Weighted tensor products of Joyal species, graphs, and charades

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Abstract
Motivated by the weighted Hurwitz product on sequences in an algebra, we produce a family of monoidal structures on the category of Joyal species. We suggest a family of tensor products for charades. We begin by seeing weighted derivational algebras and weighted Rota-Baxter algebras as monoids and semigroups, respectively, for the same monoidal structure on the category of graphs in a monoidal additive category. Weighted derivations are lifted to the categorical level.

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1 Introduction

There are monoidal structures on the category $\text{Gph}_V$ of graphs in a monoidal additive category $V$ for which weighted derivational monoids and weighted Rota-Baxter monoids (see [10], for example) can be seen as monoids (also called “algebras”) and semigroups (or “non-unital associative algebras”). We also produce a family of monoidal structures on the category of Joyal species. In particular, this defines an interesting family of tensor products for linear representations of the symmetric groups. We also suggest a family of tensor products for charades (see [8, 21]) which generalizes, in particular, the essentially classical tensor product of representations of the general linear groups over a finite field, proved braided in [16].

Weighted derivations are defined on monoidal categories with finite coproducts over which the tensor product distributes.

2 Review of $\lambda$-weighted derivations and Rota-Baxter operators

The inspiration for the family of tensor products on species came from the $\lambda$-weighted product of Hurwitz series as discussed in [10, 11] and their references. They begin by defining a derivation of weight $\lambda$ on an algebra $A$ over a commutative ring $k$, with given $\lambda \in k$, to be a $k$-module morphism $d: A \to A$ satisfying $d(1) = 0$ and

$$d(ab) = d(a)b + ad(b) + \lambda d(a)d(b).$$

They note the generalized Leibnitz rule

$$d^n(ab) = \sum_{k=0}^{n} \binom{n}{k} \binom{n-k}{j} \lambda^k d^{n-j}(a)d^{k+j}(b).$$
However, I prefer to write this in the form
\[ d^n(ab) = \sum_{n=r+s+t} \binom{n}{r,s,t} \lambda^t d^{r+t}(a)d^{s+t}(b) \tag{2.1} \]

to emphasise the relationship to the trinomial expansion rule for \((x + y + \lambda xy)^n\). Here
\[ \binom{n}{r,s,t} = \frac{n!}{r!s!t!} . \]

The \(\lambda\)-Hurwitz product on \(A^N\) can be defined by the clearly related equation
\[ (f \cdot \lambda g)(n) = \sum_{n=r+s+t} \binom{n}{r,s,t} \lambda^t f(r+t)g(s+t) . \tag{2.2} \]

**Example 1.** For \(\lambda = 0\), \(k = \mathbb{R}\) and \(A\) the algebra of smooth functions \(f: \mathbb{R} \to \mathbb{R}\) under pointwise addition and multiplication, the differentiation function \(d: A \to A\) is a 0-weighted derivation by the classical Leibnitz rule.

**Example 2.** For \(\lambda\) invertible, \(k = \mathbb{R}\) and \(A\) the algebra of functions \(f: \mathbb{R} \to \mathbb{R}\) under pointwise addition and multiplication, the function \(d: A \to A\) defined by
\[ d(f)(x) = \frac{f(x + \lambda) - f(x)}{\lambda} \]
is a \(\lambda\)-weighted derivation.

**Example 3.** Define \(d: A^N \to A^N\) by \(d(s)(n) = s(n+1) - s(n)\). This \(d\) is a 1-weighted derivation when \(A^N\) is equipped with the pointwise addition and multiplication.

**Example 4.** Define \(d: A^N \to A^N\) by \(d(f)(n) = f(n+1)\). This \(d\) is a \(\lambda\)-weighted derivation when \(A^N\) is equipped with the \(\lambda\)-Hurwitz product for any \(\lambda\). Notice that we have an algebra morphism \(d^*: A^N \to (A^N)^N\) defined by \(d^*(f)(m)(n) = f(m+n)\). This may motivate the next definition.

Define \(d^*: A \to A^N\) by \(d^*(a)(n) = d^n(a)\). We see that the Leibnitz rule (2.1) amounts to:

**Proposition 5.** \(d^*: A \to A^N\) is an algebra morphism for all \(\lambda\)-weighted derivations \(d\) on \(A\), where \(A^N\) has the \(\lambda\)-Hurwitz product.

In fact, \(A \mapsto A^N\) is a comonad
\[ G = \left( (-)^N, \varepsilon, \delta \right) \tag{2.3} \]
on the category \(\text{Alg}_k\) of \(k\)-algebras whose Eilenberg-Moore-coalgebras are \(k\)-algebras \(A\) equipped with a \(\lambda\)-derivation, so-called \(\lambda\)-derivation algebras; write \(DA\lambda\) for the category of these. The morphism \(d^*: A \to A^N\) is the coaction of the comonad.
Where there is differentiation, there should also be integration. A Rota-Baxter operator of weight $\lambda$ on a $k$-algebra $A$ is a $k$-linear morphism $P: A \to A$ satisfying
\[
P(a)P(b) = P(P(a)b + P(a)b + \lambda ab)
\]  
(2.4)
The pair $(A, P)$ is called a $\lambda$-weighted Rota-Baxter algebra. Write $\text{RBA}_\lambda$ for the category of these.

**Example 6.** For $\lambda = 0$, $k = \mathbb{R}$ and $A$ the algebra of continuous functions $f: \mathbb{R} \to \mathbb{R}$ under pointwise addition and multiplication, the integration function $P: A \to A$, defined by $P(f)(x) = \int_0^x f(t)dt$, is a 0-weighted Rota-Baxter operator by the classical integration-by-parts rule.

**Example 7.** For $\lambda = 1$ and any $k$-algebra $A$, define $P: A^\mathbb{N} \to A^\mathbb{N}$ to take a sequence $u$ in $A$ to its sequence $P(u)$ of partial sums:
\[
P(u)(n) = \sum_{i=0}^{n-1} u(i)
\]
Then $P$ is a 1-weighted Rota-Baxter operator on $A^\mathbb{N}$ with pointwise addition and multiplication. See [1, 20]. For $d$ the consecutive difference operator as defined in Example 3, notice that $d \circ P = 1_{A^\mathbb{N}}$.

**Example 8.** If $Q$ is a 1-weighted Rota-Baxter operator on $A$ then $P(a) = \lambda Q(a)$ defines a $\lambda$-weighted Rota-Baxter operator $P$ on $A$.

A $\lambda$-weighted derivation RB-algebra is a $k$-algebra $A$ equipped with a $\lambda$-weighted derivation $d$ and a $\lambda$-weighted Rota-Baxter operator $P$ such that $d \circ P = 1_A$. Write $\text{DRB}_\lambda$ for the category of these.

**Proposition 9.** [See [10]] Let $P$ be a RB-operator of weight $\lambda$ on $A$. Then $A^\mathbb{N}$ equipped with the $\lambda$-Hurwitz product, the derivation $d$ of Example 4, and $P$ defined by
\[
P(f)(n) = \begin{cases} P(f(0)) & \text{for } n = 0 \\ f(n-1) & \text{for } n > 0 \end{cases}
\]
is a $\lambda$-weighted derivation RB-algebra. Moreover, the following square commutes.

\[
\begin{array}{ccc}
A^\mathbb{N} & P & A^\mathbb{N} \\
\downarrow \text{ev}_0 & & \downarrow \text{ev}_0 \\
A & P & A
\end{array}
\]
With a little more work following Proposition 9, we see that the comonad $G$ (2.3) lifts to $\text{RBA}_\lambda$. In particular, with $V$ denoting the forgetful functor, we have a comonad $\bar{G}$ and a commutative square

$$
\begin{array}{c}
\text{RBA}_\lambda \\
\downarrow V \\
\text{Alg}_k
\end{array}
\Rightarrow
\begin{array}{c}
\text{RBA}_\lambda \\
\downarrow V \\
\text{Alg}_k
\end{array}
$$

Write $\text{RBA}_{\lambda -}$ for the category of $\lambda$-weighted Rota-Baxter algebras where we do not insist on the algebras having a unit.

