REALIZATIONS OF SLICE HYPERHOLOMORPHIC GENERALIZED
CONTRACTIVE AND POSITIVE FUNCTIONS

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ABSTRACT. We introduce generalized Schur functions and generalized positive functions in setting of slice hyperholomorphic functions and study their realizations in terms of associated reproducing kernel Pontryagin spaces.

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1. Introduction

In this paper we continue the study of Schur analysis in the hyperholomorphic setting, initiated in [5], and continued in [7, 6, 3, 2]. To set the framework we first recall a few facts on the classical case.

1.1. Schur analysis. Functions analytic and contractive in the open unit disk, or in an open half-plane, play an important role in operator theory, signal processing and related fields. Their study, and the study of their counterparts in various settings, may be called Schur analysis; see [32, 1, 24]. In the case of matrix-valued, or operator-valued, functions, contractivity is considered with respect to an indefinite metric. An important example is that of the characteristic operator function and associated operator models. More precisely, let $T$ be a (say, bounded, for the present discussion) self-adjoint operator in a Hilbert space such that $T - T^*$ has finite rank, say $m$. Let

$$T - T^* = \frac{CJC^*}{2i}$$

where $J$ is a $m \times m$ matrix which is both self-adjoint and unitary (a signature matrix). Then, the matrix-valued function

$$\Theta(z) = I + 2iC^*(zI - T)^{-1}CJ$$

is such that

$$\Theta(z)J\Theta(z)^* \geq J,$$

(1.1)

for $z$ in the intersection $\Omega(T)$ of the upper open half-plane and of the spectrum of $T$. The function $\Theta$ is the characteristic operator function of $T$.

Property (1.1) is called $J$-expansivity (or $-J$-contractivity), and is in fact equivalent to the fact that the kernel

$$K_\Theta(z, w) \overset{\text{def.}}{=} \frac{\Theta(z)J\Theta(w)^* - J}{-2i(z - w^*)} = C^*(zI - T)^{-1}(wI - T)^{-*}C$$

(1.2)

is positive definite in $\Omega(T)$.

The function $\Theta$ provides a functional model for $T$. See [22]. A key fact in the theory is the multiplicative structure of $J$-expansive functions, due to V. Potapov; see [50]. We also refer to the historical note of M. Livsic [49]. It is also worth mentioning the original papers of M. Livsic [47, 48], where the notion of characteristic operator function first appears.

Replacing $J$ by $-J$ we obtain $J$-contractive, rather than $J$-expansive functions, and this is the choice we make in the sequel (see in particular the last section).

1.2. Negative squares. One can consider functions $\Theta$ such that the associated kernel $K_\Theta$ has a finite number of negative squares (see Definition 2.3), rather than being positive definite. Such classes of operator-valued functions were introduced and studied by Krein and Langer in a long series of papers; see for instance [40, 41, 42, 43, 44]. These works are set in the framework of the open upper half-plane. To make a better connection with the quaternionic case we consider here the open right half-plane. In the complex variable case, the two cases are equivalent via a conformal map. This differs from the quaternionic setting, as will be clear in the sequel. First recall that an operator $J$ in a Hilbert space $\mathcal{H}$ is called a signature operator if it is self-adjoint and unitary. Its spectrum is then concentrated on $\pm 1$. When $-1$ is a eigenvalue of
finite order, we denote this multiplicity by \( \nu(J) \). Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be two Hilbert spaces and let \( J_1 \in \mathcal{L}(\mathcal{H}_1) \) and \( J_2 \in \mathcal{L}(\mathcal{H}_2) \) be two signature operators, such that \( \nu_-(J_1) = \nu_-(J_2) < \infty \). The \( \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) \)-valued functions \( S \) analytic in an open subset \( \Omega \) of the open right half-plane is called a generalized Schur function if the kernel

\[
\frac{J_2 - S(z)J_1S(w)^*}{z + \overline{w}}
\]

(1.3)

has a finite number of negative squares in \( \Omega \).

For instance when \( J_1 = J_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \), functions in the corresponding class are introduced and used in \[41\] to describe the set of all generalized resolvents of a given Hermitian operator. See \[41\] Satz 3.5, p. 407 and Satz 3.9, p. 409.

Similarly given a Hilbert space \( \mathcal{H} \) and a signature operator \( J \) (possibly with \( \nu_-(J) = \infty \); see \[38\] p. 358, footnote), a \( \mathcal{L}(\mathcal{H}) \)-valued function \( \Phi \) analytic in some open subset \( \Omega \) of the right open half-plane \( \Pi^+ \) is called generalized positive if the kernel

\[
\frac{J\Phi(z) + \Phi(w)^*J}{z + w^*}
\]

(1.4)

has a finite number of negative squares in \( \Omega \). The \( Q \)-function of an Hermitian operator in a Pontryagin space, introduced by Krein and Langer has such a property; see \[39\]. The function \( \Phi \) will be called positive if the kernel (1.4) is positive definite.

In both cases, Krein and Langer proved in the above mentioned works, among numerous results, realization formulas which ensure the existence of a meromorphic extension to the whole of \( \Pi^+ \). It is worth mentioning that a key result to prove this extension is that the part of the spectrum of a contraction in a Pontryagin space which lies outside the closed unit disk consists only of a finite number of eigenvalues. One proof of this fact uses the Schauder-Tychonoff fixed point theorem (see the discussion \[26\] p. 248). We also mention that a study of generalized Schur function of the open unit disk has been given in \[9\] and that unified formulas for a number of cases which include the line and circle case were developed in \[4\], based on an approach including both the disk and half-plane cases developed in \[4, 11\].

Finally we mention the works \[16, 17, 35\] to stress the interest of positive and generalized positive functions in linear system theory and operator theory.

1.3. The slice hyperholomorphic case. In previous papers we extended results of Schur analysis in the slice hyperholomorphic case, in the setting of the unit ball \( \mathbb{B}_1 \) of the quaternions. We considered in \[5\] the Schur algorithm, and the underlying counterpart of the Hardy space. Blaschke products and related interpolation problems in the Hardy space were studied in \[6\]. Nevanlinna-Pick interpolation for Schur functions is studied in \[3\] while the case of kernels having a number of negative squares was studied in \[7\].

In contrast to the above mentioned papers, we consider in this work functions which are slice hyperholomorphic in an open subset of the open half-space

\[
\mathbb{H}_+ = \{ p \in \mathbb{H} \ ; \text{Re} \, p > 0 \},
\]

which intersects the positive real axis.
We define and study the counterparts of the kernel (1.3) and (1.4) in the setting of slice hyperholomorphic functions. Here we consider the case of operator-valued generalized positive functions and generalized Schur functions, rather than scalar or matrix-valued functions.

The renowned Kalman-Yakubovich-Popov lemma establishes a connection between two presentations of generalized positive functions, as rational functions and the respective state space realization, see e.g. [25]. The extension of realization of generalized positive functions to the slice hyperholomorphic setting, introduced in this work, calls upon a corresponding extension of the Kalman-Yakubovich-Popov lemma. This will be addressed in another work.

The paper consists of nine sections, besides the Introduction. In Section 2 we recall the notion of quaternionic Pontryagin spaces, and we discuss some preliminaries on negative squares, kernels and realizations; then we provide some preliminaries on slice hyperholomorphic functions, the class of functions that we use in this paper. In Section 3 we study the Hardy space of the quaternionic half-space \( \mathbb{H}_+ \) of quaternions with real positive part. Then, in Section 4 we provide the proof of the quaternionic version of the Schauder-Tychonoff fixed point theorem whose proof is not substantially different from the one in the complex case, but we insert it for the sake of completeness. This result is crucial to show an invariant subspace theorem for contractions in Pontryagin spaces. Sections 5 and 6 deal with the study of kernels having a finite number of negative squares and with realization theorems in this setting. In Section 7 we give as an example the characteristic operator function of a quaternionic non anti-self-adjoint operator. Section 8 deals with realizations for generalized positive functions. In Section 9 we define the positive function associated to a pair of anti-self-adjoint operators. The properties of this functions will be presented in a future publication.

2. Preliminaries

In this section, which is divided into three subsections, we collect a number of facts respectively on Pontryagin spaces, slice hyperholomorphic functions and their realizations.

2.1. Negative squares and kernels. An important role in this paper is played by quaternionic Pontryagin spaces, and we first recall this notion. We refer to [8, 14] for more details. Let \( \mathcal{V} \) be a right quaternionic vector space endowed with a Hermitian form (also called inner product) \( \langle \cdot, \cdot \rangle \) from \( \mathcal{V} \times \mathcal{V} \) into \( \mathbb{H} \), meaning that:

\[
\langle ua + vb, w \rangle = \langle u, w \rangle a + \langle v, w \rangle b,
\]

\[
\langle v, w \rangle = \overline{\langle w, v \rangle},
\]

for all choices of \( u, v, w \in \mathcal{V} \) and \( a, b \in \mathbb{H} \). In particular the inner product \( \langle \cdot, \cdot \rangle \) satisfy

\[
\langle va, wb \rangle = \overline{b} \langle v, w \rangle a.
\]

When the space \( \mathcal{V} \) is two-sided, we require that

\[
\langle v, aw \rangle = \overline{\langle aw, v \rangle}, \quad a \in \mathbb{H}, \quad v, w \in \mathcal{V}.
\]

Condition (2.1) is used in particular in the proof of formula (6.9).
Definition 2.1. The space \( \mathcal{V} \) is called a right-quaternionic Pontryagin space if it can be written as \( \mathcal{V} = \mathcal{V}_+ + \mathcal{V}_- \), where:

(i) The space \( \mathcal{V}_+ \) endowed with \([\cdot, \cdot]\) is a right-quaternionic Hilbert space.

(ii) The space \( \mathcal{V}_- \) endowed with \(-[\cdot, \cdot]\) is a right-quaternionic Hilbert space.

(iii) The sum \( \mathcal{V}_+ + \mathcal{V}_- \) is direct and orthogonal, meaning that \( \mathcal{V}_+ \cap \mathcal{V}_- = \{0\} \) and \([v_+, v_-] = 0\) for every choice of \( v_+ \in \mathcal{V}_+ \) and \( v_- \in \mathcal{V}_- \).

We denote a direct and orthogonal sum by

\[
\mathcal{V} = \mathcal{V}_+[+]\mathcal{V}_-.
\] (2.2)

In general, such a decomposition will not be unique. The inner product

\[
\langle v, w \rangle = [v_+, v_+] - [v_-, v_-]
\]

makes \( \mathcal{V} \) into a Hilbert space. The inner product depends on the decomposition, but all the associated topologies are equivalent. We refer to [14] for more details on these facts in the quaternionic case, while the case of the field of complex numbers we refer to [19].

We now recall a few facts on matrices with quaternionic entries and on kernels, which we will need in the sequel. A matrix \( A \in \mathbb{H}^{m \times m} \) can be written in a unique way as

\[
A = A_1 + A_2 j,
\]

where \( A_1 \) and \( A_2 \) belong to \( \mathbb{C}^{m \times m} \). The map \( \chi : \mathbb{C}^{m \times m} \to \mathbb{C}^{2m \times 2m} \) defined by

\[
\chi(A) = \begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix}
\] (2.3)

satisfies

\[
\chi(AB) = \chi(A)\chi(B) \quad \text{and} \quad \chi(A^*) = (\chi(A))^*.
\]

See for instance [57] Theorem 4.2, p. 29 (see also [14] Proposition 3.8, p. 439). The result itself is due to Lee [46].

A key fact is that \( A \in \mathbb{H}^{m \times m} \) is Hermitian (that is, \( A = A^* \)) if and only if it can be written as \( UDU^* \), where \( U \in \mathbb{H}^{m \times m} \) is unitary and \( D \in \mathbb{R}^{m \times m} \) is diagonal. The matrix \( D \) is uniquely determined up to permutations, and one can define the signature of an Hermitian matrix with quaternionic entries as the signature of \( D \). See [57] Corollary 6.2, p. 41] and the references therein. The following result follows from the properties of \( \chi \) and can be found in [14] Proposition 3.16, p. 442].

Lemma 2.2. Assume \( A \in \mathbb{H}^{m \times m} \) Hermitian. Then \( A \) has signature \((\nu_+, \nu_-, \nu_0)\) if and only if \( \chi(A) \) has signature \((2\nu_+, 2\nu_-, 2\nu_0)\).

We now turn to the notion of kernels having a finite number of negative squares.

Definition 2.3. Let \( \mathcal{H} \) be a two-sided quaternionic Hilbert space, with inner product \( \langle \cdot, \cdot \rangle \), and let \( K(z, w) \) be a \( \mathcal{L}(\mathcal{H}, \mathcal{H}) \)-valued function defined for \( z, w \) in some set \( \Omega \). The kernel is called Hermitian if

\[
K(z, w) = K(w, z)^*, \quad z, w \in \Omega.
\]

It is said to have a finite number (say \( \kappa \)) of negative squares if for every choice of \( N \in \mathbb{N} \), of vectors \( c_1, \ldots, c_N \in \mathcal{H} \) and of points \( w_1, \ldots, w_N \in \Omega \), the \( N \times N \) Hermitian matrix with \((u, v)\) entry

\[
[K(w_u, w_v)c_u, c_v]
\]
has at most $\kappa$ strictly negative eigenvalues, and exactly $\kappa$ strictly negative eigenvalues for some choice of $N, w_1, \ldots, w_N$ and $c_1, \ldots, c_N$.

When $\kappa = 0$ we have the classical notion of positive definite function. Given a set $\Omega$, the one-to-one correspondence between positive definite functions on $\Omega$ and reproducing kernel Hilbert spaces of functions defined on $\Omega$ extends to a one-to-one correspondence between functions having a finite number of negative squares and reproducing kernel Pontryagin spaces. This fact is due to P. Sorjonen [52] and L. Schwartz [51] in the complex case, and is proved in [14] in the quaternionic case.

2.2. Slice hyperholomorphic functions. Let $\mathbb{H}$ be the real associative algebra of quaternions with respect to the basis $\{1, i, j, k\}$ satisfying the relations $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. We will denote a quaternion $p$ as $p = x_0 + ix_1 + jx_2 + kx_3$, $x_i \in \mathbb{R}$, its conjugate as $\overline{p} = x_0 - ix_1 - jx_2 - kx_3$, and $|p|^2 = p \overline{p}$. The real part $x_0$ of a quaternion will be denoted also by $\text{Re}(p)$, $\mathbb{S}$ is the 2-sphere of purely imaginary unit quaternions, i.e.

$$\mathbb{S} = \{p = ix_1 + jx_2 + kx_3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}.$$ 

Note that if $I \in \mathbb{S}$ then $I^2 = -1$ and a nonreal quaternion $p = x_0 + ix_1 + jx_2 + kx_3$ uniquely determines an element $I_p = \frac{ix_1 + jx_2 + kx_3}{|ix_1 + jx_2 + kx_3|} \in \mathbb{S}$. When $p$ is real, then $p = p + 10$ for all $I \in \mathbb{S}$.

**Definition 2.4.** Given $p \in \mathbb{H}$, $p = p_0 + Ip_1$ we denote by $[p]$ the set of all elements of the form $p_0 + Jp_1$ when $J$ varies in $\mathbb{S}$.

The set $[p]$ is a 2-sphere (we will often write that $[p]$ is a sphere, for short) which is reduced to the point $p$ when $p \in \mathbb{R}$. We now recall the definition of slice hyperholomorphic function.

**Definition 2.5.** Let $\Omega \subseteq \mathbb{H}$ be an open set and let $f : \Omega \to \mathbb{H}$ be a real differentiable function. Let $I \in \mathbb{S}$ and let $f_I$ be the restriction of $f$ to the complex plane $\mathbb{C}_I := \mathbb{R} + I\mathbb{R}$ passing through 1 and $I$ and denote by $x + Iy$ an element on $\mathbb{C}$. We say that $f$ is a left slice hyperholomorphic (or slice hyperholomorphic or slice regular) function in $\Omega$ if, for every $I \in \mathbb{S}$, we have

$$\frac{1}{2} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + Iy) = 0.$$ 

We say that $f$ is a right slice hyperholomorphic function in $\Omega$ if, for every $I \in \mathbb{S}$, we have

$$\frac{1}{2} \left( \frac{\partial}{\partial x} f_I(x + Iy) + I \frac{\partial}{\partial y} f_I(x + Iy)I \right) = 0.$$ 

Slice hyperholomorphic functions have a nice behavior on the so called axially symmetric slice domains defined below.

