MODULAR GALOIS COVERS ASSOCIATED TO SYMPLECTIC RESOLUTIONS OF SINGULARITIES

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Abstract. Let $Y$ be a normal projective variety and $\pi : X \to Y$ a projective holomorphic symplectic resolution. Namikawa proved that the Kuranishi deformation spaces $\text{Def}(X)$ and $\text{Def}(Y)$ are both smooth, of the same dimension, and $\pi$ induces a finite branched cover $f : \text{Def}(X) \to \text{Def}(Y)$. We prove that $f$ is Galois. We proceed to calculate the Galois group $G$, when $X$ is simply connected, and its holomorphic symplectic structure is unique, up to a scalar factor. The singularity of $Y$ is generically of $ADE$-type, along every codimension 2 irreducible component $B$ of the singular locus, by Namikawa’s work. The modular Galois group $G$ is the product of Weyl groups of finite type, indexed by such irreducible components $B$. Each Weyl group factor $W_B$ is that of a Dynkin diagram, obtained as a quotient of the Dynkin diagram of the singularity-type of $B$, by a group of Dynkin diagram automorphisms.

Finally we consider generalizations of the above set-up, where $Y$ is affine symplectic, or a Calabi-Yau threefold with a curve of $ADE$-singularities. We prove that $f : \text{Def}(X) \to \text{Def}(Y)$ is a Galois cover of its image. This explains the analogy between the above results and related work of Nakajima, on quiver varieties, and of Szendrői on enhanced gauge symmetries for Calabi-Yau threefolds.

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1. Introduction

An irreducible holomorphic symplectic manifold is a simply connected compact Kähler manifold $X$, such that $H^0(X, \Omega^2_X)$ is generated by an everywhere non-degenerate holomorphic two-form \[ Be1, Huy \]. A projective symplectic variety is a normal projective variety with rational Gorenstein singularities, which regular locus admits a non-degenerate holomorphic two-form. Let $\pi : X \to Y$ be a symplectic resolution of a normal projective variety $Y$. Then $Y$ has rational Gorenstein singularities and is thus a symplectic variety \[ Be2 \], Proposition 1.3. Assume that $X$ is a projective irreducible holomorphic symplectic manifold. Let $Def(X)$ and $Def(Y)$ be the Kuranishi spaces of $X$ and $Y$. Denote by $\psi : \mathcal{X} \to Def(X)$ the semi-universal deformation of $X$, by $0 \in Def(X)$ the point with fiber $X$, and let $X_t$ be the fiber over $t \in Def(X)$. Let $\tilde{\psi} : \mathcal{Y} \to Def(Y)$ be the semi-universal deformation of $Y$, $\tilde{0} \in Def(Y)$ its special point with fiber $Y$, and $Y_t$ the fiber over $t \in Def(Y)$. The following is a fundamental theorem of Namikawa:

**Theorem 1.1.** ([Nam1], Theorem 2.2) The Kuranishi spaces $Def(X)$ and $Def(Y)$ are both smooth of the same dimension. They can be replaced by open neighborhoods of $0 \in Def(X)$ and $\tilde{0} \in Def(Y)$, and denoted again by $Def(X)$ and $Def(Y)$, in such a way that there exists a natural proper surjective map $f : Def(X) \to Def(Y)$ with finite fibers. Moreover, for a generic point $t \in Def(X)$, $Y_{f(t)}$ is isomorphic to $X_t$.

$Def(X)$ is thus a branched covering of $Def(Y)$. We first observe that the covering is Galois, in analogy to the case of the resolution of a simple singularity of a $K3$ surface \[ Br \ [BW] \].
Lemma 1.2.  
1. The neighborhoods \( \text{Def}(X) \) of 0 and \( \text{Def}(Y) \) of \( \bar{0} \) in Theorem 1.1 may be chosen, so that the map \( f : \text{Def}(X) \to \text{Def}(Y) \) is a branched Galois covering.

2. The Galois group acts on \( H^*(X, \mathbb{Z}) \) via monodromy operators, which preserve the Hodge structure. The action on \( H^{1,1}(X, \mathbb{R}) \) is faithful and the action on \( H^{2,0}(X) \) is trivial.

The elementary lemma is proven in section 2. Significant generalizations of the lemma are considered in section 6. We proceed to calculate the Galois group. Let \( \Sigma \subset Y \) be the singular locus. The dissident locus \( \Sigma_0 \subset \Sigma \) is the locus along which the singularities of \( Y \) fail to be of \( ADE \) type. \( \Sigma_0 \) is a closed subvariety of \( \Sigma \).

Proposition 1.3. ([Nam1], Propositions 1.6) \( Y \) has only canonical singularities. The dissident locus \( \Sigma_0 \) has codimension at least 4 in \( Y \). The complement \( \Sigma \setminus \Sigma_0 \) is either empty, or the disjoint union of codimension 2 smooth and symplectic subvarieties of \( Y \setminus \Sigma_0 \).

Kaledin proved that the morphism \( \pi \) is semi-small, i.e., the fiber product \( X \times_Y X \) has pure dimension \( 2n \), providing further information on the dissident locus [K].

Let \( B \subset [\Sigma \setminus \Sigma_0] \) be a connected component, \( E \) the exceptional locus of \( \pi \), and \( E_B := \pi^{-1}(B) \). Then \( E_B \) is connected, of pure codimension 1 in \( X \), and each fiber \( \pi^{-1}(b), b \in B \), is a tree of smooth rational curves, whose dual graph is a Dynkin diagram of type \( ADE \), by the above proposition. We refer to this Dynkin diagram as the Dynkin diagram of the fiber. Fix \( b \in B \). The fundamental group \( \pi_1(B, b) \) acts on the set of irreducible components of the fiber \( \pi^{-1}(b) \) via isotopy classes of diffeomorphisms of the fiber, and in particular via automorphisms of the dual graph. We refer to the quotient, of the Dynkin diagram of the fiber by the \( \pi_1(B, b) \)-action, as the folded Dynkin diagram and denote by \( W_B \) the Weyl group of the folded root system (see section 3). Note that the set of irreducible components of \( E_B \) corresponds to a basis of simple roots for the folded root system. Let \( \mathcal{B} \) be the set of connected components of \( \Sigma \setminus \Sigma_0 \). Let \( G \) be the Galois group of the branched Galois cover \( f : \text{Def}(X) \to \text{Def}(Y) \), introduced in Lemma 1.2.

Theorem 1.4. \( G \) is isomorphic to \( \prod_{B \in \mathcal{B}} W_B \).

The Theorem is the combined results of Lemmas 4.16 and 4.24. The \( K3 \) case of the Theorem was proven by Burns and Wahl [BW], Theorem 2.6 and Remark 2.7. See also [Sz1]. Y. Namikawa recently extended Theorem 1.4 to the case of Poisson deformation spaces of affine symplectic varieties [Nam6].
The above group $G$, associated to the birational contraction $\pi : X \to Y$, acts on $H^*(X, \mathbb{Z})$ via local monodromy operators, by Lemma 1.2.

One can define abstractly the local monodromy group of $X$, which is larger in general ([Ma2], Definition 2.4). It includes, for example, monodromy operators induced by automorphisms and birational automorphisms. Affine Weyl groups can be constructed as subgroups of the local monodromy group, by taking the group generated by Weyl groups $G_i$, associated to two or more birational contractions $\pi_i : X \to Y_i$ as in Theorem 1.4. Consider, for example, a $K3$ surface $S$, admitting an affine Dynkin diagram of rational curves as a fiber in an elliptic fibration. Choose two Dynkin sub-diagrams of finite type, which union is the whole affine diagram. Then the elliptic fibration factors through the two different birational contractions, and the two Galois groups will generate an affine Weyl group. Affine Weyl group monodromy actions occur for quiver varieties [Nak]. A conjectural characterization of the local monodromy group is discussed in [Ma2], section 2.

The paper is organized as follows. Lemma 1.2, stating that the modular cover is Galois, is proven in section 2. In section 3 we recall the procedure of folding Dynkin diagrams of $ADE$ type by diagram automorphisms. Section 4 is dedicated to the proof of Theorem 1.4, i.e., the calculation of the modular Galois group $G$ as a product of Weyl groups. The key step is the construction of a reflection in $G$, for every codimension one irreducible component of the exceptional locus of the contraction $\pi : X \to Y$ (Lemmas 4.10 and 4.23). In section 5 we apply Theorem 1.4 to calculate modular Galois groups in specific examples. We consider, in particular, O’Grady’s 10-dimensional example of an irreducible holomorphic symplectic manifold $X$ with $b_2(X) = 24$. We apply Theorem 1.4 to explain the occurrence of the root lattice of $G_2$ as a direct summand of the lattice $H^2(X, \mathbb{Z})$ (Lemma 5.1). The above is part of an earlier result of Rapagnetta [R].

Section 6 is devoted to generalizations of Lemma 1.2. We consider birational contractions $\pi : X \to Y$, where $Y$ is affine symplectic, or the symplectic variety $Y$ does not admit a crepant resolution, or $Y$ is a Calabi-Yau threefold with a curve of $ADE$-singularities. Again we prove that the associated modular morphism $f : Def(X) \to Def(Y)$ is a Galois branched cover of it image (it need not be surjective in the non-symplectic cases). The generalization in section 6.1.2 is sufficient to explain the analogy between the results of the current paper and results of Nakajima on quiver varieties [Nak]. Arguably, the most significant generalization is Lemma 6.2. The latter is used in Lemma 6.3 to explain the apparent analogy between the current paper and the string theory phenomena of enhanced gauge symmetries associated to Calabi-Yau
threefolds with a curve of $ADE$-singularities $[\text{As}, \text{KMP}]$. These symmetries were explained mathematically by Szendrői $[\text{Sz1}, \text{Sz2}]$ and related, in a limiting set-up, to Hitchin systems of $ADE$-type in $[\text{DHP}, \text{DDP}]$.

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2. Modular branched Galois covers via local Torelli

We prove Lemma 1.2 in this section.

Part (1): We may assume that the neighborhoods $\text{Def}(X)$ and $\text{Def}(Y)$ are sufficiently small, so that the fiber of $f$ over $\bar{0}$ consists of the single point $0$ corresponding to $X$. The morphism $\pi : X \to Y$ deforms as a morphism $\nu$ of the semi-universal families, which fits in a commutative diagram

\begin{equation}
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\nu} & \mathcal{Y} \\
\psi \downarrow & & \bar{\psi} \downarrow \\
\text{Def}(X) & \xrightarrow{f} & \text{Def}(Y),
\end{array}
\end{equation}

by $[\text{KM}]$, Proposition 11.4. Let $U \subset \text{Def}(Y)$ be the largest open subset satisfying the following conditions: The semi-universal deformation $\mathcal{Y}$ restricts to a smooth family over $U$ and $f$ is unramified over $U$. Diagram (1) restricts over the subset $U$ of $\text{Def}(Y)$ to a cartesian diagram, by Theorem 1.1. Set $V := f^{-1}(U)$. The group $H^2(X, \mathbb{Z})$ is endowed with the monodromy invariant Beauville-Bogomolov symmetric bilinear pairing $[\text{Be1}]$. Set $\Lambda := H^2(X, \mathbb{Z})$ and let

$$
\Omega_\Lambda := \{ \ell \in \mathbb{P}H^2(X, \mathbb{C}) : (\ell, \ell) = 0 \text{ and } (\ell, \bar{\ell}) > 0 \}
$$

be the period domain. The local system $R^2_{\psi_*}(\mathbb{Z})$ is trivial. Choose the trivialization $\varphi : R^2_{\psi_*}(\mathbb{Z}) \to \Lambda$, inducing the identity on $H^2(X, \mathbb{Z})$. Let $p : \text{Def}(X) \to \Omega_\Lambda$ be the period map, given by $p(t) = \varphi(H^{2,0}(X_t))$. 
Recall that $p$ is a holomorphic embedding of $Def(X)$ as an open subset of $\Omega_\Lambda$, by the Local Torelli Theorem [Be1]. We get the following diagram:

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\nu} & \mathcal{Y} \\
\downarrow \psi & & \downarrow \bar{\psi} \\
\Omega_\Lambda & \xleftarrow{p} & Def(X) \\
\cup & & \cup \\
V & \longrightarrow & U.
\end{array}
\]

Choose a point $u \in U$. We get the monodromy homomorphism $\text{mon} : \pi_1(U, u) \to O(\Lambda) := O[H^2(X, \mathbb{Z})]$. Set $K := \ker(\text{mon})$ and let $f_K : \tilde{U}_K \to U$ be the Galois cover of $U$ associated to $K$.

We have a natural isomorphism

\[
\nu^* : f^* \left[ R^2_{\bar{\psi}^*}(\mathbb{Z})_{|V} \right] \cong R^2_{\tilde{\psi}^*}(\mathbb{Z})_{|V}.
\]

Hence, the local system $f^* \left[ R^2_{\tilde{\psi}^*}(\mathbb{Z})_{|V} \right]$ is trivial. It follows that the covering $f : V \to U$ factors as $f = h \circ f_K$ via a covering $h : V \to \tilde{U}_K$. The restriction of the period map $p$ to $V$ clearly factors through $h$. Now $p$ is injective. Hence, so is $h$. We conclude that $h$ is an isomorphism.

Set $G := \pi_1(U, u)/K$. It remains to extend the action of $G$ on $V$ to an action on $Def(X)$. Set $\phi := \varphi \circ \nu^* : f^* \left[ R^2_{\tilde{\psi}^*}(\mathbb{Z})_{|U} \right] \to \Lambda$. Let $\gamma : G \to O(\Lambda)$ be the homomorphism induced by the monodromy representation. Given a deck transformation $g \in G$, we get the commutative diagram of trivializations of local systems over $V$:

\[
\begin{array}{ccc}
f^* \left[ R^2_{\tilde{\psi}^*}(\mathbb{Z})_{|U} \right] & \xrightarrow{\phi} & \Lambda \\
\downarrow & & \downarrow \gamma_g \\
\left[ R^2_{\tilde{\psi}^*}(\mathbb{Z})_{|U} \right] & \xrightarrow{(g^{-1})^* \phi} & \Lambda.
\end{array}
\]

The group $O(\Lambda)$ acts on the period domain and we denote the automorphism of $\Omega_\Lambda$ corresponding to $\gamma_g$ by $\gamma_g$ as well. We have

\[
\gamma_g(p(t)) = \gamma_g \left( \varphi_t(H^{2,0}(X_t)) \right) = \gamma_g \left( \phi_t(H^{2,0}(Y_{f(t)})) \right) \equiv \phi_{g^{-1}(t)}(H^{2,0}(Y_{f(t)})) = \varphi_{g^{-1}(t)}(H^{2,0}(X_{g^{-1}(t)})) = p(g^{-1}(t)).
\]

We conclude the equality

\[
p \circ g^{-1} = \gamma_g \circ p.
\]

\[1\]I thank F. Catanese for the suggestion to consider this Galois cover. It simplifies the original proof, which considered instead the Galois closure of $f : V \to U$. 


Claim 2.1. (1) We can choose the above open neighborhood \( \text{Def}(Y) \) of \( 0 \) in the Kuranishi space of \( Y \), and similarly \( \text{Def}(X) \) in that of \( X \), to satisfy

\[
\gamma_g(p(\text{Def}(X))) = p(\text{Def}(X)).
\]

(2) \( \gamma_g(p(0)) = p(0) \).

Proof. (1) The equality (5) yields \( \gamma_g(p(V)) = p(g^{-1}(V)) = p(V) \). The period map \( p \) is open and biholomorphic onto its image, so the image \( p(\text{Def}(X)) \) is the interior of its closure in \( \Omega_\Lambda \), possibly after replacing \( \text{Def}(X) \) and \( \text{Def}(Y) \) by smaller open neighborhoods. The closure of \( p(\text{Def}(X)) \) is equal to the closure of \( p(V) \) and is thus \( \gamma_g \)-invariant.

(2) Define the homomorphism \( \alpha : G \rightarrow \text{Aut}(\text{Def}(X)) \) by

\[
\alpha_g := p^{-1} \circ \gamma_g \circ p.
\]

We have

\[
f \circ \alpha_g = f \circ p^{-1} \circ \gamma_g \circ p = f \circ p^{-1} \circ p \circ g^{-1} = f \circ g^{-1} = f.
\]

Thus \( \alpha_g(0) \) belongs to the fiber of \( f \) over \( f(0) \). This fiber consists of a single point, so \( \alpha_g(0) = 0 \).

