HODGE AND TATE CONJECTURES
FOR HYPERGEOMETRIC SHEAVES

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§0. Introduction

In this paper, we study an analog of Hodge and Tate conjectures for local systems associated to hypergeometric functions. Let \( f : X \to S, \ g : Y \to S \) be a smooth proper family of relative dimension \( n \) and \( m \) over a smooth variety \( S \) over \( k = \mathbb{C} \). Let \( K \) be an algebraic subfield of \( \mathbb{C} \). The higher direct image sheaf \( R^i f_* K \) and \( R^j g_* K \) are variations of \( K \)-Hodge structures. (For the definition of \( K \)-Hodge structure, see Section 3.) A flat proper family \( Z \) of subvarieties in \( X \times_S Y \) of codimension \( n \) defines the following morphism.

\[
R^i f_* K \to R^i (f \times g)_* K \to R^{i+2d} (f \times g)_* K \to R^i g_* K
\]

This is a morphism of variation of \( K \)-Hodge structures. A formal \( K \)-linear combination of flat families of subvarieties in \( X \times_S Y \) is called a \( K \)-algebraic correspondence from \( Y \) to \( X \). By extending \( K \)-linearly, a \( K \)-algebraic correspondence induces a morphism \( R^i f_* K \to R^i g_* K \) of variation of \( K \)-Hodge structures.

Hodge Conjecture for families of varieties. A morphism \( R^i f_* \mathbb{Q} \to R^i g_* \mathbb{Q} \) of variation of \( \mathbb{Q} \)-Hodge structures over \( S \) is induced by a \( \mathbb{Q} \)-algebraic correspondence.

If \( S \) is a point, it is classical Hodge conjecture. If there exists an action of cyclic group \( \mu_m \) on \( X \) and \( Y \) over \( S \), we can define \( \chi \)-part and \( \chi' \)-part of the cohomology group \( R^i f_* K(\chi) \) and \( R^j g_* K(\chi') \) of \( R^i f_* K \) and \( R^j g_* K \), where \( K = \mathbb{Q}(\mu_m) \). Then these groups are variations of \( K \)-Hodge structures. The \( K \)-Hodge conjecture for families of varieties is formulated as follows.

\( K \)-Hodge Conjecture for families of varieties. A morphism \( R^i f_* K(\chi) \to R^j g_* K(\chi') \) of variation of \( K \)-Hodge structures is induced by a \( K \)-algebraic correspondence.

In this paper, we study a local system arising from the theory of hypergeometric functions due to Gel’fand-Kapranov-Zelevinski. They are called hypergeometric sheaves. Let us recall the definition of hypergeometric sheaves. Let \( n \) and \( r \) be natural numbers such that \( r > n + 1 \) and \( \omega_1, \ldots, \omega_r \) be elements in \( \mathbb{Z}^n \). For an element \( v = (v_1, \ldots, v_n) \in \mathbb{Z}^n \), we define \( x^v = x_1^{v_1} \cdots x_n^{v_n} \). Let \( f(a, x) = \sum_{i=1}^r a_i x^{\omega_i} \) be a Laurent polynomial in \( \mathbb{C}[a_1^\pm, \ldots, a_r^\pm, x_1^\pm, \ldots, x_n^\pm] \) and \( U \) be the open set of \( (\mathbb{C}^\times)^{n+r} \) defined by \( U = \{(x, a) \mid f(a, x) \neq 0\} \). The covering of \( U \)
where \( f \) in inclusion is denoted by the multiplication on \( U \) defined by \( \chi \). Then we have a constructible sheaf \( F = R^nf_*K(\chi) \), where \( f : \mathcal{X} \to (\mathbb{C}^\times)^r \) is induced by the second projection \((x,a) \to a\). We can show that the sheaf \( \mathcal{F} \otimes K(a_1^{\kappa_1} \cdots a_r^{\kappa_r}) \) is defined on a quotient torus \((\mathbb{C}^\times)^{r-n-1} \) of \((\mathbb{C}^\times)^r \) for a suitable choice of \( \kappa_1, \ldots, \kappa_r \in \mathbb{Q} \). The descent sheaf is called a hypergeometric sheaf. We can prove that the hypergeometric sheaf \( \mathcal{F} \) is a local system and has a pure variation of Hodge structure of weight \( n \) for an open set \( U \) in \((\mathbb{C}^\times)^{r-n-1} \). The main theorem of this paper is stated as follows.

**Main Theorem (Theorem 4 (1) in Section 10).** Let \( \mathcal{F} \) and \( \mathcal{F}' \) be hypergeometric sheaves (for the definition of hypergeometric sheaves see Section 2) on an open set \( U \) in a torus \( T \). If there exists an isomorphism \( \phi : \mathcal{F} \to \mathcal{F}' \) of variation of \( K \)-Hodge structures on \( U \), then there exists an algebraic correspondence \( Al \) and a Hodge cycle of Fermat motive \( F \) such that \( \phi = Al \cdot F \).

On the other hand, there is an analogous conjecture, Tate conjecture for families of varieties. Let \( \mathbb{F}_q \) be a finite field with \( q \)-elements and \( p \) be its characteristic. Let \( l \) be a prime number prime to \( p \) and \( K = \mathbb{Q}_l \). Let \( S \) be a smooth variety over \( \mathbb{F}_q \) and \( \mathcal{X} \to S \) and \( \mathcal{Y} \to S \) be proper smooth varieties over \( S \). Assume that a cyclic group \( \mu_d \subset \mathbb{F}_q^\times \) acts on \( \mathcal{X} \) and \( \mathcal{Y} \) over \( S \). Then for a character \( \chi \) and \( \chi' \) of \( \mu_d \), we can consider the local system of \( K \) vector spaces \( R^if_*K(\chi) \) and \( R^ig_*(\chi') \) in etale topology. For a proper flat family \( \mathcal{Z} \) of \( \mathcal{X} \times_S \mathcal{Y} \), we can define a homomorphism \( R^if_*K(\chi) \to R^ig_*(\chi') \) as an etale sheaf on \( S \) in the same way. Therefore a \( K \)-algebraic correspondence induces a homomorphism \( R^if_*K(\chi) \to R^ig_*(\chi') \) of etale sheaves. Tate conjecture for families of varieties is formulated as follows.

**Tate Conjecture for families of varieties.** A morphism \( R^if_*K(\chi) \to R^ig_*(\chi') \) of etale sheaves is induced by a \( K \)-algebraic correspondence.

Concerning Tate conjecture, we have the following similar result.

**Main Theorem (Theorem 4 (2) in Section 10).** Let \( \mathcal{F} \) and \( \mathcal{F}' \) be hypergeometric sheaves (for the definition of hypergeometric sheaves see Section 2) on an open set \( U \) in a torus \( T \). If there exists an isomorphism \( \phi : \mathcal{F} \to \mathcal{F}' \) of etale sheaves on \( U \), then there exists an algebraic correspondence \( Al \) and a Tate cycle of Fermat motive \( F \) such that \( \phi = Al \cdot F \).

To prove the main theorem, we recover hypergeometric data from hypergeometric sheaves. In this direction, the theory of Mellin transform is useful. To apply the framework of Mellin transform, we must overcome the following points.

1. Define Mellin transform as an invariant of local systems.
2. Find enough informations from Mellin transform.
3. Construct enough algebraic correspondences to generate the equivalence relation arising from Mellin transform.

We will discuss these points for \( k = \mathbb{C} \) and \( k = \mathbb{F}_q \). The case \( k = \mathbb{C} \).

1. For a local system on an open set, one can naturally extend to a perverse sheaf. They have a structure of cohomological mixed Hodge complex. Using this, the Mellin transform is an invariant of local systems. The argument is a combination of \([\text{BBD}]\) and \([\text{D}]\).
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(2) Hodge numbers provide us enough ample informations to detect hypergeometric data up to some equivalence. This part is quite easy. (See Section 9 Proposition 9.1(1).)

(3) We construct two kinds of algebraic correspondences; one is constant correspondence and the other is multiplicative correspondence. In [GKZ], they assume the condition $\sum_{i=1}^r \kappa_i \neq 0$ which is not preserved by constant algebraic correspondences. We study in more general setting. As a consequence, we study all together 4 types of constant correspondences. It is unified in confluent hypergeometric functions. We do not take this approach since they are not algebraic.

The case $k = \mathbb{F}_q$.

(1) We should also extend the local system to torus by using perverse sheaf. Except the argument for cohomological mixed Hodge complex, it is similar to the case $k = \mathbb{C}$.

(2) We use the $p$-adic order of the Frobenius on Mellin transform. It is enough ample to detect the equivalence class for hypergeometric data. In this case, $p$ power Frobenius action on characters gives another relation. A typical example is a relation $g(\chi^p, \psi) = g(\chi, \psi)$. (see Section 8 for the notation of Gaussian sum.) Existence of this equality give rise to an subtle arithmetic relations between $p$-adic value of cohomological Mellin transforms. To control these relations, we need Proposition 9.1.(2). We prove Proposition 9.2(2) in Appendix. By the existence of this equivalence, it is not natural to assume $\sum_i l_i = 0$, which is necessary for the case $k = \mathbb{C}$.

(3) In the case $k = \mathbb{F}_q$, two kinds of correspondences; constant correspondence and multiplicative correspondence is not sufficient. We need Frobenius correspondence for hypergeometric data.

Let us explain the contents of this paper. In Section 2, we introduce hypergeometric data and hypergeometric sheaves. Hypergeometric data is a triple $D = (R, \{l_i\}, \{\kappa_i\})$ where $R$ is a finitely generated $\mathbb{Z}$-module, $l_i$ is a homomorphism $l_i : R \to \mathbb{Z}$ and $\kappa_i$ is an element of $I_m(\bar{k}, K) = \text{Hom}(\mu_m(\bar{k}), \mu_m(K))$. For a hypergeometric data $D$, we define a constructible sheaf $\mathcal{G}(D)$ (resp. $\mathcal{G}(D, \psi)$) on $\mathbb{T}(R) = \text{Spec}(k[R])$ if $k = \mathbb{C}$ (resp. $k = \mathbb{F}_q$). The sheaves $\mathcal{G}(D)$ and $\mathcal{G}(D, \psi)$ are called hypergeometric sheaves. At the end of Section 2, we prove the perversity and irreducibility of hypergeometric sheaves. The proof is analogous to that in [GKZ]. We review the definition of variation of $K$-Hodge structures in Section 3. We introduce a variation of $K$-Hodge structures on the restriction of hypergeometric sheaves $\mathcal{G}(D)$ to an open set $U$ in $\mathbb{T}(a)$. We define a multiplicative equivalence in the set of hypergeometric data in Section 4. We show that the resonance condition is stable under this equivalence relation. Multiplicative relation is related to Gauss multiplication formula (See [T].):

$$\Gamma(ns) = n^{ns-1} \prod_{i=0}^{n-1} \Gamma(s + \frac{i}{n}) \frac{\prod_{i=1}^{n-1} \Gamma(\frac{i}{n})}{\prod_{i=0}^{n-1} \Gamma(\frac{i}{n})}.$$

This relation produces non-trivial relation between hypergeometric sheaves. In Section 6, these relations are proved to be induced by an algebraic correspondence. Before constructing these algebraic correspondences, we introduce constant equivalence relation and related algebraic correspondences in Section 5. The algebraic
correspondence for constant equivalence has different aspects for \( k = \mathbb{C} \) and \( k = \mathbb{F}_q \).

In the case \( k = \mathbb{C} \), we should discuss algebraic correspondence separately according to \( \sum_i \kappa_i \neq 0 \) or \( \sum_i \kappa_i = 0 \). Here we use the irreducibility of \( \mathcal{G}(D) \) and \( \mathcal{G}(D, \psi) \). In Section 7, we give algebraic correspondences related to Frobenius action on characters. Since this equivalence relation does not respect the condition, \( \sum_{i=1}^r l_i = 0 \), we do not assume this condition if \( k = \mathbb{F}_q \). In Section 8, we define the cohomological Mellin transform. The key proposition, Proposition 9.1 is proved in Section 9 (\( k = \mathbb{C} \)) and Appendix (\( k = \mathbb{F}_q \)). The proof for the case \( k = \mathbb{F}_q \) is inspired by the work of [KO] and [A].

Hypergeometric sheaves for the case \( k = \mathbb{C} \) and \( k = \mathbb{F}_q \) have similar aspects in many points. Nevertheless, we do not work with the same category. If \( k = \mathbb{C} \), we treat hypergeometric sheaves with only regular singularities. On the other hand, if \( k = \mathbb{F}_q \), we treat hypergeometric sheaves with wild ramification.

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§1. Notations

Let \( k \) be the complex number filed \( \mathbb{C} \) or a finite field \( \mathbb{F}_q \) of \( q \) elements and \( p \) be the characteristic of \( k \). Let \( V \) be a vector space over \( \mathbb{Q} \). A free \( \mathbb{Z} \) module \( L \) of \( V \) is called a lattice if \( L \otimes \mathbb{Q} = V \). A finitely generated free \( \mathbb{Z} \) module is also called a lattice. For a lattice \( L \) of \( V \), \( T(L) \) denotes a torus defined by \( \text{Spec}(k[L]) \), where \( k[L] \) is a group ring over \( L \). Let \( \mathbb{Z}_{(p)} \) be the localization of \( \mathbb{Z} \) at \( p \). For an element \( v \in L \otimes \mathbb{Z}(p) \), the corresponding Kummer character on \( T(L) \) is denoted by \( \kappa(T; v) \) or \( \kappa(v) \) for short. \( K \) denotes the field \( \cup_N \mathbb{Q}(\mu_N) \) where \( \mu_N \) denotes the group of roots of \( N \)-th power of unity if \( k = \mathbb{C} \) and \( \mathbb{Q}_l \) if \( k = \mathbb{F}_q \), where \( l \) is a prime number prime to \( p \). The rank 1 local system on \( T(L) \) corresponding to the Kummer character \( \kappa(T(L), v) \) is denoted by \( K(T(L); v) \) or \( K(v) \). The sheaf \( K(T(L); v) \) is a trivial sheaf of \( T(L) \) if \( v \in L \) and for any sublattice \( M \subset L \), \( K(T(L); v) \) has a canonical descent data over \( T(M) \). For a subvariety \( Z \) of \( T(L) \), the restriction of \( K(T(L); v) \) to \( Z \) is denoted by \( K(Z, v) \). For a set of variable, for example \( x = (x_0, \ldots, x_n) \), \( T(x) \) and \( A(x) \) denotes a torus and an affine space \( \text{Spec} k[x_0^\pm, \ldots, x_n^\pm] \) and \( \text{Spec} k[x_0, \ldots, x_n] \) whose coordinates are given by the corresponding variables.

For a real number \( \alpha \), \( < \alpha > \) denotes an element in \( \mathbb{R} \) such that \( < \alpha > \equiv \alpha \pmod{\mathbb{Z}} \) and \( 0 \leq \alpha < 1 \). For fields \( E, F \), \( I_m(E, F) \) denotes the set \( \text{Hom}(\mu_m(E), \mu_m(F)) \) and \( I(E, F) \) denotes the inductive limit of \( I_m(E, F) \) for all \( m \). Then \( I(\mathbb{C}, \mathbb{C}) \) is canonically isomorphic to \( \mathbb{Q}/\mathbb{Z} \) and \( I(\mathbb{F}_q, \mathbb{Q}_l) \) is isomorphic to \( \mathbb{Z}_{(p)}/\mathbb{Z} \) if we fix a extension \( \bar{p} \) of the ideal \( (p) \) to the algebraic closure of \( \mathbb{Q} \) in \( \mathbb{Q}_l \).

§2 Hypergeometric data and hypergeometric sheaves

§2.1 Hypergeometric data

In this section, we define hypergeometric sheaves associated to a data \( (R, \{ l_i \}, \{ \kappa_i \}) \) called hypergeometric data. Let \( n, r \) be positive integers such that \( r - n - 1 > 0 \) and \( R \) be a lattice of rank \( r - n - 1 \). A set \( \{ l_i \}_{i=1}^r \) of homomorphism \( l_i : R \to \mathbb{Z} \) is called primitive if the induced homomorphism \( l = (l_1, \ldots, l_r) \) is a primitive injection, i.e. \( R \) is identified with a submodule of \( \mathbb{Z}^r \) via the homomorphism \( l \) whose cokernel is torsion free. The homomorphism is called separated if
for all $i \neq j$, there exists no $r \in R$ such that $l_i(r) = 1, l_j(r) = -1$ and $l_l(r) = 0$ for all $l \neq i, j$.

**Definition.** Let $k$ be $\mathbb{C}$ or $\mathbb{F}_q$. The triple $D = (R, \{l_i\}_{i=1}, \{\kappa_i\}_{i=1}, \{\kappa_i\}_{i=1}, \{\kappa_i\}_{i=1}, \psi)$ is called a hypergeometric data if

1. A set of homomorphism $l_i : R \to \mathbb{Z}$ $(i = 1, \ldots, r)$ is primitive and separated.
2. If $k = \mathbb{C}$, we assume $\sum l_i = 0$.
3. $\kappa_i \in I(k, K)$ $(i = 1, \ldots, r)$.

In this section we construct a sheaf $G(R, \{l_i\}, \{\kappa_i\}) = G(D)$ or $G(R, \{l_i\}, \{\kappa_i\}) = G(D, \psi)$ called a hypergeometric sheaf on a torus $T(R)$. Let $L$ be the cokernel of the morphism $1 = (l_1, \ldots, l_r) : R \to \mathbb{Z}^r$.

$$0 \to R \to \mathbb{Z}^r \to L \to 0.$$ 

The element in $k[\mathbb{Z}^r]$ corresponding to the $i$-th canonical generator is denoted by $a_i$. The natural homomorphism $\mathbb{Z}^r \to L$ is denoted by $q$. The image $q(e_i)$ of $e_i = (0, \ldots, 1, \ldots, 0)$ in $L$ is denoted by $\omega_i \in L$. For an element $v \in L$, the corresponding element in $k[L]$ is denoted by $x^v$. The image $(q \otimes 1)(\kappa)$ of $\kappa = (\kappa_i)_{i} \in \mathbb{Z}^r \otimes K$ under the natural homomorphism $q \otimes 1 : \mathbb{Z}^r \otimes K \to L \otimes K$ is denoted by $\alpha$.

§2.2 Hypergeometric sheaf for $k = \mathbb{C}$.

By the condition (2) of hypergeometric data, the homomorphism given by the summation $\Sigma : \mathbb{Z}^r \to \mathbb{Z} ; (u_1, \ldots, u_r) \mapsto \sum u_i$ factors through $L$. The induced map from $L$ to $\mathbb{Z}$ is denoted by $h$. The kernel of $h$ is denoted by $L_0$. Then $C[L_0]$ can be identified with a subring of $C[L]$. The element in $C[\mathbb{Z}]$ and $C[L]$, corresponding to 1 and $v \in L$ is denoted by $x_0$ and $x^v$ respectively.

**Proposition 2.1.** Let $v \in L$ such that $h(v) > 0$ and $a_1, \ldots, a_r$ be elements in $C^r$. Then for a Laurent polynomial $f = \sum a_i x^{\omega_i}$, $(f)^{h(v)} x^{-v}$ is an element in $C[L]$.

**Proof.** Since $h(v) > 0$, it is an element in $C[L]$. The subring $C[L_0]$ in $C[L]$ is characterized as follows. For a $C$-algebra homomorphism $\mu : C[\mathbb{Z}] \to C$, the map $x^w \mapsto \mu(x_0)^{h(v)} x^w$ defines a ring endomorphism $T_\mu : C[L] \to C[L]$ of $C[L]$. The subring $C[L_0]$ is characterized by

$$\{ f \in C[L] \mid T_\mu(f) = f \text{ for all } \mu : C[\mathbb{Z}] \to C \}.$$

Since

$$T_\mu(f^{h(v)} x^{-v}) = (\sum a_i T_\mu(x^{\omega_i}))^{h(v)} \mu(x_0)^{-h(v)} x^{-v}$$

$$= (\sum a_i \mu(x_0)^{h(v)} x^{\omega_i})^{h(v)} \mu(x_0)^{-h(v)} x^{-v}$$

$$= f^{h(v)} x^{-v},$$

we obtain the proposition.
Corollary. Let $f$ be as in Proposition 2.1. The ideal $(f)$ in $\mathbf{C}[L]$ is defined over $\mathbf{C}[L]$, i.e. if we put $I_f = (f) \cup \mathbf{C}[L_0]$, then $I_f \mathbf{C}[L] = (f)$.

Now we consider the relative situation. Let $a$ be a system of variables $a = (a_1, \ldots, a_r)$. Let $f$ be a polynomial in $\mathbf{C}[a_i^{\pm}] \otimes \mathbf{C}[L]$ defined by

$$f = \sum_{i=1}^{r} a_i x^{\omega_i}.$$  

**Case** $h(\alpha) \neq 0$

The subvariety of $\mathbf{T}(a) \times \mathbf{T}(L)$ defined by $\{f = 1\}$ is denote by $\mathcal{X}$ and the composite $\mathcal{X} \to \mathbf{T}(a)$ of the natural inclusion $\mathcal{X} \to \mathbf{T}(a) \times \mathbf{T}(L)$ and the first projection $\mathbf{T}(a) \times \mathbf{T}(L) \to \mathbf{T}(a)$ is denoted by $\varphi$. The element $x^\alpha$ defines a rank 1 local systems $K(\mathbf{T}(L); x^\alpha)$ and $K(\mathcal{X}; x^\alpha)$ of $K$-vector spaces over $\mathbf{T}(L)$ and $\mathcal{X}$. The $i$-th higher direct image sheaf of $K(\mathcal{X}; x^\alpha)$ by $\varphi$ is denoted by $R^i \varphi_* K(\mathcal{X}; x^\alpha)$.

