Playing Games with Multiple Access Channels

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Abstract

Communication networks have multiple users, each sending and receiving messages. A multiple access channel (MAC) models multiple senders transmitting to a single receiver, such as the uplink from many mobile phones to a single base station. The optimal performance of a MAC is quantified by a capacity region of simultaneously achievable communication rates. We study the two-sender classical MAC, the simplest and best-understood network, and find a surprising richness in both a classical and quantum context. First, we find that quantum entanglement shared between senders can substantially boost the capacity of a classical MAC. Second, we find that optimal performance of a MAC with bounded-size inputs may require unbounded amounts of entanglement. Third, determining whether a perfect communication rate is achievable using finite-dimensional entanglement is undecidable. Finally, we show that evaluating the capacity region of a two-sender classical MAC is in fact NP-hard.

1 Introduction

Information theory is a mathematical theory of communication and signal processing pioneered by Shannon [1]. A fundamental object in this theory is a point-to-point communication channel, and one of Shannon’s many insights was to realize that every channel can be characterized by a single number, the channel’s capacity, which quantifies the fundamental limit of how much information can be transmitted faithfully through the channel. More than a decade later, Shannon generalized the point-to-point communication scenario in his work on two-way channels [2], initiating the study of network information theory in which multiple parties exchange and process information. Important communication models in network information theory include the broadcast channel [3], distributed lossless source compression [4], and the multiple access channel [5, 6, 7], which is the subject of this work.

A multiple access channel (MAC) models a communication scenario involving two senders and one receiver. The two senders are spatially separated and their goal is to transmit individual messages over a common channel to a single receiver. Letting $R_1$ and $R_2$ denote the rates of information transmission for the two senders, there is now a capacity region consisting of achievable rate pairs $(R_1, R_2)$ for which faithful
information transmission through the MAC is possible. This capacity region was characterized by Ahlswede [5] and Liao [6] in terms of a so-called single-letter formula that is in principle computable. The MAC is arguably the simplest network communication scenario, and is presumably well understood in information-theoretic terms.

In quantum information theory, communication tasks can be enhanced drastically if the communicating parties are given access to quantum resources. In particular, making use of quantum correlations by sharing entangled quantum states enables communication tasks such as super-dense coding [8] or quantum teleportation [9], both of which are impossible to achieve with classical resources alone. While entanglement assistance can increase achievable rates for classical point-to-point channels in the zero-error [10, 11] and one-shot setting [12], it comes as a surprise that entanglement does not provide any advantage in the asymptotic setting with vanishing error [13].

In this work, we show that multiple access channels behave in a fundamentally different way in the presence of entanglement assistance, in contrast to the point-to-point scenario. Moreover, even unassisted classical MACs exhibit far more complex behavior than previously appreciated. We demonstrate this by constructing a family of classical MACs with surprisingly rich behavior: First, we show that entanglement shared between the senders can strictly increase the capacity region of a classical MAC. This result shows that entanglement can help in a purely classical communication scenario. Second, we exhibit examples of channels for which an unbounded amount of entanglement is needed to achieve the maximal possible increase of the achievable rate region. We also show that it is generally undecidable to determine whether the maximal rate pair can be achieved for a MAC with finite-dimensional entanglement strategies. These results demonstrate that the region of achievable rates of a MAC exhibits complex behavior when entanglement assistance is considered. As a final result, in the unassisted communication setting, we prove it is NP-hard to determine whether the maximal rate pair can be achieved for a given MAC.

The family of MACs that we consider is defined in terms of non-local games, which model interactions between a referee and two spatially separated players involving one round of communication: Each player is asked a question by the referee, and they answer independently using some pre-determined strategy on which they agree before commencing the game. The players win the game if their answers pass a certain winning condition. There are examples of non-local games for which perfect classical strategies do not exist, yet there are quantum strategies making use of shared entangled states that allow the players to always win the game [14, 15, 16, 17, 18]. Moreover, for certain non-local games defined in terms of systems of linear equations, the perfect quantum strategy can only be realized if the shared entanglement is supported on infinite-dimensional quantum systems [19, 20]. It is generally undecidable whether perfect finite-dimensional strategies exist for linear system games [19]. Finally, the well-known Boolean 3-satisfiability problem (3SAT) admits a formulation as a non-local game for which it is NP-hard to decide whether a perfect strategy exists [21, 22]. We show our results by considering a MAC communication scenario in which the two spatially separated senders play a non-local game, and the outcome of the game determines the noise level in the MAC. This construction allows us to translate the features of the non-local games mentioned above into properties of the (entanglement-assisted or unassisted) achievable rate region of the MAC.

2 Entanglement Helps a Classical Multiple Access Channel

In order to present the construction of the family of MACs used to show our main results, we first formalize the notion of a non-local game $G$. A two-player non-local game consists of finite question sets $\mathcal{X}_i$ and answer sets $\mathcal{Y}_i$ for $i = 1, 2$, and a winning condition $W \subseteq \mathcal{X}_1 \times \mathcal{Y}_1 \times \mathcal{X}_2 \times \mathcal{Y}_2$. Upon receiving questions $x_i \in \mathcal{X}_i$ from the referee, Alice and Bob answer independently with $y_i \in \mathcal{Y}_i$. They win the game if $(x_1, y_1, x_2, y_2) \in W$. While
A typical capacity region of a MAC (solid line), together with an achievable pentagonal region for a fixed input distribution (dashed lines).

The two players can agree on strategies before starting the game, they are not allowed to communicate once they have received the questions. We denote by $\omega(G, \pi)$ the maximal probability of Alice and Bob winning the game when the questions $(x_1, x_2)$ are drawn according to some probability distribution $\pi$ on $X_1 \times X_2$, and we use the shorthand $\omega_U(G)$ for the maximal winning probability with questions drawn uniformly at random.

A general MAC is shown in Figure 1. Our results concern the capacity region $C(N)$ of a MAC $N$, consisting of all the rate pairs $(R_1, R_2)$ such that sender $i$ can faithfully transmit information to the receiver at the rate $R_i$ (see Appendix A.2 for a more detailed definition). Ahlswede [5] and Liao [6] proved that $C(N)$ is given by the convex hull of all pairs $(R_1, R_2)$ satisfying

$$R_1 \leq I(A; Z|B) \quad R_2 \leq I(B; Z|A) \quad R_1 + R_2 \leq I(AB; Z)$$

for some product distribution $\pi_{A \pi_B}$ on $A \times B$.

The central object in this paper is a MAC $N_G$ defined in terms of a non-local game $G = (X_1, X_2, Y_1, Y_2, W)$ as follows. The input alphabets of the two senders Alice and Bob are the question-answer sets $X_1 \times Y_1$ and $X_2 \times Y_2$, respectively, and the output alphabet of $N_G$ is $X_1 \times X_2$. If Alice and Bob win the non-local game, $(x_1, y_1, x_2, y_2) \in W$, the channel is noiseless and outputs the question pair $(x_1, x_2)$ to the receiver. If they lose the game, $(x_1, y_1, x_2, y_2) \not\in W$, the channel outputs a question pair $(\hat{x}_1, \hat{x}_2)$ drawn uniformly at random from $X_1 \times X_2$. More formally, the MAC $N_G: (X_1 \times Y_1) \times (X_2 \times Y_2) \rightarrow X_1 \times X_2$ is defined as

$$N_G(\hat{x}_1, \hat{x}_2|x_1, y_1; x_2, y_2) := \begin{cases} \delta_{x_1, \hat{x}_1} \delta_{x_2, \hat{x}_2} & \text{if } (x_1, x_2, y_1, y_2) \in W, \\ (||X_1||X_2||)^{-1} & \text{else.} \end{cases}$$

This channel construction is inspired by [23], where the authors used a similar construction in terms of the CHSH game [24] for an interference channel consisting of two senders and two receivers.

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1Here, $I(U; V|W) = H(UW) + H(VW) - H(W) - H(UVW)$ is the conditional mutual information, $H(X) = -\sum_x p(x) \log p(x)$ is the Shannon entropy of a random variable $X \sim p$ with the logarithm taken to base 2, and $I(U; V) = H(U) + H(V) - H(UV)$ is the mutual information.
The noise in the MAC $N_G$ defined in terms of a non-local game $G$ is determined by the players’ ability to win the game. Clearly, if there exists a perfect strategy for Alice and Bob, they can select their questions uniformly at random and transmit information to the receiver at rates $R_1 = \log |X_1|$, achieving the maximal sum rate $R_1 + R_2 = \log |X_1| + \log |X_2|$. On the other hand, if they cannot win the game with certainty, then the channel necessarily adds noise to their signals, and consequently the achievable sum rate decreases. We can make this intuition precise by observing (see Proposition 2 in Appendix B) that the mutual information $I(X_1 Y_1 X_2 Y_2; Z)$ constraining the sum rate $R_1 + R_2$ in (1) can be expressed as

$$I(X_1 Y_1 X_2 Y_2; Z) = H(Z) - p_L \log |X_1| + \log |X_2|),$$

where $p_L$ denotes the probability of losing the game $G$ given a distribution $\pi_A \pi_B$ on the questions $x_i$ and a strategy producing the answers $y_i$. This relation allows us to prove a powerful bound on the capacity region of $N_G$ whenever the non-local game $G$ does not admit a perfect strategy. Informally, it can be stated as follows (see Proposition 3 for a formal statement):

**Theorem 1.** If a non-local game $G$ does not admit a perfect strategy, $\omega_U(G) < 1$, then the sum rate $R_1 + R_2$ of the MAC $N_G$ is strictly bounded away from the maximal value $\log |X_1| + \log |X_2|$. 

For non-local games $G$ with $\omega_U(G) < 1$ that have perfect quantum strategies, Theorem 1 provides a provable separation between the capacity region of the unassisted MAC $N_G$ and the entanglement-assisted achievable rate region. In entanglement-assisted strategies the two spatially separated senders can measure shared entangled states to produce the input probability distribution $\pi_{AB}$ appearing in (1) (see Appendix A.3 for a precise definition). Due to the quantum correlations present in the entangled state, the resulting probability distribution may not be of product form anymore. Depending on the non-local game, this entangled strategy may allow the players to win the game with certainty, and consequently transmit maximal information through the MAC.

