum-of-powers objective with unit simplex constraint

Problems in this class have \( f_i(x_i) = a_i|x_i - y_i|^p \) and \( g_i(x_i) = x_i \). Note that \( f_i \) is everywhere twice differentiable when \( p_i \geq 2 \). Instances were generated as follows:

- \( a_i \sim U(1, 10) \) and \( p_i \sim U(2, 4) \);
- \( \lambda_i \sim U(0, 5) \), \( u_i \sim U(l_i, l_i + 5) \) and \( y_i \sim U(u_i, u_i + 5) \);
- \( b \sim U(g(l), g(u)) \).

4.1.3 Weighted \( p \)-norm objective with unit \( p \)-norm constraint

Problems in this class have \( f_i(x_i) = a_i|x_i - y_i|^p \) and \( g_i(x_i) = |x_i|^p \). Note \( f_i \) and \( g_i \) are everywhere twice differentiable when \( p \geq 2 \). Instances with \( p \in \{2, 2.5, 3, 4\} \) were generated as follows:

- \( a_i \sim U(1, 10) \);
- \( \lambda_i \sim U(0, 5) \), \( u_i \sim U(l_i, l_i + 5) \) and \( y_i \sim U(u_i, u_i + 5) \);
- \( b \sim U(g(l), g(u)) \).

4.1.4 Decentralized inventory management

This is the operational model of Bitran and Mondschein \[1\]. The constraint uses \( g_i(x_i) = c_i x_i \) and the objective function is given by

\[
f_i(x_i) = \int_0^{x_i} d_i[\alpha x_i - (1 + \alpha)t]h_i(t) \, dt + \int_{x_i}^\infty d_i[\beta t - (1 + \beta)x_i]h_i(t) \, dt,
\]

where \( h_i \) is a probability density on the real line. In our numerical tests, we take \( h_i \) to be a normal density with mean \( \mu_i \) and standard deviation \( \sigma_i \). Instances were generated as follows:

- \( \alpha = 0.2, \beta = 0.1, c_i = 0.35 \), and \( d_i \sim U(20, 120) \);
- \( \mu_i \sim U(5000, 15000) \) and \( \sigma_i \sim U(0.05\mu_i, 0.10\mu_i) \);
- \( u_i = \min(\text{argmin} f_i, \xi_i) \), with \( \xi_i \sim U(\mu_i - 2\sigma_i, 1.1 \cdot \text{argmin} f_i) \);
- \( l_i \sim U(\mu_i - 2\sigma_i, 0.1(\mu_i - 2\sigma_i) + 0.09u_i) \);
- \( b \sim U(g(l), g(u)) \).

Because \( f'_i \) has bounded range, the Lagrange multiplier equation \( f'_i(x_i) + \rho c_i = 0 \) for the pegging subproblem often admits no solution.

4.1.5 Reliability investment

Problems in this class have \( f_i(x_i) = -\ln(1 - r_i^x) \) with domain \( x_i > 0 \) and \( g_i(x_i) = c_i x_i \), as studied by Kettelle \[15\] and Everett \[11\]. Instances for the present study were generated as follows:

- \( r_i \sim U(0, 1) \) and \( c_i \sim U(1, 10) \);
- \( l_i \sim U(0, 1) \) and \( u_i \sim U(l_i, 10) \);
- \( b \sim U(g(l), g(u)) \).
4.2 Instances with good starting guesses for separable subproblems

These problems have no closed-form subproblem solutions, but their structure provides a good starting guess for applying Newton’s method to some reformulation of the breakpoint subproblem \([14]\).

4.2.1 Resource renewal for processes with exponentially decaying throughput

Problems in this class have \(f_i(x_i) = a_i x_i (e^{-1/x_i} - 1)\) and \(g_i(x_i) = c_i x_i\) for \(x_i > 0\), as studied by Melman and Rabinowitz [18]. For convenience, we extend \(f_i\) to a \(C^\infty\) convex function on the real line by defining \(f_i(x_i) = -x_i\) for \(x_i \leq 0\). Instances were generated as follows:

- \(a_i, c_i \sim U(0.001, 1000)\) and take \(\gamma = \min_j \{a_j/c_j\}\);
- \(b = 1.1 \sum_i c_i \xi_i\), where \(\xi_i = \begin{cases} 0, & \text{if } a_i/c_i > \gamma; \\ \text{argmin}_{x_i} f_i(x_i) + \gamma g_i(x_i), & \text{if } a_i/c_i \leq \gamma; \end{cases}\)
- \(l_i = 0\) and \(u_i = b/c_i\).