**Case** $h(\alpha) = 0$

By the Corollary to Proposition 2.1, the variety $\mathcal{X}' = \{(x, a) \in \mathbf{T}(L) \times \mathbf{T}(a) \mid f(a, x) = 0\}$ is defined over $\mathbf{T}(L_0)$, i.e. there is unique subvariety of $\mathbf{T}(L_0) \times \mathbf{T}(a)$ such that $\mathcal{X}'$ is isomorphic to the pull back of $\mathcal{X}$. Under the condition $h(\alpha) = 0$, we show that the rank 1 local system $K(\prod_{i=1}^{r} (x^\omega_i)^{\kappa_i})$ corresponding to the Kummer character $\kappa(\mathbf{T}(L) \times \mathbf{T}(a); \prod_{i=1}^{r} (x^\omega_i)^{\kappa_i})$ is naturally descent to $\mathbf{T}(L_0) \times \mathbf{T}(a)$ via the the morphism $\mathbf{T}(L) \times \mathbf{T}(a) \to \mathbf{T}(L_0) \times \mathbf{T}(a)$. To prove this, it is enough to show that the element $\prod_{i=1}^{r} (x^\omega_i)^{\kappa_i}$ is invariant under the automorphism $T_\mu$ for any algebra homomorphism $\mu : \mathbf{C}[\mathbf{Z}] \to \mathbf{C}$. Since

$$T_\mu \prod_{i=1}^{r} (x^\omega_i)^{\kappa_i} = \prod_{i=1}^{r} (\mu(x_0) x^\omega_i)^{\kappa_i},$$

we have the claim. The restriction of the descent sheaf to the variety $\mathcal{X}$ is denoted by $K(\mathcal{X}; x^\alpha)$. Let $\varphi$ be the composite of the closed immersion $\mathcal{X} \to \mathbf{T}(L_0) \times \mathbf{T}(a)$ and the second projection $\mathbf{T}(L_0) \times \mathbf{T}(a) \to \mathbf{T}(a)$. The $i$-th higher direct image of $K(\mathcal{X}; x^\alpha)$ is denoted by $R^i \varphi_* K(\mathcal{X}; x^\alpha)$.

The element $a^\kappa = (a_i^{\kappa_i}, i \in \mathbf{Z}^r \otimes I(\mathbf{C}, K)$ defines a rank 1 local system of $K$-vector spaces $K(\mathbf{T}(a); a^\kappa)$. We descend the the sheaf $R^i \varphi_* K(\mathcal{X}; x^\alpha) \otimes K(\mathbf{T}(a); a^\kappa)$ to $\mathbf{T}(R)$ via the natural morphism $\mathbf{T}(a) \to \mathbf{T}(R)$ arising from the homomorphism $1 : R \to \mathbf{Z}^r$. Since we have the natural exact sequence

$$0 \longrightarrow R \longrightarrow \mathbf{Z}^r \oplus \mathbf{Z}^r \longrightarrow \mathbf{Z}^r \oplus L \longrightarrow 0$$

$$r \quad \mapsto \quad (r, -r)
(a, b) \quad \mapsto \quad (a + b, q(b))$$

The fiber product $\mathbf{T}(a) \times \mathbf{T}(a)$ is identified with $\mathbf{T}(a) \times \mathbf{T}(L)$ and the following.
Here $m$ is given by the homomorphism of the modules

$$\mathbb{Z}^r \to \mathbb{Z}^r \oplus L; u \mapsto (u, q(u)).$$

**Proposition 2.2.** Let $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}(D) = R^n \varphi_* K(\mathcal{X}; x^\alpha) \otimes K(T(a); a^\kappa)$ or $R^{n-1} \varphi_* K(\mathcal{X}; x^\alpha) \otimes K(T(a); a^\kappa)$ according to $\sum_{i=1}^r \kappa_i \neq 0$ or $\sum_{i=1}^r \kappa_i = 0$. Then there exists a natural isomorphism $\phi$

$$\phi : m^*(\tilde{\mathcal{G}}) \simeq pr_1^*(\tilde{\mathcal{G}})$$

which satisfies the cocycle condition for descent data via the isomorphism $\iota$.

**Proof.** First we consider base change diagrams of $\mathcal{X}$:

$$\begin{array}{ccc}
\mathcal{X}_1 & \longrightarrow & \mathcal{X} & \longleftarrow & \mathcal{X}_2 \\
\downarrow & & \downarrow & & \downarrow \\
T(a) \times T(L) & \longrightarrow & T(a) & \longleftarrow & T(a) \times T(L)
\end{array}$$

Here $pr_1^*, m^*$ are the ring homomorphisms

$$\mathbb{C}[a_i^\pm] \to \mathbb{C}[a_i^\pm, y^v]_{v \in L}$$

given by $pr_1^*(a_i) = a_i$ and $m^*(a_i) = a_i y^{\omega_i}$ with a new set of variable $y^v$ ($v \in L$). If $\sum_{i=1}^r \kappa_i \neq 0$ (resp. $\sum_{i=1}^r \kappa_i = 0$), the defining equation of $\mathcal{X}_1$ and $\mathcal{X}_2$ is given by

$$\begin{align*}
\mathcal{X}_1 : & \sum_{i=1}^r a_i x^{\omega_i} = 1, \quad (\text{resp. } \mathcal{X}_1 : \sum_{i=1}^r a_i x^{\omega_i} = 0) \\
\mathcal{X}_2 : & \sum_{i=1}^r a_i y^{\omega_i} x^{\omega_i} = 1, \quad (\text{resp. } \mathcal{X}_2 : \sum_{i=1}^r a_i y^{\omega_i - \omega_1} x^{\omega_i} = 0).
\end{align*}$$

Therefore the automorphisms of $\mathbb{C}[a_i^\pm, x^v, y^v]_{v \in L}$ and $\mathbb{C}[a_i^\pm, x^{v'}, y^{v'}]_{v' \in L, v' \in L}$ given by

$$\begin{align*}
\phi^* : a_i & \mapsto a_i, x^v \mapsto x^v y^v, y^v \mapsto y^v \quad (\sum_{i=1}^r \kappa_i \neq 0) \\
\phi'^* : a_i & \mapsto a_i, x^{v'} \mapsto x^{v'} y^{v'}, y^{v'} \mapsto y^{v'} \quad (\sum_{i=1}^r \kappa_i = 0)
\end{align*}$$

induce isomorphisms $\mathcal{X}_2 \to \mathcal{X}_1$. The base change of Kummer character associated to $a_1^{\kappa_1} \cdots a_r^{\kappa_r} x^\alpha \in (\mathbb{Z}^r \oplus L) \otimes I(\mathcal{C}, K)$ on $\mathcal{X}$ to $\mathcal{X}_1$ and $\mathcal{X}_2$ are given by $a_1^{\kappa_1} \cdots a_r^{\kappa_r} x^\alpha$ and

$$\begin{align*}
m^*(a_1^{\kappa_1} \cdots a_r^{\kappa_r}) x^\alpha & = (\prod_{i=1}^r a_i^{\kappa_i} y^{\kappa_i \omega_i}) x^\alpha \\
& = (\prod_{i=1}^r a_i^{\kappa_i}) y^\alpha x^\alpha
\end{align*}$$
respectively. This two Kummer character on $\mathcal{X}_1$ and $\mathcal{X}_2$ corresponds to each other via the isomorphism $\phi$ and $\phi'$ constructed as before.

Definition. The sheaf obtained by descending $\tilde{G}(D)$ to $T(R)$ is denoted by $\mathcal{G}(D)$. The sheaf $\mathcal{G}(D)$ is called hypergeometric sheaf for a hypergeometric data $D = (R_i, \{l_i\}_i, \{\kappa_i\}_i)$. Note that, $\tilde{G}(D)$ is the equal to $R^n \varphi_* K(\mathcal{X}; \prod_{i=1}^r (a_i x^\omega_i)^\kappa_i)$ or $R^{n-1} \varphi_* K(\mathcal{X}; \prod_{i=1}^r (a_i x^\omega_i)^\kappa_i)$ according to $\sum_{i=1}^r \kappa_i \neq 0$ or $\sum_{i=1}^r \kappa_i = 0$.

We compare the sheaf $\mathcal{G}(D)$ and the sheaf associated to Gel’fand-Kapranov-Zelevinski (=GKZ for short) hypergeometric functions by choosing a suitable base of $L$ under the assumption $\sum_{i=1}^r \kappa_i \neq 0$. Let $s : \mathbb{Z} \to L$ be a section of $h$ and take a base of $L_0 \simeq \oplus_{i=1}^n \mathbb{Z} v_i$. Then the ring $C[L_0]$ and $C[L]$ can be identified with Laurent polynomial rings $C[x_1^\pm, \ldots, x_n^\pm]$ and $C[x_0, x_1^\pm, \ldots, x_n^\pm]$, where $x_i$ corresponds the element $v_i$ of $L$ and $x_0$ corresponds to the image of $1 \in \mathbb{Z}$ under the section $s$. Using the base of $L$, the element $\omega_i$ is expressed as $(1, \bar{\omega}_i)$ with $\bar{\omega}_i \in \mathbb{Z}^n$. Then by the definition of $L$, it is generated by $\omega_i$. We introduce a system of variables $\bar{x} = (x_1, \ldots, x_n)$ and use the common notation for multiple index: $\bar{x}^\nu = x_1^{\nu_1} \cdots x_n^{\nu_n}$ for $\nu \in \mathbb{Z}^n$. Then $x^{\omega_i} = x_0 x_i^{\bar{\omega}_i}$ and the defining equation of $\mathcal{X}$ is $x_0 f_0 (a_1, \ldots, a_r, x_1, \ldots, x_n) = 1$, where $f_0 = \sum_{i=1}^r a_i \bar{x}^{\bar{\omega}_i}$. If we write $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in L \otimes I(C, K) \simeq \mathbb{Z}^{n+1} \otimes I(C, K)$, then the corresponding Kummer character on $\mathcal{X}$ is $x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n} = f_0^{-\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $\alpha_0 = h(\alpha) = \sum_{i=1}^r \kappa_i$. Let $\mathcal{U}$ be the open set of $T(x) \times T(a)$ defined by $\mathcal{U} = \{(x, a) \in T(x) \times T(a) \mid f_0 (a, x) \neq 0\}$, where $x = (x_1, \ldots, x_n)$. The rank 1 local system corresponding to the Kummer character $f_0^{-\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is denoted by $K_{\mathcal{U}}(x^\alpha)$. Then $R^i \varphi_* K(x^\alpha)$ is identified with $R^i \varphi_* K_{\mathcal{U}}(x^\alpha)$, where $\varphi : \mathcal{U} \to T(a)$ is the composite of the open immersion to $T(x) \times T(a)$ and the second projection. This is nothing but the local system defined by Gel’fand-Kapranov-Zelevinski. Note that is it is independent of the choice of the base $L$ given as above.

§2.3 Hypergeometric sheaf for $k = \mathbb{F}_q$

In this case, we use an extra parameter $t$. Let $\mathcal{X}$ be the subvariety of $T(x) \times A(t) \times T(a)$ defined by:

$$\mathcal{X} = \{(x, t, a) \in T(x) \times A(t) \times T(a) \mid f(a, x) = t\},$$

where $f(a, x) = \sum_{i=1}^r a_i x^{\omega_i}$. Let $\varphi : \mathcal{X} \to T(a)$ be the composite of the natural immersion $\mathcal{X} \to T(x) \times A(t) \times T(a)$ and the third projection $T(x) \times A(t) \times T(a) \to T(a)$. Let $\psi$ be a non-trivial additive character $\psi : \mathbb{F}_p \to \bar{\mathbb{Q}}$. Let $\pi : A(\tau) \to A(t)$ be the Artin-Shreier covering defined by $t = \tau^p - \tau$. The character $\psi$ of the covering transformation group $\mathbb{F}_p$ defines a rank 1 local system $L_\psi$ of $A(t)$. This local system is called the Lang sheaf. The pull back of $L_\psi$ by the projection to the $t$-line is denoted by $L_\psi(t)$. We can prove the following proposition in the same way.

Proposition 2.3. The sheaf $R^n \varphi_* (K(\mathcal{X}; x^\alpha a^\kappa) \otimes L_\psi(t)) = R^n \varphi_* (K(\mathcal{X}; x^\alpha) \otimes L_\psi(t)) \otimes K(T(a); a^\kappa)$ on $T(a)$ can be canonically descent to a constructible sheaf $\mathcal{G}(D, \psi)$ on $T(R)$.

Definition. The sheaf $\mathcal{G}(D, \psi)$ is called the hypergeometric sheaf for a hypergeometric data $D$ if $k = \mathbb{F}_q$. 
§2.4 Properties of hypergeometric sheaves

Now we recall several properties on $R^i\varphi_*K(\mathcal{X};x^\alpha)$. Let $\Delta$ be the convex hull of $\omega_i$ and 0 and $C(\Delta)$ be the convex cone generated by $\Delta$ in $L \otimes \mathbb{R}$. For a codimension 1 face $\sigma$ of $C(\Delta)$, $H_\sigma$ denotes the linear hull of $\sigma$ and $h_\sigma$ denotes a primitive linear form defining $H_\sigma$. An element $\alpha \in L \otimes I(k, K)$ is called non-resonant if $h_\alpha(\alpha)$ is not zero. (Note that if $\Delta$ contains 0 as interior, there is no codimension 1 face in $C(\Delta)$). For a face $\sigma$ of $\Delta$, we define $f_\sigma$ by

$$f_\sigma = \sum_{\omega_i \in \sigma} a_i x^{\omega_i}.$$

Let $a^{(0)} \in T(a)$. The polynomial $f^{(0)}(x) = f(a^{(0)}, x)$ is said to be non-degenerate if the variety $\{ f^{(0)} = 0 \}$ is smooth in $T[L]$ for all the faces $\sigma$ of $\Delta$. A point $a^{(0)}$ is called non-degenerated if $f^{(0)}$ is non-degenerate. The open set consisting the non-degenerate point in $T(a)$ is denoted by $U_a$. It is easy to see that for a point $r^{(0)} \in T(R)$, all the points in $I^{-1}(r^{(0)})$ is non-degenerated if a point $a^{(0)}$ in $I^{-1}(r^{(0)})$ is non-degenerate. A point $r^{(0)}$ in $T(R)$ is called non-degenerate if there exists a non-degenerate point $a^{(0)}$ in $I^{-1}(r^{(0)})$. The open set of $T(R)$ consisting of non-degenerate point is denoted by $U_R$.

For $k = \mathbb{C}$ or $\mathbb{F}$, we have the following proposition. The related statement for $D$-module, it is proved in [GKZ].

**Proposition 2.4.**

1. Let $k = \mathbb{C}$. If $\alpha$ is non-resonant, then the restriction of $R^n\varphi_*K(\mathcal{X};x^\alpha)$ (resp. $R^{n-1}\varphi_*K(\mathcal{X};x^\alpha)$) to the open set $U_a$ is a local system of $K$ vector spaces if $\sum_{i=1}^r \kappa_i \neq 0$ (resp. $\sum_{i=1}^r \kappa_i = 0$).

2. Let $k = \mathbb{F}_q$. If $\alpha$ is non-resonant, then the restriction of $R^n\varphi_*K(\mathcal{X};x^\alpha) \otimes L_\psi(t)$ to the open set $U_a$ is a local system of $K$ vector spaces.

**Proposition 2.5.** Suppose that $\alpha$ is non-resonant.

1. Let $k = \mathbb{C}$. The natural homomorphism

$$R^i \varphi_1 K(\mathcal{X};x^\alpha) \rightarrow R^i \varphi_* K(\mathcal{X};x^\alpha)$$

is an isomorphism. As a consequence, $R^i \varphi_* K(\mathcal{X};x^\alpha)$ is 0 unless $i = n$ (resp. $i = n - 1$) if $\sum_{i=1}^r \kappa_i \neq 0$ (resp. $\sum_{i=1}^r \kappa_i = 0$).

2. Let $k = \mathbb{F}_q$. The natural homomorphism

$$R^i \varphi_1 K(\mathcal{X};x^\alpha) \otimes L_\psi(t) \rightarrow R^i \varphi_* K(\mathcal{X};x^\alpha) \otimes L_\psi(t)$$

is an isomorphism. As a consequence, $R^i \varphi_* K(\mathcal{X};x^\alpha) \otimes L_\psi(t)$ is 0 unless $i = n$.

3. Let $k = \mathbb{C}$. The constructible sheaf $R^n\varphi_*K(\mathcal{X};x^\alpha)$ (resp. $R^{n-1}\varphi_*K(\mathcal{X};x^\alpha)$) is an irreducible perverse sheaf. As a consequence, the restriction of $R^n\varphi_*K(\mathcal{X};x^\alpha)$ (resp. $R^{n-1}\varphi_*K(\mathcal{X};x^\alpha)$) to $U_a$ is an irreducible local system of $K$-vector spaces.

4. Let $k = \mathbb{F}_q$. The constructible sheaf $R^n\varphi_*K(\mathcal{X};x^\alpha) \otimes L_\psi(t)$ is an irreducible perverse sheaf. As a consequence, the restriction of $R^n\varphi_*K(\mathcal{X};x^\alpha) \otimes L_\psi(t)$ to $U_a$ is an irreducible local system of $K$-vector spaces.
Proof. We use a method of Fourier transform. First we prove the following Lemma

First we prove the propositions for $k = F_q$. Let $y^v (v \in L)$ and $b = (b_1, \ldots, b_r)$ be systems of variables. Let $j : T(y) \to A(b)$ be the morphism defined by $q : N^r \to L$.

Lemma 2.6. Under the resonance condition, $R^i j_* K(T(y); y^\alpha) \to R^i j_* K(T(y); y^\alpha)$ is an isomorphism.

We use toric geometry. Let $\Delta_0$ be the image of the semi-group homomorphism $N^r \to L$. Then $A_{\Delta_0} = \text{Spec} k[\Delta_0]$ is the closure of the image of $j$. Let $\Delta$ be the intersection of the convex hull $\Delta_R$ of $q(e_i) = \omega_i$ in $L \otimes \mathbb{R}$ and $L$. Then $A_{\Delta} = \text{Spec} k[\Delta]$ is the normalization of $A_{\Delta_0}$. Let $\tilde{F}$ be a regular triangulation of the dual fan $F = \{ C_b \mid b \in \Delta_R \}$ of $\Delta_R$, where $C_b = \{ x \in L^* \otimes \mathbb{R} \mid (x, b' - b) \geq 0 \}$ for all $b' \in \Delta_R$. Then the toric variety $X(\tilde{F})$ is a smooth variety and the natural morphism $p : X(\tilde{F}) \to A_\Delta$ is proper. Therefore the composite $\pi : X(\tilde{F}) \to A(b)$ is proper. Let $\tilde{j} : T(y) \to X(\tilde{F})$ be the natural open immersion and $D$ be the complement of $T(y)$ in $X(\tilde{F})$.

$$T(y) \xrightarrow{\tilde{j}} X(\tilde{F}) \xrightarrow{p} A_\Delta \xrightarrow{\pi} A(b)$$

To prove the lemma it is enough to prove $R p_! (C) = 0$, where $C$ is the mapping cone; $C(R j_! K(T(y); y^\alpha) \to R j_* K(T(y); y^\alpha))$. Therefore, it is enough to prove that

$$H^i_c(p^{-1}(x), R^k j_* K(T(y); y^\alpha) |_{D \cap p^{-1}(x)}) = 0$$

for all $i, k$ and $x \in A_{\Delta_0}$. Let $D = \bigsqcup_{\beta \in \tilde{A}} S_\beta$ be the decomposition of $D$ by toric strata, where $\tilde{A}$ denotes a set of positive dimensional cone in $\tilde{F}$. By the long exact sequence for cohomologies with compact support, to show (1), it is enough to prove that

$$H^i(p^{-1}(x) \cap S_\beta, R^k j_* K(T(y); y^\alpha) |_{S_\beta \cap p^{-1}}) = 0.$$  

Let $L^*_\alpha$ be the intersection of $\mathbb{R}$-vector space generated by $\alpha$ and the lattice $L^*$. The element $\alpha \in L \otimes I(k, K)$ defines a homomorphism $\alpha_\beta : L^*_\alpha \to I(k, K)$. It is easy to see that

$$R^i j_* K(T(y); y^\alpha) |_{S_\beta} = \begin{cases} (\wedge^i K^{\dim(\beta)}(-1)) \otimes K(S_\beta; y^\alpha) & \text{if } \alpha_\beta = 0 \\ 0 & \text{if } \alpha_\beta \neq 0. \end{cases}$$

The restriction of $p$ to $S_\beta \to A_\Delta$ corresponds to the primitive embedding $L_{\beta'} \to L_\beta$, where $\beta'$ is the minimal cone in $F$ containing $\beta$ and $L_\beta$ and $L_{\beta'}$ is a primitive lattice of $L$ annihilated by all the elements in $\beta$ and $\beta'$ respectively. If $R^i j_* K(T(y); y^\alpha) |_{S_\beta}$ does not vanish, $\alpha \in L_{\beta'} \otimes I(k, K)$. On the other hand, by the resonance condition, $\alpha \notin L_{\beta'} \otimes I(k, K)$. Therefore, $K(S_\beta; y^\alpha)$ has not trivial monodromy along fibers. Therefore the equality (2) holds. This completes the proof of the lemma.

Proof of Proposition 2.5. Let $k : A(a) \times T(y) \to A(a) \times A(b)$ be the product of the inclusion given in Lemma 2.6. Let $\mathcal{F} = R j_! K(T(y); y^\alpha) = R j_* K(T(y); y^\alpha)$. Let $\mathcal{P}(b), H_0(b)$ and $k$ be the projective space compactifying $A(b)$, $\mathcal{P}(b) - A(b)$ and the open immersion $k : A(a) \times A(b) \to A(a) \times \mathcal{P}(b)$. The sheaf $K(T(y); y^\alpha)$
has trivial monodromy on some tame covering $\mathbb{P}(\tilde{b})$ of $\mathbb{P}(b)$. Since $\mathcal{L}_\psi(\sum_{i=1}^r a_ib_i)$ ramifies wildly along $H_0$, the natural homomorphism

$$\text{Rk}_1(pr_2^*\mathcal{F} \otimes \mathcal{L}_\psi(\sum_{i=1}^r a_ib_i)) \to \text{Rk}_1(pr_2^*\mathcal{F} \otimes \mathcal{L}_\psi(\sum_{i=1}^r a_ib_i))$$

is a quasi-isomorphism. On the other hand, $R^i\varphi_1K(\mathcal{X}; x^{\alpha}) \otimes \mathcal{L}_\psi(t)$ and $R^i\varphi_s K(\mathcal{X}; x^{\alpha}) \otimes \mathcal{L}_\psi(t)$ are isomorphic to $\text{R}pr_1(\text{Rk}_1(pr_2^*\mathcal{F} \otimes \mathcal{L}_\psi(\sum_{i=1}^r a_ib_i)))$ and $\text{R}pr_1(\text{Rk}_1(pr_2^*\mathcal{F} \otimes \mathcal{L}_\psi(\sum_{i=1}^r a_ib_i)))$ respectively, where $pr_1 : A(a) \times P(b) \to A(a)$. Since $pr_1$ is proper, we get the proposition.

