Response Calculation in Zero Memory Nonlinear System Based on Digital Filter Method

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Abstract. Analysis of dynamic characteristics of structures often relies on linear theory in many practical applications. However, almost all structures show nonlinear characteristics when experienced with large enough excitations. Firstly, calculation of forced response in linear dynamic system using digital filter method is presented. Then, linear system is extended to nonlinear system. The nonlinear restoring force is treated as external loads and used to derive a generalized mathematical model to calculate the forced response in zero memory nonlinear system based on digital filter method. Finally, the ‘Reverse Path’ and multiple coherence function are used to estimate the frequency response function of underlying linear system. Numerical simulation of a multiple degree of freedom system with multiple nonlinear elements is used to verify the presented method. The results indicate that the presented model works well for calculating the forced response in zero memory nonlinear system.

1. Introduction

Almost all structures present nonlinear dynamic characteristics when experienced with large enough excitations. Many methods have been proposed to address the nonlinear dynamic modelling and identification problem [1], including linearization method, time-domain method, frequency-domain method, time-frequency method, nonlinear modal method, black-box method and nonlinear model updating method. Among these methods, Reverse Path (RP) is a frequency domain method based on Gaussian random excitation [2]. The linear and nonlinear displacement were used as the system input, and the external force was used as the system output, to create a Multiple Input Single Output (MISO) system. Multiple Coherence Function (MCF) was used to verify the nonlinear model. Rice extended the method to Multiple degree of freedom (MDOF) system [3]. The local nonlinear force was applied to the underlying linear system together with the external force. To allow the excitation point to be away from the location of nonlinearities, Richards presented the Conditioned Reverse Path (CRP) method [4]. The nonlinear part was separated from the linear part using spectrum modulation technique, so that the responses in the frequency domain were irrelevant. The equations of this method are complicated and the computational efficiency is low. To simplify the computation process and improve the calculation efficiency, Adams presented a frequency-domain method called ‘Nonlinear Identification through Feedback of the Outputs’ (NIFO) [5]. To reduce the effects of noise on the $H_2$ estimation of frequency response function (FRF), Haroon presented the revised $H_2$ estimation to improve the accuracy of nonlinear parameter identification [6]. Corresponding to the frequency-domain CRP method, Muhamad presented the time-domain Orthogonalized Reverse Path (ORP) method [7]. Ahlin et al studied a generalized method to identify the parameters and location of nonlinear elements based on RP method and random excitations. Both simulations and tests showed good accuracy and robustness [8,9].
improve the calculation efficiency, digital filter methods were used to calculate forced response in linear dynamic systems [10,11]. Ahlin compared the aliasing error, bias error and phase error between different digital filter methods and pointed out the application prospect of digital filter methods in the forced response calculation of nonlinear system [12]. Low accuracy of nonlinear parameters and FRF estimation around the resonant frequencies was found when NIFO method was implemented. To improve the accuracy, Marchesiello presented another time-domain method called Nonlinear Subspace Identification (NSI) [13]. Peng presented a two-stage time-domain nonlinear identification method based on random subspace algorithm [14]. This method tested the responses under excitations with different amplitudes in two stages, to identify the linear and nonlinear part of the system, which was proven to be more accurate and reliable than a one-stage test when noise was introduced. Other methods using sinusoidal excitation can be found in [15-19].

The mathematical model for calculating forced response in zero memory nonlinear system using digital filter is presented in this paper. Two different types of locations of the nonlinearities are studied. A MDOF system with two nonlinearities is simulated to verify the model.

2. Theoretical modelling

2.1 Calculation of forced response in linear system using digital filter

For time history, the forced response of a linear system can be calculated using convolution integral:

\[ x(t) = \int_{0}^{t} h(t-\tau) * f(\tau) d\tau \]  

(1)

where \( x(t) \) is the output signal, \( f(t) \) is the input signal, and \( h(t) \) is the system impulse response.

