COUNTING GROWTH TYPES OF AUTOMORPHISMS OF FREE GROUPS

GILBERT LEVITT

Abstract. Given an automorphism of a free group $F_n$, we consider the following invariants: $e$ is the number of exponential strata (an upper bound for the number of different exponential growth rates of conjugacy classes); $d$ is the maximal degree of polynomial growth of conjugacy classes; $R$ is the rank of the fixed subgroup. We determine precisely which triples $(e, d, R)$ may be realized by an automorphism of $F_n$. In particular, the inequality $e \leq \frac{2n^2 - 2}{4}$ (due to Levitt-Lustig) always holds. In an appendix, we show that any conjugacy class grows like a polynomial times an exponential under iteration of the automorphism.

Introduction

Consider an automorphism $\alpha$ of a free group $F_n$ which is induced by a homeomorphism $h$ of a compact surface $\Sigma$. After replacing $h$ by a power, it becomes isotopic to a homeomorphism $h'$ with a very simple structure: $\Sigma$ is a union of invariant subsurfaces $\Sigma_i$, and on each subsurface $h'$ is either pseudo-Anosov, or a Dehn twist in an annulus, or the identity.

Three very different behaviors thus appear: exponential, linear, trivial. This qualitative analysis also has a quantitative side. Just writing that the sum of the Euler characteristics of the $\Sigma_i$’s is $\chi(\Sigma)$ gives bounds for, say, the rank of the fixed subgroup of $\alpha$, or the number of subsurfaces.

If $\alpha$ is an arbitrary automorphism of $F_n$, there is no general analogue of the Nielsen-Thurston decomposition into invariant subsurfaces. Still, one can again distinguish three different behaviors. Exponential behavior comes from exponentially growing strata in a relative train track representative of $\alpha$, or, more intrinsically, from the attracting laminations. Linear should now be replaced by polynomial: unlike in the surface case, it is possible for the length of a conjugacy class to grow as a polynomial of degree $> 1$ under iteration of $\alpha$ (the simplest example is the automorphism of $F_3$ which sends $a$ to $a$, $b$ to $ba$, $c$ to $cb$, with $c$ growing quadratically). Trivial behavior comes from the fixed subgroup of $\alpha$.

Though these three different behaviors are not separated as clearly as in the surface case, the goal of this paper is to show that one can still give precise numerical bounds for invariants that measure how much “space” each of these three behaviors occupies within $F_n$. 
The first invariant attached to an automorphism $\alpha$ is the number $e$ of attracting laminations (equal to the number of exponential strata of an improved train track representative). It is an upper bound for the number of exponential growth rates of conjugacy classes (see appendix). In the surface case, $e$ is the number of subsurfaces $\Sigma_i$ on which the map is pseudo-Anosov (it is a pleasant exercise to compute the maximal value of $e$ on a given orientable surface $\Sigma$; the answer is given in Remark 5.1).

The second invariant (which has no equivalent in the surface case) is the maximal degree $d$ such that the length of some conjugacy class grows as a polynomial of degree $d$ under iteration of $\alpha$ (it equals 2 in the example given above on $F_3$).

The third invariant is the rank $R = \text{rk Fix } \alpha$ of the fixed subgroup of $\alpha$. It is a famous theorem by Bestvina-Handel [4] that $R$ does not exceed $n$.

Our main result is a precise characterization of which triples $(e, d, R)$ may be realized by an automorphism of $F_n$. Let us first consider $e$ and $d$.

**Theorem 1.** Given $\alpha \in \text{Aut}(F_n)$, the numbers $e$ and $d$ satisfy:

$$e + d \leq n - 1$$

$$4e + 2d \leq 3n - 2 \quad (\leq 3n - 3 \text{ if } d > 0).$$

In particular, $e \leq \frac{3n - 2}{4}$ (Levitt-Lustig, [11]).

Conversely, any $(e, d)$ satisfying these inequalities may be realized by some $\alpha \in \text{Aut}(F_n)$.

The inequalities are equivalent to saying that $(e, d)$ belongs to the closed quadrilateral with vertices $(0, 0), (0, n - 1), (\frac{n - 1}{2}, \frac{n - 1}{2}), (\frac{3n - 2}{4}, 0)$ pictured on Figure 1. The equality $e = \lfloor \frac{3n - 2}{4} \rfloor$ is achieved by a surface homeomorphism with all subsurfaces $\Sigma_i$ either once-punctured tori or four-punctured spheres (see Figure 2 for a picture when $n \equiv 2 \pmod{4}$); recall that there is no pseudo-Anosov map on a pair of pants.

**Theorem 2.** Given $e$ and $d$ satisfying the conditions above, the possible values of $\text{rk Fix } \alpha$ for an automorphism $\alpha$ of $F_n$ are exactly those allowed by the following inequalities:

$$e + \max(d - 1, 0) + \text{rk Fix } \alpha \leq n$$

$$4e + 2d + 2\text{rk Fix } \alpha \leq 3n + 1 \quad (\leq 3n \text{ if } d = 0).$$

The first inequality is a strengthening of the bound $\text{rk Fix } \alpha \leq n$ proved in [4]. If the rank equals $n$, one must have $e = 0$ and $d \leq 1$, as proved in [5].

Our results have to do with growth. In an appendix, we show:

**Theorem 3.** Given $\alpha \in \text{Aut}(F_n)$ and an element (or conjugacy class) $g$ in $F_n$, there exist $\lambda \geq 1$ and $m \in \mathbb{N}$, and $C_1, C_2 > 0$, such that the length of $\alpha^p(g)$ is bounded between $C_1\lambda^pp^m$ and $C_2\lambda^pp^m$ for all $p \geq 1$. 

2
Figure 1. Possible values for \((e, d)\).

Figure 2. A surface \(\Sigma\) with \(\pi_1(\Sigma) \simeq F_n\) decomposed into \(\frac{3n-2}{4}\) subsurfaces carrying pseudo-Anosov maps.

This result is not deep (if one assumes train tracks), but has never appeared in print. Our proof uses arguments from a preliminary version of [3]. We also explain how to determine the growth types \((\lambda, m)\) with \(\lambda > 1\) from the set of attracting laminations of \(\alpha\) and the Perron-Frobenius eigenvalues. As shown in [12], the numbers \(\lambda\) may be viewed as Hölder exponents associated to periodic
points on the boundary of $F_n$.

Given $\alpha$, the number $e'$ of growth types $(\lambda, m)$ with $\lambda > 1$ is bounded by $e$, and Theorems 1 and 2 are valid with $e$ replaced by $e'$. We also show (Theorem 4.7) that the degree $m$ is bounded by $n/2 - 1$ when $\lambda > 1$ (when $\lambda = 1$, the optimal bound for $m$ is $n - 1$ for growth of conjugacy classes, $n$ for growth of elements of $F_n$).

Let us now say a few words about the proofs. As mentioned above, one of our results is a strengthening of the “Scott conjecture” $\text{rk Fix} \alpha \leq n$. As in [9], we work with the outer automorphism $\Phi$ determined by $\alpha$, and instead of $\text{rk Fix} \alpha$ we use a related invariant $r$ associated to $\Phi$. In the surface case, $r$ equals the absolute value of the Euler characteristic of the whole subsurface $\Sigma_f$ where $h'$ equals the identity, whereas $\text{rk Fix} \alpha$ can only see one component of $\Sigma_f$.

The proof is by induction on $n$. If $\Phi$ is polynomially growing (i.e. $e = 0$), some power of $\Phi$ preserves (up to conjugacy) a nontrivial decomposition of $F_n$ as a free product, or preserves a free factor of rank $n - 1$. This makes induction possible.

If $e > 0$, we consider a $\Phi$-invariant $\mathbf{R}$-tree $T$ with trivial arc stabilizers, as in [9]. Point stabilizers have rank $< n$ and we can argue by induction. The induction starts from the polynomial subgroups of $\Phi$: a canonical finite family of conjugacy classes of subgroups on which all the polynomial growth is concentrated [10]. Our techniques give precise bounds for the ranks of these subgroups (see Theorem 4.1).

Going back to the tree $T$, the inequality that would make the induction work to prove our main results is $\sum_{i=1}^{b} (3n_i - 2) \leq 3n - 6$, where $G_1, \ldots, G_b$ are representatives for conjugacy classes of non-trivial stabilizers of points of $T$, and $n_i = \text{rk } G_i$. Unfortunately, this inequality is false, as the following example shows.

Suppose that $\Phi$ is induced by a homotopy equivalence $f$ of a finite complex $Y$ obtained by attaching the boundary of a once-punctured torus $S$ onto a graph $\Gamma$ with $\pi_1 \Gamma \simeq F_{n-1}$. Assume furthermore that $S$ and $\Gamma$ are $f$-invariant, and $f|S$ is pseudo-Anosov. There is a $\Phi$-invariant tree $T$ in which $\pi_1 \Gamma$ is a stabilizer, so that $\sum (3n_i - 2) = 3(n - 1) - 2 = 3n - 5$ (there is a similar example with $S$ a four-punctured sphere).

To make the induction work, we need to have a more precise control on the outer automorphism $\Psi$ induced on $\pi_1 \Gamma$ than stated so far. The key remark is the following. The boundary of $S$ provides a nontrivial conjugacy class $\gamma$ which is fixed by $\Psi$. The element $\gamma$ is a commutator in $F_n$, but it cannot be one in $\pi_1 \Gamma$. Indeed, it must be a primitive element of $\pi_1 \Gamma$, as otherwise $\pi_1 Y$ could not be free. This motivates the introduction of an invariant $k$, computed from periodic conjugacy classes, so that an inequality $\sum_i (3n_i - 2 - k_i) \leq 3n - 6 - k$ does hold (Proposition 3.1).

In the first, preliminary, section, we explain how the induction works, and we define the polynomial subgroups. In Section 2, we define the basic invariants and we study how they behave under the induction process. In Section 3, we establish the needed inequalities about the numbers $n_i$. We prove the inequalities of our main theorems in Section 4, as Corollaries 4.5 and 4.6. In Section 5 we construct examples, showing that the inequalities are optimal. This section may
be read independently of the others. As mentioned above, we study growth in an appendix.

Acknowledgement. This research started in 1996 as joint work with Martin Lustig, and the inequality $e \leq \frac{3n-2}{4}$ is proved in [11]. Though Martin refused to coauthor the present paper, he certainly deserves a lot of credit for it.

I also thank the referee for a suggestion which simplified the proof of Proposition 3.1 a great deal.

1. Preliminaries

We view $\Phi \in \text{Out}(F_n)$ as a collection of automorphisms $\alpha \in \text{Aut}(F_n)$. We say that $\alpha$ represents $\Phi$, and we write $\Phi = \hat{\alpha}$. We write $i_x$ for the inner automorphism $g \mapsto xgx^{-1}$, so that $\hat{\beta} = \hat{\alpha}$ if and only if $\beta = i_x \circ \alpha$ for some $x \in F_n$. We say that $\alpha$ and $\beta$ are isogredient if $\beta = i_y \circ \alpha \circ (i_y)^{-1}$ for some $y$ (the word “similar” was used in [7] and [9]).

We write $\text{Fix} \alpha$ for the fixed subgroup $\text{Fix} \alpha = \{g \in F_n \mid \alpha(g) = g\}$, and $\text{rk} \text{Fix} \alpha$ for its rank. Isogredient automorphisms have conjugate fixed subgroups.

Train tracks and laminations [1].

Given $\Phi \in \text{Out}(F_n)$, there exists $q \geq 1$ such that $\Phi^q$ is represented by an improved relative train track map $f : G \to G$, as in [1, Theorem 5.1.5]. We denote by $H_i$ the $i$-th stratum. The height of a path $\gamma$ is the largest $i$ such that $\gamma$ contains an edge of $H_i$. The image of an edge of $H_i$ is a path of height $\leq i$.

There are three types of strata. If $e$ is an edge in a 0-stratum $H_i$, then $f(e)$ has height $< i$. If $H_i$ is an NEG stratum, it consists of a single edge $e_i$, and $f(e_i) = e_i u_i$ with $u_i$ of height $< i$. If $H_i$ is exponential, it has a transition matrix $M$ whose entry $M_{pq}$ records the number of times that the image of the $p$-th edge of $H_i$ crosses the $q$-th edge (in either direction). This matrix is positive, and it has a largest eigenvalue $\lambda > 1$ called the Perron-Frobenius eigenvalue of $H_i$. It is an algebraic integer.

To an exponential stratum $H_i$ is associated an attracting lamination $\Lambda_i$ of $\Phi$ [1, Section 3]. It may be defined as the set of bi-infinite paths $\gamma$ in $G$ such that any finite subpath of $\gamma$ is contained in some tightened image $f_\alpha^p(e)$, for $p \geq 1$ and $e$ an edge of $H_i$. These paths $\gamma$ are the leaves of $\Lambda_i$. Through the identification between $\pi_1(G)$ and $F_n$, one may view $\Lambda_i$ as a lamination on $F_n$ (a subset of the quotient of $\partial F_n \times \partial F_n$ minus the diagonal by the action of $\mathbb{Z}/2\mathbb{Z}$, which interchanges the factors, and the diagonal action of $F_n$).

We have described the attracting laminations through the choice of a train track representative $f$, but the set of attracting laminations on $F_n$ depends only on $\Phi$. For any $f$ representing a power of $\Phi$, there is a bijection between the set of attracting laminations of $\Phi$ and the set of exponential strata of $f$.

Growth.

