All the codeword bits in polar codes have the same BER under the SC decoder

Guodong Li  Min Ye  Sihuang Hu

Abstract

We prove that for any binary-input memoryless symmetric (BMS) channel, any code length, and any code dimension, all the codeword bits in polar codes have the same bit error rate (BER) under the successive cancellation (SC) decoder.

I. INTRODUCTION

Polar codes were proposed by Arıkan in [1], and they have been extensively studied over the last decade. In his original paper [1], Arıkan introduced the successive cancellation (SC) decoder to decode polar codes, and he proved that polar codes achieve capacity of any binary-input memoryless symmetric (BMS) channel under the SC decoder. Later, the successive cancellation list (SCL) decoder and the CRC-aided SCL decoder were proposed to further reduce the decoding error probability of polar codes [2], [3].

Although the CRC-aided SCL decoder provides the state-of-the-art performance in terms of the decoding error probability, the SC decoder still receives a lot of research attention due to the following two reasons. First, it is amenable to theoretical analysis. In fact, a large part of theoretical research on polar codes focuses on the performance of the SC decoder. Second, the running time of the SC decoder is much smaller than the (CRC-aided) SCL decoder. Therefore, SC decoder is the better choice when there is a stringent requirement on the delay of the communication system.

In an early paper [4], Arıkan observed an interesting experimental result regarding the decoding performance of the SC decoder: The average bit error rate (BER) of a subset of codeword bits is much smaller than the average BER of the message bits. Even till today, there is not a rigorous analysis that can explain this phenomenon. As we repeated the experiments in [4], we found another interesting phenomenon—the BER of each codeword bit in polar codes is surprisingly stable under the SC decoder. We conjectured that this is not a coincidence, and after some effort, we were able to prove this conjecture, which forms the main result of this paper. More precisely, we prove that for any BMS channel, any code length, and any code dimension, all the codeword bits in polar codes have exactly the same BER under the SC decoder.

After this paper was completed, we were informed that Lemma 5 and Lemma 6 were already proved in Theorem 2 of [5]. At the same time, the main focus of [5] is quite different from this paper. The main purpose of [5] is to design new decoding algorithms with smaller decoding error probability for Reed-Muller codes and polar codes. In contrast, we prove the equal BER property of the SC decoder in this paper.
The rest of this paper is organized as follows. In Section II we provide the background on polar codes and state the main result. In Section III we prove the main result. In Section IV we discuss the connection between our results and the observation in [4].

II. BACKGROUND AND OUR MAIN RESULT

For a set \( \mathcal{A} = \{i_1, i_2, \ldots, i_s\} \) of size \( s \), we use \( x_\mathcal{A} \) to denote the vector \((x_{i_1}, x_{i_2}, \ldots, x_{i_s})\), where we assume that \( i_1 < i_2 < \cdots < i_s \) are nonnegative integers. We use \( x_{[a:b]} \) to denote the vector \((x_a, x_{a+1}, x_{a+2}, \ldots, x_b)\). We say that a memoryless channel \( W \) with binary input alphabet \( \mathcal{X} = \{0, 1\} \) is a BMS channel if there is a permutation \( \pi \) on the output alphabet \( \mathcal{Y} \) satisfying i) \( \pi^{-1} = \pi \) and ii) \( W(y|1) = W(\pi(y)|0) \) for all \( y \in \mathcal{Y} \).

The construction of polar codes with code length \( n = 2^m \) involves the \( n \times n \) matrix \( \mathbf{G}_n = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^\otimes m \), where \( \otimes \) is the Kronecker product. The matrix \( \mathbf{G}_n \) serves as a linear mapping between the message vector \( u_{[0:n-1]} \) and the codeword vector \( x_{[0:n-1]} \). The message vector \( u_{[0:n-1]} \) consists of information bits and frozen bits. We use \( \mathcal{A} \subseteq \{0, 1, \ldots, n-1\} \) to denote the index set of information bits and use \( \mathcal{A}^c = \{0, 1, \ldots, n-1\} \setminus \mathcal{A} \) to denote the index set of frozen bits. A polar code has four parameters—the code length \( n \), the code dimension \( k \), the index set \( \mathcal{A} \) of information bits, and the vector \( \vec{u}_{\mathcal{A}^c} \in \{0, 1\}^{n-k} \) of frozen bits. More precisely, we define

\[
\text{Polar}(n, k, \mathcal{A}, \vec{u}_{\mathcal{A}^c}) := \{u_{[0:n-1]} \mathbf{G}_n : u_{\mathcal{A}^c} = \vec{u}_{\mathcal{A}^c}, u_{\mathcal{A}} \in \{0, 1\}^k\}.
\]

Next let us recall how the SC decoder works. For a BMS channel \( W : \{0, 1\} \to \mathcal{Y} \), we define \( W^n : \{0, 1\}^n \to \mathcal{Y}^n \) as \( W^n(y_{[0:n-1]}|x_{[0:n-1]}) = \prod_{i=0}^{n-1} W(y_i|x_i) \) for \( x_{[0:n-1]} \in \{0, 1\}^n \) and \( y_{[0:n-1]} \in \mathcal{Y}^n \). For \( 0 \leq i \leq n-1 \), we further define the bit-channel \( W_i^n : \{0, 1\} \to \mathcal{Y}^n \times \{0, 1\}^i \) as

\[
W_i^n(y_{[0:n-1]}, u_{[0:i-1]}|u_i) = \frac{1}{2^{n-1}} \sum_{u_{[i+1:n-1]} \in \{0,1\}^{n-i-1}} W^n(y_{[0:n-1]}|u_{[0:n-1]} \mathbf{G}_n).
\]

The SC decoder decodes one by one from \( u_0 \) to \( u_{n-1} \). If \( i \in \mathcal{A} \), then the decoding result of the \( i \)th bit is \( \hat{u}_i = \vec{u}_i \). If \( i \in \mathcal{A}^c \), then the decoding result of the \( i \)th bit is

\[
\hat{u}_i = \hat{u}_i(y_{[0:n-1]}, u_{[0:i-1]}|u_i) = \argmax_{u_i \in \{0, 1\}} W_i^n(y_{[0:n-1]}, u_{[0:i-1]}|u_i).
\]

If there is a tie, i.e., if \( W_i^n(y_{[0:n-1]}, u_{[0:i-1]}|0) = W_i^n(y_{[0:n-1]}, u_{[0:i-1]}|1) \), then the SC decoder outputs a random decoding result with probability \( P(\hat{u}_i = 0) = P(\hat{u}_i = 1) = 1/2 \). The decoding results of the SC decoder depend not only on the channel output vector \( y_{[0:n-1]} \), but also on the set \( \mathcal{A} \) and the values of the frozen bits \( \vec{u}_{\mathcal{A}^c} \). We write the SC decoding result of the message vector as \( \hat{u}_{[0:n-1]}(y_{[0:n-1]}, \mathcal{A}, \vec{u}_{\mathcal{A}^c}) \). The SC decoding result of the codeword vector is

\[
\hat{x}_{[0:n-1]}(y_{[0:n-1]}, \mathcal{A}, \vec{u}_{\mathcal{A}^c}) = u_{[0:n-1]}(y_{[0:n-1]}, \mathcal{A}, \vec{u}_{\mathcal{A}^c}) \mathbf{G}_n.
\]