Proof of Lemma 1.2 part (2): The monodromy action of \( G \) on \( H^2(X, \mathbb{Z}) \) is given by the homomorphism \( \gamma \). Let \( \tilde{f} : G \rightarrow GL[H^*(X, \mathbb{Z})] \) be the monodromy representation and consider the analogue of Diagram (4), where \( \tilde{f} = f \), \( \tilde{\phi} : R_{\psi*}(\mathbb{Z}) \rightarrow H^*(X, \mathbb{Z}) \) is the trivialization, inducing the identity on the fiber \( H^*(X, \mathbb{Z}) \) over 0, \( \tilde{\phi} := \tilde{\phi} \circ \nu^* : f^*[R_{\psi*}(\mathbb{Z})|_U] \rightarrow H^*(X, \mathbb{Z}) \), and \( \tilde{\gamma}_g \circ \tilde{\phi} = (g^{-1})^* \tilde{\phi} \). Given \( t \in V \), we get that

\[
\tilde{\gamma}_g^{-1}(t) \tilde{\gamma}_g \tilde{\phi}_t : H^*(X_t, \mathbb{Z}) \rightarrow H^*(X_{g^{-1}(t)}, \mathbb{Z})
\]

is induced by pullback via \( X_{g^{-1}(t)} \rightarrow Y_{f(t)} \rightarrow X_t \) and thus preserves the Hodge structure. Letting \( t \) approach 0 \( \in \text{Def}(X) \), we get that \( \tilde{\gamma}_g^{-1}(0) \tilde{\gamma}_g \tilde{\phi}_0 : H^*(X_0, \mathbb{Z}) \rightarrow H^*(X_{g^{-1}(0)}, \mathbb{Z}) \) preserves the Hodge structure as well. The statement follows, since \( g(0) = \alpha_g(0) = 0 \), by the above claim, and \( \tilde{\phi}_0 = \text{id} \). The subspace \( H^{2,0}(X) \) is a trivial representation of \( G \), since \( H^{2,0}(X) \) is contained in the trivial representation \( \pi^*H^2(Y, \mathbb{C}) \), by [K], Lemma 2.7.

\[2\] The triviality of the monodromy representation \( \pi^*H^2(Y, \mathbb{C}) \) will be proven in detail in Proposition 4.4.
3. Folding Dynkin diagrams

Consider a simply laced root system \( \Phi \) with root lattice \( \Lambda_r \) and a basis of simple roots \( \mathcal{F} := \{ f_1, \ldots, f_r \} \). We regard the simple roots also as the nodes of the Dynkin graph. The bilinear pairing on \( \Lambda_r \) is determined by the Dynkin graph as follows:

\[
(f_i, f_j) = \begin{cases} 
2 & \text{if } i = j, \\
-1 & \text{if } i \neq j \text{ and the nodes } f_i \text{ and } f_j \text{ are adjacent}, \\
0 & \text{if } i \neq j \text{ and the nodes } f_i \text{ and } f_j \text{ are not adjacent}.
\end{cases}
\]

Let \( \Gamma \) be a subgroup of the automorphism group of the Dynkin graph. If non-trivial, then \( \Gamma = \mathbb{Z}/2\mathbb{Z} \), for types \( A_n, n \geq 2, D_n, n \geq 5, \) and \( E_6 \), and \( \Gamma \) is a subgroup of the symmetric group \( \text{Sym}_3 \), for type \( D_4 \) \[\text{Hum}\], section 12.2. Given an orbit \( j := \Gamma \cdot f_j \in \mathcal{F}/\Gamma \), let \( e_j \) be the orthogonal projection of \( f_j \) from \( \Lambda_r \) to the \( \Gamma \)-invariant subspace \( [\Lambda_r \otimes \mathbb{Q}]^\Gamma \). The projection clearly depends only on the orbit \( \Gamma \cdot f_j \).

We recall the construction of the folded root system (also known as the root system of the twisted group associated to an outer automorphism, see \[C\], Ch. 13). Its root lattice is the lattice \( \Lambda\tilde{\Lambda} as the root system of the twisted group associated to an outer automorphism. Its root lattice is the lattice \( \Lambda \tilde{\Lambda} \) as the root system of the twisted group associated to an outer automorphism, see \[C\], Ch. 13). Its root lattice is the lattice \( \Lambda \tilde{\Lambda} := \text{span}_\mathbb{Z}\{e_1, \ldots, e_r\} \) where \( \tilde{r} \) is the cardinality of \( \mathcal{F}/\Gamma \). Set \( e_i^\vee := \frac{2(e_i, \bullet)}{(e_i, e_i)} \), \( i \in \mathcal{F}/\Gamma \).

Let \( f_i \) and \( f_j \) be two simple roots and set \( i := \Gamma \cdot f_i \) and \( j := \Gamma \cdot f_j \).

Examination of the Dynkin graphs of ADE type shows that there are only two types of \( \Gamma \) orbits:

(1) The orbit \( \Gamma \cdot f_j \) consists of pairwise non-adjacent roots.

(2) The original root system is of type \( A_{2n}, \Gamma = \mathbb{Z}/2\mathbb{Z}, \Gamma \cdot f_j \) consists of the two middle roots of the Dynkin graph.

The following Lemma will be used in the proofs of Lemmas 4.16 and 4.24.

**Lemma 3.1.** The following equation holds:

\[
e_j^\vee(e_i) = \begin{cases} 
2 & \text{if } i = j, \\
\sum_{f_k \in \mathcal{F} \setminus f_j} (f_k, f_i) & \text{if } i \neq j \text{ and type}(\Gamma \cdot f_j) = 1, \\
2 \sum_{f_k \in \mathcal{F} \setminus f_j} (f_k, f_i) & \text{if } i \neq j \text{ and type}(\Gamma \cdot f_j) = 2.
\end{cases}
\]

Consequently, \( e_j^\vee(e_i) \) is an integer and the reflection \( \rho_j(x) = x - e_j^\vee(x)e_j \) by \( e_j \) maps \( \Lambda_r \) onto itself.

**Proof.** \( e_j^\vee(e_i) = \frac{2(e_j, e_i)}{(e_j, e_j)} = \frac{2(e_j, f_i)}{(e_j, f_j)} = \frac{2 \left( \sum_{\gamma \in \Gamma} \gamma(f_j), f_i \right)}{\left( \sum_{\gamma \in \Gamma} \gamma(f_j), f_j \right)} \).

If the orbit \( \Gamma \cdot f_j \) is of type \[D\] the denominator \( \sum_{\gamma \in \Gamma} (\gamma(f_j), f_j) \) is equal to twice the order of the subgroup of \( \Gamma \) stabilizing \( f_j \). Thus,
\[ e^\gamma_j(e_i) = \sum_{f_k \in \Gamma \cdot f_j} (f_k, f_j) \]

If the orbit \( \Gamma \cdot f_j \) is of type 2, let \( \gamma \) be the non-trivial element of \( \Gamma \). Then the denominator is equal to \( (f_j, f_j) + (\gamma(f_j), f_j) = 2 - 1 = 1 \). Furthermore, \( \Gamma = \mathbb{Z}/2\mathbb{Z} \) acts freely on the orbit of \( f_j \).

The Weyl group \( W \), of the folded root system, is the subgroup of the isometry group of \( \Lambda_\bar{r} \), generated by the reflections with respect to \( e_i \), \( 1 \leq i \leq \bar{r} \). \( W \) is finite, since the bilinear pairing is positive definite.

The set of roots \( \Phi \) is the union of \( W \)-orbits of simple roots \( \bigcup_{i=1}^r W \cdot e_i \). We conclude that \( \Phi \) is a, possibly non-reduced, root system (i.e., it satisfies axioms R1, R3, R4 in [Hum], section 9.2, but we have not verified axiom R2, that the only multiples of \( \alpha \in \Phi \), which are also in \( \Phi \), are \( \pm \alpha \)).

Following is the list of folded root systems, which can be found in [C], section 13.3. If \( \Phi \) is of type \( A_r \), \( r \geq 2 \), \( D_r \), \( r \geq 5 \), or \( E_6 \), then the automorphism group \( \Gamma \) of the Dynkin diagram is \( \mathbb{Z}/2\mathbb{Z} \). Setting \( \Phi \) to be the folded root system, the types of the pairs \((\tilde{\Phi}, \Phi)\) are: \((A_{2n}, B_n), (A_{2n-1}, C_n), (D_r, B_{r-1}), \) and \((E_6, F_4)\). When \( \Phi \) is of type \( D_4 \), the automorphism group of the Dynkin diagram is the symmetric group \( \text{Sym}_3 \).

The folding by a subgroup \( \Gamma \) of order 2 results in \( \Phi \) of type \( C_3 \). Let us work out the remaining case explicitly.

**Example 3.2.** Consider the root system of type \( D_4 \) with simple roots \( f_1, f_2, f_3, f_4 \), whose Dynkin graph has three edges from \( f_2 \) to each of the other simple roots. Let \( \Gamma \) be either the full automorphism group \( \text{Sym}_3 \), permuting the roots \( \{f_1, f_3, f_4\} \), or its cyclic subgroup of order 3. Set \( e_1 := \frac{f_1 + f_3 + f_4}{3} \) and \( e_2 := f_2 \). Then \( e_1^\gamma = 3e_1 \) and \( e_2^\gamma = e_2 \). The intersection matrix of the root lattice span\( \mathbb{Z}\{e_1, e_2\} \) is \((e_i, e_j) = \begin{pmatrix} 2/3 & -1 \\ -1 & 2 \end{pmatrix} \)

and its Cartan matrix is \((e_i, e_j^\gamma) = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \). Hence, the folded Dynkin diagram is that of \( G_2 \). This example is revisited in section 5.2.

### 4. Galois Groups

We prove Theorem 1.4 in this section. Let \( \pi : X \to Y \) be a symplectic projective resolution of a normal projective variety \( Y \) of complex dimension \( 2n \). Assume that \( X \) is an irreducible holomorphic symplectic manifold. We keep the notation of Proposition 1.3 and Theorem 1.4.

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3See however [Hum] Section 12.2 Exercise 3, stating that the only irreducible non-reduced root systems are of type \( BC_n \).
4.1. Two Leray filtrations of $H^2(X, \mathbb{Z})$. Set $U := Y \setminus \Sigma_0$ and $\widetilde{U} := X \setminus \pi^{-1}(\Sigma_0)$. Let $\phi : \widetilde{U} \to U$ be the restriction of $\pi$ to $\widetilde{U}$. We get the commutative diagram of pull-back homomorphisms

$$
\begin{array}{ccc}
H^2(Y, \mathbb{Z}) & \xrightarrow{\iota^*_U} & H^2(U, \mathbb{Z}) \\
\pi^* \downarrow & & \downarrow \phi^* \\
H^2(X, \mathbb{Z}) & \xrightarrow{\iota^*_\widetilde{U}} & H^2(\widetilde{U}, \mathbb{Z}).
\end{array}
$$

Lemma 4.1.

1. The homomorphism $\iota^*_U : H^2(X, \mathbb{Z}) \to H^2(\widetilde{U}, \mathbb{Z})$ is bijective.
2. The homomorphism $\phi^* : H^2(U, \mathbb{Z}) \to H^2(\widetilde{U}, \mathbb{Z})$ is injective.
3. Denote the image of the composition $H^2(U, \mathbb{Z}) \xrightarrow{\phi^*} H^2(\widetilde{U}, \mathbb{Z}) \xrightarrow{(\iota^*_U)^{-1}} H^2(X, \mathbb{Z})$ by $H^2(U, \mathbb{Z})$ as well. Then $H^2(U, \mathbb{Z})$ is saturated in $H^2(X, \mathbb{Z})$.
4. Let $\mathcal{E}$ be the set of all irreducible components of codimension 1 of the exceptional locus of $\pi$. The subset $\{ [E] : E \in \mathcal{E} \}$ of $H^2(X, \mathbb{Z})$ is linearly independent. $\mathcal{E}$ maps bijectively onto a basis of $H^2(X, \mathbb{Q})/H^2(U, \mathbb{Q})$ via the map $E \mapsto [E] + H^2(U, \mathbb{Q})$.

Proof. 1) The inverse image $\widetilde{\Sigma}_0 := \pi^{-1}(\Sigma_0)$ has complex codimension $\geq 2$ in $X$, since $\Sigma_0$ has codimension $\geq 4$ in $Y$, by Proposition 1.3 and $\pi$ is semi-small [K]. $\widetilde{U}$ is thus simply connected, since $X$ is assumed so. Furthermore, the groups $H^i(\widetilde{\Sigma}_0, \mathbb{Z})$ vanish, for $i = 4n - 2$ and $4n - 3$. The top horizontal homomorphism in the following commutative diagram is thus an isomorphism.

$$
\begin{array}{ccc}
H^{4n-2}(X, \widetilde{\Sigma}_0, \mathbb{Z}) & \longrightarrow & H^{4n-2}(X, \mathbb{Z}) \\
L.D. \downarrow & & \downarrow P.D. \\
H_2(\widetilde{U}, \mathbb{Z}) & \xrightarrow{\iota^*_\widetilde{U}} & H_2(X, \mathbb{Z})
\end{array}
$$

The right arrow is the Poincare Duality isomorphism and the left is the Lefschetz Duality isomorphism (see [Mun], Theorem 70.6, as well as the argument in [V], proving equation (1.9) in the proof of Theorem 1.23). The bottom push-forward homomorphism $\iota^*_\widetilde{U}$ is thus an isomorphism as well. $H^2(X, \mathbb{Z})$ is isomorphic to $H_2(X, \mathbb{Z})^*$ and $H^2(\widetilde{U}, \mathbb{Z})$ is isomorphic to $H_2(\widetilde{U}, \mathbb{Z})^*$, by the Universal Coefficients Theorem and the fact that both spaces are simply connected. The restriction homomorphism $\iota^*_U : H^2(X, \mathbb{Z}) \to H^2(\widetilde{U}, \mathbb{Z})$ is thus an isomorphism.

2) Associated to $\pi$ we have the canonical filtration $L$ on $H^q(X, \mathbb{Z})$ and the Leray spectral sequence $E^{p,q}_r(X)$, converging to $H^{p+q}(X, \mathbb{Z})$, 

with \( E_2^{p,q}(X) = H^p(Y, R^q_{\phi} Z) \), and \( E_2^{p,q}(X) = G r_{p,q} H^{p+q}(X, Z) \) (\([\text{V}]\), Theorem 4.11). Similarly we have the Leray spectral sequence \( E_2^{p,q}(\tilde{U}) \) associated to \( \phi \) and converging to \( H^{p+q}(\tilde{U}, Z) \). We have the equalities \( E_2^{2,0}(X) = H^2(Y, Z) \) and \( E_2^{2,0}(\tilde{U}) = H^2(U, Z) \), since \( \pi \) has connected fibers, by Zariski’s Main Theorem.

The sheaf \( R^1_{\phi} Z \) vanishes, since it is supported on \( \Sigma \setminus \Sigma_0 \) and over each connected component \( B \) of the latter, \( \pi^{-1}(B) \rightarrow B \) is a topological fibration with simply connected fibers, by Proposition [\( \text{I.3} \). Consequently, \( E_2^{p,1}(\tilde{U}) \) vanishes, for all \( p \), and the differentials \( d_2^{p,1} : H^p(U, R^1_{\phi} Z) \rightarrow H^{p+2}(U, Z) \) and \( d_2^{p,2} : H^p(U, R^2_{\phi} Z) \rightarrow H^{p+2}(U, R^1_{\phi} Z) \) vanish, for all \( p \).

We get the equality \( E_2^{p,q}(\tilde{U}) = E_2^{p,q}(\tilde{U}) \), for \( p + q = 2 \), and in particular \( E_2^{0,0}(\tilde{U}) = H^2(U, Z) \). The injectivity of \( \phi^* \) follows.

\[ \square \]

We need to show that \( H^2(X, Z)/H^2(U, Z) \) is torsion free. It suffices to show that \( H^2(\tilde{U}, Z)/\phi^* H^2(U, Z) \) is torsion free, by part \([\text{I}]\). We have seen that \( \phi^* H^2(U, Z) \) is isomorphic to \( E_2^{2,0}(\tilde{U}) \). \( E_2^{1,1}(\tilde{U}) \) vanishes, by the vanishing of \( E_2^{1,1}(\tilde{U}) \) observed above. Hence, it suffices to show that \( E_2^{0,2}(\tilde{U}) \) is torsion free. Now \( E_2^{0,2}(\tilde{U}) \) is isomorphic to \( E_2^{0,2}(\tilde{U}) := H^0(U, R^2_{\phi} Z) \). The sheaf \( R^2_{\phi} Z \) is supported as a local system over \( \Sigma \setminus \Sigma_0 \). Its fiber, over a point \( b \) in a connected component \( B \) of \( \Sigma \setminus \Sigma_0 \), is the free abelian group generated by the fundamental classes of the irreducible components of the fiber \( \pi^{-1}(b) \). In particular, \( H^0(U, R^2_{\phi} Z) \) is torsion-free.

The fundamental group \( \pi_1(B, b) \) permutes the above mentioned basis of the fiber of the sheaf \( R^2_{\phi} Z \), over the point \( b \). Hence, \( H^0(U, R^2_{\phi} Z) \) has a basis consisting of the sum of elements in each \( \pi_1(B, b) \)-orbit of the original basis elements. These orbits are in one-to-one correspondence with the irreducible components of the exceptional divisor of \( \phi \).

\[ \square \]

Caution: We get the inclusions \( \pi^* H^2(Y, Z) \subset H^2(U, Z) \subset H^2(X, Z) \). The first inclusion may be strict, i.e., the pullback homomorphism \( i_\ast^\#: H^2(Y, Z) \rightarrow H^2(U, Z) \) is not surjective in general. We thank the referee for pointing out the following counter example. Let \( \pi : X \rightarrow Y \) be a contraction of a Lagrangian \( \mathbb{P}^n \), \( n \geq 2 \), to a point. Then \( \pi^* H^2(Y, Z) \rightarrow H^2(X, Z) \) is not surjective, since its image does not contain any ample line-bundle. Now \( U = \tilde{U} \), so \( i_\ast^\#: H^2(Y, Z) \rightarrow H^2(U, Z) \) must fail to be surjective. The Leray filtration of \( H^2(X, Z) \) is thus different from the image of the Leray filtration of \( H^2(\tilde{U}, Z) \) via the isomorphism \( i_\ast^\# \).
4.2. An upper estimate of the Galois group. Let $L$ be the kernel of the composition

$$H^2(X, \mathbb{Z}) \xrightarrow{P.D.} H_{4n-2}(X, \mathbb{Z}) \xrightarrow{\pi^*} H_{4n-2}(Y, \mathbb{Q}),$$

where $2n := \dim_{\mathbb{C}}(X)$. $L$ is the saturation of the sublattice generated by the irreducible components of the exceptional divisors of the map $\pi : X \to Y$.

**Lemma 4.2.** The Beauville-Bogomolov pairing restricts to a negative definite pairing on $L$. Furthermore, $L^\perp$ is equal to the subspace $H^2(U, \mathbb{Z})$ of $H^2(X, \mathbb{Z})$ defined in Lemma 4.1 part 3.

