AVERAGING PRINCIPLE FOR TWO DIMENSIONAL STOCHASTIC NAVIER-STOKES EQUATIONS

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Abstract. The averaging principle is established for the slow component and the fast component being two dimensional stochastic Navier-Stokes equations and stochastic reaction-diffusion equations, respectively. The classical Khasminskii approach based on time discretization is used for the proof of the slow component strong convergence to the solution of the corresponding averaged equation under some suitable conditions. Meanwhile, some powerful techniques are used to overcome the difficulties caused by the nonlinear term and to release the regularity of the initial value.

1. Introduction

In this paper, we shall establish the averaging principle of the following stochastic fast-slow system on the 2D torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$:

$$
\begin{cases}
    dX_t^\varepsilon = \left[\nu \Delta X_t^\varepsilon - (X_t^\varepsilon \cdot \nabla)X_t^\varepsilon + f(X_t^\varepsilon, Y_t^\varepsilon) - \nabla p\right]dt + \sigma_1(X_t^\varepsilon)dW_t^{Q_1}, \\
    dY_t^\varepsilon = \frac{1}{\varepsilon} \left[\Delta Y_t^\varepsilon + g(X_t^\varepsilon, Y_t^\varepsilon)\right]dt + \frac{1}{\sqrt{\varepsilon}}\sigma_2(X_t^\varepsilon, Y_t^\varepsilon)dW_t^{Q_2}, \\
    \nabla \cdot X_t^\varepsilon = 0, \quad \nabla \cdot Y_t^\varepsilon = 0, \\
    X_0^\varepsilon = x, \quad Y_0^\varepsilon = y,
\end{cases}
$$

where $\varepsilon > 0$ is a small parameter describing the ratio of time scale between the slow component $X_t^\varepsilon$ and the fast component $Y_t^\varepsilon$, $\Delta$ is the Laplace operator, $p$ denotes the pressure, $\nu > 0$ is the kinematic viscosity, $f, g, \sigma_1$ and $\sigma_2$ satisfy some suitable conditions. $\{W_t^{Q_1}\}_{t \geq 0}$ and $\{W_t^{Q_2}\}_{t \geq 0}$ are $L^2(\mathbb{T}^2, \mathbb{R}^2)$-valued mutually independent $Q_1$ and $Q_2$-Wiener processes on complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$.

The averaging principle for multiscale system has a long and rich history, and has wide applications in material sciences, chemistry, fluids dynamics, biology, ecology, climate dynamics, see, e.g., [1, 12, 20, 25, 28, 32] and references therein. Bogoliubov and Mitropolsky [2] first studied the averaging principle for the deterministic systems. Then Khasminskii [21] studied averaging principle for stochastic differential equations (SDEs), see, e.g., [18, 22, 24, 33, 34] for further generalization. Recently, averaging principles for stochastic partial differential equations (SPDEs) have attracted much attention. For example, Cerrai and Freidlin [6] proved the averaging principle for a general class of stochastic reaction-diffusion systems with two time-scales, which has been extended to the more general model in [4, 5, 7]. Bréhier [3] gave the strong and weak orders in averaging for stochastic evolution equation of parabolic type with slow and fast time scales. In [15], Fu, Wan and Liu proved the strong averaging principle for stochastic hyperbolic-parabolic equations with slow and
fast time-scales. For more interesting results on this topic, we refer to [14, 16, 30, 31] and references therein.

However, there are few results on the average principle for SPDEs with highly nonlinear term. Recently, the second author and his cooperators [9] have established the strong and weak averaging principle for one dimensional stochastic Burgers equation with additive noise. Averaging principle for stochastic Kuramoto-Sivashinsky equation with a fast oscillation was studied by Gao in [17]. In this paper, we focus on studying the strong averaging principle for 2D stochastic Navier-Stokes equations with multiplicative noise. To be more precise, we will prove that

$$\lim_{\varepsilon \to 0} \mathbb{E} \left( \sup_{t \in [0,T]} |X_t^\varepsilon - \bar{X}_t|^2 \right) = 0, \quad p \geq 1,$$

(1.2)

where $\bar{X}_t$ is the solution of the corresponding averaged equation (see equation (2.4) below).

The 2D stochastic Navier-Stokes equations have been studied by many authors, for instance, we refer to [8, 10, 19, 23, 26, 29] and the references therein.

The proof of our main result is based on the Khasminskii discretization introduced in [21], which is a powerful skill to study the averaging principle for different types of systems with two time-scales. More precisely, we split the interval $[0,T]$ into some subintervals of size $\delta > 0$ which depends on $\varepsilon$, and on each interval $[k\delta, (k+1)\delta]$, $k \geq 0$, we construct an auxiliary process $(\hat{X}_t^\varepsilon, \hat{Y}_t^\varepsilon)$ which associate with the system (1.1). Then (1.2) can be proved by the following two steps. Step 1, due to the highly nonlinear term in stochastic Navier-Stokes equation, we will use stopping time techniques and control the difference of $X_t^\varepsilon$ and $\bar{X}_t$ before the stopping time, which will be done by controlling $|X_t^\varepsilon - \hat{X}_t^\varepsilon|$ and $|\hat{X}_t^\varepsilon - \bar{X}_t|$ respectively. Step 2, after the stopping time term can be estimated by the priori estimates of the solution.

Comparing with some recent works on strong convergence in averaging principle for SPDEs (cf. [3, 14, 15, 16]), the main challenge in the research of the strong convergence (1.2) is the nonlinear term of the Navier-Stokes equation. Moreover, due to the dimension of space is two and multiplicative noise, the skills used in [9] don’t work in the situation for our case. In order to overcome the difficulties, we shall deal with the nonlinear term and the multiplicative noise more delicately.

Because of the approach based on time discretization, the Hölder continuity of time for $X_t^\varepsilon$ would play an important role in the proof of the average principle usually. To this purpose, the condition of the initial value $x \in H^\theta$ (the Sobolev space, see Section 2) for some $\theta > 0$ will be assumed usually, for example, see [4, Proposition 4.4], [9, Lemma 3.4] and [17, Proposition 9]. However in this paper, we would like to stress the initial value $x \in H$, then replace studying the Hölder continuity of time by proving a weak result relatively (see Lemma 3.2 below), and it would be enough to prove our main result. Hence, the techniques used here are very helpful to weaken the regularity of initial value $x$. We also believe that these techniques can be applied to more general framework of SPDEs, which will be stated in our forthcoming paper.

The rest of the paper is organized as follows. In Section 2, under some suitable assumptions, we formulate our main result. Section 3 is devoted to proving our main result. In the Appendix 4, we give some properties of the nonlinear term and the proof of the well-posedness of our system.

Throughout the paper, $C$, $C_p$ and $C_{R,T}$ will denote positive constants which may change from line to line, where $C_p$ depends on $p$, $C_{R,T}$ depends on $R, T$. 
2. Notations and main results

For \( p \geq 1 \), let \( L^p(\mathbb{T}^2, \mathbb{R}^2) \) be the space of \( p \)-th power integrable \( \mathbb{R}^2 \)-valued functions on torus \( \mathbb{T}^2 \) and \( | \cdot |_{L^p} \) be the usual norm. For \( k \in \mathbb{N} \), \( W^{k,2}(\mathbb{T}^2) \) is the Sobolev space of all functions in \( L^2(\mathbb{T}^2, \mathbb{R}^2) \) whose differentials belong to \( L^2(\mathbb{T}^2, \mathbb{R}^2) \) up to the order \( k \). Let \( L^2_0(\mathbb{T}^2, \mathbb{R}^2) \) be the space of square-integrable \( \mathbb{R}^2 \)-valued functions on the torus with vanishing mean, i.e.,

\[
L^2_0(\mathbb{T}^2, \mathbb{R}^2) := \left\{ u \in L^2(\mathbb{T}^2, \mathbb{R}^2) : \int_{\mathbb{T}^2} u(\xi) d\xi = 0 \right\}.
\]

We consider a Hilbert space \( H \) which is a closed subspace of \( L^2_0(\mathbb{T}^2, \mathbb{R}^2) \), defined by

\[
H := \left\{ u \in L^2_0(\mathbb{T}^2, \mathbb{R}^2) : \nabla \cdot u = 0 \right\}.
\]

The space \( H \) is endowed with the inner product and the norm on \( L^2(\mathbb{T}^2, \mathbb{R}^2) \), which denoted by \( \langle \cdot, \cdot \rangle \) and \( | \cdot | \) respectively.

We shall fix an orthonormal basis \( \{ e_k \}_{k \geq 1} \) of \( H \) consisting of the eigenvectors of \( \Delta \), i.e.,

\[
\Delta e_k = -\lambda_k e_k,
\]

where \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \uparrow \infty \). Moreover, we put \( \mathcal{D}(A) := W^{2,2}(\mathbb{T}^2) \cap H \), and define the linear operator

\[
Au := P_H \Delta u, \quad u \in \mathcal{D}(A),
\]

where \( P_H \) is the Helmholtz-Leray projector from \( L^2_0(\mathbb{T}^2, \mathbb{R}^2) \) onto \( H \). Furthermore, in our case it is known that \( A = \Delta \) due to the periodic boundary condition (see, e.g., [13]). For simplicity, we also assume the viscosity constant \( \nu = 1 \) in this paper.

For any \( s \in \mathbb{R} \), we define

\[
H^s := \mathcal{D}((-A)^{s/2}) := \left\{ u = \sum_k u_k e_k : u_k = \langle u, e_k \rangle \in \mathbb{R}, \sum_k \lambda_k^s u_k^2 < \infty \right\},
\]

and

\[
(-A)^{s/2} u := \sum_k \lambda_k^{s/2} u_k e_k, \quad u \in \mathcal{D}((-A)^{s/2}),
\]

with the associated norm

\[
|u|_s := |(-A)^{s/2} u| = \sqrt{\sum_k \lambda_k^s u_k^2}.
\]

It is easy to see \( H^0 = H \) and \( H^{-s} \) be the dual space of \( H^s \). Notice that the dual action is also denoted by \( \langle \cdot, \cdot \rangle \) without confusion.

It is well known that \( A \) is the infinitesimal generator of a strongly continuous semigroup \( \{ e^{tA} \}_{t \geq 0} \). For \( \theta \geq 0 \) and \( x \in H \), there exits \( C \) depends on \( \theta \) such that

\[
\|e^{tA}x\|_\theta \leq Ct^{-\frac{\theta}{2}}|x|.
\]  

(2.1)

Define the bilinear operator

\[
B(u, v) : H^1 \times H^1 \rightarrow H^{-1}, B(u, v) = P_H \left( (u \cdot \nabla) v \right)
\]

and the trilinear operator

\[
b(u, v, w) = \langle B(u, v), w \rangle = \sum_{i,j=1}^2 \int_{\mathbb{T}^2} u_i(\xi) \frac{\partial v_j(\xi)}{\partial \xi_i} w_j(\xi) d\xi, \quad \text{for } u, v, w \in H^1.
\]
Moreover, it is convenient to put $B(u) = B(u, u)$, for $u \in H^1$. The related properties of operators $b$ and $B$ are listed in the appendix.

