Nonlinear stability of sinusoidal Euler flows on a flat two-torus

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Abstract
Sinusoidal flows are important explicit stationary solutions of the incompressible Euler equation on a flat two-torus. In this paper, we prove that sinusoidal flows related to least eigenfunctions of the negative Laplacian are, up to phase translations, nonlinearly stable under $L^p$ norm of the vorticity for any $1 < p < +\infty$, which improves a classical stability result by Arnold. The key point of the proof is to distinguish least eigenstates with different amplitudes by using isovortical property of the Euler equation.

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1 Introduction and main result

1.1 Euler equation on a flat two-torus

Let $\mathbb{T}^2$ be the flat two-torus with fundamental domain
$$\mathbb{T}^2 = [0, 2\pi \nu_1] \times [0, 2\pi \nu_2],$$
where $\nu_1, \nu_2 > 0$. The motion of an ideal fluid of unit density on $\mathbb{T}^2$ is described by the following Euler equation:
$$\begin{cases}
\partial_t v + (v \cdot \nabla)v = -\nabla P, & x = (x_1, x_2) \in \mathbb{T}^2, \ t > 0, \\
\nabla \cdot v = 0.
\end{cases}$$

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where $\mathbf{v} = (v_1, v_2)$ is the velocity field, and $P$ is the scalar pressure. The scalar vorticity $\omega$ of the fluid is given by

$$\omega = \partial_{x_1} v_2 - \partial_{x_2} v_1. \quad (1.2)$$

For smooth solutions of (1.1), the following quantities are conserved (cf. [21, 22]):

(C1) total flux $F$,

$$F = \int_{T^2} \mathbf{v} d\mathbf{x}, \quad (1.3)$$

(C2) kinetic energy $E$,

$$E = \frac{1}{2} \int_{T^2} |\mathbf{v}|^2 d\mathbf{x}, \quad (1.4)$$

(C3) distribution function $d_\omega(t, \cdot) : \mathbb{R} \mapsto \mathbb{R}$ of the vorticity,

$$d_\omega(t, s) = |\{ \mathbf{x} \in T^2 \mid \omega(t, \mathbf{x}) > s \}|, \quad s \in \mathbb{R},$$

where $|\cdot|$ denotes the two-dimensional Lebesgue measure.

For $f \in L^1_{\text{loc}}(T^2)$, denote by $\mathcal{R}(f)$ the set of all equimeasurable rearrangements of $f$ on $T^2$,

$$\mathcal{R}(f) := \{ g \in L^1_{\text{loc}}(T^2) \mid d_g = d_f \}.$$  

Then the conservation of the distribution function of the vorticity can be expressed as

$$\omega(t, \cdot) \in \mathcal{R}(\omega(0, \cdot)), \quad \forall \, t \geq 0. \quad (1.5)$$

As a consequence, the $L^p$ norm of the vorticity is conserved for any $1 \leq p \leq +\infty$.

Below we introduce the vorticity-stream formulation of Euler equation (1.1). Define the normalized velocity $\tilde{\mathbf{v}}$ as

$$\tilde{\mathbf{v}} : = \mathbf{v} - \frac{1}{|T^2|} \mathbf{F}, \quad (1.6)$$

where $\mathbf{F}$ is the total flux given by (1.3), which is a constant vector. Then $\tilde{\mathbf{v}}$ is divergence-free and has zero mean over $T^2$. Consequently there is a function $\tilde{\psi} : T^2 \mapsto \mathbb{R}$ (cf. [21], p. 50), called the normalized stream function, such that $\tilde{\mathbf{v}} = \nabla \perp \tilde{\psi}$. Here and henceforth, $\mathbf{b}^\perp = (b_2, -b_1)$ denotes the clockwise rotation through $\pi/2$ of the vector $\mathbf{b} = (b_1, b_2)$, and $\nabla \perp \tilde{\psi} = (\nabla \tilde{\psi})^\perp$. Without loss of generality, by adding a suitable constant, we always assume that the normalized stream function is of zero mean over $T^2$. Then $\tilde{\psi}$ satisfies

$$\begin{cases}
-\Delta \tilde{\psi} = \omega, & \mathbf{x} \in T^2, \\
\int_{T^2} \tilde{\psi} d\mathbf{x} = 0.
\end{cases} \quad (1.7)$$

By Lemma 2.1 in Sect. 2, (1.7) has a unique solution $\tilde{\psi} = K \omega$. Thus $\mathbf{v}$ can be expressed in terms of $\omega$ and $\mathbf{F}$ via the following Biot-Savart law:

$$\mathbf{v} = \nabla \perp K \omega + \frac{1}{|T^2|} \mathbf{F}. \quad (1.8)$$

The stream function $\psi$ of the fluid is defined by

$$\psi : = K \omega - \frac{1}{|T^2|} \mathbf{F}^\perp \cdot \mathbf{x}, \quad (1.9)$$
such that $\mathbf{v} = \nabla \perp \psi$. According to the above discussion, at any time the state of the fluid can be described by $\mathbf{v}$, or $\psi$, or the pair $(\omega, \mathbf{F})$. The kinetic energy in terms of $\omega$ and $\mathbf{F}$ is

$$E = \frac{1}{2} \int_{\mathbb{T}^2} |\nabla K\omega|^2 d\mathbf{x} + \frac{1}{2|\mathbb{T}^2|} |\mathbf{F}|^2. \quad (1.10)$$

Define

$$E(\omega) := \frac{1}{2} \int_{\mathbb{T}^2} |\nabla K\omega|^2 d\mathbf{x} = \frac{1}{2} \int_{\mathbb{T}^2} \omega K \omega d\mathbf{x}, \quad (1.11)$$

which is a conserved quantity.