**Proof.** Choose a class $\sigma \in H^{2,0}(X)$, such that $\int_X (\sigma \bar{\sigma})^n = 1$. There exists a positive real number $\lambda$, such that

$$2(\alpha, \beta) = \lambda n \int_X \alpha \beta (\sigma \bar{\sigma})^{n-1} + \lambda (1-n) \left\{ \int_X \alpha |\sigma\bar{\sigma}|^{n-1} \int_X \beta |\sigma\bar{\sigma}|^{n-1} + \int_X \beta |\sigma\bar{\sigma}|^{n-1} \int_X \alpha |\sigma\bar{\sigma}|^{n-1} \right\},$$

by [Be1], Theorem 5. If $\alpha$ belongs to $L$ and $c$ belongs to $H^{4n-2}(Y)$, then $\int_X \alpha \pi^*(c) = P.D.(\alpha) \cap \pi^*(c) = \pi_*(P.D.(\alpha)) \cap c = 0$. There is a class $\sigma_Y \in H^2(Y, \mathbb{C})$, such that $\pi^*(\sigma_Y)$ spans $H^{2,0}(X)$, by Lemma 2.7 of [K]. Assume that $\beta$ belongs to $\pi^*H^2(Y)$. The classes $\sigma^n \bar{\sigma}^{n-1}$, $\sigma^{-1} \bar{\sigma}^n$, and $\beta(\sigma \bar{\sigma})^{n-1}$, all belong to $\pi^*H^{4n-2}(Y)$. Thus $(\alpha, \beta) = 0$, for all $\alpha \in L$. We conclude that $L$ is orthogonal to $\pi^*H^2(Y, \mathbb{Z})$.

The subspace $\pi^*H^2(Y, \mathbb{R})$ of $H^2(X, \mathbb{R})$ contains a positive definite three-dimensional subspace spanned by the real part of $H^{2,0}(X) \oplus H^{0,2}(X) \oplus \mathbb{R}\pi^*\alpha$, where $\alpha \in H^2(Y, \mathbb{Z})$ is the class of an ample line bundle. $L$ is negative definite, since the Beauville-Bogomolov pairing on $H^2(X, \mathbb{R})$ has signature $(3, b_2(X) - 3)$.

We prove next the inclusion $H^2(U, \mathbb{Z}) \subset L^\perp$. Let $\tilde{\beta} \in H^2(X, \mathbb{Z})$ be a class, which belongs to the image of $H^2(U, \mathbb{Z})$. Then $\int_{\tilde{U}}(\tilde{\beta}) = \phi^*(\beta)$, for a unique class $\beta \in H^2(U, \mathbb{Z})$. Let $\alpha \in H^2(X, \mathbb{Z})$ be the class Poincare-dual to an irreducible component $E_i$, of the exceptional locus of $\pi$, of codimension one in $X$. Set $B_i := \pi(E_i)$, $B_i^0 := B_i \cap U$, and $E_i^0 := E_i \cap \tilde{U}$. Then $E_i^0$ is a fibration $\phi_i : E_i^0 \to B_i^0$, with pure one-dimensional fibers, by Proposition 1.3. We get

$$\int_X \alpha \tilde{\beta}^* (\sigma_Y \bar{\sigma}_Y)^{n-1} = \int_{E_i^0} \tilde{\beta}^* (\sigma_Y \bar{\sigma}_Y)^{n-1} = \int_{E_i^0} \phi_i^* \left( [\beta(\sigma_Y \bar{\sigma}_Y)^{n-1}]_{\phi_i^*} \right).$$
The above integral vanishes, since the restriction of the $2n - 2$ form \( \beta(\sigma_Y \bar{\sigma}_Y)^{n-1} \) to the $(2n - 4)$-dimensional $B_i^0$ vanishes. The vanishing of $(\alpha, \beta)$ follows. We conclude the inclusion $H^2(U, \mathbb{Z}) \subseteq L^\perp$.

The lattices $H^2(U, \mathbb{Z})$ and $L^\perp$ have the same rank, by Lemma 4.1 part 4. The equality $H^2(U, \mathbb{Z}) = L^\perp$ follows, since $H^2(U, \mathbb{Z})$ is saturated in $H^2(X, \mathbb{Z})$, by Lemma 4.1 part 3.

Let $B$ be the set of connected components of $\Sigma \setminus \Sigma_0$. Let $\Lambda_B, B \in B$, be the sublattice of $H^2(X, \mathbb{Z})$ spanned by the classes of the irreducible components of $\pi^{-1}(B)$. Let $\mathcal{E}_B$ be the set of irreducible components of the exceptional locus of $\pi : X \to Y$, of pure co-dimension 1, which dominate $B$. Given $i \in \mathcal{E}_B$, denote by $E_i \subset X$ the corresponding divisor and let $e_i \in H^2(X, \mathbb{Z})$ be the class Poincare-dual to $E_i$. Set $E_i^0 := E_i \setminus \pi^{-1}\Sigma_0$. The fiber of $E_i^0 \to B$ over a point $b \in B$ is a union of smooth irreducible rational curves, which are all homologous in $X$. Denote by $e_i^\vee$ the class in $H^{4n-2}(X, \mathbb{Z})$ of such a rational curve. Consider $e_i^\vee$ also as a class in $H^2(X, \mathbb{Z})^*$, via Poincare-duality.

**Lemma 4.3.**

1. $\Lambda_B$ is a $G$-invariant sublattice of $H^2(X, \mathbb{Z})$.
2. For every element $g \in G$, there exists an algebraic correspondence $Z_g$ in $X \times_Y X$, of pure dimension $2n$, inducing the action of $g$ on $H^2(X, \mathbb{Z})$.
3. The endomorphism $[Z_g]^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$, induced by the correspondence $Z_g$, has the form

$$[Z_g]^* = \text{id} + \sum_{B \in B} \sum_{i,j \in \mathcal{E}_B} a_{ij} e_i \otimes e_j^\vee,$$

for some non-negative integers $a_{ij}$.

**Proof.** Part 1 follows from part 3. We proceed to prove part 2. Fix an element $g \in G$. Let $C \subset \text{Def}(X)$ be a smooth connected Riemann surface, containing 0, such that $C \setminus \{0\}$ is contained in the open subset $V$ introduced in Diagram (2). Set $C^0 := C \setminus \{0\}$. Given a subset $S$ of $\text{Def}(X)$, let $X_S$ be the restriction of $X$ to $S$ and $Y_S$ the restriction of $f^*Y$ to $S$. The morphism $\psi : X_V \to V$ is $G$-equivariant with respect to the $G$-action on $X_V$ induced by the isomorphism $\tilde{\nu} : X_V \to Y_V$. We get the isomorphism $\tilde{g} : X_{C^0} \to (g^*X)_{C^0}$. Let $\Gamma$ be the closure of the graph of the isomorphism $\tilde{g}$ in $X_C \times_C (g^*X)_C$. Then $\Gamma$ is contained in the inverse, via $\tilde{\nu} \times \tilde{\nu}$ of the diagonal in $Y_C \times_C Y_C$. Hence, the fiber $Z_g := \Gamma \cap [X \times X]$ is contained in $X \times_Y X$. We consider $Z_g$ as a subscheme of $X \times X$. The fiber $Z_g$ is of pure dimension $2n$, by the upper-semicontinuity of fiber dimension, and the irreducibility of $\Gamma$. Hence, the morphism $\Gamma \to C$ is flat. The class $[Z_g] \in H^{2n}(X \times X)$ is thus the
limit of the classes of the graph $\Gamma_t$ of the isomorphism $\tilde{g}_t : X_t \to X_{g(t)}$, $t \in C^0$. The limiting action $[Z_g]_* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is precisely the monodromy operator $\gamma_g$, given in diagram (4).

Part 3 follows from Proposition 1.3 and the above construction of $Z_g$.

Let $G_L \subset O[H^2(X, \mathbb{Z})]$ be the subgroup, which leaves invariant each of the sublattices $\Lambda_B$, $B \in \mathcal{B}$, of $H^2(X, \mathbb{Z})$, and acts as the identity on the sublattice $L^\perp$ orthogonal to $L$.

Proposition 4.4. (1) $G_L$ is a finite subgroup, which contains the image of the Galois group $G$, via the injective homomorphism $G \to O[H^2(X, \mathbb{Z})]$ constructed in Lemma 1.2.

(2) The group $G$ is isomorphic to a product of Weyl groups associated to root systems of finite type.

Caution: Part 2 of the proposition does not relate the Weyl group factors of $G$ to the Weyl groups $W_B$, $B \in \mathcal{B}$, in Theorem 1.4. At this point we do not claim even the non-triviality of $G$.

Proof. 1) Lemma 4.2 and part 1 of Lemma 4.3 reduce the proof to showing that $\gamma_g$ acts as the identity on the subgroup $H^2(U, \mathbb{Z})$ of $H^2(X, \mathbb{Z})$, for all $g \in G$. The action of $\gamma_g$ on $H^2(U, \mathbb{Z})$ is trivial, by part 3 of Lemma 4.3. Indeed, let $Z_g \subset X \times_Y X$ be the correspondence, of pure-dimension 2$n$, inducing the action of $\gamma_g$. Let $\beta \in H^2(X, \mathbb{Z})$ be a class in the image of $H^2(U, \mathbb{Z})$. Using the notation of Lemma 4.3 we have $e^\gamma_i(\beta) = 0$, for all $i \in \cup_{BE} \mathcal{E}_B$. Hence, $[Z_g]_*(\beta) = \beta$.

2) The quotient $H^{1,1}(X, \mathbb{C})/G$ is smooth, by Lemma 1.2 and Theorem 1.1 $H^{1,1}(X, \mathbb{C})$ decomposes as the direct sum of $L_C := L \otimes_{\mathbb{Z}} \mathbb{C}$ and $L_C^\perp \cap H^{1,1}(X, \mathbb{C})$, since $L$ is negative-definite, by Lemma 4.2. $G$ acts trivially on $L_C^\perp$, and hence $L_C/G$ is smooth as well. $G$ is thus a finite complex reflection group, by [Bou], Ch. V, section 5, Theorem 4, and the statement follows from the classification of complex reflection groups preserving a lattice [Bou], Ch. VI, section 2, Proposition 9.

Consider the case where $\pi : X \to Y$ is a small contraction. Then $L = (0)$, both $G_L$ and $G$ are trivial, and $f : \text{Def}(X) \to \text{Def}(Y)$ is an isomorphism. The latter fact is already a consequence of a general result of Namikawa (see Proposition 4.5 below).

4.3. The differential $df_0$. Recall that the infinitesimal deformations of $Y$ are given by the group $\text{Ext}^1(\Omega^1_y, \mathcal{O}_Y)$. The morphism $\pi : X \to Y$ induces a natural morphism $\pi_* : H^1(X, TX) \to \text{Ext}^1(\Omega^1_y, \mathcal{O}_Y)$, which coincides with the differential $df_0$ of $f : \text{Def}(X) \to \text{Def}(Y)$ at 0. Given $\xi \in H^1(X, TX)$, the class $\pi_*(\xi)$ is identified as follows. Let
0 \to \mathcal{O}_X \to F \to \Omega^1_X \to 0$ be a short exact sequence with extension class $\xi$. We get the short exact sequence
\[ 0 \to \mathcal{O}_Y \to \pi_* F \to \pi_* \Omega^1_X \to 0, \]
since $R^1_{\pi_*}(\mathcal{O}_X)$ vanishes, as $Y$ has rational singularities. Pulling back via $\pi^*: \Omega^1_Y \to \pi_* \Omega^1_X$, we get the extension with class $\pi_*(\xi)$.

Let $\Sigma$ be the singular locus of $Y$ and $\Sigma_0$ the dissident locus, as in Proposition 1.3. Set $U := Y \setminus \Sigma_0$ and $\widetilde{U} := \pi^{-1}(U)$.

**Proposition 4.5.** ([Nam1], Proposition 2.1) There is a commutative diagram
\[
\begin{array}{ccc}
H^1(X,TX) & \overset{\cong}{\longrightarrow} & H^1(\widetilde{U},T\widetilde{U}) \\
\pi_* \downarrow & & \downarrow \\
\operatorname{Ext}^1(\Omega^1_Y,\mathcal{O}_Y) & \overset{\cong}{\longrightarrow} & \operatorname{Ext}^1(\Omega^1_U,\mathcal{O}_U),
\end{array}
\]
where the restriction horizontal maps are both isomorphisms.

Let $H^{1,1}(X) = H^{1,1}(X)^G \oplus H^{1,1}(X)'$ be the $G$-equivariant decomposition, where $H^{1,1}(X)' := \oplus_{R \neq 1} \operatorname{Hom}_G[R,H^{1,1}(X)] \otimes R$, and $R$ varies over all non-trivial irreducible complex representations of $G$. Identify $H^1(X,TX)$ with $\operatorname{Hom}[H^{2,0}(X),H^{1,1}(X)]$.

**Lemma 4.6.** The differential $df_0: \operatorname{Hom}[H^{2,0}(X),H^{1,1}(X)] \to \operatorname{Ext}^1(\Omega^1_Y,\mathcal{O}_Y)$ factors as the projection onto $\operatorname{Hom}[H^{2,0}(X),H^{1,1}(X)^G]$, followed by an injective homomorphism $\operatorname{Hom}[H^{2,0}(X),H^{1,1}(X)^G] \to \operatorname{Ext}^1(\Omega^1_Y,\mathcal{O}_Y)$.

**Proof.** The statement follows immediately from Lemma 1.2, since $\operatorname{Def}(Y) = \operatorname{Def}(X)/G$. \hfill \Box

**Remark 4.7.** Once Theorem 1.4 is proven, we would get that the invariant subspace $L^G$ of $L$ vanishes. Hence, the rank of $df_0$ is equal to $\dim[H^2(U,\mathbb{C}) \cap H^{1,1}(X)] = h^{1,1}(X) - \operatorname{rank}(L)$, by Lemmas 4.2 and 4.6.

### 4.4. Pro-representable deformation functors of open subsets.
Let $\text{Art}$ be the category of local Artin algebras over $\mathbb{C}$, and $\text{Set}$ the category of sets. Let $D_X$, $D_Y$, $D_{\widetilde{U}}$, and $D_U$ be the deformation functors from $\text{Art}$ to $\text{Set}$, sending $A$ to the set of equivalence classes of deformations, of the corresponding variety, over $A$ [Sch], section (3.7). The terms hull and pro-representable functors are defined in [Sch], Definition 2.7. The reader is referred to [I], §8.1, for an excellent summary of basic definitions in formal algebraic geometry. The following is an immediate corollary of Proposition 4.5.

**Corollary 4.8.** The functors $D_X$, $D_{\widetilde{U}}$, $D_Y$ and $D_U$ are pro-representable. Denote by $\mathcal{D}\operatorname{ef}(X)$, $\mathcal{D}\operatorname{ef}(Y)$, $\mathcal{D}\operatorname{ef}(\widetilde{U})$, and $\mathcal{D}\operatorname{ef}(U)$ the corresponding
hulls. We have the commutative diagram

\[
\begin{array}{c}
\text{Def}(X) & \xleftarrow{\iota_X} & \text{Def}(X) & \xrightarrow{\rho_X} & \text{Def}(\tilde{U}) \\
\downarrow f & & \downarrow \hat{f} & & \downarrow \hat{\rho} \\
\text{Def}(Y) & \xleftarrow{\iota_Y} & \text{Def}(Y) & \xrightarrow{\rho_Y} & \text{Def}(U),
\end{array}
\]

where \( \iota_X \) factors through an isomorphism of the hull \( \text{Def}(X) \) with the completion of the Kuranishi local moduli space \( \text{Def}(X) \) at 0, and similarly for \( \iota_Y \). The morphism \( \rho_X \) is the one associated to the morphism of functors from \( D_X \) to \( D_{\tilde{U}} \) induced by restriction. The morphism \( \rho_X \) is an isomorphism. The morphism \( \rho_Y \) is defined similarly and is an isomorphism. The morphism \( \hat{f} \) is the completion of \( f \) at 0 and we set \( \hat{\rho} := \rho_Y \circ \hat{f} \circ \rho_X^{-1} \).

Proof. The deformation functor \( D_S : \text{Art} \to \text{Set} \), of a scheme \( S \) over \( \mathbb{C} \), has a hull, if and only if \( \text{Ext}^1(\Omega^1_S, \mathcal{O}_S) \) is finite dimensional, by [Sch], section 3.7. Finite dimensionality, for \( S = U \) or \( \tilde{U} \), is established in Proposition 4.5. \( \text{Def}(X) \) and \( \text{Def}(\tilde{U}) \) are pro-representable, since \( H^0(X, TX) \) and \( H^0(\tilde{U}, T\tilde{U}) \) both vanish.

The sheaf \( TY \) of derivations of \( \mathcal{O}_Y \) is torsion free. The vanishing of \( H^0(U, TU) \) would thus imply that of \( H^0(Y, TY) \), and would consequently prove that \( D_U \) and \( D_Y \) are also pro-representable. Hence, it suffices to prove that there is a sheaf isomorphism \( \pi_* T\tilde{U} \to TU \), yielding an isomorphism \( H^0(\tilde{U}, T\tilde{U}) \cong H^0(U, TU) \). The spaces of global sections are the same, regardless if we use the Zariski or analytic topology. The sheaf-theoretic question is thus local, in the analytic topology, and so reduces to the case of simple surface singularities ([Reid], section (3.4)). The latter is proven in [BW], Proposition 1.2.

The same argument shows that each of \( \rho_X \) and \( \rho_Y \) is an isomorphism. We prove it only for \( \rho_Y \). Denote by \( R_Y \) and \( R_U \) the formal coordinate rings of \( \text{Def}(Y) \) and \( \text{Def}(U) \). \( R_Y \) is a formal power series ring, by Theorem 1.1. Let \( m \) be the maximal ideal of \( R_U \). \( R_U \) is a quotient of the formal power series ring \( \mathbb{C}[[m/m^2]] \) with cotangent space \( m/m^2 \), by construction ([Sch], Theorem 2.11). The composition \( \mathbb{C}[[m/m^2]] \xrightarrow{q} R_U \xrightarrow{\rho_Y^*} R_Y \) is an isomorphism, as it induces an isomorphism of Zariski cotangent spaces, by Proposition 1.5. The homomorphism \( \rho_Y^* \) is an isomorphism, since the homomorphism \( q \) is surjective.

