Criticality in Brownian ensembles

Suchetana Sadhukhan and Pragya Shukla

Department of Physics, Indian Institute of Technology, Kharagpur, India

(Dated: September 26, 2016)

Abstract

The local statistical fluctuations in Brownian ensembles, the intermediate state of perturbation of one classical ensemble by another one, are system size-invariant if the perturbation parameter has the same size-dependence as that of the ensemble averaged local level density. The sensitivity to local spectral density however makes the measures for the critical statistics non-stationary along the spectrum.

PACS numbers: PACS numbers: 05.40.-a, 05.30.Rt, 05.10.-a, 89.20.-a
I. INTRODUCTION

Recent studies of the localization to delocalization transitions e.g many body localization, Anderson localization and random graphs indicate a common mathematical structure underlying the statistical fluctuations of their linear operators [1, 2]. The structure belongs to that of a Rosenzweig-Porter (RP) ensemble [3], undergoing a single parametric based transition from the Poisson universality class to that of Wigner-Dyson type. As the ensemble density of the Rosenzweig-Porter ensemble is analogous to that of a single parametric Brownian ensemble (BE) intermediate between Poisson and Gaussian ensembles [4], this clearly indicates a crucial but so far hidden connection of the above mentioned studies with the BEs. It is therefore natural to search for the criticality in these ensembles which motivates the present study.

A Brownian ensemble in general refers to an intermediate state of perturbation of a stationary random matrix ensemble by another one of a different universality class [5–7]. The type of a BE, appearing during the cross-over, depends on the nature of the stationary ensembles and their different pairs may give rise to different BEs [7, 8]. Similar non-stationary states may also arise in other matrix spaces e.g. unitary matrix space e.g. due to a perturbation of a stationary circular ensemble by another one [9–12]. The BEs have been focus of many studies in past decades (for example see [10, 11, 13] and the references therein) and a great deal of analytical/numerical information is already available about them. Except for a few of them [1, 4, 14], almost none of these studies probed the critical aspects of the BEs i.e the indicators of a possible phase-transition. The concept of criticality in a non-stationary matrix ensemble in general refers to a behavior different from the stationary limits in infinite matrix size limit [38]. Its analysis in BEs is important for several reasons. For example, the analytical study [16] indicate the statistical fluctuations of a wide range of complex systems are governed by a single parameter $\Lambda$ besides global-constraints on the system. Referred as the complexity parameter, $\Lambda$ turns out to be a function of the average accuracy of the matrix elements, measured in units of the mean-level spacing. The fluctuations in the systems are analogous to that of a Brownian ensemble, subjected to similar global-constraints, if their complexity parameters are equal irrespective of other system-details. The knowledge
of criticality in BE can therefore be helpful in its search in other related ensembles.

The criteria for the critical statistics of energy levels and eigenfunctions was first introduced in context of localization to delocalization transition in an ensemble of disordered Hamiltonian matrices [18]; it has long been believed that a fractional value of the spectral compressibility and multifractal behavior of the eigenfunctions are signatures of the criticality of the ensemble [19, 20]. In fact these measures were used to claim the analogy of the Anderson ensemble at metal-insulator transition with that of the Power law random banded matrix (PRBM) ensemble [21]. The study [4, 32] indicates that the statistics of both of these ensembles can be mapped to that of the Brownian ensembles appearing between Poisson and Gaussian orthogonal ensemble (GOE); a BE is therefore expected to show the critical features too. The study [4] however suggests the BE intermediate to Poisson and GOE limits as different from other ensembles appearing between these limits; this suggestion is based on a perturbative analysis of the eigenfunction fluctuations and two point spectral correlation (also see [14, 22–25] for related studies). The need for a clear answer motivates us to pursue an analytical calculation of the spectral compressibility and multifractality for the BEs. Although our approach is applicable for a generic BE of both Gaussian or Laguerre type (i.e intermediate between an arbitrary initial condition and stationary ensembles of both Gaussian and Laguerre type, these measures so far seem to be relevant in context of the ensembles undergoing localization to delocalization transition. To strengthen and support the theoretical analysis, we probe the behavior by numerical route too but that is confined to the Gaussian BEs between Poisson to GOE only. As discussed later, the seemingly contradictory claims of the studies [4] and [1] can be explained on the basis of a rate of change of the local density of states which affects the local statistical fluctuations.

The paper is organized as follows. Section II briefly introduces the Brownian ensembles in Hermitian matrix spaces. The diffusive dynamics for the eigenvalues of these types of Brownian ensembles was analyzed in detail in [10]. The similar dynamics for the eigenfunction components of the Gaussian and Laguerre type Brownian ensembles was studied in [26] and [13] respectively. This information is used in sections III and IV to derive the parametric dependence of the criticality measures i.e spectral compressibility, the multifractality
spectrum and eigenfunction correlations at two different energies. Here we also discuss the conditions under which they become critical. The results of section III and section IV are applicable for arbitrary initial conditions. But as the main interest in these measures arises from the quest to characterize the localization to delocalization transition, we numerically verify our theoretical results for the Gaussian BEs intermediate between Poisson and Gaussian orthogonal ensembles. Section V very briefly reviews the basic formulation for the BEs intermediate between Poisson and GOE and presents the details of our numerical analysis. In section VI, we discuss the reasons for the deviation of the RP ensemble results of [1] from Anderson and PRBM ensembles, and, why it does not contradict the results of [4, 16] on complexity parametric formulation. We conclude in section VII with summary of our main results and open questions.

II. BROWNIAN ENSEMBLES: THE DEFINITION

Based on the assumption of Brownian dynamics of matrix elements due to thermal noise, Brownian ensembles were originally introduced by Dyson to model the statistical behavior of systems with partially broken symmetries and/or approximate conservation laws [5, 6]. But currently a Brownian ensemble is also described as a single parameter governed diffusive state of the matrix elements of a randomly perturbed stationary ensemble [6–8, 10]. Consider an ensemble of \( N_a \times N \) rectangular matrices \( A(\lambda) = \sqrt{f}(A_0 + \lambda V) \) with \( f = (1 + \lambda^2)^{-1} \) ([10, 13]) and matrices \( A_0 \) and \( V \) distributed with probability densities \( \rho_0(A_0) \) and \( \rho_v(V) \). Here \( A = A_0 \) for \( \lambda \to 0 \), \( A \to V \) for \( \lambda \to \infty \). The ensemble of rectangular \( A \) matrices can lead to three important classes of Hermitian matrix ensembles (i) Gaussian ensembles of matrices \( H = A + A^\dagger \) with \( N = N_a \), (ii) Laguerre ensembles with matrices \( M = A^\dagger A \), and, (iii) Jacobi ensembles of matrices \( S \) which approach a form \( S = (A^\dagger A + B^\dagger B)^{-1/2} (B^\dagger B - A^\dagger A) (A^\dagger A + B^\dagger B)^{-1/2} \). Our analysis in this paper is confined only to the first two ensembles (the spectral statistics of the case (iii) is discussed in detail in [10]).

A variation of strength \( \lambda \) of the random perturbation \( V \) leads to diffusion of the matrix elements \( A_{kl} = \sqrt{f}(A_{0,kl} + \lambda V_{kl}) \) which, by a suitable choice of \( \rho_v(V) \), can be confined to a finite space. The Markovian character of the dynamics is preserved if considered in terms of
a rescaled parameter $Y$ [10]; for example, for Gaussian density of the $V$-ensemble, $Y$ is given by the relation $f = e^{-2Y}$ [10]. For $\rho_v(V) = (\frac{1}{2\pi\nu^2})^{\beta N_a N/2} e^{-\frac{1}{2\nu^2} \text{Tr}(VV^\dagger)}$, the diffusion equation for the matrix elements of $A$ can explicitly be derived. As discussed in [10] [13] [26], this in turn leads to the $Y$-governed diffusion equations for the eigenvalues and eigenfunctions of the Gaussian and Laguerre ensembles. $Y$ however does not govern the variation of the local eigenvalue/ eigenfunction fluctuations and needs to be rescaled. In case of the spectral fluctuation measures, the rescaled parameter can be given as

$$\Lambda(Y, e) = \frac{e^{\nu(Y - Y_0)}}{\Delta^2_e}$$

where $\Delta_e$ is the local mean level density, $\Delta_e = \Delta \left( \frac{N}{\xi} \right)$ for localized states, and, $\Delta_e = \Delta$ for extended states. Here $\Delta$ is the ensemble average spectral density, $\Delta(e) = (R_1(e))^{-1}$ and $\xi(e, Y)$ is the average localization length at energy $e$ and perturbation strength $Y$ [4] [27]. Note $d = 1$ for the Brownian ensemble. The local eigenfunction fluctuations on the other hand are governed by different rescaling of $Y$ sensitive to the measure under consideration, thus indicating different cross-over speeds [13] [26].

### III. SIGNATURES OF CRITICALITY IN SPECTRAL STATISTICS

As mentioned above, the criteria for spectral criticality was believed to be the spectral rigidity or compressibility, a characteristic of the long-range correlations of levels. The analytical studies later on indicated the criticality also to manifest through an asymptotically linear behavior of the number variance $\Sigma^2(r)$ (the variance in the number of levels in an interval of length $rD$) in mean number of levels $r$ [19] [20] [38]: $\Sigma^2(r) \sim r^\chi$ for large $r$ with $0 < \chi < 1$; here $\chi$ is referred as the level-compressibility and is believed to be related to $\kappa$ as $\chi = \frac{1}{2\kappa}$. In [19], $\chi$ was suggested to be related to the multifractality of eigenfunctions: $\chi = \frac{d - D_2}{2d}$ with $D_2$ as the fractal dimension and $d$ as the system-dimension. However numerical studies indicated the result to be valid only in the weak-multifractality limit [40]. Later on, another criteria was introduced in terms of the level-repulsion (an indicator of short range correlation), measured by nearest- neighbor spacing distribution. The study [18] showed that the nearest-neighbor spacing distribution $P(s)$ turns out to be a universal
hybrid of the GOE at small-$s$ and Poisson at large-$s$, with an exponentially decaying tail: \( P(s) \sim e^{-\kappa s} \) for \( s \gg 1 \) with \( \kappa \) as a constant.

