Photon—Graviton Amplitudes from the Effective Action

F. Bastianelli, O. Corradini, J. M. Dávila, and C. Schubert

Abstract—We report on the status of an ongoing effort to calculate the complete one-loop low-energy effective actions in Einstein—Maxwell theory with a massive scalar or spinor loop, and to use them for obtaining the explicit form of the corresponding M-graviton/N-photon amplitudes. We present explicit results for the effective actions at the one-graviton four-photon level, and for the amplitudes at the one-graviton two-photon level. As expected on general grounds, these amplitudes relate in a simple way to the corresponding four-photon amplitudes. We also derive the gravitational Ward identity for the 1PI one-graviton—N photon amplitude.

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1. INTRODUCTION

In string theory, the prototypical example of relations between gravity and gauge theory amplitudes are the “KL T” relations discovered by Kawai et al. [1]. Schematically, they are of the form

\[ V^\text{closed} = V^\text{open}_\text{left} V^\text{open}_\text{right} \quad (1.1) \]

These string relations finduce also relations in field theory. For example, at four and five point one has [2]

\[ M_4(1, 2, 3, 4) = -i s_{12} A_4(1, 2, 3, 4) A_4(1, 2, 4, 3) \]

\[ M_5(1, 2, 3, 4, 5) = \]

\[ = -i s_{13} s_{34} A_5(1, 2, 3, 4, 5) A_5(1, 2, 4, 3, 5) \]

\[ + i s_{13} s_{24} A_5(1, 3, 4, 2, 5) A_5(3, 4, 1, 2, 5) \quad (1.2) \]

Here the \( M_n \) are \( n \)-point tree-level graviton amplitudes, and the \( A_n \) are (colour-stripped) tree-level gauge theory amplitudes. The \( s_{ij} = (k_i + k_j)^2 \) are kinematical invariants.

Although the work of [1] was at the tree level, by unitarity those tree level relations finduce also identities at the loop level. By now, many relations between graviton and gauge amplitudes have been derived along these lines at the one loop level and beyond; see [3] and refs. therein. Presently a key issue here is the possibility that the finiteness of \( N = 4 \) SYM theory may extend to \( N = 8 \) Supergravity (see [3, 4] and P. Vanhove’s talk at this conference). Finiteness of a quantum field theory usually implies extensive cancellations between Feynman diagrams, and it is presently still not well-understood what are the precise extent and origin of such cancellations in the Supergravity case.

In this respect, gravity amplitudes are more similar to QED amplitudes than to nonabelian amplitudes, since colour factors greatly reduce the potential for cancellations between diagrams. In QED, there are many cases of surprising cancellations between diagrams. A famous case is the three-loop QED \( \beta \)-function coefficient, involving the sum of diagrams shown in Fig. 1.

As discovered by Rosner in 1967 [5], findividual diagrams give contributions to the \( \beta \)-function coefficient that involve \( \zeta(3) \), however those terms cancel out, leaving a simple rational number for the sum of diagrams. Such cancellations are usually attributed to gauge invariance, since they generally appear inside gauge invariant sets of graphs. Even for QED, little is still known about the influence of these cancellations on the large-order behaviour of the QED perturbation series [6, 7]. For recent gravity-inspired studies of the structure of QED amplitudes see [8, 9].

Considering the enormous amount of work that has been done on the structural relationships between gauge and gravity amplitudes, it is surprising that relatively few results exist for mixed graviton-gluon or gravito/N-photon amplitudes [10–12]. In this talk, I report on the status of an ongoing effort to calculate the complete one-loop low-energy effective actions in Einstein—Maxwell theory with a massive scalar or spinor loop, and to use them for obtaining the explicit form of the corresponding M-graviton/N-photon amplitudes [13–15]. The talk is organized as follows: In chapter 2 we will shortly summarize what is pres-
ently known about the QED $N$ photon amplitudes. In chapter 3 we summarize the results of [13, 14] on the one-loop effective action in Einstein—Maxwell theory, and also improve somewhat on the form of its one-graviton four-photon part as compared to [14]. Chapter 4 is devoted to the graviton—photon—photon amplitude. Our conclusions are presented in chapter 5.

2. PROPERTIES OF THE QED $N$ PHOTON AMPLITUDES

We shortly summarize what is known about the $N$ photon-amplitudes in scalar and spinor QED (results given refer to the spinor case unless stated otherwise).