We denote by $\overline{g}$ the conjugacy class of $g \in F_n$. The outer automorphism $\Phi$ acts on the set of conjugacy classes of elements, and on the set of conjugacy classes of subgroups.
If we fix a free basis for $F_n$, the length $|g|$ is the length of the reduced word representing $g$. The length $|\overline{g}|$ is the length of any cyclically reduced word representing a conjugate of $g$.

We say that two sequences $a_p$ and $b_p$ are equivalent, or that $a_p$ grows like $b_p$, if the ratio $a_p/b_p$ is bounded away from 0 and infinity. It is well-known that any sequence $|\alpha^p(g)|$ (resp. $|\Phi^p(\overline{g})|$) grows like a polynomial of degree $m \in \mathbb{N}$, or has exponential growth. We then say that $g$ (resp. $\overline{g}$) grows polynomially with degree $m$, or exponentially, under iteration of $\alpha$ (resp. $\Phi$). This does not depend on the choice of a basis.

If $g$ belongs to a finitely generated $\alpha$-invariant subgroup $H$, the growth of $g$ under $\alpha|H$ is the same as its growth under $\alpha$, because $H$ is quasiconvex in $F_n$ [14]. Similarly, the growth of $\overline{g}$ under $\Phi|H$ is the same as its growth under $\Phi$.

An automorphism $\Phi$ (or a representative $\alpha$) is polynomially growing if every conjugacy class grows polynomially. If $f$ represents a power of $\Phi$, this is equivalent to saying that $f$ has no exponential stratum.

These facts are sufficient for the proof of Theorems 1 and 2. In the appendix, we shall prove the more precise result that $g$ and $\overline{g}$ always grow like a sequence $\lambda^m p^m$ with $\lambda \geq 1$ and $m \in \mathbb{N}$. We say that $(\lambda, m)$ is the growth type. If $\lambda = 1$, the growth is polynomial. If $\lambda > 1$, the growth is exponential and we say that $\lambda$ is the exponential growth rate. If $\Phi^q$ is represented by an improved train track map, then any exponential growth rate is the $q$-th root of the Perron-Frobenius eigenvalue of an exponential stratum.

**Setting up the induction.**

We shall prove our main results by induction on $n$. First we replace $\Phi \in \text{Out}(F_n)$ by a power, so that it is represented by an improved relative train track map $f : G \to G$.

First suppose that $\Phi$ is polynomially growing. We consider the highest stratum. It consists of a single edge $E$.

**Definition 1.1.** Let $\Phi$ be polynomially growing, represented by $f : G \to G$.

1. If there is a non-trivial decomposition $F_n = G_1 \ast G_2$ such that $\Phi$ has a representative $\alpha$ which satisfies $\alpha(G_i) = G_i$, we let $n_i$ be the rank of $G_i$ and we denote $\alpha_i = \alpha|_{G_i}$, so that $\alpha = \alpha_1 \ast \alpha_2$. This happens in particular whenever the top edge $E$ separates $G$.

2. If $E$ does not separate, there is a decomposition $F_n = G_1 \ast \langle t \rangle$ (with $G_1$ of rank $n - 1$) and a representative $\alpha$ of $\Phi$ such that $\alpha(G_1) = G_1$ and $\alpha(t) = tu$ with $u \in G_1$. We assume that $u$ cannot be written $u = a\alpha(a^{-1})$ with $a \in G_1$, since otherwise $\alpha(ta) = ta$ and we reduce to the previous case $\alpha = \alpha_1 \ast \alpha_2$. We denote $\alpha_1 = \alpha|_{G_1}$.

This will allow us to deduce properties of $\alpha$ from properties of the $\alpha_i$'s.

In the remainder of this subsection, we assume that $\Phi$ is not polynomially growing. We then use the $\Phi$-invariant $\mathbb{R}$-tree $T$ associated to the highest exponential stratum $H$, in the following sense.
If $H$ is the highest of all strata, $T$ is the tree constructed in section 2 of [7]. If not, we adapt the construction of [7] as follows. Let $H'$ be the union of $H$ and all (non-exponential) strata above it. We consider the transition matrix $M_{H'}$: there is one row and one column for each edge of $H'$, and each entry records the number of times that the image of an edge crosses an edge (in either direction). The eigenvalues of $M_{H'}$ are those of the transition matrix $M_H$ of $H$, together with 0’s and 1’s. In particular, $M_{H'}$ has a non-negative eigenvector associated to the Perron-Frobenius eigenvalue $\lambda > 1$ of $M_H$. We then apply the construction of [7], using this eigenvector to give a PF-length to edges of $H'$ (edges below $H$ have PF-length 0).

In either case, the group $F_n$ acts on the $\mathbb{R}$-tree $T$ isometrically, minimally, with no global fixed point. All arc stabilizers are trivial.

We let $\ell : F_n \to [0, \infty)$ be the length function of $T$, defined by $\ell(g) = \min_{x \in T} d(x, gx)$. Its value on $g$ only depends on the conjugacy class $\overline{g}$, so we sometimes write $\ell(\overline{g})$.

An element $g$, or its class $\overline{g}$, is elliptic if $\ell(g) = 0$, hyperbolic if $\ell(g) > 0$. An elliptic element has a unique fixed point, a hyperbolic element has an axis along which it acts as translation by $\ell(g)$. Any $g \in F_n$ represented by a loop meeting only strata below $H$ is elliptic.

The tree $T$ is $\Phi$-invariant, in the sense that $\ell(\Phi(\overline{g})) = \lambda \ell(\overline{g})$, with $\lambda > 1$ the Perron-Frobenius eigenvalue of $M_H$ as above.

As in [12, §3], we note:

**Lemma 1.2.** If $\overline{g}$ grows polynomially, then $g$ is elliptic in $T$ (it fixes a unique point).

**Proof.** $\lambda^p \ell(\overline{g}) = \ell(\Phi^p(\overline{g}))$ is bounded by a constant times the word length of $\Phi^p(\overline{g})$, so $\ell(\overline{g}) = 0$. \hfill \Box

Stabilizers of points of $T$ have rank $< n$, and there are finitely many orbits of points with non-trivial stabilizer. These are general facts about trees with trivial arc stabilizers [8]. In the case at hand, they may be deduced from the description of stabilizers used in Section 3.

**Definition 1.3.** If $\Phi$ is exponentially growing, let $T$ be the $\Phi$-invariant $\mathbb{R}$-tree constructed above. We let $m_i$, $1 \leq i \leq b$, be representatives for the orbits of points with non-trivial stabilizer. We denote by $G_i$ the stabilizer of $m_i$ (it is malnormal, but not always a free factor). It has rank $n_i < n$. The conjugacy classes of the $G_i$’s are permuted by $\Phi$. Replacing $\Phi$ by a power, we may assume that $\Phi$ leaves $G_i$ invariant (up to conjugacy) and therefore induces $\Phi_i \in \text{Out}(G_i)$.

**Polynomial subgroups.**

Let $\Phi \in \text{Out}(F_n)$. A subgroup $P$ is polynomial if there exist $q \geq 1$ and $\alpha$ representing $\Phi^q$ such that $\alpha(P) = P$ and $\alpha|_P$ is polynomially growing. If $P$ is polynomial, so are its conjugates and its images by any automorphism representing a power of $\Phi$. Furthermore, $\Phi$ and all its powers have the same polynomial subgroups (for negative powers, recall that the inverse of a polynomially growing automorphism is polynomially growing).
Proposition 1.4. Let $\Phi \in \text{Out}(F_n)$.

1. The conjugacy class $\overline{g}$ of $g \in F_n$ grows polynomially if and only if $g$ belongs to a polynomial subgroup.
2. Every non-trivial polynomial subgroup is contained in a unique maximal one.
3. If $g \neq 1$ grows polynomially under $\alpha$ representing $\Phi$, the maximal polynomial subgroup $P(g)$ containing $g$ is the set of elements growing polynomially under $\alpha$.
4. Maximal polynomial subgroups have finite rank, are malnormal, and there are finitely many conjugacy classes of them.

Inequalities for the ranks of the polynomial subgroups will be given in Theorem 4.1. For the automorphism $\theta_n$ constructed in Section 5, the group $P_0$ is a maximal polynomial subgroup.

Proof. By induction on $n$. The results are true if $\Phi$ is polynomially growing, with $F_n$ the unique maximal polynomial subgroup. If not, we consider $T$, $G_i$, $\Phi_i$ as in 1.3 (after replacing $\Phi$ by a power if needed). Any polynomial subgroup of $G_i$ (relative to $\Phi_i$) is a polynomial subgroup of $F_n$ (relative to $\Phi$).

Consider $g \neq 1$ such that $\overline{g}$ grows polynomially (this is the case, in particular, if $g$ belongs to a polynomial subgroup). Let $P(g)$ be the set of $h \in F_n$ such that both $\overline{h}$ and $\overline{gh}$ grow polynomially. We prove by induction on $n$ that $P(g)$ is a polynomial subgroup.

Since $\overline{g}$ grows polynomially, the element $g$ fixes a point in $T$ by Lemma 1.2, and by conjugating $g$ we may assume $g \in G_i$. If $h \in P(g)$, then $h$ is also elliptic. It fixes the same point as $g$ because otherwise $gh$ would be hyperbolic in $T$, contradicting polynomial growth of $\overline{gh}$. This shows $P(g) \subseteq G_i$. By induction, $P(g)$ is a polynomial subgroup (relative to $\Phi_i$, hence also to $\Phi$).

Clearly $P(g)$ contains $g$, as well as every polynomial subgroup containing $g$: it is the largest polynomial subgroup containing $g$. This shows assertions (1) and (2).

Suppose $g$ and $\alpha$ are as in (3). As above, we may assume $g \in G_i$. Any element growing polynomially under $\alpha$ is in $P(g)$. For the converse, note that $\alpha(g)$ belongs to $P(g)$, hence to $G_i$. We deduce that $G_i$ and $\alpha(G_i)$ have a non-trivial intersection, hence are equal (they are point stabilizers in a tree with trivial arc stabilizers). By induction, any $h \in P(g)$ grows polynomially under $\alpha$.

The finiteness statements in (4) are immediate by induction. Let us prove malnormality. Let $P$ be a maximal polynomial subgroup. If $P$ and $gPg^{-1}$ meet non-trivially, they are equal (by the uniqueness statement in (2)), so we only have to show that $P$ equals its normalizer. As above, we may assume $P \subseteq G_i$. Since $P$ fixes a unique point $m_i$ in $T$, any $g$ normalizing $P$ belongs to $G_i$. By induction, $P$ equals its normalizer in $G_i$, so it also equals its normalizer in $F_n$. 

Lemma 1.5. Given $\Phi \in \text{Out}(F_n)$, the following are equivalent:

1. There exists a polynomial subgroup of rank $\geq 2$. 


(2) There exist \( q \geq 1 \) and \( \alpha \) representing \( \Phi^q \) with \( \text{rk Fix} \alpha \geq 2 \).

**Proof.** (2) obviously implies (1). For the converse, it suffices to prove (2) under the assumption that \( \Phi \) is polynomially growing and \( n \geq 2 \). We distinguish two cases, as in Definition 1.1. First suppose that some power of \( \Phi \) has a representative \( \alpha = \alpha_1 * \alpha_2 \). If \( n_1 \) or \( n_2 \) is \( \geq 2 \), we use induction. Otherwise, \( \alpha_2 \) is the identity of \( F_2 \) and (2) holds. In the second case, we consider a decomposition \( F_n = G_1 * (t) \). If \( n \geq 3 \), we use induction. If \( n = 2 \), then \( \alpha_2 \in \text{Aut} (F_2) \) is of the form \( a \mapsto a, t \mapsto tat^{-1} \) for some \( i \in \mathbb{Z} \), and its fixed subgroup has rank 2 (if \( i \neq 0 \), it is generated by \( a \) and \( tat^{-1} \)). \( \square \)

**Corollary 1.6.** Let \( \Phi \in \text{Out} (F_n) \). Suppose that some non-trivial conjugacy class \( \overline{g} \) grows polynomially. If \( \overline{g} \) is not periodic, there exist \( q \geq 1 \) and \( \alpha \) representing \( \Phi^q \) with \( \text{rk Fix} \alpha \geq 2 \). If \( g \) is periodic, there exists \( \alpha \) with \( \text{rk Fix} \alpha \geq 1 \).

**Proof.** Only the first assertion requires a proof. By Proposition 1.4, \( g \) belongs to a polynomial subgroup. It has rank at least 2, as otherwise \( \overline{g} \) would be periodic. We conclude by Lemma 1.5. \( \square \)

We also record the following easy fact:

**Lemma 1.7.** If every conjugacy class is \( \Phi \)-periodic, then \( \Phi \) has finite order. \( \square \)

In fact \( \Phi \) has finite order as soon as every conjugacy class of length \( \leq 2 \) is periodic.

## 2. The basic invariants

We write \( x^+ = \max(x, 0) \).

Given \( \Phi \in \text{Out} (F_n) \), we now define numbers \( e, s, d, p, r, k \). If we consider several automorphisms simultaneously, we write \( e(\Phi), s(\Phi), \ldots \) so that no confusion arises. If we consider \( \alpha \in \text{Aut} (F_n) \), we write \( e, s, \ldots \) for the invariants of the outer automorphism \( \Phi = \hat{\alpha} \) represented by \( \alpha \).

