REDUCTION OF THE KNIZHNIK - ZAMOLODCHIKOV EQUATION – A WAY OF PRODUCING VIRASORO SINGULAR VECTORS

A.Ch. Ganchev† and V.B. Petkova††

1) Istituto Nazionale di Fisica Nucleare, Sezione di Trieste
2) Institute for Nuclear Research and Nuclear Energy, Sofia 1784, Bulgaria
3) SISSA-ISAS, via Beirut 2-4, 34014 Trieste, Italy

Abstract

We prove that for (half-) integer isospins the $sl(2,\mathbb{C})$ Knizhnik - Zamolodchikov equation reduces to the decoupling equation coming from Benoit - Saint-Aubin singular vectors. In the general case an algorithm is suggested which transforms, via the Knizhnik - Zamolodchikov equation, a Kac - Moody singular vector into a Virasoro one.

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† postdoctoral fellow at 1); mail address - 3); permanent address - 2)
†† visitor at 1) and 3); permanent address and address after May 1992 - 2)
1. Recently an algorithm was proposed [1] for obtaining the singular vectors of the reducible Virasoro (Vir) algebra Verma modules $V_{c,h}$ parametrised by

$$c(k) = 13 - 6(k + 2 + \frac{1}{k + 2}), \quad h(J) = h_{r,r'} = J(J + 1)/(k + 2) - J,$$

$$J = j - j'(k + 2), \quad r = 2j + 1, \quad r' = 2j' + 1.$$  \hspace{1cm} (1.1)

with $2j, 2j' \in \mathbb{Z}_+$ and $k \in \mathbb{C}$, $k \neq -2$. Rather involved in general, the algorithm simplifies in the particular subseries with $r = 1$ (or $r' = 1$) solved earlier [2]. In the alternative reformulation of this result in [1], inspired by the Drinfeld - Sokolov formalism [3], the highest weight (vacuum) state appears embedded in an auxiliary result in [1], inspired by the Drinfeld - Sokolov formalism [3], the highest weight (vacuum) state

In an independent study [4] of the quantum reduction of the conformal WZNW models [5], based on the algebra $A_1^{(1)}$ at arbitrary level $k$ and isospin values $J$ as in (1.1), the general $n$-point chiral correlators were constructed as solutions of the Knizhnik - Zamolodchikov (KZ) system of equations [6], generalizing earlier results [7], [8]. The appropriate basis used in [4] to describe the WZNW correlators allows to recover the corresponding Dotsenko - Fateev (DF) correlators for the reduced (Virasoro) theory with central charge and scale dimensions as in (1.1). In the “thermal” case, described by integer or half-integer values of the isospin $J = j$, this basis also allows the reduction of the KZ system to a higher order differential equation – the BPZ equation [10], accounting for the decoupling of the corresponding Virasoro algebra singular vectors. It was argued in [4] that the analogy with the matrix system in [1] is not accidental.

The aim of this paper is first to prove that in the thermal case the KZ system can be recast in a matrix form that is equivalent, up to an explicit “gauge transformation”, to the matrix form of the singular vector given in [1]. Next, developing further the argumentation of [4b], we show explicitly that the (infinite in general) KZ matrix equation can be truncated, and hence reduced to a higher order differential equation, by imposing the algebraic equation, originating from a singular vector of the Kac - Moody (KM) Verma modules. This allows us to formulate – for the time being as a conjecture – an algorithm, checked on many examples, which transforms via the KZ equation the KM singular vectors, found in [11], into the general Virasoro singular vectors.

2. Let us denote by $\mathcal{A}$ the semidirect sum of the $A_1^{(1)}$ KM and the Virasoro algebra with Sugawara central charge. Our convention for the $A_1^{(1)}$ commutation relations is

$$[X^\alpha_n, X^\beta_m] = f_{\gamma}^{\alpha \beta} X^\gamma_{n+m} + k q^{\alpha \beta} n \delta_{n+m,0}, \quad f^{0 \pm}_\pm = \pm 2, \quad f_{0-}^{0+} = 1, \quad q^{00} = 2 = 2q^{+-} = 2q^{-+}. \hspace{1cm} (2.1)$$

Consider a representation of $sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})$ in a space of functions $C_J$ of two complex variables $x, z$ realized by the differential operators

$$S^- = -\frac{\partial}{\partial x}, \quad S^0 = 2x \frac{\partial}{\partial x} - 2J, \quad S^+ = x^2 \frac{\partial}{\partial x} - 2xJ, \hspace{1cm} (2.2)$$

and the generators $-L_{-1}, L_0, L_1$ of the finite dimensional subalgebra of Vir given by the above with the Sugawara dimension $\Delta_J = J(J + 1)/(k + 2)$ replacing the isospin $(-J)$ and the “space” variable $z$ replacing the “isospin” variable $x$. Furthermore $X_n^\alpha = z^n S^\alpha$ and the standard differential operators with respect to $z$ represent $\mathcal{A}$ with trivial centers.

To describe the solutions in [4] let us introduce an infinite set of functions $W_t(x, z; J)$ with $t = 0, 1, 2, \ldots$ and $x, z \in \mathbb{C}^{n-1}$. Set $S = \sum_{a=1}^{n-1} J_a - J_n$ and denote by $\Delta(X) = \sum_{a=1}^{n-1} X_a$ the action of the generator $X$ in the tensor product of $n - 1$ spaces; $X_a$ is the action of $X$ at the $a$-th place and identity everywhere else.
The functions $W_t(x,z)$ are subject to the conditions:

\[
\begin{align*}
\triangle(S^-)W_t &= 0 = \triangle(L_{-1})W_t, \\
\triangle(S^0)W_t &= -2(J_n + S - t)W_t, \\
\triangle(L_0)W_t &= (\Delta_n + S - t)W_t,
\end{align*}
\]

and the recursion relation

\[
\triangle(X^{-}_1)W_{t+1} \equiv \sum_{a=1}^{n-1} z_a S^{-}_a W_{t+1} = \nu(S - t)(c_0 - t)W_t, \quad \triangle(X^{-}_1)W_0 = 0,
\]

where $\nu = (k + 2)^{-1}$ and $c_0 = \sum_{a=1}^{n} J_a + 1 - \nu^{-1}$.

