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Uncertainties associated with position, size and shape for point cloud data

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Abstract. In this paper, we show how a variance matrix associated with a point cloud can be decomposed into components that account for uncertainty in position, size and shape. The decomposition is related to how the frame of reference of the point cloud is specified and it is shown that the decomposition corresponds to a frame of reference specification that minimises an aggregate measure of uncertainty based on the trace of a variance matrix. The decomposition provides a way of presenting uncertainty information associated with calibrated reference artefacts in which the contribution from uncertainty in position is minimised.

1. Introduction
In coordinate metrology, the outputs of the measurement are a set of point coordinates \( \{x_i, i = 1, \ldots, m\} \) along with an uncertainty statement that provides some measure of the accuracy of the coordinates. Given a model of the measuring system, for example a kinematic error model for a coordinate measurement machine, e.g., [3, 9], or a model for a network of laser trackers [5, 8], it is usually possible to construct a \( 3m \times 3m \) variance matrix associated with the point coordinates using a methodology based on the GUM [1, 2]. This variance matrix will reflect all the influence factors for the measurement. For example, in a photogrammetry, scale information can be provided by measuring a scale bar. The uncertainty associated with the calibration of the scale bar will propagate through to the variance matrix associated with the point cloud. In this paper, we show how a variance matrix associated with a point cloud can be decomposed into components that account for uncertainty in position, size and shape. The decomposition is related to how the frame of reference of the point cloud is specified and it is shown that the decomposition corresponds to a frame of reference specification that minimises an aggregate measure of uncertainty based on the trace of a variance matrix. One important application of the decomposition is that it provides a way of presenting uncertainty information associated with calibrated 3 dimensional reference artefacts in which the contribution from uncertainty in position is minimised.

2. Variance decomposition in terms of position, size and shape
Given a set of \( xyz \)-coordinates \( \{x_i, i = 1 : m\} \), let \( x = (x_1, y_1, z_1, x_2, \ldots, x_m, y_m, z_m)^T \) be \( 3m \times 1 \) vector of point coordinates and \( J \) the \( 3m \times 7 \) matrix constructed from \( J_i \) where

\[
J_i = \begin{bmatrix}
 1 & 0 & 0 & 0 & -z_i & y_i & x_i \\
 0 & 1 & 0 & z_i & 0 & -x_i & y_i \\
 0 & 0 & 1 & -y_i & x_i & 0 & z_i
\end{bmatrix}.
\]
Suppose $J$ has QR factorisation $J = QR$ where $Q$ is a $3m \times 3m$ orthogonal matrix with $Q^\top Q = QQ^\top = I$ and $R$ is a $3m \times 7$ upper-triangular matrix [7]. Partition $Q$ as $Q = [Q_1 \ Q_2 \ Q_3]$ where $Q_1$ is the submatrix comprised of columns 1 to 6, $Q_2$ corresponds to column 7 and $Q_3$ comprises columns 8 to 3$m$. Finally, let

$$V_P = P_1VP_1^\top, \quad V_Z = P_2^\top VP_2^\top, \quad V_S = P_3VP_3^\top,$$

(2)

where $P_k = Q_kQ_k^\top$. Variance matrices $V_P$, $V_Z$ and $V_S$ represent the variance components with respect to Position, Size (or scale) and Shape, respectively. Similarly, it is possible to isolate the variance components $V_{PZ}$ and $V_{ZS}$ associated with position and size, and size and shape [6], respectively, with

$$V_{PZ} = P_{12}VP_{12}^\top, \quad V_{ZS} = P_{23}VP_{23}^\top = (I - Q_1Q_1^\top)V(I - Q_1Q_1^\top)^\top,$$

(3)

where

$$P_{12} = [Q_1 \ Q_2][Q_1 \ Q_2]^\top, \quad P_{23} = [Q_2 \ Q_3][Q_2 \ Q_3]^\top.$$

The matrices $P_k$ are projections with $P_k = P_k^\top$, $P_kP_k = P_k$, $k = 1, 2, 3$. Since $Q$ is an orthogonal matrix

$$I = QQ^\top = [Q_1 \ Q_2 \ Q_3][Q_1 \ Q_2 \ Q_3]^\top = Q_1Q_1^\top + Q_2Q_2^\top + Q_3Q_3^\top = P_1 + P_2 + P_3.$$  

(4)

We can therefore write $x = (P_1 + P_2 + P_3)x = Q_1p + Q_2\lambda + Q_3s$, where $p = Q_1^\top x$, $\lambda = Q_2^\top x$ and $s = Q_3^\top x$ represent an alternative parametrisation of $x$ in terms of six position parameters $p$, one size parameter $\lambda$, and $3m - 7$ shape parameters $s$.

