On some finite dimensional complex representations of mapping class groups and Fox derivation

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2019 May 18

Abstract

We study the finite dimensional complex representations of the mapping class group $\mathcal{M}_{g,1}$ that are derived from some finite Galois coverings of the compact oriented surface with one boundary component $\Sigma_{g,1}$. The key ingredients are Fox derivation, Magnus modules and the Skolem-Noether theorem, which enable us to compute the $\mathcal{M}_{g,1}$-action on the module $L_\eta$ very explicitly, where $\eta$ is a primitive $p$th root of unity for an odd prime $p$.

1 Introduction

Let $F_m$ be the free group of rank $m$. Suppose that a subgroup $\mathcal{M} \subset \text{Aut}(F_m)$ is given. Let $N \subset F_m$ be a finite index normal subgroup preserved by the $\mathcal{M}$-action. Set the quotient group $G := F_m/N$ and denote the quotient map by $\rho_N : F_m \rightarrow G$. Then $N^{ab} \otimes_{\mathbb{Z}} \mathbb{C}$ affords a finite dimensional complex representation of $G \rtimes \mathcal{M}$, where $N^{ab}$ is the abelianization of $N$. Obviously this construction has great generality and gives rise to a plenty of finite dimensional representations (modules) of the subgroups of $\text{Aut}(F_m)$. It seems important to study their basic properties such as irreducibility, indecomposability, composition series and so on.

In the setting above, what Magnus showed in [2] is that there exists an embedding

$$F_m/[N,N] \hookrightarrow L_N \rtimes G : a \mod [N,N] \mapsto (D_N(a), a \mod N) \quad (a \in F_m)$$

, where $D_N : F_m \rightarrow L_N$ is a crossed homomorphism and $L_N$ a free left $\mathbb{Z}[G]$-module of rank $m$. We call $D_N$ Fox derivation and $L_N$ Magnus module in the present paper.

From a topological view point, $N^{ab}$ is nothing but $H_1(F_m;\mathbb{Z}[G])$ the 1st homology group of the classifying space $BF_m$ with the coefficient $\mathbb{Z}[G]$, which we regard as a right $G$-module. We have a canonical chain complex to compute $H_* (F_m;\mathbb{Z}[G])$ $(*) = 0, 1)$. In fact, there exists an exact sequence as follows;

$$0 \rightarrow N^{ab} \rightarrow L_N \rightarrow \mathbb{Z}[G] \rightarrow \mathbb{Z} \rightarrow 0.$$
The advantage of the chain complex \( 0 \to LN \to \mathbb{Z}[G] \to 0 \) is that it affords a canonical \( \mathcal{M} \)-action, which is very suitable for our purpose.

In our perspective, \( LN \) has a redundancy in the sense that, although we are merely interested in \( \mathcal{M} \)-modules, it is a \( G \rtimes \mathcal{M} \)-module. Occasionally, we can avoid this subsidiary \( G \)-action by applying the trick described below; Let \( S = e_S \mathbb{C}[G] \) be a simple component of the group algebra \( \mathbb{C}[G] \), where \( e_S \in S \) is the central idempotent corresponding to \( S \). Assume that \( S \) is preserved under the \( \mathcal{M} \)-action. Then we see that \( e_S LN \subset LN \) is a \( \mathcal{M} \)-submodule. The Skolem-Noether theorem implies that there exists a \( \mathbb{C} \)-vector space \( V_S \) that affords a projective \( \mathcal{M} \)-representation such that \( V_S^* \otimes_{\mathbb{C}} V_S \cong S \) as \( \tilde{\mathcal{M}} \)-module, where \( \tilde{\mathcal{M}} \) is a central extension of \( \mathcal{M} \). Associated with it, we can construct a \( \tilde{\mathcal{M}} \)-module \( LN,S \) such that \( e_S LN \cong V_S^* \otimes_{\mathbb{C}} LN,S \), where the latter is endowed with diagonal \( \tilde{\mathcal{M}} \)-action. Notice that \( \tilde{\mathcal{M}} \)-action on \( V_S^* \) factors through a central extension of the finite group \( \text{Aut}(G) \). It seems that we should study \( LN,S \) prior to \( e_S LN \) since \( LN,S \) must reflect more directly aspects concerning the complexity of the infinite group \( \mathcal{M} \).

Let \( \mathcal{M}_{g,1} \) be the mapping class of the compact oriented surface \( \Sigma_{g,1} \) of genus \( g \) with one boundary component. This consists of the equivalence classes of the diffeomorphisms of \( \Sigma_{g,1} \) that fix \( \partial \Sigma_{g,1} \) pointwise, with the equivalence relation being determined by isotopy. Now pick a base point \( \star \) on the boundary. Since \( \Sigma_{g,1} \) is homotopically equivalent to the bouquet of \( 2g \) circles, the fundamental group \( \pi(\Sigma_{g,1},\star) \) can be identified with the free group \( F_{2g} \). Then we see that

\[
\mathcal{M}_{g,1} \cong \langle h \in \text{Auto}(F_n) \mid h(s) = s \rangle
\]

, where \( s \in \pi(\Sigma_{g,1},\star) \) is the homotopy class of the boundary loop based at \( \star \) which travels in the direction compatible with the chosen orientation of \( \Sigma_{g,1} \). It is known that the braid group \( \text{Br}_{2g+1} \) is embedded into \( \mathcal{M}_{g,1} \), which by restriction gives rise to a functor from the category of \( \mathcal{M}_{g,1} \)-modules to that of \( \text{Br}_{2g+1} \)-modules. In the present paper, we will utilize this functor to study some finite dimensional complex representations of \( \mathcal{M}_{g,1} \), the construction of which are given by the above framework applied to the \( \mathcal{M}_{g,1} \)-equivariant surjection \( F_{2g} \to H(2g,p) \). Here \( H(2g,p) \) is a central extension by the cyclic group of order \( p \) of the elementary \( p \)-abelian group of order \( p^{2g} \).

The construction of the present paper is as follows; In Section 2, we provide basic facts on Fox derivation and Magnus modules as preliminaries. In Section 3, we give the general description of the \( \text{Br}_{2g+1} \)-action on Magnus modules. In Section 4, we enter a concrete calculation. We focus on the case where \( G = H(2g,p) \), introduce the \( \text{Br}_{2g+1} \)-module \( L^\eta_g \) and give a detailed description of the \( \text{Br}_{2g+1} \)-action. In Section 5, we prove the main theorem, which is related to the size of the part the braid group action occupies in the endomorphism algebra of the module \( L^\eta_g \).
2 Fox derivation and Magnus embedding

2.1 Fox derivation and automorphisms of free groups

We will provide preliminary results concerning Fox derivation and Magnus modules. In this subsection, we often omit the proof since it is easy or basic. See [11] or [14] for the detail.

Let $F_n := \langle x_1, x_2, \ldots, x_n \rangle$ be the free group on the $n$ letters $x_1, x_2, \ldots, x_n$. For the moment, we will fix a quotient group $G$ of $F_n$. Denote by $\rho_N : F_n \twoheadrightarrow G$ the quotient map, where $N$ stands for the kernel of this map.

**Definition 2.1.** Suppose $M$ to be a left $G$-module. A map $d : F_n \rightarrow M$ is a $G$-crossed homomorphism if

$$d(ab) = d(a) + a \cdot d(b) \quad (a, b \in F_n)$$

, where we regard $M$ as a left $F_n$-module via $\rho_N$.

**Lemma 2.2.** If $f : M \rightarrow M'$ is a left $G$-module homomorphism and if $d : F_n \rightarrow M$ is a $G$-crossed homomorphism, then $f \circ \rho$ is a $G$-crossed homomorphism.

**Lemma 2.3.** For any $G$-crossed homomorphism $d : F_n \rightarrow M$, it holds that

$$d(1) = 1, \quad d(a^{-1}) = -a^{-1}d(a) \quad (a \in F_n).$$

**Lemma 2.4.** If two $G$-crossed homomorphisms $d, d' : F_n \rightarrow M$ coincide on the generator $\{x_1, \ldots, x_n\} \subset F_n$, then $d \equiv d'$.

**Proof.** (Sketch) This follows from an inductive argument with respect to the word length $| \cdot | : F_n \rightarrow \mathbb{N} \cup \{0\}$. \hfill \qed

**Theorem 2.5.** For any left $G$-module $M$ and any $n$ elements $m_1, \ldots, m_n \in M$, there exists a unique $G$-crossed homomorphism $d : F_n \rightarrow M$ such that $d(x_i) = m_i \quad (1 \leq i \leq n)$.

**Proof.** (Sketch) Denote by $W_n$ the set of words on the $n$ letters $x_1, \ldots, x_n$ and $\pi : W_n \twoheadrightarrow F_n = W_n/ \sim$ the quotient map. Construct $\tilde{d} : W_n \rightarrow M$ inductively with respect to the word length so that it satisfies the condition

$$\tilde{d}(w_1w_2) = \tilde{d}(w_1) + \rho_N(\pi(w_1))d(w_2) \quad (w_1, w_2 \in W_n).$$

Check that $\tilde{d}$ factors through $F_n$. \hfill \qed

**Example 2.6.** When we regard $\mathbb{Z}[G]$ as a left $G$-module, we have a unique $G$-crossed homomorphism $d_{\rho_N}$ determined by $d_{\rho_N}(x_i) := \rho_N(x_i) - 1 \quad (1 \leq i \leq n)$. Then it holds that $d_{\rho_N}(a) = \rho_N(a) - 1 \quad (a \in F_n)$. 

3
For any $G$-crossed homomorphism $d : F_n \to M$, set $d_Z : \mathbb{Z}[F_n] \to M$ to be its $\mathbb{Z}$-linear extension. Regarding $M$ as a left $\mathbb{Z}[F_n]$-module, it is readily seen that
\[
d_Z(ab) = d_Z(a)\epsilon(\rho_N(b)) + ad_Z(b) \quad (a, b \in F_n)
\]
, where $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$ is the augmentation map, which is the unique $\mathbb{Z}$-algebra homomorphism determined by the condition $\epsilon(g) = 1$ for any $g \in G$.

**Lemma 2.7.** For any $G$-crossed homomorphism $d : F_n \to M$, the restriction $d|_N : N \to M$ is a group homomorphism. Recall that $G = F_n/N$.

**Proposition 2.8.** For any $G$-crossed homomorphism $d : F_n \to M$, it holds that $[N, N] \subset \text{Ker}(d)$. Recall that $G = F_n/N$.

**Definition 2.9 (Fox derivation and Magnus module [4]).** Consider a normal subgroup $N \trianglelefteq F_n$. Let $\rho_N : F_n \twoheadrightarrow G := F_n/N$ be the quotient map. Define the free left $\mathbb{Z}[G]$-module $L_N$ as
\[
L_N := \bigoplus_{1 \leq i \leq n} \mathbb{Z}[G]e_i
\]
, where $\{e_i \mid 1 \leq i \leq n\}$ is the free basis. We refer to it as the Magnus module of $N$. Define the $G$-crossed homomorphism $D_N : F_n \to L_N$ to be the one uniquely determined by the condition
\[
D_N(x_i) = e_i \quad (1 \leq i \leq n).
\]
Theorem 2.5 ensures the existence of $D_N$ and Lemma 2.4 does the uniqueness. We refer to $D_N$ as the Fox derivation with respect to $N$. Further, we will define the $\mathbb{Z}[G]$-module homomorphism $\partial_N : L_N \to \mathbb{Z}[G]$ by the condition
\[
\partial_N(e_i) = \rho_N(x_i) - 1 \quad (1 \leq i \leq n).
\]

**Proposition 2.10.** For any $a \in F_n$, it holds that
\[
\partial_N \circ D_N(a) = \rho_N(a) - 1.
\]

**Proof.** Lemma 2.2 implies that $\partial_N \circ D_N$ is a $G$-crossed homomorphism. On the other hand, it is readily seen that the map $a \mapsto \rho_N(a) - 1 (a \in F_n)$ is a $G$-crossed homomorphism. By definition, these two coincide with each other on the subset $\{x_1, \ldots, x_n\} \subset F_n$. Therefore, they are identical to each other by Lemma 2.4.

**Corollary 2.11.** As a corollary of the previous 2 propositions, we see that $[N, N] \subset \text{Ker}(D_N)$, $D_N(N) \subset \text{Ker}(\partial_N)$.

Actually, these 2 inclusions are both equalities.

**Theorem 2.12 (Magnus’s Theorem [2]. See also [4]).**

$[N, N] = \text{Ker}(D_N)$, $D_N(N) = \text{Ker}(\partial_N)$. 4
Proposition 2.13. We have the following exact sequences:

\[ 1 \rightarrow N^{ab} \rightarrow L_N \xrightarrow{\partial_N} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0, \quad \mathbb{Z}[F_n] \xrightarrow{(D_N)_{\mathbb{Z}}} L_N \rightarrow 0. \]

We often omit the subscript "\( \mathbb{Z} \)" and denote \((D_N)_{\mathbb{Z}}\) simply by \( D_N \) when there is no fear of confusion.

**Proof.** First we will treat the 1st sequence.

The exactness at the 2nd and the 3rd terms are ensured by Theorem 2.12. As for the 4th term, notice that \( \text{Ker}(\epsilon) = \langle g - 1 \mid g \in G \rangle_{\mathbb{Z}} \). Thus the construction of \( \partial_N \) immediately implies that \( \text{Image}(\partial_N) \subset \text{Ker}(\epsilon) \). On the other hand, Proposition 2.10 implies \( \text{Image}(\partial_N \circ D_N) \supset \text{Ker}(\epsilon) \). The proof for the 5th term is obvious. Thus we are done.

Secondly we will treat the 2nd sequence.

The argument just before ensures that, for any \( m \in L_N \), there exists some \( a \in \mathbb{Z}[F_n] \) such that \( \partial_N \circ D_N(a) = \partial_N(m) \in \text{Ker}(\epsilon) \). Then \( m - D_N(a) \in \text{Ker}(\partial_N) \). The 2nd assertion of Theorem 2.12 implies that there exists some \( n \in N \) such that \( m - D_N(a) = D_N(n) \), which implies that \( m = D_N(n + a) \). Thus we have shown the surjectivity of \( D_N \). \( \square \)

Definition 2.14 (Fundamental sequence, Magnus embedding). The 1st sequence in Proposition 2.13 is referred to as the fundamental sequence associated with \( N \triangleleft F_n \). Its 2nd arrow \( N^{ab} \hookrightarrow L_N \) are referred to as the Magnus embedding associated with \( D_N \).

Remark 2.15. Notice that \( N^{ab} \) is free abelian since \( N \subset F_n \) is a free group. The adjoint action of \( F_n \) on \( N \) induces the action of \( G = F_n/N \) on \( N^{ab} \). Thus \( N^{ab} \) amounts to a \( \mathbb{Z}[G] \)-module, which is free over \( \mathbb{Z} \). Actually the embedding \( N^{ab} \hookrightarrow L_N \) is a \( \mathbb{Z}[G] \)-module homomorphism. See Remark 2.31 below for the detail.

\( N^{ab} \) has an important meaning in terms of group homology;

Proposition 2.16.

\[ N^{ab} = \text{Ker}(\partial_N) \cong H_1(F_n : \mathbb{Z}[G]) \]

, where the coefficient \( \mathbb{Z}[G] \) of the R.H.S. is regarded as a right \( F_n \) module through \( \rho_N \), which gives a local system on the classifying space \( BF_n \). In fact, the sequence

\[ 0 \rightarrow L_N \xrightarrow{\partial_N} \mathbb{Z}[G] \rightarrow 0 \]

is a chain complex to compute the homology group of \( F_n \) with coefficient \( \mathbb{Z}[G] \). The left \( G \)-action on the coefficient \( \mathbb{Z}[G] \) induces the left \( G \)-action on the homology groups.

The Magnus module \( L_N \) has some sort of universal property in the sense stated below;
Theorem 2.17 (Universality of Magnus module). Suppose $M$ to be a left $G$-module and $d : F_n \to M$ a $G$-crossed homomorphism. Then there exist a unique $\mathbb{Z}[G]$-homomorphism $h : L_N \to M$ such that $d = h \circ D_N$.

Proof. • Uniqueness.) The factorization $d_Z = h \circ (D_N)_Z$ together with the surjectivity of $(D_N)_Z$ implies the uniqueness of $h$.
• Existence.) Since $L_N$ is a free $\mathbb{Z}[G]$-module on the basis $\{e_1, \ldots, e_n\}$, the map $e_i \mapsto d(x_i)$, $(1 \leq i \leq n)$ extends uniquely to a $\mathbb{Z}[G]$-module homomorphism $h : L_N \to M$. Then $h \circ D_N$ is a $G$-crossed homomorphism by Lemma 2.2. On the other hand, Lemma 2.4 implies that $d = h \circ D_N$. $\blacksquare$

Now, we will make use of Magnus embeddings for the investigation of $\text{Aut}(F_n)$.

Suppose that a subgroup $M \subset \text{Aut}(F_n)$ is given. Consider a normal subgroup $N \triangleleft F_n$.

Set $G := F_n/N$ the quotient group.

Definition 2.18 (Characteristic subgroup). The normal subgroup $N \triangleleft F_n$ is $M$-characteristic if $\phi(N) = N$ for any $\phi \in M$.

Notation. For any $\phi \in \text{End}(F_n)$ such that $\phi(N) \subset N$, denote by $\overline{\phi} \in \text{End}(G)$ the induced homomorphism.

Proposition 2.19. Suppose that $\phi \in \text{End}(F_n)$ satisfies the condition $\phi(N) \subset N$. Then there exists a unique $\hat{\phi} \in \text{End}_Z(L_N)$ that satisfies the relation $D_N \circ \overline{\phi} = \phi \circ D_N$. Further, it holds that $\hat{\phi}(g m) = \overline{\phi}(g) \hat{\phi}(m)$ ($g \in G$, $m \in M$).

Proof. Define the $\mathbb{Z}[G]$-module $L_N^\phi$ as follows;

$L_N^\phi \overset{\iota}{=} L_N$ as $\mathbb{Z}$-module.

For any $m \in M$, we denote by $m_{\overline{\phi}}$ the corresponding element of $L_N^\phi$ under this identification. We define the $G$-action on $L_N^\phi$ by

$g m_{\overline{\phi}} := \overline{\phi}(g) m$ ($g \in G$, $m \in M$).

Then $\iota^{-1} \circ D_N \circ \overline{\phi} : F_n \to L_N^\phi$ is a $G$-crossed homomorphism. Theorem 2.17 implies that there exists a unique $\mathbb{Z}[G]$-module homomorphism $\hat{\phi} : L_N \to L_N^\phi$ such that $\iota^{-1} \circ D_N \circ \overline{\phi} = \hat{\phi} \circ D_N$. Thus $\hat{\phi} := \iota \circ \hat{\phi}$ is the desired endomorphism. $\blacksquare$

Theorem 2.20 (Induced $G \times M$-action on Magnus module). Let $M \subset \text{Aut}(F_n)$ be a subgroup. Assume that $N \triangleleft F_n$ is $M$-characteristic. Then there exists a uniquely determined action of $M$ on $L_N$ such that the Fox derivation $D_N : F_n \to L_N$ is $M$-equivariant. Further, the structure map $G \times L_N \to L_N$ that determines the left action of $G$ on $L_N$ is $M$-equivariant. In other words, $L_N$ is a left $G \times M$-module by the rule

$(g, f) \cdot m := g \cdot f(m)$ ($g, f) \in G \times M, m \in L_N$).
2.2 A separation criterion

Suppose that a quotient homomorphism \( \rho_N : F_n \to G \) is given. Complexifying the fundamental sequence in Proposition 2.13 we obtain

\[
\mathbb{C}[F_n] \xrightarrow{(\mathcal{D}_N)^C} (L_N)_C \xrightarrow{(\partial_N)^C} \mathbb{C}[G] \hookrightarrow \mathbb{C} \to 0.
\]

**Convention 2.21.** From now on, unless it may cause a confusion, we omit the subscript "\( \mathbb{C} \)" for the simplicity of notation.

For any element \( h \in F_n \), let \( \text{adj}(h) \) be the conjugation action on \( F_n \) determined by \( h \), that is, \( \text{adj}(h)(k) = hkh^{-1} \) \( (k \in F_n) \). We will consider the induced endomorphism \( \hat{\text{adj}}(h) \in \text{End}_{\mathbb{C}}(L_N) \).

**Proposition 2.22 (Adjoint action formula).**

\[
\text{adj}(h)(m) = -\rho_N(h)\partial_N(m)\rho_N(h)^{-1}\mathcal{D}_N(h) + \rho_N(h)m \quad (m \in L_N).
\]

**Proof.** Since \( \mathcal{D}_N : \mathbb{C}[F_n] \to L_N \) is surjective, there exists an \( a \in \mathbb{C}[F_n] \) such that \( m = \mathcal{D}_N(a) \). Then

\[
\text{adj}(h)(m) = \mathcal{D}_N(\text{adj}(h)(a)) = \mathcal{D}_N(hah^{-1})
\]

\[
= \mathcal{D}_N(h)\epsilon(ah^{-1}) + \rho_N(h)\mathcal{D}_N(a)\epsilon(h^{-1}) - \rho_N(hah^{-1})\mathcal{D}_N(h)
\]

\[
= \rho_N(h)\{\epsilon(a) - \rho_N(a)\}\rho_N(h)^{-1}\mathcal{D}_N(h) + \rho_N(h)\mathcal{D}_N(a)
\]

\[
= R.H.S.
\]

**Corollary 2.23.** We see that

\[
\text{adj}(h)|_{\ker(\partial_N)_C} = L_{\rho_N(h)}|_{\ker(\partial_N)_C}
\]

, where \( L_g \in \text{End}_C(L_N) \) \( (g \in G) \) denotes the left action of \( g \) on \( L_N \).

**Definition 2.24.** Suppose that \( h \in F_n \) satisfies the condition \( \rho_N(h) \in Z(G) \cdots (\sharp) \). Set

\[
B_h := \text{adj}(h) - 1 \in \text{End}_C(L_N).
\]

**Proposition 2.25.** For any \( h \in F_n \) that satisfies the condition \( (\sharp) \), \( B_h \) belongs to \( \text{End}_{\mathbb{C}[G]}(L_N) \). Further, it holds that

\[
B_h \circ B_h = (L_{\rho_N(h)} - 1) \circ B_h.
\]

**Lemma 2.26.** As a direct corollary of Proposition 2.22, if \( h \in F_n \) satisfies the condition \( (\sharp) \), then we see that

\[
\text{adj}(h)(m) = -(\partial_N(m)\mathcal{D}_N(h) + \rho_N(h)m \quad (m \in L_N).
\]
Proof of Proposition 2.25 Since $h^2$ satisfies the condition (♯), it follows from the previous lemma that

$$(\text{adj}(h))^2(m) = \text{adj}(h^2) = -\partial_N(m)\mathcal{D}_N(h^2) + \rho_N(h^2)m$$

$$= -\partial_N(m)(1 + \rho_N(h))\mathcal{D}_N(h) + \rho_N(h)^2m$$

$$= -(1 + \rho_N(h))\partial_N(m)\mathcal{D}_N(h) + \rho_N(h)^2m.$$ 

Therefore, the 2nd assertion follows from the calculation below;

$$(\text{adj}(h) - 1)^2(m) = (1 - \rho_N(h))\partial_N(m)\mathcal{D}_N(h) + (\rho_N(h) - 1)^2m$$

$$= (\rho_N(h) - 1)\{-\partial_N(m)\mathcal{D}_N(h) + \rho_N(h)m - m\}$$

$$= (\rho_N(h) - 1)\{\text{adj}(h) - 1\}(m).$$

The 1st assertion follows since, for any $g \in G$, we see that

$$\text{adj}(h)(gm) = -\partial_N(gm) + \rho_N(h)gm$$

$$= -g\partial_N(m) + g\rho_N(h)m$$

$$= g\text{adj}(h)(m).$$

Proposition 2.27. Suppose that $h \in F_n$ satisfies the condition (♯). Then

$$B_h(L_N) \subset \text{Ker}\partial_N.$$ 

Proof. For any $a \in \mathbb{Z}[F_n]$, we see that

$$\partial_N \circ B_h(\mathcal{D}_N(a)) = \partial_N \circ \mathcal{D}_N((\text{adj}_h - 1)(a)) = \partial_N(hah^{-1}) - \partial_N(a)$$

$$= \partial_N(a) - \partial_N(a) = 0.$$ 

From now on, we will impose the assumption that $h \in F_n$ satisfies the condition (♯) and that $\rho_N(h) \in G$ is of finite order. The latter ensures that the left multiplication by $\rho_N(h)$ determines a semi-simple endomorphism. Denoted by $L_\lambda \ (\lambda \in \mathbb{C})$ the eigenspace of $L_{\rho_N(h)} \in \text{End}_{\mathbb{C}[G]}(L_N)$ with eigenvalue $\lambda$. Notice that $L_\lambda$ is a $\mathbb{C}[G]$-submodule. Similarly, denote by $M_\zeta \ (\zeta \in \mathbb{C})$ the eigenspace of the (left) multiplication by $\rho_N(h)$ on $\mathbb{C}[G]$ with eigenvalue $\zeta$. Then $M_\zeta$ is an ideal of $\mathbb{C}[G]$.

Proposition 2.28 (Separation criterion). If $\lambda \neq 1$, then we have the following direct sum decomposition as $\mathbb{C}[G]$-module;

$$L_\lambda = \text{Ker}(\partial_N|_{L_\lambda}) \oplus M_\lambda \mathcal{D}_N(h).$$

The 2nd summand of the R.H.S. is a free $M_\lambda$-module with the free basis $\{1_{M_\lambda}, \mathcal{D}_N(h)\}$, where $1_{M_\lambda}$ is the identity element of $M_\lambda$. 

8
Proof. Set \( A := B_\lambda|_{L_\lambda} \), \( \partial_\lambda := \partial_N|_{L_\lambda} \). Then \( A^2 - (\lambda - 1)A = 0 \). Since \( \lambda - 1 \neq 0 \), we have a \( \mathbb{C}[G] \)-module decomposition

\[
L_\lambda = \text{Ker}(A - \lambda + 1) \oplus \text{Ker}(A).
\]

Then Proposition 2.27 implies that

\[
\text{Ker}(A - \lambda + 1) = \text{Image}(A) \subset \text{Ker}(\partial_\lambda).
\]

On the other hand, Lemma 2.26 implies, for any \( m \in L_\lambda \), that

\[
(A - \lambda + 1)(m) = -\partial_\lambda(m)\mathcal{D}_N(h) + ((L_h - 1) - \lambda + 1)(m) = -\partial_\lambda(m)\mathcal{D}_N(h).
\]

Further, the exact sequence

\[
(L_N)_C \xrightarrow{(\partial_N)_C} \mathbb{C}[G] \xrightarrow{\epsilon} \mathbb{C}
\]

induces the exact sequence

\[
L_\lambda \xrightarrow{\partial} M_\lambda \rightarrow 0
\]

, which shows that \( \partial_\lambda(L_\lambda) = M_\lambda \). These two together imply that

\[
\text{Ker}(A) = \text{Image}(A - \lambda + 1) = \partial_\lambda(L_\lambda)\mathcal{D}_N(h) = M_\lambda\mathcal{D}_N(h).
\]

For any \( a \in M_\lambda \), we see that

\[
\partial_\lambda(a\mathcal{D}_N(h)) = a(\rho_N(h) - 1) = a(\lambda - 1)
\]

, which shows that \( M_\lambda\mathcal{D}_N(h) \) is free with the free basis \( \{1_M_\lambda, \mathcal{D}_N(h)\} \) and that \( \partial_\lambda|_{M_\lambda\mathcal{D}_N(h)} \) is injective. It follows that \( \text{Ker}(\partial_\lambda \cap \text{Ker}(A) = \{0\} \), which in turn implies that \( \text{Ker}(A - \lambda + 1) = \text{Ker}(\partial_\lambda) \). Thus we are done. \( \square \)

Remark 2.29. If \( \phi \in \text{Aut}(F_n) \) preservs both \( N \) and \( h \in F_n \), then the direct sum decomposition in Proposition 2.28 is preserved by \( \hat{\phi} \mid_{L_\lambda} \in \text{End}_C(L_\lambda) \).

In the end of this subsection, we add some application of Proposition 2.22.

Proposition 2.30. For any normal subgroup \( K \triangleleft F_n \) such that \( K \subset N \), the image \( \mathcal{D}_N(K) \subset L_N \) is a \( \mathbb{Z}[G] \)-submodule. Recall that \( N \triangleleft F_n \) is such that \( G = F_n/N \).

Proof. Lemma 2.7 implies that \( \mathcal{D}_N(K) \) is a \( \mathbb{Z} \)-submodule. Further, \( \mathcal{D}_N(K) \subset \mathcal{D}_N(N) = \text{Ker}(\partial_N) \). Then Corollary 2.23 together with the assumption \( K \triangleleft F_n \) implies that \( \mathcal{D}_N(K) \) is preserved by \( L_{\rho_N(a)} \) for any \( a \in F_n \). Thus we are done. \( \square \)

Remark 2.31. The reasoning above also shows that the Magnus embedding \( N^{ab} \hookrightarrow L_N \) is a \( \mathbb{Z}[G] \)-module homomorphism.
2.3 Braid Group Action

Henceforth, we fix positive integer $g$.

Let $D$ be a 2-dimensional closed disc with a base point $* \in \partial D$. Orient $D$ so that the induced orientation to $\partial D$ is clockwise. Let $l$ be the simple closed curve based at $*$ that travels on $\partial D$ clockwise. Let the subset $\mathcal{P} \subset D$ comprise $2g + 1$ distinct interior points $p_1, \ldots, p_{2g+1}$. Let $s_i \ (1 \leq i \leq 2g + 1)$ be oriented simple closed curve in $D \setminus \mathcal{P}$ based at $*$ such that

- $s_i \cap \partial D = \{\ast\}$ for $1 \leq i \leq 2g + 1$,
- $s_i \cap s_j = \{\ast\}$ for $(1 \leq i < j \leq 2g + 1)$,
- each $s_i$ encloses $p_i$ clockwise. The open 2-disc in $D$ that bounds $s_i$ does not intersects $\mathcal{P}\setminus\{p_i\}$,
- $s_1 \cdot s_2 \cdot \cdots \cdot s_{2g+1}$ is homotopic to $l$ as a based loop in $D \setminus \mathcal{P}$.

Set $y_i := [s_i] \in \pi_1(\partial D \setminus \mathcal{P}; \ast) \ (1 \leq i \leq 2g + 1)$. Since $D \setminus \mathcal{P}$ is homotopically equivalent to the bouquet of $2g + 1$ circles, the fundamental group $\pi_1(D \setminus \mathcal{P}; \ast)$ is freely generated by $\{y_1, \ldots, y_{2g+1}\}$.

**Definition 2.32.** $T_{2g+1} := \langle y_1, \ldots, y_{2g+1} \rangle \cong \pi_1(D \setminus \mathcal{P}; \ast)$.

The total space of the double branched covering over $D$ that branches at $\mathcal{P}$ is homeomorphic to $\Sigma_{g,1}$, the compact oriented surface of genus $g$ with one boundary component. Thus we denote by $\text{pr} : \Sigma_{g,1} \to D$ the covering map and $\tau$ the generator of the covering transformation group. Choose a base point $\ast \in \text{pr}^{-1}(\ast) \subset \partial \Sigma_{g,1}$. Let $\hat{D}$ be the orbifold determined by the action of $\langle \tau \rangle \cong \{\pm 1\}$ on $\Sigma_{g,1}$. The orbifold fundamental group $\pi_1^{orb}(\hat{D}; \ast)$ is the quotient of $T_{2g+1}$ by the normal closure of $\{y_1^2, \ldots, y_{2g+1}^2\} \subset T_{2g+1}$.

