THE 2-DIMENSIONAL COMPLEX JACOBIAN CONJECTURE
UNDER THE VIEWPOINT OF “PERTINENT VARIABLES”

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Abstract. Let \( F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2 \) be a polynomial map. The Jacobian matrix of \( F \) at \((x, y) \in \mathbb{C}^2\) will be denoted by \( JF(x, y) \). The 2-dimensional Complex Jacobian Conjecture, which is still open, can be expressed as follows: “if \( F \) satisfies the Non-Zero Condition \( \det(JF(x, y)) = \text{constant} \neq 0, \forall (x, y) \in \mathbb{C}^2 \), then \( F \) is non-proper”. A significant approach for the study of the Jacobian Conjecture is to remove the most possible polynomial maps that do not satisfy the Non-Zero Condition and work on the complementary set in the ring of polynomial maps. We define firstly in this paper good polynomial maps (Definition 1) which satisfy the following property: if \( F \) satisfies the Non-Zero Condition, then \( F \) is a good polynomial map. Then, with the hypothesis “\( F \) is non-proper”, we define new variables called pertinent variables and we treat \( F \) under these variables. That allows us to define a class \( C_1 \) of good, non-proper polynomial maps \( F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2 \) that are significant for the study of the Jacobian Conjecture (Theorem 3). We continue to restrict the class \( C_1 \) by expliciting a subclass \( C_2 \subset C_1 \) of polynomial maps which do not satisfy the Non-Zero Condition (Theorem 11). Then, on the one side, we get a model of a counter-example for the Conjecture if there exists (Theorem 13). On the other side, we provide a criterion for verifying the 2-dimensional Complex Jacobian Conjecture for the class of good, non-proper maps (Theorem 14). Moreover, for verifying the 2-dimensional Complex Jacobian Conjecture, it is enough to verify it for the complementary set of the set \( C_2 \) in the set of good maps (Theorem 15). On another side, the class of dominant polynomial maps is also an important class for the study of the Jacobian Conjecture. The second part of this paper is to provide a criterion to verify the dominancy of a polynomial map in general (Proposition 19). Applying this criterion, we prove that a polynomial map of the class \( C_1 \setminus C_2 \) is dominant and we describe its asymptotic set (Propositions 16 and 23). We retrieve the Jelonek’s result in [J]: “the asymptotic set of a non-proper dominant polynomial map \( F : \mathbb{C}^2 \to \mathbb{C}^2 \) is a complex curve”.

1. Introduction

Let \( F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2 \) be a polynomial map satisfying the Non-Zero Condition

\[
\det(JF(x, y)) = \text{constant} \neq 0, \forall (x, y) \in \mathbb{C}^2,
\]

where \( JF(x, y) \) is the Jacobian matrix of \( F \) at \((x, y) \). Let us recall that under the Non-Zero Condition, the fact “\( F \) is diffeomorphism” is equivalent to “\( F \) is proper” (Hadamard, 1906, see [H]).

The asymptotic set of \( F \), denoted by \( S_F \), is the set of points at which \( F \) is non-proper. Then \( S_F \) is the set of points \( a \) in the target space such that there exists a sequence \( \{\xi_k\} \) tending to
infinity in the source space but its image $F(\xi_k)$ converges to $a$, that is

$$S_F := \{ a \in \mathbb{C}^2 : \exists \{\xi_k = (x_k, y_k)\}_{k \in \mathbb{N} \setminus \{0\} \subset \mathbb{C}^2, |\xi_k| \text{ tends to infinity and } F(\xi_k) \text{ tends to } a \},$$

where $|\xi_k|$ is the Euclidean norm of $\xi_k$ in the source space $\mathbb{C}^2$. Notice that it is sufficient to define $S_F$ by considering the sequences $\{\xi_k\}$ tending to infinity in the following sense: each coordinate of $\xi_k$ either tends to infinity or converges. In this paper, when we say that a sequence $\{\xi_k\}$ tends to infinity, we mean the norm $|\xi_k|$ tends to infinity. Then, if $\xi_k$ tends to infinity, there exists at least a coordinate of $\xi_k$ tending to infinity.

The 2-dimensional Complex Jacobian Conjecture, which is still open, can be expressed as follows:

If $F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ satisfies the Non-Zero Condition, then $F$ is proper.

If one wants to prove that the Jacobian Conjecture is true, one has to prove that for any polynomial $F : \mathbb{C}^2 \to \mathbb{C}^2$ satisfying the Non-Zero Condition, then $F$ is proper. Conversely, if one wants to prove that the Jacobian Conjecture is false, one has to indicate a non-proper polynomial map $F$ satisfying the Non-Zero Condition. Therefore, a significant work for the study of the Jacobian Conjecture is to remove the most possible polynomial maps that do not satisfy the Non-Zero Condition and work on the complementary set in the ring of polynomial maps. We define firstly in this paper good polynomial maps (Definition 1) which satisfy the following property: if $F$ satisfies the Non-Zero Condition, then $F$ is a good polynomial map.

