A SECOND ORDER FRACTIONAL DIFFERENTIAL EQUATION UNDER EFFECTS OF A SUPER DAMPING

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Abstract. In this work we study asymptotic properties of global solutions for an initial value problem of a second order fractional differential equation with structural damping. The evolution equation considered includes plate equation problems. We show asymptotic profiles depending on the exponents of the Laplace operators involved in the equation and optimality of the decay rates for the associated energy and the $L^2$ norm of solutions.

1. Introduction. In this work we consider the Cauchy problem for a generalized second order evolution equation given by

$$
\begin{cases}
  u_{tt} + (-\Delta)^\delta u_{tt} + (-\Delta)^\theta u_t + (-\Delta)^\alpha u = 0, \\
  u(0, x) = u_0(x), \\
  u_t(0, x) = u_1(x),
\end{cases}
$$

with $u = u(t, x), (t, x) \in (0, \infty) \times \mathbb{R}^n (n \geq 1)$ and the exponents of the Laplace operators $\alpha, \delta$ and $\theta$ satisfying $\alpha > 0$ and $0 < \delta \leq \alpha$, $\frac{\alpha + \delta}{2} \leq \theta < \alpha + \delta$ or $\alpha < \theta$ if $\delta = 0$.

The Cauchy problem we consider in (1.1) models several physical phenomena, especially hydrodynamic problems such as propagation of waves in shallow waters. Problems of plate vibrations, wave propagation in general are also modeled by this type of equation as it appears in (1.1), depending on the choice of the fractional exponents of the Laplace operators. Several other applications can be studied by this model, for example in [19] (see also [1]) Maugin proposed a type of Boussinesq model to treat dynamics of nonlinear networks in elastic crystals. In Sander-Huter

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a brief account of the history of the evolution of these problems in hydrodynamics can be found, especially in the theory of solitary waves. The first attempt of mathematical modeling of waves in shallow water was made by Joseph Valentin Boussinesq (1842-1929) in 1871, who took into account the vertical accelerations of the fluid particles and admitted a solitary wave as a solution. However, he neglected a number of derivative product terms in his modeling, considering only nonlinear first-order effects. These equations are called classical Boussinesq equations. In order to obtain better models, some of these hydrodynamic phenomena and other equations were later developed, originating from the classical equations, considering higher order terms that Boussinesq ignored. These equations are called the Boussinesq class. Among them these are equations of Korteweg-de Vries studied by Green-Naghdi [8], Peregrine [23] and Serre [27]. Other equations related to those of Boussinesq, are the so-called IBq (improved Boussinesq) and IMBq (improved modified Boussinesq). More details on Boussinesq equations and their characteristics can be found in the articles due to Esfahani-Farah-Wang [5], Wang-Chen [33], Wang-Xue [36], Wang-Xu [34] and [35].

For the model we consider in this work, when \( \delta = \alpha = 2 \) we have a sixth order linear Boussinesq equation and when, in addition, \( \theta = 1 \), we have a Boussinesq equation under the effect of a hydrodynamic dissipation. We note that when \( \delta = 1, \alpha = 2, \theta = 0 \), we have in (1.1) a linear equation for plate vibrations under rotational inertial effects (see [6]) and frictional dissipation. Particular results on some problems associated with equations together with these properties can be found in Charão-Luz-Ikehata [2], Luz-Charão [17], Sugitani-Kawashima [29] and the references therein. Moreover, when \( \alpha = 1, \delta = \theta = 0 \) we have a linear wave equation with frictional dissipation.

The results in this paper include regularity-loss type of estimates, which have their origin in [10, 11, 21], and seem to be new for general equations considered here. For particular cases of our problem we can cite several works [2, 3, 14, 15, 16, 17, 20, 28, 29, 30, 33], and especially Horbach-Ikehata-Charão [9]. Some of our mathematical tools and ideas that are used to develop part of this work follow several ideas from those works.

We also mention that our work generalizes the works by Ikehata [13, 12] which gave us the idea to study the system with strong damping considered in this work. We note that in [13] Ikehata studies only the case \( \theta = 2, \alpha = 1 \) and \( \delta = 0 \), so even in the case of (1.1) with \( \theta = 2, \alpha = 1, \delta = 1 \) (rotational inertia effect exists):

\[
\begin{align*}
\left\{ \begin{array}{l}
\partial_{tt} u + (\Delta)^\theta u_t + (\Delta)^\alpha u = 0, \\
u(0, x) = u_0(x), \\
u_t(0, x) = u_1(x),
\end{array} \right.
\end{align*}
\]

(1.2)

the corresponding study seems to be quite new. Important works that are deserved to be mentioned are special cases of the equation (1.1), and are reported in the references of the classic works due to Matsumura [20] and Ponce [25]. Several results on the IMBq and the generalized Boussinesq equations can be seen in Wang-Chen [32, 33] and in Wang-Xue [36], respectively.

This paper is organized as follows. In Section 2 we collect several facts and notation, which will be used in the next sections. In Section 3 we study the special case of (1.1) with \( \delta = 0 \) and \( \theta > \alpha \), that is,
and mainly prove the following result, which shows the optimality of the decay rates of the solution in $L^2$-sense to problem (1.2). In this connection, one can cite recent results due to [4] and [24] on the precise $L^1 \cap L^m - L^n$ ($m \in (1, 2)$) and $L^p - L^q$ estimates of solutions to the equation (1.2) with $\theta \in (0, \alpha]$ and $\alpha \geq 1$. The case of [4] and [24] does not include such regularity-loss structure.

**Theorem 1.1.** Let $n > 2\alpha$, $\theta > \alpha > 0$, $\kappa > \frac{(n-2\alpha)(\theta-\alpha)}{2\theta}$, $\epsilon \in (0, \min\{1, \alpha\})$. If $u_0 \in L^1(\mathbb{R}^n) \cap H^\kappa(\mathbb{R}^n)$, $u_1 \in L^{1,\epsilon}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \cap W^{\kappa-\alpha, 2}(\mathbb{R}^n)$ then the solution $u(t, x)$ to problem (1.2) satisfies

$$C_1 \left| \int_{\mathbb{R}^n} u_1(x) dx \right| t^{-\frac{n-2\alpha}{2\theta}} \leq \|u(t, \cdot)\|_{L^2} \leq C_2 I_0 t^{-\frac{n-2\alpha}{2\theta}},$$

for all $t \gg 1$, where $C_1$ and $C_2$ are positive constants, and

$$I_0 := \|u_0\|_{L^1} + \|u_0\|_{H^\kappa} + \|u_1\|_{L^{1,\epsilon}} + \|u_1\|_{L^2} + \|u_1\|_{W^{\kappa-\alpha, 2}}.$$

**Remark 1.** Note that for $\kappa - \alpha \geq 0$, then $L^2(\mathbb{R}^n) \cap W^{\kappa-\alpha, 2}(\mathbb{R}^n) = H^{\kappa-\alpha}(\mathbb{R}^n)$, that is, the regularity of the initial data $u_1$ depend on the choice of $\kappa$.

In Section 4 we consider the general case $0 < \delta \leq \alpha$, and the condition to strong damping we consider is $\frac{\alpha + \delta}{2} \leq \theta < \alpha + \delta$. In fact, we can report the following general result, in the case of $\delta > 0$ and $\frac{\alpha + \delta}{2} \leq \theta < \alpha + \delta$, concerning the optimality of the $L^2$-norm of solutions to problem (1.1).

**Theorem 1.2.** Let $n > 2\alpha$, $0 < \delta \leq \alpha$, $\frac{\alpha + \delta}{2} \leq \theta < \alpha + \delta$, $\kappa \in (0, \min\{1, \delta\})$ and $\epsilon > \frac{2\kappa - \alpha}{2\theta}(n - 2\alpha)$. Assume $u_0 \in L^1(\mathbb{R}^n) \cap H^\kappa(\mathbb{R}^n)$, $u_1 \in L^{1,\epsilon}(\mathbb{R}^n) \cap H^{\delta+\kappa-\alpha}(\mathbb{R}^n) \cap W^{-\alpha, 2}(\mathbb{R}^n)$. Then there exist constants $C_1 > 0$, $C_2 > 0$ and $t_0 \gg 1$ such that

$$C_1 \left| \int_{\mathbb{R}^n} u_1(x) dx \right| t^{-\frac{n-2\alpha}{2\theta}} \leq \|u(t, \cdot)\|_{L^2} \leq C_2 I_1 t^{-\frac{n-2\alpha}{2\theta}},$$

holds for $t \geq t_0$, where $u(t, x)$ is the solution to problem (1.1), and

$$I_1 := \|u_0\|_{L^1} + \|u_0\|_{H^\kappa} + \|u_1\|_{L^{1,\epsilon}} + \|u_1\|_{W^{-\alpha, 2}} + \|u_1\|_{H^{\delta+\kappa-\alpha}}.$$

**Remark 2.** The case $0 \leq \theta \leq \frac{\alpha + \delta}{2}$ was essentially studied in [9]. The case $\theta \geq \alpha + \delta$ is still open. It should be emphasized that the results in Theorem 1.2 are essentially new under the regularity-loss type structure of the equations with rotational inertia effects. The new result just obtained above includes the case of $\delta = 1$, $\alpha = 1$, and $\theta \in (1, 2)$ in (1.1) as an example.

In order to obtain the decay rates, we have employed the method of energy in the Fourier space (see [31]) combined with the explicit solution of the associated problem in the Fourier space, and an asymptotic profile obtained from the explicit solution. Our aim is mainly concentrated on proving the optimality of decay rates for the $L^2$ norm of solutions as shown in above theorem, although we can also prove the optimality for decay rates of the $L^2$ norm of the derivatives of solutions by using the same method.

2. Notation and basic estimates. We consider the following spaces

$$L^{1,\gamma}(\mathbb{R}^n) = \left\{ u \in L^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |x|^\gamma)|u(x)| dx < \infty \right\},$$

for $\gamma > 0$;

$$\dot{W}^{m, p}(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) : \exists v \in L^p(\mathbb{R}^n), \text{tal que } u = (-\Delta)^{-\frac{m}{2}}v \right\},$$
for \( m, p \in \mathbb{R}^n \), with \( p \geq 1 \) and its norm given by
\[
\|u\|_{W^{m,p}} = \left( \int_{\mathbb{R}^n} \left| (-\Delta)^{\frac{m}{2}} u(x) \right|^p dx \right)^{\frac{1}{p}}.
\]
Note that by definition \( u \in W^{m,p}(\mathbb{R}^n) \) implies \( v = (-\Delta)^{\frac{m}{2}} u \in L^p(\mathbb{R}^n) \).

We also consider the usual Sobolev spaces \( H^s(\mathbb{R}^n), s \in \mathbb{R} \), with norms given by
\[
\|u\|_{H^s} = \left( \int_{\mathbb{R}^n} (1 + |\xi|^{2s}) |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}, \quad s \geq 0;
\]
\[
\|u\|^2_{H^{-s}} = \int_{\mathbb{R}^n} (1 + |\xi|^{2s})^{-1} |\hat{u}|^2 d\xi, \quad s > 0.
\]
where \( \mathcal{F} u = \hat{u} \) represents the usual Fourier transform of \( u \).

The following lemma justifies the equivalence between the norms given above and the usual norm in \( H^s \).

