Origin of the quantum speed-up

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Abstract

Bob chooses a function from a set of functions and gives Alice the black box that computes it. Alice is to find a characteristic of the function through function evaluations. In the quantum case, the number of function evaluations can be smaller than the minimum classically possible. The fundamental reason for this violation of a classical limit is not known. We trace it back to a disambiguation of the principle that measuring an observable determines one of its eigenvalues. Representing Bob’s choice of the label of the function as the unitary transformation of a random quantum measurement outcome shows that: (i) finding the characteristic of the function on the part of Alice is a by-product of reconstructing Bob’s choice and (ii) because of the quantum correlation between choice and reconstruction, one cannot tell whether Bob’s choice is determined by the action of Bob (initial measurement and successive unitary transformation) or that of Alice (further unitary transformation and final measurement). Postulating that the determination shares evenly between the two actions, in a uniform superposition of all the possible ways of sharing, implies that quantum algorithms are superpositions of histories in each of which Alice knows in advance one of the possible halves of Bob’s choice. Performing, in each history, only the function evaluations required to classically reconstruct Bob’s choice given the advanced knowledge of half of it yields the quantum speed-up. In all the cases examined, this goes along with interleaving function evaluations with non-computational unitary transformations that each time maximize the amount of information about Bob’s choice acquired by Alice with function evaluation.

1 Executive summary

By "quantum speed-up” one means the higher efficiency of quantum algorithms with respect their classical equivalent. Let us provide at once a simple example of speed-up. Bob hides a ball in one of four drawers, Alice is to locate it by opening drawers. In the classical case, to be sure of locating the ball, Alice should plan to open three drawers. With Grover’s quantum database search algorithm [1], only one drawer suffices.
It should be noted that Grover’s algorithm, like the seminal one of Deutsch [2], requires fewer computation steps (drawer openings in Grover’s case) than the minimum demonstrably required by any equivalent classical algorithm.

As already noted in literature [3], this violation of a limit applying to any classical time-evolution relates the speed-up to the violation of the temporal Bell inequality of Leggett and Garg [4], the information-theoretic one of Braunstein and Caves [5] and, particularly, the one formulated by Morikoshi [3] exactly in the case of Grover’s algorithm. According to this latter inequality, all is as if quantum information processing exploited unperformed computations [3]. The fundamental reason for this is not known. Here we trace it back to a disambiguation of the quantum principle – stating that the measurement of an observable determines one of its eigenvalues. As we will see, this principle becomes typically ambiguous in presence of quantum speed-up.

We focus on quantum oracle computing. Bob chooses a function from a set of functions and gives Alice the black box (oracle) that computes it. Alice is to find a characteristic of the function chosen by Bob by performing function evaluations (in Grover’s case, opening drawers amounts to evaluating the Kronecker function).

Our argument goes as follows – it is clearer to segment it by section.

2 Grover’s algorithm

We use a representation where Grover’s algorithm is the model for all the quantum algorithms based on function evaluation.

2.1 Time-symmetric representation

To the usual Alice’s register, containing the number of the drawer that Alice wants to open, we add an imaginary Bob’s register, containing the number of the drawer with the ball. We assume that the initial state of Bob’s register is maximally mixed, so that Bob’s process of choice is represented from scratch. See the far left of Fig. 1, where S. stands for state, M. for measurement, U is the unitary part of the quantum algorithm.

\[
\begin{array}{c}
\text{Initial S.} & \rightarrow & \text{Initial M.} & \rightarrow & \text{Input S.} & \rightarrow & U & \rightarrow & \text{Output S.} & \rightarrow & \text{Final M.} & \rightarrow & \text{Final S.} \\
\end{array}
\]

Quantum correlation, reading the output contributes to determining the input

Fig. 1 Time-symmetric representation of quantum algorithms

Bob measures the content of this register obtaining a drawer number uniformly at random. To start with, we assume that Bob’s choice is this very number. The corresponding eigenstate, with the usual sharp state of Alice’s register, is the input of U. Here, by performing function evaluations (by opening drawers), Alice reconstructs Bob’s choice in her register. By finally measuring

\footnote{We take the expression “imaginary register” from reference [6], which highlights the problem-solution symmetry of Grover’s and the phase estimation algorithms.}
the content of this register, she acquires the number of the drawer chosen by Bob.

In this extended representation of the quantum algorithm there is quantum correlation between the contents of Bob’s and Alice’s registers before their respective measurements. These in fact yield two identical eigenvalues whose common value (the number of the drawer chosen by Bob) is selected at random. We will see that quantum correlation remains there also when Bob unitarily changes the initial random measurement outcome into a desired number.

By time-symmetric \([7, 8]\) representation of the quantum algorithm we mean the present representation (extended to Bob’s choice), with the peculiarity that the projection of the quantum state due to Bob’s measurement is retarded to the end of \(U\). As well known, such projections can be retarded or advanced along a unitary transformation that follows or precedes the measurement. In the present case, retarding the projection relativizes the quantum state to the observer Alice in the sense of relational quantum mechanics \([9]\). Alice is in fact forbidden to observe the result of Bob’s measurement before reconstructing it through function evaluations. In this relativized representation, the maximally mixed initial state of Bob’s register remains unaltered after Bob’s measurement. Its entropy represents Alice’s ignorance of Bob’s choice.

2.2 Sharing the determination of Bob’s choice

The quantum principle, stating that the measurement of an observable determines one of its eigenvalues, becomes ambiguous when the measurement of two commuting observables yields at random two identical eigenvalues (choice and reconstruction). Which measurement determines their common value? The idea that all the determination should be ascribed to the measurement performed first is not justified. In fact Bob’s measurement can be suppressed and the determination of Bob’s choice is performed by Alice’s measurement, also at the time of the suppressed measurement – the projection of the quantum state due to Alice’s measurement (i.e. the determination) can be advanced at the time in question by applying \(U^\dagger\) to the two ends of it. Since there is no way of telling which measurement determines Bob’s choice, for reasons of symmetry we postulate that the determination shares between the two measurements (i) without over-determination (i.e. without producing twice the same information), (ii) with entropy reductions the same for each share, and (iii) in a uniform quantum superposition of all the possible ways of sharing compatible with the former conditions. Conditions (i) and (ii) imply that Alice’s measurement determines half of Bob’s choice (\(n/2\) bits in the present case where Bob’s choice is an unstructured \(n\) bit string). For condition (iii), the quantum algorithm should be seen as a uniform quantum superposition of algorithms (histories) in each of which Alice’s measurement determines one of the possible halves of Bob’s choice. We call conditions (i) through (iii) the sharing rule. This rule has been inspired by the work of Dolev and Elitzur \([10]\) on the non-sequential behavior of the wave function highlighted by partial measurement. Here partial measurements are involved in sharing the determination of Bob’s choice.

