Some new CAT(0) free-by-cyclic groups

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Abstract

We show the existence of several new infinite families of polynomially-growing automorphisms of free groups whose mapping tori are CAT(0) free-by-cyclic groups. Such mapping tori are thick, and thus not relatively hyperbolic. These are the first families comprising infinitely many examples for each rank of the nonabelian free group; they contrast strongly with Gersten’s example of a thick free-by-cyclic group which cannot be a subgroup of a CAT(0) group.

Let $F_3 = \langle a, b, c \rangle$ denote, with basis, a free group of rank 3. Consider the following automorphisms of $F_3$

$$
\Theta \begin{cases}
a \mapsto a \\
b \mapsto ba \\
c \mapsto ca^2
\end{cases}
\quad
\Phi \begin{cases}
a \mapsto a \\
b \mapsto a^{-1}ba \\
c \mapsto a^{-2}ca^2
\end{cases}
\quad
\Psi \begin{cases}
a \mapsto a \\
b \mapsto ab \\
c \mapsto a^2ca^2
\end{cases}.
$$

In [Ger94], Gersten showed that the mapping torus of $\Theta$, in other words the free-by-cyclic group $F_3 \rtimes_{\Theta} \mathbb{Z}$, cannot act properly by semi-simple isometries on a CAT(0) metric space—in particular, $F_3 \rtimes_{\Theta} \mathbb{Z}$ cannot be a subgroup of a CAT(0) group.

In contrast, the purpose of this note is to show that the mapping tori of $\Phi$, $\Psi$, and indeed all other examples of their kind are themselves CAT(0) groups.

Theorem A. Let $\Phi: F_n \to F_n$ be a polynomially-growing, symmetric automorphism. There exists an integer $k \geq 1$ such that the mapping torus of $\Phi^k$ acts geometrically on a CAT(0) 2-complex. The power $k$ satisfies $k < n!$. If the automorphism $\Phi$ is upper triangular, then the mapping torus of $\Phi$ is a CAT(0) group.

Corollary B. Let $\Phi: F_n \to F_n$ be a polynomially-growing, palindromic automorphism.

There exists an integer $k \geq 1$ such that the mapping torus of $\Phi^k$ acts geometrically on a CAT(0) 2-complex. The power $k$ satisfies $k < n!$. If the automorphism $\Phi$ is upper triangular, then the mapping torus of $\Phi$ is a CAT(0) group.

An automorphism $\Phi: F_n \to F_n$ is polynomially-growing if for all $g \in F_n$, the word length of $\Phi^k(g)$ grows at most polynomially in $k$. Fix a free basis $x_1, \ldots, x_n$ for $F_n$. The automorphism $\Phi$ is symmetric with respect to this basis if it permutes the conjugacy classes of the $x_i$. To wit, in this case there exist words $w_1, \ldots, w_n$ in the $x_i$ such that for each $i$ satisfying $1 \leq i \leq n$, we have $\Phi(x_i) = w_i^{-1}x_jw_i$ for some $j$ satisfying $1 \leq j \leq n$. Given a word $w$ in our free basis, write $\bar{w}$ for the reverse of $w$, e.g. we have $x_1x_2 = x_2x_1$. The automorphism $\Phi$ is palindromic with respect to the basis $x_1, \ldots, x_n$ if for each $i$ satisfying $1 \leq i \leq n$, we have $\Phi(x_i) = \bar{w}_ix_jw_i$ for some $j$ satisfying $1 \leq j \leq n$. In particular, elements of our free basis are sent to palindromes—words spelled the same forwards and backwards. Finally, in both of the above cases, the automorphism is upper triangular when we always have $i = j$, and for each $i$ satisfying $1 \leq i \leq n$, the word $w_i$ may be spelled using only the basis elements $x_1, \ldots, x_i$.

Corollary B is a corollary of the following theorem.
Theorem C. Let $A$ be a finite group, let $W_n = A * \cdots * A$ denote the free product of $n$ copies of $A$, and let $\Phi: W_n \to W_n$ be a polynomially-growing automorphism. There exists an integer $k \geq 1$ such that the mapping torus of $\Phi^k$ acts geometrically on a CAT(0) 2-complex.

