ASYMPTOTIC BOUNDS FOR NORI’S CONNECTIVITY THEOREM

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Abstract. Let $Y$ be a smooth complex projective variety. I study the cohomology of smooth families of hypersurfaces $X \to B$ for $B \subset \mathbb{P}^H(Y, \mathcal{O}(d))$ a codimension $c$ subvariety. I give an asymptotically optimal bound on $c$ and $k$ when $d \to \infty$ for the space $H^k(Y \times B, \mathcal{X}, \mathbb{Q})$ to vanish, thus extending the validity of Lefschetz Hyperplane section Theorem and Nori’s Connectivity Theorem [No]. Next, I construct in the limit case explicit families of higher Chow groups whose class does not vanish in $H^k(Y \times B, \mathcal{X}, \mathbb{Q})$. Some of them are indecomposable. This suggests that in the limit case the space $H^k(Y \times B, \mathcal{X}, \mathbb{Q})$ should be spanned by higher Chow groups, containing $[\mathbb{N}_0]$ and $[\mathbb{O}]$ as special cases (cf. conjecture [2]).

0. Introduction

0.1. Nori’s connectivity theorem. Let $Y$ be a smooth complex proper algebraic variety of dimension $N + 1$ with an ample invertible sheaf $\mathcal{L}$ and $\pi: \mathcal{X} \to \mathbb{P}^H(Y, \mathcal{L})$ the universal family of hypersurfaces of class $c_1(\mathcal{L})$. For any $b \in \mathbb{P}^H(Y, \mathcal{L})$, let $X_b = \pi^{-1}(b)$ be the corresponding hypersurface of $Y$.

The Lefschetz hyperplane section theorem asserts that for $k + 1 \in \{0, \ldots, N\}$,

$$H^{k+1}(Y, X_b, \mathbb{Z}) = 0.$$  

Nori’s connectivity theorem extends this result to generically defined cohomology classes of degree all the way up to $2N = \dim_{\mathbb{R}} X$. More precisely, let $T$ be a smooth variety and $\phi: T \to \mathbb{P}^H(Y, \mathcal{L})$ a differentiable map. Let $\mathcal{X}_T = \mathcal{X} \times_{\mathbb{P}^H(Y, \mathcal{L})} T$ be the universal hypersurface over $T$ and let $\mathcal{Y}_T = Y \times T$. By the Leray spectral sequence, the Lefschetz Hyperplane Section Theorem implies that for $k + 1 \in \{0, \ldots, N\}$,

$$H^{k+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{Z}) = 0.$$  

With stronger assumptions one can say more. Assume that $T$ is a smooth algebraic variety and that $\phi: T \to \mathbb{P}^H(Y, \mathcal{L})$ is a smooth (hence dominant) morphism. Let $\mathcal{O}(1)$ be a fixed very ample invertible sheaf on $Y$ and assume $\mathcal{L} = \mathcal{O}(d)$ for some positive integer $d$.

Theorem (Nori’s connectivity theorem [No]). For $d$ large enough and $k+1 \in \{0, \ldots, 2N\}$,

$$H^{k+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{Q}) = 0.$$  

Note that Nori’s connectivity theorem holds more generally for families of complete intersections in $Y$ (instead of only hypersurfaces). The more general setting is motivated by the construction of algebraic cycles which are homologically but not algebraically equivalent to zero. Since in this paper I do not consider applications to algebraic equivalence,
I restrict myself to the case of hypersurfaces, which involves all the ideas and simplifies notations.

Despite its topological nature, the proof of Nori’s connectivity theorem relies on Hodge theory (cf. [N], section 1). Namely, let $\Omega^p_{Y,T,\mathcal{X}_T}$ be the coherent sheaf of $\mathcal{O}_{Y,T}$-modules defined by the short exact sequence

$$0 \to \Omega^p_{Y,T,\mathcal{X}_T} \to \Omega^p_{Y,T} \to j^*T^p\Omega^p_{X_T} \to 0,$$

where $j^*T: \mathcal{X}_T \to \mathcal{Y}_T$ is the closed immersion. Then $H^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}) = H^{p+q+1}(\mathcal{Y}_T, \Omega^p_{Y,T,\mathcal{X}_T})$.

Following Nori, let

$$G^pH^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}) = \text{Im}(H^{p+q+1}(\mathcal{Y}_T, \Omega^p_{Y,T,\mathcal{X}_T}) \to H^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}))$$

denote the Hodge-Grothendieck filtration. There is a natural inclusion

$$F^pH^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}) \subset G^pH^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}),$$

where $F^pH^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C})$ denotes the Hodge filtration of the mixed Hodge structure on $H^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T)$. Hence if for some positive integer $k$ one has $H^{q+1}(\mathcal{Y}_T, \Omega^p_{Y,T,\mathcal{X}_T}) = 0$ for all $p + q + 1 \leq k + 1$ and $q \leq (k + 1)/2$, then by the Hodge–Fröhlicher spectral sequence $G^{k+1-[(k+1)/2]}H^{k+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}) = 0$, hence $F^{k+1-[(k+1)/2]}H^{k+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}) = 0$ and a mixed Hodge structures argument implies $H^{k+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{Q}) = 0$. Hence the connectivity theorem follows from the following more precise Hodge-theoretical statement

**Theorem** (Nori). For $d$ large enough, $p + q + 1 \leq 2N$ and $q + 1 \leq N$,

$$H^{q+1}(\mathcal{Y}_T, \Omega^p_{Y,T,\mathcal{X}_T}) = 0.$$

### 0.2. Vanishing theorems.

The first goal of this paper is to investigate the minimal assumptions for Nori’s connectivity theorem to extend. Natural questions are:

- how large does $d$ need to be?
- what happens if the map $\phi: T \to \mathbb{P}^0(Y, \mathcal{O}(d))$ fails to be dominant, i.e. if one considers families of hypersurfaces which are not general, but close to being general?
- can the above assertions be made more precise for a fixed integer $k$ (in Nori’s connectivity theorem), and for fixed integers $p$ and $q$ (in the Hodge-theoretical version)?

The first question was studied in Nagel [N] and Voisin [V2]. In this paper I give the following asymptotic answer to the above questions.

Let $T$ be a smooth algebraic variety with an algebraic morphism $\phi: T \to \mathbb{P}^0(Y, \mathcal{O}(d))$, such that $\mathcal{X}_T$ is smooth. I assume that for all $t \in T$ the cokernel of the differential map $d\phi(t): T_{T,t} \to T_{\mathbb{P}^0(Y, \mathcal{O}(d)),\phi(t)}$ is of dimension at most an integer $c$.