2.3 Advanced knowledge

By advancing (by \(U^\dagger\)) to the beginning of Alice’s action (immediately after
Bob’s measurement) the contribution of Alice’s measurement to the determination of Bob’s choice, the maximally mixed initial state of Bob’s register is projected on a less mixed state where the corresponding half of Bob’s choice is determined. Correspondingly, the entropy of the state is halved. This means that, in each history, Alice knows half of Bob’s choice in advance.

2.4 The mechanism of the speed-up in Grover’s algorithm

According to the sharing rule (the present disambiguation of the quantum principle), the quantum algorithm is a superposition of histories in each of which Alice knows in advance one of the possible halves of Bob’s choice. It should be noted that this holds for any quantum algorithm that reconstructs Bob’s choice, with or without speed-up. The quantum correlation between choice and reconstruction is anyhow there. Thus, at one extreme, the quantum algorithm can be a superposition of identical histories in each of which Alice ignores the advanced knowledge that tags the history and performs the function evaluations classically required to reconstruct Bob’s choice. At the other, in each history, Alice should be able to perform only the function evaluations required to classically identify the missing half of Bob’s choice given the advanced knowledge of the other half; in fact, this is what is needed to bring the halved entropy of Bob’s register down to zero. In Grover’s algorithm, this is made possible by interleaving function evaluations with non-computational unitary transformations applying to Alice’s register that each time maximize the amount of information about Bob’s choice acquired by Alice with function evaluation. This minimizes the number of function evaluations bringing it exactly to the number \( N_a \) required to reconstruct Bob’s choice given the advanced knowledge of half of it. This explains why Grover’s algorithm requires \( N_a = O \left( \frac{2^n}{2} \right) \) function evaluations against the \( O \left( 2^n \right) \) of the classical case and why the violation of Morikoshi’s inequality implies that it exploits unperformed computations. This is what happens in each and every history the algorithm is made of.

3 Generalizing the mechanism of the speed-up

A simple generalization of Grover’s algorithm produces all the quantum algorithms whose solution is a by-product of the reconstruction of Bob’s choice. First, we should set the non-computational unitary transformations free. Then we should determine them by maximizing each time, after the transformation that follows function evaluation, the probability of finding the solution in Alice’s register. This minimizes the number of function evaluations, bringing it to \( N_a \) in all the cases examined. Given the set of functions, this mechanism produces the quantum algorithm that yields the solution (the characteristic of the function chosen by Bob) with the maximum possible speed-up.

4 Deutsch-Jozsa’s algorithm, Simon’s and the hidden subgroup algorithms

Here Bob’s choice is a highly structured bit string. Given the advanced knowledge of half of it according to the sharing rule, finding the missing half requires a single function evaluation – against an exponential number thereof in the absence of advanced knowledge. This explains the exponential speed-up of these latter algorithms.

6 Discussion and conclusions

We have identified the fundamental reason for which some quantum algo-
rithms violate a limit applying to classical time-evolutions and/or Morikoshi’s
inequality. Although preliminary in character, the results obtained seem to open
a gap in a problem that has remained little explored. Until now there was no
fundamental explanation of the speed-up, no general mechanism for producing
it.

With respect to references [11, 12], we have reformulated the explanation of
the speed-up given for Grover’s algorithm and extended it to all the quantum
algorithms based on function evaluation.

2 Grover’s algorithm

We develop our argument in detail for Grover’s algorithm. Its time-symmetric
representation is the model for all the quantum algorithms examined in this
paper.

2.1 Time-symmetric representation

Let b and a, ranging over \{0, 1\}^n, be respectively the number of the drawer
with the ball and that of the drawer that Alice wants to open. Bob writes
his choice of the value of b in an imaginary n-qubit register B. Alice writes a
value of a in a n-qubit register A. Then the black box computes the Kronecker
function \( \delta(b, a) \), which gives 1 if \( b = a \) and 0 otherwise – tells Alice whether
the ball is in drawer a. A one-qubit register V is meant to contain the result of
the computation of \( \delta(b, a) \) – modulo 2 added to its former content for logical
reversibility.

We assume that register B is initially in a maximally mixed state, so that
the value of b is completely undetermined. We will see that this assumption just
yields a special view of the usual quantum algorithm (starting with a completely
determined value of b ). Registers A and V are prepared as usual in a sharp
state. With \( n = 2 \), the initial state of the three registers is thus:

\[
|\psi\rangle = \frac{1}{2} (e^{i\varphi_0} |00\rangle_B + e^{i\varphi_1} |01\rangle_B + e^{i\varphi_2} |10\rangle_B + e^{i\varphi_3} |11\rangle_B |00\rangle_A |1\rangle_V . \tag{1}
\]

We keep the usual state vector representation of quantum algorithms by using
the random phase representation of density operators [13]. The \( \varphi_i \) are inde-
dependent random phases each with uniform distribution in [0, 2\( \pi \)]. The density
operator is the average over all \( \varphi_i \) of the product of the ket by the bra:

\[
\langle |\psi\rangle \langle \psi| |\varphi\rangle = \frac{1}{4} (|00\rangle_B \langle 00|_B + |01\rangle_B \langle 01|_B + |10\rangle_B \langle 10|_B + |11\rangle_B \langle 11|_B

|00\rangle_A \langle 00|_A |1\rangle_V \langle 1|_V .
\]

The von Neumann entropy of the state of register B in the overall state (1)
is two bits. This is also the entropy of the overall quantum state. As we will
see, this latter entropy coincides with that of the reduced density operator of register \( B \) throughout the quantum algorithm.

We call \( B \) (\( A \)) the content of register \( B \) (\( A \)), of eigenvalue \( b \) (\( a \)). \( \hat{B} \) and \( \hat{A} \), both diagonal in the computational basis, commute. To prepare register \( B \) in the desired value of \( b \), in the first place Bob should measure \( \hat{B} \) in state \( |1\rangle \). He obtains an eigenvalue at random, say \( b = 01 \). Conventionally, state \( |1\rangle \) would be projected on:

\[
P_B |\psi\rangle = |01\rangle_B |00\rangle_A |1\rangle_V .
\]

For the time being, we assume that Bob’s choice is random, is the result of measurement itself. The case that Bob chooses a predetermined value of \( b \) is considered further on.

State \( |2\rangle \), with register \( B \) in a sharp state, is the input state of the conventional representation of the quantum algorithm. For reasons that will become clear, we retend to the end of the unitary part of the algorithm the projection of state \( |1\rangle \) on state \( |2\rangle \). Thus, the input state of the algorithm is state \( |1\rangle \) back again.

