Charged Fermions and Strong Cosmic Censorship

Kyriakos Destounis
CENTRA, Departamento de Física, Instituto Superior Técnico – IST, Universidade de Lisboa – UL, Avenida Rovisco Pais 1, 1049 Lisboa, Portugal

It was recently shown that Strong Cosmic Censorship (SCC) conjecture might be violated for near-extremally-charged black holes in de Sitter space. Here, we extend our study to charged fermionic fields in the exterior of Reissner-Nordström-de Sitter black holes. We identify three families of modes; one related to the photon sphere, a second related to the de Sitter horizon and a third which dominates near extremality. We show that for near-extremally-charged black holes there is a critical fermionic charge below which Strong Cosmic Censorship may potentially be violated. Surprisingly enough, as one approaches extremality even more, violation of Strong Cosmic Censorship may occur even beyond the critical fermionic charge.

I. Introduction. We recently studied the implications of massless neutral scalar perturbations on Strong Cosmic Censorship (SCC) in Reissner–Nordström (RN) black holes (BHs) in de Sitter (dS) spacetime [1]. Three different families of modes were identified in such spacetime; one directly related to the photon sphere, well described by standard Wentzel-Kramers-Brillouin (WKB) tools, another family whose existence and timescale is closely related to the dS horizon and a third family which dominates for near-extremally-charged BHs. Surprisingly enough, our results show that near-extremal RNdS BHs might violate SCC, leading to a severe failure of determinism in General Relativity (GR). The key quantity controlling the stability of the Cauchy horizon (CH), and therefore the fate of SCC, is given by \( \beta \equiv -\text{Im}(\omega_0)/\kappa_- \),

\[
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\]

where \( \omega_0 \) is the longest-lived/dominant non-zero quasi-normal mode (QNM [4–6]) and \( \kappa_- \) is the surface gravity of the CH. The results in [7–9] suggest that \( \beta \) remains the key quantity in the non-linear setting: the higher \( \beta \), the more stable the CH. Concretely, the modern formulation of SCC requires that

\[
\beta < 1/2
\]

in order to guarantee the breakdown of field equations at the CH [10]. In [1], a thorough linear numerical study of \( \beta \) throughout the whole parameter space of subextremal RNdS spacetimes revealed that \( 1/2 < \beta < 1 \) in the near-extremal regime, leading to a CH with enough regularity for metric extensions to be possible past it. This provides evidence for the existence of CHs which, upon perturbation, are rather singular due to the divergence of curvature invariants, but where the gravitational field can still be described by the field equations; the evolution of gravitation beyond the CH, however, is highly non-unique.

There are different ways to interpret the results of [1]. If one takes the SCC conjecture as a purely a mathematical question about GR then the results of [1] either signify a failure of SCC, or are superseded by nonlinear effects. In fact, the results of [11] proved that even nonlinear effects could not save the conjecture from failing for near-extremally-charged BHs.

An interesting suggestion to restore SCC, in the presence of a positive cosmological constant, was proposed in Ref. [12], where it was shown that the pathologies identified in [1] become non-generic if one considerably enlarges the allowed set of initial data by weakening their regularity. Even though the considered data are compatible with the modern formulation of SCC, we believe that SCC is, in essence, a formation of singularities problem which is mainly of interest for regular initial data; the mechanism of SCC becomes obscured if one considers initial data which are too "rough" (see [13] for a comparison with shock wave formation in fluid mechanics).

A subsequent study of metric fluctuations in RNdS BHs showed that such perturbations exhibit a much worse violation of SCC. In [14] it was shown that for a sufficiently large near-extremal RNdS BH, perturbations arising from smooth initial data can be extended past the CH in an arbitrarily smooth way. Nevertheless, astrophysical BHs are expected to be nearly neutral [15, 16]. Taking this into consideration, one can question the relevance of SCC violations in highly charged, non-spinning BHs. In fact, a recent study suggests that this is not the case for rapidly rotating BHs in cosmological backgrounds. According to [17], in Kerr-dS spacetime [1] remains unchanged, but now \( \beta \) seems to be bounded exactly by 1/2, at extremal rotation. Similar results were obtained in [18] for higher-dimensional Kerr-dS BHs.