**Theorem 1.** For all $\varepsilon \in [0,1[$ and for all $p \in \mathbb{N}$ and for $q + 1 \in \{0, \ldots, N\}$, there is an integer $D$ depending only on $\varepsilon$, $p$, $Y$ and $\mathcal{O}(1)$, such that for all $d \geq D$,

$$H^{q+1}(\mathcal{Y}_T, \Omega^p_{Y,T,\mathcal{X}_T}) = 0.$$
provided that
\[ c \leq (1 - \varepsilon) \frac{d^{N-q}}{(N-q)!}. \]

A straightforward adaptation of Nori’s argument sketched above shows that in the
numerical situation of Theorem 1, provided that
\[ F^pH^{p+q+1}(Y_T, \Omega, \mathbb{C}) = 0, \]
the bound given by the Corollary is asymptotically optimal.

0.3.1. **Case** \( k + 1 \in \{0, \ldots, N\} \). By the Lefschetz hyperplane section theorem and the
Leray spectral sequence one has \( H^{k+1}(Y_T, \mathcal{X}, \mathbb{Q}) = 0 \) with no assumption on \( c \). (Similarly, by the holomorphic Leray spectral sequence one has \( H^{p+1}(Y_T, \Omega^p_Y, \mathcal{X}_T) = 0 \) for \( p + q < N \) with no assumption on \( c \).

0.3.2. **Case** \( k + 1 \in \{N + 1, \ldots, 2N\} \). I show in the next section (see Theorem 3 below)
that under a weak assumption on \( Y \), the bound on \( c \) is asymptotically optimal for any
couple of positive integers \( p \) and \( q \) such that \( N/2 \leq q \leq N - 1 \) and \( p \geq q \). This implies that
the bound given by the Corollary is asymptotically optimal.

0.3.3. **Case** \( k + 1 \geq 2N + 1 \). One has \( H^{2N+1}(Y_T, X_T, \mathbb{Q}) \neq 0 \) for any positive integer \( c \),
\( d > N \) and \( T = X_B \), where \( B \subset \mathbb{P}H^0(Y, \mathcal{O}(1)) \) is any smooth codimension \( c \) subvariety
contained in the locus of smooth hypersurfaces. Indeed, the condition \( d > N \) implies
\( H^{0,N}(X_B, \mathbb{Q}) \neq 0 \) for all \( b \in B \), hence (cf. [No], section 0.2) the cohomology class of the
diagonal embedding of \( X_B \) in \( X_T = X_B \times_B X_B \) is an element of \( H^{2N}(X_T, \mathbb{Q}) \) which does
not belong to \( j_T^*H^{2N}(Y_T, \mathbb{Q}) \), hence defines a non-zero element of \( H^{2N+1}(Y_T, X_T, \mathbb{Q}) \).

0.3. **Some examples of algebraic classes.** The aim of this section is to give explicit
constructions of algebraic classes in \( F^pH^{p+q+1}(Y_T, X_T, \mathbb{C}) \) for \( q + 1 \in \{0, \ldots, N\} \), \( p \geq q \)
and \( c \) close to the bound given by theorem 1.

0.3.1. **The regulator map.** For any smooth algebraic variety \( X \) and positive integers \( p \geq q \)
the higher Chow groups \( CH^p(X, p-q) \) are defined in [B]. In this paper they are always
implicitly assumed to be with \( \mathbb{Q} \) coefficients. They come with the Deligne class map
\[ cl_{p,p-q}(D) : CH^p(X, p-q) \to H_D^{p+q}(X, \mathbb{Q}(p)), \]
where \( H_D^{p+q}(X, \mathbb{Q}(p)) \) denotes Deligne cohomology. There is a natural map
\[ r_{p,p-q} : H_D^{p+q}(X, \mathbb{Q}(p)) \to F^pH^{p+q}(X, \mathbb{C}), \]
where \( F^p H^{p+q}(X, \mathbb{C}) \) denotes the Hodge filtration. Composing \( \text{cl}_{p,p-q}(D) \) with \( r_{p,p-q} \) one gets the regulator map
\[
\text{reg}_{p,p-q} : \text{CH}^p(X, p - q) \to F^p H^{p+q}(X, \mathbb{C}).
\]
Note that this map is identically zero for \( p > q \) whenever \( X \) is proper.

Consider the case \( X = \mathcal{X}_T \) (note that \( \mathcal{X}_T \) is not proper). Let
\[
H^{p+q}(\mathcal{X}_T, \mathbb{C})_v = \text{coker} \left( j_T^* : H^{p+q}(Y_T, \mathbb{C}) \to H^{p+q}(\mathcal{X}_T, \mathbb{C}) \right)
\]
denote the \textit{vanishing} cohomology, let \( F^p H^{p+q}(\mathcal{X}_T, \mathbb{C})_v = F^p H^{p+q}(\mathcal{X}_T, \mathbb{C}) \cap H^{p+q}(\mathcal{X}_T, \mathbb{C})_v \)
and let
\[
\text{reg}_{p,p-q} : \text{CH}^p(\mathcal{X}_T, p - q) \to F^p H^{p+q}(\mathcal{X}_T, \mathbb{C})_v \subset F^p H^{p+q+1}(Y_T, \mathcal{X}_T, \mathbb{C})
\]
be the map deduced from \( \text{reg}_{p,p-q} \) by projection on \( H^{p+q}(\mathcal{X}_T, \mathbb{C})_v \).

An element of \( F^p H^{p+q}(\mathcal{X}_T, \mathbb{C})_v \) (or \( F^p H^{p+q+1}(Y_T, \mathcal{X}_T, \mathbb{C}) \)) will be called \textit{algebraic} if it lies in the image of \( \overline{\text{reg}}_{p,p-q} \).

0.3.2. \textit{Chow groups}. In the case \( p = q \) higher Chow groups are Chow groups and the regulator is the class map. I then write \( \text{CH}^p(\mathcal{X}_T) \) for \( \text{CH}^p(\mathcal{X}_T, 0) \), \( \text{cl}_p \) for \( \text{reg}_{p,0} \) and \( \overline{\text{cl}}_p \) for \( \text{reg}_{p,0} \).

The following is a variation of my result in [O].

**Theorem 2.** Let \( p, q, b \) and \( c \) be positive integers such that \( p + q = N, q \in \{(N + 1)/2, \ldots, N - 1\} \), \( b \geq 1 \) and \( c \leq b \tfrac{d^{N-q}}{N-q} \). Assume \( \phi(T) \subset \mathbb{P}^0(Y, \mathcal{O}(d)) \) lies in the locus parametrizing smooth hypersurfaces. Then then there is an integer \( D \) depending only on \( b \), \( Y \) and \( \mathcal{O}(1) \) such that for all \( d \geq D \) the following holds.

If \( N \) is even and \( p = q = N/2 \) then the space \( F^p H^{p+q}(\mathcal{X}_T, \mathbb{C})_v \) is spanned via \( \overline{\text{cl}}_p \) by classes of flat families of algebraic cycles in \( \mathcal{X}_T \) of codimension \( p \) and degree at most \( b \).

If \( q > N/2 \) then \( F^p H^{p+q}(\mathcal{X}_T, \mathbb{C})_v = 0 \).

0.3.3. \textit{Higher Chow groups}. I now fulfill the promise made in section [O].