- \( e \) is the number of exponential strata of any improved relative train track map representing a power of \( \Phi \). It is also the number of attracting laminations of \( \Phi \) [1, Subsection 3.1]. Note that \( e \) is an upper bound for the number of growth types \( (\lambda, m) \) with \( \lambda > 1 \) (see Theorem 6.2), and that \( e = 0 \) if and only if \( \Phi \) is polynomially growing. Also note that \( e(\Phi^{-1}) = e(\Phi) \) by Subsection 3.2 of [1].
- \( s \) is the maximal length of a chain of attracting laminations \( \Lambda_0 \supseteq \cdots \supseteq \Lambda_s \). Any growth type \( (\lambda, m) \) with \( \lambda > 1 \) satisfies \( m \leq s \) (see appendix).
- \( d \) is the maximal degree of polynomial growth of conjugacy classes. If \( e = d = 0 \), then \( \Phi \) has finite order by Lemma 1.7. One has \( d(\Phi^{-1}) = d(\Phi) \) by [13] (to be precise, one also needs Proposition 1.4 to reduce to the polynomially growing case studied in [13], and Lemma 2.3 below to compare \( d \) to growth of elements).
- \( p = (d - 1)^+ = \max(d - 1, 0) \).
- \( r \) is the index of \( \Phi \), computed using ranks of fixed subgroups. Namely, we write \( r_0(\alpha) = (\text{rk Fix} \alpha - 1)^+ = \max(\text{rk Fix} \alpha - 1, 0) \) for \( \alpha \) representing \( \Phi \). We then define \( r = \sum_i r_0(\alpha_i) \), the sum being taken over a set of representatives of
isogredience classes (recall that $\alpha, \alpha'$ are isogredient if there is an inner automorphism conjugating them). One has $r \leq n - 1$ by [4, Corollary 6.4] (see also [9]). If $\text{rk Fix } \alpha \geq 2$, we say that the isogredience class of $\alpha$ contributes $r_0(\alpha)$ to $r$. Only finitely many classes contribute.

- $k$ is the rank of the subgroup generated by $\Phi$-periodic conjugacy classes in the abelianization of $F_n$ (so $k \leq n$). This number plays an essential role in the proof of our main results (see the discussion in the introduction and in Section 3).

The numbers $e, s, d, p, k$ do not change if $\Phi$ is replaced by a positive power (and $e, d, p, r, k$ do not change if $\Phi$ is replaced by $\Phi^{-1}$; we do not know whether $s$ is the same for $\Phi$ and $\Phi^{-1}$). The following lemma controls $r$, so that we can always replace $\Phi$ by a power when proving upper bounds for the invariants.

**Lemma 2.1.** $r(\Phi) \leq r(\Phi^q)$ for any $q \geq 1$.

**Proof.** We have to bound any finite sum $\sum_i r_0(\alpha_i)$ as above by $r(\Phi^q)$. One always has $\text{rk Fix } \alpha \leq \text{rk Fix } \alpha^q$, since $\text{Fix } \alpha$ is a free factor of $\text{Fix } \alpha^q$ by [6], so $r_0(\alpha_i) \leq r_0(\alpha_i^q)$. But this is not enough, because $\alpha_i^q$ and $\alpha_j^q$ may be isogredient for $i \neq j$ even though $\alpha_i, \alpha_j$ aren’t.

We may assume that $\alpha_i^q$ and $\alpha_j^q$ are equal if they are isogredient (by changing automorphisms within their isogredience class), so we reduce to showing

$$\sum_{\alpha_i^q = \beta} r_0(\alpha_i) \leq r_0(\beta)$$

for any $\beta$ representing $\Phi^q$. We may also assume that $\text{Fix } \beta$ has rank $\geq 2$.

Note that $\text{Fix } \beta$ is $\alpha_i$-invariant and contains $\text{Fix } \alpha_i$. We claim that the restrictions of the $\alpha_i$’s to $\text{Fix } \beta$ represent the same outer automorphism $\tilde{\Phi}$ (of finite order), and define distinct isogredience classes. Assuming this, we conclude because

$$\sum_{\alpha_i^q = \beta} r_0(\alpha_i) \leq r(\tilde{\Phi}) \leq \text{rk Fix } \beta - 1 = r_0(\beta).$$

To prove the claim, fix distinct $i, j$. We write $\alpha_i = i_h \circ \alpha_j$ (with $i_h(g) = hgh^{-1}$) and we show $h \in \text{Fix } \beta$. We have $\alpha_i^q = i_n \circ \alpha_j^q$ with $n = h \alpha_j(h) \ldots \alpha_j^q(h)$. We deduce $n = 1$, and this implies $\alpha_i^q(h) = h$, so $h \in \text{Fix } \beta$ as required. This shows the first assertion. Next (as in [9, 2.4]), suppose that $i_m$ conjugates the restrictions of $\alpha_i$ and $\alpha_j$, with $m \in \text{Fix } \beta$. Then $\alpha_i \alpha_j^{-1} = i_m \alpha_j(m^{-1})$ on $\text{Fix } \beta$, hence everywhere because $\alpha_i \alpha_j^{-1}$ is inner and $\text{Fix } \beta$ has rank $\geq 2$. This means that $i_m$ conjugates $\alpha_i$ and $\alpha_j$ on the whole of $F_n$, so $\alpha_i, \alpha_j$ are isogredient, a contradiction to $i \neq j$. \qed

**Remark 2.2.** By Corollary 1.6, $d(\Phi) > 0$ implies that there exists $q \geq 1$ with $r(\Phi^q) > 0$.

The following lemma compares growth of conjugacy classes and growth of elements.
Lemma 2.3. Let $\Phi \in \text{Out}(F_n)$, and $d = d(\Phi)$. Suppose that some $g \in F_n$ grows polynomially with degree $d' > d$ under iteration of a representative $\alpha$ of $\Phi$. Then:

1. $d' = d + 1$.
2. If $\text{Fix} \alpha$ is non-trivial, then $d' = 1$ (and therefore $d = 0$).

Examples. If $\alpha$ is a non-trivial inner automorphism, then $d = 0$ and any non-trivial $g$ grows linearly $(d' = 1)$. In the example $a \mapsto bab^{-1}, b \mapsto b^2ab^{-1}$, due to Bridson-Groves, $b$ grows linearly and $b$ grows quadratically (the fixed subgroup is trivial).

Proof. We first prove (2). Fix a basis of $F_n$, and $h \neq 1$ in $\text{Fix} \alpha$. Write $\alpha^p(g) = w_pg_pw_p^{-1}$ with $g_p$ cyclically reduced. The length of $w_p$ grows with degree $d'$, because $g_p$ grows with degree at most $d$. We claim the cancellation between the initial segments of $w_p$ and $hw_p$ grows at most linearly. Assuming this for the moment, we consider $\alpha^p(hg) = hw_pg_pw_p^{-1}$. If $d' > 1$, the cyclic reduction of this word grows with degree $d'$, so $hg$ grows with degree $d'$ under iteration of $\Phi$. This contradicts $d' > d$.

To prove the claim, first note that the cancellation between $\alpha^p(g)$ and $h^\infty = \lim_{p \to \infty} h^p$ grows at most linearly (see [12, p. 424]). The same holds for the cancellation between $w_p$ and $h^\infty$, and therefore for that between $w_p$ and $hw_p$.

For (1), we let $\alpha_1$ be the restriction of $\alpha$ to the polynomial subgroup $P(g)$ (see item (3) in Proposition 1.4). Let $\Phi_1$ be the outer automorphism determined by $\alpha_1$. If $d(\Phi_1) = 0$, then $\Phi_1$ has finite order by Lemma 1.7, and $g$ grows at most linearly. If not, we use Corollary 1.6. After replacing $\alpha$ by a power, we write $\alpha_1 = i_a \circ \beta$, where $a \in P(g)$ and $\beta \in \text{Aut}(P(g))$ has a non-trivial fixed subgroup. Since $d(\Phi_1) > 0$, assertion (2) proved above implies that $a$ and $g$ grow with degree at most $d$ under iteration of $\beta$. Then

$$\alpha^n(g) = [a\beta(a)\beta^2(a)\ldots\beta^{n-1}(a)] [a\beta(a)\beta^2(a)\ldots\beta^{n-1}(a)]^{-1}$$

grows with degree $\leq d + 1$. \hfill \square

Our next goal will be to understand the behavior of the invariants in the induction process, so we consider $\Phi$ as in 1.1 or 1.3.

First suppose $\alpha = \alpha_1 \ast \alpha_2$ as in the first item of Definition 1.1. We denote by $k_i, r_i, d_i$ the invariants associated to $\hat{\alpha}_i$.

Lemma 2.4.

1. If $\text{Fix} \alpha_1$ or $\text{Fix} \alpha_2$ is trivial, then $r = r_1 + r_2$ and $\max(d_1, d_2) \leq d \leq 1 + \max(d_1, d_2)$.
2. If $\text{Fix} \alpha_1$ and $\text{Fix} \alpha_2$ are both non-trivial, then $r = r_1 + r_2 + 1$ and $d$ equals 1 or $\max(d_1, d_2)$.
3. $k = k_1 + k_2$.

Proof. The assertions about $r$ are proved in [9]. Here is the idea. If an isogrecedience class other than that of $\alpha$ contributes to $r$, it contributes the same amount to
either $r_1$ or $r_2$ (but not both). The class of $\alpha$ contributes $(\text{rk Fix } \alpha_i - 1)^+$ to $r_i$ and $(\text{rk Fix } \alpha - 1)^+$ to $r$. To conclude, note that $\text{Fix } \alpha = \text{Fix } \alpha_1 \ast \text{Fix } \alpha_2$.

The number $d$ is bounded from below by $\max(d_1, d_2)$, and from above by the maximal degree of growth of elements of $G_i$ under $\alpha_i$, so the assertions about $d$ follow from Lemma 2.3. The assertion about $k$ is easy and left to the reader. $\Box$

We now consider a decomposition $F_n = G_1 \ast \langle t \rangle$ and a representative $\alpha$ of $\Phi$ such that $\alpha(G_1) = G_1$ and $\alpha(t) = tu$ with $u \in G_1$, as in item (2) of Definition 1.1. Let $k_1, r_1, d_1$ be the invariants associated to the outer automorphism $\Phi_1$ of $G_1$ represented by $\alpha_1 = \alpha|_{G_1}$.

Lemma 2.5.

1. $r \leq r_1 + 1$, with equality if and only if both $\alpha_1$ and $i_u \circ \alpha_1$ have non-trivial fixed subgroups.
2. $d \leq d_1 + 1$.
3. $k \leq k_1 + 1$, with equality if and only there exists $a \in G_1$ such that $ta$ is $\alpha$-periodic.

Proof. The assertion about $r$ is proved in [9]. One now has $\text{Fix } \alpha = \text{Fix } \alpha_1 \ast t \text{Fix } (i_u \circ \alpha_1)t^{-1}$. The contribution of the isogredience class of $\alpha$ to $r$ is $(\text{rk Fix } \alpha_1 + \text{rk Fix } (i_u \circ \alpha_1) - 1)^+$, whereas its contribution to $r_1$ is $(\text{rk Fix } \alpha_1 - 1)^+ + (\text{rk Fix } (i_u \circ \alpha_1) - 1)^+$.

Let us prove (2). As in the proof of Lemma 2.3, replace $\alpha$ by a power and write $\alpha_1 = i_a \circ \beta_1$, where $a \in G_1$ and $\text{Fix } \beta_1$ is non-trivial, using Corollary 1.6; if $d_1 = 0$, we take $\beta_1$ to be the identity, using Lemma 1.7. By Lemma 2.3, all elements of $G_1$ grow with degree at most $d_1$ under $\beta_1$. Now $\beta = (i_a)^{-1} \circ \alpha$ sends $t$ to an element of the form $u'tu''$, so $t$ grows with degree at most $d_1 + 1$ under $\beta$. This implies $d \leq d_1 + 1$.

We now study $k$. Represent conjugacy classes of $F_n$ as cyclic words whose letters are either $t^\pm 1$ or non-trivial elements of $G_1$. Let $\overline{w}$ be a cyclic word not contained in $G_1$. Split it before every letter $t$ and after every $t^{-1}$. This expresses $\overline{w}$ as a product of subwords of the form $tat^{-1}$, $ta$, $at^{-1}$, $a$, with $a \in G_1$ and $a$ a non-trivial in subwords $tat^{-1}$ or $a$. This decomposition of $\overline{w}$ is preserved by $\alpha$, so that $\overline{w}$ is $\Phi$-periodic if and only if every subword is an $\alpha$-periodic element.

Let $K$ be the quotient of the abelianization of $F_n$ by the subgroup generated by all $\Phi_1$-periodic conjugacy classes (of elements of $G_1$). It has torsion-free rank $n - k_1$. To bound $k$, we need to control the image in $K$ of $\alpha$-periodic elements of the form $tat^{-1}$ or $ta$ (we treat $at^{-1}$ as the inverse of $ta^{-1}$).

If $tat^{-1}$ is $\alpha$-periodic, then $a$ is $(i_a^{-1} \circ \alpha_1)$-periodic, so $\overline{a}$ is $\Phi_1$-periodic and $tat^{-1}$ maps trivially to $K$. On the other hand, the image of $ta$ in $K$ has infinite order. But, if $ta$ and $ta'$ are both $\alpha$-periodic, then $a^{-1}a'$ is $\alpha_1$-periodic and therefore $ta$, $ta'$ have the same image in $K$. This proves the lemma. $\Box$

We now suppose that $\Phi$ has exponential growth, and we use the $\mathbb{R}$-tree $T$ as in 1.3. Recall that we have defined invariant subgroups $G_i$ and restrictions $\Phi_i$. The
following lemma allows us to bound the invariants of \( \Phi \) in terms of those of the \( \Phi_i \)'s (which we denote with the subscript \( i \)). All sums are over \( i \).

