Existence of almost global weak solution for the Euler-Poisson system in one dimension with large initial data

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Abstract

This paper deals with the global existence of BV solution for the Euler-Poisson system endowed with a $\gamma$ pressure law. More precisely, we prove the existence of almost global weak solution in the $BV$ framework with arbitrary large initial data when $\gamma = 1 + 2\varepsilon$ satisfies a smallness condition. We use the Glimm scheme combined with a splitting method as introduced in [Poupaud, Rascle and Vila, J. Differential Equations, 1995]. Existence of BV solution of 1-D isentropic Euler equation for large data and $\gamma = 1 + 2\varepsilon$ is proved in [Nishida and Smoller, Comm. Pure Appl. Math, 1973]. Due to the presence of electric field, the difficulty arises while controlling the Glimm functional for Euler-Poisson system. It requires to extend in a fine way the study of the wave interactions.

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1 Introduction

We consider the following Euler-Poisson system,
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) &= 0, \quad (1) \\
\frac{\partial}{\partial t} (\rho u) + \frac{\partial}{\partial x} (\rho u^2 + P(\rho)) &= -\sigma \rho u - \frac{q}{m} \rho \Psi, \quad (2) \\
\frac{\partial}{\partial x} \Psi &= -\frac{q}{e} (\rho - \mu), \quad (3)
\end{align*}
for \( t > 0 \) and \( x \in \mathbb{R} \). It appears in various models describing physical phenomena such as transport of electron, plasma collision (see [2, 21, 25, 26]). In the system (1)–(3) \( \rho, u \) represent respectively the concentration and mean velocity of electrons whereas the physical constants \( q, m, e \) stand for the electric charge, mass of electron and the permittivity of the medium respectively. In equation (2), \( P(\rho) \) is the pressure-density relation and we consider \( \gamma \)–law in the sequel with \( P(\rho) = \rho^\gamma \) where \( \gamma = 1 + 2\epsilon \) and \( \epsilon > 0 \). We supplement the previous system with the following Cauchy data:
\[
\begin{cases}
\rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R}.
\end{cases}
\]
In this article, we want to show the existence of BV solution to Euler-Poisson system (1)–(3) with large initial data (4). For a given time \( T \) and BV data \((\rho_0, u_0)\) we prove that there exists \( \gamma_0 \) such that for any \( \gamma \in (1, \gamma_0) \) the system (1)–(3) admits a BV solution on \([0, T]\) corresponding to the initial data \((\rho_0, u_0)\).

Degond and Markowich [8] started the study of mathematical aspects of the Euler-Poisson system. They proved the existence of solution to the steady state Euler-Poisson system in 1-D for the subsonic case, that is, \( |u| < c \), where \( c := (p'(\rho))^{1/2} \) denotes the sound speed. Gamba [11] used viscosity method to study the steady state problem of Euler-Poisson system. Local existence of strong solution to Euler-Poisson system in multi-D is known [9] for initial data in suitable Sobolev space. Global results can be found in dimension 2 and 3 in [14, 17, 19] for small perturbation of the state \((\bar{\rho}, 0)\) with \( \bar{\rho} > 0 \) and the initial velocity is assumed to be irrotational. More recently, Guo, Han and Zhang [15] have proved that global strong solution exists if the initial data is sufficiently small in Sobolev and weighted Sobolev spaces with sufficiently large index of regularity in one dimension for \( \gamma = 3 \). Concerning the existence of global solution for initial density close from the vacuum, we refer in particular to [6, 16] and the reference therein. In particular in [16] the authors prove the global existence of free-boundary Euler-Poisson solutions satisfying the physical vacuum condition. For a hyperbolic system, it is well known that smooth data in general may not give global smooth solution. In a system like (1)–(3) with hyperbolic-elliptic mixed structure, the same property holds [10, 24]. Therefore, we look for existence of weak solution.

In this article, we are interested in investigating the existence problem in BV framework. In his seminal work [13], Glimm proved the existence of weak solution to hyperbolic systems of conservation laws with genuinely nonlinear or linearly degenerate characteristic fields for initial data having small total variation bound. Later Lax and Glimm in [12] extended this previous result to \( 2 \times 2 \) hyperbolic systems with genuinely nonlinear fields providing that the initial data is only bounded with a smallness condition on the \( L^\infty \) norm. Bressan developed
in [4] a new strategy for proving Glimm’s result for $n \times n$ hyperbolic systems by using the so-called wave front tracking method. This tool is essential to prove the existence of a semigroup of contraction $L^1$ of solutions, it enables in particular to obtain the uniqueness of the Glimm’s solution in a class of uniqueness which takes into account the Lax entropy conditions (we refer to [5] and the references therein for these questions). For hyperbolic system of conservation laws in 1-D vanishing viscosity limit is obtained by Bianchini and Bressan [3] for small BV data. Nishida [22] proved the existence of global BV weak solution of $p$-system with $\gamma = 1$ for large BV data. For $\gamma > 1$, Nishida and Smoller [23] showed the existence of global weak solution to $p$-system for arbitrary large BV data by considering a smallness condition on $(\gamma - 1)/2$. The system under consideration (1)–(3) shares some similarities with hyperbolic balance laws. The existence problem for hyperbolic balance laws is studied for small BV data via Glimm scheme [7] and wave front tracking method [1].

For Euler-Poisson system BV solutions are less studied. Poupaud, Rascle and Vila [27] have obtained the existence of BV solution for large initial data in the case of the isothermal Euler-Poisson system which corresponds to the pressure law $P(\varrho) = \varrho$ that is $\gamma = 1$. They use a splitting method combined with Glimm scheme and the use of the Nishida functional introduced in [22] for getting the BV stability of the approximated solution. We refer to [20, 30] for existence of $L^\infty$ weak solutions via method of compensated compactness. Unfortunately, it hardly tells about regularity of solution.

In this article, we prove existence of BV solutions for initial data with large total variation. Proof is based on two steps: (i) approximating the solution via splitting method with an adaptation of Glimm scheme, (ii) estimating Glimm functional to get TV bound of solution. Concerning the approximate part, we modify the scheme developed in [27] with necessary changes. This has been required in order to get proper interaction estimates. Note that TV bound for $P(\varrho) = \varrho$ case is achieved by considering TV of Riemann invariants, more precisely, it has been observed in [22] that TV bounds of Riemann invariants decays in time. For $\gamma > 1$ case, the TV norm of Riemann invariants can increase for isentropic Euler system but with an order $\epsilon |\alpha| |\beta|$ where $\epsilon = (\gamma - 1)/2$ and $\alpha, \beta$ are two shocks interacting. Therefore, we need to consider Glimm functional to get bound as in Nishida and Smoller [23]. For Euler-Poisson equation, at each time step there is an increment due to electric field which has been bounded with $C_1 \Delta t$ where $C_1$ depends on the TV norm of initial data [27]. Since for $\gamma > 1$ we are working with quadratic functional, it needs a non-trivial treatment to obtain TV bound. Note that at each step of the numerical scheme, taking into account the electric field the $\varrho$ variable is unchanged compared with the Glimm scheme for the isentropic Euler system whereas the velocity unknown $u$ is translated for some $\delta$ and due to this modification, the total variation can increase in a more significant way as in [23]. With a careful analysis of Riemann problem and interaction waves, we are able to show that change in the total variation can be bounded by order of $\Delta t$.

**Far field conditions:** Before we state our main result we need to define the following far-field conditions. Throughout this article we assume the following far-field conditions:

\begin{align*}
\mu(x) := \begin{cases} 
\mu^- & \text{for } x < -L, \\
\mu^+ & \text{for } x > L,
\end{cases} & \quad \sigma(x) := \begin{cases} 
\sigma^- & \text{for } x < -L, \\
\sigma^+ & \text{for } x > L,
\end{cases} \\
\varrho_0(x) := \begin{cases} 
\mu^- & \text{for } x < -L, \\
\mu^+ & \text{for } x > L,
\end{cases} & \quad u_0(x) := \begin{cases} 
u^- & \text{for } x < -L, \\
u^+ & \text{for } x > L,
\end{cases}
\end{align*}

(5)

(6)
for some $L > 0$. We prove that these properties are preserved for any positive time $t \geq 0$:

$$
\begin{align*}
\varrho(t, x) &= \mu^-, \ t > 0, \ x < -L(t), \ \varrho(t, x) = \mu^+, \ t > 0, \ x < L(t), \\
u(t, x) &= u^-(t), \ t > 0, \ x < -L(t), \ u(t, x) = u^+(t), \ t > 0, \ x > L(t),
\end{align*}
$$

(7)

with $L(t)$ depending on the time $t$. Therefore (3) yields:

$$
\Psi(t, x) = \Psi^-(t) - \int_{-\infty}^{x} \frac{q}{e} (\varrho(t, y) - \mu(y)) dy \text{ for } t > 0, x \in \mathbb{R}.
$$

(8)

The electric field at $x = -\infty$, $\Psi^-(t)$ is assumed to be known. As it is mentioned in [27] using (7) we can explicitly compute the values at infinity of $\varrho$, $u$, and $\Psi$. We have in particular:

$$
\begin{align*}
\left\{\begin{array}{l}
u^+(t) = \nu_0^+ - \frac{q}{e} \int_0^t e^{-\sigma^+(t-s)} \Psi^+(s) ds \\
\Psi^+(t) = \Psi^+(0) + \int_0^t \frac{q}{e} (\mu^+ u^+(s) - \mu^- u^-(s)) ds + \Psi^-(t) - \Psi^-(0) \\
\Psi^+(0) = \Psi^-(0) - \int_{-\infty}^{+\infty} \frac{q}{e} (\varrho_0(y) - \mu(y)) dy.
\end{array}\right.
\end{align*}
$$

(9)

Now we are ready to state our main result.

