Remarks on the Q-curvature flow

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Abstract

The main purpose of this short note is to point out that the negative gradient flow for the prescribed Q-curvature problem on $S^n$ can be extended to handle the case that the Q-curvature candidate $f$ may change signs.

1. Various prescribing curvature problems on a manifold can be restated as follows: given a smooth function $f$ defined on $M^n$ with the metric $g$, can one find a conformal metric $g_u = e^{2u}g$ such that the aforementioned curvature is equal to $f$? The typical example is the prescribing scalar curvature problem with $(M^n, g) = (S^n, g_{S^n})$ where $g_{S^n}$ is the standard round metric. In past several decades, this problem has attracted a lot of attention. Recently, several groups of people are interested in the prescribing $Q$-curvature problem. It is well known that both problems are equivalent to solving certain partial differential equations. When the background manifold is the standard sphere, the non-compactness of the conformal group made the problem more interesting to study. We refer readers to [15] for more background materials on this type of problem. Recall the prescribing $Q$-curvature problem on $S^n$ is equivalent to the solvability of the following equation

$$P_n u + (n - 1)! = f e^{nu} \text{ on } S^n,$$

where $P_n = P_{g_{S^n}}$ is $n$-th order Paneitz operator. Notice that Equation (1) has a variational structure, hence the variational approach is a natural tool to consider. Along this line, with many people’s effort, several sufficient conditions have been found to guarantee the existence of solutions to (1), for instance, see [4], [14]-[15] and references therein.

Recently, Brendle in [3] introduced a flow method to study the problem. It seems this new method is more promising. The first and third authors of the current paper have adopted
this method to deal with the general higher order prescribing $Q$-curvature problem on $S^n$ [6]. However, the positivity of the curvature candidate $f$ plays an important role in their argument. We observe that for prescribing Gaussian curvature problem on $S^2$, Hong and Ma [9] have verified that the positivity of the curvature candidate $f$ can be removed. Some observation for the fourth-order equation on $T^4$ with changing sign curvature candidate has also appeared in [7]. The purpose of this note is to point out that, in fact, the positivity on $f$ is in general not necessary. Before stating our main result, we give the definition of non-degeneracy first. A smooth function $f$ defined on $S^n$ is called non-degenerate if it satisfies

$$(\Delta_{S^n} f)^2 + |\nabla f|_{S^n}^2 \neq 0 \text{ on } S^n.$$ 

Our main claim in this note can be stated as the following:

**Theorem 1.** Let $n \geq 4$ be an even integer. Suppose $f : S^n \to \mathbb{R}$ is a sign changing smooth function with $\int_{S^n} f(x) d\mu_{S^n} > 0$. Assume in addition that $f$ admits only isolated critical points with non-degeneracy in the set $\{x \in S^n ; f(x) > 0\}$. Let

$$\gamma_i = \#\{q \in S^n; f(q) > 0, \nabla_{S^n} f(q) = 0, \Delta_{S^n} f(q) < 0, \text{ind}(f, q) = n - i\},$$

where $\text{ind}(f, q)$ denotes the Morse index of $f$ at critical point $q$. If the system of equations

$$\gamma_0 = 1 + k_0, \gamma_i = k_{i-1} + k_i, 1 \leq i \leq n, k_n = 0,$$

has no non-negative integer solutions for $k_i$, then there exists a solution to $Q$-curvature equation (1).

Since the most part of the argument is the same as in [6], here we only indicate how to overcome the difficulty arising from the non-positivity of the curvature candidate $f$.

2. The first thing one needs to take care of is the estimate of the normalized factor $\alpha(t)$. Before we do this, let us set the stage first. Let $n = 2m \geq 4$ be an even integer and $\omega_n$ be the volume of the standard sphere $S^n$. Let $f$ be a sign changing smooth function on $S^n$. Motivated by S. Brendle [3], M. Struwe [13] and Malchiodi-Struwe [11], the first and third authors of this note introduced in [6] the following flow equation

$$2u_t = \alpha f - Q,$$

where $Q = Q_g$ is the $Q$-curvature of the conformal metric $g(t) = e^{2u(t)} g_{S^n}$ which can be calculated by the formula

$$Q e^{nu} = P_n u + (n - 1)!.$$ 

Set

$$C_f^\infty = \left\{ w \in C^\infty(S^n); g_w = e^{2u} g_{S^n} \text{ satisfies } \int_{S^n} d\mu_{g_w} = \omega_n \text{ and } \int_{S^n} f d\mu_{g_w} > 0 \right\}.$$ 

We assume the flow (4) has the initial data $u(0) = u_0 \in C_f^\infty$.