**Theorem 3.** Let \( p \) and \( q \) be positive integers such that \( p \geq q \) and \( q \in \{(N + 1)/2, \ldots, N - 1\} \). Assume that \( Y \subset \mathbb{P}^0(Y, \mathcal{O}(1)) \) contains a codimension \( q+1 \) linear subvariety \( V \) and let \( T \subset \mathbb{P}^0(Y, \mathcal{O}(d)) \) be the subvariety parametrising smooth hypersurfaces containing \( V \). Then for \( d \gg 0 \) the variety \( T \) is of codimension \( c \sim_d \tfrac{d^{N-q}}{(N-q)!} \), and there is a higher cycle \( Z \in \text{CH}^p(\mathcal{X}_T, p - q) \) such that \( \text{reg}_{p,p-q}(Z) \neq 0 \).

Hence in this situation the bound on \( c \) in Theorem [O] and its corollary are asymptotically optimal.

Note that the assumption on \( Y \) holds for \( Y = \mathbb{P}^{N+1}_\mathbb{C} \), which should be regarded as the main example.

0.3.4. \textit{Indecomposable Higher Chow groups}. For all \( i \in \{0, \ldots, p\} \) and \( j \in \{0, \ldots, p - q\} \) there are natural product maps
\[
\psi_{i,j} : \text{CH}^{p-i}(\mathcal{X}_T, p - q - j) \otimes \text{CH}^{i}(\mathcal{X}_T, j) \to \text{CH}^p(\mathcal{X}_T, p - q).
\]
I say that a higher cycle in \( \text{CH}^p(\mathcal{X}_T, p - q) \) is decomposable if it belongs to the subgroup spanned by the images of the maps \( \psi_{i,j} \) for \( (i, j) \notin \{(0, 0), (p, p - q)\} \), and that it is indecomposable otherwise.

The proof of Theorem 4 relies on an explicit construction of higher cycles in \( \text{CH}^p(\mathcal{X}_T, p - q) \). More precisely, since \( \text{CH}^1(\mathcal{X}_T, 1) \cong H^0(\mathcal{X}_T, \mathcal{O}_{\mathcal{X}_T}^*) \cong H^0(T, \mathcal{O}_T^*) \), iterating the above map \( p - q \) times I obtain the map

\[
\psi_{p-1,1}^{p-q} : \text{CH}^q(\mathcal{X}_T) \otimes H^0(T, \mathcal{O}_T^{p-q}) \to \text{CH}^p(\mathcal{X}_T, p - q).
\]

For \( p > q \) the higher cycles in \( \text{CH}^p(\mathcal{X}_T, p - q) \) I construct are in the image of \( \psi_{p-1,1}^{p-q} \). Since \( \psi_{p-1,1}^{p-q} \) commutes to the restriction to a fiber, it follows that the restriction of these higher cycles to any fiber \( \mathcal{X}_t, t \in T \), is decomposable.

My result should be compared to Voisin’s [2], who studies the case \( Y = \mathbb{P}^{N+1}_C, p = N, q = N - 1, d = 2N \) and \( T \subset \text{PH}^0(Y, \mathcal{O}(d)) \) an open subset (this implies \( c = 0 \)). She constructs a family of higher cycles in \( \text{CH}^N(\mathcal{X}_T, 1) \) that limits the validity of Nori’s connectivity theorem and whose restriction to a very general fiber \( \mathcal{X}_t \) for \( t \in T \) is indecomposable. This suggests that for all \( p, q \) such that \( p + q \geq N \) and \( p \geq q \) there might be higher cycles in \( \text{CH}^p(\mathcal{X}_T, p - q) \) for \( c \sim d \to \infty \frac{d^{N-q}}{(N-q)!} \) whose restriction to a very general fiber \( \mathcal{X}_t, t \in T \), is indecomposable. Supporting this intuition, I prove the following slightly weaker result for \( Y = \mathbb{P}^{N+1}, N \) even, \( q = N/2 \) and \( p = q + 1 \).

**Theorem 4.** Assume \( N \) even, \( q = N/2 \) and that \( Y \subset \text{PH}^0(Y, \mathcal{O}(1))^\vee \) contains a codimension \( q \) linear subvariety. Then for \( d \gg 0 \) there is a subvariety \( U \subset \text{PH}^0(Y, \mathcal{O}(d)) \) of codimension \( c \sim d \to \infty \frac{d^{N-q}}{(N-q)!} \) and a higher cycle \( Z_U \in \text{CH}^{q+1}(\mathcal{X}_U, 1) \) such that

- \( \text{reg}_{q+1,1}(Z_U) \neq 0 \), and
- the restriction of \( Z_U \) to a very general fiber \( \mathcal{X}_u, u \in U \), is indecomposable.

The cycle \( Z_U \) is constructed explicitly and the proof of indecomposability relies on Theorem 2.

I believe that Theorem 4 should generalize as follows.

**Hypothesis.** Let \( p \) and \( q \) be positive integers such that \( p > q \) and \( q \notin \{(N + 1)/2, \ldots, N - 1\} \). Assume \( Y = \mathbb{P}^{N+1} \). Then for \( d \gg 0 \) there is a subvariety \( U \subset \text{PH}^0(Y, \mathcal{O}(d)) \) of codimension \( c \sim d \to \infty (p - q + 1) \frac{d^{N-q}}{(N-q)!} \) and a higher cycle \( Z_U \in \text{CH}^p(\mathcal{X}_U, p - q) \) such that

- \( \text{reg}_{p,p-q}(Z_U) \neq 0 \), and
- the restriction of \( Z_U \) to a very general fiber \( \mathcal{X}_u, u \in U \), is indecomposable.

Theorem 4 is the case \( p = N/2 + 1, q = N/2 \).

Note that \( c \sim d \to \infty 2 \frac{d^{N-q}}{(N-q)!} \) in Theorem 4 (and \( c \sim d \to \infty (p - q + 1) \frac{d^{N-q}}{(N-q)!} \)) in its conjectural generalisation) while Theorem 3 gives examples of decomposable cycles in \( \text{CH}^p(\mathcal{X}_T, p - q) \) with a non zero vanishing cohomology class for \( c \sim d \to \infty \frac{d^{N-q}}{(N-q)!} \). This suggests that the bound on \( c \) in Theorem 4 may not be optimal.

**Question.** Is there an indecomposable higher cycle \( Z_U \in \text{CH}^p(\mathcal{X}_U, p - q) \) such that \( \text{reg}_{p,p-q}(Z_U) \neq 0 \) for \( c \sim d \to \infty \frac{d^{N-q}}{(N-q)!} \)?
0.4. **Algebraicity conjecture.** The results of Section 0.3 suggest that for \( c \) slightly bigger than the bound \( \frac{d^{N-q}}{(N-q)!} \) given by Theorem 1 the space \( F^pH^{p+q}(\mathcal{X}_T, \mathbb{Q})_v \) should be algebraic.