In general, the criticality in a joint probability distribution function (JPDF) of the eigenvalues can be defined as follows. A one-parameter scaling behavior of the distribution \( P(\{\epsilon\}) \) implies the existence of a universal distribution \( P^*(\{\epsilon\}) = \lim_{N \to \infty} P(\{\epsilon\}, \Lambda) \) at a critical point which is fixed by the critical value \( \Lambda^* = \lim_{N \to \infty} \Lambda(N) \). Thus the size-dependence of \( \Lambda \) plays a crucial role in locating the critical point of statistics. Let \( |Y - Y_0| \propto N^\alpha \) and \( D \propto N^\eta \), which gives \( \Lambda \propto N^{\alpha-2\eta} \). In finite systems, a variation of size \( N \), therefore, leads to a smooth crossover of statistics between an initial state (\( \Lambda \to 0 \)) and the equilibrium (\( \Lambda \to \infty \)); the intermediate statistics belongs to an infinite family of ensembles, parameterized by \( \Lambda \). However, for system-conditions leading to \( \alpha = 2\eta \), the statistics becomes universal for all sizes, \( \Lambda \) being \( N \)-independent; the corresponding system conditions can then be referred to as the critical conditions (or point). Further, at \( \Lambda^* = \lim_{N \to \infty} \Lambda(N) \), \( R_n(r_1, \ldots, r_n; \Lambda) \) and therefore all spectral fluctuation measures are different from the two stationary limits at \( \Lambda = 0 \) and \( \infty \). Any of them can therefore be used as a criteria for critical statistics. It should be stressed that the system conditions satisfying the critical criteria may not exist in all systems; the critical statistics therefore may not be a generic feature of all systems or all fluctuation measures of a system. (The diffusion rates of different measures being different, their criticality may occur at different values of parameters).

In general, the ensemble averaged level density \( R_1(e) \) varies along the spectrum. In case of an equilibrium ensemble, the unfolding i.e the rescaling by local mean level spacing removes the energy-dependence of the correlations in the bulk of the spectrum, resulting in their stationary behavior. For non-equilibrium cases, however, the correlations retain their energy-dependence through \( \Lambda \) even after unfolding and are non-stationary i.e vary along the spectrum. Any criteria for the criticality in the spectral statistics can then be defined only locally i.e within the energy range, say \( \delta e_c \), in which \( \Lambda \) is almost constant. The number of levels contained in \( \delta e_c \) is \( N_c = \frac{\delta e_c}{\Sigma_{\text{local}}} = \frac{\delta e_c \xi^d}{\Delta N} = \delta e_c \sqrt{\frac{\Lambda^*}{Y-Y_0}} \). As the correlation volume \( \xi^d > N \) at the critical point, \( N_c \) is although fixed but large in the thermodynamic limit \( N \to \infty \), with \( N_c \ll N \).
The Λ- governed diffusion of the eigenvalues subjects the local spectral fluctuation measures also to undergo a similar dynamics. To determine their behavior at the critical point, it is necessary to first obtain the evolution equations for the relevant measures. The compressibility believed to be a critical criteria, here we consider its evolution.

1. **Diffusion equation for compressibility** \( \chi \)

The compressibility \( \chi(r) \) in an energy range \( r \) is defined as

\[
\chi(e, r) = 1 - 2 \int_0^r (1 - R_2(e, s)) \, ds. \tag{2}
\]

where \( R_2(e, e+r) \) is the two point level density correlation at an energy \( e \). As \( R_2 \) is related to \( \Sigma_2(r) \) another 2-point measure, \( \chi \) can also be expressed as the \( r \)-rate of change of the number variance \( \Sigma_2(r) \).

For the spectrum of uncorrelated levels (no level repulsion) i.e Poisson ensemble, \( R_2(r) = 1 \) which gives \( \chi = 1 \). But for a classical ensemble (e.g. GOE or GUE), the well-known sum rule \( \int_{-N/2}^{N/2} (1 - R_2(r)) \, dr = 1 \) gives \( \lim_{r \to N/2} \chi(r) = 0 \); note this implies that a classical ensemble corresponds to the maximum level repulsion (i.e zero compressibility) in the related symmetry class \([6, 8]\). Thus if \( \lim_{r \to N/2} \chi(r) \neq 0, 1 \), it characterizes a spectrum different from classical ensembles as well as uncorrelated spectrum. This characterization however is suitable only for the stationary spectrum (where unfolded spectral correlations are independent of the location along the energy axis). In case of the non-stationarity, the statistics varies along the energy-axis and one can at best define a local compressibility within an energy range \( r \ll N \); here \( r \) corresponds to the range within which the local stationarity is valid. This led to introduction of the following criteria for criticality: the spectral statistics is believed to be critical if

\[
\begin{align*}
\lim_{r \to \infty} \lim_{N \to \infty} \chi(r) &\neq 0, \neq 1. 
\end{align*} \tag{3}
\]

(Note the order of limits on \( r \) and \( N \) are non-interchangeable i.e. \( \lim_{r \to \infty} \lim_{N \to \infty} \chi(r) \neq \lim_{N \to \infty} \lim_{r \to \infty} \chi(r) \)). This leads to technical issues in numerical search for criticality in \( \chi \): the total number of levels \( N \) in the spectrum being finite, the maximum range of allowed \( r \) is \( r \leq N_c \ll N \) and it is not easy to realize a large \( r \) limit.
To analyze a $\Lambda$-dependence of $\chi$ we proceed as follows. As discussed in [7, 10], a variation of perturbation strength of the BE subjects $R_2(r)$ to undergo diffusion which can be described as

$$\frac{\partial R_2}{\partial \Lambda} = 2 \frac{\partial}{\partial r} \left[ \frac{\partial R_2}{\partial r} - \beta \frac{R_2}{r} - \beta \int_{-N/2}^{N/2} \frac{R_3(0, x, r)}{x} \, dx \right]. \quad (4)$$

with $R_3(0, x, r)$ as the 3-point level-density correlation and $\Lambda$ given by eq.(1). By differentiating eq.(2) with respect to $\Lambda$, followed by a substitution of eq.(4) and subsequent repeated partial integrations, leads to following approximated closed form equation for $\chi$

$$\frac{\partial \chi}{\partial \Lambda} = -4 \left( \frac{\beta}{r} - \frac{\partial}{\partial r} \right) R_2(r; \Lambda) - 4 \beta \int_{-N/2}^{N/2} \frac{R_3(0, x, r; \Lambda)}{x} \, dx \quad (5)$$

An integration over $\Lambda$ of the above equation now gives

$$\chi(r; \Lambda) = \chi(r; 0) - \left[ 4 \left( \frac{\beta}{r} - \frac{\partial}{\partial r} \right) u(r; \Lambda) \right] - 4 \beta v(\Lambda) \quad (6)$$

where

$$u(r; \Lambda) = \int_0^\Lambda dt \, R_2(r, t), \quad (7)$$

$$v(\Lambda) = \int_0^\Lambda dt \int_{-N/2}^{N/2} dx \, \frac{R_3(0, x; r, t)}{x}. \quad (8)$$

Further simplification of eq.(6) is possible based on following points (i) $R_3$ can also be expressed in terms of $R_2$: $R_3(0, x, r) = Y_3(0, x, r) + R_2(x) + R_2(r) + R_2(r - x) - 2$ with $Y_3(0, r, x)$ as the 3$^{rd}$ order cluster function [6, 7]. (ii) the range of integral in eq.(10) varies from $-N/2$ to $N/2$ and our interest is in the limit $N \to \infty$ followed by $r \to \infty$, (iii) as $R_3$ varies from 0 $\to$ 1, the main contribution to the integral in eq.(10) comes from the neighborhood of $x = 0$. Thus although the range of integration $x$ varies from $-N/2$ to $N/2$, one needs to concern only with small $x$-values, (iv) the cluster function $Y_3$ vanishes if $x$ or $r$ or $|x - r|$ becomes large in comparison to the local mean level spacing. In large $r$-limit, therefore, one can approximate $R_2(r) \approx R_2(r - x) \to 1$ which leads to $R_3(0, x, r) \approx R_2(x)$. Using the latter, $v$ can be expressed in terms of $u$: $v(\Lambda) = \int_{-N/2}^{N/2} dx \, \frac{u(x; \Lambda)}{x}$. The lack of energy level correlations at large $r$ also gives

$$\lim_{r \to \infty, N \to \infty} u(r; \Lambda) = \int_{\Lambda_0}^{\Lambda^*} dt \left( \lim_{r \to \infty} R_2(r, t) \right) \approx \Lambda^*$$
In the ordered limit $r \to \infty, N \to \infty$, eq. (6) can now be reduced to following form

$$\lim_{r \to \infty, N \to \infty} \chi(r; \Lambda) = \lim_{r \to \infty, N \to \infty} \chi(r; 0) - 4\beta \int_{-\infty}^{\infty} \frac{u(x, \Lambda^*)}{x}$$

(9)

with $\Lambda^* = \lim_{N \to \infty} \Lambda$ and $u$ given by eq. (7).

Alternatively, one can also derive a diffusion equation for $\chi$ as follows. From eq. (2),

$$1 + \frac{1}{2} \frac{\partial \chi}{\partial r} \frac{\partial \chi}{\partial r} = R_2(r).$$

(10)

In large-$r$ limit, this leads to the approximation

$$\int_{-\infty}^{\infty} R_3(0, x, r; \Lambda) \frac{dx}{x} \approx \int_{-\infty}^{\infty} R_2(x, \Lambda) \frac{dx}{x} = \int_{-\infty}^{\infty} \frac{1}{2x} \frac{\partial \chi}{\partial x} dx.$$

(11)

Substitution of above relations in eq. (10) gives $\Lambda$ governed evolution of $\chi(r)$:

$$\frac{\partial \chi}{\partial \Lambda} = -\frac{4\beta}{r} - \frac{2\beta}{r} \frac{\partial \chi}{\partial r} + 2 \frac{\partial^2 \chi}{\partial r^2} - 2 \beta \int_{-\infty}^{\infty} \frac{\chi(x)}{x^2} dx$$

(12)

An integration over $\Lambda$ of the above equation now gives

$$\lim_{r \to \infty} \chi(r; \Lambda) = \lim_{r \to \infty} \chi(r; 0) - 2\beta b(\Lambda)$$

(13)

with $b(\Lambda) = \int_{0}^{\Lambda} d\Lambda \int_{-\infty}^{\infty} \frac{\chi(x, \Lambda)}{x^2} dx$.

