Geometries with the second Poincaré symmetry

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The second Poincaré kinematical group serves as one of new ones in addition to the known possible kinematics. The geometries with the second Poincaré symmetry is presented and their properties are analyzed. On the geometries, the new mechanics based on the principle of relativity with two universal constants ($c, l$) can be established.

PACS numbers: 02.90.+p, 03.30.+p, 04.20.Cv, 02.20.Sv

Keywords: the second Poincaré symmetry, geometry, degenerate, motion of a free particle

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I. INTRODUCTION

It is well known that a maximum symmetry group of a 4d non-degenerate space-time has 10 independent parameters. Minkowski (Mink), de Sitter (dS), and anti de Sitter (AdS) space-times are all the space-times of this kind. In addition, the Galilei (G) space-time has 10-parameter kinematical group even though its geometry is degenerate, splitting into 3d space and 1d time geometries. It is natural to ask: how many are there 10-parameter kinematical groups in 4d space-time? Bacry and Lévy-Leblond have answered the question [1]. Under their three assumptions and by the Inönü-Wigner contraction method [2], they show that there are 8 types of Lie algebras corresponding to 11 possible kinematical groups. They are the Poincaré (P), dS, AdS, inhomogenous SO(4) (P′+), para-Poincaré (P′), G, Newton-Hooke (NH+), anti-Newton-Hooke (NH−), para-Galilei (G′), Carroll (C), and static (S) groups. If their third assumption that inertial transformations in any given direction form a noncompact subgroup is relaxed, other 3 classical, geometrically kinematical groups will be added, corresponding to Euclid (Euc), Riemann (Riem), and Lobachevski (Lob) geometries.

On the other hand, the principle of relativity (PoR) is the foundation of physics, and it is closely related to the symmetry of space and time. Recently, it is pointed out that the triality of special relativity with Poincaré, de Sitter, anti-de Sitter invariance, respectively, can be set up based on the PoR and the postulate on two universal invariant constants c of speed dimension and l of length dimension denoted as the PoRc,l [3, 4]. It is also found in [3, 4] that there is another realization of Poincaré group being called the second Poincaré group and denoted as P2, with the corresponding realization of algebra being denoted as p2.¹ Unlike the ordinary Poincaré transformation under which the metric of Minkowski space-time is invariant, the second Poincaré transformations do not generate the automorphism of the Minkowski space-time. Instead, they preserve all straight lines in the Minkowski space-time. Furthermore, it has been shown based on the PoRc,l that every algebra in all possible kinematics revealed by Bacry and Lévy-Leblond except 4 semi-simple groups, dS, AdS,

¹ For brevity, we call the new realization of the Poincaré group as well as its algebra the second Poincaré group and the second Poincaré algebra throughout the paper.
Riem, and Lob groups has its second version [4]. Therefore, there are 24 kinematical groups with SO(3) isotropic subgroup in all. One of the reasons of the absence of the second versions of many groups, such as the second Poincaré group, in [1] is that they just consider the algebraic structure but not consider the action space of the group. A natural question appears: what are the meanings of these additional possible kinematical algebras or what do these additional possible kinematical algebras represent?

In order to clarify the question, one has to know on what kinds of 4d space-times these possible kinematics are defined. Unfortunately, more than a half of the space-times corresponding to these kinematical algebras are unknown. Our recent letter ameliorated the situation somewhat, in which we presented a geometry with the $P_2$ symmetry[5]. One of the purposes of the paper is to make the thorough investigation on the $P_2$ geometries.

Both in the treatment of Bacry and Lévy-Leblond and the approach based on $PoR_{c,l}$, which are very different from each other, SO(3) symmetry is identified as the space isotropy in algebraic sense. However, the sub-algebras in the possible kinematical algebras can be interpreted in many ways. Whether an SO(3) isotropy can be identified to the space isotropy is determined by the geometrical (as well as topological and causal) structure of the space-time. In other words, before a careful geometrical study, we cannot conclude that the space-times possess the space SO(3) isotropy in the geometries even though the corresponding kinematical algebras having SO(3) sub-algebra. The second purpose of the paper is to take the geometrical structure with $P_2$ symmetry as an example to clarify that the algebraic $\mathfrak{so}(3)$ isotropy in [1, 4] does not always imply the geometrical SO(3) space isotropy.

Once the topology and geometry are clarified, one needs to re-construct the algebras according to the understanding of the geometry. The third purpose of the paper is to show that there exists the geometrical structure which satisfies all three assumptions in [1] and the $PoR_{c,l}$ in [3, 4] even after re-construction of the algebra in terms of new space and time coordinates.

The paper will be organized in the following way. In the next section, we shall review the second Poincaré symmetries. Section III focuses on the no-go theorem that there does not exist a non-degenerate metric with the second Poincaré symmetry. In sections IV and V, we shall present degenerate metrics and connection which are $P_2$ invariant and study the structure of the space-times described by the metrics and connection, respectively. In section VI, we prove the uniqueness of the geometrical structures for the second Poincaré symmetry. Then, we show that the maximum symmetry of the new space-times is the Poincaré symmetry and re-classify the generators according to geometries in section VII. Section VIII is devoted to set up the mechanics of a free particle on the geometry. We shall study the uniform rectilinear motions in the space with degenerate metrics and present the formal Lagrangian formalism for the particle moving on the geometries. Finally, we shall conclude the paper with some concluding remarks in section IX.
II. THE SECOND POINCARÉ SYMMETRY

The ordinary Poincaré transformations

\[ x'_{\mu} = L_{\mu}^{\nu} x_{\nu} + l a_{\mu}, \quad L \in SO(1,3), \]

where \( a_{\mu} \) are dimensionless parameters, transfer the origin \( O(\alpha_{\mu}) \) to the event \( P(x_{\mu}P = l(l^{-1})_{\mu}^{\nu}a_{\nu}) \) and a generator set \( \{T\}^p = (H, P, K, J) \) \(^2\) spans a Poincaré algebra \( p \cong \text{iso}(1,3) \),

\[ H = \partial_t, \quad P_i = \partial_i, \quad K_i = t \partial_i + \frac{1}{c^2} x^i \partial_t, \quad J_i = \epsilon_{ijk} (x_j \partial_k - x_k \partial_j), \]

where the indexes are lowered or raised by \( (\eta_{\mu\nu}) = \text{diag}(1,-1,-1,-1) \) and its inverse. The transformation (2.1) can be expressed in a \( 5 \times 5 \) matrix,

\[ \left( \begin{array}{cc} L & a \\ 0 & 1 \end{array} \right) \]

With the same \( K, J \), there exists a second generator set \( \{T\}^{p_2} = (H', P', K, J) \), where

\[ H' = -c^2 l^{-2} t x^\kappa \partial_\kappa (= c P_0'), \quad P'_i = l^{-2} x^i x^\kappa \partial_\kappa. \]

They spans the second Poincaré algebra \( p_2 \),

\[ [H', P'_i] = 0, \quad [P'_i, P'_j] = 0, \quad [H', K_i] = P'_i, \quad [P'_i, K_j] = \frac{1}{c^2} H' \delta_{ij}, \]

\[ [K_i, K_j] = -\frac{1}{c^2} L_{ij}, \quad [J_i, J_j] = -\epsilon_{ijk} J_k. \]

In other words, there is no difference between the ordinary Poincaré algebra and the second Poincaré algebra in *algebraic* sense. However, the second Poincaré algebra is the different realization of \( \text{iso}(1,3) \) from the ordinary realization. The second Poincaré algebra generates the second Poincaré transformations

\[ x'^{\mu} = \frac{L_{\mu}^{\nu} x^{\nu} + l b_{\mu} x^{\nu}}{1 + l^{-1} b_\lambda x^{\lambda}}, \]

where \( b_{\mu} \) are dimensionless parameters, which can be expressed again in terms of \( 5 \times 5 \) matrix

\[ \left( \begin{array}{cc} L & 0 \\ b^t & 1 \end{array} \right) \]

where \( b^t := (\eta_{\mu\nu} b^{\nu}) \) is the transpose of \( 4 \times 1 \) matrix \( b \). Clearly, as a part of linear fractional transformations, they preserve all straight lines,

\[ \begin{aligned} x^0 &= ct, \\
 x^i &= v^i t + x^i_0, \quad v^i, x^i_0 \text{ are arbitrary constants,} \end{aligned} \]

\(^2\) \( P, K \cdots \) are the shorthands of \( P_i \), and \( K_i, \cdots \), respectively, where \( i = 1, 2, 3 \).
no matter whether the lines are causal \((c^2 - \delta_{ij}v^i v^j \geq 0)\) or not. In particular, they preserve the light cone at the origin

\[ \eta_{\mu\nu} x'^\mu x'^\nu = 0. \] (2.9)

A simple calculation shows that the second Poincaré transformations do not preserve the metric of the Minkowski space-time.\(^3\)

To be distinguished from the ordinary time and space translation generators \(H\) and \(P\), \(H'\) and \(P'\) are called the pseudo-time- and pseudo-space-translation generators because they cannot generate time or space translation in Minkowski space-time.

### III. NO-GO THEOREM

**Theorem 1** There is no tensor field \(g = g_{\mu\nu} dx^\mu \otimes dx^\nu\) with the following three conditions satisfied simultaneously: (1) \(g\) is smooth; (2) \(g\) is non-degenerate everywhere; (3) \(g\) is invariant under the \(p_2\)-translations.

