Set-valued dynamic risk measures for processes and for vectors

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Abstract
The relationship between set-valued risk measures for processes and vectors on the optional filtration is investigated. The equivalence of risk measures for processes and vectors and the equivalence of their penalty function formulations are provided. In contrast to scalar risk measures, this equivalence requires an augmentation of the set-valued risk measures for processes. We utilise this result to deduce a new dual representation for risk measures for processes in the set-valued framework. Finally, the equivalence of multi-portfolio time-consistency between set-valued risk measures for processes and vectors is provided. To accomplish this, an augmented definition for multi-portfolio time-consistency of set-valued risk measures for processes is proposed.

Keywords Set-valued risk measure · Dynamic risk measure · Time-consistency · Optional filtration

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JEL Classification C61 · D81 · G32

1 Introduction

Coherent risk measures for multi-period models were introduced by Artzner et al. [5], where time-consistency (i.e., risk preferences are consistent over time) of risk measures was shown to be equivalent to Bellman’s principle (i.e., the risk of a portfolio is equivalent to the iterated risks backwards in time). These risk measures allow the risk of cash flows to be quantified in an axiomatic framework. Though these risk measures appear distinct from the more traditional risk measures of e.g. Artzner et al. [3, 4].
Föllmer and Schied [22] and Frittelli and Rosazza Gianin [23], risk measures applied to processes can be proved to be equivalent to these more traditional risk measures via the optional filtration as demonstrated in e.g. Acciaio et al. [1].

In this paper, we are concerned with dynamic risk measures. That is, we consider those risk measures which allow updates to the minimal capital requirement over time due to revealed information as encoded in a filtration. There are many studies about dynamic risk measures for random variables which describe financial positions. For instance, Bion-Nadal [8, 9] studied the dual or robust representation and gave an equivalent characterisation of time-consistency of dynamic risk measures for random variables by a cocycle condition on the minimal penalty function (i.e., such that the penalty functions can be decomposed as a summation over time); Barrieu and El Karoui [6] studied some applications of static and dynamic risk measures under the aspects of pricing, hedging and designing derivatives; and Delbaen et al. [16] studied the representation of the penalty functions of dynamic risk measures for random variables induced by backward stochastic differential equations. There also exist many studies about dynamic risk measures applied to cash flows. For example, Riedel [30] studied the dual representation for time-consistent dynamic coherent risk measures for processes with real values; Cheridito et al. [12, 13] studied dynamic risk measures on the space of all bounded and unbounded càdlàg processes, respectively, with real or generalised real values; Frittelli and Scandolo [24] proposed dynamic risk measures for processes from a perspective of acceptance sets; Cheridito et al. [14] studied dynamic risk measures for bounded discrete-time processes and with values in the space of random variables, and provided a dual representation and an equivalent characterisation of time-consistency in terms of an additivity property of the corresponding acceptance sets; and Acciaio et al. [1] provided a supermartingale characterisation (i.e., that the sum of a risk measure and its minimal penalty function form a supermartingale with respect to the dual probability measure) of time-consistency of dynamic risk measures for processes. As highlighted in [1], as with static risk measures, these dynamic risk measures for processes can be found to be equivalent to risk measures over random variables through the use of the optional filtration. Utilising the optional $\sigma$-algebra, the equivalence of time-consistency for traditional dynamic risk measures and those for processes are also found to be equivalent.

Set-valued risk measures were introduced in Jouini et al. [28], Hamel [25], Hamel et al. [27] and extended to a dynamic framework in Feinstein and Rudloff [17, 18]. All of those works present risk measures for random vectors. Set-valued risk measures were introduced so as to consider risks in markets with frictions. Such risk measures have more recently been utilised for quantifying systemic risk in Feinstein et al. [21] and Ararat and Rudloff [2]. In Chen and Hu [10, 11], set-valued risk measures for processes were introduced. However, as demonstrated in those works, the equivalence of risk measures for processes and risk measures for random vectors no longer holds except under certain strong assumptions. These strong assumptions limit the direct application of known results from e.g. Feinstein and Rudloff [19, 20] on multi-portfolio time-consistency (i.e., a version of Bellman’s principle for set-valued risk measures) to set-valued risk measures for processes.

In this work, we prove a new relation between set-valued risk measures for processes and those for random vectors (on the optional filtration). Notably, the formulation for equivalence indicates that risk measures for processes at time $t$ need to be
augmented to capture the risks prior to time $t$ due to the non-existence of a unique “0” element in the set-valued framework; such an augmentation is not necessary in the scalar case as in e.g. [1]. By utilising this new equivalence relation, we are able to derive a novel dual representation for set-valued risk measures for processes akin to that in [1]. Additionally, we extend these results to present an equivalent formulation for multi-portfolio time-consistency; since the equivalence between risk measure settings requires an augmentation, we present a new definition for time-consistency for risk measures for processes.

The organisation of this paper is as follows. In Sect. 2, we present preliminary and background information on the set-valued risk measures of interest within this work, i.e., as functions of stochastic processes and random vectors. Section 3 demonstrates the equivalence of these risk measures both in primal and dual representations. With the focus on dynamic risk measures, the equivalence of multi-portfolio time-consistency for processes and vectors is presented in Sect. 4. Section 5 concludes.

2 Set-valued risk measures

In this section, we summarise the definitions for set-valued risk measures. We provide details of the filtrations and the spaces of claims of interest within this work in Sect. 2.1, and give the necessary background on risk measures for processes as defined in e.g. [10, 11], and for random vectors as defined in e.g. [17, 18], in Sects. 2.2 and 2.3, respectively.

2.1 Background and notation

Fix a finite time horizon $T \in \mathbb{N} := \{1, 2, \ldots\}$ and denote $\mathbb{T} := \{0, 1, \ldots, T\}$ and $\mathbb{T}_t := \{s \in \mathbb{T} : s \geq t\}$ for $t \in \mathbb{T}$. Furthermore, define the discrete time interval $[t, s) := \{t, t+1, \ldots, s-1\} \subseteq \mathbb{T}$ for any $s, t \in \mathbb{T}$ with $s > t$. Fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{P})$ with $\mathcal{F}_0$ the complete and trivial $\sigma$-algebra. Without loss of generality, take $\mathcal{F} = \mathcal{F}_T$. Let $d \geq 1$ be the number of assets under consideration and $|\cdot|$ be an arbitrary but fixed norm on $\mathbb{R}^d$. Let $L^0_\mathcal{F}(\mathbb{R}^d) := L^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^d)$ (with $L^0(\mathbb{R}^d) := L^0_T(\mathbb{R}^d)$) denote the linear space of equivalence classes of $\mathcal{F}_T$-measurable $\mathbb{R}^d$-valued functions, where we specify random vectors $\mathbb{P}$-a.s., and define the space of (equivalence classes of) $p$-integrable random $d$-vectors by

$$L^p_\mathcal{F}(\mathbb{R}^d) := L^p(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^d) \subseteq L^0_\mathcal{F}(\mathbb{R}^d)$$

for $p \in [1, \infty]$ (with $L^p(\mathbb{R}^d) := L^p_T(\mathbb{R}^d)$). In this way, $L^p_\mathcal{F}(\mathbb{R}^d)$ denotes the linear space of equivalence classes of $\mathcal{F}_T$-measurable functions $X : \Omega \to \mathbb{R}^d$ with bounded $p$-norm, where these norms are given by

$$\|X\|_p = \begin{cases} \mathbb{E}^\mathbb{P}[|X|] = \int_\Omega |X(\omega)| d\mathbb{P}, & \text{if } p = 1, \\ \text{ess sup}_{\omega \in \Omega} |X(\omega)|, & \text{if } p = \infty. \end{cases}$$
In this paper, we denote the expectation $\mathbb{E}^P[X]$ by $\mathbb{E}[X]$ and the conditional expectation $\mathbb{E}^P[X|\mathcal{F}_t]$ by $\mathbb{E}_t[X]$. Throughout this work, we consider the weak* topology on $L_1^\infty(\mathbb{R}^d)$ so that the dual space of $L_1^\infty(\mathbb{R}^d)$ is $L_1^1(\mathbb{R}^d)$.

Fix $m \in \{1, \ldots, d\}$ of the assets to be eligible for covering the risk of a portfolio. Denote by $M := \mathbb{R}^m \times \{0\}^{d-m}$ the subspace of eligible assets (those assets which can be used to satisfy capital requirements, e.g. US dollars and Euros). For any measurable set $B \subseteq \mathbb{R}^d$ and $p \in \{0, 1, \infty\}$, write

$$L_1^p(B) := \{X \in L_1^p(\mathbb{R}^d) : X \in B \text{ $\mathbb{P}$-a.s.}\}$$

for those random vectors that take $\mathbb{P}$-a.s. values in $B$. In particular, the closed convex cone of uniformly bounded $\mathbb{R}^d$-valued $\mathcal{F}_t$-measurable random vectors with $\mathbb{P}$-a.s. nonnegative components is $L_1^\infty(\mathbb{R}^d_+)$, and $L_1^\infty(\mathbb{R}^d_{++})$ consist of all $\mathcal{F}_t$-measurable random vectors with $\mathbb{P}$-a.s. strictly positive components. Additionally, we denote by $M_t := L_1^\infty(M)$ the space of time-$t$ measurable eligible portfolios. Throughout, we consider the set $M_{t,+} := M_t \cap L_1^\infty(\mathbb{R}^d_+)$ of nonnegative eligible portfolios, its positive dual cone $M^*_{t,+} := \{u \in L_1^1(\mathbb{R}^d) : \mathbb{E}[u^\top m] \geq 0 \text{ for any } m \in M_{t,+}\}$ and the perpendicular space $M_{t,+}^\perp := \{u \in L_1^1(\mathbb{R}^d) : \mathbb{E}[u^\top m] = 0 \text{ for any } m \in M_{t,+}\}$. Generally, we define $C^* := \{u \in L_1^1(\mathbb{R}^d) : \mathbb{E}[u^\top m] \geq 0 \text{ for any } m \in C\}$ for any convex cone $C \subseteq L_1^\infty(\mathbb{R}^d)$. (In)equalities between random vectors, stochastic processes and sets are always understood componentwise in the $\mathbb{P}$-a.s. sense, unless stated otherwise. The multiplication between a random variable $\lambda \in L_1^\infty(\mathbb{R})$ and a set of random vectors $B_t \subseteq L_1^\infty(\mathbb{R}^d)$ is understood in the elementwise sense, i.e., $\lambda\cdot B_t := \{\lambda X : X \in B_t\}$ with $(\lambda X)(\omega) = \lambda(\omega)X(\omega)$. Given a set $A \subseteq M_t$, $\text{co}(A)$ means the closure of $A$ with respect to the subspace topology on $M_t$ and $\text{co}(A)$ the convex hull of $A$. Denote the spaces of upper and closed-convex upper sets respectively by

$$\mathcal{U}(M_t; M_{t,+}) := \{D \subseteq M_t : D = D + M_{t,+}\}$$

and the perpendicular space $\mathcal{G}(M_t; M_{t,+}) := \{D \subseteq M_t : D = \text{cl}(\text{co}(D + M_{t,+}))\}$,

where the $+$ sign denotes the usual Minkowski addition. For $w \in L_1^1(\mathbb{R}^d) \setminus \{0\}$, we set $\Gamma_t(w) := \{u \in L_1^\infty(\mathbb{R}^d) : w^\top u \geq 0\}$. As utilised in the dual representations presented below, the Minkowski subtraction for sets $A, B \subseteq M_t$ is defined as

$$A - B := \{m \in M_t : B + \{m\} \subseteq A\}.$$

The indicator function for a set $D$ is denoted by $\mathbb{I}_D$.

We define $\mathcal{R}^\infty,d$ as the space of all $d$-dimensional adapted stochastic processes $X := (X_t)_{t \in \mathbb{T}}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{P})$ whose coordinates are in $L_1^\infty(\mathbb{R}^d)$, and $A^{1,d}$ as the space of all $d$-dimensional adapted stochastic processes $a$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{P})$ with $\sum_{i=1}^d \mathbb{E}[(\sum_{t \in \mathbb{T}} |\Delta a_{t,i}|) < +\infty$, where $a_{-1,i} := 0$ and $\Delta a_{t,i} := a_{t,i} - a_{t-1,i}$. Denote by $\mathbb{I}_{s,t}$ the element of $\mathcal{R}^\infty,d$ with $j$th component equal to 0 for $j = 0, \ldots, t - 1$ and 1 for $j \in T_t$, so that $\mathbb{I}_{s,t}$ takes at time $s$ the value $\mathbb{I}_{s,t}(s)$, and by $\mathbb{I}_{(s)}$ the element of $\mathcal{R}^\infty,d$ where only the $s$th component is equal to 1 and all others are 0. As is standard in the literature (see e.g. [14]), for any times $0 \leq r \leq s \leq T$, we define the projection $\pi_{r,s}(X)_t := \mathbb{I}_{s,t}(X)_{t \wedge s}$, $t \in \mathbb{T}$, for any $d$-dimensional adapted stochastic pro-
cess $X$. Denote $\mathcal{R}_{t}^{\infty,d} := \pi_{t,T}(\mathcal{R}_{t}^{\infty,d})$ and $A_{t}^{1,d} := \pi_{t,T}(A_{t}^{1,d})$. The space of bounded nonnegative processes is written as

$$\mathcal{R}_{\infty,d}^{+} := \{X \in \mathcal{R}_{t}^{\infty,d} : X_{t,i} \geq 0 \text{ for any } t \in T, i = 1, \ldots, d\}.$$ 

As in [1, 14], a process $X \in \mathcal{R}_{t}^{\infty,d}$ describes the evolution of a financial value or the cumulative cash flow on the interval $T$. Throughout this work, we consider the coarsest topologies (denoted by $\sigma(\mathcal{R}_{t}^{\infty,d}, A_{t}^{1,d})$, $\sigma(A_{t}^{1,d}, \mathcal{R}_{t}^{\infty,d})$ respectively) such that $\mathcal{R}_{t}^{\infty,d}$ and $A_{t}^{1,d}$ form a dual pair (and vice versa), i.e., the space of all continuous linear functionals on the topological space $(\mathcal{R}_{t}^{\infty,d}, \sigma(\mathcal{R}_{t}^{\infty,d}, A_{t}^{1,d}))$ (resp. $(A_{t}^{1,d}, \sigma(A_{t}^{1,d}, \mathcal{R}_{t}^{\infty,d}))$) is $A_{t}^{1,d}$ (resp. $\mathcal{R}_{t}^{\infty,d}$).

