New High Dimensional Expanders from Covers

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ABSTRACT
We present a new construction of high dimensional expanders based on covering spaces of simplicial complexes. High dimensional expanders (HDXs) are hypergraph analogues of expander graphs. They have many uses in theoretical computer science, but unfortunately only few constructions are known which have arbitrarily small local spectral expansion.

We give a randomized algorithm that takes as input a high dimensional expander \(X\) (satisfying some mild assumptions). It outputs a sub-complex \(Y \subseteq X\) that is a high dimensional expander and has infinitely many simplicial covers. These covers form new families of bounded-degree high dimensional expanders. The sub-complex \(Y\) inherits \(X\)’s underlying graph and its links are sparsifications of the links of \(X\). When the size of the links of \(X\) is \(O(\log |X|)\), this algorithm can be made deterministic.

Our algorithm is based on the groups and generating sets discovered by Lubotzky, Samuels and Vishne (2005), that were used to construct the first discovered high dimensional expanders. We show these groups give rise to many more “randomized” high dimensional expanders.

In addition, our techniques also give a random sparsification algorithm for high dimensional expanders, that maintains its local spectral properties. This may be of independent interest.

CCS CONCEPTS
• Theory of computation → Expander graphs and randomness extractors; Generating random combinatorial structures.

KEYWORDS
High Dimensional Expanders, HDX, Covers

1 INTRODUCTION
Expanders are graphs that are highly connected. It is a known fact that there exist families of expander graphs that are bounded-degree. These graphs play a key role in theoretical computer science, combinatorics and many other areas in mathematics [24]. High dimensional expanders (HDXs) are a hypergraph analogue of expander graphs. Loosely speaking, high dimensional expanders have the property that neighborhoods of faces are themselves expanding graphs. Lubotzky surveys the different definitions of high dimensional expanders [33].

High dimensional expanders are promising objects in theoretical computer science. They have been used for efficiently solving CSPs [1], for constructing agreement testers [10, 13] and for sampling and counting matroids and other combinatorial objects (e.g. [4], [8]).

There are only two constructions of high dimensional expanders that achieve arbitrarily good expansion, [35] and [27] (the latter was also studied by [39]). Both constructions rely on group theory. This is contrary to expander graphs, where in addition to group-theoretic constructions, elementary combinatorial constructions and random constructions are known [24].

The goal of this work is to construct new high dimensional expanders by using random simplicial covers. Graph covers (or lifts) are a key ingredient in many random expander constructions. A graph \(G'\) covers a graph \(G\) if there is a surjective graph homomorphism \(\varphi : G' \rightarrow G\) so that for every vertex \(v \in G'\), \(\deg(v) = \deg(\varphi(v))\). There is a simple algorithm (described below) that given a graph \(G\), outputs a random cover \(G'\) of \(G\). In many cases the covers will also be expander graphs, and indeed this is exploited to construct infinite families of expanders (see e.g. [5] and [36]). In this work, we manage to extend this technique to high dimensional expanders.

1.1 High Dimensional Expanders
A pure \(d\)-dimensional simplicial complex \(X\) is a set system (or hypergraph) consisting of an arbitrary collection of sets of size \(d + 1\) together with all their subsets. The sets of size \(i + 1\) in \(X\) are denoted by \(X(i)\), and in particular, the vertices of \(X\) are denoted by \(X(0)\). The 1-skeleton of \(X\) is the graph whose vertices are \(X(0)\) and edges are \(X(1)\). The sets in a simplicial complex are called faces. We say that \(X\) is connected if its 1-skeleton is connected.

Let \(s \in X(i)\). The link of \(s\) is a simplicial complex denoted by \(X_s\) whose sets are all \(t \in X\) so that \(s \cap t = \emptyset\) and \(s \cup t \in X\). For \(\lambda \geq 0\), a \(\lambda\)-two-sided high dimensional expander is a simplicial complex so that its 1-skeleton, and all the 1-skeletons of its links are \(\lambda\)-two-sided spectral expanders.

This definition is due to [13]; it concentrates on spectral properties. High dimensional expanders by this definition are useful in many applications as discussed above. However, there are other
non-equivalent definitions for high dimensional expansion. We recommend [33] for a comparison between the main definitions that interest the community.

1.2 Graph Covers and Simplicial Covers
Let $X, Y$ be pure $d$-dimensional simplicial complexes. A simplicial homomorphism is a function $\psi : X(0) \to Y(0)$ so that every $s \in X(d)$ is mapped to $\psi(s) \in Y(d)$. A homomorphism is a cover if it is surjective, and locally it behaves like an isomorphism. That is, for every face $t \in Y(i)$ and preimage $s \in X(i)$ so that $\psi(s) = t$, the restriction of $\psi$ to $X_s$ is an isomorphism between $X_s$ and $Y_t$. For graphs, this definition coincides with the previous definition discussed above.

Given a graph on $n$ vertices $G = (V = \{v_1, v_2, ..., v_n\}, E)$, we can construct a cover $G' = (V', E')$ of $G$ as follows. Let $\Gamma$ be any finite group and let $f : E \to \Gamma$ be a labeling. The vertices of $G'$ are set to be $V' = V \times \Gamma$. For every $i < j$, the vertices $(v_i, g), (v_j, h)$ are adjacent in $G'$ if $v_i v_j \in E$ and $h = g \cdot f(ij)$. The covering map is the map that projects $(i, g)$ to its left coordinate $i$. In fact, every graph cover corresponds to some group and some edge labeling $f : E \to \Gamma$.

This method has proven to be useful in constructing new expander graphs from initial expanders. Amit and Linial initiated the study of random graph covers [2]. Bilu and Linial showed that if $G$ is a bounded-degree expander, $\Gamma = \mathbb{F}_2$ and $f$ is chosen at random, then $G'$ is an expander with positive probability [5]. A breakthrough paper by Marcus, Spielman and Srivastava [36] used graph covers to create optimal (Ramanujan) bipartite expander graphs. It is tempting to try and generalize this simple construction to high dimensional expanders. Unfortunately, in simplicial complexes this method does not work as is. Given a $d$-dimensional simplicial complex $X$ and a labeling of the 1-skeleton $f : X(1) \to \Gamma$, the resulting covering graph $G'$ is not necessarily a $(1$-skeleton of a $d$-dimensional simplicial complex at all. It turns out that connected covers of a simplicial complex correspond to subgroups of its fundamental group (as a topological space) [44]. This group may be trivial so complexes may have no non-trivial connected covers.

1.3 Our Contribution
Not all is lost. In this paper we show that even if a high dimensional expander $X$ has no covers, with the help of certain groups $\Gamma$ we can find a non-trivial sub-complex $Y \subseteq X$ that is a high dimensional expander and has infinitely many covers. In more detail, we start with:

(1) An initial high dimensional expander $X$. The links of $X$ need to be (almost) regular and relatively dense, but $X$ itself may have bounded degree.

(2) An infinite group $\Gamma$ that has infinitely many normal subgroups of finite index, together with a generating set $S \subseteq \Gamma$. Links of the clique complex of the Cayley graph $\text{Cay}(\Gamma, S)$ should be themselves expanding graphs. \footnote{A clique complex of a graph is a simplicial complex whose vertices and edges are those of the graph $\text{Cay}(\Gamma, S)$. The high-dimensional faces of the complex are all cliques in the graph.}

Theorem 1.1 (Informal, for the formal theorem see Theorem 3.2). There is a randomized algorithm that takes as input a high-dimensional expander $X$ satisfying Item 1, and a group and generating set $(\Gamma, S)$ satisfying Item 2. The algorithm outputs a sub-complex $Y \subseteq X$ and a labeling $f : X(1) \to \Gamma$ so that:

1. $Y$ has infinitely many covers $(Y_n)_{n=1}^\infty$ which can be constructed in polynomial time using $f$.

2. $Y$ is a high dimensional expander.

3. $Y$ contains a constant fraction of the faces of $X$, proportional to $\frac{1}{\log |\text{Sym}(S)|}$.

We note that the cover $(Y_n)_{n=1}^\infty$ are the new family of high dimensional expanders we seek. Their vertices are $Y(0) \times (\Gamma/\Gamma_n)$ where $(\Gamma_n)_{n=1}^\infty$ are normal subgroups of $\Gamma$ of finite index. Their links are isomorphic to links of faces in the original complex $Y$. Hence links of these covers are sparsifications of links in the original complex $X$.

Groups and generating sets $(\Gamma, S)$ as required in the input appeared in [34, 35], where they were used to construct the first known high dimensional expanders. The resulting simplicial complexes we construct are however, are different and depend on the input $X$ we give the algorithm. Interestingly, we show that this new simplicial complex $Y$ inherits properties both from $X$ and from $\text{Cay}(\Gamma, S)$. Globally, the 1-skeleton of $Y$ is identical to the 1-skeleton of $X$ (i.e. it has the same vertex set and edge set). For every $\sigma \in \Gamma$, the link $Y_\sigma \subseteq X_\sigma$ is a sparsification that has a constant fraction of the faces in $X_\sigma$. The algorithm sparsifies the higher-dimensional faces so that locally its links are homomorphic to links of faces in the clique complex of $\text{Cay}(\Gamma, S)$.

There are many starting points $X$ that satisfies the requirements in Theorem 1.1. The simplest is the complete complex. We could also use a random model to sample our initial simplicial complex, such as the model suggested by [31]: its vertices are $\{1, 2, ..., m\}$, every $d$-face is sampled independently with some probability $p \in (0, 1)$. Although we don’t usually think about the complete complex or the model in [31] as “bounded-degree”, recall that $X$ is just a single complex (and thus always has bounded degree). Albeit, we can initialize our construction by taking $X$ out of a family of bounded-degree of high dimensional expanders as long as they are regular. A work by Friedgut and Iluz shows how to construct strongly regular high dimensional expanders from existing ones [17], which we can use.