A well-known example of such a non-local game is the magic square game $G_{MS}$ [14, 15, 16, 18], in which Alice and Bob receive questions $r, c \in \{0, 1, 2\}$ labeling the rows and columns of a $3 \times 3$-matrix, respectively.
They answer with bit strings of length 3, and win if a) the parity of Alice’s row is even, b) the parity of Bob’s column is odd, and c) the bit strings agree in the overlapping cell. It is easy to see that there is no valid filling of the whole square that satisfies both parity constraints above: if such a filling existed, the parity of the whole square would be even according to a), and odd according to b), a contradiction (see the left panel in Figure 2). This argument can be used to prove that there is no perfect classical winning strategy for Alice and Bob, and in fact $\omega_U(G_{MS}) \leq \frac{8}{9}$ [18]. For the MAC $N_{G_{MS}}$ defined in terms of the magic square game, we can therefore use Theorem 1 to obtain an upper bound on the achievable sum rate of 3.13694, bounding it away from the maximal value of $2\log 3 \approx 3.17$.

On the other hand, there is a well-known perfect quantum strategy in which Alice and Bob make measurements on two maximally entangled states shared between them [14, 15, 18]. Drawing the questions $r, c \in \{0, 1, 2\}$ uniformly at random and using this perfect quantum strategy allows Alice and Bob to code at rates $R_1 = R_2 = \log 3$, and hence achieve the perfect rate pair $(\log 3, \log 3)$ not achievable by any classical strategy (see Appendix C). To summarize, while the unassisted capacity region of $N_{G_{MS}}$ is separated from the point $(\log 3, \log 3)$, the entanglement-assisted achievable rate region includes this point, as plotted in Figure 6.

Our result is similar in spirit to [23] which proves a separation between classical and quantum strategies for an interference channel (with two senders and two receivers) based on the CHSH inequality. In contrast to [23], the separation between classical strategies and entanglement-assisted strategies demonstrated above works for any non-local game $G$ with $\omega_U(G) < 1$ and for which there exists a perfect quantum strategy. These games have been called “pseudo-telepathy games” [18], and each game in this class yields a separation for the corresponding MAC between the unassisted capacity region and the entanglement-assisted region. This increase of the achievable rate region due to additional resources (in the form of shared entanglement) can be compared to a similar effect for a MAC communication scenario with classical feedback from the receiver to the senders [25].

We conclude this section by noting that the particular form of entanglement assistance considered above—entanglement shared between the senders—is crucial to show our result. Specifically, consider an entanglement-assisted MAC coding scenario where each sender shares entanglement with the receiver, as for example discussed in [26] for quantum multiple access channels. Specializing the results of [26] to classical MACs shows that the achievable sum rate cannot be increased with this type of entanglement assistance (see Appendix C.3). This stands in stark contrast to the significant increase in the sum rate achieved by entanglement shared between the senders as demonstrated above.

### 3 How Much Entanglement Do You Need?

Our main result, Theorem 1 in the preceding section, can also be applied to separate achievable rate regions for entanglement-assisted strategies using different amounts of entanglement. To illustrate this, we consider the class of linear system games $G_{LS}$ [27], which are defined in terms of an $m \times n$ linear system $Ax = b$ of equations over $\mathbb{F}_2$. In the linear system game $G_{LS}$, Alice receives as a question an index $1 \leq i \leq m$ labeling a row in the linear system, and she replies with values for the variables $x$ such that the $i$-th equation in $Ax = b$ is satisfied. Bob receives as a question an index $1 \leq j \leq n$, and answers with a value for the $j$-th variable $x_j$. We assume both questions to be drawn uniformly at random. Alice and Bob win the game either if Bob’s variable does not appear in Alice’s equation or if Alice’s and Bob’s assignment of $x_j$ are consistent.

Slofstra and Vidick [20] showed that there is a particular instance $G_{SV}$ of a linear system game for which a perfect winning strategy is necessarily quantum and furthermore requires an unbounded amount of entanglement. More precisely, for the game $G_{SV}$ the local dimension $d$ of the quantum systems associated
with Alice and Bob in the quantum strategy is bounded in terms of the losing probability
\[ p_L = 1 - \omega_U(G_{SV}) \]
and positive constants \( C, C' \) as
\[ \frac{C}{p_L^{1/6}} \leq d \leq \frac{C'}{p_L^{1/2}}. \] (4)

Consider now the MAC \( N_{G_{SV}}: ([m] \times \{0, 1\}^n) \times ([n] \times \{0, 1\}) \to [m] \times [n] \) defined according to (2) in terms of the linear system game \( G_{SV} \), where we use the notation \([n] = \{1, \ldots, n\}\). If we limit Alice and Bob to entanglement assistance of local dimension at most \( d \), then their probability of losing the linear system game has to be at least \((C/d)^6\) by (4). Consequently, we can invoke Theorem 1 to conclude that the \( d \)-dimensional entanglement-assisted achievable rate region of \( N_{G_{SV}} \) considered in this paper is bounded away from the rate pair \((\log m, \log n)\) achieving the ideal sum rate \( \log m + \log n \) (see Proposition 5 for details). On the other hand, it is straightforward to define a \( d \)-dimensional entanglement-assisted coding strategy for Alice and Bob based on the quantum strategy in [20] that achieves \( p_L \leq (C'/d)^2 \). Hence, as Alice and Bob have access to larger and larger entangled states, they approach the rate pair \((\log m, \log n)\) arbitrarily well (see Proposition 6).

Our results show that linear system games give rise to a family of MACs whose \( d \)-entanglement-assisted achievable rate regions approach the rate pair \((\log m, \log n)\) in the limit \( d \to \infty \), yet they are strictly bounded away from it for any fixed finite \( d \). Moreover, considering all finite-dimensional quantum strategies for a general linear system game \( G_{LS} \), Slofstra [19] showed that it is undecidable to determine whether there is a perfect quantum strategy among them. By the arguments above, this directly translates to the MAC setting in the following way: for the MAC \( N_{G_{LS}} \) defined above in terms of a linear system game \( G_{LS} \), it is undecidable to determine whether the entanglement-assisted achievable rate region includes the rate pair \((\log m, \log n)\) (see Proposition 8).

### 4 Complexity of the Capacity Region of a Classical MAC

We now turn our focus again to the unassisted coding scenario for a discrete MAC.

In information-theoretic terms, this scenario seems well understood as the capacity region \( C(N) \) of a MAC \( N \) can be expressed in terms of a computable single-letter formula [5, 6]. However, the single-letter nature of the capacity region formula by itself does not guarantee an efficient method of computing \( C \) in, say, runtime polynomial in \( \max\{|A|, |B|, |Z|\} \), the maximal size of the input and output alphabets of \( N \). For Shannon’s mutual information formula [1] for the capacity of a point-to-point channel, there is an efficient method given by the Blahut-Arimoto algorithm [28, 29]. In contrast, for the MAC capacity region \( C \) no such efficient method is known. [30] put forward a numerical method for computing the sum capacity of MACs with binary output. However, it has been shown that for general MACs, even computing the sum capacity is equivalent to solving an instance of a rank-1 constrained optimization problem, which are notoriously hard [31, 32].

We show in the following that computing the capacity region of a MAC is hard in a precise complexity-theoretic way. Formally, using our construction of a MAC in terms of non-local games, we prove: It is NP-hard to decide whether a given point \((R_1, R_2)\) belongs to the capacity region of a MAC up to precision inverse-cubic in \( n \), where \( n \) is the size of the output alphabet. The complexity class ‘non-deterministic polynomial time’ (NP) consists of problems for which a solution can be verified in polynomial time. A problem is NP-hard if any problem in NP can be reduced to it in polynomial time. NP-hard problems are widely believed to be hard to solve (as in they cannot be solved in polynomial time), and include for example the traveling salesman problem or the Boolean 3-satisfiability problem (3SAT).
Our result is based on a promise-free non-local game version of a two prover protocol introduced by Håstad in [33]. Let \( x_1, \ldots, x_n \) be Boolean variables, and let \( C_1, \ldots, C_m \) with \( m = O(n) \) be clauses containing exactly three literals, i.e., the \( C_j \)'s are conjunctions of logical expressions of the form \( a_1 \lor a_2 \lor a_3 \) where each \( a_i \) is either a variable \( x_j \), or its negation \( \neg x_j \). In the game \( G_H \), Alice receives an index \( 1 \leq j \leq m \) labeling a clause \( C_j \) drawn uniformly at random, and answers with an assignment of the variables \( x_j, x_{j2}, x_{j3} \) appearing in \( C_j \) such that \( C_j \) is satisfied. Bob receives an index \( 1 \leq i \leq n \) labeling a variable \( x_i \) drawn uniformly at random, and answers with an assignment of \( x_i \). Alice and Bob win the game if \( x_i \) does not appear in \( C_j \), or if Alice’s and Bob’s assignment are consistent.

By the probabilistically checkable proofs (PCP) theorem [21, 22], it is NP-hard to decide if there is a perfect winning strategy for \( G_H \) or if the maximal winning probability is bounded from above by \( 1 - (1 - c)/n \) for some constant \( c < 1 \) [33]. Let us now consider \( N_{G_H} \), the MAC defined in (2) in terms of the non-local game \( G_H \). Clearly, if Alice and Bob have a perfect winning strategy for \( G_H \), they can each code at the rates \( R_1 = \log m \) and \( R_2 = \log n \) by choosing a uniform distribution over their respective question alphabets, leading to an ideal sum rate of \( R_1 + R_2 = \log m + \log n \). On the other hand, if the maximal winning probability is bounded from above by \( \omega^* = 1 - (1 - c)/n \), then Theorem 1 can be used to show that the sum rate \( R_1 + R_2 \) is bounded from above by \( \log m + \log n - (1 - \omega^*)^3 \) (see Proposition 10). In this case, the capacity region \( \mathcal{C}(N_{G_H}) \) is bounded away from the rate pair \((\log m, \log n)\). Altogether, this shows that it is NP-hard to decide if an arbitrary rate tuple \((R_1, R_2)\) belongs to \( \mathcal{C}(N_{G_H}) \) to precision inverse-cubic in \( n \). A curious corollary of our result is that the “naive” method of covering the space of product probability distribution with a net and computing an approximation of the capacity region is more or less optimal, assuming widely believed complexity-theoretic conjectures hold. For a given inverse-cubic precision, this net covering method is exponential in the alphabet size of the probability distributions.

