Abstract

We introduce in the paper a novel observability problem for a large population (in the limit, a continuum ensemble) of nonholonomic control systems with unknown population density. We address the problem by focussing on a prototype of such ensemble system, namely, the ensemble of Bloch equations which is known for its use of describing the evolution of the bulk magnetization of a collective of non-interacting nuclear spins in a static field modulated by a radio frequency (rf) field. The dynamics of the equations are structurally identical, but show variations in Larmor dispersion and rf inhomogeneity. We assume that the initial state of any individual system (i.e., individual Bloch equation) is unknown and, moreover, the population density of these individual systems is also unknown. Furthermore, we assume that at any time, there is only one scalar measurement output at our disposal. The measurement output integrates a certain observation function, common to all individual systems, over the continuum ensemble. The observability problem we pose in the paper is thus the following: Whether one is able to use the common control input (i.e., the rf field) and the single measurement output to estimate both the initial states of the individual systems and the population density? Amongst other things, we establish a sufficient condition for the ensemble system to be observable: We show that if the common observation function is any harmonic homogeneous polynomial of positive degree, then the ensemble system is observable. The main focus of the paper is to demonstrate how to leverage tools from representation theory of Lie algebras to tackle the observability problem. Although the results we establish in the paper are for the specific ensemble of Bloch equations, the approach we develop along the analysis can be generalized to investigate observability of other general ensembles of nonholonomic control systems with a single, integrated measurement output.

Key words: Ensemble observability; Ensemble system identification; Representation theory; Spherical Harmonics

1 Introduction and Main result

We consider in the paper a large population (in the limit, a continuum) of independent control systems—these individual systems are structurally identical, but show variations in system parameters. We call such a population of control systems an ensemble system.
A precise description of the system model will be given shortly. Control of an ensemble system is about broadcasting a finite-dimensional control input to simultaneously steer all the individual systems in the continuum ensemble. Questions such as whether an ensemble system is controllable and how to generate a control input to steer the entire population of systems have all been investigated to some extent in the literature. For control of linear ensembles (i.e., ensembles of linear systems), we refer the reader to [1-2] and [3, Ch. 12]. For control of nonlinear ensembles, we first refer the reader to the work [4, 5] by Li and Khaneja. The authors established controllability of a continuum ensemble of Bloch equations [6] using a Lie algebraic method. A similar controllability problem has also been addressed in [7]. But, the authors there have used a different approach that leverages tools from functional analysis. We next refer the reader to [8] in which the Rachevsky-Chow theorem (also known as the Lie algebraic rank condition) has been generalized so that it can be used a sufficient condition to check whether a continuum ensemble of control-affine systems is controllable. We have recently proposed in [9] a novel class of ensembles of control-affine systems, termed distinguished ensembles, and shown that any such ensemble system satisfies the generalized version of the Rachevsky-Chow theorem and, hence, is ensemble controllable.

We address in the paper the counterpart of the ensemble control problem, namely the ensemble estimation problem. Roughly speaking, estimation of an ensemble system is about using a single, integrated measurement output (of finite-dimension) to estimate the initial state of every individual system in the ensemble. Note that in its basic setup, the ensemble estimation problem is addressed under the assumption that the entire knowledge of the system model is available (See, for example, [9]). We consider in the paper a more challenging but realistic scenario: We assume that the underlying population density of the individual systems in the (continuum) ensemble is unknown.

The observability problem we will address in the paper is thus the problem about feasibility of estimating both the initial states and the population density of the individual systems in the ensemble. Note, in particular, that the problem can be viewed as a combination of two interrelated subproblems: One is the “usual ensemble observability problem” in which one has the complete knowledge of the ensemble model and aims to estimate the initial states of its individual systems. The other one can be related to the problem of “system identification” for which one treats the population density as an intrinsic parameter of an ensemble system.

To the best of author’s knowledge, the ensemble observability problem we posed above has not yet been addressed in the literature so far. One of the main contributions of the paper is thus to develop methods for tackling such a problem. Our methods rely on the use of representation theory of Lie algebras. To demonstrate such a connection between the observability problem and the tools from the representation theory, we focus in the paper a prototype of a general ensemble of nonholonomic control systems, namely, a continuum ensemble of Bloch equations (the system model will be given shortly). Although the observability results we establish in the paper are for the specific ensemble of Bloch equations, the methods we establish along the analysis can be extended to address other generals cases. We will address such an extension toward the end of the paper.
1.1 System model: Ensemble of Bloch equations

We introduce in the subsection the mathematical model for an ensemble of Bloch equations, which is known for its use of describing the evolution of the bulk magnetization of a collective of non-interacting nuclear spins in a static field modulated by a controlled radio frequency field. To this end, we let $S^2$ be the unit sphere embedded in $\mathbb{R}^3$. For a point $x \in S^2$, we let $x = (x_1, x_2, x_3)$ be its coordinates. Next, we define three vector fields on $S^2$ as follows:

$$
\begin{align*}
    f_0(x) &= \begin{bmatrix} x_2 \\ -x_1 \\ 0 \end{bmatrix}, \\
    f_1(x) &= \begin{bmatrix} 0 \\ x_3 \\ 0 \end{bmatrix}, \\
    f_2(x) &= \begin{bmatrix} 0 \\ 0 \\ x_2 \end{bmatrix}.
\end{align*}
$$

(1)

Then, the dynamics of an ensemble of Bloch equations, parametrized by a pair of scalar parameters $(\sigma_1, \sigma_2)$, are described by the following differential equations:

$$
\dot{x}_{\sigma}(t) = \sigma_1 f_0(x_{\sigma}(t)) + \sigma_2 \sum_{i=1}^{2} u_i(t) f_i(x_{\sigma}(t)),
$$

(2)

where $u_1(t), u_2(t)$ are scalar control inputs and the two parameters $\sigma_1, \sigma_2$ are commonly used to model Larmor dispersion and radio frequency inhomogeneity, respectively. We assume in the paper that $\sigma_1 \in [a_1, b_1]$ with $a_1 < b_1$ and $\sigma_2 \in [a_2, b_2]$ with $0 < a_2 < b_2$. We let $\sigma := (\sigma_1, \sigma_2)$ and

$$
\Sigma := [a_1, b_1] \times [a_2, b_2].
$$

We call $\Sigma$ the parametrization space.

If an individual Bloch equation is associated with the parameter $\sigma$, we call it system-$\sigma$. Note that by (2), each system-$\sigma$ is control-affine. We call $f_0$ a drifting vector field and $f_1, f_2$ control vector fields. We note here that the same model (2) has been used in [4, 5, 7] for the study of ensemble controllability problem.

For ease of notation, we let $u(t) := (u_1(t), u_2(t))$. Further, we let $x_{\Sigma}(t)$ be the collection of current states $x_{\sigma}(t)$ of all individual systems in the ensemble:

$$
x_{\Sigma}(t) := \{x_{\sigma}(t) \mid \sigma \in \Sigma\}.
$$

We call $x_{\Sigma}(t)$ a profile. Note that each profile $x_{\Sigma}(t)$ can be thought as a function from $\Sigma$ to $S^2$. Let $C^\omega(\Sigma, S^2)$ be the set of analytic functions from $\Sigma$ to $S^2$. We assume in the paper that each profile $x_{\Sigma}(t)$ belongs to $C^\omega(\Sigma, S^2)$.

Next, we let $\mu$ be a strictly positive Borel measure defined on the parametrization space $\Sigma$. The measure $\mu$ will be used to describe the population density of the individual systems. Specifically, we assume that for any given measurable subset $\Sigma'$ of $\Sigma$, the total amount of individual systems, with their parameters $\sigma$ belonging to $\Sigma'$, is proportional to $\int_{\Sigma'} d\mu$. For ease of analysis, we assume that there is a continuous function $\rho$ on $\Sigma$ such that $\rho(\sigma) > 0$ for all $\sigma$ and $d\mu = \rho(\sigma)d\sigma$. We call $\rho$ the density function.

With the measure $\mu$ defined above, we now introduce the estimation model as a counterpart of (2). Following the problem formulation in [9], we assume that there is only one scalar measurement output, denoted by $y(t)$, at our disposal. The measurement output...
integrates a certain observation function $\phi$ (common to all individual systems) over the entire parametrization space $\Sigma$. Specifically, we have that

$$y(t) := \int_{\Sigma} \phi(x_{\sigma}(t))d\mu,$$

where the observation function $\phi : S^2 \rightarrow \mathbb{R}$ is assumed to be analytic. Consider, for example, the map $\phi : x \mapsto x_i$ for some $i = 1, 2, 3$. Then, $y(t)$ can be interpreted as the projection of the bulk magnetization vector to the $x_i$-axis. We consider in the paper general observation functions that can render the ensemble system to be observable. A precise problem formulation will be given soon.

By combining the control model (2) and the above estimation model, we obtain the following ensemble system:

$$\begin{cases}
\dot{x}_{\sigma}(t) = \sigma_1 f_0(x_{\sigma}(t)) + \sigma_2 \sum_{i=1}^{2} u_i(t) f_i(x_{\sigma}(t)), \\
y(t) = \int_{\Sigma} \phi(x_{\sigma}(t))d\mu.
\end{cases}$$

We assume in the paper that $x_{\sigma}(0)$ is unknown for all $\sigma \in \Sigma$ and, moreover, the measure $\mu$ is also unknown. We note here that system (3) can be viewed as a prototype of a general ensemble of nonholonomic control systems with a single integrated measurement output.

### 1.2 Problem formulation: Ensemble observability

We formulate in the section the ensemble observability problem for system (3) with unknown population density. We start with the following definition:

**Definition 1.** Let $x_\Sigma(0), x'_\Sigma(0)$ be initial profiles and $\mu, \mu'$ be strictly positive Borel measures over $\Sigma$. Two pairs $(x_\Sigma(0), \mu)$ and $(x'_\Sigma(0), \mu')$ are output equivalent, which we denote by

$$(x_\Sigma(0), \mu) \sim (x'_\Sigma(0), \mu'),$$

if for any $T > 0$ and any integrable function $u : [0, T] \rightarrow \mathbb{R}^2$ as a control input, we have that

$$\int_{\Sigma} x_{\sigma}(t)d\mu = \int_{\Sigma} x'_{\sigma}(t)d\mu', \quad \forall t \in [0, T].$$

Following the above definition, we introduce for each pair $(x_\Sigma(0), \mu)$, the collection of its output equivalent pairs as follows:

$$O(x_\Sigma(0), \mu) := \{(x'_\Sigma(0), \mu') \mid (x'_\Sigma(0), \mu') \sim (x_\Sigma(0), \mu)\}.$$

Note that $(x_\Sigma(0), \mu)$ always belongs to $O(x_\Sigma(0), \mu)$. We next have the following definition:

**Definition 2.** System (3) is weakly ensemble observable if for any given $(x_\Sigma(0), \mu)$, the set $O(x_\Sigma(0), \mu)$ is finite. Moreover, we require that if $(x'_\Sigma(0), \mu')$ belongs to $O(x_\Sigma(0), \mu)$ and $(x'_\Sigma(0), \mu') \neq (x_\Sigma(0), \mu)$, then the following hold:
(1) The two measures \( \mu' \) and \( \mu \) are identically the same.