At this point, Alice applies the Hadamard transforms \( U_A \) and \( U_V \) to respectively registers \( A \) and \( V \):

\[
U_A U_V |\psi\rangle = \frac{1}{\sqrt{2}} \left( e^{i\delta_0} |00\rangle_B |01\rangle_A + e^{i\delta_2} |10\rangle_B |11\rangle_A \right)
\]

\[
\left( |00\rangle_A + |01\rangle_A + |10\rangle_A + |11\rangle_A \right) (|0\rangle_V - |1\rangle_V) .
\]

Then she performs the reversible computation of \( \delta (b, a) \), represented by the unitary transformation \( U_f \) (\( f \) like “function evaluation”):

\[
U_f U_A U_V |\psi\rangle = \frac{1}{\sqrt{2}} \left( e^{i\delta_0} |00\rangle_B - |01\rangle_A + |10\rangle_A + |11\rangle_A \right)
\]

\[
\left( |00\rangle_A - |01\rangle_A + |10\rangle_A - |11\rangle_A \right) (|0\rangle_V - |1\rangle_V) .
\]

\( U_f \) maximally entangles registers \( B \) and \( A \) (i. e. the observables \( \hat{B} \) and \( \hat{A} \)). Four orthogonal states of \( B \), each a value of \( b \), one by one multiply four orthogonal states of \( A \). This means that the information about the value of \( b \) has propagated to register \( A \).

If we measured \( \hat{A} \) in state \( |4\rangle \), we would obtain a value of \( a \) completely uncorrelated with that of \( b \). To make the information acquired with function evaluation accessible to measurement, we need to make correlation of entanglement. This is done by applying to register \( A \) the unitary transformation \( U_A^\prime \) (the so called inversion about the mean):

\[
U_A^\prime U_f U_A U_V |\psi\rangle = \frac{1}{\sqrt{2}} \left( e^{i\delta_0} |00\rangle_B |01\rangle_A + e^{i\delta_2} |10\rangle_B |11\rangle_A \right)
\]

\[
\left( |00\rangle_A + |01\rangle_A + |10\rangle_A + |11\rangle_A \right) (|0\rangle_V - |1\rangle_V) .
\]

Now the contents of registers \( B \) and \( A \) are identical: Alice has reconstructed Bob’s choice in register \( A \). She acquires the reconstruction by measuring \( A \).
This projects state (5) on:

\[ P_A U_A^* U_B U_A \psi = \frac{1}{\sqrt{2}} |01\>_B |01\>_A (|0\>_V - |1\>_V), \tag{6} \]

in overlap with the retarded projection due to the measurement of \( \hat{B} \) in state (1). The two projections are redundant with one another.

We call equations (1) and (3) through (6) the time-symmetric representation of the quantum algorithm. It should be noted that this representation is the conventional one, starting with a well determined value of \( b \), relativized to the observer Alice in the sense of relational quantum mechanics [9]. By definition, the projection due to measuring \( \hat{B} \) in state (1) should remain hidden to the observer Alice until she has reconstructed Bob’s choice. It should in fact be retarded until Alice measures \( \hat{A} \) in state (5).

In this representation, the two bit entropy of state (1) represents Alice’s ignorance of Bob’s choice. When Alice measures \( \hat{A} \) in state (5), the entropy of the quantum state becomes zero and she acquires full knowledge of Bob’s choice. Thus, the entropy of the quantum state – or identically that of the reduced density operator of register \( B \) – gauges Alice’s ignorance of Bob’s choice.

We can see that there is quantum correlation between the outcome of measuring \( \hat{B} \) in state (1) and that of measuring \( \hat{A} \) in state (5). In fact one obtains uniformly at random two identical eigenvalues, namely Bob’s choice – in present assumptions the value 01 of both \( b \) and \( a \). This quantum correlation plays a crucial role in the present explanation of the speed-up.

Until now we have assumed that Bob’s choice is a random quantum measurement outcome. An equally crucial point of our argument is noting that quantum correlation remains there also when Bob chooses a predetermined value of \( b \). Say that the measurement of \( \hat{B} \) in state (1) yields \( b = 11 \) and Bob wants \( b = 01 \). He applies to register \( B \) a permutation of the values of \( b \), a unitary transformation \( U_B \) such that \( U_B |11\>_B = |01\>_B \). The correlation is the same as before up to \( U_B \). The point is that, from the standpoint of quantum correlation, \( U_B \) should be considered a ”fixed” transformation.

In fact quantum correlation concerns two measurement outcomes in an ensemble of repetitions of the same experiment, consisting of the measurement of an observable in an initial state, a unitary transformation, and the measurement of another observable in the resulting state. Initial state and unitary transformation should remain unaltered throughout the ensemble of repetitions. \( U_B \), being part of the unitary transformation, should be considered always the same.

Thus, from the standpoint of quantum correlation, the predetermined value of \( b \), seen as the fixed permutation of a random measurement outcome, should be considered a random measurement outcome as well.

### 2.2 Sharing the determination of Bob’s choice

We share the determination of Bob’s choice between Bob’s and Alice’s measurements or, more exhaustively, actions. In fact Bob’s choice is determined by
either Bob’s measurement and his successive unitary action (to change the random outcome into the one desired) or Alice’s action of unitarily reconstructing Bob’s choice and finally measuring the reconstruction.

First, we introduce the tools required to perform the sharing.

We call $|\psi\rangle_B$ the state of register $B$ in the overall states (1) and (3) through (5):

$$|\psi\rangle_B = \frac{1}{2} (|00\rangle_B + e^{i\varphi_1} |01\rangle_B + e^{i\varphi_2} |10\rangle_B + e^{i\varphi_3} |11\rangle_B).$$

$|\psi\rangle_B$ is the random phase representation of the reduced density operator of register $B$:

$$\langle|\psi\rangle_B |\psi\rangle_{\varphi_{\varphi}} = \frac{1}{4} (|00\rangle_B \langle 00| + |01\rangle_B \langle 01| + |10\rangle_B \langle 10| + |11\rangle_B \langle 11|).$$

It should be noted that the unitary part of Alice’s action is the identity on $|\psi\rangle_B$ (it does not change Bob’s choice). $\mathcal{E}_B$, the entropy of $|\psi\rangle_B$, is two bits. The determination of Bob’s choice is represented by $P_B$, the projection of $|\psi\rangle_B$ on $|01\rangle_B$ due to the measurement of either $\hat{B}$ in state (1) or $A$ in state (5). We share the determination of Bob’s choice by sharing $P_B$, what can be done by resorting to the notion of partial measurement.

Let us resolve $b$ into its individual bits: $b \equiv b_0b_1$. We consider the following partial measurements and the corresponding projections of $|\psi\rangle_B$. The measurement of the content of the left cell of register $B$ of the observable $\mathcal{B}_0$ of eigenvalue $b_0$ (from now on we omit speaking of the corresponding operation on register $A$ at the end of the algorithm, which is completely redundant). A-priori, the measurement outcome is either $b_0 = 0$ or $b_0 = 1$. However, in present assumptions, the measurement of $\mathcal{B}_0$ projects $|\psi\rangle_B$ on $|01\rangle_B$, we are in fact discussing how to share this projection. Thus we should assume that the measurement of $\mathcal{B}_0$ yields $b_0 = 0$, namely projects $|\psi\rangle_B$ on $\frac{1}{\sqrt{2}} (e^{i\varphi_0} |00\rangle_B + e^{i\varphi_1} |01\rangle_B)$; we also say “on $b \in \{01,0\}$”. Similarly, the measurement of the content of the right cell of register $B$ projects $|\psi\rangle_B$ on $b \in \{01,1\}$, that of the exclusive or of the contents of the two cells projects $|\psi\rangle_B$ on $b \in \{01,1\}$.

