Long-range magnetic fields in the ground state of the Standard Model plasma

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In thermal equilibrium the ground state of the plasma of Standard Model particles is determined by temperature and exactly conserved combinations of baryon and lepton numbers. We show that at non-zero values of the global charges a translation invariant and homogeneous state of the plasma becomes unstable and the system transits into a new state, containing a large-scale magnetic field. The origin of this effect is the parity-breaking character of weak interactions and chiral anomaly. This situation can occur in the early Universe and may play an important role in its subsequent evolution.

It is generally believed that the ground state of the Standard Model at high temperatures is homogeneous and isotropic. This assumption underlies the description of all the important processes in the early Universe: baryogenesis, cosmological phase transitions, primordial nucleosynthesis, etc.44 In this work we demonstrate, however, that at finite density of lepton or baryon numbers due to parity-violating nature of the weak interactions this homogeneous “ground state” becomes unstable by developing a long-range magnetic field. The transition to the “true” ground state may depend on the details of the non-equilibrium dynamics, when various violent dissipative processes (e.g. turbulence, radiation emission, finite conductivity of plasma) play an important role.

What are the conditions for the translational invariance to be spontaneously broken by a long-range field? It is sufficient for the free energy of the gauge fields to contain an interaction term that dominates over the kinetic energy and can be both positive and negative. An example is provided by a Chern-Simons term $I_{CS} \propto A \partial A$, that has less derivatives that the kinetic term $(\partial A)^2$ and therefore can dominate over it at large scales. The presence of the Chern-Simons term in the Maxwell equations is known to lead to an instability and generation of magnetic fields.

At zero temperatures and densities the Chern-Simons term for electromagnetic fields is prohibited as a consequence of gauge invariance and Lorentz symmetry (Furry theorem [1]). At finite temperatures and densities the plasma creates a preferred reference frame and the 4-dimensional Lorentz invariance is broken down to 3-dimensional one. As a result the free energy of static gauge fields is

$$\mathcal{F}[A] = \int d^3p \left( p\partial_\mu A^\mu - e L_A \right) + O(A^3)$$ (1)

with the polarization operator

$$\Pi_{ij}(\vec{p}) = (p^2 \delta_{ij} - p_i p_j) \Pi_1(\vec{p})^2 + i e \epsilon_{ijk} p_k \Pi_2(\vec{p})^2,$$ (2)

where $i,j,k = 1,2,3$ are spacial indices; $p^2 = |\vec{p}|^2$; $\epsilon_{ijk}$ is the antisymmetric tensor. Eq. (2) is the most general form of $\Pi_{ij}$ satisfying the gauge-invariance transversality condition $p_i \Pi_{ij} = 0$. In the long wavelength limit $p^2 \rightarrow 0$ a non-zero $\Pi_2(0)$ means that the Chern-Simons term $\Pi_2(0) \vec{A} \cdot \vec{\nabla} \times \vec{A}$ appears in (1). The $3 \times 3$ matrix (2) has then a negative eigenvalue for sufficiently small momenta $p < |\Pi_2(p^2)/\Pi_1(p^2)|$ and the corresponding eigenmode grows larger and larger (until the higher order in $A$ terms would stabilize it). In the above consideration it is important that the gauge field is Abelian. Unlike the Yang-Mills fields the component of the photon field does not get screened in plasma (i.e. $\Pi_1(0)$ remains finite) and therefore the instability does not require large $\Pi_2(0)$.

In this work we demonstrate that in the Standard Model plasma in the Higgs phase an equilibrium value of $\Pi_2(0)$ for electromagnetic fields is non-zero and proportional to the values of the global charges: $baryon (B)$ and flavor lepton numbers $L_R$ (index $\alpha$ runs over flavours). Unlike the previous works [4, 8] (see discussion below) it is important that even if the anomalous charge $B + L$ is absent, $\Pi_2(0)$ remains non-zero and magnetic fields develop [43].

