First-Order Cosmological Perturbations Engendered by Point-Like Masses

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Abstract

In the framework of the concordance cosmological model, the first-order scalar and vector perturbations of the homogeneous background are derived in the weak gravitational field limit without any supplementary approximations. The sources of these perturbations (inhomogeneities) are presented in the discrete form of a system of separate point-like gravitating masses. The expressions found for the metric corrections are valid at all (sub-horizon and super-horizon) scales and converge at all points except at the locations of the sources. The average values of these metric corrections are zero (thus, first-order backreaction effects are absent). Both the Minkowski background limit and the Newtonian cosmological approximation are reached under certain well-defined conditions. An important feature of the velocity-independent part of the scalar perturbation is revealed: up to an additive constant, this part represents a sum of Yukawa potentials produced by inhomogeneities with the same finite time-dependent Yukawa interaction range. The suggested connection between this range and the homogeneity scale is briefly discussed along with other possible physical implications.

Key words: cosmological parameters – cosmology: theory – dark energy – dark matter – gravitation – large-scale structure of universe

1. Introduction

The concordance cosmological model fits very well the contemporary observational data (Hinshaw et al. 2013; Ade et al. 2014, 2015; Aubourg et al. 2015). This model assumes that the universe is filled with dominant portions of cold dark matter (CDM) and dark energy, represented by the cosmological constant and assuring the acceleration of global expansion, as well as a comparatively small portion of standard baryonic matter and a negligible portion of radiation. According to the cosmological principle, on large enough scales the universe is treated as homogeneous and isotropic, so the corresponding background Friedmann–Lemaître–Robertson–Walker (FLRW) metric is appropriate for its description. On the contrary, on sufficiently small scales the universe is highly inhomogeneous (separate galaxies, galaxy groups, and clusters stare us in the face). The thorough theoretical study of the structure formation starting from primordial fluctuations at the earliest evolution stages with the subsequent comparison of the predictions with the cosmic microwave background and other observational data may be recognized as one of the major subjects of modern cosmology.

The structure growth is usually investigated by means of two main distinct approaches, namely, the relativistic perturbation theory (see, e.g., Bardeen 1980; Durrer 2008; Gorbunov & Rubakov 2011) and N-body simulations generally based on the Newtonian cosmological approximation (see, e.g., Springel 2005; Dolag et al. 2008; Chisari & Zaldarriaga 2011). Roughly speaking, the area of the first approach may be characterized by the keywords “early universe; linearity; large scales,” while the area of the second one may be defined by the keywords of the opposite meaning: “late universe; nonlinearity; small scales.” Both approaches make great progress toward describing the inhomogeneous world within the limits of their applicability. Nevertheless, the linear cosmological perturbations theory certainly fails to describe nonlinear dynamics at small distances, while Newtonian simulations do not take into account relativistic effects becoming non-negligible at large distances (Green & Wald 2012; Adamek et al. 2013, 2014; Milillo et al. 2015). In this connection, in the latter case certain effort is required for extracting relativistic features from the large-scale Newtonian description (Chisari & Zaldarriaga 2011; Fidler et al. 2015; Hahn & Paranjape 2016).

Until now there was no developed unified scheme, which would be valid for arbitrary (sub-horizon and super-horizon) scales and treat the non-uniform matter density in a non-perturbative way, thereby incorporating its linear and nonlinear deviations from the average. This acute problem is addressed and successfully resolved in the present paper by the construction of such a self-consistent and indispensable scheme, which promises to be very useful in the precision cosmology era.

A couple of similar previous attempts deserve mentioning. First, the generalization of the well-known nonrelativistic post-Minkowski formalism (Landau & Lifshitz 2000) to the cosmological case in the form of the relativistic post-Friedmann formalism, which would be valid on all scales and include the full nonlinearity of Newtonian gravity at small distances, has been made in the recent paper by Milillo et al. (2015), but the authors resorted to expansion of the metric in powers of the parameter 1/c (the inverse speed of light). Second, the formalism for relativistic N-body simulations in the weak field regime, suitable for cosmological applications, has been developed by Adamek et al. (2013), but the authors gave different orders of smallness to the metric corrections and to their spatial derivatives (a similar “dictionary” can be found in Green & Wald 2012; Adamek et al. 2014).

The current paper also relies on the weak gravitational field regime: deviations of the metric coefficients from their background (average) values are considered first-order quantities, while the second order is completely disregarded. However, there are no additional assumptions: in the spirit of relativity, spatial and temporal derivatives are treated on an equal footing, and no dictionary giving them different orders of smallness is used (in contrast to Green & Wald 2012; Adamek et al. 2013, 2014); expansion into series with respect to the ratio 1/c is not used either (in contrast to Milillo et al. 2015);
there is no artificial mixing of first- and second-order contributions; the sub- or super-horizon regions are not singled out, and the derived formulas for the metric corrections are suitable at all scales.

The desired formalism is elaborated below within discrete cosmology (Eingorn & Zhuk 2012, 2014; Eingorn et al. 2013; Gibbons & Ellis 2014; Ellis & Gibbons 2015), based on the well-grounded idea of presenting nonrelativistic matter in the form of separate point-like particles with the corresponding energy–momentum tensor (Landau & Lifshitz 2000). At sub-horizon scales, implementation of this idea leads to nonrelativistic gravitational potentials and Newtonian equations of motion against the homogeneous background (Peebles 1980), which are commonly used in modern N-body simulations. Now, explicit expressions for potentials, applicable at super-horizon scales as well, will be made available.

The paper is structured in the following way. Section 2 is entirely devoted to solving the linearized Einstein equations for the first-order scalar and vector cosmological perturbations of the homogeneous background. In Section 3, the arresting attention properties of the derived solutions, including their asymptotic behavior, are analyzed and their role in addressing different related physical challenges is indicated. The main results are summarized laconically in the Section 4.

2. DISCRETE PICTURE OF COSMOLOGICAL PERTURBATIONS

2.1. Equations

The unperturbed FLRW metric, describing the universe being homogeneous and isotropic on the average, reads:

\[ ds^2 = a^2(d\eta^2 - \delta_{\alpha\beta}dx^\alpha dx^\beta), \quad \alpha, \beta = 1, 2, 3, \]  

where \( a(\eta) \) is the scale factor; \( \eta \) is the conformal time; \( x^\alpha, \alpha = 1, 2, 3 \), stand for the comoving coordinates, and it is supposed for simplicity that the spatial curvature is zero (the generalization to the case of non-flat spatial geometry is briefly analyzed below). The corresponding Friedmann equations in the framework of the pure \( \Lambda \)CDM model read:

\[ \frac{3\mathcal{H}^2}{a^2} = \kappa \varepsilon + \Lambda, \]  

\[ \frac{2\mathcal{H}' + \mathcal{H}^2}{a^2} = \Lambda, \]  

where \( \mathcal{H} \equiv a'/a \equiv (da/d\eta)/a; \) the prime denotes the derivative with respect to \( \eta; \kappa \equiv 8\pi G_N/c^4 \) (\( c \) is the speed of light and \( G_N \) is the Newtonian gravitational constant); \( \varepsilon \) represents the energy density of the nonrelativistic pressureless matter; the overline indicates the average value; and \( \Lambda \) is the cosmological constant. Domination of cold matter and \( \Lambda \) is at the center of attention, so contributions of radiation, relativistic cosmic neutrinos, or any warm component are negligible.