They satisfy furthermore the system of equations

\[
K_{-1,a}W_t(x,z) \equiv \left( \frac{\partial}{\partial z_a} - \nu \sum_{b(\neq a)} \frac{\Omega_{ab}}{z_{ab}} \right) W_t(x,z) = \frac{\partial}{\partial x_a} W_{t+1}(x,z),
\]

\[
a = 1, 2, \ldots, n - 1, \quad \Omega_{ab} \equiv q_{\alpha \beta} S^\alpha_a S^\beta_b.
\]

Then the (infinite unless $S$ is a positive integer) series

\[
W(x,z) = \text{const} \sum_{t=0}^{\infty} \left( \frac{1}{t!} \right)^t \left( \frac{\Gamma(c_0 - t + 1)}{\Gamma(c_0 - S + 1)} \right) W_t(x - z, z),
\]

represents the WZNW $n$-point primary fields correlators “at infinity”, i.e., the functions

\[
\lim_{x_n, z_n \to \infty} x_n^{-2J_0} z_n^{-2\Delta_0} \langle \Phi^{J_n}(x_n,z_n) \cdots \Phi^{J_1}(x_1,z_1) \rangle = \langle J_n, \Delta_n | \Phi^{J_{n-1}}(x_{n-1},z_{n-1}) \cdots \Phi^{J_1}(x_1,z_1) | 0 \rangle
\]

which determine the full $n$-point correlators up to a standard prefactor if furthermore two of the points, say $(x_1, z_1)$ and $(x_{n-1}, z_{n-1})$ are set to be $(0,0)$ and $(1,1)$ respectively.

For any set $J = (J_1, \ldots, J_n)$ of isospin values such that $S = s - s'(k + 2)$, $s' = j_1^{(\cdot)} + j_2^{(\cdot)} + \cdots + j_{n-1}^{(\cdot)} - j_n^{(\cdot)}$, $s, s' \in \mathbb{Z}_+$, a solution of (2.3)-(2.5) exists of the type

\[
W_t(x,z) = \sum_{|\alpha| = t} \left( \prod_{a} x_{a \alpha} \right) I^{\alpha}_{\alpha \Gamma}(z),
\]

where $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{Z}^{n-1}_+$, $|\alpha| \equiv \sum_{a=1}^{n-1} \alpha_a$ and $I^{\alpha}_{\alpha \Gamma}(z)$ are multiple $(s + s')$ integrals over a cycle $\Gamma$ (see [4] for the explicit expressions). In particular $I^{S}_{0,\Gamma} = W_0$ coincide exactly with the correlators (at $z_n = \infty$) of the Virasoro models with central charge $c(k)$ and scale dimensions $\{h(J_a)\}$ given by (1.1). The dependence on $\Gamma$ and $S$ will be omitted in what follows.

The integrals $I^{\alpha}$ satisfy a system of relations and a system of differential equations which can be derived in a straightforward fashion substituting (2.7) in (2.3-5). They generalize the relations in [13], see also [8]. The system of equations (2.5) is equivalent to the KZ equation for the $n$-point function $W(x,z)$. The solutions (2.7) described by the integrals $I^{\alpha}$ generalize the solutions [7], [8] of the KZ equations in the standard WZNW theory corresponding to $s' = 0$, $S = s \in \mathbb{Z}_+$; in that
case – to be referred to as the “thermal case” – the sum in (2.6) is finite and \( W(x, z) = W_s(x, z). \) The integrals in (2.7) can be seen as meromorphic modifications of the general (“two - types screening charges”) integrals of Dotsenko and Fateev [9] in the Virasoro theory described by (1.1).

The \( n \)-point correlation functions are invariant with respect to the \( sl(2,\mathcal{D}) \oplus sl(2,\mathcal{D}) \) subalgebra. The corresponding Ward identities carry over to relations for \( W \). The latter read as (2.3) if we set \( t = S \) and formally identify \( W_S \) with \( W \) (although this identification is not necessary to prove the relations for \( W \)). The validity of these relations is ensured by (2.3), (2.4) (or vice versa, together with (2.4) they imply (2.3)). *

The recursion relation originates in the Ward identity corresponding to \( X_1^- \) and the fact that the state \( (X_1^+)\{J_n, \Delta_n\} \) is a h.w. state of \( sl(2,\mathcal{D}) \oplus sl(2,\mathcal{D}) \) with highest weights \( 2(J_n + l) \) and \( \Delta_n + l \). This relation can be also recovered combining the KZ system (2.5) with the Ward identity due to \( L_0 \). Similarly (2.5) “intertwines” the Ward identities related to \( S_0^- \) and \( L_{-1} \) as is easily seen using the symmetry of \( \Omega_{ab} \).

The recursion relation plays a crucial role. Indeed already in the standard case \( s' = 0 \) it converts the conventional basis described by \( \mathbb{I}_a, |\alpha| = s \) (which originates in the Wakimoto bosonization technique) into one with an access to the “reduced” Vir theory, i.e., into a basis containing \( \mathbb{I}_0 \). The integrals are normalized in such a way that in the thermal case they vanish unless \( 0 \leq |\alpha| \leq s_0 \), \( s_0 = \min(s, 2j_1, 2j_2, ..., 2j_n) \). According to (2.4) the operator \( \Delta(X_1^-) \) acts as a raising operator in the finite set of functions \( \{W_t(x, z), \ t = 0, 1, \ldots, s_0\} \). As it is clear from (2.5) the role of the decreasing operator is played by the KZ-operator in the l.h.s. of (2.5). More precisely, multiplying both sides of (2.5) with \( x_a \) and summing over \( a = 1, \ldots, n-1 \) we obtain using (2.7)

\[
(x \cdot K_{-1})W_t = (t + 1)W_{t+1}, \quad (x \cdot K_{-1})W_{s_0} = 0.
\]

The commutator of the two differential operators \( \Delta(X_1^-) \) and \( (x \cdot K_{-1}) \) reduces to a constant on any \( W_t \). Thus the finite set \( \{W_t\}_{t=0}^{s_0} \) can be viewed as a \( sl(2) \) multiplet, the lowest and highest states of which represent the Vir and WZNW correlators \( W_0 \) and \( W_{s_0} \), respectively. In particular

\[
(x \cdot K_{-1})^{s_0+1}W_0 = s_0!(x \cdot K_{-1})W_{s_0} = 0
\]

and this is the most compact form for the BPZ equation in the thermal case. Indeed take for simplicity \( s_0 = 2J_a = 2j_a \) for a given \( a \). Expanding in powers of \( x \) and selecting all terms at the \( 2j_a+1 \)-th power of \( x_a \) in (2.9) one gets a \( 2j_a+1 \)-order partial differential equation, corresponding to the singular vector of the Virasoro Verma module labelled by the h.w. \( h_{1,2j_a+1} \).

