Propagation and ramification of a solitary pulse through an environmentally coupled qubit

Yibo Gao\textsuperscript{1}, Shijie Jin\textsuperscript{1} and Hou Ian\textsuperscript{2,3,∗}

\textsuperscript{1} College of Applied Sciences, Beijing University of Technology, Beijing, People’s Republic of China
\textsuperscript{2} Institute of Applied Physics and Materials Engineering, University of Macau, Macau, People’s Republic of China
\textsuperscript{3} Zhuhai UM Science & Technology Research Institute, Zhuhai, Guangdong, People’s Republic of China

∗ Author to whom any correspondence should be addressed
E-mail: houian@um.edu.mo

Keywords: soliton, superconducting qubit, microwave quantum optics, atom-field interaction

Abstract

We compute the relaxations experienced by a superconducting qubit and the simultaneous variation induced on the shape of a microwave pulse during the propagation of the pulse through the qubit. The environmentally affected propagation and the dressed relaxations are accounted by a microscopic-master-Maxwell equation pair. It is shown that the qubit longitudinal relaxation vanishes when the pulse envelope adopts a solitonic shape of $\pi n$ area whereas its transverse relaxation vanishes when the pulse phase has a periodic variation that is orthogonal to the spectral density of the environment. The pulse would propagate absorption-free when its area matches $2\pi n$. Otherwise, the environmental feedback decelerates the velocity of the soliton envelope and induces an monotonic increase of phase in the microwave. A pulse of non-$2\pi n$ area thus ramifies into a transparent part that travels absorption-free at incident velocity and a slowing part that decays through space. The ramification explains the environmental origin of pulse splitting observed in self-induced transparency.

1. Introduction

1.1. Qubit–pulse interactions

Superconducting circuit systems comprising one or a few qubits are controlled by microwave pulses. When coupled with a cavity field residing in a coplanar stripline resonator, these qubits form a pulse-controlled circuit quantum electrodynamic (cQED) system \cite{1, 2}, which serves as the foundation of solid-state entanglement generation \cite{3, 4} and quantum computing \cite{5, 6}. For examples, resonant and dispersive square pulses with appropriate lengths are transmitted to read out qubit states \cite{7}, to perform $x$- and $y$-rotations to achieve GHZ states \cite{8}, and recently to raise gating fidelity \cite{9}. In general, the shapes and lengths of the microwave pulses are proven to be pivotal to the desired operations on the qubits for computing purposes \cite{10, 11}. The operations involved are based on the Rabi oscillations of the Jaynes–Cummings Hamiltonian that models the cQED system and its extension to dispersive coupling regime.

From a quantum optical point of view, a superconducting qubit when biased to an optimal point is essentially a two-level artificial atom and many circuit versions of optical effects \cite{12} entail from the qubit interactions with the microwave pulses traveling in the waveguides. These effects include tunability in electromagnetically induced transparency (EIT) by dressing the artificial atom \cite{13}, coexistence of EIT and Autler–Townes splittings \cite{14}, and parametric amplification \cite{15}. The circuit EIT effect is experimentally verified \cite{16} under an improve method using polariton states \cite{17}. These studies are all considered under the umbrella of qubit-field interactions where the microwave driving field is continuous, i.e. square pulses at long-time equilibrium limit.

Less understood is the qubit-field interaction where the driving field adopts a time-varying envelope. Recent experimental advances demonstrate that engineering a slow-varying envelope can rapidly reset
Figure 1. Illustration of the model (not drawn to scale): a superconducting qubit (detail circuit model shown in the inset, typical dimension measured at 300 μm for a transmon qubit [15]) in the vicinity of a coplanar waveguide (black core line surrounded by dark gray ground strips) facing a traveling microwave field (envelope drawn in orange, wavelength measured at 2 cm for a typical 5 GHz microwave on a silicon substrate).

superconducting qubits [18]. From a control-theoretic perspective, permitting drivings to vary arbitrarily in time leads to more efficient optimal control [19]. However, a quantum optical theory of superconducting qubits interacting with finite pulses of time-dependent envelopes is not yet sufficiently developed, especially when the qubits are environmentally coupled. To address the insufficiency, we study here the time variations of both the envelope and the phase of the pulse of solitary pulses when it propagates through one single qubit while that qubit is interacting simultaneously with a bath. Meanwhile, we also study the reaction of the qubit when it faces the incoming propagation.

Regarded as a scattering problem, the qubit–pulse interaction is analogous to the classical effect of self-induced transparency (SIT) [20, 21], which investigate the propagation of narrow pulses in resonant atomic media, and its related study on backward scattering [22]. Indeed, for a coherent square pulse, part of the incident photon is reflected [23] and the energy dissipation can channel through the reflected photon [24]. Therefore, we adopt a methodology approach similar to that of SIT, i.e. deriving an equation pair that describes the intertwining motions of the pulse and the qubit and computing analytical solutions to this pair. Nevertheless, there are two major distinctions from SIT.

First, SIT studies solitary optical pulses whose widths are much shorter than the length of the atomic medium (e.g. 7 nsec pulses in a 1 mm Rb-sample at density \( N = 10^{11} \text{ cm}^{-3} \) [25]). In a superconducting circuit, as illustrated in the model figure 1, the microwave pulse is longer than the dimensions of a superconducting qubit. The pulse usually falls into the nanosecond range [7], which translates to a length on the scale of centimeter, when propagating in a silicon-substrated circuit and facing a qubit typically measured at only 300 μm [15]. Due to this consideration, backward scattered propagation is neglected, since we focus only on the envelopes with a rate of variation slower than the product of the transition frequency and the atomic polarization [22, 26]. Secondly, the original SIT studies disregard the environment, for which the relaxation times \( T_1 \) and \( T_2 \) are assumed infinite and the atomic motion part is described by a Bloch equation. In contrast, we promotes the Bloch equation to an adiabatic master equation to take qubit-bath interactions into account. Moreover, only one artificial atom is considered, rather than an atomic ensemble.

1.2. Controlling relaxation rates
The SIT effect reveals that a light pulse of solitonic shape can travel through an atomic media absorption-free when the pulse area is a multiple of \( \pi \) [27]. Following the analogy between SIT and qubit–pulse interaction, a natural question is whether microwave pulses can be used to eliminate qubit relaxations. Multiple approaches have targeted qubit decoherence. First, the problem is algebraically approached through decoherence-free subspaces [28], which is implemented in a superconducting circuit using Purcell effect [29]. The second approach is device-based. For instance, the introduction of transmon [30] permits fine tuning of the qubit anharmonicity, prolonging the decoherence times to over 2 μs [31]. With modified circuit cavity orientation, energy relaxation time of 92 μs is obtained [32]. Even with normal cavities, modifying the transmon geometry raises the coherence times to the order of tens of μs [33]. The third and most recent approach is through the technique of extrapolation [34], which given the same noisy measurements improves the effective coherence times by several folds [35].