**Theorem 1.1.** Let $T > 0$ and $(\varrho_0, u_0)$ such that $(\varrho_0, u_0) \in BV(\mathbb{R}, \mathbb{R}+) \times \mathbb{R}$ and there exists $M, \varrho, \varrho_0 \in \mathbb{R}$ such that

$$
|u_0(x)| \leq M, \ 0 < \varrho \leq \varrho_0(x) \leq \varrho \text{ for a.e. } x \in \mathbb{R}.
$$

Furthermore $\sigma, \mu$ and $\psi^-$ verify the following assumptions:

- $\sigma \geq 0, \ \sigma \in BV(\mathbb{R}), \ \mu \geq 0, \ \mu \in BV(\mathbb{R}),$
- $\Psi^- \in BV(0, +\infty)$.

Then there exists $\gamma_0 \in (1, 2)$ such that the following holds: for any $\gamma \in (1, \gamma_0]$ there exists a weak solution $(\varrho, u, \Psi)$ of (1)–(4) satisfying (5)–(6) on $[0, T]$.

**Remark 1.** We can point out that this result extend the work of Poupaud, Rascle, Villa [27] to the case $\gamma = 1 + 2\epsilon$, the counterpart is that in this more general framework we obtain almost global weak solution when $\epsilon$ is sufficiently small. In [20] Marcati and Natalini prove the existence of global weak solution in the framework of $L^\infty$ initial data without any smallness assumption on the initial data, however we think that the BV framework is relevant if we wish to prove the uniqueness of the solutions by following the arguments developed by Bressan for the homogeneous case (see [5]) at least for small initial data.

This paper is organized as follows. In the next section we precise the numerical scheme that we are going to use. In section 3, we recall the basic properties of the Riemann problems and the description of the Lax curves in terms of the Riemann invariants. In section 4, we derive uniform estimates for approximate solution in $BV$ spaces by studying carefully the wave interaction of our scheme and we give the proof of the Theorem 1.1.
2 Numerical scheme

Our aim is to show the existence of entropy solution via an adaptation of Glimm scheme and a splitting method. We slightly modify the approximate scheme defined in [27] to obtain a sequence of approximate solution \((\varrho_{\Delta x}, u_{\Delta x})_{\Delta x > 0}\). The next step will be to obtain BV bound via Glimm functional [13, 23] in order to pass to the limit when \(\Delta t > 0\) goes to 0 and to recover an almost global weak solution of the Euler-Poisson system.

Scheme:
Let \(x_i, t_n\) be defined as \(x_i = i \Delta x\) for \(i \in \mathbb{Z}\) and \(t_n = n \Delta t\) for \(n \geq 0\) where \(\Delta x > 0, \Delta t > 0\) are related as
\[
\frac{\Delta t}{\Delta x} = \lambda
\]
where \(\lambda > 0\) will be chosen later and will be sufficiently small in order to satisfy CFL condition.

Remark 2. Note that once we fix \(T\) and the initial data \(\varrho_0, u_0\), the parameter \(\lambda\) is fixed and it does not vary over time steps \(t_n\). It is the main difference between our scheme and the scheme used in [27], in [27] it seems important to ensure that \(\lambda\) varies at each step of time in order to satisfy the CFL condition at each step. In our case the CFL condition will be satisfied at each step because the smallness condition on \(\epsilon\) depend on the time \(T\).

Next, we set \(I_{n,i}^- = (x_{i-1}, x_{i+1})\) for \(i \in \mathbb{Z}\) such that \(n + i\) is even. Let \(\varrho_0^\Delta, u_0^\Delta, \mu^\Delta, \sigma^\Delta\) be defined as
\[
\begin{align*}
(\varrho_0^\Delta, u_0^\Delta)(x) &= (\varrho_0(x_i), u_0(x_i)) \quad \text{for } x \in I_{0,i}, \quad (11) \\
(\sigma^\Delta(x), \mu^\Delta(x)) &= (\sigma(x_i), \mu(x_i)) \quad \text{for } x \in I_{0,i}.
\end{align*}
\]
Furthermore these functions respect the conditions (5) and (6).

We set \(I_n^- = I_n^-(t_n)\) for \(n \geq 0\) and \(u_0^\pm = u^\pm\). Then we define
\[
\begin{align*}
\Psi_{n+1}^+ &= \Psi_n^+ + \frac{q}{e}(\mu^+ u_n^+ - \mu^- u_n^-) \Delta t + \Psi_{n+1}^- - \Psi_n^-, \quad (13) \\
\Psi_0^- &= \Psi_0^+ - \int_{-\infty}^\infty \frac{q}{e}(\varrho_0^\Delta(y) - \mu^\Delta(y)) \, dy.
\end{align*}
\]
The discretization of the velocity at the infinity is given by:
\[
u_{n+1}^\pm = u_n^\pm \exp(-\sigma^\pm \Delta t) - \frac{q}{e} \frac{1 - \exp(-\sigma^\pm \Delta t)}{\sigma^\pm} \Psi_{n+1}^\pm \quad \text{for } n \geq 0.
\]
We can bounded the sequel \((\Psi_n^\pm)_{n \in \mathbb{N}}\) and \((u_n^\pm)_{n \in \mathbb{N}}\) via the following lemma (see [27]).

Lemma 2.1 ([27]). Let \(\Psi^- \in L_{loc}^\infty([0, \infty))\) and \(T > 0\). Then there are two constants \(E_T, C_T\) such that for any \(n \in \mathbb{N}\) with \(t_n < T\) we have
\[
|\Psi_n^\pm| \leq E_T, \quad |u_n^\pm| \leq C_T.
\]
Assume that \((\varrho_n^\Delta, u_n^\Delta) = (\varrho_{n,i}, u_{n,i})_{i \in \mathbb{Z}}\) \((\varrho_n^\Delta\) and \(u_n^\Delta\) are the piecewise function with value
respectively \( q_{n,i} \) and \( u_{n,i} \) on \( I_{n,i} \) is known for some \( n \geq 1 \) and satisfy:

\[
q_{n}^\Delta(x) \geq 0, \quad (17)
\]

\[
q_{n}^\Delta(x) = \begin{cases} 
\mu^- & \text{for } x < -L - n\Delta x, \\
\mu^+ & \text{for } x > L + n\Delta x,
\end{cases} \quad (18)
\]

\[
u_{n}^\Delta(x) = \begin{cases} 
u_n^- & \text{for } x < -L - n\Delta x, \\
u_n^+ & \text{for } x > L + n\Delta x,
\end{cases} \quad (19)
\]

Next, our goal is to define \((q_{n+1}^\Delta, u_{n+1}^\Delta)\). For this purpose, we consider the homogeneous system

\[
\frac{\partial}{\partial t} \varrho + \frac{\partial}{\partial x} (\varrho u) = 0, \quad (20)
\]

\[
\frac{\partial}{\partial t} (\varrho u) + \frac{\partial}{\partial x} (\varrho u^2 + \rho^2) = 0. \quad (21)
\]

Let \( U_l = (q_l, u_l) \) and \( U_r = (q_r, u_r) \) with \( q_l > 0, q_r > 0 \). Consider the Riemann data

\[
U_0(x) = \begin{cases} 
U_l & \text{for } x < 0, \\
U_r & \text{for } x > 0.
\end{cases} \quad (22)
\]

We denote Riemann solution \( R[U_l, U_r](x,t) \) as the entropy solution \([18]\) to the homogeneous system (20)–(21) with initial data \( U_0 \) as in (22). Note that for \( U_l, U_r \) close enough \( R[U_l, U_r](x,t) \) exists and no vacuum arises in the Riemann solution (when \( U_l \) and \( U_r \) are not necessary close, we refer to [23]). Now we define

\[
(q_{n+\frac{1}{2},i}^\Delta, u_{n+\frac{1}{2},i}^\Delta) = R[U_{n,i-1}, U_{n,i+1}](\Delta x \theta_n, \Delta t) \text{ for some } \theta_n \in [-1, 1]. \quad (23)
\]

**Remark 3.** As for the Glimm scheme, our scheme is defined modulo a sequence \( \theta = (\theta_n)_{n \in \mathbb{N}} \), we will see that the numerical scheme converge for almost every sequence \( \theta \).

We define now \( \varrho_{n+\frac{1}{2}}^\Delta \theta \) as \( \varrho_{n+\frac{1}{2}}^\Delta \theta(x) := \varrho_{n+\frac{1}{2},j}^\Delta \) for \( x \in I_{n+1,j} \) with \( n+1+j \) even (we can define \( u_{n+\frac{1}{2}}^\Delta \theta \) in a similar way) and we set

\[
\varrho_{n+1}^\theta(x) = \varrho_{n+\frac{1}{2}}^\Delta \theta(x) \text{ for } x \in \mathbb{R}. \quad (24)
\]

This definition of \( \varrho_{n+1}^\Delta \) ensures that the conditions (17) and (18) are satisfied.

Next we consider a quantity \( \xi_{n+1} \) defined as follows,

\[
\xi_{n+1} := \begin{cases} 
0 & \text{for } |x| > L + (n + 1)\Delta x, \\
-\frac{q}{\rho} ((1 + \varrho_{n+1}^\Delta) \gamma_{n+1} - 1 - \mu^\Delta(x)) & \text{for } |x| < L + (n + 1)\Delta x.
\end{cases} \quad (25)
\]

As in [27] we use the corrector term \( \gamma_{n+1} \) in order to estimate in a simple way the \( L^1 \) norm.
of $\xi_n$. The term $\gamma_{n+1}$ is given by

$$\delta_0 = \int_{-L}^{L} (1 + g^\Delta_0(y)) \, dy,$$

$$\delta_{n+1} = \delta_n + (1 + \mu^-) \Delta x + (1 + \mu^+) \Delta x + \Delta t (\mu^- u^-_n - \mu^+ u^+_n),$$

$$\gamma_{n+1} = \delta_{n+1} \left[ \int_{-L-(n+1)\Delta x}^{L+(n+1)\Delta x} (1 + g^\Delta_{n+1}(y)) \, dy \right]^{-1}.$$  

With the definition above, we obtain the following estimate on the $L^1$ norm of the sequence $(\xi_n)_{n \in \mathbb{N}}$.