Now, by applying the operator $P_H$ to the first equation of the system (1.1), we remove the pressure term and consider the following abstract stochastic evolution equations:

$$
\begin{align*}
    dX_t^\varepsilon &= [AX_t^\varepsilon - B(X_t^\varepsilon) + f(X_t^\varepsilon, Y_t^\varepsilon)] dt + \sigma_1(X_t^\varepsilon) dW_t^{Q_1}, \\
    dY_t^\varepsilon &= \frac{1}{\varepsilon} [AY_t^\varepsilon + g(X_t^\varepsilon, Y_t^\varepsilon)] dt + \frac{1}{\sqrt{\varepsilon}} \sigma_2(X_t^\varepsilon, Y_t^\varepsilon) dW_t^{Q_2}, \\
    X_0^\varepsilon &= x, \quad Y_0^\varepsilon = y.
\end{align*}
$$

(2.2)

Here $W_t^{Q_i}$ ($i = 1, 2$) are $H$-valued $Q_i$-Wiener process and $Q_i$ is a positive symmetric, trace class operator on $H$.

Put $U_i = Q_i^{1/2} H$ the Hilbert space with the inner product

$$(u, v)_{U_i} = (Q_i^{-1/2} u, Q_i^{-1/2} v), \quad u, v \in U_i$$

with the norm $|u|_{U_i} = \sqrt{(u, u)_{U_i}}$. Let $L_{Q_i}(U_i, H)$ be the space of linear operators $S : U_i \to H$ such that $SQ_i^{1/2}$ is a Hilbert-Schmidt operator on the Hilbert space with the inner product $(\cdot, \cdot)_{L_{Q_i}}$. The norm on $L_{Q_i}(U_i, H)$ is defined by

$$|S|_{L_{Q_i}}^2 = \text{Tr}(SQ_iS^*) := \sum_{k \geq 1} |SQ_i^{1/2} e_k|^2,$$

where $S^*$ is the adjoint operator of $S$. In the following, we always assume that $W_t^{Q_1}$ and $W_t^{Q_2}$ are independent.

We assume that $f, g : H \times H \to H$, $\sigma_1 : H \to L_{Q_1}(U_1; H)$ and $\sigma_2 : H \times H \to L_{Q_2}(U_2; H)$ satisfy the following conditions:

**A1.** $f, g, \sigma_1$ and $\sigma_2$ are Lipschitz continuous, i.e., there exist some positive constants $L_g, L_{\sigma_2}$ and $C$ such that for any $x_1, x_2, y_1, y_2 \in H$, $\sigma_1$ and $\sigma_2$ are Lipschitz continuous, i.e., there exist some positive constants $L_g, L_{\sigma_2}$ and $C$ such that for any $x_1, x_2, y_1, y_2 \in H$,

$$|f(x_1, y_1) - f(x_2, y_2)| \leq C (|x_1 - x_2| + |y_1 - y_2|);$$

$$|g(x_1, y_1) - g(x_2, y_2)| \leq C |x_1 - x_2| + L_g |y_1 - y_2|;$$

$$|\sigma_1(x_1) - \sigma_1(x_2)|_{L_{Q_1}} \leq C |x_1 - x_2|;$$

$$|\sigma_2(x_1, y_1) - \sigma_2(x_2, y_2)|_{L_{Q_2}} \leq C |x_1 - x_2| + L_{\sigma_2} |y_1 - y_2|.$$

**A2.** There exists a constant $\zeta \in (0, 1)$, such that

$$|\sigma_2(x, y)|_{L_{Q_2}} \leq C (1 + |x| + |y|^\zeta) \quad \text{for any } x, y \in H.$$

**A3.** The smallest eigenvalue $\lambda_1$ of $-\Delta$ and the Lipschitz constants $L_g, L_{\sigma_2}$ satisfy

$$2\lambda_1 - 2L_g - L_{\sigma_2}^2 > 0.$$

**Remark 2.1.** The condition **A1** ensures the existence and uniqueness of the solution of system (2.2). The condition **A2** is used to prove all the moments of the solution $(X_t^\varepsilon, Y_t^\varepsilon)$ are finite, which could be removed if we assume the Lipschitz constant $L_{\sigma_2}$ is sufficiently small. The condition **A3** is called the dissipative condition, which can guarantee that there exits a unique invariant measure for frozen equation and the exponential ergodicity holds.

Now, we recall the following definition.
Definition 2.2. For any initial value \( x, y \in H \). The system (2.2) has a weak solution if there exist \( X^\varepsilon \in C([0, T]; H) \cap L^2(0, T; H^1) \) and \( Y^\varepsilon \in C([0, T]; H) \cap L^2(0, T; H^1) \), \( \mathbb{P} \)-a.s., such that, for any \( t \in [0, T] \) and \( \phi, \varphi \in \mathcal{D}(A) \), the following identity hold
\[
\langle X^\varepsilon_t, \phi \rangle = \langle x, \phi \rangle + \int_0^t \langle X^\varepsilon_s, A\phi \rangle ds - \int_0^t \langle B(X^\varepsilon_s), \phi \rangle ds + \int_0^t \langle f(X^\varepsilon_s, Y^\varepsilon_s), \phi \rangle ds + \int_0^t \langle \sigma_1(X^\varepsilon_s)dW^Q_1, \phi \rangle, \quad \mathbb{P}\text{-a.s.}
\]
and
\[
\langle Y^\varepsilon_t, \varphi \rangle = \langle y, \varphi \rangle + \frac{1}{\varepsilon} \int_0^t \langle Y^\varepsilon_s, A\varphi \rangle ds + \frac{1}{\varepsilon} \int_0^t \langle g(X^\varepsilon_s, Y^\varepsilon_s), \varphi \rangle ds + \frac{1}{\sqrt{\varepsilon}} \int_0^t \langle \sigma_2(X^\varepsilon_s, Y^\varepsilon_s)dw^{Q_2}_s, \varphi \rangle, \quad \mathbb{P}\text{-a.s.}
\]

Based on the local-monotonicity method, we have the following well-posedness result.

Theorem 2.3. Assume the condition \( A1 \) holds. Then for initial value \( x, y \in H \), the system (2.2) has a unique weak solution, denoted by \((X^\varepsilon, Y^\varepsilon)\).

Note that this solution is a strong one in the probabilistically meaning. Using the local-monotonicity method one can prove the existence of weak solution for 2D Navier-Stokes equations. For completeness, we will prove Theorem 2.3 in the appendix.

Now, we state our main result.

Theorem 2.4. Assume that the conditions \( A1-A3 \) hold. Then for \( x, y \in H \), \( p \geq 1 \) and \( T > 0 \), we have
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left( \sup_{t \in [0, T]} |X^\varepsilon_t - \bar{X}_t|^2 \right) = 0, \tag{2.3}
\]
where \( \bar{X}_t \) is the solution of the corresponding averaged equation:
\[
\begin{cases}
  d\bar{X}_t = A\bar{X}_t dt - B(\bar{X}_t) dt + \bar{f}(\bar{X}_t) dt + \sigma_1(\bar{X}_t) dW^{Q_1}_t, \\
  \bar{X}_0 = x,
\end{cases}
\tag{2.4}
\]
with the average \( \bar{f}(x) = \int_H f(x, y) \mu^x(dy) \). \( \mu^x \) is the unique invariant measure of the frozen equation
\[
\begin{cases}
  dY_t = [A Y_t + g(x, Y_t)] dt + \sigma_2(x, Y_t) d\bar{W}^{Q_2}_t, \\
  Y_0 = y,
\end{cases}
\]
\( \bar{W}^{Q_2}_t \) is a \( Q_2 \)-Wiener process, which is independent of \( W^{Q_1}_t \) and \( W^{Q_2}_t \).

3. Proof of Theorem 2.4

In this section, we are devoted to proving Theorem 2.4. The proof consists of the following several steps. In the subsection 3.1, we first give some priori estimates of the solution \((X^\varepsilon_t, Y^\varepsilon_t)\) to the system (2.2), then prove a weaker result than the Hölder continuity of time for \( X^\varepsilon_t \). In the subsection 3.2, following the idea inspired by Khasminskii in [21], we introduce an auxiliary process \((\hat{X}^\varepsilon_t, \hat{Y}^\varepsilon_t)\) and also give its uniform bounds. Meanwhile, making use of the skills of stopping time, we also deduce an estimate of the (difference) process \( X^\varepsilon_t - \hat{X}^\varepsilon_t \) when time \( t \) is before the stopping time. In the subsection 3.3, based on the exponential ergodicity
of frozen equation, we give the control of the difference process $\hat{X}_t - \bar{X}_t$ when time $t$ is before the stopping time. Finally, we will use the priori estimates of the solution to control the term of time $t$ after the stopping time. Note that we always assume conditions A1-A3 hold in this section.

3.1. Some priori estimates of $(X_t^\varepsilon, Y_t^\varepsilon)$. At first, we prove uniform bounds with respect to $\varepsilon \in (0, 1)$ for $p$-moment of the solutions $(X_t^\varepsilon, Y_t^\varepsilon)$ to the system (2.2).

Lemma 3.1. For any $x, y \in H$, $T > 0$, $p \geq 1$ and $\varepsilon \in (0, 1)$, there exists a constant $C_{p,T} > 0$ such that for any

$$\mathbb{E} \left( \sup_{t \in [0,T]} |X_t^\varepsilon|^{2p} \right) + \mathbb{E} \left( \int_0^T |X_t^\varepsilon|^{2p-2} \|X_t^\varepsilon\|_1^{2p-2} dt \right) \leq C_{p,T} \left( 1 + |x|^{2p} + |y|^{2p} \right)$$

(3.1)

and

$$\sup_{t \in [0,T]} \mathbb{E}|Y_t^\varepsilon|^{2p} \leq C_{p,T} \left( 1 + |x|^{2p} + |y|^{2p} \right).$$

(3.2)

Proof. According to Itô’s formula, we have

$$|Y_t^\varepsilon|^2 = |y|^2 + \frac{2}{\varepsilon} \int_0^t \langle AY_s^\varepsilon, Y_s^\varepsilon \rangle ds + \frac{2}{\varepsilon} \int_0^t \langle g(X_s^\varepsilon, Y_s^\varepsilon), Y_s^\varepsilon \rangle ds$$

$$+ \frac{1}{\varepsilon} \int_0^t |\sigma_2(X_s^\varepsilon, Y_s^\varepsilon)|_2^2 ds + \frac{2}{\sqrt{\varepsilon}} \int_0^t |\sigma_2(X_s^\varepsilon, Y_s^\varepsilon)| dW_s^{Q_2}, Y_s^\varepsilon.$$  