Denote by $\mathcal{M}(\mathbb{P})$ the set of probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ which are absolutely continuous with respect to $\mathbb{P}$. Denote those probability measures equal to $\mathbb{P}$ on $\mathcal{F}_{t}$ by $\mathcal{M}_{t}(\mathbb{P}) := \{Q \in \mathcal{M}(\mathbb{P}) : Q = \mathbb{P} \text{ on } \mathcal{F}_{t}\}$. For any given $t \leq s \in T$, define

$$D_{t,s} := \{\xi \in L_{s}^{\infty}(\mathbb{R}_{+}) : \mathbb{E}_{t}[\xi] = 1\}.$$ 

Then every $\xi \in D_{t,s}$ clearly defines a probability measure $\mathbb{Q}_{\xi}$ in $\mathcal{M}(\mathbb{P})$ with density $d\mathbb{Q}_{\xi}/d\mathbb{P} = \xi$. Conversely, every $Q \in \mathcal{M}(\mathbb{P})$ induces a collection of nonnegative random variables $\xi_{t,s}(Q) \in D_{t,s}$ for $t \leq s \in T$, where $\xi_{t,s}(Q)$ is defined as

$$\xi_{t,s}(Q) := \begin{cases} \mathbb{E}_{s}[\frac{d\mathbb{Q}}{d\mathbb{P}}], & \text{on } \{\mathbb{E}_{t}[\frac{d\mathbb{Q}}{d\mathbb{P}}] > 0\}, \\ 1, & \text{otherwise.} \end{cases}$$

As in e.g. Cheridito and Kupper [15], we use a $\mathbb{P}$-almost surely defined version of the $\mathbb{Q}$-conditional expectation of $X \in L_{\infty}(\mathbb{R}^{d})$ given by

$$\mathbb{E}_{t}^{\mathbb{Q}}[X] := \mathbb{E}^{\mathbb{Q}}[X|\mathcal{F}_{t}] := \mathbb{E}_{t}[\xi_{t,T}(Q)X]$$

for any time $t \in T$. For $Q \in \mathcal{M}(\mathbb{P})^{d}$, we utilise the vector representation

$$\xi_{t,s}(Q) := (\xi_{t,s}(Q)_{1}, \ldots, \xi_{t,s}(Q)_{d})^{\top}$$

for any times $t \leq s \in T$. We further define $w_{t,s}^{1} : \mathcal{M}(\mathbb{P})^{d} \times L_{t}^{1}(\mathbb{R}^{d}) \to L_{s}^{0}(\mathbb{R}^{d})$ for any $t, s \in T$ with $t < s$ by

$$w_{t,s}^{1}(Q, w) := \text{diag}(w)\xi_{t,s}(Q),$$

where $\text{diag}(w)$ is the diagonal matrix with the components of $w$ on the main diagonal. By convention, for any $t \in T$, $w_{t,t}^{1}(Q, w) = w$.

As we seek to relate risk measures for random vectors to those of processes, as in [1], we consider the optional filtration, i.e., $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_{t})_{t \in T}, \bar{\mathbb{P}})$ with sample space $\bar{\Omega} = \Omega \times T$, $\sigma$-algebra $\bar{\mathcal{F}} = \sigma(\{A_{t} \times \{t\} : A_{t} \in \mathcal{F}_{t}, t \in T\})$, filtration $\bar{\mathcal{F}}_{t} = \sigma(\{D_{t} \times \{r\}, D_{t} \times T_{t} : D_{r} \in \mathcal{F}_{r}, r < t, D_{t} \in \mathcal{F}_{t}\})$, $t \in T$ and probability measure $\bar{\mathbb{P}} = \mathbb{P} \otimes \mu$, where $\mu = (\mu_{t})_{t \in T}$ is some adapted reference process such that $\sum_{t \in T} \mu_{t} = 1$ and $\mathbb{P}[\min_{t \in T} \mu_{t} > \epsilon] = 1$ for some $\epsilon > 0$. For this measure
on the optional $\sigma$-algebra, the expectation is $\tilde{E}[X] := \tilde{E}^\bar{P}[X] := \mathbb{E}[\sum_{t \in T} X_t \mu_t]$ for any integrable measurable function $X = (X_t)_{t \in T}$; we emphasise that this expectation is taken over the space $\Omega$ by utilising the $\tilde{E}$ notation. A random vector $X = (X_t)_{t \in T}$ on $(\Omega, \tilde{\mathcal{F}}, \bar{P})$ is $\tilde{\mathcal{F}}_t$-measurable if and only if $X_t$ is $\mathcal{F}_t^\mathbb{P}$-measurable for all $r = 0, \ldots, t$ and $X_r = X_t$ for any $r > t$. Define $\tilde{L}_t^P(\mathbb{R}^d) := L^P(\tilde{\Omega}, \tilde{\mathcal{F}}_t, \bar{P}, \mathbb{R}^d)$ (with $L^P(\mathbb{R}^d) := \tilde{L}_T^P(\mathbb{R}^d)$) for $p \in \{0, 1, \infty\}$; similarly, denote the set of $\mathcal{F}_t^\mathbb{P}$-measurable random vectors taking values in the measurable set $B \subseteq \mathbb{R}^d$ by $\tilde{L}_t^P(B) \subseteq \tilde{L}_T^P(\mathbb{R}^d)$ for $p \in \{0, 1, \infty\}$. We often consider the nonnegative orthant $B = \mathbb{R}^d_+$ or the positive orthant $B = \mathbb{R}^d_{d+}$. Denote $\bar{M}_t = \tilde{L}_t^\infty(\mathbb{R}):= \{X \in \tilde{L}_t^\infty(\mathbb{R}^d) : X \in M \mathbb{P}$-a.s.\}. Throughout this work, we consider the weak* topology on $\tilde{L}_t^\infty(\mathbb{R}^d)$ so that the dual space of $\tilde{L}_t^\infty(\mathbb{R}^d)$ is $\tilde{L}_t^1(\mathbb{R}^d)$. As on the original probability space, denote by $\tilde{\mathcal{M}}(\tilde{P})$ the set of probability measures $\hat{Q}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ which are absolutely continuous with respect to $\tilde{\mathbb{P}}$, and by $\tilde{E}^Q[X]$ the expectation of $X \in \tilde{L}_t^\infty(\mathbb{R}^d)$ with respect to $\hat{Q}$. For $\hat{w} \in \tilde{L}_t^1(\mathbb{R}^d) \setminus \{0\}$, we further define $\tilde{\Gamma}_t(\hat{w}) := \{u \in \tilde{L}_t^\infty(\mathbb{R}^d) : \hat{w}^\top u \geq 0 \mathbb{P}$-a.s.\}. The following result, adapted from Acciaio et al. [1], provides a representation for these probability measures on the optional $\sigma$-algebra.

**Theorem 2.1** For any probability measure $\hat{Q} \in \tilde{\mathcal{M}}(\tilde{P})$ on the optional $\sigma$-algebra $\tilde{\mathcal{F}}$, there exist a probability measure $\tilde{Q} \in \mathcal{M}(\mathbb{P})$ and an optional random measure $\hat{\psi} \in \Psi(\mathbb{P}) := \{\psi \in \mathcal{R}_+^\infty : \sum_{t \in T} \psi_t = 1 \mathbb{P}$-a.s.\} such that

$$
\tilde{E}^\hat{Q}[X] = \mathbb{E}^\tilde{Q}\left[\sum_{t \in T} \hat{\psi}_t X_t\right]
$$

(2.1)

for any $X \in \tilde{L}_t^\infty(\mathbb{R})$. Conversely, any $\hat{Q} \in \tilde{\mathcal{M}}(\tilde{P})$ and $\hat{\psi} \in \Psi(\mathbb{P})$ define a probability measure $\tilde{Q} := \tilde{Q} \otimes \hat{\psi} \in \tilde{\mathcal{M}}(\tilde{P})$ such that (2.1) holds.

**Proof** This follows directly from [1, Theorem 3.4 and Remark B.1].

**Remark 2.2** We note that the decomposition of $\hat{Q} \in \tilde{\mathcal{M}}(\tilde{P})$ as $\hat{Q} = \tilde{Q} \otimes \hat{\psi}$ provided in Theorem 2.1 can be defined with arbitrary $\hat{\psi}_s \geq 0$ on the event $\{E_t(\frac{d\tilde{Q}}{d\mathbb{P}}) = 0\}$ so long as $\hat{\psi} \in \Psi(\mathbb{P})$. Without loss of generality, we always consider the modified decomposition $\tilde{Q} \otimes \hat{\psi}$ such that

$$
\hat{\psi}_t = \begin{cases} 
\hat{\psi}_t & \text{on } \{\tau(\tilde{Q}) > t\}, \\
\frac{\mu_t}{\mu_s - \sum_{s=0}^{\tau(\tilde{Q})-1} \hat{\psi}_s} (1 - \sum_{s=0}^{\tau(\tilde{Q})-1} \hat{\psi}_s) & \text{on } \{\tau(\tilde{Q}) \leq t\}, 
\end{cases}
$$

with the stopping time $\tau(\tilde{Q}) := \min\{t \in T : E_t(\frac{d\tilde{Q}}{d\mathbb{P}}) = 0\}$.

As we are often interested in those probability measures $\hat{Q}$ that agree with $\tilde{\mathbb{P}}$ along the filtration (i.e., $\hat{Q}[\tilde{D}] = \mathbb{P}[D]$ for any $\tilde{D} \in \tilde{\mathcal{F}}_t$), we present the following corollary.

**Corollary 2.3** Let $\hat{Q} \in \tilde{\mathcal{M}}(\tilde{P})$ with decomposition $\hat{Q} = \tilde{Q} \otimes \hat{\psi}$ (as in Remark 2.2). Then for any time $t \in T$, it follows that

$$
\hat{Q} = \tilde{P} \text{ on } \tilde{\mathcal{F}}_t \iff \tilde{Q} = \mathbb{P} \text{ on } \mathcal{F}_t, \text{ and } \psi_s = \mu_s, \forall s < t.
$$
Proof. This follows directly from [1, Lemma B.5].

We note that from Corollary 2.3, it follows that
\[
\mathcal{M}_t(\tilde{\mathcal{P}}) := \{ \tilde{Q} \in \mathcal{M}(\tilde{\mathcal{P}}) : \tilde{Q} = \tilde{\mathcal{P}} \text{ on } \bar{T}_t \} = \{ Q \otimes \psi : Q \in \mathcal{M}_t(\mathcal{P}), \psi \in \Psi(\mathcal{P}), \psi_s = \mu_s, \forall s < t \}.
\]

As for the original probability space, we want to use a \( \tilde{\mathcal{P}} \)-almost surely defined version of the \( \tilde{Q} \)-conditional expectation of \( X \in L^\infty(\mathbb{R}^d) \). To that end, denote by \( \tilde{E}_r^\tilde{Q}[X] := \tilde{E}_r^\tilde{Q}[X|\bar{T}_r] \) the conditional expectation of \( X \in L^\infty(\mathbb{R}^d) \) and note that this is a stochastic process indexed by \( r \). Then a \( \tilde{\mathcal{P}} \)-almost surely defined version can be provided by \( \tilde{\xi}(\tilde{Q}) \) similarly to the definition of \( \tilde{E}_r^\tilde{Q}[X] \) on the original probability space \( (\Omega, \mathcal{F}, \tilde{\mathcal{P}}) \), i.e., with
\[
\tilde{\xi}_{t,s}(\tilde{Q})_r := \begin{cases} \frac{\tilde{E}_r [d\tilde{Q}_s]}{d\tilde{E}_r [d\tilde{Q}_t]} & \text{on } \{ \tilde{E}_r [d\tilde{Q}_t] > 0 \}, \\ 1, & \text{otherwise} \end{cases}
\]
\[
\tilde{\xi}_{t,s}(\tilde{Q})_r = \begin{cases} \frac{1 - \sum_{\tau=0}^{t-1} \mu_{\tau} \psi_{\tau}}{1 - \sum_{\tau=0}^{t-1} \psi_{\tau}} \tilde{\xi}_{t,r}(\tilde{Q}) & \text{on } \{ r \in [t, s), \sum_{\tau=0}^{t-1} \psi_{\tau} < 1 \}, \\ \frac{1 - \sum_{\tau=0}^{s-1} \mu_{\tau} \sum_{\tau=s}^{t-1} \psi_{\tau}}{1 - \sum_{\tau=0}^{t-1} \psi_{\tau}} \tilde{\xi}_{t,s}(\tilde{Q}) & \text{on } \{ r \geq s, \sum_{\tau=0}^{t-1} \psi_{\tau} < 1 \}, \\ 1, & \text{otherwise} \end{cases}
\]
under the decomposition \( \tilde{Q} = Q \otimes \psi \) with \( \psi \) given as in Remark 2.2. In the following result, we provide an equivalent representation for the conditional expectation.