### 1.2 Sinusoidal flows and Arnold’s stability theorem

Stationary solutions of the Euler equation (1.1) are characterized by having $\nabla \psi$ and $\nabla \omega$ collinear. For this to hold, a sufficient condition is that $\psi$ and $\omega$ satisfy the relation

$$\omega = f(\psi). \quad (1.12)$$

for some function $f : \mathbb{R} \mapsto \mathbb{R}$. In particular, if $f$ is linear, then (1.12) becomes the eigenvalue problem of $-\Delta$ on $\mathbb{T}^2$:

$$-\Delta \psi = \lambda \psi. \quad (1.13)$$

For (1.13), any eigenvalue $\lambda$ has the form (cf. [24], p. 32)

$$\lambda = \left(\frac{k_1}{\nu_1}\right)^2 + \left(\frac{k_2}{\nu_2}\right)^2, \quad k_1, k_2 \in \mathbb{Z}, \ k_1^2 + k_2^2 \neq 0, \quad (1.14)$$

and the associated eigenfunction has the form

$$\sum_{(j_1,j_2) \in J_\lambda} A_{j_1,j_2} \sin \left(\frac{j_1}{\nu_1} x_1 + \frac{j_2}{\nu_2} x_2 + \alpha_{j_1,j_2}\right) + \sum_{(j_1',j_2') \in J_\lambda} B_{j_1',j_2'} \sin \left(\frac{j_1'}{\nu_1} x_1 - \frac{j_2'}{\nu_2} x_2 + \beta_{j_1',j_2'}\right),$$

where $A_{j_1,j_2}, B_{j_1,j_2} \geq 0, \alpha_{j_1,j_2}, \beta_{j_1,j_2} \in \mathbb{R}$, and

$$J_\lambda := \left\{ (j_1,j_2) \in \mathbb{Z}^2 \mid \left(\frac{j_1}{\nu_1}\right)^2 + \left(\frac{j_2}{\nu_2}\right)^2 = \lambda \right\}. \quad (1.15)$$

A **sinusoidal flow**, or an **eigenstate**, is an Euler flow whose stream function is an eigenfunction of $-\Delta$ on $\mathbb{T}^2$. Note that the total flux of any sinusoidal flow is 0.

The stability of sinusoidal flows is a fundamental problem in fluid dynamics and has been extensively studied in the literature. For the linear theory, the results are quite rich (although many open questions still remain). See [6, 11, 16, 19, 20, 23] and the references therein. As to nonlinear stability, the first rigorous result dates back to Arnold [2, 3], where two types of nonlinear stability criteria for plane ideal flows were proved, now usually referred to as Arnold’s first and second stability theorems. See also [25, 28, 29]. According to Arnold’s second stability theorem, the set of sinusoidal flows related to **least eigenfunctions** of $-\Delta$ on $\mathbb{T}^2$ is nonlinearly stable in $L^2$ norm of the vorticity.

To state Arnold’s result, we briefly analyze the least eigenvalue $\lambda_1$ of $-\Delta$ on $\mathbb{T}^2$. Without loss of generality, we assume that $\nu_1 \leq \nu_2$ throughout this paper (the case $\nu_1 \leq \nu_2$ is equivalent by symmetry). We distinguish two cases in view of (1.14).
(i) If \( v_1 < v_2 \), then \( \lambda_1 = v_2^{-2} \), \( J_{\lambda_1} = \{(0, 1), (0, -1)\} \), and any least eigenfunction \( \psi_1 \) takes the form

\[
\psi_1(x_1, x_2) = A \sin \left( \frac{x_1}{v_1} + \alpha \right), \tag{1.16}
\]

where \( A \geq 0 \) is called the amplitude, and \( \alpha \in \mathbb{R} \) is called the phase parameter.

(ii) If \( v_1 = v_2 = v \), then \( \lambda_1 = v^{-2} \), \( J_{\lambda_1} = \{(0, 1), (0, -1), (1, 0), (-1, 0)\} \), and any least eigenfunction \( \psi_1 \) takes the form

\[
\psi_1(x_1, x_2) = A \sin \left( \frac{x_1}{v} + \alpha \right) + B \sin \left( \frac{x_2}{v} + \beta \right) \tag{1.17}
\]

for some \( A, B \geq 0 \) and \( \alpha, \beta \in \mathbb{R} \).

Since it is more convenient to express Arnold’s result in terms of vorticity, we denote by \( \mathcal{U} \) the set of vorticity functions of the sinusoidal flows with stream function (1.16),

\[
\mathcal{U} := \left\{ \frac{A \sin \left( \frac{x_2}{v_2} + \alpha \right)}{A \geq 0, \alpha \in \mathbb{R}} \right. \tag{1.18}
\]

If \( v_1 = v_2 = v \), we denote by \( \mathcal{V} \) the set of vorticity functions of the sinusoidal flows with stream function (1.17),

\[
\mathcal{V} := \left\{ A \sin \left( \frac{x_1}{v} + \alpha \right) + B \sin \left( \frac{x_2}{v} + \beta \right) \mid A, B \geq 0, \alpha, \beta \in \mathbb{R} \right\}. \tag{1.19}
\]

**Theorem 1.1** (Arnold, [2, 3]) Let \( \mathcal{U}, \mathcal{V} \) be defined by (1.18), (1.19).

(i) If \( v_1 < v_2 \), then \( \mathcal{U} \) is nonlinearly stable in \( L^2 \) norm, i.e., for any \( \varepsilon > 0 \), there exists some \( \delta > 0 \), such that for any smooth Euler flow on \( \mathbb{T}^2 \) with vorticity \( \omega \), we have that

\[
\min_{v \in \mathcal{U}} \| \omega(t, \cdot) - v \|_{L^2(\mathbb{T}^2)} < \delta \implies \min_{v \in \mathcal{U}} \| \omega(t, \cdot) - v \|_{L^2(\mathbb{T}^2)} < \varepsilon \quad \forall t > 0. \tag{1.20}
\]

(ii) If \( v_1 = v_2 = v \), then \( \mathcal{V} \) is nonlinearly stable in \( L^2 \) norm, i.e., (1.20) holds with \( \mathcal{U} \) replaced by \( \mathcal{V} \).

**Remark 1.2** If \( v_1 < v_2 \), then the sinusoidal flow with stream function

\[
\psi(x_1, x_2) = A \sin \left( \frac{x_1}{v_1} + \alpha \right), \quad A \neq 0, \alpha \in \mathbb{R},
\]

is known to be linearly unstable in a certain sense. See [6].

The proof of Theorem 1.1 is based on the energy-Casimir method proposed by Arnold. See also [4], p. 98 and [22], p. 111.

