On the transverse mode of an atom laser

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The transverse mode of an atom laser beam that is outcoupled from a Bose–Einstein condensate is investigated and is found to be strongly determined by the mean–field interaction of the laser beam with the condensate. Since for repulsive interactions the geometry of the coupling scheme resembles an interferometer in momentum space, the beam is found show filamentation. Observation of this effect would prove the transverse coherence of an atom laser beam.

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Research on atom lasers is an active and fascinating area in atomic physics \textsuperscript{1,2}. Several laboratories around the world are now using continuous output couplers to produce atom laser beams from Bose-Einstein condensates. It is therefore important to characterize the qualities of these beams. Recently, their temporal coherence was verified \textsuperscript{3} and their transverse divergence was measured \textsuperscript{4}.

Although atom lasers and optical lasers show strong similarities, the possibility for atoms to scatter off each other leads to various effects absent in optical lasers. In the following we will show that a strong, inhomogeneous repulsive potential, as is represented by the remaining Bose-Einstein condensate, can be a source of instabilities for the beam and, in particular, can lead to its transverse filamentation. Since most experimental setups involve asymmetric traps with two stiff directions, our results are immediately applicable to these experiments.

In magnetic traps output couplers for atom laser beams are realized by coupling a fraction of the Bose-Einstein condensate into a magnetically untrapped state. This process happens inside the trapped sample along a surface where the resonance condition for output coupling is fulfilled, subjecting the output coupled atoms to the repulsive mean field potential of the condensate. Since gravity displaces the symmetry axis of the Bose-Einstein condensate with respect to the symmetry axis of the magnetic trapping field, the repulsive potential is however not homogeneous over the resonance surface. In a classical picture this situation corresponds to particles rolling off a potential from different heights, leading to a non–negligible momentum spread or dispersion in the transverse directions \textsuperscript{5}. Moreover, the finite interaction time of the falling beam with the remaining condensate leads to a non–monotonic increase of the transverse atomic position $x(t)$ as a function of the points of resonant output coupling $x_i$. In the quantum dynamics, this leads to the interference of atoms with different transverse momenta within the beam.

In the following we will investigate this interference process. First we consider an idealized, one–dimensional model in the direction perpendicular to gravity, and we examine its classical, semiclassical and quantum behaviour. After we have identified and described the relevant processes, we will include the effects of gravity and propagation in more than one dimension.

Atom lasers with radiofrequency output couplers usually couple different Zeeman substates of the trapped atoms. For a Bose-Einstein condensate in a $|F = 1, m_F = -1\rangle$ state, two output states are possible. Either $m_F = 0$ or $m_F = 1$, the first of which has a vanishing magnetic moment and the latter experiences a repulsive force by the magnetic trap. However, since the output coupling rate from the $m_F = -1$ state into the $m_F = 0$ state is usually chosen to be small, subsequent transitions into the $m_F = -1$ state can be neglected. We therefore restrict our considerations to a two–level system, where the important coupling parameters are the Rabi frequency $\Omega$ and the detuning $\Delta$ of the rf–field. For weak coupling the resonance condition is determined by the spatial dependence of $\Delta$ within the Bose-Einstein condensate. After a transformation into a co–rotating frame $\psi_{m_p}(t) \rightarrow e^{i m_p g z t} \psi_{m_p}(t)$ followed by the standard rotating wave–approximation, the equations for the condensate wave function $\psi_c$ and the atom laser beam wave function $\psi_b$ are given by a set of coupled Gross–Pitaevskii–equations \textsuperscript{6}

\begin{equation}
\text{i} \hbar \frac{\partial}{\partial t} \psi_i = -\frac{\hbar^2}{2m} \nabla^2 \psi_i + V_i(r) \psi_i - m_F \hbar \omega_{rf} \psi_i
+ U(|\psi_i|^2 + |\psi_j|^2) \psi_i + \hbar \Omega \psi_j,
\end{equation}