**Proposition 10.** For each $(A, \cdot, P) \in \text{RBA}_{\lambda -}$, there is an associative binary operation $a \diamond b$ defined on $A$ by

$$a \diamond b = P(a) \cdot b + a \cdot P(b) + \lambda a \cdot b.$$ 

Moreover, $T(A, \cdot, P) = (A, \circ, P)$ defines an endofunctor

$$T: \text{RBA}_{\lambda -} \rightarrow \text{RBA}_{\lambda -},$$

which is (well-)copointed by a natural transformation $\gamma: T \Rightarrow 1_{\text{RBA}_{\lambda -}}$ whose component at $(A, \cdot, P)$ is $P: (A, \circ, P) \rightarrow (A, \cdot, P)$.

**Remark 11.** The day after my seminar talk of 4 February 2015, on the material of this section and Section 4, Stephen Lack made the following comments.

1. Consider the category $[\Sigma N, \mathcal{Y}]$ whose objects are pairs $(M, d)$ consisting of an object $M$ of a nice monoidal additive category $\mathcal{Y}$ and an endomorphism $d: M \rightarrow M$. The forgetful functor

$$U: [\Sigma N, \mathcal{Y}] \rightarrow \mathcal{Y},$$

(2.5)

taking $(M, d)$ to $M$, has a right adjoint taking $M$ to $(M^N, d)$ where $d(f)(n) = f(n + 1)$. There is a monoidal structure on $[\Sigma N, \mathcal{Y}]$ defined by

$$(M, d) \otimes^\lambda (N, d) = (M \otimes N, d \otimes 1 + 1 \otimes d + \lambda d \otimes d).$$

The monoids in this monoidal category are precisely $\lambda$-derivation algebras. Moreover, $U$ (2.5) and its right adjoint form a monoidal adjunction which therefore defines an adjunction between the categories of monoids. This adjunction generates the comonad $G$ (2.3) on the category $\text{Mon}\mathcal{Y}$ of monoids in $\mathcal{Y}$.

2. There is a bialgebra structure on the polynomial algebra $k[x]$ with comultiplication the algebra morphism $\delta: k[x] \rightarrow k[x, y] \cong k[x] \otimes k[x]$ defined by

$$\delta(x) = x + y + \lambda xy.$$
Then the convolution product on the left-hand side of the canonical isomorphism
\[ \text{Mod}_k(k[x], A) \cong A^N \]
transports to the \( \lambda \)-Hurwitz product on \( A^N \).

3. It feels like there should be a multicategory/promonoidal/substitude structure on \([\Sigma N, \mathcal{V}]\) for dealing with RB-algebras.

3 Graphs in monoidal additive categories

Let \( \mathcal{V} \) be a monoidal additive category. We act as if the monoidal structure were strict.

Let \( \text{Gph} \mathcal{V} \) be the category of directed graphs in \( \mathcal{V} \). So an object has the form of a pair of parallel morphisms \( s, t: E \to A \) in \( \mathcal{V} \); we use \( s \) and \( t \) for source and target morphisms in all graphs. A morphism \( (f, \phi): (A, E) \to (B, F) \) in \( \text{Gph} \mathcal{V} \) consists of morphisms \( f \) and \( \phi \) making the following diagram commute.

\[
\begin{array}{ccc}
A & \xrightarrow{s} & E & \xrightarrow{t} & A \\
\downarrow f & & \downarrow \phi & & \downarrow f \\
B & \xleftarrow{s} & F & \xleftarrow{t} & B
\end{array}
\]

Write \( \text{ver}: \text{Gph} \mathcal{V} \to \mathcal{V} \) for the forgetful functor taking \( (A, E) \) to \( A \) and write \( \text{edg}: \text{Gph} \mathcal{V} \to \mathcal{V} \) for the forgetful functor taking \( (A, E) \) to \( E \).

We will use the notation \( \langle n \rangle = \{1, 2, \ldots, n\} \). For \( R \subseteq \langle n \rangle \), write
\[ \chi_R: \langle n \rangle \to \{s, t\} \]
for the characteristic function of \( R \) defined by
\[ \chi_R(i) = \begin{cases} 
  s & \text{for } i \in R \\
  t & \text{for } i \notin R 
\end{cases} \]

Choose an endomorphism \( \lambda: I \to I \) of the tensor unit \( I \) in \( \mathcal{V} \). For any \( f: A \to B \) in \( \mathcal{V} \), we define \( (\lambda f: A \to B) = (\lambda \otimes f: I \otimes A \to I \otimes B) \).

Given a list \( (A_1, E_1), \ldots, (A_n, E_n) \) of objects of \( \text{Gph} \mathcal{V} \), we define an \( n \)-fold tensor product
\[ \otimes_{1 \leq i \leq n} (A_i, E_i) = (\otimes_{1 \leq i \leq n} A_i, \otimes_{1 \leq i \leq n} E_i) \, , \]
(3.6)
where
\[ s = \sum_{\emptyset \neq R \subseteq \langle n \rangle} \lambda(\#R-1) \chi_R(1) \otimes \cdots \otimes \chi_R(n) \text{ and } t = t \otimes \cdots \otimes t \, . \]
(3.7)
For $n = 2$ this gives a binary tensor product
\[(A, E) \otimes^\lambda (B, F) = (A \otimes B, E \otimes F)\]
with
\[s = \lambda s \otimes s + s \otimes t + t \otimes s \quad \text{and} \quad t = t \otimes t.\]
The unit for this tensor is the graph $(I, I)$ with $s = 0: I \to I$ and $t = 1_I: I \to I$.

**Proposition 12.** A monoidal structure on $\text{Gph} \mathcal{V}$ is defined by (3.6) for any given $\lambda \in \mathcal{V}(I, I)$. Both $\text{ver}$ and $\text{edg}: \text{Gph} \mathcal{V} \to \mathcal{V}$ are strict monoidal.

**Proof.** Easy calculations of the source morphisms for $(A, E) \otimes^\lambda (B, F) \otimes^\lambda (C, G)$ and $(A, E) \otimes^\lambda ((B, F) \otimes^\lambda (C, G))$ show they agree with that of the triple tensor product. The target morphisms obviously agree. What this really means is that the associativity constraints for $\mathcal{V}$ lift through $\text{ver}$ and $\text{edg}$ to $\text{Gph} \mathcal{V}$ and are therefore coherent.

Let $[\Sigma \mathcal{N}, \mathcal{V}]$ denote the category whose objects $(A, e: A \to A)$ consist of an object $A$ of $\mathcal{V}$ equipped with an endomorphism $e$. Let
\[J: [\Sigma \mathcal{N}, \mathcal{V}] \to \text{Gph} \mathcal{V}\]
be the functor defined by $J(A, e) = (A, A)$ with $s = e$ and $t = 1_A$; and $Jf = (f, f)$. Notice also that a morphism $(f, \phi): (B, F) \to J(A, e)$ in $\text{Gph} \mathcal{V}$ with codomain in the subcategory amounts to a commutative diagram
\[
\begin{array}{ccc}
F & \xrightarrow{t} & B \\
\downarrow{s} & & \downarrow{f} \\
B & = & A \\
\end{array}
\]
where $\phi$ is forced to be $f \circ t: F \to A$. Clearly $J$ is fully faithful and the monoidal structure of Proposition 12 restricts to a monoidal structure on $[\Sigma \mathcal{N}, \mathcal{V}]$ yielding (3.8) as a strict monoidal functor.

