Abstract: The advent of novel nonlinear materials has stirred unprecedented interest in exploring the use of temporal inhomogeneities to achieve novel forms of wave control, amidst the greater vision of engineering metamaterials across both space and time. When the properties of an unbounded medium are abruptly switched in time, propagating waves are efficiently converted to different frequencies, and partially coupled to their back-propagating phase-conjugate partners, through a process called time-reversal. However, in realistic materials the switching time is necessarily finite, playing a central role in the resulting temporal scattering features. By identifying and leveraging the crucial role of electromagnetic momentum conservation in time-reversal processes, here we develop a general analytical formalism to quantify time-reversal due to temporal inhomogeneities of arbitrary profile. We deploy our theory to develop a formalism, dual to spatial tapering, that enables the tailoring of a desired time-reversal spectral response, demonstrating its use for the realization of broadband frequency converters and filters.

Keywords: adiabatic switching; metamaterials; tapering; temporal switching.

1 Introduction

Scattering is a key feature of wave propagation, occurring ubiquitously at the spatial interface between two media with different properties. It consists in the generation of reflected waves in the originating medium, and transmitted waves into the second medium. The temporal analogue of a spatial interface arises when the properties of an unbounded medium are abruptly and uniformly switched in time, and the associated phenomena have been attracting significant recent interest across various wave platforms [1, 2]. Such temporal interfaces exhibit remarkable differences compared to their spatial counterparts: in particular, frequency and energy are not conserved at these time boundaries, whereas momentum is. Coupling between positive and negative frequencies corresponds to time-reversal at these switching events, implying that a portion of the energy associated with waves initially propagating in one direction flips its propagation direction, travelling backwards while the wavevector is conserved [1–3]. Interference between transmitted and time-reversed waves at time-interfaces can enable highly exotic wave phenomena, at the basis of the field of time metamaterials [1, 4–8], including parametric amplification [9, 10], temporal aiming [11], topological phenomena within photonic time-crystals [12] and in synthetic frequency dimensions [13], non-Hermitian effects such as nonreciprocal gain [14], spectral causality [15] and temporal parity-time symmetry [16], temporal Anderson localization [17, 18], unitary energy transfer between resonators [19], as well as efficient [20] and broadband [21] absorbers, among several others [1]. Interesting opportunities for new physics can also be found at the interplay between temporal interfaces and material dispersion [22, 23]. While time-reversal is at the core of several of these phenomena, temporally reflected waves may be undesirable in many applications, in particular in the context of efficient frequency conversion. Antireflection coatings based on temporal multilayers, mimicking their spatial counterparts, have indeed been recently introduced to minimize the energy trapped into time-reversed waves [24–26].

Most research work in this area has so far been assuming that time interfaces are abrupt, i.e., that the time required to switch the material properties is negligible compared to the wave dynamics. However, in any realistic scenario the material response cannot be considered instantaneous, and in several instances the finite width of a temporal interface may become comparable with the period of the propagating signals, especially as we operate at higher frequencies. In addition, more interesting phenomena are observed at time interfaces involving a
large contrast of the material properties before and after the switching event, and a tradeoff between permittivity contrast and switching speed is naturally expected [27]. In this Letter, we analytically investigate temporal interfaces that follow a continuous evolution in time with arbitrary profile, and we deploy our formulation to unravel the unexplored opportunities arising when the material responses are not instantaneous. In particular, we demonstrate that the control of the temporal evolution may enable efficient frequency conversion in the temporal analogue of a Klopfenstein taper [28]. Whilst we restrict ourselves to Maxwell’s equations, the principles we invoke here are general, and our results can be extended to other wave realms. Our findings illuminate the role of momentum conservation in temporal scattering, shedding new light on the duality between spatial reflection and time-reversal in realistic settings.