IV. SIGNATURES OF CRITICALITY IN EIGENFUNCTION STATISTICS

The basis-variant nature of an ensemble, which is often the case at the critical point, implies a non-zero degree of the correlation between eigenvalues and eigenfunctions. The special features of the spectrum at the criticality are therefore expected to manifest in eigenfunctions too. As indicated by many studies of the localization → delocalization transitions, the eigenfunctions at the critical point are multifractal. The eigenfunctions in the delocalized limit are essentially structureless and overlapping almost everywhere which leads to level repulsion. In the localized limit, the wavefunctions are typically localized at different basis state with almost negligible overlap which results in uncorrelated level-statistics described by Poisson universality class. But the multifractality leads to an intimate conspiracy between correlations of the energy levels and the eigenfunctions. This has motivated three main criteria for the criticality in these fluctuations, namely, inverse participation ratio, multifractality spectrum and eigenfunction correlations at different energy. It is therefore desirable to analyze these measures in context of the Brownian ensembles.
2. Diffusion of inverse participation ratio

The criticality in wavefunctions is believed to manifest through large fluctuations of their amplitudes at all length scales and is often characterized by an infinite set of critical exponents related to the scaling of the moments of the wave-function intensity $|\Psi(r)|^2$ with system size [38, 40]. The $q^{th}$ moment $I_q$ of the wave-function intensity $|\Psi(r)|^2$, also known as $q^{th}$ inverse participation ratio is defined as $I_q = \int dr |\Psi(r)|^{2q}$. As revealed by the critical point studies of many disordered systems, an ensemble averaged $I_q$ reveals an amalous scaling with size $N$: $\langle I_q \rangle = L^d \langle |\Psi|^2 \rangle \sim L^{-\tau_q}$ with $\langle . \rangle$ implying ensemble average. The continuous set of exponents $\tau_q$ are related to the generalized fractal dimension $D_q$ of the wave-function structure: $\tau_q = (q - 1)D_q$. At critical point, $D_q$ is a non-trivial function of $q$, with $D_q = d$ and $D_q = 0$ for the eigenfunctions extended in whole space and for localized ones. Further $D_2$, a measure of the spatial correlation of the intensity $|\Psi|^2$ (with $L^{2d} \langle |\Psi(r)\Psi^2(r')| \rangle \sim (|r - r'|/L)^{D_2-d}$) is related to compressibility $\chi$.

The $\Lambda$-dependence of the ensemble averaged inverse participation ratio for a generic BE of Gaussian or Laguerre type is derived in [13, 26]. For the BE between Poisson to XOE, it can be given as (using $\langle I_q(0) \rangle = 1$ for the Poission initial condition),

$$\langle I_q(\Lambda_I) \rangle = e^{-q \tau_2 \Lambda_I} \left[ 1 + t_1 \int_0^{\Lambda_I} \langle I_{q-1}(r) \rangle e^{qt_2 r} dr \right].$$  \hspace{1cm} (14)

where

$$t_1 = \frac{(q - 1)}{2 \xi^d},$$  \hspace{1cm} (15)

$$t_2 = 1 + \frac{2}{q \beta K_e} \left( a(e) + \frac{N_e}{\Delta_e} e^{\nu} + \frac{1}{\beta \delta_{\nu1}} \right),$$  \hspace{1cm} (16)

$$\Lambda_I = (\beta^2/2) K_e (Y - Y_0)$$  \hspace{1cm} (17)

$$a(e) = \left( \frac{2}{\beta} \right)^\nu e + \frac{N - 1}{2}, \hspace{1cm} K_e = \frac{N_e (2e - \Delta_e)^\nu}{\Delta_e^2}$$  \hspace{1cm} (18)

with $\nu = 0$ and 1 for Brownian ensembles of Gaussian and Laguerre type, respectively. Here the localization length $\xi$ is assumed to be almost invariant with respect to $Y$; the assumption is justified as it leads to a well-known result i.e $\langle I_2 \rangle \propto \frac{1}{\xi^2}$ for the localization $\rightarrow$ delocalization transition (see eq.\[19\]).
For $\Lambda_I \to \infty$, eq. (14) gives a correct steady state limit, namely, XOE or XUE with $X \equiv L$ or $G$: 

$$\langle I^q \rangle = \left( \frac{2q!}{q^{q!}} \right) N^{1-q}$$ for $\beta = 1$ and $\langle I^q \rangle = q! N^{1-q}$ for $\beta = 2$. For finite nonzero $\Lambda_I$, $\langle I^q \rangle$ can be determined recursively; in case of a large but finite $\Lambda_I$, it can be approximated as

$$\langle I^q(\Lambda_I) \rangle \approx \frac{1}{q!} \left( \frac{t_1}{t_2} \right)^{q-1} + O(e^{-t\Lambda_I}) = \frac{1}{q!} \left( \frac{(q-1)}{2 \xi^d} \right)^{q-1} \tag{19}$$

The ensemble averaged localization length $\xi^d \sim (\langle u \rangle)^{-1}$ can be estimated from the ensemble averaged local intensity $\langle u \rangle$ at energy $e$. As discussed in [13], $\langle u \rangle$ depends on $Y - Y_0$ as well the ensemble averaged mean level spacing $\Delta$:

$$\langle u(Y) \rangle^2 \approx \langle u(Y_0) \rangle^2 + \frac{\beta^2 N_e (2e)^{\nu}}{\Delta^2 N^2} (Y - Y_0). \tag{20}$$

Assuming $\xi^d \sim N^b$ without any loss of generality, a comparison of the above result with $\langle I^q(\Lambda_I) \rangle \sim N^{-\tau_q}$ then gives $\tau_q \approx (q - 1)b$. This in turn implies all the fractal dimensions for large but finite $\Lambda_I$ are same: or $D_q \approx b$. A similar result for Rosenzweig-Porter ensemble (i.e. a Gaussian BE intermediate to Poisson to GOE universality class) was also obtained by a perturbative analysis of eigenfunctions (see eq.(9) in [1]) but the value of $b$ predicted by our analysis is different from their result in some cases (discussed in more detail in section V).

### 3. Diffusion of multifractal spectrum

A well-known criteria for the multifractality is the singularity spectrum $f(\alpha)$, related to $\tau_q$ by a Legendre transformation $f(\alpha) = q\alpha - \tau_q$. ($f(\alpha)$ is the fractal dimension of set of those points $r$ at which $|\psi(r)|^2 \sim L^{-\alpha}$. The number of such points in a lattice scales as $L^{f(\alpha)}$). Following from its definition, $f(\alpha)$ is a convex function and satisfies a symmetry $f(d - 2\alpha) = f(\alpha) + d - \alpha$. For delocalized wavefunctions $f(\alpha)$ is fixed: $f(\alpha) = d$ but its spread increases in going from delocalized wave limit to the localized one. In case of an ensemble, $f(\alpha)$ can be expressed in terms of the distribution $P_u(u)$ of the local intensity $u = |\psi|^2$ of a typical eigenfunction $\psi$ [1]

$$f(\alpha) = \lim_{N \to \infty} \frac{\ln(NP_u(u))}{\ln N} \tag{21}$$
where $\alpha = d - \frac{\ln u}{\ln N}$. For a classical ensemble, the eigenfunction are delocalized in the basis-space and $P_u(u)$ is chi-square distribution [6]: $P_u(u) = e^{-u} / \sqrt{2\pi u}$ for XOE and $P_u(u) = e^{-u}$ for XUE. The corresponding $f(\alpha)$ is then

$$f(\alpha) = d + \alpha/2 - (c/2)e^{\alpha/c} - c \ln(c\sqrt{2\pi/b_0}) \quad \text{for XOE}$$

$$f(\alpha) = d - \alpha - c e^{\alpha/c} - c \ln(c/b_0) \quad \text{for XUE}$$

For the systems with weak multifractality, $f(\alpha)$ is believed to be approximately parabolic:

$$f(\alpha) = d - \frac{1}{4\epsilon}(d + \epsilon - \alpha)^2 + o(e^4) \quad \text{with } \epsilon \ll 1.$$  

To derive $Y$-dependence of $f(\alpha)$ for a BE, we first invert the relation (21) which gives $P_u(u) = N^{\alpha - 2d + f} = e^{(\alpha - 2d + f)\ln N}$. Next we note that a variation of the parameter $Y$ gives rise to the diffusion of $P_u(u)$ (see [13, 26] for details):

$$\frac{\partial P_u}{\partial Y} = \frac{\beta^2 K e}{2\gamma_0} \left[ \frac{1}{\gamma_0^2} \frac{\partial^2 P_u}{\partial u^2} + \frac{\partial(u P_u)}{\partial u} \right] + L_e P_u$$

where

$$L_e \equiv \frac{\partial}{\partial e} \left[ \beta a(e) + \frac{\beta}{N e} e' + \frac{\partial}{\partial e} e' \right]$$

with $a(e)$ given by eq.(18), $\nu = 0, 1$ for Gaussian and Laguerre type Brownian ensembles respectively.