In contrast to all the above, we take a quantum-optical approach that follows from the analytical solution to the Maxwell-master equation pair to determine the relaxation rates, both longitudinal and transverse, of a qubit under pulse propagation. In particular, we seek the conditions under which these rates vanish when the propagating pulse is an SIT pulse of solitonic shape.

We find that the qubit motion contributes a complex inhomogeneous term to the Maxwell equation. The longitudinal relaxation accumulated on the qubit, which associates with the imaginary amplitude of
this term, would vanish when the pulse envelope adopts a hypersecant solution and its time integral has an asymptotic value of $n\pi$. Originated from the terminology of magnetic spins, the longitudinal relaxation associates with the diagonal elements of the density matrix for the two-level qubit. It describes the dynamic redistribution of the two elements regarded as populations of the ground and excited states, due to processes of environmental influence such as spontaneous radiation. The transverse relaxation, on the other hand, associates with the off-diagonal elements and is cognate with both the phase variations between the quantum states and the population. In the complex inhomogeneous term we derive, it corresponds to the real amplitude part and is determined by the integral of the product of two time functions. Treating the integral formula as an inner product of the space of time functions, zero transverse relaxation is theoretically obtainable when one time function (the spectral function of the bath) becomes orthogonal to the other (the sine of the pulse phase).

In addition to the properties about relaxation rates, the hypersecant solution further implies the sustenance of envelope shape during propagation when the envelope encloses a $2n\pi$ asymptotic area. In other words, an $2n\pi$-pulse propagates absorption-free through the qubit while keeping the qubit free from longitudinal relaxation. A general pulse with a non-$2n\pi$ area experiences a ramification into a $2n\pi$-hypersecant part that travels at the incident velocity and a remainder part that travels at a slower speed, being dragged by the environmental feedbacks. We remark that earlier interpretations of pulse ramifications related to SIT [26] points to a mathematical origin that double-peak solutions and hypersecant solutions are Backlund transformable to each other. This interpretation neither addresses the physical origin of ramification nor considers environmental effects. On the contrary, we focus on the environmental origin of pulse splitting observed in SIT [21, 25], which provides clues for relaxation control using microwave pulses in superconducting circuits. The similarities and differences between SIT and our treatment are summarized in table 1.

Our systematic study addresses both the concerns of the relaxation rates and the description of propagation. We begin the study by describing the qubit–pulse interaction model in section 2 and follow with the derivation of the relaxation factor in section 3. The full solutions of the pulse envelope and phase are presented in section 4 along with the discussion of pulse ramifications, before a conclusion is given in section 5.

### Table 1. Summarized comparisons of pulse propagation through a medium: atomic SIT vs our approach for microwave pulses through a superconducting qubit.

| Interaction Hamiltonian | Time-dependent semi-classical |
|-------------------------|-------------------------------|
| Propagating medium      | Natural atoms                 |
| Equation for medium     | Optical Bloch equation        |
| Propagating pulse type  | Laser pulse                   |
| Equation for pulse      | Maxwell equation              |
| Relative scale          | Wavelength $\ll$ medium dimension |
| Environmental coupling  | Ignored                       |
| Effect of pulse shape   | Pulse absorption              |
| Interpretation of pulse splitting | Mathematical Backlund-transform induced |

2. Qubit–pulse interaction in thermal environment

We begin the derivation by assuming the electric field of an incident microwave pulse take the form

$$ E(x, t) = \mathcal{E}(x, t) \cos [\varphi(x, t) - kx + \omega t] $$

(1)

where $\mathcal{E}(x, t)$ and $\varphi(x, t)$ denote, respectively, its envelope and phase during its traveling along a waveguide. $\omega$ denotes the frequency and $k$ the associated wavevector of the carrier wave. In equation (1), $x$ and $t$ denote laboratory spatial and time coordinates but they are customarily compressed [27, 36] into the single variable $\tau = t - x/v$ of local time under the reference frame that travels along with the wavefront. When dispersive effects are not considered, the velocity $v$ of the wavefront is assumed constant, whose value depends on the pulse shape, and the electric field becomes a single-variable function $E(\tau) = \mathcal{E}(\tau) \cos [\varphi(\tau) + \omega \tau]$ under the traveling reference frame. The system, illustrated in figure 1, is described by the time-dependent Hamiltonian ($\hbar = 1$)

$$ H_\text{q} = \frac{\omega_0}{2} \sigma_z + \mu E(\tau) [\sigma_+ + \sigma_-] $$

(2)
where $\omega_0$ and $\mu$ are the transition frequency and the effective dipole moment, respectively, of the qubit. In other words, $E(\tau)$ is non-zero during the propagation of the pulse through the qubit; otherwise, $E(\tau)$ vanishes, letting the qubit evolve freely and the pulse travel freely. If $E(\tau)$ remains constant, the system reduces back to the one-dimensional waveguide QED system [37, 38], on which the circuit analogue of resonant fluorescence for instance can be realized [39].

Expanding $E(\tau)$ in its amplitude and phase in the semiclassical Hamiltonian (2) and dropping the counter-rotating terms, the system Hamiltonian reads

$$H'_{\xi}(\tau) = \frac{\Delta}{2} \sigma_z + \frac{\mu E(\tau)}{2} \left[ \sigma_+ e^{-i\phi(\tau)} + \sigma_- e^{i\phi(\tau)} \right]$$

in the rotating frame $e^{i\omega_0 \tau/2}$ of the microwave carrier, i.e. after a unitary transformation of $U(\tau) = \exp \{ i(\omega \sigma_z \tau/2) \}$. The symbol $\Delta = \omega_0 - \omega$ indicates the qubit-field detuning. Then, through the dressed states

$$|\nu_+ (\tau)\rangle = e^{-i\phi(\tau)} \cos \theta(\tau) |e\rangle - \sin \theta(\tau) |g\rangle,$$

$$|\nu_- (\tau)\rangle = e^{-i\phi(\tau)} \sin \theta(\tau) |e\rangle + \cos \theta(\tau) |g\rangle,$$

$H'_{\xi}$ is diagonalized with eigenvalues $\pm \Omega/2$ where $\Omega = \sqrt{\Delta^2 + (\mu E)^2}$ and $\theta(\tau) = \frac{1}{2} \tan^{-1}(\mu E/\Delta)$ is the transformation angle. When the pulse is not overlapping with the qubit, the two dressed states assume the asymptotic $|e\rangle$ or $|g\rangle$ of the bare states.