Considering the formation of a charged BH, one would argue that charged matter has to be present. In Ref. [19]–[20], it was shown that charged scalar fields would lead to restoration of SCC in an appropriate region of the parameter space of RNdS and Kerr-Newmann-dS BHs. This implication requires working in the large-coupling regime for which \( \beta < 1/2 \). Taking into account the whole parameter space, subsequent studies [21–22] presented numerical evidence that SCC may still be violated in the setting of charged scalar perturbations in RNdS.

In Ref. [23], massive charged scalars were taken into account providing evidence that \( \beta > 1 \). Recall that this is related to bounded curvature and therefore opens the possibility to the existence of solutions to the Einstein-Maxwell-Klein-Gordon system with a scalar field satisfying Price’s law and bounded curvature across the CH.
Nonetheless, if the neutral scalar perturbations where superimposed to the charged massive ones, then the smaller of the two types of perturbations is the one relevant for SCC, thus getting $\beta < 1$, which should be enough to guarantee the blow-up of curvature components.

In [22] it was shown that even for large scalar field charge there are near-extremal BHs for which $\beta > 1/2$ and so SCC is violated. A key ingredient of the aforementioned studies was the existence of the $l = 0$ superradiantly unstable mode. This unstable mode was the dominant one for small scalar charges and was responsible for the preservation of SCC up to $Q/Q_{\text{max}} = 1 - 10^{-3}$, where $Q_{\text{max}}$ is the extremal charge of the BH.

Is it natural to question then, if the charged matter could just as well be fermionic instead of scalar. Fermions do not superradiate, leaving the entire range of fermionic charge open for the study of SCC at the linearized level. The results of Ref. [24] provide evidence that fermionic perturbations of RNdS BHs might violate SCC for sufficiently large BH charge. As a matter of fact, the family that seems to dominate the dynamics near extremality is, mostly, the photon sphere family with a very small participation of a family which is purely imaginary for zero fermionic charge $q$ and quickly becomes subdominant as $q$ increases. Unfortunately, there is no information about the classification of the latter family and if it will eventually dominate the dynamics for even higher BH charges. Moreover, since a dS horizon is present, the dS family of modes might be present as well and even dominate the dynamics for small cosmological constants in analogy with what was found in Ref. [1]. The tool used to extract the modes in [24] is time domain analysis. Although it is a very powerful tool for such calculations, there is a slight chance that long-lived modes may be missed either because of their timescale being larger than the evolution time of the system or cause of improper initial data.

In this paper, we study the propagation of charged fermions on a fixed RNdS background and extract the QNMs with the spectral method developed in Ref. [25]. After characterizing the families of modes that are present, we will examine the implications on SCC for highly near-extremal RNdS BHs.

II. Charged Fermions in RNdS. We focus on RNdS BHs, described by the metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2),$$

where $f(r) = 1 - 2Mr^{-1} + Q^2r^{-2} - \Lambda r^2/3$. Here, $M, Q$ are the BH mass and charge, respectively, and $\Lambda > 0$ is the cosmological constant. The surface gravity of each horizon is then

$$\kappa_h = \frac{1}{2} |f'(r_h)|, \quad h \in \{-, +, c\},$$

where $r_- < r_+ < r_c$ are the Cauchy, event and cosmological horizon radius. Since fermions are described by spinors, we use the tetrad formalism to accommodate them in curved space. The tetrads by definition satisfy the relations

$$e^a_{\mu} e^\nu_{\alpha} = \delta^\nu_{\alpha},$$

$$e^a_{\mu} e^b_{\nu} = \delta^a_{b},$$

The choice of the tetrad field determines the metric through

$$g_{\mu\nu} = e^a_{\mu} e^b_{\nu} \eta(a(b),$$

$$\eta(a(b) = e^a_{(a} e^b_{(b)} g_{\mu\nu},$$

where $\eta(a(b)$ and $g_{\mu\nu}$ are the Minkowski and RNdS metric, respectively. In order to write the Dirac equation, we also introduce the spacetime-dependent gamma matrices $\gamma^\mu$ which are related to the special relativity matrices, $\gamma^{(a)}$, by