with $i,j = c,b$ and $U = 4\pi \hbar^2 a_s/m$. We have assumed that all triplet scattering lengths have the same value $a_{ij} = a_s$ \textsuperscript{7}, and we will choose $a_s$ to be positive. The external potentials are given by $V_c(r) = \frac{\hbar^2}{2m} (\omega_x^2 x^2 + \omega_y^2 y^2) + mgz$ and $V_b(r) = mgz$, where $g$ is the gravitational constant, $\omega_x$ and $\omega_y$ are the trapping frequencies of the cylindrically symmetric magnetic field and $m$ denotes the mass of the atoms.

It has been found, using a separation ansatz for the spatial modes, that the atom laser beam in the direction of gravity can be almost perfectly described by an Airy–Function \textsuperscript{8,9,10}. We therefore first consider the
behavior in the transversal $x$-direction as independent of the other directions.

A naive, strictly one-dimensional treatment of the atom laser in the horizontal $x$-direction would result in exactly two resonance points $x = \pm \sqrt{2m\omega_{\perp}/\mu}$ [9]. This, however, is not a good approximation to the three dimensional situation of the experiment. Since gravity leads to a displacement of the condensate from the center of the magnetic field by an amount $z_g = -g/\omega_{\perp}^2$, the resonance shell crosses the condensate with very low curvature [2] and is therefore better approximated by a plane. This means that in the horizontal directions output coupling happens along the full Thomas–Fermi distribution and the initial beam wave function is a scaled down copy of the condensate wave function [9].

While the atoms fall under gravity, the mean field potential they experience from the condensate changes. To account for this in the one-dimensional approximation one has to diminish the mean field potential, $U_c(x,t) = U|\psi\rangle\langle\psi|_c(x,z_i)^2$ during the evolution according to the free fall of the atoms, $z_i = z_g + gt^2/2$.

However, the principal physics of the horizontal mode and its instability is most clearly demonstrated by first considering the case with $g = 0$, i.e. taking $U_c(x,t) = U_c(x,z_i)$ to be constant in time. The effective one-dimensional Gross–Pitaevskii equation for the beam is then given by

$$i\hbar \frac{\partial \psi_b}{\partial t} = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + U_c(x) + U|\psi_b|\right] \psi_b. \quad (2)$$

In the Thomas–Fermi approximation the mean field potential is given by a truncated inverted harmonic oscillator

$$U_c(x) = \begin{cases} \mu \left(1 - \frac{x^2}{x_{TF}^2}\right) & \text{for } |x| < x_{TF}, \\ 0 & \text{for } |x| > x_{TF}. \end{cases} \quad (3)$$

Here $\mu$ is the chemical potential and $x_{TF} = \sqrt{2m/\mu\omega_{\perp}^2}$ the Thomas–Fermi radius of the condensate.

Let us first analyse eq. (2) classically and neglect the nonlinear term $U|\psi_b|^2$, because it is normally three orders of magnitude smaller than $U_c(x)$. Neglecting for a moment the truncation of the potential at the condensate border, i.e., assuming the potential $U_c(x) = \mu(1 - x^2/x_{TF}^2)$, $\forall x$, the classical equation of motion

$$m\ddot{x} = -\frac{dU_c(x)}{dx} \quad (4)$$

can be exactly integrated by $x(t) = x_i\cosh(\omega_{\perp} t)$, with $x_i$ the initial position of the atoms at the time of the outcoupling. Since the coss is an exponentially increasing function of the time, the resulting evolution is a spreading of the initial distribution. This means that atoms with larger $x_i$ will also have larger $x$ at later times $t$ (see dashed line in Fig. 1b).