We get Proposition 2.5 for $k = \mathbb{C}$ by reduction mod $p$. Let $D = (R, \{l_i\}, \{\kappa_i\})$ be a hypergeometric data for $k = \mathbb{C}$. By choosing an embedding $Q(\mu_d) \to Q_l$, the sheaf $R^i\varphi_\ast K(\mathcal{X}; \prod_{i=0}^n x^{\alpha}_i) \otimes Q_l$ is considered as an etale cohomology. Let $d$ be the common denominator of $\kappa_i$ ($i = 1, \ldots, r$), $p$ be a rational prime prime to $d$ and $\mathfrak{p}$ be a prime of $Q(\mu_d)$ over $p$. The residue field $\kappa(\mathfrak{p})$ at $\mathfrak{p}$ is a finite filed $F_q$. Since the variety $\mathcal{X}$ is defined over $Q(\mu_d)$, we can consider the reduction of $R^i\varphi_\ast K(\mathcal{X}; \prod_{i=0}^n x^{\alpha}_i) \otimes Q_l(\mod p)$ at $\mathfrak{p}$ using obvious model over $\mathbb{Z}[\mu_d]$. On the other hand, the data $D$ is also a hypergeometric data for $k = F_q$. The variety defining hypergeometric sheaf over $F_q$ is denoted by $\mathcal{X}(p)$ and the natural morphism $\mathcal{X}(p) \to T(a)$ is denoted by $\varphi(p)$ for $k = F_q$. We compare the sheaf $R^i\varphi_\ast K(\mathcal{X}; \prod_{i=0}^n x^{\alpha}_i) \otimes Q_l(\mod p)$ (resp. $R^i\varphi_\ast K(\mathcal{X}; \prod_{i=0}^n x^{\alpha}_i) \otimes Q_l(\mod p)$) and $R^i\varphi(p) K(\mathcal{X}(p); \prod_{i=0}^n x^{\alpha}_i) \otimes \mathcal{L}_\psi(t)$ (resp. $R^i\varphi(p) K(\mathcal{X}(p); \prod_{i=0}^n x^{\alpha}_i) \otimes \mathcal{L}_\psi(t)$). The following proposition completes the proof of Proposition 2.5 for $k = \mathbb{C}$.

**Proposition 2.7.** We have the following isomorphism

$$R^i\varphi(p)_\ast K(\mathcal{X}(p); \prod_{i=0}^n x^{\alpha}_i)(\psi) \approx \left\{ \begin{array}{ll} R^{i-1}\varphi_\ast K(\mathcal{X}; \prod_{i=0}^n x^{\alpha}_i) \otimes Q_l(\mod p) \otimes g(\alpha_0, \psi) & (\text{if} \; \alpha_0 \neq 0) \\ R^{i-2}\varphi_\ast K(\mathcal{X}; \prod_{i=0}^n x^{\alpha}_i) \otimes Q_l(-1)(\mod p) & (\text{if} \; \alpha_0 = 0) \end{array} \right.$$ 

$$R^i\varphi_1(p)_\ast K(\mathcal{X}(p); \prod_{i=0}^n x^{\alpha}_i)(\psi) \approx \left\{ \begin{array}{ll} R^{i-1}\varphi_1(\mathcal{X}; \prod_{i=0}^n x^{\alpha}_i) \otimes Q_l(\mod p) \otimes g(\alpha_0, \psi) & (\text{if} \; \alpha_0 \neq 0) \\ R^{i-2}\varphi_1(\mathcal{X}; \prod_{i=0}^n x^{\alpha}_i) \otimes Q_l(-1)(\mod p) & (\text{if} \; \alpha_0 = 0) \end{array} \right.$$ 

where $g(\alpha_0, \psi)$ is a Gal($\bar{F}/F_q$)-module whose action of Frobenius is the multiplication by the Gaussian sum

$$g(\alpha_0, \psi) = \sum_{x \in \mathbb{F}_q^\times} \chi_{\alpha_0}(x)\psi(x),$$

where $\chi_{\alpha}$ is a character of $\mathbb{F}_q$ defined by $\chi_{\alpha}(x)(\mod p) = x^{\frac{p-1}{\alpha}}$.

**Proof.** We take a base of $L \simeq \bigoplus_{i=0}^n \mathbb{Z}$ such that the 0-th projection coincides with $h$. Under this choice of base, the ring $b[L]$ is identified with $b[x^{\frac{p-1}{\alpha}}, \ldots, x^{\frac{p-1}{\alpha}}]$. Let $\alpha$
denotes the system of variable $x = (x_1, \ldots, x_n)$. Then the image of $e_i$ under the morphism $q : \mathbb{Z}_r \to L$ is denoted by $(1, \bar{\omega}_i)$. We define $f_0 = \sum_{i=1}^{r} a_i x_i \bar{\omega}_i$. Let $\mathcal{X}$ and $\mathcal{X}^{(p)}$ be the variety over $Q(p_d)$, $F_q$ and $W$ defined by

$$
\mathcal{X} = \{(x_0, x, a) \in T(x_0) \times T(x) \times T(a) \mid x_0 f_0(a, x) = 1\}
$$
$$
\mathcal{X}^{(p)} = \{(x_0, x, a, t) \in T(x_0) \times T(x) \times T(a) \times A(t) \mid x_0 f_0(a, x) = t\}
$$
$$
W = \{(x_0, t) \in T(x_0) \times A(y) \times A(t) \mid x_0 y = t\}.
$$

Consider the following fiber product;

$$
\begin{array}{ccc}
\mathcal{X}^{(p)} & \xrightarrow{F'} & W \\
\downarrow \pi' & & \downarrow \pi \\
T(a) \times T(a) & \xrightarrow{F} & A(y) \\
(a, x) & \mapsto & f_0(a, x)
\end{array}
$$

The pull back $pr_3^* \mathcal{L}_\psi$ of the Lang sheaf $\mathcal{L}_\psi$ on $A(t)$ under the morphism $W \to A(t)$ is also denoted by $\mathcal{L}_\psi$ for short. Let $K(W; x_0^{\alpha_0})$ be the rank 1 local system corresponding to the Kummer character $x_0^{\alpha_0}$ on $W$. Then $K(W; x_0^{\alpha_0}) \otimes \mathcal{L}_\psi$ is a local system on $W$. Since the morphism $F$ is smooth, we have

$$
F^* R\pi_*(K(W; x_0^{\alpha_0}) \otimes \mathcal{L}_\psi) \simeq R\pi'_*(F')^*(K(W; x_0^{\alpha_0}) \otimes \mathcal{L}_\psi)
$$

The sheaf $R^i \pi_*(K(W; x_0^{\alpha_0}) \otimes \mathcal{L}_\psi)$ and $R^i \pi_!(K(W; x_0^{\alpha_0}) \otimes \mathcal{L}_\psi)$ are computed as follows. If $\alpha_0$ is not trivial,

$$
R^i \pi_*(K(W; x_0^{\alpha_0}) \otimes \mathcal{L}_\psi) = \begin{cases}        \bar{j}_* K(y^{-\alpha_0}) \otimes g(x_0^{\alpha_0}, \psi) & \text{(if } i = 1) \\
        0 & \text{(if } i \neq 1) \end{cases}
$$
$$
R^i \pi_!(K(W; x_0^{\alpha_0}) \otimes \mathcal{L}_\psi) = \begin{cases}        \bar{i}_! K(y^{-\alpha_0}) \otimes g(x_0^{\alpha_0}, \psi) & \text{(if } i = 1) \\
        0 & \text{(if } i \neq 1) \end{cases}
$$

and if $\alpha_0$ is trivial, we have

$$
R^i \pi_* \mathcal{L}_\psi = \begin{cases} \bar{j}_! K(-1) & \text{(if } i = 1) \\
        0 & \text{(if } i \neq 1) \end{cases}
$$
$$
R^i \pi_! \mathcal{L}_\psi = \begin{cases} \bar{i}_* K(-1) & \text{(if } i = 2) \\
        0 & \text{(if } i \neq 1, 2, \text{ or } 3) \end{cases}
$$

where $\bar{j}$ and $\bar{i}$ are the natural inclusions $T(y) \to A(y)$ and $\{0\} \to A(y)$. Note that the variety $\mathcal{X}_D$ is isomorphic to an open subset $U$ of $T(x) \times T(a)$ defined by $U = \{(x, a) \in T(x) \times T(a) \mid f(a) \neq 0\}$. The natural morphism from $T(x) \times T(a)$
to $T(a)$ is denoted by $\varphi'$. If $\alpha_0 \neq 0$, then we have

$$R\varphi^*_s(K(\mathcal{X}; x_0^{\alpha_0} \prod_{i=1}^n x_i^{\alpha_i}) \otimes \mathcal{L}_\psi(t))$$

$$\simeq R\varphi'_s[(R\pi'_*(F')^*(K(W; x_0^{\alpha_0}) \otimes \mathcal{L}_\psi(t))) \otimes K(T(x) \times T(a); \prod_{i=1}^n x_i^{\alpha_i})]$$

$$\simeq R\varphi'_s[(F^*\pi_*(K(W; x_0^{\alpha_0}) \otimes \mathcal{L}_\psi(t))) \otimes K(T(x) \times T(a); \prod_{i=1}^n x_i^{\alpha_i})]$$

$$\simeq R\varphi'_s(F^*j_*K(y^{-\alpha_0}) \otimes K(T(x) \times T(a); \prod_{i=1}^n x_i^{\alpha_i}) \otimes g(x_0^{\alpha_0}, \psi)[-1]$$

$$\simeq R\varphi_s(K(\mathcal{X}; x_0^{\alpha_0} \prod_{i=1}^n x_i^{\alpha_i}) \otimes g(x_0^{\alpha_0}, \psi)[-1].$$

Similarly, we have

$$R\varphi^*_1(K(\mathcal{X}; x_0^{\alpha_0} \prod_{i=1}^n x_i^{\alpha_i}) \otimes \mathcal{L}_\psi(t) \simeq R\varphi_1(K(\mathcal{X}; x_0^{\alpha_0} \prod_{i=1}^n x_i^{\alpha_i}) \otimes g(x_0^{\alpha_0}, \psi)[-1].$$

If $\alpha_0 = 0$, by using exact sequence and triangle

$$0 \to j_!j^*K(T(x) \times T(a); \prod_{i=1}^n x_i^{\alpha_i}) \to K(T(x) \times T(a); \prod_{i=1}^n x_i^{\alpha_i})$$

$$\to i_*i^*K(T(x) \times T(a); \prod_{i=1}^n x_i^{\alpha_i}) \to 0$$

$$K(T(x) \times T(a); \prod_{i=1}^n x_i^{\alpha_i})[-1] \to F^*\pi_1^*\mathcal{L}_\psi \otimes K(T(x) \times T(a); \prod_{i=1}^n x_i^{\alpha_i})$$

$$\to i_*i^*K(T(x) \times T(a); \prod_{i=1}^n x_i^{\alpha_i})(-1)[-2]^{\pm 1},$$

where $j : U \to T(x) \times T(a)$ and $i : T(x) \times T(a) \to U \to T(x) \times T(a)$ and

$$R\varphi'_s(K(T(x) \times T(a); \prod_{i=1}^n x_i^{\alpha_i})(-1) \simeq 0$$

and

$$R\varphi'_1(K(T(x) \times T(a); \prod_{i=1}^n x_i^{\alpha_i}) \simeq 0.$$
we have
\[ R\varphi_*^{(p)} K(X^{(p)}; \prod_{i=1}^n x_i^{\alpha_i}) \otimes \mathcal{L}_\psi(t) \]
\[ \simeq R\varphi'_* (R\pi'_*(F')^* \mathcal{L}_\psi(t)) \otimes K(T(x) \times T(a); \prod_{i=1}^n x_i^{\alpha_i}) \]
\[ \simeq R\varphi'_* (F^* j_! K) \otimes K(T(x) \times T(a); \prod_{i=1}^n x_i^{\alpha_i})(-1)[-1] \]
\[ \simeq R\varphi'_* (j_! j^* K(T(x) \times T(a); \prod_{i=1}^n x_i^{\alpha_i}))(-1)[-1] \]
\[ \simeq R\varphi'_* (i_* i^* K(T(x) \times T(a); \prod_{i=1}^n x_i^{\alpha_i}))(-1)[-2] \]
and
\[ R\varphi_1^{(p)} K(X^{(p)}; \prod_{i=1}^n x_i^{\alpha_i}) \otimes \mathcal{L}_\psi(t) \]
\[ \simeq R\varphi'_1 (R\pi'_1(F')^* \mathcal{L}_\psi(t)) \otimes K(T(x) \times T(a); \prod_{i=1}^n x_i^{\alpha_i}) \]
\[ \simeq R\varphi'_1 (F^* (R\pi_1 \mathcal{L}_\psi(t)) \otimes K(T(x) \times T(a); \prod_{i=1}^n x_i^{\alpha_i}) \]
\[ \simeq R\varphi'_1 (i_* i^* K(T(x) \times T(a); \prod_{i=1}^n x_i^{\alpha_i}))(-1)[-2]. \]

Therefore we have the proposition.

As a consequence, we have the following corollary.

**Corollary.** If \( k = \mathbb{C} \) (resp. \( k = \mathbb{F}_q \)), a hypergeometric sheaf \( \mathcal{G}(D) \) (resp. \( \mathcal{G}(D, \psi) \)) is an irreducible perverse sheaf.

**Definition.** Let \( T \) be a torus over \( k = \mathbb{C} \) (resp. \( k = \mathbb{F}_q \)). A local system \( \mathcal{F} \) on an open set \( U \) in \( T \) is called hypergeometric sheaf if there is an isomorphism \( \phi : T \rightarrow T(R) \) and a non-resonant hypergeometric data \( D \) such that the restriction \( \phi^* \mathcal{G}(D) |_U \) (resp. \( \phi^* \mathcal{G}(D, \psi) |_U \)) of \( \phi^* \mathcal{G}(D) \) (resp. \( \phi^* \mathcal{G}(D, \psi) \)) to \( U \) is isomorphic to \( \mathcal{F} \) on \( T \).

**§3 Variation of K-Hodge structure**

Throughout this section, we assume that \( k = \mathbb{C} \). First we give the definition of variation of Hodge structures. Let \( S \) be a complex manifold and \( n \) be a non-negative integer. A variation of \( K \)-Hodge structure is a datum \( (V, \mathcal{F}_*^*) \) consisting of

1. a local system \( V \) of \( K \)-vector spaces on \( S \) and
2. a decreasing filtration \( F_*^* \) by locally free \( \mathcal{O}_S \)-module on \( V \otimes_{\mathbb{C}} \mathcal{O}_S \) for all the embeddings \( \sigma : \mathbb{C} \rightarrow \Gamma \).

...
For this data, we insist the following conditions on $F_{\sigma}^*$:

1. For any point $s \in S$, the natural homomorphism

$$(F_{\sigma}^*)_s \oplus (1 \otimes c)(F_{\sigma}^{n-i})_s \to V_s \otimes K,\sigma C$$

is an isomorphism, where $\bar{\sigma}$ is the complex conjugate of the embedding $\sigma$, c the complex conjugate, and $1 \otimes c$ the natural homomorphism

$$1 \otimes c : V_s \otimes K,\bar{\sigma} C \to V_s \otimes K,\sigma C.$$

2. (Griffith transversality) Let $\nabla_\sigma$ be the connection on $V \otimes_{K,\sigma} O_S$ defined by $1 \otimes d : V \otimes_{K,\sigma} O_S \to V \otimes_{K,\sigma} \Omega^1_S$. Then $\nabla_\sigma(F_{\sigma}^i) \subset F_{\sigma}^{i-1} \otimes \Omega^1_S$.

Let $D = (R, \{l_i\}, \{\kappa_i\})$ be a hypergeometric data and $U_R$ be the open set of $T(R)$ consisting of non-degenerate points. We introduce a variation of $K$-Hodge structure on the local system $G(D)|_{U_R}$. For simplicity, we assume $\sum_{i=1}^r \kappa_i \neq 0$. For the case $\sum_{i=1}^r \kappa_i = 0$, we can define the variation of Hodge structure in the same way. Let $m$ be the minimal common denominator of $\kappa$. Then we have $\kappa \in \mathbb{Z}^r \otimes I_m(C, K)$. The super lattice $\frac{1}{m}L$ of $L$ is denoted by $M$. We identify the polytope $\Delta$ in $L \otimes \mathbb{R}$ with that in $M \otimes \mathbb{R}$. For a non-negative integer $i$, the $C$ subspace of $C[M]$ generated by the set of monomials $\{m \in M \mid m \in i\Delta\}$ is denoted by $L_M(i\Delta)$. Let $u$ be a indeterminacy. The ring $R = \oplus_{i \geq 0} L_M(i\Delta)u^i$ is a graded ring with $\deg(u) = 1$. The projective variety $\text{Proj}(R)$ is denoted by $\mathbb{P}_\Delta$. This is a toric variety associated to the fan $F_\Delta = \{C_b\}_{b \in \Delta}$, where

$$C_b = \{x \in (L)^* \otimes \mathbb{R} \mid (x, b' - b) \geq 0 \text{ for all } b' \in \Delta\}$$

We take a regular triangulation $\bar{F}$ of the fan $F_\Delta$ with respect to the lattice $M$. Then the associated toric variety $X(\bar{F})$ is smooth. The closure of the pull back of the divisor $X \in T(L) \times T(a)$ in $X(\bar{F})$ via the morphism $T(M) \times T(a) \to T(L) \times T(a)$ is denoted by $\bar{X}_M$. By the definition of non-degeneracy, the restriction $\varphi \cdot M : \bar{X}_M \mid_{U_a} \to U_a$ of the morphism $\bar{X}_M \to T(a)$ to the open set $U_a$ is smooth. Therefore $R^i \varphi \cdot M_* K$ is a variation of $K$-Hodge structure on $U_a$. By the construction, the covering group $\text{Aut}(T(M)/T(L)) \simeq \text{Hom}(\mu_m(C))$ acts on the variety $\bar{X}_M$. The element $\alpha \in L \otimes I(C, K)$ defines a character of $\text{Aut}(T(M)/T(L))$. By the non-resonance condition, $R^i \varphi \cdot M_* K(\bar{X}; x^\alpha) \mid_{U_a}$ is naturally isomorphic to the $\alpha$ part $(R^i \varphi \cdot M_* K)(\alpha)$ of $R^i \varphi \cdot M_* K$. It defines a variation of $K$-Hodge structure on $R^i \varphi \cdot K(X; x^\alpha) \mid_{U_a}$.

§4 Multiplicative equivalence

In this section, we fix a lattice $R$. Let $I = [1, r]$ and $D = (R, \{l_i\}, \{\kappa_i\})$ be a hypergeometric data. A permutation $\sigma$ of the index $I$, gives another hypergeometric data $D_\sigma = (R, \{l_{\sigma(i)}\}, \{\kappa_{\sigma(i)}\})$. The sheaf $G(D)$ and $G(D_\sigma)$ are isomorphic to each other. Therefore we use the notation $D = (l_1, \kappa_1) \oplus \cdots \oplus (l_r, \kappa_r) = \oplus_{i=1}^r (l_i, \kappa_i)$. Notice that $(l_i, \kappa_i)$ is not necessarily a hypergeometric data. More generally, for data $(\{l_i\}, \{\kappa_i\})$ and $(\{l_j\}, \{\kappa_j\})$, $(\{l_i\} \cup \{l_j\}, \{\kappa_i\} \cup \{\kappa_j\})$ is expressed as $(\{l_i\}, \{\kappa_i\}) \oplus (\{l_j\}, \{\kappa_j\})$ and so on. We use the above notation even though each terms do not satisfy the conditions for hypergeometric data.
Remark. The notation $\oplus$ corresponds to convolution product in [G-L].

If $l_1 = \cdots = l_r = l$, $\{l_i\}_{i=1,...,r}$ is denoted by $l^r$. Let $d$ be a positive natural number prime to the characteristic $p$. For an element $\kappa \in I(k, K)$, the inverse image of the multiplication by $d$ on $I(k, K)$ is denoted by $\frac{1}{d}\{\kappa\}$. The cardinality of this set is $d$. Let $D = (R, \{l_i\}, \{\kappa_i\})$ be a hypergeometric data. Suppose that $l_i$ is not equal to 0 and divisible by $d$. We define the hypergeometric data $D^{(i,d)}$ by

$$D^{(i,d)} = \oplus_{j \neq i}(l_j, \kappa_j) \oplus \left(\frac{l_i}{d}\right)^d, \frac{1}{d}\{\kappa_i\} \right)$$

By the following lemma, $D^{(i,d)}$ satisfies the conditions of hypergeometric data.

