The Mathematical Models of Lattice Functions in Modelling of Control System

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Abstract. The paper proposes an approach to constructing a mathematical model of lattice functions, which are mainly used in the study of discrete control systems in the time and domain of the Laplace transform. The proposed approach is based on the assumption of the physical absence of an impulse element. An alternative to the classical approach to the description of discrete data acquisition - the process of quantization in time, is considered. As a result, models of the lattice function in the time domain and the domain of the discrete Laplace transform are obtained. Based on the obtained mathematical models of lattice functions, a mathematical model of the time quantization element of the system is obtained. This will allow in the future to proceed to the construction of mathematical models of various discrete control systems, incl. expanding the proposed approaches to the construction of mathematical models of multi-cycle continuous-discrete automatic control systems.

1. Introduction

Discrete devices are widely used in modern control systems, in functionally complex robotic complexes such as, for example, unmanned aerial vehicles, unmanned underwater vehicles \cite{1-3}. In such systems, some part of the information (possibly all information) is formed at separate, discrete times and processed further on a computer \cite{4-8}. This process is associated with the appearance of the so-called lattice functions in the course of solving a certain range of problems, such as sampling, signal recovery, extrapolation, spectral analysis, analysis and synthesis of digital systems, construction of machine learning algorithms, and many others \cite{9-11}. Even though the problem of studying the lattice function has a long history \cite{12-17}, this problem has not lost its relevance.

In control systems using a computer, control signals are generated to influence the system (object) by the purpose and method of control. To construct mathematical models and analyze the processes occurring in the system, characteristic variables are introduced - functions of time. Let \( \dot{x}(t) \) be one of these variables, which is observed at discrete times \( t_k, k = 0,1,2,\ldots \). Usually, these points in time are equidistant, and it can be assumed that \( t_k = kT, k = 0,1,2,\ldots \). The process of forming values

\[
(1)
\]

functions at discrete times are called the quantization process. In particular, it can be carried out in a discrete sensor. The constant \( T \) is called the quantization period. Thus, the quantization process is characterized by two functions - a continuous-time \( t \) function \( \dot{x}(t) \) and a discrete-time function

\[
(1)
\]
The discrete-time function is called the lattice function. The mapping of the continuous-time function to the lattice is graphically presented in Figure 1.

Figure 1. The mapping of the continuous-time function to the lattice.

The approach to the description of lattice functions is widely known as a means of displaying discrete information [17]. It is based on the concept of impulse and impulse element, formalized with the help of \(\delta\) - Dirac function. This approach has developed under the influence of the theory of impulse systems, the impulse elements of which exist physically, as real technical objects with a rather complex mathematical description of the impulses they generate. We can mention the widespread systems with pulse amplitude and pulse-width modulation of signals.

In the case we are considering, there is no physically impulsive element. It is introduced formally as an auxiliary means of description with discrete information retrieval. The question arises, is it possible to do without such a formal reception? The article proposes a different approach to the description of discrete data retrieval - the process of quantization in time.

### 2. The straightforward approach to describing lattice functions

We will consider the lattice functions as functions of continuous-time. Let the quantization period be \(T\). Let us introduce the function of separating the whole part \(E[\frac{t}{T}]\), which we define as follows

\[
E[\frac{t}{T}] = k, \quad kT \leq t < (k+1)T, \quad k = 0,1,2,\ldots
\]  

A lattice function \(x(kT), k = 0,1,2,\ldots\), as a sequence of values of a continuous-time function \(x(t)\), formed in the process of fixing its values with a period \(T\), will be considered as a function of continuous-time \(t\) and denoted by a symbol \(x_T^*(t)\).

We define this function as follows

\[
x_T^*(t) = \begin{cases} 
  x(kT), & t = kT, \quad k = E[\frac{t}{T}]; \\
  0, & t \neq kT, \quad k = 0,1,2,\ldots
\end{cases}
\]

The function is a discontinuous function with a finite number of discontinuities of the first kind in any finite time interval.
We can extend this definition to abstract lattice functions that we need in the analysis of quantization and information processing.

Let us introduce into consideration, in particular, the unit lattice function

\[ l^*_T(t) = \begin{cases} 1, & t = kT, \quad k = E[f/T]; \\ 0, & t \neq kT, \quad k = 0,1,2,\ldots \end{cases} \quad (4) \]

Now lattice function (3) can be represented as

\[ x^*_T(t) = x(t)l^*_T(t). \quad (5) \]

Relation (5) defines an abstract modulation process without pseudo-functions. The result of modulation (5) does not pulse with infinite amplitude and finite area as in the classical approach [9], but the values of the lattice function themselves.