(2) For any \( \sigma \in \Sigma \), \( x'_{\sigma}(0) \neq x_{\sigma}(0) \).

System (3) is ensemble observable if for any \((x_{\Sigma}(0), \mu)\), \( O(x_{\Sigma}(0), \mu) = \{(x_{\Sigma}(0), \mu)\} \).

**Remark 1.** We note that the above definition about (weak) ensemble observability is stronger than the “usual” definition of ensemble observability introduced in [9]. The key difference between the two definitions is that Def. 2 takes into account the fact that one needs to identify the unknown population density as well. We also note that the two items in Def. 2 have the following implication: If system (3) is weakly ensemble observable, then by knowing the initial state \( x_{\sigma}(0) \) of any individual system-\( \sigma \), one is able to estimate the entire initial profile \( x_{\Sigma}(0) \) and the measure \( \mu \).

The problem we will address in the paper is the following: Given the control dynamics (2), what kind of observation function will guarantee that the entire system (3) is (weakly) ensemble observable? We provide below a partial solution to the above question by providing a class of observation functions that can fulfill the requirement.

### 1.3 Main result

We state in the subsection the main result of the paper. To proceed, we first introduce a few notations that are necessary to state the result. Let \( P \) be the space of all homogeneous polynomials in variables \( x_1, x_2, \) and \( x_3 \). For any nonnegative integer \( n \), we let \( P_n \) be the space of homogeneous polynomials of degree \( n \). The dimension of \( P_n \) is given by \( \lfloor n/2 \rfloor \).

We recall the following definition:

**Definition 3.** A polynomial \( p \) is harmonic if \( \Delta p = 0 \).

Let \( H_n \) be the space of homogeneous harmonic polynomials of degree \( n \). The dimension of \( H_n \) is \( 2n + 1 \). For example, for \( n = 1 \), \( H_1 \) is spanned by the basis \{\( x_1, x_2, x_3 \)\}; for \( n = 2 \), \( H_2 \) is spanned by the basis \{\( x_1^2 - x_2^2, x_2^2 - x_3^2, x_1x_2, x_1x_3, x_2x_3 \)\}.

For any real number \( r \), we let \( \lfloor r \rfloor \) be the largest integer such that \( \lfloor r \rfloor \leq r \). Then, it is known that the space of \( P_n \) can be decomposed as a direct sum as follows:

\[
P_n = H_n \oplus \|x\|^2 H_{n-2} \oplus \|x\|^4 H_{n-4} \oplus \cdots \oplus \|x\|^{2\lfloor n/2 \rfloor} H_{n-2\lfloor n/2 \rfloor},
\]

where \( \|x\|^2 := \sum_{i=1}^{3} x_i^2 \). We now state the first main result of the paper:

**Theorem 1.1.** Consider the ensemble system (3). Suppose that the observation function \( \phi \) belongs to \( H_n \) for any \( n \geq 1 \); then, the following hold:
(1) If $n$ is even, then system (3) is weakly ensemble observable. Moreover, for any pair $(x_\Sigma(0), \mu)$, the following holds:

$$O(x_\Sigma(0), \mu) = \{(x_\Sigma(0), \mu), (-x_\Sigma(0), \mu)\}.$$  

(4)

(2) If $n$ is odd, then system (3) is ensemble observable.

Remark 2. We note here that if $n$ is even, then $O(x_\Sigma, \mu)$ contains at least the two pairs in (4). We elaborate below on the fact. First, note that if two initial profiles are related by $x_\Sigma'(0) = -x_\Sigma(0)$, then for any control input $u(t)$, it always holds that $x_\Sigma'(t) = -x_\Sigma(t)$ for all $t$. Next, note that if $\phi$ is a homogeneous polynomial of even degree, then for any $x \in \mathbb{R}^3$, $\phi(-x) = (-1)^n \phi(x) = \phi(x)$. It then follows that

$$\int_\Sigma \phi(x_{\sigma(t)})d\mu = \int_\Sigma \phi(-x_{\sigma(t)})d\mu,$$

and, hence, $(-x_\Sigma(0), \mu) \sim (x_\Sigma(0), \mu)$. But then, item (1) of Theorem 1.1 says that there is no other pair $(x_\Sigma'(0), \mu')$ that can be output equivalent to $(x_\Sigma(0), \mu)$.

Organization of the paper. In the remainder of the paper, we develop methods for addressing the ensemble observability problem and prove Theorem 1.1. We will first introduce in Sec. 2 key definitions and notations that will be frequently used throughout the paper. Because our methods rely on the use of representation theory of $\mathfrak{sl}(2, \mathbb{C})$ on the space of homogeneous polynomials (where $\mathfrak{sl}(2, \mathbb{C})$ is the special Linear Lie algebra of all $2 \times 2$ complex matrices with zero trace), we present in Sec. 3 relevant results about such a representation. Then, in Sec. 4 we demonstrate how the representation theory can be used to addressed the ensemble observability problem. The proof of Theorem 1.1 will also be established along the analysis. We provide conclusions and further discussions in Sec. 5. In particular, we will discuss about connections with our earlier work [9] and extensions of the methods developed in the paper to other general ensembles of nonholonomic control systems.

2 Key Definitions and Notations

We introduce in the section key definitions and notations that will be frequently used throughout the paper.

For any two smooth vector fields $f$ and $g$ on $S^2$, we let $[f, g]$ be the Lie bracket defined as follows:

$$[f, g](x) := \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x).$$

Note that $[f, g]$ is also a vector field on $S^2$. Recall that $f_0$ is the drifting vector field and $f_1$, $f_2$ are control vector fields defined in (1). We let $\mathfrak{g}$ be the $\mathbb{R}$-span of $f_0$, $f_1$, and $f_2$. Then, $\mathfrak{g}$ is a (real) Lie algebra with the Lie bracket defined above. Note that if $(0, 1, 2)$ is a cyclic rotation of $(0, 1, 2)$, then

$$[f_i, f_j] = f_k.$$
The above structural coefficients then imply that \( \mathfrak{g} \) is isomorphic to \( \mathfrak{so}(3) \) (or simply \( \mathfrak{g} \approx \mathfrak{so}(3) \)), where \( \mathfrak{so}(3) \) is the Lie algebra of \( 3 \times 3 \) skew-symmetric matrices. We also note that \( \mathfrak{so}(3) \) is isomorphic \( \mathfrak{su}(2) \), i.e., the special unitary Lie algebra comprised of \( 2 \times 2 \) skew-Hermitian matrices with zero trace. Thus \( \mathfrak{g} \approx \mathfrak{su}(2) \).

For a given vector field \( f \in \mathfrak{g} \) and a smooth function \( \phi \) on \( S^2 \), we let \( f \phi \) be another function on \( S^2 \) defined as follows:

\[
(f \phi)(x) = \lim_{\epsilon \to 0} \frac{\phi(x + \epsilon) - \phi(x)}{\epsilon}, \quad \forall x \in S^2.
\]

Note that \( (f \phi)(x) \) is nothing but the directional derivative of \( \phi \) along \( f \) at \( x \).

Let \( \mathcal{A} \) be the collection of words over the alphabet \( \{0, 1, 2\} \), i.e., \( \mathcal{A} \) is comprised of all finite sequences \( \alpha = i_1 i_2 \cdots i_k \) where each \( i_j \) belongs to \( \{0, 1, 2\} \). The length of a word \( \alpha \) is defined to be the total number of indices \( i_j \) in it. Next, for a given word \( \alpha = i_1 \cdots i_k \) and a smooth function \( \phi \) on \( S^2 \), we let

\[ f_\alpha \phi := f_{i_1} \cdots f_{i_k} \phi. \]

Note that if \( \alpha = \emptyset \), i.e., an empty word, then we let \( f_\emptyset \phi := \phi \).

Let \( T(\mathfrak{g}) \) be the vector space spanned by \( f_\alpha \), i.e., each element \( \eta \) in \( T(\mathfrak{g}) \) is a linear combination of finitely many \( f_\alpha \) for \( \alpha \in \mathcal{A} \). Note that \( T(\mathfrak{g}) \) can be identified with the space of tensors of \( \mathfrak{g} \). Specifically, each \( f_\alpha \) can be viewed as a tensor in \( \mathfrak{g} \otimes \cdots \otimes \mathfrak{g} \), where the number of copies of \( \mathfrak{g} \) matches the length of \( \alpha \).

For an arbitrary real vector space \( V \), we let \( V^\mathbb{C} \) be the complexification of \( V \), i.e., \( V^\mathbb{C} \) is a complex vector space comprised of elements \( v + iw \) where \( i \) is the imaginary unit and \( v, w \) belong to \( V \). Recall that \( \mathfrak{g} \approx \mathfrak{su}(2) \) and, hence,

\[ \mathfrak{g}^\mathbb{C} \approx \mathfrak{sl}(2, \mathbb{C}), \]

where \( \mathfrak{sl}(2, \mathbb{C}) \) is the Lie algebra of \( 2 \times 2 \) complex matrices with zero trace.