If the theorem was incorrect, there would be certain a tensor field \(g\), which would be treated as the metric, satisfying the conditions in the theorem. Let \(\nabla\) be the Levi-Civita connection related to \(g\). The Killing equation for vector field \(P^{\mu a}\) denoted by \(-\iota^{-2} x^\mu D^a\) with \(D^a = D^\kappa (\partial_\kappa)^a = x^\kappa (\partial_\kappa)^a\) would read

\[ 0 = \nabla_a (x^\mu D_b) + \nabla_b (x^\mu D_a) = x^\mu (\nabla_a D_b + \nabla_b D_a) + (dx^\mu)_a D_b + D_a (dx^\mu)_b, \] (3.1)

where \(D_a = g_{ab} D^b = D^\mu (dx^\mu)_a\). (\(D_\mu = g_{\mu\nu} D^\nu.\)) The contraction of \(D^b\) and \(D_\mu\) with Eq. (3.1), respectively, yield

\[ x^\mu \left( D^b \nabla_b D_a + \frac{1}{2} \nabla_a (D^b D_b) + D_a \right) + (dx^\mu)_a D^b D_b = 0, \] (3.2)

\[ D_\mu D^\mu (\nabla_a D_b + \nabla_b D_a) + 2 D_a D_b = 0. \] (3.3)

The contraction of Eq. (3.2) with \(D_\mu\) gives rise to

\[ D_\mu D^\mu \left( D^b \nabla_b D_a + \frac{1}{2} \nabla_a (D^b D_b) + 2 D_a \right) = 0, \]

which is valid for an arbitrary point \(p\), either \(D_\mu D^\mu |_p = 0\) or

\[ \left( D^b \nabla_b D_a + \frac{1}{2} \nabla_a (D^b D_b) + 2 D_a \right) |_p = 0. \] (3.4)

\(^3\) It should be noted that the second Poincaré group presented here is different from the second Poincaré group presented by Aldrovandi and Pereira \([6]\). The second Poincaré group here is the semi-direct product of the pseudo-translations and Lorentz group and is a subgroup of the general projective group, while the second Poincaré group presented by Aldrovandi and Pereira is the semi-direct product of the special conformal transformations and Lorentz group and is a subgroup of the conformal transformation group.
If $D_\mu D^\mu|_p = 0$, Eq. (3.3) requires $D_a D_b|_p = 0$, which implies that $D_a|_p = 0$. Since $g_{ab}$ is non-degenerate, it is possible if and only if $D^a|_p = 0$. In other words, it is possible if and only if $p$ is the origin of the coordinate system. When $p$ is not the origin, Eq. (3.4) is always satisfied, together with $D^b D_b|_p \neq 0$. Then Eq. (3.2) results in

$$dx^\mu|_p = x^\mu D^\mu D_d D_a|_p,$$

which is absurd because $dx^0|_p$, $dx^1|_p$, $dx^2|_p$ and $dx^3|_p$ are linearly independent. $\Box$

IV. GEOMETRIES FOR $P_2$ SYMMETRY

The no-go theorem shows that the $p_2$-invariant metrics on the 4d underlying manifold must be degenerate. In order to completely fix the geometry of the space-time with a degenerate metric, more information should be assigned.

Consider a 4d manifold $M^{p_2}$ endowed with (1) a type-(0,2) degenerate symmetric tensor field

$$g^\pm = g^\pm_{\mu\nu} dx^\mu \otimes dx^\nu = \pm \frac{l^2}{(x \cdot x)^2} (\eta_{\mu\rho} \eta_{\nu\tau} - \eta_{\mu\nu} \eta_{\rho\tau}) x^\rho x^\tau dx^\mu dx^\nu,$$

where

$$x \cdot x = \eta_{\mu\nu} x^\mu x^\nu \begin{cases} < 0, & \text{for upper sign} \\ > 0, & \text{for lower sign} \end{cases}$$

(4.2)

(2) a type-(2,0) degenerate symmetric tensor field

$$h_\pm = h^\pm_{\mu\nu} \partial_\mu \otimes \partial_\nu = l^{-4} (x \cdot x) x^\mu x^\nu \partial_\mu \otimes \partial_\nu$$

(4.3)

and (3) a connection $\nabla^\pm$ compatible to $g^\pm$ and $h_\pm$, i.e.

$$\nabla^\pm_\lambda g^\pm_{\mu\nu} = \partial_\lambda g^\pm_{\mu\nu} - \Gamma^\kappa_{\lambda\mu} g^\pm_{\kappa\nu} - \Gamma^\kappa_{\mu\lambda} g^\pm_{\kappa\nu} = 0$$

(4.4)

and

$$\nabla^\pm_\lambda h^\pm_{\mu\nu} = \partial_\lambda h^\pm_{\mu\nu} + \Gamma^\kappa_{\lambda\mu} h^\pm_{\kappa\nu} + \Gamma^\kappa_{\mu\lambda} h^\pm_{\kappa\nu} = 0,$$

(4.5)

respectively, with connection coefficients in the above coordinate system,

$$\Gamma^\mu_{\pm\nu\lambda} = - \frac{x_\nu \delta^\mu_{\lambda} + \delta^\mu_{\nu} x_\lambda}{x \cdot x}.$$  

(4.6)

It is easy to check that $(M^{p_2}, g^\pm, h_\pm, \nabla^\pm)$ is invariant under $P_2$ transformation, namely, $\forall \xi \in p_2 \subset \Gamma(TM^{p_2})$, equations

$$\mathcal{L}_\xi g^\pm = \left( \xi^\lambda \partial_\lambda g^\pm_{\mu\nu} + g^\pm_{\mu\lambda} \partial_\nu \xi^\lambda + g^\pm_{\nu\lambda} \partial_\mu \xi^\lambda \right) dx^\mu \otimes dx^\nu = 0,$$

(4.7)

The abstract and component forms of tensor fields are both used in the present paper.
\[ \mathcal{L}_\xi h_\pm = (\xi^\lambda \partial_\lambda h_\pm^{\mu\nu} - h_\pm^{\mu\lambda} \partial_\lambda \xi^\nu - h_\pm^{\nu\lambda} \partial_\lambda \xi^\mu) \partial_\mu \otimes \partial_\nu = 0, \quad (4.8) \]

and

\[ [\mathcal{L}_\xi, \nabla^\pm] = 0 \quad (4.9) \]

are valid simultaneously. In other words, Eqs.\((4.1), (4.3), (4.6)\) are invariant under the coordinate transformation \((2.6)\) and its inverse transformation,

\[ x = \frac{L^{-1} x'}{1 - l^{-1}(b \cdot L^{-1} x')} = \frac{L^{-1} x'}{1 - l^{-1}(b' \cdot x')} \quad (4.10) \]

By definition, the curvature tensor is

\[ R^\sigma_{\pm \mu\nu\rho} = \partial_\nu \Gamma^\sigma_{\pm \mu\rho} - \partial_\rho \Gamma^\sigma_{\pm \mu\nu} + \Gamma^\sigma_{\pm \tau\nu} \Gamma^\tau_{\pm \mu\rho} - \Gamma^\sigma_{\pm \tau\rho} \Gamma^\tau_{\pm \mu\nu} = \pm l^{-2} (\delta^\sigma_\rho g^\pm_{\mu\nu} - \delta^\sigma_\nu g^\pm_{\mu\rho}). \quad (4.11) \]

It is antisymmetric in the latter two indexes and satisfies the Ricci and Bianchi identities. The Ricci curvature tensor is then

\[ R^\pm_{\mu\nu} = R^\sigma_{\pm \mu\sigma} = \pm 3l^{-2} g^\pm_{\mu\nu}. \quad (4.12) \]

They are obviously invariant under \(P_2\) transformation. Eqs.\((4.11)\) and \((4.12)\) are similar to those of the maximum-symmetric space-times.

### V. STRUCTURE OF THE SPACE-TIMES

The geometries in the previous section are presented in a special coordinate system \(x^\mu\). In order to see the structures of the manifolds more transparently, we consider the coordinate transformations,

\[
\begin{aligned}
&x^0 = l^2 \rho^{-1} \sinh(\psi/l) \\
x^1 = l^2 \rho^{-1} \cosh(\psi/l) \sin \theta \cos \phi \\
x^2 = l^2 \rho^{-1} \cosh(\psi/l) \sin \theta \sin \phi \\
x^3 = l^2 \rho^{-1} \cosh(\psi/l) \cos \theta \\
\end{aligned}
\quad \text{for} \ (x \cdot x) < 0, \quad (5.1)
\]

\[
\begin{aligned}
&x^0 = l^2 \eta^{-1} \cosh(r/l) \\
x^1 = l^2 \eta^{-1} \sinh(r/l) \sin \theta \cos \phi \\
x^2 = l^2 \eta^{-1} \sinh(r/l) \sin \theta \sin \phi \\
x^3 = l^2 \eta^{-1} \sinh(r/l) \cos \theta \\
\end{aligned}
\quad \text{for} \ (x \cdot x) > 0, \quad (5.2)
\]

respectively. Under the coordinate transformations, Eqs.\((4.1), (4.3), (4.6)\) become, respectively,

\[
g^\pm = \bar{g}^\pm_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu = \begin{cases} 
\frac{d\psi^2 - l^2 \cosh^2(\psi/l) d\Omega^2_2}{l^2} & \text{for} \ x \cdot x < 0 \\
-l^2 d\theta^2 - l^2 \sinh^2(r/l) d\Omega^2_2 & \text{for} \ x \cdot x > 0,
\end{cases} \quad (5.3)
\]
\[ h_{\pm} = \bar{h}_{\pm}^{\mu\nu} \partial_{\mu} \otimes \partial_{\nu} = \begin{cases} -\frac{\partial}{\partial \rho} \otimes \frac{\partial}{\partial \rho}, & \text{for } x \cdot x < 0 \\ \frac{\partial}{\partial \eta} \otimes \frac{\partial}{\partial \eta}, & \text{for } x \cdot x > 0, \end{cases} \]  