As the similarities and distinctions between the automorphisms $\Theta$, $\Phi$ and $\Psi$ above illustrate, free-by-cyclic groups form a varied and interesting class of finitely-presented groups whose properties remain far from completely understood. For instance, it is not known in general when a free-by-cyclic group admits a geometric action on a CAT(0) space. For a long time it was thought that perhaps a free-by-cyclic group would provide the first example of a hyperbolic group that cannot act geometrically on a CAT(0) space. Recently Hagen and Wise [HW16, HW15] showed that in fact hyperbolic free-by-cyclic groups act geometrically on CAT(0) cube complexes, and are thus virtually special. The free-by-cyclic groups we consider are not relatively hyperbolic: Hagen [Hag19] notes that a result of Macura [Mac02] implies that mapping tori of polynomially-growing automorphisms are thick in the sense of Behrstock–Drutu–Mosher [BDM09], and thick groups are not nontrivially relatively hyperbolic. A famous theorem originally due to Gautero and Lustig [GL07] and independently given new proofs by Ghosh [Gho18] and Dahmani–Li [DL19] says that free-by-cyclic groups are hyperbolic relative to a canonical collection of thick free-by-cyclic subgroups. These are the subgroups of polynomial growth defined in [Lev09].

The question of which free-by-cyclic groups are CAT(0) remains an interesting open problem in general [Bri]. This paper represents a major contribution to this question when the rank of the free group is allowed to vary and the free-by-cyclic group is assumed to be thick. Every $F_2$-by-$\mathbb{Z}$ group can be represented as a non-positively curved punctured-torus bundle over the circle, so every $F_2$-by-$\mathbb{Z}$ group is CAT(0). In fact, Button and Kropholler [BK16] proved that every $F_2$-by-cyclic group is the fundamental group of a non-positively curved cube complex of dimension 2.

Brady and Soroko ask whether a free-by-cyclic group is CAT(0) if and only if it is virtually special [BS19]. Our CAT(0) spaces, while 2-dimensional, are in general not cube complexes, so a reader interested in Brady–Soroko’s question may wish to investigate the following question.

**Question 1.** May these CAT(0) free-by-cyclic groups be cocompactly cubulated? Is the resulting cube complex virtually (co-)special?

In another direction, Theorem C suggests a more general statement might be true.

**Question 2.** When $W$ is a virtually free group with finite abelianization, are thick $W$-by-cyclic groups CAT(0)?

Our proof of Theorem C is somewhat tailored to the case of free products of copies of a single finite group, but perhaps there is some way to remove this restriction.

The CAT(0) 2-complex we construct is somewhat reminiscent of a graph manifold in construction. One begins at the first level with a torus and progressively attaches cylinders in such a way that the complex remains non-positively curved. This is the space considered in a special case by Samuelson in [Sam06], and the gluing is informed by Levitt’s cyclic hierarchy for thick free-by-cyclic groups. The additional assumptions needed for our theorems assure that the glued object is non-positively curved.

Here is the organization of this note. Gersten’s non-example $\Theta$ is too cute to pass up; in Section 1 we sketch his proof very briefly, explain the construction of Bridson–Haefliger and work an example of Corollary 2 to illustrate the proof of Theorem C. A reader in a great hurry could skip this section and proceed directly to the proof in Section 2.

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1 (Non-)Examples and the Construction

As a warm-up, it will be instructive to give Gersten’s non-example and compare it with an example of Corollary B. In this section we also give an exposition of the construction of the CAT(0) 2-complex.