More precisely, I believe the following should hold

**Conjecture 1.** For all integers \( b \in \mathbb{N}^* \) and \( p \in \mathbb{N} \) there is an integer \( D \) depending only on \( b, p, Y \) and on \( \mathcal{O}(1) \), such that for all \( d \geq D \) and \( 0 \leq q \leq N-1 \), if

\[
c \leq b \cdot \frac{d^{N-q}}{(N-q)!}
\]

then the map \( \overline{\mathfrak{c}}_{p,q} : \text{CH}^p(\mathcal{X}_T, p-q) \otimes \mathbb{C} \to F^pH^{p+q}(\mathcal{X}_T, \mathbb{C})_v \) is surjective.

By convention, \( \text{CH}^p(\mathcal{X}_T, p-q) = 0 \) for \( p < q \).

By Theorem 2 Conjecture 1 holds for \( p + q = N \), \( q \geq N/2 \) provided that \( \phi(T) \subset \text{PH}^0(Y, \mathcal{O}(d)) \) lies in the locus parametrizing smooth hypersurfaces.

Mimicking Nori’s argument, it is easy to see that Conjecture 1 implies the following purely topological statement.

**Conjecture 2.** For every positive integer \( b \) there is an integer \( D \) depending only on \( b, Y \) and \( \mathcal{O}(1) \) such that for all \( d \geq D \) and \( k \in \{N, \ldots, 2N-1\} \) and for

\[
c \leq b \cdot \frac{d^{N-[k/2]}}{(N-[k/2])!},
\]

– if \( k \) is even then the map \( \overline{\mathfrak{c}}_{k/2} : \text{CH}^{k/2}(\mathcal{X}_T) \otimes \mathbb{C} \to F^{k/2}H^{k}(\mathcal{X}_T, \mathbb{C})_v \) is surjective;

– if \( k \) is odd then the maps

\[
\overline{\mathfrak{c}}_{(k+1)/2,1} : \text{CH}^{(k+1)/2}(\mathcal{X}_T, 1) \otimes \mathbb{C} \to \text{H}^{k}(\mathcal{X}_T, \mathbb{C})_v,
\]

\[
\psi \circ \overline{\mathfrak{c}}_{(k+1)/2,1} : \text{CH}^{(k+1)/2}(\mathcal{X}_T, 1) \otimes \mathbb{C} \to \text{H}^{k}(\mathcal{X}_T, \mathbb{C})_v
\]

generate \( \text{H}^{k+1}(\mathcal{X}_T, \mathcal{X}_T, \mathbb{C})_v \).

Here \( \overline{\mathcal{X}}_T \) is the variety isomorphic to \( \mathcal{X}_T \) as a real manifold and endowed with the opposite complex structure and \( \psi \) is the geometric Frobenius.

Note that for \( p + q < N \) Conjecture 1 and for \( k < N \) Conjecture 2 are trivially true since the target cohomology spaces vanish (cf. section 0.2.1).

For \( k \) even Theorem 1 implies that in the numerical situation of Conjecture 2 one has \( F^{k/2+1}H^{k}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}) = 0 \), hence by a mixed Hodge structures argument the space \( \text{H}^{k}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}) \) is of pure Hodge type \((k/2, k/2)\) (like in \( [\text{No}] \), section 0.2), hence the subspace \( \text{H}^{k}(\mathcal{X}_T, \mathbb{C})_v \) is a sub-Hodge structure of pure Hodge type \((k/2, k/2)\). Hence for \( k \) even Conjecture 2 is implied by Hodge Conjecture.

0.5. **Summary.** The figures summarise the results. Namely, for \( c \leq b \cdot \frac{d^{N-q}}{(N-q)!} \)

- \( a : F^pH^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}) = 0 \) for any \( c \) by section 0.2.1
- \( b : F^pH^{p+q+1}(\mathcal{Y}_T, \mathcal{X}_T, \mathbb{C}) = 0 \) for \( c \leq (1 - \varepsilon) \cdot \frac{d^{N-q}}{(N-q)!} \) by Theorem 1 and \( F^pH^{p+q}(\mathcal{X}_T, \mathbb{C})_v \) is spanned by classes of flat families of algebraic cycles of codimension \( N/2 \) and degree at most \( b \) for \( c \leq b \cdot \frac{d^{N-q}}{(N-q)!} \) by Theorem 2 as predicted by Conjecture 1.
Proposition 1.2. For all \( \varepsilon \in \mathbb{R}_+^* \) there is an integer \( D \) depending only on \( p, \varepsilon, Y \) and on \( \mathcal{O}(1) \) such that

\[
H^{q+1}(Y_t, \Omega_{\mathcal{Y}_T, \mathcal{X}_T}^p | Y_1) = 0
\]
for all integers \( d \geq D \), \( q \in \{-1, \ldots, N - 1\} \), for all smooth analytic varieties \( T \) endowed with a morphism \( \phi: T \to \mathbb{H}^0(Y, \mathcal{O}(d)) \) such that

- for all \( t \in T \) the cokernel of \( d \phi(t) \) is of dimension at most \((1 - \varepsilon) \frac{d^{N-q}}{(N-q)!}\) and
- \( \mathcal{X}_T \) is smooth.

### 1.3. Proof of proposition 1.2

Since the proof of proposition 1.2 proceeds by induction on several parameters, it is convenient to introduce the following notation. For a fixed positive integer \( d \) and a projective space \( \mathbb{P}^A \) I say that assertion \( H(Y,p,q,c,a) \) holds for the smooth projective variety \( Y \subset \mathbb{P}^A \) of dimension \( N + 1 \) > 0 and for the integers \((p,q,c,a) \in \mathbb{N} \times \{-1, \ldots, N-1\} \times \mathbb{N} \times \mathbb{Z}\) if

\[
\mathcal{H}^{q+1}(Y_t, \Omega^p_{Y_T, \mathcal{X}_T|Y_t} \otimes \mathcal{O}(a')) = 0
\]

for all \( a' \leq a \), for all smooth analytic varieties \( T \) endowed with a morphism \( \phi: T \to \mathbb{H}^0(Y, \mathcal{O}(d)) \) such that \( \mathcal{X}_T \) is smooth, and for all \( t \in T \) such that the cokernel of \( d \phi(t) \) is of dimension at most \( c \).

The proof relies on the following four lemmas.

Recall (cf. [G1]) that for any positive integer \( c \), a \( d \)-decomposition of \( c \) is the unique series \((c_d, \ldots, c_r)\) such that \( r \in \{1, \ldots, d\} \), \( c_d > \cdots > c_r \geq r \) and

\[
c = \left( \begin{array}{c} c_d \\ d \end{array} \right) + \cdots + \left( \begin{array}{c} c_r \\ r \end{array} \right).
\]

Let

\[
c_{<d>} = \left( \begin{array}{c} c_d - 1 \\ d \end{array} \right) + \cdots + \left( \begin{array}{c} c_r - 1 \\ r \end{array} \right),
\]

with the convention \( \binom{0}{\beta} = 0 \) for \( \alpha < \beta \).

