LINEAR AND ALGEBRAIC INDEPENDENCE OF GENERALIZED EULER-BRIGGS CONSTANTS

SANOLI GUN, V. KUMAR MURTY AND EKATA SAHA

ABSTRACT. Possible transcendental nature of Euler’s constant $\gamma$ has been the focus of study for sometime now. One possible approach is to consider $\gamma$ not in isolation, but as an element of the infinite family of generalised Euler-Briggs constants. In a recent work [6], it is shown that the infinite list of generalized Euler-Briggs constants can have at most one algebraic number. In this paper, we study the dimension of spaces generated by these generalized Euler-Briggs constants over number fields. More precisely, we obtain non-trivial lower bounds (see Theorem 5 and Theorem 6) on the dimension of these spaces and consequently establish the infinite dimensionality of the space spanned. Further, we study linear and algebraic independence of these constants over the field of all algebraic numbers.

1. INTRODUCTION

In 1731, Euler introduced the following constant

$$\gamma := \lim_{x \to \infty} \left( \sum_{n \leq x} \frac{1}{n} - \log x \right)$$

and derived a number of identities involving $\gamma$, special values of the Riemann zeta function and other known constants. After Euler several other notable mathematicians including Gauss and Ramanujan have studied this constant in depth. For a beautiful account of the various aspects of research about this constant, we refer the reader to a recent article of Lagarias [11].

The appearance of $\gamma$ in its various avatars makes it a fundamental object of study in number theory. Though we expect that $\gamma$ is transcendental, it is not even known to be irrational. However there are some transcendence results involving $\gamma$. To the best of our knowledge, the first such result was due to Mahler [13]. He proved that for any non-zero algebraic number $\alpha$, the number

$$\frac{\pi Y_0(\alpha)}{2J_0(\alpha)} - \log \frac{\alpha}{2} - \gamma$$

2010 Mathematics Subject Classification. 11J81, 11J91.

Key words and phrases. Generalized Euler-Briggs constants, Baker’s theory of linear forms in logarithms, Weak Schanuel’s conjecture.
is transcendental, where \( J_0 \) and \( Y_0 \) are Bessel functions of the first and second kind of order zero. More precisely,

\[
J_0(\alpha) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left( \frac{\alpha}{2} \right)^{2n}, \quad H_n := \sum_{j=1}^{n} \frac{1}{j}
\]

and \( \frac{\pi}{2} Y_0(\alpha) := \left( \log\left( \frac{\alpha}{2} \right) + \gamma \right) J_0(\alpha) + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n}{(n!)^2} \left( \frac{\alpha^2}{4} \right)^n. \)

In the rather difficult subject of transcendence, sometimes it is more pragmatic to look at a family of numbers as opposed to a single number and derive something meaningful. There are two significant results which are worth mentioning at this point. The first one is by R. Murty and Saradha [16]. They proved the following theorem.

\textbf{Theorem 1.} (Murty and Saradha) Let \( q > 1 \) be a natural number. At most one of the numbers

\[ \gamma, \gamma(a,q), 1 \leq a \leq q, \ (a,q) = 1 \]

is algebraic. Here

\[ \gamma(a,q) := \lim_{x \to \infty} \left( \sum_{n \leq x \mod q} \frac{1}{n} - \frac{1}{q} \log x \right). \]

The constants \( \gamma(a,q) \) were introduced by Briggs [3] and later studied extensively by Lehmer [12]. We will call these constants as Euler-Briggs constants. The second result involving the transcendence of \( \gamma \) is due to Rivoal [20] and Pilehrood-Pilehrood [18] who proved the following theorem.

\textbf{Theorem 2.} (Rivoal / Pilehrood-Pilehrood) At least one of the numbers \( \gamma \) and \( \delta := \int_0^{\infty} \frac{e^{-w}}{1+w} dw \) is transcendental.

The constant \( \delta \) is known as the Euler-Gompertz constant. Note that the constants \( \gamma \) and \( \delta \) are part of a family of numbers called exponential periods (see [10] and also page 595 of [11]) introduced by Kontsevich and Zagier [9].

Another important set of numbers with \( \gamma \) as a member were introduced by Diamond and Ford. In 2008, while studying the Riemann hypothesis, they introduced the so-called generalized Euler’s constants. In order to introduce these numbers, let us set some notation.

Throughout the paper, \( P \) will denote the set of prime numbers and \( p \) will denote a prime number. For any finite subset \( \Omega \subset P \), set

\[
P_\Omega := \begin{cases} \prod_{p \in \Omega} p & \text{if } \Omega \neq \phi, \\ 1 & \text{otherwise}, \end{cases}
\]

and \( \delta_\Omega := \begin{cases} \prod_{p \in \Omega} (1 - \frac{1}{p}) & \text{if } \Omega \neq \phi, \\ 1 & \text{otherwise}. \end{cases} \)
For any finite subset $\Omega \subset P$, Diamond and Ford [4] defined the generalized Euler’s constant as follows;

$$
\gamma(\Omega) := \lim_{x \to \infty} \left( \sum_{n \leq x \atop (n, P_\Omega) = 1} \frac{1}{n} - \delta_\Omega \log x \right).
$$

Note that, when $\Omega = \phi$, then $\gamma(\Omega) = \gamma$. In a recent work, R. Murty and Zaytseva [17] noticed that at most one number in the infinite list $\gamma(\Omega)$ as $\Omega$ varies over finite subset of primes is algebraic.

Following Euler-Briggs and Diamond-Ford, one can now define for any finite subset $\Omega \subset P$ and natural numbers $a, q$ with $(q, P_\Omega) = 1$, the constants

$$
\gamma(\Omega, a, q) := \lim_{x \to \infty} \left( \sum_{n \leq x \atop n \equiv a \mod q \atop (n, P_\Omega) = 1} \frac{1}{n} - \delta_\Omega \log \frac{x}{q} \right).
$$

When $\Omega = \phi$, we see that $\gamma(\Omega, a, q) = \gamma(a, q)$, the classical Euler-Briggs constants. From now on, we refer to the constants $\gamma(\Omega, a, q)$ as generalized Euler-Briggs constants. Moreover, when $q = 1$ and $\Omega \subset P$ is a finite set, we have

$$
\gamma(\Omega, a, 1) = \gamma(\Omega) = \delta_\Omega (\gamma + \sum_{p \in \Omega} \frac{\log p}{p-1}), \text{ where } a \in \mathbb{N}.
$$

The last equality has been established by Diamond and Ford in [4]. In a recent work, the first and the third author along with Sneh Bala Sinha [6] (see also [7]) proved the following results;

**Theorem 3.** (Gun, Saha and Sinha) Let $a$ and $q > 1$ be natural numbers with $(a, q) = 1$ and $S$ be the set of prime divisors of $q$. Also let

$$
U := \{ \Omega \mid \Omega \text{ is a finite set of primes, } \Omega \cap S = \phi \}.
$$

Then the set $T := \{ \gamma(\Omega, a, q) \mid \Omega \in U \}$ is infinite and has at most one algebraic element.

