RIGHT-ANGLED COXETER QUANDLES 
AND POLYHEDRAL PRODUCTS

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ABSTRACT. To a Coxeter group $W$ one associates a quandle $X_W$ from which one constructs a group $\text{Ad}(X_W)$. This group turns out to be an intermediate object between $W$ and the associated Artin group. By using a result of Akita, we prove that $\text{Ad}(X_W)$ is given by a pullback involving $W$, and by using this pullback, we show that the classifying space of $\text{Ad}(X_W)$ is given by a space called a polyhedral product whenever $W$ is right-angled. Two applications of this description of the classifying space are given.

1. INTRODUCTION

A quandle is a set with a binary operation satisfying three conditions. Since these three conditions are thought of as algebraic abstraction of the Reidemeister moves of knots, a quandle has been intensively studied in low dimensional topology. The three conditions are also thought of as axiomatization of conjugation in a group, so it has been studied in representation theory as well. In this paper, we comprehend a quandle in the latter sense.

We look at the following connection between groups and quandles. Any conjugation closed subset $X$ of a group $G$ can be regarded as a quandle by a binary operation given by conjugation, and to any quandle $Y$ one associates a group $\text{Ad}(Y)$ which is called the adjoint group. Thus any conjugation closed subset $X$ of a group yields a new group $\text{Ad}(X)$.

We now consider a Coxeter group $W$. The set of reflections $X_W$ of $W$ is closed under conjugation, so $X_W$ is a quandle which we call the Coxeter quandle associated with $W$. Then we get a group $\text{Ad}(X_W)$ which is the object to study in this paper. The adjoint group $\text{Ad}(X_W)$ has been studied mainly in connection with representation theory [2, 8], and there are few results on its topology [12]. This paper studies the topology of $\text{Ad}(X_W)$ by applying the recent result of Akita [1], and in particular, we show that if $W$ is right-angled, then the classifying space of $\text{Ad}(X_W)$ is given by a space which is called a polyhedral product.

The following fundamental property of $\text{Ad}(X_W)$ suggests that $\text{Ad}(X_W)$ possibly gives a new direction for the study of Coxeter groups and Artin groups. The symmetric group $\Sigma_n$ of $n$ letters is a Coxeter group and its associated Artin group is the braid group $B_n$ of $n$ strands. Then there is a natural epimorphism $B_n \rightarrow \Sigma_n$. In [2], it is shown that $\text{Ad}(X_{\Sigma_n})$ is an intermediate object between $\Sigma_n$ and $B_n$ in the sense that the epimorphism $B_n \rightarrow \Sigma_n$ factors as the composite of epimorphisms $B_n \rightarrow \text{Ad}(X_{\Sigma_n}) \rightarrow \Sigma_n$. Akita generalized this
result to an arbitrary Coxeter group. Let $A_W$ be the Artin group associated with a Coxeter group $W$.

**Theorem 1.1.** For an arbitrary Coxeter group $W$, the epimorphism $A_W \to W$ factors as the composite of epimorphisms

$$A_W \to \text{Ad}(X_W) \to W.$$  

Akita [1] studied further structures of $\text{Ad}(X_W)$ and generalized Eisermann’s result [8] on $\text{Ad}(X_{\Sigma n})$ concerning abelianization. From this, we will deduce that $\text{Ad}(X_W)$ is given by a certain pullback involving $W$ and its abelianization.

When $W$ is right-angled, it is known that the classifying spaces of $W$ and $A_W$ are given by polyhedral products. Then it is natural to ask whether the classifying space of $\text{Ad}(X_W)$ is a polyhedral product or not, if $W$ is right-angled. We will give an affirmative answer to this question by using the above pullback description of $\text{Ad}(X_W)$. Then we will show two application of this description of the classifying space: a stable splitting of the classifying space of $\text{Ad}(X_W)$ and calculation of the mod 2 cohomology of $\text{Ad}(X_W)$.

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2. **Recollection on Coxeter groups**

Recall that a pair $(W, S)$ of a group $W$ and a set $S$ is called a Coxeter system if there is given a map $m : S \times S \to \mathbb{N} \cup \{\infty\}$, called the Coxeter matrix, satisfying the following conditions:

1. $m(s, t) = m(t, s)$ for any $s, t$;
2. $m(s, t) = 1$ if and only if $s = t$;
3. $W$ is defined by the presentation

$$W = \langle s \in S \mid (st)^{m(s,t)} = 1 \text{ for } m(s, t) < \infty \rangle.$$  

We call the group $W$ a Coxeter group, by which we often mean the Coxeter system $(W, S)$ too.

The Artin group associated with a Coxeter system $(W, S)$ is defined by

$$A_W = \langle a_s \mid s \in S \mid a_s a_t a_s \ldots = a_t a_s a_t \ldots \text{ for } 2 \leq m(s, t) < \infty \rangle.$$  

Since the Coxeter group is alternatively presented as

$$W = \langle s \in S \mid s t s \ldots = t s t \ldots \text{ for } m(s, t) < \infty \rangle,$$
one gets:

**Proposition 2.1.** For a Coxeter system \((W, S)\), the assignment

\[ \pi: A_W \to W, \quad a_s \mapsto s \quad (s \in S) \]

is a well-defined epimorphism.

