NEVANLINNA THEORY FOR MEROMORPHIC MAPS
FROM A CLOSED SUBMANIFOLD OF $\mathbb{C}^l$ TO A
COMPACT COMPLEX MANIFOLD

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Abstract. The purpose of this article is threefold. The first is
to construct a Nevanlina theory for meromorphic mappings from
a polydisc to a compact complex manifold. In particular, we give
a simple proof of Lemma on logarithmic derivative for nonzero
meromorphic functions on $\mathbb{C}^l$. The second is to improve the defi-
nition of the non-integrated defect relation of H. Fujimoto [7] and
to show two theorems on the new non-integrated defect relation
of meromorphic maps from a closed submanifold of $\mathbb{C}^l$ to a com-
 pact complex manifold. The third is to give a unicity theorem
for meromorphic mappings from a Stein manifold to a compact
complex manifold.

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1. Introduction

To construct a Nevanlinna theory for meromorphic mappings between complex manifolds of arbitrary dimensions is one of the most important problems of the Value Distribution Theory. Much attention has been given to this problem over the last few decades and several important results have been obtained. For instance, W. Stoll [17] introduced to parabolic complex manifolds, i.e. manifolds have exhausted functions on the ones with the same role as the radius function in \( \mathbb{C}^l \) and constructed a Nevanlinna theory for meromorphic mappings from a parabolic complex manifold into a complex projective space. In the same time, P. Griffiths and J. King [9] constructed a Nevanlinna theory for holomorphic mappings between algebraic varieties by establishing special exhausted functions on affine algebraic varieties. There is being a very interesting problem that is to construct explicitly a Nevanlinna theory for meromorphic mappings from a Stein complex manifold or a complete Kähler manifold to a compact complex manifold. The first main aim of this paper is to deal with the above mentioned problem in a special case when the Stein manifold is a polydisc. In particular, we give a simple proof of Lemma on logarithmic derivative for nonzero meromorphic functions on \( \mathbb{C}^l \) (cf. Proposition 3.7 and Remark 3.8 below).

In 1985, H. Fujimoto [7] introduced the notion of the non-integrated defect for meromorphic maps of a complete Kähler manifold into the complex projective space intersecting hyperplanes in general position and obtained some results analogous to the Nevanlinna-Cartan defect relation. We now recall this definition.

Let \( M \) be a complete Kähler manifold of \( m \) dimension. Let \( f \) be a meromorphic map from \( M \) into \( \mathbb{C}P^n, \mu_0 \) be a positive integer and \( D \) be a hypersurface in \( \mathbb{C}P^n \) of degree \( d \) with \( f(M) \not\subset D \). We denote the intersection multiplicity of the image of \( f \) and \( D \) at \( f(p) \) by \( \nu_{(f,D)}(p) \) and the pull-back of the normalized Fubini-Study metric form on \( \mathbb{C}P^n \) by \( \Omega_f \). The non-integrated defect of \( f \) with respect to \( D \) cut by \( \mu_0 \) is defined by

\[
\bar{\delta}_f^{[\mu_0]}(D) := 1 - \inf\{\eta \geq 0 : \eta \text{ satisfies condition } (*)\}.
\]

Here, the condition (*) means that there exists a bounded nonnegative continuous function \( h \) on \( M \) with zeros of order not less than \( \min\{\nu_{(f,D)}, \mu_0\} \) such that \( \eta \Omega_f + dd^c \log h^2 \geq \min\{\nu_{(f,D)}, \mu_0\} \), where
Recently, M. Ru and S. Sogome [16] generalized the above result of H. Fujimoto for meromorphic maps of a complete Kähler manifold into the complex projective space $\mathbb{C}P^n$ intersecting hypersurfaces in general position. After that, T.V. Tan and V.V. Truong [18] generalized successfully the above result of H. Fujimoto for meromorphic maps of a complete Kähler manifold into a complex projective variety $V \subset \mathbb{C}P^n$ intersecting global hypersurfaces in subgeneral position in $V$ in their sense. Later, Q. Yan [19] showed the non-integrated defect for meromorphic maps of a complete Kähler manifold into $\mathbb{C}P^n$ intersecting hypersurfaces in subgeneral position in the original sense in $\mathbb{C}P^n$. We would like to emphasize that, in the results of the above mentioned authors, there have been two strong restrictions.

- The above mentioned authors always required a strong assumption (C) that functions $h$ in the notion of the non-integrated defect are continuous. By this request, their non-integrated defect is still small.
- The above mentioned authors always asked a strong assumption as follows: (H) The complete Kähler manifold $M$ whose universal covering is biholomorphic to the unit ball of $\mathbb{C}^l$.

Motivated by studying meromorphic mappings into compact complex manifolds in [3] and from the point of view of the Nevanlinna theory on polydiscs, the second main aim of this paper is to improve the above-mentioned definition of the non-integrated defect relation of H. Fujimoto by omiting the assumption (C) (cf. Subsection 4.1 below) and to study the non-integrated defect for meromorphic mappings from a Stein manifold without the assumption (H) into a compact complex manifold sharing divisors in subgeneral position (cf. Theorems 4.3 and 4.7 below). As a direct consequence, we get the following Bloch-Cartan theorem for meromorphic mappings from $\mathbb{C}^l$ to a smooth algebraic variety $V$ in $\mathbb{C}P^m$ missing hypersurfaces in subgeneral position: a non-constant meromorphic mapping of $\mathbb{C}^l$ into an algebraic variety $V$ of $\mathbb{C}P^m$ cannot omit $(2N + 1)$ global hypersurfaces in $N$-subgeneral position in $V$. We would like to emphasize that, by using our arguments and their techniques in [16], [18], [19] we can generalize exactly their results to meromorphic mappings from a Stein manifold without the assumption (H) into a smooth complex projective variety $V \subset \mathbb{C}P^M$ (cf. Remark 4.6 below).

In [8], the author gave a unicity theorem for meromorphic mappings from a complete Kähler manifold satisfying the assumption (H) into the complex projective space $\mathbb{C}P^n$. The last aim of this paper is to give
an analogous unicity theorem for meromorphic mappings from a Stein manifold without the assumption \((H)\) to a compact complex manifold.

2. SOME FACTS FROM PLURI-POTENTIAL THEORY

2.1. Derivative of a subharmonic function. In this subsection, we give an estimation of derivative of a subharmonic function. Firstly, we recall some definitions.

For \(R > 0\), we consider the ball of radius \(R\) as follows:
\[
B_R = \{ x \in \mathbb{R}^n : |x| < R \},
\]
where \(|x|\) is the Euclidean norm in \(\mathbb{R}^n\).

For \(R = (R_1, \ldots, R_n)\) with \(R_j > 0\) for each \(1 \leq j \leq n\), we consider the polydisc with a radius \(R\) as follows:
\[
\Delta_R = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x_j| < R_j \text{ for each } 1 \leq j \leq n \}.
\]

If \(R_1 = \cdots = R_n := R > 0\), then the polydisc \(\Delta_R\) is denoted by \(\Delta := \Delta(R)\).

For \(x \in \mathbb{R}^n - \{0\}\), put
\[
E(x) = \begin{cases} 
(2\pi)^{-1} \log|x| & \text{if } n = 2, \\
-|x|^{2-n}/((n-2)c_n) & \text{if } n > 2,
\end{cases}
\]
where \(c_n\) is the area of the unit sphere in \(\mathbb{R}^n\). The classical Green function of \(B_R\) with pole at \(x \in B_R\) is
\[
G_R(x,y) = \begin{cases} 
E(x-y) - E\left(\frac{|x|}{R} \left|\frac{R^2}{|x|^2} y\right|\right) & \text{if } x \neq 0, x \neq y \\
\frac{1}{(n-2)c_n} \left(\frac{1}{|y|^{n-2}} - \frac{1}{R^{n-2}}\right) & \text{if } x = 0.
\end{cases}
\]

Note that \(G_R(x,y) = G_R(y,x) \leq 0\). The Poisson kernel of \(B_1\) is given by
\[
P(x,y) = \frac{1}{c_n} (1 - |x|^2)|y - x|^{-n} \geq 0, \ |y| = 1, \ |x| < 1.
\]

**Theorem 2.1.** (Riesz representation formula) Let \(u\) be a subharmonic function \(\neq -\infty\) in the ball \(B_R = \{ x \in \mathbb{R}^n : |x| < R \}\). Take \(0 < R' < R\). Then
\[
u(x) = \int_{B_{R'}} G_{R'}(x,y) \Delta u dm + \int_{\partial B_1} u(R'y) P\left(\frac{x}{R'}, y\right) d\omega(y), \ x \in B_R,
\]
where \(\Delta u\) is considered as a distribution (\(m\) is the Lebesgue measure on \(\mathbb{R}^n\)).
For a proof of this theorem, we refer to [2, Proposition 4.22]. The following has a crucial role in the proof of Proposition 3.7 on Logarithmic derivative lemma.

**Proposition 2.2.** Let \( u \) be a subharmonic function \( \not\equiv -\infty \) in the ball \( B_R \). Assume that \( u \) has the derivative a.e in \( B_R \). Then

\[
\int_{\Delta(R/4^n)} |\frac{\partial u}{\partial x_k}(a)| \, da \leq S(n) (R^{-5} + R^n) \left( \sup_{|x| \leq R} u(x) - \int_{\partial B_{R/2}} u \, d\omega \right)
\]

for each \( 1 \leq k \leq n \), where \( S(n) \) is a constant depending only on \( n \).

**Proof.** By considering \( u - \sup_{|x| \leq R} u(x) \) instead of \( u \) one can suppose that \( \sup_{|x| \leq R} u(x) = 0 \). By Theorem 2.1 and the hypothesis, we have

\[
|\frac{\partial u}{\partial x_k}(a)| \leq \int_{B_{R/2}} \left| \frac{\partial G_{R/2}(a, y)}{\partial x_k} \right| \Delta u \, dm + \int_{\partial B_1} \left| u(Ry/2) \right| \left| \frac{\partial P(a_{R/2}, y)}{\partial x_k} \right| \, d\omega(y)
\]

By a direct computation, we get

\[
c_n \frac{\partial G_{R/2}(a, y)}{\partial x_k} = -\frac{a_k - y_k}{|a - y|^n} + \left( \frac{R}{2} \right)^{n-2} \frac{|a|^2 - y|a|^2}{|a|^2 - y|a|^2} a_k - \frac{|a|^2 ((R/2)^2 a_k - y_k|a|^2)}{|(R/2)^2 a - y|a|^2} \frac{\partial P(a_{R/2}, y)}{\partial x_k}
\]

and

\[
\frac{\partial P(a_{R/2}, y)}{\partial x_k} = -2 \frac{a_k}{R^{n+2}} - y|a|^2 - n(1 - R^{n+2}) \frac{a_k}{R^{n+2}} - \frac{c_n |a_{R/2}| - y|a|^2}{R^{n+2}}.
\]

Take \( a \) such that \( |a| \leq R/4 \) (note that if \( a \in \Delta(R/4n) \) then \( |a| \leq R/4 \)). Then, there exists \( S(n) \) depending only on \( n \) such that

\[
|\frac{\partial G_{R/2}(a, y)}{\partial x_k}| \leq S(n) \left( \frac{1}{|a - y|^{n-1}} + \frac{1}{R^{n+3}} + \frac{1}{R^{2n+3}} \right),
\]

\[
|\frac{\partial P(a_{R/2}, y)}{\partial x_k}| \leq S(n).
\]

Therefore, for \( |a| \leq R/4 \),

\[
|\frac{\partial u}{\partial x_k}(a)| \leq S(n) \left( \int_{B_{R/2}} \left( \frac{1}{|a - y|^{n-1}} + \frac{1}{R^{n+3}} + \frac{1}{R^{2n+3}} \right) \Delta u \, dm + \int_{\partial B_1} |u(Ry/2)| \, d\omega(y) \right).
\]
In the Riesz representation formula of $u$, taking $x = 0$, we get
\[ u(0) = \int_{B_{3R/4}} G_{3R/4}(0, y) d\mu(y) + \int_{\partial B_1} u(3Ry/4) P(0, y) d\omega(y). \]
Hence
\[ \int_{B_{R/2}} \Delta u dm \leq S(n) R^n \int_{\partial B_{R/2}} -u d\omega. \]

For convenience, in this proof, $S(n)$ always stands for a constant depending only on $n$. Therefore,
\[ |\frac{\partial u}{\partial x_k}(a)| \leq S(n) \left( \int_{B_{R/2}} \frac{1}{|a-y|^{n-1}} \Delta u dm + (R^{-5} + R^{-n-5} + 1) \int_{\partial B_{R/2}} -u d\omega \right). \]

Integrating the above inequality over $\Delta(R_{4n}^n)$, we obtain
\[ \frac{1}{S(n)} \int_{\Delta(R_{4n}^n)} \left| \frac{\partial u}{\partial x_k}(a) \right| da \leq \int_{B_{R/2}} \Delta u dm(y) \int_{\Delta(R_{4n}^n)} \frac{1}{|a-y|^{n-1}} da \]
\[ + \int_{\Delta(R_{4n}^n)} (R^{-5} + R^{-n-5} + 1) \int_{\partial B_{R/2}} -u d\omega da \]
\[ \leq \int_{B_{R/2}} \Delta u dm(y) \int_{B(|y| + R/2)} \frac{1}{|a-y|^{n-1}} da \]
\[ + (R^{-5} + R^n) \int_{\partial B_{R/2}} -u d\omega \]
\[ \leq (R^{-5} + R^n) \int_{\partial B_{R/2}} -u d\omega. \]

\[ \square \]

2.2. Pluri-complex Green function. For detailed explosion of this subsection, one may consult [13, Chapter 6, Section 6.5] and [2, Chapter III, Section 6]. Firstly, we denote $d = \frac{\partial}{\partial t} + \frac{\partial}{\partial \overline{t}}$ and $d^c = \frac{1}{4\pi} (\partial - \overline{\partial})$ as usual.