Following the analysis of the first-order cosmological perturbations by Bardeen (1980), Durrer (2008), and Gorbunov & Rubakov (2011), let us fix the Poisson gauge and consider the respective metric

\[ ds^2 = a^2[(1 + 2\Phi)d\eta^2 + 2B_{\alpha\beta}dx^\alpha dx^\beta] - (1 - 2\Phi)\delta_{\alpha\beta}dx^\alpha dx^\beta, \]  

where the function \( \Phi(\eta, r) \) and the spatial vector \( B(\eta, r) \equiv (B_1, B_2, B_3) \) describe the scalar and vector perturbations, respectively. It is assumed that there is no anisotropic stress acting as a source of the difference between perturbations of the metric coefficients \( g_{00} \) and \( g_{\alpha\beta}, \alpha = \beta \). Therefore, both of these metric corrections are equated to the same expression \( 2\alpha^2\Phi \) from the very beginning. Tensor perturbations are not taken into account because their source is also of the order beyond the adopted accuracy. The first-order tensor perturbations are associated with gravitational waves, freely propagating against the FLRW background. Their propagation is governed by the well-known equation uncoupled from the equations for \( \Phi \) and \( B \) (see, e.g., Equation (11) in Noh & Hwang 2005). Here attention is concentrated solely on the perturbations with non-negligible sources, and cosmological gravitational waves are not investigated. Similar to, e.g., Adamek et al. (2013, 2014), it is demanded that

\[ \nabla B \equiv \delta^{\alpha\beta} \partial B_{\alpha} / \partial x^\beta = 0. \]  

Then the Einstein equations

\[ G_i^k = \kappa T_i^k + \Lambda \delta_i^k, \quad i, k = 0, 1, 2, 3, \]  

where \( G_i^k \) and \( T_i^k \) denote the mixed components of the Einstein tensor and the matter energy–momentum tensor, respectively, take the following form after linearization:

\[ G_0^0 = \kappa T_0^0 + \Lambda \quad \Rightarrow \quad \Delta \Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = \frac{1}{2}\kappa a^2 \delta T_0^0, \]  

\[ G_\alpha^\alpha = \kappa T_\alpha^\alpha \quad \Rightarrow \quad \frac{1}{4} \Delta B_{\alpha} + \partial^\beta (\Phi' + \mathcal{H}\Phi) = \frac{1}{2}\kappa a^2 \delta T_\alpha^\alpha, \]  

\[ G_{\alpha\beta} = \kappa T_{\alpha\beta} + \Lambda \delta_{\alpha\beta} \quad \Rightarrow \quad \Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 0, \]  

\[ \left( \frac{\partial B_{\alpha}}{\partial x^\beta} + \frac{\partial B_{\beta}}{\partial x^\alpha} \right)' + 2\mathcal{H} \left( \frac{\partial B_{\alpha}}{\partial x^\beta} + \frac{\partial B_{\beta}}{\partial x^\alpha} \right) = 0. \]  

Here \( \Delta \equiv \delta^{\alpha\beta} \partial^2 / \partial x^\alpha \partial x^\beta \) stands for the Laplace operator in the comoving coordinates; \( T_i^k = T_i^k + \delta T_i^k \), where \( T_0^0 = \varepsilon \) is the only nonzero average mixed component. In the spirit of the particle–particle method of \( N \)-body simulations, the matter constituent of the universe may be presented in the form of separate point-like massive particles. Then the deviations \( \delta T_i^k \) from the average values \( T_i^k \) can be easily determined with the help of the well-known general expression for the matter energy–momentum tensor contravariant component (Landau & Lifshitz 2000; Chisari & Zaldarriaga 2011; Eingorn & Zhuk 2012, 2014):

\[ T_i^k = \sum_n \frac{m_n c^2}{\sqrt{-g}} \frac{dx_n^i}{d\eta} \frac{dx_n^k}{d\eta} \delta(r - r_n). \]
where \( g \equiv \det(g_{\alpha \beta}) \). This expression corresponds to a system of gravitating masses \( m_\alpha \) with the comoving radius-vectors \( r_\alpha(\eta) \) and four-velocities \( u_\alpha^i = dx_\alpha^i/d\eta, \ i = 0, 1, 2, 3 \). Their rest mass density \( \rho(\eta, r) \) in the comoving coordinates reads:

\[
\rho = \sum_n m_n \delta(r - r_n) = \sum_n \rho_n, \quad \rho_n \equiv m_n \delta(r - r_n). \tag{2.12}
\]

Introducing the comoving peculiar velocities \( \bar{v}_\alpha^i = dx_\alpha^i/d\eta, \) \( \alpha = 1, 2, 3 \), and treating them as importing the first order of smallness in the right-hand side (rhs) of the linearized Einstein Equations (2.7) and (2.8), one finds that within the adopted accuracy

\[
\delta T^0_0 \equiv T^0_0 - T^0_0 = \frac{c^2}{a^2} \delta \rho + \frac{3\bar{p}c^2}{a^2} \Phi, \quad \delta \rho \equiv \rho - \bar{\rho}, \tag{2.13}
\]

and

\[
\delta T^\alpha_\alpha = -\frac{c^2}{a^2} \sum_n m_n \delta(r - r_n) \bar{v}_\alpha^i + \frac{\bar{p}c^2}{a^2} B, \tag{2.14}
\]

respectively. In addition, \( \delta T^i_0 = 0 \). In particular, in full agreement with, e.g., Adamek et al. (2013), the first-order anisotropic stress is considered to be vanishing for nonrelativistic matter. Consequently, the rhs of both Equations (2.9) and (2.10) is zero. The term arising in Equation (2.13), \( \sim \rho \Phi \), is replaced by the term \( \sim \bar{p} \Phi \) since the product \( \delta \rho \Phi \) imports the second order of smallness and therefore should be dropped (the inequality \( |\delta \rho| \gg |\delta \rho \Phi| \) certainly holds true at all scales (Chisari & Zaldarriaga 2011)). The same reasoning applies to the term \( \sim \rho B \) arising in Equation (2.14). The average rest mass density \( \bar{\rho} \) is related to the average energy density \( \bar{\varepsilon} \) by means of an evident equality: \( \bar{\varepsilon} = \bar{p}c^2/a^3 \).

It is crucial that throughout the present paper, similarly to Chisari & Zaldarriaga (2011), Eingorn & Zhuk (2012, 2014), and Adamek et al. (2013, 2014), the rest mass density \( \rho \) is treated in a non-perturbative way: fulfillment of the inequality \( |\delta \rho| \ll \bar{\rho} \) is not demanded. For instance, the intragalactic medium and dark matter halos are characterized by values of \( \rho \) that are much higher than \( \bar{\rho} \). Thus, nonlinearity with respect to the deviation of \( \rho \) from its average value \( \bar{\rho} \) at small scales is fully taken into consideration here. The sole requirement lies in the following: this deviation \( \delta \rho \) as a source of the metric correction \( \Phi \) must secure smallness of this correction, i.e., the inequality \( |\delta \rho| \ll \bar{\rho} \) is not forbidden. As regards the Einstein tensor components, their nonlinear deviation from the corresponding background values is also unforbidden. For instance, the term \( \sim \Delta \Phi \) in the expression for \( G_0^0 \) predominates at small scales not only over the other terms in this expression, but also over the background value \( 3H^2/a^2 \) of \( G_0^0 \). Nevertheless, as distinct from Adamek et al. (2013, 2014), those terms that are nonlinear with respect to the metric corrections are neglected in the expressions for the Einstein tensor components. This concerns, e.g., the product \( \Phi \Delta \Phi \), which is neglected in comparison with \( \delta \Phi \), in complete agreement with the initial well-grounded assumption \( |\Phi| \ll 1 \).