Recall, when $n$ is even, $P_n$ is given by

$$P_n = \prod_{k=0}^{(n-2)/2} (-\Delta_{S^n} + k(n - k - 1)).$$
Observe that $P_n$ is a divergent free operator, hence integrating (5) over $S^n$ yields

$$\int_{S^n} Q e^{nu} d\mu_{S^n} = (n - 1)!,$$

where $\int_{S^n}$ denotes the average of the integral over $S^n$. The energy functional associated with the equation (4) can be written as

$$E_f[u] = E[u] - (n - 1)! \log \left( \int_{S^n} f e^{nu} d\mu_{S^n} \right),$$

where

$$E[u] = \frac{n}{2} \int_{S^n} (uP_n u + 2(n - 1)!u) d\mu_{S^n}.$$ 

We remark here that the flow (4) is the negative gradient flow of the energy $E_f[u]$. The normalized factor $\alpha(t)$ is chosen to be

$$\alpha(t) = \frac{(n - 1)!}{\int_{S^n} f e^{nu} d\mu_{S^n}}. \quad (6)$$

The reason to do so is to keep the volume of the flow metric $g(t)$ unchange for all time $t$, that is,

$$\int_{S^n} e^{nu(t)} d\mu_{S^n} \equiv 1 \text{ for all } t \geq 0.$$

In view of Lemma 1.1 in [6], the energy functional $E_f[u(t)]$ is non-increasing, more explicitly, by a simple calculation, one has

$$\frac{d}{dt} E_f[u] = -\frac{n}{2} \int_{S^n} (\alpha(t) f - Q)^2 d\mu_g. \quad (7)$$

For sign changing curvature function $f$, similar to [9], we first need the following important observation.

**Lemma 1.** If $u_0 \in C^\infty_T$, then for each time $t \geq 0$, the solution $u(t) = u(t, u_0)$ is also in the class $C^\infty_T$. Moreover, there exist two positive constants $C_1$ and $C_2$ depending only on $f$ and initial data $u_0$, such that

$$0 < C_1 \leq \alpha(t) \leq C_2 \text{ for all } t \geq 0.$$ 

**Proof.** By the selection of $\alpha(t)$, we first need verify that if $u_0 \in C^\infty_T$, then $\int_{S^n} f e^{nu(t)} d\mu_{S^n} > 0$ for any time $t > 0$. In essence, with the help of (7) and Beckner’s inequality (see [1] or [6] Prop. 1.1), one has

$$-(n - 1)! \log (\max_{S^n} f) \leq -(n - 1)! \log \int_{S^n} f e^{nu} d\mu_{S^n} \leq E_f[u](t) \leq E_f[u_0] < \infty.$$

Thus there hold

$$\max_{S^n} f \geq \int_{S^n} f e^{nu(t)} d\mu_{S^n} \geq e^{\frac{-E_f[u_0]}{(n - 1)!}} > 0.$$
and
\[
0 \leq E[u] = Ef[u] + (n - 1)! \log \left( \int_{S^n} f e^{nu} d\mu_{S^n} \right)
\leq Ef[u_0] + (n - 1)! \log \left( \max_{S^n} f \right) < \infty.
\]
Furthermore, one can easily obtain:
\[
\frac{(n - 1)!}{\max_{S^n} f} \leq \alpha(t) \leq (n - 1)! e^{Ef[u_0]/(n - 1)!}.
\]
(8)

Clearly Lemma 1 follows from the equation (8) with
\[
C_1 = \frac{(n - 1)!}{\max_{S^n} f} \quad \text{and} \quad C_2 = (n - 1)! e^{Ef[u_0]/(n - 1)!}.
\]

With the help of this lemma, the integral estimate in (6) go through without any changes.

3. As usual, we have to investigate the property of the compactness and the concentration behavior along the flow. To do this, we follow the standard strategy to study its normalized flow \(v(t)\). It is well-known [for example, (8), Lemma 5.4 or (2), Proposition 6] that, for any family of smooth functions \(u(t)\), there exists a family of conformal transformations \(\phi(t) : S^n \to S^n\), smoothly depending on the time \(t\), such that
\[
\int_{S^n} x d\mu_h = 0, \quad \text{for all } t > 0,
\]
with the normalized metric
\[
h = \phi^*(e^{2u} g_{S^n}) \equiv e^{2v(t)} g_{S^n}.
\]
(10)

In view of the non-increasing property (7) of \(Ef[u(t)]\) and a sharp version of Beckner’s inequality ([15], Theorem 2.6 or [4]), the global existence of the flow (4) with any initial data \(u_0 \in C^\infty_f\) is a direct consequence of Section 2.1 in [6].

