ORDERS THAT ARE ÉTALE-LOCALLY ISOMORPHIC

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1. Introduction

Let $R$ be a regular local ring with fraction field $F$ and $G$ a group $R$-scheme. We consider the restriction map

$$H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(F, G)$$

and raise the injection question for the $R$-group scheme $G$: is the restriction map injective? That is, if two $G$-torsors defined over $R$ become isomorphic over $F$, are they isomorphic over $R$?

We will prove that, if $R$ is a semilocal Dedekind domain and $A$ is a hereditary $R$-order in a central simple $F$-algebra, then the restriction map is injective when $G$ is the $R$-group scheme $\text{Aut}_R(A)$ of automorphism of $A$, see Theorem 2.1, and has trivial kernel when $G = \text{PGL}_1(A) := \text{GL}_1(A)/\text{Gm}_R$, see Theorem 3.1. The former case is equivalent to saying that two hereditary orders in a central simple $F$-algebras that become isomorphic after base change to $F$ and a faithfully flat étale $R$-algebra are already isomorphic. However, this fails for hereditary orders in simple non-central algebras, see Example 2.9.

In contrast, we will show in Section 4 that the injectivity question has a negative answer in general for the $R$-group schemes $\text{Aut}_R(A, \sigma)$, where $A$ and $R$ are as above and $\sigma : A \to A$ is an involution fixing $R$. Equivalently, it can happen that two hereditary orders with involution which become isomorphic after base change to $F$ and a faithfully flat étale $R$-algebra are non-isomorphic. This cannot happen if $(A, \sigma)$ is Azumaya over $R$, though; see [18].

To put these results in perspective, recall the famous conjecture of Grothendieck and Serre [14, Remarque 3 pp. 26-27], [12, Remarque 1.11.a], [20, Remarque p. 31], which stipulates that the injectivity question has a positive answer if $R$ is a regular local ring and $G$ is a reductive group $R$-scheme. This conjecture is still open in full generality, but many cases have been settled. For example, see Nisnevich [19] when $R$ is a discrete valuation ring, Colliot-Thélène and Sansuc [9] when $G$ is a torus over $R$, and Fedorov and Panin [11, 17], when the ring $R$ contains a field $k$.

In [3], the injectivity question is answered on the positive for group $R$-schemes of the form $U(A, \sigma)$ with $A$ a hereditary $R$-order. This led the first two authors to formulate an extension of the Grothendieck-Serre conjecture, see [3] Question 6.4. They ask, in the case where $R$ is a semilocal Dedekind domain, whether the injectivity question has a positive answer for a certain family of $R$-group schemes larger

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Date: April 26, 2018.

2010 Mathematics Subject Classification. 16H10, 16W10, 11E57, 11E72.

Key words and phrases. Hereditary order, maximal order, Dedekind domain, group scheme, reductive group, involution, central simple algebra.

This research was supported by a Swiss National Science Foundation grant #200021_163188.
than the family of reductive groups, which, loosely speaking, arise from Bruhat–Tits theory. It seems very likely that the group schemes considered above fall into the family considered in [4, §6], but more work is required to verify this. Provided this holds, the positive results of the present paper answer [4, Question 6.4] on the positive for certain group schemes, and on the negative for others. We elaborate about this in Section 5.

2. Hereditary Orders

Throughout this section, \( R \) denotes a Dedekind domain with fraction field \( F \). For \( p \in \text{Max} \, R \), let \( R_p \) denote the localization of \( R \) at \( p \), and let \( R^h_p \) and \( R^{sh}_p \) denote the henselization and strict henselization of \( R_p \), respectively. The corresponding fraction fields are denoted \( F^h_p \) and \( F^{sh}_p \). Unadorned tensors are always assumed to be over \( R \).

Recall that an \( R \)-order in an \( F \)-algebra \( E \) is an \( R \)-subalgebra \( A \) which contains an \( F \)-basis of \( E \) and is finitely generated as an \( R \)-module. Equivalently, an \( R \)-algebra \( A \) is an \( R \)-order in some \( F \)-algebra \( E \) (necessarily isomorphic to \( A \otimes F \)) if and only if \( A \) is \( R \)-torsion-free and finitely generated as an \( R \)-module.

As usual, a ring \( A \) is called hereditary if its one-sided ideals are projective \( A \)-modules. Notable examples of hereditary rings include maximal \( R \)-orders in central simple \( F \)-algebras. Thus, every central simple \( F \)-algebra contains a hereditary order. See [19, Chapter 9] for an extensive discussion.

We shall prove:

**Theorem 2.1.** Suppose \( R \) is a semilocal Dedekind domain, let \( A \) be a hereditary \( R \)-order in a central simple \( F \)-algebra and let \( A' \) be any \( R \)-order. If \( A \) and \( A' \) become isomorphic after tensoring with \( F \) and some faithfully flat étale \( R \)-algebra, then \( A \cong A' \) as \( R \)-algebras.

As a consequence, we get:

**Corollary 2.2.** Suppose \( R \) is a semilocal Dedekind domain and let \( A \) be a hereditary \( R \)-order in a central simple \( F \)-algebra. Then the restriction map

\[ H^1_{\text{ét}}(R, \text{Aut}_R(A)) \to H^1_{\text{ét}}(K, \text{Aut}_R(A)) \]

is injective. Here, \( \text{Aut}_R(A) \) denotes the group \( R \)-scheme whose \( S \)-points are given by \( \text{Aut}_R(A)(S) = \text{Aut}_S(A \otimes S) \).

**Proof.** The cohomology set \( H^1_{\text{ét}}(R, \text{Aut}_R(A)) \) classifies isomorphism classes of \( R \)-orders which become isomorphic to \( A \) after tensoring with a faithfully flat étale \( R \)-algebra. By Theorem 2.1, it is enough to show that any such \( R \)-order \( A' \) is also hereditary.

