Bounding Ornstein-Uhlenbeck Processes and Alikes

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Abstract

In this note we consider SDEs of the type $dX_t = [F(X_t) - AX_t]dt + DdW_t$ under the assumptions that $A$’s eigenvalues are all of positive real parts and $F(·)$ has slower-than-linear growth rate. It is proved that $\lim_{t \to \infty} \frac{\|X_t\|}{\sqrt{\log t}} = \sqrt{2\lambda_1}$ almost surely with $\lambda_1$ being the largest eigenvalue of the matrix $\Sigma := \int_{-\infty}^{\infty} e^{-sA} \cdot (D \cdot D^T) \cdot e^{-sA^T} ds$; the discarded measure-zero set can be chosen independent of the initial values $X_0 = x$.

1 Introduction

It’s well known that, for a given one-dimensional stationary Ornstein-Uhlenbeck (OU for short) process $X = \{X_t : t \geq 0\}$ there exist $\lambda, \sigma > 0, \mu \in \mathbb{R}$ and a standard Brownian Motion (BM for short) $B(·)$ such that $X$ has the same distribution as $\{\sigma \cdot e^{-\frac{\lambda t}{2}} \cdot B(e^{\lambda t}) + \mu : t \geq 0\}$. Therefore the law of iterated logarithm for BM (see, e.g., [1]) leads us to the conclusion $X_t = O(\sqrt{\log t})$ almost surely. In this note we investigate what bounds can we achieve for higher dimensional OU processes $X = \{X_t : t \geq 0\}$ and alikes which may be modeled by the following SDE (of dimension $d \geq 2$)

$$dX_t = [F(X_t) - AX_t]dt + DdW_t,$$  \hspace{1cm} (1.1)

where $D$ is a constant $d$-by-$d$ matrix. And we always assume the following conditions:

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(C1) All the eigenvalues of the $d$-by-$d$ matrix $A$ have positive real parts;

(C2) $F(x) = o(\|x\|)$ (as $\|x\| \to \infty$) is a continuous $\mathbb{R}^d$-valued function. Here $\| \cdot \|$ denotes the standard Euclidean norm.

Our main result can be stated as the following.

**Theorem 1** The solution to (1.1) always satisfies

$$\lim_{t \to \infty} \frac{\|X_t\|}{\sqrt{\log t}} = \sqrt{2\lambda_1} \text{ almost surely},$$

where $\lambda_1$ is the largest eigenvalue of the matrix $\Sigma := \int_0^\infty e^{-sA} \cdot (D \cdot D^T) \cdot e^{-sA^T} ds$. Here the discarded measure-zero set can be chosen independent of the initial values $X_0 = x$.

Such result seems to be new in literature as to our knowledge and deserves a publication somewhere. The proof of the main theorem, based mainly on the well-known fact mentioned at the beginning of the introduction and on elemental linear algebra, is presented in Sect. 2 and Sect. 3; the calculation of the precise limit value in (1.2) is based mainly on [2], see Sect. 2.

2 **OU Processes Case: $F = 0$**

In this part, we consider the simpler case of $F = 0$, i.e., the follow model

$$dX_t = -AX_t dt + D dW_t. \tag{2.1}$$

Clearly the solution satisfies the follow formula

$$X_t = e^{-tA}X_0 + \int_0^t e^{-(t-s)A}D dW_s. \tag{2.2}$$

When $X_0 \sim N(0, \Sigma)$ with $\Sigma := \int_0^\infty e^{-sA} \cdot (D \cdot D^T) \cdot e^{-sA^T} ds$, $\{X_t : t \geq 0\}$ is a stationary Markov process.
Throughout this section, we will use $B_t$ in denoting one dimensional standard BM and write $W_t$ for higher dimensional standard BM.

As we have addressed in the introduction, any one dimensional stationary OU process is of growth rate $O(\sqrt{\log t})$. This result can be restated as the following lemma, whose proof is omitted.

**Lemma 2** For any $\lambda > 0$, almost surely we have

$$\int_0^t e^{-\lambda(t-s)} dB_s = O(\sqrt{\log t}).$$

Based on the above lemma, we would prove the following three lemmas.

