Nonlinear Boundary Value Problems via Minimization on Orlicz-Sobolev Spaces

J. V. Goncalves∗
Instituto de Matemática e Estatística, Universidade Federal de Goiás
74001-970 Goiânia, GO - Brasil
e-mail: goncalves.yva@pq.cnptq.br

M. L. M. Carvalho†
Departamento de Matemática, Universidade Federal de Goiás
75804-020 Jataí, GO - Brasil
e-mail: marcosleandro@jatai.ufg.br

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Abstract
We develop arguments on convexity and minimization of energy functionals on Orlicz-Sobolev spaces to investigate existence of solution to the equation

\[-\text{div}(\phi(|\nabla u|)\nabla u) = f(x, u) + h \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,\]

where \(\Omega \subset \mathbb{R}^N\) is a bounded smooth domain, \(\phi : (0, \infty) \rightarrow (0, \infty)\) is a suitable continuous function and \(f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}\) satisfies the Carathéodory conditions, while \(h : \Omega \rightarrow \mathbb{R}\) is a measurable function.

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1 Introduction
We develop arguments on convexity and minimization of energy functionals on Orlicz-Sobolev spaces to investigate existence of solution to the problem

\[-\text{div}(\phi(|\nabla u|)\nabla u) = f(x, u) + h \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,\]

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where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $\phi : (0, \infty) \rightarrow (0, \infty)$ is a continuous function satisfying

\begin{enumerate}[(i)]  \item $\lim_{s \to 0} s \phi(s) = 0$,  \item $\lim_{s \to \infty} s \phi(s) = \infty$, \end{enumerate}

\begin{enumerate}[(φ2)]  \item $s \mapsto s \phi(s)$ is nondecreasing in $(0, \infty)$, \end{enumerate}

We extend $s \mapsto s \phi(s)$ to $\mathbb{R}$ as an odd function and consider the associated even potential

$$\Phi(t) := \int_0^t s \phi(s) ds, \ t \in \mathbb{R}. $$

It follows from the continuity of $\phi$, (φ1) and (φ2) that $\Phi$ is increasing and convex. The function $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions, while $h : \Omega \rightarrow \mathbb{R}$ is assumed measurable. Further conditions will be imposed upon $f$ and $h$ in a while. Next we will introduce some notations concerning Orlicz and Orlicz-Sobolev spaces. The function $\Phi$ is said to satisfy the $\Delta_2$-condition, $\Phi \in \Delta_2$ for short, if there is a constant $K > 0$ such that

$$\Phi(2t) \leq K \Phi(t), \ t \geq 0. $$

The complementary function $\tilde{\Phi}$ associated to $\Phi$ is defined by

$$\tilde{\Phi}(t) := \max_{s \geq 0} \{ st - \Phi(s) \}, \ t \geq 0. $$

We recall, (cf. [17, thm 3, pg. 22]), that $\Phi, \tilde{\Phi} \in \Delta_2$ iff there are $\ell, m \in (1, \infty)$ such that

$$\ell \leq \frac{t^2 \phi(t)}{\Phi(t)} \leq m, \ t > 0, $$

We shall assume from now on that both $\Phi$ and $\tilde{\Phi}$ satisfy the $\Delta_2$-condition.

We recall, see e.g. Adams & Fournier [1], that the Orlicz Space associated with $\Phi$ is given by

$$L_\Phi(\Omega) := \left\{ u : \Omega \rightarrow \mathbb{R} \text{ measurable} \mid \int_\Omega \Phi \left( \frac{u(x)}{\lambda} \right) < +\infty \text{ for some } \lambda > 0 \right\}. $$

It is known, (cf. [1]), that the expression

$$\|u\|_\Phi = \inf \left\{ \lambda > 0 \mid \int_\Omega \Phi \left( \frac{u(x)}{\lambda} \right) \leq 1 \right\}$$

defines a norm in $L_\Phi(\Omega)$ named Luxemburg norm. By [4, lemma D2],

$$L_\Phi(\Omega) \hookrightarrow \text{cont} L^{\ell}(\Omega). $$

The corresponding Orlicz-Sobolev space, (also denoted $W^{1,\Phi}(\Omega)$), is defined as

$$W^{1,\Phi}(\Omega) = \left\{ u \in L_\Phi(\Omega) \mid \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega), \ i = 1, ..., N \right\}. $$
The usual Orlicz-Sobolev norm of $W^{1,\Phi}(\Omega)$ is

$$\|u\|_{1,\Phi} = \|u\|_{\Phi} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{\Phi}.$$  

