Adaptive two-stage sequential double sampling

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Abstract

In many surveys inexpensive auxiliary variables are available that can help us to make more precise estimation about the main variable. Using auxiliary variable has been extended by regression estimators for rare and cluster populations. In conventional regression estimator it is assumed that the mean of auxiliary variable in the population is known. In many surveys we don’t have such wide information about auxiliary variable. In this paper we present a multi-phase variant of two-stage sequential sampling based on an inexpensive auxiliary variable associated with the survey variable in the form of double sampling. The auxiliary variable will be used in both design and estimation stage. The population mean is estimated by a modified regression-type estimator with two different coefficient. Results will be investigated using some simulations following Medina and Thompson (2004).

Keywords and phrases: Adaptive two-stage sequential sampling, Double sampling, Multi phases sampling, Regression estimator.

1 Introduction

Adaptive cluster sampling was introduced by Thompson (1990) as an efficient sampling procedure for estimating totals and means of rare and clustered populations. Because of lack of control on final sample size and also problems that raise in performing the design to define and use neighborhood, Salehi and Smith (2005) proposed another adaptive design that does not require neighborhood and does not generate edge units in the sample, but exploit clustering in the population to find rare events with a reasonable bound for final sample size.

Panahbehagh et al. (2011) have investigated using auxiliary variable just in the design in Adaptive two-stage sequential sampling (ATS) in a real case study of fresh water mussel. Salehi et al. (2013) with assuming that the population mean for auxiliary variable is known, developed using auxiliary variable in estimation stage by two modified regression estimator. Medina and Thompson (2004) proposed a double sampling version of cluster...
sampling named adaptive cluster double sampling for using auxiliary information by regression estimator in Adaptive cluster sampling. Here we are going to introduce a double sampling version of adaptive two-stage sequential sampling for the situations which there is no complete information about auxiliary variable. We present a multi phase variant of adaptive two-stage sequential sampling that obtained by combining the ideas of adaptive two-stage sequential sampling and double sampling.

In section 2 we introduce the design and respective notation. Section 3 presents a regression type estimator with two different coefficient with respective variance and variance estimator. In section 4 we have some simulation to evaluate our design and in section 5 the paper will be finished with some conclusion about the design.

2 Notation and sampling design

Two-stage sequential sampling was initially proposed by Salehi and Smith (2005) as a sample design for sampling rare and clustered populations and then Brown et al. (2008) proposed an adaptive version of that. In adaptive two-stage sequential sampling (ATS) allocation of second-stage effort among primary units is based on preliminary information from the sampled primary units. Additional survey effort is directed to primary units where the secondary units in the initial sample have met a pre-specified criterion, or condition (e.g., an individual from the rare population is present). This design effectively over-samples primary units with high values, compared with other primary units, a method consistent with the approach recommended by Kalton and Anderson (1986) for sampling rare populations.

Suppose we have a population of $N$ units partitioned into $M$ primary sample units (PSU), each contain $N_h$ secondary sample units (SSU). Let $\{(h, j), h = 1, 2, \ldots, M, j = 1, 2, \ldots, N_h\}$ denote the $j$-th unit in the $h$-th primary unit with an associated measurement or count $y_{hj}$ and an auxiliary variables $x_{hj}$. Then, $\bar{Y}_{N_h} = \frac{1}{N_h} \sum_{j=1}^{N_h} y_{hj}$ is the mean of $y$ values for $h$-th PSU and $\bar{Y}_N = \frac{1}{N} \sum_{h=1}^{M} N_h \bar{Y}_{N_h}$ is the mean of the whole population. $\bar{X}_{N_h}$ and $\bar{X}_N$ will define the same.

The first stage of an adaptive two-stage sequential double sampling design consists of selection a simple random sample $s$ of size $m$ of $M$ PSUs. The second stage contains two phases. The first phase consists of selecting an initial conventional sample $s_{1h}$ of size $n_{1h}$ in $h$-th PSU where $h \in s$. Second phase consists of doing a sequential sampling (like the second stage of a two-stage sequential sampling) with a condition $C$, in each $s_{1h}, h \in s$ based on auxiliary information or both target and auxiliary information. The final sample in this phase named $s_{2h}$ with size $n_{2h}$.

Then for each PSU we will have 3 estimators:

- $\bar{x}_{n_{1h}}$ that is an estimator for the inexpensive variable $x$, and when $s_{1h}$ is gathered using SRSWOR we have $\bar{x}_{n_{1h}} = \frac{1}{n_{1h}} \sum_{j \in s_{1h}} x_{hj}$.

- $\hat{t}_{y{n_{2h}}}$ and $\hat{t}_{x{n_{2h}}}$ that are Murthy estimators for total of the auxiliary and target variables in the population based on doing ATS in $h$-th PSU, in the selected sample in the first phase ($s_{1h}$).
3 A regression-type estimator with two different coefficients

The common estimator in this design is Murthy estimator that is an unbiased estimator for mean of the population. In this section we will introduce a regression-type estimator for $\bar{Y}_N$ based on Murthy estimator. Following Medina and Thompson (2004) "the regression estimator will be constructed under the assumption that the relationship between $y$ and $x$ can be modelled through a stochastic regression model $\xi$ with mean $E_\xi(y_{hj} | x_{hj}) = x_{hj}\beta$ and variance $var_\xi(y_{hj} | x_{hj}) = \upsilon_{hj}\sigma^2$, $\upsilon_{hj} = \varphi(x_{hj})$ where the function $\varphi$ is assumed to be known. Throughout this paper we will consider the role of regression model $\xi$ as the model-assisted survey sampling approach (Sarndal et al., 1992); that is, we will suppose that the relationship between $y$ and $x$ is described reasonably well by $\xi$, and consequently that the model can be used as an instrument for constructing appropriate estimators of the population parameters, but inference will not depend on the assumed model and will rather be "design-based"".

Our main problem is estimating the $\bar{Y}_N$; however, because of the regression model, it will also be required to estimate the finite population regression parameter $\beta$.