In the general (nonthermal) case the action (2.4) and (2.8) of the operators \( \Delta(X_1^-) \) and \( (x \cdot K_{-1}) \) is well defined, however, there arise two different infinite multiplets, one generated by \( (x \cdot K_{-1}) \) starting from \( W_0 \) and another – by \( \Delta(X_1^-) \), starting from \( W \).

3. To recover the BPZ equations in a more transparent algebraic way let us introduce the “right” action of the algebra \( A \) in which any primary field \( \Phi^{J_a}(x_a, z_a) \) is treated as a vacuum – the h.w. state of a module of descendants [12], [10]. Namely, the subalgebra \( A_- \) spanned by \( \{X_0^-, X_{a-n}^-, L_{-n}, n > 0\} \) can be realized by the operators \( X_{-0,a}^-, \ldots, X_{-n,a}^- \) and

\[
X_{-0,a}^- = \sum_{b \neq a} S_a^b z_b^n, \quad X_{-n,a}^- = \sum_{b \neq a} S_a^b + 2x_a S_b^a z_b^n, \quad X_{-n,a}^+ = \sum_{b \neq a} S_a^b - x_a S_b^0 z_b^n + x_a^2 S_b^- z_b^n, \tag{3.1}
\]

* The conditions (2.3) can be looked as selecting l.w. states in the tensor product \( C_{J_z} \otimes C_{J_{a-1}} \otimes \ldots \otimes C_{J_{a-n-1}} \), i.e., \( W_t \) are the singular vectors generating, through the action of \( \Delta(S^+) \), l.w. representations in this tensor product. The representations in \( C_{J_z} \), defined by the “left” action generators (2.2), are not necessarily lowest or highest weight representations.
\[ L_{-n,a} = \sum_{b(\neq a)} \frac{1}{z_{ba}^{n-1}} \left( \frac{(n-1)\Delta_b}{z_{ba}} - \frac{\partial}{\partial z_b} \right), \quad \Delta_b = \nu J_b(J_b + 1). \] (3.2)

In terms of the generators (3.1) the KZ equations (2.5) read
\[ K_{-1,a} W_t \equiv (L_{-1,a} - \nu(\mathcal{X}_{a+1}^+ \mathcal{X}_{0,a}^- + J_a \mathcal{X}_{-1,a}^0)) W_t = \mathcal{X}_{0,a}^- W_{t+1}, \] (3.3)
where \( K_{-1} = K_{-1} \). We shall often omit the label \( a \). Next consider
\[ K_{-n,a} = \sum_{b(\neq a)} (-1)^n \frac{1}{z_{ab}^{n-1}} K_{-1,b}, \quad n > 1, \] (3.4)
and denote by \( M_{-n} \) the finite-sum piece of the Sugawara formula for the Vir generators that survives when acting on a h.w. state, i.e.,
\[ M_{-n,a} \equiv \frac{\nu}{2} \sum_{k=1}^{n-1} q_{ab} \mathcal{X}_{-n+k,a}^\alpha \mathcal{X}_{-k,a}^\beta + \nu(\mathcal{X}_{-n,a}^+ \mathcal{X}_{0,a}^- + J_a \mathcal{X}_{-n,a}^0). \] (3.5)

One easily obtains that \( K_{-n} \) defined in (3.4) can be rewritten as
\[ K_{-n,a} = L_{-n,a} - M_{-n,a}. \] (3.6)

Indeed, using that \( \Omega_{ab} \) is symmetric we have
\[ \sum_{b(\neq c(\neq a))} \frac{2\Omega_{bc}}{z_{ab}^{n-1} z_{bc}} = \sum_{b(\neq c(\neq a))} \frac{\Omega_{bc}}{z_{bc}} \left( \frac{1}{z_{ab}^{n-1}} - \frac{1}{z_{ac}^{n-1}} \right) = \sum_{k=1}^{n-1} \sum_{b(\neq c(\neq a))} \frac{\Omega_{bc}}{z_{ab}^{n-k} z_{ac}^{k}} - (n-1) \sum_{b(\neq a)} \frac{\Omega_{bb}}{z_{ab}^{n}}, \]
which gives (3.6) when substituted in (3.4). Inserting (3.3) into (3.4) and using (3.1) we obtain
\[ K_{-n,a} W_t = \mathcal{X}_{-n+1,a}^- W_{t+1}, \] (3.7)
a form of the KZ equation which will be the most useful in what follows. Written in terms of \( L_{-n} \) the equality (3.7) is the analogue of the Sugawara formula for the negative grade subalgebra of Vir. In terms of the set \( \{ W_t \} \) this formula acquires an additional term in the right hand side. As we shall see this allows to effectively “invert” the Sugawara construction, i.e., to convert the affine KM generators \( \mathcal{X}_{-n}^- \) into Virasoro ones.

The set of equations (2.5) (or (3.3)) can be reproduced starting from the standard form of the KZ equation
\[ K_{-1} W(x, z) = 0, \] (3.8)
with \( W \) defined by the \( x-z \) expansion (2.7) and using the linear relations (2.3a), (2.4) [4]. On the other hand one can write down a straightforward simple, although formal equation, encompassing (2.5). Namely, let us identify \( W \) with \( W_S \) by some way of analytic continuation giving meaning to “nonmeromorphic” integrals \( \Pi_\alpha \), with some \( \alpha \notin \mathbb{Z} \) (such possibilities were discussed in [14] and [4]). Then one can apply the formal \( S-t \)-th power of the intertwining operator \( \Delta(X_1^-) \), so that according to (2.4) we can write
\[ \Delta(X_1^-)^{S-t} K_{-1} W_S(x, z) = 0. \] (3.9)
Although both $W_S$ and $\Delta(X^0_1)^{S-t}$ are formal the resulting expression is well defined and reproduces (2.5). As will become clear this trick will be extremely useful in treating also the algebraic equations due to KM singular vectors in the general nonthermal case.

