On incompleteness of polynomials in some weighted spaces on half line

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Abstract

The paper studies completeness of the polynomials in weighted $L_p$-spaces on half line. It is shown that the completeness of polynomials does not hold for a wide class of weights, including the weights $\exp(-rt^q)$ with $r > 0$ and $q \in (0, 1)$.

Key words: approximation, polynomials, completeness, Krein condition

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1 Introduction

The theory of approximation of function by polynomials is well developed; however, the exact characterisation of the weights for which polynomials are complete is unknown for weighted $L_2$-spaces of functions on infinite intervals. The related questions have been studied intensively; see, e.g., [1, 2, 5, 6, 9], and the literature therein.

For example, it is known that the moment problem is indeterminate in the weighted $L_2$-space of functions defined on the entire line $\mathbb{R}$ with the weight $\rho$ such that the following Krein condition holds:

$$\int_{-\infty}^{\infty} \log \rho(\omega) \frac{d\omega}{1+\omega^2} > -\infty;$$

see, e.g., Theorem 4.14 [9]. This implies that polynomials are not complete in this weighted space; see Theorems 6.10 and 7.7 in [9].

It is also known that the Stieltjes moment problem is indeterminate in the weighted $L_2$-space of functions defined on $[0, +\infty)$ with the weight $\rho$ such that the following Krein condition holds:

$$\int_0^{\infty} \log \rho(\omega^2) \frac{d\omega}{1+\omega^2} = \int_0^{\infty} \frac{\log \rho(\omega)}{(1+\omega)^{1/2}} d\omega > -\infty;$$

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see, e.g., Theorem 4.17 [9] or Theorem 1.2 [8]. In Theorem 4.17 [9], this condition is stated as a sufficient condition of indeterminacy; in Theorem 1.2 [8], this condition is stated as a necessary and sufficient condition of indeterminacy. Some developments and historical notes can be found [1, 7, 8, 9]; see also references therein.

The present paper focuses on the problem of completeness of polynomials on half line in a framework that bypasses the moment problem and indeterminacy. It is shown directly that polynomials are incomplete on a weighted $L^p$-spaces on half line under a weaker condition than (1.2). In particular, this condition allows weights $\exp(-r\omega^q)$ with $r > 0$ and $q \in (1/2, 1)$ that are excluded by condition (1.2).

2 The result

Let $\mathbb{R}^+ \triangleq [0, +\infty)$, and let $\mathcal{R}^+$ be the set of measurable functions $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\int_0^{+\infty} \omega^k \rho(\omega) d\omega < +\infty, \text{ for all } k = 0, 1, 2, ....,$$

and

$$\int_0^{+\infty} \log \rho(\omega) \frac{d\omega}{1 + \omega^2} > -\infty.$$  \hspace{1cm} (2.1)

**Example 1** It can be verified directly that $\mathcal{R}^+$ includes $\rho(\omega) = e^{-(\log|\omega|)^2}$, $\rho(\omega) = e^{-r|\omega|^q}$, and $\rho(\omega) = e^{-r|\omega|^q/|\log|\omega||^p}$ for $r > 0$, $q \in (0, 1]$, $p \geq 2$. On the other hand, the function $\rho(\omega) = e^{-|\omega|/|\log|\omega||}$ is excluded.

For an interval $I \subset \mathbb{R}$, for a measurable function $\varrho : I \to \mathbb{R}^+$, and for $p \geq 1$, let $L_{p,\varrho}(I)$ be the Banach space of complex valued functions $u : \mathbb{R} \to \mathbb{C}$ with the norm

$$\|u\|_{L_{p,\varrho}(I)} = \left( \int_I \varrho(\omega) |u(\omega)|^p d\omega \right)^{1/p}.$$

**Theorem 1** For any $\rho \in \mathcal{R}^+$, $T > 0$, $r > 0$, and $q \in (0, 1)$, the function $e^{i\omega T}$ cannot be approximated by polynomials in the space $L_{1,\rho}(\mathbb{R}^+)$.  

**Corollary 1** For any $\rho \in \mathcal{R}^+$, $r > 0$ and $q \in (0, 1)$, the set of polynomials is incomplete in $L_{p,\rho}(D)$ for all $p \geq 1$.