(5.4)

\[
\begin{align*}
&\bar{\Gamma}^{\psi}_{+\theta\theta} = l \sinh(\psi/l) \cosh(\psi/l), \quad \bar{\Gamma}^{\psi}_{+\phi\phi} = \bar{\Gamma}^{\psi}_{+\theta\theta} \sin^2 \theta \\
&\bar{\Gamma}^{\theta}_{+\theta\theta} = \bar{\Gamma}^{\theta}_{+\phi\phi} = \bar{\Gamma}^{\phi}_{+\phi\phi} = \bar{\Gamma}^{\phi}_{+\phi\phi} = l^{-1} \tanh(\psi/l) \quad \text{for } x \cdot x < 0, \\
&\bar{\Gamma}^{\phi}_{+\alpha\beta} = -l^{-2} \rho g_{\alpha\beta}, \quad \text{others vanish,} \\
&\bar{\Gamma}^{\eta}_{-ij} = +l^{-2} \eta_{ij} \\
&\bar{\Gamma}^{r}_{-\theta\theta} = -l \sinh(r/l) \cosh(r/l), \quad \bar{\Gamma}^{r}_{-\phi\phi} = \bar{\Gamma}^{r}_{-\theta\theta} \sin^2 \theta \\
&\bar{\Gamma}^{\theta}_{-\theta\theta} = \bar{\Gamma}^{\theta}_{-r\theta} = \bar{\Gamma}^{\phi}_{-r\phi} = \bar{\Gamma}^{\phi}_{-\phi\phi} = \frac{1}{l \tanh(r/l)} \quad \text{for } x \cdot x > 0, \\
&\bar{\Gamma}^{\phi}_{-\theta\phi} = -\sin \theta \cos \theta, \quad \bar{\Gamma}^{\phi}_{-\phi\theta} = \bar{\Gamma}^{\phi}_{-\phi\phi} = \cot \theta \\
&\text{others vanish}
\end{align*}
\]

where

\[
(\bar{x}^\mu) = \begin{cases}
(\bar{x}^a; \bar{x}^3) = (\psi, \theta, \phi; \rho), & \alpha, \beta, \gamma \text{ run from 0 to 2, } x \cdot x < 0 \\
(\bar{x}^0; \bar{x}^i) = (\eta; r, \theta, \phi), & i, j, k \text{ run from 1 to 3, } x \cdot x > 0.
\end{cases}
\]  

(5.7)

All quantities in \( \bar{x}^\mu \) coordinate system are denoted by an over bar. The Ricci curvature (4.12) reads

\[
\bar{R}^{\pm}_{\mu\nu} = \begin{cases}
3l^{-2} \text{diag}(1, -\cosh^2(\psi/l), -\cosh^2(\psi/l) \sin^2 \theta, 0) & x \cdot x < 0, \\
-3l^2 \text{diag}(0, -1, -\sinh^2(r/l), -\sinh^2(r/l) \sin^2 \theta) & x \cdot x > 0.
\end{cases}
\]  

(5.8)

They show that the manifolds are, at least, local \( dS_3 \times \mathbb{R} \) for \( x \cdot x < 0 \) and local \( \mathbb{R} \times \mathbb{H}_3 \) for \( x \cdot x > 0 \), respectively. For the former case, \( \rho \in (0, \infty) \) or \( (-\infty, 0) \), \( \psi \in (-\infty, +\infty) \), \( \theta \in [0, \pi] \), and \( \phi \in [0, 2\pi) \). For the latter case, \( \eta \in (0, \infty) \) or \( (-\infty, 0) \), \( r \in [0, \infty) \), \( \theta \in [0, \pi] \), and \( \phi \in [0, 2\pi) \). Both 3d \( dS \) space-time and 3d hyperboloid have ‘radius’ \( l \).

Under the coordinate transformation (2.6), the pure Lorentz transformations \( L^\mu_{\nu} \) do not induce singular transformations and not alter \( \rho \) and \( \eta \). They will induce the transformations on \( dS_3 \) or \( \mathbb{H}_3 \). However, the points on the hypersurface satisfying

\[ 1 + l^{-1} b \cdot x = 0 \]  

(5.9)

are transformed to infinity in the new coordinate system \( x' \), meanwhile the infinity points in the coordinate system \( x \), satisfying

\[ (1 + l^{-1} b \cdot x)^{-1} = 0, \]  

(5.10)

\footnote{The antipodal identification is not taken here as in [7].}
may be transformed to finite points. In particular, when \( L_{\nu}^{\mu} = \delta_{\nu}^{\mu} \) and \( b_{\mu} \neq 0 \ \forall \mu \), the points satisfying (5.9) correspond to the points transformed from \((\psi, \theta, \phi; \rho)\) to \((\psi, \theta, \phi; 0)\) for \( x \cdot x < 0 \) and from \((\eta; r, \theta, \phi)\) to \((0; r, \theta, \phi)\) for \( x \cdot x > 0 \). It implies that the points with \( \rho = 0 \) for \( x \cdot x < 0 \) and with \( \eta = 0 \) for \( x \cdot x > 0 \) should be in the space-times. Since the geometries (5.3), (5.4), (5.5) and (5.6) are regular at \( \rho = 0 \) for \( x \cdot x < 0 \) and at \( \eta = 0 \) for \( x \cdot x > 0 \), the geometries can be extended through \( \rho = 0 \) and \( \eta = 0 \), respectively. Therefore, \( \rho^{-1} \in (-\infty, \infty) \) for \( x \cdot x < 0 \) and \( \eta^{-1} \in (-\infty, +\infty) \) for \( x \cdot x > 0 \), the manifolds are globally \( dS_3 \times \mathbb{R} \) for \( x \cdot x < 0 \) and \( \mathbb{R} \times \mathbb{H}_3 \) for \( x \cdot x > 0 \), respectively.

Furthermore, the 4d volume elements on the manifolds, defined by

\[
\epsilon = \begin{cases} 
\int l^2 \cosh^2(\psi/l) \sin \theta d\psi \wedge d\theta \wedge d\phi \wedge d\rho, & \text{for } x \cdot x < 0 \\
\int l^2 \sinh^2(r/l) \sin \eta \wedge dr \wedge d\theta \wedge d\phi, & \text{for } x \cdot x > 0
\end{cases}
\]  

(5.11)

are invariant under the \( P_2 \) transformation, so the manifolds are orientable. For \( x \cdot x < 0 \), the manifold is obviously time orientable because 3d \( dS \) space-time is. For \( x \cdot x > 0 \), the invariant tensor \( h \) defines an invariant vector field \( \partial_\eta \) which is regular on the whole manifold. Compared with the Newton-Cartan case, it gives an absolute time direction and, therefore, the space-time is also obviously time orientable.

VI. Uniqueness

A. Uniqueness of \( p_2 \)-invariant metrics

In the coordinate systems (5.1) or (5.2), \( H' \) and \( P' \) can be written as

\[
\begin{align*}
H' &= c \sinh(\psi/l) \frac{\partial}{\partial \rho}, \\
P'_1 &= \cosh(\psi/l) \sin \theta \cos \phi \frac{\partial}{\partial \rho}, \\
P'_2 &= \cosh(\psi/l) \sin \theta \sin \phi \frac{\partial}{\partial \rho}, \\
P'_3 &= \cosh(\psi/l) \cos \theta \frac{\partial}{\partial \rho}, \\
H' &= c \cosh(r/l) \frac{\partial}{\partial \eta}, \\
P'_1 &= \sinh(r/l) \sin \theta \cos \phi \frac{\partial}{\partial \eta}, \\
P'_2 &= \sinh(r/l) \sin \theta \sin \phi \frac{\partial}{\partial \eta}, \\
P'_3 &= \sinh(r/l) \cos \theta \frac{\partial}{\partial \eta},
\end{align*}
\]  

