PLETHORA OF $q$-OSCILLATORS
POSSESSING PAIRWISE ENERGY LEVEL DEGENERACY

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Abstract

Using the $q,p$-deformed oscillators as basic generating system, we obtain diverse classes
(which form distinct sectors of functional continua) of novel versions of $q$-deformed oscillators,
all of which share the property of “accidental” degeneracy within a fixed pair of energy levels
$E_m = E_{m+1}$, $m \geq 1$, occurring at the real deformation parameter fixed by an appropriate
value $q(m)$ that depends on $m$ and on particular model. Likewise, the degeneracy $E_0 = E_k$
(where $k \geq 2$) takes place, for properly fixed $q = q(k)$, in most of those models. The
formerly studied model of $q$-oscillator known as the Tamm-Dancoff cutoff deformed oscillator
is contained in the continua as isolated special case.

Keywords: $q,p$-deformed oscillators; (classes of) nonstandard $q$-oscillators; pairwise energy
levels degeneracy; two-particle correlation intercept.

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1 Introduction

For more then 15 years, diverse deformed models of quantum oscillator play the important role in
the study of modern quantum mechanical systems, see e.g., Refs. [1, 2] and references therein. Di-
verse ($q$, $q,p$, etc.) deformed oscillators, due to modified defining relations, can acquire nontrivial
and unusual properties essentially different from those of the standard quantum oscillator. In Ref.
3, general function-dependent deformations named the $f$-oscillators have been studied including
nonlinear coherent states and other physical aspects. Also, a unified framework for deformed
single-mode oscillators has been presented in Ref. 4. Recently, it has been demonstrated[5] that
the two-parameter deformed $q,p$-oscillators introduced in Ref. 6 exhibit, at appropriate values of
the deformation parameters $q$ and $p$, the unusual property of so-called “accidental” double (pair-
wise) degeneracy, within a fixed pair of energy levels. Such a pair may be of the type $E_m = E_{m+1}$,
of the type $E_0 = E_k, \ k \geq 2$, or even of more general type $E_{m_1} = E_{m_2}$, for appropriate (com-
pletely determined) set of pairs of real values $(q,p)$ such that both $q$ and $p$ depend on $m_1,m_2$.
Note that this result extends to the two-parameter case the completely analogous property (re-
vealed earlier[7] and valid for definite real values of the $q$-parameter) which is characteristic for the
rather special $q$-deformed oscillator, the so-called ‘Tamm-Dancoff (TD) cutoff’ oscillator, which
first appeared in Refs. 8, 9.

It is important to emphasize that the just mentioned degeneracy property, peculiar for the
TD-oscillator, cannot occur in principle, with real $q$, for the energy levels of the two most popular

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versions of $q$-deformed oscillator, namely the Biedenharn-Macfarlane [10, 11] (BM) $q$-oscillator and Arik-Coon [12] (AC) version of $q$-deformed oscillator.

Our goal in this paper is to demonstrate that, among one-parameter deformed $q$-oscillator models, the TD $q$-oscillator is not the unique and exotic one which possesses at real values of deformation parameter $q$ the above mentioned property of 'accidental' double (or pairwise) degeneracy in a fixed pair of energy levels. Just the contrary: we show there exist infinitely many models of $q$-deformed oscillators (constituting in fact functionally different continual sets or classes) all of which share the degeneracy properties analogous to those mentioned above. The TD $q$-oscillator is but a simplest individual case lying in one of such classes.

Few words about the plan of the paper. In Sec. 2 we recapitulate very shortly the basic setup concerning the $q, p$-deformed oscillators, as well as the main facts about two-fold (pairwise) degeneracy among their energy levels. In the next, 3rd section we construct many classes of one-parameter deformed $q$-oscillators which necessarily possess pairwise energy level degeneracy. In Sec. 4 we comment on a possible physical applications of the new $q$-oscillators. Concluding remarks are contained in the last section.

2 On the $q, p$-oscillators and their pairwise 'accidental' degeneracies

Our basic system is the two-parameter deformed $q, p$-oscillator algebra, whose generating elements $A, A^\dagger$ and $N$ obey

$$AA^\dagger - q A^\dagger A = pN, \quad AA^\dagger - p A^\dagger A = qN,$$

plus two more relations, involving $A$ (or $A^\dagger$) and $N$, similar to non-deformed case.