According to Theorem 1.1, any least eigenstate is nonlinearly stable up to phase translations and amplitude scalings. For example, in the case \( v_1 < v_2 \), given a sinusoidal flow with vorticity \( A_0 \sin(v_2^{-1}x_2 + \alpha_0) \), if a smooth Euler flow is “close” to this sinusoidal flow at initial time, then at any \( t > 0 \) the evolved flow is “close” to some sinusoidal flow with vorticity \( A(t) \sin(v_2^{-1}x_2 + \alpha(t)) \). Here “closeness” is measured in terms of the \( L^2 \) norm of the vorticity. Since \( A(t) \) and \( \alpha(t) \) may vary with time, it is not clear whether the single sinusoidal flow with vorticity \( A_0 \sin(v_2^{-1}x_2 + \alpha_0) \) is nonlinearly stable.
1.3 Main result

The nonlinear stability of a single sinusoidal flow was listed as an open problem on p. 112 of Marchioro and Pulvirenti’s book [22]. Arnold’s method cannot handle this problem since it only involves energy and enstrophy (cf. (2.10)) conservations, which is not enough to distinguish different sinusoidal flows well. In [27], Wirosoetisno and Shepherd employed high-order (cubic, quartic and quintic) Casimirs to bound the variation of the amplitudes in the case of a square torus. As a consequence, they obtained the nonlinear stability of a single sinusoidal flow only up to phase translations. However, there is a limitation in Wirosoetisno-Shepherd’s work, i.e., the stability they obtained depends on high-order Casimirs of the initial state.

Our purpose in this paper is to give an extension of Theorem 1.1 and the stability result in [27]. To state our result, for fixed \( A \geq 0 \), define

\[
U_A = \left\{ A \sin \left( \frac{x_2}{\nu_2} + \alpha \right) \mid \alpha \in \mathbb{R} \right\}.
\]

(1.21)

If \( \nu_1 = \nu_2 = \nu \), for fixed \( A \geq 0 \), define

\[
V_{A,B} = \left\{ A \sin \left( \frac{x_1}{\nu} + \alpha \right) + B \sin \left( \frac{x_2}{\nu} + \beta \right) \mid \alpha, \beta \in \mathbb{R} \right\}.
\]

(1.22)

It is easy to see that

\[
U = \bigcup_{A > 0} U_A, \quad V = \bigcup_{A, B \geq 0} V_{A,B}.
\]

The main result of this paper is the following theorem.

**Theorem 1.3** Let \( 1 < p < +\infty \) be fixed.

(i) If \( \nu_1 < \nu_2 \), then for any \( A \geq 0 \), \( U_A \) is nonlinearly stable in \( L^p \) norm, i.e., for any \( \varepsilon > 0 \), there exists some \( \delta > 0 \), such that for any smooth Euler flow on \( \mathbb{T}^2 \) with vorticity \( \omega \), it holds that

\[
\min_{v \in U_A} \| \omega(0, \cdot) - v \|_{L^p(\mathbb{T}^2)} < \delta \implies \min_{v \in U_A} \| \omega(t, \cdot) - v \|_{L^p(\mathbb{T}^2)} < \varepsilon \quad \forall t > 0.
\]

(1.23)

(ii) If \( \nu_1 = \nu_2 = \nu \), then for any \( A, B \geq 0 \), \( V_{A,B} \) is nonlinearly stable in \( L^p \) norm, i.e., (1.23) holds with \( U_A \) replaced by \( V_{A,B} \).

**Remark 1.4** In Theorem 1.3, to avoid some technical (but not essential) difficulties and illustrate the main idea clearly, we only consider smooth perturbations. However, by checking the proof carefully, Theorem 1.3 actually holds for a larger class of less regular perturbations as long as the quantities (C1)–(C3) are conserved, the vorticity is continuous in \( L^p(\mathbb{T}^2) \) with respect to the time variable, and a “follower” to the perturbed vorticity as in Sect. 3.3 exists.

**Remark 1.5** By the \( L^p \) estimate (2.3) in Section 2, the stability in Theorem 1.3 can also be equivalently measured in terms of the \( W^{1,p} \) norm of the normalized velocity, or the \( W^{2,p} \) norm of the normalized stream function.

To prove Theorem 1.3, we use a variational approach in combination with a compactness argument, which is very different from the classical energy-Casimir method used in Arnold [2, 3] and Wirosoetisno-Shepherd [27]. The proof consists of three key ingredients: suitable variational characterizations for the flows under consideration, compactness argument, and appropriate use of flow invariants of the Euler equation. These three ingredients are also
essential in the nonlinear stability analysis of many other stationary Euler flows. See [1, 5, 9, 10, 13–15, 26] for example. The variational characterization, which states that the sinusoidal flows in Theorem 1.3 are exactly the set of maximizers of the conserved functional $E$ relative to the set of all isovortical flows to them, is the most important step of the proof. The advantage of such variational characterization is that we are able to distinguish the sinusoidal flows with vorticity in $U_A$ or $V_{A,B}$ from other least eigenstates completely, which can not be achieved by merely using energy and enstrophy conservations.

It is worth mentioning that nonlinear stability up to phase translations is the optimal nonlinear stability result one can expect. For example, consider the sinusoidal flow with stream function $\bar{\psi}$ and vorticity $\bar{\omega}$,

\[
\bar{\psi} = \sin \left( \frac{x_2}{v_2} \right), \quad \bar{\omega} = -\frac{1}{v_2^2} \sin \left( \frac{x_2}{v_2} \right).
\]

We can construct a non-stationary smooth Euler flow with stream function $\psi$ and vorticity $\omega$,

\[
\psi(t, x) = \sin \left( \frac{x_2}{v_2} + t \right) + v_2 x_1, \quad \omega(t, x) = -\frac{1}{v_2^2} \sin \left( \frac{x_2}{v_2} + t \right),
\]

such that $\omega(0, \cdot) = \bar{\omega}$, but $\omega(t, \cdot) \not= \bar{\omega}$ unless $t$ is an integer multiple of $2\pi$. In this example, the total flux of the non-stationary flow is not zero, causing the phase change of the vorticity at a constant rate along the $x_2$-direction. Note that if we only consider perturbations with zero total flux, then the nonlinear stability of a single sinusoidal flow still remains open. This is an interesting further work.

This paper is organized as follows. In Sect. 2, we present some preliminaries for later use. In Sect. 3, we give the proof of Theorem 1.3.