Applying Itô’s formula for $g(z) = (z)^p$ and $z_t = |Y_t^\varepsilon|^2$, then taking expectation on both side, we obtain

$$\mathbb{E}|Y_t^\varepsilon|^{2p} = |y|^{2p} + \frac{2p}{\varepsilon} \mathbb{E} \left( \int_0^t |Y_s^\varepsilon|^{2p-2} \langle AY_s^\varepsilon, Y_s^\varepsilon \rangle ds \right)$$

$$+ \frac{2p}{\varepsilon} \mathbb{E} \left( \int_0^t |Y_s^\varepsilon|^{2p-2} \langle g(X_s^\varepsilon, Y_s^\varepsilon), Y_s^\varepsilon \rangle ds \right) + \frac{2}{\varepsilon} \mathbb{E} \left( \int_0^t |Y_s^\varepsilon|^{2p-2} |\sigma_2(X_s^\varepsilon, Y_s^\varepsilon)|_2^2 ds \right)$$

$$+ \frac{2p(p-1)}{\varepsilon} \mathbb{E} \left( \int_0^t |Y_s^\varepsilon|^{2p-4} |\sigma_2(X_s^\varepsilon, Y_s^\varepsilon)|^2 ds \right).$$

Notice that $\langle Ax, x \rangle = -\|x\|^2_1 \leq -\lambda_1 \|x\|^2$ and by conditions A1 and A2, there exists a constant $\gamma > 0$ such that

$$\frac{d}{dt} \mathbb{E}|Y_t^\varepsilon|^{2p} = -\frac{2p}{\varepsilon} \mathbb{E} \left( |Y_t^\varepsilon|^{2p-2} \|Y_t^\varepsilon\|^2_1 \right) + \frac{2p}{\varepsilon} \mathbb{E} \left[ |Y_t^\varepsilon|^{2p-2} \langle g(X_t^\varepsilon, Y_t^\varepsilon), Y_t^\varepsilon \rangle \right]$$

$$+ \frac{p}{\varepsilon} \mathbb{E} \left[ |Y_t^\varepsilon|^{2p-2} |\sigma_2(X_t^\varepsilon, Y_t^\varepsilon)|_2^2 \right] + \frac{2p(p-1)}{\varepsilon} \mathbb{E} \left[ |Y_t^\varepsilon|^{2p-4} |\sigma_2(X_t^\varepsilon, Y_t^\varepsilon)|^2 \right]$$

$$\leq -\frac{2p\lambda_1}{\varepsilon} \mathbb{E}|Y_t^\varepsilon|^{2p} + \frac{2p}{\varepsilon} \mathbb{E} \left[ |Y_t^\varepsilon|^{2p-2} \left( C \|Y_t^\varepsilon\| + L_\varphi |X_t^\varepsilon| \cdot |Y_t^\varepsilon| + L_\varphi |Y_t^\varepsilon|^2 \right) \right]$$

$$+ \frac{C_p}{\varepsilon} \mathbb{E} \left[ |Y_t^\varepsilon|^{2p-2} \left( C + |X_t^\varepsilon|^2 + |Y_t^\varepsilon|^2 \right) \right]$$

$$\leq -\frac{p\gamma}{\varepsilon} \mathbb{E}|Y_t^\varepsilon|^{2p} + \frac{C_p}{\varepsilon} \mathbb{E}|X_t^\varepsilon|^{2p} + \frac{C_p}{\varepsilon},$$
where the last inequality comes from the Young’s inequality. Hence, by comparison theorem, it is easy to see that
\[
\mathbb{E}|Y_t^\varepsilon|^{2p} \leq |y|^{2p} e^{-\frac{\varepsilon}{2} t} + \frac{C_p}{\varepsilon} \int_0^t e^{-\frac{\varepsilon}{2} (t-s)} \left( 1 + \mathbb{E}|X_s^\varepsilon|^{2p} \right) ds. \tag{3.3}
\]

On the other hand, using Itô’s formula again, we also have
\[
|X_t^\varepsilon|^{2p} = |x|^{2p} + 2p \int_0^t |X_s^\varepsilon|^{2p-2} \langle AX_s^\varepsilon, X_s^\varepsilon \rangle ds - 2p \int_0^t |X_s^\varepsilon|^{2p-2} \langle B(X_s^\varepsilon), X_s^\varepsilon \rangle ds + 2p \int_0^t |X_s^\varepsilon|^{2p-2} \langle \sigma_1(X_s^\varepsilon), dW_s^{Q_1} \rangle + p \int_0^t |X_s^\varepsilon|^{2p-2} \langle \sigma_1(X_s^\varepsilon)^2, X_s^\varepsilon \rangle ds.
\]
Then by Burkholder-Davis-Gundy’s inequality, (3.3) and Lemma 4.1, it holds that
\[
\mathbb{E} \left( \sup_{t \in [0,T]} |X_t^\varepsilon|^{2p} \right) + 2p \mathbb{E} \left( \int_0^T |X_t^\varepsilon|^{2p-2}||X_t^\varepsilon||_1^2 dt \right) \leq |x|^{2p} + C_p \int_0^T \mathbb{E}|X_t^\varepsilon|^{2p} dt + C_p \int_0^T \mathbb{E}|Y_t^\varepsilon|^{2p} dt \leq C_p(|x|^{2p} + |y|^{2p} + 1) + C_p \int_0^T \mathbb{E}|X_t^\varepsilon|^{2p} dt + \frac{C_p}{\varepsilon} \int_0^T \int_0^t e^{-\frac{\varepsilon}{2} (t-s)} \left( 1 + \mathbb{E}|X_s^\varepsilon|^{2p} \right) ds dt \leq C_p(|x|^{2p} + |y|^{2p} + 1) + C_p \int_0^T \mathbb{E}|X_t^\varepsilon|^{2p} dt.
\]
Hence, by applying Gronwall’s inequality implies
\[
\mathbb{E} \left( \sup_{t \in [0,T]} |X_t^\varepsilon|^{2p} \right) + 2p \mathbb{E} \left( \int_0^T |X_t^\varepsilon|^{2p-2}||X_t^\varepsilon||_1^2 dt \right) \leq C_{p,T}(|x|^{2p} + |y|^{2p} + 1),
\]
which also gives
\[
\mathbb{E}|Y_t^\varepsilon|^{2p} \leq C_{p,T} \left( 1 + |x|^{2p} + |y|^{2p} \right).
\]
The proof is complete. \(\square\)

We are going to use the approach of time discretization to prove our main result, so the Hölder continuity of time for \(X_t^\varepsilon\) always plays an important role. To this purpose, the condition of the initial value \(x \in H^\theta\) for some \(\theta > 0\) will be assumed, for example, see [4, Proposition 4.4], [9, Lemma 3.4] and [17, Proposition 9]. However, we will prove the following lemma instead of studying the Hölder continuity of time under the assumption of the initial value \(x \in H\), and it would be enough to prove our main result. Hence, the techniques used here are very helpful to weaken the regularity of initial value \(x\). The main idea of its proof is inspired from [27, Lemma 2.8].

**Lemma 3.2.** For any \(T > 0\), \(\varepsilon \in (0,1)\) and \(\delta > 0\) small enough, there exists a constant \(C_T > 0\) such that for any \(x, y \in H\)
\[
\mathbb{E} \left[ \int_0^T |X_t^\varepsilon - X_{t(\delta)}^\varepsilon|^2 dt \right] \leq C_T \delta^{1/2} (1 + |x|^3 + |y|^3), \tag{3.4}
\]
where \(t(\delta) := \lfloor \frac{s}{\delta} \rfloor \delta\) and \([s]\) denotes the largest integer which is no more than \(s\).
Proof. By (3.1), it is easy to get that
\[
\begin{align*}
\mathbb{E} \left[ \int_0^T |X_t^\varepsilon - X_{t(\delta)}^\varepsilon|^2 dt \right] \\
= \mathbb{E} \left( \int_\delta^t |X_t^\varepsilon - x|^2 dt \right) + \mathbb{E} \left[ \int_\delta^T |X_t^\varepsilon - X_{t(\delta)}^\varepsilon|^2 dt \right] \\
\leq C\delta + 2\mathbb{E} \left( \int_\delta^T |X_t^\varepsilon - X_{t-\delta}^\varepsilon|^2 dt \right) + 2\mathbb{E} \left( \int_\delta^T |X_{t(\delta)}^\varepsilon - X_{t-\delta}^\varepsilon|^2 dt \right). 
\end{align*}
\] (3.5)

Then, we estimate the second term on the right-hand side of (3.5) firstly. According to Itô’s formula, we have
\[
|X_t^\varepsilon - X_{t-\delta}^\varepsilon|^2 = 2\int_t^{t-\delta} \langle AX_s^\varepsilon - B(X_s^\varepsilon), X_s^\varepsilon - X_{t-\delta}^\varepsilon \rangle ds + 2\int_{t-\delta}^t \langle f(X_s^\varepsilon, X_s^\varepsilon), X_s^\varepsilon - X_{t-\delta}^\varepsilon \rangle ds
\]
\[
+ \int_t^{t-\delta} |\sigma_1(X_s^\varepsilon)|^2 \mathbb{E} ds + 2\int_{t-\delta}^t |X_s^\varepsilon - X_{t-\delta}, \sigma_1(X_s^\varepsilon)dW_s^{Q_1}
\]
\[
:= I_1(t) + I_2(t) + I_3(t) + I_4(t). 
\] (3.6)

For the first term \(I_1(t)\), by Hölder’s inequality and Corollary 4.2, there exists a constant \(C > 0\) such that
\[
\begin{align*}
\mathbb{E} \left( \int_\delta^T |I_1(t)| dt \right) \\
\leq C\mathbb{E} \left( \int_\delta^T \int_{t-\delta}^t \|AX_s^\varepsilon - B(X_s^\varepsilon)\|_1 \|X_s^\varepsilon - X_{t-\delta}^\varepsilon\|_1 ds dt \right) \\
\leq C \left[ \mathbb{E} \int_\delta^T \int_{t-\delta}^t \|AX_s^\varepsilon - B(X_s^\varepsilon)\|^2 ds dt \right]^{1/2} \left[ \mathbb{E} \int_\delta^T \int_{t-\delta}^t \|X_s^\varepsilon - X_{t-\delta}^\varepsilon\|^2 ds dt \right]^{1/2} \\
\leq C \left[ \delta\mathbb{E} \int_0^t \|X_s^\varepsilon\|^2 (1 + |X_s^\varepsilon|^2) ds \right]^{1/2} \cdot \left[ \delta\mathbb{E} \int_0^T \|X_s^\varepsilon\|^2 ds \right]^{1/2} \\
\leq C_T\delta(1 + |x|^3 + |y|^3), 
\end{align*}
\] (3.7)

where we use the Fubini theorem and (3.1) in the third and fourth inequalities respectively.