We also recall that \( P_n \) is the space of homogeneous polynomials of degree \( n \) in variables \( x_1, x_2, \) and \( x_3 \). Note that for any \( f_i \), with \( i = 0, 1, 2 \), and any \( p \in P_n \), \( f_i p \) belongs to \( P_n \). Thus, \( P_n \) is closed under directional derivative along any \( f \in \mathfrak{g} \). We now define a map \( \pi : \mathfrak{g} \times P_n \times P_n \) as follows:

\[
\pi : (f, p) \mapsto \pi(f)p := f p.
\]

The map \( \pi \) is in fact a representation of \( \pi \) on \( P_n \), i.e., each \( \pi(f) \) for, \( f \in \mathfrak{g} \), is an endomorphism of \( P_n \) and satisfies the following relationship:

\[
\pi([f, g]) = \pi(f)\pi(g) - \pi(g)\pi(f), \quad \forall f, g \in \mathfrak{g}.
\]  \hspace{1cm} (5)

Let \( P'_n \) be a subspace of \( P_n \). We say that \( P'_n \) is invariant under \( \pi(\mathfrak{g}) \) if for any \( f \in \mathfrak{g} \) and \( p \in P'_n \), we have that \( \pi(f)p \in P'_n \). Thus, if we let \( \pi' \) be defined by restricting \( \pi \) to \( \mathfrak{g} \times P'_n \), then \( \pi' \) is a representation of \( \mathfrak{g} \) on \( P'_n \). We say that \( \pi' \) is irreducible if there does not exist a nonzero, proper subspace \( P''_n \) of \( P'_n \) such that \( P''_n \) is invariant under \( \pi(\mathfrak{g}) \). We further note that the representation \( \pi \) can be naturally extended to \( \mathfrak{g}^\mathbb{C} \times P_n^\mathbb{C} \). For any \( f, g \in \mathfrak{g} \) and any \( p, q \in P_n \), we let

\[
\pi(f + ig)(p + iq) := (\pi(f)p - \pi(g)q) + i(\pi(f)q + \pi(g)p).
\]  \hspace{1cm} (6)
Then, with such an extension, \( \pi \) is a representation of \( \mathfrak{g}^\mathbb{C} \) on \( P_n^\mathbb{C} \). We will present a few relevant facts about the representation in Sec. 3.

Let \( \Phi := \{ \phi_i \}_{i=1}^1 \) be a set of functions on \( S^2 \). Let \( n_1, \ldots, n_i \) be nonnegative integers. We call \( \phi_1^{n_1} \cdots \phi_i^{n_i} \) a monomial. The degree of the monomial is \( \sum_{i=1}^i n_i \). Denote by \( S(\Phi) \) the algebra of generated by \( \Phi \), i.e., each element in \( S(\Phi) \) is a linear combination of finitely many monomials. Further, we decompose \( S(\Phi) = \bigcup_{k \geq 0} \mathcal{S}_k(\Phi) \) where each \( \mathcal{S}_k(\Phi) \) is comprised of linear combinations of monomials of degree \( k \).

For an arbitrary differential equation \( \dot{x}(t) = f(x(t)) \), we denote by \( e^{t}f \) the solution of the differential equation at time \( t \) with \( x(0) \) the initial condition.

### 3 Representation on homogeneous polynomials

Our methods for addressing the ensemble observability problem rely on the use of representation theory of \( \mathfrak{sl}(2, \mathbb{C}) \). We present in the section a few relevant results (with Prop. 3.1 the main result) that will be great use in establishing Theorem 1.1.

To proceed, we first introduce a few definitions and notations. Recall that \( \mathcal{A} \) is the collection of words over the alphabet \( \{0, 1, 2\} \) and \( T(\mathfrak{g}) \) is the vector space spanned by all \( f_\alpha \) for \( \alpha \in \mathcal{A} \). Now, for a given word \( \alpha \in \mathcal{A} \), we let

\[
\kappa(\alpha) := (\kappa_1(\alpha), \kappa_2(\alpha)) \in \mathbb{Z}^2
\]

where \( \kappa_1(\alpha) \) and \( \kappa_2(\alpha) \) are nonnegative integers defined as follows:

\[
\begin{cases}
\kappa_1(\alpha) := \text{number of appearances of “0” in } \alpha, \\
\kappa_2(\alpha) := \text{number of appearances of “1” and “2” in } \alpha
\end{cases}
\]

For example, if \( \alpha = 0121 \), then \( \kappa(\alpha) = (1, 3) \). Next, with a slight abuse of notation, we let \( \kappa(f_\alpha) := \kappa(\alpha) \). Further, we consider an element \( \xi = \sum_{j=1}^n c_i f_{\alpha_j} \) in \( T(\mathfrak{g}) \). Suppose that \( \kappa(f_{\alpha_i}) = \kappa(f_{\alpha_j}) \) for any \( i, j \in \{1, \ldots, n\} \); then, we can define without ambiguity that

\[
\kappa(\xi) := \kappa(f_{\alpha_i}), \quad \text{for some } i \in \{1, \ldots, n\}.
\]

Note that if \( \kappa(\xi) \) is defined, then the lengths of all the words \( \alpha_i \) that are involved in \( \xi \) are the same. Also, note that if we let \( \mathbb{Z}_+^2 \) be the collection of vectors \( v = (v_1, v_2) \) with \( v_1 \) and \( v_2 \) nonnegative integers, then \( T(\mathfrak{g}) \) can be decomposed as a direct sum as follows:

\[
T(\mathfrak{g}) = \bigoplus_{v \in \mathbb{Z}_+^2} T_v(\mathfrak{g}),
\]

where \( T_v(\mathfrak{g}) \) is the space of all \( \xi \in T(\mathfrak{g}) \) with \( \kappa(\xi) = v \).

With the preliminaries above, we state the main result of the section:

**Proposition 3.1.** There exist \( \xi \) and \( \zeta \) in \( T(\mathfrak{g}) \) such that the following properties are satisfied:

1. Both \( \kappa(\xi) \) and \( \kappa(\zeta) \) are well defined and nonzero, with \( \kappa_1(\xi) > 0 \) and \( \kappa_1(\zeta) = 0 \).
(2) For any $p \in H_n$ for $n \geq 1$, we have that

$$\xi p = \zeta p = \lambda p,$$

where $\lambda$ is some nonzero real number depending only on $n$.

We establish below Prop. 3.1. We will explicitly construct $\xi$ and $\zeta$ in Sec. 3.1 and show that they satisfy the two items in the above theorem toward the end of the section.

### 3.1 The Casimir element and its variants

Recall that the space $T(g)$ can be identified with the space of all tensors in $g^{\otimes k}$ for all $k \geq 0$, i.e., we identify $f_{\alpha} = f_{i_1} \cdots f_{i_k}$ with $f_{i_1} \otimes \cdots \otimes f_{i_k}$. The so-called universal enveloping algebra associated with $g$ is defined as follows:

**Definition 4.** Let $J$ be a two sided ideal in $T(g)$ generated by all $fg - gf - [f, g]$ where $f, g \in g$. Then, the universal enveloping algebra $U(g)$ is given by the following quotient:

$$U(g) := T(g)/J.$$

We also need the following definition:

**Definition 5.** The center $Z(g)$ of $U(g)$ is the collection of elements in $U(g)$ that commute with the entire $U(g)$, i.e.,

$$Z(g) := \{\eta \in U(g) \mid \eta' \eta = \eta \eta', \text{ for all } \eta' \in U(g)\}.$$

We present in the following lemma a specific element in $Z(g)$. The result is, in fact, well known. But, for completeness of presentation, we provide a short proof after the statement.

**Lemma 1.** Let $\eta^* := \sum_{i=0}^2 f_i^2$. Then, $\eta^*$ belongs to the center $Z(g)$.

**Proof.** It suffices to show that $\eta^*$ commutes with every $f_i$ for $i = 0, 1, 2$. Recall that if $(i, j, k)$ is a cyclic rotation of $(0, 1, 2)$, then $[f_i, f_j] = f_k$. Thus, by symmetry, we only need to show that $\eta^*$ commutes with $f_0$. By computation,

$$\begin{align*}
f_0 f_1^2 &= f_1^2 f_0 + [f_0, f_1] f_1 + f_1 [f_0, f_1] = f_1^2 f_0 + f_2 f_1 + f_1 f_2, \\
f_0 f_2^2 &= f_2^2 f_0 + [f_0, f_2] f_2 + f_2 [f_0, f_2] = f_2^2 f_0 - f_1 f_2 - f_2 f_1.
\end{align*}$$

It then follows that $f_0$ commutes with $(f_1^2 + f_2^2)$ and, hence, with $\eta^* = \sum_{i=0}^2 f_i^2$ as well. ■

**Definition 6.** The element $\eta^* = \sum_{i=0}^2 f_i^2$ is commonly referred as the Casimir element.

**Remark 3.** Note that if an element $\eta$ belongs to $Z(g)$, then any polynomial in $\eta$ (i.e., $\sum_{k=0}^n c_k \eta^k$) belongs to $Z(g)$ as well. The converse also holds for the case here. Precisely, it is known [10, Ch. V] that if $g \approx \mathfrak{so}(3) \approx \mathfrak{su}(2)$, then the center $Z(g)$ is exactly the space of all polynomials in $\eta^*$. We further note that for a general (complex) semi-simple Lie algebra, the center of the associated universal enveloping algebra can be characterized via the Harish-Chandra isomorphism [10] Thm. 5.44.
However, note that if we treat the Casimir element $\eta^*$ as an element in $T(\mathfrak{g})$, then $\kappa(\eta^*)$ is not well defined. To see this, we simply note that

$$
\kappa(f_0^2) = (2,0) \quad \text{and} \quad \kappa(f_1^2) = \kappa(f_2^2) = (0,2).
$$

We thus aim to find elements $\xi$ and $\zeta$ in $T(\mathfrak{g})$ that satisfy the following two conditions:

1. Both $\kappa(\xi)$ and $\kappa(\zeta)$ are well defined and satisfy item (1) of Prop. 3.1.
2. The two elements $\xi$ and $\zeta$ are the same as the Casimir element $\eta^*$ when they are treated as elements in $U(\mathfrak{g})$, i.e., all the three elements are equivalent modulo the ideal $J$ (we will simply write $\xi \equiv \zeta \equiv \eta^*$).

One way to find such elements $\xi$ and $\zeta$ is to use the commutator relations: $[f_i, f_j] = f_k$ where $(i, j, k)$ is a cyclic rotation of $(0, 1, 2)$. We have the following result:

**Lemma 2.** Let $\xi$ and $\zeta$ be defined as follows:

$$
\begin{align*}
\xi &:= f_0f_1f_2 + f_1f_2f_0 + f_2f_0f_1 - f_0f_2f_1 - f_1f_0f_2 - f_2f_1f_0, \\
\zeta &:= 3(f_1f_2f_1f_2 + f_2f_1f_2f_1 - 2(f_1f_2^2f_1 + f_2f_1^2f_2) - (f_1^2f_2^2 + f_2^2f_1^2)).
\end{align*}
$$

Then, $\kappa(\xi) = (1,2)$ and $\kappa(\zeta) = (0,4)$. Moreover, $\xi \equiv \zeta \equiv \eta^*$.