5 Conclusion

In this work we show that the capacity region of a multiple access channel displays complex behavior, both in a purely classical setting and when the senders have access to shared entangled quantum states. In particular, we prove that entanglement assistance can boost the achievable rates in a setting where two senders try to convey classical information through a common classical communication channel to a single receiver. Such an increase in capacity is impossible in the point-to-point scenario involving a single sender and a receiver. We also show that for a certain family of MACs the two senders need to share an unbounded amount of entanglement in order to achieve the ideal communication rate pair. When restricted to finite-dimensional entangled strategies, it is undecidable for this particular channel family whether the ideal rate can be achieved. Finally, we show that even in the unassisted scenario, it is in fact NP-hard to decide whether the ideal rate pair belongs to the capacity region of a MAC. This result is a strong counterpoint to the widely held belief that the availability of a computable single-letter formula for the capacity region essentially solves the MAC problem. The central tool in the proofs of all results above is the construction of a MAC in terms of a non-local game in such a way that the noise level of the channel is determined by the senders’ ability to win the game.

Our work opens up a number of interesting topics for future work. Numerical investigations for the magic square channel of Section 2 suggest that the true separation between classical and quantum coding strategies for the MACs considered in this work is considerably larger than the separation guaranteed by Theorem 1. This suggests that our bound on the sum rate could be further tightened. Moreover, the particular constants and exponents appearing in the bounds used to show the results of Sections 3 and 4 can likely be improved as well. For our results above we considered a specific achievable rate region that arises naturally when the...
two senders measure identical copies of a single entangled state. In general, the senders might have access to multipartite entangled states and implement parallel encoding strategies, which leads to the notion of an entanglement-assisted capacity region. We expect this region to be given by the regularization of the achievable region considered in this paper. For the MACs defined via our construction, this question seems to be related to parallel repetition theorems for non-local games played with quantum strategies. Furthermore, in this work we only considered entanglement shared between the two senders, and the communication setting could be generalized to one where entanglement is shared between both the senders and the receiver. Finally, our NP-hardness result for the unassisted capacity region of a MAC underlines the need for tight efficiently computable outer bounds on the unassisted capacity region. Such bounds could for example be obtained from convex relaxations of the rank-1 optimization problem describing the MAC capacity region.

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A Preliminaries

A.1 Non-local Games

A two-player non-local game is a tuple $G = (X_1, X_2, Y_1, Y_2, P, W)$, where $X_i$ and $Y_i$ are the question and answer sets for player $i = 1, 2$, respectively. The set $P \subset X_1 \times X_2$ is called the promise of the game, the set
of “allowed” questions. The set \( W \subset P \times \mathcal{Y}_1 \times \mathcal{Y}_2 \) is the winning condition, i.e., upon receiving the questions \((x_1, x_2) \in P\) and answering with \( y_i \in \mathcal{Y}_i \), the players win the game if \((x_1, x_2, y_1, y_2) \in W\), and lose otherwise.

Every game \( G = (\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2, P, W) \) with promise \( P \subseteq \mathcal{X}_1 \times \mathcal{X}_2 \) can be turned into a promise-free game \( G' = (\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2, P', W') \) by declaring \( P' = \mathcal{X}_1 \times \mathcal{X}_2 \) and \( W' = W \cup (P_e \times \mathcal{Y}_1 \times \mathcal{Y}_2) \), i.e., the players win automatically if they receive a question pair outside the promise set. We write \( G = (\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2, W) \) for a promise-free game.

While the two players Alice and Bob can agree on a strategy beforehand, they are not allowed to communicate during the game. A deterministic strategy for Alice and Bob is a pair of deterministic functions \( f_i: \mathcal{X}_i \to \mathcal{Y}_i \) for \( i = 1, 2 \). A probabilistic strategy for Alice and Bob is a probabilistic mixture of deterministic strategies. For a given probability distribution \( \pi: P \to [0, 1] \) on the promised question set \( P \subseteq \mathcal{X}_1 \times \mathcal{X}_2 \), we define \( \omega(G, \pi) \) as the maximal winning probability using probabilistic strategies. If the given probability distribution on \( P \) is the uniform distribution \( \pi_U \), we use the shorthand \( \omega_U(G) \equiv \omega(G, \pi_U) \). Note that the maximal winning probability is always achieved on an extremal point, i.e., a deterministic strategy. A strategy achieving \( \omega(G, \pi) = 1 \) is called perfect.

### A.2 Capacity Region of Multiple Access Channels

![Multiple access channel](image)

Figure 3: Multiple access channel \( N(z|a, b) \) with two senders \( A, B \) and a receiver \( Z \).

In this paper we only consider discrete memoryless multiple access channels without feedback, to which we refer simply as multiple access channels (MAC). A two-sender MAC is a tuple \((\mathcal{A}, \mathcal{B}, Z, N(z|a, b))\), where \( \mathcal{A} \) and \( \mathcal{B} \) are the input alphabets for sender 1 and 2, respectively, \( Z \) is the output alphabet for the single receiver, and \( N(z|a, b) \) is a probability distribution for all pairs \((a, b) \in \mathcal{A} \times \mathcal{B} \) (see Figure 3).

The following discussion is taken from [35]. An \((R_1^{(n)}(n), R_2^{(n)}(n))\)-code is a tuple \((\mathcal{M}_1, \mathcal{M}_2, a^n, b^n, \hat{z}^n)\), where:

- \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are message sets with \(|\mathcal{M}_i| = 2^nR_i^{(n)}\) for \( i = 1, 2 \);
- \( a^n: \mathcal{M}_1 \to \mathcal{A}^n \) and \( b^n: \mathcal{M}_2 \to \mathcal{B}^n \) are encoding functions;
- \( \hat{z}^n: Z^n \to \mathcal{M}_1 \times \mathcal{M}_2 \cup \{\epsilon\} \) is a decoding function, with \( \epsilon \) an arbitrary error message.

Without loss of generality we assume a uniform distribution over the messages \((\mathcal{M}_1, \mathcal{M}_2)\); in particular, the codewords \( a^n(M_1) \) and \( b^n(M_2) \) are independent. The average probability of error is defined as

\[
P_e^{(n)} = \text{Pr}\{|\hat{M}_1, \hat{M}_2| \neq (M_1, M_2)\},
\]

where \((\hat{M}_1, \hat{M}_2)\) is a random variable with probability mass function (pmf) \( \hat{z}^n(N^n(\hat{z}^n|a^n(M_1), b^n(M_2)))\). A rate pair \((R_1, R_2)\) is said to be achievable if there exists a sequence of codes \( \{\{R_1^{(n)}(n), R_2^{(n)}(n)\}\}_{n \in \mathbb{N}} \) such that \( \liminf_{n \to \infty} R_1^{(n)} = R_1 \) and \( \lim_{n \to \infty} P_e^{(n)} = 0 \). The capacity region \( C(N) \) of the MAC \((\mathcal{A}, \mathcal{B}, Z, N(z|a, b))\) is the closure of the set of all achievable rate pairs \((R_1, R_2)\). We also consider the sum-capacity \( S(N) \) defined as

\[
S(N) := \sup\{R_1 + R_2: (R_1, R_2) \in C(N)\}.
\]
The capacity region $\mathcal{C}(N)$ of a two-sender MAC $(A,B,Z,N(z|a,b))$ has a single-letter characterization\cite{5,6}. Let $(A,B)$ be a pair of discrete random variables jointly distributed according to the pmf $p_{AB}$, and let $Z$ be the channel output random variable with conditional pmf $N(z|a,b)$. Define $\mathcal{R}(A,B)$ as the set of rate pairs $(R_1,R_2)$ satisfying

$$R_1 \leq I(A;Z|B)$$
$$R_2 \leq I(B;Z|A)$$
$$R_1 + R_2 \leq I(A,B;Z).$$

Then the capacity region $\mathcal{C}(N)$ of the MAC $(A,B,Z,N(z|a,b))$ is the convex hull of the union of the regions $\mathcal{R}(A,B)$ over all product distributions $p_{AB}$:

$$\mathcal{C}(N) = \text{conv} \left( \bigcup \{ \mathcal{R}(A,B): (A,B) \sim p_{AB} \} \right) \quad (7)$$

### A.3 MACs and Entanglement Assistance

In this paper we consider coding strategies for classical MACs assisted by entanglement shared between the two senders.\footnote{We briefly discuss entanglement assistance where each sender shares entanglement with the receiver at the end of Appendix C.3.} To formalize this setting, let $(A,B,Z,N(z|a,b))$ be a MAC as defined in Appendix A.2, and let the two senders $A$ and $B$ share an entangled state $|\psi\rangle_{SA,S_B} \in \mathbb{C}^d \otimes \mathbb{C}^d$, where the $d$-dimensional quantum systems $S_A$ and $S_B$ with $|S_A| = |S_B| = d$ are with senders $A$ and $B$, respectively. We then consider the following coding scenario.

Let $A_1$ and $B_1$ be random variables taking values in finite alphabets $\mathcal{A}_1$ and $\mathcal{B}_1$ for sender $A$ and $B$, respectively. Depending on the value $a_1 \in \mathcal{A}_1$ of $A_1$, the first sender selects a positive operator-valued...
measure (POVM) \(L^{(a_1)} = \{ L^{(a_1)}_{a_2} \}_{a_2 \in A_2} \) with
\[
L^{(a_1)}_{a_2} \geq 0 \quad \text{for all } a_1 \in A_1, a_2 \in A_2 \text{ and}
\]
\[
\sum_{a_2 \in A_2} L^{(a_1)}_{a_2} = I_d \quad \text{for all } a_1 \in A_1.
\]

Here, \(A_2\) is some finite alphabet, and \(I_d\) denotes the identity operator on \(\mathbb{C}^d\). Likewise, for \(b_1 \in A_1\) the second sender selects a POVM \(M^{(b_1)} = \{ M^{(b_1)}_{b_2} \}_{b_2 \in B_2}\) for some finite alphabet \(B_2\) satisfying \(M^{(b_1)}_{b_2} \geq 0\) for all \(b_1 \in B_1, b_2 \in B_2\) and \(\sum_{b_2 \in B_2} M^{(b_1)}_{b_2} = I_d\) for all \(b_1 \in B_1\). Upon drawing \((a_1, b_1)\), the senders measure their respective half of the entangled state \(\ket{\psi}_{S_A S_B}\) using the measurements \(L^{(a_1)}\) and \(M^{(b_1)}\), producing a correlation
\[
P(a_2, b_2 | a_1, b_1) = \bra{\psi} L^{(a_1)}_{a_2} \otimes M^{(b_1)}_{b_2} \ket{\psi}.
\]

Finally, the senders can each post-process their measurement outcomes \(a_2\) and \(b_2\) together with their inputs \(a_1\) and \(b_1\) to produce inputs \(a\) and \(b\) to the MAC \(N\), which we summarize in a function \(f_1(a | a_1, a_2) f_2(b | b_1, b_2)\). In total, we have the classical channel
\[
E(a, b | a_1, b_1) = f_1(a | a_1, a_2) f_2(b | b_1, b_2) P(a_2, b_2 | a_1, b_1),
\]
where the correlation \(P\) is obtained from measuring the shared entangled state \(\ket{\psi}_{S_A S_B}\) through (10). The setup is depicted in Figure 4.