A substitution of $P_u(u)$ as a function of $f(\alpha)$ in eq.(24) leads to the diffusion equation for $f(\alpha)$:

$$\frac{\partial f_\alpha}{\partial Y} \approx \frac{\beta^2 K e}{2\gamma_0} \frac{N^{2(d-\alpha)}}{\xi^d} \left[ \frac{1}{\gamma_0} \frac{\partial^2 f_\alpha}{\partial \alpha^2} + \left( \frac{\partial f_\alpha}{\partial \alpha} + 1 \right) \left( \frac{\partial f_\alpha}{\partial \alpha} - \xi^d \right) \right] + \frac{\beta^2 K e}{2\gamma_0} + T_e f.$$  

with $\gamma_0 = \ln N$ and $T_e$ is the differential operator

$$T_e f \equiv \frac{b_1}{\gamma_0} + (\beta (\phi_{\nu} e + \theta_{\nu}) + 2 \delta_{\nu 1}) \frac{\partial f_\alpha}{\partial e} + e^{\nu} \frac{\partial^2 f_\alpha}{\partial e^2} + \gamma_0 e^{\nu} \left( \frac{\partial f_\alpha}{\partial e} \right)^2$$

with $\theta_{\nu}, \phi_{\nu}$ depend on the nature of BE: $\theta_0 = \frac{N e}{\Delta e}, \phi_0 = 1$ for Gaussian BE, $\theta_1 = \frac{N - 1}{2}, \phi_1 = \frac{3}{\beta} + \frac{N e}{\Delta e}$ for Laguerre BE. The appearance of $T_e f$ in eq.(26) clearly indicates an energy-sensitivity of the multifractality spectrum: it is non-stationary along the energy axis.

A desirable next step would be to solve the above equation but it is technically complicated. To gain further insight, we first simplify eq.(26) by a local spectral averaging which
gets rid of the $T ef$: integrating eq. (26) over the energy range $e - D e \rightarrow e + D e$, while assuming $f$ to be locally stationary over the region, leads to

\[
\frac{\partial \overline{f}_\alpha}{\partial Y} \approx \frac{\beta^2 K_e}{2 \gamma_0} N^{2(d-\alpha)} \left[ \frac{1}{\gamma_0} \frac{\partial^2 \overline{f}_\alpha}{\partial \alpha^2} - \left( \frac{\partial \overline{f}_\alpha}{\partial \alpha} + 1 \right) \left( \frac{\partial \overline{f}_\alpha}{\partial \alpha} - \xi^d \right) \right] + \frac{\beta^2 K_e}{2 \gamma_0} - \beta \phi \nu \overline{f}_\alpha + \beta \phi \nu \frac{\partial \overline{f}_\alpha}{\gamma_0}
\]

where $\overline{f}_\alpha = \frac{1}{2 D e} \int_{e-D e}^{e+D e} f_\alpha \, de$. Based on $\xi^d$-behavior, the above equation can further be reduced to a simple form. Noting that $\xi^d \propto \langle I^2 e \rangle = \xi_0 N \tau^2_2$, with $\tau^2_2 > 0$, and, $K_e \approx \frac{N e^{2 \xi^d(2e)^\nu}}{\Delta^2 N^2}$, we can approximate

\[
\frac{\partial \overline{f}_\alpha}{\partial \Lambda f} \approx - \frac{N^{2(d-\alpha)-\tau_2}}{2 \xi_0} \left( \frac{\partial \overline{f}_\alpha}{\partial \alpha} + 1 \right) + 1
\]

with $\Lambda f = \frac{\beta^2 K_e (Y - Y_0)}{2 \gamma_0}$. As clear from above, $\overline{f}_\alpha$ has a linear dependence on $\alpha$ for $\alpha < d - \tau_2 / 2$. It approaches a constant value with respect to $\alpha$ for $\alpha > d - \tau_2 / 2$.

For localized regimes with $\xi^d \sim 1$, the diffusion term as well as non-linear term can not be neglected. A particular solution of eq. (28) within this range can be obtained by expanding $\overline{f}_\alpha$ in the Taylor’s series in the neighborhood of $\alpha_0$ where $\overline{f}_{\alpha_0} = d$:

\[
\overline{f}_\alpha = d + t_1 (\alpha - \alpha_0) + t_2 (\alpha - \alpha_0)^2 + o(\epsilon^4)
\]

Substituting the above in eq. (28) gives $t_2, t_1, \alpha_0$.

4. Diffusion of wavefunction correlations

The criticality also manifests through the two-point energy correlation $C(e', e'')$ between two eigenstates say $Z_a$ and $Z_b$ with eigenvalues $e, e'$ respectively:

\[
C(e', e'') = \sum_{a,b} \sum_{m=1}^{N} |z_{ma}|^2 |z_{mb}|^2 \delta(e' - e_a) \delta(e'' - e_b)
\]

(with $z_{ma}$ implying $m^{th}$ component of the eigenfunction $Z_a$). As intuitively expected, its ensemble average is related to the 2-point spectral correlation $R_2(e', e'')$; this in turn connects the criticality criteria in the eigenfunction statistics to that of eigenvalues. As discussed in [13], the $Y$-governed diffusion of $\langle C(e, \omega) \rangle$ from an arbitrary initial condition, with $e' = e + \omega, e'' = e - \omega$, can be given as
\[
\frac{\partial \langle C \rangle}{\partial \nu} \approx \frac{1}{2} \left( \frac{\partial^2}{\partial e^2} + \frac{\partial^2}{\partial \omega^2} \right) e^\nu + \nu \frac{\partial}{\partial e} \omega \langle C \rangle + \beta \frac{\partial}{\partial e} \left( \left( \frac{2}{\beta} \right)^\nu e + \frac{N_e}{\Delta_e} e + a_0 - \frac{\omega}{2} \right) \langle C \rangle + \\
+ \beta \frac{\partial}{\partial \omega} \left( \left( \frac{2}{\beta} \right)^\nu \omega + \frac{N_e}{\Delta_e} \omega + a_0 - \frac{e}{2} \right) \langle C \rangle - \left( \beta + 1 \right) \frac{2e}{2\omega^2} \langle C \rangle + \frac{\beta (2e)^\nu}{2\omega^2} \langle \langle I_{e+\omega} + I_{e-\omega} \rangle \rangle R_2(e, \omega) \tag{32}
\]

A solution of the above equation for finite $\Lambda$ and Poisson initial condition gives the $\langle C(e, \omega) \rangle$ for Poisson $\rightarrow$ XOE ($\beta = 1$) or XUE ($\beta = 2$). The solution is expected to depend on both $e$ and $\omega$, thus indicating lack of stationarity in eigenfunction correlations. For local correlations i.e those for which a variation with respect to $e$ can be ignored, eq.(32) can be approximated as

\[
\frac{\partial \langle C \rangle}{\partial \nu} \approx \left[ \frac{\partial^2 \langle C \rangle}{\partial r^2} - \beta \frac{\partial \langle C \rangle}{\partial r} r - \left( \beta + 1 \right) \frac{2\nu \langle C \rangle}{r^2} \right] + \frac{2\nu \beta}{r^2} \left( \langle I_{r_0+r} \rangle + \langle I_{r_0-r} \rangle \right) R_2(r_0, r) \tag{33}
\]

where $\nu = 0, 1$ for Gaussian BE and Laguerre BE, respectively, $r_0, r$ are the rescaled energy $e = r_0 \Delta_e, \omega = r \Delta_e$ with $\Delta_e$ defined in eq.(1). In the stationarity limit $\Lambda_e \rightarrow \infty$, it can be shown that $\langle C \rangle \approx \frac{\beta}{(1+\beta) \tilde{N}} R_2(r_0, r)$ [13] (using $\langle I_{2r_0} \rangle = \frac{I_0}{\tilde{N}}$ for the stationary ensembles with delocalized eigenfunctions).

An exact solution of the above equation is complicated but its solution for small-$r$ can be given as $\langle C \rangle = r^s \sum_{n=0}^\infty \ d_n(\Lambda_e) \ r^n$ where $s, d_0, d_1$ depend on the small-$r$ behavior of $R_2(r; \Lambda_e)$:

\[
[(s-\beta)(s-1) - 2\nu (\beta + 1)] \ r^s \ d_0 + [(s-\beta)(s-1) - 2\nu (\beta + 1)] \ r^{s+1} \ d_1 = 2\nu \beta I_{2r_0} R_2
\]

Higher $d_n$ ($n > 1$) are given by the hierarchic relation

\[
\frac{dd_{n-2}}{d\Lambda_e} = [(n+s-\beta)(n+s-1) - 2\nu (\beta + 1)] \ d_n.
\]

For criticality considerations however the large-$r$ behavior is relevant which can be given as $\langle C \rangle = r^{-t} \sum_{n=0}^\infty c_n(\Lambda_e) \ r^{-n}$ with higher $c_n$ given by the equation

\[
(\mu + n + 2) \frac{dc_{n+2}}{d\Lambda_e} = [(n+t+\beta)(n+t+1)] .
\]
and \( t, c_0, c_1 \) given by the condition
\[
\left[ t r^{-t} \frac{dc_0}{d\Lambda_e} + (t + 1) r^{-(t+1)} \frac{dc_1}{d\Lambda_e} \right] = -2^{\nu-2} \beta \left( I_{2(r_0+r)} + I_{2(r_0-r)} \right).
\]

Clearly the above condition can be satisfied with \( t = 2 \) and \( c_0 \neq 0 \) only for the cases in which the inverse participation ratios \( I_{2(r_0 \pm r)} \) can be assumed to be independent of \( r \).

V. CRITICAL BE DURING POISSON → GOE TRANSITION: NUMERICAL ANALYSIS

The theoretical results in sections II-IV are applicable to the critical Brownian ensembles of both Gaussian and Laguerre type. For the numerical analysis, however, here we focus on Gaussian case only (due to its analogy with RP ensemble which has been analyzed in past in many studies).