Let us denote by $C_1$ the class of good, non-proper polynomial maps. The first part of this paper is to use the hypothesis “$F$ is non-proper” for constructing new variables called pertinent variables. Then we treat $F$ under these variables. The idea is the following: since $F$ is non-proper, then the asymptotic set of $F$ is not empty: there exists a sequence $\{\xi_k = (x_k, y_k)\}$ tending to infinity in the source space such that its image $F(\xi_k)$ does not tend to infinity. That means both of $f(\xi_k)$ and $g(\xi_k)$ do not tend to infinity. Since $\xi_k$ tends to infinity then $x_k$ tends to infinity or $y_k$ tends to infinity. Let us take the case where $x_k$ tends to infinity for expressing the idea of pertinent variables. Since $f(x_k, y_k)$ does not tend to infinity, we have to “eliminate” the infiniteness of the term $x$ in $f$ by subtracting $x$ from a monomial $\psi(x, y)$ such that $\psi(x_k, y_k)$ tends to infinity with the same velocity than the one of $x_k$. The new variable $u = x - \psi(x, y)$, where $\psi(x, y)$ is the “minimal” monomial satisfying the previous property, is called pertinent variable of $F$ with respect to the sequence considered $\{\xi_k\}$. Roughly speaking, a pertinent variable of a non-proper polynomial map with respect to a sequence tending to infinity $\{\xi_k = (x_k, y_k)\}$ is a minimal polynomial $u(x, y)$ in the two variables $x$ and $y$ such that $u(\xi_k)$ does not tend to infinity. Continuing the process for higher terms $x^t$ where $t > 1$, we obtain all the pertinent variables of $F$ with respect to the considered sequence. Notice that with a given polynomial
map, the degree \( n = \deg f(x, 0) \) is finite, then the above process ends when we reach the term \( x^n \). For that reason, the number of pertinent variables with respect to the considered sequence is finite. We obtain the description of the class \( \mathcal{C}_1 \) in Theorem 3.

We continue to restrict the class \( \mathcal{C}_1 \) to a smaller class which is significant for the study of the Jacobian Conjecture: we explicit a subclass \( \mathcal{C}_2 \subset \mathcal{C}_1 \) which does not satisfy the Non-Zero Condition (Theorem 11). Then, on the one side, we get a model of a counter-example for the 2-dimensional Complex Jacobian Conjecture if there exists (Theorem 13). On the other side, we provide a criterion for verifying the 2-dimensional Complex Jacobian Conjecture for the class of good, non-proper polynomial maps: to verify the Conjecture for the class \( \mathcal{C}_1 \), we need to verify it for the class \( \mathcal{C}_1 \backslash \mathcal{C}_2 \) only (Theorem 14). Moreover, for verifying the Conjecture, it is enough to verify it for the complementary set of the set \( \mathcal{C}_2 \) in the set of good maps (Theorem 15).

The second part of this paper is a study of the dominancy of a polynomial map. A polynomial map \( F : \mathbb{C}^2 \to \mathbb{C}^2 \) is dominant if \( F(\mathbb{C}^2) \) is dense in the target space. The class of dominant polynomial maps is an important class in the study of the Jacobian Conjecture. In fact, we have \( \overline{F(\mathbb{C}^2)} = S_F \cup F(\mathbb{C}^2) \) (see Lemma 18), where \( \overline{F(\mathbb{C}^2)} \) is the closure of \( F(\mathbb{C}^2) \) in the target space. Then, the study of dominant polynomial maps may help us to understand the behaviour of the maps at infinity. We provide in this paper a criterion to verify the dominancy of a polynomial map in general (Proposition 19). Applying this criterion, we prove that a polynomial maps in the class \( \mathcal{C}_1 \backslash \mathcal{C}_2 \) is dominant and we describe its asymptotic set (Propositions 16 and 23). We retrieve the Jelonek’s result [J]: “the asymptotic set of a non-proper dominant polynomial map \( F : \mathbb{C}^2 \to \mathbb{C}^2 \) is a complex curve”.

2. Construction of pertinent variables and the description of a good, non-proper polynomial map

**Definition 1.** A good polynomial map is a polynomial map \( F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2 \) such that the coordinate polynomials \( f \) and \( g \) have degrees greater than 1 of the form

\[
\begin{align*}
f &= \alpha x + \beta y + \text{ terms of higher degrees} \\
g &= \alpha' x + \beta' y + \text{ terms of higher degrees},
\end{align*}
\]

where \( \alpha, \beta, \alpha' \) and \( \beta' \) are non-zero complex numbers satisfying the condition

\[
\alpha \beta' - \alpha' \beta \neq 0.
\]

**Lemma 2.** A polynomial map \( F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2 \), where \( \deg f > 1 \) and \( \deg g > 1 \), satisfying the Non-Zero Condition is a good map.
Proof. Let \( F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2 \) be a polynomial map satisfying the Non-Zero Condition. Firstly, let us show that, under the hypothesis, then either \( f \) or \( g \) has non-zero terms of degree 1 with respect to \( x \) or \( y \).

Assume that both of \( f \) and \( g \) do not have non-zero terms of degree 1 with respect to \( x \) and \( y \), then \( \det(JF(x, y)) \) is a non-constant complex polynomial. Therefore, the equation \( \det(JF(x, y)) = 0 \) admits always complex solutions. This implies that the Non-Zero Condition is not satisfied, which is a contradiction with our hypothesis.

Hence, either \( f \) or \( g \) has non-zero term of degree 1 with respect to \( x \) or \( y \). We show now that both of \( f \) and \( g \) have non-zero terms of degree 1 with respect to both of \( x \) and \( y \). Without loss of generality, we can assume that \( f \) has non-zero term of degree 1 with respect to \( x \). If \( g \) does not have non-zero term of degree 1 with respect to \( y \), then the Non-Zero Condition implies that \( f \) has non-zero term of degree 1 with respect to \( y \) and \( g \) has non-zero term of degree 1 with respect to \( x \). Since \( \deg f > 1 \) and \( \deg g > 1 \), then the Abhyankar Similarity Theorem (see [E], Theorem 10.2.1, page 245) implies that the Newton polygons \( N(f) \) and \( N(g) \) are similar. Then both of \( N(f) \) and \( N(g) \) contain the two points \( (1, 0) \) and \( (0, 1) \). Consequently, both of \( f \) and \( g \) have non-zero terms of degree 1 with respect to both of \( x \) and \( y \). More precisely, \( f \) and \( g \) have the following forms:

\[
(2.2) \quad f = \alpha x + \beta y + \text{terms of higher degree}, \quad g = \alpha' x + \beta' y + \text{terms of higher degree},
\]

where \( \alpha, \beta, \alpha' \) and \( \beta' \) are non-zero complex numbers.