The asymptotic behaviour of the function \( c \mapsto c_{<d>} \) is described by the following lemma.

**Lemma 1.3.1.** For all integers \( M \geq 2 \) there exists a function \( f_M: ]0,1[ \times \mathbb{N} \to ]0,1[ \), \((\varepsilon,d) \mapsto \varepsilon'\) such that for all \( \varepsilon \in ]0,1[ \), \( \varepsilon' \to \varepsilon \frac{M-1}{M!} \) for \( d \to \infty \), and such that for all nonnegative integers

\[
c \leq (1 - \varepsilon) \frac{d^M}{M!}
\]

one has

\[
c_{<d>} \leq (1 - \varepsilon') \frac{d^{M-1}}{(M-1)!}.
\]

The following lemma is the key ingredient of my proof. It relies on Green’s hyperplane section theorem [G1].

**Lemma 1.3.2.** Let \( Y' \) be a generic hyperplane section of \( Y \). Then for \( q \leq N - 2 \)

\[
\mathcal{H}(Y', p, q, c_{<d>}, a) \Rightarrow \mathcal{H}(Y, p, q, c_{<d>}, a - 1).
\]

By convention, assumption \( \mathcal{H}(Y', -1, q, c_{<d>}, a - 1) \) holds.

The last two lemmas will only be used for \( q = N - 1 \). Both rely on an idea of Green ([G2], p. 193-194).
Lemma 1.3.2. Assume $c > 0$. Then
\[
\begin{align*}
\mathcal{H}(Y, p + 1, q, c - 1, a) \\
\mathcal{H}(Y, p + 1, q - 1, c, a)
\end{align*}
\Rightarrow \mathcal{H}(Y, p, q, c, a).
\]
By convention, assumption $\mathcal{H}(Y, p + 1, -2, c, a)$ holds.

The last lemma has been proven by Nori for $a = 0$ ([10], remark 3.10); the general case involves no new ideas. See also [1] for computations of an explicit bound for $s$.

Lemma 1.3.3. There is an integer $s$ depending only on $Y$ and $O(1)$ such that for all $d \geq p + a + s$ the assertion $\mathcal{H}(Y, p, q, 0, a)$ holds for all $q \in \{-1, \ldots, N - 1\}$.

Lemma 1.3.4 will only be used for $q = N - 1$.

1.3.5. Proof of proposition 1.2. Postponing the proofs of the above Lemmas, I now show how they imply proposition 1.2.

I show more generally that for all $\varepsilon \in [0, 1]$ there is an integer $D$ depending only on $\varepsilon$, $p$, $a$ and $Y$ such that assertion $\mathcal{H}(Y, p, q, c, a)$ holds for all $d \geq D$, $q \in \{-1, \ldots, N - 1\}$ and $c \leq (1 - \varepsilon)\frac{d^{N-q}}{(N-q)!}$. The case $a = 0$ is the proposition 1.2.

I proceed by induction on $N - q$, $c$ and $N$.

Assume $N - q = 1$. The condition on $c$ simplifies to $c \leq (1 - \varepsilon)d$. Choose $D$ such that $\varepsilon D \geq p + a + s$. Thus for $d \geq D$ one has $d \geq (1 - \varepsilon)d + \varepsilon D \geq c + p + a + s$.

The case $c = 0$, $N - q = 1$ follows from Lemma 1.3.4.

The case $c > 0$, $N = 0$ and $q = -1$ follows from Lemma 1.3.3 by increasing induction on $c$. For the case $c > 0$, $N - q = 1$ let $Y'$ be a generic hyperplane section of $Y$. Lemmas 1.3.2 and 1.3.3 provide successive implications
\[
\begin{align*}
\mathcal{H}(Y', p + 1, q - 1, c < d >, a) \\
\mathcal{H}(Y', p, q - 1, c < d >, a - 1)
\end{align*}
\Rightarrow \mathcal{H}(Y, p + 1, q - 1, c, a) \Rightarrow \mathcal{H}(Y, p, q, c, a).
\]

Assertion $\mathcal{H}(Y, p, q, c, a)$ follows by increasing induction on both $c$ and $N$.

Assume $N - q > 1$ and let $Y'$ be a generic hyperplane section of $Y$. By induction on $N$ and by Lemma 1.3.1 applied to $M = N - q$, assertions $\mathcal{H}(Y', p, q, c < d >, a)$ and $\mathcal{H}(Y', p - 1, q, c < d >, a - 1)$ hold. Hence by Lemma 1.3.2 assertion $\mathcal{H}(Y, p, q, c, a)$ holds. \(\square\)

1.4. Proof of the Lemmas.

1.4.1. Proof of Lemma 1.3.1. Let $e$ be the smallest positive integer such that
\[
(1 - \varepsilon)\frac{d^M}{M!} \leq \left(\frac{M + d}{d}\right) - \left(\frac{M + e}{e}\right)
\]
for all $q \in \{0, \ldots, N - 1\}$. One has $e \sim \varepsilon^{1/M}d$ for $d \to \infty$.

The map $c \mapsto c < d >$ is increasing. Thanks to the identities
\[
\begin{align*}
\left(\frac{M + d}{d}\right) - \left(\frac{M + e}{e}\right) &= \sum_{i=1}^{d-e} \left(\frac{M + d - i}{d - i + 1}\right) \\
\left(\frac{M - 1 + d}{d}\right) - \left(\frac{M - 1 + e}{d - e}\right) &= \sum_{i=1}^{d-e} \left(\frac{M - 1 + d - i}{d - i + 1}\right)
\end{align*}
\]
One gets
\[ c_{<d>} \leq \frac{(M - 1 + d)}{d} - \frac{(M - 1 + \epsilon)}{\epsilon}. \]

The term on the right is asymptotically equivalent to \( \varepsilon \frac{M - 1}{M - 1} \frac{d^{M-1}}{(M-1)!} \) for \( d \to \infty \). Define \( \varepsilon' \in ]0, 1[ \) by \( \varepsilon' = \varepsilon \frac{M - 1}{M - 1} \frac{d^{M-1}}{(M-1)!} \geq (M - 1 + d) - (M - 1 + \epsilon) \) and by the formula
\[ (1 - \varepsilon') \frac{d^{M-1}}{(M-1)!} = \left( \frac{M - 1 + d}{d} - \frac{M - 1 + \epsilon}{\epsilon} \right) \]
ontherwise. Thus \( \varepsilon' \in ]0, 1[ \) and \( \varepsilon' \sim \varepsilon \frac{M - 1}{M - 1} \) for \( d \to \infty \). \( \square \)

1.4.2. Proof of Lemma 1.3.2 Let \( i' : Y' \to Y \) be the projective embedding, let \( Y^p_T = Y' \times_T \) and let \( X_t = X_t \times_{\mathcal{Y}_T} Y^p_T \). Fix \( t \in T \).