Although the one-loop four-photon amplitude was calculated by Karplus and Neumann already in 1950 [16], progress towards higher leg or multiloop photon amplitudes has been extremely slow. The one-loop six-photon amplitude (recall that by Furry’s theorem amplitudes has been extremely slow. The one-loop six-photon amplitude was obtained only quite recently [17], and only there are no amplitudes with an odd number of photons). In the massless case. On-shell amplitudes for gauge bosons are nowadays generally given in the helicity eigenstate decomposition; using CP invariance, the bosons are nowadays generally given in the helicity decomposition, one finds [23]

\[
A^{(\text{EH})}\left[\varepsilon_1^+, \ldots, \varepsilon_N^+, \varepsilon_{K+1}^-, \ldots, \varepsilon_N^-ight] = \frac{m^4}{8\pi^2} \frac{(2ie)^N}{m^2} (N-3)!
\]

and a similar formula for the scalar QED case [23]. Here the $B_k$ are Bernoulli numbers, and the variables $\chi_K^\pm$ are written, in spinor helicity notation (our spinor helicity conventions follow [24])

\[
\chi_K^+ = \frac{K!}{2^k} \times \{12\}^2 \ldots \{(K-1)K\}^2 + \text{all permutations}
\]

\[
\chi_K^- = \frac{K!}{2^k} \times \{12\}^2 \ldots \{(K-1)K\}^2 + \text{all permutations}
\]

These variables appear naturally in the low energy limit. Since they require even numbers of positive and negative helicity polarizations, in this low energy limit we find a “double Furry theorem”: Only those helicity components are non-zero where both the number of positive and negative helicity photons are even. It is easy to show that this even holds true to all loop orders. For the MHV (“maximally helicity violating” =

\[
\mathcal{L}_{\text{spin}}(F) = \frac{1}{8\pi^2} \int_0^\infty \frac{dT}{T} e^{-\frac{m^2 T}{4}} \left[ \frac{(eaT)(ebT)}{\tanh(eaT)\tan(ebT)} - \frac{1}{3}(a^2 - b^2)(T^2 - 1) \right]
\]

\[
\mathcal{L}_{\text{scal}}(F) = \frac{1}{16\pi^2} \int_0^\infty \frac{dT}{T} e^{-\frac{m^2 T}{4}} \left[ \frac{(eaT)(ebT)}{\sinh(eaT)\sin(ebT)} + \frac{1}{6}(a^2 - b^2)(T^2 - 1) \right]
\]
“all +” or “all −”) case (2.2) and its scalar analogue imply that the scalar and spinor amplitudes differ only by the global factor of −2 for statistics and degrees of freedom:

\[ A^{(EH)}_{\text{spin}}[\eta^+_1; \ldots; \eta^+_N] = -2A^{(EH)}_{\text{scal}}[\eta^+_1; \ldots; \eta^+_N]. \]  

(2.4)

This well-known relation is actually true also away from the low-energy limit, and can be explained by the fact that the MHV amplitudes correspond to a self-dual background, in which the Dirac operator has a quantum-mechanical supersymmetry [25].

3. ONE-LOOP EFFECTIVE ACTION IN EINSTEIN–MAXWELL THEORY

The calculation of the one-loop effective action in Einstein–Maxwell theory is usually done using heat kernel techniques. The first calculation of relevance in the present context of the on-shell photon-graviton kernel techniques. The first calculation of relevance in the on-shell photon-graviton kernel techniques. The first calculation of relevance in the on-shell photon-graviton kernel techniques. The first calculation of relevance in the on-shell photon-graviton kernel techniques. The first calculation of relevance in the on-shell photon-graviton kernel techniques.

The result for the spinor case differs from the Drummond–Hathrell Lagrangian (3.1) by a total derivative term [13]. The parameter \( \xi \) appearing in the scalar case represents a non-minimal coupling to gravity. At the next, \( N = 4 \) level (there are no amplitudes with an odd number of photons by an extension of Furry’s theorem to the photon-graviton case) this procedure is already quite laborious. It was carried through in [14], here we give the results in a slightly more compact form than was obtained there:

\[
\mathcal{L}_{\text{spin}}^{p,47} = -\frac{1}{16 \pi^2 m^6} \left[ \frac{1}{180} R(F) \alpha^2 \right] + \frac{1}{180} \left( \frac{\xi}{12} \right) R(F) \alpha^2 \\
+ \frac{1}{945} R_{\alpha \beta}(F) \alpha^2 \left( \frac{1}{144} \left( 1 + \frac{7}{12} \right) R(F) \alpha^2 \right) \\
- \frac{1}{945} R_{\alpha \beta}(F \alpha^2 \left( \frac{1}{1080} R(F \alpha^2 \left( F \alpha^2 \right)^2 \\
- \frac{1}{270} R_{\alpha \beta \gamma}(F \alpha^2 \left( F \alpha^2 \right)^2 \\
+ \frac{1}{432} R_{\alpha \beta \gamma}(F \alpha^2 \left( F \alpha^2 \right)^2 \\
- \frac{1}{1890} R_{\alpha \beta \gamma}(F \alpha^2 \left( F \alpha^2 \right)^2 \\
+ \frac{1}{1890} R_{\alpha \beta \gamma}(F \alpha^2 \left( F \alpha^2 \right)^2 \\
(\xi = \xi - 1/4). This improvement over the formulas given in [14] is due to the following consequence of the
Bianchi identities, that had been overlooked in the list of identities used there:

\[
R_{\alpha \mu \nu \beta} (F^2)^{\alpha \beta} (F^2)^{\mu \nu} = -\frac{1}{2} F_{\mu \nu, \alpha \beta} (F^2)^{\alpha \beta} F^{\mu \nu} - \frac{1}{2} R_{\alpha \mu \nu \beta} (F^3)^{\alpha \beta} - F_{\mu \nu, \alpha \beta} (F^2)^{\alpha \beta} F^{\mu \nu} \tag{3.6}
\]