It is easy to see that if \( \alpha \beta' - \alpha' \beta = 0 \), then the Non-Zero Condition is not satisfied, which is a contradiction with our hypothesis. The Lemma is proved.

\[\square\]

**Theorem 3.** A polynomial map \( F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2 \) is a good, non-proper map if and only if each coordinate polynomial \( f \) and \( g \) is a finite sum of polynomials whose variables are finite products of finite sums of \( u_i \):

\[
(2.3) \quad \sum_{\eta=0}^{M<\infty} p_\eta \left( \prod_{j=0}^{t<\infty} \left( \sum_{i=0}^{n} \alpha_i u_i \right) \right), \quad \alpha_i \in \mathbb{C}, \quad \alpha_0 \neq 0 \text{ and } \alpha_1 \neq 0,
\]

where \( n = \max\{\deg f(x, 0), \deg g(x, 0)\} \) and

\[
u_0 = y, \quad u_i = x^i - x^{ir} y^s, \quad \text{with} \quad r, s \in \mathbb{N}\setminus\{0\},
\]

or the similar form, interchanging the roles of \( x \) and \( y \).

Here \( p_\eta \) is a polynomial, for \( \eta = 0, \ldots, M \) and \( t \) is a finite natural number depending on \( n \).
Proof. Assume that $F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ is a good polynomial map. Without loss of generality, we can assume that

$$f = x + \gamma y + \cdots, \quad g = x + \gamma' y + \cdots,$$

where $\gamma$ and $\gamma'$ are non-zero complex numbers such that $\gamma \neq \gamma'$.

Assume that $F$ is non-proper. Then its asymptotic set is non-empty. There exists a sequence $\{\xi_k = (x_k, y_k)\}$ tending to infinity such that both of $f(\xi_k)$ and $g(\xi_k)$ do not tend to infinity. Recall that it is enough to consider sequences $\xi_k$ tending to infinity in which $x_k$ tends to infinity or $y_k$ tends to infinity. Then we have three possible cases:

1) The case where both of $x_k$ and $y_k$ tend to infinity: assume that the ratio of the velocities of tending to infinity of $x_k$ and $y_k$ is $l : t$, where $l, t \in \mathbb{N}\{0\}$. Since $f(x_k, y_k)$ does not tend to infinity then it must appear in $f$ the term $(x - y^{l/t})$ to eliminate the infiniteness of $x$ in $f$. Similarly for $\gamma y$, then in $f$ appears the term $\gamma y - \gamma x^{l/t}$. But notice that $f$ is a polynomial, then $l/t$ and $t/l$ are positive integers. Hence, we have $l = t = 1$, and

$$f = (1 - \gamma)x + (\gamma - 1)y + \text{ terms of higher degree}.$$  

The similar thing happens for $g$, that means

$$g = (1 - \gamma')x + (\gamma' - 1)y + \text{ terms of higher degree}.$$  

Then $F$ is not a good polynomial map, since the coefficients of the terms of first degrees of $f$ and $g$ do not satisfy the condition (2.1). This case is excluded.

2) The case where $x_k$ tends to infinity and $y_k$ does not tend to infinity: then in $f$ must appear a polynomial of variable

$$x - x \varphi(x, y)$$

where $\varphi$ is a “minimal” monomial in the two variables $x$ and $y$ such that $\varphi(x_k, y_k)$ tends to 1.

We have two possible cases:

2.1) $\varphi(x, y) = y$ and $y_k$ tends to 1;

2.2) $\varphi(x, y) = x^d y^e$ where $d, e \in \mathbb{N}\{0\}$ and $y_k$ tends to 0 in such the way that $x_k^d y_k^e$ tends to 1 and $\gcd(d, e) = 1$.

Let us consider

$$u_1 = x - x \varphi(x, y).$$

In both of the two cases 2.1) and 2.2), we have

$$u_1 = x - x^r y^s,$$

where $r$ and $s$ are positive integers. Notice that $u_1(x_k, y_k)$ does not tend to infinity.
Similarly, if \( f \) contains the term \( x^2 \), then it appears in \( f \) a polynomial of the variable

\[
u_2 = x^2 - x^2 \varphi_2(x, y)
\]

where \( \varphi_2 \) is a minimal monomial in two variables \( x \) and \( y \) such that \( x_k^2 \varphi_2(x_k, y_k) \) tends to infinity with a velocity double than the one of \( x_k \). From the construction of the variable \( u_1 \) above, \( x_k^r y_k^s \) tends to infinity with a same velocity than the one of \( x_k \), then we have

\[
x^2 \varphi_2(x, y) = x^{2r} y^{2s}.
\]

Therefore

\[
u_2 = x^2 - x^{2r} y^{2s}.
\]

Let us denote by \( n \) and \( m \) the degrees \( \deg f(x, 0) \) and \( \deg g(x, 0) \), respectively. With a given polynomial map, we have \( n < \infty \) and \( m < \infty \). Without loss of generality, we can assume that \( n \geq m \). We finish to eliminate all the terms tending to infinity of \( f(\xi_k) \) and \( g(\xi_k) \) when we reach the term \( x^n \). Notice that if we put \( u_0(x, y) = y \), then \( u_0(\xi_k) \) does not tend to infinity. Therefore, each polynomial \( f \) and \( g \) must be a finite sum of polynomials whose variables are finite products of finite sums of \( u_0, u_1, \ldots, u_n \), where

\[
u_0 = y, \quad u_i = x^i - x^{ir} y^{is}, \quad \text{for} \ i = 1, \ldots, n.
\]

The infiteniteness of mixed terms of the form \( x^a y^b \) is eliminated by some polynomials of some variables \( u_i \), for some \( i = 1, \ldots, n \). Notice also that \( u_0 \) and \( u_1 \) must appear in \( f \) and \( g \), since \( f \) and \( g \) have non-zero terms of first degree with respect to both of \( x \) and \( y \). That explains why in the formula (2.3), we have \( \alpha_0 \neq 0 \) and \( \alpha_1 \neq 0 \). But some \( \alpha_j \) may be zero for some \( j \geq 2 \).