Chern-Simons term and axial anomaly. The origin of the $\Pi_2$ term has its roots in the axial anomaly (see e.g. [4, 6, 9–11]). Indeed, the non-conservation of the axial current at finite densities of left or right fermions $n_L, n_R$ means that one can convert fermions into gauge field configurations with a non-trivial Chern-Simons number $N_{CS} \equiv \int d^3x A \cdot B$ (where $B = \nabla \times A$ is a magnetic field):

$$\frac{d(n_L - n_R)}{dt} = \frac{e^2}{2\pi^2} \int d^3x E \cdot B = \frac{\alpha}{\pi} \frac{dN_{CS}}{dt}$$ (3)

(here $E = -\dot{A}$ is an electric field and $\alpha = e^2/4\pi$ is the fine-structure constant). Let us consider the simplest example of left and right fermions at zero temperature with different Fermi energies (chemical potentials) $\mu_L \neq \mu_R$. Infinitesimal change of the gauge field $\delta A$ will destroy (create) $\delta n_{L,R} = \pm \frac{\alpha e^2}{2\pi} \delta N_{CS}$ of real fermions around the Fermi level. The total energy of the system will change by $\delta \mathcal{F} = \left( \mu_L - \mu_R \right) \frac{\alpha e^2}{2\pi} \delta N_{CS}$, which leads to the parity-odd Chern-Simons term in the free energy: $\mathcal{F}[A] = \frac{\alpha (\mu_R - \mu_L)}{2\pi} \int d^3x A \cdot B$.

This remains true in any vector-like gauge theory at finite temperature/density where there is a difference of chemical potentials of left and right-chiral charged particles [4, 9] (also [12, 14]). Indeed, to calculate the polarization operator we need to analyze one-loop contribution from charged...
fermions described by the diagram 1a. If the left and right fermions have different chemical potentials such that
\[
G_{L,R} = \frac{1}{\gamma_0 (\omega_n + \mu_{L,R}) + \gamma \cdot p_{L,R}}
\]

where \(\omega_n = n(2n+1)T\), \(n \in \mathbb{Z}\) are the Matsubara frequencies and \(P_{L,R} = \frac{1}{2}(1 \pm \gamma_5)\) are chiral projectors; their contributions to this diagram are different. Assuming first that \(\Delta \mu \ll T\), let us consider linear in \(\Delta \mu/T\) correction to the polarization operator for this one should differentiate the fermions Greens function with respect to \(\mu\). This correction is described by the diagram 1a with \(\Delta \mu\) playing the role of a third external field. The diagram 1a thus turns into the famous triangular graph for the axial anomaly [15–17], with the third vertex containing “axial vector field” \(X_{\beta} = \partial_{\beta} \Delta \mu \gamma_5\). The resulting term in the effective action, \(\propto \epsilon_{\alpha \beta \mu} X_{\alpha} A_{\beta} \partial_{\mu} A_{\mu}\), again reduces to the Chern-Simons term with \(\Pi_2(0) = \frac{\pi}{T} \Delta \mu\).

This expression for \(\Pi_2(0)\) is actually exact in \(\Delta \mu\) and \(T\) [4].

Similar logic applies to the non-Abelian gauge fields. In the Standard Model with its chiral weak charges of fermions the coefficient in front of the SU(2) Chern-Simons term can be expressed in terms of \(\mu_{B-L}\) (\(B\) being baryon and \(L\) lepton numbers) [2] [18]. However, in this case a homogeneous state becomes unstable only at large values of chemical potential, exceeding the mass of weak bosons. Even at high temperatures in the symmetric phase the “magnetic screening” effect [2] requires \(\Delta \mu \gtrsim T\) to overcome the “magnetic mass” \(m_{mag} \sim \alpha_W T\). Moreover, anomalous non-conservation of \(B + L\) current drives the coefficient of corresponding Chern-Simons term to zero [19] and the standing wave-like configurations of the gauge fields are actually metastable (see discussion in [7]).

