The adiabatic evolution of orbital parameters in the Kerr spacetime

Noriyaka Sago\textsuperscript{1}, Takahiro Tanaka\textsuperscript{2}, Wataru Hikida\textsuperscript{3}, Katsuhiro Ganz\textsuperscript{2} and Hiroyuki Nakano\textsuperscript{4}

\textsuperscript{1} Department of Earth and Space Science, Graduate School of Science, Osaka University, Toyonaka 560-0043, Japan
\textsuperscript{2} Department of Physics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan
\textsuperscript{3} Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan
\textsuperscript{4} Department of Mathematics and Physics, Graduate School of Science, Osaka City University, Osaka 558-8585, Japan

We investigate the adiabatic orbital evolution of a point particle in the Kerr spacetime due to the emission of gravitational waves. In the case that the timescale of the orbital evolution is enough smaller than the typical timescale of orbits, the evolution of orbits is characterized by the change rates of three constants of motion, the energy $E$, the azimuthal angular momentum $L$, and the Carter constant $Q$. For $E$ and $L$, we can evaluate their change rates from the fluxes of the energy and the angular momentum at infinity and on the event horizon according to the balance argument. On the other hand, for the Carter constant, we cannot use the balance argument because we do not know the conserved current associated with it. Recently, Mino proposed a new method of evaluating the averaged change rate of the Carter constant by using the radiative field. In our previous paper we developed a simplified scheme for practical evaluation of the evolution of the Carter constant based on the Mino’s proposal. In this paper we describe our scheme in more detail, and derive explicit analytic formulae for the change rates of the energy, the angular momentum and the Carter constant.

\section{Introduction}

It is believed that supermassive black holes (SMBHs) reside in central nuclei of many galaxies, and they occasionally capture a stellar mass compact object (SMCO) which surrounds them. Gravitational waves from such binary systems with extreme mass ratios bring us information on the orbits of SMCOs and the spacetime structure near black holes. Therefore such systems are considered to be one of the most important targets of the LISA space-based gravitational wave detector.\textsuperscript{1}) In order to detect gravitational waves emitted by extreme mass ratio inspirals (EMRIs) and to extract physical information from them efficiently, we need to predict accurate theoretical waveforms in advance. Our goal along the line of this paper is to precisely calculate theoretical waveforms from EMRIs.

To investigate gravitational waves from EMRIs, our strategy is to adopt the black hole perturbation method:\textsuperscript{2}) we consider metric perturbations induced by a SMCO in a black hole spacetime governed by a SMBH. We also assume that a SMCO is described by a point particle, neglecting its internal structure. Under the above approximations, we can calculate the metric perturbation evaluated at infinity to predict gravitational waveforms. At the lowest order with respect to the
mass ratio, we may calculate the metric perturbation by approximating the particle’s orbit by a background geodesic. To step further, we consider the orbital shift from the background geodesic by taking account of the self-force induced by the particle itself.

In a Schwarzschild background, we can assume that the orbit is in the equatorial plane from the symmetry without loss of generality. Hence, the orbital velocity can be specified solely by the energy and the azimuthal angular momentum of the particle. Namely, we can evaluate the orbital evolution from the change rates of the energy and the angular momentum. Their averaged change rates can be evaluated by using the balance argument; the energy and the angular momentum that a particle loses are equal to the ones that are radiated to the infinity or across the horizon as gravitational waves because of the conservation laws. In the limit of a large mass ratio, averaged change rates will be sufficient to determine the leading order effects on the orbital evolution due to the self-force. In this sense the leading order effects can be read from the asymptotic behavior of the metric perturbation in the Schwarzschild case.

On the other hand, the third constant of motion, i.e., the Carter constant, is necessary in addition to the energy and the azimuthal angular momentum to specify a geodesic in a Kerr background. However, there is no known conserved current composed of gravitational waves that is associated with the Carter constant, and hence we cannot use the balance argument to evaluate the change rate of the Carter constant. Therefore we have to calculate the self-force acting on a particle. When we calculate the self-force, we are faced with the regularization problem. Although the formal expression for the regularized self-force had been derived, doing explicit calculation is not so straightforward.

Gal’tsov proposed a method of calculating the loss rates of the energy and angular momentum of a particle by using the radiative part of metric perturbation, which was introduced earlier by Dirac. The radiative field is defined by half retarded field minus half advanced one, which is a homogeneous solution of the field equation. It was shown that the time-averaged loss rates of the energy and angular momentum evaluated by using the radiative field are identical with the results obtained from the balance argument. Recently, Mino proved that the Gal’tsov’s scheme also gives the correct averaged change rate of the Carter constant. The Gal’tsov-Mino method has a great advantage that we do not need any regularization procedure because the radiative field is free from divergence from the beginning. In Ref. 11, we briefly reported that the formula for the adiabatic evolution of the Carter constant based on Gal’tsov-Mino method can be largely simplified. In this paper, we explain the derivation of this new formula in detail. Applying our new formula, we explicitly calculate the change rate of the Carter constant for orbits with small eccentricities and inclinations.

This paper is organized as follows. In Sec. 2, we give a brief review of the Kerr geometry and the geodesic motion. Next, we show a practical prescription to calculate the time-averaged change rates of the constants of motion in Sec. 3. We also derive a simplified expression for the change rate of the Carter constant. In Sec. 4, applying our prescription, we calculate the change rates of the constants
of motion and then show the analytic formulae of them for slightly eccentric and inclined orbits. Finally we devote Sec. 5 to summarize this paper. In Appendix A, we show the derivation of the radiative part of metric perturbation. And we also give short reviews on analytic methods of solving the radial Teukolsky equation and obtaining the spheroidal harmonics in Appendices B and C.

§2. Geodesic motion in the Kerr spacetime

In this section, we give a brief review on geodesics in the Kerr geometry. The metric of the Kerr spacetime in the Boyer-Lindquist coordinates is

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}dt d\varphi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2r}{\Sigma}\sin^2\theta\right)\sin^2\theta d\varphi^2,$$ (2.1)

where

$$\Sigma = r^2 + a^2 \cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2.$$

$M$ and $aM$ are the mass and angular momentum of the black hole, respectively. There are two Killing vectors reflecting the stationary and axisymmetric properties of the Kerr geometry:

$$\xi_{(t)}^\mu = (1, 0, 0, 0), \quad \xi_{(\varphi)}^\mu = (0, 0, 0, 1).$$ (2.2)

In addition, the Kerr spacetime possesses a Killing tensor,

$$K_{\mu\nu} = 2\Sigma l_{(\mu}n_{\nu)} + r^2\eta_{\mu\nu},$$ (2.3)

which satisfies $K_{(\mu\nu,\rho)} = 0$, where the parenthese operating on the indices is the notation for symmetric part of tensors. Here we have introduced null vectors,

$$l^\mu := \left(\frac{r^2 + a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\right), \quad n^\mu := \left(\frac{r^2 + a^2}{2\Sigma}, -\frac{\Delta}{2\Sigma}, 0, \frac{a}{2\Sigma}\right),$$

$$m^\mu := \frac{1}{\sqrt{2}(r + ia \cos \theta)} \left(i\sin \theta, 0, 1, \frac{i}{\sin \theta}\right).$$ (2.4)

We consider a point particle moving in the Kerr geometry:

$$z^\alpha(\tau) = (t_z(\tau), r_z(\tau), \theta_z(\tau), \varphi_z(\tau)),$$

where $\tau$ is the proper time along the orbit. Here we introduce quantities defined by

$$\hat{E} := -u^\alpha \xi_{(t)}^\alpha = \left(1 - \frac{2Mr_z}{\Sigma}\right)u_t + \frac{2Mar_z\sin^2\theta_z}{\Sigma}u_\varphi,$$ (2.5)

$$\hat{L} := u^\alpha \xi_{(\varphi)}^\alpha = -\frac{2Mar_z\sin^2\theta_z}{\Sigma}u_t + \frac{(r_z^2 + a^2)^2 - \Delta a^2\sin^2\theta_z}{\Sigma} \sin^2\theta_z u_\varphi,$$ (2.6)

$$\hat{Q} := K_{\alpha\beta} u^\alpha u^\beta = \frac{(\hat{L} - a\hat{E}\sin^2\theta_z)^2}{\sin^2\theta_z} + a^2 \cos^2\theta_z + \Sigma^2(u^\theta)^2,$$ (2.7)
where \( u^\alpha = dz^\alpha /d\tau \). These quantities remain constant as long as the orbit is a geodesic. \( \hat{E} \) and \( \hat{L} \) represent the energy and the (azimuthal) angular momentum per unit mass, respectively. \( \hat{Q} \) is called the Carter constant. Denoting the mass of a particle by \( \mu \), the energy, the angular momentum and the Carter constant of a particle are \( E \equiv \mu \hat{E} \), \( L \equiv \mu \hat{L} \) and \( Q \equiv \mu^2 \hat{Q} \), respectively. Another notation for the Carter constant defined by

\[
C \equiv Q - (aE - L)^2,
\]

(2.8)
is also convenient since \( C \) vanishes for orbits in the equatorial plane. We also use \( \hat{C} \equiv C/\mu^2 \).

We can specify an orbit of a particle by using three constants of motion, the total energy, angular momentum and Carter constant. Introducing a new parameter \( \lambda \) by \( d\lambda = d\tau /\Sigma \), the equations of motion are given as

\[
\frac{dr_z}{d\lambda} = -a(\hat{E} \sin^2 \theta_z - \hat{L}) + \frac{r_z^2 + a^2}{\Delta} P(r_z),
\]

(2.9)

\[
\left( \frac{d\cos \theta_z}{d\lambda} \right)^2 = R(r_z),
\]

(2.10)

\[
\left( \frac{d\varphi_z}{d\lambda} \right)^2 = \Theta(\cos \theta_z),
\]

(2.11)

\[
\frac{d\varphi_z}{d\lambda} = -\left( \frac{a\hat{E} - \hat{L}}{\sin^2 \theta_z} \right) + \frac{a}{\Delta} P(r_z),
\]

(2.12)

where

\[
P(r) := \hat{E}(r^2 + a^2) - a\hat{L},
\]

(2.13)

\[
R(r) := [P(r)]^2 - \Delta[r^2 + (a\hat{E} - \hat{L})^2 + \hat{C}],
\]

(2.14)

\[
\Theta(\cos \theta) := \hat{C} - (\hat{C} + a^2(1 - \hat{E}^2) + \hat{L}^2) \cos^2 \theta + a^2(1 - \hat{E}^2) \cos^4 \theta.
\]

(2.15)

It should be noted that the equations for \( r_z \) and \( \theta_z \) are completely decoupled by using \( \lambda \). Moreover, \( R(r) \) and \( \Theta(\cos \theta) \) are quartic functions of \( r \) and \( \cos \theta \), respectively.

We first consider the radial component of the geodesic equations. When the radial motion is bounded by the minimal and the maximal radii \( r_{\text{min}} \) and \( r_{\text{max}} \), \( r_z(\lambda) \) becomes a periodic function which satisfies \( r_z(\lambda + \Lambda_r) = r_z(\lambda) \) with period

\[
\Lambda_r = 2 \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{dr}{\sqrt{R(r)}}.
\]

(2.16)

Therefore, we can expand the radial motion in a Fourier series as

\[
r_z(\lambda) = \sum_n \tilde{r}_n e^{-in\Omega_r \lambda},
\]

(2.17)

where

\[
\Omega_r = 2\pi /\Lambda_r.
\]

(2.18)
The adiabatic evolution of orbital parameters in the Kerr spacetime

We can deal with the motion in $\theta$-direction in a similar manner. When the minimum of $\theta$ is given by $\theta_{\text{min}} (\leq \pi/2)$, the maximum is $\theta_{\text{max}} = \pi - \theta_{\text{min}}$ because of the symmetry with respect to the equatorial plane. As in the case of the radial motion, $\cos \theta_z (\lambda)$ becomes a periodic function which satisfies $\cos \theta_z (\lambda + \Lambda_{\theta}) = \cos \theta_z (\lambda)$ with period

$$\Lambda_{\theta} = 4 \int_{\cos \theta_{\text{min}}}^{\cos \theta_{\text{max}}} \frac{d(\cos \theta)}{\sqrt{\Theta(\cos \theta)}} .$$  \hspace{1cm} (2.19)$$

We can expand $\cos \theta_z (\lambda)$ in a Fourier series as

$$\cos \theta_z (\lambda) = \sum_n \tilde{z}_n e^{-in\Omega_{\theta}\lambda} ,$$  \hspace{1cm} (2.20)$$

where $\Omega_{\theta} = 2\pi/\Lambda_{\theta}$.

Next, we consider the $t$- and $\phi$-components of geodesic equations. Eqs. (2.9) and (2.12) can be integrated as

$$t_z (\lambda) = t^r (\lambda) + t^\theta (\lambda) + \left< \frac{dt_z}{d\lambda} \right> \lambda ,$$  \hspace{1cm} (2.21)$$

$$\varphi_z (\lambda) = \varphi^r (\lambda) + \varphi^\theta (\lambda) + \left< \frac{d\varphi_z}{d\lambda} \right> \lambda ,$$  \hspace{1cm} (2.22)$$

where

$$t^r (\lambda) := \int d\lambda \left[ \frac{(r_z^2 + a^2)P(r_z)}{\Delta(r_z)} - \left< \frac{(r_z^2 + a^2)P(r_z)}{\Delta(r_z)} \right> \right] ,$$

$$t^\theta (\lambda) := - \int d\lambda \left[ a^2 \hat{E} \sin^2 \theta_z - a\hat{L} - \left< a^2 \hat{E} \sin^2 \theta_z - a\hat{L} \right> \right] ,$$

$$\varphi^r (\lambda) := \int d\lambda \left[ \frac{aP(r_z)}{\Delta(r_z)} - \left< \frac{aP(r_z)}{\Delta(r_z)} \right> \right] ,$$

$$\varphi^\theta (\lambda) := \int d\lambda \left[ \frac{\hat{L}}{\sin^2 \theta_z} - a\hat{E} - \left< \frac{\hat{L}}{\sin^2 \theta_z} - a\hat{E} \right> \right] .$$

$\langle \cdots \rangle$ represents the time average along the geodesic:

$$\langle F(\lambda) \rangle := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} d\lambda' \ F(\lambda').$$

Here, $t^r (\lambda)$ and $\varphi^r (\lambda)$ are periodic functions with period $\Lambda_r$, while $t^\theta (\lambda)$ and $\varphi^\theta (\lambda)$ are those with period $\Lambda_{\theta}$.

§3. The Time Evolution of the Constants of motion

If the timescale of the orbital evolution due to gravitational radiation reaction is much longer than the typical dynamical timescale, we may be able to approximate the particle’s motion by the geodesic in the background spacetime that is momentarily tangential to the orbit (osculating geodesic approximation). Under this assumption,
we evaluate the change rates of the constants of motion at each moment. For bound orbits we can express the change rates of the constants of motion, \(I = \{E, L, Q\}\), as

\[
\frac{dI^i}{d\lambda} = \left\langle \frac{dI^i}{d\lambda} \right\rangle + \sum_{(n_r, n_\theta) \neq (0,0)} \hat{I}^{(n_r, n_\theta)} \exp \left[-i(n_r \Omega_r + n_\theta \Omega_\theta)\lambda \right].
\] (3.1)

The first term on the right hand side is a time-independent dissipative contribution due to radiation reaction, while the others are oscillating. Integrating over a long period, the first term becomes dominant. In the same spirit in Ref. 10), here we define the ‘adiabatic’ evolution as an approximation which takes account of only the first term. Namely, the adiabatic evolution is solely determined by the time averaged change rates of the constants of motion.

