A nonhomogeneous critical Kirchhoff-Schrödinger type equation in $\mathbb{R}^4$ involving vanishing potentials

Francisco S. B. Albuquerque
Universidade Estadual da Paraíba
Departamento de Matemática
CEP: 58700-070, Campina Grande - PB, Brazil
fsiberio@cct.uepb.edu.br

Marcelo C. Ferreira*
Universidade Federal de Campina Grande
Unidade Acadêmica de Matemática
CEP: 58429-900, Campina Grande - PB, Brazil
marcelo@mat.ufcg.edu.br

Abstract

In this paper we establish the existence of high energy weak solutions for a Kirchhoff-Schrödinger type problem in $\mathbb{R}^4$ involving a critical nonlinearity and a suitable small perturbation. The arisen competition between the terms due to the nonlocal coefficient and critical nonlinearity turns out to be rather interesting. The main tools used in the present work are variational methods and the Lions’ Concentration Compactness Principle.

Keywords: Kirchhoff-Schrödinger equations; Nonlocal problems; Vanishing potentials; Variational methods; Critical growth.

2010 Mathematics Subject Classification: 35B33, 35J20, 35J60.

*Corresponding author
1 Introduction and main results

In this paper we establish the existence of weak solutions for the following nonhomogeneous Kirchhoff-Schrödinger type problem:

\[
(P_{\lambda,\mu}) \begin{cases}
- \left( a + b \int_{\mathbb{R}^4} |\nabla u|^2 \, dx \right) \Delta u + \lambda V(x)u = \mu K(x)|u|^{q-2}u + u^3 + h(x), & x \in \mathbb{R}^4, \\
\end{cases}
\]

where \( a, b > 0 \) are constants, \( \lambda \) and \( \mu \) are positive parameters, \( q \in (5/2, 4) \), \( h: \mathbb{R}^4 \to \mathbb{R} \) is a small perturbation belonging to \( E^* \), the dual space of \( E \) (see the definition on page 6), and the weights \( V, K: \mathbb{R}^4 \to (0, \infty) \) satisfy the following hypotheses

\((V)\) The potential \( V \in L^\infty(\mathbb{R}^4) \cap L^2(\mathbb{R}^4) \);

\((K)\) The potential \( K \in L^\infty(\mathbb{R}^4) \) and if \( \{A_n\} \subset \mathcal{P}(\mathbb{R}^4) \) is a sequence of Borel sets such that \( |A_n| \leq R \), for all \( n \) and some \( R > 0 \), then

\[ \lim_{r \to +\infty} \int_{A_n \cap B_r(0)} K(x) \, dx = 0, \quad \text{uniformly in } n \in \mathbb{N}, \]

where \( |\cdot| \) means the Lebesgue measure in \( \mathbb{R}^4 \);

\((VK)\) The condition

\[ \frac{K}{V} \in L^\infty(\mathbb{R}^4) \]

occurs.

We point out that the hypotheses \((K)\) and \((VK)\) were firstly introduced by Alves and Souto in [5] in the frame of nonlinear Schrödinger equations and the authors observed that they are more general than that ones considered earlier by Ambrosetti, Felli and Malchiodi in [6] in order to get compactness embedding from \( E \) to \( E^p_K(\mathbb{R}^4) \) (see the definition on page 6).

Simple examples of \( V \) and \( K \) satisfying \((V)\), \((K)\) and \((VK)\) are given by

\[ V(x) = \frac{1}{1 + |x|^\alpha} \quad \text{and} \quad K(x) = \frac{1}{1 + |x|^\beta}, \]

with \( \beta > \alpha > 2 \). The potentials above belong to a class entitled vanishing at infinity (or zero mass case). After the mentioned work due to Ambrosetti et al. [6], lots of
types of stationary nonlinear Schrödinger equations involving vanishing potentials at infinity have been studied in \( \mathbb{R}^N \) \((N \geq 2)\) and, in the vast list of references in this aspect, we may cite [31, 29, 34, 35] and the references therein.

Moreover, we also assume

\((S)\) The coefficient \( b \) satisfies \( b > 1/S^2 \), where \( S \) is the best Sobolev constant for the embedding of the Sobolev space \( D^{1,2} (\mathbb{R}^4) \) into \( L^4 (\mathbb{R}^4) \), that is,

\[
S = \inf_{u \in D^{1,2} (\mathbb{R}^4) \setminus \{0\}} \frac{\int_{\mathbb{R}^4} |\nabla u|^2 \, dx}{\left( \int_{\mathbb{R}^4} u^4 \, dx \right)^{1/2}}.
\]

A problem as \((P_{\lambda,\mu})\) is called nonlocal due to the presence of the term \( \left( \int_{\mathbb{R}^4} |\nabla u|^2 \, dx \right) \Delta u \) in its formulation which implies that the equation in \((P_{\lambda,\mu})\) is no longer a pointwise identity. As we will see later, this phenomenon causes some mathematical difficulties and consequently motivates the study of such a class of problems from the mathematical viewpoint. In this sense, we would like to notice that condition \((S)\) imposes our results are rather different from the most in literature, since they are not extensions of results obtained for local Kirchhoff-Schrödinger problems to the nonlocal case. They are purely nonlocal.

Regarding to problem \((P_{\lambda,\mu})\), there are a considerable number of physical appeals. For instance, in \((P_{\lambda,\mu})\) if we set \( V(x) = 0 \), and replace \( \mu K(x)|u|^{q-2}u + u^3 + h(x) \) and \( \mathbb{R}^4 \) by \( f(x, u) \) and \( \Omega \subset \mathbb{R}^N \) a bounded domain, respectively, it reduces to the following Dirichlet problem of Kirchhoff type:

\[
\begin{cases}
- \left( a + b \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u = f(x, u), & x \in \Omega, \\
\quad u = 0, & x \in \partial\Omega,
\end{cases}
\]

which is related to the stationary analogue of the evolution problem

\[
\begin{cases}
u_{tt} - \left( a + b \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u = f(x, u), & (x, t) \in \Omega \times (0, T), \\
u = 0, & (x, t) \in \partial\Omega \times (0, T), \\u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x), & x \in \Omega.
\end{cases}
\]
Such a hyperbolic equation is a general version of the Kirchhoff equation

\[
\varrho \frac{\partial^2 u}{\partial t^2} - \left( \frac{P_0}{s} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \, dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \quad (x, t) \in (0, L) \times (0, T),
\]

which has came to light at Kirchhoff [21], in 1883, as an extension of the classical well-known D’Alembert wave equation for free vibrations of elastic strings. The Kirchhoff’s model takes into account the effects of changes in the length of the string during the vibrations. The parameters in the above equation have the following meanings: \( L \) is the length of the string, \( s \) is the area of cross-section, \( E \) is the Young modulus of the material, \( \varrho \) is the mass density and \( P_0 \) is the initial tension. We recall that nonlocal problems also appear in other fields, for instance, biological processes where the function \( u \) describes a distribution which depends on the average of itself (for example, population density), see for instance [2, 3] and its references.