The density distribution using the potential in eq. (3) is shown in Fig. 2 (dashed lines). The atoms show a temporary localization at the Thomas–Fermi radius, since the arrival time distribution for the atoms at the Thomas–Fermi edge

$$t_{TF}(\alpha) = \frac{1}{\omega_{\perp}} \ln \left(1 + \frac{1 - \sqrt{1 - \alpha^2}}{\alpha}\right), \quad (5)$$

with $\alpha = x_i/x_{TF}$, shows a plateau (see Fig. 1b). Once the atoms have passed the Thomas–Fermi radius all their initial potential energy, $E = \mu(1 - \alpha^2)$, has been transformed into kinetic energy and atoms starting closer to the condensate center ($x = 0$) therefore end up with

FIG. 1: (a) Position of atoms starting at $x_i(t = 0) < x_{TF}$ and rolling off an inverted harmonic oscillator potential for a time of $t = 2\text{ ms}$. The full line shows the results for a potential that is truncated at $x_{TF} = 4\text{ \mu m}$ (cf. eq. (3)) and the dashed line shows the results for an untruncated potential. (b) Time needed for the atoms starting at $\alpha = x_i/x_{TF}$ to reach the condensate boundary at $x_{TF}$ (cf. eq. (4)). The chemical potential in both calculations is $\mu = 1700\text{Hz}$.

FIG. 2: Transverse density distributions of the beam obtained from a classical (dashed) and a quantum-mechanical calculation for two different evolution times. The figures in the lower row show the absolute square of the Fourier transform of the quantum mechanical wave function. A strong filamentation is clearly visible in the quantum results.
higher final velocity. In the asymptotic limit, atoms originating from the center will have overtaken all other atoms and the density distribution that originally had a negative slope will have a positive slope. This can be seen from the dashed curves in Fig. 2.

The full quantum mechanical behaviour can be found by solving Eqs. (11) numerically, for which we use a standard split–operator/FFT technique. As can be seen from Fig. 2 (solid lines), although the general feature of localization is preserved, the density distribution is modulated by an interference pattern. The reason is that for finite times, \( t > t_{TF} \), atoms with different initial positions arrive at \( x \) at the same time \( t \), because the equation

\[
x(t) = x_{TF} \left( 1 + \omega \sqrt{1 - \alpha^2} (t - t_{TF}(\alpha)) \right)
\]

has more than one solution for \( \alpha \), and quantum mechanically, atoms coherently outcoupled at different \( \alpha \)'s interfere. Observation of these fringes would prove the transversal coherence of the atom laser beam.

A mathematically and conceptually very elegant method to calculate the interference pattern is by use of path integrals [13]. The unitary wave function propagator is obtained by adding phase factor contributions over all paths \( x(t) \) along which an argument \( x_i \) in the initial wave function can evolve into the argument \( x_f \) of the final state

\[
\psi(x_f, t_f) = \int D[x(t)] e^{i S[x(t)]/\hbar} \psi(x_i, t_i),
\]

The phase factor is the classical action along the path

\[
S[x(t)] = \int_{t_i}^{t_f} dt \left[ \frac{1}{2} m \dot{x}^2 - V(x(t)) \right].
\]

(8)

Path integrals are usually not exactly solvable, but they can be approximated well when for example few classical paths \( x_{cl}(t) \) are dominant. We can then restrict the integral to small deviations around them \( S[x] = S[x_{cl} + \delta x] \). This will provide a qualitative understanding of the observed fringes as well as a very good account of the quantitative results of the quantum calculations.

From Fig. 4, one can immediately make the observation that for \( x > x_{TF} \) two classical paths contribute. The atoms emerge from two different initial positions, and they will therefore arrive at \( x \) with two different momenta \( h k_1 \) and \( h k_2 \). This suggests interference fringes of width \((|k_1 - k_2|)^{-1}\), i.e., larger fringes for large values of \( x \) (see inset of Fig. 2).