Since we are assuming that $\Phi$ and $\tilde{\Phi}$ satisfy the $\Delta_2$-condition, $L_\Phi(\Omega)$ and $W^{1,\Phi}(\Omega)$ are separable, reflexive, Banach spaces, see e.g. [1]. We also set

$$W^{1,\Phi}_0(\Omega) = C_0^\infty(\Omega) \cap W^{1,\Phi}(\Omega).$$

One shows that $u \in W^{1,\Phi}_0(\Omega)$ means that $u = 0$ on $\partial \Omega$ in the trace sense, cf. Gossez [11].

In order to state our main results consider the potential function of $f$,

$$F(x, t) := \int_0^t f(x, s) ds$$

and the limit

$$A_\infty(x) := \limsup_{|s| \to \infty} \frac{F(x, s)}{|s|^\ell}.$$  

We shall assume that that there exist a number $A \geq 0$ and a nonnegative function $B \in L^1(\Omega)$ such that

$$F(x, s) \leq A|s|^\ell + B(x), \quad s \in \mathbb{R}, \quad \text{a.e. } x \in \Omega. \quad (1.4)$$

From now on suppose $1 < \ell, m < N$.

We set

$$\Phi_*^{-1}(t) := \int_0^t \frac{\Phi^{-1}(s)}{s^{\frac{N}{\ell}}} ds, \quad t > 0.$$  

The critical exponent function of $\Phi$, $\Phi_*$, is defined as the inverse function of $\Phi_*^{-1}$. It is known that $\Phi_*$ is an N-function, see e.g. Donaldson & Trudinger [5]. Moreover,

$$W^{1,\Phi}_0(\Omega) \hookrightarrow L_{\Phi_*}(\Omega). \quad (1.5)$$

**Remark 1.1** If $N \geq 3$ and $\phi(t) = 2$ then, by computing, we obtain for $t > 0$:

$$\Phi_*^{-1}(t) = \frac{t^{N-2}}{N-2} \quad \text{and} \quad \Phi_*(t) = t^{\frac{N}{N-2}}. \quad \text{The function } \Phi_* \text{ plays the role of the critical Sobolev exponent in the case of Sobolev spaces.}$$

**Remark 1.2** The operator $\text{div}(\phi(|\nabla u|)\nabla u) = \Delta_\phi u$ is referred to as the $\Phi$-Laplacian.
2 Main Results

Our main results are

**Theorem 2.1** Assume \((\phi_1), (\phi_2)\) and \((1.2)\). Suppose there is a number \(a \geq 0\) and a nonnegative function \(b \in L^1(\Omega)\) such that
\[
|f(x, s)| \leq a\Phi_*(s) + b(x), \quad s \in \mathbb{R} \quad a.e. \quad x \in \Omega. \tag{1.6}
\]
Assume also that \(F\) satisfies \((1.4)\) and
\[
\inf_{v \in W_0^{1, \Phi}(\Omega), \|v\|_\Phi = 1} \left\{ \int_\Omega \Phi(|\nabla v|)dx - \int_{\{v \neq 0\}} A_\infty(x)|v(x)|^2dx \right\} > 0. \tag{1.7}
\]
Then for \(h \in L_\Phi(\Omega)'\), there is \(u \in W_0^{1, \Phi}(\Omega)\) satisfying \((1.1)\) in the sense of distributions.

**Theorem 2.2** Assume \((\phi_1), (\phi_2), (1.2)\) and
\[
|f(x, s)| \leq a\Phi_*(s) + b(x)|s|, \quad s \in \mathbb{R} \quad a.e. \quad x \in \Omega, \tag{1.8}
\]
for some number \(a \geq 0\), and a nonnegative \(b \in L_{\Phi_*}(\Omega)\). Assume that \(F\) satisfies \((1.4)\), and \((1.7)\) holds. Then for \(h \in L_\Phi(\Omega)'\), there is a weak solution \(u \in W_0^{1, \Phi}(\Omega)\) of \((1.1)\).