Some early research on Kirchhoff equations can be found in the seminal works [10, 31]. However, the problem (1.1) received great attention of a lot of researchers only after Lions [22] proposed an abstract framework for it, more precisely, a functional analysis approach was proposed to study it, see [2, 3, 8, 9, 11, 30, 38]. Recently, many approaches involving variational and topological methods have been used in a straightforward and effective way in order to get solutions in a lot of works, see [4, 19, 25, 26, 27, 28, 29] and the references therein. The studies of Kirchhoff type equations have also already been extended to the case involving the \( p \)-Laplacian, for example [12, 15, 16, 24] and so on. Sometimes, the nonlocal term appears in generic form \( m \left( \int_\Omega |\nabla u|^2 \, dx \right) \), where \( m: \mathbb{R}_+ \to \mathbb{R}_+ \) is a continuous function that must satisfy some appropriate conditions (amongst them, monotonicity or boundedness below by a positive constant), which the typical example is given by the model considered in the original Kirchhoff equation (1.1). In [3, 18], for example, the authors have used comparison between minimax levels of energy to show that the solution of the truncated problem, that is, an auxiliary problem obtained by a truncation on function \( m \), is a solution of the original problem.

Specifically in relation to Kirchhoff-Schrödinger type problems such as \((P_{\lambda, \mu})\), it get so many attention, mainly in unbounded domains, due to the lack of compactness of the Sobolev’s embeddings, which makes the study of the problem more delicate, interesting and challenging. In order to overcome this trouble and to recover the compactness of the Sobolev’s embeddings, some authors studied the problem \((P_{\lambda, \mu})\) in a subspace consisting of radially symmetric functions (see, for example, [28, 29]). In [37], the following Kirchhoff-Schrödinger type equation
was discussed when \( h(x) = 0 \), where constants \( a > 0, b \geq 0, N = 1, 2, 3 \) and \( f, V \) are continuous with \( V \) satisfying

\[
\inf_{x \in \mathbb{R}^N} V(x) > 0 \quad \text{and for each } M > 0, \ |\{x \in \mathbb{R}^N : V(x) \leq M\}| < +\infty.
\]

The author studied the existence of nontrivial solutions and infinitely many high energy solutions by using minimax theorems. In [29], the authors studied the following Kirchhoff-Schrödinger type problem

\[
\begin{cases}
- \left( a + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \Delta u + V(|x|)u = Q(|x|)f(u), & x \in \mathbb{R}^N, \\
u(x) \to 0, & |x| \to \infty,
\end{cases}
\]

where \( a, b > 0, N \geq 2, f: \mathbb{R} \to \mathbb{R} \) is continuous, and \( V, Q \) are radial positive functions, which can be vanishing, unbounded or coercive at infinity; suitable conditions firstly introduced in [34, 35]. As in [37] and by using embedding results established in [34, 35], the authors obtained existence and multiplicity of nontrivial solutions also by using minimax methods.

Recently, in 2014, Nie in [28] finds out a nontrivial radial solution for the problem

\[
\begin{cases}
- \left( a + b \int_{\mathbb{R}^4} |\nabla u|^2 \, dx \right) \Delta u + \lambda u = |u|^{p-1}u, & x \in \mathbb{R}^4, \\
u(x) \to 0, & |x| \to \infty,
\end{cases}
\]

where \( a > 0, b \geq 0, \lambda > 0 \) and \( 1 < p < 3 \), by using variational methods. We emphasize that in four dimensions \( 2^* = 4 \) is the Sobolev critical exponent of the embeddings \( H^1(\mathbb{R}^4) \hookrightarrow L^p(\mathbb{R}^4) \). Because of that, as in [28], one of the difficulties here is investigating the boundedness of the Palais-Smale sequences. In order to circumvent this difficulty, we will use an abstract critical point theorem developed by Jeanjean [20] which has been used largely.

The role played by the nonhomogeneous term \( h \) in producing multiple solutions is crucial in our analysis. For this reason, the study of existence of multiple solutions for nonhomogeneous elliptic equations with subcritical and critical growth in bounded and unbounded euclidean domains have received much attention in recent years (see [1, 32, 33, 36]). Most of these equations are dealt with variational methods, and since
the Palais-Smale compactness condition no longer holds for this class of equations this poses an essential difficulty to the existence question. Motivated by the above works, the aim of the present paper is to continue the study of the critical nonlocal elliptic equations. To the best of our knowledge, in current literature, there are no results on the problem \((P_{\lambda,\mu})\) (neither in the nonhomogeneous nor in the homogeneous case, i.e. \(h \equiv 0\)). However, it should be emphasized that in [13], Chen and Li first studied multiple solutions for nonhomogenous Schrödinger-Kirchhoff problem \((1.2)\) by using Ekeland’s variational principle and Mountain-Pass Theorem, with the nonlinearity \(f\) satisfying the Ambrosetti-Rabinowitz condition, i.e. there exists \(\theta > 2\) such that

\[
0 < \theta F(x, t) = \theta \int_0^t f(x, s) \, ds, \quad \forall x \in \mathbb{R}^N, \, t \in \mathbb{R} \setminus \{0\}.
\]

Generalizing the Chen and Li’s results, Cheng [14] studied multiple solutions for nonhomogenous Schrödinger-Kirchhoff problem \((1.2)\) with the general nonlinearity \(F\) satisfying super-quartic condition.

We need to introduce some notations. From now on, we write \(\int u\) instead of \(\int_{\mathbb{R}^4} u(x) \, dx\) and we use \(C, C_0, C_1, C_2, \ldots\) to denote (possibly different) positive constants. We denote by \(B_R(x) \subset \mathbb{R}^4\) the open ball centered at \(x \in \mathbb{R}^4\) with radius \(R > 0\) and \(B_R^c(x) := \mathbb{R}^4 \setminus B_R(x)\). The symbols \(o_\varepsilon(1)\) and \(o_n(1)\) will denote quantities that converge to zero, as \(\varepsilon \to 0\) and \(n \to \infty\) respectively. Also, we denote the weak convergence in \(X\) by “\(\rightharpoonup\)" and the strong convergence by “\(\to\)". Besides, for each \(0 < \lambda < \infty\) and from the assumptions on \(V\), the expression

\[
\|u\|_\lambda = \left( \int (|\nabla u|^2 + \lambda V(x) u^2) \right)^{\frac{1}{2}},
\]

defines a norm over

\[
E = \left\{ u \in D^{1,2}(\mathbb{R}^4) : \int V(x) u^2 < \infty \right\}
\]

(our work space) such that \(E\) is Hilbert, \(E\) is continuously imersed in \(D^{1,2}(\mathbb{R}^4)\) and \(L^4(\mathbb{R}^4)\). Furthermore, under the hypotheses \((K)\) and \((VK)\), in the study of Alves and Souto (see [5, Proposition 2.1]), we know that \(E\) is compactly embedded into the weighted Lebesgue space

\[
L^p_K(\mathbb{R}^4) := \left\{ u : \mathbb{R}^4 \to \mathbb{R} : u \text{ is measurable and } \int K(x)|u|^p < \infty \right\},
\]

equipped with the norm

\[
|u|_{p,K} = \left( \int K(x)|u|^p \right)^{\frac{1}{p}},
\]
for all $2 < p < 4$. Also, for $u \in L^p(\mathbb{R}^4)$ we denote its $p$-norm with respect to the Lebesgue measure by $|u|_p$ and $E^*$ will designate the dual space of $E$ with the usual norm $\| \cdot \|_{E^*}$.