Let \( S \) be a faithfully flat étale \( R \)-algebra such that \( A \otimes S \cong A' \otimes S \) as \( S \)-algebras. Since \( S \) is faithfully flat and étale over \( R \), for all \( p \in \text{Max} \, R \), there exists an \( R \)-algebra homomorphism \( S \to R^h_p \). Thus, \( A \otimes R^h_p \cong A' \otimes R^h_p \) for all \( p \in \text{Max} \, R \). We shall see below, in Corollary 2.4, that this implies that \( A' \) is hereditary. \( \Box \)

The proof of Theorem 2.1 will be done in two steps. First, we will prove the theorem when \( R \) is a henselian discrete valuation ring (abbrev.: DVR). This step will rely heavily on the structure theory of hereditary orders. Then, the general case will be deduced by means of patching.

We will also show that the theorem fails for hereditary \( R \)-orders in the larger class of finite-dimensional (not necessarily central) simple \( F \)-algebras.
2A. Preliminary Results.

**Lemma 2.3.** Assume that $R$ is a DVR, and let $R'$ be a DVR which is also a faithfully flat $R$-algebra. Denote the maximal ideals of $R$, $R'$ by $m$, $m'$ respectively, and suppose that $k^\prime := R'/m'$ is a separable algebraic field extension of $k := R/m$. Let $A$ be an $R$-order. Then

(i) $\text{Jac}(A) \otimes R' = \text{Jac}(A \otimes R')$ and

(ii) $A$ is hereditary if and only if $A \otimes R'$ is hereditary.

**Proof.** Write $A' = A \otimes R'$, $J = \text{Jac}(A)$, $J' = \text{Jac}(A')$ and view $A$ as a subring of $A'$. Since $R'$ is a flat $R$-module, the map $J \otimes_R R' \to A'$ is injective, hence we may identify $J \otimes_R R'$ with $JR'$.

(i) We need to show that $J' = JR'$. By [19 Theorem 6.15], there is $n \in \mathbb{N}$ such that $J^n \subseteq mA \subseteq J$ and $J'^n \subseteq m'A' \subseteq J'$. In particular, we may view $A/J$ as a $k$-algebra, and therefore, $A'/JR' \cong (A/J) \otimes_R R' \cong (A/J) \otimes_k k'$. Since $A/J$ a semisimple finite-dimensional $k$-algebra and $k'$ is separable over $k$, the ring $(A/J) \otimes_k k'$ is semisimple, and therefore $J' \subseteq JR'$. On the other hand, $(JR')^n = J^nR' \subseteq mA' \subseteq m'A' \subseteq J'$, hence $JR' \subseteq J'$, because $J'$ is semiprime. We conclude that $J' = JR'$.

(ii) By [2] p. 5], it is enough to prove that $J := \text{Jac}(A)$ is projective as a right $A$-module if and only if $J' := \text{Jac}(A')$ is projective as a right $A'$-module. The direction ($\Rightarrow$) follows from (i) and the other direction follows from [15 Proposition 4.80(2)] (the proof in [15] is given for $R$-modules but extends verbatim to $A$-modules once noting that $\text{Hom}_{A \otimes S}(M \otimes S, N \otimes S) \cong \text{Hom}_A(M, N) \otimes S$ whenever $M$ is a finitely presented $A$-module and $S$ is a flat $R$-algebra; see [19 Theorem 2.38] for a proof of the latter).

**Corollary 2.4.** Let $A$ be an $R$-order. Then $A$ is hereditary if and only if $A \otimes R^b_p$ is hereditary for all $p \in \text{Max } R$ if and only if $A \otimes R^b_p$ is hereditary for all $p \in \text{Max } R$.

**Proof.** By [2] p. 8], $A$ is hereditary if and only if $A \otimes R_p$ is hereditary for all $p \in \text{Max } R$. Now use Lemma 2.3(ii).

2B. The Henselian Case. In this subsection, we assume that $R$ is a henselian DVR with maximal ideal $m$ and residue field $k$. We shall deduce Theorem 2.1 in this special case as a consequence of the structure theory of hereditary $R$-orders, which we now recall.

**Theorem 2.5 ([19 §12]).** Let $D$ be a finite dimensional division $F$-algebra. Then the additive valuation $\nu_F$ of $F$ extends uniquely to a discrete additive valuation $\nu_D$ on $D$. Furthermore,

$$\mathcal{O}_D := \{x \in D : \nu_D(x) \geq 0\}$$

is the unique maximal $R$-order in $D$.

The unique maximal right (and left) ideal of $\mathcal{O}_D$ is denoted $\mathfrak{m}_D$ and we write $k_D = \mathcal{O}_D/\mathfrak{m}_D$. The ring $k_D$ is a finite dimensional division $k$-algebra, the center of which may be strictly larger than $k$.

Given a ring $A$ and ideals $(a_{ij})_{i,j}$, we let

$$\begin{pmatrix} (a_{11}) & \cdots & (a_{1r}) \\ \vdots & & \vdots \\ (a_{r1}) & \cdots & (a_{rr}) \end{pmatrix}^{(n_1, \ldots, n_r)}$$

denote the set of block matrices $(X_{ij})_{i,j} \in \{1, \ldots, r\}$ for which $X_{ij}$ is an $n_i \times n_j$ matrix with entries in $a_{ij}$. If $D$ is a finite dimensional division $F$-algebra and $n_1, \ldots, n_r$
are natural numbers, we write

\[
\mathcal{O}_D^{(n_1, \ldots, n_r)} = \begin{bmatrix}
(O_D) & (m_D) & \cdots & (m_D) \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
(O_D) & \cdots & \cdots & (O_D)
\end{bmatrix}^{(n_1, \ldots, n_r)},
\]

\[
J_D^{(n_1, \ldots, n_r)} = \begin{bmatrix}
(m_D) & \cdots & \cdots & (m_D) \\
(O_D) & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
(O_D) & \cdots & \cdots & (O_D)
\end{bmatrix}^{(n_1, \ldots, n_r)}.
\]

Then \(\mathcal{O}_D^{(n_1, \ldots, n_r)}\) is an \(R\)-order in \(M_{n_1+\cdots+n_r}(D)\) and its Jacobson radical is \(J_D^{(n_1, \ldots, n_r)}\).