**Definition 2.6.** Let $\Omega$ be a domain in $\mathbb{H}$. We say that $\Omega$ is a slice domain (s-domain for short) if $\Omega \cap \mathbb{R}$ is non empty and if $\Omega \cap \mathbb{C}_I$ is a domain in $\mathbb{C}_I$ for all $I \in \mathbb{S}$. We say that $\Omega$ is axially symmetric if, for all $q \in \Omega$, the sphere $[q]$ is contained in $\Omega$. 
Theorem 4.3.2. Assume that \( f : \Omega \subseteq \mathbb{C} \cong \mathbb{C}_J \to \mathbb{H} \) is a holomorphic map. Let \( U_\Omega \) be the axially symmetric completion of \( \Omega \), i.e. \( U_\Omega = \bigcup_{x,y \in \mathbb{R}^2} (x + Jy) \). Its left slice hyperholomorphic extension \( \text{ext}(f) : U_\Omega \subseteq \mathbb{H} \to \mathbb{H} \) is computed as follows (see [23]):

\[
\text{ext}(f)(x + Jy) = \frac{1}{2} \left[ f(x + Iy) + f(x - Iy) + JI(f(x - Iy) - f(x + Iy)) \right].
\]

(4.4)

It is immediate that \( \text{ext}(f + g) = \text{ext}(f) + \text{ext}(g) \) and that if \( f(z) = \sum_{n=0}^{\infty} f_n(z) \) then \( \text{ext}(f)(z) = \sum_{n=0}^{\infty} \text{ext}(f_n)(z) \). It is also useful to recall that any function \( h \) slice hyperholomorphic on an axially symmetric s-domain \( \Omega \) satisfies the formula, see [23, Theorem 4.3.2]

\[
h(x + Jy) = \frac{1}{2} \left[ h(x + Iy) + h(x - Iy) + JI(h(x - Iy) - h(x + Iy)) \right].
\]

(5.5)

Remark 2.7. Two left (resp. right) slice hyperholomorphic functions can be multiplied, on an axially symmetric s-domain, using the so called \( \ast \)-product (resp. \( \ast_r \)-product) in order to obtain another left (resp. right) slice hyperholomorphic function. We do not repeat here the construction of such a product and we refer the reader to [23, p. 125]. In the sequel, we will consider functions \( k(p, q) \) left slice hyperholomorphic in \( p \) and right slice hyperholomorphic in \( q \). When taking the \( \ast \)-product of a function \( f(p) \) slice hyperholomorphic in the variable \( p \) with such a function \( k(p, q) \), we will write \( f(p) \ast k(p, q) \) meaning that the \( \ast \)-product is taken with respect to the variable \( p \); similarly, the \( \ast_r \)-product of \( k(p, q) \) with functions right slice hyperholomorphic in the variable \( q \) is always taken with respect to \( q \).

Let us consider a function of the form \( f(p) = \sum_{n=0}^{\infty} (p - p_0)^n a_n \) where \( a_n \in \mathbb{H} \). Following the standard nomenclature and [53] we have:

**Definition 2.9.** A function \( f \) has a pole at the point \( p_0 \) if there exists \( m \geq 0 \) such that \( a_{-k} = 0 \) for \( k > m \). The minimum of such \( m \) is called the order of the pole; \( f \) has a removable singularity at \( p_0 \) if it can be extended in a neighborhood of \( p_0 \) as a slice hyperholomorphic function.

Note the following important fact: a function \( f \) has a pole at \( p_0 \) if and only if its restriction to a complex plane has a pole. Note that there can be poles of order 0; let us consider for example the function \( (p + I)^{-\ast} = (p^2 + 1)^{-1}(p - I) \). It has a pole of order 0 at the point \(-I\) which is not a removable singularity; see [23, p.55].

**Definition 2.10.** Let \( \Omega \) be an axially symmetric s-domain in \( \mathbb{H} \). We say that a function \( f : \Omega \to \mathbb{H} \) is slice hypermeromorphic in \( \Omega \) if \( f \) is slice hyperholomorphic in \( \Omega' \subset \Omega \) such that every point in \( \Omega \setminus \Omega' \) is a pole.

The functions which are slice hypermeromorphic are called semi-regular in [53] and for these functions we have the following result, proved in [53, Proposition 7.1, Theorem 7.3]:

**Proposition 2.11.** Let \( \Omega \) be an axially symmetric s-domain in \( \mathbb{H} \) and let \( f, g : \Omega \to \mathbb{H} \) be slice hyperholomorphic. Then the function \( f^{-\ast} \ast g \) is slice hypermeromorphic in \( \Omega \). Conversely, any slice hypermeromorphic function on \( \Omega \) can be locally expressed as \( f^{-\ast} \ast g \) for suitable \( f \) and \( g \).
Remark 2.12. Since $f^{*-} = (f * f^c)^{-1} f^c$ (see [23] for the notation) it is then clear that the poles of a slice hypermeromorphic function occur in correspondence to the zeros of the function $f * f^c$ and so they are isolated spheres, possibly reduced to real points.

In this paper we are interested in slice hyperholomorphic functions with values in a left quaternionic Banach space $V$, see [2]. We then consider its dual $V^*$ and in order to have a linear structure on it, we assume that $V$ is a two-sided vector space over $\mathbb{H}$.

Definition 2.13. Let $\mathcal{V}$ be a two-sided Banach space over $\mathbb{H}$. We say that a function $f : \Omega \to \mathcal{V}$ is (weakly) slice hypermeromorphic if for any $\Lambda \in \mathcal{V}^*$ the function $\Lambda f : \Omega \to \mathbb{H}$ is slice hypermeromorphic in $\Omega$.

Remark 2.14. The previous definition means, in particular, that $f : \Omega' \to \mathcal{V}$ is slice hyperholomorphic, where the points in $\Omega \setminus \Omega'$ are the poles of $f$. In [2] we have proved that $f$ is also strongly slice hyperholomorphic in $\Omega'$. Moreover, if $\Omega'$ is an axially symmetric $s$-domain, then $f(p)$ can be written as $f(p) = f(x+Iy) = \alpha(x, y) + I\beta(x, y)$ where $\alpha, \beta$ depend on $x, y$ only, satisfy the Cauchy-Riemann system and $\alpha(x, -y) = \alpha(x, y)$, $\beta(x, -y) = \beta(x, y)$.

The $*$-product between slice hyperholomorphic functions is then defined as in [2]:

Definition 2.15. Let $\Omega$ be an axially symmetric $s$-domain in $\mathbb{H}$ and let $\mathcal{H}_i; i = 1, 2, 3$ be two sided quaternionic Hilbert spaces. Let $f : \Omega \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, $g : \Omega \to \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$ be slice hyperholomorphic and let

$$f(p) = f(x + Iy) = \alpha(x, y) + I\beta(x, y), \quad g(p) = g(x + Iy) = \gamma(x, y) + I\delta(x, y).$$

We define $f * g(p) = (\alpha(x, y)\gamma(x, y) - \beta(x, y)\delta(x, y)) + I(\alpha(x, y)\delta(x, y) + \beta(x, y)\gamma(x, y)).$

Remark 2.16. The product $\alpha(x, y)\gamma(x, y)$ (and the other three products appearing in $f * g$) is an operator belonging to $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_3)$, thus $f * g : \Omega \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_3)$. In the special case in which

$$f(p) = \sum_{n=0}^{\infty} p^n A_n, \quad A_n \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2),$$

$$g(p) = \sum_{n=0}^{\infty} p^n B_n, \quad B_n \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3),$$

then we have

$$f * g(p) = \sum_{n=0}^{\infty} p^n (\sum_{r=0}^{n} A_r B_{n-r}),$$

as expected.

We recall the following result with its proof, see [2], for the sake of completeness:

Proposition 2.17 (Identity principle). Let $\mathcal{V}$ be a two sided quaternionic Banach space, $\Omega$ be an $s$-domain and let $f, g : \Omega \subset \mathbb{H} \to \mathcal{V}$ be two slice hyperholomorphic functions. If $f = g$ on a set $Z \subset \Omega \cap \mathbb{C}$, having an accumulation point, for some $I \in \mathbb{S}$, then $f = g$ on $\Omega$. 

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Proof. The hypothesis implies $Af = Ag$ on $Z$ for every $A \in \mathcal{V}^*$ thus the slice hyperholomorphic function $A(f - g)$ is identically zero on $\Omega$ by the identity principle for quaternionic valued slice hyperholomorphic functions. By the Hahn-Banach theorem we deduce that $f - g = 0$ on $\Omega$. □

Remark 2.18. The identity principle implies that two slice hyperholomorphic functions defined on an $s$-domain and with values in a two sided quaternionic Banach space $\mathcal{V}$ coincide if their restrictions to the real axis coincide. More in general, any real analytic function $f : I \subseteq \mathbb{R} \to \mathcal{V}$ can be extended to a function $\text{ext}(f)$ slice hyperholomorphic on an axially symmetric $s$-domain $\Omega$ containing $I$. The existence of the extension is assured by the fact that for any $x_0 \in I$ the function $f$ can be written as $f(x) = \sum_{n \geq 0} x^n A_n$, $A_n \in \mathcal{V}$, and $x$ such that $|x - x_0| < \varepsilon$ and thus $\text{ext}(f) = \sum_{n \geq 0} x^n A_n$ for $|q - x_0| < \varepsilon x_0$. Thus the claim holds setting $B(x_0, \varepsilon x_0) = \{q \in \mathbb{H} : |q - x_0| < \varepsilon x_0\}$ and $\Omega = \cup_{x_0 \in I} B(x_0, \varepsilon x_0)$.

The following notions of $S$-spectrum and of $S$-resolvent operator will be used in the sequel.

Definition 2.19. Let $\mathcal{V}$ be a two sided quaternionic Banach space and let $A$ be a bounded operator on $\mathcal{V}$ into itself. We define the $S$-spectrum $\sigma_S(A)$ of $A$ as:

$$\sigma_S(A) = \{ s \in \mathbb{H} : A^2 - 2 \text{Re}(s)A + |s|^2I \text{ is not invertible} \}.$$  

The $S$-resolvent set $\rho_S(A)$ is defined by $\rho_S(A) = \mathbb{H} \setminus \sigma_S(A)$.

For $s \in \rho_S(A)$ the right $S$-resolvent operator is defined as

$$S_R^{-1}(s, A) := -(A - sI)(A^2 - 2\text{Re}(s)A + |s|^2I)^{-1}. \quad (2.6)$$

Remark 2.20. It is useful to recall that when $A$ is a matrix its (point) $S$-spectrum coincides with its right spectrum, see e.g. [7]. When $s \in \mathbb{R}$ or, more in general, when $s$ commute with an operator $A$, then $S_R^{-1}(s, A) = (sI - A)^{-1}$, see Proposition 3.1.6 in [23].

Proposition 2.21. Let $\mathcal{V}$ be a two sided quaternionic Banach space and let $f : \rho_S(A) \cap \mathbb{R} \to \mathcal{V}$ be the function $f(x) = (I - xA)^{-1}$. Then

$$p^{-1}S_R^{-1}(p^{-1}, A) = (I - pA)(I - 2\text{Re}(p)A + |p|^2A^2)^{-1}$$

is the unique slice hyperholomorphic extension to $\rho_S(A)$.

Proof. The fact that $p^{-1}S_R^{-1}(p^{-1}, A)$ is slice hyperholomorphic in $p$ outside the $S$-spectrum is trivial since it is the $S$-resolvent and it coincides with the function $f$ on the real axis. The uniqueness follows from the identity principle. □

2.3. Realizations. The notation $S_R^{-1}(p^{-1}, A)$ in the previous section comes from [23] and corresponds to the right $S$-resolvent operator. We will also write

$$p^{-1}S_R^{-1}(p^{-1}, A) = (I - pA)^{-*}.$$  

This last expression makes sense when $A$ acts on a two-sided quaternionic vector space. In a more general setting, we have the following result:
Proposition 2.22. Let $A$ be a bounded linear operator from a right-sided quaternionic Hilbert $P$ space into itself, and let $G$ be a bounded linear operator from $P$ into $Q$, where $Q$ is a two sided quaternionic Hilbert space. The slice hyperholomorphic extension of $G(I - xA)^{-1}$, $1/x \in \sigma_S(A) \cap \mathbb{R}$, is
\[(G - \overline{p}GA)(I - 2\text{Re}(p) A + |p|^2 A^2)^{-1}.\]

Proof. First we observe that $G(I - xA)^{-1} = \sum_{n=0}^{\infty} x^n GA^n$ for $|x||A| < 1$. It is immediate that, for $|p||A| < 1$, the slice hyperholomorphic extension of the series $\sum_{n=0}^{\infty} p^n GA^n$ is $\sum_{n=0}^{\infty} \overline{p}^n GA^n$ (as it is a converging power series with coefficients on the right). To show that
\[\sum_{n=0}^{\infty} p^n GA^n = (G - \overline{p}GA)(I - 2\text{Re}(p) A + |p|^2 A^2)^{-1}\]
we prove instead the equality
\[\left(\sum_{n=0}^{\infty} p^n GA^n\right)(I - 2\text{Re}(p) A + |p|^2 A^2) = (G - \overline{p}GA).\]
The left hand side gives
\[\sum_{n=0}^{\infty} p^n GA^n - 2 \sum_{n=0}^{\infty} \text{Re}(p) p^n GA^{n+1} + \sum_{n=0}^{\infty} |p|^2 p^n GA^{n+2} = G + (p - 2\text{Re}(p))GA + (p^2 - 2p\text{Re}(p) + |p|^2) \sum_{n=0}^{\infty} p^n GA^{n+2} = G - \overline{p}GA\]
where we have used the identity $p^2 - 2p\text{Re}(p) + |p|^2 = 0$. This completes the proof. \[\square\]

Remark 2.23. In analogy with the matrix case we will write, with an abuse of notation in this case, $G\ast (I - pA)^{-1}$ instead of the expression $(G - \overline{p}GA)(I - 2\text{Re}(p) A + |p|^2 A^2)^{-1}$.

Proposition 2.24. With the notation in Remark 2.23 it holds that
\[D + pC \ast (I - pA)^{-1}B = D^{-1} - pD^{-1}C \ast (I - p(A - BD^{-1}C))^{-1}BD^{-1},\] (2.7)
and
\[(D_1 + pC_1 \ast (I - pA_1)^{-1}B_1) \ast (D_2 + pC_2 \ast (I - pA_2)^{-1}B_2) =
D_1D_2 + p \left( C_1 \begin{pmatrix} A_1 & C_2 \\ 0 & A_2 \end{pmatrix} \right)^{-1} B_1D_2.\] (2.8)

Proof. When $p$ is real, the $\ast$-product is replaced by the operator product (or matrix product in the finite dimensional case) and formulas (2.7) and (2.8) are then well known; see e.g. [18]. Slice-hyperholomorphic extensions lead then to the required result. \[\square\]
3. The Hardy space of the half-space $\mathbb{H}_+$

Let $\Pi_+$ be the right open half-plane of complex numbers $z$ such that $\text{Re}(z) > 0$. The Hardy space $H^2_2(\Pi_+)$ consists of functions $f$ holomorphic in $\Pi_+$ such that

$$\sup_{x > 0} \int_{-\infty}^{\infty} |f(x + Iy)|^2 dy < \infty.$$  \hspace{1cm} (3.1)

We recall that a function $f \in H^2_2(\Pi_+)$ has nontangential limit $f(iy)$ for almost all $iy$ on the imaginary axis and $f(iy) \in L^2(\mathbb{R})$, see \cite[Theorem 3.1]{33}, moreover

$$\sup_{x > 0} \int_{-\infty}^{\infty} |f(x + Iy)|^2 dy = \int_{-\infty}^{\infty} |f(iy)|^2 dy.$$  \hspace{1cm} (3.2)

Let us consider the kernel

$$k_{\Pi_+}(z, w) = \frac{1}{2\pi} \frac{1}{z + \bar{w}},$$

which is positive definite on $\Pi_+$. Then, the associated reproducing kernel Hilbert space is the Hardy space $H^2_2(\Pi_+)$ endowed with the scalar product

$$\langle f, g \rangle_{H^2_2(\Pi_+)} = \int_{-\infty}^{+\infty} g(iy)f(iy)dy,$$

where $f, g \in H^2_2(\Pi_+)$, and the norm in $H^2_2(\Pi_+)$ is given by

$$\|f\|_{H^2_2(\Pi_+)} = \left( \int_{-\infty}^{+\infty} |f(iy)|^2 dy \right)^{\frac{1}{2}}.$$

The kernel $k_{\Pi_+}(z, w)$ is reproducing in the sense that for every $f \in H^2_2(\Pi_+)$

$$f(w) = \langle f(z), k_{\Pi_+}(z, w) \rangle_{H^2_2(\Pi_+)} = \int_{-\infty}^{\infty} k_{\Pi_+}(w, iy)f(iy)dy,$$

Let us now consider the half-space $\mathbb{H}_+$ of the quaternions $q$ such that $\text{Re}(q) > 0$ and set $\Pi_{+, I} = \mathbb{H}_+ \cap \mathbb{C}_I$. We will denote by $f_I$ the restriction of a function $f$ defined on $\mathbb{H}_+$ to $\Pi_{+, I}$. We define

$$H^2_2(\Pi_{+, I}) = \{ f \text{ slice hyperholomorphic in } \mathbb{H}_+ : \int_{-\infty}^{+\infty} |f_I(Iy)|^2 dy < \infty \},$$

where $f(Iy)$ denotes the nontangential value of $f$ at $Iy$. Note that these value exist almost everywhere, in fact any $f \in H^2_2(\Pi_{+, I})$ when restricted to a complex plane $\mathbb{C}_I$ can be written as $f_I(x + Iy) = F(x + Iy) + G(x + Iy)J$ where $J$ is any element in $S$ orthogonal to $I$, and $F, G$ are $\mathbb{C}_I$-valued holomorphic functions. Since the nontangential values of $F$ and $G$ exist almost everywhere at $Iy$, also the nontangential value of $f$ exists at $Iy$ a.e. on $\Pi_{+, I}$ and $f_I(Iy) = F(Iy) + G(Iy)J$ a.e.