Since \( Y' \) is generic, one may assume that \( X_t \) is smooth and that \( X_t \) meets \( Y_t' \) transversally in \( Y_t \). Restricting \( T \) if necessary, one may further assume that for all \( u \in T \), \( X_u \) meets \( Y_u' \) transversally in \( Y_u \). Hence there is a well-defined map \( \phi' : T \to \mathbb{P}H^0(Y', \mathcal{O}(d)) \), constructed as follows.

Let \( p \mathcal{Y} : H^0(Y, \mathcal{O}(d)) \setminus \{0\} \to \mathbb{P}H^0(Y, \mathcal{O}(d)) \) be the natural projection and let
\[ p^\ast_Y \phi : T \times_{\mathbb{P}H^0(Y, \mathcal{O}(d))} H^0(Y, \mathcal{O}(d)) \to H^0(Y, \mathcal{O}(d)) \]
be the map deduced from \( \phi \) by base change. Then \( \phi' \) is defined by the map
\[ p^\ast_Y \phi' : T \times_{\mathbb{P}H^0(Y, \mathcal{O}(d))} H^0(Y, \mathcal{O}(d)) \to H^0(Y', \mathcal{O}(d)) \]

obtained by composing \( p^\ast_Y \phi \) with the linear projection \( H^0(Y, \mathcal{O}(d)) \to H^0(Y', \mathcal{O}(d)) \) which maps to 0 the polynomials vanishing on \( Y' \).

Let \( E = p^\ast_Y \text{Im}(d\phi(t)) \), resp. \( E' = p^\ast_{Y'} \text{Im}(d\phi'(t)) \). Let
\[ c = \dim \text{coker}(d\phi(t)) = \text{codim}(E, H^0(Y, \mathcal{O}(d))), \quad \text{resp.} \]
\[ c' = \dim \text{coker}(d\phi'(t)) = \text{codim}(E', H^0(Y', \mathcal{O}(d))). \]

Let \( S = \bigoplus_{\delta \in \mathbb{N}} S_{\delta} \), resp. \( S' = \bigoplus_{\delta \in \mathbb{N}} S'_{\delta} \) denote the homogenous polynomial ring of \( H^0(Y, \mathcal{O}(1)) \), resp. \( H^0(Y', \mathcal{O}(1)) \). Let \( I \subset S \), resp. \( I' \subset S' \) denote the homogenous ideal of \( Y \), resp. \( Y' \). One has \( c = \text{codim}(E + I_d, S_d) \), resp. \( c' = \text{codim}(E' + I'_d, S'_d) \). Since \( Y' \subset Y \) is generic, Green’s hyperplane theorem \([G1]\) asserts that
\[ c' \leq c_{<d>}. \]

Hence the cokernel of \( d\phi'(t) \) is of dimension at most \( c_{<d>} \).

The sheaf \( \Omega^p_{\mathcal{Y}_T, X_T|_Y_T} \) fits into the short exact sequence
\[ 0 \to \Omega^p_{\mathcal{Y}_T, X_T|_Y_T} \otimes \mathcal{O}'_Y(-1) \overset{dH}{\to} \Omega^p_{\mathcal{Y}_T, X_T|_Y_T} \to \Omega^p_{\mathcal{Y}_T, X_T|_Y_T} \to 0 \]
where \( H \in \mathbb{H}^0(Y, \mathcal{O}(1)) \) is an equation of \( Y' \). Restricting this sequence to \( Y_t' \) and tensorising it by \( \mathcal{O}'_Y(a) \) one gets the associated long exact sequence
\[ \mathbb{H}^{q+1}(Y_t', \Omega^{p-1}_{\mathcal{Y}_T, X_T|_Y_T} \otimes \mathcal{O}'_Y(a-1)) \overset{dH}{\to} \mathbb{H}^{q+1}(Y_t', \Omega^p_{\mathcal{Y}_T, X_T|_Y_T} \otimes \mathcal{O}'_Y(a)) \to \mathbb{H}^{q+1}(Y_t', \Omega^p_{\mathcal{Y}_T, X_T|_Y_T} \otimes \mathcal{O}'_Y(a)). \]
The term on the left vanishes by assumption $H(Y', p - 1, q, c_{<d>, a - 1})$ and the term on the right vanishes by assumption $H(Y', p, q, c_{<d>, a})$. Hence the middle term vanishes as well: one has

$$H^{q+1}(Y'_t, \Omega^p_{Y'_t, X'_t|Y'_t} \otimes O'_Y(a)) = 0.$$  

For any coherent sheaf $F$ on $Y$ there is a short exact sequence

$$0 \longrightarrow F \otimes O(-1) \xrightarrow{H} F \longrightarrow i_* i^* F \longrightarrow 0,$$

Take $F = \Omega^p_{Y'_t, X'_t|Y'_t} \otimes O_Y(a)$ and consider the associated long exact sequence

$$H^{q+1}(Y_t, \Omega^p_{Y'_t, X'_t|Y'_t} \otimes O_Y(a - 1)) \xrightarrow{H} H^{q+1}(Y_t, \Omega^p_{Y'_t, X'_t|Y'_t} \otimes O_Y(a)) \longrightarrow H^{q+1}(Y'_t, \Omega^p_{Y'_t, X'_t|Y'_t} \otimes O_Y(a)).$$

The term on the right vanishes by (1). Since $q \in \{0, \ldots, N - 1\}$, by increasing induction on $a$ the term on the right vanishes as well, the case $a \ll 0$ following from Serre’s vanishing theorem. Hence the term in the middle vanishes. 

1.4.3. Proof of Lemma 1.3.3. Since the assertion depends only on the tangent space of $T$ at $t$, one may replace $T$ by a small open neighbourhood of $t$ in the tangent space of $T$ at $t$; hence one may assume $T = B \times S$, where $B \subset \mathbb{P}^d(Y, O(d))$ is an open neighbourhood of $\phi(t)$ of codimension $c$ and $S$ is an open ball.

Let $B' \subset \mathbb{P}^d(Y, O(d))$ be a smooth analytic variety containing $B$ as a codimension $1$ subvariety, and let $T' = B' \times S$. Then $\mathcal{Y}_T \subset \mathcal{Y}_{T'}$ is a smooth codimension $1$ subvariety and one has the following short exact sequence of sheaves on $\mathcal{Y}_T$:

$$0 \longrightarrow \Omega^p_{\mathcal{Y}_T, X_T} \longrightarrow \Omega^{p+1}_{\mathcal{Y}_{T'}, X_{T'}|\mathcal{Y}_T} \longrightarrow \Omega^{p+1}_{\mathcal{Y}_{T'}, X_{T'}} \longrightarrow 0.$$

Restrict this sequence to $Y$, tensorise it with $O(a)$ and consider the associated long exact sequence

$$H^q(Y_t, \Omega^p_{\mathcal{Y}_T, X_T|Y_t} \otimes O(a)) \longrightarrow H^{q+1}(Y_t, \Omega^p_{\mathcal{Y}_{T'}, X_{T'}|Y_t} \otimes O(a)) \longrightarrow H^{q+1}(Y_t, \Omega^{p+1}_{\mathcal{Y}_{T'}, X_{T'}|Y_t} \otimes O(a)).$$

The terms on the right and on the left vanish by assumption, hence the term in the middle vanishes as well. 