3) The case where \( x_k \) does not tend to infinity and \( y_k \) tends to infinity is done similarly as the above case, interchanging the roles of \( x \) and \( y \). In this case

\[
u_0 = x \quad \text{and} \quad u_i = y^i - y^{ir} x^{is}, \quad \text{for} \ i = 1, \ldots, n \quad \text{and} \quad r, s \in \mathbb{N}\setminus\{0\},
\]

where \( n \) is the maximum degree of \( \deg f(0, y) \) and \( \deg g(0, y) \).

Since the set of new variables \( \{u_0, u_1, \ldots, u_n\} \) in the second and the third cases have no intersection, then the proof of the Theorem is done.

\( \square \)

From the proof of the Theorem 3, we have the following corollary

**Corollary 4.** Let \( F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2 \) be good map. If \( F \) is non-proper then it happens exactly one possibility between the two following ones:

(1) The sequences tending to infinity such that \( F(\xi_k) \) does not tend to infinity are sequences \( \xi_k = (x_k, y_k) \) where \( x_k \) tends to infinity and \( y_k \) does not tend to infinity.
(2) The sequences tending to infinity such that $F(\xi_k)$ does not tend to infinity are sequences $\xi_k = (x_k, y_k)$ where $y_k$ tends to infinity and $x_k$ does not tend to infinity.

**Remark 5.** The variables $u_0, u_1, \ldots, u_n$ in the proof of the Theorem 3 are called “pertinent variables”. Roughly speaking, a pertinent variable of a non-proper polynomial map with respect to a sequence tending to infinity $\{\xi_k = (x_k, y_k)\}$ is a “minimal” polynomial $u(x, y)$ in two variables $x$ and $y$ such that $u(\xi_k)$ does not tend to infinity.

The notion of pertinent variables first appeared in [NT3] (Section 4.8, page 119 and Chapter 6, page 161) for characterizing and classifying the asymptotic sets associated to polynomial maps $F : \mathbb{C}^3 \to \mathbb{C}^3$ of degree 2. This notion was formalized as a definition in [NT1] for the case of dimension 3 and degree 2. The pertinent variables of a non-proper polynomial map with respect to a sequence tending to infinity reflect “the way” (called “façon” in [NT3]) in which the sequence tends to infinity. In the context of polynomial maps $F : \mathbb{C}^2 \to \mathbb{C}^2$ without any restriction of degree, the definition of pertinent variables of a good, non-proper map is the one provided in the proof of Theorem 3. Then this definition can be formalized as the following:

**Definition 6.** Let $F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ be a good, non-proper map. Then we have four possibilities:

1. The pertinent variables of $F$ with respect to the sequence $\xi_k = (x_k, y_k)$ where $x_k$ tends to infinity and $y_k$ tends to 1 are the ones of the type
   \[
   u_0 = y, \quad u_i = x^i - x^i y^i, \quad i = 1, \ldots, n,\]
   where $n = \max\{\deg f(x,0), \deg g(x,0)\}$.

2. The pertinent variables of $F$ with respect to the sequence $\xi_k = (x_k, y_k)$ where $x_k$ tends to infinity and $y_k$ tends to 0 are the ones of the type
   \[
   u_0 = y, \quad u_i = x^i - x^i y^i, \quad i = 1, \ldots, n,\]
   where $n = \max\{\deg f(x,0), \deg g(x,0)\}$, and the $d, e$ are natural numbers such that $x^d y^e$ tends to 1 and $\gcd(d, e) = 1$.

3. The pertinent variables of $F$ with respect to the sequence $\xi_k = (x_k, y_k)$ where $y_k$ tends to infinity and $x_k$ tends to 1 are the ones of the type
   \[
   u_0 = x, \quad u_i = y^i - y^i x^i, \quad i = 1, \ldots, n,\]
   where $n = \max\{\deg f(0,y), \deg g(0,y)\}$.

4. The pertinent variables of $F$ with respect to the sequence $\xi_k = (x_k, y_k)$ where $y_k$ tends to infinity and $x_k$ tends to 0 are the ones of the type
   \[
   u_0 = x, \quad u_i = y^i - y^i d x^e, \quad i = 1, \ldots, n,\]
where \( n = \max\{\deg f(0, y), \deg g(0, y)\} \), and the \( d, e \) are natural numbers such that \( y^d x^e \) tends to 1 and \( \gcd(d, e) = 1 \).