**Remark 3.1.** In alternative, we could have defined $H^2_2(\Pi_{+, I})$ as the set of slice hyperholomorphic functions $f$ such that $\sup_{x > 0} \int_{-\infty}^{+\infty} |f_I(x + Iy)|^2 dy < \infty$. However note that $f_I(x + Iy) = F(x + Iy) + G(x + Iy)J$, see the above discussion, and so
\[ |f_I(x + Iy)|^2 = |F(x + Iy)|^2 + |G(x + Iy)|^2. \] Thus, using (3.2), we have
\[
\sup_{x > 0} \int_{-\infty}^{+\infty} |f_I(x + Iy)|^2 dy = \sup_{x > 0} \int_{-\infty}^{+\infty} |F(x + Iy)|^2 dy + \sup_{x > 0} \int_{-\infty}^{+\infty} |G(x + Iy)|^2 dy = \int_{-\infty}^{+\infty} |F(Iy)|^2 dy + \int_{-\infty}^{+\infty} |G(Iy)|^2 dy = \int_{-\infty}^{+\infty} |f_I(Iy)|^2 dy. \quad (3.3)
\]

In \( H_2(\Pi_{+,I}) \) we define the scalar product
\[
\langle f, g \rangle_{H_2(\Pi_{+,I})} = \int_{-\infty}^{+\infty} g_I(Iy) f_I(Iy) dy,
\]
where \( f_I(Iy) \), \( g_I(Iy) \) denote the nontangential values of \( f, g \) at \( Iy \) on \( \Pi_{+,I} \). This scalar product gives the norm
\[
\|f\|_{H_2(\Pi_{+,I})} = \left( \int_{-\infty}^{+\infty} |f_I(Iy)|^2 dy \right)^{\frac{1}{2}},
\]
which is finite by our assumptions.

**Proposition 3.2.** Let \( f \) be slice hyperholomorphic in \( \mathbb{H}_+ \) and assume that \( f \in H_2(\Pi_{+,I}) \) for some \( I \in \mathbb{S} \). Then for all \( J \in \mathbb{S} \) the following inequalities hold
\[
\frac{1}{2} \|f\|_{H_2(\Pi_{+,I})} \leq \|f\|_{H_2(\Pi_{+,J})} \leq 2 \|f\|_{H_2(\Pi_{+,I})}.
\]

**Proof.** Formula (2.5) implies the inequality
\[
|f(x + Jy)| \leq |f(x + Iy)| + |f(x - Iy)|,
\]
and also
\[
|f(x + Jy)|^2 \leq 2(|f(x + Iy)|^2 + |f(x - Iy)|^2). \quad (3.4)
\]
Using (3.3), (2.5) and (3.4) we deduce
\[
\|f\|_{H_2(\Pi_{+,J})}^2 = \int_{-\infty}^{+\infty} |f_I(Jy)|^2 dy = \sup_{x > 0} \int_{-\infty}^{+\infty} |f_I(x + Jy)|^2 dy \leq \sup_{x > 0} \int_{-\infty}^{+\infty} 2(|f_I(x + Iy)|^2 + f_I(x - Iy)|^2) dy = 4 \int_{-\infty}^{+\infty} |f_I(Iy)|^2 dy
\]
and so \( \|f\|_{H_2(\Pi_{+,J})}^2 \leq 4\|f\|_{H_2(\Pi_{+,I})}^2 \). By changing \( J \) with \( I \) we obtain the reverse inequality and the statement follows. 

An immediate consequence of this result is:

**Corollary 3.3.** A function \( f \in H_2(\Pi_{+,I}) \) for some \( I \in \mathbb{S} \) if and only if \( f \in H_2(\Pi_{+,J}) \) for all \( J \in \mathbb{S} \).

We now introduce the Hardy space of the half space \( \mathbb{H}_+ \):
Definition 3.4. We define $H_2(\mathbb{H}_+)$ as the space of slice hyperholomorphic functions on $\mathbb{H}_+$ such that
\[
\sup_{I \in S} \int_{-\infty}^{+\infty} |f(Iy)|^2 dy < \infty. \tag{3.5}
\]
The scalar product in $H_2(\mathbb{H}_+)$ is
\[
\langle f, g \rangle_{H_2(\mathbb{H}_+)} = \sup_{I \in S} \langle f_I, g_I \rangle_{H_2(\Pi_+, I)}.
\]
We have:

Proposition 3.5. The function
\[
k(p, q) = (\bar{p} + \bar{q})(|p|^2 + 2\text{Re}(p)\bar{q} + \bar{q}^2)^{-1} \tag{3.6}
\]
is slice hyperholomorphic in $p$ and $\bar{q}$ on the left and on the right, respectively in its domain of definition, i.e. for $p \not\in [\bar{q}]$. The restriction of $\frac{1}{2\pi} k(p, q)$ to $\mathbb{C}_1 \times \mathbb{C}_1$ coincides with $k_{\Pi_+}(z, w)$. Moreover we have the equality:
\[
k(p, q) = (|q|^2 + 2\text{Re}(q)p + p^2)^{-1}(p + q). \tag{3.7}
\]
Proof. Some computations allow to obtain $k(p, q)$ as the left slice hyperholomorphic extension in $z$ of $k_q(z) = k(z, q)$, by taking $z$ on the same complex plane as $q$. The function we obtain turns out to be also right slice hyperholomorphic in $\bar{q}$. The second equality follows by taking the right slice hyperholomorphic extension in $\bar{q}$ and observing that it is left slice hyperholomorphic in $p$. 

Proposition 3.6. The kernel $\frac{1}{2\pi} k(p, q)$ is reproducing, i.e. for any $f \in H_2(\mathbb{H}_+)$
\[
f(p) = \int_{-\infty}^{\infty} \frac{1}{2\pi} k(p, Iy)f(Iy)dy.
\]
Proof. Let $q = u + Iqv$ and let $p = u + Iv$ be the point on the sphere determined by $q$ and belonging to the plane $\mathbb{C}_I$. Then we have
\[
f(p) = \int_{-\infty}^{\infty} \frac{1}{2\pi} k(p, Iy)f(Iy)dy, \quad f(\bar{p}) = \int_{-\infty}^{\infty} \frac{1}{2\pi} k(\bar{p}, Iy)f(Iy)dy.
\]
The extension formula applied to $k_{Iy}(p) = k(p, Iy)$ shows the statement. 

The following property will be useful in the sequel:

Proposition 3.7. The kernel $k(p, q)$ satisfies
\[
pk(p, q) + k(p, q)\bar{q} = 1.
\]
Proof. From the expression (3.6), and since $q$ commutes with $(|p|^2 + 2\text{Re}(p)\bar{q} + \bar{q}^2)^{-1}$, we have
\[
p(\bar{p} + \bar{q})(|p|^2 + 2\text{Re}(p)\bar{q} + \bar{q}^2)^{-1} + (\bar{p} + \bar{q})(|p|^2 + 2\text{Re}(p)\bar{q} + \bar{q}^2)^{-1}\bar{q}
\]
\[= (|p|^2 + p\bar{q} + \bar{p}q + \bar{q}^2)(|p|^2 + 2\text{Re}(p)\bar{q} + \bar{q}^2)^{-1} = 1.
\]
\[\square\]
We know that if \( \{ \phi_n(z) \} \) is an orthonormal basis for \( H_2(\Pi_{+,I}) \), for some \( I \in \mathbb{S} \), then
\[
k(z, w) = \sum_{n=1}^{\infty} \phi_n(z)\overline{\phi_n(w)},
\]
and so the kernel \( k(z, w) \) is positive definite. We now prove the following

**Proposition 3.8.** Let \( \{ \phi_n(z) \} \) be an orthonormal basis for \( H_2(\Pi_{+,I}) \), for some \( I \in \mathbb{S} \), and let \( \{ \Phi_n(q) \} = \{ \text{ext}(\phi_n(z)) \} \) be the sequence of the slice hyperholomorphic extensions of its elements. Then \( \{ \Phi_n(q) \} \) is an orthonormal basis for \( H_2(\mathbb{H}_+) \), and
\[
k(p, q) = \sum_{n=1}^{\infty} \Phi_n(p)\overline{\Phi_n(q)}.
\]

**Proof.** Let \( \{ \phi_n(z) \} \) be an orthonormal basis for \( H_2(\Pi_{+,I}) \) and let \( \{ \Phi_n(q) \} = \{ \text{ext}(\phi_n(z)) \} \) be the sequence of the slice hyperholomorphic extensions of its elements. Then \( \{ \Phi_n(q) \} \) is a generating set for \( H_2(\mathbb{H}_+) \). In fact, take any \( f \in H_2(\mathbb{H}_+) \) and consider its restriction to a complex plane \( \mathbb{C}_I \), for some \( I \in \mathbb{S} \). Then, by choosing \( J \in \mathbb{S} \) such that \( I, J \) are orthogonal, and taking \( q = x + Iy \) we have \( f_I(x + Iy) = F(x + Iy) + G(x + Iy)J \) with \( F, G \) holomorphic on \( \Pi_{+,I} \) and
\[
\int_{-\infty}^{+\infty} |f(Iy)|^2 \, dy = \int_{-\infty}^{+\infty} (|F(Iy)|^2 + |G(Iy)|^2) \, dy < \infty
\]
and, as a consequence,
\[
\int_{-\infty}^{+\infty} |F(Iy)|^2 \, dy \leq \int_{-\infty}^{+\infty} (|F(Iy)|^2 + |G(Iy)|^2) \, dy < \infty.
\]
We deduce that both
\[
\int_{-\infty}^{+\infty} |F(Iy)|^2 \, dy \quad \text{and} \quad \int_{-\infty}^{+\infty} |G(Iy)|^2 \, dy
\]
are finite and so \( F, G \) belong to \( H_2(\Pi_{+,I}) \). We can write \( F(x + Iy) = \sum_{n=1}^{\infty} \phi_n(z)a_n \) and \( G(x + Iy) = \sum_{n=1}^{\infty} \phi_n(z)b_n \), thus \( f_I(x + Iy) = \sum_{n=1}^{\infty} \phi_n(z)(a_n + b_nJ) \). By taking the extension with respect to \( z \) we finally obtain \( f(q) = \sum_{n=1}^{\infty} \Phi_n(q)(a_n + b_nJ) \). The fact that \( \{ \Phi_n(p) \} \) is made by orthonormal elements (thus linearly independent) in \( H_2(\mathbb{H}_+) \) follows from
\[
\langle \Phi_n(p), \Phi_m(p) \rangle_{H_2(\Pi_{+,I}))} = \int_{-\infty}^{+\infty} \overline{\Phi_m(Iy)}\Phi_n(Iy) \, dy
\]
\[
= \int_{-\infty}^{+\infty} \overline{\phi_m(Iy)}\phi_n(Iy) \, dy = \delta_{nm}.
\]
Then (3.8) yields
\[
k(p, w) = \text{ext}_z k(z, w) = \sum_{n=1}^{\infty} \text{ext}_z(\phi_n(z))\overline{\phi_n(w)} = \sum_{n=1}^{\infty} \Phi_n(p)\overline{\Phi_n(q)},
\]
where we have written \( \text{ext}_z \) to emphasize that we are taking the extension in the variable \( z \) (note that in this way we have obtained the kernel written in the form (3.6)). Now we observe that the function \( \sum_{n=1}^{\infty} \Phi_n(p)\overline{\Phi_n(q)} \) is slice hyperholomorphic on the left and on the right with respect to \( p \) and \( q \), respectively, and coincides with
We have \( k(p, w) \) when restricted to the plane containing \( w \). By the uniqueness of the extension we have \( k(p, q) = \sum_{n=1}^{\infty} \Phi_n(p) \Phi_n(q) \), and the statement follows. \( \square \)

We now introduce the Blaschke factors in the half space \( \mathbb{H}_+ \).

**Definition 3.9.** For \( a \in \mathbb{H}_+ \) set

\[ b_a(p) = (p + \bar{a})^{-*} \ast (p - a). \]

The function \( b_a(p) \) is called Blaschke factor at \( a \) in the half space \( \mathbb{H}_+ \).

**Remark 3.10.** The function \( b_a(p) \) is defined outside the sphere \([-a]\) as it can be easily seen by rewriting it as

\[ b_a(p) = (p^2 + 2\text{Re}(a)p + |a|^2)^{-1} (p + a) \ast (p - a) = (p^2 + 2\text{Re}(a)p + |a|^2)^{-1} (p^2 - a^2) \]

and it has a zero for \( p = a \). Note in fact that \( p = -a \) is not a zero since it is a pole (of order 0). When \( a \in \mathbb{R} \) the function \( b_a(p) = (p + a)^{-1} (p - a) \) has a pole at \( p = -a \) and a zero at \( a \). A Blaschke factor is slice hyperholomorphic where it is defined, by construction.

We have the following result which characterizes the convergence of a Blaschke product. We denote by \( \Pi^* \) the \( \ast \)-product:

**Theorem 3.11.** Let \( \{a_j\} \subset \mathbb{H}_+, j = 1, 2, \ldots \) be a sequence of quaternions such that \( \sum_{j=1}^{\infty} \text{Re}(a_j) < \infty \). Then the function

\[ B(p) := \Pi_{j \geq 1}^* (p + \bar{a}_j)^{-*} \ast (p - a_j) \]

(3.9)

converges uniformly on the compact subsets of \( \mathbb{H}_+ \).

**Proof.** We reason as in the proof of the corresponding result in the complex case (but see also the proof of Theorem 5.6 in \cite{[6]}). We note that, see Remark 5.4 in \cite{[6]}, we can write

\[ (p + \bar{a}_j)^{-*} \ast (p - a_j) = (\tilde{p} + \bar{a}_j)^{-1} (\tilde{p} - a_j) \]

(3.10)

where \( \tilde{p} = \lambda^c(p)^{-1} p \lambda^c(p) \) and \( \lambda^c(p) = p + a_j \) (note that \( \lambda^c(p) \neq 0 \) for \( p \notin [-a_j] \)) and so

\[ (p + \bar{a}_j)^{-*} \ast (p - a_j) = (\tilde{p} + \bar{a}_j)^{-1} (\tilde{p} - a_j) = 1 - 2\text{Re}(a_j)(\tilde{p} + \bar{a}_j)^{-1}. \]

(3.11)

By taking the modulus of the right hand side of (3.9), using (3.11), and reasoning as in the complex case, we conclude that the Blaschke product converges if and only if \( \sum_{j=1}^{\infty} \text{Re}(a_j) < \infty \). \( \square \)

As in the unit disk case, we have two kinds of Blaschke factors. In fact, products of the form

\[ b_a(p) \ast b_a(p) = ((p + \bar{a})^{-*} \ast (p - a)) \ast ((p + a)^{-*} \ast (p - \bar{a})) \]

can be written as

\[ b_a(p) \ast b_a(p) = (p^2 + 2\text{Re}(a)p + |a|^2)^{-1} (p^2 - 2\text{Re}(a)p + |a|^2), \]

and they admit the sphere \([a]\) as set of zeros. Note that slice regular functions which vanish at two points belonging to the same sphere in reality vanish on the whole sphere (see \cite{[23]} Corollary 4.3.7). Thus if we want to construct a Blaschke product vanishing at some prescribed points and spheres, it is convenient to introduce the following:
Lemma 3.15. Let $f \in \mathbb{H}_+$ set
\[ b_{[a]}(p) = (p^2 + 2\text{Re}(a)p + |a|^2)^{-1}(p^2 - 2\text{Re}(a)p + |a|^2). \]
The function $b_{a}(p)$ is called Blaschke factor at the sphere $[a]$ in the half space $\mathbb{H}_+$.