**Example 2.2.** The symmetric group of \(n\) letters \(\Sigma_n\) is a Coxeter group with the generating set \(\{\sigma_1, \ldots, \sigma_{n-1}\}\), where \(\sigma_i\) is the transposition \((i \ i+1)\). Indeed, \(\Sigma_n\) is generated by \(\sigma_1, \ldots, \sigma_{n-1}\) subject to the relations \(\sigma_i^2 = 1, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\) for \(i = 1, \ldots, n-1\) and \(\sigma_i\sigma_j = \sigma_j\sigma_i\) for \(|i-j| \geq 2\). Then the associated Artin group is the braid group with \(n\) strands \(B_n\) and the map \(\pi: B_n \to \Sigma_n\) maps each braid to a permutation given by its ends.

Let \(R_W\) be the complete set of representatives of \(S/\sim\), where \(\sim\) is given by conjugation by elements of \(W\), and let \(c(W)\) be the cardinality of \(R_W\). For instance, we have \(c(\Sigma_n) = 1\). Let \(G_{ab}\) denote the abelianization of a group \(G\). As in [4], one has:

**Proposition 2.3.** There are isomorphisms \(W_{ab} \cong (\mathbb{Z}/2)^{c(W)}\) and \((A_W)_{ab} \cong \mathbb{Z}^{c(W)}\).

A Coxeter system \((W, S)\) is called right-angled if the Coxeter matrix \(m\) satisfies \(m(s, t) = 1, 2, \infty\) for any \(s, t \in S\). By the definition of \(c(W)\), one gets:

**Proposition 2.4.** If a Coxeter system \((W, S)\) is right-angled, then \(c(W) = |S|\).

For a Coxeter system \((W, S)\), we define a graph \(\Gamma_W\) such that the vertex set is \(S\) and vertices \(s, t \in S\) are joined by an edge whenever \(2 \leq m(s, t) < \infty\). Then if a Coxeter group \(W\) is right-angled, all the information of \(W\) is included in the graph \(\Gamma_W\). We will see this more precisely below in terms of a graph product of groups.

### 3. The Adjoint Group of a Coxeter Quandle

A quandle is a set \(X\) with a binary operation \(*: X \times X \to X\) satisfying the three conditions:

1. \(x * x = x\);
2. \((x * y) * z = (x * z) * (y * z)\);
3. the map \(X \to X, x \mapsto x * y\) is bijective for any \(y \in X\).

A quandle is related with group theory (and representation theory) as follows. For a group \(G\), we put \(x * y = y^{-1}xy\) for \(x, y \in G\). Then one can easily check the three conditions of a quandle so that \(G\) is a quandle with this binary operation. More generally, any conjugation closed subset of a group can be regarded as a quandle in the same way.

Motivated by the above construction of a quandle from a group, for a quandle \(X\), we define a group

\[ \text{Ad}(X) = \langle e_x \mid (x \in X) \big| e_{x*y} = e_y^{-1}e_x e_y \rangle \]
which is called the adjoint group of $X$. When $X$ is a conjugation closed subset of a group $G$ regarded as a quandle, $\text{Ad}(X)$ is directly related with $G$.

**Proposition 3.1.** For a conjugation closed subset $X$ of a group $G$, the assignment

$$\phi: \text{Ad}(X) \to G, \quad e_x \mapsto x \quad (x \in X)$$

is a well-defined homomorphism. Moreover, if $X$ generates $G$, then $\phi$ is surjective.

**Proof.** For $x, y \in X$, we have $\phi(e_{xy}) = x \cdot y = y^{-1}xy = \phi(e^{-1}_yx e_y)$, implying that the map $\phi$ is a well-defined homomorphism. The remaining statement is obvious. \hfill \Box

Let $(W, S)$ be a Coxeter system. An element of $W$ of the form $w^{-1}sw$ for $w \in W$ and $s \in S$ is called a reflection of $W$. Then the set of reflections of $W$, denoted $X_W$, is a conjugation closed subset of $W$, so $X_W$ is a quandle which is called the Coxeter quandle associated with a Coxeter system $(W, S)$. Since $W$ is generated by $X_W$, we have the following by Proposition 3.1.

**Corollary 3.2.** For a Coxeter group $W$, the map $\phi: \text{Ad}(X_W) \to W$ is an epimorphism.

Akita [1] showed that $\text{Ad}(X_W)$ is related also with the Artin group $A_W$, where we reproduce its proof.

**Proposition 3.3.** For an arbitrary Coxeter system $(W, S)$, the assignment

$$\Phi: A_W \to \text{Ad}(X_W), \quad a_s \mapsto e_s \quad (s \in S)$$

is a well-defined epimorphism.

**Proof.** Suppose that $m(s,t) = k$ with $2 \leq k < \infty$ for $s, t \in S$. Then we have

$$\underbrace{\cdots (s \ast t) \ast s \cdots}_{k} = \underbrace{s^{-1}_{k}st \cdots}_{k-1} = \underbrace{\cdots s^{-1}t}_{k-1}tst \cdots = u$$

where $u = s$ for $n$ even and $u = t$ for $n$ odd, that is, $u$ is the last letter of the word $tst\cdots$.