Let \( g \) be an automorphism of \( \text{Def}(X) \), such that \( g(0) = 0 \). We use the above Corollary to obtain a sufficient criterion for \( g \) to belong to the Galois group \( G \) of \( f : \text{Def}(X) \to \text{Def}(Y) \). Given a Riemann surface \( C \) in \( \text{Def}(X) \), containing 0, denote by \( \mathcal{Y}_C \) the restriction of \( f^* \mathcal{Y} \) to \( C \).
Set $\mathcal{Y}^0_C := \mathcal{Y}_C \setminus \Sigma_0$, so that the fiber of $\mathcal{Y}^0_C$ over 0 is $U := Y \setminus \Sigma_0$. Let $V$ be the open subset of $Def(X)$ in Diagram (2). Denote by $\tilde{C}$ the completion of $C$ at 0.

**Lemma 4.9.** Assume that there exists a Zariski dense open subset $\mathcal{T}$ of $\mathbb{P} [T_0 Def(X)]$ satisfying the following property. For every connected Riemann surface $C \subset Def(X)$, containing 0, whose tangent line $T_0 C$ belongs to $\mathcal{T}$, and such that $C \setminus \{0\}$ is contained in $V$, the completions of $\mathcal{Y}^0_C$ and $g^*(\mathcal{Y}^0_{g(C)})$ along $U$ are equivalent formal deformations of $U$ over $\tilde{C}$. Then $g$ belongs to $G$.

**Proof.** It suffices to prove the equality $f \circ g|_C = f|_C$, for every curve $C$ as above, where $f|_C : C \rightarrow Def(Y)$ is the restriction of $f$ to $C$.

Let $\tilde{\mathcal{Y}}^0_C$ be the completion of $\mathcal{Y}^0_C$ along $U$. Let $\tilde{f}|_C : \tilde{C} \rightarrow \tilde{Def}(U)$ be the composition of the completion of $f|_C$ at 0 with the isomorphism $\tilde{Def}(Y) \cong Def(U)$ of Corollary 4.8. Now $\tilde{f}|_C$ is the classifying morphism of $\tilde{\mathcal{Y}}^0_C$ and $\tilde{f} \circ g|_C$ is the classifying morphism of $\tilde{g}|_C^*(\tilde{\mathcal{Y}}^0_{g(C)})$. These two formal deformations over $\tilde{C}$ are equivalent, by assumption. The equality $\tilde{f}|_C = \tilde{f} \circ \tilde{g}|_C$ thus follows from the pro-representability of the functor $\tilde{D}_U$. The desired equality $f \circ g|_C = f|_C$ follows, as $C$ is connected.  

4.5. **Galois reflections via flops.** Let $E \subset X$ be the exceptional locus of $\pi : X \rightarrow Y$, and $E_i$ an irreducible component of $E$, which intersects $\tilde{U}$ along a $\mathbb{P}^1$ bundle $E_i^0 \rightarrow \tilde{B}_i$ over an unramified cover $\tilde{B}_i \rightarrow B$ of a connected component $B$ of $\Sigma \setminus \Sigma_0$. Let $e_i \in H^2(X, \mathbb{Z})$ be the class Poincare-dual to $E_i$. Let $e_i^\vee \in H^{4n-2}(X, \mathbb{Z})$ be the class Poincare-dual to the fiber of $E_i$ over a point of $\tilde{B}_i$.

**Lemma 4.10.** The Beauville-Bogomolov degree $(e_i, e_i)$ is negative. The isomorphism $H^2(X, \mathbb{Q}) \rightarrow H^{4n-2}(X, \mathbb{Q})$, induced by the Beauville-Bogomolov pairing, maps the class $-\frac{2e_i}{(e_i, e_i)}$ to the class $e_i^\vee$. Consequently, the reflection

$$g(x) := x - \frac{2(x, e_i)}{(e_i, e_i)} e_i$$

has integral values, for all $x \in H^2(X, \mathbb{Z})$. The integral Hodge-isometry $g$ induces an automorphism of $Def(X)$, which belongs to the Galois group $G$.

**Proof.** We follow the strategy suggested by Lemma 4.9.

Step 1: (A generic one-parameter deformation). Let $C \subset Def(X)$ be a connected Riemann surface containing 0 and $\psi_C : X_C \rightarrow C$ the
restriction of the semi-universal family. Assume that \( C \setminus \{0\} \) is contained in the open subset \( V \) of \( \text{Def}(X) \), given in Diagram \([2]\). Set \( \overline{S}_0 := \pi^{-1}(S_0) \) and \( X_C^0 := \mathcal{X}_C \setminus \overline{S}_0 \), so that \( \tilde{U} \) is the fiber of \( \psi^0 : \mathcal{X}_C^0 \to C \) over \( 0 \in C \).

Denote by \( \epsilon \in \text{Ext}_U^1(N_{U/X^0}, T\overline{U}) \cong H^1(\overline{U}, T\overline{U}) \) the Kodaira-Spencer class and \( \epsilon|_E \) its restriction to \( H^1(E_1^0, T\overline{U}|_{E_1^0}) \). Let \( j : T\overline{U}|_{E_1^0} \to N_{E_1^0/\overline{U}} \) be the natural homomorphism, and \( \alpha := j_*(\epsilon|_E) \) the pushed-forward class. Denote by \( P_i^1, t \in \tilde{B}_i \), the fiber of \( E_i^0 \) over \( t \). Note that \( \alpha \) restricts to \( P_i^1 \) as a class in \( H^1(P_i^1, \omega_{P_i^1}) \).

\( H^1(\overline{U}, T\overline{U}) \) is isomorphic to \( H^1(X, TX) \), by \([Nam1]\), Proposition 2.1, and to \( H^{1,1}(X) \), via the holomorphic symplectic form. Let \( \mathbb{T} \) be the complement in \( \mathbb{P} H^1(X, TX) \) of the projectivization of the kernel of the composition \( H^1(X, TX) \to H^{1,1}(X) \to H^{1,1}(\mathbb{P}_t^1). \) \( \mathbb{T} \) is non-empty, since the homomorphism is surjective. Choose the curve \( C \) so that \( T_0C \) belongs to \( \mathbb{T} \). Then the restriction \( \alpha|_{P_1^1} \) is non-trivial.

Step 2: (A correspondence \( Z \subset X \times X \)). Let \( \Delta_{\overline{U}} \) be the diagonal in \( \overline{U} \times \overline{U} \) and set \( Z := \Delta_{\overline{U}} \cup [E_1^0 \times _{\tilde{B}_i} E_i^0] \). Let \( \overline{Z} \) be the closure of \( Z \) in \( X \times X \) and denote by \( \overline{Z}_* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}) \) the homomorphism induced by the correspondence. Let \( (e_i^0)^\perp \subset H^2(X, \mathbb{Z}) \) be the sublattice annihilated by cup product with the class of \( P_i^1 \). We clearly have that \( \int_X e_i e^\vee_i = \deg(\omega_{P_i^1}) = -2 \), \( \overline{Z}_*(e_i) = -e_i \), and \( \overline{Z}_* \) restricts to \( (e_i^0)^\perp \) as the identity. Hence, \( \overline{Z}_* \) is an involution.

Step 3: (The homomorphism \( \overline{Z}_* \) as a limit). Our choice of the family \( \mathcal{X}_C \to C \) is such that \( N_{E_i^0/X^0} \) restricts to \( P_i^1 \) as a non-trivial extension of \( \mathcal{O}_{P_i^1} \) by \( \omega_{P_i^1} \). There is a unique such extension, and so the restriction of \( N_{E_i^0/X^0} \) is isomorphic to \( \mathcal{O}_{P_i^1}(-1) \oplus \mathcal{O}_{P_i^1}(-1) \). Let \( \hat{\mathcal{X}} \) be the blow-up of \( \mathcal{X}_C^0 \) centered at \( E_i^0 \), and \( \hat{E} \) the exceptional divisor of \( \hat{\mathcal{X}} \to \mathcal{X}_C^0 \). Then \( \hat{E} \) is a \([\mathbb{P}^1 \times \mathbb{P}^1] \) bundle over \( \tilde{B}_i \). Furthermore, \( \mathcal{O}_{\hat{\mathcal{X}}}(\hat{E}) \) restricts as \( \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1) \) to each fiber\([4] \). Hence, there exists a \( C \)-morphism \( \hat{\mathcal{X}} \to \mathcal{X}' \) contracting \( \hat{E} \) along the second ruling \([4, FN] \). Note that the fiber of \( \psi' : \mathcal{X}' \to C \) over \( 0 \in C \) is naturally isomorphic to \( \tilde{U} \).

The morphism \( \hat{\mathcal{X}} \to \mathcal{X}_C^0 \times \mathcal{X}' \) is an embedding and we denote its image by \( \hat{\mathcal{X}} \) as well. Then \( \hat{\mathcal{X}} \cap [\tilde{U} \times \tilde{U}] = Z \). The restriction homomorphism \( H^2(X, \mathbb{Z}) \to H^2(\tilde{U}, \mathbb{Z}) \) is an isomorphism, by Lemma \([4, 1] \) part \([1] \). We get that both \( R^2_{\psi_0}(\mathbb{Z}) \) and \( R^2_{\psi_4}(\mathbb{Z}) \) are local systems over

---

\(^4\)We have the three relations \( \omega_{\hat{\mathcal{X}}} \cong \omega_{\mathcal{X}_C^0}(\hat{E}), \ (\omega_{\hat{\mathcal{X}}})|_{\hat{E}} \cong \omega_{\hat{E}} \otimes \mathcal{O}_{\hat{E}}(-\hat{E}), \) and \( \omega_{\hat{E}} \cong \omega_{\hat{E}/E_i^0} \otimes \omega_{E_i^0} \). Back substitution yields \( \mathcal{O}_{\hat{E}}(2\hat{E}) \cong \omega_{\hat{E}/E_i^0} \otimes \omega_{E_i^0} \).
We prove that all \( x \) and the eigenspaces of an isometry are pairwise orthogonal. Hence, \( \text{Consequently, the completion along } \tilde{\kappa} \text{ of } g \) \( \kappa \) left square commutes as well, and we get the equality \( X \) and \( X' \) follows.

We conclude that the involution \( \hat{\psi} \) is a proper morphism. Thus, the \( C \) is an embedding, by the Local Torelli Theorem [Be1]. Hence, the \( \rho \) is the reflection by \( \epsilon_i \). Let \( \epsilon_i^+ \subset H^2(X, \mathbb{Z}) \) be the orthogonal complement, of the class \( \epsilon_i \), with respect to the Beauville-Bogomolov form. We have seen that the involution \( \tilde{\psi}_* \) sends \( \epsilon_i \) to \( -\epsilon_i \) and acts as the identity on \( (\epsilon_i^i)^\perp \). Hence \( (\epsilon_i^i)^\perp = \epsilon_i^+ \), since \( \tilde{\psi}_* \) acts as an isometry and the eigenspaces of an isometry are pairwise orthogonal. Hence, \( \tilde{\psi}_* \) is the reflection by \( \epsilon_i \). On the other hand, \( \tilde{\psi}_*(x) = x + (\epsilon_i^i, x)\epsilon_i, \) for all \( x \in H^2(X, \mathbb{Z}) \), by definition of \( \tilde{\psi} \). The equality \( (\epsilon_i^i, \bullet) = -\frac{2(\epsilon_i, \bullet)}{(\epsilon_i, \epsilon_i)} \) follows.

Step 4: We prove that \( \tilde{\kappa} \) is the reflection by \( \epsilon_i \). The degree of \( \epsilon_i \) is negative, by Lemma 4.2. Let \( \epsilon_i^+ \subset H^2(X, \mathbb{Z}) \) be the orthogonal complement, of the class \( \epsilon_i \), with respect to the Beauville-Bogomolov form. We have seen that the involution \( \tilde{\psi}_* \) sends \( \epsilon_i \) to \( -\epsilon_i \) and acts as the identity on \( (\epsilon_i^i)^\perp \). Hence \( (\epsilon_i^i)^\perp = \epsilon_i^+ \), since \( \tilde{\psi}_* \) acts as an isometry and the eigenspaces of an isometry are pairwise orthogonal. Hence, \( \tilde{\psi}_* \) is the reflection by \( \epsilon_i \). On the other hand, \( \tilde{\psi}_*(x) = x + (\epsilon_i^i, x)\epsilon_i, \) for all \( x \in H^2(X, \mathbb{Z}) \), by definition of \( \tilde{\psi} \). The equality \( (\epsilon_i^i, \bullet) = -\frac{2(\epsilon_i, \bullet)}{(\epsilon_i, \epsilon_i)} \) follows.

Step 5: Let \( g \) be the automorphism of \( \text{Def}(X) \), induced by the Hodge-isometric reflection \( \tilde{\psi}_* \), and \( \tilde{\mathbf{g}} \) its completion at 0. We prove next the equivalence of two formal deformations of \( \hat{U} \): One is the completion of \( \psi' : \mathcal{X}' \to C \) along the fiber \( \hat{U} \) and the other is obtained similarly from the pullback \( g^*(\mathcal{X}'^0_{g(C)}) := C \times_{g(C)} \mathcal{X}'^0_{g(C)} \) to \( C \) via \( g \) of the \( \text{“family” } \mathcal{X}'^0_{g(C)} \to g(C) \).

Let \( \hat{C} \) be the completion of \( C \) at zero. The restrictions of \( \mathcal{X}'^0 \) and \( \mathcal{X}' \) to \( \hat{C} \) induces two classifying maps, \( \kappa_{\mathcal{X}'^0} \) and \( \kappa_{\mathcal{X}'} \), from \( \hat{C} \) to the hull \( \text{Def}(\hat{U}) \) of the deformation functor \( D_{\hat{U}} \). The isomorphism (7) of variations of Hodge structures implies that the outer square of the following diagram commutes.

\[
\begin{array}{cccccc}
\hat{C} & \xrightarrow{\kappa_{\mathcal{X}'^0}} & \text{Def}(\hat{U}) & \xrightarrow{i_{\mathcal{X}'^0} \circ g^1} & \text{Def}(X) & \xrightarrow{\rho} & \Omega \\
\downarrow & & \downarrow \hat{g} & \downarrow g & \downarrow \tilde{\psi}_* & \downarrow \\
\hat{C} & \xrightarrow{\kappa_{\mathcal{X}'}} & \text{Def}(\hat{U}) & \xrightarrow{i_{\mathcal{X}'} \circ g^1} & \text{Def}(X) & \xrightarrow{\rho} & \Omega \\
\end{array}
\]

The right and middle squares commute, by definition. The period map \( p \) is an embedding, by the Local Torelli Theorem [Be1]. Hence, the left square commutes as well, and we get the equality \( \kappa_{\mathcal{X}'} = \hat{g} \circ \kappa_{\mathcal{X}^0} \).

Consequently, the completion along \( \hat{U} \) of \( \psi' : \mathcal{X}' \to C \) is equivalent to that of \( g^*(\psi^0) : g^*(\mathcal{X}'^0_{g(C)}) \to C \) as a formal deformation of \( \hat{U} \) over \( \hat{C} \).
Step 6: A modification of a complex analytic space is a commutative diagram

\[
\begin{array}{ccc}
\tilde{U} & \subset & \tilde{A} \\
\pi \downarrow & & \downarrow \nu \\
U & \subset & A,
\end{array}
\]

where \( U \) and \( \tilde{U} \) are closed analytic subspaces of \( A \) and \( \tilde{A} \), the morphisms \( \nu \) and \( \pi \) are proper and surjective, and \( \nu \) restricts as an isomorphism from \( \tilde{A} \setminus \tilde{U} \) onto \( A \setminus U \). We construct next a modification of \( \tilde{A} := X' \) to obtain a space \( Y' := A \). There is also a notion of a formal modification \([\text{AT}], \text{Ch I Section } \S 2\). We will only use the fact that the formal completion of Diagram (8) along \( U \) and \( \tilde{U} \) is a formal modification. The following is an analytic version of an algebraic result of M. Artin \([\text{AT}]\).

**Theorem 4.11.** \([\text{AT}], \text{Theorem C}\) Let \( \tilde{U} \) be a closed analytic subspace of an analytic space \( \tilde{A} \), and \( \tilde{A} \) the formal completion of \( \tilde{A} \) along \( \tilde{U} \). Suppose that there exists a formal modification

\[
\begin{array}{ccc}
\tilde{U} & \subset & \tilde{A} \\
\pi \downarrow & & \downarrow \hat{\nu} \\
U & \subset & A,
\end{array}
\]

with the additional hypothesis that \( A \) is locally the formal completion of an analytic space along a closed analytic subset. Then there exist an analytic space \( A \), containing \( U \) as a closed analytic subset, and a modification (8), such that \( \hat{\nu} \) is the completion of \( \nu \) along \( \tilde{U} \). The pair \((A, \nu)\) is unique, up to an isomorphism.

Given a subvariety \( S \) of \( \text{Def}(X) \), containing 0, let \( Y_S \) be the restriction to \( S \) of \( f^*Y \) and \( Y^0_S := Y_S \setminus \Sigma_0 \), so that the fiber of \( Y^0_S \) over \( 0 \in S \) is \( U := Y \setminus \Sigma_0 \).

**Claim 4.12.** There exists a normal analytic space \( Y' \), admitting a morphism \( \tilde{\psi}' : Y' \to C \), and a proper surjective \( C \)-morphism \( \nu' : X' \to Y' \), having the following properties: 1) \( \nu' \) restricts to the fibers over \( 0 \in C \) as \( \pi : \tilde{U} \to U \). 2) \( \nu' \) is an isomorphism over \( Y' \setminus U_{\text{sing}} \). 3) The completion of \( \tilde{\psi}' \) along \( U \) is equivalent to that of \( g^*(Y^0_{g(C)}) \to C \).

**Proof.** The formal completion of \( X' \) along the fiber \( \tilde{U} \) was shown to be isomorphic to the completion of \( g^*(X^0_{g(C)}) \) along \( \tilde{U} \), via a \( \hat{C} \)-morphism.