Now we propose a known general form of regression estimator (Sarendal et al., 1992, p.364) as below:

$$\hat{\mu}_{reg} = \bar{y}_{n2} + \beta(\bar{x}_{n1} - \bar{x}_{n2})$$

where

$$\bar{y}_{n2} = \frac{1}{N}\sum_{h\epsilon s} a_h \hat{y}_{mn2}, a_h = \frac{N_h}{n_{1h}}$$

that $\hat{y}_{mn2}$ is Murty estimator in $s_{1h}$ and $\pi_h$ is probability of choosing h-th PSU in the first stage of sampling. $\bar{x}_{n2}$ is defined the same but for $x$. Also

$$\bar{x}_{n1} = \frac{1}{N}\sum_{h\epsilon s} a_h \frac{t_{x_{n1h}}}{\pi_h}, t_{x_{n1h}} = \sum_{j\epsilon s_{1h}} x_{hj}$$

$\beta$ is a parameter and if it is unknown we should first estimate it. Two reasonable candidates for $\beta$ are (see Salehi et al. 2013)

$$\hat{\beta}_1 = \frac{\hat{t}_{x_{n2}} - N\bar{y}_{n2}\bar{x}_{n2}}{t_{x_{n2}} - N\bar{x}_{n2}^2}$$

and

$$\hat{\beta}_o = \frac{\hat{cov}(\bar{y}_{n2}, \bar{x}_{n2})}{\hat{var}(\bar{x}_{n2})}$$

that are estimators of the conventional and the optimal regression coefficient in ATS as below

$$\beta_1 = \frac{\sum_{j=1}^{N} y_j x_j - N\bar{X}_N\bar{Y}_N}{\sum_{j=1}^{N} x_j^2 - N\bar{X}_N^2}$$

$$\beta_o = \frac{cov(\bar{y}_{n2}, \bar{x}_{n2})}{var(\bar{x}_{n2})}$$
where \( \hat{t}_{xyn_2} \) and \( \hat{t}_{x^2n_2} \) are unbiased Murthy estimators of the total of \( xy \) and \( x^2 \) in the population respectively based on the design.

### 3.1 Expectation and Variance of the estimators

With assuming \( \hat{\beta} \cong \beta \) according to the stages and the phases of the design we have

\[
E(\hat{\mu}_{reg}) = E_1 E_2 E_3(\hat{\mu}_{reg})
\]

and

\[
\text{var}(\hat{\mu}_{reg}) = V_1 E_2 E_3(\hat{\mu}_{reg}) + E_1 V_2 E_3(\hat{\mu}_{reg}) + E_1 E_2 V_3(\hat{\mu}_{reg})
\]

\[
= \text{part1} + \text{part2} + \text{part3}
\]

where \( E \) and \( V \) denote expectation and variance and the indexes \( 1, 2, 3 \), consist of first stage (\( s \)), second stage (\( s_{1h} \)) and adaptive sampling in second stage (\( s_{2h} \)) respectively. Then with SRSWOR in the first and second stage we have (see appendix A):

\[
E(\hat{\mu}_{reg}) = \bar{Y}_N
\]

and

\[
\text{var}(\hat{\mu}_{reg}) = \frac{1}{N^2}[M^2(1 - \frac{m}{M})S_{tyN}^2 + \frac{M}{m} \sum_{h=1}^{M} N_{h}^2(1 - \frac{n_{1h}}{N_{h}})S_{yN_h}^2 + \frac{1}{M} \sum_{h=1}^{M} a_{h}^2 E_2(V_3(\bar{y}_{n2h}) + \beta^2 V_3(\bar{x}_{n2h}) - 2\beta C_3(\bar{y}_{n2h}, \bar{x}_{n2h}))]
\]

where

\[
S_{tyN}^2 = \frac{1}{M - 1} \sum_{h=1}^{M} (t_{yN_h} - \bar{t}_y)^2, \quad \bar{t}_y = \frac{1}{M} \sum_{h=1}^{M} t_{yN_h}
\]

and

\[
S_{yN_h}^2 = \frac{1}{N_h - 1} \sum_{j=1}^{N_h} (y_{hj} - \bar{Y}_{N_h})^2, \quad \bar{Y}_{N_h} = \frac{1}{N_h} \sum_{j=1}^{N_h} y_{hj}
\]

### 3.2 An estimator for \( \beta_o \)

To calculate \( \beta_o \), we have (see Appendix A)

\[
\text{var}(\bar{x}_{n2}) = \frac{1}{N^2}[M^2(1 - \frac{m}{M})S_{txN}^2 + \frac{M}{m} \sum_{h=1}^{M} N_{h}^2(1 - \frac{n_{1h}}{N_{h}})S_{xN_h}^2 + \frac{1}{M} \sum_{h=1}^{M} a_{h}^2 E_2(V_3(\bar{x}_{n2h}))]
\]
and

\[
cov(\bar{x}_{n2}, \bar{y}_{n2}) = \frac{1}{N^2} \left[ M^2 \left(1 - \frac{m}{M}\right) \hat{S}_{l2N'yN} \right] + \frac{M}{m} \sum_{h=1}^{M} N_h^2 \left(1 - \frac{n_{1h}}{N_h}\right) \frac{\hat{S}_{xN'hN_h}}{n_{1h}} \\
+ \frac{M}{m} \sum_{h=1}^{M} a_h^2 E_2 C_3(\hat{t}_{xn2h}, \hat{t}_{yn2h})] \]

But \( \beta_0 \) is a parameter yet and should be estimated by sample information. For estimating variance and covariance terms, since (see Appendix B)

\[
E\left(1 - \frac{M^2}{m} \frac{\hat{S}_{l2N}}{m}\right) = \frac{1}{N^2} \left[ M^2 \left(1 - \frac{m}{M}\right) \frac{\hat{S}_{l2N}}{m} + \frac{M}{m} \sum_{h=1}^{M} N_h^2 \left(1 - \frac{n_{1h}}{N_h}\right) \frac{\hat{S}_{xN'hN_h}}{n_{1h}} \\
+ \frac{M}{m} \sum_{h=1}^{M} a_h^2 E_2 V_3(\hat{t}_{xn2h}) - \frac{M}{m} N_h^2 \frac{\hat{S}_{xN'hN_h}}{n_{1h}} - \frac{M}{m} \sum_{h=1}^{M} a_h^2 E_2 V_3(\hat{t}_{xn2h})] \]

and (see Appendix B)

\[
E(\hat{S}_{xN'h}^2) = S_{xN'h}^2 - \frac{1}{n_{1h}(n_{1h} - 1)} E_2 V_3(\hat{t}_{xn2h})
\]

where

\[
\hat{S}_{xN'h}^2 = \frac{1}{n_{1h} - 1} [\hat{\bar{x}}_{x2h}^2 - \frac{\hat{\bar{x}}_{x2h}^2}{n_{1h}}]
\]

an reasonable estimator for \( var(\bar{x}_{n2}) \) would be

\[
\hat{var}(\bar{x}_{n2}) = \frac{1}{N^2} \left[ M^2 \left(1 - \frac{m}{M}\right) \frac{\hat{S}_{l2N}}{m} + \frac{M}{m} \sum_{h\in s} N_h^2 \left(1 - \frac{n_{1h}}{N_h}\right) \frac{\hat{S}_{xN'h}}{n_{1h}} \\
+ \frac{M}{m} \sum_{h\in s} a_h^2 n_{1h}(N_h - 1) \frac{\hat{\bar{x}}_{xN'hN_h}^2}{n_{1h}(n_{1h} - 1)} - \frac{M}{m} \sum_{h\in s} a_h^2 n_{1h}(N_h - 1) \frac{\hat{S}_{xN'hN_h}}{n_{1h}(n_{1h} - 1)} - \frac{M}{m} \sum_{h\in s} a_h^2 E_2 V_3(\hat{t}_{xn2h})] \]

and for \( cov(\bar{x}_{n2}, \bar{y}_{n2}) \) in a similar way we have

\[
cov(\bar{x}_{n2}, \bar{y}_{n2}) = \frac{1}{N^2} \left[ M^2 \left(1 - \frac{m}{M}\right) \frac{\hat{S}_{l2N'yN}}{m} + \frac{M}{m} \sum_{h\in s} N_h^2 \left(1 - \frac{n_{1h}}{N_h}\right) \frac{\hat{S}_{xN'hN_h}}{n_{1h}} \\
+ \frac{M}{m} \sum_{h\in s} a_h^2 n_{1h}(N_h - 1) \frac{\hat{\bar{x}}_{xN'hN_h} \hat{\bar{y}}_{yN'hN_h}}{n_{1h}(n_{1h} - 1)} - \frac{M}{m} \sum_{h\in s} a_h^2 E_2 C_3(\hat{t}_{xn2h}, \hat{t}_{yn2h})] \]