We follow the same strategy as in the proof of Lemma 3.4 in [13] or Lemma 2.4 in [6] to obtain the asymptotic behavior of \(Q\)-curvature of the flow metric, namely,
\[
\int_{S^n} |\alpha f - Q|^2 d\mu_g \to 0 \quad \text{as } t \to \infty.
\]
(11)

Then the rough curvature convergence (11) enables us to employ Proposition 1.4 of [3] to the family of functions \(u_k = u(t_k)\) taking from the flow. We state it as the following:

**Lemma 2.** Let \(u_k = u(t_k), g_k = e^{2u_k} g_{S^n}\). Then, we have either (i) the sequence \(u_k\) is uniformly bounded in \(H^n(S^n, g_{S^n}) \hookrightarrow L^\infty(S^n)\); or (ii) there exist a subsequence of \(u_k\) and finitely many points \(q_1, \ldots, q_L \in S^n\) such that for any \(r > 0\) and any \(l \in \{1, \ldots, L\}\), there holds
\[
\liminf_{k \to \infty} \int_{B_r(q_l)} |Q_k| d\mu_k \geq \frac{1}{2} (n - 1)! \omega_n,
\]
(12)
where \(d\mu_k = d\mu_{g_k}\) and \(Q_k = Q_{g_k}\) is the \(Q\)-curvature of the metric \(g_k\); in addition, the sequence \(u_k\) is uniformly bounded on any compact subset of \((S^n \setminus \{q_1, \ldots, q_L\}, g_{S^n})\) or \(u_k \to -\infty\) locally uniformly away from \(q_1, \ldots, q_L\) as \(k \to \infty\).
Just as some previous work has shown, a refined version of Lemma 2 is much needed in late analysis.

**Lemma 3.** Let \( u_k \) be the sequence of smooth functions on \( S^n \) in Lemma 2. In addition, there exists some sign changing smooth function \( Q_\infty \) defined on \( S^n \), satisfying \( \| Q_k - Q_\infty \|_{L^2(S^n,g_k)} \to 0 \) as \( k \to \infty \). Let \( h_k = \phi_k(g_k) = e^{2u_k}g_{S^n} \) be the corresponding sequence of normalized metrics given in (9)-(10). Then up to a subsequence, either

(i) \( u_k \to u_\infty \) in \( H^n(S^n,g_{S^n}) \) as \( k \to \infty \), where \( g_{\infty} = e^{2u_\infty}g_{S^n} \) has \( Q \)-curvature \( Q_\infty \), or

(ii) there exists \( p \in S^n \), such that

\[
Q_\infty(p) = (n - 1)! \quad \text{in the weak sense of measures, and}
\]

\[
\psi_k \to 0 \quad \text{in} \quad H^n(S^n,g_{S^n}), \quad Q_{h_k} \to (n - 1)! \quad \text{in} \quad L^2(S^n,g_{S^n}).
\]

In the latter case, \( \phi_k \) converges weakly in \( H^{n/2}(S^n,g_{S^n}) \) to the constant map \( p \).

**Proof.** The proof follows the same argument as Malchiodi and Struwe did in [11] or Chen and Xu in [6]. When concentration occurs in the sense of (12), we do need to overcome some difficulties arising from the sign changing of \( f \). For each \( k \), choose \( p_k \in S^n \) and \( r_k > 0 \) such that

\[
\sup_{p \in S^n} \int_{B_{r_k}(p)} |Q_k| d\mu_k = \int_{B_{r_k}(p_k)} |Q_k| d\mu_k = \frac{1}{4} (n - 1)! \omega_n.
\]

Then by (12), \( r_k \to 0 \) as \( k \to \infty \). Also we may and will assume \( p_k \to p \) as \( k \to \infty \). For brevity, one regards \( p \) as \( N \), the north pole of \( S^n \).

Denote by \( \phi_k : S^n \to S^n \) the conformal diffeomorphisms mapping the upper hemisphere \( S^n_+ = S^n \cap \{ x^{n+1} > 0 \} \) into \( B_{r_k}(p_k) \) and the equatorial sphere \( \partial S^n_+ \) to \( \partial B_{r_k}(p_k) \). Indeed, up to a rotation, \( \phi_k \) can be written as \( \psi^{-p_k} \circ \delta_{r_k} \circ \pi^{-p_k} \), where \( \pi^{-p_k} : S^n \to \mathbb{R}^n \) is the stereographic projection from \(-p_k\) with the inverse \( \psi^{-p_k} = (\pi^{-p_k})^{-1} \) while the \( \delta_{r_k} \) is the dilation map on \( \mathbb{R}^n \) defined by \( \delta_{r_k}(y) = r_k y \). In particular, set \( \psi = \psi^p \). Consider the sequence of functions \( \tilde{u}_k : S^n \to \mathbb{R} \) defined by

\[
e^{2\tilde{u}_k}g_{S^n} = \tilde{\phi}_k^*(g_k)
\]

which solve the equation

\[
P_n \tilde{u}_k + (n - 1)! = \tilde{Q}_k e^{\tilde{u}_k} \quad \text{on} \quad S^n,
\]

