One-Dimensional Lazy Quantum Walk in Ternary System

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ABSTRACT Quantum walks play an important role for developing quantum algorithms and quantum simulations. Here, we introduce a first of its kind one-dimensional lazy quantum walk in the ternary quantum domain and show its equivalence for circuit realization in ternary quantum logic. Using an appropriate logical mapping of the position space on which a walker evolves onto the multiqutrit states, we present efficient quantum circuits for the implementation of lazy quantum walks in one-dimensional position space in ternary quantum system. We also address scalability in terms of \(n\)-qutrit ternary system with example circuits for a three-qutrit state space.

INDEX TERMS Lazy quantum walk (LQW), quantum circuit, quantum walk, ternary quantum system.

I. INTRODUCTION
As the development of quantum computers has achieved a remarkable success in recent years, everyone has shown a great interest to implement quantum algorithms, which give a potential speedup over their classical counterparts [1], [2]. Quantum walk [3] is one such quantum algorithm that can be a prospective candidate for solving search problems [4] with numerous speedup over conventional computers. For the last 25 years, researchers have been sincerely working on quantum walks and its applications [5]–[7]. Quantum walks have two main variants: one is discrete-time quantum walk (DTQW) [8], [9], and another is continuous-time quantum walk (CTQW) [10]. The DTQW is defined on the combination of coin (particle) and position Hilbert space. The evolution of this position space is driven by a position shift operator controlled by a coin flip operator. The CTQW is defined only on the position Hilbert space. In this variant, the evolution of this position space is driven by the Hamiltonian of the system. The probability distribution of particles for both the variants of the quantum walks spreads quadratically faster in position space compared to the classical random walk [11]–[13]; in DTQW, a two-state Hadamard coin flip operator has been used.

The concept of lazy quantum walk (LQW) [14] was later introduced incorporating a three-state coin flip operator. Further behavioral analysis of DTQW has been carried out by making use of three-state quantum coin on line and cycle [15]–[18]. Soon it was shown in [19] that the occupancy rate of lazy quantum walk along a one-dimensional line is better than DTQW. A small variation in LQW, where the walker has some probability of staying put, is known as lackadaisical quantum walk [20], which gives algorithmic speedup [21], [22]. In this article, we are focused on LQW, but this could be easily implemented for lackadaisical quantum walk as well while designing the coin appropriately.

Due to developments of quantum computers and the availability of these computers through the cloud with the IBM Quantum Experience, many different quantum walk experiments have been carried out on real quantum hardware [23]–[27]. But the implementation of DTQW in circuit quantum electrodynamics (QED) is arduous [28]. The superconducting qubits in circuit QED cannot move as it does in other systems. Hence, the encoding of states of the coin and the walker in superconducting qubits become tough. To overcome this problem, an efficient protocol has been proposed for the implementation of DTQW in circuit QED, in which only \(n+1\) qutrits and \(n\) assistant cavities are needed for an \(n\)-step DTQW [28]. As qutrits can be used to encode more information, more researchers have been fascinated toward working with ternary quantum system in recent years. For a ternary quantum system, the unit of information is known as a qudit, and the corresponding quantum system can be
defined using the orthonormal basis states $|0\rangle$, $|1\rangle$, $|2\rangle$ [29]. Of late, several works have been done on quantum algorithms in ternary quantum systems [30]–[32]. Search problems have also been dealt with the help of well-known quantum search algorithms [33], [34] which have been implemented using ternary elementary gates [29], [35]–[37]. In the recent past, researchers have claimed to develop a superconducting qutrit processor [38]. To add to that, researchers have also claimed to have implemented a Walsh–Hadamard gate on this superconducting qutrit processor [39]. This gives a ray of hope that ternary quantum computers will soon come into play.

For any quantum computer or hardware, the main limitations are the number of qubits/qutrits and the coherence time of the system, which limits the probable steps of DTQW/LQW that can be implemented. The challenge is to utilize the qubits/qutrits in such a manner so that maximum number of steps of DTQW/LQW can be implemented with minimum number of qubits/qutrits. This encouraged us to carry out the efficient circuit realization of a one-dimensional LQW in ternary systems in this article. The circuit realization is such that whenever ternary quantum computers are live in action, we can straightaway map them to it to get the advantage in application of LQW. In [40], it is shown that the lackadaisical quantum walk, a variation of LQW, can boost the success probability to search a marked vertex on any arbitrary dimension, strongly regular graphs, Johnson graphs, vertex-transitive graphs, such as periodic cubic lattices of arbitrary dimension, and the hypercubes, by choosing an appropriate coin. This encouraged us to carry out the proposed circuit realization for one-dimensional LQW, and it can be easily scaled up to higher spatial dimensions by introducing more coin qutrits and position qutrits. We show that the $\lceil \log_3 n \rceil + 2$ qutrits are enough for the implementation of $n$-step DTQW or $n$-step LQW and can further be scaled up to implement more steps. Our circuit design can also be implemented in any qutrit-supported quantum hardware, since all the ternary gates that are used in our design are universal, which makes our work generalized in nature.

