The noise barrier and the large signal bias of the Lasso and other convex estimators

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Abstract: Convex estimators such as the Lasso, the matrix Lasso and the group Lasso have been studied extensively in the last two decades, demonstrating great success in both theory and practice. This paper introduces two quantities, the noise barrier and the large scale bias, that provides novel insights on the performance of these convex regularized estimators.

In sparse linear regression, it is now well understood that the Lasso achieves fast prediction rates, provided that the correlations of the design satisfy some Restricted Eigenvalue or Compatibility condition, and provided that the tuning parameter is at least larger than some threshold. Using the two quantities introduced in the paper, we show that the compatibility condition on the design matrix is actually unavoidable to achieve fast prediction rates with the Lasso. In other words, the $\ell_1$-regularized Lasso must incur a loss due to the correlations of the design matrix, measured in terms of the compatibility constant, and any positive tuning parameter. We also characterize sharp phase transitions for the tuning parameter of the Lasso around a critical threshold dependent on the sparsity $k$. If $\lambda$ is equal to or larger than this critical threshold, the Lasso is minimax over $k$-sparse target vectors. If $\lambda$ is equal or smaller than this critical threshold, the Lasso incurs a loss of order $\sigma \sqrt{k}$, even if the target vector has far fewer than $k$ nonzero coefficients. This sharp phase transition highlights a minimal penalty phenomenon similar to that observed in model selection with $\ell_0$ regularization in Birgé and Massart [9].

Remarkably, the lower bounds obtained in the paper also apply to random, data-driven tuning parameters. Additionally, the results extend to convex penalties beyond the Lasso.

1. Introduction

We study the linear regression problem

$$y = X\beta^* + \varepsilon,$$

(1.1)

where one observes $y \in \mathbb{R}^n$, the design matrix $X \in \mathbb{R}^{n \times p}$ is known and $\varepsilon$ is a noise random vector independent of $X$ with $\mathbb{E}[\varepsilon] = 0$. The prediction error of an estimator $\hat{\beta}$ is given by

$$\|\hat{y} - X\beta^*\|, \quad \hat{y} := X\hat{\beta}$$

where $\| \cdot \|$ is the Euclidean norm in $\mathbb{R}^n$. This paper studies the prediction error of convex regularized estimators, that is, estimators $\hat{\beta}$ that solve the minimization problem

$$\hat{\beta} \in \arg\min_{\beta \in \mathbb{R}^p} \|X\beta - y\|^2 + 2h(\beta),$$

(1.2)
where $h : \mathbb{R}^p \to [0, +\infty)$ a semi-norm, i.e., $h$ is non-negative, $h$ satisfies the triangle inequality and $h(a\beta) = |a|h(\beta)$ for any $a \in \mathbb{R}, \beta \in \mathbb{R}^p$.

Bias and variance are well defined for linear estimators of the form $\hat{y} = Ay$ where $A \in \mathbb{R}^{n \times n}$ is a given matrix. The bias and variance of the linear estimator $\hat{y} = Ay$ are defined by
\begin{equation}
    b(A) = \| (A - I_{n \times n}) X \beta^* \|, \quad v(A) = \mathbb{E}[\|A\epsilon\|^2] \tag{1.3}
\end{equation}
and the squared prediction error of such linear estimator satisfies
\[ \mathbb{E}\|\hat{y} - X\beta^*\|^2 = b(A)^2 + v(A). \]

For linear estimators, the two quantities (1.3) characterize the squared prediction error of the linear estimator $Ay$ and these quantities can be easily interpreted in terms of the singular values of $A$. The above bias-variance decomposition and the explicit formulae (1.3) are particularly insightful to design linear estimators, as well as to understand the role of tuning parameters.

However, for nonlinear estimator such as (1.2), there is no clear generalization of the bias and the variance. It is possible to define the bias of $\hat{\beta}$ and its variance as $b = \|X(\mathbb{E}[\hat{\beta}] - \beta^*)\|$ and $v = \mathbb{E}[\|X(\hat{\beta} - \mathbb{E}[\hat{\beta}])\|^2]$. These quantities indeed satisfy $b^2 + v = \mathbb{E}[\|X(\hat{\beta} - \beta^*)\|^2]$, but $b$ and $v$ are not interpretable because of the non-linearity of $\hat{\beta}$. If the penalty $h$ is of the form $\lambda N(\cdot)$ for some norm $N$, it is not even clear whether the quantities $b$ and $v$ are monotonic with respect to the tuning parameter $\lambda$ or with respect to the noise level. These quantities appear to be of no help to study the prediction performance of $\hat{\beta}$ or to choose tuning parameters. The first goal of the present paper is to introduce two quantities, namely the noise barrier and the large signal bias, that clearly describes the behavior the prediction error of nonlinear estimators of the form (1.2), that are easily interpretable and that can be used to choose tuning parameters.

For linear estimators such as the Ridge regressor, insights on how to choose tuning parameters can be obtained by balancing the bias and the variance, i.e., the quantities $b(A)$ and $v(A)$ defined above. To our knowledge, such bias/variance trade-off is not yet understood for nonlinear estimators of the form (1.2) such as the Lasso in sparse linear regression. A goal of the paper is to fill this gap.

Although our main results are general and apply to estimators (1.2) for any semi-norm $h$, we will provide detailed consequences of these general results to the Lasso, that is, the estimator (1.2) where the penalty function is chosen as
\begin{equation}
    h(\cdot) = \sqrt{n}\lambda \cdot 1, \tag{1.4}
\end{equation}
where $\lambda \geq 0$ is a tuning parameter. The Lasso has been extensively studied in the literature since its introduction in [41], see [27, 39, 7, 28, 28, 33]. These works demonstrate that the Lasso with properly chosen tuning parameter enjoys small prediction and small estimation error, even in the high dimensional regime where $p \gg n$. To highlight the success of the Lasso in the high dimensional regime, consider first a low-dimensional setting where $n \gg p$ and $X$ has rank $p$. Then the least-squares estimator $\hat{\beta}^*$ satisfies
\[ \sigma \sqrt{p - 1} \leq \mathbb{E}\|X(\hat{\beta}^* - \beta^*)\| \leq \sigma \sqrt{p}, \]
for standard normal noise $\varepsilon \sim N(0, \sigma^2 I_{n \times n})$ and deterministic design matrix $X$. Here, $p$ is the model size, so that the above display can be rewritten informally as

$$\mathbb{E}||X(\hat{\beta} - \beta^*)|| \approx \sigma \sqrt{\text{model size}}.$$  

In the high-dimensional regime where $p \gg n$, it is now well understood that the Lasso estimator $\hat{\beta}$ with tuning parameter $\lambda = (1 + \gamma)\sigma \sqrt{2 \log p}$ for some $\gamma > 0$ satisfies

$$\mathbb{E}||X(\hat{\beta} - \beta^*)|| \leq C(X^\top X/n, T) \sigma \lambda \sqrt{||\beta^*||_0},$$  

where $C(X^\top X/n, T)$ is a constant that depends on the correlations of the design matrix and the support $T$ of $\beta^*$. Examples of such constants $C(X^\top X/n, T)$ will be given in the following subsections. In the above display, $||\beta^*||_0 = |T|$ is the number of nonzero coefficients of $\beta^*$. Even though the ambient dimension $p$ is large compared to the number of observations, the Lasso enjoys a prediction error not larger than the square root of the model size up to logarithmic factors, where now the model size is given by $||\beta^*||_0$. To put it differently, the Lasso operates the following dimension reduction: If the tuning parameter is large enough, then the Lasso acts as if the ambient dimension $p$ was reduced to $||\beta^*||_0$.

There is an extensive literature on bounds of the form (1.5) for the Lasso, see for instance [41, 27, 46, 13, 39, 7, 18, 28, 28, 33, 3, 1] for a non-exhaustive list. Despite this extensive literature, some open questions remain on the statistical performance of the Lasso. We detail such questions in the following paragraphs. They are the main motivation behind the techniques and results of the paper and behind the introduction of the noise barrier and the large signal bias defined in Sections 2.1 and 2.2 below.

**On the performance of the Lasso with small tuning parameters**  

The quantity

$$\lambda^* = \sigma \sqrt{2 \log p}$$  

is often referred to as the universal tuning parameter. Inequalities of the form (1.5) hold for tuning parameter $\lambda$ strictly larger than $\lambda^*$ [46, 7, 13]. If the sparsity $k := ||\beta||_0$ satisfies $k \geq 2$, recent results have shown that inequality (1.5) holds for tuning parameters slightly larger than

$$\sigma \sqrt{2 \log (p/k)},$$  

which leads to better estimation and prediction performance [29, 4] than the universal parameter (1.6).

However, little is known about the performance of the Lasso with tuning parameter smaller the thresholds (1.6) and (1.7). Although it is known from practice that the prediction performance of the Lasso can significantly deteriorate if the tuning parameter is too small, theoretical results to explain this phenomenon are lacking. A question of particular interest to identify the smallest tuning parameter that grants inequalities of the form

$$\mathbb{E}||X(\hat{\beta} - \beta^*)|| \lesssim \lambda \sqrt{\text{model size}},$$  

where $\lesssim$ is an inequality up to multiplicative constant, and the model size is given by $||\beta^*||_0$. Also of interest is to quantify how large becomes the risk $\mathbb{E}||X(\hat{\beta} - \beta^*)||$ for small tuning parameters.
Minimal penalty level for the Lasso  In the context of model selection [8], Birgé and Massart [9] characterized a minimal penalty. In our linear regression setting, one of the result of [9] can be summarized as follows. Let \( \mathcal{M} \) be a set of union of possible models (each model in \( \mathcal{M} \) represents a set of covariates). Then there exists a penalty of the form

\[
\text{pen}(\beta) = 2\sigma^2\|\beta\|_0 A(\|\beta\|_0)
\]

for some function \( A(\cdot) \) such that the estimator

\[
\hat{\beta}^{MS} \in \arg\min_{\beta \in \mathbb{R}^p \text{ with support in } \mathcal{M}} \|X\beta - y\|^2 + \lambda \text{ pen}(\beta)
\]

enjoys to optimal risk bounds and oracle inequalities provided that \( \lambda = (1 + \eta) \) for some arbitrarily small constant \( \eta > 0 \). On the other hand, the same optimization problem leads to disastrous results if \( \lambda = (1 - \eta) \), cf. the discussion next to equations (28)-(29) in [9]. To our knowledge, prior literature has not yet characterized such sharp transition around a minimal penalty level for \( \ell_1 \)-penalization in sparse linear regression. We will see in Section 4 that the minimal penalty level in \( \ell_1 \)-penalization for \( k \)-sparse target vectors is given by (4.1).

Necessary conditions on the design matrix for fast rates of convergence  If the Lasso satisfies inequality (1.5), then it achieves a fast rate of convergence in the sense that the prediction rate corresponds to a parametric rate with \( \|\beta^\dagger\|_0 \) parameters, up to a logarithmic factor. All existing results on fast rates of convergence for the Lasso require some assumption on the design matrix. Early works on fast rates of convergence of \( \ell_1 \)-regularized procedures [14, 46, 45] assumed that minimal sparse eigenvalue and maximal sparse eigenvalue of the Gram matrix \( X^\top X/n \) are bounded away from zero and infinity, respectively. These conditions were later weakened [7], showing that a Restricted Eigenvalue (RE) condition on the design matrix is sufficient to grant fast rates of convergence to the Lasso and to the Dantzig selector. The RE condition is closely related to having the minimal sparse eigenvalue of the Gram matrix \( X^\top X/n \) bounded away from zero [30, Lemma 2.7], but remarkably, the RE condition does not assume that the maximal sparse eigenvalue of the Gram matrix \( X^\top X/n \) is bounded. Finally, [13] proposed the compatibility condition, which is weaker than the RE condition. Variants of the compatibility condition were later proposed in [6, 20]. The following definition of the compatibility condition is from [20]. It is slightly different than the original definition of [13]. Given a subset of covariates \( T \subset \{1, \ldots, p\} \) and a constant \( c_0 \geq 1 \), define the compatibility constant by

\[
\phi(c_0, T) := \inf_{u \in \mathbb{R}^p : \|u_T\|_1 < c_0} \|Xu\| \sqrt{|T| / \sqrt{n(\|u_T\|_1 - (1/c_0)\|u_T\|_1)}},
\]

(1.8)

where for any \( u \in \mathbb{R}^p \) and subset \( S \subset \{1, \ldots, p\} \), the vector \( u_S \in \mathbb{R}^p \) is defined by \((u_S)_j = u_j \) if \( j \in S \) and \((u_S)_j = 0 \) if \( j \notin S \). For \( c_0 = 1 \), the constant \( \phi(1, T) \) is also considered in [6]. We say that the compatibility condition holds if \( \phi(c_0, T) > 0 \). If the target vector is supported on \( T \), then the Lasso estimator \( \hat{\beta} \) with tuning parameter \( \lambda = (1 + \gamma)\sigma \sqrt{2 \log p} \) for some \( \gamma > 0 \) satisfies

\[
E\|X(\hat{\beta} - \beta^\dagger)\| \leq \frac{\lambda}{\phi(c_0, T)} \sqrt{|T|} + \sigma \sqrt{|T|} + 4\sigma
\]

(1.9)
for $c_0 = 1 + 1/\gamma$. Although we have stated the above display in expectation for brevity, such results were initially obtained with probability at least $1 - \delta$ for the tuning parameter $(1 + \gamma)\sigma \sqrt{2 \log(p/\delta)}$, where $\delta$ is a predefined confidence level [7], see also the books [13, 24]. It is now understood that the tuning parameter $(1 + \gamma)\sigma \sqrt{2 \log(p)}$ enjoys such prediction guarantee for any confidence level [4], and that this feature is shared by all convex penalized least-squares estimators [1]. Results in expectation such as (1.9) are a consequence of the techniques presented in [1, 4]. We refer the reader to Proposition 3.2 in [4], which implies that for any $\delta \in (0,1)$, inequality

$$\|X(\hat{\beta} - \beta^*)\| \leq \frac{\lambda}{\phi(c_0, T)} \sqrt{|T| + \sigma \sqrt{|T|} + \sqrt{2 \log(1/\delta)}} + 2.8\sigma,$$

holds with probability at least $1 - \delta$. Inequality (1.9) is then obtained by integration. Let us mention that results of the form (1.9) are also available in the form of oracle inequalities [7, 4].

In this extensive literature, all results that feature fast rates of convergence for the Lasso require one of the condition mentioned above on the design matrix. The RE and compatibility conditions are appealing for several reasons. First, large classes of random matrices are known to satisfy these conditions with high probability, see for instance [30] for recent results on the "the small ball" method, or [4, Section 8] [2] for a survey of some existing results. Second, their introduction has greatly simplified the proofs that the Lasso achieves fast rates of convergence. But, to our knowledge, there is no evidence that the above conditions are necessary to obtain fast rates of convergence for the Lasso. It is still unknown whether these conditions are necessary, or whether they are artifacts of the currently available proofs. The following heuristic argument suggests that a minimal sparse eigenvalue condition is unavoidable, at least to obtain fast rates of estimation. Given oracle knowledge of the true support of $\beta^*$, one may consider the oracle least-squares estimator on the support $T$ of $\beta^*$, which has distribution $\hat{\beta}_{\text{oracle}} \sim N(\beta^*, \sigma^2(X_T^T X_T)^{-1})$ if $\varepsilon \sim N(0, \sigma^2 I_{n \times n})$ and $X$ is deterministic. Here, $X_T$ denotes the restriction of the design matrix to the support $T$. Then

$$\mathbb{E}\|\hat{\beta}_{\text{oracle}} - \beta^*\|^2 = \frac{\sigma^2}{n} \sum_{j=1}^{|T|} \frac{1}{\sigma_j},$$

where $\sigma_1, ..., \sigma_{|T|}$ are the eigenvalues of $\frac{1}{n} X_T^T X_T$. Hence, the estimation error diverges as the minimal eigenvalue of $\frac{1}{n} X_T^T X_T$ goes to 0. This suggests that, to achieve fast rates of estimation, the minimal sparse eigenvalue must be bounded away from 0. A counter argument is that this heuristic only applies to the estimation error, not the prediction error $\|X(\hat{\beta} - \beta^*)\|$. Also, this heuristic applies to the oracle least-squares but not to the Lasso.

Experiments suggest that the prediction performance of the Lasso deteriorates in the presence of correlations in the design matrix, but few theoretical results explain this empirical observation. Notable exceptions include [18], [20, Section 4] and [47]: These works exhibit specific design matrices $X$ for which the Lasso cannot achieve fast rates of convergence for prediction, even if the sparsity is constant. However, these results only apply to specific design matrices. One of the goal of the paper is to quantify,
for any design matrix $X$, how the correlations impact the prediction performance of the Lasso.

**Organisation of the paper**

Let us summarize some important questions raised in the above introduction.

1. How to generalize bias and variance to convex penalized estimators (1.2) that are nonlinear? If these quantities can be generalized to nonlinear estimators such as the Lasso in sparse linear regression, how is the choice of tuning parameters related to a bias/variance trade-off?
2. How large is the prediction error of the Lasso when the tuning parameter is smaller than the thresholds (1.6) and (1.7)? Does $\ell_1$ regularization present a minimal penalty phenomenon such as that observed in $\ell_0$ regularization in [9]?
3. What are necessary conditions on the design matrix to obtain fast rates of convergence for the Lasso? The RE and compatibility conditions are known to be sufficient, but are they necessary?
4. Is it possible to quantify, for a given design matrix, how the correlations impact the prediction performance of the Lasso?