The pair of relations in (1), obviously symmetric under $q \leftrightarrow p$, is satisfied with

$$A^\dagger A = [N]_{q,p}, \quad AA^\dagger = [N + 1]_{q,p},$$

where the $q, p$-bracket means

$$[X]_{q,p} = \frac{q^X - p^X}{q - p}$$

for $X$ either an operator or a real number. Note that for a non-negative integer $k$ the $q,p$-bracket in (3) reduces to the (symmetric, homogeneous) $q, p$-polynomial:

$$[k]_{q,p} = \frac{q^k - p^k}{q - p} = \sum_{r=0}^{k-1} q^{k-1-r} p^r = q^{k-1} \sum_{r=0}^{k-1} q^{-r} p^r.$$

At $p = 1$ this two-parameter system reduces to the AC-type [12] $q$-oscillator, and putting $p = q^{-1}$ yields the other distinguished case [10, 11] well known as the $q$-oscillator of Biedenharn and Macfarlane (or BM). Finally, at $p = q$ the relations (1)-(4) turn into those of the TD oscillator whose unusual properties were studied in Ref. [7].

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2The situation is however different, and some degeneracies can occur at phase-like values of deformation parameter for the BM $q$-oscillator when $q$ is taken as a root of unity, see e.g., Refs. [13, 14] and Ref. [5].
For the Hamiltonian taken in the form
\[ H = \frac{1}{2}(AA^\dagger + A^\dagger A) \] (5)
the energy spectrum \( H|n\rangle = E_n|n\rangle \) in the \( q,p\)-deformed Fock space basis reads:
\[ E_n = \frac{1}{2}\left( [n+1]_{q,p} + [n]_{q,p} \right). \] (6)

With the account of (4), the energy spectrum takes the form
\[ E_n = \frac{1}{2} \left( q^n \sum_{l=0}^{n} q^{-l} p^l + q^{n-1} \sum_{s=0}^{n-1} q^{-s} p^s \right). \] (7)

As \( q, p \to 1 \), we recover the familiar \( E_n = n + \frac{1}{2} \). Moreover, \( E_0 = \frac{1}{2} \) for any \( q, p \).

We will consider the \( q, p \)-oscillators at real \( q, p \) that belong to the intervals
\[ 0 \leq q \leq 1, \quad 0 \leq p \leq 1. \] (8)

These define the quadrant in \( q, p \)-plane from which we exclude the point \((0,0)\).

### 2.1 Degeneracy of the type \( E_m = E_0 \)

For the purposes of the present letter, of basic importance is the possibility of 'accidental' degeneracies of \( q,p\)-oscillators, obeying the relations (1)-(4). Let us first recapitulate the statements from Ref. [5] that will be needed in the sequel.

**Proposition 1.** There exists a continuum of pairs of values \((q, p)\), or continuum of points of definite curve \( F_{m,0}(p, q) = 0 \), which provide the degeneracy
\[ E_m - E_0 = 0, \quad m \geq 2. \] (9)

The degeneracy curve in \((q,p)\)-plane, got from (9) and (7), namely
\[ F_{m,0}(q, p) \equiv \sum_{r=0}^{m} p^{m-r} q^r + \sum_{s=0}^{m-1} p^{m-1-s} q^s - 1 = 0, \] (10)

implies certain implicit function \( p = f_{m,0}(q) \) on the \( q\)-interval in (8), continuous and monotonically decreasing. The latter can be shown through the derivative
\[ \frac{dp}{dq} = f'_{m,0}(q) = - \frac{\partial F_{m,0}}{\partial q} \left( \frac{\partial F_{m,0}}{\partial p} \right)^{-1}, \]
that is always negative. This continuous decreasing implicit function \( f_{m,0}(q) \) is represented by the curve (10), with endpoints on the axes, in the quadrant (8).

**Remark 1.** The values of \( q \) and \( p \) from the pairs \((q, p)\) which solve Eq. (10), in fact belong to the smaller, than in (8), intervals \( 0 < q < q_m \) and \( 0 < p < p_m \) (here \( p_m \) resp. \( q_m \) solve (10) for \( q = 0 \) resp. \( p = 0 \), and are such that \( p_m, q_m < 1 \) and \( p_m = q_m \)). Moreover, denoting \( q_\infty \equiv 1 \) (since \( q_m \to 1 \) we have
\[ q_2 < q_3 < q_4 < \ldots < q_{m-1} < q_m < \ldots < q_\infty. \] (11)
Consider for a fixed \( m, \ m \geq 2 \), the derivative \( f'_{m,0}(q) \) at the endpoints \((0, p_m), (q_m, 0)\) where \( q_m = p_m \), and at the midpoint (given by \( p = q \)) of each curve \( f_{m,0}(q) \). For any \( m \) we have \( f'_{m,0}(q)|_{q=p} = -1 \) at the midpoint, while at the endpoints the derivative depends on \( m \), is negative, and such that

\[
f'_{m,0}(q)|_{q=q_m, p=0} < -1 < f'_{m,0}(q)|_{q=0, p=p_m} < 0 .
\]

In Figure 1, we illustrate the cases of \( m = 2, 3, 4, 6 \) by the curves 1, 2, 3, 4.