In this article, we have implemented one-dimensional quantum walk using a three-state coin in ternary quantum system. Our novelty lies in the following facts.

1) We define one-dimensional DTQW using a three-state coin (LQW) in a ternary quantum system for the first time to the best of our knowledge.

2) Furthermore, we propose an efficient quantum circuit realization to implement one-dimensional DTQW using the three-state coin (LQW) in ternary quantum settings for the first of its kind using an appropriate logical mapping of the position space on which a walker evolves onto the multiqutrit states.

3) We also address scalability of the proposed circuit in terms of $n$-qutrit system, which makes this circuit realization generalized in nature for implementing more steps.

The rest of this article is as follows. Section II describes the dynamics of one-dimensional quantum walk in binary quantum system. Section III defines the dynamics of one-dimensional quantum walk using three-state quantum coin (LQW) in ternary system. Section IV proposes efficient quantum circuit implementation for one-dimensional quantum walk using three-state quantum coin (LQW) in ternary system followed by the generalization of quantum circuit. Finally, Section V concludes this article.

II. ONE-DIMENSIONAL DTQW

DTQWs take place in the product space $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_c$. $\mathcal{H}_p$ is a Hilbert space that has orthonormal basis given by the position states $\{|x\rangle, x \in \mathbb{Z}\}$. The default initial position state is $|0\rangle$. Due to the two choices of the movement, DTQWs have a two-dimensional coin. Therefore, $\mathcal{H}_c$ is a Hilbert space spanned by the orthonormal basis $\{|\uparrow\rangle, |\downarrow\rangle\} (\uparrow$ for right and $\downarrow$ for left).

Let $|x, \alpha\rangle$ be a basis state, where $x \in \mathbb{Z}$ represents the position of the particle and $\alpha \in \{\uparrow, \downarrow\}$ represents the coin state. The evolution of the whole system at each step of the walk can be described by the unitary operator denoted by $U$

$$U = S(I \otimes C)$$

(1)

where $S$ is the shift operator defined by

$$S = \sum_{x \in \mathbb{Z}} \left( \begin{array}{c|c} |\uparrow\rangle & |\uparrow\rangle \otimes |x - 1\rangle \langle x + 1| + |\downarrow\rangle \otimes |x + 1\rangle \langle x| \end{array} \right).$$

(2)

$I$ is the identity matrix that operates in $\mathcal{H}_p$, whereas $C$ is the coin operation. Hadamard ($H$) coin is an example of two-dimensional coin flip operator, which is denoted by $C$ here

$$C = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right).$$

(3)

The state of the particle in position Hilbert space after $t$ steps of the walk is given by

$$|\Psi(t)\rangle = W^t \left[ |\Psi\rangle_c \otimes |x = 0\rangle \right] = \sum_{x=-t}^{t} \begin{bmatrix} \psi_{x,t}^\dagger \psi_{x,t} \end{bmatrix}$$

(4)

where each step of the walk is realized by applying the operator, $W = SC$, where $S$ is the position shift operator and $C$ is the coin flip operator.

The probability of finding the particle at position and time $(x, t)$ is

$$P(x, t) = \left| \psi_{x,t} \right|^2 + \left| \psi_{x,t}^\dagger \right|^2.$$  

Fig. 1 shows the probability distribution after 100 steps of a DTQW using a Hadamard coin, where the initial position state is $|0\rangle$.

III. ONE-DIMENSIONAL QUANTUM WALK USING THREE-STATE COIN IN TERNARY QUANTUM SYSTEM

Usually DTQW on the line has two directions to move: right and left. But LQWs have three choices: right, left, and stay
Dynamics of the LQW are defined on the combination of particle (coin) and position Hilbert space as in DTQW, $H = H_c \otimes H_p$. A particle with internal states, $H_c = \text{span}\{|\uparrow\rangle, |.\rangle, |\downarrow\rangle\}$ ($\uparrow$ for right, . for stay put, and $\downarrow$ for left) and a one-dimensional position Hilbert space is $H_p = \text{span}\{|x\rangle\}$, where $x \in \mathbb{Z}$ represents the labels on the position states in ternary system.

