A DAMPED FLOW FOR IMAGE DENOISING

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Abstract. In this paper, we introduce a new image denoising model: the damped flow (DF), which is a second order nonlinear evolution equation associated with a class of energy functionals of image. The existence, uniqueness and regularization property of DF are proven. For the numerical implementation, based on the Störmer-Verlet method, a discrete damped flow, SV-DDF, is developed. The convergence of SV-DDF is studied as well. Several numerical experiments, as well as a comparison with other methods, are provided to demonstrate the feasibility and effectiveness of the SV-DDF.

1. Introduction. Digital images play a significant role in many fields in science, industry, and daily life, such as computer tomography, magnetic resonance imaging, geographical information systems, astronomy, satellite television, etc. Data sets collected by image sensors are always contaminated by noise. Instrument precision, the absence of some acquisition channels, and interfering natural phenomena can all degrade the data information. Moreover, noise can be introduced by transmission errors, compression and artificial editing. Therefore, it is necessary to apply a denoising technique on the original noisy image before it is analyzed.

Over the last few decades, scientists have developed numerous techniques to achieve adaptive imaging denoising, such as wavelets [15], stochastic approaches [33], and formulations based on partial differential equations [41]. We refer to [19, 38] for a review on various denoising methods.

An essential challenge for imaging denoising is to remove noise as much as possible without eliminating the most representative characteristics of the image, such as edges, corners and other sharp structures. Traditional denoising methods are given some information about the noise, but the problem of blind image denoising involves computing the denoised image from the noisy one without any knowledge of the noise. The energy functional approach has in recent years been very successful in blind image denoising, most often taking the form

\[ E(u) = \frac{1}{2} \int_\Omega (u - u_0)^2 dx + \alpha \int_\Omega \Phi(|\nabla u|) dx, \]

where \( u_0(x) \) is the observed (noisy) image, and \( \Omega \subset \mathbb{R}^N \) (\( N = 2, 3 \)) is a bounded domain with almost everywhere \( C^2 \) smooth boundary \( \partial \Omega \). The first term in (1)

2010 Mathematics Subject Classification. Primary: 35A01,35A02,65P10,65M12,65M32.
Key words and phrases. Nonlinear flow, image denoising, regularization, damped Hamiltonian system, symplectic method.

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is a fidelity term, the second term is a regularization term, and $\alpha > 0$ is the regularization parameter. The regularization term $\Phi(|\nabla u|)$ is usually assumed to be strictly convex.

A well-studied case of $E(u)$ is when the regularization term is the $p$-Dirichlet energy, i.e., $\Phi(|\nabla u|) = \frac{1}{p}|\nabla u|^p, p \geq 1$. The case $p = 1$ corresponds to the Total Variation (TV) principle [31] and $p > 1$ has been studied in, e.g., [7, 28]. Nowadays, there are many relevant extensions of the TV model, e.g., [8, 9, 10, 13]. The extension of TV to variational tensor-based formulations was investigated in [20, 30, 36]. Other relevant extensions of the energy functional can be found in [4, 5].

The Euler-Lagrange equation $\frac{\partial E}{\partial u} = 0$ associated with the functional $E(u)$ is given by

$$\begin{cases}
    u - u_0 + \alpha \cdot \text{div} \left( \frac{\Phi'(|\nabla u|)}{|\nabla u|} \nabla u \right) = 0 & \text{in } \Omega, \\
    \partial_n u = 0 & \text{on } \partial \Omega,
\end{cases}$$

(2)

where $n$ is the outward normal to the boundary $\partial \Omega$. For the $p$-Dirichlet energy, let $\alpha \to 0$, and we obtain the first order flow

$$\begin{cases}
    u_t - \Delta_p u = 0 & \text{in } (0, T) \times \Omega, \\
    u(x, 0) = u_0(x) & \text{in } \Omega, \\
    \partial_n u = 0 & \text{on } \partial \Omega,
\end{cases}$$

(3)

where the $p$-Laplace operator is defined by $\Delta_p u = \text{div} (|\nabla u|^{p-2} \nabla u)$. The $p$-parabolic equation in (3) has been studied intensively in [14, 29, 32, 35, 42], and references therein.

In this work, we are interested in the following second order flow

$$\begin{cases}
    u_{tt} + \eta u_t - \Delta_p u = 0 & \text{in } (0, T) \times \Omega, \\
    u(x, 0) = u_0(x) & \text{in } \Omega, \\
    u_t(x, 0) = 0 & \text{in } \Omega, \\
    \partial_n u = 0 & \text{on } \partial \Omega,
\end{cases}$$

(4)

where the constant $\eta > 0$ is the so-called damping parameter.

Let us explain why we are considering the second order flow (4). Denote by $V(u) = \int_\Omega \frac{1}{p}|\nabla u|^p dx$ the $p$-Dirichlet integral. Then, the first order flow in (3), i.e. $u_t + \partial_n V(u) = 0$, can be considered a classical steepest descent flow for solving the optimization problem $\min_u V(u)$. In the last two decades, there has been increasing evidence found showing that second order flows also enjoy remarkable optimization properties. Among these, a particularly important dynamical system (4), or $u_{tt} + \eta u_t + \partial_n V(u) = 0$, is called the Heavy Ball with Friction system (HBF) [6] because of its mechanical interpretation. This system is an asymptotic approximation of the equation describing the motion of a material point with positive mass, subjected to stay on the graph of $V(u)$, and which moves under the action of the gravity force, the reaction force, and the friction force ($\eta > 0$ is the friction parameter). The introduction of the inertial term $\dot{p}(t)$ in the dynamical system permits it to overcome some of the drawbacks of the steepest descent method. By contrast with steepest descent methods, the HBF system is not a descent method. It is the global energy (kinetic+potential) which decreases. The optimization properties for the HBF system have been studied in detail in [2, 3, 6], and references therein. Numerical algorithms based on the HBF system of solving some special problems, e.g., large systems of linear equations, eigenvalue problems, nonlinear Schrödinger problems, inverse source problems, etc., can be found in [16, 17, 21, 37, 43], where we can see that a second order damped system solved by a symplectic solver is far
more efficient than numerically solving a first order system. In this study, we focus on the regularity of the specific system (4) and its denoising capability.

The remainder of the paper is structured as follows: in Section 2, a regularized damped flow is introduced, and its regularity (existence and uniqueness) is proven. Section 3 briefly discusses the regularization property of the dynamical solution with (4). Based on the Störmer-Verlet method, a discrete damped flow, termed by SV-DDF, is proposed in Section 4, where the convergence property of SV-DDF is studied. Section 5 presents an algorithm for image denoising. Several numerical examples are presented in Section 6 to demonstrate the feasibility and efficiency of the proposed method. A comparison with other methods is provided as well. Finally, concluding remarks are given in Section 7.

2. A regularized damped flow and its regularity. In order to overcome the ill-posedness of the formulation (4), we introduce a regularized damped flow

\[
\begin{align*}
\left\{ \begin{array}{l}
\varepsilon u_t + \eta u_t - \text{div} \left( (\varepsilon + |G_\sigma \star u|^2)^{\frac{p-2}{2}} \nabla u \right) = 0 \quad \text{in} \ (0, T) \times \Omega, \\
u(x, 0) = u_0(x) \quad \text{in} \ \Omega, \\
u_t(x, 0) = 0 \quad \text{in} \ \Omega, \\
\partial_\nu u = 0 \quad \text{on} \ \partial \Omega.
\end{array} \right.
\end{align*}
\tag{5}
\]

where \(\varepsilon\) is a fixed small number and \(G_\sigma(x)\), \(\sigma > 0\), is the Gaussian kernel \(G_\sigma(x) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{|x|^2}{2\sigma^2}}\) and \(\nabla G_\sigma \star u = \left( \sum_{j=1}^{N} \left( \frac{\partial G_\sigma}{\partial x_j} \star u \right)^2 \right)^{1/2}\). The aim of this section is to show the existence and uniqueness of the system (5). To the best of our knowledge, it is the first time that the regularity of a nonlinear system of this type (5) has been studied.

2.1. Preliminaries. In this subsection, a brief description of the mathematical principles and some of the definitions used in this work is provided.

Let us first introduce notations for the function spaces, since the well posedness of (5) can be investigated only under the fixed functional spaces. We denote by \(H^k(\Omega)\), where \(k\) is a positive integer, and the set of all functions \(u\) defined in \(\Omega\) is such that its distributional derivatives \(D^s = \partial^s u / \partial x^s\) of order \(|s| = \sum_{i=1}^{N} s_i \leq k\) all belong to \(L^2(\Omega)\). Furthermore, \(H^k(\Omega)\) is a Hilbert space with the norm \(\|u\|_{H^k(\Omega)} = \left( \sum_{s \leq k} \int_{\Omega} |D^s u|^2 \, dx \right)^{1/2}\). The space \(L^p(0, T; H^k(\Omega))\) consists of all functions \(u\) such that for almost every \(t \in (0, T)\), the element \(u(t)\) belongs to \(H^k(\Omega)\). Hence, \(L^p(0, T; H^k(\Omega))\) is a normed space with the norm \(\|u\|_{L^p(0, T; H^k(\Omega))} = \left( \int_0^T \|u(t)\|^p_{H^k(\Omega)} \, dt \right)^{1/p}\), where \(p > 1\). We also denote by \(L^\infty(0, T; X)\) the set of all functions \(u\) such that for almost every \(t \in (0, T)\) the element \(u(t)\) belongs to \(X\). \(L^\infty(0, T; X)\) is a normed space with the norm \(\|u\|_{L^\infty(0, T; X)} = \inf\{|C|; \|u(t)\|_X \leq C, \text{ a.e. on } (0, T)\}\). We denote by \(H^1(\Omega)^*\) the dual space of \(H^1(\Omega)\). In the following, let \(C_i\) denote a constant with a different value at a different place. It does not depend on the estimated quality. Moreover, to simplify the notation, we put

\[ u^x(\nabla G_\sigma \star u) = (\varepsilon + |\nabla G_\sigma \star u|^2)^{\frac{p-2}{2}}, \quad p \in [1, 2], \tag{6} \]

and sometimes let \(u_t(t) = u_t'(t)\).