**Lemma 2.6.** The invariants of \( \Phi \) satisfy:

\[
\begin{align*}
e &\leq 1 + \sum e_i \\
s &\leq 1 + \max s_i \\
d &= \max d_i \\
r &= \sum r_i \\
k &\leq \sum k_i.
\end{align*}
\]

**Proof.** The results for \( d \) and \( k \) are direct consequences of Lemma 1.2 (note that for \( g \in G_i \) the growth of \( \overline{g} \) is the same for \( \Phi_i \) as for \( \Phi \)). The equality for \( r \) is proved in [9]: if an isogredience class contributes to \( r \), it contributes the same amount to exactly one \( r_i \).

As in the previous section, let \( H \) be the highest exponential stratum of the train track, and let \( H' \) be the union of \( H \) with all the strata above it. Every exponential stratum \( \tilde{H} \) other than \( H \) is contained in a component \( \tilde{G} \) of the closure of \( G \setminus H' \). It follows from the way \( T \) was constructed (see [7] and the previous section) that the fundamental group of \( \tilde{G} \) fixes a point of \( T \), whose stabilizer is conjugate to some \( G_i \). There is a bijection between exponential strata and attracting laminations [1, Definition 3.1.12]. By [1, Definition 3.1.5], the lamination associated to \( \tilde{H} \) is an attracting lamination of \( \Phi_i \). This shows the bound for \( e \). If \( \Lambda_0 \supseteq \cdots \supseteq \Lambda_s \) is a chain of laminations, then \( \Lambda_1, \ldots, \Lambda_s \) are attracting laminations of some \( \Phi_i \), so \( s - 1 \leq s_i \).

\[\square\]

### 3. Bounding ranks of stabilizers

In this section we assume that \( \Phi \) is not polynomially growing, and we consider the invariant \( R \)-tree \( T \) as in 1.3.

**Proposition 3.1.** Let \( n_i \) be the rank of the stabilizer \( G_i \), and \( k_i = k(\Phi_i) \). Then:

\[
\begin{align*}
\sum_{i=1}^{b} (n_i - 1) &\leq n - 2 \\
\sum_{i=1}^{b} (3n_i - 2 - k_i) &\leq 3n - 6 - k.
\end{align*}
\]

If equality holds in (1), there is only one exponential stratum.

Recall that \( k \leq \sum k_i \), so (2) is implied by the simpler inequality \( \sum_i (3n_i - 2) \leq 3n - 6 \). But this simpler inequality is not always true, for instance if \( \Phi \) is induced.
by a pseudo-Anosov homeomorphism of a punctured torus or a four-punctured sphere. This explains the introduction of the invariant \( k \).

**Proof of Proposition 3.1.**

Recall that \( T \) was constructed using the highest exponential stratum \( H \) of \( f : G \rightarrow G \). A description of the \( G_i \)'s is provided by Theorem 6.0.1 of [1], in terms of a subgraph \( Z \subseteq G \).

- First suppose that \( H \) is the highest stratum. Then \( Z \) is the union of all strata below \( H \).

If \( H \) is not geometric (in the sense of Definition 5.1.4 of [1]), then the \( G_i \)'s are (up to conjugacy) the fundamental groups of the non-contractible components of \( Z \). In particular they are free factors and \( \sum n_i \leq n \). If \( b = 0 \), we have \( n \geq 3 \) since every automorphism of \( F_2 \) is induced by a homeomorphism of a punctured torus. If \( b = 1 \) or \( b = 2 \), the existence of the exponential stratum \( H \) prevents \( n_1, n_2 \) from being too big. More precisely, we have \( n_1 < n - 1 \) if \( b = 1 \) by [1, Lemma 3.2.1], and similarly \( n_1 + n_2 < n \) if \( b = 2 \).

We get \( \sum (n_i - 1) \leq n - 3 \) and \( \sum (3n_i - 2) \leq 3n - 6 \), with equality in the second inequality possible only if \( b = 0 \) or \( b = 3 \). The proposition is true in this case since \( \sum k_i \leq k \).

If \( H \) is geometric, the components of \( Z \) account for all \( G_i \)'s but one. The exceptional one, say \( G_b \), is cyclic, generated by the homotopy class of a loop \( \rho \) (an indivisible Nielsen path) based at a point \( v \notin Z \). If \( b \) is different from 1 and 4, the previous argument yields \( \sum_{i < b} (n_i - 1) \leq n - 3 \) and \( \sum_{i < b} (3n_i - 2) \leq 3n - 7 \), and the proposition holds since \( n_b = 1 \).

If \( b = 1 \), we have \( \sum (n_i - 1) = 0 \leq n - 2 \). On the other hand, \( \sum (3n_i - 2) = 1 \). It is bounded by \( 3n - 6 \), except if \( n = 2 \). But in this case \( \Phi \) is induced by a homeomorphism of a punctured torus, so \( k = 0 \) (whereas \( k_1 = 1 \)). We get \( 3n_1 - 2 - k_1 = 0 = 3n - 6 - k \).

When \( b = 4 \), we have \( \sum (n_i - 1) \leq n - 3 \) and \( \sum_{i \leq 4} (3n_i - 2) \leq 3 \sum_{i \leq 4} n_i - 5 \). If \( \sum_{i \leq 4} n_i < n \), we are done. If not, we prove \( (2) \) by showing \( k < \sum_{i \leq 4} k_i \).

The geometric stratum \( H \) is associated to a connected surface \( S \), as in Definition 5.1.4 of [1]. This surface has four boundary components \( C_i \), with \( C_4 \) identified to \( \rho \). It is a punctured sphere because \( \sum_{i \leq 4} n_i = n \). The \( C_i \)'s represent \( \Phi \)-periodic conjugacy classes \( z_i \), and the relation \( \sum z_i = 0 \) holds in the abelianization of \( F_n \). Since \( z_4 \) may be expressed in terms of \( z_1, z_2, z_3 \), we get \( k \leq \sum_{i \leq 4} k_i = (\sum_{i \leq 4} k_i) - 1 \).

Note that \( \sum (n_i - 1) = n - 2 \) is possible only if \( n = 2 \) (and then the exponential stratum is unique).

- Now suppose that there are non-exponential strata above \( H \). Then \( Z \) contains all strata below \( H \), no edge of \( H \), and possibly edges from strata above \( H \).

Consider the union of \( H \) and all strata below it. By aperiodicity, this subgraph has a component containing \( H \), we call it \( Y^0 \). Then define connected subgraphs \( Y^0 \subset Y^1 \subset \cdots \subset Y^q = G \) such that \( Y^j \setminus Y^{j-1} \) contains exactly one edge.

If \( H \) is not geometric, the \( n_i \)'s are the Betti numbers of the non-contractible components of \( Z \). Define \( Z^j = Z \setminus Y^j \). Let \( n_i^j \) be the Betti numbers of the
non-contractible components of $Z^j$, and $n^j$ the Betti number of $Y^j$. Since $Y^0$ is $f$-invariant, the previous argument yields $\sum (n^0_i - 1) \leq n^0 - 3$ and $\sum (3n^0_i - 2) \leq 3n^0 - 6$. Using induction on $j$, one shows $\sum (n^q_i - 1) \leq n^q - 3$ and $\sum (3n^q_i - 2) \leq 3n^q - 6$ (when passing from $j$ to $j + 1$, the left hand sides cannot increase more than the right hand sides). This proves the proposition since $Z^q = Z$.

If $H$ is geometric, we again have to consider $\rho$ and $v$. We define $Z_j = (Z \cap Y_j) \cup \{v\}$. The point $v$ is an isolated point of $Z^0$. The numbers $n^j_i$ and $n^q_j$ are defined as before, except that we add 1 to the Betti number of the component $Z^i_j$ which contains $v$ (in particular, we always consider it as non-contractible). The ranks $n_i$ of the groups $G_i$ are the $n^q_j$'s (the exceptional group $G_b$ is generated by $\pi_1(Z^i_j)$ and the class of $\rho$).

The inequalities $\sum (n^j_i - 1) \leq n^j - 2$ and $\sum (3n^j_i - 2) \leq 3n^j - 5$ are true for $j = 0$, hence for $j = q$ by induction. If $\sum (3n^j_i - 2) = 3n^j - 5$ holds for $j = q$, it holds for $j = 0$ so there is a punctured sphere $S$ as above. Let $z$ be the conjugacy class represented by $\rho$. Removing $z$ from the set of periodic conjugacy classes does not change the subgroup generated in the abelianization of $F_n$. In other words, $z$ does not contribute to $k$. But it contributes to $k_b$ since it generates a free factor in $G_b$. We get $k < \sum k_i$, and the proposition is proved. \qed

4. Proof of the main results

In this section we give upper bounds for the invariants $e, s, d, p, r$ introduced in Section 2. Recall that we are free to replace $\Phi$ by a power (see Lemma 2.1 and the paragraph preceding it).

**Theorem 4.1.** Given $\Phi \in \text{Out}(F_n)$, let $P_j$ be representatives for the conjugacy classes of maximal polynomial subgroups. Then

$$e + \sum_j (\text{rk } P_j - 1)^+ \leq n - 1$$

$$4e + k + 2 \sum_j (\text{rk } P_j - 1)^+ \leq 3n - 2.$$  

Recall that $x^+ = \max(x, 0)$.

**Proof.** The result is trivially true if $\Phi$ is polynomially growing (with $e = 0$ and $k \leq n$), so we replace $\Phi$ by a power and we consider $T$, $G_i$, $n_i$, $\Phi_i$ as in 1.3.

If the action on $T$ is free, then by Lemma 1.2 there is no non-trivial polynomially growing conjugacy class (so $k = 0$) and no non-trivial polynomial subgroup. Furthermore, $e = 1$. The theorem is true in this case since $n \geq 2$, so we assume $b > 0$. We argue by induction on $n$.

We may assume that each $P_j$ is contained in a (unique) $G_{i,j}$. The groups $P_j$ contained in a given $G_i$ are non-conjugate maximal polynomial subgroups of $\Phi_i$.  

15
Successively using Lemma 2.6, the induction hypothesis, and Proposition 3.1, we now write:

\[ e + \sum_j (\text{rk } P_j - 1)^+ \leq 1 + \sum_{i=1}^b e_i + \sum_{i=1}^b (\sum_{j=i}^b (\text{rk } P_j - 1)^+) \]
\[ \leq 1 + \sum_{i=1}^b (n_i - 1) \]
\[ \leq n - 1 \]

and

\[ 4e + k + 2 \sum_j (\text{rk } P_j - 1)^+ \leq 4 + 4 \sum_{i=1}^b e_i + k + 2 \sum_{i=1}^b (\sum_{j=i}^b (\text{rk } P_j - 1)^+) \]
\[ \leq 4 + k + \sum_{i=1}^b (3n_i - 2 - k_i) \]
\[ \leq 4 + 3n - 6 \]
\[ \leq 3n - 2. \]

\[ \square \]

**Proposition 4.2.** If \( \Phi \in \text{Out}(F_n) \) is polynomially growing, then \( p + r \leq n - 1 \).

**Proof.** By induction on \( n \). After replacing \( \Phi \) by a power, we may assume that we are in the situation of Definition 1.1. There are two possibilities.

- First suppose that some \( \alpha \) representing \( \Phi \) preserves a non-trivial decomposition \( F_n = G_1 \ast G_2 \). Let \( n_i \) be the rank of \( G_i \). Define \( \alpha_i, \Phi_i, d_i, p_i, r_i \) by considering the automorphism induced on \( G_i \). Recall from Lemma 2.4 that if both \( \text{Fix } \alpha_1 \) and \( \text{Fix } \alpha_2 \) are non-trivial, \( r = r_1 + r_2 + 1 \) otherwise. Furthermore, \( d \leq \max(d_1, d_2) + 1 \) and \( p \leq \max(p_1, p_2) + 1 \).

  If \( r = r_1 + r_2 \), we write

  \[ p + r \leq p_1 + p_2 + 1 + r_1 + r_2 \leq n_1 - 1 + n_2 - 1 + 1 = n - 1, \]

  so we assume \( r = r_1 + r_2 + 1 \). By Lemma 2.4, we have \( d = 1 \) or \( d = \max(d_1, d_2) \). In both cases \( p = \max(p_1, p_2) \) and

  \[ p + r \leq p_1 + p_2 + r_1 + r_2 + 1 \leq n - 1. \]

- Now suppose that there is a decomposition \( F_n = G_1 \ast \langle t \rangle \) and a representative \( \alpha \) of \( \Phi \) which leaves \( G_1 \approx F_{n-1} \) invariant and maps \( t \) to \( tu \) with \( u \in G_1 \). As in 1.1, we assume that \( u \) cannot be written as \( a\alpha(a^{-1}) \) with \( a \in G_1 \). In particular, \( u \neq 1 \). Define \( \alpha_1, \Phi_1, d_1, p_1, r_1 \) by restricting to \( G_1 \). Recall (Lemma 2.5) that \( r \leq r_1 + 1 \), with equality if and only if both \( \alpha_1 \) and \( i_u \circ \alpha_1 \) have non-trivial fixed subgroups, and \( p \leq p_1 + 1 \).
We show that \( r = r_1 + 1 \) implies \( p = p_1 \), so that in all cases
\[
p + r \leq p_1 + r_1 + 1 \leq n_1 - 1 + 1 = n - 1.
\]

Suppose \( r = r_1 + 1 \). Since \( \text{Fix}(i_u \circ \alpha_1) \) is non-trivial, there is a non-trivial \( \alpha \)-fixed element of the form \( txt^{-1} \), and \( u \) is \( \alpha \)-fixed by property ne-(iii) of [1, Theorem 5.1.5]. In particular, the element \( t \) grows linearly under \( \alpha \). If \( d \geq 2 \), some element or conjugacy class of \( G_1 \) grows with degree \( d \), and \( d = d_1 \) by Lemma 2.3. If \( d \leq 1 \), we have \( p = p_1 \). \( \square \)

Combining Theorem 4.1 and Proposition 4.2, we deduce:

**Theorem 4.3.** Given any \( \Phi \in \text{Out} (F_n) \), we have:
\[
e + p + r \leq n - 1
4e + 2p + 2r + k \leq 3n - 2.
\]

**Proof.** By Theorem 4.1, it suffices to show \( p + r \leq \sum_j (\text{rk} P_j - 1)^+ \). We may replace \( \Phi \) by a power, and so assume that each \( P_j \) is fixed (up to conjugacy). Since \( P_j \) equals its normalizer, there is a well-defined induced \( \Phi_j \), with associated \( p_j \) and \( r_j \), and \( p = \max p_j \).