For \(I_2(t)\) and \(I_3(t)\), by condition \(A1\) and (3.1), we get
\[
\begin{align*}
\mathbb{E} \left( \int_\delta^T |I_2(t)| dt \right) \\
\leq C\mathbb{E} \left( \int_\delta^T \int_{t-\delta}^t (1 + |X_s^\varepsilon| + |Y_s^\varepsilon|)(|X_s^\varepsilon| + |X_{t-\delta}^\varepsilon|) ds dt \right) \\
\leq C\delta\mathbb{E} \left[ \sup_{s \in [0,T]} (1 + |X_s^\varepsilon|^2) \right] + C\mathbb{E} \left[ \sup_{s \in [0,T]} |X_s^\varepsilon| \int_\delta^T \int_{t-\delta}^t |Y_s^\varepsilon| ds dt \right] \\
\leq C\delta\mathbb{E} \left[ \sup_{s \in [0,T]} (1 + |X_s^\varepsilon|^2) \right] + C_T\delta^{1/2}\mathbb{E} \left[ \sup_{s \in [0,T]} |X_s^\varepsilon|^2 \right]^{1/2} \cdot \left[ \int_\delta^T \int_{t-\delta}^t |Y_s^\varepsilon|^2 ds dt \right]^{1/2} \\
\leq C_T\delta(1 + |x|^2 + |y|^2) 
\end{align*}
\] (3.8)
Also, we define the process \( \hat{X}^\varepsilon \), which is equivalent to

\[
\hat{X}^\varepsilon = x + \int_0^t A\hat{X}^\varepsilon_s ds - \int_0^t B(\hat{X}^\varepsilon_s) ds + \int_0^t f(\hat{X}^\varepsilon_s, \hat{Y}^\varepsilon_s) ds + \int_0^t \sigma_1(\hat{X}^\varepsilon_s) dW_s^{Q_1},
\]

for \( t \in [0, T] \). We remark that on each interval the fast component \( \hat{Y}^\varepsilon \) does not depend on the slow component \( \hat{X}^\varepsilon \), but only on the value of \( X^\varepsilon \) at the first point of the interval.
By the construction of \((\hat{X}_t^\varepsilon, \hat{Y}_t^\varepsilon)\), we can obtain the following estimates which will be used below. Because the proof almost follows the same steps in Lemma 3.1, we omit the proof here.

**Lemma 3.3.** For any \(x, y \in H\), \(T > 0\) and \(\varepsilon \in (0, 1)\), there exists a constant \(C_T > 0\) such that

\[
\sup_{t \in [0,T]} \mathbb{E}|\hat{Y}_t^\varepsilon|^2 \leq C_T (1 + |x|^2 + |y|^2) \tag{3.15}
\]

and

\[
\mathbb{E} \left( \sup_{t \in [0,T]} |\hat{X}_t^\varepsilon|^2 \right) + \mathbb{E} \left( \int_0^T \|\hat{X}_t^\varepsilon\|^2 dt \right) \leq C_T (|x|^2 + |y|^2 + 1). \tag{3.16}
\]

Now, we will establish the difference between \(Y_t^\varepsilon\) and \(\hat{Y}_t^\varepsilon\), and furthermore the difference \(X_t^\varepsilon\) and \(\hat{X}_t^\varepsilon\).

**Lemma 3.4.** For any \(x, y \in H\), \(T > 0\) and \(\varepsilon \in (0, 1)\), there exists a constant \(C_{R,T} > 0\) such that

\[
\mathbb{E} \left( \int_0^T |Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^2 dt \right) \leq C_T \left(1 + |x|^3 + |y|^3\right) \delta^{1/2}. \tag{3.17}
\]

**Proof.** Let \(\rho_t^\varepsilon := Y_t^\varepsilon - \hat{Y}_t^\varepsilon\). Then, it is easy to see that \(\rho_t^\varepsilon\) satisfies the following equation:

\[
\begin{cases}
    d\rho_t^\varepsilon = \frac{1}{\varepsilon} \left[ A\rho_t^\varepsilon + g(X_t^\varepsilon, Y_t^\varepsilon) - g(X_{t(\delta)}^\varepsilon, \hat{Y}_{t(\delta)}^\varepsilon) \right] dt + \frac{1}{\varepsilon} \left[ \sigma_2(X_t^\varepsilon, Y_t^\varepsilon) - \sigma_2(X_{t(\delta)}^\varepsilon, \hat{Y}_{t(\delta)}^\varepsilon) \right] dW_t Q^\varepsilon, \\
    \rho_0^\varepsilon = 0.
\end{cases}
\]

Thus, applying Itô’s formula and taking expectation, we have

\[
\mathbb{E} |\rho_t^\varepsilon|^2 = -\frac{2}{\varepsilon} \int_0^t \mathbb{E} \|\rho_s^\varepsilon\|^2 ds + \frac{2}{\varepsilon} \int_0^t \mathbb{E} \langle g(X_s^\varepsilon, Y_s^\varepsilon) - g(X_{s(\delta)}^\varepsilon, \hat{Y}_{s(\delta)}^\varepsilon), \rho_s^\varepsilon \rangle ds
\]

\[
+ \frac{1}{\varepsilon} \int_0^t \mathbb{E} \|\sigma_2(X_s^\varepsilon, Y_s^\varepsilon) - \sigma_2(X_{s(\delta)}^\varepsilon, \hat{Y}_{s(\delta)}^\varepsilon)\|^2 ds.
\]

Then by condition **A3**, there exists \(\beta > 0\) such that

\[
\frac{d}{dt} \mathbb{E} |\rho_t^\varepsilon|^2 \leq -\frac{2}{\varepsilon} \mathbb{E} \|\rho_t^\varepsilon\|^2 + \frac{2}{\varepsilon} \mathbb{E} \langle g(X_t^\varepsilon, Y_t^\varepsilon) - g(X_{t(\delta)}^\varepsilon, \hat{Y}_{t(\delta)}^\varepsilon), \rho_t^\varepsilon \rangle
\]

\[
+ \frac{1}{\varepsilon} \mathbb{E} \|\sigma_2(X_t^\varepsilon, Y_t^\varepsilon) - \sigma_2(X_{t(\delta)}^\varepsilon, \hat{Y}_{t(\delta)}^\varepsilon)\|^2
\]

\[
\leq -\frac{2\lambda_1}{\varepsilon} \mathbb{E} |\rho_t^\varepsilon|^2 + \frac{C}{\varepsilon} \mathbb{E} \left( |X_t^\varepsilon - X_{t(\delta)}^\varepsilon| \cdot |\rho_t^\varepsilon| \right) + \frac{2L_2}{\varepsilon} \mathbb{E} |\rho_t^\varepsilon|^2
\]

\[
+ \frac{1}{\varepsilon} \mathbb{E} \left( C|X_t^\varepsilon - X_{t(\delta)}^\varepsilon| + L_{\sigma_2} |\rho_t^\varepsilon| \right)^2
\]

\[
\leq -\frac{\beta}{\varepsilon} \mathbb{E} |\rho_t^\varepsilon|^2 + \frac{C}{\varepsilon} \mathbb{E} |X_t^\varepsilon - X_{t(\delta)}^\varepsilon|^2.
\]

Therefore, comparison theorem yields that

\[
\mathbb{E} |\rho_t^\varepsilon|^2 \leq \frac{C}{\varepsilon} \int_0^t e^{-\frac{\beta(t-s)}{\varepsilon}} \mathbb{E} |X_s^\varepsilon - X_{s(\delta)}^\varepsilon|^2 ds.
\]
Then by Fubini’s theorem, for any $T > 0$,
\[
\mathbb{E}\left( \int_0^T |\rho_t^\varepsilon|^2 dt \right) \leq \frac{C}{\varepsilon} \int_0^T \int_0^t e^{-\frac{\beta(t-s)}{\varepsilon}} \mathbb{E}|X_s^\varepsilon - X_{s(\delta)}^\varepsilon|^2 ds dt \\
= \frac{C}{\varepsilon} \mathbb{E}\left( \int_0^T |X_s^\varepsilon - X_{s(\delta)}^\varepsilon|^2 \left( \int_s^T e^{-\frac{\beta(t-s)}{\varepsilon}} dt \right) ds \right) \\
\leq C \mathbb{E}\left( \int_0^T |X_s^\varepsilon - X_{s(\delta)}^\varepsilon|^2 ds \right).
\]

By Lemma 3.2, we obtain
\[
\mathbb{E}\left( \int_0^T |\rho_t^\varepsilon|^2 dt \right) \leq C_T (|x|^3 + |y|^3 + 1)^{\delta^{1/2}}.
\]

The proof is complete. \qed

In order to estimate the difference process $|X_t^\varepsilon - \hat{X}_t^\varepsilon|$, we need to construct the following stopping time, i.e., for fixed $\varepsilon \in (0, 1)$, $R > 0$,
\[
\tau^\varepsilon_R := \inf \left\{ t \geq 0 : \int_0^t \|X_s^\varepsilon\|_1^2 ds \geq R \right\}.
\]

Then we will control $|X_t^\varepsilon - \hat{X}_t^\varepsilon|$ when $t$ is before this stopping time.

**Lemma 3.5.** For any $x, y \in H$, $T > 0$ and $\varepsilon \in (0, 1)$, the following fact holds.
\[
\mathbb{E}\left( \sup_{t \in [0, T]} |X_t^\varepsilon - \hat{X}_t^\varepsilon|^2 \right) \leq C_{R,T} \left( 1 + |x|^3 + |y|^3 \right)^{\delta^{1/2}}.
\]

**Proof.** Put $Z_t^\varepsilon := X_t^\varepsilon - \hat{X}_t^\varepsilon$, then we have $Z_0^\varepsilon = 0$ and
\[
dZ_t^\varepsilon = A Z_t^\varepsilon dt - \left[ B(X_t^\varepsilon) - B(\hat{X}_t^\varepsilon) \right] dt + \left[ f(X_t^\varepsilon, Y_t^\varepsilon) - f(X_{t(\delta)}^\varepsilon, \hat{Y}_t^\varepsilon) \right] dt \\
+ \left[ \sigma_1(X_t^\varepsilon) - \sigma_1(\hat{X}_t^\varepsilon) \right] dW_t^{Q_1}.
\]