If we require the senders to draw \(a_1\) and \(b_1\) independently from a product distribution \(p_{A_1}(a_1) p_{B_1}(b_1)\), then the channel \(N \circ E\) with \(E\) as defined in (11) can again be interpreted as a MAC with input alphabet \(A_1 \times B_1\) and output alphabet \(Z\). This prompts us to define the \(d\)-entanglement-assisted achievable rate region of a classical MAC \(N\) as
\[
\mathcal{C}^{(1)}_{ca,d}(N) := \bigcup E \{ \mathcal{C}(N \circ E) \},
\]
where \(\mathcal{C}(\cdot)\) is the capacity region of an ordinary MAC defined in (7), and the union is over all classical channels \(E\) as in (11) defined in terms of the following data:

- an entangled state \(\ket{\psi}_{S_A S_B} \in \mathbb{C}^d \otimes \mathbb{C}^d\);
- arbitrary finite alphabets \(A_1, B_1\) and \(A_2, B_2\);
- POVMs \(L^{(a_1)} = \{ L^{(a_1)}_{a_2} \}_{a_2 \in A_2}\) for \(a_1 \in A_1\) and \(M^{(b_1)} = \{ M^{(b_1)}_{b_2} \}_{b_2 \in B_2}\) for \(b_1 \in B_1\), defined on \(\mathbb{C}^d\);
- post-processings \(f_1: A_1 \times A_2 \ni (a_1, a_2) \mapsto a \in A\) and \(f_2: B_1 \times B_2 \ni (b_1, b_2) \mapsto b \in B\).

We also define the corresponding achievable sum rate
\[
S_{ca,d}(N) := \sup \{ R_1 + R_2: (R_1, R_2) \in \mathcal{C}^{(1)}_{ca,d}(N) \}.
\]

The coding theorem (7) for unassisted MACs implies that the region \(\mathcal{C}^{(1)}_{ca,d}(N)\) in (12) is achievable, and hence a natural inner bound on the true entanglement-assisted capacity region of a MAC. We expect that the latter is given by the regularized formula
\[
\mathcal{C}^{(1)}_{ca,d}(N) = \bigcup_{n \in \mathbb{N}} \frac{1}{n} \mathcal{C}^{(1)}_{ca,d}(N \times n),
\]
\[\text{Note that in principle this post-processing can be made part of the measurements with a potential increase of the local dimension } d. \text{ However, we choose to keep them separate in order to link the local dimension } d \text{ to the non-local games considered in Appendix D in a clean way.}\]
where $X$ denotes the closure of a set $X$. For the developments in Appendix D.3, we also define the entanglement-assisted achievable rate region

$$C^{(1)}_{en}(N) := \bigcup_{d \in \mathbb{N}} C^{(1)}_{en,d}(N),$$  \tag{15}$$

which is achievable by the two senders sharing entanglement on quantum systems of arbitrarily large but finite local dimension.

## B Encoding a Non-local Game in a MAC

The following construction of a classical multiple access channel in terms of a non-local game is our main object of study. It is inspired by a similar construction of an interference channel (two senders, two receivers) in terms of the CHSH game in [23], and also appeared in [34]. Given a promise-free non-local game $G = (X_1, X_2, Y_1, Y_2, W)$, we define the classical MAC $N_G$: $(X_1 \times Y_1) \times (X_2 \times Y_2) \rightarrow X_1 \times X_2$ as

$$N_G(x_1, x_2|y_1, y_2) := \begin{cases} \delta_{x_1, x_1'} \delta_{x_2, x_2'} & \text{if } (x_1, x_2, y_1, y_2) \in W, \\ ((|X_1||X_2|)^{-1} & \text{else.} \end{cases} \tag{16}$$

In the above construction, to each player of the game we associate a sender in the MAC scenario with input alphabet $X_i \times Y_i$ for $i = 1, 2$. If the two senders input a question-answer tuple $(x_1, x_2, y_1, y_2) \in W$ that wins the non-local game $G$, the channel outputs the question pair $(x_1, x_2)$; otherwise, the channel outputs a question pair drawn uniformly at random. In the following, for $i = 1, 2$ we denote by $X_i \sim \pi_{X_i}$ the random variables corresponding to the questions for Alice and Bob, by $Y_i \sim p_{Y_i|X_i} \pi_{X_i}$ the random variables corresponding to the answers, and by $Z$ the random variable corresponding to the output of the channel $N_G$ defined in (16) taking values in $X_1 \times X_2$.

As discussed in Appendix A.2, the capacity region of a MAC is computed in terms of a product probability distribution $p_{X_1Y_1}(x_1, y_1)p_{X_2Y_2}(x_2, y_2)$ on the set of inputs to $N$. For the MAC (16), we can think of this input distribution in the following way: Given a product probability distribution $\pi(x_1, x_2) = \pi_{X_1}(x_1)\pi_{X_2}(x_2)$ on the question set, the players produce answers $y_i$ to the game according to the probabilistic strategy $p_{Y_i|X_i}(y_1|x_1)p_{Y_2|X_2}(y_2|x_2)$ on which they agreed prior to starting the game. This allows us to connect the sum rate capacity of the channel $N_G$ to the winning probability $\omega(G, \pi)$ as follows:

**Proposition 2.** Let $G = (X_1, X_2, Y_1, Y_2, W)$ be a promise-free non-local game, $\pi(x_1, x_2) = \pi_{X_1}(x_1)\pi_{X_2}(x_2)$ a probability distribution on the questions set $X_1 \times X_2$, and $p_{Y_i|X_i}(y_1|x_1)p_{Y_2|X_2}(y_2|x_2)$ a probabilistic strategy for Alice and Bob. For the MAC $N_G$ defined in terms of $G$ according to (16), let $X_i, Y_i, Z$ be the random variables corresponding to the questions, answers, and channel output, respectively, as described above. We then have

$$I(X_1Y_1X_2Y_2; Z) = H(Z) - p_L(\log |X_1| + \log |X_2|),$$  \tag{17}$$

where $p_L = \sum_{(x_1, y_1, x_2, y_2) \in W} \Pr(x_1, y_1, x_2, y_2)$ denotes the losing probability given the distribution $\pi(x_1, x_2)$ on the questions set and the probabilistic strategy $p_{Y_i|X_i}(y_1|x_1)p_{Y_2|X_2}(y_2|x_2)$.

**Proof.** We first expand the mutual information $I(X_1Y_1X_2Y_2; Z)$ as

$$I(X_1Y_1X_2Y_2; Z) = H(Z) - H(Z|X_1Y_1X_2Y_2).$$  \tag{18}$$

Note that we can turn any game with promise into a promise-free one, as explained in Appendix A.1.
Setting $d = |X_1||X_2|$ and recalling that $W \subset X_1 \times Y_1 \times X_2 \times Y_2$ is the winning set for $G$, the conditional entropy can be expressed as

$$H(Z|X_1Y_1X_2Y_2) = \sum_{x_1, y_1, x_2, y_2} \Pr(x_1, y_1, x_2, y_2) H(Z|X_1 = x_1, Y_1 = y_1, X_2 = x_2, Y_2 = y_2)$$

$$= \sum_{(x_1, y_1, x_2, y_2) \notin W} \Pr(x_1, y_1, x_2, y_2) H(Z|X_1 = x_1, Y_1 = y_1, X_2 = x_2, Y_2 = y_2)$$

$$= \log d \sum_{(x_1, y_1, x_2, y_2) \notin W} \Pr(x_1, y_1, x_2, y_2)$$

$$= p_L \log d,$$

where the second equality follows since for $(x_1, y_1, x_2, y_2) \in W$ the channel $N_G$ outputs $(x_1, x_2)$ deterministically, and hence $H(Z|X_1 = x_1, Y_1 = y_1, X_2 = x_2, Y_2 = y_2) = 0$ in this case.

**Proposition 3.** Let $G = (X_1, Y_1, X_2, Y_2, W)$ be a promise-free non-local game with $\omega_U(G) < 1$, and consider the MAC $N_G$ defined as in (16). Using the same notation as in Proposition 2, for all $0 < \delta < -\log \omega_U(G)$ there exists an $\varepsilon > 0$ such that

$$I(X_1Y_1X_2Y_2; Z) \leq \max\{(1-\varepsilon)(\log |X_1| + \log |X_2|), \log |X_1| + \log |X_2| - \delta\}. \quad (23)$$

For a given $\delta > 0$ the maximal value of $\varepsilon$ is given by the $\varepsilon^*$ satisfying

$$\frac{\delta + h(\varepsilon^*)}{1 - \varepsilon^*} = \delta(\varepsilon^*||1 - \omega_U(G)),$$

where $h(x) = -x \log x - (1-x) \log (1-x)$ denotes the binary entropy and $\delta(x||y) := x \log \frac{x}{y} + (1-x) \log \frac{1-x}{1-y}$ denotes the binary relative entropy.

The strategy of the proof of Proposition 3 is the following: the goal of the proposition is to provide an upper bound on the sum rate capacity that separates it from the maximal value $\log |X_1| + \log |X_2|$. By the formula given in Proposition 2, the maximal value $\log |X_1| + \log |X_2|$ is attained if and only if $p_L$ vanishes and $H(Z)$ attains its maximal value. We therefore need to show that we cannot have $p_L \approx 0$ and $H(Z) \approx \log |X_1| + \log |X_2|$ at the same time.