**Definition 2.33.** $H_{2g+1} := \langle y_1, \ldots, y_{2g+1} \mid y_1^2, \ldots, y_{2g+1}^2 \rangle \cong \pi_1^{orb}(\hat{D}; \ast)$.

We have the exact sequence

$$1 \to \pi_1(\Sigma_{g,1}; \ast) \xrightarrow{\text{pr}} \pi_1^{orb}(\hat{D}; \ast) \xrightarrow{\mu} \langle \tau \rangle \to 1,$$

where $\mu$ is determined by $\mu(y_i) = \tau \ (1 \leq i \leq 2g + 1)$. Set $x_i := y_i y_{i+1} \ (1 \leq i \leq 2g)$. Then it is readily seen (e.g. by the Artin-Schreier theory) that $\pi_1(\Sigma_{g,1}; \ast) \cong \text{Ker}(\mu)$ is a free group freely generated by $\{x_1, \ldots, x_{2g}\}$. Henceforth, by abuse of notation, we denote this subgroup by $F_{2g}$. Denote by $\Delta$ the element $y_1 y_2 \cdots y_{2g+1} \in H_{2g+1}$, which is represented by the boundary loop of $\hat{D}$ based at $\ast$ with the clockwise orientation. Since $\mu(\Delta^2) = 1$, $\Delta^2 \in F_{2g}$, which is represented by the boundary loop of $\Sigma_{g,1}$ based at $\ast$ with the clockwise orientation.
Lemma 2.34. A short calculation shows that
\[ \Delta^2 = x_1 x_3 \cdots x_{2g-1} \cdot (x_1 x_2 \cdots x_{2g})^{-1} \cdot x_2 x_4 \cdots x_{2g}. \]

Lemma 2.35. The intersection form on \( H_1(\Sigma_{g,1}; \mathbb{Z}) \cong \langle [x_1], \ldots, [x_g] \rangle \mathbb{Z} \) is described as follows;
\[ [x_i] \cdot [x_j] = \begin{cases} j - i & \text{if } |i - j| = 1, \\ 0 & \text{if otherwise}. \end{cases} \]

Proof. This can be shown e.g. by counting algebraic intersection number of the oriented loops on \( \Sigma_{g,1} \) based at \( * \) that are the lifts of the loops \( s_is_{i+1} \) \( (1 \leq i \leq 2g) \) in \( D \setminus \mathcal{P} \). \( \square \)

Remark 2.36. Set \( a_j \in F_{2g} \) \( (1 \leq j \leq 2g) \) as follows;
\[ a_1 := x_1, \quad a_{2i+1} := (x_2 x_4 \cdots x_{2i})^{-1} \cdot x_1 x_2 \cdots x_{2i+1} \quad (2 \leq i \leq g), \]
\[ a_{2i} := x_{2i}^{-1} \quad (1 \leq i \leq g). \]

Then it is readily seen that
\[ \Delta^2 = [a_1, a_2][a_3, a_4] \cdots [a_{2g-1}, a_{2g}], \]
which implies that \( \{a_j\}_{1 \leq j \leq 2g} \) amounts to the generator of the standard presentation for the fundamental group of the closed orientable surface of genus \( g \). Note that we adopt here the notation that the group bracket \( [a, b] = aba^{-1}b^{-1} \). We can deduce Lemma 2.35 also as a corollary of this remark.

Now, we will introduce the braid group \( Br_{2g+1} \) of \( 2g + 1 \) strings.

Definition 2.37.
\[ Br_{2g+1} := \{ \phi \in \text{Aut}(T_{2g+1}) \mid \phi(\Delta) = \Delta \} \]

It is well-known that \( Br_{2g+1} \) has the following presentation;
\[ Br_{2g+1} = \langle \sigma_i \mid 1 \leq i \leq 2g \rangle : \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad (1 \leq i \leq 2g-1), \quad \sigma_i \sigma_j = \sigma_j \sigma_i \quad (|i-j| > 1) \rangle. \]

Proposition 2.38. The action of \( Br_{2g+1} \) on \( T_{2g+1} \) is described as follows;
\[ \sigma_i(y_j) = \begin{cases} y_{i+1} & \text{if } j = i, \\ y_{i+1}y_jy_{i+1} & \text{if } j = i + 1, \\ y_j & \text{if otherwise}. \end{cases} \]

It is readily seen that \( \sigma_i \) \( (1 \leq i \leq 2g) \) preserves the normal closure \( I_{2g+1} := N_{T_{2g+1}}(\langle y_1^2, \ldots, y_{2g+1}^2 \rangle) \). Thus we have the induced \( Br_{2g+1} \)-action on the quotient \( H_{2g+1} = T_{2g+1}/I_{2g+1} \). Further, the normal subgroup \( F_{2g} = \langle x_1, \ldots, x_{2g} \rangle \triangleleft H_{2g+1} \) is preserved by the induced action. Restricting the \( Br_{2g+1} \)-action to \( F_{2g} \) gives rise to the group homomorphism
\[ \lambda : Br_{2g+1} \rightarrow \mathcal{M}_{g,1} \cong \{ \phi \in \text{Auto}(F_n) \mid \phi(\Delta^2) = \Delta^2 \} \]

Remark 2.39. It is known that \( \lambda \) is injective, which is a consequence of the Birman-Hilden theorem. See [3] and the reference in there.
2.4 An extension to $H_{2g+1}$ of the Fox derivation $D_N$

Taking over the setting of the previous subsection, we have the $Br_{2g+1}$-equivariant exact sequence

$$1 \rightarrow F_{2g} \xrightarrow{\iota} H_{2g+1} \xrightarrow{\mu} \langle \tau \mid \tau^2 \rangle \rightarrow 1.$$ 

Suppose that a $M_{g,1}$-characteristic subgroup $N \trianglelefteq F_{2g}$ is given. At first we will see that $\iota(N)$ is $Br_{2g+1}$-characteristic. Since $\iota(N)$ is $Br_{2g+1}$-invariant, it is sufficient to show that $\iota(N)$ is normal in $H_{2g+1}$. Recall that $H_{2g+1}$ is generated by $F_2 \cup \{\Delta\}$. But $Br_{2g+1}$ includes the adjoint action determined by $\Delta$, which coincides with the generator of $Z(Br_{2g+1}) \cong \mathbb{Z}$. Thus we are done.

Set the quotient groups $G := F_N/N$, $G^+ := H_{2g+1}/\iota(N)$. Let $\rho_N : F_n \rightarrow G$ and $\rho_N^+ : H_{2g+1} \rightarrow G^+$ be the quotient maps respectively. Taking the quotient by $N$ of the sequence above, we obtain the $Br_{2g+1}$-equivariant exact sequence

$$1 \rightarrow G \xrightarrow{\tau} G^+ \xrightarrow{\pi} \langle \tau \mid \tau^2 \rangle \rightarrow 1.$$ 

Set $\tilde{N} := \pi_{2g+1}^{-1}(N)$, where $\pi_{2g+1} : T_{2g+1} \rightarrow H_{2g+1}$ is the quotient map. Then $\tilde{N} \trianglelefteq T_{2g+1}$ is $Br_{2g+1}$-characteristic. Consider the fundamental sequence associated with $\tilde{N} \trianglelefteq T_{2g+1}$, which turns out to be $Br_{2g+1}$-equivariant;

$$T_{2g+1} \xrightarrow{\partial_\tilde{N}} L_{\tilde{N}} \xrightarrow{\partial_\tilde{N}} \mathbb{Z}[G^+] \xrightarrow{\epsilon_{G^+}} \mathbb{Z} \rightarrow 0.$$ 

Recall that $T_{2g+1}$ is a free group. Thus $L_{\tilde{N}}$ is the associated Magnus module and $D_{\tilde{N}}$ the corresponding Fox derivation. See Proposition $2.13$ for the detailed construction. Theorem $2.20$ ensures the existence of the $Br_{2g+1}$-action on $L_{\tilde{N}}$ and the $Br_{2g+1}$-equivariance of $D_{\tilde{N}}$.

**Lemma 2.40.** $D_{\tilde{N}}(I_{2g+1}) \subset L_{\tilde{N}}$ is a $\mathbb{Z}[G^+]$-submodule invariant under the $Br_{2g+1}$-action.

**Proof.** The result follows from Proposition $2.30$ since $I_{2g+1} \subset \tilde{N} = \text{Ker}(\pi_{2g+1})$ and since $I_{2g+1} \subset T_{2g+1}$ is $Br_{2g+1}$-invariant. \hfill $\square$

**Definition 2.41.** Set $L^+_N := L_{\tilde{N}}/D_{\tilde{N}}(I_{2g+1})$. It amounts to a left $G^+ \rtimes Br_{2g+1}$-module due to the previous lemma. Define the $\mathbb{Z}[G^+]$-homomorphism $\partial^+_N : L^+_N \rightarrow \mathbb{Z}[G^+]$ to be the map induced from $\partial_{\tilde{N}}$ on the quotient $L^+_N$. Notice that $\partial_{\tilde{N}}$ factors through $L^+_N$ since $\partial_{\tilde{N}}(D_{\tilde{N}}(I_{2g+1})) \subset \partial_{\tilde{N}}(D_{\tilde{N}}(\tilde{N})) = \{0\}$.

**Proposition 2.42.** We have the $Br_{2g+1}$-equivariant commutative diagram as follows:

$$
\begin{array}{ccccccccc}
H_{g,1} & \xrightarrow{D_{\tilde{N}}^+} & L^+_N & \xrightarrow{\partial^+_N} & \mathbb{Z}[G^+] & \xrightarrow{\epsilon_{G^+}} & \mathbb{Z} & \rightarrow & 0 \\
\downarrow{i} & & \uparrow{i} & & \uparrow{\tau} & & \equiv & & \\
F_{2g} & \xrightarrow{D_N} & L_N & \xrightarrow{\partial_N} & \mathbb{Z}[G] & \xrightarrow{\epsilon_G} & \mathbb{Z} & \rightarrow & 0
\end{array}
$$
where $D^+_N$ is a $G^+$-crossed homomorphism, $i$ a $\mathbb{Z}[G]$-monomorphism. The 2nd row is the fundamental sequence associated with $N \triangleleft F_{2g}$, which is $\mathcal{M}_{g,1}$-equivariant. Further, it holds that $\text{Ker}(D^+_N) = \iota([N, N])$, $D^+_N \circ \iota(N) = \text{Ker}(\partial_N^+)$ and $\text{Image}(\partial_N^+) = \text{Ker}(\epsilon_{G^+})$.

**Proof.**  
- The construction of $D^+_N$.) Denote by $\text{pr} : L_\infty \twoheadrightarrow L^+_N$ the quotient map. Then $\text{pr} \circ D^+_N : T_{2g+1} \twoheadrightarrow L^+_N$ factors through $H_{2g+1} = T_{2g+1}/I_{2g+1}$ since $\text{pr} \circ D^+_N(I_{2g+1}) = 0$. It follows that there exists a unique $G^+$-crossed homomorphism $D^+_N : H_{2g+1} \twoheadrightarrow L^+_N$ such that $D^+_N \circ \pi_{2g+1} = \text{pr} \circ D^+_N$.
- The construction of $\hat{i}$.) Apply the universality property of the Magnus module $L_N$ to the $G$-crossed homomorphism $D^+_N \circ \iota : F_{2g} \twoheadrightarrow L^+_N$. (See Theorem 2.14)
- Commutativity of the diagram.) The commutativity of the leftmost square follows from the construction of $D^+_N$. As for the rectangle with vertical edges $\iota$ and $\hat{\iota}$, the upper and lower horizontal morphisms are given by

$$
H_{2g+1} \ni a \mapsto \rho_N^+(a) - 1 \in \mathbb{Z}[G^+] \\
F_{2g} \ni b \mapsto \rho_N(b) - 1 \in \mathbb{Z}[G]
$$

, which shows the results since $\rho_N^+ \circ \iota = \hat{\iota} \circ \rho_N$. As for the next to the leftmost square, the previous result together with the surjectivity of $(D^+_N)_Z$ and $(D^+_N)_{\mathbb{Z}[G]}$ shows the result.
- Asserted 3 properties.) Theorem 2.12 implies that $\text{Ker}(D^+_N) = [N, N]$, which shows that $(D^+_N)^{-1}(D^+_N(I_{2g+1})) = I_{2g+1}[N, N]$. It follows that $\text{Ker}(D^+_N \circ \pi_{2g+1}) = \text{Ker}(\text{pr} \circ D^+_N) = I_{2g+1}[N, N]$. Since $\pi_{2g+1}$ is surjective, it follows that $\text{Ker}(D^+_N) = \pi_{2g+1}(I_{2g+1}[N, N]) = [\pi_{2g+1}(N), \pi_{2g+1}(N)] = [\iota(N), \iota(N)] = \iota([N, N])$. The 2nd and the 3rd properties follow from the construction of the 1st row, which was constructed by taking some quotient of the corresponding sequence associated with $N \triangleleft T_{2g+1}$.
- The injectivity of $\hat{i}$.) This follows applying the 5-lemma to the commutative diagram below;

![Diagram](image)

, where each row is exact and $\hat{\tau}$ injective. □

**Corollary 2.43.** $L^+_N$ is a free $\mathbb{Z}[G]$-module with $\text{rank}_{\mathbb{Z}[G]}(L^+_N) = \text{rank}_{\mathbb{Z}[G]}(L_N) + 1 = 2g+1$.

**Proof.** Take any $\hat{\tau} \in \mu^{-1}(\tau)$, where $\mu : G^+ \twoheadrightarrow \langle \tau \mid \tau^2 \rangle$. Then we have a left $\mathbb{Z}[G]$-module decomposition

$$
\mathbb{Z}[G^+] = \text{Image}(\iota) \oplus \mathbb{Z}[G](\hat{\tau} - 1)
$$

since $\mathbb{Z}[G^+]$ is a free left $\mathbb{Z}[G]$-module with the free basis $\{1_{\mathbb{Z}[G^+]}, \hat{\tau}\}$. Since $\mathbb{Z}[G](\hat{\tau} - 1) \subset \text{Ker}(\epsilon^+) = \text{Image}(\partial_N^+)$, if we set $K := (\partial_N^+)^{-1}(\mathbb{Z}[G](\hat{\tau} - 1))$, we have the surjection $\partial_N^+: K \twoheadrightarrow \mathbb{Z}[G](\hat{\tau} - 1)$. Since $\mathbb{Z}[G](\hat{\tau} - 1)$ is projective as a $\mathbb{Z}[G]$-module, it follows that there exists a right inverse $\mathbb{Z}[G]$-homomorphism $\lambda : \mathbb{Z}[G](\hat{\tau} - 1) \to K$. On the other hand, the commutative diagram that appeared in the proof of Proposition 2.12 shows...
that \((\partial_N^+)\,^{-1}(\tau(Z[G])) = \tilde{i}(L_N)\). Thus we have the following direct sum decomposition as \(Z[G]\)-module:

\[
L_N^+ = \tilde{i}(L_N) \oplus \text{Image}(\lambda)
\]

, where the 1st summand in the R.H.S. is free of rank \(2g\) since \(\tilde{i}\) is injective and the 2nd summand is free of rank 1 since \(\lambda\) is injective. Thus, we are done. \(\square\)

The argument in the proof of Corollary 2.43 shows the following rather technical result.

**Corollary 2.44.** Let \(\{a_1, \ldots, a_{2g}\}\) be a free \(Z[G]\)-basis of \(L_N\) and \(b \in L_N^+\) satisfy the condition \(\partial_N(b) + 1 \in G^+ \backslash G\). Then \(\{\tilde{i}(a_1), \ldots, \tilde{i}(a_{2g}), b\}\) is a free \(Z[G]\)-basis of \(L_N^+\).

**Definition 2.45.** Set

\[
\triangle := \rho_N^+(\Delta) \in G^+.
\]

**Remark 2.46.** \(\triangle^2 \in G\). \(\triangle^2\) is central in \(G\) if and only if it is central in \(G^+\).

In the end of the present subsection, we will give an application of the separation criterion. (See Proposition 2.28.)

Suppose that \(\triangle^2\) is central in \(G\) (hence in \(G^+\)) and that \(\triangle^{2p} = 1\) where \(p \neq \pm 1\) (not necessarily prime). Then \(\triangle^2\) belongs to the center of \(G^+ \rtimes \text{Br}_{2g+1}\) since the \(\text{Br}_{2g+1}\)-action on \(G^+\) preserves \(\triangle\). Then the eigenspace of the operator \(L_{\triangle^2}\) determined by the left multiplication by \(\triangle^2\) are preserved under the \(G^+ \rtimes \text{Br}_{2g+1}\)-action. Take any \(p\)th root of unity \(\lambda \neq 1\). Complexifying the diagram in Proposition 2.42, taking its eigenspace decomposition with respect to \(L_{\triangle^2}\) and bring out the \(\lambda\)-part, we have the following \(G^+ \rtimes \text{Br}_{2g+1}\)-equivariant commutative diagram with exact rows:

\[
\begin{array}{ccc}
(L_N^+)_{\lambda} \xrightarrow{(\partial_N^+)_{\lambda}} Z[G^+]_{\lambda} & \longrightarrow & 0 \\
\tilde{i}_{\lambda} \downarrow & & \downarrow \tau_{\lambda} \\
(L_N)_{\lambda} \xrightarrow{(\partial_N)_{\lambda}} Z[G]_{\lambda} & \longrightarrow & 0
\end{array}
\]

, where the subscript ‘\(\lambda\)’ refers to the eigenspace of eigenvalue \(\lambda\) w. r. t. the \(L_{\triangle^2}\)-action. Notice that \(\tilde{i}\) and \(\tau\) are injective.

**Proposition 2.47.** Assume that \(\triangle^2\) is central in \(G\) and that \(\triangle^{2p} = 1\). Then, for any \(p\)th root of unity \(\lambda \neq 1\), we have the direct sum decomposition as \(G \rtimes \text{Br}_{2g+1}\)-module

\[
(L_N^+)_{\lambda} = \text{Ker}(\partial_N)_{\lambda} \oplus \mathbb{C}[G^+]_{\lambda} \mathcal{D}_N(\Delta^2)
\]

, where the 2nd summand of the R.H.S. is a free \(\mathbb{C}[G^+]_{\lambda}\)-module on the basis \(\{1_{\mathbb{C}[G^+]_{\lambda}} \mathcal{D}_N^+(\Delta^2)\}\). Here \(1_{\mathbb{C}[G^+]_{\lambda}} \in \mathbb{Z}[G^+]_{\lambda}\) denotes the multiplicative identity. Both summands are free as \(\mathbb{C}[G]_{\lambda}\)-module such that the former is of rank \(g\) and the latter of rank 2.
Definition 3.2. Define the module elements

\[ N \triangleright F \]

Noticing that \( N \triangleright F \) due to Proposition 2.42 and that \( D_N^+(\Delta^2) = D_N(\Delta^2) \) since \( \Delta^2 \in F_n \). Applying the proof of Proposition 2.28 to \( D_N^+ : H_{2g+1} \to L_N^+ \), we obtain the desired decomposition. Each summand is preserved under \( Br_{2g+1}-\)action due to Remark 2.29. Thus we are done.

\[ \square \]

Remark 2.48. Restricting the argument above to \( (L_N)_\lambda \subset (L_N^\pm)_\lambda \), we obtain immediately the following direct sum decomposition as \( G \rtimes Br_{2g+1} \)-module (actually as \( G \rtimes M_{g,1} \)-module);

\[ (L_N)_\lambda = \text{Ker}(\partial_N)_\lambda \oplus \mathbb{C}[G]_\lambda D_N(\Delta^2) \quad (\lambda \neq 1). \]

In other words, \( (N^{ab})_\lambda \) is a direct summand of \( L_N \) as \( G \rtimes M_{g,1} \)-module if \( \Delta^2 \in G \) is of finite order and included in \( Z(G) \).

## 3 Fundamental formulae describing the braid group action on the Magnus modules

### 3.1 A good basis of \( L_N \)

In the previous section, for any \( M_{g,1} \)-characteristic subgroup \( N \trianglelefteq F_{2g} \), we introduced the Magnus modules \( L_N \) and \( L_N^+ \) such that the former is a \( \mathbb{Z}[F_{2g}/N \rtimes M_{g,1}] \)-module and the latter a \( \mathbb{Z}[H_{2g+1}/N \times Br_{2g+1}] \)-module. Further, we gave the embedding \( i : L_N \hookrightarrow L_N^+ \) as \( \mathbb{Z}[F_{2g}/N \times Br_{2g+1}] \)-modules and defined the Fox derivation \( D_N^+ : H_{2g+1} \to L_N^+ \) so that it extends the usual Fox derivation \( D_N : F_{2g} \to L_N \).

Now we will introduce some good bases of these Magnus modules, \( L_N \) and \( L_N^+ \), for the convenience of calculation. For the moment, we will fix a \( M_{g,1} \)-characteristic subgroup \( N \trianglelefteq F_{2g} \). Denote by \( G \) and \( G^+ \) the quotient groups \( F_n/N \) and \( H_{2g+1}/N \), respectively, and denote the quotient maps by \( \rho_N : F_n \to F_n/N \) and \( \rho_N^+ : H_{2g+1} \to H_{2g+1}/N \), respectively.

**Definition 3.1.** Define the group elements

\[ x_j := \rho_N(x_j) \in G \quad (1 \leq j \leq 2g), \quad y_i := \rho_N^+(y_i) \in G^+ \quad (1 \leq i \leq 2g + 1). \]

**Definition 3.2.** Define the module elements

\[ f_j := D_N(x_j) \in L_N \quad (1 \leq j \leq 2g), \quad t_i := D_N^+(y_i) \in L_N^+ \quad (1 \leq i \leq 2g + 1). \]

Notice that

\[ f_j = D_N^+(y_j \cdot y_{j+1}) = t_i + y_i t_{i+1} \quad (1 \leq j \leq 2g). \]

**Lemma 3.3.** The sets \( \{f_1, \ldots, f_{2g}\} \) and \( \{t_1, \ldots, t_{2g+1}\} \) are free \( \mathbb{Z}[G] \)-bases of \( L_N \) and \( L_N^+ \), respectively.

**Proof.** These are the direct consequences of the construction of \( L_N \) and \( L_N^+ \). \( \square \)
Definition 3.4. Denote by $\Psi_i \in \text{Aut}(G^+) \ (1 \leq i \leq 2g)$ the automorphism induced by the action of $\sigma_i \in \text{Br}_{2g+1}^{+}$ on $H_{2g+1}$. Denote by $\hat{\Psi}_i \in \text{End}_{L}(L_N^+) \ (1 \leq i \leq 2g)$ the endomorphism induced by the action of $\sigma_i$ on $L_N^+$. (See Theorem 2.20.) Notice that each $\hat{\Psi}_i$ preserves $L_N \subseteq L_N^+$.

Lemma 3.5. As a direct consequence of Proposition 2.38 the action of $\Psi_i \ (1 \leq i \leq 2g)$ on $G^+$ is written as follows;

$$\Psi_i(y_j) = \begin{cases} y_{i+1} & \text{if } j = i, \\ y_{i+1}y_iy_{i+1} & \text{if } j = i + 1, \\ y_j & \text{if } j \neq i, i + 1. \end{cases}$$

Proposition 3.6. A short calculation shows that $\hat{\Psi}_i \ (1 \leq i \leq 2g)$ is described as follows;

$$\hat{\Psi}_i(t_j) = \begin{cases} t_{i+1} & \text{if } j = i, \\ y_{i+1}t_i + (1 + y_iy_{i+1})t_{i+1} & \text{if } j = i + 1, \\ t_j & \text{if } j \neq i, i + 1. \end{cases}$$

We will introduce a new basis of $L_N^+$.

Definition 3.7. Define a set of elements of $L_N^+$ as follows;

$$u_{2i} := y_1y_2 \ldots y_{2i-1}D_N^+(y_2,y_{2i+1} \ldots y_{2g+1}) \ (1 \leq i \leq g),$$

$$u_{2i-1} := -y_{2g+1}y_{2g} \ldots y_{2i+1}D_N^+(y_2,y_{2i} \ldots y_1) \ (1 \leq i \leq g),$$

$$w_0 := D_N^+(y_1y_2 \ldots y_{2g+1}) = D_N^+(\Delta).$$

Remark 3.8. It is readily seen that $u_i, w_0 \notin \ell(L_N)$ for $1 \leq i \leq 2g$.

Recall that $\Delta \equiv y_1 \ldots y_{2g+1} = \rho_N^+(\Delta)$, where $\Delta = y_1y_2 \ldots y_{2g+1}$.

Theorem 3.9 (Good basis of $L_N^+$). Assume that the group $G$ is finite. Then the set

$$\{u_i \mid 1 \leq i \leq 2g\} \cup \{w_0\}$$

is a free $\mathbb{Z}[G]$-basis of $L_N^+$.

Proof. It follows from the 2 lemmas just below that $\{y_1u_1, \ldots y_{2g}, \Delta^{-1}w_0\}$ is a free $\mathbb{Z}[G]$-basis of $L_N^+$. Since $y_1\Delta \in G$, we see that $\{\Delta^{-1}u_1, \ldots \Delta^{-1}u_{2g}, \Delta^{-1}w_0\}$ is another free $\mathbb{Z}[G]$-basis of $L_N^+$, which implies that $\{u_i, u_{2g}, w_0\}$ is a free $\Delta(\mathbb{Z}[G])\Delta^{-1}$-basis of $L_N^+$. But $\Delta(\mathbb{Z}[G])\Delta^{-1} = \mathbb{Z}[G]$ since $G \triangleleft G^+$. Thus we are done. □

Lemma 3.10. The set $\{f_1, \ldots, f_{2g}, \Delta^{-1}w_0\}$ is a free $\mathbb{Z}[G]$-basis of $L_N^+$.

Proof. Since $\partial_N^+(w_0) = \partial_N^+ \circ D_N^+(\Delta) = \rho_N^+(\Delta) - 1$, we see that $\partial_N^+(-\Delta^{-1}w_0) + 1 = -\Delta^{-1}(\rho_N^+(\Delta) - 1) + 1 = \Delta^{-1} \in G^+ \setminus G$. Therefore Corollary 2.44 shows the assertion. □
Lemma 3.11. The set \( \{y_1u_1, y_1u_2, \ldots, y_1u_{2g}\} \) is a free \( \mathbb{Z}[G] \)-basis of \( L_N \).

**Proof.** Since \( G \) is finite, it is sufficient to show that \( \{y_1u_1, y_1u_2, \ldots, y_1u_{2g}\} \) generates the \( \mathbb{Z}[G] \)-module \( L_N \). By definition, it holds that
\[
y_1u_{2i} = x_2x_4 \ldots x_{2i-2} D_+^N(x_{2i} \ldots x_{2g})
\]
which implies that
\[
f_{2i} = D_+^N(x_{2i}) \in \langle f_{2i+2}, \ldots, f_{2g}, y_1u_{2i} \rangle_{\mathbb{Z}[G]}.
\]
In particular, \( f_{2g} \in \langle y_1u_{2g} \rangle_{\mathbb{Z}[G]} \). Thus a downward inductive argument w.r.t. \( i \) shows that
\[
f_{2i} \in \langle y_1u_{2i}, \ldots, y_1u_{2g} \rangle_{\mathbb{Z}[G]} \quad (1 \leq i \leq g).
\]
Similarly, it holds that
\[
y_1u_{2i-1} = (x_1x_2 \ldots x_{2g})(x_{2+1}x_{2i+3} \ldots x_{2g-1})^{-1} D_+^N((x_1x_3 \ldots x_{2i-1})^{-1})
\]
which implies that
\[
f_{2i-1} \in \langle f_1, \ldots f_{2i-3}, y_1u_{2i-1} \rangle_{\mathbb{Z}[G]}.
\]
In particular, \( f_1 \in \langle y_1u_1 \rangle_{\mathbb{Z}[G]} \). Thus an upward inductive argument w.r.t. \( i \) shows that
\[
f_{2i-1} \in \langle y_1u_1, \ldots, y_1u_{2i-1} \rangle_{\mathbb{Z}[G]} \quad (1 \leq i \leq g).
\]
Now that we have shown that \( \{f_i \mid 1 \leq i \leq 2g\} \subset \langle y_1u_i \mid 1 \leq i \leq 2g \rangle_{\mathbb{Z}[G]} \), we are done. \( \square \)

Now we will give a set of formulae describing the \( \text{Br}_{2g+1} \) action on \( L_N^+ \) with respect to the basis introduced in Theorem 3.9. This is a point of departure for the explicit calculation in the present paper.