Owing to the argument given in Ref. 10), we can evaluate the averaged change rates of the constants of motion by using the radiative field of the metric perturbation

\[
\left\langle \frac{dI^i}{d\lambda} \right\rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} d\lambda \sum \frac{\partial I^i}{\partial u^\alpha} f^\alpha[h^{\text{rad}}_{\mu\nu}],
\] (3.2)

where \(h^{\text{rad}}_{\mu\nu}\) is the radiative part of the metric perturbation defined by half retarded field minus half advanced field, i.e., \(h^{\text{rad}}_{\mu\nu} := (h^{\text{ret}}_{\mu\nu} - h^{\text{adv}}_{\mu\nu})/2\). Radiative field is a solution of source-free vacuum Einstein equation. The singular parts contained in both retarded and advanced fields cancel out. Therefore we can avoid the tedious issue of regularizing the self-force. \(f^\alpha\) is a differential operator,

\[
f^\alpha[h_{\mu\nu}] := -\frac{1}{2} (g^{\alpha\beta} + u^\alpha u^\beta) (h_{\beta\gamma;\delta} + h_{\beta\delta;\gamma} - h_{\gamma\delta;\beta}) u^\gamma u^\delta.
\] (3.3)

This operator with its index lowered reduces to

\[
f_\alpha[h_{\mu\nu}] = g_{\alpha\beta} f^\beta[h_{\mu\nu}]
\]

\[
= \frac{1}{2} \left( \partial_\alpha h_{\gamma\delta} \right) u^\gamma u^\delta - \frac{d}{dt} (h_{\alpha\gamma} u^\gamma) - \frac{1}{2} u_\alpha \frac{d}{dt} \left( h_{\gamma\delta} u^\gamma u^\delta \right) + O(\mu^2),
\] (3.4)

ignoring the second order terms.

3.1. Calculation of \(dE/dt\) and \(dL/dt\)

From Eq. (3.2), we obtain

\[
\left\langle \frac{dE}{d\lambda} \right\rangle = \lim_{T \to \infty} \frac{\mu}{2T} \int_{-T}^{T} d\lambda \left( -\xi^{(t)}_\alpha \right) f^\alpha[h^{\text{rad}}_{\mu\nu}]
\]

\[
= \lim_{T \to \infty} \frac{-\mu}{2T} \int_{-T}^{T} d\lambda \left[ \frac{\Sigma}{2} \left( \partial_\alpha h^{\text{rad}}_{\gamma\delta} \right) u^\gamma u^\delta - \frac{d}{d\lambda} \left( h^{\text{rad}}_{\gamma\delta} u^\gamma u^\delta \right) + \frac{\hat{E}}{2} \frac{d}{d\lambda} \left( h^{\text{rad}}_{\gamma\delta} u^\gamma u^\delta \right) \right]
\]

\[
= \lim_{T \to \infty} \frac{-\mu}{2T} \int_{-T}^{T} d\lambda \left[ \frac{\Sigma}{2} \left( \partial_\alpha h^{\text{rad}}_{\gamma\delta} \right) u^\gamma u^\delta \right].
\] (3.5)

In the last equality, the total derivative terms are neglected.
Next, we introduce a vector field \( \tilde{u}^\mu(x) \) by\(^{11} \)
\[
(\tilde{u}_t, \tilde{u}_r, \tilde{u}_\theta, \tilde{u}_\varphi) := \left( -\dot{E}, \pm \sqrt{\frac{R(r)}{\Delta(r)}} \Theta(\cos \theta), \pm \frac{\Theta(\cos \theta)}{\sin \theta}, \dot{L} \right). \tag{3.6}
\]

This vector field is a natural extension of the four-velocity of a particle. In fact, it satisfies \( \tilde{u}_\mu(z(\lambda)) = u_\mu(\lambda) \). \( \tilde{u}_\mu \) depends only on \( r \) and \( \theta \). Furthermore, since \( \tilde{u}_r \) and \( \tilde{u}_\theta \) depend only on \( r \) and \( \theta \), respectively, we have the relation \( u_{\mu;\nu} = u_{\nu;\mu} \).

Using this vector field, we can rewrite Eq. (3.5) as
\[
\left\langle \frac{dE}{d\lambda} \right\rangle = \lim_{T \to \infty} \frac{-\mu}{2T} \int_{-T}^{T} d\lambda \left[ \partial_\lambda \left( \frac{\sum \Lambda_{\gamma\delta} \tilde{u}^\gamma \tilde{u}^\delta}{2} \right) \right]_{x \to z(\lambda)}, \tag{3.7}
\]
where we used the fact that \( \Sigma \) and \( \tilde{u}_\mu \) are independent of \( t \) (and \( \varphi \)).

As shown in Appendix A (Eq. (A.54)), the radiative field of metric perturbation is given by
\[
h^{\text{rad}}_{\mu\nu}(x) = \mu \int d\omega \sum_{\ell m} \frac{1}{2i\omega^3} \left\{ |N^\text{out}_s|^2 \Lambda^\text{out}(x) \int d\lambda \left[ \sum \Lambda_{\alpha\beta}^{\text{out}}(z(\lambda)) u^\alpha u^\beta \right] \right.
\]
\[
+ \omega k |N^\text{down}_s|^2 \Lambda^\text{down}(x) \int d\lambda \left[ \sum \Lambda_{\alpha\beta}^{\text{down}}(z(\lambda)) u^\alpha u^\beta \right] \} + (\text{c.c.}), \tag{3.8}
\]
where \( \Lambda = \{ \ell m \omega \} \), \( k = \omega - ma/2Mr_+ \) and \( r_+ = M + \sqrt{M^2 - a^2} \). \( s \Lambda^\text{out}(x) \) and \( s \Lambda^\text{down}(x) \) are the out-going and down-going mode solutions for \( h_{\mu\nu} \), respectively. \( N^\text{out}_s \) and \( N^\text{down}_s \) are normalization factors, given by Eqs. (A.52) and (A.53). A bar represents complex conjugation. Using this formula, we obtain
\[
\psi^{\text{rad}}(x) := \frac{1}{2} \sum_{\ell m} \frac{1}{4i\omega^3} \left[ \phi^{\text{out}}(x) \int d\lambda' \phi^{\text{out}}(z(\lambda')) \right.
\]
\[
+ \omega k \phi^{\text{down}}(x) \int d\lambda' \phi^{\text{down}}(z(\lambda')) \} \tag{3.9}
\]
where
\[
\phi^{\text{out/(down)}}(x) := \sum_{\ell m} N^\text{out/(down)}(x) \Lambda^{\text{out/(down)}}(x) u^\gamma(x) u^\delta(x). \tag{3.10}
\]
For a bound orbit, we can expand \( \phi^{\text{out}} \) in a Fourier series as:
\[
\phi^{\text{out/(down)}}(z(\lambda)) = \frac{1}{2\pi} \left( \frac{dt_z}{d\lambda} \right) \sum_{n_r, n_\theta} \hat{z}^{\text{out/(down)}}_{\ell m, n_\theta}(x) \exp \left[ i \left( \frac{dt_z}{d\lambda} \right) (\omega - \omega_{mn,n_\theta}) \right], \tag{3.11}
\]
where
\[
\omega_{mn,n_\theta} := \left( \frac{dt_z}{d\lambda} \right)^{-1} \left( m \frac{d\varphi_z}{d\lambda} + n_r \Omega_r + n_\theta \Omega_\theta \right). \tag{3.12}
\]
Substituting Eqs. (3.9) and (3.11) into (3.7), we obtain:

$$\left\langle \frac{dE}{dt} \right\rangle = -\mu^2 \sum_{\ell m n} \frac{1}{4\pi \omega_{mn}^2} \left( |\mathcal{Z}_{\ell m n}|^2 + \frac{\omega_{mn}}{k_{mn} \omega_{mn}} |\mathcal{Z}_{\ell m n}|^2 \right),$$

(3.13)

where $k_{mn} = \omega_{mn} - ma/2M r_+$. After all, we find

$$\langle \mathcal{Z}_{\ell m n}^{\text{out/down}} \rangle = \tilde{\mathcal{Z}}_{\ell m n}^{\text{out/down}} ( \omega_{mn} ) \rangle, \quad (3.14)$$

In a similar manner, the formula for the loss rate of the angular momentum is given by

$$\left\langle \frac{dL}{dt} \right\rangle = -\mu^2 \sum_{\ell m n} \frac{m}{4\pi \omega_{mn}^2} \left( |\mathcal{Z}_{\ell m n}|^2 + \frac{\omega_{mn}}{k_{mn} \omega_{mn}} |\mathcal{Z}_{\ell m n}|^2 \right). \quad (3.15)$$

### 3.2. Calculation of $dQ/dt$

To obtain the change rate of the Carter constant, we need to evaluate

$$\left\langle \frac{dQ}{dt} \right\rangle = \lim_{T \to \infty} \frac{\mu^2}{2T} \int_{-T}^T d\lambda 2 \Sigma K_{\beta}^{\alpha} u^\beta f_\alpha [r^\text{rad}], \quad (3.16)$$

Using the vector field $\tilde{u}^a (x)$, which was introduced in the previous subsection, we obtain

$$2K_{\beta}^{\alpha} u^\beta f_\alpha = \lim_{x \to z} \left[ K_{\beta}^{\alpha} u^\beta \partial_\alpha (h_{\gamma \delta} \tilde{u}^\gamma \tilde{u}^\delta) + 2h_{\gamma \delta} \tilde{u}^\gamma (K_{\beta}^{\delta} \tilde{u}^\alpha - K_{\alpha}^{\delta} \tilde{u}^\beta) \right], \quad (3.17)$$

to the first order in perturbation, excluding total derivative terms with respect to $\tau$. Those total derivative terms do not contribute after taking a long-time average. Furthermore, one can show that the second term also vanishes by using $K_{(\alpha \beta; \gamma)} = 0$ and $\tilde{u}_{\alpha; \beta} = \tilde{u}_{\beta; \alpha}$. After all, we find

$$\left\langle \frac{dQ}{d\lambda} \right\rangle = \lim_{T \to \infty} \frac{\mu^2}{2T} \int_{-T}^T d\lambda \left[ 2 \Sigma K_{\beta}^{\alpha} u^\beta \partial_\alpha \left( \frac{\psi^\text{rad}(x)}{\Sigma} \right) \right] \xrightarrow{x \to z(\lambda)}$$

$$= \lim_{T \to \infty} \frac{-\mu^2}{T} \int_{-T}^T d\lambda \times \left[ \left\{ \frac{P(r)}{\Delta} ((r^2 + a^2) \partial_t + a \partial_\phi) + \frac{dr_z}{d\lambda} \partial_r \right\} \psi^\text{rad}(x) \right] \xrightarrow{x \to z(\lambda)} \quad (3.18)$$

To obtain the last term in the last line, the term with $\tilde{u}^\mu \partial_\mu$ was rewritten into $\Sigma^{-1} d/d\lambda$, and integration by parts was applied.

Substituting Eqs. (3.9) and (3.11) into Eq. (3.15), we obtain:

$$\left\langle \frac{dQ}{d\lambda} \right\rangle = \lim_{T \to \infty} \frac{-\mu^3}{2T} \int_{-T}^T d\omega \sum_{\ell m n} \frac{1}{2\mu \omega^2} \delta (\omega - \omega_{mn} \omega_{mn})$$

$$\times \left[ Z_{\ell m n}^{\text{out}} \left\{ \frac{P(r)}{\Delta} ((r^2 + a^2) \partial_t + a \partial_\phi) + \frac{dr_z}{d\lambda} \partial_r \right\} \phi_{\ell m n}^{\text{out}} (x) \right] \quad (3.19)$$
The adiabatic evolution of orbital parameters in the Kerr spacetime

\[
\frac{\omega}{k} Z_{\text{down}}^{mn,n_0} \left\{ \frac{P(r)}{\Delta} (r^2 + a^2) \partial_t + a \partial_\phi \right\} \phi_A^{\text{down}}(x) \bigg|_{x \rightarrow z(\lambda)} + (\text{c.c.}) \tag{3.19}
\]

Now we focus on the \( r \)-derivative term in the curly brackets. Since \( \phi_A^{\text{out}} \) and \( \phi_A^{\text{down}} \) depend on \( t \) and \( \varphi \) only through an exponential function \( e^{\omega t + im \varphi} \), we can write

\[
\phi_A(z(\lambda)) \delta(\omega - \omega_{mn,n_0}) = f(r_z(\lambda), \cos \theta_z(\lambda)) \delta(\omega - \omega_{mn,n_0}) \times \exp \left[ -i \omega_{mn,n_0} \int \frac{dr_z}{d\lambda} \partial_t \phi_A(z(\lambda)) \right]
\]

\[
= f(r_z(\lambda), \cos \theta_z(\lambda)) \delta(\omega - \omega_{mn,n_0}) \times \exp \left[ -i \omega_{mn,n_0} t^{(r)}(\lambda) + im \varphi^{(r)}(\lambda) \right], \tag{3.20}
\]

where \( f(r, \cos \theta) \) represents the dependence on \( r \) and \( \cos \theta \) in \( \phi_A(x) \). \( r_z(\lambda), t^{(r)}(\lambda) \) and \( \varphi^{(r)}(\lambda) \) are periodic functions with period \( \Lambda_r \), while \( \theta_z(\lambda), t^{(\theta)}(\lambda) \) and \( \varphi^{(\theta)}(\lambda) \) are those with period \( \Lambda_\theta \). We introduce two different time variables \( \lambda_r \) and \( \lambda_\theta \). We use them instead of \( \lambda \) for functions with period \( \Lambda_r \) and \( \Lambda_\theta \). Then, by using these new variables, we can replace the infinitely long time average with a double integral over a finite region:

\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} d\lambda \delta(\omega - \omega_{mn,n_0}) \frac{dr_z}{d\lambda} \partial_t \phi_A(z(\lambda)) = \frac{1}{\Lambda_r \Lambda_\theta} \int_0^{\Lambda_r} d\lambda_r \int_0^{\Lambda_\theta} d\lambda_\theta \delta(\omega - \omega_{mn,n_0}) \frac{dr_z}{d\lambda_r} \partial_t \left\{ f(r_z(\lambda_r), \cos \theta_z(\lambda_\theta)) \right. \times \exp \left[ -i \omega_{mn,n_0} t^{(r)}(\lambda_r) + im \varphi^{(r)}(\lambda_r) \right]
\]

\[
\left. -i \omega_{mn,n_0} t^{(\theta)}(\lambda_\theta) + im \varphi^{(\theta)}(\lambda_\theta) \right\} \right]. \tag{3.21}
\]

We only need to integrate over one cycle for each of \( \lambda_r \) and \( \lambda_\theta \). Using the relation

\[
\frac{d}{d\lambda_r} \left\{ f(r_z(\lambda_r), \cos \theta_z(\lambda_\theta)) \exp \left[ -i \omega_{mn,n_0} t^{(r)}(\lambda_r) + im \varphi^{(r)}(\lambda_r) \right] \right\} = \left[ \frac{dt^{(r)}}{d\lambda_r} \partial_t + \frac{dr_z}{d\lambda_r} \partial_r + \frac{d\varphi^{(r)}}{d\lambda_r} \partial_\varphi + \partial_{\lambda_r} \right]
\]

\[
\times f(r_z(\lambda_r), \cos \theta_z(\lambda_\theta)) \exp \left[ -i \omega_{mn,n_0} t^{(r)}(\lambda_r) + im \varphi^{(r)}(\lambda_r) \right],
\]

\( \lambda_r \)-integral in (3.21) can be rewritten as

\[
\int_0^{\Lambda_r} d\lambda_r \frac{dr_z}{d\lambda_r} \partial_r \left\{ f(r_z(\lambda_r), \cos \theta_z(\lambda_\theta)) \right. \]
3.3. Consistency of our formulae in simple cases

In this subsection, we examine our formulae in a few simple cases. First, we consider circular orbits. We know that a circular orbit remains circular under radiation reaction. This condition fixes \(dQ/dt\) for circular orbits as

\[
\frac{dQ}{dt} = \frac{2\mu(r^2 + a^2) P \frac{dE}{dt}}{\Delta} - 2\mu \frac{aP}{\Delta} \frac{dL}{dt} + \mu^2 \frac{n_r \Omega_r}{2\pi \omega_{\ell mn, n_g}^3} \left( |Z_{\ell mn, n_g}^{\text{out}}|^2 + \frac{\omega_{\ell mn, n_g}}{k_{mn, n_g}} |Z_{\ell mn, n_g}^{\text{down}}|^2 \right).
\]

Since \(Z_{\ell mn, n_g}^{\text{out/down}} = 0\) for \(n_r \neq 0\) in the case of a circular orbit, the last term in Eq. (3.24) vanishes. Thus Eq. (3.24) is consistent with the above condition that a circular orbit remains circular.

