Wave propagation in a weak gravitational field and the validity of the thin lens approximation

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Wave effects can be important for the gravitational lensing of gravitational waves. In such a case, wave optics must be used instead of geometric optics. We consider a plane wave entering a lens object and solve numerically the wave equation for three lens models: the uniform density sphere, the singular isothermal sphere, and the Hernquist model. By comparing our numerical solutions with the analytical solutions under the thin lens approximation, we evaluate the error of this approximation. The results show that the relative error of the thin lens approximation is small if the geometrical thickness of the lens is much smaller than the distance between the lens and the observer.

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I. INTRODUCTION

Gravitational waves from coalescing compact binaries composed of neutron stars or black holes are the most promising targets for ground-based as well as space-based detectors. By applying the matched filtering technique which uses our theoretical predictions of wave forms obtained by the post-Newtonian computations, we can extract the binary parameters such as the masses of each compact object, distance to the source, spatial positions of the source and so on.

One possibility which alters the predicted wave forms calculated with high precision is the gravitational lensing of gravitational waves. If a massive object lies suitably between the source and the observer, gravitational lensing of gravitational waves occurs. One important point is that since the wavelength of gravitational waves we are interested in is much larger than that of light, a situation where the geometrical optics approximation breaks down can be realized in some cases. As is discussed by many authors, if the wavelength is larger than the Schwarzschild radius of the lens object, the diffraction effect becomes important and the magnification approaches to unity. Therefore we must use wave optics rather than the geometric optics for

\[ M_L \lesssim 10^8 M_\odot \left( \frac{f}{\text{mHz}} \right)^{-1}, \]

where \( M_L \) is a mass of the lens and \( f \) is the frequency of the gravitational waves. This frequency (mHz) is the case for the planned detector LISA.

Further because the gravitational waves from a compact binary are coherent, interference between lensed waves is important. Note that this situation is not in general realized in the case of gravitational lensing of electro-magnetic wave such as visible light. Since light is emitted from microscopic region (usually atomic size) which is much smaller than the size of the source, each phase of the electro-magnetic wave emitted from different points has no correlation and thus interference effect vanishes.

If we assume the coalescence of SMBHs of mass \( 10^4 \sim 10^7 M_\odot \) as the source of the gravitational waves, it can be detected even if the sources are located at the cosmological distance (\( z > 5 \)). Event rate of SMBH-SMBH merger for LISA is estimated as \( 0.1 \sim 10^2 \text{ event/yr} \) and lensing probability becomes several percent. Hence, some lensing events per year will be detected by LISA.

Motivated by the fact that wave effects can be detected for the gravitational lensing of gravitational waves, there are now growing interests in the wave optics in gravitational lensing. However we have to solve wave equation which is generally partial differential equation between the source and the observer in order to obtain the lensed wave form at the observer. Except for a few special cases, exact solutions of wave equation are not known at present. Several authors have used thin lens approximation which reduces wave equation to double integral for single lens object (for multi-lens objects, integration becomes multi-integral, see [12]) and thus makes the problem easier. In geometric optics the trajectory of light ray is obtained by solving geodesic equation, and it is known that thin lens approximation is valid. However there has been no studies or comments about the validity of the thin lens approximation in the framework of wave optics.

In this paper, we develop a formulation to solve the wave equation for a spherically symmetric lens, where a partial differential equation reduces to a set of ordinary differential equations. We also solve those equations for simple lens models: the uniform density sphere, the singular isothermal sphere and the Hernquist model and evaluate the error of the thin lens approximation.

This paper is organized as follows. In section II, we briefly review wave optics in gravitational lensing under the thin lens approximation. In section III, we develop a formalization to solve the scattering problem of gravitational waves by a lens. In section IV, we present our numerical results and discuss the validity of the thin lens approximation. Section V is a summary. We use unit of \( c = 1 \).
II. GRAVITATIONALLY LENSED WAVEFORM UNDER THE THIN LENS APPROXIMATION

We consider the wave propagation under the gravitational fields of a lens. We assume that the spacetime metric is a Minkowski spacetime plus a small perturbation due to the existence of a static lensing object. Then the metric can be written as
\[
ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(1 + 2U)dt^2 + (1 - 2U)dx^2,
\]
where \(U\) is a Newtonian potential of the lensing object. We consider a propagation of scalar waves \(\phi\), instead of gravitational waves, since the wave equation for \(\phi\) is the same as that for gravitational waves \(\Box \phi = 0\). The scalar wave equation, \(\Box \phi = 0\), with the metric \(\Box = (\Delta + \omega^2)\phi = 4\omega^2 U \phi\), is rewritten as
\[
(\Delta + \omega^2)\phi(\vec{x}) = 4\omega^2 U(\vec{x})\phi(\vec{x}),
\]
where we assume that the wave is monochromatic with the angular frequency \(\omega\). The above equation was solved by using the Kirchhoff diffraction integral (see Ref. 17, Sec. 4.7 and 7) under the thin lens approximation.

