A Broken Circuit Ring

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Abstract. Given a matroid $M$ represented by a linear subspace $L \subset \mathbb{C}^n$ (equivalently by an arrangement of $n$ hyperplanes in $L$), we define a graded ring $R(L)$ which degenerates to the Stanley-Reisner ring of the broken circuit complex for any choice of ordering of the ground set. In particular, $R(L)$ is Cohen-Macaulay, and may be used to compute the $h$-vector of the broken circuit complex of $M$. We give a geometric interpretation of $\text{Spec } R(L)$, as well as a stratification indexed by the flats of $M$.

1 Introduction

Consider a vector space with basis $\mathbb{C}^n = \mathbb{C}\{e_1, \ldots, e_n\}$, and its dual $(\mathbb{C}^n)^\vee = \mathbb{C}\{x_1, \ldots, x_n\}$. Let $L \subset \mathbb{C}^n$ be a linear subspace of dimension $d$. We define a matroid $M(L)$ on the ground set $[n] := \{1, \ldots, n\}$ by declaring $I \subset [n]$ to be independent if and only if the composition $\mathbb{C}\{x_i \mid i \in I\} \hookrightarrow (\mathbb{C}^n)^\vee \twoheadrightarrow L^\vee$ is injective. Recall that a minimal dependent subset $C \subset [n]$ is called a circuit; in this case there exist scalars $\{a_c \mid c \in C\}$, unique up to scaling, such that $\sum_C a_c x_c$ vanishes on $L$. Conversely, the support of every linear form that vanishes on $L$ contains a circuit.

The central object of study in this paper will be the ring $R(L)$ generated by the inverses of the restrictions of the linear functionals $\{x_1, \ldots, x_n\}$ to $L$. More formally, let

$$\mathbb{C}[x,y] := \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]/\langle x_i y_i - 1 \rangle,$$

and let $\mathbb{C}[x]$ and $\mathbb{C}[y]$ denote the polynomial subrings generated by the $x$ and $y$ variables, respectively. Let $\mathbb{C}[L]$ denote the ring of functions on $L$, which is a quotient of $\mathbb{C}[x]$ by the ideal generated by the linear forms $\{ \sum_C a_c x_c \mid C \text{ a circuit} \}$. We now set

$$R(L) := \left( \mathbb{C}[L] \otimes_{\mathbb{C}[x]} \mathbb{C}[x,y] \right) \cap \mathbb{C}[y].$$

1Partially supported by the Clay Mathematics Institute Liftoff Program
Geometrically, Spec $R(L)$ is a subscheme of Spec $\mathbb{C}[y]$, which we will identify with $(\mathbb{C}^n)^\vee$. Using the isomorphism between $\mathbb{C}^n$ and $(\mathbb{C}^n)^\vee$ provided by the dual bases, Spec $R(L)$ may be obtained by intersecting $L$ with the torus $(\mathbb{C}^\ast)^n$, applying the involution $t \mapsto t^{-1}$ on the torus, and taking the closure inside of $\mathbb{C}^n$. If $C$ is any circuit of $M(L)$ with $\sum_{c \in C} a_c x_c$ vanishing on $L$, then we have the relation

$$f_C := \sum_{c \in C} a_c \prod_{c' \in C \setminus \{c\}} y_{c'} = 0 \quad \text{in} \quad R(L).$$

Our main result (Theorem 4) will be that the elements $\{f_C \mid C \text{ a circuit}\}$ are a universal Gröbner basis for $R(L)$, hence this ring degenerates to the Stanley-Reisner ring of the broken circuit complex of $M(L)$ for any choice of ordering of the ground set $[n]$. It follows that $R(L)$ is a Cohen-Macaulay ring of dimension $d$, and that the quotient of $R(A)$ by a minimal linear system of parameters has Hilbert series equal to the $h$-polynomial of the broken circuit complex. In Proposition 7 we identify a natural choice of linear parameters for $R(L)$.