**Remark 2.1** When \(\phi(t) \equiv 1\) one has
(i) \(W_0^{1, \Phi}(\Omega) = H_0^1(\Omega)\),
(ii) equation \((1.1)\) and condition \((1.8)\) become respectively
\[
-\Delta u = f(x, u) + h(x) \quad \text{in} \quad \Omega, \tag{1.9}
\]
and
\[
|f(x, s)| \leq a|s|^{2^* - 1} + b(x), \quad a.e. \quad x \in \Omega, \quad s \in \mathbb{R},
\]
(iii) condition \((1.7)\) becomes
\[
\inf_{v \in H_0^1(\Omega), \|v\|_2 = 1} \left\{ \int_\Omega |\nabla v|^2dx - \int_{\{v \neq 0\}} A_\infty(x)v^2dx \right\} > 0, \tag{1.10}
\]
(iv) finding a weak solution of \((1.1)\) means finding a weak solution of \((1.9)\),
(v) when \(A_\infty(x) \leq \alpha(x)\) for some \(\alpha \in L^\infty(\Omega)\) with \(\alpha \leq \lambda_1\) in \(\Omega\), and \(\alpha < \lambda_1\) on a subset of \(\Omega\) with positive measure, where \(\lambda_1\) is the principal eigenvalue of \((-\Delta, H_0^1(\Omega))\), then \((1.10)\) holds, (cf. [10]).
A classical result on integral equations which goes back to Hammerstein \[13\] shows that

\[-\Delta u = f(x, u) + h(x) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,\]

(1.11)
is solvable provided \(f(x, s)\) grows at most linearly in \(s\) and a condition such as

\[A_\infty(x) \leq \mu, \text{ a.e. } x \in \Omega \text{ for some } \mu \in \mathbb{R},\]

holds, with \(\mu < \lambda_1\).

Mawhin, Willem & Ward in \[15\] allowed subcritical growth on \(f(x, s)\) and a solution of (1.11) was shown to exist under the additional condition

\[A_\infty(x) \leq \alpha(x) \text{ a.e. } x \in \Omega \text{ for some } \alpha \in L^\infty(\Omega), \]

with \(\alpha \leq \lambda_1\) in \(\Omega\), \(\alpha < \lambda_1\) on a subset of \(\Omega\) with positive measure.

Goncalves in \[10\] allowed critical growth condition on \(f(x, s)\), obtaining solutions in the distribution sense under condition (1.10) which was introduced by Brézis & Oswald \[2\].

In this paper we go back to the setting above regarding problem (1.1), this time in the framework of Orlicz-Sobolev spaces. We refer the reader to the papers \[9, 3, 8, 7, 6, 14, 12, 16\] and their references for nonlinear boundary value problems on Orlicz-Sobolev spaces.

Problems involving the \(\Phi\)-Laplacian operator appear in nonlinear elasticity, plasticity and generalized Newtonian fluids, see e.g. \[6, 8\] and their references.

Consider the problem, (where the operator is an example of the general \(\Delta_\Phi\) above),

\[-\text{div} \left( \gamma \frac{\left(1 + |\nabla u|^2 - 1\right)^{\gamma-1}}{\sqrt{1 + |\nabla u|^2}} \nabla u \right) = f(x, u) + h \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,\]

(1.12)

where \(1 \leq \gamma < \infty\).

**Remark 2.2** We shall use the notation \(\gamma^* = N\gamma/(N-\gamma)\) for \(\gamma \in (1, N)\).

The results below will be proved by applying theorems 2.1 and 2.2.

**Theorem 2.3** Let \(1 < \gamma < N\). Assume that

\[|f(x, s)| \leq a|s|^\gamma + b(x), \quad s \in \mathbb{R} \text{ a.e. } x \in \Omega,\]

(1.13)

where \(a \geq 0\) is some constant, \(b \in L^1(\Omega)\) is nonnegative and

\[F(x, s) \leq A|s|^\gamma + B(x), \quad s \in \mathbb{R}, \text{ a.e. } x \in \Omega,\]

(1.14)

for some constant \(A \geq 0\) and some nonnegative function \(B \in L^1(\Omega)\). If in addition,

\[
\inf_{v \in W^{1,\gamma}_0, \|v\|_{L^\gamma} = 1} \left\{ \int_{\Omega} \left( \frac{1}{\sqrt{1 + |\nabla v|^2}} - 1 \right)^\gamma \, dx - \int_{\{v \neq 0\}} A_\infty(x)|v(x)|^\gamma \, dx \right\} > 0,
\]

(1.15)

then for each \(h \in L^{\infty}(\Omega)\) problem (1.12) admits a solution \(u \in W^{1,\gamma}_0(\Omega)\), in the distribution sense.
The result below is a variant of theorem 2.3 for the case of weak solutions.