**Lemma 2.1.** For all \( \xi \in \mathbb{R}^n \) and \( s \geq 0 \) it holds that

i) \( \frac{1}{2}(1 + |\xi|^{2s}) \leq (1 + |\xi|^2)^s \leq 2(1 + |\xi|^{2s}), \)

ii) \( 2^{-s}(1 + |\xi|^{2s})^{-1} \leq (1 + |\xi|^2)^{-s} \leq 2(1 + |\xi|^{2s})^{-1}. \)

For \( u \in H^s(\mathbb{R}^n) \) the operator \( (-\Delta)^{\frac{s}{2}} \) is defined via the Fourier transform by
\[
(-\Delta)^{\frac{s}{2}} u(x) = \mathcal{F}^{-1} \left[ (1 + |\xi|^2)^s \hat{u}(\xi) \right](x), \quad x \in \mathbb{R}^n.
\]

With this definition we can characterize the Fourier transform of \( (-\Delta)^{\frac{s}{2}} u \), \( u \in H^s(\mathbb{R}^n) \), as
\[
\mathcal{F}((-\Delta)^{\frac{s}{2}} u)(\xi) = |\xi|^{2s} \hat{u}(\xi), \quad \xi \in \mathbb{R}^n.
\]

The next two lemmas can be proved in a standard way (see Horbach-Ikehata-Charão [9]).

**Lemma 2.2.** Let \( f \in L^1(\mathbb{R}^n) \). Then it holds that \( \hat{f}(\xi) = A_f(\xi) - iB_f(\xi) + P_f \), for all \( \xi \in \mathbb{R}^n \), where \( i := \sqrt{-1} \),

- \( A_f(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (\cos(x \cdot \xi) - 1) f(x) dx \),
- \( B_f(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \sin(x \cdot \xi) f(x) dx \),
- \( P_f = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) dx \).

**Lemma 2.3.** i) If \( f \in L^1(\mathbb{R}^n) \), then for all \( \xi \in \mathbb{R}^n \) it is true that
\[
|A_f(\xi)| \leq L \|f\|_{L^1} \quad \text{and} \quad |B_f(\xi)| \leq N \|f\|_{L^1}.
\]

ii) If \( 0 < \kappa \leq 1 \) and \( f \in L^{1,\kappa}(\mathbb{R}^n) \) then for all \( \xi \in \mathbb{R}^n \) it is true that
\[
|A_f(\xi)| \leq K|\xi|^\kappa \|f\|_{L^{1,\kappa}} \quad \text{and} \quad |B_f(\xi)| \leq M|\xi|^\kappa \|f\|_{L^{1,\kappa}}
\]
with \( L, N, K \) and \( M \) positive constants depending only on the dimension \( n \) and \( \kappa \).

The next lemma is well known.

**Lemma 2.4.** Let \( k > -n, \vartheta > 0 \) and \( C > 0 \). Then there exists a constant \( K > 0 \) depending on \( n \) such that
\[
\int_{|\xi| \leq 1} e^{-\vartheta|\xi|^k} |\xi|^k d\xi \leq K (1 + t)^{-\frac{n+k}{\vartheta}},
\]
for all \( t > 0 \).
The proofs of next two lemmas appear in [9] for the case $\alpha = 1$. The case $\alpha > 0$ is quite similar.

**Lemma 2.5.** Let $n > 2\alpha$ and $\theta > \frac{\alpha}{2}$. Then there exists $t_0 > 0$ such that
\[
\int_{\mathbb{R}^n} e^{\frac{|\xi|^2}{1+|\xi|^2}} \frac{\sin(|\xi|^\alpha t)^2}{|\xi|^{2\alpha}} \geq C t^{-\frac{n-2\alpha}{2\theta}},
\]
for $t \geq t_0$ with $C$ a positive constant depending only on $n$, $\theta$ and $\alpha$.

**Lemma 2.6.** Let $n \geq 1$ and $\theta > \frac{\alpha}{2}$. Then there exists $t_0 > 0$ such that
\[
\int_{\mathbb{R}^n} e^{-\frac{|\xi|^2}{1+|\xi|^2}} |\cos(|\xi|^\alpha t)|^2 \geq C t^{-\frac{n}{2\theta}},
\]
for $t \geq t_0$, where $C$ is a positive constant depending only on $n$, $\theta$ and $\alpha$.

Our goal in the next sections is to obtain decay rates, an asymptotic expansion (profile) and to prove the optimality of the decay rates to the Cauchy problem (1.1) with the exponent $\theta$ of the damping term to be large.

Now we are going to study optimal decay rates for the $L^2$ norm of solutions. Decay rates for $L^2$ norms of the derivatives are also obtained in this work, but we do not prove the optimality. Indeed, by using the same method and similar estimates to the ones for $L^2$-norm of solutions the optimality can be proved more easily. We begin with considering two cases on the exponents in (1.1):

- **Case** $\delta = 0$, $0 < \alpha < \theta$.
- **Case** $0 < \delta \leq \alpha$, $\frac{\alpha + \delta}{2} \leq \theta < \alpha + \delta$.

We note that in case $\delta > 0$ the restriction $\theta < \alpha + \delta$ is considered to get the optimality of the decay rates. The optimality for the case $\alpha + \delta \leq \theta$ we leave it open, however, to get the decay rates such additional restriction is not necessary.

3. **Case** $\delta = 0$ and $\theta > \alpha$. In this section we prove $L^2$ decay rates to the linear problem (1.1) when $\delta = 0$, that is to a Cauchy problem of type
\[
\begin{aligned}
&u_{tt} + (-\Delta)^\alpha u + (-\Delta)^\theta u_t = 0, \\
&u(0,x) = u_0(x), \\
&u_t(0,x) = u_1(x),
\end{aligned}
\]
with $(t,x) \in (0,\infty) \times \mathbb{R}^n$ and the assumption $\theta > \alpha > 0$.

The results of this section generalize the work by Ikehata-Iyota [13] that considered the particular case $\alpha = 1$ and $\theta = 2$.

To get our estimates we work with the Cauchy problem in the Fourier space given by
\[
\begin{aligned}
&\hat{u}_{tt} + |\xi|^{2\alpha} \hat{u} + |\xi|^{2\theta} \hat{u}_t = 0, \\
&\hat{u}(0,\xi) = \hat{u}_0(\xi), \\
&\hat{u}_t(0,\xi) = \hat{u}_1(\xi),
\end{aligned}
\]
where $\hat{u} = \hat{u}(t,\xi)$, $(t,\xi) \in (0,\infty) \times \mathbb{R}^n$ and $0 < \alpha < \theta$.

**Remark 3** (Existence of solutions). On the existence of solutions to the Cauchy problem (3.1) we note that we can write it in an abstract form as follows
\[
\begin{aligned}
&u_{tt} + Au + 2a A^\nu u_t = 0, \\
&u(0,x) = u_0(x) \in D(A^\alpha), \\
&u_t(0,x) = u_1(x) \in D(A^\nu),
\end{aligned}
\]
where $a = \frac{1}{2} > 0$, $v = \frac{\theta}{a} > 1$, $A = (-\Delta)^{\alpha}$ and $\alpha_0 \geq 0$, $\alpha_1 \geq 0$.

From the definition of the operator $A$ we may see that it is self-adjoint and non-negative. In addition, for initial data in the energy space $X = H^\alpha(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ we have $\alpha_0 = 1/2$, $\alpha_1 = 0$. Then, considering $\tau = \max\{\frac{1}{2}, v\} = v = \frac{\theta}{\alpha} > 1$ due to $\theta > \alpha$, we see that

$$1 - \tau < 0 < \alpha_0 - \alpha_1 = 1/2 < \tau.$$ 

Thus, by Theorem 2.1 in [7] we have the existence of a unique solution $u = u(t, x)$, which verifies

(a) $(u, u_t) \in C([0, \infty), H^{\alpha}(\mathbb{R}^n) \times L^2(\mathbb{R}^n))$;

(b) for $m \geq 1$ integer, the $m$-th derivative in time $D_t^m u(t, x)$ satisfies

$$D_t^m u \in C((0, \infty), D(A^{1+m-\theta})),$$

This regularity says that $u_t \in C((0, \infty), D((-\Delta)^\theta))$. Note that the item (b) is the regularizing effect of the strong damping.

3.1. Decay Estimates. We apply an improvement (cf. [9]) of the energy method in the Fourier space (cf. [31]) to obtain the decay estimates. For this purpose, set

$$e_0(t, \xi) = \frac{1}{2} |\hat{u}_t|^2 + \frac{1}{2} |\xi|^{2\alpha} |\hat{u}|^2,$$  \hspace{1cm} (3.3)

which defines the associated density of energy in the Fourier space, defined for $t \geq 0$ and $\xi \in \mathbb{R}^n$. Furthermore, one defines the important function $\rho : \mathbb{R}^n \to [0, \infty)$ by

$$\rho(\xi) = \frac{|\xi|^{2\theta}}{1 + |\xi|^{4\theta - 2\alpha}}.$$  \hspace{1cm} (3.4)

A way to choose the best function $\rho(\xi)$ can be seen in [18]. Then as in the similar argument to [13] and/or [18] one can arrive at the following significant energy estimate in the Fourier space.

Lemma 3.1. Let $\theta > \alpha > 0$. Then, there exist positive constants $C = C(b)$ and $\omega = \omega(b)$ depending on $0 < b < 1$ such that

$$e_0(t, \xi) \leq Ce^{-\omega(\xi)^t}e_0(0, \xi)$$

holds for all $\xi \in \mathbb{R}^n$ and $t \geq 0$.

Applying the above Lemma 3.1 we can prove the next result on the total energy. The result below is a generalization to $\alpha > 0$ of that already obtained for the case of $\alpha = 1$ in [13]. We state it without proof because the proof is almost the same as that of [13].

Theorem 3.2. Assume $n \geq 1$, $\theta > \alpha > 0$ and $\kappa \geq 0$. If $u_0 \in H^{\kappa+1}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, $u_1 \in H^{\kappa}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ then there exists a constant $C > 0$ such that

$$\|u(t, \cdot)\|^2 + \|(-\Delta)^{\frac{\theta}{2}} u(t, \cdot)\|^2 \leq C(1 + t)^{-\frac{\theta}{2}} \|u_1\|^2_{L^1} + C(1 + t)^{-\frac{n-2\alpha}{2}} \|u_0\|^2_{L^1} + C(1 + t)^{-\frac{n}{4\theta}} (\|u_1\|^2_{H^\kappa} + \|u_0\|^2_{H^{\kappa+1}}), \hspace{1cm} t > 0.$$

On the asymptotic behavior of the $L^2$-norm of solutions one has the following result by basing on Lemma 3.1. One also mentions without proof (see [13]).