**Definition 1.1** We say that $u: \mathbb{R}^4 \to \mathbb{R}$ is a weak solution of $(P_{\lambda,\mu})$ if $u \in E$ and it holds the identity

$$
\left( a + b \int |\nabla u|^2 \right) \int \nabla u \cdot \nabla \varphi + \lambda \int V(x)u \varphi = \mu \int K(x)|u|^{q-2}u \varphi + \int u^3 \varphi + \int h \varphi,
$$

for all $\varphi \in E$.

The main results of this work can be stated as follows.

**Theorem 1.2** Assume that $(V)$, $(K)$, $(VK)$ and $(S)$ hold. Then, there exists $\mu^* > 0$ sufficiently large and $\lambda^* = \lambda_{\mu^*} > 0$ sufficiently small such that if $0 < \lambda < \lambda^*$, then problem $(P_{\lambda,\mu})$ has a high energy positive weak solution in $D^{1,2}(\mathbb{R}^4)$ for almost everywhere $\mu > \mu^*$, whenever $0 < \|h\|_{E^*}$ is sufficiently small.

The heart of the proof of Theorem 1.2 is inspired by some arguments presented in [20], as we said above, mainly in order to prove the boundedness of the Palais-Smale sequences, which cannot be proved directly in our case.

**Theorem 1.3** Assume that $(V)$, $(K)$, $(VK)$ and $(S)$ hold. Then, there exists $\mu^{**} > 0$ sufficiently small, such that if $0 < \mu < \mu^{**}$ and $\lambda > 0$, problem $(P_{\lambda,\mu})$ has a negative energy weak solution in $D^{1,2}(\mathbb{R}^4)$, whenever $0 < \|h\|_{E^*}$ is sufficiently small.

The proof of Theorem 1.3 is based on Ekeland’s variational principle (see [17]) to prove the existence of a local minimum type solution.

The outline of the paper is as follows: Section 2 contains the variational setting in which our problem will be treated and allow us to follow a variational approach. Section 3 is devoted to study the behaviour of the Palais-Smale sequences. The proofs of the main results are established in Section 4.

## 2 The variational framework

Following the line firstly introduced by Alves et al. in [3] to solve the Kirchhoff problem, we establish now the necessary functional framework where solutions are
naturally studied by variational methods. The energy functional associated to problem \((P_{\lambda,\mu})\), that is, \(I_{\lambda,\mu} : E \to \mathbb{R}\) where

\[
I_{\lambda,\mu}(u) = \frac{1}{2} \int \left( a|\nabla u|^2 + \lambda V(x)u^2 \right) + \frac{b}{4} \left( \int |\nabla u|^2 \right)^2 - \frac{\mu}{q} \int K(x)|u|^q - \frac{1}{4} \int u^4 - \int hu
\]

is well defined and \(I_{\lambda,\mu} \in C^1(E, \mathbb{R})\) with derivative given by

\[
I'_{\lambda,\mu}(u)v = \left( a + b \int |\nabla u|^2 \right) \int \nabla u \cdot \nabla v + \lambda \int V(x)uv - \mu \int K(x)|u|^{q-2}uv - \int u^3v
\]

for all \(u, v \in E\). So that, any critical point of the functional \(I_{\lambda,\mu}\) is a weak solution to problem \((P_{\lambda,\mu})\).

2.1 The Mountain-Pass Geometry

Next two lemmas describe the geometric structure of the functional \(I_{\lambda,\mu}\) required by the Mountain-Pass Theorem due to Ambrosetti and Rabinowitz in [7]. We would like to point out that hypothesis \((S)\) plays an important role for the proof of the second one.

**Lemma 2.1** Let \(\mu > 0\). Then, there exists \(\delta_\mu > 0\) such that for \(h \in E^*\) with \(\|h\|_{E^*} < \delta_\mu\), it holds that

\[
I_{\lambda,\mu}(u) \geq \sigma, \text{ for } \|u\|_\lambda = \tau,
\]

for some \(\sigma > 0\) and \(0 < \tau < 1\).

**Proof.** The continuous embeddings \(E \hookrightarrow L^q_K(\mathbb{R}^4)\) and \(E \hookrightarrow L^4(\mathbb{R}^4)\), yields

\[
I_{\lambda,\mu}(u) = \frac{1}{2} \int \left( a|\nabla u|^2 + \lambda V(x)u^2 \right) + \frac{b}{4} \left( \int |\nabla u|^2 \right)^2 - \frac{\mu}{q} \int K(x)|u|^q - \frac{1}{4} \int u^4
- \int hu
\geq \frac{\min\{a,1\}}{2} \|u\|_\lambda^2 - \mu C_0 \|u\|_\lambda^2 - C_1 \|u\|_\lambda^4 - \|h\|_{E^*} \|u\|_\lambda
= \|u\|_\lambda \left( \frac{\min\{a,1\}}{2} - \mu C_0 \|u\|_\lambda^{q-2} - C_1 \|u\|_\lambda^{4-q} - \|h\|_{E^*} \right).
\]
Taking $0 < \tau < 1$ such that $\mu C_0 \tau^{q-1} + C_1 \tau^3 \leq \frac{\min\{a, 1\}}{4} \tau$, then for $\|u\|_\lambda = \tau$ we have

$$I_{\lambda, \mu}(u) \geq \tau \left( \frac{\min\{a, 1\}}{4} \tau - \|h\|_{E^*} \right).$$

Thus, for $\|h\|_{E^*} < \delta_\mu := \frac{\min\{a, 1\}}{4} \tau$, we derive

$$I_{\lambda, \mu}(u) \geq \sigma := \tau \left( \frac{\min\{a, 1\}}{4} \tau - \|h\|_{E^*} \right),$$

if $\|u\|_\lambda = \tau$,

which concludes the proof of the lemma.

**Lemma 2.2** Let $\mu > 0$ sufficiently large. Then, there exists $\lambda_\mu > 0$ such that for each $0 < \lambda < \lambda_\mu$, we can choose $w \in E$ satisfying

$$\|w\|_\lambda > 1 \text{ and } I_{\lambda, \mu}(w) < 0.$$

**Proof.** Fix $u \in C^\infty_0(\mathbb{R}^4) \setminus \{0\}$. Setting

$$u_t(x) = u \left( \frac{x}{t^2} \right), \ t \in (0, \infty),$$

we obtain

$$I_{\lambda, \mu}(u_t) \leq \frac{a}{2} \left( \int |\nabla u|^2 \right) t^4 + \left( S^{-1/2} \|h\|_{E^*} \|u\|_\lambda \right) t^2$$

$$+ \left( \frac{\lambda M}{2} \int u^2 - \frac{\mu}{q} \int K(t^2x)|u|^q + \frac{1}{4} \left( b \left( \int |\nabla u|^2 \right)^2 - \int u^4 \right) \right) t^8,$$

where $M = \text{ess sup}_{\mathbb{R}^4} V$. Let

$$A = \frac{a}{2} \left( \int |\nabla u|^2 \right) \text{ and } B = S^{-1/2} \|h\|_{E^*} \|u\|_\lambda.$$