where \( \tilde{Q}_k = Q_k \circ \tilde{\phi}_k \). From the selection of \( r_k, p_k \) and (14), by applying Lemma 2 to \( \tilde{u}_k \), we conclude that \( \tilde{u}_k \to \tilde{u}_\infty \) in \( H^n_\text{loc}(S^n \setminus \{ S \},g_{S^n}) \) as \( k \to \infty \), where \( S \) is the south pole on \( S^n \). Meanwhile, \( \tilde{Q}_k \to Q_\infty(p) \) almost everywhere as \( k \to \infty \). Introducing the sequence of functions \( \tilde{u}_k : S^n \to \mathbb{R} \) by

\[
e^{2\tilde{u}_k}g_{S^n} = (\psi^{-p_k})^*(e^{2\tilde{u}_k}g_{S^n}) = \tilde{\psi}_k^*(g_k),
\]

where \( \tilde{\psi}_k = \psi^{-p_k} \circ \delta_{r_k} \), namely,

\[
\tilde{u}_k = u_k \circ \tilde{\psi}_k + \frac{1}{n} \log(\det d\tilde{\psi}_k),
\]
we find that $\tilde{u}_k$ converges in $H^1_{\text{loc}}(\mathbb{R}^n)$ to a function $\tilde{u}_\infty$, satisfying

$$(-\Delta_{\mathbb{R}^n})^{n/2} \tilde{u}_\infty = Q_\infty(p)e^{n\tilde{u}_\infty} \text{ in } \mathbb{R}^n. \quad (15)$$

Moreover, by Fatou’s lemma we get

$$\int_{\mathbb{R}^n} e^{n\tilde{u}_\infty} \, dz \leq \liminf_{k \to \infty} \int_{\mathbb{R}^n} e^{n\tilde{u}_k} \, dz = \omega_n. \quad (16)$$

Based on the proof of Lemma 3.2 of [6], we need a preliminary lemma to finish the proof of Lemma 3.

**Lemma 4.** Under assumptions on $u_k$ as in Lemma 3, there holds $Q_\infty(p) > 0$ and the solution $\tilde{u}_\infty$ to equations (15)-(16) has the form

$$\tilde{u}_\infty(z) = \log 2 + \frac{2\lambda}{1 + |\lambda(z - z_0)|^2} - \frac{1}{n} \log \frac{Q_\infty(p)}{(n - 1)!} \quad (17)$$

for some $\lambda > 0$ and $z_0 \in \mathbb{R}^n$.

**Proof.** For brevity, one uses $u_\infty$ instead of $\tilde{u}_\infty$. Let

$$\bar{w}(\rho) = \int_{\partial B_{\rho}(0)} w(z) \, d\sigma(z), \rho > 0$$

denote the spherical average of the function $w$ defined in $\mathbb{R}^n$. Due to the proof of Lemma 3.3 in [6], we only need rule out the case of $Q_\infty(p) \leq 0$. Arguing by negation, we assume $Q_\infty(p) \leq 0$. The argument heavily relies on the following important estimate obtained through the analysis on Green’s function of some $(n - 2)$-order differential operator in [6], Lemma 3.3.

For convenience, we restate it here: for any $r > 0$ and $q \in S^n$, there holds

$$|\int_{B_r(q)} \Delta_{S^n} u_k d\mu_{S^n}| \leq B_0 r^{n-2} \quad (18)$$

for all $k$, where $B_0 > 0$ is a constant.

Let $m = n/2 \geq 2$ and $w_i(z) = (-\Delta)^i u_\infty(z), i = 1, 2, \ldots, m$. Then, we claim that for $1 \leq i \leq m - 1$, there holds

$$w_{m-i} \leq 0 \text{ in } \mathbb{R}^n. \quad (19)$$

For $w_{m-1}$, by negation, we assume there exists $z_0 \in \mathbb{R}^n$, such that $w_{m-1}(z_0) > 0$. Without loss of generality, assume $z_0 = 0$. From (15) and Jensen’s inequality, we have

$$\begin{cases}
-\Delta \tilde{u}_\infty = \bar{w}_1, \\
-\Delta \bar{w}_1 = \bar{w}_2, \\
\cdots \\
-\Delta \bar{w}_{m-1} = \bar{w}_m \leq Q_\infty(p)e^{n\bar{u}_\infty} \leq 0.
\end{cases} \quad (20)$$

Thus $\bar{w}_{m-1}(\rho) \geq 0$, which indicates $\bar{w}_{m-1}(\rho) \geq \bar{w}_{m-1}(0) = w_{m-1}(0) > 0$. Observe that

$$\bar{w}_{m-2}(\rho) = \frac{-\rho}{n} |B_\rho(0)|^{-1} \int_{B_\rho(0)} w_{m-1}(z) \, dz \leq \frac{(-w_{m-1}(0))}{n} \rho < 0.$$
Thus it follows that
\[ -\bar{w}_{m-2}(\rho) \geq B_2 \rho^2 \] for \( \rho \geq \rho_1 > 0, B_2 > 0. \]