The Hilbert series of $R(L)$ has already been computed by Terao [Te], using different methods. The main novelty of our paper lies in our geometric approach, and our interpretation of $R(L)$ as a deformation of another well-known ring. The ring $R(L)$ also appears as a cohomology ring in [PW], and as the homogeneous coordinate ring of a projective variety in [Lo, 3.1].

**Acknowledgment.** Both authors would like to thank Ed Swartz for useful discussions.

## 2 The broken circuit complex

Choose an ordering $w$ of $[n]$. We define a broken circuit of $M(L)$ with respect to $w$ to be a set of the form $C \setminus \{c\}$, where $C$ is a circuit of $M(L)$ and $c$ the $w$-minimal element of $C$. We define the broken circuit complex $bc_w(L)$ to the simplicial complex on the ground set $[n]$ whose faces are those subsets of $[n]$ that do not contain any broken circuit. Note that all of the singletons will be faces of $bc_w(L)$ if and only if $M(L)$ has no parallel pairs, and the empty set will be a face if and only if $M(L)$ has no loops. We will not need to assume that either of these conditions holds.

Consider the $f$-vector $(f_0, \ldots, f_d)$ of $bc_w(L)$, where $f_i$ is the number of faces of order $i$. Then $f_i$ is equal to the rank of $H^i(A(L))$, where $A(L) = L \setminus \bigcup_{i=1}^{n} \{x_i = 0\}$ is the
complement of the restriction of the coordinate arrangement from \( \mathbb{C}^n \) to \( L \) (see for example [OT]). In particular, the \( f \)-vector of \( \text{bc}_w(L) \) is independent of the ordering \( w \). The \( h \)-vector \((h_0, \ldots, h_{d-1})\) of \( \text{bc}_w(L) \) is defined by the formula \( \sum h_iz^i = \sum f_iz^i(1-z)^{d-i} \).

The *Stanley-Reisner ring* \( \text{SR}(\Delta) \) of a simplicial complex \( \Delta \) on the ground set \([n]\) is defined to be the quotient of \( \mathbb{C}[e_1, \ldots, e_n] \) by the ideal generated by the monomials \( \prod_{i \in N} e_i \), where \( N \) ranges over the nonfaces of \( \Delta \). The complex \( \text{bc}_w(L) \) is shellable of dimension \( d-1 \) [Bj], which implies that \( \text{Spec } \text{SR}(\text{bc}_w(L)) \) is Cohen-Macaulay and pure of dimension \( d \). Let \( \mathbb{C}[L^\vee] \) denote the ring of functions on \( L^\vee = (\mathbb{C}^n)^\vee/L^\perp \), which we may think of as the symmetric algebra on \( L \). The inclusion of \( L \) into \( \mathbb{C}^n \) induces an inclusion of \( \mathbb{C}[L^\vee] \) into \( \mathbb{C}[e_1, \ldots, e_n] \), which makes \( \text{SR}(\text{bc}_w(L)) \) into an \( \mathbb{C}[L^\vee] \)-algebra. Let \( \text{SR}_0(\text{bc}_w(L)) = \text{SR}(\text{bc}_w(L)) \otimes_{\mathbb{C}[L^\vee]} \mathbb{C} \), where each linear function on \( L^\vee \) acts on \( \mathbb{C} \) by 0. The following proposition asserts that \( L \) constitutes a linear system of parameters (l.s.o.p.) for \( \text{SR}(\text{bc}_w(L)) \).

**Proposition 1.** The Stanley-Reisner ring \( \text{SR}(\text{bc}_w(L)) \) is a free \( \mathbb{C}[L^\vee] \)-module, and the ring \( \text{SR}_0(\text{bc}_w(L)) \) is zero-dimensional with Hilbert series \( \sum h_iz^i \).