1.1 Gersten’s non-Example

Gersten’s example concerns the following automorphism Θ: \( F_3 = \langle a, b, c \rangle \rightarrow F_3 \). We write

\[
\begin{align*}
\Theta & : a \mapsto a \\
& : b \mapsto ba \\
& : c \mapsto ca^2
\end{align*}
\]

\( F_3 \rtimes_{\Theta} \mathbb{Z} = \langle a, b, c, t \mid tat^{-1} = a, \ tbt^{-1} = ba, \ tct^{-1} = ca^2 \rangle \)

Gersten’s first observation is to rewrite this group as a double HNN extension of \( \langle a, t \rangle \approx \mathbb{Z}^2 \) with \( b \) and \( c \) as stable letters. He does this by rewriting the relators.

\[
\begin{align*}
tbt^{-1} = ba \rightsquigarrow b^{-1}tb = at \\
tct^{-1} = ca^2 \rightsquigarrow c^{-1}tc = a^2t
\end{align*}
\]

This observation generalizes: every thick free-by-cyclic group has a finite-index subgroup that admits a cyclic hierarchy, a repeated graph-of-groups decomposition with cyclic edge stabilizers and thick free-by-cyclic groups of lower rank as vertex stabilizers, terminating with \( \mathbb{Z}^2 \). Levitt records this fact as [Lev09, Definition 1.1]. This allows for arguments by induction.

We return to Gersten’s proof. Suppose, aiming for a contradiction, that \( F_3 \rtimes_{\Theta} \mathbb{Z} \) acts properly by semi-simple isometries on a CAT(0) metric space \( X \). By the Flat Torus Theorem [BH99, Theorem II.7.1, p. 244], there is an isometrically embedded Euclidean plane \( Y \subset X \). This plane is preserved by \( H = \langle a, t \rangle \), which acts on \( Y \) by translation, and the quotient \( Y/H \) is a 2-torus.

Fix a point \( p \in Y \). The content of the HNN extension is that in \( F_3 \rtimes_{\Theta} \mathbb{Z} \), \( t, at \) and \( a^2t \) are all conjugate, so in the action of \( H \) on \( Y \), these elements have the same translation length. Thus there is a circle of radius \( d(p, t.p) \) in \( Y \) centered at \( p \) that meets the points \( t.p, at.p \) and \( a^2t.p \). But on the other hand, these three points lie on an axis for the translation action of \( a \). But in Euclidean geometry, a straight line cannot meet a circle in three points. See Figure 1.

![Figure 1: Gersten’s example would force a line to intersect a circle in three points.](image)

The lesson here is that in order for an HNN extension of a CAT(0) group with cyclic associated subgroups to be a CAT(0) group, there must be a “good reason” for the generators of the associated cyclic subgroups to have the same translation length.
1.2 Bridson–Haefliger’s Construction

Bridson and Haefliger give a general construction providing a sufficient condition for an HNN extension of a CAT(0) group to be a CAT(0) group. Because the geometry of this space will be important to our arguments, we describe their construction in the setting where the associated subgroups are infinite cyclic.

**Theorem 1** ([BH99] Proposition II.11.21, p. 358). Let \( H \) be a group acting properly and cocompactly on a CAT(0) space \( X \). Given \( x \) and \( y \) infinite order elements of \( H \) whose translation lengths on \( X \) are equal, there is a CAT(0) space \( Y \) on which the HNN extension \( G = H \ast_{x \sim y} \) acts properly and cocompactly.

Informally, the construction proceeds by “blowing up” the Bass–Serre tree \( T \) for the HNN extension. For each vertex of \( T \), \( Y \) contains an isometric copy of \( X \). When two vertices share an edge, there is a strip, that is, a space \( S := \mathbb{R} \times [0,1] \) glued in, with \( \mathbb{R} \times \{0\} \) glued to one copy of \( X \) along an axis for \( x \), and \( \mathbb{R} \times \{1\} \) glued to another copy of \( X \) along an axis for \( y \). See Figure 2.

In the case where \( X \) is the universal cover of a space with fundamental group \( H \), one might imagine attaching a cylinder to \( X/H \) with one end attached along a loop representing \( x \) and the other along a loop representing \( y \) in \( \pi_1(X/H) \). If this is done carefully, the universal cover of the resulting space is \( Y \).