**Definition 13.** A $\lambda$-weighted derivational monoid in $\mathcal{V}$ is a monoid $(A, d)$ in $[\Sigma \mathcal{N}, \mathcal{V}]$ equipped with the monoidal structure obtained as the restriction through (3.8) of that of Proposition 12 on $\text{Gph} \mathcal{V}$.

More explicitly, a $\lambda$-weighted derivational monoid is a monoid $A$ in $\mathcal{V}$ equipped with an endomorphism $d: A \to A$ satisfying the $\lambda$-weighted equation:
\[d \circ \mu = \mu \circ (\lambda d \otimes d + d \otimes 1 + 1 \otimes d),\]  
(3.10)
and the equation $d \circ \eta = 0$ (where $\eta: I \to A$ is the unit of $A$).

There is an isomorphism of categories

$$\text{op}: \text{Gph} \to \text{Gph}$$

(3.11)

taking $(A, E)$ to $(A, E)\text{op}$ for which $A$ and $E$ are unchanged but $s$ and $t$ have been interchanged.

Put

$$J\text{op} = \left(\Sigma \mathbb{N}, \mathbb{V} \to \text{Gph} \to \text{Gph} \right).$$

(3.12)

Like $J$, this composite $J\text{op}$ is fully faithful.

However, the image of $J\text{op}$ is not closed under the monoidal structure of Proposition 12. All we obtain on $\Sigma \mathbb{N}, \mathbb{V}$ is a structure of multicategory (sometimes called a “coloured operad”). The sets of multimorphisms are defined by

$$P_\lambda((A_1, p_1), \ldots, (A_n, p_n); (B, p)) = \text{Gph} \left(\bigotimes_{1 \leq i \leq n} J\text{op}(A_i, p_i), J\text{op}(B, p) \right).$$

(3.13)

To be more explicit, for $R \subseteq \langle n \rangle$ and $i \in \langle n \rangle$, put

$$R(i) = \left\{ \begin{array}{ll}
1_A & \text{for } i \in R \\
p_i & \text{for } i \notin R.
\end{array} \right.$$

Then, an element of the set (3.13), a multimorphism, is a morphism

$$f: A_1 \otimes \cdots \otimes A_n \to B$$

satisfying the equation

$$f \circ (p_1 \otimes \cdots \otimes p_n) = p \circ f \circ \sum_{\emptyset \neq R \subseteq \langle n \rangle} \lambda(#R - 1)R(1) \otimes \cdots \otimes R(n).$$

Definition 14. A $\lambda$-weighted Rota-Baxter monoid in $\mathbb{V}$ is an object $(A, p)$ of $\Sigma \mathbb{N}, \mathbb{V}$ (that is, $p: A \to A$ in $\mathbb{V}$) equipped with the structure of semigroup on $J\text{op}(A, p)$ in the monoidal category Gph$\mathbb{V}$ of Proposition 12, and a unit $\eta: I \to A$ for the underlying semigroup $A$ in $\mathbb{V}$.

This definition should make the calculation of free weighted Rota-Baxter monoids possible; compare [20, 2] for $\lambda = 1, -1$ for the case of commutative algebras.

To make Definition 14 a little more explicit, as expected, a $\lambda$-weighted Rota-Baxter monoid $(A, p)$ is a monoid $A$ in $\mathbb{V}$ equipped with an endomorphism $p: A \to A$ satisfying (3.14):

$$\mu \circ (p \otimes p) = p \circ \mu \circ (\lambda 1 \otimes 1 + 1 \otimes p + p \otimes 1).$$

(3.14)

Derivations and Rota-Baxter operators are not the only sources of semigroups and monoids for the monoidal structure of Proposition 12. The forgetful functor

$$\text{ve}: \text{Gph}\mathbb{V} \to \mathbb{V} \otimes \mathbb{V},$$

(3.15)
taking the graph \((A, E)\) to the pair \((A, E)\), is strict monoidal and has a right adjoint \(R\) defined by
\[
R(X, Y) = (X, X \oplus X \oplus Y)
\] (3.16)
with \(s = \text{pr}_1\) (the first projection) and \(t = \text{pr}_2\) (the second projection). It follows that \(R\) is monoidal and hence takes monoids to monoids.

**Example 15.** Take \(\mathcal{V} = \text{Mod}_k\), the category of modules over a commutative ring \(k\). For a graph \((A, E)\) in this \(\mathcal{V}\), we can write \(e: a \rightarrow b\) to mean \(a, b \in A, e \in E\) with \(s(e) = a, t(e) = b\). For \(k\)-algebras \(A\) and \(B\), we obtain a monoid \(R(A, B)\) in \(\text{Gph}\mathcal{V}\): the graph is \(\text{pr}_1, \text{pr}_2: A \oplus A \oplus B \rightarrow A\) and the multiplication is defined by:
\[
((a_1, a_2, b): a_1 \rightarrow a_2) \cdot ((c_1, c_2, d): c_1 \rightarrow c_2)
= (\lambda a_1 c_1 + a_1 c_2 + a_2 c_1, a_2 c_2, bd): a_1 c_1 \rightarrow a_2 c_2
\]

Of course the \(\mathcal{V}\)-functor \(J\) (3.8) has both adjoints if \(\mathcal{V}\) is complete and cocomplete enough. In particular, the right adjoint
\[
K: \text{Gph}\mathcal{V} \longrightarrow [\Sigma N, \mathcal{V}]
\] (3.17)
is defined by taking \(K(A, E)\) to be the equalizer of the two morphisms
\[
s^N, t^{\text{succ}}: E^N \rightarrow A^N
\]
equipped with the endomorphism \(e: K(A, E) \rightarrow K(A, E)\) induced by \(E^{\text{succ}}\). Here \(\text{succ}: \mathbb{N} \rightarrow \mathbb{N}\) is the successor function \(n \mapsto n + 1\). Since \(J\) (3.8) is strong monoidal for the monoidal structures under discussion, the adjunction \(J \dashv K\) is monoidal. So \(K\) takes semigroups to semigroups and monoids to monoids.

In particular, if \((A, p)\) is a \(\lambda\)-weighted Rota-Baxter monoid in \(\mathcal{V}\), then \(K\) takes the graph \((A, A)\) with \(s = 1_A\) and \(t = p\) to a \(\lambda\)-weighted derivational monoid in \(\mathcal{V}\). The underlying monoid is the limit of the diagram
\[
A \leftarrow^p A \leftarrow^p A \leftarrow^p \ldots
\]
of monoids in \(\mathcal{V}\).

**Example 16.** Taking \(\mathcal{V} = \text{Vect}_k\) and a \(\lambda\)-weighted Rota-Baxter \(k\)-algebra \(p: A \rightarrow A\), we have the \(\lambda\)-weighted derivational \(k\)-algebra
\[
K(J(A, p)^{\text{op}}) = \{a \in A^\mathbb{N} \mid p(a_{n+1}) = a_n\}
\]
with \(d(a)_n = a_{n+1}\). Moreover, \(K(J(A, p)^{\text{op}})\) supports a \(\lambda\)-weighted Rota-Baxter operator \(p\) defined by \(p(a)_n = p(a_n)\). Notice too that \(d \circ p = 1\).
We conclude this section by describing the promonoidal structure in the sense of Day [4] with respect to which the monoidal structure of Proposition 12 is convolution.

Let $G$ denote the category whose only objects are 0 and 1, with the only non-identity morphisms $\sigma, \tau: 1 \to 0$. Write $I_*G$ for the free $\mathcal{V}$-category on $\mathcal{V}$.