Let $\Omega$ be a connected open subset of $\mathbb{C}^n$ and let $a$ be a point in $\Omega$. If $u$ is a plurisubharmonic function in a neighborhood of $a$, we shall say that $u$ has a logarithmic pole at $a$ if
\[ u(z) - \log|z-a| \leq O(1), \text{ as } z \to a, \]
where $|z-a|$ is the Euclidean norm in $\mathbb{C}^n$. The pluricomplex Green function of $\Omega$ with pole at $a$ is
\[ g_{\Omega,a}(z) = \sup \{ u(z) : u \in \mathcal{P}SH(\Omega, [-\infty, 0)) \text{ and } u \text{ has a logarithmic pole at } a \}. \]
(It is assumed here that $\sup \mathcal{O} = \infty$). We would like to notice that if $V$ is a plurisubharmonic function, then the $\ddc V \wedge (\ddc g_{\Omega, a})^{n-1}$ is well-defined (see [2, Proposition 4.1]). For $r \in (-\infty, 0]$, put

$$g_r(z) = \max\{g_{\Omega, a}(z), r\}, \quad S(r) = g_{\Omega, a}^{-1}(r).$$

Define

$$\mu_r = (\ddc g_r(z))^n - 1_{\{g_{\Omega, a} \geq r\}}(\ddc g_{\Omega, a})^n, \quad r \in (-\infty, 0).$$

Then the measure $\mu_r$ is supported on $S(r)$ and $r \to \mu_r$ is weakly continuous on the left. Denote by $\mu_{\Omega, a}$ the weak-limit of $\mu_r$ as $r \to 0$. We now consider $\Omega = \Delta_R = \{(z_1, z_2, \cdots, z_n) \in \mathbb{C}^n : |z_1| < R_1, \cdots, |z_n| < R_n\}$ is a polydisc in $\mathbb{C}^n$. For brevity, we will denote the polydisc $\Delta_R$ by $\Delta$ in the end of this subsection. Then, we have

$$g_{\Delta, a} = \max_{1 \leq j \leq n} \log \left| \frac{R_j(z_j - a_j)}{R_j^2 - z_j \bar{a}_j} \right|.$$

Moreover, $\mu_{\Delta, a}$ is concentrated on the distinguished boundary $\partial' \Delta$ of $\Delta$ and

$$(1) \quad d\mu_{\Delta, a} = \prod_{j=1}^n \frac{R_j^2 - |a_j|^2}{|a_j - R_j e^{i\theta}|^2} dt_1 \cdots dt_n.$$

**Theorem 2.3.** Let $V$ be a plurisubharmonic function on an open neighborhood of a polydisc $\Delta$ of $\mathbb{C}^n$. Let $g_{\Delta, a}$ be a pluricomplex Green function of $\Delta$ with pole at $a = (a_1, \cdots, a_n) \in \Delta$. Then

$$\int_{\partial' \Delta} V \prod_{j=1}^n \frac{R_j^2 - |a_j|^2}{|a_j - R_j e^{i\theta}|^2} dt_1 \cdots dt_n - (2\pi)^n V(a) = \int_{-\infty}^0 \int_{\{g_{\Delta, a} < t\}} \ddc V \wedge (\ddc g_{\Delta, a})^{n-1} \, dt$$

*Proof.* By the Lelong-Jensen formula (see [13, Chapter 6, Section 6.5] and [2, Chapter III, Section 6]) and the fact that $(\ddc g_{\Delta, a})^n = (2\pi)^n \delta_{\{a\}}$, we obtain for $r < 0$

$$\mu_r(V) - \int_{\{g_{\Delta, a} < r\}} V(\ddc g_{\Delta, a})^n = \int_{-\infty}^r \int_{\{g_{\Delta, a} < t\}} \ddc V \wedge (\ddc g_{\Delta, a})^{n-1} \, dt.$$

It is clear that the right-handed side converges to

$$\int_{-\infty}^0 \int_{\{g_{\Delta, a} < t\}} \ddc V \wedge (\ddc g_{\Delta, a})^{n-1} \, dt$$

as $r$ tends to 0. Now suppose that $V$ is continuous. Since the supports of $\mu_r \subset \Delta$ and $\mu_r$ weakly converge to $\mu_{\Omega, a}$, we get $\mu_r(V) \to \mu_{\Omega, a}(V)$ as
r tends to 0. In general, by taking a decreasing sequence of continuous plurisubharmonic functions $V_n$ converging to $V$, we get the desired equality. Notice that $\mu_{\Omega,a}(V)$ is finite by (1) and
\[ \int V(dd^c g_{\Delta,a})^n = (2\pi)^n \delta_{(a)}(V) = (2\pi)^n V(a) \]
is finite or $-\infty$. \hfill \Box

2.3. Pluri-subharmonic functions on complex manifolds. This subsection is devoted to prove a version of [12, Theorem A] in the case where $M$ is Stein and $u$ is a (not necessary continuous) plurisubharmonic function. Throughout this subsection $M$ will denote an $m$-dimensional closed complex submanifold of $\mathbb{C}^n$ and the Kähler metric of $M$ is induced from the canonical one of $\mathbb{C}^n$.

**Definition 2.4.** Let $N$ be a complex manifold and $f$ be a locally integrable real function in $N$. We say that $f$ is plurisubharmonic function (or psh function, for brevity) if $dd^c f \geq 0$ in the sense of currents.

**Lemma 2.5.** (see [12, Lemma, p.552]) Let $N_1$ be a Kähler manifold and $N_2$ be a complex manifold. Let $g$ be a holomorphic map of $N_1$ to $N_2$. Then for each $C^2$-psh function $f$ in $N_2$, $f \circ g$ is subharmonic in $N_1$.

**Lemma 2.6.** The volume of $M$ is infinite.

*Proof.* Take a point $a \in M$. Let $B_M(a, R)$ be the ball centered at $a$ of $M$ and of radius $R$. Put $u = |z - a|$. Then $u$ is a psh function on $\mathbb{C}^n$ and hence, it is a subharmonic function on $M$. Since the Kähler metric on $M$ is induced from the canonical one of $\mathbb{C}^n$, it implies that $B_M(a, R) \subset B(a, R)$, where $B(a, R)$ is the usual ball centered at $a$ and of radius $R$ in $\mathbb{C}^n$. Therefore $u \leq R$ in $B_M(a, R)$. By [12, Theorem A], we get
\[ \liminf_{r \to \infty} \frac{1}{R^2} \int_{B_M(a, R)} u^2 \ dvol = \infty. \]
From this we deduce that $\int_M dvol = \infty$. \hfill \Box

**Proposition 2.7.** Let $u$ be a psh function on $M$ and $K$ be a compact subset of $M$. For each open subset $U$ of $M$ such that $K \subset U \subset M$, there exists a decreasing sequence of $C^\infty$-psh functions $u_k$ in $U$ such that $u_k$ converge to $u$, a.e in $U$. Moreover, if $u$ is non-negative then $u_k$ is non-negative.

*Proof.* By [10, Chapter VIII, Theorem 8], there exists a holomorphic retraction $\alpha$ of an open subset $V$ of $\mathbb{C}^n$ containing $M$ to $M$, i.e $\alpha$ is
holomorphic and \( \alpha|_M = id_M \). Then \( u \circ \alpha \) is a psh function on \( V \). The conclusion now is deduced immediately from this fact. \( \square \)

As a direct consequence, we get the following.

**Corollary 2.8.** Let \( \xi \) be an increasing convex function in \( \mathbb{R} \). Let \( u \) be a psh function on \( M \). Then \( \xi \circ u \) is a psh function. Specially, if \( u \) is non-negative then \( u^p \) \( (p \geq 1) \) is in the Sobolev space \( \mathcal{H}_0(M) \) of degree 0 of \( M \).

**Theorem 2.9.** Let \( u \) be a non-negative psh function on \( M \) and \( p \) be a positive number greater than 1. Take a point \( a \in M \). Let \( B_M(a,R) \) (or \( B(R) \) for brevity) be the ball centered at \( a \) of \( M \) and of radius \( R \). Then one of the following two statements holds:

(i) \[
\liminf_{r \to \infty} \frac{1}{R^2} \int_{B_M(a,R)} u^p \, d\text{vol} = \infty.
\]

(ii) \( u \) is constant a.e in \( M \).

**Proof.** Suppose that \( u \) is not constant a.e in \( M \) and

\[
\liminf_{r \to \infty} \frac{1}{R^2} \int_{B_M(a,R)} u^p \, d\text{vol} = A < \infty.
\]

Then, there exists a sequence \( \{r_j\} \) such that

\[
\frac{1}{r_j^2} \int_{B_M(a,r_j)} u^p \, d\text{vol} = A.
\]

By Proposition \[2.7\] there is a decreasing sequence \( u_k \) of \( C^\infty \)-nonnegative functions such that \( u_k \) is psh in \( B(r_{k+2}) \) and \( u_k \) converge to \( u \), a.e in \( B(r_{k+1}) \). By the monotone convergence theorem, there exists a subsequence of \( u_k \), without loss of generality we may assume that this subsequence is \( u_k \), satisfying

\[
\frac{1}{r_j^2} \int_{B_M(a,r_j)} u^p \, d\text{vol} \leq \frac{1}{r_j^2} \int_{B_M(a,r_j)} u^p \, d\text{vol} + 1.
\]

For each \( j \geq 1 \), let \( \varphi_j \) be a Lipschitz continuous function such that \( \varphi_j(x) \equiv 1 \) on \( B(a,r_j) \) and \( \varphi_j(x) \equiv 0 \) in \( M \setminus B(a,r_{j+1}) \) and \( \text{grad} \varphi_j \leq \frac{C}{r_{j+1}} \) a.e on \( M \), where \( C \) is a constant which does not depend on index \( j \) (see \[12\], Lemma 1). Put

\[
I_j^N(\epsilon) = \int_{B(r_{j+1})} \varphi_j^2(u_N^2 + \epsilon)^{\frac{p-2}{2}} \|\text{grad} u_N\|^2 \, d\text{vol}, N \geq j
\]
and $I_j^N = \lim_{\epsilon \to 0} I_j^N(\epsilon)$. By the proof of [12, Theorem 2.1], $I_j^N < +\infty$ (by [2]) and there exists a constant $C = C(p) > 0$ such that

$$I_{j+1}^N I_j^N \leq (I_{j+1}^N)^2 \leq C(I_{j+1}^N - I_j^N), \quad 1 \leq j \leq N.$$  

The rest of the proof is proceeded as in the one of [12, Theorem 2.4]. For convenience we sketch it here.

It is easy to see that for some $j$, $I_j^N > 0$ for an infinite number of values of $N$. Since $I_j^N \leq I_k^N$ for $j \leq k \leq N$, it follows that there exist an index $j_0$ and a sequence $N_k \to +\infty$ such that for each $m \geq j_0$, $N_k \geq m$, $I_m^{N_k} > 0$. Divide (3) by $I_{j+1}^{N_k} I_j^N$, $m \leq j \leq N_k$ and summing over $j$ (from $m$ to $N_k$), we obtain $1/I_{m}^{N_k} \geq C(N_k - m)$ for a constant $C$. Hence,

$$\lim_{k \to +\infty} \int_{B(r_m)} (u_{N_k}^2 + \epsilon)^{p-2/2} ||\text{grad } u_{N_k}||^2 d\text{vol} = 0.$$

Now, let $\varphi \in C_0^2(M)$ and $q$ be the smallest integer greater than $(p - 2)/2$. Then,

$$\int_M u^{q+1} \Delta \varphi d\text{vol} = \lim_{k \to +\infty} u_{N_k}^{q+1} \Delta \varphi d\text{vol}$$

$$= -(q + 1) \lim_{k \to +\infty} u_{N_k}^q < \text{grad } u_{N_k}, \text{grad } \varphi >$$

$$= 0.$$

In the other words, $\Delta u = 0$ in the sense of currents. Hence, $u^{q+1} \in C^\infty$ by the regularity theorem (so that ”$\text{grad } u^{q+1}$” makes sense). Put $X = \text{grad } u^{q+1}$. Then,

$$\int ||X||^2 \varphi d\text{vol} = \int < \text{grad } u^{q+1}, \varphi X > d\text{vol}$$

$$= -\int u^{q+1} \text{div}(\varphi X) d\text{vol}$$

$$= -\lim_{k \to +\infty} \int u_{N_k}^{q+1} \text{div}(\varphi X) d\text{vol}$$

$$= (q + 1) \lim_{k \to +\infty} \int u_{N_k}^{q+1} < \text{grad } u_{N_k}, \varphi X > d\text{vol}$$

$$= 0.$$

Therefore, $X = 0$. That means $u$ is constant, a contradiction. \qed

**Corollary 2.10.** Let $u$ be a psh function on $M$. Then

$$\int_M e^u d\text{vol} = \infty.$$
3. NEVANLINNA THEORY FOR POLYDISCS

First of all we notice that the Euclidean ball is not biholomorphic to the polydisks so that construct a Nevanlinna theory in polydisks is not trivial task. Another important point is that by Theorem 1 [5] any complex manifold \( M \) is the union of a polydisc and a subset of \( M \) of Lebesgue zero measure. Hence understanding the value distribution of a holomorphic map departing from a polydisc will give some sense for the one from a general complex manifold, for example see Theorem 4.3 and Theorem 4.7.