An example of the situation when the non-zero coefficient in front of the Chern-Simons is realized in the Standard Model at high temperatures, \(T \gg m_f\), when the smallness of the electron’s Yukawa coupling makes the number of right electrons conserved at classical level. If the initial conditions have non-zero \(\mu_{\gamma 5}\), the Chern-Simons term for the U(1) hyperfield is then generated (with \(\Pi_2(0) \propto \mu_{\gamma 5}\) and the generation of long wave-length magnetic fields occurs [8] until the chirality-flipping reactions, suppressed as \((m_e/T)^2\) do not destroy the \(\mu_{\gamma 5}\). In the early Universe where such a situation can be realized, the rate of these reactions becomes comparable to the Hubble expansion rate at high temperatures \(T \sim 80\) TeV.

Magnetic fields, generated in such a way are rather short-wavelength (much smaller than the horizon size at that epoch) and are probably erased during the subsequent evolution due to plasma dissipative processes. At lower temperatures all the chirality-flipping reactions are in thermal equilibrium and naively the chemical potentials of all left and right-chiral particles are equal. However, it was shown in [20] that if strong helical magnetic fields are initially present in the plasma, then the relaxation rate both for \(\Delta \mu\) for electrons and for helical fields significantly increases and they both can survive down to \(T \sim 10\) MeV. The ground state that the system eventually reaches contains neither fields nor \(\Delta \mu\).

We demonstrate below that although these considerations are true for electrodynamics, in the Standard Model where fermions are also involved in parity-violating weak interactions, the difference of chemical potentials of all left- and right-chiral fermions is actually present (with all chirality-flipping reactions taken into account) and leads to the generation of magnetic fields.

In this paper we analyze the simplest situation when this effect is present: the case \(T \ll m_W\) (mass of the \(W\)-boson) when weak interactions can be described by the Fermi theory:

\[
\mathcal{L}_F = \frac{4G_F}{\sqrt{2}} \left( (J_\mu^W)^2 + 2(J_\mu^{EM})^2 \right).
\]

The full Hamiltonian of the theory \(\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_F + \mathcal{H}_{EM}\) has a free part for fermions and photons, \(\mathcal{H}_0\), and terms describing electromagnetic \(\mathcal{H}_{EM}\) and Fermi \(\mathcal{H}_F\) interactions [46].

\textbf{Dispersion relation of fermions and chemical potentials.}

To describe the equilibrium plasma at \(T \ll m_W\) we introduced the density matrix, \(\hat{\rho} = Z^{-1} \exp \left( -\beta (\mathcal{H} - \sum_{\alpha} \lambda_{\alpha} L_{\alpha} - \right) \)
\[ \lambda_Q Q - \lambda_B B \]. Five global charges commute with the Hamiltonian \( \mathcal{H} \): three \( L_\alpha, B \) and \( Q (\lambda_\alpha, \lambda_Q, \lambda_B) \) are the corresponding Lagrange multipliers); and the partition function \( Z \) ensures that \( \text{tr} (\hat{\rho}) = 1 \).

To find the distribution functions of left and right-chiral particles we compute the correlators \( \langle \bar{\psi} P_{LR} \psi \rangle = \text{tr} (\hat{\rho} \bar{\psi} P_{LR} \psi) \). We expand the density matrix in interactions to get

\[ \hat{\rho} \approx \hat{\rho}_0 \left( 1 - \beta \mathcal{H}_F - \beta \mathcal{H}_{\text{EM}} - \frac{\beta^2}{2} \mathcal{H}_F \mathcal{H}_F + \ldots \right). \] (6)

At zeroth order in interactions one gets \( \langle \bar{\psi} P_L \psi \rangle_0 = \frac{1}{2} \langle \bar{\psi} \psi \rangle_0 \). The propagators of left and right particles (up to corrections of the order \( m/T \ll 1 \)) have the form \( \Sigma \) with \( \mu_L = \mu_R \). This conclusion remains true if we take into account parity-preserving electromagnetic interactions.

