CUPLENGTH ESTIMATES FOR LAGRANGIAN INTERSECTIONS-REVISITED

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ABSTRACT. In this note we give a novel proof of Arnold’s conjecture for the zero section of a cotangent bundle of a closed manifold. The proof is based on some basic properties of Lagrangian spectral invariants.

1. INTRODUCTION

Let $M$ be an $n$-dimensional closed manifold. We denote by $\omega$ the canonical symplectic structure on the cotangent bundle $T^*M$, which is given by $\omega = -d\theta$ with the Liouville one-form $\theta = pdq$. Given $H \in C^\infty([0,1] \times T^*M)$, the Hamiltonian vector field $X_H$ is determined by $dH = -\omega(X_H, \cdot)$. The flow of $X_H$ is denoted by $\varphi^t_H$ and its time-one map by $\varphi_H := \varphi^1_H$. Denote by $\Ham_c(M, \omega)$ the set of all Hamiltonian diffeomorphisms with compact support. Clearly, for any asymptotically constant Hamiltonian $H$ we have $\varphi_H \in \Ham_c(M, \omega)$. Fix a ground field $\mathbb{F}$, e.g., $\mathbb{Z}_2$, $\mathbb{R}$, or $\mathbb{Q}$. The singular homology of a topological space $X$ with coefficients in $\mathbb{F}$ is denoted by $H_\ast(X)$. The $\mathbb{F}$-cuplength $\text{cl}(M)$ of $M$ is by definition the maximal integer $k$ such that there exist homology classes $u_1, \ldots, u_{k-1}$ in the homology $H_\ast(M)$ as a ring (with the intersection product) with $\deg(u_i) < \dim(M)$ such that

$$u_1 \cap \cdots \cap u_{k-1} \neq 0,$$

where $\cap$ denotes the intersection product.

The goal of this note is to give a new proof of the cuplength estimate:

**Theorem 1.** Let $O_M$ denote the zero section of $T^*M$. Then we have

$$\sharp(\varphi(O_M) \cap O_M) \geq \text{cl}(M), \quad \forall \varphi \in \Ham_c(M, \omega).$$

The above estimate is a special case of the Arnold conjectures [1]. This case has already been solved by different approaches, for instance, by Chaperon [2] for the cotangent bundle of torus using variational methods given by Conley and Zehnder [4], and for general cotangent bundles by Hofer [10] applying Lyusternik-Shnirelman category theory, and Laudenbach and Sikorav [13] employing a finite-dimensional method of “broken extremals”.

**Remark 1.** The method of our proof of Theorem 1 is making good use of the properties of the Lagrangian spectral invariants defined by Oh [16, 17].

**Remark 2.** By a modification of the method used here, one can prove a slightly general case of Arnold’s conjectures: If $L$ is a closed Lagrangian submanifold of a smooth tame symplectic manifold $(P, \omega)$ satisfying $\pi_2(M, L) = 0$, then for any $\varphi \in \Ham_c(P, \omega)$, $L \cap \varphi(L)$ has at least $\text{cl}(L)$ many points. This case was independently proved by Floer [7] and Hofer [11]. For the case that $L$ is a closed monotone Lagrangian submanifold of of a smooth tame symplectic manifold $P$, we refer to [9] for partial results about the Arnold conjecture on Lagrangian intersections in degenerate (non-transversal) sense.
2. Spectral invariants

2.1. The minmax critical values. Let \( X \) be a closed \( n \)-dimensional manifold and let \( f \in C^\infty(X) \). For any \( \mu \in \mathbb{R} \) we put

\[
X^\mu := \{ x \in X | f(x) < \mu \}.
\]

To a non-zero singular homology class \( \alpha \in H_*(X) \), we associate a numerical invariant by

\[
c_{LS}(\alpha, f) = \inf \{ \mu \in \mathbb{R} | \alpha \in \text{Im}(i^\mu_*) \},
\]

where \( i^\mu : H_*(X^\mu) \to H_*(X) \) is the map induced by the natural inclusion \( i^\mu : X^\mu \to X \).

This number is a critical value of \( f \). The function \( c_{LS} : H_*(X) \setminus \{ 0 \} \times C^\infty(X) \) is often called a minmax critical value selector. The following proposition summarizing the properties of the resulting function, which can be easily extracted from the classical Ljusternik–Schnirelman theory, see, e.g., \([3, 12, 19, 5, 8]\).

Proposition 3. The minmax critical value selector \( c_{LS} \) satisfies the following properties.