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and then an reasonable and asymptotically unbiased estimator for \( \text{var}(\hat{\mu}_{reg}) \) is (see appendix B)

\[
\text{var}(\hat{\mu}_{reg}) = \frac{1}{N^2} \left[ M^2 \left( 1 - \frac{m}{M} \right) \hat{S}_{yn}^2 + \frac{M}{m} \sum_{hes} N_h (1 - \frac{n_{1h}}{N_h}) \hat{S}_{ynh}^2 \right] + \frac{M}{m} \sum_{hes} a_h^2 n_{1h} (N_h - 1) \hat{V}_3 (\hat{t}_{yn2h}) + \frac{M^2}{m^2} \sum_{hes} a_h^2 (\hat{\beta}^2 \hat{V}_3 (\bar{t}_{n2h}) - 2 \hat{\beta} \hat{C}_3 (\hat{t}_{yn2h}, \hat{t}_{x2h}))
\]

where

\[
\hat{S}_{yn}^2 = \frac{1}{m - 1} \sum_{hes} (a_h \hat{t}_{yn2h} - \bar{\hat{t}}_{yn2h})^2, \quad \bar{\hat{t}}_{yn2h} = \frac{1}{m} \sum_{hes} a_h \hat{t}_{yn2h},
\]

\[
\hat{S}_{ynh}^2 = \frac{1}{n_{1h} - 1} (\hat{\hat{t}}_{yn2h} - \frac{\hat{t}_{yn2h}^2}{n_{1h}});
\]

\[
\hat{V}_3 (\hat{t}_{yn2h}) = n_{1h} \left\{ p_{2h1} \left[ (n_{1h} - 1) (l_{2h1} - 1) \right] + \frac{l_{2h} - 1}{l_{2h}} \left[ (1 - p_{2h1}) \frac{n_{1h} - n_{2h1}}{n_{2h1} - 1} - n_{1h} p_{2h1} \right] \right\} S_{x2hc}
\]

\[
+ p_{2h1} (1 - p_{2h1}) \frac{n_{1h} - n_{2h1}}{n_{2h1} - 1} (\bar{x}_{2hc} - \bar{x}_{2hc})^2 + (1 - p_{2h1}) \left[ \frac{(n_{1h} - 1) (n_{2h1} - l_{2h1} - 1)}{n_{2h1} - 1} \right] + \frac{n_{2h} - l_{2h} - 1}{n_{2h} - l_{2h}} \left[ p_{2h1} \frac{n_{1h} - n_{2h1}}{n_{2h1} - 1} - n_{1h} (1 - p_{2h1}) \right] \right\} S_{x2hc}^2
\]

and

\[
\hat{C}_3 (\hat{t}_{yn2h}, \hat{t}_{x2h}) = n_{1h} \left\{ p_{2h1} \left[ (n_{1h} - 1) (l_{2h1} - 1) \right] + \frac{l_{2h} - 1}{l_{2h}} \left[ (1 - p_{2h1}) \frac{n_{1h} - n_{2h1}}{n_{2h1} - 1} - n_{1h} p_{2h1} \right] \right\} S_{xy2hc},
\]

\[
+ p_{2h1} (1 - p_{2h1}) \frac{n_{1h} - n_{2h1}}{n_{2h1} - 1} (\bar{y}_{2hc} - \bar{y}_{2hc}) (\bar{x}_{2hc} - \bar{x}_{2hc}) + (1 - p_{2h1}) \left[ \frac{(n_{1h} - 1) (n_{2h1} - l_{2h1} - 1)}{n_{2h1} - 1} \right] + \frac{n_{2h} - l_{2h} - 1}{n_{2h} - l_{2h}} \left[ p_{2h1} \frac{n_{1h} - n_{2h1}}{n_{2h1} - 1} - n_{1h} (1 - p_{2h1}) \right] \right\} S_{xy2hc}^2
\]

where \( l_{2h} \) and \( l_{2h1} \) are number of SSU in h-th PSU in the total sample and the primary sample that satisfy in condition C respectively and \( n_{2h1} \) is the size of the primary sample in h-th PSU to perform an ATS.

Also \( \hat{\beta} \) can be even \( \hat{\beta}_y \) or \( \hat{\beta}_1 \) with respect to the coefficient that is used in the regression estimator.

### 4 Monte Carlo study

In this section, following Medina and Thompson (2004), to investigate the design and the estimators, we simulated two populations with different features, and each of them with different auxiliary variables. Each population obtained by dividing a 20 × 20 unit square
into N=400 unit quadrates, partitioned in 4 PSUs with equal size. We associated with the unit or quadrat (SSU) \( u_{hj} \) a vector \((y_{hj}, x_{hj}, z_{hj})\), where \( y_{hj}, x_{hj} \) and \( z_{hj} \) denote the j-th value of the survey variable \( y \) in h-th PSU, and the values of two auxiliary variables \( x \) and \( z \). Information about populations are in table 1. We generated the spatial pattern following the Poisson cluster process. The number of clusters was selected from a Poisson distribution, and cluster centers were randomly located throughout the site. Individuals within the cluster were located around the cluster center at a random distance following an exponential distribution and a random direction following a uniform distribution. Also we used another variable in simulations, say \( w \), where \( w_{hj} = 1 \) if \( y_{hj} > 0 \) and \( w_{hj} = 0 \) otherwise.

### 4.1 Expectation of the designs costs

To compare fairly the design, we have derived analytic formula for the expectation of adaptive two-stage sequential double sampling cost and its conventional sampling counterpart with equal effort. The sampling designs considered in this study were compared using the expected value of the cost function, \( \text{Cost}_T = c_{aux}n_{aux} + c_{tar}n_y \), where \( \text{Cost}_T \) is the total cost, \( c_{aux} \) and \( c_{tar} \) were the per element costs of measuring the auxiliary variable and the target variable, respectively, and \( n_{aux} \) and \( n_y \) were the total numbers of measurements of the auxiliary variable and the target variable.