**Theorem 3.12 (Formulae for the braid group action).** For \( 1 \leq k \leq 2g \), it holds that
\[
\hat{\Psi}_{2i-1}(u_j) - u_j = \begin{cases} 
-x_1x_3 \ldots x_{2i-3} \cdot x_{2i}x_{2i+2} \ldots x_{2g} & \text{if } j = 2i, \\
0 & \text{if } j \neq 2i.
\end{cases}
\]
\[
\hat{\Psi}_{2i}(u_j) - u_j = \begin{cases} 
\{x_1x_3 \ldots x_{2i-1} \cdot x_{2i}x_{2i+2} \ldots x_{2g}\}^{-1}(u_{2i} - u_{2i+2}) & \text{if } j = 2i - 1, \\
0 & \text{if } j \neq 2i - 1.
\end{cases}
\]
\[
\hat{\Psi}_k(w_0) - w_0 = 0.
\]

**Proof.** At first, we will prove the 1st formula. Since \( D_+^N \) commutes with the \( \text{Br}_{2g+1} \)-action, we see that
\[
\hat{\Psi}_j(u_{2i}) = \Psi_j(y_1y_2 \ldots y_{2i-1}) \hat{\Psi}_j(D_+^N(y_2y_{2i+1} \ldots y_{2g+1}))
= \Psi_j(y_1y_2 \ldots y_{2i-1})D_+^N(\sigma_j(y_2y_{2i+1} \ldots y_{2g+1})).
\]

17
If $j \neq 2i - 1$, then $\hat{\Psi}_j(u_{2i}) = u_{2i}$ since $\Psi_j$ fixes $y_1y_2 \ldots y_{2i-1}$ and since $\sigma_j$ fixes $y_{2i}y_{2i+1} \ldots y_{2g+1}$. If $j = 2i - 1$, then it follows that
\[
\hat{\Psi}_{2i-1}(u_{2i}) = \Psi_{2i-1}(y_1y_2 \ldots y_{2i-1}) D_N^+ \left( \sigma_{2i-1}(y_{2i}y_{2i+1} \ldots y_{2g+1}) \right) \\
= y_1y_2 \ldots y_{2i-2}y_{2i} D(y_{2i}y_{2i-1}y_{2i}y_{2i+1} \ldots y_{2g+1}) \\
= y_1y_2 \ldots y_{2i-2}y_{2i}(t_{2i} + y_{2i}t_{2i-1}) \\
+ y_1y_2 \ldots y_{2i-2}y_{2i}(y_{2i}y_{2i-1}) D_N^+ (y_{2i}y_{2i+1} \ldots y_{2g+1}) \\
= y_1y_2 \ldots y_{2i-2}(t_{2i-1} + y_{2i}t_{2i}) + u_{2i}.
\]

On the other hand, we see that $u_{2i-1} - u_{2i-3} = -y_{2g+1}y_{2g} \ldots y_{2i+1} D_N^+ (y_{2i}y_{2i-1})$, which implies that
\[
y_1y_2 \ldots y_{2i-2} \cdot y_{2i}y_{2i+1} \ldots y_{2g+1}(u_{2i-1} - u_{2i-3}) \\
= -y_1y_2 \ldots y_{2i-2} \cdot y_{2i} D_N^+ (y_{2i}y_{2i-1}) \\
= -y_1y_2 \ldots y_{2i-2}(y_{2i}t_{2i} + y^2_{2i}t_{2i-1}) \\
= -y_1y_2 \ldots y_{2i-2}(t_{2i-1} + y_{2i}t_{2i}).
\]

Thus we have
\[
\hat{\Psi}_{2i-1}(u_{2i}) = -y_1y_2 \ldots y_{2i-2} \cdot y_{2i}y_{2i+1} \ldots y_{2g+1}(u_{2i-1} - u_{2i-3}) + u_{2i}.
\]

Now we will prove the 2nd formula. The same reasoning as in the case of 1st formula shows that
\[
\hat{\Psi}_j(u_{2i-1}) = -\Psi_j(y_{2g+1}y_{2g} \ldots y_{2i+1}) D_N^+ \left( \sigma_j(y_{2i}y_{2i-1} \ldots y_{1}) \right) .
\]

If $j \neq 2i$, then $\hat{\Psi}_j(u_{2i-1}) = u_{2i-1}$ since $\Psi_j$ fixes $y_{2g+1}y_{2g} \ldots y_{2i+1}$ and since $\sigma_j$ fixes $y_{2i}y_{2i-1} \ldots y_{1}$. If $j = 2i$, it follows that
\[
\hat{\Psi}_{2i}(u_{2i-1}) = -y_{2g+1}y_{2g} \ldots y_{2i+1}y_{2i+1} D_N^+ (y_{2i+1}y_{2i-1} \ldots y_{1}) \\
= -y_{2g+1}y_{2g} \ldots y_{2i+1} D_N^+ (y_{2i+1}y_{2i}y_{2i-1} \ldots y_{1}) \\
= -y_{2g+1}y_{2g} \ldots y_{2i+1}y_{2i+1}(t_{2i+1} + y_{2i+1}t_{2i}) \\
= -y_{2g+1}y_{2g} \ldots y_{2i+1} D_N^+ (y_{2i}y_{2i-1} \ldots y_{1}) \\
= -y_{2g+1}y_{2g} \ldots y_{2i+1}(t_{2i} + y_{2i+1}t_{2i+1}) + u_{2i-1}.
\]

On the other hand, we see that $u_{2i} - u_{2i+2} = y_1y_2 \ldots y_{2i-1} D_N^+(y_{2i}y_{2i+1})$, which implies that
\[
y_{2g+1}y_{2g} \ldots y_{2i+1}y_{2i-1} \ldots y_1(u_{2i} - u_{2i+2}) = y_{2g+1}y_{2g} \ldots y_{2i+1} D_N^+(y_{2i}y_{2i+1}) \\
= y_{2g+1}y_{2g} \ldots y_{2i+1}(t_{2i} + y_{2i+1}t_{2i+1}) \\
= y_{2g+1}y_{2g} \ldots y_{2i+1}y_{2i+1}(y_{2i}t_{2i} + t_{2i+1}) \\
= -y_{2g+1}y_{2g} \ldots y_{2i+1}y_{2i+1}(t_{2i} + y_{2i+1}t_{2i+1})
\]
Thus we have
\[
\hat{\Psi}_{2i}(u_{2i-1}) = y_{2g+1}y_{2g} \cdots y_{2i+1} \cdot y_{2i-1} \cdots y_1(u_{2i} - u_{2i+2}) + u_{2i-1}.
\]

The proof of the 3rd formula is fairly easy. In fact,
\[
\hat{\Psi}_k(w_0) = D_N^+(\sigma_k(y_1y_2 \cdots y_{2g+1})) = D_N^+(y_1y_2 \cdots y_{2g+1}) = w_0.
\]

### 3.2 Multiplicative Jordan decomposition of \( \hat{\Psi}_i \)

Consider the multiplicative Jordan decomposition of the invertible endomorphism \( \hat{\Psi}_i (1 \leq i \leq 2g) \) as follows;
\[
\hat{\Psi}_i = \hat{\Psi}_{i,ss} \circ \hat{\Psi}_{i,uni} = \hat{\Psi}_{i,uni} \circ \hat{\Psi}_{i,ss}
\]
where \( \hat{\Psi}_{i,ss} \) is semisimple and \( \hat{\Psi}_{i,uni} \) unipotent, respectively.

**Proposition 3.13.** Suppose that \( x_i^p = 1 \) \((1 \leq i \leq 2g)\) for some \( p \in \mathbb{N} \). For any \( h \in \mathbb{C}[G] \), it holds that
\[
\begin{align*}
\hat{\Psi}_{2k-1,ss}(hu_i) &= \hat{\Psi}_{2k-1}(h) \left\{ u_i - \delta_{i,2k} \{ x_1x_3 \cdots x_{2k-3} \{ 1 - \phi_p(x_{2k-1}) \} x_{2k}x_{2k+2} \cdots x_{2g} \} (u_{2k-1} - u_{2k-3}) \right\}, \\
\hat{\Psi}_{2k-1,uni}(hu_i) &= h \left\{ u_i - \delta_{i,2k} \{ x_1x_3 \cdots x_{2k-3} \phi_p(x_{2k-1})x_{2k}x_{2k+2} \cdots x_{2g} \} (u_{2k-1} - u_{2k-3}) \right\}, \\
\hat{\Psi}_{2k,ss}(hu_i) &= \hat{\Psi}_{2k}(h) \left\{ u_i + \delta_{i,2k-1} \{ x_{2g}^{-1}x_{2g-2} \cdots x_2^{-1} (1 - \phi_p(x_{2k})) x_{2k-1}^{-1}x_{2k-3}^{-1} \cdots x_1^{-1} \} (u_{2k} - u_{2k+2}) \right\}, \\
\hat{\Psi}_{2k,uni}(hu_i) &= h \left\{ u_i + \delta_{i,2k-1} \{ x_{2g}^{-1}x_{2g-2} \cdots x_2^{-1} \phi_p(x_{2k})x_{2k-1}^{-1}x_{2k-3}^{-1} \cdots x_1^{-1} \} (u_{2k} - u_{2k+2}) \right\}.
\end{align*}
\]
where \( \phi_p \) is the polynomial defined by
\[
\phi_p(t) := \frac{1}{p} \sum_{i=1}^{p-1} t^i.
\]

**Proof.** Applying the 1st formula in Theorem 3.12 \( m \) times, we see that
\[
\begin{align*}
(\hat{\Psi}_{2k-1,ss})^m(hu_{2k}) - (\hat{\Psi}_{2k-1})^m(h)u_{2k} \\
= -(\hat{\Psi}_{2k-1})^m(h) \left\{ x_1x_3 \cdots x_{2k-3} \{ 1 + x_{2k-1}^{-1} + \cdots + x_{2k-1}^{-(m-1)} \} x_{2k}x_{2k+2} \cdots x_{2g} \} (u_{2k-1} - u_{2k-3}) \right\}.
\end{align*}
\]
Taking into account the assumption that \( x_{2k-1}^p = 1 \), a short calculation shows that \( (\hat{\Psi}_{2k-1})^p - \text{id} \) divides \( (t^p - 1)^2 \). Thus Lemma 3.11 below shows the result for \( i = 2k - 1 \). The case where \( i \) is even is left to the reader since the argument is very similar. \( \square \)
Lemma 3.14. If the minimal polynomial of a linear transformation $A$ divides $(t^p - 1)^2$, the multiplicative Jordan decomposition of $A$ is given by $A_{ss} = A(1 - \frac{1}{p}(A^p - 1))$, $A_{uni} = 1 + \frac{1}{p}(A^p - 1)$.

Proof. By the assumption, $A^p = 1 + N$ such that $N^2 = 0$. Write the multiplicative Jordan decomposition of $A$ as $A = S(1 + L)$, where $S$ is semisimple and $L$ nilpotent such that $SL = LS$. Then $A^p = S^p + pS^pL(1 + LM)$ where $M$ is some nilpotent element that commutes with both $S$ and $L$. The uniqueness of additive Jordan decomposition for $A^p$ implies that $1 = S^p$, $N = pS^pL(1 + LM)$. Therefore, $N = pL(1 + LM)$. But, $N^2 = 0$ implies $L^2 = 0$, which in turn implies $N = pL$. Therefore, we have $L = \frac{1}{p}(A^p - 1)$ and $S = A(1 + L)^{-1} = A(1 - L)$. Thus, we are done. \hfill \Box

Notice that, in the above proof, we have deduced the following fact which we will fully make use of in the subsequent sections.

Corollary 3.15. Under the same assumption as Proposition 3.13, $(\hat{\Psi}_{i,ss})^p = \text{id}_{L_N}$.

4 Concrete Calculation

4.1 $H(p, 2g)$

Let $p$ be an odd prime. Consider the free group $F_{2g} = \langle x_1, x_2, \ldots, x_{2g} \rangle$. The abelianization $F_{2g}^{ab}$ is a free abelian group of rank $2g$ generated by $\{[x_1], [x_2], \ldots, [x_{2g}]\}$. The intersection form on $F_{2g}^{ab}$ was defined in Subsection 2.3 which we will denote by $\omega(\cdot, \cdot) : F_{2g}^{ab} \times F_{2g}^{ab} \to \mathbb{Z}$. Since the induced action of $\mathcal{M}_{g,1}$ on $F_{2g}^{ab}$ preserves $\omega$, we have the group homomorphism

$$\mu : \mathcal{M}_{g,1} \to \text{Aut}(F_{2g}^{ab}, \omega) := \{\phi \in \text{Aut}(F_{2g}^{ab}) \mid \phi(\omega) = \omega\} \cong \text{Sp}(2g, \mathbb{Z}).$$

Set $A_{2g} := F_{2g}^{ab} \otimes_{\mathbb{Z}} \mathbb{F}_p$. Denote by $\overline{\omega}$ the induced skew-symmetric $\mathbb{F}_p$-bilinear form on $A_{2g}$, which is non-degenerate since $\omega$ does so.

Notation. For any subset $S$ of a group $G$, we denote by $S^p$ the normal closure in $G$ of the subset $\{s^p \mid s \in S\}$.

Notation. For any $h \in F_{2g}$, we denote

$$\overline{h} := h \mod [F_{2g}, F_{2g}], \quad \overline{h} := h \mod (F_{2g}^{ab})^p.$$ 

Notation. Denote by $F_{2g}^{(k)}$ the $k$-th member of the lower central series of $F_{2g}$, that is, $F_{2g}^{(1)} = F_{2g}$, $F_{2g}^{(2)} = [F_{2g}, F_{2g}]$, $F_{2g}^{(3)} = [F_{2g}^{(2)}, F_{2g}]$, and so on.

Lemma 4.1. We have the canonical isomorphisms

$$A_{2g} \cong F_{2g}^{(1)} / F_{2g}^{(2)}(F_{2g}^{(1)})^p, \quad 2 \wedge A_{2g} \cong F_{2g}^{(2)} / F_{2g}^{(3)}(F_{2g}^{(2)})^p.$$
Proof. The 1st statement is rather trivial. As for the 2nd, we have the canonical isomorphism

\[ F_{2g}^{(2)} / F_{2g}^{(3)} \cong \wedge^2 F_{2g}^{ab} : [a, b] \mod F_{2g}^{(3)} \leftrightarrow \bar{a} \wedge \bar{b}. \]

Applying the functor \( \otimes_{\mathbb{Z}} \mathbb{F}_p \) to this isomorphism, we are done. \[\square\]

**Definition 4.2.** Define the contraction map

\[ \text{Cont}_\omega : F_{2g}^{(2)} / F_{2g}^{(3)} (F_{2g}^{(2)})^p \rightarrow \mathbb{F}_p : [a, b] \mod F_{2g}^{(3)} (F_{2g}^{(2)})^p \rightarrow \omega(a, b) \mod p\mathbb{Z}. \]

Since \( \omega \) is non-degenerate, \( \text{Cont}_\omega \) is surjective. Since \( \omega \) is invariant with respect to the \( M_{g,1} \)-action, we have the \( M_{g,1} \)-equivariant exact sequence as follows;

\[ 1 \rightarrow \text{Ker}(\text{Cont}_\omega) \rightarrow F_{2g}^{(1)} / F_{2g}^{(3)} (F_{2g}^{(2)})^p \xrightarrow{\text{Cont}_\omega} \mathbb{F}_p \rightarrow 0. \] (1)

**Proposition 4.3.** We have the \( M_{g,1} \)-equivariant central extension

\[ 1 \rightarrow C_p \rightarrow \text{H}(p, 2g) \rightarrow A_{2g} \rightarrow 1. \]

Proof. The statement is rather trivial except for the injectivity of \( \iota \).

Set \( H := F_{2g}^{(1)} / F_{2g}^{(3)} \). It is sufficient to show the injectivity of the natural map

\[ \iota' : H^{(2)} / (H^{(2)})^p \rightarrow H^{(1)} / (H^{(1)})^p \]

, which is equivalent to show \( H^{(2)} \cap (H^{(1)})^p \subset (H^{(2)})^p \).

**SubLemma.** \( a^p b^p \equiv (ab)^p \mod (H^{(2)})^p \) for any \( a, b \in H \).

Proof.

\[ a^p b^p = (ab)^p [a, b] \frac{1}{2} p(p-1) \equiv (ab)^p \mod [H, H]^p \]

, where at the congruence on the right we have used the assumption that \( p \) is odd, which implies that \( 2 \mod (p) \) is invertible in \( \mathbb{F}_p \). Thus we are done. \[\square\]

Returning to the proof of the present proposition, let \( a \in H^{(2)} \cap (H^{(1)})^p \). By the sublemma above, we may assume that \( a = b^p \) for some \( b \in H \). Then \( (b^p)^p = 0 \in H^{ab} \) since \( b^p \in H^{(2)} \). But \( H^{ab} \) is a free abelian group, which implies that \( b = 0 \), that is, \( b \in H^{(2)} \). It follows that \( a = b^p \in (H^{(2)})^p \). Thus we are done. \[\square\]

**Definition 4.4.** Set the group

\[ \text{H}(p, 2g) := \left( F_{2g}^{(1)} / F_{2g}^{(3)} (F_{2g}^{(1)})^p \right) / \iota'(\text{Ker}(\text{Cont}_\omega)). \]

**Corollary 4.5.** We have the \( M_{g,1} \)-equivariant central extension

\[ 1 \rightarrow C_p \rightarrow \text{H}(p, 2g) \xrightarrow{p} A_{2g} \rightarrow 1 \]

, where \( C_p \) is the cyclic group of order \( p \).
Proof. Consider the exact sequence in Proposition 4.3. Take the quotients of the middle and the next to the left terms by the subgroup Ker(Cont), which is contained in the center, so that we get the desired exact sequence.

By the construction, $H(p, 2g)$ is a quotient group of $F_{2g} = \langle x_1, x_2, \ldots, x_{2g} \rangle$. We will add an extra generator $c$ together with the relation $c = [x_1, x_2] \equiv x_1x_2x_1^{-1}x_2^{-1}$.

Proposition 4.6. We have the presentation

$$H(p, 2g) = \langle x_1, x_2, \ldots, x_{2g}, c \mid x_i^p (1 \leq i \leq 2g), [x_i, x_j]c^{-\omega_{i,j}} (1 \leq i, j \leq 2g), c \text{ is central.} \rangle$$

, where

$$\omega_{i,j} = \begin{cases} j - i & \text{if } |i - j| = 1, \\ 0 & \text{if otherwise.} \end{cases}$$

Proof. The exact sequence in Corollary 4.5 implies that the order of $H(p, 2g)$ is equal to $p^{2g+1}$. On the other hand, it is readily seen that $H(p, 2g)$ is some quotient of the group represented on the R.H.S., which we denote by $G$ temporarily. It is sufficient to show that the order of $G$ is no more than $p^{2g+1}$. But using the relation appropriately, every element of $G$ can be represented by some monomial as $x_1^{n_1}x_2^{n_2} \cdots x_{2g}^{n_{2g}} c^{n_{2g}+1}$ ($0 \leq n_i \leq p - 1; (1 \leq i \leq 2g + 1)$).

But the number of such monomials is $p^{2g+1}$. Thus we are done.

Corollary 4.7. The map $p : H(p, 2g) \to A_{2g}$ in Corollary 4.5 is the abelianization map of $H(p, 2g)$. The number of the conjugacy classes of $H(p, 2g)$ is equal to $p^{2g} - 1 + p$.

Proof. The 1st assertion is readily seen. As for the 2nd, for any $a \in A_{2g} \setminus \{1\}$, $p^{-1}(a)$ constitutes a conjugacy class of $H(p, 2g)$. On the other hand $p^{-1}(1)$ constitutes the center $Z(H(p, 2g)) \cong C_p$, each element of which constitutes a conjugacy class. Therefore, we have $(p^{2g} - 1) + p$ conjugacy classes.

Remark 4.8. We can see that the $Br_{2g+1}$-action on $H(p, 2g)$ factors through $Sp(2g; \mathbb{F}_p)$. But this statement is false if we replace $Br_{2g+1}$ by $\mathcal{M}_{g,1}$.

4.2 Finite Fourier Analysis and the group algebra $\mathbb{Z}[H(p, 2g)]$

For the moment, we will fix an odd prime $p$. Let $G_{2g}$ be $H(p, 2g)$. By the construction given in the previous subsection, we have the natural projection $\rho_N : F_{2g} = \langle x_1, \ldots, x_{2g} \rangle \to G_{2g}$, where $N = \text{Ker}(\rho_N)$. Recall that $N \triangleleft F_{2g}$ is $\mathcal{M}_{g,1}$-characteristic as we have seen before. Set $x_i := \rho_N(x_i)$ ($1 \leq i \leq 2g$), which generate $G_{2g}$.

Since $G_{2g}$ is finite, its group $\mathbb{C}$-algebra $\mathbb{M}_{2g} := \mathbb{C}[G_{2g}]$ is semisimple and decomposes uniquely as the direct product of simple algebras. As is well-known, the number of
semisimple components is equal to the number of the conjugacy classes of $G_{2g}$, which is equal to $p^{2g} - 1 + p$ as we saw in the previous subsection. Let $c$ be the generator of the center $Z(G_{2g}) \cong C_p$ that appeared in the presentation in Proposition 4.6. Then we see that $M_{2g}$ decomposes into the direct sum of non-zero ideals as

$$M_{2g} = C[A_{2g}] \oplus \bigoplus_{\eta, p = 1, \eta \neq 1} M_\eta^{2g}$$

, where we set

$$M_\eta^{2g} := \{ x \in M_{2g} \mid cx = \eta x \}.$$  

We have $p^{2g}$ 1-dimensional group representations which factor through the abelianization $A_{2g}$. The corresponding $p^{2g}$ simple components span the ideal $C[A_{2g}]$. Thus $p - 1$ simple components remain. But the complement to $A_{2g}$ decomposes as the direct sum of $p - 1$ non-zero components. Therefore each component $M_\eta^{2g}$ must be simple. The Galois($Q(ζ_p)/Q$)-action on $M^{2g}$ induces the one on the set of simple components, where $Q(ζ_p)$ is the cyclotomic field of degree $p$. Since $\{M_\eta^{2g} \mid \eta^p = 1, p \neq 1 \}$ forms an orbit, these $p - 1$ components are isomorphic to each other as $Q$-algebras. But they are isomorphic to the matrix algebras over $C$ since $C$ is algebraically closed. Thus they are isomorphic to each other as $C$-algebras.

For the moment we will fix a primitive $p$-th root of unity $η$ and focus on the simple component $M_\eta^{2g}$ corresponding to $η$. Notice that the $M_{g,1}$-action on $M_{2g}$ preserves $M_\eta^{2g}$ since it fixes $c$. We will introduce a basis of $M_\eta^{2g}$, which behaves as the set of matrix units, that is, the square matrices each of which has zero entries except for one that is 1. In other words, we will construct an explicit isomorphism between $M_\eta^{2g}$ and a matrix algebra over $C$.

Let $π_η : M_{2g} \rightarrow M_\eta^{2g}$ be the projection. We adopt the following convention;

**Convention 4.9.** • Henceforth as an abuse of notation, we denote $π_η(x_i) \in M_\eta^{2g}$, the image by $π_η$ of $x_i$, by the same symbol $x_i$ unless otherwise specified.
• An alphabet with hat, say $\hat{i}$, means the multi-index $(i_1, i_2, \ldots, i_g)$ of possibly $g$ entries, with each $i_l \in F_p$. We often regard $\hat{i}$ as an element of the $F_p$-vector space $F^g_p$.
• $e_k$ for $1 \leq k \leq g$ denotes the kth coordinate vector of $F^g_p$.

With this understood, we will define a set of elements in $M_\eta^{2g}$ as follows;

**Definition 4.10.** For any $\hat{i}, \hat{j} \in (F_p)^g$, set

$$x_{\text{odd}} := x_1^{i_1} x_2^{i_2} \cdots x_{2g-1}^{i_{2g-1}}, \quad x_{\text{even}} := x_2^{i_2} x_4^{i_4} \cdots x_{2g}^{i_{2g}}$$

$$\phi(\hat{i}; x_{\text{odd}}) := \frac{1}{p^g} \sum_{\hat{a} \in (Z/(p))^g} \eta^{-\hat{a}} x_{\text{odd}}^{\hat{a}}, \quad \phi(\hat{i}; x_{\text{even}}) := \frac{1}{p^g} \sum_{\hat{a} \in (Z/(p))^g} \eta^{-\hat{a}} x_{\text{even}}^{\hat{a}}.$$  

23
Proposition 4.11. It is readily seen that
\[\sum_{\hat{a} \in (\mathbb{Z}/(p))^g} \phi(\hat{i}; x_{\text{odd}}) = 1_{M_2^g} = \sum_{\hat{a} \in (\mathbb{Z}/(p))^g} \phi(\hat{i}; x_{\text{even}}),\]
\[\phi(\hat{i}; x_{\text{odd}}) \phi(\hat{j}; x_{\text{odd}}) = \delta_{\hat{i}, j} \phi(\hat{i}; x_{\text{odd}}),\]
\[\phi(\hat{m}; x_{\text{even}}) \phi(\hat{n}; x_{\text{even}}) = \delta_{\hat{m}, \hat{n}} \phi(\hat{m}; x_{\text{even}}).\]

Definition 4.12. For any \(\hat{i}, \hat{j} \in (\mathbb{F}_p)^g\), set
\[E_{\hat{i}, \hat{j}} = \phi(\hat{i}; x_{\text{odd}}) \phi(\hat{j}; x_{\text{odd}}) \in M_2^g.\]

Definition 4.13. Define the square matrix \(T = (T_{i,j})_{1 \leq i, j \leq g}\) of degree \(g\) as
\[T_{i,j} := \begin{cases} 1 & \text{if } i - j = 1, \\ 0 & \text{otherwise}. \end{cases}\]

Set \(\Omega := 1 + T + \cdots + T^{g-1}\). Notice that \(T\) is nilpotent, that \(1 - T = \Omega^{-1}\) and that \(\Omega = (\Omega_{i,j})_{1 \leq i, j \leq g}\) is a lower triangular matrix all of whose entries on the diagonal and in the lower left part are equal to 1;
\[\Omega_{i,j} = \begin{cases} 1 & \text{if } i \geq j, \\ 0 & \text{otherwise}. \end{cases}\]

Proposition 4.14. It holds that
\[E_{\hat{i}, \hat{j}} = \frac{1}{p^g} x_{\text{even}}^{\Omega(\hat{i} - \hat{j})} \phi(\hat{j}; x_{\text{odd}}).\]

Note that in the expression "\(\Omega(\hat{i} - \hat{j})\)”, we regard \(\hat{i}\) and \(\hat{j}\) as column vectors.

Proof.
\[\phi(\hat{i}; x_{\text{odd}}) \phi(\hat{j}; x_{\text{odd}})\]
\[= \frac{1}{p^{2g}} \sum_{\hat{a} \in (\mathbb{Z}/(p))^g} \eta^{-\hat{a} \cdot \hat{i}} x_{\text{odd}}^{\hat{a}} \sum_{\hat{b} \in (\mathbb{Z}/(p))^g} x_{\text{even}}^{\hat{b}} \phi(\hat{j}; x_{\text{odd}})\]
\[= \frac{1}{p^{2g}} \sum_{\hat{a} \in (\mathbb{Z}/(p))^g} \sum_{\hat{b} \in (\mathbb{Z}/(p))^g} \eta^{-\hat{a} \cdot \hat{b} - \hat{a} \cdot \hat{i}} x_{\text{even}}^{\hat{b}} x_{\text{odd}}^{\hat{a}} \phi(\hat{j}; x_{\text{odd}})\]
\[= \frac{1}{p^{2g}} \sum_{\hat{a} \in (\mathbb{Z}/(p))^g} \sum_{\hat{b} \in (\mathbb{Z}/(p))^g} \eta^{\hat{a} \cdot \hat{b} - \hat{a} \cdot \hat{i} + \hat{b} \cdot \hat{j}} x_{\text{even}}^{\hat{b}} \phi(\hat{j}; x_{\text{odd}})\]
\[= \frac{1}{p^{2g}} \sum_{\hat{b} \in (\mathbb{Z}/(p))^g} \delta_{\hat{b} \cdot \hat{i} + \hat{b} \cdot \hat{j} + \hat{b} \cdot \hat{0}} x_{\text{even}}^{\hat{b}} \phi(\hat{j}; x_{\text{odd}})\]
\[= \frac{1}{p^{2g}} x_{\text{even}}^{\Omega(\hat{i} - \hat{j})} \phi(\hat{j}; x_{\text{odd}}).\]
where in the last equality we have used the fact that
\[(1 - T)b = \hat{i} - \hat{j} \iff \hat{b} = \Omega(\hat{i} - \hat{j}).\]

\[\square\]

As a corollary of the previous 2 propositions, a short calculation shows the following result, whose proof is safely left to the reader;

**Proposition 4.15.** For any \(\hat{i}, \hat{j}, \hat{k}, \hat{l} \in (\mathbb{Z}/(p))^g\), it holds that \(E^g_{\hat{i}, \hat{j}} E^g_{\hat{k}, \hat{l}} = \delta_{\hat{j}, \hat{k}} E^g_{\hat{i}, \hat{l}}\), where the \(\delta_{\hat{j}, \hat{k}}\) is the Kronecker delta. Further, it holds that \(1_{M^g_{2g}} = \sum_{\hat{i} \in (\mathbb{Z}/(p))^g} E^g_{\hat{i}, \hat{i}}\).

**Corollary 4.16.** \(\{E^g_{\hat{i}, \hat{j}} \mid \hat{i}, \hat{j} \in (\mathbb{Z}/(p))^g\}\) is a \(\mathbb{C}\)-basis of \(M^g_{2g}\).

**Proof.** The previous proposition gives rise to the surjective \(\mathbb{C}\)-algebra homomorphism \(\mu : M^g_{2g} \to \text{Mat}(p^{2g}; \mathbb{C})\). But \(\dim \mathbb{C} M^g_{2g} = (p^{2g+1} - p^g)/(p - 1) = p^{2g}\), which shows that \(\mu\) is isomorphism. Thus we are done. \(\square\)

Recall that the Skolem-Noether theorem states that any automorphism of a finite dimensional central simple algebra over \(\mathbb{C}\) is an inner one. We will determine the inner automorphisms of \(M^g_{2g}\) corresponding to the action of \(\Psi_l\) \((1 \leq l \leq 2g)\) on \(M^g_{2g}\).

**Notation (Quadratic Gauss Sum).** We denote the quadratic Gauss sum w. r. t. the prime \(p\) as
\[G(a) := \sum_{t \in \mathbb{Z}/(p)} \eta^{at^2} (a \in \mathbb{Z}/(p)).\]

As is well-known, \(G(a) \cdot G(-a) = |G(a)|^2 = p\) for \(a \neq 0 \mod (p)\). Before proceeding to the next step, we put the following convention for the notational simplicity.

**Convention 4.17 (Ignoring e_{g+1} convention).** Throughout the present paper, whenever the symbol \(e_{g+1}\) appears, we will ignore it and presume \(e_{g+1} = 0\) unless otherwise stated.