**Proof.** The lemma follows directly from computation. Specifically, we note that

$$
\begin{align*}
f_0^2 &\equiv f_0f_1f_2 - f_0f_2f_1, \\
f_1^2 &\equiv f_1f_2f_0 - f_1f_0f_2, \\
f_2^2 &\equiv f_2f_0f_1 - f_2f_1f_0.
\end{align*}
$$

The element $\xi$ is then obtained by replacing $f_i^2$ for $i = 0, 1, 2$ in $\eta^*$ with the terms on the right hand side of the above expression. Further, by replacing each $f_0$ in the expression of $\xi$ with $(f_1f_2 - f_2f_1)$, we obtain $\zeta$.  

We note that there are many other (in fact, infinitely many) different $\xi$ and $\zeta$ in $T(\mathfrak{g})$ that satisfy the above two items. These elements can again be obtained by recursively applying the commutator relations $[f_i, f_j] = f_k$ for $(i, j, k)$ a cyclic rotation of $(0, 1, 2)$.

Toward the end of the section, we will show that the two elements $\xi$ and $\zeta$ satisfy item (2) of Prop. 3.1. For that, we need to have a few preliminaries about irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$. This is done in the next subsection.

### 3.2 Irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$

Recall that $P_n$ is the (real) vector space of all homogeneous polynomials of degree $n$ in variables $x_1$, $x_2$, and $x_3$. The space $P_n$ is closed under directional derivative along any vector field $f \in \mathfrak{g}$. The map $\pi : \mathfrak{g} \times P_n \to P_n$ defined by

$$
\pi : (f, p) \mapsto \pi(f)p := fp
$$

...
is a Lie algebra representation of $\mathfrak{g}$ on $P_n$. We also recall that $P_n^\mathbb{C}$ is the complexification of $P_n$, i.e., $P_n^\mathbb{C}$ is the space of homogeneous polynomials in (real) variables $x_1, x_2, x_3$ with complex coefficients.

One can extend $\pi$ to $\mathfrak{g}^\mathbb{C} \times P_n^\mathbb{C}$ using (6) so that $\pi$ is now a representation of $\mathfrak{g}^\mathbb{C}$ on $P_n^\mathbb{C}$. Note that $\mathfrak{g}^\mathbb{C} \approx \mathfrak{sl}(2, \mathbb{C})$. Representation of $\mathfrak{sl}(2, \mathbb{C})$ is extensively investigated in the literature. We review in the section a few basic facts that are relevant for establishing Prop. 3.1. To this end, we define a triplet $(h, e_+, e_-)$ of elements in $\mathfrak{g}^\mathbb{C}$ using the three elements $\{f_i\}_{i=0}^2$ from $\mathfrak{g}$ as follows:

$$h := 2if_0, \quad e_+ := f_1 + if_2, \quad e_- := -f_1 + if_2,$$

Then, by computation, we have the following standard commutator relationship for the triplet $(h, e_+, e_-)$ in $\mathfrak{sl}(2, \mathbb{C})$:

$$[h, e_+] = 2e_+, \quad [h, e_-] = -2e_-, \quad [e_+, e_-] = h.$$ 

Denote by $\mathfrak{Ch}, \mathfrak{Ce}_+$, and $\mathfrak{Ce}_-$ the vector spaces (over $\mathbb{C}$) spanned by $h, e_+$, and $e_-$, respectively. Then, $\mathfrak{Ch}$ is known as a Cartan subalgebra of $\mathfrak{sl}(2, \mathbb{C})$ while $\mathfrak{Ce}_+$ and $\mathfrak{Ce}_-$ are the two root spaces. Recall that a representation $\pi : \mathfrak{sl}(2, \mathbb{C}) \times V \to V$ is irreducible if there does not exist a nonzero, proper subspace $V'$ of $V$ such that $\pi(V') \subseteq V'$. The following result is well-known [11] for finite-dimensional irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$:

**Lemma 3.** Let $\pi : \mathfrak{sl}(2, \mathbb{C}) \times V \to V$ be an arbitrary irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ on a (complex) vector space $V$ of dimension $(n + 1)$ for $n \geq 0$. Then, $V$ can be decomposed as a direct sum of one-dimensional subspaces $V = \bigoplus_{k=0}^n V_{n-2k}$. These subspaces satisfy the following conditions:

$$\pi(e_+) V_{n-2k} = V_{n-2k+2}, \quad \pi(e_-) V_{n-2k} = V_{n-2k-2}.$$ 

Moreover, for any $v \in V_{n-2k}$, $\pi(h)v = (n - 2k)v$.

**Definition 7.** The subspaces $V_{n-2k}$ in the above lemma are weight spaces, and the integers $(n - 2k)$ are weights. The weight $n$ (i.e., $k = 0$) is called the highest weight and, correspondingly, any nonzero vector $v$ in $V_n$ is called a highest weight vector.

Note that by Lemma 3 if $v$ is a highest weight vector (of weight $n$), then the set of vectors $\{v, \pi(e_-)v, \ldots, \pi^n(e_-)v\}$ is a basis of $V$. Each one-dimensional weight space $V_{n-2k}$ is spanned by the vector $\pi^k(e_-)v$. Conversely, we have the following fact:

**Lemma 4.** Let $\pi : \mathfrak{sl}(2, \mathbb{C}) \times V \to V$ be an arbitrary representation (not necessarily irreducible). Suppose that there is a nonzero vector $v \in V$ and an integer $n \geq 0$ such that

$$\pi(h)v = nv \quad \text{and} \quad \pi(e_+)v = 0;$$

then, the subspace $V'$ spanned by $\{v, \pi(e_-)v, \ldots, \pi^n(e_-)v\}$ is an invariant subspace of $V$ under $\pi(\mathfrak{sl}(2, \mathbb{C}))$. Let $\pi'$ be defined by restricting $\pi$ to $\mathfrak{sl}(2, \mathbb{C}) \times V'$, then $\pi'$ is an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ on $V'$ with $n$ the highest weight and $v$ a highest weight vector.
The above lemma is an application of the Theorem of Highest Weight \cite[Thm. 5.5]{10}.

We now return to the representation $\pi : g^C \times P_n^C \to P_n^C$. We will see soon that the $\pi$ is not irreducible. But, by the unitarian trick (see, for example, \cite[Thm. 5.29]{10}), any finite-dimensional representation of a complex semi-simple Lie algebra is completely reducible. Specifically, we first recall that $H_n$ is the (real) vector space of harmonic homogeneous polynomials of degree $n$. We let $H_n^C$ be its complexification. Then,

$$P_n^C = H_n^C \oplus \|x\|^2 H_{n-2}^C \oplus \cdots \oplus \|x\|^{2[n/2]} H_{n-2[n/2]}^C,$$

where $\|x\|^2 = \sum_{i=1}^{3} x_i^2$. The following fact is well-known \cite[Ch. 17]{12}. For completeness of the presentation, we provide a proof after the statement.

**Lemma 5.** For any $k = 0, \ldots, [n/2]$, the subspace $\|x\|^{2k} H_{n-2k}^C$ is invariant under $\pi(g^C)$. Let $\pi_k$ be defined by restricting $\pi$ to $g^C \times \|x\|^{2k} H_{n-2k}^C$. Then, $\pi_k$ is an irreducible representation with $2(n - 2k)$ the highest weight and

$$p_k^* := \|x\|^{2k}(x_1 + ix_2)^{n-2k}$$

a highest weight vector.

**Proof.** Let $h, e_+, e_-$ be defined in (8). Then, by computation, we obtain that

$$\pi(h)p_k^* = 2(n-2k)p_k^* \quad \text{and} \quad \pi(e_+)p_k^* = 0.$$  

Let $V_k$ be a subspace of $P_n^C$ spanned by $\pi^l(e_-)p_k^*$ for $l = 0, \ldots, 2(2n-k)$. Then, by Lemmas\cite[3]{4} and \cite[4]{4} it suffices to show that $V_k = \|x\|^{2k} H_{n-2k}^C$.

First, note that the dimension of $H_{n-2k}^C$ is $2(n-2k) + 1$, which is the same as $V_k$. Thus, we only need to show that each $\pi^l(e_-)p_k^*$, for $l = 0, \ldots, 2(n-2k)$, belongs to $\|x\|^{2k} H_{n-2k}^C$. Note that for any $i = 1, 2, 3$, $f_i \|x\|^2 = 0$ and, hence, $\pi(e_-)\|x\|^2 = 0$. Thus, for any $l = 0, \ldots, 2(n-2k)$, we have that

$$\pi^l(e_-)p_k^* = \|x\|^{2k}\pi^l(e_-)(x_1 + ix_2)^{n-2k}.$$  

It remains to show that each $\pi^l(e_-)(x_1 + ix_2)^{n-2k}$ for $l = 0, \ldots, 2(n-2k)$ belongs to $H_{n-2k}^C$. First, by computation,

$$\Delta(x_1 + ix_2)^{n-2k} = 0.$$  

Next, note that the Laplacian $\Delta$ commutes with every $f_i$, i.e., $\Delta f_i = f_i \Delta$ for all $i = 1, 2, 3$. In particular, it commute with $\pi(e_-)$. Thus,

$$\Delta\pi^l(e_-)(x_1 + ix_2)^{n-2k} = \pi^l(e_-)\Delta(x_1 + ix_2)^{n-2k} = 0$$

for all $l = 0, \ldots, 2(n-2k)$.
3.3 Proof of Prop. 3.1

We establish in the subsection Prop. 3.1. With slight abuse of notation, we will now let
\[ \pi : g \times H_n \to H_n \]
be the representation of \( g \) on \( H_n \). By Lemma 5, \( \pi \) is irreducible. The map \( \pi \) can be naturally extended to \( T(g) \times H_n \), which we have implicitly used throughout the paper. Specifically, for any \( p \in H_n \) and any \( \eta \in T(g) \), we define \( \pi(\eta)p := \eta p \). Further, note that the relationship (5) which we reproduce below:
\[ \pi([f, g]) = \pi(f)\pi(g) - \pi(g)\pi(f), \quad \forall f, g \in g. \]
allows us to pass the map \( \pi \) to the quotient \( U(g) \times H_n \), i.e., if two elements \( \eta \) and \( \eta' \) in \( T(g) \) are equivalent (i.e., \( \eta \equiv \eta' \)), then \( \eta p = \eta' p \) for any \( p \in H_n \).