The integrand in condition (2.1) is the same as the one in the Krein condition for the entire real line (1.1). This condition is less restrictive than condition (1.2). For example, for $\rho(\omega) = \exp(-r\omega^q)$ with $r > 0$ and $q \in [1/2, 1)$, condition (2.1) holds but condition (1.2) does not hold.
3 Proofs

The proof of Theorem 1 below is based on the approach [3] developed for analysis of predictability of processes with fast decaying Fourier transform.

3.1 Some background notations

Let $R^- \triangleq \{ \omega \in R : \omega < 0 \}$, $C^+ \triangleq \{ z \in C : \Re z > 0 \}$, $C^- \triangleq \{ z \in C : \Re z < 0 \}$, $i = \sqrt{-1}$.

For $p \in [1, +\infty]$ and intervals $I \subset R$, we denote by $L_p(I)$ the usual $L_p$-spaces of functions $x : I \to C$.

For $x \in L_p(R)$, $p = 1, 2$, we denote by $X = Fx$ the function defined on $iR$ as the Fourier transform of $x$:

$$X(i\omega) = (Fx)(i\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}x(t)dt, \; \omega \in R.$$  

If $x \in L_2(R)$, then $X$ is defined as an element of $L_2(iR)$, i.e., $X(i\cdot) \in L_2(R)$.

For $x(\cdot) \in L_p(R)$, $p = 1, 2$, such that $x(t) = 0$ for $t < 0$, we denote by $Lx$ the Laplace transform

$$X(z) = (Lx)(z) \triangleq \int_{0}^{\infty} e^{-zt}x(t)dt, \; z \in C^+.$$  

In this case, $X|_{iR} = Fx$.

Let $H_r^+$ be the Hardy space of holomorphic on $C^+$ complex valued functions $h(z)$ with finite norm $\|h\|_{H^+_r} = \sup_{s>0} \|h(s+i\cdot)\|_{L^r(R)}$, $r \in [1, +\infty]$; see, e.g., [3].

Similarly to $R^+$, we denote by $R$ the set of measurable functions $\rho : R \to [0, +\infty)$ such that, for $k = 0, 1, 2, \ldots$, $\int_{-\infty}^{+\infty} |\omega|^k \rho(\omega)d\omega < +\infty$, and

$$\int_{-\infty}^{\infty} \log \frac{\rho(\omega)}{1 + \omega^2}d\omega > -\infty.$$  

3.2 A supposition

We assume below that we are given $T > 0$ and $\rho \in R^+$.

Suppose that the theorem statement is incorrect. Then there exists sequence of polynomials $\{\tilde{\psi}_d(\omega)\}_{d=1}^{\infty}$ in $\omega \in R$ of order $d$ such that

$$\varepsilon_d \triangleq \|e^{iT\cdot} - \tilde{\psi}_d(\cdot)\|_{L_{1,\rho}(R^+)} = \int_{0}^{\infty} |e^{iT\omega} - \tilde{\psi}_d(\omega)|\rho(\omega)d\omega \to 0 \; \text{as} \; d \to +\infty. \quad (3.2)$$

3.3 Some featured functions

3.3.1 Weights $\rho_d$

We presume that the given $\rho \in R^+$ is extended to $\rho \in R$; a possible choice is such that $\rho$ is an even function.
Let
\[ L_d \triangleq \| e^{iT} - \tilde{\psi}_d(\cdot) \|_{L^2([-T, T])}^2 = \int_{-\infty}^{0} |e^{iT\omega} - \tilde{\psi}_d(\omega)|\rho(\omega)d\omega. \]

Consider a sequence of functions \( \{\rho_d(\omega)\}_{d=1}^{\infty} \) defined as
\[ \rho_d(\omega) = \mathbb{1}_{\{\omega < 0\}} \rho(\omega) \frac{\varepsilon_d}{L_d} + \mathbb{1}_{\{\omega \geq 0\}} \rho(\omega). \]
Since \( \varepsilon_d > 0 \) for all \( d \), we have that \( \{\rho_d(\omega)\}_{d=1}^{\infty} \subset \mathcal{R} \).