Let $|x, \alpha\rangle$ be a basis state, where $x \in \mathbb{Z}$ represents the position of the particle and $\alpha \in \{\uparrow, ., \downarrow\}$ represents the coin state. The evolution of the whole system at each step of the walk can be described by the unitary operator, denoted by $\hat{U}$

$$\hat{U} = \hat{S}(I \otimes \hat{C})$$

where shift operator ($\hat{S}$) is defined by

$$\hat{S} = \sum_{x \in \mathbb{Z}} \left( |\uparrow\rangle\langle\uparrow| \otimes |x - 1\rangle\langle x | + |.\rangle\langle.| \otimes |x\rangle\langle x + 1 | \right).$$

The shift operator at time $t$ translates the position conditioned on the internal state of the particle. During each step of the LQW, the particle remains at the same position and also moves to left and right. $I$ is the identity matrix, which operates in $H_p$, whereas $\hat{C}$ is the coin operation. In this article, we consider two kinds of coin flip operators. The first kind is the discrete Fourier transform (DFT) coin flip operator

$$G(\rho) = \begin{pmatrix}
-\rho^2 & \rho \sqrt{2-2\rho^2} & 1 - \rho^2 \\
\rho \sqrt{2-2\rho^2} & 2\rho^2 - 1 & \rho \sqrt{2-2\rho^2} \\
1 - \rho^2 & \rho \sqrt{2-2\rho^2} & -\rho^2
\end{pmatrix}$$

with the coin parameter $\rho \in (0, 1)$

$$C = \frac{1}{3} \begin{pmatrix}
-1 & 2 & 2 \\
2 & -1 & 2 \\
2 & 2 & -1
\end{pmatrix}.$$
IV. QUANTUM CIRCUIT FOR IMPLEMENTING THE ONE-DIMENSIONAL LQW IN TERNARY QUANTUM SYSTEM

In this section, we present logical realization of quantum circuits to implement one-dimensional quantum walk using three-state coin (LQW) in ternary quantum system. To implement any ternary quantum circuit, we require some dedicated ternary gates, which are described in the following section.

A. TERNARY GATES

This section gives a brief description of the ternary gates that are required for the circuit synthesis proposed in this article.

1) TERNARY SHIFT GATES

In ternary logic, there are five unitary one-qutrit gates namely, $X_{+1}$, $X_{+2}$, $X_{01}$, $X_{12}$, and $X_{02}$ [29], [41]. These gates were proposed in [29] where they were realized using ion trap model. Each of these gates can be represented using a $3 \times 3$ unitary matrix. The truth tables for each of these gates are given in Table 1. Fig. 4 shows the matrix representations of these gates.

2) TERNARY MUTHUKRISHNAN–STROUD GATE

Muthukrishnan–Stroud gate [29] is a two-qutrit gate defined as follows:

$$M - S_X(A, B) = \begin{cases} 
\text{apply } X \text{ gate to } c & \text{if } a_1 = a_2 = \ldots = a_n = 2 \\
B & \text{otherwise}
\end{cases}$$

where $A$ is the control, $B$ is the target, and $X$ can be any of the five one-qutrit gates defined earlier, i.e., $X_{+1}$, $X_{+2}$, $X_{01}$, $X_{12}$, and $X_{02}$. The truth tables for all possible two-qutrit M–S gates are given in Table 2. The general circuit for an M–S gate is given in Fig. 5.

B. IMPLEMENTATION

To implement an LQW in one-dimensional position Hilbert space of size $3^q$, $(q + 1)$ qutrits are required, one qutrit to represent the particle’s internal state (coin qutrit) and $q$ qutrits to represent its position. The coin operation can be implemented by applying a single qutrit rotation gate on the coin qutrit, and the position shift operation is implemented subsequently with the help of multiqutrit gates where the coin qutrit acts as the control. Quantum circuits for implementing LQW depend on how the position space is represented. Example circuits for a four-qutrit system are given in this section.

For $q = 4$, the number of steps of LQW that can be implemented is $\left\lfloor \frac{3^q - 1}{2} \right\rfloor = 13$. We choose the position state mapping given in Tables 3–5 with a fixed initial position state $|000\rangle$. Fixing the initial state of the walker helps in reducing the gate count in the quantum circuit and, hence, reduces the overall error. For example, if the initial state is not fixed to...
Position State Mapping With the Multiqutrits States for Quantum Circuits Presented in Fig. 7

| \( x = 0 \) | \( = 000 \) |
| \( x = 1 \) | \( = 001 \) |
| \( x = -1 \) | \( = 010 \) |

Position State Mapping With the Multiqutrits States for Quantum Circuits Presented in Fig. 8

| \( x = 2 \) | \( = 021 \) |
| \( x = 3 \) | \( = 020 \) |
| \( x = 4 \) | \( = 022 \) |
| \( x = -2 \) | \( = 012 \) |
| \( x = -3 \) | \( = 010 \) |
| \( x = -4 \) | \( = 011 \) |

Position State Mapping With the Multiqutrits States for Quantum Circuits Presented in Fig. 9

| \( x = 5 \) | \( = 211 \) |
| \( x = 6 \) | \( = 210 \) |
| \( x = 7 \) | \( = 212 \) |
| \( x = 8 \) | \( = 201 \) |
| \( x = 9 \) | \( = 200 \) |
| \( x = 10 \) | \( = 202 \) |
| \( x = 11 \) | \( = 221 \) |
| \( x = 12 \) | \( = 220 \) |
| \( x = 13 \) | \( = 222 \) |

Quantum circuit for the first step of LQW on four qutrits as given in Table 3.