**Theorem 2.4** Let $1 < \gamma < N$. Assume that
\[ |f(x,s)| \leq a|s|^\gamma - 1 + b(x), \quad s \in \mathbb{R} \text{ a.e. } x \in \Omega, \quad (1.16) \]
where $a \geq 0$ is some constant, $b \in L^{\gamma'}(\Omega)$ is nonnegative. If in addition, (1.14) and (1.15) hold then for each $h \in L^{\gamma'}(\Omega)$ problem (1.12) admits a weak solution $u \in W^{1,\gamma}_0(\Omega)$.

**Remark 2.3** The function
\[ \phi(t) = pt^p - 2 \ln(1 + t) + t^{p-1} + 1, \quad t > 0. \]
where $1 < p < N - 1$, satisfies $(\phi_1), (\phi_2)$ and (1.2). However, in this case $L_\Phi(\Omega)$ is not a Lebesgue space $L^q(\Omega)$. This follows by applying a result in [17, pg 156].

**Remark 2.4** In our arguments, $C$ will denote a positive (cumulative) constant.

## 3 Proofs of Theorems 2.1 and 2.2

At first, we recall that $L_\Phi(\Omega)' = L_{\Phi^*}(\Omega)$ and moreover,
\[ \langle h, u \rangle = \int_{\Omega} hudx, \quad u \in L_{\Phi}(\Omega), \]
(cf. [11 thm 8.19]). Consider the energy functional associated to (1.1),
\[ I(u) = \int_{\Omega} \Phi(|\nabla u|)dx - \int_{\Omega} F(x, u)dx - \int_{\Omega} hudx, \quad u \in W^{1,\Phi}_0(\Omega). \]
It follows by using $\Phi \in \Delta_2$ and (1.3) – (1.4) that $I : W^{1,\Phi}_0(\Omega) \rightarrow \mathbb{R}$ is defined. Next we state and prove some technical lemmas.

**Lemma 3.1** Assume $(\phi_1), (\phi_2)$, (1.2) and (1.4). Then $I$ is weakly lower semicontinuous, wise for short.

**Proof.** Let $(u_n) \subseteq W^{1,\Phi}_0(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{1,\Phi}_0(\Omega)$. Then $u_n \rightarrow u$ in $L_\Phi(\Omega)$ and, by eventually passing to subsequences, $u_n \rightarrow u$ a.e. in $\Omega$ and there is $\theta_2 \in L^\ell(\Omega)$ such that $|u_n| \leq \theta_2$ a.e. in $\Omega$. By (1.4) we have
\[ F(x, u_n) \leq A|\theta_2|^\ell + B(x). \]
Since $F$ is a Carathéodory function,
\[ F(x, u_n(x)) \rightarrow F(x, u(x)) \text{ a.e. } x \in \Omega. \]
By Fatou’s lemma,
\[
\limsup \int_{\Omega} F(x, u_n)dx \leq \int_{\Omega} F(x, u)dx.
\]

Hence
\[
I(u) \leq \liminf \left\{ \int_{\Omega} \Phi(|\nabla u_n|)dx - \int_{\Omega} F(x, u_n)dx - \int_{\Omega} h u_n dx \right\} = \liminf I(u_n)
\]
showing that I is wloc.

**Lemma 3.2** Assume \((\phi_1), (\phi_2), (1.2), (1.4) \) and \((1.6)\). Let \(u \in W_0^{1,\Phi}(\Omega)\) such that
\[
I(u) = \min_{v \in W_0^{1,\Phi}(\Omega)} I(v). \tag{1.17}
\]
Then \(u\) satisfies \((1.1)\) in the sense of distributions.