Taking into account chiral Fermi interactions, \( \hat{\rho}_F \approx \hat{\rho}_0 (1 - \beta \mathcal{H}_F) \) one finds that \( \langle \bar{\psi} P_L \psi \rangle_0 \neq \langle \bar{\psi} P_R \psi \rangle_0 \). The Green’s function is found via \( G^{-1} = G_0^{-1} - \Sigma \) where for example the self-energy of left electron, \( \Sigma_{eL} \), is

\[ \Sigma_{eL} = \frac{4G_F}{\sqrt{2}} \left[ g_F^c \bar{\gamma} \mu P_L \langle \bar{\psi} e \rangle_0 \gamma_\mu P_L + \sum_\psi g_L^c \bar{\gamma} \mu P_L \langle \bar{\psi} \gamma_\mu P_L \psi \rangle_0 \right] + \frac{1}{2} \bar{\gamma} \mu P_L \langle \bar{\psi} \psi \rangle_0 \gamma_\mu P_L \equiv \delta \mu_{eL} \gamma_0 P_L. \] (7)

Expression (7) does not depend on momentum, the thermal averages \( \langle \bar{\psi} \psi \rangle_0 \) and \( \langle \bar{\psi} \gamma_\mu P_L \psi \rangle_0 \) are proportional to the particle-antiparticle asymmetry (see e.g. [21]). To compute \( \delta \mu_{eL} \) one should substitute in (7) the thermal averages, summarized in the Table 1 in Appendix B. As a result, for example the electron propagator becomes

\[ G = \frac{1}{\gamma_0 (\delta \mu_{eL} P_L + \delta \mu_{eR} P_R + \mu_{\text{tree}}) + \bar{\rho} + m_e}. \] (8)

i.e. the dispersion relation of electrons change when taking into account Fermi corrections (c.f. [21], [47]). Indeed, from \( (\rho - m - \Sigma) \psi = 0 \) we see that the “on-shell conditions”:\n\( (\omega - \mu_{\text{tree}})^2 = p^2 + m^2 \) gets shifted for left (right) particles by \( 2(\omega - \mu_{\text{tree}}) \delta \mu_{L,R} \) (in the limit \( \delta \mu_{L,R} \ll \omega \), where \( \mu_{\text{tree}} = (\lambda_Q - \lambda_e) \). In the limit \( m_e/T \rightarrow 0 \), Eq. (8) splits into the sum of free propagators in the form (4) with different chemical potentials \( \mu_{L,R} \).

This difference can give rise to a parity-odd term in polarization operator of photons [4]. Indeed the polarization operator that was parity-even when computed with respect to the density matrix \( \hat{\rho}_0 \) acquires a parity-odd part when averaged with respect to the \( \hat{\rho}_F \). The lowest order weak corrections are represented by two diagrams [18] and [19]. The computation of the diagram [18] is quite similar to that of [4] and gives a non-zero \( \Pi_2(0) \) given by

\[ \Pi_2(0) = \frac{\alpha}{2\pi} \frac{4G_F}{\sqrt{2}} \left[ c_L \alpha_c + c_B B \right], \] (9)

where coefficients \( c_L, c_B \sim O(1) \) depend on the fermionic content of the plasma (details are summarized in Appendix 48). The diagram [22] does not contribute to the \( \Pi_2(0) \) as it can be cut into two diagrams [18] along the vertical dotted line, each of which is at least first order in momentum (Eq. 3).

Although the Fermi theory [5] is not renormalizable, the result [9] is given by the non-divergent part of the diagram [3] and is expressed in terms of well-defined physical quantities (c.f. 21).

**Chern-Simons coefficient at two loops and “non-renormalization theorems”.** The diagram [18] is similar to the triangular diagram, responsible e.g. for \( \pi^0 \rightarrow 2\gamma \) decay (with \( \Delta \mu \gamma_0 \gamma_5 \psi \) playing the role of the only non-zero component of the chiral current, describing pion) [15–17]. It is well known that the axial anomaly should be calculated at one loop only and is not renormalized by higher-loop corrections [22]–[25], also at finite temperature and density. At the same time our result becomes non-zero only at two loops. There is, however, no contradiction. What is non-renormalized for the chiral anomaly is the numerical coefficient in front of the proper combination of external fields, (e.g. \( \delta \mu \) in Eq. 3). In our case this coefficient is also not renormalized. The structure of the parity-odd one loop term has the same form at tree-level and at one-loop in \( G_F^c \): \( \Pi_2(0) = \frac{\alpha}{2\pi} (\delta \mu_{\text{tree}} + \delta \mu) A \cdot B \). The numerical coefficient is dictated by the axial anomaly; \( \delta \mu_{\text{tree}} \) is a possible difference of chemical potentials present at tree-level (zero in our case); and \( \delta \mu \) is the shift generated by the diagram [18]. The structure of the \( \Pi_2(0) \) term therefore remains the same as in [4] with the total difference of chemical potential \( (\Delta \mu_{\text{tree}} + \delta \mu) \).