The algorithm in Theorem 1.1 can also be derandomized when $X$ has logarithmic sized links.

Though our main result deals with covers, our random sparsification technique has another application of independent interest. We show that we can sparsify a high dimensional expander $X$, so that it will be homomorphic to some fixed smaller high dimensional expander $C_0$, while maintaining its high dimensional expansion.

Theorem 1.2 (Informal, for the formal theorem see the full version of the paper [9]). There is a randomized algorithm that takes as input a simplicial complex $C_0$, and simplicial complex $X$ that satisfies Item 1, and outputs a sub-complex $Y \subseteq X$ so that:

1. $Y$ is homomorphic to $C_0$. 

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(2) $Y$ has a constant fraction of the faces of $X$ (the constant depends on $C_0$).
(3) If $X$ and $C_0$ are high dimensional expanders then $Y$ is also a high dimensional expander.

Note that $Y \subseteq X$ could be viewed also as a sparsifying algorithm for $X$, that produces a sparser simplicial complex that still has high dimensional expansion properties. We also show that when $X$ has small enough links, then we can derandomize the algorithm in Theorem 1.2.

1.4 Constructing Covers in Higher Dimension
Let us understand how to generalize the graph cover construction to higher dimensional simplicial complexes.

Let $X$ be a $d$-dimensional high dimensional expander where $d \geq 2$ with vertex set $X(0) = \{v_1, v_2, ..., v_n\}$ (we order of the vertices arbitrarily). Let $f : X(1) \rightarrow \Gamma$ be a labeling to some group $\Gamma$. Let $G'$ be the graph cover of the 1-skeleton of $X$, generated by $f$. In order to extend $G'$ to a simplicial cover of $f$, we need the property that for every triangle $uvw \in X$ where $i < j < k$,

$$f(u) f(v) f(w) = f(uw).$$  

(1.1)

Labelings $f$ with this property are called $\Gamma$-cosystole. If $f$ is a $\Gamma$-cosystole, then whenever $(u, g) \sim (v, g f(uv))$ and $(v, g) \sim (w, g f(vw))$, it also holds that

$$(u, g f(uv)) \sim (v, g f(vw)) \sim (w, g f(vw)).$$

I.e. all edges of the triangle $\{(u, g), (v, g f(uv)), (w, g f(vw))\}$ are in $G'$. We can then extend $G'$ to a simplicial complex $X'$ with

$$X'(2) = \{(u, g), (v, g f(uv)), (w, g f(vw)) : u, v, w \in X(2), i < j < k < g \in \Gamma\}$$

and similarly for $t > 2$,

$$X'(t) = \{(u_0, g), (u_1, g f(u_0 u_1)), ..., (u_t, g f(u_0 u_1 ... u_t)) : u_0, u_1, ..., u_t \in X(t), 0 < i_0 < i_1 < ... < i_t \in \Gamma, g \in \Gamma\}$$

1.5 Overview of our Construction
The cover constructed from a labeling $f : X(1) \rightarrow \Gamma$ may be trivial, that is, that $X'$ may consist of $|\Gamma|$ disjoint copies of $X$. Recall that connected covers are determined by the fundamental group of $X$, and it could be the case that all covers $X'$ of $X$ are trivial.

To overcome this in our construction, we “go the other way around”. We sample a labeling $f : X(1) \rightarrow \Gamma$ by sampling the label of each edge independently. We then wish to remove from $X$ all faces that contain triangles $uvw \in X(2)$ where $i < j < k$ and $f(uv) f(vw) \neq f(uw)$. If we do so, we get a sub-complex $Y \subseteq X$ so that $f$ is one of its $\Gamma$-cosyntols.

The main problem with this naive triangle removal, is that with high probability some links of faces in $Y$ will not be expanders anymore. It could be, for example, that all triangles adjacent to some vertex are removed (and its link becomes disconnected). Denote by $B_\sigma$ the event that the link of $\sigma$ is not an expander graph.

Fortunately, Lovasz’s Local Lemma [15] guarantees us that if the probability of each $B_\sigma$ is small, and that every event $B_\sigma$ is independent of all but a few of the $B_\tau$, then the event where no $B_\sigma$ occurs has positive probability (i.e. $Y$ is a high dimensional expander with positive probability). The heart of our analysis is to properly define these “local bad events”, and to show that we can indeed apply Lovasz’s Local Lemma to them. We use similar techniques to also promise that the $f$-cover of $Y$ is indeed connected.

Moser and Tardos proved that there is a procedure to turn the existential proof based on Lovasz’s Local Lemma to a randomized algorithm that finds $Y$ in polynomial time [38]. When the links of $X$ are not too small, we can derandomize this algorithm using the work of [6].

As $Y$ is obtained by removing faces from $X$, it is not surprising that 1-skeletons of links of faces in $Y$ are obtained by removing vertices and edges from links of faces in $X$. To show that these links still expand with high probability, we analyze random sparsifications of expander graphs with logarithmic degree, and show that with high enough probability a graph remains an expander after such a sparsification. To do so we use the Inverse Expander Mixing Lemma proven by Bilu and Linial [5]. This analysis is quite general and may be of independent interest.

1.6 Related Works
Properties of high dimensional expanders were first used in the work of Garland [20], even before they were explicitly defined. Local spectral properties of links were used there to show that the real cohomology of a simplicial complex vanishes. Later on, a strong notion of coboundary expansion was discovered by Gromov [23], and also by Linial, Meshulam and Wallach [31], [37]. The definition we use of high dimensional expanders is due to Dinur and Kaufman [13], who also defined the one-sided version of high-dimensional expansion (before their work, a weaker notion of expansion in the links was used in e.g. [16, 26]). The first construction of bounded-degree high dimensional expanders was done before the definition we use today, by Lubotzky, Samuels and Vishne [34]. A second algebraic construction was discovered by Kaufman and Oppenheim [27, 28]. Recently, [39] construct new high dimensional expanders using the method of [27] with Chevalley groups. Friedgut and Iluz modified existing constructions for creating hyper-regular high dimensional expanders [17]. There have been combinatorial constructions of high dimensional expanders by Chapman, Linial and Peled [7], Liu, Mohanty and Yang [32] and Golowich [21]. However, the combinatorial constructions currently produce $\lambda$-HDXs for $\lambda \geq \frac{1}{2}$.

As mentioned above, there is a long line of works on random graph covers and their properties e.g. [2, 3, 5, 18, 36, 41]. This work lead to the breakthrough construction of bipartite Ramanujan graphs of all degrees by Marcus, Spielman and Srivastava. The works that study expansion usually focused on analysis of the eigenvalues of the covering graph. In higher dimension however, every connected cover is also a high dimensional expander. This fact is a corollary of a work of Oppenheim [40], which we discuss more precisely in Section 2. However, to analyze the links in our construction, we use the techniques developed by Bilu and Linial [5] towards analyzing random graph covers.

Dinur and Meshulam also studied on simplicial covers from a TCS perspective [14]. They study a topological property test on...
a class of simplicial complexes called cosystolic expanders. They show that if \( X \) is such a complex, then every surjective mapping \( f : Y \rightarrow X \) that satisfies almost all the local conditions of being a cover is close to a genuine cover \( g : Y \rightarrow X \).

Random sparsification of graphs and their adjacency matrices is also a well studied problem, and there are known randomized and deterministic algorithms to sparsify graphs while keeping their spectral properties, e.g. \([42],[43],[30],[25]\) and \([29]\). Furedi and Kolmos initiated the study of random sparsified matrices \([19]\) (see also the work of Vu \([45]\)). A celebrated result by Spielman and Teng, studied sparsification of spectral expanders by randomly removing edges \([43]\). However, the probability of removing an edge in \([43]\) varies according to the degrees of its vertices, and we could not use its analysis as is. Hence, we prove a graph sparsification result that is tailored for our needs.

1.7 Open Questions

The construction we give in this paper is for two sided high dimensional expansion. Is there also an extension of our algorithm or our analysis that works on one sided high dimensional expanders \( X \)?

All constructions of bounded-degree high dimensional expanders with \( \lambda < \frac{1}{2} \) rely on group theory (either deterministic or randomized). Finding a combinatorial construction that does not rely on a group for arbitrarily small \( \lambda > 0 \) or even \( \lambda = 0.49 \) is an open problem. It is interesting to note that the theorem of Oppenheim mentioned above \([40]\) only applies for \( \lambda < \frac{1}{2} \). This seems to show that \( \lambda \)-high dimensional expanders for \( \lambda < \frac{1}{2} \) must have some non-trivial structure.

Our construction has properties inherited by both the complex \( X \), and the clique complex of \( Cay(\Gamma,S) \). There are many other interesting properties that could be of interest in our construction. For example, one can repeat this construction with groups and generating sets so that the links of the clique complex of \( Cay(\Gamma,S) \) are small set expanders, but not expanders themselves. Will the resulting complexes be small set expanders (either locally or globally)? Another potentially interesting property is coboundary and cosystolic expansion. Will the resulting complexes be cosystolic expanders? Our construction relies on infinite groups \( \Gamma \) that are residually finite and have expanding links. Besides the groups in \([35]\), we are not aware of groups that have these properties. \(^2\) Finding new examples of such groups is interesting. The groups that were used in \([27],[28]\) are possible candidates, as they already showed potential for construction of high dimensional expanders. Note that \([46]\) showed that any such \( \Gamma \) has Kazhdan’s property-T. This already constrains the types of groups we can consider. \([22]\) defined a random model of groups that seems to have expanding links, but it is not clear whether they are residually finite or not. It will be interesting to understand if this model, or some variation to it, could produce new examples of groups useful for our construction.