Next, we introduce the solution space \(U\) for the problem (5).
Definition 2.1. We say that an element $u$ belongs to the solution space $\mathcal{U}$ for the problem (5) if $u \in L^\infty(0,T;H^1(\Omega))$, and its derivatives $u'$ and $u''$, with respect to $t$ in the sense of distributions to the spaces $L^\infty(0,T;L^2(\Omega))$ and $L^\infty(0,T;H^1(\Omega)^*)$ respectively.

It is easily seen that $\mathcal{U}$ is a Banach space equipped with the norm

$$
\|u\|_\mathcal{U} = \|u\|_{L^\infty(0,T;H^1(\Omega))} + \|u'\|_{L^\infty(0,T;L^2(\Omega))} + \|u''\|_{L^\infty(0,T;H^1(\Omega)^*)}.
$$

The solutions for the problem (5) are considered in the weak sense, as follows

**Definition 2.2.** A function $u$ is called a weak solution of the problem (5) if $u \in \mathcal{U}$ and satisfies (5) for almost every $t \in (0,+\infty)$ with derivatives of $u$ in the sense of distributions.

We will show the existence of weak solutions for problem (5) by using the Schauder fixed point theorem; see [11, 12]. In the sequel, we need the following results for the corresponding linear problem [18].

$$
\begin{cases}
\begin{align*}
&u_t + \eta u_t - \text{div}(b(x,t)\nabla u) = 0 \quad \text{in } (0,T) \times \Omega, \\
&u(x,0) = u_0(x) \quad \text{in } \Omega, \\
&u_t(x,0) = 0 \quad \text{in } \Omega, \\
&\partial_n u = 0 \quad \text{on } \partial\Omega.
\end{align*}
\end{cases}
$$

where $b(x,t)$ is a given function such that $b(x,t) \geq c > 0$ and $c$ is a constant.

**Theorem 2.3.** Suppose that $b(x,t)$ is bounded, and let $u_0 \in H^1(\Omega)$. Then the problem in (7) has a unique solution $u \in \mathcal{U}$. Moreover, if $u_0 \in H^2(\Omega)$, then it follows that $u_t \in L^\infty(0,T;H^1(\Omega))$.

The linear problem (7) is by now well-studied and Theorem 2.3 can be proven by the Galerkin method [18].

2.2. Existence and uniqueness of the damped flow. Before providing the main result, let us consider the following lemma.

**Lemma 2.4.** Assume that for all $x \in \Omega$ and a.e. $t \in (0,T)$ there exists a positive constant $C$ depending on $G_\sigma$ and $\Omega$ such that

$$
\begin{align*}
&\|(\nabla G_\sigma \ast v)\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C\|u_0\|_{H^1(\Omega)}, \\
&\|(\nabla G_\sigma \ast v_t)\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C\|u_0\|_{H^1(\Omega)}.
\end{align*}
$$

Then, the following inequalities hold for $p \in [1,2]$ and

$$
\begin{align*}
&(\varepsilon + C^2\|u_0\|_{H^1(\Omega)}^2)\frac{p^2-2}{p} \leq a^\varepsilon (\nabla G_\sigma \ast v) \leq \varepsilon \frac{p^2-2}{p}, \quad (8) \\
&|a^\varepsilon (\nabla G_\sigma \ast v)| \leq (2-p)NC^2\varepsilon^\frac{p^4}{p^2-2}\|u_0\|_{H^1(\Omega)}^2. \quad (9)
\end{align*}
$$

**Proof.** Inequalities (8) hold obviously by noting the definition of $a^\varepsilon$ in (6) and the following inequalities

$$
\varepsilon^{\frac{2-p}{p}} \leq (\varepsilon + |\nabla G_\sigma \ast v|^2)^{\frac{2-p}{2}} \leq (\varepsilon + C^2\|u_0\|_{H^1(\Omega)}^2)^{\frac{2-p}{p}}
$$

for $p \in [1,2]$, and for all $x \in \Omega$ and a.e. $t \in (0,T)$.

Now, consider the inequality (9). Since

$$
a^\varepsilon (\nabla G_\sigma \ast v) = \frac{p-2}{2} (\varepsilon + |\nabla G_\sigma \ast v|^2)^{\frac{p^4}{p}} \sum_{j=1}^N 2 \left( \frac{\partial G_\sigma}{\partial x_j} \ast v \right) \left( \frac{\partial G_\sigma}{\partial x_j} \ast v_t \right),
$$
we can deduce that
\[
|\alpha(\nabla G_{\sigma} \ast v)| = (2 - p) \left( \varepsilon + |\nabla G_{\sigma} \ast v|^2 \right)^{\frac{p-2}{2}} \left| \sum_{j=1}^{N} \left( \frac{\partial G_{\sigma}}{\partial x_j} \ast v \right) \left( \frac{\partial G_{\sigma}}{\partial x_j} \ast v_{t} \right) \right|
\leq (2 - p) \varepsilon^{\frac{p-2}{2}} \cdot N \cdot C^2 \|u_0\|_{H^1(\Omega)}^2,
\]
by noting the conditions of the lemma.

Now, we are in a position to show our main result.

**Theorem 2.5.** Assume that \( p \in [1, 2] \) and \( u_0 \in H^2(\Omega) \). Then a unique weak solution to problem (5) exists if \( T \) is sufficiently small with an upper bound depending on \( \|u_0\|_{H^1(\Omega)}, G_{\sigma} \) and \( \Omega \).

**Proof.** We use the Schauder fixed point theory \([11, 12]\) to prove the existence. Let \( v \in U \) be such that
\[
\|v\|_{L^\infty(0,T;L^2(\Omega))} + \|v_t\|_{L^\infty(0,T;L^2(\Omega))} \leq C_1 \|u_0\|_{H^1(\Omega)}
\]  
(10)
where the positive constant \( C_1 \) will be determined later. Then the elements \( \nabla G_{\sigma} \ast v \) and \( \nabla G_{\sigma} \ast v_{t} \) belong to \( L^\infty(0,T;C^\infty(\Omega)) \), and for all \( x \in \Omega \) and a.e. \( t \in (0,T) \) a positive constant \( C_2 \) depending on \( G_{\sigma} \) and \( \Omega \) exists such that
\[
\|\nabla G_{\sigma} \ast v\|_{L^\infty(0,T;L^2(\Omega))} \leq C_1 C_2 \|u_0\|_{H^1(\Omega)},
\]
\[
\|\nabla G_{\sigma} \ast v_{t}\|_{L^\infty(0,T;L^2(\Omega))} \leq C_1 C_2 \|u_0\|_{H^1(\Omega)}.
\]
By Lemma 2.4, for all \( x \in \Omega \) and a.e. \( t \in (0,T) \), it follows that
\[
(\varepsilon + C_1^2 C_2^2 \|u_0\|_{H^1(\Omega)}^2)^{\frac{p-2}{2}} \leq \alpha(\nabla G_{\sigma} \ast v) \leq \varepsilon^{\frac{p-2}{2}},
\]
(11)
\[
|\alpha(\nabla G_{\sigma} \ast v)| \leq (2 - p)N(C_1 C_2)^2 \varepsilon^{\frac{p-2}{2}} \|u_0\|_{H^1(\Omega)}^2.
\]
(12)
Let \( v \in U \) satisfy (10) and consider the problem \( P_v \):
\[
\langle u_{tt}, \varphi \rangle_{H^1(\Omega)^* \times H^1(\Omega)} + \int_{\Omega} (\eta u_t \varphi + a^\sigma(\nabla G_{\sigma} \ast v) \nabla u \cdot \nabla \varphi) \, dx = 0
\]
(13)
for every element \( \varphi \in H^1(\Omega) \), a.e. \( t \in (0,T) \). The linear problem \( P_v \) in (13) is well-posed \([18]\) and has a solution \( u_v \) which satisfies
\[
\|u_v\|_{H^1(\Omega)} \leq C_3 \|u_0\|_{H^1(\Omega)},
\]
(14)
where \( C_3 \) is a positive constant, depending only on the domain \( \Omega \).

Now, let us consider two cases when \( \varphi = u_v \) and \( \varphi = u \) respectively.

**Case 1.** For \( \varphi = u_v \), we have
\[
\langle u_{tt}, u_v \rangle_{H^1(\Omega)^* \times H^1(\Omega)} + \int_{\Omega} (\eta u_t u_v + a^\sigma(\nabla G_{\sigma} \ast v) \nabla u \cdot \nabla u_v) \, dx = 0,
\]
which gives
\[
\frac{1}{2} \int_{\Omega} (u_t u_v') dx + \int_{\Omega} a^\sigma(\nabla G_{\sigma} \ast v) \nabla u \cdot \nabla u_v' \, dx = 0,
\]
or equivalently
\[
\frac{d}{dt} \|u_v\|_{L^2(\Omega)}^2 = -2 \int_{\Omega} (\nabla u_v \nabla u_v') \, dx - 2 \int_{\Omega} a^\sigma(\nabla G_{\sigma} \ast v) \nabla u \cdot \nabla u_v' \, dx.
\]
Denote by $I_1 = -2 \int_\Omega a^\varepsilon (\nabla G_\sigma \ast v) \nabla u_v \cdot \nabla u'_v \, dx$ and, integrated with the above equation, we obtain

$$
||u'_v(t)||^2_{L^2(\Omega)} = -||\nabla u'_v(t)||^2_{L^2(\Omega)} - 2\eta \int_0^t ||u'_v||^2_{L^2(\Omega)} \, d\tau + \int_0^t I_1 \, d\tau,
$$

where we used $u_{vt}(x,0) = 0$ and $\nabla u_{vt}(x,0) = 0$. Hence,

$$
||u'_v(t)||^2_{L^2(\Omega)} \leq -2\eta \int_0^t ||u'_v||^2_{L^2(\Omega)} \, d\tau + \int_0^t I_1 \, d\tau \quad (15)
$$