Formally, fix geodesic axes \( \gamma \) and \( \eta \) for the actions of \( x \) and \( y \) on \( X \), respectively. We think of \( \gamma \) and \( \eta \) as isometric embeddings of \( \mathbb{R} \) into \( X \). Let \( \alpha \) be the translation length of both \( x \) and \( y \) in \( X \). Recall that the vertices of the Bass–Serre tree \( T \) correspond to cosets of \( H \) in \( G \) and edges of \( T \) correspond to cosets of \( K = \langle x \rangle \). The vertices \( gH \) and \( gtH \) are connected by the edge \( gK \) in \( T \). Let \( K \) act on \( S \) by translation by \( \alpha \) in the first factor. The CAT(0) space \( Y \) is a quotient of the disjoint union \( G \times X \cup G \times S \) by the equivalence relation generated by the following, where \( g \in G \), \( h \in H \), \( x \in K \), \( p \in X \), \( t \in \mathbb{R} \) and \( \theta \in [0,1] \).

1. \((gh,p) \sim (g,h,p)\)
2. \((gx,t,\theta) \sim (g,x,t,\theta)\)
3. \((g,\gamma(t)) \sim (g,t,0)\)
4. \((gt,\eta(t)) \sim (g,t,1)\)

The group \( G \) acts on \( Y \) by multiplication in the labels, and it is easy to see that \( Y \) contains distinct, isometrically embedded copies of \( X \) for each coset \( G/H \), and likewise for copies of \( S \) indexed by the cosets \( G/K \).

1.3 A Palindromic Automorphism of \( F_3 \)

Let \( W_n \) be the free product of \( n \) copies of a cyclic group of order 2,

\[
W_n = C_2 \ast \cdots \ast C_2 = \langle a_1, \ldots, a_n \mid a_i^2 = 1, 1 \leq i \leq n \rangle.
\]

We have a homomorphism \( W_n \rightarrow C_2 \) sending each \( a_i \) to the generator of \( C_2 \). This map splits by sending \( C_2 \) to \( a_1 \), and the kernel is free of rank \( n-1 \), so \( W_n = F_{n-1} \rtimes C_2 \). A free basis for the kernel is \( a_1 a_2, \ldots, a_1 a_n \). In our example, \( n = 4 \); we will write \( F_3 = \langle x, y, z \rangle \), and write \( a \) for the generator of the \( C_2 \) factor. We have \( a^{-1} xa = x^{-1} \), and similarly for \( y \) and \( z \). Automorphisms of \( F_{n-1} \) that commute with the conjugation action of \( a \) send basis elements to palindromes. Consider the following palindromic automorphism of \( F_3 \).

\[
\Phi \begin{cases}
  x \mapsto x \\
y \mapsto xyx \\
z \mapsto yzy
\end{cases}
\]

\[
F_3 \rtimes \Phi, Z = \langle x, y, z, t \mid [x,t] = 1, (x^{-1}t)^y = xt, (y^{-1}t)^z = yt \rangle
\]
We use the exponential notation for conjugation $x^y = y^{-1}xy$. Let $b = ax$, $c = ay$, and $d = az$ be the generators for $W_4$ as a free product of finite groups. Setting $\Phi(a) = a$, we get an automorphism, still called $\Phi$

$$
\Phi = \begin{cases}
  a &\mapsto a \\
  b &\mapsto b \\
  c &\mapsto bacab \\
  d &\mapsto cadac.
\end{cases}
$$

Our aim is to inductively apply Theorem 1 to show that $F_3 \rtimes \Phi \mathbb{Z}$ is the fundamental group of a CAT(0) 2-complex. Along the way we will also show the complex admits a compatible action of $a$, so the resulting mapping torus, $W_4 \rtimes \Phi \mathbb{Z}$, acts properly and cocompactly on the same space. Above we have rewritten the presentation for $F_3 \rtimes \Phi \mathbb{Z}$ to make the hierarchy clearer. Write $G_0 = \langle x,t \rangle \cong \mathbb{Z}^2$, $G_1 = G_0 \ast_{(xt)^{y^{-1}=x^{-1}}} (xt)$, and $G_2 = G_1 \ast_{(yt)^{y^{-1}=y^{-1}}} (yt) = F_3 \rtimes \Phi \mathbb{Z}$. Write $K_0 = \langle xt \rangle$ and $K_1 = \langle yt \rangle$, respectively.