The environmental effects to the qubit are modeled on a multi-mode-resonator bath with free Hamiltonian $H_B = \sum \omega_j a_j^\dagger a_j$. Paired with the diagonalized $H'_{\xi} = \Omega/2 \{ |\nu_+ \rangle \langle \nu_+ | - |\nu_- \rangle \langle \nu_- | \}$, the Universe has the Hamiltonian $H = H'_{\xi} + H_B + H_I$, where the system-bath coupling is the tensor product

$$H_I = h_S \otimes h_B = |e\rangle \langle g| + \langle g| e\rangle \otimes \sum_j g_j(a_j + a_j^\dagger).$$

Since the system eigenstates do not remain static but rather follow the change in the amplitude of the microwave pulse, the basis for system-bath coupling should align with the time-dependent basis in equations (4) and (5) to take into account of dressed relaxations [13, 40]. While the dressed system and the bath mutually affect each other during the process of qubit-field interaction, we consider the slow-varying envelope case $dE/d\tau \ll \omega_0 E$ as illustrated in figure 1, making the evolution adiabatic on the system side [41]. To validate the adiabaticity, consider the downward transition rate $\langle \nu_- | H'_{\xi} | \nu_+ \rangle / \Omega$ between the two dressed states (4) and (5) as dictated by the Fermi golden rule, where the denominator is the difference between the eigenenergies of the states. For the numerator, $H'_{\xi}$ under the bare-state basis has $(\dot{E} + i \dot{\phi} E) e^{i\phi}$ and its complex conjugate as the off-diagonal elements. Since the phase variation $\dot{\phi}$ in time is environmentally induced during the qubit–pulse interaction, its magnitude is secondary to $\dot{E}$ and its effect can be omitted at first-order expansion of $H'_{\xi}$. Then transforming to the dressed-state basis,

$$\langle \nu_- | H'_{\xi} | \nu_+ \rangle = \frac{i}{2} \dot{E} \cos 2\theta.$$ For a resonant pulse with a slow-varying envelope, both the transformation angle ($2\theta \approx \pi/2$) and the time derivative ($\dot{E} \ll \omega_0 E$) guarantee a vanishing downward transition rate between the dressed levels. The same considerations also apply to the upward transition and thus the adiabaticity is validated. Following the reference frame of the transient pulse, we therefore consider the evolution of the system density matrix $\rho' = U_{ad} \rho U_{ad}^\dagger$ under the picture of adiabatic transformation, where

$$U_{ad}(\tau) = |\nu_+(\tau)\rangle \langle \nu_+(0)| e^{-i\phi(\tau)} + |\nu_-(\tau)\rangle \langle \nu_-(0)| e^{-i\phi(\tau)}$$

denotes the unitary matrix up to time $\tau$. In the matrix,

$$\phi_\pm(\tau) = \int_0^\tau ds \left[ \pm \Omega(s) - i \langle \nu_\pm(s)| \dot{\nu}_\pm(s) \rangle \right]$$

expresses the total phase, i.e. both the dynamic (the first term) and the geometric phase (the second term), culminated in each dressed state.

Our considerations begin with the Liouville equation for the system density matrix $\rho$ in the Schrödinger picture, which reads

$$\frac{d\rho'}{d\tau} = -\int_0^\tau ds \left[ [H_I(\tau), [H_I(s), \rho'(\tau) \otimes \rho]] \right]$$

where the integral corresponds to the first nontrivial term in the perturbative expansion of time-ordered evolution of the Universe. The density matrix $\rho$ of the bath will be partial-traced out when taking the
ensemble average. This integral represents the feedback from the bath onto the system during the pulse propagation under the Born–Markov approximation. Expanding the double commutator will give four terms involving both \( \tau \) and \( \tau - s \). Those related to the bath part only involves double-time correlations when taking the trace and read \[ (h_\text{B}(\tau)h_\text{B}(s)) = \sum_j g_j^2 e^{-i\omega_j(\tau-s)}. \] (10)

Those related to the system can be considered separately. Following the method \[42\], the feedback can be recorded by reversing the direction of time \((\tau - s \rightarrow s)\) such that \(U_{\text{sd}}(s) = \exp \{i(\tau - s)H_5(\tau)\} U_{\text{sd}}(\tau)\).

### 3. Relaxation factor

Expanding the commutators and tracing out the bath operators in the Liouville equation yields the microscopic master equation in the Lindblad form

\[
\frac{d\rho}{d\tau} = -i[H_s, \rho] + \gamma(\Omega) \left[ \cos^2 2\theta \cos^2 \varphi + \sin^2 \varphi \right] \left[ \hat{\sigma}_- \rho \hat{\sigma}_+ - \frac{1}{2} \{\hat{\sigma}_+ \hat{\sigma}_-, \rho\} \right] \tag{11}
\]
after converting to the Schroedinger picture. The Pauli matrices are hatted to indicate that the dressed basis is assumed, e.g. \(\hat{\sigma}_x = |\nu_+),(\tau)\rangle \langle \nu_-,(\tau)| + |\nu_-),(\tau)\rangle \langle \nu_+,(\tau)|\).

\[
\gamma(\Omega) = 2\pi \sum_j g_j^2 \delta(\omega_j - \Omega) \tag{12}
\]
denotes the spectral density distribution of the bath stemming from the integration, which is essentially the Fourier transform of equation (10). The detailed derivations of equation (11) is given in appendix A. It can already be observed from equation (11) that the environmental influence described by the second term is highly dependent on the transformation angle \(\theta\) and the phase \(\varphi\), where the resonance regime \((\theta = \pi/4)\) and the dispersive regime \((\theta \approx 0)\) represents two limiting cases.

To quantify this influence, we consider the qubit polarization \(P(\tau) = \mu \text{ tr} \{\hat{\rho}(\tau)\} \) as a time-dependent response to the incident pulse, where the trace is taken over the dressed system basis. From equation (11), one can derive that \(P(\tau) = \mu F e^{i(\varphi + \omega \tau)/2} + \text{ h.c.} \) where \(F\) indicates a \(\rho(\tau_0)\)-dependent complex factor. For a resonant pulse \((\delta = 0)\) interacting with a ground-state qubit akin to the case of SIT \[20\], the real and imaginary parts read

\[
\Re\{F\} = 1 - e^{-\Gamma(\tau)}, \tag{13}
\]
\[
\Im\{F\} = -e^{-\Gamma(\tau)/2} \sin \int_0^\tau \Omega(s)ds. \tag{14}
\]

In the two parts of the factor,

\[
\Gamma(\tau) = \int_0^\tau ds \gamma(\Omega) \sin^2 \varphi(s) \tag{15}
\]
converts the spectral function \(\gamma(\Omega)\) into the time domain to determine the effective decay in the response of the polarization. It can be regarded as the bath-spectrum-weighted transform of equation (12) and thus a decay factor corresponding to the bath correlations prescribed in equation (10). Since the polarization derived from the density matrix is proportional to the qubit susceptibility \[13\], the two parts in equations (13) and (14) associates with, respectively, the transverse and the longitudinal relaxations of the qubit. If the resonant pulse is interacting rather with a qubit of total population inversion, the case is akin to that of quantum amplification \[36\], whence the sign of \(\Im\{F\}\) flips whilst the rest remains unchanged.