\( |000\rangle \), then at first we have to bring the initial state to \( |000\rangle \) with the help of one-qutrit \( M-S \) gates. We denote the initial state as \( |x = 0\rangle \equiv |000\rangle \) in Table 3. After each step of the LQW, two new position states have to be considered along with stay put. In Tables 3–5, we show that the mapping of these new position states onto the multiqutrit states is in such a way that optimal number of gates are used to implement the shift operation, we consider the nearest neighbor position space so as to make the circuit efficient.

After first step of LQW, if the coin state is \( |2\rangle \), particle moves to the right, \( |x = 1\rangle \equiv |002\rangle \), if the coin state is \( |1\rangle \), particle moves to the left, \( |x = -1\rangle \equiv |001\rangle \), and if the coin state is \( |0\rangle \), particle stays at the initial state, \( |x = 0\rangle \equiv |000\rangle \), as shown in Table 3. We are following the nearest neighbor approach to determine the position states, as shown in Table 3, so that the least significant qutrit (LSQ) only changes, rest of the two qutrits remain unchanged. The mathematical formulation of this step can be described as follows.

Suppose at \( t = 0 \), initial state: \( |0\rangle \otimes |000\rangle \).

At \( t = 1 \), after first step: \( a_0 \times |0\rangle \otimes |000\rangle + b_0 \times |1\rangle \otimes |001\rangle + c_0 \times |2\rangle \otimes |002\rangle \), where \( a_0 \), \( b_0 \), and \( c_0 \) are the amplitudes.

This logic is mapped into the circuit, as shown in Fig. 7. In Fig. 7, the first qutrit is used for coin operation and rest of the three qutrits for position states on which shift operation will be performed as per the coin state. First, all the four qutrits are initialized with \( |0\rangle \)

\[ \psi_0 \rightarrow |0\rangle \otimes |000\rangle. \]

A three-state coin \( C \) has been applied on the first qutrit so that we can have a superposition state

\[ \psi_1 \rightarrow a_0 \times |0\rangle \otimes |000\rangle + b_0 \times |1\rangle \otimes |000\rangle + c_0 \times |2\rangle \otimes |000\rangle. \]

Next, we have to perform the shift operation. For that, we have applied \( X_{+1} \) gate on first qutrit so that we can perform shift operation for the coin state \( |1\rangle \). Hence, the quantum state becomes

\[ \psi_2 \rightarrow a_0 \times |1\rangle \otimes |000\rangle + b_0 \times |2\rangle \otimes |000\rangle + c_0 \times |0\rangle \otimes |000\rangle. \]

For the shift operation for the coin state \( |1\rangle \), we have applied a two-qutrit \( M-S_{+1} \) gate on first qutrit as control and last qutrit as target. Now, the quantum state becomes

\[ \psi_3 \rightarrow a_0 \times |1\rangle \otimes |000\rangle + b_0 \times |2\rangle \otimes |001\rangle + c_0 \times |0\rangle \otimes |000\rangle. \]

To take back the coin state to its previous state for further operation, we have to apply \( X_{+2} \) gate on first qutrit, such that the quantum state becomes

\[ \psi_4 \rightarrow a_0 \times |0\rangle \otimes |000\rangle + b_0 \times |1\rangle \otimes |001\rangle + c_0 \times |2\rangle \otimes |000\rangle. \]

Then, we have applied a two-qutrit \( M-S_{+2} \) gate on the first qutrit as control and last qutrit as the target for the shift operation for the coin state \( |2\rangle \). Thus, the quantum state becomes

\[ \psi_5 \rightarrow a_0 \times |0\rangle \otimes |000\rangle + b_0 \times |1\rangle \otimes |001\rangle + c_0 \times |2\rangle \otimes |002\rangle. \]

We do not have to apply any gate to perform shift operation for the coin state \( |0\rangle \) as the position state remains unchanged.

Now, we are at superposition state

\[ a_0 \times |0\rangle \otimes |000\rangle + b_0 \times |1\rangle \otimes |001\rangle + c_0 \times |2\rangle \otimes |002\rangle, \]

where the walker is at position state \( |000\rangle \) with coin state \( |0\rangle \), position state \( |001\rangle \) with coin state \( |1\rangle \), and position state \( |002\rangle \) with coin state \( |2\rangle \). Following quantum walk logic, for further movement, the walker needs to toss the coin again. At this stage with coin toss, we move to the superposition state
a_{11} \otimes |0\rangle \otimes |000\rangle + a_{12} \otimes |1\rangle \otimes |000\rangle + a_{13} \otimes |2\rangle \otimes |000\rangle + b_{11} \otimes |0\rangle \otimes |001\rangle + b_{12} \otimes |1\rangle \otimes |001\rangle + b_{13} \otimes |2\rangle \otimes |001\rangle + c_{11} \otimes |0\rangle \otimes |002\rangle + c_{12} \otimes |1\rangle \otimes |002\rangle + c_{13} \otimes |2\rangle \otimes |002\rangle.