**Theorem 4.** (Gun, Saha and Sinha) Let $\Omega$ be a finite set of primes and $S = \{q_1, q_2, \ldots \}$ be an infinite set of mutually co-prime natural numbers $q_i > 1$ with $(q_i, P_\Omega) = 1$ for all $i \in \mathbb{N}$. Then for any $a \in \mathbb{N}$ with $(a, q_i) = 1$ $\forall i$, the set

$$
T := \{ \gamma(\Omega, a, q_i) \mid q_i \in S \}
$$

has at most one algebraic element.

In order to prove these theorems, one needs to find a closed formula for generalized Euler-Briggs constants. In [6], it was proved that

$$
(2) \quad \gamma(\Omega, a, q) = \frac{1}{\varphi(q)} \sum_{\chi \mod q \atop \chi \neq \chi_0} \bar{\chi}(a) L(1, \chi) \prod_{p \in \Omega} \left( 1 - \frac{\chi(p)}{p} \right) + \frac{\delta_\Omega}{q} (\gamma + \sum_{p \mid q} \log \frac{p}{p-1} + \sum_{p \in \Omega} \log \frac{p}{p-1}),
$$

where $\Omega$ is a finite subset of primes, $a, q$ are co-prime natural numbers with $(q, P_\Omega) = 1$. 

The articles [6, 7] can be thought of a generalization of the work of R. Murty and Zaytseva [17] on generalized Euler constants. Whereas the results of R. Murty and Zaytseva are obtained by Hermite and Lindemann theorem, the proofs in [6] require careful analysis of units in cyclotomic fields and Baker’s theorem on linear forms in logarithms.

The above theorems do not answer the question of linear independence of these constants over a number field or over \( \mathbb{Q} \).

In this article, we establish non-trivial lower bounds for the dimension of these spaces. To start with, we have the following theorem over the field of rational numbers \( \mathbb{Q} \).

**Theorem 5.** Let \( \Omega \subset P \) be a finite subset of primes and \( P_\Omega \) be as in (1). Consider the \( \mathbb{Q} \)-vector space

\[
V_{\mathbb{Q}, N} := \mathbb{Q} \langle \gamma(\Omega, m, n) \mid 1 \leq m \leq n \leq N, \ (m, n) = 1 = (n, P_\Omega) \rangle.
\]

Then for \( N \) sufficiently large, we have

\[
N \ll_{\Omega} \dim_{\mathbb{Q}} V_{\mathbb{Q}, N},
\]

where the implied constant depend on \( \Omega \). In particular, the dimension of the \( \mathbb{Q} \)-vector space

\[
V_{\mathbb{Q}} := \mathbb{Q} \langle \gamma(\Omega, m, n) \mid m, n \in \mathbb{N}, \ (m, n) = 1 = (n, P_\Omega) \rangle
\]

is infinite.

In fact, one has the following general theorem about linear independence of these constants over number fields.

**Theorem 6.** Let \( K \) be a number field with discriminant \( d > 1 \), \( \Omega \subset P \) be a finite subset of primes, \( P_\Omega \) be as in (1) such that \( K \cap \mathbb{Q}(\zeta_{P_\Omega}) = \mathbb{Q} \), where \( \zeta_{P_\Omega} := e^{2\pi i / P_\Omega} \). Consider the \( K \)-vector space

\[
V_{K, N} := K \langle \gamma(\Omega, m, n) \mid 1 \leq m \leq n \leq N, \ (m, n) = 1 = (n, dP_\Omega) \rangle.
\]

Then for \( N \) sufficiently large, we have

\[
N \ll_{K, \Omega} \dim_{K} V_{K, N},
\]

where the implied constant depend on \( \Omega \) and \( K \). In particular, the \( K \)-vector space

\[
V_{K} := K \langle \gamma(\Omega, m, n) \mid m, n \in \mathbb{N}, \ (m, n) = 1 = (n, dP_\Omega) \rangle
\]

is infinite dimensional.

**Remark 1.1.** Note the trivial upper bounds for \( \dim_{\mathbb{Q}} V_{\mathbb{Q}, N} \) in Theorem 5 and for \( \dim_{K} V_{K, N} \) in Theorem 6 are \( N^2 \).

Next we study the linear independence of these constants over the field of algebraic numbers. In order to do so, let us set

\[
C(q) := \{ \Omega \subset P \mid |\Omega| < \infty, \ (q, P_\Omega) = 1 \}, \quad \text{where } q \in \mathbb{N}.
\]
We define an equivalence relation on the set of all \( \gamma(\Omega, a, q) \)'s as \( \Omega \) varies over elements of \( C(q) \) and \( a, q \) are co-prime natural numbers. We say that \( \gamma(\Omega_1, a, q) \) and \( \gamma(\Omega_2, a, q) \) are equivalent, denoted by \( \gamma(\Omega_1, a, q) \sim \gamma(\Omega_2, a, q) \), if there exists \( \lambda \in \mathbb{Q} \setminus \{0\} \) such that \( \gamma(\Omega_1, a, q) = \lambda \gamma(\Omega_2, a, q) \).

In this set-up, we prove the following theorem.

**Theorem 7.** Let \( a, q \) be natural numbers with \( (a, q) = 1 \). Consider the set

\[
M_1 := \{ \gamma(\Omega, a, q) \mid \Omega \in C(q) \}.
\]

Then each equivalence class \( [\gamma(\Omega, a, q)] \) in \( M_1 \) has at most two elements.

Next let \( \Omega \subset P \) be a finite set, \( P_\Omega \) be as in (1), \( a \in \mathbb{N} \) and

\[
C(a, \Omega) := \{ q \in \mathbb{N} \mid (a, q) = 1 = (q, P_\Omega) \}.
\]

As before, one can define an equivalence relation on the set \( \gamma(\Omega, a, q) \)'s, where \( q \in C(a, \Omega) \). In this set-up, we prove;

**Theorem 8.** Let \( \Omega \) be a finite set of primes, \( \{q_i\} \) be a sequence of mutually co-prime natural numbers and \( a \) be a natural number such that \( (a, q_i) = 1 \) for all \( i \). Consider the set

\[
M_2 := \{ \gamma(\Omega, a, q_i) \mid q_i \in C(a, \Omega) \}.
\]

Then each equivalence class \( [\gamma(\Omega, a, q_i)] \) in \( M_2 \) has at most two elements.

We see that Theorem 7 and Theorem 8 give information about pairwise \( \mathbb{Q} \)-independence of the generalized Euler-Briggs constants. However, they do not say that the vector space generated by these constants over \( \mathbb{Q} \) is infinite dimensional. In this regard, we have the following theorem;

**Theorem 9.** Let \( a, q \) be natural numbers with \( (a, q) = 1 \). Then the dimension of the \( \mathbb{Q} \)-vector space

\[
V_\mathbb{Q} := \mathbb{Q} \langle \gamma(\Omega, a, q) \mid \Omega \in C(q) \rangle
\]

is infinite over \( \mathbb{Q} \).

We end this section with a brief outline of the structure of the paper. In §2, we list the inputs from transcendence theory relevant to our work. We also state a general non-vanishing result (Theorem 12) which is integral to our investigation. The proof of this theorem is detailed in §3. We devote §4 to prove all the linear independence results indicated in the introduction. Finally, in §5, we state and prove some algebraic independence results assuming the Weak Schanuel conjecture.
2. Preliminaries

In this section, we state the theorems which will be required to prove our results. The first and the third author proved the following theorem about the existence of an infinite sum (see [8]).