For a polar code \( \mathcal{C} = \text{Polar}(n, k, \mathcal{A}, \vec{u}_{\mathcal{A}^c}) \), a BMS channel \( W \), and a specific choice of information vector \( \vec{u}_{\mathcal{A}} \in \{0, 1\}^k \), we write the transmitted codeword as \( \bar{x}_{[0:n-1]} = \vec{u}_{[0:n-1]} \mathbf{G}_n \). In this case, the BER of the \( j \)th codeword bit under the SC decoder is

\[
\text{BER}_j(\mathcal{C}, W, \vec{u}_{\mathcal{A}}) = \sum_{y_{[0:n-1]} \in \mathcal{Y}^n} \sum_{x_{[0:n-1]} \in \{0, 1\}^n} W^n(y_{[0:n-1]}|x_{[0:n-1]}) P(\hat{x}_{[0:n-1]}(y_{[0:n-1]}, \mathcal{A}, \vec{u}_{\mathcal{A}^c}) = x_{[0:n-1]}(1|x_j \neq \bar{x}_j).
\]

\(^1\)We use \( \bar{u} \) and its variations to denote the true value. We use \( \hat{u} \) and its variations to denote the decoded value.
The term \( P(\hat{x}_{[0:n-1]}(y_{[0:n-1]}), A, \bar{u}_{A'}) \neq x_{[0:n-1]} \) appears because the SC decoder involves randomness when there is a tie in \( (1) \). The average BER of the \( j \)th codeword bit over all choices of the information vector is

\[
\text{BER}_j(C, W) = \frac{1}{2^k} \sum_{\bar{u}_{A} \in \{0,1\}^k} \text{BER}_j(C, W, \bar{u}_{A}).
\]

For an integer \( 0 \leq i \leq 2^m - 1 \), we write its binary expansion as

\[
i = 2^{m-1}b_{m-1}(i) + 2^{m-2}b_{m-2}(i) + \cdots + 2b_1(i) + b_0(i),
\]
where \( b_{m-1}(i), b_{m-2}(i), \ldots, b_1(i), b_0(i) \in \{0,1\} \). For two integers \( 0 \leq i, j \leq 2^m - 1 \), we say that \( i \geq j \) if \( b_r(i) \geq b_r(j) \) for all \( 0 \leq r \leq m - 1 \). Now we are ready to state our main result.

**Theorem 1.** Let \( n = 2^m \) and \( k \) be two positive integers satisfying \( k \leq n \). Let \( A \subseteq \{0,1,\ldots,n-1\} \) be a set of size \( |A| = k \) satisfying the following condition:

\[
\text{If } j \in A \text{ and } i \geq j, \text{ then } i \in A.
\]

Let \( \bar{u}_{A'} \in \{0,1\}^{n-k} \) be a binary vector of length \( n-k \). We write \( C = \text{Polar}(n,k,A,\bar{u}_{A'}) \). Then for any BMS channel \( W \), we have

\[
\text{BER}_0(C, W) = \text{BER}_1(C, W) = \text{BER}_2(C, W) = \cdots = \text{BER}_{n-1}(C, W).
\]

We need the following lemma to apply Theorem 1 to polar codes.

**Lemma 1.** The set \( A \) in polar code construction always satisfies the condition \( (4) \).

**Proof.** The set \( A \) in polar codes is constructed as follows: We first reorder the indices \( \{0,1,\ldots,n-1\} \) according to the channel capacity of the bit-channels. Let \( \{j_0, j_1, j_2, \ldots, j_{n-1}\} \) be a permutation of \( \{0,1,\ldots,n-1\} \) satisfying that \( I(W_{j_0}^{(n)}) \geq I(W_{j_1}^{(n)}) \geq I(W_{j_2}^{(n)}) \geq \cdots \geq I(W_{j_{n-1}}^{(n)}) \), where \( I(\cdot) \) is the channel capacity. Then we choose \( A \) to be \( A = \{j_0, j_1, j_2, \ldots, j_k-1\} \).

To prove this lemma, we only need to show that if \( i \geq j \), then \( I(W_i^{(n)}) \geq I(W_j^{(n)}) \). Recall from the polar coding literature that the bit-channel \( W_0^{(2)} \) is the \( \cdots \) transform of \( W \), and the bit-channel \( W_1^{(2)} \) is the \( \cdots \) transform of \( W \), i.e., \( W_0^{(2)} = W^- \) and \( W_1^{(2)} = W^+ \). Moreover, the following recursive relation holds between the bit-channels:

\[
W_{2i}^{(n)} = (W_i^{(n/2)})^-, \quad W_{2i+1}^{(n)} = (W_i^{(n/2)})^+ \quad \text{for } 0 \leq i \leq n/2 - 1.
\]

Therefore, each bit-channel \( W_i^{(n)} \) is obtained from \( m \) consecutive \(+/-\) transforms of \( W \), where \( n = 2^m \). More precisely, let \( b(i) = (b_{m-1}(i), b_{m-2}(i), \ldots, b_1(i), b_0(i)) \in \{0,1\}^m \) be the binary expansion of \( i \) defined in \( (3) \). We then define \( s(i) = (s_{m-1}(i), s_{m-2}(i), \ldots, s_1(i), s_0(i)) \in \{-,-,+,+\}^m \) as follows: For \( 0 \leq r \leq m - 1 \), if \( b_r(i) = 0 \), then we set \( s_r(i) = -; \) if \( b_r(i) = 1 \), then we set \( s_r(i) = +. \) The recursive relation \( (5) \) implies that \( W_i^{(n)} = W_{s(i)} \) for \( 0 \leq i \leq n-1 \). Now assuming that \( i \geq j \). By definition, it means that for \( 0 \leq r \leq m - 1 \), if \( s_r(i) \neq s_r(j) \), then \( s_r(i) = + \) and \( s_r(j) = - \). Since the \(+/-\) transform results in a better channel and the \(-/+\) transform results in a worse channel, we have \( I(W_i^{(n)}) \geq I(W_j^{(n)}) \). \( \square \)

Note that the condition \( (4) \) is necessary for Theorem 1 to hold. Let us consider the simplest case of \( n = 2 \) and \( k = 1 \). The choice of \( A = \{1\} \) satisfies the condition \( (4) \). Under this choice, \( u_0 \) is the frozen bit, so its decoding result \( \hat{u}_0 \) is equal to the true value \( u_0 \). The decoding results of the codeword bits are \( \hat{x}_0 = \hat{u}_0 + \hat{u}_1 \) and \( \hat{x}_1 = \hat{u}_1 \). Therefore, the three events \( \{\hat{x}_0 \neq \bar{u}_0\}, \{\hat{x}_1 \neq \bar{u}_1\} \) and \( \{\hat{u}_0 \neq \bar{u}_1\} \) are the