**Theorem 2.6.** Let \(D\) be a finite dimensional division \(F\)-algebra and let \(A\) be a hereditary \(R\)-order in \(M_n(D)\). Then there are natural numbers \(n_1, \ldots, n_r\) with \(\sum_i n_i = n\) such that

\[A \cong \mathcal{O}_D^{(n_1, \ldots, n_r)}.\]

Conversely, any \(R\)-order of this form is hereditary. The tuple \((n_1, \ldots, n_r)\) is uniquely determined by \(A\) up to a cyclic permutation.

**Proof.** When \(\Cent(D) = F\), this is [19, Theorem 39.14, Theorem 39.24]. The general case follows by observing that the integral closure of \(R\) in \(\Cent(D)\) is itself a henselian DVR which is finitely generated as an \(R\)-module; see [19, §12,§13], for instance. \(\Box\)

For \(A\) as in the theorem, we denote the equivalence class of \((n_1, \ldots, n_r)\) under cyclic permutations by

\[\text{inv}(A)\]

The class \(\text{inv}(A)\) and the simple \(F\)-algebra \(A \otimes F\) determine \(A\) up to isomorphism.

For brevity, we shall write \((n_1, \ldots, n_r)^s\) to denote the concatenation of \(s\) copies of \((n_1, \ldots, n_r)\), e.g. \((n) = (n, \ldots, n)\) \((s\) times).

**Proposition 2.7.** Let \(D\) be a central division \(F\)-algebra, let \(A = \mathcal{O}_D^{(n_1, \ldots, n_r)}\) and let \(k'' = \mathcal{O}_D^{(n_1, \ldots, n_r)} / \mathcal{O}_D^{(n_1, \ldots, n_r)}\). Let \(k'' = \Cent(k''D)\) and let \(k'\) denote the maximal subfield of \(k''\) which is separable over \(k\). Write \(t = [k' : k]\) and \(s = \deg k_D = \sqrt{|k_D : k''|}\). Then

\[\text{inv}(A \otimes R^{sh}) = (sn_1, \ldots, sn_r)^t\]

In particular, if \(\text{inv}(A \otimes R^{sh}) = (m_1, \ldots, m_t)\), then \(\text{inv}(A) = (\frac{m_1}{s}, \ldots, \frac{m_t}{s})\).

**Proof.** We first claim that \(\text{inv}(O_D \otimes R^{sh}) = (s)^t\). Let \(k_s\) denote the residue field of \(R^{sh}\), which is also a separable closure of \(k\). By Lemma 2.3, \(\text{Jac}(O_D \otimes R^{sh}) = m_D \otimes R^{sh}\), hence

\[\frac{(O_D \otimes R^{sh})}{\text{Jac}(O_D \otimes R^{sh})} \cong k_D \otimes R^{sh} \cong k_D \otimes_k k_s \cong k_D \otimes_{k'} (k'' \otimes_{k'} (k' \otimes_k k_s))\]

By assumption, \(k' \otimes_k k_s \cong k_s \times \cdots \times k_s\) \((t\) times). Since \(k''\) is a purely inseparable finite extension of \(k'\), the tensor product \(\overline{k} := k'' \otimes_{k'} k_s\) is a separably closed field, and hence \(k_D \otimes_{k'} \overline{k} \cong M_s(\overline{k})\). Putting this into \((2.1)\), we get

\[\frac{O_D \otimes R^{sh}}{\text{Jac}(O_D \otimes R^{sh})} \cong k_D \otimes_{k'} (\underbrace{k \times \cdots \times k}_{\text{\(t\) times}}) \cong M_s(\overline{k}) \times \cdots \times M_s(\overline{k})\]
On the other hand, writing \( \mathcal{O}_D \otimes R^h \cong \mathcal{O}_E^{(m_1, \ldots, m_n)} \) for a suitable central division \( F^h \)-algebra \( E \), we see that

\[
\frac{\mathcal{O}_D \otimes R^h}{\text{Jac}(\mathcal{O}_D \otimes R^h)} \cong \frac{\mathcal{O}_E^{(m_1, \ldots, m_n)}}{J_{E}^{(m_1, \ldots, m_n)}} \cong M_{m_1}(k_E) \times \cdots \times M_{m_n}(k_E).
\]

The claim follows by comparing the right hand sides of (2.2) and (2.3).

We now prove the proposition. The previous paragraph implies that \( \mathcal{O}_D \otimes R^h \cong \mathcal{O}_E^{(s)^r} \), and by Lemma 2.8 ii), this isomorphism restricts to an isomorphism \( m_D \otimes R^h = J_{E}^{(s)^r} \). As a result, we have

\[
\mathcal{O}_D^{(n_1, \ldots, n_r)} \otimes R^h \cong \begin{bmatrix}
(O_E^{(s)}) & (J_E^{(s)}) & \cdots & (J_E^{(s)}) \\
(O_E^{(s)}) & (O_E^{(s)}) & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
(O_E^{(s)}) & \cdots & \cdots & (O_E^{(s)})
\end{bmatrix}^{(n_1, \ldots, n_r)}.
\]

Conjugating the right hand side by a suitable permutation matrix in \( M_{n_1} \), the required permutation sends \( i + s(j - 1) + st(k - 1) \) with \( i \in \{1, \ldots, s\}, j \in \{1, \ldots, t\}, k \in \{1, \ldots, n_i\} \), to \( i + s(k - 1) + sn(j - 1) \).

\[\square\]

**Corollary 2.8.** Suppose \( R \) a henselian DVR and let \( A, A' \) be two hereditary \( R \)-orders in central simple \( F \)-algebras. If \( A \) and \( A' \) become isomorphic after extending scalars to \( F \) and some faithfully flat étale \( R \)-algebra, then \( A \cong A' \).