We show lensing configuration in Fig. 1. We choose a position of the lens as the origin of the coordinate system \((r, \theta, \phi)\). The observer position is \((r, \theta)\) with \(\theta \ll 1\). The incident wave is a plane wave propagating in the \(z\)-direction. Denoting the position of the lens as the origin of the polar coordinate axes, while the observer position is \((r, \theta)\), with \(r \ll 1\), the observer position is \((r, \theta)\).

In this section, we assume the thin lens approximation, in which the wave is scattered only on the thin lens plane at \(z = 0\), and the lens is characterized by the surface mass density \(\Sigma(\vec{s})\), where \(\vec{s} = (x, y)\). The two dimensional potential \(\psi(\vec{s})\) in \(z = 0\) plane is defined as
\[
\psi(\vec{s}) = 2 \int_{-\infty}^{\infty} dz U(\vec{s}, z).
\]
Here, \(\psi\) is also obtained from the surface density by using \(\nabla^2 \psi(\vec{s}) = 8\pi \Sigma(\vec{s})\).

It is useful to define the amplification factor \(F\) (which is called the transmission factor in Ref. 17) as \(F = \phi/\phi_0\), where \(\phi\) is the gravitationally lensed waveform obtained by solving Eq. (3) and \(\phi_0\) is the incident wave. The amplification factor at the observer under the thin lens approximation is given by
\[
F_{\text{thin}}(r, \theta) = \frac{\omega}{2\pi r} \int d^2 \vec{s} e^{i\vec{s} \cdot \vec{r}} |r - \vec{s}^2| - i\omega \psi(\vec{s}).
\]
Here, \(F_{\text{thin}}\) is normalized so that \(F_{\text{thin}} = 1\) for \(\psi = 0\).

III. FORMULATION OF NUMERICAL CALCULATION

In this section, we develop a formulation to solve the scattering problem of gravitational waves by lensing object which is applicable to the case where the lens potential is spherically symmetric. A situation we will consider is that plane wave is entering weak gravitational field which is spherically symmetric. When the lensing object is spherically symmetric, a scattering problem can be reduced to a problem of determining so-called phase shift which is used to probe the nature of nuclear physics and is also useful for the scattering by BHs where the assumption of weak gravitational field breaks down 21.

We have to solve Eq. (3) in order to evaluate quantities such as the amplification factor which can be compared with the one derived under the thin lens approximation. To solve Eq. (3), we choose a center of the lens as the origin of coordinate system. Then \(U\) becomes a function which depends only on the radius coordinate \(r\). Now let us expand \(\phi\) in terms of Legendre function
\[
\phi(\xi, \theta) = \sum_{\ell=0}^{\infty} \frac{g_\ell(\xi)}{\xi} P_\ell(\cos \theta),
\]
where \(\xi\) is a dimensionless variable defined by \(\xi = r\omega\). Then equations for \(g_\ell(\xi)\) are
\[
\left(\ell^2 - 1 + 4\tilde{U}(\xi) - \frac{\ell(\ell + 1)}{\xi^2}\right) g_\ell(\xi) = 0,
\]
where we have used \(\tilde{U}\) in stead of \(U\) in order to stress that \(\tilde{U}\) is a function of \(\xi\).