**Step One.** The first CAT(0) space, $X_0$, for $G_0$ to act on is the Euclidean plane by translation. Letting $\vec{x}$ and $\vec{t}$ be the translation vectors for $x$ and $t$, notice that the translation lengths of $xt$ and $x^{-1}t$ are equal to the lengths of the diagonals of the parallelogram determined by $\vec{x}$ and $\vec{t}$. This implies that $xt$ and $x^{-1}t$ have the same translation length exactly when $\vec{x}$ and $\vec{t}$ are orthogonal.

Choosing $X_0$ so that $\vec{x}$ and $\vec{t}$ are orthogonal, $X_0$ admits an isometric action of $a$ by reflection across a fixed geodesic axis for $t$. Choose an axis $\gamma$ for $xt$ and $\eta := a, \gamma$ for $x^{-1}t$. With this data, we apply Theorem 1 to yield a new CAT(0) space $X_1$ on which $G_1$ acts properly and cocompactly.

**Step Two.** We extend $a$ to an isometry of $X_1$: $a$ acts on the copy of $X_0$ corresponding to the identity coset of $G_1/G_0$ as in the previous paragraph. If $h \in G_0$, $a$ takes $h, \gamma$ to $h^a, \eta$, and vice versa, so we extend our definition of $a$ so that it swaps the associated strips $S = \mathbb{R} \times [0,1]$ and sends $(s, \theta)$ to $(s, 1 - \theta)$. More generally, $a$ takes $gG_0 \times X_0$ to $g^aG_0 \times X_0$, takes $gK_0 \times S$ to $g^a y^{-1} K_0$ and acts as above in the $X_0$ and $S$ factors. One checks that because $a$ is an isometry of the pieces and respects the gluing, this defines an isometry of $X_1$. 
Step Three. This done, notice that $(yt)^n = y^{-1}t$, so these elements must have the same translation length in $X_1$. Now we repeat: applying Theorem 1 one more time yields a CAT(0) space on which $G_2$ acts properly and cocompactly. In fact, an identical argument as above allows us to again extend $a$ to an isometry of $X_2$, as desired.

Thus we see that our example satisfies the conclusions of Corollary B and Theorem C.

2 Proof of the Main Theorem

Let $A$ be a finite group or $\mathbb{Z}$, and write $W_n$ for the free product of $n$ copies of $A$. We are interested in polynomially-growing automorphisms $\Phi: W_n \to W_n$ which permute the conjugacy classes of the $A$ in $W_n$. (This is automatic if $A$ is finite, and is the assumption that $\Phi$ is symmetric if $A = \mathbb{Z}$.) The key technical lemma we need is the following.

Lemma 2. Given a polynomially-growing automorphism $\Phi: W_n \to W_n$ as above, there is an automorphism $\Phi': W_n \to W_n$ in the same outer class as $\Phi$ which is a root of an automorphism which is upper triangular with respect to some free product decomposition of $W_n$.

Let us recall that if $\Phi$ and $\Phi'$ define the same outer automorphism of $W_n$, then $W_n \rtimes_{\Phi'} \mathbb{Z}$ and $W_n \rtimes_{\Phi} \mathbb{Z}$ are isomorphic, so there is no loss in passing from one to the other. Upper triangular here means that there is a free product decomposition $W_n = B_1 \ast \cdots \ast B_n$, where each $B_i$ is conjugate to one of the original $A$, and there exist $w_i \in B_1 \ast \cdots \ast B_{i-1}$ such that $\Phi(b_i) = w_i^{-1}b_iw_i$ for $b_i \in B_i$.

Assuming the lemma for now, we prove the main theorems.