Theorem 3.3. Let $n > 2\alpha$, $\theta > \alpha > 0$, $\kappa \geq \alpha$. If $u_0 \in H^{\kappa}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, $u_1 \in H^{\kappa-\alpha}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that

$$\|u(t, \cdot)\|^2 \leq C(1 + t)^{-\frac{n-2\alpha}{2}} \|u_1\|^2_{L^1} + C(1 + t)^{-\frac{n}{4\theta}} \|u_0\|^2_{L^1}.$$
\[ + C(1 + t)^{-\frac{n}{\alpha}} \left( \|u_1\|_{H^{s_\nu, \alpha}}^2 + \|u_0\|_{H^{s_\nu}}^2 \right), \quad t > 0. \]

**Remark 4.** In order to impose less regularity on the initial data we may only consider \( u_1 \in \dot{W}^{\kappa - \alpha, 2}(\mathbb{R}^n) \) with \( \kappa > 0 \). In this case we get the following estimate on the zone of high frequency

\[
\int_{\|\xi\| \geq 1} |\hat{u}(t, \xi)|^2 d\xi \leq C(1 + t)^{-\frac{n}{\alpha}} \left[ \|\hat{u}_1\|_{\dot{W}^{\kappa - \alpha, 2}}^2 + \|\hat{u}_0\|_{\dot{W}^{\kappa, 2}}^2 \right] d\xi
\]

\[
\leq C(1 + t)^{-\frac{n}{\alpha}} \left[ \int_{\mathbb{R}^n} \left\| (-\Delta)^{\frac{\kappa - \alpha}{2}} u_1(\xi) \right\|^2 d\xi + \|u_0\|^2_{H^s} \right]
\]

\[
= C(1 + t)^{-\frac{n}{\alpha}} \left( \|u_1\|_{\dot{W}^{\kappa - \alpha, 2}}^2 + \|u_0\|_{H^s}^2 \right), \quad t > 0.
\]

#### 3.2. Asymptotic expansion: estimates for low frequencies.

In this subsection we shall consider the asymptotic expansion of the solution in the \( L^2 \)-norm. The result in this subsection is a generalization of [13], which dealt with the case of \( \delta = 0, \alpha = 1, \) and \( \theta = 2 \). For the sake of reader’s convenience we shall state its full proof based on Lemma 3.1.

We first observe that the characteristic equation associated to the Cauchy problem (3.2) in the Fourier space is given by

\[ \lambda^2 + |\xi|^{2\theta} \lambda + |\xi|^{2\alpha} = 0, \]

for \( \alpha > 0 \) and \( \theta \geq 0 \).

We are interested in the case \( \theta > \alpha > 0 \). The associated characteristics roots are

\[ \lambda_{\pm} = \frac{-|\xi|^{2\theta} \pm |\xi|^{\alpha} \sqrt{|\xi|^{4\theta - 2\alpha} - 4}}{2}. \]

We fix \( 0 < \delta_0 < 1 \). In this case we have \( |\xi|^{4\theta - 2\alpha} - 4 \leq \delta_0^{4\theta - 2\alpha} - 4 \leq -3 < 0 \) for \( \xi \in \mathbb{R}^n \) such that \( |\xi| \leq \delta_0 < 1 \). Thus, the the characteristics roots are complex and we can write them as

\[ \lambda_{\pm} = -a(\xi) + ib(\xi), \quad |\xi| \leq \delta_0 < 1, \]

where

\[ a(\xi) = \frac{|\xi|^{2\theta}}{2}, \quad b(\xi) = \frac{|\xi|^{\alpha} \sqrt{4 - |\xi|^{4\theta - 2\alpha}}}{2}. \]

The explicit solution of (3.2) is given by

\[ \hat{u}(t, \xi) = \hat{H}(t, \xi)\hat{u}_0 + \hat{G}(t, \xi)\hat{u}_1, \quad |\xi| \leq \delta_0 < 1, \]

where

\[ \hat{H}(t, \xi) = \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} = e^{-a(\xi)t} \frac{a(\xi)}{b(\xi)} \sin(b(\xi)t) + e^{-a(\xi)t} \cos(b(\xi)t), \]

\[ \hat{G}(t, \xi) = \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} = e^{-a(\xi)t} \frac{1}{b(\xi)} \sin(b(\xi)t). \]

(3.5)

Now, using Lemma 2.2 for \( \hat{u}_0 \) and \( \hat{u}_1 \) we define the functions

\[ K_1(t, \xi) = P \frac{e^{-a(\xi)t} a(\xi)}{b(\xi)} \sin(b(\xi)t), \]

\[ K_2(t, \xi) = \left( A_1(\xi) - iB_1(\xi) \right) \hat{G}(t, \xi), \]

\[ K_3(t, \xi) = \left( A_0(\xi) - iB_0(\xi) \right) \hat{H}(t, \xi), \]
where \( \hat{u}_j = A_j(\xi) - iB_j(\xi) + P_j, \ i = 0, 1, \) and

\[
P_j := \int_{\mathbb{R}^n} u_j(x) \, dx \quad (j = 0, 1).
\]

Applying the mean value theorem on the functions \( \sin(b(\xi)t) \) and \( \cos(b(\xi)t) \) we get that

\[
\hat{u}(t, \xi) = P_1 e^{-a(\xi)t} \frac{1}{b(\xi)} \sin(|\xi|^\alpha t) + P_0 e^{-a(\xi)t} \cos(|\xi|^\alpha t)
\]
\[
+ P_1 e^{-a(\xi)t} \frac{1}{b(\xi)} t \left( b(\xi) - |\xi|^\alpha \right) \cos(\epsilon(\xi)t) - P_0 e^{-a(\xi)t} t \left( b(\xi) - |\xi|^\alpha \right) \sin(\eta(\xi)t)
\]
\[
+ \sum_{j=1}^{3} K_j(t, \xi), \ |\xi| \leq \delta_0 < 1, \ t > 0.
\]

To obtain an asymptotic expansion for \( \hat{u}(t, \xi) \) we need to deal with \( b^{-1}(\xi) \) in the first term of the right hand side of \( \hat{u}(t, \xi) \). Using the mean value theorem we can see that

\[
\frac{2}{\sqrt{4 - |\xi|^{4\theta-2\alpha}}} - 1 = |\xi|(4\theta - 2\alpha) \frac{\mu^{4\theta-2\alpha-1}}{(4 - \mu^{4\theta-2\alpha})^2}, \ (\mu = \beta'|\xi|, \beta' \in (0, 1)).
\]

That is

\[
\frac{2}{\sqrt{4 - |\xi|^{4\theta-2\alpha}}} = 1 + (4\theta - 2\alpha)(\beta')^{4\theta-2\alpha-1} \frac{|\xi|^{4\theta-2\alpha}}{(4 - (\beta')^{4\theta-2\alpha}|\xi|^{4\theta-2\alpha})^2}.
\]

Therefore, we have for \( |\xi| \leq \delta_0 < 1, \ t > 0 \)

\[
\hat{u}(t, \xi) = P_1 e^{-a(\xi)t} \frac{\sin(|\xi|^\alpha t)}{|\xi|^\alpha} + P_0 e^{-a(\xi)t} \cos(|\xi|^\alpha t)
\]
\[
+ P_1 e^{-a(\xi)t} (4\theta - 2\alpha) \frac{(\beta')^{4\theta-2\alpha-1}|\xi|^{4\theta-2\alpha}}{(4 - (\beta')^{4\theta-2\alpha}|\xi|^{4\theta-2\alpha})^2} \sin(|\xi|^\alpha t)
\]
\[
+ P_1 e^{-a(\xi)t} \frac{1}{b(\xi)} t \left( b(\xi) - |\xi|^\alpha \right) \cos(\epsilon(\xi)t)
\]
\[
- P_0 e^{-a(\xi)t} t \left( b(\xi) - |\xi|^\alpha \right) \sin(\eta(\xi)t) + \sum_{j=1}^{3} K_j(t, \xi).
\]

To simplify the notation we define the functions

\[
K_4(t, \xi) = P_1 e^{-a(\xi)t} \frac{1}{b(\xi)} t(b(\xi) - |\xi|^\alpha) \cos(\epsilon(\xi)t),
\]
\[
K_5(t, \xi) = -P_0 e^{-a(\xi)t} t(b(\xi) - |\xi|^\alpha) \sin(\eta(\xi)t),
\]
\[
K_6(t, \xi) = P_1 e^{-a(\xi)t} (4\theta - 2\alpha) \frac{(\beta')^{4\theta-2\alpha-1}|\xi|^{4\theta-2\alpha}}{(4 - (\beta')^{4\theta-2\alpha}|\xi|^{4\theta-2\alpha})^2} \sin(|\xi|^\alpha t).
\]

We need \( L^2 \) estimates for \( K_j(t, \xi), \ (i = 1, \ldots, 6) \) on the zone of low frequency \( |\xi| \leq \delta_0 < 1 \) and \( t > 0 \). We use Lemmas 2.3 and 2.4 and the fact that \( a(\xi) = \frac{|\xi|^{2\theta}}{2} \).

From the estimate

\[
\left| \frac{a(\xi)}{b(\xi)} \right|^2 = \frac{|\xi|^{2\theta-\alpha}}{\sqrt{4 - |\xi|^{4\theta-2\alpha}}}^2 \leq \frac{1}{3} |\xi|^{4\theta-2\alpha},
\]
we can easily obtain

\[ \int_{|\xi| \leq \delta_0} |K_1(t, \xi)|^2 d\xi \leq \frac{1}{3} \|u_0\|_{L^2}^2 t^{-\frac{n+4\theta-2\alpha}{2\theta}}. \]

To estimate \( K_2 \) we note that \( |\hat{G}(t, \xi)|^2 \leq \frac{4}{3} |\xi|^{-2\alpha} e^{-|\xi|^{2\theta} t} \). Then for \( \epsilon \in (0, \min\{1, \alpha\}) \) it holds

\[ \int_{|\xi| \leq \delta_0} |K_2(t, \xi)|^2 d\xi \leq \frac{4}{3} \int_{|\xi| \leq \delta_0} |A_1(\xi) - iB_1(\xi)|^2 |\xi|^{-2\alpha} e^{-|\xi|^{2\theta} t} d\xi \]

\[ \leq C \|u_1\|_{L^2}^2 t^{-\frac{n+4\theta-2\alpha}{2\theta}} \leq C \|u_2\|_{L^2}^2 t^{-\frac{n+4\theta-2\alpha}{2\theta}}. \]

To estimate \( K_3 \) we get first that \( |\hat{H}(t, \xi)|^2 \leq C e^{-|\xi|^{2\theta} t} |\xi|^{4\theta-2\alpha} + e^{-|\xi|^{2\theta} t} \). Then

\[ \int_{|\xi| \leq \delta_0} |K_3(t, \xi)|^2 d\xi \leq C \|u_0\|_{L^2}^2 \int_{|\xi| \leq \delta_0} e^{-|\xi|^{2\theta} t} |\xi|^{4\theta-2\alpha} d\xi + C \|u_0\|_{L^2}^2 \int_{|\xi| \leq \delta_0} e^{-|\xi|^{2\theta} t} d\xi \]

\[ \leq C \|u_0\|_{L^2}^2 t^{-\frac{n+4\theta-2\alpha}{2\theta} + C \|u_0\|_{L^2}^2 t^{-\frac{n+4\theta-2\alpha}{2\theta}}}. \]

To estimate \( K_4 \) we use

\[ \left| \frac{b(\xi) - |\xi|^{\alpha}}{b(\xi)} \right|^2 \leq \frac{1}{3} \left( \frac{|\xi|^{4\theta-2\alpha}}{\sqrt{4} - |\xi|^{4\theta-2\alpha}} \right)^2 \leq C |\xi|^{8\theta-4\alpha}. \]

In fact,

\[ \int_{|\xi| \leq \delta_0} |K_4(t, \xi)|^2 d\xi \leq C \|P_1\|^2 t^2 \int_{|\xi| \leq \delta_0} e^{-|\xi|^{2\theta} t} |\xi|^{8\theta-4\alpha} d\xi \]

\[ \leq C \|u_1\|_{L^2}^2 t^2 t^{-\frac{n+4\theta-2\alpha}{2\theta}} = C \|u_1\|_{L^2}^2 t^{-\frac{n+4\theta-2\alpha}{2\theta}}. \]