Sections 2.1 and 2.2 define two quantities, the noise barrier and the large signal bias, that will be useful to describe the performance of convex regularized estimators of the form (1.2). Section 3 establishes that, due to the large signal bias, the compatibility condition is necessary to achieve fast prediction rates. Section 4 studies the performance of the Lasso estimator for tuning parameters smaller than (1.6) and (1.7). In particular, Section 4 describe a phase transition and a bias/variance trade-off around a critical tuning parameter smaller than (1.7). In Section 5, we consider data-driven tuning parameters and show that the lower bounds induced by the large scale bias and the noise barrier also hold for estimators (1.2) with a random, possibly data-driven choice of tuning parameter. Finally, Section 6 extends some results on the Lasso to nuclear norm penalization in low-rank matrix estimation.

**Notation**

We use $\| \cdot \|$ to denote the Euclidean norm in $\mathbb{R}^n$ or $\mathbb{R}^p$. The $\ell_1$-norm of a vector is denoted by $\| \cdot \|_1$ and the matrix operator norm is denoted by $\| \cdot \|_{op}$. The number of nonzero coefficients of $\beta \in \mathbb{R}^p$ is denoted by $\| \beta \|_0$. The square identity matrices of size $n$ and $p$ are denoted by $I_{p \times p}$ and $I_{n \times n}$, and $(e_1, ..., e_p)$ is the canonical basis in $\mathbb{R}^p$.

Throughout the paper, $[p] = \{1, ..., p\}$ and $T \subset [p]$ denotes a subset of covariates. We will often take $T = \{j \in \mathbb{R}^p : \beta^*_j \neq 0\}$ to be the support of $\beta^*$. If $u \in \mathbb{R}^p$, the vector $u_T$ is the restriction of $u$ to $T$ defined by $(u_T)_j = u_j$ if $j \in T$ and $(u_T)_j = 0$ if $j \notin T$.

Some of our results will be asymptotic. When describing an asymptotic result, we implicitly consider a sequence of regression problems indexed by some implicit integer $q \geq 0$. The problem parameters and random variables of the problem (for instance,
\((n, p, k, X, \beta^*, \hat{\beta})\) are implicitly indexed by \(q\) and we will specify asymptotic relations between these parameters, see for instance (4.4) below. When such asymptotic regime will be specified, the notation \(a \asymp b\) for deterministic quantities \(a, b\) means that \(a/b \to 1\), and \(o(1)\) denotes a deterministic sequence that converges to 0.

2. The noise barrier and the large signal bias

2.1. The noise barrier

Consider the linear model (1.1) and let \(\hat{\beta}\) be defined by (1.2). We define the noise barrier of the penalty \(h\) by
\[
\text{NB}(\varepsilon) := \sup_{u \in \mathbb{R}^p: \|Xu\| \leq 1} (\varepsilon^T Xu - h(u)).
\] (2.1)

**Proposition 2.1.** Assume that the penalty \(h\) is a semi-norm. The noise barrier enjoys the following properties.

- For any realization of the noise \(\varepsilon\) we have
  \[
  \text{NB}(\varepsilon) \leq \|X(\hat{\beta} - \beta^*)\|.
  \] (2.2)

- If \(X\beta^* = 0\) then (2.2) holds with equality.

- If the penalty function \(h\) is of the form \(h(\cdot) = \lambda N(\cdot)\) for some norm \(N(\cdot)\) and \(\lambda \geq 0\), then \(\text{NB}(\varepsilon)\) is non-increasing with respect to \(\lambda\).

The proof is given in Appendix A. The lower bound (2.2), which holds with probability 1, is equivalent to
\[
\forall u \in \mathbb{R}^p, \quad \varepsilon^T Xu - h(u) \leq \|Xu\|\|X(\hat{\beta} - \beta^*)\|.
\]

Intuitively, the noise barrier captures how well the penalty represses the noise vector \(\varepsilon\). If the penalty dominates the noise vector uniformly then the noise barrier is small. On the contrary, a weak penalty function (in the sense that for some \(u \in \mathbb{R}^p\), the quantity \(h(u)\) is too small compared to \(\varepsilon^T Xu\)) will induce a large prediction error because of (2.2).

The noise barrier for norm-penalized estimators shares similarities with the variance for linear estimators defined by \(v(A)\) in (1.3). In the absence of signal \((X\beta^* = 0)\), for norm-penalized estimators we have \(\mathbb{E}[\text{NB}(\varepsilon)^2] = \mathbb{E}[\|X(\hat{\beta} - \beta^*)\|^2]\), while for linear estimators \(v(A) = \mathbb{E}[\|X(\hat{\beta} - \beta^*)\|^2]\). The noise-barrier is non-increasing with respect to \(\lambda\) if \(h(\cdot) = \lambda N(\cdot)\) for some norm \(N\), and a similar monotonicity property holds for linear estimators such as Ridge regressors or cubic splines, given by
\[
\hat{\beta}^{lin} = \arg\min_{\beta \in \mathbb{R}^p} \|y - X\beta\|^2 + 2\lambda x^T K\beta,
\]
where \(K \in \mathbb{R}^{p \times p}\) is a positive semi-definite penalty matrix.

The noise barrier defined above depends on the noise vector and the penalty function, but not on the target vector \(\beta^*\). The next section defines a quantity that resembles the bias for linear estimators: it depends on \(\beta^*\) but not on the noise random vector \(\varepsilon\).
2.2. The large signal bias

We will study the linear model (1.1) in the high-dimensional setting where $n$ may be smaller than $p$. In the high-dimensional setting where $n > p$, the design matrix $X$ is not of rank $p$ and $\beta^*$ is possibly unidentifiable because there may exist multiple $\beta \in \mathbb{R}^p$ such that $X\beta = \mathbb{E}[y]$. Without loss of generality, we will assume throughout the rest of the paper that $\beta^*$ has minimal $h$-seminorm, in the sense that $\beta^*$ is a solution to the optimization problem

$$\min_{\beta \in \mathbb{R}^p: X\beta = X\beta^*} h(\beta).$$

(2.3)

The large signal bias of a vector $\beta^* \neq 0$ is defined as

$$\text{LSB}(\beta^*) := \sup_{\beta \in \mathbb{R}^p: X\beta \neq X\beta^*} h(\beta^*) - h(\beta) \quad \|X(\beta^* - \beta)\|$$

(2.4)

with the convention $\text{LSB}(\beta^*) = 0$ for $\beta^* = 0$.

**Proposition 2.2.** Assume that the penalty $h$ is a semi-norm. The large signal bias enjoys the following properties.

- For any $\beta^* \neq 0$ we have $h(\beta^*)/\|X\beta^*\| \leq \text{LSB}(\beta^*) < +\infty$ provided that $\beta^*$ is solution of the optimization problem (2.3).
- For any scalar $t \in \mathbb{R}$, $\text{LSB}(t\beta^*) = \text{LSB}(\beta^*)$.
- For any small $\gamma > 0$, if $X$ is deterministic and if the noise satisfies $\mathbb{E}[\varepsilon] = 0$, $\mathbb{E}[\|\varepsilon\|^2] < +\infty$ and if $\|X\beta^*\|$ is large enough then

$$\left(1 - \gamma\right) \text{LSB}(\beta^*) \leq \mathbb{E}[\|X(\hat{\beta} - \beta^*)\|].$$

(2.5)

- In the noiseless setting ($\varepsilon = 0$), then $\|X(\hat{\beta} - \beta^*)\| \leq \text{LSB}(\beta^*)$ for any $\beta^* \in \mathbb{R}^p$.
- If the penalty function $h$ is of the form $h(\cdot) = \lambda N(\cdot)$ for some norm $N(\cdot)$ and $\lambda \geq 0$, then $\text{LSB}(\beta^*)$ is increasing with respect to $\lambda$.

The proof is given in Appendix B. Note the lower bound (2.5) is tight in the noiseless setting.

Inequality (2.5) above requires that the signal strength $\|X\beta^*\|$ is large enough in the following sense. Given any direction $v \in \mathbb{R}^p$ with $\|Xv\| = 1$, inequality (2.5) holds for any target vector $\beta^* = tv$ with $t \geq t_0$ where $t_0$ is a quantity that depends on $v$, $\gamma$, $\mathbb{E}[\|\varepsilon\|^2]$ and $h$. That is, given any direction $v$, an arbitrarily small constant $\gamma > 0$, a penalty function $h$ and $\mathbb{E}[\|\varepsilon\|^2]$, it is possible to find a target vector $\beta^*$ positively proportional to $v$ such that (2.5) holds.

3. Compatibility conditions are necessary for fast prediction rates

This section explores some consequence of inequality (2.5). Consider the Lasso penalty (1.4) and the compatibility constant defined in (1.8). The next result shows that the compatibility constant is necessarily bounded from below if the Lasso estimator enjoys fast prediction rates over a given support.
Theorem 3.1. Let $X$ be a deterministic design matrix, let $T \subset [p]$ be any given support and assume that $\phi(1, T) > 0$. Consider the estimator (1.2) with penalty (1.4), for any $\lambda > 0$. Let $\gamma > 0$ be any arbitrarily small constant. Assume that the noise satisfies $E[\varepsilon] = 0$ and $E[||\varepsilon||^2] < +\infty$. Then there exists $\beta^* \in \mathbb{R}^p$ supported on $T$ such that

$$(1 - \gamma) \frac{\lambda |T|^{1/2}}{\phi(1, T)} \leq E[||X(\hat{\beta} - \beta^*)||].$$ \hspace{0.5cm} (3.1)

An equivalent statement of the previous theorem is as follows: If the Lasso estimator has a prediction error bounded from above by $C(T)\lambda |T|^{1/2}$ for some constant $C(T)$ uniformly over all target vectors $\beta^*$ supported on $T$, then

$$\phi(1, T)^{-1} \leq C(T).$$ \hspace{0.5cm} (3.2)

This is formalized in the following.

Theorem 3.2. Let $X$ be deterministic, let $T \subset [p]$ and assume that $\phi(1, T) > 0$. Assume that the noise satisfies $E[\varepsilon] = 0$ and $E[||\varepsilon||^2] < +\infty$. If the Lasso estimator (1.2) with penalty (1.4) satisfies

$$\sup_{\beta^* \in \mathbb{R}^p: \text{supp}(\beta^*) \subset T} E_{\beta^*}[||X(\hat{\beta} - \beta^*)||] \leq C(T)\lambda |T|^{1/2}$$

for some constant $C(T) > 0$ that may depend on $T$ but is independent of $\beta^*$, then $C(T)$ is bounded from below as in (3.2).

The above results are a direct consequence of the definition of the large signal bias and inequality (2.5). The formal proof of these results is given in Section 3 and a self-contained sketch of the proof is given in the present section.

Let us emphasize that, in the above theorems, the design matrix $X$ and the support $T$ are not specific: The above result applies to any $X$ and any support $T$ such that $\phi(1, T)$ is nonzero.

A sketch of the proof of (3.1) is as follows. Set $c_0 = 1$ and assume for simplicity that the infimum in (1.8) is attained at some $u \in \mathbb{R}^p$. By homogeneity, we may assume that $||Xu|| = 1$. Next, let $r > 0$ be a deterministic scalar that will be specified later, set $\beta^* = ru_T$ and $\beta = -ru_{T^c}$ so that, for every scalar $r > 0$ we have $\beta^* - \beta = ru$ and

$$\frac{h(\beta^*) - h(\beta)}{r} = \frac{h(\beta^*) - h(\beta)}{||X(\beta - \beta^*)||} = \frac{h(u_T) - h(u_{T^c})}{||Xu||} = \frac{\lambda \sqrt{|T|}}{\phi(1, T)}.$$ \hspace{0.5cm} (3.3)

Let $\hat{r} = ||X(\hat{\beta} - \beta^*)||$ for brevity, and notice that $r = ||X(\beta - \beta^*)||$. The optimality conditions of the optimization problem (1.2) for the Lasso implies that

$$\frac{\varepsilon^T X(\beta - \beta^*)}{r} + \frac{h(\beta^*) - h(\beta)}{r} \leq \frac{(X(\beta - \beta^*)^T X(\beta^* - \beta^*)}{r} + \frac{\varepsilon^T X(\hat{\beta} - \beta^*) + h(\beta^*) - h(\hat{\beta}) - \hat{r}^2}{r}.$$
Cauchy-Schwarz inequality. For the second term of the right hand side, using again the Cauchy-Schwarz inequality, the definition of \( \phi(1, T) \) and simple algebra, one gets

\[
\mathbb{E} \left[ \varepsilon^\top X(\hat{\beta} - \beta^*) + h(\hat{\beta}^*) - h(\hat{\beta}) - r^2 \right] \leq \mathbb{E} \left[ \|\varepsilon\|_2 \sqrt{|T|\lambda^2 - \ell^2} \right] \leq \mathbb{E} \left[ \|\varepsilon\|_2^2 / 2 \right] + |T|\lambda^2 / (2\phi(1, T)^2).
\]

The right hand side is independent of \( r > 0 \), so that \( \mathbb{E} \left[ \varepsilon^\top X(\hat{\beta} - \beta^*) + h(\hat{\beta}^*) - h(\hat{\beta}) - r^2 \right] \) can be made arbitrarily small for \( r > 0 \) large enough, which completes the proof.

Known upper bounds on the prediction error of the Lasso include the so-called “slow-rate” upper bound, of the form \( \|X(\hat{\beta} - \beta^*)\|^2 \leq 4\sqrt{n} \|\beta^*\|_1 \) with high probability, see [39, 43, 25, 20] for more precise statements. In Theorem 3.1 above, the target vector \( \beta^* \) has large amplitude, so that \( \|\beta^*\|_1 \) is large and the slow-rate upper bound is not favorable compared to the fast-rate bound (1.9). This also explains that the lower bound (3.1) is not in contradiction with the slow-rate upper bound.

Several conditions have been proposed to provide sufficient assumptions for fast prediction rates: the Restricted Isometry property \([7, 14, 15]\), the Sparse Riesz condition \([46]\), the Restricted Eigenvalue condition \([7]\), the Compatibility condition \([13]\), the Strong Restricted Convexity condition \([35]\) and the Compatibility Factor \([6, 20]\), to name a few. The two theorems above are of a different nature. They show that for any design matrix \( X \), the Lasso may achieve fast prediction rates only if \( \phi(1, T) \) is bounded away from 0. Hence, the compatibility condition with constant \( c_0 = 1 \), i.e., the fact that \( \phi(1, T) \) is bounded from 0, is necessary to obtain fast prediction rates over the support \( T \).

If the diagonal elements of \( \frac{1}{n}X^\top X \) are no larger than 1, i.e.,

\[ \max_{j=1, \ldots, p} (1/n)\|Xe_j\| \leq 1, \tag{3.4} \]

which corresponds to column normalization, then the compatibility constant \( \phi(1, T) \) is less than 1. To see this, consider a random vector in \( \{-1, 1\}^p \) with iid Rademacher coordinates and let \( Z \) be the restriction of this random vector to the support \( T \) so that \( \|Z\|_1 = \|T\| \). Then by independence of the coordinates,

\[ \mathbb{E}_Z \left[ \frac{|T|\|XZ\|_2^2}{\sqrt{n}\|Z\|_1} \right] \leq 1. \]

This proves that the compatibility constant \( \phi(c, T) \) is less than 1 for any \( c > 0 \), provided that the columns are normalized as in (3.4). For orthogonal designs or equivalently the Gaussian sequence model, we have \( \phi(1, T) = 1 \). As one moves away from orthogonal design, the compatibility constant \( \phi(1, T) \) decreases away from 1 and the lower bound (3.1) becomes larger. This shows that the performance of the Lasso is worse than that of soft-thresholding in the sequence model, which is of order \( \lambda \sqrt{|T|} \). So there is always a price to pay for correlations in non-orthogonal design matrices compared to orthogonal designs.

Lower bounds similar to (3.1) exist in the literature \([7, 33, 5]\), although these results do not yield the same conclusions as the above theorems. Namely, \([7, (B.3)]\), \([33, \text{Theorem 7.1}]\) and \([5, (A.1)]\) state that the lower bound

\[ \frac{\lambda\sqrt{n}}{2\Phi_{\text{max}}} \leq \|X(\hat{\beta} - \beta^*)\| \quad \text{holds whenever} \quad \lambda > 2\|\varepsilon^\top X\|_{\infty} / \sqrt{n}. \tag{3.5} \]
where $\Phi_{\text{max}}$ is a *maximal* sparse eigenvalue and $\hat{s}$ is the sparsity of the Lasso. These papers also provide assumptions under which $\hat{s}$ is of the same order as $|T|$, the support of $\beta^*$, so that results such as (3.5) resemble the above theorems. However, since $\Phi_{\text{max}}$ is a maximal sparse eigenvalue, it is greater than 1 if the normalization (3.4) holds with equality. The left hand side of (3.5) is thus smaller than $\lambda \sqrt{\hat{s}}$ which is the performance of soft-thresholding in the sequence model. Furthermore, as one moves away from orthogonal design, $\Phi_{\text{max}}$ increases and the lower bound (3.5) becomes weaker. In contrast, as one moves away from orthogonal design, the compatibility constant $\phi(1, T)$ decreases and the lower bound (3.1) becomes larger. Thus inequality (3.1) from Theorem 3.1 explains the behavior observed in practice where correlations in the design matrix deteriorate the performance of the Lasso. Finally, upper bounds on the risk of Lasso involve the Restricted Eigenvalue constant or the Compatibility constant, which resemble *minimal* sparse-eigenvalues. Thus, existing results that involve the maximal sparse-eigenvalue such as (3.5) do not match the known upper bounds.