**Proof.** Write \( A \otimes F = M_{n_1}(D) \) and \( A' \otimes F = M_{n_2}(D') \). Since \( A \otimes F \cong A' \otimes F \), we have \( D \cong D' \). Since \( A \) and \( A' \) become isomorphic over a faithfully flat étale \( R \)-algebra, and since any faithfully flat étale \( R \)-algebra admits an \( R \)-algebra morphism into \( R^h \), we have \( A \otimes R^h \cong A' \otimes R^h \). Now, by Proposition 2.4, \( \text{inv}(A) = \text{inv}(A') \), so \( A \cong A' \).

\[\square\]

**2C. Proof of Theorem 2.1** Recall that \( R \) is assumed to be a semilocal Dedekind domain, \( A \) is a hereditary \( R \)-order and \( A' \) is any \( R \)-order.

Arguing as in the proof of Corollary 2.2, we see that \( A' \) is hereditary. Now, Corollary 2.3 implies that \( A \otimes R_p^h \cong A' \otimes R_p^h \) for all \( p \in \text{Max} R \). We finish by using a patching argument to show that \( A \cong A' \).

Let \( S = \prod_{p \in \text{Max} R} R_p^h \) and \( K = \prod_{p \in \text{Max} R} E_p^h \). Then there are algebra isomorphisms \( \phi : A \otimes F \cong A' \otimes F \) and \( \psi : A \otimes S \cong A' \otimes S \). Consider \( \psi^{-1}\phi_K : A \otimes K \rightarrow A \otimes K \).

Since \( A \otimes F \) is a central simple \( F \)-algebra, the Skolem-Noether Theorem implies that there exists \( a \in (A \otimes K)^{\times} \) such that \( \psi^{-1}\phi_K = \text{Int}(a) := \{x \mapsto axa^{-1}\} \). A standard density argument (e.g. see the second paragraph in the proof of 3 Theorem 5.1) implies that there are \( b \in (A \otimes F)^{\times}, c \in (A \otimes S)^{\times} \) such that \( a = c^{-1}b \). Then \( \psi^{-1}\phi_K = \text{Int}(c) \text{Int}(b)_K \), or rather \( \text{Int}(c) \psi^{-1} = \text{Int}(b) \phi^{-1} \). Now, a patching argument (e.g. see 3 Lemma 4.2(i)) implies that there is an \( R \)-module isomorphism \( \eta : A' \rightarrow A \) such that \( \eta S = \text{Int}(c) \psi^{-1} \) and \( \eta_F = \text{Int}(b) \phi^{-1} \). It is easy to see that \( \eta \) is an isomorphism of \( R \)-algebras, which completes the proof.

\[\square\]

**2D. Counterexamples.** We finish with noting that Theorem 2.1 may fail if some of the assumptions are dropped.

We begin with observing that Theorem 2.1 can fail for hereditary orders in simple, non-central, \( F \)-algebras.
Example 2.9. Let $R$ be a DVR with fraction field $F$. For brevity, denote the henselization of $R$ and its fraction field by $R'$ and $F'$, respectively. Suppose that there exists a cubic field extension $K/F$ such that:

- $\text{Gal}(K/F) = \{\text{id}_K\}$, and
- $K \otimes F' \cong F' \times F' \times F'$.

(Explicit choices with these properties are $R = \mathbb{Z}_p$, $F = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt[3]{6})$.)

Let $m'$ denote the maximal ideal of $R'$. Define $R'$-orders $B_1$ and $B_2$ in $M_2(K) \otimes F' \cong M_2(F')^3$ as follows:

$$B_1 = \left[ \begin{array}{c|c} R' & m' \\ \hline R' & R' \end{array} \right] \times \left[ \begin{array}{c|c} R' & m' \\ \hline R' & R' \end{array} \right] \times \left[ \begin{array}{c|c} R' & R' \\ \hline R' & R' \end{array} \right],$$

$$B_2 = \left[ \begin{array}{c|c} R' & m' \\ \hline R' & R' \end{array} \right] \times \left[ \begin{array}{c|c} R' & R' \\ \hline R' & R' \end{array} \right] \times \left[ \begin{array}{c|c} R' & m' \\ \hline R' & R' \end{array} \right],$$

and let

$$A_i = M_2(K) \cap B_i \quad (i = 1, 2).$$

Observe that $B_1$ and $B_2$ are isomorphic as $R'$-algebras, but not as $R' \times R' \times R'$-algebras.

It is easy to see that $A_1$ and $A_2$ are orders in $M_2(K)$ satisfying $A_i \otimes R' \cong B_i$ ($i = 1, 2$). Since $B_1$ and $B_2$ are hereditary (Theorem 2.6), so are $A_1$ and $A_2$ (Corollary 2.8).

Now, $A_1 \otimes R' \cong A_2 \otimes R'$ as $R'$-algebras. Since $R'$ is a direct limit of faithfully flat étale $R$-algebras, the $R$-orders $A_1$ and $A_2$ become isomorphic after base change to some faithfully flat étale $R$-algebra. In addition, we have $A_1 \otimes F \cong M_2(K) \cong A_2 \otimes F$. However, we claim that $A_1 \not\cong A_2$.