Also a four dimensional theory at finite temperatures can be regarded as a three dimensional Euclidian model albeit with the infinite number of particles – each Matsubara mode of a fermion becomes a “particle” with the mass \( \omega_n = \pi (2n+1)T, n \in \mathbb{Z} \). Therefore (as it was argued in [26]) our result may seem to be in contradiction with the “Coleman and Hill (CH) theorem” [27] that states that in any Euclidian three-dimensional gauge theory without massless particles \( \Pi_2(0) = \int f \frac{d^4 \omega}{(2\pi)^4} \frac{m_f}{|m_f|} \) and is exact at one loop. However, the presence of the infinite number of modes changes the situation, as can be seen already in the simplest chiral gauge theory, if one computes the \( \Pi_2(0) \) in the Matsubara formalism (see
generation of magnetic fields. The Chern-Simons number fixed vector

\[ \frac{\omega_n - i\mu R}{\sqrt{(\omega_n - i\mu L)^2}}. \]

The reason why this happens is clear: the degree of divergence of the diagrams is different in 3 and 4 dimensions (hence the infinite sum over \( n \)). In particular, if we first summed over the Matsubara frequencies and then integrating over momentum (or if one uses dimensional regularization in 3-momentum integral and then takes a limit \( d \to 3, 4 \)), one obtains a well-defined answer of \( \mathcal{O}(\alpha^2) \).

Moreover, the CH theorem uses the fact that the 3-point

\[ \Gamma^{(3)}(p_1, \ldots) = \mathcal{O}(p_1). \]

This is not true in our case, as the diagram \( \mathcal{D} \) becomes linearly divergent in 4 dimensions and therefore the shift of integration momentum by any fixed vector \( k \) changes its parity-odd part by a finite amount \( \propto \Delta \mu e^{i\eta R} A_1 A_1 [28, 29] \).

**Ground state.** The presence of \( \Pi_2(0) \neq 0 \) leads to the generation of magnetic fields. The Chern-Simons number

\[ N_{CS} \sim kA^2 \]

will increase until it reaches

\[ \frac{\alpha}{\pi} N_{CS} \sim (n_L - n_R) \sim G_F L_{tot} \] (see e.g. [6]). At fixed \( N_{CS} \) the magnetic field tends to increase its wavelength to decrease the total energy

\[ (B^2 \sim kN_{CS}). \]

As a result, the system does not have a thermodynamic (infinite volume) limit (c.f. [8]): the value of the field and the scale of the inhomogeneity will be determined by the size of the system. It is clear, however, that in realistic systems establishing of the long-range field is a complicated process (see e.g. [20]), greatly affected by the dissipative processes and by existence of different relaxation channels of \( N_{CS} \) (resistivity of plasma, energy radiation, turbulence, etc., see e.g. [30, 32]). This may significantly affect subsequent evolution and the final state of the system.

**Discussion.** In this work we demonstrated that the Standard Model plasma at finite densities of lepton and baryon numbers becomes unstable and tends to develop large scale magnetic fields. We considered electrodynamics plus Fermi theory (5), a description of weak interactions valid when

\[ e^{-m_W^2/T} \lesssim (T/m_W)^2, \]

i.e. at \( T \lesssim 40 \text{ GeV} \). At higher temperatures one should consider full electroweak theory and perform two-loop computations of \( \Pi_2(0) \). At even higher temperatures (in the symmetric phase) one should analyze hypermagnetic fields. We leave these analyses for future works. We expect however that our conclusion about the instability of a homogeneous state will hold.