$$G^\mu = e^a_{(a}\gamma^{(a)},$$

and are chosen in a proper way to satisfy the anti-commutation relations

$$\{\gamma^{(a)}, \gamma^{(b)}\} = -2\eta^{(a)(b)},$$

$$\{G^\mu, G^\nu\} = -2g^{\mu\nu}.$$ Consequently, we define $G^\mu$ with respect to a fixed tetrad as

$$G^t = e^t_{(a}\gamma^{(a)}, \quad G^r = e^r_{(a}\gamma^{(a)} = \sqrt{f(r)} \gamma^r,$$

$$G^\theta = e^\theta_{(a}\gamma^{(a)} = \gamma^\theta, \quad G^\varphi = e^\varphi_{(a}\gamma^{(a)} = \gamma^\varphi,$$

where $\gamma^t, \gamma^r, \gamma^\theta$ and $\gamma^\varphi$ are the $\gamma-$matrices in "polar coordinates" [26]

$$\gamma^t = \gamma^{(0)},$$

$$\gamma^r = \sin \theta \cos \varphi \gamma^{(1)} + \sin \theta \sin \varphi \gamma^{(2)} + \cos \theta \gamma^{(3)},$$

$$\gamma^\theta = \frac{1}{r} \left( \cos \theta \cos \varphi \gamma^{(1)} + \cos \theta \sin \varphi \gamma^{(2)} - \sin \theta \gamma^{(3)} \right),$$

$$\gamma^\varphi = \frac{1}{r \sin \theta} \left( -\sin \varphi \gamma^{(1)} + \cos \varphi \gamma^{(2)} \right)$$

and

$$\gamma^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^{(k)} = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$$

the standard Dirac $\gamma$-matrices, where $\sigma^k, k = 1, 2, 3$ the Pauli matrices. The propagation of a spin 1/2 particle of mass $m_f$ on a fixed RNdS background is then described by the Dirac equation in curved spacetime [27]

$$(iG^\mu D_\mu - m_f)\psi = 0,$$

with the covariant derivative

$$D_\mu = \partial_\mu + iqA_\mu + \Gamma_\mu,$$
where \( q \) the charge of the Dirac particle, \( A = (Q/r)dt \) the electrostatic potential and \( \Gamma_\mu \) the spin connection coefficients defined as

\[
\Gamma_\mu = - \frac{1}{8} \omega_{(a)(b)\mu} \left[ \gamma^a, \gamma^b \right].
\]

The spin connection \( \omega_{(a)(b)\mu} \) is defined as

\[
\omega_{(a)(b)\mu} = \eta_{(a)(c)} \left( e^c_{(b)} \epsilon^\lambda_{(c)\mu} \lambda \right) - \epsilon_{(b)\mu} e^c_{(a)},
\]

with \( \Gamma^\mu_{\lambda} \) the Christoffel symbols. By choosing the ansatz \( \psi = f(r)^{-1/4}r^{-1}\Psi \), (5) can be written as

\[
\left[ i\gamma^\mu \frac{\partial}{\partial r} + i\sqrt{f(r)}\gamma^\tau \frac{\partial}{\partial r} - \frac{i\gamma^\tau}{r} + i \left( \gamma^\theta \frac{\partial}{\partial \theta} + \gamma^\varphi \frac{\partial}{\partial \varphi} \right) \right] \Psi = 0.
\]

Since the external fields are spherically symmetric and time-independent, we can separate out the angular and time dependence of the wave functions via spherical harmonics and plane waves, respectively. For the Dirac wavefunctions, we choose the ansatzs

\[
\Psi_{j,kw}^{+} = e^{-i\omega t} \left( \phi_{j+1/2}^{+}(r) \right), \quad (7)
\]

\[
\Psi_{j,kw}^{-} = e^{-i\omega t} \left( \phi_{j-1/2}^{-}(r) \right), \quad (8)
\]

where we introduced the spinor spherical harmonics [20]

\[
\phi_{j+1/2}^{+} = \left( \sqrt{\frac{\Gamma}{2}} Y^{k+1/2}(\theta, \varphi) \right), \quad \text{for } j = l + \frac{1}{2}, \quad (19)
\]

\[
\phi_{j-1/2}^{-} = \left( \sqrt{\frac{\Gamma}{2}} Y^{k-1/2}(\theta, \varphi) \right), \quad \text{for } j = l - \frac{1}{2}. \quad (20)
\]