To see this, suppose $\varphi : A_1 \to A_2$ is an isomorphism of $R$-algebras. Then $\varphi$ extends to an $F$-automorphism of $M_2(K)$. Since $\text{Gal}(K/F) = \{\text{id}_K\}$, the isomorphism $\varphi$ is $K$-linear, hence $\varphi_{R'} : A_1 \otimes R' \to A_2 \otimes R'$ is an $R' \times R' \times R'$-linear isomorphism, which cannot exist by our choice of the $R'$-orders $B_1$ and $B_2$.

In the previous example, the orders are isomorphic over the henselizations, but this fails to descend to the original ring. The next example shows that problems can also occur in passing from the strict henselization to the henselization if one allows orders in semisimple $F$-algebras; compare with Corollary 2.8.

Example 2.10. Suppose $R$ is a henselian DVR and $F$ admits a non-commutative central division algebra $D$ such that $D \otimes F^{sh} \cong M_n(F^{sh})$. (For example, take $R = \mathbb{Z}_p$, $F = \mathbb{Q}_p$, and let $D$ be the unique quaternion division $F$-algebra.) Using Corollary 2.8, write $\text{inv}(O_D \otimes F^{sh}) = (s)^t$. Let

$$E = M_2(D) \times M_{2n}(F)$$

and define orders in $E$ as follows:

$$A_1 = O_D^{(1,1)} \times O_F^{(2n)^t},$$

$$A_2 = O_D^{(2)} \times O_F^{(s)^t}. $$

Proposition 2.7 implies that $A_1 \otimes R^{sh} \cong O_F^{(s)^t} \times O_F^{(2n)^t} \cong A_2 \otimes R^{sh}$, and $A_1 \otimes F \cong A_2 \otimes F$ is clear. Since $R^{sh}$ is a direct limit of faithfully flat étale $R$-algebras, this means that $A_1$ and $A_2$ become isomorphic over $F$ and some faithfully flat étale $R$-algebra. On the other hand, an $R$-algebra isomorphism $A_1 \to A_2$ will necessarily induce an isomorphism $O_D^{(1,1)} \to O_D^{(2)}$, which is impossible by Theorems 2.6.

Finally, we note that Theorem 2.4 may fail if $R$ is a Dedekind domain which is not semilocal.
Example 2.11. Let \( R \) be a Dedekind domain whose class group is a nontrivial 2-torsion group, let \( I \) and \( I' \) be two non-isomorphic fractional ideals of \( R \), and let \( A = \text{End}_R(R \oplus I) \) and \( A' = \text{End}_R(R \oplus I') \). It is easy to check that the \( R \)-modules \( R \oplus I \) and \( R \oplus I' \) become isomorphic over \( F \) and some faithfully flat \( \text{ét} \) algebra (e.g. a suitable Zariski covering of \( \text{Spec} \ R \)), so the same holds for \( A \) and \( A' \). However, \( A \not\cong A' \) as \( R \)-algebras. Indeed, for the sake of contradiction, assume that there is an isomorphism of \( R \)-algebra \( \phi : A' \to A \). Then we may view \( R \oplus I' \) as a left \( A \)-module via \( \phi \) and form \( J := \text{Hom}_A(R \oplus I, R \oplus I') \). Morita Theory implies that \( J \) is a fractional ideal of \( R \) satisfying \( (R \oplus I) \otimes J \cong R \oplus I' \). However, the latter implies \( IJ^2 = I'J^2 \) in \( \text{Cl}(R) \), which is impossible by our choice of \( I \), \( I' \) and the fact that \( \text{Cl}(R) \) is a 2-torsion group.

3. \( \text{PGL}_1(A) \)-Torsors

Let \( R \) an commutative ring and let \( A \) be a finite projective \( R \)-algebra. As usual, let \( \text{GL}_1(A) \) denote the \( R \)-group scheme determined by \( \text{GL}_1(A)(S) = (A \otimes S) \times \). The group \( R \)-scheme \( \text{PGL}_1(A) \) is defined to be the \( R \)-scheme representing the quotient sheaf \( \text{GL}_1(A)/\text{G}_m,R \) on the flat (fpqc) site of \( \text{Spec} \ R \). To see that it exists, apply \([10, \text{XVI, Corollaire 2.3}]\) to the morphism \( \text{GL}_1(A) \to \text{Aut}_R(A) \) sending a section to its corresponding inner automorphism. The group \( \text{PGL}_1(A) \) is smooth over \( R \) by \([10, \text{VI}, \text{Proposition 9.2[xiii]})\), because \( \text{GL}_1(A) \) is smooth and \( \text{G}_m,R \) is flat over \( R \).

Theorem 3.1. Let \( R \) be a regular semilocal domain with fraction field \( F \) and let \( A \) be a finite projective \( R \)-algebra. Then the restriction map

\[
\text{H}^1_{\text{ét}}(R, \text{PGL}_1(A)) \to \text{H}^1_{\text{ét}}(F, \text{PGL}_1(A))
\]

has trivial kernel.

The injectivity of \( \text{H}^1_{\text{ét}}(R, \text{PGL}_1(A)) \to \text{H}^1_{\text{ét}}(F, \text{PGL}_1(A)) \) is more delicate and so far we were unable to establish it.

Proof. We first observe that since \( \text{PGL}_1(A) \) is smooth over \( R \), the canonical map \( \text{H}^1_{\text{ét}}(R, \text{PGL}_1(A)) \to \text{H}^1_{\text{fpqc}}(R, \text{PGL}_1(A)) \) is an isomorphism; see \([13, \text{Théorème 11.7(1), Remarque 11.8(3)]})\). It is therefore enough to prove the theorem with fpqc cohomology. To this end, we first establish two independent facts.