From \( |b(\xi) - |\xi|^{\alpha}|^2 \leq C |\xi|^{8\theta-4\alpha} \), we obtain the following estimate for \( K_5 \).

\[ \int_{|\xi| \leq \delta_0} |K_5(t, \xi)|^2 d\xi \leq C \|P_0\|^2 t^2 \int_{|\xi| \leq \delta_0} e^{-|\xi|^{2\theta} t} |\xi|^{8\theta-2\alpha} d\xi \leq C \|u_0\|_{L^2}^2 t^{-\frac{n+4\theta-2\alpha}{2\theta}}. \]

Finally, to estimate \( K_6(t, \xi) \) we prepare the estimate

\[ \left( 4\theta - 2\alpha \right) \frac{\left( \beta' \right)^{4\theta-2\alpha-1}|\xi|^{4\theta-2\alpha}}{\left( 1 - \beta' \right)^{4\theta-2\alpha} |\xi|^{4\theta-2\alpha}} \right)^2 \leq \frac{(4\theta - 2\alpha)^2}{\sqrt{27}} |\xi|^{8\theta-4\alpha}. \]

It holds because of \( \theta > \alpha, 0 < \beta' < 1, |\xi| \leq \delta_0 \) and \( 0 < \delta_0 < 1 \). Thus, we get

\[ \int_{|\xi| \leq \delta_0} |K_6(t, \xi)|^2 d\xi \leq C \|P_1\|^2 \int_{|\xi| \leq \delta_0} e^{-|\xi|^{2\theta} t} |\xi|^{8\theta-6\alpha} d\xi \leq C \|u_1\|_{L^2}^2 t^{-\frac{n+8\theta-6\alpha}{2\theta}}. \]

The above estimates imply the following estimate for an asymptotic profile to the solution \( \hat{u} \) on the low frequency zone in the Fourier space.

**Lemma 3.4.** Let \( n \geq 1, \epsilon \in (0, \min\{1, \alpha\}) \). Then there exists a constant \( C > 0 \) such that

\[ \int_{|\xi| \leq \delta_0} \left\| \hat{u}(t, \xi) - \left\{ P_1 e^{-\alpha(\xi) t} \frac{\sin(|\xi|^{\alpha} t)}{|\xi|^{\alpha}} + P_0 e^{-\alpha(\xi) t} \cos(|\xi|^{\alpha} t) \right\} \right\|^2 d\xi \]

\[ \leq C \|u_1\|_{L^2}^2 t^{-\frac{n+2\alpha+2\theta}{2\theta}} + C \|u_0\|_{L^2}^2 t^{-\frac{n+4\theta-2\alpha}{2\theta}}, \quad t > 0. \]
3.3. Estimates for high frequencies. Considering $\delta_0$ as in the previous subsection and the definition of $\rho(\xi)$ in (3.4) we have

$$\rho(\xi) = \frac{\|\xi\|^{2\theta}}{1 + \|\xi\|^{4\theta - 2\alpha}} \geq \frac{|\xi|^{2\theta}}{2} \geq \frac{\delta_0^{2\theta}}{2}, \quad \delta_0 \leq |\xi| \leq 1,$$

$$\rho(\xi) = \frac{\|\xi\|^{2\theta}}{1 + |\xi|^{4\theta - 2\alpha}} \geq \frac{|\xi|^{2\theta}}{2|\xi|^{4\theta - 2\alpha}} = \frac{1}{2|\xi|^{2\theta - 2\alpha}}, \quad |\xi| \geq 1.$$ 

Thus, from Lemma 3.1 we get

$$\int_{|\xi| \geq \delta_0} |\hat{u}(t, \xi)|^2 d\xi \leq C \int_{|\xi| \leq 1} \frac{e^{-\frac{\delta_0^{2\theta}}{t}}}{|\xi|^{2\alpha}} \left( \frac{|\hat{u}_1|^2}{|\xi|^{2\alpha}} + |\hat{u}_0|^2 \right) d\xi + C \int_{|\xi| \geq 1} \frac{e^{-\frac{\delta_0^{2\theta}}{t}}}{|\xi|^{2\alpha}} \left( \frac{|\hat{u}_1|^2}{|\xi|^{2\alpha}} + |\hat{u}_0|^2 \right) d\xi \leq C e^{-\frac{\delta_0^{2\theta}}{t}} (\|u_1\|^2 + \|u_0\|^2) + C t^{-\frac{\sigma}{\theta}} (\|u_1\|^2_{W^{\kappa - \alpha, 2}} + \|u_0\|^2_{H^n}),$$

where the integral on the high frequency was estimated easily by the same method as that of Theorem 3.3.

On the other hand, it is easy to see that

$$\int_{|\xi| \geq \delta_0} |P_1|^2 e^{-|\xi|^{2\theta} \sin^2(|\xi|^{\alpha} t)} d\xi + \int_{|\xi| \geq \delta_0} |P_0|^2 e^{-|\xi|^{2\theta} \cos^2(|\xi|^{\alpha} t)} d\xi \leq C \left( \|u_1\|_{L^1}^2 + \|u_0\|_{L^1}^2 \right) \int_{|\xi| \geq \delta_0} e^{-\frac{|\xi|^{2\theta}}{2} t^{-\frac{2\alpha}{\theta}}} d\xi \leq C (\|u_1\|_{L^1}^2 + 2\|u_0\|_{L^1}^2) e^{-dt},$$

with constants $C > 0$ and $d > 0$, and $t \geq 1$.

3.4. Proof of Theorem 1.1. The previous results on the zones of low and high frequency imply the following result on the asymptotic profile to the solution $\hat{u}$.

**Proposition 1.** Let $n < 2\alpha$, $\epsilon \in (0, \min\{1, \alpha\})$, $\kappa \geq 0$, $u_0 \in L^1(\mathbb{R}^n) \cap H^\kappa(\mathbb{R}^n)$ and $u_1 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \cap W^{\kappa - \alpha, 2}(\mathbb{R}^n)$. Then, the following asymptotic estimate for the solution $\hat{u}(t, \xi)$ of the Cauchy problem (3.2) holds for $t \geq 1$.

$$\int_{\mathbb{R}^n} \left| \hat{u}(t, \xi) - \left\{ P_1 e^{-\alpha(\xi) t \sin(|\xi|^{\alpha} t)} \left| \frac{\xi}{|\xi|} \right|^2 + P_0 e^{-\alpha(\xi) t \cos(|\xi|^{\alpha} t)} \right\} \right|^2 d\xi \leq C \left( \|u_1\|_{L^1}^2 + \|u_0\|_{L^1}^2 \right) t^{-\frac{2\alpha}{\theta} t^{-\frac{2\alpha}{\theta}}} + C \left( \|u_0\|_{L^1}^2 + \|u_1\|_{L^1}^2 + \|u_0\|^2 + \|u_1\|^2 \right) e^{-dt} + C t^{-\frac{\sigma}{\theta}} \left( \|u_0\|^2_{H^n} + \|u_1\|^2_{W^{\kappa - \alpha, 2}} \right),$$

with constants $C > 0$ and $d > 0$, that can be calculated explicitly.

Next step is to prove Theorem 1.1 by relying on Proposition 1. For this we need the upper bound for the following quantities.

- For $n > 2\alpha$

$$\int_{\mathbb{R}^n} e^{-|\xi|^{2\theta} \sin^2(|\xi|^{\alpha} t)} |\xi|^{-2\alpha} d\xi \leq C t^{-\frac{2\alpha}{\theta}}, \quad t \geq 1.$$ (3.7)
\[ \int_{\mathbb{R}^n} e^{-|\xi|^{2\alpha}} \cos^{2}(|\xi|^{\alpha}) d\xi \leq C t^{-\frac{\alpha}{\theta}}, \quad t \geq 1. \]  

(3.8)

**Proof of Theorem 1.1.** By using the Plancherel theorem, Lemmas 2.5, 2.6 and Proposition 1 we first obtain

\[ \|u(t, \cdot)\| \geq C|P_1| t^{-\frac{n-2\alpha}{\theta} - \frac{2^{\alpha+\delta}}{2^{\theta+\alpha}}} - C|P_2| t^{-\frac{2^{\alpha+\delta}}{2^{\theta+\alpha}}} - C\|u_1\|_{L_1} t^{-\frac{n-2\alpha}{\theta} - \frac{2^{\alpha+\delta}}{2^{\theta+\alpha}}} - C\|u_0\|_{L_1} t^{-\frac{2^{\alpha+\delta}}{2^{\theta+\alpha}}} \]

(3.9)

for \( t \geq \max\{1, t_0\} \), where \( t_0 \) is fixed by the above mentioned lemmas.

Now, for \( \kappa > \frac{(n-2\alpha)(\theta-\alpha)}{2^{\theta+\alpha}} \) we can see from (3.9) that there exists a constant \( C_1 > 0 \) such that the estimate from below

\[ \|u(t, \cdot)\| \geq C_1|P_1| t^{-\frac{n-2\alpha}{\theta} - \frac{2^{\alpha+\delta}}{2^{\theta+\alpha}}} \]

holds for \( t \gg 1 \).

On the other side, using again the Plancherel theorem, (3.7), (3.8) and Proposition 1 we can get the following estimate from above

\[ \|u(t, \cdot)\| \leq C\|u_1\|_{L_1} t^{-\frac{n-2\alpha}{\theta} - \frac{2^{\alpha+\delta}}{2^{\theta+\alpha}}} + C\|u_0\|_{L_1} t^{-\frac{2^{\alpha+\delta}}{2^{\theta+\alpha}}} + C\|u_1\|_{L_1} t^{-\frac{n-2\alpha}{\theta} - \frac{2^{\alpha+\delta}}{2^{\theta+\alpha}}} \]

(4.1)

provided that \( \kappa > \frac{(n-2\alpha)(\theta-\alpha)}{2^{\theta+\alpha}} \) and \( t \geq 1 \), where \( I_0 \) is a constant defined in the statement of Theorem 1.1. Hence, we have just proved Theorem 1.1, which shows the optimality of the decay rates of the solution to problem (1.2). \( \square \)

4. **Case \( \delta > 0 \) and \( \frac{\alpha + \delta}{2} < \theta < \alpha + \delta \).** These conditions under consideration on the exponents of the Laplace operator include the case we consider as a strong damping \( \theta > \alpha \). An equivalent problem for the case \( \frac{\alpha + \delta}{2} \geq \theta \) with \( \alpha = 2 \) was studied in [9].

The aim of this section is to get optimal decay rates to the Cauchy problem (1.1) under the assumption \( 0 < \delta \leq \alpha, \frac{\alpha + \delta}{2} \leq \theta < \alpha + \delta \). The results of this section essentially new under the effect of rotational inertia terms.

**Remark 5.** We briefly mention how one can constructs a unique solution to problem (1.1) under the assumptions about the exponents of the Laplace operator in this section. Applying the linear operator \((I + (-\Delta)^{\delta})^{-1}\) in both sides of the equation in (1.1) it follows that

\[ u_t + (I + (-\Delta)^{\delta})^{-1}(-\Delta)\theta u_t + (I + (-\Delta)^{\delta})^{-1}(-\Delta)^{\alpha} u = 0. \]

If we define \( P = (I + (-\Delta)^{\delta})^{-1}, A = (-\Delta)^{\alpha} \) we may rewrite this equation as

\[ u_t + PA^\top u_t + P A u = 0, \]  

(4.1)

where \( v = \frac{\theta}{\alpha} > 0 \). The equation (4.1) is written in an abstract form described in Theorem 2.1 of [7], except for the linear and continuous operator \( P \). In fact, in [7] \( P = Id \) the identity operator. But reviewing the proof of existence and regularity in the article by [7], it can be proved that the bounded linear operator \( P \) does not
affect the existence of solution to the Cauchy problem associated with the equation (4.1) and thus to problem (1.1).