A similar decomposition can be undertaken for point clouds in 2 dimensions.

3. Properties associated with the variance decomposition

3.1. Traces of the variance matrices

Using (4), we can write

$$V = (P_1 + P_2 + P_3)V(P_1 + P_2 + P_3)^\top = V_P + V_Z + V_S + \sum_{k \neq j} P_kVP_j^\top.$$  

This means that in general $V \neq V_P + V_Z + V_S$. However, we recall that for matrices for which $AB$ and $BA$ can be formed, trace$(AB) =$ trace$(BA)$, [7], so that if $k \neq j$ then trace$(P_kVP_j^\top) =$ trace$(P_jVP_k^\top V) = 0$, since $P_kP_j^\top =$ $0$. Regarding the trace of a variance matrix as an aggregate measure of the total variance, we have

$$\text{trace}(V) = \text{trace}(V_P) + \text{trace}(V_Z) + \text{trace}(V_S) = \text{trace}(V_P) + \text{trace}(V_{ZS}),$$

so that in terms of this aggregate measure, no information is lost in the decomposition.

3.2. Consistency of the decomposition

The projections $P_k$ are determined by $x$ and, applying the process twice, we have $P_kP_jVP_j^\top P_k^\top = P_kVP_k^\top$, if $k = j$, and is zero otherwise. Thus, $V_P$ has no component of variance relating to size or shape, $V_Z$ has no component of variance relating to position or shape, etc.
3.3. Decomposition for specific classes of variance matrix $V$

If $V = \sigma^2 I$, then $\text{trace}(V_P) = 6\sigma^2$, $\text{trace}(V_Z) = \sigma^2$ and $\text{trace}(V_S) = (3m - 7)\sigma^2$. For large $m$, the variance is dominated by the uncertainty in shape; random, uncorrelated perturbations will have only a small position and size component.

Suppose $\tilde{x}_i = R(\alpha)(x_i - x_0)$ is a rigid body transformation of $x_i$ depending on three rotation angles $\alpha$ and three translation parameters $x_0$. Assuming $t = \begin{bmatrix} x_0 \\ \alpha \end{bmatrix}$ is associated with variance matrix $V_t$, let $G$ be the $3m \times 6$ matrix of partial derivatives of $\tilde{x}$ with respect to $t$ and set $V = GV_tG^\top$. Then $V$ is the variance matrix associated with the $3m \times 1$ vector $\tilde{x}$ derived by propagating the variance associated with $t$ through to $\tilde{x}$. Then the variance decomposition for $V$ has $V_P = V$ and $V_Z = V_S = 0$.

Now suppose $V = \sigma^2 x x^\top$ so that $V$ represents a variance matrix consisting solely in the uncertainty contribution from a scale parameter $\lambda$. In general the variance decomposition of $V$ will have a non-zero position component $V_P$ as well as a non-zero size component $V_Z$; the shape component $V_S$ will be zero. However, for mean centred data with $\sum x_i = \sum y_i = \sum z_i = 0$, then $V_Z = V$ and $V_P = V_S = 0$.

3.4. Uncertainties associated with distances

If $d_{ij}$ is the distance between $x_i$ and $x_j$ and $g_{ij}$ the $3m$-vector of partial derivatives of $d_{ij}$ with respect to $x$, then

$$u^2(d_{ij}) = g_{ij}^\top V g_{ij} = g_{ij}^\top V_Z g_{ij}, \text{ } g_{ij}^\top V_P g_{ij} = 0;$$

uncertainty in position does not contribute to uncertainty in distance.

3.5. Uncertainties associated with angles

If $\alpha_{ijk}$ is the angle between $x_i - x_j$ and $x_i - x_k$ and $g_{ijk}$ the $3m$-vector of partial derivatives of $\alpha_{ijk}$ with respect to $x$, then

$$u^2(\alpha_{ijk}) = g_{ijk}^\top V g_{ijk} = g_{ijk}^\top V_Z g_{ijk}, \text{ } g_{ijk}^\top V_P g_{ijk} = g_{ijk}^\top V_Z g_{ijk} = g_{ijk}^\top V_P g_{ijk} = 0;$$

uncertainty in angle depends only on the uncertainty in shape.