For a point mass lens, i.e. \(\tilde{U}(\xi) = -p/(2\xi)\), where \(p = 2GM\omega\) and \(M\) is a lens mass, analytic solutions of Eq. (7) are known as Coulomb wave functions 22. The
solution which is regular at $\xi = 0$ is
\[ F_\ell(-p, \xi) = e^{\frac{\pi i}{2} \frac{\Gamma(\ell + 1 - ip)}{\Gamma(2(\ell + 1))}} \xi^{\ell + 1} e^{i\xi + i\sigma_\ell} \times F(\ell + 1 - ip; 2(\ell + 1); -2i\xi), \]
\[ \sigma_\ell \equiv \arg \Gamma(\ell + 1 + ip), \]
where $F$ is the confluent hypergeometric function. The solution which is singular at $\xi = 0$ is
\[ G_\ell(-p, \xi) = \xi^{\ell + 1} 2^\ell e^{-\frac{\pi i}{2} p(\ell + 1)} e^{i\xi - i\sigma_\ell} \times U(\ell + 1 - ip; 2(\ell + 1); -2i\xi) + c.c., \]
where $U$ is defined as
\[ U(a; b; z) = \frac{1}{\Gamma(a)} \int_0^\infty dt e^{-zt} t^{a-1}(1 + t)^{b-a-1}. \]
The asymptotic form of these functions are
\[ F_\ell(-p, \xi) \underset{\xi \gg 1}{\longrightarrow} \sin(\xi + p \log 2\xi - \frac{\pi}{2} \ell - \sigma_\ell), \]
\[ G_\ell(-p, \xi) \underset{\xi \gg 1}{\longrightarrow} \cos(\xi + p \log 2\xi - \frac{\pi}{2} \ell - \sigma_\ell). \]
The term $p \log 2\xi$ in the phase of trigonometric functions represents the nature of long range force which is characteristic of Coulomb force.

On the contrary, a solution of Eq. (8) that a plane wave is entering a point mass lens is well known and is given by
\[ \phi_{p} = e^{\frac{\pi i}{2} p}(1 - ip)e^{i\xi \cos \theta} F(ip; 1; i\xi (1 - \cos \theta)). \]
Because the solution Eq. (12) is regular at $\xi = 0$, it is written as a partial wave sum of regular Coulomb wave function,
\[ \phi_{p} = \sum_{\ell=0}^{\infty} a_\ell \frac{F_\ell(-p, \xi)}{\xi} P_\ell(\cos \theta), \]
\[ a_\ell = i^\ell (2\ell + 1) e^{-i\sigma_\ell}. \]

For an extended lensing object, analytic solution of Eq. (8) does not exist usually. However if the lensing object exists only in a finite region, then the solution of Eq. (8) outside the lens can be written as a summation of partial waves which are now a linear combination of two independent Coulomb wave functions. By determining a coefficient of each Coulomb wave function, we can calculate the wave form $\phi$ outside the lens. In the aim of only determining the wave form far from the lensing object, we don’t need to know the expression of $\phi$ inside the lens.

Now let us write the solution of Eq. (8) as
\[ \phi = \phi_{p} + \phi_{s}. \]
Thus $\phi_{s}$ represents the scattered wave which arises due to the deviation of lens from a point mass. There may be no incoming scattered wave to the lens from infinity, so we assume the asymptotic form of $\phi_{s} as
\[ \phi_{s}(\xi, \theta) = \sum_{\ell=0}^{\infty} \frac{e^{i\xi + ip \log 2\xi - i\pi \ell}}{2i\xi} s_\ell P_\ell(\cos \theta), \]
where $s_\ell$ are undetermined complex numbers, but not arbitrary. In order that $\phi$ in Eq. (15) satisfy the wave equation (8), $s_\ell$ must be related to $a_\ell$ as
\[ s_\ell = a_\ell e^{2i\delta_\ell} - 1, \]
where $\delta_\ell$ are real numbers. In terms of $\delta_\ell$, $\phi$ is written as
\[ \phi(\xi, \theta) = \sum_{\ell=0}^{\infty} a_\ell e^{i\delta_\ell} \left( \cos \delta_\ell \frac{F_\ell(-p, \xi)}{\xi} + \sin \delta_\ell \frac{G_\ell(-p, \xi)}{\xi} \right) P_\ell(\cos \theta). \]

Thus we can calculate wave form outside the lens object by determining the phase shift $\delta_\ell$. Here, $\delta_\ell$ are determined by matching a solution of Eq. (12) with Eq. (8) at a radius $\xi_0$ being outside the lens. Then, we have
\[ \tan \delta_\ell = \frac{g(\xi_0) F'(\xi_0) - g'(\xi_0) F(\xi_0)}{g(\xi_0) F'(-\xi_0) - g'(\xi_0) F(-\xi_0)}, \]
where $g_\ell$ is a solution of Eq. (7) which is to be calculated numerically. We set the initial condition of Eq. (7) at $\xi = 0$ is that $g_\ell$ is regular. The range of $\delta_\ell$ is from $-\pi/2$ to $\pi/2$.