4. THE GRAVITON–PHOTON–PHOTON AMPLITUDE

We proceed to the simplest amplitude case, the graviton–photon–photon amplitude shown in Fig. 2.

Getting its low energy limit from the three-point Lagrangians (3.2), (3.3) (or equivalently from (3.1) in the spinor case) is straightforward. In the helicity basis, and using the standard factorization of the graviton polarization tensor in terms of vector polarizations, \(\varepsilon_{\mu \nu}^{\alpha \beta} (k) = \varepsilon_{\mu \nu}^{\alpha} (k) \varepsilon_{\alpha \beta}^{\gamma} k_\gamma\), one finds that only the \(\text{“all +”}\) and \(\text{“all -”}\) components are nonzero:

\[
A^{(+++)_{\text{spin}}} = \frac{\kappa e^2}{90 (4\pi)^2 m^2} [01]^2 [02]^2 \tag{4.1}
\]

Here the first upper findex pair refers to the graviton polarization, and \(\kappa\) is the gravitational coupling constant. Moreover, those components fulfill the MHV relation (2.4),

\[
A^{(++)_{\text{spin}}} = (2) A^{(++)_{\text{skal}}} \tag{4.2}
\]

Also, these graviton–photon–photon amplitudes relate to the (low energy) four photon amplitudes in the following way: From (2.2), (2.3) the only non-vanishing components of those are

\[
A^{+++}[k_1, k_2, k_3, k_4] \sim \langle 12 \rangle [34]^2
\]

\[
+ \langle 13 \rangle [24]^2 + \langle 14 \rangle [23]^2 \tag{4.3}
\]

\[
A^{-++}[k_1, k_2, k_3, k_4] \sim \langle 12 \rangle \langle 34 \rangle^2
\]

\[
+ \langle 13 \rangle \langle 24 \rangle^2 + \langle 14 \rangle \langle 23 \rangle^2.
\]

Replacing \(k_1 \rightarrow k_0\), \(k_2 \rightarrow k_0\) in the 4 photon amplitudes, the middle one of these three components becomes zero, and the remaining ones become proportional to the corresponding components of (4.1),

\[
A^{+++}[k_0, k_0, k_3, k_4] \sim 2 \langle 03 \rangle [04]^2
\]

\[
\sim A^{++-}[k_0, k_0, k_3, k_4] \tag{4.4}
\]

\[
A^{---}[k_0, k_0, k_3, k_4] \sim 2 \langle 03 \rangle \langle 04 \rangle^2
\]

\[
\sim A^{--+}[k_0, k_3, k_4].
\]

Thus effectively two photons have coalesced to form a graviton, clearly a result in the spirit of the KLT relations.

At the next level of one graviton and four photons, the conversion of the effective action into amplitudes becomes already extremely laborious. Moreover, here there are already one-particle reducible contributions to the amplitudes, with the graviton attached to a photon, and those are essential to arrive at a well-defined helicity decomposition. This is because the 1PI amplitudes are transversal in the photon findices, but not in the graviton ones; rather, one has the inhomogeneous Ward identity [15]

\[
2k_{0\mu} A^{\mu \nu \alpha_1 \alpha_N}[k_0, \ldots, k_N] = -\sum_{i=1}^{N} A^{\mu \nu \alpha_1 \ldots \alpha_N} [k_0 + k_i, k_1, \ldots, k_i, \ldots, k_N] \times (\delta_{\alpha_i}^{\eta} k_{i
})
\]

where a ‘hat’ means omission which connects the one graviton–\(N\) photon amplitudes to the \(N\) photon amplitudes.

5. CONCLUSIONS

We have presented here the first results of a systematic study of the mixed one-loop photon–graviton amplitudes with a scalar or spinor loop in the low energy limit. At the one graviton–two photon level, we find a KLT like factorization of the graviton into two photons. If this type of factorization persist for higher points, it would imply that, in the low energy limit, the full information on the \(M\) graviton–\(N\) photon amplitudes is contained in the \(N + 2M\) photon amplitudes. However, the three-point result may not be representative due to the absence of one-particle reducible contributions. The situation will be clearer after the
completion of the one graviton—four photon calculation, which is presently in progress.

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