Then it follows that

$$\Phi(a_s a_t a_s \cdots) = e_s e_t e_s \cdots = e_t e_s e_t e_s \cdots = e_t e_s e_t e_s \cdots ((s \ast t) \ast s) \ast \cdots = e_t e_s e_t \cdots,$$

implying that $\Phi$ is a well-defined homomorphism. To see that $\Phi$ is surjective, we shall show that $\text{Ad}(X_W)$ is generated by $e_s$ for $s \in S$. For $s_1, \ldots, s_n, s \in S$ and $w = s_1 \cdots s_n$, one has $w^{-1}sw = (\cdots (s \ast s_1) \ast s_2 \cdots) \ast s_n$, implying that $e^{-1}_{w^{-1}sw} = e_{(s \ast s_1) \ast s_2 \cdots \ast s_n} = e^{-1}_{s_n} \cdots e^{-1}_{s_1} e_s e_{s_1} \cdots e_{s_n}$. Thus the proof is completed. \hfill \Box
Combining Corollary 3.2 and Proposition 3.3, we obtain Theorem 1.1.

We now recall the structure theorem of $\text{Ad}(X_W)$ due to Akita [1]. Eisermann [8] showed that there is a short exact sequence $1 \rightarrow A_n \rightarrow \text{Ad}(X_{\Sigma_n}) \rightarrow \mathbb{Z} \rightarrow 0$, where $A_n$ is the alternating group of $n$ letters. Note that $A_n \cong [\Sigma_n, \Sigma_n]$ and $c(\Sigma_n) = 1$. Akita [1] generalized this result to an arbitrary Coxeter group.

**Theorem 3.4.** For any Coxeter group $W$, the following hold:

1. there is an isomorphism $\text{Ad}(X_W)_{ab} \cong \mathbb{Z}^{c(W)}$;
2. the map $\phi: \text{Ad}(X_W) \rightarrow W$ induces an isomorphism $[\text{Ad}(X_W), \text{Ad}(X_W)] \cong [W, W]$.

**Corollary 3.5.** For any Coxeter group $W$, there is a short exact sequence

$$1 \rightarrow [W, W] \rightarrow \text{Ad}(X_W)_{ab} \xrightarrow{\phi} \mathbb{Z}^{c(W)} \rightarrow 0.$$

By [4], $W_{ab}$ is a $\mathbb{Z}/2$-vector space with a basis $\{[x] \in W_{ab} \mid x \in R_W\}$, from which the first isomorphism of Proposition 2.3 follows. On the other hand, Akita [1] showed that $\text{Ad}(X_W)_{ab}$ is a free abelian group with a basis $\{[e_x] \in \text{Ad}(X_W)_{ab} \mid x \in R_W\}$, which yields Theorem 3.4 (1). Then we identify the abelianization of the map $\phi: \text{Ad}(X_W) \rightarrow W$ as:

**Lemma 3.6.** The map $\phi_{ab}: \text{Ad}(X_W)_{ab} \rightarrow W_{ab}$ is identified with the canonical projection $\mathbb{Z}^{c(W)} \rightarrow (\mathbb{Z}/2)^{c(W)}$.

Thus we get a commutative diagram

$$(3.1) \quad \begin{array}{ccc}
\text{Ad}(X_W) & \xrightarrow{\phi} & \mathbb{Z}^{c(W)} \\
\downarrow & & \downarrow \text{proj} \\
W & \xrightarrow{\text{ab}} & (\mathbb{Z}/2)^{c(W)}.
\end{array}$$

We show that this diagram is a pullback.

**Lemma 3.7.** Suppose that there is a commutative square of groups

$$(3.2) \quad \begin{array}{ccc}
G_1 & \xrightarrow{f_1} & H_1 \\
\downarrow g & & \downarrow h \\
G_2 & \xrightarrow{f_2} & H_2
\end{array}$$

where $f_1$ are surjective. Then the square is a pullback if and only if the canonical map $\text{Ker} f_1 \rightarrow \text{Ker} f_2$ is an isomorphism.

**Proof.** Let $G = \{(x, y) \in G_2 \times H_1 \mid f_2(x) = h(y)\}$ which is the pullback of the triad $G_2 \xrightarrow{f_2} H_2 \xrightarrow{h} H_1$. Let $p_1: G \rightarrow G_2$ and $p_2: G \rightarrow H_1$ be the projections, and define a map $e: G_1 \rightarrow G$ by $e(x) = (g(x), f_1(x))$ for $x \in G_1$. Then we have $p_1 \circ e = g$ and since $f_1$
is surjective, the projection $p_2$ is surjective too. Moreover, the commutative square (3.2) extends to the following commutative diagram with exact columns and rows

\[
\begin{array}{ccccccccc}
1 & \rightarrow & \ker f_1 & \rightarrow & G_1 & \overset{f_1}{\rightarrow} & H_1 & \rightarrow & 1 \\
\downarrow & & \downarrow \bar{e} & & \downarrow e & & \downarrow & & \\
1 & \rightarrow & \ker p_2 & \rightarrow & G & \overset{p_2}{\rightarrow} & H_1 & \rightarrow & 1 \\
\downarrow & & \downarrow \bar{p}_1 & & \downarrow p_1 & & \downarrow h & & \\
1 & \rightarrow & \ker f_2 & \rightarrow & G_2 & \overset{f_2}{\rightarrow} & H_2 & \rightarrow & 1 \\
\end{array}
\]

