THE SECONDARY PERIODIC ELEMENT $\beta_{p^2/p^2-1}$ AND ITS APPLICATIONS

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Abstract. In this paper, we prove that $\beta_{p^2/p^2-1}$ survives to $E_\infty$ in the Adams-Novikov spectral sequence (ANSS) for all $p \geq 5$. As an easy consequence, we prove that $\beta_{sp^{n+1}/j}$ are permanent cycles for $s \geq 1$, $j \leq p^2 - 1$. From the Thom map $\Phi : Ext^i_{BP_*BP}(BP_*, BP_*) \to Ext^i_{BP_*BP}(BP_*, BP_*)$, we also see that $h_0 h_3$ survives to $E_\infty$ in the classical Adams spectral sequence.

1. Introduction

Let $p \geq 5$ be an odd prime. The Adams-Novikov spectral sequence (ANSS) based on the Brown-Peterson spectrum is one of the most powerful tools to compute the $p$-component of the stable homotopy groups of spheres $\pi_* S^0$ (cf. [11, 4, 8, 20]). The $E_2$-term of the ANSS is $Ext_{BP_*BP}^i(BP_*, BP_*)$.

From [10, 8] we know that $Ext^1_{BP_*BP}(BP_*, BP_*) = H^1 BP_*$ is generated by $\alpha_{sp^n/n+1}$ for $n \geq 0$, $p \nmid s \geq 1$, where $\alpha_{sp^n/n+1}$ has order $p^{n+1}$. $Ext^2_{BP_*BP}(BP_*, BP_*) = H^2 BP_*$ is the direct sum of cyclic groups generated by $\beta_{sp^n/j,i+1}$ for suitable $(n, s, j, i)$ (cf. [8, 20, 21]).

It is known that each element $\alpha_{sp^n/n+1}$ in $H^1 BP_*$ is a permanent cycle in the ANSS which represents an element of $\text{Im} J$ having the same order. But we are far from fully determining which element of $\beta_{sp^n/j,i+1}$ in $H^2 BP_*$ survives to $E_\infty$.

Let $\beta_{sp^n/j}$ denote $\beta_{sp^n/j,1}$. H. Toda [21, 25] proved $\alpha_1 \beta^p_1$ is zero in $\pi_* S^0$. This relation supports a non-trivial Adams-Novikov differential $d_{2p-1}(\beta_{p/p}) = a_1 \beta^p_1$, which is called the Toda differential. Based on Toda differential, D. Ravenel [17] proved that

$$d_{2p-1}(\beta_{sp^n}) \equiv a_1 \beta_{sp^n-1/p^n-1} \mod \ker \beta_1^{(p^{n-1}-1)/(p-1)}$$

for $n \geq 1$. That is to say, $\beta_{sp^n}$ cannot survive to $E_\infty$ in the Adams-Novikov spectral sequence. From this one can see that only $\beta_{sp^n/j} \in H^2 BP_*$ for $s \geq 2$, $1 \leq j \leq p^n$ or $s = 1$, $1 \leq j \leq p^n - 1$ might survive to $E_\infty$ in the ANSS. The following are some known results in this area:

Oka [14] proved that for $s = 1$, $1 \leq j \leq p - 1$ or $s \geq 2$, $1 \leq j \leq p$, $\beta_{sp/j}$ is a permanent cycle in the ANSS.

Oka [13] proved that for $s = 1$, $1 \leq j \leq 2p - 2$ or $s \geq 2$, $1 \leq j \leq 2p_1$, $\beta_{sp^2/j}$ is a permanent cycle in the ANSS.

Later Oka [15, 16] generalized the result to $n \geq 2$, i.e. for $n \geq 2$, $s = 1$, $1 \leq j \leq 2^{n-1}(p-1)$ or $s \geq 2$, $1 \leq j \leq 2^{n-1}p$, $\beta_{sp^n/j}$ survives to $E_\infty$ in the ANSS.

Shimomura [23] proved that for $s \geq 1$, $1 \leq j \leq p^2 - 2$, $\beta_{sp^2/j}$ survives to $E_\infty$ in the ANSS.

In this paper, we proved:

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**Theorem A** Let \( p \geq 5 \) be an odd prime. Then \( \beta_{p^2/p^2-1} \) is a permanent cycle in the Adams-Novikov spectral sequence.

Let \( M \) be the cofiber of the degree \( p \) map \( p : S^0 \to S^0 \),

\[
S^0 \xrightarrow{p} S^0 \xrightarrow{} M.
\]

There exists the Smith-Toda map \( v_j^i : \Sigma^{|v_j^i|} M \to M \) and its cofiber is denoted by \( M(1; j) \).

D. Ravenel proved that

**Theorem 7.12** \[22\] Let \( p \geq 5 \) be an odd prime. If for some fixed \( n \geq 1 \),

(i) the spectrum \( M(1, p^n - 1) \) is a ring spectrum,

(ii) \( \beta_{p^n/p^n-1} \) is a permanent cycle and

(iii) the corresponding homotopy element has order \( p \),

then \( \beta_{p^2/p^2} \) is a permanent cycle (and the corresponding homotopy element has order \( p \)) for all \( s \geq 1 \) and \( 1 \leq j \leq p^n - 1 \).

From [14] [15] [16], we know that \( M(1, p^n - 1) \) is a ring spectrum for \( n \geq 1, p \geq 5 \). Thus from the theorem above and Theorem A, we have:

**Proposition B** Let \( p \geq 5 \) be an odd prime. Then for \( s \geq 1, j \leq p^2 - 1, \beta_{sp^2/j} \) is a permanent cycle in the ANSS.

It is not yet known whether \( \beta_{sp^2/p^2} \) for \( s \geq 2 \) is a permanent cycle.