In \( \text{Cost}_T \) formula, \( c_{aux}, c_{tar} \) and \( n_{aux} \) are constant and just \( n_y \) is variable. Then we have

\[
E(\text{Cost}_T) = c_{aux}n_{aux} + c_{tar}E(n_y)
\]

Let \( L_h, l_{1h} \) and \( l_{2h} \) be the number of units satisfying condition \( C \) in the h-th PSU, in the first phase and the second phase of the second stage of sampling respectively. Furthermore, let \( l_{1h}^{(r)}, l_{1h}^{(r)} \) and \( l_{2h}^{(r)} \) are the number of rare units in the h-th PSU, in the first phase and the second phase of the second stage of sampling, respectively. Then

\[
n_y = \sum_{h=1}^{M} (n_1 + dl_{1h})I_h
\]

where \( n_1 \) is the size of initial sample in ATSD in the h-th PSU and \( I_h \) is an indicator function that takes “1” when the h-th PSU is selected in the first stage and “0” otherwise. Since \( l_{1h} | I_h = 1, n_{1h}, L_{1h} \sim HG(n_{1h}, n_{1h}, L_{1h}) \) where HG denote Hypergeometric distribution and \( L_{1h} \) denotes number of units that satisfy in condition \( C \) in the selected sample of size \( n_{1h} \), we have (with \( p_h = \frac{L_h}{N_h} \))

\[
E(l_{1h}) = E(E(l_{1h}|I_h = 1, n_{1h}, L_{1h})) = E(n_1 \frac{L_{1h}}{n_{1h}}) = n_1 \frac{E(L_{1h})}{n_{1h}} = n_1 \frac{n_{1h}p_h}{n_{1h}} = n_1 p_h
\]

and

\[
E(I_h) = \frac{m}{M},
\]

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therefore

\[ E(n_y) = \frac{m}{M} \sum_{h=1}^{M} (n_1 + d_n p_h) \]

Then to have a fair comparison,

- when we want to execute a ATS with just target variable with equal effort, we should set \( E(\text{Cost}_T) = E(\text{Cost}_{\text{ATS}}) = n_{\text{ATS}}c_{\text{tar}} \), therefore we should set the initial sample size in each PSU as

\[ n_1 \simeq \frac{E(\text{Cost}_T)}{c_{\text{tar}} \left[ \frac{m}{M} \sum_{h=1}^{M} (1 + d_{\text{ATS}} p_h) \right]} \]

where \( p_h \) is percentage of rare units in h-th PSU

- for two-stage sampling we set

\[ n \simeq \frac{E(\text{Cost}_T)}{mc_{\text{tar}}} \]

- for SRSWOR we set

\[ n \simeq \frac{E(\text{Cost}_T)}{c_{\text{tar}}} \]

- and for regression in Two-stage Double sampling because this design use as much as ATSD of auxiliary variables (i.e. \( mn_{h1} \)) then it is enough to set number of target sample size that should be taken in each selected PSU as

\[ n_{yR} \simeq \frac{E(n_y)}{m} \]

(note that the symbols \( n, c, \) and \( \text{Cont.} \), define the same as ATSD, but "." is replaced with a proper symbol according to respective design).

We used 4 designs and 7 estimators in the simulations:

- Adaptive two-stage sequential double sampling that used both target and auxiliary variables for condition \( C \) and the estimators in this design were

  - Two regression estimators with \( \beta_o \) and \( \hat{\beta}_o \) named \( \text{RegO} \) and \( \text{Regopt} \)
  - Two regression estimator with \( \beta_1 \) and \( \hat{\beta}_1 \) named \( \text{Reg1} \) and \( \text{Regb1} \)

- Adaptive two-stage sequential sampling that used just target variables as condition \( C \) with Murthy estimator named \( \text{ATS} \)

- Double sampling that used simple random sampling for both two phases with regression estimator that used sample mean named \( \text{Regs} \)
Simple random sampling without replacement named $\bar{y}_s$

Efficiency was defined as $\text{eff}(\hat{y}_u) = \frac{\text{var}(\hat{y}_u)}{\text{MSE}(\hat{y}_u)}$ and relative bias was defined as $\text{rbias}(\hat{y}_u) = \frac{E(\hat{y}_u) - \bar{y}_N}{\bar{y}_N}$. Condition $C$ was defined as ” the respective SSU is nonempty” and it depends on the design that used just target or both target and auxiliary variables. Also in the iterations when it was not possible to calculate the respective $\hat{\beta}$, we used $\bar{y}_{n_2}$ for respective regression estimator.

Two values for the ratio of costs $c_{aux}/c_{tar}$ were considered, $c_{tar}/c_{aux} = 5$ and $c_{tar}/c_{aux} = 10$. In each case the parameters were chosen such that total cost for all the designs be almost the same.

Table 1: the feature of the populations.

|           | population1 | population2 |
|-----------|-------------|-------------|
|           | rare and cluster | not so rare but cluster |
| mean      | $0.19$      | $0.50$      |
| variance  | $1.68$      | $4.85$      |
| correlation with $y$ | $0.89$      | $0.93$      |

The results for Population1 are in table 2 and 3 and can be summarized as follow. For efficiency in the case of $c_{tar}/c_{aux} = 10$, adaptive two-stage sequential double sampling (ATSD) is appropriate (albeit $w$ shows no regular pattern). In the case of $c_{tar}/c_{aux} = 5$, just for enough high correlation (using $x$) we can trust ATSD and for low correlations ordinary ATS (that expense all the costs for sampling target variable using ATS) is more appropriate than others. Gain in efficiency for $\text{Regb}$ and $\text{Regopt}$ relative to $\text{Regs}$ is considerable.

Also results show that $n_{1h}$ and portion of $\frac{n_{aux}}{n_{1h}}$ are two important factors to improve the efficiency of the estimators in the design. It is expected that with increasing $n_{1h}$ the efficiency increases, but it is interesting that bigness of $\frac{n_{aux}}{n_{1h}}$ can amend smallness of $n_{1h}$. For comparing $\text{Regopt}$ and $\text{Regb1}$ according to the results for high correlation $\text{Regopt}$ has better performance than $\text{Regb1}$ and when the correlations are low we can trust $\text{Regb1}$ more than $\text{Regopt}$. It could be a result of complexity of the $\text{Regopt}$ formula (see Salehi et. al. 2013).

For unbiasedness the results can be summarized as follows. $n_{1h}$ is one of the important factor that with increasing it, the bias decreases and the next important factors are $m$ and $n_{2h1}$. In the cases that ATSD is better than ATS, $\text{Regopt}$ has better (or at least equivalent) performance than $\text{Regb1}$ and in some cases (for example the cases that the correlations are low and $\text{Regb1}$ has good performance in efficiency) bias of $\text{Regopt}$ is substantially smaller than $\text{Regb1}$ and bias of $\text{Regb1}$ is almost unacceptable. Also the amount of bias for both $\text{Regopt}$ and $\text{Regb1}$ are unacceptable when our auxiliary variable is $w$ with $n_{1h} = 50$. 