**Proposition 4.18.** For \(1 \leq k \leq g\) and for any \(\hat{i}, \hat{j} \in (\mathbb{Z}/(p))^g\), it holds that
\[
\Psi_{2k-1}(E^g_{\hat{i}, \hat{j}}) = \eta^{-(\frac{i_k}{2} + \frac{j_k}{2})} E^g_{\hat{i}, \hat{j}},
\]
\[
\Psi_{2k}(E^g_{\hat{i}, \hat{j}}) = \frac{1}{p^g} \sum_{t, s \in \mathbb{Z}/(p)} \eta^{-(\frac{t}{2} + \frac{s}{2})} E^g_{1 + t(e_k - e_{k+1}) + s(e_k - e_{k+1})}.
\]

**Proof.**
\[
\Psi_{2k-1}(\phi(\hat{\alpha}_2; x_{\text{even}})) = \frac{1}{p^g} \sum_{\hat{\alpha} \in (\mathbb{Z}/(p))^g} \Psi_{2k-1}(x_{\text{even}})
\]
\[
= \frac{1}{p^g} \sum_{\hat{\alpha} \in (\mathbb{Z}/(p))^g} (x_{2k-2} x_{2k-1})^{\alpha_k} \cdots x_{2k-4} x_{2k-2} x_{2k-1} \cdots x_{2g}
\]
\[
= \frac{1}{p^g} \sum_{\hat{\alpha} \in (\mathbb{Z}/(p))^g} \eta^{-(\frac{i_k}{2} + \frac{j_k}{2})} E^g_{\hat{i}, \hat{j}} x_{\text{even}}
\]
\[
= \frac{1}{p^g} \sum_{\hat{\alpha} \in (\mathbb{Z}/(p))^g} \eta^{-(\frac{i_k}{2} + \frac{j_k}{2})} E^g_{\hat{i}, \hat{j}} x_{\text{even}}
\]
\[
= \frac{1}{p^g} \sum_{\hat{\alpha} \in (\mathbb{Z}/(p))^g} \eta^{-(\frac{i_k}{2} + \frac{j_k}{2})} E^g_{\hat{i}, \hat{j}} x_{\text{even}}
\]
Therefore, it follows that
\[
\Psi_{2k-1}(E_{i,j}^g) = \phi(\hat{i}; x_{\text{odd}}) \Psi_{2k-1}(\phi(\hat{0}; x_{\text{even}})) \phi(\hat{j}; x_{\text{odd}})
\]
\[
= \frac{1}{p^g} \sum_{\hat{a} \in (\mathbb{Z}/(p))^g} \eta^{-a_{k-1}a_k + \left(\frac{a_k}{2}\right) + \left(\frac{a_k}{2}\right) + i_k(a_{k-1} - a_k)} \phi(\hat{i}; x_{\text{odd}}) x_{\text{even}}^{\hat{a}} \phi(\hat{j}; x_{\text{odd}}).
\]

Notice that any term in the sum above contributes trivially unless the multi-index \( \hat{a} \) satisfies the condition that \( \hat{a} = \Omega(\hat{i} - \hat{j}) \), that is,
\[
a_l = (i_1 - j_1) + \cdots + (i_l - j_l) \quad (1 \leq l \leq g).
\]
See the proof of Proposition 4.14. If \( \hat{a} \) meets this condition, the exponent to \( \eta \) is
\[
- a_{k-1}a_k + \left(\frac{a_k}{2}\right) + \left(\frac{a_k}{2}\right) + i_k(a_{k-1} - a_k)
\]
\[
= \frac{1}{2}(a_k - a_{k-1})(a_k - a_{k-1} - 2x_k - 1)
\]
\[
= \frac{1}{2}(i_k - j_k)((i_k - j_k) - 2x_k - 1)
\]
\[
= -\left(\frac{i_k + 1}{2}\right) + \left(\frac{j_k + 1}{2}\right)
\]
Thus the 1st equation has been proven.
Next, we will prove the 2nd equation. Similar calculation shows the following;
\[
\Psi_{2k}(\phi(\hat{i}; x_{\text{odd}})) = \frac{1}{p^g} \sum_{\hat{a} \in (\mathbb{Z}/(p))^g} \eta^{-\hat{a} + a_k a_{k+1} - \left(\frac{a_k}{2}\right) - \left(\frac{a_k}{2}\right) + (a_{k+1} + 1)} x_{\text{odd}}^{\hat{a}} x_{2k}^{(a_k - a_{k+1})}
\]
\[
= \frac{1}{p^g} \sum_{\hat{a} \in (\mathbb{Z}/(p))^g} \eta^{-\hat{a} + a_k a_{k+1} + \left(\frac{a_k}{2}\right) + \left(\frac{a_k}{2}\right) + (a_{k+1})} x_{2k}^{(a_k - a_{k+1})} x_{\text{odd}}^{\hat{a}}
\]
\[
\Psi_{2k}(E_{i,j}^g) = \Psi_{2k}(\phi(\hat{i}; x_{\text{odd}})) \phi(\hat{0}; x_{\text{even}}) \Psi_{2k}(\phi(\hat{j}; x_{\text{odd}}))
\]
\[
= \frac{1}{p^{2g}} \sum_{\hat{a}, \hat{b} \in (\mathbb{Z}/(p))^g} \eta^{-\hat{a} + a_k a_{k+1} - \left(\frac{a_k}{2}\right) - \left(\frac{a_k}{2}\right) + (a_{k+1})} \eta^{-\hat{b} - b_k b_{k+1} + \left(\frac{b_k}{2}\right) + \left(\frac{b_k}{2}\right)} \cdot x_{\text{odd}}^{\hat{a}} \phi(\hat{0}; x_{\text{even}}) x_{\text{odd}}^{\hat{b}}
\]
Thus we have obtained the following factorization;
\[
\phi(\hat{m}; x_{\text{odd}}) \Psi_{2k}(E_{i,j}^g) \phi(\hat{n}; x_{\text{odd}})
\]
\[
= \left\{ \frac{1}{p^g} \sum_{\hat{a} \in (\mathbb{Z}/(p))^g} \eta^{(\hat{m} - \hat{i}) + a_k a_{k+1} - \left(\frac{a_k}{2}\right) - \left(\frac{a_k}{2}\right) + (a_{k+1})} \right\} \cdot \left\{ \frac{1}{p^g} \sum_{\hat{b} \in (\mathbb{Z}/(p))^g} \eta^{(\hat{n} - \hat{j}) - b_k b_{k+1} + \left(\frac{b_k}{2}\right) + \left(\frac{b_k}{2}\right)} \right\} \cdot E_{\hat{m}, \hat{n}}^g
\]
We will calculate each factor on the R.H.S. separately using the sublemma below;
**SubLemma.** Define the quadratic form $Q$ on $(\mathbb{Z}/(p))^{\oplus 4}$ as follows:

$$Q(\alpha, \beta; \lambda, \mu) := \frac{\alpha(\alpha - 1)}{2} + \frac{\beta(\beta + 1)}{2} - \alpha \beta - \lambda \alpha - \mu \beta.$$  

Then for any $(\lambda, \mu) \in (\mathbb{Z}/(p))^{\oplus 2}$ it holds that

$$\sum_{\alpha, \beta \in \mathbb{Z}/(p)} \eta^{\pm Q(\alpha, \beta; \lambda, \mu)} = pG(\pm \frac{1}{2})\delta_{\lambda+\mu,0} \eta^{\pm \frac{1}{2}(\lambda+\frac{1}{2})^2}$$  

(double-sign corresponds).

**Proof.**

L.H.S. = \[
\sum_{\alpha, \beta \in \mathbb{Z}/(p)} \eta^{\pm \left(\frac{1}{2}(\alpha - \beta - \lambda - \frac{1}{2})^2 - (\lambda + \mu)\beta - \frac{1}{2}(\lambda + \frac{1}{2})^2\right)} = G(\pm \frac{1}{2}) \sum_{\beta \in \mathbb{Z}/(p)} \eta^{\pm (\lambda + \mu)\beta + \frac{1}{2}(\lambda + \frac{1}{2})^2} = \text{R.H.S}
\]

Returning to the proof of the present proposition,

1st factor = \[
\prod_{l \neq k,k+1} \delta_{m_{l,i_l},i_l} \cdot p^{-2} \sum_{\alpha, \beta \in \mathbb{Z}/(p)} \eta^{-\frac{\alpha(\alpha - 1)}{2} - \frac{\beta(\beta + 1)}{2} + \alpha \beta + (m_k - i_k)\alpha + (m_{k+1} - i_{k+1})\beta} = \prod_{l \neq k,k+1} \delta_{m_{l,i_l},i_l} \cdot p^{-2} \sum_{\alpha, \beta \in \mathbb{Z}/(p)} \eta^{-Q(\alpha, \beta; m_k - i_k, m_{k+1} - i_{k+1})} = \frac{1}{p}G(\frac{1}{2}) \cdot \prod_{l \neq k,k+1} \delta_{m_{l,i_l},i_l} \cdot \delta_{m_k - i_k + m_{k+1} - i_{k+1},0} \eta^{\frac{1}{2}(m_k - i_k + \frac{1}{2})^2}
\]

Similarly,

2nd factor = \[
\prod_{l \neq k,k+1} \delta_{n_{l,j_l},j_l} \cdot p^{-2} \sum_{\alpha, \beta \in \mathbb{Z}/(p)} \eta^{Q(\alpha, \beta; -(n_k+1 - j_k+1), - (n_k - j_k))} = \frac{1}{p}G(\frac{1}{2}) \cdot \prod_{l \neq k,k+1} \delta_{n_{l,j_l},j_l} \cdot \delta_{n_k+1 - j_k+1 + n_k - j_k,0} \eta^{-\frac{1}{2}(n_k+1 - j_k+1 + \frac{1}{2})^2} = \frac{1}{p}G(\frac{1}{2}) \cdot \prod_{l \neq k,k+1} \delta_{n_{l,j_l},j_l} \cdot \delta_{n_k - j_k + n_{k+1} - j_{k+1},0} \eta^{-\frac{1}{2}(n_k - j_k + \frac{1}{2})^2}
\]

The product of the 1st and the 2nd factor is equal to the coefficient of $E_{m,n}$ in $\Psi_2(E_{i,j})$. Thus rewriting the consequence, the result follows.

\[\blacksquare\]
4.3 The tensor product decomposition of $M^\eta_{2g}$

With Proposition 4.118 in mind, in the present subsection, we will introduce the new $\mathbb{C}$-vector spaces, $V^\eta_g$ and its dual $V^{\eta*}_g$, endowed with $B_{2g}$-action.

**Definition 4.19.** Let $V^\eta_g$ be the $\mathbb{C}$-vector space spanned by the basis $\{e_j \mid \hat{j} \in (\mathbb{Z}/(p))^g\}$. The right $G_{2g}$-module structure is determined by the rule
\[ e_j x_{2k-1} := \eta^k e_j, \quad e_j x_{2k} = e_j e_{k+1} \quad (1 \leq k \leq g). \]

(Due to Convention 4.111, if $k = g$, we understand that $e_j x_{2g} = e_{-g}$.)

Define the left action of the symbol $\Psi$ on $(\mathbb{Z}/(p))^g$ by the following rule;
\[ \Psi_{2k-1}(e_j) = \eta^{(k+1)\frac{1}{2}} e_j, \quad \Psi_{2k}(e_j) = \frac{G(\frac{k}{2})}{p} \sum_{s \in \mathbb{Z}/(p)} \eta^{-\frac{1}{2}(s+\frac{1}{2})^2} e_{j+s(e_k-e_{k+1})} \]
for $1 \leq k \leq g$ and for any $\hat{j} \in (\mathbb{Z}/(p))^g$.

It can be readily seen that the multiplication rule above surely reduces to a right $G_{2g}$-module structure on $V^\eta_g$. Recall that $\{x_m \mid 1 \leq m \leq 2g\}$ generates $G_{2g}$. This right $G_{2g}$-module structure and the left $\Psi$-action on $V^\eta_g$ are compatible in the sense that
\[ \Psi_l(e_j x_m) = \Psi_l(e_j)\Psi_l(x_m) \quad (\hat{j} \in (\mathbb{Z}/(p))^g, \ l, m \in \{1, 2, \ldots, 2g\}). \]

**Definition 4.20.** Let $V^{\eta*}_g$ be the $\mathbb{C}$-vector space spanned by the dual basis $\{e^*_i \mid \hat{i} \in (\mathbb{Z}/(p))^g\}$. The induced left $G_{2g}$-module structure are given by the rule
\[ x_{2k-1} e^*_i = \eta^i e^*_i, \quad x_{2k} e^*_i = e^*_{i+e_k-e_{k+1}} \quad (1 \leq k \leq g). \]

The induced left action of the symbol $\Psi$ on $V^{\eta*}_g$ is determined as follows;
\[ \Psi_{2k-1}(e^*_i) = \eta^{-(i+1)\frac{1}{2}} e^*_i, \quad \Psi_{2k}(e^*_i) = \frac{G(-\frac{k}{2})}{p} \sum_{t \in \mathbb{Z}/(p)} \eta^{\frac{1}{2}(t+\frac{1}{2})^2} e^*_{i+t(e_k-e_{k+1})} \]
for $1 \leq k \leq g$ and for any $\hat{i} \in (\mathbb{Z}/(p))^g$.

By definition, we have the canonical $\mathbb{C}$-bilinear pairing $\langle \cdot, \cdot \rangle : V^\eta_g \times V^{\eta*}_g \rightarrow \mathbb{C}$ such that
\[ \langle e_i, e^*_j \rangle = \delta_{i,j} \quad (\hat{i}, \hat{j} \in (\mathbb{Z}/(p))^g). \]

Further, for any $(v, w) \in V^\eta_g \times V^{\eta*}_g$, it holds that
\[ \langle v, hw \rangle = \langle vh, w \rangle \quad (h \in G), \quad \langle \Psi_l(v), \Psi_l(w) \rangle = \langle v, w \rangle \quad (1 \leq l \leq 2g). \]

Consider the tensor product $V^{\eta*}_g \otimes_{\mathbb{C}} V^\eta_g$, which has the induced $G_{2g}$-$\mathbb{C}$-module structure. We have the canonical $\mathbb{C}$-linear isomorphism
\[ \iota : V^{\eta*}_g \otimes_{\mathbb{C}} V^\eta_g \rightarrow M^\eta_{2g} : e^*_i \otimes e_j \mapsto E^\eta_{i,j}. \]
which turns out to be a $\mathbb{C}[G_{2g}]$-bimodule map. Further it holds that

$$\iota(\Psi_l(u) \otimes \Psi_l(v)) = \Psi_l(\iota(u \otimes v)) \quad (u \otimes v \in V_g^{\eta^*} \otimes_{\mathbb{C}} V_g^{\eta}, \ 1 \leq l \leq 2g).$$

In other words, the map $\iota$ intertwines the $\Psi_l$-actions on both sides, the diagonal one on $V_g^{\eta^*} \otimes_{\mathbb{C}} V_g^{\eta}$ and the usual one on $M_2^{\eta^*}$. Thus we have

**Proposition 4.21.** $\iota : V_g^{\eta^*} \otimes_{\mathbb{C}} V_g^{\eta} \rightarrow M_2^{\eta^*}$ is an isomorphism both as $\mathbb{C}[G_{2g}]$-bimodules and as $B_{2g}$-modules.

Three remarks here are in order.

- First, a short calculation shows that the action of $\Psi_l$ ($1 \leq l \leq 2g$) on $M_2^{\eta^*}$ coincides with the adjoint action of the invertible element

$$\frac{G(-\frac{1}{2})}{p} \sum_{t \in \mathbb{Z}/(p)} \eta^{\frac{1}{2}(t+\frac{1}{2})^2} \pi_{\eta}(x_t)^t \in M_2^{\eta^*}.$$

- Secondly, since the map determined by the adjoint action

$$\text{adj} : \text{Unit}(M_2^{\eta^*}) \rightarrow \text{Aut}_{\mathbb{C}}\text{-alg}(M_2^{\eta^*})$$

has non trivial kernel $\text{Unit}(M_2^{\eta^*}) \cap \text{Cent}(M_2^{\eta^*}) \cong \mathbb{C}^*$, it is not obvious at a first glance whether the action given in Definition 4.20 amounts to a (true) $\text{Br}_{2g+1}$-action or merely a projective representation. But the former is the case as shown in the next proposition.

- Thirdly, both $V_g^{\eta}$ and $V_g^{\eta^*}$ afford projective representations over $\mathbb{C}$ of $\mathcal{M}_{g,1}$, which extend the $\text{Br}_{2g+1}$-actions above, since $\mathcal{M}_{g,1}$ acts on $M_2^{\eta^*}$ as $\mathbb{C}$-algebra automorphisms.

**Proposition 4.22.** The action of $\Psi_l$ ($1 \leq l \leq 2g$) on $V_g^{\eta}$ satisfies the defining relation of $\text{Br}_{2g}$.

**Proof.** A short calculation shows that, for any $\hat{j} \in (\mathbb{Z}/(p))^g$ and for $1 \leq k \leq g$,

$$\Psi_{2k-1} \Psi_{2k} \Psi_{2k-1}(e_j) = \frac{G(\frac{1}{2})}{p} \sum_{s \in \mathbb{Z}/(p)} \eta^{-\frac{1}{2}(s+\frac{1}{2})^2 + \left(\frac{j_k+1}{2}\right)^2 + \left(\frac{j_k+1}{2}\right)} e_{\hat{j}+s(e_k-e_{k+1})}$$

$$= \frac{G(\frac{1}{2})}{p} \sum_{s \in \mathbb{Z}/(p)} \eta^{j_k(s+j_k+1)-\frac{k}{p}} e_{\hat{j}+s(e_k-e_{k+1})}.$$
On the other hand, we see that
\[
\Psi_{2k}\Psi_{2k-1}\Psi_{2k}(e_j) = \frac{G(\frac{1}{2})^2}{p^2} \sum_{s,t \in \mathbb{Z}/(p)} \eta^{-\frac{1}{2}(s+\frac{1}{2})^2-\frac{1}{2}(t+\frac{1}{2})^2+(j_k+\frac{1}{2})} e_{j+s} e_{e_k-e_{k+1}}
\]

\[
= \frac{G(\frac{1}{2})^2}{p^2} \sum_{s,t \in \mathbb{Z}/(p)} \eta^{-\frac{1}{2}(-2j_k s^2+t^2-j_k^2)} e_{j+s} e_{e_k-e_{k+1}}
\]

\[
= \frac{G(\frac{1}{2})^2}{p^2} \sum_{s,t \in \mathbb{Z}/(p)} \eta^{-\frac{1}{2}((t+j_k)^2-2j_k s-2j_k^2)} e_{j+s} e_{e_k-e_{k+1}}
\]

\[
= \frac{G(\frac{1}{2})}{p} \sum_{s \in \mathbb{Z}/(p)} \eta^j e_{j+s} e_{e_k-e_{k+1}}.
\]

Thus we have shown that \(\Psi_{2k-1}\Psi_{2k}\Psi_{2k-1} = \Psi_{2k}\Psi_{2k-1}\Psi_{2k}\) holds on \(V_g^\eta\).

Similarly, a short calculation shows that, for any \(j \in (\mathbb{Z}/(p))^g\) and for \(1 \leq k \leq g - 1\),

\[
\Psi_{2k+1}\Psi_{2k}\Psi_{2k+1}(e_j) = \frac{G(\frac{1}{2})}{p} \sum_{s \in \mathbb{Z}/(p)} \eta^{-j_k} e_{j+s} e_{e_k-e_{k+1}}
\]

\[
= \Psi_{2k}\Psi_{2k+1}\Psi_{2k}(e_j).
\]

Thus we have shown that \(\Psi_{2k+1}\Psi_{2k}\Psi_{2k+1} = \Psi_{2k}\Psi_{2k+1}\Psi_{2k}\) holds on \(V_g^\eta\). The remaining braid relations can be ensured more easily and left to the reader. \(\square\)

**Corollary 4.23.** The action of \(\Psi_l\) \((1 \leq l \leq 2g)\) on \(V_g^\eta\) satisfies

\[(\Psi_l|_{V_g^\eta})^p = \text{id}_{V_g^\eta}.
\]

**Proof.** If \(l = 2k - 1\) \((1 \leq k \leq g)\), the assertion follows immediately from Definition 4.20. If \(l = 2k\) \((1 \leq k \leq g)\), the braid relation implies that the action of \(\Psi_{2k}\) is conjugate to that of \(\Psi_{2k-1}\) in \(\text{End}_C(V_g^\eta)\). Thus we are done. \(\square\)

We will provide a version of finite Fourier transformation needed in the subsequent part of the present paper.

**Definition 4.24** (finite Fourier transformation). For any \(\hat{n} \in (\mathbb{Z}/(p))^g\), set

\[e_{\hat{n}}^* := \alpha_g \sum_{j \in (\mathbb{Z}/(p))^g} \eta^{\hat{n} \cdot \Omega \cdot j} e_j \in V_g^\eta\]

, where we set the constant \(\alpha_g := (\frac{G(\frac{1}{2})}{p})^g\).

Then the inverse Fourier transformation takes the following form;
Proposition 4.25. For any $\hat{j} \in (\mathbb{Z}/(p))^g$, it holds that

$$e_{\hat{j}} = \bar{\alpha}_g \sum_{\hat{n} \in (\mathbb{Z}/(p))^g} \eta^{-\hat{n} \cdot \hat{j}} e_{\hat{n}}^*.$$ 

Notice that $\alpha_g \bar{\alpha}_g = \frac{1}{p^g}$ since $\bar{\alpha}_g = \left( \frac{G(-\frac{1}{2})}{p} \right)^g$.

Proposition 4.26. For $1 \leq k \leq g$ and for any $\hat{n} \in (\mathbb{Z}/(p))^g$, it holds that

$$\Psi_{2k}(e_{\hat{n}}^*) = \eta^{(nk+1)} e_{\hat{n}}^*.$$ 

Proof.

$$\Psi_{2k}(e_{\hat{n}}^*) = \alpha_g \sum_{\hat{j} \in (\mathbb{Z}/(p))^g} \eta^{\hat{n} \cdot \hat{j}} \Psi_{2k}(e_{\hat{j}})$$

$$= \frac{G(\frac{1}{2})}{p} \alpha_g \sum_{s \in \mathbb{Z}/(p)} \sum_{\hat{j} \in (\mathbb{Z}/(p))^g} \eta^{\hat{n} \cdot \hat{j} - \frac{1}{2}(s+\frac{1}{2})^2} e_{\hat{j} + s(e_k - e_{k+1})}$$

$$= \frac{G(\frac{1}{2})}{p} \alpha_g \sum_{s \in \mathbb{Z}/(p)} \sum_{\hat{j} \in (\mathbb{Z}/(p))^g} \eta^{\hat{n} \cdot \hat{j} - s(e_k - e_{k+1}) - \frac{1}{2}(s+\frac{1}{2})^2} e_{\hat{j}}$$

$$= \frac{G(\frac{1}{2})}{p} \alpha_g \sum_{s \in \mathbb{Z}/(p)} \sum_{\hat{j} \in (\mathbb{Z}/(p))^g} \eta^{\hat{n} \cdot \hat{j} - nk^2 - \frac{1}{2}(s+\frac{1}{2})^2} e_{\hat{j}}$$

$$= \frac{G(\frac{1}{2})}{p} \alpha_g \sum_{s \in \mathbb{Z}/(p)} \sum_{\hat{j} \in (\mathbb{Z}/(p))^g} \eta^{\hat{n} \cdot \hat{j} - \frac{1}{2}(s+\frac{1}{2}+nk)^2 + \frac{1}{2}nk(nk+1)} e_{\hat{j}}$$

$$= \alpha_g \sum_{\hat{j} \in (\mathbb{Z}/(p))^g} \eta^{\hat{n} \cdot \hat{j} + \frac{1}{2}nk(nk+1)} e_{\hat{j}}$$

$$= \eta^{nk(nk+1)} e_{\hat{n}}^*$$

\[\square\]

4.4 The $\text{Br}_{2g+1}$-module $L_{g}^\eta$

We have the canonical identifications as $\mathbb{C}$-vector space as follows;

$$L_{N}^+ \cong \mathbb{C}[G_{2g}] \otimes_{\mathbb{C}} \langle u_1, u_2, \ldots, u_{2g}, w_0 \rangle / C, \quad L_{N} \cong \mathbb{C}[G_{2g}] \otimes_{\mathbb{C}} \langle f_1, f_2, \ldots, f_{2g} \rangle / C.$$ 

Keeping these in mind, we introduce the new $\text{Br}_{2g+1}$-module $L_{g}^\eta$ associated with the latter.

Definition 4.27. Define the $\mathbb{C}$-vector space $L_{g}^\eta$ as follows;

$$L_{g}^\eta := V_{g}^\eta \otimes_{\mathbb{C}} \langle u_i \mid 1 \leq i \leq 2g \rangle / C$$
Define the action of the symbol $\hat{\Psi}_l$ ($1 \leq l \leq 2g$) on $L^n_g$ by the formula

\[
\hat{\Psi}_{2k-1}(v \otimes u_l) = \Psi_{2k-1}(v) \otimes u_l - \delta_{l,2k} \Psi_{2k-1}(v) \{x_1 x_3 \cdots x_{2k-3},x_{2k} x_{2k+2} \cdots x_{2g}\} \otimes (u_{2k-1} - u_{2k-3}),
\]
\[
\hat{\Psi}_{2k}(w \otimes u_l) = \Psi_{2k}(w) \otimes u_l + \delta_{l,2k-1} \Psi_{2k}(w) \{x_1 x_3 \cdots x_{2k-1},x_{2k} x_{2k+2} \cdots x_{2g}\}^{-1} \otimes (u_{2k} - u_{2k+2}).
\]

**Definition 4.28.** Set

\[
\phi(c; \eta) := \frac{1}{p} \sum_{n \in \mathbb{Z}/(p)} \eta^{-n} c^n \in \mathbb{C}[G_{2g}]
\]

, where $c$ is the generator of the center of $G_{2g} \equiv H(p, 2g)$ that appeared in the presentation in Proposition 4.29. Recall that $c$ is invariant under the $\text{Br}_{2g+1}$-action on $G_{2g}$.

**Proposition 4.29.** We have a canonical $\text{Br}_{2g+1}$-module isomorphism

\[
\phi(c; \eta)L_N \cong M^0_{2g} \otimes_{\mathbb{C}} \langle u_l \mid 1 \leq l \leq 2g \rangle_\mathbb{C}.
\]

**Proof.** Consider the self map of $L^+_N$ determined by the left multiplication by $\Delta \in G_{2g}^+$:

\[
L_\Delta : L^+_N \to L^+_N : w \mapsto \Delta w.
\]

Then we have the induced $\mathbb{C}$-linear bijective map

\[
L_\Delta|_{(L_N)} : \iota(L_N) \xrightarrow{\sim} \mathbb{C}[G_{2g}] \otimes_{\mathbb{C}} \langle u_l \mid 1 \leq l \leq 2g \rangle_\mathbb{C} \subset L^+_N
\]

, which turns out to be a $\text{Br}_{2g+1}$-module isomorphism since $\Delta$ is invariant under the $\text{Br}_{2g+1}$-action. It follows that

\[
\phi(c; \eta)L_N \cong \phi(c; \eta)\mathbb{C}[G_{2g}] \otimes_{\mathbb{C}} \langle u_l \mid 1 \leq l \leq 2g \rangle_\mathbb{C}
\]

\[
\cong M^0_{2g} \otimes_{\mathbb{C}} \langle u_l \mid 1 \leq l \leq 2g \rangle_\mathbb{C}
\]

, where the congruences mean \"isomorphic as $\text{Br}_{2g+1}$-modules\". At the 1st congruence, we have used the fact $c$ and $\Delta$ commute with each other. \hfill $\square$

**Theorem 4.30.** The action of $\hat{\Psi}_l$ ($1 \leq l \leq 2g$) on $L^n_g$ amounts to a $\text{Br}_{2g+1}$-action. Further, the tensor product $V^n_g \otimes_{\mathbb{C}} L^n_g$ endowed with the diagonal $\text{Br}_{2g+1}$-action is canonically isomorphic to $\phi(c; \eta)L_N$ as a $\text{Br}_{2g+1}$-module.

**Proof.** The composition of the canonical $\mathbb{C}$-linear bijective maps

\[
V^n_g \otimes_{\mathbb{C}} L^n_g \cong V^n_g \otimes_{\mathbb{C}} V^n_g \otimes_{\mathbb{C}} \langle u_l \mid 1 \leq l \leq 2g \rangle_\mathbb{C}
\]

\[
\cong M^0_{2g} \otimes_{\mathbb{C}} \langle u_l \mid 1 \leq l \leq 2g \rangle_\mathbb{C}
\]

\[
\cong \phi(c; \eta)L_N
\]

32
intertwines the action of \( \hat{\Psi} \) on the L.H.S. and that on the R.H.S., giving rise to a \( \text{Br}_{2g+1} \)-module isomorphism. But since the \( \hat{\Psi}_l \)-action on the 1st factor \( V_{g}^{\ast} \) of the L.H.S. (via \( \Psi_l \)-action) amounts to a \( \text{Br}_{2g+1} \)-action, that on the 2nd factor \( L_{g} \) of the L.H.S. reduces to a \( \text{Br}_{2g+1} \)-action. Thus we are done. \( \square \)

**Theorem 4.31.** For \( 1 \leq l \leq 2g \), the semisimple and the unipotent parts of the \( \hat{\Psi}_l \)-action on \( L_{g} \) are respectively described by the very same formulae as in Theorem 3.13 with the modified assumption that \( h \in V_{g}^{\ast} \) replacing the one that \( h \in Z[G_{2g}] \) there.

**Proof.** Since the \( \Psi_l \)-action on \( V_{g}^{\ast} \) satisfies the condition \((\Psi_l|_{V_{g}^{\ast}})^{p} = \text{id}_{V_{g}^{\ast}} \) (See Corollary 4.23), exactly the same reasoning that deduced Theorem 3.13 shows the result. \( \square \)

**Remark 4.32.** In the same spirit as the argument so far, define the \( \mathbb{C} \)-vectors space \( L_{g} \) by

\[
L_{g} := V_{g}^{\ast} \otimes_{\mathbb{C}} \langle f_i \mid 1 \leq i \leq 2g \rangle_{\mathbb{C}}.
\]

We can endow \( L_{g} \) with a projective \( \mathcal{M}_{g,1} \)-module structure over \( \mathbb{C} \) such that it is accompanied by a canonical \( \mathcal{M}_{g,1} \)-module isomorphism

\[
\xi : V_{g}^{\ast} \otimes_{\mathbb{C}} L_{g} \xrightarrow{\cong} \phi(c; \eta)L_{N}.
\]

Further, there exists a canonical \( \text{Br}_{2g+1} \)-module isomorphism \( \tau : L_{g} \xrightarrow{\cong} L_{g} \) such that the composition \( (\text{id}_{V_{g}^{\ast}} \otimes \tau) \circ \xi^{-1} \) coincides with the isomorphism given in Proposition 4.30.

### 4.5 A concrete description of the \( B_{2g} \)-action on \( L_{g} \) : unipotent part

We will describe the unipotent part of the action of \( \hat{\Psi}_i \) \( (1 \leq i \leq 2g) \) on \( L_{g} \).

**Theorem 4.33.** For \( 1 \leq k \leq g \), \( 1 \leq m \leq 2g \) and for any \( \hat{i} \in (\mathbb{Z}/(p))^{g} \), it holds that

\[
\begin{align*}
\left( \hat{\Psi}_{2k-1, \text{uni}} - 1 \right) (e_i u_m) &= -\delta_{m,2k} \delta_{i,0} \eta^{(i_1+\cdots+i_{k-1})} e_{i-e_k} (u_{2k-1} - u_{2k-3}), \\
\left( \hat{\Psi}_{2k, \text{uni}} - 1 \right) (e_i u_m) &= \delta_{m,2k-1} \frac{1}{p} \eta^{-s} \sum_{s \in \mathbb{Z}/(p)} \sum_{e_k+1} (u_{2k} - u_{2k+2}).
\end{align*}
\]

**Proof.** At first we will show the 1st equation.

If \( m \neq 2k \), it follows from Theorem 3.12 that \( \hat{\Psi}_{2k-1}(e_i u_m) = \Psi_{2k-1}(e_i) u_m \). But the action of \( \Psi_{2k-1} \) on \( V_{g} \) is semisimple. Thus the nilpotent part of its unipotent part contributes trivially, that is, the L.H.S. of the 1st equation is equal to zero.

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33
If \( m = 2k \), it follows from Theorem 4.31 that
\[
\left( \hat{\Psi}_{2k-1, \text{uni}} - 1 \right) (e_i u_{2k})
= -e_i \left\{ x_1 x_3 \ldots x_{2k-3} \phi_p(x_{2k-1}) x_{2k} x_{2k+2} \ldots x_{2g} \right\} (u_{2k-1} - u_{2k-3}),
\]
\[
= -\eta^{(i_1 + \cdots + i_k)} e_i \left\{ \phi_p(x_{2k-1}) x_{2k} x_{2k+2} \ldots x_{2g} \right\} (u_{2k-1} - u_{2k-3})
\]
\[
= -\delta_{i,0} \eta^{(i_1 + \cdots + i_k-1)} e_i \left\{ x_{2k-2k+2} \ldots x_{2g} \right\} (u_{2k-1} - u_{2k-3})
\]
\[
= -\delta_{i,0} \eta^{(i_1 + \cdots + i_k-1)} e_i (u_{2k-1} - u_{2k-3}).
\]

Then we will show the 2nd equation. If \( m \neq 2k - 1 \), it follows from Theorem 3.12 that \( \hat{\Psi}_{2k}(e_i u_m) = \hat{\Psi}_{2k}(e_i) u_m \). Since \( \hat{\Psi}_{2k} \) is semisimple, the nilpotent part of the unipotent part is equal to zero.