Between the ANSS and the classical Adams spectral sequence (ASS), there is the Thom reduction map

\[
\Phi : Ext^*_BP(BP_s, BP_s) \to Ext^*_A(\mathbb{Z}/p, \mathbb{Z}/p)
\]

and \( \Phi(\beta_{p^2/p^2-1}) = h_0h_3 \). Thus

**Corollary C** Let \( p \geq 5 \) be an odd prime. Then \( h_0h_3 \) is a permanent cycle in the Adams spectral sequence.

This paper is arranged as follows. In section 2 we recall the construction of the topological small descent spectral sequence (TSDSS) which is a spectral sequence that converges to \( \pi_*S^0 \) started from the homotopy groups of a complex with \( p \)-cells. Then we describe the generators of its \( E_1 \)-term. In section 3 we compute a differential in the TSDSS, which is used in proving theorem A. Then in section 4, we prove our main theorem by the topological small descent spectral sequence.

### 2. The small descent spectral sequence and the ABC Theorem

In 1985, D. Ravenel [18] [19] [20] [21] introduced the method of infinite descent and used it to compute the first thousand stems of the stable homotopy groups of spheres at the prime 5. This method is an approach to finding the \( E_2 \)-term of the ANSS by the following spectral sequence referred to as the small descent spectral sequence (SDSS).

Hereafter we set that \( q = 2p - 2 \). Let \( T(n) \) be the Ranevel spectrum (cf. [20] Section 5, Chapter 6) characterized by

\[
BP_*T(n) = BP_*[t_1, t_2, \cdots, t_n].
\]

Then we have the following diagram

\[
S^0 = T(0) \xrightarrow{} T(1) \xrightarrow{} T(2) \xrightarrow{} \cdots \xrightarrow{} T(n) \xrightarrow{} \cdots \xrightarrow{} BP,
\]
where $S^0$ denote the sphere spectrum localized at an odd prime $p \geq 5$. Let $T(0)_{p-1}$ and $T(0)_{p-2}$ denote the $q(p-1)$ and $q(p-2)$ skeletons of $T(1)$ respectively, they are denoted by $Y$ and $\overline{Y}$ for simple. Then

$$Y = S^0 \cup_{\alpha_1} e^q \cup \cdots \cup_{\alpha_1} e^{(p-2)q} \cup_{\alpha_1} e^{(p-1)q} \quad \text{and} \quad \overline{Y} = S^0 \cup_{\alpha_1} e^q \cup \cdots \cup_{\alpha_1} e^{(p-2)q}.$$ 

The $BP$-homologies of them are

$$BP_*(Y) = BP_*[t_1]/(t_1^p) \quad \text{and} \quad BP_*(\overline{Y}) = BP_*[t_1]/(t_1^{p-1}).$$

From the definition above we get the following cofibre sequences

$$0 \longrightarrow BP_* S^0 \longrightarrow BP_* Y \longrightarrow BP_* \Sigma^q Y \longrightarrow 0,$$

and the short exact sequences of $BP_*$ homologies

$$0 \longrightarrow BP_* S^0 \longrightarrow BP_* Y \longrightarrow BP_* \Sigma^q Y \longrightarrow 0.$$

Put (2.3) and (2.4) together, one has the following long exact sequence

$$0 \longrightarrow BP_* S^0 \longrightarrow BP_* Y \longrightarrow BP_* \Sigma^q Y \longrightarrow BP_* \Sigma^q Y \longrightarrow \cdots.$$

Put (2.1) and (2.2) together, one has the following Adams diagram of cofibres

$$0 \longrightarrow S^0 \longrightarrow Y \longrightarrow \Sigma^q Y \longrightarrow \Sigma^{p-2} Y \longrightarrow \Sigma^{(p+1)q-3} Y \longrightarrow \cdots$$

Then one has:

**Proposition 2.1** [Ravenel [20] 7.4.2 Proposition] Let $Y$ be as above. Then

(a) There is a spectral sequence converging to $Ext_{BP_*BP_*}^{s,t,u}(BP_*, BP_*(S^0))$ with $E_1$-term

$$E_1^{s,t,u} = Ext_{BP_*BP_*}^{s,t}(BP_*, BP_*) \otimes E[\alpha_1] \otimes P[\beta_1], \quad \text{where} \quad \alpha_1 \in E_1^{0,q,1}, \quad \beta_1 \in E_1^{0,pq,2}$$

and $d_r : E_r^{s,t,u} \longrightarrow E_r^{s-r+1,t,u+r}$. Where $E[-]$ denotes the exterior algebra and $P[-]$ denotes the polynomial algebra on the indicated generators. This spectral sequence is referred as the small descent spectral sequence (SDSS).

(b) There is a spectral sequence converging to $\pi_*(S^0)$ with $E_1$-term

$$E_1^{s,t} = \pi_*(Y) \otimes E[\alpha_1] \otimes P[\beta_1], \quad \text{where} \quad \alpha_1 \in E_1^{q}, \quad \beta_1 \in E_1^{2pq}$$

and $d_r : E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}$. This spectral sequence is referred as the topological small descent spectral sequence (TSDSS).
The above two spectral sequences produce the 0-line and the 1-line $\text{Ext}_{BPBP}^0(BP_*, BP_*(S^0))$, $\text{Ext}_{BPBP}^1(BP_*, BP_*(S^0))$ or the corresponding elements in $\pi_*(S^0)$ by $\text{Ext}^0_{BPBP}(BP_*, BP_*Y)$ and $\text{Ext}^1_{BPBP}(BP_*, BP_*Y)$. $\text{Ext}^s_{BPBP}(BP_*, BP_*(S^0)) (s \geq 2)$ or the corresponding elements in $\pi_*(S^0)$ is produced by $\text{Ext}^s_{BPBP}(BP_*, BP_*Y) (s \geq 2)$ as described as the following ABC Theorem.