**Proposition 4.1.** The data $D^{(i,d)}$ is separated. If $D$ is non-resonant, then $D^{(i,d)}$ is also non-resonant.

**Proof.** We assume $d > 1$, $i = 1$. If $(l_1/d)(r) = 1$, this $r$ does not give non separatedness criterion. If $l_1(r) = 1$ and $l_2(r) = 1$ and $l_k(r) = 0$ for $k \neq 1, i, j$ and $l_1(r)/d = 0$. Then this contradicts to the separateness of the data $D$. Before proving the resonance of $D^{(1,d)}$, we give a description of $L^{(1,d)} = \text{Coker}(I^{(1,d)} : R \to \mathbb{Z}^d \oplus \mathbb{Z}^{r-1})$, where $I^{(1,d)} = (l_i^{(1,d)})_{i}$ is given by the data $D^{(1,d)} = (R, \{l_i^{(1,d)}\}, \{\kappa_i^{(1,d)}\})$. Let $q^{(1,d)}$ be the natural projection $\mathbb{Z}^d \oplus \mathbb{Z}^{r-1}$ and the image of $((0, \ldots, 1, \ldots, 0), (0, \ldots, 0))) \oplus \mathbb{Z}^{r-1}$ and $((0, \ldots, 0), (0, \ldots, 1, \ldots, 0))) \oplus \mathbb{Z}^{r-1}$ under the homomorphism $q^{(1,d)}$. By the definition, $L^{(1,d)}$ is generated by $\eta_i$ and $\omega_j$ ($i = 1, \ldots, d, j = 2, \ldots, r$). By the following commutative diagram, the submodule $Q$ of $L$ generated by $\omega_i$ ($i = 2, \ldots, r$) and $d\omega_1$ is isomorphic to the submodule of $L^{(1,d)}$ generated by $\omega_i$ ($i = 2, \ldots, r$) and $\eta_1 + \cdots + \eta_d$.

\[
\begin{array}{ccc}
I^{(1,d)} & \text{Coker}(I^{(1,d)} : R \to \mathbb{Z}^d \oplus \mathbb{Z}^{r-1}) \\
\mathbb{Z}^d \oplus \mathbb{Z}^{r-1} & \mathbb{Z}^d \oplus \mathbb{Z}^{r-1} \\
(\Delta, 1) & (d, 1) \\
\mathbb{Z} \oplus \mathbb{Z}^{r-1} & \\
\end{array}
\]

Therefore $L^{(1,d)}$ is isomorphic to $(Q \oplus \oplus_{i=1}^d \mathbb{Z}\eta_i)/(\eta_1 + \cdots + \eta_d - d\omega_1)\mathbb{Z}$. Then by the definition of $\alpha = q(k)$ and $\alpha^{(1,d)} = q^{(1,d)}(\kappa^{(1,d)})$,

$$\alpha = \kappa_1 \omega_1 + \sum_{i=2}^r \kappa_i \omega_i$$

$$\alpha^{(1,d)} = \sum_{i=1}^d \frac{\kappa_1 + \cdots + \kappa_i}{d} \eta_i + \sum_{i=2}^r \kappa_i \omega_i.$$
Therefore the linear hull of a codimension 1 face of $C(\Delta^{(1,d)})$ is the linear hull $H_{i,\sigma}$ of $\eta_j \ (j \neq i)$ and a codimension 1 face $\sigma$ of $C(\Delta)$. Let $h_\sigma$ be a primitive linear form vanishing on $\sigma$. Let $\eta^*_i$ be the linear form on $Q \oplus \bigoplus_{i=1}^d \mathbb{Z} \eta_i$ such that $\eta^*_i(Q) = 0$ and $\eta^*_i(\eta_j) = \delta_{i,j}$. The linear form defined by

$$h_{i,\sigma} = dh_\sigma(\omega_1) \eta^*_i + h_\sigma$$

vanishes on $\sigma$ and $\eta_j \ (j \neq i)$. Since $h_{i,\sigma}(d\omega_1 - \sum_{i=1}^d \eta_i) = 0$, it defines a linear form on $L^{(1,d)}$ vanishing on the hyperplane $H_{i,\sigma}$. Since

$$h_{i,\sigma}(\alpha^{(1,d)}) \equiv h_\sigma(\alpha) \pmod{\mathbb{Z}},$$

and the linear form $h_{i,\sigma}$ is an integral on $L^{(1,d)}$, $\alpha^{(1,d)}$ is non-resonant.

**Definition.** The equivalence relation generated by $D \sim D^{(1,d)}$ is called multiplicative equivalence.

§5 **Constant equivalence and algebraic correspondences**

§5.1 **Reduced part and constant correspondence for $k = \mathbb{C}$**

Let $D = (R, \{l_i\}, \{\kappa_i\})$ be a hypergeometric data and $S = \{i \mid l_i = 0\}$. Then it is easy to see that the homomorphism $R \to \bigoplus_{i \notin S} \mathbb{Z}$ is also a primitive separate embedding. Let $L_{\text{red}} = \text{Coker}(R \to \bigoplus_{i \notin S} \mathbb{Z})$. Then we have $L = \text{Coker}(R \to \mathbb{Z}^r) \simeq L_{\text{red}} \bigoplus_{i \in S} \mathbb{Z} \omega_i$.

**Proposition 5.1.** If $D$ is non-resonant, then $D_{\text{red}} = (R, \{l_i\}_{i \notin S}, \{\kappa_i\}_{i \notin S})$ is non-resonant and $\kappa_i \notin \mathbb{Z} \ (i \in S)$.

**Proof.** Let $C(\Delta_{\text{red}})$ be the convex cone generated by the $\omega_i \ (i \notin S)$. Then the codimension 1 face of the convex cone $C(\Delta)$ generated by $\Delta$ is either

1. the cone generated by $C(\Delta_{\text{red}})$ and $\omega_i \ (i \in S - \{k\})$ for some $k \in S$ or
2. the cone generated by $\sigma$ and $\omega_i \ (i \in S)$, where $\sigma$ is a codimension 1 face of $C(\Delta_{\text{red}})$.

The defining equation $\tilde{h}_\sigma$ of the linear hull of $\sigma$ and $\omega_i \ (i \in S)$ is the same as the equation $h_\sigma$ of that of $\sigma$ under the decomposition. Therefore $h_\sigma(\alpha_{\text{red}}) = \tilde{h}_\sigma(\alpha)$, where $\alpha_{\text{red}} = \sum_{i \notin S} \kappa_i \omega_i$. The defining equation of the hyperplane of type (1) is the dual base $\omega_k^* \ (k \in S)$ of $\{\omega_i\}_{i \in S}$ and $L_{\text{red}}$. Therefore $\kappa_k \notin \mathbb{Z}$.

**Definition.** The above hypergeometric data $D_{\text{red}}$ is called the reduced part of $D$. On the set of hypergeometric data over a lattice $R$, we introduce the constant equivalence as the equivalence relation generated by $D \sim D_{\text{red}}$.

We define Fermat motif and Artin-Shreier motif as follows. Let $\kappa_1, \ldots, \kappa_r$ be an element of $I(k, K) - \{0\}$. Suppose that $\sum_{i=1}^r \kappa_i \notin \mathbb{Z}$ if $k = \mathbb{C}$. Positive Fermat motif (resp. Positive Fermat-Artin-Shreier motif) is a $K$-Hodge structure ($l$-adic representation) of the form $H^{r-1}(F, K(\kappa_1, \ldots, \kappa_r))$ (resp. $H^n(FAS, K(\kappa_1, \ldots, \kappa_r))(\psi)$), where $F$ and $FAS$ is defined by

$$F : x_1 + \cdots + x_r = 1$$

$$FAS : x_1 + \cdots + x_r = \sigma^n - \sigma$$
and the local system $K(\kappa_1, \ldots, \kappa_r)$ on $F$ and $FAS$ correspond to the Kummer character $\kappa(\prod_{i=1}^{r} x_i^{\kappa_i})$. We omit the expression for dimension of $F$ and $FAS$, because it is determined by the number of the components of the Kummer characters. It is well known that they are stable under the tensor product;

$$H^{r-1}(F)(\kappa_1, \ldots, \kappa_r) \otimes H^{s-1}(F)(\lambda_1, \ldots, \lambda_s)(-1)$$

$$\simeq H^{r+s}(F)(\kappa_1, \ldots, \kappa_r, \lambda_1, \ldots, \lambda_s, -\sum_{i=1}^{r} \kappa_i) \otimes K((-1)^{-\sum_{i=1}^{r} \kappa_i})$$

$$H^{r}(FAS)(\kappa_1, \ldots, \kappa_r, \psi) \otimes H^{s}(FAS)(\lambda_1, \ldots, \lambda_s, \psi)$$

$$\simeq H^{r+s}(FAS)(\kappa_1, \ldots, \kappa_r, \lambda_1, \ldots, \lambda_s, \psi),$$

where $K((-1)^{-\sum_{i=1}^{r} \kappa_i})$ is the Hodge structure on $\text{Spec} \mathbb{C}$ corresponding to the Kummer character $(-1)^{-\sum_{i=1}^{r} \kappa_i}$. As Hodge structures, $K((-1)^{-\sum_{i=1}^{r} \kappa_i})$ is isomorphic to $K$ over $\text{Spec} \mathbb{C}$. Moreover these isomorphisms are induced by algebraic correspondences. Positive Fermat motif contains $K(-1) \simeq H(F)(\kappa_1, -\kappa_1, \kappa_2)$ and Fermat motif is generated by tensoring positive Fermat motif and $K(1)$. Using $K(-1) \simeq FAS(-\kappa_1, \kappa_1)$, we define Fermat-Artin-Shreier motif in the same way. An element in $H(F)(\kappa_1, \ldots, \kappa_r)$ is called Hodge cycle if the Hodge type of $H(F)(\kappa_1, \ldots, \kappa_r)$ is $(m, m)$ for all the embedding $\sigma : K \subset \mathbb{C}$. An element in $H(FAS)(\kappa_1, \ldots, \kappa_r, \psi)$ is called Tate cycle if the action of Frobenius of $F_q$ is $q^m \times \zeta$, where $\zeta$ is a root of unity. In many case, Hodge cycles and Tate cycles of Fermat hypersurface is known to be algebraic cycle. But it remains still open in general case ([A][S]).

We introduce an algebraic correspondence associated to $G(D)$ (resp. $G(D, \psi)$) and $G(D_{red})$ (resp. $G(D_{red}, \psi)$). First we consider the case $k = \mathbb{C}$.

**Theorem 1.** Let $D$ and $D_{red}$ be hypergeometric data defined in Proposition 5.1. We renumber the index such that $S = \{1, \ldots, s\}$. Then there exists an isomorphism of variation of Hodge structures:

$$G(D) \simeq G(D_{red}) \otimes H^s(F)(\kappa_1, \ldots, \kappa_s, \sum_{i=s+1}^{r} \kappa_i) \quad (\sum_{i=1}^{r} \kappa_i \neq 0, \sum_{i=s+1}^{r} \kappa_i \neq 0)$$

$$G(D) \simeq G(D_{red}) \otimes H^s(F)(\kappa_1, \ldots, \kappa_s)(-1) \quad (\sum_{i=1}^{r} \kappa_i \neq 0, \sum_{i=s+1}^{r} \kappa_i = 0)$$

$$G(D) \simeq G(D_{red}) \otimes H^s(F)(\kappa_1, \ldots, \kappa_s) \otimes K((-1)^{\sum_{i=1}^{s} \kappa_i})$$

$$\left(\sum_{i=1}^{r} \kappa_i = 0, \sum_{i=s+1}^{r} \kappa_i \neq 0\right)$$

$$G(D) \simeq G(D_{red}) \otimes H^s(F)(\kappa_1, \ldots, \kappa_{s-1})(-1) \otimes K((-1)^{\sum_{i=1}^{s-1} \kappa_i})$$

$$\left(\sum_{i=1}^{r} \kappa_i = 0, \sum_{i=s+1}^{r} \kappa_i = 0\right)$$

defined by an algebraic correspondence.

We use systems of coordinates $y = (y_1, \ldots, y_s)$ and $\bar{x} = (x_{s+1}, \ldots, x_n)$. We take a base of $I$ such that $x_i^{\kappa_i} = x_i$ for $i = 1, \ldots, s$ and $x_i^{\kappa_i} = x_i^{\kappa_i}(i = s+1, \ldots, n)$.
For a constructible sheaf $\mathcal{F}$ on a torus $T$ is said to be of weight less (resp. more) than $w$ if there exists an open set $U$ of $T$ such that the fiber of $\mathcal{F}$ at $p$ in $U$ is mixed Hodge structure of weight less (resp. more) than $w$ and it is denoted by $\text{wt}(\mathcal{F}) \leq w$ (resp. $\text{wt}(\mathcal{F}) \geq w$).

**Case (A) $\sum_{i=1}^{r} \kappa_{i} \neq 0$**

The variety $\mathcal{X}_{D}$ associated to $D$ and the rank 1 local system is given by

$$\mathcal{X}_{D} = \{(y, z, a) \in T \times T(y) \times T(z) \mid f_{0} \neq 0\}$$

and

$$K(x^{\alpha} a^{\kappa}) = K(\mathcal{X}_{D}; \prod_{i=1}^{s} (x_{0} a_{i} y_{i})^{\kappa_{i}} \prod_{i \geq s+1} (x_{0} a_{i} x_{i}^{\kappa})^{\kappa_{i}})$$

We put $f_{0} = \sum_{i=s+1}^{r} a_{i} x_{i}^{\kappa}$. Let $\mathcal{X}_{D}^{0}$ and $\mathcal{X}_{D}^{1}$ be the open set and closed set of $\mathcal{X}_{D}$ defined by

$$\mathcal{X}_{D}^{0} = \{(y, z, a) \in \mathcal{X}_{D} \mid f_{0} \neq 0\}$$

and

$$\mathcal{X}_{D}^{1} = \{(y, z, a) \in \mathcal{X}_{D} \mid f_{0} = 0\}$$

Let $\varphi^{0}$ (resp. $\varphi^{1}$) be the composite of the open immersion $\mathcal{X}_{D}^{0} \rightarrow \mathcal{X}_{D}$ (resp. the closed immersion $\mathcal{X}_{D}^{1} \rightarrow \mathcal{X}_{D}$) and $\varphi : \mathcal{X}_{D} \rightarrow T(a)$. We get the following long exact sequence

$$\cdots \rightarrow R^{n} \varphi_{*} K(x^{\alpha} a^{\kappa}) \xrightarrow{i_{*}} R^{n} \varphi_{*} K(x^{\alpha} a^{\kappa}) \rightarrow R^{n} \varphi_{*} K(x^{\alpha} a^{\kappa})$$

$$\delta \rightarrow R^{n+1} \varphi_{*} K(x^{\alpha} a^{\kappa}) \simeq R^{n-1} \varphi_{*} K(x^{\alpha} a^{\kappa})(-1) \rightarrow \cdots,$$

where $R^{i} \varphi_{*}$ denotes the $i$-th higher direct image with a support in $\mathcal{X}_{D}^{1}$.

**Case (A-1) $\sum_{i=1}^{r} \kappa_{i} \neq 0$ AND $\sum_{i=s+1}^{r} \kappa_{i} \neq 0$**

Let us introduce sets of variables $Y = (Y_{1}, \ldots, Y_{s})$, $X_{0}$, and $t_{0}$ and change coordinate by $Y_{i} = x_{0} a_{i} y_{i}$, $X_{0} = x_{0} f_{0}$ and $t_{0} f_{0} = 1$. Then $\mathcal{X}_{D}^{0}$ is isomorphic to $\mathcal{X}_{D_{red}}^{0} \times F$, where

$$\mathcal{X}_{D_{red}}^{0} = \{(\bar{x}, t_{0}, a) \in T(\bar{x}) \times T(t_{0}) \times T(a) \mid t_{0} f_{0}(a, \bar{x}) = 1\}$$

and

$$F = \{(Y, X_{0}) \in T(Y) \times T(X_{0}) \mid Y_{1} + \cdots + Y_{s} + X_{0} = 1\}$$

Under the above isomorphism, the corresponding rank 1 local system is $p_{1}^{*}(\mathcal{F}^{(1)}) \otimes p_{2}^{*}(\mathcal{F}^{(2)})$, where

$$\mathcal{F}^{(1)} = K(\mathcal{X}_{D_{red}}; \prod_{i=s+1}^{r} (a_{i} x_{i}^{\kappa})^{\kappa_{i}} \sum_{i=s+1}^{r} \kappa_{i})$$

and

$$\mathcal{F}^{(2)} = K(F; \prod_{i=1}^{s} Y_{i}^{\kappa_{i}} X_{0}^{\sum_{i=s+1}^{r} \kappa_{i}}).$$

The natural homomorphism $\mathcal{X}_{D_{red}} \rightarrow T(a_{s+1}, \ldots, a_{r})$ is denoted by $\varphi_{red}$. Therefore we have

$$R^{i} \varphi_{*} K(x^{\alpha} a^{\kappa}) \simeq R^{n-s} \varphi_{red}^{*} \mathcal{F}^{(1)} \otimes H^{s}(F)(\kappa_{1}, \ldots, \kappa_{s}, \sum_{i=s+1}^{r} \kappa_{i}).$$

We have $\text{wt}(R^{n} \varphi_{*} K(x^{\alpha} a^{\kappa})) = n$ and $\text{wt}(R^{n+1} \varphi_{*} K(x^{\alpha} a^{\kappa})) \geq n + 1$. Therefore the connecting homomorphism $\delta$ vanishes. Since both $R^{n} \varphi_{*} K(x^{\alpha} a^{\kappa})$ and $R^{n+1} \varphi_{*} K(x^{\alpha} a^{\kappa})$ are perverse and irreducible, we get the required isomorphism.
Case (A-2) $\sum_{i=1}^r \kappa_i \neq 0$ and $\sum_{i=s+1}^r \kappa_i = 0$

We introduce an coordinate $Y = (Y_1, \ldots, Y_s)$ and change coordinates $(y, x_0, x, a) \rightarrow (y, x_0, \tilde{x}, a)$ by $Y_i = x_0 a_i y_i$. Then the variety $X_D^1$ is isomorphic to $\mathcal{X}_{D, red} \times F \times T(x_0)$, where

$$X_{D, red} = \{ \tilde{x} \in T(\tilde{x}) \mid f_0 = 0 \}$$

$$F = \{ Y \in T(Y) \mid Y_1 + \cdots Y_s = 1 \}.$$ 

The corresponding rank 1 local system is $pr_1^*(\mathcal{F}(1)) \otimes pr_2^*(\mathcal{F}(2))$, where $\mathcal{F}(1)$ and $\mathcal{F}(2)$ is the local system on $X_{D, red}$ and $F$ defined by

$$\mathcal{F}(1) = K(X_{D, red}; \prod_{i=s+1}^r (a_i \tilde{x}_i)^{\kappa_i}) \quad \text{and} \quad \mathcal{F}(2) = K(F; \prod_{i=1}^s Y_i^{\kappa_i}).$$

Therefore we have

$$R^n \varphi_{X^1} K(x^0 a^\kappa) = R^{n-1} \varphi_{red*} \mathcal{F}(1) \otimes H^{s-1}(F)(\kappa_1, \ldots, \kappa_s)(-1).$$

Here $\varphi_{red}$ denotes the natural morphism $X_{D, red} \rightarrow T(a_{s+1}, \ldots, a_r)$.

We show that $\text{wt}(R^n \varphi_* K(x^0 a^\kappa)) = n+2$. In fact, under the change of coordinate $Y_i = x_0 a_i y_i$ and $X_0 = x_0 f_0$, the variety $X_D^0$ is isomorphic to $U_1 \times U_2 \times T(x_0)$, where

$$U_1 = \{ Y \in T(Y) \mid Y_1 + \cdots Y_s - 1 \neq 0 \}$$

$$U_2 = \{ \tilde{x} \in T(\tilde{x}) \times T(a) \mid f_0(a, \tilde{x}) \neq 0 \}$$

and the corresponding Kummer character is $pr_1^* \mathcal{F}(2) \otimes pr_2^* \mathcal{F}(1)$ defined as above. Therefore $\text{wt}(H^s(U_1, \mathcal{F}(2))) = s + 1$ and $\text{wt}(R^{n-s} \pi_2^* \mathcal{F}(1)) = n - s + 1$, where $\pi_2 : U_2 \rightarrow T(a_{s+1}, \ldots, a_r)$ is the natural projection. Therefore the morphism $i_* i_!$ is surjective on the open set $U_a$. The perversity of $R^n \varphi_{X^1} K(x^0 a^\kappa)$ and $R^n \varphi_* K(x^0 a^\kappa)$ implies the required isomorphism.

Case (B) $\sum_{i=1}^r \kappa_i = 0$

In this case, the variety $X_D$ and the rank 1 local system associated to $D$ are given by

$$X_D = \{ (y, \tilde{x}, a) \in T(y) \times T(\tilde{x}) \times T(a) \mid a_1 y_1 + \cdots a_s y_s + \sum_{i=s+1}^r a_i \tilde{x}_i = 0 \}$$

and

$$K(x^0 a^\kappa) = K(X_D; \prod_{i=1}^s (a_i y_i)^{\kappa_i} \prod_{i \geq s+1} (a_i \tilde{x}_i)^{\kappa_i}).$$

We put $f_0 = \sum_{i=s+1}^r a_i \tilde{x}_i$. Let $X_D^0$ and $X_D^1$ be the open set and closed set of $X_D$ defined by

$$X_D^0 = \{ (y, \tilde{x}, a) \in X_D \mid f_0 \neq 0 \} \quad \text{and} \quad X_D^1 = \{ (y, \tilde{x}, a) \in X_D \mid f_0 = 0 \}.$$ 

Let $\varphi^0$ (resp. $\varphi^1$) be the composite of the open immersion $X_D^0 \rightarrow X_D$ (resp. the closed immersion $X_D^1 \rightarrow X_D$) and $\varphi$. On the other hand, we get the following long exact sequence

$$\cdots \rightarrow R^{n-1} \varphi_{X^1} K(x^0 a^\kappa) \rightarrow R^{n-1} \varphi_* K(x^0 a^\kappa) \rightarrow R^{n-1} \varphi^0_* K(x^0 a^\kappa)$$

$$\rightarrow \delta R^n \varphi_{X^1} K(x^0 a^\kappa) \simeq R^{n-2} \varphi^1_* K(x^0 a^\kappa)(-1) \rightarrow \cdots,$$

where $R^i \varphi_* K$ denotes the $i$th higher direct image with a support in $X^1$. 