On the other hand, by the inequalities (11), (12) and (14), we have

$$
\int_0^t \int_\Omega a^\varepsilon (\nabla G_\sigma \ast v) \nabla u_v \cdot \nabla u_v \, dx \, d\tau \\
\leq (2 - p) N(C_1 C_2)^2 \varepsilon^{\frac{p-1}{p}} ||u_0||^2_{H^1(\Omega)} T ||\nabla u_v||^2_{L^2(\Omega)} \\
\leq (2 - p) NT(C_1 C_2)^2 \varepsilon^{\frac{p-1}{p}} ||u_0||^2_{H^1(\Omega)} ||u_v||^2_{H^1(\Omega)} \\
\leq (2 - p) NT(C_1 C_2 C_3)^2 \varepsilon^{\frac{p-1}{p}} ||u_0||^4_{H^1(\Omega)} \quad (16)
$$

and

$$
- \int_0^t \int_\Omega \frac{d}{dt} [a^\varepsilon (\nabla G_\sigma \ast v) \nabla u_v \cdot \nabla u_v] \, dx \, dt \\
= [a^\varepsilon (\nabla G_\sigma \ast v) \nabla u_v \cdot \nabla u_v]_{\tau=t} - [a^\varepsilon (\nabla G_\sigma \ast v) \nabla u_v \cdot \nabla u_v]_{\tau=0} \\
\leq 2\varepsilon^{\frac{p-2}{2}} C_3^2 ||u_0||^2_{H^1(\Omega)} \quad (17)
$$

Since

$$
I_1 = -2 \int_\Omega a^\varepsilon (\nabla G_\sigma \ast v) \nabla u_v \cdot \nabla u'_v \, dx \\
= \int_\Omega a^\varepsilon (\nabla G_\sigma \ast v) \nabla u_v \cdot \nabla u_v \, dx - \int_\Omega \frac{d}{dt} [a^\varepsilon (\nabla G_\sigma \ast v) \nabla u_v \cdot \nabla u_v] \, dx,
$$

by combining (16) and (17), one can deduce that

$$
\int_0^t I_1 \, d\tau \leq (2 - p) NT(C_1 C_2 C_3)^2 \varepsilon^{\frac{p-1}{p}} ||u_0||^4_{H^1(\Omega)} + 2\varepsilon^{\frac{p-2}{2}} C_3^2 ||u_0||^2_{H^1(\Omega)} \quad (18)
$$

Furthermore, using Grönwall’s lemma in (15), we get

$$
||u'_v(t)||^2_{L^2(\Omega)} \leq \int_0^t I_1 \, d\tau + \int_0^t \left[ \left( \int_0^\tau I_1 \, ds \right) (-2\eta)e^{\int_0^{\tau} (-2\eta) \, ds} \right] \, d\tau \quad (19)
$$

The second term in the right-hand side of the above inequality can be rewritten as

$$
2\eta \int_0^t \int_\Omega \frac{d}{dt} [a^\varepsilon (\nabla G_\sigma \ast v) \nabla u_v \cdot \nabla u_v] \, dx - \int_\Omega a^\varepsilon (\nabla G_\sigma \ast v) \nabla u_v \cdot \nabla u_v \, dx \right] ds) \\
\cdot e^{-2\eta \tau} \, d\tau = J_1 + J_2.
$$

It follows that

$$
|J_1| = 2\eta \int_0^t \left[ \int_\Omega a^\varepsilon (\nabla G_\sigma \ast v) \nabla u_v \cdot \nabla u_v \, dx \right] e^{-2\eta \tau} \, d\tau \\
\leq 4\eta \int_0^t \left[ \varepsilon^{\frac{p-2}{2}} C_3^2 ||u_0||^2_{H^1(\Omega)} \right] e^{-2\eta \tau} \, d\tau \leq 2\varepsilon^{\frac{p-2}{2}} C_3^2 ||u_0||^2_{H^1(\Omega)}
$$
by noting inequality (11). Similarly, by (12) and (14), one can deduce that

\[ |J_2| = 2\eta \int_0^T \left( \int_0^T \left( \int_0^T \nabla G_\sigma \cdot \nabla u_v \cdot \nabla u_v \, dx \right) \right) ds e^{-2\eta r} \, dt \]
\[ \leq 2\eta \int_0^T \left[ \left( 2(p - 2)N(C_1C_2)\varepsilon \frac{e^{2s}}{s} \|u_0\|^2_{H^1(\Omega)} \|\nabla u_v\|^2_{L^2(\Omega)} \right) \right] e^{-2\eta r} \, dt \]
\[ \leq 2(2 - p)N\eta(C_1C_2C_3)^2\varepsilon \frac{e^{2s}}{s} \|u_0\|^4_{H^1(\Omega)} \int_0^T \tau e^{-2\eta r} \, dt \]
\[ \leq 2(2 - p)N\eta(C_1C_2C_3)^2\varepsilon \frac{e^{2s}}{s} \|u_0\|^4_{H^1(\Omega)}. \]

Putting the above two inequalities for \(|J_i|, i = 1, 2\), and the inequality (18) in the estimate (19), we derive

\[ \|u_v'(t)\|^2_{L^2(\Omega)} \leq (2 - p)N(TC_1C_2C_3)^2\varepsilon \frac{e^{2s}}{s} \|u_0\|^4_{H^1(\Omega)} + 2\varepsilon \frac{e^{2s}}{s} C_3^2 \|u_0\|^2_{H^1(\Omega)} \]
\[ + 2\varepsilon \frac{e^{2s}}{s} C_3^2 \|u_0\|^2_{H^1(\Omega)} + 2(2 - p)N\eta(C_1C_2C_3)^2\varepsilon \frac{e^{2s}}{s} \|u_0\|^4_{H^1(\Omega)} \]
\[ \leq 4\varepsilon \frac{e^{2s}}{s} C_3^2 + (2 - p)N(C_1C_2C_3)^2\varepsilon \frac{e^{2s}}{s} \|u_0\|^2_{H^1(\Omega)}(T + 2\eta), \] (20)

and we obtain

\[ \|u_v'(t)\|^2_{L^2(\Omega)} \leq C\|u_0\|^2_{H^1(\Omega)}. \] (21)

**Case 2.** Let \( \varphi = u_v \) in (13), and we obtain

\[ \langle u_v', u_v \rangle_{H^1(\Omega)} + \int_\Omega \eta u_v' u_v \, dx + \int_\Omega \alpha^2 (\nabla G_\sigma \ast v) \nabla u_v \cdot \nabla u_v \, dx = 0, \]

which is equivalent to

\[ \frac{1}{2} \frac{d^2}{dt} \|u_v\|^2_{H^1(\Omega)} - \|u_v'\|^2_{L^2(\Omega)} + \frac{1}{2} \eta \int_\Omega \frac{d}{dt} |u_v|^2 \, dx + \int_\Omega \alpha^2 (\nabla G_\sigma \ast v) \nabla u_v \cdot \nabla u_v \, dx = 0 \]

by noting

\[ \langle u_v', u_v \rangle_{H^1(\Omega)} = \frac{1}{2} \frac{d^2}{dt} \|u_v\|^2_{H^1(\Omega)} - \|u_v'\|^2_{L^2(\Omega)}. \]

Denote \( I_2 = -2 \int_\Omega \alpha^2 (\nabla G_\sigma \ast v) \nabla u_v \cdot \nabla u_v \, dx \), and we obtain

\[ \frac{d^2}{dt} \|u_v\|^2_{H^1(\Omega)} = 2\|u_v'\|^2_{L^2(\Omega)} - \eta \frac{d}{dt} \|u_v\|^2_{L^2(\Omega)} + I_2 \]

We integrate the above identity twice and get

\[ \|u_v\|^2_{H^1(\Omega)} = 2 \int_0^t \int_0^\tau \|u_v'\|^2_{L^2(\Omega)} \, ds \, d\tau - \eta \int_0^t (\|u_v(\tau)\|^2_{L^2(\Omega)} - \|u_v(0)\|^2_{L^2(\Omega)}) \, d\tau \]
\[ + \int_0^t \int_0^\tau I_2 \, ds \, d\tau \] (22)

or equivalently

\[ \|u_v\|^2_{L^2(\Omega)} = -\|\nabla u_v\|^2_{L^2(\Omega)} + 2 \int_0^t \int_0^\tau \|u_v'\|^2_{L^2(\Omega)} \, ds \, d\tau \]
\[ - \eta \int_0^t (\|u_v(\tau)\|^2_{L^2(\Omega)} + 2\|u_v(0)\|^2_{L^2(\Omega)} + \int_0^t \int_0^\tau I_2 \, ds \, d\tau. \]