The square diagram (3.2) is a pullback if and only if the map $e: G_1 \to G$ is an isomorphism. By the above diagram, the latter is equivalent to that the map $\bar{e}$ is an isomorphism. One sees that $\ker p_2 \cong \ker f_2$ and the map $\bar{p}_1$ is identified with the identity map. Thus the map $\bar{e}$ is identified with the canonical map $\ker f_1 \to \ker f_2$, completing the proof. \hfill $\square$

**Theorem 3.8.** The commutative square (3.1) is a pullback.

**Proof.** Combine Theorem 3.4 (2) and Lemma 3.7. \hfill $\square$

In [2] it is proved that there is a short exact sequence $0 \to \mathbb{Z} \to \Ad(X_{\Sigma_n}) \to \Sigma_n \to 1$, and this was also generalized by Akita [1] to an arbitrary Coxeter group. We can reprove this by applying Theorem 3.8

**Corollary 3.9.** For any Coxeter group $W$, there is a short exact sequence

\[0 \to \mathbb{Z}^c(W) \to \Ad(X_W) \overset{\phi}{\to} W \to 1.\]

**Proof.** By Theorem 3.8, the kernel of $\phi$ is isomorphic to the kernel of the projection $\mathbb{Z}^c(W) \to (\mathbb{Z}/2)^c(W)$, which is isomorphic to $\mathbb{Z}^c(W)$. Thus the proof is done. \hfill $\square$

4. Classifying spaces and polyhedral products

Let $K$ be a simplicial complex on the vertex set $[m] = \{1, \ldots, m\}$ and $(X, A)$ be topological pair. The polyhedral product of $(X, A)$ with respect to $K$ is defined by

\[Z(K; (X, A)) = \bigcup_{\sigma \in K} (X, A)^\sigma\]

where $(X, A)^\sigma = Y_1 \times \cdots \times Y_m$ such that $Y_i = X, A$ according to $i \in \sigma$ and $i \notin \sigma$. Note that $Z(K; (X, A))$ is natural with respect to $(X, A)$ and inclusions of simplicial complexes. Polyhedral products were introduced as a generalisation of the moment-angle complex and the Davis-Januszkiewicz space which are fundamental in toric topology [3, 5], and connect algebraic geometry, combinatorics, commutative algebra, geometry, group theory, and topology. Among others, their homotopy theory is rapidly developing (cf. [11]).
We will use the following property of polyhedral products, which is immediately deduced from the definition. For $\emptyset \neq I \subset [m]$, the full subcomplex of $K$ on $I$ is defined by $K_I = \{ \sigma \in K \mid \sigma \subset I \}$.

**Proposition 4.1.** For $\emptyset \neq I \subset [m]$, $Z(K_I; (X, A))$ is a retract of $Z(K; (X, A))$.

For an acyclic space $X$, the acyclicity of $Z(K; (X, *))$ is completely characterized in terms of $K$. The necessity of the acyclicity of $Z(K; (X, *))$ has been often referred to [6] although it is an easy consequence of the old result of Whitehead [13]. So we here give a simple proof using the result of Whitehead. A simplicial complex $K$ is called flag if $\sigma \subset [m]$ is a simplex of $K$ whenever any two elements of $\sigma$ are joined by an edge of $K$.

**Proposition 4.2.** For an acyclic space $X$, $Z(K; (X, *))$ is acyclic if and only if $K$ is flag.

**Proof.** Whitehead [13] proved the following. Suppose that there is a homotopy push out of path-connected spaces

\[ X_{12} \xrightarrow{\alpha_1} X_1 \] \[ \xrightarrow{\alpha_2} X_2 \xrightarrow{\alpha_2} X \]

in which $X_1, X_2, X_{12}$ are acyclic and $\alpha_1, \alpha_2$ are injective in $\pi_1$. Then $X$ is acyclic. We apply this to prove the if part of the proposition.

For a vertex $v$ of $K$, let $lk(v)$ and $dl(v)$ denote the link and the deletion of a vertex $v$ in $K$. Then there is a pushout of simplicial complexes

\[ lk(v) \longrightarrow lk(v) \ast \{v\} \] \[ \xrightarrow{\text{injection}} \] \[ dl(v) \longrightarrow K \]

which induces a homotopy pushout

\[ Z(lk(v); (X,*)) \longrightarrow Z(lk(v); (X,*)) \times X \] \[ \xrightarrow{\text{injection}} \] \[ Z(dl(v); (X,*)) \longrightarrow Z(K; (X,*)) \]