**Lemma 2.2.** We have for $T > 0$ and $\lambda \leq \frac{1}{C_T}$ with $C_T > 0$ defined in Lemma 2.1:

$$0 \leq \delta_n \leq \delta_T + (2 + \mu^- + \mu^+) n \Delta x \quad \text{for } t_n \leq T,$$

where $\delta_T = \delta_0 + (\mu^- + \mu^+) T C_T$. Moreover,

$$\int_{-\infty}^{\infty} \xi_{n+1}(y) \, dy = \int_{-\infty}^{\infty} \xi_n(y) \, dy + \frac{q \Delta t}{e} (\mu^+ u^+_n - \mu^- u^-_n),$$

$$||\xi_n||_{L^1} \leq \xi_T + 4 \frac{q}{e} (1 + ||\mu^\Delta||_{L^\infty})(L + n \Delta x), \quad \text{for } t_n \leq T \quad (31)$$

where $\xi_T = ||\xi_0||_{L^1} + \frac{q}{e} T (\mu^- + \mu^+) C_T$.

For a detailed proof of Lemma 2.2, we refer to [27]. To make it convenient to reader, here we briefly discuss some of the key steps.

The proof of (29) yields from the formula (27) and the Lemma 2.1. In addition we use the fact that using Lemma 2.1, we have:

$$(1 + \mu^-) \Delta x + (1 + \mu^+) \Delta x + \Delta t (\mu^- u^-_n - \mu^+ u^+_n) = 2 \Delta x + \mu^- \Delta t (\frac{\Delta x}{\Delta t} + u^-_n) + \mu^+ \Delta t (\frac{\Delta x}{\Delta t} - u^+_n) \geq 2 \Delta x + \mu^- \Delta t \frac{1}{\lambda} - C_T + \mu^+ \Delta t \frac{1}{\lambda} - C_T \geq 0.$$ 

From (25) and (28), we deduce that:

$$\int_{-\infty}^{+\infty} \xi_{n+1}(y) \, dy = -\frac{q}{e} (\delta_{n+1} - \int_{-L-(n+1)\Delta x}^{L+(n+1)\Delta x} (1 + \mu^\Delta(y)) \, dy).$$

Using (27) and the fact that $\mu^\Delta = \mu^+$ for $y > L$ and $\mu^\Delta = \mu^-$ for $y < L$, we get (30). Now from (25) we have the following since $\gamma_n \geq 0$ (this is a direct consequence of the definition (28) and the fact that $\delta_n \geq 0$):

$$|\xi_n| \leq -\xi_n + \frac{2q}{e} (1 + \mu^\Delta).$$
It implies that:
\[
\int_{-\infty}^{+\infty} |\xi_n(y)|dy \leq - \int_{-\infty}^{+\infty} \xi_n(y)dy + \frac{2q}{e} \int_{-L-nh}^{L+nh} (1 + \mu^\Delta(y))dy.
\]
But from (30) we get:
\[
\int_{-\infty}^{+\infty} \xi_n(y)dy \leq \|\xi_0\|_{L^1} + \frac{q}{e} n\Delta t(N^- + N^+)C_T.
\]
The last two estimates lead to (31). □

We define \(\Psi_n^\Delta : \mathbb{R} \to \mathbb{R}\) for \(n \geq 0\) as follows
\[
\Psi_n^\Delta(x) := \Psi_{n,i} \quad \text{for} \quad x \in I_{n,i} \quad \text{where} \quad \Psi_{n,i} = \Psi^- + \int_{-\infty}^{x_{i+1}} \xi_n(y)dy.
\]
From (13), (25) and (30) we deduce as in [27] the following Lemma.

**Lemma 2.3 ([27]).** For \(x > L + (n+1)\Delta x\), \(\Psi_{n+1}^\Delta(x) = \Psi_{n+1}^+\) and \(\Psi_{n+1}^\Delta(x) = \Psi_{n+1}^-\) for \(x < -L - (n+1)\Delta x\).

Now we are ready to define \(u_{n+1,i}\) as follows,
\[
u_{n+1,i} = u_{n+1,i}^{\rho} \exp(-\sigma_i \Delta t) - \frac{q}{m} \frac{1 - \exp(-\sigma_i \Delta t)}{\sigma_i^\Delta} \Psi_{n+1,i}.
\]
As previously we set:
\[
u_{n+1,j} = u_{n+1,j} \quad \text{on} \quad I_{n+1,j} \quad \text{with} \quad n + 1 + j \text{ even}.
\]
We can note that the Lemma 2.3 implies the assumption (19). Finally we can define the solution \((\varrho^\Delta, u^\Delta, \Psi^\Delta)\) of our numerical scheme for any time \(t \in [0, T]\) with \(T > 0\) defined in the Theorem 1.1. First for \(t_n \leq T\) we set:
\[
(\varrho^\Delta, u^\Delta, \Psi^\Delta) \text{ satisfies (20)-(21) on } (t_n, t_{n+1}).
\]
Indeed we can solve (20)-(21) on \((t_n, t_{n+1})\) at the condition that there is no interaction between the solutions of each Riemann problem (we will see later that it will be the case by imposing a CFL condition on the parameter \(\lambda\)).

The electric field is now defined by:
\[
\begin{aligned}
\gamma^\Delta(t) &= \gamma_n, \quad t_n \leq t < t_{n+1}, \\
\xi^\Delta(t, x) &= -\frac{q}{e}((1 + \varrho^\Delta)\gamma^\Delta(t) - 1 - \mu^\Delta), \quad x \in \mathbb{R}, \quad t > 0, \\
\Psi^\Delta(t, x) &= \int_{-\infty}^{x} \xi^\Delta(y)dy, \quad x \in \mathbb{R}, \quad t \in [0, T].
\end{aligned}
\]

### 3 Riemann Invariant and Lax curves

In this section, we discuss some basic facts on the homogeneous problem for the Euler system and recall some results on Riemann invariant from Nishida-Smoller [23]. We can rewrite the
system (20)–(21) in density and momentum variables \( \rho, m \) as follows

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \left( m \right) = 0, \tag{37}
\]

\[
\frac{\partial m}{\partial t} + \frac{\partial}{\partial x} \left( \frac{m^2}{\rho} + \rho^\gamma \right) = 0. \tag{38}
\]

This system is hyperbolic for \( \rho > 0 \) and eigenvalues are

\[
\lambda_1(\rho, u) = \frac{m}{\rho} - \sqrt{\gamma} \rho^{\frac{\gamma-1}{2}} = u - \sqrt{\gamma} \rho^{\frac{\gamma-1}{2}},
\]

\[
\lambda_2(\rho, u) = \frac{m}{\rho} + \sqrt{\gamma} \rho^{\frac{\gamma-1}{2}} = u + \sqrt{\gamma} \rho^{\frac{\gamma-1}{2}}. \tag{39}
\]

1. **Shock waves:** We denote the admissible shock curve via the Lax criterion \( S_i(\rho_-, m_-) \) starting from \( (\rho_-, m_-) \in (0, \infty) \times \mathbb{R} \) defined as

\[
m - m_- = \frac{m_-(\rho - \rho_-) + (-1)^i(\rho - \rho_-) \sqrt{\rho(\rho_- - \rho^2)}}{\rho(\rho_- - \rho)} \text{ if } (-1)^i(\rho - \rho_-) < 0 \text{ for } i = 1, 2. \tag{40}
\]

In \( (\rho, u) \) variables, we can rewrite (40) as

\[
u - u_- = (-1)^i \sqrt{\frac{(\rho_- - \rho^2)(\rho - \rho_-)}{\rho}} \text{ if } (-1)^i(\rho - \rho_-) < 0 \text{ for } i = 1, 2. \tag{41}
\]

2. **Rarefaction waves:** We denote rarefaction curve \( R_i(\rho_-, m_-) \) starting from \( (\rho_-, m_-) \in (0, \infty) \times \mathbb{R} \) defined as

\[
m - m_- = \frac{m_-(\rho - \rho_-) + (-1)^i \frac{2}{\gamma - 1} \sqrt{\gamma} \left( \rho^{\frac{\gamma-1}{2}} - \rho^{\frac{\gamma-1}{2}} \right)}{\rho(\rho_- - \rho)} \text{ if } (-1)^i(\rho - \rho_-) > 0 \text{ for } i = 1, 2. \tag{42}
\]

In \( (\rho, u) \) variables, we can rewrite (42) as

\[
u - u_- = (-1)^i \frac{2}{\gamma - 1} \sqrt{\gamma} \left( \rho^{\frac{\gamma-1}{2}} - \rho^{\frac{\gamma-1}{2}} \right) \text{ if } (-1)^i(\rho - \rho_-) > 0 \text{ for } i = 1, 2. \tag{43}
\]

**Riemann invariant:** The corresponding Riemann invariants \( r, s \) are as follows

\[
r(\rho, u) = u - \sqrt{\gamma} \rho^\epsilon - 1 \text{ and } s(\rho, u) = u + \sqrt{\gamma} \rho^\epsilon - 1 \text{ where } \epsilon = \frac{\gamma - 1}{\epsilon}. \tag{44}
\]

We recall the following result on the Riemann problem solved by Riemann [28].