Note that the definition is well posed since it does not depend on the choice of the point $a$. As a consequence of Theorem 3.11 we have

Corollary 3.13. Let $\{c_j\} \subset \mathbb{H}_+$, $j = 1, 2, \ldots$ be a sequence of quaternions such that $\sum_{j \geq 1} \text{Re}(c_j) < \infty$. Then the function
\[ B(p) := \Pi_{j \geq 1}(p^2 + 2\text{Re}(c_j)p + |c_j|^2)^{-1}(p^2 - 2\text{Re}(c_j)p + |c_j|^2), \]converges uniformly on the compact subsets of $\mathbb{H}_+$.

Proof. It is sufficient to write $B(p) = \prod_{j \geq 1} b_{[c_j]}(p) = \prod_{j \geq 1} b_{c_j}(p) \ast b_{c_j}(p)$ and to observe that $2 \sum_{j \geq 1} \text{Re}(c_j) < \infty$ by hypothesis.

Thus we can prove the following:

Theorem 3.14. A Blaschke product having zeros at the set
\[ Z = \{(a_1, \mu_1), (a_2, \mu_2), \ldots, ([c_1], \nu_1), ([c_2], \nu_2), \ldots\} \]
where $a_j \in \mathbb{H}_+$, $a_j$ have respective multiplicities $\mu_j \geq 1$, $[c_i] \neq [c_j]$ if $i \neq j$, $c_i \in \mathbb{H}_+$, the spheres $[c_j]$ have respective multiplicities $\nu_j \geq 1$, $j = 1, 2, \ldots$, $[c_i] \neq [c_j]$ if $i \neq j$ and
\[ \sum_{i,j \geq 1} \left( \mu_i (1 - |a_i|) + 2\nu_j (1 - |c_j|) \right) < \infty \]

is given by
\[ \prod_{i \geq 1} (b_{[c_i]}(p))^\nu_i \prod_{j \geq 1} (b_{a_j}(p))^* \mu_j, \]
where $a'_1 = a_1$ and $a'_j \in [a_j]$ are suitably chosen elements, $j = 2, 3, \ldots$.

Proof. The Blaschke product converges and defines a slice hyperholomorphic function by Theorem 3.11 and its Corollary 3.13. It has zeros of assigned multiplicities at the assigned spheres since the Blaschke factors $b_{[c]}$ are functions with real coefficients. The rest of the statement follows from the fact that $a_1$ is the zero corresponding to the leftmost factor $b_{a_1}$ while the zero of the factor $b_{a'_j}(p)$, $j > 1$, belongs to the sphere $[a'_j]$ and $a'_j$ can be chosen in order that the zero is exactly the assigned point, as we did in Proposition 5.12.

We conclude this section by proving that the operator of multiplication by a Blaschke factor is an isometry. In the proof we are in need of the notion of conjugate of a function $f$. Given a slice hyperholomorphic function $f$ consider its restriction to a complex plane $\mathbb{C}_I$ and write it, as customary, in the form $f_I(z) = F(z) + G(z)J$ where $J$ is an element in $\mathbb{S}$ orthogonal to $I$ and $F, G$ are $\mathbb{C}_I$-valued holomorphic functions. Define $f^c(p) = \text{ext}(\overline{F(z)} - G(z)J)$ where the extension operator is defined in (2.4).

Note that if $f(p) = \sum_{n \geq 0} p^n a_n$ then $f^c(p) \sum_{n \geq 0} p^n \overline{a}_n$. We have the following:

Lemma 3.15. Let $f \in \mathcal{H}_2(\mathbb{H}_+)$. Then $\|f\|_{\mathcal{H}_2(\mathbb{H}_+)} = \|f^c\|_{\mathcal{H}_2(\mathbb{H}_+)}$. 

Proof. By definition we have
\[
\|f\|_{H_2(\Pi_+,I)}^2 = \int_{-\infty}^{+\infty} |f(Iy)|^2 dy = \int_{-\infty}^{+\infty} (|F(Iy)|^2 + |G(Iy)|^2) dy
\]
and
\[
\|f^c\|_{H_2(\Pi_+,I)}^2 = \int_{-\infty}^{+\infty} |f^c(Iy)|^2 dy = \int_{-\infty}^{+\infty} (|\bar{F}(-Iy)|^2 + |G(Iy)|^2) dy
\]
Thus \(\|f\|_{H_2(\Pi_+,I)}^2 = \|f^c\|_{H_2(\Pi_+,I)}^2\) and taking the supremum for \(I \in \mathbb{S}\) the statement follows. \(\square\)

Theorem 3.16. Let \(b_a\) be a Blaschke factor. The operator
\[
M_{b_a} : f \mapsto b_a * f
\]
is an isometry from \(H_2(\mathbb{H}_+)\) into itself.

Proof. Recall that, by (3.10), we can write \(b_a(p) = (\bar{p} + a)^{-1}(\bar{p} - a)\) for \(\bar{p} = \lambda^c(p)^{-1}p\lambda(p)\). Let us set \(t = Iy\) where \(I \in \mathbb{S}\). We have
\[
|b_a(Iy)| = |(Iy + \bar{a})^{-1}(Iy - a)| = |(Iy + a)^{-1}(Iy + a)| = 1.
\]
Similarly, \(|b_a^c(Iy)| = 1\). We now observe that for any two functions \(f\) and \(g\) we have \((f \ast g)^c = g^c * f^c\). We prove this equality by showing that the two functions \((f \ast g)^c\) and \(g^c * f^c\) coincide on a complex plane (so the needed equality follows from the identity principle). Using the notation introduced above, let us write \(f_I(z) = F(z) + G(z)J\) and \(g_I(z) = H(z) + L(z)J\). We have
\[
(f * g)_I(z) = f_I(z) * g_I(z) = (F(z)H(z) - G(z)L(z)) + (F(z)L(z) + G(z)\bar{H}(z))J
\]
so, by definition of \((f \ast g)^c\), we have
\[
(f \ast g)^c_I(z) = (\bar{F}(z)\bar{H}(z) - \bar{G}(z)L(z)) - (F(z)L(z) + G(z)\bar{H}(z))J
\]
and
\[
(g^c \ast f^c)_I(z) = (\bar{H}(z) - L(z)J) \ast (\bar{F}(z) - G(z)J)
\]
\[
= (\bar{H}(z) \bar{F}(z) - L(z)G(z)) - (\bar{H}(z)G(z) + L(z)F(z))J
\]
the two expressions coincide since the functions \(F, G, H, L\) are \(C_I\)-valued and thus they commute. To compute \(\|b_a \ast f\|_{H_2(\mathbb{H}_+)}\), where \(f \in H_2(\mathbb{H}_+)\), we follow an idea used in [3] and we compute \(\|(b_a \ast f)^c\|_{H_2(\mathbb{H}_+)}^2\). Note that \((f^c \ast b_a^c)(x + Iy) = 0\) where \(f^c(x + Iy) = 0\), i.e. on a set of isolated points on \(\Pi_+,I\) while, if \(q = f^c(x + Iy) \neq 0\), \((f^c \ast b_a^c)(x + Iy) = f^c(x + Iy)b_a^c(q^{-1}(x + Iy)q)\), see [23] Proposition 4.3.22, where \(q^{-1}(x + Iy)q = x + I'y\), see [34] Proposition 2.22. Thus we have \((f^c \ast b_a^c)(Iy) = f^c(Iy)b_a^c(I'y)\) almost everywhere.
and
\[ \|b \ast f\|^2_{H_2(H_+)} = \|(b \ast f)^c\|^2_{H_2(H_+)} = \sup_{I \in \mathbb{S}} \int_{-\infty}^{+\infty} |(f^c \ast b^c_a)(Iy)|^2 \, dy = \sup_{I \in \mathbb{S}} \int_{-\infty}^{+\infty} |f^c(Iy)b^c_a(I'y)|^2 \, dy = \sup_{I \in \mathbb{S}} \int_{-\infty}^{+\infty} |f^c(Iy)||b^c_a(I'y)|^2 \, dy = \sup_{I \in \mathbb{S}} \int_{-\infty}^{+\infty} |f^c(Iy)|^2 \, dy = \|f^c\|^2_{H_2(H_+)}. \]

By the previous lemma, we have \( \|f^c\|^2_{H_2(H_+)} = \|f\|^2_{H_2(H_+)} \) and this concludes the proof.

Blaschke factors will provide a concrete example of the functions studied in Section 6. See Example 6.5 there.

4. The Schauder-Tychonoff fixed point theorem

In this section we extend the Schauder-Tychonoff fixed point theorem to the quaternionic setting. The proof is similar to that of the classical case given in [27], but we include it for the sake of completeness. This results is crucial to prove an invariant subspace theorem for contractions in a Pontryagin spaces.

4.1. The Schauder-Tychonoff fixed point theorem. In the sequel we will use a consequence of the Ascoli-Arzelà theorem that we state in this corollary.

**Lemma 4.1** (Corollary of Ascoli-Arzelà theorem). Let \( \mathcal{G}_1 \) be a compact subset of a topological group \( \mathcal{G} \) and let \( \mathcal{K} \) be a bounded subset of the space of continuous functions \( C(\mathcal{G}_1) \). Then \( \mathcal{K} \) is conditionally compact if and only if for every \( \varepsilon > 0 \) there is a neighborhood \( U \) of the identity in \( \mathcal{G} \) such that \( |f(t) - f(s)| < \varepsilon \) for every \( f \in \mathcal{K} \) and every pair \( s, t \in \mathcal{G}_1 \) with \( t \in U s \).

**Proof.** It is Corollary 9 p. 267 in [27] and its proof can be obtained in the same arguments.

**Definition 4.2.** We say that a quaternionic topological vector space \( \mathcal{V} \) has the fixed point property if for every continuous mapping \( T : \mathcal{V} \to \mathcal{V} \) there exists \( u \in \mathcal{V} \) such that \( u = T(u) \).

To show our result we need the following Lemmas:

**Lemma 4.3.** Let \( \mathcal{C} \) be the subspace of \( \ell^2(\mathbb{H}) \) defined by
\[ \mathcal{C} = \{ \{\xi_n\} \in \ell^2(\mathbb{H}) : |\xi_n| \leq 1/n, \ \forall n \in \mathbb{N} \}. \]
Then \( \mathcal{C} \) has the fixed point property.
Proof. Let $P_n : \mathfrak{C} \to \mathfrak{C}$ be the map

$$P_n(\xi_1, \xi_2, \ldots, \xi_n, \xi_{n+1} \ldots) = (\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots).$$

Then $\mathfrak{C}_n = P_n(\mathfrak{C})$ is homeomorphic to the closed sphere in $\mathcal{H} \cong \mathbb{R}^{4n}$. Let now $T : \mathfrak{C} \to \mathfrak{C}$ be a continuous map. Then $P_nT : \mathfrak{C}_n \to \mathfrak{C}_n$ is continuous. Brower theorem implies that there is a fixed point $\zeta_n \in \mathfrak{C}_n \subseteq \mathfrak{C}$ and so

$$|\zeta_n - T(\zeta_n)| \leq \left( \sum_{i=n+1}^{\infty} \frac{1}{i^2} \right)^{\frac{1}{2}}.$$

Since $\mathfrak{C}$ is compact, then $\{\zeta_n\}$ contains a subsequence converging to a point which is a fixed point of $T$. \hfill $\square$

Lemma 4.4. Let $\mathcal{K}$ be a compact convex subset of a locally convex linear quaternionic space $\mathcal{V}$ and let $T : \mathcal{K} \to \mathcal{K}$ be continuous. If $\mathcal{K}$ contains at least two points, then there exists a proper closed convex subset $\mathcal{K}_1 \subset \mathcal{K}$ such that $T(\mathcal{K}_1) \subseteq \mathcal{K}_1$.

Proof. It is possible to assume that $\mathcal{K}$ has the $\mathcal{V}^*$ topology.

We will say that a set of continuous linear functionals $F$ is determined by another set $G$, if for every $f \in F$ and $\varepsilon > 0$ there exists a neighborhood

$$\mathcal{N}(0; \gamma, \delta) = \{v \in \mathcal{V} : |g(v)| < \delta, \ g \in \gamma\},$$

where $\gamma$ is a finite subset of $G$ with the property that if $u, v \in \mathcal{K}$ and $u - v \in \mathcal{N}(0; \gamma, \delta)$ then $|f(Tu) - f(Tv)| < \varepsilon$. It is clear that if $F$ is determined by $G$, then $g(u) = g(v)$ for $g \in G$ implies that $f(Tu) = f(Tv)$ for $f \in F$. Each continuous linear functional $f$ is determined by some countable set of functional $G = \{g_m\}_{m \in \mathbb{N}}$.

Thanks to Lemma 4.1 the scalar function $f(Tu)$ is uniformly continuous on the compact set $\mathcal{K}$. Hence for every integer $n$ there is a neighborhood $\mathcal{N}(0; \gamma_n, \delta_n)$ of the origin in $\mathcal{V}$, given by a set of linear continuous functionals $\gamma_n$ and a $\delta_n > 0$, such that if $u, v \in \mathcal{K}$ and $u - v \in \mathcal{N}(0; \gamma_n, \delta_n)$ then $|f(Tu) - f(Tv)| < 1/n$. Let $G = \bigcup_{n=1}^{\infty} \gamma_n$ then $f$ is determined by $G$. It follows that if $F$ is a countable subset of $\mathcal{V}^*$, there exists a countable subset $G_F$ of $\mathcal{V}^*$ such that each $f \in F$ is determined by $G_F$. We claim that each continuous linear functional $f$ can be included in a countable self-determined set $G$ of functionals. In fact, if $f$ is determined by the countable set $G_1$, let each functional in $G_1$ be determined by the countable set $G_2$; then let each functional in $G_2$ be determined by the countable set $G_3$, and so on. We obtain a sequence $\{G_i\}$ and we set $G = \{f\} \cup \bigcup_{i=1}^{\infty} G_i$. Assume now that $\mathcal{K}$ contains two points $u, v, u \neq v$ and let $f \in \mathcal{V}^*$ be such that $f(u) \neq f(v)$. Let $G = \{g_i\}$ be a countable self-determined set of continuous linear functionals containing $f$. Since $\mathcal{K}$ is compact, $g_i(\mathcal{K})$ is a bounded set of scalars for every $i$ and since we can multiply $g_i$ by a suitable constant we may suppose that $g_i(\mathcal{K}) \leq 1/i$. In this case the mapping $H : \mathcal{K} \to l^2(\mathbb{H})$, defined by

$$H(k) := [g_i(k)]$$

is a continuous mapping of $\mathcal{K}$ onto a compact convex subset $\mathcal{K}_0$ of the subspace $\mathfrak{C}$ of $l^2(\mathbb{H})$. Then $\mathfrak{C}$ contains trivially at least two points since there are at least two points in $l^2(\mathbb{C})$, see [27]. Consider the mapping

$$T_0 = HTH^{-1} : \mathcal{K}_0 \to \mathcal{K}_0$$
since $G$ is self determined $T_0$ is single-valued. To see that $T_0$ is continuous, let $b_0 \in K_0$ and $\varepsilon \in (0, 1)$. Choose $N$ such that $\sum_{i=N+1}^{\infty} 1/i^2 < \varepsilon$. Then $G$ is self-determined, there exists a $\delta > 0$ and an $m$ such that if $|g_j(u) - g_j(v)| < \delta$, $j = 1, \ldots, m$ then

$$|g_i(Tu) - g_i(Tv)| < \varepsilon/N, \quad i = 1, \ldots, N.$$  \hfill (4.1)

Thus if $|b - b_0| < \delta$ and $u$ and $v$ are point in $K$ with $b = [g_i(u)]$ and $b_0 = [g_i(v)]$ then (4.1) holds and

$$|T_0(b) - T_0(b_0)|^2 = |HTH^{-1}(b) - HTH^{-1}(b_0)|^2$$

$$\leq \sum_{i=1}^{N} |g_i(Tu) - g_i(Tv)|^2 + 2 \sum_{i=N+1}^{\infty} 1/i^2$$

$$< 3\varepsilon.$$

So $T_0$ is a continuous mapping of $K_0$ into itself. From the fixed point property of $C$, see Lemma 4.4, it follows that $T_0$ has has a fixed point $k_0$. Thus

$$TH(k_0) \subseteq H^{-1}T_0(k_0) = H^{-1}(k_0).$$

Setting $K_1 = H^{-1}(k_0)$ we note that $K_1$ is a proper closed subset of $K$, and that $T(K_1) \subseteq K_1$. The linearity of $H$ implies that $K_1$ is convex. This concludes the proof. \hfill \Box

\textbf{Theorem 4.5} (Schauder-Tychonoff). A compact convex subset of a locally convex quaternionic linear space has the fixed point property.