1. \( c_{LS}(\alpha, f) \) is a critical value of \( f \), and \( c_{LS}(k\alpha, f) = c_{LS}(\alpha, f) \) for any nonzero \( k \in \mathbb{R} \).
2. \( c_{LS}(\alpha, f) \) is Lipschitz in \( f \) with respect to the \( C^0 \)-topology.
3. Let \( [pt] \) and \( [X] \) denote the class of a point and the fundamental class respectively. Then

\[
c_{LS}([pt], f) = \min f \leq c_{LS}(\alpha, f) \leq \max f = c_{LS}([X], f).
\]

4. \( c_{LS}(\alpha \cap \beta, f) \leq c_{LS}(\alpha, f) \) for any \( \beta \in H_*(X) \) with \( \alpha \cap \beta \neq 0 \).
5. If \( \beta \neq [X] \) and \( c_{LS}(\alpha \cap \beta, f) = c_{LS}(\alpha, f) \), then the set \( \Sigma = \{ x \in \text{Crit}(f) | f(x) = c_{LS}(\alpha, f) \} \) is homologically non-trivial.

Here a subset \( S \) of a topological space \( X \) is called homologically non-trivial in \( X \) if for every open neighborhood \( U \) of \( S \) the map \( i_* : H_k(U) \to H_k(X) \) induced by the inclusion \( i : U \hookrightarrow X \) is non-trivial.

2.2. The Lagrangian spectral invariants. In this subsection we briefly recall the construction of Lagrangian spectral invariants for Hamiltonian diffeomorphism mainly following Oh \([16, 17]\), see also \([15, 18]\). Denote \( \mathcal{H}_{ac} \) the set of Hamiltonians \( H \in C^\infty([0, 1] \times T^*M) \) which are asymptotically constant at infinity.

For \( H \in \mathcal{H}_{ac} \), the action functional is defined as

\[
A_H(\gamma) = \int_0^1 H(t, \gamma(t))dt - \int \gamma^*\theta
\]

on the space of paths in \( T^*M \)

\[
\mathcal{P} = \{ \gamma : [0, 1] \to T^*M | \gamma(0), \gamma(1) \in O_M \}.
\]

Set \( L = \varphi^*_H(O_M) \). We define the Lagrangian action spectrum of \( H \) on \( T^*M \) by

\[
\text{Spec}(L, H) = \{ A_H(\gamma) | \gamma \in \text{Crit}(A_H) \}.
\]

This is a compact subset of \( \mathbb{R} \) of measure zero, see for instance \([16]\).

Given a generic \( H \in \mathcal{H}_{ac} \), the intersection \( \varphi^*_H(O_M) \cap O_M \) is transverse and hence \( \text{Crit}(A_H) \) is finite. There is an integer-valued index, called the Maslov-Viterbo index, \( \mu_{MV} : \text{Crit}(A_H) \to \mathbb{Z} \) which is normalized so that if \( H : T^*M \to \mathbb{R} \) is a lift of a Morse function \( f \) then \( \mu_{MV} \) coincides with the Morse index of \( f \).
Denote by $CF_k^<a(L, H)$, where $a \in (-\infty, \infty]$ is not in $\text{Spec}(L, H)$, the vector space of formal sums
\[ \sum_{x_i \in \mathcal{P}} \sigma_i x_i, \]
where $\sigma_i \in \mathbb{F}$, $\mu_M(x_i) = k$ and $\mathcal{A}_H(x_i) < a$. The graded $\mathbb{F}$-vector space $CF_k^<a(L, H)$ has the Floer differential counting the anti-gradient trajectories of the action functional in the standard way whenever a time-dependent almost complex structure compatible with $\omega$ is fixed and the regularity requirements are satisfied, see i.e., [6, 16]. As a consequence, we have a filtration of the total Lagrangian Floer complex $CF_\ast(L, H) := CF_{[-\infty, \infty]}(L, H)$. Since the resulting homology, the filtered Lagrangian Floer homology of $H$, does not depend on $H \in \mathcal{H}_{ac}$ (due to continuation isomorphisms), one can extend this construction to all asymptotically constant Hamiltonians. Let $H \in \mathcal{H}_{ac}$ be an arbitrary Hamiltonian and let $a$ be outside of $\text{Spec}(L, H)$. We define
\[ HF^<a(L, H) = HF^<a(L, \tilde{H}), \]
where $\tilde{H}$ is a small perturbation of $H$ so that $\varphi_H^\ast(O_M) \cap O_M$ is transverse. It is not hard to see that $HF^<a(L, \tilde{H})$ is independent of $\tilde{H}$ provided that $\tilde{H}$ is sufficiently close to $H$.