By (20) and mathematical induction, in general, for \( 2 \leq i \leq m - 1 \), we have
\[ (-1)^{i-1} \bar{w}_{m-i}(\rho) \geq B_i \rho^{2(i-1)} \] for \( \rho \geq \rho_{i-1} > 0, B_i > 0. \)

Apply this to \( i = m - 1 \) to get
\[ (-1)^{m-2} \int_{\partial B_{\rho}(0)} (-\Delta u_\infty(z))d\sigma(z) \geq B_{m-1} \rho^{2(m-2)+n-1} \] for \( \rho \geq \rho_{m-1}. \) (21)

For sufficiently large \( k \) and all \( \rho \geq \rho_{m-1} \), one has
\[ (-1)^{m-2} \int_{\partial B_{\rho}(0)} (-\Delta \tilde{u}_k(z))dz \geq A_1 \rho^{2(m-2)+n-1} \] (22)

where \( A_1 > 0 \) is a universal constant. By a similar argument on pages 951-953 of [6], using (22) and the expression
\[ \tilde{u}_k(z) = u_k \circ \tilde{\psi}_k + \log \frac{2r_k}{1 + |r_k z|^2}, \]

one obtains that for some fixed \( L > 0 \) and any \( d > L \), there holds
\[ (-1)^{m-2} \int_{B_{dr_k(p_k)}} (-\Delta S^n u_k)d\mu_{S^n} \geq A_2 r_k^{n-2}(d^{2(m-2)}+n-2-L^{2(m-2)+n-L^{n-2}}) \] (23)

for sufficiently large \( k \), where \( A_2 > 0 \) is a constant.

On the other hand, by choosing \( r = r_k d \) and \( q = p_k \) in (18), with a uniform constant \( A_3 > 0 \) it yields
\[ (-1)^{m-2} \int_{B_{dr_k(p_k)}} (-\Delta S^n u_k)d\mu_{S^n} \leq A_3 r_k^{n-2}d^{n-2}. \] (24)

Hence, for any fixed \( L > 0 \) as above and sufficiently large \( k \), (23) and (24) yield a contradiction by choosing \( d \) sufficiently large.

Next, we prove (19) by the induction argument. The case \( i = 1 \) has been settled above. If \( m = 2 \), we are done. Thus we assume \( m > 2 \). Suppose for some \( i \) with \( 1 \leq i \leq m - 2 \) and all \( 1 \leq k \leq i \), \( w_{m-k} \leq 0 \) in \( \mathbb{R}^n \). Then one needs to show \( w_{m-i-1} \leq 0 \) in \( \mathbb{R}^n \). By negation, we may assume \( w_{m-i-1}(0) > 0 \). Since \( \bar{w}'_{m-i-1}(\rho) = \frac{-1}{|\partial B_{\rho}(0)|} \int_{B_{\rho}(0)} w_{m-i}(z)dz \geq 0 \), it follows that
\[ -\bar{w}_{m-i-1}(\rho) \geq \bar{w}_{m-i-1}(0) = w_{m-i-1}(0) > 0. \]

If \( i \leq m - 3 \), by \(-\Delta \bar{w}_{m-i-2} = \bar{w}_{m-i-1} \), one has
\[ -\bar{w}_{m-i-2}(\rho) \geq B_2 \rho^2 \] for \( \rho \geq \rho_{j-1} > 0, B_2 > 0. \)

In general, by (20) one obtains
\[ (-1)^{i+j} \bar{w}_{m-i-j}(\rho) \geq B_j \rho^{2(i-1)} \] for \( \rho \geq \rho_{j-1} > 0, B_j > 0, i + j \leq m - 1. \)
Choosing \( j = m - 1 - i \), we have
\[
(-1)^{m-i} \int_{\partial B_\rho(0)} (-\Delta u_\infty(z)) d\sigma(z) \geq B_{m-1-i} \rho^{2(m-i-2)+n-1} \quad \text{for } \rho \geq \rho_{m-2-i}.
\] (25)

Fixing \( L \geq \rho_{m-2-i} \) and for any \( d > L \), by a similar argument on \( (23) \), with a constant \( A_5 > 0 \), one gets
\[
(-1)^{m-i} \int_{B_{dr_k}(p_k)} (-\Delta S_n u_k) d\mu_{S^n} \geq A_5 r_k^{n-2} (d^{2(m-i-2)+n} - L^{2(m-i-2)+n} - L^{-2})
\] (26)
for all sufficiently large \( k \). On the other hand, choosing \( r = r_k d \) and \( q = p_k \) in \( (18) \), with another constant \( A_6 > 0 \) one has
\[
(-1)^{m-i} \int_{B_{dr_k}(p_k)} (-\Delta S_n u_k) d\mu_{S^n} \leq A_6 r_k^{n-2} d^{n-2}.
\] (27)
Thus \( (26) \) and \( (27) \) would contradict each other if \( d \) is chosen to be sufficiently large.