We fix $t_0 \approx \infty$ such that

$$At_0^4 + Bt_0^2 - t_0^8 < 0$$

and

$$\|u_{t_0}\|_\lambda > 1, \ \forall \lambda > 0.$$
From hypothesis (S), we can choose $\mu_0 > 0$ such that

$$-1 = -\frac{\mu_0}{q} \int K(t_0^2 x)|u|^q + \frac{1}{4} \left( b \left( \int |\nabla u|^2 \right)^2 - \int u^4 \right).$$

If $\mu > \mu_0$, then

$$-\frac{\mu}{q} \int K(t_0^2 x)|u|^q + \frac{1}{4} \left( b \left( \int |\nabla u|^2 \right)^2 - \int u^4 \right) < -1.$$

Now, taking $0 < \lambda < \lambda_\mu < \infty$ such that

$$\frac{\lambda_\mu M}{2} \int u^2 = -1 - \left( -\frac{\mu}{q} \int K(t_0^2 x)|u|^q + \frac{1}{4} \left( b \left( \int |\nabla u|^2 \right)^2 - \int u^4 \right) \right),$$

for each $0 < \lambda < \lambda_\mu$, we get

$$I_{\lambda,\mu}(u_{t_0}) < At_0^4 + Bt_0^2 - t_0^8 < 0.$$

So that, if we take $w = u_{t_0}$, then $\|w\|_\lambda > 1$ and $I_{\lambda,\mu}(w) < 0$, which finishes the proof of the lemma. ■

Next, we summarized the last two lemmas in the following:

**Proposition 2.3** Let $\mu > 0$ sufficiently large. Then, there exist $\lambda_\mu, \delta_\mu > 0$ such that for each $0 < \lambda < \lambda_\mu$, the energy $I_{\lambda,\mu}$ satisfies the Mountain Pass Geometry, whenever $\|h\|_{E^*} < \delta_\mu$.

**Remark 2.4** Let $\mu_1 > \mu > 0$.

(i) It is easily seen in the proof of Lemma 2.2 that we can take $\lambda_{\mu_1} > \lambda_\mu$. So, a fortiori, the functional $I_{\lambda_{\mu_1}}$ satisfies the Mountain Pass Geometry for $0 < \lambda < \lambda_{\mu_1}$;

(ii) For each $0 < \lambda < \lambda_\mu$ and $w$ given by Lemma 2.2 we have $I_{\lambda,\mu_1}(w) < I_{\lambda,\mu}(w)$. Hence, we can take the mountain pass levels

$$c_{\lambda,\mu} = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_{\lambda,\mu}(\gamma(t)) \quad \text{and} \quad c_{\lambda,\mu_1} = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_{\lambda,\mu_1}(\gamma(t))$$

over the same class of paths $\Gamma$, that is,

$$\Gamma = \{ \gamma \in C([0,1], E) : \gamma(0) = 0, \; \gamma(1) = w \}.$$
2.2 The geometry of $I_{\lambda,\mu}$ at the origin

In what follows, $B_{\tau} := \overline{B_{\tau}(0)} \cap E$, where $\tau > 0$ is given by Lemma 2.1. Evidently, $B_{\tau}$ is a complete metric subspace of $E$.

**Lemma 2.5** The functional $I_{\lambda,\mu}$ is bounded below in $B_{\tau}$.

**Proof.** Due to the continuous $E \hookrightarrow L^q_K(\mathbb{R}^4)$ and $E \hookrightarrow L^4(\mathbb{R}^4)$, we have

$$|I_{\lambda,\mu}(u)| \leq \max\{a, 1\} \|u\|_\lambda^2 + \frac{b}{4} \|u\|_\lambda^4 + \mu C_1 \|u\|_\lambda^q + C_2 \|u\|_\lambda^4 + \|h\|_{E^*} \|u\|_\lambda.$$  

Then, for $\|u\|_\lambda \leq \tau$,

$$|I_{\lambda,\mu}(u)| \leq \max\{a, 1\} \tau^2 + \frac{b}{4} \tau^4 + \mu C_1 \tau^q + C_2 \tau^4 + \|h\|_{E^*} \tau := C,$$

which shows

$$I_{\lambda,\mu}(u) \geq -C, \forall u \in B_{\tau}.$$  

**Lemma 2.6** Let $\nu_{\lambda,\mu} = \inf_{B_{\tau}} I_{\lambda,\mu}$. Then, $\nu_{\lambda,\mu} < 0$.

**Proof.** Fix $u \in C_0^\infty(\mathbb{R}^4)$ with $\int hu > 0$ and $t > 0$. Then

$$I_{\lambda,\mu}(tu) = \frac{t^2}{2} \int (a \|\nabla u\|^2 + \lambda V(x)u^2) + \frac{t^4}{4} \left(b \left( \int |\nabla u|^2 \right)^2 - \int u^4 \right)$$

$$- \frac{\mu t^q}{q} \int K(x)|u|^q - t \int hu,$$

yields $I_{\lambda,\mu}(tu) < 0$, for any sufficiently small value of $t$. Since $tu \in B_{\tau}$ for these values, so do it is also true that

$$\nu_{\lambda,\mu} \leq I_{\lambda,\mu}(tu),$$

for any sufficiently small value of $t$, which shows the desired result. ■
3 On Palais-Smale sequences

Firstly we recall that \((u_n)\) in \(E\) is a Palais-Smale sequence at level \(d \in \mathbb{R}\) (briefly \((PS)_d\)) for the functional \(I_{\lambda,\mu}\) if

\[
I_{\lambda,\mu}(u_n) \to d \text{ in } \mathbb{R} \quad \text{and} \quad I'_{\lambda,\mu}(u_n) \to 0 \text{ in } E^* \quad \text{as} \quad n \to +\infty.
\]

In order to prove the boundedness of Palais-Smale sequences, we will use the following result due to Jeanjean [20].

**Proposition 3.1 (Jeanjean, [20])** Let \((X, \| \cdot \|)\) be a Banach space, \(J \subset \mathbb{R}^+\) an interval and \((\varphi_{\mu})\) be a family of \(C^1\) functionals on \(X\) of the form

\[
\varphi_{\mu}(u) = A(u) - \mu B(u), \quad \mu \in J,
\]

where \(B(u) \geq 0, \forall u \in X, \) and such that

\[
A(u) \to \infty \quad \text{or} \quad B(u) \to \infty, \quad \text{as} \ \|u\| \to \infty.
\]

If there exist two points \(v_1, v_2 \in X\) such that setting

\[
\Gamma = \{ \gamma \in C([0, 1], X) : \gamma(0) = v_1, \ \gamma(1) = v_2 \},
\]

for all \(\mu \in J\) there hold

\[
\beta_{\mu} := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \varphi_{\mu}(\gamma(t)) > \max\{\varphi_{\mu}(v_1), \varphi_{\mu}(v_2)\},
\]

then, for almost every \(\mu \in J\), there is a bounded \((PS)_{\beta_{\mu}}\) sequence \((u_n)\) for \(\varphi_{\mu}\) in \(X\).