**Theorem 10.** (Gun and Saha) Let \( f \) be a periodic arithmetic function with period \( q \geq 1 \) and \( M \) be a natural number co-prime to \( q \). Then

\[
\sum_{n \geq 1 \atop (n,M)=1} f(n) \frac{n}{n} \text{ exists if and only if } \sum_{a=1}^{q} f(a) = 0.
\]

Moreover, whenever the above sum exists, we have

\[
\sum_{n \geq 1 \atop (n,M)=1} f(n) \frac{n}{n} = \sum_{a=1}^{q} f(a) \gamma(\Omega, a, q),
\]

where \( \Omega \) is the set of prime divisors of \( M \).

An important ingredient to prove Theorem 6 is the following non-vanishing result of Baker, Birch and Wirsing (see [2], see also chapter 23 of [15]).

**Theorem 11.** (Baker, Birch and Wirsing). Let \( f \) be a non-zero algebraic valued periodic function with period \( q \) defined on the set of integers. Also let \( f(n) = 0 \) whenever \( 1 < (n,q) < q \) and the \( q \)-th cyclotomic polynomial \( \Phi_q(X) \) be irreducible over \( \mathbb{Q}(f(1), \ldots, f(q)) \), then

\[
\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.
\]

Other important theorems that are required to prove our results are the following.

**Theorem 12.** Let \( q_1, q_2, q_3 > 1 \) be mutually co-prime natural numbers. Then for any algebraic numbers \( \alpha_p, \beta_\chi, \beta_\phi, \beta_\psi \), the number

\[
\sum_{p \mid q_1 q_2 q_3} \alpha_p \log p + \sum_{\chi \mod q_1 \atop \chi \neq 0} \beta_\chi L(1, \chi) + \sum_{\phi \mod q_2 \atop \phi \neq 0} \beta_\phi L(1, \phi) + \sum_{\psi \mod q_3 \atop \psi \neq 0} \beta_\psi L(1, \psi)
\]

is transcendental provided not all \( \alpha_p, \beta_\chi, \beta_\phi, \beta_\psi \) for even characters \( \chi, \phi, \psi \) are zero.

This result is new and we will give a proof of this result in the next section. A particular case of Theorem 12 was noticed in [6]. In the same paper, the authors also proved the following theorem which will be required to prove Theorem 9.

**Theorem 13.** (Gun, Saha and Sinha) Let \( q > 1 \) be a natural number and \( \Omega_1, \ldots, \Omega_t \in C(q) \) be disjoint subsets of prime numbers. Then for any algebraic numbers \( \alpha_p, \beta_\chi, \epsilon_{\Omega_i, p} \), the number

\[
\sum_{p \mid q} \alpha_p \log p + \sum_{i=1}^{t} \sum_{p \in \Omega_i} \epsilon_{\Omega_i, p} \log p + \sum_{\chi \mod q \atop \chi \neq 0} \beta_\chi L(1, \chi)
\]
is transcendental provided not all \( \alpha_p, \epsilon_{\Omega,p}, \beta_\chi \) for even characters \( \chi \) are zero.

We shall be using the following result of Baker (see pages 10 and 11 of [1], see also chapter 19 of [15]).

**Theorem 14.** (Baker) Let \( \alpha_1, \cdots, \alpha_n \in \mathbb{Q} \setminus \{0\} \) and \( \beta_1, \cdots, \beta_n \in \mathbb{Q} \), then

\[
\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n
\]

is either zero or transcendental. The latter case arises if \( \log \alpha_1, \cdots, \log \alpha_n \) are linearly independent over \( \mathbb{Q} \) and not all \( \beta_1, \cdots, \beta_n \) are zero.

Finally, we will be using the following result about the non-vanishing of certain special linear forms in logarithms of non-zero algebraic numbers.

**Theorem 15.** (R. Murty and Saradha [16], see also R. Murty and K. Murty [14]) Let \( \alpha_1, \cdots, \alpha_n \) be positive algebraic numbers. If \( \beta_0, \cdots, \beta_n \) are algebraic numbers with \( \beta_0 \neq 0 \), then

\[
\beta_0 \pi + \sum_{i=1}^{n} \beta_i \log \alpha_i
\]

is a transcendental number and hence non-zero.

3. **Proof of Theorem 12**

We will prove this theorem by contradiction. We know that for any even Dirichlet character \( \chi \neq \chi_0 \), one can write \( L(1, \chi) \) as a non-zero algebraic multiple of

\[
\sum_{1 < a < q/2 \atop (a,q) = 1} \overline{\chi}(a) \log \xi_a,
\]

where \( \xi_a \)'s are real multiplicatively independent units in the cyclotomic field \( \mathbb{Q}(\zeta_q) \), known as Ramachandra units (see pages 147 to 149 of [19], page 149 of [21] as well as page 1728 of [14]). For any odd Dirichlet character \( \chi \), we know that \( L(1, \chi) \) is a non-zero algebraic multiple of \( \pi \) (see page 38 of [21]). Using these results and Theorem 15, we can therefore ignore the odd characters. In order to complete the proof of the theorem, we will now show that

1. \( \log p \) : for all primes \( p | q_1q_2q_3 \)
2. \( L(1, \chi) \) : for all even non-principal characters \( \chi \) modulo \( q_1 \)
3. \( L(1, \phi) \) : for all even non-principal characters \( \phi \) modulo \( q_2 \)
4. \( L(1, \psi) \) : for all even non-principal characters \( \psi \) modulo \( q_3 \)

are linearly independent over \( \overline{\mathbb{Q}} \). Suppose not. Then there exists algebraic numbers \( \alpha_p \) for \( p | q_1q_2q_3 \) and \( \beta_\chi, \beta_\phi, \beta_\psi \), where \( \chi, \phi, \psi \) vary over non-principal even Dirichlet characters modulo
\[ q_1, q_2 \text{ and } q_3 \text{ respectively, not all zero, such that} \]
\[
\sum_{p|q_1q_2q_3} \alpha_p \log p + \sum_{\chi \neq \chi_0} \beta_\chi L(1, \chi) + \sum_{\phi \neq \phi_0} \beta_\phi L(1, \phi) + \sum_{\psi \neq \psi_0} \beta_\psi L(1, \psi) = 0.
\]