2. Proof of Theorem 2.2

Consider the exact sequence

$$R^k \pi_*(C) \rightarrow R^k \pi_*(j_{T*}C) \rightarrow R^k \pi_*(C \rightarrow j_{T*}C) \rightarrow R^{k+1} \pi_*(C \rightarrow R^{k+1} \pi_*(j_{T*}C)$$

and let $\mathcal{H}^k_{X_T, v} = \text{coker} (R^k \pi_*(C \rightarrow R^k \pi_*(j_{T*}C))$ denote the locally trivial subsheaf of $R^k \pi_*(C \rightarrow j_{T*}C)$ whose fiber at any point $t \in T$ is canonically isomorphic to $H^k(X_t, C)_v$.

Let $F^p H^0 \left(T, \mathcal{H}^k_{X_T, v}\right)$ denote the space of sections of $\mathcal{H}^k_{X_T, v}$, whose fiber at any point $t \in T$ belongs to $F^p H^0(X_t, C)_v$.

In [O] I have shown
Theorem ([Q]). For all integers \( b \in \mathbb{N}^+ \) there is an integer \( D \) depending only on \( b \), \( Y \) and on \( \mathcal{O}(1) \), such that for all \( d \geq D \), for all \( q \in \{[(N + 1)/2], \ldots, N - 1\} \), for all

\[
c \leq b^d \frac{d^{N-q}}{(N-q)!}
\]

for all codimension \( c \) algebraic varieties \( T \subset \mathbb{P}^0(Y, \mathcal{O}(d)) \) lying in the locus parametrizing smooth hypersurfaces and for all \( t \in T \), the following holds.

If \( N \) is even and \( q = N/2 \) then the space \( F^{N-q}\mathbb{H}^0(T, \mathcal{H}_{X_T, v}^N) \) is spanned by families of algebraic cycles of codimension \( q \) and degree at most \( b \).

If \( q > N/2 \) then \( F^{N-q}\mathbb{H}^0(T, \mathcal{H}_{X_T, v}^N) = 0 \).

The proof of [Q] gives in fact a stronger result, namely one can replace the subspace \( T \subset \mathbb{P}^0(Y, \mathcal{O}(d)) \) by any algebraic variety \( T \) endowed with a morphism \( \phi: T \to \mathbb{P}^0(Y, \mathcal{O}(d)) \), such that

- \( \phi(T) \) lies in the locus parametrizing smooth hypersurfaces and
- for all \( t \in T \) the cokernel of the differential \( d\phi(t): T_{T,t} \to T_{\mathbb{P}^0(Y, \mathcal{O}(d)), \phi(t)} \) is of dimension at most \( c \);

This stronger version of the theorem of [Q] is equivalent to Theorem 2. Indeed, one only needs to show

\[
F^{N-q}\mathbb{H}^0(N\mathcal{X}_T, \mathbb{C})_v \simeq F^{N-q}\mathbb{H}^0(T, \mathcal{H}_{X_T, v}^N).
\]

But the Lefschetz hyperplane Theorem and the Leray spectral sequence give a canonical isomorphism

\[
\mathbb{H}^N(N\mathcal{X}_T, \mathbb{C})_v \simeq \mathbb{H}^0(T, \mathcal{H}_{X_T, v}^N).
\]

Since for any \( t \in T \), the restriction map \( \mathbb{H}^N(N\mathcal{X}_T, \mathbb{C})_v \to \mathbb{H}^N(\mathcal{X}_T, \mathbb{C})_v \) is a homomorphism of Hodge structures, hence is strict, the result follows. \( \square \)

3. Proof of Theorem 3

Since \( q \geq N/2 \), the generic hypersurface of degree \( d \) containing \( V \) is smooth; let \( T \subset \mathbb{P}^0(Y, \mathcal{O}(d)) \) be a smooth open affine subvariety of the space of all smooth hypersurfaces containing \( V \). One has \( c \sim_d \to \infty \frac{d^{N-q}}{(N-q)!} \).

For all \( i \in \{1, \ldots, p - q\} \) choose hyperplane sections \( H_i \subset T \) in general position and and let \( f_i \in \mathbb{H}^0(T, \mathcal{O}_T) \) be the equations of \( H_i \); Let \( P_i = \bigcap_{j \geq i} H_j \), \( Q_i = \bigcup_{j \leq i} H_j \) and \( T_i = P_i \setminus (P_i \cap Q_i) \). For \( j \leq i \) the functions \( f_j \) do not vanish on \( T_i \), hence belong to \( \mathbb{H}^0(T, \mathcal{O}^*_T) \) and there are well-defined higher cycles in \( \mathbb{H}^q+i(\mathcal{X}_{T_i}, i) \)

\[
Z_i = \varphi^i_{1,1}([V \times T_i], f_1, \ldots, f_i).
\]

I show by increasing induction on \( i \) that \( \varphi^{q+i,i}_* (Z_i) \neq 0 \). Case \( i = p-q \) implies Theorem 3.

Case \( i = 0 \) a straightforward adaptation of [V2], Proposition 4: for \( d \gg 0 \) and for any open subset \( U_0 \subset T_0 \) the cohomology class of the cycle \( [V \times U_0] \in \mathbb{H}^q(\mathcal{X}_{U_0}) \) does not belong to \( j_{U_0}^*(\mathbb{H}^{2q}((\mathcal{Y}_{U_0}), \mathbb{C})) \).
Assume $i > 0$. Then $T_{i-1} \cup T_i = P_t \setminus (P_t \cap Q_{i-1})$, hence $T_{i-1}$ is a hyperplane section of $T_{i-1} \cup T_i$ and $T_i$ is the complementary open set. Hence there is a commutative diagram
\[
\begin{array}{ccc}
\text{CH}^{q+i}(X_t, i) & \longrightarrow & \text{CH}^{q+i-1}(X_{t-1}, i - 1) \\
\downarrow l_{q+i,i} & & \downarrow l_{q+i-1,i-1} \\
F^{q+i}H^{2q+i+1}(Y_t, X_t, \mathbb{C}) & \longrightarrow & F^{q+i-1}H^{2q+i}(Y_{t-1}, X_{t-1}, \mathbb{C})
\end{array}
\]
where res is the residue map and $l_{q+i,i}$ is the linking map for the long exact sequence of higher Chow groups. One has
\[
l_{q+i,i}(Z_i) = (l_{q,0} \circ \psi_{1,1}^i)([V \times T_i], f_1, \ldots, f_i) = \psi_{1,1}^i([V \times T_{i-1}], f_1, \ldots, f_{i-1}) = Z_{i-1}.
\]
The induction assumption $\text{reg}_{q+i-1,i-1}(Z_{i-1}) \neq 0$, and the commutativity of the diagram implies $\text{reg}_{q+i,i}(Z_i) \neq 0$.