The aforementioned logic is mapped into the circuit, as shown in Fig. 8. So, after the first step of LQW, we have superposition state, $|a_0\rangle \otimes |0\rangle \otimes |000\rangle + b_{02} \otimes |1\rangle \otimes |001\rangle + c_0 \otimes |2\rangle \otimes |002\rangle$. Again three-state coin $C$ has been applied on the first qutrit so that we can have a superposition state

$$
\psi_6 \rightarrow a_{11} \otimes |0\rangle \otimes |000\rangle + a_{12} \otimes |1\rangle \otimes |000\rangle + a_{13} \otimes |2\rangle \otimes |000\rangle + b_{11} \otimes |0\rangle \otimes |001\rangle + b_{12} \otimes |1\rangle \otimes |001\rangle + b_{13} \otimes |2\rangle \otimes |001\rangle + c_{11} \otimes |0\rangle \otimes |002\rangle + c_{12} \otimes |1\rangle \otimes |002\rangle + c_{13} \otimes |2\rangle \otimes |002\rangle.
$$

At this stage to move forward, the circuit design for $|000\rangle$ position state remains same, as shown in Fig. 7. As shown in Fig. 8, the dotted block shows the circuit design for $|000\rangle$ position state. After applying this block of gates, the quantum state becomes

$$
\psi_7 \rightarrow a_{11} \otimes |0\rangle \otimes |000\rangle + a_{12} \otimes |1\rangle \otimes |001\rangle + a_{13} \otimes |2\rangle \otimes |002\rangle + b_{11} \otimes |0\rangle \otimes |001\rangle + b_{12} \otimes |1\rangle \otimes |002\rangle + b_{13} \otimes |2\rangle \otimes |002\rangle + c_{11} \otimes |0\rangle \otimes |002\rangle + c_{12} \otimes |1\rangle \otimes |000\rangle + c_{13} \otimes |2\rangle \otimes |001\rangle.
$$

Now, for further shift operation, we have to extend the circuit for $|001\rangle$ and $|002\rangle$ states. For that, we have applied $X_{+1}$ gate on first qutrit so that we can perform shift operation for the coin state $|1\rangle$. Hence, the quantum state becomes

$$
\psi_8 \rightarrow a_{11} \otimes |1\rangle \otimes |000\rangle + a_{12} \otimes |2\rangle \otimes |001\rangle + a_{13} \otimes |0\rangle \otimes |002\rangle + b_{11} \otimes |1\rangle \otimes |001\rangle + b_{12} \otimes |2\rangle \otimes |002\rangle + b_{13} \otimes |0\rangle \otimes |000\rangle + c_{11} \otimes |1\rangle \otimes |002\rangle + c_{12} \otimes |2\rangle \otimes |000\rangle + c_{13} \otimes |0\rangle \otimes |001\rangle.
$$

For the shift operation for the coin state $|1\rangle$, we have applied a three-qutrit $M - S_{+1}$ gate on first qutrit and last qutrit as control and third qutrit as target. Now, the quantum state becomes

$$
\psi_9 \rightarrow a_{11} \otimes |1\rangle \otimes |000\rangle + a_{12} \otimes |2\rangle \otimes |001\rangle + a_{13} \otimes |0\rangle \otimes |002\rangle + b_{11} \otimes |1\rangle \otimes |001\rangle + b_{12} \otimes |2\rangle \otimes |002\rangle + b_{13} \otimes |0\rangle \otimes |000\rangle + c_{11} \otimes |1\rangle \otimes |002\rangle + c_{12} \otimes |2\rangle \otimes |000\rangle + c_{13} \otimes |0\rangle \otimes |001\rangle.
$$

To take back the coin state to its previous state for further operation, we have to apply $X_{+2}$ gate on the first qutrit, such that the quantum state becomes

$$
\psi_{10} \rightarrow a_{11} \otimes |0\rangle \otimes |000\rangle + a_{12} \otimes |1\rangle \otimes |001\rangle + a_{13} \otimes |2\rangle \otimes |002\rangle + b_{11} \otimes |0\rangle \otimes |001\rangle + b_{12} \otimes |1\rangle \otimes |002\rangle + b_{13} \otimes |2\rangle \otimes |002\rangle + c_{11} \otimes |0\rangle \otimes |002\rangle + c_{12} \otimes |1\rangle \otimes |000\rangle + c_{13} \otimes |2\rangle \otimes |001\rangle.
$$