If \( m = 2k - 1 \), it follows from Theorem 4.31 that
\[
\left( \hat{\Psi}_{2k, \text{uni}} - 1 \right) (e_i u_{2k-1})
= e_i \left\{ x_{2g-2} x_{2g-2} \ldots x_{2k-3} x_{2k} x_{2k+2} \ldots x_{2g} \right\} (u_{2k} - u_{2k+2})
\]
\[
= e_i \left\{ \phi_p(x_{2k-1}) x_{2k} x_{2k+2} \ldots x_{2g} \right\} (u_{2k} - u_{2k+2})
\]
\[
= \frac{1}{p} \sum_{s \in \mathbb{Z}/(p)} e_i + e_{k+1} + (e_{k+1}) \left\{ x_{2k-2} x_{2k-3} \ldots x_{2k-1} \right\} (u_{2k} - u_{2k+2})
\]
\[
= \frac{1}{p} \sum_{s \in \mathbb{Z}/(p)} \eta^{-(i_1 + \cdots + i_k)} e_i + e_{k+1} + s \left( \eta - 1 \right) (u_{2k} - u_{2k+2}).
\]

\[ \square \]

4.6 A concrete description of the \( B_2 \)-action on \( L^g_L \) : semisimple part

We will describe the semisimple part of the action of \( \hat{\Psi}_i \) (\( 1 \leq i \leq 2g \)) on \( L^g_L \).

Recall that \( p \) is an odd prime and that \( \eta \) is a primitive \( p \)-th root of unity. Corollary 4.23 states that \( (\hat{\Psi}_{i, \text{ss}} | L^g_L)^p = 1 \) (\( 1 \leq i \leq 2g \)). Henceforth, we omit the subscript \( \text{ss} \) because it is apparent from the context. With these understood, we will define the finite Fourier transformation of \( \hat{\Psi}_{i, \text{ss}} \) as follows;

**Definition 4.34.** For any \( a \in \mathbb{Z}/(p) \) and for \( 1 \leq i \leq 2g \), set
\[
\Phi \left( \hat{\Psi}_{i, \text{ss}}; a \right) = \frac{1}{p} \sum_{n \in \mathbb{Z}/(p)} \eta^{-an} (\hat{\Psi}_{i, \text{ss}})^n.
\]
Notice that these are idempotents of \( \text{End}_C(L^g_L) \).
**Theorem 4.35.** For \(1 \leq k \leq g, 1 \leq n \leq 2g\) and for any \(a \in \mathbb{Z}/(p)\), \(j \in (\mathbb{Z}/(p))^g\), it holds that

\[
\Phi \left( \hat{\Psi}_{2k-1,ss} a \right) (e_j u_n) = \delta_{a,(j+1)} \left\{ e_j u_n - \delta_{n,2k} \delta_{j,k} \neq 0 (1 - \eta^{-jk})^{-1} \eta^{(j_1 + \ldots + j_{k-1})} e_{j-e_k} (u_{2i-1} - u_{2i-3}) \right\}.
\]

where the symbol \(\delta_{a \neq b}\) is determined by

\[
\delta_{a \neq b} := \begin{cases} 
1 & \text{if } a \neq b, \\
0 & \text{if otherwise}.
\end{cases}
\]

**Proof.** The proof in the case where \(n \neq 2k\) is easy. In fact,

\[
\Phi \left( \hat{\Psi}_{2k-1,ss} a \right) (e_j u_n) = \frac{1}{p} \sum_{m \in \mathbb{Z}/(p)} \eta^{-am} \hat{\Psi}_m (e_j u_n) = \frac{1}{p} \sum_{m \in \mathbb{Z}/(p)} \eta^{-am} \Psi_m (e_j u_n) = \frac{1}{p} \sum_{m \in \mathbb{Z}/(p)} \eta^{-am} \eta^{(j+1)m} e_j = \delta_{a,(j+1)} e_j u_n.
\]

Next we treat the remaining case where \(n = 2k\). Applying the 1st formula in Theorem 4.31 successively, it follows that, for each \(m \in \mathbb{Z}/(p)\),

\[
(\hat{\Psi}_{2k-1,ss})^m (e_j u_{2k}) = (\Psi_{2k-1})^m (e_j) \left\{ u_{2k} - \{ x_1 x_3 \ldots x_{2i-3} (1 + x_{2k-1}^{-1} + \ldots + x_{2k-1}^{-m+1} - m \phi(x_{2k-1})) x_{2i} x_{2i+2} \ldots x_{2g} (u_{2i-1} - u_{2i-3}) \right\} = \eta^{(j+1)m} \left\{ e_j u_{2k} - e_j \{ x_1 x_3 \ldots x_{2i-3} (1 + x_{2k-1}^{-1} + \ldots + x_{2k-1}^{-m+1} - m \phi(x_{2k-1})) x_{2i} x_{2i+2} \ldots x_{2g} (u_{2i-1} - u_{2i-3}) \right\} = \eta^{(j+1)m} \left\{ e_j u_{2k} - \delta_{j,k} \neq 0 \frac{1 - \eta^{-mjk}}{1 - \eta^{-jk}} \eta^{(j_1 + \ldots + j_{k-1})} e_{j-e_k} (u_{2i-1} - u_{2i-3}) \right\}.
\]

Multiplying both sides by \(\eta^{-am}\) and summing up the results with \(m \) running through \(\mathbb{Z}/(p)\), we obtain the desired formula. \(\square\)

**Definition 4.36.** Define the map

\[
\tau_p : \mathbb{Z}/(p) \to \mathbb{Z}/(p) : i \mapsto \left( \frac{i + 1}{2} \right).
\]

Further, set \(I_p := \text{Image}(\tau_p) \subset \mathbb{Z}/(p)\).

**Remark 4.37.** \( \square \)

- \( \triangledown I_p = \frac{p+1}{2} \).
- \( 0, -\frac{1}{2} \in I_p, \tau_p^{-1}(0) = \{0, -1\}, \tau_p^{-1}(-\frac{1}{2}) = \{-\frac{1}{2}\}. \)
- \( \tau_p(a) = \tau_p(b) \iff a + b = 1. \)
Definition 4.38. Define the map
\[
\hat{\tau}_p : (\mathbb{Z}/(p))^g \to I_p^g : \hat{i} \mapsto (\tau_p(i_1), \ldots, \tau_p(i_g)).
\]

Corollary 4.39. The spectrum of the action $\hat{\Psi}_l$ $(1 \leq l \leq 2g)$ on $L^g$ coincides with $\{\eta^a \mid a \in I_p\} \subset \mathbb{C}$. Thus one may safely say that, for each $\hat{\Psi}_l$, a copy of $I_p$ parametrizes its spectrum.

Proof. Theorem [4.35] shows the assertion for $\hat{\Psi}_{2k-1}$ $(1 \leq k \leq g)$. But the braid relation implies that $\hat{\Psi}_{2k-1}$ and $\hat{\Psi}_{2k}$ are conjugate to each other in $\text{End}_C(L^g)$, which completes the proof. \(\square\)

Definition 4.40. For each $a, s \in \mathbb{Z}_p$, define the constant $\hat{B}_a^s$ as follows:
\[
\hat{B}_a^s := \begin{cases} 
0 & \text{if } a \not\in I_p, \\
\eta^{ms} + \eta^{-ms} & \text{if } a \in I_p
\end{cases}
\]
, where $m \in \{0, 1, \ldots, \frac{p-1}{2}\}$ is the element such that $I_p(m - \frac{1}{2}) = a$, that is, $m^2 = 2a + \frac{1}{4}$.

Remark 4.41. We have the symmetry $\hat{B}_a^{1-s} = \hat{B}_a^s$.

Definition 4.42. For each $a, s \in \mathbb{Z}_p$, define the constant $\hat{C}_a^s$ as follows:
\[
\hat{C}_a^s := \begin{cases} 
0 & \text{if } a \not\in I_p, \\
\{\eta^{\frac{s}{2}} + \eta^{-(\frac{s}{2})}\}(\eta^{-\frac{1}{2}} - \eta^{\frac{1}{2}})^{-1} & \text{if } a = 0, \\
\{\eta^{\frac{s}{2}} + \eta^{-(\frac{s}{2})}\}(\eta^{-\frac{1}{2}} - \eta^{\frac{1}{2}})^{-1} & \text{if } a = -\frac{1}{8}, \\
\{\eta^{m+\frac{1}{2}}(s-\frac{1}{2}) + \eta^{-m+\frac{1}{2}}(s-\frac{1}{2})\}(\eta^{\frac{1}{2}}(m+\frac{1}{2}) - \eta^{-\frac{1}{2}}(m+\frac{1}{2}))^{-1} & \text{if otherwise}
\end{cases}
\]
, where $m \in \{0, 1, \ldots, \frac{p-1}{2}\}$ is the one such that $m^2 = 2a + \frac{1}{4}$.

Remark 4.43. We have the symmetry $\hat{C}_{1-s} = \hat{C}_a^s$.

Proposition 4.44. For any $a \in \mathbb{Z}/(p)$ and for any $\hat{n} \in (\mathbb{Z}/(p))^g$, it holds that
\[
\Phi(\Psi_{2k}; a)(e^{\hat{n}}) = \delta_{a,(\eta_{k+1})} e^{\hat{n}}
\]
Proof. The result follows immediately from Proposition [4.26]. \(\square\)

Proposition 4.45. For any $a \in \mathbb{Z}/(p)$ and for any $\hat{i} \in (\mathbb{Z}/(p))^g$, it holds that
\[
\Phi(\Psi_{2k}; a)(e^{\hat{i}}) = \frac{1}{p} \sum_{s \in \mathbb{Z}/(p)} \eta^{-\frac{1}{8}} \hat{B}_a^s e_{\hat{i} + s(e_k - e_{k+1})}.
\]

36
Proof.

\[ \Phi (\Psi_{2k}; a) (e_i) \]
\[ = \alpha_g \sum_{\hat{n} \in \mathbb{Z}/(p)^g} \eta^{-\hat{n} \cdot \hat{\Omega}} \Phi (\Psi_{2k}; a) (e^*_n) \]
\[ = \alpha_g \sum_{\hat{n}' \cdot \hat{n}' \perp \hat{v}_k} \sum_{m_1 = n_2} \eta^{-(\hat{n}' + m_1 \cdot \hat{v}_k)} \hat{\Omega} e^*_n \]
\[ = \alpha_g \hat{\alpha}_g \sum_{\hat{n}' \cdot \hat{n}' \perp \hat{v}_k} \sum_{m_1 = n_2} \sum_{\hat{j} \in \mathbb{Z}/(g)} \eta^{-(\hat{n}' + m_1 \cdot \hat{v}_k)} \hat{\Omega}(\hat{i} - \hat{j}) e^*_n \]
\[ = \frac{1}{p} \sum_{m_1 = n_2} \sum_{\hat{j} \in \mathbb{Z}/(g) \setminus \mathbb{Z}/(p)} \eta^{-(\hat{n}' + m_1 \cdot \hat{v}_k)} \hat{\Omega}(\hat{i} - \hat{j}) e^*_n \]
\[ = \frac{1}{p} \sum_{m_1 = n_2} \sum_{\hat{j} \in \mathbb{Z}/(g) \setminus \mathbb{Z}/(p) \setminus \mathbb{Z}/(1-T)\hat{v}_k} \eta^{n_1 \cdot \hat{a}} e^*_n \]
\[ = \frac{1}{p} \sum_{s \in \mathbb{Z}/(p)} \sum_{m_1 = n_2} \eta^{n_1 \cdot \hat{a}} e^*_n \]

, where we have used the result of the previous proposition at the 2nd equality and the fact that \( \hat{\Omega}^{-1} = 1 - T \) at the 3rd to the last equality. Thus the result follows from Lemma 4.46 below. \( \square \)

Lemma 4.46. For any \( a \in \mathbb{Z}/(p) \), it holds that

\[ \sum_{m_1 = n_2} \eta^{n_1 \cdot \hat{a}} = \eta^{-\hat{2}} \hat{B^a}_s. \]

Proof. Put \( m = n + \frac{1}{2} \). Then,

\[ L.H.S. = \sum_{m: m^2 = 2a + \frac{1}{4}} \eta^{(m - \frac{1}{2}) s} = \eta^{-\hat{2}} \sum_{m: m^2 = 2a + \frac{1}{4}} \eta^{m s} = R.H.S. \]

\( \square \)

Theorem 4.47. For \( 1 \leq k \leq g \), \( 1 \leq m \leq 2g \) and for any \( a \in \mathbb{Z}/(p) \), \( i \in (\mathbb{Z}/(p))^g \), it holds that

\[ \Phi (\hat{\Psi}_{2k, ss}; a) (e_i; u_m) = \frac{1}{p} \sum_{s \in \mathbb{Z}/(p)} \eta^{-\hat{2}} \hat{B^a}_s e^*_n \]
\[ + \delta_{i, 2k-1} \frac{1}{p} \sum_{s \in \mathbb{Z}/(p)} \eta^{-(i_1 + \cdots + i_k) s} \hat{c^a}_s e^*_n \]
\[ + \delta_{i, 2k-1} \frac{1}{p} \sum_{s \in \mathbb{Z}/(p)} \eta^{-(i_1 + \cdots + i_k) s} \hat{c^a}_s e^*_n (u_{2k} - u_{2k+2}). \]
Proof. In the case where $m \neq 2k - 1$, we see that

$$\Phi \left( \hat{\Psi}_{2k,s}; a \right) (e_i u_m) = \Phi \left( \hat{\Psi}_{2k,s}; a \right) (e_i) u_m.$$ 

Thus the result follows directly from Proposition 4.45.

Now we are going to handle the case where $m = 2k - 1$. Applying the 3rd formula in Theorem 4.31 successively, it follows that, for each $l \in \mathbb{Z}/(p)$,

$$(\hat{\Psi}_{2k,s})^l (e_n^* u_{2k-1})
= (\Psi_{2k})^l (e_n^*)
\cdot \left\{ u_{2k-1} + x_{2g}^{-1} \cdots x_{2k+2}^{-1} x_{2k}^{-1} \left( 1 + x_{2k}^{-1} + \cdots + x_{2l-1}^{-1} - l \phi(x_{2l}) \right) x_{2k-1}^{-1} x_{2k-3}^{-1} \cdots x_1^{-1} (u_{2k} - u_{2k+2}) \right\}
= \eta^{(n_k + 1)} l \left\{ e_n^* u_{2k-1} + \delta_{n_k \neq 0} \frac{1 - \eta^{-ln} \eta^{-(n+\cdots+g)}}{1 - \eta^{-n_k}} e_n^*(u_{2k} - u_{2k+2}) \right\}
= \eta^{(n_k + 1)} l e_n^* u_{2k-1}
+ \delta_{n_k \neq 0} \left( \eta^{(n_k + 1)} l - \eta^{(n_k + 2)} l \right) \eta^{n_k - 1} \eta^{-(n_k+1+\cdots+g)} e_n^*(u_{2k} - u_{2k+2}).$$

Multiplying both sides by $p^{-1} \eta^{-al}$ and summing up the results with $l$ running through $\mathbb{Z}/(p)$, we obtain

$$\Phi \left( \hat{\Psi}_{2k,s}; a \right) (e_n^* u_{2k-1})
= \Phi \left( \Psi_{2k}; a \right) (e_n^*) u_{2k-1}
+ \delta_{n_k \neq 0} \left( \delta_{a,(n_k + 1)} - \delta_{a,(n_k + 2)} \right) \eta^{n_k - 1} \eta^{-(n_k+1+\cdots+g)} e_n^*(u_{2k} - u_{2k+2}).$$

Using the Fourier inversion formula and linearity, we see that

$$\Phi \left( \hat{\Psi}_{2k,s}; a \right) (e_i u_{2k-1})
= \Phi \left( \psi_{2k}; a \right) (e_i) u_{2k-1}
+ \delta_{g} \sum_{\hat{n}:n_k \neq 0, (n_k + 1)} \eta^{\hat{n} \cdot \Omega} (\eta^{n_k - 1} \eta^{-(n_k+1+\cdots+g)} e_{n}^*(u_{2k} - u_{2k+2})
- \delta_{g} \sum_{\hat{n}:n_k \neq 0, (n_k + 2)} \eta^{\hat{n} \cdot \Omega} (\eta^{n_k - 1} \eta^{-(n_k+1+\cdots+g)} e_{n}^*(u_{2k} - u_{2k+2}).$$

The 1st term has been calculated just before. We will manage the 2nd and the 3rd
Thus, when one executes the summation above with respect to the variable $\hat{n}'$ first, each term could give non trivial contribution only if $\Omega(\hat{i} - \hat{j} + e_{k+1}) \in \mathbb{F}_p e_k$, which is equivalent to the condition that $\hat{j} = \hat{i} + e_{k+1} + s(1 - T)e_k = (\hat{i} + s e_k + (1 - s)e_{k+1})$ for some $s \in \mathbb{Z}/(p)$. Thus, R.H.S.

$$= p^{-1} \sum_{s \in \mathbb{Z}/(p)} \sum_{n: n \neq 0, \binom{n}{2} = a} \eta^{n - s - e_k \cdot \Omega(\hat{i} + e_{k+1} + s(e_k - e_{k+1}))} (\eta^n - 1)^{-1} e_{i + s e_k + (1 - s)e_{k+1}} (u_{2k} - u_{2k+2})$$

Almost the same calculation shows that

3rd term

$$= -p^{-1} \sum_{s \in \mathbb{Z}/(p)} \sum_{n: n \neq 0, \binom{n}{2} = a} \eta^{(n-1)s - (i_1 + \cdots + i_k)} (\eta^n - 1)^{-1} e_{i + s e_k + (1 - s)e_{k+1}} (u_{2k} - u_{2k+2})$$

Thus the result follows from Lemma 4.48 below, whose proof is a tedious calculation.

\[ \square \]

**Lemma 4.48.** For any $a, s \in \mathbb{Z}/(p)$, it holds that

$$\sum_{n: \binom{n}{2} = a} \{ \delta_{n \neq 0} \eta^{(n-1)s} (\eta^n - 1)^{-1} - \delta_{n = -1} \eta^{ns} (\eta^{n+1} - 1)^{-1} \} = \eta^{-s} \tilde{C}_a.$$
5 The subalgebra of $\text{End}_C(L^n_\eta)$ generated by the $B_{2g+1}$ action

5.1 Setting

The $B_{2g+1}$-action on $L^n_\eta$ induces an algebra morphism $\hat{\Psi}_C : \mathbb{C}[B_{2g+1}] \to \text{End}_C(L^n_\eta)$. The important problem is to describe the image of this morphism as explicitly as possible. For example, the Burnside theorem states that $L^n_\eta$ is an irreducible $B_{2g+1}$-module if and only if $\hat{\Psi}_C$ is surjective.

Since the endomorphisms $\hat{\Psi}_{2k-1} = \hat{\Psi}_C(\sigma_{2k-1}) \in \text{End}_C(L^n_\eta)$ for $1 \leq k \leq g$ commute to each other, we can consider their simultaneous generalized eigenspaces. Set the projections to these generalized eigenspaces as

$$\text{Proj}(\hat{\Psi}_\text{odd,ss}; \hat{a}) := \prod_{k=1}^{g} \Phi \left( \hat{\Psi}_{2k-1,\text{ss}}; a_k \right) \quad (\hat{a} \in I^g_p)$$

, which are idempotents of the algebra $\text{End}_C(L^n_\eta)$.

Definition 5.1. For any $\hat{a}, \hat{b} \in I^g_p$, set

$$\text{Hom}(\eta; \hat{a}, \hat{b}) := \text{Proj}(\hat{\Psi}_\text{odd,ss}; \hat{a}) \circ \text{End}_C(L^n_\eta) \circ \text{Proj}(\hat{\Psi}_\text{odd,ss}; \hat{b}),$$

$$\text{End}(\eta; \hat{a}) := \text{Hom}(\eta; \hat{a}, \hat{a}),$$

$$L^n_\eta(\hat{a}) := \text{Image}(\text{Proj}(\hat{\Psi}_\text{odd,ss}; \hat{a})).$$

Thus by definition, we see that

$$v \in L^n_\eta(\hat{a}) \iff \hat{\Psi}_{2k-1}(v) = \eta^{a_k}v \quad (1 \leq k \leq g).$$

Obviously we have the direct sum decompositions

$$L^n_\eta = \bigoplus_{\hat{a} \in I^g_p} L^n_\eta(\hat{a}), \quad \text{End}_C(L^n_\eta) = \bigoplus_{\hat{a}, \hat{b} \in I^g_p} \text{Hom}(\eta; \hat{a}, \hat{b}) \equiv \bigoplus_{\hat{a}, \hat{b} \in I^g_p} \text{Hom}_C(L^n_\eta(\hat{a}), L^n_\eta(\hat{b})).$$

Thus it holds that

$$\text{Hom}(\eta; \hat{a}, \hat{b})\text{Hom}(\eta; \hat{c}, \hat{d}) = \begin{cases} \text{Hom}(\eta; \hat{a}, \hat{d}) & \text{if } \hat{b} = \hat{c}, \\ \{0\} & \text{if otherwise.} \end{cases}$$

In particular, $\text{End}(\eta; \hat{a}) \subset \text{End}_C(L^n_\eta)$ is a subalgebra and $L^n_\eta(\hat{a})$ a faithful left $\text{End}(\eta; \hat{a})$-module.

Denote by $B^n_\eta$ the subalgebra $\hat{\Psi}_C(\mathbb{C}[B_{2g+1}]) \subset \text{End}_C(L^n_\eta)$. Since $B^n_\eta$ contains the set of the projections $\{\text{Proj}(\hat{\Psi}_\text{odd,ss}; \hat{a}) \mid \hat{a} \in I^g_p\}$, $B^n_\eta$ decomposes into the direct sum of the homogeneous components as follows;

$$B^n_\eta = \bigoplus_{\hat{a}, \hat{b} \in I^g_p} B^n_{\hat{a}, \hat{b}}$$

40
Now we will introduce a useful new basis of $L^g_\hat{a}$.  

**Definition 5.2.** For $1 \leq k \leq g$ and for any $\hat{j} \in \langle \mathbb{Z}_p \rangle^g$, set 
\[ \tilde{w}^{\hat{j}}_{2k-1} := e^{\hat{j}}(u_{2k-1} - u_{2k-3}), \]
\[ \tilde{v}^{\hat{j}}_{2k} := \eta^{-(j_1 + \cdots + j_k - 1)} \Phi \left( \hat{\Psi}_{2k-1, ss}; \left( j_k + 1 \right) \over 2 \right) (e^{\hat{j}} u_{2k}) \]
\[ = \eta^{-(j_1 + \cdots + j_k - 1)} e^{\hat{j}} u_{2k} - \delta_{j_k \neq 0} (1 - \eta^{-j_k})^{-1} e^{\hat{j} - e_k} (u_{2k-1} - u_{2k-3}). \]

Note that in the case where $k = 1$ we presume $u_{-1} \equiv 0$. Further, notice that we have used the result of Theorem 4.3 at the 2nd equality in the 2nd equation.

By the very definition, it holds that for $1 \leq k \leq g$ and for any $\hat{j} \in \hat{\tau}_p^{-1}(\hat{a})$,
\[ \tilde{w}^{\hat{j}}_{2k-1}, \tilde{v}^{\hat{j}}_{2k} \in L^g_\hat{a}. \]
In fact, these vectors compose a basis of $L^g_\hat{a}$.

**Definition 5.3 (Distinguished Basis).** The distinguished basis of $L^g_\hat{a} (\hat{a} \in \mathbb{I}^g_p)$ is
\[ \left\{ \tilde{w}^{\hat{j}}_{2k-1}, \tilde{v}^{\hat{j}}_{2k} \mid 1 \leq k \leq g, \hat{j} \in \hat{\tau}_p^{-1}(\hat{a}) \right\}. \]

**Definition 5.4 (Distinguished Subspace).** A subspace $V \subset L^g_\hat{a}$ is distinguished if $V$ is spanned by some subset $F$ of the distinguished basis of $L^g_\hat{a}$. If it is the case, we say that $V$ has the distinguished basis $F$.

**Definition 5.5 (Odd and Even Subspaces).** For any $\hat{a} \in \mathbb{I}^g_p$, define the odd and the even subspaces of $L^g_\hat{a}$ by
\[ L^g_\hat{a} \text{odd}(\hat{a}) := \left\langle \tilde{w}^{\hat{j}}_{2k-1} \mid 1 \leq k \leq g, \hat{j} \in \hat{\tau}_p^{-1}(\hat{a}) \right\rangle_C, \]
\[ L^g_\hat{a} \text{even}(\hat{a}) := \left\langle \tilde{v}^{\hat{j}}_{2k} \mid 1 \leq k \leq g, \hat{j} \in \hat{\tau}_p^{-1}(\hat{a}) \right\rangle_C. \]
, respectively. Thus we have
\[ L^g_\hat{a}(\hat{a}) = L^g_\hat{a} \text{odd}(\hat{a}) \oplus L^g_\hat{a} \text{even}(\hat{a}). \]

**Definition 5.6 (Distinguished Morphism).** A $\mathbb{C}$-linear endomorphism $\phi$ of $L^g_\hat{a}(\hat{a})$ is a distinguished morphism if $\text{Ker}(\phi)$ is distinguished and if, for any distinguished subspace $V \subset L^g_\hat{a}(\hat{a})$, $\phi(V)$ is distinguished.

Recall that we are subject to Convention 4.17 so that we presume the symbol $e_{g+1}$ to be identically zero. Now we add the following convention.
Corollary 5.10. Whenever we meet the symbol \( \tilde{w}_{2g+1} \) or \( \tilde{v}_{2g+2} \), we ignore it, that is, we presume that \( \tilde{w}_{2g+1} = 0 = \tilde{v}_{2g+2} \).

The following formulae are essential ingredients in the subsequent part of the present paper.

**Proposition 5.8.** For any \( k, l \in \{1, 2, \ldots, g\} \), \( a \in \mathbb{Z}/(p) \), \( \hat{j} \in (\mathbb{Z}/(p))^g \), it holds that

\[
\Phi \left( \hat{\Psi}_{2k,ss} ; a \right) (e_j u_{2l-1}) = \frac{1}{p} \sum_{s \in \mathbb{Z}/(p)} \eta^{-\hat{s}} \hat{B}_s a e_j^s (e_k - e_{k+1}) u_{2l-1} + \delta_{l,k} \frac{1}{p} \sum_{s \in \mathbb{Z}/(p)} \hat{C}_s a \delta_s^j + s(j_k + 1) \neq 0 \left( \eta^{s(j_k + 1)} - 1 \right)^{-1} \tilde{w}_{2l-1}^j + s(e_k - e_{k+1}) \\
+ \delta_{l,k} \frac{1}{p} \sum_{s \in \mathbb{Z}/(p)} \hat{C}_s a \delta_s^j - s(j_k + 1 - 1) \neq 0 \left( \eta^{s(j_k + 1)} - 1 \right)^{-1} \tilde{w}_{2l+1}^j + s(e_k - e_{k+1}) \\
+ \delta_{l,k} \frac{1}{p} \sum_{s \in \mathbb{Z}/(p)} \hat{C}_s a \left\{ \eta^{-(j_k + s)} \tilde{v}_{2k}^j + s(e_k + (1-s)e_{k+1}) - \tilde{v}_{2k+2}^j + s(e_k + (1-s)e_{k+1}) \right\}
\]

**Proof.** The result follows plugging in the defining equations of \( \tilde{v} \)s for the formula in Theorem 4.47. \( \square \)

**Definition 5.9.** For \( 1 \leq k \leq g \) and for any \( s \in \mathbb{Z}/(p) \), \( \hat{j} \in (\mathbb{Z}/(p))^g \), set

\[
\nu(j, k, s) := \eta^{-(j_k + s)} \tilde{v}_{2k}^j + s(e_k - e_{k+1}) - \tilde{v}_{2k+2}^j + s(e_k - e_{k+1}).
\]

**Corollary 5.10.** As a direct corollary of the previous proposition, we see that

\[
\Phi \left( \hat{\Psi}_{2k,ss} ; a \right) (e_j u_{2l-1}) \equiv \frac{1}{p} \sum_{s \in \mathbb{Z}/(p)} \hat{C}_s a \nu(\hat{j} + e_{k+1}, k, s) \mod L_{g, odd}^\eta.
\]

**Proposition 5.11.** For \( 1 \leq k, l \leq g \) and for any \( a \in \mathbb{Z}/(p) \), \( \hat{j} \in (\mathbb{Z}/(p))^g \), the following equation modulo \( L_{g, odd}^\eta \) holds:

\[
\Phi \left( \hat{\Psi}_{2k,ss} ; a \right) (\tilde{v}_{2l}) \equiv \frac{1}{p} \sum_{s \in \mathbb{Z}/(p)} \eta^{-\varepsilon(l, k)} \hat{B}_s a \tilde{v}_{2l}^{j + s(e_k - e_{k+1})} \\
- \delta_{l,k} \frac{1}{p} \sum_{s \in \mathbb{Z}/(p)} \hat{C}_s a \nu(\hat{j}, k, s) + \delta_{l,k+1} \frac{1}{p} \sum_{s \in \mathbb{Z}/(p)} \hat{C}_s a \nu(\hat{j}, k, s)
\]

, where

\[
\varepsilon(l, k) = \begin{cases} -1 & \text{if } l = k + 1, \\ 1 & \text{if otherwise}. \end{cases}
\]

42
Proof. Suppose that \( l \neq k + 1 \). Then we see that
\[
\Phi \left( \hat{\Psi}_{2k,ss} \bigg| a \right) \left( \hat{v}_{2l}^j \right)
\]
\[
= \eta^{-(j_1 + \cdots + j_l - 1)} \Phi \left( \hat{\Psi}_{2k,ss} \bigg| a \right) \left( e_j u_{2l} \right)
\]
\[
- \delta_{j_l \neq 0} (1 - \eta^{-j_l})^{-1} \Phi \left( \hat{\Psi}_{2k,ss} \bigg| a \right) \left( e_{j - e_l} (u_{2l - 1} - u_{2l - 3}) \right)
\]
\[
\equiv \frac{1}{p} \sum_{s \in \mathbb{Z}/(p)} \eta^{-2} \tilde{B}_s^a e_{j + s(e_k - e_{k+1})} u_{2l}
\]
\[
- \delta_{l,k} \delta_{j_l \neq 0} (1 - \eta^{-j_l})^{-1} \Phi \left( \hat{\Psi}_{2k,ss} \bigg| a \right) \left( e_{j - e_k} u_{2k - 1} \right)
\]
\[
\equiv \frac{1}{p} \sum_{s \in \mathbb{Z}/(p)} \eta^{-2} \tilde{B}_s^a \hat{v}_{2l}^j \cdot \delta_{j_l \neq 0} (1 - \eta^{-j_l})^{-1} \frac{1}{p} \sum_{s \in \mathbb{Z}/(p)} \hat{c}_s^a v(j, k, s - 1).
\]

Next, suppose that \( l = k + 1 \). Then we see that
\[
\Phi \left( \hat{\Psi}_{2k,ss} \bigg| a \right) \left( \hat{v}_{2k+2}^j \right)
\]
\[
= \eta^{-(j_1 + \cdots + j_k)} \Phi \left( \hat{\Psi}_{2k,ss} \bigg| a \right) \left( e_j u_{2k+2} \right)
\]
\[
- \delta_{j_k \neq 0} (1 - \eta^{-j_k})^{-1} \Phi \left( \hat{\Psi}_{2k,ss} \bigg| a \right) \left( e_{j - e_k} (u_{2k+1} - u_{2k-1}) \right)
\]
\[
\equiv \frac{1}{p} \sum_{s \in \mathbb{Z}/(p)} \eta^{-2} \tilde{B}_s^a e_{j + s(e_k - e_{k+1})} u_{2k+2}
\]
\[
- \delta_{j_k \neq 0} (1 - \eta^{-j_k})^{-1} \Phi \left( \hat{\Psi}_{2k,ss} \bigg| a \right) \left( -e_{j - e_k} u_{2k-1} \right)
\]
\[
\equiv \frac{1}{p} \sum_{s \in \mathbb{Z}/(p)} \eta^{-2} \tilde{B}_s^a \hat{v}_{2k+2}^j \cdot \delta_{j_k \neq 0} (1 - \eta^{-j_k})^{-1} \frac{1}{p} \sum_{s \in \mathbb{Z}/(p)} \hat{c}_s^a v(j, k, s). \]

\( \square \)

**Definition 5.12.** Set \( \hat{\Psi}_{l,\text{unil}} := \hat{\Psi}_{l,\text{uni}} - 1 \) \((1 \leq l \leq 2g)\), which is the nilpotent part of the unipotent part \( \hat{\Psi}_{l,\text{uni}} \) of \( \hat{\Psi}_{l} \).