\textbf{Proof.} By the Zorn lemma there exists a minimal convex subset of $K_1$ of $K$ with the property that $TK_1 \subseteq K_1$. By Lemma 4.4 this minimal subset contains only one point. \hfill \Box

\subsection*{4.2. An invariant subspace theorem.}

As we explained at the beginning of the section, the Schauder-Tychonoff theorem is now used to prove an invariant subspace theorem for contractions in quaternionic Pontryagin spaces. This theorem is used in the realization theorems to prove the existence of slice hyperholomorphic extensions of certain functions defined in a neighborhood of a point on the positive axis. In the complex numbers case, this theorem can be found in [26, Theorem 1.3.11]. We also refer to [26, Notes on chapter 1] for historical notes on the theorem.

\textbf{Theorem 4.6.} A contraction in a quaternionic Pontryagin space has a unique maximal invariant negative subspace, and it is one-to-one on it.

\textbf{Proof.} The proof of [26] carries up to the quaternionic setting, and we recall the main lines for the convenience of the reader. Let $A$ be a contraction in the Pontryagin space $P$. To prove that $A$ has a maximal negative invariant subspace we first recall a well known fact in the theory of linear fractional transformations (see for instance [28] for more details). Let $P = P_{[+]}P_{[-]}$ be a fundamental decomposition of $P$. Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\
A_{21} & A_{22} \end{pmatrix}$$

be the block decomposition of $A$ along $P_{[+]}P_{[-]}$. Since $A$ is a contraction we have

$$A_{21}A_{21}^* - A_{22}A_{22}^* \leq -I,$$
and it follows that $A_{22}^{-1}$ and $A_{22}^{-1}A_{21}$ are strict contractions. Thus the map

$$L(X) = (A_{11}X + A_{12})(A_{21}X + A_{22})^{-1}$$

is well defined, and sends in fact the closed unit ball $B_1$ of $\mathcal{L}(\mathcal{P},\mathcal{P}_+)$ into itself. The main point in the proof of the theorem is to show that the map $L$ is continuous in the weak operator topology from $B_1$ into itself. Since $B_1$ is compact in this topology (and of course convex) the Schauder-Tychonoff theorem implies that $L$ has a unique fixed point, say $X$. To conclude one notes (see Theorem [26, 1.3.10]) that the space spanned by the elements

$$f + Xf, \quad f \in \mathcal{P}_- \quad (4.1)$$

is then negative. It is maximal negative because $X$ cannot have a kernel (any $f$ such that $Xf = 0$ will lead to a strictly positive element of (4.1)). □

5. The spaces $\mathcal{P}(S)$

We now introduce the counterparts of the kernels (1.3) in the slice hyperholomorphic setting. In the quaternionic case signature operators are defined as in the complex case. We here consider real signature operators, that is, which are unitarily equivalent to an operator of the form

$$\begin{pmatrix}
I_+ & 0 \\
0 & -I_-
\end{pmatrix}.$$

It is clear that the $S$-spectrum is concentrated on $\pm 1$, so we define $\nu_-(J)$ as in the complex case. This follows by simple computations, that is $1 \pm 2s_0 + |s|^2 = 0$ which give $\pm 1$.

**Definition 5.1.** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two quaternionic two-sided Hilbert spaces and let $J_1 \in \mathcal{L}(\mathcal{H}_1)$ and $J_2 \in \mathcal{L}(\mathcal{H}_2)$ be two real signature operators such that $\nu_-(J_1) = \nu_-(J_2) < \infty$. The $\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$-valued function $S$ slice hypermeromorphic in an axially symmetric s-domain $\Omega$ which intersects the positive real line belongs to the class $\mathcal{S}_\kappa(J_1, J_2)$ if the kernel

$$K_S(p,q) = J_2k(p,q) - S(p) * k(p,q) * r, J_1S(q)^*$$

has $\kappa$ negative squares in $\Omega$, where $k(p,q)$ is defined in (3.6).

We do not mention the dependence of the class on $\Omega$. As we will see, every element of these classes has a unique meromorphic extension to $\mathbb{H}_+$. To reduce the case of arbitrary signature operators (with same number of negative squares) to the case of the identity, we define the Potapov-Ginzburg transform in the present setting. We refer to the book [22] for the classical case, even though some formulas are also recalled in [1].

We begin with a lemma. A proof in the classical case can be found in [9] Lemma 4.4.3, p. 164] but we provide an argument for completeness. First a remark: a matrix $A \in \mathbb{H}^{m \times m}$ is not invertible if and only if there exists $c \neq 0 \in \mathbb{H}^m$ such that $c^*A = 0$. This fact can be seen for instance from [45, Theorem 7, p. 202], where it is shown that a matrix over a division ring has row rank equal to the column rank, or [50, Corollary 1.1.8].
Lemma 5.2. Let $T$ be a $\mathbb{H}^{m \times m}$-valued function slice hyperholomorphic in an axially symmetric s-domain $\Omega$ which intersect the positive real line, and such that the kernel
\[ T(p) \star k(p, q) \star_r T(q)^* - k(p, q)I_m \]
has a finite number of negative squares, say $\kappa$, in $\Omega$. Then $T$ is invertible in $\Omega$, with the possible exception of a countable number of spheres.

Proof. We first show that $T$ is invertible on $\Omega \cap \mathbb{R}_+$ with the possible exception of a countable number of points. Let $x_1, \ldots, x_M$ be zeros of $T$. Then, there exist vectors $c_1, \ldots, c_M$ such that
\[ c_j^* T(x_j) = 0, \quad j = 1, \ldots, M. \]
Thus
\[ m_{jk} = c_j^* K(x_j, x_k) c_k = -\frac{c_j^* c_k}{x_j + x_k} \]
To conclude we use [10, Lemma 2.1, p. 30] to see that the $M \times M$ matrix with $jk$ entry $m_{jk}$ is strictly negative, and so $M \leq \kappa$. The result in [10, Lemma 2.1, p. 30] is proved for the case of complex numbers, but extends to the quaternionic case, as is seen by using the map $\chi$ defined in (2.3) and Lemma 2.2.

Let now $S \in \mathcal{S}_\kappa(J_1, J_2)$ and let
\[
S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}
\] (5.1)
be the decomposition of $S$ according to fundamental decompositions of the coefficient spaces. In the statement of the following theorem, we denote by $I_{2+}$ the identity of the positive space in the fundamental decomposition of $\mathcal{H}_2$.

Theorem 5.3. Let $S \in \mathcal{S}_\kappa(J_1, J_2)$, defined in an axially symmetric s-domain $\Omega$ intersecting the real positive axis, and with decomposition (5.1). Then the function $S_{22}$ is \(\star\)-invertible in $\Omega$, with the possible exception of a countable number of spheres. Let
\[
A(p) = \begin{pmatrix} I_{2+} & S_{12}(p) \\ 0 & S_{22}(p) \end{pmatrix} \quad \text{and} \quad \Sigma(p) = \begin{pmatrix} S_{11} - S_{12} \star S_{22}^* \star S_{21} & S_{12} \star S_{22}^* \\ S_{21}^* \star S_{22}^* & S_{22}^* \end{pmatrix}(p). \]
(5.2)
Then,
\[
J_2 k(p, q) - S(p) \star k(p, q) \star_r J_1 S(q)^* = \] 
\[
= A(p) \star (k(p, q) - \Sigma(p) \star k(p, q) \star_r \Sigma(q)^*) \star_r A(q)^*, \]
(5.3)
and the kernel
\[
k(p, q) - \Sigma(p) \star k(p, q) \star_r \Sigma(q)^* \]
(5.4)
has a finite number of negative squares on the domain of definition of $\Sigma$ in $\Omega$ and hence has a slice hyperholomorphic extension to the whole of the right half-space, with the possible exception of a finite number of spheres.

The function $\Sigma$ is called the Potapov-Ginzburg transform of $S$; see e.g. [10] (i), p. 25.

Proof of Theorem 5.3. To show that $S_{22}$ is \(\star\)-invertible, we note that
\[
\begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} J_2 k(p, q) - S(p) \star k(p, q) \star_r J_1 S(q)^* \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} = S_{22}(p) \star k(p, q) \star_r S_{22}(q)^* - k(p, q)I_m. \]
This last kernel has therefore a finite number of negative squares, and Lemma 5.2 allows to conclude that $S_{22}$ is $\ast$-invertible, and the definition of the Potapov-Ginzburg transform makes sense.

When $p \in \Omega \cap \mathbb{R}_+$, the star product is replaced by the pointwise product and the (5.3) then follow from [9, p. 156]. The case of $p \in \Omega$ follows by slice hyperholomorphic extension. The claim on the number of negative squares of (5.4) follows from an application of [7, Proposition 5.3]. □

**Definition 5.4.** Let $S \in S_\kappa(J_1, J_2)$. We denote by $\mathcal{P}(S)$ the associated reproducing kernel Pontryagin space of $\mathcal{H}_2$-valued functions defined in $\Omega$ and with reproducing kernel $K_S(p, q)$.

6. **Realization for elements in $S_\kappa(J_1, J_2)$**

In this section we present a realization theorem for elements in $S_\kappa(J_1, J_2)$, where the state space is the reproducing kernel Pontryagin space $\mathcal{P}(S)$ (see Definition 5.4 for the latter). In the case $\kappa = 0$ one could get the existence of a realization using a Cayley transform in the variable and use our previous results in [5]. Here we give a direct proof to get a realization defined in $\mathcal{P}(S)$, taking into account that $\kappa$ may be strictly positive. We begin with a definition:

**Definition 6.1.** Let $\mathcal{G}$ and $\mathcal{H}$ be two quaternionic right Pontryagin spaces. A pair of operators $(G, A) \in \mathcal{L}(\mathcal{H}, \mathcal{G}) \times \mathcal{L}(\mathcal{G})$ is called controllable (or closely outer connected) if

$$\bigcap_{n=0}^\infty \ker GA^n = \{0\}.$$ 

The terminology controllable is the one from the theory of linear systems, while closely outer connected has been used in operator theory in particular by Krein and Langer; see [9].

**Theorem 6.2.** Let $x_0$ be a strictly positive real number. A function $S$ slice hyperholomorphic in an axially symmetric $s$-domain $\Omega$ containing $x_0$ is the restriction to $\Omega$ of an element of $S_\kappa(J_1, J_2)$ if and only if it can be written as

$$S(p) = H - (p - x_0) \left( G - (\overline{p} - x_0)(\overline{p} + x_0)^{-1}GA \right) \times$$

$$\times \left( \frac{|p-x_0|^2}{|p+x_0|^2} A^2 - 2\text{Re} \left( \frac{p-x_0}{p+x_0} \right) A + I \right)^{-1} F,$$

where $A$ is a linear bounded operator in a right-sided quaternionic Pontryagin space $\Pi_\kappa$ of index $\kappa$, and, with $B = -(I + x_0A)$, the operator matrix

$$\begin{pmatrix} B & F \\ G & H \end{pmatrix} : \begin{pmatrix} \Pi_\kappa \\ \mathcal{H} \end{pmatrix} \longrightarrow \begin{pmatrix} \Pi_\kappa \\ \mathcal{H} \end{pmatrix}$$

is co-isometric. In particular $S$ has a unique slice hyperholomorphic extension to $\mathbb{H}_+$. Furthermore, when the pair $(G, A)$ is observable, the realization is unique up to a unitary isomorphism of Pontryagin right quaternionic spaces.
Remark 6.3. When the operators are finite matrices we note that formula (6.1) can be rewritten as:

\[ S(p) = H - (p - x_0)G \star ((x_0 + p)I + (p - x_0)B)^{\star}\star F. \]

Sometimes, and by abuse of notation, we will use this expression also for the infinite dimensional case. See Proposition 2.22 for more information.

Proof of Theorem 6.2. We proceed in a number of steps, and first prove in Steps 1-8 that a realization of the asserted type exists with \( \Pi_k = \mathcal{P}(S) \). We denote by \( \mathcal{H}_2(J_2) \) the space \( \mathcal{H}_2 \) endowed with the indefinite inner product

\[ [u, v]_{J_2} = [u, J_2v] \]

and similarly \( \mathcal{H}_1(J_1) \). Both \( \mathcal{H}_1(J_1) \) and \( \mathcal{H}_2(J_2) \) are quaternionic Pontryagin spaces, and they have the same index.

Following [1] pp. 51-52] we introduce a relation \( R \) in \((\mathcal{P}(S) \oplus \mathcal{H}_2(J_2)) \times (\mathcal{P}(S) \oplus \mathcal{H}_1(J_1))\) by the linear span of the vectors

\[
\left( \begin{array}{c}
K_S(\cdot, q)(x_0 - \overline{q})u \\
(x_0 - \overline{q})v
\end{array} \right), \left( \begin{array}{c}
K_S(\cdot, q)(x_0 + \overline{q})u - 2x_0K_S(\cdot, x_0)u + \sqrt{2x_0}K_S(\cdot, x_0)(x_0 - \overline{q})v \\
\sqrt{2x_0}(S(q)^* - S(x_0)^*)u + S(x_0)^*(x_0 - \overline{q})v
\end{array} \right)
\]

(6.2)

STEP 1: The relation \( R \) is isometric.

Indeed, let \((F_1, G_1)\) and \((F_2, G_2)\) be two elements in the relation, corresponding to \( q_1 \in \Omega, u_1, v_1 \in \mathcal{H}_1 \) and to \( q_2 \in \Omega, u_2, v_2 \in \mathcal{H}_2 \) respectively. On the one hand we have

\[ [F_2, F_1] = [(x_0 - q_1)K_S(q_1, q_2)(x_0 - \overline{q}_2)u_2, u_1] + [(x_0 - q_1)J_2(x_0 - \overline{q}_2)v_2, v_1]. \]

On the other hand, with \( G_1 = \left( \begin{array}{c} g_1 \\ h_1 \end{array} \right) \) where

\[
g_1(\cdot) = K_S(\cdot, q_1)(x_0 + \overline{q}_1)u_1 - 2x_0K_S(\cdot, x_0)u_1 + \sqrt{2x_0}K_S(\cdot, x_0)(x_0 - \overline{q}_1)v_1,
\]

\[
h_1 = \sqrt{2x_0}(S(q_1)^* - S(x_0)^*)u_1 + S(x_0)^*(x_0 - \overline{q}_1)v_1
\]

(and similarly for \( G_2 \)) we have

\[ [G_2, G_1] = [g_2, g_1] + [h_2, h_1]. \]

We want to show that

\[ [F_2, F_1] = [g_2, g_1] + [h_2, h_1]. \]

(6.3)

In the computations of these inner products, there are terms which involve only \( u_1, u_2 \), terms which involve only \( v_1, v_2 \) and similarly for \( u_1, v_2 \) and \( v_1, u_2 \). We now write these inner terms separately:

Terms involving \( u_1, u_2 \). To show that these terms are the same on both sides of (6.3) we have to check that

\[
[(x_0 - q_1)K_S(q_1, q_2)(x_0 - \overline{q}_2)u_2, u_1] = [(x_0 + q_1)(K_S(q_1, q_2)(x_0 + \overline{q}_2) - 2x_0K_S(x_0, q_2)(x_0 + \overline{q}_2) - 2x_0(x_0 + q_1)K_S(q_1, x_0) + 4x_0^2K_S(x_0, x_0)) u_2, u_1] + 2x_0[(S(q_1) - S(x_0))J_1(S(q_2)^* - S(x_0))u_2, u_1].
\]
Using

\[ k(x_0, x_0) = \frac{1}{2x_0} \quad \text{and} \quad K_S(x_0, x_0) = \frac{1}{2x_0} (J_2 - S(x_0)J_1S(x_0)^*), \quad (6.4) \]

we see that this is equivalent to prove that

\[ (x_0 - q_1)J_2k(q_1, q_2)(x_0 - \overline{q_2}) - (x_0 - q_1)S(q_1)J_1k(q_1, q_2)S(q_2)^*(x_0 - \overline{q_2}) = \\
= (x_0 + q_1)J_2k(q_1, q_2)(x_0 + \overline{q_2}) - (x_0 + q_1)S(q_1)J_1k(q_1, q_2)S(q_2)^*(x_0 + \overline{q_2}) - \\
- 2x_0(J_2 - S(q_1)J_1S(x_0)^*) - 2x_0(J_2 - S(x_0)J_1S(q_2)^*) + \\
+ 2x_0(J_2 - S(x_0)J_1S(x_0)^*) + \\
+ 2x_0(S(q_1) - S(x_0)J_1S(q_2)^* - S(x_0)^*). \]

But this amounts to check that

\[ q_1k(q_1, q_2) + k(q_1, q_2)\overline{q_2} = 1, \]

which has been seen to hold in Proposition 3.7.