Corollary 2.4. For \( \tilde{Q} \in \mathcal{M}(\tilde{\mathcal{P}}) \) with decomposition \( \tilde{Q} = Q \otimes \psi \), the conditional expectation of \( X \in L^\infty(\mathbb{R}) \) given \( \bar{T}_t \) takes the form
\[
\tilde{E}_t^\tilde{Q}[X] = \sum_{s=0}^{t-1} X_s I_{s \leq t} + \left( I_{\sum_{\tau=0}^{s-1} \psi_{\tau} < 1} \tilde{E}_t^{\tilde{Q}} \left[ \sum_{s \in \mathbb{T}_t} \psi_s \right] \sum_{s \in \mathbb{T}_t} \frac{\psi_s}{1 - \sum_{\tau=0}^{t-1} \psi_{\tau}} X_s \right) I_{s \leq t},
\]
for any \( X \in L^\infty(\mathbb{R}) \). By matching terms,
\[
\tilde{E}_t^\tilde{Q}[X]_s = X_s \quad Q\text{-a.s. on } \{ \psi_s > 0 \} \text{ for } s < t,
\]
\[
\tilde{E}_t^\tilde{Q}[X]_t = \frac{\tilde{E}_t^\tilde{Q}[\sum_{s \in \mathbb{T}_t} \psi_s X_s]}{\sum_{s \in \mathbb{T}_t} \psi_s} \quad Q\text{-a.s. on } \left\{ \sum_{s \in \mathbb{T}_t} \psi_s > 0 \right\}.
\]
Noting that $\sum_{s \in \mathbb{T}} \psi_s = 1 - \sum_{s=0}^{t-1} \psi_s$ by construction, we first recover the $\bar{Q}$-almost surely defined version of the conditional expectation as presented in e.g. [1, Corollary B.3]. Finally, we recover the representation of the conditional expectation presented within this corollary by taking the $\bar{P}$-almost surely defined version of the $\bar{Q}$-conditional expectation through the use of $\bar{\xi}(\bar{Q})$ as presented in (2.2); in particular, $\bar{\xi}_{t,T}(\bar{Q})_r = 1$ for every time $r$ on $\{\sum_{t=0}^{t-1} \psi_t = 1\}$. \hfill \Box

We further define the functions $\bar{\omega}_t^s : \bar{\mathcal{M}}(\bar{P})^d \times \bar{L}_1^1(\mathbb{R}^d) \to \bar{L}_0^0(\mathbb{R}^d)$ for any $t, s \in \mathbb{T}$ with $t < s$ by

$$\bar{\omega}_t^s(\bar{Q}, \bar{w}) := \sum_{r=1}^{t-1} \bar{w}_r \mathbb{I}_{[r]} + \text{diag}(\bar{w}_t) \left( \sum_{r=t}^{s-1} \bar{\xi}_{t,s}(\bar{Q}) \mathbb{I}_{[r]} + \bar{\xi}_{t,s}(\bar{Q}) \mathbb{I}_{\mathbb{T}_r} \right),$$

where $\bar{\xi}_{t,s}(\bar{Q})$ is defined in (2.2). By convention, for any $t \in \mathbb{T}$, $\bar{\omega}_t^t(\bar{Q}, \bar{w}) = \bar{w}$ for any $\bar{Q} \in \bar{\mathcal{M}}(\bar{P})^d$ and $\bar{w} \in \bar{L}_1^1(\mathbb{R}^d)$. We note that through the decomposition of $\bar{Q}$ in Theorem 2.1, an equivalent formulation for $\bar{\omega}_t^s(\bar{Q}, \bar{w})$ can be given in the style of Corollary 2.4.

We conclude this section with a brief result relating the topologies of $\bar{L}_\infty^\infty(\mathbb{R}^d)$ and $\bar{\mathcal{R}}_\infty^1$. This is utilised later.

**Lemma 2.5**

(1) Suppose that $X^n \to X$ in $\sigma(\bar{L}_\infty^\infty(\mathbb{R}^d), \bar{L}_1^1(\mathbb{R}^d))$. Then $X^n_t \to X_t$ in $\sigma(L_\infty^\infty(\mathbb{R}^d), L_1^1(\mathbb{R}^d))$ and $X^n \mathbb{I}_{\mathbb{T}_r} \to X \mathbb{I}_{\mathbb{T}_r}$ in $\sigma(\mathcal{R}_\infty^1, \mathcal{A}_1^1)$, for every time $t \in \mathbb{T}$.

(2) Fix $t \in \mathbb{T}$. Suppose that $X^n_s \to X_s$ in $\sigma(L_\infty^\infty(\mathbb{R}^d), L_1^1(\mathbb{R}^d))$ for every $s < t$ and $X^n \mathbb{I}_{\mathbb{T}_r} \to X \mathbb{I}_{\mathbb{T}_r}$ in $\sigma(\mathcal{R}_\infty^1, \mathcal{A}_1^1)$. Then $X^n \to X$ in $\sigma(\bar{L}_\infty^\infty(\mathbb{R}^d), \bar{L}_1^1(\mathbb{R}^d))$.

**Proof**

(1) Let $X^n \to X$ in $\sigma(\bar{L}_\infty^\infty(\mathbb{R}^d), \bar{L}_1^1(\mathbb{R}^d))$ and fix $t \in \mathbb{T}$.

(a) Let $Z_t \in L_1^1(\mathbb{R}^d)$ and define $Y := \mu_t^{-1} Z_t \mathbb{I}_{[t]} \in \bar{L}_1^1(\mathbb{R}^d)$. It follows that

$$E[Z_t^T X^n] = E \left[ \sum_{s \in \mathbb{T}} \mu_s Y_s^T X^n_s \right] = \bar{E}[Y^T X^n] \longrightarrow \bar{E}[Y^T X] = E[Z_t^T X_t].$$

(b) Let $a \in \mathcal{A}_1^1$. Define $Y := \sum_{s \in \mathbb{T}_t} \mu_s^{-1} \Delta a_s \mathbb{I}_{\{s\}} \in \bar{L}_1^1(\mathbb{R}^d)$. Then we find

$$E \left[ \sum_{s \in \mathbb{T}_t} \Delta a_s^T X^n_s \right] = E \left[ \sum_{s \in \mathbb{T}} \mu_s Y_s^T X^n_s \right] = \bar{E}[Y^T X^n]$$

$$\longrightarrow \bar{E}[Y^T X] = E \left[ \sum_{s \in \mathbb{T}_t} \Delta a_s^T X_s \right].$$

(2) Fix $t \in \mathbb{T}$. Suppose that $X^n_s \to X_s$ in $\sigma(L_\infty^\infty(\mathbb{R}^d), L_1^1(\mathbb{R}^d))$ for every $s < t$ and $X^n \mathbb{I}_{\mathbb{T}_r} \to X \mathbb{I}_{\mathbb{T}_r}$ in $\sigma(\mathcal{R}_\infty^1, \mathcal{A}_1^1)$. Let $Y \in \bar{L}_1^1(\mathbb{R}^d)$ and $Z_t := \mu_t Y_t \in L_1^1(\mathbb{R}^d)$ for every $s < t$ and $\Delta a_s := \mu_s Y_s$ for every $s \in \mathbb{T}_t$ (with $\Delta a_s := 0$ for $s < t$) so that
\[ a_r = \sum_{s=0}^{r} \Delta a_r \] defines \( a \in A^1_d \). Then we obtain that
\[
\mathbb{E}[Y^\top X^n] = \mathbb{E} \left[ \sum_{s \in \mathcal{T}} \mu_s Y_s^\top X_s^n \right] = \sum_{s=0}^{t-1} \mathbb{E} [Z_s^\top X_s^n] + \mathbb{E} \left[ \sum_{s \in \mathcal{T}_t} \Delta a_s^\top X_s^n \right]
\rightarrow \sum_{s=0}^{t-1} \mathbb{E} [Z_s^\top X_s] + \mathbb{E} \left[ \sum_{s \in \mathcal{T}_t} \Delta a_s^\top X_s \right] = \mathbb{E}[Y^\top X].
\]

### 2.2 Risk measures for processes

In this section, we provide a quick overview of the definition of set-valued risk measures for processes as defined in Chen and Hu [10, 11]. Here we present an axiomatic framework for such functions in Definition 2.6. We then summarise prior results on the primal representation with respect to an acceptance set. We conclude this section with considerations for a novel dual representation for these conditional risk measures.

**Definition 2.6** A function \( \rho_t : \mathcal{R}^{\infty,d}_t \rightarrow \mathcal{U}(M_t; M_{t,+}) \) for \( t \in \mathbb{T} \) is called a **set-valued conditional risk measure for processes** if it satisfies the following properties for all \( X, Y \in \mathcal{R}^{\infty,d}_t \):

1. (cash-invariance) for any \( m \in M_t \),
   \[ \rho_t(X + m I_{\mathcal{T}_t}) = \rho_t(X) - m ; \]
2. (monotonicity) \( \rho_t(X) \subseteq \rho_t(Y) \), if \( X \leq Y \) component-wise;
3. (finiteness at zero) \( \rho_t(0) \neq \emptyset \) is closed (with respect to the subspace topology on \( M_t \)) and \( \mathbb{P}[\tilde{\rho}_t(0) = M] = 0 \), where \( \tilde{\rho}_t(0) \) is an \( \mathcal{F}_t \)-measurable random set such that \( \rho_t(0) = L^{\infty}_t(\tilde{\rho}_t(0)) \).

Recall that \( \rho_t(0) \) is \( \mathcal{F}_t \)-decomposable if and only if \( \mathbb{I}_A \rho_t(0) = \mathbb{I}_A \mathbb{I}_{A^c} \rho_t(0) \subseteq \rho_t(0) \) for any \( A \in \mathcal{F}_t \); see e.g. Molchanov [29, Chap. 2]. Furthermore, \( \tilde{\rho}_t(0) \) is an \( \mathcal{F}_t \)-measurable random set if and only if
\[
\text{graph} \tilde{\rho}_t(0) := \{ (\omega, x) \in \Omega \times \mathbb{R}^d : x \in \rho_t(0; \omega) \}
\]
is \( (\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)) \)-measurable for the Borel \( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}^d) \) of \( \mathbb{R}^d \); see e.g. Ben Tahar and Lépinette [7]. A conditional risk measure for processes at time \( t \in \mathbb{T} \) is said to be

4. normalised if \( \rho_t(X) = \rho_t(X) + \rho_t(0) \) for every \( X \in \mathcal{R}^{\infty,d}_t \);
5. conditionally convex if for all \( X, Y \in \mathcal{R}^{\infty,d}_t \)
   \[ \rho_t(\lambda X + (1 - \lambda)Y) \supseteq \lambda \rho_t(X) + (1 - \lambda) \rho_t(Y) \]
for all \( \lambda \in L^{\infty}_t([0, 1]) \);
(6) conditionally positively homogeneous if for all $X \in \mathcal{R}_t^{\infty,d}$,
\[
\rho_t(\lambda X) = \lambda \rho_t(X)
\]
for all $\lambda \in L_t^\infty(\mathbb{R}_{++})$;
(7) conditionally coherent if it is conditionally convex and conditionally positively homogeneous;
(8) closed if the graph of $\rho_t$,
\[
\text{graph } \rho_t := \{(X, u) \in \mathcal{R}_t^{\infty,d} \times M_t : u \in \rho_t(X)\},
\]
is closed in the product topology;
(9) conditionally convex upper continuous (c.u.c.) if
\[
\rho_t^-(D) := \{X \in \mathcal{R}_t^{\infty,d} : \rho_t(X) \cap D \neq \emptyset\}
\]
is closed for any $\mathcal{F}_t$-conditionally convex set $D \in \mathcal{G}(M_t; -M_t, +)$.

A dynamic risk measure for processes is a sequence $(\rho_t)_{t \in \mathbb{T}}$ of conditional risk measures for processes. It is said to have one of the above properties if $\rho_t$ has the corresponding property for any $t \in \mathbb{T}$.

For the interpretation of these risk measures, recall that $X \in \mathcal{R}_t^{\infty,d}$ denotes the evolution of a financial value or a cumulative cash flow over time. As such, for instance, cash-invariance implies that an addition of $m$ eligible assets at time $t$ to $X \in \mathcal{R}_t^{\infty,d}$ transforms the value into $X + mI_T^t$ and the risk is decreased by $m$ at time $t$.

We now consider the primal representation of a set-valued conditional risk measure for processes. For any such risk measure, let the acceptance set be defined as the set of those claims that do not require the addition of any eligible asset in order to be “acceptable” to the risk manager, i.e.,
\[
A_{\rho_t} := \{X \in \mathcal{R}_t^{\infty,d} : 0 \in \rho_t(X)\}.
\]
As shown in [10, 11], these acceptance sets uniquely define the risk measure via the relation
\[
\rho_t^{A_t}(X) = \{m \in M_t : X + mI_T^t \in A_t\}
\]
for any $X \in \mathcal{R}_t^{\infty,d}$. Furthermore, as provided by [10, Proposition 2.1], there is a one-to-one relation between the risk measure and its acceptance set so that
\[
\rho_t = \rho_t^{A_t} \quad \text{ and } \quad A_t = A_{\rho_t}^{A_t}.
\]

Throughout this work, we take the risk measure/acceptance set pair $(\rho_t, A_t)$ without use of the explicit sub- or superscript notation.