**Lemma 3.1.** The Cauchy problem (37)–(38) with Riemann data (22) has a piecewise continuous solution in \( \mathbb{R} \times \mathbb{R}^+ \) satisfying

\[
r(x, t) = r(\rho(x, t), u(x, t)) \geq \min\{r_-, r_+\}, \quad s(x, t) = s(\rho(x, t), u(x, t)) \leq \max\{s_-, s_+\} \tag{45}
\]

where \( r_\pm = r(\rho_\pm, u_\pm), s_\pm = s(\rho_\pm, u_\pm) \) with \( s_- - r_+ > -\sqrt{\gamma} \).

**Remark 4.** The condition \( s_- - r_+ > -2\sqrt{\gamma}/\epsilon \) is equivalent to \( u_+ - u_- < \sqrt{\gamma}/\epsilon (\rho_-^{\epsilon} + \rho_+^{\epsilon}) \), in particular for any state \( U_-, U_+ \) we can solve the Riemann problem provided that \( \epsilon > 0 \) is sufficiently small.
In terms of Riemann coordinate \((r, s)\) we rephrase the shock curves \(S_i \equiv S_i(r_-, s_-) = S_i(\varrho_-, u_-)\) for \(i = 1, 2\) as below (see [23])

\[
S_1:\begin{cases}
    r_- - r = \varrho_+^{-}\sqrt{\frac{(\alpha - 1)(\alpha^\gamma - 1)}{\alpha} + \frac{\sqrt{\alpha^\gamma - 1}}{\epsilon}}, & \text{where } \alpha = \frac{\varrho}{\varrho_-} \geq 1, \\
    s_- - s = \varrho_+^{-}\sqrt{\frac{(1 - \alpha)(1 - \alpha^\gamma)}{\alpha} + \frac{\sqrt{1 - \alpha^\gamma}}{\epsilon}}.
\end{cases}
\]

\[
S_2:\begin{cases}
    s_- - s = \varrho_+^{-}\sqrt{\frac{(1 - \alpha)(1 - \alpha^\gamma)}{\alpha} + \frac{\sqrt{1 - \alpha^\gamma}}{\epsilon}}, & \text{where } 0 < \alpha = \frac{\varrho}{\varrho_-} \leq 1.
\end{cases}
\]

Borrowing notation from [23] we write respectively for the \(S_1\) and \(S_2\) admissible shocks \(s_- - s = g_1(r_- - r, \varrho_-)\) for \(r \leq r_-\) and \(r_- - r = g_2(s_- - s, \varrho_-)\) for \(s \leq s_-\).

**Lemma 3.2** (Nishida-Smoller, [23]). Let \(g_i(\alpha, \varrho_-)\) be defined above for \(i = 1, 2, \varrho_- > 0\),

Then

\[
0 \leq g'_i(\alpha, \varrho_-) < 1 \text{ and } g''_i(\alpha, \varrho_-) \geq 0 \text{ for } i = 1, 2,
\]

where we use the notation \(g'_i = \frac{\partial g_i}{\partial \alpha}, g''_i = \frac{\partial^2 g_i}{\partial^2 \alpha}\) for \(i = 1, 2\).

**Remark 5.** Now we know that for a shock wave with left state \((r_-, s_-)\) and right state \((r_+, s_+)\), we have for example if we consider a 1 shock \(|r_- - r_+| = |\gamma|\) and \(|s_- - s_+| \leq |\gamma|\) using (48). In particular it implies that:

\[
|u_- - u_+| \leq \frac{1}{2}(|r_- - r_+| + |s_- - s_+|) \leq |\gamma|.
\]

We have a similar result for a 2 shock wave.

We denote \(\Omega_I[r_0, s_0], \Omega_{II}[r_0, s_0], \Omega_{III}[r_0, s_0], \Omega_{IV}[r_0, s_0]\) as follows (see Figure 1 for a demonstration)

\[
\Omega_I[r_0, s_0] := \{(r, s); r \leq r_0, s \leq s_0 \text{ and } r_0 - g_2(s_0 - s, \varrho_0) \geq r, s_0 - g_1(r_0 - r, \varrho_0) \geq s\},
\]

\[
\Omega_{II}[r_0, s_0] := \{(r, s); r \leq r_0, s \leq s_0 \text{ and } r_0 - g_2(s_0 - s, \varrho_0) \leq r\} \cup \{(r, s); r \geq r_0, s \leq s_0\},
\]

\[
\Omega_{III}[r_0, s_0] := \{(r, s); r \geq r_0, s \geq s_0\},
\]

\[
\Omega_{IV}[r_0, s_0] := \{(r, s); r \leq r_0, s \leq s_0 \text{ and } s_0 - g_1(r_0 - r, \varrho_0) \leq s\} \cup \{(r, s); r \leq r_0, s \geq s_0\}.
\]

**Remark 6.** It is important to note (see [29]) that if \((r_1, s_1) \in \Omega_I[r_0, s_0]\) and that \((r_0, s_0), (r_1, s_1)\) satisfy the assumptions of Lemma 3.1, then the Riemann problem issue of the left state \((r_0, s_0)\) and right state \((r_1, s_1)\) is described by three different state which are relied by a 1 shock and a 2 shock. Similarly if \((r_1, s_1) \in \Omega_{II}[r_0, s_0]\), then the solution of the Riemann problem is the composition of 1 rarefaction and a 2 shock. We have similar properties when \((r_1, s_1)\) is respectively in \(\Omega_{III}[r_0, s_0]\) or \(\Omega_{IV}[r_0, s_0]\).

We give now a useful Lemma for the sequel which allows to identify the position of translated states in the different regions \(\Omega_I, \cdots, \Omega_{IV}\).
Lemma 3.3. Let \((r_0, s_0) \in \mathbb{R}^2\) be a point in \(r-s\) plane. Suppose \((r_0, s_0)\) corresponds to \((\varrho_0, u_0)\). Consider a point \((r_1, s_1) \in \Omega_I[r_0, s_0]\) corresponds to \((\varrho_1, u_1)\). Let \(\delta_0, \delta_1 > 0\) and \(\bar{u}_0, \bar{u}_1\) be defined as \(\bar{u}_i = u_i + \delta_i\) for \(i = 0, 1\). Then we have

1. If \((r_1, s_1) \in \Omega_I[r_0, s_0]\), then \((r_1, s_1) \in \Omega_I[\bar{r}_0, \bar{s}_0]\). Furthermore, if \(\delta_0 = \delta_1\), then we have \((\bar{r}_1, \bar{s}_1) \in \Omega_I[\bar{r}_0, \bar{s}_0]\).

2. If \((r_1, s_1) \in \Omega_{II}[r_0, s_0]\), then \((\bar{r}_1, \bar{s}_1) \notin \Omega_{I}[r_0, s_0] \cup \Omega_{IV}[r_0, s_0]\) and \((r_1, s_1) \notin \Omega_{III}[\bar{r}_0, \bar{s}_0] \cup \Omega_{IV}[\bar{r}_0, \bar{s}_0]\).

3. If \((r_1, s_1) \in \Omega_{III}[r_0, s_0]\), then \((\bar{r}_1, \bar{s}_1) \in \Omega_{III}[r_0, s_0]\).

4. If \((r_1, s_1) \in \Omega_{IV}[r_0, s_0]\), then \((\bar{r}_1, \bar{s}_1) \notin \Omega_{I}[r_0, s_0] \cup \Omega_{II}[r_0, s_0]\) and \((r_1, s_1) \notin \Omega_{II}[\bar{r}_0, \bar{s}_0] \cup \Omega_{III}[\bar{r}_0, \bar{s}_0]\).

Proof. We just prove here the first case. Assume that \((r_1, s_1) \in \Omega_I[r_0, s_0]\), it means in particular that:

\[ r_1 \leq r_0 \text{ and } r_1 \leq r_0 - g_2(s_0 - s_1, \varrho_0). \]

From the definition of \(\bar{r}_0\), \(r_1 \leq \bar{r}_0\). Now we write,

\[ r_1 \leq r_0 + \delta_0 - g_2(s_0 + \delta_0 - s_1, \varrho_0) + (g_2(s_0 + \delta_0 - s_1, \varrho_0) - g_2(s_0 - s_1, \varrho_0) - \delta_0). \]

Using the fact that \(0 \leq g_2(\cdot, \varrho_0) < 1\) we deduce that \(g_2(s_0 + \delta_0 - s_1, \varrho_0) - g_2(s_0 - s_1, \varrho_0) - \delta_0 \leq 0\) and that:

\[ r_1 \leq r_0 + \delta_0 - g_2(s_0 + \delta_0 - s_1, \varrho_0). \]

In a similar way we prove that \(s_1 \leq \bar{s}_0\) and \(s_1 \leq \bar{s}_0 - g_1(\bar{r}_0 - r_1, \varrho_0)\). Similarly when \(\delta_0 = \delta_1\), if \((r_1, s_1) \in \Omega_I[r_0, s_0]\) then we have:

\[ r_1 \leq r_0 \text{ and } r_1 \leq r_0 - g_2(s_0 - s_1, \varrho_0). \]
We deduce using the fact that $\varrho_0 = \varrho_0$:

$$r_1 + \delta_1 \leq r_0 + \delta_0 \text{ and } r_1 + \delta_1 \leq r_0 + \delta_0 - g_2(s_0 + \delta_0 - s_1 - \delta_1, \varrho_0).$$

Similarly we have $\bar{s}_1 \leq \bar{s}_0 - g_1(\bar{r}_0 - \bar{r}_1, \varrho_0)$ \(\square\)

Now we recall the following result from Nishida-Smoller, [23]. This plays a crucial role in controlling Glimm functional for large data.