**Functions** \( X_d \)

Let us construct a sequence of functions \( \{X_d(\omega)\}_{d=1}^{\infty} \subset \mathbb{H}^2 \) such that \( |X_d(i\omega)| = \mu_d(\omega) \), where
\[ \mu_d(\omega) \triangleq \frac{\rho_d(\omega)}{1 + \omega^2}. \]
Existence of such functions follows from Theorems 11.6 and 11.7 from [4]. For example, one can select
\[ X_d(z) = \exp \left[ \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{(1 - isz) \log \mu_d(s)}{(s + iz)(1 + s^2)} ds \right], \quad z \in \mathbb{C}. \]
We used here equation (11) from Theorem 11.6 [4] stated for the Hardy spaces on the upper complex half-plane; in the present paper, it is adjusted to the Hardy spaces \( \mathbb{H}^p \) on the right half-plane. In particular, we have that
\[ \int_{-\infty}^{\infty} \frac{\log |X_d(i\omega)|}{1 + \omega^2} d\omega = \int_{-\infty}^{\infty} \frac{\log \rho_d(\omega)}{1 + \omega^2} d\omega + \int_{-\infty}^{\infty} \frac{\log((1 + \omega^2)^{-1})}{1 + \omega^2} d\omega < +\infty. \]

Let \( x_d \triangleq \mathcal{F}^{-1}X_d|_{\mathbb{R}} \); clearly, \( x_d(t) = 0 \) for \( t < 0 \).

Clearly,
\[ \sup_d \|X_d(i\cdot)\|_{L^2(\mathbb{R})} < +\infty, \quad \sup_d \|x_d\|_{L^2(\mathbb{R})} < +\infty. \tag{3.3} \]

**Convolution kernels** \( h_d \) and functions \( y_d \)

Let us construct functions \( h_d : \mathbb{R} \to \mathbb{C} \) such that \( h_d(t) = 0 \) for \( t \notin [-T, 0] \), \( h_d \in C^\infty(\mathbb{R}) \), and
\[ \inf_d \left\| \int_{0}^{T} h_d(-t)x_d(t)dt \right\| > 0. \tag{3.4} \]

First, let us define \( \kappa_\varepsilon(t) \triangleq \varepsilon^{-1} \kappa_1(t/\varepsilon) \), where \( \kappa_1(t) \triangleq \exp(t^2(1 - t^2)^{-1}) \) is the so-called Sobolev kernel. Let
\[ g_d(t) \triangleq \|x_d|_{[0,T]}\|_{L^2(0,T)}^{-2} x_d(-t). \]
Finally, let \( h_{d,\varepsilon} \) be defined as the convolution
\[
h_{d,\varepsilon}(t) = \int_{-\infty}^{\infty} \kappa_{\varepsilon}(t-s) \mathbb{I}_{[-T+\varepsilon, -\varepsilon]}(s) g_d(s) ds, \quad \varepsilon > 0.
\]
In this case, we have that \( h_{d,\varepsilon} \to g_d \) in \( L_2(\mathbb{R}) \) as \( \varepsilon \to 0 \).

It follows from (3.3) that
\[
\sup_d \| h_d(i \cdot) \|_{L_p(\mathbb{R})} < +\infty, \quad p = 1, 2.
\]

Furthermore, let
\[
y_d(t) \triangleq \int_t^{t+T} h_d(t-s) x_d(s) ds.
\]
Let \( \xi_{d,\varepsilon}(t) \triangleq h_{d,\varepsilon}(-t) - g_d(-t) \). We have that \( h_d(-t) = h_{d,\varepsilon}(-t) = g_d(-t) + \xi_{d,\varepsilon}(t) \),
\[
y_d(0) = \int_0^T h_d(-t) x_d(t) ds
\]
\[
= \| x_d \mathbb{I}_{[0,T]} \|_{L_2(0,T)}^{-2} \int_0^T |x_d(t)|^2 dt + \| x_d \mathbb{I}_{[0,T]} \|_{L_2(0,T)}^{-2} \int_0^T \xi_{d,\varepsilon}(t) x_d(t) dt
\]
\[
= 1 + \| x_d \mathbb{I}_{[0,T]} \|_{L_2(0,T)}^{-2} \int_0^T \xi_{d,\varepsilon}(t) x_d(t) dt.
\]
Furthermore,
\[
|y_d(0) - 1| \leq \| x_d \mathbb{I}_{[0,T]} \|_{L_2(0,T)}^{-2} \| x_d \mathbb{I}_{[0,T]} \|_{L_2(0,T)} \| \xi_{d,\varepsilon} \mathbb{I}_{[0,T]} \|_{L_2(0,T)} = \| x_d \mathbb{I}_{[0,T]} \|_{L_2(0,T)}^{-1} \| \xi_{d,\varepsilon} \mathbb{I}_{[0,T]} \|_{L_2(0,T)}.
\]
We have that the sequence \( \{X_d\} \) has a limit in \( L_2(\mathbb{R}) \), therefore, the sequence \( \{x_d\} \) has a limit in \( L_2(\mathbb{R}) \), and
\[
\| x_d \mathbb{I}_{[0,T]} \|_{L_2(0,T)}^{-1} \| \xi_{d,\varepsilon} \mathbb{I}_{[0,T]} \|_{L_2(0,T)} \leq C
\]
for all \( d, \varepsilon \) for some \( C > 0 \).