By Itô’s formula and Corollary 4.2, we have
\[
|Z_t^\varepsilon|^2 = \int_0^t -2\|Z_s^\varepsilon\|_1^2 ds - 2 \int_0^t \langle B(X_s^\varepsilon) - B(\hat{X}_s^\varepsilon), Z_s^\varepsilon \rangle ds \\
+ 2 \int_0^t \langle f(X_s^\varepsilon, Y_s^\varepsilon) - f(X_{s(\delta)}^\varepsilon, \hat{Y}_s^\varepsilon), Z_s^\varepsilon \rangle ds \\
+ 2 \int_0^t \langle Z_s^\varepsilon, \sigma_1(X_s^\varepsilon) - \sigma_1(\hat{X}_s^\varepsilon) \rangle dW_s^{Q_1} + \int_0^t \|\sigma_1(X_s^\varepsilon) - \sigma_1(\hat{X}_s^\varepsilon)\|_{L_2}^2 ds \\
\leq - \int_0^t \|Z_s^\varepsilon\|_1^2 ds + C \int_0^t |Z_s^\varepsilon|^2 \|X_s^\varepsilon\|_1^2 ds + C \int_0^t |X_s^\varepsilon - X_{s(\delta)}^\varepsilon|^2 ds \\
+C \int_0^t |Y_s^\varepsilon - \hat{Y}_s^\varepsilon|^2 ds + C \sup_{t \in [0, T]} |\langle Z_s^\varepsilon, \sigma_1(X_s^\varepsilon) - \sigma_1(\hat{X}_s^\varepsilon) \rangle W_s^{Q_1}|,
\]

which yields
\[
\sup_{t \in [0, T]} |Z_t^\varepsilon|^2 \leq C \int_0^{T \wedge \tau^\varepsilon_R} (1 + \|X_s^\varepsilon\|_1^2) |Z_s^\varepsilon|^2 ds + C \int_0^{T \wedge \tau^\varepsilon_R} |X_s^\varepsilon - X_{s(\delta)}^\varepsilon|^2 ds \\
+ C \int_0^{T \wedge \tau^\varepsilon_R} |Y_s^\varepsilon - \hat{Y}_s^\varepsilon|^2 ds + 2 \sup_{t \in [0, T \wedge \tau^\varepsilon_R]} |\langle Z_s^\varepsilon, \sigma_1(X_s^\varepsilon) - \sigma_1(\hat{X}_s^\varepsilon) \rangle W_s^{Q_1}|.
\]
Using Gronwall’s inequality and the definition of $\tau^e_R$, we can deduce
\[
\sup_{t \in [0, T \wedge \tau^e_R]} |Z^e_t|^2 \leq C_{R,T} \left[ \int_0^{T \wedge \tau^e_R} |X^e_s - X^e_{s(\delta)}|^2 ds + \int_0^{T \wedge \tau^e_R} |Y^e_s - \dot{Y}^e_s|^2 ds \right. \\
+ \left. \sup_{t \in [0, T \wedge \tau^e_R]} \left[ \int_0^t \langle Z^e_s, [\sigma_1(X^e_s) - \sigma_1(\dot{X}^e_s)]dW^Q_1 \rangle \right] \right].
\]

Then by Burkholder-Davis-Gundy’s inequality, we obtain
\[
E \left( \sup_{t \in [0, T \wedge \tau^e_R]} |Z^e_t|^2 \right) \leq C_{R,T} \left( 1 + |x|^3 + |y|^3 \right) \delta^{1/2} + \frac{1}{2} E \left( \sup_{t \in [0, T \wedge \tau^e_R]} |Z^e_t|^2 \right)
+ C_{R,T} E \left( \int_0^{T \wedge \tau^e_R} |Z^e_t|^2 dt \right),
\]
which implies
\[
E \left( \sup_{t \in [0, T \wedge \tau^e_R]} |Z^e_t|^2 \right) \leq C_{R,T} \left( 1 + |x|^3 + |y|^3 \right) \delta^{1/2} + C_{R,T} \int_0^T E \left( \sup_{s \in [0, t \wedge \tau^e_R]} |Z^e_s|^2 \right) dt.
\]

By Gronwall’s inequality again, we get
\[
E \left( \sup_{t \in [0, T \wedge \tau^e_R]} |Z^e_t|^2 \right) \leq C_{R,T} \left( 1 + |x|^3 + |y|^3 \right) \delta^{1/2}.
\]

The proof is complete. \qed

### 3.3. Frozen and averaged equation

We first recall the frozen equation associate to fast motion for fixed slow component $x \in H$, i.e.,
\[
\begin{aligned}
\left\{ \begin{array}{l}
dY_t = [AY_t + g(x, Y_t)]dt + \sigma_2(x, Y_t)d\bar{W}_t^{Q_2}, \\
Y_0 = y,
\end{array} \right.
\end{aligned}
\tag{3.18}
\]
where $\bar{W}_t^{Q_2}$ is $Q_2$-Wiener process independent of $W_t^{Q_1}$ and $W_t^{Q_2}$. Notice that $g(x, \cdot)$ and $\sigma_2(x, \cdot)$ is Lipschitz continuous, it is easy to prove for any fixed $x \in H$ and any initial data $y \in H$, equation (3.18) has a unique mild solution $Y_t^{x,y}$. Let $P^x_t$ be the transition semigroup of $Y_t^{x,y}$, that is, for any bounded measurable function $\varphi$ on $H$,
\[
P^x_t \varphi(y) = E [\varphi (Y_t^{x,y})] , \quad y \in H, \quad t > 0.
\]

Similar as the argument in [4, Section 2.1], under the condition A3, it is easy to prove that $E|Y_t^{x,y}|^2 \leq C(1 + |x|^2 + e^{-\delta_1 t} |y|^2)$ for some $\delta_1 > 0$, and $P^x_t$ has unique invariant measure $\mu^x$. We here give the following asymptotic behavior of $P^x_t$ proved in [4, (2.13)].

**Theorem 3.6.** For any given value $x, y \in H$, there exists a unique invariant measure $\mu^x$ for (3.18). Moreover, there exist $C > 0$ and $\eta > 0$ such that for any Lipschitz function $\varphi : H \rightarrow \mathbb{R}$,
\[
|P^x_t \varphi (y) - \int_H \varphi (z) \mu^x (dz)| \leq C(1 + |x| + |y|) e^{-\frac{\eta t}{2}} |\varphi|_{Lip},
\tag{3.19}
\]
where $|\varphi|_{Lip} = \sup_{x, y \in H} \frac{|\varphi(x) - \varphi(y)|}{|x - y|}$. 

Next, we recall the corresponding averaged equation, i.e.,
\[
\begin{aligned}
&d\bar{X}_t = A\bar{X}_t dt - B(\bar{X}_t) dt + \bar{f}(\bar{X}_t) dt + \sigma_1(\bar{X}_t) dW_t^{Q_1}, \\
&\bar{X}_0 = x,
\end{aligned}
\]  
(3.20)

with the average
\[
\bar{f}(x) = \int_H f(x,y)\mu^x(dy), \quad x \in H,
\]
where \(\mu^x\) is the unique invariant measure for equation (3.18).

Due to the Lipschitz of \(f\), by a standard method, it is easy to check \(\bar{f}\) is Lipschitz and equation (3.20) has a unique solution. Then similar with the argument in Lemma 3.1, we also have the following estimate.

**Lemma 3.7.** For any \(T > 0, p \geq 1\), there exists a positive constant \(C_{p,T}\) such that for any \(x \in H\)
\[
\mathbb{E}\left(\sup_{t \in [0,T]}|\bar{X}_t|^{2p}\right) + \mathbb{E}\left(\int_0^T |\bar{X}_t|^{2p-2}\|\bar{X}_t\|_2^2 dt\right) \leq C_{p,T}(1 + |x|^{2p}).
\]

In the next lemma, we shall deal with the difference process \(\hat{X}_t - \bar{X}_t\). To this end, we shall construct another stopping time, i.e., for fixed \(\varepsilon \in (0, 1)\), \(R > 0\),
\[
\tilde{\tau}_R := \inf \left\{ t \geq 0 : \int_0^t \|X_s^\varepsilon\|_1^2 ds + \int_0^t \|\hat{X}_s^\varepsilon\|_1^2 ds + \int_0^t \|\bar{X}_s\|_1^2 ds + \int_0^t \|J_s^\varepsilon\|_2^2 ds \geq R \right\},
\]
where \(J_t^\varepsilon\) is the solution of the following equation:
\[
\begin{aligned}
&dJ_t^\varepsilon = AJ_t^\varepsilon dt + \left[f(X_t^{\varepsilon(\delta)}), \hat{Y}_t^\varepsilon\right] - \left[f(X_t^{\varepsilon(\delta)}), \bar{f}(X_t^{\varepsilon(\delta)})\right] dt, \\
&J_0^\varepsilon = 0,
\end{aligned}
\]  
(3.21)

which satisfies
\[
J_t^\varepsilon = \int_0^t e^{(t-s)A} \left[f(X_{s(\delta)}^{\varepsilon(\delta)}), \hat{Y}_s^\varepsilon\right] - \left[f(X_{s(\delta)}^{\varepsilon(\delta)}), \bar{f}(X_{s(\delta)}^{\varepsilon(\delta)})\right] ds.
\]

We remark here that the reason why we introduce \(J_t^\varepsilon\) in \(\tilde{\tau}_R\) is a technical treatment in the proof of following Lemma.

**Lemma 3.8.** For any \(x, y \in H, T > 0\) and \(\varepsilon \in (0, 1)\), then there exists a constant \(C_{R,T} > 0\) such that
\[
\mathbb{E}\left(\sup_{t \in [0,T]}|\hat{X}_t^\varepsilon - \bar{X}_t|^2\right) \leq C_{R,T}(1 + |x|^3 + |y|^3) \left(\frac{\varepsilon^2}{\delta} + \delta^{1/2}\right).
\]

**Proof.** The proof is divided into two steps.

**Step 1. (Splitting \(\hat{X}_t^\varepsilon - \bar{X}_t\) into two terms):** Let \(L_t^\varepsilon := \hat{X}_t^\varepsilon - \bar{X}_t\) and \(V_t^\varepsilon := L_t^\varepsilon - J_t^\varepsilon\). From equations (3.14), (3.20) and (3.21), we can write
\[
\begin{aligned}
dV_t^\varepsilon &= AV_t^\varepsilon dt - [B(\hat{X}_t^\varepsilon) - B(\bar{X}_t)] dt \\
&\quad + [\bar{f}(X_t^{\varepsilon(\delta)}) - \bar{f}(\bar{X}_t)] dt + [\sigma_1(\hat{X}_t^\varepsilon) - \sigma_1(\bar{X}_t)]dW_t^{Q_1},
\end{aligned}
\]

\[
\begin{aligned}
dV_t^\varepsilon &= AV_t^\varepsilon dt - [B(\hat{X}_t^\varepsilon) - B(\bar{X}_t)] dt + [\bar{f}(X_t^{\varepsilon(\delta)}) - \bar{f}(\hat{X}_t^\varepsilon)] dt \\
&\quad + [\bar{f}(X_t^\varepsilon) - \bar{f}(\hat{X}_t^\varepsilon)] + [\bar{f}(\hat{X}_t^\varepsilon) - \bar{f}(\bar{X}_t)] dt + [\sigma_1(\hat{X}_t^\varepsilon) - \sigma_1(\bar{X}_t)]dW_t^{Q_1}.
\end{aligned}
\]
Applying Itô’s formula, we have
\[
|V_t^\epsilon|^2 = -2 \int_0^t \|V_s^\epsilon\|^2 ds - 2 \int_0^t \langle B(\tilde{X}_s^\epsilon) - B(\tilde{X}_s), V_s^\epsilon\rangle ds \\
+ 2 \int_0^t \langle \tilde{f}(X_{s(\delta)}^\epsilon) - \tilde{f}(X_s^\epsilon) + \tilde{f}(X_s^\epsilon) - \tilde{f}(X_s^\epsilon) - \tilde{f}(X_s^\epsilon), V_s^\epsilon\rangle ds \\
+ \int_0^t |\sigma_1(\tilde{X}_s^\epsilon) - \sigma_1(\tilde{X}_s^\epsilon)|^2 ds + 2 \int_0^t \langle V_s^\epsilon, [\sigma_1(\tilde{X}_s^\epsilon) - \sigma_1(\tilde{X}_s^\epsilon)]dW_s^Q\rangle \\
:= -2 \int_0^t \|V_s^\epsilon\|^2 ds + \mathcal{I}_1(t) + \mathcal{I}_2(t) + \mathcal{I}_3(t) + \mathcal{I}_4(t). \tag{3.22}
\]