Next, we consider orbits in the equatorial plane. An orbit in the equatorial plane should not leave the plane by symmetry. This can be confirmed by rewriting the above formula in terms of \(C\). From the definition of \(\omega_{\ell mn, n_g}\) (3.12), we obtain the following identity:

\[
\mu^2 \sum_{\ell mn, n_g} \frac{n_r \Omega_r}{4\pi \omega_{\ell mn, n_g}} \left( |Z_{\ell mn, n_g}^{\text{out}}|^2 + \frac{\omega_{\ell mn, n_g}}{k_{mn, n_g}} |Z_{\ell mn, n_g}^{\text{down}}|^2 \right)
\]
The adiabatic evolution of orbital parameters in the Kerr spacetime

\[ = \mu^2 \sum_{\ell,m,n,\theta} \frac{1}{4\pi\omega_{mn,\theta}^2} \left( \left\langle \frac{dt_z}{d\lambda} \right\rangle - \frac{m}{\omega_{mn,\theta}} \left\langle \frac{d\varphi_z}{d\lambda} \right\rangle - \frac{n_\theta \Omega}{4\pi\omega_{mn,\theta}} \right) \times \left( \left| Z_{\ellmn,\theta}^{\text{out}} \right|^2 + \omega_{mn,\theta}^2 \left| Z_{\ellmn,\theta}^{\text{down}} \right|^2 \right) \]

\[ = -\left\langle \frac{dt_z}{d\lambda} \right\rangle \left\langle \frac{dE}{dt} \right\rangle_t + \left\langle \frac{d\varphi_z}{d\lambda} \right\rangle \left\langle \frac{dL}{dt} \right\rangle_t - \mu^2 \sum_{\ell,m,n,\theta} \frac{n_\theta \Omega}{4\pi\omega_{mn,\theta}^3} \left( \left| Z_{\ellmn,\theta}^{\text{out}} \right|^2 + \omega_{mn,\theta} \left| Z_{\ellmn,\theta}^{\text{down}} \right|^2 \right), \quad (3.26) \]

where we used the the expressions of \( \left\langle \frac{dE}{dt} \right\rangle_t \) and \( \left\langle \frac{dL}{dt} \right\rangle_t \) given in Eqs. (3.13) and (3.15). Using this identity, we have

\[ \left\langle \frac{dC}{dt} \right\rangle_t = \left\langle \frac{dQ}{dt} \right\rangle_t - 2(aE - L) \left( a \left\langle \frac{dE}{dt} \right\rangle_t - \left\langle \frac{dL}{dt} \right\rangle_t \right) \]

\[ = -2 \left\langle a^2 E \cos^2 \theta \right\rangle + 2 \left( L \cot^2 \theta \right) \left\langle \frac{dL}{dt} \right\rangle_t - \mu^3 \sum_{\ell,m,n,\theta} \frac{n_\theta \Omega}{2\pi\omega_{mn,\theta}^3} \left( \left| Z_{\ellmn,\theta}^{\text{out}} \right|^2 + \omega_{mn,\theta} \left| Z_{\ellmn,\theta}^{\text{down}} \right|^2 \right), \quad (3.26) \]

where we have used the following relations:

\[ \left\langle \frac{dt_z}{d\lambda} \right\rangle = -a(a\hat{E} - \hat{L}) + \left\langle a^2 \hat{E} \cos^2 \theta \right\rangle + \left\langle \frac{r_z^2 + a^2}{\Delta} P \right\rangle, \]

\[ \left\langle \frac{d\varphi_z}{d\lambda} \right\rangle = -a\hat{E} + \hat{L} + \left\langle \hat{L} \cot^2 \theta \right\rangle + \left\langle \frac{aP}{\Delta} \right\rangle. \]

From this equation, it is found that \( \left\langle \frac{dC}{dt} \right\rangle_t = 0 \) when \( \theta = \pi/2 \). Note that we have \( Z_{\ellmn,\theta}^{\text{out/down}} \neq 0 \) only for \( n_\theta = 0 \) in the case of equatorial orbits.

§4. Application of our formulation to orbits with small eccentricity and inclination

In this section, as an application of our formulation, we consider a slightly eccentric orbit with small inclination from the equatorial plane. Since, in this case, we can expand an orbit with respect to the eccentricity and inclination, we can analytically calculate the change rates of the constants of motion.

4.1. Orbits

Here we define \( r_0 \) so that the potential in \( r \)-direction \( R(r) \) takes its minimum at \( r = r_0 \):

\[ \left. \frac{dR}{dr} \right|_{r=r_0} = 0. \quad (4.1) \]

We denote the outer turning point by \( r_0(1 + e) \). Namely,

\[ R(r_0(1 + e)) = 0, \quad (4.2) \]
which gives the definition of the eccentricity $e$. We also define a parameter $y = C/L^2$, which is related to the inclination angle. For orbits in the equatorial plane, we have $y = 0$. Further, we introduce a new parameter $v = \sqrt{M/r_0}$. For circular orbits $v$ represents the orbital velocity at the Newtonian order. Hence, we regard $v$ as a parameter whose power indicates twice the post-Newtonian (PN) order.

Solving (4.1) and (4.2) for $\dot{E}$ and $\dot{L}$, they are expressed in terms of $e$ and $y$ as

$$
\dot{E} = 1 - \frac{1}{2}v^2 + \frac{3}{8}v^4 - qv^5 - \left(\frac{1}{2}v^2 - \frac{1}{4}v^4 + 2qv^5\right)e^2 + \frac{1}{2}qv^5y + qv^5e^2y, \quad (4.3)
$$

$$
\dot{L} = r_0v\left[1 + \frac{3}{2}v^2 - 3qv^3 + \frac{27}{8}v^4 + q^2v^4 - \frac{15}{2}qv^5
\right.
\left. + \left(-1 + \frac{3}{2}v^2 - 6qv^3 + \frac{81}{8}v^4 + \frac{7}{2}q^2v^4 - \frac{63}{2}qv^5\right)e^2
\right.
\left. + \left(-\frac{1}{2} - \frac{3}{4}v^2 + 3qv^3 - \frac{27}{16}v^4 - \frac{3}{2}q^2v^4 + \frac{15}{2}qv^5\right)y
\right.
\left. + \left(\frac{1}{2} - \frac{3}{4}v^2 + 6qv^3 - \frac{81}{16}v^4 - \frac{19}{4}q^2v^4 + \frac{63}{2}qv^5\right)e^2y\right], \quad (4.4)
$$

where $q := a/M$. Hereafter we keep terms up to $O(v^5e^2y)$ relative to the leading order.

With the initial condition set to $r_z(\lambda = 0) = r_0(1 + e)$, the solution for $r_z(\lambda)$ is obtained in an expansion with respect to $e$ as

$$
r_z(\lambda) = r_0[1 + er^{(1)} + e^2r^{(2)}], \quad (4.5)
$$

where

$$
r^{(1)} = \cos \Omega_r \lambda,
$$

$$
r^{(2)} = p^{(1)}(1 - \cos \Omega_r \lambda) + p^{(2)}(1 - \cos 2\Omega_r \lambda),
$$

$$
\Omega_r = r_0v\left[1 - \frac{3}{2}v^2 + 3qv^3 - \frac{45}{8}v^4 - \frac{3}{2}q^2v^4 + \frac{33}{2}qv^5
\right.
\left. - \left(1 + \frac{3}{2}v^2 - 6qv^3 + \left(\frac{165}{8} + \frac{9}{2}q^2\right)v^4 - \frac{165}{2}qv^5\right)e^2
\right.
\left. - \left(\frac{3}{2}qv^3 - 2q^2v^4 + \frac{19}{4}qv^5\right)y - \left(3qv^3 - \frac{27}{4}q^2v^4 + \frac{165}{4}qv^5\right)e^2y\right],
$$

$$
p^{(1)} = -1 - v^2 + 2q^2v^4 - 6v^4 - q^2v^4 + 20qv^5 - \left(qv^3 - 2q^2v^4 + 10q^5\right)y,
$$

$$
p^{(2)} = -\frac{1}{2} - \frac{3}{4}v^2 + qv^3 - 3v^4 - \frac{1}{2}q^2v^4 + 10qv^5 - \left(\frac{1}{2}qv^3 - q^2v^4 + 5q^5\right)y.
$$

We also compute $\cos \theta_z(\lambda)$ in a series expansion in $y$ as

$$
\cos \theta_z(\lambda) = \sqrt{y}\xi^{(1)}(\lambda) + yc^{(1)}(\lambda), \quad (4.6)
$$

where

$$
\xi^{(0)} = (1 - \frac{1}{2}q^2v^4 - \frac{3}{2}q^2v^4e^2)\sin \Omega_\phi \lambda,
$$

$$
\xi^{(1)} = \left(-\frac{1}{2} + \frac{13}{16}q^2v^4 + \frac{39}{16}q^2v^4e^2\right)\sin \Omega_\phi \lambda + \left(\frac{1}{16}q^2v^4 + \frac{3}{16}q^2v^4e^2\right)\sin 3\Omega_\phi \lambda,
$$

$$
c^{(0)} = \left(1 - qv^3 - q^2v^4y\right)\sin \Omega_\phi \lambda + \frac{1}{16}q^2v^4\sin 3\Omega_\phi \lambda,
$$

$$
c^{(1)} = \left(-\frac{1}{2} + \frac{13}{16}q^2v^4 + \frac{39}{16}q^2v^4e^2\right)\sin \Omega_\phi \lambda + \left(\frac{1}{16}q^2v^4 + \frac{3}{16}q^2v^4e^2\right)\sin 3\Omega_\phi \lambda,
$$

$$
c^{(2)} = \left(-1 - v^2 + 2q^2v^4 - 6v^4 - q^2v^4 + 20qv^5 - \left(qv^3 - 2q^2v^4 + 10q^5\right)y\right)\sin \Omega_\phi \lambda + \left(1 - \frac{3}{2}v^2 + qv^3 - 3v^4 - \frac{1}{2}q^2v^4 + 10qv^5\right)\sin 3\Omega_\phi \lambda.
$$
\[ \Omega_\theta = r_0 v \left[ 1 + \frac{3}{2} v^2 - 3q v^3 + \frac{27}{8} v^4 + \frac{3}{2} q^2 v^4 - \frac{15}{2} q v^5 \right. \\
+ \left( -1 + \frac{3}{2} v^2 - 6q v^3 + \frac{81}{8} v^4 + \frac{9}{2} q^2 v^4 - \frac{63}{2} q v^5 \right) e^2 \\
+ \left( \frac{3}{2} q v^3 - \frac{7}{4} q^2 v^4 + \frac{15}{4} q v^5 \right)y + \left( 3q v^3 - \frac{9}{2} q^2 v^4 + \frac{63}{4} q v^5 \right)e^2 y \].

Here the solution satisfies the condition, \( \cos \theta_z(\lambda = 0) = 0 \).

Substituting \( r_z \) and \( \cos \theta_z \) into Eqs. (2.10), (2.12) and (2.23), we obtain

\[ \ell^{(r)} = \frac{r_0 e}{v} \left\{ \left( 2 + 4v^2 - 6q v^3 + 17v^4 + 3q^2 v^4 - 54 q v^5 \right) \\
+ \left( 2 + 6v^2 - 10q v^3 + 33v^4 + 5q^2 v^4 - 108 q v^5 \right)e \\
+ \left( 3q v^3 - 4q^2 v^4 + 27 q v^5 \right)y \\
+ \left( 5q v^3 - 8q^2 v^4 + 54 q v^5 \right)e y \right\} \sin \Omega_r \lambda \\
\times \left[ \left( \frac{3}{4} + \frac{7}{4} q v^2 - \frac{13}{4} q v^3 + \frac{81}{8} v^4 + \frac{13}{8} q^2 v^4 - \frac{135}{4} q v^5 \right) e \\
+ \left( \frac{13}{8} q v^3 - \frac{5}{2} q^2 v^4 + \frac{135}{8} q v^5 \right)e y \right] \sin 2\Omega_r \lambda, \tag{4.7} \]

\[ \ell^{(\theta)} = q^2 v^3 r_0 y \left\{ \left( \frac{3}{4} + \frac{7}{4} q v^2 - \frac{13}{4} q v^3 + \frac{5}{8} q^2 v^4 + q v^5 \right) \\
+ \left( \frac{3}{4} + \frac{11}{8} v^2 - 3q v^3 + \frac{1}{2} v^4 + \frac{23}{8} q^2 v^4 + \frac{9}{2} q v^5 \right)e^2 \right\} \sin 2\Omega_\theta \lambda, \tag{4.8} \]

\[ \left\langle \frac{dt_z}{d\lambda} \rightangle = r_0^2 \left[ 1 + \frac{3}{2} v^2 + \frac{27}{8} v^4 - 3q v^5 - \left( \frac{5}{2} + \frac{21}{4} v^2 - 6q v^3 + \frac{315}{16} v^4 + 3q^2 v^4 - \frac{123}{2} q v^5 \right)e^2 \right. \\
+ \left( \frac{1}{2} q^2 v^4 + \frac{3}{2} q v^5 \right)y + \left( -3q v^3 + 6q^2 v^4 - \frac{123}{4} q v^5 \right)e^2 y \right] \tag{4.9} \]

\[ \varphi^{(r)} = q v^3 e \left\{ \left( -2 + 2q v - 10v^2 + 18q v^3 \right) + \left( -2 + 2q v - 12v^2 + 24q v^3 \right)e \right. \\
\left. - (q v + 9q v^3) y - (q v + 12q v^3) e y \right\} \sin \Omega_r \lambda \\
\times \left\{ \left( \frac{3}{4} v + \frac{3}{8} q v^3 \right) e \left( \frac{1}{8} q v + \frac{3}{8} q v^3 \right) e y \right\} \sin 2\Omega_r \lambda, \tag{4.10} \]

\[ \varphi^{(\theta)} = y \left[ \left( \frac{3}{4} + \frac{1}{2} q v - \frac{3}{4} q v^3 \right) e + \left( \frac{1}{8} q v + \frac{3}{8} q v^3 \right) e y \right] \sin 2\Omega_\theta \lambda, \tag{4.11} \]

\[ \left\langle \frac{d\varphi_z}{d\lambda} \rightangle = r_0 v \left[ 1 + \frac{3}{2} v^2 - q v^3 + \frac{27}{8} v^4 - \frac{9}{2} q v^5 - \left( 1 - \frac{3}{2} v^2 + 2q v^3 - \frac{81}{8} v^4 + \frac{27}{2} q v^5 \right)e^2 \right. \\
+ \left( \frac{3}{2} q v^3 - q^2 v^4 + \frac{15}{4} q v^5 \right)y + \left( 3q v^3 - \frac{9}{4} q^2 v^4 + \frac{63}{4} q v^5 \right)e^2 y \right] \tag{4.12} \]