Consider the transition in Gaussian ensembles with an initial state \( H = H_0 \) described by the ensemble density \( \rho_0(H_0) \propto e^{-\sum_i H_{0,ii}^2} \). Assuming a complete localization of its eigenfunctions in the basis in which \( H_0 \) is represented, the initial spectral statistics belongs to the Poisson universality class. The perturbation, of strength \( \lambda \), by a matrix \( V \) taken from a GOE (when represented in the unperturbed basis and of variance \( v^2 = 1 \), subjects eigenfunctions to increasingly delocalize as a function of \( \lambda \). The ensemble of matrices \( H = \sqrt{f}(H_0 + \lambda V) \), with \( f = (1 + \lambda^2)^{-1} \) is then referred as the Brownian ensemble during Poisson → GOE transition; it is described by the probability density [14, 23, 25, 29–31].

\[
\rho(H) \propto \exp \left[ -\frac{\eta}{2} \sum_{i=1}^{N} H_{ii}^2 - \eta(1 + \mu) \sum_{i,j=1; i<j}^{N} |H_{ij}|^2 \right]
\]

with \( \eta \) as an arbitrary parameter and \( (1 + \mu) = (\lambda^2 f)^{-1} \); here \( H = H_0 \) for \( \lambda \to 0 \) or \( \mu \to \infty \) and \( H \to V \) for \( \lambda \to \infty \). An ensemble \( H \) given by the above measure, is also known as the Rosenzweig-Porter (RP) ensemble [3].

The standard route for the spectral statistical analysis is based on the fluctuations around the average level density. In the present case, the ensemble averaged level density \( R_1 \), also known as 1st order correlation, changes from a Gaussian to a semi-circular form at the scale.
of $N\mu \sim R_1^2$:

\[
R_1(e) = \frac{N}{\sqrt{\pi}} e^{-e^2} \quad \text{for } (\mu/N) \to \infty \\
= \frac{1 + \mu}{\pi} \sqrt{\frac{2N}{1 + \mu} - e^2} \quad \text{for } (\mu/N) \to 0 \\
= NF(e, a) \quad \text{for } (\mu/N) = a
\]

with $a$ as an $N$-independent constant. Although the exact form of the function $F(e)$ is not known, our numerical analysis, displayed in figure 1, suggests a semicircle behavior in the spectral bulk i.e. $F(e) \approx (Nb\pi)^{-1} \sqrt{2bN - e^2}$ with Gaussian tails and $b$ as a constant independent of $N$. (Note the results (35,36) are given in 29 for $H$ as a complex Hermitian matrix only but the numerical evidence given in 4 and in the present study confirms its validity also for the real-symmetric $H$.)

An important point clearly indicated by eq.(37) is that $R_1(e)$ is non-stationary as well as non-ergodic; as discussed below, this plays a crucial role in compressibility calculation.

As discussed in 4, 29, 30, the spectral fluctuations around $R_1(e)$ are governed by the parameter $\Lambda$ [4], given by eq.(38), which in this case becomes (using mean level spacing $\Delta(e) = (R_1(e))^{-1}$ and $d = 1$ for a BE)

\[
\Lambda(e) = \frac{\xi^2}{N^2} \frac{R_1^2(e)}{(1 + \mu)} \quad (38)
\]

Note however for extended eigenstates, $\Lambda_e$ is given as

\[
\Lambda(e) = \frac{R_1^2(e)}{(1 + \mu)} \quad (39)
\]

For finite $N$, the $\Lambda$-variation due to changing $\mu$ at a fixed energy $e$ results in a cross-over of the BE statistics from Poisson ($\Lambda \to 0$) to GOE ($\Lambda \to \infty$) statistics. In limit $N \to \infty$ and for arbitrary $\mu$, $\Lambda(e)$ varies abruptly between 0 or $\infty$, ruling out possibility of any intermediate statistics. But for the case in which $\mu$ is such that it conspires to an existence of the limit $\Lambda^*(e) \equiv \lim_{N \to \infty} \Lambda(e)$, the statistics is size-independent and belongs to a new universality class, different from the two end-points and is referred as the critical Brownian ensemble.

As clear, eq.(38) alongwith eq.(37) implies the existence of two critical points during the transition (instead of one as previously believed 14, 29):
\( \mu = c_2 N \): as mentioned above, \( R_1(e) \) for this case behaves as a semi-circle in the bulk: 
\[ R_1(e) = (b\pi)^{-1} \sqrt{2bN - e^2}. \]
Eq. (20) gives \( \xi \sim (\langle u \rangle)^{-1} \approx \sqrt{N} \) and this case therefore 
Corresponds to extended eigenstates. (Note the relation of averaged localization length to 
Averaged inverse participation ratio gives \( \langle I_2 \rangle \sim \xi \sim N^{-1/2} \), implying \( \tau_2 = D_2 = 0.5 \) 
which is also confirmed by the behavior depicted in figure 5(c). But the study \[ 1 \] gives 
\( D_2 = 1 \). Eq. (39) then gives \( \Lambda(e) = \frac{2bN - e^2}{\pi^2 b^2 N c_2} \) with \( b \sim 2 \). Note however although \( \Lambda(e) \) is 
Size-independent near the band-center \( e \sim 0 \), it is still quite large (\( \Lambda \approx \frac{b}{\pi c_2} \)), indicating the 
Level-statistics to be close to the GOE. An intermediate statistics between Poisson and GOE 
can however be seen near \( e \sim \sqrt{e_0 N} \) for \( e_0 \ll b \).

\( \mu = c_1 N^2 \): \( R_1 \) is now given by eq. (35) and eq. (20) gives \( \xi \sim N^0 \sim 1 \). From eq. (38), \( \Lambda \) is 
Again size-independent: \( \Lambda(e) = \frac{1}{4\pi c_1} e^{-e^2} \). (Note our analysis predicts \( D_2 \sim 0 \) for this case 
which is in agreement with study \[ 1 \]). For \( c_1 \sim 1, e \sim 0, \Lambda \sim \frac{1}{4\pi} \) and the statistics lies 
between Poisson and GOE even for energy ranges near \( e \approx 0 \). The numerical evidence for 
the criticality in this case has already been presented in \[ 4, 32 \].

The theoretical formulations of the spectral compressibility and multifractal spectrum 
discussed in previous sections are based on some approximations at various stages of the 
derivation. It is therefore desirable to verify the results by numerical route. The latter can 
also give an insight in critical point behavior of some other measures e.g nearest neighbor 
space distribution. A finite size scaling analysis given in \[ 4 \] has already confirmed the 
critical behavior of level statistics for \( \mu \propto N^2 \). The criticality of BE for this case but 
\( H \) taken from a real-quaternion ensemble was numerically verified in \[ 32 \] (see figure 3 of 
\[ 32 \]). In the present work, we pursue a numerical analysis of the criticality for the case 
\( \mu \propto N \). Note, this case was numerically analyzed in \[ 1 \] too but mainly in the context of 
the eigenfunction in the bulk (around the eigenvalues at maximum level density) regime. 
To understand non-stationary aspects of critical statistics, we analyze three energy regime 
i.e. edge, bulk \( (e \sim 0) \) or at intermediate energies (the region where \( R_1(e) \) is half of its 
maximum value). Although, due to rapid change in \( R_1(e) \), edge results are believed to be 
error-prone and thus a bit unreliable, but our results show a systematic trend.
A. Critical spectral statistics

As our claim about criticality of BE at $\mu = N$ is based on a $\sqrt{N}$-dependence of average level density, our first step is to numerically confirm its size-dependence. At this stage, an important question is regarding the ergodicity of the level density for the BE which implies $\rho_{sm}(e) = R_1(e)$, with $\rho_{sm}$ as the spectral averaged level density; $R_1(e)$ can then be used as a substitute for $\rho_{sm}(e)$ for various analytical purposes. The ergodicity has already been confirmed in a previous study [36] (by a numerical comparison of the ensemble and the spectral averaging of the level density). It is therefore sufficient to analyze the size-dependence of $R_1(e)$. For this purpose, we consider the ensembles consisting of a large number of real-symmetric matrices of many system sizes; the spectrum for each such ensemble is numerically generated using LAPACK subroutine based on exact diagonalization approach. As shown in figure 1, $R_1(e)$ is indeed semi-circle in the bulk but deviating from it near the edge. As clear from the behavior of curves for different $N$, the $N$-dependence is same for all energy ranges including edge as well as bulk.

As a next step, we analyze the spectral statistics which requires a careful unfolding of the spectrum. Due to unavailability of the analytical form of $R_1(e)$ for all energy ranges, we apply the local unfolding procedure [35] based on following steps: the smoothed level density $\rho_{sm}$ for each spectrum is first determined by a histogram technique, and then integrated numerically to obtain the unfolded eigenvalues $r_n = \int_{-\infty}^{e_n} \rho_{sm} \, de$. The spectrum being non-stationary with energy-sensitive fluctuations (see figures 2,3 of [36]), it is necessary to analyze the statistics at different energy-ranges. For $\Lambda$-based comparisons, ideally one should consider an ensemble averaged fluctuation measure at a given energy-point $r$ without any spectral averaging. But in the regions where $\Lambda$ varies very slowly, it is possible to choose an optimized range $\Delta e$, sufficiently large for good statistics but keeping mixing of different statistics at minimum. We analyze 5% of the total eigenvalues taken from a range $\Delta e$, centered at the energy-scale of interest i.e. edge, bulk and intermediate energies. (As for $\mu = cN$, $\rho_{sm}$ in the bulk is almost constant, the statistics is locally stationary and one can take levels within larger energy ranges without mixing the statistics. A rapid variation of $\rho_{sm}$ in the edge however permits one to consider the levels only within very small spectral
ranges. For edge-bulk comparisons, it is preferable to choose the same number of levels for both spectral regimes. The number of matrices $M$ in the ensemble for each matrix size $N$ is chosen so as to give approximately $10^5$ eigenvalues and their eigenfunctions for the analysis.

To verify size-independence of the spectral statistics for $\mu \propto N$, we consider $P(s)$ and $\Sigma^2(r)$ for the BE with $\mu \propto N$ for many system sizes. For comparison, it is useful to give their behavior in the two stationary limits:

(i) GOE: $P(s) = \frac{s}{2} \exp\left(-\pi s^2/2\right)$, $\Sigma^2(r) = \frac{2}{\pi r} (\ln r + C)$, with $C \approx 2.18$,

(ii) Poisson: $P(s) = \exp(-s)$, $\Sigma^2(r) = r$.