Then for population 1 with looking at efficiency and unbiasedness together, in the cases that ATSD is better than ATS, we can trust Regopt more than Regb1 and Regopt can be our first candidate to estimate the parameters.

The results of population 2 are in table 4. In the case of high correlation \((x)\) and also for \(w\) (with enough sample size) with \(c_{tar}/c_{aux} = 10\), ATSD is the proper design to investigate the population. But for the other cases the results shows SRSWOR with \(y_{s}\) is more appropriate. It seems if there is weak correlation between target and auxiliary variable, because the target variable is not rare, it is better to expand all the costs on finding and investigating target variable with SRSWOR.

For unbiasedness the results can be summarized as follow. The amount of unbiasedness is acceptable for almost all the cases. Also in the cases that ATSD is better than ATS in efficiency, again bias of Regopt is substantially smaller (or at least equivalent) than Regb1. In high correlation cases, that ATSD is the proper design, Regb1 is a little better than Regopt in efficiency, but as we discussed before, Regopt is better than Regb1 according to bias. Then if we look at efficiency and unbiasedness simultaneously, we prefer to use Regopt is such cases.

5 Conclusion

ATSD is double sampling version of ATS that can be useful to investigate rare and cluster population with presenting auxiliary variables. The results in the simulations are conditional on the data sets that we used but they should apply to any population with similar features. In the case of high correlation the proposed design has good performance and for middle amount of correlations it is depend on structure of target variable and relative costs of target and auxiliary variables. Simulations show when the variables are rare and relative costs is reasonably high, the proper strategy is ATSD.

6 Appendix

6.1 Appendix A

We have

\[ \hat{\mu}_{reg} = E_{3}(\bar{y}_{n2}) + \beta(\bar{x}_{n1} - E_{3}(\bar{x}_{n2})) = \bar{y}_{n1} + \beta(\bar{x}_{n1} - \bar{x}_{n1}) = \bar{y}_{n1} \]

and

\[ E_{2}E_{3}(\hat{\mu}_{reg}) = E_{2}(\bar{y}_{n1}) = \frac{M}{N} \frac{1}{m} \sum_{hes} t_{yN_{h}} \]

and then

\[ E_{1}E_{2}E_{3}(\hat{\mu}_{reg}) = \frac{M}{N} E_{1}(\frac{1}{m} \sum_{hes} t_{yN_{h}}) = \bar{Y}_{N} \]
Also for part 1 we have

\[
part1 = V_1 E_2 E_3(\hat{\mu}_{reg}) = \frac{1}{N_2 M^2 (1 - \frac{m}{M})} \frac{S_{1N}^2}{m}.
\]

For part 2 we have

\[
V_2 E_3(\hat{\mu}_{reg}) = V_2(\bar{y}_{n_1}) = \frac{M^2}{m^2 N^2} \sum_{h \in s} N_h^2 V_2(\bar{y}_{n_1 h}) = \frac{M^2}{m^2 N^2} \sum_{h \in s} N_h^2 (1 - \frac{n_{1h}}{N_h}) \frac{S_{yN}^2}{n_{1h}}
\]

and then

\[
part2 = E_1 V_2 E_3(\hat{\mu}_{reg}) = \frac{M}{m N^2} \sum_{h=1}^{M} N_h^2 (1 - \frac{n_{1h}}{N_h}) \frac{S_{yN}^2}{n_{1h}}.
\]

For part 3 we have

\[
V_3(\hat{\mu}_{reg}) = V_3(\bar{y}_{n_2}) + \beta^2 V_3(\bar{x}_{n_2}) - 2\beta C_3(\bar{y}_{n_2}, \bar{x}_{n_2})
\]

and

\[
V_3(\bar{y}_{n_2}) = \frac{M^2}{m^2 N^2} \sum_{h \in s} a_h^2 V_3(\hat{t}_{yn_2 h})
\]

where \(V_3(\hat{t}_{yn_2 h})\) is variance of Murthy estimator in ATS in \(h\)-th PSU under \(s_{1h}\). Then

\[
E_1 E_2 V_3(\bar{y}_{n_2}) = \frac{M}{m N^2} \sum_{h=1}^{M} a_h^2 E_2 V_3(\hat{t}_{yn_2 h})
\]

And then

\[
part3 = E_1 E_2 V_3(\hat{\mu}_{reg}) = \frac{M}{m} \sum_{h=1}^{M} a_h^2 E_2 \left(V_3(\hat{t}_{yn_2}) + \beta^2 V_3(\hat{t}_{xn_2}) - 2C_3(\hat{t}_{yn_2}, \hat{t}_{xn_2})\right)
\]

6.2 Appendix B

For \(E(\hat{S}_{1Ny}^2)\) we have

\[
\hat{S}_{1Ny}^2 = \frac{1}{2m(m-1)} \sum_{h \in s} \sum_{h' \neq h} (a_h \hat{t}_{yn_2 h} - a_{h'} \hat{t}_{yn_2 h'})^2
\]

and with

\[
E_{2,3}(a_h \hat{t}_{yn_2 h} - a_{h'} \hat{t}_{yn_2 h'})^2 = V_{2,3}(a_h \hat{t}_{yn_2 h} - a_{h'} \hat{t}_{yn_2 h'}) + E_{2,3}^2(a_h \hat{t}_{yn_2 h} - a_{h'} \hat{t}_{yn_2 h'})
\]

\[
= V_2 E_3(a_h \hat{t}_{yn_2 h} - a_{h'} \hat{t}_{yn_2 h'}) + E_2 V_3(a_h \hat{t}_{yn_2 h} - a_{h'} \hat{t}_{yn_2 h'}) + (E_2 E_3(a_h \hat{t}_{yn_2 h} - E_2 E_3(a_h \hat{t}_{yn_2 h}) - E_2 E_3(a_{h'} \hat{t}_{yn_2 h}))^2
\]

\[
= V_2(a_h \hat{t}_{yn_1 h} - a_{h'} \hat{t}_{yn_1 h'}) + E_2 V_3(a_h \hat{t}_{yn_2 h}) + V_3(a_{h'} \hat{t}_{yn_2 h'}) + (E_2(a_h \hat{t}_{yn_1 h} - E_2(a_{h'} \hat{t}_{yn_1 h}))^2
\]

\[
= (N_h^2 - \frac{n_{1h}}{N_h} S_{yN}^2_{h}) + N_h^2 (1 - \frac{n_{1h'}}{N_h}) \frac{S_{yN}^2_{h'}}{n_{1h'}} + (a_h^2 E_2 V_3(\hat{t}_{yn_2 h}) + a_{h'}^2 E_2 V_3(\hat{t}_{yn_2 h'})) + (t_{yN_1 h} - t_{yN_1 h'})^2
\]