The recent preprint [42], contemporaneous of the present paper, also investigates lower bounds on the Lasso involving the compatibility constant $\phi(1, T)$. The main result of [42] requires strong assumptions on the design matrix, the noise level and the tuning parameter: $\lambda$ must be large enough and the maximal entry of the design covariance matrix must be bounded and satisfy relations involving $\lambda, n$ and $p$. In contrast, Theorem 3.1 above makes no assumption on $\lambda$ or the design matrix: (3.1) applies to any tuning parameter $\lambda$, any noise distribution with finite second moment, any design $X$ and any support $T$.

If a lower bound holds for some noise distribution then it should also hold for heavier-tailed distributions, because intuitively, heavier-tails make the problem harder. Existing results cited above hold on an event of the form $\{ \|X^T \varepsilon\|_\infty 2/\sqrt{n} \leq \lambda \}$. This event is of large probability only for large enough $\lambda$ and light-tailed noise distributions. In contrast, an appealing feature of the lower bound (3.1) is that it holds for any tuning parameter and any centered noise distribution with finite second moment.

Information-theoretic lower bounds (see, e.g. [36] or [4, Section 7]) involve an exponential number of measures, each corresponding to a different support $T$ with $|T| = k$. The results above involve a single measure supported on any given support $T$. The lower bound (3.1) adapts to $T$ through the constant $\phi(1, T)$, while minimax lower bounds are not adaptive to a specific support.

Information-theoretic arguments lead to minimax lower bounds of the order of

$$(k \log(p/k))^{1/2} \phi_{\text{min}}(2k),$$

where $\phi_{\text{min}}(2k)$ is a lower sparse eigenvalue of order $2k$ and $k$ is the sparsity [4, Section 7]. Lower sparse eigenvalues become smaller as one moves away from orthogonal design. Hence, minimax lower bounds do not explain that the prediction performance deteriorates with correlations in the design, while (3.1) does. For the defense of information-theoretic lower bounds, they apply to any estimators. The lower bounds derived above are of a different nature and only apply to the Lasso.
3.1. On the signal strength condition: How large is large enough?

The large-scale bias lower bound (2.5) and its consequences given above in the current section requires the signal $\|X\beta^*\|$ to be large enough. This subsection shows that this signal strength requirement can be relaxed with reasonable assumptions on the noise vector provided that the components of $\beta^*$ are greater than $\lambda/\sqrt{n}$ and that the tuning parameter $\lambda$ is large enough.

**Proposition 3.3.** Assume that $\varepsilon^TX$ is symmetric, i.e., that $\varepsilon^TX$ and $-\varepsilon^TX$ have the same distribution. Let $\hat{\beta}$ be the Lasso estimator with penalty (1.4) and tuning parameter $\lambda > 0$. Assume that the $k$ nonzero coordinates of $\beta^*$ are separated from 0, i.e.,

$$
\min_{j=1,\ldots,p: |\beta^*_j| \neq 0} |\beta^*_j| \geq \lambda/\sqrt{n} \tag{3.6}
$$

and define $\bar{s} \in \{-1, 0, 1\}^p$ as the sign vector of $\beta^*$, so that $\beta^*$ and $\bar{s}$ both have sparsity $k$ and same sign on every nonzero coordinate. We also assume that the tuning parameter $\lambda$ is large enough such that for some constant $\gamma > 0$ we have

$$
P \left( \sup_{u \in \mathbb{R}^p} \frac{1}{\sqrt{n}} \lambda^X u + \lambda \|\beta^*\|_1 - \lambda \|\beta^* + u\|_1}{\|Xu\|} \leq (1 + \gamma) \lambda \sqrt{k} \right) \geq 5/6. \tag{3.7}
$$

Then with probability at least $1/3$ we have

$$
\lambda \sqrt{k}(1 + \nu/2)(1 - \sqrt{\nu}) \leq \|X(\hat{\beta} - \beta^*)\| \tag{3.8}
$$

where $\nu = 2 \max \left( \frac{\gamma}{\sqrt{n}} \|X\bar{s}\| - 1 \right)$. On the same event, if $\nu \in (0, 1/2)$ then

$$
\|X(\hat{\beta} - \beta^* + (\lambda/\sqrt{n})\bar{s})\|^2 \leq 40\sqrt{\nu} \|X(\hat{\beta} - \beta^*)\|^2.
$$

The proof is similar to that of the lower bound (4.12) and the above phenomenon is due to the bias of the Lasso on the coordinates of $\beta^*$ that satisfy (3.6). The proof is given in Appendix I.

The assumption (3.7) has been established under several assumptions to prove oracle inequalities or upper bound on the prediction error of the Lasso. Inequality (3.7) is often an intermediate step to derive such upper bounds, since the KKT conditions of the lasso imply that $\|Xu\|/\sqrt{n} \leq (1/\sqrt{n}) \varepsilon^TXu + \lambda \|\beta^*\|_1 - \lambda \|\beta^* + u\|_1$ for $u = \hat{\beta} - \beta^*$. For instance, we prove a version of (3.7) at the end of Appendix D.

If $\nu$ is small, then (3.8) yields a lower bound of the form $\lambda \sqrt{k} \lesssim \|X(\hat{\beta} - \beta^*)\|$, which extends the lower bounds Theorem 3.1 under the beta-min condition (3.6), instead of the signal strength requirement of Theorem 3.1. On the other hand, Proposition 3.3 only shows a lower bound on the form $\lambda \sqrt{k}(1 - \sqrt{\nu}) \leq \|X(\hat{\beta} - \beta^*)\|$ which does not imply a lower bound involving the compatibility constant, as opposed to Theorem 3.1. Hence, we refer the reader to Theorem 3.1 for a clear argument that the compatibility condition with constant $\phi(T, 1)$ is necessary to achieve fast prediction rates. The goal of Proposition 3.3 is to show the signal strength condition ("$\|X\beta^*\|$ large enough") made in (2.5) and Theorem 3.1 can be weakened, and to present a proof argument to do so.
The quantity \( \nu \) tends to 0 provided that
\[
\gamma \to 0, \quad \frac{1}{\sqrt{ns}} \| X \bar{s} \| \leq 1 + o(1).
\]
In this case, the conclusion of the above Theorem can thus be rewritten informally as
\[
\| X(\hat{\beta} - \beta^*) + (\lambda/\sqrt{n}) \bar{s} \| = o(1) \| X(\hat{\beta} - \beta^*) \|.
\]
Thus, the distance between \( X \hat{\beta} \) and \( X(\beta^* - (\lambda/\sqrt{n}) \bar{s}) \) is an order of magnitude smaller than the prediction error \( \| X(\hat{\beta} - \beta^*) \| \).

4. Noise barrier and phase transitions for the Lasso

In this section, we again consider the Lasso, i.e., the estimator (1.2) with penalty (1.4). Throughout the section, let also \( k \in \{1, \ldots, p\} \) denote an upper bound on the sparsity of \( \beta^* \), i.e., \( \| \beta^* \|_0 \leq k \). The previous section showed that the Lasso estimator incurs an unavoidable prediction error of order at least \( \lambda \sqrt{k} \) for some target vector \( \beta^* \) with \( k \) nonzero coefficients, provided that the signal is large enough. Define the critical tuning parameter of sparsity level \( k \), denoted \( L_0(P_k) \), by
\[
L_0(P_k) := \sigma \sqrt{2 \log(p/k) - 5 \log \log(p/k) - \log(4\pi)}.
\]
(4.1)
The goal of the present section is to show that \( L_0(P_k) \) corresponds to a minimal penalty in \( \ell_1 \) regularization: We will see that the Lasso with tuning parameter equal or larger to \( L_0(P_k) \) is successful, while any tuning parameter smaller than \( L_0(P_k) \) will lead to terrible results.

The next result holds under the following probabilistic assumptions. Define the random variables
\[
g_j := \frac{1}{\sqrt{n}} \varepsilon^\top X e_j, \quad j = 1, \ldots, p, \quad \text{and set } g := (g_1, \ldots, g_p)\top.
\]
(4.2)
where \((e_1, \ldots, e_p)\) is the canonical basis in \( \mathbb{R}^p \).

**Assumption 4.1.** The noise vector \( \varepsilon \) has distribution \( N(0, \sigma^2 I_{n \times n}) \) and the design \( X \) is deterministic satisfying the normalization (3.4).

**Assumption 4.2.** The noise vector \( \varepsilon \) is deterministic with \( \sigma^2 = \| \varepsilon \|^2 / n \) and the design \( X \) has iid rows distributed as \( N(0, \Sigma) \) with \( \max_{j=1,\ldots,p} \Sigma_{jj} \leq 1 \).

Note that under Assumption 4.2 we have \( \mathbb{E}[gg\top] / \sigma^2 = \Sigma \) and \( \mathbb{E}[(g\top u)^2] / \sigma^2 = u\top \Sigma u \) for any \( u \in \mathbb{R}^p \), while under Assumption 4.1 we have \( \mathbb{E}[gg\top] / \sigma^2 = \frac{1}{n} X\top X \) and \( \mathbb{E}[(g\top u)^2] / \sigma^2 = \frac{1}{n} \| Xu \|^2 \). Hence the matrix \( \bar{\Sigma} \) defined by
\[
\bar{\Sigma} := \mathbb{E}[gg\top] / \sigma^2 = \begin{cases} 
\Sigma & \text{under Assumption 4.2}, \\
\frac{1}{n} X\top X & \text{under Assumption 4.1}
\end{cases}
\]
(4.3)
lets us describe the sparse or restricted eigenvalue properties of the design simultaneously under both assumptions.
Theorem 4.1. Let either Assumption 4.2 or Assumption 4.1 be fulfilled. Let \( k, n, p \) be positive integers and consider an asymptotic regime with

\[
k, n, p \to +\infty, \quad k/p \to 0, \quad k \log^{3}(p/k)/n \to 0. \tag{4.4}
\]

Let \( h \) be the penalty (1.4) and let \( \hat{\beta} \) be the Lasso (1.2) with tuning parameter \( \lambda = L_{0}(\frac{p}{k}) \).

(i) Assume that for some constant \( C_{\min} > 0 \) independent of \( k, n, p \) we have \( \| \bar{\Sigma}^{-1} \|_{op} \leq 1/C_{\min}^{2} \). Then, if \( \| \beta^{*} \|_{0} \leq k \) we have

\[
\mathbb{P} \left( \| X(\hat{\beta} - \beta^{*}) \| \leq \frac{1 + o(1)}{C_{\min}} \sqrt{2} \lambda \sqrt{k} \right) \to 1. \tag{4.5}
\]

(ii) Assume that \( \bar{\Sigma}_{i,j} = 1 \) for each \( j = 1, \ldots, p \), and assume that \( \| \Sigma \|_{op} \leq C_{\max}^{2} \) for some constant \( C_{\max} \) independent of \( k, n, p \). Then for any \( \beta^{*} \in \mathbb{R}^{p} \) we have

\[
\mathbb{P} \left( \frac{1 - o(1)}{C_{\max}} \lambda \sqrt{k} \leq \| X(\hat{\beta} - \beta^{*}) \| \right) \geq 1/3. \tag{4.6}
\]

In (4.5) and (4.6), the quantity \( o(1) \) is a deterministic positive sequence that converges to 0 in the asymptotic regime (4.4).

The lower bound (4.6) above holds even for target vectors \( \beta^{*} \) with \( \| \beta^{*} \|_{0} \ll k \).

Let us emphasize that the same tuning parameter \( L_{0}(\frac{p}{k}) \) enjoys both the upper bound (4.5) and the lower bound (4.6). Furthermore, the right hand side of (4.5) and the left hand side of (4.6) are both of order \( \sigma \sqrt{2k \log(p/k)} \approx \lambda \sqrt{k} \). Hence, in the asymptotic regime (4.4), the tuning parameter \( L_{0}(\frac{p}{k}) \) satisfies simultaneously the two following properties:

- The Lasso with tuning parameter \( \lambda = L_{0}(\frac{p}{k}) \) achieves the minimax rate over the class of \( k \)-sparse target vectors, which is of order \( \sigma \sqrt{2k \log(p/k)} \).
- The Lasso with tuning parameter \( \lambda = L_{0}(\frac{p}{k}) \) suffers a prediction error of order at least \( \sigma \sqrt{k \log(p/k)} \), even if the sparsity of the target vector is negligible compared to \( k \).

These two simultaneous properties illustrate the critical behavior of the Lasso with tuning parameter \( L_{0}(\frac{p}{k}) \):

- The tuning parameter \( L_{0}(\frac{p}{k}) \) is too small for target vectors with \( \| \beta^{*} \|_{0} \ll k \) since it will incur a prediction error of order \( \sqrt{k \log(p/k)} \) although one hopes to achieve an error of order \( \sqrt{\| \beta^{*} \|_{0} \log(p/\| \beta^{*} \|_{0})} \).
- The tuning parameter \( L_{0}(\frac{p}{k}) \) is too large for target vectors with \( \| \beta^{*} \|_{0} \gg k \) since it will incur a prediction error of order \( \sqrt{\| \beta^{*} \|_{0} \log(p/\| \beta^{*} \|_{0})} \) although one hopes to achieve an error of order \( \sqrt{\| \beta^{*} \|_{0} \log(p/\| \beta^{*} \|_{0})} \).

The fact that \( L_{0}(\frac{p}{k}) \) corresponds to a minimal penalty in the sense of [9] will be explained in Section 4.3.

Theorem 4.1 above features two results: an upper bound and a lower bound. Section 4.1 studies upper bounds on the prediction error \( \| X(\hat{\beta} - \beta^{*}) \| \) that imply the upper
bound (4.5) above. The formal proof of (4.5) is given at the end of Section 4.1. Section 4.2 provides several lower bounds on \(|X(\hat{\beta} - \beta^*)| \) when the tuning parameter is smaller or equal to \(L_0(\frac{p}{k})\). Similarly, the results of Section 4.2 imply (4.6) and the formal proof of (4.6) is given at the end of Section 4.2.

Although one goal of Sections 4.1 and 4.2 is to prove Theorem 4.1, some results of the next subsections are of independent interest. For instance, Theorem 4.2 of the next subsection shows that the Lasso with tuning parameter

\[
\sigma \sqrt{2 \log(p/k) - 5 \log \log(p/k) + \log \log \log(p/k)}
\]
satisfies the upper bound (4.5) with the constant \(\sqrt{2}(1 + o(1)) \) replaced by \((1 + o(1))\). Also, the assumptions on the eigenvalues of \(\Sigma\) made in the above result will be weakened.

4.1. Upper bounds

For \(x > 0\), define

\[
L_f(x) = \sigma \sqrt{2 \log(x) - 5 \log \log(x) - \log(4\pi) + 2f(x)} \tag{4.7}
\]

for some non-decreasing \(f\) s.t. \(0 \leq 2f(x) \leq \log(4\pi) + 5 \log \log(x)\).

We will study tuning parameters of the form \(L_f(p/k)\), where \(k\) is a sparsity level. Note that if \(f = 0\), then \(L_f(\frac{p}{k})\) is equal to \(L_0(\frac{p}{k})\), the critical tuning parameter defined in (4.1) above, so that the notations (4.1) and (4.7) are coherent. Nonzero functions \(f\) will be useful to illustrate the behavior of the Lasso when the tuning parameter slightly deviates from \(L_0(\frac{p}{k})\).

**Theorem 4.2.** There exist absolute constants \(C, C' > 0\) such that the following holds. Assume that \(p/k > C\). Let either Assumption 4.2 or Assumption 4.1 be fulfilled, and let \(\Sigma\) be the matrix (4.3). Let the tuning parameter \(\lambda\) of the Lasso be equal to \(L_f(\frac{p}{k})\) where \(L_f\) and \(f\) are as in (4.7). Let \(T\) be the support of \(\beta^*\) and assume that \(|\beta^*|_0 = |T| \leq k\). Define the constant \(\theta\) by

\[
\theta = \inf_{u \neq 0: ||u||_{\Sigma} \leq \sqrt{\tilde{k}||u||}} \frac{||\Sigma^{1/2}u||}{||u||} \text{ where } \tilde{k} = k (24 \log(p/k))^2.
\]

and assume that \(\theta > 0\). Define \(r_n\) by \(r_n = 0\) under Assumption 4.1 and \(r_n = (C'/\theta)\sqrt{\log(2p/k)/n}\) under Assumption 4.2. Assume that \(r_n < 1\).

For any \(t \in [0, \lambda \sqrt{\tilde{k}/\sigma}]\), the Lasso with tuning parameter \(\lambda = L_f(\frac{p}{k})\) satisfies with probability at least \(1 - 3 \exp(-t^2/2)\),

\[
\theta ||u|| \leq ||\Sigma^{1/2}u|| \leq \frac{||Xu||}{\sqrt{n(1 - r_n)}} \leq \frac{\sqrt{2}E\lambda \sqrt{1 + \frac{\sigma^2}{X^2} + \eta(\frac{p}{k}) \exp[-f(\frac{p}{k})] + \sigma \theta t}}{\theta(1 - r_n)^2}
\]

where \(u = \hat{\beta} - \beta^*\) and where \(\eta : [0, +\infty) \to [0, +\infty)\) is a function such that \(\eta(x) \to 1\) as \(x \to +\infty\).
Theorem 4.2 is proved at the end of Appendix E. Consider a function $f$ such that

$$f(x) \rightarrow +\infty \quad \text{as} \quad x \rightarrow +\infty,$$

for instance $f(x) = \sqrt{\log \log(x)}$. For such choice of $f$ and $t = \lambda/\sigma$, Theorem 4.2 implies that

$$\mathbb{P}\left(\sqrt{\lambda^2 n \log \log(1/n) \log \log(p/k)} \leq \frac{\sigma \sqrt{2k \log(p/k)}}{\lambda} \right) \rightarrow 1$$

in the asymptotic regime (4.4). This is a consequence of the fact that $r_n \rightarrow 0$, $\sigma^2 / \lambda^2 \rightarrow 0$, $\exp[-f(\xi)] \rightarrow 0$ and $\lambda \asymp \sigma \sqrt{2 \log(p/k)}$ in the asymptotic regime (4.4), where $a \asymp b$ means that $a/b \rightarrow 1$. The rate $\sigma \sqrt{2k \log(p/k)}/\lambda$ is known to be asymptotically minimax (with exact multiplicative constant) under Assumption 4.2 with $\Sigma = I_{p \times p}, \theta = 1$, that is, if $X$ has iid standard normal entries [38, Theorem 5.4]. Theorem 4.2 thus shows that the Lasso with tuning parameter $L_{f}(\mathbf{\xi})$ is minimax over $k$-sparse target vectors, with exact asymptotic constant, for isotropic normal designs in the asymptotic regime (4.4).