It is also desirable to compare the BE-numerics with theoretical BE results but the exact $P(s)$ results for the BE with matrices of arbitrary size $N$ is not known. It is however easy to derive the $P(s)$ for $N = 2$ case [17, 31]:

$$P(s, \Lambda) = \frac{s}{4\Lambda} \exp\left(-\frac{s^2}{8\Lambda}\right) \int_0^\infty dx \, e^{-(x^2/8\Lambda)-x} I_0\left(\frac{xs}{4\Lambda}\right) \tag{40}$$

with $I_0$ as the modified Bessel function. As $P(s)$ is dominated by the nearest neighbor pairs of eigenvalues, this result is a good approximation also for $N \times N$ case, especially in small-$s$ and small-$\Lambda$-result [31].

Figures 2-3 display the behavior of $P(s)$ and $\Sigma^2(r)/r$ for many system sizes ranging from $N = 500$ to $N = 25000$ for $P(s)$ and $N = 500$ to $N = 10000$ for $\Sigma^2(r)/r$, and, for the three energy regions. As for $e \approx 0$, $\Lambda$ is $N$-independent and also large, both $P(s)$ as well as $\Sigma^2(r)$ curves are close to those of GOE and it is not easy to resolve the deviation from the numerical error. For $e$ near the edge, $\Lambda$ is almost zero and, as expected, figures 2 and 3 confirm the statistics to be that of a Poisson. For intermediate $e$, $\Lambda$ is $N$-independent (see eq.(1)), thus implying a size-independent statistics, as well as distinguishable from both Poisson and GOE. As shown in figure 2(b), $P(s)$ in this regime is indeed size-independent and is well-fitted by the curve $a \, s \, \exp(-bs^2 - \kappa s)$. (Note, the fit is a close approximation of the theoretical formulation for $P(s)$, known only for a $2 \times 2$ BE. The $\ln P(s)$ behavior displayed in figure 2.(d)- 2.(f) reconfirms the exponential tail, $P(\text{large } s) \approx \exp(-\kappa s)$, where $\kappa \approx 2$; a non-zero $\kappa$ is believed to be an indicator of the critical spectral statistics [18]. Its relation with compressibility $\chi$ (defined as the large $r$-limit of $\Sigma^2(r)/r$) further suggests a
fractional value for the latter: \( \chi = 1/2\kappa \). Note \( \chi = 1, 0 \) for Poisson and GOE respectively. But the large-\( r \) behavior of \( \Sigma^2(r)/r \) curves in figure 3 seems to be less than the expected value. The reason for this deviation seems to be the spurious fluctuations due to finite size effects which affect the long-range statistics more severely. The true fluctuations can only be seen by going to \( N \to \infty \) limit. As can be seen from figure 3, the large \( r \)-limit of \( \Sigma^2(r)/r \) is indeed increasing with system size \( N \) which encourages one to believe that the expected value \( \chi = 0.25 \) will be approached in the limit \( N \to \infty \). (It is worth emphasizing here that our analysis confirms a fractional compressibility through two measures, one spectral, i.e. \( P(s) \) and, other based on eigenfunctions i.e fractal dimension \( D_2 = d(1-2\chi) \); see discussion in section V.B. Although the \( \chi \)-value given by a third measure i.e number variance is lower than the previous two for the system-sizes considered here but the errors due to finite-size effect in \( \Sigma^2(r) \) for large \( r \) are well known. For reliable results, one needs to go to very large \( N \); this is also evident from \( \chi \) increasing with \( N \).)

The critical point analysis in figures 2 and 3 is based on an ensemble of finite size matrices. As the critical behavior in physical systems manifests in thermodynamic limit \( N \to \infty \), one should confirm the persistence of finite \( N \) behavior in infinite \( N \) limit. The standard route for this purpose is the finite-size scaling analysis of the fluctuation measures. According to single parameter scaling (SPS) hypothesis of disordered systems, a measure \( F \) at disorder \( W \), energy \( e \) and sizes \( N \) follows a scaling law \( F(N,W,e) = f(\xi(W,e)/N) \) where the scaling function \( f \) is a generalized homogeneous function and \( \xi \) denotes the correlation length [34, 38]. A more generalized version of SPS for a function \( F \), dependent on \( n \) parameters \( x_1, x_2, \ldots, x_n \) and size \( N \) is \( F(x_1, x_2, \ldots, x_n; N) = F(\Lambda) \) with \( \Lambda = \Lambda(x_1, \ldots, x_n; N) \). One traditionally used measure in this context is the relative behavior of the tail of nearest-neighbor spacing distribution \( P(s) \), defined as

\[
\gamma(\delta; \Lambda) = \frac{\int_0^\delta (P(s;\Lambda) - P(s;\infty))ds}{\int_0^\delta (P(s;0) - P(s;\infty))ds}
\]

with \( \delta \) as any one of the crossing points of \( P_o(s) = P(s;\infty) \) and \( P_p(s) = P(s;0) \) (here the subscripts \( o \) and \( p \) refer to the GOE and Poisson cases respectively) \[33\]. As obvious, \( \gamma = 0 \) and 1 for GOE and Poisson limit respectively and a fractional value of \( \gamma \) indicates the probability of small-spacings different from the two limits. In the region around critical
parameter $\Lambda_c$, $\gamma$ is expected to scale as $\gamma = \gamma_c + \left(\frac{N}{\xi}\right)^{1/\nu}$ with $\nu$ as the critical exponent.

Now for a BE representing an intermediate state in localization to delocalization transition, it is natural to seek the validity of SPS for BE too. Note this is already implied by the $\Lambda$-formulation of the BE-fluctuation measures, including $\gamma$. Expanding $\gamma(\Lambda)$ in the critical region around its critical value $\gamma_c$ at $\Lambda = \Lambda_c$ gives

$$\gamma = \gamma_c + a \left( \frac{1}{\Lambda} - \frac{1}{\Lambda_c} \right)$$

$$= \gamma_c + \frac{a}{R_1^2(\mu_c)} (\mu - \mu_c)$$

(42)

(43)

with $a$ as a constant. Now as for $\mu_c = N$, $R_1(\mu_c) \propto \sqrt{N}$, it is tempting to conclude that the correlation length $\xi$ for the BE behaves as $|\mu - \mu_c|$; (note the dimensionality $d = 1$ in this case). But as $\mu_c$ itself is $N$-dependent, an interpretation of $\xi$ from the above result is not obvious. The above clearly indicates $\Lambda$ as a more generic parameter for scaling analysis as compared to $\xi/N$. The figure 4 shows the numerically obtained behavior of $\gamma$ (for $\delta = 0.473$) (for 40% of total eigenvalues at edge-intermediate and bulk) with respect to $|z - c|$ ($=|\mu - \mu_c|N^{-1}$) for a fixed $c$ (arbitrarily chosen) with $z$ as a variable; Here $z$ and $c$ are the values of the parameter $\mu N^{-1}$ for a general BE and a critical BE respectively. The constant value of $\gamma$ at $|z - c| = 0$ for different $N$-values confirms the size-independence of the level-statistics of BE with parameter $\mu = cN$ at $c = 1$ and therefore its critical nature. Further the convergence of $\gamma$-values for BEs with different $\mu$ and $N$-values on two branches indicates the presence of a scaling with $|z - c|$ ($=|\mu - \mu_c|N^{-1}$) as the scaling parameter. The corresponding behavior for $\mu = cN^2$ is displayed in figure 1. of [4].

B. Multifractal analysis of wavefunctions

Our next step is to investigate the criticality in wavefunctions. Multifractality of wavefunctions at criticality can be shown on the basis of scaling of inverse participation ratio and singularity spectrum. We first analyze the distribution of inverse participation ratio. It has been conjectured that the distribution of $I_q$ normalized to its typical value $\langle \ln I_q \rangle$ has a scale-invariance at the localization-delocalization transition. This corresponds to a shape-invariance of $P(\ln I_q)$ with increasing system size $N$, the latter causing only a shift
of the distribution along $I_q$ axis \[40\]. The above conjecture was questioned at first but confirmed later by numerical studies on Anderson transition in $d > 2$ (with $d$ as dimension) and critical power law random banded matrix (PRBM). This provokes the curiosity about its validity in case of the critical BEs and we attempt to answer it by a numerical analysis of the eigenstates for the case $\mu = N$. To overcome finite size effects, an essential drawback of numerics, one has to consider averages over different realizations of disorder as well as a narrow energy range. As these fluctuations in bulk are analyzed in detail in \[1\], here we confine ourselves to intermediate regime only. For this purpose, we consider the eigenstates in a narrow energy range 5% around intermediate energy for each matrix of the ensemble with $\mu = N$, consisting of $M$ matrices, with $M = 8000, 6000, 5000, 3000, 2500, 1500$ for $N = 500, 750, 1000, 1500, 2000, 3000$ respectively; ( note, for the scaling exponent and singularity spectrum calculation, we have taken $M = 20$ for each $N$ cases, a large ensemble size $M$ not required for their analysis). Figure 5(a) shows the distribution $P(\ln I_2)$ for the critical BE with $\mu = N$ for six system sizes; the scale invariance of the distribution is clearly indicated from the figure. As indicated by previous studies \[40\], the $I_q$-distribution is expected to show a power-law tail at the transition: $P(I_q/I_q^{typ}) \propto (I_q/I_q^{typ})^{-1-x_q}$ for $I_q \gg I_q^{typ}$; the behavior is confirmed in figure 5(b) for $q = 2$ with $x_{q=2} \gg 1$. Furthermore the change in peak-position of $P(\ln I_q)$ with changing system size indicates a power-law dependence of $\langle I_q \rangle$ on system size $N$, governed by a continuous set of exponents: $\langle I_q \rangle \sim N^{-\tau^{typ}_q}$ where $\tau^{typ}_q = \tau_q$ for $x_q > 1$.