With \( j = 1/2, 3/2, \ldots, k = -j, -j + 1, \ldots, j \) and \( Y^m \) the ordinary spherical harmonics. By substituting (7) and (8) into (9) and (10) and utilizing the identities

\[
K = \tilde{\sigma} \tilde{L} + 1 = -r \sigma^\tau \left( \sigma^\theta \partial_\theta + \sigma^\varphi \partial_\varphi \right) + 1,
\]

\[
K \phi_{j+1/2}^{\pm} = (\pm(j + \frac{1}{2}) \phi_{j+1/2}^{\pm}, \quad \sigma^\tau \phi_{j+1/2}^{\pm} = \phi_{j+1/2}^{\pm},
\]

with \( \tilde{\sigma}, \tilde{L} \) the Pauli and angular momentum vectors, respectively, and \( \sigma^\tau, \sigma^\theta, \sigma^\varphi \) the Pauli matrices in “polar coordinates” [29], we end up with the coupled Dirac equations

\[
\frac{\partial F}{\partial r} = \frac{\xi \sqrt{f(r)}}{r} F + \left( \omega - \frac{qQ}{r} \right) G + m_f \sqrt{f(r)} G = 0, \quad (9)
\]

\[
\frac{\partial G}{\partial r} = \frac{\xi \sqrt{f(r)}}{r} G - \left( \omega - \frac{qQ}{r} \right) F + m_f \sqrt{f(r)} F = 0, \quad (10)
\]

where \( \xi = \pm(j + 1/2) = \pm1, \pm2, \ldots \) and \( dr_\ast = f/dr \). Since the charge-to-mass ratio of the electron is of order \( 10^{11} \text{C/kg} \), it is reasonable to explore massless fermions. By setting \( m_f = 0 \) we can decouple (9), (10) by introducing a new coordinate

\[
d\tilde{r}_\ast = \frac{1 - \frac{qQ}{f^2}}{f} dr,
\]

to get

\[
\frac{dF}{d\tilde{r}_\ast} - WF + \omega G = 0, \quad (11)
\]

\[
\frac{dG}{d\tilde{r}_\ast} + WG - \omega F = 0, \quad (12)
\]

and subsequently

\[
\frac{d^2 F}{d\tilde{r}_\ast^2} + (\omega^2 - V_+) F = 0, \quad (13)
\]

\[
\frac{d^2 G}{d\tilde{r}_\ast^2} + (\omega^2 - V_-) G = 0, \quad (14)
\]

with

\[
V_{\pm} = \pm \frac{dW}{d\tilde{r}_\ast} + W^2,
\]

where

\[
W = \frac{\xi \sqrt{f}}{r (1 - \frac{qQ}{f^2})}. \quad (15)
\]

It can be shown that potentials related in this manner and subjected to Sommerfeld conditions are isospectral, thus, allowing us to work only with the field \( F \) [3, 29]. Since we are interested in the characteristic frequencies of this spacetime, we impose the boundary conditions

\[
F(r \to r_+) \sim e^{-i\omega \tilde{r}_\ast}, \quad F(r \to r_-) \sim e^{i\omega \tilde{r}_\ast},
\]

which select a discrete set of frequencies \( \omega \) called the QNMs. The QN frequencies are characterized, for each \( \xi \), by an integer \( n \geq 0 \) labeling the mode number. The fundamental mode \( n = 0 \) corresponds, by definition, to the non-vanishing frequency with the smallest (in absolute value) imaginary part and will be denoted by \( \omega \neq 0 \). It is apparent from (11), (12) and (13) that the symmetry \( \omega \to -\omega, q \to -q, \xi \to -\xi \) holds, enabling us to only study positive \( \xi \). As shown in Appendix A for \( q \neq 0 \) the stability of the CH continues to be determined by (1). The results shown in the following sections were obtained with the Mathematica package of [29] and checked in various cases with a WKB approximation [29] and with a code developed based on the matrix method [30].

**III. QNMs of massless, charged fermionic fields: the three families.** In [11, 21], we found three qualitatively different families of QNMs: the photon sphere (PS) family, the dS family and the near-extremal (NE) family.
The first two connect smoothly to the modes of asymptotically flat Schwarzschild and of empty dS, respectively, while the last family cannot be found in either of these spacetimes. Here we, again, distinguish three families of modes.