Recall that \( \eta^* = \sum_{i=0}^{2} f_i^2 \) is the Casimir element. Let \( \xi \) and \( \zeta \) be defined in (7) and we have that \( \eta^* \equiv \xi \equiv \zeta \). Then, by the above arguments,
\[ \eta^* p = \xi p = \zeta p, \quad \forall p \in H_n. \quad (9) \]

The following fact is a straightforward consequence of Schur’s Lemma (see, for example, Lemma 1.69 in [10]). We provide a short proof after the statement.

**Lemma 6.** The Casimir element \( \eta^* \) acts on \( H_n \) as a scalar multiple of the identity operator. Specifically, for any \( p \in H_n \),
\[ \eta^* p = -n(n + 1)p. \]

**Proof.** Because \( \eta^* \) belongs to the center of \( U(g) \), \( \eta^* f = f \eta^* \) for all \( f \in g \). Then, by Schur’s Lemma, there exists a constant \( \lambda \in \mathbb{C} \) such that \( \eta^* p = \lambda p \) for all \( p \in H_n^C \). To evaluate \( \lambda \), we let \( p^* := (x_1 + ix_2)^n \) be a highest weight vector in \( H_n^C \) (with the highest weight vector being \( 2n \)). Next, let \( h, e_+ \), and \( e_- \) be defined in (8). Note that
\[ \eta^* = \sum_{i=0}^{2} f_i^2 = -\frac{1}{4} h^2 - \frac{1}{2} (e_+ e_- + e_- e_+). \]

Then, using the fact that \( e_+ p^* = 0 \), we obtain that
\[ \eta^* p^* = -\frac{1}{4} h^2 p^* - \frac{1}{2} (e_+ e_- + e_- e_+) p^* = -\frac{1}{4} h^2 p^* - \frac{1}{2} (e_+ e_- - e_- e_+) p^* \]
Further, note that \([e_+, e_-] = h \) and \( h p^* = 2n p^* \). It follows that
\[ \eta^* p^* = -\left( \frac{1}{4} h^2 + \frac{1}{2} h \right) p^* = -n(n + 1)p^*. \]

We thus conclude that \( \lambda = -n(n + 1) \). This completes the proof. ■

Prop. 3.1 then follows from Lemmas 2 and 6.
4 Analysis and Proof of Theorem 1.1

We establish in the section Theorem 1.1. The proof will be built upon several relevant facts, which are summarized in three items of Theorem 4.1 stated below. With these facts, we will see that Theorem 1.1 is a straightforward consequence of Theorem 4.1. In fact, we will establish the three items of Theorem 4.1. In particular, we demonstrate how the representation theory of \( \mathfrak{sl}(2, \mathbb{C}) \) will be used in the analysis along the proof of Theorem 4.1.

Then, the main focus of the remainder of the section (after the proof of Theorem 1.1) is to establish the three items of Theorem 4.1. In particular, we demonstrate how the representation theory of \( \mathfrak{sl}(2, \mathbb{C}) \) will be used in the analysis along the proof of Theorem 4.1.

Now, to state Theorem 4.1, we first recall that for a given set of functions \( \Phi := \{ \phi_i \}_{i=1}^l \) defined on \( S^2 \), \( S(\Phi) \) is the algebra generated by the set \( \Phi \), i.e., it is comprised of all linear combinations of finitely many monomials \( \phi_1 \cdots \phi_l \). Also, recall that \( S_2(\Phi) \) is the space of quadratic forms in \( \phi_i \) for \( i = 1, \ldots, l \). We further let \( 1_{S^2} \) be the identity function on \( S^2 \), i.e., \( 1_{S^2}(x) := 1 \) for all \( x \in S^2 \). We establish in the section the following result:

**Theorem 4.1.** The following items hold under the assumption of Theorem 1.1:

1. If \( (x_\Sigma^*(0), \mu') \sim (x_\Sigma(0), \mu) \), then for any \( p \in H_n \), we have that
   \[
   p(x_\sigma(0))\rho(\sigma) = p(x_\sigma'(0))\rho'(\sigma), \quad \forall \sigma \in \Sigma.
   \]

2. Let \( \Phi \) be any basis of \( H_n \). Then, \( S_2(\Phi) \) contains the identity function \( 1_{S^2} \).

3. For two points \( x \) and \( x' \) in \( S^2 \), if \( p(x) = p(x') \) for all \( p \in H_n \), then \( x' \in \{ x, (-1)^{n-1}x \} \).

With the above result, we will be able to prove Theorem 1.1.

**Proof of Theorem 1.1** We fix a pair \( (x_\Sigma(0), \mu) \) and let \( (x_\Sigma^*(0), \mu') \) be chosen such that \( (x_\Sigma^*(0), \mu') \sim (x_\Sigma(0), \mu) \). We need to show that

\[
(x_\Sigma^*(0), \mu') \in \{(x_\Sigma(0), \mu), ((-1)^{n-1}x_\Sigma(0), \mu)\}.
\]

Let \( \Phi := \{ p_i \}_{i=1}^{2n+1} \) be an arbitrary basis of \( H_n \). Then, by item (1) of Theorem 4.1, we obtain that for any \( \sigma \in \Sigma \) and any \( i = 1, \ldots, 2n + 1 \),

\[
p_i(x_\sigma(0))\rho(\sigma) = p_i(x_\sigma'(0))\rho'(\sigma).
\]

Next, by item (2) of Theorem 4.1, there exists a quadratic form \( q \) in \( p_i \) such that

\[
q(x) = \sum_{1 \leq i \leq l} c_{ij} p_i(x)p_j(x) = 1, \quad \forall x \in S^2.
\]

It then follows from (10) that

\[
\rho^2(\sigma) = \rho^2(\sigma)q(x_\sigma(0)) = \rho^2(\sigma)q(x_\sigma'(0)) = \rho^2(\sigma), \quad \forall \sigma \in \Sigma.
\]

Because the two density functions \( \rho(\sigma) \) and \( \rho'(\sigma) \) are positive everywhere, we obtain that

\[
\rho(\sigma) = \rho'(\sigma), \quad \forall \sigma \in \Sigma.
\]
It then follows from (10) that for any \( \sigma \in \Sigma \) and any \( i = 1, \ldots, 2n + 1 \),

\[
p_i(x_\sigma(0)) = p_i(x'_\sigma(0)).
\]

(11)

Since \( \Phi \) is a basis of \( H_n \), we obtain by item (3) of Theorem 4.1 that

\[
x_\sigma(0) \in \{x_\sigma(0), (-1)^{n-1}x_\sigma(0)\}, \quad \forall \sigma \in \Sigma.
\]

(12)

Note, in particular, that if \( n \) is odd, then \( x'_\sigma(0) = x_\sigma(0) \) for all \( \sigma \in \Sigma \). Thus, in this case, system (3) is ensemble observable. We now assume that \( n \) is even and show that \( x'_2(0) \) is either \( x_\Sigma(0) \) or \( -x_\Sigma(0) \). But, this follows from the fact that both \( x_\sigma(0) \) and \( x'_\sigma(0) \) are analytic (and hence, continuous) in \( \sigma \). More specifically, consider a map \( \delta : \Sigma \to \mathbb{R} \) defined by sending \( \sigma \) to the Euclidean distance between \( x_\sigma(0) \) and \( x'_\sigma(0) \), i.e.,

\[
\delta : \sigma \mapsto \|x_\sigma(0) - x'_\sigma(0)\|.
\]

Because \( x_\sigma(0) \) and \( x'_\sigma(0) \) are analytic in \( \sigma \), the map \( \delta \) is continuous. On the other hand, note that by (12), there are only two cases:

1. If \( x'_\sigma(0) = x_\sigma(0) \), then \( \delta(\sigma) = 0 \);
2. If \( x'_\sigma(0) = -x_\sigma(0) \), then \( \delta(\sigma) = 2 \).

Thus, if \( x'_\sigma(0) = x_\sigma(0) \) (resp. \( x'_\sigma(0) = -x_\sigma(0) \)) for a certain \( \sigma \in \Sigma \), then by continuity of \( \delta \), we conclude that \( x'_2(0) = x_\Sigma(0) \) (resp. \( x'_2(0) = -x_\Sigma(0) \)). This completes the proof.

The remainder of the section will be devoted to the proof of Theorem 4.1. We establish subsequently the items in the next three subsections.

4.1 Proof of item (1) of Theorem 4.1

We establish in the subsection item (1) of Theorem 4.1. Recall that for an element \( f_\alpha \in T(\mathfrak{g}) \), we have that \( \kappa(f_\alpha) = (\kappa_1(f_\alpha), \kappa_2(f_\alpha)) \) where \( \kappa_1(f_\alpha) \) (resp. \( \kappa_2(f_\alpha) \)) counts the number of “0” (resp. “1” and “2”) in the word \( \alpha \). For convenience, we introduce the following notation:

\[
\sigma^{\kappa(\alpha)} := \sigma_1^{\kappa_1(\alpha)}\sigma_2^{\kappa_2(\alpha)}, \quad \forall \sigma \in \Sigma,
\]

which is a polynomial function over \( \Sigma \). We first have the following fact:

Lemma 7. Let \( \phi \) be any smooth observation function. If \( (x_\Sigma(0), \mu) \sim (x'_\Sigma(0), \mu') \), then for any \( f_\alpha \) with \( \alpha \in \mathcal{A} \),

\[
\int_\Sigma \sigma^{\kappa(\alpha)}(f_\alpha\phi)(x_\sigma(0))d\mu = \int_\Sigma \sigma^{\kappa(\alpha)}(f_\alpha\phi)(x'_\sigma(0))d\mu'.
\]

(13)

Proof. Let \( n \) be an arbitrary nonnegative integer number. We prove the lemma for any word \( \alpha \) of length \( n \). The arguments used in the proof will be similar to the one used in [9]: We will appeal to the class of piecewise constant control inputs to establish (13).
Define a piecewise constant control input $u(t)$ as follows: First, let $0 < t_1 < \cdots < t_n$ be switching times. Then, we let $u(t) := (u_{i_1}, u_{i_2})$ for $t \in [t_{i-1}, t_i)$ where $t_0 := 0$. Next, for ease of notation, we define for each $i = 1, \ldots, n$, the duration $\tau_i := t_i - t_{i-1}$ and the corresponding vector field over the period $[t_{i-1}, t_i)$:

$$f_i := \sigma_1 f_0 + \sigma_2 (u_{i_1} f_1 + u_{i_2} f_2). \quad (14)$$

Recall that for an arbitrary differential equation $\dot{x}(t) = f(x(t))$, we use $e^{\tau f} x(0)$ to denote the solution of the equation at time $t$ with $x(0)$ the initial condition. In the context here, we have that for any system-$\sigma$ with $\sigma \in \Sigma$, the following hold with respect to the piecewise constant control input:

$$\begin{align*}
    x_\sigma(t) &= e^{\tau_1 f_1} \cdots e^{\tau_i f_1} x_\sigma(0), \\
    x'_\sigma(t) &= e^{\tau_1 f_1} \cdots e^{\tau_i f_1} x'_\sigma(0).
\end{align*}$$