With this construction of risk measures for processes, we also provide a dual representation. The following dual representation – Corollary 2.7 – is novel in the literature for set-valued risk measures for processes; we postpone the proof until Sect. 3.2. Although this result can be proved using the duality theory of Hamel et al. [26] and following the same logic as the proof of Feinstein and Rudloff [19, Corollary 2.4],

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we take a different approach by utilising the prior results on risk measures for vectors and the yet-to-be-proved equivalence between these formulations. For this result, we first need to consider the set of dual variables

$$
\mathcal{W}_t := \{(Q, w) \in \mathcal{D}_t : w^s(Q_s, w_s) \in L^1_s(\mathbb{R}_+^d), \forall s \in T_t\},
$$

$$
\mathcal{D}_t := M_t(P^d_t) \times (T - t + 1) \times (M^*_t, + \setminus M^\perp_t)^T - t + 1.
$$

Note that \((Q, w) \in \mathcal{W}_t\) implies \(w_s \geq 0\) \(Q_s\)-a.s. since \(w^s(Q_s, w_s) \in L^1_s(\mathbb{R}_+^d)\).

**Corollary 2.7** A function \(\rho_t : \mathcal{R}^\infty_{0}, d \to \mathcal{G}(M_t; M_t, +)\) is a closed conditionally convex risk measure if and only if

$$
\rho_t(X) = \bigcap_{(Q, w) \in \mathcal{W}_t} \left( \sum_{s \in T_t} \left( (E^Q_s[-X_s] + \Gamma_t(w_s)) \cap M_t \right) - \alpha_t(Q, w) \right),
$$

(2.3)

where \(\alpha_t\) is the minimal conditional penalty function given by

$$
\alpha_t(Q, w) = \bigcap_{Y \in A_t} \sum_{s \in T_t} \left( (E^Q_s[-Y_s] + \Gamma_t(w_s)) \cap M_t \right).
$$

The function \(\rho_t\) is additionally conditionally coherent if and only if

$$
\rho_t(X) = \bigcap_{(Q, w) \in \mathcal{W}_t^{\max}} \sum_{s \in T_t} \left( (E^Q_s[-X_s] + \Gamma_t(w_s)) \cap M_t \right)
$$

for

$$
\mathcal{W}_t^{\max} = \left\{ (Q, w) \in \mathcal{W}_t : \sum_{s \in T_t} w^s \cdot E^Q_s[Z_s] \geq 0 \text{ for any } Z \in A_t \right\}.
$$

We conclude this section with a brief introduction to a time-consistency notion, called multi-portfolio time-consistency, for these set-valued risk measures for processes. This notion was introduced in Chen and Hu [10]. We revisit this definition in Sect. 4.

**Definition 2.8** A dynamic risk measure for processes \((\rho_t)_{t \in T}\) is multi-portfolio time-consistent if for any times \(t < s \in T\),

$$
\rho_s(X) \subseteq \bigcup_{Y \in B} \rho_s(Y) \quad \Longrightarrow \quad \rho_t(Z \mathbb{P}[t, s] + X \mathbb{P}_T) \subseteq \bigcup_{Y \in B} \rho_t(Z \mathbb{P}[t, s] + Y \mathbb{P}_T)
$$

for any \(X \in \mathcal{R}^\infty_{0}, d, Z \in \mathcal{R}^\infty_{0}, d\) and \(B \subseteq \mathcal{R}^\infty_{0}, d\).

Conceptually, consider a multi-portfolio time-consistent risk measure. Let \(X\) be a cumulative cash flow process and \(B\) a collection of cumulative cash flows such that all of these cash flows are identical up to time \(s\). If \(X\) is guaranteed (almost surely) to be more risky than \(B\) at time \(s\), then \(X\) is also more risky at all earlier times. This can most easily be understood when the collection \(B\) is a singleton, i.e., when comparing two cumulative cash flows \(X\) and \(Y\) directly.
2.3 Risk measures for vectors

In this section, we provide a quick overview of the definition of set-valued risk measures for random vectors as defined in [17, 18]. Here we present an axiomatic framework for such functions in Definition 2.9 with emphasis on the special case on the optional filtration. We then summarise prior results on the primal representation with respect to an acceptance set. We conclude this section with considerations for a dual representation for these conditional risk measures.

Definition 2.9 A function $\tilde{\mathcal{R}}_t : \check{L}^\infty(\mathbb{R}^d) \to \mathcal{U}(\tilde{M}_t; \tilde{M}_{t,+})$ for $t \in T$ is called a set-valued conditional risk measure for vectors on $\check{L}^\infty(\mathbb{R}^d)$ if it satisfies the following properties for all $X,Y \in \check{L}^\infty(\mathbb{R}^d)$:

1. (cash-invariance) for any $m \in \tilde{M}_t$,
   \[ \tilde{\mathcal{R}}_t(X + m) = \tilde{\mathcal{R}}_t(X) - m; \]

2. (monotonicity) $\tilde{\mathcal{R}}_t(X) \subseteq \tilde{\mathcal{R}}_t(Y)$, if $X \leq Y$ component-wise;

3. (finiteness at zero) $\tilde{\mathcal{R}}_t(0) \neq \emptyset$ is closed and $\tilde{\mathcal{F}}_t$-decomposable and has the property that $\tilde{\mathbb{P}}[\tilde{\mathcal{R}}_t(0) = M] = 0$, where $\tilde{\mathcal{R}}_t(0)$ is an $\tilde{\mathcal{F}}_t$-measurable random set such that $\tilde{\mathcal{R}}_t(0) = \tilde{L}^\infty(\tilde{\mathcal{R}}_t(0))$.

A conditional risk measure for vectors at time $t \in T$ is said to be

4. normalised if $\tilde{\mathcal{R}}_t(X) = \tilde{\mathcal{R}}_t(X) + \tilde{\mathcal{R}}_t(0)$ for every $X \in \check{L}^\infty(\mathbb{R}^d)$;

5. time-decomposable if
   \[ \tilde{\mathcal{R}}_t(X) = \sum_{s=0}^{t-1} \tilde{\mathcal{R}}_t(X I_{[s]} I_{\{s\}}) + \tilde{\mathcal{R}}_t(X I_{\{T\} t} I_{\{T\}}); \]

for any $X \in \check{L}^\infty(\mathbb{R}^d)$;

6. conditionally convex if for all $X, Y \in \check{L}^\infty(\mathbb{R}^d)$,
   \[ \tilde{\mathcal{R}}_t(\lambda X + (1 - \lambda) Y) \supseteq \lambda \tilde{\mathcal{R}}_t(X) + (1 - \lambda) \tilde{\mathcal{R}}_t(Y) \]

for all $\lambda \in L^\infty_t([0, 1])$;

7. conditionally positively homogeneous if for all $X \in \check{L}^\infty(\mathbb{R}^d)$,
   \[ \tilde{\mathcal{R}}_t(\lambda X) = \lambda \tilde{\mathcal{R}}_t(X) \]

for all $\lambda \in L^\infty_t(\mathbb{R}^{++})$;

8. conditionally coherent if it is conditionally convex and conditionally positively homogeneous;

9. closed if the graph of $\tilde{\mathcal{R}}_t$,
   \[ \text{graph } \tilde{\mathcal{R}}_t := \{(X, u) \in \check{L}^\infty(\mathbb{R}^d) \times \tilde{M}_t : u \in \tilde{\mathcal{R}}_t(X)\}, \]
   is closed in the product topology;
(10) conditionally convex upper continuous (c.u.c.) if
\[ \tilde{R}_t(D) := \{ X \in \bar{L}^\infty(\mathbb{R}^d) : \tilde{R}_t(X) \cap D \neq \emptyset \} \]
is closed for any \( \tilde{F}_t \)-conditionally convex set \( D \in \mathcal{G}(\bar{M}_t; -\bar{M}_{t,+}) \).

A dynamic risk measure for vectors is a sequence \((\tilde{R}_t)_{t \in \mathbb{T}}\) of conditional risk measures for vectors. It is said to have one of the above properties if \( \tilde{R}_t \) has the corresponding property for any \( t \in \mathbb{T} \).

**Remark 2.10** We note that the time-decomposable property presented above is, as far as the authors are aware, novel to this work. Consider a conditional risk measure at time \( t \in \mathbb{T} \) and let \( s < t \) be an earlier time point. Time-decomposability conceptually means that the time-\( s \)-eligible assets required to compensate for the risk of the cumulative cash flows at time \( s \) only depend on the realised portfolio at time \( s \). This is similarly true for eligible assets required to compensate for the random future cash flows at time \( t \). Notably, by Feinstein and Rudloff [17, Proposition 2.8], any conditionally convex risk measure is time-decomposable.

We now consider the primal representation of a set-valued conditional risk measure for vectors. For any such risk measure, let the acceptance set be defined as the set of those claims that do not require the addition of any eligible asset in order to be “acceptable” to the risk manager, i.e.,
\[ \bar{A}_{\tilde{R}_t} := \{ X \in \bar{L}^\infty(\mathbb{R}^d) : 0 \in \tilde{R}_t(X) \} . \]

As shown in [17], these acceptance sets uniquely define the risk measure via the relation
\[ \tilde{R}_t^{\bar{A}_t}(X) = \{ m \in \bar{M}_t : X + m \in \bar{A}_t \} \]
for any \( X \in \bar{L}^\infty(\mathbb{R}^d) \). Furthermore, as with the risk measures for processes and as provided by [17, Remark 2], there is a one-to-one relation between the risk measure and its acceptance set so that
\[ \tilde{R}_t = \tilde{R}_t^{\bar{A}_t} \quad \text{and} \quad \bar{A}_t = \bar{A}_{\tilde{R}_t^{\bar{A}_t}} . \]

**Throughout this work**, we take the risk measure/acceptance set pair \((\tilde{R}_t, \bar{A}_t)\) without use of the explicit sub- or superscript notation.

We now provide a dual representation for risk measures for vectors. For comparison to the dual representation presented above for risk measures for processes, we present the dual representation provided first in Feinstein and Rudloff [19]. For this result, we first need to consider the set of dual variables
\[ \tilde{W}_t := \{ (\tilde{Q}, \tilde{w}) \in \bar{M}_t(\mathbb{P})^d \times (\bar{M}_{t,+}^* \setminus \bar{M}_t^+) : \tilde{w}^T_t(\tilde{Q}, \tilde{w}) \in \bar{L}^1(\mathbb{R}^d_+) \} . \]

As above, \((\tilde{Q}, \tilde{w}) \in \tilde{W}_t\) implies \( \tilde{w} \geq 0 \) \( \tilde{Q} \)-a.s.
Corollary 2.11 ([19, Corollary 2.4]) A function \( \bar{R}_t : \bar{L}^\infty(\mathbb{R}^d) \to G(\bar{M}_t; \bar{M}_{t,+}) \) is a closed conditionally convex risk measure if and only if

\[
\bar{R}_t(X) = \bigcap_{(\bar{Q}, \bar{w}) \in \bar{W}_t} \left( (\bar{E}_{\bar{Q}} X + \bar{\Gamma}_t(\bar{w})) \cap \bar{M}_t - \bar{a}_t(\bar{Q}, \bar{w}) \right),
\]

where \( \bar{a}_t \) is the minimal conditional penalty function given by

\[
\bar{a}_t(\bar{Q}, \bar{w}) = \bigcap_{Y \in \bar{A}_t} (\bar{E}_{\bar{Q}} Y + \bar{\Gamma}_t(\bar{w})) \cap \bar{M}_t.
\]

The function \( \bar{R}_t \) is additionally conditionally coherent if and only if

\[
\bar{R}_t(X) = \bigcap_{(\bar{Q}, \bar{w}) \in \bar{W}_t^{\max}} (\bar{E}_{\bar{Q}} X + \bar{\Gamma}_t(\bar{w})) \cap \bar{M}_t
\]

for

\[
\bar{W}_t^{\max} = \{ (\bar{Q}, \bar{w}) \in \bar{W}_t : \bar{w}^T (\bar{Q}, \bar{w}) \in \bar{A}_t^* \}.
\]

We conclude this introduction to risk measures for vectors with a consideration of multi-portfolio time-consistency. This notion for risk measures for vectors is a modification of that introduced in Feinstein and Rudloff [17, 18]; we demonstrate in Proposition 2.13 that this coincides with the typical definition for time-decomposable risk measures.

Definition 2.12 A dynamic risk measure \( (\bar{R}_t)_t \in \mathbb{T} \) for vectors is multi-portfolio time-consistent if for any times \( t < s \in \mathbb{T} \),

\[
\bar{R}_s(X) \subseteq \bigcup_{Y \in B} \bar{R}_s(Y) \implies \bar{R}_t(X) \subseteq \bigcup_{Y \in B} \bar{R}_t(Y)
\]

for any \( X \in \bar{L}^\infty \) and \( B := \{(b_0, b_1, \ldots, b_{s-1}, b_s) : b_r \in B_r, r < s, b_s \in B_s \} \) for any sequence of sets \( B_r \subseteq L_r^\infty(\mathbb{R}^d) \) for \( r < s \) and \( B_s \subseteq \mathbb{R}^\infty_{s,d} \).