**ABC Theorem** (7.4.3 ABC Theorem [20]) For $p > 2$ and $t - s < q(p^3 + p - 1) - 3$, $s \geq 2$ $\text{Ext}_{BPBP}^{s,t}(BP_*, BP_*Y) = A \oplus B \oplus C,$

where $A$ is the $\mathbb{Z}/p$-vector space spanned by

$$A = \{ \beta_{jp}, \beta_{jp+1} | 0 \leq p - 1 \} \cup \{ \beta_{jp/p^2-j} | 0 \leq j \leq p - 1 \},$$

$$B = R \otimes \{ \gamma_k | k \geq 2 \},$$

where

$$R = P[b_{20}^p] \otimes E[h_{20}] \otimes \mathbb{Z}/p \{ \{ b_{11}^k | 0 \leq k \leq p - 1 \} \cup \{ h_{11} b_{20}^k | 0 \leq k \leq p - 2 \} \},$$

and

$$C^{s,t} = \bigoplus_{i \geq 0} R^{s+2i,t+i(p^2-1)q}.$$

From the generators of $R$, we can obtain the generators of $C$ as follows:

Let $i = jp + m$. Then from $R^{s+2i,t+i(p^2-1)} \subset C^{s,t}$ we have

1. $b_{20}^{p,m-1}u_{jp+m}$ for $p - 1 \geq m \geq 1$. From which we have

$$b_{20}^{p,m-1}u_{jp+m} \otimes E[h_{20}] \otimes \{ b_{11}^k | 0 \leq k \leq p - 1 \} \cup \{ h_{11} b_{20}^k | 0 \leq k \leq p - 2 \},$$

where

$$u_{jp+m} \in C^{2,q[(j+1)p^2+(j+m+1)p+m]}.$$

2. $b_{11}^{k-1}b_{20}^{p} \in R^{2(k-m)+2(jp+m)+t+(jp+m)(p^2-1)q} \subset C^{2(k-m),t}$ is represented by

$$b_{11}^{k-1} \beta_{(j+1)p/p-m}$$

for $p - 1 \geq k \geq m + 1 \geq 1$. From which we have

$$b_{11}^{k-1} \beta_{(j+1)p/p-m} \otimes E[h_{20}],$$

where

$$\beta_{(j+1)p/p-m} \in C^{2,q[(j+1)p^2+jp+m]}.$$

- Especially $h_{20} b_{11}^{p-1} b_{20}^{p} \in R^{3+2(jp+p-2)+t+(jp+p-2)(p^2-1)q} \subset C^{3,t}$ is represented by

$h_{11} \beta_{(j+1)p/1,2}$, which is an element of order $p^2$.

3. $h_{11} b_{20}^{p} \in R^{2(k-m)+1+2(jp+m)+t+(jp+m)(p^2-1)q} \subset C^{2(k-m)+1,t}$ is represented by

$b_{20}^{k-1} \eta_{jp+m+1}$

for $p - 2 \geq k \geq m + 1 \geq 1$, where

$$\eta_{jp+m+1} = h_{11} u_{jp+m} \in C^{3,q[(j+1)p^2+(j+m+2)p+m]}.$$
\( b_{20}^{k/2} \) for \( p - 2 \geq k \geq m \geq 0 \), where
\[
\beta_{jp+m+2} = \beta_{jp+p+2} \in C^2, \text{ } (jp+(j+m+2)p+m+1).
\]

- Especially \( h_{20}b_{20}^{k-1}h_{20}^{b_{20}} \in R^{2(k-m+1)+2(jp+m)), t+(jp+m)(p^2-1)}q \subset C^{2(k-m+1)}t \) is represented by \( b_{20}^{k-m} \beta_{jp+m+2} \).

From the ABC Theorem above, we compute that \( \text{Ext}_{BP, BP}(BP, BP, Y) \) for \( s \geq 2, t - s < q(p^3 + p - 1) - 3 \) is the \( \mathbb{Z}_{(p)} \)-module generated by the following generators, here the generators are listed as generators, total degree \( t - s \) and \( t - s \text{ mod } pq - 2 \), range of index; where \( pq - 2 \) is the total degree of \( \beta_1 \in E_1^{b_{20}^k} \) in the SDSS.

**Generators of A**

| Generators | \( t - s \text{ and } t - s \text{ mod } pq - 2 \) | Range of index |
|------------|-----------------------------------------------|----------------|
| \( \beta_{ip} \) | \( q[ip^2 + ip - 1] - 2 \) | \( 2(i-1)p + 2i \) \text{ if } i \leq p - 2 \text{ if } i = p - 1 |
| \( \beta_{ip+1} \) | \( q[ip^2 + (i+1)p] - 2 \) | \( 2ip + 2i \) \text{ if } i \leq p - 2 \text{ if } i = p - 1 |
| \( \beta_{ip^2/p^2-j} \) | \( q[p^3 + j] - 2 \) | \( 2(j+1)p - 2j_{ap} \) \text{ if } j \leq p - 2 \text{ if } j = p - 1 |

**Generators of B**

| Generators | \( t - s \text{ and } t - s \text{ mod } pq - 2 \) | Range of index |
|------------|-----------------------------------------------|----------------|
| \( h_{20}b_{20}^{k} \gamma_i \) | \( q[(i+k)p^2 + (i+k)p + i - 1] - 2k - 4 \) | \( 2(k + 2i - 2)p \) \text{ if } 2 \leq i, k + 2i \leq p \text{ if } k + 2i > p |
| \( h_{20}b_{20}^{k} \gamma_i \) | \( q[(i+k)p^2 + (i+k+1)p + i - 1] - 2k - 5 \) | \( 2(k + 2i - 1)p - 1 \) \text{ if } k \leq 2i < p \text{ if } k + 2i \geq p |
| \( b_{11}^{k} \gamma_i \) | \( q[(i+k)p^2 + (i+1)p + i - 2] - 2k - 3 \) | \( 2(k + 2i - 2)p - 2k - 1 \) \text{ if } k \leq 2i \leq p + 1 \text{ if } k = 0, 2i = p + 1 \text{ if } k + 2i \geq p + 2 |
| \( h_{20}b_{11}^{k} \gamma_i \) | \( q[(i+k)p^2 + ip + i - 1] - 2k - 4 \) | \( 2(k + 2i - 1)p - 2k - 1 \) \text{ if } k \leq 2i < p \text{ if } k + 2i > p |