2 Results

We are commonly used to writing Maxwell’s equations in the frequency domain by assuming harmonic time dependence, since in static linear media frequency is conserved across spatial interfaces. At time interfaces, on the contrary, spatial momentum is conserved and not frequency, therefore it is convenient to consider spatially harmonic $e^{ikz}$ fields in space, where $k$ is the wavenumber and $z$ is the propagation coordinate. Under this assumption, the displacement field $D$ and magnetic induction $B$ in a generally time-varying homogeneous medium obey at any point in space the temporal analogue of the telegrapher’s equations:

$$\begin{align*}
\frac{\partial B}{\partial t} &= -Z(t)D, \\
\frac{\partial D}{\partial t} &= -Y(t)B,
\end{align*}$$

(1)

where $Z = ik/\varepsilon(t)$ and $Y = ik/\mu(t)$, and $\varepsilon$ and $\mu$ are the permittivity and permeability of the material. At a time interface, $D$ and $B$ are continuous [29], hence

$$\begin{align*}
D(t^+ &= (T + R)D(t^-) \\
B(t^+) &= Z_0(t)(T - R)B(t^-),
\end{align*}$$

(2)

where $R$ and $T$ are the scattering coefficients, respectively, associated with the backward (time-reversed) and forward waves generated at the time interface, and $Z_0(t) = \sqrt{Z(t)/Y(t)}$ is the wave impedance. We can invert Eq. (2) to yield the ratio $R/T$ as a function of the local wave impedance $B/D$ at instant $t$:

$$\rho(t) = R(t)/T(t) = \frac{Z_0(t) - B(t)/D(t)}{Z_0(t) + B(t)/D(t)}.$$  

(3)

Using this result, we can derive a general solution for $\rho(t)$ as a function of an arbitrary time-modulation profile of the material properties. Dividing the first of Eq. (1) by $B$ and the second by $D$ and taking their difference, we find

$$\frac{\partial}{\partial t}(\ln B/D) = -\frac{Z(t)}{(B/D)} + Y(t)(B/D).$$  

(4)

Combining (3) and (4), after some algebra we obtain

$$\frac{\partial}{\partial t} \ln Z_0 - \frac{2}{1 - \rho^2} \frac{\partial \rho}{\partial t} + \frac{4\gamma \rho}{1 - \rho^2} = 0,$$  

(5)

with $\gamma = \sqrt{2ZY} = ik/\sqrt{\varepsilon \mu}$. Eq. (5) can be linearized assuming $\rho^2 \ll 1$. This is a safe assumption in most realistic scenarios, and it does not impose a limit on the switching speed itself, which can in principle be arbitrarily fast, but more generally on the ratio between $R$ and $T$, which can only approach 1 in the limits where the final impedance $Z_0(t \to \infty) \to \{0, \infty\}$. This yields

$$\frac{\partial \rho}{\partial t} - 2\gamma(t)\rho = F(t),$$  

(6)

with $F(t) = \frac{1}{2}\frac{\partial}{\partial t} \ln Z_0$. The general solution is

$$\rho(t) = \int_{-\infty}^{t} F(t')e^{2\gamma(t')\rho}dt',$$  

(7)

which defines the ratio $R/T$ of time-reversed over transmitted signals in time for arbitrary variations of the material properties through $Z_0(t)$, with $i\Phi(t) = \frac{1}{2} \int_{-\infty}^{t} \varepsilon(t')\mu(t')|{-}1/2|dt'$. In contrast with the case of a spatial interface, here we are concerned with the ratio $\rho = R/T$ because at time-interfaces the energy is not conserved, so $|T|$ and $|R|$ can both become arbitrarily large (1). Conservation of the total electromagnetic momentum $P$ in the medium, which is ensured by translational invariance, allows us to derive the actual time-reversal and transmission magnitudes. Assuming, without loss of generality, that only forward propagating waves are initially present, the total momentum density before and after an arbitrary time variation going from $Z_1$ to $Z_2$ is $P_1 = Z_1|D_1|^2$ and $P_2 = Z_2(|T|^2 - |R|^2)|D_1|^2$, respectively. Conservation of momentum therefore requires

$$|T|^2 - |R|^2 = Z_1/Z_2.$$  

(8)
As a result, whilst |T| and |R| can both change arbitrarily, their difference must remain constant. Note that we made no assumption here on the temporal variation \( Z_s(t) \), so this result holds for any form of temporal switching as long as it is carried uniformly across the spatial extent of the wave. Combining this result with Eq. (3) yields

\[
|T|^2 = \frac{Z_0}{Z_2} \frac{1}{1 - |\rho_2|^2};
|R|^2 = \frac{Z_0}{Z_2} \frac{|\rho_2|^2}{1 - |\rho_2|^2},
\]

where \( \rho_2 \) is the ratio \( \rho(t) \) at the end of the switching process.

