Three-coloring statistical model with ‘domain wall’ boundary conditions

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1 Introduction
   - Three-colorings
   - Combinatorial problems
   - Three-colorings and six vertex model

2 Six vertex model with domain wall boundary conditions
   - Domain wall boundary conditions
   - Recursion relations
   - Functional equations

3 Three-coloring model with ‘domain wall’ boundary conditions
   - ‘Domain wall’ boundary conditions
   - Gauge transformation
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1 Introduction
   - Three-colorings
   - Combinatorial problems
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   - Domain wall boundary conditions
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1 Introduction
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- Domain wall boundary conditions
- Recursion relations
- Functional equations

3 Three-coloring model with ‘domain wall’ boundary conditions
- ‘Domain wall’ boundary conditions
- Gauge transformation
- Recursion relations
## Contents

1. **Introduction**
   - Three-colorings
   - Combinatorial problems
   - Three-colorings and six vertex model

2. **Six vertex model with domain wall boundary conditions**
   - Domain wall boundary conditions
   - Recursion relations
   - Functional equations

3. **Three-coloring model with ‘domain wall’ boundary conditions**
   - ‘Domain wall’ boundary conditions
   - Gauge transformation
   - Recursion relations
Three-colorings

Three-coloring statistical model with ‘domain wall’ boundary conditions
Three-coloring statistical model with ‘domain wall’ boundary conditions
Three-colorings statistical model with ‘domain wall’ boundary conditions

The diagram shows a grid of squares colored in three colors: red, green, and blue. The colors correspond to different states, indicated by the symbols $\bar{0}$, $\bar{1}$, and $\bar{2}$. The grid illustrates how the three-coloring problem can be visualized with 'domain wall' boundary conditions.
Three-colorings

Three-coloring statistical model with ‘domain wall’ boundary conditions
Three-colorings

Colors

The colors of adjacent faces are **different**.

Elements of $\mathbb{Z}_3$

The ‘colors’ of adjacent faces differ by $+1$ or $-1 = +2$. 

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Three-colorings

Colors

The colors of adjacent faces are **different**.

Elements of $\mathbb{Z}_3$

The ‘colors’ of adjacent faces differ by $+\bar{1}$ or $-\bar{1} = +\bar{2}$. 
Enumerations

- Enumerate three-colorings — $C_{n,m}$
- Find refined enumerations — $C_{n,m}(k_0, k_1, k_2)$

Generating function

$$Z_{n,m}(z_0, z_1, z_2) = \sum_{k_0, k_1, k_2} \frac{z_0^{k_0} z_1^{k_1} z_2^{k_2}}{k_0 + k_1 + k_2 = nm} C_{n,m}(k_0, k_1, k_2),$$

Statistical mechanics interpretation

- The numbers $z_0, z_1, z_2$ — the Boltzmann weights of faces
- The product of the Boltzmann weights of the faces — the Boltzmann weight of a state
- $Z_{n,m}(z_0, z_1, z_2)$ — the partition function (state sum) of the model
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with $k_0 + k_1 + k_2 = nm$.

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### Combinatorial problems

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Contents

1 Introduction
   - Three-colorings
   - Combinatorial problems
   - Three-colorings and six vertex model

2 Six vertex model with domain wall boundary conditions
   - Domain wall boundary conditions
   - Recursion relations
   - Functional equations

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In 1961 Lenard remarked that there is a correspondence between the three-colorings and the states of the six vertex model.

**Ice condition**

At every vertex there are two arrows pointing in and two arrows pointing out.
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At every vertex there are two arrows pointing in and two arrows pointing out.
Consider **four edges** containing a fixed four-valent vertex of the lattice and **four faces** containing these edges. The possible color combinations for the four face sets are given below.

Visit the selected faces moving anticlockwise. If intersecting an edge we see that the color changes by $\pm 1$ we place on the edge a pointing in arrow, if the color changes by $-1$ we place a pointing out arrow.
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Weight as a product over vertices

\[(z_\mu z_\nu z_\rho z_\sigma)^{1/4}\]

For the toroidal boundary conditions each face enters four vertex configurations.

In 1970 Baxter found the partition function in the thermodynamic limit.
Three-colorings and six vertex model

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Three-colorings and six vertex model

The spectral parameter $u$ is associated with the central vertex of a four face configuration.
Three-colorings and six vertex model

Stroganov’s solution of star-triangle relation (1982)

The weights are expressed via standard elliptic $\theta$-functions of nome $p$:

$$a_\mu(u|\alpha, p) = \zeta^{3u/4\pi}_\mu(\alpha, p) \frac{\theta_1(2\pi/3 - u|p)}{\theta_1(2\pi/3|p)},$$

$$b_\mu(u|\alpha, p) = \zeta^{1/2-3u/4\pi}_\mu(\alpha, p) \frac{\theta_1(u|p)}{\theta_1(2\pi/3|p)},$$

$$c_\mu(u|\alpha, p) = \frac{\zeta^{u/2\pi}_\mu(\alpha, p)}{\zeta^{u/2\pi}_\mu(\alpha, p)} \frac{\theta_4(\alpha + 2\pi\bar{\mu}/3 + u|p)}{\theta_4(\alpha + 2\pi\bar{\mu}/3|p)},$$

$$c'_\mu(u|\alpha, p) = \frac{\zeta^{u/2\pi}_{\mu-1}(\alpha, p)}{\zeta^{u/2\pi}_\mu(\alpha, p)} \frac{\theta_4(\alpha + 2\pi\bar{\mu}/3 - u|p)}{\theta_4(\alpha + 2\pi\bar{\mu}/3|p)}.$$