Considering the complex factor \(F\) as a function of \(\Omega(\tau)\) and \(\varphi(\tau)\), we observe that the qubit relaxations are controlled by the incident microwave pulse. As a typical case, the qubit–pulse resonance scenario, i.e. \(\Omega = \mu E\), shows that the longitudinal part is determined by both the pulse envelope \(E(\tau)\) and the pulse phase \(\varphi(\tau)\), the latter implied in the decay factor \(\Gamma(\tau)\). Since \(E(\tau)\) appears as the integrand in the argument of a sine function, extending the upper limit of integration to asymptotic values (i.e. \(\lim_{s \to \infty} \int_0^s \Omega(s)ds\)) shows that if the full area under the envelope is \(\pi\), \(\Im\{F\}\) and thus longitudinal relaxation of the qubit would vanish. The condition for zero transverse relaxation \(\Re\{F\}\), on the other hand, requires a vanishing \(\Gamma(\tau)\). The integral formula (15) for \(\Gamma(\tau)\) has a product of two factors \(\gamma(\Omega)\) and \(\sin^2 \varphi\) as its integrand. Since both of the factors are time functions, where the time dependence of \(\gamma\) deduces from that of \(\Omega(\tau)\), equation (15) can be regarded as the definition of an inner product of the function space over the real-number field of time \(s\). Consequently, an orthogonality condition is established between pairs of time...
functions, i.e. the pairs that render the inner product to vanish. It signifies the vanishment of transverse relaxation when one has matching time variations of the pulse phase \( \varphi(t) \) in \( \sin^2 \varphi \) and of the eigenvalue \( \Omega(t) \) in \( \gamma(\Omega) \) to obey the orthogonality. In other words, if the spectral distribution of the environment is determinable, a pulse with appropriate phase variation can be designed to cancel out the dephasing of the qubit during the pulse propagation. Most of the existing experimental investigations rely on a constant-phase carrier \( \cos(\omega \tau) \), typically generated by an oven-based fixed frequency microwave generator. With the advent of high-speed arbitrary wave generators (AWG) of effective bandwidth over 10 GHz, a carrier \( \cos(\omega \tau + \varphi(\tau)) \) of variable phase can be either directly synthesized by the AWG digital-to-analog conversion circuit and generated in combination with oven signal generators through IQ modulations. For the latter, the AWG generates the baseband IQ signals that are fed into modulators to mix with the fixed carrier generated by the oven signal generator.

Equipped with the expression of \( P(\tau) = P(t - x/v) \), we can determine how the microwave pulse responds to the qubit. Consider the standard Maxwell equation

\[
\frac{\partial^2 E}{\partial t^2} + \kappa \frac{\partial E}{\partial t} - c^2 \frac{\partial^2 E}{\partial x^2} = -\frac{1}{\epsilon_0} \frac{\partial^2 P}{\partial x^2} \tag{16}
\]

where \( \kappa \) is the classical decay factor of the electric field and \( c \) is the velocity of light in the medium, which is typically silicon for a superconducting circuit. Since \( E(t) \) assumes the form of equation (1), in which the envelope \( E(t) \) and the phase \( \varphi(t) \) are the slow variables compared to \( \omega \), and the precession of \( P(t) \) follows \( F(t) \), which is also slow compared to \( \omega \), the terms not on the order of \( \omega \) can be ignored [21, 43] after substituting the expressions of \( E(t) \) and \( P(t) \) into the derivatives and equation (16) can be linearized.

Comparing the coefficients of the carrier \( e^{i(\varphi(t) + \omega(t - x/v))} \) and its conjugate, we obtain the coupled equations

\[
E \left( \frac{1}{v} - \frac{1}{c} \right) \frac{d\varphi}{d\tau} = \frac{\mu k}{2\epsilon_0} \Re \{F\}, \tag{17}
\]

\[
\left( \frac{1}{v} - \frac{1}{c} \right) \frac{d\epsilon}{d\tau} = -\frac{\mu k}{2\epsilon_0} \Im \{F\}, \tag{18}
\]

about the envelope and the phase, respectively, under the local time frame \( \tau = t - x/v \). In the equations, \( v \) is the velocity of the envelope wavefront and not necessarily equal to the phase velocity \( c \). The detailed derivations are given in appendix B.

With equations (17) and (18), it becomes clear that the factor \( F \) affects \( E(t) \) only through its imaginary part and \( \varphi(t) \) through both its real and imaginary parts, which concords with equations (13) and (14) since the longitudinal relaxation corresponds to the energy loss of the traveling pulse. The two equations obtained show that not only the qubit relaxations are determined by the pulse shape, but the pulse shape variations are also decided by the relaxation factor. Consequently, the master-Maxwell equation pair given by equations (11) and (16) serves as a microscopic foundation for the evolutions of both the environmentally coupled qubit and the pulse. Substituting equation (14) into equation (18) leads to an integro-differential equation about the pulse envelope, under the influence of the thermal environment through the factor \( \Gamma(\tau) \). We give its analytical solution in the next section and show that pulses of asymptotic \( 2\pi \) areas are transparent to the qubit. Combining the observations above, we note that a \( 2\pi \)-pulse would experience no absorption by the qubit, which coincides with the SIT effect, on one hand and drive the qubit to be immune from longitudinal relaxation on the other.

4. Absorption-free propagation and ramification

To find a general solution to equation (18) for \( E(t) \) with arbitrary initial area, we convert the integro-differential equation of \( E \) into the second-order differential equation

\[
\ddot{A} = M^2 e^{-\Gamma/2} \sin A \tag{19}
\]

of the enveloped area \( A(\tau) = \mu \int_0^\tau ds \ E(s) \) up to the wavefront, which is an inverted pendulum equation augmented with a decay factor. We have used the positive factor \( M = \sqrt{\mu k/c} \) to abbreviate the equation valid for the initial setting of \( \rho(0) = |g\rangle \langle g| \).

To find the analytic expression for \( A \), the pendulum equation is first reduced to the first-order equation:

\[
\dot{A} = 2M e^{-\Gamma/4} \sin(A/2). \]

Taking the time limit \( \tau \rightarrow \infty \) on both sides show that a solitonic \( 2\pi \)-pulses (i.e. asymptotically smooth envelopes with vanishing slopes at two ends) experience no area loss. For pulses of arbitrary enveloping area, we retain the phase variable \( \varphi \) in the expression of \( \Gamma(\tau) \) and solve equation (19)
part being absorbed. Meanwhile, the odd-number along a path, along which the qubit state evolves effectively according to ramification. As a particular example, a \((2^n + A)\) pulse traveling absorption-free while driving the qubit relaxation-free and one \(n\pi\)-pulse attenuating over time. System parameters are taken from experiments of superconducting qubit circuits.