To get several estimates to (1.1), for the case in consideration, we again consider the associated Cauchy problem in the Fourier space given by

\[
\begin{aligned}
(1 + |\xi|^{2\delta})\hat{u}_{tt} + |\xi|^{2\theta}\hat{u}_t + |\xi|^{2\alpha}\hat{u} = 0, & \quad \xi \in \mathbb{R}^n, \ t > 0, \\
\hat{u}(0, \xi) = \hat{u}_0(\xi),
\end{aligned}
\tag{4.2}
\]

4.1. Decay estimates. In this subsection we also use the energy method in the Fourier space. This framework is standard nowadays and we develop this section in a similar way to the previous sections. To do that, we define the functionals of energy

\[
e_0(t, \xi) = \frac{1}{2}(1 + |\xi|^{2\delta})|\hat{u}_t|^2 + \frac{1}{2}|\xi|^{2\alpha}|\hat{u}|^2,
\]

\[
e_1(t, \xi) = (1 + |\xi|^{2\delta})Re(\hat{u}_t\overline{\xi}) + \frac{1}{2}|\xi|^{2\theta}|\hat{u}|^2,
\tag{4.3}
\]

which satisfy the following identities of energy.

\[
\frac{d}{dt}e_0(t, \xi) + |\xi|^{2\theta}|\hat{u}_t|^2 = 0, \ \forall \xi \in \mathbb{R}^n, \ \forall t > 0,
\]

\[
\frac{d}{dt}e_1(t, \xi) + |\xi|^{2\alpha}|\hat{u}|^2 = (1 + |\xi|^{2\delta})|\hat{u}_t|^2, \ \forall \xi \in \mathbb{R}^n, \ \forall t > 0.
\tag{4.4}
\]

Considering (4.3) we define the functional

\[
e(t, \xi) = e_0(t, \xi) + b\rho(\xi)e_1(t, \xi),
\tag{4.5}
\]

for \(b > 0\), where the function \(\rho : \mathbb{R}^n \to [0, \infty)\) is given by

\[
\rho(\xi) = \varepsilon \frac{|\xi|^{2\theta}}{2 + |\xi|^{4\delta - 2\alpha}},
\tag{4.6}
\]

with \(\varepsilon > 0\) to be chosen. This function \(\rho(\xi)\) is quite similar to the one defined in (3.4).

The time derivative of \(e(t, \xi)\) combined with (4.4) says that the next identity follows.

\[
\frac{d}{dt}e(t, \xi) + F(t, \xi) = R(t, \xi), \ \xi \in \mathbb{R}^n, \ t > 0,
\tag{4.7}
\]

where

\[
F(t, \xi) = |\xi|^{2\theta}|\hat{u}_t|^2 + b\rho(\xi)|\xi|^{2\alpha}|\hat{u}|^2 \quad \text{and} \quad R(t, \xi) = b\rho(\xi)(1 + |\xi|^{2\delta})|\hat{u}_t|^2.
\tag{4.8}
\]

On the other hand,

- for \(|\xi| \leq 1\) and because \(|\xi|^{2\delta} - 1 \leq 0 \leq |\xi|^{4\delta - 2\alpha}\) it follows that \(\frac{1 + |\xi|^{2\delta}}{2 + |\xi|^{4\delta - 2\alpha}} \leq 1,

- for \(|\xi| \geq 1\) since \(\theta \geq \frac{\alpha + \delta}{2}\) it holds that \(1 + |\xi|^{2\delta} \leq 2(2 + |\xi|^{4\delta - 2\alpha})\). Then, one has

\[
\frac{1 + |\xi|^{2\delta}}{2 + |\xi|^{4\delta - 2\alpha}} \leq 2.
\]

Hence, for \(0 < \varepsilon \leq \frac{1}{2}\) it follows from the definition of \(\rho(\xi)\) we have

\[
\rho(\xi)(1 + |\xi|^{2\delta}) = \varepsilon \frac{|\xi|^{2\theta}}{2 + |\xi|^{4\delta - 2\alpha}}(1 + |\xi|^{2\delta}) \leq |\xi|^{2\theta}, \ \forall \xi \in \mathbb{R}^n,
\tag{4.9}
\]
and from (4.8) we get \( R(t, \xi) \leq bF(t, \xi) \). Thus, combining with (4.7) we deduce that

\[
\frac{d}{dt} e(t, \xi) + (1 - b)F(t, \xi) \leq 0, \quad \forall t > 0, \forall \xi \in \mathbb{R}^n, \quad b \in (0, 1).
\]  \hspace{1cm} (4.10)

Now we need the next lemma.

**Lemma 4.1.** Let \( \theta \geq \frac{\alpha + \delta}{2} \) and \( 0 < \varepsilon \leq \frac{1}{2} \). Then there exists a constant \( M = M(b) > 0 \) such that

\[
\rho(\xi)e(t, \xi) \leq MF(t, \xi).
\]

**Proof.** For \( \xi = 0 \) the result is trivial.

For \( \xi \neq 0 \) we have \( \text{Re}(\hat{u} \bar{u}) \leq \frac{|\hat{u}|^2}{2|\xi|^2} + \frac{|\xi^\alpha| |\hat{u}|^2}{2} \). Then

\[
\rho(\xi)e(t, \xi) \leq |\xi|^{2\theta} |\hat{u}|^2 \left( \frac{\rho(\xi)(1 + |\xi|^{2\theta})}{2|\xi|^{2\theta}} + \frac{b\rho(\xi)^2}{2|\xi|^{2\theta + \alpha}} \right)
+ b\rho(\xi)|\xi|^{2\theta} |\hat{u}|^2 \left( \frac{1}{2\theta} + \frac{\rho(\xi)}{2|\xi|^\alpha} + \frac{\rho(\xi)}{2|\xi|^{2\theta - 2\alpha}} \right). \hspace{1cm} (4.11)
\]

Due to (4.9) we note that

\[
\frac{\rho(\xi)}{2|\xi|^{2\theta}}(1 + |\xi|^{2\theta}) \leq \frac{1}{2}, \quad \xi \neq 0.
\]

Using \( p \geq 0, r > 0 \), and the elemental inequality

\[
\frac{1}{\frac{1}{r} + \frac{r}{p}} \leq \frac{1}{2},
\]

and considering \( 0 < \varepsilon \leq \frac{1}{2} \) one can get the estimate

\[
\frac{b\rho(\xi)^2}{2|\xi|^{2\theta + \alpha}} \leq \frac{b}{2} \cdot \frac{|\xi|^{2\theta - \alpha}}{2 + |\xi|^{4\theta - 2\alpha}} \cdot \frac{1}{2 + |\xi|^{4\theta - 2\alpha}} \leq \frac{b}{4}, \quad \xi \neq 0.
\]

For \( 0 < \varepsilon \leq \frac{1}{2} \) the definition of the function \( \rho(\xi) \) in (4.6) implies that

\[
\frac{\rho(\xi)}{2|\xi|^\alpha} \leq \frac{1}{4}, \quad \frac{\rho(\xi)|\xi|^{2\theta - 2\alpha}}{2} \leq \frac{1}{2}, \quad \xi \neq 0.
\]

Substituting the last four estimates in (4.11) we prove the lemma. \( \square \)

Finally, from Lemma 4.1 and inequality (4.10) for a fixed \( b \in (0, 1) \) we obtain

\[
\frac{d}{dt} e(t, \xi) + (1 - b)\rho(\xi)M^{-1}e(t, \xi) \leq 0, \quad \forall t > 0, \forall \xi \in \mathbb{R}^n.
\]

This important differential inequality admits the solution

\[
e(t, \xi) \leq e^{-\omega(\xi)t}e(0, \xi), \quad \forall \xi \in \mathbb{R}^n, \hspace{1cm} (4.12)
\]

with \( \omega = (1 - b)M^{-1} > 0 \) for fixed \( b \in (0, 1) \).

For \( \xi \neq 0 \) the next inequality holds.

\[
\pm (1 + |\xi|^{2\theta}) \text{Re}(\hat{u} \bar{u}) \leq \frac{1}{2} (1 + |\xi|^{2\theta})^2 |\xi|^{-2\theta} |\hat{u}|^2 + \frac{1}{2} |\xi|^{2\theta} |\hat{u}|^2. \hspace{1cm} (4.13)
\]

We should use this inequality to prove the next important lemma.

**Lemma 4.2.** Let \( \theta \geq \frac{\alpha + \delta}{2} \) and \( 0 < \varepsilon \leq \frac{1}{2} \). Then there exist positive constants \( C = C(b) \) and \( \omega = \omega(b) \) such that

\[
e_0(t, \xi) \leq Ce^{-\omega(\xi)t}e_0(0, \xi), \quad \forall t > 0, \forall \xi \in \mathbb{R}^n.
\]
Therefore, for \( \xi \neq 0 \) the result follows with \( C = 1 \).

For \( \xi \neq 0 \) using the inequality (4.13) with minus sign, it follows

\[
e(t, \xi) \geq e_{0}(t, \xi) - \frac{b}{2} \rho(\xi)(1 + |\xi|^{2\theta})^{2} |\xi|^{-2\theta} |\hat{u}_{t}|^{2}
\]

\[
= \frac{1}{2} |\xi|^{2\alpha} |\hat{u}|^{2} + \frac{1}{2} (1 + |\xi|^{2\delta}) |\hat{u}_{t}|^{2} (1 - b\rho(\xi)(1 + |\xi|^{2\delta})|\xi|^{-2\theta}).
\]

Due to \( \rho(\xi)(1 + |\xi|^{2\delta}) \leq |\xi|^{2\theta} \) for \( 0 < \varepsilon \leq \frac{1}{2} \), we can estimate

\[
\rho(\xi)(1 + |\xi|^{2\delta})|\xi|^{-2\theta} \leq b|\xi|^{2\theta} = b.
\]

Therefore, for \( b \in (0, 1) \) we obtain

\[
e(t, \xi) \geq \frac{1}{2} |\xi|^{2\alpha} |\hat{u}|^{2} + \frac{(1 - b)}{2} (1 + |\xi|^{2\delta}) |\hat{u}_{t}|^{2} \geq (1 - b)e_{0}(t, \xi), \quad \forall \xi \in \mathbb{R}^{n}.
\]  

Combining (4.12) and (4.14) we arrive at

\[
e_{0}(t, \xi) \leq \frac{1}{1 - b} e^{-\omega(\xi)t}e_{0}(0, \xi), \quad \forall \xi \in \mathbb{R}^{n}.
\]  

On the other side, using the inequality (4.13) with plus sign we get

\[
e(t, \xi) \leq e_{0}(t, \xi) + \rho(\xi)\||\xi|^{2\theta}|\hat{u}|^{2} + \frac{b}{2}\rho(\xi)(1 + |\xi|^{2\delta})^{2}|\xi|^{-2\theta} |\hat{u}_{t}|^{2}, \quad \xi \neq 0.
\]

Now, due to \( \rho(\xi)(1 + |\xi|^{2\delta}) \leq |\xi|^{2\theta} \) for \( 0 < \varepsilon \leq \frac{1}{2} \), the following estimate is true:

\[
\rho(\xi)(1 + |\xi|^{2\delta})^{2}|\xi|^{-2\theta} \leq |\xi|^{2\theta}(1 + |\xi|^{2\delta}(|\xi|^{-2\theta} = 1 + |\xi|^{2\delta}
\]

for \( \xi \neq 0 \), and as consequence for all \( \xi \in \mathbb{R}^{n} \).