Another open question that may have algorithmic applications, is embedding simplicial complexes in high dimensional expanders. That is, given a simplicial complex \( X \), is it possible to embed it in a high dimensional-expander \( X' \supseteq X \), so that \( |X'|/|X| \) is constant. For graphs this is easy, since given \( G = (V,E) \), we can find an expander graph with the same vertex set \( H = (V',E') \) and embed \( G \) in \( G' = (V,E \cup E') \). This simple embedding in graphs is useful, for example, in the PCP-theorem proof by \([12]\). Understanding when we also have such an embedding in simplicial complexes could potentially lead to new algorithms for solving constraint satisfaction problems.

Finally, \([11]\) generalized the notion of high dimensional-expanders to expanding posets. Another open direction of research is whether one could use topological covers to construct new expanding posets.

1.8 Organization of this Paper

Section 2 contains the preliminaries necessary for this paper. We formally state and prove our main theorem about simplicial covers in Section 3, while deferring some technical claims to the full version of the paper \([9]\). In the full version we also show that for every fixed simplicial complex \( C \), there is an algorithm that receives a simplicial complex \( X \) and outputs a sub-complex \( Y \subseteq X \) that is homomorphic to \( C \), and also derandomize the results when the input simplicial complex is bounded-degree.

2 PRELIMINARIES

2.1 Classical Probability

Theorem 2.1 (Lovász Local Lemma, \([15]\), Algorithmic Version by \([38]\)). Let \( X_1,\ldots,X_d \) be independent random variables. Let \( E_1,E_2,\ldots,E_n \) be events determined by these random variables. For every \( E_i \) we denote by \( A_i = \{ j \mid E_i,E_j \text{ are not independent} \} \). Assume that there are \( \alpha_i \in (0,1) \) so that for every event

\[
P\left[ E_i \right] \leq \alpha_i \prod_{j \in A_i} (1 - \alpha_j).
\]

Then \( P [ \bigwedge E_1 \wedge \bigwedge E_2 \wedge \cdots \wedge \bigwedge E_n ] > 0 \). Moreover, there exists a randomized algorithm that finds assignments to \( X_1,X_2,\ldots,X_d \) so that all the events \( E_1,E_2,\ldots,E_n \) don’t occur. The algorithm runs in an expected time of \( \sum_{i=1}^n |A_i|/|A_i| \).

Note that when all \( |A_i| < R \), if one takes \( \alpha_i = 1/R \) and shows that \( P [ E_i ] \leq 1/(R^{d+1}) \), then this theorem promises us there exists an assignment so that \( P [ \bigwedge E_1 \wedge \bigwedge E_2 \wedge \cdots \wedge \bigwedge E_n ] > 0 \). Moreover, it gives us a polynomial time algorithm to find such an assignment to \( X_1,X_2,\ldots,X_d \).

2.2 Expander Graphs

Let \( G = (V,E) \) be a graph. The measure of a set of edges \( E' \subseteq E \) is \( \gamma(G)(E') = |E'| / |V| \) and the measure of a set of vertices \( V' \subseteq V \) is \( \gamma(G)(V') = \frac{1}{|V'|} \sum_{v' \in V'} \deg(v') \). This is generalized in a straightforward manner to weighted graphs.

Definition 2.2 (spectral expanders). Let \( G \) be a graph and let \( 0 < \lambda < 1 \). \( G \) is a \( \lambda \)-two sided spectral expander, if

\[
\max(|\lambda_2|,|\lambda_d|) \leq \lambda.
\]

In this paper, when saying a graph is a \( \lambda \)-spectral expander, we mean it is a \( \lambda \)-two sided spectral expander.
Bilu and Linial gave a converse to the expander mixing lemma:

**Theorem 2.3 ([5]).** Let $A$ be a normalized adjacency-operator of a bipartite graph $H' = (A, B, E')$. Assume that for all $S \subseteq A, T \subseteq B$ it holds that

$$|\mathbb{E}(S, T) - \mathbb{E}(S)\mathbb{E}(T)| \leq \alpha \sqrt{\mathbb{E}(S)\mathbb{E}(T)}.$$ 

Then $\lambda(A) \leq 260\alpha(1 + \log_2(3/\alpha))$.

Theorem in [5] assumes $H'$ is regular. We show their proof extends to arbitrary the full version of the paper [9].

### 2.3 Graph Homomorphisms

Let $G = (V, E)$ and $H = (V', E')$ be graphs. We say that $f : V \to V'$ is a graph homomorphism if for any edge $\{u, v\} \in E$ it holds that $\{f(u), f(v)\} \in E'$. If $f$ is surjective we say that $f$ an $H$-coloring of $G$, and that $G$ is $H$-colorable. In addition, if for every edge $\{a, b\} \in E'$ there is $\{u, v\} \in E$ so that $f(u) = a, f(v) = b$ then we say that $G$ is non-degenerate $H$-colorable and that $f$ is a non-degenerate $H$-coloring. We depict this in Figure 1.

Let $G = (V, E), H = (V', E')$ be graphs. We say that $G$ covers $H$ if there is a graph homomorphism $f : V \to V'$ so that for every $v \in V$, $f$ restricted to a neighbourhood of $v$ a bijection to the neighbourhood of $f(v)$.

When $(H, v_H), (G, v_G)$ have probability measures, then a non-degenerate $H$-coloring defines a new measure $v_f$ on $G$ as follows.

For $uw \in E$ so that $f(u) = a, f(b) = b$,

$$v_f(uw) := v_H(ab) \sum_{f'(u') = a, f'(v') = b} v_G(u'v').$$

In other words, we sample (oriented or non oriented) edges in $E$ by the following process:

1. Sample an edge $(a, b) \in E'$.
2. Sample an edge $(u, v) \in E$ conditioned on its image being $(a, b)$.

We stress that $v_f$ could be different than $v_G$, the original measure on $G$. For example it always holds that $v_f(f^{-1}(a)) = v_H(a)$, regardless of $v_G(f^{-1}(a))$.

When $G$ is an $H$-colorable graph, we can partition the vertices of $G$ by $V = \bigsqcup_{a \in V} S_a$ where $S_a = \{v \in V \mid f(v) = a\}$. In the full version of the paper [9] we show that when $H$ is an expander and the bipartite graphs between the parts in the partition of $G$ are bipartite expanders, then $(G, \nabla_f)$ is also an expander:

**Claim 2.4.** Let $H = (V', E')$ be a $\lambda$-two sided (one-sided) spectral expander. Let $G = (V, E)$ and let $f : V \to V'$ be a non-degenerate $H$-coloring of $G$. Assume that for every $(a, b) \in E'$ the bipartite graph between $S_a, S_b$ is $\eta$-bipartite spectral expander (with respect to the conditional weights $\frac{v_G(uw)}{\sum_{uw' f'(u') = u} v_G(uw')}$). Then the weighted graph $(G, v_f)$ is a $\max(\lambda, \eta)$-two sided (one sided) spectral expander.

### 2.4 Simplicial Complexes and High Dimensional Expanders

We include here the basic definitions needed for our results. For a more comprehensive introduction to this topic we refer the reader to [13] and the references therein.

A simplicial complex is a hypergraph that is downward closed with respect to containment. It is called $d$-dimensional if the largest hyperedge has size $d + 1$. We refer to $X(t)$ as the hyperedges (also called faces) of size $t + 1$. $X(0)$ are the vertices. For a face $s$ and $t \leq d$ we denote by $\text{deg}_t(s) = |\{t \in X(t) \mid s \subseteq t\}|$.

It will sometimes be useful to consider oriented faces. We denote by

$$\overrightarrow{X(t)} = \{(u_0, u_1, ..., u_t) \mid \{u_0, u_1, ..., u_t\} \in X(t)\}.$$

We define a weighted simplicial complex. Suppose we have a $d$-dimensional simplicial complex $X$ and a probability distribution $v : X(d) \to [0, 1]$. We consider the following sampling for choosing lower dimensional faces:

1. Choose some $d$-face $s_d = \{v_0, v_1, ..., v_d\} \in X(d)$ with probability $v(s_d)$.
2. Choose an ordering $\overrightarrow{s}_d = (v_0, v_1, ..., v_d)$ uniformly at random.

For any $(v_0, v_1, ..., v_k) \in X(k)$ we denote by

$$P\left[\{(u_0, u_1, ..., u_k) \mid (u_0, u_1, ..., u_k) \in X(d) \\text{ and } v(s)\right]\left[\sum_{s' \in X(d)} v(s').\right],$$

the probability of sampling an ordered face so that $(v_0, v_1, ..., v_k)$ is its prefix. For an unordered face $s \in X(k)$, we denote by

$$P[s] = \frac{1}{\binom{k+1}{d+1}} \sum_{s' \in X(d), s' \supset s} v(s').$$

From here throughout the rest of the paper, when we refer to a simplicial complex $X$, we always assume that there is a probability measure on it constructed as above.

**Definition 2.5** (link of a face). Let $s \in X(k)$ be some $k$-face. The *link* of $s$ is a $d - (k + 1)$-dimensional simplicial complex defined by:

$$X_0 = \{t \mid s \subseteq t \subseteq X\}.$$

The associated probability measure $v_{X_0}$, for the link of $s$ is defined to be the probability of sampling $t$ in the process above given that we sampled $s$. It is proportional equal to $v_{X_0}(t) = \frac{v_{X}(t / s)}{v_{X}(s)}$.

For an oriented face $(v_0, v_1, ..., v_t) \in \overrightarrow{X(t)}$ we denote by $X_0(v_0, v_1, ..., v_t) = X_{(v_0, v_1, ..., v_t)}$ the link of its underlying set.