We can rewrite the above expression as
\[
\sum_{p|q_1q_2q_3} \alpha_p \log p + \sum_{1<a<q_1/2} \delta_a \log \xi_a + \sum_{1<b<q_2/2} \delta_b \log \xi_b + \sum_{1<c<q_3/2} \delta_c \log \xi_c = 0,
\]
where \( \xi_a, \xi_b, \xi_c 's \) are multiplicatively independent units in \( \mathbb{Q}(\zeta_{q_1}), \mathbb{Q}(\zeta_{q_2}) \) and \( \mathbb{Q}(\zeta_{q_3}) \) respectively. Now by Baker’s Theorem, we have
\[
(4) \quad \prod_{p|q_1q_2} p^{c_p} = \prod_{1<a<q_1/2} \xi_a^{d_a} \prod_{1<b<q_2/2} \xi_b^{e_b} \prod_{1<c<q_3/2} \xi_c^{f_c}
\]
where \( c_p, d_a, e_b, f_c 's \) are integers. By taking norms on both sides of (4), we get \( c_p = 0 \) for all \( p \). Hence
\[
(5) \quad \prod_{1<a<q_1/2} \xi_a^{d_a} = \prod_{1<b<q_2/2} \xi_b^{e_b} \prod_{1<c<q_3/2} \xi_c^{f_c}
\]
Since \( q_1, q_2, q_3 \) are mutually co-prime, \( \mathbb{Q}(\zeta_{q_1}) \cap \mathbb{Q}(\zeta_{q_2q_3}) = \mathbb{Q} \). So we see that both sides of (5) are rational numbers and hence equal to \( \pm 1 \). Now squaring both sides, we get
\[
(6) \quad \prod_{1<a<q_1/2} \xi_a^{2d_a} = \prod_{1<b<q_2/2} \xi_b^{2e_b} \prod_{1<c<q_3/2} \xi_c^{2f_c} = 1.
\]
This forces that \( d_a = 0 \) for all \( a \) since \( \xi_a 's \) are multiplicatively independent. Again going back to (5) and following the same argument, we get \( e_b = 0, f_c = 0 \) for all \( b, c \). This completes the proof.

4. Proofs of Linear independence results

4.1. Proof of Theorem 9 It is sufficient to show that given any natural number \( n \), there exist disjoint subsets \( \Omega_1, \cdots, \Omega_n \subset C(q) \) such that \( \gamma(\Omega_1, a, q), \cdots, \gamma(\Omega_n, a, q) \) are linearly independent over \( \mathbb{Q} \). Suppose that our claim is not true. Then there exists an \( n \in \mathbb{N} \) such that for any disjoint sets \( \Omega_1, \cdots, \Omega_n \subset C(q) \) and \( \Omega'_1, \cdots, \Omega'_n \subset C(q) \), we can find \( \alpha_i, \beta_j \in \overline{\mathbb{Q}}, 1 \leq i, j \leq n \), not all zero such that
\[
\alpha_1 \gamma(\Omega_1, a, q) + \cdots + \alpha_n \gamma(\Omega_n, a, q) = 0 \quad \text{and} \quad \beta_1 \gamma(\Omega'_1, a, q) + \cdots + \beta_n \gamma(\Omega'_n, a, q) = 0.
\]
Further assume that \( \Omega_i \)'s are disjoint from \( \Omega'_j \)'s for all \( 1 \leq i, j \leq n \). Then by (2), we have

\[
(7) \quad \gamma \sum_{i=1}^{n} \alpha_i \delta_{\Omega_i} = \frac{-q}{\varphi(q)} \sum_{\chi \mod q \atop \chi \not= \chi_0} \chi(a) L(1, \chi) \sum_{i=1}^{n} \alpha_i \prod_{p \in \Omega_i} (1 - \frac{\chi(p)}{p}) - \sum_{p \mid q} \log \frac{p}{p-1} \sum_{i=1}^{n} \alpha_i \delta_{\Omega_i}
\]

and

\[
\gamma \sum_{j=1}^{n} \beta_j \delta_{\Omega'_j} = \frac{-q}{\varphi(q)} \sum_{\chi \mod q \atop \chi \not= \chi_0} \chi(a) L(1, \chi) \sum_{j=1}^{n} \beta_j \prod_{p \in \Omega'_j} (1 - \frac{\chi(p)}{p}) - \sum_{p \mid q} \log \frac{p}{p-1} \sum_{j=1}^{n} \beta_j \delta_{\Omega'_j}
\]

Applying Theorem 13 we see that \( A := \sum_{i=1}^{n} \alpha_i \delta_{\Omega_i} \neq 0 \) and \( B := \sum_{j=1}^{n} \beta_j \delta_{\Omega'_j} \neq 0 \). Hence from (7), we have

\[
\frac{q}{\varphi(q)} \sum_{\chi \mod q \atop \chi \not= \chi_0} \chi(a) L(1, \chi) \left( \sum_{i=1}^{n} \frac{\alpha_i}{A} \prod_{p \in \Omega_i} (1 - \frac{\chi(p)}{p}) - \sum_{j=1}^{n} \frac{\beta_j}{B} \prod_{p \in \Omega'_j} (1 - \frac{\chi(p)}{p}) \right) + \sum_{i=1}^{n} \frac{\alpha_i \delta_{\Omega_i}}{A} \sum_{p \in \Omega_i} \frac{\log p}{p-1} - \sum_{j=1}^{n} \frac{\beta_j \delta_{\Omega'_j}}{B} \sum_{p \in \Omega'_j} \frac{\log p}{p-1} = 0,
\]

a contradiction to Theorem 13. This completes the proof of Theorem 9.

4.2. Proofs of Theorem 5 and Theorem 6. Now we will give a proof of Theorem 5. For any finite subset \( \Omega \subset P \) and \( P_\Omega \) as in (1), define

\[
S_\Omega := \{ u \in \mathbb{N} \mid (u, P_\Omega) = 1 \}
\]

and for any natural number \( u \in S_\Omega \), let us set

\[
\Gamma_{\Omega, u} := \{ \gamma(\Omega, v, u) \mid 1 \leq v \leq u, (v, u) = 1 \}.
\]

Note that the cardinality of \( \Gamma_{\Omega, u} \) is \( \varphi(u) \). We claim that for any two pairwise co-prime natural numbers \( q, r \in S_\Omega \), either the set of numbers \( \Gamma_{\Omega, q} \) is linearly independent over \( \mathbb{Q} \) or the set of numbers \( \Gamma_{\Omega, r} \) is linearly independent over \( \mathbb{Q} \).

Suppose that our claim is not true. Then there exists \( \alpha_a, \beta_b \in \mathbb{Q} \), not all zero, for \( 1 \leq a < q \) and \( 1 \leq b < r \) with \( (a, q) = 1 = (b, r) \) such that

\[
(8) \quad \sum_{1 \leq a < q \atop (a, q) = 1} \alpha_a \gamma(\Omega, a, q) = 0 \quad \text{and} \quad \sum_{1 \leq b < r \atop (b, r) = 1} \beta_b \gamma(\Omega, b, r) = 0.
\]
Define two arithmetic functions as follows;

\[
f(n) := \begin{cases} 
\alpha_a & \text{if } n \equiv a \pmod{q}, \ (a,q) = 1, \\
- \sum_{1 \leq a \leq q \atop (a,q) = 1} \alpha_a & \text{if } n \equiv 0 \pmod{q}, \\
0 & \text{otherwise}, 
\end{cases}
\]

and

\[
g(n) := \begin{cases} 
\beta_b & \text{if } n \equiv b \pmod{r}, \ (b,r) = 1, \\
- \sum_{1 \leq b \leq r \atop (b,r) = 1} \beta_b & \text{if } n \equiv 0 \pmod{r}, \\
0 & \text{otherwise}. 
\end{cases}
\]