(6.1)

respectively. Obviously, if the vectors in \( p_2 \)-translation subalgebra \( T' \) spanned by \( H' \) and \( P' \) are denoted by \( \xi_{(\sigma)} \), where the subscript in parenthesis \( (\sigma) \) is used to distinguish different vectors, their components have the form of \( \xi^{\lambda}_{(\sigma)} = f_{(\sigma)}(\psi, \theta, \phi)\delta_{3}^{\lambda} \) for \( x \cdot x < 0 \) or \( \xi^{\lambda}_{(\sigma)} = f_{(\sigma)}(r, \theta, \phi)\delta_{0}^{\lambda} \) for \( x \cdot x > 0 \), respectively. The direct calculations show that all \( \xi_{(\sigma)} \) in Lorentz algebra \( \mathfrak{L}_p \) only depend on \( (\psi, \theta, \phi) \) or \( (r, \theta, \phi) \) and \( \xi^{3}_{(\sigma)} = 0 \) or \( \xi^{0}_{(\sigma)} = 0 \) for \( x \cdot x \leq 0 \), respectively.
Suppose $g^\pm$ are $p_2$-invariant metrics of the covariant form and $h_\pm$ are $p_2$-invariant metrics of the contravariant form. They satisfy

\begin{align}
    \mathcal{L}_\xi g^\pm_{\mu\nu} &= \xi^\lambda \partial_\lambda g^\pm_{\mu\nu} + \bar{g}^\pm_{\mu\lambda} \partial_\nu \xi^\lambda + \bar{g}^\pm_{\nu\lambda} \partial_\mu \xi^\lambda = 0, \quad \forall \xi \in p_2, \\
    \mathcal{L}_\xi h^\pm_{\mu\nu} &= \xi^\lambda \partial_\lambda h^\pm_{\mu\nu} - \bar{h}^\pm_{\mu\lambda} \partial_\nu \xi^\lambda - \bar{h}^\pm_{\nu\lambda} \partial_\mu \xi^\lambda = 0, \quad \forall \xi \in p_2.
\end{align}

Consider the case $x \cdot x < 0$ first. Eq. (6.3) for $\xi_{(\sigma)} \in T'$ reads

\begin{align}
    0 &= f(\sigma) \partial_3 \bar{g}^+_{33} + 2\bar{g}^+_{3\alpha} \partial_3 f(\sigma) = f(\sigma) \partial_3 \bar{g}^+_{33} \\
    0 &= f(\sigma) \partial_3 \bar{g}^+_{\alpha\beta} + \bar{g}^+_{\alpha\gamma} \partial_3 f(\sigma) = f(\sigma) \partial_3 \bar{g}^+_{\alpha\beta} + \bar{g}^+_{\gamma\alpha} \partial_3 f(\sigma) \\
    0 &= f(\sigma) \partial_3 \bar{g}^+_{\alpha\beta} + \bar{g}^+_{\alpha\gamma} \partial_3 f(\sigma) + \bar{g}^+_{\beta\gamma} \partial_3 f(\sigma). \quad \text{(no summation for } \alpha) \\
    0 &= f(\sigma) \partial_3 \bar{g}^+_{\alpha\beta} + \bar{g}^+_{\alpha\gamma} \partial_3 f(\sigma) + \bar{g}^+_{\beta\gamma} \partial_3 f(\sigma). \quad \text{(no summation for } \alpha) \\
\end{align}

Eq. (6.5) gives $\partial_3 \bar{g}^+_{33} = 0$ right away. The validity of Eq. (6.6) for all $(\sigma)$ at the same time requires $\partial_3 \bar{g}^+_{\alpha\beta} = \bar{g}^+_{33} = 0$. Similarly, Eq. (6.7) results in $\partial_3 \bar{g}^+_{\alpha\beta} = \bar{g}^+_{33} = 0$. Then, Eq. (6.8) leads to $\partial_3 \bar{g}^+_{\alpha\beta} = 0$. The nontrivial equations of Eq. (6.3) for $\xi_{(\sigma)} \in \Sigma_p$ are

\begin{align}
    \xi^\gamma \partial_\gamma \bar{g}^+_{\alpha\beta} + \bar{g}^+_{\alpha\gamma} \partial_\beta \xi^\gamma + \bar{g}^+_{\beta\gamma} \partial_\alpha \xi^\gamma = 0. \quad (6.9)
\end{align}

This is nothing but the Killing equation on $dS_3$ on which the 3d metric tensor $^3g$ is unique up to an overall constant scale factor. Thus, the 4d degenerate metric in the coordinate system (5.1) takes the form

\begin{align}
    (\bar{g}_{\mu\nu}) &= \text{diag}(1, -\cosh^2(\psi/l), -\cosh^2(\psi/l) \sin^2 \theta, 0), \quad (6.10)
\end{align}

in which the overall constant scale factor has been chosen as 1. Similarly, Eq. (6.4) for $\xi_{(\sigma)} \in T'$ reads

\begin{align}
    0 &= f(\sigma) \partial_3 \bar{h}^+_{33} - 2\bar{h}^+_{3\alpha} \partial_3 f(\sigma) \\
    0 &= f(\sigma) \partial_3 \bar{h}^+_{\alpha\beta} - \bar{h}^+_{\alpha\gamma} \partial_3 f(\sigma) \\
    0 &= f(\sigma) \partial_3 \bar{h}^+_{\alpha\beta}. \quad (6.13)
\end{align}

They demands that $\partial_3 \bar{h}^+_{33} = \bar{h}^+_{33} = \partial_\gamma \bar{h}^+_{\alpha\beta} = \bar{h}^+_{\alpha\beta} = \partial_3 \bar{h}^+_{\alpha\beta} = 0$. Eq. (6.4) for $\xi_{(\sigma)} \in \Sigma_p$ reads

\begin{align}
    0 &= \xi^\gamma(\sigma) \partial_\gamma \bar{h}^+_{33}, \quad (6.14)
    0 &= \xi^\gamma(\sigma) \partial_\gamma \bar{h}^+_{\alpha\beta} - \bar{h}^+_{\alpha\gamma} \partial_\beta \xi^\gamma(\sigma) = \xi^\gamma(\sigma) \partial_\gamma \bar{h}^+_{\alpha\beta}, \quad (6.15)
    0 &= \xi^\gamma(\sigma) \partial_\gamma \bar{h}^+_{\alpha\beta} - \bar{h}^+_{\alpha\gamma} \partial_\beta \xi^\gamma(\sigma) - \bar{h}^+_{\beta\gamma} \partial_\alpha \xi^\gamma(\sigma) = \xi^\gamma(\sigma) \partial_\gamma \bar{h}^+_{\alpha\beta}. \quad (6.16)
\end{align}

They constrains $\partial_\gamma \bar{h}^+_{33} = \partial_\gamma \bar{h}^+_{\alpha\beta} = \partial_\gamma \bar{h}^+_{\alpha\beta} = 0$. Therefore, $h_+ = -\partial_\rho \otimes \partial_\rho$ is unique up to a constant scale factor.

Next, consider the case $x \cdot x > 0$. Eq. (6.3) for $\xi_{(\sigma)} \in T'$ reads

\begin{align}
    00 & : \quad 0 = f(\sigma) \partial_0 \bar{g}^-_{00} + 2\bar{g}^-_{00} \partial_0 f(\sigma) = f(\sigma) \partial_0 \bar{g}^-_{00} \\
    0i & : \quad 0 = f(\sigma) \partial_0 \bar{g}^-_{0i} + \bar{g}^-_{00} \partial_i f(\sigma) + \bar{g}^-_{i0} \partial_0 f(\sigma) = f(\sigma) \partial_0 \bar{g}^-_{0i} + \bar{g}^-_{00} \partial_i f(\sigma) \\
    ii & : \quad 0 = f(\sigma) \partial_0 \bar{g}^-_{ii} + 2\bar{g}^-_{i0} \partial_i f(\sigma) \quad \text{(no summation for } i) \\
    ij & : \quad 0 = f(\sigma) \partial_0 \bar{g}^-_{ij} + \bar{g}^-_{i0} \partial_j f(\sigma) + \bar{g}^-_{j0} \partial_i f(\sigma). \quad (6.20)
\end{align}
They give rise to $\partial_0 \bar{g}^{i0} = \partial_0 \bar{g}_{0i} = \bar{g}^{i0} = \partial_0 \bar{g}_{0i} = \bar{g}_{i0} = \partial_0 \bar{g}_{ij} = 0$. Eq. (6.3) for $\xi \in L_p$ requires
\[ 0 = \xi^k \partial_k \bar{g}^{-ij} + \bar{g}^{-ik} \partial_j \xi^k + \bar{g}^{-jk} \partial_i \xi^k, \] (6.21)
which is again just the Killing equation on $\Sigma$. So, $\bar{g}_{ij}$ is unique up to a scale factor. Without loss of generality,
\[ (\bar{g}^{-\mu\nu}) = \text{diag}(0, -1, -\sinh^2 r, -\sinh^2 r \sin^2 \theta). \] (6.22)
Similarly, Eq. (6.4) for $\xi \in p_2$ sets $\partial_0 \bar{h}^{00} = \bar{h}_{0i} = \partial_0 \bar{h}^{-0i} = \bar{h}^{-ik} = \partial_i \bar{h}^{00} = \partial_i \bar{h}^{-0j} = \partial_i \bar{h}^{-jk} = 0$.
Therefore, $h_\perp = \bar{h}_{00} \partial_0 \otimes \partial_0$ up to a scale factor.

Therefore, we come to the following theorem.

**Theorem 2** (The uniqueness of $p_2$-invariant metrics)
Up to an overall constant scale factor,
1. The type-(0,2) degenerate symmetric tensor fields $g^\pm$ (4.1) are unique $p_2$-invariant metrics of the covariant form for $x \cdot x < 0$ and $x \cdot x > 0$, respectively; and
2. the type-(2,0) degenerate symmetric tensor fields $h_\pm$ (4.3) are unique $p_2$-invariant metrics of the contravariant form for $x \cdot x < 0$ and $x \cdot x > 0$, respectively.