For the shift operation for the coin state $|2\rangle$, we have applied a $X_{+1}$ gate on the last qutrit followed by three-qutrit $M - S_{+2}$ gate on the first qutrit and the last qutrit as control and third qutrit as target. So, the generated quantum state after applying $X_{+1}$ gate on the last qutrit is

$$
\psi_{11} \rightarrow a_{11} \otimes |0\rangle \otimes |001\rangle + a_{12} \otimes |1\rangle \otimes |002\rangle + a_{13} \otimes |2\rangle \otimes |000\rangle + b_{11} \otimes |0\rangle \otimes |002\rangle + b_{12} \otimes |1\rangle \otimes |010\rangle + b_{13} \otimes |2\rangle \otimes |001\rangle + c_{11} \otimes |0\rangle \otimes |000\rangle + c_{12} \otimes |1\rangle \otimes |001\rangle + c_{13} \otimes |2\rangle \otimes |002\rangle.
$$

The quantum state after applying three-qutrit $M - S_{+2}$ is

$$
\psi_{12} \rightarrow a_{11} \otimes |0\rangle \otimes |001\rangle + a_{12} \otimes |1\rangle \otimes |002\rangle + a_{13} \otimes |2\rangle \otimes |000\rangle + b_{11} \otimes |0\rangle \otimes |002\rangle + b_{12} \otimes |1\rangle \otimes |010\rangle + b_{13} \otimes |2\rangle \otimes |001\rangle + c_{11} \otimes |0\rangle \otimes |000\rangle + c_{12} \otimes |1\rangle \otimes |001\rangle + c_{13} \otimes |2\rangle \otimes |022\rangle.
$$

To take back the last qutrit state to its previous state, we have to apply $X_{+2}$ gate on the last qutrit, such that the quantum state becomes
Quantum circuit for first two steps of the LQW on a four-qutrit ternary system with a fixed initial state $|\uparrow\rangle \otimes |x = 0\rangle \equiv |\uparrow\rangle \otimes |000\rangle$. The position state mapping is shown in Table 8.

For further next two steps of LQW, the same circuit works as no new qutrit is required to represent the position states, since we are following nearest neighbor approach.

We have seen that in the first step of LQW, only one qutrit (last qutrit) change was enough to describe the position states as after first step of LQW, there are only three different position states in the one-dimensional line. Similarly, for the next three steps, the change is required on two qutrits (third and last qutrits) to get the all possible states to describe all the four steps of LQW. Now, for the fifth step of LQW, two new position states are introduced. For that, the change in three qutrits is required. At this stage to move forward, the circuit design for previous position states remains same, as shown in Fig. 8 (dotted in Fig. 9). Now, for further shift operation, we have to extend the circuit for $|011\rangle$ and $|022\rangle$ states, as shown in Fig. 9, to get the new position states $|211\rangle$ and $|122\rangle$, as shown in Table 5. For further next eight steps of LQW, this same circuit works as no new qutrit is required to represent the position states since we are following nearest neighbor approach.

Quantum circuit for first 13 steps of the LQW on line by considering the position state is mapping shown in Tables 3–5 on a four-qutrit ternary system with a fixed initial state $|\uparrow\rangle \otimes |x = 0\rangle \equiv |\uparrow\rangle \otimes |000\rangle$, they have been illustrated in Fig. 12. The output of the quantum circuit shown in Fig. 9 for first three steps of LQW using three-state DFT coin is illustrated in Table 6.

An alternative quantum circuit is shown in Fig. 14 for different mapping choice of position states (as we will always have two alternatives nearest neighbor position spaces due to three orthonormal basis states in ternary system) onto multi-qutrits states is shown in Table 7. These two mapping choices are the only appropriate mapping of qutrit states with the nearest neighbor position space, which makes the circuit not only efficient but also generalized for $n$ qutrit systems. These appropriate position state mapping using nearest neighbor approach makes our implementation efficient in terms of gate cost and the number of qutrits. As an example in Table 10, we illustrate that the LSQ changes in every step of quantum walk by following nearest neighbor approach. But the next qutrit toward left of LSQ and the most significant qutrit changes for the minimum possible time by following nearest neighbor approach.
The gates in quantum circuit are required for the change in qutrit states. Hence, the gate count in each step of quantum walk can also be minimized. Apart from these mapping choices, any naive mapping choices of the position states onto the qutrit states will lead into an inefficient quantum circuit with higher number of quantum gates as they do not follow nearest neighbor logic. One such example is given in Table 8 and Fig. 10. In this example, due to the configuration of mapped position state, only two steps of LQW can be performed, which is shown in Fig. 10, whereas using our position state mapping approach, 13 steps of LQW can be realized in the same system. Thus, the naive mapping choices of the position states based circuit consists of many additional ternary M–S gates compared to the nearest neighbor position space based circuits, as shown in Fig. 13. The next section will discuss about the generalization of quantum circuit for the LQW on a n-qutrit ternary system.