Let \( \alpha \) be a representative of \( \Phi \). If \( \text{Fix} \alpha \) is non-trivial, it is contained in a conjugate of some \( P_j \), and by changing \( \alpha \) in its isogredience class we may assume \( \text{Fix} \alpha \subset P_j \). This implies that \( P_j \) is \( \alpha \)-invariant (by uniqueness of maximal polynomial subgroups). Furthermore, if \( \text{Fix} \alpha \) and \( \text{Fix} \alpha' \) are contained in \( P_j \), then the restrictions of \( \alpha \) and \( \alpha' \) to \( P_j \) represent the same outer automorphism (if \( \alpha' = i_h \circ \alpha \), then \( P_j \) is \( i_h \)-invariant, so \( h \in P_j \) because \( P_j \) equals its normalizer); if \( P_j \) has rank \( \geq 2 \), and \( \alpha, \alpha' \) are not isogredient, the restrictions are not isogredient (see [9] or the proof of Lemma 2.1).

Write \( r = \sum r_0(\alpha_m) \), the sum being taken over non-isogredient automorphisms \( \alpha_m \). We may assume \( \text{Fix} \alpha_m \subset P_{j_m} \), with \( P_{j_m} \) of rank \( \geq 2 \). Now
\[
p + r \leq \sum_j p_j + \sum_j \sum_{\text{Fix} \alpha_m \subset P_j} r_0(\alpha_m) \leq \sum_j p_j + \sum_j r_j \leq \sum (\text{rk} P_j - 1)^+
\]
by Proposition 4.2. \( \square \)

The inequality \( 4e + 2p + 2r + k \leq 3n - 2 \) may be an equality, for instance for the automorphism \( \alpha \) of \( F_3 \) defined by \( a \mapsto a, b \mapsto b, c \mapsto aca^{-1} \), with \( n = k = 3, d = 1, r = 2 \). Note that \( r(\hat{\alpha}) = 2 \), but the contribution of \( \alpha \) is \( r_0(\alpha) = (\text{rk Fix} \alpha - 1)^+ = 1 \), as \( \text{Fix} \alpha \) has rank 2. The other non-zero contribution is from \( \alpha' = i_{a^{-1}} \circ \alpha \), whose fixed subgroup also has rank 2. The point of the next theorem is that this is a general phenomenon: if \( \Phi \in \text{Out} (F_n) \) satisfies \( 4e + 2p + 2r + k = 3n - 2 \), and \( d > 0 \), then \( r \) has to be carried by at least two isogredience classes (no representative \( \alpha \) of \( \Phi \) satisfies \( \text{rk} \text{Fix} \alpha = r + 1 \)).
Theorem 4.4. Given any $\alpha \in \mathrm{Aut}(F_n)$, we have

$$4e + 2d + 2\mathrm{rk}\, \mathrm{Fix}\, \alpha + k \leq 3n + 1.$$ 

Note that Theorem 4.3, together with the inequalities $\mathrm{rk}\, \mathrm{Fix}\, \alpha \leq r + 1$ and $d \leq p + 1$, implies $4e + 2d + 2\mathrm{rk}\, \mathrm{Fix}\, \alpha + k \leq 3n + 2$.

Proof. It suffices to show

$$4e + 2p + 2\mathrm{rk}\, \mathrm{Fix}\, \alpha + k = 3n \implies d = 0,$$  

(*) 

since any $\alpha$ satisfying $4e + 2d + 2\mathrm{rk}\, \mathrm{Fix}\, \alpha + k = 3n + 2$ must satisfy $d = p + 1$ and therefore $4e + 2p + 2\mathrm{rk}\, \mathrm{Fix}\, \alpha + k = 3n$. The proof of (*) is by induction on $n$.

Consider $\alpha$ satisfying $4e + 2p + 2\mathrm{rk}\, \mathrm{Fix}\, \alpha + k = 3n$. Note that $\mathrm{Fix}\, \alpha$ must have rank $r + 1$. We argue by way of contradiction, assuming $d > 0$. By Remark 2.2, we may also assume $r > 0$ (we are free to replace $\alpha$ by a power since $e, p, k$ do not change and $\mathrm{rk}\, \mathrm{Fix}\, \alpha$ may only increase). In particular, $\mathrm{Fix}\, \alpha$ has rank $r + 1 \geq 2$.

First suppose that $\alpha$ is polynomially growing. Note that in this case $4e + 2p + 2\mathrm{rk}\, \mathrm{Fix}\, \alpha + k = 3n$ together with $d = 0$ imply that $\alpha$ is the identity, since $\alpha$ has a power which is inner by Lemma 1.7, and $\mathrm{rk}\, \mathrm{Fix}\, \alpha = k = n$.

After raising $\alpha$ to a power, we are in the situation of Definition 1.1. The representative of $\hat{\alpha}$ introduced in Definition 1.1 is not necessarily equal to $\alpha$, so we denote it by $\beta$. As usual, we distinguish two cases.

- First suppose that there is a decomposition $F_n = G_1 \ast G_2$ such that $\beta$ leaves each $G_i$ invariant. The main difficulty is to show that we may assume $\beta = \alpha$.

  Consider the action of $F_n$ on the simplicial tree $T_0$ associated to the free product $G_1 \ast G_2$. This action is $\hat{\alpha}$-invariant, and there is an isometry $H$ of $T_0$ representing $\alpha$ in the sense that $\alpha(g)H = Hg$ for all $g \in F_n$ (see [9]). We have to show that $H$ fixes an edge $e$: if it does, the stabilizers $G_1', G_2'$ of the endpoints of $e$ are $\alpha$-invariant, and $\alpha$ preserves the free product decomposition $F_n = G_1' \ast G_2'$.

  Since $\mathrm{Fix}\, \alpha$ has rank $\geq 2$, the map $H$ has a fixed point [9, 1.1]. The stabilizer of this point is $\alpha$-invariant, so we may assume $\alpha(G_1) = G_1$ (possibly exchanging the roles of $G_1$ and $G_2$ and replacing $\alpha$ by an isogredient automorphism). By an argument given in the proof of 4.3, $\alpha$ and $\beta$ induce the same outer automorphism $\Phi_1$ of $G_1$ because $G_1$ equals its normalizer. The numbers $k_i, p_i, r_i$ used below refer to $\Phi_1$ and to $\Phi_2 = \beta|_{G_2}$.

  First suppose $\mathrm{Fix}\, \beta \subset G_1$ (this holds in particular if $H$ has a unique fixed point). Then

$$3n = 2\mathrm{rk}\, \mathrm{Fix}\, \alpha + k + 2p$$

$$\leq 2\mathrm{rk}\, \mathrm{Fix}\, \alpha|_{G_1} + k_1 + k_2 + 2p_1 + 2p_2 + 2$$

$$\leq 3n_1 + 3n_2 - 2 + 2$$

$$= 3n,$$

using Lemma 2.4 and Theorem 4.3.
All inequalities must be equalities. In particular $d(\Phi_1) = 0$ because $(\ast)$ is true on $G_1$ by induction, and as pointed out earlier this implies that $\alpha$ is the identity on $G_1$. Also $2p = 2p_1 + 2p_2 + 2$, so $d \geq 2$. But $r_2 = 0$, because $r_1 + r_2 \leq r = \text{rk Fix}_\alpha - 1 = r_1$ since $\text{Fix} \alpha = G_1$. Remark 2.2 implies $d_2 = 0$, contradicting Lemma 2.4.

This proves that $\text{Fix} \alpha \subset G_1$ cannot hold. Therefore $H$ fixes an edge, so we may indeed assume $\alpha = \alpha_1 \ast \alpha_2$. Then $\text{Fix} \alpha = \text{Fix} \alpha_1 \ast \text{Fix} \alpha_2$. Since we have ruled out $\text{Fix} \alpha \subset G_i$, both $\text{Fix} \alpha_1$ and $\text{Fix} \alpha_2$ are nontrivial. As in the proof of Theorem 4.2, this implies $p = \max(p_1, p_2)$.

We now write:

$$3n = 2 \text{rk Fix} \alpha + k + 2p$$

$$\leq 2 \text{rk Fix} \alpha_1 + 2 \text{rk Fix} \alpha_2 + k_1 + k_2 + 2p_1 + 2p_2$$

$$\leq 3n_1 + 3n_2$$

$$\leq 3n.$$

Applying $(\ast)$ to $\alpha_1$ and $\alpha_2$ (using the induction hypothesis), we find that $\alpha_1$ and $\alpha_2$ are the identity, so $\alpha$ is the identity.

- Now consider a decomposition $F_n = G_1 \ast \langle t \rangle$ as in Definition 1.1, with $\beta$ representing $\hat{\alpha}$ such that $\beta(G_1) = G_1$ and $\beta(t) = tu$. The numbers $k_1, p_1$ refer to $\beta|_{G_1}$. We have shown the inequality $p + r \leq p_1 + r_1 + 1$ in the proof of Proposition 4.2. Using Theorem 4.3, we deduce

$$3n - 2 = 2r + k + 2p$$

$$\leq 2r_1 + k_1 + 2p_1 + 2 + k - k_1$$

$$\leq 3n_1 - 2 + 2 + k - k_1$$

$$\leq 3n - 3 + k - k_1,$$

showing $k \geq k_1 + 1$. By Lemma 2.5, some decomposition $F_n = G_1 \ast \langle ta \rangle$ is invariant under a power of $\beta$, and we reduce to the previous case.

The proof of Theorem 4.4 is now complete for polynomially growing automorphisms. In the general case, we consider the invariant $R$-tree $T$ as in 1.3.

All elements growing polynomially under $\alpha$ are contained in an $\alpha$-invariant stabilizer, say $G_1$. Let $\alpha_1 = \alpha|_{G_1}$, so that $\text{Fix} \alpha = \text{Fix} \alpha_1$ and $r(\hat{\alpha}) = r(\hat{\alpha}_1)$. All $\Phi_i$’s with $i \geq 2$ have $r_i = 0$ by Lemma 2.6, hence $d_i = 0$ by Remark 2.2. Since $d = \max d_i$, it suffices to show $d_1 = 0$.

Writing

$$4e + 2 \text{rk Fix} \alpha + k + 2p \leq 4 + 4 \sum e_i + 2 \text{rk Fix} \alpha_1 + k - \sum k_i + \sum k_i + 2p_1$$

$$\leq (4e_1 + 2 \text{rk Fix} \alpha_1 + k_1 + 2p_1) + \sum_{i > 1} (4e_i + k_i) + 4 + k - \sum_{i \geq 1} k_i$$

$$\leq (4e_1 + 2 \text{rk Fix} \alpha_1 + k_1 + 2p_1) + \sum_{i > 1} (3n_i - 2) + 4 + k - \sum_{i \geq 1} k_i$$

$$\leq (4e_1 + 2 \text{rk Fix} \alpha_1 + k_1 + 2p_1) + 3n - 3n_1,$$
we get $4e_1 + 2\text{rk} \text{Fix} \alpha_1 + k_1 + 2p_1 = 3n_1$, and $d_1 = 0$ by the induction hypothesis.

By Corollary 1.6, any $\Phi$ with $d > 0$ has a power $\Phi^q$ represented by an automorphism $\alpha$ with $\text{rk} \text{Fix} \alpha \geq 2$. In particular, $\Phi^q$ satisfies $d \leq p + r$. We thus get from Theorems 4.3 and 4.4:

**Corollary 4.5.** Given $\Phi \in \text{Out}(F_n)$, we have

- $e + d \leq n - 1$
- $4e + 2d \leq 3n - 2$
- $4e + 2d \leq 3n - 3$ if $d > 0$.

As is easily checked, this corollary is equivalent to saying that $(e, d)$ belongs to the closed quadrilateral with vertices $(0, 0), (0, n-1), (\frac{n-1}{2}, \frac{n-1}{2}), (\frac{3n-2}{4}, 0)$ pictured on Figure 1.

We also get:

**Corollary 4.6.** Given $\alpha \in \text{Aut}(F_n)$, we have

- $e + (d-1)^+ + \text{rk} \text{Fix} \alpha \leq n$
- $4e + 2d + 2\text{rk} \text{Fix} \alpha \leq 3n + 1$ (≤ 3n if $d = 0$).