\(^{2}\)A more rigorous argument involves the concept of channel degradation, whose definition is given in Section III of [6] and Definition 1.7 of [2], among many other places. It is well known that if a BMS channel \( W \) is degraded with respect to another BMS channel \( V \), then \( I(W) \leq I(V) \); see for example Lemma 3 in [6]. Furthermore, the degradation relation is preserved by the \( \cdots \) \( \cdots \) transforms; see for example Lemma 5 in [6] and Lemma 4.7 in [7]. One can also show that \( W^- \) is degraded with respect to \( W \), and \( W \) is degraded with respect to \( W^+ \). Combining these three facts, we are able to obtain our conclusion in a more rigorous way. Since this argument is quite standard in polar coding literature, we omit the details.
same, so the two codeword bits have the same BER. On the other hand, the choice of \( A = \{0\} \) does not satisfy the condition (4). Under this choice, \( u_1 \) is the frozen bit, so its decoding result \( \hat{u}_1 \) is equal to the true value \( \bar{u}_1 \), which implies that \( \hat{x}_1 = \bar{x}_1 \). However, the event \( \{ \hat{x}_0 \neq \bar{x}_0 \} \) happens with positive probability unless the channel \( W \) is noiseless. Therefore, the two codeword bits do not have the same BER when the condition (4) is not satisfied.

### III. Proof of the main result

#### A. Restricting to the all-zero codeword

Recall that there is a permutation \( \pi \) on the output alphabet \( Y \) of a BMS channel \( W \) satisfying i) \( \pi^{-1} = \pi \) and ii) \( W(y|1) = W(\pi(y)|0) \) for all \( y \in Y \). These two conditions further imply that \( W(y|0) = W(\pi(y)|1) \). For \( x \in \{0, 1\} \) and \( y \in Y \), we define \( x \cdot y \) as follows: If \( x = 0 \), then \( x \cdot y = y \); if \( x = 1 \), then \( x \cdot y = \pi(y) \). Then it is easy to see that \( W(y|x \oplus a) = W(a \cdot y|x) \) for all \( a, x \in \{0, 1\} \), where \( \oplus \) is the addition over the binary field. For two vectors \( x_{[0:n-1]} \in \{0, 1\}^n \) and \( y_{[0:n-1]} \in Y^n \), we define \( x_{[0:n-1]} \cdot y_{[0:n-1]} = (x_0 \cdot y_0, x_1 \cdot y_1, \ldots, x_{n-1} \cdot y_{n-1}) \). Then we have \( W^n(x_{[0:n-1]} \cdot y_{[0:n-1]}) = W^n(a_{[0:n-1]} \cdot y_{[0:n-1]}) \) for all \( a_{[0:n-1]}, x_{[0:n-1]} \in \{0, 1\}^n \), where \( \cdot \) is the coordinatewise addition over the binary field.

We first prove that the BER of each codeword bit is independent of the true value \( \bar{u}_{[0:n-1]} \) of the message vector. Note that the proof technique of Lemma 2 below is quite standard in polar coding literature. In fact, this proof technique was already used in Arikan’s original paper; see Section VI of [1]. The only difference is that the proof in [1] focused on the word error rate while we focus on BER.

**Lemma 2.** For any \( \bar{u}_{[0:n-1]} \in \{0, 1\}^n \), any \( \bar{u}'_{[0:n-1]} \in \{0, 1\}^n \), and any \( 0 \leq j \leq n - 1 \), we have

\[
\text{BER}_j(\text{Polar}(n, k, \bar{u}_A, \bar{u}'_A), \bar{u}, \bar{u}_A) = \text{BER}_j(\text{Polar}(n, k, \bar{u}_A, \bar{u}'_A), \bar{u}, \bar{u}_A).
\]

**Proof.** Define \( a_{[0:n-1]} = \bar{u}_{[0:n-1]} \oplus \bar{u}'_{[0:n-1]} \cdot x_{[0:n-1]} = a_{[0:n-1]} G_n, \bar{x}_{[0:n-1]} = \bar{u}_{[0:n-1]} G_n, \) and \( \bar{x}'_{[0:n-1]} = \bar{u}'_{[0:n-1]} G_n \). Then we have

\[
W^n(y_{[0:n-1]} \bar{x}_{[0:n-1]}) = W^n(x_{[0:n-1]} \cdot y_{[0:n-1]} \bar{x}'_{[0:n-1]}) \quad \text{for all } y_{[0:n-1]} \in Y^n,
\]

\[
1[z \neq \bar{x}_j] = 1[z \oplus x'_j \neq x_j] \quad \text{for all } z \in \{0, 1\} \text{ and all } 0 \leq j \leq n - 1.
\]

In this proof, we use the shorthand notation

\[
\hat{u}_{[0:n-1]}(y_{[0:n-1]}) = \hat{u}_{[0:n-1]}(y_{[0:n-1]}, \bar{u}_A, \bar{u}'_A), \quad \hat{u}'_{[0:n-1]}(y_{[0:n-1]}) = \hat{u}_{[0:n-1]}(y_{[0:n-1]}, \bar{u}_A, \bar{u}'_A).
\]

We first prove that

\[
P(\hat{u}_{[0:n-1]}(y_{[0:n-1]}) = u_{[0:n-1]}) = P(\hat{u}'_{[0:n-1]}(x_{[0:n-1]} \cdot y_{[0:n-1]}) = u_{[0:n-1]} \oplus a_{[0:n-1]})
\]

for all \( u_{[0:n-1]} \in \{0, 1\}^n \). The randomness here comes from the random decision of the SC decoder when there is a tie in (1). Note that

\[
P(\hat{u}_{[0:n-1]}(y_{[0:n-1]}) = u_{[0:n-1]}) = \prod_{i=0}^{n-1} P(\hat{u}_i(y_{[0:n-1]}, u_{[0:i-1]}) = u_i),
\]

\[
P(\hat{u}'_{[0:n-1]}(x_{[0:n-1]} \cdot y_{[0:n-1]}) = u_{[0:n-1]} \oplus a_{[0:n-1]})
\]

\[
= \prod_{i=0}^{n-1} P(\hat{u}_i(x_{[0:n-1]} \cdot y_{[0:n-1]}, u_{[0:i-1]} \oplus a_{[0:i-1]}) = u_i \oplus a_i).
\]

Therefore, in order to prove (8), we only need to show that

\[
P(\hat{u}_i(y_{[0:n-1]}, u_{[0:i-1]}) = u_i) = P(\hat{u}'_i(x_{[0:n-1]} \cdot y_{[0:n-1]}, u_{[0:i-1]} \oplus a_{[0:i-1]}) = u_i \oplus a_i)
\]
for all $0 \leq i \leq n - 1$ and all $u_{[0:n-1]} \in \{0, 1\}^{i+1}$. If $i \in \mathcal{A}^c$, then $\hat{u}_i = \bar{u}_i$ and $\bar{u}_i' = \bar{u}_i$. Since $\bar{u}_i' = \bar{u}_i \oplus a_i$, the equality \( (9) \) clearly holds. If $i \in \mathcal{A}$, then we need to analyze the bit-channel in \( (1) \). More specifically, we have

\[
W_i(n) (y_{[0:n-1]} | u_{[0:i-1]} | u_i) = \frac{1}{2^{n-1}} \sum_{u_{[i+1:n]} \in \{0,1\}^{n-i-1}} W^n(y_{[0:n-1]} | u_{[0:n-1]} G_n)
\]

\[
= \frac{1}{2^{n-1}} \sum_{u_{[i+1:n]} \in \{0,1\}^{n-i-1}} W^n(x^a_{[0:n-1]} \cdot y_{[0:n-1]} | (u_{[0:n-1]} \oplus a_{[0:n-1]}) G_n)
\]

\[
=W_i(n) (x^a_{[0:n-1]} \cdot y_{[0:n-1]} | u_{[0:i-1]} \oplus a_{[i+1:n]} | u_i \oplus a_i).
\]