**FIGURE 13.** Quantum circuit for the first 13 steps of the LQW on a four-qutrit ternary system with a fixed initial state |↑⟩ ⊗ |x = 0⟩ = |↑⟩ ⊗ |000⟩. The position state mapping is shown in Table 3–5.
TABLE 6. Output After Each step (First Three Steps) of LQW Using Three-State DFT Coin and Output of Quantum Circuit Shown in Fig. 9

| Steps | Lazy quantum walk output | Circuit (Fig. 9) output |
|-------|--------------------------|------------------------|
| 0     | (0) ⊗ (|x = 0⟩)         | (0) ⊗ (|000⟩)          |
| 1     | (0) ⊗ (|x = 0⟩)       + (1) ⊗ (|x = −1⟩) + (2) ⊗ (|x = 1⟩)            | (0) ⊗ (|000⟩) + (1) ⊗ (|001⟩) + (2) ⊗ (|002⟩)) / √3 |
| 2     | (0) ⊗ (|x = 0⟩) + (1) ⊗ (|x = −1⟩) + (2) ⊗ (|x = 1⟩) + (0) ⊗ (|x = −2⟩) + e^{2πi/3} * (1) ⊗ (|x = x = −2⟩) + e^{2πi/3} * (0) ⊗ (|x = x = 0⟩) + e^{2πi/3} * (2) ⊗ (|x = x = 1⟩) + e^{2πi/3} * (1) ⊗ (|x = x = 0⟩) + e^{2πi/3} * (2) ⊗ (|x = x = 2⟩)/3 | (0) ⊗ (|000⟩) + (1) ⊗ (|001⟩) + (2) ⊗ (|002⟩) + (0) ⊗ (|000⟩) + e^{2πi/3} * (1) ⊗ (|101⟩) + e^{2πi/3} * (2) ⊗ (|000⟩) + (0) ⊗ (|000⟩) + e^{2πi/3} * (1) ⊗ (|001⟩) + e^{2πi/3} * (2) ⊗ (|000⟩) + e^{2πi/3} * (1) ⊗ (|001⟩) + e^{2πi/3} * (1) ⊗ (|000⟩) + e^{2πi/3} * (1) ⊗ (|010⟩) + e^{2πi/3} * (1) ⊗ (|100⟩) + e^{2πi/3} * (1) ⊗ (|001⟩) + e^{2πi/3} * (2) ⊗ (|000⟩) + e^{2πi/3} * (1) ⊗ (|010⟩) + e^{2πi/3} * (1) ⊗ (|100⟩) + e^{2πi/3} * (1) ⊗ (|001⟩) + e^{2πi/3} * (2) ⊗ (|000⟩) + e^{2πi/3} * (1) ⊗ (|010⟩) + e^{2πi/3} * (1) ⊗ (|100⟩) + e^{2πi/3} * (1) ⊗ (|001⟩) + e^{2πi/3} * (2) ⊗ (|000⟩) + e^{2πi/3} * (1) ⊗ (|010⟩) + e^{2πi/3} * (1) ⊗ (|100⟩) + e^{2πi/3} * (1) ⊗ (|001⟩) + e^{2πi/3} * (2) ⊗ (|000⟩)) / 3√3 |

TABLE 7. Alternative Position State Mapping Onto the Multiqutrits States for Quantum Circuits Presented in Fig. 14

| Position State | Multiqutrits State |
|---------------|--------------------|
| x = 0         | |000⟩ |
| x = 1         | |001⟩ |
| x = 2         | |012⟩ |
| x = 3         | |010⟩ |
| x = 4         | |011⟩ |
| x = 5         | |122⟩ |
| x = 6         | |120⟩ |
| x = 7         | |212⟩ |
| x = 8         | |202⟩ |
| x = 9         | |200⟩ |
| x = 10        | |201⟩ |
| x = 11        | |112⟩ |
| x = 12        | |110⟩ |
| x = 13        | |222⟩ |

TABLE 8. Example of Mapping of Position State Onto the Multiqutrits States for Quantum Circuits Presented in Fig. 10

| Position State | Multiqutrits State |
|---------------|--------------------|
| x = 0         | |000⟩ |
| x = 1         | |222⟩ |
| x = 2         | |012⟩ |
| x = 3         | |212⟩ |

C. GENERALIZATION OF QUANTUM CIRCUIT FOR IMPLEMENTING ONE-DIMENSIONAL LQW IN TERNARY SYSTEM

The proposed circuits can be scaled to implement more steps of one-dimensional LQW on a larger ternary system with the help of higher controlled M–S gates. As shown in Table 9, using n + 1-qutrit system, implementation of ⌊3^n/2⌋-steps of an LQW can be performed. In other words, to implement n-step of LQW, at most ([log_3 n] + 1) + 1 qutrits are required. Mathematically, it can be described as follows.