The subproblem \([14]\) is solved efficiently by taking \(x_i = 0\) when \(1 - \rho c_i/a_i \leq 0\) and otherwise taking \(x_i = 1/y_i\), where \(y_i > 0\) is obtained by applying Newton’s method to the equation \((1 + y_i) e^{-y_i} - 1 + \rho c_i/a_i = 0\) with initial value \(y_i = 1\).

4.2.2 Weighted \(p\)-norm objective with unit \(r\)-norm constraint

Problems in this class have \(f_i(x_i) = a_i |x_i - y_i|^p\) and \(g_i(x_i) = |x_i|^r\). Note that \(f_i\) and \(g_i\) are everywhere twice differentiable when \(p, r \geq 2\). Instances with \(p, r \in \{2, 2.5, 3, 4\}\) and \(p \neq r\) were generated as follows:

- \(a_i \sim U(1, 10);\)
- \(l_i \sim U(0, 5), u_i \sim U(l_i, l_i + 5)\) and \(y_i \sim U(u_i, u_i + 5);\)
- \(b \sim U(g(l), g(u)).\)

The subproblem \([14]\) is solved efficiently by applying Newton’s method to \(f'_i(x_i) + \rho g'_i(x_i) = 0\), initialized at \(x_i = y_i\).

4.2.3 Sum-of-powers objective with sum-of-powers constraint

Problems in this class have \(f_i(x_i) = a_i |x_i - y_i|^{p_i}\) and \(g_i(x_i) = |x_i|^{r_i}\). Instances were generated as follows:

- \(a_i \sim U(1, 10)\) and \(p_i, r_i \sim U(2, 4);\)
- \(l_i \sim U(0, 5), u_i \sim U(l_i, l_i + 5)\) and \(y_i \sim U(u_i, u_i + 5);\)
- \(b \sim U(g(l), g(u)).\)

The subproblem \([14]\) is solved efficiently by applying Newton’s method to \(f'_i(x_i) + \rho g'_i(x_i) = 0\), initialized at \(x_i = y_i\).
4.3 No special subproblem solution available

The following problems don’t appear to have a simple, specialized approach for solving the breakpoint subproblems.

4.3.1 Convex quartic objective with unit simplex constraint

This class of problems has $f_i(x_i) = a_i x_i^4 + b_i x_i^3 + c_i x_i^2 + d_i x_i$ and $g_i(x_i) = x_i$. Instances of these problems were generated as follows:

- $a_i = (\xi_i^2 + \eta_i^2) / \sqrt{8}$, $b_i = (\xi_i \zeta_i + \eta_i \chi_i) / \sqrt{3}$ and $c_i = (\zeta_i^2 + \chi_i^2) / \sqrt{8}$, with $\xi_i, \eta_i, \zeta_i, \chi_i \sim N(0, 1)$;
- $d_i = -f_i'(\tau_i)$, with $\tau_i \sim U(0, 10)$;
- $u_i = \min(\tau_i, \lambda_i)$, with $\lambda_i \sim U(0, \tau_i)$;
- $l_i \sim U(0, u_i)$;
- $b \sim U(g(l), g(u))$.

The choice of coefficients for $f_i$ guarantees that $f_i$ is strictly convex on the real line, which is true if and only if $8a_i c_i > 3b_i^2$, $a_i > 0$, and $c_i > q0$. Equivalently, the matrix

$$
\begin{bmatrix}
\sqrt{8a_i} & \sqrt{3b_i} \\
\sqrt{3b_i} & \sqrt{8c_i}
\end{bmatrix}
$$

must be positive definite. This can be ensured by selecting $a_i, b_i, c_i$ to be the rescaled entries of a matrix formed as $A^T A$, where the entries of $A$ are given by $\xi_i, \eta_i, \zeta_i, \chi_i$. The choice of $d_i$ guarantees that the critical point of $f_i$ is positive, after which the bounds are chosen so that each $f_i$ has the same monotonicity.