We denote by $i^a_\ast : HF^<a(L, H) \rightarrow HF_\ast(L, H)$ the induced inclusion maps. It is well known that for $H_f = \pi^\ast f$ where $\pi : T^*M \rightarrow M$ is the projection map and $f$ is a Morse function on $M$, $HF_\ast(L, H_f)$ is canonically isomorphic to the singular homology $H_\ast(M)$, and hence $H_\ast(M) \cong HF(L, H)$ for all $H \in \mathcal{H}_{ac}$. Using this identification, for $\alpha \in H_\ast(M)$ and $H \in \mathcal{H}_{ac}$ we define
\[ \ell(\alpha, H) = \inf \{ a \in \mathbb{R} \setminus \text{Spec}(L, H) | \alpha \in \text{Im}(i^a_\ast) \}. \]

By convention, we have $\ell(0, H) = -\infty$.

**Proposition 4.** The Lagrangian spectral invariant $\ell : H_\ast(M) \setminus \{0\} \times \mathcal{H}_{ac} \rightarrow \mathbb{R}$ has the following properties:

(a) $\ell(\alpha, H) \in \text{Spec}(L, H)$, in particular it is a finite number.
(b) $\ell$ is Lipschitz in $H$ in the $C^0$-topology.
(c) $\ell([M], H) = -\ell([pt], \overline{T}(t, H))$ with $\overline{T}(t, H) = -H(-t, H)$.
(d) $\ell([pt], H) \leq \ell(\alpha, H) \leq \ell([M], H)$ for all $\alpha \in H_\ast(M) \setminus \{0\}$.
(e) $\ell(\alpha, H) = \ell(\alpha, K)$, when $\varphi_H = \varphi_K$ in the universal covering of the group of Hamiltonian diffeomorphisms, and $H, K$ are normalized.
(f) $\ell(\alpha \cap \beta, H^aK) \leq \ell(\alpha, H) + \ell(\beta, K)$, where $(H^aK)(t, x) = H(t, x) + K(t, (\varphi_H^t)^{-1}(x))$.
(g) If $\varphi_H(O_M) = \varphi_K(O_M)$, then there exists $C \in \mathbb{R}$ such that $\ell(\alpha, H) = \ell(\alpha, K) + C$ for all $\alpha \in H_\ast(M) \setminus \{0\}$.
(h) Let $f : M \rightarrow \mathbb{R}$ be a smooth function, and let $H_f : T^*M \rightarrow \mathbb{R}$ denote a compactly supported Hamiltonian so that $H_f = f \circ \pi$ on a ball bundle $T^*_R M := \{(q, p) \in T^* M | |p| \leq R \}$ containing $L_1 := \{(q, \partial_q f(q)) \in T^* M | q \in M \}$, and $H_f = 0$ outside $T^*_R M$ in $M$, where $| \cdot |$ is the norm induced by a metric $p$ on $M$, and $\pi : T^* M \rightarrow M$ is the natural projection map. Then $\ell(\alpha, H_f) = c_{LS}(\alpha, f)$ for all $\alpha \in H_\ast(M) \setminus \{0\}$.

3. THE PROOF OF THE MAIN THEOREM

Our main theorem follows immediately from the following lemma.
Lemma 5. Let $H \in \mathcal{H}_{ac}$ and $\alpha, \beta \in H_*(M) \setminus \{0\}$ with $\deg(\alpha) < n$. If the intersections of $O_M$ and $\varphi_H(O_M)$ are isolated, then

$$\ell(\alpha \cap \beta, H) < \ell(\beta, H).$$

Proof. Since the intersections of $O_M$ and $\varphi_H(O_M)$ are isolated, we can pick a small open neighborhood $U$ of $O_M \cap \varphi_H(O_M) = \{0\}$ so that $H_k(U) = 0$ for all $k > 0$. Let $f : M \to \mathbb{R}$ be a $C^2$-small function such that $f = 0$ on $U$ and $f < 0$ on $M \setminus U$, and let $H_f$ be the lift of $f$ to the cotangent bundle $T^*M$ as in Proposition 4(h). We claim that for any $\alpha \in H_{<n}(M)$, it holds that

$$\ell(\alpha, H_f) < 0. \quad (3.1)$$

For this end, we first prove that $c_{LS}(\alpha, f) < 0$ for all $\alpha \in H_{<n}(M)$. In fact, if there exists a homology class $\alpha_1 \in H_{<n}(M)$ such that $c_{LS}(\alpha_1, f) = 0$, then we have $c_{LS}(\alpha_1 \cap [M], f) = c_{LS}(\alpha_1, f) = 0$. It follows from Proposition 3(3) that $c_{LS}([M], f) = \max_{H_f} f = 0$. So we have $c_{LS}(\alpha_1 \cap [M], f) = c_{LS}([M], f)$ with $\alpha_1 \in H_{<n}(M) \setminus \{0\}$. Then by Proposition 3(5) the zero level set $O$ of $f$ is homologically non-trivial – a contradiction. Therefore, for any $\alpha \in H_{<n}(M)$, we have $c_{LS}(\alpha, f) < 0$. This, together with Proposition 4(h), yields $\ell(\alpha, H_f) < 0$.