\textbf{Case (b-1)} \sum_{i=1}^{r} \kappa_i = 0 \text{ and } \sum_{i=s+1}^{r} \kappa_i \neq 0

We introduce systems of coordinates \( t_0 = (t_0) \) and \( Y = (Y_1, \ldots, Y_s) \). By changing coordinate, \( t_0 f_0 = 1 \) \( Y_i = -t_0 a_i y_i \), the variety \( \mathcal{X}_D^0 \) is isomorphic to the product \( \mathcal{X}_{D_{red}} \times F \), where

\[
\mathcal{X}_{D_{red}} = \{ (t_0, \bar{x}, a) \in \mathbb{T}(t_0) \times \mathbb{T}(\bar{x}) \times \mathbb{T}(a) \mid t_0 f_0 = 1 \}
\]

\[
F = \{ y \in \mathbb{T}(Y) \mid \sum_{i=1}^{s} Y_i = 1 \},
\]

and under isomorphism, the corresponding rank 1 local system is \( pr_1^* \mathcal{F}(1) \otimes pr_2^* \mathcal{F}(2) \), where

\[
\mathcal{F}(1) = K(\mathcal{X}_{D_{red}}; \prod_{i=s+1}^{r} (t_0 a_i \bar{x}^{\omega_i})^{\kappa_i}) \text{ and } \mathcal{F}(2) = K(F; \prod_{i=1}^{s} (-Y_i)^{\kappa_i}).
\]

The natural morphism \( \mathcal{X}_{D_{red}} \rightarrow \mathbb{T}(a_{s+1}, \ldots, a_r) \) is denoted by \( \varphi_{red} \). Therefore we have

\[
R^{n-1} \varphi_*^0 K(x^\alpha a^\kappa) = R^{n-s} \varphi_{D_{red}*} \mathcal{F}(1) \otimes H^{s-1}(F)(\kappa_1, \ldots, \kappa_s) \otimes K((-1)^{\sum_{i=1}^{s} \kappa_i}).
\]

Since \( \text{wt}(R^n \varphi_{X_1*} K(x^\alpha a^\kappa)) \geq n \), the morphism \( \delta \) is zero on the open set \( U_a \). Therefore \( R^{n-1} \varphi_*^0 K(x^\alpha a^\kappa) \rightarrow R^{n-1} \varphi_*^0 K(x^\alpha a^\kappa) \) is an isomorphism by the perversity and irreducibility argument.

\textbf{Case (b-2)} \sum_{i=1}^{r} \kappa_i = 0 \text{ and } \sum_{i=s+1}^{r} \kappa_i = 0

We introduce a variable \( \eta = (\eta_1, \ldots, \eta_{s-1}) \) and \( Y_s = (Y_s) \). By changing coordinate, \( (y, \bar{x}, a) \rightarrow (\bar{x}, a, \eta, Y_s) \) with \( \eta_i = -\frac{a_i y_i}{a_s y_s} \) \( (i = 1, \ldots, s-1) \), the variety \( \mathcal{X}_D^1 \) is isomorphic to \( \mathcal{X}_{D_{red}} \times F \times \mathbb{T}(Y_s) \), where

\[
\mathcal{X}_{D_{red}} = \{ (\bar{x}, a) \in \mathbb{T}(\bar{x}) \times \mathbb{T}(a) \mid f(a, \bar{x}) = 0 \}
\]

\[
F = \{ \eta \in \mathbb{T}(\eta) \mid \eta_1 + \ldots + \eta_{s-1} = 1 \},
\]

and the rank 1 local system is \( pr_1^* \mathcal{F}(1) \otimes pr_2^* \mathcal{F}(2) \), where

\[
\mathcal{F}(1) = K(\mathcal{X}_{D_{red}}; \prod_{i=s+1}^{r} (a_i \bar{x}^{\omega_i})^{\kappa_i}) \text{ and } \mathcal{F}(2) = K(F; \prod_{i=1}^{s-1} (-\eta_i)^{\kappa_i}).
\]

The natural map \( \mathcal{X}_{D_{red}} \rightarrow \mathbb{T}(a_{s+1}, \ldots, a_r) \) is denoted by \( \varphi_{red} \). Therefore we have

\[
R^{n-1} \varphi_{X_1*} K(x^\alpha a^\kappa) = R^{n-s-1} \varphi_{D_{red}*} \mathcal{F}(1) \otimes H^{s-2}(F)(\kappa_1, \ldots, \kappa_{s-1})(-1) \otimes K((-1)^{\sum_{i=1}^{s-1} \kappa_i}).
\]

As in the case (a-2), we can show that \( \text{wt}(R^{n-1} \varphi_*^0 K(x^\alpha a^\kappa)) \geq n + 1 \). Therefore \( i_* i^! \) is surjective. Perversity and irreducibility implies the required isomorphism. We complete all the cases.
§5.2 Constant correspondence for $k = \mathbf{F}_q$

Next we study the case $k = \mathbf{F}_q$.

**Theorem 2.** Let $D = (R, \{l_i\}_i, \{\kappa_i\}_i)$ be a non-resonant hypergeometric data and $D_{red}$ be the hypergeometric data defined in Proposition 5.1. We renumber the index such that $S = \{1, \ldots, s\}$. Then there exists the following isomorphism of constructible sheaves induced by an algebraic correspondence:

$$\mathcal{G}(D) \simeq \mathcal{G}(D_{red}) \otimes H^s(FAS)(\kappa_1, \ldots, \kappa_s, \psi).$$

**Proof.** The proof is simpler compared to the case $k = \mathbf{C}$. Let $y_i$ denote the element corresponding to $q(e_i) = \omega_i$ in $k[L]$ for $i = 1, \ldots, s$. Take a base of $L_{red} \simeq \mathbf{Z}^{n-r}$. Then $k[L_{red}]$ is identified with $k[x_{s+1}^{\pm}, \ldots, x_n^{\pm}]$. We use a system of coordinate $\tilde{x} = (x_{s+1}, \ldots, x_n)$. If we put $f_0 = \sum_{i=s+1}^{n} a_i \tilde{x}^{\omega_i}$, the variety $X_D$ is denoted by

$$X_D = \{(y, \tilde{x}, t) \in T(y) \times T(\tilde{x}) \times A(t) | a_1 y_1 + \cdots + a_r y_r + f_0 = t\}.$$

We define the variety $\tilde{X}_D$ by

$$\tilde{X}_D = \{(y, \tilde{x}, t, s, a) \in T(y) \times T(\tilde{x}) \times A(t) \times A(s) \times T(a) |$$

$$a_1 y_1 + \cdots + a_s y_s + f_0(a, \tilde{x}) = t, f_0 = s\}.$$

It is easy to see that this is an etale covering of $X_D$ with the Galois group $\mathbf{F}_p$. We introduce a new coordinate $r = (r)$ and $Y = (Y_1, \ldots, Y_s)$. By changing coordinate $(y, \tilde{x}, t, s, a) \to (Y, \tilde{x}, s, r, a)$ given by $a_i y_i = Y_i$ and $r = t - s$, the variety $X_D$ is isomorphic to $X_{D_{red}} \times FAS$, where

$$X_{D_{red}} = \{(x, a) \in T(x) \times A(a) | f_0(a, x) = s\},$$

$$FAS = \{(Y, \rho) \in T(Y) \times A(\tau) | Y_1 + \cdots + Y_s = r\}.$$

The corresponding rank 1 local system is $pr_1^*(\mathcal{F}(1)) \otimes pr_2^*(\mathcal{F}(2))$, where

$$\mathcal{F}(1) = K(X_{D_{red}}; \prod_{i=s+1}^{r} (a_i x^{\omega_i})^{\kappa_i}) \otimes \mathcal{L}_\psi(s) \text{ and } \mathcal{F}(2) = K(FAS; \prod_{i=1}^{s} Y_i^{\kappa_i}) \otimes \mathcal{L}_\psi(r).$$

Let $\varphi_{red}$ be the natural projection $X_{D_{red}} \to T(a_{s+1}, \ldots, a_r)$. Since $\tilde{X}_{D_{red}}$ is etale over $X_{D_{red}}$, we have

$$R^n \varphi_* K(X_D; \prod_{i=1}^{s} Y_i^{\kappa_i} \prod_{i=s+1}^{r} (a_i x^{\omega_i})^{\kappa_i}) \otimes \mathcal{L}_\psi(t) \simeq R^{n-s} \varphi_{red,*} K(\mathcal{F}(1)) \otimes H^s(FAS; \kappa_1, \ldots, \kappa_s, \psi).$$

Therefore we have the proposition.
§6 Algebraic correspondences associated to the multiplicative equivalence

§6.1 Existence of good base

In this section we introduce an algebraic correspondence associated to multiplicative equivalence. Let \( D = (R, \{ l_i \}_i, \{ \kappa_i \}_i) \) be a hypergeometric data and suppose that \( l_1(R) = d\mathbb{Z} \) with \( d > 0 \).

**Proposition 6.1.** There exists a base of \( L \cong \mathbb{Z}^{n+1} \) such that

\[
< \omega_i >_{i=2,...,r} = < d\omega_1, \omega_i >_{i=2,...,r} = d\mathbb{Z} \oplus \mathbb{Z}^n
\]

and \( \omega_1 = (k,0,\ldots,0) \), with \((k,d) = 1\).

**Proof.** Let \( Q = < \omega_i >_{i=2,...,r} \). By the following commutative diagram, we have \( L/Q = \mathbb{Z}/d\mathbb{Z} \).

\[
\begin{array}{cccccc}
0 & 0 & 0 & & & \\
\downarrow & & & & & \\
0 & \mathbb{Z}^{r-1} \cap R & \mathbb{Z}^{r-1} & Q & 0 \\
\downarrow & & & & & \\
0 & R & \mathbb{Z}^r & L & 0 \\
\downarrow & & & & & \\
0 & \text{Im}(l_1) & \mathbb{Z} & Q/L & \\
\downarrow & & & & & \\
0 & 0 & 0 & 0 & \\
\end{array}
\]

Here, columns and rows are exact. If we put \( Q' = Q + d\omega_1 \mathbb{Z} \), then by the following commutative diagram, we have \( L/Q' \cong \mathbb{Z}/d\mathbb{Z} \), and as consequence, we have \( Q = Q' \).

\[
\begin{array}{cccccc}
0 & R & d\mathbb{Z} \oplus \mathbb{Z}^{r-1} & Q' & 0 \\
\downarrow & & & & & \\
0 & R & \mathbb{Z}^r & L & 0 \\
\end{array}
\]

Let \( \omega_1 = kl \) with a primitive element \( l \) in \( L \). If we write \( l = a_1\omega_1 + \ldots a_r\omega_r \), then \((1-ka_1)\omega_1 \in Q, (k,d)=1\). The natural map \( \mathbb{Z}\omega_1 \rightarrow L/Q \) is surjective, the homomorphism \( \mathbb{Z}l \rightarrow L/Q \) is also surjective. Therefore \( dl \) is a primitive element in \( Q \), and we can take a base of \( Q; Q = \mathbb{Z}dl \oplus \oplus_{i=1}^n \mathbb{Z}u_i \).

We construct an algebraic correspondence using a coordinate given in Proposition 6.1. We take a base of \( L : L \cong \mathbb{Z}^{n+1} \) which satisfies the condition of Proposition 6.1. Under this base, the first coordinate of \( \omega_i \) is divisible by \( d \) for \( i = 2,\ldots,r \). Therefore \( f_0 = \sum_{i=2}^r a_i x^{\omega_i} \) can be written as \( f_0(x_1^d, x_2, \ldots, x_{n+1}) \). The polynomial \( f \) can be written as a \( \phi^k \) of \( f_0(x_1^d, x_2, \ldots, x_{n+1}) \). Using the base
as in Proposition 6.2, the lattice \( L^{(1,d)} = (\mathbb{Z}^d \oplus \mathbb{Z}^{r-1})/I^{(1,d)}(R) \) and the morphism \( \mathbb{Z}^d \oplus \mathbb{Z}^{r-1} \rightarrow L^{(1,d)} \) is expressed as follows. We write the structure of \( L^{(1,d)} \) multiplicatively. Let \( u_1, \ldots, u_d \) be a system coordinates. \( L^{(1,d)} \) is generated by \( Q \) and \( u_1, \ldots, u_d \) with the relation \( u_1 \cdot \cdots \cdot u_d = z^k \), where \( z = x^l \) and \( l = (1,0, \ldots, 0) \). Note that \( \alpha_1 = k\kappa_1 \) under this base. (For the definition of \( I^{(1,d)} \) and \( L^{(1,d)} \) see Section 4.) Before constructing algebraic correspondences, we note the following lemma.

**Lemma 6.2.** Let \( d \) be a natural number prime to the characteristic of \( k \). Let \( \xi_1, \ldots, \xi_d \) be a set of variables and \( s_1, \ldots, s_d \) be the elementary symmetric functions of \( \xi_i \) \((i = 1, \ldots, d)\) of degree \( 1, \ldots, d \). Then \( \xi_1^d + \cdots + \xi_d^d \) is expressed as a polynomial \( F(s_1, \ldots, s_d) \) of \( s_1, \ldots, s_d \) and we have

\[
F(s_1, \ldots, s_d) = F(s_1, \ldots, s_{d-1}, 0) + (-1)^{d-1} ds_d.
\]

Moreover, if \( s_1, \ldots, s_{d-1} \) are elementary symmetric functions in \( \gamma_1, \ldots, \gamma_{d-1} \), then

\[
F(s_1, \ldots, s_{d-1}, 0) = \gamma_1^d + \cdots + \gamma_{d-1}^d.
\]

**Proof.** See [T].

§6.2 Multiplicative correspondences for \( k = \mathbb{C} \).

The main theorem of this section is the following.

**Theorem 3.** Let \( D \) be a hypergeometric data with \( l_i(R) = d\mathbb{Z} \) (if \( k = \mathbb{C} \) or \( l_i(R) = dp^i\mathbb{Z} \) with \((d, p) = 1\) (if \( k = \mathbb{F}_q \)) and \( D^{(i,d)} \) be the hypergeometric data defined in Section 3. We put \( \tilde{D}^{(d)} = D \oplus \oplus_{i=1}^{d-1}(0, \frac{i}{d}) \). Then we have the following isomorphism of hypergeometric sheaves induced by an algebraic correspondence.

1. If \( k = \mathbb{C} \), \( G(\tilde{D}^{(d)}) \otimes K((\frac{(-1)^{d-1}}{d})^{\kappa_1}) \approx T_{(i,d)}^* G(\tilde{D}^{(i,d)}), \) where \( K((\frac{(-1)^{d-1}}{d})^{\kappa_1}) \) is the Hodge structure on \( \text{Spec } \mathbb{C} \) corresponding to the Kummer character \((\frac{(-1)^{d-1}}{d})^{\kappa_1})

2. If \( k = \mathbb{F}_q \), \( G(\tilde{D}^{(d)}, \psi) \otimes K((\frac{(-1)^{d-1}}{d})^{\kappa_1}) \approx T_{(i,d)}^* G(\tilde{D}^{(i,d)}, \psi), \) Here \( T_{(i,d)}^* \) denotes the translation of \( T(R) \) given by \( T_{(i,d)}^* (a_i) = (-1)^{d-1} a_i/d \) and \( K((\frac{(-1)^{d-1}}{d})^{\kappa_1}) \) is the \( l \)-adic sheaf on \( \text{Spec } \mathbb{F}_q \) corresponding to the Kummer character \((\frac{(-1)^{d-1}}{d})^{\kappa_1})

**Proof.** In this subsection, we prove (1). We choose a base of \( L \) satisfying the condition of Proposition 6.1. Let \( \kappa = (\kappa_1, \ldots, \kappa_r) \) and \( \alpha = (\alpha_1, \ldots, \alpha_{n+1}) \in L \otimes I(\mathbb{C}, K) \) be the image of \( \kappa \) under the homomorphism \( q \). We prove the above theorem for the case \( \sum_{i=1}^{n+1} \kappa_i \neq 0 \). In the case \( \sum_{i=1}^{n+1} \kappa_i = 0 \), we can prove the theorem by similar argument. We introduce systems of variables

\[
x_1, x = (x_2, \ldots, x_{n+1}), \xi = (\xi_1, \ldots, \xi_d), \gamma = (\gamma_1, \ldots, \gamma_{d-1}),
\]

\[
u = (u_1, \ldots, u_d), g = (g_1, \ldots, g_{d-1}), s^0 = (s_1, \ldots, s_{d-1}), s = (s_1, \ldots, s_d), z,
\]

\[
\alpha = (\alpha_1, \ldots, \alpha_n), b = (b_1, \ldots, b_n), e = (e_1, \ldots, e_n)
\]
We use the notation $a_i' = (-1)^{d-1}a_i/d$ for short. We define five varieties $X_{D(1,d)}$, $X_1$, $Y$, $X_2$ and $X'_{D(d)}$.

$$X_{D(1,d)} = \{(u, x, z, b, a \geq 2) \in T(u) \times T(x) \times T(z) \times T(b) \times T(a \geq 2) \mid b_1u_1 + \cdots + b_dud + f_0(z, x_2, \ldots, x_{n+1}) = 1, u_1 \cdots u_d = z^k\}$$

$$X_1 = \{((\xi, x, z, a) \in T(\xi) \times T(x) \times T(z) \times T(a) \mid a_1(\xi_1^d + \cdots + \xi_d^d) + f_0(z, x_2, \ldots, x_{n+1}) = 1, (\xi_1 \cdots \xi_d)^d = z^k\}$$

$$Y = \{(s^0, s_d, x, z, a) \in A(s^0) \times T(s_d) \times T(x) \times T(z) \times T(a) \mid a_1'(s_1, \ldots, s_{d-1}, 0) + (-1)^{d-1} ds_d + f_0(z, x_2, \ldots, x_{n+1}) = 1, s_d = z^k\}$$

$$X_2 = \{((\gamma, x_1, x, a) \in A(\gamma) \times T(x_1) \times T(x) \times T(a) \mid a_1'(\gamma_1^d + \cdots + \gamma_{d-1}^d) + a_1x_1^k + f_0(x_1^d, x_2, \ldots, x_{n+1}) = 1\}$$

$$X'_{D(d)} = \{(g, x_1, x, c, a) \in T(g) \times T(x_1) \times T(x) \times T(c) \times T(a) \mid c_1g_1 + \cdots + c_{d-1}g_{d-1} + a_1x_1^k + f_0(x_1^d, x_2, \ldots, x_{n+1}) = 1\}$$

Let $(\xi, x, z, a)$ (resp. $(\gamma, x_1, x, a)$) be a point in $X_1$ (resp. $X_2$). By substituting $s_i (i = 1, \ldots, d)$ (resp. $\gamma_i (i = 1, \ldots, d-1)$) by the $i$-th symmetric polynomial of $\xi_i$ (resp. $\gamma_i$ (i = 1, ..., d-1)), we get a finite homomorphism $\pi_1$ (resp. $\pi_2$) from $T'_{(1,d)}(X_1)$ to $Y$ (resp. $X_2$ to $Y$) using Lemma 6.2. Then $\pi_1$ and $\pi_2$ are quotient by finite groups $G_d$ and $G_{d-1}$ respectively. The natural morphisms $X_1 \rightarrow T(a)$, $Y \rightarrow T(a)$ and $X_2 \rightarrow T(a)$ are denoted by $\phi_1$, $\phi_y$ and $\phi_2$ respectively.

We consider the following sheaves on each variety.

$$F^{(1,d)} = K(X_{D(1,d)}; \prod_{i=1}^{d} (b_iu_i)^{(i+\kappa_1)/d} \prod_{i=2}^{n+1} x_i^{\alpha_i} \prod_{i=2}^{r} a_i^{\kappa_i})$$

$$F_1 = K(X_1; \prod_{i=1}^{d} (a_1\xi_i^d)^{(i+\kappa_1)/d} \prod_{i=2}^{n+1} x_i^{\alpha_i} \prod_{i=2}^{r} a_i^{\kappa_i})$$

$$F_y = K(Y; (a')^{d-1} = (a_1s_d)^{\kappa_1} \prod_{i=2}^{n+1} x_i^{\alpha_i} \prod_{i=2}^{r} a_i^{\kappa_i})$$

$$F_2 = K(X_2; \prod_{i=1}^{d-1} (a_1\gamma_i^d)^{i/d}(a_1x_1^k)^{\kappa_1} \prod_{i=2}^{n+1} x_i^{\alpha_i} \prod_{i=2}^{r} a_i^{\kappa_i})$$

$$F^{(d)} = F(X'_{D(d)}; \prod_{i=1}^{d-1} (c_ig_i)^{i/d}(a_1x_1^k)^{\kappa_1} \prod_{i=2}^{n+1} x_i^{\alpha_i} \prod_{i=2}^{r} a_i^{\kappa_i})$$

By the argument in §2, the variety $X_{D(1,d)} \rightarrow T(b) \times T(a \geq 2)$ and $X'_{D(d)} \rightarrow T(c) \times T(a)$ descend to variety $X_R \rightarrow T(R)$ and $X''_R \rightarrow T(R)$ respectively. Under this descent, the sheaves $F^{(1,d)}$ and $F^{(d)}$ descend to sheaves $\mathcal{F}^{(1,d)}$ and $\mathcal{F}^{(d)}$ on $X_R$ and $X''_R$ respectively. By the definition of hypergeometric sheaves, we have

$$R^n \varphi_{R*} F^{(1,d)} = \mathcal{G}(D^{(1,d)})$$

$$R^n \varphi_{R*} F^{(d)} = \mathcal{G}((D'_{(d)}))$$
We consider the following diagram:

\[
\begin{array}{ccc}
X_1 & \longrightarrow & X_{D^{(1,d)}} \\
\varphi_1 \downarrow & & \varphi_R \\
T(a) & \overset{\Sigma}{\longrightarrow} & T(b) \times T(a_{\geq 2}) \\
(a_1, \ldots, a_r) & \mapsto & (b_1, \ldots, b_r, a_2, \ldots, a_r) \\
\varphi_2 \downarrow & & \varphi_R \\
X_2 & \longrightarrow & X_{\tilde{D}^{(d)}} \\
T(a) & \overset{\Sigma'}{\longrightarrow} & T(c) \times T(a) \\
(a_1, \ldots, a_r) & \mapsto & (c_1, \ldots, c_{r-1}, a_1, \ldots, a_r)
\end{array}
\]

where \(X_1 \to X_{D^{(1,d)}}\) and \(X_2 \to X_{\tilde{D}^{(d)}}\) are given by

\[
(u, x, z, b, a_{\geq 2}) = ((\xi^d_1, \ldots, \xi^d_2), x, z, (a_1, \ldots, a_1), a_{\geq 2})
\]

and

\[
(g, x_1, x, c, a) = ((\gamma^d_1, \ldots, \gamma^d_2), x_1, x, (a_1, \ldots, a_1), a).
\]

Let \(I^* : T(a) \to T(R)\) be the morphism induced by \(I\). Since \((\sum X_{D^{(1,d)}}) \times T(a)\)
\(T(\xi) = X_1\) and \(((\sum') X_{\tilde{D}^{(d)}}) \times T(g)\) \(T(\gamma) = X_2\), we have

\[
I^* \mathcal{G}(D^{(1,d)}) \simeq R^n \varphi_1 \mathcal{F}_1 \\
I^* \mathcal{G}(\tilde{D}^{(d)}) \simeq R^n \varphi_2 \mathcal{F}_2.
\]

Therefore to prove the theorem, it is enough to show the following theorem.