Figure 2. Plot of the pulse envelope \(\mathcal{E}(\tau)\) scale to arbitrary unit as a function of local time \(\tau = t - x/c\), illustrating the scenario of pulse ramification. The single pulse at the initial moment \(\tau = 0\) splits into one \(2\pi\)-pulse traveling absorption-free while the rest \(A\) part travel freely and the \(\pi\) part being absorbed. Meanwhile, the odd-number \(\pi\)-area pulse would guide the evolution of the qubit along a path, along which the qubit state evolves effectively according to \(H_S\) in equation (2) only if the environmental coupling \(H_I\) in equation (6) is non-existent.

Since the initial qubit inversion, i.e. \(\rho(0) = |e\rangle \langle e|\), only affects the sign of equation (14), the line of derivation up to equation (20) remains valid except that the sign flipping should be carried to rhs of equation (19). Hence, the solitonic solution still applies as long as the negative sign can be absorbed into the expression back to the typical SIT solution. For an arbitrary solitonic pulse with \((2n\pi + A)\) area where \(A < 2\pi\), its own shape determines its degree \(\Gamma(\tau)\) of relaxations experienced and determines its speed through the qubit. The \(2\pi\) transparent part would travel at normal speed \(v\) while the rest \(A\) at a reduced speed (cf the numerical analysis below and figure 2), the difference of which gives rise to the pulse ramification. As a particular example, a \((2n + 1)\pi\) pulse would have the \(2\pi\) part travel freely and the \(\pi\) part being absorbed. Meanwhile, the odd-number \(\pi\)-area pulse would guide the evolution of the qubit along a path, along which the qubit state evolves effectively according to \(H_S\) in equation (2) only if the environmental coupling \(H_I\) in equation (6) is non-existent.

Formally as a pendulum equation. The envelope as a time derivative of \(A\) reads

\[
\mathcal{E}(\tau) = \frac{2M}{\mu} e^{-(\Gamma(\tau)/4)} \text{sech} \left( \int_{\tau_0}^{\tau} ds \ e^{-(\Gamma(s)/4 + \Gamma_D)} \right),
\]

where \(\Gamma_D\) is a delay time. The derivation is given in appendix C. Note that equation (20) retains the characteristic hypersecant hump of a soliton. The environment accumulates an attenuation on the pulse ramification. The single pulse at the initial moment \(\tau = 0\) splits into one \(2\pi\)-pulse traveling absorption-free while driving the qubit relaxation-free and one \(n\pi\)-pulse attenuating over time. System parameters are taken from experiments of superconducting qubit circuits.
variation, which allows the approximation $\Gamma(\tau) \approx 4C_0(\tau - \tau_0)$ and leads to an exponential decay of the pulse peak in equation (20) (the scale factor 4 is added to simplify expressions below).

Under such premise, the integral in equation (20) is computed numerically, giving rise to the propagation of a decaying pulse as illustrated in figure 2. Note that $\mathcal{E}(\tau)$ under the local time frame $\tau$ would appear simply as a decaying envelope peaking at $\tau = 0$. To appreciate the propagation process in the figure, we have returned the reference frame to the separate laboratory axes $x/v$ and $t$, where parameters are set to values accessible by typical qubits in superconducting circuits: $\omega = 5$ GHz and a Q-factor of $10^3$ [15, 44]. Since $\tau_0$ is an arbitrary initial time point, it is set to zero to simplify the analysis.

We observe that a pulse of arbitrary enveloping area ramifies into two: one of area a multiple of $2\pi$ travels freely, shown as one ridge that converges to a constant height, and one of non-integral area attenuates over $t$, shown as the other ridge in figure 2. Following the wavefronts of the two peaks, one observe that the slopes of their projections onto the $x-t$ plane differs. The one traveling absorption-free pertains to a constant slope and therefore travels at a constant velocity of light while the other has a curving slope. The separation of wavefronts increases monotonically over time, showing that the attenuating pulse is decelerating. This can be proved analytically by taking the $\tau$-derivative of the argument of the hyper-secant function in equation (20), giving effective velocity $v_{\text{eff}} = v \exp\{ -C_0(t - t_0) \}$. 

The approximations taken in giving figure 2 is essentially a first-order perturbative expansion of $\mathcal{E}(\tau)$, from which the envelope and phase governed by equations (17) and (18) are decoupled. Consequently, the dynamic phase accumulated when propagating through the qubit is computed by integrating equation (17), which reads

$$
\varphi(\tau) = \varphi_0 + M \int_{\tau_0}^{\tau} ds \left[ e^{-C_0(s - \tau_0)} \sinh \left\{ 2C_0(s - \tau_0) \right\} \cosh \left\{ -\frac{M}{C_0} \left( e^{-C_0(s - \tau_0)} - C_0 \tau_D - 1 \right) \right\} \right].
$$

(22)
The second term contributed by the environment feedback generates an advancement to the phase. Using the same system parameters as in figure 2, the integrals of equation (22) can be numerically computed, giving rise to the typical carrier wave oscillations as plotted in figure 3, where the phase advancement over a duration of 20 periods is shown against a carrier of no phase variation.

Since the three factors in the integrands of equation (22) are either exponential or variants of exponential functions, the phase culminates on a pulse is a monotonically increasing function of time. This is evidenced by the plots of $\varphi(\tau)$ given as dashed curves in figure 4 for four different constant spectral densities $\gamma$. The rate of phase accumulation increases along with the increment of $\gamma$ when $\Gamma$, regarded as a measure of rate of feedback from the environment is accordingly increased. The envelope area $A(\tau)$ computed from the integral of $E(\tau)$ in equation (20) is plotted in the same figure, showing that the area variation accentuates on the range of time where the phase variation is minimal, thereby ratifying the assumptions we took above when arriving at the explicit solution of $E(\tau)$.

5. Conclusion

To conclude, we have taken an adiabatic master equation approach to analyze the propagation of a pulse through an environmentally coupled qubit. The qubit would be free from longitudinal relaxation when the pulse area is $\pi \gamma$. Further, when an orthogonal condition between the pulse phase and the bath spectral density is satisfied, the transverse relaxation also vanish during propagation. If the pulse area is $2\pi \gamma$, the propagation becomes transparent. It is also proved that when relaxations are not vanishing, the thermal environment is responsible for inducing pulse ramifications which are frequently observed in SIT experiments. The approach has also enabled us for the first time to compute analytically the phase variation in the pulse during its propagation through a two-level system. Modeled on superconducting qubit circuits, the exact knowledge on pulse–qubit interactions would benefit the designs of more sophisticated microwave control pulses for quantum information processing.