Then $G\mathcal{P}V = [G, \mathcal{V}] = [I_*G, \mathcal{V}]$ where the first set of square brackets means the ordinary functor category while the second means the $\mathcal{V}$-enriched functor category. The promonoidal structure in question is technically on $I_*G$ in the $\mathcal{V}$-enriched sense. However, we can look at it as consisting of an ordinary a functor $P: G^{\text{op}} \times G^{\text{op}} \to G\mathcal{P}V$ (3.18) and an object $J \in G\mathcal{P}\mathcal{V}$. Of course $J$ is just the graph $0, 1: I \to I$ which is the tensor unit. We can regard $P$ as a “cograph of cographs of graphs” (although a cograph looks just like a graph):

$$
\begin{array}{ccc}
I & \xrightarrow{(1,0)} & I \oplus I = 2 \cdot I \\
\downarrow^{(0,1)} & & \downarrow^{(0,1)} \\
(1,0) & \xrightarrow{(1000)} & (0010)
\end{array}
$$

$2 \cdot I = I \oplus I \xrightarrow{(1000)} ((\lambda, 1, 1, 0), (0, 0, 0, 1): I \to 4 \cdot I)$

4 The $L$-tensor product of species

Let $\mathcal{G}$ denote the groupoid whose objects are finite sets and whose morphisms are bijective functions. We write $U + V$ for the disjoint union of sets $U$ and $V$; this is the binary coproduct as objects of the category Set of sets and all functions. It is not the coproduct in $\mathcal{G}$; yet it does provide the symmetric monoidal structure on $\mathcal{G}$ of interest here. When we write $X = A + B$ for $A$ and $B$ subsets of a set $X$, we mean $X = A \cup B$ and $\emptyset = A \cap B$.

We have the particular finite sets $(n) = \{1, 2, \ldots, n\}$.

Let $\mathcal{V}$ denote a monoidal category with finite coproducts which are preserved by tensoring on either side by an object. The tensor product of $V, W \in \mathcal{V}$ is denoted by $V \otimes W$ and the unit object by $I$. Justified by coherence theorems (see [15] for example), we write as if the monoidal structure on $\mathcal{V}$ were strictly associative and strictly unital. For any set $S$, write $S \cdot V$ for the coproduct of $S$ copies of $V \in \mathcal{V}$, when it exists (as it does for $S$ finite).

The category of $\mathcal{V}$-valued Joyal species, after [12, 13], is the functor category $[\mathcal{G}, \mathcal{V}]$. The objects will simply be called species when $\mathcal{V}$ is understood.
Suppose \( L : \mathcal{S} \to \mathcal{Z} \mathcal{V} \) is a braided strong monoidal functor into the monoidal centre (in the sense of \([14]\)) of \( \mathcal{V} \). We have natural isomorphisms

\[
u_{X,V} : LX \otimes V \cong V \otimes LX
\]
such that

\[
\begin{array}{c}
LX \otimes V \otimes W \xrightarrow{u_{X,V} \otimes 1_W} V \otimes LX \otimes W \\
V \otimes W \otimes LX \xrightarrow{1_V \otimes u_{X,W}} \end{array}
\]

If \( \mathcal{V} \) itself is braided (a fortiori symmetric), we can take a braided strong monoidal functor \( L : \mathcal{S} \to \mathcal{V} \) since then there is a canonical braided strong monoidal functor \( \mathcal{V} \to \mathcal{Z} \mathcal{V} \).

By way of example, we could have any finite set \( \Lambda \) and \( LX = \Lambda^X \cdot I \) with \( L\sigma = \Lambda^{\sigma^{-1}} \cdot I \) for any bijective function \( \sigma \).

Define the \( L \)-tensor product \( F \otimes^L G \) of species \( F \) and \( G \) on objects \( X \in \mathcal{S} \) by

\[
(F \otimes^L G)_X = \sum_{X = U \cup V} L(U \cap V) \otimes FU \otimes GV .
\]

The definition of \( F \otimes^L G \) on morphisms is clear since any bijective function \( \sigma : X \to Y \) restricts to bijections

\[
U \to \sigma U, \ V \to \sigma V, \ U \cup V \to \sigma U \cup \sigma V, \ U \cap V \to \sigma U \cap \sigma V .
\]

Let \( J : \mathcal{S} \to \mathcal{V} \) be the species whose value at \( X \) is the unit \( I \) for tensor in \( \mathcal{V} \) when \( X \) is empty and is initial in \( \mathcal{V} \) otherwise. Clearly \( J \) is a unit for the \( L \)-tensor product in the sense that we have canonical isomorphisms

\[
\lambda_G : J \otimes^L G \to G \quad \text{and} \quad \rho_F : F \to F \otimes^L J .
\]

Associativity isomorphisms

\[
\alpha_{F,G,H} : (F \otimes^L G) \otimes^L H \cong F \otimes^L (G \otimes^L H) \quad (4.20)
\]

are obtained using the following fact easily proved by Venn diagrams.

**Lemma 17.** \((U \cup V) \cap W + U \cap V \cong U \cap (V \cup W) + V \cap W\)

In the case \( L = J \), we recover from (4.19) the usual convolution (Cauchy) product of species appearing in \([12]\). For a general \( L \), the term \( L(U \cap V) \) can be considered a measure of the failure of \( U \) and \( V \) to be disjoint.
5 A combinatorial interpretation

We consider the case where \( \mathcal{V} = \text{Set} \) so that \([\mathcal{S}, \text{Set}]\) is the category of species as studied in [12]. Fix any set \( \Lambda \). Define the species \( L \) by

\[
L_X = \{ S = (S_\lambda)_{\lambda \in \Lambda} \mid S_\lambda \subseteq X, \sum_{\lambda \in \Lambda} S_\lambda = X \}
\]

(5.21)

and \((L\sigma)S = (\sigma S_\lambda)_{\lambda \in \Lambda}\). In other words, a structure of the species \( L \) on the set \( X \) is a partition of \( X \) into a \( \Lambda \)-indexed family of disjoint (possibly empty) subsets.

A structure of the species \( F \otimes^L G \) on the set consists of a quintuplet \((U, V, S, \phi, \gamma)\) where \( U, V \) are subsets of \( X \) such that \( X = U \cup V \), and \( S, \phi, \gamma \) are \( L-, F-, G- \) structures on \( U \cap V, U, V \), respectively.

We write \#\( S \) for the cardinality of the set \( S \). We assume \( \Lambda \) is finite and put \( \lambda = \#\Lambda \).

The cardinality sequence of a species \( F \) is the sequence \( \#F : \mathbb{N} \to \mathbb{Z} \) defined by

\[
(#F)(n) = \#F\langle n \rangle.
\]

We consider the \( \lambda \)-Hurwitz product (2.2) on \( \mathbb{Z}^\mathbb{N} \).

Proposition 18. \( \#(F \otimes^L G) = \#F \cdot \lambda \cdot \#G \)

6 The iterated tensor and coherence

Proposition 19. An alternative definition of \( F \otimes^L G \) is

\[
(F \otimes^L G)X = \sum_{X = A + B + C} L(C) \otimes F(A + C) \otimes G(B + C).
\]

Proof. Given \( X = A + B + C \), put \( U = A + C \) and \( V = B + C \). Given \( X = U \cup V \), put \( A = U \setminus V, B = V \setminus U, \) and \( C = U \cap V. \)

The \( n \)-fold version of this tensor product is

\[
\otimes_n^L(F_1, \ldots, F_n)X = \sum_{X = \sum a \neq \sum (\langle n \rangle) A_S} L\left( \sum_S (\#S - 1) \cdot A_S \right) \otimes F_1\left( \sum_{1 \in S} A_S \right) \otimes \cdots \otimes F_n\left( \sum_{n \in S} A_S \right).
\]

(6.22)

This yields the formula in Proposition 19 for \( n = 2 \) by taking \( A = A_1, B = A_2, C = A_{\{1,2\}} \). Note that (6.22) is unchanged if we replace \( \langle n \rangle \) by any set of cardinality \( n \).