Since $C_{\min} \leq \theta$ and $r_n \rightarrow 0$ in the asymptotic regime (4.4), the previous result implies the upper bound of Theorem 4.1 by taking $f = 0$ so that $\lambda = L_{0}(\mathbf{\xi})$.

A novelty of Theorem 4.2 lies in how small the tuning parameter is allowed to be while still achieving a prediction error of order at most $\lambda \sqrt{k}$. The literature already features upper bound such as (1.9) of order $\lambda \sqrt{k}$ on the prediction error of the Lasso under restricted eigenvalue or compatibility conditions. However, existing results such as (1.9) require tuning parameters of the form $(1+c)\sigma \sqrt{2 \log p}$ for some constant $c > 0$ [7, 18, 13, 35, 20], or of the form $(1+c)\sigma \sqrt{2 \log(p/k)}$ where again $c > 0$ is constant [39, 29, 4, 23]. The result above shows that one can use tuning parameters even smaller than $\sigma \sqrt{2 \log(p/k)}$, up to the log log correction visible in (4.1) and (4.7). The fact that the upper bound given in Theorem 4.2 holds for tuning parameter as small as $L_{0}(\mathbf{\xi})$ is important to show that the Lasso with parameter $L_{0}(\mathbf{\xi})$ enjoy both the upper bound (4.5) and the lower bound (4.6). As we will see in the next section, tuning parameters proposed in past results of the form $(1+c)\sigma \sqrt{2 \log(p/k)}$ or $(1+c)\sigma \sqrt{2 \log(p/k)}$ are too large to satisfy simultaneously an upper bound and a lower bound such as (4.5)-(4.6).

To put this novelty in perspective, consider the asymptotic regime

$$p \rightarrow +\infty, \quad n = p^b, \quad k = p^a \quad \text{for constants} \quad 0 < a < b < 1.$$ 

Then (4.4) clearly holds and $L_{0}(\mathbf{\xi}) \asymp \sigma \sqrt{2 \log(p/k)} \asymp \sigma \sqrt{2(1-a) \log p}$ where $a \asymp b$ means $a/b \rightarrow 1$. On the other hand, the tuning parameter $(1+c)\sigma \sqrt{2 \log(p/k)}$ for some constant $c > 0$ can be rewritten as $\sigma \sqrt{\log(p/k')}$ where $k'$ is the sparsity level $k' = p^{a'}$ for the constant $a' = a - (2c + c^2)(1-a)$ which satisfies $a' < a$. Hence the tuning parameter $(1+c)\sigma \sqrt{2 \log(p/k)}$ corresponds here to a sparsity level $k'$ which is negligible compared to the true sparsity level $k$.

A drawback of tuning parameters larger or equal to $(1+c)\sigma \sqrt{2 \log(p/k)}$ for constant $c > 0$ is the following. By the results of Section 3, the Lasso with such tuning
parameter suffers a prediction error of order at least $(1 + c)\sigma \sqrt{2k \log (p/k)}$ for some $k$-sparse large enough target vectors. Hence the prediction error is separated from the minimax risk $\sigma \sqrt{2k \log (p/k)}$ by the multiplicative constant $1 + c$. In contrast, the tuning parameter $L_f(\frac{c}{k})$ discussed after (4.8) achieves the minimax risk over $k$-sparse vectors.

4.2. Lower bounds

The previous section shows that the Lasso with tuning parameter equal or larger than $L_0(\frac{c}{k})$ achieves a prediction error not larger than $\lambda \sqrt{k}$ if $\beta^*$ is $k$-sparse. Since $\lambda$ is of logarithmic order, the Lasso thus enjoys a dimensionality reduction property of order $k$: Its prediction error is of the same order as that of the Least-Squares estimator of a design matrix with only $k = \|\beta^*\|_0$ covariates. The present subsection answers the dual question: Given an integer $k \geq 1$, for which values of the tuning parameter does the Lasso lack the dimensionality reduction property of order $k$?

**Theorem 4.3.** There exists an absolute constant $C > 0$ such that the following holds. Let either Assumption 4.2 or Assumption 4.1 be fulfilled and assume that the matrix $\Sigma$ defined in (4.3) satisfies $\Sigma_{jj} = 1$ for each $j = 1, ..., p$. Consider the asymptotic regime (4.4). Then for $k, p, n$ large enough, we have for any $\beta^*$ and any tuning parameter $\lambda \leq L_0(\frac{c}{k})$ the lower bound

$$\mathbb{P}\left( Z \leq [1 + o(1)] \psi \|X(\hat{\beta} - \beta^*)\| \right) \geq 5/6,$$

where $Z$ is a random variable such that $\mathbb{E}[Z^2] = \sigma^2 2k \log (p/k)$, where

$$\psi = \sup_{u \in \mathbb{R}^p : \|u\| = 1, \|u\|_0 \leq Ck \log^2 (p/k)} \|\bar{\Sigma}^{1/2} u\|$$

is the maximal sparse eigenvalue of order $Ck \log^2 (p/k)$, and where $o(1)$ is a deterministic positive sequence that only depends on $(n, p, k)$ and converges to 0 in the asymptotic regime (4.4). Furthermore, under the additional assumption that $\|\Sigma\|_{op} = o(k \log (p/k))$ we have

$$\mathbb{P}\left( \sigma \sqrt{2k \log (p/k)} \leq [1 + o(1)] \psi \|X(\hat{\beta} - \beta^*)\| \right) \geq 1/3.$$

The proof is given in Appendix G. It is a consequence of the definition of the noise barrier and inequality (2.2). Theorem 4.3 implies the lower bound for the tuning parameter $L_0(\frac{c}{k})$ given in Theorem 4.1, since $\psi \leq C_{\text{max}}$.

The Lasso with tuning parameter satisfying $\lambda \leq L_0(\frac{c}{k})$ incurs a prediction error of order at least $\sigma \sqrt{2k \log (p/k)}$, hence it does not enjoy the dimensionality reduction property at any sparsity level negligible compared to $k$. Indeed, if $\|\beta^*\|_0 \ll k$, then the Lasso with parameter $L_0(\frac{c}{k})$ incurs a prediction error of order $\sigma \sqrt{2k \log (p/k)}$ and the prediction error is much larger than $\sigma \sqrt{2\|\beta^*\|_0 \log (p/\|\beta^*\|_0)}$ when $\|\beta^*\|_0 \ll k$. The tuning parameter $\lambda = L_0(\frac{c}{k})$ enjoys the dimension reduction property for $k$-sparse target vectors, but it cannot enjoy the dimension reduction property for target vectors with $\|\beta^*\|_0 \ll k$. 
A result similar to Theorem 4.3 was obtained in [40, Proposition 14] in a random design setting. Theorem 4.3 is different from this result of [40] in at least two ways. First the assumption on $\|\Sigma\|_{\text{op}}$ is allowed to be larger in Theorem 4.3. Second, and more importantly, Theorem 4.3 is sharper in the sense that it applies to the critical tuning parameter $L_0(\frac{p}{k})$ that achieves the upper bounds of Theorem 4.2, while the lower bound in [40] only applies to tuning parameters strictly smaller than $L_0(\frac{p}{k})$.

The above result requires that the maximal sparse eigenvalue of $\bar{\Sigma}$ is bounded from above. The following result shows that it is possible to draw similar conclusions when the minimal sparse eigenvalue of $\bar{\Sigma}$ is bounded away from 0.

**Theorem 4.4 (Lower bound, deterministic design with minimal sparse eigenvalue).** Let $d$ be a positive integer with $d \leq p/5$. Let Assumption 4.1 be fulfilled and assume that the design matrix satisfies

$$
\inf_{u \in \mathbb{R}^p: \|u\|_0 \leq 2d, u \neq 0} \frac{\|Xu\|}{\sqrt{n} \|u\|} \geq (1 - \delta_{2d}) \quad \text{for some } \delta_{2d} \in (0, 1). \tag{4.10}
$$

Let $h$ be the penalty function (1.4). If the tuning parameter satisfies

$$
\lambda \leq \frac{\sigma(1 - \delta_{2d})}{8} \sqrt{\log(p/(5d))}, \tag{4.11}
$$

then we have

$$
\lambda \sqrt{d} \leq \mathbb{E}\|X(\hat{\beta} - \beta^*)\|.
$$

The proof is given in Appendix G. It is a consequence of the definition of the noise barrier and the lower bound (2.2).

Again, Theorem 4.4 makes no sparsity assumption on the target vector $\beta^*$ and its implications are better understood for target vectors such that $\|\beta^*\|_0 \ll d$; the discussion following Theorem 4.3 applies here as well: Even though the size of the true model is sparse and of size $\|\beta^*\|_0 \ll d$, the Lasso with small tuning parameter (as in (4.11)) suffers a prediction error of order at least $\sqrt{d \log(p/d)}$ and cannot satisfy the dimensionality reduction property of order $\|\beta^*\|_0$.

Theorems 4.3 and 4.4 have two major differences. First, Theorem 4.3 requires that the maximal sparse eigenvalue is bounded from above, while Theorem 4.4 requires that the minimal sparse eigenvalue is bounded away from 0. Second, the multiplicative constant $(1 - \delta_{2d})/8$ of the right hand side of (4.11) is not optimal, while the multiplicative constant in Theorem 4.3 is optimal since it applies to the critical tuning parameter $L_0(\frac{p}{k})$.

**Remark 4.1 (Combining lower bounds from the large signal bias and the noise barrier).** The lower bounds that we have obtained so far for the Lasso are of two different nature. Section 2.2, inequality (2.5) and Section 3 establish a lower bound of the form

$$
(1 - \gamma)\lambda \sqrt{k} C(\beta^*) \leq \mathbb{E}\|X(\hat{\beta} - \beta^*)\| \quad \text{where } C(\beta^*) = \sup_{\beta \in \mathbb{R}^p} \frac{\sqrt{n}(\|\beta^*\|_1 - \|\beta\|_1)}{\sqrt{k}\|X(\beta^* - \beta)\|} \tag{4.12}
$$

for any arbitrarily small $\gamma > 0$ provided that $\|X\beta^*\|$ is large enough. On the other hand, Section 4 shows that if the tuning parameter of the Lasso is smaller or equal
to (4.1) for some \( k > 0 \), then the prediction error of the Lasso is at least of order 
\[ \sigma \sqrt{k \log(p/k)}, \]
even if the sparsity \( \| \beta^* \|_0 \) of \( \beta^* \) is negligible compared to \( k \). These 
two lower bounds illustrate the trade-off for the choice of tuning parameters: For large 
tuning parameter the lower bound (4.12) is stronger than that of Theorem 4.3, while for 
tuning parameters smaller than \( L_0(\frac{p}{k}) \), the lower bound given in Theorem 4.3 becomes 
stronger.

4.3. The bias-variance trade-off in sparse linear regression

With the above results of the previous subsections, we are now equipped to exhibit a 
bias-variance trade-off involving \( \text{NB}(\varepsilon) \) and \( \text{LSB}(\beta^*) \). Let \( L_0(\frac{p}{k}) \) be the critical tuning 
parameter defined in (4.1) and let \( \zeta \) be a decreasing sequence depending on \( k, n, p \) such 
that \( \zeta \in (0, 1) \), \( \zeta \to 0 \) and \( \log(1/\zeta) = o(\log(p/k)) \). For instance, one may take
\[
\zeta = 1/ \log \log(p/k).
\]

Define the sparsity levels \( k_{\oplus} := k/\zeta \) and \( k_{\ominus} := k\zeta \), so that \( k_{\ominus} \leq k \leq k_{\oplus} \) as well as 
\( k = o(k_{\ominus}) \) and \( k_{\ominus} = o(k) \). In this subsection, we consider the asymptotic regime 
\[
k, n, p \to +\infty, \quad k_{\ominus}/p \to 0, \quad k_{\ominus} \log^2(p/k_{\ominus})/n \to 0.
\]
(Note that this asymptotic regime is very close to (4.4) for the choice (4.13)). We will 
study the behavior of the Lasso with the three tuning parameters \( L_0(\frac{p}{k_{\ominus}}) \), \( L_0(\frac{p}{k}) \) and 
\( L_0(\frac{p}{k_{\oplus}}) \), where the function \( L_0(\cdot) \) is defined in (4.7) for \( f = 0 \). The role of the sequence 
\( \zeta \) is to study the behavior of the Lasso over \( k \)-sparse target vectors when the tuning 
parameter slightly deviates from \( L_0(\frac{p}{k}) \). The tuning parameter \( L_0(\frac{p}{k_{\ominus}}) \) is slightly larger 
than \( L_0(\frac{p}{k}) \) while \( L_0(\frac{p}{k_{\oplus}}) \) is slightly smaller.

In this paragraph, let either Assumption 4.2 or Assumption 4.1 be fulfilled and let 
\( C_{\text{min}}, C_{\text{max}} \) be as in Theorem 4.1. Consider the asymptotic regime (4.14) and assume 
that the constants \( C_{\text{min}}, C_{\text{max}} \) are independent of \( n, p, k \). In order to study the behavior 
of the large signal bias, the noise barrier and the prediction error relatively to the 
opimal rate, we define the random quantities
\[
B_\lambda = \frac{\text{LSB}(\beta^*)}{\sigma \sqrt{2k \log(p/k)}}, \quad V_\lambda = \frac{\text{NB}(\varepsilon)}{\sigma \sqrt{2k \log(p/k)}}, \quad R_\lambda = \frac{\|X(\hat{\beta} - \beta^*)\|}{\sigma \sqrt{2k \log(p/k)}},
\]
for a given tuning parameter \( \lambda \). We now describe the behavior of \( B_\lambda, V_\lambda \) for each tuning 
parameter \( \lambda \in \{ L_0(\frac{p}{k}), L_0(\frac{p}{k_{\ominus}}), L_0(\frac{p}{k_{\oplus}}) \} \). The following results are consequences of 
results derived so far; their formal proof is deferred to Appendix L.

- The Lasso with tuning parameter \( \lambda = L_0(\frac{p}{k}) \) satisfies for any \( k \)-sparse target 
  vectors
\[
\mathbb{P} \left( \{ B_\lambda, V_\lambda, R_\lambda \} \subset [C^{-1}, C] \right) \geq c
\]
for some absolute constant \( c > 0 \) and some constant \( C \) depending only on \( C_{\text{min}} \)
and \( C_{\text{max}} \). Here, the large signal bias, the noise barrier and the prediction error 
are of the same order as the optimal rate \( \sigma \sqrt{2k \log(p/k)} \).
• The Lasso with tuning parameter $\lambda = L_0(\frac{p}{k})$ satisfies for any $k$-sparse target vector $\beta^*$

$$\mathbb{P}\left(C^{-1} \leq B_\lambda \leq C, \quad B_\lambda \leq o(1) V_\lambda \right) \geq c$$

(4.16)

for some absolute constant $c > 0$ and some constant $C$ depending only on $C_{\min}$ and $C_{\max}$, where $o(1)$ denotes a positive deterministic sequence that only depends on $n, p, k, C_{\min}, C_{\max}$ and that converges to 0 in the asymptotic (4.14). Here, the large signal bias is of the same order as the optimal rate over $k$-sparse vectors. Because the tuning parameter $L_0(\frac{p}{k})$ is too small, the noise barrier dominates the large signal bias and the optimal rate $\sigma \sqrt{2 \log(p/k)}$ is negligible compared to the prediction error $\|X(\hat{\beta} - \beta^*)\|$. This phenomenon is similar the minimal penalty in $\ell_0$-regularization studied in [9]: For tuning parameters slightly smaller than $L_0(\frac{p}{k})$ such as $L_0(\frac{p}{k} \oplus)$, the Lasso will lead to terrible results in the sense that its prediction error will be much larger than $\sigma \sqrt{2 \log(p/k)}$.

The phase transition is remarkably sharp: Although $L_0(\frac{p}{k}) > L_0(\frac{p}{k} \ominus)$, the two tuning parameters are both asymptotically equal to $\sigma \sqrt{2 \log(p/k)}$.

• The Lasso with tuning parameter $\lambda = L_0(\frac{p}{k} \ominus)$ satisfies for any $k$-sparse target vector $\beta^*$

$$\mathbb{P}\left(C^{-1} \leq B_\lambda \leq C, \quad V_\lambda \leq o(1) B_\lambda \right) \geq c$$

(4.17)

where $c, C$ and $o(1)$ are as in the previous point. Here, the tuning parameter is large enough so that the noise barrier is negligible compared to the large signal bias. This is in contrast with (4.15) where $NB(\varepsilon)$ and $LSB(\beta^*)$ are of the same order.

Hence, in order to reveal the bias/variance trade-off intrinsic to the class of $k$-sparse target vectors, one has to “zoom in” on the critical tuning parameter $L_0(\frac{p}{k})$ and study tuning parameters that slightly deviates from $L_0(\frac{p}{k})$ such as $L_0(\frac{p}{k} \oplus)$ and $L_0(\frac{p}{k} \ominus)$ defined above.