The last equality holds because when $u_{[i+1:n]}$ ranges over all values in $\{0, 1\}^{n-i-1}$, the sum $u_{[i+1:n]} \oplus a_{[i+1:n]}$ also ranges over all values in $\{0, 1\}^{n-i-1}$. This equality together with the decoding rule \( (1) \) immediately implies that \( (9) \) holds for $i \in \mathcal{A}$. This completes the proof of \( (8) \) and \( (9) \).

Recall the notation for the decoding result of the codeword vector in \( (2) \). Similarly to \( (7) \), we use the shorthand notation

\[
\hat{x}_{[0:n-1]}(y_{[0:n-1]}) = \hat{x}_{[0:n-1]}(y_{[0:n-1]}, \mathcal{A}, \bar{u}_A), \quad \hat{x}'_{[0:n-1]}(y_{[0:n-1]}) = \hat{x}_{[0:n-1]}(y_{[0:n-1]}, \mathcal{A}, \bar{u}'_A).
\]

With this notation, Equation \( (8) \) is equivalent to

\[
\mathbb{P}(\hat{x}_{[0:n-1]}(y_{[0:n-1]})) = \mathbb{P}(\hat{x}_{[0:n-1]}(x^a_{[0:n-1]} \cdot y_{[0:n-1]})) = x_{[0:n-1]} \oplus x^a_{[0:n-1]})
\]

for all $x_{[0:n-1]} \in \{0, 1\}^n$. Combining this with \( (6) \), we have

\[
\text{BER}_j(\text{Polar}(n, k, \mathcal{A}, \bar{u}_A), W, \bar{u}_A)
\]

\[
= \sum_{y_{[0:n-1]} \in \mathcal{Y}^n} \mathbb{P}(\hat{x}_{[0:n-1]}(y_{[0:n-1]})) \mathbb{P}(\hat{x}_{[0:n-1]}(y_{[0:n-1]} | y_{[0:n-1]})) = x_{[0:n-1]} \oplus x^a_{[0:n-1]}) \mathbb{1}[x_j \neq \bar{x}_j]
\]

\[
= \sum_{y_{[0:n-1]} \in \mathcal{Y}^n} \sum_{x_{[0:n-1]} \in \{0,1\}^n} \mathbb{P}(\hat{x}_{[0:n-1]}(y_{[0:n-1]} | x_{[0:n-1]})) \mathbb{P}(\hat{x}_{[0:n-1]}(y_{[0:n-1]} | x_{[0:n-1]})) = x_{[0:n-1]} \oplus x^a_{[0:n-1]}) \mathbb{1}[x_j \neq \bar{x}_j]
\]

\[
= \text{BER}_j(\text{Polar}(n, k, \mathcal{A}, \bar{u}_A'), W, \bar{u}_A').
\]

The equality \( (\ast) \) is obtained from replacing $x^a_{[0:n-1]} \cdot y_{[0:n-1]}$ with $y_{[0:n-1]}$ and replacing $x_{[0:n-1]} \oplus x^a_{[0:n-1]}$ with $x_{[0:n-1]}$. These two replacements are eligible because when $y_{[0:n-1]}$ ranges over $\mathcal{Y}^n$, $x^a_{[0:n-1]} \cdot y_{[0:n-1]}$ also ranges over all values in $\mathcal{Y}^n$; similarly, when $x_{[0:n-1]}$ ranges over $\{0, 1\}^n$, $x_{[0:n-1]} \oplus x^a_{[0:n-1]}$ also ranges over all values in $\mathcal{Y}^n$. Moreover, since $x_j \oplus x^a_j$ is the $j$th coordinate of $x_{[0:n-1]} \oplus x^a_{[0:n-1]}$, we also replace $x_j \oplus x^a_j$ with $x_j$ when we replace $x_{[0:n-1]} \oplus x^a_{[0:n-1]}$ with $x_{[0:n-1]}$. This completes the proof of the lemma.

This lemma implies that in the analysis of BER of the SC decoder, we can always assume that the true value $\bar{u}_{[0:n-1]}$ of the message vector (including both information bits and frozen bits) is all-zero, or equivalently, the transmitted codeword is all-zero. More specifically, we have the following expression for the BER:

\[
\text{BER}_j(\text{Polar}(n, k, \mathcal{A}, \bar{u}_A), W) = \text{BER}_j(\text{Polar}(n, k, \mathcal{A}, 0^{n-k}), W, 0^k)
\]

\[
= \sum_{y_{[0:n-1]} \in \mathcal{Y}^n} \sum_{x_{[0:n-1]} \in \{0,1\}^n} W^n(y_{[0:n-1]} | 0^n) \mathbb{P}(\hat{x}_{[0:n-1]}(y_{[0:n-1]} | \mathcal{A}, 0^{n-k}) = x_{[0:n-1]} \oplus 0^n) \mathbb{1}[x_j \neq 0], \tag{11}
\]

(\text{BER}_j(\text{Polar}(n, k, \mathcal{A}, \bar{u}_A), W) = \text{BER}_j(\text{Polar}(n, k, \mathcal{A}, 0^{n-k}), W, 0^k))
where $0^i$ is the all-zero vector of length $i$.

Note that (10) immediately implies the following lemma, which will be used later to prove the main theorem.

**Lemma 3.** Let $x^a_{[0:n-1]} = a_{[0:n-1]}G_n$ be a codeword of Polar$(n, k, A, 0^{n-k})$, i.e., $a_i = 0$ for all $i \in A^c$. Then
\[
\mathbb{P}(\hat{x}_{[0:n-1]}(y_{[0:n-1]}), A, 0^{n-k}) = x_{[0:n-1]}(y_{[0:n-1]}), A, 0^{n-k}) = x_{[0:n-1]} \oplus x^a_{[0:n-1]}
\]
for all $x_{[0:n-1]} \in \{0, 1\}^n$ and all $y_{[0:n-1]} \in \mathcal{Y}^n$.

As a final remark, we note that we do not need condition (4) for the set $A$ in Lemma 2 and Lemma 3, but we will need this condition later in the proof of the main theorem.