We finally prove:

**Theorem 4.7.** If $\Phi \in \text{Out}(F_n)$ is not polynomially growing, one has $2s + p + r \leq n - 2$. In particular, $s \leq \frac{n}{2} - 1$.

It follows from the appendix that $m \leq s$ if a conjugacy class grows like $\lambda^pp^n$, with $\lambda > 1$, under iteration of $\Phi$. The optimality of $s \leq \frac{n}{2} - 1$ will be shown in the next section.

**Proof.** The result is true if $s = 0$, by Theorem 4.3, so we assume $s > 0$. We argue by induction on $n$, using the invariant $\mathbb{R}$-tree $T$ as in 1.3. Since $s > 0$, there is at least one $i_0 \in \{1, \ldots, b\}$ such that $\Phi_{i_0}$ is not polynomially growing. By Lemma 2.6, there is such an $i_0$ with $s \leq 1 + s_{i_0}$.

We then get

$$2s + p + r \leq 2s_{i_0} + 2 + \sum_{i=1}^{b} p_i + \sum_{i=1}^{b} r_i \leq n_{i_0} + \sum_{i \neq i_0}(n_i - 1) = 1 + \sum_{i=1}^{b}(n_i - 1)$$

by the induction hypothesis and Theorem 4.3. Since $s > 0$, Proposition 3.1 yields $\sum_{i=1}^{b}(n_i - 1) < n - 2$ and the theorem is proved.

**Corollary 4.8.** If $\alpha \in \text{Aut}(F_n)$ is not polynomially growing, one has $2s + d + \text{rk} \text{Fix} \alpha \leq n$ and $2s + d \leq n - 2$.

**Proof.** The first inequality is clear. The second follows from Remark 2.2.
5. Examples

We give examples, and we show that the inequalities of Corollaries 4.5 and 4.6 are optimal.

**Automorphisms of $F_2$.**

Any $\Phi \in \text{Out} \ (F_2)$ is induced by a homeomorphism of a punctured torus. Some power of $\Phi$ is either the identity, or a Dehn twist, or a pseudo-Anosov map.

The simplest pseudo-Anosov automorphism is $a \mapsto ab, \ b \mapsto a$. For future reference, we note that its square $\tau$, which sends $a$ to $aba$ and $b$ to $ba$, fixes the commutator $[a, b] = aba^{-1}b^{-1}$.

**Geometric automorphisms.**

An automorphism $\Phi \in \text{Out} \ (F_n)$ is geometric if it is induced by a homeomorphism of a compact surface $\Sigma$ with fundamental group $F_n$. For $\Phi$ geometric, it follows from Nielsen-Thurston theory that the growth of any non-periodic conjugacy class under iteration of $\Phi$ (always equivalent to some $\lambda^p m^m$) is either linear or purely exponential: $(\lambda, m) = (1, 1)$, or $\lambda > 1$ and $m = 0$.

We now construct a geometric automorphism $\Phi_n$ of $F_n$ with $e$ equal to the maximal value $e_n = \left\lfloor \frac{3n-2}{4} \right\rfloor$. It has a representative $\varphi_n \in \text{Aut} \ (F_n)$ with non-trivial fixed subgroup.

Write $n = 4\ell + 3 + \delta$ with $0 \leq \delta \leq 3$ (for $n = 2$, we take $\Phi_2 = \hat{\tau}$). Construct a compact surface $\Sigma_n$ with fundamental group $F_n$ by gluing $2\ell + \delta$ once-punctured tori and $\ell + 1$ four-punctured spheres (see Figure 2 for a picture with $\delta = 3$). The number of subsurfaces is $e_n = 3\ell + 1 + 1 = \left\lfloor \frac{3n-2}{4} \right\rfloor$. Consider an orientation-preserving homeomorphism of $\Sigma_n$ inducing a pseudo-Anosov map on each of the subsurfaces. The induced automorphism $\Phi_n \in \text{Out} \ (F_n)$ satisfies $e(\Phi_n) = e_n$.

If all pseudo-Anosov maps used in the construction have distinct dilation factors $\lambda_i$, then $\Phi_n$ has $e_n$ different exponential growth types $(\lambda_i, 0)$.

This example explains the appearance of four-punctured spheres in the proof of Proposition 3.1.

**Remark 5.1.** If $\Phi \in \text{Out} \ (F_n)$ is induced by a homeomorphism of a compact orientable surface of genus $g$ with $b$ boundary components, one has $n = 2g + b - 1$ and $k \geq b - 1$, so Theorem 4.1 yields $e \leq \frac{3g+b-2}{2}$. It is easy to see that this bound is optimal.

**Nested laminations.**

The inequality $s \leq \frac{n}{2} - 1$ of Theorem 4.7 is an equality for the automorphism $\alpha$ of $F_{2\ell}$ defined by:
The length of $\alpha^p(a_\ell)$ (and of its conjugacy class) is equivalent to $p^{\ell-1} \lambda^p$, with $\lambda$ the Perron-Frobenius eigenvalue of \[
abla = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.
\]

**Polynomial growth.**

Let $n \geq 2$. For the automorphism $\alpha_n$ of $F_n$ defined by $\alpha_n(a_1) = a_1$ and $\alpha_n(a_i) = a_ia_{i-1}$ for $2 \leq i \leq n$, both the element $a_i$ and the conjugacy class $\bar{a}_i$ grow polynomially with degree $i-1$. In particular, $d(\alpha_n)$ equals the maximal value $n-1$ (so $\alpha$ is not geometric for $n \geq 3$). The rank of Fix $\alpha_n$ is 2 (it is generated by $a_1$ and $a_2a_1a_2^{-1}$).

We need a slightly different example, with every generator but one mapped to a conjugate. We write $x^y$ for $yxy^{-1}$.

**Lemma 5.2.** Let $\ell \geq 1$. For the automorphism $\beta_\ell$ of $F_{\ell+2} = \langle a, a_0, a_1, \ldots, a_\ell \rangle$ defined by:

\[
\begin{align*}
    a &\mapsto a \\
    a_0 &\mapsto a_0a \\
    a_1 &\mapsto a_1{a_0}a \\
    a_2 &\mapsto a_2{a_1}a \\
    \quad &\vdots \\
    a_\ell &\mapsto a_\ell{a_{\ell-1}}a,
\end{align*}
\]

the conjugacy class of $aa_\ell$ grows with degree $\ell + 1$. If one adds a generator $t$ with $t \mapsto t a_\ell a$, the class of $t$ grows with degree $\ell + 2$.

**Remark.** If an automorphism of $F_n$ maps every generator to a conjugate, no conjugacy class grows polynomially with degree $n - 1$. This follows from Theorem 4.4 and Corollary 1.6.

**Proof.** One first shows by induction on $i \geq 1$ that no cancellation occurs when computing iterates $\beta^p_\ell(a_i)$, because the initial letter of $\beta^p_\ell(a_i)$ is $a_{i-p}$ for $p \leq i - 1$, and $a_0$ for $p \geq i$ (and the final letter is the inverse of the initial one). The length
of the (non cyclically reduced word) $\beta^p_\ell(a_\ell)$ is the $\ell^1$-norm of the vector

$$
\begin{pmatrix}
  n_a \\
  n_0 \\
  n_1 \\
  \vdots \\
  n_\ell
\end{pmatrix} =
\begin{pmatrix}
  1 & 1 & 2 & \ldots & 2 \\
  1 & 2 & & & \\
  & & \ddots & \ddots & \\
  & & & 1 & 2 \\
  & & & & 1
\end{pmatrix}^p
\begin{pmatrix}
  0 \\
  \vdots \\
  \vdots \\
  0 \\
  1
\end{pmatrix}.
$$

It grows with degree $\ell + 1$. The word $\beta^p_\ell(aa_\ell) = a\beta^p_\ell(a_\ell)$ is cyclically reduced, so the class of $aa_\ell$ grows with degree $\ell + 1$.

The $p$-th iterate of the new generator $t$ is a cyclically reduced word containing $a_\ell, \beta_\ell(a_\ell), \ldots, \beta^{p-1}_\ell(a_\ell)$ as disjoint subwords, so the class of $t$ grows with degree $\ell + 2$.

\[\square\]

**Mixed growth.**

For $n \geq 3$, we construct an automorphism $\theta_n$ of $F_n$ with $e$ and $d$ as close as possible to $(n - 1)/2$. Its fixed subgroup has rank 2.

First assume $n$ odd, and write $n = 2\ell+3$. Consider $F_n = \langle a, b, a_0, a_1, b_1, \ldots, a_\ell, b_\ell \rangle$. Let $u = [a, b]$ and $u_i = [a_i, b_i]$. Recall that $\tau : (a, b) \mapsto (aba, ba)$ is an exponentially growing automorphism fixing $u = [a, b]$. Define $\theta_n \in \text{Aut}(F_n)$ by:

$$
\begin{align*}
  a &\mapsto aba \\
  b &\mapsto ba \\
  a_0 &\mapsto a_0u \\
  a_1 &\mapsto (a_1b_1a_1)^{a_0u} \\
  b_1 &\mapsto (b_1a_1)^{a_0u} \\
  a_2 &\mapsto (a_2b_2a_2)^{u_1u} \\
  b_2 &\mapsto (b_2a_2)^{u_1u} \\
  \vdots \\
  a_\ell &\mapsto (a_\ell b_\ell a_\ell)^{u_{\ell-1}u} \\
  b_\ell &\mapsto (b_\ell a_\ell)^{u_{\ell-1}u}
\end{align*}
$$

Geometrically, $\theta_n$ is represented by a homotopy equivalence $\psi$ on a 2-complex $X_n$, built as follows. Take disjoint punctured tori $T, T_1, \ldots, T_\ell$, with points $v, v_1, \ldots, v_\ell$ on the boundary. Glue a circle to $v$, and add edges $vv_i$. The map $\psi$ induces a pseudo-Anosov homeomorphism on each punctured torus.

The automorphism $\theta_n$ has $\ell + 1$ exponential strata. Furthermore, consider the subgroup $P_0$ generated by $u, a_0, u_1, \ldots, u_\ell$ (it is the fundamental group of the 1-complex obtained from $X_n$ by removing the interior of each torus). It is $\theta_n$-invariant, and the restriction of $\theta_n$ to $P_0$ is $\beta_\ell$. Thus $\theta_n$ satisfies $e = d = \ell + 1 = \dfrac{n-1}{2}$ by Lemma 5.2. Both inequalities of Corollary 4.5 are equalities.
If \( n \) is even, we write \( n = 2\ell + 4 \) and we add a generator \( t \) mapped to \( tu\ell u \). We get an automorphism with \( e = \frac{n}{2} - 1 \) and \( d = \frac{n}{2} \) by Lemma 5.2.

The inequality \( e + d \leq n - 1 \) of Corollary 4.5 is an equality.

These examples have only one exponential growth type \((\lambda, 0)\), with \( \lambda \) the Perron-Frobenius eigenvalue of \( \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \). It is easy to modify the construction so that there are \( e \) distinct exponential growth types \((\lambda^i, 0)\), \( 0 \leq i \leq \ell \), by using \( \tau^i+1 \) rather than \( \tau \) when defining the image of \( a_i \) and \( b_i \).

**Optimality.**

We can now show:

**Theorem 5.3.** Given \((\varepsilon, \delta)\) belonging to the closed quadrilateral with vertices \((0,0), (0,n-1), (n/2,n/2), (3n/4,0)\), there exists \( \alpha \in \text{Aut} \left( F_n \right) \) such that:

1. Any improved relative train track map representing a power of \( \hat{\alpha} \) has \( \varepsilon \) exponential strata. Equivalently, there are \( \varepsilon \) attracting laminations.
2. There are \( \varepsilon \) distinct exponential growth rates \((\lambda_i, 0)\) with \( \lambda_i > 1 \).
3. There is a conjugacy class whose growth is polynomial of degree \( \delta \).
4. The rank of \( \text{Fix} \alpha \) is the maximal value \( \rho_0 \) permitted by the inequalities of Corollary 4.6.

**Proof.** We write \( \rho \) for \( \text{rk} \text{Fix} \alpha \). The inequalities of Corollary 4.6 are

\[
\varepsilon + (\delta - 1) + \rho \leq n
\]
\[
4\varepsilon + 2\delta + 2\rho \leq 3n + 1 \quad (\leq 3n \text{ if } \delta = 0).
\]

Since \((\varepsilon, \delta)\) is in the quadrilateral, the maximal value \( \rho_0 \) is always \( \geq 1 \). Which of the two inequalities is the limiting one depends on the position of \( \varepsilon \) with respect to \( n/2 \) if \( \delta = 0 \), with respect to \( (n - 1)/2 \) if \( \delta > 0 \).

We shall construct \( \alpha \) satisfying (1), (3), (4), using the automorphisms \( \tau, \varphi_n, \alpha_n, \theta_n \) introduced above. As in the construction of \( \theta_n \), one achieves (2) by using varying powers of \( \tau \).

We write \( I_\ell \) for the identity automorphism of \( F_\ell \).

- If \( \varepsilon = 0 \) and \( \delta > 0 \), we define \( \alpha = \alpha_{\delta+1} \circ I_{n-\delta-1} \), with \( \alpha_{\delta+1} \in \text{Aut} \left( F_{\delta+1} \right) \) defined above.
- Suppose \( \delta = 0 \). The inequalities are \( \varepsilon + \rho \leq n \) and \( 4\varepsilon + 2\rho \leq 3n \).