Remark 20. As Joachim Kock reminded me, if we replace \( \langle n \rangle \) by the ‘(\( n - 1 \))-simplex’ \( [n - 1] = \{0,1,\ldots, n-1\} \), then the non-empty subsets \( S \) correspond to the non-degenerate faces of \( [n-1] \) and \( \#S - 1 \) is the dimension of the face.
Let us consider the effect of inserting one pair of parentheses in a multiple tensor (6.22). We look at

\[ \otimes^L_{p+1+r} (F_1, \ldots, F_p, \otimes^L_{q} (F_{p+1}, \ldots, F_{p+q}), F_{p+q+1}, \ldots, F_{p+q+r}) X \ . \tag{6.23} \]

Using (6.22) twice, once with \( n = p + 1 + r \) and once with \( n = q \), we obtain the expression

\[
L \left( \sum_T (\#T - 1) \cdot B_T \right) \otimes F_1 \left( \sum_{1 \in T} B_T \right) \otimes \cdots \otimes F_p \left( \sum_{p \in T} B_T \right)
\]

\[ \otimes L \left( \sum_R (\#R - 1) \cdot C_R \right) \otimes F_{p+1} \left( \sum_{p+1 \in R} C_R \right) \otimes \cdots \otimes F_{p+q} \left( \sum_{p+q \in R} C_R \right) \]

\[ \otimes F_{p+q+1} \left( \sum_{p+q+1 \in T} B_T \right) \otimes \cdots \otimes F_{p+q+r} \left( \sum_{p+q+r \in T} B_T \right) \]

summed over all families

\[ B = (B_T \mid \emptyset \neq T \subseteq \{1, \ldots, p, q + 1, \ldots, p + q + r\}) \]

providing a partition \( X = \sum_T B_T \) of \( X \), together with all families

\[ C = (C_R \mid \emptyset \neq R \subseteq \{p+1, \ldots, p+q\}) \]

providing a partition \( \sum_{\ast \in T} B_T = \sum_R C_R \) of \( \sum_{\ast \in T} \). Using the fact that \( L \) lands in \( Z^\mathcal{V} \) and that \( L \) is strong monoidal, we obtain the isomorphic expression

\[ L \left( \sum_T (\#T - 1) \cdot B_T + \sum_R (\#R - 1) \cdot C_R \right) \]

\[ \otimes F_1 \left( \sum_{1 \in T} B_T \right) \otimes \cdots \otimes F_p \left( \sum_{p \in T} B_T \right) \]

\[ \otimes F_{p+1} \left( \sum_{p+1 \in R} C_R \right) \otimes \cdots \otimes F_{p+q} \left( \sum_{p+q \in R} C_R \right) \]

\[ \otimes F_{p+q+1} \left( \sum_{p+q+1 \in T} B_T \right) \otimes \cdots \otimes F_{p+q+r} \left( \sum_{p+q+r \in T} B_T \right) \]

summed over the same families \((B, C)\). For \( \ast \in T \), we have \( B_T = \sum_R C_R \cap B_T \). On the other hand, \( C_R = \sum_{\ast \in T} C_R \cap B_T \). Put

\[ Q = \{p + 1, \ldots, p + q\} \text{ and } N = \{1, \ldots, p\} \cup \{p + q + 1, \ldots, p + q + r\} \text{,} \]
and obtain a family
\[ A = (A_S \mid \emptyset \neq S \subseteq (p + q + r)) \]
partitioning \( X \) by defining
\[
A_S = \begin{cases} B_S & \text{for } S \cap Q = \emptyset \\ C_{S \cap Q} \cap B_{S \cap \mathbb{N}^\circ \cup \{\ast\}} & \text{for } S \cap Q \neq \emptyset. \end{cases}
\]
Then we can recover the \( B \) and \( C \) families via
\[
B_T = \begin{cases} A_T & \text{for } \ast \notin T \\ \sum_R A_{R \cup (T \setminus \ast)} & \text{for } \ast \in T \end{cases} \quad \text{and} \quad C_R = \sum_{\ast \in T} A_{R \cup (T \setminus \ast)}.
\]
We have the following equations:

(i) \[ \sum_S (\#S - 1) \cdot A_S = \sum_T (\#T - 1) \cdot B_T + \sum_R (\#R - 1) \cdot C_R \]

(ii) \[ \sum_{k \in S} A_S = \begin{cases} \sum_{k \in T} B_T & \text{for } 1 \leq k \leq p \text{ or } p + q + 1 \leq k \leq p + q + r \\ \sum_{k \in R} C_R & \text{for } p + 1 \leq k \leq p + q. \end{cases} \]

This shows that the sum of the expressions (6.25) over the pairs \((B, C)\) is equal to (6.22) with \( n = p + q + r \). Remember however that the tensor product \(+\) on \( \mathcal{S} \) is not strict symmetric; the symmetry on \( \mathcal{S} \) provides canonical bijections between the left- and right-hand sides of (i) and (ii). Since \( L \) is braided, we have constructed a natural isomorphism
\[ a_{p,q,r} : \otimes_n^L (F_1, \ldots, F_{p+q+r}) \cong \otimes_{p+1+r}^L (F_1, \ldots, F_p, \otimes_q^L (F_{p+1}, \ldots, F_{p+q}), F_{p+q+1}, \ldots, F_{p+q+r}). \]

Now consider the Mac Lane-Stasheff pentagon for 2-fold bracketings of \( F_1 \otimes^L F_2 \otimes^L F_3 \otimes^L F_4 \) as the vertices. Let \( a : H \to K \) denote one of the edges of the pentagon obtained using the associativity isomorphisms (4.20). There is a composite \( b \) of two isomorphisms, each using one instance of an isomorphism (6.26), which goes from \( \otimes_4^L (F_1, F_2, F_3, F_4) \) to \( H \), and another one \( c : \otimes_4^L (F_1, F_2, F_3, F_4) \to H \). By coherence of the braided strong monoidal functor \( L \), it follows that \( a \circ b = c \). Commutativity of the pentagon is a consequence of commutativity of all these triangular sides of the so-formed pentagonal cone.

### 7 Promonoidal structures on \( \mathcal{S} \)

For finite sets \( A, B \) and \( X \), let \( \text{Cov}(A,B;X) \) denote the set of jointly surjective pairs \((\mu, \nu)\) of injective functions
\[ A \xrightarrow{\mu} X \xleftarrow{\nu} B. \]
We write $A \times_X B$ for the pullback of $\mu$ and $\nu$.
Define a functor
\[ P : \mathcal{S}^{\text{op}} \times \mathcal{S}^{\text{op}} \times \mathcal{S} \to \mathcal{V} \]
by
\[ P(A, B; X) = \sum_{(\mu, \nu) \in \text{Cov}(A, B; X)} L(A \times_X B) . \quad (7.27) \]

**Proposition 21.** $(F \otimes^L G)X \cong \int^{A, B} P(A, B; X) \otimes FA \otimes GB$

**Proof.** A universal dinatural transformation
\[ \theta_{A,B} : P(A, B; X) \otimes FA \otimes GB \to \sum_{X=U \cup V} L(U \cap V) \otimes FU \otimes GV \]
is defined by taking its composite with the injection at $(\mu, \nu) \in \text{Cov}(A, B; X)$ to be obtained from the $(\mu(A), \nu(B))$ injection and the bijections $A \cong \mu(A)$, $B \cong \nu(B)$, $A \times_X B \cong \mu(A) \cap \nu(B)$, noting $X = \mu(A) \cup \nu(B)$. \hfill \Box

By Day’s general theory of promonoidal categories [3, 4], we have:

**Corollary 22.** If moreover $\mathcal{V}$ is (left and right) closed and sufficiently complete then $\otimes^L$ defines a (left and right) closed monoidal structure on $[\mathcal{S}, \mathcal{V}]$.