Here $\alpha$ is an arbitrary parameter and

$$\zeta_\mu(\alpha, p) = \frac{\theta_4(\alpha + 2\pi(\bar{\mu} - 1)/3|p)\theta_4(\alpha + 2\pi(\bar{\mu} + 1)/3|p)}{\theta_4^2(\alpha + 2\pi\bar{\mu}/3|p)}.$$
For $u = \pi/3$ we have

$$a_\mu(\pi/3|\alpha, p) = \zeta^{1/4}_\mu(\alpha, p), \quad c_\mu(\pi/3|\alpha, p) = \zeta^{1/2}_{\mu+1}(\alpha, p)\zeta^{1/2}_\mu(\alpha, p),$$

$$b_\mu(\pi/3|\alpha, p) = \zeta^{1/4}_\mu(\alpha, p), \quad c'_\mu(\pi/3|\alpha, p) = \zeta^{1/2}_{\mu-1}(\alpha, p)\zeta^{1/2}_\mu(\alpha, p),$$

and we come to the three-coloring model by Baxter with

$$z_\mu = \zeta_\mu(\alpha, p).$$
Three-colorings and six vertex model

Standard solution of star-triangle relation

\[ a(u|\eta) = \frac{\sin(u + \eta/2)}{\sin \eta}, \quad b(u|\eta) = \frac{\sin(u - \eta/2)}{\sin \eta}, \quad c(u|\eta) = 1, \]

where \( \eta \) is the crossing parameter.

For \( \eta = 2\pi/3 \) and the domain wall boundary conditions the partition function of the inhomogeneous six vertex model satisfies some simple functional equations. The partition function of the three-coloring model for appropriate boundary condition satisfies a similar equation.
Three-colorings and six vertex model

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Proof of functional equation for six vertex model

- Recursion relations (Korepin, 1982)
- Determinant representation of the partition function (Izergin, 1987)
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Contents

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   • Three-colorings
   • Combinatorial problems
   • Three-colorings and six vertex model

2 Six vertex model with domain wall boundary conditions
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Inhomogeneous model

The spectral parameter associated with a vertex is $x_i - y_j$. 

Domain wall boundary conditions
The spectral parameter associated with a vertex is $x_i - y_j$. 
Contents

1 Introduction
   - Three-colorings
   - Combinatorial problems
   - Three-colorings and six vertex model

2 Six vertex model with domain wall boundary conditions
   - Domain wall boundary conditions
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Recursion relations

\[
\begin{align*}
\sin(x_n - y_n + \eta/2) &= \frac{1}{\sin \eta} \\
x_n - y_n + \eta/2 &= 0
\end{align*}
\]
Recursion relations

\[
\frac{\sin(x_n - y_n + \eta/2)}{\sin \eta} = 1
\]

\[x_n - y_n + \eta/2 = 0\]
Recursion relations

\[ \sin \left( x_n - y_n + \eta / 2 \right) \frac{1}{\sin \eta} = 0 \]

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\[
\sin(x_n - y_n + \eta/2) = \frac{\sin \eta}{2}
\]

\[
x_n = y_n - \eta/2
\]
Recursion relations

\[ x_n = y_n - \eta / 2 \]

\[ \frac{\sin(x_n - y_n + \eta / 2)}{\sin \eta} \]

Three-coloring statistical model with ‘domain wall’ boundary conditions
Recursion relations

\[
Z_n(x_1, \ldots, x_{n-1}, x_n; y_1, \ldots, y_{n-1}, y_n) \bigg|_{x_n = y_n - \eta/2} = \sin^{2-2n} \eta \prod_{i=1}^{n-1} \sin(x_i - y_n - \eta/2) \prod_{i=1}^{n-1} \sin(y_n - y_i - \eta) \\
\times Z_{n-1}(x_1, \ldots, x_{n-1}; y_1, \ldots, y_{n-1}).
\]

Since the function \( Z_n \{x\}; \{y\} \) is symmetric in the variables \( x_i \) and \( y_i \) we have \( n^2 \) recursion relations

\[
Z_n(x_1, \ldots, x_k, \ldots, x_n; y_1, \ldots, y_\ell, \ldots, y_n) \bigg|_{x_k = y_\ell - \eta/2} = \sin^{2-2n} \eta \prod_{i=1}^{n} \sin(x_i - y_\ell - \eta/2) \prod_{i=1}^{n} \sin(y_\ell - y_i - \eta) \\
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Recursion relations

\[ Z_n(x_1, \ldots, x_{n-1}, x_n; y_1, \ldots, y_{n-1}, y_n) \bigg|_{x_n = y_n - \eta / 2} \]