**Definition 2.6** (1-skeleton). The 1-skeleton of a simplicial complex $X$ with some probability measure as defined above, is the graph whose vertices are $X(0)$ and edges are $X(1)$, with (the restriction of) the probability measures of $X$ to the vertices and edges.
We are ready to define our notion of high dimensional expanders.

**Definition 2.7** (one-sided and two-sided high dimensional expander). Let $0 \leq \lambda < 1$. A simplicial complex $X$ is a \( \lambda \)-two sided link expander (or \( \lambda \)-two sided HDX) if for every \( -1 \leq k \leq d-2 \) and every $s \in X(k)$, the underlying graph of the link $X_s$ is a \( \lambda \)-two sided spectral expander.

When $X$ is a graph, this definition coincides with the definition of a spectral expander.

A theorem by [40] shows that expansion “trickles down”, that is, expansion of links of $d-2$-faces imply expansion of links of $d-1$-faces (if they are connected).

**Theorem 2.8** (Theorem 5.2 in [40]). If for all $s \in X(k)$, \( \lambda(X_s) \leq \lambda \), for some $\lambda \in (0, \frac{1}{2})$, then for any $r \in X(k-1)$, s.t. $X_r$’s 1-skeleton is connected, $\lambda_2(X_r) \leq \frac{1}{1-\lambda}$.

### 2.5 Simplicial Homomorphisms

Let $X, Y$ be two pure $d$-dimensional simplicial complexes. A simplicial homomorphism is a function $f : X(0) \to Y(0)$ so that every $k$-face $s \in X$ is mapped by $f$ to a face $k$-face $f(s) \in Y(k)$. We say that $f$ is a non-degenerate $Y$-coloring if for every $d$-face $t \in Y(d)$ there exists some $d$-face $s \in X(d)$ so that $f(s) = t$.

A surjective function (not necessarily a coloring) $f : X(0) \to Y(0)$ induces a measure $\mu_f$ on $X$ defined by the following sampling process:

1. Sample a $d$-face $t \in Y(d)$.
2. Sample a $d$-face $s \in X(d)$ given that $f(s) = t$.

We note that by taking a subcomplex $X' \subseteq X$ that contains all faces in $X$ that are in the support of $\mu_f$, then $f : X'(0) \to Y(0)$ is a non-degenerate coloring (we can also induce a measure on $X'$ so that for every $d$-face $s \in X'(d)$, $\nu_{X'}(s) = \nu_X(s) / \nu_X(X'(d))$).

We say that two simplicial complexes $X, Y$ are isomorphic if there exists a bijection $f : X(0) \to Y(0)$ so that $s \in X$ if and only if $f(s) \in Y$. We then denote $X \cong Y$.

**Simplicial homomorphisms and oriented faces.** Let $\vec{s} = (v_0, v_1, ..., v_d) \in X(d)$, whose underlying set $s = \{v_0, v_1, ..., v_d\}$. Let $f(\vec{s}) = (f(v_0), f(v_1), ..., f(v_d))$ and let $f(s) = \{f(v_0), f(v_1), ..., f(v_d)\}$. Its probability is

$$\Pr_{\mu_f}[\vec{s}] = \frac{1}{(d+1)!} \prod_{i=0}^{d} \Pr_{\nu_X}[s_i]$$

where $s_i = s$ or $f(s_i)$ or $f^{-1}(s)$.

For every $s'$ so that $f(s') = f(s)$ there is a single orientation $\vec{s}'$ of $s'$ so that $f(s') = f(\vec{s})$ so this is equal to

$$\Pr_{\mu_f}[\vec{s}'] = \frac{1}{(d+1)!} \prod_{i=0}^{d} \Pr_{\nu_X}[s_i']$$

Hence we can also describe $\mu_f$ via the sampling process of oriented faces:

1. Sample an oriented $d$-face $\vec{t} \in Y(d)$.
2. Sample an oriented $d$-face $\vec{s} \in X(d)$ given that $f(\vec{s}) = \vec{t}$.

### 2.6 Simplicial Covers

**Definition 2.9** (Simplicial cover). Let $X, Y$ be two pure $d$-dimensional simplicial complexes. We say $Y$ covers $X$ if there exists a surjective simplicial homomorphism $f : Y \to X$ so that for every non-empty face $s \in Y$, $f|_{Y_1(0)}$ is a simplicial isomorphism between $Y_s$ and $X_f$.

Simplicial covers are the combinatorial equivalent of topological covers. These are usually classified and analyzed by the fundamental group of a simplicial complex as a topological space, and the universal cover. For more on this, see [44]. In this paper however, we use an alternative description of covers via the non-abelian cosystols (defined below). We define what we need below, however we will not give full proofs (but we refer the reader again to [44]).

Let $X$ be a connected finite $d$-dimensional simplicial complex (for $d \geq 2$, where $X(0) = \{v_1, v_2, ..., v_n\}$ (the vertices are ordered in some arbitrary order). Let $\Gamma$ be a group (not necessarily finite). Let $f : X(1) \to \Gamma$ be some labeling. We say that $f$ is a $\Gamma$-cosystol if for every triangle $v_i v_j v_k \in X(2)$ where $i < j < k$ it holds that

$$f(v_i) f(v_j) f(v_k) = e$$

where $e$ is the identity in $\Gamma$. We denote the set of cosystols by $Z_1(X, \Gamma)$.

For a cosystol $f \in Z_1(X, \Gamma)$ the $f$-cover of $X$ is the following simplicial complex.

**Definition 2.10** (Group cover construction). Let $X$ be a pure $d$-dimensional simplicial complex, and $\Gamma$ be a finite group. Let $f \in Z_1(X, \Gamma)$. The $f$-cover of $X$, denoted by $\tilde{X} = \tilde{X}^f$ is a pure $d$-dimensional simplicial complex defined as follows. The vertex set $\tilde{X}(0) = X(0) \times \Gamma$. The $d$-faces are all sets $s = \{(v_i, g_0), ..., (v_i, g_d)\}$ so that $\{(v_i, g_0), ..., (v_i, g_d)\} \in X(d)$ and for every two distinct $i_1, i_2$ so that $i_1 < i_2$, $f(v_{i_1}, g_{i_1}) = f(v_{i_2}, g_{i_2})$.

If $X$ is measured, then $\mu_{\tilde{X}}(s) = \mu_X(s | X) \frac{1}{|\Gamma|}$ where $s | X$ is a face $s = \{(v_i, g_0), ..., (v_i, g_d)\}$ is $s | X = \{(v_i, g_0), ..., (v_i, g_d)\} \in X(k)$.

We call covers that come from $f \in Z_1(X, \Gamma)$ as $\Gamma$-covers.

One can show that $\tilde{X}$ indeed covers $X$ with the map $\phi : \tilde{X} \to X$, $\phi(v_i, g) = v_i$. We note for the curious reader that there is also an inverse to this statement, see [44] Proposition 2.3 for a precise statement.

We are interested in simplicial complexes that are connected. The following claim characterizes the connected components of $X^f$.

Let $\tilde{X}(1) = \{(v_i, v_j), (v_j, v_i) | (v_i, v_j) \in X(1)\}$ be the set of directed edges on $X$. For a labeling $f : X(1) \to \Gamma$, we denote by $\tilde{f} : \tilde{X}(1) \to \Gamma$ the function

$$\tilde{f}(v_i, v_j) = \begin{cases} f(v_i v_j) & i < j \ \text{or} \ f(v_i v_j)^{-1} & i > j \end{cases}$$

**Claim 2.11.** Let $X$ be a $d$-dimensional simplicial complex whose 1-skeleton is connected. Fix some arbitrary $v \in X(0)$ and denote by $H_0 \subseteq \Gamma$ to be the set of all elements $g \in \Gamma$ for
which there is a cycle \( a = u_0, v_1, u_1, v_2, ..., u_m = v \) such that \( g = f(u_0, v_1) f(u_1, v_2) f(u_2, v_3) ... f(u_{m-1}, u_m) \).

Then \( H_G \) is a subgroup of \( \Gamma \). Moreover, \( X \) has one connected component for every \( [g] \in \Gamma / H_G \).

We prove the claim in the full version of the paper [9], but the following corollary is all we need for our result:

**Corollary 2.12.** Let \( X \) be a \( d \)-dimensional simplicial complex whose 1-skeleton is connected. \( X^f \) is connected if and only if \( H_G = G \) for some \( v \in X(0) \).

Finally, we note that if \( f \in Z_1(X, \Gamma) \) and \( N \leq \Gamma \) is a normal subgroup, then \( f_N \in Z_1(X, \Gamma / N) \) that is defined by \( f_N(v) = f(v)N \) is also a cosystol. Thus if \( \Gamma \) has an infinite sequence of finite-indexed normal subgroups \( N_1, N_2, ... \) whose index goes to infinity, then any \( f \in Z_1(X, \Gamma) \) gives rise to an infinite family of simplicial complexes that cover \( X \), namely, \( \{X^f_{N_m}\}_m \).

### 2.7 The Clique Complex of a Cayley Graph

Let \( \Gamma \) be a finitely presented group (not necessarily finite), and let \( S = \{s_1, s_2, ..., s_m\} \) a finite symmetric set of generators. We define \( C(\Gamma, S) \), as the clique complex that arises from the Cayley graph \( Cay(\Gamma, S) \). More explicitly, the vertex set is \( \Gamma \), the labeled edges are all pairs \( (s_1, s_2)_s \) so that \( s_1 s_2 = s_3 \), and the \( d \)-faces are all \( d \)-cliques in this graph. While this complex is infinite, the neighbourhood of every vertex is of finite size [5].

We limit ourselves to groups and generating sets where there exists some \( d \geq 1 \) so that every edge is in some \( (d + 1) \)-clique. In this case, the resulting complex is pure and \( (d + 1) \)-dimensional. We abuse definitions and say that an infinite complex is an \( \eta \)-high dimensional expander if all its links (not including the complex itself) are \( \eta \)-spectral expanders.