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we have

\[ E_{2,3}(\hat{S}_{yN}^2) = \frac{1}{m} \sum_{h=1}^{M} N_h^2 (1 - \frac{n_{1h}}{N_h}) \frac{S_{yN}^2}{n_{1h}} I_h + \frac{1}{m} \sum_{h=1}^{M} a_h^2 E_2 V_3(\hat{t}_{y_{2h}}) I_h \]

\[ + \frac{1}{2m(m-1)} \sum_{h=1}^{M} \sum_{h' \neq h} (t_{yN_h} - t_{yN_{h'}})^2 I_{hh'} \]

then

\[ E(\hat{S}_{yN}^2) = \frac{1}{M} \sum_{h=1}^{M} N_h^2 (1 - \frac{n_{1h}}{N_h}) \frac{S_{yN}^2}{n_{1h}} + \frac{1}{M} \sum_{h=1}^{M} a_h^2 E_2 V_3(\hat{t}_{y_{2h}}) \]

\[ + \frac{1}{2m(m-1)} \frac{m(m-1)}{M(M-1)} 2M \sum_{h=1}^{M} (t_{yN_h} - \bar{t})^2 \]

and finally we have

\[ E\left( \frac{1}{N^2} M^2 (1 - \frac{m}{M}) \hat{S}_{yN}^2 \right) = \frac{1}{N^2} \left[ M^2 (1 - \frac{m}{M}) \frac{S_{yN}^2}{m} + M \sum_{h=1}^{M} N_h^2 (1 - \frac{n_{1h}}{N_h}) \frac{S_{yN}^2}{n_{1h}} \right] \]

\[ + \frac{M}{m} \sum_{h=1}^{M} a_h^2 E_2 V_3(\bar{y}_{n_{2h}}) - \sum_{h=1}^{M} N_h^2 (1 - \frac{n_{1h}}{N_h}) \frac{S_{yN}^2}{n_{1h}} - \sum_{h=1}^{M} a_h^2 E_2 V_3(\hat{t}_{y_{2h}}) \]}

Now for estimating \( S_{xN_h}^2 \) with

\[ \hat{S}_{xN_h}^2 = \frac{1}{n_{1h} - 1} \left[ \hat{t}_{x_{n_{2h}}}^2 - \frac{\hat{t}_{x_{n_{2h}}}^2}{n_{1h}} \right] \]

we have (for \( r = 1, 2 \))

\[ E(\hat{t}_{x_{n_{2h}}}^r) = E_{2,3}(\hat{t}_{x_{n_{2h}}}^r) = E_2 \left( \sum_{j=1}^{n_{1h}} x_{hj}^r \right) = \frac{n_{1h}}{N_h} \sum_{j=1}^{N_h} x_{hj}^r \]

also

\[ E(\hat{t}_{x_{n_{2h}}}^2) = E_{2,3}(\hat{t}_{x_{n_{2h}}}^2) = V_{2,3}(\hat{t}_{x_{n_{2h}}}^2) + E_{2,3}(\hat{t}_{x_{n_{2h}}}^2) = V_2 E_3(\hat{t}_{x_{n_{2h}}}^2) + E_2 V_3(\hat{t}_{x_{n_{2h}}}^2) + \left( \frac{n_{1h}}{N_h} \sum_{j=1}^{N_h} x_{hj} \right)^2 \]

\[ = n_{1h}^2 (1 - \frac{n_{1h}}{N_h}) \frac{S_{yN}^2}{n_{1h}} + E_2 V_3(\hat{t}_{x_{n_{2h}}}^2) + \left( \frac{n_{1h}}{N_h} \sum_{j=1}^{N_h} x_{hj} \right)^2 \]
therefore we have

\[ E(\hat{S}_{x Nh}^2) = S_{x Nh}^2 - \frac{1}{n_{1h}(n_{1h} - 1)} E_2 V_3(\hat{t}_{x_{n2h}}) \]

Now with all above computation, an asymptotic unbiased estimator for variance of the estimator is (if we set \( \beta \) instead of \( \hat{\beta} \) the estimator will be unbiased):

\[
\text{var}(\hat{\mu}_{\text{reg}}) = \frac{1}{N^2}[M^2(1 - \frac{m}{M}) \hat{S}_{y Nh}^2] + \frac{M}{m} \sum_{hes} N_h^2(1 - \frac{n_{1h}}{N_h}) \frac{\hat{S}_{y Nh}^2}{n_{1h}} \\
+ \frac{M}{m} \sum_{hes} a_h^2 \frac{n_{1h}(N_h - 1)}{N_h(n_{1h} - 1)} \hat{V}_3(\hat{t}_{y_{n2h}}) + \frac{M^2}{m^2} \sum_{hes} a_h^2(\hat{\beta}^2 \hat{V}_3(\hat{t}_{x_{n2h}}) - 2\hat{\beta}\hat{C}_3(\hat{t}_{y_{n2h}}, \hat{t}_{x_{n2h}}))
\]

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Table 2: efficiency and relative bias of the estimators in population1. \( n_{2h1} \) and \( d \) belong to first phase of executing a ATS in \( s_{1h} \) and \( n_1 \) and \( d_1 \) belong to first phase of executing a ATS in all a PSU. \( m \) is number of PSU that is selected in the first stage.