Substitution of the expressions (2.13) and (2.14) into Equations (2.7) and (2.8), respectively, gives

\[
\Delta \Phi = -\frac{3}{2} \mathcal{H}(\Phi' + \gamma \Phi) - \frac{3\bar{p}c^2}{2a} \Phi = \frac{\kappa a^2}{2a} \delta \rho, \tag{2.15}
\]

\[
\frac{1}{4} \Delta B + \nabla (\Phi' + \gamma \Phi) - \kappa \mathcal{H}^2 B = -\frac{\kappa a^2}{2a} \sum_n \rho_n \bar{v}_n = -\frac{\kappa a^2}{2a} \sum_n \rho_n \bar{v}_n, \tag{2.16}
\]

where \( \bar{v}_n(\eta) \equiv dr_n/d\eta \equiv (v^1_n, v^2_n, v^3_n) \). With the help of the continuity equation

\[
\rho_n' + \nabla (\rho_n \bar{v}_n) = 0, \tag{2.17}
\]

which is satisfied identically for any \( n \)th particle, it is not difficult to split the vector \( \sum \rho_n \bar{v}_n \) into its grad and curl parts:

\[
\sum \rho_n \bar{v}_n = \nabla \Xi + \left( \sum \rho_n \bar{v}_n - \nabla \Xi \right), \tag{2.18}
\]

where

\[
\Xi = \frac{1}{4\pi} \sum_n \frac{m_n}{|r - r_n|^3} \bar{v}_n. \tag{2.19}
\]

Really, this function satisfies the Poisson equation

\[
\Delta \Xi = \nabla \sum \rho_n \bar{v}_n = -\sum \rho_n' =, \tag{2.20}
\]

and this fact can be easily checked alternatively with the help of the Fourier transform

\[
-k^2 \hat{\Xi}(\eta, k) \equiv \int \Xi(\eta, r) \exp(-ikr)dr, \tag{2.22}
\]

\[
\hat{\rho}_n(\eta, k) \equiv \int \rho_n(\eta, r) \exp(-ikr)dr = m_n \int \delta(r - r_n) \exp(-ikr)dr = m_n \exp(-ikr_n). \tag{2.23}
\]

In other words, one can demonstrate that the function \( \hat{\Xi}(\eta, k) \) derived from Equation (2.22) after the substitution of Equation (2.19) satisfies Equation (2.21) and, consequently, reads:

\[
\hat{\Xi} = -\frac{i}{k^2} \sum_n m_n (k \bar{v}_n) \exp(-ikr_n). \tag{2.24}
\]

Taking into account (2.18), from (2.16) one gets

\[
\Phi' + \gamma \Phi = -\frac{\kappa a^2}{2a} \Xi \tag{2.25}
\]

and

\[
\frac{1}{4} \Delta B - \frac{\kappa \bar{p}c^2}{2a} B = -\frac{\kappa a^2}{2a} \left( \sum \rho_n \bar{v}_n - \nabla \Xi \right). \tag{2.26}
\]

Substitution of Equation (2.25) into Equation (2.15) gives

\[
\Delta \Phi = \frac{3\kappa \bar{p}c^2}{2a} \Phi = \frac{\kappa a^2}{2a} \delta \rho - \frac{3\kappa a^2}{2a} \mathcal{H} \Xi. \tag{2.27}
\]

Thus, Equations (2.26) and (2.27) are derived for the vector and scalar perturbations, respectively. Below their solutions are found and the fulfillment of Equations (2.5), (2.9), (2.10), and (2.25) is verified.
2.2. Solutions

In the Fourier space Equation (2.26) takes the following form:
\[
-\frac{k^2}{4} \dot{\hat{B}} = -\frac{\kappa c^2}{2a}\left(\sum_n \tilde{\rho}_n \tilde{\gamma}_n - i k \hat{z} \right).
\] (2.28)

Substituting (2.23) and (2.24) into (2.28), one immediately obtains
\[
\dot{\hat{B}} = \frac{2\kappa c^2}{a}\left(k^2 + \frac{2\kappa c^2}{a}\right)^{-1} \sum_n m_n \exp(-ikn) \times \left(\tilde{\gamma}_n - \frac{(k\tilde{\gamma}_n)}{k^2}\right). \tag{2.29}
\]

The condition (2.5) is evidently satisfied since \(k\dot{\hat{B}} = 0\). It is also not difficult to verify that Equation (2.10) is fulfilled within the adopted accuracy. In fact, it is enough to show that
\[
\dot{\hat{B}}' + 2\gamma \dot{\hat{B}} = 0. \tag{2.30}
\]

For this purpose let us write down the spacetime interval for the \(n\)th particle
\[
d_{sn} = a[1 + 2\Phi + 2B_n \tilde{\gamma}_n - (1 - 2\Phi)\delta_{n,0} \tilde{\gamma}_0 \tilde{\gamma}_0^{1/2}]d\eta
\]
and the corresponding Lagrange equations of motion
\[
[a(B_{\mid \eta=\tau_n} - \tilde{\gamma}_n)]' = a\nabla \Phi_{\mid \eta=\tau_n}, \tag{2.32}
\]
where the contributions of the considered particle itself to \(B\) and \(\Phi\) are excluded as usual, so divergences are absent (in other words, the particle moves in the gravitational field produced by the other particles). Since Equation (2.32) and its consequences will be used exclusively in linearized Einstein equations (e.g., Equation (2.10)), all terms being nonlinear with respect to \(\tilde{\gamma}_n\), \(B_n\), and \(\Phi\) have been dropped in order to avoid exceeding the adopted accuracy. Multiplying (2.32) by \(\rho_n\) and summing up, one gets
\[
\rho(aB)' - \sum_n \rho_n (a\tilde{\gamma}_n)' = a\rho \nabla \Phi. \tag{2.33}
\]

Furthermore, in the terms containing \(B\) and \(\Phi\) the rest mass density \(\rho\) should be replaced by its average value \(\bar{\rho}\) as discussed before. Consequently,
\[
\sum_n \rho_n (a\tilde{\gamma}_n)' = -a\bar{\rho} \nabla \Phi + \bar{\rho}(aB)'. \tag{2.34}
\]

Of course, in order to study dynamics of the \(N\)-body system one should include the removed nonlinear terms \(-a\bar{\rho} \nabla \Phi\) and \(\delta \rho(aB)'\). However, for analyzing the Einstein equations for the first-order cosmological perturbations it is apparently enough to keep only linear terms in the equations of motion. In the Fourier space, Equation (2.34) reads:
\[
\sum_n \rho_n (a\tilde{\gamma}_n)' = \sum_n m_n \exp(-ikr_n)(a\tilde{\gamma}_n)',
\]

Now, expressing \(\dot{\hat{B}}'\) from Equation (2.29) with the help of Equation (2.35) and substituting the result into Equation (2.30), one arrives at the identity.