IV. RESULTS

We investigate the validity of the thin lens approximation for three lens models; the uniform density sphere, the singular isothermal sphere and the Hernquist model.

A. Uniform density sphere

We first present the results for uniform density sphere which has the simplest structure next to the point mass. The gravitational potential for the uniform density sphere is given by
\[ U(r) = \begin{cases} \frac{GM}{r} - \frac{GM}{R} & (r \leq R) \\ \frac{GM}{r} & (r \geq R) \end{cases}, \]
where $r$ is the distance from the center of the sphere, $R$ is the radius of the sphere and $M$ is the lens mass. Here we consider the case that $R$ is larger than the Einstein radius $r_E$, in which case the effect of the size of the lens object is expected to be important. For $R < r_E$, the result is almost the same as for point lens mass and it is known that amplitude of the amplification factor for
shows that the thin lens approximation is valid for the geometric thickness of lens ratio of the radius to distance between lens and observer is much smaller than unity. (The dashed lines denote the error ∆ defined as, ∆ = \frac{1}{r} \left| \frac{i \omega r}{2} \int_0^\infty ds J_0(\omega s) e^{i \omega s} - \psi(s) \right|,

(20)

where \( J_0 \) is the 0-th Bessel function and \( \psi \) is given by

\[
\psi(s) = \begin{cases} 
  4M \log \left[ 1 + \sqrt{1 - (s/R)^2} \right] - \frac{4M}{3} (4 - (s/R)^2) \sqrt{1 - (s/R)^2} & (s \leq R) \\
  4M \log(s/R) & (s \geq R).
\end{cases}
\]

Next let us calculate \( F \) numerically by using the method developed in the previous section. We first show \( \delta \ell \) as a function of \( \ell \) in fig. 2. We see that \( \delta \ell \) becomes larger. This is because for large \( \ell \)-th partial wave can be interpreted as an incident particle with impact parameter \( \ell/\omega \). Partial waves of \( \ell \gtrsim \omega R \) pass through the lens potential outside the lens object and the gravitational effects on these partial waves are the same as point mass, which implies \( \delta \ell \) becomes zero.

Fig. 2 shows the error of the amplification factor for the uniform density sphere as a function of lens parameters \( \omega R, \omega r, GM\omega \) and \( \theta \) respectively. The vertical axis is the relative error of the amplification factor in the thin lens approximation \[15\].

The amplification factor in the thin lens approximation Eq. 5 for the case of uniform density sphere is written as

\[
F_{\text{thin}}(r, \theta) = -i \frac{\omega}{r} e^{\frac{i}{2} \omega r^2} \int_0^\infty ds J_0(\omega s) e^{i \omega s} - \psi(s),
\]

(20)

where \( J_0 \) is the 0-th Bessel function and \( \psi \) is given by

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\psi(s) = \begin{cases} 
  4M \log \left[ 1 + \sqrt{1 - (s/R)^2} \right] - \frac{4M}{3} (4 - (s/R)^2) \sqrt{1 - (s/R)^2} & (s \leq R) \\
  4M \log(s/R) & (s \geq R).
\end{cases}
\]

The top left and right panels show \( \Delta \) as a function of \( \omega R \) and \( \omega r \) respectively. We find that averaged in all range of frequencies \( \omega r \theta \) becomes same as for the point mass lens for which it is known that thin lens approximation is valid. Except for a small region of \( \theta \) around where \( \Delta \) takes maximum value, \( \Delta \) is smaller than \( R/r \).

To summarize, for the uniform density sphere, the relative error of the amplification factor in the thin lens approximation is suppressed within the ratio of its radius to distance \( R/r \) which is much smaller than unity in a realistic astrophysical situations and is the largest for \( \lambda \sim GM \) (\( \lambda \) is the wavelength).