Let \( m = 2 \) (the case given by the curve 1 in Fig. 1) The degeneracy implies

\[
F_{2,0}(q,p) = p^2 + pq + q^2 + p + q - 1 = 0 ,
\]

which yields the (explicit in this case) function

\[
p = f_{2,0}(q) = \frac{-1 - q + \sqrt{(1 + q)(1 - 3q) + 4}}{2}
\]

monotonically decreasing for \( 0 < q < q_2 \) where \( q_2 = (\sqrt{5} - 1)/2 \), and \( p_2 = q_2 \). Then,

\[
f'_{2,0}(q) = \frac{-p + 2q + 1}{2p + q + 1} = \begin{cases} 
\frac{-p_2 + 1}{2p_2 + 1} & q = 0 , \\
-1 & p = q , \\
\frac{2q_2 + 1}{q_2 + 1} & p = 0.
\end{cases}
\]

\[2.2\, \text{Degeneracy of the type} \ E_{m+1} = E_m\]

Now consider the case with pairs of energy levels which are nearest neighbors.

*Proposition 2.* For each \( m, \ m \geq 1 \), there exists a continuum of pairs of values \((q, p)\), or continuum of points of definite curve \( F_{m+1,m}(q,p) = 0 \) in the \((q,p)\)-plane, for which the degeneracy

\[
E_{m+1} - E_m = 0
\]

does hold[3]. The curve is given as

\[
F_{m+1,m}(q,p) \equiv \sum_{r=0}^{m-1} p^{m+1-r} q^r - \sum_{s=0}^{m-1} p^{m-1-s} q^s = 0 .
\]

Eq. (14), due to polynomial form of its l.h.s., determines a *continuous* implicit function \( p = f_{m+1,m}(q) \). This function monotonically decreases on the \( q \)-interval in (8) as follows from the inequality

\[
\frac{dp}{dq} = f'_{m+1,m}(q) = -\frac{\partial}{\partial q} F_{m+1,m}(q,p) < 0 .
\]

Indeed, the both partial derivatives (being polynomials) are continuous and positive functions of two variables. E.g., in the case of \( m = 3 \) due to eq. (14) we have

\[
F_{4,3} \equiv p^4 + p^3 q + p^2 q^2 + pq^3 + q^4 - p^2 - pq - q^2 = 0
\]

from which we deduce that the inequality

\[
\frac{dp}{dq} = f_{4,3}'(q) = \frac{-p^3 + 2qp^2 + 3q^2 p + 4q^3 - p - 2q}{q^3 + 2pq^2 + 3p^2 q + 4p^3 - q - 2p} < 0
\]
Figure 1: Diverse cases of pairwise degeneracies of energy levels of $q,p$-oscillator: the curves 1, 2, 3, 4 correspond to $E_0 = E_2$, $E_0 = E_3$, $E_0 = E_4$ and $E_0 = E_6$, see eq. (9); the curves 5, 6, 7 correspond to $E_1 = E_2$, $E_2 = E_3$ and $E_4 = E_5$, see eq. (13).

is valid at each point of the curve. For more details concerning the proof that $p = f_{m+1,m}(q)$ is continuous, monotonically decreasing implicit function see Ref. [5].

Remark 2. The case $m = 1$ (or $E_1 = E_2$) obviously differs from the rest $m \geq 2$ cases since in the endpoints $(0,1)$ and $(1,0)$ the derivative $f'_{m+1,m}(q)$ acquires the values distinct from those in $m \geq 2$ cases. Indeed, at $m = 1$, $f'_{2,1}(q)|_{q=0} = -\frac{1}{2}$ and $f'_{2,1}(q)|_{q=1} = -2$ (i.e., the derivative $f'_{2,1}(q)$ continuously drops from $-\frac{1}{2}$ to $-2$ as $q$ runs from zero to one). In contrast, for each $m \geq 2$ we have $f'_{m+1,m}(q)|_{q=0} = 0$ and $f'_{m+1,m}(q)|_{q=1} \rightarrow -\infty$, i.e., $f'_{m+1,m}(q)$ decreases from 0 to $-\infty$ as $q$ grows from zero to one. The distinction of $E_1 = E_2$ case from the rest $m \geq 2$ cases, e.g., $E_2 = E_3$ or $E_4 = E_5$, is evident in Fig. 1 (the curve 5 versus the curves 6 and 7).