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\[ \times Z_{n-1}(x_1, \ldots, \hat{x}_k, \ldots, x_n; y_1, \ldots, \hat{y}_\ell, \ldots, y_n). \]
Recursion relations

One can start with the top rightmost vertex. This gives

\[
Z_n(x_1, \ldots, x_k, \ldots, x_n; y_1, \ldots, y_\ell, \ldots, y_n) \bigg|_{x_k = y_\ell + \eta / 2}
\]

\[
= \sin^{2-2n} \eta \prod_{i=1 \atop i \neq k}^{n} \sin(x_i - y_\ell + \eta / 2) \prod_{i=1 \atop i \neq \ell}^{n} \sin(y_\ell - y_i + \eta)
\]

\[
\times Z_{n-1}(x_1, \ldots, \hat{x}_k, \ldots, x_n; y_1, \ldots, \hat{y}_\ell, \ldots, y_n).
\]

Thus we have \(2n^2\) recursion relations.
Recursion relations

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\[ Z_n(x_1, \ldots, x_k, \ldots, x_n; y_1, \ldots, y_\ell, \ldots, y_n) \big|_{x_k = y_\ell + \eta/2} \]

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Contents

1 Introduction
   - Three-colorings
   - Combinatorial problems
   - Three-colorings and six vertex model

2 Six vertex model with domain wall boundary conditions
   - Domain wall boundary conditions
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Assume now that $\eta = 2\pi/3$ and define the functions

$$F_n(\{x\}, \{y\}) = \prod_{i,j=1, i<j}^{n} \sin(x_i - x_j) \prod_{i,j=1}^{n} \sin(x_i - y_j) \prod_{i,j=1, i<j}^{n} \sin(y_i - y_j) Z_n(\{x\}, \{y\}) .$$

These functions satisfy $2n^2$ recursion relations

$$F_n(x_1, \ldots, x_k, \ldots, x_n; y_1, \ldots, y_\ell, \ldots, y_n) |_{x_k = y_\ell \pm \pi/3} = \mp (-1)^n 4^{2-2n} \sin^{3-2n}(2\pi/3) \prod_{i=1}^{n} \sin[3(y_\ell - x_i)] \prod_{i=1}^{n} \sin[3(y_\ell - y_i)]$$

$$\times F_{n-1}(x_1, \ldots, \hat{x}_k, \ldots, x_n; y_1, \ldots, \hat{y}_\ell, \ldots, y_n).$$
Assume now that $\eta = 2\pi / 3$ and define the functions

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These functions satisfy $2n^2$ recursion relations

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$$

$$
\times F_{n-1}(x_1, \ldots, \hat{x}_k, \ldots, x_n; y_1, \ldots, \hat{y}_\ell, \ldots, y_n).
$$
Now define the functions

\[ S_{n,k}(x_1, \ldots, x_k, \ldots, x_n; \{y\}) = \sum_{r=0}^{2} F_n(x_1, \ldots, x_k + 2\pi r / 3, \ldots, x_n; \{y\}). \]

and prove by induction that \( S_{n,k}(\{x\}; \{y\}) = 0. \)

Actually it suffices to prove that \( S_{n,1}(\{x\}; \{y\}) = 0. \)

The base case of induction is \( n = 1. \) It is not difficult to show that

\[ S_{1,1}(x_1; y_1) = \sum_{r=0}^{2} \sin(x_1 - y_1 + 2\pi / 3) = 0. \]
Now define the functions

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\[ S_{1,1}(x_1; y_1) = \sum_{r=0}^{2} \sin(x_1 - y_1 + 2\pi /3) = 0. \]
Assume now that $S_{n-1,1}({x};{y}) = 0$ for some $n > 1$. The recursion relations give

\[
S_{n,1}(x_1, \ldots, x_{n-1}x_n; y_1, \ldots, y_\ell, \ldots, y_n) \big|_{x_n = y_\ell \pm \pi/3} = \mp (-1)^n 4^{2-2n} \sin^{3-2n}(2\pi/3) \prod_{i=1}^{n-1} \sin[3(y_\ell - x_i)] \prod_{i=1, i \neq \ell}^{n} \sin[3(y_\ell - y_i)]
\]

\[
\times S_{n-1,1}(x_1, \ldots, x_{n-1}; y_1, \ldots, \hat{y}_{\ell}, \ldots, y_n) = 0.
\]
Counting orders

With respect to the variable $x_n$ the partition function $Z_n(\{x\}; \{y\})$ is a trigonometric polynomial of order less or equal to $n - 1$, and $S_{n,1}(\{x\}; \{y\})$ is a trigonometric polynomial of order less or equal to $3n - 2$. It has at most $6n - 4$ zeros in the interval $0 \leq x_n < 2\pi$.