**B. The uniqueness of $p_2$-invariant connection**

**Theorem 3** (The uniqueness of $p_2$-invariant connection)
Suppose $\nabla$ is a connection which satisfies
1. $\nabla g^\pm = 0$;
2. $\nabla h_\pm = 0$;
3. $[\mathcal{L}_\xi, \nabla]v = 0$, $\forall \xi \in p_2$ and $\forall v \in TM$,
then $\nabla$ is unique.

**Proof:** Taking $x \cdot x > 0$ as an example. In coordinate system $\bar{x}^\mu$, the 3-d induced connection is uniquely determined by $\bar{g}^{-ij}$ and the unknown components of connection are $\bar{\Gamma}_{\eta\eta}^\eta$, $\bar{\Gamma}_{-\eta\eta}^\eta$, $\bar{\Gamma}_{-\eta\eta}^\eta$, and $\bar{\Gamma}_{-\eta\eta}^\eta$ because of the first condition. The second condition requires $\bar{\Gamma}_{\eta\eta}^\eta = \bar{\Gamma}_{-\eta\eta}^\eta = 0$. The third condition for $\xi \in L_p$ and $v = \partial_\eta$ is
\[ 0 = [\mathcal{L}_\xi, \nabla]v' = \mathcal{L}_\xi \nabla_{\mu} v' - \nabla_{\mu} [\xi, \nabla] v' = \mathcal{L}_\xi \nabla_{\mu} (\frac{\partial}{\partial \eta})^\nu \]
\[ = f_\sigma (\frac{\partial}{\partial \eta})^\nu \bar{\Gamma}_{-\mu\eta}^\nu - \bar{\Gamma}_{-\mu\eta}^\nu (\partial_i f_\sigma) \delta_\eta^\nu + \bar{\Gamma}_{-\mu\eta}^\nu (\partial_{\mu} f_\sigma). \] (6.23)
When $\mu = \nu = \eta$, it reads
\[ \bar{\Gamma}_{-\eta\eta}^\eta \partial_i f_\sigma = 0. \] (6.24)
This is an over-determined set of linear homogeneous equations for $\bar{\Gamma}_{-\eta\eta}^\eta$, which has only zero solution, $\bar{\Gamma}_{-\eta\eta}^\eta = 0$. When $\mu = k$ and $\nu = \eta$, the equation becomes over-determined sets of linear homogeneous equations for $\bar{\Gamma}_{-k\eta}^i$:
\[ \bar{\Gamma}_{-k\eta}^i \partial_i f_\sigma = 0, \] (6.25)
which have only zero solutions, $\bar{\Gamma}^{\mu}_{i\eta} = 0$, too. The third condition for $\xi \in T^p$ and $v = \partial_i$ is

$$0 = [L_{\xi(o)}(\nabla_\mu), v^\nu] = L_{\xi(o)}(\nabla_\mu v^\nu) - \nabla_\mu [\xi(o), \partial_i]^\nu$$

$$= f(\sigma) \frac{\partial}{\partial \eta} \hat{\Gamma}^\nu_{-\mu} - \hat{\Gamma}^k_{-\mu}(\partial_k f(\sigma))\delta^\nu_\eta + \hat{\Gamma}^\nu_{\eta}(\partial_\mu f(\sigma)) + (\partial_\mu \partial_i f(\sigma))\delta^\nu_\eta + (\partial_\nu f(\sigma))\hat{\Gamma}^\nu_{-\eta}.$$(6.26)

When $\mu = j$, $\nu = \eta$, it reads

$$f(\sigma) \frac{\partial}{\partial \eta} \hat{\Gamma}^\eta_{-ji} - \hat{\Gamma}^k_{-ji}(\partial_k f(\sigma)) + \partial_j \partial_i f(\sigma) = 0,$$

which leads to

$$\frac{\partial}{\partial \eta} \hat{\Gamma}^\eta_{-ji} = \hat{\Gamma}^k_{-ji}(\partial_k \ln f(\sigma)) - \frac{\partial_j \partial_i f(\sigma)}{f(\sigma)} = +g_{ij}.$$(6.27)

Thus,

$$\hat{\Gamma}^\eta_{-ji} = \eta g_{ij} + \gamma^0_{ij}$$

where $\gamma$ is independent on $\eta$. The third condition for $\xi \in L_p$ and $v = \partial_i$ is

$$0 = [L_{\xi(o)}(\nabla_\mu), v^\nu] = L_{\xi(o)}(\nabla_\mu v^\nu) - \nabla_\mu [\xi(o), \partial_i]^\nu$$

$$= \xi(o) \partial_k \hat{\Gamma}^\nu_{\mu} - \Gamma^\lambda_\mu \partial_\lambda \xi(o)^\nu + \Gamma^\nu_{\lambda\mu} \partial_\mu \xi(o)^\lambda + \partial_\mu \partial_i \xi(o)^\nu + \Gamma^\nu_{\mu k} \partial_i \xi(o)^k.$$(6.29)

When $\mu = j$ and $\nu = \eta$, it reads

$$0 = \xi(o) \partial_k \hat{\Gamma}^\eta_{-ji} + \hat{\Gamma}^\eta_{-ki} \partial_j \xi(o)^k + \hat{\Gamma}^\eta_{-jk} \partial_i \xi(o)^k.$$(6.30)

Since $\eta g_{ij}$ satisfies the equation, $\gamma^0_{ij}$ should also satisfy the equation. It is just the Killing equation if $\gamma^0_{ij}$ acts as a $(0, 2)$-type tensor. Thus, the general form of $\hat{\Gamma}^\eta_{-ij}$ should be

$$\hat{\Gamma}^\eta_{-ij} = (\eta + C)g_{ij}.$$ 

It differs from Eq.(5.6) trivially by a simple coordinate transformation $\eta \rightarrow \eta + C$, which corresponds to the coordinate transformation Eq.(2.6) with Eq.(5.9) and $L^\mu_{\nu} = \delta^\mu_{\nu}$. \hfill\Box

### VII. SYMMETRIES

#### A. Maximum symmetry of the geometries

In the above section, we have shown that the geometries $(M^{p2}, g^\pm, h_\pm, \nabla^\pm)$ are the unique geometries which are invariant under the $P_2$ transformation. In this subsection, we shall show that the Killing vector field $\xi$ satisfying Eqs.(4.7), (4.8) and (4.9) simultaneously must belong to $p_2$.

Now, suppose $v$ be an arbitrary vector field. Eq.(4.9) acting on $v^\nu$ gives

$$[L_{\xi(o)}(\nabla_\mu), v^\nu] = \xi^\lambda(\nabla^\mu_\gamma \nabla^\nu_\mu v^\nu - \nabla^\mu_\gamma \nabla^\nu_\mu v^\nu) + v^\lambda \nabla^\mu_\gamma \nabla^\nu_\lambda \xi^\nu$$

$$= -\xi^\lambda R^\nu_{\mu\kappa\lambda} v^\kappa + v^\lambda \nabla^\mu_\gamma \nabla^\nu_\lambda \xi^\nu = 0,$$(7.1)
which implies
\[ \nabla^\pm_\mu \nabla^\pm_\lambda \xi^\nu = R^\nu_{\pm\lambda\mu\kappa} \xi^\kappa = \pm l^{-2}(\delta^\nu_\mu g^\pm_{\lambda\kappa} - g^\pm_{\lambda\mu} \xi^\nu). \] (7.2)

On the other hand,
\[ \nabla^\pm_\mu \nabla^\pm_\lambda \xi^\nu = \partial_\mu(\partial_\lambda \xi^\nu + \Gamma^\nu_{\pm\lambda\kappa}) - \Gamma^\kappa_{\pm\mu\lambda}(\partial_\kappa \xi^\nu + \Gamma^\nu_{\pm\kappa\sigma} \xi^\sigma) + \Gamma^\nu_{\pm\mu\kappa}(\partial_\lambda \xi^\kappa + \Gamma^\kappa_{\pm\lambda\sigma} \xi^\sigma) \]
\[ = \partial_\mu \partial_\lambda \xi^\nu - \frac{x_\kappa \delta^\nu_\lambda \partial_\mu \xi^\kappa + \partial_\kappa \delta^\nu_\lambda \partial_\mu \xi^\kappa}{x \cdot x} \pm l^{-2}(\delta^\nu_\mu g^\pm_{\kappa\sigma} + \delta^\nu_\kappa g^\pm_{\mu\sigma}) \xi^\sigma + \frac{x_\mu \delta^\nu_\lambda + x_\lambda \delta^\nu_\mu}{(x \cdot x)^2} x_\sigma \xi^\sigma = 0. \] (7.3)

They give rise to the PDE
\[ \partial_\mu \partial_\lambda \xi^\nu - \frac{x_\kappa \delta^\nu_\lambda \partial_\mu \xi^\kappa + \partial_\kappa \delta^\nu_\lambda \partial_\mu \xi^\kappa}{x \cdot x} \pm l^{-2}(\delta^\nu_\mu g^\pm_{\kappa\sigma} + \delta^\nu_\kappa g^\pm_{\mu\sigma}) \xi^\sigma + \frac{x_\mu \delta^\nu_\lambda + x_\lambda \delta^\nu_\mu}{(x \cdot x)^2} x_\sigma \xi^\sigma = 0. \] (7.4)

Multiplication with \( x^\mu x^\lambda x_\nu \), it reduces to
\[ x_\mu \partial_\mu(x^\lambda \partial_\lambda(x_\nu \xi^\nu)) - 5x_\mu \partial_\mu(x_\kappa \xi^\kappa) + 6x_\kappa \xi^\kappa = 0. \] (7.5)

It has the following general solution
\[ x_\kappa \xi^\kappa = C_1(x^i/x^0)(\pm x \cdot x) + C_2(x^i/x^0)(\pm x \cdot x)^{3/2}, \] (7.6)

where \( C_1 \) and \( C_2 \) are the functions of the ratio of \( x^i \) to \( x^0 \) to be determined. Therefore, the Killing vector field should have the form
\[ \xi^\kappa = \pm C_1(x^i/x^0)x^\kappa \pm C_2(x^i/x^0) \cdot (\pm x \cdot x)^{1/2} x^\kappa + C_3(x)(x_\lambda \delta^\kappa_\sigma - x_\sigma \delta^\kappa_\lambda) + C^\mu(x) g^\pm_{\mu\nu} \eta^{\nu\kappa}, \] (7.7)

where \( C_3(x) \) and \( C_\mu(x) \) are the functions of \( x \) to be determined.