### 2.1 Temporal scattering from a sigmoidal step

In the case of a step-like switching profile of permittivity, modeled as a sigmoid function with rise time \( \tau \) of the form

\[
\varepsilon(t) = \left( \varepsilon_1 + \frac{\varepsilon_2 - \varepsilon_1}{2} \right) \tanh(t/\tau) + \frac{\varepsilon_2 - \varepsilon_1}{2},
\]

and a static permeability \( \mu = 1 \), the phase in Eq. (7) can be explicitly written as

\[
i\Phi_s(t) = \int_t^\infty \frac{2ik}{\sqrt{A+B}\tanh(\tau/\tau')} \frac{1}{\sqrt{A+B}} dt',
\]

where \( A = \varepsilon_1 + \delta \varepsilon / 2 \), \( B = \delta \varepsilon / 2 \).

![Figure 1](image)

**Figure 1:** Top row: Permittivity profile \( \varepsilon(t) \) and \( \frac{1}{\varepsilon} \frac{d\varepsilon}{dt} \) for a sigmoidal step with rise time \( \tau/\Theta = 0.05 \) (left), 0.2 (center) and 0.35 (right), where \( \Theta = 2\pi/\omega_1 \) is the period of the incoming wave. The input frequency is \( \omega_1 = 1 \). Second row: total displacement field \( \Re[D_{tot}] \) and squared modulus \( |D|^2 \) through the switching process. Third row: amplitude of the forward (blue, \( \Re[e^{i\omega(t-\omega_2t)}] \)) and backward (red, \( \Re[e^{-i\omega(t-\omega_2t)}] \)) wave at the end of the modulation process. In order to be able to demonstrate the sharp-step case with high accuracy, the results in this figure are computed numerically using the routine in [30]. The two methods are compared in Figure 2.
exact Maxwell’s Equations. Indeed, the decay in amplitude of the time-reversed wave, associated with a more adiabatic temporal transition, is accompanied by a corresponding reduction in forward amplitude. The phenomenon is the dual of an adiabatic spatial interface, as in a tapered waveguide transition that suppresses unwanted reflections. Indeed, we find that the reflection coefficient rapidly converges to zero in the range in which the transition time is comparable to the period of the input wave, and for slower temporal transitions, for which no backward wave is excited, |T(τ → ∞)|^2 → Z_1/Z_2 following Eq. (8). Details of our efficient numerical scheme used here are provided in [30]. Minor discrepancies between analytical and numerical results <2% can be observed for larger reflections, due to the small-reflection approximation in Eq. (6).

2.2 Temporal taper design

Our analytical formulation yields a particularly interesting result in the isorefractive scenario, i.e., as we vary the impedance Z_0 but not γ. Such a scenario may be envisioned, for instance, if we vary in time the distance between two parallel plates for transverse electromagnetic wave propagation. In this case, the frequency of the wave remains constant through the temporal transition, and Φ(τ) = 2ωt in Eq. (7). Thus, assuming an arbitrary switching profile occurring from time t = 0 to t = T, we can write

\[ ρ = \int_0^T F(t')e^{2iωt'} dt', \]  

which can be inverted to yield

\[ F(t) = \frac{1}{π} \int_{-∞}^{∞} \rho(ω)e^{2iωt} dω. \]  

Equation (12) explicitly returns the temporal impedance profile required to synthesize a desired frequency dependence of ρ(ω) [28, 31], ideally suited, for instance, to tailor the bandwidth over which the time-reversed wave is minimized (or maximized) at will.

We can use this result to explore the optimal temporal profile that maximizes the bandwidth over which temporal reflections are suppressed for a given duration of the switching profile. The spatial analogue of this problem is known as the Klopfenstein taper [28], which describes the optimal spatial profile that maximizes the bandwidth over which reflections stay below a desired minimum value for a given taper length. In our temporal scenario, we rigorously solve this problem in [30], deriving the optimal temporal profile

\[ \ln Z_0 = \frac{1}{2} \ln(Z_1Z_2) + \frac{ρ_0}{\cos h(A)} A^2 φ(2t/T - 1, A), \]  