In \( \mathbb{C}^n \), consider a polydisc
\[
\Delta(a, R) = \{(z_1, z_2, \cdots, z_n) \in \mathbb{C}^n : |z_1 - a_1| < R_1, \cdots, |z_n - a_n| < R_n\},
\]
where \( 0 < R_1, \cdots, R_n \leq \infty \). In case of \( a = 0 \), we simply denote \( \Delta(a, R) \) by \( \Delta_R \).

We now construct definitions in the case where \( R_j < \infty \) for each \( j \). The construction in the case where \( R_j = \infty \) for some \( j \) is similar.

3.1. First main theorem. Let \( \mathcal{L} \xrightarrow{\pi} X \) be a holomorphic line bundle over a compact complex manifold \( X \) and \( d \) be a positive integer. Let \( E \) be a \( \mathbb{C} \)-vector subspace of dimension \( m + 1 \) of \( H^0(X, \mathcal{L}^d) \). Take a basis \( \{c_k\}_{k=1}^{m+1} \) a basis of \( E \). Put \( B(E) = \cap_{\sigma \in E} \{\sigma = 0\} \). Then
\[
\cap_{1 \leq i \leq m+1} \{c_i = 0\} = B(E)
\]
and
\[
\omega = dd^c \log(|c_1|^2 + \cdots + |c_{m+1}|^2)^{1/d}
\]
is well-defined on \( X \setminus B(E) \).

Assume that \( R = (R_1, \cdots, R_n) \) and \( R' = (R'_1, \cdots, R'_n) \), where \( R_j > 0 \) and \( R'_j > 0 \) for each \( 1 \leq j \leq n \). Recall that \( R < R' \) (\( R \leq R' \) resp.) if \( 0 < R_j < R'_j \) (\( 0 < R_j \leq R'_j \) resp.) for each \( 1 \leq j \leq n \).

As usual, we say that the assertion \( P \) holds for a.e \( r \leq R \) if the assertion \( P \) holds for each \( r \leq R \) such that \( r_j \) is excluded a Borel subset \( E_j \) of the interval \( [0, R_j] \) with \( \int_{E_j} ds < \infty \) for each \( 1 \leq j \leq n \).

Let \( f \) be a meromorphic mapping of a polydisc \( \Delta_R \) of radius \( R = (R_1, \cdots, R_n) \) into \( X \) such that \( f(\Delta_R) \cap B(E) = \emptyset \). We define the characteristic function of \( f \) with respect to \( E \) as follows
\[
T_f(r, E) = \frac{1}{m(\Delta_{r_0})} \int_{\Delta_{r_0}} dm(a) \int_{-\infty}^{0} \int_{g_{\Delta_r,a} < s} f^* \omega \wedge (dd^c g_{\Delta_r,a})^{n-1} ds
g_{\Delta_r,a} = g_{\Delta_r,a}^{n-1},
\]
and
\[
\omega = dd^c \log(|c_1|^2 + \cdots + |c_{m+1}|^2)^{1/d}
\]
where \( r_0 < R \) is fixed, \( r_0 < r \leq R \) and \( dm \) is the Lebesgue measure in \( \mathbb{C}^n \). Remark that the definition does not depend on choosing basic of \( E \). Take a section \( \sigma \) of \( \mathcal{L}^s \) for some \( s \). Let \( D \) be its zero divisor. Put

\[
||\sigma|| := \frac{|\sigma|}{(|c_1|^2 + \cdots + |c_{m+1}|^2)^{\frac{1}{m}}}
\]

For \( 1 \leq k \leq \infty \), the truncated counting function of \( f \) to level \( k \) with respect to \( D \) is

\[
N_{f}^{[k]}(r, D) = \int_{\Delta_{r_0}} \frac{d\mu(a)}{m(\Delta_{r_0})} \int_{-\infty}^{0} \int_{g_{\Delta_{r_0}} < s} \min\{|[f^*D], k\} \land (dd^c g_{\Delta_{r_0}, a})^{n-1} \, ds \\
= \frac{1}{m(\Delta_{r_0})} \int_{\Delta_{r_0}} d\mu(a) \int_{\Delta_r} |g_{\Delta_{r_0}, a}| \min\{|[f^*D], k\} \land (dd^c g_{\Delta_{r_0}, a})^{n-1}
\]

and the proximity function is

\[
m_f(r, D) = \frac{1}{m(\Delta_{r_0})} \int_{\Delta_{r_0}} d\mu(a) \int_{\partial\Delta_r} \log \frac{1}{||\sigma \circ f||^2} \, d\mu_{\Delta_{r_0}, a}.
\]

For brevity, we will omit the character \([k]\) if \( k = \infty \). Now, take holomorphic functions \( f_0, f_1, \cdots, f_m \) in \( \Delta_R \) such that \( (f)_0 = f^*(c_1)_0, f_{i+1} = f^*_{c_{i+1}/c_i(f)} \) for \( i \geq 0 \). We get a reduced representation \( (f_0, f_1, \cdots, f_m) \) of \( f \). By the Lelong-Jensen formula, we get

**Theorem 3.1.** *(First main theorem)*

\[
sT_f(r, E) = N_f(r, D) + m_f(r, D) - \int_{\Delta_{r_0}} \log \frac{1}{||\sigma \circ f||^2} \, d\mu(a).
\]

**Proposition 3.2.**

\[
T_f(r, E) = \int_{\partial\Delta_r} \log(|f_0|^2 + \cdots + |f_m|^2)^{1/4} dt_1 dt_2 \cdots dt_n + O(1),
\]

where \( \partial\Delta_r \) is the distinguished boundary of \( \Delta_r \). Hence, \( T_f(r, E) \) is a convex increasing function of \( \log r_i \) for each \( i \).

We also have an analogue for \( N_f(r, D) \).

**Remark 3.3.** Put \( \log^+ s = \max\{\log s, 0\} \) for all \( s \geq 0 \). Let \( g \) be a meromorphic function on \( \Delta_R \). Then \( g \) is considered as a mapping of \( \Delta_R \) into \( \mathbb{C}P^1 \) by sending \( z \) to \( [g_1(z), g_2(z)] \), where \( g = g_1/g_2 \). Let \( H_1 \) be the hyperplane bundle on \( \mathbb{C}P^1, \) \( D = [0, 1] \) (the divisor consists of only one component \([0, 1]\) with coefficient 1). Put

\[
m_g(r) = \int_{\partial\Delta_r} \log^+ |g(re^{it})| \, dt_1 \cdots dt_n.
\]

Then \( m_g(r) = m_g([0, 1]) + O(1) \).
In fact, we have
\[
\int_{\partial \Delta_r} \log |g(re^{it})| \, dt = \int_{\partial \Delta_r} \log \left(1 + \left| \frac{g_1(re^{it})}{g_2(re^{it})} \right| \right) \, dt + O(1)
\]
\[= m_g(r, [0, 1]) + O(1).\]

3.2. Second main theorem. Let \( g \) be a meromorphic function on a polydisc \( \Delta_R \). For an \( n \)-tuple \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) of non-negative integers, we put
\[
|\alpha| = \sum_{i=1}^{n} \alpha_i, \quad D^n g = \frac{\partial^{|\alpha|} g}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}.
\]

First of all, we need some auxiliary lemmas.

Lemma 3.4. Let \( f \in L^1(\Delta_R) \). Then \( f \in L^1(\partial \Delta_r) \) for e.a \( r \leq R \). Put
\[
f_1(r) = \int_{\partial \Delta_r} f \, dt_1 \cdots dt_n \quad (r \leq R)
\]
and
\[
f_2(r) = \int_{\Delta_r} f(z) \, dm(z) \quad (r \leq R).
\]
Then
\[
\frac{\partial^n f_2}{\partial r_1 \partial r_2 \cdots \partial r_n}(r) = r_1 r_2 \cdots r_n f_1(r) \quad \text{for a.e } r \leq R.
\]

Proof. It is known that for a disc \( \Delta_r \) in \( \mathbb{C} \) and \( h(z) \in L^1(\Delta_r) \), we have
\[
\frac{d}{ds} \int_{\Delta_s} h(z) \, dm(z) = s \int_{\partial \Delta_s} h \, dt
\]
for a.e \( s \leq r \). The proof now is deduced from the Fubini theorem and the above formula. \( \square \)

Lemma 3.5. Let \( \phi(r) \geq 0 \) be a monotone increasing function for \( 0 < r \leq R \). Let \( \delta \) be a positive real number. Then
\[
\frac{\partial^n \phi}{\partial r_1 \partial r_2 \cdots \partial r_n}(s) \leq \prod_{j=1}^{n} \frac{1}{R_j - s_j} \phi(s)^{1+\delta}
\]
for all \( s \leq R \) such that \( s_j \) does not belong to a set \( E_j \subset [0, R_j] \) with
\[
\int_{E_j} \frac{1}{R_j - s_j} \, dt < \infty \quad (1 \leq j \leq n).
\]

Proof. Put \( r_j = R_j - R_j e^{-\rho_j} \), i.e. \( \rho_j = -\log \left( \frac{R_j - r_j}{R_j} \right) \geq 0 \). Then
\[
\frac{\partial^n \phi}{\partial r_1 \partial r_2 \cdots \partial r_n}(s) = \frac{\partial^n \phi}{\partial \rho_1 \partial \rho_2 \cdots \partial \rho_n}(s) \prod_{j=1}^{n} \frac{1}{R_j - s_j}.
\]
The proof is easily deduced from applying [15, Lemma 1.2.1]. \( \square \)
Lemma 3.6. ([11, Lemma 2.4]) Assume that $T(r)$ is a continuous and increasing function for $r_0 \leq r < R_0 < \infty$ and $T(r) \geq 1$. Then we have

$$T\left(r + \frac{R_0 - r}{eT(r)}\right) < 2T(r)$$

outside a set $E_0$ of $r$ such that $\int_{E_0} \frac{1}{R_0 - s} \, ds < \infty$.

Proposition 3.7. (Lemma on logarithmic derivative) Let $g$ be a meromorphic function in $\Delta_R$. Then, there exists a constant $C$ depending only on $n$ such that

$$m_{D^{\alpha}g}(r, [0, 1]) \leq C \left(\log T_g(r, E) + \sum_{j=1}^{n} \log \frac{1}{R_j - r_j}\right)$$

for all $\alpha \in \mathbb{Z}^n_+$ and for all $r \leq R$ such that $r_j$ does not belong to a set $E_j \subset [0, R_j]$ with $\int_{E_j} \frac{1}{R_j - s} \, ds < \infty$ ($1 \leq j \leq n$).

Proof. We consider the case that $|\alpha| = 1$. The general case follows easily from this case. Take $r < r'$. Write $g = g_1/g_2$. Notice that $\partial g/g = \partial g_1/g_1 - \partial g_2/g_2$ and $|\partial \log |g_j||/|g_j| = |\partial g_j/g_j|$ for $j = 1, 2$. Applying Proposition 2.2 for $\log |g_1|, \log |g_2|$ we get

$$\int_{\Delta(a, r' - r/2)} |\partial z_k g/g| \, dm(z) \leq \sup_{|z-a| \leq r' - r/2} \log |g_1 g_2| - \int_{|z-a| = r' - r/2} \log |g_1 g_2| \, d\omega.$$ 

Hence, for $a \in \Delta_r$,

$$\int_{\Delta(a, r' - r/2\alpha)} \left|\frac{\partial g/\partial z_k}{g}\right| \, dm(x) \leq \left(\prod_{j=1}^{n} \frac{2r_j'}{r_j' - r_j}\right) T_f(r', E).$$

By dividing the polydisc $\Delta_r$ into many polydiscs which have the form $\Delta(a, r' - r/2\alpha)$, we obtain

$$\int_{\Delta_r} \left|\frac{\partial g/\partial z_k}{g}\right| \, dm(x) \leq C(n) \prod_{j=1}^{n} \left(\frac{r_j}{R_j - r_j}\right)^3 T_f(r', E),$$

where $C(n)$ is a constant depending only on $n$. Moreover, by Lemma 3.6 we deduce that

$$\int_{\partial^n \Delta_r} \left|\frac{\partial g/\partial z_k}{g}\right| \, dt_1 \cdots dt_n \leq C(n) \prod_{j=1}^{n} \left(\frac{r_j}{R_j - r_j}\right)^3 T_f^{3n+1}(r, E).$$
for all \( r \leq R \) such that \( r_j \) does not belong to a set \( E_j \subset [0, R_j] \) with \( \int_{E_j} \frac{1}{R_j-s} \, ds < \infty \). Hence,

\[
\int_{\partial \Delta_r} \log^{+} \left| \frac{\partial g/\partial z_k}{g} \right| \, dt_1 \cdots dt_n \leq \sum_{j=1}^{n} \log \left( \frac{r_j}{R_j-r_j} \right) + (3n + 1) \log T_f(r', E) + O(1)
\]

for all \( r \leq R \) such that \( r_j \) does not belong to a set \( E_j \subset [0, R_j] \) with \( \int_{E_j} \frac{1}{R_j-s} \, ds < \infty \). The proof is completed. \( \square \)

**Remark 3.8.** We can apply the above argument to the characteristic function in the usual sense (i.e. over the balls) of a nonzero meromorphic function \( g \) in \( \mathbb{C}^n \) to get a simple new proof of Lemma on logarithmic derivative.