By the property of kernels \( \kappa_{\varepsilon} \), for any \( d \),
\[
\| x_d \mathbb{I}_{[0,T]} \|_{L_2(0,T)}^{-1} \| \xi_{d,\varepsilon} \mathbb{I}_{[0,T]} \|_{L_2(0,T)} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]
Hence, for each \( d \), for some choice of \( \bar{\varepsilon} = \bar{\varepsilon}(d) > 0 \), the kernels selected as \( h_d = h_{d,\bar{\varepsilon}} \) are such that
\[
\inf_d |y_d(0)| > 0.
\]
It follows that (3.4) holds for this choice of \( h_{d} \).
3.4 Proof of Theorem 1: final steps

Let \( H_d \triangleq L h_d \) and \( Q_d = L q_d \), where \( q_d(t) \equiv h_d(t - T) \). Clearly, \( H_d(i \omega) \equiv Q_d(i \omega) e^{i \omega T} \), \( q_d(t) = 0 \) for \( t < 0 \), \( q_d \in C^\infty(\mathbb{R}) \), and \( Q_d \in \mathbb{H}^2 \cap \mathbb{H}^\infty \).

For our purposes, it would be more convenient to use a sequence of polynomials \( \{ \psi_d(z) \}_{d=1}^\infty \) of order \( d \) such that

\[
\|e^{iT \cdot} - \psi_d(i \cdot)\|_{L_2, \rho(\mathbb{R})} = \int_{-\infty}^{\infty} |e^{iT \omega} - \psi_d(i \omega)|^2 \rho(\omega) \, d\omega \to 0 \quad \text{as} \quad d \to +\infty. \quad (3.8)
\]

The coefficients \( a_k \) of the polynomials \( \psi_d(z) = \sum_{k=0}^d a_k z^k \) can be constructed by adjustment the signs of the coefficients for the polynomials \( \tilde{\psi}_d(\omega) = \sum_{k=0}^d \tilde{a}_k \omega^k \) such that \( \psi_d(i \omega) \equiv \tilde{\psi}_d(i \omega) \), i.e., \( \tilde{a}_k = a_k i^k \) and \( a_k = \tilde{a}_k i^{-k} \). In this case, statements (3.2) and (3.8) are equivalent.

For \( d = 1, 2, \ldots \), set

\[
\tilde{H}_d(z) \triangleq e^{-T} \tilde{\psi}_d(z) H_d(z) = \psi_d(z) Q_d(z), \quad \tilde{h}_d \triangleq \mathcal{F}^{-1} \tilde{H}_d|_{\mathbb{R}}. \quad (3.9)
\]

Since \( q_d(t) = 0 \) for \( t < 0 \) and \( q_d \in C^\infty(\mathbb{R}) \), we have that \( z^n Q_d(z) \in \mathbb{H}^2 \cap \mathbb{H}^\infty \) for any integer \( n \geq 0 \). It follows that

\[
\tilde{H} \in \mathbb{H}^2 \cap \mathbb{H}^\infty.
\]

Let \( \tilde{Y}_d(i \omega) \triangleq \tilde{H}_d(i \omega) X_d(i \omega) \) and \( \tilde{y}_d \triangleq \mathcal{F}^{-1} \tilde{Y}_d(i \omega) \). It follows that

\[
\tilde{y}_d(t) = \int_{-\infty}^{t} \tilde{h}_d(t - s) x_d(s) \, ds. \quad (3.10)
\]

It can be noted that, by the definitions, \( \tilde{h}_d(t) \triangleq \sum_{k=0}^d a_k \frac{d^k}{dt^k} (t + T) \).