where A quantifies the bandwidth ωT ≥ A over which the reflection coefficient is below ρ_{max} = ρ_0/cosh(A), and ρ_0 = 1/2 ln(Z_1/Z_2), and the special function φ(x, y) is defined in [30]. More details on the derivation and implication of this explicit formula for the optimal switching profile can also be found in [30]. In the more general non-isorefractive scenario, we can use our general formulation to derive numerically the optimal switching profile, yielding a similar result to (13) when the index contrast is small.
Generally, our formulation enables the design of ultrafast switching profiles for optimal broadband frequency conversion with minimal back-reflections. In Figure 3 we compare the response of a non-isoreflective temporal Klopfenstein taper (blue) with $A/2\pi T_0 \approx 3$, where $T_0 = (\varepsilon_1 \varepsilon_3)^{1/4}/4$ is the duration of the switching profile, and maximum ripples $R_{\text{max}} \approx 0.0014$, to the one of a quarter-wave (QW) anti-reflection temporal coating (red) [26] – the temporal analogue of a conventional anti-reflection coating, consisting of two abrupt temporal interfaces delayed by a quarter period (within the middle-layer) of the lowest possible target frequency (see [30] for details). As seen in the figure, both temporal profiles feature the same initial and final permittivity $\varepsilon_1 = 1$ and $\varepsilon_2 = 2$, and same total duration, and they are both aimed at suppressing the time-reversed signals within the same frequency range. Panel (a) shows the two temporal profiles, while (b) shows the calculated $|R|$ for the QW (red) and Klopfenstein (blue) cases as a function of input frequency. The two filters are designed to work for an incoming frequency $\omega/2\pi = 5$. In [30] we further investigate the optimal trade-off between bandwidth and reflection suppression with equal reflection peaks in the pass-band enabled by the Klopfenstein taper.

The difference between the QW temporal slab and the Klopfenstein taper can be appreciated when the input wave is a broadband pulse. In Figure 3c we show the normalized spectral distribution of the incoming (dashed black line, plotted for reference out of scale) and time-reversed (continuous lines) waves computed via FDTD for a relatively broadband pulse with carrier frequency $\omega_c/2\pi = 5$ and full-width-at-half-maximum (FWHM) $\Delta \omega/2\pi = 1$ (shaded in yellow), for the two scenarios. The Klopfenstein taper produces hardly any backward wave for this pulse, leading to a pure frequency translation of $\approx 30\%$, while the QW temporal slab can only suppress time-reversal over a much narrower bandwidth around the target frequency, clearly yielding a lower efficiency. Panels (d, e) show the initial (d) and final (e) spatial field distributions for this pulse excitation, while panels (f, g) show the temporal signal at

![Figure 3](image-url)
the two probes shown in panel (d), demonstrating the superior performance of the Klopfenstein taper. The employed profile is indeed optimal to maximize the bandwidth over which reflection is minimal for this time interface and its duration, as detailed in [30], where we study other taper profiles and compare their performance.

3 Conclusions

In this work, we introduced a rigorous and general analytic formulation to model temporal scattering for arbitrary switching profiles of homogeneous media, and deployed it to investigate the interplay between the finite timescale of a continuous time-switching process and the temporal variations of the impinging wave. As an application, we demonstrated how the profile of temporal switching can be tailored to control the temporal reflection in order to maximize the efficiency of time-reversal processes in ultrafast modulation setups at any frequency. Amidst the current multidisciplinary interest in exploiting mixing processes in time varying media, our findings outline the importance of considering, and possibly controlling, the switching speed and its temporal profile: the consecutive accumulation of reflection amplitude and phase throughout a continuous switching process can dictate whether time-reversal will be maximized or suppressed. We envisage potential implementations of temporal tapers by using varicaps at radio frequencies, piezoelectric elements in elastodynamics, and gravity modulation through vertical acceleration for water waves. Our findings set the stage for future investigations of time-reversal, frequency conversion and mixing in photonics, electromagnetics, acoustics and other wave systems undergoing temporal switching of arbitrary form, with relevant implications also in the growing area of Floquet condensed matter.

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Supplemental document: See [30] for supporting content on fast numerical solutions, Klopfenstein taper design and the associated trade-offs between bandwidth and reflection ripples, analytical details on our derivation, and comparisons between different tapering strategies.

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