**Example 2.13.** [34] showed that for any \( d > 0 \) and prime power \( q^e \), the group \( PGL_d(F_{q^e}) \) has a generating set \( S \) of size \( m = m(d) \) so that \( C(\Gamma, S) \) is a pure \( d \)-dimensional simplicial complex. Moreover, all links of \( C(\Gamma, S) \) are \( \eta \)-one-sided spectral expanders for \( \eta \) that goes to 0 as \( q^e \) tends to infinity. By using Theorem 2.8 one can show that low dimensional skeletons of these complexes are also two-sided spectral expanders.

### 3 RANDOM HIGH DIMENSIONAL EXPANDERS FROM COVERS

In this section we present our construction of high dimensional expanders that are based on covers. We first define the simplicial complexes that are suitable for our construction. We stress that in this section, we think of the dimension \( d \) as being a fixed constant.

**Definition 3.1.** Let \( c, r > 1 \) and \( \eta > 0 \). Let \( X \) be a \( d \)-dimensional pure simplicial complex and denote by \( Q = \max_{v \in X(0)} \deg_{\partial Q}(v) \). We say that \( X \) is \((c, r, \eta)\)-suitable if

1. \( X \) is an \( \eta \)-two-sided high dimensional expander.
2. For every \( 0 \leq t \leq d - 2 \), \( \sigma \in X(t) \) and every \( v \in X_{\sigma}(0) \), the degree of \( v \) in the 1-skeleton of \( X_{\sigma}(0) \) is at least \( c(1 + \log Q) \).
3. For every \( 0 \leq t \leq d - 2 \) and \( \sigma \in X(t) \), the weight of every edge \( e \in X_{\sigma}(1) \) is between \( \frac{1}{|X_{\sigma}(1)|} \) and \( \frac{r}{|X_{\sigma}(1)|} \).

Moreover, the weight of every vertex \( v \in X_{\sigma}(0) \) is between \( \frac{1}{|X_{\sigma}(1)|} \) and \( \frac{r}{|X_{\sigma}(1)|} \).

Note that a \((c, r, \eta)\)-suitable simplicial complex is also \((c', r', \eta')\)-suitable for any \( c' \leq c \), \( r' \geq r \) and \( \eta' \geq \eta \).

**Theorem 3.2 (MAIN).** For every pair of integers \( d, m \) there exists some \( c, r > 1, \eta > 0 \) so that the following holds. Let \( \Gamma \) be a group with a generating set \( S \) of size \( m \). Assume that \( C(\Gamma, S) \) is a \( d \)-dimensional \( \frac{1}{|\Gamma|} \)-high dimensional expander for some \( \lambda < 1 \). Let \( X \) be a \((c, r, \eta)\)-suitable \( d \)-dimensional simplicial complex. Then there exists some pure \( Y \subset X \) so that:

1. \( Y \) is a \( d \)-dimensional \( \lambda \)-high dimensional expander.
2. \( Y \) contains \( \Omega_m d(1) \) fraction of the faces of \( X \).
3. \( Y \) has a connected \( \Gamma \)-cover.
4. Moreover, for every finite indexed subgroup \( N < \Gamma \), if the ball of radius 2 of the identity \( e \in Cay(\Gamma, S) \) is isomorphic to that of \( e \in Cay(\Gamma / N, S / N) \), then \( Y \) has a \( (\Gamma / N) \)-cover.

There is a randomized algorithm that outputs \( Y \) in expected polynomial time.

**Remark 3.3.** We prove Theorem 3.2 in an existential manner, using the Lovasz Local Lemma. Once existence is established, we use the algorithmic version Theorem 2.1 by [38] as a black box to obtain an algorithm that outputs \( Y \).

### 3.1 Proof Outline

As discussed in the introduction, we prove Theorem 3.2 via the probabilistic method. Fix a group \( \Gamma \), and a high dimensional expander \( X \) that satisfy the assumptions of the theorem. We sample a function \( f : X(1) \to \{s_1, ..., s_m\} \) uniformly at random, and remove all faces \( s \in X \) that contain a triangle \( v_1v_2v_3 \in X \) so that \( f(v_1)v_2f(v_3) \) is not in \( \Gamma \) (the edges whose distance from \( f \) is at most 2). Thus, the probability that \( Y \) stays constant and \( |X(d)| \) grows to infinity, the number of faces with a disconnected link will grow in expectation.

This is where the Lovasz Local Lemma comes into play. For every face \( \tau \in X \) we will define a “bad event” \( E_{\tau} \), that informally, says that the link of \( \tau \) is not an expander graph.

To use the Lovasz Local Lemma, we show that these events \( E_{\tau} \) are independent whenever the distance between every pair of vertices \( v, u \in \tau \) is greater than 2. Thus every \( E_{\tau} \) depends on \( poly(Q) \) other \( E_{\tau'} \). This makes sense since the structure of the link of a face \( e \in X \), depends only on the value of \( f \) on edges \( u \) so that \( \tau \cup u \in X \).

Furthermore, we show that the probability of \( E_{\tau} \) is at most \( 1/Q^t \) for \( t \) as large as we need (this is where the fact that \( X \) is \((c, r, \eta)\)-suitable for appropriate \( c, r, \eta \) comes into play). Now we can apply...
the Lovasz Local Lemma, and get that $\bigwedge_{\tau \in X} \neg E_\tau$ occurs with positive probability. This implies that the resulting sub-complex $Y$ is a high dimensional expander (the same argument will show the other properties required from $Y$ occur as well).

3.2 Notation for the Proof

Definition 3.4. Let $f : X(1) \to \{s_1, ..., s_m\}$.

(1) We say a face $\sigma \in X$ is satisfied by $f$ if for every triangle $v_i v_j v_k \subseteq \sigma$ so that $i < j < k$ it holds that $f(v_i v_j f(v_j v_k))^{-1} = e$.

(2) For a face $\sigma$ in $X$, we say that $\sigma$ is maximally satisfied if there exists $\tau \in X(d)$ so that $\sigma \subseteq \tau$ and $\tau$ is satisfied by $f$.

Note that vertices and edges are (vacuously) satisfied by any $f$. We observe the following about satisfied faces.

Observation 3.5. A face $\{v_0, v_1, ..., v_\ell\}$ is satisfied if and only if $f(v_0 v_1, f(v_0 v_2), ..., f(v_0 v_\ell)) \cup \{e\}$ is a face in $C(\Gamma, S)$, if and only if $f(v_0 v_1, f(v_0 v_2), ..., f(v_0 v_\ell)) \cup \{e\}$ is a clique in $C_{\gamma}(\Gamma, S)$.

This observation follows from the fact that whenever $f(v_0 v_1 f(v_0 v_2))^{-1} = e$ this implies that $f(v_0 v_1, f(v_0 v_2), ..., f(v_0 v_\ell)) \cup \{e\}$ is a clique in $C_{\gamma}(\Gamma, S)$.

Definition 3.6. Let $f : X(1) \to \{s_1, ..., s_m\}$. The $f$-pruning of $X$ is the simplicial complex $Y$ whose vertices are $Y(0) = X(0)$ and faces are all $t \in X$ that are maximally satisfied by $f$.

We imbue $\bar{Y}(d)$ with the following measure, where we sample an oriented $(d - 1)$-face in the link of the unit $e$ in $C(\Gamma, S)$, $(s_0, s_1, ..., s_d) \in C_{\Gamma}(\Gamma, S)$, and then sample an oriented $d$-face $(u_0, u_1, ..., u_d) \in Y$ so that $f(u_0 u_1) = s_1$ and for every $j_1, j_2 \neq 0, f(u_{j_1} u_{j_2}) = s_{j_1}^{-1} s_{j_2}$. This induces a measure over $\bar{Y}(d)$ by taking the underlying set of $(u_0, u_1, ..., u_d)$. As $\mathbb{P}[\{u_0, u_1, ..., u_d\}] = \mathbb{P}[\{u_{\pi(0)}, u_{\pi(1)}, ..., u_{\pi(d)}\}]$ for any permutation $\pi$, it holds that $\mathbb{P}[\{u_0, u_1, ..., u_d\}] = (d + 1)! \mathbb{P}[\{u_0, u_1, ..., u_d\}]$. This measure is only possible if for every $C(\Gamma, S)$, $(s_0, s_1, ..., s_d) \in C_{\gamma}(\Gamma, S)$ there exists a face $Y$ so that $f(u_0 u_1) = s_1$ and for every $j_1, j_2 \neq 0, f(u_{j_1} u_{j_2}) = s_{j_1}^{-1} s_{j_2}$. When this is not possible we do not consider $Y$ as measured.

Definition 3.7. Fix $f : X(1) \to \{s_1, ..., s_m\}$, and let $\sigma \in X(d)$ be a satisfied face for $\ell \leq d - 2$. The saturation graph of $\sigma$, denoted by $G_\sigma = (V_\sigma, E_\sigma)$ is the following:

- The vertices are all $u \in X(0)$ so that $\sigma \cup u$ is satisfied.
- The edges are all $uw \in X(1)$ so that $uw \cup \sigma$ is satisfied.