As for risk measures for processes, conceptually, multi-portfolio time-consistency allows the comparison of a single (cumulative) cash flow with a collection of portfolios over time in which a guaranteed ordering must hold backwards in time.

Proposition 2.13 Let \( (\bar{R}_t)_t \in \mathbb{T} \) be a normalised, time-decomposable risk measure. Then \( (\bar{R}_t)_t \in \mathbb{T} \) is multi-portfolio time-consistent if and only if for any times \( t < s \in \mathbb{T} \),

\[
\bar{R}_s(X) \subseteq \bigcup_{Y \in B} \bar{R}_s(Y) \implies \bar{R}_t(X) \subseteq \bigcup_{Y \in B} \bar{R}_t(Y)
\]

for any \( X \in \bar{L}^\infty(\mathbb{R}^d) \) and \( B \subseteq \bar{L}^\infty(\mathbb{R}^d) \).
Proof Consider a normalised, time-decomposable risk measure. Then multi-portfolio time-consistency (as in Definition 2.12) is equivalent to the recursive relation

$$\bar{R}_t(X) = \bigcup_{Z \in \bar{R}_s(X)} \bar{R}_t(-Z)$$

for any $X \in \bar{L}^\infty(R^d)$ and times $t < s$. This is proved similarly to [17, Theorem 3.4]. Furthermore, due to time-decomposability, this recursive relation is likewise equivalent to the implication provided within this proposition. □

Remark 2.14 Throughout much of the remainder of this work, the restriction of set-valued risk measures for vectors to a sub-$\sigma$-algebra is utilised. To that end, we define $R_t : L^\infty_t(\mathbb{R}^d) \to \mathcal{U}(M_t; M_t, +)$ to be the restriction of a set-valued risk measure (with filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \in T}, \mathbb{P})$) to $L^\infty_t(\mathbb{R}^d)$. The primal and dual representation of such risk measures are utilised extensively below and are defined in analogous ways to those provided above; for closed and convex risk measures – which are those utilised for the dual representation –, the codomain of $R_t$ is $\tilde{G}(M_t; M_t, +) \subseteq \mathcal{U}(M_t; M_t, +)$. Notably, the penalty function $\alpha_{R_t}$ for these restricted risk measures only depends on a single dual variable $w \in M^*_t, +$, i.e., $\alpha_{R_t}(w)$ does not depend on a vector $Q \in M_t(\mathbb{P})^d$ of probability measures.

3 Equivalence of risk measures for processes and for vectors

In this section, we focus on the relation between set-valued risk measures for processes and vectors. This was studied under restrictive conditions by Chen and Hu [10]; these conditions match many of the same properties of the scalar setting (i.e., with a full space of eligible assets, $M = \mathbb{R}^d$, and with the strong normalisation property $\rho_t(0) = L^\infty_t(\mathbb{R}^d_+)$). Here we drop these restrictions and determine the equivalence for both the primal and dual representations. Notably, this requires the introduction of an augmentation of the risk measure for processes with a series of risk measures for vectors on the original filtered probability space; when the aforementioned strong assumptions are imposed, these augmenting risk measures can be dropped, as detailed in Corollary 3.2.

3.1 Relation for primal representations

First, we compare the relation between set-valued risk measures for processes and for vectors through their axiomatic definitions. This is akin to the results presented in Acciaio et al. [1] for scalar risk measures. Notably, in contrast to that work, the set-valued risk measures for vectors on the optional filtration are a larger class of functions than those for processes. In particular, we find that any risk measure for vectors on the optional filtration is equivalent to a risk measure for processes augmented with a collection of restricted set-valued risk measures for vectors. (These restricted set-valued risk measures are discussed in Remark 2.14.) Intuitively, this augmentation is necessary to account for the possible risk prior to the current time.
that cannot trivially be accounted for by eligible assets; that is, the augmentation is required for the $M \neq \mathbb{R}^d$ setting. In the full eligible asset setting $M = \mathbb{R}^d$, the equivalence of risk measures for processes and vectors no longer requires an augmentation by restricted set-valued risk measures, as is detailed in Corollary 3.2 below.

**Theorem 3.1** Fix a time $t \in \mathbb{T}$.

1. Any conditional risk measure $\rho_t : \mathcal{R}_t^\infty,d \to \mathcal{U}(M_t; M_{t,+})$ for processes and series of conditional risk measures $R_s : L_s^\infty(\mathbb{R}^d) \to \mathcal{U}(M_s; M_{s,+})$ restricted to $\mathcal{F}_s$-measurable random vectors, $s = 0, 1, \ldots, t - 1$, define a time-decomposable conditional risk measure $\tilde{R}_t : \bar{L}^\infty(\mathbb{R}^d) \to \mathcal{U}(\bar{M}_t; \bar{M}_{t,+})$ for random vectors on the optional filtration via

$$\tilde{R}_t(X) := \sum_{s=0}^{t-1} R_s(X_s)^{I_{[s]}} + \rho_t(\pi_t,T(X))^{I_{T_t}}. \quad (3.1)$$

Additionally, if $(R_s)_{s=0}^{t-1}$ and $\rho_t$ are normalised, conditionally convex or conditionally coherent, the corresponding risk measure $\tilde{R}_t$ has the same property.

2. Conversely, any conditional risk measure $\rho_t : \mathcal{R}_t^\infty,d \to \mathcal{U}(M_t; \bar{M}_{t,+})$ for random vectors on the optional filtration defines a conditional risk measure

$$\rho_t : \mathcal{R}_t^\infty,d \to \mathcal{U}(M_t; M_{t,+})$$

for processes and a series of conditional risk measures $R_s : L_s^\infty(\mathbb{R}^d) \to \mathcal{U}(M_s; M_{s,+})$ restricted to $\mathcal{F}_s$-measurable random vectors, $s = 0, 1, \ldots, t - 1$, via

$$\rho_t(X) := \bar{R}_t(X^{I_{T_t}})^{I_{T_t}}, \quad (3.2)$$

$$R_s(Z) := \tilde{R}_t(Z^{I_{[s]}})^{I_{[s]}}, \quad s = 0, 1, \ldots, t - 1. \quad (3.3)$$

Additionally, if $\bar{R}_t$ is normalised, conditionally convex or conditionally coherent, the corresponding risk measures $(R_s)_{s=0}^{t-1}$ and $\rho_t$ have the same property.

3. If $\bar{R}_t$ is time-decomposable, then it can be decomposed into the form (3.1) such that $\rho_t$ and $(R_s)_{s=0}^{t-1}$ are defined as in (3.2) and (3.3), respectively. Additionally, if $\bar{R}_t$ is conditionally convex and either closed or conditionally convex upper continuous, then $\rho_t$ and $(R_s)_{s=0}^{t-1}$ have the same property, and vice versa.

**Proof** (1) It is easy to check that $\tilde{R}_t$ defined in (3.1) is a time-decomposable conditional risk measure on the optional filtration as in Definition 2.9. This is done directly by considering the equivalent definitions for the risk measure for processes and the sequence of conditional risk measures. For brevity, we omit the direct constructions.

(2) It is easy to check that $R_s$ defined in (3.3) for $s < t$ is a conditional risk measure on $L_s^\infty(\mathbb{R}^d)$ for vectors. To prove that $\rho_t$ defined in (3.2) is a risk measure for processes, we apply Chen and Hu [10, Proposition 3.4], with minor modifications to account for the space of eligible portfolios (i.e., $M \subseteq \mathbb{R}^d$), for all properties except finiteness at zero; due to the simplicity of these results, we omit those proofs. Due to
the updated definition for finiteness at zero within the present paper, we cannot apply [10, Proposition 3.4] here. However, finiteness at zero follows trivially by construction (recalling $\mathbb{P}[\min_{t \in \mathbb{T}} \mu_t > \epsilon] = 1$ for some $\epsilon > 0$), and thus we omit the details as well.

(3) If $\tilde{R}_t$ is time-decomposable and $X \in \tilde{L}^\infty(\mathbb{R}^d)$, then

$$\tilde{R}_t(X) = \sum_{s=0}^{t-1} \tilde{R}_t(XI_{[s]})I_{[s]} + \tilde{R}_t(XII_{T})I_{T} = \sum_{s=0}^{t-1} R_s(Xs)I_{[s]} + \rho_t(\pi_{t,T}(X))II_{T}.$$ 

Finally, we want to study closedness properties for conditionally convex risk measures.

(a) Closedness

– Assume $\tilde{R}_t$ is closed. First, we show that $R_s$ is closed for any $s < t$. Write

$$\text{graph } R_s = \{(X_s, m_s) \in L^\infty_s(\mathbb{R}^d) \times M_s : m_s \in R_s(X_s)\}$$

$$= \{(X_s, m_s) \in L^\infty_s(\mathbb{R}^d) \times M_s : m_s \in \tilde{R}_t(XI_{[s]})s\}$$

$$= \{(X, m) \in \tilde{L}^\infty(\mathbb{R}^d) \times \tilde{M}_t : m \in \tilde{R}_t(X)\}_s$$

$$= (\text{graph } \tilde{R}_t)_s.$$ 

Let $(X^n_s, m^n_s)_{n \in \mathcal{N}} \subseteq \text{graph } R_s \rightarrow (X_s, m_s)$ in $\sigma(L^\infty_s(\mathbb{R}^d), L^1_s(\mathbb{R}^d))$ be a convergent net indexed by the set $\mathcal{N}$. By Lemma 2.5 and the assumption, it must follow that $(X^n_sI_{[s]}, m^n_sI_{[s]} + m^0I_{[s]}\emptyset) \rightarrow (X_sI_{[s]}, m_sI_{[s]} + m^0I_{[s]}\emptyset) \in \text{graph } \tilde{R}_t$ for any $m^0 \in \tilde{R}_t(0)$. As $(X_s, m_s) \in \text{graph } R_s$, closedness of $R_s$ is proved. Consider now the closedness of $\rho_t$. By the same logic as above,

$$\text{graph } \rho_t = \{(XI_{T}, m_t) : (X, m) \in \text{graph } \tilde{R}_t\}.$$ 

Therefore, as above by concatenation with $(0, m^0) \in \text{graph } \tilde{R}_t$, graph $\rho_t$ is closed.

– Assume $\rho_t$ and $(R_s)_{s=0}^{t-1}$ are all closed risk measures. Write

$$\text{graph } \tilde{R}_t = \{(X, m) \in \tilde{L}^\infty(\mathbb{R}^d) \times \tilde{M}_t : m \in \tilde{R}_t(X)\}$$

$$= \{(X, m) \in \tilde{L}^\infty(\mathbb{R}^d) \times \tilde{M}_t : m \in R_s(X_s), \forall s < t, m_t \in \rho_t(\pi_{t,T}(X))\}$$

$$= \sum_{s=0}^{t-1} (\text{graph } R_s)I_{[s]} + (\text{graph } \rho_t)II_{T}.$$ 

Let $(X^n, m^n) \subseteq \text{graph } \tilde{R}_t \rightarrow (X, m)$ in the product topology. By the above construction and Lemma 2.5, it must follow that $(X_s, m_s) \in \text{graph } R_s$ for every $s < t$ and $(XI_{T}, m_t) \in \text{graph } \rho_t$. Therefore closedness of $\tilde{R}_t$ is proved as $(X, m) \in \text{graph } \tilde{R}_t$. 

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(b) Conditional convex upper continuity

Assume \( \bar{R}_t \) is conditionally convex upper continuous. First we show that \( R_s \) is conditionally convex upper continuous. Let \( D_s \in \mathcal{G}(M_s; -M_s, +) \) be closed and conditionally convex, and write

\[
R_s^-(D_s) = \{ X_s \in L^\infty_s(\mathbb{R}^d) : R_s(X_s) \cap D_s \neq \emptyset \}
\]

\[
= \{ X_s \in L^\infty_s(\mathbb{R}^d) : \bar{R}_t(X_s) \cap D_s \neq \emptyset \}
\]

\[
= \{ X \in \bar{L}^\infty_s(\mathbb{R}^d) : \bar{R}_t(X) \cap D_s \neq \emptyset \}
\]

Let \( X^n_s \subseteq R_s^-(D_s) \rightarrow X_s \) in \( \sigma(L^\infty_s(\mathbb{R}^d), L^1_s(\mathbb{R}^d)) \). By Lemma 2.5 and the assumption, it must follow that \( X^n_s \bar{R}_t(X) \rightarrow X_s \bar{R}_t(X) \in \bar{R}_t^-(D_s) \). Therefore conditional convex upper continuity of \( R_s \) is proved as \( X_s \in R_s^-(D_s) \).

Let \( X^n_s \subseteq \bar{R}_t^-(D_s) \rightarrow X_s \) in \( \sigma(\bar{R}_t^-(\mathbb{R}_t^d), \bar{R}_t^+(\mathbb{R}_t^d)) \). By Lemma 2.5 and the assumption, \( X^n_s \bar{R}_t(X) \rightarrow X_s \bar{R}_t(X) \in \bar{R}_t^-(D_s) \). Therefore conditional convex upper continuity of \( \bar{R}_t \) is proved because \( X_s \in \bar{R}_t^-(D_s) \).