**Example 7.** Let us consider the polynomial map \( F : \mathbb{C}^2 \to \mathbb{C}^2 \) defined by
\[
F = (x + y - xy, x + 2y - xy^2 - 3x^6 + 6x^6 y^3 - 3y^6).
\]

By an easy calculation, following the above procedure, one shows that
\[
F = (y + (x - xy), 2y + (x - xy) - 3(x^3 - x^3 y^3)^2 + y(x - xy)).
\]

Then \( F \) is non-proper. In fact, the sequences tending to infinity such that \( F(\xi_k) \) does not tend to infinity are sequences \( \xi_k = (x_k, y_k) \) where \( x_k \) tends to infinity and \( y_k \) tends to 1. The pertinent variables of \( F \) are
\[
u_0 = y, \quad u_1 = x - xy \quad \text{and} \quad u_3 = x^3 - x^3 y^3.
\]

Then \( F = (f, g) \) where
\[
f = u_0 + u_1, \quad g = 2u_0 + u_1 - 3u_3^2 + u_0 u_1.
\]

3. **An approach to the 2-dimensional Complex Jacobian Conjecture via pertinent variables**

Without loss of generality, in this section, let us assume that \( n = \deg f(x, 0) \geq m = \deg g(x, 0) \) if \( f \) and \( g \) have the form (2.3). Otherwise, we assume that \( n = \deg f(0, y) \geq m = \deg g(0, y) \) if \( f \) and \( g \) have the similar form than the one of (2.3), interchanging the roles of \( x \) and \( y \).

**Definition 8** (of the class \( C_1 \)). One says that \( F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2 \) is a polynomial map of the class \( C_1 \) if \( F \) is written under the form (2.3) or the similar form, interchanging the roles of \( x \) and \( y \).

**Proposition 9.** A polynomial map \( F : \mathbb{C}^2 \to \mathbb{C}^2 \) is good, non-proper if and only if it belongs to the class \( C_1 \).

We continue now to restrict the class \( C_1 \) to a class \( C_1 \setminus C_2 \) which is significant for the study of the 2-dimensional Complex Jacobian Conjecture. That means, we will explicit a subclass \( C_2 \subset C_1 \) of polynomial maps which do not satisfy the Non-Zero Condition.

**Definition 10** (of the class \( C_2 \)). One says that \( F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2 \) is a polynomial map of the class \( C_2 \) if \( F \) belongs to the class \( C_1 \) with the integer \( r = 1 \), that is \( u_i = x^i (1 - y^i) \) or \( u_i = y^i (1 - x^i) \), for \( i = 1, \ldots, n \).
Theorem 11. A polynomial map belonging to the class $C_2$ is not a good polynomial map.

Before proving the Theorem 11, we prove the following lemma:

Lemma 12. Let $F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial map belonging to the class $C_1$.

1. If $u_0 = y$ and $u_i = x^i - x^r y^s$, for $i = 1, \ldots, n$, then the condition for the system
   \begin{equation}
   \frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} = \cdots = \frac{\partial u_n}{\partial x} = 0
   \end{equation}
   having solution is $r = 1$.

2. If $u_0 = x$ and $u_i = y^i - y^r x^s$, for $i = 1, \ldots, n$, then the condition for the system
   \begin{equation}
   \frac{\partial u_1}{\partial y} = \frac{\partial u_2}{\partial y} = \cdots = \frac{\partial u_n}{\partial y} = 0
   \end{equation}
   having solution is $r = 1$.

Proof. Assume that $F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ belongs to the class $C_1$ for the case $u_0 = y$ and $u_i = x^i - x^r y^s$, for $i = 1, \ldots, n$. By an easy calculation, we have:

\[
\frac{\partial u_1}{\partial x} = 1 - rx^{r-1}y^s, \quad \frac{\partial u_2}{\partial x} = 2x[1 - (\sqrt[r]{rx^{r-1}y^s})^2], \quad \cdots, \quad \frac{\partial u_n}{\partial x} = nx^{n-1}[1 - (\sqrt[r]{rx^{r-1}y^s})^n].
\]

In general, we have

\[
\frac{\partial u_i}{\partial x} = ix^{i-1}[1 - (\sqrt[r]{rx^{r-1}y^s})^i],
\]

for $i = 1, \ldots, n$, with the convention $\sqrt[r]{r} = r$.

Recall that $r$ is a positive integer. Then the first equation $\partial u_1/\partial x = 0$ implies that

\[x^{r-1}y^s = \frac{1}{r}.
\]

Therefore, the second equation $\partial u_2/\partial x = 0$ implies that $2x(1 - 1/r) = 0$. If $r > 1$, then $x = 0$ and that provides a contradiction with the fact of $x^{r-1}y^s = \frac{1}{r} \neq 0$. That implies $r = 1$ and this is the condition for the system (3.5) having solutions. More precisely, a solution of the system (3.5) has the form $(x, \sqrt[r]{r})$, where $x \in \mathbb{C}$. The second case of the Lemma is proved similarly, interchanging the roles of $x$ and $y$. \hfill \Box

We provide now a proof of the Theorem 11.

Proof [of Theorem 11]. Let $F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial map belonging to the class $C_2$. Assume that the map $F$ admits the form (2.3) with the integer $r = 1$ for the case $u_0 = y$ and $u_i = x^i(1 - y^s)$, for $i = 1, \ldots, n$. By the Lemma 12, in this case, the system (3.5) has solutions. Let $x_0 = (x_0, y_0)$ be a solution of this system, then

\[
\frac{\partial f}{\partial x}(x_0) = \frac{\partial g}{\partial x}(x_0) = 0.
\]
That implies
\[ \det(JF(x_0)) = \left( \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \right)(x_0) = 0. \]
Therefore, \( F \) does not satisfy the Non-Zero Condition. Hence, \( F \) is not a good polynomial map.

The case where \( u_0 = x \) and \( u_i = y^i(1 - x^i) \) for \( i = 1, \ldots, n \) is proved similarly.