**Proposition 6.3.** On the torus \(T(a)\), the morphism \(\pi_1 \) and \(\pi_2\) induce isomorphisms of sheaves:

\[
\begin{align*}
(1) & \quad T^{*}_{(1,d)}(R^n \varphi_1 \mathcal{F}_1) \simeq R^n \varphi_y \mathcal{F}_y \otimes K((\frac{(-1)^{d-1}}{d})^{\kappa_1}) \quad \text{and} \\
(2) & \quad R^n \varphi_1 \mathcal{F}_2 \simeq R^n \varphi_y \mathcal{F}_y
\end{align*}
\]

Moreover they are compatible with the descent data.

**Proof.** We prove the isomorphism (1). The isomorphism (2) can be proved similarly.

Let

\[
B = \{(T, D, x, a) \in A(T) \times T(D) \times T(x) \times T(a) \mid -a'_1 T + f_0(D, x_2, \ldots, x_{n+1}) = 1\}
\]

be a variety over \(T(a)\). The natural map \(B \to T(a)\) is denoted by \(\varphi_B\). Then the morphism

\[
(\xi^d_1, \ldots, \xi^d_2, z, x, a) \mapsto (\xi^d_1, \ldots, \xi^d_2, z, x, a) \quad \text{and} \\
(\xi^d_1, \ldots, \xi^d_2, z, x, a) \mapsto (F(a_1, \ldots, a_1, z, x, a))
\]
induces a morphism from \( h'_1 : T^*_{(1,d)}X_1 \to B \) and \( h_y : \mathcal{Y} \to B \). Consider the following commutative diagram:

\[
\begin{array}{ccc}
T^*_{(1,d)}X_1 & \xrightarrow{\pi_1} & \mathcal{Y} \\
\downarrow h'_1 & & \downarrow h_y \\
B & \xrightarrow{\varphi_B} & T(a)
\end{array}
\]

Let \( \mathcal{F}_B \) be a sheaf on \( B \) defined by

\[
\mathcal{F}_B = K(B; (a_1')^{d+1} (a_1 D^\frac{1}{2})^{\kappa_1} \prod_{i=2}^{n} x_i^{\alpha_i} \prod_{i=2}^{r} a_i^{\kappa_i}).
\]

We have

\[
\begin{align*}
(1) \quad & R\varphi_B! (Rh'_{1!} K(1, \ldots, d) \otimes \mathcal{F}_B) \otimes K((\frac{-1}{d})^{d-1}) \simeq T^*_{(1,d)} R\varphi_{1!} \mathcal{F}_1 \\
(2) \quad & R\varphi_B! (Rh_{y!} K \otimes \mathcal{F}_B) \simeq R\varphi_{y!} \mathcal{F}_y.
\end{align*}
\]

By the Poincare duality, it is enough to prove that the morphism \( \pi_1 \) induces an isomorphism between (1) and (2). By the comparison theorem, it is enough to show the isomorphism by reduction mod \( p \). Suppose that all the varieties, sheaves and automorphisms are defined over a finite field \( \mathbb{F}_q \) and \( d \mid (q - 1) \). Since \( T^*_{(1,d)} X_1 \to \mathcal{Y} \) is a finite \( \mathcal{G}_d \) covering, we have \( Rh_{y!} K \simeq (Rh'_{1!} K)^{\mathcal{G}_d} \) and the action of \( (\mu_d)^d \) on \( X_1 \) induces a decomposition \( Rh'_{1!} K \simeq \oplus_{(t_1, \ldots, t_d) \mathcal{G}_d} Rh'_{1!} K(t_1, \ldots, t_d) \) according to the characters \( (t_1, \ldots, t_d) \) of \( (\mu_d)^d \). For an element \( \sigma \in \mathcal{G}_d \), we have \( \sigma^*(Rh'_{1!} K(t_1, \ldots, t_d)) = Rh'_{1!} K(t_{\sigma(1)}, \ldots, t_{\sigma(d)}) \).

**Lemma 6.4.** Suppose that \( t_p = t_q \) for some \( 1 \leq p \neq q \leq d \). Let \( \sigma_{pq} \) be the transposition of \( p \) and \( q \). Then for a closed point \( p = (T, D) \) of \( B \), we have

\[
\text{tr}(Frob_{\kappa(p)}^m(\sigma_{pq} + 1) \mid Rh'_{1!} K(t_1, \ldots, t_d))_{\kappa(p)} = 0
\]

for all \( m \geq 1 \).

The proof of this lemma is given later. Let \( \mathcal{F}_1(t_1, \ldots, t_d) = Rh'_{1!} K(t_1, \ldots, t_d) \). If \( t_p = t_q \), by Lemma 6.4 and Lefschetz trace formula for etale cohomology, we have

\[
\begin{align*}
\text{tr}(Frob_{\kappa(a)}^m(\sigma_{pq} + 1) \mid Rh'_{1!} K(t_1, \ldots, t_d))_{\kappa(a)} &= \sum_{p \in \varphi_B^{-1}(a)(\kappa^m(a))} \text{tr}(Frob_p(\sigma_{pq} + 1) \mid Rh'_{1!} K(t_1, \ldots, t_d))_{\kappa(p)}) \cdot Frob_p | (\mathcal{F}_B)_{\kappa(p)} \\
&= 0.
\end{align*}
\]

Here \( \kappa^m(a) \) denotes the degree \( m \) extension of \( \kappa(a) \). Therefore by Proposition 2.4, we have

\[
\text{tr}(Frob_{\kappa(a)}^m(\sigma_{pq} + 1) \mid Rh'_{1!} K(t_1, \ldots, t_d))_{\kappa(a)} = 0
\]
for all positive $m$ and $\sigma_{p,q}$ acts as $-1$ on $R^n\varphi_{B!}\mathcal{F}'_1(t_1,\ldots,t_d)$. As a consequence, we have

$$(R^n\varphi_{B!}\mathcal{R}h'_1K)^{\mathcal{G}_a}_{\bar{\kappa}(a)} \simeq R^n\varphi_{B!}\mathcal{F}'_1(1,\ldots,d))_{\bar{\kappa}(a)}$$

and we obtain the proposition.

**Proof of Lemma 6.4.** We may assume that $\kappa(p) = F_q$. Let $X_{T,D_0}$ be the variety defined by

$$X_{T,D_0} = \{ \xi \in T(\xi) \mid \xi^d_1 + \cdots + \xi^d_d = T, (\xi_1 \cdots \xi_d)^d = D_0 \}.$$  

It is easy to see that the fiber of $h'_1$ at $(T,D)$ is isomorphic to $X_{T,D^+}$. Though this variety is not geometrically connected, we consider the cohomology with compact support. For a character $t = (t_1,\ldots,t_d)$ of $(\mu_d)^d$, $H^i_c(X_{T,D},K)(t_1,\ldots,t_d)$, denotes the $t$-part of the corresponding cohomology. Define $\Phi(T)$ by

$$\Phi(T) = \sum_{i=0}^{2d-4} (-1)^i \text{tr}(\text{Frob}_{F_\xi}(\sigma_{p,q} + 1) \mid H^i_c(X_{T,D},K)(t_1,\ldots,t_d)).$$

First we assume that $T \neq 0$. Let $Z_{D_0}$ be the variety defined by

$$Z_{D_0} = \{ (\xi,T) \in T(\xi) \times T(T) \mid \xi^d_1 + \cdots + \xi^d_d = T, (\xi_1 \cdots \xi_d)^d = D_0 \}$$

and $u : Z_{D_0} \to T(T)$ be the natural homomorphism. For a multiplicative character $\gamma : F_q^\times \to K$, $K(\gamma)$ denotes the corresponding rank 1 local system on $T(T)$. Then by the Lefschetz trace formula, we have

$$\text{tr}(\text{Frob}_{F_\xi}(\sigma_{p,q} + 1) \mid H^*_c(T(T),R u_1 K(t_1,\ldots,t_d) \otimes K(\gamma)) = \sum_{T \in F_q^\times} \Phi(T)\gamma(T).$$

On the other hand,

$$H^i_c(T(T),R u_1 K(t_1,\ldots,t_d) \otimes K(\gamma)) = H^i_c(Z_{D_0},K)(t_1,\ldots,t_d,\gamma)$$

$$= \left\{ \begin{array}{ll} 1\text{-dimensional vector space over } K & \text{ if } i = d - 1 \\ 0 & \text{ if } i \neq d - 1 \end{array} \right.$$ 

since $Z_{D_0}$ is a quotient of a $d - 1$-dimensional Fermat hypersurface. It is easy to see that $\sigma_{p,q}$ acts as $(-1)$-multiplication on $H^{d-1}(Z_{D_0},K)(t_1,\ldots,t_d,\gamma)$. Therefore we have $\sum_{T \in F_q^\times} \Phi(T)\gamma(T) = 0$ for all $\gamma$. As a consequence, we have $\Phi(T) = 0$. For the case, $T = 0$, the variety $X_{0,D}$ is a quotient of a Fermat hypersurface. In this case we also have $\Phi(0) = 0$.

### §6.3 The case $k = F_q$

In this subsection, we prove (2) of Theorem 3. We use the same systems of coordinates. Let $\#\mathbf{Z}/\text{Im}(R) = dp^e$, where $(d,e) = 1$. As in §6.2, we define five
varieties $\mathcal{X}_{D(1,d)}$, $\mathcal{X}_1$, $\mathcal{Y}$, $\mathcal{X}_2$ and $\mathcal{X}_{D(d)}$ by
\[
\mathcal{X}_{D(1,d)} = \{(u, x, b, z, t, a_{\geq 2}) \in \mathbf{T}(u) \times \mathbf{T}(x) \times \mathbf{T}(b) \times \mathbf{T}(z) \times \mathbf{A}(t) \times \mathbf{T}(a_{\geq 2}) \mid \begin{align*}
&b_1u_1 + \cdots + b_du_d + f_0(t^p, x_2, \ldots, x_{n+1}) = t, u_1\cdots u_d = z^k \end{align*} \}
\]
\[
\mathcal{X}_1 = \{(\xi, x, z, t, a) \in \mathbf{T}(\xi) \times \mathbf{T}(x) \times \mathbf{T}(z) \times \mathbf{A}(t) \times \mathbf{T}(a) \mid \begin{align*}
a_1(\xi_1^d + \cdots + \xi_d^d) + f_0(t^p, x_2, \ldots, x_{n+1}) = t, u_1\cdots u_d = z^k \end{align*} \}
\]
\[
\mathcal{Y} = \{(s^0, s_d, x, z, t, a) \in \mathbf{A}(s^0) \times \mathbf{T}(s_d) \times \mathbf{T}(x) \times \mathbf{T}(z) \times \mathbf{A}(t) \times \mathbf{T}(a) \mid \begin{align*}
a_1'(F(s_1, \ldots, s_{d-1}, 0) + (-1)^{d-1}ds_d) + f_0(z^p, x_2, \ldots, x_{n+1}) = t, s_d^d = t^k \end{align*} \}
\]
\[
\mathcal{X}_2 = \{(\gamma, x_1, x, t, a) \in \mathbf{A}(\gamma) \times \mathbf{T}(x_1) \times \mathbf{T}(x) \times \mathbf{A}(t) \times \mathbf{T}(a) \mid \begin{align*}
a_1'(\gamma_1^d + \cdots + \gamma_{d-1}^d) + a_1x_1^k + f_0(x_1^d, x_2, \ldots, x_{n+1}) = t \end{align*} \}
\]
\[
\mathcal{X}_{D(d)} = \{(g, x_1, x, t, c, a) \in \mathbf{T}(g) \times \mathbf{T}(x_1) \times \mathbf{T}(x) \times \mathbf{A}(t) \times \mathbf{T}(c) \times \mathbf{T}(a) \mid \begin{align*}
c_1g_1 + \cdots + c_{d-1}g_{d-1} + a_1x_1^k + f_0(x_1^d, x_2, \ldots, x_{n+1}) = t \end{align*} \}
\]
We define a morphism $\pi_1 : T(1,d)\mathcal{X}_1 \to \mathcal{Y}$ and $\pi_2 : \mathcal{X}_2 \to \mathcal{Y}$ as in §6.2. Then $\pi_1$ and $\pi_2$ are quotient by the symmetric group $\mathcal{S}_d$ and $\mathcal{S}_{d-1}$. We define sheaves on each varieties:
\[
\mathcal{F}^{(1,d)} = \kappa(\mathcal{X}_{1,d}) = \prod_{i=1}^d (b_iu_i)^{(i+\kappa_1)/d} \prod_{i=2}^{n+1} x_i^{\alpha_i} \prod_{i=2}^r a_i^{\kappa_i} \otimes \mathcal{L}_\psi(t)
\]
\[
\mathcal{F}_1 = \kappa(\mathcal{X}_1) = \prod_{i=1}^d (a_i\xi_i^d)^{(i+\kappa_1)/d} \prod_{i=2}^{n+1} x_i^{\alpha_i} \prod_{i=2}^r a_i^{\kappa_i} \otimes \mathcal{L}_\psi(t)
\]
\[
\mathcal{F}_y = \kappa(\mathcal{Y}) = (a')_{d-1}^{-1} (a_1s_1)^{\kappa_1} \prod_{i=2}^{n+1} x_i^{\alpha_i} \prod_{i=2}^r a_i^{\kappa_i} \otimes \mathcal{L}_\psi(t)
\]
\[
\mathcal{F}_2 = \kappa(\mathcal{X}_2) = \prod_{i=1}^{d-1} (a_1\gamma_i^d)^{i/d} (a_1x_1^k)^{\kappa_1} \prod_{i=2}^{n+1} x_i^{\alpha_i} \prod_{i=2}^r a_i^{\kappa_i} \otimes \mathcal{L}_\psi(t)
\]
\[
\mathcal{F}^{(d)} = \kappa(\mathcal{X}_{D(d)}) = \prod_{i=1}^{d-1} (c_1g_1)^{i/d} (a_1x_1^k)^{\kappa_1} \prod_{i=2}^{n+1} x_i^{\alpha_i} \prod_{i=2}^r a_i^{\kappa_i} \otimes \mathcal{L}_\psi(t)
\]
Let $l^*$ be the morphism $\mathbf{T}(a) \to \mathbf{T}(R)$ induced by $l$. Then we have
\[
l^*G(D^{(1,d)}, \psi) \simeq R^n\varphi_{1*}\mathcal{F}_1
\]
\[
l^*G(D^{(d)}, \psi) \simeq R^n\varphi_{2*}\mathcal{F}_2.
\]
As in §6.2, it is enough to prove the following proposition.

**Proposition 6.5.** On the torus $\mathbf{T}(a)$, the morphism $\pi_1$ and $\pi_2$ induce isomorphisms of sheaves:
\[
(1) \quad T_{(1,d)}^* (R^n\varphi_{1*}\mathcal{F}_1) \simeq R^n\varphi_y \mathcal{F}_y \otimes K((-1)^{d-1}D)/D \alpha_1)
\]
\[
(2) \quad R^n\varphi_{1*}\mathcal{F}_2 \simeq R^n\varphi_y \mathcal{F}_y
\]
Moreover they are compatible with the descent data.

**Proof.** The proof is similar to that of Proposition 6.3 and omit the proof.
§7 Multiplication by $p$ for $k = \mathbb{F}_q$

Let $D = (R, \{l_i\}_i, \{\kappa_i\}_i)$ be a hypergeometric data and we put $l_1(R) = d\mathbb{Z}$. Suppose that $d$ is divisible by $p$; $d = pd'$. Since $p$ is invertible in $I(\mathbb{F}_q, \mathbb{Q}_l)$, there exists unique $\kappa'_i$ such that $p\kappa'_i = \kappa_i$. Let $l'_1 = \frac{1}{p}l_1$ then the non-resonance condition for $D = (R, \{l_i\}_i, \{\kappa_i\}_i)$ is equivalent to that for $D^{(1,p)} = (R, \{l'_i, l_i\}_{i \geq 2}, \{\kappa'_i, \kappa_i\}_{i \geq 2})$ by the following commutative diagram.

\[
\begin{array}{ccc}
R \xrightarrow{(l'_1, l_2, \ldots, l_r)} \mathbb{Z} \oplus \mathbb{Z}^{r-1} & \longrightarrow & L' \\
\downarrow & & \downarrow \\
R \xrightarrow{(l_1, \ldots, l_r)} \mathbb{Z} \oplus \mathbb{Z}^{r-1} & \longrightarrow & L
\end{array}
\]

By choosing a sufficiently good base as in the last section, we may assume that $\omega_1 = (k, 0, \ldots, 0)$ and the first component of $\omega_i$ $(i = 2, \ldots, n)$ is divisible by $d = pd'$. Therefore we can write $\sum_{i=2}^{r} a_i x^{\omega_i} = f_0(x^p_1, x_2, \ldots, x_{n+1})$ and the defining equation of $\mathcal{X}$ and $\mathcal{X}'$ for the hypergeometric data $D$ and $D'$ is

\[
\mathcal{X} = \{(x, \tau, a) \in \mathbf{T}(x) \times \mathbf{A}(\tau) \times \mathbf{T}(a) \mid a_1 x_1^k + f_0(x^p_1, x_2, \ldots, x_{n+1}) = \tau^p - \tau \}
\]
\[
\mathcal{X}' = \{(y, x_2, \ldots, x_n, \tau, b_1, a_2, \ldots, a_r) \in \mathbf{T}(x) \times \mathbf{A}(\tau) \times \mathbf{T}(b_1) \times \mathbf{T}(a_2, \ldots, a_r) \mid b_1 y^k_1 + f_0(y_1, x_2, \ldots, x_{n+1}) = \sigma^p - \sigma \}.
\]

The corresponding rank 1 local system is given by

\[
K(\prod_{i=1}^{r} (a_i x^{\omega_i})^{\kappa_i}) \quad \text{and} \quad K((b_1 y^k_1)^{\kappa'_i} \prod_{i=2}^{r} (a_i x^{\omega_i})^{\kappa_i}).
\]

Define a homomorphism $\mathcal{X} \to \mathcal{X}'$ by sending

\[
(x, \tau, a) \to (y_1, x_2, \ldots, x_{n+1}, \sigma, a) = (x^p_1, x_2, \ldots, x_{n+1}, \tau + a_1 x_1^k, a^p_1, a_2, \ldots, a_r).
\]

This is equivariant under the action of $\mathbb{F}_q$. Then it easy to see that the corresponding character coincides. Taking account into the above commutative diagram and the definition of descent data, we have an isomorphism of hypergeometric sheaves on $\mathbf{T}(R)$: $\mathcal{G}(D) \simeq \mathcal{G}(D^{(1,p)})$ by taking the Artin-Shreier character part of the higher direct images. As a consequence, we have the following proposition.

**Proposition 7.1.** Let $D = (R, \{l_i\}_i, \{\kappa_i\}_i)$ be a non-resonant hypergeometric data and such that $l_1(R)$ is divisible by $p$. If we put $D^{(1,p)} = (R, \{l'_1, l_i\}_{i \geq 2}, \{\kappa'_i, \kappa_i\}_{i \geq 2})$, where $p\kappa'_i = \kappa_i$ and $l'_1 = \frac{1}{p}l_1$. Then $D^{(1,p)}$ is also non-resonant and on $\mathbf{T}(R)$, we have an isomorphism of constructible sheaves $\mathcal{G}(D) \simeq \mathcal{G}(D^{(1,p)})$ induced by an algebraic correspondences.

**Definition.** The equivalence relation generated by $D \sim D^{(1,p)}$ is called the Frobenius equivalence.
§8 Cohomological Mellin Transform

In this section, we give a definition of cohomological Mellin transform. This notion is also introduced in [L-S] and [G-L] in different presentation. First we consider the case $k = \mathbb{C}$.

**Proposition 8.1.** Let $G(D)$ be the hypergeometric sheaf of a hypergeometric data $D$ and $\chi \in R \otimes I(\mathbb{C}, K)$ be a character of $\pi_1(T(R))$ of finite order. Then the cohomology group $H^{r-n-1}(T(R), G(D) \otimes K(\chi))$ has a mixed Hodge structure. Moreover if $\kappa_i + l_i(\chi)$ are not zero for all $i = 1, \ldots, r$, it is pure of weight $r$ (resp. $r-1$) and one dimensional. Moreover the Hodge type is given by

$$\left(\sum_{i=1}^{r} <\sigma(\kappa_i + l_i(\chi))> + <\sum_{i=1}^{r} \sigma \kappa_i>, \sum_{i=1}^{r} < -\sigma(\kappa_i + l_i(\chi))> + <\sum_{i=1}^{r} \sigma \kappa_i>\right)$$

(resp. $\sum_{i=1}^{r} \kappa_i \neq 0$ (resp. $\sum_{i=1}^{r} \kappa_i = 0$).

**Definition.** For a constructible sheaf $\mathcal{F}$ which is a mixed Hodge complex on a torus $T$ and an element $\chi$ in Hom($T, G_m$) $\otimes I(\mathbb{C}, K)$, the mixed Hodge structure $H^{r-n-1}(T(R), \mathcal{F} \otimes K(\chi))$ is denoted by $C(\mathcal{F}, \chi)$. This “function “ from Hom($T, G_m$) $\otimes I(\mathbb{C}, K)$ to the category of mixed Hodge structure is called the cohomological Mellin transform of $\mathcal{F}$.