The upper horizontal arrow is obviously injective in $\pi_1$. $K$ is flag if and only if $K_V = lk(v)$ for any vertex $v$, where $V$ is the vertex set of $lk(v)$. Then it follows from Proposition 5.4 that if $K$ is flag, the left vertical arrow of (4.1) is injective in $\pi_1$. Moreover, if $K$ is flag, then $K_I$ is flag too for any $\emptyset \neq I \subset [m]$. Then we can apply the above result of Whitehead to (4.1) inductively on the number of vertices. Thus we obtain that if $K$ is flag, then $Z(K; (X,*))$ is acyclic for an acyclic space $X$. 
Next we conversely suppose that $Z(K; (X, *))$ is acyclic for an acyclic space $X$. Assume that $K$ is not flag. Then there is $I \subset [m]$ with $|I| \geq 3$ such that $K_I$ is the boundary of the $(|I| - 1)$-dimensional simplex. Recall from [11] that there is a homotopy fibration $Z(K_I; (C\Omega X, \Omega X)) \rightarrow Z(K_I; (X, *)) \rightarrow X^{|I|}$ and a homotopy equivalence $Z(K_I; (C\Omega X, \Omega X)) \simeq \Sigma^{|I| - 1} \Omega X \wedge \cdots \wedge \Omega X$, where the number of $\Omega X$ in the wedge is $|I|$. Since $\Omega X$ is discrete, $Z(K_I; (C\Omega X, \Omega X))$ is a wedge of spheres of dimension $|I| - 1 \geq 2$.

Then $Z(K_I; (X, *))$ is not acyclic. On the other hand, by Proposition 5.4, $Z(K_I; (X, *))$ is a retract of $Z(K; (X, *))$, implying that $Z(K_I; (X, *))$ is acyclic. This is a contradiction, so $K$ is flag. Therefore the proof is completed. □

Let $G$ be a group and $\{G_s\}_{s \in S}$ be a family of groups such that $G_s = G$ for all $s \in S$. We denote the free product of $G_s$ for $s \in S$ by $F_S(G)$. Let $\Gamma$ be a simple graph (a graph without loops and multiple edges) with the vertex set $S$. The graph product of $G$ with respect to $\Gamma$, denoted $G^\Gamma$, is defined by dividing out $F_S(G)$ by the commuting relations $[G_s, G_t] = 1$ for edges $\{s, t\}$ of $\Gamma$. Note that $G^\Gamma$ is natural with respect to homomorphisms of groups and inclusions of graphs.

**Lemma 4.3.** For a path-connected space $X$, there is an isomorphism

$$\pi_1(Z(K; (X, *))) \cong \pi_1(X)K^{(1)}$$

where $K^{(n)}$ denotes the $n$-skeleton of $K$.

**Proof.** By the cellular approximation theorem, the inclusion $Z(K^{(1)}; (X, *)) \rightarrow Z(K; (X, *))$ is an isomorphism in $\pi_1$. Since $Z(K^{(0)}; (X, *))$ is a wedge of $m$ copies of $X$, we have $\pi_1(Z(K^{(0)}; (X, *))) \cong F_{[m]}(\pi_1(X))$. By the van Kampen theorem, attaching the edge $\{i, j\}$ adds the commutator relation of $i$-th and $j$-th $\pi_1(X)$ in $F_{[m]}(\pi_1(X))$. Thus we have proved the lemma. □

For a simple graph $\Gamma$, let $C(\Gamma)$ be the flag complex whose 1-skeleton is $\Gamma$.

**Proposition 4.4.** For a group $G$ and a finite simple graph $\Gamma$, there is a homotopy equivalence

$$B(G^\Gamma) \simeq Z(C(\Gamma); (BG, *))$$

which is natural with respect to $G$ and $\Gamma$.

**Proof.** The homotopy equivalence is obtained by Proposition 4.2 and Lemma 4.3, and the naturality is obvious by the construction. □

We now consider Coxeter groups. If a Coxeter group $W$ is right-angled, there are isomorphisms $W \cong (Z/2)^W$ and $A_W \cong Z^W$. Then by Proposition 4.4, one gets:
**Corollary 4.5.** If a Coxeter system \((W, S)\) is finitely generated and right-angled, then there are homotopy equivalences

\[ BW \simeq Z(C(\Gamma_W); (\mathbb{R}P^\infty, *)) \quad \text{and} \quad BA_W \simeq Z(C(\Gamma_W); (S^1, *)). \]

We are going to show that Ad\((X_W)\) is also given by a certain polyhedral product when \(W\) is right-angled. To this end, we need several lemmas. The following lemma is well known and useful, which is proved, for example, in [9, Proposition, pp.180].

**Lemma 4.6.** Let \( \{F_i \to E_i \to B\}_{i \in I} \) be an \( I \)-diagram of homotopy fibration with a fixed base \( B \). Then the sequence

\[ \text{hocolim}_i F_i \to \text{hocolim}_i E_i \to B \]

is a homotopy fibration.

The following is an up-to-homotopy version of [7, Lemma 2.3.1].

**Lemma 4.7.** Let \((F, F') \to (E, E') \to (B, B)\) be a pair of homotopy fibrations such that \((F, F'), (E, E')\) are NDR pairs. Then

\[ Z(K; (F, F')) \to Z(K; (E, E')) \to B^m \]

is a homotopy fibration.