For potentials whose second spatial derivative is constant, the propagator can be written as

\[
\int D[x(t)] e^{i S[x(t)]} = e^{i S[x_{cl}]} \left( 2\pi i k_f k_i \int_{x_i}^{x_f} \frac{dx}{k(x)^3} \right)^{-\frac{1}{2}}
\]

with \( k_i = k(x_i) \), \( k_f = k(x_f) \) and the classical action is given by

\[
S[x_{cl}] = -E (t_f - t_i) + \hbar \int_{x_i}^{x_f} dx \ k(x).
\]

(10)

Inserting the inverted and truncated harmonic oscillator potential of eq. (3), \( V(x) = U_c(x) \), the action can be calculated to give

\[
S[x_{cl}] = \frac{\mu}{\omega} \left[ -\alpha^2 \omega_{TF}^2 + \frac{2x_f - x_{TF}}{x_{TF}} \sqrt{1 - \alpha^2} \right] - \mu(t_f - t_i)(1 - \alpha^2)
\]

(11)

for \( x_f > x_{TF} \) and the bracketed expression in eq. (10) can be calculated to be

\[
\left( 2\pi i k_f k_i \int_{x_i}^{x_f} \frac{dx}{k(x)^3} \right)^{-\frac{1}{2}} = \frac{1}{\sqrt{2\pi i a_0^2} \sqrt{1 - \alpha^2}}
\]

(12)

where \( a_0 = \sqrt{\hbar/m\omega} \). In Fig. 3 the evolution of the wave function according to the semiclassical approximation to eq. (7) is shown (dashed line). Comparison with the full quantum mechanical evolution of eq. (2) shows almost perfect agreement.

To justify and understand the semiclassical approximation, let us note that restriction to classical paths and the assumption of quadratic potentials is formally equivalent to a WKB approximation [14] and the condition of validity can be written as

\[
\frac{1}{k^2} \left| \frac{dk}{dx} \right| = \frac{x}{a_0^2 k(x)^3} < 1
\]

(13)

where we have used the momentum of the atoms given by \( \hbar k(x) = \sqrt{2m(E - V(x))} \). This condition is well fulfilled as long as \( V(x) \) is an inverted harmonic oscillator for all atoms. One may, however, question the validity close to the Thomas–Fermi radius, where the slope of the potential changes strongly within the healing length.
The Thomas–Fermi radius is larger than the healing length of the de-Broglie-wavelength of the atoms when reaching the reflectivity of the potential step, we note that the quantum reflection effects at this point. To estimate the density distribution at $z = -15 \mu m$, corresponding to an evolution time of 1.3 ms. (c) The corresponding Fourier transform. Comparison with Fig. 2 shows good agreement.

The beam after an evolution time of 2 ms. (b) A cut through the density distribution at $z = -15 \mu m$, corresponding to an evolution time of 1.3 ms. (c) The corresponding Fourier transform. Comparison with Fig. 2 shows good agreement.

In summary we have shown that the transverse mode of an atom laser is strongly determined by the interaction of the beam with the mean-field of the residing condensate. Due to the finite time of this interaction, the system resembles an interferometer in momentum space and the beam shows filamentation in the transverse directions.

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$R = \left| \frac{1}{1 - 20 \sqrt{1 - \alpha^2}} \right|^2 \lesssim 10^{-2} \quad (15)$

justifying very well the semiclassical treatment.

Let us finally consider a two dimensional situation including gravity. The beam atoms are subjected to the mean-field potential for a time of the order of $0.5 - 1 \text{ms}$ assuming a typical condensate radius of $4 \mu m$. Most of the atoms therefore do not experience a complete horizontal roll-off from the mean-field potential (compare Fig. 1b), however qualitatively the above picture remains unchanged. We have solved the two-dimensional version of eq. (14) in the $x$– and $z$–plane numerically and in Fig. 4b, the atom laser beam for short evolution times is shown. Once the beam has left the overlap area with the Bose-Einstein condensate its evolution within the transverse direction is completely determined by a free evolution and the far–field result can be simply calculated by the Fourier transform.

A cut through the density distribution is shown in Fig. 4b and the far field of this distribution is shown in Fig. 4c. Both pictures show good qualitative agreement with the results of the one-dimensional analysis.

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