TABLE 9. Number of Steps and Maximum Number of Control Qutrits Required to Control a Target Qutrit in the LQW for a Ternary System of Upto n Qutrits Using a Circuit Similar to the One Presented in Fig. 13

| No. of qutrits | No. of steps | Max. No. of controls in M–S gates |
|----------------|--------------|----------------------------------|
| 2              | 1            | 1                                |
| 3              | 4            | 2                                |
| 4              | 13           | 3                                |
| 5              | 40           | 4                                |
| n + 1          | [3^n/2]      | n                                |

- For 1-step of LQW, ([log_3 1] + 1) + 1 = 2 qutrits are required, where [log_3 1] + 1 for position states and 1 for coin.
- For 2-steps of LQW, ([log_3 2] + 1) + 1 = 3 qutrits are required, where [log_3 2] + 1 for position states and 1 for coin.
- For 3-steps of LQW, ([log_3 3] + 1) + 1 = 3 qutrits are required, where [log_3 3] + 1 for position states and 1 for coin.
- For 9-steps of LQW, ([log_3 9] + 1) + 1 = 4 qutrits are required, where [log_3 9] + 1 for position states and 1 for coin.
- For 27-steps of LQW, ([log_3 27] + 1) + 1 = 5 qutrits are required, where [log_3 27] + 1 for position states and 1 for coin.

Similarly, for n-step of LQW, ([log_3 n] + 1) + 1 qutrits are required, where [log_3 n] + 1 for position states and 1 for coin.

Fig. 11 shows the generalized quantum circuit for ⌊3^n/2⌋-steps of the LQW on a n + 1-qutrit ternary system with a fixed initial state |↑⟩ ⊗ |x = 0⟩ ≡ |↑⟩ ⊗ |0^(n−1)⟩. As shown in Table 10, if the position state mapping onto multiqutrits
TABLE 10. Position State Mapping Onto the Multiqutrits States for Quantum Circuits Presented in Fig. 11

| x = 5⟩ | x = 4⟩ | x = 3⟩ | x = 2⟩ | x = 1⟩ | x = 0⟩ | x = −1⟩ | x = −2⟩ | x = −3⟩ | x = −4⟩ | x = −5⟩ |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
|        | 00⟩    | 01⟩    | 10⟩    | 11⟩    | 000⟩   | 001⟩   | 012⟩   | 010⟩   | 011⟩   | 122⟩   |
|        | 02⟩    | 02⟩    | 02⟩    | 02⟩    | 000⟩   | 001⟩   | 012⟩   | 010⟩   | 011⟩   |        |
|        |        |        |        |        |        |        |        |        |        |        |
|        |        |        |        |        |        |        |        |        |        |        |
|        |        |        |        |        |        |        |        |        |        |        |

FIGURE 14. (a) Quantum circuit for the first steps of LQW on four qutrits, as shown in Table 7. (b) Quantum circuit for the second-fourth steps of LQW on four qutrits, as shown in Table 7. (c) Quantum circuit for the fifth-thirteenth steps of LQW on four qutrits, as shown in Table 7.

states can be scaled up to ⌊3^n/2⌋-steps, the generalized quantum circuit on n + 1-qutrit ternary system shown in Fig. 11 will be required to implement LQW. Similarly, for binary quantum system, using n + 1-qubit, implementation of ⌊2^n/2⌋-steps of an LQW can be performed. The Fig. 12 shows the variation in number of steps for the binary and ternary quantum system and it clearly shows that the ternary system outperforms the binary system quite comprehensibly.

As shown in our proposed circuit, for an n + 1-qutrit system after every 3^{q−1} (where q is the number of qutrits and q ranges from 1 to n) steps, two new gates are added to realize the new position states along with the previous set of gates (when q > 1) due to stay put. In this way, we can also implement LQW in two-dimensional position space, for this, the same circuit can be scaled with an appropriate mapping of qutrit states with the nearest neighbor position space in both dimensions by introducing appropriate coin qutrit into the ternary system. This approach can also be scaled up to n-dimensional position space. In such cases, the control over the target or position qutrit increases with the number of coin qutrits.

V. CONCLUSION

In this article, we have defined LQW in ternary quantum system for the first time to the best of our knowledge. Furthermore, we have proposed an efficient quantum circuit realization to implement LQW in ternary quantum system using an appropriate logical mapping of the position space on which a walker evolves onto the multiqutrit states. Later, we also address scalability of the proposed circuit to n + 1-qutrit system. We have also verified our proposed circuits through simulation. In future, we will try to reduce the number of gates to implement LQW by adding additional ancilla qutrits. This will help one to implement more number of steps of the LQW keeping the circuit depth constant. The results are very promising to pave the way for further research works in qutrit-assisted quantum computing. With the evolution of qutrit-supported quantum hardware, we would like to validate our designs in near future.

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