The application of Proposition 3.1 to functional \(I_{\lambda,\mu}\) yields bounded Palais-Smale sequences for large values of \(\mu\) and small values of \(\lambda\). Here, once again, we stress the role played by hypothesis \((S)\).

**Lemma 3.2** Let \(\mu^* > 0\) sufficiently large and \(\mu > \mu^*\). If \(0 < \lambda < \lambda_{\mu^*}\) and \(\|h\|_{E^*} < \delta_{\mu}\) (see lemmas 2.1 and 2.2), there exists (except for a zero measure set of \(\mu\)'s) a bounded Palais-Smale sequence for \(I_{\lambda,\mu}\) at level

\[
c_{\lambda,\mu} := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I_{\lambda,\mu}(\gamma(t)),
\]

where

\[
\Gamma = \{ \gamma \in C([0, 1], E) : \gamma(0) = 0, \ \gamma(1) = w \},
\]

with \(w\) given by Lemma 2.2 (see also Remark 2.4 (ii)).
**Proof.** Let $\mu^* > 0$ large enough and $0 < \lambda < \lambda_{\mu^*}$. Setting 
\[ A(u) := \frac{1}{2} \int (a|\nabla u|^2 + \lambda V(x)u^2) + \frac{b}{4} \left( \int |\nabla u|^2 \right)^2 - \frac{1}{4} \int u^4 - \int hu \]
and 
\[ B(u) := \frac{1}{q} \int K(x)|u|^q, \]
we can consider 
\[ I_{\lambda,\mu}(u) = A(u) - \mu B(u), \mu \in (\mu^*, \infty). \]
From hypothesis $(S)$, it follows that 
\[ A(u) \to \infty, \text{ as } ||u||_\lambda \to \infty. \]
Since 
\[ c_{\lambda,\mu} > \max \{I_{\lambda,\mu}(0), I_{\lambda,\mu}(w)\}, \forall \mu \in (\mu^*, \infty), \]
the result follows from Proposition 3.1. 

3.1 The behaviour of the Palais-Smale sequences

**Lemma 3.3** Let $(u_n)$ be a bounded $(PS)_d$ sequence for $I_{\lambda,\mu}$ and $u_0 \in E$ its weak limit. Then $u_n \to u_0$ in $L^4(\mathbb{R}^4)$.

**Proof.** It is sufficient to show that $|u_n|_4 \to |u_0|_4$, as $n \to \infty$. By using Lions’ Second Concentration Compactness Lemma (cf. [23, Lemma 2.1]), there exist at most a countable set $\mathcal{I}$, $\{x_k\}_{k \in \mathcal{I}} \subset \mathbb{R}^4$ and $\{\eta_k\}_{k \in \mathcal{I}}$, $\{\nu_k\}_{k \in \mathcal{I}} \subset (0, \infty)$ such that 
\[ |\nabla u_n|^2 dx \rightharpoonup \eta \geq |\nabla u_0|^2 dx + \sum_{k \in \mathcal{I}} \eta_k \delta_{x_k}, \]
\[ u^4_n dx \rightharpoonup \nu = u^4_0 dx + \sum_{k \in \mathcal{I}} \nu_k \delta_{x_k}, \]
\[ \eta_k \geq Su_k^{\frac{1}{4}} (k \in \mathcal{I}). \tag{3.3} \]
Our task now is to show that $\mathcal{I} = \emptyset$. By way of contradiction, assume that $\mathcal{I} \neq \emptyset$. 

13
For each $k \in \mathcal{I}$ and $\varepsilon > 0$, we consider a smooth function $\phi = \phi_{k,\varepsilon} : \mathbb{R}^4 \to \mathbb{R}$ such that
\[
\begin{cases}
\phi = 1, & \text{in } B_\varepsilon(x_k), \\
\phi = 0, & \text{in } B_{2\varepsilon}(x_k), \\
0 \leq \phi \leq 1, & \text{in the remaining case,} \\
|\nabla\phi| \leq \frac{2}{\varepsilon}, & \text{in } \mathbb{R}^4.
\end{cases}
\]
Noticing that $I'_{\lambda,\mu}(u_n)(u_n\phi) \to 0$ in $E^*$, we have
\[
\lim_n \left[ \left( a + b \int |\nabla u_n|^2 \right) \int |\nabla u_n|^2 \phi - \int u_n^4 \phi \right] + o_\varepsilon(1) = 0, 
\] (3.4)
where
\[
o_\varepsilon(1) = \lim_n \left[ \lambda \int_{B_{2\varepsilon}(x_k)} V(x) u_n^2 \phi - \int_{B_{2\varepsilon}(x_k)} h u_n \phi - \mu \int_{B_{2\varepsilon}(x_k)} K(x) |u_n|^q \phi \right. \\
+ \left( a + b \int |\nabla u_n|^2 \right) \int_{B_{2\varepsilon}(x_k)} (\nabla u_n \cdot \nabla \phi) u_n \right].
\]
In fact, combining Schwarz’s inequality, Hölder’s inequality and the compact embedding $E \hookrightarrow L^2_{\text{loc}}(\mathbb{R}^4)$, we get
\[
\left| \lim_{n \to \infty} \left( a + b \int |\nabla u_n|^2 \right) \int_{B_{2\varepsilon}(x_k)} (\nabla u_n \cdot \nabla \phi) u_n \, dx \right|
\leq C \lim_{n \to \infty} \left( \int_{B_{2\varepsilon}(x_k)} |\nabla u_n|^2 \, dx \right)^{1/2} \left( \int_{B_{2\varepsilon}(x_k)} |u_n|^2 |\nabla \phi|^2 \, dx \right)^{1/2}
\leq C \left( \int_{B_{2\varepsilon}(x_k)} u_0^4 \, dx \right)^{1/4} \left( \int_{B_{2\varepsilon}(x_k)} |\nabla \phi|^4 \, dx \right)^{1/4}
\leq C \left( \int_{B_{2\varepsilon}(x_k)} u_0^4 \, dx \right)^{1/4},
\]
where in the last inequality we use that $|\nabla \phi| \leq \frac{2}{\varepsilon}$. In addition, by using analogous arguments as the previous above, we obtain
\[
\left| \lim_{n \to \infty} \left( \lambda \int_{B_{2\varepsilon}(x_k)} V(x) u_n^2 \phi - \int_{B_{2\varepsilon}(x_k)} h u_n \phi - \mu \int_{B_{2\varepsilon}(x_k)} K(x) |u_n|^q \phi \right) \right|
\leq \lambda \int_{B_{2\varepsilon}(x_k)} V(x) u_0^2 + \int_{B_{2\varepsilon}(x_k)} h u_0 + \mu \int_{B_{2\varepsilon}(x_k)} K(x) |u_0|^q.
\]
Thus, formula (3.4) is justified. But then, by applying Lions’ Concentration Compactness Lemma on this formula, we derive

\[ 0 \geq \left( a + b \int_{B_{2\varepsilon}(x_k)} \phi \, d\eta \right) \int_{B_{2\varepsilon}(x_k)} \phi \, d\eta - \int_{B_{2\varepsilon}(x_k)} \phi \, d\nu + o_\varepsilon(1) \].