Let $\sigma = \{v_0, v_1, ..., v_\ell\}$ be some $f$-satisfied face. Denote by $f(v_0 v_\ell) = s_\ell$ and let $a = \{e\} \cup \{s_j\}_{j=0}^{\ell}$. Recall that $C_\sigma$ is the link of $a$ in the complexes induced by cliques in $C_{\gamma}(\Gamma, S)$. The satisfaction graph $G_\sigma$ has a $C_\sigma$-coloring $\psi_\sigma : V_\sigma(0) \to C_\sigma(0)$ by $\psi_0(u) = f(u_0 u)$. The fact that this is a coloring following from Observation 3.5. We consider the measure on $G_\sigma$ to be the measure induced by the coloring $\psi_\sigma$ as described in Section 2.5. More explicitly, we sample an edge $uw \in E_\sigma$ by sampling $s_i s_j \in C_\sigma$ and then we sample an edge $uw \in E_\sigma$ so that

3.3 The Proof

Proof of Theorem 3.2. Consider the probabilistic experiment where we sample $f : X(1) \to \{s_1, ..., s_m\}$ uniformly at random and set $Y \subseteq X$ to be the $f$-pruning of $X$. Define the following events for faces in $\tau \in X$:

- $AT(\tau)$, the event where for some face $\tau = \{v_0, v_1, ..., v_\ell\} \in X(\ell)$, there exist an $\ell + 1$ tuple $(s_{i_0}, s_{i_1}, ..., s_{i_\ell})$ so that the probability (in $X_\tau$) of $u \in X_\tau(0)$ so that $f(v_0 u) = s_{i_0}, f(v_0 v_1) = s_{i_1}, ..., f(v_0 v_\ell) = s_{i_\ell}$ is either greater than $\frac{\epsilon^2}{m^{2\omega}}$ or smaller than $\leq \frac{1}{m^{2\omega}}$ (AT stands for “atypical tuple”). Note that as this should hold for all tuples, then this doesn’t depend on the chosen orientation of $\tau$.
- $NE(\tau)$, the event where the satisfaction graph $G_\tau$ is not a $\frac{\epsilon}{\omega}$-spectral expander (NE stands for “not expander”).
- $BC(\tau)$, for $\tau \in X(0)$, the event where there exists some $s_i \in S$ so that for every triangle raw in $X(2)$, it holds that $\bar{f}(\bar{w}) f(u) f(wr) \neq s_i^4$ (BC stands for “bad cover”).

These events allow us to define $\{E_\tau \mid \tau \in X \setminus X(d)\} \subseteq \{E_\tau \mid \tau \in X(d)\}$ as in the proof outline:

$$E_\tau = \begin{cases} AT(\tau) & \tau \in X(d - 1) \\ AT(\tau) \cap NE(\tau) & \tau \in X(\ell), 1 \leq \ell \leq d - 2. \end{cases}$$

The formula is ambiguous when $d = 2$. In this case we define $E_\tau = AT(\tau) \cap NE(\tau) \cap BC(\tau)$ for all vertices $\tau \in X(0)$.

Finally let $G = \bigwedge_{\tau \in X} \neg E_\tau$. We claim the following:

Claim 3.8. When $G$ occurs, then

(1) $Y$ is an $\omega$-dimensional $\lambda$-high dimensional expander.
(2) $Y$ has a connected $\Gamma$-cover.
(3) $Y$ contains $\Omega(m^d)\lambda(1)$ fraction of the faces of $X$.

Claim 3.9. The event $G$ occurs with positive probability.

The proof is complete given these claims.

Proof of Claim 3.8. We assert that when $\neg AT(\tau)$ holds for all $\tau \in X \setminus X(d)$ the measure of the 1-skeleton in every link $X_\sigma$ is similar to the measure induced by $G_\sigma$ for every $\sigma \in X$.
(2) For every $-1 \leq j \leq d-2$, every $\sigma \in X(j)$ and $\overrightarrow{w} \in \overrightarrow{Y}_\sigma(1)$ or $w \in Y_\sigma(0)$ it holds that
\[
\frac{\mathbb{P}_G [\overrightarrow{w}]}{\mathbb{P}_G [w]} \leq r^{15d}.
\]

We prove this claim in the full version of the paper [9]. In particular, the first item implies that every satisfied face is also maximally satisfied since it has positive probability. This implies that the 1-skeleton of $Y_\tau$ and $G_\tau$ are the same as sets. The second item implies that for every $\tau$, the $\overrightarrow{\psi}_\tau$ coloring is non-degenerate and the the measures of $G_\tau$ and $Y_\tau$ are the same as sets.

(1) $Y$ is an $d$-dimensional $\lambda$-high dimensional expander.

If $\mathcal{G}$ occurs, then for every $\tau \in X$ of dimension $\leq d-2$, the 1-skeleton of a link $Y_\tau$ is the satisfaction graph $G_\tau$ (since every satisfied face is also maximally satisfied). Moreover, fix some $\tau$ and let $A$ be the adjacency matrix of the 1-skeleton of $Y_\tau$ and $\tilde{A}$ is the adjacency matrix of $G_\tau$ defined via the non-degenerate coloring $\overrightarrow{\psi}_\tau$. The spectral norm of $A$ is the operator norm of the matrix $\mathbb{B} = A - I$ where $(A - I)_{u,v} = \mathbb{P}_{\{w \in \overrightarrow{Y}_\tau(1) \mid w = v \wedge w \sim u\}} - \mathbb{P}_{\{w \in \overrightarrow{Y}_\tau(0) \mid w = v \wedge w \sim u\}}$ and similarly the spectral norm of $\tilde{A}$ is the operator norm of $\mathbb{B} = A - I$ where $(A - I)_{u,v} = \mathbb{P}_{\{w \in G_\tau \mid w = v \wedge w \sim u\}} - \mathbb{P}_{\{w \in G_\tau \mid w = v \wedge w \sim u\}}$.

By the second item of Claim 3.10 and direct calculations, the difference matrix $\mathbb{E} = \mathbb{B} - \tilde{\mathbb{B}}$ is a matrix so that in every entry $(u,v)$
\[
|\mathbb{E}_{\tau}(v,u)| \leq (r^{30d} - 1) \cdot \left( \mathbb{P}_{\{w \in \overrightarrow{G}_\tau \mid w = v \wedge w \sim u\}} + \mathbb{P}_{\{w \in G_\tau \mid w = v \wedge w \sim u\}} \right).
\]

Thus the $l_1$ norm of every row is at most $2(r^{30d} - 1)$. This implies that $\|\mathbb{E}\|_{l_1} \leq 2r^{30d} - 1$, and hence
\[
\lambda(\mathbb{Y}_\tau) = \|\mathbb{B}\|_{l_1} \leq 2\|\tilde{\mathbb{B}}\|_{l_1} + 2(\|\mathbb{B}\|_{l_1} - 1) = \lambda(G_\tau) + 2(\|\mathbb{B}\|_{l_1} - 1).
\]

By $-\mathcal{NE}(\tau)$, we have that $\lambda(G_\tau) \leq \frac{\lambda}{2}$ and so if we choose $r$ so that $2(r^{30d} - 1) = \frac{\lambda}{2}$ we get that $\lambda(\mathbb{Y}_\tau) \leq \lambda$.

(2) $Y$ contains $\Omega_{m,d}(1)$ fraction of the faces of $X$.

If $\mathcal{G}$ occurs, if an $\ell$-face satisfies $f$ there are at least $\frac{1}{2m^d}$ fraction of the $\ell + 1$-faces that contain it that also satisfy $f$. Thus by induction, at least
\[
\left( \frac{1}{2m^d} \right)^{d-\ell} - \tau$-face of $X$ that appear in $Y$ (where the base of the induction is by the fact that all edges of $X$ appear in $Y$).

(3) $Y$ has a connected $1$-cover.

The cover is the one induced by $f$ itself. By Corollary 2.12 it is enough to fix some $v \in Y(0)$ and show that for every $g \in \Gamma$ there is a cycle in $Y(1)$, $(v_0, v_0, v_1, \ldots, v_{d-1} = v_0)$ so that
\[
\overrightarrow{f}(v_0, v_1) \overrightarrow{f}(v_1, v_2) \cdots \overrightarrow{f}(v_{d-2}, v_{d-1}) = g.
\]

The cycle need not be simple. If $\mathcal{G}$ occurs the 1-skeleton of $X$ and of $Y$ are the same, so its enough to find such a cycle in $X(1)$. Fix an arbitrary vertex $v \in X(0)$. When $-\mathcal{BC}(v)$ holds, for every generator $s_i$ there is a path $(v, u, w, v)$ (in the 1-skeleton) so that $\overrightarrow{f}(v, u) \overrightarrow{f}(u, w) \overrightarrow{f}(w, v) = s_i$. In particular, this implies that for every $g = s_{i_1} s_{i_2} \cdots s_{i_t} \in \Gamma$, we can concatenate the cycles for $s_{i_1}, s_{i_2}, \ldots, s_{i_t}$ to get a cycle that generates $g$ as required.

(4) Covers from normal subgroups Let $N < \Gamma$. Take $f_N : X(1) \rightarrow \Gamma/N$ defined by $f_N(aw) = f(aw)N$. Following the previous item in the proof, one may verify that this function induces a connected $\Gamma/N$-cover on $Y$.

Proof of Claim 3.9. To show that $\mathcal{G}$ has positive probability we use Lovasz Local Lemma Theorem 2.1. First, for a given $\tau \in X$ we bound the number $r^\tau$ so that $E_\tau$ and $E_r$ are not independent.

Note that $\Delta T(\tau), \mathcal{NE}(\tau)$ and $\mathcal{BC}(\tau)$ depend only on the value $f(aw)$ for edges $aw \in X(1)$ that either intersect $\tau$, or so that at least one of $u, v \in X(\tau, 0)$, $E_r$ is independent of the value of $f(aw)$ for the rest of the edges. Thus, for every $\tau$, $\tau \in X$, if $E_r, E_r$ are not independent, then there is a path of length at most 2 edges from a vertex in $\tau$ to a vertex in $r^\tau$.