**Proof of Proposition 3.** We again set $d = |X_1||X_2|$. For the purpose of bounding the sum rate capacity $I(X_1Y_1X_2Y_2; Z)$ away from the maximal value $\log d$, we can assume without loss of generality that the losing probability $p_L = 1 - \omega(G, \pi)$ for Alice and Bob is strictly positive, $p_L > 0$: In case $p_L = 0$, the probability distribution $\pi$ on $X_1 \times X_2$ necessarily has support $\text{supp} \pi$ strictly contained in $X_1 \times X_2$, as by assumption the game $G$ cannot be won with certainty on receiving any one of the full set $X_1 \times X_2$ of questions. Hence, $I(X_1Y_1X_2Y_2; Z) \leq \log |\text{supp} \pi| \leq \log (d - 1) < \log d$.

in this case, as Alice and Bob have to lose on at least one question pair. Furthermore, we can assume w.l.o.g. that $p_L \leq 1 - \omega_U(G)$, since $p_L > 1 - \omega_U(G)$ and $\omega_U(G) < 1$ imply that

$$I(X_1Y_1X_2Y_2; Z) < \omega_U(G) \log d < \log d.$$

Therefore, we may assume that $0 < p_L \leq 1 - \omega_U(G)$ for the remainder of the proof.

We prove the assertion of the theorem by contradiction. To this end, assume that

$$H(Z) \geq \log d - \delta$$

(27)
for some $0 < \delta < -\log \omega_U(G)$. Define a random variable $W$ by

$$W = \begin{cases} 1 & \text{Alice and Bob win the game;} \\ 0 & \text{Alice and Bob lose the game,} \end{cases}$$

(taking values 1 and 0 with probability $1 - p_L$ and $p_L$, respectively. By the non-negativity of conditional entropy and (27) we have

$$H(W) + H(Z|W) = H(ZW) \geq H(Z) \geq \log d - \delta.$$  

(29)

Expanding the left-hand side of (29) gives

$$h(p_L) + (1 - p_L)H(X_1X_2) + p_L \log d \geq \log d - \delta,$$

(30)

which can be rearranged to

$$H(X_1X_2) \geq \log d - \gamma$$

(31)

with $\gamma := \frac{\delta + h(p_L)}{1 - p_L}$. Observe that $D(\pi_{X,1}\pi_{X,2}\|\pi_U) = \log d - H(X_1X_2)$, and hence

$$\gamma \geq D(\pi_{X,1}\pi_{X,2}\|\pi_U).$$

(32)

Let now $Q = Q_{Y_1|X_1}Q_{Y_2|X_2}$ be the optimal probabilistic strategy for $G$ given the distribution $\pi_{X,1}\pi_{X,2}$ on the questions, and denote by $q_L$ the losing probability of the same strategy $Q$ with questions drawn uniformly at random. Furthermore, let $\chi_W : \mathcal{X}_1 \times \mathcal{Y}_1 \times \mathcal{X}_2 \times \mathcal{Y}_2 \to \{0,1\}$ be the characteristic function of the winning set $W \subset \mathcal{X}_1 \times \mathcal{Y}_1 \times \mathcal{X}_2 \times \mathcal{Y}_2$. Applying the data-processing inequality with respect to $\chi_W \circ Q$ to (32), we obtain

$$\gamma \geq \delta(p_L\|q_L) \geq \delta(p_L\|1 - \omega_U(G)),$$

(33)

where $\delta(x\|y) := x \log \frac{x}{y} + (1 - x) \log \frac{1-x}{1-y}$ denotes the binary relative entropy, and the second inequality follows from the monotonicity of $y \to \delta(x\|y)$ for $y \geq x$ and the fact that $q_L \geq 1 - \omega_U(G)$.

The function $\gamma(x) = \frac{\delta + h(x)}{1 - x}$ is monotonically increasing for all $x > 0$, and $\lim_{x \to 0} \gamma(x) = \delta < -\log \omega_U(G)$ by assumption. On the other hand, $x \to \delta(x\|1 - \omega_U(G))$ is monotonically decreasing for $x \in [0,1 - \omega_U(G)]$, and $\lim_{x \to 0} \delta(x\|1 - \omega_U(G)) = -\log \omega_U(G)$. Hence, there exists an $\epsilon > 0$ such that (33) is violated for all $p_L < \epsilon$, which means that we either have $p_L \geq \epsilon$ or the assumption (27) is false. In the first case,

$$I(X_1Y_1X_2Y_2; Z) = H(Z) - p_L \log d \leq (1 - \epsilon) \log d,$$

(34)

while in the second case,

$$I(X_1Y_1X_2Y_2; Z) < (1 - p_L) \log d - \delta < \log d - \delta.$$  

(35)

By the arguments above, for a given $\delta > 0$ the maximal value of $\epsilon$ is given by the $\epsilon^*$ satisfying

$$\frac{\delta + h(\epsilon^*)}{1 - \epsilon^*} = \delta(\epsilon^\ast\|1 - \omega_U(G)),$$

(36)

which concludes the proof.

**Remark 4.** In applications of Proposition 3, the optimal (minimal) upper bound in Proposition 3 can be obtained by optimizing the right-hand side of (23) over $\delta \in (0, -\log \omega_U(G))$ and computing $\epsilon$ via (24).
C Magic Square Game

Consider a 3×3-matrix whose rows and columns are labeled by \( r, c \in \{0, 1, 2\} \), respectively. The magic square game \( G_{MS} \) [14, 15, 16, 18] is a two-player game in which Alice and Bob receive questions \( r, c \in \{0, 1, 2\} \) respectively, labeling a row \( r \) for Alice and a column \( c \) for Bob. They answer with 3-bit strings \( s, t \in \{0, 1\}^3 \), where the bits in \( s, t \) correspond to the cells in \( r, c \), respectively. Alice and Bob win the game if the following three conditions are satisfied:

1. the parity of Alice’s bit string \( s \) is even: \( s_0 \oplus s_1 \oplus s_2 = 0 \);
2. the parity of Bob’s bit string \( t \) is odd: \( t_0 \oplus t_1 \oplus t_2 = 1 \);
3. the bit strings agree in the overlapping cell \( (r, c) \): \( s_c = t_r \).

C.1 Classical Strategies

The two parity constraints for Alice’s and Bob’s bit strings \( s \) and \( t \) render any deterministic perfect classical strategy for \( G_{MS} \) impossible, since the latter corresponds to a fixed valid filling of the nine cells of the magic square with bits such that conditions 1-3 above are satisfied. However, according to condition 1 the parity of all cells is even, while according to condition 2 this parity should be odd.

If the questions \( (r, c) \) are drawn uniformly at random, the best deterministic strategy for Alice and Bob consists in filling 8 of the 9 cells with valid bits. Hence, the optimal deterministic strategy has winning probability 8/9, and in fact \( \omega_U(G_{MS}) = 8/9 \) [18].

C.2 A Perfect Quantum Strategy

Brassard et al. [18] described the following perfect quantum strategy for the magic square game \( G_{MS} \) that is equivalent to the commuting observables strategy devised by Mermin [14] and Peres [15] in Figure 2(b):

Consider the 4-qubit entangled state

\[
|\psi\rangle_{A_1A_2B_1B_2} = \frac{1}{2} (|00\rangle_{A_1A_2}|11\rangle_{B_1B_2} + |11\rangle_{A_1A_2}|00\rangle_{B_1B_2} - |01\rangle_{A_1A_2}|10\rangle_{B_1B_2} - |10\rangle_{A_1A_2}|01\rangle_{B_1B_2}), \tag{37}
\]

where qubits \( A_1A_2 \) are with Alice, and \( B_1B_2 \) are with Bob. Furthermore, consider the following 2-qubit unitaries:

\[
U_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 & 0 & i \\ 0 & -i & 1 & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & 0 & i \end{pmatrix}, \quad U_1 = \frac{1}{2} \begin{pmatrix} i & 1 & 1 & i \\ -i & 1 & -1 & i \\ i & 1 & -i & -i \\ -i & 1 & 1 & -i \end{pmatrix}, \quad U_2 = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix},
\]

\[
V_0 = \frac{1}{2} \begin{pmatrix} i & -i & 1 & 1 \\ -i & -i & 1 & -1 \\ 1 & 1 & -i & i \\ -i & 1 & i & 1 \end{pmatrix}, \quad V_1 = \frac{1}{2} \begin{pmatrix} -1 & i & 1 & i \\ 1 & i & 1 & -i \\ 1 & -i & 1 & i \\ -1 & -i & 1 & -i \end{pmatrix}, \quad V_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}. \tag{38}
\]

Upon receiving the questions \( (r, c) \), Alice applies the unitary \( U_r \) to her qubits \( A_1A_2 \), while Bob applies \( V_c \) to his qubits \( B_1B_2 \). They each measure their respective qubits of the resulting state \( U_r \otimes V_c |\psi\rangle \) in the computational basis and obtain measurement outcomes \( s_0s_1 \) and \( t_0t_1 \). As a last step, they complete their 2-bit outcome with a third bit such that the parity conditions of the magic square game are satisfied: Alice chooses \( s_2 \) such that \( s_0 \oplus s_1 \oplus s_2 = 0 \), while Bob chooses \( t_2 \) such that \( t_0 \oplus t_1 \oplus t_2 = 1 \). A lengthy but
straightforward computation shows that this strategy indeed produces a valid answer pair \((s,t)\) for every possible question pair \((r,c)\).

### C.3 MAC Based on the Magic Square Game

Specializing definition (16) to the magic square game \(G_{MS}\) described in Appendix C, we set \(\mathcal{R} = \{0, 1, 2\}\), \(\mathcal{C} = \mathcal{R}\), \(\mathcal{S} = \{0, 1\}\), \(\mathcal{T} = \mathcal{S}\), and consider the following channel:

\[
N_{G_{MS}} : (\mathcal{R} \times \mathcal{S}) \times (\mathcal{C} \times \mathcal{T}) \rightarrow \mathcal{R} \times \mathcal{S}
\]

\[
N_{G_{MS}}(\hat{r}, \hat{s}|r, s; c, t) := \begin{cases} 
\delta_r \delta_s & \text{if } (r, s; c, t) \in W, \\
\frac{1}{9} & \text{else,}
\end{cases}
\]

where \(W \subset \mathcal{R} \times \mathcal{S} \times \mathcal{C} \times \mathcal{T}\) is the subset of instances \((r, s; c, t)\) winning the magic square game.