Thus, if $(x_\Sigma(0), \mu) \sim (x'_\Sigma(0), \mu')$, then for any $\tau_i$ with $i = 1, \ldots, n$, the following holds:

$$\int_{\Sigma} \phi \left( e^{\tau_1 f_1} \cdots e^{\tau_i f_1} x_\sigma(0) \right) d\mu = \int_{\Sigma} \phi \left( e^{\tau_1 f_1} \cdots e^{\tau_i f_1} x'_\sigma(0) \right) d\mu'.$$

We next take partial derivative $\partial^\alpha / \partial \tau_1 \cdots \partial \tau_n$ on both sides of the above expression and let them be evaluated at $\tau_1 = \cdots = \tau_n = 0$. Then, by computation, we obtain

$$\int_a^b (\tilde{f}_n \cdots \tilde{f}_1 \phi)(x_\sigma(0))d\mu = \int_a^b (\tilde{f}_n \cdots \tilde{f}_1 \phi)(x'_\sigma(0))d\mu'.$$

Note that by (14), each $\tilde{f}_i$ depends on $(u_{i_1}, u_{i_2})$ and the above expression holds for all $(u_{i_1}, u_{i_2}) \in \mathbb{R}^2$ with $i = 1, \ldots, n$. Also, note that by expanding each $\tilde{f}_i$ using (14), we have that $\tilde{f}_n \cdots \tilde{f}_1 \phi$ is a linear combination of $\sigma^{\kappa(\alpha)} f_\alpha \phi$ for $\alpha$ any word of length $n$. It then follows that (13) holds.

Recall that a set of functions $\{\psi_i\}_{i=1}^n$ on $\Sigma$ is said to separate points if for any two distinct points $\sigma$ and $\sigma'$ in $\Sigma$, there exists a function $\psi_i$ out of the set such that $\psi_i(\sigma) \neq \psi_i(\sigma')$. We also recall that by Lemma 2, $\kappa(\xi) = (1, 2)$ and $\kappa(\zeta) = (0, 4)$. We define monomials $m_\xi$ and $m_\zeta$ in variables $\sigma_1$ and $\sigma_2$ as follows:

$$\begin{align*}
    m_\xi(\sigma) := \sigma^{\kappa(\xi)} &= \sigma_1^2 \sigma_2^0, \\
    m_\zeta(\sigma) := \sigma^{\kappa(\zeta)} &= \sigma_1^0 \sigma_2^4.
\end{align*} \quad (15)$$

We next have the following fact:

**Lemma 8.** The set $\{m_\xi, m_\zeta\}$ separates points. Moreover, $m_\xi$ is everywhere nonzero.

**Proof.** Let $\sigma = (\sigma_1, \sigma_2)$ and $\sigma' = (\sigma'_1, \sigma'_2)$ be two distinct points in $\Sigma$. If $\sigma_2 \neq \sigma'_2$, then $m_\zeta(\sigma) \neq m_\zeta(\sigma')$. If $\sigma_2 = \sigma'_2$, then $\sigma_1 \neq \sigma'_1$ and, hence, $m_\xi(\sigma) \neq m_\xi(\sigma')$. Since $\sigma_2 \in [a_2, b_2]$ and $b_2 > a_2 > 0$, we have that $m_\xi$ is everywhere nonzero.

With the lemmas above, we prove item (1) of Theorem 4.1.
Proof of item (1) of Prop 4.1. Recall that $U(g)$ is the universal enveloping algebra associated with $g$. Let $p$ be any nonzero polynomial in $H_n$ and $H_n' := U(g)p = \{ \eta p \mid \eta \in U(g) \}$. Let $\pi : g \times H_n \to H_n$ be the representation defined in Sec. 3.3 i.e.,

$$
\pi : (f, p) \in g \times H_n \mapsto \pi(f)p := fp.
$$

Because $H_n$ is closed under $\pi(g)$, $H_n'$ is a subspace of $H_n$. Also, note that by the definition, $H_n'$ itself is closed under $\pi(g)$. Thus, by the fact that $\pi$ is an irreducible representation (Lemma 5), we must have that $H_n = H_n' = U(g)p$. Since $U(g)$ is spanned by $f_\alpha$ for $\alpha \in \mathcal{A}$ and dim $H_n = 2n + 1$, there exist $f_\alpha i$, for $i = 1, \ldots, 2n + 1$, such that $f_\alpha i, p$ form a basis $H_n$. For convenience, we let

$$
p_i := f_\alpha i, p, \quad \forall i = 1, \ldots, 2n + 1.
$$

Let $(x_\Sigma'(0), \mu') \sim (x_\Sigma(0), \mu)$. Let $\rho$ and $\rho'$ be the density functions corresponding to $\mu$ and $\mu'$, respectively. By Lemma 7, we have that for any $i = 1, \ldots, 2n + 1$ and any word $\alpha$ over the alphabet $\{0, 1, 2\}$, the following hold:

$$
\int_{\Sigma} \sigma^{\kappa(f_\alpha) + \kappa(f_\alpha)}(f_\alpha i)(x_\Sigma(0))d\mu = \int_{\Sigma} \sigma^{\kappa(f_\alpha) + \kappa(f_\alpha)}(f_\alpha i)(x_\Sigma'(0))d\mu'.
$$

It then implies that for any $\eta \in T(g)$ such that $\kappa(\eta)$ is well defined, the following holds:

$$
\int_{\Sigma} \sigma^{\kappa(\eta) + \kappa(f_\alpha)}(\eta p_i)(x_\Sigma(0))d\mu = \int_{\Sigma} \sigma^{\kappa(\eta) + \kappa(f_\alpha)}(\eta p_i)(x_\Sigma'(0))d\mu'.
$$

Now, let $\xi$ and $\zeta$ be defined in (7) and let $\lambda := -n(n + 1)$. Then, by (9) and Lemma 6, we have that for any $N \geq 0$ and $i = 1, \ldots, 2n + 1$,

$$
\xi^N p_i = \zeta^N p_i = \lambda^N p_i.
$$

Thus, by replacing $\eta$ in (16) with $\xi^N$ or $\zeta^N$, we obtain the following:

$$
\left\{ \begin{array}{c}
\int_{\Sigma} m_\xi^N(\sigma)\psi_i(\sigma)d\sigma = \int_{\Sigma} m_\xi^N(\sigma)\psi'_i(\sigma)d\sigma, \\
\int_{\Sigma} m_\zeta^N(\sigma)\psi_i(\sigma)d\sigma = \int_{\Sigma} m_\zeta^N(\sigma)\psi'_i(\sigma)d\sigma,
\end{array} \right. \quad \text{(18)}
$$

where $m_\xi, m_\zeta$ are monomials given by (15) and $\psi_i, \psi'_i$ are defined as follows:

$$
\left\{ \begin{array}{c}
\psi_i(\sigma) := \sigma^{\kappa(f_\alpha)}(x_\Sigma(0))\rho(\sigma), \\
\psi'_i(\sigma) := \sigma^{\kappa(f_\alpha)}(x_\Sigma'(0))\rho'(\sigma).
\end{array} \right.
$$

Let $C^0(\Sigma)$ be the space of continuous functions on $\Sigma$ and $L^2(\Sigma)$ be the space of square integrable functions $\psi$ on $\Sigma$, i.e., $\int_{\Sigma} \|\psi\|^2 d\sigma < \infty$. Note that $L^2(\Sigma)$ is an inner-product space: For any $\psi$ and $\psi'$ in $L^2(\Sigma)$, we let their inner-product be defined as follows:

$$
\langle \psi, \psi' \rangle_{L^2} := \int_{\Sigma} \psi(\sigma)\psi'(\sigma)d\sigma.
$$
By Lemma 8, the set \( \{m_{\xi}, m_{\xi}'\} \) separates points and, moreover, \( m_{\xi} \) is everywhere nonzero on \( \Sigma \). Thus, by the Stone-Weierstrass Theorem [13], the algebra generated by \( m_{\xi} \) and \( m_{\xi}' \) is dense in \( C^0(\Sigma) \). Furthermore, since \( \Sigma \) is compact, \( C^0(\Sigma) \) is dense in \( L^2(\Sigma) \). It then follows from (18) that \( \psi_i(\sigma) = \psi'_i(\sigma) \) for almost all \( \sigma \in \Sigma \). Since \( \psi_i \) and \( \psi'_i \) are continuous over \( \Sigma \), the two functions are identical:

\[
\sigma^{\kappa(f_{i, q})} p_i(x_{\sigma}(0)) \rho(\sigma) = \sigma^{\kappa(f_{i, q}')} p_i(x'_{\sigma}(0)) \rho'(\sigma), \quad \forall \sigma \in \Sigma,
\]

which, further, implies that

\[
p_i(x_{\sigma}(0)) \rho(\sigma) = p_i(x'_{\sigma}(0)) \rho'(\sigma), \quad \forall \sigma \in \Sigma.
\]

Note that the above holds for all \( i = 1, \ldots, 2n + 1 \). Since \( \{p_i\}_{i=1}^{2n+1} \) is a basis of \( H_n \), we conclude that item (1) of Theorem 4.1 holds. \( \blacksquare \)

**Remark 4.** Note that the two items of Prop. 3.1 are critical in the above proof: Item (1) of Prop. 3.1 guarantees that \( \{m_{\xi}, m_{\xi}'\} \) separates points and item (2) of Prop. 3.1 implies that (18) holds.

### 4.2 Proof of item (2) of Theorem 4.1

Let \( \Phi = \{p_i\}_{i=1}^{2n+1} \) be a basis of \( H_n \). Recall that each \( q \in S_2(\Phi) \) is a quadratic form in \( p_i \) for \( p_i \in \Phi \). Each \( p_i \) is a homogeneous polynomial of degree \( n \) in \( x_1, x_2, x_3 \). Thus, each \( q \in S_2(\Phi) \) is a homogeneous polynomial of degree \( 2n \) in variables \( x_1, x_2, \) and \( x_3 \). We establish item (2) of Theorem 4.1 by proving the following result:

**Proposition 4.2.** For any basis \( \Phi \) of \( H_n \), the set \( S_2(\Phi) \) contains \( (\sum_{i=1}^3 x_i^2)^n \) and, hence, the identity function \( 1_{S^2} \).