4. Frame of reference constraints for point clouds and $V_{ZS}$

In the calibration of 2 or 3 dimensional reference artefacts such as hole and ball-plates [4], it is usual to specify the frame of reference of the ball centres by having one ball centred at the origin, $x_1 = 0$, a second centred on the $x$-axis, $y_2 = z_2 = 0$, and a third positioned in the $xy$-plane, $z_3 = 0$, six constraints in all. These constraints can usually be written as $C^\top x = c_0$, where $C$ is a $3m \times 6$ matrix.

Let $x$, $V$, constraint matrix $C$ and $c_0$ be given with the assumption that $C^\top x \approx c_0$, that is, $x$ approximately satisfies the frame of reference constraints. Define $\tilde{x}$ by $\tilde{x}_i(t) = R(\alpha)(x_i - x_0)$, a rigid body transformation. We look for $t$ such that $\tilde{x}$ satisfies the frame of reference constraints exactly: $C^\top \tilde{x} = c_0$. Since $x$ approximately satisfies the constraints, to first order approximation, $\tilde{x} = x + Gt$ where $G$ is the $3m \times 6$ matrix of partial derivatives of $\tilde{x}$ with respect to $t$ evaluated at $t$ specifying the identity transformation. We note here that $G$ is exactly the same matrix as the first 6 columns of $J$ defined by (1). To first order, $t$ is defined by the equation $C^\top(x + Gt) = c_0$ so that $t = (C^\top G)^{-1}(c_0 - C^\top x)$ and

$$\tilde{x} = x + (C^\top G)^{-1}(c_0 - C^\top x).$$

This last equation defines $\tilde{x}$ as a linear function of $x$ and allows us to propagate the variance $V$ associated with $x$ through to that, $\tilde{V}$ associated with $\tilde{x}$ [2]:

$$\tilde{V} = (I - G(C^\top G)^{-1}C^\top) V (I - G(C^\top G)^{-1}C^\top)^\top.$$
Let $G = Q_1 S_1$ and $C = U_1 T_1$ be QR decompositions were $Q_1$ and $U_1$ are $3n \times 6$ matrices and $S_1$ and $T_1$ are $6 \times 6$ upper triangular matrices [7]. The matrix $Q_1$ is exactly the same matrix as that used to define $V_0$ in (2) and $V_{ZS}$ in (3). With these factorisations, $G(C^T G)^{-1} C^T = Q_1 F_1^T$ where $F_1^T = (U_1^T Q_1)^{-1} U_1^T$, so that $\hat{V} = (I - Q_1 F_1^T) V (I - Q_1 F_1^T)^\top$. If in fact, $C = G$, i.e., we use $G$ to specify the frame of reference, then $\hat{V} = V_{ZS}$, since for this case $F_1 = Q_1$.

We now show that $\text{trace}(\hat{V}) \geq \text{trace}(V_{ZS})$, i.e., the choice $C = G$ leads to a variance matrix $\hat{V}$ with the minimal aggregate variance, as measured by its trace, amongst all possible choices of constraint matrix $C$. Suppose that $V$ has Cholesky factorisation $V = LL^\top$ [7]. Then, using the identity $\text{trace}(AB) = \text{trace}(BA)$,

$$\text{trace}(\hat{V}) = \text{trace}\left((I - Q_1 F_1^T) LL^\top (I - Q_1 F_1^T)^\top\right) = \text{trace}\left((I - Q_1 F_1^T)^\top (I - Q_1 F_1^T) LL^\top\right),$$

while, similarly, $\text{trace}(V_{ZS}) = \text{trace}\left(L^\top (I - Q_1 Q_1^\top) L\right)$. Hence,

$$\text{trace}(\hat{V}) - \text{trace}(V_{ZS}) = \text{trace}\left(L^\top \left[(I - Q_1 F_1^T)^\top (I - Q_1 F_1^T) - (I - Q_1 Q_1^\top)\right] L\right),$$

$$= \text{trace}\left(L^\top \left[Q_1 F_1^T - Q_1 F_1^T - F_1 Q_1^\top - F_1 F_1^T\right] L\right),$$

$$= \text{trace}\left(L^\top (Q_1 - F_1)(Q_1 - F_1)^\top L\right) \geq 0.$$

The last inequality holds because the matrix inside the brackets can be written as $KK^\top$, a positive, semi-definite matrix, and $\text{trace}(KK^\top) = \text{trace}(K^\top K) = \sum_{ij} k_{ij}^2$, the sum of the squares of the elements of $K$. (In general, $\hat{V} - V_{ZS}$ need not be positive semi-definite.)

A very practical application of the decomposition is that it gives a way of presenting uncertainty information about calibrated reference artefacts in which the contribution from uncertainty in position is minimised.

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