Similarly, multiplying the basic system of equations (2.5) with $z_n^a$ and summing over $a = 1, 2, \ldots, n - 1$, one gets an expression for $\Delta(L_n)$, $n \geq 0$ analogous to (3.7), (3.6) with $\mathcal{X}_{-p+1}$ replaced by $\Delta(X_{-p+1})$ and $\mathcal{X}_a^\alpha_p$ in $\mathcal{M}_{-m,a}$, (3.5), replaced everywhere by $\Delta(X_{-p})$.

4. Our next objective is to obtain from the above a matrix equation for the integrals $\mathbb{I}_t \equiv \mathbb{I}_{t,a}$ for a fixed $a$ (here $(\varepsilon_a)_b = \delta_{ab}$). Obviously $(\mathcal{X}_0^\alpha)^t W_t = \mathbb{I}_t$. Thus applying $(\mathcal{X}_0^\alpha)^t$ to both sides of (3.7) and commuting it through $\mathcal{K}_{-n}$ we obtain

\[ (\mathcal{X}_0^\alpha)^t \mathcal{X}_{-n+1,a} W_{t+1} = (\mathcal{K}_{-n,a} + \nu t \mathcal{X}_{-n,a}^0) \mathbb{I}_t - \nu \gamma_t(J_a)(\mathcal{X}_0^\alpha)^{t-1} \mathcal{X}_{-n,a} W_t \]  

(4.1) where $\gamma_t(J) = t(2J - t + 1)$. Eliminating recursively the terms $(\mathcal{X}_0^\alpha)^t \mathcal{X}_{-n+1,a} W_{t+1}$ we get

\[ \mathbb{I}_{t+1} = (\mathcal{K}_{-1} + \nu t \mathcal{X}_{-1}^0) \mathbb{I}_t + \sum_{p=1}^{t} (-\nu p) \prod_{i=0}^{p-1} \gamma_{t-i}(J) (\mathcal{K}_{-p} + \nu(t-p) \mathcal{X}_{-p}^0) \mathbb{I}_{t-p} \]  

(4.2)

The above equation can be alternatively obtained by using directly the KZ equation written in terms of the integrals $\mathbb{I}_n$.

The relation (4.2) is valid for any $t = 0, 1, 2 \ldots$ and obviously can be cast in a triangular (in general infinite) matrix form. Introduce generators $\mathcal{J}^{0, \pm}$ with the same commutation relations as $S^{0, \pm}$ acting on a basis $\{\nu_t, t = 0, 1, \ldots\}$ as $\mathcal{J}^- v_t = v_{t+1}$, $\mathcal{J}^+ v_t = \gamma_t(J)v_{t-1}$, $\mathcal{J}^0 v_t = 2(j-t)v_t$, and set

\[ \mathbb{I}(v, z) = \sum_{t=0}^{2j} v_{2j-t} \mathbb{I}_t(z) \]  

(4.3)

Let us now restrict ourselves to the case $2J_a = 2j_a \in \mathbb{Z}_+$ in which $\mathcal{J}^{0, \pm}$ can be realised by finite matrices (assuming $\mathcal{J}^- v_{2j} = 0$) and the set $\{\mathbb{I}_t\}_{t=0}^{j_a}$ can be turned into a $2j_a + 1$-dimensional $\mathfrak{sl}(2, \mathbb{C})$ module. In this basis (4.2) can be written as

\[ \mathcal{K} \mathbb{I} = \left( -\mathcal{J}^- + \sum_{p=0}^{2j} (-\nu \mathcal{J}^+) p (\mathcal{K}_{-p+1} + \nu \frac{\mathcal{X}_{-p}^0}{2} (\mathcal{J}^0 + 2j)) \right) \mathbb{I} = v_0 \mathbb{I}_{2j+1} \]  

(4.4)

We keep the r.h.s. although it is identically zero; the same formula with summation not restricted from above and with zero r.h.s. holds for arbitrary $J$.

Following [1] let us introduce the matrix system

\[ LF = \left( -\mathcal{J}^- + \sum_{p=0}^{2j} (-\nu \mathcal{J}^+) p L_{-p-1} \right) F = v_0 F_{2j+1}, \quad F = \sum_{t=0}^{2j} v_{2j-t} F_t \]  

(4.5)

or, in components

\[ F_{t+1} = \sum_{p=0}^{t} (-\nu p) \prod_{i=0}^{p-1} \gamma_{t-i}(J) L_{t-p} F_{t-p} = \mathcal{N}_{t+1}(J) F_0, \]  

(4.6)

where $\mathcal{N}_t(j)$ is a polynomial in the (negative grade) Vir generators,

\[ \mathcal{N}_t(j) = \prod_{i=1}^{t-1} \gamma_i(j) \sum_{r=1}^{t} (-\nu)^{t-r} \sum_{k_i \geq 0: \sum_{i=1}^{t} k_i = t-r} \frac{L_{-1-k_1} \cdots L_{-1-k_{k_1+2j}}}{\gamma_{k_1+\cdots+k_{r-1}+r+1(j)} \cdots \gamma_{k_1+k_2+2(j)} \gamma_{k_1+1(j)}}. \]
In particular if \( F_0 \) is the highest weight state of the reducible Vir Verma module \( V_{c(k) h(j)} \), \( 2j \in \mathbb{Z}_+ \) (i.e., \( L_0 F_0 = h_{2j+1,1} F_0 \), \( L_n F_0 = 0 \) for \( n > 0 \) then \( F_{2j+1} \) reproduces \([1]\) the expression found in \([2]\) for the singular vector of weight \( h(j)+2j+1 \). We can realize the abstract Vir generators \( L_- \), \( n > 0 \) by the differential operators \( \mathcal{L}^{(h)}_{-n,a} \) defined as in (3.2) with the Sugawara conformal dimensions \( \Delta_{J} \) replaced by \( h(J) \). Clearly the system (4.4) ( (4.2) ) derived from the KZ system (4.4) ( (4.2) ) of equations has almost the same structure as (4.5) ( (4.6) ). Indeed on \( \mathbb{II}_t \), which are independent of \( x \), the operator \( K_{-n} \) reduces to