**Proof.** We choose a base of $L \simeq \bigoplus_{i=0}^{n} \mathbb{Z} v_i$. We consider the variety over $T(a)$. By changing coordinate, $\mathcal{X}$ is actually defined over $T(R)$. To study the descent variety $\mathcal{X}_R$, we choose a section $s : L \to \mathbb{Z}$ of $q : \mathbb{Z}^{r} \to L$. Let $u_i = a^{s(v_i)}$ ($i = 0, \ldots, n$) and put $y_i = u_i x_i$. Then the equation for $\mathcal{X}_R$ is $\sum a^{e_i - (s q)(v_i)} y^{\omega_i} = 1$. Therefore the coefficient $a^{e_i - (s q)(v_i)} \in \mathbb{C}[R]$. Using new coordinate $y = (y_0, \ldots, y_n)$,

$$\mathcal{X}_R = \{ (r, y) \in T(R) \times T(y) | \sum a^{e_i - (s q)(v_i)} y^{\omega_i} = 1 \}.$$  

Again we change coordinate given by $b_i = a^{e_i - (s q)(v_i)} y^{\omega_i}$. This change of coordinate gives an isomorphism $T(R) \times T(y) \simeq T(b)$, where $b = (b_0, \ldots, b_n)$. In fact, we have an equality

$$\left\{ \begin{array}{l}
 b^{e_i - (s q)(v_i)} = a^{e_i - (s q)(v_i)} \\
 b^{-s(v_i)} = y_i.
\end{array} \right.$$  

Therefore $\mathcal{X}_R \simeq \mathcal{X}_b = \{ b \in T(b) | \sum_{i=1}^{r} b_i = 1 \}$. Under this isomorphism, the character $\kappa(\mathcal{X}_R, x^\alpha a^\kappa)$ corresponds to $\kappa(\mathcal{X}_b, b^\kappa)$. We compute the cohomology $H^i(\mathcal{X}_b, \mathcal{F}_\chi)$ using Leray spectral sequence for $\tilde{f} : \mathcal{X}_b \simeq \mathcal{X}_R \to T(R)$, where $\mathcal{F}_\chi = K(\kappa(\mathcal{X}_b; b^{\kappa + 1}(\chi)))$. By Proposition 2.5, $R^i \tilde{f}_* \mathcal{F}_\chi = 0$ if $i \neq n$. Therefore we have

$$H^{r-1}(\mathcal{X}_b, \mathcal{F}_\chi) \simeq H^{n-r-1}(T(R), \mathbb{C}^n \tilde{f}_* \mathcal{F}_\chi)$$

$$H^{n-r-1}(T(R), G(D) \otimes K(\chi)).$$

To show the last part of the theorem, we can compute the Hodge number of the cohomology of $H^{r-1}(\mathcal{X}_b, \mathcal{F}_\chi)$ via that of Fermat hypersurfaces (See [S]).
To state the similar results for $k = \mathbb{F}_q$, we recall the definition of Gaussian sum. Let $\overline{p}$ be an extension of valuation of $p$ to $\overline{Q} \subset \overline{Q}_l$. We normalize additive valuation by $\text{ord}_{\overline{p}}(p) = 1$. Via the reduction mod $\overline{p}$, $\mu_m(\overline{Q}_l)$ is naturally identifies with $\mu_m(\mathbb{F}_q)$ if $(m, p) = 1$. The inverse of the reduction map is denoted by $\omega : \overline{F}_q \to \bigcup_{(m, p) = 1} \mu_m(\overline{Q}_l)$. Therefore $I_n(\mathbb{F}_q, \overline{Q}_l)$ is identified with $\frac{1}{m}\mathbb{Z}/\mathbb{Z}$, and $q = p^e$ such that $m \mid q - 1$, we define a Gaussian sum by

$$g(\kappa, \psi) = g(\mathbb{F}_q, \kappa, \psi) = \sum_{x \in \mathbb{F}_q^\times} \chi_{\kappa}(x)^{\frac{q^{-1}}{m}} \psi(\text{tr}_{\mathbb{F}_q/\mathbb{F}_p}(x)),$$

where $\chi_{\kappa}(x) = \omega(x^\frac{m}{(q-1)m})$. It is easy to see that $g(0, \psi) = -1$. The order of $g(\mathbb{F}_q, \kappa, \psi)$ at $\overline{p}$ is given by $\text{ord}_{\overline{p}}(g(\mathbb{F}_q, \kappa, \psi)) = \sum_{i=0}^{e-1} < p^i \kappa >$. Therefore $\frac{1}{e} \sum_{i=0}^{e-1} < p^i \kappa >$ is independent of the choice of $q = p^e$ such that $m \mid q - 1$. Moreover if $\kappa = 0$, we have $\text{ord}_{\overline{p}}(g(\mathbb{F}_q, 0, \psi) = 0$. For $\sigma \in \text{Gal}(\overline{Q}/Q)$

$$\text{ord}_{\overline{p}}(g(\mathbb{F}_q, \sigma \kappa, \psi) = \sum_{i=0}^{e-1} < tp^i \kappa >,$$

where $t \in (\mathbb{Z}/m\mathbb{Z})^\times$ is defined by the equality $\sigma(\zeta) = \zeta^t$ for all $\zeta \in \mu_m$. For an element $\mathbb{Z}_l(\overline{p})/\mathbb{Z}$, we use the following notation $\langle <x> \rangle = \frac{1}{e} \sum_{i=0}^{e-1} < p^ix > - \frac{1}{2}$, where $e$ is the minimal positive integer such that $p^e x = x$. For $k = \mathbb{F}_q$, we have the following proposition.

**Proposition 8.2.** Let $G(D, \psi)$ be the hypergeometric sheaf of a hypergeometric data $D$. Let $\chi \in R \otimes I(\mathbb{F}_q, \overline{Q}_l)$ be a character of $\pi_1(\text{Tr}(R))$ of finite order. We put

$$C(G(D, \psi), \chi) = \sum_{i=0}^{2(r-n-1)} (-1)^i (\text{tr}_{\mathbb{F}_q/\mathbb{F}_p}(H^i_\psi(\text{Tr}(R) \otimes \mathbb{F}_q, G(D, \psi) \otimes K(\chi))))$$

Then we have $C(G(D, \psi), \chi) = \prod_{i=1}^{r} g(l_i(\chi) + \kappa_i, \psi)$.

**Definition.** For a constructible sheaf $\mathcal{F}$ of $\text{Tr}$, the function from $\text{Hom}(\text{Tr}, G_m) \otimes I(\mathbb{F}_q, \overline{Q}_l)$ to $\mathbb{Q}_l$ given by

$$\chi \mapsto C(\mathcal{F}, \chi) = \sum_{i=0}^{2(r-n-1)} (-1)^i \text{tr}_{\mathbb{F}_q/\mathbb{F}_p}(H^i_\psi(\text{Tr}(R) \otimes \mathbb{F}_q, \mathcal{F} \otimes K(\chi)))$$

is called the cohomological Mellin transform of $\mathcal{F}$.

**Proof.** We can prove the proposition in the same way. The variety $\mathcal{X}$

$$\mathcal{X} = \{(x, t, a) \mid \text{Tr}(x) \times A(t) \times \text{Tr}(a) \mid f(a, x) = t\}$$

and the sheaf $R^n \varphi_* K(\mathcal{X}; x^\alpha a^\kappa) \otimes L_\psi(t)$ on $\text{Tr}(a)$ is descent to $\text{Tr}(R)$. Via the same change of coordinate, the descent variety $\mathcal{X}_b$ is isomorphic to $\mathcal{X}_b$:

$$\mathcal{X}_b = \{(b, t) \in \text{Tr}(b) \times A(t) \mid \sum_{i=1}^{r} b_i = t\}$$
The descent sheaf on \( X_b \) corresponds to the Kummer character \( \kappa(b^\kappa) = \kappa(X_b; \prod_{i=1}^r b_i^\kappa) \). Since \( R^i\varphi_!K(\chi; x^a a^\nu) \otimes \mathcal{L}_\psi(t) \simeq R^i\varphi_*K(\chi; x^a a^\nu) \otimes \mathcal{L}_\psi(t) = 0 \) for \( i \neq n \), we have \( H^r_{\text{c}}(\mathcal{T}(R), G(D, \psi) \otimes K(\chi)) \). The alternating sum of the trace of Frobenius substitution on the cohomologies of \( X_b \) is given by

\[
C(G(D, \psi), \chi) = \sum_{b_i \in \mathbb{F}_q^r} \chi_{\kappa_i + l_i(\chi)}(b_i) \psi(\sum_{i=1}^r \text{tr}_{\mathbb{F}_q/\mathbb{F}_p}(b_i))
= \prod_{i=1}^r \left( \sum_{b_i \in \mathbb{F}_q} \chi_{\kappa_i + l_i(\chi)}(b_i) \psi(\text{tr}_{\mathbb{F}_q/\mathbb{F}_p}(b_i)) \right)
= \prod_{i=1}^r g(l_i(\chi) + \kappa_i, \psi).
\]

**Corollary.** The order of \( C(G(D, \psi)) \) is given by

\[
\text{ord}_p C(G(D, \psi)) = [\mathbb{F}_q : \mathbb{F}_p] \sum_{i=1}^r (\langle l_i(\chi) + \kappa_i \rangle) + \frac{1}{2}
\]

§9 A LEMMA FOR INDEPENDENCE

A hypergeometric data \( D = (R, \{l_i\}_i, \{\kappa_i\}_i) \) is called primitive if \( l_i(R) = \mathbb{Z} \). Let \( D = (R, \{l_i\}_i, \{\kappa_i\}_i) \) be a primitive hypergeometric data. The data \( D \) is called non-divisorial if there exists no \( i, j \) \((i \neq j)\) such that \( l_i + l_j = 0 \) and \( \kappa_i + \kappa_j = 0 \). In this section, we prove the following proposition.

**Proposition 9.1.** Let \( D = (R, \{l_i\}_i, \{\kappa_i\}_i) \) and \( D' = (R, \{l'_j\}_j, \{\kappa'_j\}_j) \) be primitive reduced non-divisorial hypergeometric data and \( c, c' \) be elements in \( \mathbb{Q} \).

(1) Suppose that the equality

\[
\sum_{i=1}^r (\langle l_i(\chi) + \kappa_i \rangle > -\frac{1}{2}) + c = \sum_{i=1}^{r'} (\langle l'_i(\chi) + \kappa'_i \rangle > -\frac{1}{2}) + c'
\]

holds for a dense open set in \( \chi \in R \otimes (\mathbb{R}/\mathbb{Z}) \). (We say that the equality holds for almost all \( r \) for short.) Then \( D \) is equal to \( D' \) up to permutation and \( c = c' \).

(2) Suppose that \( \kappa_i, \kappa'_j \in I(\bar{\mathbb{F}}_q, \bar{\mathbb{Q}}_l) \) and that the equality

\[
\sum_{i=1}^r (\langle \sigma(l_i(\chi) + \kappa_i) \rangle) + c = \sum_{i=1}^{r'} (\langle \sigma(l'_i(\chi) + \kappa'_i) \rangle) + c'
\]

holds for all \( \chi \in R \otimes I(\bar{\mathbb{F}}_q, \bar{\mathbb{Q}}_l) \) and \( \sigma : \bar{\mathbb{Q}}_l \to \bar{\mathbb{Q}}_l \). Then \( D \) is equal to \( D' \) up to permutation and \( c = c' \).

To prove the proposition, we prove the following lemma.
Lemma 9.2.

(1) Let \( \kappa_i \in \mathbb{Q}/\mathbb{Z} \) be distinct elements and \( c \in \mathbb{Q} \). If
\[
\sum_{i=1}^{r} m_i (\langle x + \kappa_i \rangle - \frac{1}{2}) + c = 0
\]
holds for almost all \( x \in \mathbb{R}/\mathbb{Z} \). Then \( m_i = 0 \) for all \( i \) and \( c = 0 \).

(2) Let \( \kappa_i \in \mathbb{Z}_{(p)}/\mathbb{Z} \) be distinct elements and \( c \in \mathbb{Q} \). If
\[
\sum_{i=1}^{r} m_i (\langle t(x + \kappa_i) \rangle) + c = 0
\]
for all \( x \in \mathbb{Z}_{(p)}/\mathbb{Z} \) and \( t \in (\hat{\mathbb{Z}}')^\times \), then \( m_i = 0 \) for all \( i \) and \( c = 0 \). Here \( \hat{\mathbb{Z}}' = \lim_{(m,p) \to 1} \mathbb{Z}/m\mathbb{Z} \).

Proof. (1) Let \( f(x) = \sum_{i=1}^{r} m_i (\langle x + \kappa_i \rangle - \frac{1}{2}) + c. \) Since \( m_i = \lim_{\epsilon \to 0} f(x - \kappa_i + \epsilon) + f(x - \kappa_i - \epsilon) \), we get the theorem. (2) To prove this theorem, we use an arithmetic argument. We prove this statement in Appendix.

Proof of Proposition 9.1. (1) We prove the proposition by the induction on rank \( R \). If we take a base of \( R \cong \mathbb{Z} \), \( l_i(\chi) \) and \( l'_j(\chi) \) can be written as \( \chi \) or \( -\chi \) by the primitivity condition. Let \( S_\pm \) and \( S'_\pm \) be \( S_\pm = \{ i \mid l_i = \pm \chi \} \) and \( S'_\pm = \{ j \mid l'_j = \pm \chi \} \). By the assumption of non-divisoriality, \( K'_+ = \{ \kappa_i \mid i \in S'_+ \} \) (resp. \( K'_- = \{ \kappa_j \mid j \in S'_- \} \) ) and \( -K_- = \{ -\kappa_i \mid i \in S_- \} \) (resp. \( -K'_- = \{ -\kappa_j \mid j \in S'_- \} \) ) does not intersect to each other. Since \( \langle x \rangle > -\frac{1}{2} = -(< -x > - \frac{1}{2}) \) for almost all \( x \), we have
\[
\sum_{i=1}^{r} (\langle l_i(\chi) + \kappa_i \rangle - \frac{1}{2}) = \sum_{i \in S_+} (\langle \chi + \kappa_i \rangle - \frac{1}{2}) - \sum_{i \in S_-} (\langle \chi - \kappa_i \rangle - \frac{1}{2})
\]
\[
= \sum_{\kappa_i \in K_+} m_i (\langle \chi + \kappa_i \rangle - \frac{1}{2}) - \sum_{\kappa_i \in K_-} m_i (\langle \chi - \kappa_i \rangle - \frac{1}{2})
\]
for almost all \( \chi \), where \( m_i = \# \{ i \mid \kappa_i \in K_+ \cup K_- \} \) is the multiplicity of \( \kappa_i \). By the assumption,
\[
\sum_{\kappa_i \in K_+} m_i (\langle \chi + \kappa_i \rangle - \frac{1}{2}) - \sum_{\kappa_i \in K_-} m_i (\langle \chi - \kappa_i \rangle - \frac{1}{2}) + c
\]
\[
= \sum_{\kappa_j \in K'_+} m'_j (\langle \chi + \kappa'_j \rangle - \frac{1}{2}) - \sum_{\kappa_j \in K'_-} m'_j (\langle \chi - \kappa'_j \rangle - \frac{1}{2}) + c'
\]
for almost all \( \chi \). Here \( m'_j \) is the multiplicity of \( \kappa'_j \) defined in the same way. Since \( m_i, m'_j > 0 \) for all \( i, j \), we have \( c = c' \), \( K_+ = K'_+ \), \( K_- = K'_- \) and \( m_i = m'_j \) if \( \kappa_i = \kappa'_j \) by Lemma 9.2.

Assume that rank \( R = r - n - 1 \geq 2 \). Since \( l_1 \) is primitive, we choose a base \( \kappa_1, \kappa_2, \ldots, \kappa_{r-1} \) such that \( \ker(l_1) = \langle \kappa_1, \kappa_2, \ldots, \kappa_{r-1} \rangle \). The coordinate is written
as $\chi_1, \ldots, \chi_{r-n-1}$. Let $S = \{i \mid l_i = \pm l_1\}$ and $S' = \{i \mid l'_j = \pm l_1\}$. For $i \notin S$ and $j \notin S'$, $\bar{l}_i$ and $\bar{l}'_j$ be the restriction of $l_i$ and $l'_j$ to $\text{Ker}(l_1)$. Since $\bar{l}_i$ and $\bar{l}'_j$ is not zero, we have $\bar{l}_i = d_i l_i$ and $\bar{l}'_j = d'_j l'_j$ with primitive forms $l_i$ and $l'_j$. Therefore for $r_1 v_1 \in Q v_1$, we have $\langle \bar{l}_i + \bar{l}_i(\chi_1) + \kappa_i \rangle > -\frac{1}{2} = \sum_{k=1}^{d_i} (\langle \bar{l}_i + \frac{1}{d_i} (\bar{l}_i(\chi_1) + \kappa_i + k) \rangle > -\frac{1}{2})$. Therefore $\sum_{i \notin S}(\langle \bar{l}_i + \bar{l}_i(\chi_1) + \kappa_i \rangle > -\frac{1}{2})$ can be expressed as a sum of $\langle L_i + K_i \rangle$, where $L_i$ is a primitive linear form on $\text{Ker}(l_1)$. If necessary, by canceling terms, we may assume that the sum does not contain divisorial pair in this expression. Since the restriction of $\sum_{i \in S}(\langle l_i + \kappa_i \rangle > -\frac{1}{2})$ to $\text{Ker}(l_1) + \chi_1 v_1$ is a constant function, we have

$$\sum_{i \in S}(\langle l_i(\chi_1 v_1) + \kappa_i \rangle > -\frac{1}{2}) + c = \sum_{j \in S'}(\langle l'_j(\chi_1 v_1) + \kappa'_j \rangle > -\frac{1}{2}) + c'$$

for almost all $\chi_1$ by using the assumption of induction for $\text{Ker}(l_1)$. (Note that non-divisoriality condition inherits to the restriction to $\text{Ker}(l_1) + \chi_1 v_1$.) Applying the assumption for $v_1 Q$, $c = c'$, $\{\kappa_i \mid l_i = l_1\} = \{\kappa'_j \mid l'_j = l_1\}, \{\kappa_i \mid l_i = -l_1\} = \{\kappa'_j \mid l'_j = -l_1\}$ and $\# \{i \in S \mid \kappa_i = \kappa\} = \# \{j \in S' \mid \kappa'_j = \kappa\}$ for all $\kappa \in Q/Z$.

Replacing $l_1$ to another primitive linear form on $R$, we get the proposition.

For (2), we can prove the same argument.

**Lemma 9.3.** Under the non-resonance condition, the primitive hypergeometric data is non-divisorial

**Proof.** Assume that $l_i + l_j = 0$ and $\kappa_i + \kappa_j = 0$. For simplicity, we may assume that $i = 1, j = 2$. Let $R' = \text{Ker}(l_1)$. Consider the following commutative diagram, where $L' = Z^{r-2}/R'$.

```
0 \rightarrow R' \rightarrow Z^{r-2} \rightarrow L' \rightarrow 0
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow
0 \rightarrow R \rightarrow Z^r \rightarrow L \rightarrow 0
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow
0 \rightarrow Z \rightarrow Z^2 \rightarrow Z \rightarrow 0
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow
0 \quad 0
```

Then we have $L/L' \simeq Z$ and

$$L = L' \oplus \omega_1. \quad (*)$$

Take an element $\rho \in R$ such that $l_1(\rho) = 1$ and $l_2(\rho) = -1$. Put $l_k(k) = a_k$. Then we have

$$\omega_1 - \omega_2 + \sum_{i=1}^{r} a_i \omega_k = 0. \quad (***)$$
The cone $C'$ generated by $\omega_i$ for $i = 3, \ldots, r$ is contained in codimension 1 subspace $L' \otimes R$ and the cone $C(\Delta)$ generated by $\omega_i$ $(i = 1, \ldots, r)$ is a cone of $\omega_1$ and $\omega_2$ over $C'$ by the equality (**). Therefore $C'$ is a codimension 1 face of $C(\Delta)$. It is easy to see that the equation of for the face $C'$ is given by the projection $pr$ to the $L'$ component via the decomposition (*). Since the image $\alpha$ of $\kappa$ is given by

$$\alpha = \kappa_1 \omega_1 + (-\kappa_1) \omega_2 + \sum_{i=3}^{r} \kappa_i \omega_i = -\kappa_1 \sum_{i=3}^{r} a_i \omega_i + \sum_{i=3}^{r} \kappa_i \omega_i,$$

we have $pr(\alpha) = 0$. This contradicts to the non-resonance condition.

§10 Main Theorem

In this section we state and prove the main theorem. First we prove the following lemma

**Lemma 4.1.** Let $\mathcal{G}(D)$ and $\mathcal{G}(D, \psi)$ be hypergeometric sheaves on $T(R)$ for $k = \mathbb{C}$ and $\mathbb{F}_q$ respectively. Let $\alpha$ be an element in $T(R)(\mathbb{C})$ (resp. $T(R)(\mathbb{F}_q)$) such that $T^*_\alpha \mathcal{G}(D) = \mathcal{G}(D)$ (resp. $T^*_\alpha \mathcal{G}(D, \psi) = \mathcal{G}(D, \psi)$) if $k = \mathbb{C}$ (resp. $k = \mathbb{F}_q$). Then we have $\alpha = 1$.

**Proof.** First we assume $k = \mathbb{C}$. The subgroup

$$\text{Stab}(\mathcal{G}(D)) = \{ \alpha \in T(R)(\mathbb{C}) \mid T^*_\alpha \mathcal{G}(D) \simeq \mathcal{G}(D) \}$$

is an algebraic subgroup because it is isomorphic to

$$\text{Stab}(\mathcal{G}(D)) = \{ \alpha \in T(R)(\mathbb{C}) \mid T^*_\alpha DR(\mathcal{G}(D) \mid U) \simeq DR(\mathcal{G}(D) \mid U) \},$$

where $DR$ is a de Rham functor for local system on an open set $U$ of $T$. Therefore if $\text{Stab}(\mathcal{G}(D))$ is not zero, it contains a non-trivial finite subgroup $S$. Let $k = \mathbb{F}_q$. If

$$\text{Stab}(\mathcal{G}(D, \psi)) = \{ \alpha \in T(R)(\mathbb{F}_q) \mid T^*_\alpha \mathcal{G}(D, \psi) \simeq \mathcal{G}(D, \psi) \}$$

is non-trivial group, it contains a non-trivial finite subgroup $S$. Therefore the sheaves $\mathcal{G}(D)$ (resp. $\mathcal{G}(D, \psi)$) descend to a sheaf $\mathcal{G}(D)'$ (resp. $\mathcal{G}(D, \psi)'$) on $T(R)/S$, since $\mathcal{G}(D)$ and $\mathcal{G}(D, \psi)$ are irreducible. Therefore we have

$$\chi_c(\mathcal{G}(D)) = \#(S) \cdot \chi_c(\mathcal{G}(D))$$

$$\chi_c(\mathcal{G}(D, \psi)) = \#(S) \cdot \chi_c(\mathcal{G}(D, \psi))'.$$

This contradicts to the fact that $\chi_c(\mathcal{G}(D)) = 1$ (resp. $\chi_c(\mathcal{G}(D, \psi)) = 1$).