Now, put \( \Delta(a) = \{(z_1, \ldots, z_n) : |z_k| \leq a \text{ for all } k\} \) and \( \Delta'(a) = \{(z_1, \ldots, z_n) : |z_k| \leq a \text{ for } k \neq i, |z_i| = a\} \). The measure \( m_i \) on \( \Delta(a) \) is the product of the \((n-1)\) dimensional Lebesgue measure and the usual measure on a circle in \( \mathbb{C} \).

**Proposition 3.9.** Let \( g \) be a meromorphic function in \( \Delta_R \). Let \( p, p' \) be positive real numbers such that \( p < p' \). Assume \( R = (R_1, \ldots, R_0) \) and \( p |\alpha| \leq 1 \). Then, there exist a constant \( C \) depending only on \( n \) and \( \alpha \) and a set \( E \subset [0, R_0] \) satisfying \( \int_E \frac{1}{R_0-t} \, dt < \infty \) such that for all \( a \in [0, R_0] \) outside the set \( E \),

\( \begin{align*}
(i) & \quad |D^\alpha g|^p \text{ is integrable in } \Delta(a) \text{ with the given measure.} \\
(ii) & \quad \sum_{i=1}^{n} \int_{\Delta \setminus E} \left( \left| \frac{D^\alpha g}{g} \right| \right)^p \, dm_i(z) \leq \frac{C}{R_0-a} \left( \frac{R_0^3}{(R_0-a)^3} T_f((a, \ldots, a), E) \right)^p |\alpha|^n .
\end{align*} \)

**Proof.** Let \( \alpha_k, k = 1, 2, \ldots, |\alpha| \) be a sequence of \( n \)-tuples satisfying: \( \alpha_1 = 0, |\alpha_k| = |\alpha_{k-1}| + 1 \), for all \( k \geq 2 \). By the proof of Proposition 3.7 and \( p < 1 \),

\[
\int_{\Delta \setminus E} \left| \frac{D^{\alpha_k} g}{D^{\alpha_{k-1}} g} \right|^{p|\alpha|} \, dm(z) \leq C(n) \prod_{j=1}^{n} \left( \frac{r_j}{R_j-a} \right)^3 T_f^{(3n+1)p|\alpha|}((a, \ldots, a), E)^p
\]

for all \( a \leq R \) outside a set \( E \subset [0, R_0] \) with \( \int_{E_0} \frac{1}{R_0-s} \, ds < \infty \). Put

\[
h(a) = \int_{\Delta \setminus E} \left| \frac{D^\alpha g}{g} \right|^p \, dm(z) \quad (a \in [0, R_0]).
\]
Then
\[
h(a) = \int_{\Delta(a)} \prod_{k=2}^{[\alpha]} \frac{D^{\alpha_k} g}{D^{\alpha_k-1} g}|^p dm(z) \leq \frac{1}{|\alpha|} \sum_{k=2}^{[\alpha]} \int_{\Delta(a)} \left| \frac{D^{\alpha_k} g}{D^{\alpha_k-1} g}|^p \right| dm(z)
\]
\[
\leq \frac{1}{|\alpha|} \sum_{k=2}^{[\alpha]} C(n) \prod_{j=1}^n \left( \frac{r_j}{R_j - a} \right)^3 T_f^{(3n+1)p[\alpha]}((a, \cdots, a), E)^p
\]
On the other hand, we have
\[
\frac{\partial h}{\partial a} = a \sum_{i=1}^n \frac{\partial h}{\partial \alpha_i} = a \sum_{i=1}^n \int_{\Delta(a)} \left( \left| \frac{D^\alpha g}{g} \right|^p \right) dm_i(z).
\]
From this and Lemma 3.5 we get the desired conclusion. ■

Now, we need the generalized Wronskian of a meromorphic mapping which is due to H. Fujimoto [7].

**Proposition 3.10.** Let \( F : \Delta \rightarrow \mathbb{C}^m \) be a linearly non-degenerate mapping. Assume that \( F = (F_0, \cdots, F_{m+1}) \) is a reduced representation of \( F \). Then there exist \( n \)-tuples \( \alpha_1, \cdots, \alpha_{m+1} \) such that
\[
|\alpha_1| + \cdots + |\alpha_m| \leq \frac{m(m+1)}{2} \quad \text{and} \quad |\alpha_k| \leq m \quad (1 \leq k \leq m+1)
\]
and the generalized Wronskian of \( F \)
\[
W_{\alpha_1, \cdots, \alpha_{m+1}}(F) := \det(D^{\alpha_i} F_j : 1 \leq i, j \leq m+1) \neq 0.
\]
Moreover, for such \( \alpha_1, \cdots, \alpha_{m+1} \), we have
\[
\left( \frac{W_{\alpha_1, \cdots, \alpha_{m+1}}(F)}{F_1 \cdots F_{m+1}} \right)_\infty \leq \sum_{i=1}^{m+1} \min\{(F_i)_0, m\}.
\]

**Proof.** See [7] Proposition 4.5 and Proposition 4.10. ■

Let \( L \rightarrow X \) be a holomorphic line bundle over a compact complex manifold \( X \) of dimension \( n \) and \( E \) be a \( \mathbb{C} \)-vector subspace of \( H^0(X, L) \) of dimension \( m+1 \). Let \( \{c_k\}_{k=1}^{m+1} \) be a basis of \( E \) and \( B(E) \) be the base locus of \( E \). Define a mapping \( \Phi : X \setminus B(E) \rightarrow \mathbb{C}^m \) by
\[
\Phi(x) := [c_1(x) : \cdots : c_{m+1}(x)].
\]
Denote by rank\( E \) the maximal rank of Jacobian of \( \Phi \) on \( X \setminus B(E) \). It is easy to see that this definition does not depend on choosing a basis of \( E \). Take \( \sigma_j \in H^0(X, L), D_j = \{\sigma_j = 0\} \ (1 \leq j \leq q) \). Assume that \( N \geq n \) and \( q \geq N + 1 \).
Definition 3.11. The hypersurfaces $D_1, D_2, \ldots , D_q$ is said to be located in $N$-subgeneral position with respect to $E$ if for any $1 \leq i_0 < \cdots < i_N \leq q$, we have $\cap_{j=0}^{N} D_{i_j} = B(E)$.

Assume that $\{D_j\}$ is located in $N$-subgeneral position with respect to $E$. Put $u = \text{rank}E, b = \text{dim}B(E) + 1$ if $B(E) \neq \emptyset$ and $b = -1$ if $B(E) = \emptyset$. Assume that $u > b$. We set $\sigma_i = \sum_{1 \leq j \leq m+1} a_{ij} c_j$, where $a_{ij} \in \mathbb{C}$. Define a mapping $\Phi : X \to \mathbb{CP}^m$ by

$$\Phi(x) := [c_1(x) : \cdots : c_{m+1}(x)].$$

It is a meromorphic mapping. Let $G(\Phi)$ be the graph of $\Phi$. Define

$$p_1 : G(\Phi) \to X, p_2 : G(\Phi) \to \mathbb{CP}^m$$

by $p_1(x, z) = x, p_2(x, z) = z$. Since $X$ is compact, $p_1$ and $p_2$ are proper. Hence, $Y = \Phi(X) = p_2(p_1^{-1}(X))$ is an algebraic variety of $\mathbb{CP}^m$. Moreover, by definition of rank $E$, $Y$ is of dimension $\text{rank}E = u$. Denote by $\mathcal{H}$ the hyperplane line bundle of $\mathbb{CP}^m$. Put $H_i := \sum_{1 \leq j \leq m+1} a_{ij} z_{j-1}$, where $[z_0, z_1, \cdots , z_m]$ is the homogeneous coordinate of $\mathbb{CP}^m$.

For each $K \subset Q$, put $c(K) = \text{rank}\{H_i\}_{i \in K}$. We also set

$$n_0(\{D_j\}) = \max\{c(K) : K \subset Q \text{ with } |K| \leq N + 1\} - 1,$$

and

$$n(\{D_j\}) = \max\{c(K) : K \subset Q\} - 1.$$

Then $n(\{D_j\}), n_0(\{D_j\})$ are independent of the choice the $\mathbb{C}$-vector subspace $E$ of $H^0(X, L)$ containing $\sigma_j(1 \leq j \leq q)$. We see that

$$u \leq n_0(\{D_j\}) \leq n(\{D_j\}) \leq m.$$

Proposition 3.12. Let notations be as above. Assume that $D_1, \cdots , D_q$ are in $N$-subgeneral position with respect to $E$ and $q \geq 2N - u + 2 + b$. Put $k_N = 2N - u + 2 + b, s_N = n_0(\{D_j\})$ and $t_N = \frac{u - b}{n(\{D_j\}) - u + 2 + b}$.

Then, there exist Nochka weights $\omega(j)$ for $\{D_j\}$, i.e there exist constants $\omega(j)$ ($j \in Q$) and $\Theta$ satisfying the following conditions:

(i) $0 < \omega(j) \leq \Theta \leq 1$ ($j \in Q$) and $\Theta \geq t_N/k_N$.

(ii) $\sum_{j \in Q} \omega(j) \geq \Theta(q - k_N) + t_N$.

(iii) Let $E_j$ ($j \in Q$) be arbitrary positive real numbers and $R$ be a subset of $Q$ with $|R| = N + 1$. Then, there exist $j_1, \cdots , j_{s_N+1}$ in $R$ such that

$$\cap_{1 \leq i \leq s_N+1} H_{j_i} \cap Y = \cap_{j \in R} H_{j} \cap Y$$

and

$$\sum_{j \in R} \omega(j) E_j \leq \sum_{1 \leq i \leq s_N+1} E_{j_i}.$$
Proof. See [3] Section 2. \qed

By repeating the argument in [3], we get an analogous version of the Second Main Theorem in [3]. We state the following theorem without its proof.

**Theorem 3.13.** Let $X$ be a compact complex manifold. Let $\mathcal{L} \to X$ be a holomorphic line bundle over $X$. Fix a positive integer $d$. Let $E$ be a $\mathbb{C}$-vector subspace of dimension $m+1$ of $H^0(X, \mathcal{L}^d)$. Put $u = \text{rank}E$ and $b = \dim B(E) + 1$ if $B(E) \neq \emptyset$, otherwise $b = -1$. Take positive divisors $d_1, d_2, \cdots, d_q$ of $d$. Let $\sigma_j$ ($1 \leq j \leq q$) be in $H^0(X, \mathcal{L}^{d_j})$ such that $\sigma_1^{1/d}, \cdots, \sigma_q^{1/d} \in E$. Set $D_j = (\sigma_j)_0$ ($1 \leq j \leq q$). Assume that $D_1, \cdots, D_q$ are in $N$-subgeneral position with respect to $E$ and $u > b$. Let $f : \Delta_R \to X$ be an analytically non-degenerate meromorphic mapping with respect to $E$, i.e $f(\Delta_R) \not\subset \text{supp}(\langle \sigma \rangle)$ for any $\sigma \in E \setminus \{0\}$ and $f(\Delta_R) \cap B(E) = \emptyset$. Then, for all $r \leq R$ such that $r_j$ does not belong to a set $E_j \subset [0, R_j]$ with $\int_{E_j} \frac{1}{R_j - s} ds < \infty$, we have

$$(q - (m + 1)K(\mathcal{E}, N, \{D_j\}))T_f(r, E) \leq \sum_{i=1}^{q} \frac{1}{d_i} N_f^{[md_i/d]}(r, D_i) + S_f(r),$$

where $k_N, s_N, t_N$ are defined as in Proposition 3.12 and

$$K(\mathcal{E}, N, \{D_j\}) = \frac{k_N(s_N - u + 2 + b)}{t_N}.$$

4. Non-integrated defect relation

4.1. Definitions and basic properties. Let all notations be as in Section 3. The defect of $f$ with respect to $D$ truncated by $k$ in $E$ is defined by

$$\delta^{[k]}_{f,E}(D) = \liminf_{r \to R} \left( 1 - \frac{N^{[k]}_f(r, D)}{sT_f(r, E)} \right).$$

We now assume that $f$ is a meromorphic mapping of a connected complex manifold $M$ into $X$ such that $f(M) \cap B(E) = \emptyset$. Let $D$ be a divisor of $H^0(X, \mathcal{L}^s)$ for some $s > 0$. For $0 \leq k \leq \infty$, denote by $D^{[k]}_{f,E}$ the set of real numbers $\eta \geq 0$ such that there exists a bounded measurable nonnegative function $h$ on $M$ such that

$$\eta f^*dd^c \log(|c_1|^2 + \cdots + |c_{m+1}|^2)^{1/d} + dd^c \log h^2 \geq \frac{1}{s} \min\{k, f^*D\}$$

in the sense of currents. The non-integrated defect of $f$ with respect to $D$ in $E$ truncated by $k$ is defined by

$$\tilde{\delta}^{[k]}_{f,E}(D) := 1 - \inf\{\eta : \eta \in D^{[k]}_{f,E}\}.$$
Note that this definition does not depend on choosing a base of $E$.