Then \( f \) and \( g \) are periodic functions with periods \( q \) and \( r \) respectively. Further

\[
\sum_{1 \leq a \leq q} f(a) = 0 \quad \text{and} \quad \sum_{1 \leq b \leq r} g(b) = 0.
\]

Hence by Theorem 10, we have

\[
\sum_{n \geq 1 \atop (n,P_{\Omega})=1} \frac{f(n)}{n} = \frac{1}{q} \gamma(\Omega,a,q) \quad \text{and} \quad \sum_{m \geq 1 \atop (m,P_{\Omega})=1} \frac{g(m)}{m} = \frac{1}{r} \gamma(\Omega,b,r).
\]

Now equations (8), (9), (10) and the fact

\[
\gamma(\Omega,q,q) = \frac{1}{q} \left( \gamma(\Omega) - \delta_{\Omega} \log q \right),
\]

imply that

\[
\sum_{n \geq 1 \atop (n,P_{\Omega})=1} \frac{f(n)}{n} = \frac{f(q)}{q} \left( \gamma(\Omega) - \delta_{\Omega} \log q \right)
\]

and

\[
\sum_{m \geq 1 \atop (m,P_{\Omega})=1} \frac{g(m)}{m} = \frac{g(r)}{r} \left( \gamma(\Omega) - \delta_{\Omega} \log r \right).
\]

Note that \( f(q) \) and \( g(r) \) can not be zero. Indeed, if for example \( f(q) = 0 \), then

\[
\sum_{n \geq 1 \atop (n,P_{\Omega})=1} \frac{f(n)}{n} = 0.
\]

However, viewing \( f_{\chi_0} \) as a periodic function modulo \( qP_{\Omega} \), where \( \chi_0 \) is the trivial character modulo \( P_{\Omega} \), we have

\[
(f_{\chi_0})(n) = 0 \quad \text{for all} \quad 1 < (n,qP_{\Omega}) < qP_{\Omega}.
\]
Hence by Theorem 11, we have
\[ \sum_{n \geq 1} \left( \frac{f(n)}{n} \right)_{(n,P) = 1} \neq 0, \]
a contradiction to (12). As \( f(q) \neq 0 \) and \( g(r) \neq 0 \), we have
\[ \frac{q}{f(q)} \sum_{n \geq 1} \left( \frac{f(n)}{n} \right)_{(n,P) = 1} + \delta_\Omega \log q - \frac{r}{g(r)} \sum_{m \geq 1} \left( \frac{g(m)}{m} \right)_{(m,P) = 1} - \delta_\Omega \log r = 0. \]
By Theorem 10 and equations (2), (9), (11), we have
\[ \frac{q}{f(q)} \sum_{1 \leq a < q} f(a) \gamma(\Omega, a, q) + \gamma(\Omega) - \delta_\Omega \log q = \frac{q}{f(q)} \sum_{\chi \mod q} L(1, \chi) \prod_{p \in \Omega} \left( 1 - \frac{\chi(p)}{p} \right) \sum_{1 \leq a \leq q, \,(a,qP) = 1} f(a) \chi(a) - \delta_\Omega \sum_{p \mid q} \frac{\log p}{p-1}. \]
Similarly, we have
\[ \frac{r}{g(r)} \sum_{1 \leq b < r} g(b) \psi(b) - \delta_\Omega \sum_{p \mid r} \frac{\log p}{p-1}. \]
Replacing these two expressions in (13), we see that the left hand side of the above expression is a non-trivial algebraic linear combinations of \( L(1, \chi) \) as \( \chi \) varies over non-principal characters modulo \( q \), \( L(1, \psi) \) as \( \psi \) varies over non-principal character modulo \( r \), logarithms of prime divisors of \( q \) and logarithms of prime divisors of \( r \). Then by Theorem 12 this can not be equal to zero, a contradiction.

Thus there exists a natural number \( r_0 \in S_\Omega \) such that for any \( q \in \mathbb{N} \) with \( (q, r_0 P_\Omega) = 1 \), the family of numbers \( \Gamma_{\Omega, q} \) are linearly independent over \( \mathbb{Q} \). Using this, we will calculate the dimension of the space
\[ V_{\mathbb{Q}, N} := \mathbb{Q} \langle \gamma(\Omega, m, n) \mid 1 \leq m \leq n \leq N \in \mathbb{N}, \,(m,n) = 1 = (n, P_\Omega) \rangle, \]
where \( N \) is sufficiently large. To get a non-trivial lower bound on the dimension of \( V_{\mathbb{Q}, N} \), we will try to find a pair of prime numbers \( p, \ell \) in terms of \( N \).
Let \( t \) be the number of primes in \( \Omega \). Now using Bertrand’s Postulate, we get that there are at least \( t+2 \) primes between \( \frac{N}{2^t+2} \) and \( N \), where \( N > 2^{t+2} \). Hence there exist two primes \( p, \ell \geq \frac{N}{2^t+2} \) with \((p\ell, P_{\Omega}) = 1\). Thus

\[
\dim V_{\mathbb{Q},N} \geq \min\{\varphi(p), \varphi(\ell)\} = \min\{p - 1, \ell - 1\} \geq \frac{N}{2^{t+2}} - 1 \gg_{\Omega} N.
\]

This completes the proof of Theorem 5.

We now indicate the required modifications to derive Theorem 6. Let \( K \) be a number field with discriminant \( d > 1 \) and \( K \cap \mathbb{Q}(\zeta_{P_{\Omega}}) = \mathbb{Q} \). We claim that there exists a natural number \( r_0 \in S'_{\Omega} \) such that for any \( q \in \mathbb{N} \) with \((q, r_0 dP_{\Omega}) = 1\), a set of numbers \( \Gamma_{\Omega, q} \) (defined below) consisting of suitable \( \gamma(\Omega, a, q) \)'s with \(|\Gamma_{\Omega, q}| = \varphi(q)\) is linearly independent over \( K \).

In order to prove the claim, we replace the set \( S_{\Omega} \) in Theorem 5 by \( S'_{\Omega} := \{u \in \mathbb{N} \mid (u, P_{\Omega}) = 1, \Phi_{u P_{\Omega}}(X) \text{ is irreducible over } K\} \).

Since \( K \cap \mathbb{Q}(\zeta_{u P_{\Omega}}) = \mathbb{Q} \) if and only if \( \Phi_{u P_{\Omega}}(X) \text{ is irreducible over } K \), for any natural number \( u \) with \((u, dP_{\Omega}) = 1\), one has \( u \in S'_{\Omega} \). Consider the set

\[
\Gamma_{\Omega, u} := \{\gamma(\Omega, v, u) \mid 1 \leq v \leq u, (v, u) = 1\} \text{ for } u \in S'_{\Omega}.
\]

In order to complete the proof of the claim, we now define \( f, q \) as before. The proof of the claim now follows mutatis mutandis as in Theorem 5 except when we need to show that neither \( f(q) = 0 \) nor \( g(r) = 0 \). Once again we will use the theorem of Baker, Birch and Wirsing, but over number fields \( K \) with \([K : \mathbb{Q}] > 1\). This forces us to have the additional condition that \( \Phi_{u P_{\Omega}}(X) \text{ is irreducible over } K \). This is why we have replaced our set \( S_{\Omega} \) in Theorem 5 by \( S'_{\Omega} \) in Theorem 6.