**Proof.** By [St, 5.9], it is enough to prove that \( \text{SR}_0(\text{bc}_w(L)) \) is a zero-dimensional ring. Let \( \pi \) denote the composition \( \text{Spec } \text{SR}(\text{bc}_w(L)) \hookrightarrow (\mathbb{C}^n)^\vee \twoheadrightarrow L^\vee \). The variety \( \text{Spec } \text{SR}(\text{bc}_w(L)) \) is a union of coordinate subspaces, one for each face of \( \text{bc}_w(L) \). Let \( F \) be such a face, with vertices \((v_1, \ldots, v_{|F|})\). The broken circuit complex is a subcomplex of the matroid complex, hence \((v_1, \ldots, v_{|F|})\) is an independent set, which implies that \( \pi \) maps the corresponding coordinate subspace injectively to \( L^\vee \). Thus \( \pi^{-1}(0) = \text{Spec } \text{SR}_0(\text{bc}_w(L)) \) is supported at the origin, and we are done. \( \square \)

### 3 A degeneration of \( R(L) \)

In this section we show that \( R(L) \) degenerates flatly to the Stanley-Reisner ring \( \text{SR}(\text{bc}_w(L)) \) for any choice of \( w \).

**Lemma 2.** The spaces \( \text{Spec } R(L) \) and \( \text{Spec } \text{SR}(\text{bc}_w(L)) \) are both pure \( d \)-dimensional homogeneous varieties of degree \( t_{M(L)}(1,0) \), where \( t_M(w,z) \) is the Tutte polynomial of \( M \).

**Proof.** The broken circuit complex is pure of dimension \( d-1 \), hence \( \text{Spec } \text{SR}(\text{bc}_w(L)) \) is union of \( d \)-dimensional coordinate subspaces of \((\mathbb{C}^n)^\vee \). Its degree is the number of facets of \( \text{bc}_w(L) \), which is equal to \( \sum h_i = t_{M(L)}(1,0) \) [Bj].
The variety $\text{Spec } R(L)$ is equal to the closure inside of $(\mathbb{C}^n)^\vee \cong \mathbb{C}^n$ of $L \cap (\mathbb{C}^*)^n$, and is therefore $d$ dimensional. We will now show that $\deg \text{Spec } R(L)$ obeys the same recurrence as $t_{M(L)}(1,0)$. First, suppose that $i \in [n]$ is a loop of $M(L)$. Then $L$ lies in a coordinate subspace of $\mathbb{C}^n$, $L \cap (\mathbb{C}^*)^n$ is empty, and $\text{Spec } R(L)$ is thus empty and has degree 0. In this case, we also have $t_{M(L)}(1,0) = 0$. Next, suppose that $i$ is a coloop of $M(L)$. Then $L$ is invariant under translation by $e_i$, and $\text{Spec } R(L)$ is similarly invariant under translation by $x_i$. Write $L/i$ for the quotient of $L$ by this translation, so that $\text{Spec } R(L) = \text{Spec } R(L/i) \times \mathbb{C}$ and $\deg \text{Spec } R(L) = \deg \text{Spec } R(L/i)$. It is clear that $M(L/i) = M(L)/i$, and indeed $t_{M}(1,0) = t_{M/i}(1,0)$ when $i$ is a coloop.

Now consider the case where $i$ is neither a loop nor a coloop, hence we have

$$t_{M(L)}(1,0) = t_{M(L)/i}(1,0) + t_{M(L) \setminus i}(1,0).$$

In this case, we may apply the following theorem.

**Theorem 3.** [KMY, 2.2] Let $X$ be a homogeneous irreducible subvariety of $\mathbb{C}^n = H \oplus \ell$, with $H$ a hyperplane and $\ell$ a line such that $X$ is not invariant under translation in the $\ell$ direction. Let $X_1$ be the closure of the projection along $\ell$ of $X$ to $H$, and let $X_2$ be the flat limit in $H \times \mathbb{P}^1$ of $X \cap (H \times \{t\})$ as $t \to \infty$. Then $X$ has a flat degeneration to a scheme supported on $(X_1 \times \{0\}) \cup (X_2 \times \ell)$. In particular, $\deg X \geq \deg X_1 + \deg X_2$, with equality if the projection $X \to X_1$ is generically one to one.