**B. Recursive implementation of the SC decoder based on LLR**

In practice, the SC decoder is usually implemented recursively based on the log likelihood ratio (LLR). Below we recap this recursive structure since it is needed in the next step of our proof. The LLR of an output symbol $y \in \mathcal{Y}$ is
\[
\text{LLR}(y) = \ln \frac{W(y|0)}{W(y|1)}.
\]
For a channel output vector $y_{[0:n-1]} \in \mathcal{Y}^n$, we write its LLR vector as
\[
L_{[0:n-1]} = L_{[0:n-1]}(y_{[0:n-1]}) = (L_0, L_1, \ldots, L_{n-1}), \quad \text{where } L_i = \text{LLR}(y_i) \text{ for } 0 \leq i \leq n-1.
\]
We write $L_{[0:n-1]}(y_{[0:n-1]})$ when we want to emphasize its dependence on $y_{[0:n-1]}$. In most scenarios this dependence is clear from the context, and we will simply write $L_{[0:n-1]}$.

It is well-known that the decoding result of the SC decoder only depends on the LLR of the channel outputs. In other words, if two channel output vectors have the same LLR, then their decoding results have the same probability distribution under the SC decoder. Below we will write $\hat{u}_i(y_{[0:n-1]}, u_{[i-1]})$ and $\hat{u}_i(L_{[0:n-1]}, u_{[i-1]})$ interchangeably. We also write $\hat{u}_i(y_{[0:n-1]}, A, u_{A^c})$ and $\hat{u}_i(L_{[0:n-1]}, A, u_{A^c})$ interchangeably.

At the beginning of Section II-A, we defined an operation $x \cdot y$ for $x \in \{0, 1\}$ and $y \in \mathcal{Y}$. Now we define an analogous operation for LLR. For $x \in \{0, 1\}$ and a real number $L \in \mathbb{R}$, we define $x \odot L$ as follows: If $x = 0$, then $x \odot L = L$; if $x = 1$, then $x \odot L = -L$. It is easy to verify that LLR($x \cdot y$) = $x \odot $LLR($y$) for all $x \in \{0, 1\}$ and $y \in \mathcal{Y}$. For two vectors $x_{[0:n-1]} \in \{0, 1\}^n$ and $L_{[0:n-1]} \in \mathbb{R}^n$, we define $x_{[0:n-1]} \odot L_{[0:n-1]} = (x_0 \odot L_0, x_1 \odot L_1, \ldots, x_{n-1} \odot L_{n-1})$. With this new notation, we can restate Lemma 3 as follows.

**Lemma 4.** Let $x^a_{[0:n-1]} = a_{[0:n-1]}G_n$ be a codeword of Polar$(n, k, A, 0^{n-k})$, i.e., $a_i = 0$ for all $i \in A^c$. Let $L_{[0:n-1]} = L_{[0:n-1]}(y_{[0:n-1]})$ be the LLR vector of $y_{[0:n-1]}$. Then
\[
\mathbb{P}(\hat{x}_{[0:n-1]}(L_{[0:n-1]}), A, 0^{n-k}) = \mathbb{P}(\hat{x}_{[0:n-1]}(x^a_{[0:n-1]} \oplus L_{[0:n-1]}), A, 0^{n-k}) = x_{[0:n-1]} \oplus x^a_{[0:n-1]}
\]
for all $x_{[0:n-1]} \in \{0, 1\}^n$ and all $y_{[0:n-1]} \in \mathcal{Y}^n$.

We need some more notation to describe the recursive structure of the SC decoder. For $y_0, y_1 \in \mathcal{Y}$ and $u \in \{0, 1\}$, we define
\[
L^-(y_0, y_1) = \ln \left(\frac{1 + \exp(\text{LLR}(y_0) + \text{LLR}(y_1))}{\exp(\text{LLR}(y_0)) + \exp(\text{LLR}(y_1))}\right),
\]
\[
L^+(y_0, y_1, u) = (-1)^u y_0 + y_1.
\]

\(^{3}\text{Recall that the SC decoder produces a random decoding result when there is a tie in } (1).

For $y_{[0:n-1]} \in \mathcal{Y}^n$ and $u_{[0:n/2-1]} \in \{0,1\}^{n/2}$, we define

$$L_{[0:n/2-1]}^{-}(y_{[0:n-1]}) = (L_0^-, L_1^-, \ldots, L_{n/2-1}^-), \quad \text{where } L_i^- = L^-(y_i, y_{i+n/2}) \text{ for } 0 \leq i \leq n/2 - 1,$$

$$L_{[0:n/2-1]}^{+}(y_{[0:n-1]}, u_{[0:n/2-1]}) = (L_0^+, L_1^+, \ldots, L_{n/2-1}^+),$$

where $L_i^+ = L^+(y_i, y_{i+n/2}, u_i)$ for $0 \leq i \leq n/2 - 1$.

For a set $\mathcal{A} \subseteq \{0,1,\ldots,n-1\}$, we define

$$\mathcal{A}^- = \mathcal{A} \cap \{0,1,\ldots,n/2-1\} \quad \text{and} \quad \mathcal{A}^+ = \{i : i + n/2 \in \mathcal{A} \cap \{n/2,n/2 + 1,\ldots,n-1\}\}.$$

The encoding procedure $x_{[0:n-1]} = u_{[0:n-1]}G_n$ for polar codes can be decomposed in the following way: Let $z_{[0:n/2-1]} = u_{[0:n/2-1]}G_n/2$ and $z_{[n/2-n-1]} = u_{[n/2-n-1]}G_n/2$, i.e., we first encode the two halves separately. Then $x_{[0:n/2-1]} = z_{[0:n/2-1]} \oplus z_{[n/2-n-1]}$ and $x_{[n/2-n-1]} = z_{[n/2-n-1]}$. This recursive structure can be summarized as

$$u_{[0:n-1]}G_n = ((u_{[0:n/2-1]} \oplus u_{[n/2-n-1]})G_n/2, u_{[n/2-n-1]}G_n/2).$$

From this point on, we assume that all the frozen bits take value 0, which is justified by Lemma 2.

We use the shorthand notation

$$\hat{u}_{[0:n-1]}(L_{[0:n-1]}, \mathcal{A}) = \hat{u}_{[0:n-1]}(L_{[0:n-1]}, \mathcal{A}, 0^{n-k}), \quad \hat{x}_{[0:n-1]}(L_{[0:n-1]}, \mathcal{A}) = \hat{x}_{[0:n-1]}(L_{[0:n-1]}, \mathcal{A}, 0^{n-k}).$$

Given a channel output vector $y_{[0:n-1]}$, the SC decoder first decodes $u_{[0:n/2-1]}$ as

$$\hat{u}_{[0:n/2-1]}(L_{[0:n/2-1]}^{-}(y_{[0:n-1]}), \mathcal{A}^-).$$

Then it calculates $\hat{z}_{[0:n/2-1]} = \hat{u}_{[0:n/2-1]}(L_{[0:n/2-1]}^{-}(y_{[0:n-1]}), \mathcal{A}^-)G_n/2$. In the next step, the SC decoder decodes $u_{[n/2-n-1]}$ as

$$\hat{u}_{[0:n/2-1]}(L_{[0:n/2-1]}^{+}(y_{[0:n-1]}, \hat{z}_{[0:n/2-1]}), \mathcal{A}^+).$$