We may thus identify the two completions and denote both by \( \tilde{A} \). Let \( \tilde{\nu} : X \to f^*Y \) be the natural lift of \( \nu \), given by the universal property of the fiber product \( f^*Y := \text{Def}(X) \times_{\text{Def}(Y)} Y \). Now \( \tilde{\nu} \) restricts to
a contraction \(g'((\mathcal{X}_g^0)') \to g^*(\mathcal{Y}_g^0))\). Let \(\hat{\nu} : \tilde{A} \to A\) be the formal completion of the latter contraction along \(\tilde{U}\) and \(U\). We obtain a formal modification \(\hat{\nu}\). We apply Theorem 4.11 with \(\tilde{A} = X'\) and conclude the existence of the morphism \(\nu' : \mathcal{X}' \to \mathcal{Y}'\), satisfying property 1). \(\mathcal{Y}'\) is normal, since \(g^*(\mathcal{Y}_g^0)\) is (see [Ha], Ch. III, Lemma 9.12). Property 2) follows from the smoothness of \(X'\) and \(\mathcal{Y}'\setminus U_{\text{sing}}\), and the fact that \(\nu' : \mathcal{X}' \setminus \tilde{U} \to \mathcal{Y}' \setminus U\) is an isomorphism. Property 3) is clear, by construction.

Step 7: We are now ready to apply Lemma 4.9. The assumptions of Lemma 4.9 would be verified, once we prove the equivalence of the completions along \(U\) of \(\mathcal{Y}_0\) and \(\mathcal{Y}'\) (here we use property 3 in Claim 4.12). The composite morphism \(\hat{\nu} : \tilde{X} \to \mathcal{X}_C \dashrightarrow \mathcal{Y}_C^0\) is clearly also the composition of \(\hat{\nu} : \tilde{X} \to \mathcal{X}'\) and a \(C\)-morphism \(\nu' : \mathcal{X}' \to \mathcal{Y}_C^0\).

Set \(a := \nu' \times \nu'' : \mathcal{X}' \to \mathcal{Y}' \times \mathcal{Y}_C^0\). We show that \(a\) maps \(\mathcal{X}'\) onto the graph of a \(C\)-isomorphism between \(\mathcal{Y}'\) and \(\mathcal{Y}_C^0\). The fiber \(\mathcal{Y}_{f(t)}\) is smooth, for all \(t \in C \setminus \{0\}\), by our choice of \(C\). The statement is thus clear over \(C \setminus \{0\}\). The restriction of \(a\) to the special fiber is the morphism \(\pi' \times \pi : \tilde{U} \to U \times U\), which maps \(\tilde{U}\) onto the diagonal in \(U \times U\). The morphism \(a\) is proper, so its image is supported by a closed subvariety \(\hat{Y}\), which maps bijectively onto each of \(\mathcal{Y}'\) and \(\mathcal{Y}_C^0\). It remains to show that each of these bijective morphisms induces an isomorphism of the structure sheaves. This question is infinitesimal, and so we may pass to the algebraic category. Note that \(\mathcal{Y}_C^0\) is normal, by a lemma of Hironaka [Ha], Ch. III, Lemma 9.12. Each of the projections from \(\hat{Y}\), to each of the factors \(\mathcal{Y}'\) and \(\mathcal{Y}_C^0\), is a proper morphism, since \(\nu'\) and \(\nu''\) are. \(\hat{Y}\) is thus the graph of an isomorphism, by Lemma 4.13. This completes the proof of Lemma 4.10.

**Lemma 4.13.** Let \(Y_1, Y_2\) be normal schemes of finite type over \(\mathbb{C}\) and \(\hat{Y} \subset Y_1 \times Y_2\) a closed integral subscheme. Assume that the projection \(\pi_i : \hat{Y} \to Y_i\) is a proper and bijective morphism, for \(i = 1, 2\). Then \(\hat{Y}\) is the graph of an isomorphism from \(Y_1\) onto \(Y_2\).

**Proof.** (See the proof of [Ha], Corollary III.11.4) The projection \(\pi_i : \hat{Y} \to Y_i\) is clearly a homeomorphism, since \(\pi\) is bijective continuous and closed. It suffices to prove that the sheaf homomorphism \(O_{Y_i} \to \pi_i_*O_{\hat{Y}}\) is an isomorphism. The question is local, so we may assume that \(Y_i = \text{Spec}(A_i)\) and \(\hat{Y} = \text{Spec}(B)\). \(B\) is a finitely generated \(A_i\)-module, since \(\pi_i_*O_{\hat{Y}}\) is a coherent \(O_{Y_i}\) module. \(A_i\) and \(B\) have the same quotient field, and \(A_i\) is integrally closed. Consequently \(A_i = B\).
Keep the notation of Step 1 of the proof of Lemma 4.10.

Corollary 4.14. The complex manifold $g^*(\mathcal{X}_{g(C)}^0) := C \times_{g(C)} \mathcal{X}_{g(C)}^0$ is isomorphic to the flop of $\mathcal{X}_{C}^0$ along $E_0^i$ via a $C$-isomorphism.

**Proof.** Let $\mathcal{X}'$ be the flop of $\mathcal{X}_{C}^0$ along $E_0^i$, constructed in Step 3 of the proof of Lemma 4.10. We have constructed the following two modifications in the proof of Lemma 4.10:

\[
\begin{align*}
\tilde{U} & \subset g^*(\mathcal{X}_{g(C)}^0) \\
\pi \downarrow & \quad \tilde{\nu} \downarrow \\
U & \subset g^*(\mathcal{Y}_{g(C)}^0)
\end{align*}
\]

the left in Step 6 and the right in Step 7. Their completions, along $\tilde{U}$ and $U$, are isomorphic formal modifications, by Claim 4.12. The desired isomorphism $\mathcal{X}' \cong g^*(\mathcal{X}_{g(C)}^0)$ follows from the uniqueness part of the statement of the existence and uniqueness of dilations [AT], Theorem D. \(\square\)

4.6. Weyl groups as Galois groups. We prove Theorem 1.4 in this section, provided that all connected components of $\Sigma \setminus \Sigma_0$ satisfy the technical Assumption 4.15 introduced below. The proof of the Theorem is completed in section 4.8, dropping the assumption.

Let $B$ be a connected component of $\Sigma \setminus \Sigma_0$. Then $Y$ has singularities along $B$ of type $A_r$, $D_r$, or $E_r$, by Proposition 1.3. Set $E_B^i := \pi^{-1}(B)$. Choose a point $b$ in $B$. We adopt, throughout this section, the following:

**Assumption 4.15.** If $Y$ has $A_n$ singularities along $B$, and $n$ is even, then $\pi_1(B, b)$ acts trivially on the set of irreducible components of $\pi^{-1}(b)$.

Each irreducible component $E_i^0$ of $E_B$ is a $\mathbb{P}^1$-bundle $\pi_i : E_i^0 \to \tilde{B}_i$ over an unramified cover $u_i : \tilde{B}_i \to B$, such that $u_i \circ \pi_i$ is equal to the restriction of $\pi$ to $E_i^0$. This follows from Assumption 4.15 and the classification of the automorphism groups of Dynkin diagrams ([Hum], Section 12.2 Table 1).

Let $\tau_B$ be the type of the Dynkin diagram of the singularity of $Y$ along $B$. Then $\tau_B$ is also the type of the Dynkin diagram of the fiber $\pi^{-1}(b)$. The fundamental group $\pi_1(B, b)$ acts on the dual graph, via graph automorphism. Let $\tilde{\tau}_B$ be the type of the Dynkin diagram obtained by folding the Dynkin diagram of the fiber, via the action of $\pi_1(B, b)$, following the procedure recalled in section 3. The dual type is denoted by $\tau_B$. Let $E_i$ be the closure of $E_i^0$ in $X$ and set $e_i$ to be its class in $H^2(X, \mathbb{Z})$. Let $e_i'$ be the class in $H^{4n-2}(X, \mathbb{Z})$ Poincare-dual to the fiber of $E_i^0$ over a point in $\tilde{B}_i$. Set $\Lambda_B := \text{span}_{\mathbb{Z}}\{e_1, \ldots, e_r\}$ and
Λ_B := \text{span}_\mathbb{Z}\{e_1^\vee, \ldots, e_r^\vee\}, where \(r\) is the rank of the root system of type \(\tau_B\).

Lemma 4.16. (1) We can rearrange the irreducible components \(E_i\), so that the matrix \(-\int_X e_i^\vee e_j\) is a Cartan matrix of type \(\tau_B^\vee\).

(2) The lattices \(\Lambda_B\) and \(\Lambda_B^\vee\) both have rank \(r\).

(3) The isomorphism \(H^2(X, \mathbb{Q}) \to H^{4n-2}(X, \mathbb{Q})\), induced by the Beauville-Bogomolov pairing, restricts to an isomorphism from \(\Lambda_B \otimes \mathbb{Z} \mathbb{Q}\) onto \(\Lambda_B^\vee \otimes \mathbb{Z} \mathbb{Q}\), mapping the class \(-2e_i\) to the class \(e_i^\vee\).

(4) Let \(W_B\) be the subgroup of \(G\), generated by the reflections with respect to \(e_i\), as in Lemma 4.10. Then \(W_B\) is isomorphic to the Weyl group of type \(\tau_B \vee\).

(5) Let \(B_0 \subset B\) be the subset consisting of connected components \(B\) of \(\Sigma \setminus \Sigma_0\) satisfying Assumption 4.15. The subgroup \(W\) of \(G\), generated by \(\cup_{B \in B_0} W_B\), is isomorphic to \(\prod_{B \in B_0} W_B\).

(6) If \(B_0 = B\), then \(G = \prod_{B \in B} W_B\).

Given a lattice \(M\) of maximal rank in \(\Lambda_B \otimes \mathbb{Z} \mathbb{Q}\), we denote by \(M^*\) the sublattice of \(\Lambda_B^\vee \otimes \mathbb{Z} \mathbb{Q}\), consisting of classes \(x\), such that \((x, y)\) is an integer, for all \(y \in M\). The fundamental group of the root system is, by definition, the group \(\Pi_B := (\Lambda_B^* / \Lambda_B^\vee)\).

Proof. This part follows immediately from the definition of an ADE-singularity along \(B\), if \(\pi_1(B, b)\) acts trivially on the dual graph of the fiber \(\pi^{-1}(b)\). More generally, let \(\mathcal{F} := \{F_1, \ldots, F_r\}\) be the set of irreducible components of \(\pi^{-1}(b)\). Denote by \(f_i\) the node, corresponding to \(F_i\), of the dual graph. We regard \(\{f_1, \ldots, f_r\}\) as a basis of simple roots of the root lattice of type \(\tau_B\), which bilinear pairing was recalled in Section 3. Let \([F_j] \in H^{4n-2}(X, \mathbb{Z})\) be the cohomology class of \(F_j\). Let \(\Gamma\) be the image of \(\pi_1(B, b)\) in the automorphism group of the graph dual to the fiber \(\pi^{-1}(b)\). Note that if \(F_i\) and \(F_j\) belong to the same \(\Gamma\)-orbit, then \([F_i] = [F_j]\).

Let \(E_{i0}\) be an irreducible component of \(E_B\) and \(F_i\) an irreducible component of the fiber of \(E_{i0}^\vee \to B\) over \(b\). Let \(E_{j0}\) and \(F_j\) be another such pair. Then \(\int_X [E_j][F_i] = -2\), if \(i = j\), and

\[
\int_X [E_j][F_i] = -\sum_{f_k \in \Gamma \cdot F_j} (f_k, f_i), \quad \text{if } i \neq j,
\]

\footnote{The Cartan matrix of a root system with fundamental basis \(\{e_1^\vee, \ldots, e_r^\vee\}\) has entry \(e_i^\vee(e_j)\) in the \(i\)-th row and \(j\)-th column. Note that we are folding the root system of the fiber.}
by Assumption 4.15. Thus $-\int_X [E_j][F_i]$ is equal to the right hand side of the equation in Lemma 3.1. Hence, we can order the orbit $\mathcal{F}/\Gamma$, so that the matrix $-\left(\int_X [E_j][F_i]\right)$ is the Cartan matrix of the folded Dynkin diagram.

2) The sets $\{e_1, \ldots, e_r\}$ and $\{e_1', \ldots, e_r'\}$ are linearly independent, since the rank of the Cartan matrix is $\bar{r}$.

3) This part was proven in Lemma 4.10, which applies by Assumption 4.15.

4) Part 3 implies that the basis $\{e_1, \ldots, e_r\}$ of $\Lambda_B$, endowed with minus the Beauville-Bogomolov pairing, is a basis of simple roots for a root system of type $\bar{\tau}_B$.

5) The lattice $L$ is negative definite and the sub-lattices $\Lambda_B$, $B \in \mathcal{B}$, are pairwise orthogonal, yielding an orthogonal direct sum decomposition $L \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{B \in \mathcal{B}} \Lambda_B \otimes_{\mathbb{Z}} \mathbb{Q}$. $W_B$ acts on $\Lambda_B$ via the reflection representation and it acts trivially on $\Lambda_B'$, if $B \neq B'$.

6) The inclusion $W \subset G$ follows from part 5. We prove the inclusion $G \subset W$. The inclusion $G \subset G_L$ was proven in Proposition 4.4. Let $G_B$ be the image of $G$ in $O(\Lambda_B)$ via the composition $G \to G_L \to O(\Lambda_B)$. It suffices to prove the inclusion $G_B \subset W_B$, since $W = \prod_{B \in \mathcal{B}} W_B$. The inclusion $G_B \subset W_B$ would follow, if we can prove that $G$ acts trivially on the fundamental group $\Pi_B$, since $W_B$ is equal to the subgroup of $O(\Lambda_B)$, which acts trivially on $\Pi_B$, by [Hum], Exercise 5 in Section 13.4. Fix $g \in G$. Lemma 4.3 implies that $g$ acts on $\Lambda_B$ via the class $[Z_g] = I + \sum_{i=1}^r \sum_{j=1}^{\bar{r}} a_{ij} e_i \otimes e_j'$, where $I$ is the identity linear transformation, and $a_{ij}$ are non-negative integers. Let $\lambda$ be an element of $\Lambda^*_B$. Then $g(\lambda) = \lambda + \sum_{i=1}^{\bar{r}} \sum_{j=1}^r a_{ij} \lambda(e_i)e_j'$, which is congruent to $\lambda$ modulo $\Lambda_B'$.

\textbf{Definition 4.17.} Let $B \in \mathcal{B}$ be a connected component satisfying Assumption 4.15. The root system $\Phi_B \subset \Lambda_B$ is the union of the $W_B$-orbits of the classes $e_i$, $1 \leq i \leq \bar{r}$, of the irreducible components of the closure of $\pi^{-1}(B)$. The root system $\Phi_B' \subset \Lambda_B'$ is the union of $W_B$-orbits of the classes $e_i'$, $1 \leq i \leq \bar{r}$, of the irreducible components of the fiber $\pi^{-1}(b)$, $b \in B$.

Root systems, as defined in section 9.2 of [Hum], are reduced. See [Hum], Section 12.2 Exercise 3, for the non-reduced root system of type $BC_n$. As a corollary of Lemma 4.16 we get:

\textbf{Corollary 4.18.} (1) $\Lambda_B$ is the root lattice of the root systems $\Phi_B \subset \Lambda_B$ and $\Lambda_B'$ is the root lattice of the root system $\Phi_B' \subset \Lambda_B'$. Both root systems are reduced and are dual to each other. $\Phi_B$ has type $\bar{\tau}_B$. 

(2) If the type $\bar{\tau}_B$ is $E_8$, $F_4$, or $G_2$, then $H^2(X, \mathbb{Z})$ decomposes as the orthogonal direct sum $\Lambda_B \oplus \Lambda^\perp_B$. If, furthermore, $B = \Sigma \setminus \Sigma_0$, then $\Lambda^\perp_B$ is equal to the subspace $H^2(U, \mathbb{Z})$ of $H^2(X, \mathbb{Z})$, defined in Lemma 4.1 part 3.

Proof. $\Phi_B$ spans $\Lambda_B$ and $\Phi^\perp_B$ spans $\Lambda^\perp_B$, by construction. The foldings of every finite root system of type ADE, by any subgroup of its automorphism group, produces a reduced root system, by the classification of the automorphism groups of Dynkin diagrams of ADE type (see \cite{C}, Section 13.3).

If $\bar{\tau}_B$ is $E_8$, $F_4$, or $G_2$, then the fundamental group $\Pi_B$ of the root system is trivial, so that the sublattice $\Lambda^\perp_B$ of $H^2(X, \mathbb{Z})$ is isometric to $\Lambda_B$. The extension $0 \to \Lambda_B \to H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})/\Lambda_B \to 0$ is split, by the Poincare-Duality isomorphism:

$$H^2(X, \mathbb{Z}) \cong H^{4n-2}(X, \mathbb{Z})^* \to (\Lambda^\perp_B)^* \cong \Lambda_B.$$ 

The kernel of the above homomorphism is the annihilator of $\Lambda^\perp_B$, which is equal to $\Lambda_B$, by Lemma 4.16 part 3. If $\Sigma \setminus \Sigma_0 = B$, then $\Lambda_B = H^2(U, \mathbb{Z})$, by Lemma 4.2. $\square$

4.6.1. The saturation of the root lattice $\Lambda_B$. Let $L_B$ be the saturation of the sublattice $\Lambda_B$ of $H^2(X, \mathbb{Z})$. Let $L_B^\perp$ be the saturation of the sublattice $\Lambda^\perp_B$ of $H^2(X, \mathbb{Z})$. Let $|\Pi_B|$ be the cardinality of the fundamental group $\Pi_B$ of the root system. We clearly have the flag

$$\Lambda_B^\perp \subset L_B^\perp \subset L_B^* \subset \Lambda_B^*.$$ 

We get the equality

$$|L_B^\perp/\Lambda_B| \cdot |L_B/\Lambda_B| \cdot |L_B^*/L_B^\perp| = |\Pi_B|.$$ 

$\Pi_B$ is determined by the type $\bar{\tau}_B$ of the folded root system as follows:

- $A_r$: $\mathbb{Z}/(r+1)\mathbb{Z}$, $B_r$, $C_r$: $\mathbb{Z}/2\mathbb{Z}$, $D_r$: $\mathbb{Z}/4\mathbb{Z}$, if $r$ is odd, and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, if $r$ is even, $E_6$: $\mathbb{Z}/3\mathbb{Z}$, $E_7$: $\mathbb{Z}/2\mathbb{Z}$, $E_8$, $F_4$, $G_2$: trivial \cite{Hum}, section 13.1.