First, we show that the set \( \text{H}^1_{\text{fpqc}}(R, \text{GL}_1(A)) \) is trivial. The flat cohomology set \( \text{H}^1_{\text{fpqc}}(R, \text{GL}_1(A)) \) parametrizes the isomorphism classes of left \( A \)-modules \( P \) which become isomorphic to \( A \) itself after a faithfully flat extension of \( R \). As such an \( A \)-module \( P \) is necessarily a finitely generated and projective \( A \)-module (see the first paragraph of the proof of \([4, \text{Proposition 5.1}]\)), one can apply \([4, \text{Proposition 2.11}]\) to conclude that \( P \) is isomorphic to \( A \). Consequently, there is only one such isomorphism class of \( A \)-modules and the set \( \text{H}^1_{\text{fpqc}}(R, \text{GL}_1(A)) \) is a singleton.

Second, as \( R \) is a regular domain, the morphism \( \text{H}^2_{\text{fpqc}}(R, \text{G}_m) \to \text{H}^2_{\text{fpqc}}(F, \text{G}_m) \) is injective by \([12, \text{Corollaire 1.8}]\).

Now, consider the exact sequence

\[
1 \to \text{G}_m,R \to \text{GL}_1(A) \to \text{PGL}_1(A) \to 1
\]
of sheaves on the flat (fpqc) site of Spec $R$. It induces the following diagram of pointed cohomology sets with exact rows.

$$
\begin{array}{ccc}
\text{H}^1_{\text{fpqc}}(R, \text{GL}_1(A)) & \longrightarrow & \text{H}^1_{\text{fpqc}}(R, \text{PGL}_1(A)) \longrightarrow \text{H}^2_{\text{fpqc}}(R, \text{G}_m) \\
\downarrow & & \downarrow \\
\text{H}^1_{\text{fpqc}}(F, \text{GL}_1(A)) & \longrightarrow & \text{H}^1_{\text{fpqc}}(F, \text{PGL}_1(A)) \longrightarrow \text{H}^2_{\text{fpqc}}(F, \text{G}_m)
\end{array}
$$

A straightforward diagram chasing using the two facts proved above gives the result.

\[\square\]

**Remark 3.2.** Theorem 2.1 and Theorem 3.1 require independent proofs because in general, the morphism $\text{PGL}_1(A) \to \text{Aut}_R(A)$ sending a section $x$ to conjugation by $x$ is not an isomorphism, even when $A$ is a hereditary $R$-order in a central simple $F$-algebra. For example, let $R$ be a DVR with maximal ideal $m = \pi R$ and let $A = \left[ \begin{smallmatrix} R & 0 \\ R & R \end{smallmatrix} \right]$. Then $a \mapsto \left[ \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix} \right] a \left[ \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right]^{-1}$ is an automorphism of $A$ which is not inner.

4. **Hereditary Orders with Involution**

Throughout, $R$ is a semilocal Dedekind domain with fraction field $F$. As in Section 2, unadorned tensors are taken over $R$.

We now show that for suitably chosen $R$, there exist non-isomorphic hereditary $R$-orders with involution which become isomorphic (as algebras with involution) over $F$ and over a faithfully flat étale $R$-algebra. We shall see that the counterexample can be chosen so that base change of the involution to $F$ is either orthogonal, symplectic or unitary. However, this cannot happen for Azumaya orders with involution, at least when $R$ is local, by [13].

**Example 4.1.** Let $R = \mathbb{R}[t]$. The complex conjugation on $C$ and the canonical symplectic involution on the real quaternions $\mathbb{H}$ are both denoted $\bar{}$. Define the $R$-order with involution $(\mathcal{O}, \tau)$ to be any of the following:

1. $\mathcal{O} = R$ and $\tau = \text{id}_R$,
2. $\mathcal{O} = R \otimes_R \mathbb{C} = \mathbb{C}[t]$ and $\tau = \text{id}_R \otimes \bar{}$,
3. $\mathcal{O} = R \otimes_R \mathbb{H} = \mathbb{H}[t]$ and $\tau = \text{id}_R \otimes \bar{}$.

Theorem 2.3 implies that $\mathcal{O}$ is the unique maximal $R$-order in $D := \mathcal{O} \otimes_R \mathbb{R}((t))$, which is either $\mathbb{R}((t))$, $\mathbb{C}((t))$ or $\mathbb{H}((t))$.

Let $m = t\mathcal{O}$ denote the maximal ideal of $\mathcal{O}$, and, with the notation of [23], define

$$A = \mathcal{O}^{(4,2)}_D = \left[ \begin{array}{cc} \mathcal{O} & m \\ \mathcal{O} & \mathcal{O} \end{array} \right]^{(4,2)}.$$

By Theorem 2.6, $A$ is a hereditary $R$-order in $M_6(D)$.

The involution $\tau$ induces an involution $(a_{ij})_{i,j} \mapsto (\tau(a_{ji}))_{i,j} : M_6(D) \to M_6(D)$, which we also denote by $\tau$. Let

$$a_1 = \text{diag}(1, -1, 1, -1, t, t),$$

$$a_2 = \text{diag}(1, -1, 1, 1, t, -t),$$

and define involutions $\sigma_1, \sigma_2 : A \to A$ by

$$\sigma_i(x) = a_i^{-1} \tau(x) a_i \quad (i = 1, 2).$$

We claim that $(A, \sigma_1)$ and $(A, \sigma_2)$ become isomorphic after tensoring with $F$ and some faithfully flat étale $R$-algebra, but are nevertheless non-isomorphic as $R$-algebras with involution.
To see this, we note that if \( S \) is an \( R \)-algebra and \( u \in A \otimes S \) satisfies \( \tau(u)\sigma_2 u = \alpha u_1 \) for some \( \alpha \in S^\times \), then \( x \mapsto uxu^{-1} \) determines an isomorphism \( (A \otimes S, \sigma_1 \otimes \text{id}_S) \rightarrow (A \otimes S, \sigma_2 \otimes \text{id}_S) \). For \( S = F \), one can take

\[
\begin{bmatrix}
\frac{1}{2}t + \frac{1}{2} & \frac{1}{2}t - \frac{1}{2} \\
\frac{1}{2}t - \frac{1}{2} & \frac{1}{2}t + \frac{1}{2}
\end{bmatrix} + \begin{bmatrix}
t & \alpha \\
\alpha & t
\end{bmatrix}
\]

with \( \alpha = t \). For the faithfully flat étale \( R \)-algebra \( S = R[\sqrt{-1}] \), one can take

\[
\begin{bmatrix}
1 & \alpha \\
\alpha & 1
\end{bmatrix} + \begin{bmatrix}
\sqrt{-1} & 0 \\
0 & \sqrt{-1}
\end{bmatrix}
\]

with \( \alpha = 1 \).