As mentioned in section IV, the multifractal behavior of eigenfunction is described by a continuous set of scaling exponents $\tau_q$ \[40\]. The latter can be computed by standard box-size scaling approach. This is based on first dividing the system of $L^d$ basis states into $N_l = (L/l)^d$ boxes ($d$ is the dimension of the system and for our case, $d=1$) and computing the box-probability $\mu_k$ of $\psi$ in the $i^{th}$ box: $\mu_k(l) = \sum_n |\psi_n|^2$; here $\sum_n$ is over over basis-states within the $k^{th}$ box. This gives the scaling exponent $\tau_q$ for the typical average of $I_q(l) = \sum_{k=1}^{N_l} I_q^k(l)$:

$$\tau^{typ}_q = \frac{\langle \ln I_q(\lambda) \rangle}{\ln \lambda}$$

where $\langle \cdot \rangle$ is the average over many wavefunction at the criticality. For numerical calculation
of $\tau_{q}^{typ}$, one usually considers the limit $\lambda \equiv l/L \to 0$ which can be achieved either by making $L \to \infty$ or $l \to 0$. For our analysis, we consider $\lambda = 0.1$.

At criticality, $\tau_q$ is a nontrivial function of $q$ and can be written as $\tau_q = d(q - 1) + \Delta_q$ [40]. By definition of anomalous dimensions $\Delta_q$, $\Delta_0 = \Delta_1 = 0$ which makes $\tau_q$ as a nondecreasing convex function with $\tau_0 = -d$, $\tau_1 = 0$. As mentioned in section IV, $\tau_q$ is also related to the fractal dimension $D_q$. For the critical BE with $\mu = N$, our numerics gives $D_2 = \tau_2 \approx 0.5$ (see figure 5(c)) which is different from the prediction based on study [1] for this case (latter gives $D_2 \approx 1$). Note, based on the relation $D_2 = d(1 - 2\chi)$, we get $\chi = 0.25$ which is not only same as the value given by $\chi - \kappa$ relation but is also in agreement with our theoretical prediction [4].

The multifractality of eigenfunctions can also be characterized by the singularity spectrum $f(\alpha_q) = q\alpha_q - \tau(q)$ with $f'(\alpha_q) = q$, $\alpha_q = \tau'_q$. For completely delocalized wavefunctions $f(\alpha)$ is fixed: $f(\alpha) = d$ but its spread increases in going from delocalized wave limit to the localized one. For our numerical calculation using box-approach, we use following expression for $f(\alpha)$ and $\alpha$ [41]

$$\alpha_{q}^{typ} = \lim_{\lambda \to 0} \frac{1}{\ln \lambda} \ln \left( \frac{1}{I_q(\lambda)} \sum_{k=1}^{N_\lambda} \mu_k(q, \lambda) \ln \mu_k(1, \lambda) \right)$$

$$f(\alpha_{q}^{typ}) = \lim_{\lambda \to 0} \frac{1}{\ln \lambda} \left[ q \left( \frac{1}{I_q(\lambda)} \sum_{k=1}^{N_\lambda} \mu_k(q, \lambda) \ln \mu_k(1, \lambda) \right) - \langle I_q(\lambda) \rangle \right]$$

with superscript "typ" on a variable implying its typical value. It is believed that the typical spectra is equal to the average spectra (i.e. $\tau_{q}^{typ} = \tau_q$ and $f_{q}^{typ}(\alpha) = f(\alpha)$) in the regime $q_- < q < q_+$ [? ]. Here $q_\pm$ correspond to the values of $q$ such that $f(\alpha_q) = 0$; the corresponding value of $\alpha_q$ are referred as $\alpha_\pm$, respectively. Our numerics of $f(\alpha)$ is confined within this regime.

As displayed in figure 5 for six system sizes, $f(\alpha)$ behavior for the BE with $\mu = cN$ is intermediate between the localized and delocalized limit. The symmetry relation $\Delta_q = \Delta_{1-q}$ in the spectrum of $\Delta_q$ can also be seen from the figure. Also clear from the figure, $\alpha$ is
contained in the interval \((0, 2)\) and the values of \(f(\alpha)\) for the region \(\alpha \leq 1\) can be mapped to the region \(\alpha > 1\) by the mapping \(f(2 - \alpha) = f(\alpha) + 1 - \alpha\). Our analysis also gives \(\alpha_0 = 1.3 > d, \alpha_1 = 0.74, f(\alpha_0) = d = 1, f(\alpha_1) = \alpha_1\). Above results are consistent with expected multifractal characteristics of the critical eigenstates [40][41]. Theoretical value of \(x_q > 1\) for \(q_- < q < q_+\) is also confirmed by our numerics.

VI. CONNECTION WITH OTHER ENSEMBLES

A Gaussian Brownian ensemble is a special case of a multi-parametric Gaussian ensemble. As indicated by the studies [4][8][16], the eigenvalue distributions of a wide range of ensembles with single well potential e.g those with a multi-parametric Gaussian measure and independent matrix elements, appear as a non-equilibrium stages of a Brownian type diffusion process [16]. Here the eigenvalues evolve with respect to a single parameter, say \(Y\), which is a function of the distribution parameters of the ensemble. The parameter is related to the complexity of the system represented by the ensemble and can therefore be termed as the spectral ”complexity” parameter. The solution of the diffusion equation for a given value of the complexity parameter gives the distribution of the eigenvalues, and thereby their correlations, for the corresponding system. As the local spectral fluctuations are defined on the scale of local mean level spacing, their diffusion is governed by a competition between \(Y - Y_0\) and local mean level spacing. Consequently the evolution parameter \(\Lambda_e\) for the local spectral statistics is again given by eq. (1) but with a more generic definition of \(Y\); (note so far the complexity parameter formulation has been analyzed in detail only in context of Gaussian ensembles although the studies [12][16] indicate its validity for more generic cases). A single parameter formulation is also possible for the eigenfunction fluctuations but, contrary to spectral case, the parameter is not same for all of them.

The implication of complexity parametric formulation is significant: as the system dependence enters through a single parameter in a fluctuation measure, its behavior for different systems with same value of the complexity parameter (although may be consisting of different combinations of the system parameters) wil be analogous (valid for same global constraints; see [16] for details). An important point worth emphasizing here is the following: although
the unfolding (rescaling by local spectral density) of the eigenvalues removes their dependence on the local spectral scale, the latter is still contained in $\Lambda_e$. The spectral dependence of $\Lambda_e$ varies from system to system. Thus two systems in general may have same spectral statistics at a given spectrum-point but the analogy need not extend for a spectral range of sufficient width. It could however happen in case the two systems have same local rate of change of $\Lambda_e$ along the spectrum which usually requires a similar behavior for the local spectral density. The analogy implied by the complexity parameter formulation is therefore strictly valid only in case of the ensemble averaging. It can however be extended to include spectral averaging within the range in which the local density is almost stationary.

The Anderson ensemble (AE) consisting of Anderson Hamiltonians, the power law random banded matrix (PRBM) ensemble and the Brownian ensemble appearing during Poisson $\rightarrow$ GOE transition belong to same global symmetry class (time-reversal symmetry preserved). Based on the complexity parameter formulation, therefore, the critical point statistics of an AE or PRBME can be mapped to that of the P $\rightarrow$ G Brownian ensemble. The validity of the mapping was indeed confirmed by a number of numerical studies [4, 32]. The study [1] however claims that the critical point behavior for an Anderson ensemble and a PRBM ensemble differ from that of a Rosenzweig-Porter ensemble (same as the Brownian ensemble between Poisson $\rightarrow$ GOE cross-over). For example, the study shows that the correlation $C(\omega)$ between two wavefunctions, at energies $e$ and $e + \omega$ decays as $\omega^{-\mu}$ for $\omega \gg E_{th}$, with $\mu = 2$ for Rosenzweig-Porter ensemble and $\mu = D_2 - 1$ for Anderson Hamiltonian and PRBM ensemble. Here $E_{th} \sim N^{-z}$ is the Thouless energy, with $z = 1$ for AE and PRBME and $z < 1$ for the BE. These results are however based on the assumption of local stationarity of the spectral density around which the fluctuations are measured. The seeming contradiction of the results between [1] and [4] originates in the range of validity of the assumption. As indicated by previous studies, the ensemble averaged spectral density of both Anderson as well as PRBM ensemble in the bulk is almost similar but is different from that of the Poisson $\rightarrow$ GOE Brownian ensemble. In the latter case, it varies more rapidly along the spectrum (see section V); the range $r$ of local stationarity in case of the BE is therefore much smaller than the AE and PRBME and the measures which are based on
large $r$-limit considerations can not be compared appropriately. (Note, as mentioned in [1], $C(\omega)$ for AE and PRBME indeed behaves as $\omega^{-2}$ for $\omega$ of the order of half spectrum width. But this is also the case for the BE as $E_{th}$ for a BE is of the order of half spectrum width.

(iii) The critical BE analog of a critical AE is unique. Further, it is different for critical AE with and without time-reversal symmetry. Similar to AEs, the level-statistics of BEs shows a scaling behavior as well as a critical point with fractional level-compressibility. However, unlike AEs, the $\chi$ turns out to be symmetry dependent for BEs, their parameter $\Lambda$ being symmetry independent.