If \( \varepsilon \leq \frac{n}{2} \), we write \( F_n \) as the free product of \( F_{n-2\varepsilon} \) and \( \varepsilon \) copies of \( F_2 \), and we define \( \alpha = I_{n-2\varepsilon} \circ \tau \circ \cdots \circ \tau \). Then \( \rho = n - 2\varepsilon + \varepsilon = \rho_0 \).

If \( \varepsilon > \frac{n}{2} \), first suppose \( n \) is even, say \( n = 2\ell \). The required value of \( \rho \) is \( \rho_0 = \frac{3n-4\varepsilon}{2} = 3\ell - 2\varepsilon \). Write \( F_n \) as the free product of \( F_{4\varepsilon-4\ell+2} \) with \( 3\ell - 2\varepsilon - 1 \) copies of \( F_2 \) and define \( \alpha = \varphi_{4\varepsilon-4\ell+2} \circ \tau \circ \cdots \circ \tau \). The number of exponential strata is \( 3\varepsilon - 3\ell + 1 + 3\ell - 2\varepsilon - 1 = \varepsilon \), and \( \rho = 1 + 3\ell - 2\varepsilon - 1 = \rho_0 \). If \( n = 2\ell + 1 \), we take the free product of \( \varphi_{4\varepsilon-4\ell+1} \) with \( 3\ell - 2\varepsilon \) copies of \( \tau \). Then \( \varepsilon = 3\varepsilon - 3\ell + 3\ell - 2\varepsilon = \varepsilon \) and \( \rho = 1 + 3\ell - 2\varepsilon = \rho_0 \).

We now suppose \( \delta, \varepsilon \geq 1 \). This implies \( n \geq 3 \) and \( \rho_0 \geq 2 \).
• Suppose \( \delta, \varepsilon \geq 1 \), and \( \varepsilon \leq (n - 1)/2 \). Then \( \rho_0 = n - \varepsilon - \delta + 1 \).

The construction uses auxiliary parameters \( w, x, y, z \geq 0 \), to be determined in terms of \( \delta \) and \( \varepsilon \).

Starting with \( \theta_{2w+3} \), for which \( d = e = w + 1 \) and \( \rho = 2 \), we add \( x \) generators so as to obtain an automorphism of \( F_{2w+3+x} \) with \( e = w + 1 \) and \( d = w + 1 + x \) (map the first generator \( t_1 \) to \( t_1u \) as in Lemma 5.2, then \( t_i \) to \( t_it_{i-1} \) as in the definition of \( \alpha_n \)). We then take the free product with \( I_y \) and with \( z \) copies of \( \tau \).

We get an automorphism of a group of rank \( 2w+3+x+y+2z \), with \( d = w+1+x \), \( e = w + 1 + z \), and \( \rho = 2 + y + z \). We must therefore find \( w, x, y, z \geq 0 \) satisfying

\[
\begin{align*}
w + 1 + x &= \delta \\
w + 1 + z &= \varepsilon \\
2 + y + z &= \rho_0 = n - \varepsilon - \delta + 1 \\
2w + 3 + x + y + 2z &= n.
\end{align*}
\]

Note that the last equation is the sum of the others. If we know \( w \), we get \( x, y, z \) by

\[
\begin{align*}
x &= \delta - w - 1 \\
y &= \rho_0 - \varepsilon + w - 1 \\
z &= \varepsilon - w - 1.
\end{align*}
\]

We have to choose \( w \) so that \( w, x, y, z \) are non-negative. This is equivalent to \( w \geq 0, w \geq \varepsilon - \rho_0 + 1, w \leq \delta - 1, w \leq \varepsilon - 1 \). Since \( \delta \) and \( \varepsilon \) are \( \geq 1 \), and \( \rho_0 \geq 2 \), we only need to check \( \varepsilon - \rho_0 + 1 \leq \delta - 1 \). This holds because \( \rho_0 = n - \varepsilon - \delta + 1 \) and \( \varepsilon \leq (n - 1)/2 \).

• Finally, suppose \( \delta, \varepsilon \geq 1 \), and \( \varepsilon > (n - 1)/2 \). Then \( \rho_0 = \left\lceil \frac{3n + 1 - 4\varepsilon - 2\delta}{2} \right\rceil \).

First assume that \( n \) is odd. We use three parameters \( w, x, z \). We first combine \( \varphi_{4x+2} \) with \( \theta_{2w+3} \), in the following sense. We consider \( F_{4x+2w+3} = F_{4x+2} \ast \langle a_0, a_1, b_1, \ldots, a_w, b_w \rangle \). Let \( u \) be a generator for the fixed subgroup of \( \varphi_{4x+2} \). We define an automorphism of \( F_{4x+2w+3} \) as being equal to \( \varphi_{4x+2} \) on the first factor and mapping \( a_0, a_1, b_1, \ldots, a_w, b_w \) by the same formulas as in the definition of \( \theta_{2w+3} \). This automorphism has \( e = 3x + 1 + w \), \( d = w + 1 \), and \( \rho = 2 \). We then take the free product with \( z \) copies of \( \tau \), so as to increase \( e \) and \( \rho \) by \( z \).

We now have to solve:

\[
\begin{align*}
w + 1 &= \delta \\
3x + 1 + w + z &= \varepsilon \\
2 + z &= \rho_0 \\
4x + 2w + 3 + 2z &= n.
\end{align*}
\]

We have assumed \( n \) to be odd. Setting \( n = 2\ell + 1 \), we have \( \rho_0 = 3\ell + 2 - 2\varepsilon - \delta \). It is easy to check that \( w = \delta - 1, z = \rho_0 - 2, x = \varepsilon - \ell \) is a non-negative solution.
For $n = 2\ell + 2$, we use the same construction with $\varphi_{4\ell+1}$ rather than $\varphi_{4\ell+2}$, defining $w, z, x, y$ by the exact same formulas (now $\rho_0 = 3\ell + 3 - 2\varepsilon - \delta$, the second equation is $3x + w + z = \varepsilon$, and the fourth one is $4x + 2w + 2 + 2z = n$). \hfill \Box

6. Appendix: Growth

More on train tracks.

Let $f : G \to G$ be an improved relative train track map. We recall some more definitions from [1].

We write $f^p(\gamma)$ for the tightened image of $\gamma$ (the reduced path homotopic to $f(\gamma)$ rel. endpoints). A decomposition $\gamma = \gamma_1 \ldots \gamma_q$ is a splitting if $f^p(\gamma) = f^p(\gamma_1) \ldots f^p(\gamma_q)$ for all $p \geq 1$ (i.e. there is no cancellation between $f^p(\gamma_j)$ and $f^p(\gamma_{j+1})$). The subpaths $\gamma_j$ are the pieces of the splitting, and we say that $\gamma$ splits over each $\gamma_j$.

If $H_i$ is an NEG stratum, it consists of a single edge $e_i$, and $f(e_i)$ splits as $e_i, u_i$ with $u_i$ of height $< i$. If $e$ is an edge in an exponential stratum $H_i$, then $f(e)$ has a splitting whose pieces are edges of $H_i$ or paths of height $< i$.

Let $H_i$ be exponential. By aperiodicity, every edge of $H_i$ appears in $f(e)$, for $e$ any edge of $H_i$. Any subpath of $f^p(e)$ is $i$-legal (for the purposes of this appendix, this may be taken as the definition of $i$-legal). We call $\Delta_i$ the (finite) collection of maximal subpaths of height $< i$ which appear in $f(e)$, for $e$ an edge of $H_i$. If $\delta \in \Delta_i$, then no $f^p(\delta)$ is a point. It follows from bounded cancellation that there exists a constant $K_i$ such that, if a path $\gamma$ contains an $i$-legal subpath with more than $K_i$ edges in $H_i$, then $\gamma$ splits over an edge of $H_i$ (see 4.2 in [1]).

As mentioned earlier, there is a bijection between the set of attracting laminations of the automorphism represented by $f$ and the set of exponential strata of $f$. A bi-infinite path $\gamma$ in $G$ is a leaf of the lamination associated to $H_i$ if and only if any finite subpath is contained in some $f^p(\gamma)$, with $\epsilon$ an edge of $H_i$.

**Lemma 6.1.** Let $f$ be an improved relative train track map. Let $H_i, H_j$ be exponential strata, with associated laminations $\Lambda_i, \Lambda_j$. The following are equivalent:

1. $\Lambda_j \subset \Lambda_i$.
2. There exist edges $e_i, e_j$ of $H_i, H_j$, and $p \geq 1$, such that $f^p(e_i)$ splits over $e_j$.
3. Given edges $e_i, e_j$ of $H_i, H_j$, there exists $p \geq 1$ such that $f^p(e_i)$ splits over $e_j$.

**Proof.** (2) $\iff$ (3) follows from aperiodicity. If $f^p(e_i)$ splits over $e_j$, every $f^q(e_j)$ is contained in $f^{p+q}(e_i)$, so every leaf of $\Lambda_j$ is a leaf of $\Lambda_i$. Conversely, if $\Lambda_j \subset \Lambda_i$, let $\gamma_0$ be a segment in a leaf of $\Lambda_j$. It is contained in some $f^p(e_i)$, and by bounded cancellation (see above) $f^p(e_i)$ splits over some edge of $H_j$ contained in $\gamma_0$ if $\gamma_0$ is long enough. \hfill \Box
Remark. In particular, \(j \leq i\) if \(\Lambda_j \subset \Lambda_i\). But the total order on the set of exponential strata of \(f\) defined by \(H_i \leq H_j\) if \(i \leq j\) does not have an intrinsic meaning.

The Perron-Frobenius eigenvalue \(\lambda_i\) of \(H_i\) is called the expansion factor of \(\Lambda_i\) (for the automorphism represented by \(f\)).

Given \(\Phi \in \text{Out}(F_n)\), it is only true that some power \(\Phi^q\) is represented by \(f : G \to G\) as above. The attracting laminations of \(\Phi\) are those of \(\Phi^q\), and we define the expansion factor of \(\Lambda\) for \(\Phi\) as \(\lambda^{1/q}\), where \(\lambda\) is the expansion factor of \(\Lambda\) for \(\Phi^q\).

**Growth types.**

Given \(\lambda \geq 1\) and \(m \in \mathbb{N}\), we say that a conjugacy class \(\overline{g}\) grows like \(\lambda^p p^m\) under \(\Phi\), or has growth type \((\lambda, m)\), if the length \(|\Phi^p(\overline{g})|\) grows like \(\lambda^p p^m\) in the sense that there exist constants \(C_1, C_2 > 0\) with \(C_1 \lambda^p p^m \leq |\Phi^p(\overline{g})| \leq C_2 \lambda^p p^m\) for all \(p \geq 1\). The set of growth types is ordered lexicographically, so that \((\lambda, m) \leq (\lambda', m')\) if \(\lambda^p p^m \leq \lambda'^p p^{m'}\) as \(p \to \infty\).

We define similarly the growth type of an element \(g \in F_n\) under \(\alpha \in \text{Aut}(F_n)\), and of an edge-path (or a loop) \(\gamma\) in \(G\) under a relative train track map \(f : G \to G\) by considering the simplicial length \(|f^n(\gamma)|\).

If \(g\) or \(\overline{g}\) grows like \(\lambda^p p^m\) under some positive power \(\alpha^q\) or \(\Phi^q\), it grows like \((\lambda^{1/q})^p p^m\) under \(\alpha\) or \(\Phi\). This allows us to replace an automorphism by a power whenever convenient.

We consider the set of attracting laminations of \(\Phi\), ordered by inclusion. Each attracting lamination \(\Lambda\) has an expansion factor \(\lambda_0 > 1\) (see above). From this data, we shall now associate to each \(\Lambda\) a growth type \(c = (\lambda, m)\), with \(\lambda > 1\) and \(m \in \mathbb{N}\). The definition will ensure that, if \(\Lambda\) is associated to an exponential stratum \(H_i\) of \(f : G \to G\) representing \(\Phi\), and \(e\) is an edge of \(H_i\), then the length of \(f^n(e)\) grows like \(\lambda^p p^m\) (see Proposition 6.4).

The definition is by induction on the number of laminations contained in \(\Lambda\), using the following rules. Let \(\lambda_0\) be the expansion factor of \(\Lambda\). If \(\Lambda\) is minimal (for inclusion), then \(c = (\lambda_0, 0)\). If not, let \((\lambda', m')\) be the maximum growth type for \(\Lambda' \subset \Lambda\). If \(\lambda' < \lambda_0\), then \(c = (\lambda_0, 0)\). If \(\lambda' > \lambda_0\), then \(c = (\lambda', m')\). If \(\lambda' = \lambda_0\), then \(c = (\lambda_0, m' + 1)\). In all cases, \(c(\Lambda) = \max_{\Lambda' \subset \Lambda} c(\Lambda')\). Also note that, if \(m > 0\), there is a decreasing chain of laminations \(\Lambda = \Lambda_0 \supseteq \Lambda_1 \supseteq \cdots \supseteq \Lambda_m\), so that \(m \leq s\).

**Example.** Consider the automorphism of \(F_4\) defined by \(a \mapsto abaa', b \mapsto ba, a' \mapsto a'b', b' \mapsto a'\). There are two attracting laminations \(\Lambda\) and \(\Lambda'\), with \(\Lambda' \subset \Lambda\).