We will see afterwards that $P_B$ should be shared into any two of the three projections of $|\psi\rangle_B$ on: $b \in \{01,0\}$, $b \in \{01,1\}$, and $b \in \{01,1\}$. One share (either one) should be ascribed to the action of Bob, the other to that of Alice.

Until now we have introduced the tools to share the determination of Bob’s choice. Now we introduce some conditions that, reasonably, should be satisfied by the sharing.

First, we get rid of all redundancy between the two measurements. We resort to Occam’s razor; in Newton’s formulation, it states “We are to admit no more causes of natural things than such that are both true and sufficient to explain their appearances” [14]. This requires that, together, the two shares of $P_B$ (the corresponding partial measurements) tightly determine the value of $b$, namely without determining twice any Boolean function of $b$. This is condition (i) of the sharing rule.

We apply it to Grover’s algorithm. Here, the $n$ bits that specify the value of $b$ are independently selected in a random way. Thus, condition (i) requires
that the determination of \( p \) of these bits \((0 \leq p \leq n)\) is ascribed to the action of Bob, that of the other \( n - p \) bits to that of Alice.

Condition (i) does not constrain the value of \( p \). This is up to the following condition (ii). Let \( \Delta \mathcal{E}_B^{(B)} \) (\( \Delta \mathcal{E}_B^{(A)} \)) be the reduction of the entropy of the state of register \( B \) associated with the share of \( P_B \) ascribed to Bob’s (Alice’s) action. Here we have \( \Delta \mathcal{E}_B^{(B)} = p \) bit, \( \Delta \mathcal{E}_B^{(A)} = (n - p) \) bit. Since Bob’s choice is indistinguishably determined by either Bob’s or Alice’s action, for reasons of symmetry we require:

\[
\Delta \mathcal{E}_B^{(B)} = \Delta \mathcal{E}_B^{(A)}.
\]

Here this becomes \( p = n - p = n/2 \) – the \( n \) bits of \( \mathcal{E}_B \) share evenly between the two actions.

We can see that sharing \( P_B \) into any two of the above said three projections satisfies conditions (i) and (ii). Any pair of projections, corresponding to the measurement of a pair of observables among \( \hat{B}_0, \hat{B}_1, \) and \( \hat{B}_X \), tightly selects a value of \( b \). Any projection reduces the entropy of the state of register \( B \) by one bit, so that equation (8) is always satisfied. We can also see that there is no other way of satisfying the sharing rule.

Sharing between Bob’s and Alice’s actions the determination of Bob’s choice is equivalent to saying that Alice’s action contributes to this determination. Thus, in Grover’s algorithm, Alice’s action determines half of the bits that specify Bob’s choice.

This faces us with the problem that half of Bob’s choice can be taken in many ways. A natural way of solving this problem is requiring that the sharing is done in a uniform quantum superposition of all the possible ways of taking half of the choice. This is condition (iii) of the sharing rule. It implies seeing the quantum algorithm as a uniform superposition of algorithms (or ”histories”), in each of which Alice determines one of the possible halves of Bob’s choice.

### 2.3 Advanced knowledge

We show that ascribing to Alice’s action the determination of part of Bob’s choice implies that Alice knows in advance, before running the algorithm, that part of the choice.

For example, we ascribe to Alice’s action the determination \( b_0 = 0 \), namely the projection of state (5) on

\[
\frac{1}{2} \left(e^{i\varphi_0} |00\rangle_B |0\rangle_A + e^{i\varphi_1} |01\rangle_B |1\rangle_A\right) (|0\rangle_V - |1\rangle_V).
\]

We advance this projection to the beginning of Alice’s action, immediately after Bob’s measurement. This is done by applying \( U_V^U A U_A^U U_A^{U'} \) to the two ends of it, namely to states (9) and (9). This yields the projection of the input state of the quantum algorithm (1) on

\[
\frac{1}{\sqrt{2}} \left(e^{i\varphi_0} |00\rangle_B + e^{i\varphi_1} |01\rangle_B\right) |00\rangle_A |1\rangle_V.
\]
Thus, the entropy of the state of register $B$ (or identically of the overall quantum state) in the input state of the quantum algorithm is halved. Since this entropy represents Alice’s ignorance of Bob’s choice (Section 2.1), this means that Alice, before running the algorithm, knows $n/2$ of the bits that specify Bob’s choice, here one bit – in fact $b_0 = 0$.

According to the sharing rule, the quantum algorithm is a superposition of histories in each of which Alice determines half of Bob’s choice. Now this becomes a superposition of histories in each of which Alice knows in advance half of Bob’s choice before performing any computation.

### 2.4 The mechanism of the speed-up in Grover’s algorithm

We have seen that Grover’s algorithm is a superposition of histories in each of which Alice knows in advance one of the possible halves of Bob’s choice. We note that this holds for any quantum algorithm that reconstructs Bob’s choice, no matter whether with or without speed-up – the maximally entangled state \( |\psi\rangle \) is evidently the end state in any case. Thus, at one extreme, the quantum algorithm can be a superposition of identical histories in each of which Alice classically reconstructs Bob’s choice without benefitting of the advanced knowledge that tags the history. At the other extreme, in each history, Alice should be able to perform only the \( (N_a) \) function evaluations required to classically reconstruct Bob’s choice given the advanced knowledge of half of it. In fact, this is what is needed to bring the halved entropy of Bob’s register down to zero. This is what Grover’s algorithm does. It goes along with interleaving function evaluations with non-computational unitary transformations that each time maximize the amount of information about Bob’s choice acquired by Alice with function evaluation. This minimizes the number of function evaluations required to reconstruct Bob’s choice, bringing it exactly to \( N_a \).

We show how things go in detail, starting with the function evaluation part of the algorithm.

Let us assume that Bob’s choice is $b = 01$. Alice’s advanced knowledge can be: $b \in \{01, 00\}$, or $b \in \{01, 11\}$, or $b \in \{01, 10\}$ (Section 2.2). We assume it is $b \in \{01, 00\}$ (we are pinpointing one of the possible histories). To identify the value of $b$ Alice should compute $\delta(b, a)$ (for short ”$\delta$”) for either $a = 01$ or $a = 00$. We assume it is for $a = 01$. The outcome of the computation, $\delta = 1$, tells Alice that $b = 01$. This corresponds to two classical computation histories, one for each possible sharp state of register $V$: we represent each classical computation history as a sequence of sharp quantum states. The initial state of history 1 is $e^{i\theta_1} |01\rangle_B |01\rangle_A |0\rangle_V$, what means that the input of the computation of $\delta(b, a)$ is $b = 01$, $a = 01$; $|0\rangle_V$ is one of the two possible sharp states of register $V$. The state after the computation of $\delta$ is $e^{i\theta_1} |01\rangle_B |01\rangle_A |1\rangle_V$ – the result of the computation is modulo 2 added to the former content of $V$. We are using the history amplitudes that reconstruct the quantum algorithm; our present aim is to show that the quantum algorithm is a superposition of histories where Alice classically reconstructs Bob’s choice given the advanced knowledge of one of the possible halves of it.
In history 2, the states before/after the computation of $\delta$ are $-e^{i\varphi_1}|01\rangle_B|01\rangle_A|1\rangle_V \rightarrow -e^{i\varphi_1}|01\rangle_B|01\rangle_A|0\rangle_V$.