**Lemma 3.4** (Nishida-Smoller, [23]). Let $0 \leq \epsilon < 1/2, s_+ > s'_-$ and $\varrho_-, \varrho'_- \in [\varrho, \bar{\varrho}]$ with $0 < \varrho < \bar{\varrho} < \infty$. Suppose the two $S_1$ curves starting from $(r_-, s_-) = (\varrho_-, u_-)$ and $(r'_-, s'_-) = (\varrho'_-, u'_-)$ which are continued to $(r_+, s_+)$ and $(r'_+, s'_+)$ respectively. Then we get

$$0 \leq (s'_- - s'_+) - (s_- - s_+) \leq C \epsilon (s_- - s'_-)(r_- - r'_+)$$

where $C$ is independent of $\epsilon, \varrho_-, \varrho'_-$ and depending on $\varrho, \bar{\varrho}$. Similar results are true for $S_2$ curve as well.

In the sequel we denote by $S'_1$ and $S'_2$ the non physical inverse shock wave curves, $S'_2$ consists of those states $(r, s)$ which can be connected to the state $(r_0, s_0)$ on the right by an $S_2$ shock. In particular as previously (see [23]), we can represent $S'_2$ by the following equation:

$$r - r_0 = g_1(s - s_0, \varrho_0) \text{ for } s > s_0.$$  \(50\)

Similarly we have for $S'_1$:

$$s - s_0 = g_2(r - r_0, \varrho_0) \text{ for } r > r_0.$$  \(51\)

### 4 Proof of the Theorem 1.1

#### 4.1 New estimates on the wave interactions

**Lemma 4.1.** Let $(r_0, s_0) \in \mathbb{R}^2$ be a point in r-s plane. Consider a point $(r_1, s_1) \in \Omega_1[r_0, s_0]$. Let $\delta > 0$ and $(r_2, s_2)$ be defined as $(r_2, s_2) = (r_0 + \delta, s_0 + \delta)$. Let $\beta + \gamma$ be the outgoing wave from Riemann data $U_l = (r_0, s_0)$ and $U_r = (r_1, s_1)$ and $\beta' + \gamma'$ be the outgoing wave from Riemann data $U_l = (r_2, s_2)$ and $U_r = (r_1, s_1)$. Then we have

$$|\beta'| \leq |\beta| + \delta \text{ and } |\gamma'| \leq |\gamma| + \delta.$$  \(52\)

**Remark 7.** Due to $2 \sqrt{\gamma \frac{\varrho_0^d - 1}{\epsilon}} = s - r$ we have in the previous Lemma that $g_2 = \varrho_0$.

**Proof of Lemma 4.1:** Using the Lemma 3.3, we know that the Riemann problems associated to the states $[(r_0, s_0), (r_1, s_1)]$ and $[(r_2, s_2), (r_1, s_1)]$ are solved via a 1 shock and a 2 shock. Let the $S'_2$ curve from $(r_1, s_1)$ and the $S_1$ curve from $(r_0, s_0)$ intersect at $(r_3, s_3)$. Let the $S'_2$ curve from $(r_1, s_1)$ and the $S_1$ curve from $(r_2, s_2)$ intersect at $(r_4, s_4)$. Note that $(r_3, s_3), (r_4, s_4) \in \mathbb{R}$ lie on the curve $r - r_1 = g_1(s - s_1, \varrho_1)$. Suppose the line $L_1 = \{(r_3 + d, s_3 + d); d \geq 0\}$ intersects $S_1$ curve from $(r_2, s_2)$ at $(\bar{r}, \bar{s})$. As $S_1$ curve from $(r_2, s_2)$ is represented by $s - s_2 = g_1(r - r_2, \varrho_0)$ we have for $r \leq \bar{r}$ since $g_1' < 1$:

$$s - s_2 = \bar{s} - s_2 - \int_{r-r_2}^{\bar{r}-r_2} g_1'(\beta, \varrho_0) d\beta$$

$$s - \bar{s} \geq r - \bar{r}.$$  

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It implies that the curve \( s - s_2 = g_1(r - r_2, g_0) \) lies in \( \{(r, s); s - \bar{s} > r - \bar{r}, r \leq \bar{r}, s \leq \bar{s}\} \). Now if \( s_4 < s_3 \) it implies necessary that \( r_4 \leq r_3 \) since \((r_4, s_4)\) and \((r_3, s_3)\) are on the curve \( r - r_1 = g_1(s - s_1, g_1) \) with \( s \geq s_1 \) and \( g_1'(\cdot, g_1) \geq 0 \), we can prove again since \( g_1' < 1 \) that \((r_3, s_3)\) lies in \( \{(r, s); s - s_4 > r - r_4, s \geq s_4, r \geq r_4\} \) and it is not possible since \((r_4, s_4)\) is in \( \{(r, s); s - \bar{s} > r - \bar{r}, r \leq \bar{r}, s \leq \bar{s}\} \). Therefore we have \( s_4 > s_3 \). Since \( 0 \leq g_1' \) and \((r_3, s_3), (r_4, s_4)\) are on the curve \( S'_2 \) and \( s_4 > s_3 \), we have \( r_3 \leq r_4 \) (see Figure 2 for clear illustration). Hence we have:

\[
|\beta'| = r_2 - r_4 = r_0 - r_3 + \delta + r_3 - r_4 \leq |\beta| + \delta
\]

Next, we observe that \( \bar{r} = r_3 + \delta, \bar{s} = s_3 + \delta \) since \( S_1 \) curve remains unchanged under translation along \( g = g_0 \) line (and we have \( g_2 = g_0 \)). Since \((r_4, s_4)\) is in \( \{(r, s); s - \bar{s} > r - \bar{r}, r \leq \bar{r}, s \leq \bar{s}\} \) we have \( s_4 \leq \bar{s} = s_3 + \delta \). Therefore, we obtain

\[
|\gamma'| \leq |\gamma| + \delta.
\]

Figure 2: The \( S_1 \) curve starting from \((r_0, s_0)\) intersects the \( S'_2 \) curve from \((r_1, s_1)\) at the point \((r_3, s_3)\). Also, the \( S_1 \) curve starting from \((r_2, s_2)\) intersects the \( S'_2 \) curve from \((r_4, s_4)\) at the point \((r_0, s_0)\). Here, \( r_0 - r_3 = |\beta'|, r_2 - r_4 = |\beta'| \) and \( s_3 - s_1 = |\gamma|, s_4 - s_1 = |\gamma'| \). In this figure, \((\bar{r}, \bar{s}) = (r_3 + \delta, s_3 + \delta)\).

By a similar argument we get

**Lemma 4.2.** Let \((r_0, s_0) \in \mathbb{R}^2\) be a point in \(r-s\) plane. Consider a point \((r_1, s_1) \in \Omega_I[r_0, s_0]\). Let \(\delta > 0\) and \((r_2, s_2)\) be defined as \((r_2, s_2) = (r_1 + \delta, s_1 + \delta)\) with \(\delta > 0\) such that \((r_2, s_2) \in \Omega_I[(r_0, s_0)]\). Let \(\beta + \gamma\) be the outgoing wave from Riemann data \(U_I = (r_0, s_0)\) and \(U_r = (r_1, s_1)\) and \(\beta' + \gamma'\) be the outgoing wave from Riemann data \(U_I = (r_0, s_0)\) and \(U_r = (r_2, s_2)\). Then we have

\[
|\beta'| \leq |\beta| + \delta \quad \text{and} \quad |\gamma'| \leq |\gamma| + \delta.
\]

**Lemma 4.3.** Let \((\delta_-, \delta_+) \in \mathbb{R}^2\) and \((r_+, s_+), (r_-, s_-) \in \mathbb{R}^2\) satisfying the assumptions of Lemma 3.1 such that we can get a Riemann problem solution with \(U_I = (r_-, s_-) = \ldots\)
\((q_-, u_-), U_r = (r_+, s_+) = (q_+, u_+)\). Let \(u'_\pm = u_\pm + \delta_\pm\) and \((r'_\pm, s'_\pm)\) be Riemann invariant corresponding to \((q_\pm, u_\pm)\). We assume again that we can solve the Riemann problem associated to \((r'_\pm, s'_\pm)\). Then we have the following

1. when \(\Omega_1 = [r'_-, s'_-]\),
   
   1.1 If \((r_+, s_+) \in \Omega_1\) then
   
   \(|\beta'| \leq |\beta| + |\delta_+ - \delta_-|\) and \(|\gamma'| \leq |\gamma| + |\delta_+ - \delta_-|\).

   1.2 If \((r_+, s_+) \in \Omega_2\) then
   
   \(|\beta'| \leq |\beta| + |\delta_+ - \delta_-|\) and \(|\gamma'| \leq |\delta_+ - \delta_-|\).

   1.3 If \((r_+, s_+) \in \Omega_3\) then
   
   \(|\gamma'| \leq |\gamma| + |\delta_+ - \delta_-|\) and \(|\beta'| \leq |\delta_+ - \delta_-|\).

   1.4 If \((r_+, s_+) \in \Omega_4\) then
   
   \(|\beta'| + |\gamma'| \leq 2|\delta_+ - \delta_-|\).

2. When \((r'_+, s'_+) \in \Omega_1\)
   
   2.1 If \((r_+, s_+) \in \Omega_1\) then
   
   \(|\gamma'| \leq |\beta| + |\gamma| + |\delta_+ - \delta_-|\).

   2.2 If \((r_+, s_+) \in \Omega_2\) then
   
   \(|\gamma'| \leq |\beta| + |\gamma| + |\delta_+ - \delta_-|\).

   2.3 If \((r_+, s_+) \in \Omega_3\) then
   
   \(|\gamma'| \leq |\beta| + |\gamma| + |\delta_+ - \delta_-|\).

3. When \((r'_+, s'_+) \in \Omega_4\)
   
   3.1 If \((r_+, s_+) \in \Omega_1\) then
   
   \(|\beta'| \leq |\beta| + |\gamma| - |\delta_+ - \delta_-|\).

   3.2 If \((r_+, s_+) \in \Omega_2\) then
   
   \(|\beta'| \leq |\delta_+ - \delta_-|\).