Below we discuss several realistic systems in which the effects discussed here can become important. As a first example, let us consider the primordial plasma at radiation dominated epoch. Eq. (9) gives

\[ \Pi_2(0) \sim \frac{4\pi}{G_F} \eta_{LB}. \]

\( \eta_{LB} \) is the ratio of the total lepton (baryon) number to the number of photons, \( \eta_g = \frac{\eta_{LB}}{\pi^2} T^3 \) and the numerical coefficient \( c \approx 2.5 \times 10^{-2} \). We see that the CS coefficient decreases with temperature fast and therefore the effect is the strongest at high temperatures \( T \lesssim m_W [49] \).

The instability starts to develop at scales \( k \approx \Pi_2(0) \) and the magnetic field initially growth as \( e^{kT/\sigma} \) where \( \sigma \equiv k^2 T/\sigma \) (see e.g. [8, 20]). The conductivity of the plasma is \( \sigma \sim \mathcal{O}(10^3 T) [33] \). The requirement for an instability to develop over the characteristic time of temperature change (i.e. Hubble time) is:

\[ \beta(T) \approx 2.0 \left( \frac{m_W}{T} \right)^3 (\eta_{LB})^2 > 1. \]

We see that the measured value of **baryon asymmetry** \( \eta_B \sim 6.0 \times 10^{-10} [34] \) is too small to trigger any instability. The situation is different for lepton asymmetry where only the upper bounds at the epoch of primordial nucleosynthesis exist:

\[ |\eta_L| \lesssim \text{few} \times 10^{-2} [35]. \]

At earlier epochs even \( \eta_L \sim 1 \) is possible (if this lepton asymmetry disappears later). This is the case e.g. in the \( \nu \)MSM (see [36] for review), where the lepton asymmetry keeps being generated below the sphaleron freeze-out temperature and as a result may reach the levels

\[ \eta_L \sim 10^{-2} \lesssim 1 \text{ before it disappears at } T \sim \text{ few GeV} [37]. \]

We see that significant magnetic fields can develop in this case, which can play an important role for analysis of the cosmological implications of the \( \nu \)MSM.

As a next application we consider a high density degenerate electron plasma (appearing e.g. in white dwarfs and neutron stars, [38]). Notice, that our consideration remains valid in this regime, as Eq. (7) makes no assumption about the relation between mass, temperature and chemical potential of the particles. Only the numerical coefficient in Eq. (9) changes and we checked that it is non-zero and \( O(1) \). The same relation

\[ \Pi_2(0) \sim \frac{4\pi}{G_F} G_F L_{tot} \]

holds, however now \( L_{tot} = n_e \) (density of electrons) that can be quite essential, reaching

\[ 10^{30} \div 10^{35} \text{ cm}^{-3} \text{ in the crust of neutron stars} [39]. \]

The corresponding scale of the instability \( k \sim \Pi_2(0) \) is then in (sub)km size and the time of its development is much shorter than the lifetime of the star.

**To summarize:** in this work we discussed a previously unknown effect that occurs in the Standard Model at finite temperature and density. It implies that a number of processes in the early Universe can be affected, including cosmological phase transition, baryogenesis, dark matter production. This effect may in particular lead to the generation of horizon-scale helical cosmic magnetic fields purely within the Standard Model. Such fields may survive till present and serve as seeds for the observed magnetic fields in galaxies and clusters. The effect may also be important for explanation of physics of compact stars.