It can be shown that the subproblem (11) can be solved for these instances by applying Newton’s method to $f_i'(x_i) + \rho g_i'(x_i) = 0$, initialized at the inflection point $x_i = -b_i/(4a_i)$ of $f_i'$. However, such an approach turns out to be slower than using the general-purpose solver, which is why we have listed this class of problems here.

4.3.2 Log-exponential objective with linear constraint

Problems in this class have $g_i(x_i) = c_i x_i$ and

$$f_i(x_i) = \ln \left[ \sum_{j=1}^{5} \exp(a_{ij} x_i + d_{ij}) \right].$$

Instances were generated as follows:

- $d_{ij} \sim N(0, 1)$ and $c_i \sim U(0, 10)$;
- $\xi_{ij} \sim N(0, 1)$ and $\zeta_i \sim \begin{cases} U(0, 1), & \text{if } i \in I, \\ N(0, 1), & \text{if } i \notin I; \end{cases}$
- $a_{ij} = \begin{cases} |\xi_{ij}|, & \text{if } \xi_{ij} > 0 \text{ for all } j, \\ \xi_{ij}, & \text{otherwise}; \end{cases}$
\[ u_i = \begin{cases} \min(\arg\min f_i, 1.2\zeta_i \cdot \arg\min f_i), & \text{if } i \in I, \\ 5\zeta_i, & \text{if } i \notin I; \end{cases} \]
\[ l_i = u_i - 0.05|u_i| - 5|\eta_i|, \text{ with } \eta_i \sim N(0, 1); \]
\[ b \sim U(g(l), g(u)). \]

The rules used to generate \( a_{ij}, u_i \) and \( l_i \) ensure that \( f_i \) is decreasing on \([l_i, u_i]\), while also allowing for \( u_i \) to be a critical point of \( f_i \) occasionally. In these instances, \( f_i' \) might have multiple inflection points, so initialization of Newton’s method for the breakpoint subproblem (4) is not straightforward.

## 5 Computational results

The procedures of §3 were coded in MATLAB 7.12 and their performance compared on randomly generated instances of the problems described in §4. As noted, most of those problem classes admit some additional algebraic structure that potentially improves the performance of breakpoint search. Therefore, the results below refer to three optimization frameworks: interior point method, general breakpoint search, and specialized breakpoint search. All tests were performed on a dedicated 2×quad-core Intel 64-bit (2.26 GHz) platform with 24GB RAM running CentOS Linux.

We attempted to solve instances of dimension \( 10^k \) for \( k \in \{1, 2, 3, 4, 5, 6\} \) with each of the applicable methods for each problem class. One hundred random instances were generated at each dimension for eight of the ten problem classes. The other two classes are those described in §4.1.3 and §4.2.2, for which we generated 100 instances at each dimension for each value of \( p, r \in \{2, 2.5, 3, 4\} \). Performance differences among these combinations of \( p \) and \( r \) were detectable, but too small to warrant separating out the results for discussion. We therefore report them in aggregate over all \( p \) and \( r \), and remark that the higher values of \( p \) or \( r \) tended to require slightly longer running times than did the smaller values.

Based on preliminary testing, an \emph{a priori} time limit of \( 10^{k-2} \) seconds was imposed on each attempt at solution. The interior point method was run first and never exceeded this time limit. Consequently, for higher dimensional instances on some problem types, the breakpoint searches were limited to at most a factor of ten over the worst runtime for the interior point method on problems of the same type and size.

The results of the tests are presented for each problem class in Figure 3. Because the breakpoint searches often exceeded the given time limits, the graphs consist mainly of the median runtimes. Mean runtimes are drawn separately whenever they can be visually distinguished from the medians on the scale shown; the mean curves are always the upper branches when two curves of the same line type are shown. Moreover, means that include runtimes at their limits are specially marked.

Three notable features stand out in the graphs of Figure 3. First, the interior point method dominates the general breakpoint search whenever the dimension is at least 100. Second, the specialized breakpoint search dominates the general version by roughly an order of magnitude in many cases, but certainly not all (especially at higher dimensions). Third, the advantage of the specialized search over the interior point method tends to decrease with dimension, so that the interior point method emerges as dominant for high-dimensional instances of all but one of the ten problem classes. Table 1 displays the frequency with which such dominance occurs.