\[
K_{-n} \mathbb{II}_t = \left[ \mathcal{L}^{(h)}_{-n} - \frac{n-1}{2} \lambda_{-n}^0 - \nu \left( \sum_{k=1}^{n-1} \lambda_{-n+k}^0 \lambda_{-k}^0 - 2(n-1-2J) \lambda_{-n}^0 \right) \right] \mathbb{II}_t , \tag{4.7}
\]

and furthermore \( \lambda_{-n}^0 \) reduces to \( \hat{\lambda}_{-n}^0 = \sum_{b(\neq a)} \frac{-2h}{z_{b,a}} \), \( \hat{\lambda}_{-n}^0 \mathbb{II}_t = \lambda_{-n}^0 \mathbb{II}_t \). This suggests that the functions \( \{F_n\} \) can be realized as \( z \)-dependent linear combinations of the integrals \( \{\mathbb{II}_t\} \). Indeed define

\[
F = g \mathbb{II} , \tag{4.8}
\]

where

\[
g_a = \left[ \exp(-\nu \mathbb{J}^+(L_{-1,a} - \frac{1}{2} \lambda_{-1,a}^0)) \right] \cdot 1 \tag{4.9}
\]

\[
= \mathbb{II} + \nu \mathbb{J}^+ \hat{\lambda}_{-1}^0 \frac{1}{2} + \frac{(\nu \mathbb{J}^+)^2}{2} \left[ \left( \frac{\hat{\lambda}_{-1}^0}{2} \right)^2 - \hat{\lambda}_{-2}^0 \right] + \ldots
\]

and the sum is finite for \( r = 2j + 1 \) - positive integer, since \((\mathbb{J}^+)^{2j+1} = 0\).

**Proposition 1.** The matrix systems (4.4) and (4.5) are connected by the ”gauge transformation” \( g \), i.e.,

\[
gK \mathbb{II} = \mathbb{L} g \mathbb{II} . \tag{4.10}
\]

The proof consists of the following two Lemmas.

**Lemma 1.**

\[
g \left( -\mathbb{J}^- + \sum_{k=1}^{\infty} (\nu \mathbb{J}^+)^{k-1} (-M_{-k,a} + \nu \lambda_{-k,a}^0 (\mathbb{J}^0 + 2ja)) \right) \mathbb{II} = -\mathbb{J}^- g \mathbb{II} . \tag{4.11}
\]

Denote \( R_{-n} = \left( \frac{1}{2} \lambda_{-1,a}^0 - L_{-1,a} \right)^n \cdot 1 \) and use the \( sl(2) \) commutation relations of the generators \( \mathbb{J}_0, \pm \) to get

\[
[-\mathbb{J}^-, g] = \nu \sum_{k=1}^{\infty} (\nu \mathbb{J}^+)^{k-1} \frac{R_{-k}}{(k-1)!} \left( \mathbb{J}^0 + k - 1 \right) .
\]

The lemma is obtained using twice the following identity proved by induction

\[
\frac{R_{-k}}{(k-1)!} = \sum_{l=0}^{k-1} \frac{(-1)^{l+k-1}}{l!} R_{-l} \frac{\hat{\lambda}_{-k+1,a}^0}{2} .
\]

**Lemma 2.**

\[
g(u) \left( \sum_{n=1}^\infty \mathcal{L}_{-n,a} u^{n-1} \right) g(u)^{-1} = \sum_{n=1}^\infty \left( \mathcal{L}_{-n,a} + (n-1) \hat{\lambda}_{-n,a}^0 \right) u^{n-1} \equiv \sum_{n=1}^\infty \mathcal{L}^{(h)}_{-n,a} u^{n-1} . \tag{4.12}
\]
Here $g(u) = [\exp(u(L_{-1} - \frac{1}{2} X_{-1}^0))] \cdot 1$. To prove the lemma one has to use

$$
\frac{\partial}{\partial z_0} R_{n,a} = \sum_{k=1}^{\ell} (-1)^{k+1} \binom{n}{k} (k-1)! R_{n+k,a} \frac{\partial}{\partial z_0} \hat{X}_{-k,a}^{0}. 
$$

In terms of the coefficient functions $\Pi_t$ and $F_t$ (4.8) reads

$$
\mathcal{N}_t(J) \Pi_0 \equiv F_t = \Pi_t + \sum_{p=1}^{t} \nu^p \left( \prod_{i=0}^{p-1} \gamma_{t-i}(J) \right) (R_{-p}) \Pi_{t-p}. \quad \tag{4.13}
$$

The gauge transformation (4.8) keeps invariant the first and the last elements of the multiplet $\{F_t\}$, i.e., $F_0 = \Pi_0$, $F_{2j+1} = \Pi_{2j+1}$. For $t = 2j+1$ each term in the sum in (4.13) vanishes since $\gamma_{2j+1}(j) = 0$ so that $\Pi_{2j+1}$ (which is identically zero in our realization) can be identified with the singular vector, given explicitly by the l.h.s., i.e., $\mathcal{N}_{2j+1}(j) \Pi_0$. The splitting (4.13) of any $\Pi_t$, $t = 0, 1, \ldots$, as a sum of two terms, hold for arbitrary values of $J$ - then $\Pi$ and $F$ become infinite series while the r.h.s. of (4.4) and (4.5) should be replaced by zero.