The expansion factor \(\lambda_0\) of \(\Lambda\) is the Perron-Frobenius eigenvalue \(\mu\) of \(\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\), the expansion factor \(\lambda'_0\) of \(\Lambda'\) is the eigenvalue \(\nu\) of \(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\). One has \(\mu > \nu\); the growth type of \(\Lambda\) is \((\mu, 0), \) corresponding to the fact that \(|\Phi^p(\overline{\pi})|\) grows like \(\mu^p\). For the automorphism \(a \mapsto ab\), \(b \mapsto a, a' \mapsto a'b'a', b' \mapsto a'b'\), one still has two laminations \(\Lambda' \subset \Lambda\), but now \(\lambda_0 = \nu\) and \(\lambda'_0 = \mu\), so \(\lambda_0 < \lambda'_0\). The growth type of \(\Lambda\) is \((\mu, 0), \) and \(|\Phi^p(\overline{\alpha})|\) grows like \(\mu^p\). Finally, consider \(a \mapsto ab\), \(b \mapsto a, a' \mapsto a'b'\),
b' → a'. One has λ₀ = λ'₀ = ν. The growth type of Λ is (ν, 1), and |Φ^p(γ)| grows like \( pν^p \).

**Theorem 6.2.** Let \( Φ ∈ \text{Out}(F_n) \).

1. Given \( g ∈ F_n \), there exist \( λ ≥ 1 \) and \( m ∈ N \) such that \( |Φ^p(γ)| \) grows like \( λ^p p^m \).
2. If \( |Φ^p(γ)| \) grows like \( λ^p p^m \), and \( m' < m \), there exists \( g' \) such that \( |Φ^p(γ')| \) grows like \( λ^p p^{m'} \).
3. Given \( (λ, m) \) with \( λ > 1 \), there exists \( g \) such that \( |Φ^p(γ)| \) grows like \( λ^p p^m \) if and only if \( (λ, m) \) is the growth type of some attracting lamination of \( Φ \).

When \( λ > 1 \), the power \( m \) which appears in the growth of \( |Φ^p(γ)| \) satisfies \( m ≤ n/2 - 1 \) by Theorem 4.7. When \( λ = 1 \), the maximum value of \( m \) is the number that we have called \( d \). It satisfies \( d ≤ n - 1 \) (see Corollary 4.5).

Before proving Theorem 6.2, we also note:

**Corollary 6.3.** Given \( α ∈ \text{Aut}(F_n) \) and \( g ∈ F_n \), there exist \( λ ≥ 1 \) and \( m ∈ N \) such that \( |α^p(g)| \) grows like \( λ^p p^m \).

**Proof.** Extend \( α \) to an automorphism \( β \) of \( F_{n+1} \) by mapping the new generator \( t \) to itself. The growth of \( g \) under \( α \) is that of the conjugacy class \( \overline{γ} \) under \( β \). \( \square \)

The set of growth types of elements of \( F_n \) under \( α \) is the same as the set of growth types of conjugacy classes under \( α \), except that there may be elements with growth \( p^{d+1} \) (see Lemma 2.3).

**Proof of Theorem 6.2.**

The rest of this section is devoted to the proof of Theorem 6.2. See [2, 4, 12] for partial results. Our proof elaborates on an argument due to Bridson-Groves.

After replacing \( Φ \) by a power, we may assume that it is represented by an improved relative train track map \( f : G → G \). The heart of the proof is to show that any edge-path or loop \( γ \) in \( G \) has a growth type \( (λ, m) \), in the sense that the length of \( f^p_γ(γ) \) is bounded between \( C_1 λ^p p^m \) and \( C_2 λ^p p^m \) for some \( C_1, C_2 > 0 \).

Recall that we have defined a growth type \( c = (λ, m) \) for an attracting lamination \( Λ \), hence also for an exponential stratum \( H_i \). We write \( c_i \) for the growth type attached to \( H_i \).

**Proposition 6.4.** Given \( γ \), let \( C_γ \) be the set of all \( j \) such that some \( f^p_γ(γ) \) splits over an edge belonging to an exponential stratum \( H_j \). If \( C_γ = ∅ \), then \( γ \) grows like \( p^m \) for some \( m ∈ N \). Otherwise, the growth type of \( γ \) is the maximal \( c_j \), for \( j ∈ C_γ \).

In particular, the growth type of an edge in an exponential stratum \( H_i \) is \( c_i \) by Lemma 6.1 and the equality \( c(Λ) = \max_{Λ' \subseteq Λ} c(Λ') \).

The proof of the proposition is by induction on the height of \( γ \). First suppose that \( γ \) is a single edge \( e \) in an exponential stratum \( H_i \), and the proposition is true for paths of height < \( i \). Let \( c_i = (λ, m_i) \), and let \( λ_i ≤ λ \) be the Perron-Frobenius eigenvalue attached to \( H_i \). We show that \( |f^p_γ(e)| \) grows like \( λ^p p^m \).
As mentioned above, \( f(e) \) splits over edges of \( H_i \) and paths \( \delta \in \Delta_i \) (recall that we have defined \( \Delta_i \) as the finite set of maximal subpaths of height \( < i \) which appear in \( f(e') \) for \( e' \) an edge of \( H_i \)). Thus \( f^q_{-1}(\delta) \) splits over edges of \( H_i \) and paths of the form \( f^q_{-1}(\delta) \) with \( q \leq p - 1 \) and \( \delta \in \Delta_i \). Up to constants (which we will not write), the number of edges in \( f^q_{-1}(\delta) \cap H_i \) is \( \lambda_i^p \), and for given \( q \leq p - 1 \) and \( \delta \in \Delta_i \) the number of subpaths \( f^q_{-1}(\delta) \) is \( \lambda_i^{p-q-1} \) (as they are created by edges in \( f^p_{-1}(\delta) \cap H_i \)).

We first show that the length of \( f^p_{-1}(\delta) \) grows at most like \( \lambda^p p^m \). It suffices to show that, for a given \( \delta \) in the set \( \Delta_i \), the sum \( \sum_{q=1}^{p} \lambda_i^{p-q} | f^q_{-1}(\delta) | \) grows at most like \( \lambda^p p^m \). If some \( f^q_{-1}(\delta) \) splits over an edge in an exponential stratum \( H_j \), then so does some \( f^q_{-1}(\delta) \). By the induction hypothesis, \( | f^q_{-1}(\delta) | \) grows either polynomially or with growth type \( c_j = (\lambda_j, m_j) \), with \( j \in \Gamma \), so we have to show that \( S_p = \sum_{q=1}^{p} \lambda_i^{p-q} | f^q_{-1}(\delta) | \) grows at most like \( \lambda^p p^m \). But the growth type of a laminations was defined in such a way that this holds, since \( S_p \) grows like \( \lambda_i^p \) if \( \lambda_j < \lambda_i \), like \( \lambda_j^p p^{m_j} \) if \( \lambda_j > \lambda_i \), like \( \lambda_j^p p^{m_j+1} \) if \( \lambda_j = \lambda_i \).

We now show that \( \gamma \) grows at most like \( \lambda^p p^m \). This is clear if \( \Lambda_i \) is minimal (for inclusion). If \( \lambda > \lambda_i \), there is \( \Lambda_j \subset \Lambda_i \) with \( c_j = c_i \). By Lemma 6.1, some \( f^q_{-1}(\delta) \) splits over an edge \( e_j \) of \( H_j \), so \( \gamma \) grows as fast as \( e_j \), whose growth type is \( c_j \) by induction. The case when \( \lambda = \lambda_i \) is harder.

The result is clear if \( m = 0 \), so assume \( m > 0 \). Then there is \( \Lambda_j \subset \Lambda_i \) with \( c_j = (\lambda, m-1) \). Some \( f^q_{-1}(\delta) \) splits over an edge \( e_j \) of \( H_j \). This edge is contained in some \( f^q_{-1}(\delta) \), with \( q' \leq q-1 \) and \( \delta \in \Delta_i \). The splitting of \( f^q_{-1}(\delta) \) over \( e_j \) may not be compatible with the splitting of \( f^q_{-1}(\delta) \) over \( f^q_{-1}(\delta) \). But further iterates of \( \gamma \) split over long \( j \)-legal paths, and by bounded cancellation we may therefore assume (after increasing \( q \)) that a subpath \( \delta' = f^q_{-1}(\delta) \) of \( f^q_{-1}(\delta) \) splits over an edge of \( H_j \), hence has growth type at least \( (\lambda, m-1) \) by induction.

Fix \( q \) and \( \delta' \), and consider \( f^{q^p}_{-1}(\delta') \). For each \( \ell < p \) it contains (up to constants) at least \( \lambda^{q(p-\ell)} \) subpaths \( f^{q^\ell}_{-1}(\delta') \), each with length \( \lambda^{q\ell} q^{\ell-1} \). It follows that the length of \( f^{q^p}_{-1}(\delta') \) is bounded below by \( \sum_{\ell=1}^{p-1} \lambda^{q(p-\ell)} \lambda^{q\ell} q^{\ell-1} \), hence by \( \lambda^{q^p} q^p \).

This completes the proof of the induction step for \( \gamma \) an edge in an exponential stratum. In general, we now know that \( \gamma \) grows at least like the maximal \( c_j \). For the upper bound, we use the following fact.

**Lemma 6.5** [1, Lemmas 4.1.4, 4.2.6, 5.5.1]. If \( \gamma \) has height \( i \), there exists \( p_0 \) such that \( f^{p_0}_{-1}(\gamma) \) has a splitting whose pieces are edges of \( H_i \), paths of height \( < i \), Nielsen paths, and exceptional paths. \( \square \)

Nielsen paths do not grow, and exceptional paths grow linearly. Paths of height \( < i \) grow at most like the maximal \( c_j \) by the induction hypothesis. We know how edges in an exponential stratum grow, so there only remains the case when \( \gamma \) is an edge in an NEG stratum. We have \( f^{p_i}_{-1}(\gamma) = \gamma u f^q_{-1}(u) f^q_{-1}(u) \ldots f^q_{-1}(u) \) for some path \( u \) of height \( < i \). By the induction hypothesis, \( u \) grows either polynomially or
with growth type \( c_j \) with \( j \in C_\gamma \). Thus \( \gamma \) grows at the same speed as \( u \) if \( u \) grows exponentially, like \( p^{m+1} \) if \( u \) grows like \( p^m \).

The proof of Proposition 6.4 is now complete. To prove the first assertion of Theorem 6.2, we simply observe that the type of growth of a conjugacy class \( \overline{g} \) (under \( \Phi \)) is the same as that of the loop representing \( \overline{g} \) in \( G \) (under \( f \)). If \( H_i \) is an exponential stratum, any conjugacy class represented by an \( i \)-legal loop meeting \( H_i \) has growth type \( c_i \). This shows assertion (3), and assertion (2) when \( \lambda > 1 \).

Assertion (2) in the polynomial case is well-known, but we sketch a proof for completeness. If \( \overline{g} \) grows like \( p^m \), the arguments given above imply the existence of a path \( u \) which grows like \( p^{m-1} \). This path is a loop by [1], but images \( f^p(u) \) may fail to be reduced as loops. It is easy to check, however, that for each \( p \) one of the loops \( f^p(u), f^{p+1}(u), f^p(u)f^{p+1}(u) \) is reduced, and assertion (2) follows.

References

1. M. Bestvina, M. Feighn, M. Handel, *The Tits alternative for Out(\( F_n \))*, I, Dynamics of exponentially-growing automorphisms, Ann. of Math. 151 (2000), 517–623.
2. M. Bestvina, M. Feighn, M. Handel, *The Tits alternative for Out(\( F_n \))*, II, A Kolchin type theorem, Ann. of Math. 161 (2005), 1–59.
3. M. Bridson, D. Groves, *The quadratic isoperimetric inequality for mapping tori of free group automorphisms*, Memoirs of the AMS (to appear).
4. M. Bestvina, M. Handel, *Train tracks for automorphisms of the free group*, Ann. Math. 135 (1992), 1–51.
5. D.J. Collins, E.C. Turner, *All automorphisms of free groups with maximal rank fixed subgroups*, Math. Proc. Camb. Phil. Soc. 119 (1996), 615–630.
6. J.L. Dyer, P.G. Scott, *Periodic automorphisms of free groups*, Comm. Algebra 3 (1975), 195–201.
7. D. Gaboriau, A. Jaeger, G. Levitt, M. Lustig, *An index for counting fixed points of automorphisms of free groups*, Duke Math. Jour. 93 (1998), 425–452.
8. D. Gaboriau, G. Levitt, *The rank of actions on \( R \)-trees*, Ann. Sc. ENS 28 (1995), 549–570.
9. D. Gaboriau, G. Levitt, M. Lustig, *A dendrological proof of the Scott conjecture for automorphisms of free groups*, Proc. Edinburgh Math. Soc. 41 (1998), 325–332.
10. F. Gautero, M. Lustig, *The mapping-torus of a free group automorphism is hyperbolic relative to the canonical subgroups of polynomial growth*, arXiv:0707.0822.
11. G. Levitt, M. Lustig, Unpublished notes, 1998.
12. G. Levitt, M. Lustig, *Periodic ends, growth rates, Hölder dynamics for automorphisms of free groups*, Comm. Math. Helv. 75 (2000), 415–430.
13. A. Piggott, *Detecting the growth of free group automorphisms by their action on the homology of subgroups of finite index*, math.GR/0409319.
14. H. Short, *Quasiconvexity and a theorem of Howson’s*, Group theory from a geometrical viewpoint (Ghys, Haefliger, Verjovsky, eds.), World Scientific, 1991, pp. 168–176.

Gilbert Levitt: LMNO, UMR CNRS 6139, BP 5186, Université de Caen, 14032 Caen Cedex, France.
E-mail address: levitt@math.unicaen.fr

30