---

### 8 The weighted bimonoidale structure on $\text{fam}\mathcal{S}$

Consider the 2-category $\text{Cat}_+$ of (small) categories admitting finite coproducts, and finite-coproduct-preserving functors. This becomes a symmetric closed monoidal bicategory (see [5]) with tensor product $\mathcal{A} \boxtimes \mathcal{B}$ representing functors $H : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ for which each $H(A, -)$ and each $H(-, B)$ is finite coproduct preserving. Clearly the monoidal category $\mathcal{V}$ of Section 4 is a monoidale (= pseudomonoid) in $\text{Cat}_+$.

For any category $\mathcal{C}$, we write $\text{fam}\mathcal{C}$ for the free finite coproduct completion of $\mathcal{C}$. That is, $\text{fam}$ provides the left biadjoint to the forgetful 2-functor $\text{Cat}_+ \to \text{Cat}$. Indeed, $\text{fam}$ is a strong monoidal pseudofunctor; in particular, there is a canonical equivalence
\[ \text{fam}\mathcal{C} \boxtimes \text{fam}\mathcal{D} \simeq \text{fam}(\mathcal{C} \times \mathcal{D}) . \]

Every monoidal category $\mathcal{C}$ determines a monoidale $\text{fam}\mathcal{C}$ in $\text{Cat}_+$.

Explicitly, the objects of $\text{fam}\mathcal{C}$ can be written formally as $\sum_{s \in S} C_s$ where $S$ is a finite set and $C_s \in \mathcal{C}$. Then, if $\mathcal{C}$ is monoidal, the monoidale structure on $\text{fam}\mathcal{C}$ is defined by
\[ \sum_{s \in S} C_s \otimes \sum_{t \in T} D_t = \sum_{(s, t) \in S \times T} C_s \otimes D_t . \]
We are interested in \( \text{fam}\mathcal{S} \). By what we have just said, this is a monoidale in \( \text{Cat}_+ \):

\[
\sum_{s \in S} U_s \otimes \sum_{t \in T} V_t = \sum_{(s,t) \in S \times T} (U_s + V_t) .
\]

Fix a finite set \( \Lambda \) and define \( L : \mathcal{S} \rightarrow \text{Set} \) by \( LX = \Lambda^X \) and \( L\sigma = \Lambda^{\sigma^{-1}} \). Define a coproduct-preserving functor

\[
\Delta : \text{fam}\mathcal{S} \rightarrow \text{fam}(\mathcal{S} \times \mathcal{S}) \simeq \text{fam}\mathcal{S} \boxtimes \text{fam}\mathcal{S}
\]

by

\[
\Delta(X) = \sum_{X = A + B + C} L(C) \cdot (A + C, B + C)
\]

for \( X \in \mathcal{S} \).

**Proposition 23.** The functor \( \Delta \) of (8.28) is strong monoidal.

**Proof.** In \( \Delta(X + Y) = \sum_{X + Y = A + B + C} L(C) \cdot (A + C, B + C) \) we can put

\[ P = X \cap A, Q = X \cap B, R = X \cap C, U = Y \cap A, V = Y \cap B, W = Y \cap C \]

to obtain

\[
\Delta(X + Y) = \sum_{X = P + Q + R, Y = U + V + W} L(R + W) \cdot (P + U + R + W, Q + V + R + W)
\]

\[
\simeq \sum_{X = P + Q + R} L(R) \cdot (P + R, Q + R) \times \sum_{Y = U + V + W} L(W) \cdot (U + W, V + W)
\]

\[
\simeq \Delta X \times \Delta Y ,
\]

as required. \( \square \)

### 9 Weighted categorical derivations

Harking back to Remark 11, we are prompted to consider the 2-category

\[
\mathcal{E} = \text{Hom}(\Sigma\mathcal{S}, \mathcal{V}^{\text{-Cat}_{L,+}}) .
\]

Here \( \Sigma\mathcal{S} \) denotes the bicategory with one object (denoted \( * \)) whose homcategory is the symmetric groupoid \( \mathcal{S} \); composition is provided by the monoidal structure \( + \) on \( \mathcal{S} \). Also \( \mathcal{V}^{\text{-Cat}_{L,+}} \) denotes the 2-category of \( \mathcal{V} \)-categories admitting finite coproducts and tensoring with the object \( L(X) \) of \( \mathcal{V} \); the morphisms are \( \mathcal{V} \)-functors preserving these colimits; the 2-cells are \( \mathcal{V} \)-natural transformations. The objects of (9.29) are pseudofunctors \( T : \Sigma\mathcal{S} \rightarrow \mathcal{V}^{\text{-Cat}_{L,+}} \), the morphisms are pseudonatural transformations, and the 2-cells are modifications (in terminology of [18]). Such
an object $T$ determines a $\mathcal{V}$-category $T^\bullet = \mathcal{M} \in \mathcal{V}$-$\text{Cat}_{L,+}$ and a strong monoidal functor $T^\bullet : \mathcal{S} \rightarrow \mathcal{V}$-$\text{Cat}_{L,+}$. This $T^\bullet$ is determined up to equivalence by an endomorphism $D : \mathcal{M} \rightarrow \mathcal{M}$ in $\mathcal{V}$-$\text{Cat}_{L,+}$ and an involutive Yang-Baxter operator $\rho : D \circ D \Rightarrow D \circ D$ on $D$ (for example, see [14] for terminology). Then $T^\bullet(n) \cong D^n$ and, for the non-identity bijection $\tau : (2) \rightarrow (2), \tau$ transports to $\rho$. Therefore we shall write the object $T$ of $\mathcal{C}$ (9.29) as a pair $(\mathcal{M}, D^\bullet)$ where $T^\bullet = \mathcal{M}$ and $T^\bullet = D^\bullet$. The morphisms of $\mathcal{C}$ are then squares

$$
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{K} & \mathcal{N} \\
D^\bullet X & \xrightarrow{\kappa X \cong} & E^\bullet X \\
\mathcal{M} & \xrightarrow{K} & \mathcal{N}
\end{array}
$$

in $\mathcal{V}$-$\text{Cat}_{L,+}$ which are $\mathcal{V}$-natural in $X$ and, stacking vertically, respect the tensor in $\mathcal{S}$. Generalizing the tensor $\boxtimes$ on $\text{Cat}_{\omega}$ as in Section 8, we have a tensor, also denoted by $\boxtimes$, on $\mathcal{V}$-$\text{Cat}_{L,+}$, where the tensor product $\mathcal{S} \boxtimes \mathcal{R}$ represents $\mathcal{V}$-functors $H : \mathcal{S} \otimes \mathcal{R} \rightarrow \mathcal{X}$ for which each of $H(A,-)$ and $H(-,B)$ preserves finite coproducts and tensoring with each $L(X)$.