Finally, the vector perturbation \(B\) can be determined by multiplying Equation (2.29) by \(\exp(ikr)/(2\pi)^3\) and integrating over \(k\). A cumbersome, but straightforward calculation gives
\[
B = \frac{\kappa c^2}{8\pi a} \sum_n \left[\frac{m_n}{|r - r_n|}\right] \left(3 + 2\sqrt{3}q_n + 4q_n^2\exp(-2q_n/\sqrt{3}) - 3q_n^2\right) - \frac{m_n[\tilde{\gamma}_n(r - r_n)]}{|r - r_n|^3} (r - r_n) - \frac{9 - (9 + 6\sqrt{3}q_n + 4q_n^2\exp(-2q_n/\sqrt{3})}{q_n^2} \right]. \tag{2.36}
\]

where the following convenient spatial vector is introduced:
\[
q_n(\eta, r) = \left[\frac{3\kappa c^2}{2a}\right](r - r_n), \quad q_n = |q_n|. \tag{2.37}
\]

Let us now switch over to Equation (2.27). In the Fourier space this equation takes the following form:
\[
-k^2\hat{\Phi} - \frac{3\kappa c^2}{2a} \hat{\Phi} = \frac{\kappa c^2}{2a} \sum_n \rho_n - \frac{\kappa c^2}{2a}(2\pi)^3 \delta(k)
\]
and substituting the result into Equation (2.38), one immediately obtains
\[
\hat{\Phi} = -\frac{\kappa c^2}{2a}\left(k^2 + \frac{3\kappa c^2}{2a}\right)^{-1} \times \left[\sum_n m_n \exp(-ikr_n)\right] \left[1 + 3i\hat{H}(k\tilde{\gamma}_n)\right]
\]
where the well-known presentation \((2\pi)^3 \delta(k) = \int \exp(-ikr)dr\) is taken into account. Substituting Equations (2.23) and (2.24) into Equation (2.38), one immediately obtains
\[
\hat{\Phi} = \frac{1}{3} - \frac{3\kappa c^2}{8\pi a} \sum_n \frac{m_n}{|r - r_n|} \exp(-q_n)
\]
\[
+ \frac{3\kappa c^2}{8\pi a} \hat{H}\sum_n \frac{m_n[\tilde{\gamma}_n(r - r_n)]}{|r - r_n|} \left(1 + (1 + q_n)\exp(-q_n/\sqrt{3})\right). \tag{2.40}
\]

Thus, the explicit analytical expressions (2.36) and (2.40), for the first-order vector and scalar cosmological perturbations, respectively, are determined for the first time. Let us accentuate the irrefutable fact that the dictionary-based approach compels us to say goodbye to all hope of finding analytical solutions. Really, let us momentarily return to Equation (2.27). In Adamek et al. (2013) the order \(O(\epsilon)\) is assigned to \(\Phi\) while every spatial derivative is treated as importing the order
in Landau & Lifshitz (2000) for the metric correction $h_{00}$, as it certainly should be. In addition, the term containing $\ddot{v}_i$ disappears from Equation (2.40) in the considered limit (in view of the factor $H$), and at the same time there is no term that is linear with respect to $\ddot{v}_i$ in the expression (106.13) (Landau & Lifshitz 2000). This fact also serves as confirmation of coincidence.

Regarding Equation (3.2), the only difference between this expression and (106.15) in Landau & Lifshitz (2000) is that the integers $4; 4$ are replaced by $7; 1$ (one should keep in mind that the comoving peculiar velocities $\ddot{v}_i$ defined with respect to the conformal time $\eta$ as $d\dot{v}_i/d\eta$ are related to those defined with respect to the synchronous time $t$ as $\ddot{v}_i = d\dot{v}_i/dt$ by means of an evident equality: $c\rho = a\dot{\rho} = \frac{c}{c_0}$. Apparently, this difference in integers represents nothing other than a result of different gauge conditions here and in Landau & Lifshitz (2000). Indeed, the condition (2.5) is not demanded and, of course, does not hold true in Landau & Lifshitz (2000). Correspondence between Equation (3.2) and (106.15) (Landau & Lifshitz 2000) lies in the fact that the sum of these integers is the same: $4 + 4 = 7 + 1$. One can show that it equals 8 for the other appropriate gauge choices as well. Therefore, Equation (3.2) exactly coincides with the purely vector part of (106.15) (Landau & Lifshitz 2000), as one can easily see by finding the curl of both expressions.

### 3.2. Newtonian Approximation and Homogeneity Scale

Now let us switch over to the Newtonian cosmological approximation: $q_n \ll 1$, i.e., $|r - r_i| \ll \frac{\sqrt{2\alpha}}{3\kappa c^2}$, and peculiar motion as a source of the gravitational field is completely ignored (Chisari & Zaldarriaga 2011), so the summands directly proportional to the velocities $v_i$ are omitted. Then only the scalar perturbation $\Phi$ survives in the same form (Equation (3.3)), where the constant 1/3 has been dropped for another reason: only the gravitational potential gradient enters into equations of motion describing dynamics of the considered system of gravitating masses. These equations for any $j$th particle follow directly from Equation (2.31) and take the form

$$R_j - \frac{\ddot{a}}{a} R_j = -G_N \sum_{n \neq j} \frac{m_n (R_j - R_n)}{|R_j - R_n|^3}$$

in the physical coordinates $X^3 = ax^3$, $\beta = 1, 2, 3$, being in accordance with the corresponding equations in the papers by Springel (2005), Dolag et al. (2008), Labini (2013), Warren (2013), Eingorn (2014), and Ellis & Gibbons (2015) devoted to cosmological simulations. Here, the dots denote the derivatives with respect to $t$. Let us consider two important questions. First, what are the applicability bounds for the above-mentioned inequality, which may be rewritten in the form $|R - R_i| \ll \frac{\sqrt{2\alpha}}{3\kappa c^2}$? In order to answer, one should simply calculate the rhs of this inequality:

$$\lambda \equiv \frac{\sqrt{2a^3}}{3\kappa c^2} \sqrt{9H_0^2 \Omega_m} \left( \frac{a^2}{a_0} \right)^3, \quad \Omega_m \equiv \frac{\kappa c^4}{3H_0^2 a_0^2},$$

where $a_0$ and $H_0$ are the current values of the scale factor $a$ and the Hubble parameter $H \equiv a/da/dt = c\dot{H}/a$, respectively. According to Ade et al. (2014, 2015), $H_0 \approx 68$ km s$^{-1}$ Mpc$^{-1}$ and $\Omega_m \approx 0.31$. Therefore, the current
value of $\lambda$ is $\lambda_0 \approx 3700$ Mpc $\approx 12$ Gly. It is very interesting that this Yukawa interaction range and the sizes of the largest known cosmic structures (Clowes et al. 2013; Horvath et al. 2014; Balazs et al. 2015) are of the same order, thereby hinting at the opportunity to resolve the formidable challenge lying in the fact that their sizes essentially exceed the previously reported epoch-independent scale of homogeneity $\sim 370$ Mpc (Yadav et al. 2010). The authors arrived at this underestimate by comparing the deviation of the fractal dimension, characterizing the distribution of matter, from 3 (dimensionality of space) to its statistical dispersion. Along with fractal analysis, their approach relies on the weak clustering limit and cosmological simulations driving $512^3$ particles in a cube with the edge $\sim 1.5$ Gpc. Incidentally, this edge is less than half $\lambda_0$, and from the very beginning of such a volume-restricted simulation it is difficult to expect any definite and reliable indications of structuring in bigger volumes. Now, if one associates the scale of homogeneity with $\lambda$ instead, then the cosmological principle, asserting that the universe is homogeneous and isotropic when viewed at a sufficiently large scale, is saved and reinstated when this typical averaging scale is greater than $\lambda$. The proposed association does not mean that the homogeneity scale is equated exactly to $\lambda$ but rather describes $\lambda$ as an approximate upper bound to the cosmic structure size, and the homogeneity scale as a distance exceeding $\lambda$ in a few times while remaining of the same order. It is remarkable that this reasoning is actually confirmed by Li & Lin (2015). The authors defined the scale of homogeneity as a distance at which the correlation dimension is within 1% of 3 (and, consequently, equals 2.97) and fixed an upper bound to such a distance $\sim 3\lambda_0$. The dependence $\lambda \sim a^{3/2}$ is noteworthy as well: the earlier the evolution stage, the smaller the scale of homogeneity. Naturally, this is closely related to the hierarchical clustering process.