Acknowledgments

Y-BG acknowledges the support of the National Natural Science Foundation of China under Grant No. 11674017. HI acknowledges the support by FDCT of Macau under Grants 065/2016/A2 and 0130/2019/A3, University of Macau under Grant MYRG2018-00088-IAPME, and National Natural Science Foundation of China under Grant No. 11404415.

Appendix A. Adiabatic quantum master equation

To construct a master equation for the system when taking into account the geometric phases, we consider customarily a quantum heat bath consisting of a multimode resonator $H_B = \sum \omega_j a_j^\dagger a_j$ having dipole-field interaction $H_I = \sum g_j (a_j^\dagger + a_j) \sigma_z$ with the qubit.

Since the incident pulse with envelope $E(x,t)$ is a weak driving field to the qubit, the qubit evolution under the coupling strength $\mu E$ could be regarded as an adiabatic process under Born–Oppenheimer approximation when compared to the thermal relaxing process under couplings $\{g_j\}$. Under the total Hamiltonian

$$H(\tau) = H_I(\tau) + H_B + H_I$$

with $H_I(\tau) + H_B$ regarded as the free energy, the adiabatic process is described by the master equation for the Universe $\mathcal{U}$:

$$\frac{d\mathcal{U}}{d\tau} = -\int_{\tau_0}^\tau ds \{ [H_I(\tau), [H_I(\tau-s), \rho(\tau) \otimes \hat{\rho}]] \}$$

in interaction picture, $\hat{\rho}$ stands for the density matrix for the thermal bath. The system-bath interaction $H_I(\tau) = U_{ad}(\tau) H_I U_{ad}(\tau)\dagger$ has been transformed to the adiabatic evolution frame with the double-time transformation $U_{ad}(\tau-s) = e^{iH_E(\tau)} U_{ad}(\tau)$ being used for the historic Hamiltonian $H_I(\tau-s)$ following the convention [41, 42].

As given in equation (5) in the main text, the unitary transformations involve the dressed basis defined at both the initial moment $\tau_0$ and the current moment $\tau$ or the historic moment $\tau-s$. In practice, we unify the basis reference to time $\tau$ by inverting eigenstate definitions of equations (4) and (5), finding equation (7), i.e.

$$U_{ad}(\tau) = |\mu_+(\tau)\rangle \langle \mu_+(\tau_0)| e^{-i\phi_+(\tau)} + |\mu_-(\tau)\rangle \langle \mu_-(\tau_0)| e^{-i\phi_-(-\tau)},$$

(A.3)
Based on which we can also derive

$$U_{ad}(\tau - s) = e^{i\Omega(t\tau)}|\nu_+(\tau)\rangle\langle\nu_+(\tau)| e^{-i\Omega(t\tau)} + e^{i\Omega(t\tau)}|\nu_-(\tau)\rangle\langle\nu_-(\tau)| e^{-i\Omega(t\tau)}. \quad (A.4)$$

During the adiabatic process, the bath variables $a_i$ and $a_j^\dagger$ stay relatively static while the operator $\sigma_x$ relevant to the system is rotated. Therefore, the static $\sigma_x$ in the dressed basis reads

$$\sigma_x = |e\rangle\langle g| + |g\rangle\langle e| = -\sin 2\theta \cos \varphi [\hat{\sigma}_+ \hat{\sigma}_- - \hat{\sigma}_- \hat{\sigma}_+] + [\cos 2\theta \cos \varphi + i \sin \varphi] \hat{\sigma}_+ + \text{h.c.}. \quad (A.5)$$

Then following the qubit-field interaction using the unitary transformation equation (A.4), the operator during the interaction reads for any historic moment $s$:

$$\hat{\sigma}_x(\tau - s) = U_{ad}^\dagger(\tau - s)\sigma_x U_{ad}(\tau - s)$$

$$= U_{ad}^\dagger(\tau) \{ [\cos 2\theta \cos \varphi + i \sin \varphi] e^{-i\Omega} \hat{\sigma}_+ + \text{h.c.} - \sin 2\theta \cos \varphi [\hat{\sigma}_+ \hat{\sigma}_- - \hat{\sigma}_- \hat{\sigma}_+] \} U_{ad}(\tau) \quad (A.6)$$

where $\Omega = \sqrt{\delta^2 + (\mu E)}$, which includes the special case with $s = 0$.

The derivation of the microscopic master equations begins with tracing out the bath variables in the Liouville equation (A.2) of the Universe, giving

$$\frac{d\rho(\tau)}{dt} = -\int_0^{\tau - s} ds \operatorname{tr}_B \left\{ [H_i(\tau), [H_i(\tau - s), \rho(\tau) \otimes \hat{\rho}]] \right\}, \quad (A.7)$$

where the integration limit is modified following equation (9). We note that evoking the commutator generates the double-time operators $\hat{\sigma}_x(\tau)\hat{\sigma}_x(\tau - s)$ according to equation (6). Therefore, equipped with equation (A.6), we compute

$$\hat{\sigma}_x(\tau)\hat{\sigma}_x(\tau - s) = U_{ad}^\dagger(\tau) \left\{ \sin^2 2\theta \cos^2 \varphi \left[ \hat{\sigma}_+ \hat{\sigma}_- - \hat{\sigma}_- \hat{\sigma}_+ \right] + \cos^2 2\theta \cos^2 \varphi + \sin^2 2\varphi \right\} e^{i\Omega} \hat{\sigma}_+ + \text{h.c.} \right\} U_{ad}(\tau) \quad (A.8)$$

and $\hat{\sigma}_x(\tau - s)\hat{\sigma}_x(\tau) = [\hat{\sigma}_x(\tau)\hat{\sigma}_x(\tau - s)]^\dagger$. At the long-term limit $\tau \to \infty$, equation (A.7) expands into four terms. In dealing with the first term, we consider

$$\int_0^{\infty} ds \operatorname{tr}_B \{H_i(\tau)H_i(\tau - s)\rho(\tau) \otimes \hat{\rho}\} = -\int ds \hat{\sigma}_x(\tau)\hat{\sigma}_x(\tau - s)\rho(\tau) \operatorname{tr}_B \{h_i(\tau) \hat{h}_i(\tau - s) \hat{\rho}\}$$

$$= -\int ds \hat{\sigma}_x(\tau)\hat{\sigma}_x(\tau - s)\rho(\tau) \sum_j g_j^2 e^{-i\omega_j s}. \quad (A.9)$$