For a simple analog system:

\[ H(s) = \frac{1}{s+a} \]  

(2)

where \( s \) is the Laplace variable. The impulse response is:

\[ h(t) = \exp(-at) \]  

(3)

From equation (1) and (3) we can calculate \( x(nT+T) \), where \( T \) is the sampling interval:

\[ x(nT+T) = \exp(-aT) * x(nT) + \exp(-aT) \int_{0}^{T} \exp(au) * f(u+nT) du \]  

(4)

From equation (4) we can see that \( x(nT+T) \) can be calculated using \( x(nT) \) and the input between \([nT, nT+T]\), the system influence shows in the term \( \exp(-aT) \) and \( \exp(au) \). The way we use samples of \( f(t) \) to evaluate the last integral in equation (4) defines the digital filter methods. Equation (4) can be transformed into a digital filter:

\[ x(n) = \sum_{i=0}^{N_b} b_i * f(n-i) - \sum_{j=1}^{N_a} a_j * x(n-j) \]  

(5)

where \( b_i \) and \( a_j \) are digital filter coefficients, \( N_b \) and \( N_a \) are orders of filter.

The transfer function of a linear MDOF system with \( Q \) inputs and \( P \) outputs is:

\[ [H]_{P,Q}(s) = \sum_{r=1}^{N_m} \left[ [R_r]_{P,Q} \right] \frac{s - p_r}{s - p_r^{*}} \]  

(6)

where \( N_m \) is the mode order, \( [R_r]_{P,Q} \) and \( P_r \) are residue matrix and poles for mode \( r \), superscript * stands for complex conjugate.
From equation (4)-(6), we can calculate the forced response of DOF \( q \) under the excitation of DOF \( p \):

\[
x_q[n] = \sum_{i=0}^{N_q} x_{qp}^p[n]
\]

\[
x_{qp}^p[n] = \sum_{i=0}^{N_q} b_{i}^{qp} f_q[n-i] - \sum_{i=1}^{N_q} a_{i}^{qp} x_{qp}^p[n-i]
\]

where \( b_{i}^{qp} \) and \( a_{i}^{qp} \) are the digital filter coefficients for mode \( r \). The ramp invariant digital filter method is used in this paper, the coefficients can be found in [20].

2.2 Calculation of forced response in nonlinear system

The dynamic equation of a MDOF nonlinear system can be expressed as:

\[
[M][\dot{x}(t)] + [C][\ddot{x}(t)] + [K][x(t)] = \{f(t)\} - \{g(x, \dot{x}, \ddot{x})\}
\]

where \([M], [C]\) and \([K]\) are mass matrix, damping matrix and stiffness matrix respectively. \( f(t) \) is the input force, \( g(x, \dot{x}, \ddot{x}) \) is the zero memory nonlinear force, \( x(t) \) is the displacement.

Assuming the excitation is applied to DOF \( q \), the response is calculated at DOF \( p \), there exists two different types of locations of nonlinearities:

1. Between DOF \( p \) of mode \( r \) under the excitation of \( f(t) \):

\[
x_{pr1}^p[n] = \sum_{i=0}^{N_q} b_{i}^{qp} f_q[n-i] - \sum_{i=1}^{N_q} a_{i}^{qp} x_{pr1}^p[n-i]
\]

2. Between DOF \( p \) of mode \( r \) under the excitation of \( g(x, \dot{x}, \ddot{x}) \):

\[
x_{pr2}^p[n] = \sum_{i=0}^{N_q} b_{i}^{pk} (-g_k[n-i]) - \sum_{i=1}^{N_q} a_{i}^{pk} x_{pr2}^p[n-i]
\]

The total response of DOF \( p \) of mode \( r \) is:

\[
x_{pr}^p[n] = x_{pr1}^p[n] + x_{pr2}^p[n] = Q_{pr} - b_{0}^{pk} g_k[n]
\]

where \( Q_{pr} \) is known at time step \( n \), depending on the input force at and before time step \( n \), nonlinear force and displacement before time step \( n \), and digital filter coefficients corresponding to \( H_{pk}^r \). The only unknown parameter is \( g_k[n] \) at time step \( n \). \( g_k[n] \) is a zero memory nonlinearity and can be expressed as a polynomial function of \( x_k[n] \). For mode \( r \):

\[
x_{kr}^p[n] = x_{kr1}^p[n] + x_{kr2}^p[n] = Q_{kr} - b_{0}^{kl} g_k[n]
\]