Let $X = \text{Spec } R(L)$, $\ell = \mathbb{C} x_i$, and $H = \mathbb{C}\{x_j | j \neq i\}$. Then in the notation of Theorem 3, we have $X_1 = \text{Spec } R(L \setminus i)$, where $L \setminus i$ is the projection of $L$ onto $H$, and $X_2 = \text{Spec } R(L/i)$. The projection of $\text{Spec } R(L)$ onto $H$ is one to one because the corresponding projection of $L$ in the $x_i$ direction is one to one. Thus the degree of $\text{Spec } R(L)$ is additive.

We are now ready to prove our main theorem, which asserts that $R(L)$ degenerates flatly to $\text{SR}(\text{bc}_w(L))$ for any choice of $w$.

**Theorem 4.** The set $\{f_C \mid C$ a circuit of $M(L)\}$ is a universal Gröbner basis for $R(L)$. Given any ordering $w$ of $[n]$, with the induced term order on $\mathbb{C}[y]$, we have $\text{In}_w R(L) = \text{SR}(\text{bc}_w(L))$.

**Proof.** Suppose given an ordering $w$ of $[n]$ and a circuit $C$ of $M(L)$. Let $c_0$ denote the $w$ minimal element of $C$, so that $\prod_{c' \in C \setminus \{c_0\}} y_{c'}$ is the leading term of $f_C$ with respect to $w$. Every monomial of this form vanishes in $\text{In}_w R(L)$, hence we deduce that $\text{Spec } \text{In}_w(R(L))$ is
a subscheme of \( \text{Spec} \, SR(bc_w(L)) \). However, Lemma 2 tells us that these two schemes have the same dimension and degree, and \( \text{Spec} \, SR(bc_w(L)) \) is reduced. Thus they are equal.

Let \( R \) be the quotient ring of \( \mathbb{C}[y] \) generated by the polynomials \( \{f_C\} \). It is clear that \( \text{In}_w \text{Spec} \, R(L) \subseteq \text{In}_w \text{Spec} \, R \subseteq \text{Spec} \, SR(bc_w(L)) \). Since the two ends of this chain are equal, we have \( \text{In}_w R = \text{In}_w R(L) \), and thus \( R \) and \( R(L) \) have the same Hilbert series. As \( R(L) \) is a quotient ring of \( R, R = R(L) \).

\[ \square \]

4 A stratification of \( \text{Spec} \, R(L) \)

Let \( I \) be a subset of \( [n] \). The rank of \( I \) is defined to be the cardinality of the largest independent subset of \( I \). If any strict superset of \( I \) has strictly greater rank, then \( I \) is called a flat of \( M(L) \). If \( I \) is a flat, let \( L_I \subset \mathbb{C}^I \) be the projection of \( L \) onto the coordinate subspace \( \mathbb{C}^I \subset \mathbb{C}^n \), and let \( L^I \subset \mathbb{C}^{I^c} \) be the intersection of \( L \) with the complimentary coordinate subspace \( \mathbb{C}^{I^c} \). The matroid \( M(L_I) \) is called the localization of \( M(L) \) at \( I \), while \( M(L^I) \) is called the deletion of \( I \) from \( M(L) \).

For any \( I \subset [n] \), let \( U_I = \{ y \in (\mathbb{C}^n)^\vee \mid y_i = 0 \iff i \notin I \} \), and let \( A_I = \text{Spec} \, R(L) \cap U_I \).

**Proposition 5.** The variety \( A_I \) is nonempty if and only if \( I \) is a flat of \( M(L) \). If nonempty, \( A_I \) is isomorphic to \( A(L_I) = L_I \setminus \bigcup_{i \in I} \{ y_i = 0 \} \).

**Proof.** First suppose that \( I \) is not a flat of \( M(L) \). Then there exists some circuit \( C \) of \( M(L) \) and element \( c_0 \in C \) such that \( C \cap I = C \setminus \{c_0\} \). On one hand, the polynomial \( f_C = \sum_{c \in C} a_c \prod_{c' \in C \setminus \{c\}} y_{c'} \) vanishes on \( A_I \). On the other hand, \( f_C \) has a unique nonzero term \( \prod_{c \in C \setminus \{c_0\}} y_{c'} \) on \( U_I \), and therefore cannot vanish on this set. Hence \( A_I \) must be empty.