The groups $L_B/\Lambda_B$ and $L_B^\perp/\Lambda_B^\perp$ are invariants of the singularity, which need not be determined by the type of the folded root system, if the fundamental group $\Pi_B$ is non-trivial.

Example 4.19. For the singularity type $A_1$ we have three possibilities $|L_B/\Lambda_B| = 2$, $|L_B^\perp/\Lambda_B^\perp| = 2$, or $|L_B^*/L_B^\perp| = 2$. Following are two examples.

a) Let $S$ be a $K3$ surface, $S^{[n]}$ the Hilbert scheme of length $n$ subschemes of $S$, $S^{(n)}$ the $n$-th symmetric product, and $\pi : S^{[n]} \to S^{(n)}$ the Hilbert-Chow morphism. Then $\pi$ is a contraction of type $A_1$, $B := \Sigma \setminus \Sigma_0$ is irreducible, the exceptional divisor $E$ has class $e = 2\delta$, 

...
where \( \delta \) is integral, primitive, and \( (\delta, \delta) = 2 - 2n \) \( [\text{Bel}] \). Hence, 
\[ L_B/\Lambda_B \cong \mathbb{Z}/2\mathbb{Z}. \]

b) Let \( \pi : X \to Y \) be a contraction of type \( A_1 \), such that the class \( e \) of the exceptional divisor satisfies \( (e, e) = -2 \), and \( (e, x) = 1 \), for some \( x \in H^2(X, \mathbb{Z}) \). Then \( (e^\vee, x) = (e, x) \), so \( |L_B/\Lambda_B| = |L_B^\vee/\Lambda_B^\vee| = 1 \), and thus \( |L_B^\vee/L_B| = 2 \). The contraction of a \(-2\) curve on a \( K3 \) surface \( X \) provides such an example. Higher dimensional examples can be found in \( [\text{YI}] \), or \( [\text{Mal}] \), Corollary 3.19 in the case of Mukai surface \( X \) with \( \chi(v) = 0 \).

4.7. Weyl group actions via flops. Let \( B \in \mathcal{B} \) be a connected component satisfying Assumption \( [\text{I.15}] \). Let \( g \in W_B \) and write \( g \) as a word in the reflections with respect to the simple roots \( \{ e_1, \ldots, e_r \} \)

\[ g = \rho_{e_{i_k}} \rho_{e_{i_{k-1}}} \cdots \rho_{e_{i_1}}. \]

Let \( C \subset \text{Def}(X) \) be a connected Riemann surface containing 0, such that \( C \setminus \{ 0 \} \) is contained in the subset \( \mathcal{V} \) given in Diagram \( [\text{2}] \). Denote by \( \Phi_B \subset \Lambda_B \) the root system (Definition \( [\text{4.17}] \)). Assume further that \( T_0C \) is not contained in the \(+1\) eigenspace in \( T_0 \text{Def}(X) \) of any reflection with respect to a root in \( \Phi_B \). Notice that the special fiber of the flop of \( \mathcal{X}_C^0 \) centered along \( E_i^0 \), \( 1 \leq i \leq \tilde{r} \), is naturally identified with the fiber \( \tilde{U} \) of \( \mathcal{X}_C^0 \). Hence, we can talk about sequences of such flops. Furthermore, the following equality holds for every \( h \in W_B \) and for every \( E_{i}^0 \):

\[ h^*(\text{flop of } \mathcal{X}_{h(C)}^0 \text{ along } E_{i}^0) = \text{flop of } h^*(\mathcal{X}_{h(C)}^0) \text{ along } E_{i}^0. \]

**Lemma 4.20.** The complex manifold \( g^*(\mathcal{X}^0_{g(C)}) \) is isomorphic to the one obtained from \( \mathcal{X}^0_C \) by the sequence of \( k \) flops, starting with \( E_{i_1}^0 \) and ending with \( E_{i_k}^0 \).

**Proof.** The proof is by induction on \( k \). The case \( k = 1 \) was proven in Corollary \( [\text{4.14}] \): \( \rho_{e_i}(\mathcal{X}^0_{\rho_{e_i}(C)}) \) is isomorphic to the flop of \( \mathcal{X}^0_C \) along \( E_i^0 \). Assume that \( k \geq 2 \) and the statement holds for \( k - 1 \). Set \( \tilde{g} := \rho_{e_{i_{k-1}}} \cdots \rho_{e_{i_1}} \). Then \( \tilde{g}^*(\mathcal{X}^0_{C}) \) is obtained from \( \mathcal{X}^0_{C} \) by the sequence of flops starting with \( E_{i_1}^0 \) and ending with \( E_{i_{k-1}}^0 \). Apply next the case \( k = 1 \) of the Lemma with the Riemann surface \( \tilde{g}(C) \) and the reflection \( \rho_{e_{i_k}} \) to get

\[ g^*(\mathcal{X}^0_{g(C)}) = \tilde{g}^* \rho_{e_{i_k}}^*(\mathcal{X}^0_{\rho_{e_{i_k}}(g(C))}) = \tilde{g}^*(\text{the flop of } \mathcal{X}^0_{g(C)} \text{ along } E_{i_k}^0) \]

\[ \equiv \text{the flop of } \tilde{g}^*(\mathcal{X}^0_{\tilde{g}(C)}) \text{ along } E_{i_k}. \]

**Example 4.21.** Choose a connected component \( B \in \mathcal{B} \), a point \( b \in B \), and assume that the type \( \tau_B \), of the Dynkin diagram of the fiber \( \pi^{-1}(b) \), is \( A_2 \), and that \( \pi_1(B, b) \) acts trivially on the Dynkin graph, so that \( \pi^{-1}_B = \) [Details of the example are omitted for brevity.]
The folded $A_{2k}$ case of Theorem 1.4. Drop Assumption 4.15. Fix $B \in \mathcal{B}$, along which $Y$ has an $A_{r}$-singularity, $r$ even, a point $b \in B$, and assume that the image $\Gamma$ of $\pi_1(B, b)$, in the automorphism group of the graph of the fiber $\pi^{-1}(b)$, is $\mathbb{Z}/2\mathbb{Z}$. Let $\Lambda_B \subset H^2(X, \mathbb{Z})$ be the

Choose a smooth connected Riemann surface $C \subset \text{Def}(X)$, as in Lemma 4.20. We see that flopping $\mathcal{X}^0_C$ first along $E_1^0$, then along $E_2^0$, and finally along $E_1^0$ again, yields the same “family” $\rho_{e_1+e_2}(\mathcal{X}^0_{\rho_{e_1}+e_2}(C))$, as flopping first along $E_0^0$, then along $E_0^0$, and finally along $E_0^0$ again.

The above Example will enable us to complete the proof of Theorem 1.4, dropping Assumption 4.15. We will further need the following Lemma. Keep the notation of the Example. Let $Z \subset \left[\mathcal{X}^0_C \times \rho^*_{e_1+e_2}(\mathcal{X}^0_{\rho_{e_1}+e_2}(C))\right]$ be the closure of the graph of the composite isomorphism

$[\mathcal{X}^0_C \setminus \mathcal{U}] \cong \left[Y^0_C \setminus U\right] \cong [\rho^*_{e_1+e_2}(\mathcal{Y}^0_{\rho_{e_1}+e_2}(C)) \setminus U] \cong [\rho^*_{e_1+e_2}(\mathcal{X}^0_{\rho_{e_1}+e_2}(C)) \setminus \mathcal{U}]$.

Let $Z$ be the fiber of $Z$ over $0 \in C$.

**Lemma 4.22.** The reduced induced subscheme structure of $Z$ consists of the union of the following five irreducible components: The diagonal $\Delta_{\mathcal{U}}$, and the four components $\mathcal{E}_1^0 \times_B E_0^0$, $E_2^0 \times_B E_0^0$, $E_1^0 \times_B E_0^0$, $E_2^0 \times_B E_1^0$ of $E_B \times_B E_B$. $Z$ is generically reduced along each of these components.

**Proof.** Choose the equality $\rho_{e_1+e_2} = \rho_{e_1} \rho_{e_2} \rho_{e_1}$ and consider the corresponding sequence of flops

$\mathcal{X}^0_C \xrightarrow{\rho^*_{e_1}(\mathcal{X}^0_{\rho_{e_1}+e_2}(C))} \mathcal{Z}_2 \xrightarrow{\rho^*_{e_1+e_2}(\mathcal{X}^0_{\rho_{e_1}+e_2}(C))} \mathcal{Z}_3$.

Each $\mathcal{Z}_i$ embeds as a correspondence in the product of its two successive blow-downs. $\mathcal{Z}$ is the composite correspondence $Z = \mathcal{Z}_3 \circ \mathcal{Z}_2 \circ \mathcal{Z}_1$. Let $Z_i$ be the fiber of $Z_i$ over $0 \in C$. For $i = 1, 2$, we have $Z_i := \Delta_\mathcal{U} \cup [E_1^0 \times_B E_1^0]$ and $Z_3 = Z_1$. $Z$ is generically equal to the composition $\mathcal{Z}_3 \circ \mathcal{Z}_2 \circ \mathcal{Z}_1$ of the three correspondences, which is the union of the five irreducible components as stated. □

4.8. The folded $A_{2k}$ case of Theorem 1.4. Drop Assumption 4.15. Fix $B \in \mathcal{B}$, along which $Y$ has an $A_r$-singularity, $r$ even, a point $b \in B$, and assume that the image $\Gamma$ of $\pi_1(B, b)$, in the automorphism group of the graph of the fiber $\pi^{-1}(b)$, is $\mathbb{Z}/2\mathbb{Z}$. Let $\Lambda_B \subset H^2(X, \mathbb{Z})$ be the
lattice generated by the classes $e_j$ of the irreducible components $E_j$ of the closure of $E_B$. Define $\Lambda_B^\vee$ as the lattice spanned by the classes $[F_j]$, where $F_j$ is one of the irreducible components of the fiber of $E^0_j$ over $b$. We will see below that $\Lambda_B$ is a unimodular (non-reduced) root lattice of type $BC_{\tilde{r}}$, $\tilde{r} = r/2$, and $\Lambda_B^\vee$ is isometric to the dual lattice $\Lambda_B^\ast$ (Remark 4.25).

We define a sublattice $\tilde{\Lambda}_B$ of $\Lambda_B$. The irreducible components $F_j$ of the fiber $\pi^{-1}(b)$ come with two natural orderings, one the reverse of the other. The two are interchanged by the non-trivial element of $\Gamma$. Choose one of the orderings and let $E^0_j$ be the irreducible component of $E_B$ containing the fiber $F_j$, $1 \leq j \leq \tilde{r}$. Then $E^0_j$ is the irreducible component, such that the fiber of $E^0_j$ over $b$ consists of the pair of middle components $F_t, F_{t+1}$ in the graph of the fiber. Set $\tilde{e}_j := [E_j]$, if $j \neq \tilde{r}$, $\tilde{e}_\tilde{r} := 2[E_\tilde{r}]$, and let $\tilde{\Lambda}_B := \text{span}\{\tilde{e}_1, \ldots, \tilde{e}_\tilde{r}\}$. We set $\tilde{\Lambda}_B^\vee := \text{span}\{\tilde{e}_1^\vee, \ldots, \tilde{e}_\tilde{r}^\vee\}$. Then $\tilde{\Lambda}_B^\vee = \Lambda_B^\vee$.

**Lemma 4.23.** The Beauville-Bogomolov degree $(\tilde{e}_\tilde{r}, \tilde{e}_\tilde{r})$ is negative. The isomorphism $H^2(X, \mathbb{Z}) \to H^{4n-2}(X, \mathbb{Z})$, induced by the Beauville-Bogomolov pairing, maps the class $\frac{-c_\tilde{r}}{(e_\tilde{r}, e_\tilde{r})} = \frac{-2\tilde{e}_\tilde{r}}{\tilde{e}_\tilde{r}, \tilde{e}_\tilde{r}}$ to the class $\tilde{e}_\tilde{r}^\vee$. Consequently, the reflection

$$g(x) := x - \frac{2(x, \tilde{e}_\tilde{r})}{\tilde{e}_\tilde{r}, \tilde{e}_\tilde{r}} \tilde{e}_\tilde{r}$$

has integral values, for all $x \in H^2(X, \mathbb{Z})$. The Hodge-isometry $g$ induces an automorphism of $\text{Def}(X)$, which belongs to the Galois group $G$.

**Proof.** We indicate the modifications needed in the proof of the analogous Lemma 4.10.

Step 1: Choose $C \subset \text{Def}(X)$ as in Step 1 of the proof of Lemma 4.10. Let $\tilde{B}_\tilde{r} \to B$ be the double cover corresponding to the choice of a line in a fiber of $E^0_\tilde{r} \to B$. We get a natural $\mathbb{P}^1$-bundle $\tilde{E}^0_\tilde{r} \to \tilde{B}_\tilde{r}$ and a morphism $\eta : \tilde{E}^0_\tilde{r} \to \mathcal{X}^0_\tilde{r}$ with an injective differential, and so the normal bundle $N_\eta$ of the morphism is locally free. Step 1 of the proof of Lemma 4.10 goes through, with $\mathbb{P}^1_t, t \in \tilde{B}_\tilde{r}$.

Step 2: In Step 2 of the proof of Lemma 4.10 we defined a subscheme $Z \subset \tilde{U} \times \tilde{U}$. We need to redefine $Z$ as the subscheme $\Delta_{\tilde{U}} \cup [E^0_\tilde{r} \times_B E^0_\tilde{r}]$. Note that $Z$ has three irreducible components. Let $\overline{Z} \subset X \times X$ be the
We use Example 4.21 and Lemma 4.22 to construct the flop space of the irreducible components, and the reduced induced subscheme of the 


to show that the bimeromorphic isomorphism of the two irreducible components of \( \tilde{W} \).

Example 4.21 shows that the bimeromorphic isomorphism \( \tilde{W}' \to \tilde{W} \to \tilde{W}'' \) is actually biregular. Hence, \( \tilde{W}' \) is independent of the choice of ordering of the two irreducible components of \( E \). We can thus glue \( \{ \tilde{W}' \} \) to a smooth analytic space \( \mathcal{X}' \), locally isomorphic to \( \mathcal{X}^0 \), admitting morphisms \( \psi' : \mathcal{X}' \to C \), and \( \tilde{\nu}' : \mathcal{X}' \to \mathcal{Y}^0_C \). Let \( \tilde{X} \subset \mathcal{X}^0_C \times \mathcal{X}' \) be the closure of the graph of the isomorphism

\[
[\mathcal{X}_C^0 \setminus \tilde{U}] \cong [\mathcal{Y}_C^0 \setminus U] \cong [\mathcal{X}' \setminus \tilde{U}].
\]

Set \( Z' := \tilde{X} \cap (\tilde{U} \times \tilde{U}) \). Then \( Z' \) is generically reduced along each of its irreducible components, and the reduced induced subscheme of \( Z' \) is isomorphic to \( \Delta_{\tilde{U}} \cup [E^0_B \times_B E^0_B] \), by Lemma 4.22. The proof of Step 3 of Lemma 4.110 now goes through to show that \( Z_s : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}) \) is a Hodge-isometry.

The rest of the proof is identical to that of Lemma 4.110.

The following lemma completes the proof of Theorem 1.4. Let \( B \in \mathcal{B} \) and \( E^0_B \subset E_B \) be as above. We indicate the changes required in the statement and proof of Lemma 4.110 once we drop Assumption 4.115.

**Lemma 4.24.**

1. The matrix with entry \( \int_X \bar{e}_i \bar{e}_j \) in the i-th row and j-th column is the Cartan matrix of type \( B_r \).

2. The lattices \( \Lambda_B \) and \( \Lambda_B^\vee \) both have rank \( r \).

3. The isomorphism \( H^2(X, \mathbb{Q}) \to H^{4n-2}(X, \mathbb{Q}) \), induced by the Beauville-Bogomolov pairing, restricts to an isometry from \( \Lambda_B \otimes \mathbb{Q} \) onto \( \Lambda_B^\vee \otimes \mathbb{Q} \), mapping the class \( -2\bar{e}_i \) of \( \bar{e}_i \) to the class \( \bar{e}_i^\vee \).
(4) Let $W_B$ be the subgroup of $G$ generated by the reflections with respect to $\tilde{e}_i$, as in Lemmas 4.10 and 4.23. Then $W_B$ is isomorphic to the Weyl group of type $B_r$.

(5) $G = \prod_{B \in B} W_B$.

**Proof.**

1. We have $\tilde{e}_i^\vee(\tilde{e}_r) = 2 \int_X |E_r||F_i| = -2 \sum_{f_i \in \Gamma F_r} (f_k, f_i)$. We see that $-\tilde{e}_i^\vee(\tilde{e}_r)$ is equal to the right hand side of the equation in Lemma 3.1, in the case $(i, j) = (i, \bar{r})$. The equality of $-\tilde{e}_i^\vee(\tilde{e}_j)$ with the right hand side of the equation in Lemma 3.1 for $j \neq \bar{r}$, is proven in part 1 of Lemma 4.16. Consequently, the matrix $-\tilde{e}_i^\vee(\tilde{e}_j)$ is the Cartan matrix of the root system $\tilde{B}_r$, the type of the Dynkin diagram obtained by folding that of $A_r$ (see [C], Section 13.3).