### 2 Preliminaries

First we collect some definitions and basic facts that will be used in subsequent sections. For $1 \leq p \leq +\infty$ and $k \in \mathbb{Z}^+$, denote by $L^p(\mathbb{T}^2)$ and $W^{k,p}(\mathbb{T}^2)$ the usual real $L^p$ space and Sobolev space on $\mathbb{T}^2$, respectively. Define

\[
\dot{L}^p(\mathbb{T}^2) = \left\{ f \in L^p(\mathbb{T}^2) \mid \int_{\mathbb{T}^2} f \, dx = 0 \right\},
\]

\[
\dot{W}^{k,p}(\mathbb{T}^2) = \left\{ f \in W^{k,p}(\mathbb{T}^2) \mid \int_{\mathbb{T}^2} f \, dx = 0 \right\}.
\]

It is clear that $\dot{L}^p(\mathbb{T}^2)$ is a closed subspace of $L^p(\mathbb{T}^2)$, and $\dot{W}^{k,p}(\mathbb{T}^2)$ is a closed subspace of $W^{k,p}(\mathbb{T}^2)$. Denote by $L^p(\mathbb{T}^2; \mathbb{C})$ the complex $L^p$ space on $\mathbb{T}^2$. Note that $L^2(\mathbb{T}^2; \mathbb{C})$ is a Hilbert space endowed with the inner product

\[
< f, g > = \int_{\mathbb{T}^2} f(x) \overline{g(x)} \, dx, \quad \forall f, g \in L^2(\mathbb{T}^2; \mathbb{C}),
\]

where $\overline{g(x)}$ is the complex conjugate of $g(x)$. Denote by $\mathbb{Z}^2$ the set of all points in $\mathbb{R}^2$ with integer coordinates. For $k = (k_1, k_2) \in \mathbb{Z}^2$, define

\[
\xi_k(x) = \frac{1}{|\mathbb{T}^2|^{1/2}} e^{i \left( \frac{k_1}{v_1} x_1 + \frac{k_2}{v_2} x_2 \right)}, \quad x = (x_1, x_2) \in \mathbb{T}^2.
\]
Then \( \{ \xi_k \}_{k \in \mathbb{Z}^2} \) is an orthonormal basis of \( L^2(\mathbb{T}^2; \mathbb{C}) \) (cf. [17], p. 186). For \( f \in L^1(\mathbb{T}^2; \mathbb{C}) \) and \( k \in \mathbb{Z}^2 \), the Fourier series of \( f \) is \( f \sim \sum_{k \in \mathbb{Z}^2} \hat{f}_k \xi_k \), where \( \hat{f}_k \) is the \( k \)-th Fourier coefficient,
\[
\hat{f}_k = \langle f, \xi_k \rangle = \int_{\mathbb{T}^2} f(x) \xi_k(x) dx.
\]
For \( f \in L^1(\mathbb{T}^2; \mathbb{C}) \) and \( N \in \mathbb{Z}^+ \), denote by \( f_N \) the \( N \)-th square partial sum of the Fourier series of \( f \), i.e.,
\[
f_N = \sum_{k \in \mathbb{Z}^2 : |k|_\infty \leq N} \hat{f}_k \xi_k, \quad \text{where } |k|_\infty := \max(|k_1|, |k_2|). \tag{2.1}
\]
Note that if \( f \) is real-valued, then \( \hat{f}_{-k} = \overline{\hat{f}_k} \) for any \( k \in \mathbb{Z}^2 \), thus \( f_N \) is also real-valued for any \( N \in \mathbb{Z}^+ \). Also note that for fixed \( 1 < p < +\infty \), if \( f \in L^p(\mathbb{T}^2; \mathbb{C}) \), then \( f_N \to f \) in \( L^p(\mathbb{T}^2) \) as \( N \to +\infty \) (cf. [17], Theorem 4.1.8).

The following lemma may be known to some extent, but we can not find it in the literature.

**Lemma 2.1** Let \( 1 < p < +\infty \) and \( f \in \hat{L}^p(\mathbb{T}^2) \). Then the following Poisson equation
\[
\begin{aligned}
-\Delta u &= f, & x &\in \mathbb{T}^2, \\
u &\in \hat{W}^{2,p}(\mathbb{T}^2),
\end{aligned} \tag{2.2}
\]
has a unique solution. Moreover, the following \( L^p \) estimate holds:
\[
\|u\|_{W^{2,p}(\mathbb{T}^2)} \leq C \|f\|_{L^p(\mathbb{T}^2)}, \tag{2.3}
\]
where \( C > 0 \) depends only on \( v_1, v_2 \) and \( p \).

**Proof** First we prove existence. Consider the following approximate equation:
\[
\begin{aligned}
-\Delta u_N &= f_N, & x &\in \mathbb{T}^2, \\
u_N &\in \hat{W}^{2,p}(\mathbb{T}^2),
\end{aligned} \tag{2.4}
\]
where \( f_N \) is \( N \)-th square partial sum of the Fourier series of \( f \) defined by (2.1). Since \( f \in \hat{L}^p(\mathbb{T}^2) \), we have \( f_0 = 0 \). Then it is easy to check that (2.4) admits an explicit solution:
\[
u_N = \sum_{k \in \mathbb{Z}^2 : 0 < |k|_\infty \leq N} \frac{\hat{f}_k}{(\frac{k_1}{v_1})^2 + (\frac{k_2}{v_2})^2} \xi_k.
\]
Moreover, we have the following uniform estimate for \( u_N \) (cf. [12], Theorem 10):
\[
\|\partial_{x_i x_j} u_N\|_{L^p(\mathbb{T}^2)} \leq C \|f_N\|_{L^p(\mathbb{T}^2)}, \quad \forall i, j = 1, 2, \tag{2.5}
\]
where \( C > 0 \) depends only on \( v_1, v_2 \) and \( p \). Applying the Poincaré inequality (notice that \( u_N \in \hat{L}^p(\mathbb{T}^2) \) and \( \partial_i u_N \in \hat{L}^p(\mathbb{T}^2) \), \( i = 1, 2 \)), we further have that
\[
\|u_N\|_{W^{2,p}(\mathbb{T}^2)} \leq C \|f_N\|_{L^p(\mathbb{T}^2)}, \tag{2.6}
\]
Similarly, for any \( N_1, N_2 \in \mathbb{Z}^+ \),
\[
\|u_{N_1} - u_{N_2}\|_{W^{2,p}(\mathbb{T}^2)} \leq C \|f_{N_1} - f_{N_2}\|_{L^p(\mathbb{T}^2)}. \tag{2.7}
\]
From (2.7), taking into account the fact that \( f_N \to f \) in \( L^p(\mathbb{T}^2) \) as \( N \to +\infty \), we see that \( \{u_N\} \) is a Cauchy sequence in \( \hat{W}^{2,p}(\mathbb{T}^2) \), thus \( u_N \) converges to some \( u \) in \( \hat{W}^{2,p}(\mathbb{T}^2) \) as...
$N \to +\infty$. It is clear that $-\Delta u = f$ a.e. $x \in \mathbb{T}^2$, hence $u$ solves (2.2). Moreover, passing to the limit $N \to +\infty$ in (2.6) gives

$$\|u\|_{W^{2,p}(\mathbb{T}^2)} \leq C\|f\|_{L^p(\mathbb{T}^2)}.$$

Next we prove uniqueness. Suppose (2.2) has two solutions $u_1$, $u_2$. By integration by parts,

$$\int_{\mathbb{T}^2} |\nabla (u_1 - u_2)|^2 dx = 0,$$

which implies that $u_1 - u_2 = c$ for some constant $c$. Taking into account the fact that both $u_1$ and $u_2$ have zero mean, we obtain $u_1 \equiv u_2$. \hfill $\Box$