4.2. Calculation of \( Z_{\text{out/} \text{down}}^{\text{out/} \text{down}} \)

In order to obtain the averaged change rates of the energy, angular momentum and Carter constant, we have to calculate \( Z_{\text{out/} \text{down}}^{\text{out/} \text{down}} \) defined by Eq. (3.14) with
Eq. (3.11). Integrating Eq. (3.10) with respect to \( \lambda \), we obtain
\[
\hat{Z}_A^{(\text{out}/\text{down})} \equiv \int d\lambda \mathcal{F}_A^{(\text{out}/\text{down})}(z(\lambda)) \\
= N_s^{(\text{out}/\text{down})} \int d^4x \sqrt{-g(x)} \tilde{F}_{A,\alpha\beta}^{(\text{out}/\text{down})}(x) \int d\tau \frac{\tilde{u}^\alpha(x)\tilde{u}^\beta(x)}{\sqrt{-g(x)}} \delta^{(4)}(x-z(\lambda)) \\
= N_s^{(\text{out}/\text{down})} \int d^4x \sqrt{-g(x)} \tilde{F}_{A,\alpha\beta}^{(\text{out}/\text{down})}(x) T^\alpha\beta(x), \quad (4.13)
\]
where
\[
T^\alpha\beta(x) = \mu \int d\tau \frac{1}{\sqrt{-g(x)}} u^\alpha u^\beta \delta^{(4)}(x-z(\tau)). \quad (4.14)
\]
is the energy momentum tensor of a mono-pole particle of mass \( \mu \). Using the relation given in Eq. (A.30), \( \hat{Z}_A^{(\text{out}/\text{down})} \) can be also expressed in the familiar form which appears as an integration over the source term in the standard Teukolsky formalism as
\[
\hat{Z}_A^{(\text{out}/\text{down})} = \frac{N_s^{(\text{out}/\text{down})}}{\mu} \tilde{\zeta}_s \int d^4x \sqrt{-g(x)} \tilde{F}_{A,\alpha\beta}^{(\text{in}/\text{up})}(r) \tilde{Z}_A(\theta, \varphi) e^{i\omega t} s \tilde{T}(x), \quad (4.15)
\]
where \( s \tilde{T}(x) \) is a projected energy momentum tensor defined by \( s \tilde{T} := s \tilde{\tau}_{\mu\nu} T^{\mu\nu} \) with (A.9), and \( s \tilde{F}_{A,\alpha\beta}^{(\text{in}/\text{up})}(r) := -s \tilde{F}_{A,\alpha\beta}^{(\text{out}/\text{down})}(r) \) and \( s Z_A(\theta, \varphi) \) are, respectively, the radial mode functions and the spheroidal harmonics introduced in Appendix A2.

In the following discussion we concentrate on the case with \( s = -2 \). Substituting the explicit forms of the energy momentum tensor and the projection operator \( -2 \tilde{\tau}_{\mu\nu} \), we obtain
\[
\hat{Z}_A^{(\text{out}/\text{down})} = 2N_s^{(\text{out}/\text{down})} \tilde{\zeta}_s \int_{-\infty}^{\infty} dt e^{i \omega t - i m \varphi(t)} I_A^{(\text{in}/\text{up})}(r(t), \theta(t)), \quad (4.16)
\]
with
\[
I_A = \left[ R_A(A_{nn0} + A_{nm0} + A_{mm0}) - \frac{dR_A}{dr}(A_{nm1} + A_{nm1}) + \frac{d^2R_A}{dr^2}A_{nm2} \right]_{r=r(t), \theta=\theta(t)},
\]
\[
A_{nn0} = \frac{-2}{\sqrt{2\pi \Delta^2}} C_{mn} \tilde{z}^2 \tilde{z} \mathcal{L}_1^1 \left\{ \tilde{z}^4 \mathcal{L}_2^1(\tilde{z}^{-3} S_A) \right\},
\]
\[
A_{nm0} = \frac{2}{\sqrt{2\pi \Delta}} C_{mn} \tilde{z}^3 \left\{ \left( \frac{iK}{\Delta} + 1 - \frac{1}{\tilde{z}} \right) \mathcal{L}_2^1 S_A - \frac{K}{\Delta} (\tilde{z}^{-1} - \tilde{z}) a \sin \theta S_A \right\},
\]
\[
A_{mm0} = \frac{-1}{\sqrt{2\pi \Delta}} \tilde{z}^3 \tilde{z}^{-1} C_{mm} S_A \left[-i \left( \frac{K}{\Delta} \right)_r - \frac{K^2}{\Delta^2} + \frac{2i K}{\tilde{z}} \right],
\]
\[
A_{mm1} = \frac{2}{\sqrt{2\pi \Delta}} \tilde{z}^3 C_{mn} \left[ \mathcal{L}_2^1 S_A + i a \sin \theta (\tilde{z}^{-1} - \tilde{z}) S_A \right],
\]
\[
A_{mm1} = \frac{2}{\sqrt{2\pi \Delta}} \tilde{z}^3 C_{mn} \left[ \mathcal{L}_2^1 S_A + i a \sin \theta (\tilde{z}^{-1} - \tilde{z}) S_A \right],
\]
\[
A_{mm1} = \frac{2}{\sqrt{2\pi \Delta}} \tilde{z}^3 C_{mn} \left[ \mathcal{L}_2^1 S_A + i a \sin \theta (\tilde{z}^{-1} - \tilde{z}) S_A \right],
\]
\[
A_{mm1} = \frac{2}{\sqrt{2\pi \Delta}} \tilde{z}^3 C_{mn} \left[ \mathcal{L}_2^1 S_A + i a \sin \theta (\tilde{z}^{-1} - \tilde{z}) S_A \right],
\]
The adiabatic evolution of orbital parameters in the Kerr spacetime

\[ A_{m\bar{m}1} = -\frac{2}{\sqrt{2\pi}} \bar{z}^3 z^{-1} C_{m\bar{m}S_A} \left( \frac{r}{\Delta} + \bar{z}^{-1} \right), \]

\[ A_{m\bar{m}2} = -\frac{1}{\sqrt{2\pi}} \bar{z}^3 z^{-1} C_{m\bar{m}S_A}, \]

\[ C^{\mu\nu} = \frac{u^\mu u^\nu}{\Sigma u^t}, \]

where \( S_A \) represents \( -2S_A(\theta) \) defined in Appendix A, and

\[ z = r + ia \cos \theta, \]
\[ K = (r^2 + a^2)\omega - ma, \]
\[ \mathcal{L}_s = \partial_\theta + \frac{m}{\sin \theta} - a\omega \sin \theta + s \cot \theta. \]

Here dagger (\( ^\dagger \)) means an operation that transforms \((m, \omega)\) to \((-m, -\omega)\). The radial functions and the spheroidal harmonics appearing in the above equations can be evaluated analytically, as shown in Appendices B and C. For a bound orbit, since \( e^{-im\varphi(t)} Z^{(in/up)}_{\Lambda}(r(t), \theta(t)) \) is a double periodic function, \( \hat{Z}_A^{(out/down)} \) has a discrete spectrum as

\[ \hat{Z}_A^{out/down} = \sum_{n_r, n_\theta} Z_{\ell mn_r n_\theta}^{out/down} \delta(\omega - \omega_{mn_r n_\theta}), \] (4.17)

where the coefficients \( Z_{\ell mn_r n_\theta}^{out/down} \) are those already introduced in Eq. (3.14) with Eq. (3.11). Although we cannot show all the processes explicitly here, it is straightforward to calculate \( Z_{\ell mn_r n_\theta}^{out/down} \) for each \( \omega_{mn_r n_\theta} \) by substituting the analytic expansions of the orbits, the radial functions and the spheroidal harmonics.

### 4.3. Results

Substituting \( Z_{\ell mn_r n_\theta}^{out/down} \) obtained by following the scheme explained in the preceding subsection into Eqs. (3.13), (3.15) and (3.24), we obtain:

\[ \left\langle \frac{dE}{dt} \right\rangle_t = -\frac{32}{5} \left( \frac{\mu}{M} \right)^2 v^{10} \]
\[ \times \left\{ -1 - \frac{1247}{336} v^2 - \frac{33}{12} q - 4\pi \right\} v^3 \]
\[ - \left( \frac{44711}{9072} - \frac{33}{16} q^2 \right) v^4 + \left( \frac{3749}{336} q - \frac{8191}{672} \right) v^5 \]
\[ + \left( \frac{277}{24} - \frac{4001}{84} v^2 + \left( \frac{3583}{48} \pi - \frac{457}{4} q \right) v^3 \right) \]
\[ + \left( \frac{42q^2}{9072} \right) v^4 + \left( \frac{58487}{672} q - \frac{364337}{1344} \pi \right) v^5 \right\} e^2 \]
\[ + \left( \frac{73}{24} q v^3 - \frac{527}{96} q^2 v^4 - \frac{3749}{672} q v^5 \right) y \]
\[ + \left( \frac{457}{8} q v^3 - \frac{5407}{48} q^2 v^4 - \frac{58487}{1344} q v^5 \right) e^2 y \}, \] (4.18)

\[ \left\langle \frac{dL}{dt} \right\rangle_t = -\frac{32}{5} \left( \frac{\mu^2}{M} \right) v^7 \]
\[
\begin{align*}
& \times \left[ 1 - \frac{1247}{336} v^2 - \left( \frac{61}{12} q - 4 \pi \right) v^3 \\
& - \left( \frac{44711}{9072} - \frac{33}{16} q^2 \right) v^4 + \left( \frac{417}{56} q - 8191 \pi \right) v^5 \\
& + \left\{ \frac{51}{8} - \frac{17203}{672} v^2 + \left( - \frac{781}{12} q + \frac{369}{8} \pi \right) v^3 \\
& + \left( \frac{929}{32} q^2 - \frac{1680185}{18144} \right) v^4 + \left( \frac{1809}{224} q - \frac{48373}{336} \pi \right) v^5 \right\} e^2 \\
& + \left\{ - \frac{1}{2} + \frac{1247}{672} v^2 + \left( \frac{61}{8} q - 2 \pi \right) v^3 \\
& - \left( \frac{213}{32} q^2 - \frac{44711}{18144} \right) v^4 - \left( \frac{4301}{224} q - \frac{8191}{1344} \pi \right) v^5 \right\} y \\
& + \left\{ - \frac{51}{16} + \frac{17203}{1344} v^2 + \left( \frac{1513}{16} q - \frac{369}{16} \pi \right) v^3 \\
& + \left( \frac{1680185}{36288} - \frac{5981}{64} q^2 \right) v^4 - \left( 168 q - \frac{48373}{672} \pi \right) v^5 \right\} e^2 y, \quad (4.19)
\end{align*}
\]

From the above results we can compute
\[
\langle \frac{dQ}{dt} \rangle_t = \frac{64}{5} \mu^3 v^6 \\
\begin{align*}
& \times \left[ 1 - qv - \frac{743}{336} v^2 - \left( \frac{1637}{336} q - 4 \pi \right) v^3 \\
& + \left( \frac{439}{48} q^2 - \frac{129193}{18144} - 4 \pi q \right) v^4 + \left( \frac{151765}{18144} q - \frac{4159}{672} \pi - \frac{33}{16} q^3 \right) v^5 \\
& + \left\{ \frac{43}{8} + \frac{51}{8} q v - \frac{2425}{224} v^2 - \left( \frac{14869}{224} q - \frac{337}{8} \pi \right) v^3 \\
& - \left( \frac{453601}{4536} q^2 + \frac{369}{8} \pi q \right) v^4 \\
& - \left( \frac{151049}{9072} q - \frac{38029}{672} \pi - \frac{929}{32} q^3 \right) v^5 \right\} e^2 \\
& + \left\{ \frac{1}{2} - \frac{1637}{672} q v^3 - \left( \frac{1355}{96} q^2 - 2 \pi q \right) v^4 \\
& - \left( \frac{151765}{36288} q - \frac{213}{32} q^3 \right) v^5 \right\} y \\
& + \left\{ \frac{51}{16} + \frac{14869}{448} q v^3 + \left( \frac{369}{16} \pi q - \frac{33257}{192} q^2 \right) v^4 \\
& - \left( \frac{141049}{18144} q + \frac{5981}{64} q^3 \right) v^5 \right\} e^2 y \right]. \quad (4.20)
\end{align*}
\]
The adiabatic evolution of orbital parameters in the Kerr spacetime

\[ - \left( \frac{335}{16} q v^3 - \frac{7559}{192} q^2 v^4 - \frac{1355}{1344} q v^5 \right) y \right]. \tag{4.21} \]

The left hand side of this equation vanishes for circular orbits. In fact, the right hand side vanishes for \( e = 0 \). When we did not know how to compute \( \langle dQ/dt \rangle_t \), the best guess that we could do for \( \langle dQ/dt \rangle_t \) was to assume that the left hand side vanishes for general orbits. Therefore this combination represents the errors coming from this hand-waving working hypothesis. We can also compute

\[
\langle dC \rangle_t = \langle dQ \rangle_t - 2(aE - L) \left( a \langle dE \rangle_t - \langle dL \rangle_t \right)
= - \frac{64}{5} \mu^2 v^6 \left[ 1 - \frac{743}{36} q^2 - \left( \frac{85}{8} q - 4\pi \right) v^3 \right. \\
- \left( \frac{129193}{18144} - \frac{307}{96} q^2 \right) v^4 + \left( \frac{2553}{224} q - \frac{4159}{672} \pi \right) v^5 \\
+ \left( \frac{43}{8} \frac{2425}{224} q^2 + \left( \frac{337}{8} \pi - \frac{1793}{16} q \right) v^3 \right. \\
- \left( \frac{453601}{4536} - \frac{7849}{192} q^2 \right) v^4 \\
+ \left( \frac{3421}{224} q - \frac{38029}{672} \pi \right) v^5 \right] \tag{4.22} \]

Since \( y = 0 \) (i.e., \( C = 0 \)) corresponds to \( \theta = \pi/2 \), \( C \) does not evolve for equatorial orbits. This is consistent with the requirement from the symmetry that an orbit in the equatorial plane stays in the equatorial plane.

Finally, we consider the evolution of an inclination angle \( \iota \) defined by,

\[ \cos \iota = \frac{L}{\sqrt{L^2 + C}}. \tag{4.23} \]

which, roughly speaking, represents an angle between the normal vector of an orbital plane and the rotational axis of the central black hole, but this is not the unique definition. Although the definition of inclination angle can be changed at will to some extent in Kerr case, thus defined inclination angle reduces correctly to the usual one in the \( q = 0 \) Schwarzschild limit. Taking the average of the time derivative of \( \cos \iota \), we obtain

\[
\langle d\cos \iota \rangle_t = \frac{1}{2(L^2 + C)^{3/2}} \left( C - L \langle dC \rangle_t \right)
= - \frac{32 \mu^2 v^6}{5L^2(1 + y)^2} \left[ \left( - \frac{61}{24} v^3 + \frac{13}{96} q v^4 + \frac{1779}{224} v^5 \right) \\
- \left( \frac{431}{16} v^3 - \frac{775}{192} q v^4 - \frac{22431}{224} v^5 \right) e^2 \right]. \tag{4.24} \]

Substituting \( q = 0 \) into this equation, we can confirm that \( \iota \) does not change in the case of Schwarzschild limit, which must be so because of the spherical symmetry of Schwarzschild spacetime.
§5. Summary

In this paper, we have considered a scheme to evaluate the change rates of the orbital parameters of a particle orbiting a Kerr black hole under the adiabatic approximation. We have adopted the method proposed by Mino,\(^\text{10}\) in which we use the radiative field instead of the retarded field in order to compute the change rates for “the constants of motion” due to radiation reaction approximately. Based on Mino’s method, we have developed a simplified scheme to evaluate the long term average of the change rates. Applying our new scheme, we have performed explicit calculations to present analytic formulas of change rates, \(\langle dE/dt \rangle_t\), \(\langle dL/dt \rangle_t\) and \(\langle dQ/dt \rangle_t\), for orbits with small eccentricity and inclination angle.