**Lemma 3** For $\lambda > 0$ and any $k \in \mathbb{N}$, almost surely we have

$$\int_0^t \frac{(t-s)^k}{k!} \cdot e^{-\lambda(t-s)} dB_s = O(\sqrt{\log t}).$$

**Lemma 4** For $\lambda > 0, \mu \neq 0$, almost surely we have

$$\int_0^t e^{-\lambda(t-s)} \cdot \cos \mu (t-s) dB_s = O(\sqrt{\log t}),$$

$$\int_0^t e^{-\lambda(t-s)} \cdot \sin \mu (t-s) dB_s = O(\sqrt{\log t}).$$

**Lemma 5** For $\lambda > 0, \mu \neq 0$ and any $k \in \mathbb{N}$, almost surely we have

$$\int_0^t \frac{(t-s)^k}{k!} \cdot e^{-\lambda(t-s)} \cdot \cos \mu (t-s) dB_s = O(\sqrt{\log t}),$$

$$\int_0^t \frac{(t-s)^k}{k!} \cdot e^{-\lambda(t-s)} \cdot \sin \mu (t-s) dB_s = O(\sqrt{\log t}).$$

**Proof of Lemma**

Put $Y_t := \int_0^t e^{-\lambda(t-s)} dB_s$ and

$$L(t) := \sup_{u \in [0,t]} |Y_u|, \quad I_t^{(k)} := \int_0^t \frac{(t-s)^k}{k!} \cdot e^{-\lambda(t-s)} dB_s.$$

Clearly

$$I_t^{(1)} = \int_0^t e^{-\lambda(t-u)} Y_u du, \quad I_t^{(k+1)} = \int_0^t e^{-\lambda(t-u)} I_u^{(k)} du, k \geq 1.$$
And

\[ |I_t^{(1)}| \leq \int_0^t e^{-\gamma(t-u)}|Y_u|du \leq \int_0^t e^{-\gamma(t-u)}L(u)du \leq L(t)/\lambda. \]

Lemma \( \text{[2]} \) tells us \( L(t) = O(\sqrt{\log t}) \). Hence \( I_t^{(1)} = O(\sqrt{\log t}) \). Inductively \( I_t^{(k)} = O(\sqrt{\log t}) \) for all \( k \in \mathbb{N} \).

\[ \square \]

Proof of Lemma \( \text{[2]} \). For any \( \theta \in \mathbb{R} \), we write

\[ R_\theta := \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}. \]

These are rotations which preserve the distance induced by the standard norm \( \| \cdot \| \) on \( \mathbb{R}^2 \).

Consider the following diffusion process

\[ X_t := \int_0^t e^{-\gamma(t-s)} \cdot R_{-\mu(t-s)} dW_s, \]

where \( W = (W^1, W^2)^T \) is a 2-dimensional standard BM. Define

\[ \widetilde{W}_t := \int_0^t R_{\mu t} dW_s. \]

It is easy to see that \( \widetilde{W} \) is still a 2-dimensional standard BM. And

\[ X_t = \int_0^t e^{-(t-s)} \cdot R_{-\mu t} \widetilde{W}_s. \]

Now in view of Lemma \( \text{[2]} \) it is clear that

\[ \| X_t \| = \| \int_0^t e^{-(t-s)} \widetilde{W}_s \| = O(\sqrt{\log t}). \]

Thus

\[ \int_0^t e^{-\gamma(t-s)} \cdot \left[ \cos \mu(t-s) dW^1_s - \sin \mu(t-s) dW^2_s \right] = O(\sqrt{\log t}). \]

Similarly,

\[ \int_0^t e^{-\gamma(t-s)} \cdot \left[ \cos \mu(t-s) dW^1_s + \sin \mu(t-s) dW^2_s \right] = O(\sqrt{\log t}). \]

The lemma follows from the above equations. \[ \square \]
Proof of Lemma 5. Now for any $k \geq 1$ (fixed), consider

$$X_t := \int_0^t \frac{(t-s)^k}{k!} \cdot e^{-\lambda (t-s)} \cdot R_{\mu(t-s)} \, dW_s,$$

where $R$ is introduced in the proof of Lemma 4. It is easy to see that

$$\|X_t\| = \left\| \int_0^t \frac{(t-s)^k}{k!} \cdot e^{-\lambda (t-s)} \, \widetilde{d}W_s \right\|,$$

where $\widetilde{W}$ is also introduced in the proof of Lemma 4. Now Lemma 3 tells us $\|X_t\| = O(\sqrt{\log t})$ and the rest proof follows smoothly as in the proof of Lemma 4.

From the above four lemmas we easily prove the bound $O(\sqrt{\log t})$ for the solutions to (2.1) via exploiting the standard Jordan form of $A$ (and hence of $e^{-(t-s)A}$) in formula (2.2). The fact that the $\lim$ in (1.2) is constant almost surely follows from the ergodic property of the stationary OU process.