In order to fix \( C_1, C_2, C_3 \) and \( C^\mu \), we study the above expression term by term. The PDE (7.4) for the first term reads
\[ \partial_\mu \partial_\nu C_1 = 0. \] (7.8)

Thus, \( C_1 \) is, at most, the linear function of \( x \). However, \( C_1 \) is independent of \( x \cdot x \). Therefore, \( C_1 \) can only be a non-zero constant, at most. Note that
\[ \mathcal{L}_{x^\nu \partial_\nu} h^\ab_{\pm} = l^{-4} \mathcal{L}_{x^\nu \partial_\nu} [(x \cdot x)x^\lambda x^\sigma(\partial_\lambda)^a(\partial_\sigma)^b] = 2 h^\ab_{\pm} \neq 0. \] (7.9)

Therefore, \( C_1 \) must be zero.

The PDE (7.4) for the second term reads
\[ \partial_\mu \partial_\nu((\pm x \cdot x)^{1/2} C_2) = 0 \] (7.10)

It means that \((\pm x \cdot x)^{1/2} C_2\) is a linear function of \( x \), at most. Since \( C_2 \) does not contain the factor \((\pm x \cdot x)^{1/2}\). The above result implies that \( C_2 \) should be the homogeneous linear function of \( x^\kappa/(\pm x \cdot x)^{1/2} \). Therefore, the possible linear-independent vector fields \( \xi^\mu_{(\nu)} = x_\mu x^\kappa \partial_\kappa \), which are proportional to the pseudo-translation generators in \( \mathfrak{p}_2 \).
The PDE (7.4) for the fourth term reduces to
\[ \partial_\mu \partial_\nu (C^\lambda(x)g^\pm_{\lambda\kappa}y^{\kappa\sigma}) = 0, \] (7.11)
which implies
\[ C^\lambda(x)g^\pm_{\lambda\kappa} = A_{\kappa\nu}x^\nu + B_\kappa, \] (7.12)
where \( A_{\kappa\nu} \) and \( B_\kappa \) are constants. Since
\[ A_{\kappa\nu}x^\kappa x^\nu + B_\kappa = 0 \quad \forall x, \] (7.13)
there is no nonzero solution for \( A_{\kappa\nu} \) and \( B_\kappa \). It implies
\[ C^\kappa(x)g^\pm_{\kappa\nu} = 0. \] (7.14)

Finally, Eq. (4.7) for the third requires
\[ C_{3,\mu}(x)g^\pm_{\mu[\sigma}x_{\lambda]} + C_{3,\mu}(x)g^\pm_{\nu[\sigma}x_{\lambda]} = 0 \] (7.15)
because \( x_\lambda \partial_\sigma - x_\sigma \partial_\lambda \) are Killing vectors. When \( \mu = \nu \), it reduces to
\[ C_{3,\mu}(x)g^\pm_{\nu[\sigma}x_{\lambda]} = 0. \] (7.16)
Since \( g^\pm_{\nu[\sigma}x_{\lambda]} \) does not always vanish, \( C_3 \) must be a constant.

Then, we come to the theorem.

**Theorem 4** The maximum symmetry of the geometries \((M^{p,2}, g^\pm, h^\pm, \nabla^\pm)\) with Eqs. (4.1), (4.3) and (4.6) is the second Poincaré group.

### B. Re-classification of the symmetry

In the algebraic point of view, the so-called pseudo-translation generators, \( H' \) and \( P' \), spanning the Abelean ideal of \( \text{iso}(1,3) \), take the role of the time and space translation ones, respectively, and \( K \) and \( J \) span the \( \mathfrak{so}(1,3) \) algebras as usual, generating the \( SO(1,3) \) isotropy of space-time. Its subalgebra \( \mathfrak{so}(3) \) generates the \( SO(3) \) isotropy of space. However, the decomposition does not fit the above structure of space-time.

For the case \( x \cdot x < 0 \), the manifold is \( dS_3 \times \mathbb{R} \). The metric of the \( dS_3 \) space-time can be written as
\[ ds^2 = \frac{\eta_{\alpha\beta}dz^\alpha z^\beta}{\sigma_3(z)} + \frac{(\eta_{\alpha\beta}z^\alpha z^\beta)^2}{l^2 \sigma_3^2(z)} \] (7.17)
in terms of a 3d Beltrami coordinate system, say, on the chart \( U_3 \) [7],
\[ z^0 = \frac{l x^0}{x^3}, \quad z^1 = \frac{l x^1}{x^3}, \quad z^2 = \frac{l x^2}{x^3}, \] (7.18)
where

\[ \sigma_3(z) = 1 - l^{-2} \eta_{\alpha \beta} z^\alpha z^\beta > 0, \]  

(7.19)

and \( \alpha, \beta \) run over 0, 1, 2. On the \( dS_3 \) space-time, there are 3d Beltrami translations, which take the role of translation in the neighborhood of the origin on the 3d manifolds. They are

\[
\begin{cases}
3H^+ = c\partial_3 - cl^{-2}z^\beta \partial_\beta =: \mathcal{H}, \\
3P_1^+ = \partial_1 - l^{-2}z_1z^\beta \partial_\beta =: \mathcal{P}_1, \\
3P_2^+ = \partial_2 - l^{-2}z_2z^\beta \partial_\beta =: \mathcal{P}_2,
\end{cases}
\]

(7.20)

where the superscript 3 stands for the quantity being defined on the 3d space-time. The pseudo-translation generator in \( x^3 \) defines the translation in direction \( z^3 = l^2/x^3 = \rho \text{sech}(\psi/l) \text{sec} \theta \)

\[ P_3' = \partial z^3 =: \mathcal{P}_3. \]

(7.21)

The boost generators in 3d dS space-time are

\[
\begin{cases}
3K_1 = \frac{1}{c}(z_0 \partial_1 - z_1 \partial_0) = K_1 =: \mathcal{K}_1, \\
3K_2 = \frac{1}{c}(z_0 \partial_2 - z_2 \partial_0) = K_2 =: \mathcal{K}_2.
\end{cases}
\]

(7.22)

The Galilei boost in the direction \( z^3 \) is

\[ \frac{1}{c}z_0 \partial z^3 = \frac{l}{c^2}H' =: \mathcal{K}_3. \]

(7.23)

The three space ‘rotation’ generators

\[
\begin{cases}
\mathcal{J}_1 := z_1 \partial z_3 = lP_1', \\
\mathcal{J}_2 := z_2 \partial z_3 = lP_2', \\
\mathcal{J}_3 := z_1 \partial z_2 - z_2 \partial z_1 = J_3,
\end{cases}
\]

(7.24)

spans an \( \text{iso}(2) \) subalgebra,

\[ [\mathcal{J}_1, \mathcal{J}_2] = 0, \ [\mathcal{J}_1, \mathcal{J}_3] = -\mathcal{J}_2, \ [\mathcal{J}_2, \mathcal{J}_3] = \mathcal{J}_1. \]

(7.25)

Finally, it can be shown that

\[
\begin{cases}
\frac{c^2}{l^2}K_3 = \mathcal{H} - \frac{c^2}{l^2}z^3 \mathcal{K}_3 \\
-l^{-1}J_2 = \mathcal{P}_1 - l^{-2}z^3 \mathcal{J}_1 \\
l^{-1}J_1 = \mathcal{P}_2 - l^{-2}z^3 \mathcal{J}_2
\end{cases}
\quad \text{or} \quad
\begin{cases}
\frac{c^2}{l^2}K_3 + \frac{1}{x^3}H' = \mathcal{H} \\
l^{-1}J_2 + \frac{1}{x^3}P_1' = \mathcal{P}_1 \\
l^{-1}J_1 + \frac{1}{x^3}P_2' = \mathcal{P}_2
\end{cases}
\]

(7.26)

The set of generators \((\mathcal{H} - (c^2/l^2)z^3 \mathcal{K}_3, \mathcal{P}_1 - l^{-2}z^3 \mathcal{J}_1, \mathcal{P}_2 - l^{-2}z^3 \mathcal{J}_2, \mathcal{P}_3, \mathcal{K}, \mathcal{J})\) defines an alternative decomposition of \( \text{iso}(1, 3) \) algebra, different from the Poincaré algebra. Clearly, the alternative decomposition fits to the geometrical structure. The decomposition together the geometrical structure
gives a new realization of \( \mathfrak{iso}(1,3) \) algebra. It defines a new possible kinematics without the space \( \text{SO}(3) \) isotropy.