In the case that Alice computes $\delta(b,a)$ for $a = 00$ instead, she obtains $\delta = 0$, which of course tells her again that $b = 01$. This originates other two histories. History 3: $e^{i\varphi_1}|01\rangle_B|00\rangle_A|0\rangle_V \rightarrow e^{i\varphi_1}|01\rangle_B|00\rangle_A|0\rangle_V$; history 4: $-e^{i\varphi_1}|01\rangle_B|00\rangle_A|1\rangle_V \rightarrow -e^{i\varphi_1}|01\rangle_B|00\rangle_A|1\rangle_V$. Etc.

The function evaluation step of Grover’s algorithm, namely the transformation of state 3 into state 4, is the superposition of all such histories.

Function evaluation is preceded and followed by two non-computational unitary transformations, respectively $U_A U_V$ and $U_A'$. The first transformation branches the initial sharp state of registers $A$ and $V$ into the superposition of the inputs of the function evaluation part of the histories. This superposition maximizes the amount of information acquired by Alice with function evaluation – i.e. entanglement between $\hat{B}$ and $\hat{A}$. The second branches the output states of function evaluation into a superposition of states that interfere with one another making correlation of entanglement. As already noted in Section 2.1, these transformations (together) maximize the correlation between the outcomes of measuring $\hat{B}$ and $\hat{A}$ respectively at the beginning and the end of the unitary part of the algorithm. In other words, they maximize the probability of finding Bob’s choice in register $A$.

Summing up, Grover’s algorithm for $n = 2$ can be decomposed into a superposition of histories in each of which Alice knows in advance half of the result of the computation and utilizes this information to identify the other half in a classical way. This clarifies why, according to the information-theoretic temporal Bell inequality derived by Morikoshi [5], all is as if Grover’s algorithm exploited unperformed computations. This is what happens in each and every one of the histories Grover’s algorithm is made of.

Let us now consider the case $n > 2$. As well known, the sequence ’function evaluation-inversion about the mean’ (the algorithm’s iterate) should be repeated $\frac{\pi}{2^{n/2}}$ times. This maximizes the probability of finding the solution leaving a probability of error $\leq \frac{1}{2^n}$. This goes along with the present explanation of the speed-up in the order of magnitude. In fact, according to it, one should perform $O\left(2^{n/2}\right)$ computations of $\delta$ – this is the number of classical computations required to find the missing half of Bob’s choice given the advanced knowledge of the other half.

3 Generalizing the mechanism of the speed-up

In all the quantum algorithms examined in this paper, finding the solution of the problem (a deterministic or stochastic function of Bob’s choice) is a by-product of reconstructing Bob’s choice. Because of this commonality, all these algorithms can be generated by a simple generalization of Grover’s algorithm.

Given the problem, let $p_S$ be the probability of finding the (or a) solution with a potential measurement of $\hat{A}$. To generate the quantum algorithm that solves the problem, we set the matrix elements of the non-computational trans-
formations of Grover’s algorithm free up to unitarity; then, after the transformation that follows each function evaluation, maximize $p_S$. For the time being we give the generalized algorithm. In the following sections we will check that it unifies all the quantum algorithms considered in this paper.

I) Start with some set of functions $f_b(a)$, with $f$, $b$, and $a$ ranging over some sets of values. For example, in Grover’s algorithm, we have $f_b(a) \equiv \delta(b,a)$, with $f$ ranging over $\{0,1\}$ and $b,a$ over $\{0,1\}^n$. The imaginary register $B$ contains $b$, the label of the function, register $A$ the argument of the function, and register $V$ the result of function evaluation reversibly added to its former content.

II) Assume that $B$ is in a maximally mixed state, prepare $A$ and $V$ in a sharp state.

III) Apply to $A$ a unitary transformation whose matrix elements are free variables up to unitarity. Do the same with $V$.

IV) Perform function evaluation.

V) Apply to $A$ another free unitary transformation.

VI) Maximize $p_S$, what can be done in principle by zeroing its partial derivatives with respect to the free variables we are dealing with.

VII) Points (IV), (V), and (VI) constitute the algorithm’s iterate. Iterate until $p_S = 1$. In all the cases examined, this sets the algorithm to a superposition of histories in each of which Alice classically reconstructs Bob’s choice given the advanced knowledge of one of the possible halves of it. The number of function evaluations is always that ($N_a$) foreseen by the sharing rule.

VIII) Acquire the characteristic by measuring $\hat{A}$.

It should be noted that the present mechanism diverges from Grover’s algorithm if we over-iterate. Having replaced the inversion about the mean by the unitary transformation that maximizes $p_S$, it is never the case that we reduce this probability – this transformation becomes the identity if we over-iterate.

A slight modification of this mechanism can be applied to the search for new speed-ups even if we do not know beforehand which is the characteristic of the function that leaks to register $A$ with function evaluation (as necessary to compute $p_S$). Let $|\psi\rangle_A$ be the state (reduced density operator in random phase representation) of register $A$, $\mathcal{E}_A$ its entropy. Clearly, $\mathcal{E}_A$ gauges the amount of information about Bob’s choice leaked to register $A$ with function evaluation – for example, it is zero bit in states $\mathsf{1}$ and $\mathsf{3}$ and two bits in states $\mathsf{4}$ and $\mathsf{5}$. We should perform steps (I) through (IV) and maximize $\mathcal{E}_A$. At this point, we should try to identify the characteristic of the function leaked – what the information leaked is about. For example, this is relatively simple in Grover’s and Deutsch&Jozsa’s algorithms. We note that this characteristic is fully there, in the part of the state of Alice’s register entangled with Bob’s choice, after the first function evaluation. Eventually, provided that we have succeeded in identifying the characteristic in question, we can perform steps (V) through (VIII).

Reference [11] provides the example of a new speed-up that can be obtained in this way (finding a certain characteristic of a permutation). Reasonably, given any set of functions, this mechanism generates with the maximum possible speed-up a characteristic of the function chosen by Bob. Naturally, we should look for set
of functions where knowing in advance half of Bob’s choice yields an interesting advantage.