Here we used the expansions with respect to the post-Newtonian order, the eccentricity and the inclination angle in evaluating \(\langle dE/dt \rangle_t\), \(\langle dL/dt \rangle_t\) and \(\langle dQ/dt \rangle_t\). As a next step therefore we need to examine how large parameter region is covered by our formulae with a sufficient accuracy. As for the inclination, we recently found a formulation to obtain the analytic formulae for the change rates without assuming a small inclination angle.\(^\text{14}\) On the other hand, it is almost certain that we need numerical calculation for the cases with a large eccentricity. Drasco and Hughes\(^\text{15}\) developed a numerical code to calculate the gravitational wave fluxes of energy and azimuthal angular momentum evaluated at infinity and at the event horizon for general geodesic orbits. Fujita and Tagoshi also developed a numerical code based on an analytic method of solving the radial Teukolsky equation. By applying such codes to our scheme, we can evaluate the time-averaged change rate of the Carter constant for general orbits, although computational cost will not be small because we need to take into account a large number of frequency modes.

Once we obtain the change rates of “the constants of motion”, as a next step, we want to use them to trace the evolution of orbits. Some strategies to solve the orbital evolution taking into account the radiation reaction effects were proposed in Refs. 17) and 18). However, it should be noted that the adiabatic approximation used here contains only the dissipative part of the self-force on a particle, and it does not contain the conservative part. In general, the conservative part also contributes to the secular evolution of orbits, though it is not the dominant part in the limit \(\mu \to 0\). Therefore the adiabatic approximation may not be sufficient to evaluate the orbital evolution.

Recently, Pound, Poisson and Nickel showed that the conservative part of the self-force can produce significant shifts in orbital phases in an analogous problem with a charged particle in electromagnetism.\(^\text{19}\) They suggested that the conservative contribution to the phase shift is relatively large in weak field, slow motion cases, while it is suppressed in strong field, rapid motion cases. Furthermore, there are different types of effects higher order in \(\mu\) which may produce significant shifts in phases. Therefore it is important to quantify the range of validity of the adiabatic approximation for appropriate applications of the results obtained in this paper. Although it requires computing second order perturbations in \(\mu\) in order to understand the whole effects which potentially give phase shifts greater than \(O(1)\), some of effects can be evaluated by studying the first order self-force at each moment.
without averaging over a long period. We will come back to this issue in one of our forthcoming papers.\textsuperscript{20)}

\section*{Acknowledgments}

We would like to thank S. Drasco, S. Jhingan, Y. Mino, T. Nakamura, M. Sasaki and H. Tagoshi for invaluable discussions. NS, WH and HN would like to thank all participants of the 8th Capra Meeting at the Rutherford Appleton Laboratory in UK for useful discussions. This work was supported by Monbukagaku-sho Grant-in-Aid for Scientific Research of the Japanese Ministry of Education, Culture, Sports, Science and Technology, Nos. 14047212 and 14047214. HN and WH are supported by a JSPS Research Fellowship for Young Scientists, No. 5919 and No. 1756, respectively.

\section*{Appendix A}

\textbf{Radiative solution for the metric perturbation}

In this Appendix, we give a brief review on Teukolsky formalism, followed by a derivation of the radiative Green function of the linearized Einstein equations. This derivation is based on Refs. 22), 23) and 8).

\subsection*{A.1. Teukolsky equation}

As a master variable we consider the Teukolsky functions defined by

\begin{equation}
 s\Psi := s D^{\mu\nu} h_{\mu\nu} = \left\{ \begin{array}{ll}
 - C_{\alpha\beta\gamma\delta}^\ast m^\alpha l^\beta m^\gamma l^\delta, & s = 2, \\
 - z^4 C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta, & s = -2,
\end{array} \right. \tag{A.1}
\end{equation}

where

\begin{equation}
 2 D^{\mu\nu} = -\frac{1}{2z} \left[ \frac{1}{2} \mathcal{L}_0 \mathcal{L}_0 \mathcal{L}_0^{\dagger} \Lambda z^2 \partial_{\mu} \partial_{\nu} + D_0^2 z m^\mu m^\nu \right.
\end{equation}

\begin{equation}
 - \frac{1}{2\sqrt{2}} \left( D_0 \mathcal{L}_0 \mathcal{L}_0^{\dagger} \Lambda z^2 + \mathcal{L}_0 \mathcal{L}_0 \mathcal{L}_0^{\dagger} \Lambda z^2 \right) (l^\mu m^\nu + m^\mu l^\nu),
\end{equation}

\begin{equation}
 -2 D^{\mu\nu} = -\frac{1}{2z} \left[ \frac{1}{2} \mathcal{L}_0 \mathcal{L}_0 \mathcal{L}_0^{\dagger} \Lambda z^2 n^\mu n^\nu + \frac{1}{4} \Lambda^2 D_0^2 \Lambda z^2 \bar{m}^\mu \bar{m}^\nu \right.
\end{equation}

\begin{equation}
 + \frac{\Delta^2}{4\sqrt{2}} \left( D_0 \mathcal{L}_0 \mathcal{L}_0 \mathcal{L}_0^{\dagger} \Lambda z^2 + \mathcal{L}_0 \mathcal{L}_0 \mathcal{L}_0^{\dagger} \Lambda z^2 \right) (n^\mu \bar{m}^\nu + \bar{m}^\mu n^\nu) \right], \tag{A.2}
\end{equation}

where $z := r + ia \cos \theta$, $\Delta := r^2 - 2Mr + a^2$, and $\Sigma := r^2 + a^2 \cos^2 \theta$. $\mathcal{D}_n$ and $\mathcal{L}_s$ are the differential operators defined by

\begin{equation}
 \mathcal{D}_n := \partial_r + \frac{(r^2 + a^2)}{\Delta} \partial_t + \frac{a}{\Delta} \partial_\varphi + \frac{2m(r-M)}{\Delta}, \tag{A.3}
\end{equation}

\begin{equation}
 \mathcal{L}_s := \partial_\theta - \frac{i}{\sin \theta} \partial_\varphi - ia \sin \theta \partial_t + s \cot \theta, \tag{A.4}
\end{equation}

and a dagger (\dagger) acting on an operator means transformation of $(\partial_t, \partial_\varphi) \rightarrow (-\partial_t, -\partial_\varphi)$, which reduces to the one defined in the main text by $(\omega, m) \rightarrow (-\omega, -m)$ under the
assumption of Fourier expansion. The Teukolsky functions satisfy a separable partial differential equation

\[ s\mathcal{O}^s \Psi = 4\pi \Sigma \hat{s}^T, \quad (A.5) \]

where

\[ \hat{s}^T := s_{\mu\nu} T^{\mu\nu}, \quad (A.6) \]

and \( s\mathcal{O} \) is the Teukolsky differential operator,

\[ s\mathcal{O} := s\mathcal{O}_r + s\mathcal{O}_\theta, \quad (A.7) \]

with

\[ s\mathcal{O}_r := -(\frac{r^2+a^2}{\Delta}) \partial_t^2 + \Delta^{-s} \partial_r (\Delta^{s+1} \partial_r) - \frac{a^2}{\Delta} \partial_\varphi^2 - \frac{4Mar}{\Delta} \partial_t \partial_\varphi + \frac{2sa(r-M)}{\Delta} \partial_\varphi \]

\[ + 2s \left( \frac{M(r^2-a^2)}{\Delta} - r \right) \partial_t + s, \]

\[ s\mathcal{O}_\theta := a^2 \sin^2 \theta \partial_t^2 + \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin^2 \theta} \partial_\varphi^2 + \frac{2is \cos \theta}{\sin^2 \theta} \partial_\varphi \]

\[ - 2isa \cos \theta \partial_t - s^2 \cot^2 \theta, \quad (A.8) \]

and

\[
2\tau_{\mu\nu} := \frac{1}{\bar{z}^4} \left[ \frac{1}{2\sqrt{2}} \left( L_{-1}^\dagger \frac{\bar{z}^4}{\bar{z}^2} D_0 + D_0 \frac{\bar{z}^4}{\bar{z}^2} L_{-1}^\dagger \right) \right] z^2 (\ell_\mu m_\nu + m_\mu l_\nu) \\
- \frac{1}{2\sqrt{2}} \Delta \left( L_{-1}^\dagger \frac{\bar{z}^4}{\bar{z}^2} D_0 + D_0 \frac{\bar{z}^4}{\bar{z}^2} L_{-1}^\dagger \right) \right] z^2 (\bar{m}_\mu \bar{m}_\nu + \bar{m}_\nu \bar{m}_\mu) \\
+ \frac{1}{2} \Delta^2 D_0^\dagger \bar{z}^4 D_0^\dagger \frac{\bar{z}^2}{\bar{z}} \bar{m}_\mu \bar{m}_\nu \right]. \quad (A.9)
\]

We consider the following forms of expansions for \( s\Psi \) and \( \hat{s}^T \):

\[
s\Psi = \int_{-\infty}^{\infty} d\omega \sum_{\ell m} e^{-i\omega t} sX_A(r) sS_A(\theta) \frac{e^{im\varphi}}{\sqrt{2\pi}},
\]

\[
4\pi \Sigma \hat{s}^T = \int_{-\infty}^{\infty} d\omega \sum_{\ell m} e^{-i\omega t} sT_A(r) sS_A(\theta) \frac{e^{im\varphi}}{\sqrt{2\pi}},
\]

where \( A := \{lm\omega \} \). Substituting these expressions into the Teukolsky equation \( (A.5) \), we obtain equations separated for the radial and angular parts as

\[
\begin{bmatrix}
\Delta^{-s} \frac{d}{dr} \\
\sin \theta \frac{d}{d\theta}
\end{bmatrix}
\begin{bmatrix}
\Delta^{s+1} \frac{d}{dr} \\
\sin \theta \frac{d}{d\theta}
\end{bmatrix}
+ \frac{K^2 - 2is(r-M)K}{\Delta} + 4is\omega r - \lambda
\end{bmatrix}
\]

\[ sX_A(r) = sT_A, \quad (A.10) \]
The adiabatic evolution of orbital parameters in the Kerr spacetime

\[-2aωs \cos θ + λ + s + 2amω \] \( sS_A(θ) = 0, \) (A.11)

where \( K := (r^2 + a^2)ω - ma \) and \( λ := sE_{ℓm}(aω) - s(s + 1) + a^2ω^2 - 2amω. \) The eigenvalue \( sE_{ℓm}(aω) \) is determined by solving Eq. (A.11) as an eigenvalue problem imposing regular boundary conditions on \( sS_A(θ) \) at \( θ = ±π/2. \) Here \( ℓ \) is an index that labels different eigenvalues. We also give a brief review on how to solve this equation analytically in Appendix C.

A.2. Mode functions

We write mode functions for the Teukolsky equation (A.5) in the form

\[ sΩ_A := sR_A( r) sZ_A(θ, ϕ)e^{-iωt}, \] (A.12)

where \( sR_A( r) \) is a homogeneous solution of the radial Teukolsky equation (A.10), and \( sZ_A(θ, ϕ) \) is the spheroidal harmonics

\[ sZ_A(θ, ϕ) = \frac{1}{\sqrt{2π}} sS_A(θ)e^{imϕ}, \] (A.13)

normalized as

\[ \int_0^π dθ \sin θ |sS_A(θ)|^2 = 1. \] (A.14)

Using the symmetry of the radial equation (A.10) under the simultaneous operations of the complex conjugation and the transformation of \( (m, ω) \rightarrow (−m, −ω) \), we impose

\[ sR_A = sR_A^{†}, \] (A.15)

where a dagger \( (\dagger) \) acting on a mode function means transformation of \( (ω, m) \rightarrow (−ω, −m). \) In a similar manner, by virtue of the symmetries of Eq. (A.11), we arrange the spheroidal harmonics to satisfy

\[ sZ_A = (-1)^m sZ_A^{†}. \] (A.16)

In our later discussions, we also need the well-known Teukolsky-Starobinsky identities:

\[ -sR_A = sU_sR_A, \] (for \( |s| = 2), \] (A.17)

with

\[ -2U := \frac{A}{C} D_0^{†}, \quad 2U := \frac{1}{AC} Δ^2 D_0^{†} Δ^2, \] (A.18)

where

\[ C = \left[ (λ + s(s + 1))^2 + 4aωm - 4a^2ω^2 \right] \left[ (λ + s(s + 1) - 2)^2 + 36aωm - 36a^2ω^2 \right] + (2λ + 2s(s + 1) - 1)(96a^2ω^2 - 48aωm) - 144a^2ω^2]^{1/2} + 12iωM, \] (A.19)

and \( A \) is a factor which depends on how we normalize the radial functions. In this paper, we simply adopt \( A = 1. \) (This convention is the one used in Ref. 24).

Now we discuss how to construct mode functions for metric perturbations from mode functions of the Teukolsky equation. The basic idea owes to Chrzanowski.\(^{22}\)
Here we follow a more rigorous approach taken by Wald.\textsuperscript{23) Using the relation (A.1), the Teukolsky equation (A.5) is rewritten as

\[
\frac{1}{4\pi} \Sigma O s^\mu h_{\mu \nu} = \hat{T}.
\]

(A.20)

On the other hand, operating \( s^{\tau}_{\alpha \beta} \) on the linearized Einstein equation, which we schematically denote as \( G^{\alpha \beta}_{\mu \nu} h_{\mu \nu} = 4\pi T_{\alpha \beta} \), we obtain

\[
\frac{1}{4\pi} s^{\tau}_{\alpha \beta} G^{\alpha \beta}_{\mu \nu} h_{\mu \nu} = \hat{T}.
\]

(A.21)

From the comparison of these equations, we find an identity at the operator level:

\[
\frac{1}{\Sigma} O s^\mu = s^{\tau}_{\alpha \beta} G^{\alpha \beta}_{\mu \nu}.
\]

(A.22)

Here we define \( O^* \mu \nu \), the adjoint of an operator \( O^\mu \nu \), so as to satisfy

\[
\int \sqrt{-g} X O^\mu \nu Y_{\mu \nu} d^4x = \int \sqrt{-g} Y_{\mu \nu} O^* \mu \nu X d^4x,
\]

for arbitrary scalar field \( X \) and tensor field \( Y_{\mu \nu} \). The definition of the adjoint operators for different types of tensor operators is a straightforward generalization of this definition. It will be worth noting \( \sqrt{-g} d^4x = \sin \theta \Sigma dt dr d\theta d\varphi \), and

\[
(AB)^* = B^* A^*, \quad D^* = -\Sigma^{-1} D_{-n}^\dagger \Sigma, \quad L^* = -\Sigma^{-1} L_{1-s}^\dagger \Sigma.
\]

(A.24)

By taking adjoint of each side in Eq. (A.22), we obtain

\[
s D^* \mu \nu (\Sigma^{-1} s O)^* = G^{\alpha \beta}_{\mu \nu} s^{\tau}_{\alpha \beta}.
\]

(A.25)

Here we used the fact that the linearized Einstein operator \( G^{\alpha \beta}_{\mu \nu} \) is self-adjoint, i.e., \( G^{\alpha \beta}_{\mu \nu} = G^{\alpha \beta}_{\mu \nu} \). Then, from the definition of \( s O_r \) and \( s O_\theta \) given in Eqs. (A.8), it is easy to see that

\[
(\Sigma^{-1} s O_r)^* = \Sigma^{-1} s O_r, \quad (\Sigma^{-1} s O_\theta)^* = \Sigma^{-1} s O_\theta.
\]