By Lemma 2.1, the negative Laplacian on $\mathbb{T}^2$ under the zero mean condition has an inverse, denoted by $K$. The estimate (2.3) indicates that $K$ is a bounded operator from $\dot{W}^{2,p}(\mathbb{T}^2)$ to $\dot{W}^{2,p}(\mathbb{T}^2)$. The following lemma, asserting that $K$ is symmetric and positive-definite, can be easily verified by integration by parts.

**Lemma 2.2** Let $1 < p < +\infty$ be fixed.

(i) For any $f$, $g \in \dot{L}^p(\mathbb{T}^2)$, it holds that

$$\int_{\mathbb{T}^2} fKg dx = \int_{\mathbb{T}^2} gKf dx; \quad (2.8)$$

(ii) For any $f \in \dot{L}^p(\mathbb{T}^2)$, it holds that

$$\int_{\mathbb{T}^2} fKf dx \geq 0, \quad (2.9)$$

and the equality holds if and only if $f \equiv 0$.

Denote by $\mathcal{U}^\perp$ the orthogonal complement of $\mathcal{U}$ in $\dot{L}^2(\mathbb{T}^2)$, and $\mathcal{V}^\perp$ the orthogonal complement of $\mathcal{V}^\perp$ in $\dot{L}^2(\mathbb{T}^2)$ if $\nu_1 = \nu_2 = \nu$. The following lemma can also be easily proved by integration by parts.

**Lemma 2.3** It holds that

$$\int_{\mathbb{T}^2} \nabla Kg \cdot \nabla Kh dx = 0, \quad \forall \ g \in \mathcal{U}, \ h \in \mathcal{U}^\perp.$$ 

If additionally $\nu_1 = \nu_2 = \nu$, then

$$\int_{\mathbb{T}^2} \nabla Kg \cdot \nabla Kh dx = 0, \quad \forall \ g \in \mathcal{V}, \ h \in \mathcal{V}^\perp.$$ 

Recall that $E$ is defined by (1.11). Define the *enstrophy* $Z(\omega)$ of the fluid with vorticity $\omega$ as follows:

$$Z(\omega):= \frac{1}{2} \int_{\mathbb{T}^2} \omega^2 dx. \quad (2.10)$$

It is easy to check that $E$ is well-defined in $\dot{L}^p(\mathbb{T}^2)$ for any $1 < p < +\infty$, and $Z$ is well-defined in $L^2(\mathbb{T}^2)$. The following energy-enstrophy type inequalities can be proved by Fourier series expansion.
Lemma 2.4 (i) If $v_1 < v_2$, then

$$E(f) = v_2^2 Z(f) \ \forall \ f \in \mathcal{U}, \ E(f) \leq \max \left\{ v_1^2, \frac{v_2^2}{4} \right\} Z(f) \ \forall \ f \in \mathcal{U}^\perp.$$ 

(ii) If $v_1 = v_2 = v$, then

$$E(f) = v^2 Z(f) \ \forall \ f \in \mathcal{V}, \ E(f) \leq \frac{v^2}{4} Z(f) \ \forall \ f \in \mathcal{V}^\perp.$$ 

The following two lemmas will be used in the proof of Proposition 3.3.

Lemma 2.5 ([7], Theorem 6) Let $R(f_0)$ be the set of rearrangements of some $f_0 \in L^p(T^2)$ on $T^2$, and $\overline{R(f_0)}$ be the weak closure of $R(f_0)$ in $L^p(T^2)$. Then $\overline{R(f_0)}$ is convex, i.e., $\theta f_1 + (1 - \theta) f_2 \in \overline{R(f_0)}$ whenever $f_1, f_2 \in R(f_0)$ and $\theta \in [0, 1]$.

Lemma 2.6 ([7], Theorem 4) Let $p^* = \frac{p}{p - 1}$ be the Hölder conjugate of $p$. Let $R(f_0), R(g_0)$ be sets of rearrangements on $T^2$ of some $f_0 \in L^p(T^2)$ and some $g_0 \in L^q(T^2)$, respectively. Then for any $\tilde{g} \in R(g_0)$, there exists $\tilde{v} \in R(f_0)$, such that

$$\int_{T^2} \tilde{f} \tilde{g} dx \geq \int_{T^2} f g dx, \ \forall \ f \in R(f_0), \ g \in R(g_0).$$

3 Proof

In this section, we prove the main theorem. Throughout this section, let $A, B \geq 0$ and $1 < p < +\infty$ be fixed.

3.1 Variational characterizations

Our aim in this subsection is to establish suitable variational characterizations for $\mathcal{U}_A$ and $\mathcal{V}_{A,B}$.

Denote

$$v_A = A \sin \left( \frac{x_2}{v_2} \right). \quad (3.1)$$

If additionally $v_1 = v_2 = v$, denote

$$v_{A,B} = A \sin \left( \frac{x_1}{v} \right) + B \sin \left( \frac{x_2}{v} \right). \quad (3.2)$$

Denote by $\mathcal{R}_A, \mathcal{R}_{A,B}$ the set of rearrangements of $v_A, v_{A,B}$ on $T^2$, respectively,

$$\mathcal{R}_A = R(v_A), \quad (3.3)$$

$$\mathcal{R}_{A,B} = R(v_{A,B}) \text{ if } v_1 = v_2 = v. \quad (3.4)$$

It is easy to check that $\mathcal{U}_A \subset \mathcal{R}_A$, and $\mathcal{V}_{A,B} \subset \mathcal{R}_{A,B}$ if $v_1 = v_2 = v$.