Our main theorem is the following.

**Theorem 4.**

1. Let $k = \mathbb{C}$. Let $\mathcal{F}$ and $\mathcal{F}'$ be hypergeometric sheaves on a torus $T$. If there exists an isomorphism $\phi : \mathcal{F} \mid_U \rightarrow \mathcal{F}' \mid_U$ as a variation of $K$-Hodge structures on an open set $U$ in $T$, then there exists an algebraic correspondence $A_l$ and a Hodge cycle of Fermat-Artin-Shreier motif $F$ such that $\phi = A_l \cdot F$.

2. Let $k = \mathbb{F}_q$. Let $\mathcal{F}$ and $\mathcal{F}'$ be hypergeometric sheaves on a torus $T$. If there exists an isomorphism $\phi : \mathcal{F} \mid_U \rightarrow \mathcal{F}' \mid_U$ as a $\mathbb{Q}_l$ local system on an open set $U$ in $T$, then there exists an algebraic correspondence $A_l$ and a Tate cycle of Fermat-Artin-Shreier motif $FAS$ such that $\phi = A_l \cdot FAS$.
(1) Let $D$ be a hypergeometric data. By using multiplicative correspondence successively, there exist a primitive hypergeometric data $D^{(p)}$ and elements $\lambda_1, \ldots, \lambda_a$ such that there exists an isomorphism $\mathcal{G}(D) \simeq T_{\alpha}^{*}\mathcal{G}(D^{(p)})$ for some $\alpha \in \text{bold}T(R)$ induced by an algebraic correspondence, where $\mathcal{D} = D \oplus \oplus_{i=1}^{a}(0, \lambda_i)$. Let $D^{(p)}_{\text{red}}$ be the reduced part of $D^{(p)}$. Then there exist Fermat motives $F_1, F_2$ such that $\mathcal{G}(D) \otimes F_1 \simeq T_{\alpha}^{*}\mathcal{G}(D^{(p)}_{\text{red}}) \otimes F_2$ for some $\alpha \in \text{bold}T(R)$. Therefore there exist a Fermat motif $F_3$ and the following isomorphism induced by the composite of algebraic correspondences

\[(*)\]  

\[\mathcal{G}(D) \otimes F_3 \simeq T_{\alpha}^{*}\mathcal{G}(D^{(p)}),\]

for some $\alpha \in \text{bold}T(R)$. By the isomorphism $(*)$, the difference of Mellin transform of $\mathcal{G}(D)$ and $\mathcal{G}(D^{(p)}_{\text{red}})$ is a constant. Now we compare cohomological Mellin transform of sheaves $\mathcal{F}$ and $\mathcal{F}'$ on $T = \text{bold}T(R')$. Since $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{F}|_U$ and $\mathcal{F}'|_U$ are cohomological mixed Hodge complexes, there is natural mixed Hodge structure on $\text{Hom}(\mathcal{F}, \mathcal{F}') \simeq \mathbb{Q}$ and $\text{Hom}(\mathcal{F}|_U, \mathcal{F}'|_U) \simeq \mathbb{Q}$. Therefore the restriction map

\[\text{Hom}(\mathcal{G}(D), \mathcal{G}(D')) \to \text{Hom}(\mathcal{G}(D)|_U, \mathcal{G}(D')|_U)\]

is an isomorphism of Hodge structure. Since the right hand side is pure of weigh 0 with type $(0, 0)$, $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic as a mixed Hodge complex. Therefore the Mellin transform $C(\mathcal{F}, \chi)$ and $C(\mathcal{F}', \chi)$ of $\mathcal{F}$ and $\mathcal{F}'$ coincide for all $\chi \in R'' = \text{Hom}(T, C^\times) \otimes \mathbb{Q}$. By the definition of hypergeometric sheaves, there exists an isomorphism $\Phi : T \simeq T(R)$ and $\Phi' : T \simeq T(R')$ and hypergeometric data $D = (R, \{l_i\}, \{\kappa_i\})$ and $D' = (R', \{l'_i\}, \{\kappa'_i\})$ such that that $\mathcal{F} = \Phi^*\mathcal{G}(D)$ and $\mathcal{F}' = \Phi'^*\mathcal{G}(D')$. Via the isomorphism $\Phi$ and $\Phi'$, we have an isomorphism of $R \simeq R' \simeq R''$. Under this isomorphism, we identify $R$, $R'$ and $R''$. Via this identification, cohomological Mellin transform $\mathcal{G}(D)$ and $\mathcal{G}(D')$ coincide with those of $\mathcal{F}$ and $\mathcal{F}'$. Therefore the Mellin transform of $\mathcal{G}(D^{(p)}_{\text{red}})$ and $\mathcal{G}(D'^{(p)}_{\text{red}})$ coincide up to constant. By using Proposition 8.1, the explicit calculation of the Hodge type of Mellin transform of $\mathcal{G}(D^{(p)}_{\text{red}})$ and $\mathcal{G}(D'^{(p)}_{\text{red}})$, we have

\[\left(\sum_{i=1}^{r} <\sigma(\kappa_i + l_i(\chi)) > + <\sum_{i=1}^{r} \sigma \kappa_i \right)\]

\[\left(\sum_{i=1}^{r} <\sigma(\kappa'_i + l'_i(\chi)) > + <\sum_{i=1}^{r} \sigma \kappa'_i \right) + c,\]

for some constant $c$, where $D^{(p)}_{\text{red}} = (R, \{l_i\}, \{\kappa_i\})$ and $D'^{(p)}_{\text{red}} = (R, \{l'_i\}, \{\kappa'_i\})$. Therefore by Proposition 9.1, hypergeometric data $D^{(p)}_{\text{red}}$ is equal to $D'^{(p)}_{\text{red}}$ up to permutation. Again using $(*)$, there exists a Fermat motif $F_4$ and an isomorphism

\[\mathcal{G}(D) \otimes F_4 \simeq T_{\alpha}^{*}\mathcal{G}(D')\]

induced by an algebraic correspondence. Comparing to the assumption, $F_1$ is a Hodge cycle. Since $\text{Hom}(\mathcal{G}(D), \mathcal{G}(D'))$ is one dimensional, this is equal to the original isomorphism up to constant. Since $T_{\alpha}^{*}\mathcal{G}(D) = \mathcal{G}(D)$, implies $\alpha = 1$, we have the theorem.
(2) The proof is similar. The isomorphism $G(D^{(p)}_{red}, \psi) |_U \simeq G(D^{(p)}_{red}, \psi) |_U$ implies
\[
\left(\sum_{i=1}^r \langle \langle t(\kappa_i + l_i(\chi)) \rangle \rangle \right) + \left(\sum_{i=1}^r \langle \langle -t\kappa_i \rangle \rangle \right) = \left(\sum_{i=1}^r \langle \langle t(\kappa'_i + l'_i(\chi)) \rangle \rangle \right) + \left(\sum_{i=1}^r \langle \langle -t\kappa'_i \rangle \rangle \right),
\]
for all $t \in (\hat{\mathbb{Z}}')^\times$ and $\chi \in R \otimes (\mathbb{Z}/p\mathbb{Z})$ by taking $p$-adic order of Mellin transform (Corollary to Proposition 8.2). This implies $D^{(p)}_{red}$ is equal to $D^{(p)}_{red}$ up to permutation by Proposition 9.1 (2). The rest of the proof is similar to that of (1).

§ Appendix The proof of Proposition 9.1 (2)

The following lemma is well known.

Lemma A.1. Let $N$ be a natural number greater than 2, $\psi$ be a additive character of $\mathbb{Z}/N\mathbb{Z}$ given by $\psi(x) = \exp(2\pi\sqrt{-1}x/N)$. Then for an odd multiplicative character $\chi$,
\[
L_N(\chi, 1) = -\frac{2\pi\sqrt{-1}}{N} \sum_{m,l=0}^{N-1} \psi(ml)\chi(m)(< \frac{l}{N} > - \frac{1}{2})
\]
is non-zero.

Proof. See [BS].

From now on we assume that $p \neq 2$ and $N$ is even. We introduce a set $S$ (super singular divisor of $N$ for $p$) and $S^c$ of divisor of $N$ by
\[
S = \{ M \mid M \text{ divides } N, \text{ the subgroup } <p>^\times \text{ of } (\mathbb{Z}/M\mathbb{Z})^\times \text{ generated by } p \text{ contains } -1, M \neq 1, 2 \},
\]
\[
S^c = \{ M \mid M \text{ divides } N, \text{ the subgroup } <p>^\times \text{ of } (\mathbb{Z}/M\mathbb{Z})^\times \text{ generated by } p \text{ does not contain } -1, M \neq 1, 2 \}.
\]
By definition of $\langle \langle \rangle \rangle$, we can check the following property.

1. $\langle \langle 0 \rangle \rangle = -\frac{1}{2}$
2. If $x \neq 0$, $\langle \langle -x \rangle \rangle = -\langle \langle x \rangle \rangle$.
3. If $x \in \frac{1}{N}\mathbb{Z}/\mathbb{Z} - 0$ with $M \in S$, then $\langle \langle x \rangle \rangle = 0$.

The properties (1) and (2) is direct consequence from the definition. To show (3), use the equality $\langle \langle px \rangle \rangle = \langle \langle x \rangle \rangle$ and (2). Let $\text{Func}(S)$ be the vector space of $\mathbb{C}$-valued function on the set $S$. The natural linear map from $\text{Func}(\frac{1}{N}\mathbb{Z}/\mathbb{Z})$ to $\text{Func}(\mathbb{Z}/N\mathbb{Z})^\times \times (\frac{1}{N}\mathbb{Z}/\mathbb{Z})$ is induced by the second projection. Let $V$ be the quotient space
\[
\text{Func}(\mathbb{Z}/N\mathbb{Z})^\times \times (\frac{1}{N}\mathbb{Z}/\mathbb{Z})/\text{Func}(\frac{1}{N}\mathbb{Z}/\mathbb{Z}).
\]

We define a linear map $\phi$ from $\bigoplus_{\kappa \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}} \mathbb{C}v_\kappa$ to $V$ defined by $\phi(v_\kappa) = \langle \langle t(x + \kappa) \rangle \rangle$. Here $\langle \langle t(x + \kappa) \rangle \rangle$ is the class of function on $(t, x) \in (\mathbb{Z}/N\mathbb{Z})^\times \times (\frac{1}{N}\mathbb{Z}/\mathbb{Z})$ in $V$. For $\kappa \in \frac{1}{N}\mathbb{Z}/\frac{M}{N}\mathbb{Z}, M \in S \cup \{2\}$ we define an element $\sigma_{M, \kappa}$ by
\[
\sigma_{M, \kappa} = \sum_{l=1}^{l-1} v_{\kappa + \frac{l}{N}}.
\]
where \( l = N/M \). Then \( \sigma_{M,\kappa} \in \text{Ker}(\phi) \). In fact,
\[
\phi(\sigma_{M,\kappa}) = \sum_{i=0}^{l-1} \langle \langle t(x + \kappa + \frac{i}{l}) \rangle \rangle
= \langle \langle lt(x + \kappa) \rangle \rangle.
\]
Since \( lt(x + \kappa) \in \frac{1}{M} \mathbb{Z}/\mathbb{Z} \), it is 0 if \( x \neq -\kappa \mod \frac{1}{l} \mathbb{Z} \) and \(-\frac{1}{2} \) if \( x = -\kappa \mod \frac{1}{l} \mathbb{Z} \). Therefore the class of this function in \( V \) is 0. Now we choose a numbering of \( S \cup \{2\} = \{M_1, \ldots, M_a\} \) such that the the divisibility \( M_i \mid M_j \) implies \( j \leq i \). Let \( t_j = \sum_{i=1}^{j} \varphi(M_i) \).

**Proposition A.2.**

(1) \( \text{dim Im}(\phi) \geq \sum_{M \in S^c} \varphi(M) \).

(2) Let \( W \) be the subspace of \( \text{Ker}(\phi) \) generated by \( \sigma_{M_i, t_i-1+s} \) \( (i = 1, \ldots, a, s = 1, \ldots, \varphi(M_i)) \), \( \sigma_{2,0} \) and \( \sigma_{2,1} \). Then \( \text{dim } W \geq \sum_{M \in S} \varphi(M) + 2 \).

As a consequence, the equalities holds for both (1) and (2) and \( \sigma_{M,\kappa} \) \( (\kappa \in \frac{1}{N} \mathbb{Z}/\frac{1}{l} \mathbb{Z}, M \in S, \sigma_{2,0} \) and \( \sigma_{2,1} \) generates \( \text{Ker}(\psi) \).

**Proof.** (1). For an element \( M \in S^c \), there exists an odd character \( \chi \) of \( (\mathbb{Z}/M\mathbb{Z})^\times \) such that \( \chi(p) = 1 \). For an element \( l' \in \mathbb{Z}/N\mathbb{Z} \) such that \( l'(\frac{1}{N} \mathbb{Z}/\mathbb{Z}) = \frac{1}{M} \mathbb{Z}/\mathbb{Z} \), we define a linear map \( \theta_{l'} \) from \( V \) to \( \mathbb{C} \) by
\[
\theta_{l'}(f) = N_{M,l'} \sum_{t \in (\mathbb{Z}/N\mathbb{Z})^\times, x \in \frac{1}{N} \mathbb{Z}/\mathbb{Z}} \chi(t) \psi(l'x) f(t, x),
\]
where \( N_{M,l'} \) is the normalizing factor;
\[
N_{M,l'} = \chi(l/l') \frac{M \varphi(M)}{2\pi \sqrt{-1} \varphi(N)} L_M(\chi, 1)^{-1}.
\]
Then \( \theta_{l'} \circ \phi(v_\kappa) \) is calculated as follows.
\[
\theta_{l'} \circ \phi(v_\kappa) = N_{M,l'} \sum_{t \in (\mathbb{Z}/N\mathbb{Z})^\times, x \in \frac{1}{N} \mathbb{Z}/\mathbb{Z}} \chi(t) \psi(l'x) \langle \langle t(x + \kappa) \rangle \rangle
= N_{M,l'} \psi(-l'\kappa) \sum_{t \in (\mathbb{Z}/N\mathbb{Z})^\times, x \in \frac{1}{N} \mathbb{Z}/\mathbb{Z}} \chi(t) \psi(l'x) \langle \langle tx \rangle \rangle
= N_{M,l'} \psi(-l'\kappa) \sum_{t \in (\mathbb{Z}/N\mathbb{Z})^\times, x \in \frac{1}{N} \mathbb{Z}/\mathbb{Z}} \chi(t) \psi(l'x) \sum_{y \mod \frac{1}{N} \mathbb{Z}/\mathbb{Z}} \langle \langle ty \rangle \rangle
= N_{M,l'} \psi(-l'\kappa) \sum_{t \in (\mathbb{Z}/N\mathbb{Z})^\times, x \in \frac{1}{N} \mathbb{Z}/\mathbb{Z}} \chi(t) \psi(l'x) \langle \langle tlx \rangle \rangle.
\]
By changing summation \( t = t_0 \tau, t_0x = \xi \) where \( t_0l = l' \), it is equal to
\[
N_{M,l'} \psi(-l'\kappa) \sum_{\tau \in (\mathbb{Z}/N\mathbb{Z})^\times, \xi \in \frac{1}{N} \mathbb{Z}/\mathbb{Z}} \chi(t_0 \tau) \psi(l\xi) \langle \langle \tau \xi \rangle \rangle
= \psi(l'\kappa) N_{M,l'} \chi(l'/l) \frac{2\pi \sqrt{-1} \varphi(N)}{M \varphi(M)} L_M(\chi, 1)
= \psi(l'\kappa).
\]
For $M \in S^c$, there exist $\varphi(M)$ elements $l'$ in $\mathbb{Z}/N\mathbb{Z}$ satisfying (*). By considering $\theta_{l'}$, for all $M \in S^c$ and $l'$ with the condition, we get $\sum_{M \in S^c} \varphi(M)$ linear homomorphism which is independent on the image of $\psi$. In fact the determinant of $(\theta_{l'}(v_\kappa))_{(M,l'),\kappa} = (\psi(-l'\kappa))_{(M,l'),\kappa}$ does not vanish by Van der Monde determinant.

(2) We have already shown that $\sigma_{M,\kappa}$ is contained in $\text{Ker}(\phi)$. Let $M \in S$ and $l' \in \mathbb{Z}/N\mathbb{Z}$ such that $l'(\frac{1}{N}\mathbb{Z}/\mathbb{Z}) = \frac{1}{M}\mathbb{Z}/\mathbb{Z}$. To show that there exist enough independent $\sigma_{M,\kappa}$, we define linear forms $\gamma_{M,l'}$ for $(M,l')$ satisfying the above condition. Since $-1 \neq 1 \in (\mathbb{Z}/M\mathbb{Z})^\times$, we choose an odd character $\chi$ of $\mathbb{Z}/M\mathbb{Z}$

Let us define $\gamma_{M,l'}$ by

$$
\gamma_{M,l'}(v_\kappa) = N_{M,l'} \sum_{t \in (\mathbb{Z}/M\mathbb{Z})^\times, x \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}} \chi(t) \psi(l'x)(<t(x + \kappa)> - \frac{1}{2}),
$$

where the normalizing factor $N_{M,l'}$ is given by the same formula as before. Then we can show that

$$
\gamma_{M,l'}(v_\kappa) = \psi(-l'\kappa).
$$

Let $\tilde{M} \in S$ and $\tilde{l}M = N$. Then we have

$$
\gamma_{M,l'}(\sigma_{\tilde{M},\bar{\kappa}}) = \sum_{i=1}^{l-1} \psi(-l'(\kappa + \frac{\tilde{M}}{N}i))
$$

$$
= \begin{cases} 
\tilde{l}\psi(-l'\kappa) & \text{if } M | \tilde{M} \\
0 & \text{otherwise}
\end{cases}
$$

Then for $M_j \in S$, the determinant of $\gamma_{M_j,l'}(\sigma_{M_j,t_j-1+s})_{l',s=1,...,\varphi(M_j)}$ is equal to

$$
\prod_{l'} \psi(-l'(t_{i-1} + 1)) \prod_{l'<l''} (\psi(-l') - \psi(-l''))
$$

and non-zero. Since

$$
(\gamma_{M_j,l'}(\sigma_{M_j,t_j-1+s}))_{l',s=1,...,\varphi(M_j)}
$$

is a zero matrix if $i < j$. Therefore the matrix

$$
(\gamma_{M,l'}(\sigma_{M,i}))_{(M,l),i=1,...,\sum_{M \in S} \varphi(M)}
$$

is a non singular matrix. Moreover, we can see $\sigma_{2,0}$ and $\sigma_{2,1}$ is annihilated by all $\gamma_{M,l'}$ ($M \in S$) and independent. Therefore we have prove the proposition. last statement is the consequence of (1) and (2).

Now we take a sufficiently large $N$ to prove the independenstiy of $\langle\langle t(x + \kappa)\rangle\rangle$.

Proof of Proposition 9.1 (2). Let $N$ be an even integer such that $\kappa_i \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}$ ($i = 1, \ldots, r$). We choose a numbering of $S = \{M_1, \ldots, M_a\}$ satisfying the condition given just before Proposition 9.2. Suppose that $\sum_{i=1}^a m_i \langle\langle t(x + \kappa_i)\rangle\rangle + c = 0$. Since the equality holds as a function of $t \in (\mathbb{Z}/N\mathbb{Z})^\times, x \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}$, modulo constant, $\sum_{i=1}^a m_i v_{\kappa_i}$ is contained in the submodule generated by $\sigma_{M,i,j}$ ($i = 1, \ldots, a, j = t_{i-1} + 1, \ldots, t_i$) and $\sigma_{2,1}$ and $\sum_{i=0}^{N-1} v_i$ by Proposition A.2. Suppose that $\sum_{i=1}^a m_i c = \sum_{i=0}^{N-1} \sum_{j=t_{i-1}+1}^{t_i} \sigma_{M,i,j} \text{ and } \sum_{i=0}^{N-1} \langle\langle t(x + \kappa_i + M_i b)\rangle\rangle = 0$. The contradiction is proved.
for $\frac{1}{C_N}Z/Z$ for sufficiently large $C$. Let $M_1 \geq 2$ be a maximal element where $a_{M_1,j} \neq 0$. We choose $C$ such that the subgroup generated by $p$ in $(Z/CM_1Z)^\times$ does not contain $-1$. We apply the linear map $\theta'_l$ for $l'$ such that $l'(\frac{1}{C_N}Z/Z) = \frac{1}{C_{M_1}}$. Let $M_1 \geq 2$ be a maximal element where $a_{M_1,j} \neq 0$. We choose $C$ such that the subgroup generated by $p$ in $(Z/CM_1Z)^\times$ does not contain $-1$. We apply the linear map $\theta'_l$ for $l'$ such that $l'(\frac{1}{C_N}Z/Z) = \frac{1}{C_{M_1}}$ by choosing an odd character $(Z/NCZ)^\times$ vanishing on the subgroup generated by $p$. Then $\theta'_l(\sum a_{M_1,j}\sigma_{M_1,j}) = \sum a_{M_1,j}\psi_{CN}(-l'j)$. Here we used $\psi_{NC}(x) = \exp(2\pi\sqrt{-1}x/NC)$. Since the determinant $(\psi_{CN}(-l'j))_{l'j=t_i-1+1,\ldots,t_i}$ is not zero, $a_{M_1,j} (j = t_i-1+1,\ldots,t_i)$ must be 0. Suppose that $a\sum_{i \in \frac{1}{2}Z/Z} \langle t(x+i) \rangle + c = 0$. This implies $a\langle t(x+i) \rangle + c = 0$ for all $t$ and $x$. By putting $x = \frac{1}{2N}t$ (resp. $x = 0$), we have $c = 0$ (resp. $-\frac{1}{2}a + c = 0$). Therefore $a = c = 0$. This is contradiction.

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