Denote by $Q$ the maximal number of $d$-faces that contain some fixed vertex $v \in X(0)$. Let $R$ be the number of edges adjacent to a vertex $v \in X(0)$. For a fixed $\tau$, there are at most $2^{d^2}Q(1 + R^2)$ faces $r^\tau \in X$ so that $E_r, E_r^\tau$ depend on the same edge. This (crude) bound is obtained as follows: we have at most $d^{2d}$ choices for $v \in \tau$, then we have $1 + R^2$ choices for $u \in X(0)$ so that $u$ is connected to $v$ by a path of length $\leq 2$. There are at most $Q$ $d$-faces that contain $u$ hence there are at most $Q2^{d^2}$ faces on all levels that contain $u$. Denote $K = d^2Q(1 + R^2)$, and we record that $K = poly(Q)$ (when $d$ is a fixed constant).

If we show that for all $\ell < d$ and $\tau \in X(\ell)$ it holds that
\[
\mathbb{P}[E_r] < \frac{1}{(K + 1)e},
\]
then by the Lovasz Local Lemma, we will be promised that $\mathcal{G}$ occurs with some positive probability. The $e$ in (3.2) is Euler’s number.

To show (3.2), we need the following assertions: Recall that if $X$ is $(c, r, \eta)$-suitable, then it is also $(c', r', \eta')$-suitable for $c' \leq c, r' > r$ and $\eta' > \eta$.

Lemma 3.11. There exists $c, r > 1$ and $\eta > 0$ so that the following guarantee holds for any $(c, r, \eta)$-suitable $d$-dimensional simplicial complex $X$ with maximal $d$-degree $Q$. Let $\tau \in Y(\ell)$. Then
\[
\mathbb{P}[\mathcal{NE}(\tau)] \leq 8m^3|X_\tau(0)|^2e\log|X_\tau(0)| - \mu c e \log Q + (|X_\tau(0)| + 1)m^d e^{-0.03md-(r-1)^2c\log Q}.
\]

Here $\mu = \mu(r, \lambda, m, d)$ is some positive constant. That is, the satisfaction graph $G_\tau$ is a $\frac{1}{2}$-spectral expander with probability at least $1 - 8m^3|X_\tau(0)|^2e\log|X_\tau(0)| - \mu c e \log Q - (|X_\tau(0)| + 1)m^d e^{-0.03md-(r-1)^2c\log Q}$.

Claim 3.12. Let $X$ be $(c, r, \eta)$-suitable with maximal $d$-degree $Q$. Let $\tau \in X(\ell)$ for $\ell < d - 1$. Then $\mathcal{P}[\mathcal{AT}(\tau)] \leq m^d e^{-0.03md-(r-1)^2c\log Q}$.

Claim 3.13. Let $X$ be $(c, r, \eta)$-suitable with maximal $d$-degree $Q$, let $\ell < d$. Fix $v \in X(0)$. Then
\[
\mathbb{P}[\mathcal{BC}(v)] < m^{1 \leq \ell \leq c\log Q}.
\]
That is, the probability that there exists some $s_i \in S$ so that for every cycle $(v, u, w, v)$, it holds that $f(uv) f(uw) f(vw) \neq s_i$ is at most $m \left(1 - \frac{1}{m^r}\right)^{\log Q}$.

For every $\tau \in X$, $|X_\tau(0)| = poly(Q)$: the size of a link of a vertex $v$, is just the number of edges connected to it (which is polynomial to the number of $d$-faces connected to it). The size of links of higher-dimensional faces is only smaller. Thus in the assertions above, the upper bounds on the probabilities of $AT(\tau)$, $NE(\tau)$ and $BC(\tau)$ are all of the form $poly(Q) \cdot e^{-\Omega(c \log Q)}$ where the hidden constants depend on $m, d, \lambda, r$. Hence by fixing some values of $m, d, \lambda, r$ one can take large enough $c$ and small enough $\eta$ so that the bounds are all $\leq \frac{1}{\eta}$ for any $(c, r, \eta)$-suitable simplicial complexes $X$.

Thus (3.2) holds. □

We first prove Claim 3.12 and Claim 3.13. Then we prove Lemma 3.11.

**Proof of Claim 3.12.** Fix $\tau = \{u_0, u_1, \ldots, u_r\} \in X(f)$. Recall that the measure of every vertex in the link is $\frac{1}{r!|X_{u_0}(0)|} \leq P[v] \leq \frac{r!}{r!|X_{u_0}(0)|}$. Thus for every tuple $s = (s_{i_0}, s_{i_1}, \ldots, s_{i_r})$, the number of vertices $v \in X_\tau(0)$ so that $f(u_0v) = s_{i_0}, f(u_1v) = s_{i_1}, \ldots, f(u_rv) = s_{i_r}$ is between $r^{-m}d|X_{u_0}(0)|$ and $r^{-m}d|X_{u_0}(0)|$, then $\neg AT(\tau)$ doesn’t hold.

For a fixed tuple $s = (s_{i_0}, s_{i_1}, \ldots, s_{i_r})$ the probability that $v \in X_\tau(0)$ has that $f(u_0v) = s_{i_0}, f(u_1v) = s_{i_1}, \ldots, f(u_rv) = s_{i_r}$ is $m^{-d}$. Let $E(v, s)$ be this event. As $\{E(v, s) \mid v \in X_\tau(0)\}$ are independent. Let $R_v$ be the random variable counting the number of $E(v, s)$. The fraction of $v \in X_\tau(0)$ so that $E(v, s)$ holds is greater than $r^{-m}d$ or smaller than $r^2^{-m}d$ when $|R_v - m^{-d}|X_{u_0}(0)|| \geq (r - 1)|X_{u_0}(0)|$. By Chernoff’s bound,

$$\mathbb{P}\left[|R_v - m^{-d}|X_{u_0}(0)|| \geq (r - 1)|X_{u_0}(0)|\right] \leq e^{-0.03m^{-d}(r-1)^2|X_{u_0}(0)|}.$$ 

Thus by a union bound over all tuples $s$, the probability that $AT(\tau)$ occurs is at most $m^d e^{-0.03m^{-d}(r-1)^2|X_{u_0}(0)|}$. □

**Proof of Claim 3.13.** Fix $\tau \in X(f)$, and let $u \in X_{u_0}(0)$. (1) There are at least $c \log Q$ triangles that contain $u$. (2) All triangles share only the edge $u_0u$. The rest of the edges $uv, uw$ appear only in one three-cycle each. (3) Thus the probability that $f(uv) f(uw) f(vw) \neq s_i$ is greater or equal to the probability that $f(uv) = f(uw)^{-1}$ and $f(vw) = s_i$, which is $\frac{1}{m^r}$. Furthermore, these events are all independent from one another (given any value $f(uv)$). (4) Hence the probability that this doesn’t occur for any triangle as above is at most $\left(1 - \frac{1}{m^r}\right)^{\log Q}$. We multiply by $m = |S|$ as a union bound on all $s_i \in S$.

□

3.4 Structure of a Satisfaction Graph

In order to prove Lemma 3.11, we give a description of the satisfaction graph of $G_\tau$ for $\tau \in X(f), \ell \leq d - 2$. We already saw that as sets $G_\tau$ is equal to the 1-skeleton of $Y_\tau$ when $\bigcap_{\tau} \neg AT(\sigma)$ holds. Recall the definition of $C(G \tau)$, to be the (possibly infinite) simplicial complex whose faces are cliques in the Cayley graph of $G$ with generators $S$.

Order $\tau = \{u_0, u_1, \ldots, u_r\}$ arbitrarily, and denote by $s_{ij} = f(u_0u_j)$. Denote by $a = \{e, s_{i_1}, s_{i_2}, \ldots, s_{i_r}\}$, and let $C_a$ be the link of $a \in C(G \tau)$. Fixing the values of $f$ on all the edges in $\tau$, we use Observation 3.5 to describe the satisfaction graph $G_\tau$ according to the following probabilistic experiment:

(1) **Vertex sampling:** Sample $f(u_0v)$ for every $v \in X_{u_0}(0)$ and $u_i \in \tau$. We sample $v$ into $G_\tau$ if and only if $\tau \cup v$ is satisfied by $f$.

This step partitions $X_{u_0}(0)$ to distinct sets according to the values of $f(u_0v)$. Denote by $S_i$ all vertices $v$ so that $f(u_0v) = s_i$.

(2) **Edge sampling:** Sample $f(u_0v')$ for all edges $v' \in X_{u_0}(1)$. For every edge $s_{ij} \in C_a(1)$ there exists some $s_{ij} \in S_i$ so that $s_{ij} = j$. In the edge sample step an edge $v' \in X_{u_0}(1)$ is sampled into $Y_\tau(1)$ if and only if $v \in S_i, v' \in S_j$, and $f(u_0v') = s_{ij}$ for some edge $s_{ij}$ in the link of $a$.

For every $s \in C_a(0)$, the probability that a vertex $v \in X_{u_0}(0)$ is sampled into $S_i$ is $m^{-r-1}$; this is the probability that $f(u_0v) = s_i$, and the rest of the edges $f(u_0v')$ are chosen so that all triangles in $\tau \cup v$ are satisfied. Every two $v, v' \in X_{u_0}(0)$ are sampled independently in this step.

Now consider the edge sample step. Let $v \in S_i, v' \in S_j$ be so that $v' \in X_{u_0}(1)$. If $s_{ij}$ and $s_{ij}$ are not connected in $C_a$ then the edge $v' \in Y_{u_0}(1)$ (as $a \cup s_{ij}$ is not a clique in $C(G \tau)$).

Finally, let $s_{ij} \in C_a(1)$, i.e. $a \cup s_{ij}$ is a clique in $C(G \tau)$. Denote by $s_{ij}$ the element so that $s_{ij} = s_j$ (there is such an element since $a \cup s_{ij}$ is a clique in $C(G \tau)$). Then for every $v' \in X_{u_0}(1)$ so that $v \in S_i, v' \in S_j$, the edge $v' \in Y_{u_0}(1)$ if and only if $f(u_0v') = s_{ij}$.