(A.26)

Therefore we have \((\Sigma s O)^* = -s R_A s Z^A e^{-i\omega t} = 0\), which means that

\[
G^{\alpha \beta}_{\mu \nu} s^{\tau}_{\alpha \beta} - s R_A s Z^A e^{-i\omega t} = 0.
\]

(A.27)

Here the explicit form of the adjoint operators \( s^{\tau}_{\mu \nu} \) are

\[
2 s^{\tau}_{\mu \nu} = \left[ \frac{1}{\sqrt{2}} (l_{\mu} \bar{m}_{\nu} + m_{\mu} \bar{l}_{\nu}) \bar{\xi} \left( D_{0} z^4 \frac{\bar{L}_2}{z^2} + \bar{L}_2 z^4 \frac{D_{0}}{z^2} \right) \right. \\
- l_{\mu} \bar{l}_{\nu} \frac{1}{z^2} \bar{L}_1 z^4 L_2 - 2 \bar{m}_{\mu} \bar{m}_{\nu} \frac{1}{z} D_{0} z^4 D_{0} \left. \right] \frac{1}{z^3},
\]

(A.28)
The adiabatic evolution of orbital parameters in the Kerr spacetime

\[ n_\mu n_\nu z L_1^\dagger z^4 L_2^\dagger + \frac{1}{2} m_\mu m_\nu \frac{z}{z^2} D_0^\dagger z^4 D_0^\dagger \Delta^2 \frac{1}{z^3}. \]  

(A.29)

Hence,

\[ s \Pi_{A,\mu \nu} := \zeta_s s \tau^\dagger_{\mu \nu} s \tilde{\Omega}_A, \]  

(A.30)

with

\[ s \tilde{\Omega}_A = -s R_A s Z_A e^{-i\omega t}, \]  

(A.31)

is a complex-valued homogeneous solution of the linearized Einstein equations. Here \( \zeta_s \) is a numerical coefficient which we determine so as to satisfy

\[ s D_{\mu \nu} \sum_A (A_{A} s \Pi_{A,\mu \nu} + \bar{A}_{A} s \bar{\Pi}_{A,\mu \nu}) = \sum_A A_A \bar{s} \Omega_A, \]  

(A.32)

for any complex-valued amplitude of each mode, \( A_A \). Here the complex conjugate term in parentheses is necessary to make the metric perturbation real. Using Eqs. (A.2) and (A.29), we can verify

\[ 2 D_{\mu \nu} - 2 \tau^\dagger_{\mu \nu} = \frac{1}{4} \bar{\mathcal{L}}_{\dagger 1} \mathcal{L}_0 \mathcal{C}_{\dagger 1} \mathcal{L}_2^\dagger, \quad -2 D_{\mu \nu} - 2 \tau^\dagger_{\mu \nu} = \frac{1}{16} \Delta^2 D_{0}^\dagger \Delta^2, \]  

(A.33)

\[ -2 D_{\mu \nu} - 2 \bar{\tau}^\dagger_{\mu \nu} = 0, \quad 2 D_{\mu \nu} 2 \bar{\tau}^\dagger_{\mu \nu} = D_0^4, \]  

(A.34)

In literature \( s \tau^\dagger_{\mu \nu} \) is used to represent what we denote here by \( s \bar{\tau}^\dagger_{\mu \nu} \). The difference arises because we use the notation for the differential operators without assuming that they always act on a single Fourier mode. Namely, instead of writing \((-i\omega, im)\), we are using here \((\partial_t, \partial_\phi)\). The complex conjugation of the former gives rise a flip of signature, while that of the latter does not. With the aid of the above relations (A.17) and (A.34), we find that the complex conjugate terms vanish to obtain

\[ 2 D_{\mu \nu} 2 \Pi_{A,\mu \nu} = \zeta_2 \mathcal{C} \mathcal{Z}_2 \mathcal{O}_A, \]  

\[ -2 D_{\mu \nu} - 2 \Pi_{A,\mu \nu} = \zeta_{-2} \bar{\mathcal{C}} \mathcal{Z}_2 \mathcal{O}_A. \]  

(A.35)

Thus the normalization constants are fixed as

\[ \zeta_2 = \frac{1}{\mathcal{C}}, \quad \zeta_{-2} = \frac{16}{\mathcal{C}}. \]  

(A.36)

A.3. Radiative field

Here we explain a method of constructing radiative field for metric perturbations. Radiative field is a homogeneous solution of field equations. Hence, once we obtain the radiative field for the Teukolsky function, it can be easily transformed into that for metric perturbations by using the relations established in the preceding subsection. We therefore first derive the radiative field for the Teukolsky function.

The retarded Green function of the Teukolsky function is defined as a solution of

\[ s \mathcal{O}_s G(x, x') = \frac{\delta^{(4)}(x - x')}{\Delta_s}, \]  

(A.37)
with the retarded boundary condition: \( sG(x, x') = 0 \) for \( t < t' \). We write the retarded Green function of the Teukolsky equation in the form of Fourier-harmonic expansion as

\[
sG(x, x') = \int \frac{d\omega}{2\pi} \sum_{\ell m} s g_A(r, r') s \mathcal{Z}_A(\theta, \varphi) s \bar{\mathcal{Z}}_A(\theta', \varphi') e^{-i\omega(t-t')} . \tag{A.38}
\]

Then the radial part of the Green function \( s g_A(r, r') \) is given by

\[
s g_A(r, r') = \frac{1}{W(s R_A^{\text{in}}, s R_A^{\text{up}})} \left[ s R_A^{\text{up}}(r) s R_A^{\text{in}}(r') \theta(r-r') + s R_A^{\text{in}}(r) s R_A^{\text{up}}(r') \theta(r'-r) \right] , \tag{A.39}
\]

with the Wronskian defined by

\[
W(s R_A^{\text{in}}, s R_A^{\text{up}}) := \Delta^{s+1} \left[ s R_A^{\text{in}}(r) \frac{d}{dr} s R_A^{\text{up}}(r) - s R_A^{\text{up}}(r) \frac{d}{dr} s R_A^{\text{in}}(r) \right] . \tag{A.40}
\]

The advanced Green function can be constructed in a similar manner just by replacing “in” and “up” with “out” and “down”, respectively. Then it is easy to show that the radiative Green function has a simple structure which does not contain any step function \( \theta(r-r') \). To show this, let us start with the following expression for the radial part of the radiative Green function for \( r > r' \): 

\[
s g_A^{\text{rad}}(r, r') = \frac{1}{2} \left[ s R_A^{\text{up}}(r) s R_A^{\text{in}}(r') - \frac{s R_A^{\text{down}}(r) s R_A^{\text{out}}(r')} {W(s R_A^{\text{out}}, s R_A^{\text{down}})} \right] . \tag{A.41}
\]

We rewrite this expression in terms of the down-field and the out-field, eliminating \( s R_A^{\text{up}}(r) \) and \( s R_A^{\text{out}}(r') \) in Eq. \((A.41)\). Hence we expand \( s R_A^{\text{up}} \) and \( s R_A^{\text{out}} \) as

\[
s R_A^{\text{up}} = \alpha s R_A^{\text{out}} + \beta s R_A^{\text{down}} , \quad s R_A^{\text{out}} = \gamma s R_A^{\text{up}} + \delta s R_A^{\text{in}} . \tag{A.42}
\]

Taking the Wronskians of both sides of Eqs. \((A.42)\) with appropriate radial functions, one can easily obtain

\[
W(s R_A^{\text{up}}, s R_A^{\text{down}}) = \alpha W(s R_A^{\text{out}}, s R_A^{\text{down}}) , \quad W(s R_A^{\text{up}}, s R_A^{\text{out}}) = \beta W(s R_A^{\text{down}}, s R_A^{\text{out}}) , \quad W(s R_A^{\text{out}}, s R_A^{\text{up}}) = \gamma W(s R_A^{\text{in}}, s R_A^{\text{up}}) , \quad W(s R_A^{\text{out}}, s R_A^{\text{in}}) = \delta W(s R_A^{\text{down}}, s R_A^{\text{in}}) .
\]

Substituting these relations, the expression \((A.41)\) reduces to

\[
s g_A^{\text{rad}}(r, r') = \frac{\Delta^{-s}(r')}{2 W(s R_A^{\text{in}}, s R_A^{\text{up}}) W(s R_A^{\text{out}}, s R_A^{\text{down}})} \times \left[ W(s R_A^{\text{out}}, s R_A^{\text{in}}) s R_A^{\text{down}}(r) - s R_A^{\text{down}}(r') + W(s R_A^{\text{up}}, s R_A^{\text{out}}) s R_A^{\text{out}}(r) - s R_A^{\text{out}}(r') \right] . \tag{A.43}
\]
We can do an analogous reduction for $r < r'$, and the result turns out to be the same as that for $r > r'$. Namely, the step functions which was present in the retarded and advanced Green functions do not appear in the radiative Green function. This is consistent with the fact that the radiative field is a source-free homogeneous solution.

Since the radiative field is a homogeneous solution, we can use the method for reconstruction of metric perturbation explained in the preceding subsection. When we consider the metric perturbation by a point mass, the energy-momentum tensor is given by \( T^\mu_\nu \). In this case it is easy to verify that the radiative field of the metric perturbations is given by

\[
\bar{h}_{\mu\nu}^{\text{rad}}(x) = \mu \int d\omega \sum_{\ell m} \left\{ \mathcal{N}_s \Pi_{s,\mu\nu}^\text{out}(x) \int d\tau \left[ s \Pi_{s,\alpha\beta}^\text{out}(z(\lambda)) u^\alpha u^\beta \right] \\
+ \mathcal{N}_s \Pi_{s,\mu\nu}^\text{down}(x) \int d\tau \left[ s \Pi_{s,\alpha\beta}^\text{down}(z(\lambda)) u^\alpha u^\beta \right] \right\} + (\text{c.c.}), \quad (A.44)
\]

with

\[
\mathcal{N}_s^\text{out} = \frac{W(sR_A^\text{up}, sR_A^\text{down})}{\zeta_s W(sR_A^\text{up}, sR_A^\text{up}) W(sR_A^\text{up}, sR_A^\text{down})}, \\
\mathcal{N}_s^\text{down} = \frac{W(sR_A^\text{up}, sR_A^\text{down})}{\zeta_s W(sR_A^\text{up}, sR_A^\text{up}) W(sR_A^\text{up}, sR_A^\text{down})}. \quad (A.45)
\]

In fact, if we apply \( sD^\mu_\nu \), we correctly recover \( s\Psi^{\text{rad}}(x) = 4\pi \int G^{\text{rad}}(x, x') \Sigma(x') \Delta^s(x') \hat{T}(x') d^4x \). To show this, we also used

\[
\bar{\zeta}_s \int \sqrt{-g} \bar{\Omega}_s \hat{T} d^4x = \bar{\zeta}_s \int \sqrt{-g} (s\tau^s_\mu_\nu, s\bar{\Omega}_s) T^\mu_\nu d^4x \\
= \mu \int d\tau u^\mu u^\nu \bar{\Pi}_{s,\mu\nu}(z(\tau)). \quad (A.46)
\]

It is more convenient to rewrite \( \mathcal{N}_s \) written in terms of Wronskians by using the coefficients in the asymptotic forms of radial functions. The radial functions take the asymptotic forms,

\[
sR_A^\text{in} := \left\{ \begin{array}{ll}
sB_A^{\text{inc}} r^{-1} e^{-i\omega r^*} + sB_A^{\text{ref}} r^{-2s-1} e^{i\omega r^*}, & \text{for } r^* \to \infty, \\
sB_A^{\text{trans}} \Delta^{-s} e^{-ikr^*}, & \text{for } r^* \to -\infty, \end{array} \right. \quad (A.47)
\]

\[
sR_A^\text{up} := \left\{ \begin{array}{ll}
sC_A^{\text{trans}} r^{-2s-1} e^{i\omega r^*}, & \text{for } r^* \to \infty, \\
sC_A^{\text{inc}} e^{ikr^*} + sC_A^{\text{ref}} \Delta^{-s} e^{-ikr^*}, & \text{for } r^* \to -\infty, \end{array} \right. \quad (A.48)
\]

where \( r^* \) is the tortoise coordinate defined by \( dr^*/dr = (r^2 + a^2)/\Delta \). Using the relations \( sR_A^\text{out} = \Delta^{-s} sR_A^\text{out} \) and \( sR_A^\text{down} = \Delta^{-s} sR_A^\text{down} \), we can describe the asymptotic forms of out- and down- fields with the same coefficients that appear in Eqs. \( A.47 \) and \( A.48 \). Then, the Wronskians that we need to evaluate are

\[
W(sR_A^\text{in}, sR_A^\text{up}) = 2i\omega sB_A^{\text{inc}} sC_A^{\text{trans}}, \\
W(sR_A^\text{out}, sR_A^\text{down}) = -2i\omega sC_A^{\text{trans}} sB_A^{\text{inc}},
\]
where \( \kappa_s := 1 - is(r_+ - M)/2kMr_+ \). The coefficients with \((-s)\)-spin can be erased by using the Teukolsky-Starobinsky identities (A.47). Substituting the asymptotic forms (A.47) and (A.48) into Eqs. (A.17), we obtain

\[
-2B^\text{inc}_A = \frac{C}{(2\omega)^4} 2B^\text{inc}_A, \quad -2B^\text{trans}_A = \left( \frac{1}{4Mr_+k} \right)^4 \frac{C}{\kappa_2\kappa_1} 2B^\text{trans}_A, \\
-2C^\text{trans}_A = \left( \frac{C}{2} \right)^4 2C^\text{trans}_A, \quad -2B^\text{ref}_A = \left( \frac{2\omega}{C} \right)^4 2B^\text{ref}_A. 
\]  

(A.50)

Using the above relations, the coefficients \( N_s \) are rewritten as

\[
N^\text{out}_s = \frac{1}{2i\omega^3} |N^\text{out}_s|^2, \quad N^\text{down}_s = \frac{1}{2i\omega^2k} |N^\text{down}_s|^2, 
\]  

(A.51)

with

\[
|N^\text{out}_s|^2 = \frac{2^{3s-2} \omega^{2s+2}}{|C|^{s+1}} \frac{1}{|sB^\text{inc}_A|^2}, \quad |N^\text{down}_s|^2 = \frac{2^{3s-2} \omega^{2s+2} |C|^s |2s+1|}{|\kappa_2|^{s+1} |\kappa_1|^{s(2Mr_+)^2-1}} \frac{1}{|sB^\text{trans}_A|^2 |sC^\text{trans}_A|^2}. 
\]  

(A.52)

Hence, we finally obtain

\[
h^\text{rad}_{\mu\nu} = \mu \int d\omega \sum_{lm} \frac{1}{2i\omega^3} \left( N^\text{out}_s \Pi^\text{out}_{\mu\nu}(x) \int \frac{d\tau}{\Sigma} \bar{\phi}^\text{out}_A(\tau) \\
+ \frac{\omega}{k} N^\text{down}_s \Pi^\text{down}_{\mu\nu}(x) \int \frac{d\tau}{\Sigma} \bar{\phi}^\text{down}_A(\tau) \right) + (c.c.), 
\]  

(A.54)

where

\[
\phi^\text{(out/down)}_A(\tau) := N^\text{(out/down)}_s \Sigma(z(\tau))_s \Pi^\text{(out/down)}_{\alpha\beta}(z(\tau)) u^\gamma(\tau) u^\delta(\tau), 
\]  

(A.55)

whose extension to a field is \( \phi^\text{(out/down)}_A(x) \) defined in Eq. (3.10).