(23)
By inequalities (11) and (14), we have
\[ |I_2| = 2 \int_\Omega \alpha^\varepsilon (\nabla G_\sigma \ast v) \nabla u_v \cdot \nabla u_v \, dx \leq 2\varepsilon \frac{\varepsilon^2}{2} C_3^2 \| u_0 \|^2_{H^1(\Omega)}. \]
Furthermore, it follows by (21) that
\[ \sup_{0 \leq t \leq T} \| u'_v(t) \|^2_{L^2(\Omega)} \leq C \| u_0 \|^2_{H^1(\Omega)}. \]  
(24)
Hence, by ignoring the non-positive terms in the right-hand side of (23) we obtain
\[ \| u_v \|^2_{L^2(\Omega)} \leq 2T^2 C \| u_0 \|^2_{H^1(\Omega)} + \eta T \| u_0 \|^2_{H^1(\Omega)} + 2\varepsilon \frac{\varepsilon^2}{2} C_3^2 T^2 \| u_0 \|^2_{H^1(\Omega)} \]
\[ = T \left( 2TC + \eta + 2\varepsilon \frac{\varepsilon^2}{2} C_3^2 T \right) \| u_0 \|^2_{H^1(\Omega)}. \]  
(25)
Inserting \( u_v \) into (10) to obtain
\[ \| u_v \|_{L^\infty(0,T;L^2(\Omega))} + \| u_{\text{ext}} \|_{L^\infty(0,T;L^2(\Omega))} \leq C_1 \| u_0 \|_{H^1(\Omega)}. \]  
(26)
By combining (21), (25) and (27), we obtain that
\[ \| u_v \|_{L^\infty(0,T;L^2(\Omega))} + \| u_{\text{ext}} \|_{L^\infty(0,T;L^2(\Omega))} \leq C_1 \| u_0 \|_{H^1(\Omega)} \]  
(27)
provided that
\[ \sqrt{2T^2 C + \eta T + 2\varepsilon \frac{\varepsilon^2}{2} C_3^2 T^2} + \sqrt{C} \leq C_1. \]
Hence, it is sufficient to show that there exists \( C_1 > 0 \) such that
\[ 2T^2 C + \eta T + 2\varepsilon \frac{\varepsilon^2}{2} C_3^2 T^2 + C \leq C_1^2, \]
which is equivalent to
\[ \eta T + 2(5T^2 + 2)\varepsilon \frac{\varepsilon^2}{2} C_3^2 \]
\[ \leq \left[ 1 - (2 - p)N(C_2 C_3)^2 \varepsilon \frac{\varepsilon^2}{2} \right] \| u_0 \|^2_{H^1(\Omega)} (2T^2 + 1)(T + 2\eta) \]
by noting the definition of \( C \) in (20). Hence, we have to require that
\[ (2 - p)N(C_2 C_3)^2 \varepsilon \frac{\varepsilon^2}{2} \| u_0 \|^2_{H^1(\Omega)} (2T^2 + 1)(T + 2\eta) < 1. \]
This is easily fulfilled if we choose
\[ T < \frac{1}{\sqrt{2(2-p)N(C_2 C_3)^2 \varepsilon \frac{\varepsilon^2}{2} \| u_0 \|^2_{H^1(\Omega)}}}. \]
On the other hand, by equation (22) we obtain
\[ \| \nabla u_v \|^2_{L^2(\Omega)} = -\| u_v \|^2_{L^2(\Omega)} + 2 \int_0^T \int_0^\tau \| u'_v \|^2_{L^2(\Omega)} \, ds \, d\tau - \eta \int_0^\tau \| u_v(\tau) \|^2_{L^2(\Omega)} \, d\tau + T\eta \| u_0 \|^2_{L^2(\Omega)} + \int_0^\tau \int_0^\tau I_2 \, ds \, d\tau. \]
Since \( I_2 = -2 \int_\Omega \alpha^\varepsilon (\nabla G_\sigma \ast v) \nabla u_v \cdot \nabla u_v \, dx \leq -2\varepsilon \| \nabla u_v \|^2_{L^2(\Omega)} \leq 0 \), using inequalities (24) and (25), we can deduce that
\[ \| \nabla u_v \|^2_{L^2(\Omega)} \leq 2 \int_0^\tau \int_0^\tau \| u'_v \|^2_{L^2(\Omega)} \, ds \, d\tau + T\eta \| u_0 \|^2_{L^2(\Omega)} \]
\[ \leq 2T^2 C \| u_0 \|^2_{H^1(\Omega)} + \eta T \left( 2T^2 C + \eta T + 2\varepsilon \frac{\varepsilon^2}{2} C_3^2 T^2 \right) \| u_0 \|^2_{H^1(\Omega)} \]
\[ = \left( 2(1 + \eta T^2)T^2 C + \eta^2 T^2 + 2\varepsilon \frac{\varepsilon^2}{2} \eta T^3 C_3^2 \right) \| u_0 \|^2_{H^1(\Omega)} \]  
(28)
Hence, we can define
\[ C_4 = \sqrt{2(1 + \eta T^2)T^2C + \eta^2 T^2 + 2\varepsilon \frac{\eta T^3}{\lambda^2} C_3^2}, \]
and note that \( C_4 \) is independent on \( t \), and we obtain
\[ \|\nabla u_0\|_{L^\infty(0, T; L^2(\Omega))} \leq C_4. \]  
(29)

Now, let \( \varphi \) in (13) such that \( \|\varphi\|_{H^1(\Omega)} = 1 \), and we obtain
\[ \langle u''_0, \varphi \rangle_{H^1(\Omega) \times H^1(\Omega)} = -\eta \int_\Omega u'_0 \varphi \, dx - \int_\Omega a^c (\nabla G_\sigma + v) \nabla u_0 \cdot \nabla \varphi \, dx \leq -\eta \int_\Omega u'_0 \varphi \, dx - (\varepsilon + C^2 \|u_0\|_{H^1(\Omega)}^2) \frac{\varepsilon + 2}{\lambda^2} \int_\Omega \nabla u_0 \cdot \nabla \varphi \, dx \leq \eta \|u'_0\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} + (\varepsilon + C^2 \|u_0\|_{H^1(\Omega)}^2) \frac{\varepsilon + 2}{\lambda^2} \|\nabla u_0\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)} \leq \|u_0\|_{H^1(\Omega)}^2 \left( \eta C + (\varepsilon + C^2 \|u_0\|_{H^1(\Omega)}^2) \frac{\varepsilon + 2}{\lambda^2} \right). \]

by noting inequalities (21) and (28). This implies that
\[ \|u''_0\|_{H^1(\Omega)} = \sup_{\|\varphi\|_{H^1(\Omega)} = 1} \langle u''_0, \varphi \rangle_{H^1(\Omega) \times H^1(\Omega)} \leq C_5. \]

Since constant \( C_5 \) is independent on \( t \), we obtain
\[ \|u''_0\|_{L^\infty(0, T; H^1(\Omega))} \leq C_5. \]  
(30)

From (27), (29) and (30), we introduce the subspace \( U_0 \) of \( U \) defined by
\[ U_0 = \{ v \in U; v \text{ satisfies (5) in the sense of distribution,} \]
\[ \|v\|_{L^\infty(0, T; L^2(\Omega))} + \|v'\|_{L^\infty(0, T; L^2(\Omega))} \leq C_1 \|u_0\|_{H^1(\Omega)}, \]
\[ \|\nabla u_0\|_{L^\infty(0, T; L^2(\Omega))} \leq C_4, \quad \|u''_0\|_{L^\infty(0, T; H^1(\Omega))} \leq C_5 \} \]

It follows by construction that \( P : v \mapsto u_v \) is a mapping from \( U_0 \) to \( U_0 \). Furthermore, it can be shown that \( U_0 \) is a nonempty, convex and weakly compact subset of \( U \). We want to use Schauder’s fixed point theorem and need to prove that \( P : v \mapsto u_v \), with a weakly continuous mapping from \( U_0 \) to \( U_0 \). Let \( v_j \) be a sequence that converges weakly to some \( v \) in \( U_0 \) and let \( u_j = u_{v_j} \). We have to prove that \( u_j = P(v_j) \) converges weakly to \( u_v = P(v) \). From (29) and (30), classical results of compact inclusion in Sobolev spaces [1], we can deduce from \( v_j \) and \( u_j \) respectively a subsequence such that for some \( u \), we have
- \( v_j \to v \) in \( L^2(0, T; L^2(\Omega)) \) and a.e. on \( \Omega \times (0, T) \),
- \( \partial_{x_k} G_\sigma * v_j \to \partial_{x_k} G_\sigma * v \) in \( L^2(0, T; L^2(\Omega)) \) and a.e. on \( \Omega \times (0, T) \), \( k = 1, 2, \ldots, n \),
- \( a^c (\partial_{x_k} G_\sigma * v_j) \to a^c (\partial_{x_k} G_\sigma * v) \) in \( L^2(0, T; L^2(\Omega)) \) and a.e. on \( \Omega \times (0, T) \),
- \( u_j \to u \) weakly * in \( L^\infty(0, T; H^1(\Omega)) \),
- \( u_j' \to u' \) weakly * in \( L^\infty(0, T; L^2(\Omega)) \),
- \( u_j'' \to u'' \) weakly * in \( L^2(0, T; H^1(\Omega)) \),
- \( u_j \to u \) in \( L^2(0, T; L^2(\Omega)) \) and a.e. on \( \Omega \times (0, T) \),
- \( \partial_{x_k} u_j \to \partial_{x_k} u \) weakly * in \( L^\infty(0, T; L^2(\Omega)) \), \( k = 1, 2, \ldots, n \),
- \( u_j(0) \to u_0 \) in \( L^2(\Omega) \),
- \( u_j'(0) \to 0 \) in \( H^1(\Omega) \).

Hence, we can define \( u = P(v) \) as the limit in the problem \( P_{v_j} \). Moreover, \( P \) is weakly continuous, since the sequence \( u_j = P(v_j) \) converges weakly in \( U_0 \) to a unique element \( u = P(v) \). By the Schauder fixed point theorem, there exists \( v \in U_0 \)
such that \( v = P(v) = u_v \) showing that the element \( u_v \) solves the problem (5).

**Uniqueness.** We proceed as in [18, 11]. Let \( u_1 \) and \( u_2 \) be two weak solutions of (5). Denote by \( a_i = a^\varepsilon(\nabla G_\varepsilon * u_i) \), where \( a^\varepsilon(\cdot) \) is defined in (6). Then for a.e. \( t \in (0, T) \) we obtain

\[
(u_1 - u_2)'' + \eta(u_1 - u_2)' - \text{div}(a_1 \nabla (u_1 - u_2)) = \text{div}((a_1 - a_2) \nabla u_2),
\]

subject to the initial condition

\[
(u_1 - u_2)(x, 0) = 0, \quad (u_1 - u_2)_t(x, 0) = 0, \quad x \in \Omega,
\]

and the boundary condition

\[
\partial_n (u_1 - u_2) = 0, \quad x \in \partial\Omega, \quad t \in (0, T)
\]

in the distribution sense.

It suffices to show that \( u_1 - u_2 = 0 \). Now, fix \( s \in (0, t) \) and let

\[
v_k(t) = \begin{cases} \int_t^s u_k(\tau) \, d\tau, & 0 < t \leq s \\ 0, & s \leq t < T \end{cases}
\]

for \( k = 1, 2 \). Then for every \( t \in (0, T) \), we have that \( v_k(t) \in H^1(\Omega) \) and \( \partial_n v_i = 0 \) on \( \partial\Omega \). Multiplying (31) by \( v_1 - v_2 \) and integrating, we obtain

\[
\int_0^s \int_\Omega (u_1 - u_2)''(v_1 - v_2) + \eta(u_1 - u_2)'(v_1 - v_2) - \text{div}(a_1 \nabla (u_1 - u_2))(v_1 - v_2) \, dx \, dt
\]

\[= \int_0^s \int_\Omega \text{div}((a_1 - a_2) \nabla u_2)(v_1 - v_2) \, dx \, dt.\]

Applying integration by parts with respect to the time variable, we obtain

\[-\int_0^s \int_\Omega (u_1 - u_2)'(v_1 - v_2)' \, dx \, dt - \eta \int_0^s \int_\Omega (u_1 - u_2)(v_1 - v_2)' \, dx \, dt
\]

\[+ \int_0^s \int_\Omega (a_1 \nabla (u_1 - u_2)) \cdot \nabla (v_1 - v_2) \, dx \, dt
\]

\[= -\int_0^s \int_\Omega (a_1 - a_2) \nabla u_2 \cdot \nabla (v_1 - v_2) \, dx \, dt\]

by noting the initial condition (32) and the fact \( v_i(s) \equiv 0 \).