Counting zeros

- Recursion relations give $2n$ zeros at the points $x_n = y_\ell \pm \pi/3$, $\ell = 1, \ldots, n$.
- By construction there are $2n - 2$ zeros at the points $x_n = x_\ell$, $\ell = 2, \ldots, n - 1$ and at the points $x_n = y_\ell$, $\ell = 1, \ldots, n$.
- The relation $S_{n,1}(x_1, \ldots, x_n + \pi; \{y\}) = (-1)^n S_{n,1}(x_1, \ldots, x_n + \pi; \{y\})$ doubles zeros.
- Thus, $S_{n,1}(\{x\}; \{y\})$ has $8n - 4$ zeros in the interval $0 \leq x_n < 2\pi$. 
Counting orders

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Three-coloring statistical model with ‘domain wall’ boundary conditions
Counting orders

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- Thus, $S_{n,1}(\{x\}; \{y\})$ has 8$n - 4$ zeros in the interval $0 \leq x_n < 2\pi$. 

Counting orders

With respect to the variable \( x_n \) the partition function \( Z_n(\{x\}; \{y\}) \) is a trigonometric polynomial of order less or equal to \( n - 1 \), and \( S_{n,1}(\{x\}; \{y\}) \) is a trigonometric polynomial of order less or equal to \( 3n - 2 \). It has at most \( 6n - 4 \) zeros in the interval \( 0 \leq x_n < 2\pi \).

Counting zeros

- Recursion relations give \( 2n \) zeros at the points \( x_n = y_\ell \pm \pi/3, \ell = 1, \ldots, n \).
- By construction there are \( 2n - 2 \) zeros at the points \( x_n = x_\ell, \ell = 2, \ldots, n - 1 \) and at the points \( x_n = y_\ell, \ell = 1, \ldots, n \).
- The relation \( S_{n,1}(x_1, \ldots, x_n + \pi; \{y\}) = (-1)^n S_{n,1}(x_1, \ldots, x_n + \pi; \{y\}) \) doubles zeros.
- Thus, \( S_{n,1}(\{x\}; \{y\}) \) has \( 8n - 4 \) zeros in the interval \( 0 \leq x_n < 2\pi \).
Counting orders

With respect to the variable $x_n$ the partition function $Z_n(\{x\}; \{y\})$ is a trigonometric polynomial of order less or equal to $n - 1$, and $S_{n,1}(\{x\}; \{y\})$ is a trigonometric polynomial of order less or equal to $3n - 2$. It has at most $6n - 4$ zeros in the interval $0 \leq x_n < 2\pi$.

Counting zeros

- Recursion relations give 2$n$ zeros at the points $x_n = y_\ell \pm \pi/3$, $\ell = 1, \ldots, n$.
- By construction there are 2$n - 2$ zeros at the points $x_n = x_\ell$, $\ell = 2, \ldots, n - 1$ and at the points $x_n = y_\ell$, $\ell = 1, \ldots, n$.
- The relation $S_{n,1}(x_1, \ldots, x_n + \pi; \{y\}) = (-1)^n S_{n,1}(x_1, \ldots, x_n + \pi; \{y\})$ doubles zeros.
- Thus, $S_{n,1}(\{x\}; \{y\})$ has 8$n - 4$ zeros in the interval $0 \leq x_n < 2\pi$. 
Conclusion

\[\sum_{r=0}^{2} F_n(x_1, \ldots, x_k + 2\pi r/3, \ldots, x_n; \{y\}) = 0,\]

where

\[F_n(\{x\}, \{y\}) = \prod_{i,j=1}^{n} \sin(x_i - x_j) \prod_{i,j=1}^{n} \sin(x_i - y_j) \prod_{i,j=1}^{n} \sin(y_i - y_j) Z_n(\{x\}, \{y\}).\]
1 Introduction
   • Three-colorings
   • Combinatorial problems
   • Three-colorings and six vertex model

2 Six vertex model with domain wall boundary conditions
   • Domain wall boundary conditions
   • Recursion relations
   • Functional equations

3 Three-coloring model with ‘domain wall’ boundary conditions
   • ‘Domain wall’ boundary conditions
   • Gauge transformation
   • Recursion relations
Partial partition functions

The total partition sum can be naturally represented as

$$Z_n(\{x\}; \{y\}) = \sum_{\mu \in \mathbb{Z}_3} Z_n^\mu(\{x\}; \{y\}),$$

where $\mu$ is the color of the left topmost vertex of the lattice.
If one starts with any boundary face and walks anticlockwise along the boundary then the color changes by $+\bar{1}$ from face to face for the vertical boundaries, and by $-\bar{1}$ for the horizontal boundaries.

The spectral parameter associated with a vertex configuration is equal to $x_i - y_j$. 
‘Domain wall’ boundary conditions

If one starts with any boundary face and walks anticlockwise along the boundary then the color changes by $+1$ from face to face for the vertical boundaries, and by $-1$ for the horizontal boundaries.