We also have for \( 0 < \varepsilon \leq \frac{1}{2} \)

\[
\rho(\xi)|\xi|^{2\theta} \leq \frac{\|\xi|^{2\theta}}{2 + \|\xi|^{2\theta}} |\xi|^{2\theta} \leq \frac{|\xi|^{4\theta - 2\alpha}}{1 + |\xi|^{4\theta - 2\alpha}} |\xi|^{2\alpha} \leq |\xi|^{2\alpha} \quad \xi \in \mathbb{R}^{n}.
\]

Therefore, one has

\[
\begin{align*}
e(t, \xi) & \leq \frac{1}{2} (1 + |\xi|^{2\delta}) |\hat{u}_{t}|^{2} + \frac{1}{2} |\xi|^{2\alpha} |\hat{u}|^{2} + \frac{b}{2} (1 + |\xi|^{2\delta}) |\hat{u}_{t}|^{2} + b|\xi|^{2\alpha} |\hat{u}|^{2} \\
& \leq (2b + 1)e_{0}(t, \xi), \quad t > 0, \xi \in \mathbb{R}^{n}.
\end{align*}
\]  

Finally, making \( t = 0 \) in (4.16) and using the estimate (4.15) we conclude the proof of lemma, that is

\[
e_{0}(t, \xi) \leq Ce^{-\omega(\xi)t}e_{0}(0, \xi), \quad \forall t > 0, \forall \xi \in \mathbb{R}^{n},
\]

with \( C = C(b) = \frac{b+1}{b-1}, 0 < b < 1 \) and \( 0 < \varepsilon \leq \frac{1}{2} \).

The next result gives decay estimates on the norms of the energy of the system.

**Theorem 4.3.** Let \( n \geq 1, \theta \geq \frac{\alpha+1}{2} \) and \( \epsilon \geq 0 \). If \( u_{0} \in L^{1}(\mathbb{R}^{n}) \cap H^{\alpha+\epsilon}(\mathbb{R}^{n}), u_{1} \in L^{1}(\mathbb{R}^{n}) \cap H^{\delta+\epsilon}(\mathbb{R}^{n}), \) then the solution \( u(t, x) \) to problem (1.1) satisfies

\[
\|u_{t}(t, \cdot)\|_{H^{\alpha}} + \|(-\Delta)\frac{\theta}{2} u(t, \cdot)\|^{2} \leq C(1 + t)^{-\frac{\delta}{2\theta}} \|u_{1}\|_{L^{2}}^{2} + C(1 + t)^{-\frac{\alpha+1}{2\theta}} \|u_{0}\|_{L^{2}}^{2} + C(1 + t)^{-\frac{\alpha+1}{2\theta}} \|u_{1}\|_{H^{\alpha+\epsilon}}^{2} + \|u_{0}\|_{H^{\alpha+\epsilon}}^{2}, \quad t > 0,
\]

with \( C > 0 \) a constant that is possible to be calculated explicitly.
Proof. By the Plancherel theorem and Lemma 4.2, for \( t > 0 \) it holds that
\[
\|u_t(t, \cdot)\|_{\dot{F}^\alpha_{2, 1}}^2 + \|(-\Delta)^{\frac{\alpha}{2}} u(t, \cdot)\|^2 \leq C \int_{\mathbb{R}^n} e^{-\omega \rho(\xi) t} \left[ (1 + |\xi|^{2\delta}) |\hat{u}_1|^2 + |\xi|^{2\alpha} |\hat{u}_0|^2 \right] d\xi.
\]
We now estimate the integral on the right hand side of the above inequality on the zones of low and high frequency.

- For \( |\xi| \leq 1 \) we have \( 2 + |\xi|^{4\theta - 2\alpha} \leq 3 \) due to \( \theta > \alpha/2 \). Thus
  \[
  \rho(\xi) = \varepsilon \cdot \frac{|\xi|^{2\theta}}{2 + \frac{3}{5} |\xi|^{4\theta - 2\alpha}} \geq \frac{\varepsilon}{3} |\xi|^{2\theta} = c_1 |\xi|^{2\theta}.
  \]
Therefore, from Lemma 2.4 one has
\[
\int_{|\xi| \leq 1} e^{-\omega \rho(\xi) t} \left[ (1 + |\xi|^{2\delta}) |\hat{u}_1|^2 + |\xi|^{2\alpha} |\hat{u}_0|^2 \right] d\xi \\
\leq \int_{|\xi| \leq 1} e^{-c_1 \omega |\xi|^{2\theta} t} \left[ (1 + |\xi|^{2\delta}) |\hat{u}_1|^2 + |\xi|^{2\alpha} |\hat{u}_0|^2 \right] d\xi \\
\leq \left( C(1 + t)^{-\frac{1}{2\theta}} + (1 + t)^{-\frac{4\theta + 2\alpha}{3\theta}} \right) \|u_t\|_{L^1}^2 + C(1 + t)^{-\frac{4\theta + 2\alpha}{3\theta}} \|u_0\|_{L^1}^2.
\]

- For \( |\xi| \geq 1 \) we have \( 2 + |\xi|^{4\theta - 2\alpha} \leq 3|\xi|^{4\theta - 2\alpha} \) because \( \theta > \alpha/2 \). Thus
  \[
  \rho(\xi) = \varepsilon \cdot \frac{|\xi|^{2\theta}}{2 + \frac{3}{5} |\xi|^{4\theta - 2\alpha}} \geq \frac{\varepsilon}{3} |\xi|^{2\theta} = \frac{c_2}{|\xi|^{4\theta - 2\alpha}} \geq \frac{c_2}{|\xi|^{4\theta - 2\alpha}}.
  \]
Hence,
\[
\int_{|\xi| \geq 1} e^{-\omega \rho(\xi) t} \left[ (1 + |\xi|^{2\delta}) |\hat{u}_1|^2 + |\xi|^{2\alpha} |\hat{u}_0|^2 \right] d\xi \\
\leq \int_{|\xi| \geq 1} e^{-\frac{5}{4\theta} |\xi|^{2\delta} t} \left[ (1 + |\xi|^{2\delta}) |\hat{u}_1|^2 + |\xi|^{2\alpha} |\hat{u}_0|^2 \right] d\xi \\
\leq \int_{|\xi| \geq 1} \left( e^{-\frac{5}{4\theta} |\xi|^{2\delta} t} \right) (2|\xi|^{2\delta + 2\alpha} |\hat{u}_1|^2 + |\xi|^{2\alpha + 2\alpha} |\hat{u}_0|^2) d\xi.
\]

Now, let us control the quantity \( e^{-\frac{5}{4\theta} |\xi|^{2\delta} t} |\xi|^{-2\delta} \), which has a close relation to the so-called regularity-loss structure (cf. [10, 11, 13]) such that one can obtain the crucial estimate as follows:
\[
\sup_{|\xi| \geq 1} \left( e^{-\frac{5}{4\theta} |\xi|^{2\delta} t} |\xi|^{-2\delta} \right) \leq C(1 + t)^{-\frac{\theta}{\alpha + b}}, \quad t > 0,
\]
where \( C = C(\varepsilon, \theta, \alpha, b) \) is a positive constant. Thus,
\[
\int_{|\xi| \geq 1} e^{-\omega \rho(\xi) t} \left[ (1 + |\xi|^{2\delta}) |\hat{u}_1|^2 + |\xi|^{2\alpha} |\hat{u}_0|^2 \right] d\xi \\
\leq C(1 + t)^{-\frac{\theta}{\alpha + b}} \int_{|\xi| \geq 1} \left( |\xi|^{2\delta + 2\alpha} |\hat{u}_1|^2 + |\xi|^{2\alpha + 2\alpha} |\hat{u}_0|^2 \right) d\xi \\
= C(1 + t)^{-\frac{\theta}{\alpha + b}} \left( \|u_1\|_{H^{\alpha + \theta}}^2 \|u_0\|_{H^{\alpha + \theta}}^2 \right), \quad t > 0.
\]
Combining the above two estimates the proof follows. \( \square \)

To the \( L^2 \) norm of the solution itself we prepare the next result.
Theorem 4.4. Let $n > 2\alpha > 0$, $\theta \geq \frac{\alpha + \delta}{2}$, $\epsilon > \alpha - \delta$. If $u_0 \in L^1(\mathbb{R}^n) \cap H^\epsilon(\mathbb{R}^n)$, $u_1 \in L^1(\mathbb{R}^n) \cap H^{\delta + \epsilon - \alpha}(\mathbb{R}^n)$ then

$$
\|u(t, \cdot)\|^2 \leq C(1 + t)^{-\frac{n - 2\alpha}{2\theta}} \|u_1\|^2_{L^1} + C(1 + t)^{-\frac{n - 2\alpha}{2\theta}} \|u_0\|^2_{L^1} \\
+ C(1 + t)^{-\frac{n - 2\alpha}{2\theta}} \left(\|u_1\|^2_{H^{\delta + \epsilon - \alpha}} + \|u_0\|^2_{H^\epsilon}\right), \quad t > 0.
$$

Proof. By Lemma 4.2 we have for $\xi \neq 0$,

$$
|\hat{u}(t, \xi)|^2 \leq Ce^{-\omega t(|\xi|^2)} \left(\frac{1 + |\xi|^2}{|\xi|^2} |\hat{u}_1(\xi)|^2 + |\hat{u}_0(\xi)|^2\right).
$$

- On the zone of low frequency $|\xi| \leq 1$ one can see as in the proof of Theorem 4.3 that $\rho(\xi) \geq c_1|\xi|^{2\theta}$. Thus

$$
\int_{|\xi| \leq 1} |\hat{u}(t, \xi)|^2 d\xi \leq C \int_{|\xi| \leq 1} e^{-\omega t|\xi|^2} \left(|\xi|^{-2\alpha} |\hat{u}_1|^2 + |\hat{u}_0|^2\right) d\xi \\
\leq C(1 + t)^{-\frac{n - 2\alpha}{2\theta}} \|u_1\|^2_{L^1} + C(1 + t)^{-\frac{n - 2\alpha}{2\theta}} \|u_0\|^2_{L^1}.
$$

- On the region of high frequency $|\xi| \geq 1$ we can also observe as in the proof of Theorem 4.3 that the estimate

$$
\rho(\xi) \geq \frac{c_2}{|\xi|^{4\theta - 2\alpha}}, \quad \sup_{|\xi| \geq 1} \left(e^{-\frac{1}{|\xi|^{4\theta - 2\alpha}}} - \frac{1}{|\xi|^{2\theta}}\right) \leq C(1 + t)^{-\frac{n - 2\alpha}{2\theta}}
$$

is true for $t > 0$.