**Generators of C**
Generators & \( t - s \) and \( t - s \mod pq - 2 \) & Range of index

| Expression | Description |
|------------|-------------|
| \( b_{11}^k b_{20}^{p-m-1} u_{jp+m} \) | \( q[(p - m + j + k + 1)p^2 + j p + m] - 2(p - m + k) \) if \( 1 \leq m \leq p - 1 \) |
| \( h_{20} b_{11}^k b_{20}^{p-m-1} u_{jp+m} \) | \( q[(p - m + j + k + 1)p^2 + (j + 1)p + m + 1] - 2(p - m + k) - 1 \) if \( 1 \leq m \leq p - 1 \) |
| \( h_{11} b_{20}^{k+p-m-1} u_{jp+m} \) | \( q[(p - m + j + k + 1)p^2 + (j + k + 1)p + m + 1] - 2(p - m + k) - 1 \) if \( 1 \leq m \leq p - 1 \) |
| \( h_{20} h_{11} b_{20}^{k+p-m-1} u_{jp+m} \) | \( q[(p - m + j + k + 1)p^2 + (j + k + 2)p + m + 1] - 2(p - m + k + 1) \) if \( 1 \leq m \leq p - 1 \) |
| \( b_{11}^{k-m-1} \beta_{(j+1)p/p-m} \) | \( q[(j + k - m)p^2 + j p + m] - (2k - 2m) \) if \( j + k \leq p - 2 \) |
| \( h_{20} b_{11}^{k-m-1} \beta_{(j+1)p/p-m} \) | \( q[(j + k - m)p^2 + (j + 1)p + m + 1] - (2k - 2m + 1) \) if \( j + k \leq p - 3 \) |
| \( h_{11} \beta_{(j+1)p/1,2} \) | \( q[(j + 1)p^2 + (j + 2)p - 1] - 3 \) if \( j \leq p - 3 \) |
| \( b_{2,0}^{k-m-1} \eta_{jp+m+1} \) | \( q[(j + k - m)p^2 + (j + k + 1)p + m] - (2k - 2m + 1) \) if \( j + k \leq p - 2 \) |
| \( b_{2,0}^{k-m} \beta_{jp+m+2} \) | \( q[(j + k - m)p^2 + (j + k + 2)p + m + 1] - 2(k - m + 1) \) if \( j + k \leq p - 3 \) |
| \( \beta_{(j+1)p/1,2} \) | \( q[(j + 1)p^2 + (j + 1)p - 1] - 2 \) if \( j \leq p - 3 \) |
Remark. The Adams-Novikov spectral sequence for the spectrum $Y$ collapses from $E_2$-term $\text{Ext}^{s,t}_{BP, BP}(BP_{*}, BP, Y)$ in the range $t - s < q(p^3 + p - 1) - 3$, since there are no elements with filtration $> 2p$. Thus we actually get the homotopy groups $\pi_{t-s}(Y)$ in this range.

3. A DIFFERENTIAL IN THE TSDSS

This section is armed at showing that

$$d_{2p-1}(b_{20}b_{11} \gamma_s) = \alpha_1 \beta_1^p h_{20} \gamma_s$$

in the TSDSS, which will be used in proving Theorem A in section 4.

We begin from showing that $\pi_{q(p^2+2p+2)-2}V(2) = 0$. From which we show that the Toda bracket $\langle \alpha_1 \beta_1, p, \gamma_s \rangle = 0$ and the Toda bracket $\langle \alpha_1 \beta_1^{p-1}, \alpha_1 \beta_1, p, \gamma_s \rangle$ is well defined. Then from that relation

$$\langle \alpha_1 \beta_1^{p-1}, \alpha_1 \beta_1, p, \gamma_s \rangle = \alpha_1 \beta_1^{p-1} h_{20} \gamma_s = \beta_{p/p-1} \gamma_s.$$ 

and $d(h_{20}b_{11}) = \beta_1 \beta_{p/p-1}$ we get the desired differential in the TSDSS.

Let $p \geq 5$ be an odd prime and $V(2)$ be the Smith-Toda spectrum characterized by

$$BP_{*}V(2) = BP_{*}/I_3$$

where $I_3$ is the invariant ideal of $BP_{*} = \mathbb{Z}_p[v_1, v_2, \ldots, v_i, \ldots]$ generated by $\langle p, v_1, v_2 \rangle$. To compute the homotopy groups of $V(2)$, one has the ANSS $\{E_{r,s}V(2), d_r\}$ that converges to $\pi_{*}V(2)$. The $E_2$-term of this spectral sequence is

$$E_2^{s,t}V(2) = \text{Ext}_{BP, BP}^{s,t}(BP_{*}, BP_{*})M$$

Let $(BP_{*}, \Gamma)$ be a Hopf algebroid and $M$ be a $\Gamma$-comodule with coaction $\psi : M \to M \otimes_{BP_{*}} \Gamma$, one has cobar complex of $M$ given by

$$C^s_{\Gamma}M = M \otimes_{BP_{*}} \Gamma \otimes_{BP_{*}} \Gamma \otimes_{BP_{*}} \cdots \otimes_{BP_{*}} \Gamma$$

(with $s$ factors of $\Gamma$). The differential $d : C^s_{\Gamma}M \to C^{s+1}_{\Gamma}M$ is of degree +1 given by

$$d(m \otimes x_1 \otimes \cdots \otimes x_s) = \psi(m) \otimes x_1 \otimes \cdots \otimes x_s$$

$$+ \sum_{i=1}^{s} (-1)^i m \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes \Delta(x_i) \otimes x_{i+1} \otimes \cdots \otimes x_s,$$

where $\Delta : \Gamma \to \Gamma \otimes BP_{*} \Gamma$ is the coproduct of $\Gamma$. The cohomology of $(C^s_{\Gamma}M, d)$ is $\text{Ext}_{\Gamma}^{s,*}(BP_{*}, M)$. Thus we have the cobar complex

$$C^s_{BP, BP}BP_{*}/I_3 = BP_{*}/I_3 \otimes_{BP_{*}} BP_{*}BP \otimes_{BP_{*}} \cdots \otimes_{BP_{*}} BP_{*}BP_{*}BP_{*}$$

whose cohomology is $\text{Ext}_{BP, BP}^{s,*}(BP_{*}, BP_{*}V(2))$.