Using the perfect quantum strategy for the magic square game detailed in Appendix C.2, for any question pair \((r, c)\) Alice and Bob can produce answers \((s, t)\) such that \((r, s; c, t) \in W\). Hence, with a uniform distribution over the questions \(\mathcal{R} \times \mathcal{C}\) they can achieve the maximal sum rate of \(\log 9 \approx 3.16993\) for the magic-square-MAC (39). To bound the sum rate achievable by classical strategies corresponding to product input distributions on \((\mathcal{R} \times \mathcal{S}) \times (\mathcal{C} \times \mathcal{T})\), our goal is to find the smallest upper bound on \(I(\mathcal{R} \times \mathcal{S} ; Z)\) given by Proposition 3 (we again use capital Latin letters for the random variables corresponding to the question and answer sets, as well as \(Z\) for the channel output random variable):

\[
I(\mathcal{R} \times \mathcal{S} ; Z) \leq \max \{ (1 - \varepsilon^*) \log 9, \log 9 - \delta \} =: u(\delta)
\]

for some \(\delta \in (0, \log \frac{9}{8})\) and the corresponding optimal \(\varepsilon^*\) determined through (24). As explained in Remark 4, we find the optimal \(\delta^* = 0.03299\) (using, e.g., Mathematica), which yields \(\varepsilon(\delta^*) = 0.01040\) and \(I(\mathcal{R} \times \mathcal{S}; Z) \leq u(\delta^*) = 3.13694\). In Figure 5 we plot the upper bound (40) as a function of \(\delta \in [0, \log 9/8]\).

We can compare the upper bound \(u(\delta^*) = 3.13694\) to a lower bound on the sum rate computed by numerically maximizing the mutual information \(I(\mathcal{R} \times \mathcal{S}; Z)\) with respect to product probability distributions (see (6)). Carrying out this optimization in MATLAB in repeated runs using different random starting
points gives a lower bound of 2.84195 on the true maximum. Assuming that this value is close to the true maximum, this result suggests that our upper bound \( u(\delta^*) = 3.13694 \) on the sum rate can likely be further improved. We also computed an inner bound on the capacity region \( \mathcal{C} \) of the MAC (39) using the method detailed in [31, Sec. II.A]. This inner bound on the capacity region is plotted in Figure 6.

![Figure 6: Inner bound on the capacity region](image)

We briefly comment on a different type of entanglement assistance for a MAC where each sender shares entanglement with the receiver. This communication scenario was discussed by Hsieh et al. [26] for quantum multiple access channels \( \mathcal{N}: A'B' \rightarrow C \) mapping quantum systems \( A' \) in Alice’s possession and \( B' \) in Bob’s possession to a quantum system \( C \) in possession of the receiver Charlie. In addition to entanglement assistance, quantum MACs have been studied in various other scenarios [36, 37, 38, 39, 40].

The following capacity region for entanglement-assisted quantum MACs is proved in [26]: Let \( |\phi\rangle_{AA'} \) and \( |\psi\rangle_{BB'} \) be pure quantum states, and set \( \omega_{ABC} = (\text{id}_A \otimes \text{id}_B \otimes \mathcal{N}_{A'B'ightarrow C})(|\phi\rangle_{AA'} \otimes |\psi\rangle_{BB'}) \). Let \( \mathcal{C}_E(\mathcal{N}, \phi, \psi) \) be the set of all non-negative rate pairs \( (R_1, R_2) \) satisfying

\[
\begin{align*}
R_1 &\leq I(A;C|B) \\
R_2 &\leq I(B;C|A) \\
R_1 + R_2 &\leq I(AB;C),
\end{align*}
\]

where the quantum (conditional) mutual informations on the right-hand sides are evaluated on the state \( \omega_{ABC} \). Define \( \bar{\mathcal{C}}_E(\mathcal{N}) \) as the union over all states \( \phi \) and \( \psi \). Then the entanglement-assisted capacity region
\( C_E(N) \) of a quantum MAC \( N \) is equal to

\[
C_E(N) = \bigcup_{n \in \mathbb{N}} \frac{1}{n} \tilde{C}_E(N^\otimes n). \tag{44}
\]

Moreover, we have the following single-letter upper bound on the sum rate:

\[
R_1 + R_2 \leq \max_{\phi_{AA'}^* \otimes N_{BB'}^*} I(AB; C). \tag{45}
\]

We now specialize the above entanglement-assisted setting to classical MACs \( N : A \times B \rightarrow Z \) as introduced in Appendix A.2. Any classical channel necessarily completely dephases a quantum system with respect to some fixed basis. Hence, choosing bases \( \{ |i\rangle_A \} \) and \( \{ |i\rangle_B \} \) and fixing pure quantum states \( \phi_{AA'}^* \) and \( \psi_{BB'}^* \), the joint input state of Alice and Bob for a classical MAC is of the form

\[
\sum_{i,j} p_i p_j \phi_A^i \otimes |i\rangle_A \otimes \psi_B^j \otimes |j\rangle_B^r.
\tag{46}
\]

where \( \{ p_i \} \) with \( p_i = \text{tr}(|i\rangle_A \langle i| \otimes \phi_{AA'}) \) and \( \{ p_j \} \) with \( p_j = \text{tr}(|j\rangle_B^r \langle j| \otimes \psi_{BB'}) \) are probability distributions, and \( \phi_A^i = \frac{1}{p_i} |i\rangle_A \langle i| \otimes \phi_{AA'}^* \) and \( \phi_A^j = \frac{1}{p_j} |j\rangle_B^r \langle j| \otimes \psi_{BB'}^* \). The MAC \( N \) maps the joint input state in (46) to a state

\[
\omega_{ABZ} = \sum_{i,j} p_i p_j N(k|i,j\rangle \langle i|) \phi_A^i \otimes \psi_B^j \otimes |k\rangle_{Z}. \tag{47}
\]

On the other hand, consider a classical state

\[
\theta_{ABZ} = \sum_{i,j} p_i p_j N(k|i,j\rangle \langle i|) \langle i| \otimes |j\rangle_B \otimes |k\rangle_{Z}, \tag{48}
\]

and observe that \( \omega_{ABZ} \) can be obtained from \( \theta_{ABZ} \) by a quantum operation that first measures the (classical) systems \( AB \) in \( \theta \) and depending on the outcome \((i,j)\) prepares the state \( \phi_A^i \otimes \psi_B^j \). Hence, by the data processing inequality for the quantum mutual information we have

\[
I(AB; Z)_{\omega} \leq I(AB; Z)_{\theta}, \tag{49}
\]

and \( I(AB; Z)_{\theta} \) is the classical mutual information with respect to the product probability distribution \( p_i p_j \) appearing in the sum rate constraint for the classical MAC \( N \) given in (6).

From the above discussion and (45), we conclude that for a classical MAC entanglement shared between each sender and the receiver cannot increase the achievable sum rate \( R_1 + R_2 \). In contrast, we showed in this section that entanglement shared between the senders can indeed increase the sum rate up to the maximal value.

### D Linear System Games

In this section we discuss non-local games \( G_{LS} \) based on linear systems of equations \([27]\). Let \( Ax = b \) be an \( m \times n \) linear system of equations over \( \mathbb{F}_2 \). We denote by \( V_i = \{ j \in [n] : A_{ij} \neq 0 \} \) the indices of variables appearing the \( i \)-th equation of the linear system. In the linear system game, Alice receives as a question an index \( i \in [m] \) labeling a row in the linear system. She replies with a vector \( y \in \mathbb{F}_2^n \) of values for \( x \) such that \( \sum_{j \in V_i} y_j = b_i \). Bob receives as a question an index \( j \in [n] \), and he answers with a bit \( x_j \) corresponding to an assignment of the variable \( x_j \). Alice and Bob win the game if either \( j \notin V_i \) or \( y_j = x_j \).
A linear system game $G_{LS}$ defined in terms of a linear system $Ax = m$ can be associated with a certain finitely-presented group $\Gamma(A, b)$ called a solution group. The maximal winning probability using quantum strategies can then be related to approximate representations of $\Gamma(A, b)$ [19]. Slofstra and Vidick [20] showed that suitable approximate representations of $\Gamma(A, b)$ (giving rise to near-perfect quantum strategies) do exist provided the dimension of the representation space, called the hyperlinear profile, is large enough. They exhibited a particular example $G_{SV}$ of a linear system game based on a suitable solution group $\Gamma(A, b)$, for which the above observations can be translated into lower and upper bounds on the local dimension $d$ of any quantum strategy for $G_{SV}$. In terms of the losing probability $p_L = 1 - \omega_U(G_{SV})$ and constants $C, C'$, the following bounds are proved in [20]:

$$\frac{C}{p_L^{1/6}} \leq d \leq \frac{C'}{p_L^{1/2}}. \tag{50}$$

D.1 Limiting the Entanglement Assistance

**Proposition 5.** If Alice and Bob are constrained to quantum strategies with dimension at most $d$, then the sum rate capacity of $N_{G_{SV}}$ is bounded away from perfect, i.e., $\log m + \log n$, by $\Theta(1/d^9)$.

**Proof.** Let $G_{SV}$ be the linear system game defined in [20]. By the discussion above and (50), we have the following lower bound for the losing probability if Alice and Bob only use $d$-dimensional quantum strategies:

$$1 - \omega_U(G_{SV}) \geq \frac{C_1}{d^6}, \tag{51}$$

for some constant $C_1 > 0$. In order to use (3), we let $\delta = \frac{C_1}{d^6}$ and assume that $\varepsilon^* < \delta$. For large $d$, we can upper-bound the left-hand side of eq. (24) by

$$\frac{\delta + h(\delta)}{1 - \delta} \geq \delta(\varepsilon^* \| 1 - \omega_U(G_{SV})),$$  \tag{52}

where we used $h(\varepsilon^*) \leq h(\delta)$ whenever $\delta < \frac{1}{2}$. Next, observe that for $\delta \in [0, \frac{1}{2}]$ the binary entropy term $h(\delta)$ is upper-bounded by $a\delta^\alpha$ for $\alpha < 1$ and $a$ large enough. Letting $\alpha = \frac{25}{26}$, we can underestimate the right-hand side via Pinsker’s inequality and get

$$\delta(\varepsilon^* \| 1 - \omega_U(G_{SV})) \geq \frac{2}{\ln 2} \left[ \varepsilon^* - (1 - \omega_U(G_{SV})) \right]^2 \geq \frac{2}{\ln 2} \left[ \varepsilon^* - \frac{C_1}{d^6} \right]^2 \geq \frac{2}{\ln 2} \left[ \frac{C_1^2}{d^{12}} - 2\delta \frac{C_1}{d^6} \right].$$

Putting it all together, we conclude the following inequality

$$\frac{\delta + a\delta^\alpha}{1 - \delta} \geq \frac{2C_1^2}{\ln 2} \left[ \frac{1}{d^{12}} - 2\frac{1}{d^6} \right]. \tag{53}$$

Observe that as $d$ goes to infinity, the right-hand side goes as $1/d^{12}$, while the left-hand side goes as $1/d^{12.5}$. At some large enough $d$, (54) is violated. $\varepsilon^*$ cannot be smaller than $\delta$ for large enough $d$. Hence, by (3) we have the following upper bound on the sum rate capacity:

$$I(X_1Y_1X_2Y_2; Z) \leq \log n + \log m - \frac{C_1}{d^8}. \tag{55}$$

Since $d$-dimensional quantum strategies subsume all lower dimensional strategies, this converse provides a limit, if implicit, on how well $N_{G_{SV}}$ can be used for strategies with small dimension.
D.2 Achievable Strategies Using $d$-dimensional Maximally Entangled States

In this section we prove the existence of a sequence of coding strategies for the MAC $N_{G_{SV}}$ defined in terms of the $m \times n$-linear system game $G_{SV}$ described above that achieves the rate pair $(\log m, \log n)$ in the achievable rate region in the limit $d \to \infty$.