We have that

\[
\| \tilde{y}_d - y_d \|_{L_\infty(\mathbb{R})} \leq \frac{1}{2\pi} \left\| (\tilde{H}_d(i \cdot) - H_d(i \cdot)) X(i \cdot) \right\|_{L_1(\mathbb{R})}. \quad (3.11)
\]

Furthermore,

\[
\| (\tilde{H}_d(i \cdot) - H_d(i \cdot)) X_d(i \cdot) \|_{L_1(\mathbb{R})} = \int_{-\infty}^{\infty} \left| (e^{-i\omega T} \tilde{\psi}_d(i \omega) - 1) e^{i\omega T} Q_d(i \omega) X_d(i \omega) \right| \, d\omega
\]

\[
= \int_{-\infty}^{\infty} \rho_d(\omega) \left| (e^{-i\omega T} \tilde{\psi}_d(i \omega) - 1) \rho_d(\omega) e^{i\omega T} Q_d(i \omega) X_d(i \omega) \right| \, d\omega
\]

\[
= \int_{-\infty}^{\infty} \rho_d(\omega) \left| (\tilde{\psi}_d(i \omega) - e^{i\omega T}) \rho_d(\omega) e^{i\omega T} Q_d(i \omega) X_d(i \omega) \right| \, d\omega \leq \alpha_d \beta_d. \quad (3.12)
\]

Here

\[
\alpha_d \triangleq \int_{-\infty}^{\infty} \rho_d(\omega) |\tilde{\psi}_d(i \omega) - e^{i\omega T}| \, d\omega \leq \|\tilde{\psi}_d(i \cdot) - e^{iT}\|_{L_1, \rho_d(\mathbb{R}^-)} + \|\tilde{\psi}_d(i \cdot) - e^{iT}\|_{L_1, \rho_d(\mathbb{R}^+)},
\]
and
\[ \beta_d \triangleq \operatorname{ess sup}_{\omega} \rho_d(\omega)^{-1} |e^{i\omega T} Q_d(i\omega) X_d(i\omega)|. \]

By the choice of \( \rho_d \), it follows that
\[ \| \psi_d(i\cdot) - e^{i\cdot T} \|_{L_1, \rho_d(\mathbb{R}^-)} = \| \psi_d(i\cdot) - e^{i\cdot T} \|_{L_1, \rho_d(\mathbb{R}^+)} = \varepsilon_d. \]

It gives that
\[ \alpha_d \to 0 \quad \text{as} \quad d \to +\infty. \quad (3.13) \]

It follows from estimate (3.5) with \( p = 1 \) that
\[ \sup_{\omega, d} |Q_d(i\omega)| < +\infty. \]

Hence, by the choice of \( X_d \), we obtain that
\[ \sup_{d} |\beta_d| \leq \operatorname{ess sup}_{\omega, d} |Q_d(i\omega)| \rho_d(\omega)^{-1} |X_d(i\omega)| = \operatorname{ess sup}_{\omega, d} |Q_d(i\omega)|(1 + \omega^2)^{-1} < +\infty. \quad (3.14) \]

Then estimates (3.11)–(3.14) imply that
\[ \sup_t |\hat{y}_d(t) - y_d(t)| \to 0 \quad \text{as} \quad d \to +\infty. \quad (3.15) \]

On the other hand, since \( x_d(t) \) and \( \hat{h}_d(t) \) are both vanishing for \( t < 0 \), we have that
\[ \hat{y}_d(t) = 0, \quad t \leq 0. \quad (3.16) \]

Hence (3.15) cannot hold simultaneously with (3.7) and (3.16). This means that the supposition that the theorem statement is incorrect leads to a contradiction. This completes the proof of Theorem 1. \( \square \).

The proof of Corollary 1 follows from the fact that \( L_{p, \rho}(\mathbb{R}^+) \subseteq L_{1, \rho}(\mathbb{R}^+) \) for any \( p \geq 1 \), and that this embedding is continuous.

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