Next we show that for sufficiently small $\varepsilon > 0$

$$\ell(\alpha \cap \beta, H) = \ell(\alpha \cap \beta, \varepsilon H_f \sharp H). \quad (3.2)$$

Observe that $\varphi_{H_f}^t(q, p) = (q, p + t \partial_q f(a)) \in T^*M$ for $t \in [0, 1]$ and $(q, p) \in T^*_R M$. Set $L_R^H = \varphi_H^1(O_M) \cap T^*_R M$. Then we have

$$\varphi_{H_f}(L_R^H) = \{(q, p + \varepsilon d_f(q))(q, p) \in L_R^H\}. \quad (3.3)$$

Since $L_R^H \cap \pi^{-1}(O_M \setminus U)$ is compact and has no intersections with $O_M$, we deduce that for small enough $\varepsilon > 0$, $\varphi_{H_f}(L_R^H) \cap \pi^{-1}(O_M \setminus U)$ has no intersections with $O_M$ as well. For $(q, p) \in T^*M$ with $R \leq \|(q, p)\| \leq R + 1$ we have

$$d_g(\varphi_{H_f}(q, p), (q, p)) \leq \left\| \int_0^1 \frac{d}{dt} \varphi_{H_f}^t(q, p) \right\| \leq \varepsilon \sup_{R \leq \|(q, p)\| \leq R+1} \|X_{H_f}\|,$$

where $d_g$ is the distance function induced by some Riemannian metric $g$ on $M$. Therefore, for sufficiently small $\varepsilon > 0$, $\varphi_{H_f}(T^*_{R+1} M \setminus T^*_R M)$ does not intersect $O_M$. Note that the Hamiltonian diffeomorphism $\varphi_{H_f}$ is supported in $T^*_{R+1} M$, we conclude that $\varphi_{H_f} \varphi_H(O_M) \cap \pi^{-1}(O_M \setminus U)$ does not intersect $O_M$ provided that $\varepsilon > 0$ is sufficiently small. On the other hand, we have that $\varphi_{H_f} \varphi_H(O_M) \cap \pi^{-1}(U) = \varphi_H(O_M) \cap \pi^{-1}(U)$ because $f = 0$ on $U$. So if $\varepsilon > 0$ is sufficiently small then the Lagrangians $\varphi_{H_f} \varphi_H(O_M)$ and $\varphi_H(O_M)$ have the same intersections with $O_M$. A direct calculation shows that for every such intersection point, the two action values corresponding to $\varepsilon H_f \sharp H$ and $H$ are the same. Indeed, there is a one-to-one correspondence between the set $\varphi_H(O_M) \cap O_M$ and the set $\mathcal{P}(H) := \{x \in \mathcal{P} | \dot{x} = X_H(x(t))\}$ of Hamiltonian chords by sending $q \in \varphi_H(O_M) \cap O_M$ to $x = \varphi_{H_f}^{-1}(q)$. So we get a bijective map defined by

$$\Upsilon : \text{Crit}(A_H) \to \text{Crit}(A_{\varepsilon H_f \sharp H}), \quad x(t) \mapsto \varphi_{H_f}^t(x(t)).$$

Notice that the Hamiltonian flow has the following property

$$(\varphi_{H_f})^* \theta - \theta = dF_t,$$
where the function $F : [0, 1] \times T^* M \to \mathbb{R}$ is given by $F_t = \int_0^1 (\theta(X_{\varepsilon H_f}) - \varepsilon H_f) \circ \varphi_{\varepsilon H_f}^s ds$, see for instance [14, Proposition 9.19]. As a consequence, for any $x \in \mathcal{P}(H)$ we have

$$\frac{d}{dt} F_t(x(t)) = dF_t(\dot{x}(t)) + (\theta(X_{\varepsilon H_f}) - \varepsilon H_f) \circ \varphi_{\varepsilon H_f}^t(x(t)),$$

which implies

$$(\varphi_{\varepsilon H_f})^* \theta(\dot{x}(t)) = \theta(\dot{x}(t)) + \frac{d}{dt} F_t(x(t)) - (\theta(X_{\varepsilon H_f}) - \varepsilon H_f) \circ \varphi_{\varepsilon H_f}^t(x(t)).$$