Terms involving \( v_1, v_2 \). To show that these terms are the same on both sides of (6.3) we have to check that

\[ (x_0 - q_1)J_2(x_0 - \overline{q_2}) = [(x_0 - q_1)S(x_0)J_1S(x_0)^*(x_0 - \overline{q_2})v_2, v_1] + \\
+ 2x_0[(x_0 - q_1)K_S(x_0, x_0)(x_0 - \overline{q_2})v_2, v_1]. \]

This follows directly from the formula for \( K_S(x_0, x_0) \). See (6.4).

Terms involving \( u_2, v_1 \). There are no such terms on the left side of (6.3) and so we need to show that the terms on the right add up to 0. This is the case since

\[ \sqrt{2x_0}[(x_0 - q_1)S(x_0)J_1(S(q_2)^* - S(x_0)^*)u_2, v_1] + \\
+ \sqrt{2x_0}[(x_0 - q_1)(K_S(x_0, q_2)(x_0 + \overline{q_2}) - 2x_0K_S(x_0, x_0)) u_2, v_1] = \\
= [Xu_2, v_1] = 0 \]

with

\[ X = \sqrt{2x_0}(x_0 - q_1)S(x_0)J_1(S(q_2)^* - S(x_0)^*) + \sqrt{2x_0}(x_0 - q_1)J_2 - S(x_0)J_1S(q_2)^* - \\
- \sqrt{2x_0}(x_0 - q_1)J_2 - S(x_0)J_1S(x_0)^* \\
= 0 \]

since

\[ K_S(x_0, q_2)(x_0 + \overline{q_2}) = J_2 - S(x_0)J_1S(q_2)^*. \]

Terms involving \( u_1, v_2 \). These form a symmetric expression to the previous one, and will not be written down.

**STEP 2:** The domain of \( R \) is dense.

To prove this step, let \( \begin{pmatrix} f \\ w \end{pmatrix} \in (P(S) \oplus H_2(J_2)) \) be orthogonal to \( \text{Dom} \ R \). Then, for all \( q \in \Omega \) and \( u, v \in H_2 \) we have

\[ [(x_0 - q)f(q), u] + [(x_0 - q)w, v]_{J_2} = 0. \]
It follows that $w = 0$ and that 

$$(x_0 - q)f(q) \equiv 0, \quad q \in \Omega,$$

and so $f \equiv 0$ in $\Omega$.

**STEP 3:** The relation $R$ extends to the graph of an isometry

Indeed, the spaces $\mathcal{P}(S) \oplus \mathcal{H}_2(J_2)$ and $\mathcal{P}(S) \oplus \mathcal{H}_1(J_1)$ are Pontryagin spaces with same index. By the quaternionic version of a theorem of Shumlyan (see [9, Theorem 1.4.1, p. 27] for the classical case and [6, Theorem 7.2] for the quaternionic case) a densely defined contractive relation defined on a pair of Pontryagin spaces with same index extends to the graph of a contraction.

In preparation to the next step we introduce an operator $R_{x_0}$ as follows. Let $\mathcal{H}$ be a two-sided quaternionic Hilbert space. A $\mathcal{H}$-valued function slice hyperholomorphic in a neighborhood of $x_0 > 0$ can be written as a convergent power series

$$f(p) = \sum_{n=0}^{\infty} (p - x_0)^n f_n,$$

where the coefficients $f_n \in \mathcal{H}$. We define

$$(R_{x_0}f)(p) = (p - x_0)^{-1}(f(p) - f(x_0)) = \begin{cases} \sum_{n=1}^{\infty} (p - x_0)^{n-1} f_n, & p \neq x_0, \\ f_1, & p = x_0. \end{cases} \quad (6.5)$$

**STEP 4:** Let $V$ denotes the isometry in the previous step. We compute $V^*$ and show that, with

$$V^* = \begin{pmatrix} B & F \\ G & H \end{pmatrix} : \mathcal{P}(S) \oplus \mathcal{H}_2(J_2) \rightarrow \mathcal{P}(S) \oplus \mathcal{H}_2(J_1), \quad (6.6)$$

we have $H = S(x_0)$ and

$$Bf = -(I + 2x_0R_{x_0})f, \quad (6.7)$$
$$Fu = -\sqrt{2x_0}R_{x_0}Su, \quad (6.8)$$
$$Gf = \sqrt{2x_0}f(x_0). \quad (6.9)$$

To compute (6.7) let $f \in \mathcal{P}(S)$ and $(p, u) \in \Omega \times \mathcal{H}_2$. We have

$$[(x_0 - p)(Bf(p)), u] = [Bf, K_S(\cdot, p)(x_0 - \overline{p})u] = [f, B^*(K_S(\cdot, p)(x_0 - \overline{p}))u] = [f, K_S(\cdot, p)(x_0 + \overline{p})u - 2x_0K_S(\cdot, x_0)u] = [(p + x_0)f(p) - 2x_0f(x_0), u],$$

and so

$$(x_0 - p)(Bf(p)) = (p + x_0)f(p) - 2x_0f(x_0), \quad p \in \Omega,$$

which can be rewritten as (6.7).
Similarly, to compute (6.8) let \( v \in \mathcal{H}_2 \). We have:

\[
[(x_0 - p)((Fv)(p)), u] = [Fv, K_S(\cdot, p)(x_0 - \overline{p})u] \\
= [v, \sqrt{2x_0}(S(p)^* - S(x_0)^*)u] \\
= [\sqrt{2x_0}(S(p) - S(x_0))v, u],
\]

and so

\[
(x_0 - p)(Fv(p)) = \sqrt{2x_0}(S(p) - S(x_0))v, \quad p \in \Omega.
\]

Finally, we have:

\[
[(x_0 - p)Gf, v] = [Gf, (x_0 - \overline{p})v] \\
= [f, G^*(x_0 - \overline{p})v] \\
= [f, \sqrt{2x_0}K_S(\cdot, x_0)(x_0 - \overline{p})v] \\
= \sqrt{2x_0}[(x_0 - p)f(x_0), v].
\]

where we have used (2.1) to get the first equality.

**STEP 5:** We prove (6.1) for real \( p \) near \( x_0 \): The operator \( I + 2x_0R_{x_0} \) is bounded and so is the operator \( R_{x_0} \) (with \( x_0 > 0 \)). Let \( f \in \mathcal{P}(S) \), with power series expansion

\[
f(p) = \sum_{n=0}^{\infty} (p - x_0)^nf_n, \quad f_0, f_1, \ldots \in \mathcal{H}_2,
\]

around \( x_0 \). We have for real \( p = x \) near \( x_0 \):

\[
f(x) = \sum_{n=0}^{\infty} (x - x_0)^nf_n
\]

\[
= \frac{1}{\sqrt{2x_0}} \sum_{n=0}^{\infty} (x - x_0)^nG\Omega^n f
\]

\[
= \frac{1}{\sqrt{2x_0}} G(I - (x - x_0)R_{x_0})^{-1}f.
\]

Applying this formula to \( f = R_{x_0}Su = -\frac{1}{\sqrt{2x_0}} Fu \) where \( u \in \mathcal{H}_1 \) we have

\[
(R_{x_0}Su)(x) = -G(2x_0I - 2(x - x_0)x_0R_{x_0})^{-1}Fu
\]

and so, since \( B = -I - 2x_0R_{x_0} \),

\[
S(x)u = S(x_0)u + (x - x_0)(R_{x_0}Su)(x)
\]

\[
= S(x_0)u - (x - x_0)G(2x_0I - 2(x - x_0)x_0R_{x_0})^{-1}Fu
\]

\[
= S(x_0)u - (x - x_0)G(2x_0I + (x - x_0)(B + I))^{-1}Fu
\]

\[
= S(x_0)u - (x - x_0)G((x + x_0)I + (x - x_0)B)^{-1}Fu.
\]

**STEP 6:** Assume that \( J_1 = I_{\mathcal{H}_1} \) and \( J_2 = I_{\mathcal{H}_2} \). Then, the operator \((x_0 + x)I + (x - x_0)B\) is invertible for all real \( x \), with the possible exception of a finite set in \( \mathbb{R} \).
Assume first the kernel $K_S$ to be positive definite. Then, the operator matrix (6.6) is a contraction between Hilbert spaces and so $B$ is a Hilbert space contraction, and the operator 
\[ I - \frac{x_0 - x}{x_0 + x} B \]
is invertible for all $x > 0$ since $|\frac{x_0 - x}{x_0 + x}| < 1$ for such $x$.

Assume now that $\mathcal{P}(S)$ is a Pontryagin space. The operator $V^*$ is a contraction between Pontryagin spaces of same index, and so its adjoint $V$ is a contraction (see [7, Theorem 7.2]). So it holds that 
\[ B^* B + G^* G \leq I. \]

But 
\[ \langle G^* Gf, f \rangle = \langle Gf, Gf \rangle_{\mathcal{H}_2} \geq 0 \]
since $J^2 = I_{\mathcal{H}_2}$ and so $B$ is a contraction. It admits a maximal strictly negative invariant subspace, say $\mathcal{M}$ (see [26, Theorem 1.3.11] for the complex case and Theorem 4.6 for the quaternionic case). Writing 
\[ \mathcal{P}(S) = \mathcal{M}[+] \mathcal{M}[\perp], \]
the operator matrix representation of $B$ is upper triangular with respect to this decomposition,
\[ B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}. \]
The operator $B_{22}$ is a contraction from the Hilbert space $\mathcal{M}[\perp]$ into itself, and so has $I - \frac{x_0 - x}{x_0 + x} B_{22}$ is invertible for every $x > 0$. The operator $B_{11}$ is a contraction from the finite dimensional anti-Hilbert space $\mathcal{M}$ onto itself, and so has right eigenvalues outside the open unit ball. So the operator $I - \frac{x_0 - x}{x_0 + x} B_{11}$ is invertible in $x > 0$, except the points $x \neq x_0$ such that $\frac{x_0 + x}{x - x_0}$ is a real eigenvalue of $B_{11}$ of modulus greater or equal to 1. There is a finite number of such points since, see [57, Corollary 5.2, p. 39], a $n \times n$ quaternionic matrix has exactly $n$ right eigenvalues up to equivalence (in other words, it has exactly $n$ spheres of eigenvalues).

It follows that the operator 
\[ I - \frac{x_0 - x}{x_0 + x} B = \begin{pmatrix} I - \frac{x_0 - x}{x_0 + x} B_{11} & -\frac{x_0 - x}{x_0 + x} B_{12} \\ 0 & I - \frac{x_0 - x}{x_0 + x} B_{22} \end{pmatrix} \]
is invertible for all $x > 0$, with the possible exception of a finite number of points.

**STEP 7:** Assume that $J_1 = I_{\mathcal{H}_1}$ and $J_2 = I_{\mathcal{H}_2}$. The function $S$ admits a slice hypermeromorphic extension to $\mathbb{H}_+$, with the possible exception of a finite number of spheres.

We note that, for $p \in \mathbb{H}$ near $x_0$ we can extend $S(x)u$ computed in STEP 5 to a slice hyperholomorphic function:
\[
S(p)u = S(x_0)u + \frac{x_0 - p}{x_0 + p} G \star \left( I - \frac{x_0 - p}{x_0 + p} B \right)^{-*} Fu \\
= S(x_0)u + \frac{p - x_0}{p + x_0} \star \left( G - \frac{x_0 - p}{x_0 + p} GB \right) \left( \frac{|x_0 - p|^2}{|x_0 + p|^2} B^2 - 2Re \left( \frac{x_0 - p}{x_0 + p} B + I \right) \right)^{-1} Fu.
\]
Let \( t = \frac{\text{Re} q}{|q|^2} \) where \( q = \frac{x_0 - p}{|x_0 + p|} \). We have

\[
|q|^2 B^2 - 2(\text{Re} q) B + I = |q|^2 \begin{pmatrix} B^2_{11} - 2tB_{11} + \frac{1}{|q|^2} & B_{11}B_{12} + B_{12}B_{22} - 2tB_{12} + \frac{1}{|q|^2} \\ 0 & B^2_{22} - 2tB_{22} + \frac{1}{|q|^2} \end{pmatrix}.
\]

By the property of the resolvent, the operator \( B^2_{22} - 2tB_{22} + \frac{1}{|q|^2} \) is invertible for \( q \) such that \( \frac{1}{|q|^2} \) is in the resolvent of \( B_{22} \). Since \( B_{22} \) is a Hilbert space contraction, this happens in particular when \( |q| < 1 \), see [23], proof of Theorem 4.8.11. Similarly the operator \( B^2_{11} - 2tB_{11} + \frac{1}{|q|^2} \) is invertible if and only if \( \frac{1}{|q|^2} \) is in the resolvent of \( B_{11} \). Since \( B_{11} \) is a finite dimensional Hilbert space expansion, it has just point S-spectrum which is inside the closed unit ball. The point S-spectrum coincides with the set of right eigenvalues, see Remark 2.20 and it consists of a finite number of (possibly degenerate) spheres.

We now consider the case of arbitrary signature matrices, with same negative index.

STEP 8: We use the Potapov-Ginzburg transform to show that \( S \) has a meromorphic extension.

This follows from computing \( S \) from its Potapov-Ginzburg transform.

STEP 9: Any \( S \) with a realization of the form (6.1) is in a class \( \mathcal{S}_n(J_1, J_2) \).

Indeed, for real \( p = x \) and \( q = y \) near \( x_0 \), the existence of the realization leads to

\[
\frac{J_2 - S(x)J_1 S(y)^*}{x + y} = G(I(x_0 + x) - (x + x_0)B)^{-1}(I(y + x_0) - (y - x_0)B)^{-*}G^*,
\]

where \( B = -(I + x_0 A) \). Thus, with \( K(x, y) = G(I(x_0 + x) - (x + x_0)B)^{-1}(I(y + x_0) - (y - x_0)B)^{-*}G^* \),

\[
J_2 - S(x)J_1 S(y)^* = xK(x, y) + K(x, y) y
\]

and the result follows by observing that (6.1) is the hyperholomorphic extension.

STEP 10: The realization is unique up to isomorphism when it is observable.