In summary, we have

$$\mathbb{P}(\hat{u}_{[0:n-1]}(y_{[0:n-1]}), \mathcal{A}) = u_{[0:n-1]}$$

$$=\mathbb{P}(\hat{u}_{[0:n/2-1]}(L_{[0:n/2-1]}^{-}(y_{[0:n-1]}), \mathcal{A}^-) = u_{[0:n/2-1]})$$

$$\times \mathbb{P}(\hat{u}_{[0:n/2-1]}(L_{[0:n/2-1]}^{+}(y_{[0:n-1]}, u_{[0:n/2-1]}G_n/2), \mathcal{A}^+) = u_{[n/2-n-1]})$$

for all $y_{[0:n-1]} \in \mathcal{Y}^n$ and all $u_{[0:n-1]} \in \{0,1\}^n$. In light of (14), Equation (15) is equivalent to

$$\mathbb{P}(\hat{x}_{[0:n-1]}(y_{[0:n-1]}), \mathcal{A}) = (z_{[0:n/2-1]} \oplus z_{[n/2-n-1]}; z_{[n/2-n-1]}))$$

$$=\mathbb{P}(\hat{x}_{[0:n/2-1]}(L_{[0:n/2-1]}^{-}(y_{[0:n-1]}), \mathcal{A}^-) = z_{[0:n/2-1]})$$

$$\times \mathbb{P}(\hat{x}_{[0:n/2-1]}(L_{[0:n/2-1]}^{+}(y_{[0:n-1]}, \hat{z}_{[0:n/2-1]}), \mathcal{A}^+) = z_{[n/2-n-1]})$$

for all $y_{[0:n-1]} \in \mathcal{Y}^n$ and all $z_{[0:n-1]} \in \{0,1\}^n$.

C. Proof of Theorem 7

For $n = 2^m$, we define $m$ permutations $\delta_0^{(m)}, \delta_1^{(m)}, \ldots, \delta_{m-1}^{(m)}$ on the set $\{0,1,\ldots,n-1\}$. For $0 \leq i \leq n-1$, let $(b_{m-1}(i), b_{m-2}(i), \ldots, b_1(i), b_0(i))$ be its binary expansion defined in (3). For $0 \leq r \leq m-1$, $\delta_r^{(m)}(i)$ is obtained from flipping the $r$th digit in the binary expansion of $i$. In other words, the binary expansion of $\delta_r^{(m)}(i)$ is

$$(b_{m-1}(i), b_{m-2}(i), \ldots, b_{r+1}(i), b_r(i) \oplus 1, b_{r-1}(i), \ldots, b_1(i), b_0(i)).$$

As a concrete example, if we apply the permutation $\delta_1^{(3)}$ to each coordinate of $(0,1,2,3)$, then we obtain $(2,3,0,1)$; if we apply the permutation $\delta_0^{(2)}$ to each coordinate of $(0,1,2,3)$, then we obtain $(1,0,3,2)$. 

We further define \( m \) mappings \( \xi^{(m)}_0, \xi^{(m)}_1, \ldots, \xi^{(m)}_{m-1} \) on vectors of length \( n \). For \( 0 \leq r \leq m-1 \) and a length-\( n \) vector \( x_{[0:n-1]} \), we define

\[
\xi^{(m)}_r(x_{[0:n-1]}) = (x_{\delta^{(m)}_r(0)}, x_{\delta^{(m)}_r(1)}, \ldots, x_{\delta^{(m)}_r(n-1)}).
\]

In particular, we have

\[
\xi^{(m)}_{m-1}(x_{[0:n-1]}) = (x_{[n/2:n-1]}, x_{[0:n/2-1]}).
\]

**Lemma 5.** Let \( n = 2^m \). Suppose that all the frozen bits take value 0. Suppose that the index set \( A \) of information bits satisfies the condition (14). Then

\[
\mathbb{P}(\hat{x}_{[0:n-1]}(y_{[0:n-1]}, A) = x_{[0:n-1]}) = \mathbb{P}(\hat{x}_{[0:n-1]}(\xi^{(m)}_{m-1}(y_{[0:n-1]}), A) = \xi^{(m)}_{m-1}(x_{[0:n-1]}))
\]

for all \( y_{[0:n-1]} \in \mathcal{Y}^n \) and all \( x_{[0:n-1]} \in \{0,1\}^n \).

**Proof.** We will prove another equation that is equivalent to (17). Specifically, we will prove that

\[
\mathbb{P}(\hat{x}_{[0:n-1]}(y_{[0:n-1]}, A) = (z_{[0:n/2-1]} \oplus z_{[n/2:n-1]}, z_{[n/2:n-1]})) = \mathbb{P}(\hat{x}_{[0:n-1]}(\xi^{(m)}_{m-1}(y_{[0:n-1]}), A) = (z_{[n/2:n-1]} \oplus z_{[n/2:n-1]}))
\]

for all \( y_{[0:n-1]} \in \mathcal{Y}^n \) and all \( z_{[0:n-1]} \in \{0,1\}^n \). Equation (18) is obtained from replacing \( x_{[0:n-1]} \) with \((z_{[0:n/2-1]} \oplus z_{[n/2:n-1]}, z_{[n/2:n-1]}) \) in (17), so these two equations are clearly equivalent.

By definition (12), \( L^-(y_0, y_1) = L^-(y, y_1) \) for all \( y, y_1 \in \mathcal{Y} \). Definition (13) further tells us that \( L^-_{[0:n/2-1]}(y_{[0:n-1]}) = L^-_{[0:n/2-1]}(\xi^{(m)}_{m-1}(y_{[0:n-1]})) \) for all \( y_{[0:n-1]} \in \mathcal{Y}^n \). Therefore,

\[
\mathbb{P}(\hat{x}_{[0:n-1]}(L^-_{[0:n/2-1]}(y_{[0:n-1]}), A^-) = z_{[0:n/2-1]})
\]

\[
= \mathbb{P}(\hat{x}_{[0:n-1]}(L^-_{[0:n/2-1]}(\xi^{(m)}_{m-1}(y_{[0:n-1]})), A^-) = z_{[0:n/2-1]})
\]

for all \( y_{[0:n-1]} \in \mathcal{Y}^n \) and all \( z_{[0:n/2-1]} \in \{0,1\}^{n/2} \).

By definition (12), \( L^+(y_0, y_1) = L^+(y_1, y_0) \) for all \( y, y_1 \in \mathcal{Y} \); \( L^+(y_0, y_1, 1) = -L^+(y_1, y_0, 1) \) for all \( y, y_1 \in \mathcal{Y} \). Definition (13) further tells us that

\[
L^+_{[0:n/2-1]}(\xi^{(m)}_{m-1}(y_{[0:n-1]}), z_{[0:n/2-1]}) = z_{[0:n/2-1]} \oplus L^+_{[0:n/2-1]}(y_{[0:n-1]}, z_{[n/2:n-1]})
\]

for all \( y_{[0:n-1]} \in \mathcal{Y}^n \) and all \( z_{[n/2:n-1]} \in \{0,1\}^{n/2} \).