Note from the expression of equation (A.8) the double-time operator product would contribute multiple terms in the integral, but only those with the factor $e^{i(\Omega - \omega_j)s}$ would remain since the other terms containing the fast-oscillating exponential factors would vanish after the integration over long period. Consequently, since only this exponential factor involves in the integration over the variable $s$, all other factors about time $\tau$ do not participate in the integration and

$$\sum_j g_j^2 \int ds e^{i(\Omega - \omega_j)s} = \pi \sum_j g_j^2 \delta(\Omega - \omega_j) = \frac{1}{2} \gamma(\Omega) \quad (A.10)$$

where the last equation employs the definition of equation (12). The integral would become

$$-\frac{1}{2} \gamma(\Omega) (\cos^2 2\theta \cos^2 \varphi + \sin^2 \varphi) U_{ad}^\dagger(\tau)\hat{\sigma}_+ U_{ad} \rho' \quad (A.11)$$

When applying the same arguments to the other three terms in the expansion of the rhs of equation (A.7), we arrive at

$$\frac{d\rho'_{\tau}}{d\tau} = -\gamma(\Omega) (\cos^2 2\theta \cos^2 \varphi + \sin^2 \varphi) \left\{ \frac{1}{2} \left[ U_{ad}^\dagger(\tau)\hat{\sigma}_+ U_{ad} \rho' \right] - U_{ad}^\dagger(\tau)\rho - U_{ad} \rho' U_{ad}^\dagger(\tau)\hat{\sigma}_+ U_{ad} \right\}. \quad (A.12)$$
Finally, seeing that \( \rho' = U_{ad}^{\dagger} \rho U_{ad} \) is given in the interaction picture of the adiabatic evolution, we note that for the conversion back to Schroedinger picture,

\[
\frac{d\rho}{d\tau} = \frac{d}{d\tau} \left\{ U_{ad} \rho U_{ad}^{\dagger} \right\}
\]

\[
= U_{ad} \frac{d\rho'}{d\tau} U_{ad}^{\dagger} - iH_S U_{ad} \rho' U_{ad}^{\dagger} + iU_{ad} \rho' U_{ad}^{\dagger} H_S
\]

\[
= -i [H_S, \rho] - \gamma(\Omega) \left( \cos^2 2\theta \cos^2 \varphi + \sin^2 \varphi \right) \left[ \frac{1}{2} \{ \sigma+, \sigma- \}, \rho \right] - \dot{\sigma} - \rho \dot{\sigma}
\]

(A.13)

which is the Lindblad form of the master equation as in equation (11). Therefore, at resonance with zero detuning, \( \theta = \pi/4 \) and the Lindbladian term assumes a minimal coefficient \( \gamma(\mu E(\tau)) \sin^2 \varphi(\tau) \), which can be further reduced depending on the historic values of \( \varphi(\tau) \) as demonstrated in the main text. On the other hand, the absence of interference from the incident pulse corresponds to the limiting case of large detuning \( \delta \) with \( E \to 0 \), giving \( \theta = 0 \) and thus a maximal Lindbladian coefficient \( \gamma(\omega_0) \). With this coefficient, the master equation resumes its standard form in the bare-state basis.

Appendix B. Equations of envelope and phase

From the original Maxwell equation (16), the loss \( \kappa \) in the waveguide is assumed negligible. Then the so-called slowly-varying envelope approximation [36], that is \( \partial E/\partial t \ll \omega E, \partial E/\partial x \ll kE, \partial \varphi/\partial t \ll \omega, \) and \( \partial \varphi/\partial x \ll k \) for the electric field \( E(t) \); \( \partial F/\partial t \ll \omega F \) and \( \partial F/\partial x \ll kF \) for the polarization \( P(t) \) can be taken. Thus when substituting the expressions of \( E(t) \) and \( P(t) \) (i.e. in the form of equation (1)), the second-order partial derivatives with respect to both time and space are regarded as negligible terms in comparison to the linear terms \( \partial^2 E/\partial t^2 \), etc. The Maxwell equation is essentially reduced to a first-order PDE.

When comparing the real and the imaginary parts of both sides of this PDE, one arrives at the coupled equations

\[
\frac{\partial E}{\partial x} + \frac{1}{c} \frac{\partial E}{\partial t} = -\frac{k\mu}{2\epsilon_0} 2\{F\}, \quad \text{(B.1)}
\]

\[
\frac{\partial \varphi}{\partial x} + \frac{1}{c} \frac{\partial \varphi}{\partial t} = -\frac{k\mu}{2\epsilon_0} 2R\{F\}, \quad \text{(B.2)}
\]

for the envelope variable and the phase variable. For the local time \( \tau = t - x/v \), the two derivative operators on the left-hand side can be combined, i.e.

\[
\frac{\partial}{\partial t} + \frac{\partial}{\partial x} = \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \tau} \frac{\partial}{\partial \tau} = (1 - \frac{\varepsilon}{v}) \frac{\partial}{\partial \tau}, \quad \text{(B.3)}
\]

under which equations (B.1) and (B.2) become ODEs of one variable:

\[
\frac{dE}{d\tau} = \frac{\omega}{2(1 - \frac{\varepsilon}{v})} 2\{F\}, \quad \text{(B.4)}
\]

\[
\frac{d\varphi}{d\tau} = -\frac{\omega}{2(1 - \frac{\varepsilon}{v})} 2R\{F\}. \quad \text{(B.5)}
\]

Rearranging the factors on two sides lead to equations (17) and (18). The equation of \( E(\tau) \) still couples to that of \( \varphi(\tau) \) through an implicit dependence in \( F \). To effectively decouple them, we consider the perturbative expansion of \( \Gamma(\tau) \) of equation (15) in \( 2\{F\} \), i.e. letting \( \varphi \) assume the initial value \( \varphi(0) \) in the integral, which is valid under the adiabatic approximation. Consequently, the expression of equation (20) for the envelope can be regarded as its zeroth-order solution and its higher-order corrections can be obtained by substituting the solution of \( \varphi \) in equation (22) back into \( \Gamma(\tau) \) in a consecutive manner.