We pinpoint a limit of the result obtained. Maximizing each time the probability of finding the solution in Alice’s register minimizes the number of function evaluations required to reach it. Whether this number is always \( N_a \) – the number foreseen by the sharing rule – is of course an important question in the present context. For the time being, we must leave this question open in the general case. This work is exploratory in character and we limit ourselves to checking that the two numbers coincide with one another in all the quantum algorithms examined.

It might be interesting to underline the kernel of the present mechanism, which is maximizing in a suitable quantum context input-output correlation. Quantum retroaction of the output on the input (Alice’s action contributing to Bob’s choice) is what allows building this correlation with a speed-up.

4 Deutsch&Joza’s algorithm

In Deutsch&Joza’s [15] algorithm, the set of functions is all the constant and balanced functions (with the same number of zeroes and ones) \( f_b : \{0,1\}^n \rightarrow \{0,1\} \). Array (11) gives (part of) the set of eight functions for \( n = 2 \).

| \( a \) | \( f_{0000}(a) \) | \( f_{1111}(a) \) | \( f_{0011}(a) \) | \( f_{1100}(a) \) | \( f_{0101}(a) \) |
|-----|-----------|-----------|-----------|-----------|-----------|
| 00  | 0         | 1         | 0         | 1         | 0         |
| 01  | 0         | 1         | 0         | 1         | 1         |
| 10  | 0         | 1         | 1         | 0         | 0         |
| 11  | 0         | 1         | 1         | 0         | 1         | etc. |

(11)

The bit string \( b \equiv b_0, b_1, ..., b_{2n-1} \) is both the suffix and the table of the function \( f_b(a) \) – the sequence of function values for increasing values of the argument. Specifying the choice of the function by means of the table of the function simplifies the discussion. Alice is to find whether the function selected by Bob is balanced or constant by computing \( f_b(a) \equiv f(b,a) \) for appropriate values of \( a \). In the classical case this requires, in the worst case, a number of computations of \( f(b,a) \) exponential in \( n \); in the quantum case one computation.

4.1 Time-symmetric representation

Register \( B \) (\( A \)) contains \( b \) (\( a \)), register \( V \) the result of function evaluation reversibly added to its former content. The input and output states of the quantum algorithm are respectively:

\[
|\psi\rangle = \frac{1}{2\sqrt{2}} \left( e^{i\varphi_0} |0000\rangle_B + e^{i\varphi_1} |1111\rangle_B + e^{i\varphi_2} |0011\rangle_B + e^{i\varphi_3} |1100\rangle_B + ... \right) |00\rangle_A |1\rangle_V ,
\]

(12)

\[
U_A U_f U_A U_V |\psi\rangle = \frac{1}{4} \left( (e^{i\varphi_0} |0000\rangle_B - e^{i\varphi_1} |1111\rangle_B) |00\rangle_A + (e^{i\varphi_2} |0011\rangle_B - e^{i\varphi_3} |1100\rangle_B) |10\rangle_A + ... \right) (|0\rangle_V - |1\rangle_V).
\]

(13)
$U_A$ and $U_V$ are the Hadamard transforms on respectively registers $A$ and $V$, $U_f$ is function evaluation, namely the computation of $f(b,a)$. Measuring $\hat{B}$ in state (12) yields Bob’s choice, a value of $b$. Measuring $\hat{A}$ in state (13) yields the characteristic of the function: "constant" if $a$ is all zeros, "balanced" otherwise.

This time the result of Alice’s measurement is not Bob’s choice but a function thereof. However, as we will show in sections 4.3 and 4.4, the determination of this result is a by-product of reconstructing Bob’s choice. This can be explicitly represented by adding another imaginary register $A'$ of the same size of $B$.

Besides reversibly writing in $V$ the result of function evaluation, the black box should reversibly write in $A'$ the corresponding reconstruction of Bob’s choice.

States (12) and (13) should be replaced respectively by

$$\frac{1}{2\sqrt{2}} (e^{i\varphi_0} |0000\rangle_B + e^{i\varphi_1} |1111\rangle_B + \ldots) |0000\rangle_{A'} |00\rangle_A |1\rangle_V$$  

and

$$\frac{1}{4} [e^{i\varphi_0} (|0000\rangle_B |0000\rangle_{A'} - e^{i\varphi_1} |1111\rangle_B |1111\rangle_{A'} ) |00\rangle_A + \ldots] (|0\rangle_V - |1\rangle_V).$$  

(14)

4.2 Sharing the determination of Bob’s choice

The determination of Bob’s choice should be shared evenly between the measurements of $\hat{B}$ and $\hat{A}'$ exactly as we did with $\hat{B}$ and $\hat{A}$ in the case of Grover’s algorithm. The fact Alice does not really measure $\hat{A}'$ but a function thereof (i.e. $\hat{A}$) is irrelevant. The important thing is that Alice would acquire Bob’s choice by measuring $\hat{A}'$.

The state of register $B$ in states (12) and (13) – or (14) and (15) – is:

$$|\psi\rangle_B = \frac{1}{2\sqrt{2}} (e^{i\varphi_0} |0000\rangle_B + e^{i\varphi_1} |1111\rangle_B + e^{i\varphi_2} |0011\rangle_B + e^{i\varphi_3} |1100\rangle_B + \ldots).$$  

(16)

Say that Bob’s choice is $b = 0011$. This bit string is the table of the function chosen by Bob, more explicitly: $f_b(00) = 0$, $f_b(01) = 0$, $f_b(10) = 1$, $f_b(11) = 1$. $P_B$ is the projection of $|\psi\rangle_B$ on $|0011\rangle_B$, namely on the table of the function. We can share $P_B$ by taking two shares of the table such that the projections of $|\psi\rangle_B$ on them satisfy the sharing rule (of course such projections can be related to partial measurements of the content of register $B$). We show further below that such two shares of the table should be two complementary half tables in each of which all the values of the function are the same. We call each share of this kind a good half table.

This leaves us with only one way of sharing the table $b = 0011$; the two shares should be $f_b(00) = 0$, $f_b(01) = 0$ and respectively $f_b(10) = 1$, $f_b(11) = 1$. The former half table corresponds to the projection of $|\psi\rangle_B$ on $b \in \{0011,0000\}$, the latter on $b \in \{0011,1111\}$. Either half table represents the contribution of Alice’s action to the determination of Bob’s choice.
We show that there is no other way of satisfying the sharing rule. First, let us assume that one of the two complementary half tables is not good (the values of the function are not all the same). Because of the structure of the table, also the other half would not be good. Thus, the two corresponding shares of $P_B$ would both determine the fact that the function is balanced (a Boolean function of $b$). This would violate the no over-determination condition of the sharing rule. If one or both shares were less than half table, this would either not satisfy equation (8) or not determine the value of $b$, as readily checked.

4.3 Advanced knowledge

Also in the present case, the fact that Alice contributes to the determination of Bob’s choice implies that she knows that contribution in advance. This can be seen more quickly as follows. Since the state of register $B$ remains unaltered throughout the unitary part of Alice’s action, also its projection on the half table remains unaltered. At the end of the unitary part of Alice’s action, this projection represents the contribution of Alice’s action to the determination of Bob’s choice. Advanced at the beginning, it changes Alice’s complete ignorance of Bob’s choice into knowledge of the half table.