Let

$$\Gamma = BP_{*}/I_3 \otimes_{BP_{*}} BP_{*}BP \otimes_{BP_{*}} BP_{*}/I_3 = BP_{*}/I_3[t_1, t_2, \cdots].$$

Then $(BP_{*}, \Gamma)$ is a Hopf algebroid. The structure maps of $(BP_{*}, \Gamma)$ is given by

$$\eta_R(v_3) = v_3;$$

$$\Delta(t_1) = t_1 \otimes 1 + 1 \otimes t_1;$$

$$\Delta(t_2) = t_2 \otimes 1 + t_1 \otimes t_1^p + 1 \otimes t_2;$$

$$\Delta(t_3) = t_3 \otimes 1 + t_2 \otimes t_1^p + t_1 \otimes t_2^p + 1 \otimes t_3.$$
One can easily see that
\[
C_{BP, BP}^2 BP, BP, V(2) = BP^*/I_3 \otimes_{BP^*} BP, BP, BP \otimes_{BP^*} BP, BP \otimes_{BP^*} BP, BP
\]
\[
= BP, BP \otimes_{BP^*} \Gamma \otimes_{BP^*} \Gamma \otimes_{BP^*} \Gamma
\]
\[
= C_1^2 BP^*/I_3,
\]
and then
\[
\text{Ext}_{BP, BP}^{s,t}(BP^*, BP, V(2)) = \text{Ext}_{BP}^{s-t}(BP^*, BP^*/I_3)
\]

**Lemma 3.1.** The \(q(p^2 + 2p + 2) - 2\) dimensional stable homology group of \(V(2)\) is trivial, i.e.,
\[
\pi_{q(p^2 + 2p + 2) - 2} V(2) = 0.
\]

**Proof.** Fix \(t - s = q(p^2 + 2p + 2) - 2\), we know that the Adams-Novikov \(E_2\)-term
\[
\text{Ext}_{BP, BP}^{s+s+q(p^2+2p+2)-2}(BP^*, BP, V(2)) = \text{Ext}_{BP}^{s+q(p^2+2p+2)-2}(BP^*, BP^*/I_3)
\]
converges to \(\pi_{q(p^2+2p+2)-2} V(2)\). We will prove that \(\pi_{q(p^2+2p+2)-2} V(2) = 0\) by showing that
\[
\text{Ext}_{BP, BP}^{s+q(p^2+2p+2)-2}(BP^*, BP, V(2)) = 0.
\]

In the cobar complex \(C_1^2 BP^*/I_3\), the inner degree of \(|\ell| = |t_i| \geq q(p^3 + p^2 + p + 1)\) for \(i \geq 4\). It follows that in the range \(t - s \leq q(p^3 + p^2 + p + 1) - 1\),
\[
\text{Ext}_{BP, BP}^{s,t}(BP^*, BP^*/I_3) = \text{Ext}_{BP}^{s-t}(BP^*, BP^*/I_3) = \text{Ext}_{BP}^{s-t}(BP^*, BP^*/I_3),
\]
where \(I' = \mathbb{Z}/p[v_3][t_1, t_2, t_3]\). From \(H_{BP}(v_3) = v_3\) in (3.2), we see that
\[
\text{Ext}_{BP, BP}^{s,s+q(p^2+2p+2)-2}(BP^*, BP, V(2)) \cong \text{Ext}_{\mathbb{Z}/p[t_1, t_2, t_3]}(\mathbb{Z}/p, \mathbb{Z}/p) \otimes \mathbb{Z}/p[v_3].
\]

To compute the \(\text{Ext}\) groups \(\text{Ext}_{BP}^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p)\), we can use the modified May spectral sequence (MSS) introduced in [B6, B7, 21].

There is the May spectral sequence \(\{E_r^{s,t,*}, \delta_r\}\) that converges to \(\text{Ext}_{\mathbb{Z}/p[t_1, t_2, t_3]}(\mathbb{Z}/p, \mathbb{Z}/p)\). The \(E_1\)-term of this spectral sequence is
\[
E_1^{s,t,*} = E[h_{ij} | 0 \leq j, i = 1, 2, 3,] \otimes P[b_{ij} | 0 \leq j, i = 1, 2, 3],
\]
where
\[
h_{ij} \in E_1^{1, q(1+\cdots+p^{i-1})p^j, 2i-1} \quad \text{and} \quad b_{ij} \in E_1^{2, q(1+\cdots+p^{i-1})p^{j+1}, p(2i-1)}.
\]
The first May differential is given by
\[
\delta_1(h_{i,j}) = \sum_{0 < k < i} h_{i-k,k+j} b_{k,j} \quad \text{and} \quad \delta_1(b_{i,j}) = 0.
\]

For the reason of the total degree, to compute \(\text{Ext}_{BP, BP}^{s,s+q(p^2+2p+2)-2}(BP^*, BP, I_3)\) we only need to consider the sub-module generated by \(h_{30}, h_{20}, h_{10}, h_{11}, h_{12}\) and \(b_{20}, b_{10}, b_{11}\), i.e. the subcomplex
\[
E[h_{ij} | 1 \leq i, j \leq 3] \otimes E[b_{20}, b_{11}] \otimes P[b_{10}].
\]