**Proposition 5.13.** For \( 1 \leq k, l \leq g \) and for any \( \hat{j} \in (\mathbb{Z}/(p))^g \), we have
\[
\hat{\Psi}_{2k-1,\text{unil}} (\hat{w}_{2l-1}^j) = 0, \quad \hat{\Psi}_{2k-1,\text{unil}} (\hat{v}_{2l}^j) = -\delta_{k,l} \delta_{j_k,0} \hat{w}_{2k-1}^{j - e_k}.
\]
Proposition 5.14. For $1 \leq k, l \leq g$ and for any $\hat{j} \in (\mathbb{Z}/(p))^g$, the following equation modulo $L_{g, odd}^\eta$ holds;

$$\Psi_{2k, \text{unil}}(e_j u_{2l-1}) \equiv \frac{1}{p} \sum_{s \in \mathbb{Z}/(p)} \left\{ \eta^{-(j_k + s) + s e_{k+1} + (1-s)e_{k+1}} - \Psi_{2k}^{j} + s e_{k} + (1-s)e_{k+1} \right\} \mod L_{g, odd}^\eta$$

Proof. Proposition 5.13 and Proposition 5.14 are both direct consequences of Theorem 4.33.

5.2 Useful operators of $\mathcal{B}_{0, \hat{0}}^\eta$

Caution. In the present subsection, we halt both Convention 4.17 and Convention 5.7 and will take care of the difference between the ”$k < g$” case and the ”$k = g$” case.

Notation. Denote $\text{Proj}(\hat{\Psi}_{\text{odd}, s}; \hat{0})$ by the symbol $\text{Proj}(\text{odd}; \hat{0})$ for the simplicity of notation.

We introduce several useful operators in $\mathcal{B}_{0, \hat{0}}^\eta$. Recall that $\mathcal{B}_{0, \hat{0}}^\eta$ was defined to be $\mathcal{B}^\eta \cap \text{Hom}(\eta; \hat{0}, \hat{0})$ in the previous subsection. First of all, note that $\hat{j} \in \hat{\tau}_p^{-1}(\hat{0})$ if and only if $j_k \in \{0, -1\}$ $(1 \leq k \leq g)$.

Definition 5.15. Define $B_k, D_l \in \mathcal{B}_{0, \hat{0}}^\eta$ $(1 \leq k \leq g, 1 \leq l \leq g - 1)$ as follows;

$$B_k := -p(\hat{C}_0\eta)^{-1} \text{Proj}(\text{odd}; \hat{0}) \circ \Psi_{2k-1, \text{unil}} \circ \text{Proj}(\text{odd}; \hat{0}) \circ \Psi_{2k, \text{unil}} \circ \text{Proj}(\text{odd}; \hat{0}),$$

$$D_l := p(\hat{C}_0\eta)^{-1} \text{Proj}(\text{odd}; \hat{0}) \circ \Psi_{2l+1, \text{unil}} \circ \text{Proj}(\text{odd}; \hat{0}) \circ \Psi_{2l, \text{unil}} \circ \text{Proj}(\text{odd}; \hat{0}).$$

Proposition 5.16. Suppose that $\hat{j} \in \hat{\tau}_p^{-1}(\hat{0})$. We have the following formulae;

When $k < g$, $B_k(e_j u_{2l-1}) = \delta_{k,l} \begin{cases} w_{2k-1}^j & \text{if } j_k = -1, \\
 -w_{2k-1}^j + e_{k+1} & \text{if } (j_k, j_{k+1}) = (0, -1), \\
 0 & \text{if } (j_k, j_{k+1}) = (0, 0), \\
 \end{cases}$

When $k = g$, $B_g(e_j u_{2l-1}) = \delta_{g,l} \begin{cases} w_{2g-1}^j & \text{if } j_g = -1, \\
 -w_{2g-1}^j + e_k & \text{if } j_g = 0, \\
 \end{cases}$

For $k < g$, $D_k(e_j u_{2l-1}) = \delta_{k,l} \begin{cases} w_{2k+1}^j & \text{if } j_{k+1} = -1, \\
 -w_{2k+1}^j + e_{k+1} & \text{if } (j_k, j_{k+1}) = (-1, 0), \\
 0 & \text{if } (j_k, j_{k+1}) = (0, 0). \\
\end{cases}$

In particular, both $B_k$ and $D_k$ preserve $L_{g, odd}^\eta(\hat{0})$.

Definition 5.17. Define $B_k^\dagger, D_l^\dagger \in \mathcal{B}_{0, \hat{0}}^\eta$ $(1 \leq k \leq g, 2 \leq l \leq g)$ as follows;

$$B_k^\dagger := -p\text{Proj}(\text{odd}; \hat{0}) \circ \Psi_{2k, \text{unil}} \circ \text{Proj}(\text{odd}; \hat{0}) \circ \Psi_{2k-1, \text{unil}} \circ \text{Proj}(\text{odd}; \hat{0}),$$

$$D_l^\dagger := -p\text{Proj}(\text{odd}; \hat{0}) \circ \Psi_{2l-2, \text{unil}} \circ \text{Proj}(\text{odd}; \hat{0}) \circ \Psi_{2l-1, \text{unil}} \circ \text{Proj}(\text{odd}; \hat{0})$$

44
**Proposition 5.18.** For $\hat{j} \in \hat{\tau}^{-1}(\hat{0})$, we have the following formulae modulo $L^\eta_{g_{\text{odd}}} (\hat{0})$;

For $1 \leq k < g$, $B^\dagger_k (\hat{\nu}_{2g}^j) \equiv_{L^\eta_{g_{\text{odd}}} (\hat{0})} \begin{cases} \hat{v}_{2k}^j - \hat{v}_{2k+2}^j & \text{if } (j_k, j_{k+1}) = (0, 0), \\
\hat{v}_{2k}^j - \hat{v}_{2k+2}^j + \eta \hat{v}_{2k}^{j-e_k} - \hat{v}_{2k+2}^{j-e_k} & \text{if } (j_k, j_{k+1}) = (1, 1), \\
0 & \text{if } j_k = -1.
\end{cases} $

For $2 \leq k < g$, $B^\dagger_k (\hat{\nu}_{2g}^j) \equiv_{L^\eta_{g_{\text{odd}}} (\hat{0})} \begin{cases} \hat{v}_{2g}^j + \eta \hat{v}_{2g}^{j-e_k} & \text{if } j_g = 0, \\
0 & \text{if } j_g = -1.
\end{cases} $

Further, both $B^\dagger_k$ and $D^\dagger_k$ vanish on $L^\eta_{g_{\text{odd}}} (\hat{0})$.

**Proof.** Proposition 5.16 and 5.18 above follow immediately from Proposition 5.13 and 5.14 respectively. □

**Corollary 5.19.** It holds that

$$B_*(L^\eta_{g_{odd}} (\hat{0})), D_*(L^\eta_{g_{odd}} (\hat{0})) \subset L^\eta_{g_{odd}} (\hat{0}),$$

$$B^\dagger_*(L^\eta_{g_{odd}} (\hat{0})) = D^\dagger_*(L^\eta_{g_{odd}} (\hat{0})).$$

Further, $B_*$ and $D_*$ are all idempotents. The endomorphisms induced by $B^\dagger_*$ and $D^\dagger_*$ on $L^\eta_{g_{odd}} (\hat{0})/L^\eta_{g_{odd}} (\hat{0})$ are idempotents

**Remark 5.20.** In the case where $g = 1$, we have neither $D_*$ nor $D^\dagger_*$.

**Definition 5.21.** We introduce some important constants for $s \in \mathbb{Z}/(p)$:

$$\hat{\Lambda}_s := \hat{C}_s^0 - \frac{1 + \eta}{(1 + \eta^\frac{1}{2})^2} \hat{C}_s^{-\frac{1}{2}}, \quad \hat{\Gamma}_s := \hat{B}_s^0 - \frac{1 + \eta}{(1 + \eta^\frac{1}{2})^2} \hat{B}_s^{-\frac{1}{2}}.$$ 

Notice that $\hat{\Lambda}_s = \hat{\Lambda}_{1-s}$, $\hat{\Gamma}_s = \hat{\Gamma}_{-s}$ by the symmetry of $\hat{C}_s^0$ and $\hat{B}_s^0$.

**Lemma 5.22.** It holds that

$$\hat{\Lambda}_0 = \hat{\Lambda}_1 = 0, \quad \hat{\Lambda}_{-1} = \hat{\Lambda}_2 = \frac{(1 + \eta)}{(1 - \eta)} \left\{ (\eta^{-\frac{1}{2}} - \eta^\frac{1}{2})^2 - (\eta^{-\frac{1}{2}} - \eta^\frac{1}{2})^2 \right\},$$

$$\hat{\Gamma}_{\pm 1} = (\eta^{-\frac{1}{2}} + \eta^\frac{1}{2})^2 / (\eta^{-\frac{1}{2}} + \eta^\frac{1}{2})^2.$$ 

**Proof.** By the very definition of the constants $\hat{C}_s^a$, we see that

$$\hat{C}_0^a / \hat{C}_0^{-\frac{1}{2}} = \hat{C}_1^a / \hat{C}_1^{-\frac{1}{2}}$$

$$= \frac{\eta^{-\frac{1}{2}} + \eta^\frac{1}{2}}{\eta^{-\frac{1}{2}} - \eta^\frac{1}{2}} \cdot \frac{\eta^{-\frac{1}{2}} - \eta^\frac{1}{2}}{\eta^{-\frac{1}{2}} + \eta^\frac{1}{2}} = \frac{(\eta^{-\frac{1}{2}} + \eta^\frac{1}{2})}{(\eta^{-\frac{1}{2}} + \eta^\frac{1}{2})^2}.$$
Thus, the 1st assertion follows. Next,

\[ \hat{\Lambda}_1 = \hat{\Lambda}_2 = \frac{\eta^{-\frac{1}{2}} + \eta^{\frac{3}{2}}}{\eta^{-\frac{1}{2}} - \eta^{\frac{3}{2}}} - \frac{\eta^{-\frac{1}{2}} + \eta^{\frac{3}{2}}}{(\eta^{-\frac{1}{4}} + \eta^{\frac{1}{4}})^2} \cdot \frac{\eta^{-\frac{1}{2}} + \eta^{\frac{1}{2}}}{\eta^{-\frac{1}{2}} - \eta^{\frac{1}{2}}} \]

\[ = \frac{\eta^{-\frac{1}{2}} + \eta^{\frac{3}{2}}}{\eta^{-\frac{1}{2}} - \eta^{\frac{3}{2}}} \cdot \frac{\eta^{-\frac{1}{2}} + \eta^{\frac{3}{2}}}{\eta^{-\frac{1}{2}} + \eta^{\frac{1}{2}}} - \frac{\eta^{-\frac{1}{2}} + \eta^{\frac{3}{2}}}{\eta^{-\frac{1}{2}} - \eta^{\frac{3}{2}}} \cdot \frac{\eta^{-\frac{1}{2}} + \eta^{\frac{3}{2}}}{\eta^{-\frac{1}{2}} + \eta^{\frac{1}{2}}} \]

\[ = \frac{\eta^{-\frac{1}{2}} + \eta^{\frac{3}{2}}}{\eta^{-\frac{1}{2}} - \eta^{\frac{3}{2}}} \cdot ((\eta^{-1} - \eta) - (\eta^{-\frac{1}{2}} - 1 + \eta^{\frac{1}{2}})) \]

\[ = \frac{\eta^{-\frac{1}{2}} + \eta^{\frac{3}{2}}}{\eta^{-\frac{1}{2}} - \eta^{\frac{3}{2}}} \cdot ((\eta^{-1} - 2 + \eta) - (\eta^{-\frac{1}{2}} - 2 + \eta^{\frac{1}{2}})) \]

\[ = \frac{\eta^{-\frac{1}{2}} + \eta^{\frac{1}{2}}}{\eta^{-\frac{1}{2}} - \eta^{\frac{3}{2}}} \cdot ((\eta^{-\frac{1}{2}} - \eta^{\frac{1}{2}})^2 - (\eta^{-\frac{1}{2}} - \eta^{\frac{3}{2}})^2) \].

\[ \square \]

**Definition 5.23** (The Constant \( h_\eta \)). Define the constant \( h_\eta := (\eta^{-\frac{1}{2}} + \eta^{\frac{3}{2}})^{-1} \).

**Remark 5.24.** Recal that \( \eta \) is a primitive \( p \)th root of unity. Then we see that

\( h_\eta = \pm 1 \iff h_\eta = 1 \iff \eta^3 = 1 \iff p = 3 \)

since \( p \) is odd and since

\( (\eta^{-\frac{1}{2}} + \eta^{\frac{3}{2}} - 1)(\eta^{-\frac{1}{2}} + \eta^{\frac{3}{2}} + 1) = \eta + \eta^{-1} + 1 \).

**Corollary 5.25.** It holds that

\( \hat{\Lambda}_2 = \hat{\Lambda}_1 = (\eta^{-\frac{1}{2}} - \eta^{\frac{1}{2}}) \{1 + h_\eta\} \hat{\Gamma}_{\pm 1} \).

**Proof.**

\[ \hat{\Lambda}_2(\eta^{-\frac{1}{2}} - \eta^{\frac{1}{2}})^{-1}/\hat{\Gamma}_{\pm 1} = (\eta^{-\frac{1}{2}} + \eta^{\frac{1}{2}})^{-1} \cdot \left( \frac{\eta^{-\frac{1}{2}} + \eta^{\frac{3}{2}}}{\eta^{-\frac{1}{2}} - \eta^{\frac{3}{2}}} \right)^2 \cdot \left( (\eta^{-\frac{1}{2}} - \eta^{\frac{3}{2}})^2 - (\eta^{-\frac{1}{2}} - \eta^{\frac{1}{2}})^2 \right) \]

\[ = (\eta^{-\frac{1}{2}} + \eta^{\frac{1}{2}})^{-1} \left( (\eta^{-\frac{1}{2}} + \eta^{\frac{1}{2}})^2 - 1 \right) \]

\[ = (\eta^{-\frac{1}{2}} + \eta^{\frac{1}{2}})^{-1} \left( \eta^{-\frac{1}{2}} + \eta^{\frac{1}{2}} + 1 \right) \]

\[ = 1 + h_\eta. \]

\[ \square \]

**Definition 5.26.** Denote by \( L_g^n(\hat{0})^\perp \) the complementary distinguished subspace to \( L_g^n(\hat{0}) \) in \( L_g \), that is,

\[ L_g^n(\hat{0})^\perp := \bigoplus_{\hat{a} \in \mathbb{F}_q \setminus \{\hat{0}\}} L_g^n(\hat{a}). \]
We introduce the other useful operators as follows.

**Definition 5.27.** Define the operator $A_k \in \mathcal{B}^n_{q,0}$ for $1 \leq k \leq g$ as follows:

$$A_k := \frac{p}{\Gamma_1} \cdot \text{Proj}(\text{odd}; \hat{\Theta}) \circ \left\{ \Phi \left( \hat{\Psi}_{2k,ss}; 0 \right) - \frac{(1 + \eta)}{(1 + \eta^2)^2} \Phi \left( \hat{\Psi}_{2k,ss}; -\frac{1}{8} \right) - \hat{\Theta}_0 \right\} \circ \text{Proj}(\text{odd}; \hat{\Theta}).$$

**Proposition 5.28.** Suppose that $\hat{j} \in \tilde{\tau}_p^{-1}(\hat{0})$. If $1 \leq k \leq g - 1$, then we have

$$A_k(e_j u_{2l-1}) \equiv \eta^{-\frac{1}{2}} e_{j+e_k-e_{k+1}} u_{2l-1} + \eta^2 e_{j-e_k+e_{k+1}} u_{2l-1}$$

$$+ \delta_k \cdot (1 + h_\eta) \eta^{-\frac{1}{2}} \hat{w}_{2k-1}^{j+e_k+e_{k+1}}$$

$$+ \delta_k \cdot (1 + h_\eta) \eta^2 \hat{w}_{2k+1}^{j-e_k+e_{k+1}}.$$ 

If $k = g$, then we have

$$A_g(e_j u_{2l-1}) \equiv \eta^{-\frac{1}{2}} e_{j+e_g} u_{2l-1} + \eta^2 e_{j-e_g} u_{2l-1}$$

$$+ \delta_g \cdot (1 + h_\eta) \eta^{-\frac{1}{2}} \hat{w}_{2g-1}^{j+e_g}$$

$$+ \delta_g \cdot (1 + h_\eta) (\eta^{-\frac{1}{2}} - \eta^2) \eta \hat{v}_{2g}^{j-e_g}.$$ 

In particular, $A_k$ preserves $L_{g,\text{odd}}^\eta(\hat{0}) \subset L_{g}^\eta(\hat{0})$ if $1 \leq k \leq g - 1$.

**Proof.** We will make use of Prop 5.28. By linearity, if $1 \leq k \leq g - 1$, then we have

$$\hat{\Gamma}_1 \cdot A_k(e_j u_{2l-1}) \equiv \eta^{-\frac{1}{2}} \hat{\Gamma}_1 e_{j+e_k-e_{k+1}} u_{2l-1} + \eta^2 \hat{\Gamma}_1 e_{j-e_k+e_{k+1}} u_{2l-1}$$

$$+ \delta_k \cdot \hat{\Lambda}_2 (\eta - 1)^{-1} \hat{w}_{2k-1}^{j+e_k-e_{k+1}}$$

$$+ \delta_k \cdot \hat{\Lambda}_1 (\eta - 1)^{-1} \hat{w}_{2k+1}^{j-e_k+e_{k+1}}.$$ 

If $k = g$, then we have

$$\hat{\Gamma}_1 \cdot A_g(e_j u_{2l-1}) \equiv \eta^{-\frac{1}{2}} \hat{\Gamma}_1 e_{j+e_g} u_{2l-1} + \eta^2 \hat{\Gamma}_1 e_{j-e_g} u_{2l-1}$$

$$+ \delta_g \cdot \hat{\Lambda}_2 (\eta - 1)^{-1} \hat{w}_{2g-1}^{j+e_g}$$

$$+ \delta_g \cdot \hat{\Lambda}_1 \eta \hat{v}_{2g}^{j-e_g}.$$ 

In either case, Corollary 5.25 shows the result.

The reason why the coefficients of all the $\hat{v}$-terms vanish in the case where $1 \leq k \leq g - 1$ is as follows; Corollary 5.10 implies that

$$\hat{\Gamma}_1 \cdot A_k(e_j u_{2l-1}) \equiv \delta_{t,k} \sum_{s \in \mathbb{Z}/(p)} \hat{\Lambda}_s \hat{v}(\hat{j} + e_{k+1}, k, s) \mod L_{g,\text{odd}}^\eta + L_{g}^\eta(\hat{0})^\perp.$$ 

Each term in the sum vanishes if $s = 0, 1$ since $\hat{\Lambda}_0 = 0 = \hat{\Lambda}_1$. On the other hand, if $s \neq 0, 1$, a moment thought shows that $\hat{v}(\hat{j} + e_{k+1}, k, s) \subset L_{g,\text{odd}}^\eta(\hat{0})^\perp$ since $(\hat{j}_k, \hat{j}_{k+1}) + (s, 1 - s) \not\in \{0, -1\}^2$. \[\square\]
As a corollary, we obtain the following formulae;

**Proposition 5.29.** Suppose that \( \hat{j} \in \hat{\tau}_p^{-1}(\hat{0}) \). If \( 1 \leq k \leq g - 1 \), then it holds that, for \( 1 \leq l \leq g \),

\[
A_k(\tilde{w}_{2l-1}^j) \equiv \begin{cases} 
\eta^{-\frac{1}{2}} \tilde{w}_{2l-1}^j + \eta^{\frac{1}{2}} \tilde{w}_{2l-1}^j + (1 + \eta)(\eta^{-\frac{1}{2}} - \eta^{\frac{1}{2}})\eta \tilde{v}_{2g}^j & \text{if } l \neq k, k + 1, \\
\eta^{-\frac{1}{2}} \tilde{w}_{2l-1}^j - \eta^{\frac{1}{2}} \tilde{w}_{2l-1}^j + (1 + \eta)(\eta^{-\frac{1}{2}} - \eta^{\frac{1}{2}})\eta \tilde{v}_{2g}^j & \text{if } l = k, \\
\eta^{-\frac{1}{2}} \tilde{w}_{2l-1}^j - \eta^{\frac{1}{2}} \tilde{w}_{2l-1}^j + (1 + \eta)(\eta^{-\frac{1}{2}} - \eta^{\frac{1}{2}})\eta \tilde{v}_{2g}^j & \text{if } l = k + 1 
\end{cases}
\]

If \( k = g \), then it holds that, for \( 1 \leq l \leq g \),

\[
A_g(\tilde{w}_{2l-1}^j) \equiv \begin{cases} 
\eta^{-\frac{1}{2}} \tilde{w}_{2l-1}^j + \eta^{\frac{1}{2}} \tilde{w}_{2l-1}^j & \text{if } l \neq g, \\
\eta^{-\frac{1}{2}} \tilde{w}_{2l-1}^j - \eta^{\frac{1}{2}} \tilde{w}_{2l-1}^j + (1 + \eta)(\eta^{-\frac{1}{2}} - \eta^{\frac{1}{2}})\eta \tilde{v}_{2g}^j & \text{if } l = g.
\end{cases}
\]

**Corollary 5.30.** If \( 1 \leq k \leq g - 1 \), then it holds that

\[ A_k(L^\eta_{g, \text{odd}}(\hat{0})) \subset L^\eta_{g, \text{odd}}(\hat{0}). \]

Unfortunately, \( A_g \) does not preserve \( L^\eta_{g, \text{odd}}(\hat{0}) \). To compensate this disadvantage, we will introduce the other useful operators, which are the key in the subsequent subsection.

**Definition 5.31.** Set

\[ T_g := (-\eta^{-\frac{1}{2}}h_\eta)^{-1}A_g \circ B_g, \quad T_g^\dagger := (-\eta^{\frac{1}{2}}h_\eta)^{-1}A_g \circ B_g^\dagger. \]

**Proposition 5.32.** For \( \hat{j} \in \hat{\tau}_p^{-1}(\hat{0}) \) and for \( 1 \leq l \leq g \), it holds that

\[
T_g(\tilde{w}_{2l-1}^j) = \delta_{g,l} \begin{cases} 
\tilde{w}_{2g-1}^j & \text{if } j_g = 0, \\
\tilde{w}_{2g+1}^j & \text{if } j_g = -1.
\end{cases}
\]

**Proof.** This follows from Proposition 5.16 and 5.29. \( \square \)

**Proposition 5.33.** For \( \hat{j} \in \hat{\tau}_p^{-1}(\hat{0}) \) and for \( 1 \leq l \leq g \), it holds that

\[
T_g^\dagger(\tilde{w}_{2l-1}^j) = \begin{cases} 
\tilde{v}_{2g}^j - h_\eta^{-1}\tilde{v}_{2g}^j - \eta \tilde{v}_{2g}^j & \text{if } j_g = 0, \\
0 & \text{if } j_g = -1.
\end{cases}
\]

**Proof.** This follows from Proposition 5.18 and 5.35 below. \( \square \)

**Corollary 5.34.** It is readily seen that

\[
T_g(L^\eta_{g, \text{odd}}(\hat{0})) \subset L^\eta_{g, \text{odd}}(\hat{0}), \quad T_g^\dagger(L^\eta_{g, \text{odd}}(\hat{0})) = 0.
\]

Further, both \( T_g \) and the induced action of \( T_g^\dagger \) on \( L^\eta_{g, \text{odd}}(\hat{0}) \) are idempotents.
Proposition 5.35. Suppose that $\hat{j} \in \hat{z}_p^{-1}(\hat{0})$. If $1 \leq k \leq g - 1$, then it holds that, for $1 \leq l \leq g$,
\[
A_k(\hat{v}_{2l}) \equiv \begin{cases} \eta^{-\frac{1}{2}}\hat{v}_{2l}^\eta + \eta\frac{1}{2}\hat{v}_{2l}^\eta_\eta & \text{if } l \neq k, k + 1, \\ \eta\frac{1}{2}\hat{v}_{2k}^\eta - \eta\frac{1}{2}\hat{v}_{2k}^\eta_\eta + (1 + \eta)\hat{v}_{2k+1}^\eta & \text{if } l = k, \\ \eta\frac{1}{2}\hat{v}_{2k+2}^\eta + (1 + \eta)\hat{v}_{2k+1}^\eta_\eta & \text{if } l = k + 1 \end{cases} \mod L_g^{\eta, \hat{0}} + L_{\eta}^{\eta, \hat{0}}.
\]

If $k = g$, then it holds that, for $1 \leq l \leq g$,
\[
A_g(\hat{v}_{2l}) \equiv \begin{cases} \eta^{-\frac{1}{2}}\hat{v}_{2l}^\eta + \eta\frac{1}{2}\hat{v}_{2l}^\eta_\eta & \text{if } l \neq k, \\ \eta\frac{1}{2}\hat{v}_{2g}^\eta - \eta\frac{1}{2}\hat{v}_{2g}^\eta_\eta & \text{if } l = k. \end{cases} \mod L_g^{\eta, \hat{0}} + L_{\eta}^{\eta, \hat{0}}.
\]

**Proof.** The result follows from Proposition 5.11 in almost the same fashion as in the proof of Proposition 5.29. 

Since $A_k$ ($1 \leq k \leq g - 1$) preserves the odd subspace $L_g^{\eta, \hat{0}} \subset L_{\eta}^{\eta, \hat{0}}$, we can consider the induced action of $A_k$ ($1 \leq k \leq g - 1$) on the ”even quotient” $L_{\eta}^{\eta, \hat{0}}$. A short calculation shows that

**Corollary 5.36.** For $1 \leq k \leq g - 1$, the minimal polynomials of the actions of $A_k$ on $L_g^{\eta, \hat{0}}$ and on $L_{\eta}^{\eta, \hat{0}}$ are equal to
\[
t(t^2 - h_\eta^2)(t^2 - 1).
\]

### 5.3 Proof of the Main Theorem

We will devote the whole of the present subsection to prove the following theorem. Recall that $B^{\eta}_{0, 0} \subset \text{End}(\eta; \hat{0})$ is a subalgebra with $1_{g^\eta_{0, 0}} = 1_{\text{End}(\eta; \hat{0})}$.

**Theorem 5.37** (Main Theorem). If $p$ is a prime greater than 3 and $g \geq 2$, then the subalgebra $B^{\eta}_{0, 0}$ coincides with $\text{End}(\eta; \hat{0})$.

First of all, we agree with the following convention;

**Convention.** If we refer to an algebra, say, $A$, we did not assume that $A$ is unital unless otherwise specified. If we have a subset $X$ of an algebra $A$, the subalgebra $B$ that $X$ generates means the minimal subalgebra that contains $X$. Therefore, whether $B$ is unital or not depends on the situation.

**Notation.** For simplicity, we adopt the following notation;
\[
\mathcal{R} := B^{\eta}_{0, 0}, \quad L_g := L_g^{\eta, \hat{0}}, \quad L_{\eta, \text{odd}} := L_{g, \text{odd}}^{\eta, \hat{0}}.
\]
Consider the filtration $F : \{0\} \subset L_{g, \text{odd}} \subset L_g$. Set $\mathcal{R}^F$ to be the subalgebra of $\mathcal{R}$ consisting of the elements that preserve $F$. The actions of $\mathcal{R}^F$ on $L_{g, \text{odd}}$ and on $L_g/L_{g, \text{odd}}$ induce the following two maps respectively:

$$\pi : \mathcal{R}^F \rightarrow \text{End}_C(L_{g, \text{odd}}), \quad \pi' : \mathcal{R}^F \rightarrow \text{End}_C(L_g/L_{g, \text{odd}}).$$

**Definition 5.38.** Define the two algebras $\mathcal{R}_{\text{odd}}$ and $\mathcal{R}_{\text{even}}$ respectively as follows:

$$\mathcal{R}_{\text{odd}} := \pi(\mathcal{R}^F) \subset \text{End}_C(L_{g, \text{odd}}), \quad \mathcal{R}_{\text{even}} := \pi'(\mathcal{R}^F) \subset \text{End}_C(L_g/L_{g, \text{odd}}).$$

Under the same assumption as Theorem 5.37, we propose two assertions;

**Theorem 5.39.** The subalgebra $\mathcal{R}_{\text{odd}}$ coincides with $\text{End}_C(L_{g, \text{odd}})$.

**Theorem 5.40.** The subalgebra $\mathcal{R}_{\text{even}}$ coincides with $\text{End}_C(L_g/L_{g, \text{odd}})$.

The proof of these two are almost the same. We will prove Theorem 5.39 first. As for Theorem 5.40, we will state the outline with some minor changes needed there and omit the detail to avoid unnecessary repetition. Theorem 5.37 follows as a corollary of Theorem 5.39 and Theorem 5.40.

**Definition 5.41.** For the distinguished basis

$$\left\{ \check{w}_{2k-1}^j \mid 1 \leq k \leq g, \check{j} \in \check{\tau}^{-1}(\hat{0}) \right\}$$

of $L_{g, \text{odd}}$, the corresponding distinguished endomorphisms $H_{i, l}^{j, k} \in \text{End}_C(L_g)$ are determined by

$$H_{2l-1,2m-1}^{\hat{i}, \hat{j}}(\check{w}_{2n-1}^k) := \delta_{m,n} \delta^{\hat{i}, \hat{j}} \check{w}_{2l-1}^{\hat{i}} (\hat{i}, \hat{j}, \hat{k} \in \check{\tau}^{-1}(\hat{0}), 1 \leq l, m, n \leq g).$$

In particular, denote the corresponding idempotents as follows;

$$E_l^i \equiv H_{2l-1,2l-1}^{\hat{i}, \hat{i}} (\hat{i} \in \check{\tau}^{-1}(\hat{0}), 1 \leq l \leq g).$$

**Notation.** For two multi-indexes $\hat{j}_1$ of length $k$ and $\hat{j}_2$ of length $l$, the juxtaposition $\hat{j}_1 \hat{j}_2$ is regarded as a multi-index of length $k + l$.

We will add some notations.

**Definition 5.42.** For $h$ $(1 \leq h \leq g)$, set

$$W_{\leq h} := \langle \check{w}_{2l-1}^j \mid 1 \leq l \leq h, \check{j} \in \{0, -1\}^g \rangle_C,$$

$$W_h := \langle \check{w}_{2l-1}^j \mid \check{j} \in \{0, -1\}^g \rangle_C,$$

$$W_{\geq h} := \langle \check{w}_{2l-1}^j \mid h \leq l \leq g, \check{j} \in \{0, -1\}^g \rangle_C.$$
The subspaces $W_{<h}$ and $W_{≥h}$ are defined in the same fashion. For example, we see that $L_{g_{\text{odd}}} = W_{<h} \oplus W_{≥h}$. The distinguished projection to $W_{≥h}$ and $W_h$ are written respectively as

$$P_{≥h} := \sum_{l=h}^{g} \sum_{\hat{\alpha} \in \{0, -1\}^g} H_{2l-1,2l-1}^{\hat{\alpha}} \hat{\alpha},$$

$$P_h := \sum_{\hat{\alpha} \in \{0, -1\}^g} H_{2h-1,2h-1}^{\hat{\alpha}} \hat{\alpha}.$$

We will use the Burnside theorem below everywhere without notice.