**Photon sphere modes.** The PS is a spherical trapping region of space where gravity is strong enough that photons are forced to travel in unstable circular orbits around a BH. This region has a strong pull in the control of decay of perturbations and the QNMs with large frequencies. For asymptotically dS BHs, we find a family that can be traced back to the photon sphere and refer to them as PS modes. These modes are shown with blue colors in Figs. 13 and 6. They satisfy the symmetry \( \omega \to -\omega \) for \( q = 0 \) and the symmetry breaks as the fermionic charge is turned on, according to (15). For very small \( \Lambda \), \( q \) and \( Q \), \( \xi \to \infty \) defines the dominant mode which can be very well approximated by a WKB approximation and asymptotic to the Schwarzschild BH Dirac QNMs [31]. The lowest lying PS modes are weekly dependent on the BH charge as it is apparent for the case presented in Fig. 1. For sufficiently large \( \Lambda \) the former does not hold. For large BH charges the \( \xi = 1 \) PS modes dominate the family (see Appendix B).

**de Sitter modes.** In pure dS space solutions of the Dirac equation with purely imaginary \( \omega \) exist [32]

\[
\omega_{\text{pure dS}}/\kappa^{\text{dS}} = -i \left( \xi + n + \frac{1}{2} \right)
\]

where \( \xi = 1, 2, \ldots \). The second family of modes we find are the Dirac BH dS QNMs, which are deformations of pure dS QNMs [16]. The dominant BH dS mode (\( \xi = 1, n = 0 \)) is almost identical to (16) and higher overtones have increasingly larger deformations.

These modes have weak dependence on the BH charge and are described by the surface gravity \( \kappa^{\text{dS}} = \sqrt{\Lambda/3} \) of the cosmological horizon of pure dS space, as opposed to that of the cosmological horizon in the RN\(\text{dS} \) BH in study. This could be explained by the fact that the accelerated expansion of RN\(\text{dS} \) spacetimes is also governed by \( \kappa^{\text{dS}} [33, 34] \).

To the best of our knowledge, this family of Dirac BH dS modes has been identified here for the first time. The scalar equivalent of these modes has been identified for the first time in the QNM calculations of [11, 24]. Moreover, as \( \Lambda M^2 \to 0 \), these modes converge to the exact pure dS modes (16) (see Table 1).

**Near-extremal modes.** In the limit where the Cauchy and event horizon approach each other, a third NE family dominates the dynamics. In the extremal limit and for sufficiently small fermionic charges this family approaches

\[
\omega_{\text{NE}} \approx \frac{qQ}{r_+ - i\kappa_-} \left( \xi + n + \frac{1}{2} \right) \approx \frac{qQ}{r_+ - i\kappa_-} \left( \xi + n + \frac{1}{2} \right),
\]

where \( \xi = 1, 2, \ldots \), with weak dependence on \( \Lambda \) as shown by our numerics. As indicated by (17), the dominant mode of this family is the \( \xi = 1, n = 0 \).

In the asymptotically flat case, such modes have been identified in [35]. Here, we show that these modes exist in RN\(\text{dS} \) BHs, and that they are the limit of a new family of modes.

**IV. Dominant modes and SCC.** Since our purpose is to investigate the implications of charged fermions in SCC, we will restrict ourselves to choices of NE RN\(\text{dS} \) BH parameters which are problematic, since in this region \( \kappa_- \) becomes comparable to \( \text{Im}(\omega) \). For the region of interest, \( \xi = 1 \) modes dominate all three families (see Appendix B).

In [24] it was shown that for the choice of \( \Lambda M^2 = 0.06 \) and \( Q/Q_{\text{max}} = 0.996 \) only the \( \xi = 1 \) PS mode is relevant for SCC and there is a region in the parameter space.
where \( \beta > 1/2 \) (for \( q \approx 0.53 \)) implying that violation of SCC indeed occurs. Quietly interesting, there was no participation of the NE modes to the determination of \( \beta \) for these parameters. On the other hand, for \( \Lambda M^2 = 0.06 \) and \( Q/Q_{\text{max}} = 0.999 \) a family that originates from purely imaginary modes (for \( q = 0 \)) comes into play to dominate for very small fermionic charges and quickly becomes subdominant to give its turn to the \( \xi = 1 \) PS mode. Again, \( \beta > 1/2 \) (for \( q \approx 0.85 \)) so SCC may be violated.