3 Novel $q$-oscillators with double degeneracies

Thus, for $q,p$-deformed oscillators we have the property of double degeneracy (within a fixed pair) of energy levels. Due to this fact, a possibility arises to obtain a host of one-parameter $q$-deformed oscillators all inheriting, like the TD-oscillator studied in Ref. [7], the property of double (pairwise) degeneracy. Although there exist a plenty of continual classes of $q$-oscillators, below we present only few of them.

$q$-Oscillators obtained by imposing $p = q^l$, $0 < l < \infty$. The first large class of non-standard $q$-oscillators can be obtained from the $q,p$-oscillators if we impose $p = q^l$ with $l$ such that $0 < l < \infty$. 

![Diagram of energy levels](image-url)
Then we have
\[
AA^\dagger - q A^\dagger A = q^N, \quad AA^\dagger - q_l A^\dagger A = q^N,
\]
with the bracket as in (3), and the energy spectrum takes the form
\[
E_n^{(l)} = \frac{1}{2} \left( q^{nl} + (1 + q)^{n^{(n-1)}} \sum_{s=0}^{n-1} q^{s(1-l)} \right).
\]
Clearly, this large class naturally splits in three subclasses:
(i) the one for which \(1 < l < \infty\);
(ii) the one for which \(0 < l < 1\);
(iii) the isolated case of \(l = 1\), equivalent to the TD-oscillator.
Any of these \(q\)-oscillators (see Ref. [7] for the TD case) acquires the property of double degeneracy at certain \(q\), within a pair of energy levels. This fact follows immediately since, in the \((q, p)\)-plane, each line \(p = q^l\) with \(l\) fixed so that \(0 < l < \infty\), crosses just once the \(q, p\)-curve \(E_{m_i} - E_{m_j} = 0\) of the corresponding degeneracy.

\[\text{Figure 2: Intersection of } E_0 = E_2, E_0 = E_5, E_3 = E_4 \text{ (degeneracy curves of } q,p\text{-oscillator) by the lines } p = q^l \text{ leading to } q\text{-oscillators. Here } l = 0.25, 0.5, 1, 2.5, 5.7 \text{ for A, B, C, D, E respectively.}\]

In Fig. 2, the representatives of all three subclasses are shown, including the \(l = 1\) (or the TD) case. Any representative of each subclass (i), (ii) and (iii) does intersect exactly once with each of the degeneracy curves \(E_{m_i} - E_{m_j} = 0\) at certain value of \(q\). As a consequence of the symmetry \(q \leftrightarrow p\), the first and the second subclasses are 'dual' (or inverse) to each other.

\(q\)-Oscillators deduced by setting \(p = 1 + \alpha \ln q\), \(\alpha > 0\), and those deduced by setting \(p = \exp \left( \alpha (q-1) \right)\), \(\alpha > 0\). The next two classes of \(q\)-deformed oscillators arise from \(q,p\)-oscillators if we impose logarithmic or exponential type of relation \(p = f(q)\). Let us first put
\[
p = 1 + \alpha \ln q, \quad \alpha > 0.
\]
Then we arrive at the logarithmic class of $q$-oscillators, whose members are labeled by some real $\alpha, \alpha > 0$. Likewise, by imposing on the $q,p$-oscillators the relation

$$p = \exp(\alpha(q-1)), \quad \alpha > 0,$$

we get the exponential class of $q$-oscillators whose members are also labeled by $\alpha$. Typical representatives of the two classes are shown in Fig. 3 and Fig. 4.