Let \( |A| = k, |A^-| = k^- \), and \( |A^+| = k^+ \). The condition (14) implies that \( A^- \subseteq A^+ \). As a consequence, \( \text{Polar}(n, k^-, A^-, 0^{n/2-k^-}) \subseteq \text{Polar}(n, k^+, A^+, 0^{n/2-k^+}) \). In other words, if \( z_{[0:n/2-1]} \) is a codeword in \( \text{Polar}(n, k^-, A^-, 0^{n/2-k^-}) \), then it must also be a codeword in \( \text{Polar}(n, k^+, A^+, 0^{n/2-k^+}) \).

We divide the proof into two cases.

**Case 1** If \( z_{[0:n/2-1]} \) is not a codeword in \( \text{Polar}(n, k^-, A^-, 0^{n/2-k^-}) \), then the probability on both sides of (19) is 0. This is simply because the SC decoder can only output a valid codeword as the decoding result. In this case, (16) implies that the probability on both sides of (18) is 0.

**Case 2** If \( z_{[0:n/2-1]} \) is a codeword in \( \text{Polar}(n, k^-, A^-, 0^{n/2-k^-}) \), then it is also a codeword in \( \text{Polar}(n, k^+, A^+, 0^{n/2-k^+}) \). In this case, we have

\[
\mathbb{P}(\hat{x}_{[0:n/2-1]}(L^+_{[0:n/2-1]}(\xi^{(m)}_{m-1}(y_{[0:n-1]}), z_{[0:n/2-1]}), A^+) = z_{[0:n/2-1]} \oplus z_{[n/2:n-1]})
\]

\[
= \mathbb{P}(\hat{x}_{[0:n/2-1]}(L^+_{[0:n/2-1]}(y_{[0:n-1]}), A^+) = z_{[0:n/2-1]} \oplus z_{[n/2:n-1]})
\]

for all \( z_{[n/2:n-1]} \in \{0,1\}^{n/2} \), where the first equality follows from (20), and the second equality follows from Lemma 4. Combining this with (16) and (19), we complete the proof of (18). Since (18) is equivalent to (17), this completes the proof of the lemma. \( \Box \)
Lemma 6. Let \( n = 2^m \). Suppose that all the frozen bits take value 0. Suppose that the index set \( \mathcal{A} \) of information bits satisfies the condition (4). Then

\[
P(\hat{x}_{[0:n-1]}(y_{[0:n-1]}, \mathcal{A}) = x_{[0:n-1]}) = P(\hat{x}_{[0:n-1]}((m)_{m-r}(y_{[0:n-1]}), \mathcal{A}) = (m)_{m-r}(x_{[0:n-1]}))
\]

for all \( y_{[0:n-1]} \in \mathcal{Y}^n \), all \( x_{[0:n-1]} \in \{0,1\}^n \), and all \( 1 \leq r \leq m \).

Proof. We prove by induction on \( r \). The base case \( r = 1 \) is already proved in Lemma 5. Now we assume that (21) holds for \( r - 1 \) and all values of \( m \), and we prove it for \( r \). Observe that

\[
L_{[0:n/2-1]}^{-}(\xi_{m-r}(y_{[0:n-1]})) = \xi_{m-r}^{-1}(L_{[0:n/2-1]}^{-}(y_{[0:n-1]})).
\]

Since \( \mathcal{A} \) satisfies the condition (4), both \( \mathcal{A}^+ \) and \( \mathcal{A}^- \) also satisfy the condition (4). By the induction hypothesis, (21) holds for \( r - 1 \) and \( m - 1 \), so

\[
P(\hat{x}_{[0:n/2-1]}(L_{[0:n/2-1]}^{-}(y_{[0:n-1]}), \mathcal{A}^-) = z_{[0:n/2-1]})
\]

\[
= P(\hat{x}_{[0:n/2-1]}(L_{[0:n/2-1]}^{-}(y_{[0:n-1]}), \mathcal{A}^-) = (m-1)_{m-r}(z_{[0:n/2-1]}))
\]

\[
= P(\hat{x}_{[0:n/2-1]}(L_{[0:n/2-1]}^{-}(y_{[0:n-1]}), \mathcal{A}^-) = (m-1)_{m-r}(z_{[0:n/2-1]}))
\]

for all \( z_{[0:n/2-1]} \in \{0,1\}^{n/2} \). It is easy to verify that

\[
L_{[0:n/2-1]}^{-}(\xi_{m-r}(y_{[0:n-1]}), (m-1)_{m-r}(z_{[0:n/2-1]})) = (m-1)_{m-r}(L_{[0:n/2-1]}^{-}(y_{[0:n-1]}, z_{[0:n/2-1]})).
\]

Again by the induction hypothesis,

\[
P(\hat{x}_{[0:n/2-1]}(L_{[0:n/2-1]}^{+}(y_{[0:n-1]}), \mathcal{A}^+) = z_{[0:n/2-1]})
\]

\[
= P(\hat{x}_{[0:n/2-1]}(L_{[0:n/2-1]}^{+}(y_{[0:n-1]}), \mathcal{A}^+) = (m-1)_{m-r}(z_{[0:n/2-1]}))
\]

\[
= P(\hat{x}_{[0:n/2-1]}(L_{[0:n/2-1]}^{+}(y_{[0:n-1]}), (m-1)_{m-r}(z_{[0:n/2-1]}), \mathcal{A}^+) = (m-1)_{m-r}(z_{[0:n/2-1]}))
\]

for all \( z_{[0:n/2-1]} \in \{0,1\}^{n/2} \). Combining this with (16) and (22), we obtain that

\[
P(\hat{x}_{[0:n-1]}(y_{[0:n-1]}), \mathcal{A}) = (z_{[0:n/2-1]} \oplus z_{[n/2:n-1]}, z_{[n/2:n-1]}))
\]

\[
= P(\hat{x}_{[0:n-1]}((m)_{m-r}(y_{[0:n-1]}), \mathcal{A}) = (m-1)_{m-r}(z_{[0:n/2-1]} \oplus (m-1)_{m-r}(z_{[0:n/2-1]})))
\]

for all \( z_{[0:n-1]} \in \{0,1\}^n \). Since

\[
(m-1)_{m-r}(z_{[0:n/2-1]} \oplus (m-1)_{m-r}(z_{[0:n/2-1]}))
\]

\[
= (m-1)_{m-r}(z_{[0:n/2-1]} \oplus (m-1)_{m-r}(z_{[0:n/2-1]}))
\]

we further obtain that

\[
P(\hat{x}_{[0:n-1]}(y_{[0:n-1]}), \mathcal{A}) = (z_{[0:n/2-1]} \oplus z_{[n/2:n-1]}, z_{[n/2:n-1]}))
\]

\[
= P(\hat{x}_{[0:n-1]}((m)_{m-r}(y_{[0:n-1]}), \mathcal{A}) = (m)_{m-r}(z_{[0:n/2-1]} \oplus z_{[n/2:n-1]}, z_{[n/2:n-1]}))
\]

for all \( z_{[0:n-1]} \in \{0,1\}^n \). Finally, (21) follows from replacing \((z_{[0:n/2-1]} \oplus z_{[n/2:n-1]}, z_{[n/2:n-1]}))\) with \(x_{[0:n-1]}\). This completes the proof of the lemma. \( \square \)

Lemma 7. Let \( n = 2^m \). Let \( W \) be a BMS channel. Suppose that the index set \( \mathcal{A} \) of information bits satisfies the condition (4). Then

\[
\text{BER}_j(\text{Polar}(n, k, \mathcal{A}, \bar{u}_A), W) = \text{BER}_{\mathcal{A}_{\bar{u}_A}}^{(n,j)}(\text{Polar}(n, k, \mathcal{A}, \bar{u}_A), W)
\]

for all \( 0 \leq j \leq n - 1 \), all \( 0 \leq r \leq m - 1 \), and all \( \bar{u}_A \in \{0,1\}^{n-k} \).
where the second equality follows from $W_x^{\xi}$ is $\frac{1}{2}$.