The second important question is, what are the applicability bounds for peculiar motion ignoring? In order to answer, one can consider the ratio of the third term in Equation (2.40) to the second one. For a single gravitating mass $m_1$ momentarily located at the origin of coordinates ($\bar{r}_1 = 0$) with the velocity $\bar{v}_1$ collinear to $r$ (for ensuring the maximum value of the scalar product $\bar{v}_1 r = \bar{v}_1 r$, where $\bar{v}_1 \equiv [\bar{v}_1]$) this ratio amounts (up to a sign) to $3H\bar{v}_1 r = 3\dot{H}v_1 R/(2c^2)$, where $v_1 \equiv [v_1] = c\bar{v}_1/a$, $R \equiv |R| = ar$ and $q_\ell \ll 1$ as before. Actually the product $av_1$ is none other than the absolute value of the particle’s physical peculiar velocity. For example, with the help of today’s typical values ($250 \div 500$) km s$^{-1}$ and the inequality $R \ll 3700$ Mpc, one finds that the considered ratio is much less than $(1 \div 2) \times 10^{-3}$. Exactly the same estimate can be made for the ratio of derivatives of the considered terms with respect to $r$. This means that at the scales under consideration, the gravitational force originating from the second term in the gravitational potential (2.40), which does not contain particle velocities, is much stronger than that coming from the third one, which contains them.

Thus, the Newtonian cosmological approximation may be used when $|R - R_\ell| \ll \lambda$. Otherwise, at scales comparable or greater than $\lambda$, one should use the complete expressions for the metric corrections obtained in the previous section. In particular, the derived Yukawa-type potentials should be used instead of the Newtonian ones in order to study the formation and evolution of the largest structures in the universe. It is necessary to understand that the elaborated formalism results in Newtonian behavior of the considered physical system at sufficiently small distances without any relativistic corrections. Hence, the accuracy of the developed theory is limited in this region by the standard Newtonian approach. However, the predicted Yukawa behavior at greater distances may be considered a relativistic effect since it follows directly from Einstein equations of General Relativity.

It should be emphasized that despite the presence of those terms in Equations (2.36) and (2.40), which do not contain exponential functions, the influence of any particle on the motion of its neighbors does drop exponentially when the distance increases. Really, with the help of Equation (2.30), the equations of motion (2.32) may be rewritten in the form

\[
(a\bar{v}_1)' = - a(\nabla \Phi |_{x=r} + \mathcal{H} \mathcal{B} |_{x=r}),
\]

and this peculiar acceleration of a given particle, caused by all other gravitating masses, contains solely terms with exponential functions. Indeed, the direct substitution of Equations (2.36) and (2.40) into the rhs of Equation (3.6) demonstrates that all terms without exponential functions exactly cancel each other. This fact confirms the revealed Yukawa nature of universal gravitation. In addition, it indicates that for sufficiently small values of $a$ and nonzero separation distances between particles they almost do not interact gravitationally (all terms containing exponential functions may be dropped under such conditions), so the system behaves as a perfect gas undergoing global expansion. It is also interesting that the physical screening length $\sqrt{3}\lambda/2$ from Equation (2.36) is less than the counterpart $\lambda$ from Equation (2.40), meaning that vector modes diminish with distance faster than scalar modes. The equations of motion (3.6) are ready to be used in a new generation of cosmological simulation codes (see, however, the discussion of coordinate transformations below, indicating the possibilities of the reinterpretation of Newtonian simulations from a relativistic perspective). It would be quite reasonable to confront the outputs of relativistic simulations with those of various Newtonian predecessors, thereby discriminating between them and the proposed Yukawa modification, especially regarding predictions of peculiarities of the hugest gravitationally bound objects in the universe.

One more important detail consists of the fact that $\lambda$ does not coincide with the Hubble radius $c/H$, in contrast to the Yukawa interaction range proposed by Signore (2005) in order to limit gravitational effects of a particle outside its causal sphere. Really, in terms of the Hubble parameter $H$ and the deceleration parameter $q \equiv -\ddot{a}/(aH^2)$,

\[
\frac{1}{\lambda^2} = \frac{3H^2}{c^2}(1 + q) = -\frac{3H}{c^2}, \quad \lambda = \frac{1}{\sqrt{3(1 + q)}} H, \quad \frac{3H^2}{c^2} = \frac{\kappa m c^2}{a^3} + \Lambda
\]

following directly from Equation (3.5) and the Friedmann Equations (2.2) and (2.3), which may be rewritten in the form

\[
\frac{d^2a}{d\tau^2} = 3\frac{H^2}{c^2}(1 + q) = -\frac{3H}{c^2}
\]
\[
\frac{H^2}{c^2}(1 - 2q) + \frac{3H^2}{c^2} + \frac{2H}{c^2} = \Lambda, \tag{3.9}
\]

respectively. One obtains from Equation (3.7) that \( \lambda = c/H \) in the unique moment of time when \( q = -2/3 \). According to Equations (3.8) and (3.9), \( 2\Lambda/\lambda = \kappa c^2/a^3 \) at this moment, or \( 2\Omega_\Lambda/\lambda = \Omega_\Lambda(\alpha_0/a)^3 \), where \( \Omega_\Lambda \equiv \Lambda c^2/(3H_0^2) \approx 0.69 \) (Ade et al. 2014, 2015). Hence, \( \lambda = c/H \) in the near future when \( a/a_0 \approx 1.16 \). Before this moment \( \lambda < c/H \), while afterward the opposite inequality takes place.

Likewise \( \lambda \) does not coincide with the shielding length introduced by Hahn & Paranjape (2016). Those authors resorted to the dominant growing mode in the framework of linear relativistic perturbation theory (see their Equation (15), which is actually a predetermined approximate solution but, nevertheless, serves as an assumed starting point) and presented \( \Phi \) in the standard form of a product of a function of time and a function of spatial coordinates. This allowed expressing \( 3H(\Phi' + \dot{\Phi}) \) as \( l^{-2}\Phi \), where \( l \) is a certain time-dependent parameter, and then, after substitution into the linearized Einstein equation \( G^0_0 = \kappa T^0_0 + \Lambda \), declaring \( l \) to be a shielding length. It should be mentioned that the same shielding mechanism may be also discerned in the preceding paper by Eingorn & Brilenkov (2015), where continuous matter sources are the focus of attention instead of the discrete ones investigated here. In this connection, it makes sense to confront in brief the approaches by Hahn & Paranjape (2016) and Eingorn & Brilenkov (2015). First, at the same level of linear energy–momentum fluctuations, the velocity-dependent term introduced by Eingorn & Brilenkov (2015) in the equation for \( \Phi \) (see, e.g., their Equation (16)) can be also easily reduced to \( l^{-2}\Phi \) for the considered growing mode. Of course, this is a foreseeable coincidence because the aforementioned velocity-dependent term coincides exactly with \( 3H(\Phi' + \dot{\Phi}) \) owing to the linearized Einstein equation \( G^0_0 = \kappa T^0_0 \). Second, in contrast to the current paper, Hahn & Paranjape (2016) did not single out the very important contribution to \( \kappa T^0_0 \), namely, the second term in the rhs of (2.13), which is directly proportional to \( \Phi \) (see, however, their Appendix C, where the authors address this issue along with the connection to the approach of Chisari & Zaldarriaga 2011). The aforementioned term is absolutely necessary for satisfying the perturbed energy conservation equation (Eingorn & Brilenkov 2015) and leads to the screening length \( \lambda \) (3.5) irrespective of the velocity-dependent contribution.