5. For spins of the form $J = j - j'/\nu$ with $r = 2j + 1$, $r' = 2j' + 1 \in \mathbb{Z}_+$, $\nu \in \mathcal{G}$, $\nu \neq 0$, the KM Verma module of highest weight $2J$ contains a singular vector of weight $2(J - r)$ (this is part of the Kac-Kazhdan theorem [15]). There is an explicit expression for the singular vector $P_{(r,r')}|J\rangle$ given by the Malikov Feigin Fuchs (MFF) formula [11]:

$$
P_{(r,r')} = f_{1}^{r+1/(r-1)/\nu} f_{0}^{r-1/(r-2)/\nu} \ldots f_{1}^{r-1/(r-3)/\nu} f_{0}^{r-1/(r-2)/\nu} f_{1}^{r-1/(r-1)/\nu}, \quad \tag{5.1}
$$

where for short we denote $f_{1} = X_{0}^{-}$, $f_{0} = X_{-1}^{+}$. This expression is a monomial in $f_1, f_0$ but they are raised to, in general, complex powers. It can be viewed as a compact notation for a polynomial in $X_{0}^{-}, X_{n}^{\pm}$, $n = 1, 2, \ldots$, with coefficients that are polynomials in $1/\nu$. In practice, to obtain such an expanded form one can start from the middle of (5.1) where $f_0$ or $f_1$, depending on whether $r'$ is even or odd respectively, is raised to the integer power $r$. With the help of

$$
\begin{align*}
& f_{0}^{1+q} X_{m}^{-} f_{0}^{q} = f_{0}^{1+q} X_{m}^{-} + q f_{0}^{1} X_{m}^{0} + q(q-1) X_{m}^{-}, \quad f_{0}^{1+q} X_{m}^{-} f_{0}^{-q} = f_{0}^{1+q} X_{m}^{-} - 2q X_{m-1}, \\
& f_{1}^{1+q} X_{m}^{-} f_{1}^{q} = X_{m}^{+} f_{1}^{2} - q f_{1}^{1} X_{m}^{-} + q(q-1) X_{m}^{-}, \quad f_{1}^{1+q} X_{m}^{-} f_{1}^{-q} = X_{m}^{+} f_{1} + 2q X_{m}^{-} \quad \tag{5.2}
\end{align*}
$$

we can move from the middle of (5.1) outwards eliminating all noninteger powers. For a different algorithm to obtain the singular vectors using fusion see [16]. It is in principle also possible to write (5.1) as a homogeneous polynomial in the generators $f_1, f_0$, of degree $r r'$, and $r (r' - 1)$, respectively (see [17] for such explicit expression in the case $(r,r') = (r,2)$). Substituting in $P$ (to be denoted by $\mathcal{P}$) the realization (3.1) of the generators $f_0, f_1$ we impose the condition

$$
\mathcal{P}_{(r,r'),a} W(x, z) = 0, \quad \tag{5.3}
$$

which expresses the decoupling of the corresponding descendant of the primary field $\Phi^{a}(x_{a}, z_{a})$ – we can extend (3.1) setting $X_{0,a}^{-} W = 0, X_{n,a}^{\pm} W = 0$, for $n > 0, \alpha = 0, \pm$, $X_{0,a}^{q} W = 2J_{0} W$. **

** Alternatively, one can impose the algebraic condition on the state $\langle J_{n}, \Delta_{n} \rangle$; then the condition on $W$ looks like (5.3) but $X_{0}^{-}$ and $X_{+}^{\pm}$ in $\mathcal{P}_{(r,r')}$ are replaced by $\Delta(S^{+})$ and $\Delta(X_{-}^{\pm})$ respectively.

7
The algebraic condition (5.3) is equivalent to linear (with rational in $z$ coefficients) relations for the set of functions $\{W_t(x,z)\}$, or, equivalently for the integrals $\{J_\mu\}$. Namely for any $t \geq rr'$ we have

$$\frac{(X_0^-)^{rr'}}{(r(r'-1))!}W_t + \sum_{k=0}^{r(r'-1)-1} \frac{\nu^{r(r'-1)-k}}{k!} P_{(r,r')}^{(k)} W_{t-r(r'-1)+k} = 0, \quad P^{(k)} = \left( \frac{\partial}{\partial f_0} \right)^k P. \quad (5.4)$$

Note that $P^{(k)}$ is well defined, i.e., does not depend on the form chosen to represent $P$. Indeed, starting from $P$ written as a polynomial only in $f_1, f_0$ we can obtain any other form by multiple commutations. At each step we will use $f_0 X_{-n} = X_{-n} f_0 + X_0^0 n - 1$, $n \geq 0$, or $f_0 X_{-m} = X_0^0 m f_0 - 2X_m^0 m - 1, m \geq 1$, which remains an equality after “differentiating” in $f_0$ (it is important that $m \neq 0$). In particular, the highest derivative reproduces the first term in (5.4). In the thermal case $r' = 1$ this is the only term and the equation (5.3) is automatically satisfied since $W(x,z)$ is a polynomial of highest degree $r - 1 = 2j$ in $x$.

The easiest way to obtain (5.4) is to start, as in the derivation of the KZ equation (2.5) for $W_t$, from the “nonmeromorphic” $W_S$ and then “intertwine” it to a “meromorphic” one, i.e., consider

$$\Delta(X^-_1)^{S-q} P_{(r,r')} W_S = 0, \quad q = t - r(r' - 1) \geq r. \quad (5.5)$$

With the help of

$$[\Delta(X^-_1)^A, f_0^B] = \sum_{k=1}^\infty \left( \frac{\partial f_0^k}{\partial f_0^0} \right)_k \left( \prod_{j=0}^{k-1} (A - j)(2k - j - 2J - A - B - \Delta(S^0)) \right) \Delta(X^-_1)^{A-k} \quad (5.6)$$

computed with $f_0$ being realized as $X_+^1$, we can move $\Delta(X^-_1)$ through $P_{(r,r')}$. For example, to obtain the term with first “derivative” in (5.4) apply (5.6) to each factor $f_0^B_i$ of (5.1), $(B_i \equiv r - (r' - 2i)/\nu, i = 1, \ldots, r' - 1)$, keeping only the first term of (5.6). Commute $(\Delta(S^0) + B_i + \ldots)$ to the right through the powers of $X_+^1$, and $X^-_1$, and apply at the end the second equality in (2.3). The dependence on $i$ cancels and (5.5) becomes

$$0 = \mathcal{P} \Delta(X^-_1)^{S-q} W_S + (S-q)(c_0-q) \mathcal{P}' \Delta(X^-_1)^{S-q-1} W_S + \ldots$$

and it remains to use the recursion relation (2.4) to recover the “first derivative” term in (5.4). The higher “derivative” terms are obtained in a similar fashion.