**Proof.** Let \(v \in C^\infty_0(\Omega)\) and \(0 < t < 1\). Then \(u + tv \in W_0^{1,\Phi}(\Omega)\) and
\[
0 \leq \frac{I(u+tv) - I(u)}{t} = \int_{\Omega} \left[ \frac{\Phi(|\nabla u + t \nabla v|)}{t} - \frac{\Phi(|\nabla u|)}{t} - \frac{F(x, u+tv) - F(x, u)}{t} - hv \right]dx \tag{1.18}
\]
We claim that
\[
\lim_{t \to 0^+} \int_{\Omega} \frac{\Phi(|\nabla u + t \nabla v|) - \Phi(|\nabla u|)}{t} dx = \int_{\Omega} \phi(|\nabla u|) \nabla u \nabla v dx. \tag{1.19}
\]
Indeed, take a function \(\theta_t\) such that
\[
\Phi(|\nabla u + t \nabla v|) - \Phi(|\nabla u|) = \phi(\theta_t) \theta_t [||\nabla u + t \nabla v| - |\nabla u||] \text{ a.e. in } \Omega \tag{1.20}
\]
and
\[
\min\{|\nabla u + t \nabla v|, |\nabla u|\} \leq \theta_t \leq \max\{|\nabla u + t \nabla v|, |\nabla u|\} \text{ a.e. in } \Omega. \tag{1.21}
\]
By \((1.21)\), \(\theta_t \to |\nabla u| \) a.e. in \(\Omega\) as \(t \to 0_+\). We infer that
\[
\lim_{t \to 0^+} \frac{\Phi(|\nabla u + t \nabla v|) - \Phi(|\nabla u|)}{t} = \phi(|\nabla u|) \nabla u \nabla v \text{ a.e. in } \Omega. \tag{1.22}
\]
By \((1.20), (1.21)\) and \((\phi_2)\) we get
\[
\left| \frac{\Phi(|\nabla u + t \nabla v|) - \Phi(|\nabla u|)}{t} \right| \leq \phi(|\nabla u| + |\nabla v|)(|\nabla u| + |\nabla v|)|\nabla v|. \tag{1.23}
\]
By [7, lemma A.2] one has $\Phi(t\phi(t)) \leq \Phi(2t)$ for $t \in \mathbb{R}$, so that
$$\phi(|\nabla u| + |\nabla v|)(|\nabla u| + |\nabla v|) \in L_2(\Omega)$$
and by the H"older Inequality (cf. [1]),
$$\phi(|\nabla u| + |\nabla v|)(|\nabla u| + |\nabla v|) |\nabla v| \in L^1(\Omega).$$

By [122], [123] and Lebesgue’s theorem, (1.19) follows.

We claim that
$$\lim_{t \to 0^+} \int_\Omega \frac{F(x, u + tv) - F(x, u)}{t} dx = \int_\Omega f(x, u)v dx. \quad (1.24)$$
Indeed, take a function $\rho_t$ such that
$$F(x, u + tv) - F(x, u) = f(x, \rho_t(x))v \text{ a.e. in } \Omega$$
and
$$\min\{u + tv, u\} \leq \rho_t \leq \max\{u + tv, u\} \text{ a.e. in } \Omega. \quad (1.25)$$

Using (1.5) we infer that $\Phi_*([u] + |v|) \in L^1(\Omega)$. Using (1.6) and (1.25) we have
$$|f(x, \rho_t)v| \leq a\Phi_*([u] + |v|) |v| + b|v| \leq (a\Phi_*([u] + |v|) + b) |v|_\infty.$$

By Lebesgue theorem, (1.24) follows. Passing to the limit in (1.18) we infer that $u$ is a distribution solution of (1.1). This proves lemma 3.2.

Proof. (of Theorem 2.1) At first we show that $I$ is coercive. Indeed, assume by the way of contradiction, that there is $(u_n) \subseteq W_0^{1, \phi}(\Omega)$ such that
$$\|\nabla u_n\|_\phi \to \infty \text{ and } I(u_n) \leq C.$$
Using (1.4) and the H"older Inequality we have
$$\int_\Omega \Phi(|\nabla u_n|) dx \leq A \int_\Omega |u_n|^\ell dx + 2\|h\|_\Phi\|u_n\|_\phi + C \quad (1.26)$$
We claim that $\int_\Omega |u_n|^\ell dx \to \infty$. Indeed, assume on the contrary, that
$$\int_\Omega |u_n|^\ell dx \leq C.$$

By (1.26) and Poincaré’s Inequality (cf. [11]),
$$\int_\Omega \Phi(|\nabla u_n|) dx \leq C(1 + \|\nabla u_n\|_\phi),$$
which is impossible because by [11, lemma 3.14],
\[
\int_{\Omega} \frac{\Phi(|\nabla u_n|) dx}{\|\nabla u_n\|_\Phi} \to \infty,
\]
showing that \(\int_{\Omega} |u_n| dx \to \infty\). We infer, using (1.3) that \(\|u_n\|_\Phi \to \infty\).