**Proof.** For any \( \sigma \subset [m] \), the sequence \((F, F')^\sigma \to (E, E')^\sigma \to B^m\) is a homotopy fibration, where \((X, A)^\sigma\) is as in the definition of a polyhedral product in the previous section. Since this homotopy fibration is natural with respect to \( \sigma \), we get a \( F(K) \)-diagram of homotopy fibrations \( \{(F, F')^\sigma \to (E, E')^\sigma \to B^m\}_{\sigma \in F(K)} \), where \( F(K) \) is the face poset of \( K \). Then it follows from Lemma 4.6 that the sequence

\[ \text{hocolim}_{F(K)} (F, F')^\sigma \to \text{hocolim}_{F(K)} (E, E')^\sigma \to B^m \]

is a homotopy fibration. Since \((F, F')\) is an NDR pair, the projection \( \text{hocolim}_{F(K)} (F, F')^\sigma \to \text{colim}_{F(K)} (F, F')^\sigma = Z(K; (F, F'))\) is a homotopy equivalence. Similarly we get a natural homotopy equivalence \( \text{hocolim}_{F(K)} (E, E')^\sigma \simeq Z(K; (E, E'))\). Thus the proof is completed. \( \square \)

**Lemma 4.8.** Let \( F \to E \to B \) be a homotopy fibration such that \( F \to E \) is a cofibration. Then the following commutative square is a homotopy pullback.

\[
\begin{array}{ccc}
Z(K; (F, E)) & \longrightarrow & E^m \\
\downarrow & & \downarrow \\
Z(K; (B, *)) & \longrightarrow & B^m
\end{array}
\]
Proof. To see that (4.2) is a homotopy pullback, it is sufficient to show that the natural map between the homotopy fibers of the horizontal arrows is a homotopy equivalence. By Lemma 4.7, the homotopy fibers of the both horizontal arrows are homotopy equivalent to $Z(K; (C\Omega B, \Omega B))$ and the natural map between them is identified with the identity map. Thus (4.2) is a homotopy pullback. □

Lemma 4.9. Suppose that there is a pullback of groups

$$
\begin{array}{ccc}
G_1 & \xrightarrow{g} & G_2 \\
\downarrow & & \downarrow \\
H_1 & \xrightarrow{h} & H_2
\end{array}
$$

where $g, h$ are surjective. Then the induced square

$$
\begin{array}{ccc}
BG_1 & \longrightarrow & BG_2 \\
\downarrow & & \downarrow \\
BH_1 & \longrightarrow & BH_2
\end{array}
$$

is a homotopy pullback.

Proof. It is sufficient to show that the natural map between the homotopy fibers of the horizontal arrows in the second square is a homotopy equivalence. Since $g, h$ are surjective, the homotopy fibers are $B\text{Ker } g$ and $B\text{Ker } h$, respectively, and the natural map between them is induced from the canonical map $\text{Ker } g \to \text{Ker } h$ which is an isomorphism by Lemma 3.7. Thus the proof is completed. □

Let $M$ be the closed Möbius band and $(M, S^1)$ be the pair of $M$ and its boundary circle.

Theorem 4.10. For a finitely generated right-angled Coxeter group $W$, there is a homotopy equivalence

$$
B\text{Ad}(X_W) \simeq Z(C(\Gamma_W); (M, S^1)).
$$

Proof. By Theorem 3.8 and Lemma 4.9, for any Coxeter group $W$, we have a homotopy pullback

$$
\begin{array}{ccc}
B\text{Ad}(X_W) & \longrightarrow & (S^1)^{c(W)} \\
\downarrow & & \downarrow \\
B W & \longrightarrow & (\mathbb{R}P^\infty)^{c(W)}
\end{array}
$$

Consider a homotopy fibration $S^1 \to M \to \mathbb{R}P^\infty$ where the first map is the boundary inclusion and the second map is equivalent to the bottom cell inclusion. Then by Lemma
there is a homotopy pullback
\[
\begin{array}{ccc}
Z(K; (M, S^1)) & \longrightarrow & (S^1)^m \\
\downarrow & & \downarrow \\
Z(K; (\mathbb{R}P^\infty, *)) & \longrightarrow & (\mathbb{R}P^\infty)^m.
\end{array}
\]
Thus the proof is done by Corollary 4.5. □

By Corollary 4.5 and Theorem 4.10 together with the naturality of Proposition 4.4, one gets:

**Corollary 4.11.** Let $W$ be a finitely generated right-angled Coxeter group.

1. The map $\Phi: BA_W \to B\text{Ad}(X_W)$ is identified with
   \[
   Z(C(\Gamma_W); (S^1, *)) \to Z(C(\Gamma_W); (M, S^1))
   \]
   which is induced from the composite $(S^1, *) \simeq (M, *) \to (M, S^1)$.
2. The map $\phi: B\text{Ad}(X_W) \to BW$ is identified with
   \[
   Z(C(\Gamma_W); (M, S^1)) \to Z(C(\Gamma_W); (\mathbb{R}P^\infty, *))
   \]
   which is induced from the composite $(M, S^1) \to (M/S^1, *) = (\mathbb{R}P^2, *) \to (\mathbb{R}P^\infty, *)$.

5. Applications

We give two applications of Theorem 4.10 and first show a stable splitting of $B\text{Ad}(X_W)$. We will use the following stable splitting of a polyhedral product, which was proved in [3] (See [10] for a more precise proof of the naturality). Define the polyhedral smash product $\hat{Z}(K; (X, A))$ in the same way as $Z(K; (X, A))$ by replacing the direct product with the smash product.

**Theorem 5.1.** There is a homotopy equivalence
\[
\Sigma Z(K; (X, A)) \simeq \Sigma \bigvee_{\emptyset \neq I \subset [m]} \hat{Z}(K_I; (X, A))
\]
which is natural with respect to $(X, A)$.