The total response \( x_k[n] \) is:

\[
x_k[n] = \sum_{i=1}^{N} Q_{ir} - g_k[n] \sum_{i=1}^{N} b_{0}^{kl}
\]

Equation (14) can be rearranged as:
where $B_n$ and $A_n$ are known at time step $n$. The problem is transformed into a nonlinear equation, and Newton-Raphson method is used to solve the equation in this paper. The response and nonlinear force calculated at time step $n$ is used as initial value for the next time step. $g_k[n]$ can be calculated and used in equation (12) to get the total response at DOF $p$:

$$x_p[n] = \sum_{r=1}^{N} Q_{pr} - g_k[n] \sum_{r=1}^{N} b_0^{plr}$$  \hspace{1cm} (16)

(2) Between DOF $l$ and DOF $m$:

The response of DOF $p$ for mode $r$ under the excitation of $f(t)$ is the same as equation (10). The response of DOF $p$ for mode $r$ under the excitation of $g\left[\{x\}_r, \{x\}_r, \{x\}_r\right]$ is:

$$x_{pr2}[n] = \sum_{i=0}^{N^p} b_i^{plr} (-g_i[n-i]) - \sum_{i=1}^{N^p} a_i^{ltr} x_{pr2}[n-i]$$  \hspace{1cm} (17)

$$x_{pr3}[n] = \sum_{i=0}^{N^p} b_i^{pmr} g_m[n-i] - \sum_{i=1}^{N^p} a_i^{pmr} x_{pr3}[n-i]$$  \hspace{1cm} (18)

The total response of DOF $p$ for mode $r$ is:

$$x_{pr}[n] = x_{pr1}[n] + x_{pr2}[n] + x_{pr3}[n] = Q_{pr} - b_0^{plr} g_i[n] + b_0^{pmr} g_m[n]$$  \hspace{1cm} (19)

where $g_i[n]$ and $g_m[n]$ are functions of $x_i[n]$ and $x_m[n]$. For mode $r$:

$$x_{lr}[n] = x_{lr1}[n] + x_{lr2}[n] + x_{lr3}[n] = Q_{lr} - b_0^{lr} g_l[n] + b_0^{lmr} g_m[n]$$  \hspace{1cm} (20)

$$x_{mr}[n] = x_{mr1}[n] + x_{mr2}[n] + x_{mr3}[n] = Q_{mr} - b_0^{mnr} g_l[n] + b_0^{mnr} g_m[n]$$  \hspace{1cm} (21)

The total response $x_i[n]$ and $x_m[n]$ are:

$$x_i[n] = \sum_{r=1}^{N} Q_{ir} - g_i[n] \sum_{r=1}^{N} b_0^{lir} + g_m[n] \sum_{r=1}^{N} b_0^{lir}$$  \hspace{1cm} (22)

$$x_m[n] = \sum_{r=1}^{N} Q_{mr} - g_l[n] \sum_{r=1}^{N} b_0^{mlr} + g_m[n] \sum_{r=1}^{N} b_0^{mlr}$$  \hspace{1cm} (23)

Equation (22) and (23) can be rearranged as:

$$x_i[n] + g_i[n] B_{n1} - g_m[n] B_{n2} - A_{n1} = 0$$  \hspace{1cm} (24)

$$x_m[n] + g_m[n] B_{n3} - g_l[n] B_{n4} - A_{n2} = 0$$  \hspace{1cm} (25)

where $B_{ni}$ and $A_{ni}$ are similar to $B_n$ and $A_n$ in equation (15). $g_i[n]$ and $g_m[n]$ can be calculated and used in equation (19) to get the total response at DOF $p$:

$$x_p[n] = \sum_{r=1}^{N} Q_{pr} - g_i[n] \sum_{r=1}^{N} b_0^{plr} + g_m[n] \sum_{r=1}^{N} b_0^{pmr}$$  \hspace{1cm} (26)