Our numerics completely agree with this picture. Here, we will be mostly interested in the case of even higher BH charges and the classification of the families originating from purely imaginary modes. To do so, we will study various choices of \( \Lambda \). The BH charges we consider are:

\[
1 - Q/Q_{\text{max}} = 10^{-3}, 10^{-4}, 10^{-5}.
\]

According to our results (see Figs. 2, 3) for small BHs \( (\Lambda M^2 = 0.005) \) with \( 1 - Q/Q_{\text{max}} = 10^{-3} \) we see that \( \beta \) is defined by the dS mode up to \( q \approx 0.5 \); for larger \( q \) the PS mode becomes dominant. Interestingly enough, for \( q < 0.5 \), the NE mode lies very close to the dS one being the first subdominant mode in this range. For larger BHs \( (\Lambda M^2 = 0.06) \) with \( 1 - Q/Q_{\text{max}} = 10^{-3} \) the dS mode moves rapidly to the subdominant side, giving its place to the NE mode to dominate up until \( q \approx 0.35 \); for larger \( q \) the PS mode dominates again. For BHs with \( 1 - Q/Q_{\text{max}} \geq 10^{-4} \) the NE mode always dominate the dynamics, while the rest of the families lie out of the range of interest.

For all cases presented, there is always a critical fermionic charge \( q_c \) above which \( \beta < 1/2 \) and SCC is preserved. In Fig. 4 we display the dependence of \( q_c \) on the \( \Lambda \) and \( Q \). We observe that as the BH becomes extremal a larger violation gap occurs in the parameter space. A larger \( q_c \) is also obtained for smaller cosmological constants. Similar results were obtained in [21][23]

for the case of charged scalar perturbations, although the absence of superradiance effect in fermionic fields leads to even larger regimes in the parameter space where violation of SCC may occur.

By observing the cases with \( 1 - Q/Q_{\text{max}} = 10^{-5} \) we see that above \( q_c \), \( \beta \) lies very close to 1/2. To examine if non-perturbative effects are present (like the “wiggles” in [22]) we plot \( \beta \) for \( \Lambda M^2 = 0.005, 0.06 \) and \( 1 - Q/Q_{\text{max}} = 10^{-8} \) versus the fermionic charge. In Fig. 3 we observe the existence of “wiggles” in highly extremal RNdS BHs. These wiggles are small “oscillations” of the imaginary part of the dominant mode around \( \beta = 1/2 \). They decrease exponentially fast in size with increasing \( q \). We believe that these wiggles were missed from the analysis of [24] because they did not consider highly NE BHs. The consequences of the wiggles for SCC are fierce. Our results indicate that, even for fermionic fields with \( q > q_c \), there are still NE BHs for which \( \beta > 1/2 \) and SCC is violated, regardless of the cosmological constant, in contrast with the results in [24].

V. Conclusions.

We recently presented evidence in [1][21] for the potential failure of SCC in NE RNdS BHs under neutral and charged scalar perturbations. By utilizing (1) we performed thorough numerical analyses of \( \beta \) through the calculation of QNMs of the system. Here, we extend our analysis to charged fermionic fields. First, we provide justification that Eq. (1) remains valid for charged fermionic perturbations. Then, we perform a detailed numerical computation of the dominant modes of RNdS BHs and distinguish three families of QNMs. The first family is closely related to the PS of the BH while the second is related to the existence and timescale of the dS
horizon of pure dS space. The final family dominates the dynamics when NE BH charges are considered. According to our study, the only relevant region for SCC is the NE, where the surface gravity of the CH, \( \kappa \), becomes comparable with the decay rates of the dominant QNMs. We show that all families admit their dominant modes comparable with the decay rates of the dominant QNMs.

Our main results are shown in Figs. 1 - 5 and our conclusions are summarized here. For all choices of \( \Lambda \) we always find a region of fermionic charges for which \( \beta > 1/2 \) which predicts a potential failure of SCC, since the CH can be seen as singular due to the blow-up of curvature but maintain enough regularity for metric extensions to be possible beyond it. For sufficiently large fermionic charges the conjecture seems to be initially restored for highly charged RNdS BHs. After examining BHs even closer to extremality, we realize that even beyond the critical fermionic charge, violation can still occur due to the existence of wiggles.