\section*{Appendix B}

Mano-Suzuki-Takasugi method

Mano, Suzuki and Takasugi formulated a method of constructing a homogeneous solution for the radial Teukolsky equation in two kinds of series by using the Coulomb wave function and the hypergeometric functions.\cite{24,25,26} By applying this method under slow motion approximation, we can express homogeneous solutions in an analytic form. Furthermore, this method determines the asymptotic amplitudes of homogeneous solutions without numerical integration. This allows us to compute the gravitational wave flux at infinity and on the horizon with a high accuracy.\cite{16}

We summarize this method in this appendix.
B.1. Outer solution of radial Teukolsky equation

According to (24)–(26), we can expand \( s_R^\nu \), a homogeneous solution of the radial Teukolsky equation (A.10), in terms of the Coulomb wave functions as

\[
s_R^\nu = \frac{\Gamma(\nu + 1 - s + i\epsilon)}{\Gamma(2\nu + 2)} z^{-s}(2\hat{z})^\nu e^{-i\hat{z}} \left(1 - \frac{\epsilon\kappa}{\hat{z}}\right)^{-s-i\epsilon+} \times \sum_{n=-\infty}^{\infty} (-2i\hat{z})^n \frac{(\nu + 1 + s - i\epsilon)n}{(2\nu + 2)n} a_{n,s}^{\nu,s} \times _1F_1(n + \nu + 1 - s + i\epsilon, 2n + 2\nu + 2, 2i\hat{z}), \tag{B.1}
\]

where \( \epsilon = 2M\omega, \epsilon_+ = \epsilon + \tau, \tau = \kappa^{-1}(\epsilon - ma/M), \kappa = \sqrt{1 - (a/M)^2}, (x)_n := \Gamma(x + n)/\Gamma(x), \hat{z} := \omega(r - r_\ast), \) and \( r_\ast = M - \sqrt{M^2 - a^2} \). The coefficients \( a_{n,s}^{\nu,s} \) satisfies the following three term recurrence relation,

\[
a_{n,s}^{\nu,s}a_{n+1,s}^{\nu,s} + \beta_{n,s}^{\nu,s}a_{n,s}^{\nu,s} + \gamma_{n,s}^{\nu,s}a_{n-1,s}^{\nu,s} = 0, \tag{B.2}
\]

where

\[
\alpha_n^{\nu} = \frac{i\epsilon\kappa(n + \nu + 1 + s + i\epsilon)(n + \nu + 1 + s - i\epsilon)(n + \nu + 1) + i\tau}{(n + \nu + 1)(2n + 2\nu + 3)},
\]

\[
\beta_n^{\nu} = -\lambda - s(s + 1) + (n + \nu)(n + \nu + 1) + \epsilon^2 + \epsilon(\epsilon - mq) + \frac{\epsilon(\epsilon - mq)(s^2 + \epsilon^2)}{(n + \nu)(n + \nu + 1)},
\]

\[
\gamma_n^{\nu} = \frac{-i\epsilon\kappa(n + \nu - s + i\epsilon)(n + \nu - s - i\epsilon)(n + \nu - i\tau)}{(n + \nu)(2n + 2\nu - 1)}, \tag{B.3}
\]

and \( q = a/M \). The renormalized angular momentum \( \nu \) is determined by the conditions

\[
\lim_{n \to \infty} n \frac{a_{n,s}^{\nu,s}}{a_{n-1,s}^{\nu,s}} = \frac{i\epsilon\kappa}{2}, \quad \lim_{n \to \infty} n \frac{a_{n,s}^{\nu,s}}{a_{n+1,s}^{\nu,s}} = \frac{i\epsilon\kappa}{2}. \tag{B.4}
\]

Under this condition, the series of Coulomb wave functions (B.1) converges for any \( r > r_\ast \).

From the equations in (B.3), we can show that \( a_{n,s}^{\nu-1} = \gamma_n^{\nu} \) and \( \beta_{n,s}^{\nu-1} = \beta_n^{\nu} \). By using these relations, we can find that \( a_{n,s}^{\nu-1,s} = a_{n,s}^{\nu,s} \) and

\[
\lim_{n \to \infty} n \frac{a_{n,s}^{\nu-1,s}}{a_{n-1,s}^{\nu-1,s}} = \frac{i\epsilon\kappa}{2}, \quad \lim_{n \to \infty} n \frac{a_{n,s}^{\nu-1,s}}{a_{n+1,s}^{\nu-1,s}} = -\frac{i\epsilon\kappa}{2}. \tag{B.5}
\]

This fact shows that \( s_R^C^{-\nu-1} \) is also a solution of the radial Teukolsky equation, which converges within the region \( r > r_\ast \).

B.2. In-going and up-going solutions

The in-going solution of the radial Teukolsky equation is given in terms of the Coulomb type solutions (B.1) as

\[
s_R^{in} = A_\nu e^{i\epsilon\kappa}(K_{s,\nu,s}R_C^\nu + K_{s,-\nu-1,s}R_C^{-\nu-1}), \tag{B.6}
\]
where

\[ A_2 = \tilde{C} \left( \frac{\omega}{\epsilon K} \right)^4 \frac{\Gamma(3 - 2\epsilon \omega_+)}{\Gamma(-1 - 2\epsilon \omega_+)} \left| \frac{\Gamma(\nu - 1 + i\epsilon)}{\Gamma(\nu + 3 + i\epsilon)} \right|^2, \quad A_{-2} = 1, \]  

(B.7)

\[ K_{s,\nu} = \frac{(2\epsilon K)^{s - \nu - 2 - s} r}{(\nu + 1 + i\tau)_r (\nu + 1 + s + i\epsilon)_r \Gamma(1 - s - 2\epsilon \omega_+) \Gamma(n + 2\nu + 2) \Gamma(n + 2\nu + 1)} \times \frac{\Gamma(r + \nu + 1 - s + i\epsilon) \Gamma(\nu + 1 - s - i\epsilon) \Gamma(\nu + 1 - i\tau)}{\Gamma(\nu + 1 - s + i\epsilon) \Gamma(\nu + 1 - i\tau)} \times \left[ \sum_{n=r}^{\infty} (-1)^n (r + 2\nu + 1) (\nu + 1 + s + i\epsilon)_n (\nu + 1 + i\tau)_n a^\nu,s_n \right] \times \left[ \sum_{n=-\infty}^{r} (\nu + 1 - s - i\epsilon)_n a^\nu,s_n \right]^{-1}. \]  

(B.8)

Here \( r \) is an arbitrary integer and \( K_{s,\nu} \) is independent of the choice of \( r \).

Next, we consider the up-going solution. \( s R^\nu_C \) can be divided into two parts as

\[ s R^\nu_C = s R^\nu_+ + s R^\nu_-, \]  

(B.9)

where

\[ s R^\nu_+ = e^{-\pi \epsilon} e^{i\pi (\nu + 1 - s)} e^{-i(2z\nu + 1)(2iz)\nu} \left( 1 - \frac{\epsilon K}{\zeta} \right)^{-s - i\epsilon} \frac{\Gamma(\nu + 1 - s + i\epsilon)}{\Gamma(\nu + 1 + s - i\epsilon)} \times \sum_{n=-\infty}^{\infty} (2i\zeta)^n a^\nu,s_n \Psi(n + \nu + 1 - s + i\epsilon, 2n + 2\nu + 2, 2i\zeta), \]  

(B.10)

\[ s R^\nu_- = e^{-\pi \epsilon} e^{-i\pi (\nu + 1 + s)} e^{i(2z\nu + 1)(2iz)\nu} \left( 1 - \frac{\epsilon K}{\zeta} \right)^{-s - i\epsilon} \times \sum_{n=-\infty}^{\infty} (2i\zeta)^n (\nu + 1 + s - i\epsilon)_n a^\nu,s_n \Psi(n + \nu + 1 + s - i\epsilon, 2n + 2\nu + 2, -2i\zeta), \]  

(B.11)

and \( \Psi(a, c; x) \) is the irregular confluent hypergeometric function. From the asymptotic form of \( \Psi(a, c; x) \),

\[ \Psi(a, c; x) \rightarrow x^{-a}, \quad (|x| \rightarrow \infty), \]  

(B.12)

the asymptotic forms of \( s R^\nu_+ \) and \( s R^\nu_- \) become

\[ s R^\nu_+ = s A^\nu_+ z^{-1} e^{-i(z + \epsilon \ln z)}, \quad s R^\nu_- = s A^\nu_- z^{-1} e^{i(z + \epsilon \ln z)}, \]  

(B.13)

where

\[ s A^\nu_+ = e^{-\pi \epsilon/2} e^{i\pi (\nu + 1 - s)/2} \zeta^{\nu - 1 - i\epsilon} \frac{\Gamma(\nu + 1 - s + i\epsilon)}{\Gamma(\nu + 1 + s - i\epsilon)} \sum_{n=-\infty}^{\infty} a^\nu,s_n, \]  

(B.14)

\[ s A^\nu_- = e^{-\pi \epsilon/2} e^{-i\pi (\nu + 1 + s)/2} \zeta^{-1 + i\epsilon} \sum_{n=-\infty}^{\infty} (-1)^n (\nu + 1 + s - i\epsilon)_n a^\nu,s_n. \]  

(B.15)
This shows that \( sR^\nu_{\pm} \) satisfies the up-going (down-coming) boundary condition at infinity. So we can take the up-going solution as

\[
sR_{A}^{\text{up}} = B_{sA}R^\nu,
\]

where \( B_2 = \tilde{C}\omega^{2s} \) and \( B_{-2} = 1 \). Taking the limit \( r^* \to \pm\infty \) in Eqs. (B.6) and (B.16) by means of the asymptotic form of \( r^* \),

\[
\omega r^* \to \hat{z} + \epsilon \ln \hat{z} - \epsilon \ln \epsilon \quad (r \to \infty),
\]

\[
k r^* \to \epsilon_+ \ln(-x) + \kappa \epsilon_+ + \frac{2\epsilon_+}{1 + \kappa} \ln \kappa \quad (r \to r_+),
\]

we find that the coefficients which appear in the asymptotic forms of Eqs. (A.47) and (A.48) are given by

\[
sB_A^{\text{inc}} = \frac{A_s e^{i\epsilon \kappa}}{\omega} \left[ K_{s\nu} - i e^{-i\epsilon \nu} \sin(\nu - s + i\epsilon) K_{s,-\nu-1} \right] A_\nu^{\nu},
\]

\[
sB_A^{\text{trans}} = \frac{A_s (\epsilon \kappa)}{\omega} 2s \sum_{n=-\infty}^{\infty} a_{n,s}^{\nu,\nu},
\]

\[
sC_A^{\text{trans}} = \omega^{-1 - 2s} e^{i\epsilon \ln \epsilon} sA_\nu^{\nu}.
\]

**Appendix C**

---

**Spheroidal harmonics**

Here, we review the formalism to represent the spin-weighted spheroidal harmonics in a series of Jacobi polynomials based on Ref. 27, which was slightly improved in Ref. 16).

We first transform the angular part of the Teukolsky equation (A.11) as

\[
\left[ (1 - x^2) \frac{d^2}{dx^2} - 2x \frac{d}{dx} + \xi^2 x^2 - \frac{m^2 + s^2 + 2msx}{1 - x^2} - 2s\xi x + sE_{\ell m}(\xi) \right] sS_{\ell m}^\xi(x) = 0,
\]

where \( \xi = a\omega, x = \cos \theta \) and \( sE_{\ell m}(\xi) = \lambda + s(s + 1) - \xi^2 + 2m\xi \). The angular function \( sS_{\ell m}^\xi(x) \) is called the spin-weighted spheroidal harmonics. Equation (C.1) is a Sturm-Liouville type eigenvalue equation with regular boundary conditions at \( x = \pm1 \). Since there are a countable number of eigenvalues for fixed parameters \( s, m \) and \( \xi \), we introduced an index \( \ell \) starting with \( \max(|m|, |s|) \) as such a label that sorts the eigenvalues \( sE_{\ell m}(\xi) \) in an ascending order. When \( \xi = 0 \), \( sS_{\ell m}^\xi(x) \) is reduced to the spin-weighted spherical harmonics, and the eigenvalue \( sE_{\ell m}(\xi) \) becomes \( \ell(\ell + 1) \). We normalize the amplitude of \( sS_{\ell m}^\xi(x) \) as

\[
\int_0^\pi \left| sS_{\ell m}^\xi \right|^2 \sin \theta d\theta = 1.
\]
The differential equation (C.1) has singularities at $x = \pm 1$ and at $x = \infty$. We transform the angular function as

$$sS_{\ell m}^\xi(x) \equiv e^{\xi x} \left( \frac{1 - x}{2} \right)^{\frac{\alpha}{2}} \left( \frac{1 + x}{2} \right)^{\frac{\beta}{2}} sU_{\ell m}(x),$$  \hspace{1cm} (C.3)

and

$$sS_{\ell m}^\xi(x) \equiv e^{-\xi x} \left( \frac{1 - x}{2} \right)^{\frac{\alpha}{2}} \left( \frac{1 + x}{2} \right)^{\frac{\beta}{2}} sV_{\ell m}(x),$$  \hspace{1cm} (C.4)

where $\alpha = |m + s|$ and $\beta = |m - s|$. Then, Eq. (C.1) becomes

$$(1 - x^2) sU''_{\ell m}(x) + [\beta - \alpha - (2 + \alpha + \beta)x] sU'_{\ell m}(x)$$
$$+ \left[ sE_{\ell m}(\xi) - \frac{\alpha + \beta}{2} \left( \frac{\alpha + \beta}{2} + 1 \right) \right] sU_{\ell m}(x)$$
$$= \xi \left[ -2(1 - x^2) sU'_{\ell m}(x) + (\alpha + \beta + 2s + 2)x sU_{\ell m}(x) \right.$$
$$\left. - (\xi - \beta - \alpha) sU_{\ell m}(x) \right],$$  \hspace{1cm} (C.5)

and

$$(1 - x^2) sV''_{\ell m}(x) + [\beta - \alpha - (2 + \alpha + \beta)x] sV'_{\ell m}(x)$$
$$+ \left[ sE_{\ell m}(\xi) - \frac{\alpha + \beta}{2} \left( \frac{\alpha + \beta}{2} + 1 \right) \right] sV_{\ell m}(x)$$
$$= \xi \left[ 2(1 - x^2) sV'_{\ell m}(x) - (\alpha + \beta + 2s + 2)x sV_{\ell m}(x) \right.$$
$$\left. - (\xi - \beta + \alpha) sV_{\ell m}(x) \right].$$  \hspace{1cm} (C.6)

From Eqs. (C.3) and (C.4), we find

$$sV_{\ell m}(x) = \exp(2\xi x) sU_{\ell m}(x).$$  \hspace{1cm} (C.7)

When $\xi = 0$, the right-hand sides of Eqs. (C.5) and (C.6) are zero, and they reduce to the differential equation satisfied by the Jacobi polynomials,

$$(1 - x^2) P_n^{(\alpha, \beta)}''(x) + [\beta - \alpha - (2 + \alpha + \beta)x] P_n^{(\alpha, \beta)'}(x)$$
$$+ n(n + \alpha + \beta + 1) P_n^{(\alpha, \beta)}(x) = 0.$$  \hspace{1cm} (C.8)