This tensor product $\boxtimes$ on $\mathcal{V}$-$\text{Cat}_{L,+}$ lifts to one, denoted by $\hat{\boxtimes}$, on $\mathcal{C}$ (9.29):

$$(\mathcal{M}, D^\bullet) \hat{\boxtimes} (\mathcal{N}, E^\ast) = (\mathcal{M} \boxtimes \mathcal{N}, D^\bullet \hat{\boxtimes} E^\ast) \quad \text{where}$$

$$(D^\bullet \hat{\boxtimes} E^\ast) X = \sum_{X = U \cup V} L(U \cap V) \otimes D^\bullet U \boxtimes E^\ast V. \quad (9.31)$$

To see that $D^\bullet \hat{\boxtimes} E^\ast : \mathcal{S} \rightarrow \mathcal{V}$-$\text{Cat}_{L,+}(\mathcal{M} \boxtimes \mathcal{N}, \mathcal{M} \boxtimes \mathcal{N})$ is strong monoidal, we calculate:

$$(D^\bullet \hat{\boxtimes} E^\ast)(X + Y) \cong \sum_{X + Y = U \cup V} L(U \cap V) \otimes D^\bullet U \boxtimes E^\ast V$$

$\cong \sum_{X = U_1 \cup V_1 \cup U_2 \cup V_2} L(U_1 \cap V_1 + U_2 \cap V_2) \otimes (D^\bullet U_1 \circ D^\bullet U_2) \boxtimes (E^\ast V_1 \circ E^\ast V_2)$$

$\cong \sum_{X = U_1 \cup V_1 \cup U_2 \cup V_2} L(U_1 \cap V_1 \cap U_2 \cap V_2) \otimes (D^\bullet U_1 \boxtimes E^\ast V_1) \circ (D^\bullet U_2 \boxtimes E^\ast V_2)$$

$\cong \sum_{X = U_1 \cup V_1 \cup U_2 \cup V_2} L(U_1 \cap V_1 \cap U_2 \cap V_2) \circ (D^\bullet U_1 \boxtimes E^\ast V_1) \otimes (D^\bullet U_2 \boxtimes E^\ast V_2)$$

$\cong (D^\bullet \hat{\boxtimes} E^\ast) X \circ (D^\bullet \hat{\boxtimes} E^\ast) Y.$

---

1This is Rodney Baxter <http://en.wikipedia.org/wiki/Rodney_Baxter>, not the author of [1].
In this way, $\mathcal{E}$ (9.29) becomes a monoidal bicategory.

An $L$-weighted derivation $D^*$ on a monoidale $\mathcal{M}$ in $\mathcal{V}$-$\text{Cat}_{L,+}$ is a lifting of the monoidale structure on $\mathcal{M}$ to a monoidale structure on $(\mathcal{M}, D^*)$ in $\mathcal{E}$ (9.29).

**Example 24.** An $L$-weighted derivation $D^* : \mathfrak{S} \to \mathcal{V}$-$\text{Cat}_{L,+}$ is defined by $D^* X = F(X + -)$. The main point is the canonical isomorphism below.

\[
\begin{array}{ccc}
[D^{\mathfrak{S}} \circ D^*] & \cong & [D^* X] \\
\downarrow & & \downarrow \\
[D^{\mathfrak{S}} \circ D^*] & \cong & [D^* X]
\end{array}
\]

**Remark 25.** The first item of Remark 11 has a categorical version. The forgetful 2-functor $U : \mathcal{E} \to \mathcal{V}$-$\text{Cat}_{L,+}$ has a right biadjoint $JS$ taking the $\mathcal{V}$-category $A$ to the object of $\mathcal{E}$ determined by the $\mathcal{V}$-category $JS_A = [\mathfrak{S}, A]$ of species in $\mathcal{A}$, equipped with $L$-weighted derivation the $D^*$ just as in Example 24 with the codomain $\mathcal{V}$ replaced by $\mathcal{A}$. Since $U$ is strong monoidal, the biadjunction $U \dashv \text{bi}JS$ is monoidal. Consequently the biadjunction lifts to one between the 2-categories of monoidales in $\mathcal{E}$ and $\mathcal{V}$-$\text{Cat}_{L,+}$. Indeed $U$ is pseudocomonadic.

### 10 The iterated tensor product again

Observe the following simple reindexing of (4.19).

**Proposition 26.** An alternative definition of $F \otimes^L G$ is

\[
(F \otimes^L G)X = \sum_{V \subseteq U \subseteq X} L(U \setminus V) \otimes F(U) \otimes G(X \setminus V).
\]

This leads us to another formula for the $n$-fold $L$-weighted tensor product. Define the modified $n$-filtration set on any finite set $X$ by:

\[m\text{Fil}_nX = \{(U, V) \mid U = (0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{n-1} \subseteq U_n = X), \\
V = (V_0, V_1, \ldots, V_{n-1}) \text{ with } V_i \subseteq U_i \text{ for } 0 \leq i < n\} .\] (10.32)

**Proposition 27.** An alternative definition of the $n$-fold tensor product (6.22) is

\[
\otimes_L^n(F_1, \ldots, F_n)X = \sum_{(U, V) \in m\text{Fil}_nX} L(U_1 \setminus V_1) \otimes \cdots \otimes L(U_{n-1} \setminus V_{n-1}) \\
\otimes F_1(U_1 \setminus V_0) \otimes \cdots \otimes F_n(U_n \setminus V_{n-1}).
\] (10.33)
Proof. The formula follows by repeated application of the formula of Proposition 26 in evaluating the left bracketing

\[ ((\ldots (F_1 \otimes^L F_2) \otimes^L \ldots) \otimes^L F_n) \]

at \( X \).

Let us relate the formulas (6.22) and (10.33) in the case \( n = 3 \). A modified 3-filtration \((U, V) \in \text{mFil}_3 X\) of \( X \) amounts to subsets \( U_1 \subseteq U_2 \subseteq X \) and \( V_1 \subseteq U_1, V_2 \subseteq U_2 \). With this we can define

\[
A_1 = V_1 \cap V_2, \quad A_2 = V_2 \setminus U_1 \cap V_2, \quad A_3 = X \setminus U_2
\]

\[
A_{12} = U_1 \cap V_2 \setminus A_1, \quad A_{13} = V_1 \setminus A_1, \quad A_{23} = (U_2 \setminus U_1) \setminus A_2, \quad A_{123} = (U_1 \setminus V_1) \setminus A_{12}
\]

and verify that \( X = A_1 + A_2 + A_3 + A_{12} + A_{13} + A_{23} + A_{123} \). Conversely, given the partition \( A \) of \( X \), we can define

\[
U_1 = X \setminus (A_2 + A_3 + A_{23}) , \quad U_2 = X \setminus A_3 , \quad V_1 = A_1 + A_{13} , \quad V_2 = A_1 + A_2 + A_{12} .
\]

11 Tensor products for charades

Motivated by Proposition 26, we consider generalizing the tensor product of [16]. Let \( \mathcal{G}_q \) be the groupoid of finite vector spaces over the field \( \mathbb{F}_q \) of cardinality \( q \); the morphisms are linear bijections. We write \( V \leq U \) to mean \( V \) is an \( \mathbb{F}_q \)-linear subspace of \( U \), and we write \( U/V \) for the quotient space.

To be specific, take \( \mathcal{V} = \text{Vect}_\mathbb{C} \) to be the category of complex vector spaces with all linear functions.

Let \( L: \mathcal{G}_q \to \mathcal{V} \) be a suitable functor: we will consider conditions on it later.

For functors \( F, G: \mathcal{G}_q \to \mathcal{V} \), define \( F \otimes^L G: \mathcal{G}_q \to \mathcal{V} \) by

\[
(F \otimes^L G)X = \sum_{V \leq U \leq X} L(U/V) \otimes F(U) \otimes G(X/V) .
\]

This leads us to an \( n \)-fold tensor product in a manner analogous to (10.33).

Define the modified \( n \)-flag set on any finite \( \mathbb{F}_q \)-vector space \( X \) by:

\[
\text{mFlg}_n X = \{ (U, V) \mid U = (0 = U_0 \leq U_1 \leq \cdots \leq U_{n-1} \leq U_n = X) , \\
V = (V_0, V_1, \ldots, V_{n-1}) \text{ with } V_i \leq U_i \text{ for } 0 \leq i < n \} .
\]

Now we put

\[
\otimes^L_n (F_1, \ldots, F_n) X = \sum_{(U, V) \in \text{mFlg}_n X} L(U_1 \setminus V_1) \otimes \cdots \otimes L(U_{n-1} \setminus V_{n-1}) \otimes F_1(U_1 \setminus V_0) \otimes \cdots \otimes F_n(U_n \setminus V_{n-1}) .
\]
The formula follows by repeated application of (11.35) in evaluating the left bracketing
\[((\ldots (F_1 \otimes^L F_2) \otimes^L \ldots) \otimes^L F_n)\]
at \(X\).