Finally, we point out that for all cases presented, all dominant modes from the dS, PS or NE family admit \( \beta > 1 \) for a small but significant regime of fermionic charges. This result is even more alarming for SCC since it is related to bounded curvature and therefore opens the possibility to the existence of solutions with even higher regularity across the CH. Nevertheless, if we superimpose all perturbations, then the smallest of all types of perturbations is the one relevant for SCC. Thus, the neutral scalar modes admit \( 1/2 < \beta < 1 \), which is enough to guarantee the blow-up of curvature components at the CH.

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Appendix A: The definition of \( \beta \) for charged fermions.

In [21] a justification of searching for \( \beta > 1/2 \) was provided, leading to potential violation of SCC in RNdS BHs under charged scalar perturbations. Here, we prove that the same holds for charged fermions. To determine the regularity of the metric up to the CH we study the regularity of QNMs at the CH. To do so, we change to outgoing Eddington-Finkelstein coordinates which are regular there. The outgoing Eddington-Finkelstein coordinates are obtained by replacing \( u = t - r_\ast \) in (3) to get

\[
ds^2 = -f(r)du^2 - 2du dr + r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]

with the associated electromagnetic potential

\[
A = \frac{Q}{r} du = \frac{Q}{r} \left( dt - \frac{dr}{f(r)} \right).
\]

A straightforward way to write down the Dirac equation in this new coordinates is to choose a new tetrad that reproduces (A1). An alternative way is to transform in the new coordinates \((u, r)\) with the associated transformed electromagnetic potential [A2]. To do so, we write [0] with \( m_f = 0 \) and perform the transformation
It can be shown that the mode solutions of (A8) are conormal at $r = r_-$, meaning that they grow at the same rate $|r - r_-|^\lambda$. Thus, if $F \sim |r - r_-|^\lambda$ then the first four terms have higher regularity than the rest, where $f \sim |r - r_-|$ near the CH modulo irrelevant terms. This means that these terms can be neglected, which leads to a regular-singular ordinary differential equation near $r = r_-$ of the form $PF = 0$ with the operator

$$P = (f\partial_r)^2 + 2i\omega f\partial_r - \frac{2iq\omega}{r}f\partial_r.$$  \hspace{1cm} (A9)

It is convenient to use $f$ as a radial coordinate instead of $r$, so $\partial_r = f'\partial_f = f'(r_-)\partial_f$ near the CH modulo irrelevant terms. Moreover, the surface gravity at the CH is $\kappa_- = -f'(r_-)/2$ so $f\partial_r = -2\kappa_-(f\partial_f)$. Thus, (A9) becomes

$$P = (f\partial_f)^2 - \frac{i\omega}{\kappa_-}(f\partial_f) + \frac{iq\omega}{\kappa_- - \kappa_-}(f\partial_f)$$

$$= f\partial_f \left( f\partial_f - \left( \frac{i\omega}{\kappa_-} - \frac{iq\omega}{\kappa_- - \kappa_-} \right) \right).$$  \hspace{1cm} (A10)

It remains to calculate the allowed growth rates $\lambda$, i.e. indicial roots of the quadratic polynomial (A11), namely

$$\lambda_1 = 0, \quad \lambda_2 = \frac{i\omega}{\kappa_-} - \frac{iq\omega}{\kappa_- - \kappa_-}. \hspace{1cm} (A12)$$

The root $\lambda_1 = 0$ corresponds to mode solutions which are approximately constant, i.e. remain smooth at the CH and are irrelevant for SCC, while $\lambda_2$ corresponds to asymptotics

$$|f|^{\lambda_2} \sim |r - r_-|^\frac{\kappa_-}{\kappa_- - \kappa_-} \frac{iq\omega}{\kappa_- - \kappa_-}.$$

If we consider QNMs of the form $\omega = \omega_R - i\omega_I$ then

$$|f|^{\lambda_2} \sim |r - r_-|^\frac{\omega_I}{\kappa_-} |r - r_-|^\left(\omega_R - \frac{\omega_I}{\kappa_-} \right).$$

The second factor is purely oscillatory, so the only relevant factor for SCC is $|r - r_-|^{\frac{\omega_I}{\kappa_-}}$ with $\alpha := -\Im \omega$ the spectral gap define in [1]. This function lies in the Sobolev space $H^s$ for all $s < \frac{1}{2} + \frac{\alpha}{\kappa_-}$. This provides the justification for our search for BH parameters for which $\beta > 1/2$.  