We can see that the quantum algorithm requires the number of function evaluations of a classical algorithm that has to reconstruct Bob’s choice starting from the advanced knowledge of a good half table. In fact, the value of $b$ is always identified by computing $f_b(a)$ for only one value of $a$ (anyone) outside the half table. Thus, both the quantum algorithm and the advanced knowledge classical algorithm require just one function evaluation.

4.4 Mechanism of the speed-up

Let us group the histories with the same value of $b$. Starting with $b = 0011$, we assume that Alice’s advanced knowledge is, e. g., $b \in \{0011, 0000\}$. In order to determine the value of $b$ and thus the characteristic of the function, Alice should perform function evaluation for either $a = 10$ or $a = 11$. We assume it is for $a = 10$. Since we are under the assumption $b = 0011$, the result of the computation is 1. This, besides telling Alice that $b = 0011$, originates two classical computation histories, each consisting of a state before and one after function evaluation. History 1: $e^{i\varphi_2} |0011\rangle_B |10\rangle_A |0\rangle_V \rightarrow e^{i\varphi_2} |0011\rangle_B |10\rangle_A |1\rangle_V$. History 2: $-e^{i\varphi_2} |0011\rangle_B |10\rangle_A |1\rangle_V \rightarrow -e^{i\varphi_2} |0011\rangle_B |10\rangle_A |0\rangle_V$. If she performs function evaluation for $a = 11$ instead, this originates other two histories, etc.

As readily checked, the superposition of all these histories is the function evaluation stage of the quantum algorithm. Then, Alice applies the Hadamard transform to register $A$. Each history branches into four histories. The end states of such branches interfere with one another to yield state (13).

We can see that Deutsch&Jozsa algorithm is generated by the mechanism of the speed-up of Section 3. We should replace the Hadamard transforms before and after function evaluation by free unitary transformations and then maximize $p_S$ (the probability of finding the solution in register $A$).
It is easy to see that the present analysis, like the notion of sharing the table of the function into two complementary good halves, holds unaltered for \( n > 2 \).

5 Simon’s and the hidden subgroup algorithms

In Simon’s [16] algorithm, the set of functions is all the \( f_b : \{0, 1\}^n \rightarrow \{0, 1\}^{n-1} \) such that \( f_b(a) = f_b(c) \) if and only if \( a = c \) or \( a = c \oplus h(b) \); \( \oplus \) denotes bitwise modulo 2 addition; the bit string \( h(b) \), depending on \( b \) and belonging to \( \{0, 1\}^n \) excluded the all zeroes string, is a sort of period of the function. Array (17) gives (part of) the set of six functions for \( n = 2 \). The bit string \( b \) is both the suffix and the table of the function. Since \( h(b) \oplus h(b) = 0 \) (the all zeros string), each value of the function appears exactly twice in the table, thus 50% of the rows plus one always identify \( h(b) \).

| \( a \) | \( f_{0011}(a) \) | \( f_{1100}(a) \) | \( f_{0101}(a) \) | \( f_{1010}(a) \) |
|---|---|---|---|---|
| 00 | 0 | 1 | 0 | 1 |
| 01 | 0 | 1 | 1 | 0 |
| 10 | 1 | 0 | 0 | 1 |
| 11 | 1 | 0 | 1 | 0 |

And etc. (17)

Bob selects a value of \( b \). Alice’s problem is finding the value of \( h(b) \), ”hidden” in \( f_b(a) \), by computing \( f_b(a) = f(b, a) \) for different values of \( a \). In present knowledge, a classical algorithm requires a number of computations of \( f(b, a) \) exponential in \( n \). The quantum algorithm solves the hard part of this problem, namely finding a string \( s_j^{(b)} \) orthogonal to \( h(b) \), with one computation of \( f(b, a) \); ”orthogonal” means that the modulo 2 addition of the bits of the bitwise product of the two strings is zero. There are \( 2^{n-1} \) such strings. Running the quantum algorithm yields one of these strings at random (see further below). The quantum algorithm is iterated until finding \( n-1 \) different strings. This allows us to find \( h(b) \) by solving a system of modulo 2 linear equations.

We check that the history superposition picture and the mechanism of the speed-up for the present algorithm.

5.1 Time-symmetric representation

Register \( B \) (\( A \)) contains \( b \) (\( a \)), register \( V \) the result of function evaluation reversibly added to it former content. The input and output states of the quantum algorithm are respectively:

\[
|\psi\rangle = \frac{1}{\sqrt{6}} (e^{i\varphi_0} |0011\rangle_B + e^{i\varphi_1} |1100\rangle_B + e^{i\varphi_2} |0101\rangle_B + e^{i\varphi_3} |1010\rangle_B + \ldots ) |00\rangle_A |0\rangle_V ,
\]

\[
U_A U_f U_A |\psi\rangle =
\]

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$$\frac{1}{\sqrt{6}} \left\{ (e^{i\varphi_0}|0011\rangle_B + e^{i\varphi_1}|1100\rangle_B)\left[ (|00\rangle_A + |10\rangle_A)\langle 0\rangle_V + (|00\rangle_A - |10\rangle_A)\langle 1\rangle_V \right] + \\
(e^{i\varphi_2}|0101\rangle_B + e^{i\varphi_3}|1010\rangle_B)\left[ (|00\rangle_A + |01\rangle_A)\langle 0\rangle_V + (|00\rangle_A - |01\rangle_A)\langle 1\rangle_V \right] + ... \right\}.$$ (19)

In state (18), $V$ is prepared in the all zeros string (just one zero for $n = 2$). $U_A$ is Hadamard on $A$, $UV$ – being the identity here – does not appear, $U_f$ is function evaluation. In state (19), for each value of $b$, register $A$ (no matter the content of $V$) hosts even weighted superpositions of the $2^{n-1}$ strings $s_j^{(b)}$ orthogonal to $h^{(b)}$. By measuring $\hat{A}$ in this state, Alice obtains at random one of these $s_j^{(b)}$. Then she repeats the "right part" of the algorithm (preparation of registers $A$ and $V$, computation of $f(b, a)$, and measurement of $\hat{A}$) until obtaining $n - 1$ different $s_j^{(b)}$.

As we will see in sections 5.3 and 5.4, finding the characteristic of the function is a by-product of reconstructing Bob’s choice. We omit the explicit representation of this reconstruction, completely similar to that of Section 4.