**Proposition 6.** Let $G_{SV}$ be the linear system game from [20] associated with the $m \times n$-linear system $Ax = b$, and let $N_{G_{SV}}$ be the MAC defined in terms of $G_{SV}$ via (16). Assume that the two players share a maximally entangled state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ of Schmidt rank $d$ sufficiently large. Then there is a coding strategy that achieves the rate pair $(\log m, \log n)$ in the limit $d \to \infty$.

For this coding strategy, both the losing probability $p_L$ and the function $f(d)$ vanish in the limit $d \to \infty$.

**Proof.** In order to prove the claim of the proposition, we make use of the following easily-verifiable entropic inequalities:

$$I(A; B) - I(A; B|D) = -H(A|B) + I(A; D) + H(A|BD) \geq -H(A|B)$$

$$I(A; B|C) - I(A; B|CD) = I(A; D|C) + H(D|ABC) - H(D|BC) \geq -H(D|BC)$$

$$I(A; B|CD) - I(A; BC|D) = -I(A; C|D)$$

Let $X_i, Y_i$ be the random variables associated to the questions and answers for players $i = 1, 2$, let $Z$ be the random variable associated to the output of the MAC $N_{G_{SV}}$, and let $W$ be the random variable indicating a win defined in (28). We fix the following coding strategy: Alice and Bob draw the questions $x_1$ and $x_2$ uniformly at random, and produce $y_1$ and $y_2$ using the quantum strategy detailed in [20] based on measuring a maximally entangled state $\psi$, as depicted in Figure 7. In terms of the general entanglement-assisted coding scenario described in Appendix A.3, this corresponds to setting $A_i = X_i$, $X'_i = Y_i$, and using the trivial post-processing $f_i(x_j, y_j|x_i, y_i) = \delta_{x_i, x_j}\delta_{y_i, y_j}$. By the right-hand inequality in (50) (which is proved in Theorem 1.1 in [20]), the above strategy has losing probability

$$p_L \leq \left(\frac{C'}{d}\right)^2$$

for some constant $C'$. 

---

Figure 7: Coding strategy for the MAC $N_{G_{SV}}$ associated with the linear system game $G_{SV}$ defined in Appendix D, where $P$ is the correlation produced by the quantum strategy detailed in [20].
Due to (61), the winning probability satisfies

\[ I(Z; X_1|X_2) \geq I(Z; X_1|X_2W) - H(W|X_1X_2) \]

\[ = I(Z; X_1X_2|W) - I(Z; X_2|W) - H(W|X_1X_2) \]

\[ \geq I(Z; X_1X_2|W) - I(Z; X_2|W) - h(p_L) \]

\[ = (1 - p_L)[H(X_1X_2|W = 1) - H(X_2|W = 1)] - h(p_L) \]

\[ \geq (1 - p_L)[H(X_1X_2|W = 1) - \log n] - \frac{p_L}{2} \log(nm - 1) - h(p_L). \]  

In the second line we used (60) and in the third line we used \( H(W|X_1X_2) \leq H(W) \leq h(p_L) \). In the fourth line we used that, if Alice and Bob win the game (\( W = 1 \)), then the variable \( Z \) is a deterministic function of \( X_1X_2 \) and hence \( I(Z; X_1Y_1X_2Y_2|W = 1) = H(X_1X_2|W = 1) \), together with the fact that \( I(Z; X_1Y_1X_2Y_2|W = 0) = 0 \). Finally, in the last line we used the trivial bound \( H(X_2|W = 1) \leq \log |X_2| = n \) as well as the fact that \( \frac{p_L}{2} \log(nm - 1) \geq 0 \).

We now bound the entropy \( H(X_1X_2|W = 1) \) in (66) by considering the probability distribution

\[ \pi_{X_1X_2}^W = \{ \Pr(X_1X_2 = x_1x_2|W = 1) \}_{x_1,x_2}. \]  

Our goal is to show that \( \pi_{X_1X_2}^W \) converges to the uniform distribution \( \pi_U \) on \( X_1 \times X_2 \) in total variation distance as \( d \to \infty \). By continuity of entropy, this then implies that \( H(X_1X_2|W = 1) \approx \log m + \log n \) with the approximation error vanishing in the limit \( d \to \infty \).

To show this claim, we use Bayes’ theorem to express \( \Pr(X_1X_2 = x_1x_2|W = 1) \) as

\[ \Pr(X_1X_2 = x_1x_2|W = 1) = \frac{\Pr(W = 1|X_1X_2 = x_1x_2) \Pr(X_1X_2 = x_1x_2)}{\Pr(W = 1)} \]

\[ = \frac{1}{nm} \frac{\Pr(W = 1|X_1X_2 = x_1x_2)}{\Pr(W = 1)}. \]  

Due to (61), the winning probability satisfies

\[ \Pr(W = 1) = 1 - p_L \geq 1 - (C'/d)^2. \]  

Moreover, by Lemma 4.2 in [20] every strategy that achieves a winning probability of at least \( 1 - p_L \) wins with probability \( 1 - nmp_L \) on any question \( (x_1, x_2) \), and hence

\[ \Pr(W = 1|X_1X_2 = x_1x_2) \geq 1 - nmp_L \geq 1 - nm(C'/d)^2. \]  

For the total variation distance \( d_{TV}(\pi_{X_1X_2}^W, \pi_U) \), the bounds (70) and (71) imply that

\[ d_{TV}(\pi_{X_1X_2}^W, \pi_U) = \frac{1}{2} \sum_{x_1, x_2} \left| \Pr(X_1X_2 = x_1x_2|W = 1) - \frac{1}{nm} \right| =: f(d) \]  

for some non-negative function \( f(d) \) that converges to zero as \( d \to \infty \). By the continuity of entropy [41],

\[ H(X_1X_2|W = 1) \geq \log m + \log n - f(d) \log(nm - 1) - h(f(d)), \]  

and substituting this in (66) yields \( R_1 \leq I(Z; X_1|X_2) \) with

\[ R_1 := (1 - p_L) \log m - (1 - p_L)(f(d) \log(nm - 1) + h(f(d))) - \frac{p_L}{2} \log(nm - 1) - h(p_L). \]
Using similar steps as above, we can also show that \( R_2 \leq I(Z; X_2|X_1) \) with
\[
R_2 := (1 - p_L) \log n - (1 - p_L)(f(d) \log(nm - 1) + h(f(d))) - \frac{p_L}{2} \log(nm - 1) - h(p_L). \tag{75}
\]

For the rate pair \((R_1, R_2)\) to be achievable, it remains to be shown that \( R_1 + R_2 \) satisfies the sum rate constraint \( R_1 + R_2 \leq I(Z; X_1X_2) \). To this end, we use (58) with the choices \( A = Z, B = X_1X_2, D = W \) to obtain
\[
I(Z; X_1X_2) \geq I(Z; X_1X_2|W) - H(Z|X_1X_2)
= (1 - p_L)H(X_1X_2|W = 1) - H(Z|X_1X_2)
\geq (1 - p_L)(\log m + \log n) - (1 - p_L)(f(d) \log(nm - 1) + h(f(d))) - H(Z|X_1X_2),
\tag{76}
\]
which follows from the discussion above and (73). To bound the conditional entropy \( H(Z|X_1X_2) \), note that \( \Pr(Z \neq X_1X_2) = p_L \frac{nm-1}{nm} \leq p_L \), and hence we can apply Fano’s inequality to obtain the bound
\[
H(Z|X_1X_2) \leq p_L \log(nm - 1) + h(p_L). \tag{77}
\]
Substituting this in (78) yields
\[
I(Z; X_1X_2) \geq (1 - p_L)(\log m + \log n) - (1 - p_L)(f(d) \log(nm - 1) + h(f(d))) - p_L \log(nm - 1) - h(p_L)
\geq (1 - p_L)(\log m + \log n) - 2(1 - p_L)(f(d) \log(nm - 1) + h(f(d))) - p_L \log(nm - 1) - 2h(p_L)
\tag{80}
= R_1 + R_2 \tag{81}
\]
with \( R_1 \) and \( R_2 \) as in (74) and (75), respectively. This finishes the proof. \( \square \)

By Proposition 6, the achievable rate region \( C_{ea,d}^{(1)}(N_{GSV}) \) gets arbitrarily close to the rate pair \((\log m, \log n)\) in the limit \( d \to \infty \). Hence, we have the following result:

**Corollary 7.** Let \( G_{SV} \) be the linear system game from [20] associated with the \( m \times n \)-linear system \( Ax = b \), and let \( N_{GSV} \) be the MAC defined in terms of \( G_{SV} \) via (16). Then the rate pair \((\log m, \log n)\) is contained in the closure of \( C_{ea,d}^{(1)}(N_{GSV}) \).

### D.3 Undecidability of the Rate Region of a MAC

Propositions 5 and 6 show that there is a MAC \( N_{GSV} \) defined in terms of the \( m \times n \)-linear system game \( G_{SV} \) such that the sum rate capacity \( S_{ea,d}(N_{GSV}) \) is bounded away from \((\log m, \log n)\) for any finite \( d \), but the boundary of the \( d \)-entanglement-assisted single-letter capacity region \( C_{ea,d}^{(1)}(N_{GSV}) \) gets arbitrarily close to \((\log m, \log n)\) in the limit \( d \to \infty \).