Proof of Theorem A Suppose $\Phi: F_2 \to F_n$ is a polynomially-growing symmetric automorphism. Then after replacing $\Phi$, Lemma 2 yields a basis $x_1, \ldots, x_n$ for $F_n$ and an automorphism $\Phi: F_n \to F_n$ which is upper-triangular. We have

$$F_n \rtimes_{\Phi} \mathbb{Z} = \langle x_1, \ldots, x_n, t \mid tx_kt^{-1} = w_k^{-1}x_kw_k \rangle,$$

where each $w_k \in \langle x_1, \ldots, x_{k-1} \rangle$, and $w_1 = 1$. Note that the relation $tx_kt^{-1} = w_k^{-1}x_kw_k$ can be rewritten as $x_k^{-1}w_ktx_k = w_kt$, yielding a hierarchy for $F_n \rtimes_{\Phi} \mathbb{Z}$ as an iterated HNN extension. The first stage is the base group $\langle x_1, t \rangle \cong \mathbb{Z}^2$. At each stage, the hypothesis of Theorem 1 are obviously satisfied, so iteratingly applying Theorem 1 beginning with any geometric action of $\langle x_1, t \rangle$ on the Euclidean plane proves the result.

To prove Theorem C we need a bit of group theory. Let $A$ be a finite group and write $W_n$ for the free product of $n$ copies of $A$. If $a \in A$, write $a_i$ for $a$ in the $i$th free factor. There is a surjective homomorphism $W_n \to A$ sending each $a_i$ to $a$. This map splits: send $a \in A$ to $a_1$. The kernel is free of rank $(|A| - 1)(n - 1)$, a free basis is given by

$$a_1^{-1}a_2, \ldots, a_1^{-1}a_n$$

as $a \in A \setminus \{1\}$ varies. Thus $W_n \cong F \rtimes A$ for a free group $F$.

Proof of Theorem C Fix a finite group $A$. We proceed by induction on $n$, the Kurosh rank of $W_n$. The base case is $n = 2$. Since upper triangular automorphisms of $W_2$ are inner, we may consider the action on the (metric) product $T \times \mathbb{R}$, where $T$ is a regular $|A|$-valent tree on which $W_2$ acts geometrically with two orbits of vertices.

So assume that for $k < n$ and for all polynomially-growing automorphisms $\Phi: W_k \to W_k$, the conclusions of the theorem hold. By Lemma 2 we may without loss of generality assume that $\Phi$ is upper triangular. We assume (perhaps after replacing $\Phi$ by a power) that the mapping torus of $\Phi|_{W_1 \ast \cdots \ast W_{n-1}}$ acts properly and cocompactly on a CAT(0) 2-complex $X$.

Recall our notation from above: for $a \in A$, we write $a_i$ for the image of $a$ in $A_i$. We have $W_n = F \rtimes A$, where $F = \langle a_1^{-1}a_i \mid 2 \leq i \leq n \text{ and } a \in A \setminus \{1\} \rangle$. By Lemma 2 there
is \( w_n \in A_1 \ast \cdots \ast A_{n-1} \) such that \( \Phi(a_n) = w_n^{-1} a_n w_n \) for all \( a_n \in A_n \). We may also assume \( \Phi(a_1) = a_1 \) for all \( a_1 \in A_1 \). We need that \( w_n \in F \cap W_{n-1} \). This can be arranged by composing \( \Phi \) with the inner automorphism corresponding to conjugation by some \( a \in A_1 \). This does not change the isomorphism type of the mapping torus of \( \Phi|_{A_1 \ast \cdots \ast A_{n-1}} \), and we continue to work with \( X \).

If \( A \) is not abelian, it may no longer be the case that \( \Phi(a_1) = a_1 \) for all \( a_1 \in A_1 \). Restore this property by replacing \( \Phi \) by a power.