5.2 Sharing the determination of Bob’s choice

This time a good half table should not contain a same value of the function twice, what would over-determine $h^{(b)}$, namely a Boolean function of $b$ (also the other half would contain a same value twice). Assume Bob’s choice is $b = 0011$. There are two ways of sharing this table. One is $f_b(00) = 0, f_b(10) = 1$ and $f_b(01) = 0, f_b(11) = 1$; the corresponding shares of $P_B$ are the projections of $|\psi\rangle_B$ on $b \in \{0011,0110\}$ and $b \in \{0011,1001\}$. The other is $f_b(00) = 0, f_b(11) = 1$ and $f_b(01) = 0, f_b(10) = 1$, etc.

We should note that sharing each table into two halves is accidental to the present algorithm. In the quantum part of Shor’s [17] factorization algorithm (finding the period of a periodic function), taking two parts of the table that do not contain a same value of the function twice implies that each part is less than half table if the domain of the function spans more than two periods.

5.3 Advanced knowledge

Ascribing to Alice’s action the determination of a good half table implies that she knows it in advance – as in Section 4.3. Also in the present case the quantum algorithm requires the number of function evaluations of a classical algorithm that has to determine Bob’s choice starting from the advanced knowledge of a good half table. In fact, since no value of the function appears twice in the half table, the value of $b$ is always identified by computing $f(b, a)$ for only one value of $a$ (anyone) outside the half table.

5.4 Mechanism of the speed-up

The history superposition picture can be developed as in Section 4.4: given the advanced knowledge of, say, $b \in \{0011,0110\}$, in order to determine the value of $b$, Alice should perform function evaluation for either $a = 01$ or $a = 11$, etc.
etc. We can see that Simon’s algorithm is generated by the mechanism of the speed-up of Section 3 (here the solution, any $s_j^{(b)}$ orthogonal to $h^{(b)}$, is stochastic in character). We should replace the transformations before and after function evaluation (comprising the identity on register $V$) by free unitary transformations and then maximize $p_s$ (the probability of finding the solution in register $A$).

The present analysis – like the notion of sharing the table into two good halves – holds unaltered for $n > 2$. It also applies to the generalized Simon’s problem and to the Abelian hidden subgroup problem. In fact the corresponding algorithms are essentially the same as the algorithm that solves Simon’s problem. In the hidden subgroup problem, the set of functions $f_b : G \to W$ map a group $G$ to some finite set $W$ with the property that there exists some subgroup $S \leq G$ such that for any $a, c \in G$, $f_b(a) = f_b(c)$ if and only if $a + S = c + S$. The problem is to find the hidden subgroup $S$ by computing $f_b(a)$ for various values of $a$. Now, a large variety of problems solvable with a quantum speed-up can be re-formulated in terms of the hidden subgroup problem [18, 19]. Among these we find: the seminal Deutsch’s problem, finding orders, finding the period of a function (thus the problem solved by the quantum part of Shor’s factorization algorithm), discrete logarithms in any group, hidden linear functions, self shift equivalent polynomials, Abelian stabilizer problem, graph automorphism problem.

6 Discussion and conclusions

We have pinpointed the fundamental reason for which quantum algorithms can require fewer function evaluations than the minimum required by any equivalent classical algorithm and/or violate Morikoshi’s information-theoretic temporal Bell inequality. The quantum principle, stating that the measurement of an observable determines one of its eigenvalues, becomes ambiguous when the measurement of two commuting observables yields at random two identical eigenvalues, which in our case are Bob’s choice and Alice’s reconstruction of it. Which measurement determines their common value? Postulating that the projection of the quantum state induced by either measurement (i.e. the determination) shares between the two measurements (i) with no over-projection, (ii) with entropy reductions the same for each share, and (iii) in a uniform quantum superposition of all the possible ways of sharing compatible with the former conditions, implies that the quantum algorithm is a uniform superposition of algorithms (histories) in each of which Alice determines one of the possible halves of Bob’s choice. Advancing this determination to the beginning of Alice’s action shows that Alice, in each history, knows in advance half of Bob’s choice. In all the cases examined, she can perform only the $N_a$ function evaluations required to classically reconstruct Bob’s choice given the advanced knowledge of half of it.

To this end, function evaluations should be interleaved with non-computational unitary transformations that each time maximize the probability of finding the
solution in Alice’s register. This also maximizes the amount of information about Bob’s choice acquired by Alice with function evaluation. The number of function evaluations is correspondingly minimized and brought in fact to $N_a$ in all the cases examined.

We discuss these results.

The history superposition picture highlights an essential difference between quantum and classical causality. The former can host a loop of the latter. The causal quantum process is for example the unitary transformation of $|\Psi\rangle = |01\rangle_B |00\rangle_A |0\rangle_V$ into $U'_A U_f U_A U_V |\Psi\rangle = \frac{1}{\sqrt{2}} |01\rangle_B |01\rangle_A (|0\rangle_V - |1\rangle_V)$—equations (2) and (6). This is a superposition of histories in each of which Alice knows in advance half of the result of her computation and exploits this information to reach that same result with fewer function evaluations. Alice’s partial knowledge of the result of a computation before performing it (a causality loop and in fact the reason for the violation of Morikoshi’s inequality) would be impossible if histories were isolated with respect to one another. However, quantum superposition and interference (as generated by the maximization procedure) allow this. The half choice known in advance in one history becomes the missing half in another one, where it is computed. Thus, all the possible halves of Bob’s choice are computed, in quantum superposition. Moreover, histories are not isolated from one another, as quantum interference provides cross-talk between them.

It is natural to think that such loops of classical causality, besides the violation of Morikoshi’s inequality in the case of Grover’s algorithm, explain the violation of temporal Bell inequalities on the part of quantum mechanics. A way of investigating this prospect is trying and extend the present explanation of the speed-up to more general quantum processes that yield a speed-up, like for example quantum random walks [20] or mixed state quantum computing [21]. In a way, we should go back to the original Feynman’s observation that the classical simulation of a quantum process can require an essentially higher amount of resources [22]. As it is, the explanation requires seeing a problem in the input of the quantum process and the solution of the problem in the output. To apply it to more general quantum processes, we should decouple it from problem-solving. This would seem to be possible. The basic concept of the explanation is the possibility that the quantum process builds a stronger than classical input-output correlation thanks to the fact that (from the standpoint of quantum correlation) the final measurement of the output contributes to determining the input. This concept of quantum retroaction of the output on the input is not committed to problem-solving.

From a technical standpoint, the present work can be used in the search for new speed-ups. Given a set of functions, one should: (a) interleave function evaluations with free non-computational unitary transformations, (b) after the unitary transformation that follows the first function evaluation, maximize the amount of information about Bob’s choice leaked to Alice’s register, (c) identify the characteristic of the function obtained and (d) iterate function evaluation and the successive unitary transformation maximizing each time the probability
of finding that characteristic in Alice’s register. The number of function evaluations should be that required to reconstruct Bob’s choice given the advanced knowledge of half of it according to the sharing rule. Reference [11] provides the example of a new quantum speed-up that can be obtained in this way.

In conclusion, although preliminary in character, these results seem to open a gap in a problem that has remained little explored. Until now there was no fundamental explanation of the speed-up, no general mechanism for producing it.

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