Inhomogeneous model

The spectral parameter associated with a vertex configuration is equal to $x_i - y_j$. 
1. Introduction
   - Three-colorings
   - Combinatorial problems
   - Three-colorings and six vertex model

2. Six vertex model with domain wall boundary conditions
   - Domain wall boundary conditions
   - Recursion relations
   - Functional equations

3. Three-coloring model with ‘domain wall’ boundary conditions
   - ‘Domain wall’ boundary conditions
   - Gauge transformation
   - Recursion relations
Gauge transformation

Notation for weights

\[ W_{\mu \nu}^{\sigma \rho}(u) \]

\[
\tilde{W}_{\mu \nu}^{\sigma \rho}(u) = \Phi_{\rho}^{-1}(u) \Phi_{\nu}(u) \Phi_{\sigma}^{-1}(u) \Phi_{\mu}(u) W_{\mu \nu}^{\sigma \rho}(u)
\]

\[
\Phi_{\mu}(u - u') = \Phi_{\mu}(u) \Phi_{\mu}^{-1}(u')
\]
Gauge transformation

Notation for weights

\[ W_{\mu \nu}^{\sigma \rho}(u) \]

Transformation

\[ \widetilde{W}_{\mu \nu}^{\sigma \rho}(u) = \Phi^{-1}_{\mu}(u) \Phi_{\nu}(u) \Phi^{-1}_{\rho}(u) \Phi_{\sigma}(u) W_{\mu \nu}^{\sigma \rho}(u) \]

\[ \Phi_{\mu}(u - u') = \Phi_{\mu}(u) \Phi^{-1}_{\mu}(u') \]
Gauge transformation

Notation for weights

Transformation

\[ \tilde{W}_{\mu\nu}^{\sigma\rho}(u) = \Phi_{\mu}^{-1}(u)\Phi_{\nu}(u)\Phi_{\rho}^{-1}(u)\Phi_{\sigma}(u)W_{\mu\nu}^{\sigma\rho}(u) \]

\[ \Phi_{\mu}(u - u') = \Phi_{\mu}(u)\Phi_{\mu}^{-1}(u') \]
Three-coloring model

$$\Phi_\mu(u|\alpha, p) = \zeta^u_{\mu}/4\pi(\alpha, p)$$

$$\tilde{a}_\mu(u|\alpha) = \frac{\theta_1(2\pi/3 - u)}{\theta_1(2\pi/3)},$$  
$$\tilde{b}_\mu(u|\alpha) = \zeta^{1/2}_\mu(\alpha) \frac{\theta_1(u)}{\theta_1(2\pi/3)},$$  
$$\tilde{c}_\mu(u|\alpha) = \frac{\theta_4(\alpha + 2\pi \mu/3 + u)}{\theta_4(\alpha + 2\pi \mu/3)},$$  
$$\tilde{c'}_\mu(u|\alpha) = \frac{\theta_4(\alpha + 2\pi \mu/3 - u)}{\theta_4(\alpha + 2\pi \mu/3)}.$$  

The partial partition functions $$\tilde{Z}_n^\mu(\{x\}; \{y\})$$ are symmetric in the variables $$x_i$$ and $$y_i.$$
### Three-coloring model

\[
\Phi_{\mu}(u|\alpha, p) = \zeta_{\mu}^{u/4\pi}(\alpha, p)
\]

\[
\tilde{a}_{\mu}(u|\alpha) = \frac{\theta_1(2\pi/3 - u)}{\theta_1(2\pi/3)},
\]

\[
\tilde{b}_{\mu}(u|\alpha) = \zeta_{\mu}^{1/2}(\alpha) \frac{\theta_1(u)}{\theta_1(2\pi/3)},
\]

\[
\tilde{c}_{\mu}(u|\alpha) = \frac{\theta_4(\alpha + 2\pi \bar{\mu}/3 + u)}{\theta_4(\alpha + 2\pi \bar{\mu}/3)},
\]

\[
\tilde{c}'_{\mu}(u|\alpha) = \frac{\theta_4(\alpha + 2\pi \bar{\mu}/3 - u)}{\theta_4(\alpha + 2\pi \bar{\mu}/3)}.
\]

The partial partition functions \(\tilde{Z}_{n}^{\mu}([x]; [y])\) are symmetric in the variables \(x_i\) and \(y_i\).
Gauge transformation

Three-coloring model

\[ \Phi_\mu(u|\alpha, p) = \zeta^{u/4\pi}_\mu(\alpha, p) \]

\[ \tilde{a}_\mu(u|\alpha) = \frac{\theta_1(2\pi/3 - u)}{\theta_1(2\pi/3)}, \quad \tilde{c}_\mu(u|\alpha) = \frac{\theta_4(\alpha + 2\pi\bar{\mu}/3 + u)}{\theta_4(\alpha + 2\pi\bar{\mu}/3)}, \]

\[ \tilde{b}_\mu(u|\alpha) = \zeta^{1/2}_\mu(\alpha) \frac{\theta_1(u)}{\theta_1(2\pi/3)}, \quad \tilde{c'}_\mu(u|\alpha) = \frac{\theta_4(\alpha + 2\pi\bar{\mu}/3 - u)}{\theta_4(\alpha + 2\pi\bar{\mu}/3)} \]

The partial partition functions \( \tilde{Z}_n^\mu(\{x\}; \{y\}) \) are symmetric in the variables \( x_i \) and \( y_i \).
Contents

1 Introduction
   • Three-colorings
   • Combinatorial problems
   • Three-colorings and six vertex model

2 Six vertex model with domain wall boundary conditions
   • Domain wall boundary conditions
   • Recursion relations
   • Functional equations