5. Data-driven tuning parameters are subject to the same limitations as deterministic parameters

Remarkably, the lower bounds derived in the previous sections also apply to penalty functions with random tuning parameters. To illustrate this, consider the Lasso penalty (1.4) with $\lambda$ replaced by an estimator $\hat{\lambda} \geq 0$ so that

$$h(\cdot) = \sqrt{n} \hat{\lambda} \cdot \|\cdot\|_1, \quad \hat{\lambda} \geq 0. \quad (5.1)$$

5.1. Noise barrier lower bounds

The lower bounds of Theorems 4.3 and 4.4 apply to the $\ell_1$ penalty with data-driven parameters.

Theorem 5.1 (Extension of Theorem 4.4 to data-driven tuning parameter). Let Assumption 4.1 be fulfilled and let $d = p/5$ be an integer such that (4.10) holds. Let $h$ be
the penalty function (5.1). If \( \mathbb{E}[\hat{\lambda}] \) is no larger than the right hand side of (4.11) then \( \mathbb{E}[\hat{\lambda}] \sqrt{d} \leq \mathbb{E}[\|\hat{\beta} - \beta^*\|] \) holds.

**Theorem 5.2** (Extension of Theorem 4.3 to data-driven tuning parameter). Let \( k, p, n \) be integers with \( 8k \leq p \) and \( p \geq 3k \log(p/k)^2 \). Let Assumption 4.2 be fulfilled with \( \Sigma = I_{p \times p} \). If \( \mathbb{E}[\hat{\lambda}] \leq L_0(\frac{p}{k}) \) then the Lasso estimator with tuning parameter \( \hat{\lambda} \) satisfies

\[
ca^2 2k\log(p/k) \leq \mathbb{E}[\|X(\hat{\beta} - \beta^*)\|^2] \left(1 + c_1 \sqrt{k\log(p/k)}\right)
\]

for some absolute constants \( c, c_1 > 0 \).

These two theorems are proved in Appendix J. To understand why the noise barrier lower bounds extend to data-driven tuning parameters, consider a penalty of the form

\[
h(\cdot) = \hat{\lambda} N(\cdot) \text{ for some seminorm } N \text{ and a data-driven tuning parameter } \hat{\lambda} \geq 0.
\]

Then the noise barrier lower bound (2.2) implies, for instance, that for any deterministic \( \rho > 0 \),

\[
\mathbb{E}\left[\sup_{u \in \mathbb{R}^p: \|Xu\| \leq 1, N(u) \leq \rho} e^TXu\right] - \mathbb{E}[\hat{\lambda}] \rho \leq \mathbb{E}[\|X(\hat{\beta} - \beta^*)\|] \tag{5.2}
\]

as well as

\[
\mathbb{E}\left[\sup_{u \in \Omega} e^TXu\right] - \mathbb{E}[\hat{\lambda}] \rho \leq \mathbb{E}[\|X(\hat{\beta} - \beta^*)\|^2]^{1/2}\mathbb{E}[\sup_{u \in \Omega} \|Xu\|]^{1/2} \tag{5.3}
\]

for any deterministic set \( \Omega \) such that \( \sup_{u \in \Omega} N(u) \leq \rho \). Hence, one can obtain lower bounds on \( \mathbb{E}[\|X(\hat{\beta} - \beta^*)\|] \) or \( \mathbb{E}[\|X(\hat{\beta} - \beta^*)\|^2]^{1/2} \) when \( \mathbb{E}[\hat{\lambda}] \) is small enough, provided that one can bound the expected suprema appearing in (5.2) and (5.2).

Note that the lower bounds (5.2) and (5.3) apply to any seminorm \( N(\cdot) \) and any data-driven tuning parameter \( \hat{\lambda} \). Hence the lower bounds (5.2)-(5.3) can be used beyond the \( \ell_1 \) penalty.

### 5.2. Large scale bias lower bounds

The lower bound (4.12) also extends to data-driven tuning parameters. This is the content of the following proposition.

**Proposition 5.3.** Let \( M, M' > 0 \) be constants and \( X \) be a deterministic design. Let \( N(\cdot) \) be a nonrandom semi-norm on \( \mathbb{R}^p \) and let \( \hat{\lambda} \geq 0 \) be any data-driven tuning parameter. Assume that \( \mathbb{E}[e] = 0 \), that \( \mathbb{E}[\|e\|^2] \leq M \) and that \( 1/M' \leq \mathbb{E}[\hat{\lambda}] \) and \( \mathbb{E}[\hat{\lambda}^2] \leq M \) for any target vector \( \beta^* \). Consider the random penalty function \( h(\cdot) = \hat{\lambda} N(\cdot) \) and the estimator (1.2). For any \( \gamma > 0 \), if \( \|X\beta^*\| \) is large enough then

\[
(1 - \gamma)\mathbb{E}[\hat{\lambda}] b \leq \mathbb{E}[\|X(\hat{\beta} - \beta^*)\|] \quad \text{where} \quad b = \sup_{\beta \in \mathbb{R}^p: X\beta \neq X\beta^*} \frac{N(\beta^*) - N(\beta)}{\|X(\beta^* - \beta)\|}.
\]
This result generalizes (2.5) to penalty with data-driven tuning parameters. The proof is given in Appendix B. For the $\ell_1$ penalty, i.e., $N(\cdot) = \sqrt{n}\cdot \|\cdot\|_1$, if the assumptions of the above proposition are fulfilled, we get that for any arbitrarily small $\gamma > 0$,

$$(1 - \gamma)E[\hat{\lambda}] \sup_{\beta \in R^p: X \beta \neq X \beta^*} \frac{\|\beta^*\|_1 - \|\beta\|_1}{\|X(\beta^* - \beta)\|/\sqrt{n}} \leq E[\|X(\hat{\beta} - \beta^*)\|]$$

if $\|X \beta^*\|$ is large enough. We also have the analog of Theorem 3.1 for data-driven tuning parameters. It can be proved in the same fashion as Theorem 3.1.

**Theorem 5.4.** Let $M, M’ > 0$. Consider the estimator (1.2) with penalty (5.1) for some data-driven tuning parameter $\hat{\lambda} \geq 0$. Let $T \subset [p]$ and assume that $\phi(1, T) > 0$. Let $\gamma > 0$ be any arbitrarily small constant. Assume that the noise satisfies $E[\|e\|^2] < +\infty$ and the data driven parameter $\hat{\lambda}$ satisfies $1/M’ \leq E[\hat{\lambda}]$ and $E[\hat{\lambda}^2] \leq M$ for any target vector $\beta^*$. Then there exists $\beta^* \in R^p$ supported on $T$ such that

$$(1 - \gamma) \frac{E[\hat{\lambda}] |T|^{1/2}}{\phi(1, T)} \leq E[\|X(\hat{\beta} - \beta^*)\|].$$

6. Extension to penalties different than the $\ell_1$ norm

In this section, we consider penalty functions different than the $\ell_1$ norm. We will see that the lower bounds induced by the noise barrier and the large scale bias are also insightful for these penalty functions and that most of our results on the Lasso can be extended.

**6.1. Nuclear norm penalty**

Let $p = mT$ for integers $m \geq T > 0$. We identify $R^p$ with the space of matrices with $m$ rows and $T$ columns. Let $\beta’$ be the transpose of a matrix $\beta \in R^{m \times T}$ and denote by $\text{tr}(\beta_1’\beta_2)$ the scalar product of two matrices $\beta_1, \beta_2 \in R^{m \times T}$. Assume that we observe pairs $(X_i, y_i)_{i=1,\ldots,n}$ where $X_i \in R^{m \times T}$ and

$$y_i = \text{tr}(X_i’\beta^*) + \varepsilon_i, \quad i = 1, \ldots, n, \quad (6.1)$$

where each $\varepsilon_i$ is a scalar noise random variable and $\beta^* \in R^{m \times T}$ is an unknown target matrix. The model (6.1) is sometimes referred to as the trace regression model. Define $y = (y_1, \ldots, y_n), \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ and define the linear operator $X : R^{m \times T} \rightarrow R^n$ by $(X\beta)_i = \text{tr}(X_i’\beta)$ for any matrix $\beta$ and any $i = 1, \ldots, n$, so that the model (6.1) can be rewritten as the linear model

$$y = X\beta^* + \varepsilon.$$

Define the nuclear norm penalty by

$$h(\cdot) = \sqrt{\sigma(\hat{\lambda}) \cdot \|S_1\|}, \quad (6.2)$$

where $\hat{\lambda} \geq 0$ may be random and $\|\cdot\|_{S_1}$ is the nuclear norm in $R^{m \times T}$. Define also the Frobenius norm of a matrix $\beta$ by $\|\beta\|_{S_2} = \text{tr}(\beta’\beta)^{1/2}$. 

Theorem 6.1. Assume that \( \varepsilon \sim N(0, \sigma^2 I_{n \times n}) \) and let \( r \in \{1, \ldots, T\} \) be an integer such that for some \( \delta_r \in (0, 1) \) we have

\[
(1 - \delta_r) \| \beta \|_{S_2} \leq \| X \beta \| / \sqrt{n} \leq (1 + \delta_r) \| \beta \|_{S_2}, \quad \forall \beta \in \mathbb{R}^{m \times T} : \text{rank}(\beta) \leq 2r
\]

Let \( h \) be the penalty function (6.2). Then the nuclear norm penalized least-squares estimator (1.2) satisfies

\[
\sqrt{r} \left( c\sigma(1 - \delta_r) \sqrt{m} - \mathbb{E}[\hat{\lambda}] \right) \leq (1 + \delta_r) \mathbb{E}\|X(\hat{\beta} - \beta^*)\|.
\]

for some absolute constant \( c > 0 \).

The proof is given in Appendix H. If the tuning parameter is too small in the sense that \( \mathbb{E}[\hat{\lambda}] \leq c\sigma(1 - \delta_r) \sqrt{m}/2 \), then the prediction error \( \mathbb{E}\|X(\hat{\beta} - \beta^*)\| \) is bounded from below by \( c\sigma \sqrt{m}/(2(1 + \delta_r) \sqrt{m}) \). For common random operators \( X \), the Restricted Isometry condition of Theorem 6.1 is granted with \( \delta_r = 1/2 \) for any \( r \leq T/C \) where \( C > 0 \) is some absolute constant, see for instance [16] and the references therein. In this case, the above result with \( r = T/C \) yields that for some absolute constant \( c' > 0 \),

\[
\mathbb{E}[\hat{\lambda}] \leq (c/4)\sigma \sqrt{m} \quad \text{implies} \quad c' \sigma \sqrt{Tm} \leq \mathbb{E}\|X(\hat{\beta} - \beta^*)\|.
\]

For tuning parameters smaller than \( c\sigma \sqrt{m}/4 \) in expectation, the performance of the nuclear penalized least-squares estimator is no better than the performance of the unpenalized least-squares estimator for which \( \mathbb{E}\|X(\hat{\beta}^* - \beta^*)\| \) is of order \( \sigma \sqrt{p} = \sigma \sqrt{mT} \).

The large signal bias lower bound of Proposition 5.3 yields that for any target vector \( \beta^* \) such that \( \|X\beta^*\| \) is large enough and under the mild assumptions of Proposition 5.3, we have

\[
0.99 \mathbb{E}[\hat{\lambda}] \sup_{\beta \in \mathbb{R}^{m \times T}, X\beta \neq X\beta^*} \frac{\|\beta^*\|_{S_1} - \|\beta\|_{S_1}}{\|X(\beta^* - \beta)\| / \sqrt{n}} \leq \mathbb{E}\|X(\hat{\beta} - \beta^*)\|.
\]

As in Theorem 3.1 for the Lasso, the previous display shows that the nuclear norm penalized estimator will incur a prediction error due to correlations in the design matrix \( X \).

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Appendix A: Proofs: Properties the noise barrier

Proof of Proposition 2.1. The optimality conditions of the optimization problem (1.2) yields that there exists \( \hat{s} \) in the subdifferential of \( h \) at \( \hat{\beta} \) such that

\[
X^T(\epsilon + X(\beta^* - \hat{\beta})) = \hat{s}. \tag{A.1}
\]

The subdifferential of a seminorm at \( \hat{\beta} \) is made of vectors \( \hat{s} \in \mathbb{R}^p \) such that \( \hat{s}^T \hat{\beta} = h(\hat{\beta}) \) and \( \hat{s}^T u \leq h(u) \) for any \( u \in \mathbb{R}^p \). For any \( u \in \mathbb{R}^p \), we thus have

\[
\epsilon^T X u - h(u) \leq u^T X(\hat{\beta} - \beta^*). \tag{2.2}
\]

If \( \|X u\| \leq 1 \), then applying the Cauchy-Schwarz inequality to the right hand side yields (2.2).

If \( X \beta^* = 0 \), to prove the second bullet it is enough to prove that \( \|X \hat{\beta}\| \leq \text{NB}(\epsilon) \). Multiplication of (A.1) by \( \hat{\beta} \) yields \( \epsilon^T X \hat{\beta} - \|X \hat{\beta}\|^2 = h(\hat{\beta}) \). If \( X \hat{\beta} \neq 0 \) then dividing by \( \|X \hat{\beta}\| \) yields (2.2) with equality. If \( X \hat{\beta} = 0 \) then \( \|X \hat{\beta}\| \leq \text{NB}(\epsilon) \) trivially holds. Regarding the third bullet, the monotonicity with respect to \( \lambda \) is a direct consequence of definition (2.1). \( \square \)
Appendix B: Proofs: Properties the large scale bias

Proof of Proposition 2.2. Inequality \( h(\beta^*)/||X\beta^*|| \leq \text{LSB}(\beta^*) \) is a direct consequence of the definition (2.4). Using Lagrange multipliers, since \( \beta^* \) is solution of the minimization problem (2.3), there is a subdifferential \( d \) of \( h \) at \( \beta^* \) such that \( d = \lambda^T X \) for some \( \lambda \in \mathbb{R}^p \). Then \( \text{LSB}(\beta^*) \) is bounded from above by \( ||\lambda|| \) since

\[
\frac{h(\beta^*) - h(\beta)}{||X(\beta^* - \beta)||} \leq \frac{d^T (\beta^* - \beta)}{||X(\beta^* - \beta)||} = \frac{\lambda^T X (\beta^* - \beta)}{||X(\beta^* - \beta)||} \leq ||\lambda|| \leq +\infty
\]

where we used that \( h(\beta) - h(\beta^*) \geq d^T (\beta - \beta^*) \) by convexity.

For the second line of the right hand side is bounded from above by \( \text{LSB}(\beta^*) \).

Proof of Proposition 5.3. Let \( \hat{\beta} = ||X(\beta^* - \beta)|| \) for brevity. Let \( \beta \in \mathbb{R}^p \) be deterministic vector that will be specified later. Multiplication of (A.1) by \( \beta - \hat{\beta} \) yields

\[
(\beta - \hat{\beta})^T X^T (\varepsilon + X(\beta^* - \hat{\beta})) = \tilde{s}^T \beta - \tilde{s}^T \hat{\beta} \leq h(\beta) - h(\hat{\beta}).
\]

The last inequality is a consequence of \( \tilde{s} \) being in the subdifferential of the seminorm \( h \) at \( \hat{\beta} \), hence \( h(\hat{\beta}) = \tilde{s}^T \hat{\beta} \) and \( \tilde{s}^T \beta \leq h(\beta) \). The previous display can be rewritten as

\[
\varepsilon^T X(\beta - \beta^*) + h(\beta^*) - h(\beta) \leq (\beta^* - \beta) X^T X(\beta^* - \hat{\beta}) + \varepsilon^T X(\beta^* - \hat{\beta}) + h(\beta^*) - h(\hat{\beta}) - \tilde{s}^2. \tag{B.1}
\]

The second line of the right hand side is bounded from above by

\[
(||\varepsilon|| + \tilde{\lambda} hl) \hat{\beta} - \tilde{s}^2 \leq (1/2)(||\varepsilon||^2 + b^2 \tilde{\lambda}^2),
\]

thanks to the elementary inequality \((a + a')\hat{\beta} - \tilde{s}^2 \leq a^2/2 + a'^2/2 \). The first line of the right hand side of (B.1) is bounded from above by \( ||X(\beta^* - \beta)|| \hat{\beta} \) by the Cauchy-Schwarz inequality. Dividing by \( ||X(\beta^* - \beta)|| \) and taking expectations, since \( \max(E||\varepsilon||^2, E[\tilde{\lambda}^2]) \leq M \) we have established that

\[
E(\tilde{\lambda}) \frac{N(\beta^*) - N(\beta)}{||X(\beta^* - \beta)||} \leq E[\hat{\beta}] + \frac{(E||\varepsilon||^2 + b^2 E[\tilde{\lambda}^2])}{2||X(\beta^* - \beta)||} \leq E[\hat{\beta}] + \frac{M(1 + b^2)}{2||X(\beta^* - \beta)||}.
\]
Let \( v = \beta^*/\|X\beta^*\| \). By homogeneity, \( b = \sup_{u \in \mathbb{R}^p} \frac{N(v) - N(u)}{\|X(v-u)\|} = \sup_{\beta \in \mathbb{R}^p} \frac{N(\beta^*) - N(\beta)}{\|X(\beta - \beta)\|} \).

By definition of the supremum, there exists \( u \in \mathbb{R}^p \) such that \( \frac{N(v) - N(u)}{\|X(v-u)\|} \geq (1 - \gamma/2)b \).