**Lemma 5.2.** The inclusion $Z(K; (X, *)) \to Z(K; (X, A))$ has a left homotopy inverse after a suspension.

**Proof.** By definition, if $L$ is a full simplex with $n$ vertices, then $\hat{Z}(L; (Y, B))$ is the smash product of $n$ copies of $Y$. Then $\hat{Z}(K_I; (X, *)) = \hat{Z}(K_I; (X, A))$ if $I \subset [m]$ is a simplex of $K$. On the other hand, if $I$ is not a simplex of $K$, we have $\hat{Z}(K_I; (X, *)) = *$. Thus by Theorem 5.1, $\Sigma Z(K; (X, *))$ is a wedge summand of $\Sigma Z(K; (X, A))$, up to homotopy, completing the proof. □
Theorem 5.3. For a finitely generated right-angle Coxeter group \( W \), the map \( \Phi: BA_W \to \text{BAd}(X_W) \) has a left homotopy inverse after a suspension. In particular, there is a simply connected space \( X \) such that
\[
\Sigma \text{BAd}(X_W) \simeq \Sigma BA_W \vee \Sigma X.
\]

Proof. By Corollary 4.11 and Lemma 5.2, the first assertion follows. If we put \( X \) to be the cofiber of \( \Phi: BA_W \to \text{BAd}(X_W) \), then the second assertion follows from the first one, where \( X \) is simply connected by the van Kampen theorem.

Let \( W \) be a right-angled Coxeter group with the generating set \([m]\). We next calculate the mod 2 cohomology of \( \text{Ad}(X_W) \). By Corollary 3.9, there is a homotopy fibration
\[
(S^1)^m \to \text{BAd}(X_W) \to BW.
\]

We calculate the Serre spectral sequence of (5.1), where we denote its \( E_r \) by \( E_r(W) \). For a subset \( T \subset [m] \), let \( W_T \) be the subgroup of \( W \) generated by \( T \). Then \( W_T \) is a right-angled Coxeter group with the generating set \( T \) such that \( \Gamma_{W_T} = (\Gamma_W)_T \). By the naturality of Theorem 3.8, one sees that for \( V \subset U \subset [m] \), (5.1) for \( W_V \) is a retract of (5.1) for \( W_U \), implying the following.

Lemma 5.4. For \( V \subset U \subset [m] \), \( E_r(W_V) \) is a retract of \( E_r(W_U) \).

By Corollary 4.5, one has \( BW \simeq Z(C(\Gamma_W);(\mathbb{RP}^\infty, *)) \). Then quite similarly to the calculation of the cohomology of \( Z(K;(\mathbb{CP}^\infty, *)) \) in [5], we see that
\[
H^*(BW;\mathbb{F}_2) = \mathbb{F}_2[x_1, \ldots, x_m]/(x_ix_j | \{i, j\} \notin E(\Gamma_W)), \quad |x_i| = 1
\]
where \( E(\Theta) \) denotes the edge set of a graph \( \Theta \). By Theorem 3.8, the homotopy fibration (5.1) is a homotopy pullback of the homotopy fibration \( (S^1)^m \xrightarrow{2} (S^1)^m \to (\mathbb{RP}^\infty)^m \), implying that its local system of coefficients is trivial. Then one gets
\[
E_2(W) \cong \mathbb{F}_2[x_1, \ldots, x_m]/(x_ix_j | \{i, j\} \notin E(\Gamma_W)) \otimes \Lambda(y_1, \ldots, y_m)
\]
such that \( d_2x_i = 0 \) and \( d_2y_i = x_i^2 \), where \( |y_i| = 1 \). For \( I \subset [m] \), put \( z_{i,I} = x_iy_I \), where \( y_I = \prod_{i \in I} y_i \) for \( I \neq \emptyset \) and \( y_\emptyset = 1 \). Then for \( I \subset N_i \), we have \( d_2z_{i,I} = 0 \) and \( E_3(W) \) is generated by such \( z_{i,I} \) as an algebra, where \( N_i = \{j | \{i, j\} \notin E(\Gamma_W)\} \). By definition, we have
\[
z_{i,I}z_{j,J} = \begin{cases} 
0 & \text{if } \{i, j\} \notin E(\Gamma_W) \text{ or } I \cap J \neq \emptyset \\
z_{i,I-k}z_{j,J∪\{k\}} & \text{if } \{i, j\} \in E(\Gamma_W), \ I \cap J = \emptyset, k \in N_j
\end{cases}
\]
in \( E_2(W) \). Moreover, since \( d_2y_I = \sum_{i \in I} z_{i,y}z_{i-I,i} \), we have
\[
\sum_{i \in I} z_{i,y}z_{i,I-i} = 0 \quad \text{if } I - i \subset N_i \text{ for any } i \in I
\]
in \( E_3(W) \). Since these are all relations that \( z_{i,I} \in E_3(W) \) satisfy, one gets:
Lemma 5.5. \(E_3(W)\) is an algebra generated by \(z_{i,I}\) for \(i = 1, \ldots, m\) and \(I \subset N_i\) subject to the relations (5.2) and (5.3), where \(|z_{i,I}| = i + |I|\).