**Remark 4.1.** In the original definition of H. Fujimoto [7], when $X = \mathbb{C}P^k$, $\mathcal{L}$ is the hyperplane bundle and $s = 1$ he required that functions $h, \frac{h}{\varphi}$ are continuous, where $\varphi$ is a holomorphic function in $M$ such that $(\varphi)_0 = \min\{k, f^* D\}$.

By [3, Theorem 1], there exists an open subset $U$ of $M$ such that $U$ is biholomorphic to a polydisc $\Delta_R$ and $M \setminus U$ has a zero measure, i.e. if $(V, \varphi)$ is a local coordinate then $\varphi((M \setminus U) \cap V)$ is of zero Lebesgue measure.

**Proposition 4.2.** We have the following properties of the non-integrated defect:

(i) $0 \leq \delta_{f,E}^{[k]}(D) \leq 1$.

(ii) $\delta_{f,E}^{[k]}(D) = 1$ if $f(M) \cap D = \emptyset$.

(iii) $\delta_{f,E}^{[k]}(D) \geq 1 - \frac{k}{k_0}$ if $f^* D \geq k_0 \min\{f^* D, 1\}$.

(iv) Denote by $f_U$ the restriction of $f$ to $U$ and assume that $\lim_{r \to R} T_{f_U}(r, E) = \infty$.

Then $0 \leq \delta_{f,E}^{[k]}(D) \leq \delta_{f,U,E}^{[k]}(D) \leq 1$.

**Proof.** The properties (i) and (ii) are evident. To prove (iii), put

$$h = \left(\frac{\sigma(f)}{(\sum |c_1(f)|^2 + \cdots + |c_{m+1}(f)|^2)^{\frac{1}{2}}}\right)^{\frac{k}{k_0}} \text{ and } \eta = \frac{k}{k_0}.$$

Since $f^* D \geq k_0 \min\{f^* D, 1\}$, we get (iii).

We now prove (iv). Take a holomorphic function $\varphi$ in $\Delta_R$ such that $(\varphi) = \min\{f_U^* D, k\}$. For $\eta \in D_{f_U,E}^{[k]}$, put

$$v = \eta \log(|c_1(f_U)|^2 + \cdots + |c_{m+1}(f_U)|^2)^{\frac{1}{2}} + \log h - \frac{1}{s} \log \varphi.$$

Then $dd^c \eta \geq 0$ and hence, by Corollary 2.3, we get

$$0 \leq \int_{\Delta_R} \frac{dm(a)}{m(\Delta_R)} \int_{\Theta_{\Delta}} v d\mu_{\Delta,a} - \int_{\Delta_R} \frac{dm(a)}{m(\Delta_R)} \int_{\Delta} v (dd^c g_{\Delta,a})^n$$

$$= \eta \int_{\Theta_{\Delta}} \log(|c_1(f_U)|^2 + \cdots + |c_{m+1}(f_U)|^2)^{\frac{1}{2}} d\mu_{\Delta,0} + \int_{\Theta_{\Delta}} \log h d\mu_{\Delta,0}$$

$$- \int_{\Theta_{\Delta}} \frac{1}{s} \log \varphi d\mu_{\Delta,0} - \int_{\Delta_R} v dm(a) \leq \eta T_{f_U}(r, E) - \frac{1}{s} N_{f_U}^{[k]}(r, D) + K,$$
where $K$ is a constant, because $h$ is bounded from above. This implies that

$$1 - \frac{N^{[k]}_{f,U}(r, D)}{sT_{f,U}(r, E)} \geq 1 - \gamma + \frac{K}{T_{f,U}(r, E)}.$$ 

Letting $r \to R$, we obtain $\delta^{[k]}_{f,U}(D) \leq \delta^{[k]}_{f,U}(E)$. \hfill \Box

4.2. Defect relation with a truncation. Now we give the non-integrated defect with a truncation for meromorphic mappings from a submanifold of $\mathbb{C}^l$ to a compact complex manifold.

**Theorem 4.3.** Let $M$ be an $n$-dimensional closed complex submanifold of $\mathbb{C}^l$ and $\omega$ be its Kähler form that is induced from the canonical Kähler form of $\mathbb{C}^l$. Let $L \to X$ be a holomorphic line bundle over a compact manifold $X$. Fix a positive integer $d$ and let $d_1, d_2, \cdots, d_q$ be positive divisors of $d$. Let $E$ be a $\mathbb{C}$-vector subspace of dimension $m + 1$ of $H^0(X, L^d)$. Put $u = \text{rank} E$ and $b = \dim B(E) + 1$ if $B(E) \neq \emptyset$, otherwise $b = -1$. Let $\sigma_j$ $(1 \leq j \leq q)$ be in $H^0(X, L^d)$ such that $\sigma_1^q, \cdots, \sigma_q^q \in E$. Set $D_j = (\sigma_j)_0$ $(1 \leq j \leq q)$. Assume that $D_1, \cdots, D_q$ are in $N$-subgeneral position with respect to $E$ and $u > b$. Let $f : M \to X$ be an analytically non-degenerate meromorphic mapping with respect to $E$, i.e $f(M) \not\subset \text{supp}((\sigma))$ for any $\sigma \in E \setminus \{0\}$ and $f(M) \cap B(E) = \emptyset$. Assume that, for some $\rho \geq 0$ and for some basis $\{c_k\}_{k=1}^{m+1}$ of $E$, there exists a bounded measurable function $h \geq 0$ on $M$ such that

$$\rho f^*dd^c \log(|c_1|^2 + \cdots + |c_{m+1}|^2)^{1/d} + dd^c \log h^2 \geq \text{Ric} \omega.$$ 

Then,

$$\sum_{i=1}^{q} \delta^{[md_i/d]}_{f,E}(D_i) \leq k_N + K'(E, N, \{D_j\}),$$

where $k_N, s_N, t_N$ are defined as in Proposition 3.12 and $K'(E, N, \{D_j\})$ is the constant given in the end of the proof.

**Proof.** Put $\sigma^{\hat{i}}_i = \sum_{1 \leq j \leq m+1} a_{ij}c_j$, where $a_{ij} \in \mathbb{C}$. We define a meromorphic mapping $\Phi : X \to \mathbb{C}P^m$ by $\Phi(x) := [c_1(x) : \cdots : c_{m+1}(x)]$. Also since $X$ compact, $Y = \Phi(X)$ is an algebraic variety of $\mathbb{C}P^m$. Moreover, by definition of rank $E$, $Y$ is of dimension $\text{rank} E = u$. Put $F = \Phi \circ f$. Since $f(M) \cap B(E) = \emptyset$ and $f$ is non-degenerate with respect to $E$, $F$ is meromorphic and linearly non-degenerate. Denote by $H_m$ the hyperplane bundle of $\mathbb{C}P^m$. Put $H_i := \sum_{1 \leq j \leq m+1} a_{ij}z_j$, where $[z_0, z_1, \cdots, z_m]$ is the homogeneous coordinate of $\mathbb{C}P^m$. By [5]...
Theorem 1], there exists an open subset $U$ of $M$ such that $U$ is biholomorphic to a polydisc $\Delta_R$ and $M \setminus U$ has a zero measure. For convenience, we still denote by $f, F$ their restrictions to $U$. It is easy to see that

$$T_f(r, \mathcal{L}) = \frac{1}{d} T_F(r, \mathcal{H}_m)\text{ and } N_f(r, D_i) = N_F(r, H_i).$$

Moreover, we get the following.

- $N^{\mathcal{F}}(r, H_i) = N^{[kd_i/d]}(r, D_i), \bar{\delta}_{F,H_m}(H_i) = \bar{\delta}_{F,E}(D_i)$. 
- $H_{j_1} \cap \cdots \cap H_{j_t} \cap Y \subset \Phi(B(E))$ if $D_{j_1} \cap \cdots \cap D_{j_t} = B(E)$.

Put $K_1 = \{ R' \subset \{1, 2, \cdots, q + m - u + b + 1\} : |R'| = \text{rank}(R') = m + 1\}$.

Without loss of generality we may assume $R_j = R^* (1 \leq j \leq n)$. From now on, we just consider $n$-tuples $r = (r_1, \cdots, r_n)$ such that $r_j = r^* (1 \leq j \leq n)$. Suppose that

$$\limsup_{r \to R} \frac{T_f(r, E)}{-\log (R^* - r^*)} = \infty.$$

Then by Theorem 3.13 we have

$$\sum_{j=1}^{q} \delta^{[md_i/d]}_{f,E}(D_j) \leq (m+1)K(E, N, \{D_j\}) \leq \frac{k_N}{l_N} (m+1)(s_N+m-2(u-b-1)).$$

By Proposition 4.2, we get

$$\sum_{j=1}^{q} \delta^{[md_i/d]}_{f,E}(D_j) \leq (m+1)K(E, N, \{D_j\}) \leq \frac{k_N}{l_N} (m+1)(s_N+m-2(u-b-1)).$$

Hence, we can assume

$$\limsup_{r \to R} \frac{T_f(r, E)}{-\log (R^* - r^*)} < \infty.$$

Let $\alpha_1, \cdots, \alpha_{m+1}$ be as in Proposition 3.10. Set $l_0 = |\alpha_1| + \cdots + |\alpha_{m+1}|$ and take $t, p$ with $0 < l_0 t \leq p < 1$. Put

$$l_N = \sum_{j \in Q} \omega(j) - (m+1)(s_N - u + 2 + b) + m - u + 1 + b.$$

By Proposition 3.9 and the proof of Theorem A in [3], we get the following.

**Claim 1.** Let $p$ be a real positive number such that

$$p \sum_{i=1}^{m+1} |\alpha_i| \leq \frac{p(m+1)}{2} < 1.$$
Then there exists a positive constant $K$ such that
\[
\sum_{i=1}^{n} \int_{\Delta} \left( \frac{|W(F)(z)|^{s_N-u+2+b}}{\prod_{j=1}^{q+m-u+b+1} |H_j(F(z))|^{\omega(j)}} \right)^p \|F(z)\|^p dm_i(z) \leq C \left( \frac{(R^*)^{3n}}{(R^*-r^*)^{3n}} T_F(r^*, L) \right)^{p(m+1)/2}
\]
for each $0 < r^* < R^*$ and $r^*$ outside a set $E$ satisfying $\int_E \frac{1}{R^*-r} dt < \infty$.

Claim 2.
\[
\sum_{1 \leq j \leq q+m-u+b+1} \omega(j) (\nu_{H_j(F)} - \nu_{H_j(F)}^{[m]}) \leq (s_N - u + 2 + b) \nu_{W(F)}.
\]

By definition of the non-integrated defect, there exist $\eta_i \geq 0$ ($1 \leq i \leq q + m - u + b + 1$) and a nonnegative functions $h_i$ such that
\[
\eta_i F^* \log(|z_1|^2 + \cdots + |z_{m+1}|^2) + \ddc log h_i^2 \geq \min\{m, F^* H_i\}
\]
and $1 - \eta_i \leq \delta_{F,H_m}^m(H_i) \leq 1$. Take a holomorphic function $\varphi_i$ in $M$ such that $(\varphi_i) = \min\{m, F^* H_i\}$. Put
\[
u_i = \Theta \left( \log \frac{h_i^2}{K_i} + \log(|F_1|^2 + \cdots |F_{m+1}|^2)^{\eta_i/2} \right),
\]
where $K_i$ is a constant which is greater than $h_i^2$ and $\Theta$ is the constant in Proposition 3.12. By the above inequality, we see that $\nu_i - \Theta \log \varphi_i$ is a plurisubharmonic function on $M$ and
\[
ed^{\nu_i} \leq \|F\|^{\eta_i \Theta}.
\]

Put
\[
s = \sum_{i=1}^{q+m-u+b+1} (1 - \eta_i)
\]
and
\[
v := \log \frac{|W(F)(z)|^{s_N-u+2+b}}{\prod_{j=1}^{q+m-u+b+1} |H_j(F(z))|^{\omega(j)}} + \sum_{i=1}^{q+m-u+b+1} u_i.
\]
Since $\nu_i - \Theta \log \varphi_i$ ($\Theta \geq \omega(j)$) is a psh function and by virtue of Claim 2, it implies that $v$ is a psh function. On the other hand, by the hypothesis, there exist $\rho > 0$ and a nonnegative bounded function $h$ such that
\[
\rho \frac{1}{d^f} \Omega_F + \ddc log h^2 \geq \text{Ric} \omega,
\]
where $\omega$ is the Kähler form on $M$. 
Put \( w = \log \frac{||F||^{\rho/d}h^2}{K_0 \det(h_{i\bar{j}})} \), where \( \omega = \sqrt{-1} \sum h_{i\bar{j}}dz_i \wedge d\bar{z}_j \) in \( U \), \( K_0 \) is a constant which is greater than \( h^2 \). Then we have

\[
e^w \omega^n \leq ||F||^{\rho/d}dx^1 \wedge \cdots \wedge dx^n
\]

and \( \omega \) is a psh function. Hence,

\[
e^w d\text{vol} \leq ||F||^{\rho/d}dx^1 \wedge \cdots \wedge dx^n \text{ in } U,
\]

where \( d\text{vol} \) stands for the volume form of \( M \) with respect to the given Kähler metric. Put

\[
\alpha = \frac{\rho/d}{l_N + \Theta(s-q)},
\]

\[
\chi = \frac{|W(F)(z)|^{sN-u+2+b}}{\prod_{j=1}^{q+m-u+b+1}|H_j(F(z))|^{\omega(j)}},
\]

and \( w_1 = w + \alpha \nu \). Then \( w_1 \) is plurisubharmonic. Hence, \( e^{w_1} \) is also plurisubharmonic. We have

\[
\int_{\Delta_r} e^{w_1} d\text{vol} \leq \int_{\Delta_r} |\chi|^\alpha ||F||^{\rho/d+\Theta \alpha (q+m-u+b+1-s)} dm(z)
\]

\[
= \int_{\Delta_r} |\chi|^\alpha ||F||^{\alpha l_N} dm(z)
\]