3 Three-coloring model with ‘domain wall’ boundary conditions
   • ‘Domain wall’ boundary conditions
   • Gauge transformation
   • Recursion relations
Recursion relations

\[ \begin{align*}
\theta_1 \left( \frac{2\pi}{3} - u \right) & \quad \theta_4 \left( \alpha + \frac{2\pi \bar{\mu}}{3} + u \right) \\
\theta_1 \left( \frac{2\pi}{3} \right) & \quad \theta_4 \left( \alpha + \frac{2\pi \bar{\mu}}{3} \right)
\end{align*} \]
Recursion relations

Three-coloring statistical model with ‘domain wall’ boundary conditions

\[
\begin{align*}
\frac{\theta_1(2\pi/3 - u)}{\theta_1(2\pi/3)} & \quad \frac{\theta_4(\alpha + 2\pi\mu/3 + u)}{\theta_4(\alpha + 2\pi\mu/3)}
\end{align*}
\]
Recursion relations

\[ \theta_1 \left( \frac{2\pi}{3} - u \right) \quad \theta_1 \left( \frac{2\pi}{3} \right) \]

\[ \frac{\theta_1 \left( \frac{2\pi}{3} - u \right)}{\theta_1 \left( \frac{2\pi}{3} \right)} \quad \frac{\theta_4 \left( \alpha + 2\pi \mu / 3 + u \right)}{\theta_4 \left( \alpha + 2\pi \mu / 3 \right)} \]
Recursion relations

\[ x_1 \begin{array}{cccc} 2 & 0 & 1 & 2 \\ 0 & \overline{1} & 0 & 2 \\ \overline{1} & 0 & \overline{1} & 2 \\ 2 & \overline{0} & 2 & 0 \end{array} \]

\[ y_1 \begin{array}{cccc} 2 & 0 & 1 & 2 \\ 0 & \overline{1} & 0 & 2 \\ \overline{1} & 0 & \overline{1} & 2 \\ 2 & \overline{0} & 2 & 0 \end{array} \]

\[ \frac{\theta_1(2\pi/3 - u)}{\theta_1(2\pi/3)} = \frac{\theta_4(\alpha + 2\pi \mu/3 + u)}{\theta_4(\alpha + 2\pi \mu/3)} \]

\[ 2\pi/3 - u = 2\pi/3 - x_n + y_n = 0 \]

A. V. Razumov, Yu. G. Stroganov

Three-coloring statistical model with ‘domain wall’ boundary conditions
Recursion relations

\[ x_n = y_n + \frac{2\pi}{3} \]

\[ \frac{\theta_1(2\pi/3 - u)}{\theta_1(2\pi/3)} = \frac{\theta_4(\alpha + 2\pi\mu/3 + u)}{\theta_4(\alpha + 2\pi\mu/3)} \]
Recursion relations

\begin{align*}
\theta_1(2\pi/3 - u) & \quad \theta_4(\alpha + 2\pi\mu/3 + u) \\
\theta_1(2\pi/3) & \quad \theta_4(\alpha + 2\pi\mu/3)
\end{align*}

\[ x_n = y_n + 2\pi/3 \]
Recursion relations

Total set of relations

\[ \tilde{Z}_n^\mu (x_1, \ldots, x_k, \ldots, x_n; y_1, \ldots, y_\ell, \ldots, y_n) \bigg|_{x_k = y_\ell + 2\pi / 3} = \frac{\theta_4 (\alpha + 2\pi (\bar{\mu} + n) / 3)}{\theta_4 (\alpha + 2\pi (\bar{\mu} + n - 1) / 3)} \cdot \theta_1^{2-2n} (2\pi / 3) \prod_{i=1}^{n} \theta_1 (x_i - y_\ell) \prod_{i=1}^{n} \theta_1 (y_\ell - y_i + 2\pi / 3) \times \tilde{Z}_{n-1}^\mu (x_1, \ldots, \hat{x}_k, \ldots, x_n; y_1, \ldots, \hat{y}_\ell, \ldots, y_n) \]
Recursion relations

Total set of relations

\[
\tilde{Z}_n^\mu(x_1, \ldots, x_k, \ldots, x_n; y_1, \ldots, y_\ell, \ldots, y_n) \bigg|_{x_k = y_\ell + 2\pi/3} = \frac{\theta_4(\alpha + 2\pi(\mu + n)/3)}{\theta_4(\alpha + 2\pi(\mu + n - 1)/3)} \theta_1^{2-2n}(2\pi/3) \prod_{i=1\atop i\neq k}^n \theta_1(x_i - y_\ell) \prod_{i=1\atop i\neq \ell}^n \theta_1(y_\ell - y_i + 2\pi/3)
\]

\[
	imes \tilde{Z}_{n-1}^\mu(x_1, \ldots, \hat{x}_k, \ldots, x_n; y_1, \ldots, \hat{y}_\ell, \ldots, y_n),
\]

\[
\tilde{Z}_n^\mu(x_1, \ldots, x_k, \ldots, x_n; y_1, \ldots, y_\ell, \ldots, y_n) \bigg|_{x_k = y_\ell}
\]