By passing to the limit as \( \varepsilon \to 0 \) and using relation (3.3), we get

\[ 0 \geq (a + b\eta_k)\eta_k - \nu_k \geq b\eta_k^2 - \nu_k \geq \nu_k (bS^2 - 1). \]

Hence

\[ b \leq 1/S^2, \]

which contradicts the hypothesis (S). Therefore, \( \mathcal{I} = \emptyset \) and the lemma is proved.

\[ \Box \]

**Lemma 3.4** Let \( (u_n) \) be a bounded \((PS)_d\) sequence for \( I_{\lambda,\mu} \) and \( u_0 \in E \) its weak limit. Then

\[ I'_{\lambda,\mu}(u_0) = 0. \]

**Proof.** From the boundedness of \( (u_n) \) in \( E \), there exists \( l \geq 0 \) such that

\[ \int |\nabla u_n|^2 \to l^2. \]

Now, if we set

\[ J_{\lambda,\mu}(u) = \left( \frac{a + bl^2}{2} \right) \int |\nabla u|^2 + \frac{\lambda}{2} \int V(x)u^2 - \frac{\mu}{q} \int K(x)|u|^q - \frac{1}{4} \int u^4 - \int hu, \]

we can easily infer that

\[ J_{\lambda,\mu}(u_n) \to d + \frac{bl^4}{4} \quad \text{and} \quad J'_{\lambda,\mu}(u_n) \to 0. \]

Combining the weak convergence \( u_n \to u_0 \) in \( E \) with the second convergence above, we can deduce that

\[ J'_{\lambda,\mu}(u_0) = 0, \]

that is, \( u_0 \) is a solution in \( E \) to the equation

\[ - (a + bl^2) \Delta u + \lambda V(x)u = \mu K(x)|u|^{q-2}u + u^3 + h(x), \quad x \in \mathbb{R}^4, \]
in the weak sense. So, we must show that
\[ l^2 = \int |\nabla u_0|^2. \]

We notice that \( I'_\lambda,\mu(u_n)(u_n - u_0) = o_n(1) \), that is,
\[
\left(a + b \int |\nabla u_n|^2\right) \int \nabla u_n \cdot \nabla (u_n - u_0) + \lambda \int V(x)u_n(u_n - u_0)
- \mu \int K(x)|u_n|^{q-2}u_n(u_n - u_0) - \int u_n^3(u_n - u_0) - \int h(u_n - u_0) = o_n(1). \quad (3.5)
\]

For what follows, we set
\[
I^1_n = \int V(x)u_n(u_n - u_0), \quad I^2_n = \int K(x)|u_n|^{q-2}u_n(u_n - u_0), \quad I^3_n = \int u_n^3(u_n - u_0)
\]
and
\[
I^4_n = \int h(u_n - u_0).
\]

We claim that \( I^1_n, I^2_n, I^3_n, I^4_n \to 0 \), as \( n \to +\infty \). The last convergence follows immediately from the weak convergence \( u_n \rightharpoonup u_0 \) in \( E \). Next, we shall verify the first three in the following steps.

**Step 1:** \( I^1_n = o_n(1) \). Firstly, we observe that \( Vu_n \in L^{\frac{4}{3}}(\mathbb{R}^4) \), that is, \( V^{\frac{4}{3}}u_n^\frac{4}{3} \in L^1(\mathbb{R}^4) \). In fact, take into account that \( V \in L^2(\mathbb{R}^4) \) (hypothesis \( (V) \)), we get
\[
V^{\frac{4}{3}} \in L^{\frac{4}{3}}(\mathbb{R}^4) \quad \text{and} \quad u_n^{\frac{4}{3}} \in L^3(\mathbb{R}^4).
\]

Then, by Hölder inequality \( V^{\frac{4}{3}}u_n^{\frac{4}{3}} \in L^1(\mathbb{R}^4) \), with
\[
|V u_n|^{\frac{4}{3}} = |V^{\frac{4}{3}}u_n^{\frac{4}{3}}|_1 \leq |V^{\frac{4}{3}}|_2^\frac{4}{3}|u_n^{\frac{4}{3}}|_3 = |V|_2^{\frac{4}{3}}|u_n|_4^{\frac{4}{3}}.
\]

Now, from the boundedness of \((u_n)\) in \( L^4(\mathbb{R}^4) \), one has
\[
|V u_n|_4 \leq |V|_2|u_n|_4 \leq C_1, \quad \forall n \in \mathbb{N}.
\]

By using Hölder inequality once again, we derive
\[
|I^1_n| \leq \int V(x)|u_n||u_n - u_0| \leq C_1|u_n - u_0|_4.
\]
Since by Lemma 3.3, \( u_n \to u_0 \) in \( L^4(\mathbb{R}^4) \) and we get \( I_1^n = o_n(1) \) as claimed.

**Step 2:** \( I_2^n = o_n(1) \). Firstly, we notice that \( K^{\frac{2}{3}}|u_n|^{q-1} \in L^\frac{4}{3}(\mathbb{R}^4) \), since by the condition on the exponent \( q \), we have \( 2 \frac{4}{3} (q - 1) < 4 \). Hence, by Hölder inequality and inasmuch as \( K \in L^\infty(\mathbb{R}^4) \), we get

\[
|I_2^n| \leq \int K(x)^\frac{2}{3}|u_n|^{q-1}K(x)^\frac{2}{3}|u_n - u_0| \\
\leq |K^{\frac{2}{3}}|u_n|^{q-1}| \frac{4}{3} |K^{\frac{2}{3}}(u_n - u_0)|_4 \leq C_2|u_n|^{q-1}K|u_n - u_0|_4.
\]

From the boundedness of \((u_n)\) in \( L^{\frac{4}{3}(q-1)}(\mathbb{R}^4) \), we conclude

\[
|I_2^n| \leq C_3|u_n - u_0|_4
\]

and \( I_2^n = o_n(1) \), as we claimed.

**Step 3:** \( I_3^n = o_n(1) \). In fact, by using Hölder inequality

\[
|I_3^n| \leq |u_n|^{\frac{4}{3}}|u_n - u_0|_4 = |u_n|^{\frac{2}{3}}|u_n - u_0|_4 \leq C_4|u_n - u_0|_4
\]

and also \( I_3^n = o_n(1) \), which concludes the verification of the claim.