If \( i = m - 2 \), by inductive assumption, \( w_2(z) \leq 0 \) in \( \mathbb{R}^n \) and \( w_1(0) = 0 \). Since \( \bar{w}_1'(\rho) \geq 0 \),
\[
\bar{w}_1(\rho) \geq \bar{w}_1(0) = w_1(0) > 0.
\]

Given any \( d > 0 \), from the above inequality, it is easy to derive
\[
\int_{\partial B_\rho(0)} (-\Delta u_\infty) dz \geq 2A_7 \rho^{n-1} \quad \text{for } 0 \leq \rho \leq d,
\]
where \( A_7 > 0 \) depends on \( w_1(0) \) and \( n \) only. Repeating the argument for \( (23) \) and scaling back to \( S^n \), one gets
\[
\int_{B_{dr_k}(p_k)} (-\Delta S_n u_k) d\mu_{S^n} \geq A_7 r_k^{n-2} d^n,
\] (28)
for all sufficiently large \( k \). Now, the equation \( (18) \) with \( r = r_k d \) and \( q = p_k \) gives
\[
\int_{B_{dr_k}(p_k)} (-\Delta S_n u_k) d\mu_{S^n} \leq A_8 r_k^{n-2} d^{n-2}
\] (29)
where \( A_8 > 0 \) is a constant. Equations \( (28) \) and \( (29) \) contradict each other if \( d > 0 \) is sufficiently large.

Therefore, we conclude that \( w_{m-i-1} \geq 0 \) in \( \mathbb{R}^n \) and the induction argument is complete.

Finally, it follows from the inequality \( (19) \) that \( -\Delta u_\infty = w_1 \leq 0 \) in \( \mathbb{R}^n \), that is, \( u_\infty \) is a subharmonic function. By the mean value property for subharmonic functions, for any \( z \in \mathbb{R}^n \) and \( r > 0 \), there holds
\[
nu_{\infty}(z) \leq |\partial B_r(0)|^{-1} \int_{\partial B_r(z)} nu_{\infty}(y) dy.
\]
By this inequality and Jensen’s inequality, one gets
\[
e^{nu_{\infty}(z)} \leq |\partial B_r(0)|^{-1} \int_{\partial B_r(z)} e^{nu_{\infty}(y)} dy
\]
for any \( r > 0 \). In view of (16), by letting \( r \to \infty \), the inequality (31) indicates that \( u_\infty \equiv -\infty \) in \( \mathbb{R}^n \), which is obviously impossible. The proof is complete.

**Proof of Lemma 3 (completed).** With the help of Lemma 4, we can show that there is the unique concentration point \( p \) of \( \{g_k\} \) such that \( Q_\infty(p) = (n - 1)! \). To see this, first set \( Q_\infty^+ = \max \{Q_\infty, 0\}, Q_\infty^- = \min \{Q_\infty, 0\} \). Notice that by Lemma 2 there are only finitely many blow-up points, say \( p_1, p_2, \ldots, p_l \). By previous two Lemmas, we know that at each \( p_i, Q_\infty(p_i) > 0 \). Now for each \( i \), choose a sufficiently small \( r_i > 0 \) so that \( Q_\infty \geq 0 \) in \( B_r(p_i) \) and \( B_{r_i}(p_i) \cap B_{r_j}(p_j) = \emptyset \) if \( i \neq j \). Then follow the same argument on page 957 of [6] (or similar one in [11]) to conclude that

\[
(n - 1)!l \omega_n = \sum_{i=1}^{l} \int_{\mathbb{R}^n} Q_\infty(p_i) e^{n\omega_\infty} \, dz \leq \sum_{i=1}^{l} \int_{B_{r_i}(p_i)} Q_\infty^+ \, d\mu_k + o(1),
\]

for all sufficiently large \( k \), where \( o(1) \to 0 \) as \( k \to \infty \). Thus, there holds \( \lim_{k \to \infty} \int_{\mathbb{S}^n} Q_\infty^- \, d\mu_k = \sum_{i=1}^{l} Q_\infty(p_i) \omega_n = 0 \) since concentration phenomena only occur at points \( p_i \) where \( Q_\infty(p_i) > 0, 1 \leq i \leq l \). From this identity and the selection of \( r_i \), one has

\[
\sum_{i=1}^{l} \int_{B_{r_i}(p_i)} Q_\infty^+ \, d\mu_k = \sum_{i=1}^{l} \int_{B_{r_i}(p_i)} Q_\infty \, d\mu_k
\]