B. Singular Isothermal Sphere (SIS) model

We also calculated \( \Delta \) for singular isothermal sphere which is a model of galaxies, dark matter haloes and
star clusters. The density profile of SIS model is \( \rho(r) = v^2 / (2\pi r^2) \) where \( v \) is the velocity dispersion. For numerical calculation, we have to introduce cutoff radius \( r_c \) because gravitational potential does not approaches to \( \propto r^{-1} \) far from the lens without cutoff. Thus we here assume that the density vanishes for \( r > r_c \). Newton potential becomes

\[
U(r) = \begin{cases} 
\frac{GM}{r_c} \left( \ln \frac{r}{r_c} - 1 \right) & (r \leq r_c) \\
\frac{GM}{r} & (r \geq r_c)
\end{cases}
\]

where \( M \) is the mass inside the cutoff radius \( r_c : M = 2v^2r_c \).

\[ F_{\text{thin}} \text{ is given by Eq. (20), and } \psi(s) \text{ is given by} \]

\[
\psi(s) = \begin{cases} 
-\frac{GM}{r_c} \sqrt{r_c^2 - s^2} + \frac{4GM}{r_c} \arctan \frac{\sqrt{r_c^2 - s^2}}{s} + 4GM \ln \left( 1 + \sqrt{1 - (s/r_s)^2} \right) & (s \leq r_c) \\
8GM \ln(s/r_s) & (s \geq r_c)
\end{cases}
\]

Fig. 3 presents dependence of \( \Delta \) on parameters, \( \omega r_c \), \( \omega r \), \( \omega GM \) and \( \theta \). We find that the qualitative behavior
is the same as the case of uniform density sphere. We see from the top left panel and the top right panel that thin lens approximation becomes worse as cutoff approaches to observer’s distance $r$ from the center of SIS.

### C. Hernquist model

We also calculated $\Delta$ for Hernquist model [19], which fits well the luminosity distribution of many elliptical galaxies and bulges. Its density profile is given by

$$\rho(r) = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^3}, \quad (24)$$

where $r_s$ is a scale length and $\rho_s$ is a characteristic density. For numerical calculation, we introduce cutoff radius $r_c$ for the same reason as SIS. For $r < r_c$ the density is given in Eq.(24), while for $r > r_c$ the density vanishes. Then, Newton potential becomes

$$U(r) = \begin{cases} \frac{-GM}{r + r_s} \left( \frac{r + r_c}{r_c} \right)^2 + \frac{r}{r_c}GM & (r \leq r_c) \\ \frac{-GM}{r} & (r \geq r_c) \end{cases}, \quad (25)$$

where $M$ is the mass inside the cutoff radius $r_c$ : $M = 2\pi \rho_s r_s^3 [r_c/(r_s + r_c)]^2$.

$F_{\text{thin}}$ is given by Eq.(20), and $\psi(s)$ is given by

$$\psi(s) = \begin{cases} -4Mr_s^2 + 2r_s r_c \arccosh \frac{r}{s} + 4Mr_s^3 M \sqrt{s^2 - r_s^2} - s^2 + \frac{4Mr_c(r_s + r_c)^2}{s^2 + r_c r_s} \arctanh \left( \frac{\sqrt{(r_s^2 - s^2)}(r_s^2 - r_c^2)}{s^2 + r_c r_s} \right), & (s \leq r_s) \\ -4Mr_s^2 + 2r_s r_c \arccosh \frac{r_c}{r} + 4Mr_s^3 M \sqrt{s^2 - r_c^2} - s^2 + \frac{4Mr(r_c + r_c)^2}{s^2 + s r_c} \arctanh \left( \frac{\sqrt{(s^2 - r_c^2)(s^2 - r_s^2)}}{s^2 + s r_c} \right), & (r_s \leq s \leq r_c) \\ 4M \ln s, & (r \geq r_c), \end{cases} \quad (26)$$
Fig. 5 shows $\Delta$ for Hernquist model. We see that the behavior of $\Delta$ is almost the same as SIS.

V. SUMMARY

In this paper we discussed the validity of the thin lens approximation in the framework of wave optics.

In sec III, we developed a formalism to solve the wave equation for the spherically symmetric potential. In this case, the partial differential equation can be reduced to a set of ordinary differential equations. The method we used is to determine the so-called phase-shift which represents the difference of scattered waves between the point mass lens and an arbitrary spherically symmetric potential. This formalism is only applicable to the spherically symmetric lens whose size is finite. For lens models such as the SIS profile which extends to infinity, we have to introduce cutoff to make the total mass finite.

We also solved the wave equation numerically for the spherically symmetric potential. By numerical calculations, we found that the error of the thin lens approximation for the simple lens models is the same as or smaller than the geometric thickness of the lens, $s/r$, where $s$ is the size of the lens and $r$ is the distance between the lens and the observer. The error is the largest for the wavelength comparable to the Schwarzschild radius of the lens.

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