Given the above results on the equivalence of risk measures for processes and vectors, we consider the special case in which every asset is eligible to cover risk, i.e., \( M = \mathbb{R}^d \). This setting provides a simpler equivalent relation insofar as the restricted risk measures \( R_s \) utilised in Theorem 3.1 above can be fully defined by \( R_s(0) \). This setting more clearly demonstrates the relationship with scalar risk measures, as
Risk measures for processes and for vectors 523

Corollary 3.2 Let \( M = \mathbb{R}^d \). The equivalences in Theorem 3.1 can be simplified insofar as the conditional risk measures \( R_s \) can be replaced with (conditionally convex) upper sets \( C_s \), i.e.,

\[
C_s := \tilde{R}_t(0)_s,
\]

\[
\tilde{R}_t(X) = \sum_{s=0}^{t-1} (-X_s + C_s)_I + \rho_t(\pi_t,T(X))_I.
\]

Proof This follows from Theorem 3.1 by noting that \( R_s(X)I_{\{s\}} = R_s(0) - X_s \) by cash-invariance for this choice of eligible assets. Therefore the full risk measure \( R_s \) is no longer required, but only the risk measure at 0 which we denote by \( C_s \).

\( \square \)

We conclude this section on the equivalence of primal representations by providing the relation between acceptance sets for processes and vectors on the optional filtration.

Corollary 3.3 Fix a time \( t \in \mathbb{T} \).

(1) Given a conditional risk measure \( \rho_t : \mathcal{R}_t^\infty, d \to \mathcal{U}(M_t; M_{t,+}) \) for processes and a series of conditional risk measures \( R_s : L_s^\infty(\mathbb{R}^d) \to \mathcal{U}(M_s; M_{s,+}) \) for \( s = 0, 1, \ldots, t-1 \), define \( \tilde{R}_t \) via (3.1). Then \( \tilde{A}_t = \sum_{s=0}^{t-1} A_{R_s} I_s + A_I I_T \).

(2) Given a conditional risk measure \( \tilde{R}_t : \mathcal{L}_t^\infty(\mathbb{R}^d) \to \mathcal{U}(\bar{M}_t; \bar{M}_{t,+}) \) for random vectors on the optional filtration, define \( \rho_t \) and \( R_s \) via (3.2) and (3.3), respectively. Then \( A_t = \{ X \in \mathcal{R}_t^\infty, d : X I_T \in \tilde{A}_t \} \) and \( A_{R_s} = (\tilde{A}_t)_s \) for every \( s = 0, 1, \ldots, t-1 \).

Proof This follows directly by the construction of the equivalences in Theorem 3.1.

\( \square \)

3.2 Relation for dual representations

Before presenting the following results, we recall the dual representations we have given for risk measures for processes in Corollary 2.7 and for vectors in Corollary 2.11. We also highlight the form for the penalty functions for risk measures \( R_s : L_s^\infty(\mathbb{R}^d) \to \mathcal{G}(M_s; M_{s,+}) \) as provided in Remark 2.14. Finally, we recall that any conditionally convex risk measure for vectors is time-decomposable by construction; see Remark 2.10.

Lemma 3.4 Fix a time \( t \in \mathbb{T} \).

(1) Let \( \rho_t : \mathcal{R}_t^\infty, d \to \mathcal{G}(M_t; M_{t,+}) \) be a closed conditionally convex risk measure for processes and \( R_s : L_s^\infty(\mathbb{R}^d) \to \mathcal{G}(M_s; M_{s,+}) \), \( s < t \), a series of closed conditionally convex risk measures. The associated closed conditionally convex risk measure

\[ R_t(0) = L_s^\infty(\mathbb{R}^d) \] in the \( d = 1 \) asset setting for any choice of normalised risk measure with closed values.
\( \bar{R}_t : L^\infty(\mathbb{R}^d) \to \bar{G}(\bar{M}_t; \bar{M}_t, +) \) defined in (3.1) on the optional filtration has a dual representation with respect to the penalty function

\[
\bar{\alpha}_t(\bar{Q}, \bar{w}) = \bar{\alpha}_t(Q \otimes \psi, \bar{w}) = \sum_{s=0}^{t-1} \alpha_{R_t}(\bar{w}_s)1_{\{s\}} + \alpha_t(W_t(Q \otimes \psi, \bar{w}))1_{T_t},
\]

where \( W_t(Q \otimes \psi, \bar{w}) := (\hat{Q}, \hat{w}) \in \bar{W}_t \) is defined as

\[
d\hat{Q}_{s,i} = \frac{\psi_{s,i}}{E_t[\psi_{s,i}]} d\hat{\bar{P}}_t[\psi_{s,i}]1_{\{\hat{Q}_{s,i} > 0\}} + \frac{\mu_s}{E_t[\mu_s]} d\hat{\bar{P}}_t[\psi_{s,i} > 0],
\]

\[
w_{s,i} := \frac{E_t[\psi_{s,i}]}{1 - \sum_{r=0}^{t-1} \mu_r} \bar{w}_{t,i},
\]

for \( i = 1, 2, \ldots, d \). and \( s \in T_t \).

(2) Conversely, consider a closed conditionally convex risk measure

\( \bar{R}_t : L^\infty(\mathbb{R}^d) \to \bar{G}(\bar{M}_t; \bar{M}_t, +) \)

for random vectors on the optional filtration. The associated closed conditionally convex risk measure \( R_t : \mathcal{R}_t^\infty(\mathbb{R}^d) \to \mathcal{G}(\bar{M}_t; \bar{M}_t, +) \) for processes and series of closed conditionally convex risk measures \( R_s : L^\infty(\mathbb{R}^d) \to \mathcal{G}(\bar{M}_s; \bar{M}_s, +) \) defined in (3.2) and (3.3), respectively, have dual representations with respect to the penalty functions

\[
\alpha_{R_t}(w_s) = \bar{\alpha}_t(\bar{P}, w_s1_{\{s\}}),
\]

\[
\alpha_t(Q, w) = \bar{\alpha}_t(\bar{W}_t(Q, w)),
\]

where \( \bar{W}_t(Q, w) := (\hat{Q}, \hat{w}) \in \bar{W}_t \) is defined as

\[
\bar{w} := \sum_{s \in T_t} \text{diag}(1_{\{w_s > 0\}})w_s1_{T_t},
\]

\[
\left( \frac{d\bar{Q}_{s,i}}{d\bar{P}} \right)_s := 1_{\{s \in [0,t)\}} + \frac{1 - \sum_{r=0}^{t-1} \mu_r}{\mu_s} \left( \frac{w_{s,i}}{\bar{w}_{t,i}}1_{\{\bar{w}_{t,i} > 0\}} + 1_{\{\bar{w}_{t,i} = 0\}} \right) \xi_{s,i}(\bar{Q}_{s,i})1_{T_t},
\]

for \( i = 1, 2, \ldots, d \).

Proof (1) We first exhibit the decomposition of the penalty function \( \bar{\alpha}_t ; \) then we can guarantee the decomposition is well defined by showing that \( W_t(Q \otimes \psi, \bar{w}) \in \bar{W}_t \). Recall the definition of the minimal penalty function \( \bar{\alpha}_t \) from Corollary 2.11, i.e.,

\[
\bar{\alpha}_t(\bar{Q}, \bar{w}) = \bigcap_{Z \in \bar{A}_t} \left( \bar{E}_t^{\bar{Q}}[-Z] + \bar{\Gamma}_t(\bar{w}) \right) \cap \bar{M}_t
\]

\[
= \left\{ u \in \bar{M}_t : \bar{w}^T u \geq \text{ess sup}_{Z \in \bar{A}_t} \bar{w}^T \bar{E}_t^{\bar{Q}}[-Z] \bar{P}-a.s. \right\}.
\]
Consider now the decomposition of the acceptance set $\tilde{A}_t$ given in Corollary 3.3 as $\tilde{A}_t = \sum_{s=0}^{t-1} A_{R_s} I_{[s]} + A_{t \perp T_t}$. Additionally, assume $\tilde{Q}_i = Q_i \otimes \psi_i$ to be the decomposition provided in Corollary 2.3 for each component $i$ of the vector of measures $\tilde{Q} \in \tilde{M}_t(\mathbb{P})^d$. Recall from Corollary 2.3 that $\psi_{s,i} = \mu_s$ for every asset $i$ and time $s < t$. Additionally, and as a direct consequence of this equality, $\sum_{s=0}^{t-1} \psi_{s,i} < 1$ for every asset $i$. From this relation, it follows that

$$\tilde{\alpha}_t(\tilde{Q}, \tilde{w})$$

$$= \left\{ u \in \tilde{M}_t : \tilde{w}^T u \geq \sum_{s=0}^{t-1} \text{ess sup}_{Z_s \in A_{R_s}} \tilde{w}^T I_{I_{[s]}} \left[ -Z_s I_{[s]} \right] + \text{ess sup}_{Z \in A_t} \tilde{w}^T E_t \left[ -Z I_{T_t} \right] \right\}$$

$$= \left\{ u \in \tilde{M}_t : \sum_{s=0}^{t-1} \tilde{w}_s^T u_s I_{[s]} + \tilde{w}_t^T u_t I_{T_t} \right\} \geq \sum_{s=0}^{t-1} \text{ess sup}_{Z \in A_{R_s}} \tilde{w}_s^T (-Z_s) I_{[s]}

+ \text{ess sup}_{Z \in A_t} \tilde{w}_t^T E_t \left[ \sum_{s \in T_t} \frac{1}{1 - \sum_{r=0}^{t-1} \mu_r} \text{diag}(\psi_s)(-Z_s) \right] I_{T_t} \text{ a.s.}$$

$$= \sum_{s=0}^{t-1} \left\{ u_s \in M_s : \tilde{w}_s^T u_s \geq \text{ess sup}_{Z \in A_{R_s}} \tilde{w}_s^T (-Z_s) I_{[s]} \right\}

+ \left\{ u_t \in M_t : \tilde{w}_t^T u_t \geq \text{ess sup}_{Z \in A_t} \tilde{w}_t^T E_t \left[ \sum_{s \in T_t} \frac{1}{1 - \sum_{r=0}^{t-1} \mu_r} (\psi_s)(-Z_s) \right] I_{T_t} \right\}$$

Furthermore, by the construction of $(\tilde{Q}, w) = W_t(\tilde{Q}, \tilde{w})$,

$$\sum_{s \in T_t} w_s^T E_s[Y_s] = \tilde{w}_t^T E_t \left[ \sum_{s \in T_t} \frac{1}{1 - \sum_{r=0}^{t-1} \mu_r} \text{diag}(\psi_s)Y_s \right]$$

for any $Y \in \mathcal{R}_t^{\infty,d}$. Therefore the desired result follows if $(\tilde{Q}, w) \in \mathcal{W}_t$. But this holds by the relation $w_s^T(\tilde{Q}, w) = \frac{\mu_s}{1 - \sum_{r=0}^{t-1} \mu_r} \tilde{w}_s^T(\tilde{Q}, \tilde{w})_s \in L_s^1(\mathbb{R}_+^d)$.

(2) We first show that $\alpha_{R_s}$ is the minimal penalty function for $R_s$. We then consider the form of $\alpha_t$ as provided in the statement of the lemma; we accomplish this by proving that the appropriate representation holds and that $\tilde{W}_t(\tilde{Q}, w) \in \mathcal{W}_t$ so that the representation is well defined. First, recall the definition of the minimal penalty function $\alpha_{R_s}$ for the risk measure $R_s : L_\infty(\mathbb{R}^d) \to \mathcal{G}(M_s; M_{s,+})$ provided in Remark 2.14 as

$$\alpha_{R_s}(w) = \bigcap_{Z \in A_{R_s}} \left\{ (\ -Z + \Gamma_s(w)) \cap M_s \right\} = \left\{ u \in M_s : w^T u \geq \text{ess sup}_{Z \in A_{R_s}} w^T (-Z) \right\}$$
for any \( w \in L_1^d(\mathbb{R}^d) \setminus M_s^+ \). Using the construction of \( A_{R_s} = (\bar{A}_t)_s \) given in Corollary 3.3, the construction of \( \alpha_{R_s} \) in that lemma trivially holds because \((\bar{\mathcal{P}}, w[I_{\{s\}}]) \in \tilde{\mathcal{W}}_{t}\) by inspection. Second, recall the definition of the minimal penalty function \( \alpha_t \) for the risk measure \( \rho_t \) for processes given in Corollary 2.7 as

\[
\alpha_t(Q, w) = \bigcap_{Z \in A_t} \sum_{s \in T_t} \left( E_t^{Q_s} [-Z_s] + \Gamma_t (w_s) \right) \cap M_t
\]

\[
= \left\{ u \in M_t : \sum_{s \in T_t} w_s^T u \geq \text{ess sup} \sum_{Z \in A_t} w_s^T E_t^{Q_s} [-Z_s] \right\}
\]

\[
= \left\{ u \in M_t : \sum_{s \in T_t} w_s^T \text{diag}(I_{\{ws > 0\}}) u \geq \text{ess sup} \sum_{Z \in A_t} w_s^T \text{diag}(I_{\{ws > 0\}}) E_t^{Q_s} [-Z_s] \right\}
\]

for any \((Q, w) \in \mathcal{W}_t\). The last equality above follows from the construction of the eligible assets and \( w_s \geq 0 Q_s\)-a.s. By the construction of \((\bar{Q}, \bar{w}) = \bar{W}_t(Q, w)\),

\[
(\bar{w}^T \bar{E}_t^{\bar{Q}} [Y])_r = \sum_{s \in \mathbb{T}_t} w_s^T E_t^{Q_s} [Y_s] \quad \text{for all } r \in T_t,
\]

for any \( Y \in \bar{L}_\infty(\mathbb{R}^d)\). Therefore, using the construction of \( A_t \) given in Corollary 3.3 as \( A_t = \{ X \in \mathcal{R}_{\infty,d} : X[I_{\{T_s\}}] \in \bar{A}_t \} \), the definition of \( \alpha_t \) provided in that lemma holds as long as \( \mathcal{W}_t(Q, w) \in \mathcal{W}_t \). But the latter holds by the relation

\[
\bar{w}_t^T (\bar{W}_t(Q, w)) = \left( 1 - \sum_{r=0}^{t-1} \mu_r \right) \sum_{s \in \mathbb{T}_t} \mu_s^{-1} w_t^r (Q_s, w_s) \in \bar{L}_1(\mathbb{R}^d_+).
\]

**Remark 3.5** We highlight that in the full eligible space setting \( M = \mathbb{R}^d \) with the notation in Corollary 3.2, the penalty function representation simplifies; indeed, we obtain \( \alpha_{R_s}(\bar{w}_s) = \Gamma_s(\bar{w}_s) \) if \( \bar{w}_s \in C^*_s \), and \( \emptyset \) otherwise.