We have the following variational characterizations for $\mathcal{U}_A$ and $\mathcal{V}_{A,B}$.

Proposition 3.1 (i) If $v_1 < v_2$, then

$$\mathcal{U}_A = \{ v \in \mathcal{R}_A \mid E(v) = m_A \}, \ m_A := \sup_{v \in \mathcal{R}_A} E(v).$$
(ii) If \( \nu_1 = \nu_2 = \nu \), then
\[
\mathcal{V}_{A,B} = \{ v \in \mathcal{R}_{A,B} \mid E(v) = m_{A,B} \}, \quad m_{A,B} := \sup_{v \in \mathcal{R}_{A,B}} E(v)
\]

**Proof** First we prove (i). Fix \( v \in \mathcal{R}_{A} \). Let \( v_A \) be given by (3.1). Denote \( Z_A = Z(v_A) \). Then \( Z(v) = Z_A \). Write \( v = \tilde{v} + \check{v} \), where \( \tilde{v} \in \mathcal{U}, \check{v} \in \mathcal{U}^\perp \). By orthogonality,
\[
Z(\tilde{v}) + Z(\check{v}) = Z_A. \tag{3.5}
\]

Using Lemma 2.3, we have that
\[
E(v) = \frac{1}{2} \int_{L^2(T^2)} |\nabla K v|^2 dx
\]
\[
= \frac{1}{2} \int_{L^2(T^2)} |\nabla K \tilde{v}|^2 dx + \int_{L^2(T^2)} \nabla K \tilde{v} \cdot \nabla K \check{v} dx + \frac{1}{2} \int_{L^2(T^2)} |\nabla K \check{v}|^2 dx \tag{3.6}
\]
\[
= E(\tilde{v}) + E(\check{v}).
\]

Recalling Lemma 2.4(i), it holds that
\[
E(\tilde{v}) = \nu_2^2 Z(\tilde{v}), \quad E(\check{v}) \leq \max \left\{ \nu_1^2, \nu_2^2 \right\} Z(\check{v}). \tag{3.7}
\]

Combining (3.5)–(3.7), we obtain
\[
E(v) \leq \nu_2^2 Z_A, \tag{3.8}
\]
and the equality holds if and only if \( v \in \mathcal{U} \). In other words, we have proved that
\[
m_A = \nu_2^2 Z_A,
\]
and \( E(v) = m_A \) if and only if \( v \in \mathcal{U} \). To finish the proof, it is sufficient to show that
\[
\mathcal{U} \cap \mathcal{R}_{A} = \mathcal{U}_A.
\]

The inclusion \( \mathcal{U}_A \subset \mathcal{U} \cap \mathcal{R}_{A} \) is quite obvious. Below we prove the inverse inclusion. Suppose \( v \in \mathcal{U} \cap \mathcal{R}_{A} \) has the form
\[
v = B \sin \left( \frac{x_2}{v_2} + \beta \right), \quad B \geq 0, \beta \in \mathbb{R}.
\]

It suffices to show that \( B = A \). This is obvious since
\[
B = \|v\|_{L^\infty(T^2)} = \|v_A\|_{L^\infty(T^2)} = A.
\]

Next we prove (ii) in an analogous way. Fix \( v \in \mathcal{R}_{A,B} \). Denote \( Z_{A,B} = Z(v_{A,B}) \). It is clear that \( Z(v) = Z_{A,B} \). Write \( v = \tilde{v} + \check{v} \), where \( \tilde{v} \in \mathcal{V}, \check{v} \in \mathcal{V}^\perp \). Then
\[
Z(\tilde{v}) + Z(\check{v}) = Z_{A,B}. \tag{3.9}
\]

Repeating the argument in (3.6), we can prove that
\[
E(v) = E(\tilde{v}) + E(\check{v}). \tag{3.10}
\]

Moreover, by Lemma 2.4(ii),
\[
E(\check{v}) = \nu^2 Z(\tilde{f}), \quad E(\check{v}) \leq \frac{\nu^2}{4} Z(\check{v}). \tag{3.11}
\]
From (3.9)–(3.11), we deduce that
\[ E(v) \leq \nu^2 Z_{A,B}, \]
and the equality holds if and only if \( v \in \mathcal{V} \). Hence we have proved that
\[ m_{A,B} = \nu^2 Z_{A,B}, \]
and \( v \) is a maximizer of \( E \) relative to \( \mathcal{R}_{A,B} \) if and only if \( v \in \mathcal{V} \). To finish the proof, it is sufficient to show that
\[ \mathcal{V} \cap \mathcal{R}_{A,B} = \mathcal{V}_{A,B} \cup \mathcal{V}_{B,A}. \]
We only prove the inclusion \( \mathcal{V} \cap \mathcal{R}_{A,B} \subset \mathcal{V}_{A,B} \cup \mathcal{V}_{B,A} \). To this end, suppose \( v \in \mathcal{V} \cap \mathcal{R}_{A,B} \) has the form
\[ v = C \sin \left( \frac{x_1}{\nu} + \alpha \right) + D \sin \left( \frac{x_2}{\nu} + \beta \right), \quad C, D \geq 0, \; \alpha, \beta \in \mathbb{R}. \]
It suffices to show that
\[ A = C, \quad B = D \]
(3.13)
or
\[ A = D, \quad B = C. \]
(3.14)
Since \( v \in \mathcal{R}_{A,B} \), we have that
\[ \|v\|_{L^\infty(\mathbb{T}^2)} = \|v_{A,B}\|_{L^\infty(\mathbb{T}^2)}, \quad \|v\|_{L^2(\mathbb{T}^2)} = \|v_{A,B}\|_{L^2(\mathbb{T}^2)}, \]
(3.15)
which yields
\[ A + B = C + D, \quad A^2 + B^2 = C^2 + D^2, \]
(3.16)
or equivalently,
\[ A - C = D - B, \quad (A - C)(A + C) = (D - B)(D + B). \]
(3.17)
If \( A = c \), then \( B = D \), and thus (3.13) holds; if \( A \neq C \), then \( A + C = D + B \), which together with \( A - C = D - B \) yields (3.14).