Now we calculate the $\lim$ in (1.2) explicitly via [2]: Without loss of generality, assume $\Sigma$ to be invertible. Take $V(x) = \frac{1}{2} x^T \Sigma^{-1} x$, the result in [2] tells us that $\lim_{t \to \infty} \frac{V(X_t)}{\sqrt{\log t}} \leq 1$ almost surely, which implies

$$\lim_{t \to \infty} \frac{\|X_t\|}{\sqrt{\log t}} = c \leq \sqrt{2\lambda_1} \text{ almost surely.}$$

Now let $\alpha$ be a unit eigen-vector of $\Sigma$ corresponding to $\lambda_1$. It is easy to see that $Y := \{Y_n := \alpha^T X_n/\sqrt{\lambda_1} : n \geq 0\}$ is a stationary Gaussian process with steady distribution $N(0,1)$; This process inherits the exponential mixing property from $X$. A standard result says that for i.i.d. standard normal random variables $\{Z_n : n \geq 0\}$, we always have $\lim_{n \to \infty} \frac{|Z_n|}{\sqrt{2 \log n}} = 1$ almost surely. With a tedious but routine effort (which we omit the details here), it is not hard to see that we still have $\lim_{n \to \infty} \frac{|Y_n|}{\sqrt{2 \log n}} = 1$ almost surely for the new process $Y$. Therefore

$$c = \lim_{t \to \infty} \frac{\|X_t\|}{\sqrt{\log t}} \geq \lim_{n \to \infty} \frac{\|X_n\|}{\sqrt{\log n}} = \sqrt{2\lambda_1}.$$ 

Hence $c = \sqrt{2\lambda_1}$. And (1.2) follows.
3 General Case: $F \neq 0$

Now we consider the general case with $F \neq 0$. As is known, the solution to (1.1) satisfies

$$X_t = e^{-tA}X_0 + \int_0^t e^{-(t-s)A}F(X_s)ds + \int_0^t e^{-(t-s)A}DdW_s. \quad (3.1)$$

Define

$$L(t) = \sup_{u \in [0,t]} \| \int_0^u e^{-(u-s)A}DdW_s \|.$$ 

Clearly $L(t) = O(\sqrt{\log t})$.

Since $A$ satisfies condition (C1), there exist $\lambda_0 > 0$ and $K > 0$ such that

$$\|e^{-tA}\| \leq K \cdot e^{-\lambda_0 t}, \quad \forall t \geq 0. \quad (3.2)$$

Fix an arbitrarily small $\varepsilon > 0$ (with $\varepsilon < \frac{\lambda_0}{K}$), by assumption (C2) there exists $C = C(\varepsilon) > 0$ such that

$$\|F(x)\| \leq C + \varepsilon \|x\|, \quad \forall x \in \mathbb{R}^d. \quad (3.3)$$

In view of (3.1) we have

$$\|X_t\| \leq Ke^{-\lambda_0 t}\|X_0\| + \int_0^t Ke^{-\lambda_0(t-s)}\left(C + \varepsilon \|X_s\|\right)ds + L(t), \quad \forall t \geq 0.$$ 

Define

$$f(t) := K\|X_0\| + L(t) + KC/\lambda_0, \quad \varepsilon_0 := K\varepsilon, \quad u(t) := \|X_t\|.$$ 

Then $u(\cdot)$ can be regarded as a continuous positive function (almost surely) which satisfies the following inequality

$$u(t) \leq f(t) + \varepsilon_0 \int_0^t e^{-\lambda_0(t-s)}u(s)ds, \quad \forall t \geq 0. \quad (3.4)$$

Put $\varphi(t) := \int_0^t e^{\lambda_0 s}u(s)ds$, we have

$$\frac{d\varphi}{dt}(t) \leq f(t) \cdot e^{\lambda_0 t} + \varepsilon_0 \varphi(t), \quad t \geq 0.$$
which implies (where $\lambda := \lambda_0 - \varepsilon_0$)

$$\varphi(t) \leq e^{\varepsilon_0 t} \cdot \int_0^t f(s) \cdot e^{\lambda s} ds, \quad t \geq 0.$$ 

Thus, noting (3.4) and the monotonicity of $f$, we have

$$u(t) \leq f(t) + \varepsilon_0 e^{-\lambda_0 t} \cdot \varphi(t) \leq f(t) + \varepsilon_0 \int_0^t f(s)e^{-\lambda(t-s)} ds \leq [1 + \frac{\varepsilon_0}{\lambda_0 - \varepsilon_0}] \cdot f(t) = \frac{\lambda_0}{\lambda_0 - \varepsilon_0} \cdot f(t).$$

This implies that the solution $X_t$ has almost the same growth rate as $\int_0^t e^{-(t-s)A} D dW_s$.

Specifically we always have $\|X_t\| = O(\sqrt{\log t})$ almost surely. Clearly the limit value in (1.2) is coincident with that of the OU case (i.e., the case $F = 0$).

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