Instead of the Laplace transform

\[ x^*(s) = \int_0^\infty e^{-st}x^*(t)dt = \sum_{k=0}^\infty x(kT)e^{-skT}, \quad (6) \]

where \( x^*(t) \) is a sequence of infinite amplitude modulated pulses \( x(t)\delta(t-kT) \) with an area

\[ \int_{-\infty}^{+\infty} x(t)\delta(t-kT)dt = x(kT) \quad (7) \]

we now immediately introduce the discrete Laplace transforms \( \{x^*_T(t)\} = x^*_T(s) \) as

\[ x^*_T(s) = \sum_{k=0}^\infty x(kT)e^{-skT}. \quad (8) \]

Relation (8) repeats the results of the classical approach [9] but is formed directly (as an infinite sum) without recourse to the Laplace transform of pulse modulation.

The connection between the functions \( x^*_T(s) \) and \( x(s) \) is established by (5) in a more natural, in our opinion, and rigorous way without the need for justification with actions on pseudo functions.

Let \( x(t) \) be an original function for which representation

\[ x(t) = \frac{1}{2\pi} \int_{C+j\infty}^{C+j\infty} e^{-pt}x(p)dp. \quad (9) \]

and remark that the real number \( C \) is greater than the abscissa of the convergence of the Laplace integral for \( x(t) \) are valid.

Then

\[ x(kT) = \frac{1}{2\pi} \int_{C-j\infty}^{C+j\infty} e^{bkT}x(p)dp. \quad (10) \]

Substituting into (8), we find
$x_T^*(s) = \frac{1}{2\pi j} \int_{C^-}^{C^+} x(p) \sum_{k=0}^{\infty} e^{-(s-p)kT} dp.$ \hfill (11)

Under the condition, $\text{Re}(s-p) > 0$ the series under the integral sign converges absolutely. As a result, we get a representation for the discrete Laplace transform of the lattice function.

3. **The mathematical model of the time quantization element**

Having different versions of mathematical models for describing lattice functions (the classical approach and the approach proposed in this article), we can proceed to construct models of various discrete systems, including multi-cycle and continuous-discrete systems [2, 7, 18-20].

Consider a signal, time-sliced with a tact $T$

$$x^*(t) = x(t)\delta_T^*(t),$$ \hfill (13)

where $x(t)$ is a quantized function of continuous-time, $\delta_T^*(t)$ is an unmodulated sequence of infinite amplitude pulses with an area equal to 1. We will assume that the signal is considered at some discrete times $t \in \Theta, \Theta = \{t_k : t_k = k\tau; \ k \text{ - integer}\}$.

In general, $\tau \neq T$. Consider the case

$$T = N\tau,$$ \hfill (14)

where $N$ is an integer.

Let us introduce a complex function $\sigma_N(k)$ that defines the function $\delta_T^*(t)$ at times $t_k$. A periodic function $\sigma_N(k) = \sigma_N(k+N)$ defined on a set of integers $k$ for a natural number $N$ can be expressed in terms of a finite sum

$$\sigma_N(k) = \frac{1}{N} \sum_{l=0}^{N-1} e^{\frac{2\pi j kl}{N}} = \begin{cases} 1, & \left\{\frac{k}{N}\right\} = 0, \\ 0, & \left\{\frac{k}{N}\right\} \neq 0, \end{cases}$$ \hfill (15)

where $\left\{\frac{k}{N}\right\}$ is the remainder of division $k$ by $N$.

Example. When $N = 3$, $\sigma_3(k)$ is determined through the sum

$$\sigma_3(k) = \frac{1}{3} \left(1 + e^{\frac{2\pi j}{3}} + e^{\frac{2\pi j}{3}}\right).$$ \hfill (16)

Then, using the introduced function $\sigma_N(k)$, the lattice function $\delta_T^*(t)$ can be represented in the form

$$\delta_T^*(t) = \delta_{N\tau}^*(t) = \sum_{p=-\infty}^{+\infty} \delta(t - pN\tau) = \begin{cases} 0, & t \in I | \Theta, \\ \sigma_N(k), & t \in \Theta, \end{cases}$$ \hfill (17)

where $I = [0, \infty), \ \Theta = \{t_k : t_k = k\tau; \ k \text{ - integer}\}$.

$$I = [0, \infty), \ \Theta = \{t_k : t_k = k\tau; \ k \text{ - integer}\}$$ \hfill (18)
4. Conclusion
Various options for the mathematical description of lattice functions as an element of constructing mathematical models of discrete automatic control systems (both in the classical representation and in the form proposed in the article) are based, on the one hand, on the representation of lattice functions as a means of displaying discrete information, and on the other hand, on the representation of lattice functions as a function of time. Having different versions of mathematical models for describing lattice functions, we can further proceed to the construction of mathematical models of various discrete control systems, incl. expanding the proposed approaches to the construction of mathematical models of multicycle continuous-discrete automatic control systems [18-20].

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