By (1.26) and lemma 5.1 in the Appendix, we have
\[
\|\nabla u_n\|_\Phi \leq \int_{\Omega} \Phi(|\nabla u_n|) dx \leq C\|u_n\|_\Phi^\ell + 2\|h\|_{\tilde{\Phi}}\|u_n\|_\Phi + C. \tag{1.27}
\]
Dividing in (1.27) by \(\|u_n\|_\Phi^\ell\) we get
\[
\|\nabla v_n\|_\Phi^\ell \leq C + 2\|h\|_{\tilde{\Phi}} + \frac{C}{\|u_n\|_\Phi^\ell}.
\]
where \(v_n = \frac{u_n}{\|u_n\|_\Phi}\). It follows that \((\|\nabla v_n\|_\Phi)\) is bounded. Passing to a subsequence, we have,

- \(v_n \rightharpoonup v\) in \(W^{1,\Phi}(\Omega)\) and \(\int_{\Omega} \Phi(|\nabla v|) dx \leq \liminf_{n \to \infty} \int_{\Omega} \Phi(|\nabla v_n|) dx\),

- \(v_n \to v\) in \(L_\Phi(\Omega)\) and there is \(\theta_3 \in L^\ell(\Omega)\) such that \(|v_n| \leq \theta_3\) a.e. in \(\Omega\).

We claim that
\[
\limsup \int_{\Omega} \frac{F(x, \|u_n\|_\Phi v_n)}{\|u_n\|_\Phi^\ell} dx \leq \int_{\{v \neq 0\}} A_\infty(x) |v(x)|^\ell dx. \tag{1.28}
\]
Indeed, it follows using (1.4) that
\[
\frac{F(x, \|u_n\|_\Phi v_n(x))}{\|u_n\|_\Phi^\ell} \leq A\theta_3^\ell(x) + B(x). \tag{1.29}
\]
In addition,
\[
\limsup \frac{F(x, \|u_n\|_\Phi v_n)}{\|u_n\|_\Phi^\ell} \leq \limsup \frac{F(x, \|u_n\|_\Phi v_n)}{\|u_n\|_\Phi^\ell |v_n(x)|^\ell \chi_{\{v_n \neq 0\}}}
\]
and hence
\[
\limsup \frac{F(x, \|u_n\|_\Phi v_n)}{(\|u_n\|_\Phi v_n(x))^{\ell}} |v_n(x)|^\ell \chi_{\{v_n \neq 0\}} \leq A_\infty(x) |v(x)|^\ell, \ v(x) \neq 0. \tag{1.30}
\]
By (1.29), (1.30) and Fatou’s Lemma, (1.28) follows. Using again the fact that \( \Phi \) is convex and continuous, lemma 5.1 \( I(u) \leq C \) and (1.28) it follows that
\[
\int_\Omega \Phi(|\nabla v|) \, dx \leq \liminf_{n \to \infty} \int_\Omega \Phi(|\nabla v_n|) \, dx
\]
\[
\leq \liminf_{n \to \infty} \frac{1}{\|u_n\|_\Phi} \int_\Omega \Phi(|\nabla u_n|) \, dx
\]
\[
\leq \liminf \left\{ \int_\Omega \left[ F(x, \|u_n\|_\Phi v_n) + \frac{2\|h\|_\Phi}{\|u_n\|_\Phi^{-1}} \right] \, dx + \frac{C}{\|u_n\|_\Phi} \right\}
\]
\[
\leq \limsup \int_\Omega \frac{F(x, \|u_n\|_\Phi v_n)}{\|u_n\|_\Phi} \, dx
\]
\[
\leq \int_{\{v \neq 0\}} A_\infty(x)|v(x)| \, dx,
\]
which contradicts (1.7). Therefore, \( I \) is coercive. By lemma 3.1 there is \( u \in W^{1,\Phi}_0(\Omega) \) satisfying (1.17) and by lemma 3.2 \( u \) satisfies (1.11) in the sense of distributions. This proves theorem 2.1.