2. The proof is identical to that of Lemma 4.16 part 2.

3. Follows from Lemma 4.23, if $i = \bar{r}$, and from Lemma 4.10, for $1 \leq i \leq \bar{r} - 1$.

4. The proof is identical to that of Lemma 4.16 part 4.

5. Let $G_B$ be the image of $G$ in $O(\Lambda_B)$ via the composition $G \rightarrow G_L \rightarrow O(\Lambda_B)$. It suffices to prove the inclusion $G_B \subset W_B$, for every component $B$ violating Assumption 4.15, by the reduction argument provided in Lemma 4.16 part 6.

We rephrase first the above results in the language of root systems. Let $\tilde{\Phi}_B$ be the union of all $W_B$ orbits of the simple roots $\tilde{e}_i$, $1 \leq i \leq \bar{r}$. Define $\tilde{\Phi}_B$ similarly, with respect to $\tilde{e}_i^\vee$. We get that $\Lambda_B$ is a root lattice of the root system $\tilde{\Phi}_B$ of type $C_r$, and $\Lambda_B^\vee$ is the root lattice of the root system $\tilde{\Phi}_B^\vee$ of type $B_r$.

We define next $\Lambda_B$ as a root lattice of type $B_r$. Set $\Lambda_B = \Lambda_B$, with a basis of simple roots $\text{span}_\mathbb{Z}\{\tilde{e}_1, \ldots, \tilde{e}_\bar{r}\}$, where $\tilde{e}_i = e_i$, $1 \leq i \leq \bar{r}$. Set $\tilde{e}_i^\vee = [F_i]$, for $1 \leq i \leq \bar{r} - 1$, $\tilde{e}_r^\vee = 2[F_r]$, and $\Lambda_B^\vee := \text{span}_\mathbb{Z}\{\tilde{e}_1^\vee, \ldots, \tilde{e}_\bar{r}^\vee\}$. We get the root systems $\tilde{\Phi}_B \subset \Lambda_B$ and $\tilde{\Phi}_B^\vee \subset \Lambda_B^\vee$. Part 3 still holds, replacing $\tilde{e}_i$ with $\tilde{e}_i^\vee$. Thus $\Lambda_B$ is the root lattice of the root system $\tilde{\Phi}_B$ of type $B_r$ and $\Lambda_B^\vee$ is the root lattice of the root system $\tilde{\Phi}_B^\vee$ of type $C_r$. The proof of part 6 of Lemma 4.16 applies to the pair of dual root systems $\tilde{\Phi}_B \subset \Lambda_B$ and $\tilde{\Phi}_B^\vee \subset \Lambda_B^\vee$, and shows the the inclusion $G_B \subset W_B$.

**Remark 4.25.** The union $\tilde{\Phi}_B \cup \tilde{\Phi}_B$ in $\Lambda_B$ is a root system of type $BC_r$, by the proof of part 3 above. Similarly, the union $\tilde{\Phi}_B^\vee \cup \tilde{\Phi}_B^\vee$ in $\Lambda_B^\vee$ is a root system of type $BC_r$.

5. **Examples**

### 5.1. Hilbert schemes.

Let $\tilde{S}$ be a projective $K3$ surface with $ADE$-singularities at $Q := \{Q_1, \ldots, Q_k\} \subset \tilde{S}$, and $c : S \rightarrow \tilde{S}$ the crepant
resolution, so that \( S \) is a smooth \( K3 \) surface. Let \( S^{[n]} \) be the Hilbert scheme of length \( n \) subschemes of \( S \), \( n \geq 2 \), and \( \pi : S^{[n]} \to \bar{S}^{(n)} \) the composition, of the Hilbert-Chow morphism \( S^{[n]} \to S^{(n)} \) onto the symmetric product of \( S \), with the natural morphism \( S^{(n)} \to \bar{S}^{(n)} \) induced by \( c \). Set \( W_0 := \mathbb{Z}/2\mathbb{Z} \) and let \( W_i \) be the Weyl group of the root system of the type of the singularity of \( \bar{S} \) at \( Q_i \), \( 1 \leq i \leq k \). We apply Theorem \[\ref{1.4}\] to show that \( W := \prod_{i=0}^{k} W_i \) is isomorphic to the Galois group \( G \) of \( f : \text{Def}(S^{[n]}) \to \text{Def}(\bar{S}^{(n)}) \), introduced in Lemma \([1.2]\).

A point \( P \) of \( \bar{S}^{(n)} \) is a function \( P : \bar{S} \to \mathbb{Z}_{\geq 0} \), with finite support \( \text{supp}(P) \), satisfying \( \sum_{Q \in S} P(Q) = n \). Set \( \Sigma := \{ P : \sum_{i=1}^{k} P(Q_i) + \sum_{Q \in \text{supp}(P) \setminus Q} [P(Q) - 1] > 0 \} \), \( \Sigma_0 := \{ P : \sum_{i=1}^{k} P(Q_i) + \sum_{Q \in \text{supp}(P) \setminus Q} [P(Q) - 1] > 1 \} \). \( \Sigma \setminus \Sigma_0 \) is the singular locus of \( \bar{S}^{(n)} \) and \( \Sigma_0 \) is its dissident locus. The set \( \Sigma \setminus \Sigma_0 \) is smooth, with \( 2n - 2 \) dimensional connected components: \( B_0 := \{ P : \sum_{i=1}^{k} P(Q_i) = 0 \text{ and } \sum_{Q \in \text{supp}(P) \setminus Q} [P(Q) - 1] = 1 \} \), \( B_j := \{ P : P(Q_j) = 1 \text{ and } \sum_{i=1}^{k} P(Q_i) + \sum_{Q \in \text{supp}(P) \setminus Q} [P(Q) - 1] = 0 \} \).

\( \bar{S}^{(n)} \) has a singularity of type \( A_1 \) along \( B_0 \) and the singularity along \( B_j, j \geq 1 \), has the same type as that of \( \bar{S} \) at \( Q_j \). Set \( U := \bar{S}^{(n)} \setminus \Sigma_0 \) and \( \tilde{U} := \pi^{-1}(U) \). Let \( E \) be the exceptional divisor of \( \pi \). The intersection \( E^0 := E \cap \tilde{U} \) has \( k+1 \) connected components. One, a \( \mathbb{P}^1 \) bundle over \( B_0 \), is irreducible. The connected component of \( E^0 \) over \( B_j, j \geq 1 \), is isomorphic to the product of the fiber of \( S \) over \( Q_j \in \bar{S} \) with \( B_j \). Choose a point \( b_j \in B_j, 0 \leq j \leq k \). We see that \( \pi_1(B_j, b_j) \) acts trivially on the Dynkin graph of the fiber \( \pi^{-1}(b_j) \), \( 0 \leq j \leq k \). Theorem \[\ref{1.4}\] implies the equality \( G = W \).

5.2. O’Grady’s 10-dimensional example. Let \( S \) be a \( K3 \) surface, \( K(S) \) its topological \( K \)-group, and \( v \in K(S) \) the class of an ideal sheaf of a length 2 zero dimensional subscheme of \( S \). There is a system of hyperplanes in the ample cone of \( S \), called \( v \)-walls, that is countable but locally finite \( \mathbb{H} \), Ch. 4C. An ample class is called \( v \)-generic, if it does not belong to any \( v \)-wall. Choose a \( v \)-generic ample class \( H \). Let \( M := M_H(2v) \) be the moduli space of Gieseker \( H \)-semi-stable sheaves with class \( 2v \). \( M \) is singular, but it admits a projective symplectic resolution \( \beta : X \to M \) \([Q]\). Let \( Y \) be the Ulenbeck-Yau compactification of the moduli space of \( H \)-stable locally free sheaves with class \( 2v \). There is a natural morphism \( \phi : M \to Y \), which is an isomorphism along the locus of stable locally free sheaves \([\mathbb{L}]\). Let \( \pi : X \to Y \) be the composition

\[
X \xrightarrow{\beta} M \xrightarrow{\phi} Y.
\]
Lemma 5.1.  

(1) The Galois group of \( f : \text{Def}(X) \to \text{Def}(Y) \) is the Weyl group of type \( G_2 \).

(2) The lattice \( H^2(X, \mathbb{Z}) \) is the orthogonal direct sum \( \Lambda \oplus H^2(U, \mathbb{Z}) \), where \( \Lambda \) is the (negative definite) root lattice of type \( G_2 \), and \( H^2(U, \mathbb{Z}) \) is defined in Lemma 4.1 part 3.

Proof. The proof of Lemma 5.1 consists of two steps. In step 1 we recall that the singular locus \( \Sigma_Y \) of \( Y \) is irreducible, and \( Y \) has \( D_4 \) singularities along the complement \( B_Y := \Sigma_Y \setminus \Sigma_{Y,0} \) of the dissident locus. This fact is implicitly proven in [O]. I thank A. Rapagnetta and M. Lehn for explaining this fact to me. In the short step 2 we observe that the image of \( \pi_1(B_Y, b), b \in B_Y \), in the automorphism group of the Dynkin diagram of the fiber \( \pi^{-1}(b) \), is the full automorphism group; i.e., the symmetric group \( \text{Sym}_3 \). The folded Dynkin diagram is thus of type \( G_2 \), by Example 3.2, and the Galois group is the Weyl group of \( G_2 \), by Theorem 1.4. The orthogonal direct sum decomposition follows, by Corollary 4.18.

Step 1: While \( M = M_H(2v) \), the moduli space \( M_H(v) \) is the Hilbert scheme \( S^{[2]} \) of length 2 zero dimensional subschemes of \( S \). The singular locus \( \Sigma_M \) of \( M \) is isomorphic to the symmetric square \( S^2(v) \) and the dissident locus \( \Sigma_{M,0} \) is the diagonal of \( M(v)^{(2)} \) [O]. \( \Sigma_Y \) is isomorphic to the fourth symmetric power \( S^4 \), and \( \Sigma_{Y,0} \) consists of the big diagonal in \( S^4 \) (see [O], paragraph following the proof of Prop. 3.1.1). Note that \( B_Y := \Sigma_Y \setminus \Sigma_{Y,0} \) is connected.

We review first the main ingredients of the proof that \( \Sigma_Y = S^4 \). Any \( H \)-semi-stable locally free sheaf of class \( 2v \) is \( H \)-stable, by [O], Lemma 1.1.5. \( H \)-stable sheaves correspond to smooth points of \( M \) [Muk]. Hence, the singular locus \( \Sigma_M \) is contained in the sublocus \( D \subset M \) parametrizing equivalence classes of \( H \)-semi-stable sheaves, which are not locally free. Each equivalence class is represented by a unique isomorphism class of an \( H \)-poly-stable sheaf. The reflexive hull \( E^* \), of every \( H \)-poly-stable sheaf \( E \) corresponding to a point in \( D \), is necessarily isomorphic to \( \mathcal{O}_S \oplus \mathcal{O}_S \) [O], Prop. 3.0.5. We get a short exact sequence

\[
0 \to E \to \mathcal{O}_S \oplus \mathcal{O}_S \to Q_E \to 0,
\]

where \( Q_E \) is a sheaf of length 4. \( Q_E \) determines a point of \( S^4 \). The construction yields a surjective morphism \( \phi|_D : D \to S^4 \), which is the restriction of the morphism \( \phi \), by the results of Jun Li [Li].

We need to recall the proof of the surjectivity of \( \phi|_D : D \to S^4 \). It suffices to prove that \( \phi|_D \) is dominant. Let \( D^0 \subset D \) be the Zariski dense

---

6Lehn and Sorger showed that \( X \) is the blow-up of \( M \) centered at \( \Sigma_M \) [LS].
We have seen that polystability implies that $\pi_0(D) \to B_Y$. O’Grady proves that the latter is a $\mathbb{P}^1$-bundle [O], Prop. 3.0.5. Following is the identification of the fiber. Let $b \subset S$ be a length four subscheme consisting of four distinct points $(q_1, q_2, q_3, q_4)$. A poly-stable sheaf $E$ in $D$, with $Q_E$ isomorphic to $O_b$, consists of a choice of a one dimensional subspace of each of the fibers $E_{q_i}^{**}$. These four fibers can be identified, since $E^{**}$ is trivial. Hence, a point $E$ in the fiber $D_b$ of $M \to Y$ over $b$ corresponds to a choice of a point $(\ell_1, \ell_2, \ell_3, \ell_4)$ of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, modulo the action of $\text{Aut}(E^{**})$. If $\ell_1 = \ell_2 = \ell_3$, then the ideal sheaf $I_{q_i}$ is a subsheaf of $E$, contradicting the semi-stability of $E$. Consequently, $H$-semi-stability of $E$ implies that each point of $\mathbb{P}^1$ appears at most twice in $(\ell_1, \ell_2, \ell_3, \ell_4)$. The two conditions are in fact equivalent. The quotient $D_b$ is thus isomorphic to the GIT quotient $\overline{M}_{0,4}$ of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by the diagonal action of $PGL(2)$. Now $\overline{M}_{0,4}$ is isomorphic to $\mathbb{P}^1$.

We are ready to identify the fiber of $\pi : X \to Y$ over $b \in B_Y$. Let $D_b$ be the fiber $\phi^{-1}(b)$. Let $E \in D_b$ be the polystable sheaf, such that $\ell_i = \ell_j$ and $i \neq j$. Set $Z := \{q_i, q_j\}$ and $W := b \setminus Z$. We get the inclusion $I_W \subset E$ of the ideal sheaf of the length two subscheme of $S$ supported on $W$, and the quotient $E/I_W$ is the ideal sheaf $I_Z$. Polystability implies that $E = I_W \oplus I_Z$. The intersection $D_b \cap \Sigma_M$ thus consists precisely of three points $I_W \oplus I_Z$, where $W \cup Z = \{q_1, q_2, q_3, q_4\}$ is a decomposition of $b$ as the union of two disjoint subsets of cardinality two. Set $\tilde{B}_Y := D^0 \cap \Sigma_M$. Then $\tilde{B}_Y$ is a Zariski dense open subset of $B_M := \Sigma_M \setminus \Sigma_{M,0}$. $M$ has $A_1$-singularities along $\tilde{B}_Y$, by [O], Proposition 1.4.1. Hence, the total transform of $D_b$ in $X$ consists of the union of the strict transform of $D_b$ and three smooth rational curves over the three points of intersection $D_b \cap \Sigma_M$. The graph, dual to the fiber of $\pi : X \to Y$ over $b$, is thus a Dynkin graph of type $D_4$. We conclude that $Y$ has $D_4$ singularities along $B_Y$, by Namikawa’s classification [Nam1], section 1.8.

Step 2: We have seen that $\tilde{B}_Y$ is a connected unramified cover of $B_Y$ of degree 3. Let $E_D$ be the proper transform of $D$ in $X$ and $E_\Sigma$ the exceptional divisor of $X_\Sigma \to M$. Then $E_\Sigma^0 \to \tilde{B}_Y$ is the composition of a $\mathbb{P}^1$-bundle $E_\Sigma^0 \to \tilde{B}_Y$ and the covering map $\tilde{B}_Y \to B_Y$. The fundamental group of $B_Y$ is the symmetric group $Sym_4$, and it acts on $\tilde{B}_Y$ via the natural homomorphism $Sym_4 \to Sym_3$, permuting the three decompositions $b = W \cup Z$ of $b$. We conclude that the image $\Gamma$ of $\pi_1(B_Y, b)$, in the automorphisms group of the Dynkin graph of type
$D_4$, is the full automorphism group of the latter. This completes the proof of Lemma 5.1.

Remark 5.2. Part 2 of Lemma 5.1 falls short of determining the Beauville-Bogomolov pairing on $H^2(X, \mathbb{Z})$, which is the main result of [R]. Missing still is the determination of the rank of $H^2(U, \mathbb{Z})$ and of two constants $\lambda_1, \lambda_2$ defined below. The bilinear pairing on the root lattice $\Lambda$ of $G_2$ is determined only up to a constant. Rapagnetta proved that the matrix of the bilinear pairing on $\Lambda$ is the multiple of the one in Example 3.2 by a factor of $\lambda_1 = -3$. Rapagnetta furthermore shows that the direct summand $H^2(U, \mathbb{Z})$ is Hodge-isometric to $H^2(S, \mathbb{Z})$.

The proof of the latter statement goes as follows: Donaldson’s homomorphism $\mu : H^2(S, \mathbb{Q}) \to H^2(Y, \mathbb{Q})$ is injective, by [O], Claim 5.2. Compose with restriction to $U$ to get an injective homomorphism $\tilde{\mu} : H^2(S, \mathbb{Q}) \to H^2(U, \mathbb{Q})$. The homomorphism $\tilde{\mu}$ is surjective, since $b_2(X) = 24$, by [R], Theorem 1.1. There exists a rational number $\lambda_2$, such that $\lambda_2\tilde{\mu}$ is an isometric embedding, with respect to the intersection pairing on $S$ and the Beauville-Bogomolov pairing induced on the direct summand $H^2(U, \mathbb{Z})$ of $H^2(X, \mathbb{Z})$. The existence of $\lambda_2$ follows from the calculation of the degree 10 Donaldson polynomial in [O], equation (5.1). A short argument, using the polarization of that Donaldson polynomial, shows that $\lambda_2 = 1$ (see the first part of the proof of [R], Theorem 3.1). The unimodularity of the $K3$ lattice, shows that $\pi^* (\mu[H^2(S, \mathbb{Z})])$ is saturated in $H^2(X, \mathbb{Z})$, and hence equal to $H^2(U, \mathbb{Z})$.