Thereby, from the above convergences, \( (3.5) \) and taking into account that

\[
0 < \left( a + b \int |\nabla u_n|^2 \right) < C, \text{ for some } C > 0,
\]

it follows that

\[
\int |\nabla u_n|^2 = o_n(1) + \int \nabla u_n \cdot \nabla u_0.
\]

Then, the weak convergence \( u_n \rightharpoonup u_0 \) in \( E \) ensures

\[
\int |\nabla u_n|^2 \to \int |\nabla u_0|^2,
\]

and thus \( l^2 = \int |\nabla u_0|^2 \), which finishes the proof of the lemma.

\[
4 \quad \text{Proofs of the main results}
\]

In this section, we will prove Theorems 1.2 and 1.3.
4.1 The proof of Theorem 1.2

Let $\mu^*$ and $\lambda_{\mu^*}$ given by Lemma 3.2. For $\mu > \mu^*$, $0 < \lambda < \lambda_{\mu^*}$ and $0 < \|h\|_{E^*} < \delta_{\mu}$, there exists (almost everywhere) a bounded Palais-Smale sequence $(u_n)$ in $E$ for $I_{\lambda,\mu}$ at mountain pass level $c_{\lambda,\mu}$. Let $u_0 \in E$ be the weak limit of $(u_n)$. By Lemma 3.4, $u_0$ is a non-trivial solution for $(P_{\lambda,\mu})$. Moreover, having in mind that $I_{\lambda,\mu}'(u_0)u_0 = 0$, we get

$$\int (a|\nabla u_0|^2 + \lambda V(x)u_0^2) = \mu \int K(x)|u_0|^q + \int u_0^4 + \int h u_0 - b \left( \int |\nabla u_0|^2 \right)^{2/4}.$$  

Hence

$$I_{\lambda,\mu}(u_0) = \mu \left( \frac{1}{2} - \frac{1}{q} \right) \int K(x)|u_0|^q + \frac{1}{4} \int u_0^4 - \frac{1}{2} \int h u_0 - b \left( \int |\nabla u_0|^2 \right)^{2/4} > 0, \quad (4.6)$$

since $\mu > \mu^* > 0$ is sufficiently large. Thus, the Theorem 1.2 is proved.

\textbf{Remark 4.1} In fact, the final part of the proof of Theorem 1.2 gives an important additional generalized information. Let us denote by $N_{\lambda,\mu}$ the Nehari manifold associated to $I_{\lambda,\mu}$ defined by

$$N_{\lambda,\mu} := \{ u \in E \setminus \{0\} : I_{\lambda,\mu}'(u)u = 0 \}.$$  

Thus, by using hypothesis $(S)$ and (4.6), for each $u \in N_{\lambda,\mu}$, we infer that

$$I_{\lambda,\mu}(u) > 0, \text{ if and only if, } \mu \approx +\infty \text{ and } I_{\lambda,\mu}(u) < 0, \text{ if and only if, } \mu \approx 0^+.$$  

4.2 The proof of Theorem 1.3

We shall use the following well known result due to Ekeland (see [17]), applying it to $X = B_{\tau}$ and $\Phi = I_{\lambda,\mu}$.

\textbf{Proposition 4.2} (Ekeland's variational principle - weak form) Let $(X, d)$ be a complete metric space and $\Phi : X \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous functional, which is bounded below. Then, for each given $\varepsilon > 0$, there exists $u_\varepsilon \in X$ such that

$$\Phi(u_\varepsilon) < \inf_X \Phi + \varepsilon$$

and

$$\Phi(u_\varepsilon) < \Phi(u) + \varepsilon d(u_\varepsilon, u), \quad u \neq u_\varepsilon.$$
In light of the Lemma 2.5 and since $I_{\lambda,\mu}$ is lower semicontinuous in $B_\tau$, using 4.2 provides a sequence $(v_n)$ in $B_\tau$ such that

$$I_{\lambda,\mu}(v_n) < \nu_{\lambda,\mu} + \frac{1}{n}$$

and

$$I_{\lambda,\mu}(v_n) < I_{\lambda,\mu}(u) + \frac{1}{n}||v_n - u||_\lambda, u \in B_\tau, u \neq u_n.$$

The proof of Theorem 1.3 will following lemmas:

**Lemma 4.3** If $||h||_{E^*} < \delta_\mu$, then $(v_n)$ is a $(PS)_{\nu_{\lambda,\mu}}$ sequence for $I_{\lambda,\mu}$.

**Proof.** Since $(v_n)$ is a sequence in $B_\tau$, it follows that $I_{\lambda,\mu}(v_n) \to \nu_{\lambda,\mu} < 0$. Without loss of generality we can assume that $I_{\lambda,\mu}(v_n) < 0$, $\forall n \in \mathbb{N}$. From Lemma 2.1 we have $v_n \in \bar{B}_\tau$, where $\bar{B}_\tau$ denotes the interior of $B$. Therefore, for $v \in E$ such that $||v||_\lambda \leq 1$, and for any small positive value of $\theta \in \mathbb{R}$, we get

$$v_n + \theta v \in \bar{B}_\tau \text{ and } v_n + \theta v \neq v_n.$$

But then

$$I_{\lambda,\mu}(v_n) < I_{\lambda,\mu}(v_n + \theta v) + \frac{\theta}{n}||v||_\lambda.$$

The differentiability of $I_{\lambda,\mu}$ implies that

$$I_{\lambda,\mu}'(v_n)v \geq -\frac{1}{n}.$$

Replacing $v$ by $-v$ we obtain

$$I_{\lambda,\mu}'(v_n)v \leq \frac{1}{n}.$$

Thus, $||I_{\lambda,\mu}'(v_n)||_{E^*} \to 0$. 

**Lemma 4.4** Let $v_0 \in E$ the weak limit of $(v_n)$ (passing to a subsequence if necessary). Then, it is fulfilled

$$I_{\lambda,\mu}'(v_0) = 0 \text{ and } I_{\lambda,\mu}(v_0) < 0.$$
Proof. By Lemma 3.4, we have

\[ I'_{\lambda,\mu}(v_0) = 0. \]

So, we will show

\[ I_{\lambda,\mu}(v_0) < 0. \]

First of all, we notice that

\[ \|v_0\|_{\lambda} \leq \lim \inf \|v_n\|_{\lambda} \leq \tau. \]

Then

\[ I_{\lambda,\mu}(v_0) \geq \nu_{\lambda,\mu}. \quad (4.7) \]

On the other hand, writing

\[ I_{\lambda,\mu}(v_n) = I_{\lambda,\mu}(v_n) - \frac{1}{4} I'_{\lambda,\mu}(v_n)v_n + \frac{1}{4} I'_{\lambda,\mu}(v_n)v_n, \]

we derive

\[
I_{\lambda,\mu}(v_n) = \left(\frac{1}{2} - \frac{1}{4}\right) \int (a |\nabla v_n|^2 + \lambda V(x)v_n^2)
- \mu \left(\frac{1}{q} - \frac{1}{4}\right) \int K(x)|v_n|^q - \frac{3}{4} \int hv_n + o_n(1).
\]

So, passing to the limit as \( n \to \infty \) and by using the compact embedding \( E \hookrightarrow L^q(\mathbb{R}^4) \) (hypothesis \((VK)\)), we obtain

\[
\nu_{\lambda,\mu} \geq \left(\frac{1}{2} - \frac{1}{4}\right) \int (a |\nabla v_0|^2 + \lambda V(x)v_0^2) - \mu \left(\frac{1}{q} - \frac{1}{4}\right) \int K(x)|v_0|^q - \frac{3}{4} \int hv_0
= I_{\lambda,\mu}(v_0) - \frac{1}{4} I'(v_0)v_0 = I_{\lambda,\mu}(v_0).
\]

This way,

\[ I_{\lambda,\mu}(v_0) = \nu_{\lambda,\mu}, \]

which proves

\[ I_{\lambda,\mu}(v_0) < 0. \]

\[ \blacksquare \]

Remark 4.5 For the solution obtained in Theorem 1.3 we must have \( \mu > 0 \)
sufficiently small, since otherwise, by (4.6), we get a contradiction. Hence, the \( v_0 \)'s
negative energy information could be obtained directly via (4.6) (or Remark 4.1),
but in an isolated way, under an additional restriction on the parameter \( \mu \) and without
to explicit the energy value of \( v_0 \).
References

[1] S. Adachi, K. Tanaka, *Four positive solutions for the semilinear elliptic equation: \(-\Delta u + u = a(x)u^p + f(x)\) in \(\mathbb{R}^N\)*, Calc. Var. Partial Differ. Equ. 11 (2000), 63-95.