We proceed by showing that there is no isomorphism \( (A, \sigma_1) \rightarrow (A, \sigma_2) \). Let \( \overline{A} := A / \text{Jac}(A) \cong M_4(k_D) \times M_2(k_D) \), where \( k_D = \mathcal{O}/\mathfrak{m} \) is either \( \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \). For \( i = 1, 2 \), the involution \( \sigma_i \) induces an involution \( \overline{\sigma}_i : \overline{A} \rightarrow \overline{A} \), and direct computation shows that \( (\overline{A}, \overline{\sigma}_1) = (M_4(k_D), \sigma'_1) \times (M_2(k_D), \sigma'_2) \) for suitable \( \mathbb{R} \)-involutions \( \sigma'_1, \sigma'_2 \).

In fact, \( \sigma'_1 \) and \( \sigma'_2 \) are given by

\[
\sigma'_1[\frac{z}{w}] = [\frac{\overline{z}}{\overline{w}}] \quad \text{and} \quad \sigma'_2[\frac{z}{w}] = [\frac{1}{\overline{z}}]^{-1}[\frac{\overline{w}}{\overline{w}}][\frac{1}{\overline{w}}] \cdot
\]

An isomorphism \( \psi : (A, \sigma_1) \rightarrow (A, \sigma_2) \) would induce an isomorphism of \( \mathbb{R} \)-algebras with involution \( \overline{\psi} : (\overline{A}, \overline{\sigma}_1) \rightarrow (\overline{A}, \overline{\sigma}_2) \). Considering the action on \( \overline{\mathfrak{a}} \) on the central idempotents of \( \overline{A} \), we see that \( \overline{\psi} \) must further restrict to an isomorphism \( (M_2(k_D), \sigma''_1) \rightarrow (M_2(k_D), \sigma''_2) \). However, \( \sigma''_1 \) is easily seen to be anisotropic (meaning that \( \sigma''_1(x)x = 0 \) implies \( x = 0 \)) while \( \sigma''_2 \) is isotropic, so \( \sigma''_2 \) cannot exist.

Let \( \text{Aut}_R(A, \sigma_1) \) denote the group \( R \)-scheme representing the functor \( S \mapsto \text{Aut}_S(A \otimes S, \sigma_1 \otimes \text{id}_S) \). From the previous discussion, we conclude that the restriction map

\[
H^1_{\text{ét}}(R, \text{Aut}_R(A, \sigma_1)) \rightarrow H^1_{\text{ét}}(F, \text{Aut}_R(A, \sigma_1))
\]

is not injective. Indeed, the left hand side classifies \( R \)-orders with involution which become isomorphic to \((A, \sigma_1)\) over some faithfully flat étale \( R \)-algebra, and \((A, \sigma_2)\) represents a nontrivial class which is mapped to the trivial element of \( H^1_{\text{ét}}(F, \text{Aut}_R(A, \sigma_1)) \).

We finally note that in cases (1), (2) and (3) above, \( \sigma_1 \otimes \text{id}_F : A \otimes F \rightarrow A \otimes F \) is orthogonal, unitary and symplectic, respectively.

Call an \( R \)-order with involution \((A, \sigma)\) residually anisotropic if the induced involution \( \overline{\sigma} : A / \text{Jac}(A) \rightarrow A / \text{Jac}(A) \) is anisotropic, i.e. \( \overline{\sigma}(x)x = 0 \) implies \( x = 0 \) for all \( x \in A / \text{Jac}(A) \). The reader will notice that in Example 4.1 both \((A, \sigma_1)\) and \((A, \sigma_2)\) are not residually anisotropic, and the isotropicity plays a crucial role. We therefore ask:

**Question 4.2.** Let \( R \) be a semilocal Dedekind domain and let \((A, \sigma), (A', \sigma')\) be two hereditary \( R \)-orders with involution which become isomorphic over the fraction field of \( R \) and over some faithfully flat étale \( R \)-algebra. Suppose \((A, \sigma)\) is residually anisotropic. Is \((A, \sigma)\) isomorphic to \((A', \sigma')\) as \( R \)-algebras with involution?

The question is also motivated by the work of Bruhat and Tits on the cohomology of reductive groups over henselian discretely valued fields [7]. Specifically, it seems likely that a positive answer should follow from [4, Lemma 3.9] in case \( R \) is a complete DVR; we elaborate about this in the next section. That said, we hope that a more direct proof can be found.
5. Discussion

We finish with explaining how the results of the previous sections relate to a question asked by the first two authors, [4, Question 6.4].

Let $R$ be a semilocal Dedekind domain with fraction field $F$ and suppose that $R/p$ is perfect for all $p \in \text{Max } R$. We let $\hat{R}_p$ and $\hat{F}_p$ denote the completion of $R_p$ and its corresponding fraction field. We shall use the notation of Section 2 for henselizations and strict henselizations.