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Charged leptons: \[ \langle \bar{\ell} \gamma^\mu P \ell \rangle = \delta^{\mu 0} \frac{1}{2} \Delta n_{\ell} \]

\[ \langle \ell \ell \rangle = \frac{1}{4} \gamma_0 (\Delta n_{\ell}) \]

Neutrinos: \[ \langle \bar{\nu} \gamma^\mu P \nu \rangle = \delta^{\mu 0} \Delta n_{\nu} \]

\[ \langle \nu \nu \rangle = \frac{1}{2} \gamma_0 (\Delta n_{\nu}) \]

Quarks: \[ \langle \bar{q} \gamma^\mu P q \rangle = \delta^{\mu 0} \frac{1}{2} \Delta n_q \]

\[ \langle q \bar{q} \rangle = \frac{1}{12} \gamma_0 (\Delta n_q) \]

| TABLE I: Thermal averages of fermions. Particle-antiparticle asymmetry is defined as \( \Delta n = \frac{g}{2} \int \frac{d^3 \vec{p}}{(2\pi)^3} [f_p - \bar{f}_p] \), where \( f_p \) and \( \bar{f}_p \) is the Fermi-Dirac distribution for particles (anti-particles) and the number of internal degrees of freedom, \( g = 2 \) for neutrinos \( \nu \); \( g = 4 \) for charged leptons \( \ell \); \( g = 12 \) for quarks \( q \). |

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**Appendix A: Fermi theory**

For completeness we summarize in this Appendix the definitions of charged and neutral currents in the Fermi theory (see e.g. Chapter 20 in [40]). The **charged currents** are defined as

\[ J_{\mu}^{CC} = \frac{1}{\sqrt{2}} (\bar{e}_L \gamma^\mu \nu_L + \bar{d}_L \gamma^\mu u_L) + \text{other generations} \quad \text{(A1)} \]

and **neutral currents** are

\[ J_{\mu}^{NC} = \sum_{\psi} \bar{\psi} \gamma^\mu \left( g_L^\psi P_L + g_R^\psi P_R \right) \psi \quad \text{(A2)} \]

Here \( P_L = \frac{1}{2} (1 \pm \gamma_5) \) are chiral projectors, charges \( g_L^\psi = (T_3 - \sin^2 \theta_W Q) \), \( g_R^\psi = (-\sin^2 \theta_W Q) \), \( T_3 = \pm \frac{1}{2} \) is the 3rd generator of \( SU(2) \), \( Q \) is the electric charge and the sum in \( \text{(A2)} \) goes over fermions in all flavours.

**Appendix B: General expression for the Chern-Simons coefficient \( \Pi_2(0) \)**

A general expression for \( \Pi_2(0) \) is given through the asymmetries of all fermions (if some fermions are absent in the plasma, their asymmetry should be put to zero).

\[ \Pi_2(0) = \frac{\alpha}{2\pi} \frac{4G_F}{\sqrt{2}} \left( -\frac{2}{9} \sum_{\alpha} \Delta n_{\nu_{\alpha}} - \frac{31}{36} (1 - 2 \cos(2\theta_W)) \sum_{\alpha} \Delta n_{\ell_{\alpha}} \right. \]

\[ + \frac{1}{81} \left( 17 - 62 \cos(2\theta_W) \right) (\Delta n_u + \Delta n_c) \]

\[ + \frac{1}{324} \left( 91 + 134 \cos(2\theta_W) \right) (\Delta n_d + \Delta n_s) \]

\[ + \frac{67}{324} (1 + 2 \cos(2\theta_W)) \Delta n_b \right) \quad \text{(B1)} \]

(here \( \theta_W \) is the Weinberg’s angle).

Once the asymmetries of all particles \( \Delta n \) are expressed through the conserved charges \( B \) and \( L_\alpha \) under the condition of electric neutrality of the plasma [43], the expression \( \text{(B1)} \) reduces to the form \( \text{(9)} \):

\[ \Pi_2(0) = \frac{\alpha}{2\pi} \frac{4G_F}{\sqrt{2}} \left[ c_{L_\alpha} L_\alpha + c_B B \right] \quad \text{(B2)} \]

where the values of coefficients depend on the fermionic content of the plasma. For example, if plasma contains 5 quarks (except for the top quark) and all leptons, then

\[ c_{L_\alpha} = \frac{8 (22 \cos(2\theta_W) - 45)}{621} \quad \text{and} \quad c_B = \frac{53 \cos(2\theta_W) + 430}{621} \quad \text{(B3)} \]