Note that for any two bases \( \Phi \) and \( \Phi' \) of \( H_n \), \( S_2(\Phi) = S_2(\Phi') \). Thus, it suffices to establish the proposition for only one particular basis \( \Phi \).

**Example 1.** We demonstrate Prop. 4.2 for the cases where \( n = 1, 2, 3 \):

1. If \( n = 1 \), then \( H_1 \) is 3-dimensional and is spanned by

\[
\Phi := \{x_1, x_2, x_3\}.
\]

It should be clear that \( S_2(\Phi) \) contains \( \sum_{i=1}^3 x_i^2 \).

2. If \( n = 2 \), then \( H_2 \) is 5-dimensional and is spanned by

\[
\Phi := \{p_1 := x_1^2 - x_2^2, p_2 := x_2^2 - x_3^2, p_3 := x_1 x_2, p_4 := x_1 x_3, p_5 := x_2 x_3\}.
\]

Then, we obtain that

\[
\left( \sum_{i=1}^3 x_i^2 \right)^2 = p_1^2 + p_2^2 + p_1 p_2 + 2 \left( p_3^2 + p_4^2 + p_5^2 \right).
\]
(3) If \( n = 3 \), then \( H_3 \) is 7-dimensional and is spanned by

\[
\Phi := \{ p_1 := x_1(2x_1^2 - 3x_2^2 - 3x_3^2), \; p_2 := x_2(2x_1^2 - 3x_2^2 - 3x_3^2), \; p_3 := x_3(2x_1^2 - 3x_2^2 - 3x_3^2), \; p_4 := x_1(x_1^2 - x_2^2), \; p_5 := x_2(x_1^2 - x_3^2), \; p_6 := x_3(x_1^2 - x_3^2), \; p_7 := x_1x_2x_3 \}
\]

Then, by computation, we obtain that

\[
\left( \sum_{i=1}^{3} x_i^2 \right)^3 = \frac{1}{4} (p_1^2 + p_2^2 + p_3^2) + \frac{15}{4} (p_4^2 + p_5^2 + p_6^2) + 15p_7^2.
\]

We establish below proposition 4.2. There are multiple methods for proving the result. The approach we use below leverages again the representation theory of \( \mathfrak{sl}(2, \mathbb{C}) \). But, we note that one can also use the Addition Theorem for spherical harmonics [14, Ch. 12] to prove the proposition. For the latter approach, we refer the reader to the Appendix for details.

To proceed, we first recall that the polynomial \((x_1 + ix_2)^n\) is a highest weight vector (with the highest weight being \(2n\)) associated with the irreducible representation \(\pi : \mathfrak{g}^C \times H_n^C \to H_n^C\). Let \(h, e_+, \) and \(e_-\) be defined in (8). We next define

\[
p_k(x) := \pi^k(e_-(x_1 + ix_2))^n, \quad \forall k = 0, \ldots, 2n.
\]

Then, by Lemma 3 each \(p_k\) is a weight vector and

\[
\pi(h)p_k = (2n - 2k)p_k.
\]

It should be clear from the definition that \(\pi(e_-)p_k = p_{k+1}\) for all \(k = 0, \ldots, 2n - 1\). Conversely, for any \(k = 1, \ldots, 2n\), the following holds (see, for example, [12, Ch. 17]):

\[
\pi(e_+)p_k = k(2n - k + 1)p_{k-1}.
\]

Furthermore, by computation, we obtain the following fact:

**Lemma 9.** For any \(k = 0, \ldots, n\),

\[
p_{2n-k} = (-1)^{n-k} \frac{(2n - k)!}{k!} \bar{p}_k,
\]

where \(\bar{p}_k\) is the complex conjugate of \(p_k\).

Note, in particular, that by (22), \(p_n = \bar{p}_n\) and, hence, \(p_n\) is real.

Now, let \(\Phi := \{ p_k \}_{k=0}^{2n}\). Then, \(\Phi\) is a basis of \(H_n^C\). Let \(S_2^C(\Phi)\) be the complexification of \(S_2(\Phi)\), i.e., \(S_2^C(\Phi)\) is the space of all quadratic forms in \(p_k\) with complex coefficients. To establish Prop. 4.2 it now suffices to show that \(S_2^C(\Phi)\) contains \((\sum_{i=1}^{3} x_i^2)^n\). It should be clear that \(S_2^C(\Phi)\) is a subspace of \(P_2^C\) and is spanned by \(p_ip_j\) for \(0 \leq i \leq j \leq 2n\). Let \(\tilde{\pi}\) be the representation of \(\mathfrak{g}^C\) on \(P_2^C\), i.e.,

\[
\tilde{\pi} : (f, \phi) \in \mathfrak{g}^C \times P_2^C \mapsto \tilde{\pi}(f)\phi := f\phi \in P_2^C.
\]

We have the following fact:
Lemma 10. The subspace $S_2^C(\Phi)$ of $P_{2n}^C$ is invariant under $\tilde{\pi}(g)$.

Proof. It suffices to show that for any $f \in g^C$ and any $p_i p_j$ with $0 \leq i, j \leq 2n$, $\tilde{\pi}(f)(p_i p_j)$ belongs to $S_2^C(\Phi)$. By Leibniz rule,

$$\tilde{\pi}(f)(p_i p_j) = (fp_i)p_j + p_i(f p_j).$$

Note that both $fp_i$ and $f p_j$ belong to $H_n^C$ because $H_n^C$ is invariant under $\pi(g)$. Thus, the right hand side of the above expression belongs to $S_2^C(\Phi)$. ■

By Lemma 10, one can obtain a representation of $g^C$ on $S_2^C(\Phi)$ by restricting $\tilde{\pi}$ to $g C \times S_2^C(\Phi)$. With slight abuse of notation, we will still use $\tilde{\pi}$ to denote such a representation. The representation $\tilde{\pi}$ is, in general, not irreducible. But, by Lemma 5, we know that there exists an integer $N > 0$ and nonnegative integers $0 \leq k_1 < \cdots < k_N \leq n$ such that

$$S_2^C(\Phi) = \|x\|^{2k_1} H_{2n-2k_1}^C \oplus \cdots \oplus \|x\|^{2k_N} H_{2n-2k_N}^C$$

and, moreover, $\tilde{\pi}$ is an irreducible representation when it is restricted to every subspace $\|x\|^{2k_i} H_{2n-2k_i}^C$, for $i = 1, \ldots, N$. Note, in particular, that if $k_N = n$, then $S_2^C(\Phi)$ contains the desired polynomial. We show below that this is indeed the case:

Proof of Prop. 4.2. Let $q^* := \sum_{k=0}^{2n} (-1)^{n+k} p_k p_{2n-k}$, which belongs to $S_2^C(\Phi)$. We show below that $q^* = \lambda \left( \sum_{i=1}^{3} x_i^2 \right)^n$ for some $\lambda > 0$. First, note that by (22), $q^*$ can be re-written as follows:

$$q^* = p_n^2 + \sum_{k=0}^{n-1} \frac{(2n-k)!}{k!} |p_k|^2.$$ 

In particular, $q^*$ is strictly positive (and, hence, nonzero).

We now show that both $\tilde{\pi}(h)q^*$ and $\tilde{\pi}(e_+)q^*$ are zero. By (20), we have that for each $k = 0, \ldots, 2n$, $hp_k = (2n-2k)p_k$ and $hp_{2n-k} = -(2n-2k)p_{2n-k}$. It then follows that

$$\tilde{\pi}(h)(p_k p_{2n-k}) = (hp_k)p_{2n-k} + p_k(hp_{2n-k}) = 0,$$

and, hence, $\tilde{\pi}(h)q^* = 0$. Next, for $\tilde{\pi}(e_+)q^*$, we obtain by computation that

$$\tilde{\pi}(e_+)q^* = \sum_{k=0}^{2n-1} (-1)^{n+k} (p_k(e_+ p_{2n-k}) - (e_+ p_{k+1})p_{2n-k-1})$$

It follows from (21) that

$$\begin{cases} e_+ p_{2n-k} = (2n-k)(k+1)p_{2n-k-1}, \\ e_+ p_{k+1} = (k+1)(2n-k)p_{n-2k}, \end{cases}$$

and, hence, each addend on the right hand side of (23) is 0.

Let $Cq^*$ be the one-dimensional subspace of $S_2^C(\Phi)$ spanned by $q^*$. Because $\tilde{\pi}(h)q^* = 0$ and $\tilde{\pi}(e_+)q^* = 0$, we obtain by Lemma 4 that $\tilde{\pi}$ is an irreducible representation when restricted to $q^C \times Cq^*$. Moreover, its highest weight of the representation is 0. But then, by Lemma 5 $Cq^* = \|x\|^{2n} H_0^C$. Since $q^*$ is positive, we conclude that $q^* = \lambda \|x\|^{2n}$ for some $\lambda > 0$. ■
4.3 Proof of item (3) of Theorem 4.1

To establish Theorem 4.1, it now remains to prove item (3). This is done below:

Proof of item (3) of Theorem 4.1 Recall that $H_n^C$ is the complexification of $H_n$. We fix an arbitrary $x \in S^2$ and show that if $p(x') = p(x)$ for all $p \in H_n^C$, then $x' \in \{ x, (-1)^{n+1}x \}$. Since the $x_i$’s cannot be zero simultaneously, we assume without loss of generality that $x_3 \neq 0$. Then, consider the following three homogeneous polynomials in $H_n^C$:

$$p_1(x) := (x_1 + ix_2)^n, \quad p_2(x) := x_3(x_1 + ix_2)^{n-1}, \quad p_3(x) := (x_3 + ix_1)^n.$$ 

We assume that the values of the above polynomials at the given $x$ are the following:

$$p_1(x) = a_1, \quad p_2(x) = a_2, \quad p_3(x) = a_3$$

for some $a_1, a_2, a_3 \in \mathbb{C}$. We provide below solutions $x'$ to the above polynomial equations.