\[
= \theta_1^{2-2n}(2\pi/3) \prod_{i=1\atop i\neq k}^n \theta_1(x_i - y_\ell - 2\pi/3) \prod_{i=1\atop i\neq \ell}^n \theta_1(y_\ell - y_i - 2\pi/3)
\]

\[
	imes \tilde{Z}_{n-1}^{\mu+\tilde{1}}(x_1, \ldots, \hat{x}_k, \ldots, x_n; y_1, \ldots, \hat{y}_\ell, \ldots, y_n)
\]
Define the functions

\[
F_{n}^{\mu}(x_{1}, \ldots, x_{n}; y_{1}, \ldots, y_{n}) = \frac{1}{\theta_{4}(\alpha + 2\pi(m + n)/3)}
\]

\[
\times \prod_{i \neq j=1}^{n} \theta_{1}(x_{i} - x_{j}) \prod_{i \neq j=1}^{n} \theta_{1}(x_{i} - y_{j} - \pi/3) \prod_{i \neq j=1}^{n} \theta_{1}(y_{i} - y_{j})
\]

\[
\times Z_{n}^{\mu}(x_{1}, \ldots, x_{n}; y_{1}, \ldots, y_{n}),
\]

and use the relation

\[
\theta_{1}(u|p)\theta_{1}(u + \pi/3|p)\theta_{1}(u + 2\pi/3|p) = D(p)\theta_{1}(3u|p^{3}),
\]

\[
D(p) = \frac{\theta_{1}'(0|p)\theta_{1}(\pi/3|p)\theta_{1}(2\pi/3|p)}{3\theta_{1}'(0|p^{3})}.
\]
Define the functions

\[ F^\mu_n(x_1, \ldots, x_n; y_1, \ldots, y_n) = \frac{1}{\theta_4(\alpha + 2\pi(m + n)/3)} \times \prod_{i<j}^{n} \theta_1(x_i - x_j) \prod_{i<j}^{n} \theta_1(x_i - y_j - \pi/3) \prod_{i<j}^{n} \theta_1(y_i - y_j) \times Z^\mu_n(x_1, \ldots, x_n; y_1, \ldots, y_n), \]

and use the relation

\[ \theta_1(u|p)\theta_1(u + \pi/3|p)\theta_1(u + 2\pi/3|p) = D(p)\theta_1(3u|p^3), \]

\[ D(p) = \frac{\theta'_1(0|p)\theta_1(\pi/3|p)\theta_1(2\pi/3|p)}{3\theta'_1(0|p^3)}. \]
Functional equations

Recursion relations

\[
F_n^\mu(x_1, \ldots, x_k, \ldots, x_n; y_1, \ldots, y_\ell, \ldots, y_n | \alpha, p) \bigg|_{x_k = y_\ell + 2\pi/3} = D^{2n-2}(p) \theta_1^{3-2n}(2\pi/3|p) \prod_{i=1 \atop i \neq k}^n \theta_1 (3(x_i - y_\ell)|p^3) \prod_{i=1 \atop i \neq \ell}^n \theta_1 (3(y_\ell - y_i)|p^3) \times F_{n-1}^\mu(x_1, \ldots, \hat{x}_k, \ldots, x_n; y_1, \ldots, \hat{y}_\ell, \ldots, y_n | \alpha, p),
\]

\[
F_n^\mu(x_1, \ldots, x_k, \ldots, x_n; y_1, \ldots, y_\ell, \ldots, y_n | \alpha, p) \bigg|_{x_k = y_\ell} = -D^{2n-2}(p) \theta_1^{3-2n}(2\pi/3|p) \prod_{i=1 \atop i \neq k}^n \theta_1 (3(x_i - y_\ell)|p^3) \prod_{i=1 \atop i \neq \ell}^n \theta_1 (3(y_\ell - y_i)|p^3) \times F_{n-1}^{\mu+1}(x_1, \ldots, \hat{x}_k, \ldots, x_n; y_1, \ldots, \hat{y}_\ell, \ldots, y_n | \alpha, p)
\]
\[ S_{n,k}^\mu (x_1, \ldots, x_k, \ldots, x_n; \{y\}) = \sum_{r=0}^{2} F_{n}^{\mu + r} (x_1, \ldots, x_k + 2\pi r / 3, \ldots, x_n; \{y\}). \]

**Counting zeros**

- Recursion relations give 2\(n\) zeros at the points \(x_{n} = y_\ell + 2\pi / 3\) and \(x_{n} = y_\ell, \ell = 1, \ldots, n\).
- By construction there are 2\(n\) - 2 zeros at the points \(x_{n} = x_\ell, \ell = 2, \ldots, n - 1\) and at the points \(x_{n} = y_\ell + \pi / 3, \ell = 1, \ldots, n\).
- The relation \(S_{n,1}^\mu (x_1, \ldots, x_n + \pi; \{y\}) = (-1)^n S_{n,1}^\mu (x_1, \ldots, x_n + \pi; \{y\})\) doubles zeros.
- Thus, \(S_{n,1}^\mu (\{x\}; \{y\})\) has 8\(n\) - 4 zeros in the interval 0 \(\leq x_{n} < 2\pi\).
Functional equations

\[ S_{n,k}^{\mu}(x_1, \ldots, x_k, \ldots, x_n; \{y\}) = \sum_{r=0}^{2} F_{n}^{\mu+r}(x_1, \ldots, x_k + 2\pi r/3, \ldots, x_n; \{y\}). \]