   3.3 If \((r_+, s_+) \in \Omega_3\) then
   
   \(|\beta'| \leq |\delta_+ - \delta_-|\).

Proof. We first prove for \((r'_+, s'_+) \in \Omega_1\) and split the proof in four sub-cases depending on the fact that the Riemann problem associated to \((r_-, s_-)\) and \((r_+, s_+)\) is solved by combining 1 shock or 1 rarefaction and 2 shock or 2 rarefaction. In order to prove these different cases, we will use a generic notation \((\tilde{r}_-, \tilde{s}_-), (\tilde{r}_+, \tilde{s}_+)\) for the following.

1. When \(\delta_- \geq \delta_+\) we shift \((r'_+, s'_+)\) to \((r_+, s_+)\) and \((r'_-, s'_-)\) is shifted to the point \((\tilde{r}_-, \tilde{s}_-) := (r'_- - \delta_+, s'_- - \delta_-) = (r_- + \delta_1, s_- + \delta_1)\) where \(\delta_1 = \delta_- - \delta_+ \geq 0\).

2. When \(\delta_+ > \delta_-\) we shift \((r'_-, s'_-)\) to \((r_-, s_-)\) and \((r'_+, s'_+)\) is shifted to the point \((\tilde{r}_+, \tilde{s}_+) := (r'_+ - \delta_-, s'_+ - \delta_-) = (r_+ + \delta_2, s_+ + \delta_2)\) where \(\delta_2 = \delta_+ - \delta_- \geq 0\).
Case-1.1 We first consider the case when \((r_+, s_+) \in \Omega_I[r_-, s_-]\). Then there arise two possibilities

(a) \(\delta_- \geq \delta_+\). From Lemma 3.3 and since \(\delta_1 \geq 0\) we know that \((r_+, s_+) \in \Omega_I(\bar{r}_-, \bar{s}_-). We suppose now that the \(S_1\) curves from \((r_-, s_-), (\bar{r}_-, \bar{s}_-) \) intersect with \(S'_2\) curve from \((r_+, s_+) \) at \((r_1, s_1), (r_2, s_2)\) respectively. Then by Lemma 4.1 we get

\[|r_2 - \bar{r}_-| \leq |r_1 - r_-| + \delta_1\]

and since \(|s_2 - \bar{s}_-| \leq |s_1 - s_-| + \delta_1\) this gives us the required estimate because we have \((\bar{r}_-, \bar{s}_-):= (r'_- - \delta_+, s'_- - \delta_+)\) and \((r_+, s_+) = (r'_+ - \delta_+ , s'_+ - \delta_+)\). For a clear illustration of this case see Figure 3.

(b) Now since \((\bar{r}_+, \bar{s}_+):= (r'_+ - \delta_+ , s'_+ - \delta_+)\), \((r_-, s_-):= (r'_- - \delta_+, s'_- - \delta_-)\) and \((r'_+, s'_+) \in \Omega_I[r'_-, s'_-]\) we deduce from Lemma 3.3 that \((\bar{r}_+, \bar{s}_+) \in \Omega_I(r_-, s_-)\). Now consider \(\delta_+ > \delta_-\). This case can be handled similarly as in case \(\delta_- \geq \delta_+\) and we use Lemma 4.2 instead of Lemma 4.1 to get the required estimate.

Figure 3: From the Riemann data \((r_-, s_-)\) and \((r_+, s_+)\) two shocks of strengths \(|\beta|, |\gamma|\) arise. Two shocks situation arises even for Riemann data \((r'_-, s'_-)\) and \((r'_+, s'_+)\) where \(r'_\pm = r_\pm + \delta_\pm, s'_\pm = s_\pm + \delta_\pm\) with \(\delta_- > 0 > \delta_+\). In the later case, shock strengths are \(|\beta'|, |\gamma'|\) which remains same even after a translation \((r, s) \mapsto (r + \delta, s + \delta)\) where \(\delta = |\delta_+|\).

Case-1.2 Now we consider the case when \((r_+, s_+) \in \Omega_{IV}[r_-, s_-]\). Then there arise two possibilities.

(a) \(\delta_- \geq \delta_+\). Since \((r'_+, s'_+) \in \Omega_I[r'_-, s'_-]\) and \((\bar{r}_-, \bar{s}_-) = (r'_- - \delta_+, s'_- - \delta_+)\) we deduce from Lemma 3.3 that \((r_+, s_+) \in \Omega_I[\bar{r}_-, \bar{s}_-]. Since \(\delta_1 := \delta_- - \delta_+ \geq 0\), the point \((r_+, s_+)\) lies between two \(S_1\) curves starting from \((r_-, s_-)\) and \((\bar{r}_-, \bar{s}_-)\). Hence there exists a \(\eta \in (0, \delta_1]\) such that we can shift the point \((r_-, s_-)\) to \((r_1, s_1) = (r_- + \eta, s_- + \eta)\) such that \((r_+, s_+)\) lies on the \(S_1\) curve of \((r_1, s_1)\). It follows of the transversality of the \(S_1\) curves with the segment \([r_-, s_-], (\bar{r}_-, \bar{s}_-)\]. Let \(\beta_0\) denotes the shock connecting \((r_1, s_1)\) and \((r_+, s_+)\).
Then by Lemma 4.1 we obtain $|\gamma'| \leq (\delta_1 - \eta)$ and $|\beta'| \leq |\beta_0| + (\delta_1 - \eta)$. Since $|\beta_0| = |\beta| + \eta$ we get $|\beta'| \leq |\beta| + \delta_1$. This case is demonstrated in Figure 4.

(b) $\delta_+ > \delta_-$. Note that by 4) Lemma 3.3, we get $(\bar{r}_+, \bar{s}_+) \notin \Omega_I[r_-, s_-]$. This gives a contradiction since we know that $(\bar{r}_+, \bar{s}_+) \in \Omega_I[r_-, s_-]$. It comes from the 1) Lemma 3.3 and the fact that $(\bar{r}_+, \bar{s}_+) = (r'_+, s'_+) - (\delta_-, \delta_-)$ and $(r_-, s_-) = (r'_-, s'_-) - (\delta_-, \delta_-)$ with $(r'_+, s'_+) \in \Omega_I[r'_-, s'_-].$

![Figure 4: Two points $(r_-, s_-), (r_+, s_+)$ are related as $(r_+, s_+) \in \Omega IV[r_-, s_-]$ which means the Riemann data $(r_+, s_-), (r_+, s_+)$ gives 1-shock of strength $|\beta|$ and 2-rarefaction. Under the translation $(r, s) \mapsto (r + \delta, s + \delta)$, the point $(r_-, s_-)$ goes to $(r'_-, s'_-)$. For the Riemann data $(r'_-, s'_-), (r_+, s_+)$, two shocks arise with strengths $|\beta'|, |\gamma'|$.](image)

Case-1.3 The case when $(r_+, s_+) \in \Omega_{II}[r_-, s_-]$ can be handled in the similar way.

Case-1.4 Suppose $(r_+, s_+) \in \Omega_{III}[r_-, s_-].$

(a) Suppose $\delta_- \geq \delta_+$. Since we have seen that $(r_+, s_+) \in \Omega_I[\bar{r}_-, \bar{s}_-]$, we have $0 \leq \bar{r}_- - r_+ \leq \delta_1$ and $0 \leq \bar{s}_- - s_+ \leq \delta_1$. Hence we get the required estimate. See Figure 5 for a clear illustration of this case.

(b) Consider $\delta_+ > \delta_-$. By applying Lemma 3.3 we get $(\bar{r}_+, \bar{s}_+) \notin \Omega_I[r_-, s_-]$ which is contradiction to our assumption $(r'_+, s'_+) \in \Omega_I[r'_-, s'_-].$

Next we prove the case $(r'_+, s'_+) \in \Omega_{II}[r'_-, s'_-]$. Proof of this case follows in a similar way as in the previous case. For proof of this case, we use a generic notation $(\bar{r}_-, \bar{s}_-), (\bar{r}_+, \bar{s}_+)$ for the following.

1. When $\delta_- \geq \delta_+$ we shift $(r'_+, s'_+) \rightarrow (r_+, s_+)$ and $(r'_-, s'_-)$ is shifted to the point $(\bar{r}_-, \bar{s}_-) := (r'_- - \delta_+, s'_- - \delta_+) = (r_- + \delta_1, s_- + \delta_1)$ where $\delta_1 = \delta_- - \delta_+ \geq 0$.

2. When $\delta_+ > \delta_-$ we shift $(r'_-, s'_-) \rightarrow (r_-, s_-)$ and $(r'_+, s'_+)$ is shifted to the point $(\bar{r}_+, \bar{s}_+) := (r'_+ - \delta_-, s'_+ - \delta_-) = (r_+ + \delta_2, s_+ + \delta_2)$ where $\delta_2 = \delta_+ - \delta_- > 0$.

Case-2.1 Consider the case when $(r_+, s_+) \in \Omega_I[r_-, s_-]$. As before, we divide into two cases: (i) $\delta_- \geq \delta_+$, (ii) $\delta_+ > \delta_-.$

(a) $\delta_- \geq \delta_+$. In this case, by Lemma 3.3 we note that $(r_+, s_+) \notin \Omega_{III}[\bar{r}_-, \bar{s}_-].$
Figure 5: Three points \((r_-, s_-), (r_+, s_+), (r'_-, s'_-)\) are considered in \(r-s\) plane such that \(r'_- = r_- + \delta, s'_- = s_- + \delta\). From Riemann data \((r_-, s_-), (r_+, s_+)\) two rarefaction waves arise whereas the Riemann data \((r'_-, s'_-), (r_+, s_+)\) gives two shocks of strengths \(|\beta|, |\gamma|\).