Define \( \beta = \|X\beta^*\|u \). Then

\[
E[\hat{\lambda}(1-\gamma/2)b] \leq E[\hat{\lambda}] \frac{N(v) - N(u)}{\|X(v-u)\|} = E[\hat{\lambda}] \frac{N(\beta^*) - N(\beta)}{\|X(\beta - \beta)\|} \leq E[\hat{\lambda}] + \frac{M(1 + b^2)}{2\|X(\beta - \beta)\|},
\]

Note that by definition of \( u \) and \( v \), \( \|X(\beta^* - \beta)\| = \|X\beta^*\|\|X(v-u)\| \). Since \( E[\hat{\lambda}] \geq 1/M' \), the right hand side of the previous display is bounded from above by \( E[\hat{\lambda}] + E[\hat{\lambda}]b\gamma/2 \) provided that \( \|X\beta^*\| \geq \frac{1}{(\gamma/2)b} \frac{MM'(1 + b^2)}{2\|X(v-u)\|} \).

\[ \square \]

**Appendix C: Proof of the compatibility constant lower bound**

*Proof of Theorem 3.1.* by definition of the infimum in (1.8) with \( c_0 = 1 \), there exists \( u \in \mathbb{R}^p \) such that \( \|u_{T^c}\|_1 < \|u_{T}\|_1 \) and

\[
\frac{\sqrt{T\|\|Xu\|}}{\sqrt{n}(\|u_{T}\|_1 - \|u_{T^c}\|_1)} \leq \frac{\phi(1, T)}{\sqrt{1 - \gamma}} .
\]

(C.1)

By homogeneity, we can choose \( \|Xu\| = 1 \). Now let \( \beta^* = tu_{T^c} \) for some \( t > 0 \). If \( t > 0 \) is large enough, by (2.5), for any \( \beta \in \mathbb{R}^p \) such that \( X(\beta - \beta^*) \neq 0 \) we have

\[
\sqrt{1 - \gamma} \frac{n\lambda(\|\beta^*\|_1 - \|\beta\|_1)}{|X(\beta - \beta^*)|} \leq E[\|X(\beta - \beta^*)\|].
\]

We now set \( \beta = -tu_{T^c} \) so that \( \beta^* - \beta = tu \). By (C.1), the left hand side of the previous display is bounded from above by \( (1 - \gamma)\lambda\sqrt{T}/\phi(1, T) \) and the proof is complete. \[ \square \]

**Appendix D: Preliminary probabilistic bounds**

In this subsection, we require the following subgaussian assumption on the noise vector \( \varepsilon \) and the design matrix \( X \). This assumption is implied by either Assumption 4.2 or 4.1.

**Assumption D.1.** Given \( \varepsilon \in \mathbb{R}^n \) and \( X \in \mathbb{R}^{n \times p} \), assume that the random variables \( (g_j)_{j=1,...,p} \) in (4.2) satisfy

\[
\forall a \in \{-1, 1\}, \forall j \in [p], \ \forall t > 0, \ \text{Pr}(ag_j > \sigma t) \leq \int_t^{+\infty} \frac{e^{-u^2/2}}{\sqrt{2\pi}} du. \quad (D.1)
\]

Inequality (D.1) means that \( ag_j/\sigma \) is stochastically dominated by a standard normal random variable. Under Assumption 4.2 and Assumption 4.1 above, the random variables \( g_j \) defined in (4.2) are centered, normal with variance at most \( \sigma^2 \) so that (D.1) holds.
We will need the following probabilistic results. Define the pdf and survival function of the standard normal distribution by

\[ \varphi(u) = \frac{e^{-u^2/2}}{\sqrt{2\pi}}, \quad Q(\mu) = \int_{\mu}^{\infty} \varphi(u) \, du \]  

for any \( u, \mu \in \mathbb{R} \).

**Proposition D.1.** Let \( \sigma > 0 \). Let \((g_1, \ldots, g_p)\) be random variables that need not be independent such that (D.1) holds. Let \( \mu > 0 \), let \( d \) be an integer in \([p]\) and denote by \((g^1_1, g^2_1, \ldots, g^d_1)\) a non-increasing rearrangement of \((|g_1|, \ldots, |g_p|)\). Then

the random variable \( Z^2 := \frac{1}{2d\sigma^2} \sum_{j=1}^{d} (g^1_j - \mu)^2_+ \)

satisfies \( \mathbb{E}[Z^2] \leq \log \mathbb{E}e^{Z^2} \leq \log \left( 1 + \frac{2p \varphi(\frac{\mu}{\sigma})}{d} \right) \leq \frac{2p \varphi(\frac{\mu}{\sigma})}{d} \) \[(D.3)\]

where \( \varphi \) is defined in (D.2). Consequently,

\[ \mathbb{E} \sum_{j=1}^{p} (|g_j| - \mu)^2_+ \leq \frac{4p\varphi(\frac{\mu}{\sigma})}{(\frac{\mu}{\sigma})^3}. \tag{D.4} \]

**Proof.** Without loss of generality, by scaling (i.e., replacing \( \mu/\sigma \) and \( g_j \) by \( g_j/\sigma \)) we can assume that \( \sigma^2 = 1 \).

The first inequality in (D.3) is a consequence of Jensen’s inequality while the third inequality is simply \( \log(1 + t) \leq t \) for \( t \geq 0 \). We now prove the second inequality.

By Jensen’s inequality, \( \mathbb{E}\exp(\frac{1}{d} \sum_{j=1}^{d} (g^1_j - \mu)^2_+ / 2) \) is bounded from above by

\[ \frac{1}{d} \sum_{j=1}^{d} \mathbb{E}[e^{(g^1_j - \mu)^2_+ / 2}] = 1 + \frac{1}{d} \sum_{j=1}^{d} \mathbb{E}[e^{(g^1_j - \mu)^2_+ / 2} - 1] \leq 1 + \frac{1}{d} \sum_{a=\pm 1}^{d} \sum_{j=1}^{p} \mathbb{E}[e^{(ag_j - \mu)^2_+ / 2} - 1]. \]

The double sum on the right hand side has \( 2p \) terms, each of which can be bounded as follows. Let \( \varphi, Q \) be defined in (D.2). For each \( a = \pm 1 \) and all \( j = 1, \ldots, p \), since we assume that the tails of \( ag_j \) are dominated by the tails of a standard normal random variable, by stochastic dominance, if \( g \sim \mathcal{N}(0, 1) \) we have

\[ \mathbb{E}[e^{(ag_j - \mu)^2_+ / 2} - 1] \leq \mathbb{E}[e^{(g - \mu)^2_+ / 2} - 1] = \mathbb{E}[I_{g > \mu}(e^{(g - \mu)^2_+ / 2} - 1)] = \frac{\varphi(\mu)}{\mu} - Q(\mu). \]

By integration by parts, inequality \( Q(\mu) \geq \varphi(\mu)(\frac{1}{\mu} - \frac{1}{\mu^3}) \) holds for any \( \mu > 0 \) so that the right hand side of the previous display is bounded from above by \( \varphi(\mu)/\mu^3 \).

For the second part of the proposition, we apply the previous result to \( d = p \) and we obtain (D.4). \( \Box \)
Proposition D.2. Let $T \subset [p]$ with $|T| = k$ and let $\lambda > \mu > 0$ and define

$$
Rem(\lambda, \mu) := \sqrt{1 + \frac{\sigma^2}{\lambda^2} + \frac{4p}{k} \frac{\varphi(\mu/\sigma)}{(\lambda^2 \mu^3)/\sigma^5}}
$$

(D.5)

where $\varphi$ is the standard normal pdf as in (D.2). Let $g_1, \ldots, g_p$ be centered random variables such that (D.1) holds. Define the random variable $S(\lambda, \mu)$ by

$$
S(\lambda, \mu) := \sup_{u \neq 0} \frac{\sum_{j=1}^{p} g_j u_j + \lambda \sum_{j \in T} (|\beta_j^*| - |\beta_j^* + u_j|) - \mu \sum_{j \in T^c} |u_j|}{\|u\|}.
$$

Then $\mathbb{E}[S(\lambda, \mu)] \leq \lambda \sqrt{k} \ Rem(\lambda, \mu)$.

Proof. Let $s_j \in \{\pm 1\}$ be the sign of $\beta_j^*$ for $j \in T$. By simple algebra on each coordinate, we have almost surely

$$
S(\lambda, \mu) \leq \sup_{u \neq 0} \frac{\sum_{j \in T^c} (g_j - s_j \lambda) u_j + \sum_{j \in T^c} (|g_j| - \mu)^+ |u_j|}{\|u\|}.
$$

Hence by the Cauchy-Schwarz inequality,

$$
S(\lambda, \mu)^2 \leq \sum_{j \in T^c} (g_j - s_j \lambda)^2 + \sum_{j \in T^c} (|g_j| - \mu)^2_+.
$$

Thanks to the assumption on $g_1, \ldots, g_p$, we have $\mathbb{E}[\sum_{j \in T^c} (g_j - s_j \lambda)^2] \leq k(\lambda^2 + \sigma^2)$ and (D.4) provides an upper bound on $\mathbb{E} \sum_{j \in T^c} (|g_j| - \mu)^2_+$. Jensen’s inequality yields $\mathbb{E}[S(\lambda, \mu)] \leq \mathbb{E}[S(\lambda, \mu)^2]^{1/2}$ which completes the proof.

Proposition D.3 (From a bound on the expectation to subgaussian tails). Let $\beta^* \in \mathbb{R}^p$ and let $T$ be its support with $|T| = k$. Let $g_1, \ldots, g_p$ be centered normal random variable with variance at most $\sigma^2$, let $g = (g_1, \ldots, g_p)^\top$ and $\Sigma = \mathbb{E}[gg^\top]/\sigma^2$. Let $\lambda > \mu > 0$ and let $\theta_1 > 0$. Then for any $t > 0$, with probability at least $1 - e^{-t^2/2}$,

$$
\sup_{u \in \mathbb{R}^p: u \neq 0} \left[ \frac{\sum_{j=1}^{p} g_j u_j + \lambda \sum_{j \in T} (|\beta_j^*| - |\beta_j^* + u_j|) - \mu \sum_{j \in T^c} |u_j|}{\max \left( \|u\|, \frac{1}{\sqrt{\pi}} \|\Sigma^{1/2} u\| \right)} \right] \leq \lambda \sqrt{k} Rem(\lambda, \mu) + \sigma \theta_1 t
$$

(D.6)

where $Rem(\cdot, \cdot)$ is defined in (D.5).

Proof. Let $x \sim N(0, I_{p \times p})$ and write $g = \sigma \Sigma^{1/2} x$. Denote by $G(x)$ the left hand side of (D.6). Observe that $G$ is $(\sigma \theta_1)$-Lipschitz, so that by the Gaussian concentration theorem [12, Theorem 5.6], with probability at least $1 - e^{-t^2/2}$ we have $G(x) \leq \mathbb{E}[G(x)] + \sigma \theta_1 t$. Proposition D.2 provides an upper bound on $\mathbb{E}G(x)$. 

\qed
Appendix E: Lasso upper bounds

Theorem E.1. There exists an absolute constant $C > 1$ such that, if $p/k \geq C$ then the following holds. Let $T$ be the support of $\beta^*$ and assume that $\|\beta^*\|_0 \leq k$. Let either Assumption 4.2 of Assumption 4.1 be fulfilled. Let $\lambda > \mu > 0$ and let $\text{Rem}(\cdot, \cdot)$ be defined by (D.5). Define the constant $\theta_1$ and the sparsity level $s_1$ by

$$
\theta_1 \triangleq \inf_{v \in \mathbb{R}^p : \|v_T\|_1 \leq \sqrt{s_1}} \frac{\|\Sigma^{1/2}v\|}{\|v\|} \quad \text{where} \quad \sqrt{s_1} = \frac{\lambda \sqrt{k} \text{Rem}(\lambda, \mu) + \sigma t}{\lambda - \mu}
$$

and assume that $\theta_1 > 0$. Let $t > 0$. Define $\rho_n(t)$ by $\rho_n(t) = 0$ under Assumption 4.1 and $\rho_n(t) = \frac{c}{\sqrt{n}} (t + \sqrt{s_1 \log(2p/s_1)})$ under Assumption 4.2. Assume that $\rho_n(t) < 1$. Then the Lasso with tuning parameter $\lambda$ satisfies with probability at least $1 - 3 \exp(-t^2/2)$

$$
\theta_1 \|u\| \leq \|\Sigma^{1/2}u\| \leq \frac{\|Xu\|}{\sqrt{n}(1 - \rho_n(t))} \leq \frac{\lambda \sqrt{k} \text{Rem}(\lambda, \lambda) + \sigma t}{\lambda \sqrt{n}(1 - \rho_n(t))^2} \quad \text{where} \quad u = \hat{\beta} - \beta^*.
$$

Proof of Theorem E.1 under Assumption 4.1. Let $u = \hat{\beta} - \beta^*$. By standard calculations, or, for instance, Lemma A.2 in [4] we have

$$
\|Xu\|^2 \leq e^T Xu + h(\beta^*) - h(\hat{\beta}) = \sqrt{n} \sum_{j=1}^p g_j u_j + \sqrt{n} \lambda (\|\beta^*\|_1 - \|u + \beta^*\|_1)
$$

where $g_1, \ldots, g_p$ are defined in (4.2). Proposition D.3 with $\mu = \lambda$ implies that with probability $1 - e^{-t^2/2}$ we have

$$
\|Xu\|^2 \leq \sqrt{n} N(u) \left[ \lambda \sqrt{k} \text{Rem}(\lambda, \lambda) + \sigma t \right]
$$

where $N(u) = \|u\| \vee \frac{\|\Sigma^{1/2}u\|}{\|\hat{\beta}\|}$.

We use again Proposition D.3 with $\mu \in (0, \lambda)$, which implies on an event of probability at least $1 - e^{-t^2/2}$ we have

$$
\|Xu\|^2 \leq \sqrt{n} \left( N(u) \left[ \lambda \sqrt{k} \text{Rem}(\lambda, \mu) + \sigma t \right] - (\lambda - \mu) \|u_T\|_1 \right)
$$

where we used that $\theta_1 \leq 1$ so that $\sigma \theta_1 t \leq \sigma t$. We claim that on this event, $\theta_1 \|u\| \leq \|\Sigma^{1/2}u\|$ must hold. If this was not the case, then $N(u) = \|u\|$ and by the previous display we have $\|u_T\|_1 \leq \sqrt{s_1} \|u\|$, which implies $\theta_1 \|u\| \leq \|\Sigma^{1/2}u\|$ by definition of $\theta_1$. Hence

$$
N(u) = \frac{\|\Sigma^{1/2}u\|}{\theta_1}, \quad \|u\| \leq \frac{\|\Sigma^{1/2}u\|}{\theta_1} \quad \text{and} \quad \frac{\|u_T\|_1}{\sqrt{s_1}} \leq \frac{\|\Sigma^{1/2}u\|}{\theta_1}
$$

must hold. Inequalities (E.1) and (E.3) hold simultaneously on an event of probability at least $1 - 2e^{-t^2/2}$ thanks to the union bound. Combined with (E.1), this completes the proof under Assumption 4.1 with $\rho_n(t) = 0$ and $\Sigma = \frac{1}{n} X^T X$. \qed
To prove the result under Assumption 4.2, we will need the following which can be deduced from the results of [22, 32].

**Proposition E.2.** There exists an absolute constant $C > 0$ such that the following holds. Let Assumption 4.2 be fulfilled. Consider for some $s_1 > 0$ and some $\theta_1$ the cone

$$ C(s_1) = \left\{ u \in \mathbb{R}^p : \max \left( \frac{\|u\|_1}{\sqrt{s_1}}, \frac{\|u\|_1}{\sqrt{2}} \right) \leq \frac{\|\Sigma^{1/2}u\|}{\theta_1} \right\}. \quad (E.4) $$

Then with probability at least $1 - e^{-t^2/2}$,

$$ \sup_{u \in C, \Sigma^{1/2}u \neq 0} \frac{\frac{1}{\sqrt{n}}\|Xu\| - \|\Sigma^{1/2}u\|}{\|\Sigma^{1/2}u\|} \leq C \left( \frac{t}{\sqrt{n}} + r_n(s_1) \right) \quad (E.5) $$

where $r_n(s_1) := \sqrt{s_1 \log(2p/s_1)}/\theta_1 \sqrt{n}$.