We see that if there exists a polynomial map \( F \) belonging to the class \( C_1 \setminus C_2 \) satisfying the Non-Zero Condition then \( F \) is a counter-example for the 2-dimensional Complex Jacoban Conjecture. From Theorem 3, Proposition 9 and Theorem 11, we get the following Theorem, which is a model of a counter-example for the 2-dimensional Complex Jacobian Conjecture if there exists:

**Theorem 13** (A model of a counter-example for the 2-dimensional Complex Jacobian Conjecture if there exists). If there exists a polynomial map \( F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2 \) where \( f \) and \( g \) have the form
\[
\sum_{\eta=0}^{M<\infty} \sum_{j=0}^{n} \sum_{\alpha_i \in \mathbb{C}} p_{\eta} \left( \prod_{i=0}^{n} \alpha_i u_i \right), \quad \alpha_i \in \mathbb{C}, \quad \alpha_0 \neq 0 \text{ and } \alpha_1 \neq 0
\]
where
\[ u_0 = y \quad \text{and} \quad u_i = x^i - x^{ir} y^i, \quad \text{for } i = 1, \ldots, n, \quad r \geq 2, s \geq 1 \]
or
\[ u_0 = x \quad \text{and} \quad u_i = y^i - y^{ir} x^i, \quad \text{for } i = 1, \ldots, n, \quad r \geq 2, s \geq 1 \]
satisfying the Non-Zero Condition, then the 2-dimensional Complex Jacobian Conjecture is false.

Otherwise, we get a criterion for verifying the 2-dimensional Complex Jacobian Conjecture for the class of good, non-proper polynomial maps: in this case, we need to verify the Conjecture for the class \( C_1 \setminus C_2 \) only.

**Theorem 14** (A criterion for verifying the 2-dimensional Complex Jacobian Conjecture for the class of good, non-proper polynomial maps). If every polynomial map \( F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2 \) where \( f \) and \( g \) have the form
\[
\sum_{\eta=0}^{M<\infty} \sum_{j=0}^{n} \sum_{\alpha_i \in \mathbb{C}} p_{\eta} \left( \prod_{i=0}^{n} \alpha_i u_i \right), \quad \alpha_i \in \mathbb{C}, \quad \alpha_0 \neq 0 \text{ and } \alpha_1 \neq 0
\]
where
\[ u_0 = y \quad \text{and} \quad u_i = x^i - x^{ir} y^i, \quad \text{for } i = 1, \ldots, n, \quad r \geq 2, s \geq 1 \]
or
\[ u_0 = x \quad \text{and} \quad u_i = y^i - y^{ir} x^i, \quad \text{for } i = 1, \ldots, n, \quad r \geq 2, s \geq 1 \]
does not satisfy the Non-Zero Condition, then the 2-dimensional Complex Jacobian Conjecture is true for the class of good, non-proper map.
Finally, for verifying the 2-dimensional Complex Jacobian Conjecture, it is enough to verify it for the complementary set of the set \( C_2 \) in the set of good maps:

**Theorem 15** (A criterion for verifying the 2-dimensional Complex Jacobian Conjecture). *The 2-dimensional Complex Jacobian Conjecture is true if and only if every polynomial map \( F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2 \) where*

\[
    f = \alpha x + \beta y + \text{terms of higher degrees},
\]
\[
    g = \alpha' x + \beta' y + \text{terms of higher degrees},
\]

*such that \( \alpha \beta' - \alpha' \beta \neq 0 \) and \( f, g \) are not of the form*

\[
    \sum_{\eta=0}^{M<\infty} \prod_{j=0}^{t<\infty} \left( \sum_{i=0}^{n} \alpha_i u_i \right), \quad \alpha_i \in \mathbb{C}, \quad \alpha_0 \neq 0 \text{ and } \alpha_1 \neq 0
\]

*with*

\[
    u_0 = y \quad \text{and} \quad u_i = x^i - x^i y^i, \quad \text{for } i = 1, \ldots, n,
\]

*or*

\[
    u_0 = x \quad \text{and} \quad u_i = y^i - y^i x^i, \quad \text{for } i = 1, \ldots, n,
\]

*satisfies the Non-Zero Condition.*

**Proof.** If \( \deg f = 1 \) or \( \deg g = 1 \), the Jacobian Conjecture is true (see, for example, Theorem 1.8.1 of [NT2]). Now, if \( \deg f > 1 \) and \( \deg g > 1 \), and \( F \) is not a good map, then by the Lemma 2, \( F \) does not satisfy the Non-Zero Condition. Moreover, by the Theorem 11, a polynomial map belonging to the class \( C_2 \) does not satisfy the Non-Zero Condition. We conclude that for verifying the 2-dimensional Complex Jacobian Conjecture, it is sufficient to verify it for the complementary set of the set \( C_2 \) in the set of good maps only. The Theorem is proved.

\[ \square \]

4. **Asymptotic set and Dominancy of a map belonging to the class \( C_1 \setminus C_2 \)**

4.1. **Asymptotic set of a map belonging to the class \( C_1 \setminus C_2 \).** Let \( F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2 \) be a map belonging to the class \( C_1 \setminus C_2 \). Then \( f \) and \( g \) can be written under the form (2.3) or the similar form, interchanging the roles of \( x \) and \( y \). Let us assume that we have the form (2.3). We can separate the polynomial of the pertinent variable \( u_0 = y \) and the polynomial of the pertinent variable of highest degree from the form (2.3) of \( f \) and \( g \), then we can write (4.6)

\[
    f = p_0(y) + \sum_{\eta=0}^{M<\infty} p_\eta \left( \prod_{j=0}^{t<\infty} \left( \sum_{i=0}^{n} \alpha_i u_i \right) \right) + h(u_n), \quad g = g_0(y) + \sum_{\eta=1}^{M'<\infty} q_\eta \left( \prod_{j=0}^{t'<\infty} \left( \sum_{i=0}^{m} \beta_i u_i \right) \right) + k(u_m).
\]
Proposition 16. Let $F = (f, g) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be a map belonging to the class $C_1 \setminus C_2$. With the notations in (4.6), the asymptotic set $S_F$ of $F$ is given as follows:

(1) if $n = m$, then $S_F$ is the curve

$$\left\{ \left( p_0(0) + \sum_{\eta=1}^{M} p_\eta(0) + h(-nz), g_0(0) + \sum_{\eta=1}^{M'} q_\eta(0) + k(-nz) \right), z \in \mathbb{C} \right\}.$$ 

(2) if $n > m$, then $S_F$ is the curve

$$\left\{ \left( p_0(0) + \sum_{\eta=1}^{M} p_\eta(0) + h(-nz), g_0(0) + \sum_{\eta=1}^{M'} q_\eta(0) + k(0) \right), z \in \mathbb{C} \right\}.$$ 

(3) if $n < m$, then $S_F$ is the curve

$$\left\{ \left( p_0(0) + \sum_{\eta=1}^{M} p_\eta(0) + h(0), g_0(0) + \sum_{\eta=1}^{M'} q_\eta(0) + k(-mz), z \in \mathbb{C} \right) \right\}.$$ 

Proof. Let $F$ be a polynomial map of the class $C_1 \setminus C_2$ of the form (4.6). Recall that we denote by $n = \deg f(x, 0)$ and $m = \deg g(x, 0)$. Notice that $n \geq 1$. Let us consider a sequence $\{\xi_k = (x_k, y_k)\}$ tending to infinity in the source space such that $F(\xi_k)$ does not tend to infinity. Without loss of generality, we can assume that $x_k = k$, with $k \in \mathbb{N}\setminus\{0\}$. Let us remind that $u_i = x^i - x^r y^{i\lambda}$, for $i = 1, \ldots, n$. We have

$$u_i(x_k, y_k) = x_k^i (1 - x_k^{(r-1)i} y^{i\lambda}).$$ 

Since $F \notin C_1 \setminus C_2$, then $r > 1$. Since $u_i(x_k, y_k)$ does not tend to infinity, hence $y_k$ tends to zero in such a way that $(1 - x_k^{(r-1)i} y_k^{i\lambda})$ tends to zero with a velocity less than ou equal to the one of $x_k^i$. Consequently, $y_k$ must have the form

$$y_k = \sqrt[1-k^{r-1}]{1 + \frac{z}{k^n}},$$ 

where $z$ is a complex number. In fact,

$$u_i(x_k, y_k) = k^i \left( 1 - 1 - \sum_{\lambda=1}^{i} C_i^\lambda \left( \frac{z}{k^n} \right)^\lambda \right) = -\sum_{\lambda=1}^{i} C_i^\lambda \frac{z^\lambda}{k^{n\lambda - 1}},$$ 

where

$$C_i^\lambda = \frac{i!}{\lambda!(i-\lambda)!}.$$ 

If $i < n$, then $n\lambda - i > 0$, for all $\lambda = 1, \ldots, i$. In this case, $u_i(\xi_k)$ tends to 0, for $i = 1, \ldots, n - 1$. Consider now the case $i = n$, then

$$u_n(x_k, y_k) = -\sum_{\lambda=1}^{n} C_n^\lambda \frac{z^\lambda}{k^{n\lambda - n}}.$$
In that case, $n\lambda - n = 0$ when $\lambda = 1$ and $n\lambda - n > 0$ when $\lambda > 1$. Therefore $u_n(x_k, y_k)$ tends to $-nz$. We get the parametrized equation of the asymptotic set of $F$ for the cases (1) and (2) as in the statement of the Proposition. The case where $n < m$ is proved similarly.

4.2. Dominance of the class $C_1 \backslash C_2$.

4.2.1. A criterion for the dominancy of a polynomial map.

**Definition 17.** Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. Let $\overline{F(\mathbb{C}^n)}$ be the closure of $F(\mathbb{C}^n)$ in $\mathbb{C}^n$. The map $F$ is called **dominant** if $\overline{F(\mathbb{C}^n)} = \mathbb{C}^n$, i.e., $F(\mathbb{C}^n)$ is dense in $\mathbb{C}^n$.

Before proving the Proposition 19, that is a criterion for the dominancy of a polynomial map, we need the following lemma that appeared in [NT3].

**Lemma 18 ([NT3]).** Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. Then we have

$$\overline{F(\mathbb{C}^n)} = S_F \cup F(\mathbb{C}^n).$$

**Proof.** Notice that $S_F$ is contained in $\overline{F(\mathbb{C}^n)}$, then

$$S_F \cup F(\mathbb{C}^n) \subset \overline{F(\mathbb{C}^n)}.$$

Now let us take $a \in \overline{F(\mathbb{C}^n)}$. Then $a$ belongs to $F(\mathbb{C}^n)$ or $a$ belongs to $\overline{F(\mathbb{C}^n)} \backslash F(\mathbb{C}^n)$. In order to prove $\overline{F(\mathbb{C}^n)} \subset S_F \cup F(\mathbb{C}^n)$, we need to consider only the case $a \in \overline{F(\mathbb{C}^n)} \backslash F(\mathbb{C}^n)$. In that case, there exists a sequence $\{\xi_k\}$ in $\mathbb{C}^n$ such that $F(\xi_k)$ tends to $a$. Let us assume that $\xi_k$ does not tend to infinity. Then there exists a point $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ in the source space such that $\xi_k$ converges to $x$. Since $F$ is a polynomial map, then $F$ is continuous and $F(\xi_k)$ tends to $F(x)$. Since $\mathbb{C}^n$ is a Hausdorff space, then $F(x) = a$. Consequently, $a$ belongs to $F(\mathbb{C}^n)$, which contradicts to the fact that $a \in \overline{F(\mathbb{C}^n)} \backslash F(\mathbb{C}^n)$. Then $\xi_k$ tends to infinity and $a \in S_F$. We conclude that $\overline{F(\mathbb{C}^n)} \subset S_F \cup F(\mathbb{C}^n)$. The Lemma is proved. □

**Proposition 19** (Criterion for the dominancy of a polynomial map). Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping. Then $F$ is dominant if and only if for any $a \in \mathbb{C}^n \backslash S_F$, the set $F^{-1}(a)$ is not empty.