**Remark 3.2** From the above proof, we see that \( \mathcal{U}_A \) is in fact the set of maximizers of \( E \) relative to
\[ \{v \in \dot{L}^p(\mathbb{T}^2) \mid Z(v) = Z_A\}. \]
However, when \( \nu_1 = \nu_2 = \nu \), there is no similar characterization in terms of only \( E \) and \( Z \) for \( \mathcal{V}_{A,B} \). To distinguish \( \mathcal{V}_{A,B} \) from other sinusoidal flows, it is necessary to introduce the notion of rearrangements.

### 3.2 Compactness

Our purpose in this subsection is to prove the following proposition.

**Proposition 3.3** Let \( \mathcal{R}_A, \mathcal{R}_{A,B} \) be defined by (3.3), (3.4), and \( m_A, m_{A,B} \) be defined as in Proposition 3.1.
(i) If $v_1 < v_2$, then any sequence $\{v_n\} \subset \mathcal{R}_A$ satisfying
\[
\lim_{n \to +\infty} E(v_n) = m_A
\]
has a subsequence converging to some $\hat{v} \in \mathcal{U}_A$ strongly in $L^p(\mathbb{T}^2)$.

(ii) If $v_1 = v_2 = v$, then any sequence $\{v_n\} \subset \mathcal{R}_{A,B}$ satisfying
\[
\lim_{n \to +\infty} E(v_n) = m_{A,B},
\]
has a subsequence converging to some $\hat{v} \in \mathcal{V}_{A,B}$ strongly in $L^p(\mathbb{T}^2)$.

**Proof** We only prove (i). The proof of (ii) is analogous. Fix a sequence $\{v_n\} \subset \mathcal{R}_A$ satisfying (3.18). Obviously $\{v_n\}$ is bounded in $L^p(\mathbb{T}^2)$. Without loss of generality, we can assume, up to a subsequence, that $v_n$ converges weakly to some $\hat{v} \in \overline{\mathcal{R}_A}$ in $L^p(\mathbb{T}^2)$, where $\overline{\mathcal{R}_A}$ is the weak closure of $\mathcal{R}_A$ in $L^p(\mathbb{T}^2)$. By using Sobolev embedding theorem and the fact that $K$ is bounded from $\tilde{L}^p(\mathbb{T}^2)$ to $\tilde{W}^{2,p}(\mathbb{T}^2)$, it is easy to check that $E$ is sequentially weakly continuous in $\tilde{L}^p(\mathbb{T}^2)$. Taking into account (3.18), we have that
\[
E(\hat{v}) = m_A \geq E(v), \quad \forall v \in \overline{\mathcal{R}_A}.
\]
(3.20)
Here we used the fact that
\[
m_A = \sup_{v \in \mathcal{R}_A} E(v) = \sup_{v \in \overline{\mathcal{R}_A}} E(v).
\]
Since $\overline{\mathcal{R}_A}$ is convex by Lemma 2.5, we have that
\[
\theta v + (1 - \theta)\hat{v} \in \overline{\mathcal{R}_{A,2}}, \quad \forall v \in \mathcal{R}_A, \quad \theta \in [0, 1].
\]
Hence $E(\theta v + (1 - \theta)\hat{v})$ attains its maximum value at $\theta = 0$. Then
\[
\frac{d}{d\theta} E(\theta v + (1 - \theta)\hat{v}) \bigg|_{\theta=0^+} \leq 0, \quad \forall v \in \mathcal{R}_A,
\]
which gives
\[
\int_{\mathbb{T}^2} vK\hat{v}dx \leq \int_{\mathbb{T}^2} \hat{v}K\hat{v}dx, \quad \forall v \in \mathcal{R}_A.
\]
(3.21)
Applying Lemma 2.6, there exists some $\tilde{v} \in \mathcal{R}_{A,2}$ such that
\[
\int_{\mathbb{T}^2} vK\hat{v}dx \leq \int_{\mathbb{T}^2} \tilde{v}K\tilde{v}dx, \quad \forall v \in \mathcal{R}_A.
\]
(3.22)
By a simple density argument, it is easy to show that (3.22) actually holds for any $v \in \overline{\mathcal{R}_A}$. In particular,
\[
\int_{\mathbb{T}^2} \tilde{v}K\hat{v}dx \leq \int_{\mathbb{T}^2} \tilde{v}K\tilde{v}dx.
\]
(3.23)
By \((3.20), (3.23)\), and the fact that \(K\) is symmetric (cf. Lemma 2.2), it holds that
\[
E(\tilde{v} - \hat{v}) = E(\tilde{v}) + E(\hat{v}) - \int_{T^2} \tilde{v} K \hat{v} d\mathbf{x}
\leq E(\tilde{v}) + E(\hat{v}) - \int_{T^2} \hat{v} K \hat{v} d\mathbf{x}
= \frac{1}{2} E(\tilde{v}) - \frac{1}{2} E(\hat{v})
\leq 0.
\]
(3.24)

Taking into account the fact that \(K\) is positive definite (cf. Lemma 2.2), we deduce that \(\tilde{v} = \hat{v}\).

In particular, \(\hat{v} \in R_A\). By uniform convexity of the \(L^p\) norm, we further deduce that \(v_n\), up to a subsequence, converges strongly to \(\hat{v}\) in \(L^p(T^2)\). Furthermore, since \(E(\hat{v}) = m_A\) (cf. (3.20)), we deduce from Proposition 3.1(i) that \(\hat{v} \in U_A\).

\[
\square
\]

3.3 Proof of Theorem 1.3

Now we are ready to prove Theorem 1.3.

**Proof of Theorem 1.3(i)** Suppose by contradiction that \(U_A\) is not stable in the sense of (1.23). Then there exist some \(\varepsilon_0 > 0\), a sequence of smooth Euler flows on \(T^2\) with vorticity \(\{\omega^n\}\), and a sequence of times \(\{t_n\}\), such that
\[
\lim_{n \to +\infty} \min_{v \in U_A} \|\omega^n_0 - v\|_{L^p(T^2)} = 0,
\]
(3.25)
\[
\min_{v \in U_A} \|\omega^n_{t_n} - v\|_{L^p(T^2)} \geq \varepsilon_0, \quad \forall n.
\]
(3.26)

Here \(\omega^n_{t_n} := \omega^n(t, \cdot)\). Since \(U_A\) is obviously compact in \(L^p(T^2)\), there exist some subsequence of \(\{\omega^n_0\}\), still denoted by \(\{\omega^n_0\}\), and some \(\bar{\omega} \in U_A\), such that
\[
\lim_{n \to +\infty} \|\omega^n_0 - \bar{\omega}\|_{L^p(T^2)} = 0.
\]
(3.27)

Consequently,
\[
\lim_{n \to +\infty} E(\omega^n_0) = E(\bar{\omega}) = m_A.
\]
(3.28)

By energy conservation,
\[
\lim_{n \to +\infty} E(\omega^n_{t_n}) = \lim_{n \to +\infty} E(\omega^n_0) = m_A.
\]
(3.29)

If \(\{\omega^n_{t_n}\} \subset R_{A,2}\), then we can apply Proposition 3.3(i) to deduce that \(\omega^n_{t_n}\), up to a subsequence, converges to some element in \(U_A\) strongly in \(L^p(T^2)\), which is a contradiction to (3.26).