[2] C. O. Alves, F. J. S. A. Corrêa, *On existence of solutions for a class of problem involving a nonlinear operator*, Comm. Appl. Nonlinear Anal. 8 (2001), 43-56.

[3] C. O. Alves, F. J. S. A. Corrêa, T. F. Ma, *Positive solutions for a quasilinear elliptic equation of Kirchhoff type*, Comput. Math. Appl. 49 (2005), 85-93.

[4] C. O. Alves, G. M. Figueiredo, *Nonlinear perturbations of a periodic Kirchhoff equation in \(\mathbb{R}^N\)*, Nonlinear Anal. 75 (2012), 2750-2759.

[5] C. O. Alves, M. A. S. Souto, *Existence of solutions for a class of nonlinear Schrödinger equations with potential vanishing at infinity*, J. Differential Equations 254 (2013), 1977-1991.

[6] A. Ambrosetti, V. Felli, A. Malchiodi, *Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity*, J. Eur. Math. Soc. 7 (2005), 117-144.

[7] A. Ambrosetti, P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. 14 (1973), 349-381.

[8] P. D’Ancona, S. Spagnolo, *Global solvability for the degenerate Kirchhoff equation with real analytic data*, Invent. Math. 108 (1992), 247-262.

[9] A. Arosio, S. Panizzi, *On the well-posedness of the Kirchhoff string*, Trans. Am. Math. Soc. 348 (1996), 305-330.

[10] S. Bernstein, *Sur une classe d’équations fonctionelles aux dérivées partielles*, Bull. Acad. Sci. URSS. Sér 4 (1940), 17-26.

[11] M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. A. Soriano, *Global existence and uniform decay rates for the Kirchhoff-Carrier equation with nonlinear dissipation*, Adv. Differential Equations 6 (6) (2001), 701-730.

[12] F. J. S. A. Corrêa, G. M. Figueiredo, *On a elliptic equation of p-Kirchhoff type via variational methods*, Bull. Aust. Math. Soc. 74 (2006), 263-277.
[13] S. J. Chen, L. Li, *Multiple solutions for the nonhomogeneous Kirchhoff equation on* $\mathbb{R}^N$, Nonlinear Anal. RWA 14 (2013), 1477-1486.

[14] B. Cheng, *A new result on multiplicity of nontrivial solutions for the nonhomogeneous Schrödinger-Kirchhoff type problem in* $\mathbb{R}^N$, Mediterr. J. Math. doi:10.1007/s00009-015-0527-1.

[15] M. Dreher, *The Kirchhoff equation for the p-Laplacian*, Rend. Semin. Mat. Univ. Politec. Torino 64 (2006), 217-238.

[16] M. Dreher, *The wave equation for the p-Laplacian*, Hokkaido Math. J. 36 (2007), 21-52.

[17] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl. 47 (1974), 324-353.

[18] G. M. Figueiredo, *Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument*, J. Math. Anal. Appl. 401 (2013), 706-713.

[19] G. M. Figueiredo, U. B. Severo, *Ground State Solution for a Kirchhoff Problem with Exponential Critical Growth*, Milan J. Math. 84 (2016), 23-39.

[20] L. Jeanjean, *On the existence of bounded Palais-Smale sequences and applications to a Landesman-Lazer type problem set on* $\mathbb{R}^N$, Proc. Roy. Soc. Edinburgh Sect. A 129 (1999), 787-809.

[21] G. Kirchhoff, Mechanik, Teubner, Leipzig, 1883.

[22] J. L. Lions, On some questions in boundary value problems of mathematical physics, in: Contemporary Developments in Continuum Mechanics and Partial Differential Equations, Proc. Internat. Sympos., Inst. Mat. Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977, in: North-Holland Math. Stud., vol 30, North-Holland, Amsterdam, 1978, pp. 284-346.

[23] P. L. Lions, The concentration-compactness principle in the calculus of variations. The limit case. Part 1, Rev. Mat. Iberoam. 1 (1985), 145-201.

[24] D. Liu, P. Zhao, *Multiple nontrivial solutions to a p-Kirchhoff equation*, Nonlinear Anal. TMA 75 (2012), 5032-5038.
[25] A. Mao, Z. Zhang, *Sign-changing and multiple solutions of Kirchhoff type problems without the P.S. condition*, Nonlinear Anal. TMA 70 (2009), 1275-1287.

[26] D. Naimen, *Positive solutions of Kirchhoff type elliptic equations involving a critical Sobolev exponent*, NoDEA Nonlinear Differential Equations Appl. 21 (2014), 885-914.

[27] D. Naimen, *The critical problem of Kirchhoff type elliptic equations in dimension four*, J. Differential Equations 257 (2014), 1168-1193.

[28] J. Nie, *Existence and multiplicity of nontrivial solutions for a class of Schrödinger-type equations*, J. Math. Anal. Appl. 417 (2014), 65-79.

[29] J. Nie, X. Wu, *Existence and multiplicity of non-trivial solutions for Kirchhoff-Schrödinger-type equations with radial potential*, Nonlinear Anal. TMA 75 (2012), 3470-3479.

[30] K. Perera, Z. Zhang, *Nontrivial solutions of Kirchhoff-type problems via the Yang index*, J. Differential Equations 221 (2006), 246-255.

[31] S. I. Pohožaev, *A certain class of quasilinear hyperbolic equations*, Mat. Sb. (N.S.) 96 (138) (1975), 152-166, 168 (in Russian).

[32] P. H. Rabinowitz, *Multiple critical points of perturbed symmetric functionals*, Trans. Am. Math. Soc. 272 (1982), 753-769.

[33] V. Radulescu, D. Smets, *Critical singular problems on infinite cones*, Nonlinear Anal. 54 (2003), 1153-1164.

[34] J. Su, Z.-Q. Wang, M. Willem, *Nonlinear Schrödinger equations with unbounded and decaying radial potentials*, Commun. Contemp. Math. 9 (2007), 571-583.

[35] J. Su, Z.-Q. Wang, M. Willem, *Weighted Sobolev embedding with unbounded and decaying radial potentials*, J. Differential Equations 238 (2007), 201-219.

[36] G. Tarantello, *On nonhomogeneous elliptic equations involving critical Sobolev exponents*, Ann. Inst. H. Poincaré Anal. Non Linéaire 9 (1992), 281–304.

[37] X. Wu, *Existence of nontrivial solutions and high energy solutions for Kirchhoff-Schrödinger-type equations in $\mathbb{R}^N$*, Nonlinear Anal. RWA 12 (2011), 1278-1287.
[38] Z. Zhang, K. Perera, *Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow*, J. Math. Anal. Appl. *317* (2006), 456-463.