If we set \( v'_k = -u_k \) in the above equation, we obtain

\[
\int_0^s \int_\Omega (u_1 - u_2)'(u_1 - u_2) \, dx \, dt + \eta \int_0^s \int_\Omega |u_1 - u_2|^2 \, dx \, dt
\]

\[+ \int_0^s \int_\Omega (a_1 \nabla (v_1 - v_2)') \cdot \nabla (v_1 - v_2) \, dx \, dt
\]

\[= -\int_0^s \int_\Omega (a_1 - a_2) \nabla u_2 \cdot \nabla (v_1 - v_2) \, dx \, dt,\]
or equivalently
\[
\frac{1}{2} \int_{\Omega} |u_1 - u_2|^2 \, dx + \eta \int_0^s \int_{\Omega} |u_1 - u_2|^2 \, dx \, dt \\
- \frac{1}{2} \int_{\Omega} a_1 |\nabla (v_1 - v_2)|^2 \big|_{t=s} \, dx + \frac{1}{2} \int_{\Omega} a_1 |\nabla (v_1 - v_2)|^2 \big|_{t=0} \, dx \\
+ \frac{1}{2} \int_0^s \int_{\Omega} a'_1 |\nabla (v_1 - v_2)|^2 \, dx \, dt \\
= - \int_0^s \int_{\Omega} (a_1 - a_2) \nabla u_2 \cdot \nabla (v_1 - v_2) \, dx \, dt
\]

Since \( \nabla v_k(s) \equiv 0 \), one can deduce that
\[
\frac{1}{2} \int_{\Omega} |u_1 - u_2|^2 \, dx + \eta \int_0^s \int_{\Omega} |u_1 - u_2|^2 \, dx \, dt + \frac{1}{2} \int_{\Omega} a_1 |\nabla (v_1 - v_2)|^2 \big|_{t=0} \, dx \\
= - \int_0^s \left( \int_{\Omega} (a_1 - a_2) \nabla u_2 \cdot \nabla (v_1 - v_2) \, dx \right) \, dt - \frac{1}{2} \int_0^s \int_{\Omega} a'_1 |\nabla (v_1 - v_2)|^2 \, dx \, dt
\]

Denote \( C_6 = (\varepsilon + C_1^2 C_2^2 \|u_0\|_{H^1(\Omega)}^2)^{\frac{p-2}{2}} \) and \( C_7 = (2-p)N(C_1 C_2)^2 \varepsilon^{\frac{p-2}{2}} \|u_0\|_{H^1(\Omega)}^2 \)
and by inequalities (11) and (12) we get
\[
\frac{1}{2} \int_{\Omega} |u_1 - u_2|^2 \, dx + \eta \int_0^s \int_{\Omega} |u_1 - u_2|^2 \, dx \, dt + \frac{C_6}{2} \int_{\Omega} |\nabla (v_1 - v_2)|^2 \big|_{t=0} \, dx \\
\leq \int_0^s \left( \|a_1 - a_2\|_{L^\infty(\Omega)} \left( \int_{\Omega} |\nabla u_2|^2 \, dx \right)^{1/2} \left( \int_{\Omega} |\nabla (v_1 - v_2)|^2 \, dx \right)^{1/2} \right) \, dt \\
+ \frac{C_7}{2} \int_0^s \int_{\Omega} |\nabla (v_1 - v_2)|^2 \, dx \, dt
\]

Since \( G_\sigma \) is smooth, there is for every \( p \geq 1 \) a positive constant \( C_{p,\sigma} \) depending only on \( p \) and \( \sigma \) such that
\[
\|a_1 - a_2\|_{L^\infty(\Omega)} \leq C_{p,\sigma} \|u_1 - u_2\|_{L^2(\Omega)}
\]

We have
\[
\frac{1}{2} \int_{\Omega} |u_1 - u_2|^2 \, dx + \eta \int_0^s \int_{\Omega} |u_1 - u_2|^2 \, dx \, dt + \frac{C_6}{2} \int_{\Omega} |\nabla (v_1 - v_2)|^2 \big|_{t=0} \, dx \\
\leq \int_0^s \left( C_{p,\sigma} \left( \int_{\Omega} |u_1 - u_2|^2 \, dx \right)^{1/2} \left( \int_{\Omega} |\nabla u_2|^2 \, dx \right)^{1/2} \left( \int_{\Omega} |\nabla (v_1 - v_2)|^2 \, dx \right)^{1/2} \right) \, dt \\
+ \frac{C_7}{2} \int_0^s \int_{\Omega} |\nabla (v_1 - v_2)|^2 \, dx \, dt
\]
Since $u \in H^1(\Omega)$, then a positive constant $C_8$ exists such that
\[
\frac{1}{2} \int_\Omega |u_1 - u_2|^2 \, dx + \eta \int_0^s \int_\Omega |u_1 - u_2|^2 \, dx \, dt + \frac{C_6}{2} \int_\Omega |\nabla(v_1 - v_2)|^2|t=0 \, dx \\
\leq \int_0^s \left( C_{p,c} C_8 \left( \int_\Omega |u_1 - u_2|^2 \, dx \right)^{1/2} \left( \int_\Omega |\nabla(v_1 - v_2)|^2 \, dx \right)^{1/2} \right) \, dt \\
+ \frac{C_7}{2} \int_0^s \int_\Omega |\nabla(v_1 - v_2)|^2 \, dx \, dt
\]

Applying Young’s inequality, a positive constant $C_{10}$ exists such that
\[
\frac{1}{2} \int_\Omega |u_1 - u_2|^2 \, dx + \eta \int_0^s \int_\Omega |u_1 - u_2|^2 \, dx \, dt + \frac{C_6}{2} \int_\Omega |\nabla(v_1 - v_2)|^2 \, dx \\
\leq C_{10} \int_0^s \left( \int_\Omega |u_1 - u_2|^2 \, dx + \int_\Omega |\nabla(v_1 - v_2)|^2 \, dx \right) \, dt \tag{34}
\]

Define
\[
w_k(t) = \int_0^t u_k(\tau) \, d\tau, \quad 0 < t < T, \quad k = 1, 2.
\]

Then (34) can be rewritten as
\[
\frac{1}{2} \int_\Omega |u_1 - u_2|^2 \, dx + \eta \int_0^s \int_\Omega |u_1 - u_2|^2 \, dx \, dt + \frac{C_6}{2} \int_\Omega |\nabla(w_1 - w_2)|^2 \, dx \tag{35}
\]

Using the inequality
\[
||\nabla(w_1-w_2)(s)-\nabla(w_1-w_2)(t)||_{L^2(\Omega)}^2 \leq 2||\nabla(w_1-w_2)(s)||_{L^2(\Omega)}^2+2||\nabla(w_1-w_2)(t)||_{L^2(\Omega)}^2
\]
we have
\[
\frac{1}{2} \int_\Omega |u_1 - u_2|^2 \, dx + \eta \int_0^s \int_\Omega |u_1 - u_2|^2 \, dx \, dt + \frac{C_6}{2} \int_\Omega |\nabla(w_1 - w_2)|^2 \, dx \tag{36}
\]

Using the inequality
\[
0 \leq C_{10} \int_0^s \left( \int_\Omega |u_1 - u_2|^2 \, dx + \int_\Omega |\nabla(w_1 - w_2)|^2 \, dx \right) \, dt \\
+ 2sC_{10} ||\nabla(w_1-w_2)(s)||_{L^2(\Omega)}^2
\]

If we choose $T = T_1$ sufficiently small and $\varepsilon_1 > 0$ such that $C_6/2 - 2T_1C_{10} \geq \varepsilon_1$, then, for $0 \leq s \leq T_1$, we have
\[
\frac{1}{2} \int_\Omega |u_1 - u_2|^2 \, dx + \eta \int_0^s \int_\Omega |u_1 - u_2|^2 \, dx \, dt + \varepsilon_1 \int_\Omega |\nabla(w_1 - w_2)(s)|^2 \, dx \tag{37}
\]

which implies that
\[
\frac{1}{2} \int_\Omega |u_1 - u_2|^2 \, dx + \varepsilon_1 \int_\Omega |\nabla(w_1 - w_2)(s)|^2 \, dx \\
\leq C_{10} \int_0^s \left( \int_\Omega |u_1 - u_2|^2 \, dx + \int_\Omega |\nabla(w_1 - w_2)|^2 \, dx \right) \, dt \tag{38}
\]
and finally, if we define \( C = (2 + 1/\varepsilon)C_{10} \), we obtain
\[
\int_{\Omega} |u_1 - u_2|^2 \, dx + \int_{\Omega} |\nabla (w_1 - w_2)(t)|^2 \, dx \\
\leq C \int_0^T \left( \int_{\Omega} |u_1 - u_2|^2 \, dx + \int_{\Omega} |\nabla (w_1 - w_2)(t)|^2 \, dx \right) \, dt
\]

Using Grönwall’s inequality, we obtain that \( u_1 - u_2 = 0 \) on \((0, T_1]\). By applying the argument on the intervals \((T_1, T_2]\), \((T_2, T_3]\), and so on, we see that \( u_1 - u_2 = 0 \) on \((0, T]\).

3. Regularization property of the damped flow. On the first glimpse, the PDE-based formulation (5) looks better than the original variational formulation (1) because of the absence of the regularization parameter \( \alpha \), which is always an obstacle for solving an ill-posed inverse problem. Unfortunately, the ill-posedness remains. Indeed, the terminating time \( T \) of the damped flow (5), instead of \( \alpha \) in the original problem (1), plays the role of the regularization parameter for the image denoising problem. If the damped flow is discretized, then the formulation (5) presents a second order iteration scheme; see Section 4 for details. The choice of the terminating time \( T \) for the damped flow exactly coincides with the stopping rule for the (nonlinear) Landweber iteration [23].

In this section, we devote our researches to the method of choosing the terminating time \( T \). First, let us consider the long-term behavior of the damped flow (5). Though we only proved the local well-posedness of the dynamical system (5), we still assume the global existence and uniqueness of the solution to (5), which will be used for the stability analysis with respect to the noisy image.