The lemma below is needed in order to prove theorem 2.2.

**Lemma 3.3** Assume \((\phi_1), (\phi_2), (1.2), (1.4)\) and (1.8). If \( u \in W^{1,\Phi}_0(\Omega) \) satisfies (1.17) then \( u \) is a weak solution of (1.1).

The proof is similar to that of lemma 3.2. In the present case one must show (1.19) and (1.24) for \( v \in W^{1,\Phi}_0(\Omega) \) which follows basically the same lines as in the case \( v \in C_0^{\infty}(\Omega) \).

As earlier there is a function \( \rho_t(x) \) such that
\[
\frac{F(x, u + tv) - F(x, u)}{t} = f(x, \rho_t(x))v, \text{ a.e. in } \Omega
\]
and
\[
\min\{u + tv, u\} \leq \rho_t \leq \max\{u + tv, u\} \text{ a.e. in } \Omega.
\]

Using (1.8) and the fact that \( t \mapsto \frac{\Phi_t(v)}{t} \) is increasing (cf. [11]) we get to
\[
|f(x, \rho_t)v| \leq a\Phi_\ast(|u| + |v|) + b|v|.
\]

Since \( \Phi_\ast(|u| + |v|) \in L^1(\Omega) \) we get \( f(x, \rho_t)v \in L^1(\Omega) \). Applying Lebesgue’s theorem we get (1.24). As in the proof of lemma 3.2 we infer that \( u \) is a weak solution of (1.1). This proves lemma 3.3.

**Proof. (of Theorem 2.2)** As in the proof of theorem 2.1 one shows that \( I \) is coercive. Since by lemma 3.1 \( I \) is wlsic, there is \( u \in W^{1,\Phi}_0(\Omega) \) such that satisfying (1.17). By lemma 3.3 \( u \) is a weak solution of (1.1). This proves theorem 2.2.
4 Proofs of Theorems 2.3 and 2.4

We shall need some preliminary results. Set
\[ \phi(t) = \gamma \frac{(\sqrt{1 + t^2} - 1)^{\gamma-1}}{\sqrt{1 + t^2}}, \quad t \geq 0. \]

**Lemma 4.1** The function \( \phi \) satisfies (\( \phi_1 \)), (\( \phi_2 \)), and
\[ \gamma \leq \frac{t^2 \phi(t)}{\Phi(t)} \leq 2\gamma, \quad t > 0. \] (1.31)

In addition, when \( 1 < \gamma < N \),
\[ L_\Phi(\Omega) = L^\gamma(\Omega), \]
\[ |u|_\Phi \leq |u|_\gamma, \quad u \in L_\Phi(\Omega), \] (1.32)
and
\[ W_0^{1,\Phi}(\Omega) = W_0^{1,\gamma}(\Omega). \]

**Proof.** Of course \( \phi \in C(0, \infty) \). By a direct computation we infer that for \( t > 0 \),
\[ \lim_{t \to 0} t \phi(t) = 0, \quad \lim_{t \to \infty} t \phi(t) = \infty, \quad (t \phi(t))' > 0 \quad \text{and} \quad \gamma \leq \frac{t^2 \phi(t)}{\Phi(t)} \leq 2\gamma, \]
showing (\( \phi_1 \)), (\( \phi_2 \)) and (1.31). To prove that \( L_\Phi(\Omega) = L^\gamma(\Omega) \), we point out that
\[ \Phi(t) \leq t^\gamma, \quad t \geq 0 \quad \text{and} \quad \Phi(t) \geq \frac{1}{2^\gamma} t^\gamma, \quad t \geq 2. \]

By a result in [17, p 156], \( L_\Phi(\Omega) = L^\gamma(\Omega) \). As a consequence, \( W_0^{1,\Phi}(\Omega) = W_0^{1,\gamma}(\Omega) \).