We now complete the proof of Theorem 6. For a lower bound, we let \( s \) be the number of distinct prime divisors of \( d \) and \( t \) be the number of primes in \( \Omega \). Then again by Bertrand’s Postulate, we get that there are at least \( s + t + 2 \) many primes between \( \frac{N}{2^{s+t+2}} \) and \( N \) with \( N > 2^{s+t+2} \). Thus we can get two distinct primes \( p, \ell \geq \frac{N}{2^{s+t+2}} \) such that they are co-prime to \( dP_{\Omega} \). Then

\[
\dim V_{K,N} \geq \min\{\varphi(p), \varphi(\ell)\} = \min\{p - 1, \ell - 1\} \geq \frac{N}{2^{s+t+2}} - 1 \gg_{\Omega,K} N.
\]

4.3. **Proof of Theorem 7** Suppose that \( \gamma(\Omega_2, a, q), \gamma(\Omega_3, a, q) \in [\gamma(\Omega_1, a, q)] \), where \( \Omega_1, \Omega_2 \) and \( \Omega_3 \) are distinct elements in \( C(q) \). Then there exist non-zero algebraic numbers \( \beta, \lambda \) such that

\[
\gamma(\Omega_1, a, q) = \beta \gamma(\Omega_2, a, q) \quad \text{and} \quad \gamma(\Omega_1, a, q) = \lambda \gamma(\Omega_3, a, q).
\]
For a Dirichlet character $\chi$ modulo $q$ and a finite set $\Omega$ consisting of primes co-prime to $q$, we define

$$a_{\Omega} := \frac{\delta_{\Omega}}{q} \neq 0, \quad \gamma_1 := \gamma + \sum_{p \mid q} \frac{\log p}{p - 1}$$

and $\alpha_{\chi, \Omega} := \frac{\chi(a)}{\varphi(q)} \prod_{p \in \Omega} (1 - \frac{\chi(p)}{p}).$

Using (2) and (14), we get

$$\gamma_1 (a_{\Omega_1} - \beta a_{\Omega_2}) + a_{\Omega_1} \sum_{p \in \Omega_1} \frac{\log p}{p - 1} - \beta a_{\Omega_2} \sum_{p \in \Omega_2} \frac{\log p}{p - 1} + \sum_{\chi \mod q, \chi \neq \chi_0} L(1, \chi) (\alpha_{\chi, \Omega_1} - \beta \alpha_{\chi, \Omega_2}) = 0. \quad (15)$$

Similarly, we have

$$\gamma_1 (a_{\Omega_1} - \lambda a_{\Omega_3}) + a_{\Omega_1} \sum_{p \in \Omega_1} \frac{\log p}{p - 1} - \lambda a_{\Omega_3} \sum_{p \in \Omega_3} \frac{\log p}{p - 1} + \sum_{\chi \mod q, \chi \neq \chi_0} L(1, \chi) (\alpha_{\chi, \Omega_1} - \lambda \alpha_{\chi, \Omega_3}) = 0. \quad (16)$$

Since $\Omega_1, \Omega_2, \Omega_3$ are distinct sets of primes, applying Theorem 13 to equations (15) and (16), we get

$$a_{\Omega_1} - \beta a_{\Omega_2} \neq 0, \quad a_{\Omega_1} - \lambda a_{\Omega_3} \neq 0.$$

By the same reasoning, we see that

$$a_{\Omega_3} - \frac{\beta}{\lambda} a_{\Omega_2} \neq 0.$$

Again from (15) and (16), it follows that

$$\frac{a_{\Omega_1}(\beta a_{\Omega_2} - \lambda a_{\Omega_3})}{(a_{\Omega_1} - \beta a_{\Omega_2})(a_{\Omega_1} - \lambda a_{\Omega_3})} \sum_{p \in \Omega_1} \frac{\log p}{p - 1} - \frac{\beta a_{\Omega_2}}{(a_{\Omega_1} - \beta a_{\Omega_2})} \sum_{p \in \Omega_2} \frac{\log p}{p - 1} + \frac{\lambda a_{\Omega_3}}{(a_{\Omega_1} - \lambda a_{\Omega_3})} \sum_{p \in \Omega_3} \frac{\log p}{p - 1} + \sum_{\chi \mod q, \chi \neq \chi_0} L(1, \chi) A(\chi) = 0,$$

where

$$A(\chi) = \frac{(\alpha_{\chi, \Omega_1} - \beta \alpha_{\chi, \Omega_2})}{(a_{\Omega_1} - \beta a_{\Omega_2})} - \frac{(\alpha_{\chi, \Omega_3} - \lambda \alpha_{\chi, \Omega_3})}{(a_{\Omega_1} - \lambda a_{\Omega_3})}.$$

Since $\Omega_1, \Omega_2, \Omega_3$ are distinct sets of primes, without loss of generality, one can assume that there exists a prime $p_1 \in \Omega_1$ such that either $p_1 \notin \Omega_2 \cup \Omega_3$ or $p_1 \in \Omega_2$ but not in $\Omega_3$. The coefficient of $\log p_1$ in the first case is

$$\frac{a_{\Omega_1}(\beta a_{\Omega_2} - \lambda a_{\Omega_3})}{(a_{\Omega_1} - \beta a_{\Omega_2})(a_{\Omega_1} - \lambda a_{\Omega_3})(p_1 - 1)} \neq 0$$

and in the second case is

$$\frac{\lambda a_{\Omega_3}}{(\lambda a_{\Omega_3} - a_{\Omega_1})(p_1 - 1)} \neq 0.$$

Hence in both cases we arrive at a contradiction by Theorem 13.
4.4. Proof of Theorem

Suppose that \( \gamma(\Omega, a, q_2), \gamma(\Omega, a, q_3) \in [\gamma(\Omega, a, q_1)] \), where \( q_1, q_2, q_3 \) are distinct elements in \( C(a, \Omega) \). Then there exist non-zero algebraic numbers \( \beta, \lambda \) such that

\[
\gamma(\Omega, a, q_1) = \beta \gamma(\Omega, a, q_2), \quad \text{and} \quad \gamma(\Omega, a, q_1) = \lambda \gamma(\Omega, a, q_3).
\]

Write

\[
a_{q_i} := \frac{\delta_{\Omega}}{q_i} \neq 0, \quad \gamma_1 := \gamma + \sum_{p|\Omega} \frac{\log p}{p - 1}
\]

and

\[
\alpha_{\chi,q_i} := \frac{\chi(a)}{\varphi(q_i)} \prod_{p \in \Omega} (1 - \frac{\chi(p)}{p}).
\]

Using (2) and (17), we get

\[
\gamma_1(a_{q_1} - \beta a_{q_2}) + a_{q_1} \sum_{p|q_1} \frac{\log p}{p - 1} - \beta a_{q_2} \sum_{p|q_2} \frac{\log p}{p - 1} + \sum_{\chi \mod q_1, \chi \neq \chi_0} \alpha_{\chi,q_1} L(1, \chi)
\]

\[
- \beta \sum_{\chi \mod q_2, \chi \neq \chi_0} \alpha_{\chi,q_2} L(1, \chi) = 0.
\]