From (3.4), we know that the May’s \(E_2\)-term
\[
E_2^{s,t,*} = H^{s,t,*}(E_1^{s,t,*}, \delta_1) = H^{s,t,*}(E[h_{ij} | 0 \leq j, i + j \leq 3], \delta_1) \otimes E[b_{20}, b_{11}] \otimes P[b_{10}]
\]
within \(t - s \leq q(p^2 + 2p + 2) - 1\). H. Toda in [25] computed the cohomology of \((E[h_{ij} | 0 \leq j, i + j \leq 3], \delta_1)\) which is the \(\mathbb{Z}/p\)-module generated by the following elements:

| dim 0 | 1 |
|-------|--|
| 1 | \(h_{10}\) \(h_{11}\) \(h_{12}\) |
| 2 | \(h_{10} h_{12}\) \(h_{20} h_{10}\) \(h_{20} h_{11}\) \(h_{21} h_{11}\) \(h_{21} h_{12}\) |
Suppose we have a generator \( y \) in \( \text{Ext}^{x,p+q(p^2+2p+2)-2}(BP_*,BP_*V(2)) \). Then \( y \) is the form of \( x \) or \( v_2x \) where \( x \) is an even dimensional generator in \( H^*(E[h_{ij}|i+j\leq 3]) \otimes E[b_{20},b_{11}] \). From the total degree \( t-s \) of \( h_{10} \), one can easily see that except for \( h_{20}h_{10} \) and \( h_{20}h_{11} \), for any generator \( x \) listed above the total degree \( t-s \) of \( xb_{10}^k \) and \( v_3xb_{10}^k \) is not \( q(p^2+2p+2)-2 \). Suppose we have a generator \( h_{20}h_{10}b_{10}^k,h_{20}h_{11}b_{10}^k \) or \( b_{10}^k \) whose total degree is \( q(p^2+2p+2)-2 \), then the total degree of \( h_{20}h_{10},h_{20}h_{11} \) or \( 1 \) should be \( q(p^2+2p+2)-2 \) modulo the total of \( b_{10} \).

From
\[
\begin{align*}
q(p^2 + 2p + 2) - 2 &\equiv 6p - 2 & \text{mod } pq - 2 \\
q(p + 2) - 2 &\equiv 4p - 4 & \text{mod } pq - 2 \\
q(2p + 1) - 2 &\equiv 2p & \text{mod } pq - 2
\end{align*}
\]

we know that the total degree of any generator of the form \( h_{20}h_{10}b_{10}^k,h_{20}h_{11}b_{10}^k \) or \( b_{10}^k \) is not \( q(p^2+2p+2)-2 \). Thus
\[
\text{Ext}^{x,p+q(p^2+2p+2)-2}(BP_*,BP_*V(2)) = \text{Ext}^{x,p+q(p^2+2p+2)-2}(BP_*,BP_*/I_3) = 0.
\]

The Lemma follows. \( \square \)

It is easily showed that the following theorem holds from the lemma above.

**Theorem 3.2.** For \( p \geq 7, \ s \geq 1 \), the Toda bracket \( (\alpha_1\beta_1, p, \gamma_s) = 0 \)

**Proof.** Let \( \overline{v_3} \) be the composition of the following maps
\[
S^{q(p^2+p+1)} \overset{j}{\longrightarrow} S^{q(p^2+p+1)}V(2) \overset{\overline{v_3}}{\longrightarrow} V(2)
\]
where the first map is inclusion to the bottom cell.

It is known that \( \overline{v_3} \) is an order \( p \) elements in \( \pi_{q(p^2+p+1)}V(2) \). Thus the Toda bracket \( (\alpha_1\beta_1, p, \overline{v_3}) \) is well defined and \( (\alpha_1\beta_1, p, \overline{v_3}) \in \pi_{q(p^2+2p+2)-2}V(2) = 0 \). It follows that the Toda bracket
\[
(\alpha_1\beta_1, p, \overline{v_3}) = 0.
\]

Let \( j : V(2) \longrightarrow S^{q(p^2+p+2)+3} \) be the collapsing lower cells map from \( V(2) \), then \( \gamma_s = \overline{v_3} \cdot v_3^{-1} \cdot \overline{j} \).

As a result,
\[
(\alpha_1\beta_1, p, \gamma_s) = (\alpha_1\beta_1, p, \overline{v_3} \cdot v_3^{-1} \cdot \overline{j}) = (\alpha_1\beta_1, p, \overline{v_3}) \cdot v_3^{-1} \cdot \overline{j} = 0
\]

because \( (\alpha_1\beta_1, p, \overline{v_3}) = 0 \in \pi_{q(p^2+2p+2)-2}V(2) = 0 \). \( \square \)

**Proposition 3.3.** (cf. [20] 7.5.11 and [21] 7.6.11 ) Let \( p \geq 7 \) be an odd prime. Then in \( \pi_*(S^0) \), the Toda bracket \( (\alpha_1\beta_1^{-1}, \alpha_1\beta_1, p, \gamma_s) \) is well defined and
\[
(\alpha_1\beta_1^{-1}h_{20}\gamma_s = (\alpha_1\beta_1^{-1}, \alpha_1\beta_1, p, \gamma_s) = \beta_{p/p-1}\gamma_s.
\]
Proof. From $\langle \beta_p^{-1}, \alpha_1 \beta_1, p \rangle = 0$, we know that the following four fold Toda bracket is well defined and

$$\beta_p^{p-1} = (\beta_1^{p-1}, \alpha_1 \beta_1, p, \alpha_1); \quad \alpha_1 h_2 \gamma_s = (\alpha_1, \alpha_1 \beta_1, p, \gamma_s).$$