VII. CONCLUSION

Based on a non-perturbative diffusion route, we find that the fluctuation measures for the critical Brownian ensemble are sensitive to both energy as well as the perturbation strength. Our results are applicable for both Gaussian Brownian ensembles as well as Laguerre Brownian ensembles of the Hermitian matrices. An understanding of critical BEs lying between the pairs of stationary ensemble subjected to other global constraints e.g. non-Hermiticity (e.g. circular ensembles), chirality, column constraints still remains an open question.
[1] V.E. Kravtsov, I.M. Khaymovich, E.Cuevas and M. Amini, New. J. Phys (IOP), (2016).
[2] P. Shukla, New. J. Phys. (IOP), 18, 021004, (2016).
[3] N. Rosenzweig and C.E.Porter, Phys. Rev. 120, 1698 (1960).
[4] P. Shukla, J. Phys.: Condens. Matter 17, 1653, (2005); Phys. Rev. E, 62, 2098, (2000).
[5] F. Dyson, J. Math. Phys. 3, 1191 (1962).
[6] M. L. Mehta, *Random Matrices*, Academic Press, (1991).
[7] A. Pandey, Chaos, Solitons, Fractals, 5, 1275, (1995).
[8] P. Shukla, Int. J. Mod. Phys. B (WSPC) 26, 12300008, (2012).
[9] A. Pandey and P. Shukla, J. Phys. A, 24, 3907, (1991).
[10] S. Kumar and A. Pandey, Ann. Phys. 326, 1877, (2011).
[11] Vinayak and A. Pandey, Phys. Rev. E, 81, 036202 (2010).
[12] P. Shukla, Phys. Rev. Lett., 87, 19, 194102, (2001).
[13] P. Shukla, arXiv/submit/1673866.
[14] A. Altland, M. Janssen and B. Shapiro, Phys. Rev. E, 56, 1471, (1997).
[15] M. Janssen, Phys. Rep. 295, 1, (1998).
[16] P. Shukla, J. Phys. A, 41, 304023, (2008); P. Shukla, Phys. Rev. E, (71), 026226, (2005); Phys. Rev. E, 62, 2098, (2000);
[17] M.V. Berry and P. Shukla, J. Phys. A, Math. Theo. 42, 485102, (2009).
[18] B.I. Shklovskii, B. Shapiro, B.R. Sears, P. Lambrianides and H.B. Shore, Phys. Rev. B, 47, 11487 (1993).
[19] J.T. Chalker, V.E. Kravtsov and I.V. Lerner, Pis’ma Zh. Eksp. Teor. Fiz. 64, 355 (1996) [JETP Lett. 64, 386, (1996)].
[20] B.L. Altshuler, I.Kh. Zharekeshev, S.A. Kotochigova and B. Shklovskii, Sov. Phys. JETP 67, 625, (1988).
[21] A.D. Mirlin, Y.V. Fyodorov, F.-M. Dittes, J. Quezada and T.H. Seligman, Phys. Rev. E, 54, 3221, (1996).
[22] J.B. French, V.K.B. Kota, A. Pandey and S. Tomsovic, Ann. Phys., (N.Y.) 181, 198 and 235 (1988).
[23] F. Leyvraz and T.H. Seligman, J. Phys. A: Math. Gen. 23, 1555, (1990).
[24] H.Kunz and B.Shapiro, Phys. Rev. E, 58, 400, (1998).
[25] K.M.Frahm, T.Guhr, A.Muller-Groeling, Ann. Phys. (N.Y.) 270, 292 (1998).
[26] P. Shukla, Phys. Rev. E, 75, 051113, (2007).
[27] A.G. Aronov, V.E.Kravtsov and I.V.Lerner, Phys. Rev. Lett., 74, 1174, (1995).
[28] J.M.G. Gomez, R.A.Molinas, A. Relano and J. Retamosa, Phys. Rev. E, 66, 036209, (2002);
    O. Bohiga and M.J.Giannoni, Ann. Phys. 89, 422, (1975); I.O.Morales, E.Landa, P.Stransky
    and A.Frank, Phys. Rev. E, 84, 016203 (2011).
[29] M. Krenin and B. Shapiro, Phys. Rev. Lett., 74, 4122, (1995); B. Shapiro, Int. J. Mod. Phys.
    B, 10, 3539, (1996).
[30] J-L. Pichard and B. Shapiro, J. Phys. I: France 4, 623, (1994).
[31] S.Tomsovic, Ph.D Thesis, University of Rochester (1986); G.Lenz and F.Haake, Phys. Rev.
    Lett. 67, 1, (1991); V.K.B.Kota and S.Sumedha, Phys. Rev. E, 60, 3405, (1999).
[32] R. Dutta and P.Shukla, Phys. Rev. E, 76, 51124, (2007).
[33] E.Hofstetter and M.Schreiber, Phys. Rev. B, 49, 14726, (1994).
[34] B.Kramer and A. MacKinnon, Rep. Prog. Phys. 56, (1469) (1993).
[35] J.M.G. Gomez, R.A.Molinas, A. Relano and J. Retamosa, Phys. Rev. E, 66, 036209, (2002);
    O. Bohiga and M.J.Giannoni, Ann. Phys. 89, 422, (1975); I.O.Morales, E.Landa, P.Stransky
    and A.Frank, Phys. Rev. E, 84, 016203 (2011).
[36] P.Shukla and S. Sadhukhan, J. Phys. A, , 48, 415003, (2015); J. Phys A, 48, 415002, (2015).
[37] Y.V.Fyodorov and A.D.Mirlin, Int. J. Mod. Phys. B, 8, 3795, (1994).
[38] M. Janssen, Phys. Rep. 295, 1, (1998).
[39] F. Evers, A. Mildenberger and A.D. Mirlin, Phys. Rev. B 64, 241303, (2001).
[40] F. Evers and A.D. Mirlin, Rev. Mod. Phys, 80, 1355, (2008).
[41] A. Rodriguez, L.J. Vasquez and R.A.Romer, \texttt{arXiv:0807.4854v1} (2008).
FIG. 1. **Ensemble averaged level density** $R_1(x)$: Behavior of a Brownian ensemble (BE) eq.\([34]\) with $\mu = N$ for nine system sizes $N = 500, 750, 1000, 1500, 2000, 3000, 10000, 15000, 25000$. Here, $x = \epsilon/\sqrt{N}$. Graphs for different $N$ are scaled by $R_1(x) \rightarrow R_1(x)/N$. The solid line corresponds to the semi-circle fit as given in eq.\([37]\) with $b \approx 2$. A comparison of $R_1(\epsilon)$ with spectral level density $\rho_{sm}(\epsilon)$ is given in \([36]\). The above figure confirms the semicircle behaviour at bulk and same $N$-dependence for all energy ranges.
FIG. 2. $P(s)$ for three energy regimes: The behavior for BE eq. (34) with $\mu = N$ for nine sizes $N$ in three energy ranges: (a) and (d)- edge, (b) and (e)- intermediate, (c) and (f)- bulk. The parts (d), (e), (f) compare the tail behavior with the fit $a s \exp(-b s^2 - \kappa s)$. As clear from fig.(a), the statistics is approaching poisson behavior as $N$ increases. The bulk behavior coincides with GOE limit but intermediate regime is different from both poisson and GOE limit; this is in agreement with theoretically expectation as $\Lambda_{\text{bulk}} \sim 1 > \Lambda_{\text{edge}}$. As expected for critical statistics, $P(s)$ in (c) approaches to an invariant form as the system size $N$ increases. The criticality of the case (b) is again confirmed in (e) by the exponential decay which gives $\kappa \approx 2$. 

\[ \text{FIG. 2. } P(s) \text{ for three energy regimes: The behavior for BE eq. (34) with } \mu = N \text{ for nine sizes } N \text{ in three energy ranges: (a) and (d)- edge, (b) and (e)- intermediate, (c) and (f)- bulk.} \]
FIG. 3. **Number-variance $\Sigma_2(r)/r$ in three energy regimes:** The behavior of the variance of number of levels in a distance of $r$ mean level spacings for BE eq. (34) with $\mu = N$ for seven system sizes in three energy ranges: (a) edge, (b) intermediate, (c) bulk. The dotted line in (a, b, c) correspond to theoretical prediction of GOE given by eq. (V A). The value of $\chi$ is less than the expected value of $\chi = \frac{1}{25} = 0.25$. As indicated by the parts (a), (b), the critical behavior of $\chi$ (i.e. $0 < \chi < 1$) is not evident for small $N$ cases but appears only in large $N$ limit. This is due to spurious fluctuations due to finite size effects which is even more in the large $r$-limit.
FIG. 4. Finite size scaling of spectral statistics: To study the behavior of $\gamma$ (eq. (42)) for BE described by eq. (34) as a function of $|z - c|$ for $\mu = cN$ with $c = 1$, unlike the other fluctuations in this work, we take 40% levels for the statistics in the energy range of interest. Consequently, the intermediate regime almost overlaps with that of edge, leading only two energy ranges for analysis namely "edge-intermediate and "bulk". Fig. (a) and fig. (b) correspond to edge-intermediate and bulk regime respectively. Constant value of $\gamma$ at $|z - c| = 0$ for different $N$ confirms the critical nature. Also, two branches of $\gamma$ values for different $N$ and $\mu$ value points out the presence of a scaling parameter of the BE under consideration. The corresponding behavior for $\mu = cN^2$ is given in [4].
FIG. 5. Multifractality of eigenfunctions at intermediate regime: The figures displays the distribution as well as multifractality spectrum for BE eq.\(^{(34)}\), with \(\mu = N\), for six system sizes, \(N = 500, 750, 1000, 1500, 2000, 3000\) at intermediate energy regime: (a) \(P(\ln I_2)\) - distribution shifts along \(\ln <I_2>\) axis preserving their form as \(N\) increases, (b) \(P(I_2/I_2^{typ})\) - the fit \(f(I_2) = (I_2/I_2^{typ})^{−1}−x_2\) at \(I_2 \gg I_2^{typ}\) in this part gives value of \(x_2 \gg 1\), which in its turn says the typical spectra and averaged spectra are equal. (c) \(\tau_q\) - As, \(D_q = \frac{\tau_q}{q-1}\), we get the value of \(D_2 = \tau_2 = 0.5\), which is again corroborated from the other studies. As clear from the figure, although \(D_q\) is constant but not 1 as reported in \([?]\). (d) Multifractal exponent \(\Delta_q\) - proof of symmetricity around \(q = 0\). It also implies the symmetricity of singularity spectrum, (e) \(\text{avg}IPR\) and \(I_2^{typ}\), (f) \(f(\alpha_q)^{typ}\) - This singularity spectrum depicts a comparison with parabolic approximation which is expected to be valid in weak-multifractality limit.