It remains an open problem to prove, using the explicit expressions for the integrals $J_\alpha$ found in [4], that indeed the relations (5.4), or equivalently, the condition on the correlators (5.3) hold. For the time being we are able to check this in the simplest examples so the validity of (5.4) will be taken as an assumption.

The equality (5.4) implies that $J_t$ for every $t \geq rr'$ can be expressed in terms of $J_\alpha$ with $r \leq |\alpha| < t$. The important relation is the one for $t = rr'$, which for $r' > 1$ is transformed into the corresponding BPZ equation as we will argue below. Plugging the expressions for the right action into (5.4), (5.1) for $t = rr'$ the dependence on $x$ disappears; the first term reproduces $J_{rr'}$. For example, in the simplest nonthermal case $(r, r') = (1, 2)$, i.e., $2J_a = -1/\nu$, the linear relation is:

$$J_{2\epsilon_a} = (1 + \nu)(X_0^0 \cdot X^-_1 - \gamma_1(J) X^-_1) W_1 = (1 + \nu) \sum_{b(\neq a)} \frac{2J_a J_b J_c - 2J_b J_d J_a}{z_{ba}}. \quad (5.7)$$
Another example is provided by \((r, r') = (2, 2)\), i.e., \(2J = 1 - 1/\nu\):

\[
\Pi_4 + \nu \mathcal{P}^{(1)}_{(2, 2)} W_3 + \frac{\nu^2}{2} \mathcal{P}^{(0)}_{(2, 2)} W_2 = 0, \quad \mathcal{P}^{(1)}_{(2, 2)} W_3 = \frac{\gamma_4}{2}[\mathcal{X}_0^0 \Pi_3 - \frac{\gamma_3}{3}(\mathcal{X}_0^-)^2 \mathcal{X}_1^- W_3],
\]

in (5.8)

Recalling the realization of \(\mathcal{X}_0^0, \mathcal{X}_1^-\) in (3.1) we see that (5.8) is a linear combination of \(\Pi_\alpha\) for \(2 \leq |\alpha| \leq 4\).

It is convenient to rewrite in general the singular vector in “+0−”-ordered form with \(X^+\) to the left, \(X^0\) in the middle and \(X^-\) on the right. Then just by counting the degree in \(x\) one sees that for \(t = rr'\) in (5.4) all terms with \(X^+\) give zero and we are left with terms of the type “0−” as in (5.7), (5.8). Now solving (5.4) for the first term \(\Pi_{rr'}\) and inserting into (4.2) (or (4.13)) one effectively truncates the infinite KZ system. Exploiting repeatedly (3.7) for \(t < rr'\) (or the relations (4.1), (4.2), (4.13) derived as a consequence of it) it can be furthermore reduced to higher \((rr' - )\) order partial differential equation for \(\Pi_0\) – a BPZ-type equation, accounting for the decoupling of the singular vector of the Virasoro module \(V_{(k), h(J)}\). With the help of (3.7), (4.7) these equations can be written in terms of the generators \(\mathcal{L}^{(h)}_{-n}\) acting on the minimal model integral \(\Pi_0\). It seems natural to expect, although technically difficult to prove, that the rest terms, similarly to the gauge degrees of freedom in Proposition 1, cancel in the final equation for \(\Pi_0\). Thus we conjecture the following

**Proposition 2.** For any \(2J + 1 = r - (r' - 1)/\nu\), \(\nu \neq 0, \infty\), and \(rr' - 1 \in \mathbb{N}\), the set of equations (5.4) for \(t = rr'\), accounting for the decoupling of the KM null vector, and (3.7) for \(t = 0, 1, ..., rr' - 1\), equivalent to the KZ system, determine the singular vector at level \(rr'\) of the Virasoro Verma module \(V_{(k), h(J)}\).

We recall that in the thermal case \(r' = 1\) the KZ system (3.7) itself reduces to the corresponding Virasoro singular vector, while the KM algebraic condition (5.4) is trivially satisfied, i.e., in this case the two conditions decouple.

Let us illustrate the mechanism of reduction on the examples considered above. Using the splitting (4.13) for \(\Pi_2\) and (3.7), (4.7) to convert \(X^-_{-1}\) we obtain that (5.7) is equivalent to

\[
\mathcal{N}_2(J) \Pi_0 \equiv \left(\mathcal{L}^{(h)}_{-1}\right)^2 - \nu \gamma_1(J) \mathcal{L}^{(h)}_{-2} \Pi_0 = -(1 + \nu) \gamma_1(J) \mathcal{L}^{(h)}_{-2} \Pi_0,
\]

Indeed, apart from the term proportional to \(\mathcal{L}^{(h)}_{-2} \Pi_0\), the r.h.s of (5.7) exactly compensates the term containing gauge degrees of freedom in the r.h.s. of (4.13). Taking into account that \(\gamma_1(J) = 2J = 1 - 1/\nu\) (5.9) recovers the corresponding Vir singular vector.