For the case \( x \cdot x > 0 \), the manifold is \( \mathbb{R} \times \mathbb{H}_3 \). In terms of the Beltrami coordinates, the metric of \( \mathbb{H}_3 \) space is
\[
d s^2 = -\frac{\delta_{ij} dz^i dz^j}{\sigma^E_3(z)} - \frac{(\delta_{ij} z^i dz^j)^2}{(l \sigma^E_3(z))^2},
\]
where
\[
z^i = l \frac{x^i}{x^0},
\]
and
\[
\sigma^E_3(z) = 1 - l^{-2} \delta_{ij} z^i z^j > 0.
\]
On \( \mathbb{H}_3 \) space, the 3d Beltrami translations
\[
^3P_i^\perp = \partial z_i + l^{-2} z_i z^j \partial z_j = \frac{c}{l} K_i =: \tilde{P}_i,
\]
play the role of translation in the neighborhood of the origin on the 3d manifold, and the space rotation generators, defined by
\[
\tilde{J}_i := \frac{1}{2} \epsilon_{ijk} (z_j P_k - z_k P_j) = J_i,
\]
span the \( \mathfrak{so}(3) \) subalgebra
\[
[\tilde{J}_i, \tilde{J}_j] = -\epsilon_{ijk} \tilde{J}_k.
\]
Define \( z^0 = l^2 / x^0 = \eta \text{sech}(r/l) \). Then, the Carroll boosts
\[
\tilde{K}_i := \frac{1}{c} z_i \partial z^0 = \frac{l}{c} P_i'.
\]
The pseudo-time translation
\[
H' = c \partial z^0 =: \tilde{H}
\]
defines the time translation in \( z^0 \) direction. Finally, it can be shown that
\[
\frac{c}{l} K_i = \tilde{P}_i + \frac{c}{l^2} z^0 \tilde{K}_i \quad \text{or} \quad \frac{c}{l} K_i = \frac{l}{x^0} \tilde{P}_i.
\]
(\( \tilde{H}, \tilde{P}_i + (c/l^2) z^0 \tilde{K}_i, \tilde{J}_i \)) gives another alternative decomposition of \( \mathfrak{iso}(1,3) \) algebra,
\[
\begin{align*}
[H, \tilde{P}_i + \frac{c}{l^2} z^0 \tilde{K}_i] &= \frac{c^2}{l^2} \tilde{K}_i, \\
[H, \tilde{K}_i] &= 0, \\
[H, \tilde{J}_i] &= -\epsilon_{ijk} \tilde{K}_k, \\
[\tilde{P}_i + \frac{c}{l^2} z^0 \tilde{K}_i, \tilde{J}_j] &= -\epsilon_{ijk} (\tilde{P}_k + \frac{c}{l^2} z^0 \tilde{K}_k),
\end{align*}
\]
\[
\begin{align*}
[H, \tilde{J}_i] &= \epsilon_{ijk} \tilde{J}_k, \\
[H, \tilde{P}_i] &= 0, \\
[H, \tilde{K}_i] &= 0, \\
[H, \tilde{K}_i, \tilde{J}_j] &= -\epsilon_{ijk} \tilde{K}_k, \\
[\tilde{K}_i, \tilde{J}_j] &= -\epsilon_{ijk} \tilde{K}_k,
\end{align*}
\]
\[
\begin{align*}
[H, \tilde{P}_i + \frac{c}{l^2} z^0 \tilde{K}_i, \tilde{J}_j] &= -\epsilon_{ijk} (\tilde{P}_k + \frac{c}{l^2} z^0 \tilde{K}_k),
\end{align*}
\]
different from the Poincaré algebra. The decomposition fits the geometrical structure on $\mathbb{R} \times \mathbb{H}_3$. The decomposition with the geometrical structure gives another new realization of $\text{iso}(1,3)$ algebra. It is easy to see that the new realization has the space $SO(3)$ isotropy and is invariant under the parity ($z^i \rightarrow -z^i$) and time-reversal ($z^0 \rightarrow -z^0$), i.e.

$$\Pi : \tilde{\mathcal{H}} \rightarrow \tilde{\mathcal{H}}, \tilde{\mathcal{P}} \rightarrow -\tilde{\mathcal{P}}, \tilde{\mathcal{J}} \rightarrow \tilde{\mathcal{J}}, \tilde{\mathcal{K}} \rightarrow -\tilde{\mathcal{K}}, \quad (7.37)$$

$$\Theta : \tilde{\mathcal{H}} \rightarrow -\tilde{\mathcal{H}}, \tilde{\mathcal{P}} \rightarrow \tilde{\mathcal{P}}, \tilde{\mathcal{J}} \rightarrow \tilde{\mathcal{J}}, \tilde{\mathcal{K}} \rightarrow -\tilde{\mathcal{K}}. \quad (7.38)$$

In addition, each $\tilde{\mathcal{K}}_i$ ($i = 1, 2, 3$) generates a noncompact subgroup. In other words, the generator set $(\tilde{\mathcal{H}}, \tilde{\mathcal{P}}, \tilde{\mathcal{K}}, \tilde{\mathcal{J}})$ satisfies all three assumptions in Ref. [1].

Since the algebra relation is the same as the para-Poincaré algebra if $\tilde{\mathcal{P}}$ are replaced by $-\tilde{\mathcal{P}}$ [1], the new realization of $\text{iso}(1,3)$ algebra is actually the para-Poincaré algebra.

In brief, the Beltrami translations on the 3d manifolds are different from the algebraic (pseudo) space translations $P'$ assigned a priori. In the new sets of generators fitting the geometrical structure, the space-time $SO(1,3)$ isotropy and even space $SO(3)$ isotropy are absent. Based on the above analysis, the space-times are the homogeneous spaces, respectively,

$$M^p_{p^2} = ISO(1,3)/ISO(1,2), \quad x \cdot x < 0, \quad (7.39)$$

$$M^p_{\dot{p}^2} = ISO(1,3)/ISO(3), \quad x \cdot x > 0. \quad (7.40)$$

### VIII. MOTIONS ON THE GEOMETRY

Since the second Poincaré symmetry is found based on the $PoR_{c,l}$ [3, 4], the motion for free particles should be uniform rectilinear. In this section, we shall confine ourselves in the 4d degenerate space-time $(M^p_{p^2}, g^-, h_-, \nabla^-)$ and study the motion of free particles in it, because it possesses the space isotropy.

#### A. Geodesic equation

The geodesic equation

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\lambda} \frac{dx^\lambda}{d\lambda} = 0, \quad (8.1)$$

gives rise to

$$\frac{d}{d\lambda} \left( \frac{1}{x \cdot x} \frac{dx^\mu}{d\lambda} \right) = 0. \quad (8.2)$$

It solutions is

$$\frac{1}{x \cdot x} \frac{dx^\mu}{d\lambda} = C^\mu. \quad (8.3)$$

Therefore,

$$\frac{dx^i}{dx^0} = \frac{C^i}{C^0}, \quad \Rightarrow \quad x^i = a^i x^0 + b^i, \quad (8.4)$$
where \( a^i = C^i/C^0 \) and \( b^i \) are two dimensionless constants. In other words, if the motion for free particles is still determined by the geodesic equation, the motion for free particles is a ‘uniform rectilinear motion’ as required if \( x^0/c \) is interpreted as the time and \( x^i \) are interpreted as coordinates of space. \( ca^i \) and \( lb^i \) play the roles of the uniform velocity and the initial position respectively. However, Eq. (8.4) reads

\[
\dot{z}^i = b^i z^0 + la^i \tag{8.5}
\]

in terms of the Beltrami coordinates on the \( \mathbb{H}_3 \) space, \( \dot{z}^i = l x^i / x^0 \), and \( z^0 = l^2 / x^0 \) introduced in the previous section. Again, it has the form of ‘uniform rectilinear motion’ if \( z^0/c \) is interpreted as the time coordinates. But now, \( cb^i \) and \( la^i \) play the roles of the uniform velocity and the initial position, respectively. The discrepancy raises a question: which is the genuine velocity of the free particle moving in the space-time?

It should be noted that the geometric structure of the space-time shows that \( x^0/c \) and \( z^0/c \) are not the coordinate of time and that \( x^i \) are not the coordinates of space with respect to the degenerate metric. In the space-time, \( \eta/c \) is the absolute time and \( z^i \) is the coordinates of the space \( \mathbb{H}_3 \). In terms of \( \eta \) and \( z^i \), Eq. (8.4) reads

\[
\dot{z}^i = \frac{b^i}{\cosh(r/l)} \eta + la^i, \tag{8.6}
\]

where \( r = l \tanh^{-1}(\sqrt{(z^1)^2 + (z^2)^2 + (z^3)^2}/l) \). When \( r \ll l \), it reduces to \( \dot{z}^i = b^i \eta + la^i \). This is a uniform rectilinear motion in the conventional sense. Therefore, \( cb^i \) is the genuine velocity of the free particle moving in a neighborhood of the origin of the space.