Denote by \( U^* \) the equilibrium solutions of (5), namely, \( \text{div}(\varepsilon \frac{w}{\varepsilon^2} + |G_\sigma * u|^2)^{\frac{p}{p-2}} \nabla u^* = 0 \) for all \( u^* \in U^* \). Note that \( \text{div}(\varepsilon \frac{w}{\varepsilon^2} + |G_\sigma * u|^2)^{\frac{p}{p-2}} \nabla \cdot \) is a monotone operator in a Banach space, therefore, using the Galerkin method, it is not difficult to show that there exists a solution to the equation \( \text{div}(\varepsilon \frac{w}{\varepsilon^2} + |G_\sigma * u|^2)^{\frac{p}{p-2}} \nabla u = 0 \) with the Neumann boundary condition. Moreover, such a solution is unique up to an overall additive constant. Obviously, \( u = \text{constant} \) is a solution to the equation \( \text{div}(\varepsilon \frac{w}{\varepsilon^2} + |G_\sigma * u|^2)^{\frac{p}{p-2}} \nabla u = 0 \). Hence, we conclude that \( U^* = \{ u : u' = 0 \} \). Then, based on the results from [25] and [24] (Theorem 2.1 in [25] and Theorem 3.1 in [24]), we have the convergence result of the global and bounded solutions of problem (5), i.e., the following theorem holds.

**Theorem 3.1.** Let \( \Omega \) be a bounded, open, and connected set in \( \mathbb{R}^N \) \((N = 2, 3)\) having a boundary \( \partial \Omega \) of class \( \mathcal{C}^2 \). Then, there exists a constant \( u_\infty \) such that the solution \( u \) to the equation (5) satisfies
\[
\|u_t\|_{L^2(\Omega)} + \|u(x, t) - u_\infty\|_{L^2(\Omega)} \leq C(u_0, \Omega)e^{-c(\Omega)t},
\]
where \( C(u_0, \Omega) > 0 \) depends on the initial data \( u_0 \) and the geometry of domain \( \Omega \), while \( c(\Omega) > 0 \) depends only on \( \Omega \).

Suppose that instead of the exact image \( \bar{u} \) we are given approximate one, \( u_0^\delta \), such that \( \|u_0^\delta - \bar{u}\|_{L^2(\Omega)}/\|\bar{u}\|_{L^2(\Omega)} \leq \delta \), where the positive number \( \delta \) denotes the degree of difference between the accurate image \( \bar{u} \) and polluted image \( u_0^\delta \). Obviously, if there is no noise, i.e. \( \delta = 0 \), no denoising algorithm is needed. In this case, \( T(\delta) = 0 \).
Theorem 3.2. (A priori selection method for \( T(\delta) \))

Denote by \( u(x, t) \) the solution of the damped flow (5) with the initial data \( u_0^\delta(x) \). Then, if the terminating time point is chosen as \( T(\delta) = C_1 \ln(1 + C_2 \delta^\gamma) \), where \( C_1, C_2, \gamma \) are positive constants independent of \( \delta \), the approximate solution \( u(T(\delta), x) \) converges to the exact image \( \tilde{u}(x) \) as \( \delta \to 0 \).

Proof. Using the estimate (39), we obtain

\[
\|u(T, x) - \tilde{u}(x)\|_{L^2(\Omega)} \leq \|u(T, x) - u_0^\delta(x)\|_{L^2(\Omega)} + \|u_0^\delta(x) - \tilde{u}(x)\|_{L^2(\Omega)} \\
\leq \int_0^T \|u_t\|_{L^2(\Omega)} \, dt + \bar{\delta} \|\tilde{u}\|_{L^2(\Omega)} \leq \int_0^T C(u_0, \Omega) e^{-c(\Omega)\gamma} \, dt + \bar{\delta} \|\tilde{u}\|_{L^2(\Omega)} \\
= \frac{C(u_0, \Omega)}{c(\Omega)} \left( 1 - e^{-c(\Omega)\gamma T} \right) + \bar{\delta} \|\tilde{u}\|_{L^2(\Omega)}.
\]

Note that the terminating time point is chosen as \( T(\delta) = C_1 \ln(1 + C_2 \delta^\gamma) \). By combining the above inequalities, we can deduce that

\[
\|u(T(\delta), x) - \tilde{u}(x)\|_{L^2(\Omega)} \leq \frac{C(u_0, \Omega)}{c(\Omega)} \left( 1 - (1 + C_2 \delta^\gamma)^{-c(\Omega)C_1} \right) + \bar{\delta} \|\tilde{u}\|_{L^2(\Omega)}. \tag{40}
\]

On the other hand, for a sufficiently small \( \delta \), the inequality \((1 + C_2 \delta^\gamma)^{-c(\Omega)C_1} \geq 1 - c(\Omega)C_1 C_2 \delta^\gamma \) holds. Therefore, by (40), we can deduce that

\[
\|u(T(\delta), x) - \tilde{u}(x)\|_{L^2(\Omega)} \leq C(u_0, \Omega)C_1 C_2 \delta^\gamma + \|\tilde{u}\|_{L^2(\Omega)} \delta,
\]

which implies the convergence of the obtained approximate solution \( u(T(\delta), x) \). \( \square \)

By the proof of the above theorem, we know that under the a priori selection method for the final time point \( T(\delta) = C_1 \ln(1 + C_2 \delta^\gamma) \), the convergence rate of the method is \( \min(\gamma, 1) \). However, an a priori parameter choice is not suitable in practice, since a good terminating time point \( T \) requires knowledge of the unknown image \( \tilde{u}(x) \). Moreover, there are intractable factors, \( C_1, C_2 \), around the parameter. This knowledge is not necessary for a posteriori parameter choice. Here, we develop a modified Morozov’s discrepancy principle of choosing the terminating time point \( T \).

Define by

\[
\sigma(T) = \|u(T, x) - u_0^\delta(x)\|_{L^2(\Omega)}/\|u_0^\delta(x)\|_{L^2(\Omega)},
\]

the tolerability ratio of the difference between the estimated and noisy images.

Introduce the discrepancy function

\[
\chi(T) = \sigma(T) - \delta,
\]

which describes the difference between the tolerability ratio of the denoised image and the degree of the measured noisy image. Obviously, by Theorem 2.3, \( \chi(T) \) is a continuous function.

Theorem 3.3. (A posteriori selection method for \( T(\delta) \))

Suppose that the noisy image \( u_0^\delta \) is not an “almost-constant”, i.e.

\[
\inf_{c \text{ is a constant}} \|c - u_0^\delta(x)\| = \mu > 0. \tag{41}
\]

Then, there exists a positive number \( \delta_0 > 0 \) such that for all \( \delta \in (0, \delta_0] \), the discrepancy function \( \chi(T) \) admits at least one positive root. Moreover, the approximate solution \( u(T(\delta), x) \), with the terminating time point chosen as the positive root of \( \chi(T) \), converges to the exact image \( \tilde{u}(x) \) as \( \delta \to 0 \).
Proof. Combine the estimate (39) and the assumption of $u_0^\delta(x)$ in (41), and one can deduce that for $\delta < \mu/\|u_0^\delta(x)\|$:  
$$\lim_{T \to \infty} \chi(T) \geq \frac{\mu}{\|u_0^\delta(x)\|} - \delta > 0.$$  
On the other hand, by the definition of the approximate solution $u$ (the solution to (5)), we have $\chi(0) = -\delta < 0$. Since $\chi(T)$ is a continuous function, $\chi(T)$ must admit at least one positive root.

Now, consider the convergence property of the solution $u(T(\delta), x)$. By the definition of the final time point in selection (the positive root of $\chi(T)$), we obtain  
$$\|u(T, x) - \bar{u}(x)\| \leq \|u(T(\delta), x) - u_0^\delta(x)\| + \|u_0^\delta(x) - \bar{u}(x)\| \leq (\|u_0^\delta(x)\| + 1)\delta,$$
which implies the convergence of the desired approximate solution $u(T, x)$ immediately. \qed

Remark 1. If function $\chi(T)$ has more than one positive root, then, any of root $T$ gives a stable approximate image $u(T, x)$. In practice, one can choose $T^* = \inf_{\chi(T)} T$, i.e. $\sigma(T) < \delta$ for all $T < T^*$ and $\sigma(T^*) = \delta$. In other words, $T^*$ is the first time point for which the tolerability ratio $\sigma(T)$ coincides with the data error.

4. A discrete damped flow. Loosely speaking, the damped flow (5) with an appropriate numerical discretization yields a discrete second order regularization method. Just like the Runge-Kutta integrators [34] or the exponential integrators [26] for solving first order equations, the damped symplectic integrators are extremely attractive for solving second order equations (5), since the schemes are closely related to the canonical transformations [22], and the trajectory of the discretized second flows are usually more stable. In this section, based on the Störmer-Verlet method, we develop a discrete damped flow for image denoising.

For simplicity and clarity of statements, let $\Omega$ denote a rectangle region in $\mathbb{R}^2$, and let us consider a uniform grid $\Omega_{MN} = \{(x_i, y_j)\}_{i,j=1}^{M,N}$ in $\Omega$ with the uniform step size $h = x_{i+1} - x_i = y_{j+1} - y_j$. Define $u(t) = [u(x_i, y_j, t)]_{i,j=1}^{M,N},$. Denote $u^k$ as the projection of $u(x, y, t)$ at the spacial grid $\Omega_{MN}$ and time point $t = t_k$. We approximate the $\text{div}(a^\epsilon(u) \nabla u)$ by a linear one – $\text{div}(a^\epsilon(u^{k-1}) \nabla u^k)$, where $a^\epsilon(u)$ is defined in (6). Using the central difference discretization rule, we have

$$\text{div}(a^\epsilon(u^{k-1}) \nabla u^k) = D_x \frac{1}{2} \left( a_{i,j}^{\epsilon,k-1} D_{x,j} u_{i,j}^k + D_{y,j} \frac{1}{2} \left( a_{i,j}^{\epsilon,k-1} D_{y,j} u_{i,j}^k \right) \right)$$

$$= D_x \frac{1}{2} \left( a_{i,j}^{\epsilon,k-1} \frac{1}{h} \left( u_{i,j+1}^k - u_{i,j}^k \right) + D_{y,j} \frac{1}{2} \left( a_{i,j}^{\epsilon,k-1} \frac{1}{h} \left( u_{i,j+1}^k - u_{i,j}^k \right) \right) \right)$$

$$= \frac{1}{h^2} \left\{ a_{i,j}^{\epsilon,k-1} u_{i,j+1}^k + a_{i,j}^{\epsilon,k-1} u_{i,j}^k - \left( a_{i,j}^{\epsilon,k-1} + a_{i,j}^{\epsilon,k-1} + a_{i,j}^{\epsilon,k-1} + a_{i,j}^{\epsilon,k-1} \right) u_{i,j}^k \right\}$$

$$+ a_{i,j}^{\epsilon,k-1} u_{i,j+1}^k + a_{i,j}^{\epsilon,k-1} u_{i,j}^k,$$

where

$$a_{i,j}^{\epsilon,k-1} = \left( \varepsilon + \|\nabla G_\sigma \| u_{i,j}^{k-1} \right) \frac{1}{2} \left( u_{i,j}^{k-1} + u_{i,j}^k \right),$$

and $\nabla G_\sigma$ is the project of function $\nabla G_\sigma$ on the same grid $\Omega_{MN}$.