Suppose \( \alpha' = \frac{3n(m+1)}{4} \alpha < 1 \). Then by Claim 1, for each \( r^* \) outside a set \( E \) with \( \int_E \frac{1}{R^*-r} ds < \infty \), we have

\[
\sum_{i=1}^{n} \int_{\Delta_r(r^*)} |\chi|^\alpha ||F||^{\alpha l_N} dm_i(z) \leq C \left( \frac{(R^*)^{3n}}{(R^*-r^*)^{3n}} T_F(r^*, \mathcal{L}) \right)^{\alpha m(m+1)}.
\]

Combining with the fact that \( \limsup_{r \to R} \frac{T_F(r,E)}{-\log(R^*-r)} < \infty \), we get

\[
\sum_{i=1}^{n} \int_{\Delta_r(r^*)} |\chi|^\alpha ||F||^{\alpha l_N} dm_i(z) \leq K_1 \left( \frac{1}{(R^*-r^*)^{\alpha}} \left( \log \frac{1}{R^*-r^*} \right)^{\alpha'/4} \right).
\]

for some \( K_1 \) and \( r^* \in [0, R^*) - \setminus E \). Varying \( K_1 \) slightly, we may assume the above inequality holds for all \( r^* \in [0, R^*) \) by [6, Proposition 5.5]. From this, we conclude that

\[
\int_U e^{w_1} d\text{vol} \leq K_1 \int_0^{R^*} \frac{1}{(R^*-t)^{\alpha'}} \left( \log \frac{1}{R^*-t} \right)^{\alpha'/4} dt + O(1) < \infty
\]

Combining with the fact that \( M \setminus U \) has zero measure, we get

\[
\int_M e^{w_1} d\text{vol} < \infty.
\]
By Proposition 2.10 we get a contradiction. Hence, $3nm(m+1)\alpha \geq 1$. This means

$$\frac{3\rho nm(m+1)}{\Theta d} + q + m - u + b + 1 - \frac{l_N}{\Theta} \geq s.$$ 

Put

$$K'(E, N, \{D_j\}) = \frac{t_N - (m+1)(s_N - u + 2 + b) + m - u + b + 1}{\Theta} + \frac{3\rho nm(m+1)}{\Theta d}.$$ 

By a direct computation and note that

$$|s - \sum_{i=1}^{q+m-u+b+1} \delta_{f,E}^{[md_i/d]}(D_i)| < \epsilon q$$

for $\epsilon > 0$ small enough and $\Theta \geq t_N/k_N$, and

$$l_N \geq \Theta(q - k_N) + t_N - (m+1)(s_N - u + 2 + b) + m - u + b + 1.$$ 

we obtain the desired inequality. □

In the case where $X$ is the complex projective space, $L$ is the hyperplane bundle of $X$ and $D_j$ are hyperplanes in $N$-subgeneral position, we get the following.

**Corollary 4.4.** Let $M$ be an $n$-dimensional closed complex submanifold of $\mathbb{C}^l$ and $\omega$ be its Kähler form that is induced from the canonical Kähler form of $\mathbb{C}^l$. Let $f : M \to \mathbb{C}P^m$ be a linear non-degenerate meromorphic map. Let $\{H_j\}$ be a family of hyperplanes in $N$-subgeneral position in $\mathbb{C}P^m$. Denote by $\Omega_f$ the pull-back of the Fubini-Study form of $\mathbb{C}P^m$ by $f$. Assume that, for some $\rho \geq 0$, there exists a bounded measurable function $h \geq 0$ on $M$ such that

$$\rho \Omega_f + \ddc \log h^2 \geq \text{Ric} \omega.$$ 

Then,

$$\sum_{i=1}^q \delta_{f,E}^{[m]}(D_i) \leq (2N - m + 1) + 2\rho nm(2N - m + 1).$$ 

**Corollary 4.5.** Let $M$ be an $n$-dimensional closed complex submanifold of $\mathbb{C}^l$ and $\omega$ be its Kähler form that is induced from the canonical Kähler form of $\mathbb{C}^l$. Let $f : M \to \mathbb{C}P^m$ be a meromorphic mapping. Denote by $\Omega_f$ the pull-back of the Fubini-Study form of $\mathbb{C}P^m$ by $f$. Assume that $f$ satisfies the following two conditions:
(i) Assume that, for some $\rho \geq 0$, there exists a bounded measurable function $h \geq 0$ on $M$ such that
\[ \rho \Omega_f + \partial \bar{\partial} \log h^2 \geq \text{Ric} \omega, \]
(ii) $f$ omits $((2N - m + 2) + 2\rho nm(2N - m + 1))$ hyperplanes in $N$-subgeneral position in $\mathbb{C}P^m$.

Then $f$ is linearly degenerate.

**Remark 4.6.** By using our arguments and their techniques in [16], [18], [19], we can generalize exactly their results to meromorphic mappings from a Stein manifold without the assumption (H) into a smooth complex projective variety $V \subset \mathbb{C}P^M$.

Let $D_1, \cdots, D_q$ be hypersurfaces in $\mathbb{C}P^n$, where $q > n$. Also, the hypersurfaces $D_1, \cdots, D_q$ are said to be in general position in $\mathbb{C}P^n$ if for every subset $\{i_0, \cdots, i_n\} \subset \{1, \cdots, q\}$,
\[ D_{i_0} \cap \cdots \cap D_{i_n} = \emptyset. \]

We now can prove the following improvement of [16, Theorem 1.1]).

**Theorem 4.3'.** Let $M$ be an $n$-dimensional closed complex submanifold of $\mathbb{C}^l$ and $\omega$ be its Kähler form that is induced from the canonical Kähler form of $\mathbb{C}^l$. Let $f : M \to \mathbb{C}P^n$ be a meromorphic map which is algebraically nondegenerate (i.e. its image is not contained in any proper subvariety of $\mathbb{C}P^n$). Denote by $\Omega_f$ the pull-back of the Fubini-Study form of $\mathbb{C}P^n$ by $f$. Let $D_1, \cdots, D_q$ be hypersurfaces of degree $d_j$ in $\mathbb{C}P^n$, located in general position. Let $d = \text{l.c.m.}\{d_1, \cdots, d_q\}$ (the least common multiple of $\{d_1, \cdots, d_q\}$). Assume that, for some $\rho \geq 0$, there exists a bounded continuous function $h \geq 0$ on $M$ such that
\[ \rho \Omega_f + \partial \bar{\partial} \log h^2 \geq \text{Ric} \omega. \]

Then for every $\epsilon > 0$,
\[ \sum_{i=1}^q \delta_f^{[l-1]}(D_i) \leq (n + 1) + \epsilon + \frac{pl(l-1)}{d}, \]
where $l \leq 2^{n^2+4n}e^{n}d^{2n}(nI(\epsilon^{-1}))^n$ and $I(x) := \min\{k \in \mathbb{N} : k > x\}$ for a positive real number $x$.

We now recall the definition of the subgeneral position in the sense of [18, Theorem 1.2].

Let $V \subset \mathbb{C}P^N$ be a smooth complex projective variety of dimension $n \geq 1$. Let $n_1 \geq n$ and $q \geq 2n_1 - n + 1$. Hypersurfaces $D_1, \cdots, D_q$ in $\mathbb{C}P^N$ with $V \not\subseteq D_j$ for all $j = 1, \cdots, q$ are said to be in $n_1$-subgeneral position in $V$ if the two following conditions are satisfied:
Theorem 4.3”. Let $V \subset \mathbb{C}^N$ be a smooth complex projective variety of dimension $n \geq 1$. Let $n_1 \geq n$ and $q \geq 2n_1 - n + 1$. Let $D_1, \ldots, D_q$ be hypersurfaces in $\mathbb{C}^N$ of degree $d_j$, in $n_1$-subgeneral position in $V$. Let $d = \text{l.c.m.}\{d_1, \ldots, d_q\}$ (the least common multiple of $\{d_1, \ldots, d_q\}$). Let $\epsilon$ be an arbitrary constant with $0 < \epsilon < 1$. Set

$$m := \left[4d^n(2n + 1)(2n_1 - n + 1)\deg V \cdot \frac{1}{\epsilon}\right] + 1,$$

where $[x] := \max\{k \in \mathbb{Z} : k \leq x\}$ for a real number $x$. Let $M$ be an $n$-dimensional closed complex submanifold of $\mathbb{C}^l$ and $\omega$ be its Kähler form that is induced from the canonical Kähler form of $\mathbb{C}^l$. Let $f$ be an algebraically nondegenerate meromorphic map of $M$ into $V$. Denote by $\Omega_f$ the pull-back of the Fubini-Study form of $\mathbb{C}^N$ by $f$. For some $\rho \geq 0$, if there exists a bounded continuous function $h \geq 0$ on $M$ such that

$$\rho \Omega_f + dd^c \log h^2 \geq \text{Ric } \omega,$$

then we have

$$\sum_{j=1}^{q} \delta_{f^*h\mathcal{I}_N}(D_j) \leq 2n_1 - n + 1 + q\epsilon + \rho T$$

for some positive integers $l, T$ satisfying

$$l \leq \binom{N + md}{md} \text{ and } T \leq \frac{(2n_1 - n + 1) \cdot \binom{N + md}{md}}{d(m - (n + 1)(2n + 1)d^n \deg V)}.$$

With the same definition of hypersurfaces in subgeneral position as in Definition 3.11, we can also prove the following improvement of [19, Theorem 1.1]).

Theorem 4.3”. Let $M$ be an $n$-dimensional closed complex submanifold of $\mathbb{C}^l$ and $\omega$ be its Kähler form that is induced from the canonical Kähler form of $\mathbb{C}^l$. Let $f$ be an algebraically nondegenerate meromorphic map of $M$ into $\mathbb{C}^N$. Let $D_1, \ldots, D_q$ be hypersurfaces in $\mathbb{C}^N$ of degree $d_j$, in $k$-subgeneral position in $\mathbb{C}^N$. Let $d = \text{l.c.m.}\{d_1, \ldots, d_q\}$
(the least common multiple of \(\{d_1, \cdots, d_q\}\)). Denote by \(\Omega_f\) the pull-back of the Fubini-Study form of \(\mathbb{CP}^n\) by \(f\). Assume that for some \(\rho \geq 0\), there exists a bounded continuous function \(h \geq 0\) on \(M\) such that

\[
\rho \Omega_f + dd^c \log h^2 \geq \text{Ric } \omega.
\]

Then, for each \(\epsilon > 0\), we have

\[
\sum_{j=1}^{q} \delta^{[l-1]}_{f,\mathcal{H}_n}(D_j) \leq k(n+1) + \epsilon + \frac{\rho l(l-1)}{d},
\]

where \(l = \left(\frac{N+n}{n}\right) \leq (3ekdI(\epsilon^{-1}))^{n(n+1)^3}n \) for \(N := 2kdn(n+1)^2I(\epsilon^{-1})\).

4.3. **Defect relation with no truncation.** In this case, we get the following sharp defect relation.