For \(\mathcal{I}_1(t)\), we can use Corollary 4.2 and Young’s inequality to get that
\[
\mathcal{I}_1(t) = -2 \int_0^t \langle B(\tilde{X}_s^\epsilon) - B(\tilde{X}_s + J_s^\epsilon), V_s^\epsilon\rangle ds - 2 \int_0^t \langle B(\tilde{X}_s + J_s^\epsilon) - B(\tilde{X}_s), V_s^\epsilon\rangle ds \\
\leq C \int_0^t \|V_s^\epsilon\| \|V_s^\epsilon\|_1 \|\tilde{X}_s\|_2 ds + 2 \int_0^t \|B(\tilde{X}_s + J_s^\epsilon) - B(\tilde{X}_s)\|_1 \|V_s^\epsilon\|_1 ds \\
\leq \int_0^t \|V_s^\epsilon\|^2 ds + C \int_0^t \|V_s^\epsilon\|^2 ds + C \int_0^t \|J_s^\epsilon\|^2_1 \|\tilde{X}_s\|^2_2 ds + C \int_0^t \|V_s^\epsilon\|^2 ds. \tag{3.23}
\]

For \(\mathcal{I}_2(t)\), we have by using the Lipschtiz continuous property of \(\tilde{f}\) and Young’s inequality that
\[
\mathcal{I}_2(t) \leq C \int_0^t (|X_{s(\delta)}^\epsilon - X_s^\epsilon|^2 + |X_s^\epsilon - \tilde{X}_s^\epsilon|^2 + |L_s^\epsilon|^2) ds + C \int_0^t \|V_s^\epsilon\|^2 ds. \tag{3.24}
\]

According to condition A1, it is easy to see that
\[
\mathcal{I}_3(t) \leq C \int_0^t \|L_s^\epsilon\|^2 ds. \tag{3.25}
\]

Combining estimates (3.22)-(3.25) together, we obtain
\[
|V_t^\epsilon|^2 + \int_0^t \|V_s^\epsilon\|^2 ds \leq C \int_0^t (1 + \|\tilde{X}_s^\epsilon\|^2) \|V_s^\epsilon\|^2 ds + C \int_0^t \|J_s^\epsilon\|^2_1 (\|\tilde{X}_s + J_s^\epsilon\|^2_1 + \|\tilde{X}_s\|^2_2) ds \\
+ C \int_0^t (|X_{s(\delta)}^\epsilon - X_s^\epsilon|^2 + |X_s^\epsilon - \tilde{X}_s^\epsilon|^2 + |L_s^\epsilon|^2) ds + \mathcal{I}_4(t).
\]

As a result, it follows from Gronwall’s inequality that
\[
|V_t^\epsilon|^2 + \int_0^t \|V_s^\epsilon\|^2 ds \leq C \left[ \int_0^t \left( |X_{s(\delta)}^\epsilon - X_s^\epsilon|^2 + |X_s^\epsilon - \tilde{X}_s^\epsilon|^2 + |L_s^\epsilon|^2 \right) ds + \int_0^t \|J_s^\epsilon\|^2_1 (\|\tilde{X}_s + J_s^\epsilon\|^2_1 + \|\tilde{X}_s\|^2_2) ds + \mathcal{I}_4(t) \right] e^{\int_0^t (\|\tilde{X}_s\|^2_2 + 1) ds}.
\]
By the definition of stopping time $\tilde{\tau}_R$ and Burkholder-Davis-Gundy’s inequality, we can deduce that
\[
\mathbb{E}\left( \sup_{t \in [0,T \wedge \tilde{\tau}_R]} |V_t^\varepsilon|^2 \right) \leq C_{R,T} \mathbb{E}\left[ \sup_{t \in [0,T \wedge \tilde{\tau}_R]} \|J_t^\varepsilon\|_{1/2}^2 \int_0^{T \wedge \tilde{\tau}_R} (\|J_t^\varepsilon\|_{1/2}^2 + \|\hat{X}_t^\varepsilon\|_1^2) dt \right. \\
+ \int_0^{T \wedge \tilde{\tau}_R} (|X_{t(\delta)}^\varepsilon - X_t^\varepsilon|^2 + |X_t^\varepsilon - \hat{X}_t^\varepsilon|^2 + |L_t^\varepsilon|^2) dt + \left. \sup_{t \in [0,T \wedge \tilde{\tau}_R]} \mathcal{I}_t(t) \right]
\]

Then it follows
\[
\mathbb{E}\left( \sup_{t \in [0,T \wedge \tilde{\tau}_R]} |V_t^\varepsilon|^2 \right) \leq C_{R,T} \mathbb{E}\left( \sup_{t \in [0,T \wedge \tilde{\tau}_R]} \|J_t^\varepsilon\|_{1/2}^2 \right) + C_{R,T} \mathbb{E}\left( \int_0^{T \wedge \tilde{\tau}_R} |L_t^\varepsilon|^2 dt \right) \\
+ C_{R,T} \mathbb{E}\left[ \int_0^{T \wedge \tilde{\tau}_R} (|X_{t(\delta)}^\varepsilon - X_t^\varepsilon|^2 + |X_t^\varepsilon - \hat{X}_t^\varepsilon|^2) dt \right].
\]

Notice that $\tilde{\tau}_R^\varepsilon \leq \tilde{\tau}_R$, we can get by Lemmas 3.2 and 3.5 that
\[
\mathbb{E}\left( \sup_{t \in [0,T \wedge \tilde{\tau}_R]} |L_t^\varepsilon|^2 \right) \leq 2 \mathbb{E}\left( \sup_{t \in [0,T \wedge \tilde{\tau}_R]} |V_t^\varepsilon|^2 \right) + 2 \mathbb{E}\left( \sup_{t \in [0,T]} |J_t^\varepsilon|^2 \right) \\
\leq C_{R,T} \mathbb{E}\left( \sup_{t \in [0,T]} \|J_t^\varepsilon\|_{1/2}^2 \right) + C_{R,T} \mathbb{E}\left( \int_0^{T \wedge \tilde{\tau}_R^\varepsilon} |L_t^\varepsilon|^2 dt \right) \\
+ C_{R,T} \left[ \int_0^{T \wedge \tilde{\tau}_R} (|X_{t(\delta)}^\varepsilon - X_t^\varepsilon|^2 + |X_t^\varepsilon - \hat{X}_t^\varepsilon|^2) dt \right] \\
\leq C_{R,T} \mathbb{E}\left( \sup_{t \in [0,T]} \|J_t^\varepsilon\|_{1/2}^2 \right) + C_{R,T} \mathbb{E}\left( \int_0^{T \wedge \tilde{\tau}_R^\varepsilon} |L_t^\varepsilon|^2 dt \right) \\
+ C_{R,T} \left( 1 + |x|^3 + |y|^3 \right)^{\delta^{1/2}}.
\]

Then by Gronwall’s inequality, we obtain
\[
\mathbb{E}\left( \sup_{t \in [0,T \wedge \tilde{\tau}_R]} |L_t^\varepsilon|^2 \right) \leq C_{R,T} \mathbb{E}\left( \sup_{t \in [0,T]} \|J_t^\varepsilon\|_{1/2}^2 \right) + C_{R,T} \left( 1 + |x|^3 + |y|^3 \right)^{\delta^{1/2}}.
\] (3.26)

Hence, the proof will be finished by the following estimates (3.27), which will be done in the next step.

**Step 2. (The estimate for $J_t^\varepsilon$:** By the discussion above, it suffices to show that the following estimate holds, i.e.,
\[
\mathbb{E}\left( \sup_{t \in [0,T]} \|J_t^\varepsilon\|_{1/2}^2 \right) \leq C_T (1 + |x|^2 + |y|^2)^{\varepsilon^{\frac{1}{2}}}.
\] (3.27)
To this end, we decompose $J^ε_1$ by

$$
J^ε_1 = \sum_{k=0}^{[t/\delta]-1} \int_{k\delta}^{(k+1)\delta} e^{(t-s)A} \left[ f(X^ε_{k\delta}, \hat{Y}^ε_s) - \hat{f}(X^ε_{k\delta}) \right] ds + \int_{[t/\delta]}^t e^{(t-s)A} \left[ f \left( X^ε_{[t/\delta]}, \hat{Y}^ε_s \right) - \hat{f} \left( X^ε_{[t/\delta]} \right) \right] ds
:= J^ε_1(t) + J^ε_2(t).
$$

For $J^ε_2(t)$, by Lemmas 3.1 and 3.3, it is easy to see

$$
\mathbb{E} \left( \sup_{t \in [0,T]} \| J^ε_2(t) \|_{1/2}^2 \right) \leq C \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_t^T (t-s)^{-\frac{1}{2}} (1 + |X^ε_{[t/\delta]}| + |\hat{Y}^ε_s|) ds \right|^2 \right]
\leq C \left( \int_0^T s^{-\frac{1}{2}} ds \right) \cdot \mathbb{E} \left[ \int_0^T (1 + |X^ε_{[t/\delta]}|^2 + |\hat{Y}^ε_s|^2) ds \right]
\leq C T (1 + |x|^2 + |y|^2) \delta^2.
$$

For $J^ε_1(t)$, by the construction of $\hat{Y}^ε_{s+\delta}$, we obtain that, for any $k \in \mathbb{N}$ and $s \in [0, \delta)$,

$$
\hat{Y}^ε_{s+k\delta} = \hat{Y}^ε_{k\delta} + \frac{1}{\varepsilon} \int_{k\delta}^{k\delta+s} A\hat{Y}^ε_r dr + \frac{1}{\varepsilon} \int_{k\delta}^{k\delta+s} g \left( X^ε_{k\delta}, \hat{Y}^ε_r \right) dr
+ \frac{1}{\varepsilon} \int_{k\delta}^{k\delta+s} \sigma_2 \left( X^ε_{k\delta}, \hat{Y}^ε_r \right) dW^ε_{r+k\delta}
= \hat{Y}^ε_{k\delta} + \frac{1}{\varepsilon} \int_0^s A\hat{Y}^ε_{r+k\delta} dr + \frac{1}{\varepsilon} \int_0^s g \left( X^ε_{k\delta}, \hat{Y}^ε_{r+k\delta} \right) dr
+ \frac{1}{\varepsilon} \int_0^s \sigma_2 \left( X^ε_{k\delta}, \hat{Y}^ε_{r+k\delta} \right) dW^ε_{r+k\delta},
$$

where $W^ε_{t+k\delta} := W^ε_{t+k\delta} - W^ε_{k\delta}$ is the shift version of $W^ε_{t+k\delta}$.