We follow [9]. Let \( p \) real and set \( x = \frac{p - x_0}{p + x_0} \). When \( p \) varies in a real neighborhood of \( x_0 \) then \( x \) varies in a real neighborhood \( \mathcal{I}_0 \) of the origin. For \( x, y \in \mathcal{I}_0 \) we have

\[
\frac{J_2 - S(x)J_1 S(y)^*}{1 - xy} = G_1(I_{P_1} - xB_1)^{-1}(I_{P_1} - yB_1)^{-*}G_1^* = G_2(I_{P_2} - xB_2)^{-1}(I_{P_2} - yB_2)^{-*}G_2^*,
\]

where the indices 1 and 2 correspond to two observable and coisometric realizations, with state spaces \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) respectively. Then the domain and range of the relation \( R \) spanned by the pairs

\[
((I_{P_1} - yB_1)^{-*}G_1^*h, (I_{P_2} - yB_2)^{-*}G_2^*k), \quad h, k \in \mathcal{H}_2,
\]

are dense. By the quaternionic version of a theorem of Shmulyan (see [6, Theorem 7.2]) \( R \) is the graph of a unitary map, which provides the desired equivalence. The arguments are as in [9]. \( \square \)
Remark 6.4. In the case \( x_0 \to 0 \), we can rewrite the computations with
\[
G_0 f = f(x_0) \\
F_0 u = R_{x_0} S u \\
I + B = -x_0 R_{x_0}, \\
I - B = 2I + x_0 R_{x_0};
\]
we have that
\[
S(p)u = S(x_0)u + (p - x_0)G \ast (M_{x_0+p}^\ell + M_{p-x_0}^\ell B)^{-1} F_0 u \\
= S(x_0) + (p - x_0)2x_0G_0 \ast (x_0(I - B) + p(I + B))^{-1} F_0 u \\
= S(x_0) + (p - x_0)G_0 \ast (I + x_0 R_{x_0} - pR_{x_0})^{-1} F_0 u
\]
which tends formally to the backward shift realization as \( x_0 \to 0 \).

Example 6.5. We now show how to obtain a realization for a Blaschke factor \( b_a(p) \). For real \( p = x \), using formula \((3.10)\) we obtain that \( b_a(x) = (x - \bar{a})^{-1}(x - a) \), moreover
\[
b_a(x) = b_a(1) + b_a(x) - b_a(1) \\
= \frac{1 - a}{1 + \bar{a}} + (x - 1) \frac{2\text{Re}(a)}{(x + \bar{a})(1 + \bar{a})},
\]
where \( B = \frac{1 - \pi}{1 + \bar{a}} \)
\[
= \frac{1 - a}{1 + \bar{a}} + (x - 1) \frac{2\text{Re}(a)(1 + B)}{(x + \bar{a})(1 + \bar{a})},
\]
\[
= \frac{1 - a}{1 + \bar{a}} + (x - 1) \frac{2\text{Re}(a)}{(x(1 + B) + (1 - B))(1 + \bar{a})^2},
\]
\[
= \frac{1 - a}{1 + \bar{a}} + (x - 1) \frac{2\text{Re}(a)(x + 1 + (x - 1)B)^{-1} \frac{2}{1 + \bar{a}}}{1 + \bar{a}}
\]
since
\[
\frac{1 + B}{1 + \bar{a}} = \frac{2}{(1 + \bar{a})^2}.
\]
Now note that
\[
b_a(p) = H - (p - 1)G \ast ((p + 1) + (p - 1)B)^{-*} F
\]
is slice hyperholomorphic, extends \( S(x) \), and
\[
\begin{pmatrix} B & F \\ G & H \end{pmatrix} = \begin{pmatrix}
1 - \bar{a} & 2\sqrt{\text{Re} a} \\
1 + \bar{a} & 1 + \bar{a} \\
2\sqrt{\text{Re} a} & 1 - a \\
1 + \bar{a} & 1 + \bar{a}
\end{pmatrix}.
\]
This matrix is unitary.
7. Characteristic operator function

We now present an example of functions in a class $S_0(J, J)$. Consider a linear bounded operator $A$ in a right quaternionic Hilbert space $H$, and assume that $A + A^*$ is finite dimensional, say of rank $m$. We can thus write:

$$A + A^* = -CJC^*,$$

where $J \in \mathbb{H}^{m \times m}$ is a real signature matrix, and where $C$ is linear bounded operator from $H$ into $H$. We will assume $(C, A)$ observable.

**Lemma 7.1.** The pair $(C, A)$ is observable if and only if there is no non trivial invariant subspace of $A$ on which $A + A^* = 0$.

The proof of this lemma is as in the complex case, and will be omitted.

**Definition 7.2.** The function

$$S(p) = I - pC^* \ast (I - pA)^{-*} C J$$

is called the characteristic operator function of the operator $A$.

**Theorem 7.3.** The characteristic operator function belongs to $S_0(J, J)$.

**Proof.** Let

$$K(p, q) = C^* \ast (I - pA)^{-*} (I - \overline{q}A^*)^{-*} \ast_r C.$$

Then it holds that

$$J - S(p)JS(q)^* = pK(p, q) + K(p, q)\overline{q}.$$

This formula is proved by first considering the case of real $p$ and $q$, and taking the slice hyperholomorphic extension, and proves that $S \in S_0(J, J)$.

We note that formula (7.1) corresponds to a realization centered at 0, as in Remark 6.4, and not to a realization of the form (6.1). It is interesting to find a functional model for the operator $A$ in terms of $S$. The special case where $S$ is a (possibly infinite convergent) Blaschke product is of special interest. The case of general $S$ leads to the question of finding the $\ast$-multiplicative structure of elements in $S_0(J, J)$, that is the counterpart of the paper [50] in the present setting.

8. The space $\mathcal{L}(\Phi)$ and realizations for generalized positive functions

8.1. The indefinite case

**Definition 8.1.** Let $\mathcal{H}$ be a quaternionic Hilbert space, and let $J \in \mathcal{L}(\mathcal{H})$ be a real signature. A $\mathcal{L}(\mathcal{H})$-valued function $\Phi$ slice hyperholomorphic in an axially symmetric $s$-domain $\Omega$ which intersects the positive real line belongs to the class $\text{GP}_\kappa(J)$ if the kernel

$$K_\Phi(p, q) = J\Phi(p) \ast k(p, q) + k(p, q) \ast_\kappa \Phi(q) J$$

has $\kappa$ negative squares in $\Omega$.

**Lemma 8.2.** The kernel $K_\Phi$ satisfies

$$pK_\Phi(p, q) + K_\Phi(p, q)\overline{q} = J\Phi(p) + \Phi(q) J.$$

**Proof.** It follows with immediate computations from Proposition 7.7.
As in the case of generalized Schur functions, we do not mention the dependence of the class on $\Omega$ since, as we prove later, every element of a class $\text{GP}_\kappa(J)$ has a unique slice hypermeromorphic extension to $\mathbb{H}_+$. 

We note that $J$ does not play a role, as noted in [38, p. 358, footnote], and could be set to be the identity. We denote by $\mathcal{H}$ a two sided quaternionic Hilbert space, and set $L(\mathcal{H}) \overset{\text{def}}{=} L(\mathcal{H}, \mathcal{H})$.

**Theorem 8.3.** A $L(\mathcal{H})$-valued function $\Phi$ slice hyperholomorphic in an axially symmetric s-domain $\Omega$ containing $x_0 > 0$ is in the class $\text{GP}_\kappa(J)$ if and only if there exists a right quaternionic Pontryagin space $\Pi_\kappa$ of index $\kappa$ and operators

$$
\begin{pmatrix}
B & F \\
G & H
\end{pmatrix}: \Pi_\kappa \to \Pi_\kappa
$$

verifying

$$(I + 2x_0B)(I + 2x_0B)^* = I$$

and such that $\Phi$ can be written as

$$\Phi(p) = H - (p - x_0)G \ast ((p + x_0)I + (p - x_0)B)^{-1}F. \quad (8.3)$$

Furthermore, $\Phi$ has a unique slice hypermeromorphic extension to $\mathbb{H}_+$. Finally, when the pair $(G, B)$ is observable, the realization is unique up to a unitary isomorphism of Pontryagin right quaternionic spaces.

**Proof.** Given $\Phi \in \text{GP}_\kappa(J)$, we denote by $\mathcal{L}(\Phi)$ associated right reproducing kernel Pontryagin space of $\mathcal{H}$-valued functions with reproducing kernel $K_\Phi$. We proceed in a number of steps to prove the theorem.

**STEP 1:** The formula

$$(p - x_0)(Bh(p)) = (p + x_0)h(p) - 2x_0h(x_0), \quad h \in \mathcal{L}(\Phi). \quad (8.4)$$

defines a (continuous) coisometry in $\mathcal{L}(\Phi)$.

Indeed, define a relation $\mathcal{R}_{x_0}$ on $\mathcal{L}(\Phi) \times \mathcal{L}(\Phi)$ generated by the linear span of the pairs

$$\mathcal{R}_{x_0} = (K_\Phi(\cdot, p)(\overline{p} - x_0)u, (K_\Phi(\cdot, p) - K_\Phi(\cdot, x_0))u). \quad (8.5)$$

Then the following holds:

$$\langle f, g \rangle \in \mathcal{R}_{x_0} \implies [f, f] = [f + 2x_0g, f + 2x_0g] \quad (8.6)$$

We first prove that

$$[f, g] + [g, f] + 2x_0[g, g] = 0. \quad (8.7)$$

An element in $\mathcal{R}_{x_0}$ can be written as $(f, g)$ with

$$f(p) = \sum_{j=1}^{m} K_\Phi(p, p_j)(\overline{p_j} - x_0)u_j$$

$$g(p) = \sum_{j=1}^{m} K_\Phi(p, p_j)u_j - K_\Phi(p, x_0)d, \quad \text{where} \quad d = \sum_{j=1}^{m} u_j. \quad (8.8)$$
With \( f \) and \( g \) as in (8.8) we have:

\[
[f, g] = \left( \sum_{i,j=1}^{m} u_i^* K_{\Phi}(p_i, p_j) (\overline{p_j} - x_0) u_j \right) - d^* \left( \sum_{j=1}^{m} K_{\Phi}(x_0, p_j) (\overline{p_j} - x_0) u_j \right),
\]

\[
[g, f] = \left( \sum_{i,j=1}^{m} u_i^* (p_i - x_0) K_{\Phi}(p_i, p_j) u_j \right) - \left( \sum_{i=1}^{m} u_i^* (p_i - x_0) K_{\Phi}(p_i, x_0) \right) d.
\]

Thus

\[
[f, g] + [g, f] = -2x_0 \left( \sum_{i,j=1}^{m} u_i^* K_{\Phi}(p_i, p_j) u_j \right) +
\]

\[
+ \sum_{i,j=1}^{m} u_i^* \left\{ p_i K_{\Phi}(p_i, p_j) + K_{\Phi}(p_i, p_j) \overline{p_j} \right\} u_j - d^* \left( \sum_{j=1}^{m} K_{\Phi}(x_0, p_j) \overline{p_j} u_j \right) +
\]

\[
x_0 d^* \left( \sum_{j=1}^{m} K_{\Phi}(x_0, p_j) u_j \right) - \left( \sum_{i=1}^{m} u_i^* p_i K_{\Phi}(p_i, x_0) \right) d +
\]

\[
x_0 \left( \sum_{j=1}^{m} u_j^* K_{\Phi}(p_j, x_0) \right) d.
\]

Taking into account (8.2) we have

\[
[f, g] + [g, f] = -2x_0 \left( \sum_{i,j=1}^{m} u_i^* K_{\Phi}(p_i, p_j) u_j \right) +
\]

\[
+ \left( \sum_{i=1}^{m} u_i^* J\Phi(p_i) \right) d + d^* \left( \sum_{j=1}^{m} \Phi(p_j)^* J u_j \right) -
\]

\[
d^* \left( \sum_{j=1}^{m} K_{\Phi}(x_0, p_j) \overline{p_j} u_j \right) + x_0 d^* \left( \sum_{j=1}^{m} K_{\Phi}(x_0, p_j) u_j \right) -
\]

\[
- \left( \sum_{i=1}^{m} u_i^* p_i K_{\Phi}(p_i, x_0) \right) d + x_0 \left( \sum_{i=1}^{m} u_i^* K_{\Phi}(p_i, x_0) \right) d.
\]

We now turn to \([g, g]\). We have:

\[
[g, g] = \left( \sum_{i,j=1}^{m} u_i^* (K_{\Phi}(p_i, p_j) u_j \right) - d^* \left( \sum_{j=1}^{m} K_{\Phi}(x_0, p_j) u_j \right) -
\]

\[
- \left( \sum_{i=1}^{m} u_i^* K_{\Phi}(p_i, x_0) \right) d + d^* K_{\Phi}(x_0, x_0) d.
\]
Thus
\[
[f, g] + [g, f] + 2x_0[g, g] = \left( \sum_{i=1}^{m} u_i^* J\Phi(p_i) \right) d + d^* \left( \sum_{j=1}^{m} \Phi(p_j)^* J u_j \right) - \\
- d^* \left( \sum_{j=1}^{m} K\Phi(x_0, p_j) u_j \right) - x_0 d^* \left( \sum_{j=1}^{m} K\Phi(x_0, p_j) u_j \right) - \\
- \left( \sum_{i=1}^{m} u_i^* K\Phi(p_i, x_0) \right) d - x_0 \left( \sum_{i=1}^{m} u_i^* K\Phi(p_i, x_0) \right) d + \\
+ 2x_0 d^* K\Phi(x_0, x_0) d
\]

using
\[
K\Phi(x_0, x_0) = \frac{1}{2x_0} (J\Phi(x_0) + \Phi(x_0)^* J),
\]
we obtain
\[
[f, g] + [g, f] + 2x_0[g, g] = \left( \sum_{i=1}^{m} u_i^* J\Phi(p_i) \right) d + d^* \left( \sum_{j=1}^{m} \Phi(p_j)^* J u_j \right) - \\
- d^* \left( \sum_{j=1}^{m} (J\Phi(x_0) + \Phi(p_j)^* J) u_j \right) - \\
- \left( \sum_{j=1}^{m} u_i^* (J\Phi(p_i) + \Phi(x_0)^* J) \right) d + 2d^* x_0 K\Phi(x_0, x_0) d
\]

and so we have proved (8.7). Equation (8.6) follows since
\[
[f + 2x_0 g, f + 2x_0 g] = [f, f] + 2x_0 ([f, g] + [g, f] + 2x_0 [g, g]).
\]

Equation (8.7) expresses that the linear space of functions \((f, f + 2x_0 g)\) with \(f, g\) as in (8.8) define an isometric relation \(\mathcal{R}\) from the Pontryagin space \(\mathcal{L}(\Phi)\) into itself. Let now \(h \in \mathcal{L}(\Phi)\) be such that
\[
[h, K\Phi(\cdot, p) (\overline{p} - x_0) u] = 0 \quad \forall p \in \Omega \quad \text{and} \quad u \in \mathcal{H}.
\]
Then
\[
(p - x_0) h(p) = 0, \quad \forall p \in \Omega
\]
and \(h \equiv 0\) in \(\Omega\) (recall that the elements of \(\mathcal{L}(\Phi)\) are slice hyperholomorphic in \(\Omega\)). Thus the domain of this relation is dense. By the quaternionic version of Shmulyan’s
theorem (see \cite[Theorem 7.2]{[6]}), $\mathcal{R}$ extends to the graph of a (continuous) isometry, say $B^*$, on $\mathcal{L}(\Phi)$. We have for $h \in \mathcal{L}(\Phi)$
\[
u^*(p - x_0)(Bh(p)) = [Bh, K_\Phi(\cdot, p)(\overline{p} - x_0)u] = [h, B^*(K_\Phi(\cdot, p)(\overline{p} - x_0)u)] = [h, K_\Phi(\cdot, p)(\overline{p} - x_0)u + 2(K_\Phi(\cdot, p) - K_\Phi(\cdot, p))u] = \nu^*(p - x_0)h(p) + 2h(x_0) - 2h(x_0)) = \nu^*(p + x_0)h(p) - 2h(x_0)) \]

We note that $\mathcal{R}_{x_0}$ extends to the graph of $R_{x_0}^*$.

**STEP 2:** The function $p \mapsto R_{x_0} \Phi \eta$ belongs to $\mathcal{L}(\Phi)$ for every $\eta \in \mathcal{H}$ and the operator $F$ from $\mathcal{H}$ into $\mathcal{L}(\Phi)$ defined by
\[F \eta = R_{x_0} \Phi \eta\]
is bounded.