| \( m \) | \( c_{tar}/c_{aux}=10 \) | \( c_{tar}/c_{aux}=5 \) |
|---|---|---|
| \( (n_{2h1}, d, n_1, d_1) \) | (10,4,13,10) | (10,4,13,10) | (6,4,9,10) | (10,4,16,10) | (10,4,16,10) | (5,4,12,10) |
| \( RegO_x \) | 2.697 | 1.724 | 2.757 | 2.021 | 1.405 | 1.793 |
| \( Reg1_x \) | 2.610 | 1.692 | 2.660 | 1.956 | 1.376 | 1.730 |
| \( Regopt_x \) | 2.260 | 1.654 | 2.145 | 1.668 | 1.351 | 1.320 |
| \( Regb_x \) | 2.208 | 1.652 | 2.082 | 1.628 | 1.343 | 1.279 |
| \( ATSC_x \) | 1.087 | 1.010 | 1.070 | 1.008 | 1.000 | 1.086 |
| \( Regs_x \) | 1.766 | 1.557 | 1.588 | 1.258 | 1.254 | 1.010 |
| \( (n_{2h1}, d, n_1, d_1) \) | (10,3,13,12) | (10,3,13,12) | (6,3,9,12) | (9,3,16,12) | (9,3,16,12) | (5,3,12,12) |
| \( RegO_z \) | 1.403 | 1.218 | 1.158 | 1.025 | 0.925 | 0.712 |
| \( Reg1_z \) | 1.438 | 1.242 | 1.186 | 1.053 | 0.946 | 0.730 |
| \( Regopt_z \) | 1.051 | 1.151 | 0.961 | 0.740 | 0.827 | 0.523 |
| \( Regb_z \) | 1.391 | 1.328 | 1.582 | 1.080 | 1.057 | 1.094 |
| \( ATSC_z \) | 1.236 | 1.159 | 1.182 | 1.372 | 1.186 | 1.266 |
| \( Regs_z \) | 1.116 | 1.168 | 1.372 | 0.850 | 0.903 | 0.987 |
| \( (n_{2h1}, d, n_1, d_1) \) | (10,4,12,10) | (10,4,12,9) | (6,4,8,10) | (10,4,16,9) | (10,4,16,10) | (5,4,11,10) |
| \( RegO_w \) | 1.481 | 1.26 | 1.253 | 1.071 | 1.017 | 0.741 |
| \( Reg1_w \) | 1.489 | 1.264 | 1.256 | 1.074 | 1.018 | 0.745 |
| \( Regopt_w \) | 1.012 | 1.083 | 1.134 | 0.736 | 0.868 | 0.732 |
| \( Regb_w \) | 1.014 | 1.079 | 1.137 | 0.738 | 0.865 | 0.734 |
| \( ATSC_w \) | 1.096 | 1.015 | 1.036 | 1.134 | 1.091 | 1.049 |
| \( Regs_w \) | 0.854 | 0.919 | 0.949 | 0.630 | 0.753 | 0.648 |
| \( (n_{2h1}, d, n_1, d_1) \) | (10,3,13,12) | (10,3,13,12) | (6,3,9,12) | (9,3,16,12) | (9,3,16,12) | (5,3,12,12) |
| \( RegO_x \) | 0.002 | 0.016 | -0.007 | -0.002 | -0.001 | 0.000 |
| \( Reg1_x \) | 0.003 | 0.017 | -0.007 | -0.002 | -0.001 | 0.000 |
| \( Regopt_x \) | -0.066 | -0.066 | -0.171 | -0.074 | -0.079 | -0.216 |
| \( Regb_x \) | -0.067 | -0.073 | -0.174 | -0.076 | -0.085 | -0.219 |
| \( ATSC_x \) | -0.009 | 0.011 | 0.001 | -0.006 | 0.003 | -0.006 |
| \( Regs_x \) | -0.110 | -0.119 | -0.209 | -0.119 | -0.127 | -0.285 |
| \( (n_{2h1}, d, n_1, d_1) \) | (10,4,13,10) | (10,4,13,10) | (6,4,9,10) | (10,4,16,10) | (10,4,16,10) | (5,4,12,10) |
| \( RegO_z \) | 0.024 | -0.003 | -0.007 | -0.003 | -0.008 | -0.001 |
| \( Reg1_z \) | 0.022 | -0.002 | -0.008 | -0.003 | -0.007 | -0.001 |
| \( Regopt_z \) | -0.038 | -0.052 | -0.111 | -0.066 | -0.057 | -0.115 |
| \( Regb_z \) | -0.094 | -0.124 | -0.235 | -0.138 | -0.153 | -0.279 |
| \( ATSC_z \) | 0.009 | -0.012 | 0.007 | -0.01 | -0.001 | 0.001 |
| \( Regs_z \) | -0.182 | -0.208 | -0.312 | -0.203 | -0.225 | -0.355 |
| \( (n_{2h1}, d, n_1, d_1) \) | (10,4,12,10) | (10,4,12,9) | (6,4,8,10) | (10,4,16,9) | (10,4,16,10) | (5,4,11,10) |
| \( RegO_w \) | -0.001 | -0.005 | 0.000 | 0.001 | 0.010 | -0.009 |
| \( Reg1_w \) | -0.002 | -0.006 | 0.000 | 0.001 | 0.010 | -0.008 |
| \( Regopt_w \) | -0.111 | -0.147 | -0.280 | -0.113 | -0.125 | -0.342 |
| \( Regb_w \) | -0.113 | -0.144 | -0.282 | -0.114 | -0.123 | -0.344 |
| \( ATSC_w \) | -0.013 | 0.003 | 0.014 | 0.003 | 0.002 | -0.006 |
| \( Regs_w \) | -0.084 | -0.089 | -0.254 | -0.098 | -0.104 | -0.312 |
Table 3: efficiency and relative bias of the estimators, Population1.

|               | eff, $n_{th}$=70 | $c_{tar}/c_{aux}=10$ | $c_{tar}/c_{aux}=5$ |
|---------------|------------------|----------------------|----------------------|
|               | $m=4$            | $m=3$                | $m=4$                | $m=3$                |
| $(n_{th}, d, n_1, d_1)$ | (11,5,15,12) | (11,5,15,12) | (14,5,16,13) | (9,4,17,12) | (9,4,17,12) |
| Reg$O_x$      | 3.595            | 1.765                | 3.217                | 2.074                | 1.368                |
| Reg$1_x$      | 3.174            | 1.726                | 3.082                | 1.999                | 1.338                |
| Regopt$O_x$   | 2.594            | 1.655                | 2.755                | 1.542                | 1.271                |
| Reg$1_y$      | 2.454            | 1.646                | 2.770                | 1.488                | 1.268                |
| ATSC$O_x$     | 1.044            | 0.988                | 0.905                | 1.028                | 0.975                |
| Regs$O_x$     | 1.839            | 1.541                | 1.890                | 1.104                | 1.175                |
| $(n_{th}, d, n_1, d_1)$ | (10,4,15,13) | (10,4,15,13) | (12,4,16,14) | (8,3,17,13) | (8,3,17,13) |
| Reg$O_x$      | 1.500            | 1.254                | 1.625                | 0.766                | 0.789                |
| Reg$1_x$      | 1.550            | 1.287                | 1.689                | 0.789                | 0.804                |
| Regopt$O_x$   | 1.277            | 1.049                | 1.243                | 0.589                | 0.636                |
| Reg$1_y$      | 1.377            | 1.291                | 1.380                | 0.850                | 1.001                |
| ATSC$O_x$     | 1.363            | 1.186                | 1.302                | 1.319                | 1.207                |
| Regs$O_x$     | 1.068            | 1.151                | 0.973                | 0.714                | 0.861                |
| $(n_{th}, d, n_1, d_1)$ | (15,5,13,12) | (11,5,13,11) | (14,5,14,14) | (9,4,16,12) | (9,4,16,12) |
| Reg$O_z$      | 1.559            | 1.241                | 1.570                | 0.839                | 0.896                |
| Reg$1_z$      | 1.566            | 1.242                | 1.574                | 0.842                | 0.898                |
| Regopt$O_z$   | 0.963            | 1.025                | 1.032                | 0.603                | 0.770                |
| Reg$1_w$      | 0.964            | 1.022                | 1.034                | 0.604                | 0.767                |
| ATSC$O_z$     | 0.990            | 0.964                | 0.960                | 1.007                | 1.013                |
| Regs$O_z$     | 0.790            | 0.893                | 0.822                | 0.509                | 0.661                |
| $(n_{th}, d, n_1, d_1)$ | (10,4,15,13) | (10,4,15,13) | (12,4,16,14) | (8,3,17,13) | (8,3,17,13) |
| Reg$O_z$      | -0.011           | 0.002                | 0.004                | -0.002               | -0.004               |
| Reg$1_z$      | -0.001           | 0.003                | 0.003                | -0.003               | -0.005               |
| Regopt$O_z$   | -0.060           | -0.070               | -0.027               | -0.096               | -0.108               |
| Reg$1_y$      | -0.065           | -0.078               | -0.032               | -0.100               | -0.119               |
| ATSC$O_z$     | 0.006            | 0.004                | 0.005                | 0.003                | -0.007               |
| Regs$O_z$     | -0.106           | -0.120               | -0.080               | -0.157               | -0.154               |
| $(n_{th}, d, n_1, d_1)$ | (11,5,13,12) | (11,5,13,11) | (14,5,14,14) | (9,4,16,12) | (9,4,16,12) |
| Reg$O_z$      | 0.007            | 0.014                | 0.001                | 0.006                | 0.002                |
| Reg$1_z$      | 0.008            | 0.015                | 0.001                | 0.006                | 0.005                |
| Regopt$O_z$   | -0.038           | -0.017               | -0.028               | -0.074               | -0.061               |
| Reg$1_y$      | -0.098           | -0.105               | -0.075               | -0.17                | -0.197               |
| ATSC$O_z$     | -0.005           | 0.012                | -0.001               | 0.008                | -0.005               |
| Regs$O_z$     | -0.210           | -0.202               | -0.002               | -0.245               | 0.001                |
| $(n_{th}, d, n_1, d_1)$ | (11,5,13,12) | (11,5,13,11) | (14,5,14,14) | (9,4,16,12) | (9,4,16,12) |
| Reg$O_z$      | 0.005            | 0.014                | 0.001                | -0.002               | 0.002                |
| Reg$1_z$      | 0.005            | 0.014                | 0.001                | -0.001               | 0.002                |
| Regopt$O_z$   | -0.088           | -0.111               | -0.048               | -0.150               | -0.173               |
| Reg$1_y$      | -0.090           | -0.108               | -0.050               | -0.152               | -0.171               |
| ATSC$O_z$     | 0.003            | -0.001               | -0.007               | -0.004               | -0.011               |
| Regs$O_z$     | -0.070           | -0.0815               | -0.038               | -0.120               | -0.114               |
Table 4: efficiency and relative bias of the estimators, Population2.