Appendix C. Solving for envelope and phase

Noticing that the sinusoidal factor in equation (14) is simply \( \sin A \), given the definition of the envelope area \( A \), we can write equation (19) as

\[
\frac{dE}{d\tau} = -\frac{\mu c k}{2(c/v - 1) \epsilon_0} e^{-\Gamma/2} \sin A. \quad \text{(C.1)}
\]
Recognizing $dE/d\tau = d^2A/d\tau^2$ and using the abbreviation $M = \sqrt{\mu^2 k c_0 / (c - v)}$, we arrive at the inverted pendulum equation
\[ \ddot{A} = M^2 e^{-\Gamma / 2} \sin \dot{A}. \tag{C.2} \]
Since $\ddot{A} = dA/d\tau = \dot{A} \left( d\dot{A}/d\tau \right)$, the equation can be rewritten as
\[ \dot{A} \, d\dot{A} = M^2 e^{-\Gamma / 2} \sin \dot{A} \, dA. \tag{C.3} \]
Since two time variables $A(\tau)$ and $\Gamma(\tau)$ are on the rhs, we adopt the Born–Oppenheimer approximation to regard $\Gamma(\tau)$ as a slow-varying variable compared to $A(\tau)$ such that formal integration can be carried out on both sides. We note that in current experiments the longitudinal relaxation time $T_1$ have entered microsecond range [31–33] while the microwave pulse width in concern is in the range of tens of nanoseconds. The latter can be synthesized with modern waveform generators without difficulty. Hence, the validity of the assumption is experimentally guaranteed and carrying out the integration gives
\[ \dot{A}^2 = 2M^2(1 - \cos A)e^{-\Gamma / 2} = 4M^2 e^{-\Gamma / 2} \sin^2 \frac{A}{2}. \tag{C.4} \]
Then taking the square root on both sides, one arrives at a first-order equation
\[ \frac{dA}{d\tau} = 2M e^{-\Gamma/4} \sin \frac{A}{2}, \tag{C.5} \]
whereby the $\sin A/2$ factor can be moved to lhs and we can formally solve for $A$:
\[ \ln \tan \frac{A}{4} = M \int_{\tau_0}^\tau e^{-\Gamma/4} \, d\tau + C \tag{C.6} \]
where $\tau_0$ is an arbitrary initial time of integration and $C$ is the integration constant. Hence,
\[ \tan \frac{A}{4} = C \exp \left\{ M \int_{\tau_0}^\tau e^{-\Gamma/4} \, d\tau \right\}, \tag{C.7} \]
and letting $A_0 = A(\tau_0)$ shows $C = \tan A_0/4$. Absorbing $C$ into the exponential, we can simplify the above into
\[ \tan \frac{A}{4} = \exp \left\{ M \int_{\tau_0}^\tau e^{-\Gamma/4} \, d\tau + \tau_D \right\}, \tag{C.8} \]
where $\tau_D = \ln \frac{\tan(A_0/4)}{M}$ can be regarded as the delay time. Then, using the identity
\[ \sin 2\theta = 2[\tan \theta + 1/\tan \theta]^{-1}, \tag{C.9} \]
which gives equation (20).

For the phase $\varphi(\tau)$, we substitute the decay factor equation (13) into equation (17) to get
\[ \frac{d\varphi}{d\tau} = -\frac{\mu k}{2\mathcal{E}(c/v - 1)\epsilon_0} (1 - e^{-\tau}) = \frac{M^2}{\mu \mathcal{E}} (1 - e^{-\Gamma}), \tag{C.10} \]
which is an integro-differential equation, considering the expression of $\mathcal{E}$ in equation (20). Like described in the text, we simplify the consideration by reducing $\Gamma(\tau)$ to a linear dependence on time, writing $\Gamma(\tau) = 4C_0(\tau - \tau_0)$ and hence
\[ \int_{\tau_0}^\tau e^{-\Gamma(\tau)/4} \, d\tau = \int_{\tau_0}^\tau e^{-C_0(s-\tau_0)} \, d\tau = -\frac{1}{C_0} \left[ e^{-C_0(\tau_0 - \tau_0)} - 1 \right]. \tag{C.11} \]
Therefore, equation (C.10) reads
\[ \frac{d\varphi}{d\tau} = \frac{M}{4} \frac{e^{\Gamma/4} - e^{-\Gamma/4}}{\text{sech} \left( \int_{\tau_0}^\tau ds \, e^{-\Gamma(s)/4} + \tau_D \right)} = \frac{M}{4} \left[ e^{C_0(\tau_0 - \tau)} - e^{-3C_0(\tau_0 - \tau)} \right] \cosh \left\{ -\frac{M}{C_0} \left( e^{C_0(\tau_0 - \tau)} - C_0\tau_D - 1 \right) \right\}. \tag{C.12} \]
Then integrating both sides with respect to the local time $\tau$ yields equation (22).
References

[1] Wallraff A, Schuster D I, Blais A, Frunzio L, Huard R S, Majer J, Kumar S, Girvin S M and Schoelkopf R J 2004 Nature 431 162
[2] Blais A, Huard R-S, Wallraff A, Girvin S M and Schoelkopf R J 2004 Phys. Rev. A 69 062320
[3] Neeley M et al 2010 Nature 467 570
[4] Eichler C, Lang C, Fink J M, Goeniuus J, Filipp S and Wallraff A 2012 Phys. Rev. Lett. 109 240501
[5] Mariantoni M et al 2011 Science 334 61
[6] Lucero E et al 2012 Nat. Phys. 8 719
[7] Mallet F, Ong F R, Palacios-Laloy A, Nguyen F, Pivert P, Vion D and Esteve D 2009 Nat. Phys. 5 791
[8] Dharmar L et al 2010 Nature 467 570
[9] Neeley M et al 2010 Nature 467 570
[10] Chow J M, DiCarlo L, Gambetta J M, Motzoi F, Frunzio L, Girvin S M and Schoelkopf R J 2010 Phys. Rev. A 82 040305
[11] You J Q and Nori F 2011 Nature 474 589
[12] McCall S L and Hahn E L 1969 Phys. Rev. 183 457
[13] Marzak R and Österberg U L 2014 Phys. Rev. A 89 023828
[14] Lindkvist J and Johansson G 2014 New J. Phys. 16 035018
[15] Nguyen D V, Catelani G and Basko D M 2017 Phys. Rev. B 96 214508
[16] Slusher R E and Gibbs H M 1972 Phys. Rev. A 5 1634
[17] Lamb G L 1971 Rev. Mod. Phys. 43 99
[18] Schwab J et al 2008 Phys. Rev. B 77 180502
[19] Rigetti C et al 2012 Phys. Rev. B 86 100506
[20] Barredo R et al 2014 Nature 508 500
[21] Temme K, Bravyi S and Gambetta J M 2017 Phys. Rev. Lett. 119 180509
[22] Basov N G, Ambartsumyan A D, Mezzacapo A, Chow J M and Gambetta J M 2019 Nature 567 491
[23] Zhou B, Liu X and Shabesh K 1998 Phys. Rev. Lett. 81 2594
[24] Gambetta J M, Houck A A and Blais A 2011 Phys. Rev. Lett. 106 030502
[25] Koch J et al 2007 Phys. Rev. A 76 042319
[26] Breuer H-P and Petruccione F 2002 The Theory of Open Quantum Systems (Oxford: Oxford University Press)
[27] Scully M O and Zubairy M S 1997 Quantum Optics (Cambridge: Cambridge University Press)