This done, notice that \( ta_1^{-1} a_n t^{-1} = a_1^{-1} w_n^{-1} a_n w_n = (w_n^{-1})^{a_1} a_1^{-1} a_n w_n \). Recall our exponential notation \( x^y = y^{-1}xy \). This implies that \( a_1^{-1} a_n \), thought of as the stable letter for our HNN extension, conjugates \( (w_n t)^{a_1} \) to \( w_n t \). Since \( a_1 \) is an isometry of \( X \), as in Section \([3]\), we may apply Theorem \([1]\) for each \( a_1 \in A_n \). We do this by first fixing an axis \( \gamma \) for the action of \( w_n t \) on \( X \), and then using \( a_1, \gamma \) as \( a_1 \in A_1 \) varies to work as the geodesic axes of interest. This yields a CAT(0) space \( Y \) that \( F \times \mathbb{Z} \) acts on geometrically. We check that once again, the isometric actions of \( a_1 \in A_1 \) on \( X \) also extend to isometries of \( Y \), proving the claim. 

**Proof of Corollary**\([2]\) As we saw in the example in Section \([1]\) palindromic automorphisms of free groups arise as the restriction to a finite-index free subgroup of automorphisms \( \Phi : W_n \to W_n \) in the case where \( A \) is cyclic of order two.

### 2.1 Train Track Maps

To complete the proof, it remains to prove Lemma\([2]\). In this subsection we assume knowledge of the relative train track maps of \([BH92]\) as generalized to graphs of groups in \([Lym20]\).

Let \((\Gamma_n, \mathcal{G}_n)\) denote the following graph of groups. The graph \( \Gamma_n \) has \( n+1 \) vertices and \( n \) edges: one vertex has valence \( n \) and the \( n \) edges connect this vertex, call it \( * \), to the remaining \( n \) vertices. The graph of groups structure \( \mathcal{G}_n \) assigns the trivial group to the vertex \( * \) and to the edges and assigns the group \( A \) to each vertex. Identify the fundamental group \( \pi_1(\Gamma_n, \mathcal{G}_n, *) \) with \( W_n \).

Because the automorphism \( \Phi \) preserves the conjugacy classes of the \( A \) in \( W_n \), the automorphism \( \Phi \) can be realized as a map \((\Gamma_n, \mathcal{G}_n, *) \to (\Gamma_n, \mathcal{G}_n, *)\) in the sense of \([Lym20]\) Chapter 2] as a subdivision followed by a morphism of graphs of groups. Therefore by \([Lym20]\) Theorem 3.9.3, which uses the algorithm of \([BH92]\), there is a relative train track map \( f : (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G}) \) representing the outer class \( \varphi \) of \( \Phi \) in \( \text{Out}(W_n) \). Because \( \Phi \) is polynomially-growing by assumption, the irreducible strata of \( f \) have Perron-Frobenius eigenvalue \( \lambda = 1 \). By passing to an iterate of \( f \), we may assume each irreducible stratum consists of a single edge \( E \), and by subdividing and choosing orientations, we may assume that \( f(E) = Eu \), where \( u \) is a path in lower strata of \( (\Gamma, \mathcal{G}) \).

The graph of groups \((\Gamma, \mathcal{G})\) has \( n \) vertices with stabilizer \( A \). Since \( \Gamma_n \) was a tree, \( \Gamma \) is a tree. Each leaf of \( \Gamma \) is one of the \( n \) vertices with stabilizer \( A \). Having passed to an iterate, each such vertex is fixed by \( f : (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G}) \), and the action of \( f \) induces an automorphism of \( A \). By inspecting the action of \( f \) on these vertices, we see the iterate required is bounded by \( n! \). By passing to a further iterate, since \( \text{Aut}(A) \) is finite, we may assume this automorphism is the identity for each such vertex. This latter step is not needed for Theorem \([A]\) nor Corollary \([B]\).

**Proof of Lemma**\([2]\) The idea is to use the inductive hypothesis of \([Lev09]\) Definition 1.1]. Namely, consider the top stratum of \((\Gamma, \mathcal{G})\). It is irreducible and thus consists of a single edge \( E \). If \( E \) separates \( \Gamma \), collapsing the complement of \( E \) determines a free product decomposition \( W_n = G_1 \ast G_2 \), and basing the fundamental group at the initial vertex of \( E \) provides a lift of \( f \) to an automorphism of \( \pi_1(\Gamma, \mathcal{G}) \), call it \( \Phi \), satisfying \( \Phi(G_i) = G_i \).