**Proof.** Assume that $F : \mathbb{C}^n \to \mathbb{C}^n$ be a dominant polynomial map, then $\overline{F(\mathbb{C}^n)} = \mathbb{C}^n$. By the Lemma 18, we have

$$\mathbb{C}^n = S_F \cup F(\mathbb{C}^n).$$

Let us take $a \in \mathbb{C}^n \backslash S_F$, then $a \in F(\mathbb{C}^n)$. There exists $x \in \mathbb{C}^n$ such that $F(x) = a$. That means $x \in F^{-1}(a)$, and we conclude that $F^{-1}(a) \neq \emptyset$. □
We assume now that $F : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map such that for any $a \in \mathbb{C}^n \setminus S_F$, the set $F^{-1}(a)$ is not empty. We prove that $F$ is dominant. We need to prove $F(\mathbb{C}^n) = \mathbb{C}^n$. It is clear that $\overline{F(\mathbb{C}^n)} \subseteq \mathbb{C}^n$. Let us take $a$ in the target $\mathbb{C}^n$. If $a \in S_F$, then $a \in \overline{F(\mathbb{C}^n)}$. Assume that $a \in \mathbb{C}^n \setminus S_F$. By the hypothesis of the Proposition, we have $F^{-1}(a) \neq \emptyset$. Then there exists $x \in \mathbb{C}^n$ such that $F(x) = a$, and therefore, $a \in F(\mathbb{C}^n)$. We conclude that $\mathbb{C}^n \subseteq \overline{F(\mathbb{C}^n)}$. The Proposition is proved.

Remark 20. The Proposition 19 says that for proving that a polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$ is dominant, it is sufficient to prove that for any $a \in \mathbb{C}^n \setminus S_F$, the equation $F(x) = a$ admits at least one solution $x \in \mathbb{C}^n$.

Remark 21. The Definition 17 is also valid for polynomial maps $F : \mathbb{K}^m \to \mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and the dimension of the source space can be bigger than the dimension of the target space. The necessary and sufficient conditions for the dominancy in the Proposition 19 hold also for polynomial maps $F : \mathbb{K}^m \to \mathbb{K}^n$.

We recall here an important result of Jelonek on the asymptotic set of a dominant polynomial map.

Theorem 22 ([J]). Let $F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. If $F$ is dominant, then $S_F$ is either an empty set or an algebraic hypersurface.

4.2.2. Dominancy of the class $\mathcal{C}_1 \setminus \mathcal{C}_2$.

Proposition 23. A polynomial map of the class $\mathcal{C}_1 \setminus \mathcal{C}_2$ is dominant.

Proof. Let $F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial map belonging to the class $\mathcal{C}_1 \setminus \mathcal{C}_2$. Assume that each coordinate polynomial $f$ and $g$ is a finite sum of polynomials whose variables are finite products of finite sums of $u_0, u_1, \ldots, u_n$ where

$$u_0 = y, \quad u_i = x^i - x^r y^s, \quad \text{with} \quad r, s \in \mathbb{N} \setminus \{0\}.$$  

Separate the polynomial of the pertinent variable $u_1$ and the polynomial of the pertinent variable $u_0 = y$ from the form of $f$ and $g$, we can write

$$f(x, y) = p(u_1) + q(y) + \cdots, \quad g(x, y) = h(u_1) + k(y) + \cdots.$$  

Since $F$ is a good map, then $p, h, q$ and $k$ are non-zero polynomials.

Let us take $a = (\alpha, \beta) \in \mathbb{C}^2 \setminus S_F$. Consider the system of equations

$$f(x, y) = \alpha, \quad g(x, y) = \beta.$$  

(4.7)
Since \((\alpha, \beta) \notin S_F\), then by the Theorem 16, there exists \(\mu \geq 1\) such that \(u_\mu(x, y) \neq 0\). We have
\[
u(x, y) = x^\mu(1 - x^{\mu r - \mu s} y^{\mu s}) \neq 0.
That implies \(x \neq 0\). Since \(p(u_1(x, y))\) is a non-zero polynomial, then the first equation of the system (4.7) admits at least a non-constant solution with respect to the variable \(y\), written as a function in the variable \(x\). Let us assume that \(y_0(x)\) is a non-constant solution of this equation.

One knows that \(h \neq 0\) and \(h\) is a polynomial in the variable \(u_1 = x - x^r y^s\), where \(r\) and \(s\) are positive integers. Then \(h(u_1)\) provides for \(g\) a non-zero term of degree 1 with respect to \(x\). Now, notice that the other elements of \(g\) are polynomials whose variables are combinations of \(u_j\), for \(j = 2, \ldots, n\) and each \(u_j\) does not have the term of first degree with respect to \(x\). By replacing \(y_0(x)\) to the second equation of the system (4.7), we get a non-constant complex equation in \(x\). This equation has always complex solutions. Let \(x_0\) be a (complex) solution of this equation. Then \((x_0, y_0(x_0))\) is a solution of the system (4.7). By the Proposition 19, we have \(F^{-1}(\alpha, \beta) \neq \emptyset\), and hence, \(F\) is dominant.

The case where \(F\) admits the similar form than the form (2.3), interchanging the roles of \(x\) and \(y\), is proved similarly.

\[\square\]

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