Definition 4.1. Given a matrix $u \in \mathbb{R}^M \times \mathbb{R}^N$, one can obtain a vector $\bar{u} \in \mathbb{R}^{MN}$ by stacking the columns of $u$. This defines a linear operator $\text{vec} : \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}^{MN}$,

$$\text{vec}(u) = (u_{1,1}, u_{2,1}, \cdots, u_{M,1}, u_{1,2}, u_{2,2}, \cdots, u_{M,2}, \cdots, u_{1,N}, u_{2,N}, \cdots, u_{M,N})^T,$$

$$\bar{u} = \text{vec}(u), \quad \bar{u}_q = u_{i,j}, \quad q = (i-1)M + j.$$
This corresponds to a lexicographical column ordering of the components in the matrix \( \mathbf{u} \). The symbol \( \text{array} \) denotes the inverse of the \( \text{vec} \) operator. That means the following equalities hold
\[
\text{array}(\text{vec}(\mathbf{u})) = \mathbf{u}, \quad \text{vec}(\text{array}(\vec{\mathbf{u}})) = \vec{\mathbf{u}},
\]
whenever \( \mathbf{u} \in \mathbb{R}^{M \times N} \) and \( \vec{\mathbf{u}} \in \mathbb{R}^{MN} \).

Based on the above definition, rewrite (42) as the matrix form, \( \mathbf{F}^{k-1}\vec{\mathbf{u}}^{k} \), where the matrix \( \mathbf{F}^{k-1} \) is dependent only on \( \vec{\mathbf{u}}^{k-1} \).

**Proposition 1.** All eigenvalues of \( \mathbf{F}^{k} \) \((k = 2, \ldots, MN)\) are non-positive.

**Proof.** By the definition of \( \mathbf{F}^{k} \), it is not difficult to show that \( \mathbf{F}^{k} \) is a symmetrical and diagonally dominant matrix. Then, all eigenvalues of \( \mathbf{F}^{k} \) \((k = 1, \ldots, MN)\) are real and, by Gershgorin’s circle theorem, for each eigenvalue \( \lambda \) an index \( \nu \) exists such that:
\[
\lambda \in \left[ |F^{k}_{\nu,\nu}| - \sum_{i \neq \nu} |F^{k}_{\nu,i}|, |F^{k}_{\nu,\nu}| + \sum_{i \neq \nu} |F^{k}_{\nu,i}| \right],
\]
which implies, by definition of the diagonal dominance, \( \lambda \leq 0 \). Here, \( [F^{k}]_{\nu,i} \) denotes the element of the matrix \( \mathbf{F}^{k} \) at the position \((\nu, i)\). \(\square\)

Denote \( \bar{\mathbf{u}}^{k} = \frac{\partial \mathbf{u}^{k}}{\partial t} \). In this work, the Störmer-Verlet method is employed to solve PDE (5), namely
\[
\begin{aligned}
\bar{\mathbf{v}}^{k+\frac{1}{2}} &= \mathbf{v}^{k} + \frac{\Delta t_{k}}{2} \left( \mathbf{F}^{k-1} \bar{\mathbf{u}}^{k} - \eta \mathbf{v}^{k+\frac{1}{2}} \right), \\
\bar{\mathbf{u}}^{k+1} &= \mathbf{u}^{k} + \Delta t_{k} \bar{\mathbf{v}}^{k+\frac{1}{2}}, \\
\mathbf{v}^{k+1} &= \mathbf{v}^{k+\frac{1}{2}} + \frac{\Delta t_{k}}{2} \left( \mathbf{F}^{k} \bar{\mathbf{u}}^{k+1} - \eta \mathbf{v}^{k+\frac{1}{2}} \right), \\
\bar{\mathbf{u}}_{0} &= \bar{\mathbf{u}}^{0}, \mathbf{v}_{0} = 0,
\end{aligned}
\]  
(44)

where \( \bar{\mathbf{u}}^{0} = \text{vec}(\mathbf{u}^{0}) \) and \( \mathbf{u}^{0} \) is the project of \( u_{0}^{0}(x) \) on the grid \( \Omega_{MN} \).

Now, we are in a position to give a numerical analysis for the scheme (44).

Denote by \( \mathbf{z}^{k} = (\bar{\mathbf{u}}^{k}; \mathbf{v}^{k}) \), and \( \mathbf{E} \) the identity matrix of size \( MN \), then, equation (44) can be rewritten as
\[
\mathbf{z}^{k+1} = \mathbf{B}^{k} \mathbf{A}^{k-1} \mathbf{z}^{k},
\]  
(45)

where
\[
\mathbf{A}^{k-1} = \frac{2}{2 + \eta \Delta t_{k}} \begin{pmatrix}
1 + \frac{\eta \Delta t_{k}}{2} & \mathbf{E} + \frac{\Delta t_{k}^{2}}{2} \mathbf{F}^{k-1} \\
\mathbf{E} & \mathbf{E}
\end{pmatrix}
\]  
(46)

and
\[
\mathbf{B}^{k} = \begin{pmatrix}
\mathbf{E} & 0 \\
\frac{\Delta t_{k}}{2} \mathbf{F}^{k} & \left(1 - \frac{\eta \Delta t_{k}}{2}\right) \mathbf{E} + \frac{\Delta t_{k}^{2}}{2} \mathbf{F}^{k}
\end{pmatrix}.
\]  
(47)

**Theorem 4.2.** (Boundedness) If
\[
\Delta t_{k} \leq \eta / \sqrt{\lambda^{(k)}_{\max}},
\]  
(48)
then, the scheme (44) is uniformly bounded.

**Proof.** By Proposition 1, all the eigenvalues of \( \mathbf{F}^{k} \) are non-positive. By noting that \( \mathbf{F}^{k} \) is a symmetrical matrix, there exists a decomposition \( \mathbf{F}^{k} = \Phi^{k} \Lambda^{k} (\Phi^{k})^{T} \), where \( \Phi^{k} \) is an unitary matrix and \( \Lambda^{k} = -\text{diag}(\lambda_{i}^{(k)}) \), where \( \lambda_{i}^{(k)} \geq 0, i = 1, \ldots, MN \). 


It is well known that, a sufficient condition for the boundedness of a dynamical system is \( \|B^k A^{k-1}\|_2 \leq 1 \), i.e. the composite mapping \( B^k A^{k-1} \) is non-expansive. By the calculation, the maximal eigenvalues of \( B^k \) are always the unit, i.e. \( \nu_{\text{max}}(B^k) \equiv 1 \) for all \( k \). Therefore, using the relation \( \|B^k A^{k-1}\|_2 \leq \|B^k\|_2 \|A^{k-1}\|_2 = \|A^{k-1}\|_2 \) it is sufficient to show that for the given time step size \( \Delta t_k \) in (48), the corresponding eigenvalues of \( A^{k-1} \) are not greater than the unit.

The eigenvalues of matrices \( A^{k-1} \) are

\[
\mu_{i,\pm}^{(k-1)}(A^{k-1}) = \frac{2}{2 + \eta \Delta t_k} \left( 1 + \frac{\Delta t_k}{4} \left( \eta - \Delta t_k \lambda_i^{(k-1)} \pm \sqrt{\left( \eta - \Delta t_k \lambda_i^{(k-1)} \right)^2 - 8 \lambda_i^{(k-1)}} \right) \right),
\]

Now, we have to show that for all \( k \): \( \mu_{\text{max}}^{(k-1)}(\eta, \Delta t_k^{(k-1)}) \leq 1 \) for the parameter \( \Delta t_k^{(k-1)} \) defined by (48).

For simplicity, we ignore the superscript \(^{(k-1)}\) from now on. Denote by \( i_* \) the index of \( \lambda_{i_*} \), corresponding the maximal absolute value of \( \mu_{i,\pm}^{(k-1)}(A^{k-1}) \), i.e.

\[
|\mu_{\text{max}}| = \frac{2}{2 + \eta \Delta t_k} \left( 1 + \frac{\Delta t_k}{4} \left( \eta - \Delta t_k \lambda_{i_*} \pm \sqrt{\left( \eta - \Delta t_k \lambda_{i_*} \right)^2 - 8 \lambda_{i_*}} \right) \right).
\]

If \( \lambda_{i_*} = 0 \), the theorem holds, obviously, since \( |\mu_{\text{max}}| \equiv 1 \) in this case.

Now, consider the case when \( \lambda_{i_*} > 0 \). There are three possible cases here: the overdamped case \( (\eta - \Delta t_k \lambda_{i_*})^2 > 8 \lambda_{i_*} \), the underdamped case \( (\eta - \Delta t_k \lambda_{i_*})^2 < 8 \lambda_{i_*} \), and the critical damped case \( (\eta - \Delta t_k \lambda_{i_*})^2 = 8 \lambda_{i_*} \). Let us consider these cases respectively.