Let us look at the ternary tensor product
\(\otimes^L_3 (F, G, H) X = ((F \otimes^L G) \otimes^L H) X\)
It is a direct sum over modified 3-flags \((U, V)\) on \(X\); that is, subspaces \(U_1 \leq U_2 \leq X\), \(V_1 \leq U_1\) and \(V_2 \leq U_2\). From these we can uniquely define vector spaces \(A_S\) for each \(\emptyset \neq S \subseteq (3)\) via the following diagrams of short exact sequences:

\[
\begin{align*}
A_1 > & \quad U_1 \cap V_2 \quad A_{12} \\
V_1 > & \quad U_1 \quad U_1/V_1 \\
A_{13} > & \quad U_1/U_1 \cap V_2 \quad A_{123} \\
U_1 \cap V_2 > & \quad V_2 \quad A_2 \\
U_1 > & \quad U_2 \quad U_2/U_1 \\
U_1/U_1 \cap U_2 > & \quad U_2/V_2 \quad A_{23} \\
U_2 > & \quad X \quad A_3, & (11.38)
\end{align*}
\]

from which we see
\[X \cong A_1 \oplus A_2 \oplus A_3 \oplus A_{12} \oplus A_{23} \oplus A_{13} \oplus A_{123} .\] (11.40)

Note also the isomorphisms
\[
\begin{align*}
U_1/V_1 & \cong A_{12} \oplus A_{123} , \\
U_2/V_2 & \cong A_{13} \oplus A_{23} \oplus A_{123} , \\
U_1 & \cong A_1 \oplus A_{12} \oplus A_{13} \oplus A_{123} , \\
U_2/V_1 & \cong A_2 \oplus A_{12} \oplus A_{23} \oplus A_{123} , \\
X/V_2 & \cong A_3 \oplus A_{13} \oplus A_{23} \oplus A_{123} . & (11.42)
\end{align*}
\]

On the other hand, we can see that the formula for the right bracketing is
\[(F \otimes^L (G \otimes^L H)) X\] (11.43)
\[= \sum_{M \leq N \leq X, M \leq I \leq J \leq X} L(N/M) \otimes L(J/I) \otimes F N \otimes G(J/M) \otimes H(X/I)\]
We can see that this indexing set also leads to a direct sum decomposition (11.41) from the following diagrams of short exact sequences.

\[
\begin{array}{cccc}
A_1 & \rightarrow & U_1 \cap V_2 & \rightarrow A_{12} \\
V_1 & \rightarrow & U_1 & \rightarrow U_1/V_1 \\
A_{13} & \rightarrow & U_1/U_1 \cap V_2 & \rightarrow A_{123}
\end{array}
\]

\[
\begin{array}{cccc}
I \cap N & \rightarrow & I & \rightarrow A_2 \\
J \cap N & \rightarrow & J & \rightarrow J/J \cap N \\
A_{123} & \rightarrow & J/I & \rightarrow A_{23}
\end{array}
\]

\[
J + N \rightarrow X \rightarrow A_3, \quad J \cap N \rightarrow N \rightarrow A_{13}.
\]

Note also the isomorphisms

\[
\begin{align*}
N/M & \cong A_{12} \oplus A_{13} \oplus A_{123}, \quad J/I \cong A_{23} \oplus A_{123} \\
N & \cong A_1 \oplus A_{12} \oplus A_{13} \oplus A_{123}, \quad J/M \cong A_2 \oplus A_{12} \oplus A_{23} \oplus A_{123} \\
X/I & \cong A_3 \oplus A_{13} \oplus A_{23} \oplus A_{123}.
\end{align*}
\]

(11.46)

In order to have an associativity isomorphism we at least need a canonical isomorphism

\[
L(A_{12} \oplus A_{123}) \otimes L(A_{13} \oplus A_{23} \oplus A_{123}) \cong L(A_{12} \oplus A_{13} \oplus A_{123} \oplus A_{23} \oplus A_{123}).
\]

(11.47)

We do have such an isomorphism if \( L: \mathfrak{G}_q \rightarrow \mathcal{V} \) takes direct sums to tensor products; of course, direct sum of \( \mathbb{F}_q \)-vector spaces is neither product nor coproduct in \( \mathfrak{G}_q \).

Recall from [16] that \( \mathfrak{G}_q \) has a braided promonoidal structure. The convolution structure on \([\mathfrak{G}_q, \mathcal{V}]\) arising from this (as per Day [3, 4]) is precisely the tensor product \( F \otimes^J G \) where \( JX = I \) for \( X = 0 \) and \( JX = 0 \) for \( X \neq 0 \).

**Conjecture.** If \( L \) is braided strong promonoidal then (11.35) defines a monoidal structure \( \otimes^L \) on \([\mathfrak{G}_q, \mathcal{V}]\)
Should this be the case, the tensor \( \otimes^L \) on \([\mathcal{S}_q, \mathcal{V}]\) would be obtained from quite an interesting promonoidal structure on \( \mathcal{S}_q \). A short sequence

\[
A \xrightarrow{f} X \xrightarrow{g} B
\]

(11.50)
in \( \text{Vect}_{\mathcal{F}_q} \) might be called \textit{short pre-exact} when \( f \) is a monomorphism, \( g \) is an epimorphism and \( \ker f \leq \text{im} g \). Write \( \text{Spes}(A, B; X) \) for the set of such \((f, g)\). Put

\[
P(A, B; X) = \sum_{(f, g) \in \text{Spes}(A, B; X)} L(\text{im}(g \circ f)) .
\]

(11.51)

This \( P : \mathcal{S}_q^{\text{op}} \times \mathcal{S}_q^{\text{op}} \times \mathcal{S}_q \rightarrow \mathcal{V} \), defined on morphisms in the obvious way, would give the promonoidal structure in question. The term \( L(\text{im}(g \circ f)) \) measures the failure of the sequence (11.50) to be exact.

12 The dimension sequence

Following on from Section 11, we take \( F \in [\mathcal{S}_q, \text{Vect}_C] \) and define its \textit{dimension sequence} \( \dim F \in \mathbb{Z}^N \) by

\[
(\dim F)n = \dim \left( F(\mathbb{F}_q^n) \right) .
\]

(12.52)

This inspires an algebra structure on \( A^N \) for any \( k \)-algebra \( A \). We assume we have \( \lambda \in k \) as before, but also some integer \( q \) (not necessarily a prime power). As in [16], we use

\[
\phi_n(q) = (q^n - 1)(q^{n-1} - 1) \ldots (q - 1) .
\]

We define

\[
\left[ \begin{array}{c} n \\ r, s \end{array} \right] = \frac{\phi_n(q)}{\phi_r(q) \phi_s(q)} , \quad \left[ \begin{array}{c} n \\ r, s, t \end{array} \right] = \frac{\phi_n(q)}{\phi_r(q) \phi_s(q) \phi_t(q)} , \ldots .
\]

For \( f, g \in A^N \), put

\[
f \cdot \lambda \_ q g = \sum_{r+s+t=n} \left[ \begin{array}{c} n \\ r, s, t \end{array} \right] \lambda^t f(r + t) g(s + t) .
\]

(12.53)

The calculations of Section 11 show that this is associative at least when \( A = \mathbb{Z} \), \( q \) is a prime power and \( \lambda = \dim L(\mathbb{F}) \).

More generally, I claim \( A^N \) is an associative \( k \)-algebra.

**Proposition 28.** \( \dim (F \otimes^L G) = \dim F \cdot \lambda \_ q \dim G \)
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