To deal with the general perturbations, we need to introduce a sequence of “followers” to \(\{\omega^n\}\) as in [8, 18]. For fixed \(n\), denote by \(v^n\) the velocity of the Euler flow with vorticity \(\omega^n\). Then \(\omega^n\) satisfies the following nonlinear transport equation (see [21], p. 20):
\[
\begin{align*}
\partial_t \omega^n + v^n \cdot \nabla \omega^n &= 0, \quad t > 0, \ x \in T^2, \\
\omega^n(0, \cdot) &= \omega^n_0.
\end{align*}
\]
(3.30)
Let $\xi^n$ be the solution of the following linear transport equation:

$$
\begin{aligned}
\frac{\partial_t \xi^n + v^n \cdot \nabla \xi^n}{\xi^n(0, \cdot) = \tilde{\omega}} = 0, \quad t > 0, \quad x \in \mathbb{T}^2,
\end{aligned}
$$

(3.31)

For simplicity, denote $\xi^n_t = \xi^n(t, \cdot)$. Since $v^n$ is divergence-free, by the Liouville theorem (see [22], p. 48), it holds that

$$
\xi^n_t \in \mathcal{R}(\tilde{\omega}) = \mathcal{R}_A, \quad \forall \, t \geq 0,
$$

(3.32)

On the other hand, from (3.30) and (3.31), we see that $\xi^n - \omega^n$ satisfies

$$
\begin{aligned}
\left\{ \begin{array}{l}
\frac{\partial_t (\xi^n - \omega^n) + v^n \cdot \nabla (\xi^n - \omega^n)}{\xi^n(0, \cdot) = \tilde{\omega} - \omega^n_0} = 0, \quad t > 0, \quad x \in \mathbb{T}^2, \\
(\xi^n - \omega^n)(0, \cdot) = \tilde{\omega} - \omega^n_0,
\end{array} \right.
\end{aligned}
$$

(3.33)

Again, by the Liouville theorem,

$$
\xi^n_t - \omega^n_t \in \mathcal{R}(\tilde{\omega} - \omega^n_0), \quad \forall \, t \geq 0.
$$

(3.34)

By (3.27) and (3.34),

$$
\lim_{n \to +\infty} \| \xi^n_t - \omega^n_t \|_{L^p(\mathbb{T}^2)} = 0,
$$

(3.35)

which together with (3.29) implies that

$$
\lim_{n \to +\infty} E(\xi^n_t) = \mathfrak{m}_A.
$$

(3.36)

From (3.32), (3.36), we can apply Proposition 3.3(i) to deduce that $\xi^n_t$, up to a subsequence, converges to some $\tilde{\omega} \in \mathcal{U}_A$ in $L^p(\mathbb{T}^2)$. In combination with (3.35), we further deduce that $\omega^n_t$ converges to $\tilde{\omega}$ in $L^p(\mathbb{T}^2)$, which obviously contradicts (3.26).

To prove Theorem 1.3(ii), we need the following lemma.

**Lemma 3.4** If $v_1 = v_2 = v$ and $A \neq B$, then

$$
\min_{u \in \mathcal{V}_{A,B}, v \in \mathcal{V}_{B,A}} \| u - v \|_{L^p(\mathbb{T}^2)} > 0.
$$

(3.37)

**Proof** It is easy to see that both $\mathcal{V}_{A,B}$ and $\mathcal{V}_{B,A}$ are compact in $L^p(\mathbb{T}^2)$. Therefore, to finish the proof, it is sufficient to show that $\mathcal{V}_{A,B} \cap \mathcal{V}_{B,A} = \emptyset$. Fix $u \in \mathcal{V}_{A,B}$ and $v \in \mathcal{V}_{B,A}$,

$$
\begin{aligned}
&u = A \sin \left( \frac{x_1}{v} + \alpha \right) + B \sin \left( \frac{x_2}{v} + \beta \right), \quad v = B \sin \left( \frac{x_1}{v} + \alpha' \right) + A \sin \left( \frac{x_2}{v} + \beta' \right),
\end{aligned}
$$

where $\alpha, \beta, \alpha', \beta' \in \mathbb{R}$. Then $u, v$ can be equivalently written as

$$
\begin{aligned}
u = A \cos \alpha \sin \left( \frac{x_1}{v} \right) + A \sin \alpha \cos \left( \frac{x_1}{v} \right) + B \cos \beta \sin \left( \frac{x_2}{v} \right) + B \sin \beta \cos \left( \frac{x_2}{v} \right),
\end{aligned}
$$

$$
\begin{aligned}
v = B \cos \alpha' \sin \left( \frac{x_1}{v} \right) + B \sin \alpha' \cos \left( \frac{x_1}{v} \right) + A \cos \beta' \sin \left( \frac{x_2}{v} \right) + A \sin \beta' \cos \left( \frac{x_2}{v} \right).
\end{aligned}
$$

If $u = v$, then

$$
A \cos \alpha = B \cos \alpha', \quad A \sin \alpha = B \sin \alpha', \quad B \cos \beta = A \cos \beta', \quad B \sin \beta = A \sin \beta',
$$

which implies that $A = B$, a contradiction.

**Proof of Theorem 1.3(ii)** Repeating the proof of Theorem 1.3(i), we can show that $\mathcal{V}_{A,B} \cup \mathcal{V}_{B,A}$ is stable. We distinguish two cases: (i) if $A = B$, then $\mathcal{V}_{A,B} = \mathcal{V}_{A,B} \cup \mathcal{V}_{B,A}$, which is stable; (ii) if $A \neq B$, then (3.37) holds, hence by continuity both $\mathcal{V}_{A,B}$ and $\mathcal{V}_{B,A}$ are stable.

\[\square\]
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