We construct a process $Y^ε_{X^ε_{k\delta}, Y^ε_{k\delta}}$ by means of $Y^ε_{X^ε_{k\delta}, Y^ε_{k\delta}} | (x,y) = (X^ε_{k\delta}, Y^ε_{k\delta})$, where $Y^ε_{X^ε, Y^ε}$ is the solution to equation (3.18), i.e.,

$$
Y^ε_{X^ε_{k\delta}, Y^ε_{k\delta}} = \hat{Y}^ε_{k\delta} + \int_0^{\frac{t}{\delta}} A Y^ε_{X^ε_{k\delta}, Y^ε_{k\delta}} dr + \int_0^{\frac{t}{\delta}} g \left( X^ε_{k\delta}, Y^ε_{X^ε_{k\delta}, Y^ε_{k\delta}} \right) dr
+ \int_0^{\frac{t}{\delta}} \sigma_2 \left( X^ε_{k\delta}, Y^ε_{X^ε_{k\delta}, Y^ε_{k\delta}} \right) dW^ε_{t+k\delta}
= \hat{Y}^ε_{k\delta} + \frac{1}{\varepsilon} \int_0^s A Y^ε_{X^ε_{k\delta}, Y^ε_{k\delta}} dr + \frac{1}{\varepsilon} \int_0^s g \left( X^ε_{k\delta}, Y^ε_{X^ε_{k\delta}, Y^ε_{k\delta}} \right) dr
+ \frac{1}{\varepsilon} \int_0^s \sigma_2 \left( X^ε_{k\delta}, Y^ε_{X^ε_{k\delta}, Y^ε_{k\delta}} \right) dW^ε_{t+k\delta},
$$

where $W^ε_{t+k\delta} := \varepsilon^{1/2} W^ε_{t+k\delta}$. Then the uniqueness of the solution to equations (3.28) and (3.29) implies that the distribution of $\left( X^ε_{k\delta}, Y^ε_{X^ε_{k\delta}, Y^ε_{k\delta}} \right)_{0 \leq s \leq \delta}$ coincides with the distribution of $\left( X^ε_{k\delta}, Y^ε_{X^ε_{k\delta}, Y^ε_{k\delta}} \right)_{0 \leq s \leq \delta}$. 


Now we try to control $\|J_1^e(t)\|_{1/2}$:

$$
\mathbb{E}\left( \sup_{t \in [0,T]} \|J_1^e(t)\|_{1/2}^2 \right) \\
= \mathbb{E}\left\{ \sup_{t \in [0,T]} \left[ \sum_{k=0}^{[t/\delta]-1} e^{(t-(k+1)\delta)} \left[ -A \right]^{1/4} e^{((k+1)\delta-s)A} \left[ f \left( X^e_{k\delta}, \hat{Y}^e_s \right) - \tilde{f} \left( X^e_{k\delta} \right) \right] ds \right]^2 \right\} \\
\leq \mathbb{E}\left\{ \sup_{t \in [0,T]} \left[ \sum_{k=0}^{[t/\delta]-1} e^{(t-(k+1)\delta)} \left[ -A \right]^{1/4} e^{((k+1)\delta-s)A} \left[ f \left( X^e_{k\delta}, \hat{Y}^e_s \right) - \tilde{f} \left( X^e_{k\delta} \right) \right] ds \right]^2 \right\} \\
\leq \frac{T}{\delta} \sum_{k=0}^{\left[ \frac{T}{\delta} \right]-1} \mathbb{E}\left[ \int_{k\delta}^{(k+1)\delta} \left[ -A \right]^{1/4} e^{((k+1)\delta-s)A} \left[ f \left( X^e_{k\delta}, \hat{Y}^e_s \right) - \tilde{f} \left( X^e_{k\delta} \right) \right] ds \right]^2 \\
\leq \frac{C_T}{\delta^2} \max_{0 \leq k \leq \left[ \frac{T}{\delta} \right]-1} \mathbb{E}\left[ \int_{k\delta}^{(k+1)\delta} \left[ -A \right]^{1/4} e^{((k+1)\delta-s)A} \left[ f \left( X^e_{k\delta}, \hat{Y}^e_s \right) - \tilde{f} \left( X^e_{k\delta} \right) \right] ds \right]^2 \\
= \frac{2C_T}{\delta^2} \max_{0 \leq k \leq \left[ \frac{T}{\delta} \right]-1} \int_0^{\delta} \int_r^{r+\delta} \Psi_k(s, r) ds dr,
$$

where

\[
\Psi_k(s, r) := \mathbb{E}\left\langle \left( -A \right)^{1/4} e^{(\delta-s)A} \left[ f \left( X^e_{k\delta}, \hat{Y}^e_{s+k\delta} \right) - \tilde{f} \left( X^e_{k\delta} \right) \right], \right. \\
\left. \left( -A \right)^{1/4} e^{(\delta-r)A} \left[ f \left( X^e_{k\delta}, \hat{Y}^e_{r+k\delta} \right) - \tilde{f} \left( X^e_{k\delta} \right) \right] \right\rangle.
\]

Denote the $\sigma$-field generated by $\{Y^x,y_u; u \leq s\}$ by

\[
\tilde{\mathcal{F}}_s := \sigma(Y^x,y_u; u \leq s).
\]

Then for $0 \leq r < s \leq \frac{\delta}{\varepsilon}$, notice that the distribution of $(X^e_{s+k\delta}, \hat{Y}^e_{s+k\delta})_{0 \leq s \leq \delta}$ coincides with the distribution of $(X^e_{k\delta}, \hat{Y}^e_{k\delta})_{0 \leq s \leq \delta}$, we can get that

\[
\Psi_k(s, r) = \mathbb{E}\left\langle \left( -A \right)^{1/4} e^{(\delta-s)A} \left[ f \left( x, Y^x,y_s \right) - \tilde{f} \left( x \right) \right], \right. \\
\left. \left( -A \right)^{1/4} e^{(\delta-r)A} \left[ f \left( x, Y^x,y_r \right) - \tilde{f} \left( x \right) \right] \right\rangle \bigg| (x,y) = (X^e_{k\delta}, \hat{Y}^e_{k\delta}) \\
= \mathbb{E}\left\langle \left( -A \right)^{1/4} e^{(\delta-s)A} \mathbb{E}\left[ f \left( x, Y^x,y_s \right) - \tilde{f} \left( x \right) \big| \tilde{\mathcal{F}}_r \right], \right. \\
\left. \left( -A \right)^{1/4} e^{(\delta-r)A} \mathbb{E}\left[ f \left( x, Y^x,y_r \right) - \tilde{f} \left( x \right) \big| \tilde{\mathcal{F}}_r \right] \right\rangle \bigg| (x,y) = (X^e_{k\delta}, \hat{Y}^e_{k\delta})
\]
Then by the Markov property of $Y^x_t$, (2.1) and Theorem 3.6, we have

$$
\Psi_k(s, r) \leq C(\delta - s\varepsilon)^{-1/4}(\delta - r\varepsilon)^{-1/4}
\cdot \mathbb{E} \left[ \mathbb{E} \left( \left| f(x, Y^x_{s-r}) - f(x) \right| \mathbb{I}_{\{z=\gamma^x_t\}} (1 + |x| + |Y^x_r|) \right) \mathbb{I}_{(x,y)=(X^x_{k\delta},Y^x_{k\delta})} \right]
\leq C(\delta - s\varepsilon)^{-1/4}(\delta - r\varepsilon)^{-1/4} e^{-(s-r)\eta} \mathbb{E} \left( 1 + |x|^2 + |Y^x_r|^2 \right) \mathbb{I}_{(x,y)=(X^x_{k\delta},Y^x_{k\delta})}
\leq C \mathbb{E} \left( 1 + |X^\varepsilon_{k\delta}|^2 \right) (\delta - s\varepsilon)^{-1/4}(\delta - r\varepsilon)^{-1/4} e^{-(s-r)\eta}
\leq C_T (1 + |x|^2 + |y|^2) (\delta - s\varepsilon)^{-1/4}(\delta - r\varepsilon)^{-1/4} e^{-(s-r)\eta},
$$

where the last two inequalities are deduced by (3.1) and (3.15). Then we get

$$
\mathbb{E} \left[ \sup_{t \in [0,T]} ||J^\varepsilon_t(t)||^2_{1/2} \right] \leq C_T (1 + |x|^2 + |y|^2) \frac{\varepsilon^2}{\delta^2}
\cdot \int_0^\frac{\delta}{\varepsilon} \int_r^\frac{\delta}{\varepsilon} (\delta - s\varepsilon)^{-1/4}(\delta - r\varepsilon)^{-1/4} e^{-(s-r)\eta} ds dr
\leq C_T (1 + |x|^2 + |y|^2) \frac{\varepsilon^2}{\delta^2}
\cdot \int_0^\frac{\delta}{\varepsilon} \left[ \int_r^\frac{\delta}{\varepsilon} (\delta - s\varepsilon)^{-1/2} ds \right]^{1/2} \left[ \int_r^\frac{\delta}{\varepsilon} e^{-2\eta s} ds \right]^{1/2} (\delta - r\varepsilon)^{-1/4} e^{\eta r} dr
\leq C_T (1 + |x|^2 + |y|^2) \frac{\varepsilon^2}{\delta^2} \int_0^{\frac{\delta}{\varepsilon} \frac{1}{\varepsilon^{1/2}}} (\delta - r\varepsilon)^{1/4} e^{-\eta r} (\delta - r\varepsilon)^{-1/4} e^{\eta r} dr
\leq C_T (1 + |x|^2 + |y|^2) \frac{\varepsilon^{1/2}}{\delta},
$$

which completes the estimate (3.27). The proof is complete.