On the other hand, one has

$$\beta_1^{p-1} \alpha_1 h_2 \gamma_s = \beta_1^{p-1} \langle \alpha_1, \alpha_1 \beta_1, p, \gamma_s \rangle = \langle \alpha_1, \alpha_1 \beta_1, p, \gamma_s \rangle = \alpha_1 \langle \beta_1^{p-1}, \alpha_1 \beta_1, p, \gamma_s \rangle = \alpha_1 \langle \beta_1^{p-1}, \alpha_1 \beta_1, p, \alpha_1 \gamma_s \rangle = \langle \beta_1^{p-1}, \alpha_1 \beta_1, p, \alpha_1 \gamma_s \rangle = \beta_1^{p-1} \alpha_1 \gamma_s.$$

The proposition holds. \hfill \qed

**Theorem 3.4.** Let $p \geq 7$ be an odd prime and $2 \leq s \leq p - 2$. Then in the topological small descent spectral sequence (TSDSS), we have the following differential

$$d_{2p-1}(h'_2, \beta_1 \gamma_s) = \alpha_1 \beta_1^{p-1} h_2 \gamma_s.$$

Proof. Recall from [21] page ?, one has the differential in the small descent spectral sequence (SDSS)

$$d_1(h_2 b_1) = \beta_1^{p-1}.$$

Then from $\beta_1 \beta_1^{p-1} \alpha_1 \gamma_s \in \text{Ext}_B^{1,s}(BP_*, BP, Y)$ and the relation $\alpha_1 \beta_1^{p-1} h_2 \gamma_s = \beta_1^{p-1} \alpha_1 \gamma_s$, we have the differential in the TSDSS

$$d_{2p-1}(h'_2, \beta_1 \gamma_s) = \beta_1 \beta_1^{p-1} \alpha_1 \gamma_s = \beta_1^{p-1} \alpha_1 \gamma_s.$$

The theorem follows. \hfill \qed

**4. The Proof of Theorem A**

In this section we prove our main theorem by showing that $\beta_2^{p^2/p^2-1}$ survives to $E_\infty$ in the topological small descent spectral sequence.

Consider the TSDSS introduced in section 2 Proposition 2.1 (b), the element $\beta_2^{p^2/p^2-1} \in E_0^{0,q(p^3+1)-2}$ To prove that $\beta_2^{p^2/p^2-1}$ survives to $E_\infty$ in the TSDSS, it is sufficient to prove that all the differentials $d_r(\beta_2^{p^2/p^2-1}) = 0$ for $r \geq 1$. From $\beta_2^{p^2/p^2-1} \in E_1^{0,q(p^3+1)-2}$, we see that

$$d_r(\beta_2^{p^2/p^2-1}) \in E_r^{q(p^3+1)-3+r},$$

in the TSDSS.

**Lemma 4.1.** For $r \geq 1$, the $E_1$-term $E_1^{r,q(p^3+1)-3+r}$ of the TSDSS is a $\mathbb{Z}/p$ module generated by the following 8 elements

$$g_1 = \beta_1^{p^2/p^2-1} h_2 \gamma_s \in E_1^{p^2-6p+1}; \quad g_2 = \beta_1^{p^2/p^2-1} h_2 \beta_1^{p^2/p^2-1}; \quad g_3 = \beta_1^{p^2/p^2-1} \eta_{(p-3)p^2}; \quad g_4 = \alpha_1 \beta_1^{p^2/p^2-1} \in E_1^{1,q(p^3+1)-2}; \quad g_5 = \alpha_1 \beta_1^{p^2/p^2-1} \in E_1^{1,q(p^3+1)-2};$$

$$g_6 = \alpha_1 \beta_1^{p^2/p^2-1} h_2 \gamma_s \in E_1^{p^2}; \quad g_7 = \alpha_1 \beta_1^{p^2/p^2-1} h_2 \gamma_s \in E_1^{p^2}; \quad g_8 = \alpha_1 \beta_1^{p^2/p^2-1} \beta_1 \in E_1^{p^2-1}.$$
Proof. From the ABC Theorem, we know that the generators of $E^{r,q(p^3+1)−3+r}_1$ are of the form $W = \beta^1 w$ or $W = \alpha_1 \beta^k w$, where $w$ is an element listed in the ABC Theorem.

1. If a generator of $E^{r,q(p^3+1)−3+r}_1$ is of the form $W = \beta^k w$, then the total degree of $\beta^1 w$ is $q(p^3 + 1) - 3$ and the total degree of $w$ is $q(p^3 + 1) - 3$ module the total degree of $\beta^1$ $(q p - 2)$. From $q(p^3 + 1) - 3 \equiv 4 p - 3 \ mod \ qp - 2$, we list all the generators whose total degree is $4p − 3$ $mod \ qp − 2$ (marked with underline and subscript $4p − 3$ in the ABC Theorem).

2. If a generator of $E^{r,q(p^3+1)−3+r}_1$ is of the form $W = \alpha_1 \beta^k w_1$, then the total degree of $\alpha_1 w_1$ must be congruent to $4p − 3$ modulo $|\beta^1| = qp − 2$, and the total degree of $w_1$ is $2p$ modulo $qp − 2$ from $|\alpha_1| = 2p − 3$. We can find all such $w_1’s$ as follows (marked with underline and subscript $2p$ in the ABC Theorem):

From which we get the following generators in $E^{r,q(p^3+1)−3+r}_1$:

$g_1 = \beta_1^{2-2 \epsilon_{p+1}} b^{2-2 \epsilon_{p+1}}_{11} \gamma_{\epsilon_{p+1}}$; $g_2 = \beta_1^{2-p} h_{20} \beta_{p/p}$; $g_3 = \beta_1^{p-1} \eta_{(p-3)p+3}$.

Compute the filtration of the corresponding generators, we get the lemma. □

Theorem 4.2. In the topological small descent spectral sequence, the element

$\beta_{p^2/p^2-1} \in E^0_{1,q(p^3+1)-2}$

is a permanent cycle.