(b) \(\delta_+ > \delta_-\). Since \((\bar{r}_+, \bar{s}_+) = (r_+ + \delta_2, s_+ + \delta_2) \in \Omega_{II}[r_-, s_-]\) and \((r_+, s_+) \in \Omega_I[r_-, s_-]\), we can find a \(\delta \in [0, \delta_2]\) such that \((r_-, s_-)\) lies on the \(S_2'\) curve starting from \((r_+ + \delta, s_+ + \delta)\). By Lemma 4.2 we get \(|\gamma'| \leq \delta + |\beta| + |\gamma|\). Hence, we prove the estimate \(|\gamma'| \leq |\beta| + |\gamma| + |\delta_+ - \delta_-|\).

Case-2.2 Consider the case when \((r_+, s_+) \in \Omega_{II}[r_-, s_-]\). As before, we divide into two cases: (i) \(\delta_- \geq \delta_+\), (ii) \(\delta_+ > \delta_-\).

(a) \(\delta_- \geq \delta_+\). In this case we note that \(\bar{s}_- - s_+ = s_- - s_+ + \delta_1\) where \(\delta_1 = \delta_- - \delta_+\). Hence we have \(|\gamma'| = |\gamma| + \delta_1\).

(b) \(\delta_+ > \delta_-\). Observe that \(\bar{s}_- - s_+ = s_- - s_+ - \delta_2\) where \(\delta_2 = \delta_+ - \delta_-\). Hence we have \(|\gamma'| = |\gamma| - \delta_2\).

Case-2.3 Consider the case when \((r_+, s_+) \in \Omega_{IV}[r_-, s_-]\). As before, we divide into two cases: (i) \(\delta_- \geq \delta_+\), (ii) \(\delta_+ > \delta_-\).

(a) \(\delta_- \geq \delta_+\). In this case, by invoking Lemma 3.3 we get \((r_+, s_+) \notin \Omega_{II}[\bar{r}_-, \bar{s}_-]\). It is a contradiction.

(b) \(\delta_+ > \delta_-\). Again by Lemma 3.3 we get \((\bar{r}_+, \bar{s}_+) \notin \Omega_{II}[r_-, s_-]\). This gives a contradiction.

Case-2.4 Consider the case when \((r_+, s_+) \in \Omega_{III}[r_-, s_-]\). As before, we divide into two cases: (i) \(\delta_- \geq \delta_+\), (ii) \(\delta_+ > \delta_-\).

(a) \(\delta_- \geq \delta_+\). Since \((r_+, s_+) \in \Omega_{III}[r_-, s_-]\), we have \(s_- \leq s_+\). Hence \(\bar{s}_- - s_+ = s_- + \delta_1 - s_+\) and this gives the required estimate.

(b) \(\delta_+ > \delta_-\). In this case we apply Lemma 3.3 to get \((\bar{r}_+, \bar{s}_+) \notin \Omega_{II}[r_-, s_-]\) which gives a contradiction.
that the decreasing total variation on
\(\Omega\) that: 

\[ TV(r_{\Delta}^{\theta}(t_n, \cdot)) \leq 2V(O^n) + |r_n^+ - r_n^-|, \quad TV(s_{\Delta}^{\theta}(t_n, \cdot)) \leq 2V(O^n) + |s_n^+ - s_n^-| \]

It implies in particular that we have with our definition of \(r_{\Delta}^{\theta}\) and \(s_{\Delta}^{\theta}\) that:

\[ TV(r_{\Delta}^{\theta}(t_n, \cdot)) \leq 2V(O^n) + |r_n^+ - r_n^-|, \quad TV(s_{\Delta}^{\theta}(t_n, \cdot)) \leq 2V(O^n) + |s_n^+ - s_n^-| \]

Remark 8. It is important to point out that for a \(J\) curve there is no shock wave crossing \(J\) at infinity since the solution of the Riemann problem on time interval \((t_n, t_{n+1})\) is trivial since the solution \((\rho^\Delta, u^\Delta)\) is constant at infinity at the time \(t_n\).

Remark 9. As in [23] we can estimate the total variation of the solution \((\rho^\Delta, u^\Delta)\) along \(O^n\) by using the quantity \(V(O^n)\). When a shock wave cross \(O^n\), the Riemann invariants \(r\) and \(s\) decrease whereas \(r\) and \(s\) increase when a rarefaction wave cross \(O^n\). We deduce using Remark 3 that the decreasing total variation on \(r\) and \(s\) is controlled by \(V(O^n)\), in particular if we consider the restriction of the functions \(r_{\Delta}^{\theta}\), \(s_{\Delta}^{\theta}\) on \(O^n\) that we note \(r_{\Delta}^{\theta}|_{O^n}\), \(s_{\Delta}^{\theta}|_{O^n}\) where we assume that \(r_{\Delta}^{\theta}|_{O^n}\) and \(s_{\Delta}^{\theta}|_{O^n}\) take the same values as \(r_{\Delta}^{\theta}\) and \(s_{\Delta}^{\theta}\) except on the points \((t_{n+1}, x_k + \theta_{n+1})\) where \(r_{\Delta}^{\theta}|_{O^n}\) and \(s_{\Delta}^{\theta}|_{O^n}\) take the values \(r_{n+\frac{1}{2}, k}, s_{n+\frac{1}{2}, k}\). It implies that the total variation of \(r_{\Delta}^{\theta}|_{O^n}\), \(s_{\Delta}^{\theta}|_{O^n}\) along \(O^n\) is bounded by:

\[ TV(r_{\Delta}^{\theta}|_{O^n}) \leq 2V(O^n) + |r_n^+ - r_n^-| \]

\[ TV(s_{\Delta}^{\theta}|_{O^n}) \leq 2V(O^n) + |s_n^+ - s_n^-| \]

4.2 Estimates of Glimm functional

Definition 4.4. 1. An I-curve is a piecewise linear, Lipschitz continuous function such that each linear part coincides either with the line joining \((x_i + \theta_n, n\Delta t), (x_i + \Delta x + \theta_{n+1}, (n + 1)\Delta t))\), or with the line joining \((x_j + \theta_k, k\Delta t), (x_j - \Delta x + \theta_{k+1}, (k + 1)\Delta t))\).

2. For \(n \geq 0\), we define \(O^n\)-curve is an I-curve contained in \(\{(x,t); x \in \mathbb{R}, n\Delta t \leq t \leq (n + 1)\Delta t\}\), we simply write \(O\)-curve instead of \(O^0\).

Glimm functional: For an I-curve \(J\) we define

\[ V(J) = \sum \{|\alpha| : \alpha \text{ is a shock wave crossing } J\}, \quad (54)\]

\[ Q(J) = \sum \{|\beta| : |\beta| \text{ cross } J \text{ and approach}\}, \quad (55)\]

\[ F(J) = V(J) + KQ(J). \quad (56)\]

with \(K > 0\) which will be defined later.

This completes the proof for case \((r'_+, s'_+) \in \Omega_{II}[r'_-, s'_-]\). By a similar argument we can show the case \((r'_+, s'_+) \in \Omega_{IV}[r'_-, s'_-]\). This ends the proof of Lemma 4.3.
Using Lemma 2.1 and the definition of the Riemann invariant we deduce that:
\[
TV(u^{\Delta, \theta}(t_n, \cdot)) \leq 4V(O^n) + C'_T, \\
TV(u^{\Delta, \theta}(t_{n+\frac{1}{2}}, \cdot)) \leq 4V(O^n) + C'_T. \tag{60}
\]
with $C'_T$ a positive constant depending only on $T$.

We recall the following Lemma for the $O$ curve (see [23] p197).

**Lemma 4.5.** Let $\epsilon_1$ such that $4C\epsilon_1 TV(r_0(\cdot), s_0(\cdot)) \leq 1$. Let $0 \leq \epsilon \leq \min\{\epsilon_0, \epsilon_1\}$ where $\epsilon_0$ is as in Lemma 3.1 for Riemann problems issue of the values defined by $(\varrho_0, u_0)$ with $0 < \varrho < \varrho_0(x) \leq \bar{\varrho} < \infty$ for $x \in \mathbb{R}$. Suppose $F$ is defined as in (56) for $K = 4C\epsilon$ where $C$ is the constant as in (49). Then we have
\[
F(O) = V(O) + KQ(O) \leq 2V(O) \leq 2TV(r_0(\cdot), s_0(\cdot)). \tag{61}
\]

**Lemma 4.6.** Let $T > 0$. Then for $n \geq 1$ such that $t_n \leq T$, we have
\[
TV(\Psi_n^\Delta) \leq \xi_T + 4\frac{q}{e} (1 + ||\mu^\Delta||_{L^\infty}),
\]
\[
||\Psi_n^\Delta||_{L^\infty(\mathbb{R})} \leq ||\Psi^-||_{L^\infty(O,T)} + \xi_T + 4\frac{q}{e} (1 + ||\mu^\Delta||_{L^\infty}),
\]
\[
||\varrho_n^\Delta||_{L^\infty(\mathbb{R})} \leq C(\rho^-, V(O^n)),
\]
\[
||u_n^\Delta||_{L^\infty(\mathbb{R})} \leq C_T + 4V(O^n),
\]
where $C(\cdot, \cdot)$ depends only on $\varrho, \bar{\varrho}, M$ and $\xi_T = ||\xi_0||_{L^1} + T(\mu^- + \mu^+)C_T$ with $C_T$ is the same constant as in Lemma 2.1.