So, in a similar way to the previous theorem we can conclude that

$$
\int_{|\xi| \geq 1} |\hat{u}(t, \xi)|^2 d\xi \leq \int_{|\xi| \geq 1} e^{-\frac{1}{|\xi|^{4\theta - 2\alpha}}} |\xi|^{-2\alpha} \left(|\xi|^{2\delta + 2\epsilon - 2\alpha} |\hat{u}_1|^2 + |\xi|^{2\epsilon} |\hat{u}_0|^2\right) d\xi \\
\leq C(1 + t)^{-\frac{n - 2\alpha}{2\theta}} \left(\|u_1\|^2_{H^{\delta + \epsilon - \alpha}} + \|u_0\|^2_{H^\epsilon}\right), \quad t > 0.
$$

Remark 6. Note that in the proof, the assumption $u_1 \in H^{\delta + \epsilon - \alpha}(\mathbb{R}^n)$ can be changed by $u_1 \in \dot{W}^{\delta + \epsilon - \alpha, 2}(\mathbb{R}^n)$ in the case $\epsilon > 0$.

4.2. Asymptotic expansion - low frequencies. We fix $0 < \delta_0 < 1$ and we work on the zone of low frequency $0 < |\xi| \leq \delta_0$. The characteristic roots associated to the equation (4.2) are given by

$$
\lambda_{\pm} = -|\xi|^{2\theta} \pm |\xi|^\alpha \sqrt{|\xi|^{4\theta - 2\alpha} - 4(1 + |\xi|^2)}
$$

For $|\xi| \leq \delta_0$ we have $|\xi|^{4\theta - 2\alpha} - 4(1 + |\xi|^2) \leq 0$. Then we can rewrite these roots in the form

$$
\lambda_{\pm} = -a(\xi) \pm i b(\xi),
$$

where

$$
a(\xi) = \frac{|\xi|^{2\theta}}{2(1 + |\xi|^2)}, \quad b(\xi) = \frac{|\xi|^\alpha \sqrt{4(1 + |\xi|^2)} - |\xi|^{4\theta - 2\alpha}}{2(1 + |\xi|^2)}.
$$

As in the previous section, the explicit solution for the case $\delta > 0$ in low frequencies is given by $\hat{u}(t, \xi) = \hat{H}(t, \xi) \hat{u}_0 + \hat{G}(t, \xi) \hat{u}_1$, with $\hat{G}(t, \xi)$ and $\hat{H}(t, \xi)$ like in (3.5).

From the mean value theorem, we can write

$$
\cos(b(\xi)t) = \cos(|\xi|^2 t) - t(\xi - |\xi|^2) \sin(\xi t),
$$

$$
\sin(b(\xi)t) = \sin(|\xi|^2 t) + t(\xi - |\xi|^2) \cos(\xi t),
$$

where $\eta(\xi) = \beta_1 b(\xi) + (1 - \beta_1)|\xi|^\alpha$, $\mu(\xi) = \beta_2 b(\xi) + (1 - \beta_2)|\xi|^\alpha$ for some $\beta_1, \beta_2 \in (0, 1)$. 

Therefore, applying Lemma 2.2, we can rewrite the explicit solution for \( \hat{u}(t, \xi) \) as

\[
\hat{u}(t, \xi) = e^{-a(\xi) t} P_0 \cos(|\xi|^\alpha t) + e^{-a(\xi) t} P_1 \frac{\sin(|\xi|^\alpha t)}{|\xi|^\alpha} + K_1(t, \xi) + K_2(t, \xi) + K_3(t, \xi) + K_4(t, \xi) + K_5(t, \xi) + K_6(t, \xi); \tag{4.17}
\]

where one has just used the fact that \( \alpha < \frac{\alpha + \delta}{2} \), where \( P_0 \) and \( P_1 \) are again defined as (see Lemma 2.2)

\[
P_j := \int_{\mathbb{R}^n} u_j(x) dx \quad (j = 0, 1). \tag{4.18}
\]

The first two terms in the right side of (4.17) give the leading term of the solutions. In fact, in the next step we are going to prove that the functions \( K_i \), \( i = 1, \ldots, 6 \), decay faster than such leading term.

Similarly to the previous section we get \( L^2 \) decay estimates for low frequencies to the functions \( K_i(t, \xi) \), \( i = 1, \ldots, 6 \), using the assumptions on the exponents and Lemma 2.4.

By definition of \( a(\xi) \) we may estimate the exponential function in the definition \( K_i(t, \xi) \), \( i = 1, \ldots, 6 \), as follows:

\[
e^{-2a(\xi) t} \leq e^{-\frac{1}{2} |\xi|^{2\theta} t}, \quad t > 0, \quad |\xi| \leq \delta_0.
\]

To get estimate for \( K_1(t, \xi) \) we consider a fixed number \( \kappa \) such that \( 0 < \kappa < \min\{1, \delta\} \) and Lemma 2.3. Thus, one has

\[
\int_{|\xi| \leq \delta_0} |K_1(t, \xi)|^2 d\xi \leq C \|u_1\|_{L^1_t, t^{-\frac{2-a+2\alpha}{2\theta}}} L^2, \quad t > 0.
\]

Applying Lemma 2.3, we get an estimate for \( K_2(t, \xi) \):

\[
\int_{|\xi| \leq \delta_0} |K_2(t, \xi)|^2 d\xi \leq C \|u_0\|_{L^2_t, t^{-\frac{4\theta}{2\theta}}} L^2, \quad t > 0.
\]

Now we note that for \( |\xi| \leq \delta_0 \) it holds \( 4(1 + |\xi|^{2\theta}) - |\xi|^{4\theta - 2\alpha} \geq 4 - \delta_0^{4\theta - 2\alpha} \geq 0 \).

Then

\[
\frac{|a(\xi)|}{|b(\xi)|} = \frac{|\xi|^{2\theta - \alpha}}{\sqrt{4(1 + |\xi|^{2\theta}) - |\xi|^{4\theta - 2\alpha}}} \leq \frac{|\xi|^{2\theta - \alpha}}{\sqrt{4 - \delta_0^{4\theta - 2\alpha}}}.
\]

Therefore, we may get the following estimate for \( K_3(t, \xi) \)

\[
\int_{|\xi| \leq \delta_0} |K_3(t, \xi)|^2 d\xi \leq \frac{1}{4 - \delta_0^{4\theta - 2\alpha}} \|u_0\|_{L^1_t, t^{-\frac{2-a+2\alpha}{2\theta}}} L^2 \leq C \|u_0\|_{L^2_t, t^{-\frac{4\theta}{2\theta}}} L^2, \quad t > 0,
\]

where one has just used the fact that \( \alpha < \theta \).
Therefore we can estimate $K$ for $t > 0$

$$
\int_{|\xi| \leq \delta_0} |K_4(t, \xi)|^2 d\xi \leq C\|u_0\|_{L^2}^2 \int_{|\xi| \leq \delta_0} e^{-|\xi|^{2\alpha}} |\xi|^{4\alpha + 2\alpha} d\xi \leq C\|u_0\|_{L^2}^2 t^{-\frac{n+2\alpha+4\delta}{2\theta}}, \quad t > 0.
$$

For $|\xi| \leq \delta_0$ working similarly to the previous estimate for $K_4(t, \xi)$ we can see that

$$
|b(\xi) - |\xi|^{\alpha}|^2 \leq C\frac{\xi^{2\alpha}}{|(4 - \delta_0^{4\alpha - 2\alpha} + 2\alpha)|^2} \left( |\xi|^{2\alpha} + |\xi|^{4\alpha - 2\alpha} \right)^2 \leq C|\xi|^{4\alpha + 2\alpha}.
$$

Thus, from the definition of $K_4(t, \xi)$ we obtain for $t > 0$

$$
\int_{|\xi| \leq \delta_0} |K_4(t, \xi)|^2 d\xi \leq C\|u_0\|_{L^2}^2 t^{-\frac{n+2\alpha+4\delta}{2\theta}}.
$$

Now, using the assumption that $\theta < \alpha + \delta$ we have $n + 4\delta + 2\alpha - 4\theta > n - 2\alpha$. Then, we may fix $\kappa_1 > 0$ such that $n + 4\delta + 2\alpha - 4\theta \geq n - 2\alpha + \kappa_1$. Thus, substituting this fact in the previous estimate for $K_4(t, \xi)$, we arrive at

$$
\int_{|\xi| \leq \delta_0} |K_4(t, \xi)|^2 d\xi \leq C\|u_0\|_{L^2}^2 t^{-\frac{n+2\alpha+4\delta}{2\theta}}, \quad t > 0.
$$

For $|\xi| \leq \delta_0$ working similarly to the previous estimate for $K_4(t, \xi)$ we can see that

$$
\frac{|b(\xi) - |\xi|^{\alpha}|^2}{b(\xi)} \leq C\frac{\xi^{2\alpha}}{|(4 - \delta_0^{4\alpha - 2\alpha} + 2\alpha)|^2} \left( |\xi|^{2\alpha} + |\xi|^{4\alpha - 2\alpha} \right)^2 \leq C|\xi|^{4\alpha + 2\alpha}.
$$

Therefore we can estimate $K_5(t, \xi)$ as follows:

$$
\int_{|\xi| \leq \delta_0} |K_5(t, \xi)|^2 d\xi \leq C\frac{t^2}{4 - \delta_0^{4\alpha - 2\alpha}} \int_{|\xi| \leq \delta_0} e^{-|\xi|^{2\alpha}} |\xi|^{4\alpha + 2\alpha} d\xi \leq C\|u_1\|_{L^2}^2 t^{-\frac{n+2\alpha+4\delta}{2\theta}}, \quad t > 0 \quad (C > 0 \text{ constant}).
$$

To estimate $K_6(t, \xi)$, we assume the additional hypothesis $u_1 \in \mathring{W}^{\alpha, 2}(\mathbb{R}^n)$ and the assumption that $\theta < \alpha + \delta$. By choosing $\varepsilon_2 > 0$ such that $4\delta + 4\alpha - 4\theta \geq \varepsilon_2$ one obtains

$$
\int_{|\xi| \leq \delta_0} |K_6(t, \xi)|^2 d\xi \leq C t^2 \int_{|\xi| \leq \delta_0} e^{-|\xi|^{2\alpha}} |\xi|^{4\alpha + 2\alpha} |u_1(\xi)|^2 d\xi \leq C t^2 \int_{|\xi| \leq \delta_0} e^{-|\xi|^{2\alpha}} |\xi|^{4\alpha + 2\alpha} |u_1^{\alpha}(\xi)|^2 d\xi \leq C\|u_1\|_{W^{\alpha, 2}}^2 t^{-\frac{n+2\alpha+4\delta}{2\theta}}, \quad t > 0.
$$

The above estimates imply that the next result is true.

**Lemma 4.5.** Let $0 < \delta \leq \alpha$ and $\frac{\alpha + \delta}{2} \leq \theta < \alpha + \delta$. Consider the initial data

$$
u_0 \in L^1(\mathbb{R}^n), \quad u_1 \in L^1(\mathbb{R}^n) \cap \mathring{W}^{\alpha, 2}(\mathbb{R}^n)
$$

with $\kappa \in (0, \min\{1, \delta\})$. Then there exists a number $\epsilon_0 > 0$ such that the solution $\hat{u}(t, \xi)$ of (4.2) satisfies

$$
\int_{|\xi| \leq \delta_0} \left( \hat{u}(t, \xi) - P_0 e^{-\alpha(\xi)t} \cos (|\xi|\alpha t) - P_1 \frac{e^{-\alpha(\xi)t} \sin (|\xi|\alpha t)}{|\xi|^{\alpha}} \right)^2 d\xi \leq C \left[ \|u_1\|_{L^1}^2 + \|u_1\|_{W^{\alpha, 2}}^2 \right] t^{-\frac{n+2\alpha+4\delta}{2\theta}} + C\|u_0\|_{L^2}^2 t^{-\frac{n+2\alpha+4\delta}{2\theta}}, \quad t > 0,
$$

and
where $C$ is a positive constant and $P_0$ and $P_1$ are defined in (4.18).