1 The same operator arises for the field $G$ with $PG = 0$ by following exactly the same steps.
Appendix B: Higher $\kappa$ modes

In this appendix we verify the expectation that the higher $\xi$ QNMs do not affect Strong Cosmic Censorship. In Fig. 6 we show the $\xi = 1, 2, 3$ modes. The ones depicted with blue colors belong to the PS family, since they originate from complex modes for $q = 0$ and follow their pattern. The ones depicted with green colors belong to the NE family, since they originate from purely imaginary modes for $q = 0$ and follow their pattern. The dS modes are not present in the range of interest since they are too subdominant for the chosen cosmological constant. We clearly see that the modes defining $\beta$ according to (I) will be the $\xi = 1$ QNMs. The same holds for other choices of $\Lambda$.

For completeness, in Table I we show various modes from different families with $\xi = 1, 10$ for various choices of $Q, q$ and $\Lambda$. We compare the neutral $\xi = 10$ PS modes with a WKB approximation for arbitrarily large $\xi$ and verify that indeed the imaginary parts lie very close. It is apparent that for NE charges $\xi = 1$ modes always dominate. It is also apparent that the only way for $\xi \to \infty$ modes to be the dominant ones of the PS family is for very small cosmological constants. Specifically, for $\Delta M^2 = 0.001$, $\xi \to \infty$ modes are dominant up to a critical BH charge $Q_c \approx 0.866$. Above $Q_c$, $\xi = 1$ modes dominate the PS family. E.g. for $Q = 0.865$ and $q = 0$ the dominant ($\xi \to \infty$) PS mode admits $\text{Im}(\omega_{PS})/\kappa_{-} = -0.0479$, while the dominant ($\xi = 1$) dS mode admits $\text{Im}(\omega_{dS})/\kappa_{-} = -0.0135$; the NE family is too subdominant for this BH charge. None of those modes can potentially violate SCC so it becomes a necessity to search closer to extremality, where we are aware that $\kappa_{-}$ becomes comparable to $\text{Im}(\omega)$ even smaller.

Finally, for larger $\Lambda$, $Q_c$ decreases, moving even further away from extremality.

Considering the above, we are convinced that throughout the parameter space in study, $\xi = 1$ indeed gives the dominant modes for all families.

| $Q = 10^{-1}$ | $\Delta M^2 = 0.005$ |
|-------------|----------------|
| $\xi$ | $q=0$ | $q=0.1$ |
| 1 | $\omega_{PS} = 0.1795 - 0.0947 i$ | $\omega_{PS} = -0.1760 - 0.0941 i$ |
| | $\omega_{dS} = -0.0614 i$ | $\omega_{dS} = 0.0003 - 0.0614 i$ |
| 10 | $\omega_{PS} = 1.8831 - 0.0941 i$ | $\omega_{PS} = -1.8797 - 0.0940 i$ |
| | $\omega_{dS} = -0.0941 i$ | $\omega_{dS} = -0.0941 i$ |

| $Q/Q_{max} = 1 - 10^{-2}$ | $\Delta M^2 = 0.005$ |
|-------------|----------------|
| $\xi$ | $q=0$ | $q=0.1$ |
| 1 | $\omega_{PS} = 0.1280 - 0.0650 i$ | $\omega_{PS} = -0.1247 - 0.0647 i$ |
| | $\omega_{dS} = -0.2170 i$ | $\omega_{dS} = -0.0003 - 0.2170 i$ |
| 10 | $\omega_{PS} = 1.3097 - 0.0654 i$ | $\omega_{PS} = -1.3064 - 0.0654 i$ |
| | $\omega_{dS} = -0.0654 i$ | $\omega_{dS} = -0.0654 i$ |

TABLE I. Lowest lying fermionic QNMs of RNdsBH for various $Q, q, \Lambda$ and $\xi$.

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