2.3 Estimation of underlying linear FRF

To verify the calculated forced response, RP method is used to estimate the FRF and compared with theoretical value. MCF is used to verify the nonlinear model. Transfer equation (9) to frequency domain:
\[
[H(w)]\{[F(w)]-[G(w)]\} = \{X(w)\}
\]  
(27)

By assuming the excitation is applied to DOF \( q \), the nonlinearity is between DOF \( k \) and the ground, row \( n \) of equation (27) can be written as:
\[
H_{mq}(w)F_q(w) - H_{mk}(w)G_k(w) = X_n(w)
\]  
(28)

Zero memory nonlinearity can be expressed as polynomial function:
\[
g_k(t) = \{P\}_k^T \{z(t)\}_k = [P_1 P_2 \ldots P_{Q-1}] [x_k^1(t)x_k^2(t) \ldots x_k^Q(t)]^T
\]  
(29)

where \( P_i \) is the coefficients. Transfer equation (29) to frequency domain:
\[
G_k(w) = \{P\}_k^T \{Z(w)\}_k
\]  
(30)

Substitute equation (30) to (28) and multiply \( H_{mk}^{-1}(w) \) on both sides:
\[
[H_{mk}^{-1}(w) H_{mk}(w) H_{mk}^{-1}(w) \{P\}_k^T] \{X_n(w)\} = F_q(w)
\]  
(31)

Equation (31) is a MISO system, the \( H_i \) estimation of FRF is:
\[
[H_{mq}^{-1}(w) H_{mk}(w) H_{mk}^{-1}(w) \{P\}_k^T] = \left[G_{FX}\right]^{-1} \left[G_{XX}\right]^{-1}
\]  
(32)

where \( \left[G_{FX}\right] \) is the cross power spectrum matrix, and \( \left[G_{XX}\right] \) is the power spectrum matrix of input.

MCF can be used to verify the nonlinear model:
\[
\gamma_{n,k}^2 = \frac{\left[G_{FX}\right]^{-1} \left[G_{XX}\right]}{G_{FF}}
\]  
(33)

For \( \gamma_{n,k}^2 \), the closer to 1 at each frequency point, the better the assumed input-output correlation.

3. Numerical example
A four DOF system with two cubic springs is used to calculate the response and identify the FRF and nonlinear parameters, as shown in figure 1. The values for \( p_1 \) and \( p_2 \) are assumed to be \( 1.7 \times 10^{15} \) and \( 5 \times 10^{15} \). The parameters of underlying linear system are assumed to be known. A random force with a frequency range of 0-80Hz is applied to DOF4, and the response is calculated at each DOF. The sampling frequency is 800Hz.

![Figure 1. Four DOF nonlinear system with two cubic springs.](image-url)

Comparison between theoretical FRF, linear estimation of FRF and FRF using equation (32) is shown in figure 2. From figure 2 we can see that good agreement is obtained between theoretical and estimated
FRF using the model presented in this paper. The linear estimation of FRF shows higher resonant frequencies.

Comparison between ordinary and multiple coherence function is shown in figure 3. We can see that the ordinary coherence function deviates from 1 at most frequencies, meaning that the response cannot be explained by the input assuming the system is linear. The MCF function has a value of 1 at all frequencies, meaning that the nonlinear model is properly formulated and the forced response is accurately calculated. The theoretical and estimated value of the nonlinear parameters are shown in Figure 4. From Figure 4 we can see that the nonlinear parameters are accurately identified.

4. Conclusion
The mathematical model for calculating forced response in zero memory nonlinear system using digital filter is presented in this paper. Two different types of locations of the nonlinearities are studied. Numerical example shows good agreement between calculated response, FRF, nonlinear parameters and theoretical ones.
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