**Remark 7.** Note that $\epsilon_0 = \min\{2\kappa, \kappa_1, \varepsilon_2, 4\delta\}$, where $\varepsilon_2$ appears at the estimate for $K_0(t, \xi)$.

### 4.3. Estimates for high frequencies.

For $\xi \neq 0$ the estimate in Lemma 4.2 implies

$$|\hat{u}(t, \xi)|^2 \leq Ce^{-\omega\rho(\xi)t} \left( \frac{(1 + |\xi|^{2\alpha})}{|\xi|^{2\alpha}} |\hat{u}_1(\xi)|^2 + |\hat{u}_0(\xi)|^2 \right). \quad (4.19)$$

To the case $\delta_0 \leq |\xi| \leq 1$ we have $\rho(\xi) = \frac{\varepsilon_0^{2\theta}}{2 + |\xi|^{2\alpha}} \geq \frac{\varepsilon}{3} |\xi|^{2\theta} \geq \varepsilon \theta_0^{2\theta}$. Therefore we get the following exponential decay result:

$$\int_{|\xi|^{\delta_0}}^{1} |\hat{u}(t, \xi)|^2 d\xi \leq C \int_{|\xi|^{\delta_0}}^{1} e^{-\varepsilon_0^{2\theta} t} \left( (1 + |\xi|^{2\alpha}) |\xi|^{-2\alpha} |\hat{u}_1(\xi)|^2 + |\hat{u}_0(\xi)|^2 \right) d\xi \leq \frac{2C}{\delta_0} e^{-\varepsilon_0^{2\theta} t} \left[ \|u_1\|^2 + \|u_0\|^2 \right], \quad t > 0,$$

On the other side, for $|\xi| \geq 1$, it follows that $\rho(\xi) = \varepsilon \cdot \frac{|\xi|^{2\theta}}{2 + |\xi|^{2\alpha}} \geq \varepsilon \theta_0^{2\theta}$. Thus, from the pointwise estimate (4.19) we get the polynomial decay

$$\int_{|\xi| \geq 1} |\hat{u}(t, \xi)|^2 d\xi \leq CT^{-\sigma \theta_0} \left[ \|u_1\|^2_{H^{\delta + \alpha}} + \|u_0\|^2_{L^\infty} \right], \quad t > 0,$$

where the estimate (4.20) is obtained for $\varepsilon \geq \alpha - \delta$.

Moreover, the asymptotic profile in (4.17) itself can be estimate easily. Thus

$$\int_{|\xi| \geq \delta_0} |P_1|^2 e^{-|\xi|^{2\theta} t} \frac{\sin^2(|\xi|^{\alpha} t)}{|\xi|^{2\alpha}} d\xi + \int_{|\xi| \geq \delta_0} |P_0|^2 e^{-|\xi|^{2\theta} t} \cos^2(|\xi|^{\alpha} t) d\xi \leq C \left( \|u_0\|^2_{L^1} + \|u_1\|^2_{L^1} \right) \int_{|\xi| \geq \delta_0} e^{-|\xi|^{2\theta} t} e^{-|\xi|^{2\theta} t} d\xi \leq C \left( \|u_0\|^2_{L^1} + \|u_1\|^2_{L^1} \right) e^{-\frac{\delta_0^{2\theta}}{2\theta_0} t}, \quad t > 0, \quad (C > 0 \text{ a constant}).$$

### 4.4. Optimal decay rates.

Finally, combining the estimates for low and high frequency in previous subsections we get the next lemma.

**Lemma 4.6.** Let $n > 2\alpha > 0$, $\frac{a + \delta}{2} \leq \theta < \alpha + \delta$, $0 < \kappa < \min\{1, \delta\}$, $\varepsilon \geq \alpha - \delta$, $u_0 \in L^1(\mathbb{R}^n) \cap H^\delta(\mathbb{R}^n)$ and $u_1 \in L^{1,\kappa}(\mathbb{R}^n) \cap H^{\delta + \varepsilon - \alpha}(\mathbb{R}^n) \cap \dot{W}^{-\alpha, 2}(\mathbb{R}^n)$. Then, there exist $\epsilon_0 > 0$ such that the difference between the solution $\hat{u}(t, \xi)$ of (4.2) and its asymptotic profile satisfies

$$\int_{\mathbb{R}^n} \left| \hat{u}(t, \xi) - \left\{ P_1 e^{\theta(\xi)^{\alpha \theta}} \frac{\sin(|\xi|^{\alpha} t)}{|\xi|^{\alpha \theta}} + P_0 e^{-\alpha(\xi)^{\alpha \theta}} \cos(|\xi|^{\alpha} t) \right\} \right|^2 d\xi \leq C \left( \|u_0\|^2_{L^{1,\kappa}} + \|u_1\|^2_{H^{\delta - \alpha} \cap \dot{W}^{-\alpha, 2}} \right) \int e^{-\frac{\delta_0^{2\theta}}{2\theta_0} t} + C \left( \|u_0\|^2_{L^1} + \|u_1\|^2_{L^1} + \|u_0\|^2 + \|u_1\|^2 \right) e^{-\frac{\delta_0^{2\theta}}{2\theta_0} t} + C T^{-\theta_0^{2\theta}} \left( \|u_0\|^2_{H^\delta} + \|u_1\|^2_{H^{\delta + \varepsilon - \alpha}} \right), \quad t > 0,$$

with $C > 0$ a constant and $d := \frac{\delta_0^{2\theta}}{2\theta_0}$ for a fixed $0 < \delta_0 < 1$. 

**Remark 8.** We notice that it is easy to see that the hypothesis \( u_1 \in \tilde{W}^{-\alpha,2}(\mathbb{R}^n) \) does not need if we consider only the case \( \frac{\alpha+\delta}{2} \leq \theta < \frac{\alpha+3\delta}{2} \) (See the estimate for \( K_6 \) before Lemma 4.5).

Now we are in a position to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let \( n > 2\alpha \). Then, by the Plancherel theorem, Lemma 2.5, the estimate (3.7) and Lemma 4.6 it follows

\[
\|u(t,\cdot)\| \geq |P_1| \left| \frac{e^{-a(\xi)t} \sin(\|\xi\|^\alpha t)}{|\xi|^\alpha} - |P_0| \left| e^{-a(\xi)t} \cos(\|\xi\|^\alpha t) \right| \right| \\
- \left| \hat{u}(t,\cdot) - \left\{ P_1 e^{-a(\xi)t} \frac{\sin(\|\xi\|^\alpha t)}{|\xi|^\alpha} + P_0 e^{-a(\xi)t} \cos(\|\xi\|^\alpha t) \right\} \right| \\
\geq C|P_1| t^{-\frac{n-2\alpha}{2\theta}} - C|P_0| t^{-\frac{\theta}{2}} \\
- C \left( \|u_1\|_{L^1} + \|u_1\|_{W^{\alpha,2}} \right) t^{-\frac{n-2\alpha+\kappa}{4\theta}} - C\|u_0\|_{L^1} t^{-\frac{\theta}{2}} \\
- C \left( \|u_0\|_{H^1} + \|u_1\|_{H^{1+\kappa}} \right) t^{-\frac{n-2\alpha+\kappa}{4\theta}}, \quad t \geq t_0 > 0.
\]

From the above estimate, there exists a constant \( C_1 > 0 \) independent of the time \( t \) and \( t_1 > t_0 > 0 \) such that

\[
\|u(t,\cdot)\| \geq C_1 |P_1| t^{-\frac{n-2\alpha}{2\theta}}, \quad t \geq t_1.
\]

On the other side, applying the Plancherel theorem and Lemma 4.6 one can conclude that

\[
\|u(t,\cdot)\| \leq |P_1| \left| \frac{e^{-a(\xi)t} \sin(\|\xi\|^\alpha t)}{|\xi|^\alpha} + |P_0| \left| e^{-a(\xi)t} \cos(\|\xi\|^\alpha t) \right| \right| \\
+ \left| \hat{u}(t,\cdot) - \left\{ P_1 e^{-a(\xi)t} \frac{\sin(\|\xi\|^\alpha t)}{|\xi|^\alpha} + P_0 e^{-a(\xi)t} \cos(\|\xi\|^\alpha t) \right\} \right| \\
\leq C \left( \|u_1\|_{L^1} + \|u_1\|_{W^{\alpha,2}} \right) t^{-\frac{n-2\alpha+\kappa}{4\theta}} + C\|u_0\|_{L^1} t^{-\frac{\theta}{2}} \\
+ C|P_1| t^{-\frac{n-2\alpha}{2\theta}} + C|P_0| t^{-\frac{\theta}{2}} \\
+ C \left( \|u_0\|_{L^1} + \|u_1\|_{L^1} + \|u_0\|_{H^1} + \|u_1\|_{H^{1+\kappa}} \right) t^{-\frac{n-2\alpha+\kappa}{4\theta}}, \quad t \gg 1,
\]

where \( \kappa, \epsilon_0, \epsilon \) and \( d \) are numbers defined in previous estimates. Therefore, for \( \epsilon \) such that \( \frac{n-2\alpha}{2\theta} < \frac{\epsilon}{2\theta - \alpha} \) we conclude that

\[
\|u(t,\cdot)\| \leq C_2 I_1 t^{-\frac{n-2\alpha}{2\theta}}, \quad t > 0,
\]

with a constant \( C_2 > 0 \) and \( I_1 \) defined in the statement of Theorem 1.2. The proof of Theorem 1.2 is now completed. \( \square \)

**Remark 9.** In order to impose less regularity on the initial data we can consider only \( u_1 \in \tilde{W}^{\delta+\kappa-\alpha,2}(\mathbb{R}^n) \cap W^{-\alpha,2}(\mathbb{R}^n) \) for \( \kappa \geq 0 \) in the estimate (4.20) analogous to the case \( \delta = 0 \).

**Remark 10.** Throughout the same type of estimates above combined with the derivatives of the explicit solution of \( \hat{u}(t, \xi) \) one can also prove the optimality of the
decay rates of the total energy appeared in Theorems 3.2 and 4.3. In fact, taking the derivatives of the explicit solution and the associated asymptotic profile, using Lemmas 2.5, 2.6 and the estimates (3.7), (3.8) we can prove the following optimal decay rates of the total energy appeared in Theorems 3.2 and 4.3. In fact, taking
\[ \|u(t, \cdot)\| \leq C t^{-\frac{\theta}{2}} \quad \text{and} \quad \|(-\Delta)^{\alpha/2} u(t, \cdot)\| \leq C t^{-\frac{\theta}{2}}, \quad t \gg 1, \]
for the case \( \delta = 0 \) and \( \kappa \geq \frac{n(\theta-\alpha)}{2\theta} \) (\( \kappa \) as in Theorem 3.2).

Similarly, for the case \( \delta > 0 \) we can obtain the same optimal decay rates for \( \epsilon \geq \frac{n(\theta-\alpha)}{2\theta} \) as given in Theorem 4.3. Moreover, for \( \epsilon \geq \frac{(n+2\alpha)(2\theta-\alpha)}{2\theta} \) we can prove
\[ \|(-\Delta)^{\delta/2} u(t, \cdot)\| \leq C t^{-\frac{n+2\delta}{2\theta}}, \quad t \gg 1. \]

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