If both $x_1$ and $x_2$ are 0, then $a_1 = a_2 = 0$ and $a_3$ is a (nonzero) real number. It follows that $x_1' = x_2' = 0$ and $x_3'' = x_3'' = a_3$. Thus, in this case, $x' \in \{ x, (-1)^{n+1}x \}$. Next, we assume that $x_1^2 + x_2^2 \neq 0$. Since $x_3 \neq 0$, every $a_i$ is nonzero. Then,

$$\frac{p_1(x')}{p_2(x')} = \frac{x_1' + ix_2'}{x_3'} = \frac{a_1}{a_2}.$$ 

Since $x_1', x_2'$, and $x_3'$ are real, we have that

$$x_1' = \text{re}(a_1/a_2)x_3' \quad \text{and} \quad x_2' = \text{im}(a_1/a_2)x_3'.$$ 

(24)

where $\text{re}(\cdot)$ and $\text{im}(\cdot)$ denote the real and imaginary part of a complex number, respectively. On the other hand, we also have that $\sum_{i=1}^3 x_i'^2 = 1$. Thus, (24) determines $x'$ up to sign, i.e., $x' = \pm x$. If, further, $n$ is odd, then $p(-x) = -p(x) = -a_1 \neq a_1$ and, hence, $x'$ can only be $x$. Combining the above arguments, we conclude that $x' \in \{ x, (-1)^{n+1}x \}$. ■

5 Conclusions and Further Discussions

We address in the paper the problem about observability of a continuum ensemble of Bloch equations (3). The problem formulation is given as follows: We assume that the initial states $x_\sigma(0)$ of the individual systems in the ensemble are unknown and, moreover, the measure $\mu$ that describes the overall population density of the individual systems over the parametrization space $\Sigma$ is also unknown. Then, the observability problem we address in the paper is about whether one is able to estimate $x_\sigma(0)$ for all $\sigma \in \Sigma$ and the measure $\mu$ using only a scalar measurement output $y(t)$.

Given the ensemble system dynamics (3) and the integrated measurement output $y(t)$, we have established a sufficient condition (Theorem 1.1) on the common observation function $\phi$ for system (3) to be (weakly) ensemble observable. Specifically, we have shown that if $\phi$ is any harmonic homogeneous polynomial of positive degree, then two pairs $(x_\Sigma(0), \mu)$ and $(x_\Sigma'(0), \mu')$ are output equivalent if and only if $\mu = \mu'$ and $x_\Sigma'(0) \in \{ x_\Sigma(0), (-1)^{n+1}x_\Sigma(0) \}$. In particular, if $n$ is odd, then system (3) is ensemble observable.

The proof of Theorem 1.1 relies on the use of representation theory of $g^C \approx \mathfrak{sl}(2, \mathbb{C})$:
(1) We have used the fact that the representation of $g^C$ on $H^C_n$ is irreducible.

(2) We have introduced the Casimir element $\eta^*$ (and its variants $\xi$ and $\zeta$ defined in Lemma 2) which acts on the space of harmonic homogeneous polynomials as a scalar multiple of the identity operator. This fact is instrumental in establishing item (1) of Theorem 4.1.

(3) We have further used the fact that any finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$ is reducible and then, decomposed the space $S^C_2(\Phi)$ (with $\Phi$ a basis of $H^C_n$) into a direct sum of invariant subspaces. In particular, we have shown that $S_2(\Phi)$ contains the one-dimensional subspace spanned by $\|x\|^{2n}$ and, hence, the identity function $1_{S^2}$ on $S^2$. This fact is crucial in establishing the fact that $\mu = \mu'$ (see the proof of Theorem 1.1) given after the statement of Theorem 4.1.

We note that the ensemble observability result and the approach established in the paper can be extended to other ensemble system dynamics defined on more general homogenous spaces. The extension will revolve around the above three items, especially item (2). For that particular item, we note here the following fact [10, Ch. V]: For any general (complex) semisimple Lie algebra $g^C$, the Casimir element can be defined via the Killing form $B(X, Y) := \text{tr}(\text{ad}_X \, \text{ad}_Y)$. Specifically, if $\{X_i\}_{i=1}^n$ is a basis of $g^C$ and $\{\tilde{X}_i\}_{i=1}^n$ be the dual basis, i.e., $B(\tilde{X}_i, X_j) = \delta_{ij}$ where $\delta_{ij}$ is the Kronecker delta, then $\sum_{i,j} B(X_i, X_j) \tilde{X}_i \tilde{X}_j$ is the corresponding Casimir element.

Finally, we note that the idea of using the Casimir element to establish item (1) of Theorem 4.1 originated from the definition of “co-distinguished set” introduced in [9]. We elaborate below on the similarity and the difference. To proceed, we first recall that a set of functions $\{\phi_j\}_{j=1}^l$ on an arbitrary manifold $M$ is said to be codistinguished to a set of vector fields $\{f_i\}_{i=1}^m$ over $M$ if the following hold:

(1) The one forms $\{d_x \phi_j\}_{j=1}^l$ span the cotangent space $T^*_x M$ for all $x \in M$.

(2) For any $\phi_j$ and any $f_i$, there exist a $\phi_k$ such that

$$f_i \phi_j = \lambda \phi_k. \quad (25)$$

Conversely, for any $\phi_k$, there exist $f_i, \phi_j$, and a nonzero $\lambda$ such that (25) holds.

(3) For any $x$ and $x'$, if $\phi_j(x) = \phi_j(x')$ for all $j = 1, \ldots, l$, then $x = x'$.

In the case here, we have that $M$ is the unit sphere $S^2$, $\{f_i\}_{i=0}^2$ is the set of vector fields defined in (1), and $\{\phi\}_{j=1}^l$ can be any spanning set of $H_n$ for $n \geq 1$. We note without a proof that any such spanning set $\{\phi_j\}_{j=1}^l$ satisfies item (1). Moreover, if $n$ is odd, then by Theorem 4.1, $\{\phi_j\}_{j=1}^l$ satisfies item (3). However, we do not require that item (2) holds for $\{\phi_j\}_{j=1}^l$ and $\{f_i\}_{i=0}^2$. Instead, we let $\{\phi_j\}_{j=1}^l$ be “codistinguished” to the Casimir element $\eta^*$ in a sense that (25) is now replaced with the condition that $\eta^* \phi_j = \lambda \phi_j$ for a nonzero $\lambda$.

The above arguments also suggest that one can extend the definition of codistinguished set by replacing a set of vector fields $\{f_i\}_{i=1}^m$ with a set $\{\eta_i\}_{i=1}^m$ where each $\eta_i$ is a linear
combination of $f_\alpha$ for $\alpha$ a word over the alphabet $\{1, \ldots, m\}$. More specifically, we let $\mathfrak{g}$ be the Lie algebra generated by a set of vector fields $\{f_i\}_{i=1}^m$ and $U(\mathfrak{g})$ be the associated universal enveloping algebra. If $\{f_i\}_{i=1}^m$ spans $\mathfrak{g}$, then the collection of $f_\alpha$ span $U(\mathfrak{g})$. Thus, the above extension is nothing but to replace the condition that $f_i \in \mathfrak{g}$ with a more relaxed condition that $\eta_i \in U(\mathfrak{g})$. The Casimir element (or any element in the center $Z(U(\mathfrak{g}))$) is thus always a valid candidate. Note that by such a relaxation from $\mathfrak{g}$ to $U(\mathfrak{g})$, we have an increase chance of finding a set of functions $\Phi := \{\phi_j\}_{j=1}^l$ that is codistinguished to $\{\eta_i\}_{i=1}^m$, especially, to the Casimir element $\eta^*$; indeed, as is demonstrated in the paper, the remaining task is to make sure that $\Phi$ is contained in an invariant space $H$ under $\mathfrak{g}$ such that the representation $\pi : \mathfrak{g} \times H \to H$ is irreducible. However, we note that if $\{f_i\}_{i=1}^m$ does not span $\mathfrak{g}$, then the collection of $f_\alpha$ does not necessarily span $U(\mathfrak{g})$. In particular, the Casimir element may not be available, i.e., it cannot be obtained as a certain linear combination of $f_\alpha$. We will address such a scenario in our future works.

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Appendix

We provide here another proof of Prop. 4.2 using the Addition Theorem (see, for example, [14, Ch. 12]). Recall that the Cartesian coordinate system \((x_1, x_2, x_3)\) and the spherical coordinate system \((r, \theta, \phi)\) are related by

\[
x_1 = r \sin \theta \cos \phi, \quad x_2 = r \sin \theta \sin \phi, \quad x_3 = r \cos \theta.
\]

(26)

We next recall that spherical harmonics \(Y^k_n(\theta, \phi)\) are defined as follows: For a given a nonnegative integer \(n\) and an integer \(k\) with \(|k| \leq n\), we have that

\[
Y^k_n(\theta, \phi) := (-1)^k \sqrt{\frac{2n+1}{4\pi}} \frac{(n-k)!}{(n+k)!} L^k_n(\cos \theta) e^{in\phi},
\]

where \(L^k_n\) is the associated Legendre polynomial defined by

\[
L^k_n(x) := \frac{(-1)^k}{2^n n!} (1 - x^2)^{k/2} \frac{d^{n+k}}{dx^{n+k}} (x^2 - 1)^n.
\]

It is known that \(\{Y^k_n\}_{k=-n}^n\) is a basis of \(H^n_C\) (after change of coordinates (26)). In other words, each harmonic homogeneous polynomial \(p \in H^n_C\) can be expressed as a linear combination of the spherical harmonics and vice versa. In fact, we note here that each \(Y^k_n\) for \(|k| \leq n\) is linearly proportional to \(p_{n-k}\) where \(p_{n-k}\) is defined in (19).

We now reproduce the Additional Theorem for spherical harmonics: First, recall that the ordinary Legendre polynomial \(L_n\) is given by the Rodrigues’ formula:

\[
L_n(x) := \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.
\]

Next, for two points \((1, \theta, \phi)\) and \((1, \theta', \phi')\) on the unit sphere \(S^2\), we let \(\gamma\) be the angle between these two points, i.e.,

\[
\cos \gamma := \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi').
\]

Then, the Addition Theorem for spherical harmonics is the following:
Lemma 11 (Addition Theorem). For any two points \((1, \theta, \varphi)\) and \((1, \theta', \varphi')\) in \(S^2\),

\[
L_n(\cos \gamma) = \frac{4\pi}{2n + 1} \sum_{k=-n}^{n} Y_n^k(\theta, \varphi)Y_n^k(\theta', \varphi')
\]

Prop. 4.2 is then a corollary to the above result. To see this, we let \((\theta, \varphi) = (\theta', \varphi')\). Then, by the Addition Theorem, we have that for any \((\theta, \varphi)\),

\[
\frac{4\pi}{2n + 1} \sum_{k=-n}^{n} |Y_n^k(\theta, \varphi)|^2 = L_n(1).
\]

Finally, note that \(L_n(1) = 1\) for any \(n \geq 1\), which then completes the proof of Prop. 4.2. ■