### 3.3. Yukawa Interaction and Zero Average Values

It is important to stress that, as a manifestation of the superposition principle, the second term in Equation (2.40) represents the sum of Yukawa potentials

\[
\phi_n = -\frac{\kappa c^2}{8\pi a} \frac{m_n}{|r - r_n|} \exp(-q_n) = -\frac{G_n m_n}{c^2|\vec{r} - \vec{r}_n|} \exp\left(-\frac{|\vec{\vec{r}} - \vec{\vec{r}}_n|}{\lambda}\right) \tag{3.10}
\]

coming from each single particle, with the same interaction radius \( \lambda \). Such a favorable situation is possible owing to the last term in the left-hand side (lhs) of Equation (2.15), which has been disregarded in (Eingorn & Zhuk 2012) by mistake and erroneously compensated in Eingorn & Zhuk (2014) by inhomogeneous radiation of an unknown nature. Actually, such radiation must not only possess negligible average energy density (requiring additional questionable reasoning), but also exchange the momentum with the nonrelativistic pressureless matter, despite the fact that no non-gravitational interaction between these two constituents has been assumed, and therefore the energy–momentum interchange is strictly forbidden. Here the aforementioned unpardonable omission is rectified: the ill-starred term is reinstated, and there is no necessity in any additional interacting universe components at all.

The sum \( \sum_n \phi_n \) is certainly convergent at all points except at positions of the gravitating masses, and computational obstacles do not come into existence. In particular, the order of adding terms corresponding to different particles is arbitrary and does not depend on their locations. On the contrary, there are certain obstacles when calculating the sum of Newtonian potentials or their gradients. Let us address the well-known formulas (8.1) and (8.3) in the textbook by Peebles (1980) for the gravitational potential and the peculiar acceleration, respectively, derived in the Newtonian approximation (see above):

\[
\Phi \sim \int dr' \frac{3 \rho \dot{r} \cdot r'}{|r - r'|^3} (r - r'), \tag{3.11}
\]

up to space-independent factors being of no interest here. Substituting Equation (2.12) into the second integral in Equation (3.11), one gets formula (8.5) in (Peebles 1980):

\[
-\nabla \Phi \sim \sum_n \frac{m_n}{|r - r_n|} (r - r_n). \tag{3.12}
\]

According to Peebles (1980), this sum is well-defined and depends on the order of adding terms, and if one adds them in order of increasing distances \( |r - r_n| \) and assumes that the distribution of particles corresponds to a spatially homogeneous and isotropic random process with the correlation length being much less than the Hubble radius \( c/H \), then this sum converges. Regarding the first integral in Equation (3.11), the argumentation by Peebles (1980) again relies on the random process assuring convergence; however, the substitution of Equation (2.12) splits this integral into two divergent parts: the sum of an infinite number of the Newtonian potentials of the same sign (see also the paper by Norton (1999) devoted to the related famous Neumann–Seelig gravitational paradox) and the integral of the pure Newtonian kernel.

Of course, the enumerated difficulties are absent when summing up the Yukawa-type potentials. In addition, it is interesting that in this case the particles’ distribution may be nonrandom and anisotropic. The lattice universe model with the toroidal topology \( T \times T \times T \) represents a striking example. As explicitly demonstrated by Brilenkov et al. (2015), in the framework of this model the gravitational potential has no definite values on the straight lines joining identical point-like masses in neighboring cells if the last term in the lhs of Equation (2.15) is not taken into account. Evidently, the finite Yukawa interaction range \( \lambda \) arising due to this term easily resolves this challenge as well as any similar ones related to the choice of periodic boundary conditions. Incidentally, if the
space is supposed to have the usual, non-toroidal topology \( R \times R \times R \), but the choice of periodic boundary conditions is made for \( N \)-body simulation purposes, then the dimensions of a cell should normally be greater than \( \lambda \), thereby weakening the undesirable impact of periodicity on simulation outputs.

A noteworthy feature of the Yukawa potentials (3.10) consists of assuring the zero average value of the scalar perturbation \( \Phi \) (2.40). Let us determine the average value of a single one of them:

\[
\begin{align*}
\langle \phi_n \rangle &= \frac{1}{V} \int_V d\phi_n \\
&= -\frac{\kappa^2}{8\pi a} \int_V \frac{d\mathbf{r}}{V |\mathbf{r} - \mathbf{r}_n|} \exp\left( -\frac{a|\mathbf{r} - \mathbf{r}_n|}{\lambda} \right) \\
&= -\frac{\kappa^2}{8\pi a} \frac{4\pi^2}{a^2} = -\frac{m_n}{V} \frac{1}{3p},
\end{align*}
\]

where the comoving averaging volume \( V \) tends to infinity. Here the definition of \( \lambda (3.5) \) has been used. Consequently,

\[
\sum_n \langle \phi_n \rangle = -\frac{1}{3p} \cdot \frac{1}{V} \sum_n m_n = \frac{1}{3},
\]

since \((1/V)\sum m_n \equiv p\). Combining Equation (3.14) with the first term in Equation (2.40), one immediately achieves the desired result \( \Phi = 0 \) (the third term in Equation (2.40) is apparently zero on average in view of the different directions of particle velocities, and the same applies to the vector perturbation \( \mathbf{B} \)). This result means that the first-order backreaction effects are absent, as they certainly should be. Zero average values of the first-order cosmological perturbations are expected from the very beginning, since these metric corrections are none other than linear deviations from the unperturbed average values of the metric coefficients. Nevertheless, as shown by Eingorn & Zhuk (2014) and Eingorn et al. (2015), there exists a concrete example of the mass distribution, which gives the nonzero average value of the gravitational potential determined by the standard prescription (3.11). This problem is solved by Eingorn et al. (2015) through manually introducing the abrupt cutoff of the gravitational interaction range with the help of the Heaviside step function. One can see now that the same problem is strictly solved with the help of the finite Yukawa range, and the potential remains smooth together with its gradient thanks to the smoothness of the exponential function. Obviously, the established equality \( \Phi = 0 \) takes place for an arbitrary mass distribution including that investigated by Eingorn et al. (2015). In addition, the well-grounded equalities \( \delta T_0^0 = 0 \) and \( \delta T_0^i = 0 \) are valid as well, following from Equations (2.13) and (2.14), respectively.

Let us bring up and settle a related issue consisting of the following. One can easily prove that in the limiting case considered, the equation of motion of a test cosmic body reads:

\[
\ddot{\mathbf{R}} = \frac{\dot{a}}{a} \mathbf{R},
\]

so the acceleration of the body is reasonably connected with the acceleration of the universe expansion. At the same time, the described simple, but crucial, test cannot be passed by Newtonian gravity. Indeed, in the framework of the Newtonian cosmological approximation the contribution from the outer region of the considered sphere is absent, while the contribution from its inner region generates an additional force in the rhs of Equation (3.15), spoiling the established correspondence between the accelerations. This demonstrates once again the superiority of formula (2.40) for all scales.