**Proof.** Let $C = C(s_1)$ and $s = s_1$ for brevity. We apply [32, Theorem 1.4 with $A = X\Sigma^{-1/2}$] which yields that for any $t > 0$, with probability at least $1 - e^{-t^2/2}$ we have

$$ \sup_{u \in C, \|\Sigma^{1/2}u\| = 1} \frac{\frac{1}{\sqrt{n}}\|Xu\| - \|\Sigma^{1/2}u\|}{\|\Sigma^{1/2}u\|} \leq C \left( \frac{t + \gamma(\Sigma^{1/2}C)}{\sqrt{n}} \right) $$

where $\gamma(\Sigma^{1/2}C) := \mathbb{E}\sup_{u \in C, \|\Sigma^{1/2}u\| = 1} |G^T \Sigma^{1/2}u|$ where the expectation is with respect to $G \sim N(0, I_{p \times p})$. Define $n_1, ..., n_p$ by $n_j = G^T \Sigma^{1/2}e_j$ so that $n_j$ is centered normal with variance at most 1. Let $(n_{j1}, ..., n_{jp})$ be a non-increasing rearrangement of $(|n_1|, ..., |n_p|)$. Then for any $u = (u_1, ..., u_p) \in C$ we have

$$ G^T \Sigma^{1/2}u \leq \sum_{j=1}^p u_j^+ n_{j1}^+ \leq \left( n_{j1}^+ \right)^2 + \cdots + \left( n_{jp}^+ \right)^2 \frac{1}{s} \|u\|_1 \|u\|_1, $$

$$ \leq 3\sqrt{s}/\theta_1 \left( \frac{n_{j1}^+}{s} + \cdots + \frac{n_{jp}^+}{s} \right)^{1/2}. $$

To bound the expectation of $Z := \sqrt{3/8} \left( n_{j1}^2 + \cdots + n_{jp}^2 \right)^{1/2}$ from above, we proceed as in [4, Proposition E.1]. By Jensen’s inequality,

$$ \mathbb{E}[Z]^2 \leq \mathbb{E}[Z]^2 \leq \log(\mathbb{E}[\exp(Z^2)]) \leq \log(\frac{1}{s} \sum_{j=1}^p \mathbb{E}[\exp((3/8)n_{j1}^2)]) \leq \log(2p/s) $$

where we used that $\mathbb{E}[\exp(3n_{j1}^2/8)] \leq 2$ if $n_{j1}$ is centered normal with variance at most 1. We conclude that $\gamma(\Sigma^{1/2}C) \leq Cr_n(s_1)\sqrt{n}$ for some large enough absolute constant $C > 0$ and the proof is complete. \(\square\)
Proof of Theorem E.1 under Assumption 4.2. Since the random vector \( g \) defined in (4.2) satisfies \( \mathbb{E}[(g^T u)^2] = \sigma^2 \| \Sigma^{1/2} u \|_2 \), inequality (E.1) holds with \( \Sigma = \Sigma \) which is the covariance matrix of the rows of \( X \). By the union bound, there is an event of probability at least \( 1 - 3e^{-t^2/2} \) on which (E.1), (E.2), (E.3) and (E.5) must all hold. In particular, (E.3) implies that \( u \in C(s_1) \) where \( C(s_1) \) is the cone (E.4).

By (E.5), inequality \( \| Xu \| / \sqrt{n} \geq \| \Sigma^{1/2} u \|_2 (1 - \rho_n(t)) \) holds for \( u \in C(s_1) \). Combining this with (E.1) completes the proof under Assumption 4.2.

For \( x > 0 \) and some function \( f \) as in (4.7), define the quantity

\[
\mu_f(x) = \sigma \sqrt{2 \log(x) - 5 \log \log(x) - \log(8\pi) + 2f(x)} \tag{E.6}
\]

The only difference between \( L_f \) in (4.7) and \( \mu_f \) in (E.6) is the term \( \log(8\pi) \) replaced by \( \log(4\pi) \), so that \( L_f(x)^2 - \mu_f(x)^2 = \log(2) \).

Proof of Theorem 4.2. Recall that \( L_f \) is defined in (4.7) for some function \( f \) as in (4.7). We have \( \lambda = L_f(\frac{\mu}{K}) \) and for brevity, let \( \mu = \mu_f(\frac{\mu}{K}) \).

By simple algebra, using that \( 1/\lambda \leq 1/\mu_f(\frac{\mu}{K}) \leq 1/\mu_0(\frac{\mu}{K}) \), the quantity \( \text{Rem}(\cdot, \cdot) \) defined in (D.5) satisfies

\[
\text{Rem}(\lambda, \lambda)^2 \leq 1 + \frac{\sigma^2}{\lambda^2} + \eta(\frac{\mu}{K}) \exp[-f(\frac{\mu}{K})] \quad \text{where} \quad \eta(x) := \left( \frac{\sqrt{2 \log(x)}}{\mu_0(x)} \right)^5
\]

as well as

\[
\text{Rem}(\lambda, \mu)^2 \leq 1 + \frac{\sigma^2}{\lambda^2} + \sqrt{2} \eta(\frac{\mu}{K}) \exp[-f(\frac{\mu}{K})] \leq 1 + \frac{\sigma^2}{\lambda^2} + \sqrt{2} \eta(\frac{\mu}{K}).
\]

It is clear that \( \eta(x) \to 1 \) as \( x \to +\infty \), and that \( \text{Rem}(\lambda, \mu)^2 \leq 4 \) if \( p/k \geq C > 0 \) for some large enough absolute constant \( C > 0 \).

We now bound \( s_1 \) from above. By definition of \( L_f \) and \( \mu_f \),

\[
\lambda - \mu = \frac{\lambda^2 - \mu^2}{\mu + \lambda} = \frac{\sigma^2 \log 2}{\mu + \lambda} \geq \frac{\sigma^2 \log 2}{2\lambda}, \quad \tag{E.7}
\]

so that, using that \( \lambda \leq \sigma \sqrt{2 \log(p/k)} \),

\[
\frac{(\lambda - \mu)}{\lambda} \geq \frac{\log 2}{4 \log(p/k)} \geq \frac{1}{6 \log(p/k)}.
\]

Since \( \sigma t \leq \sqrt{k}\lambda \),

\[
\sqrt{s_1} \leq \frac{\lambda \sqrt{k}(\text{Rem}(\lambda, \mu) + 1)}{\lambda - \mu} \leq \frac{\lambda \sqrt{k} 3}{\lambda - \mu} \leq \sqrt{k} 24 \log(p/k)
\]

provided that \( p/k \geq C \) for some large enough absolute constant \( C > 0 \).

This calculation shows that \( \theta, \tilde{k} \) from Theorem 4.2 and \( \theta_1, s_1 \) from Theorem E.1 satisfy \( s_1 \leq \tilde{k} \) and \( 1/\theta_1 \leq 1/\theta \). Finally, for \( t \leq \lambda \sqrt{k}/\sigma \), the quantity \( \rho_n(t) \) from Theorem E.1 satisfies \( \rho_n(t) \leq C' r_n \) where \( r_n \) is defined in Theorem 4.2. Hence Theorem 4.2 is a consequence of Theorem E.1. \( \square \)
Appendix F: RIP property of sparse vectors

Proposition F.1. There exists an absolute constant $C > 0$ such that the following holds. Let Assumption 4.2 be fulfilled. Then

$$\mathbb{E} \left[ \sup_{u \in \mathbb{R}^p : \|u\|_0 \leq d, \|\Sigma^{1/2} u\| = 1} \left| \frac{1}{n} \|Xu\|^2 - \|\Sigma^{1/2} u\|^2 \right| \right] \leq C \frac{\sqrt{d \log(ep/d)}}{\sqrt{n}}$$

holds and with probability at least $1 - e^{-t^2/2}$ we have

$$\sup_{u \in \mathbb{R}^p : \|u\|_0 \leq d, \|\Sigma^{1/2} u\| = 1} \left| \frac{1}{\sqrt{n}} \|Xu\| - \|\Sigma^{1/2} u\| \right| \leq C \frac{\sqrt{d \log(ep/d) + t}}{\sqrt{n}}.$$

Proof. We apply the second part of Theorem D in [34] which yields that the left hand side of the first claim is bounded from above by $C_1 \sqrt{\frac{d}{n}}$ for some absolute constant $C_1 > 0$ where $\Gamma = \mathbb{E} \sup_{u \in \mathbb{R}^p : \|u\|_0 \leq d, \|\Sigma^{1/2} u\| = 1} |G^T \Sigma^{1/2} u|$ where the expectation is with respect to $G \sim N(0, I_{p \times p})$. By the Cauchy-Schwarz inequality, Jensen’s inequality and the fact that the maximum of a set of positive numbers is smaller than the sum of all the elements of the set, we have for any $\lambda > 0$

$$\Gamma \leq \mathbb{E} \sup_{T \subseteq \{1, \ldots, p\} : |T| = d} \|\Pi_T G\| \leq \frac{1}{\lambda} \log \left( \sum_{T \subseteq \{1, \ldots, p\} : |T| = d} \mathbb{E} \exp \lambda \|\Pi_T G\| \right)$$

where for any support $T$, the matrix $\Pi_T$ is the orthogonal projection onto the subspace $\{\Sigma^{1/2} u, u \in \mathbb{R}^p \text{ s.t. } u_T = 0\}$. The random variable $\|\Pi_T G\|^2$ has chi-square distribution with degrees of freedom at most $d$ and the function $G \rightarrow \|\Pi_T G\|$ is 1-Lipschitz so that $\mathbb{E} \exp \lambda \|\Pi_T G\| \leq \exp(\lambda \sqrt{d} + \lambda^2/2)$ by Theorem 5.5 in [12]. There are $N = \binom{p}{d}$ supports of size $d$ so that

$$\Gamma \leq \frac{1}{\lambda} \log \left( N \exp(\lambda \sqrt{d} + \lambda^2/2) \right) = \sqrt{d} + \frac{\lambda^2}{2} + \frac{1}{\lambda} \log N.$$

Now set $\lambda = \sqrt{2 \log N} = 1/2$ so that $\Gamma \leq \sqrt{d} + \sqrt{2 \log N}$. By a standard bound on binomial coefficients, $\log N \leq d \log(ep/d)$ and $\Gamma \leq C_2 \sqrt{d \log(ep/d)}$ for some absolute constant $C_2 > 0$.

The second part of the proposition follows from [32, Theorem 1.4 with $A = X \Sigma^{-1/2}$] and the upper bound on $\Gamma$ derived in the previous paragraph.

Appendix G: Proof of Lasso lower bounds

Proof of Theorem 4.4. Taking expectations in (2.2), we obtain

$$\mathbb{E} \sup_{u \in \mathbb{R}^p : \|u\| \leq 1} [x^T Xu - h(u)] \leq \mathbb{E} \|X(\hat{\beta} - \beta^*)\|.$$

Let $\Omega \subseteq \{-1, 0, 1\}^p$ be given by Lemma K.2. For any $w \in \Omega$, define $u_w$ as $u_w = (1/\sqrt{dn})w$. Then, thanks to the properties of $\Omega$ in Lemma K.2, $\|Xu_w\| = \|Xw\|/\sqrt{nd} \leq 1$.
Next, notice that $h(u_w) = \lambda \sqrt{d}$ for all $w \in \Omega$. Thus

$$
\mathbb{E} \sup_{w \in \Omega} \epsilon^T X u_w - \lambda \sqrt{d} \leq \mathbb{E} \sup_{u \in \mathcal{V} : \|X u\| \leq 1} [\epsilon^T X u - h(u)]. \tag{G.1}
$$

For any two distinct $w, w' \in \Omega$, by Lemma K.2 we have $\mathbb{E}[(\epsilon^T X (u_w - u'_w))^2] \geq \sigma^2(1 - \delta_{2d})^2$. By Sudakov’s lower bound (see for instance Theorem 13.4 in [12]) we get

$$
\mathbb{E}[\sup_{w \in \Omega} \epsilon^T X u_w] \geq \frac{\sigma}{2} (1 - \delta_{2d}) \sqrt{\log |\Omega|} \geq \frac{\sigma}{4} (1 - \delta_{2d}) \sqrt{d \log (p/(5d))}. \tag{G.2}
$$

Combining (G.1) and the previous display, we obtain the desired lower bound provided that $\lambda$ satisfies (4.11).

**Proof of Theorem 4.3.** We will bound from below the noise barrier lower bound (2.2). By monotonicity with respect to $\lambda$, it is enough to prove the result for $\lambda = L_0(\frac{p}{k})$; hence we assume that $\lambda = L_0(\frac{p}{k})$. By scaling, we can also assume that $\sigma^2 = 1$. Let $(g_j)_{j=1}^p$ be defined by (4.2) and define $u \in \mathbb{R}^p$ with $u_j = \text{sgn}(g_j)(|g_j| - \lambda)_+$ be the soft-thresholding operator applied to $(g_1, \ldots, g_p)$. The lower bound (2.2) implies

$$
Z := \left[ \sum_{j=1}^p (|g_j| - \lambda)_+^2 \right]^{1/2} = \frac{1}{\|u\|} \sum_{j=1}^p g_j u_j - \lambda |u_j| \leq \|X(\beta - \beta^*)\|_2 \frac{\|X u\|}{\|u\| \sqrt{n}}. 
$$

Since $\Sigma_{jj} = 1$ for all $j = 1, \ldots, p$, each $g_j$ has $N(0, 1)$ distribution, an integration by parts reveals

$$
\mathbb{E}[Z^2] = \mathbb{E}[(|g_j| - \lambda)^2_+] = 2 \left[ -\lambda \varphi(\lambda) + (\lambda^2 + 1) Q(\lambda) \right]
$$

where $\varphi, Q$ are defined in (D.2) (this formula for $\mathbb{E}[(|g_j| - \lambda)^2_+]$ for $g_j \sim N(0, 1)$ is obtained in [19, equation (68)] or [26, Appendix 11]). By differentiating, one can readily verify that

$$
Q(\lambda) = \varphi(\lambda) \left[ \frac{1}{\lambda} - \frac{1}{\lambda^3} + \frac{3}{\lambda^5} - \frac{15}{\lambda^7} \right] + \int_{\lambda}^{+\infty} \frac{105 \varphi(u)}{u^8} du.
$$

Since the rightmost integral is positive, this yields a lower bound on $Q(\lambda)$. This implies that, as $p/k \to +\infty$,

$$
\mathbb{E}[Z^2] \geq 2p \varphi(\lambda) \left[ \frac{2}{\lambda^3} - \frac{12}{\lambda^5} - \frac{15}{\lambda^7} \right] \simeq \frac{p 4 \varphi(\lambda)}{\lambda^3} \simeq \frac{k_4 \log (p/k) \sqrt{\pi}}{2 \sqrt{2}} \sqrt{\frac{2}{2 \pi}} \simeq 2k \log (p/k),
$$

where $a \asymp b$ means $a/b \to 1$ and we used that $\lambda = L_0(\frac{p}{k})$ and that $\sqrt{2 \log (p/k)}/\lambda \to 1$. To complete the proof of the first part of the Theorem, it remains to bound $\|X u\|/\sqrt{n}\|u\|$ from above in (G.3). We now prove that $u$ is $d$-sparse with high probability, where $d \in [1, p]$ is an integer that will be specified later. Denote by $g_1^*, g_2^*, \ldots, g_p^*$ a non-increasing rearrangement of $(|g_1|, \ldots, |g_p|)$. We have $\lambda = L_0(\frac{p}{k})$ and for brevity, let
\[ \mu = \mu_0(\xi) \] where the notation \( L_f \) and \( \mu_f \) are defined in (4.7) and (E.6). Since \( u \) is the soft-thresholding operator with parameter \( \lambda \) applied to \( g \), the following implication holds

\[ (g_d^1 - \mu)^2_+ \leq (\lambda - \mu)^2 \Rightarrow g_d^1 \leq \lambda \Rightarrow u \text{ is } d\text{-sparse}. \]

Hence, by (E.7), in order to show that \( u \) is \( d\)-sparse, it is enough to prove that \( (g_d^1 - \mu)_{+} \leq \frac{\log^2(2g)}{4\lambda^2} \) holds. By (D.3) applied to \( \mu \) we have

\[ \lambda^2 \mathbb{E}(g_d^1 - \mu)_{+}^2 \leq \lambda^2 \frac{4p \varphi(\mu)}{d} \leq \lambda^2 \frac{2\sqrt{2}k \log(p/k)}{d} \approx \frac{4k \log^2(p/k)}{d}, \]

where \( a \approx b \) means \( a/b \to 1 \). Thus, if \( d = Ck \log^2(p/k) \) for a large enough absolute constant \( C > 0 \) and \( k, p, n \) large enough, by Markov inequality, we have \( (g_d^1 - \mu)_{+}^2 \leq \frac{\log^2(2g)}{4\lambda^2} \) with probability at least \( 11/12 \), hence \( u \) is \( d\)-sparse with probability at least \( 11/12 \).

- This completes the proof of (4.9) under Assumption 4.1, because in this case \( \frac{n\|\bar{X}u\|}{\sqrt{\|u\|}} = \frac{\|\Sigma^{1/2}u\|}{\|u\|} \leq \psi \) on the event of probability at least \( 11/12 \) where \( u \) is \( d\)-sparse.

- In the random design setting, i.e., under Assumption 4.2, we have \( \bar{\Sigma} = \Sigma \) and \( \frac{n\|\bar{X}u\|}{\sqrt{\|u\|}} \leq \psi \frac{\|\bar{X}u\|}{\|u\|} \). Furthermore, by Proposition F.1 and the definition of \( d \) we have

\[ \mathbb{P} \left( \sup_{\|u\|_{op} \leq d} \frac{\|u\|_{op}}{\sqrt{n}} \leq 1 + C_3 \frac{\sqrt{k \log^3(p/k)}}{n} \right) \geq 11/12 \]

for some absolute constant \( C_3 > 0 \). The quantity \( k \log^3(p/k)/n \) converges to 0 in (4.4). Combining the two events, each of probability at least \( 11/12 \), we obtain (4.9).