\[
= \sum_{i=1}^{l} \left[ \int_{B_{r_i}(p_i)} Q_k \, d\mu_k + \int_{B_{r_i}(p_i)} (Q_\infty - Q_k) \, d\mu_k \right]
\]

\[
\leq \int_{\mathbb{S}^n} Q_k \, d\mu_k + 2 \int_{\mathbb{S}^n} |Q_\infty - Q_k| \, d\mu_k + \int_{\mathbb{S}^n \setminus \bigcup_{i=1}^{l} B_{r_i}(p_i)} |Q_\infty| \, d\mu_k
\]

\[
= (n - 1)! \omega_n + o(1),
\]

for all sufficiently large \( k \) and where we have used the local volume concentration property in the last term and uniform bound of \( Q_\infty \). Thus, it follows from (31) and (32) that \( l = 1 \) and \( Q_\infty(p) = (n - 1)! \). Finally, the rest part of the proof of Lemma 3 is the same as the proof of Lemma 3.2 in [6].

**Remark 1.** We should point out that, one can not apply Theorem 9 in [12] to derive Lemma 3 directly. The assumption in [12]: \( Q_k \to Q_\infty \) in \( C^0(\mathbb{S}^n) \) is much stronger than the one in Lemma 3. Similar blow-up analysis as in [12] has also been done by Malchiodi [10]. However those estimates seem not suitable for \( Q \)-curvature flow since it is hard to have \( C^0 \) convergence. So we have to seek another reasonable procedure to do blow-up analysis in the flow setting.

The remainder of the proof of Theorem 1 will be completed through a contradictive argument. From now on, we assume \( f \) can not be realized as a \( Q \)-curvature of any metric in the conformal class of \( g_{\mathbb{S}^n} \). Following the standard scheme in [6], in particular Sections 4-5, along with Lemma 3 one eventually obtains the asymptotic behavior of the flow \( u(t) \) and the so-called shadow flow

\[
\Theta = \Theta(t) = \int_{\mathbb{S}^n} \phi(t) \, d\mu_{\mathbb{S}^n}.
\]
Lemma 5. Let $u(0) = u_0 \in C^\infty_f$ be the initial data of the flow \((4)\) and \((5)\). Then the flow metrics $g(t)$ concentrate at a critical point $p$ of $f$ with $f(p) > 0$, $\Delta_{S^n} f(p) \leq 0$ and the energy $E_f[u(t)]$ converges to $-(n-1)! \log f(p)$, that is

$$E_f[u(t)] \to -(n-1)! \log f(p), \text{ as } t \to \infty.$$ 

Moreover, the critical point $p$ is also the unique limit of the shadow flow $\Theta(t)$ associated with $u(t)$, in other words, $p = \lim_{t \to \infty} \Theta(t)$.

4. In this and next part, we will briefly sketch the proof of our main result. For $q \in S^n$, $0 < \epsilon < \infty$, denote by $\phi_{q,\epsilon} = \psi^{-q} \circ \delta_\epsilon \circ \pi^{-q}$ the stereographic projection with $-q$ at infinity, that is, $q$ becomes the north pole in the stereographic coordinates. It is relatively easy to verify that $\phi_{q,\epsilon}$ converges weakly in $H^{n/2}(S^n, g_{S^n})$ to $q$ as $\epsilon \to 0$. Define a map

$$j : S^n \times (0, \infty) \ni (q, \epsilon) \mapsto u_{q,\epsilon} = \frac{1}{n} \log \det(d\phi_{q,\epsilon}) \in C^\infty_*.$$ 

And set $g_{q,\epsilon} = \phi_{q,\epsilon}^*(g_{S^n}) = e^{2u_{q,\epsilon}} g_{S^n}$. Then we have

$$d\mu_{g_{q,\epsilon}} = e^{2u_{q,\epsilon}} d\mu_{S^n} \to \omega_n \delta_q,$$

in the weak sense of measures as $\epsilon \to 0$. For $\gamma \in \mathbb{R}$, denote by

$$L_\gamma = \{u \in C^\infty_f; E_f[u] \leq \gamma\},$$

the sub-level set of $E_f$. Under our assumptions on $f$, label all critical points of $f$ with positive critical values by $q_1, \cdots, q_N$ such that $0 < f(q_i) \leq f(q_j)$ for $1 \leq i \leq j \leq N$ and set

$$\beta_i = -(n-1)! \log f(q_i) = \lim_{\epsilon \to 0} E_f[u_{q,\epsilon}], 1 \leq i \leq N.$$ 

Without loss of generality, we assume all critical levels $f(q_i)$, $1 \leq i \leq N$ are distinct, so there exists a $\nu_0 > 0$ such that $\beta_i - 2\nu_0 > \beta_{i+1}$, in fact we can take $\nu_0 = \frac{1}{2} \min_{1 \leq i \leq N-1} \{\beta_i - \beta_{i+1}\} > 0$.