\section{Proof of Theorem 4}

4.1. Construction of the cycle $Z_U$. Let $P \subset Y$ be a codimension $q$ linear subspace, let $R \subset Y$ be a codimension $q+2$ linear subspace such that $R \subset P$ and let $S$ be the blow-up of $P$ along $R$. There are natural maps $\psi: S \to \mathbb{P}^1$ (where $\mathbb{P}^1$ parametrizes hyperplanes of $P$ containing $R$) and $s: S \to Y$. For any $x \in \mathbb{P}^1$ let $L_x = s(\psi^{-1}(x))$ denote the codimension $q+1$ linear subspace of $Y$. For any subvariety $B \subset \mathbb{P}^0(Y, \mathcal{O}(d))$ let $C_B = S \times_Y X_B$ and let $\psi_B: C_B \to \mathbb{P}^1$ and $s_B: C_B \to X_B$ be the morphisms deduced from $\psi$ and $s$ by base change.

Let $U \subset \mathbb{P}^0(Y, \mathcal{O}(d))$ be the locus of hypersurfaces $X_t$ such that $[L_0 \cap X_t] = dR$ and $[L_\infty \cap X_t] = dR$. One has codim $(U, \mathbb{P}^0(Y, \mathcal{O}(d))) \sim_d \infty \frac{2d^{q+2}}{(q+2)^q}$. By construction, one has $s_U(\psi_U^{-1}(0)) = R \times U = s_U(\psi_U^{-1}(\infty))$, hence the couple $(C_U, \psi_U)$ defines an element $Z_U \in \text{CH}^{q+1}(U, 1)$.

4.2. Proof of $\text{reg}_{q+1,1}(Z_U) \neq 0$. Let $T \subset \mathbb{P}^0(Y, \mathcal{O}(d))$ be the locus of hypersurfaces $X_t$ such that $L_0 \subset X_t$ and $[L_\infty \cap X_t] = dR$. Then $T \subset T \cup U$ is a hyperplane section and $U \subset T \cup U$ is the complementary open set. Hence there is a natural linking homomorphism $l_{q+1,1}: \text{CH}^{q+1}(X_U, 1) \to \text{CH}^{q}(X_T)$. Since $C_T$ is the union of a variety dominating $\mathbb{P}^1$ and of the variety $\psi_T^{-1}(0) \simeq L_0 \times T$, one has $l_{q+1,1}(Z_U) = d[s_T(\psi_T^{-1}(0))] = d[L_0 \times T]$.

I now proceed as in section 3. A straightforward adaptation of the argument of [V2], Proposition 3 shows that for $d \gg 0$ the class $\text{cl}_q(d[L_0 \times T]) \in H^{2q}(X_T, \mathbb{C})$ does not belong to $j_T^*(H^{2q}(Y_T, \mathbb{C})$, hence $\text{cl}_q(d[L_0 \times T]) \in H^{2q+1}(Y_T, X_T, \mathbb{C})$ is non-zero. Since the diagram
\[
\begin{array}{ccc}
\text{CH}^{q+1}(X_U, 1) & \longrightarrow & \text{CH}^q(X_T) \\
\downarrow l_{q+1,1} & & \downarrow l_q \\
H^{2q+2}(Y_U, X_U, \mathbb{C}) & \longrightarrow & H^{2q+1}(Y_T, X_T, \mathbb{C})
\end{array}
\]
is commutative, it follows that $\text{reg}_{q+1,1}(Z_U) \neq 0$. 

4.3. **Indecomposability of the cycle** $Z_U$. Since $\text{reg}_{q+1,1}(Z_U) \neq 0$, it is enough to show that the space of decomposable cycles is contained in the kernel of $\text{reg}_{q+1,1}$.

Let $i \in \{1, \ldots, q\}$, $Z'_U \in \text{CH}^i(\mathcal{X}_U)$ and $Z''_U \in \text{CH}^{q-i}(\mathcal{X}_U, 1)$. One has to show $\text{reg}_{q+1,1}(\phi_i(Z'_U \otimes Z''_U)) \in \text{ker} F^{q+1}\text{H}^{2q+1}(\mathcal{Y}_U, \mathbb{C})$. Since the diagram

$$
\begin{array}{ccc}
\text{CH}^i(\mathcal{X}_U) \otimes \text{CH}^{q+1-i}(\mathcal{X}_U, 1) & \xrightarrow{\phi_i, 0} & \text{CH}^{q+1}(\mathcal{X}_U, 1) \\
\downarrow \text{cl}_i \otimes \text{reg}_{q+1,1-i, 1} & & \downarrow \text{reg}_{q+1,1} \\
F^i\text{H}^{2i}(\mathcal{X}_U, \mathbb{C}) \otimes F^{q-i+1}\text{H}^{2q-2i+1}(\mathcal{X}_U, \mathbb{C}) & \xrightarrow{\sim} & F^{q+1}\text{H}^{2q+1}(\mathcal{X}_U, \mathbb{C}),
\end{array}
$$

is commutative, it is enough to show $\text{cl}_i(Z'_U) \in \text{H}^{2i}(\mathcal{Y}_U, \mathbb{C})$ and $\text{reg}_{q+1,1-i, 1}(Z''_U) \in \text{H}^{2q-2i+1}(\mathcal{Y}_U, \mathbb{C})$.

If $i < q$ this follows from the isomorphisms given by Theorem \[\Box\]

$$
F^i\text{H}^{2i}(\mathcal{X}_U, \mathbb{C}) \simeq F^i\text{H}^{2i}(\mathcal{Y}_U, \mathbb{C}),
$$

$$
F^{q-i+1}\text{H}^{2q-2i+1}(\mathcal{X}_U, \mathbb{C}) \simeq F^{q-i+1}\text{H}^{2q-2i+1}(\mathcal{Y}_U, \mathbb{C}).
$$

If $i = q$, one has

$$
\text{CH}^1(\mathcal{X}_U, 1) \simeq \text{H}^0(U, \mathcal{O}^*(U)) \simeq \text{CH}^1(\mathcal{Y}_U, 1),
$$

hence $\text{reg}_{1,1}(Z''_U) \in \text{H}^1(\mathcal{Y}_U, \mathbb{C})$; on the other hand, since $\mathcal{X}_U$ does not contain any flat family of cycles of codimension $q$ and degree less than two, by Theorem \[\Box\] one has $F^q\text{H}^{2q}(\mathcal{X}_U, \mathbb{C}) \simeq F^q\text{H}^{2q}(\mathcal{Y}_U, \mathbb{C})$, hence $\text{cl}_q(Z'_U) \in \text{H}^q(\mathcal{Y}_U, \mathbb{C})$. \[\Box\]

4.4. **Indecomposability of the restriction of** $Z_U$ **to a very general fiber** $\mathcal{X}_u, u \in U$.

This follows from the indecomposability of $Z_U$ by Proposition 5 of \[\Box\].

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