**Theorem 5.43 (Burnside Theorem).** Let $A$ be a $\mathbb{C}$-algebra and $V$ a $A$-module. Denote by $\text{res}_V : A \rightarrow \text{End}_\mathbb{C}(V)$ the map induced by the $A$-action on $V$. Assume that $\text{res}_V \neq 0$. Then $V$ is an irreducible $A$-module if and only if $\text{res}_V$ is surjective.

We provide some useful lemmas below, which are corollaries of the Burnside theorem.

**Lemma 5.44.** Let $V_i$ ($i = 1, 2, 3$) be finite dimensional $\mathbb{C}$-vector spaces. Consider the direct sum decomposition

$$\text{End}_\mathbb{C}(V_1 \oplus V_2 \oplus V_3) = \bigoplus_{1 \leq i \neq j \leq 3} \text{Hom}_\mathbb{C}(V_i, V_j).$$

Suppose that

$$f \in \text{Hom}_\mathbb{C}(V_1, V_2) \oplus \text{End}_\mathbb{C}(V_3, V_3) \quad \text{and} \quad g \in \text{Hom}_\mathbb{C}(V_2, V_1) \oplus \text{End}_\mathbb{C}(V_3, V_3)$$

satisfy the condition that $f|_{V_1} : V_1 \rightarrow V_2$ is surjective and $g|_{V_2} : V_2 \rightarrow V_1$ is injective. Then the subalgebra $\mathcal{S}$ generated by the subset $\text{End}_\mathbb{C}(V_1) \cup \{f, g\} \subset \text{End}_\mathbb{C}(V_1 \oplus V_2 \oplus V_3)$ contains $\text{End}_\mathbb{C}(V_1 \oplus V_2)$.

**Proof.** The statement is obviously true if either $V_1$ or $V_2$ is equal to $\{0\}$. Thus we may assume that neither of them be $\{0\}$. The two conditions on $f$ and $g$ together imply that

$$\text{End}_\mathbb{C}(V_2) = f \circ \text{End}_\mathbb{C}(V_1) \circ g.$$

Therefore, $\mathcal{S}$ contains the subalgebra $\mathcal{T} := \text{End}_\mathbb{C}(V_1) \oplus \text{End}_\mathbb{C}(V_2)$. It is enough to show that $V_1 \oplus V_2$ is irreducible as a $\mathcal{S}$-module. Let $M \subset V_1 \oplus V_2$ be a minimal non zero $\mathcal{S}$-submodule. Since $V_1$ and $V_2$ are irreducible $\mathcal{T}$-modules not isomorphic to each other, $M$ must contain at least one of them. But none of them can be a $\mathcal{S}$-submodule because $f(V_1) \setminus V_1 \neq \emptyset \neq g(V_2) \setminus V_2$. Thus $M$ must contain both of them, which implies that $M = V \oplus W$. Thus we are done. □

**Lemma 5.45.** Let $U, V_1, \ldots, V_k, W$ be finite dimensional $\mathbb{C}$-vector spaces. Suppose that we have

$$f_i \in \text{Hom}_\mathbb{C}(U, V_i) \oplus \text{End}_\mathbb{C}(W), \quad g_i \in \text{Hom}_\mathbb{C}(V_i, U) \oplus \text{End}_\mathbb{C}(W) \quad (1 \leq i \leq l)$$

51
satisfying the condition that all \( f_i|_U : U \to V_i \) are surjective and all \( g_i|_{V_i} : V_i \to U \) are injective. Then the subalgebra \( \mathcal{S} \) generated by the subset

\[
\text{End}_\mathbb{C}(U) \cup \{ f_i, g_i \mid 1 \leq i \leq k \} \subset \text{End}_\mathbb{C}(U \oplus \bigoplus_{1 \leq i \leq k} V_i \oplus W)
\]

contains

\[
\text{End}_\mathbb{C}(U \oplus \bigoplus_{1 \leq i \leq k} V_i).
\]

**Proof.** Applying Lemma 5.44 successively, one can show that

\[
\text{End}_\mathbb{C}(U \oplus \bigoplus_{1 \leq i \leq l} V_i) \subset \mathcal{S}
\]

by an upward inductive argument with respect to \( l = 1, 2, \ldots, k \).

**Proof of Theorem 5.39.**) The proof divides into 2 steps.

1st Step.) We will show the following assertion:

**Proposition 5.46.** \( \mathcal{R}_{\text{odd}} \) contains the 4-dimensional subalgebra spanned by the distinguished morphisms below;

\[
\sum_{\hat{\beta} \in \{0, -1\}^{g-1}} H_{2g-1, 2g-1}^{\hat{\beta}(i), \hat{\beta}(j)} (i, j \in \{0, -1\}).
\]

In particular, \( \mathcal{R}_{\text{odd}} \) includes the distinguished projection \( P_g : L_{g\text{odd}} = W_{g<} \oplus W_g \to W_g \).

**Remark 5.47.** Proposition 5.46 implies that, for each \( \hat{\beta} \in \{0, -1\}^{g-1} \),

\[
W_g(\hat{\beta}) := \langle \tilde{w}_{2g-1}^{\hat{\beta}(i)} \mid i \in \{0, -1\} \rangle_{\mathbb{C}}
\]

is an irreducible and faithful \( P_g \circ \mathcal{R}_{\text{odd}} \circ P_g \)-module.

**Definition 5.48.** For any multi-index \( \hat{\alpha} \in \{0, -1\}^{g-2} \), define the following distinguished subspaces of \( L_{g\text{odd}} \);

\[
W_g^+(\hat{\alpha}) := \langle \tilde{w}_{2g-3}^{\hat{\alpha}(-1,0)}, \tilde{w}_{2g-3}^{\hat{\alpha}(0,-1)}, \tilde{w}_{2g-1}^{\hat{\alpha}(-1,0)}, \tilde{w}_{2g-1}^{\hat{\alpha}(0,-1)}, \tilde{w}_{2g-1}^{\hat{\alpha}(-1,-1)}, \tilde{w}_{2g-1}^{\hat{\alpha}(0,0)} \rangle_{\mathbb{C}},
\]

\[
W_g^+ := \bigoplus_{\hat{\alpha} \in \{0, -1\}^{g-2}} W_g^+(\hat{\alpha}) \subset W_{g-1} \oplus W_g.
\]

**Definition 5.49.** Denote by \( \mathcal{W} \) the subalgebra of \( \mathcal{R}_{\text{odd}} \) generated by \( \{ \pi(A_{g-1}), \pi(B_g), \pi(T_g) \} \).

Notice that each \( W_g^+(\hat{\alpha}) \) is a \( \mathcal{W} \)-submodule and that the distinguished complement \( W_g^+ \) to \( W_g^+ \subset L_{g\text{odd}} \) is preserved under the \( \mathcal{W} \)-action.

52
Definition 5.50. Let
\[ \text{res}_\alpha^\dagger : \mathcal{W} \to \text{End}_C(W_g^\dagger(\hat{\alpha})) \]
be the map induced by the \( \mathcal{W} \)-action on \( W_g^\dagger(\hat{\alpha}) \).

Notice that there is the canonical (obvious) identification \( \tau_{\hat{\beta}, \hat{\alpha}} : W_g^\dagger(\hat{\alpha}) \cong W_g^\dagger(\hat{\beta}) \) for any \( \hat{\alpha}, \hat{\beta} \in \{0, -1\}^{g-2} \), which turns out to be a \( \mathcal{W} \)-module isomorphism.

Definition 5.51. For each \( \hat{\alpha} \in \{0, -1\}^{g-2} \), define the distinguished subspace \( W_g^\circ(\hat{\alpha}) \subset W_g^\dagger(\hat{\alpha}) \) to be that spanned by the distinguished subbasis
\[ \{ \hat{w}_2g_{-3}^{\hat{\alpha}(0,-1)}, \hat{w}_2g_{-3}^{\hat{\alpha}(0,-1)}, \hat{w}_2g_{-3}^{\hat{\alpha}(0,-1)} \}. \]

Notice that we have the canonical inclusion \( \text{End}_C(W_g^\circ(\hat{\alpha})) \subset \text{End}_C(W_g^\dagger(\hat{\alpha})) \) due to the existence of the distinguished compliment \( W_g^\circ(\hat{\alpha})^\perp \subset W_g^\dagger(\hat{\alpha}) \to W_g^\circ(\hat{\alpha}) \).

Lemma 5.52. It holds that \( \text{End}_C(W_g^\circ(\hat{\alpha})) \subset \text{Image}(\text{res}_\alpha^\dagger) \).

Proof. Fix an \( \hat{\alpha} \in \{0, -1\}^{g-2} \). Set
\[ A := \text{res}_\alpha^\dagger(\pi(A_{g-1})), B := \text{res}_\alpha^\dagger(\pi(B_g)), T := \text{res}_\alpha^\dagger(\pi(T_g)) \in \text{End}_C(W_g^\dagger(\hat{\alpha})). \]

Notice that \( W_g^\circ(\hat{\alpha}) \) is \( A \)-invariant and that \( A|_{W_g^\circ(\hat{\alpha})^\perp} = 0 \).

The representation matrix \( M_A \) of \( A|_{W_g^\circ(\hat{\alpha})} \) is
\[ M_A = \begin{pmatrix} 0 & -h_\eta & 0 & 0 \\ 1 & 0 & 1 + h_\eta & 0 \\ 0 & 1 + h_\eta & 0 & 1 \\ 0 & 0 & -h_\eta & 0 \end{pmatrix} \begin{pmatrix} \eta^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & \eta^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \eta^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & \eta^{-\frac{1}{2}} \end{pmatrix}. \]

The characteristic polynomial of \( M_A \) is \( (t^2 - 1)(t^2 - h_\eta^2) \), which has no multiple root due to the assumption that \( p \neq 3 \). Thus \( M_A \) is diagonalized by the invertible matrix
\[ P = \begin{pmatrix} \eta^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & \eta^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \eta^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & \eta^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} h_\eta & h_\eta & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & -1 & -1 \\ h_\eta & -h_\eta & 1 & -1 \end{pmatrix}, \]
as follows;
\[ P^{-1}M_A P = \text{diag}(1, -1, h_\eta, -h_\eta). \]

A short calculation shows that
\[ P^{-1} = \frac{1}{2(1 - h_\eta)} \begin{pmatrix} -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & h_\eta & h_\eta & 1 \\ 1 & h_\eta & -h_\eta & -1 \end{pmatrix} \begin{pmatrix} \eta^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & \eta^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \eta^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & \eta^{-\frac{1}{2}} \end{pmatrix}. \]
Since the representation matrix of $A$ is semisimple, there exists some polynomial $f_\lambda(t)$ of complex coefficients for each $\lambda \in \{0, \pm 1, \pm h_\eta\}$ such that the projection $P(\lambda) : W^+_g(\hat{\alpha}) \to E_A(\lambda)$ is equal to $f_\lambda(A)$, where $E_A(\lambda)$ denotes the eigenspace of $A$ for the eigenvalue $\lambda$. In particular, the distinguished projection $P^\circ(\hat{\alpha}) : W^+_g(\hat{\alpha}) \to W^\circ_g(\hat{\alpha})$ coincides with the sum $\sum_{\lambda=\pm 1, \pm h_\eta} f_\lambda(A)$, which shows that $P^\circ(\hat{\alpha}) \in \text{Image}(\text{res}_\alpha^\dagger)$.

Set $T' := P^\circ(\hat{\alpha}) \circ T \circ P^\circ(\hat{\alpha})$. It is readily seen that the representation matrix of $T'$ is equal to $\text{diag}(0, 0, 0, 1, 0, 0)$. It follows that $T'$ is an idempotent of rank 1 such that

- $T' \circ P(\lambda)$ does not vanish for $\lambda = \pm 1, \pm h_\eta$.
- $P(\lambda) \circ T'$ does not vanish for $\lambda = \pm 1, \pm h_\eta$.

In fact, all the entries of the square matrix $C$ below are non-zero;

$$C := P^{-1} \text{diag}(0, 0, 0, 1) P = (4\text{th column of } P^{-1})(4\text{th row of } P)$$

Denote by $c_{\lambda, \mu}$ the $(\lambda, \mu)$-entry of the matrix $C$ and set

$$E_{\lambda, \mu} := c_{\lambda, \mu}^{-1} P(\lambda) \circ T' \circ P(\mu) = c_{\lambda, \mu}^{-1} P(\lambda) \circ T \circ P(\mu) \quad (\lambda, \mu = \pm 1, \pm h_\eta).$$

It follows that

$$E_{\kappa, \lambda} \circ E_{\mu, \nu} = \delta_{\mu, \nu} E_{\kappa, \nu}$$

, which implies that $\{E_{\lambda, \mu} \mid \lambda, \mu \in \{\pm 1, \pm h_\eta\}\}$ spans the 16-dimensional subspace $\text{End}_C(W^\circ_g(\hat{\alpha})) \subset \text{End}_C(W^r_g(\hat{\alpha}))$. Notice that each $E_{\lambda, \mu}$ belongs to $\text{Image}(\text{res}_\alpha^\dagger)$ by its construction. Thus we are done. \hfill \Box

Define the distinguished subspace

$$W^\circ_g := \bigoplus_{\hat{\alpha} \in \{0, -1\}^g-2} W^\circ_g(\hat{\alpha}) \subset W_{g-1} \oplus W_g.$$ 

**Corollary 5.53.** There exists a 16-dimensional subalgebra $M^\circ \subset W$ such that

- $W^\circ_g \perp$ vanishes under the $M^\circ$-action, where $W^\circ_g \perp$ is the distinguished complement in $L_{g \text{ odd}}$ of $W^\circ_g$.
- It holds that $\text{res}_\alpha^\dagger(M^\circ) = \text{End}_C(W^\circ_g(\hat{\alpha}))$ for any $\hat{\alpha} \in \{0, -1\}^g-2$.

**Remark 5.54.** The corollary above implies that $W^\circ_g(\hat{\alpha})$ is an irreducible and faithful $M^\circ$-module.

**Proof.** Notice that the whole argument in the proof of Lemma 5.32 is independent of the multi-index $\hat{\alpha}$. We have obtained some subalgebra $W' \subset W$ there that satisfies the following three properties:

- Both $W^+_{g}$ and $W^+_{g} \perp$ are preserved under the $W'$-action, where the latter is the distinguished complement in $L_{g \text{ odd}}$ of $W^+_{g}$.
- $\text{res}_\alpha^\dagger(W') = \text{End}_C(W^\circ_g(\hat{\alpha})) \subset \text{End}_C(W^+_{g})$.
- $W'$ is contained in the ideal $M := W \circ \pi(T_g) \circ W \subset W$.

The 3rd condition together with the 1st one implies that $W^\perp_g$ vanishes under the $M$-action. In fact, $W^\perp_g$ is preserved under the $W'$-action and killed by $\pi(T_g)$. Thus it
follows that
• \( W_g^{\perp} \) vanishes under the \( \mathcal{W}' \)-action, where \( W_g^{\perp} \) is the distinguished complement of \( W_g^\circ \) in \( L_{\text{odd}} \).

Set \( \mathcal{M}^\circ := \mathcal{W}' \), which is what we seek for. Thus we are done .

**Definition 5.55.** For each \( \hat{\alpha} \in \{0, -1\}^{g-2} \), define the distinguished subspace \( W_g^\perp(\hat{\alpha}) \subset W_g^\dagger(\hat{\alpha}) \) to be the one spanned by the distinguished subbasis

\[
\left\{ \hat{w}_{2g-1}^{\hat{\alpha}(-1,0)}, \hat{w}_{2g-1}^{\hat{\alpha}(0,1)}, \hat{w}_{2g-1}^{\hat{\alpha}(-1,-1)}, \hat{w}_{2g-1}^{\hat{\alpha}(0,0)} \right\}.
\]

**Lemma 5.56.** It holds that \( \text{End}_C(W_g^\perp(\hat{\alpha})) \subset \text{Image}(\text{res}_\hat{\alpha}^\dagger) \).

**Proof.** Fix an \( \hat{\alpha} \in \{0, -1\}^{g-2} \). Corollary 5.53 implies that the set of the distinguished endomorphisms

\[
\left\{ E_{2g-1}^{\hat{\alpha}(-1,0)}, E_{2g-1}^{\hat{\alpha}(0,1)}, H_{2g-1,2g-1}^{\hat{\alpha}(-1,0), \hat{\alpha}(0,1)}, H_{2g-1,2g-1}^{\hat{\alpha}(0,1), \hat{\alpha}(-1,0)} \right\} \subset \text{End}_C(W_g^\dagger(\hat{\alpha}))
\]

is contained in \( \text{res}_\hat{\alpha}^\dagger(\mathcal{M}^\circ) \). Consider the distinguished direct sum decomposition

\[
W_g^\perp(\hat{\alpha}) = W_g^-(\hat{\alpha}) \oplus W_g^+(\hat{\alpha})
\]

, where we set

\[
W_g^-(\hat{\alpha}) := \langle \hat{w}_{2g-1}^{\hat{\alpha}(-1,0)}, \hat{w}_{2g-1}^{\hat{\alpha}(-1,-1)} \rangle_C, \quad W_g^+(\hat{\alpha}) := \langle \hat{w}_{2g-1}^{\hat{\alpha}(0,1)}, \hat{w}_{2g-1}^{\hat{\alpha}(0,0)} \rangle_C.
\]

**SubLemma 5.57.** \( \text{End}_C(W_g^\pm(\hat{\alpha})) \subset \text{Image}(\text{res}_\hat{\alpha}^\dagger) \)

**Proof.** Set \( X := E_{2g-1}^{\hat{\alpha}(-1,0)}, X' := E_{2g-1}^{\hat{\alpha}(0,1)} \in \text{End}_C(W_g^\dagger(\hat{\alpha})) \). Further, set \( B := \text{res}_\hat{\alpha}^\dagger(\pi(B_g)), T := \text{res}_\hat{\alpha}^\dagger(\pi(T_g)) \) as in the proof of Lemma 5.52.

• Set \( Y := B - X'B, Z := XT \). Then it is readily seen that \( X, Y, Z \) belong to the subalgebra \( \text{End}_C(W_g^-(\hat{\alpha})) \subset \text{End}_C(W_g^\dagger(\hat{\alpha})) \) and that their representation matrices are

\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
1 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
1 & 1
\end{pmatrix}
\]

, which shows that \( X, Y, Z, YX \) spans the 4-dimensional space \( \text{End}_C(W_g^-(\hat{\alpha})) \).

• Set \( Y' := X'B, Z' := T - XT \). Then it is readily seen that \( X', Y', Z' \) belong to the subalgebra \( \text{End}_C(W_g^+(\hat{\alpha})) \subset \text{End}_C(W_g^\dagger(\hat{\alpha})) \) and that their representation matrices are

\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
1 & 1
\end{pmatrix}
\]

, which shows that \( X', Y', Z', Z'X' \) spans the 4-dimensional space \( \text{End}_C(W_g^+(\hat{\alpha})) \). □
Definition 5.58. Set
\[ W_g^\pm := \bigoplus_{\hat{\alpha} \in \{0, -1\}^{g-2}} W_g^\mp(\hat{\alpha}). \]
Notice that \( W_g^- \oplus W_g^+ = W_g \).

Corollary 5.59. \( W \) includes the distinguished projection \( P^\pm : L_{g, \text{odd}} \rightarrow W_g^\pm \).

Proof. Denote by \( T \) the ideal of \( W \) generated by \( \{\pi(B_g), \pi(T_g)\} \). The proof of SubLemma 5.57 shows that
\[ \text{End}_C(W_g^-(\hat{\alpha})) \oplus \text{End}_C(W_g^+(\hat{\alpha})) \subset \text{res}_\hat{\alpha}^\dagger(T) \subset \text{End}_C(W_g^\dagger(\hat{\alpha})). \]

It follows that there exists the unique \( P^\pm \in T \) such that \( \text{res}_\hat{\alpha}^\dagger(P^\pm) \) is the distinguished projection \( W_g^\pm(\hat{\alpha}) \rightarrow W_g^\mp(\hat{\alpha}) \). This in turn implies that \( P^\pm|_{W_g^\pm} \) is the distinguished projection \( W_g^\dagger \rightarrow W_g^\perp \) since the argument so far is independent of the choice of the multi-index \( \hat{\alpha} \). On the other hand, \( W_g^\perp \) vanishes under the \( T \)-action since it is preserved under the \( W \)-action and since it vanishes under both \( \pi(B_g) \)- and \( \pi(T_g) \)-action. It follows that \( P^\pm(W_g^\perp) = \{0\} \). Thus we are done.

Now we will return to the proof of Lemma 5.56. Thanks to Corollary 5.59, \( W \) includes the distinguished projection
\[ P_g = P^- + P^+ : L_{g, \text{odd}} \rightarrow W_g^- \oplus W_g^+ = W_g. \]
Set \( U := P_g \circ W \circ P_g \). To complete the proof, it is sufficient to show that \( \text{res}_\hat{\alpha}^\dagger(U) = \text{End}_C(W_g^\gamma(\hat{\alpha})) \) for some (therefore all) \( \hat{\alpha} \in \{0, -1\}^{g-2} \). This is equivalent to show that \( W_g^\gamma(\hat{\alpha}) \) is an irreducible \( U \)-module. Assuming to the contrary that we had a nonzero minimal \( U \)-submodule \( M \neq W_g^\gamma(\hat{\alpha}) \). SubLemma 5.57 states that \( \text{res}_\hat{\alpha}^\dagger(U) \) contains the subalgebra \( V_\hat{\alpha} := \text{End}_C(W_g^-(\hat{\alpha})) \oplus \text{End}_C(W_g^+(\hat{\alpha})) \) and that, in the direct sum decomposition
\[ W_g^\gamma(\hat{\alpha}) = W_g^-(\hat{\alpha}) \oplus W_g^+(\hat{\alpha}) \]
, the two components on the R.H.S. are irreducible \( V_\hat{\alpha} \)-modules, which are not isomorphic to each other. Therefore, regarded as a \( V_\hat{\alpha} \)-module, \( M \) must be either \( W_g^-(\hat{\alpha}) \) or \( W_g^+(\hat{\alpha}) \). But \( \text{res}_\hat{\alpha}^\dagger(U) \) includes the distinguished endomorphisms \( H_{2g-1, 2g-1}^{(-1, 0)}, H_{2g-1, 2g-1}^{(0, -1)}, H_{2g-1, 2g-1}^{(0, 0)} \in \text{End}_C(W_g^\dagger(\hat{\alpha})) \), which implies that, if \( M \) contains one of the two, say, \( W_g^-(\hat{\alpha}) \), then it must contain the other, say, \( W_g^+(\hat{\alpha}) \), and vise versa. This leads to a contradiction. Thus we are done.

Now we will complete the proof of Proposition 5.46. Notice that the argument so far is independent of the multi-index \( \hat{\alpha} \). Taking this into account, the subalgebra \( U \subset \mathcal{R}_{\text{odd}} \) that appeared in the last part of the proof of Lemma 5.56 is nothing but what we seek for. Thus we are done.
2nd Step.)

**Definition 5.60.** For $1 \leq h \leq g$, let $\mathcal{R}_{\text{odd},h} \subset \mathcal{R}_{\text{odd}}$ be the subalgebra generated by

$$\{ \pi(A_l) \mid h \leq l \leq g-1 \} \cup \{ \pi(B_g), \pi(T_g) \}.$$ 

**Definition 5.61.** For any multi-index $\hat{\alpha} \in \{0, -1\}^{h-1}$, define the distinguished subspace

$$W_{\geq h}(\hat{\alpha}) := \left\langle \tilde{w}_{2l-1}^{\hat{\gamma}} \mid h \leq l \leq g, \hat{\gamma} \in \{0, -1\}^{g-h+1} \right\rangle \subset L_{g\text{ odd}},$$

which turns out to be a $\mathcal{R}_{\text{odd},h}$-submodule. Note that, if $h = g$, then we denote it by $W_g(\hat{\alpha})$.

Notice that $W_{<h}, W_{\geq h}$ and each $W_{\geq h}(\hat{\alpha})$ are $\mathcal{R}_{\text{odd},h}$-submodules and that we have the direct sum decomposition

$$W_{\geq h} = \bigoplus_{\hat{\alpha} \in \{0, -1\}^{h-1}} W_{\geq h}(\hat{\alpha}).$$

**Definition 5.62.** Denote by $P_{\geq h}$ the distinguished projection $P_{\geq h} : L_{g\text{ odd}} = W_{\geq h} \oplus W_{<h} \twoheadrightarrow W_{\geq h}$.

**Remark 5.63.** Obviously we have the decreasing filtration by subalgebra as follows;

$$\mathcal{R}_{\text{odd}} = \mathcal{R}_{\text{odd},1} \supset \mathcal{R}_{\text{odd},2} \supset \cdots \supset \mathcal{R}_{\text{odd},g} \supset \{0\}.$$ 

For any $\hat{\alpha} = (a_1, \ldots, a_{g-1}) \in \{0, -1\}^{g-1}$, we have the decreasing sequence

$$L_{g\text{ odd}} = W_{\geq 1}(\emptyset) \supset W_{\geq 2}(a_1) \supset W_{\geq 3}(a_1, a_2) \cdots \supset W_g(a_1, \ldots, a_{g-1}) \supset \{0\}.$$ 

, which is compatible with the filtration above in the sense that each $W_{\geq h}(a_1, \ldots, a_{h-1})$ is a $\mathcal{R}_{\text{odd},h}$-submodule for $1 \leq h \leq g$.

**Remark 5.64.** Notice that the isomorphism class of the $\mathcal{R}_{\text{odd},h}$-module $W_h(\hat{\alpha})$ is independent of the choice of $\hat{\alpha}$. In fact, for any two multi-indices $\hat{\alpha}, \hat{\beta}$, we have the canonical $\mathbb{C}$-linear isomorphism

$$\tau_{\hat{\beta} \hat{\alpha}}^{\geq h} : W_{\geq h}(\hat{\alpha}) \xrightarrow{\cong} W_{\geq h}(\hat{\beta}) : \tilde{w}_{2l-1}^{\hat{\gamma}} \mapsto \tilde{w}_{2l-1}^{\hat{\gamma}} \quad (h \leq l \leq g, \hat{\gamma} \in \{0, -1\}^{g-h+1})$$

, which turns out to be a $\mathcal{R}_{\text{odd},h}$-module homomorphism.

**Definition 5.65.** Set

$$\text{res}_{\hat{\alpha}}^{\geq h} : \mathcal{R}_{\text{odd},h} \to \text{End}_\mathbb{C}(W_{\geq h}(\hat{\alpha}))$$

to be the map induced by the $\mathcal{R}_{\text{odd},h}$-action on $W_{\geq h}(\hat{\alpha})$.

We will propose the following assertion $A(h)$ for $1 \leq h \leq g$, which will be proven by the downward inductive argument w.r.t. $h$.  

57
**Assertion. A(h).** For any \( \hat{\alpha} \in \{0, -1\}^{h-1} \), \( W_{\geq h}(\hat{\alpha}) \) is an irreducible \( \mathcal{R}_{\text{odd},h} \)-submodule. Further, \( \mathcal{R}_{\text{odd},h} \) contains the distinguished projection \( P_{\geq h} : L_{g_{\text{odd}}} \to W_{\geq h} \).

**Remark 5.66.** The assertion A(h) implies that \( P_{\geq h} \) is a central idempotent of \( \mathcal{R}_{\text{odd},h} \) and that \( W_{\geq h} = P_{\geq h}(L_{g_{\text{odd}}}) \) is an "isotypic" component of the \( \mathcal{R}_{\text{odd},h} \)-module \( L_{g_{\text{odd}}} \). In fact, we see that \( \mathcal{R}_{\text{odd},h} \) decomposes into the direct sum of ideals as
\[
\mathcal{R}_{\text{odd},h} = (1 - P_{\geq h}) \cdot \mathcal{R}_{\text{odd},h} \cdot (1 - P_{\geq h}) \oplus P_{\geq h} \cdot \mathcal{R}_{\text{odd},h} \cdot P_{\geq h}
\]
since \( L_{g_{\text{odd}}} = W_{<h} \oplus W_{\geq h} \) is faithful with the direct summands being \( \mathcal{R}_{\text{odd},h} \)-submodules.

**Proof of A(g)** This has already been proven by Proposition 5.46.

**Proof of A(h).** Suppose \( h < g \). Assume that A(h+1), . . . , A(g) be true. We will prove A(h). Two remarks here are in order;

**Remark 5.67.** It follows from the induction assumption that the distinguished projection \( P_{h+1} : L_{g_{\text{odd}}} \to W_{h+1} \) is included in \( \mathcal{R}_{\text{odd},h+1} \). In fact, if \( h < g - 1 \), then \( P_{h+1} = P_{\geq h+1} - P_{\geq h+2} \), where the two terms of the R.H.S. are included in \( \mathcal{R}_{\text{odd},h+1} \) by A(h+1) and by A(h+2), respectively. If \( h = g - 1 \), then A(g) implies that \( P_{g} \subset \mathcal{R}_{\text{odd},g} \).

**Remark 5.68.** Notice that \( P_{\geq h+1} \cdot \mathcal{R}_{\text{odd},h+1} \cdot P_{\geq h+1} \) is a subalgebra of \( \mathcal{R}_{\text{odd},h+1} \). The 1st statement of A(h+1) implies that, for any \( \hat{\beta} \in \{0, -1\}^{h} \), the subspace \( W_{\geq h+1}(\hat{\beta}) \) is an irreducible and faithful \( P_{\geq h+1} \cdot \mathcal{R}_{\text{odd},h+1} \cdot P_{\geq h+1} \)-module. Thus the Burnside theorem together with Remark 5.64 implies that, for any distinguished subspace \( U \subset W_{\geq h+1}(\hat{\beta}_{0}) \), the distinguished projection of \( L_{g_{\text{odd}}} \) to the distinguished subspace
\[
\bigoplus_{\hat{\beta} \in \{0, -1\}^{h}} \tau_{\hat{\beta},\hat{\beta}_{0}}^{h}(U) \subset L_{g_{\text{odd}}}
\]
is included in \( P_{\geq h+1} \cdot \mathcal{R}_{\text{odd},h+1} \cdot P_{\geq h+1} \subset \mathcal{R}_{\text{odd},h} \).

We will define a useful subalgebra \( A_h \subset \mathcal{R}_{\text{odd},h} \).

**Definition 5.69.** Let \( P_{h+1}^{(-1)} : L_{g_{\text{odd}}} \to U_{h+1} \subset L_{g_{\text{odd}}} \) be the distinguished projection where we set
\[
U_{h+1} := \langle \bar{w}_{2h+1}^{(\hat{\beta}^{(-1)}\hat{\gamma})} | \hat{\beta} \in \{0, -1\}^{h}, \hat{\gamma} \in \{0, -1\}^{g-h-1} \rangle_{C}.
\]

**Lemma 5.70.** It holds that \( P_{h+1}^{(-1)} \in P_{\geq h+1} \cdot \mathcal{R}_{\text{odd},h+1} \cdot P_{\geq h+1} \).