5.3. Constructions of moduli spaces of sheaves on a $K3$. I thank K. Yoshioka for kindly explaining to me the following examples. Fix an ample line bundle $H$ on a $K3$ surface $S$. The notion of Gieseker $H$-stability admits a refinement, due to Matsuki and Wentworth, called $\omega$-twisted $H$-stability, where $\omega$ is a class in $\text{Pic}(S) \otimes_\mathbb{Z} \mathbb{Q}$ [MW]. Yoshioka extended this notion for sheaves of pure one-dimensional support [Y3]. The moduli space $\mathcal{M}_h^\omega(r, c_1, c_2)$, of $S$-equivalence classes of $\omega$-twisted $H$-semistable sheaves with rank $r$ and Chern classes $c_i$, is a projective scheme [MW, Y3]. There is a countable, but locally finite, collection of walls, defining a chamber structure on $\text{Pic}(S) \otimes_\mathbb{Z} \mathbb{Q}$, once we fix $H, r,$ and $c_i, i = 1, 2$. A class $\omega$ is called generic, if it does not lie on a wall.

When $\omega$ is generic, it is often the case that $\omega$-twisted $H$-semistability, for sheaves with the specified rank and Chern classes, is equivalent to $\omega$-twisted $H$-stability. The moduli space $\mathcal{M}_h^\omega(r, c_1, c_2)$ is smooth and holomorphic symplectic, for such a generic $\omega$ [Muk]. The moduli space is in fact deformation equivalent to $S^{[n]}$ and is thus a projective irreducible holomorphic symplectic manifold [Y2]. If we deform $\omega$ to a class $\bar{\omega}$ in the closure of the chamber of $\omega$, then there exists a morphism
\[ \pi^\omega: M_H^\omega(r, c_1, c_2) \to M_H^\omega(r, c_1, c_2) \] [MW]. One can carefully analyze the exceptional locus of the contraction in many examples. This is done for two dimensional moduli spaces in [OK]. They obtain many examples where the morphism \( \pi^\omega \) is the crepant resolution of a normal K3 surface with ADE singularities. These methods can be extended to the higher dimensional examples considered in [Y4], Example 3.2 and Proposition 4.3, to obtain divisorial contractions with Weyl groups of irreducible root systems of ADE-type. A detailed explanation would, regrettably, take us too far afield.

6. Generalizations

6.1. Symplectic generalizations. We thank Y. Namikawa for pointing out the following generalizations of Lemma [L1.2] allowing the singular symplectic \( Y \) to be affine, or considering only partial resolutions of \( Y \).

6.1.1. Partial resolutions. Let \( Y \) be a projective symplectic variety. A \( \mathbb{Q} \)-factorial terminalization of \( Y \) is a morphism \( \pi : X \to Y \) from a projective \( \mathbb{Q} \)-factorial symplectic variety \( X \) with only terminal singularities, which is an isomorphism over the non-singular locus \( Y_{reg} \) of \( Y \). \( Y \) need not admit a symplectic resolution [KLS], but it always admits a \( \mathbb{Q} \)-factorial terminalization, by [Nam2], Remark 2, and recent results in the minimal model program [BCHM]. The Hodge structure of \( H^2(X_{reg}, \mathbb{C}) \) is pure, any flat deformation of \( X \) is locally trivial, i.e., it preserves all the singularities of \( X \) [Nam2], the Kuranishi space \( Def(X) \) is smooth, and the Local Torelli Theorem holds for \( X \) [Nam3], Theorem 8. The analogue of Theorem [L1.1] above for a symplectic \( \mathbb{Q} \)-factorial terminalization \( \pi : X \to Y \) was proven in [Nam2], Theorem 1. The analogue of Lemma [L1.2] above extends, by exactly the same argument. We expect Theorem [L1.4] to generalize as well.

6.1.2. Affine examples. Theorem [L1.1] was further extended by Namikawa to the the case where \( \pi : X \to Y \) is a crepant resolution of an affine symplectic variety \( Y \) with a good \( \mathbb{C}^* \)-action and with a positively weighted Poisson structure ([Nam5], Theorems 2.3 and 2.4). A \( \mathbb{C}^* \)-action on an affine variety \( Y \) is good, if there is a closed point \( y_0 \in Y \) fixed by the action, and \( \mathbb{C}^* \) acts on the maximal ideal of \( y_0 \) with positive weights. Examples include Springer resolutions of the closure of a nilpotent orbit of a complex simple Lie algebra. More generally, Nakajima’s quiver varieties fit into the above set-up [Nak].

The analogue of Lemma [L1.2] above extends. Infinitesimal Poisson deformations of \( X \) are governed by \( H^2(X, \mathbb{C}) \), by [Nam4], Corollary 10.
The only change to the proof of Lemma 1.2 is that we replace the period domain by \( H^2(X, \mathbb{C}) \), on which the monodromy group acts, and we do not need to consider any pairing on \( H^2(X, \mathbb{C}) \). The Kuranishi Poisson deformation space \( PDef(X) \) is an open analytic neighborhood of 0 in \( H^2(X, \mathbb{C}) \), and the period map is replaced by the inclusion \( PDef(X) \subset H^2(X, \mathbb{C}) \). The analogue of Theorem 1.4 for quiver varieties follows from the results in [Nak].

6.2. **Modular Galois covers in the absence of simultaneous resolutions.** Let \( Y \) be a normal compact complex variety, \( X \) a connected compact Kähler manifold, and \( \pi : X \to Y \) a morphism, which is bimeromorphic. Assume that \( R^1_{\pi*}O_X = 0 \). Then \( \pi \) induces a morphism \( f : Def(X) \to Def(Y) \) and \( \pi \) deforms as a morphism \( \nu \) of the semi-universal families, fitting in a commutative diagram (1), by [KM], Proposition 11.4. Let \( M \subset Def(Y) \) be the image of \( f \). We keep the notation of diagram (1) and impose the following additional assumptions:

**Assumption 6.1.**

1. \( Def(X) \) is smooth.
2. The morphism \( f \) is proper with finite fibers.
3. The Local Torelli Theorem holds for \( Def(X) \) in the following sense. There exists a positive integer \( i \), such that the local system \( R_i \psi_* (Q) \) is a polarized\(^7\) variation of Hodge structures, with period domain \( \Omega \) and period\(^8\) map \( p : Def(X) \to \Omega \). Furthermore, the differential \( dp_0 \) at \( 0 \in Def(X) \) is injective.
4. There exists a closed analytic proper subset \( \Delta \subset M \), such that if we set \( U := M \setminus \Delta \) and \( V := Def(X) \setminus f^{-1}(\Delta) \), then the restriction \( \psi|_U : \mathcal{Y}|_U \to U \) is a topological fibration. Furthermore, for every \( t \in U \) and for every two points \( t_1, t_2 \) in \( V \), satisfying \( f(t_1) = f(t_2) = t \), the pull-back homomorphisms \( \nu^*_t : H^i(Y_t, Q) \to H^i(X_{t_j}, Q) \), \( j = 1, 2 \), are surjective and have the same kernel, which we may denote by \( L_t \).

Part 4 of the assumption implies that the homomorphism of local systems

\[ \nu^* : f^* \left( [R^i_{\psi*} Q]|_U \right) \to R^i_{\psi*}(Q)|_U \]

is surjective and its kernel is the pullback \( f^*L \) of a local subsystem \( L \subset [R^i_{\psi*} Q]|_U \) over \( U \). Consequently, the local system \( R^i_{\psi*}(Q)|_U \) is

\(^7\)A canonical polarization exists, for example, if \( i \) is the middle cohomology, or if \( i = 2 \) and \( X \) is an irreducible holomorphic symplectic manifold.

\(^8\)See the reference [G] for the definitions of variations of Hodge structures and period maps.
isomorphic to the pullback of the quotient local system \([R^i_{\overline{\psi}} \mathbb{Q}]|_U/L\) over \(U\). This isomorphism is a generalization of the isomorphism (3) used in the proof of Lemma 1.2. The image \(M\) of \(f\) is a closed irreducible analytic subset of \(\text{Def}(Y)\), by part 2 of the assumption. We endow \(M\) with the reduced structure sheaf, and let \(\widetilde{M} \to M\) be its normalization. The morphism \(f\) factors through \(\widetilde{M}\), since \(\text{Def}(X)\) is smooth. The proof of Lemma 1.2 is easily seen to generalize yielding the following result.

**Lemma 6.2.** Under the above assumptions, there exist neighborhoods of 0 in \(\text{Def}(X)\) and of 0 in \(M\), which we denote also by \(\text{Def}(X)\) and \(M\), and a finite group \(G\), acting faithfully on \(\text{Def}(X)\), such that the morphism \(f\) is the composition of the quotient \(\text{Def}(X) \to \text{Def}(X)/G\), an isomorphism \(\text{Def}(X)/G \to \widetilde{M}\), and the normalization \(\widetilde{M} \to M\). The group \(G\) acts on \(H^*(X, \mathbb{Z})\) via monodromy operators preserving the Hodge structure.

### 6.3. Calabi-Yau threefolds with a curve of ADE singularities.

We provide examples of the set-up of section 6.2. Let \(\pi : X \to Y\) be a crepant resolution, by a Calabi-Yau threefold \(X\), of a normal projective variety \(Y\) with a uniform ADE singularity along a smooth curve \(C\) of genus \(g \geq 1\). The morphism \(\pi\) is assumed to restrict to an isomorphism over \(Y \setminus C\).

**Lemma 6.3.** Assume\(^9\) that for a generic \(t \in M\), \(Y_t\) is normal with only isolated singular points, and \(\nu_t : X_t \to Y_t\) is a small resolution. Then the conclusion of Lemma 6.2 holds.

**Proof.**\(^10\) We verify the conditions of Assumption 6.1. \(\text{Def}(X)\) is smooth, by the Bogomolov-Tian-Todorov Theorem \([Bo, Ti, To]\). Condition 1 follows. The Local Torelli Condition 3 (with \(i = \dim_C(X)\)) is well known for Kähler manifolds with trivial canonical bundle. Condition 2 follows from the assumption that the singularities along \(C\) are of \(ADE\)-type, by the same argument used in the proof of Theorem 1.1 (see \([Nam1]\), Claim 2 in the proof of Theorem 2.2). The key is the uniqueness of the crepant resolution of \(ADE\)-singularities.\(^9\)

\(^9\)The assumption is automatically satisfied if \(g = 1\), since \(Y_t\) is in fact smooth. In the \(g = 1\) case \(Y\) is deformation equivalent to \(X\), and \(\text{Def}(X)\) and \(\text{Def}(Y)\) have the same dimension, by \([W]\), Proposition 4.4 and the erratum of \([W]\). In that case, \(M = \text{Def}(Y)\). If \(g > 1\), then \(Y\) is not deformation equivalent to \(X\), and the expected dimension of \(\text{Def}(Y)\) is larger. The variety \(Y_t\) would thus be singular, if \(g > 1\).

\(^10\)I thank T. Pantev for explaining to me the results of \([DDP]\), leading to the formulation of this Lemma.
It remains to prove Condition [4] We endow \( M \) with the reduced structure sheaf, so that \( M \) is reduced and irreducible. Let \( \Delta \subset M \) be the union of the singular locus of \( M \), the branch locus of \( f \), the locus where \( Y_t \) has singularities, which are not isolated singular points, and the locus where \( Y_t \) has more than the number of isolated singular points of \( Y_{t'} \), of a given topological type, for a generic \( t' \in M \). Set \( U := M \setminus \Delta \). \( U \) is non-empty, by assumption, open, and \( V \) clearly restricts to a topological fibration over \( U \). Set \( V := f^{-1}(U) \). Fix \( s \in V \) and set \( t := f(s) \). The surjectivity of \( \nu^*_s : H^3(Y_t, \mathbb{Z}) \to H^3(X_s, \mathbb{Z}) \) follows from the following general fact.

**Claim 6.4.** Let \( N \) be a three dimensional normal complex variety, smooth away from a finite set of isolated singular points, admitting a small resolution \( \phi : M \to N \). Then \( \phi^* : H^3(N, \mathbb{Z}) \to H^3(M, \mathbb{Z}) \) is surjective.

**Proof.** [3] Associated to \( \phi \) we have the canonical Leray filtration \( L \) on \( H^q(M, \mathbb{Z}) \), and the Leray spectral sequence \( E^p_q \) converging to \( H^{p+q}(M, \mathbb{Z}) \), with \( E^0_{2,q} = H^p(N, R^q_\phi \mathbb{Z}) \) and \( E^\infty_{p,q} = \text{Gr}^p_H H^{p+q}(M, \mathbb{Z}) \) (see [V], Theorem 4.11).

Denote by \( d^r_{p,q} : E^r_{p,q} \to E^{r+1}_{p,q} \) the differential at the \( r \) term. Note the equality \( E^3_{2,0} = H^3(N, \mathbb{Z}) \), since \( N \) is normal and thus the fibers of \( \phi \) are connected. We have a surjective homomorphism

\[
H^3(N, \mathbb{Z}) = E^3_{2,0} \to E^3_{\infty} = \text{Gr}^3_H H^3(M, \mathbb{Z}),
\]

since \( d^r_{3,0} : E^3_{2,0} \to E^4_{2,0} \) vanishes for \( r \geq 2 \). Hence, it suffices to prove that \( E^\infty_{p,q} = \text{Gr}^p_H H^3(M, \mathbb{Z}) \) vanishes, for \( p \neq 3 \). Now \( E^\infty_{p,q} \) is a sub-quotient of \( E^p_{2,q} \), so it suffices to prove the vanishing of \( E^2_{2,1} = H^2(N, R^1_\phi \mathbb{Z}) \), \( E^1_{2,2} = H^1(N, R^2_\phi \mathbb{Z}) \), and \( E^{0,3}_{2} = H^0(N, R^3_\phi \mathbb{Z}) \). The first two vanish, since for \( q > 0 \) the sheaf \( R^q_\phi \mathbb{Z} \) is supported on the zero dimensional set of isolated singular points, and so \( H^p(Y, R^q_\phi \mathbb{Z}) \) vanishes, for \( p > 0 \). \( E^{0,3} \) vanishes, since the sheaf \( R^3_\phi \mathbb{Z} \) vanishes, because the fibers of \( \phi \) have real dimension at most 2. \[ \square \]

Fix \( t \in U \) and let \( s_1, s_2 \) be two points of \( V \) with \( f(s_1) = f(s_2) = t \). We prove \[ 12 \] next that \( \ker(\nu^*_{s_1}) = \ker(\nu^*_{s_2}) \). Let \( X \) be the closure in \( X_{s_1} \times X_{s_2} \) of the graph of the birational isomorphism \( X_{s_1} \to Y_t \to X_{s_2} \). Choose a resolution of singularities \( X \to X \). Let \( p_i : X \to X_{s_i} \) be the projection. We have the equality \( \nu_{s_1} \circ p_1 = \nu_{s_2} \circ p_2 \). Set \( \mu := \nu_{s_1} \circ p_1 \).

\[ ^{11} \text{I thank Y. Namikawa for kindly providing this proof.} \]

\[ ^{12} \text{I thank B. Hassett for kindly providing this proof.} \]
Let $[\tilde{X}] \in H_6(\tilde{X}, \mathbb{Q})$ be the fundamental class. The homomorphisms $p_i^* : H^3(X_{s_i}, \mathbb{Q}) \to H^3(\tilde{X}, \mathbb{Q})$ satisfy

$$p_{i*} ([\tilde{X}] \cap p_i^* \alpha) = p_{i*} ([\tilde{X}]) \cap \alpha = [X_{s_i}] \cap \alpha,$$

for all $\alpha \in H^3(X_{s_i}, \mathbb{Q})$, by the projection formula. The homomorphisms $p_i^*$ are thus injective, since the cap product $[X_{s_i}] \cap : H^3(X_{s_i}, \mathbb{Q}) \to H_3(X_{s_i}, \mathbb{Q})$ is an isomorphism, by Poincare Duality. Hence, $\ker(\nu_{s_i}^*) = \ker(\mu^*), i = 1, 2$. This completes the proof of Lemma 6.3. □

Example 6.5. The following simple example is discussed in [DHP], section 3.2, in connection with large $N$ duality for Dijkgraaf-Vafa geometric transitions in string theory. Consider the complete intersection of a quadric $Q$ and a quartic $F$ hypersurfaces in $\mathbb{P}^5$. Let $\tilde{Q}$ be the quadratic form with zero locus $Q$. If $\tilde{Q}$ has rank 3, so that $Q$ has an $A_1$-singularity along a plane $P$, then for a generic $F$, the complete intersection $Y := Q \cap F$ has an $A_1$-singularity along the smooth plane quartic $C := P \cap F$. The blow-up $\pi : X \to Y$, of $Y$ along $C$, yields a Calabi-Yau threefold $X$. If the quadratic form $\tilde{Q}_t$ has rank 4, so that the quadric $\tilde{Q}_t$ is singular along a line $\ell \subset Q$, then for a generic $F$, the complete intersection $Y_t := Q_t \cap F$ has four ordinary double points along $\ell \cap F$. The proper transforms of $Y_t$, in each of the two small resolutions of $Q_t$, is a crepant resolution $X_{s_i} \to Y_t, i = 1, 2$. Both $X_{s_i}$ are deformation equivalent to $X$. The two corresponding points $s_i \in \text{Def}(X), i = 1, 2$, are interchanged by the Galois involution of $f : \text{Def}(X) \to M$. In this case $\text{Def}(X)$ and $M$ are 86-dimensional, while $\text{Def}(Y)$ is 89-dimensional. For a generic pair of smooth quadric $Q_t$ and quartic $F$, the complete intersection $Q \cap F$ is a smooth Calabi-Yau threefold, which is deformation equivalent to $Y$ but not to $X$.

Additional examples of contractions, of the type considered in Lemma 6.3 with $A_n$ singularities along a curve, are provided in [KMP]. An example of type $C_2$ singularities along a curve is provided in [Sz1], Example 8. Szendrői states Lemma 6.3 as well as the analogue of Theorem 1.4 in this set-up, as a meta-principle in [Sz1], Proposition 6. He proceeds to prove that proposition in a beautiful family of examples, where $X$ is a rank two vector bundle over $C$ and $Y$ is its quotient by a finite group of vector bundle automorphisms, which acts on each fiber as a subgroup of $SL(2)$ (see [Sz2], Proposition 2.9 part iii). Szendrői’s main emphasis is the construction of a braid group action on the level of derived categories, which he carries out in general [Sz1, Sz2].
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