**B. Formal Lagrangian and Euler-Lagrangian equation for a free particle**

Consider the Lagrangian

\[
L = \frac{1}{x \cdot x} \sqrt{(\eta_{\mu\nu} \eta_{\rho\tau} - \eta_{\mu\rho} \eta_{\nu\tau}) x^\mu x^\nu \dot{x}^\rho \dot{x}^\tau}, \tag{8.7}
\]

with \( \dot{x}^\mu := dx^\mu/d\lambda \), where \( \lambda \) is the affine parameter along the trajectory of a particle. The Euler-Lagrangian equation reads

\[
\frac{d}{d\lambda} \left( (x \cdot \dot{x}) x_\kappa - (x \cdot x) \dot{x}_\kappa \right) = \frac{2(x \cdot \dot{x})^2 x_\kappa - (x \cdot x)(\dot{x} \cdot \dot{x}) x_\kappa - (x \cdot x)(x \cdot \dot{x}) \dot{x}_\kappa}{(x \cdot x)^2 \sqrt{(x \cdot x)(\dot{x} \cdot \dot{x}) - (x \cdot \dot{x})^2}} = 0. \tag{8.8}
\]

After some manipulation, it reduces to

\[
[(x \cdot x)(\dot{x} \cdot \dot{x}) - (x \cdot \dot{x})^2] \ddot{x}_\kappa + (\dot{x} \cdot \dot{x}) [(x \cdot x) x_\kappa - (x \cdot x) \dot{x}_\kappa] + (x \cdot \dot{x}) [(x \cdot \dot{x}) \dot{x}_\kappa - (\dot{x} \cdot \dot{x}) x_\kappa] = 0. \tag{8.9}
\]

This is a system of homogeneous equations for \( \ddot{x} \). Since its coefficient determinant

\[
\left|[(x \cdot x)(\dot{x} \cdot \dot{x}) - (x \cdot \dot{x})^2] \delta_\kappa^\lambda + \dot{x}_\lambda [(x \cdot x) x_\kappa - (x \cdot x) \dot{x}_\kappa] + x_\lambda [(x \cdot \dot{x}) \dot{x}_\kappa - (\dot{x} \cdot \dot{x}) x_\kappa]\right|, \tag{8.10}
\]

is not equal to 0, it has only zero solution \( \ddot{x}_\kappa = 0 \). It is equivalent to \( \dot{x}_\kappa = \text{const} \). Thus,

\[
\frac{dx_\kappa}{dx^9} = \text{const}. \tag{8.11}
\]

In other words, the generalized inertial motion can be obtained from the Lagrangian.
IX. CONCLUDING REMARKS

Bacry and Lévy-Leblond focus their attention on the algebraic relation in [1]. Their theorem says that under the three assumptions there exist only 11 kinds of kinematical algebraic relations. If the third assumption is relaxed, 3 kinds of geometrical algebraic relations will be added. In comparison, the approach from the principle of relativity with two universal constant, $PoR_{c,l}$, not only the algebraic relations but also the realization of the generators are concerned. Therefore, more possible kinematics than Bacry and Lévy-Leblond revealed are obtained. The kinematics with the second Poincaré symmetry is one of them. Obviously, the second Poincaré algebra is isomorphic to the ordinary Poincaré algebra algebraically, but the geometric realization of the two algebras are different. The second Poincaré group no longer preserves the metric of Minkowski space-time, but preserves the (non-vanishing-identically) geometry $(M^{p^2}, g, h, \nabla)$.

The geometrical analysis will, no doubts, provide a new view on all possible kinematics. In the algebraic analysis, $H, H'$, and $H^\pm$ take the role of the time translations, and $P, P'$, and $P^\pm$ serves as the space translations. The geometrical analysis, however, shows that they may have very different meaning. For example, in the geometry with $x \cdot x > 0$, the pseudo-space translations $P'$ (relating to $\tilde{K}$) actually generate the new kind of the boost transformations on $\mathbb{R} \times \mathbb{H}_3$, while the Beltrami space translations on the $\mathbb{H}_3$ space are generated by $\tilde{P}$ which is proportional to the Lorentz boost $K$. This can be seen in another way. In this case, we have $p_2$-invariant degenerate metric $g^-$ and absolute ‘time’ direction $\partial_\eta$. Because $\partial_\eta$ is unique and $g^-$ is independent of $\eta$, the manifold $M^{p^2}$ has a line bundle structure $\pi: M^{p^2} \to \Sigma = \mathbb{H}_3, (\eta, z^i) \mapsto (z^i)$, where $\partial_\eta$ is the tangent direction of the fiber. The $\mathbb{R}^4$ ideal of $iso(1,3)$ algebra are all along the fiber direction and $(\mathbb{H}_3, g^-)$ can be seen as an “absolute space”. Combine the $p_2$ action on $M^{p^2}$ and $\pi: M^{p^2} \to \Sigma$, we can define the $p_2$ action on $\Sigma$ as

$$g(z) = \pi \circ g \circ \pi^{-1}(z), \ \forall z \in \Sigma \text{ and } \forall g \in P_2.$$ 

Under this definition, the actions of the $\mathbb{R}^4$ ideal are trivial on $\Sigma$, i.e. they are no longer ‘space translations’. The $p_2$ action defined above is equivalent to the $L_p$ action on $\Sigma$. And the three boosts $\{K_i\}$ combined with $(1/x^0)P_i$, respectively, take the place of ‘space translations’, like the original space translation, spanning a representation space of the $so(3)$ sub-algebra on $\Sigma$.

The difference between the two Poincaré algebras should be further remarked on. In the above $p_2$ algebra, the $so(3)$ on $\Sigma$ is unique. In contrast, in the ordinary Poincaré algebra, the choice of the $so(3)$ in $L_p$ is not canonical. The division of the ideal of $p$ into $\mathbb{R} \oplus \mathbb{R}^3$ based on the irreducible representation of the $so(3)$ depends on the choice. The different choice of the $so(3)$ corresponds to different sets of inertial observers.

Like the Galilei and Carroll space-times, the space and time of the new geometry $\{M^{p^2}, g, h, \nabla\}$ are split. For the $x \cdot x < 0$ case, 1-d space is split out. There is a special direction in space. The kinematics on the 3d space-time is still relativistic, but is dramatically different from the kinematics in the special direction. It should be noted that $z^3$ is not the intrinsic coordinates for the split-out
space. In terms of the intrinsic coordinates $z^\alpha, \rho$, Eq. (7.21) and Eq. (7.23) become

$$P'_3 = -\frac{1}{\sqrt{1 - l^{-2} \eta_{\alpha\beta} z^\alpha z^\beta}} \partial_\rho = -P_3,$$

$$\frac{l}{c^2} H' = \frac{1}{c} \frac{z_0}{\sqrt{1 - l^{-2} \eta_{\alpha\beta} z^\alpha z^\beta}} \partial_\rho = K_3,$$

(9.1)

(9.2)

When $|z^i| \ll l$, they tend to the ordinary translation $P_3 \approx \partial_\rho$ and Galilei boost $K_3 \approx c^{-1} z_0 \partial_\rho$, respectively. For the $x \cdot x > 0$, the time is split out, which fixes a special time direction and an absolute space. In terms of the intrinsic coordinates $\eta, z^i$, Eq. (7.34) and Eq. (7.33) become, respectively,

$$H' = \frac{c}{\sqrt{1 - l^{-2} \delta_{ij} z^i z^j}} \partial_\eta = \tilde{H},$$

$$\frac{l}{c} P'_i = \frac{1}{c} \frac{z_i}{\sqrt{1 - l^{-2} \delta_{jk} z^j z^k}} \partial_\eta = \tilde{K}_i,$$

(9.3)

(9.4)

When $|z^j| \ll l$, they reduce to the ordinary time translation $\tilde{H} \approx c^{-1} \partial_\eta$ and the Carroll boosts $\tilde{K}_i \approx -c^{-1} z^i \partial_\eta$, respectively. The latter situation is very similar to the Carroll algebra and Carroll space-time, in which there is a special time direction and an absolute space. The difference between the Carroll space-time and the new space-time is that the absolute space in Carroll space-time is flat while the absolute space in the new space-time is Lobachevskian. In this sense, the new kinematics is non-relativistic.

If the space isotropy is required on the both algebraic and geometrical levels, only the space-time with $x \cdot x > 0$ remains. On the new space-time, the motions of free particles can be well defined. The mechanics, field theories and even gravity on the space-time should be further investigated in order to clarify the application of the new space-time. In the higher dimensional theories, there may be the second Poincaré group as its subgroup of symmetry. Hence, the geometric structure may appear in a higher dimension.

The reason that only the geometries for $x \cdot x < 0$ and $x \cdot x > 0$ cases are presented is that $x \cdot x = 0$ defines a three dimensional hypersurface, while the possible kinematics we are interested in is defined on a 4-d manifold.

**Acknowledgments**

We are very grateful to Prof. H.-Y. Guo for his valuable suggestions and comments. We would also like to thank Z.-N. Hu, W.-T. Ni and Dr. H.-T. Wu for their helpful discussion. This work is supported by NSFC under Grant Nos. 10775140, 10705048, 10731080, 10975141, the President
Fund of GUCAS, and the Fundamental Research Funds for the Central Universities under Grant No. 105116.

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