### 3.4. Transformation of Spatial Coordinates

When writing down the perturbed metric (2.4), the gauge choice is made in favor of the so-called Poisson/longitudinal/ conformal-Newtonian gauge, by analogy with Adamek et al. (2013, 2014) and Milillo et al. (2015). However, it is common knowledge that there is no preferable coordinate system, so other gauges are admissible as well. The chosen gauge is characterized, in particular, by the coincidence of the found function \( \Phi \) (2.40) with the corresponding gauge-invariant Bardeen potential (Bardeen 1980). The introduced energy–momentum fluctuations \( \delta T_i^k \) also coincide with the corresponding gauge-independent quantities. For instance, let us verify that the expression (2.13) for \( \delta T_0^0 \) remains unchanged for the analog of the so-called \( N \)-body gauge (Fidler et al. 2015). This particular gauge features the unperturbed comoving volume, providing the chance of eliminating the second term in the rhs of Equation (2.13) and, hence, of rehabilitating the Newtonian description. In this connection, it is necessary to show directly that this chance does not contradict the Yukawa screening of the gravitational interaction established in the Poisson gauge. For this purpose, let us rewrite the metric (2.4), except for the vector perturbation \( \mathbf{B} \):

\[
ds^2 = a^2 \left[ (1 + 2\Phi) \eta^2 - (1 - 2\Phi) \delta_{\alpha\beta} dx^\alpha dx^\beta \right],
\]

where the scale factor \( a \) is a function of the conformal time \( \eta \) while the scalar perturbation \( \Phi (2.40) \) is a function of \( \eta \) and comoving coordinates \( x^\alpha, \alpha = 1, 2, 3 \). The transformation of coordinates

\[
\eta = \tau + A, \quad x^\alpha = \xi^\alpha + \frac{\partial L}{\partial \xi^\alpha},
\]

where \( A \) and \( L \) are (first-order) functions of the new conformal time \( \tau \) and new comoving coordinates \( \xi^\alpha, \alpha = 1, 2, 3 \), gives

\[
d\tau^2 = a^2 \left[ (1 + 2\Phi + 2A') + 2H(A) \right] d\eta^2 + 2 \left( \frac{\partial \Omega}{\partial \xi^\alpha} - \frac{\partial L}{\partial \xi^\alpha} \right) d\xi^\alpha d\xi^\beta - \left( (1 - 2\Phi + 2H(A)) \delta_{\alpha\beta} + 2 \frac{\partial^2 L}{\partial \xi^\alpha \partial \xi^\beta} \right) d\xi^\alpha d\xi^\beta.
\]
fluctuations of the mixed energy–momentum tensor components with the corresponding gauge-invariant perturbations. Despite the fact that this choice differs from that made by Fidler et al. (2015; where \( A = 0 \)), this does not affect the following main idea of the \( N \)-body gauge. In accordance with the general definition (2.11), in the new coordinates \((\tau, \xi^a)\) instead of Equation (2.13) one has

\[
\delta T^0_0 = \frac{c^2}{a} \delta \rho_\xi + \frac{\kappa a^2}{a^3} (3\Phi - \Delta \xi L),
\]

where \(\Delta \xi \equiv \delta^{\alpha\beta} \frac{\partial}{\partial \xi^\alpha} \frac{\partial}{\partial \xi^\beta} ; \quad \delta \rho_\xi \equiv \rho_\xi - \bar{\rho}, \quad \rho_\xi = \sum_n m_n \delta (\xi - \xi_n).
\]

Next, fixing \(\Delta \xi L = 3\Phi\), one may present the energy density fluctuation (3.19) conformably in the form

\[
\delta T^0_0 = \frac{c^2}{a^3} \delta \rho_\xi.
\]

Thus, it may seem that the proper use of gauge freedom ensures the disappearance of the second term in the rhs of Equation (2.13). Nevertheless, the expressions (2.13) and (3.21) for \(\delta T^0_0\) are equal. In order to prove this, let us use the fact that the perturbation \(\delta \rho_\xi\) entering into Equation (2.21) is not equal to the counterpart \(\delta \rho\) entering into Equation (2.13). Indeed, the rest mass density \(\rho\) (2.12) is connected with \(\rho_\xi\) (3.20) by means of the relationship

\[
\rho = \frac{1}{1 + \Delta \xi L} \rho_\xi,
\]

where the denominator represents the Jacobian \(\det (\partial x^a / \partial \xi^b)\) of the comoving coordinates transformation. Since \(\rho = \bar{\rho} + \delta \rho\) and \(\rho_\xi = \bar{\rho} + \delta \rho_\xi\), recalling that \(L\) is the first-order quantity, from Equation (3.22) one gets

\[
\delta \rho_\xi = \delta \rho + \bar{\rho} \Delta \xi L = \delta \rho + 3 \bar{\rho} \Phi.
\]

Substitution of Equation (3.23) into Equation (3.21) revives the gauge-independent perturbation (2.13). It is important to remember that positions of the gravitating masses are described by radius-vectors which certainly depend on the choice of comoving coordinates. For instance, apparently, \(r_\xi = \xi_\xi\) in the case of the nontrivial function \(L\) in Equation (3.17).

The initial displacement of particles proposed by Chisari & Zaldarriaga (2011) can be studied in the same vein. Restricting themselves to the linear relativistic perturbation theory for large enough scales where the failure of Newtonian dynamics is expected and striving for the absorption of relativistic effects into the initial conditions for Newtonian simulation codes, the authors took advantage of the transformation of spatial coordinates

\[
x^a = \xi^a + \xi^a_{\xi_\xi}, \quad \frac{\partial}{\partial \xi^a} (\xi^a_{\xi_\xi}) = 3 \xi_{\xi_\xi},
\]

where \(\xi_{\xi_\xi}\) stands for the initial value of the so-called comoving curvature, or curvature perturbation variable (Durrer 2008),

\[
\zeta = \frac{2a \mathcal{H} (\Phi' + \mathcal{H} \Phi)}{\kappa \rho c^2} + \Phi.
\]

Then, the substitution of Equation (3.19), where now \(\Delta \xi L\) is replaced by \(3 \xi_{\xi_\xi}\), into Equation (2.7) gives

\[
\Delta \Phi - 3 \mathcal{H}(\Phi' + \mathcal{H} \Phi) = \frac{\kappa c^2}{2a} [\delta \rho_\xi + 3 \bar{\rho} (\Phi - \zeta_{\xi_\xi})].
\]

Taking into account that the introduced comoving curvature does not evolve at large scales under consideration, one can replace \(\zeta_{\xi_\xi}\) in Equation (2.26) by \(\zeta\) (3.25), and the subsequent cancellation of terms in the obtained equation reduces it to the following form:

\[
\Delta \Phi = \frac{\kappa c^2}{2a} \delta \rho_\xi.
\]

Once again, as it follows from the first equality in Equation (3.23), \(\delta \rho_\xi = \delta \rho + 3 \bar{\rho} \zeta\). Then Equation (2.27) is reduced to its original form before the transformation (2.24), in complete agreement with the gauge invariance of the Bardeen potential.