Similarly, we have

\[
\gamma_1(a_{q_1} - \lambda a_{q_3}) + a_{q_1} \sum_{p|q_1} \frac{\log p}{p - 1} - \lambda a_{q_3} \sum_{p|q_3} \frac{\log p}{p - 1} + \sum_{\chi \mod q_1, \chi \neq \chi_0} \alpha_{\chi,q_1} L(1, \chi)
\]

\[
- \lambda \sum_{\chi \mod q_3, \chi \neq \chi_0} \alpha_{\chi,q_3} L(1, \chi) = 0.
\]

Since \( q_1, q_2 \) and \( q_3 \) are mutually co-prime natural numbers, applying Theorem[12] to equations (18) and (19), we get

\[
a_{q_1} - \beta a_{q_2} \neq 0, \quad a_{q_1} - \lambda a_{q_3} \neq 0.
\]

Similar reasoning show that

\[
\beta a_{q_2} - \lambda a_{q_3} \neq 0.
\]

Hence we have

\[
Ca_{q_1} \sum_{p|q_1} \frac{\log p}{p - 1} - \frac{\beta a_{q_2}}{(a_{q_1} - \beta a_{q_2})} \sum_{p|q_2} \frac{\log p}{p - 1} + \frac{\lambda a_{q_3}}{(a_{q_1} - \lambda a_{q_3})} \sum_{p|q_3} \frac{\log p}{p - 1} + C \sum_{\chi \mod q_1, \chi \neq \chi_0} \alpha_{\chi,q_1} L(1, \chi)
\]

\[
+ \frac{\lambda}{(a_{q_1} - \lambda a_{q_3})} \sum_{\chi \mod q_3, \chi \neq \chi_0} \alpha_{\chi,q_3} L(1, \chi) - \frac{\beta}{(a_{q_1} - \beta a_{q_2})} \sum_{\chi \mod q_2, \chi \neq \chi_0} \alpha_{\chi,q_2} L(1, \chi) = 0,
\]

where

\[
C := \frac{\beta a_{q_2} - \lambda a_{q_3}}{(a_{q_1} - \beta a_{q_2})(a_{q_1} - \lambda a_{q_3})} \neq 0,
\]

a contradiction to Theorem[12] This completes the proof of the theorem.
5. Consequences of Weak Schanuel Conjecture

In this section, we state some conditional results on algebraic independence of Euler-Briggs constants assuming the Weak Schanuel conjecture (see page 111 of [15], see also [5]).

Conjecture 16. (Weak Schanuel) Let $\alpha_1, \cdots, \alpha_n$ be non-zero algebraic numbers such that the numbers $\log \alpha_1, \cdots, \log \alpha_n$ are $\mathbb{Q}$-linearly independent. Then $\log \alpha_1, \cdots, \log \alpha_n$ are algebraically independent.

Before we proceed further, let us fix few more notation. For $a, q \in \mathbb{N}$ with $1 \leq a \leq q$ and $(a, q) = 1$, we define

$$\gamma^*(\Omega, a, q) := \frac{q \gamma(\Omega, a, q)}{\delta_{\Omega}}.$$  

We say $\gamma^*(\Omega_1, a, q) \sim \gamma^*(\Omega_2, a, q)$ if there exists a non-zero algebraic number $\alpha$ such that

$$\gamma^*(\Omega_1, a, q) = \alpha \gamma^*(\Omega_2, a, q).$$

Note that $\gamma(\Omega_1, a, q) \sim \gamma(\Omega_2, a, q)$ if and only if $\gamma^*(\Omega_1, a, q) \sim \gamma^*(\Omega_2, a, q)$. Hence we will study $\gamma^*(\Omega, a, q)$ in place of $\gamma(\Omega, a, q)$ whenever convenient.

We call a finite sequence of sets $\{\Omega_1, \cdots, \Omega_n\}$ an irreducible sequence if

$$\bigcup_{i=1}^n \Omega_i \neq \bigcup_{j \in J} \Omega_j$$

for any proper subset $J \subset \{1, \cdots, n\}$. We call an infinite sequence of distinct sets $\{\Omega_n | n \in \mathbb{N}\}$ an irreducible sequence if every finite subsequence is irreducible. It is easy to see if

$$p_1 < p_2 < \cdots$$

is a sequence of distinct prime numbers and $\Omega_i = \{p_i\}$, then $\{\Omega_i\}$ is an irreducible sequence. On the other hand, the sequence

$$\{p_1\}, \{p_2\}, \{p_1, p_2\}, \{p_3\}, \{p_1, p_2, p_3\}, \cdots$$

where $p_i$’s are distinct prime numbers is not an irreducible sequence though it contains an irreducible subsequence. Here we have the following theorem.

Theorem 17. Suppose that the Weak Schanuel conjecture is true. Further, suppose that $T := \{\Omega_n\}_{n \in \mathbb{N}}$ is an infinite sequence of non-empty finite subsets of prime numbers co-prime to $q$. Consider the set

$$S_1 := \{ \gamma^*(\Omega_n, a, q) - \gamma - \sum_{\chi \neq \chi_0} \alpha^*_{\chi, \Omega_n, q} L(1, \chi) | \Omega_n \in T\},$$

where $\chi$ is a Dirichlet character modulo $q$ and

$$\alpha^*_{\chi, \Omega, q} := \chi(a) \prod_{p \in \Omega} (1 - \frac{\chi(p)}{p})(1 - \frac{1}{p})^{-1} \prod_{p | q} (1 - \frac{1}{p})^{-1}.$$  

Then the elements of $S_1$ are algebraically independent if the infinite sequence $T$ is irreducible.
Proof of Theorem 17. Let \( T = \{\Omega_n\}_{n \in \mathbb{N}} \) be an irreducible sequence where no \( \Omega_n \) contains any prime divisors of \( q \). By (2), we know that
\[
A_n := \gamma^*(\Omega_n, a, q) - \gamma - \sum_{\chi \neq \chi_0} \alpha_{\chi, \Omega_n, q}^* L(1, \chi) = \sum_{p \in \Omega_n} \frac{\log p}{p - 1} + \sum_{p|q} \frac{\log p}{p - 1}.
\]
Hence by Weak Schanuel’s conjecture 16, it is sufficient to show that the elements \( A_n \)'s for \( \Omega_n \in T \) are linearly independent over \( \mathbb{Q} \). If not, then there exists a finite subsequence \( T' = \{\Omega_{n_1}, \cdots, \Omega_{n_k}\} \) of \( T \) and integers \( m_1, \cdots, m_k \), not all zero, such that
\[
m_1 A_{n_1} + \cdots + m_k A_{n_k} = 0 \tag{20}
\]
Write \( \Omega := \bigcup_{i=1}^k \Omega_{n_i} \). Then applying (2) in (20), we get
\[
\sum_{p \in \Omega} t_p \log p + \sum_{\ell|q} r_{\ell} \log \ell = 0, \tag{21}
\]
where \( t_p, r_{\ell} \in \mathbb{Q} \) and \( p \in \Omega \) with \( (p, q) = 1 \). Since \( T' \) is an irreducible sequence and not all \( m_i \)'s are zero, it follows that not all \( t_p \)'s are zero, a contradiction to (21). This completes the proof of Theorem 17.