![Figure 3: Crossing of $E_0 = E_2$, $E_0 = E_5$ and $E_3 = E_4$ (degeneracy curves of $q,p$-oscillator) by the lines $p = 1+\alpha \ln q$ leading to $q$-oscillators. Here A, B, C, D correspond to $\alpha = 0.35$, 1, 1.85, 6.05.](image)

**Remark 3.** As seen from Fig. 3, the $q$-oscillator obtained from the ‘master’ $q,p$-oscillators with $p = 1+\alpha \ln q$ at $\alpha = 6.05$, see line D, does not admit the degeneracies $E_0 = E_2$ and $E_0 = E_5$ (and the ‘intermediate’ ones $E_0 = E_3$, $E_0 = E_4$): indeed, the line D does not cross those degeneracy curves. Likewise, the $q$-oscillator obtained from the ‘master’ $q,p$-oscillators by imposing $p = \exp(\alpha(q-1))$ with $\alpha = 0.1653$, see the curve A in Fig. 4, does not admit the degeneracies $E_0 = E_2$ and $E_0 = E_5$ (as well as $E_0 = E_3$ and $E_0 = E_4$). This peculiarity is in sharp contrast with the class of $q$-oscillators produced by $p = q^l$, see above, which admit all the degeneracies occurring in the TD-oscillator and in $q,p$-oscillators.

**Remark 4.** The expressions for energy $E_n = E(n)$ in the cases of exp-class and ln-class of $q$-oscillators can be explicitly given in complete analogy with Eq. (16) of the $p = q^l$ class. Next, it is clear that due to the symmetry $q \leftrightarrow p$ encoded in (3), (7), the roles of the deformation parameters $q$ and $p$ are interchangeable. Therefore, the exp-class and the ln-class are dual (or inverse) to each other.

**Remark 5.** Of course there exist, besides those given above, many other non-standard classes of one-parameter $q$-oscillators exhibiting analogous degeneracy properties. To obtain any such class or a concrete desired nonstandard $q$-oscillator by imposing an appropriate relation $p = f(q)$, we should guarantee: (i) monotonic character of $p = f(q)$ in the quadrant of $(q,p)$-plane given in (8); (ii) validity of the property $f(1) = 1$. Say, the relation $p = \frac{3}{q+1+q}$ gives admissible example.
Figure 4: Crossing of $E_0 = E_2$, $E_0 = E_5$ and $E_3 = E_4$ (degeneracy curves of $q,p$-oscillator) by the lines $p = \exp(\alpha(q-1))$ leading to $q$-oscillators. Here A, B, C, D correspond to $\alpha = 0.165$, 0.54, 1, 2.

4 Towards applications

The first point to be emphasized is the behavior of $n$-th level energy $E_n = E(n)$ as function of the particle number $n$. We find qualitative similarity between the shape of $E(n)$ for the TD $q$-oscillator and that of $E(n)$ for the nonstandard versions of $q$-oscillator considered in the present paper. Instead of trying to illustrate all possible cases, we present in Fig. 5, for different $q$, the picture of $E(n)$ for one typical member of the $\exp$-class of $q$-oscillators, namely the one got through setting $p = \exp(\alpha(q-1))$. As the Fig. 5 demonstrates, the basic features of $E(n)$ are the same as for the TD deformed oscillator: (i) existence of maximum and hence the ability to realize a degeneracy within certain pair of levels; (ii) location of the maximum more and more to the right for successively larger values of $q$ (compare the $q = 0.4$ curve with those for $q = 0.7$ or 0.88), with the tendency to turn into straight line $E_n = n + \frac{1}{2}$ (i.e., equal spacing of levels) of the standard oscillator in the limit of $q = 1$.

Let us emphasize once more the peculiar behavior, see Fig. 5, of energy spectrum of the considered $q$-deformed oscillators versus standard one. Namely, while the lowest value of energy is $E_0 = \frac{1}{2}$ for all the oscillators, both standard and $q$-deformed, the energy $E_n$ of the considered $q$-oscillators, after growing for few values $n \leq n_0$ of the quantum number $n$, becomes a decreasing function of $n$, with $E_n \to 0$ when $n \to \infty$. This very special behavior of energy spectrum can be of use in describing some exotic physical systems (maybe, relativistic, and with complicated interaction).