In order to show (1.32), take \( u \in L^\gamma(\Omega) \) and \( k > 0 \) and notice that
\[ \int_\Omega \frac{|u(x)|^\gamma}{k^{\gamma}} dx \leq 1 \quad \text{iff} \quad |u|_\gamma \leq k. \]

Since \( \Phi(t) \leq t^\gamma \) for \( t \geq 0 \) we have
\[ \int_\Omega \Phi \left( \frac{|u|}{k} \right) dx \leq \frac{|u|_\gamma^2}{k^{\gamma}}. \]

Setting \( k = |u|_\gamma \), we get (1.32). This proves lemma 4.1.

**Proposition 4.1** Assume (1.15). Then (1.7) holds.
Proof. By \((1.15)\) and \(W_0^{1,\Phi}(\Omega) = W_0^{1,\gamma}(\Omega)\),
\[
\inf_{u \in W_0^{1,\Phi}(\Omega), \|u\|_{L^\gamma} = 1} \left\{ \int_{\Omega} \left(1 + |\nabla u|^2 - 1\right)^\gamma dx - \int_{\{u \neq 0\}} A_\infty(x)|u|^\gamma dx \right\} > 0,
\]
Recalling that \(\Phi(t) = (\sqrt{1 + t^2} - 1)^\gamma\), there is \(\delta > 0\) such that
\[
\delta \leq \int_{\Omega} \Phi\left(\frac{|\nabla u|}{|u|_\gamma}\right) dx - \frac{1}{|u|_\gamma} \int_{\Omega} A_\infty(x)|u|^\gamma dx, \quad u \in W_0^{1,\Phi}(\Omega).
\]
Using the convexity of \(\Phi\) and \(|u|_{\Phi} \leq |u|_\gamma\) we have
\[
\delta \leq \int_{\Omega} \Phi(|\nabla u|) dx - \int_{\Omega} A_\infty(x)|u|^\gamma dx, \quad u \in W_0^{1,\Phi}(\Omega), \quad |u|_\Phi = 1.
\]
This proves proposition 4.1.

**Proposition 4.2** Assume \(1 < \gamma < N\) and \((1.13)\). Then \((1.16)\) holds.

Proof. Set \(\Phi(t) = t^\gamma\). By lemma 5.2 (in the Appendix), we have for \(t \geq 1\),
\[
\Phi_*(1) t^\gamma \leq \Phi_*(t).
\]
Using the inequality above and \((1.13)\) we get \((1.16)\). This proves proposition 4.2.

Proof. (of Theorem 2.3) As a consequence of propositions 4.1 and 4.2 theorem 2.1 applies ending the proof of theorem 2.3.

Proof. (of Theorem 2.4) Similarly to the proof of the theorem above it suffices to apply theorem 2.2.

5 Appendix

We refer the reader to [7] for the lemmas below whose proofs are elementary.

**Lemma 5.1** Assume that \(\phi\) satisfies \((\phi_1) - (\phi_2)\) and \((1.2)\). Set
\[
\zeta_0(t) = \min\{t^s, t^m\}, \quad \zeta_1(t) = \max\{t^s, t^m\}, \quad t \geq 0.
\]
Then \(\Phi\) satisfies
\[
\zeta_0(t)\Phi(\rho) \leq \Phi(\rho t) \leq \zeta_1(t)\Phi(\rho), \quad \rho, t > 0,
\]
\[
\zeta_0(\|u\|_\Phi) \leq \int_{\Omega} \Phi(u) dx \leq \zeta_1(\|u\|_\Phi), \quad u \in L_\Phi(\Omega).
\]

**Lemma 5.2** Assume that \(\phi\) satisfies \((\phi_1) - (\phi_2)\) and \((1.2)\). Set
\[
\zeta_2(t) = \min\{t^s, t^m\}, \quad \zeta_3(t) = \max\{t^s, t^m\}, \quad t \geq 0.
\]
Then
\[
\ell^s \leq \frac{t^2\Phi'(t)}{\Phi_*(t)} \leq m^*, \quad t > 0,
\]
\[
\zeta_2(t)\Phi_*(\rho) \leq \Phi_*(t^p\rho) \leq \zeta_3(t)\Phi_*(\rho), \quad \rho, t > 0,
\]
\[
\zeta_2(\|u\|_\Phi) \leq \int_{\Omega} \Phi_*(u) dx \leq \zeta_3(\|u\|_\Phi), \quad u \in L_\Phi(\Omega).
\]
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