Now suppose that \( I \) is a flat. If \( I = [n] \), then we are simply repeating the observation that \( \text{Spec} \, R(L) \cap (\mathbb{C}^*)^n \cong L \cap (\mathbb{C}^*)^n = A(L) \). In the general case, Theorem 4 tells us that \( \text{Spec} \, R(L) \) is cut out of \( (\mathbb{C}^n)^\vee \) by the polynomials \( f_C \), so we need to understand the restrictions of these polynomials to the set \( U_I \). If \( C \) is not contained in \( I \), then \( C \setminus I \) has size at least 2, and therefore \( f_C \) vanishes on \( U_I \). Thus we may restrict our attention to those circuits that are contained in \( I \). Proposition 5 then follows from the fact that the circuits of \( M(L_I) \) are precisely the circuits of \( M(L) \) that are supported on \( I \).

\[ \square \]

**Remark 6.** The stratification of \( \text{Spec} \, R(L) \) given by Proposition 5 is analogous to the standard stratification of \( L \) into pieces isomorphic to \( A(L^I) \), again ranging over all flats of \( M(L) \).

The identification of \( e_i \) with \( y_i \) makes \( R(L) \) into an algebra over \( \mathbb{C}[L^\vee] \). We conclude by showing that, as in Proposition 1, \( L \) provides a natural linear system of parameters for \( R(L) \).
Proposition 7. The ring $R(L)$ is a free module over $\mathbb{C}[L^\vee]$. The zero dimensional quotient $R_0(L) := R(L) \otimes_{\mathbb{C}[L^\vee]} \mathbb{C}$ has Hilbert series $\sum h_i z^i$.

Proof. The fact that $R(L)$ is Cohen-Macaulay follows from Theorem 4, which asserts that it is a deformation of the Cohen-Macaulay ring $SR(bc_w(L))$. Furthermore, Theorem 4 tells us that any quotient of $R(L)$ by $d$ generic parameters has the same Hilbert series of $SR_0(bc_w(L))$. Therefore, as in Proposition 1, we let $\pi$ denote the composition $\text{Spec } R(L) \rightarrow (\mathbb{C}^n)^\vee \rightarrow L^\vee$, and observe that it is enough to show that $\pi^{-1}(0)$ is supported at the origin.

Let $I \subset [n]$ and suppose that $y = (y_1, \ldots, y_n) \in A_I = \text{Spec } R(L) \cap U_I$. By Proposition 5, $A_I$ is obtained from $A(L_I)$ by applying the inversion involution of $(\mathbb{C}^*)^I$, hence there exists $x_I \in A(L_I) \subset L_I$ such that $x_i = y_i^{-1}$ for all $i \in I$. Extend $x_I$ to an element $x \in L$. Then $\langle x, y \rangle = \sum x_i y_i = |I|$, hence if $y$ projects trivially onto $L^\vee$, we must have $I = \emptyset$. \qed

Remark 8. It is natural to ask the question of whether $R_0(L)$ has a $g$-element; that is an element $g \in R(L)$ in degree 1 such that the multiplication map $g^{r-2i} : R_0(L)_i \rightarrow R_0(L)_{r-i}$ is injective for all $i < r/2$, where $r$ is the top nonzero degree of $R_0(L)$. This property is known to fail for the ring $SR_0(bc_w(L))$ [Sw, §5], but the inequalities that it would imply for the $h$-numbers are not known to be either true or false. In fact, the ring $R_0(L)$ fares no better than its degeneration; Swartz’s counterexample to the $g$-theorem for $SR_0(bc_w(L))$ is also a counterexample for $R_0(L)$.

Remark 9. All of the constructions and results in this paper generalize to arbitrary fields with the exception of Proposition 7, which uses in an essential manner the fact that $\mathbb{C}$ has characteristic zero.

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