**Theorem 4.7.** Let \(M\) be an \(n\)-dimensional closed complex submanifold of \(\mathbb{C}^l\) and \(\omega\) be its Kähler form that is induced from the canonical Kähler form of \(\mathbb{C}^l\). Let \(\mathcal{L} \to X\) be a holomorphic line bundle over a compact manifold \(X\). Fix a positive integer \(d\) and let \(d_1, d_2, \cdots, d_q\) be positive divisors of \(d\). Let \(E\) be a \(\mathbb{C}\)-vector subspace of dimension \(m+1\) of \(H^0(X, \mathcal{L}^d)\). Let \(\sigma_j\) (\(1 \leq j \leq q\)) be in \(H^0(X, \mathcal{L}^d)\) such that \(\sigma_1^d, \cdots, \sigma_q^d \in E\). Set \(D_j = (\sigma_j)_0\) (\(1 \leq j \leq q\)). Assume that \(D_1, \cdots, D_q\) are in \(N\)-subgeneral position with respect to \(E\). Let \(f: M \to X\) be a meromorphic mapping satisfying \(f(M) \not\subset D_j\) for \(1 \leq j \leq q\) and \(\overline{f(M)} \cap B(E) = \emptyset\). Assume that, there exists a holomorphic section \(\nu\) of \(K^{-1}_M\) such that for some basis \(\{c_1, c_2, \cdots, c_{m+1}\}\) of \(E\) and \(l\) large enough,

\[
dd^c \log(|c_1(f)|^2 + \cdots + |c_{m+1}(f)|^2)^{l/d} \geq dd^c(\nu \mathcal{O})^n).
\]

Then,

\[
\sum_{i=1}^{q} \delta_{f,E}(D_i) \leq 2N.
\]

Before proving the above theorem, we will give a modification of [1, Theorem 2']

**Proposition 4.8.** Let \(M, \delta_1, \delta_2 > 0\) and \(q, n \in \mathbb{N}, q > 2n\). Then, there is a number \(\alpha = \alpha(\delta_1, \delta_2, M, q, n) > 0\) with the following property: If \(u, u_1, \cdots, u_q\) are subharmonic functions in an open neighborhood of \(\Delta_1 \subset \mathbb{C}\) with Riesz charges \(\nu, \nu_1, \cdots, \nu_q\), respectively such that

\[
(\nu + \sum_{i=1}^{q} \nu_i)(\Delta_1) \leq M,
\]

then
then
\[
\int_{\Delta_1} \left| \sup_{1 \leq k \leq n+1} u_{i_k} - u \right| dx dy \leq \alpha, \text{ for all } 1 \leq i_1 < \cdots < i_{n+1} \leq q.
\]
Moreover, there exists \( r \in [1 - \delta_1, 1] \) such that
\[
\left( \sum_{i=1}^{q} \nu_i - (q - 2n)\nu \right)(\Delta_r) > -\delta_2.
\]

Proof. Suppose the theorem is false. Then there are a number \( \delta > 0 \) and a sequence \((u^j, u^j_1, \cdots, u^j_q), j \in \mathbb{N}\) with Riesz charge \( \nu^j, \nu^j_1 \) respectively such that
\[
\left( \sum_{i=1}^{q} \nu_i - (q - 2n)\nu \right)(\Delta_1) \leq M,
\]
and for all \( r \in [1 - \delta_1, 1] \) one has
\[
\left( \sum_{i=1}^{q} \nu_i - (q - 2n)\nu \right)(\Delta_r) \leq -\delta_2.
\]
Hence, by passing to a subsequence if necessary, we can assume
\[
\nu^j_i \to \nu_i, \quad \nu^j \to \nu,
\]
Let \( G * \lambda \) be the Green potential of the charge \( \lambda \) in the disk \( \Delta_1 \). By the Riesz representation formula, we have
\[
u^j_i - u^j = h^j_i + G * (\nu^j_i - \nu^j),
\]
where \( h^j_i \) is harmonic in \( \Delta_1 \). By the proof of [4, Theorem 2'], we get the followings:

(i) \( G * \nu^j_i \to G * \nu_i, \quad G * \nu^j \to G * \nu \) in \( L^1(\Delta_1) \).
(ii) \( h^j_i \to h_i \) uniformly on compact subsets, some of \( h_i \) may be identical \(-\infty\). We can suppose \( h_i \neq -\infty \) for \( 1 \leq i \leq q' \) and \( h_i = -\infty \) for \( i > q' \). Note that \( q' - q \leq n \).
(iii) Put \( u_i = h_i + G * \nu_i \) (\( 1 \leq i \leq q' \)), \( u = G * \nu, n' = n - (q - q') \).
Then \( u = \sup_{1 \leq k \leq n+1} u_{i_k} \) for all \( 1 \leq i_1 < \cdots < i_{n+1} \leq q' \).
Hence, by [4, Theorem 2'], we get
\[
\sum_{i=1}^{q'} \nu_i - (q' - 2n')\nu \geq 0.
\]
Consequently,
\[
\kappa := \sum_{i=1}^{q} \nu_i - (q - 2n)\nu
\]
\[
= \sum_{i=1}^{q'} \nu_i - (q' - 2n)\nu + \sum_{i=q'+1}^{q} \nu_i + \{(q' - 2n') - (q - 2n)\}\nu.
\]
Obviously, the expression in the braces is nonnegative. Therefore, \(\kappa \geq 0\).

Finally, for a Radon measure \(\lambda\) in a neighborhood of \(\overline{\Delta}_1\), we have \(\lambda(\partial \Delta_r) = 0\) for all \(r\) outside a countable subset of \([0, 1]\).

By the Jensen formula and Corollary 4.9, we have the Eremenko-Sodin second main theorem.

**Corollary 4.9.** Let \(M, \delta > 0\) and \(q, n \in \mathbb{N}, q > 2n\). Let \(u, u_1, \cdots, u_q\) be subharmonic functions in an open neighborhood of \(\overline{\Delta}_R \subset \mathbb{C}\) with Riesz charges \(\nu, \nu_1, \cdots, \nu_q\), respectively such that the following two statements satisfied

(i) \(\nu(\Delta_r) \to +\infty\) as \(r\) tends to \(R\),

(ii) For all \(1 \leq i_1 < \cdots < i_{n+1} \leq q\), we have
\[
\frac{1}{r^2} \int_{\Delta_r} \left| \sup_{1 \leq k \leq n+1} u_{i_k} - u \right| dx dy = O(1)
\]
as \(r\) tends to \(R\).

Then for each \(\delta > 0\),
\[
\left( \sum_{i=1}^{q} \nu_i - (q - 2n)\nu \right)(\Delta_r) \geq -\delta \left( \nu(\Delta_r) + \sum_{i=1}^{q} \nu_j(\Delta_r) \right)
\]
for \(r\) close enough to \(R\).

**Proof.** Put
\[
w(z) = \frac{u(rz)}{\nu(\Delta_r) + \sum_{i=1}^{q} \nu_j(\Delta_r)}, \quad w_i(z) = \frac{u_i(rz)}{\nu(\Delta_r) + \sum_{i=1}^{q} \nu_j(\Delta_r)}
\]
for \(z\) in a neighborhood of \(\overline{\Delta}_1\) and \(r < R\). By the condition (i) and Proposition 4.8, we obtain the assertion. \(\square\)

By the Jensen formula and Corollary 4.9, we have the Eremenko-Sodin second main theorem.
Corollary 4.10. Let the notations and the hypothesis be as in Corollary 4.9. Then for each $\delta > 0$,
\[
\int_0^{2\pi} \left( \sum_{i=1}^q u_i(re^{it}) - (q - 2n)u(re^{it}) \right) dt > -\delta \int_0^{2\pi} u(re^{it}) dt + O(1)
\]
for all $r$ close enough to $R$. Here the term $O(1)$ is a constant as $r \to R$, but depends on $u, u$.

In high dimension, we have

Corollary 4.11. Let $M, \delta > 0$ and $q, n \in \mathbb{N}, q > 2n$. Let $u, u_1, \cdots, u_q$ be psh functions in an open neighborhood of $\overline{\Delta}_R \subset \mathbb{C}^l$ such that the following two statements are satisfied

(i) $\int_{\partial \Delta_r} u dt_1 \cdots dt_l \to +\infty$ as $r$ tends to $R$;

(ii) For all $1 \leq i_1 < \cdots < i_{n+1} \leq q$,
\[
\left| \sup_{1 \leq k \leq n+1} u_{i_k}(z) - u(z) \right| = O(1)
\]
as $|z|$ tends to $R$.

Then for each $\delta > 0$,
\[
\int_{\partial \Delta_r} \left( \sum_{i=1}^q u_i - (q - 2n)u \right) dt_1 \cdots dt_l > -\delta \int_{\partial \Delta_r} u dt_1 \cdots dt_l + O(1)
\]
for $r$ close enough to $R$.

Proof. Put
\[
w(\cdot) = \int_0^{2\pi} \cdots \int_0^{2\pi} u(\cdot, r_2 e^{it_2}, \cdots, r_l e^{it_l}) dt_2 \cdots dt_l.
\]
And we define $w_i$ ($1 \leq i \leq q$) in the similar manner. The radius $r = (r_1, \cdots, r_n)$ is chosen close enough to $R$. It is easy to see that $w, w_i$ satisfy conditions in Corollary 4.9.

Proof of Theorem 4.7. We still use notations as in the first paragraph of the proof of Theorem 4.3. We now suppose on the contrary. By definition of the non-integrated defect, there exist $\eta_i \geq 0$ ($1 \leq i \leq q$) and nonnegative functions $h_i$ such that
\[
\eta_i F^*dd^c \log(\sum_{1}^{2} z_i^2 + \cdots + |z_{m+1}|^2) + dd^c \log h_i^2 \geq F^*H_i
\]
and $1 - \eta_i \leq \delta_{F,H_0}(H_i) \leq 1$. Put $\eta = \sum_{i=1}^q (1 - \eta_i)$. Therefore,
\[
(q - \eta)dd^c \log ||F||^2 + dd^c \log h^{2} \geq \sum_{i=1}^q F^*H_i,
\]
where \( h' \) is measurable and bounded. Subtracting \((q - 2n)\ddc \log||F||^2\) from the two sides of the above inequality, we get
\[
\ddc \log h'^2 \geq \left( \sum_{i=1}^{q} F^*H_i - (q - 2n)\ddc \log||F||^2 \right) + (\eta - 2n)\ddc \log||F||^2.
\]
Note that by \( \overline{f(M)} \cap B(E) = \emptyset \), we have
\[
\log||F||^2 - \max_{1 \leq j \leq N+1} \log|H_i(F)|^2 = O(1)
\]
for all \( 1 \leq i_1 < \cdots < i_{N+1} \leq q \).

**Claim 1.**
\[
\int_{\Delta_R} \nu \omega^n = +\infty
\]
Indeed, denote by \( |\nu|^2 \) the trivialization of \( \nu \omega \) on \( \Delta_R \). If \( |\nu|^2 \) is not integrable over \( M \) then we are done. Otherwise, it is integrable hence since \( \nu \) holomorphic \( |\nu|^2 \) is plurisubharmonic function on \( M \). Taking into account of Theorem 2.9 one gets Claim 1.

**Claim 2.**
\[
\lim_{r\to R} \int_{\partial \Delta_r} \log||F(z)|| = +\infty.
\]
Applying Theorem 2.3 for \( \frac{1}{2} \log||F|| \) and \( \nu \omega^n \), and integrating in \( a \) over \( \Delta_{r_0} \) (for some \( r_0 \) fixed) and the inequality in the hypothesis of Theorem 4.7, we get
\[
\int_{\partial \Delta_r} \log||F(z)|| \geq \frac{d}{t} \int_{\partial \Delta_r} \nu \omega^n + C,
\]
where \( R' < R \) and \( C \) is a constant that does not depend on \( R' \). On the other hand,
\[
+\infty = \int_{\Delta_R} \nu \omega^n = \int_0^{R_1} r_1 dr_1 \cdots \int_0^{R_n} r_n dr_n \int_{\partial \Delta_r} \nu \omega^n
\]
Hence, \( \int_{\partial \Delta_r} \nu \omega^n \) tends to \( +\infty \) as \( r \) closes to \( R \). That yields the claim 2. Now, by applying Corollary 4.11 for \( u = \log||F(z)|| \), \( u_i = H_i(F) \) and Jensen’s formula, we get the desired conclusion. \( \square \)

We recall the following version of the Bloch-Cartan theorem which plays an essential role in Geometric Function Theory.

**Theorem 4.12.** (see [14, Corollary 3.10.8, p.137]) If a holomorphic map \( f : \mathbb{C} \to \mathbb{C}P^m \) misses \( 2m + 1 \) or more hyperplanes in general position, then it is a constant map.

The Bloch-Cartan theorem is generalized to hypersurfaces in general position in \( \mathbb{C}P^m \) by Babets and Eremenko-Sodin.
Theorem 4.13. (see [1] and [11]) If a holomorphic map \( f : \mathbb{C} \rightarrow \mathbb{C}P^m \) misses \( 2m + 1 \) or more hypersurfaces in general position, then it is a constant map.

From Theorem 4.7 we have the following Bloch-Cartan theorem for meromorphic mappings from \( \mathbb{C} \) to a smooth algebraic variety \( V \) in \( \mathbb{C}P^m \) missing \( 2N + 1 \) or more hypersurfaces in \( N \)-general position.

Corollary 4.14. Let \( f \) be a meromorphic mapping of \( \mathbb{C} \) to a smooth algebraic variety \( V \) in \( \mathbb{C}P^m \). Let \( D_1, \ldots, D_{2N+1} \) be hypersurfaces of \( \mathbb{C}P^m \) such that \( V \not\subset D_j \) and \( D_j \cap V \) are in \( N \)-subgeneral position in \( V \). Assume that \( f \) omits \( D_j \) \((1 \leq j \leq 2N + 1)\). Then \( f \) is constant.

5. A unicity theorem

In [8], the author gave a unicity theorem for meromorphic mappings from a complete Kähler manifold satisfying the assumption \((H)\) into the complex projective space \( \mathbb{C}P^n \). The last aim of this paper is to give an analogous unicity theorem for meromorphic mappings from a Stein manifold without the assumption \((H)\) to a compact complex manifold.

Denote by \( A_\rho(M, X) \) the set of holomorphic mappings \( f : M \rightarrow X \) satisfying the following condition: \( \text{There exist } \rho > 0 \text{ and a bounded measurable function } h \geq 0 \text{ on } M \text{ such that} \)
\[
\rho f^*dd^c \log(|c_1|^2 + \cdots + |c_{m+1}|^2)^{1/d} + dd^c \log h^2 \geq \text{Ric}\omega,
\]
where \( \{c_k\}_{k=1}^{m+1} \) are a basis of \( E \).

In this section, we assume the hypothesis as in the statement of Theorem 4.3 and also keep the notations as in the first part of the proof of Theorem 4.3. Let \( f, g \) be in \( A_\rho(M, X) \). Set \( F = \Phi \circ f, G = \Phi \circ g \).