Using a recent result by Slofstra [19], we can even prove the following: for a general linear system game \( G_{LS} \) and the corresponding MAC \( N_{GLS} \), it is undecidable to determine if the maximal rate \((\log m, \log n)\) is achievable with finite-dimensional entanglement assistance:

**Proposition 8.** Let \( N_{GLS} \) be the MAC defined via (16) in terms of an \( m \times n \)-linear system game \( G_{LS} \). Then it is undecidable to determine if the rate pair \((\log m, \log n)\) belongs to \( C_{ea}^{(1)}(N_{GLS}) \).

**Proof.** Let \( G_{LS} \) be the game associated to the \( m \times n \)-linear system \( Ax = b \). Then Corollary 1.3 in [19] proves that it is undecidable to determine if \( G_{LS} \) has a perfect strategy in the set of finite-dimensional quantum
correlations as defined in (10). If there is a perfect strategy, then by the construction of \( N_{G_{LS}} \) the two senders can code at the rate pair \((\log m, \log n)\) by drawing the questions \( x_i \) uniformly at random and using the perfect strategy to produce \( y_i \) such that \((x_1, y_1, x_2, y_2) \in W\). Conversely, if there is no perfect strategy and hence \( \omega_U(G_{LS}) < 1 \), then for any finite \( d \) the sum rate capacity \( S_{ea,d}(N_{G_{LS}}) \) can be bounded away from \( \log m + \log n \) using Proposition 3, and this separates the point \((\log m, \log n)\) from the entanglement-assisted rate-region \( C_{ea}^{(1)}(N_{G_{LS}}) \). Hence, the pair \((\log m, \log n)\) belongs to \( C_{ea}^{(1)}(N_{G_{LS}}) \) if and only if there is a perfect strategy for \( G_{LS} \), which is undecidable.

**E Hardness of Computing the Capacity Region of MACs**

Despite the availability of a single-letter characterization, as given by (6), computing the capacity region of an arbitrary multiple access channel is a difficult task [32]. The difficulty lies in the inherent non-convexity of the problem, i.e., the optimization is constrained to be over product distributions [31]. In this section, we show that deciding if a MAC can be used perfectly or not (up to \( \Theta(\frac{1}{n^3}) \)) is NP-hard. This implies that deciding if an arbitrary point \((R_1, R_2)\) belongs to the capacity region to \( \Theta(\frac{1}{n^3}) \) precision is NP-hard.

**E.1 The PCP Theorem**

The results to follow rely on the probabilistically checkable proofs (PCP) theorem, which says that any language in the class NP admits a characterization via probabilistically checkable proofs [42, 43, 44]. More formally, let \( \text{PCP}_{c,s}[r(n),q(n)] \) be the class of all languages \( L \) such that there exists a verifier \( V \), which is free to use \( O(r(n)) \) random bits and query a given proof \( O(q(n)) \) times, with the following properties:

1. Completeness: If \( x \in L \), then there exists a proof \( P \) such that \( V \) accepts with probability at least \( c \).
2. Soundness: If \( x \notin L \), then \( V \) accepts with probability at most \( s \).

Note that this can be considered a generalization of NP as \( \text{NP} = \text{PCP}_{1,0}[0,\text{poly}(n)] \). The original PCP theorem says that \( \text{NP} \subseteq \text{PCP}_{1,1/2}[\log n,1] \) [45]. To illustrate its implications, consider the canonical NP-complete language 3SAT for example. Take a Boolean formula \( \psi \) in 3-conjunctive normal form (3CNF), i.e., it is a conjunction of clauses that are disjunctions of three literals. Note that a literal can be a Boolean variable or its negation. Say \( \psi \notin 3\text{SAT} \). A verifier exists such that, with access to logarithmic randomness and a constant number of queries to a given proof or witness, it will reject with non-trivially high probability. This suggests that proving a falsehood, e.g., \( \psi \in 3\text{SAT} \) when that is not the case, typically involves making many errors.

The PCP theorem can be equivalently formulated as a statement about the hardness of approximating NP-complete problems [46, 33]. We will restrict our attention to the following formulation.

**Theorem 9** (PCP theorem; [21, 22]). Given a 3-CNF-5 Boolean formula \( \psi \), to decide whether \( \psi \) has a satisfying assignment or that every assignment violates at least \((1 - c)\) fraction of the clauses in \( \psi \) is NP-hard, for some constant \( c < 1 \).

Here, a formula \( \psi \) is called 3-CNF-5 if it is a conjunction of \( m \) clauses and each clause is a disjunction of exactly three distinct literals and each of the \( n \) Boolean variables appears in exactly five clauses. Remark that the number of clauses \( m \) is \( \Omega(n) \). We call \( \psi \) at most \( c \)-satisfiable for some \( c \in [0, 1] \) if some assignment satisfies \( f \) fraction of its clauses, for \( f \in [0, c] \), and no assignments satisfies more than \( c \) fraction of its clauses.
E.2 The Basic Two-Prover Game

We denote by $G_H$ the non-local game version of the basic two-prover protocol introduced in [33]. Namely, given a 3-CNF-5 Boolean formula $\psi = C_1 \land C_2 \land ... \land C_m$ as input, where $C_j = y_{a_j} \lor y_{b_j} \lor y_{c_j}$, the referee does the following:

1. Choose an integer $j \in \{1,...,m\}$ uniformly at random and send $j$ to Alice. Choose $k \in \{a_j, b_j, c_j\}$ uniformly at random and send $k$ to Bob.

2. Receive an assignment for $C_j$ from Alice and a truth value for $x_k$ from Bob. They win if Alice’s answer satisfies $C_j$ and the two agree on the value of $x_k$, otherwise they lose.

Let $\psi$ be at most $c$-satisfiable. Because the optimal strategy is deterministic, Bob will have an assignment to $\psi$. If the clause in the question to Alice is violated by Bob’s assignment, then the best Alice can do is disagree with Bob on the value of one Boolean variable in the clause and hope that Bob did not receive it as a question. This implies that $\omega(G_H) \leq \frac{2+\varepsilon}{3}$. Conversely, $\omega(G_H) \leq \frac{2+\varepsilon}{3}$ implies that $\psi$ is at most $c$-satisfiable.

E.3 Hardness Result

If the game is made promise-free, then it follows that it is NP-hard to decide if $\omega_U(G_H) = 1$ or $\omega_U(G_H) \leq 1 - (\frac{1-c}{n})$.

Proposition 10. It is NP-hard to decide if the sum capacity of the MAC associated with the promise-free version of $G_H$ is equal to its maximum value $\log m + \log n$ or it is bounded away from it by $\Theta(\frac{1}{n^2})$.

Proof of Proposition 10. Observe that if $\psi$ has a satisfying assignment, then the two senders can use the channel perfectly, i.e., $R_1 = \log m$ and $R_2 = \log n$. On the other hand, if $\psi$ has no satisfying assignment, then $\omega_U(G_H)$ is strictly less than 1. Hence, we can use Proposition 3 to make statements about the sum capacity in a manner similar to (5). Let $\delta = (\frac{1-c}{n^2})^2$ and assume that $\varepsilon^* < \delta$. For large $n$, we can overestimate the left-hand side of eq. (24) by

$$\frac{\delta + b\delta^\beta}{1 - \delta} \geq \delta(\varepsilon^* || 1 - \omega(U(G_H))),$$

(83)

where $\beta = \frac{5}{6}$ and $b$ is taken to be large enough. Again, we use Pinsker’s inequality to lower bound the right hand side.

$$\frac{\delta + b\delta^\beta}{1 - \delta} \geq \frac{2(1 - c)^2}{\ln 2} \left[ \frac{1}{n^2} - 2 \frac{1}{n^4} \right]$$

(84)

As $n$ goes to infinity, the right-hand side goes as $1/n^2$, while the left-hand side goes as $1/n^{2.5}$. $\varepsilon^*$ cannot be smaller than $\delta$ for large enough $n$. Therefore, whenever $\omega_U(G_H) < 1$, i.e., $\psi$ has no satisfying assignment, we conclude from eq. (23) that for all large enough $n$,

$$I(X_1Y_1X_2Y_2; Z) \leq \log m + \log n - \frac{(1 - c)}{n^3}$$

(85)

The proposition follows from here via the PCP theorem. □
It is instructive to compare this hardness result with the time complexity of the popular Airmoto-Blahut (AB) algorithm for computing the point-to-point discrete channel capacity [28, 29]. If we consider the two senders together, then the channel capacity is the solution to a convex program. The number of iterations needed in order to have $\frac{1}{\epsilon^3}$ additive precision for the capacity using the AB algorithm is $O(n^3 \log n)$ in the worst case. Assuming P $\neq$ NP, there is no polynomial-time algorithm to get to within the same precision for the boundary of the capacity region of an arbitrary discrete MAC. Moreover, assuming the exponential time hypothesis, there is no sub-exponential algorithm to compute the boundary of the region to inverse cubic precision. In such a case, the “naive” method of covering the space of product probability distributions with a net and computing an approximation of the capacity region is more or less optimal, which can be seen as follows.

Let $K \subseteq \mathbb{R}^n$ be a subset of the Euclidean space $\mathbb{R}^n$ and $\epsilon > 0$. An $\epsilon$-net for $K$ is a subset $N \subseteq K$ such that every point of $K$ is within distance $\epsilon$ of a net point in $N$. We denote by $C(K, \epsilon)$ the covering number of $K$, defined as the smallest possible cardinality of an $\epsilon$-net $N$ for $K$. By a standard volume argument, $C(K, \epsilon)$ is bounded from below as

$$C(K, \epsilon) \geq \frac{|K|}{|B^n_\epsilon|},$$

where $|K|$ denotes the (Euclidean) volume of $K$ embedded in $\mathbb{R}^n$, and $B^n_\epsilon$ is the $n$-ball with radius $\epsilon$. Let now $K = \Delta_n$ be the $n$-probability simplex, and recall that

$$|\Delta_n| = \frac{\sqrt{n}}{(n-1)!} \quad \text{and} \quad |B^n_\epsilon| = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} n^\epsilon.$$

Here, $\Gamma(\cdot)$ is the well-known Gamma function, satisfying $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$. Using the Stirling approximation $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$ as well as $\epsilon = \text{poly}(n)^{-1}$, we obtain from (86) that $C(K, \epsilon) = \Omega(\text{poly}(n)^n)$. 

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