We additionally consider the relation between the dual representations for conditionally coherent risk measures. This is provided in the following result utilising the maximal set of dual variables.

**Corollary 3.6** Fix a time \( t \in \mathbb{T} \).

1. Let \( \rho_t : \mathcal{R}_{\infty,d}^t \rightarrow \mathcal{G}(M_t; M_{t,+}) \) be a closed conditionally coherent risk measure for processes and \( R_s : L_\infty^d(\mathbb{R}^d) \rightarrow \mathcal{G}(M_s; M_{s,+}), s < t \), a series of closed conditionally coherent risk measures. The associated closed conditionally coherent risk measure on the optional filtration, \( \bar{R}_t : \bar{L}_\infty^d(\mathbb{R}^d) \rightarrow \mathcal{G}(\bar{M}_t; \bar{M}_{t,+}) \) defined in (3.1), has

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a dual representation with respect to the set of dual variables
\[ \mathcal{W}^{\max}_t = \{(\bar{Q}, \bar{w}) \in \bar{W}_t : \bar{w}_s \in \mathcal{W}^{\max}_s, \forall s = 0, \ldots, t-1, W_t(\bar{Q}, \bar{w}) \in \mathcal{W}^{\max}_t \} . \]

(2) Conversely, consider a closed conditionally coherent risk measure
\[ \bar{R}_t : \bar{L}^\infty(\mathbb{R}^d) \to \mathcal{G}(\bar{M}_t; \bar{M}_t,+) \]
for vectors on the optional filtration. The associated closed conditionally coherent risk measure \( \rho_t : \mathcal{R}^{\infty,d}_s \to \mathcal{G}(M_s; M_s,+) \) for processes and series of closed conditionally coherent risk measures \( R_s : L^\infty_s(\mathbb{R}^d) \to \mathcal{G}(M_s; M_s,+) \) defined in (3.2) and (3.3), respectively, have dual representations with respect to the set of dual variables
\[ \mathcal{W}^{\max}_t = \{(Q, w) \in \bar{W}_t : \bar{W}_t(Q, w) \in \mathcal{W}^{\max}_t \} , \]
\[ \mathcal{W}^{\max}_s = \bigcup_{\bar{Q} \in \bar{W}_s(\bar{P})^d} \{ \bar{w}_s : (\bar{Q}, \bar{w}) \in \mathcal{W}^{\max}_s \} . \]

Proof This is an immediate consequence of Lemma 3.4 using the indicator formulation of the penalty functions, i.e.,
\[ \alpha_t(Q, w) = \begin{cases} \sum_{s \in \mathbb{T}_t} \Gamma_t(w_s) \cap M_t, & \text{if } (Q, w) \in \mathcal{W}^{\max}_t, \\ \emptyset, & \text{else}, \end{cases} \]
\[ \alpha_{R_s}(w_s) = \begin{cases} \Gamma_s(w_s) \cap M_s, & \text{if } w_s \in \mathcal{W}^{\max}_{R_s}, \\ \emptyset, & \text{else}, \end{cases} \]
\[ \bar{\alpha}_t(\bar{Q}, \bar{w}) = \begin{cases} \tilde{\Gamma}_t(\bar{w}) \cap \bar{M}_t, & \text{if } (\bar{Q}, \bar{w}) \in \mathcal{W}^{\max}_t, \\ \emptyset, & \text{else}. \end{cases} \]

We now use the results of this section to prove the dual representation for risk measures for processes presented in Corollary 2.7. Specifically, we take advantage of the known dual representation for set-valued risk measures for vectors (Corollary 2.11) and the equivalence of forms for the set-valued risk measures for processes and vectors presented above to prove this new dual representation for risk measures for processes.

Proof of Corollary 2.7 Let \( \rho_t : \mathcal{R}^{\infty,d}_t \to \mathcal{G}(M_t; M_t,+) \) be a closed conditionally convex risk measure for processes, and fix a sequence of closed and conditionally convex risk measures \( R_s : L^\infty_s(\mathbb{R}^d) \to \mathcal{G}(M_s; M_s,+) \) for \( s < t \). The existence and uniqueness of a closed conditionally convex risk measure \( \bar{R}_t : \bar{L}^\infty(\mathbb{R}^d) \to \mathcal{G}(\bar{M}_t; \bar{M}_t,+) \) on the optional filtration such that \( \rho_t(X) = \bar{R}_t(X|\mathbb{T}_t) \), for any \( X \in \mathcal{R}^{\infty,d}_t \) is guaranteed by Theorem 3.1. Consider also the decomposition of the penalty function from Lemma 3.4 to determine that \( \alpha_t(Q, w) = \bar{\alpha}_t(\bar{Q}, \bar{w}) \), for specifically constructed \( (\bar{Q}, \bar{w}) = \bar{W}_t(Q, w) \). Utilising the dual representation for risk measures for vectors as

\( \bar{\alpha}_t(\bar{Q}, \bar{w}) \).
provided in Corollary 2.11 gives

\[ \rho_t(X) = \bigcap_{(\bar{Q}, \bar{w}) \in \bar{W}_t} \left( \left( E_t^{\bar{Q}}[-X_{tI}] + \bar{\Gamma}_t(\bar{w}) \right) \cap M_t \right) \]

\[ = \bigcap_{(Q \otimes \psi, \bar{w}) \in \bar{W}_t} \left( \left( E_t^Q\left[-\sum_{s \in T_t} -\frac{\psi_s}{1 - \sum_{r=0}^{t-1} \psi_r} X_s \right] + \Gamma_t(w_s) \right) \cap M_t \right) \]

\[ - \cdot \bar{\alpha}_t(Q \otimes \psi, \bar{w}) \]

\[ = \bigcap_{(Q \otimes \psi, \bar{w}) \in \bar{W}_t} \left( \sum_{s \in T_t} \left( E_t^{\bar{Q}^s}[-X_s] + \Gamma_t(w_s) \right) \cap M_t \right) \]

where \((\hat{Q}, \hat{w}) = W_t(\hat{Q}, \hat{w})\) is defined as in Lemma 3.4, (1). To recover (2.3), it remains to show that

\[ \{(w^s_t(Q_s, w_s))_{s \in T_t} : (Q, w) \in \bar{W}_t \} = \{(w^s_t(W_t(\hat{Q}, \hat{w}), s))_{s \in T_t} : (\hat{Q}, \hat{w}) \in \bar{W}_t \} \]

First, by construction, we have \(\{W_t(\hat{Q}, \hat{w}) : (\hat{Q}, \hat{w}) \in \bar{W}_t\} \subseteq \bar{W}_t\), which implies “\(\supseteq\)” in the desired equality. Second, for any \((Q, w) \in \bar{W}_t\), there exists \(W_t(Q, w) \in \bar{W}_t\) (as constructed in Lemma 3.4, (2)), so that \(w^s_t(Q_s, w_s) = w^s_t(W_t(\hat{Q}, \hat{w}), s)\) for any \(s \in T_t\), which implies “\(\subseteq\)” in the desired equality. Thus the proof in the convex case is complete.

For the conditionally coherent case, define

\[ \beta_t(Q, w) = \begin{cases} \sum_{s \in T_t} \Gamma_t(w_s) \cap M_t, & \text{if ess inf}_{Z \in A_t} \sum_{s \in T_t} w^s_t E_t^{\bar{Q}^s} [Z_s] = 0 \mathbb{P}\text{-a.s.,} \\ \emptyset, & \text{else.} \end{cases} \]

Then from the positive homogeneity of \(\rho\), we have \(\beta_t(Q, w) \subseteq \alpha_t(Q, w)\) which yields

\[ \sum_{s \in T_t} \left( (E_t^{\bar{Q}^s}[-X_s] + \Gamma_t(w_s)) \cap M_t \right) \subseteq \sum_{s \in T_t} \left( (E_t^{\bar{Q}^s}[-X_s] + \Gamma_t(w_s)) \cap M_t \right) \]

Note that

\[ \sum_{s \in T_t} \left( E_t^{\bar{Q}^s}[-X_s] + \Gamma_t(w_s) \right) \cap M_t \]

\[ = \sum_{s \in T_t} \left( (E_t^{\bar{Q}^s}[-X_s] + \Gamma_t(w_s)) \cap M_t \right) \]

\[ \sum_{s \in T_t} \left( E_t^{\bar{Q}^s}[-X_s] + \Gamma_t(w_s) \right) \cap M_t \subseteq \emptyset = M_t \]
and therefore
\[ \rho_t(X) \subseteq \bigcap_{(Q,w) \in \mathcal{W}^\text{max}_t} \sum_{s \in T_t} \left( \mathbb{E}^Q_t \left[ -X_s + \Gamma_t(w_s) \right] \right) \cap M_t. \]

Finally, from (2.3), it follows that \( u_t \in \rho_t(X) \) if and only if
\[
\text{ess inf}_{(Q,w) \in \mathcal{W}_t} \left\{ \sum_{s \in T_t} w_s^T u_t + \text{ess sup}_{Z \in A_t} \sum_{s \in T_t} w_s^T \mathbb{E}^Q_t \left[ -Z_s \right] \right\} \geq 0.
\]

Assume \( u \in \bigcap_{(Q,w) \in \mathcal{W}^\text{max}_t} \sum_{s \in T_t} \left( \mathbb{E}^Q_t \left[ -X_s + \Gamma_t(w_s) \right] \right) \cap M_t \). Then immediately,
\[
\text{ess inf}_{(Q,w) \in \mathcal{W}^\text{max}_t \setminus \mathcal{W}_t} \left\{ \sum_{s \in T_t} w_s^T u_t + \text{ess sup}_{Z \in A_t} \sum_{s \in T_t} w_s^T \mathbb{E}^Q_t \left[ -Z_s \right] \right\} \geq 0.
\]

Additionally and clearly,
\[
\text{ess inf}_{(Q,w) \in \mathcal{W}^\text{max}_t \setminus \mathcal{W}_t} \left\{ \sum_{s \in T_t} w_s^T u_t + \text{ess sup}_{Z \in A_t} \sum_{s \in T_t} w_s^T \mathbb{E}^Q_t \left[ -Z_s \right] \right\} \geq 0.
\]

Hence the essential infimum over the full set \( \mathcal{W}_t \) of dual variables must also be bounded from below by 0, which implies that \( u_t \in \rho_t(X) \) and the equivalence of \( \rho_t(X) \) and its dual representation is shown. \( \square \)

## 4 Equivalence of multi-portfolio time-consistency for set-valued risk measures for processes and for vectors

Though multi-portfolio time-consistency of set-valued risk measures for processes was presented in Chen and Hu [10], we find in this work that a slight variation of that definition is useful. In particular, motivated by Theorem 3.1, we consider a joint definition of both a risk measure \( \rho_t \) for processes and a series of restricted conditional risk measures \( R_s \) for vectors.

**Definition 4.1** The pair \( (\rho_t, (R^r_s)_{s=0}^{r-1})_{r \in \mathbb{T}} \) is jointly multi-portfolio time-consistent if
1. \( \rho \) is multi-portfolio time-consistent as in Definition 2.8;
2. for any times \( t < s \), any \( X_r \in L^\infty_r(\mathbb{R}^d) \) and \( B_r \subseteq L^\infty_r(\mathbb{R}^d) \) for \( r \in [t, s) \), and any \( Z \in \mathcal{R}_r^\infty \), we have that
\[
R^s_r(X_r) \subseteq \bigcup_{Y_r \in B_r} R^s_r(Y_r), \quad \forall r \in [t, s)
\]
\[
\implies \rho_t \left( \sum_{r=t}^{s-1} X_r \mathbb{I}_{[r]} + Z \mathbb{I}_{[s]} \right) \subseteq \bigcup_{Y_{t-1} \in B_{t-1}} \cdots \bigcup_{Y_{s-1} \in B_{s-1}} \rho_t \left( \sum_{r=t}^{s-1} Y_r \mathbb{I}_{[r]} + Z \mathbb{I}_{[s]} \right);
\]
(3) for any times \( r < t < s \) and any \( X_r \in L^\infty_r(\mathbb{R}^d) \) and \( B_r \subseteq L^\infty_r(\mathbb{R}^d) \), we have

\[
R^s_r(X_r) \subseteq \bigcup_{Y_r \in B_r} R^s_r(Y_r) \implies R^t_r(X_r) \subseteq \bigcup_{Y_r \in B_r} R^t_r(Y_r).
\]

Conceptually, \((\rho, R)\) is jointly multi-portfolio time-consistent if in addition to (1) multi-portfolio time-consistency, it satisfies two additional consistency properties, namely (2) if two claims are identical after time \( s \in \mathbb{T} \) and the restricted risk measures (associated with \( \rho_s \)) on the claims at \( r \in [t, s) \) are ordered such that one is guaranteed to be (almost surely) more risky, then the claims as measured by \( \rho_t \) at time \( t \) must also satisfy the ordering of risks, i.e., the restricted risk measures are consistent in time with the risk measures for processes, and (3) the ordering of portfolios induced by the restricted risk measures is consistent at all times. This trivially holds if the restricted risk measure at time \( r \) is independent of the risk measure for processes with which it is associated, i.e., if \( R^s_r = R^t_r \) for \( r < t < s \); such a setting is discussed in Remark 4.3 below.