Before we state our next theorem, let us introduce a notation and a definition. For \( I \subseteq \mathbb{N} \), let \( P(I) \) be the set of all prime divisors of the elements of \( I \). A finite subset \( I \) of \( \mathbb{N} \) is called irreducible if and only if
\[
P(I) \neq \bigcup_{J \subseteq I} P(J). \tag{22}
\]
An infinite subset \( T \subseteq \mathbb{N} \) is called irreducible if all finite subset of \( T \) are irreducible. In this context, we have the following theorem.

Theorem 18. Suppose that the Weak Schanuel conjecture is true. Let \( \Omega \) be a finite set of primes. Further, suppose that \( T = \{q_i\}_{i \in \mathbb{N}} \) be an infinite irreducible sequence of natural numbers co-prime to the primes in \( \Omega \). Let \( a \in \mathbb{N} \) be such that \( (a, q_i) = 1 \) for all \( i \in \mathbb{N} \). Consider the set
\[
S_2 := \{\gamma^*(\Omega, a, q) - \gamma - \sum_{\chi \neq \chi_0} \alpha_{\chi, \Omega, q}^* L(1, \chi) \mid q \in T\},
\]
where \( \alpha_{\chi, \Omega, q}^* \) is as in Theorem 17. Then the elements of \( S_2 \) are algebraically independent.

Proof of Theorem 18. We have
\[
\gamma^*(\Omega, a, q) - \gamma - \sum_{\chi \neq \chi_0} \alpha_{\chi, \Omega, q}^* L(1, \chi) = \sum_{p \in \Omega} \frac{\log p}{p - 1} + \sum_{p|q} \frac{\log p}{p - 1}.
\]
Hence by Weak Schanuel’s conjecture [16] it is sufficient to show that the elements of $S_2$ are linearly independent over $\mathbb{Q}$. Suppose not, then there exists a finite subset $\{q_1, \cdots, q_n\}$ of $T$ and integers $m_1, \cdots, m_n$, not all 0, such that

$$\sum_{i=1}^{n} m_i \{ \gamma^* (\Omega, a, q_i) - \gamma - \sum_{\chi \neq \chi_0} \alpha_{\chi, \Omega, q_i} L(1, \chi) \} = 0.$$ 

So

$$\sum_{p \in \Omega} \frac{\log p}{p - 1} \sum_{i=1}^{n} m_i + \sum_{i=1}^{n} m_i \sum_{p | q_i} \frac{\log p}{p - 1} = 0.$$ 

Without loss of generality let $m_1$ be non-zero. Since $T$ is irreducible, by definition all finite subsets of $T$ are irreducible. Then by (22) we know that $P(q_1, q_2, \cdots, q_n) \neq P(q_2, q_3 \cdots, q_n)$ and hence there exists a prime $p$ such that $p | q_1$ but $p \nmid q_j$ for all $j \neq 1$. This implies that the coefficient of $\log p$ is $m_1/(p - 1) \neq 0$, a contradiction by Baker’s theorem.

Acknowledgments. Part of the work of the first and third author was supported by a DAE number theory grant. The work of the first author was also supported by her SERB grant. Some part of the work was done when the first author was visiting the Mathematics department of University of Bordeaux as an ALGANT scholar. She would like to thank the members of the institute for their hospitality and the ALGANT program for the financial support. We would also like to thank the referee for his/her suggestions which improved the content as well as presentation of the paper.

References

[1] A. Baker, Transcendental number theory, 2nd edn, Cambridge University Press, Cambridge, 1990.
[2] A. Baker, B. J. Birch and E. A. Wirsing, On a problem of Chowla, J. Number Theory 5 (1973), 224–236.
[3] W. E. Briggs, The irrationality of $\gamma$ or of sets of similar constants, Norske Vid. Selsk. Forh. (Trondheim) 34 (1961), 25–28.
[4] H. Diamond and K. Ford, Generalized Euler constants, Math. Proc. Cambridge Philos. Soc. 145 (2008), no. 1, 27–41.
[5] S. Gun, M. Ram Murty and P. Rath, Transcendence of the log gamma function and some discrete periods, J. Number Theory 129 (2009), no. 9, 2154–2165.
[6] S. Gun, E. Saha and S. B. Sinha, Transcendence of Generalized Euler-Lehmer constants, J. Number Theory 145 (2014), 329–339.
[7] S. Gun, E. Saha and S. B. Sinha, A generalisation of an identity of Lehmer, to appear in Acta. Arithmetica.
[8] S. Gun and E. Saha, A note on generalized Euler-Briggs constants, to appear in RMS Lecture Notes Series in Mathematics.
[9] M. Kontsevich and D. Zagier, Periods, In: Mathematics Unlimited, 2001 and Beyond, Springer, Berlin, 2001, pp. 771–808.
[10] M. Kontsevich, Periods, Mathématique et physique, 28–39, SMF Journ. Annu., 1999, Soc. Math. France, Paris, 1999.
[11] J. C. Lagarias, Euler’s constant: Euler’s work and modern developments, Bull. Amer. Math. Soc. 50 (2013), 527–628.
[12] D. H. Lehmer, Euler constants for arithmetical progressions, Acta Arith. 27 (1975), 125–142.
[13] K. Mahler, Applications of a theorem by A. B. Shidlovski, Proc. Roy. Soc. Ser. A 305 (1968) 149–173.
[14] M. Ram Murty and V. Kumar Murty, A problem of Chowla revisited, J. Number Theory 131 (2011), no. 9, 1723–1733.
[15] M. Ram Murty and P. Rath, Transcendental numbers, Springer, New York, 2014.
[16] M. Ram Murty and N. Saradha, Euler-Lehmer constants and a conjecture of Erdős, J. Number Theory 130 (2010), 2671–2682.
[17] M. Ram Murty and A. Zaytseva, Transcendence of generalized Euler constants, Amer. Math. Monthly 120 (2013), no. 1, 48–54.
[18] Kh. Hessami Pilehrood and T. Hessami Pilehrood, On a continued fraction expansion for Euler’s constant, J. Number Theory 133 (2013), 769–786.
[19] K. Ramachandra, On the units of cyclotomic fields, Acta Arith. 12 (1966/1967) 165–173.
[20] T. Rivoal, On the arithmetic nature of the values of the gamma function, Euler’s constant, and Gompertz’s constant, Michigan Math. J. 61 (2012), no. 2, 239–254.
[21] L. C. Washington, Introduction to cyclotomic fields, Graduate Texts in Mathematics 83, 2nd edn, Springer-Verlag, New York, 1997.

SANOLI GUN AND EKATA SAHA,
INSTITUTE OF MATHEMATICAL SCIENCES, C.I.T. CAMPUS, TARAMANI, CHENNAI, 600 113, INDIA.
E-mail address: sanoli@imsc.res.in
E-mail address: ekatas@imsc.res.in

V. KUMAR MURTY,
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, 40 ST. GEORGE STREET, TORONTO, ON, CANADA, M5S 2E4.
E-mail address: murty@math.toronto.edu