Summarizing, there are two consistent options for cosmological simulations. On the one hand, one can resort to the initial displacement of particles (Chisari & Zaldarriaga 2011) or the \(N\)-body gauge (Fidler et al. 2015) and reinterpret the large-scale Newtonian \(N\)-body outputs as the relativistic ones. On the other hand, one can remain faithful to the Poisson gauge and calculate the gravitational potential from the Helmholtz equation, in harmony with the reasoning by Hahn & Paranjape (2016; see also the paper by Rampf & Rigopoulos 2013 where the Helmholtz equation links the potential to the density perturbation at scales comparable to the horizon).

### 3.5. Nonzero Spatial Curvature and Screening of Gravity

The promised generalization to both cases of curved spatial geometry can be made straightforwardly. For simplicity and illustration purposes, let us restrict ourselves to Equation (2.27) and rewrite it dropping the velocity contributions (i.e., the second term in the rhs) and taking into consideration the nonzero spatial curvature:

\[
\Delta \Phi + \left( 3 \mathcal{K} - \frac{3 \kappa \rho c^2}{2a} \right) \Phi = \frac{\kappa c^2}{2a} \delta \rho,
\]

where \(\mathcal{K} = +1\) for the spherical (closed) space and \(\mathcal{K} = -1\) for the hyperbolic (open) space, and the Laplace operator is redefined appropriately (see Eingorn & Zhuk 2012; Burgazli et al. 2015). This equation is equivalent to the Equation (2.25) in (Burgazli et al. 2015) up to designations. Hence, one can make use of its solutions derived by Burgazli et al. (2015), simply adjusting the notation. There seems no sense in reproducing these solutions here, but it should be emphasized that they are smooth at any point except at particle positions (where the Newtonian limits are reached) and characterized by zero average values, similarly to the flat space case \(\mathcal{K} = 0\).

One more important detail lies in the fact that the definition of \(\lambda (3.7)\) remains valid not only in the curved space case, but also in the presence of an arbitrary number of additional universe components in the form of perfect fluids with constant or varying parameters in the equations of state like \(p = \omega \rho\) (e.g., radiation with the parameter 1/3), as one can prove (Eingorn & Brilenkov 2015). This hints at the universality of the presentation (3.7). In particular, the gravitational potentials derived by Burgazli et al. (2015) may be interpreted as valid for the universe filled with quintessence with the parameter \(-1/3\).
in the presence of the cosmological constant as well as the nonrelativistic pressureless matter with negligible average energy density.

Returning to the conventional cosmological model, from Equation (3.7) one gets the dependence $\lambda \sim a^2$ at the radiation-dominated stage of the universe evolution. Since $\lambda$ may be associated with the homogeneity scale, as stated above, the asymptotic behavior $\lambda \to 0$ when $a \to 0$ supports the idea of the homogeneous Big Bang.

Finally, it seems almost impossible to overcome the irresistible temptation of associating the Yukawa interaction range $\lambda$ with the graviton Compton wavelength $h/(m_g c)$, where $h$ is the Planck constant and $m_g$ is the graviton mass, in the particle physics spirit. However, one should act with proper circumspection when discussing the massive graviton (see the reasoning by Faraoi & Cooperstock (1998) as well as argumentation by Gazeau & Novello (2011) with respect to Minkowski and de Sitter spacetimes). It is remarkable that setting $\lambda$ equal to $h/(m_g c)$, $h \equiv h/(2\pi)$, gives $m_g = h/(\lambda c) \approx 1.7 \times 10^{-33}$ eV today (when $a = a_0$), and the ratio $1/\lambda^2 = m_g^2 c^2/h^2$ turns out to be numerically equal to $2\Lambda/3$. And vice versa, if one does not initially resort to the known numerical value of $\Omega_M$ and, hence, does not estimate $\lambda$ and $m_g$, the conjectural relationship $m_g^2 c^2/h^2 = 2\Lambda/3$ (see Haranas & Gikigitzis 2014 and references therein) when $a = a_0$ may be rewritten with the help of Equation (3.5) as $9\Omega_M = 4\Omega_\Lambda$, whence in the case of the negligible spatial curvature ($\Omega_M + \Omega_\Lambda = 1$) one gets $\Omega_M = 4/13 \approx 0.31$, in solid agreement with Ade et al. (2014, 2015). It is noteworthy as well that since $\lambda \sim a^{-3/2}$ (3.5), one has $m_g \sim a^{-3/2}$, so $m_g \sim 1/\lambda$ at the matter-dominated stage of the universe evolution (when $a \sim t^2/3$). This dependence on time agrees with that found by Haranas & Gikigitzis (2014). At the radiation-dominated stage $\lambda \sim a^2$, $m_g \sim a^{-2}$. Thus, $m_g \to 0$ when $a \to +\infty$ ($\Lambda$-dominance prevents screening of gravity) and formally $m_g \to +\infty$ when $a \to 0$.

The established finite Yukawa range of the gravitational interaction may potentially pretend to play a key role in resolving the coincidence and cosmological constant problems as well as developing the holographic and inflationary scenarios. Clarification and rigorous substantiation of this role overstep the limits of the current paper.

4. CONCLUSION

The following main results have been obtained in the present paper in the framework of the concordance cosmological model:

1. the first-order scalar (2.40) and vector (2.36) cosmological perturbations, produced by inhomogeneities in the discrete form of a system of separate point-like gravitating masses, are derived in the weak gravitational field limit without any supplementary approximations (no 1/e series expansion, no “dictionaries”);
2. the obtained explicit analytical expressions (2.35) and (2.40) for the metric corrections are valid at all (subhorizon and super-horizon) scales, converge at all points except at locations of the sources (where the appropriate Newtonian limits are reached), and average to zero (no first-order backreaction effects);
3. both the Minkowski background limit (see Equations (3.2) and (3.3)) and the Newtonian cosmological approximation (see Equation (3.4)), which is widely used in modern $N$-body simulations, represent particular limiting cases of the constructed scheme and serve as its corroboration;
4. the velocity-independent part of the scalar perturbation (2.40) contains a sum of Yukawa potentials produced by inhomogeneities with the same finite time-dependent Yukawa interaction range (3.5), which may be connected with the scale of homogeneity, thereby explaining the existence of the largest cosmic structures; and
5. the general Yukawa range definition (3.7) is given for various extensions of the $\Lambda$CDM model (nonzero spatial curvature, additional perfect fluids), and advantages of the established gravity screening are briefly discussed.

Based on the obtained results, it should not be too difficult to construct similarly an appropriate scheme for the second-order cosmological perturbations including the tensor ones. Accomplishment of this quite possible technical mission would predict, in particular, the backreaction effects. It is expected that the second-order metric corrections will be much smaller than the first-order ones at arbitrary scales. Besides, the direct generalization of the elaborated approach to the case of alternative (nonconventional) cosmological models, for example, those replacing the $\Lambda$-term by some other dynamical physical substance, serving as dark energy and also fitting all data, is straightforward and can be made with hardly any trouble. Then, simulating nonlinear dynamics at arbitrary scales, predicting the formation and evolution of large cosmic structures, determining the influence of metric corrections on the propagation of photons through the simulation volume, etc, one can actually probe cosmology and potentially distinguish among different competing representations of the dark sector. Of course, extra effort and care are required for constituting a link between physical quantities extracted from relativistic simulations and observables measured in galaxy surveys such as redshifts and positions in the sky (see, e.g., Bonvin & Durrer 2011; Yoo & Zaldarriaga 2014).

Thus, the developed cosmological perturbations theory covering the whole space, in combination with such future high-precision surveys as Euclid (Scaramella et al. 2015), approaching the Hubble horizon scale, may essentially deepen our knowledge about the amazing world we live in.

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