Theorem 5.1. Assume that the following are satisfied.
i) \( f = g \) on \( \bigcup_{i=1}^q (f^{-1}(D_i) \cup g^{-1}(D_i)) \);
ii) \( q > (m+1)K(E, N, \{D_j\}) + \frac{k_N}{t_N} (3\rho(\gamma_F + \gamma_G) + m_F + m_G + m - u + b + 1) \)
(for the definition of \( m_F, m_G, \) see below).
Then \( f \equiv g \).

Firstly, some notations and auxiliary lemmas in [7], [8] are re-used. Let \( \xi \) be a holomorphic mapping of \( M \) to \( \mathbb{C}P^m \). Take a point \( p \in M \) and a reduced representation of \( \xi \) as \( \xi = (\xi_0, \ldots, \xi_m) \) in a neighborhood of \( p \). Denote by \( \mathcal{M}_p \) the field of germs of meromorphic functions in an open subset containing \( p \). Let \( \mathcal{F}_p^k \) be the submodule of \( \mathcal{M}_p^{m+1} \) generated by \( \partial^{\alpha} \xi / \partial z^{\alpha} \) with \( |\alpha| \leq k \), where \( z = (z_1, z_2, \ldots, z_n) \) is a holomorphic local coordinate around \( p \). Clearly, this definition does not depend on
the coordinate \( z \) and the reduced representation of \( f \). The \( k \)-th rank of \( f \) is defined by

\[
r_\xi(k) = \text{rank}_M z^k - \text{rank}_M z^{k-1}
\]

which is independent of the choice of \( p \in M \), if \( M \) is connected. Set

\[
\gamma_\xi = \sum_k k r_\xi(k),
\]

\[
m_\xi = \sum_{k,l} (k - l)^+ \min\{A_i^{n-1}, (r_\xi(k) - \sum_{\lambda=1}^{l-1} A_\lambda^{n-1})\},
\]

where \( A_i^{n-1} \) denotes the number of solutions of the equation

\[
t_1 + t_2 + \cdots + t_{n-1} = l,
\]

where \( t_i \) (\( 1 \leq i \leq n-1 \)) is a non-negative integer.

**Lemma 5.2.** (see [8, Def. 3.1, Example 3.3 and Paragraph (3.5)])

(i) We have \( 0 \leq m_\xi \leq \gamma_\xi \leq m (m+1)/2 \).

(ii) Suppose that \( n = m \) and \( \text{rank} \xi = n \). Then \( m_\xi = 1, \gamma_\xi = n \).

(iii) Let \( \alpha_i \) (\( 1 \leq i \leq m+1 \)) be \( n \)-tuples satisfying the properties given in Proposition 3.10 with respect to \( \xi \). Put \( \alpha_i = (\alpha_1^i, \alpha_2^i, \cdots, \alpha_n^i) \). Then \( \sum_{j=1}^{m+1} \alpha_j^i \leq m_\xi \) for each \( 1 \leq i \leq n \).

For convenience, we denote by \( W(\xi) \) one of the generalized Wronskians

\[
W_{\alpha_1^{m+1}}(\xi)
\]

for some \( \{\alpha_1, \cdots, \alpha_{m+1}\} \) being as in the statement iii) of Lemma 5.2.

**Remark 5.3.** Let notations and hypothesis be as in the statement of Theorem 3.13 and its proof. Then, for all \( r \leq R \) such that \( r_j \) does not belong to a set \( E_j \subset [0, R_j] \) with \( \int_{E_j} \frac{1}{r_j-s} ds < \infty \), we have

\[
( q - (m + 1)K(E, N, \{D_j\}))T_f(r, E) \leq \sum_{j=1}^{a+m-u+b+1} \omega(j) N_{H_i(F)}(r, 0) - \sum_{j=1}^{a+m-u+b+1} \omega(j) N_{W_i(F)}(r, 0) + S_f(r),
\]

where \( K(E, N, \{D_j\}) = k_N(s_N - u + 2 + b)/t_N \).

**Remark 5.4.** By [7, Proposition 4.5 and Proposition 4.10], the right-handed side of the last inequality in Proposition 3.10 can be improved to be

\[
\sum_{i=1}^{m+1} \min\{(F_i)_0, m_F\}.
\]

and the right-handed side of its first inequality can be chosen to be \( \gamma_F \).
Lemma 5.5. The coefficients of the divisor
\[ \sum_{j=1}^{q+m-u+b+1} \nu_{H_j(F)} - (s_N - u + 2 + b)\nu_W(F) \]
are smaller than \( m_F \).

Proof. Denote by \( \mathcal{K} \) the set of all subsets \( K \) of \( \{1, \cdots, q\} \) such that \( |K| = s_N + 1 \) and \( \cap_{j \in K} D_j = B(E) \). Then \( \mathcal{K} \) is the set of all subsets \( K \subset \{1, 2, \cdots, q\} \) such that \( |K| = s_N + 1 \) and \( \cap_{j \in K} H_j \cap Y = \Phi(B(E)) \). By [3, Lemma 4.1 and 4.3], there are \((m-u)\) hyperplanes \( H_{q+1}, \cdots, H_{q+m-u+b+1} \in \mathbb{C}P^m \) such that
\[
\{H_j, H_{q+i} : j \in R, 1 \leq i \leq m-u+b+1\}
\]
are in \((s_N+m-u+b+1)\)-subgeneral position in the usual sense, where \( R \in \mathcal{K} \). Put
\( \mathcal{K}_1 = \{R \subset \{1, 2, \cdots, q + m - u + b + 1\} : |R| = \text{rank}(R) = m + 1\} \).

By Proposition [3,12] for any \( z \in M \) and for any \( J \subset \{1, 2, \cdots, q\} \) with \( |J| = N + 1 \), there exists a subset \( K'(J, z) \in \mathcal{K} \) such that
\[
\sum_{j \in J} \omega(j)\nu_{H_j(F)}(z) \leq \sum_{j \in K'(J, z)} \nu_{H_j(F)}(z) \leq \max_{K \in \mathcal{K}} \sum_{j \in K} \nu_{H_j(F)}(z).
\]

Hence,
\[
(9) \quad \max_{|J| = N+1} \sum_{j \in J} \omega(j)\nu_{H_j(F)}(z) \leq \max_{K \in \mathcal{K}} \sum_{j \in K} \nu_{H_j(F)}(z).
\]

On the other hand, we have
\[
(10) \quad \sum_{j=1}^{q} \omega(j)\nu_{H_j(F)}(z) = \max_{|J| = N+1} \sum_{j \in J} \omega(j)\nu_{H_j(F)}(z).
\]

Put \( LH = \sum_{j=1}^{q+m-u+b+1} \omega(j)\nu_{H_j(F)}(z) \). Combining (10) and (9) and by Lemma 4.2 [3], we have
\[
(11) \quad LH \leq \max_{K \in \mathcal{K}} \sum_{j \in K} \nu_{H_j(F)}(z) + \sum_{j=q+1}^{q+m-u+b+1} \nu_{H_j(F)}(z)
\]
\[
\leq \max_{R \in \mathcal{K}_1} (s_N - u + 2 + b) \left( \sum_{j \in R} \nu_{H_j(F)}(z) \right).
\]

On the other hand, for \( R \in \mathcal{K}_1 \) we deduce from [8, Lemma 3.4] that
\[
\sum_{j \in R} \nu_{H_j(F)}(z) - \nu_W(F)(z) \leq m_F.
\]
It follows that
\[(12)\quad \nu_{W(F)}(z) + m_F \geq \max_{R \in K_1} \sum_{j \in R} \nu_{H_j(F)}(z).\]

Therefore we get the conclusion. \(\square\)

**Proof of Theorem 5.1.** Suppose that \(f \not\equiv g\). We consider two cases:

**Case 1.**
\[
\limsup_{r \to R} \frac{T_f(r, E) - \log(R - r)}{- \log(R - r)} = \infty \text{ or } \limsup_{r \to R} \frac{T_g(r, E) - \log(R - r)}{- \log(R - r)} = \infty.
\]
By the remark above and by noting that \(N_{H_j(F)H_j(G)}(r, 0) \leq T_f(r, E) + T_g(r, E)\), one can see that
\[
\left(q - (m+1)K(E, N, \{D_j\}) - (m-u+b+1)\right) (T_f(r, E) + T_g(r, E)) \leq \sum_{j=1}^q \omega(j) N_{H_j(F)H_j(G)}(r, 0) - (s_N - u + 2 + b)N_{W(F)W(G)}(r, 0) + (S_f(r) + S_g(r)),
\]
Choose two indices \(i_0, j_0\) such that \(\chi := F_{i_0}G_{j_0} - F_{j_0}G_{i_0} \not\equiv 0\). By the assumption, we have
\[
\operatorname{supp} \cup_{j=1}^q H_j(F) \cup H_j(G) \subset \operatorname{supp} \nu_{\chi}.
\]
By Lemma 5.5 it implies that
\[
\sum_{j=1}^q \omega(j) N_{H_j(F)H_j(G)}(r, 0) - (s_N - u + 2 + b)N_{W(F)W(G)}(r, 0) \leq (m_F + m_G)N_{\chi}(r, 0) + (S_f(r) + S_g(r)).
\]
For \(|\chi| \leq ||F|| ||G||\), one deduces
\[
\left(q - (m+1)K(E, N, \{D_j\}) - (m-u+b+1) - m_F - m_G\right) (T_f(r, E) + T_g(r, E)) \leq S_f(r) + S_g(r).
\]
This is a contradiction by the condition posed in this case.

**Case 2.**
\[
\limsup_{r \to R} \frac{T_f(r, E)}{- \log(R - r)} < \infty \text{ and } \limsup_{r \to R} \frac{T_g(r, E)}{- \log(R - r)} < \infty.
\]
By assumption, we can take psh functions \(u_1, u_2\) such that
\[
e^{u_1} \det(h_{ij})^{1/2} \leq ||F||^{p/d}, e^{u_2} \det(h_{ij})^{1/2} \leq ||G||^{p/d}.
\]
Put
\[ l_N = \sum_{j \in Q} \omega(j) - (m + 1)(s_N - u + 2 + b) + m - u + 1 + b, \]
\[ t = \frac{\rho}{l_N - m_F - m_G}. \]
Also set
\[ \phi(F) = \frac{|W(F)(z)|^{s_N-u+2+b}}{\prod_{j=1}^{q+m-u+b+1}|H_j(F(z))|^{\omega(j)}}, \]
and a similar notation for \( G \). By Lemma 5.5, the function
\[ v := t \log \phi(F)\phi(G) + t(m_F + m_G) \log |\chi| \]
is plurisubharmonic. Since \(|\chi| \leq 2||F||||G||\), we have
\[ \int e^{v+u_1+u_2} d\text{vol} \leq \int ||F||^{plN} \phi(F)^t ||G||^{plN} \phi(G)^t dx^1 \wedge \cdots \wedge dx^n \text{ in } U, \]
where \( d\text{vol} \) stands for the volume form of \( M \) with respect to the given Kähler metric. Put \( p_1 = (\gamma_F + \gamma_G)/\gamma_F \), \( p_2 = (\gamma_F + \gamma_G)/\gamma_G \). Applying Holder’s inequality with the powers \((p_1, p_2)\), it implies that
\[ \int_{\Delta_r} e^{v+u_1+u_2} d\text{vol} \leq \left( \int_{\Delta_r} ||F||^{plN} \phi(F)^{tp_1} \right)^{1/p_1} \times \left( \int_{\Delta_r} ||G||^{plN} \phi(G)^{tp_2} \right)^{1/p_2} \]
By Proposition 3.12 we get \( \Theta \geq t_N/k_N \) and
\[ \sum_{j \in Q} \omega(j) \geq \Theta(q - k_N) + t_N \]
\[ \geq (q - k_N) \frac{t_N}{k_N} + t_N \]
\[ \geq 3\rho(\gamma_F + \gamma_G) + m_F + m_G. \]
This yields that \( 3p_1 t\gamma_F = 3\frac{\rho(\gamma_F + \gamma_G)}{l_N - m_F - m_G} < 1 \), and similarly \( 3p_2 t\gamma_G < 1 \). Now, proceeding as the last part of the proof of Theorem 4.3 we get a contradiction. Hence, \( f \equiv g \).

We have a nice corollary in case of equi-dimension.

\textbf{Corollary 5.6.} Let \( M \) be a Stein manifold of dimension \( m \). Let \( f, g \) be two holomorphic mappings of \( M \) to \( \mathbb{C}P^m \) such that they have rank \( m \) and belong to \( A_\rho(M, \mathbb{C}P^m) \). Let \{ \( D_j \}_{j=1}^q \) be a family of hyperplanes in general position. Suppose that
\begin{enumerate} 
  \item \( f = g \) on \( \cup_{i=1}^q f^{-1} D_i \cup g^{-1} D_i \). 
\end{enumerate}
\( ii) \; q > m + 3 + 6m\rho. \)

Then \( f \equiv g. \)

**Remark 5.7.** In the case where \( X = \mathbb{CP}^m \) and \( \{D_j\} \) is a family of hyperplanes in general position, the difference between our result and the unicity theorem of Fujimoto in \([8]\) only is the coefficient 3 corresponding \( \rho(\gamma_F + \gamma_G) \). That is caused by the power in the left-handed side of the inequality in Proposition 3.9.

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