The other point is the application of new versions of $q$-oscillators, or $q$-bosons, in the framework of respective versions of $q$-Bose gas model and their use for description of non-Bose features of multi-pion (and multi-kaon) correlations observed in experiments on relativistic heavy-ion collisions. For illustration, we give the picture confirming as well efficient description using the presented classes of $q$-bosons. Namely, in Fig. 6 we show the behavior of the intercept (maximal
value) \( \lambda^{(2)}(K) \) of two-pion momentum correlation function derived\(^3\) within the version of \( q \)-Bose gas model based on the same version of \( q \)-oscillator obtained through \( p = \exp(\frac{1}{2}(q - 1)) \). Three pairs of curves are shown, for the two temperatures 120 MeV and 180 MeV in each pair. Like in Refs. \[16, 17\], the intercept \( \lambda^{(2)}(K) \) rises with growing mean momenta and its large momentum asymptotics shows saturation determined by the parameter \( q \). However, unlike the case of Arik-Coon \( q \)-bosons where asymptotical value of intercept is just the \( q \) by itself \[16, 20\], in the present version of \( q \)-Bose gas model the intercept tends to its asymptotical value given as \( \lambda_{\text{asym}} = -1 + q + \exp(\frac{1}{2}(q - 1)) \). Such feature, we hope, opens new possibilities for adjustment to the appearing experimental data.

5 Concluding remarks

In Ref. \[7\] we have revealed the unusual ability of Tamm-Dancoff \( q \)-oscillator to exhibit, at appropriate real values of \( q \), the accidental double degeneracy within a fixed pair of energy levels. Subsequently, similar type of pairwise degeneracy of energy levels has been demonstrated\(^5\) in more general setting of two-parameter \( q,p \)-oscillators if the admissible values of \( q,p \) constitute certain continual set.

It appears natural to exploit the \( q,p \)-oscillators with their property of pairwise energy level degeneracies in order to obtain, by definite reduction, many other unusual one-parameter deformed oscillators with similar properties. In this paper we presented a simple procedure that really allows to infer, from the "master" \( q,p \)-oscillators, the innumerable sets (classes) of new one-parameter models of \( q \)-deformed oscillators all of which inherit, almost completely (cf. Remark 3), the same pairwise degeneracies, e.g., the degeneracy \( E_2 = E_0 \), or \( E_2 = E_1 \), or \( E_5 = E_4 \), or others.

Our results witness that the TD oscillator is not the unique one with such exotic degeneracies, but only a separate case from a plethora of continual classes of \( q \)-oscillators exhibiting the special property of the pairwise degeneracy of energy levels within a fixed pair of them. As in the previous

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\(^3\)The results from the papers\[19, 20\] are of use in order to gain relevant formulas.
Figure 6: Intercept $\lambda^{(2)}(K)$ of two-pion correlation function in the $q$-Bose gas model that uses $q$-bosons based on setting $p = \exp\left(\frac{1}{2}(q-1)\right)$. The pairs A, B, C of curves are given for $q = 0.4, 0.65, 0.9$ respectively. Solid (dotted) curves correspond to the temperature 180 MeV (120 MeV).

paper\cite{7}, the degeneracy occurs at a properly fixed value of the $q$-parameter, it concerns single pair of levels and is an ‘accidental’ one (without underlying symmetry).

Of course, there exist many other possibilities yielding non-standard $q$-oscillators which can have even more complicated patterns of degeneracy, e.g., those realized in Ref. \cite{15} where the derived models exhibit two double (two pairwise) degeneracies that involve two degenerate pairs: $E_{m_1} = E_{m_2}$ and $E_{m_3} = E_{m_4}$. That means that the procedure developed in this paper and in Ref. \cite{15} is really able to produce myriads of one-parameter deformed nonlinear oscillators with more and more nontrivial and unusual properties (patterns) of degeneracy.

We expect for the considered $q$-oscillators many interesting physical applications. Say, it is of interest to explore in more detail the applicability of such $q$-oscillators ($q$-bosons) in the context of effective description of non-Bose properties of the two- and multi-pion (-kaon) correlations observed in the experiments on relativistic nuclear collisions. In Sec. 4, we have made a small step towards possible applications. It was based on the version of $q$-bosons and $q$-Bose gas involving one special member of the obtained $exp$-class of nonstandard $q$-oscillators. Note in this context that the earlier results on multi-particle correlations of $q,p$-bosons in the two-parameter $q,p$-Bose gas model, including the explicit formulas for (intercepts of) general $n$-particle momentum correlation functions\cite{19, 20} are very useful. Our conclusion is that the usage of the unusual $q$-oscillators, the main subject of our present paper, in the framework of the corresponding $q$-Bose gas model is, at least, on an equal footing with the other distinguished versions of $q$-bosons (and $q$-Bose gas model), such as Arik-Coon\cite{12}, Tamm-Dancoff \cite{8, 9} or Biedenharn-Macfarlane \cite{10, 11} ones.

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