This means that $\beta_{p^2/p^2-1}$ is a permanent cycle in the ANSS by the relation between the TSDSS and the ANSS.

Proof. We know that $\beta_{p^2/p^2-1} \in Ext^{2*}_{BP_*BP}(BP_*, BP_*)$ and in the TSDSS $d_1(\beta_{p^2/p^2-1}) = 0$. Thus $g_1$ and $g_5$ have too low filtration to to be the target of $d_1 \beta_{p^2/p^2-1}$.

In the TSDSS we have the following differentials:

1. From $d_{2p-1}(b_{11}) = \alpha_1 \beta_1^p$, we have

\[ d_{2p-1}(g_1) = d_{2p-1}(\beta_1^{2-2 \epsilon_{p+1}} b^{2-2 \epsilon_{p+1}}_{11} \gamma_{\epsilon_{p+1}}) = 2\alpha_1 \beta_1^{2-2 \epsilon_{p+1}} b_{11} \gamma_{\epsilon_{p+1}}. \]

2. From $d_1(h_{20} \beta_{p/p}) = \beta_1 \beta_{p/p-1}$ (cf. [21] section 5 of Chapter 7), we have

\[ d_1(g_2) = d_1(\beta_1^{2-p} h_{20} \beta_{p/p}) = \beta_1^{2-p} h_{20} \beta_{p/p}. \]

3. From the Toda differential $d_{2p-1}(\beta_{p/p}) = \alpha_1 \beta_1^p$ and the relation $\alpha_1 \eta_{(p-3)p+3} = \beta_{(p-3)p+4} \beta_{p/p}$ (21 7.5.7), we have

\[ d_{2p-1}(\alpha_1 \eta_{(p-3)p+3}) = \alpha_1 d_{2p-1} \eta_{(p-3)p+3} \]

\[ = d_{2p-1}(\beta_{(p-3)p+4} \beta_{p/p}) = \beta_{(p-3)p+4} d_{2p-1}(\beta_{p/p}) = \alpha_1 \beta_{(p-3)p+4} \beta_1^p. \]
Thus \( d_{2p-1}(\eta_{(p-3)p+3}) = \beta_{(p-3)p+4}\beta_1^p \) and
\[
d_{2p-1}(g_3) = d_{2p-1}(\beta_1^{p-1}\eta_{(p-3)p+3}) = \beta_1^{2p-1}\beta_{(p-3)p+4}.\]

(4) From Theorem 3.4, we have
\[
d_{2p-1}(\beta_1^{p^2-p-1} h_{20b_{11}^2 p+1}) = \beta_1^{p^2-4p-1} \alpha_1 \beta_1^p h_{20} \gamma_{p+1} = \alpha_1 \beta_1^{p^2-2p-1} h_{20} \gamma_{p+1} = g_6.
\]

(5) For \( 1 \leq m \leq \frac{p-1}{2} \), from \( d_1(h_{20}\beta_1^{m+1}p/p-m+1) = \beta_1\beta_1^{m+1}p/p-m \) (cf. [21] section 5 of Chapter 7), we have
\[
d_1(g_7) = d_1(\alpha_1 \beta_1^{m+1-p} \frac{m+1}{p} h_{20} \beta_1^{m+1}p/p-m+1) = \alpha_1 \beta_1^{m+1-p} \frac{m+1}{p} b_{11}^{m+1-m} \beta_1^{m+1}p/p-m.
\]

(6) We have
\[
d_{2p-1}(\beta_1^{p^2-p-1} b_{11}^2) = \alpha_1 \beta_1^{p^2-1} \beta_2 = g_8
\]

From the discussion above, we known that all the differentials \( d_r(\beta_{p^2/p^2-1}) = 0 \), for \( r \geq 1 \). This show that \( \beta_{p^2/p^2-1} \) is a permanent cycle in the TSDSS.

**Proposition 4.3.** Let \( p \geq 5 \) be an odd prime. For \( s \geq 1 \), \( j \leq p^2-1 \), \( \beta_{sp^2/j} \) is a permanent cycle in the ANSS and the corresponding homotopy element has order \( p \).

**Proof.** This proposition is easily got from the theorem above and the [22] Theorem 7.12.

**Corollary 4.4.** Let \( p \geq 5 \) be an odd prime. \( h_0h_3 \) is a permanent cycle in the Adams spectral sequence.

**Proof.** Let \( \phi : BP \to K\mathbb{Z}/p \) be the Thom map which induces the Thom reduction map
\[
\Phi : Ext^*_{BP_\mathbb{Z}/p}(BP_\mathbb{Z}/p, BP_\mathbb{Z}/p) \to Ext^*_{\mathbb{Z}/p}(\mathbb{Z}/p, \mathbb{Z}/p),
\]
then \( \Phi(\beta_{p^2/p^2-1} + x) = h_0h_3, \ x \in \ker \Phi \) (cf. [8] Theorem 9.4).
Hence \( h_0h_3 \) is the permanent cycle and the corresponding homotopy class has order \( p \) since \( x = 0 \) by degree reason.
Remark At prime 5, D. Ravenel in [20] page 304 proved that $\beta_{5/5/5}^5 = \beta_1 x_{952} \in \pi_{950} S^0$ survives to $E_\infty$ in the ANSS. Then from $\langle \beta_{5/5/5}^5, \alpha_1, 5 \rangle = 0$ and $\langle \alpha_1, 5, \alpha_1 \rangle = 0$, we know that the four-fold Toda bracket

$$\langle \beta_{5/5/5}^5, \alpha_1, 5, \alpha_1 \rangle$$

is well defined and then $\beta_{5/5/5} = \langle \beta_{5/5/5}^5, \alpha_1, 5, \alpha_1 \rangle$ survives to $E_\infty$ in the ANSS.

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