In this limit, the eigenvalue $sE_{\ell m}(\xi)$ in the equation (C.5) becomes $\ell(\ell + 1)$, where $n = \ell - (\alpha + \beta)/2 = \ell - \max(|m|, |s|)$. Here, the Jacobi polynomials are defined by the Rodrigue’s formula by

$$P_n^{(\alpha, \beta)}(x) := \frac{(-1)^n}{2^n n!} (1 - x)^{-\alpha} (1 + x)^{-\beta} \left( \frac{d}{dx} \right)^n \left[ (1 - x)^{\alpha + n} (1 + x)^{\beta + n} \right].$$  \hspace{1cm} (C.9)

Now, we expand $sU_{\ell m}(x)$ and $sV_{\ell m}(x)$ in a series of Jacobi polynomials:

$$sU_{\ell m}(x) = \sum_{n=0}^{\infty} sA_{\ell m}^{(n)}(\xi) P_n^{(\alpha, \beta)}(x),$$  \hspace{1cm} (C.10)

$$sV_{\ell m}(x) = \sum_{n=0}^{\infty} sB_{\ell m}^{(n)} P_n^{(\alpha, \beta)}(x).$$  \hspace{1cm} (C.11)
The expansion coefficients \( sA_{tm}^{(n)}(\xi) \) and \( sB_{tm}^{(n)}(\xi) \) satisfy the recurrence relations

\[
\begin{align*}
\alpha^{(0)} sA_{tm}^{(1)}(\xi) + \beta^{(0)} sA_{tm}^{(0)}(\xi) &= 0, \\
\alpha^{(n)} sA_{tm}^{(n+1)}(\xi) + \beta^{(n)} sA_{tm}^{(n)}(\xi) + \gamma^{(n)} sA_{tm}^{(n-1)}(\xi) &= 0, \quad (n \geq 1),
\end{align*}
\]

with

\[
\begin{align*}
\alpha^{(n)} &= \frac{4\xi(n + \alpha + 1)(n + \beta + 1)(n + (\alpha + \beta)/2 + 1 - s)}{(2n + \alpha + \beta + 2)(2n + \alpha + \beta + 3)}, \\
\beta^{(n)} &= sE_{lm}(\xi) + \xi^2 - \left(n + \frac{\alpha + \beta}{2}\right) \left(n + \frac{\alpha + \beta}{2} + 1\right)
\quad + \frac{2\xi s(\alpha - \beta)(\alpha + \beta)}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)}, \\
\gamma^{(n)} &= -\frac{4\xi n(n + \alpha + \beta)(n + (\alpha + \beta)/2 + s)}{(2n + \alpha + \beta - 1)(2n + \alpha + \beta)},
\end{align*}
\]

and

\[
\begin{align*}
\tilde{\alpha}^{(0)} sB_{tm}^{(1)}(\xi) + \tilde{\beta}^{(0)} sB_{tm}^{(0)}(\xi) &= 0, \\
\tilde{\alpha}^{(n)} sB_{tm}^{(n+1)}(\xi) + \tilde{\beta}^{(n)} sB_{tm}^{(n)}(\xi) + \tilde{\gamma}^{(n)} sB_{tm}^{(n-1)}(\xi) &= 0, \quad (n \geq 1),
\end{align*}
\]

with

\[
\begin{align*}
\tilde{\alpha}^{(n)} &= -\frac{4\xi(n + \alpha + 1)(n + \beta + 1)(n + (\alpha + \beta)/2 + 1 + s)}{(2n + \alpha + \beta + 2)(2n + \alpha + \beta + 3)}, \\
\tilde{\beta}^{(n)} &= sE_{lm}(\xi) + \xi^2 - \left(n + \frac{\alpha + \beta}{2}\right) \left(n + \frac{\alpha + \beta}{2} + 1\right)
\quad + \frac{2\xi s(\alpha - \beta)(\alpha + \beta)}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)}, \\
\tilde{\gamma}^{(n)} &= \frac{4\xi n(n + \alpha + \beta)(n + (\alpha + \beta)/2 - s)}{(2n + \alpha + \beta - 1)(2n + \alpha + \beta)}.
\end{align*}
\]

The eigenvalues \( sE_{lm}(\xi) \) are determined in a way similar to the renormalized angular momentum \( \nu \). The three-term recurrence relation Eq. (C.13) has two independent solutions, which respectively behave for large \( n \) as

\[
\begin{align*}
A_{(1)}^{(n)} &\sim \frac{(\text{const.}) (-\xi)^n}{\Gamma(n + (\alpha + \beta + 3)/2 - s)}, \\
A_{(2)}^{(n)} &\sim (\text{const.}) \xi^n \Gamma(n + (\alpha + \beta + 1)/2 + s).
\end{align*}
\]

The first one, \( A_{(1)}^{(n)} \), is the minimal solution, and the second one, \( A_{(2)}^{(n)} \), is a dominant solution, since \( \lim_{n \to \infty} A_{(1)}^{(n)}/A_{(2)}^{(n)} = 0 \). In the case of the dominant solution these coefficients \( A_{(2)}^{(n)} \) increase with \( n \), and the series (C.10) diverges for all values of \( x \). In the case of the minimal solution this series converges. Thus, we have to choose
For a general \( sE_{\ell m}(\xi) \), \( A_{(1)}^{(n)} \) does not satisfy the relation (C.12). Hence, the requirement to satisfy this condition determines the discrete eigen values \( sE_{\ell m}(\xi) \).

As a practical way to obtain \( A_{(1)}^{(n)} \) as well as \( sE_{\ell m}(\xi) \), we introduce

\[
R_n = \frac{A_{(1)}^{(n)} - A_{(1)}^{(n-1)}}{A_{(1)}^{(n-1)}}, \quad L_n = \frac{A_{(1)}^{(n)} + A_{(1)}^{(n+1)}}{A_{(1)}^{(n+1)}}.
\]

The ratio \( R_n \) can be expressed as a continued fraction,

\[
R_n = -\frac{\gamma^{(n)}}{\beta^{(n)} + \alpha^{(n)} R_{n+1}} = -\frac{\gamma^{(n)}}{\beta^{(n)} - \frac{\alpha^{(n)} \gamma^{(n+1)}}{\beta^{(n+1)} - \frac{\alpha^{(n+1)} \gamma^{(n+2)}}{\beta^{(n+2)} - \cdots}}}
\]

We can also express \( L_n \) in a similar way as

\[
L_n = -\frac{\alpha^{(n)}}{\beta^{(n)} + \gamma^{(n)} L_{n-1}} = -\frac{\alpha^{(n)}}{\beta^{(n)} - \frac{\alpha^{(n-1)} \gamma^{(n)}}{\beta^{(n-1)} - \frac{\alpha^{(n-2)} \gamma^{(n-1)}}{\beta^{(n-2)} - \cdots}}}
\]

This expressions for \( R_n \) and \( L_n \) are valid if the continued fraction (C.20) converge. (Notice that the last step of Eq. (C.21) is not a continued fraction, but just a rational function.) By using the properties of the three-term recurrence relations, it is proved that the continued fraction (C.20) converges as long as the eigenvalue \( sE_{\ell m}(\xi) \) is finite.

Dividing Eq. (C.13) by the expansion coefficients \( sA_{\ell m}^{(n)} \), we obtain

\[
\beta^{(n)} + \alpha^{(n)} R_{n+1} + \gamma^{(n)} L_{n-1} = 0.
\]

We replace \( R_{n+1} \) and \( L_{n-1} \) by Eqs. (C.20) and (C.21). Then we can determine the eigenvalue \( sE_{\ell m} \) as a root of Eq. (C.22). There are many roots, and the above equations for all value of \( n \) are equivalent. In practice, however, we truncate the continued fractions at finite lengths. In this case the most efficient way is to choose the equation with \( n = n_\ell := \ell - (\alpha + \beta)/2 \). With this choice all terms in Eq. (C.22) become \( O(\xi^2) \), and the length of the continued fractions that we must keep to achieve a given accuracy goal is the shortest.

As was done in Fujita and Tagoshi’s paper, in general, we can adopt Brent’s algorithm\(^{28}\) in order to determine \( sE_{\ell m}(\xi) \). However, when \( |\xi| \) is not large, we can derive an analytic expression for \( sE_{\ell m}(\xi) \). The result is

\[
sE_{\ell m}(\xi) = \ell(\ell + 1) - \frac{2s^2 m}{\ell(\ell + 1)} \xi + [H(\ell + 1) - H(\ell) - 1] \xi^2 + O(\xi^3),
\]

with

\[
H(\ell) = \frac{2(\ell^2 - m^2)(\ell^2 - s^2)^2}{(2\ell - 1)\ell^3(2\ell + 1)}.
\]

\[
(C.24)
\]
Because the Jacobi polynomials are orthogonal, we have

\[ \alpha^{(n)} / (1) \gamma^{(n)} / (1) = \beta^{(n+1)} / (1) \gamma^{(n+2)} / (1) \cdots, \quad (C.25) \]

Then, Eq. (C.28) reduces to

\[ B^{(n)} / (1) \frac{B^{(n-1)} / (1)}{B^{(n+1)} / (1)} = \frac{\tilde{\alpha}^{(n)} / (1) \tilde{\gamma}^{(n)} / (1) \tilde{\alpha}^{(n-1)} / (1) \tilde{\gamma}^{(n-1)} / (1) \cdots \tilde{\alpha}^{(1)} / (1) \tilde{\gamma}^{(1)} / (1)}{\tilde{\beta}^{(n)} / (1) \tilde{\gamma}^{(n+2)} / (1) \cdots \tilde{\beta}^{(n+2)} / (1) \tilde{\gamma}^{(n+5)} / (1)}. \quad (C.26) \]

From these equations, we can determine the ratios of all coefficients, \( B^{(n)} / (1) / B^{(n-1)} / (1) \).

Now, we determine the values of the two coefficients \( A^{(n)} / (1) / (n) \) and \( B^{(n)} / (1) \) that determines the overall normalization. Since Eq. (C.27) must hold for any value of \( x \), we can set \( x = 1 \) in it to obtain

\[ s B^{(n)} / (1) / (n) = \sum_{n=0}^{\infty} s B^{(n)} / (1) / (n) \Gamma(n + \alpha + 1) \Gamma(n + 1) \Gamma(\alpha + 1) \]

\[ = \exp(2\xi) s A^{(n)} / (1) / (n) \sum_{n=0}^{\infty} s A^{(n)} / (1) / (n) \Gamma(n + \alpha + 1) \Gamma(n + 1) \Gamma(\alpha + 1). \quad (C.27) \]

On the other hand, from the normalization condition (C.22), we find

\[ \int_{-1}^{1} dx \left( \frac{1 - x}{2} \right)^{\alpha} \left( \frac{1 + x}{2} \right)^{\beta} \sum_{n=0}^{\infty} s A^{(n)} / (1) / (n) / (n) P_{n_1 / (2)}^{(\alpha, \beta)}(x) \sum_{n=2}^{\infty} s B^{(n)} / (1) / (n) P_{n_2 / (2)}^{(\alpha, \beta)}(x) = 1. \quad (C.28) \]

Because the Jacobi polynomials are orthogonal, we have

\[ \int_{-1}^{1} dx \left( \frac{1 - x}{2} \right)^{\alpha} \left( \frac{1 + x}{2} \right)^{\beta} P_{n_1 / (2)}^{(\alpha, \beta)}(x) P_{n_2 / (2)}^{(\alpha, \beta)}(x) \]

\[ = \frac{2 \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1) \delta_{n_1, n_2}}{(2n + \alpha + \beta + 1) \Gamma(n + 1) \Gamma(n + \alpha + \beta + 1)}. \quad (C.29) \]

Then, Eq. (C.28) reduces to

\[ \sum_{n=0}^{\infty} \left[ s A^{(n)} / (1) / (n) / (n) \right] \left[ s B^{(n)} / (1) / (n) / (n) \right] \frac{2 \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{(2n + \alpha + \beta + 1) \Gamma(n + 1) \Gamma(n + \alpha + \beta + 1)} \]

\[ = \frac{1}{s A^{(n)} / (1) / (n) / (n) \cdot s B^{(n)} / (1) / (n) / (n)}. \quad (C.30) \]
Combining Eqs. (C.27) and (C.30), we can determine the squares of $s^{A(n)}_{\ell m}$ and $s^{B(n)}_{\ell m}$. Finally, we fix the signatures of $s^{A(n)}_{\ell m}$ and $s^{B(n)}_{\ell m}$ so that $s^{S}_{\ell m}(x)$ reduces to the spin-weighted spherical harmonics in the limit $\xi \to 0$.

References

1) K. Danzman et al, LISA – Laser Interferometer Space Antenna, Pre-Phase A Report, Max-Planck-Institute für Quantenoptic, Report MPQ 233 (1998).
2) Y. Mino, M. Sasaki, M. Shibata, H. Tagoshi and T. Tanaka, Prog. Theor. Phys. Suppl. 128, 1 (1997) [arXiv:gr-qc/9712057].
3) A. Ori, Phys. Lett. A 202, 347 (1995) [arXiv:gr-qc/9507048].
4) A. Ori, Phys. Rev. D 55, 3444 (1997).
5) Y. Mino, M. Sasaki and T. Tanaka, Phys. Rev. D 55, 3457 (1997) [arXiv:gr-qc/9606018].
6) T. C. Quinn and R. M. Wald, Phys. Rev. D 56, 3381 (1997) [arXiv:gr-qc/9610053].
7) S. Detweiler and B. F. Whiting, Phys. Rev. D 67, 024025 (2003) [arXiv:gr-qc/0202086].
8) D.V. Gal’tsov, J. Phys. A 15, 3737 (1982); Tanaka’s note entitled ‘Radiation Reaction by Gravitational Wave Emission’.
9) P. A. M. Dirac, Proc. Roy. Soc. Lond. A 167, 148 (1938).
10) Y. Mino, Phys. Rev. D 67, 084027 (2003) [arXiv:gr-qc/0302075].
11) N. Sago, T. Tanaka, W. Hikida and H. Nakano, Prog. Theor. Phys. 114, 509 (2005) [arXiv:gr-qc/0506092].
12) D. Kennefick and A. Ori, Phys. Rev. D 53, 4319 (1996) [arXiv:gr-qc/9512018].
13) S. A. Hughes, Phys. Rev. D 61, 084004 (2000) [Erratum-ibid. D 63, 049902 (2001)] [arXiv:gr-qc/9910091].
14) K. Ganz, W. Hikida, H. Nakano, N. Sago and T. Tanaka, in preparation.
15) S. Drasco and S. A. Hughes, arXiv:gr-qc/0509101.
16) R. Fujita and H. Tagoshi, Prog. Theor. Phys. 112, 415 (2004) [arXiv:gr-qc/0410018].
17) Y. Mino, Prog. Theor. Phys. 113, 733 (2005) [arXiv:gr-qc/0506003].
18) T. Tanaka, arXiv:gr-qc/0508114.
19) A. Pound, E. Poisson and B. G. Nickel, arXiv:gr-qc/0509122.
20) W. Hikida, S. Sanjay, H. Nakano, N. Sago, M. Sasaki and T. Tanaka, in preparation.
21) S. A. Teukolsky, Astrophys. J. 185, 635 (1973).
22) P. L. Chrzanowski, Phys. Rev. D 11, 2042 (1975).
23) R. M. Wald, Phys. Rev. Lett. 41, 203 (1978).
24) S. Mano and E. Takasugi, Prog. Theor. Phys. 97, 213 (1997) [arXiv:gr-qc/9611014].
25) S. Mano, H. Suzuki and E. Takasugi, Prog. Theor. Phys. 95, 1079 (1996) [arXiv:gr-qc/9603020].
26) M. Sasaki and H. Tagoshi, Living Rev. Rel. (2003), to appear. arXiv:gr-qc/0306120.
27) E. D. Fackerell and R. G. Crossman, J. Math. Phys. 9, 1849 (1977).
28) W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery, Numerical Recipes in C (Cambridge University Press, Cambridge, England, 1992).