| $n_{1h}=50$ | $c_{tar}/c_{aux}=10$ | | $c_{tar}/c_{aux}=5$ | |
|---|---|---|---|
| $m$ | 4 | 3 | 4 | 4 | 3 |
| $(n_{2h1}, d, n_{1d})$ | (7,5,7,11) | (7,5,7,11) | (4,5,5,11) | (7,5,10,10) | (7,5,10,10) |
| $RegO_x$ | 3.182 | 1.919 | 3.469 | 2.331 | 1.523 |
| $Reg1_x$ | 3.211 | 1.919 | 3.527 | 2.348 | 1.522 |
| $RegOpt_x$ | 2.691 | 1.605 | 1.893 | 1.900 | 1.260 |
| $Regb1_x$ | 2.730 | 1.637 | 2.196 | 2.041 | 1.312 |
| $ATS_x$ | 0.919 | 0.941 | 0.976 | 1.056 | 0.975 |
| $Regs_x$ | 2.220 | 1.499 | 1.870 | 1.653 | 1.217 |
| $(n_{2h1}, d, n_{1d})$ | (7,5,7,10) | (7,5,7,10) | (4,5,5,10) | (7,5,9,10) | (7,5,9,10) |
| $RegO_z$ | 1.241 | 1.103 | 0.983 | 0.918 | 0.932 |
| $Reg1_z$ | 1.142 | 1.053 | 0.927 | 0.846 | 0.890 |
| $RegOpt_z$ | 0.666 | 0.549 | 0.289 | 0.512 | 0.510 |
| $Regb1_z$ | 0.744 | 0.823 | 0.797 | 0.746 | 0.845 |
| $ATS_z$ | 0.601 | 0.600 | 0.322 | 0.461 | 0.605 |
| $Regs_z$ | 0.783 | 0.832 | 0.540 | 0.569 | 0.634 |

| rbias | $(n_{2h1}, d, n_{1d})$ | (7,5,7,11) | (7,5,7,11) | (4,5,5,11) | (7,5,10,10) | (7,5,10,10) |
|---|---|---|---|---|---|---|
| $(n_{2h1}, d, n_{1d})$ | (7,5,7,10) | (7,5,7,10) | (4,5,5,10) | (7,5,9,10) | (7,5,9,10) |
| $RegO_z$ | 0.002 | 0.000 | -0.003 | 0.000 | -0.001 |
| $Reg1_z$ | 0.002 | 0.000 | -0.003 | 0.000 | -0.001 |
| $RegOpt_z$ | -0.004 | -0.02 | -0.038 | -0.003 | -0.019 |
| $Regb1_z$ | -0.015 | -0.035 | -0.063 | -0.015 | -0.035 |
| $ATS_z$ | 0.000 | -0.005 | -0.007 | 0.002 | 0.002 |
| $Regs_z$ | -0.022 | -0.048 | -0.047 | -0.020 | -0.041 |
| $(n_{2h1}, d, n_{1d})$ | (7,5,7,10) | (7,5,7,10) | (4,5,5,10) | (7,5,9,10) | (7,5,9,10) |
| $RegO_z$ | 0.001 | -0.011 | 0.001 | -0.004 | -0.002 |
| $Reg1_z$ | 0.001 | -0.011 | 0.003 | -0.005 | -0.001 |
| $RegOpt_z$ | 0.055 | 0.061 | 0.143 | 0.055 | 0.066 |
| $Regb1_z$ | 0.018 | 0.000 | 0.036 | 0.013 | 0.008 |
| $ATS_z$ | 0.003 | -0.001 | -0.002 | 0.004 | 0.006 |
| $Regs_z$ | 0.034 | 0.024 | 0.072 | 0.034 | 0.030 |
| $(n_{2h1}, d, n_{1d})$ | (7,5,7,10) | (7,5,7,10) | (4,5,5,10) | (7,5,9,10) | (7,5,9,10) |
| $RegO_w$ | 0.001 | -0.001 | 0.006 | 0.002 | -0.005 |
| $Reg1_w$ | 0.001 | -0.001 | 0.008 | 0.002 | -0.005 |
| $RegOpt_w$ | -0.008 | -0.032 | -0.055 | -0.007 | -0.035 |
| $Regb1_w$ | -0.007 | -0.033 | -0.059 | -0.006 | -0.035 |
| $ATS_w$ | -0.004 | 0.006 | 0.004 | 0.009 | 0.009 |
| $Regs_w$ | -0.002 | -0.0246 | -0.006 | 0.004 | -0.015 |