We show that \(E_r(W)\) collapses for \(r = 3\) by considering the special case that \(\Gamma_W = K_{m-1} \sqcup \{m\}\), where \(K_n\) is the complete graph with \(n\) vertices. If \(\Gamma_W = K_{m-1} \sqcup \{m\}\), then \(C(\Gamma_W) = \Delta^{[m-1]} \sqcup \{m\}\), where \(\Delta^S\) is the full simplex on the vertex set \(S\).

Lemma 5.6. For a right-angled Coxeter group \(W\) with \(\Gamma_W = K_{m-1} \sqcup \{m\}\), the dimension of \(H^m(\text{Ad}(X_W); \mathbb{F}_2)\) is \(m + 1\).

Proof. By Theorems 4.10 and 5.1, one has

\[
\Sigma B\text{Ad}(X_W) \simeq \bigvee_{\emptyset \neq I \subset [m]} \hat{Z}(C(\Gamma_W)_I; (M, S^1)).
\]

Note that \(C(\Gamma_W)_I = \Delta^I\) for \(m \notin I\) and \(C(\Gamma_W)_I = \Delta^{I-m} \sqcup \{m\}\) for \(m \in I\). By definition, \(\hat{Z}(\Delta^I; (M, S^1)) = \bigwedge_{i \in I} M \simeq S^{[I]}\). When \(m \in I\), we have \(\hat{Z}(\Delta^{I-m} \sqcup \{m\}; (M, S^1)) = U \cup V\) for \(U = (\bigwedge_{i \in I-m} M) \land S^1\) and \(V = (\bigwedge_{i \in I-m} S^1) \land M\) such that \(U \cap V = \bigwedge_{|I|} S^1 = S^{|I|}\). Since the inclusions of \(U \cap V\) into \(U\) and \(V\) are trivial in the mod 2 cohomology, by the Mayer-Vietoris sequence of this cover, we get

\[
H^*(\hat{Z}(\Delta^{I-m} \sqcup \{m\}; (M, S^1)); \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & * = 0, |I| + 1 \\ \mathbb{F}_2 \oplus \mathbb{F}_2 & * = |I| \\ 0 & \text{otherwise.} \end{cases}
\]

Thus \(H^m(\text{Ad}(X_W); \mathbb{F}_2) \cong \bigoplus_{J \subset [m-1], |J| \geq m-2} H^m(\hat{Z}(\Delta^J \sqcup \{m\}; (M, S^1)); \mathbb{F}_2) \cong \mathbb{F}_2^{m+1}\), completing the proof. \(\square\)

Suppose that \(\Gamma_W = \Delta^{[m-1]} \sqcup \{m\}\). By Lemma 5.5, \(E_3^m(W)\) is spanned by \(z_{m,[m-1]}\), \(z_{1,0} \cdots z_{m-2,0} z_{m-1,1}(m)\) and \(z_{m,0} z_{m,[m-1]}\), for \(i = 1, \ldots, m-1\). Then it follows from Lemma 5.6 that these elements are permanent cycles. Thus one gets:

Lemma 5.7. If \(\Gamma_W = \Delta^{[m-1]} \sqcup \{m\}\), then \(z_{m,[m-1]} \in E_3(W)\) is a permanent cycle.

Let \(W\) be an arbitrary right-angled Coxeter group on the vertex set \([m]\). Let \(I \subset N_i\) and \(W'\) be the right-angled Coxeter group with the generating set \(I \cup \{i\}\) such that \(\Gamma_{W'} = K_{|I|} \sqcup \{i\}\). Since \(\Gamma_{W_{I\cup\{i\}}} \subset \Gamma_{W'}\), by the naturality of graph products of groups, there is a homotopy commutative diagram

\[
\begin{array}{ccc}
(S^1)^{|I|+1} & \longrightarrow & \text{Ad}(X_{W_{I\cup\{i\}}}) \\
\downarrow & & \downarrow \\
(S^1)^{|I|+1} & \longrightarrow & \text{Ad}(X_{W'})
\end{array}
\begin{array}{ccc}
& \longrightarrow & \text{BW}_{I\cup\{i\}} \\
& & \downarrow \\
& \longrightarrow & \text{BW'}
\end{array}
\]
Then we get a map $E_r(W') \to E_r(W')_{W \cup \{i\}}$ which maps $z_{i,I} \in E_3(W')$ to $z_{i,I} \in E_3(W'_{W \cup \{i\}})$, so by Lemma 5.7, $z_{i,I} \in E_3(W'_{W \cup \{i\}})$ is a permanent cycle. Thus by Lemma 5.4, $z_{i,I} \in E_3(W)$ is a permanent-cycle too and we obtain:

**Lemma 5.8.** The Serre spectral sequence of (5.1) collapses at $E_3$.

For a degree reason, the extension of $E_\infty(W)$ to $H^*(\text{Ad}(X_W);\mathbb{F}_2)$ is trivial. Therefore by Lemmas 5.5 and 5.8, we finally obtain:

**Theorem 5.9.** For a right-angled Coxeter group $W$ with the generating set $[m]$, the mod 2 cohomology of $\text{Ad}(X_W)$ is an algebra generated by $z_{i,I}$ for $i = 1, \ldots, m$ and $I \subset N_i$ subject to the relations (5.2) and (5.3), where $|z_{i,I}| = i + |I|$.

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