Counting zeros

- Recursion relations give **2n zeros** at the points \( x_n = y_\ell + 2\pi/3 \) and \( x_n = y_\ell, \, \ell = 1, \ldots, n \).
- By construction there are **2n − 2 zeros** at the points \( x_n = x_\ell, \, \ell = 2, \ldots, n − 1 \) and at the points \( x_n = y_\ell + \pi/3, \, \ell = 1, \ldots, n \).
- The relation \( S_{n,1}^{\mu}(x_1, \ldots, x_n + \pi; \{y\}) = (-1)^n S_{n,1}^{\mu}(x_1, \ldots, x_n + \pi; \{y\}) \) doubles zeros.
- Thus, \( S_{n,1}^{\mu}(\{x\}; \{y\}) \) has **8n − 4 zeros** in the interval \( 0 \leq x_n < 2\pi \).
\[ S_{n,k}^\mu (x_1, \ldots, x_k, \ldots, x_n; \{y\}) = \sum_{r=0}^{2} F_{n}^{\mu+r} (x_1, \ldots, x_k + 2\pi r/3, \ldots, x_n; \{y\}). \]

### Counting zeros

- Recursion relations give **2n zeros** at the points \( x_n = y_\ell + 2\pi/3 \) and \( x_n = y_\ell, \; \ell = 1, \ldots, n \).
- By construction there are **2n – 2 zeros** at the points \( x_n = x_\ell, \; \ell = 2, \ldots, n-1 \) and at the points \( x_n = y_\ell + \pi/3, \; \ell = 1, \ldots, n \).
- The relation \( S_{n,1}^\mu (x_1, \ldots, x_n + \pi; \{y\}) = (-1)^n S_{n,1}^\mu (x_1, \ldots, x_n + \pi; \{y\}) \) doubles zeros.
- Thus, \( S_{n,1}^\mu (\{x\}; \{y\}) \) has **8n – 4 zeros** in the interval \( 0 \leq x_n < 2\pi \).
\( S_{n,k}^{\mu}(x_1, \ldots, x_k, \ldots, x_n; \{y\}) = 2 \sum_{r=0}^{2} F_n^{\mu+r}(x_1, \ldots, x_k + 2\pi r / 3, \ldots, x_n; \{y\}). \)

### Counting zeros

- Recursion relations give **2n zeros** at the points \( x_n = y_\ell + 2\pi / 3 \) and \( x_n = y_\ell, \ell = 1, \ldots, n \).

- By construction there are **2n − 2 zeros** at the points \( x_n = x_\ell, \ell = 2, \ldots, n − 1 \) and at the points \( x_n = y_\ell + \pi / 3, \ell = 1, \ldots, n \).

- The relation \( S_{n,1}^{\mu}(x_1, \ldots, x_n + \pi; \{y\}) = (-1)^n S_{n,1}^{\mu}(x_1, \ldots, x_n + \pi; \{y\}) \) doubles zeros.

- Thus, \( S_{n,1}^{\mu}(\{x\}; \{y\}) \) has **8n − 4 zeros** in the interval \( 0 \leq x_n < 2\pi \).
Functional equations

\[
S_{n,k}^\mu(x_1, \ldots, x_k, \ldots, x_n; \{y\}) = \sum_{r=0}^{2} F_{n}^{\mu+r}(x_1, \ldots, x_k + 2\pi r / 3, \ldots, x_n; \{y\}).
\]

Counting zeros

- Recursion relations give 2\(n\) zeros at the points \(x_n = y_\ell + 2\pi / 3\) and \(x_n = y_\ell, \ell = 1, \ldots, n\).
- By construction there are 2\(n - 2\) zeros at the points \(x_n = x_\ell, \ell = 2, \ldots, n - 1\) and at the points \(x_n = y_\ell + \pi / 3, \ell = 1, \ldots, n\).
- The relation \(S_{n,1}^\mu(x_1, \ldots, x_n + \pi; \{y\}) = (-1)^n S_{n,1}^\mu(x_1, \ldots, x_n + \pi; \{y\})\) doubles zeros.
- Thus, \(S_{n,1}^\mu(\{x\}; \{y\})\) has 8\(n - 4\) zeros in the interval \(0 \leq x_n < 2\pi\).
Conclusion

\[
\sum_{r=0}^{2} F^\mu_n (x_1, \ldots, x_k + 2\pi r/3, \ldots, x_n; \{y\}) = 0,
\]

where

\[
F^\mu_n (x_1, \ldots, x_n; y_1, \ldots, y_n) = \frac{1}{\theta_4(\alpha + 2\pi (m + n)/3)} \times \prod_{i,j=1}^{n} \theta_1(x_i - x_j) \prod_{i,j=1}^{n} \theta_1(x_i - y_j - \pi/3) \prod_{i,j=1}^{n} \theta_1(y_i - y_j) \times Z^\mu_n (x_1, \ldots, x_n; y_1, \ldots, y_n),
\]