By Observation 3.5 this is the only possible assignment for $v'$ so that $\tau \cup v'$ is satisfied by $f$, and since $\tau \cup v'$ is a $d$-dimensional face, it is sampled into $Y$ if and only if it is satisfied.

To summarize, $Y_\tau$ is obtained by vertex removal, partitioning the remaining vertices to random sets $S_i$, and for every pair of sets $S_i, S_j$ so that $s_{ij}, s_{ij}$ are in the link of $a$, random sub-sampling of the edges between $S_i, S_j$. From this explanation we can conclude the following:

**Claim 3.14.** There is a $C_a$-coloring of $Y_\tau$, $\psi_\tau : Y_{u_0}(0) \to C_a(0), \psi_\tau(v) = f(u_0v)$. □

3.5 Proof of Lemma 3.11

We shall use the following assertions whose proofs are in the full version of the paper [9]

**Claim (Restatement of Claim 2.4).** Let $H = (V', E')$ be a $\lambda$-two sided (one-sided) spectral expander. Let $G = (V, E)$ and let $f : V \to V'$ be a non-degenerate $H$-coloring of $G$. Assume that for every $(a, b) \in E'$ the bipartite graph between $S_a, S_b$ is $k$-bipartite spectral expander (with respect to the conditional weights $\sum_{v \in S_a} |f(vw)| \sum_{w \in S_b} |f(vw)|$). Then the weighted graph $(G, \psi_f)$ is a max-$\lambda$-two sided (one sided) spectral expander.
Proposition 3.15. Let $G = (V, E)$ be a $\lambda$-two-sided spectral expander over $n$ vertices for $\lambda \leq 0.0001$. Let $r > 1$ be so that every edge has weight between $\frac{1}{r^2}$ and $\frac{r}{|V|}$. Assume further that every vertex has weight between $\frac{1}{r|V|}$ and $\frac{r}{|V|}$. Let $p \in (0, \frac{1}{2})$. Consider the sampling of a random bipartite graph $H = (A, B, E')$ as follows:

1. Sample a set $A \subseteq V$ by adding every $v \in V$ to $A$ independently with probability $p$.
2. Sample a set $B \subseteq V \setminus A$ by adding every $v \in V \setminus A$ to $B$ independently with probability $\frac{p}{1-p}$.
3. (take the edges of $H$ to be $E' = E(A, B)$.

Then $H$ is a $\frac{100}{P}$ $\lambda$-bipartite spectral expander with probability $\geq 1 - 3n^{-0.2\lambda^2r^{-2}D}$, where $D$ is the minimal degree of a vertex in $G$.

Lemma 3.16. Let $p \in (0, 1)$. Let $H = (A, B, E)$ be an $n$-vertex bipartite graph so that every edge has weight between $\frac{1}{r|E|}$ and $\frac{r}{|E|}$ for some $r > 1$. Let $0 < \epsilon < 0.05$ and $0 < \lambda < (1 - 0.25\epsilon)^2 \log^3 n$. Denote by $\mu = 0.5r^{-2}3^2/2^3\lambda(1 - 0.25\epsilon)\left((1 - 0.25\epsilon)^2 \lambda^2 - \lambda\right)$. Let $D \in \mathbb{N}$ be such that $\mu D > 2 \log n + 5$.

Assume that $\min_{A \cup B} \deg(v) \geq D$ and that $H$ is a $\lambda$-spectral expander. Let $H' = (A, B, E')$ be a subgraph where every edge is sampled in $H'$ with probability $p$. Then with probability at least $1 - 5n^2e^{-DpD}$, $H'$ is a $260e(1 + \ln(3/r))$-spectral expander.

Proof of Lemma 3.11. Recall that by Claim 3.14 there is a $C_d$ coloring of $Y_r$, $\psi_{Y_r}(v) = f(u_0)$.

The probability that $AT(\tau)$ or $AT(\tau \cup \{v\})$ occurs for some $v \in X_r(0)$ is at most $(|X_r(0)| + 1)m^d e^{-0.03m^{-d}(r-1)\epsilon \log Q}$ (by Claim 3.12). When all these events do not occur, the $C_d$ coloring is non-degenerate; every possible label $s_i \in C_d$ is given to a vertex (due to $\neg AT(\tau)$), and for a vertex $v$ labeled $s_i$, every possible edge also appears (due to $\neg AT(\tau \cup \{v\})$). Thus we can use Claim 2.4.

By assumption $C_d$ is a $\frac{1}{2}$-spectral expander, so we need to verify that for every edge $s_i s_j \in C_d$, the bipartite subgraph between elements labeled $s_i$ and those labeled $s_j$ is also a $\frac{1}{2}$-bipartite expander.

There are at most $m^2$ edges in the link of $Cay(\Gamma, S)$. For each edge $s_i s_j$, we bound the probability that the induced bipartite subgraph between $S_i$ and $S_j$ is not a $\frac{1}{2}$-bipartite expander. If we show that this probability is bounded by $8m|X_r(0)|^{\log |X_r(0)| - \mu \log Q}$ then by a union bound over the $m^2$ edge we can conclude. Denote by $s_i s_j$ the generator so that $s_i s_j s_i^{-1} = e$.

Set $p = m^{-1}$. From the discussion above, the links are random $p$-sparsifications of edges and vertices. By Proposition 3.15 the bipartite graph between the $S_i$ and the preimage of $S_j$ in $X_r$ is not an $O(m^{1.5d^2})$-expander with probability $\leq 3|X_r(0)| \epsilon e^{-0.2\lambda^2r^{-2}D}$.

By Lemma 3.16, whenever the graph between $S_i$ and $S_j$ in $X_r$ is an $O(m^{1.5d^2})$-expander, the graph in $Y_r$ is not a $\frac{1}{2}$-expander with probability at most $3|X_r(0)|^{2^{-\mu \log Q - \log |X_r(0)|}}$ for some $\mu > 0$.

Summing up the probability that the said events mentioned, and taking union bounds over all $m^2$ subgraphs induced by $s_i s_j$ we get the desired bound.

3.6 Instantiating the Construction

The groups. As stated in Section 2.7, for every $\lambda > 0$ and $d > 0$, [34] constructed infinite groups $G$ such that the Cayley complex $Cay(G, S)$ is a $\lambda$-expanding links. These groups are residually finite, and thus they have infinite finite index subgroups $G_1, G_2, G_3, \ldots \leq G$. It holds that $S \cap G_i = \emptyset$ for all $i$. Thus for every $f : Y \to S$ obtained by Theorem 3.2, the function $f_i : Y(1) \to \Gamma / G_i$, $f_i(u) = f(u)G_i$ gives rise to a cover $Y_f$ which is finite.

The complexes $X$. - The complete $d$-dimensional complex with $n$ vertices, has all the properties needed from $X$ in Theorem 3.2 for large enough $n$. We don’t usually think of the complete complex as being bounded-degree. However, recall that $X$ is a single complex and the sequence of covers grows to infinity in size.

- The complexes defined by [34] or [27] don’t suffice for this theorem since the weights in their links are not uniform or close to uniform as required by Theorem 3.2. However, a work by Friedgut and Iluz [17] shows how to transform some high dimensional expanders to high dimensional expanders in which the degree is regular in every link, and the weights are uniform. This work applies to the high dimensional expanders in [27].

- By tensoring, we can also satisfy the requirement that the links be locally dense, i.e. that if the maximal number of $d$-faces covering a vertex is $Q$, then every $1$-skeleton of a link has a minimal degree of at least $c \log Q$ (for a large enough $c$). If the degree in every link is too low, then instead of $X$ we can take the complex $X'$, which is a tensor product between $X$ and the complete complex over $t$-vertices. This complex is explicitly defined as follows:

$(1)$ $X'(0) = \{1, 2, \ldots, t\} \times X(0)$.

$(2)$ The $d$-faces are all matchings between sets in $X(d)$ and subsets of size $d + 1$ of the set $\{1, 2, \ldots, t\}$. Namely,

$$X'(d) = \{\{(i_0, 0), (i_1, 1), (i_2, 2), \ldots, (i_d, d)\} : \{v_1, \ldots, v_d\} \in X(d), (i_0, i_1, \ldots, i_d) \subseteq \{1, 2, \ldots, t\} \text{ of size } d + 1\}.$$ (3.4)

The measure on $d$ faces is by sampling a $d$-face $\{v_0, v_1, \ldots, v_d\} \in X(d)$, sampling a set of size $d + 1$ from $\{i_0, i_1, \ldots, i_d\} \subseteq \{1, 2, \ldots, t\}$ uniformly, and then sampling a matching $\{(i_0, v_0), \ldots, (i_d, v_d)\}$.

Claim 3.17.

$(1)$ $X'$ has $t \cdot |X(0)|$ vertices.

$(2)$ If the maximal degree (i.e. $d$-faces covering a vertex) in $X$ was $Q$, then the maximal degree in $X'$ is $dt^2(Q)$.

$(3)$ Let $d = 2$-face $\sigma' = \{(i_0, 0), \ldots, (i_{d-2}, d_2)\}$ where $\{v_0, \ldots, v_{d-2}\} = \sigma \subseteq X(d-2)$. The $1$-skeleton of its link $X'_{\sigma'}$ is the tensor product of the link of $X_{\sigma'}$ and the complete graph over $t - (d - 1)$ vertices. In particular, if $X_{\sigma'}$ was a $\lambda$-spectral expander, then $X'_{\sigma'}$ is a $\lambda$-spectral expander for a large enough $t$.

$(4)$ If the faces in $X$ had uniform weights, then so do the faces in $X'$.
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