**Proof.** For any \( \hat{\beta} \in \{0, -1\}^{h} \), set the distinguished subspace
\[
U_{h+1}(\hat{\beta}) := \langle \bar{w}_{2h-1}^{(\hat{\gamma}^{(-1)}\hat{\gamma})} | \gamma \in \{0, -1\}^{g-h-1} \rangle_{C} \subset W_{\geq h+1}(\hat{\beta}).
\]

Notice that
\[
U_{h+1} = \bigoplus_{\hat{\beta} \in \{0, -1\}^{h}} U_{h+1}(\hat{\beta}) = \bigoplus_{\hat{\beta} \in \{0, -1\}^{h}} \tau_{\hat{\beta},\hat{\beta}_{0}}^{h}(U_{h+1}(\hat{\beta}_{0})).
\]

Applying the argument in Remark 5.68 to \( U_{h+1}(\hat{\beta}) \subset W_{\geq h+1}(\hat{\beta}) \), the result follows. \( \square \)
Definition 5.71. Let $\mathcal{A}_h \subset \mathcal{R}_{\text{odd}, h}$ be the subalgebra generated by $\{\pi(A_h), P_{h+1}^{(-1)}\}$.

When regarded as a $\mathcal{A}_h$-module, $L_{g\text{odd}}$ decomposes into the direct sum

$$W_{<h} \oplus W_{>h+1} \oplus \bigoplus_{\hat{\alpha} \in \{0, -1\}^{h-1}} \bigoplus_{\hat{\gamma} \in \{0, -1\}^{g\text{-}h-1}} \{W_{h+1}^\circ (\hat{\alpha}, \hat{\gamma}) \oplus W_{h+1}^* (\hat{\alpha}, \hat{\gamma}) \oplus W_{h+1}^*(\hat{\alpha}, \hat{\gamma})\}$$

, where we set

$$W_{h+1}^\circ (\hat{\alpha}, \hat{\gamma}) := \begin{pmatrix} \hat{\alpha}(0, 0) \hat{\gamma} \\ \hat{\alpha}(0, 1) \hat{\gamma} \\ \hat{\alpha}(1, 0) \hat{\gamma} \\ \hat{\alpha}(1, 1) \hat{\gamma} \end{pmatrix}_C,$$

$$W_{h+1}^* (\hat{\alpha}, \hat{\gamma}) := \begin{pmatrix} \hat{\alpha}(0, 0) \hat{\gamma} \\ \hat{\alpha}(0, 1) \hat{\gamma} \\ \hat{\alpha}(1, 0) \hat{\gamma} \\ \hat{\alpha}(1, 1) \hat{\gamma} \end{pmatrix}_C,$$

$$W_{h+1}^*(\hat{\alpha}, \hat{\gamma}) := \begin{pmatrix} \hat{\alpha}(0, 0) \hat{\gamma} \\ \hat{\alpha}(0, 1) \hat{\gamma} \\ \hat{\alpha}(1, 0) \hat{\gamma} \\ \hat{\alpha}(1, 1) \hat{\gamma} \end{pmatrix}_C.$$

Further, set

$$W_{h+1}^\circ (\hat{\alpha}) := \bigoplus_{\hat{\gamma} \in \{0, -1\}^{g\text{-}h-1}} W_{h+1}^\circ (\hat{\alpha}, \hat{\gamma})$$

$$W_{h+1}^* (\hat{\alpha}) := \bigoplus_{\hat{\gamma} \in \{0, -1\}^{g\text{-}h-1}} W_{h+1}^* (\hat{\alpha}, \hat{\gamma}).$$

Let $\text{res}_{\hat{\alpha}, \hat{\gamma}}^\circ : \mathcal{A}_h \rightarrow \text{End}_C(W_{h+1}^\circ (\hat{\alpha}, \hat{\gamma}))$ be the map induced by the action of $\mathcal{A}_h$ on $W_{h+1}^\circ (\hat{\alpha}, \hat{\gamma})$. Notice that the representation matrices of $\text{res}_{\hat{\alpha}, \hat{\gamma}}^\circ$ w. r. t. the distinguished basis is independent of the the choice of the pair$(\hat{\alpha}, \hat{\gamma})$.

Lemma 5.72. $\text{res}_{\hat{\alpha}, \hat{\gamma}}^\circ$ is surjective.

Proof. We can prove the result in exactly the same manner as Lemma 5.52. In fact, we should replace $A$ and $T$ in the proof of that lemma by $\text{res}_{\hat{\alpha}, \hat{\gamma}}^\circ(\pi(A_h))$ and $\text{res}_{\hat{\alpha}, \hat{\gamma}}^\circ(P_{h+1}^{(-1)})$ here, respectively.

Definition 5.73. Let $\mathcal{A}_h' \subset \mathcal{A}_h$ be the ideal generated by $\{P_{h+1}^{(-1)} \circ \pi(A_h)\}$.

Lemma 5.74. $W_{<h} \oplus W_{>h+1}$ and each $W_{h+1}^\circ (\hat{\alpha}, \hat{\gamma}) \oplus W_{h+1}^* (\hat{\alpha}, \hat{\gamma})$ vanish under the $\mathcal{A}_h'$-action.

Proof. Since $W_{<h} \oplus W_{>h+1}$ is a $\mathcal{A}_h$-submodule and since it vanishes under the $P_{h+1}^{(-1)}$-action, it vanishes under the $\mathcal{A}_h'$-action. Similarly, since each $W_{h+1}^\circ (\hat{\alpha}, \hat{\gamma}) \oplus W_{h+1}^* (\hat{\alpha}, \hat{\gamma})$ is a $\mathcal{A}_h$-submodule and since it vanishes under the $\pi(A_h)$-action, it does so under the $\mathcal{A}_h'$-action.

Lemma 5.75. $\text{res}_{\hat{\alpha}, \hat{\gamma}}^\circ|_{\mathcal{A}_h'}$ is bijective.

Proof. Due to the result of Lemma 5.74 and the fact that $L_{g\text{odd}}$ is a faithful $\mathcal{R}_{g\text{odd}, h}$-module, the injectivity of $\text{res}_{\hat{\alpha}, \hat{\gamma}}^\circ|_{\mathcal{A}_h'}$ follows. Thus it is enough to show the surjectivity. Since $\text{res}_{\hat{\alpha}, \hat{\gamma}}^\circ(P_{h+1}^{(-1)} \circ \pi(A_h)) \neq 0$ and since $\text{res}_{\hat{\alpha}, \hat{\gamma}}^\circ|_{\mathcal{A}_h'}$ is surjective, we see that $\text{res}_{\hat{\alpha}, \hat{\gamma}}^\circ(\mathcal{A}_h') \subset \text{End}_C(W_{h+1}^\circ (\hat{\alpha}, \hat{\gamma}))$ is a non zero ideal. But the former must coincide with the latter since the latter is a simple algebra. Thus we are done.

59
Recall that, for each \( \hat{\beta} \in \{0, -1\}^h \), \( W_{\geq h+1}(\hat{\beta}) \) is an irreducible and faithful \( P_{\geq h+1} \circ \mathcal{R}_{odd,h+1} \circ P_{\geq h+1} \)-module. Further, any two of them are canonically isomorphic to each other as modules of this subalgebra. Thus \( P_{\geq h+1} \circ \mathcal{R}_{odd,h+1} \circ P_{\geq h+1} \) includes the four operators \( \phi_{h+1}^{i,j} (i, j \in \{0, -1\}) \) determined by

\[
\phi_{h+1}^{i,j}(\mu_{2l-1}^{m}) := \delta_{l,h+1}^{i} \delta_{m}^{j} \mu_{2h+1}^{i,j}
\]

, where \( 1 \leq l \leq g \), \( m \in \{0, -1\} \), \( \mu \in \{0, -1\}^h \), \( \hat{\gamma} \in \{0, -1\}^{g-h-1} \). We remark that the set \( \{\phi_{h+1}^{i,j} | i, j \in \{0, -1\}\} \) composes an 4-dimensional subalgebra.

**Definition 5.76.** Let \( \mathcal{B}'_h \subset \mathcal{R}_{odd,h} \) be the subalgebra generated by

\[
P_{\geq h+1} \circ \mathcal{A}'_h \circ P_{\geq h+1} \cup \{\phi_{h+1}^{i,j} | i, j \in \{0, -1\}\}.
\]

**Definition 5.77.** For any \( \hat{\alpha} \in \{0, -1\}^{h-1} \) and \( \hat{\gamma} \in \{0, -1\}^{g-h-1} \), set

\[
W_{h+1}^{\gamma}(\hat{\alpha}, \hat{\gamma}) := \left\langle \tilde{w}_{2h+1}^{\hat{\alpha}(-1,0)\hat{\gamma}}, \tilde{w}_{2h+1}^{\hat{\alpha}(0,-1)\hat{\gamma}}, \tilde{w}_{2h+1}^{\hat{\alpha}(-1,-1)\hat{\gamma}}, \tilde{w}_{2h+1}^{\hat{\alpha}(0,0)\hat{\gamma}} \right\rangle_{\mathbb{C}}
\]

, which turns out to be a \( \mathcal{B}'_h \)-submodule. Set

\[
W_{h+1}^{\gamma}(\hat{\alpha}) := \bigoplus_{\hat{\gamma} \in \{0, -1\}^{g-h-1}} W_{h+1}^{\gamma}(\hat{\alpha}, \hat{\gamma}).
\]

Further, let

\[
\text{res}_{\hat{\alpha}, \hat{\gamma}}^{\gamma} : \mathcal{B}'_h \to \text{End}_{\mathbb{C}}(W_{h+1}^{\gamma}(\hat{\alpha}, \hat{\gamma}))
\]

be the map induced by the \( \mathcal{B}'_h \)-action on \( W_{h+1}^{\gamma}(\hat{\alpha}, \hat{\gamma}) \).

**Lemma 5.78.** \( W_{h+1}^{\gamma}(\hat{\alpha}, \hat{\gamma}) \) is an irreducible \( \mathcal{B}'_h \)-module.

**Proof.** Set \( \mathcal{A}''_h := P_{\geq h+1} \circ \mathcal{A}'_h \circ P_{\geq h+1} \). Lemma [5.15] implies that it is the subalgebra of \( \mathcal{B}'_h \) spanned by the 4 operators \( \psi_{h+1}^{i,j} (i, j \in \{0, -1\}) \) determined as follows;

\[
\psi_{h+1}^{i,j}(\mu_{2l-1}^{m}) := \begin{cases} 
\delta_{l,h+1} \delta_{m} \mu_{2h+1}^{i,j} & \text{if } m + n = -1 \\
0 & \text{if } m + n \neq -1
\end{cases}
\]

, where \( 1 \leq l \leq g \), \( (m, n) \in \{0, -1\}^2 \), \( \hat{\alpha} \in \{0, -1\}^{h-1} \), \( \hat{\gamma} \in \{0, -1\}^{g-h-1} \). It follows that \( W_{h+1}^{\gamma}(\alpha, \gamma) \) decomposes into the direct sum of \( \mathcal{A}''_h \)-submodules as

\[
W_{h+1}^{\gamma}(\hat{\alpha}, \hat{\gamma}) \oplus W_{h+1}^{\gamma}(\hat{\alpha}, \hat{\gamma})
\]

, where the 1st summand and the 2nd one are spanned respectively by

\[
\left\{ \tilde{w}_{2h+1}^{\hat{\alpha}(-1,0)\hat{\gamma}}, \tilde{w}_{2h+1}^{\hat{\alpha}(0,-1)\hat{\gamma}} \right\}, \quad \left\{ \tilde{w}_{2h+1}^{\hat{\alpha}(0,0)\hat{\gamma}}, \tilde{w}_{2h+1}^{\hat{\alpha}(-1,-1)\hat{\gamma}} \right\}.
\]
Notice that the 1st summand is an irreducible $A'_h$-module and that the 2nd one vanishes under the $A'_h$-action. Let $1_{A'_h}$ be the (multiplicative) identity of the algebra $A'_h$. Set
\[
f := 1_{A'_h} \circ (\phi_{h+1}^{-1,0} + \phi_{h+1}^{0,-1})
\]
\[
g := (\phi_{h+1}^{-1,0} + \phi_{h+1}^{0,-1}) \circ 1_{A'_h}
\]
Then the actions of $f$ and $g$ on $W_{h+1}^\gamma(\hat{\alpha}, \hat{\gamma})$ are described as follows;
\begin{itemize}
  \item $f(W_{h+1}^\gamma(\hat{\alpha}, \hat{\gamma})) = \{0\} = g(W_{h+1}^\gamma(\hat{\alpha}, \hat{\gamma}))$.
  \item $f|_{W_{h+1}^\gamma(\hat{\alpha}, \hat{\gamma})} : W_{h+1}^\gamma(\hat{\alpha}, \hat{\gamma}) \to W_{h+1}^\gamma(\hat{\alpha}, \hat{\gamma})$ is bijective.
  \item $g|_{W_{h+1}^\gamma(\hat{\alpha}, \hat{\gamma})} : W_{h+1}^\gamma(\hat{\alpha}, \hat{\gamma}) \to W_{h+1}^\gamma(\hat{\alpha}, \hat{\gamma})$ is bijective.
\end{itemize}
Applying Lemma 5.45, we see that the subalgebra $B'_h \subset B'_h$ generated by
\[
\{\psi_{h+1}^{i,j} | i, j \in \{0, -1\}\} \cup \{f, g\}
\]
generates the endomorphism algebra $\text{End}_C(W_{h+1}^\gamma(\hat{\alpha}, \hat{\gamma}))$ via the map $\text{res}_{\hat{\alpha}, \hat{\gamma}}$. Thus we are done. \hfill \Box

**Definition 5.79.** Denote by $B_h$ the subalgebra $P_{\geq h+1} \circ R_{\text{odd}, h} \circ P_{\geq h+1} \subset R_{\text{odd}, h}$. Notice that $B_h \supset B'_h$.

**Definition 5.80.** For any $\hat{\alpha} \in \{0, -1\}^{h-1}$, define the subspace
\[
Y_{h+1}(\hat{\alpha}) := W_{\geq h+1}(\hat{\alpha}(0)) \oplus W_{\geq h+1}(\hat{\alpha}(-1))
\]
, which turns out to be a $B_h$-submodule.

**Proposition 5.81.** $Y_{h+1}(\hat{\alpha})$ is an irreducible $B_h$-module.

**Proof.** Notice that the 2 direct summands $W_{\geq h+1}(\hat{\alpha}(0))$ and $W_{\geq h+1}(\hat{\alpha}(-1))$ are irreducible $P_{\geq h+1} \circ R_{\text{odd}, h+1} \circ P_{\geq h+1}$-modules by $A(h+1)$. Assume to the contrary that $Y_{h+1}(\hat{\alpha})$ were not irreducible as a $B_h$-module. Let $M \subset Y_{h+1}(\hat{\alpha})$ be a minimal nonzero submodule. Then $M \neq Y_{h+1}(\hat{\alpha})$. Since $M$ is a $P_{\geq h+1} \circ R_{\text{odd}, h+1} \circ P_{\geq h+1}$-submodule, $M$ must coincide with one of the direct summands above. On the other hand, $B_h$ includes the operators $\psi_{h+1}^{i,j}$ ($i, j \in \{0, -1\}$) in the proof of Lemma 5.78. It is readily seen that, for any $a \in \{0, -1\}$,
\[
\{0\} \neq \psi_{h+1}^{-a, -a}(W_{\geq h+1}(\hat{\alpha}(a))) \subset W_{\geq h+1}(\hat{\alpha}(-a - 1))
\]
, which implies that neither summand can be $B_h$-submodule. This leads to a contradiction. Thus we are done. \hfill \Box

**Definition 5.82.** Let $C_h \subset R_{\text{odd}, h}$ be the subalgebra generated by $A'_h \cup B_h$.

**Definition 5.83.** For any $\hat{\alpha} \in \{0, -1\}^{h-1}$, set
\[
W'_{\geq h}(\hat{\alpha}) := W_{h+1}^\circ(\hat{\alpha}) + Y_{h+1}(\hat{\alpha})
\]
, which turns out to be a faithful $C_h$-submodule.
Proposition 5.85. $W'_{\geq h}(\hat{\alpha})$ is an irreducible $C_h$-submodule. Further, $C_h$ includes the distinguished projection $P'_{\geq h}$ of $L_{g^{\text{odd}}}$ to the subspace

$$W'_{\geq h} := \bigoplus_{\hat{b} \in \{0, -1\}^{h-1}} W'_{\geq h}(\hat{b}).$$

Proof. Set

$$M^\delta_h(\hat{\alpha}) := \left\langle \tilde{w}^{\hat{\alpha}((0, -1)^{\hat{\gamma}}}, \tilde{w}^{\hat{\alpha}((0, -1)^{\hat{\gamma}})} \mid \hat{\gamma} \in \{0, -1\}^{g-h-1} \right\rangle_C.$$

Then we have the direct sum decomposition

$$W'_{\geq h}(\hat{\alpha}) = M^\delta_h(\hat{\alpha}) \oplus Y_{h+1}(\hat{\alpha}).$$

Lemma 5.75 implies that $\mathcal{A}'_h$ contains the operators $\chi_{h+1}^R$, $\chi_{h+1}^L$ determined by

$$\chi_{h+1}^R(\tilde{w}^{\hat{\alpha}(m,n)^{\hat{\gamma}}}) = \delta_{l,h} \delta_{m+n,-1} \tilde{w}^{\hat{\alpha}(m,n)^{\hat{\gamma}}},$$

$$\chi_{h+1}^L(\tilde{w}^{\hat{\alpha}(m,n)^{\hat{\gamma}}}) = \delta_{l+1,h} \delta_{m+n,-1} \tilde{w}^{\hat{\alpha}(m,n)^{\hat{\gamma}}},$$

where $1 \leq l \leq g$, $\hat{\alpha} \in \{0, -1\}^{h-1}$, $(m,n) \in \{0, -1\}^2$, $\hat{\gamma} \in \{0, -1\}^{g-h-1}$. Then it is readily seen that

$$\text{Ker}(\chi_{h+1}^R) \supset W_{<h} \oplus W_{\geq h+1}, \quad \chi_{h+1}^R(W_h) \subset W_{h+1},$$

$$\text{Ker}(\chi_{h+1}^L) \supset W_{\leq h} \oplus W_{>h+1}, \quad \chi_{h+1}^L(W_{h+1}) \subset W_h.$$ 

Thus we are in the following situation;

- $Y_{h+1}(\hat{\alpha})$ is an irreducible and faithful $B_h$-module.
- $M^\delta(\hat{\alpha})$ vanishes under the $B_h$-action.
- $\chi_{h+1}^R|M^\delta(\hat{\alpha}) : M^\delta_h(\hat{\alpha}) \to Y_{h+1}(\hat{\alpha})$ is injective.

The 1st property implies that there exists a unique element $b_h \in B_h$ such that the induced $b_h$-action on $Y_{h+1}(\hat{\alpha})$ coincides with the distinguished projection $Y_{h+1}(\hat{\alpha}) \to \chi_{h+1}^R(M^\delta(\hat{\alpha}))$. With this understood, we see that

- $b_h \circ \chi_{h+1}^R|M^\delta(\hat{\alpha}) : M^\delta_h(\hat{\alpha}) \to Y_{h+1}(\hat{\alpha})$ is injective.
- $\chi_{h+1}^L \circ b_h : (Y_{h+1}(\hat{\alpha})) = M^\delta_h(\hat{\alpha}).$
- The distinguished subspace $W_{\geq h+1} \subset L_{g^{\text{odd}}}$ complementsary to $W'_{\geq h+1}$ vanishes under the action of $B_h \cup \{b_h \circ \chi_{h+1}^R, \chi_{h+1}^L \circ b_h\}$.

Thus Lemma 5.45 together with all the six properties above except the 3rd implies that the subalgebra of $C_h$ generated by $B_h \cup \{b_h \circ \chi_{h+1}^R, \chi_{h+1}^L \circ b_h\} \subset C_h$ contains some subalgebra $C'_h$ such that

- $W'_{\geq h}(\hat{\alpha})$ is an irreducible and faithful $C'_h$-module.
- $W_{\geq h+1} \perp$ vanishes under the $C'_h$-action.

Notice that $C_h = C'_h$ since $W_{\geq h}(\hat{\alpha})$ is a faithful $C_h$ module and because of the next to the last property above. Thus the identity element of $C_h$ is nothing but the distinguished projection $P'_{\geq h} : L_{g^{\text{odd}}} \to W_{\geq h}$, which is what we seek for. Thus we are done. \qed
Now we will finish the proof of the assertion $A(h)$. Recall that, for any $\hat{\alpha} \in \{0, -1\}^h$,
\[
W_{\geq h}(\hat{\alpha}) = W_{h+1}^\ast(\hat{\alpha}) \oplus W'_{\geq h}(\hat{\alpha}).
\]
With this in mind, define the elements $S^{(i)}_h$ ($h + 2 \leq i \leq g$) of $R_{\text{odd}, h}$ by
\[
S^{(i)}_h := (1 - P'_{\geq h}) \circ \pi(A_i) \circ (1 - P'_{\geq h}).
\]
(Of course, this definition is in vein when $h = g - 1$. But still we put it here for the uniform treatment.) Notice that $W'_{\geq h}$ vanishes and $W_{\leq h}$ and each $W_{h+1}^\ast(\hat{\alpha})$ are preserved under the $S^{(i)}_h$ -action.

We will introduce some grading on $W_{h+1}^\ast(\hat{\alpha})$. For this purpose, consider the direct sum decomposition
\[
W_{h+1}^\ast(\hat{\alpha}) = W_{h+1,-1}^\ast(\hat{\alpha}) \oplus W_{h+1,0}^\ast(\hat{\alpha})
\]

, where
\[
W_{h+1,a}^\ast(\hat{\alpha}) := \left\{ \frac{\tilde{w}_{2h-1}^{\hat{\alpha}(a,0)^j}}{\gamma_j \in \{0, -1\}^{g-h-1}} \right\} \quad (a \in \{0, -1\}).
\]

The grading of these subspaces are determined by setting their degree $f$ -part as
\[
W_{h+1,a}^\ast(\hat{\alpha}; f) := \left\{ \frac{\tilde{w}_{2h-1}^{\hat{\alpha}(a,0)^j}}{\gamma_j \geq 1, \gamma_{f+1} = -a - 1} \right\} \quad (a \in \{0, -1\})
\]

, where we understand the multi-index $\hat{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_{g-h-1})$. The degree $f$ takes values in $\{0, 1, \ldots, g - h - 1\}$. For example, if $f = 0$ and $a = -1$, we see that
\[
W_{h+1,-1}^\ast(\hat{\alpha}; 0) = \left\{ \frac{\tilde{w}_{2h-1}^{\hat{\alpha}(-1,-1,0)^j}}{\gamma' \in \{0, -1\}^{g-h-2}} \right\}.
\]

If $f = g - h - 1$ and $a = -1$, we see that
\[
W_{h+1,-1}^\ast(\hat{\alpha}; g - h - 1) = \left\{ \frac{\tilde{w}_{2h-1}^{\hat{\alpha}(-1,-1,0)^j}}{\gamma' \in \{0, -1\}^{g-h-2}} \right\}.
\]

The induced grading on $W_{h+1}^\ast(\hat{\alpha})$ and $W_{h+1}^\ast$ are determined as follows;
\[
W_{h+1}^\ast(\hat{\alpha}; f) := W_{h+1,-1}^\ast(\hat{\alpha}; f) \oplus W_{h+1,0}^\ast(\hat{\alpha}; f), \quad W_{h+1}^\ast(f) := \bigoplus_{\hat{\alpha} \in \{0, -1\}^{h-1}} W_{h+1}^\ast(\hat{\alpha}; f).
\]

**Definition 5.86.** Define some useful elements in $R_{\text{odd}, h}$ as below;
\[
T_{h,f} := \begin{cases} S_h^{(h+2)} \circ S_h^{(h+3)} \circ \cdots \circ S_h^{(h+f+1)} & \text{if } f \geq 1, \\ (1 - P'_{\geq h}) & \text{if } f = 0. \end{cases}
\]
\[
\bar{T}_{h,f} := \begin{cases} S_h^{(h+f+1)} \circ S_h^{(h+f)} \circ \cdots \circ S_h^{(h+2)} & \text{if } f \geq 1, \\ (1 - P'_{\geq h}) & \text{if } f = 0, \end{cases}
\]
where $0 \leq f \leq g - h - 1$. 

63
Notice that these operators preserve $W_{<h}$ and kill $W_{\geq h}$.

**Lemma 5.87.** For $1 \leq f \leq g - h - 1$ and for $0 \leq f' \leq g - h - 1$, it holds that
- $T_{h,f}|_{W_{>h+1}(f')} = 0$ if $f \neq f'$.
- $T_{h,f}(W_{h+1}(\hat{\alpha}; f)) \subset W_{h+1}(\hat{\alpha}; 0)$
- $T_{h,f}|_{W_{h+1}(f)}$ is injective.
- $T_{h,f}|_{W_{h+1}(f')} = 0$ if $f' \neq 0$.
- $T_{h,f}(W_{h+1}(\hat{\alpha}; 0)) = W_{h+1}(\hat{\alpha}; f)$.

**Definition 5.88.** For $0 \leq f \leq g - h - 1$, define the elements $Q_{h,f}, Q_{h,f}^* \in \mathcal{R}_{\text{odd}, h}$ as follows:

\[
Q_{h,f} := (1 - P_{\geq h+1}) \circ \pi(A_{h+1}) \circ (1 - P_{\geq h+1}) \circ T_{h,f},
\]

\[
Q_{h,f}^* := T_{h,f}^* \circ (1 - P_{\geq h+1}) \circ \pi(A_{h+1}) \circ (1 - P_{\geq h+1}).
\]

**Lemma 5.89.** For $0 \leq f, f' \leq g - h - 1$ and for any $\hat{\alpha} \in \{0, -1\}^{h-1}$, it holds that
- $Q_{h,f}$ preserves $W_{<h}$ and kills $W_{\geq h+1}$.
- $Q_{h,f}|_{W_{h+1}(f')} = 0$ if $f \neq f'$.
- $Q_{h,f}(W_{h+1}(\hat{\alpha}; f)) \subset W_{h+1}(\hat{\alpha})$ such that $Q_{h,f}|_{W_{h+1}(\hat{\alpha}; f)}$ is injective.
- $Q_{h,f}^*$ preserves $W_{<h}$ and kills $W_{h+1}$.
- $Q_{h,f}^*(W_{h+1}(\hat{\alpha})) = W_{h+1}(\hat{\alpha}; f)$.

Lemma 5.87 and Lemma 5.89 are readily deduced from a moment inspection on those operators. Now we will complete the proof of the assertion A(h). Applying Lemma 5.45 to the result of Lemma 5.89, the subalgebra $\mathcal{D}_h \subset \mathcal{R}_{\text{odd}, h}$ generated by

\[
\mathcal{C}_h \cup \{ Q_{h,f}, Q_{h,f}^* \mid 0 \leq f \leq g - h - 1 \}
\]

contains some subalgebra $\mathcal{D}'_h$ such that
- each $W_{\geq h}(\hat{\alpha})$ is an irreducible and faithful $\mathcal{D}'_h$-submodule,
- $W_{<h}$ vanishes under the $\mathcal{D}'_h$-action.

The 1st conclusion above implies that each $W_{\geq h}(\hat{\alpha})$ is an irreducible $\mathcal{R}_{\text{odd}, h}$-module. Since the argument so far is independent of the choice of the multi-index $\hat{\alpha}$ and since

\[
W_{\geq h} = \bigoplus_{\hat{\alpha} \in \{0, -1\}^{h-1}} W_{\geq h}(\hat{\alpha})
\]

, the 2nd conclusion above implies that the identity element of $\mathcal{D}'_h$ is nothing but the distinguished projection $P_{\geq h} : L_{g,\text{odd}} \twoheadrightarrow W_{\geq h}$. Thus we are done. \hfill $\Box$

**Proof of Theorem 5.40.** Define the distinguished basis of $L_g/L_{g,\text{odd}}$ as follows:

\[
\{ \pi'(\tilde{v}_{2l}^j) \mid 1 \leq l \leq g, \ j \in \{0, -1\} \}
\]

Consider the following correspondence between the basis vectors;

\[
L_g/L_{g,\text{odd}} \ni \pi'(\tilde{v}_{2l}^j) \leftrightarrow \tilde{w}_{2l-1}^j \in L_{g,\text{odd}}
\]
where we set \( \hat{j}^* := (-j_1 - 1, -j_2 - 1, \ldots, -j_g - 1) \) for \( \hat{j} = (j_1, j_2, \ldots, j_g) \).

Accordingly, define the distinguished subspaces of \( L_g/L_{g, \text{odd}} \) such as \( V_{\leq h}, V_{\geq h}, W_{\geq h}(\hat{\alpha}^*) \subset L_{g, \text{odd}} \). We propose the Assertion \( A'(h) \) for \( 1 \leq h \leq g \) analogously, whose statement is the same as \( A(h) \) except that we replace \( R_{\text{odd}} \) by \( R_{\text{even}} \), \( \tilde{w}^{j^*}_{2l-1} \) by \( \tilde{v}^{j^*}_{2l} \), \( V_{\geq h}(\hat{\alpha}^*) \) by \( V_{\geq h}(\hat{\alpha}^*) \) and so on. In the proof, we should replace \( \pi(A_k) \) (\( 1 \leq k \leq g - 1 \)) by \( \pi'(A_k) \), \( \pi(B_g) \) by \( \pi'(B_g) \) and \( \pi(T_g) \) by \( \pi'(T_g) \). Then the same inductive argument as in the proof of Theorem 5.39 works word for word to prove \( A'(h) \).

Proof of Theorem 5.37. It follows from Theorem 5.39 and Theorem 5.40 that the filtration

\[
F : \{0\} \subset L_{g, \text{odd}} \subset L_g
\]

is a composition series of \( L_g \) as a \( R^F \)-module. Set

\[
\Theta := \text{Proj}(\text{odd} : 0) \circ \tilde{\Psi}_{2g-1, \text{unil}} \circ \text{Proj}(\text{odd} : 0),
\]

\[
\Theta^\dagger := \text{Proj}(\text{odd} : 0) \circ \tilde{\Psi}_{2g, \text{unil}} \circ \text{Proj}(\text{odd} : 0).
\]

(1.) It follows from Proposition 5.13 that

\[
\text{Image}(\Theta) \subset L_{g, \text{odd}} \subset \text{Ker}(\Theta)
\]

and that the induced map \( \bar{\Theta} : L_g/L_{g, \text{odd}} \to L_{g, \text{odd}} \) is non zero.

(2.) It follows from Proposition 5.14 that the composition

\[
L_{g, \text{odd}} \xrightarrow{\Theta^\dagger} L_g \to L_g/L_{g, \text{odd}}
\]

is nonzero.

It is sufficient to show that \( L_g \) is an irreducible \( R \)-module. Suppose to the contrary that there were a proper non-zero \( R \)-submodule \( M \), which we may assume to be minimal. If we regard \( M \) as a \( R^F \)-submodule, then one of the following two cases occurs without fail:

Case 1.) \( M \) coincides with \( L_{g, \text{odd}} \). But if it were the case, \( M \) can not be \( R \)-submodule because of (1).

Case 2.) \( M \) projects isomorphically to \( L_g/L_{g, \text{odd}} \). But if it were the case, \( M \) can not be \( R \)-submodule because of (2).

Therefore neither case can occur, which leads to a contradiction. Thus we are done.

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