For the chosen time step size \( \Delta t_k \) in (48), we have \( \eta - \Delta t_k \lambda_{i_*} \geq 0 \). Therefore, for the overdamped case,

\[
|\mu_{\text{max}}| = \frac{2}{2 + \eta \Delta t_k} \left( 1 + \frac{\Delta t_k}{4} \left( \eta - \Delta t_k \lambda_{i_*} + \sqrt{\left( \eta - \Delta t_k \lambda_{i_*} \right)^2 - 8 \lambda_{i_*}} \right) \right).
\]

Define \( \eta - \Delta t_k \lambda_{i_*} = a \sqrt{8 \lambda_{i_*}} \) \((a > 1)\), and we have

\[
|\mu_{\text{max}}| = \frac{1 + \frac{\Delta t_k}{2} \left( a + a \sqrt{a^2 - 1} \right) \sqrt{8 \lambda_{i_*}}}{1 + \frac{\Delta t_k}{2} \eta}.
\]

Substituting \( \eta = \Delta t_k \lambda_{i_*} + a \sqrt{8 \lambda_{i_*}} \) in the above equation, we can deduce that

\[
|\mu_{\text{max}}| = \frac{1 + \frac{\Delta t_k}{2} \left( a + a \sqrt{a^2 - 1} \right) \sqrt{8 \lambda_{i_*}}}{1 + \frac{\Delta t_k}{2} \left( \Delta t_k \lambda_{i_*} + a \sqrt{8 \lambda_{i_*}} \right)} < 1.
\]

Now, consider the underdamped case. The complex eigenvalue \( \mu_{\text{max}} \) satisfies

\[
|\mu_{\text{max}}|^2 = \frac{1 + \frac{\Delta t_k}{2} \left( \eta - \Delta t_k \lambda_{i_*} \right) + (\frac{\Delta t_k}{2})^2 8 \lambda_{i_*}}{(1 + \frac{\Delta t_k}{2} \eta)^2}.
\]

Similarly, if we define \( \eta - \Delta t_k \lambda_{i_*} = a \sqrt{8 \lambda_{i_*}} \) with \( a < 1 \), we have

\[
|\mu_{\text{max}}|^2 = \frac{1 + \frac{\Delta t_k}{2} a \sqrt{8 \lambda_{i_*}} + (\frac{\Delta t_k}{2})^2 8 \lambda_{i_*}}{(1 + \frac{\Delta t_k}{2} \left( \Delta t_k \lambda_{i_*} + a \sqrt{8 \lambda_{i_*}} \right))^2} = \frac{1 + a \Delta t_k \sqrt{2 \lambda_{i_*}} + \frac{\Delta t_k^2}{2} \lambda_{i_*}}{1 + a \Delta t_k \sqrt{2 \lambda_{i_*}} + \frac{\Delta t_k^2}{2} \lambda_{i_*}} < 1.
\]

Finally, consider the critical damped case. In this case,\n
\[
|\mu_{\text{max}}| = \frac{1 + \frac{\Delta t_k}{4} \sqrt{8 \lambda_{i_*}}}{1 + \frac{\Delta t_k}{2} \eta} = \frac{1 + \frac{\Delta t_k}{4} \sqrt{8 \lambda_{i_*}}}{1 + \frac{\Delta t_k}{2} \left( \Delta t_k \lambda_{i_*} + \sqrt{8 \lambda_{i_*}} \right)} < 1,
\]
which completes the proof.

By Taylor’s theorem and the finite difference formula, it is not difficult to show the consistency of the scheme (44). It is well known that boundedness implies the convergence of consistent schemes for any (especially nonlinear) problem, namely, the following theorem holds [39].

**Theorem 4.3.** (Convergence) The scheme (44) is convergent if the time step size is chosen by the criterion (48).

5. **An algorithm.** In this section, we propose an algorithm for image denoising. Various stopping criteria exist for an iteration algorithm [19, 38, 27]. In principle, the stopping criterion for image denoising problems should be proposed case by case. In real world problems, in order to obtain a high qualified denoised image, a manual stopping criterion is always required, especially for the PDE-based denoising technique. Nevertheless, an automatic stopping criterion can definitely help people to select a good initial guess of the denoised image.

In this paper, we adapt a frequency domain threshold method based on the fact that noise is usually represented by high frequencies in the frequency domain. Define high frequencies energy by

\[
\Delta_{N_0}(u) = \sum_{i+j \geq N_0} |\mathcal{F}(u)(i,j)|^2,
\]

where \(\mathcal{F}(u)\) denotes a 2D discrete Fourier transform of an image \(u\), and \(N_0\) presents the high frequencies index. In the simulation, we set \(N_0 = \lfloor 0.6N^2 \rfloor\), where \(\lfloor \cdot \rfloor\) denotes the floor function. Define by

\[
RDE(k) = \frac{|\Delta_{N_0}(u^k) - \Delta_{N_0}(u^{k-1})|}{\Delta_{N_0}(u^{k-1})},
\]

the relative denoising efficiency. Then, the value of \(RDE\) at every iteration can be used as a stopping criterion. Based on this stopping criterion, an algorithm of SV-DDF for imaging denoising is proposed in Algorithm 1.

**Algorithm 1** The SV-DDF for imaging denoising.

**Input:** Observed noisy image \(u_0^\delta\). Parameters \(\eta\) and \(p\). Tolerance \(\varepsilon\).

**Output:** A denoised image \(\bar{u}\) ← array(\(\bar{u}^k\)).

1: \(\bar{u}_0 ← \text{vec}(u_0), \bar{v}_0 ← 0, \Delta t_0 ← \lambda_{\text{max}}(F^0), F^{-1} ← F^0, RDE(0) ← 1, k ← 0\)
2: while \(RDE(k) > \varepsilon\) do
3: \(\bar{v}^{k+\frac{1}{2}} ← (1 + \frac{\Delta t_k}{2}\eta)^{-1}(\bar{v}^k + \frac{\Delta t_k}{2}F^{-1}\bar{u}^k)\)
4: \(\bar{u}^{k+1} ← \bar{u}^k + \Delta t_k\bar{v}^{k+\frac{1}{2}}\)
5: \(\bar{v}^{k+1} ← \bar{v}^{k+\frac{1}{2}} + \frac{\Delta t_k}{2}(F^k\bar{u}^{k+1} - \eta\bar{v}^{k+\frac{1}{2}})\)
6: \(k ← k + 1\)
7: \(RDE(k) ← |\Delta_{N_0}(u^k) - \Delta_{N_0}(u^{k-1})|/\Delta_{N_0}(u^{k-1})\)
8: \(\Delta t_k ← \lambda_{\text{max}}(F^k)\)
9: end while

6. **Numerical experiments.** In this section, several numerical examples are implemented to show the feasibility and effectiveness of our proposed image denoising approach – SV-DDF (Algorithm 1).

Let \(\bar{u}\) be the noise-free image, see Fig. 1. Uniformly distributed noise with the level \(\delta\) is added to the original image \(\bar{u}\) to get a noisy image \(u_0^\delta: u_0^\delta(x) = \ldots\)
\[1 + \delta \cdot (2 \cdot \text{rand}(x) - 1)\] \(\bar{u}(x), x \in \Omega,\) where “rand” returns a pseudo-random value drawn from a uniform distribution on \([0, 1].\) To assess the accuracy of the denoised images we use the structural similarity error measure SSIM to obtain a quantitative estimate of the denoising performance [40].

6.1. **Example 1.** The purpose of the first example is to explore the dependence of the accuracy of the denoised image on the damping parameter \(\eta\) and the method parameter \(p,\) and thus to give a guide on the choices of them in practice. The noisy image and the reconstructed image by SV-DDF are presented in Fig. 2. In Tab. 6.1, we displayed the results by using different values of damping parameter \(\eta\) and the method parameter \(p.\) The results show that we obtain the best result for \(p = 1\) and \(\eta = 300.\) According to (48) in Theorem 2, \(\eta\) should be chosen such that \(\eta \geq \Delta t \sqrt{\lambda_{\text{max}}^{(k)}}.\) In our experiment in the case of \(p = 1\) we obtained the result that the bound \(\eta \geq 15.\)

![Figure 1](image1.png)

**Figure 1.** Test pictures: the exact image.

![Figure 2](image2.png)

**Figure 2.** Test pictures: (a): the noisy image with \(\delta = 54\%\) (b) the denoised image.
Table 1. Results with different damped parameters $\eta$ and $p$; the table shows the SSIM value.

| $p \backslash \eta$ | 0.001 | 1 | 100 | 300 | 600 | 1500 | 3000 |
|-------------------|-------|---|-----|-----|-----|------|------|
| 1                 | 0.108 | 0.108 | 0.538 | 0.549 | 0.548 | 0.548 | 0.548 |
| 1.5               | 0.116 | 0.116 | 0.114 | 0.530 | 0.532 | 0.532 | 0.532 |
| 2                 | 0.032 | 0.032 | 0.032 | 0.032 | 0.081 | 0.135 |       |

6.2. Example 2. In order to show the advantages of our algorithm over the existing approaches, we solve the same problem by the following methods: total variation (TV), Modified telegraph (MTele) and telegraph (Tele). The test degraded image is given in the first picture of Fig. 3 while the denoised images are displayed in the last five pictures in Fig. 3. We see that the SV-DDF, as well as the TV methods, gives the highest SSIM value 0.549, and also gives a smoother result than the other methods. Note that in Fig. 3 we have chosen the terminating time which produces the highest SSIM value. We also indicate that in general, with an appropriate choice of damping parameter $\eta$, the SV-DDF method exhibits an acceleration phenomenon, i.e. with an earlier termination time, the reconstructed image by SV-DDF presents...
a higher SSIM value than the denoised image by TV. However, a rigorous theoretical analysis on the acceleration of the damped flow (5) will be performed in later work.

7. Conclusion. In this paper, we introduce a new image denoising model – the damped flow. The existence and uniqueness of the solution to the model are both proven under certain assumptions. For the numerical implementation, based on the Störmer-Verlet method, a discrete damped flow, SV-DDF, is developed. The convergence of SV-DDF is discussed as well. A numerical algorithm with automatic stopping criterion is provided. A comparison with three existing methods has shown that the SV-DDF appears to be very competitive with respect to its image denoising capabilities. Obviously, except in the $p$-Dirichlet energy denoising, the damped flow can be used for solving a general nonlinear optimization problem of the type (1).

Finally, we emphasize that the aim of the paper is to introduce the SV-DDF method to image denoising. Since it is comparable to the conventional TV method, it is a promising approach which merits further theoretical and numerical development as well as more extensive comparison to state-of-the-art methods.

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