Then we compute

$$A_{\varepsilon H_f}(\mathcal{Y}(x(t))) = \int_0^1 \varepsilon H_f(\varphi_{\varepsilon H_f}^t(x(t))) dt + \int_0^1 H_t \circ (\varphi_{\varepsilon H_f}^t)^{-1}(\varphi_{\varepsilon H_f}^t(x(t))) dt - \int_0^1 \theta(\varphi_{\varepsilon H_f}^t(x(t))) dt$$

$$= \int_0^1 \varepsilon H_f(\varphi_{\varepsilon H_f}^t(x(t))) dt + \int_0^1 H_t(x(t)) dt - \int_0^1 (\varphi_{\varepsilon H_f})^* \theta(\dot{x}(t)) dt$$

$$= \mathcal{A}_H(x(t)) + \int_0^1 \varepsilon H_f(\varphi_{\varepsilon H_f}^t(x(t))) dt - \frac{d}{dt} F_t(x(t)) dt$$

$$+ \int_0^1 (\theta(X_{\varepsilon H_f}) - \varepsilon H_f) \circ \varphi_{\varepsilon H_f}^t(x(t)) dt - (\theta(X_{\varepsilon H_f}) \circ \varphi_{\varepsilon H_f}^t(x(t))) dt$$

$$= \mathcal{A}_H(x(t)) + F_t(x(1)) - F_t(x(0))$$

$$= \mathcal{A}_H(x(t)),$$  

(3.3)

where in the last equality we have used the fact that the value of an autonomous Hamiltonian $H_f$ is constant along its Hamiltonian flow, and $f = 0$ on $U$ which contains $x(1)$. Therefore, the action spectra $\text{Spec}(L, \varepsilon H_f^s \mathbb{R})$ and $\text{Spec}(L, H_f^s \mathbb{R})$ are the same. Now fix a sufficiently small $\varepsilon > 0$ and consider the family of Lagrangians $\varphi_{\varepsilon H_f^s \mathbb{R}} \varphi_{\varepsilon H_f^s \mathbb{R}}(O_L)$ with $s \in [0, 1]$. As before, the action spectra $\text{Spec}(L, s \varepsilon H_f^s \mathbb{R})$, $s \in [0, 1]$ are all the same. Since the action spectrum is a closed nowhere dense subset of $\mathbb{R}$, it follows from Proposition 4(b) that $\ell(\alpha \cap \beta, s \varepsilon H_f^s \mathbb{R})$ do not depend on $s$. So we have $\ell(\alpha \cap \beta, H) = \ell(\alpha \cap \beta, \varepsilon H_f^s \mathbb{R})$.

Combining (3.1) and (3.2), it follows from Proposition 4(f) that

$$\ell(\alpha \cap \beta, H) = \ell(\alpha \cap \beta, \varepsilon H_f^s \mathbb{R}) \leq \ell(\alpha, \varepsilon H_f^s) + \ell(\beta, H) < \ell(\beta, H).$$

This completes the proof of the lemma. 

The proof of Theorem 1. Without loss of generality we may assume that the intersections of $O_M$ and $\varphi_H(O_M)$ are isolated, otherwise, nothing needs to prove. Set $cl(M) = k + 1$. By definition there exist $u_i \in H_{<\varepsilon}(M)$, $i = 1, \ldots, k$ such that $u_1 \cap \cdots \cap u_k = [pt]$. We put

$$[M] = \alpha_0, \alpha_1, \ldots, \alpha_k \in H_k(M), \quad \alpha_i = u_{k-i+1} \cap \alpha_{i-1}.$$

For any $\varphi \in \mathcal{H}_{\text{crit}}(M, \omega)$, there exists a Hamiltonian $H \in \mathcal{H}_{\text{ac}}$ such that $\varphi = \varphi_{\varepsilon H_f}^1$. It follows from Lemma 5 and Proposition 4(a) that there exist $k + 1$ elements $x_i \in \text{Crit}(A_H)$ such that

$$\ell(\alpha_k, H) = \mathcal{A}_H(x_k) < \ell(\alpha_{k-1}, H) = \mathcal{A}_H(x_{k-1}) < \cdots < \ell(\alpha_0, H) = \mathcal{A}_H(x_0).$$
Hence, all $x_i$, $i = 0, \ldots, k$ are different. The one-to-one correspondence between the intersection points of $O_M$ and $\varphi_H^1(O_M)$ and the critical points of $A_H$ concludes the desired result.

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