Remark 1. The permutations $\xi_r^{(m)}, 0 \leq r \leq m - 1$ used in this section flip one bit in the binary expansion of the indices. We would like to mention that a class of somewhat related permutations, which permute/reorder the bits in the binary expansion of the indices, was used to improve the decoding performance of polar codes; see Section 6.2.2 of [7] and [8].

IV. THE CONNECTION TO [4]

As mentioned in the Introduction, we observed the simulation results in Fig. 1 as we repeated the experiments in [4]. From Fig. 1 we can see that the variance of the BERs of message bits is very large.

Proof. By (11), we have

$$\text{BER}_j(\text{Polar}(n, k, A, \bar{u}_{A^c}), W)$$

$$= \sum_{y_{[0:n-1]} \in \mathcal{Y}^n} \sum_{x_{[0:n-1]} \in \{0,1\}^n} W^n(y_{[0:n-1]}|0^n)\mathbb{P}(\hat{x}_{[0:n-1]}(y_{[0:n-1]}, A) = x_{[0:n-1]}1[x_j \neq 0]$$

$$= \sum_{y_{[0:n-1]} \in \mathcal{Y}^n} \sum_{x_{[0:n-1]} \in \{0,1\}^n} W^n(\xi_r^{(m)}(y_{[0:n-1]})|0^n)\mathbb{P}(\hat{x}_{[0:n-1]}(\xi_r^{(m)}(y_{[0:n-1]}), A) = \xi_r^{(m)}(x_{[0:n-1]})1[x_j \neq 0])$$

$$= \sum_{y_{[0:n-1]} \in \mathcal{Y}^n} \sum_{x_{[0:n-1]} \in \{0,1\}^n} W^n(y_{[0:n-1]}|0^n)\mathbb{P}(\hat{x}_{[0:n-1]}(y_{[0:n-1]}, A) = x_{[0:n-1]}1[x^{(m)}_{\delta(j)}(j) \neq 0]$$

$$= \text{BER}_j(\text{Polar}(n, k, A, \bar{u}_{A^c}), W),$$

where the second equality follows from $W^n(y_{[0:n-1]}|0^n) = W^n(\xi_r^{(m)}(y_{[0:n-1]})|0^n)$ and Lemma 6; the third equality is obtained from replacing $\xi_r^{(m)}(y_{[0:n-1]})$ with $y_{[0:n-1]}$ and replacing $\xi_r^{(m)}(x_{[0:n-1]})$ with $x_{[0:n-1]}$.

Lemma 7 immediately implies Theorem 1 because for any $j \in \{0, 1, \ldots, n - 1\}$, we can always apply a subset of $\delta_0^{(m)}, \delta_1^{(m)}, \ldots, \delta_{n-1}^{(m)}$ to $j$ and obtain 0. This means that $\text{BER}_j(\text{Polar}(n, k, A, \bar{u}_{A^c}), W) = \text{BER}_0(\text{Polar}(n, k, A, \bar{u}_{A^c}), W)$ for all $j \in \{0, 1, \ldots, n - 1\}$ and completes the proof of Theorem 1.

Fig. 1: Simulation results of the $(n = 256, k = 128)$ polar code over binary-input AWGN channel with $E_b/N_0 = 2\text{dB}$
In contrast, the BERs of codeword bits are extremely stable, to the extent that they form a straight line. At this point, it makes sense to discuss the connection between our results and [4].

We still use $u[0:n-1]$ to denote the message vector and use $x[0:n-1]$ to denote the codeword vector. The set $\mathcal{A}$ is still the index set of information bits. The main observation in [4] is that the average BER of $x_\mathcal{A}$ is much smaller than the average BER of $u_\mathcal{A}$ under the SC decoder. This is somewhat counter-intuitive because the SC decoder directly decodes $u_\mathcal{A}$, and $x_\mathcal{A}$ is obtained from multiplying the decoding result of $u[0:n-1]$ with the encoding matrix $G_n$. Even till today, no rigorous analysis is available to explain this phenomenon.

The results in this paper imply that the BERs of all the codeword bits are equal to each other, and they all equal to the BER of the last message bit because the last codeword bit is the same as the last message bit. On the bright side, the last message bit is the best-protected message bit if we assume that all the previous message bits are decoded correctly. However, we also have an argument in the opposite direction: The decoding error in the SC decoder accumulates, and the last message bit takes the most damage from decoding errors in previous message bits. In fact, if we look at Fig. 1 closely, we can see that there is an increasing trend in the BERs of message bits (increasing with the indices). The BER of the last message bit is an outlier because it drops abruptly compared to the previous bits.

To conclude, although our result is somewhat correlated with the observation in [4], we are still not able to explain the phenomenon in [4]. That calls for future research effort.

ACKNOWLEDGEMENT

The second author would like to thank Dr. Ling Liu for bringing the paper [4] to our attention and providing feedback to the draft.

We also thank Henry Pfister for informing us that Lemma 5 and Lemma 6 were already proved in Theorem 2 of [5].

REFERENCES

[1] E. Arikan, “Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels,” IEEE Transactions on Information Theory, vol. 55, no. 7, pp. 3051–3073, 2009.
[2] I. Tal and A. Vardy, “List decoding of polar codes,” IEEE Transactions on Information Theory, vol. 61, no. 5, pp. 2213–2226, 2015.
[3] K. Niu and K. Chen, “CRC-aided decoding of polar codes,” IEEE Communications Letters, vol. 16, no. 10, pp. 1668–1671, 2012.
[4] E. Arikan, “Systematic polar coding,” IEEE Communications Letters, vol. 15, no. 8, pp. 860–862, 2011.
[5] M. Geiselhart, A. Elkelesh, M. Ebada, S. Cammerer, and S. ten Brink, “Automorphism ensemble decoding of Reed–Muller codes,” IEEE Transactions on Communications, vol. 69, no. 10, pp. 6424–6438, 2021.
[6] I. Tal and A. Vardy, “How to construct polar codes,” IEEE Transactions on Information Theory, vol. 59, no. 10, pp. 6562–6582, 2013.
[7] S. B. Korada, “Polar codes for channel and source coding,” Ph.D. dissertation, ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE, 2009.
[8] S. A. Hashemi, N. Doan, M. Mondelli, and W. J. Gross, “Decoding Reed-Muller and polar codes by successive factor graph permutations,” in 2018 IEEE 10th International Symposium on Turbo Codes Iterative Information Processing (ISTC), 2018, pp. 1–5.