If \( E \) does not separate \( \Gamma \), then the initial vertex of \( E \) is a leaf of \( \Gamma \) and thus one of the \( n \) vertices with stabilizer \( A \). Base the fundamental group of \((\Gamma, \mathcal{G})\) as the terminal vertex of \( E \), call it \( v \), and choose a path \( \sigma \) in \( \Gamma \setminus E \) from \( v \) to \( f(v) \). Collapsing the complement of \( E \) determines a free product decomposition \( W_n = G_1 \ast A \), and the action of \( f \) on \( \pi_1(\Gamma, \mathcal{G}, v) \)
as $\gamma \mapsto \sigma f(\gamma)\sigma$ defines an automorphism $\Phi$ satisfying $\Phi(G_1) = G_1$ and $\Phi(A) = w^{-1}Aw$ for some $w \in G_1$. The proof follows by induction on $n$. 

References

[BDM09] Jason Behrstock, Cornelia Drutu, and Lee Mosher. Thick metric spaces, relative hyperbolicity, and quasi-isometric rigidity. *Math. Ann.*, 344(3):543–595, 2009.

[BH92] Mladen Bestvina and Michael Handel. Train tracks and automorphisms of free groups. *Ann. of Math. (2)*, 135(1):1–51, 1992.

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[BK16] J. O. Button and R. P. Kropholler. Nonhyperbolic free-by-cyclic and one-relator groups. *New York J. Math.*, 22:755–774, 2016.

[Bri] Martin R. Bridson. Problems concerning hyperbolic and CAT(0) groups. Available at https://docs.google.com/file/d/0B-tup63120-GVZqLFIteEJmMmc/edit.

[BS19] Noel Brady and Ignat Soroko. Dehn functions of subgroups of right-angled Artin groups. *Geom. Dedicata*, 200:197–239, 2019.

[DL19] François Dahmani and Ruoyu Li. Relative hyperbolicity for automorphisms of free products. Available at arXiv:1901.06760 [math.GR], January 2019.

[Ger94] S. M. Gersten. The automorphism group of a free group is not a CAT(0) group. *Proc. Amer. Math. Soc.*, 121(4):999–1002, 1994.

[Gho18] Pritam Ghosh. Relative hyperbolicity of free-by-cyclic extensions. Available at arXiv:1802.08670 [math.GR], February 2018.

[GL07] Francois Gautero and Martin Lustig. The mapping-torus of a free group automorphism is hyperbolic relative to the canonical subgroups of polynomial growth. Available at arXiv:0707.0822 [math.GR], July 2007.

[Hag19] Mark Hagen. A remark on thickness of free-by-cyclic groups. *Illinois J. Math.*, 63(4):633–643, 2019.

[HW15] Mark F. Hagen and Daniel T. Wise. Cubulating hyperbolic free-by-cyclic groups: the general case. *Geom. Funct. Anal.*, 25(1):134–179, 2015.

[HW16] Mark F. Hagen and Daniel T. Wise. Cubulating hyperbolic free-by-cyclic groups: the irreducible case. *Duke Math. J.*, 165(9):1753–1813, 2016.

[Lev09] Gilbert Levitt. Counting growth types of automorphisms of free groups. *Geom. Funct. Anal.*, 19(4):1119–1146, 2009.

[Lyn20] Rylee Alanza Lyman. *Train Tracks on Graphs of Groups and Outer Automorphisms of Hyperbolic Groups*. ProQuest LLC, Ann Arbor, MI, 2020. Thesis (Ph.D.)–Tufts University.

[Mac02] Nataša Macura. Detour functions and quasi-isometries. *Q. J. Math.*, 53(2):207–239, 2002.

[Sam06] Peter Samuelson. On CAT(0) structures for free-by-cyclic groups. *Topology Appl.*, 153(15):2823–2833, 2006.