In the next example \((r, r') = (2, 2)\), i.e., \(2J = 1 - 1/\nu\), let us neglect for the time being all terms in (5.8), which in the “0−”-ordered form start with some \(X^-_{-n}\), i.e.,

\[
\Pi_4 = \frac{\nu}{3!} \gamma_4 \gamma_3 (X_0^-)^2 \mathcal{X}_1^- W_3 - \frac{\nu^2}{4!} \gamma_4 \gamma_3 \gamma_2 (2X_0^- \mathcal{X}_2^- W_2 - \gamma_1(X^-_{-1})^2) W_2 \approx 0.
\]

We now apply the KZ equation in the form (3.7) repeatedly until we turn all \(X^-_{-n}\) into \(K_{-n-1}\). For the terms of the form \((X_0^-)^n \mathcal{X}_k^- W_{n+1}\) we can use directly (4.1). For the term involving \((X^-_{-1})^2\) we need the commutator

\[
[X_{-1}, K_{-2}] = -(\nu 2J + 1) X_{-3} - \nu (X^-_{-3} X_0^- + X^-_{-2} X_{-1}^- - X^-_{-1} X^-_{-2}),
\]
so that \((X^-_1)^2 W_2 \approx (K^{c2}_2 - \nu K_{-4}) I_0\) up to the neglected terms. Using (4.7) keep for any \(K_{-n}\)
in the l.h.s. of (5.10) only the term \(L^{(h)}_{-n}\) while for the integrals \(I_0\) one has to use the splitting (4.13) putting aside the second term in the r.h.s of (4.13). Finally a tedious computation shows that all the neglected terms combine and give zero, so that we get

\[
\left[ L^{(h)}_{-1} + (\nu - \frac{1}{\nu})^2 L^{(h)}_{-2} + 2(\nu - \frac{1}{\nu})L^{(h)}_{-2} L^{(h)}_{-1} + 2(3 - \nu - \frac{1}{\nu})L^{(h)}_{-3} L^{(h)}_{-1} - 3\nu(1 - \frac{1}{\nu})^2 L^{(h)}_{-4} \right] I_0 = 0 ,
\]

(5.11)

The l.h.s. of (5.11) recovers the expression for the Virasoro singular vector at level 4 of the Virasoro Verma module \(V_{c(k)h_2,x}\).

The procedure illustrated on these examples is in fact an algorithm, which can be implemented into a computer code (e.g., we have used REDUCE). As generators of the subalgebra \(A_-\) we choose \(X_0^-, X^a_0\) and the “improved” Vir generators \(L^{(h)}_{-n} \equiv L_{-n} + \frac{(n-1)}{2} X^{-n}_0\) – which reduce to \(L^{(h)}_{-n}\) on \(x\)-independent functions in the realization for \(L_{-n}\) and \(X^{-n}_0\) given by (3.2), (3.1). Let us say that a monomial in these generators is ordered if it has the form \(X^+ \ldots X^0 \ldots L^{(h)} \ldots X^- \ldots\). The instructions we have to provide are: 1) using the commutation relations order any expression, 2) use (3.7), i.e., substitute \(X_{-n+1}^- W_{t+1}\) by \(L^{(h)}_{-n} - \frac{(n-1)}{2} X^{-n}_0/2 - M_{-n}\) and \(X_{-n+1}^- W_{0}\) by 0. Taking a particular KM singular vector we feed (5.4) into the program and let it apply the instructions. It can be easily seen that the ordered terms containing \(X^+\) actually do not survive. One can speed up the algorithm using along with (3.7) its consequences – the relations (4.1), (4.2) or (4.13). We have checked the above algorithm on several examples. The result is a polynomial in the \(L^{(h)}_{-n}\) acting on \(W_0\) – all “gauge” terms, i.e., containing \(X^0\), cancel. Extending the negative grade subalgebra by setting \(L^{(h)}_{0,a} W_0 = h(J_a) W_0\), \(L^{(h)}_{n} W_0 = 0\) for \(n > 0\), and assuming the standard Vir commutation relations with central charge \(c(k)\) it is straightforward to check that \(L^{(h)}_{1,a}\), \(L^{(h)}_{1,b}\) annihilate the final result.

The proof of Proposition 2 could be simplified if we were able to rewrite (5.4) for \(t = rr'\) in a matrix form which can be combined with the matrix form of the KZ equation. As in the proof of the Proposition 1 above this would allow to handle the gauge degrees of freedom and probably to obtain more explicit expressions for the Virasoro singular vectors. The matrix representation of [16] might be of use if rewritten in the basis containing \(I_0\) exploited here.

There is one special case in which such a matrix form appears naturally as a consequence of the duality discussed in [4]. Namely, in the “quasithermal” case, described by \(r = 1\) (i.e., \(J = -j'(k + 2)\)) one can use the dual WZNW theory – defined in general by replacing \(J \rightarrow \bar{J} = -j/(k + 2), (k + 2) \rightarrow (k + 2) = 1/(k + 2)\), which keeps invariant \((c(k), h(J))\). In this case the dual theory is a thermal one, with \(J = j'\), and since the integral \(I_0\) is the same in both theories one can compare the corresponding KZ systems. This produces a linear relation for the quasithermal integrals \(I_{a}, |a| \leq 2j' + 1\) which can be rewritten in a matrix form in terms of the two sets of vectors \(\{I_{t}, t = 0, 1, \ldots, 2j' + 1\}\) and \(\{I_{t}, t = 0, 1, \ldots, 2j'\}\), using the matrix differential operators \(\mathcal{K}, \mathcal{\bar{K}}\) (4.4) and the gauge transformations \(g, \bar{g}\).

6. We end with a few remarks. The marked difference between the thermal \((r' = 1)\) and nonthermal cases appears also in [5] where the thermal BRST cohomology is trivial *** and in [18] where going from KM characters one takes a limit in the thermal case and a residue in the nonthermal one to recover the Virasoro characters.

Clearly the mechanism of reduction described here is essentially different from the standard procedure discussed in [5] which requires fermions to implement the action of the BRST operator

*** We thank P. Bouwknegt for emphasizing this fact to us.
on Fock modules. Nevertheless, as discussed in [4], there is certainly some analogy in the role played by the operator $\Delta(X_1^-)$ and the standard BRST operator used to impose the constraint in [5].

Although the result in the general nonthermal case lacks so far the rigour and the explicitness of the thermal one it looks conceptually important that the problem of finding the singular vectors of the Virasoro Verma modules is reduced to the analogous simpler problem for the KM algebra $A_1^{(1)}$. Generalizations seem possible since the singular vectors for higher rank algebras are also known [11] and there are natural analogues of the “intertwining” operator $\Delta(X_1^-)$ in the various reduction schemes discussed in [5], [19]. In any case, the generalization of the result (Proposition 1) for the thermal subseries, i.e., dominant integral weights, should be easier.

Finally it would be interesting to find a connection between the algorithm discussed above and the recently proposed [20] compact (though rather formal) general formula for the Virasoro singular vectors, inspired by the MFF expression.

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