We now turn to the final main result of this work—the equivalence of multi-portfolio time-consistency for set-valued risk measures for processes and for vectors. This is again akin to the results presented for scalar risk measures in Acciaio et al. [1]. Fundamentally, these results can be used to construct the various equivalent formulations of multi-portfolio time-consistency as well; in particular, we highlight the recursive formulation presented in Feinstein and Rudloff [17] and Chen and Hu [10] and the cocycle condition for penalty functions in Feinstein and Rudloff [18].

**Theorem 4.2** (1) Consider a jointly multi-portfolio time-consistent pair \((\rho, R)\) of risk measures. The associated time-decomposable risk measure on the optional filtration, \( \tilde{R}_t : L^\infty(\mathbb{R}^d) \to \mathcal{U}(\tilde{M}_t; \tilde{M}_t, +) \) defined in (3.1), is multi-portfolio time-consistent.

(2) Conversely, consider a multi-portfolio time-consistent, time-decomposable risk measure \( \bar{R} \) for random vectors on the optional filtration. The associated risk measure \( \rho_t : \mathcal{R}^\infty_{t,d} \to \mathcal{U}(M_t; M_t, +) \) for processes and series of risk measures

\[
R_t^s : L^\infty_s(\mathbb{R}^d) \to \mathcal{U}(M_s; M_s, +),
\]

\( s = 0, \ldots, t - 1 \) for \( t = 0, \ldots, T \), defined in (3.2) and (3.3) are jointly multi-portfolio time-consistent.

**Proof** Throughout the proof, we fix times \( t < s \).

(1) Let \( B := \{(b_0, b_1, \ldots, b_{s-1}, b_s) : b_r \in B_r, r < s, b_s \in B_s\} \) for any sequence of sets \( B_r \subseteq L^\infty_r(\mathbb{R}^d) \) for \( r < s \) and \( B_s \subseteq \mathcal{R}^\infty_{s,d} \) and \( X \in L^\infty(\mathbb{R}^d) \) such that

\[
\bar{R}_s(X) \subseteq \bigcup_{Y \in B} \bar{R}_s(Y).
\]

By construction, this implies that \( R^s_r(X_r) \subseteq \bigcup_{Y_r \in B_r} R^s_r(Y_r) \) for every \( r < s \) and \( \rho_s(X|\mathcal{T}_r) \subseteq \bigcup_{Y_r \in B_r} \rho_s(Y_r|\mathcal{T}_r) \). Immediately, from the third condition of joint multi-portfolio time-consistency, we know that \( R^t_r(X_r) \subseteq \bigcup_{Y_r \in B_r} R^t_r(Y_r) \) for every \( r < t \),
and so it remains to show that $\rho_t(X_{[t,T]}^T) \subseteq \bigcup_{Y \in B} \rho_t(Y_{[t,T]}^T)$. But we have

$$\rho_t(X_{[t,T]}^T) \subseteq \bigcup_{Y_t \in B_t} \cdots \bigcup_{Y_{t-1} \in B_{t-1}} \rho_t\left( \sum_{r=t}^{s-1} Y_r \mathbb{I}_{[r]} + X_{T_s} \right)$$

$$\subseteq \bigcup_{Y_t \in B_t} \cdots \bigcup_{Y_{t-1} \in B_{t-1}} \bigcup_{Y_s \in B_s} \rho_t\left( \sum_{r=t}^{s-1} Y_r \mathbb{I}_{[r]} + Y_s \mathbb{I}_{[T_s]} \right) = \bigcup_{Y \in B} \rho_t(Y_{[t,T]}^T),$$

where the first inclusion follows from the second condition of joint multi-portfolio time-consistency, and the second inclusion from the first condition of joint multi-portfolio time-consistency. Therefore, by the construction of $\tilde{R}_t$, it immediately follows that $\tilde{R}_t(X) \subseteq \bigcup_{Y \in B} \tilde{R}_t(Y)$.

(2) Note first that since $(\tilde{R}_t)_{t \in \mathbb{T}}$ is time-decomposable, we have from Theorem 3.1 that

$$\tilde{R}_t(X) = \sum_{r=0}^{t-1} \tilde{R}_r(X_{[t,T]}^r) + \rho_t(\pi_{t,T}(X))_{[t,T]}$$

for any $t \in \mathbb{T}$. Here we only directly demonstrate the first condition of joint multi-portfolio time-consistency; the latter two properties follow from identical arguments. Fix $X \in \mathcal{R}_{\infty,d}$ and $B \subseteq \mathcal{R}_{\infty,d}$ such that

$$\rho_s(\pi_{s,T}(X)) \subseteq \bigcup_{Y \in B} \rho_s(\pi_{s,T}(Y)).$$

This implies by the decomposition of $\tilde{R}_s$ that

$$\tilde{R}_s(Z_{[t,s]}^T + X_{T_s}^T) \subseteq \bigcup_{Y \in B} \tilde{R}_s(Z_{[t,s]}^T + Y_{T_s}^T)$$

for any $Z \in \mathcal{R}_{\infty,d}$. Utilising multi-portfolio time-consistency of $\tilde{R}$, we recover

$$\tilde{R}_t(Z_{[t,s]}^T + X_{T_s}^T) \subseteq \bigcup_{Y \in B} \tilde{R}_t(Z_{[t,s]}^T + Y_{T_s}^T)$$

for any $Z \in \mathcal{R}_{\infty,d}$. By the decomposition of $\tilde{R}_t$, we immediately conclude that

$$\rho_t(Z_{[t,s]}^T + X_{T_s}^T) \subseteq \bigcup_{Y \in B} \rho_t(Z_{[t,s]}^T + Y_{T_s}^T)$$

for any $Z \in \mathcal{R}_{\infty,d}$, i.e., $\rho$ is multi-portfolio time-consistent.

From Theorems 3.1 and 4.2 and using results in Feinstein and Rudloff [18, 19], one can deduce some equivalent characterisations of multi-portfolio time-consistency for set-valued dynamic risk measures for processes, such as the cocycle condition on the sum of minimal penalty functions and a supermartingale relation.
Remark 4.3 Consider the setting in which all assets are eligible, i.e., $M = \mathbb{R}^d$. Let $\rho$ be a multi-portfolio time-consistent risk measure for processes (see Definition 2.8) and define $R_s : L_s^\infty(\mathbb{R}^d) \to \mathcal{U}(L_s^\infty(\mathbb{R}^d); L_s^\infty(\mathbb{R}^d_+))$ by

$$R_s(X_s) := -X_s + L_s^\infty(\mathbb{R}^d_+), \quad X_s \in L_s^\infty(\mathbb{R}^d).$$

By monotonicity, $(\rho, R)$ is jointly multi-portfolio time-consistent. Therefore, by Theorem 4.2, $\rho$ is multi-portfolio time-consistent if and only if the associated risk measure $\bar{R}$ for vectors, as defined in (3.1), is multi-portfolio time-consistent.

We conclude this discussion by remarking that in the present discrete-time setting, it is sufficient to consider multi-portfolio time-consistency defined for single time steps only through a sequential application of the recursive definition (i.e., by setting $s = t + 1$). This is discussed in e.g. Feinstein and Rudloff [18] as well.

5 Conclusion

In this work, we have demonstrated the equivalence between set-valued risk measures for processes with those for random vectors on the optional filtration. Such considerations allow the application of results from one stream of literature to the other. In particular, we highlight that set-valued risk measures for vectors constitute a more mature field with more results on dual representations and equivalent forms for multi-portfolio time-consistency. Herein, we have used this equivalence to prove a new dual representation for risk measures for processes. We also highlight that the equivalence of these risk measures allows the generalisation of e.g. the cocycle condition on the sum of minimal penalty functions or the supermartingale relation for multi-portfolio time-consistency for risk measures for vectors to risk measures for processes. We caution the reader that such extensions, while following the results of this work, require the use of a series of conditional risk measures for vectors on the original filtration, in addition to the risk measure for processes, which is not necessary if considering scalar risk measures as presented in Acciaio et al. [1].

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References

1. Acciaio, B., Föllmer, H., Penner, I.: Risk assessment for uncertain cash flows: model ambiguity, discounting ambiguity, and the role of bubbles. Finance Stoch. 16, 669–709 (2012)
2. Ararat, Ç., Rudloff, B.: Dual representations for systemic risk measures. Math. Financ. Econ. 14, 139–174 (2020)
3. Artzner, P., Delbaen, F., Eber, J.M., Heath, D.: Thinking coherently. Risk 10, 68–71 (1997)
4. Artzner, P., Delbaen, F., Eber, J.M., Heath, D.: Coherent measures of risk. Math. Finance 9, 203–228 (1999)
5. Artzner, P., Delbaen, F., Eber, J.M., Heath, D., Ku, H.: Coherent multiperiod risk adjusted values and Bellman’s principle. Ann. Oper. Res. 152, 5–22 (2007)
6. Barrieu, P., El Karoui, N.: Pricing, hedging, and designing derivatives with risk measures. In: Carmona, R. (ed.) Indifference Pricing: Theory and Applications, pp. 77–144. Princeton University Press, Princeton (2009)

7. Ben Tahar, I., Lépinette, E.: Vector-valued coherent risk measure processes. Int. J. Theor. Appl. Finance 17, 1450011 (2014)

8. Bion-Nadal, J.: Dynamic risk measures: time consistency and risk measures from BMO martingales. Finance Stoch. 12, 219–244 (2008)

9. Bion-Nadal, J.: Time consistent dynamic risk processes. Stoch. Process. Appl. 119, 633–654 (2009)

10. Chen, Y., Hu, Y.: Time consistency for set-valued dynamic risk measures for bounded discrete-time processes. Math. Financ. Econ. 12, 305–333 (2018)

11. Chen, Y., Hu, Y.: Set-valued dynamic risk measures for bounded discrete-time processes. Int. J. Theor. Appl. Finance 23, 2050017 (2020)

12. Cheridito, P., Delbaen, F., Kupper, M.: Coherent and convex risk measures for bounded càdlàg processes. Stoch. Process. Appl. 112, 1–22 (2004)

13. Cheridito, P., Delbaen, F., Kupper, M.: Coherent and convex risk measures for unbounded càdlàg processes. Finance Stoch. 9, 349–367 (2005)

14. Cheridito, P., Delbaen, F., Kupper, M.: Dynamic monetary risk measures for bounded discrete-time processes. Electron. J. Probab. 11, 57–106 (2006)

15. Cheridito, P., Kupper, M.: Composition of time-consistent dynamic monetary risk measures in discrete time. Int. J. Theor. Appl. Finance 14, 137–162 (2011)

16. Delbaen, F., Peng, S., Gianin, E.R.: Representation of the penalty term of dynamic concave utilities. Finance Stoch. 14, 449–472 (2010)

17. Feinstein, Z., Rudloff, B.: Time consistency of dynamic risk measures in markets with transaction costs. Quant. Finance 13, 1473–1489 (2013)

18. Feinstein, Z., Rudloff, B.: Multi-portfolio time consistency for set-valued convex and coherent risk measures. Finance Stoch. 19, 67–107 (2015)

19. Feinstein, Z., Rudloff, B.: A supermartingale relation for multivariate risk measures. Quant. Finance 18, 1971–1990 (2018)

20. Feinstein, Z., Rudloff, B.: Time consistency for scalar multivariate risk measures. Stat. Risk. Model. 38, 71–90 (2021)

21. Feinstein, Z., Rudloff, B., Weber, S.: Measures of systemic risk. SIAM J. Financ. Math. 8, 672–708 (2017)

22. Föllmer, H., Schied, A.: Convex measures of risk and trading constraints. Finance Stoch. 6, 429–447 (2002)

23. Frittelli, M., Rosazza Gianin, E.: Putting order in risk measures. J. Bank. Finance 26, 1473–1486 (2002)

24. Frittelli, M., Scandolo, G.: Risk measures and capital requirements for processes. Math. Finance 16, 589–612 (2006)

25. Hamel, A.H.: A duality theory for set-valued functions I: Fenchel conjugation theory. Set-Valued Var. Anal. 17, 153–182 (2009)

26. Hamel, A.H., Heyde, F., Löhne, A., Rudloff, B., Schrage, C.: Set optimization – a rather short introduction. In: Hamel, A., et al. (eds.) Set Optimization and Applications in Finance. Springer Proceedings in Mathematics & Statistics, vol. 151, pp. 65–141. Springer, Berlin (2015)

27. Hamel, A.H., Heyde, F., Rudloff, B.: Set-valued risk measures for conical market models. Math. Financ. Econ. 5, 1–28 (2011)

28. Jouini, E., Meddeb, M., Touzi, N.: Vector-valued coherent risk measures. Finance Stoch. 8, 531–552 (2004)

29. Molchanov, I.: Theory of Random Sets. Springer, Berlin (2005)

30. Riedel, F.: Dynamic coherent risk measures. Stoch. Process. Appl. 112, 185–200 (2004)

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