Additional information regarding the distribution of solution times for the interior point method is given in Figure 4. Comparing that figure with Figure 3, we see that the variability among running times for the interior point method is less than the typical difference between it and the other two methods. Moreover, Table 2 suggests that running times for the interior point method do not depend too heavily on the type of problem being solved.
Figure 3: Mean and median running times (seconds) for three methods on ten problem classes.

6 Conclusions

By incorporating a particularly efficient linear-system solution in the Newton update, a generic interior point method can perform as well as, or better than, special-purpose methods on convex resource allocation problems. This addresses two of the questions posed by Patriksson in his survey [25]. First, it shows that it is possible to exploit the particular sparsity of resource allocation problems within the setting of a general-purpose optimizer. In particular, we note that this special structure could be readily identified during the pre-processing phase of a general solver. Second, Patriksson notes that the pegging and breakpoint search methods impose assumptions regarding
Table 1: Win percentage of IPM over best breakpoint method

| problem class                  | $n = 10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|-------------------------------|------------|--------|--------|--------|--------|--------|
| lot-sizing                    | 0          | 0      | 0      | 0      | 0      | 0      |
| powers over simplex           | 0          | 0      | 0      | 0      | 0      | 41     |
| p-norm over p-ball            | 0          | 0      | 0      | 0      | 0      | 75     |
| inventory management          | 0          | 0      | 0      | 0      | 1      | 100    |
| reliability investment        | 0          | 0      | 0      | 0      | 78     | 84     |
| resource renewal              | 0          | 0      | 73     | 86     | 100    | 100    |
| p-norm over r-ball            | 3          | 4      | 95     | 99     | 100    | 100    |
| powers s.t. powers            | 0          | 0      | 97     | 100    | 100    | 100    |
| quartic s.t. linear           | 27         | 86     | 100    | 100    | 100    | 97     |
| log-exponential               | 97         | 100    | 100    | 100    | 100    | 100    |

Table 2: Mean IPM solution times (in milliseconds) by dimension

| problem class                  | $n = 10^1$ | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|-------------------------------|------------|--------|--------|--------|--------|--------|
| lot-sizing                    | 8.75       | 9.46   | 20.17  | 125.8  | 1022   | 8045   |
| powers over simplex           | 8.20       | 11.02  | 24.20  | 141.0  | 1168   | 12281  |
| p-norm over p-ball            | 9.07       | 10.81  | 21.86  | 131.1  | 1011   | 9009   |
| inventory management          | 9.32       | 11.28  | 28.08  | 147.7  | 1404   | 11756  |
| reliability investment        | 8.23       | 9.60   | 21.65  | 131.7  | 991    | 10448  |
| resource renewal              | 8.18       | 9.88   | 24.00  | 131.7  | 1118   | 12082  |
| p-norm over r-ball            | 8.87       | 10.76  | 20.80  | 135.1  | 1073   | 9235   |
| powers s.t. powers            | 11.55      | 13.66  | 36.23  | 215.1  | 1693   | 17989  |
| quartic s.t. linear           | 8.69       | 13.58  | 24.49  | 171.2  | 1651   | 19390  |
| log-exponential               | 8.88       | 12.35  | 31.83  | 233.7  | 2807   | 38778  |

domain, monotonicity or strict convexity of the objective and/or constraint functions (cf., Assumptions A1–A3 in §2). Interior point methods require little beyond convexity and second-order continuous differentiability. By operating in the interior of the intervals $[l_i, u_i]$, they are largely unaffected by asymptotes outside those intervals and could potentially even contend with asymptotes at the endpoints. Also, the implementation used here seems to handle a nonlinear inequality constraint just as easily as a linear equality.

Two main directions warrant further investigation. We have provided computational evidence of the interior point method’s usefulness, but we do not address theoretical issues of convergence or complexity. Also, because interior point methods are not easily warm-started, no conclusions are drawn here regarding the use of such methods for applications in which the problem data change slightly from call to call, as happens in a branch-and-bound algorithm.

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Figure 4: Extreme and median running times (seconds) for the interior point method.

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