\hfill \Box

### 3.4. Proof of Theorem 2.4

Taking $\delta = \varepsilon^{\frac{1}{4}}$, we can get by Lemmas 3.5 and 3.8 that

$$
\mathbb{E} \left( \sup_{t \in [0,T]} |X^\varepsilon_t - \bar{X}_t|^2 \mathbb{I}_{\{t \leq \tau_R\}} \right) \leq C \mathbb{E} \left( \sup_{t \in [0,T \wedge \tau_R]} |X^\varepsilon_t - \bar{X}_t|^2 \right) + \sup_{t \in [0,T \wedge \tau_R]} |\bar{X}_t - X^\varepsilon_t|^2
\leq C_{R,T} (1 + |x|^3 + |y|^3) \left( \frac{\varepsilon^{1/2}}{\delta} + \delta^{1/2} \right)
\leq C_{R,T} (1 + |x|^3 + |y|^3) \varepsilon^{\frac{1}{4}}. \tag{3.30}
$$
By Chebyshev’s inequality, Lemmas 3.1, 3.3 and 3.7, we have
\[
\mathbb{E} \left( \sup_{t \in [0,T]} |X^\varepsilon_t - \bar{X}_t|^2 1_{\{T > \tau^\varepsilon_R\}} \right) \\
\leq \left[ \mathbb{E} \left( \sup_{t \in [0,T]} |X^\varepsilon_t - \bar{X}_t|^4 \right) \right]^{1/2} \cdot \left[ \mathbb{P} \left( T > \tau^\varepsilon_R \right) \right]^{1/2} \\
\leq \frac{C_T (1 + |x|^2 + |y|^2)}{\sqrt{R}} \left[ \mathbb{E} \left( \int_0^T \|\bar{X}_s\|^2 ds + \int_0^T \|X^\varepsilon_s\|^2 ds + \int_0^T \|\bar{X}^\varepsilon_s\|^2 ds + \int_0^T \|J^\varepsilon_s\|^2_{1/2} ds \right) \right]^{1/2} \\
\leq \frac{C_T (1 + |x|^3 + |y|^3)}{\sqrt{R}}. \tag{3.31}
\]

Hence, combining (3.30) and (3.31) together yields
\[
\mathbb{E} \left( \sup_{t \in [0,T]} |X^\varepsilon_t - \bar{X}_t| \right) \leq C_{R,T} (1 + |x|^3 + |y|^3)\varepsilon + \frac{C_T (1 + |x|^3 + |y|^3)}{\sqrt{R}}.
\]

Now, letting \( \varepsilon \to 0 \) firstly and then \( R \to \infty \), we have
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left( \sup_{t \in [0,T]} |X^\varepsilon_t - \bar{X}_t|^2 \right) = 0. \tag{3.32}
\]

Then for any \( p > 1 \), we have
\[
\mathbb{E} \left( \sup_{t \in [0,T]} |X^\varepsilon_t - \bar{X}_t|^{2p} \right) = \mathbb{E} \left[ \sup_{t \in [0,T]} \left( |X^\varepsilon_t - \bar{X}_t||X^\varepsilon_t - \bar{X}_t|^{2p-1} \right) \right] \\
\leq C_p \left[ \mathbb{E} \left( \sup_{t \in [0,T]} |X^\varepsilon_t - \bar{X}_t|^2 \right) \right]^{1/2} \left[ \mathbb{E} \left( \sup_{t \in [0,T]} |X^\varepsilon_t - \bar{X}_t|^{4p-2} \right) \right]^{1/2}.
\]

As a result, by (3.32), Lemmas 3.1 and 3.7, it is easy to see
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left( \sup_{t \in [0,T]} |X^\varepsilon_t - \bar{X}_t|^{2p} \right) = 0,
\]
which completes the proof. \( \square \)

4. Appendix

The following properties of \( b(\cdot, \cdot, \cdot) \) and \( B(\cdot, \cdot) \) are well-known (for example see \([10, 13, 19]\)).

**Lemma 4.1.** For any \( u, v, w \in H^1 \), we have
\[
b(u, v, w) = -b(u, w, v), \quad b(u, v, v) = 0.
\]
Moreover, if \( \alpha_i \geq 0 \) \((i = 1, 2, 3)\) satisfies one of the following conditions
\((1)\) \( \alpha_i \neq 1(i = 1, 2, 3), \alpha_1 + \alpha_2 + \alpha_3 \geq 1, \)
\((2)\) \( \alpha_i = 1 \) for some \( i, \alpha_1 + \alpha_2 + \alpha_3 > 1, \)
then \( b(u, v, w) \) is continuous from \( H^{\alpha_1} \times H^{\alpha_2+1} \times H^{\alpha_3} \) to \( \mathbb{R} \), i.e.
\[
|b(u, v, w)| \leq C \|u\|_{\alpha_1} \|v\|_{\alpha_2+1} \|w\|_{\alpha_3}.
\]

From the definition of \( b \), interpolation inequality and Lemma 4.1 above, the following inequalities can be derived.

**Corollary 4.2.** For any \( u, v \in H^1 \), there exists a constant \( C > 0 \) such that
(1) \( \langle B(u), v \rangle \leq C \| u \|_{L^4}^{3/2} \| u \|_{L^4}^{1/2} \| v \|_{L^4} ; \)
(2) \( \| B(u) \|_{-1} \leq C \| u \|_{1} ; \)
(3) \( \langle B(u) - B(v), u - v \rangle \leq C \| u - v \|_{1} \| u \|_{1} ; \)
(4) \( \| B(u) - B(v) \|_{-1} \leq C \| u - v \|_{1/2} \left( \| u \|_{1/2} + \| v \|_{1/2} \right) . \)

At the end of this section, we give the proof of Theorem 2.3 based on the techniques used in [11].

**Proof of Theorem 2.3:** Let \( \mathcal{H} := H \times H \) be the product Hilbert space. For any \( \phi = (\phi_1, \phi_2), \varphi = (\varphi_1, \varphi_2) \in \mathcal{H} \), we denote the scalar product and the induced norm by
\[
\langle \phi, \varphi \rangle_{\mathcal{H}} = \int_{\mathbb{T}^2} \phi_1(\xi) \varphi_1(\xi) d\xi + \int_{\mathbb{T}^2} \phi_2(\xi) \varphi_2(\xi) d\xi, \quad |\phi|_{\mathcal{H}} = \sqrt{\langle \phi, \phi \rangle_{\mathcal{H}}} = \sqrt{|\phi_1|^2 + |\phi_2|^2}.
\]
Similarly, we also define \( \mathcal{V} := H^1 \times H^1 \). Then \( \mathcal{V} \) is a product Hilbert space with the scalar product and the induced norm,
\[
\langle \phi, \varphi \rangle_{\mathcal{V}} = \int_{\mathbb{T}^2} \nabla \phi_1(\xi) \cdot \nabla \varphi_1(\xi) d\xi + \int_{\mathbb{T}^2} \nabla \phi_2(\xi) \cdot \nabla \varphi_2(\xi) d\xi, \quad \| \phi \|_{\mathcal{V}} = \sqrt{\langle \phi, \phi \rangle_{\mathcal{V}}} = \sqrt{\| \phi_1 \|_{1}^2 + \| \phi_2 \|_{1}^2}.
\]
Now we rewrite the system (2.2) for \( Z^\varepsilon = (X^\varepsilon, Y^\varepsilon) \) as
\[
dZ^\varepsilon = \tilde{A}Z^\varepsilon dt + F(Z^\varepsilon) dt + \sigma(Z^\varepsilon) dW_t, \quad Z_0^\varepsilon = (x, y) \in \mathcal{H}, \quad (4.1)
\]
where \( W_t := (W_t^{Q_1}, W_t^{Q_2}) \), which is a \( \mathcal{H} \)-valued \( Q \)-Wiener process with \( Q := (Q_1, Q_2) \) and \( Q \) is a positive symmetric, trace class operate on \( \mathcal{H} \) and
\[
\tilde{A}Z^\varepsilon = \left( AX^\varepsilon, \frac{1}{\varepsilon} AY^\varepsilon \right), \quad F(Z^\varepsilon) = \left( -B(X^\varepsilon) + f(X^\varepsilon, Y^\varepsilon), \frac{1}{\varepsilon} g(X^\varepsilon, Y^\varepsilon) \right), \quad \sigma(Z^\varepsilon) = \left( \sigma_1(X^\varepsilon), \frac{1}{\sqrt{\varepsilon}} \sigma_2(X^\varepsilon, Y^\varepsilon) \right).
\]
Moreover, \( \sigma \) is an operator from \( \mathcal{H} \rightarrow \mathcal{L}_Q(Q^{1/2} \mathcal{H}, \mathcal{H}) \), here \( \mathcal{L}_Q(Q^{1/2} \mathcal{H}, \mathcal{H}) \) is the space of linear operate \( S : Q^{1/2} \mathcal{H} \rightarrow \mathcal{H} \) such that \( SQ^{1/2} \) is a Hilbert-Schmidt operator from \( \mathcal{H} \) to \( \mathcal{H} \). The norm in \( \mathcal{L}_Q(Q^{1/2} \mathcal{H}, \mathcal{H}) \) is defined by
\[
|\sigma(z)|_{\mathcal{L}_Q} = \sqrt{|\sigma_1(x)|_{\mathcal{L}_{Q_1}}^2 + \frac{1}{\varepsilon} |\sigma_2(x, y)|_{\mathcal{L}_{Q_2}}^2}, \quad z = (x, y) \in \mathcal{H}.
\]
Let \( \mathcal{V}' \) be the dual space of \( \mathcal{V} \) and we consider the following Gelfand triple \( \mathcal{V} \subset \mathcal{H} \equiv \mathcal{H}' \subset \mathcal{V}' \). It is easy to see that the following mappings
\[
\tilde{A} : \mathcal{V} \rightarrow \mathcal{V}', \quad F : \mathcal{V} \rightarrow \mathcal{V}'
\]
are well defined. To complete the proof, it remains to take finite-dimensional Galerkin approximation, and apply the weak convergence approach. The details are similar to the arguments in [11, 26], where the authors deal with the 2D stochastic Boussinesq equation and 2D stochastic Navier-Stokes equation, respectively. So we only check the new coefficients in equation (4.1) satisfy the local monotonicity and coercivity properties and further details are omitted here.
Indeed, for any \( w = (u, v) \in \mathcal{V} \), by Lemma 4.1, there exist constants \( C_\varepsilon > 0 \) and \( C > 0 \) such that
\[
\langle \tilde{A}w + F(w), w \rangle + |\sigma(w)|^2_{Q_2} = \langle Au - B(u) + f(u, v), u \rangle + \frac{1}{\varepsilon} \langle Av + g(u, v), v \rangle + C(1 + |u|^2 + |v|^2)
\]
\[
\leq -\|u\|^2_1 - \frac{1}{\varepsilon}|v|^2_1 + C(1 + |u|^2 + |v|^2)
\]
\[
\leq -C_\varepsilon\|w\|^2_{Q} + C(1 + |w|^2_H),
\]
which implies that the coercivity condition holds. For any \( w_1 = (u_1, v_1), w_2 = (u_2, v_2) \in \mathcal{V} \), by Corollary 4.2 and the condition A1, we have
\[
\langle F(w_1) - F(w_2), w_1 - w_2 \rangle
\]
\[
= \langle -B(u_1 - u_2, u_2) + f(u_1, v_1) - f(u_2, v_2), u_1 - u_2 \rangle + \frac{1}{\varepsilon} \langle g(u_1, v_1) - g(u_2, v_2), v_1 - v_2 \rangle
\]
\[
\leq 2\|u_1 - u_2\|^{3/2}_1 |u_1 - u_2|^{1/2}_1 |u_2|_{L^4} + C(|u_1 - u_2|^2 + |v_1 - v_2|^2)
\]
\[
\leq \frac{1}{2} \|u_1 - u_2\|^2_1 + C(1 + |u_2|^4 |u_1 - u_2|^2 + C|v_1 - v_2|^2).
\]
Therefore, we get
\[
\langle \tilde{A}w_1 + F(w_1) - \tilde{A}w_2 - F(w_2), w_1 - w_2 \rangle + |\sigma(w_1) - \sigma(w_2)|^2_{Q_2}
\]
\[
\leq -|u_1 - u_2|^2_1 - \frac{1}{\varepsilon}|v_1 - v_2|^2_1 + \frac{1}{2} \|u_1 - u_2\|^2_1 + C(1 + |u_2|^4 |u_1 - u_2|^2 + C|v_1 - v_2|^2
\]
\[
\leq C(1 + |u_2|^4 |u_1 - u_2|^2 + C|v_1 - v_2|^2).
\]

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