Let $G$ be a group scheme over $R$ and let $G = G \times_R F$ denote its generic fiber, which we assume to be reductive and connected. In [4, §6], the group scheme $G$ was called a point-stabilizer (resp. parahoric) group scheme for $G$ if for every $p \in \text{Max } R$, the group $\hat{R}_p$-scheme $G \times_R \hat{R}_p$ coincides with one of the group schemes that Bruhat and Tits [5] associate with stabilizers of points of the affine building of $G \times F \hat{F}_p$ (resp. parahoric subgroups of $G(\hat{F}_p)$). Briefly, letting $B(G, \hat{F}_p)$ denote the (extended) affine Bruhat–Tits building of $G \times F \hat{F}_p$, the point stabilizer group scheme associated with a point $y \in B(G, \hat{F}_p)$ is the smooth affine group $\hat{R}_p$-scheme $G_y$ with generic fiber $G \times F \hat{F}_p$ characterized by the condition that $G_y(\hat{R}_p)$ is the stabilizer of $y \in B(G, \hat{F}_p)$ under the action of $G(\hat{R}_p)$. The neutral component of $G_y \to \text{Spec } \hat{R}_p$ is then called the parahoric group scheme associated with $y$.

Let $A$ be a hereditary $R$-order in a simple $F$-algebra and let $\sigma : A \to A$ be an $R$-involution. We write $A_F = A \otimes F$, $\sigma_F = \sigma \otimes \text{id}_F$ and let $U(A, \sigma)$ denote the affine group $R$-scheme determined by $U(A, \sigma)(S) = U(A_S, \sigma_S) := \{a \in A_S | a^{\sigma}a = 1\}$ for any $R$-algebra $S$. In order to guarantee that $G := U(A_F, \sigma_F) \to \text{Spec } F$ is connected, we assume further that either $\text{Cent}(A_F) = F$ and $\sigma_F$ is symplectic, or $\text{Cent}(A_F)$ is a quadratic étale $F$-algebra and $\sigma_F$ is unitary. The excluded case where $\text{Cent}(A_F) = F$ and $\sigma_F$ is orthogonal can be handled similarly after few modifications.

In [4, §5–6], the first two authors showed that

$$H^1_{\text{ét}}(R, U(A, \sigma)) \to H^1_{\text{ét}}(F, U(A, \sigma))$$

is injective, and moreover, that the point stabilizer group schemes of $G$ are all of the form $U(A, \sigma)$ as $A$ varies over the hereditary orders in $A_F$ stable under $\sigma_F$. This has led the first two authors to ask whether $H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(F, G)$ is injective for any point stabilizer (resp. parahoric) group scheme $G$ of a connected reductive group scheme over $F$, [4, Question 6.4]. We explain briefly how the results and counterexamples of the previous sections relate to this question.

The generic fibers of $\text{Aut}_R(A, \sigma)$ and $\text{Aut}_R(A)$ are the reductive groups $\text{Aut}_F(A_F, \sigma_F)$ and $\text{Aut}_F(A_F)$ (which is just $\text{PGL}_1(A_F)$ if $\text{Cent}(A) = F$), the buildings of which are well-understood; see [4] or [3, 8], for instance. Provided that $\text{Aut}_R(A, \sigma)$ and $\text{Aut}_R(A)$ are smooth over $R$, one can use these sources to show that these are indeed point-stabilizer group schemes of their corresponding generic fibers. Then, the results of this paper give a mixed answer to the question above: By Theorem 2.1, the answer is positive for some of the point-stabilizer group schemes of $\text{Aut}_F(A_F) = \text{PGL}_1(A_F)$ when $\text{Cent}(A) = F$, whereas by Example 4.1 the answer can be negative for some point-stabilizer group schemes of $\text{Aut}_F(A_F, \sigma_F)$. If moreover $\text{PGL}_1(A) \to \text{Spec } R$ is the neutral component of $\text{Aut}_R(A) \to \text{Spec } R$ when $\text{Cent}(A) = F$, then Theorem 5.3 provides a partial positive answer for the question in the case of parahoric group schemes of $\text{PGL}_1(A_F)$.

The examples in Subsection 2.3 do not relate to [4, Question 6.4] because, in these examples, the generic fiber of $\text{Aut}_R(A)$ is reductive but not connected.

We remark that the smoothness of $\text{Aut}_R(A)$ and $\text{Aut}_R(A, \sigma)$ over $R$ as well as the condition that $\text{PGL}_1(A) \to \text{Spec } R$ is the neutral component of $\text{Aut}_R(A) \to$
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Spec \( R \) are false for arbitrary \( R \)-orders in central simple algebras. We hope to address both of these problems in the case of hereditary orders in subsequent work.

We finally note that in the case where \( R \) is a complete DVR with residue field \( k \), [4, Question 6.4] reduces to asking whether a result of Bruhat and Tits, [7, Lemma 3.9], can be strengthened. Indeed, the latter result asserts the injectivity of \( H^1_\text{ét}(R, \mathcal{G}_y) \to H^1_\text{ét}(F, \mathcal{G}_y) \) for any point stabilizer group scheme \( \mathcal{G}_y \) in which \( y \) is the barycenter of a cell in the affine building of \( \mathbf{G} := \mathcal{G}_y \times_R F \) and such that the closed fiber \( \mathcal{G}_y \times_R k \) is almost anisotropic in the sense that it has no proper parabolic subgroups.

Since in Example 4.1, the base ring is a complete DVR, it is possible that the reason for the apparent negative answer to [4, Question 6.4] is that arbitrary point-stabilizer group schemes are allowed. Returning to the case where \( R \) is any semilocal Dedekind domain with perfect residue fields, one can ask instead whether \( H^1_\text{ét}(R, \mathcal{G}) \to H^1_\text{ét}(F, \mathcal{G}) \) is injective when \( \mathcal{G} \to \text{Spec } R \) is barycentric point-stabilizer group scheme whose closed fibers are almost anisotropic. This statement already follows from [7, Lemma 3.9] when \( R \) is a complete DVR.

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