First of all, we shall characterize the homotopy types of the sub-level sets. We state them as a proposition, which has analogous counterpart in \([11]\) or \([6]\).

**Proposition 1.** (i) If $\delta_0 > \max\{-(n-1)! \log(\int_{S^n} f(x) d\mu_{S^n}), \beta_1\}$, the set $L_{\delta_0}$ is contractible.

(ii) For any $0 < \nu \leq \nu_0$ and each $1 \leq i \leq N$, the sets $L_{\beta_i - \nu}$ and $L_{\beta_{i+1} + \nu}$ are homotopy equivalent.

(iii) For each critical point $q_i$ of $f$ where $\Delta_{S^n} f(q_i) > 0$ and $f(q_i) > 0$, the sets $L_{\beta_i + \nu_0}$ and $L_{\beta_i - \nu_0}$ are homotopy equivalent.

(iv) For each critical point $q_i$ where $\Delta_{S^n} f(q_i) < 0$ and $f(q_i) > 0$, the set $L_{\beta_i + \nu_0}$ is homotopic to the set $L_{\beta_i - \nu_0}$ with $(n - \text{ind}(f, q_i))$-cell attached.
Proof: (i) Let δ₀ be chosen as above. For 0 ≤ s ≤ 1 and u₀ ∈ C_f, define

\[ H_1(s, u_0) = \frac{1}{n} \log((1 - s)e^{nu_0} + s), \]

that is \( e^{nH_1(s,u_0)} = (1 - s)e^{nu_0} + s \), then one easily obtains

\[
\int_{S^n} e^{nH_1(s,u_0)} d\mu_{S^n} = 1 \quad \text{and} \quad \int_{S^n} f e^{nH_1(s,u_0)} d\mu_{S^n} = (1 - s) \int_{S^n} f e^{nu_0} d\mu_{S^n} + s \int_{S^n} f d\mu_{S^n} > 0,
\]

in view of the assumption that \( \int_{S^n} f(x) d\mu_{S^n} > 0 \), \( H_1(s, u_0) \) provides a homotopic deformation within the set \( C_f^\infty \). Given such \( u_0 \) and 0 ≤ s ≤ 1, by Lemma 3 and the selection of \( \delta_0 \), there exists a minimal time \( T = T(s, u_0) \), such that \( E_f[u(T, H_1(s, u_0))] \leq \delta_0 \), where the continuity of \( T(s, u_0) \) on s and \( u_0 \) can be deduced by (7) and the expression of \( H_1(s, u_0) \). Thus the map \( H : (s, u_0) \mapsto u(T(s, u_0), H_1(s, u_0)) \) is the desired contraction of \( L_{\delta_0} \) within itself. To see this, first, by lemma 11 one knows that \( H(s, u_0) \in C_f^\infty \); next notice that \( T(0, u_0) = 0 \), hence \( u(T(0, u_0), H(0, u_0)) = u(0, u_0) = u_0 \) and \( u(T(1, u_0), H(1, u_0)) = 0 \) since \( H(1, u_0) = 0 \) with \( E_f[0] = -(n - 1)! \log(\int_{S^n} f(x) d\mu_{S^n}) < \delta_0 \), \( T(1, u_0) = 0 \).

The proofs of (ii)-(iv) are identical to the corresponding ones of Proposition 6.1 (ii)-(iv) in [6].

5. We are now in position to complete the proof of our main theorem.

Proof of Theorem 11: By negation, suppose the flow is divergent for any initial data in \( C_f^\infty \) and there is no conformal metric of \( g_{S^n} \) with \( Q \)-curvature \( f \). Proposition 11 shows that for some suitable \( \delta_0 \), \( L_{\delta_0} \) is contractible and homotopically equivalent to a set \( E_\infty \) whose homotopy type consists of a point \( \{p\} \) with \( (n - \text{ind}(f, q)) \)-dimensional cell attached for each critical point \( q \) of \( f \) with \( \Delta_{S^n} f(q) < 0 \) and \( f(q) > 0 \). By applying [5], Theorem 4.3 on page 36 to \( L_{\delta_0} \), we conclude that the identity

\[
\sum_{j=0}^{n} s^j\gamma_j = 1 + (1 + s) \sum_{j=0}^{n} s^j k_j
\]

holds for Morse polynomials of \( L_{\delta_0} \) and \( E_\infty \), where \( k_j ≥ 0 \) are integers and \( \gamma_j \) is defined in (2). Thus we achieve a contradiction with the assumption that the system (3) has no nonnegative integer solutions \( k_j \) and this contradiction completes the proof.

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**Added notes by Xuezhang Chen:** This is a second version of this paper, which is essentially completed in January of 2013. The first version is completed in April of 2012.