We now prove the second part of the Theorem, under the additional assumption that \( \|\Sigma\|_{op} = o(k \log(p/k)) \). Since \( g \sim N(0, \sigma^2 \Sigma) \) (cf. (4.2)), let \( x \sim N(0, I_p) \) be such that \( g = \sigma \Sigma^{1/2}x \). Define \( F(x) = Z = \|u\| \), where, as above, \( u \) is the soft-thresholding operator applied to \( g \). Then for two \( x_1, x_2 \in \mathbb{R}^p \), by the triangle inequality and the fact that the soft-thresholding operator is 1-Lipschitz, we have

\[ |F(x_1) - F(x_2)| \leq \|\Sigma^{1/2}(x_1 - x_2)\| \]

Hence \( F \) is \( \|\Sigma^{1/2}\|_{op}\)-Lipschitz. By Gaussian concentration and the Poincaré inequality (see for instance [11, Theorem 3.20]), the variance of \( F(x) \) is at most \( \|\Sigma\|_{op} \), so that \( \mathbb{E}[F(x)^2] \geq \mathbb{E}[F(x)^2] - \|\Sigma\|_{op} \). If \( \text{Med}[Y] \) denotes the median of any random variable \( Y \) then we have \( |\mathbb{E}[F(x)] - \text{Med}[F(x)]| \leq \|\Sigma^{1/2}\|_{op} \sqrt{\pi/2} \), cf. the discussion after (1.6) in [31, page 21]. Combining these inequalities with the definition of the median, with obtain that with probability at least \( 1/2 \) we have

\[ Z = F(x) \geq \text{Med}[F(x)] \geq \sqrt{\mathbb{E}[F(x)^2] - \|\Sigma\|_{op} - \|\Sigma^{1/2}\|_{op} \sqrt{\pi/2}}. \]
Since we have established above that \( \mathbb{E}[F(x)^2] = \mathbb{E}[Z^2] \approx 2\sigma^2 k \log(p/k) \), this shows that
\[
P \left[ Z \geq \sqrt{2k \log(p/k)} | 1 - o(1) \right] \geq 1/2.
\]
Since \( 5/6 - 1/2 = 1/3 \), the union bound and (4.9) complete the proof. □

**Appendix H: Proof of lower bounds for nuclear norm penalized estimators**

**Proof of Theorem 6.1.** Inequality (2.2) implies that
\[
\mathbb{E} \sup_{u \in \mathbb{R}^{m \times 1} : \|Xu\| \leq 1} \left[ \sum_{i=1}^{n} \varepsilon_i \text{tr}(X_i'u) - \sqrt{n} \bar{\lambda} \|u\|_S \right] \leq \mathbb{E}\|X(\hat{\beta} - \beta^*)\|
\]
For any \( \gamma > 0 \), [37, Section 7, proof of Theorem 5] proves the existence of a finite set \( \mathcal{A}^0 \) of matrices such that \( 0 \in \mathcal{A}^0 \), the cardinality of \( \mathcal{A}^0 \) is at least \( 2^{rm/8} \) and
\[
\|u\|_{S_2} = \gamma \sqrt{rm/n}, \quad \text{rank} (u) \leq r, \quad \|u - v\|_{S_2} \geq (\gamma/8) \sqrt{rm/n}
\]
for any \( u, v \in \mathcal{A}^0 \). Set \( \gamma = 1/(1 + \delta_r) \sqrt{rm/n} \) for the remaining of this proof. Then for any \( u \in \mathcal{A}^0 \) we get \( \|Xu\| \leq 1 \) and \( \sqrt{n} \|u\|_{S_1} \leq \sqrt{rm} \|u\|_{S_2} \leq \sqrt{r/(1 + \delta_r)} \). By restricting the above supremum to matrices in \( \mathcal{A}^0 \), we get
\[
\mathbb{E} \sup_{u \in \mathcal{A}^0} \left[ \sum_{i=1}^{n} \varepsilon_i \text{tr}(X_i'u) - \sqrt{n} \mathbb{E}[\bar{\lambda}] \right] \leq \mathbb{E}\|X(\hat{\beta} - \beta^*)\|.
\]
By Sudakov inequality (see for instance Theorem 13.4 in [12]), the properties of \( \mathcal{A}^0 \) and the Restricted Isometry property, we obtain a lower bound on the expected supremum which yields the desired result. □

**Appendix I: Bias lower bound under beta-min condition**

**Proof of Proposition 3.3.** Let \( \hat{r} = \|X(\hat{\beta} - \beta^*)\| \) for brevity and set \( \bar{C} = 1 + \gamma \). Denote by \( \hat{\beta} \) the deterministic vector \( \beta := \beta^* - \lambda \bar{s}/\sqrt{n} \), i.e., each of the \( k \) nonzero coordinates of \( \beta^* \) is shrunk by \( \lambda/\sqrt{n} \). If \( \psi^2 = \|X\bar{s}\|^2/(nk) \) we have
\[
\|X(\beta^* - \beta)\|^2 = \lambda^2 \|X\bar{s}\|^2 / n = \lambda^2 k \psi^2 \cdot
\]
Let \( u = \hat{\beta} - \beta^* \). By assumption, we have \( \varepsilon^T X u + h(\beta^*) - h(\hat{\beta}) \leq \bar{C} \lambda \sqrt{k} \hat{r} \) with probability at least \( 5/6 \). Since \( \varepsilon^T X (\beta - \beta^*) \) is a symmetric random variable, it is non-negative with probability \( 1/2 \). By the union bound, the intersection of these two events has probability at least \( 1/3 \) and on this intersection, (B.1) implies
\[
h(\beta^*) - h(\beta) \leq (X u)^T X (\beta - \beta^*) + \hat{r} \bar{C} \lambda \sqrt{k} - \hat{r}^2 \leq \hat{r} \lambda \sqrt{k} + \bar{C} \lambda \sqrt{k} - \hat{r}. \quad (I.1)
\]
Let \( R = \lambda \sqrt{k} \max(\psi, \bar{C}) \). By definition of \( \beta \), \( h(\beta^*) - h(\beta) = \lambda^2 k \). This yields
\[
\lambda^2 k \leq \hat{r} (2R - \hat{r}) = - (\hat{r} - R)^2 + R^2,
\]
hence $(\hat{r} - R)^2 \leq (R - \lambda\sqrt{k})(R + \lambda\sqrt{k}) \leq (R - \lambda\sqrt{k})^2 R$ and

$$|\hat{r} - 1| \leq \sqrt{2(1 - \lambda\sqrt{k}/R)}.$$  

By definition of $\nu$ we have $R/\lambda\sqrt{k} = 1 + \nu/2$. Using $\sqrt{2(1 - 1/(1 + \nu/2))} \leq \sqrt{\nu}$ yields the desired result.

Next, we bound $\|X(\hat{\beta} - \beta)\|$ from above. Using (1.1), on the above event of probability at least $1/3$ we get

$$\|X(\hat{\beta} - \beta)\|^2 = \hat{r}^2 + \|X(\beta - \beta^\ast)\|^2 + 2(Xu)^\top(X(\beta^\ast - \beta)),$$

$$= \hat{r}^2 + \lambda^2 k\psi^2 + 2(\hat{r}C\lambda\sqrt{k} - \lambda^2 k - \hat{r}^2),$$

$$= \lambda^2 k\psi^2 + 2(\hat{r}C\lambda\sqrt{k} - \lambda^2 k - \hat{r}^2),$$

$$\leq R^2 + 2\hat{r}R - 2\lambda^2 k - \hat{r}^2.$$  

Using the bound $R(1 - \sqrt{\nu}) \leq \hat{r}$ derived above as well as $-\lambda^2 k \leq -\hat{r}^2/(1 + \nu)^2 \leq -\hat{r}^2/(1 + \nu/2)^2$ we obtain

$$\frac{\|X(\hat{\beta} - \beta)\|^2}{\hat{r}^2} \leq \frac{1}{(1 - \sqrt{\nu})^2} + \frac{2}{1 - \sqrt{\nu}} - \frac{2}{(1 + \nu/2)^2} - 1 \leq 40\sqrt{\nu}$$

where we used that $\nu \in [0, 1/2]$ for the last inequality. 

\[\]  

**Appendix J: Data-driven tuning parameters**

**Proof of Theorem 5.1.** To prove Theorem 5.1, we apply (5.2) to $N(\cdot) = \sqrt{n} \cdot ||\cdot||_1$ and $\rho = \sqrt{d}$, and use the lower bound on the expected supremum (G.2) obtained in the proof of Theorem 4.4. The details are omitted. 

**Proof of Theorem 5.2.** By scaling, we may assume that $\sigma^2 = 1$. Let $d = k \log^2(p/k)$ so that $p/d \geq 3$. Define $(g_1, ..., g_p)$ by (4.2) so that $\varepsilon^\top Xu = \sqrt{n} \sum_{j=1}^p g_j u_j$ for any $u \in \mathbb{R}^p$. Since the entries of $X$ are iid $N(0, 1)$ and $\varepsilon$ is deterministic (cf. Assumption 4.2), it is clear that the random vector $(g_1, ..., g_p)^\top$ is jointly Gaussian, centered, with covariance matrix $I_{p \times p}$.

By (2.2), for any $u \in \mathbb{R}^p$ the inequality $\varepsilon^\top Xu - \hat{\lambda}\sqrt{n} \|u\|_1 \leq \|Xu\| \|X(\hat{\beta} - \beta^\ast)\|$ holds. Applying this inequality to every $d$-sparse vector $u \in \{\pm \frac{1}{\sqrt{nd}}, 0, \frac{1}{\sqrt{nd}}\}^p$, taking expectations and using the Cauchy-Schwarz inequality on the right hand side, we get

$$\mathbb{E}\left(\frac{1}{\sqrt{d}} \sum_{k=1}^d g_k^2\right) - \sqrt{d}\mathbb{E}[\hat{\lambda}] \leq \mathbb{E}\left[\|X(\hat{\beta} - \beta^\ast)\|^2\right]^{1/2} \left(\mathbb{E}\sup_{u \in \{\pm \frac{1}{\sqrt{nd}}, 0, \frac{1}{\sqrt{nd}}\}^p} \|Xu\|^2\right)^{1/2}$$

where $(g_1^+, ..., g_p^+)$ is a non-increasing rearrangement of $(|g_1|, ..., |g_p|)$. The left hand side is bounded from below by $\sqrt{d}(\mathbb{E}[g_k^2] - \mathbb{E}[\hat{\lambda}])$. By assumption $\mathbb{E}[\hat{\lambda}] \leq L_0(\frac{p}{k})$ holds, so it is enough to prove the result in the case $\mathbb{E}[\hat{\lambda}] = L_0(\frac{p}{k})$. Denote by $L$ the value $L = L_0(\frac{p}{k})$ and we assume in the rest of the proof that $\mathbb{E}[\hat{\lambda}] = L$. 


Let $F$ be the cdf of the absolute value of a standard normal random variable. Then $t \to 1/F(t)$ is convex so that by [21, (4.5.6), Section 4.5] with $r = p + 1 - d$ we have $F^{-1}(1 - d/p) \leq \mathbb{E}[g_d^1]$. Using the lower bound [10, Proposition 4.1(iii)], we get

$$\sqrt{2\log(2p/d) - \log(2p/d) - \log(4\pi)} \leq F^{-1}(1 - d/p) \leq \mathbb{E}[g_d^1].$$

provided that $p/d \geq 3$. Hence, using $a - b = \frac{a^2 - b^2}{a+b}$ with $a = \mathbb{E}[g_d^1]$ and $b = L$ we get

$$\mathbb{E}[g_d^2] - L \geq \frac{2\log(2p/d) - \log(2p/d) - \log(4\pi) - (L)^2}{L + \mathbb{E}[g_d^1]}$$

We now use that $d = k\log^2(p/k)$. Inequality $k \leq d/2$ holds thanks to the assumption $p/k \geq 8$. Since $L = L_0(\frac{k}{P})$ is given by (4.1), the numerator of the previous display is equal to

$$2\log(2) + 5\log(p/k) - \log(2p/d) - 4\log(p/k)$$

which is larger than $2\log(2)$ since $k \leq d/2$. For the denominator, since $\sqrt{2\log(p/k)} \geq L$ as well as $\sqrt{2\log(p/k)} \geq \sqrt{2\log(2p/d)} \geq \mathbb{E}[g_d^1]$ we have $1/(L + \mathbb{E}[g_d^1]) \geq 1/\sqrt{8\log(p/k)}$. Combining the above pieces, we have proved that for some absolute constant $c > 0$ we have

$$c\sqrt{2k\log(p/k)} \leq \sqrt{d(\mathbb{E}[g_d^1] - \mathbb{E}[\hat{\lambda}])} \leq \mathbb{E}^{1/2}[||X(\hat{\beta} - \beta^*)||^2] q$$

where $q^2 = \mathbb{E}\sup_{u \in (-1,+1)^p} |u|_0 = d \ |Xu|^2/(nd)$. To complete the proof, it remains to show that $q^2$ is bounded from above by $1 + c_1 \sqrt{d\log(p/d)/n}$ for some absolute constant $c_1 > 0$ which is proved in Proposition F.1.

### Appendix K: Signed Varshamov-Gilbert extraction Lemma

**Lemma K.1** (Lemma 2.5 in [24]). For any positive integer $d \leq p/5$, there exists a subset $\Omega_0$ of the set $\{w \in \{0, 1\}^p : |w|_0 = d\}$ that fulfills

$$\log(|\Omega_0|) \geq (d/2)\log\left(\frac{p}{\max_j |Xe_j|^2}\right), \quad \sum_{j=1}^p w_j \neq w'_j = ||w - w'||^2 > d, \quad (K.1)$$

for any two distinct elements $w$ and $w'$ of $\Omega_0$, where $|\Omega_0|$ denotes the cardinality of $\Omega_0$.

**Lemma K.2** (Implicitly in [44] and in [4]. The proof below is provided for completeness.). Let $X \in \mathbb{R}^{n \times p}$ be any matrix with real entries. For any positive integer $d \leq p/5$, there exists a subset $\Omega$ of the set $\{w \in \{-1, 0, 1\}^p : |w|_0 = d\}$ that fulfills both (K.1) and

$$\frac{1}{n} ||Xw||^2 \leq d \max_{j=1,\ldots,p} \frac{||Xe_j||^2}{n},$$

for any two distinct elements $w$ and $w'$ of $\Omega$. 

/ 39
Proof. By scaling, without loss of generality we may assume that \( \max_{j=1,...,p} \|Xe_j\|^2 = n. \) Let \( \Omega_0 \) be given from (K.1). For any \( u \in \Omega_0, \) define \( w_u \in \{-1, 0, 1\}^d \) by

\[
w_u = \arg\min_{v \in \{-1,0,1\}^d: |v_j| = u_j \text{ for } j=1,...,p} \|Xv\|^2
\]

and breaking ties arbitrarily. Next, define \( \Omega \) by \( \Omega := \{w_u, u \in \Omega_0\}. \) It remains to show that \( \frac{1}{n} \|Xw_u\|^2 \leq d. \) Consider iid Rademacher random variables \( r_1, ..., r_p \) and define the random vector \( T(u) \in \mathbb{R}^p \) by \( T(u)_j = r_j u_j \) for each \( j = 1,...,p. \) Taking expectation with respect to \( r_1, ..., r_p \) yields

\[
\frac{1}{n} E[\|XT(u)\|^2] = \frac{1}{n} \sum_{j=1}^p |u_j| \|Xe_j\|^2 \leq d.
\]

By definition of \( w_u, \) the quantity \( (1/n) \|Xw_u\|^2 \) is bounded from above by the previous display and the proof is complete. \( \square \)

Appendix L: Behavior of the Lasso around the critical tuning parameter

In the proof below, \( o(1) \) denotes a positive deterministic sequence that may only depend on \( n, p, k, C_{\text{min}} \) and \( C_{\text{max}} \) and that converges to 0 in the asymptotic regime (4.14).

Proof of (4.15). By (2.2) we have \( V_\lambda \leq R_\lambda \) with probability one. By the upper bound of Theorem 4.1 we get \( R_\lambda \leq (1/C_{\text{min}}) 1.01 \sqrt{2} \) with probability approaching one. The lower bound of Theorem 4.1 yields \( (1 - o(1))/C_{\text{max}} \leq V_\lambda \) with probability at least \( 1/3. \) By simple algebra, if \( \hat{s} \) is the vector of signs of \( \beta^* \) (so that \( \|\hat{s}\|_0 = \|\beta^*\|_0 = k \)), the large signal bias satisfies

\[
\frac{\sqrt{k}}{\sqrt{n} \|X\hat{s}\|} \leq \frac{\text{LSB}(\beta^*)}{\lambda \sqrt{k}} \leq \sup_{u \in \mathbb{R}^p: \|u\|_1 \leq \sqrt{2} \|X\|} \frac{\|u\|}{\sqrt{n} \|Xu\|}
\]

and the leftmost and rightmost quantities can be bounded with probability approaching one using only \( C_{\text{min}} \) and \( C_{\text{max}} \) using for instance Proposition E.2. Combining these inequalities with the union bound proves (4.15). \( \square \)

Proof of (4.16). The fact that \( B_\lambda \in [C^{-1}, C] \) can be established as in the proof of (4.15) above, since \( L_0(\frac{p}{k\lambda}) \asymp L_0(\frac{p}{k\lambda}) \) as \( k/p \to 0. \) By taking \( \beta^* = 0 \) in the lower bound of Theorem 4.1 (so that \( \text{NB}(\epsilon) = \|X(\hat{\beta} - \beta^*)\| \)) applied to the tuning parameter \( L_0(\frac{p}{k\lambda}) \), it follows that \( 2k\lambda \log(p/k\lambda) \leq [1 + o(1)]C_{\text{max}} \text{NB}(\epsilon) \) with probability at least \( 1/3. \) Combining these inequalities together with the union bound and the fact that \( k = o(k_{\lambda}) \) yields (4.16). \( \square \)

Proof of (4.17). Again, the fact that \( B_\lambda \in [C^{-1}, C] \) can be established as in the proof of (4.15) above, since \( L_0(\frac{p}{k\lambda}) \asymp L_0(\frac{p}{k\lambda}) \) as \( k/p \to 0. \) To prove that \( \text{NB}(\epsilon) = o(1) B_\lambda \) with constant probability, by the upper bound of Theorem 4.1 applied to the tuning parameter \( L_0(\frac{p}{k\lambda}) \) we get that

\[
\Pr \left( \text{NB}(\epsilon) \leq (1/C_{\text{min}})[1 + o(1)] \sqrt{k\lambda} L_0(\frac{p}{k\lambda}) \right) \to 1.
\]
Since \( k = o(k) \) and \( L_0(\frac{k}{n}) \approx L_0(\frac{p}{k_{\Theta}}) \), we obtain the desired result.