We note that $B = I + 2x_0R_{x_0}$ and so $R_{x_0}$ is a bounded operator in $\mathcal{L}(\Phi)$. From \eqref{[8.2]} we have for $\xi \in \mathcal{H}$
\[J\Phi(p)\xi + \Phi(x_0)^*J\xi = pK_\Phi(p, x_0)\xi + K_\Phi(p, x_0)\xi x_0. \quad (8.9)\]
Apply $R_{x_0}$ on both sides (as an operator on slice hyperholomorphic functions; the two sides of \eqref{[8.9]} will no belong to $\mathcal{L}(\Phi)$ in general). Note that
\[R_{x_0}(pf(p)) = f(p) + x_0(R_{x_0}f)(p), \]
and so we obtain
\[R_{x_0} \Phi J\xi = K_\Phi(p, x_0)\xi + x_0(R_{x_0}K_\Phi(\cdot, x_0)\xi)(p) + (R_{x_0}K_\Phi(p, x_0)\xi x_0)(p), \]
and this expresses $R_{x_0} \Phi J\xi$ as an element of $\mathcal{L}(\Phi)$ since $R_{x_0}$ is bounded in $\mathcal{L}(\Phi)$ and so the elements on the right side of the above equality belong to $\mathcal{L}(\Phi)$. This ends the proof of the first claim since $J$ is invertible. Finally, to see that the operator $F$ is bounded we remark that it is closed and everywhere defined.

We remark that the argument is different from the one for the corresponding operator $F$ (defined by \eqref{[6.8]}) in the spaces $\mathcal{P}(S)$. In the classical case, the argument we are aware of, uses a Cayley transform to go back to the case of generalized Schur functions. The argument we presented here is probably known in the classical case, but we are not aware of any reference for it.

**STEP 3:** The realization formula \eqref{[8.3]} holds.

The proof is the same as the one in STEP 6 for $S$.

**STEP 4:** The function $\Phi$ admits a slice hypermeromorphic extension to $\mathbb{H}_+.$

Recall that $T = I + 2x_0B$ is co-isometric. Using Theorem\eqref{[4.6]} we can thus write $T$ as
\[T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \]
where $T_{11}$ is a bijective contraction from a anti-Hilbert space onto itself, and $T_{22}$ is a contraction from a Hilbert space into itself. Thus for $x > 0$ in a neighborhood of $x_0$,

$$(1 + x)I + (x - 1)B = (1 + x)I + (x - 1)\left(\frac{T - I}{2}\right)$$

$$= (3 + x) + (x - 1)T$$

$$= (3 + x) \left( I + \left( \begin{array}{cc} \frac{x-1}{3+x}T_{11} & \frac{x-1}{3+x}T_{12} \\ \frac{1}{3+x}T_{22} & 0 \end{array} \right) \right)$$

and hence the result by slice hyperholomorphic extension since $\frac{q-1}{3+q}$ sends $\mathbb{H}_+$ into $\mathbb{B}$.

**STEP 5:** A function $\Phi$ admitting a realization of the form (8.3) is in a class $\text{GP}_\kappa(J)$. The proof is as in the case of the functions $S$ and is based on the identity

$$J\Phi(x) + \Phi(y)^*J = (x + y)G(I(x_0 + x) - (x_0 - x)B)^{-1}(I(x_0 + y) - (x_0 - y)B)^{-*},$$

where $x, y$ are real and in a neighborhood of $x_0$.

**STEP 6:** An observable realization of the form (8.3) is unique up to a isomorphism of quaternionic Pontryagin spaces.

We note that the relation (8.5) is inspired from [15, p. 708] and more generally, by the constructions of the "$\epsilon$-method" developed in the papers of Krein and Langer; see for instance [38, 39] for the latter.

**Corollary 8.4.** In $\mathcal{L}(\Phi)$ it holds that

$$R_{x_0} + R_{x_0}^* = -2x_0R_{x_0}^* R_{x_0}. \quad (8.10)$$

**Proof.** This is a rewriting of (8.9). \hfill $\square$

We note that (8.10) is a special case of the structural identity characterizing $\mathcal{L}(\Phi)$ spaces in the complex case; see [20]. To ease the notation we consider the case $J = I$.

**Theorem 8.5.** Let $\mathcal{L}(\mathcal{H})$-valued function $\Phi$ slice hyperholomorphic in an axially symmetric s-domain $\Omega$ containing $p = 0$, such that the associated space does not contain non zero constants, and has its elements slice-hyperholomorphic in a neighborhood of the origin. Assume that $\Phi \in \text{GP}_\kappa(I)$. Then there exists a right quaternionic Hilbert space $\mathcal{H}_1$ and operators

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mathcal{H}_1 \oplus \mathcal{H} \rightarrow \mathcal{H}_1 \oplus \mathcal{H}$$

such that

$$\Phi(p) = D + pC \ast (I_{\mathcal{H}_1} - pA)^{-*}B \quad (8.11)$$

and

$$\text{Re} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I_{\mathcal{H}_1} & 0 \\ 0 & -I_{\mathcal{H}} \end{pmatrix} = 0$$
Proof. We first define a linear relation \( R_\Phi \) in \((\mathcal{L}(\Phi) \oplus \mathcal{H}) \times (\mathcal{L}(\Phi) \oplus \mathcal{H})\) via the formulas

\[
\left( \begin{array}{c} -K_\Phi(\cdot, q)u \\ u \end{array} \right), \left( \begin{array}{c} K_\Phi(\cdot, q)u \\ \Phi(q)^*u \end{array} \right),
\]

\( (8.12) \)

STEP 1: The relation \( R_\Phi \) satisfies

\[
\text{Re} \left( \begin{array}{c} f \\ -g \end{array} \right), \left( \begin{array}{c} F \\ G \end{array} \right) = 0.
\]

\( (8.13) \)

Furthermore, it has dense domain since the space \( \mathcal{L}(\Phi) \) contains no non zero constant functions.

Let

\[
\left( \begin{array}{c} f \\ -g \end{array} \right) = -\sum_{n=1}^t \left( \begin{array}{c} K_\Phi(\cdot, q_n)u_n \\ u_n \end{array} \right),
\]

and

\[
\left( \begin{array}{c} F \\ G \end{array} \right) = \sum_{n=1}^t \left( \begin{array}{c} K_\Phi(\cdot, q_n)^*u_n \\ \Phi(q_n)^*u_n \end{array} \right).
\]

Then

\[
\left\langle \left( \begin{array}{c} f \\ -g \end{array} \right), \left( \begin{array}{c} F \\ G \end{array} \right) \right\rangle = -\sum_{n,m=1}^t u_n^*K_\Phi(q_m, q_n)u_m + u_m^*\Phi(q_n)u_m
\]

so that, using \( (8.2) \), we obtain

\[
\text{Re} \left( \begin{array}{c} f \\ -g \end{array} \right), \left( \begin{array}{c} F \\ G \end{array} \right) = 0.
\]

STEP 2: The relation \( R_\Phi \) is the graph of a densely defined operator which has a continuous extension, and its adjoint is the backward-shift realization

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]

where

\[
pAf(p) = f(p) - f(0),
pBu(p) = (\Phi(p) - \Phi(0))u,
Cf = f(0),
Du = \Phi(0)u.
\]

We only have to consider the operator \( B \). Consider a family \( T \) of pairs \((q, u) \in \Omega \times \mathcal{H}\) such that the functions \( K_\Phi(\cdot, q)u \) are linearly independent and span the space of all the functions \( K_\Phi(\cdot, p)v \), where \( p \) runs through all of \( \Omega \) and \( v \) runs through all of \( \mathcal{H} \). Define a densely defined operator from \( \mathcal{L}(\Phi) \) into \( \mathcal{H} \) by

\[
X(K_\Phi(\cdot, p)u) = (\Phi(p)^* - \Phi(0)^*)u, \quad (p, u) \in A.
\]

We claim that \( X \) has an adjoint which is the operator \( B \) above. To see that, we remark that \( (8.13) \) can be rewritten as

\[
\left\langle \left( \begin{array}{c} f \\ -g \end{array} \right), \left( \begin{array}{c} A^* \\ X \end{array} \right) \left( \begin{array}{c} f \\ g \end{array} \right) \right\rangle + \left( A^* \\ X \end{array} \right) \left( \begin{array}{c} f \\ g \end{array} \right) = 0.
\]

\( (8.14) \)
Using the quaternionic polarization formula it follows that for any \( (f_1, g_1) \) and \( (f_2, g_2) \) in the domain of \( R_\Phi \) we have
\[
\langle \left( \begin{array}{c} f_1 \\ -g_1 \end{array} \right), \left( \begin{array}{ccc} A^* & B^* & X D^* \\ X & D^* & X^* \\ 1 & 0 & 1 \end{array} \right) \left( \begin{array}{c} f_2 \\ g_2 \end{array} \right) \rangle + \langle \left( \begin{array}{c} A^* & B^* & X D^* \\ X & D^* & X^* \\ 1 & 0 & 1 \end{array} \right) \left( \begin{array}{c} f_1 \\ -g_1 \end{array} \right), \left( \begin{array}{c} f_2 \\ -g_2 \end{array} \right) \rangle = 0.
\]
(8.15)
and so \( R_\Phi \) has an adjoint and so does \( X \).

\[\square\]

**Example 8.6.** As an illustration of the previous theorem consider the function
\[
\varphi(p) = (p + a)^{-*},
\]
where \( a \in \mathbb{H} \) is such that \( \Re a = 0 \). Set
\[
M(p, q) = (p + a)^{-*}(q + a)^{-*}.
\]

Since \( a + \bar{a} = 0 \) we have
\[
pM(p, q) + M(p, q)\bar{q} = \varphi(p) + \overline{\varphi(q)},
\]
and so \( \varphi \) is a positive function. For \( p = x > 0 \) we have
\[
\varphi(x) = a^{-1} - \frac{x}{(1 + xa^{-1})a^2},
\]
which leads to the realization (8.11) with
\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
-a^{-1} & a^{-1} \\
-a^{-1} & a^{-1}
\end{pmatrix}
\]
So
\[
\Re \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} = -(a^{-1} + a^{-1} + a^{-1}) \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\]
since \( a + \bar{a} = 0 \).

8.2. **The positive case.** In this section we prove one theorem in the case \( \kappa = 0 \) and \( J = I \). We say that the function \( \Phi \) is positive rather that writing \( \Phi \in \text{GP}_0(I) \).
The proof uses the existence of a squareroot of a positive operator in a quaternionic Pontryagin space. In the indefinite case, such a result still exists in the complex case (this is called the Bognar-Kramli theorem; see [19, Theorem 2.1 p. 149], [26, Theorem 1.1.2]). A quaternionic version of this factorization theorem is not available at present.

**Theorem 8.7.** Let \( \Phi \) be slice-hyperholomorphic in an axially symmetric s-domain of the origin with realization (8.11) such that
\[
\Re \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \begin{pmatrix}
I_{\mathcal{H}_1} & 0 \\
0 & -I_{\mathcal{H}}
\end{pmatrix} \leq 0
\]
Then \( \Phi \) is positive.
Proof. We first note that a positive operator $T$, in a quaternionic Hilbert space has a square root, that is, there exists a positive operator $X$ such that $X^2 = T$. The proof uses the spectral theorem, which holds for Hermitian operators in quaternionic Hilbert spaces. The theorem is mentioned without proof in a number of papers (see for instance [31, 54, 55]). The spectrum used in these works is not the $S$-spectrum, see [23, p. 141]); a proof is given in the preprint [8]. Another way to prove the existence of a square root is to define (assuming first $\|T\| \leq 1$), as in the complex case, a sequence of operators $X_0, X_1, \ldots$ by

$$X_{n+1} = \frac{1}{2}((I - T) + X_n^2), \quad n = 0, 1, \ldots,$$

(see for instance [36, p. 64]) and check that:

1. A weakly convergent increasing sequence of positive operators converges strongly.
2. An increasing family $(X_n)_{n \in \mathbb{N}}$ of bounded positive operators such that
   $$\lim_{n \to \infty} \langle X_n f, f \rangle < \infty, \quad \forall f \in \mathcal{H}$$

converges strongly to a positive operator. Since the arguments do not differ from the complex case we omit them.

Let $X$ be the square root of $-\text{Re} \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \left( \begin{array}{cc} I_{\mathcal{H}_1} & 0 \\ 0 & -I_{\mathcal{H}} \end{array} \right)$. We write

$$X = \left( \begin{array}{c} L \\ K \end{array} \right)$$

where $L$ is a linear operator from $\mathcal{H} \times \mathcal{H}_1$ into $\mathcal{H}$ and $K$ is a a linear operator from $\mathcal{H} \times \mathcal{H}_1$ into $\mathcal{H}_1$.

Let now

$$\text{Re} \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \left( \begin{array}{cc} I_{\mathcal{H}_1} & 0 \\ 0 & -I_{\mathcal{H}} \end{array} \right) = - \left( \begin{array}{c} L \\ K \end{array} \right) \left( \begin{array}{c} L \\ K \end{array} \right)^*,$$

Then

$$\Phi(x) + \Phi(y)^* = D + D^* + xC(I - xA)^{-1}B + yB^*(I - yA)^{-*}C^*$$

$$= KK^* + xC(I - xA)^{-1}(C^* - LK^*) +$$

$$+ y(C - KL^*)(I - yA)^{-*}C^*$$

$$= (K - xC(I - xA)^{-1}L)(K - yC(I - yA)^{-1}L)^* +$$

$$+ xC(I - xA)^{-1}C^* + yC(I - yA)^{-*}C^* -$$

$$- xyC(I - xA)^{-1}LL^*(I - yA)^{-*}C^*.$$

But, using $A + A^* + LL^* = 0$, we have

$$xC(I - xA)^{-1}C^* + yC(I - yA)^{-*}C^* - xyC(I - xA)^{-1}LL^*(I - yA)^{-*}C^* =$$

$$(x + y)C(I - xA)^{-1}(I - yA)^{-*}C^*.$$

The claim follows by slice-hyperholomorphic extension. \hfill \square

We note that the computations are classical; see for instance [30], [29, Theorem 3.3, p. 26].
9. Operator models and positive functions

Elements of \( \text{GP}_0(I) \) (that is, positive functions) play an important role in models for pairs of self-adjoint operators. This originates with the paper of de Branges and Rovnyak [21]. We refer to [12, 13, 37] for examples and applications of the model of de Branges and Rovnyak. In this section, we briefly outline how a positive function also appears in the present setting. We follow the approach of [13], and consider bounded operators for the sake of illustration. The proof of the following lemma is as in [13, p. 18] and is omitted.

**Lemma 9.1.** Let \( T_+ \) and \( T_- \) be two anti-self-adjoint operators in the quaternionic space \( \mathcal{H} \). Then:

1. The space
   \[
   \bigcap_{n=1}^{\infty} \ker(T_+^n - T_-^n) \tag{9.1}
   \]
   is the largest subspace, invariant under \( T_+ \) and \( T_- \) and on which they coincide.

2. Assume that \( \text{rank } T_+ - T_- = n < \infty \). Then there exists a \( n \times n \) matrix \( J \in \mathbb{H}^{n \times n} \) such that \( J^2 = -I_n \) and \( J^* = -J \), and a linear bounded operator \( C \) from \( \mathcal{H} \) into \( \mathbb{H}^n \) such that
   \[
   T_+ - T_- = -C^*JC.
   \]

**Theorem 9.2.** The function
   \[
   \Phi(p) = J + C \ast (pI - T_+)^{-*}C^*
   \]
   is positive and its inverse is equal to
   \[
   \Phi^{-*}(p) = -J - JC \ast (pT - T_-)^{-*}C^*J
   \]

*Proof.* For \( x, y \) on the positive real axis we have (recall that \( T_+^* = -T_+ \))
\[
\Phi(x) + \Phi(y) = J + J^* + C(xI - T_+)^{-1}C^* + C(yI - T_+)^{-*}C^*
\]
\[
= C(xI - T_+)^{-1}C^* + C(yI + T_+)^{-1}C^*
\]
\[
= xK(x, y) + K(x, y)y,
\]
where \( K(x, y) = C(xI - T_+)^{-1}(yI - T_+)^{-*}C^* \). The result follows then by slice hyperholomorphic extension.

Still for positive \( x \) and using for instance formula (2.7) we have
\[
\Phi(x)^{-1} = J^{-1} - J^{-1}C(xI - (T_+ - C^*J^{-1}C^*)^{-1}C^*)J^{-1}
\]
\[
= -J - JC(xI - T_-)^{-1}C^*J.
\]

The formula for \( \Phi^{-1} \) follows then by slice hyperholomorphic extension. \( \square \)

Models for pairs of (possibly unbounded) anti-selfadjoint operators in a quaternionic Hilbert space in terms of the reproducing kernel Hilbert spaces \( \mathcal{L}(\Phi) \) and \( \mathcal{L}(\Phi^{-1}) \), and related trace formulas similar to the ones presented in the papers [21, 13] will be considered elsewhere.
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