**Proof.** From the definition of $\Psi_n^\Delta$ and Lemma 2.2 we have
\[
TV(\Psi_n^\Delta) = \sum_{n+j} |\Psi_{n,j} - \Psi_{n,j+2}| \leq \int_\mathbb{R} |\xi_n^\Delta| \, dx \leq \xi_T + 4\frac{q}{e} (1 + ||\mu^\Delta||_{L^\infty})
\]
where $\xi_T = ||\xi_0||_{L^1} + T(\mu^- + \mu^+)C_T$. To get (63) note the following
\[
||\Psi_n^\Delta||_{L^\infty(\mathbb{R})} \leq \Psi^- + TV(\Psi_n^\Delta) \leq ||\Psi^-||_{L^\infty(O,T)} + TV(\Psi_n^\Delta)
\]
\[
\leq ||\Psi^-||_{L^\infty(O,T)} + \xi_T + 4\frac{q}{e} (1 + ||\mu^\Delta||_{L^\infty}). \tag{66}
\]
The estimate (65) is a direct consequence of the Remark 9 and of the Lemma 2.1. Similarly since we can write $\varrho$ in terms of $r(\varrho, u)$ and $s(\varrho, u)$, and using the fact that the Riemann invariant defined $C^1$ diffeomorphism with respect to $\varrho, u$ we obtain (63) with $C(\cdot, \cdot)$ depending only on $\varrho, \bar{\varrho}, M$. Now with a similar argument as in (66) we get (64), (65). \qed

We Suppose $A_1, B_1$ be two constants such that for $L_1 = 2L + \frac{2T}{A}$
\[
A_1 \geq 4e^{\lambda A L_1} TV(\sigma^\Delta) + 16 ||\sigma^\Delta||_\infty \tag{67}
\]
\[
B_1 \geq 4C_T TV(\sigma^\Delta) + 4B \frac{e^{\lambda A L_1}}{A} TV(\sigma^\Delta) + 4C'_T ||\sigma^\Delta||_\infty \geq 4q \frac{1}{m} ||\Psi_{n+1}^\Delta||_\infty TV(\sigma^\Delta) \Delta t, \tag{68}
\]
\[
\geq 4q \frac{1}{m} TV(\Psi_{n+1}^\Delta).\]
where $A, B > 0$ are defined as
\[
A \geq 32 \|\sigma^\Delta\|_\infty \quad \text{and} \quad B \geq 8(C_T + C'_T) \|\sigma^\Delta\|_\infty + \frac{8q}{m} \left( \|\Psi^\Delta\|_\infty \right) |\sigma^\Delta| \Delta t + \|\Psi^\Delta\|_\infty. \quad (69)
\]

Lemma 4.7. Let $A_1, B_1 > 0$ be defined as in (67) and (68). Let $\epsilon_2 > 0$ satisfies
\[
4C\epsilon_2 \left[ e^{A\Delta t} - \frac{1}{A} e^{A_1 n \Delta t} TV(r_0, s_0) + B \frac{e^{A_2} - 1}{A} e^{A_1 n \Delta t} - B_1 \frac{e^{A_1 n \Delta t} - 1}{A_1} \right] \leq \min\{1, C_0\}. \quad (70)
\]
where $C > 0$ is a constant appeared in (49) and $C_0 > 0$ is the constant as in [23, Lemma 4, pgae-193]. Let $\epsilon_0, \epsilon_1$ be as in Lemma 4.5. Then for $0 \leq \epsilon \leq \min\{\epsilon_i, i = 0, 1, 2\}$ we have
\[
F(O^n) \leq e^{A_1 n \Delta t} F(O) + B_1 \sum_{k=0}^{n-1} (1 + A_1 \Delta t)^k \quad \text{for} \quad 0 \leq n + 1 \leq \frac{T}{\Delta t}. \quad (71)
\]

Remark 4.8. In approximation process, we choose $\sigma^\Delta, \mu^\Delta$ such that $\|\sigma^\Delta\|_\infty \leq \|\sigma\|_\infty, TV(\sigma^\Delta) \leq TV(\sigma)$ and $\|\mu^\Delta\|_\infty \leq \|\mu\|_\infty$. By (62) and (63), we get that the constants $A_1, B_1$ are independent of mesh size.

Proof. We prove (71) for $0 \leq n \leq T/\Delta t$ by induction. For $n = 0$, the estimate (71) trivially follows. We assume that the inequality (71) is true for $n$ then we prove it for $n + 1$. By (19) we know that for a sufficiently large $i_n \geq 1$,
\[
(\theta_{n+1,i}, u_{n+1,i}) = (\theta_{n,i+1}, u_{n,i+1}) \quad \text{for} \quad i \Delta x \leq -L - (n + 1) \Delta x,
\]
\[
(\theta_{n+1,i}, u_{n+1,i}) = (\theta_{n,i-1}, u_{n,i-1}) \quad \text{for} \quad i \Delta x \geq L + (n + 1) \Delta x.
\]

Therefore, we can reach $O^{n+1}$ from $O^n$ in finitely many steps by considering consecutively immediate successor (indeed there is nothing to do on $O^{n+1}$ when $x \leq -L - (n + 1) \Delta x$ and when $x \geq L + (n + 1) \Delta x$ since we know that there is no shock wave crossing $O^{n+1}$), that is, there are $\bar{J}_i, 0 \leq i \leq m$ such that $O^n = \bar{J}_0 \leq \bar{J}_1 \leq \cdots \leq \bar{J}_m = O^{n+1}$ where $\bar{J}_{i+1}$ is immediate successor of $\bar{J}_i$ for $i \geq 0$. We also observe that $m \Delta x \leq L_1 = 2L + T$.

Claim 4.9. Let $A, B > 0$ be defined as in (69). Then we have,
\[
F(\bar{J}_i) \leq e^{A_1 \Delta x} F(O^n) + B \Delta t \sum_{l=0}^{i-1} (1 + A \Delta t)^l \quad \text{for} \quad 0 \leq i \leq m. \quad (72)
\]

Proof of Claim 4.9. We will also prove this claim by induction. Note that $i = 0$ case is trivial. Next we assume that (72) is true for $i = j$, then we show for $i = j + 1$.

First, we assume that $J_2$ is an immediate successor to $J_1$ and we wish to evaluate $F(J_2)$ in terms of $F(J_1)$. Let $J_1 \setminus J_2$ be consisting lines $L[(x_k + \theta_{n+1}, t_{n+1}), (x_{k+1} + \theta_n \Delta x, t_n)]$ and $L[(x_{k+2} + \theta_{n+1} \Delta x, t_{n+1}), (x_{k+1} + \theta_n \Delta x, t_n)]$. Let $J_2 \setminus J_1$ be consisting lines $L[(x_k + \theta_{n+1} \Delta x, t_{n+1}), (x_{k+1} + \theta_{n+2} \Delta x, t_{n+2})]$ and $L[(x_{k+2} + \theta_{n+1} \Delta x, t_{n+1}), (x_{k+1} + \theta_{n+2} \Delta x, t_{n+2})]$. We can observe that we define a diamond-shaped region $D_{n+1,k}$ with vertices at the four surrounding sampling points $(x_k + \theta_{n+1}, t_{n+1}), (x_{k+1} + \theta_n \Delta x, t_n), (x_{k+2} + \theta_{n+1} \Delta x, t_{n+1})$ and $(x_{k+1} + \theta_{n+2} \Delta x, t_{n+2})$ with respectively the following state $(\theta_{n+1,k}, u_{n+1,k}), (\theta_{n+1,k+1}, u_{n+1,k+1}), (\theta_{n+1,k+2}, u_{n+1,k+2})$ and $(\theta_{n+2,k+1}, u_{n+2,k+1})$ (see Figure 6 for an illustration). From (33) we have
\[
u_{n+1,k} = u_{n+1,k+1} \exp(-\sigma_k \Delta t) - \frac{q}{m} \frac{1 - \exp(-\sigma_k^\Delta \Delta t)}{\sigma_k^\Delta} \Psi_{n+1,k}.
\]
Figure 6: This illustrates the diamond (in dotted lines) formed by \( (J_2 \setminus J_1) \cup (J_1 \setminus J_2) \) where \( J_i, i = 1, 2 \) are I-curves and \( J_2 \) is immediate successor of \( J_1 \).

We rewrite \( u_{n+1,k} \) as follows

\[
u_{n+1,k} = u_{n+\frac{1}{2},k} + \delta_{n+1,k}\]

where \( \delta_{n+1,k} \) is defined as

\[
\delta_{n+1,k} = u_{n+\frac{1}{2},k} \left[ \exp(-\sigma_k^\Delta \Delta t) - 1 \right] - \frac{q}{m} \frac{1 - \exp(-\sigma_k^\Delta \Delta t)}{\sigma_k^\Delta} \Psi_{n+1,k}.
\]

Similarly we write \( u_{n+1,k+2} = u_{n+\frac{1}{2},k+2} + \delta_{n+1,k+2} \) where \( \delta_{n+1,k+2} \) is defined as

\[
\delta_{n+1,k+2} = u_{n+\frac{1}{2},k+2} \left[ \exp(-\sigma_{k+2}^\Delta \Delta t) - 1 \right] - \frac{q}{m} \frac{1 - \exp(-\sigma_{k+2}^\Delta \Delta t)}{\sigma_{k+2}^\Delta} \Psi_{n+1,k+2}.
\]

We can observe in particular using the definition of the scheme (34) and (35) that the strength of the shock waves which cross \( J_1 \setminus J_2 \) are issue of the two Riemann problem with left and right states \([\left( \bar{\nu}_{n+\frac{1}{2},k}, u_{n+\frac{1}{2},k} \right), \left( \bar{\nu}_{n,k+1}, u_{n,k+1} \right)]\) and \([\left( \bar{\nu}_{n,k+1}, u_{n,k+1} \right), \left( \bar{\nu}_{n+\frac{1}{2},k+2}, u_{n+\frac{1}{2},k+2} \right)]\). We remark also that the strength of the shock waves which cross \( J_2 \setminus J_1 \) are issue of the Riemann problem with left and right state \([\left( \bar{\nu}_{n+1,k}, u_{n+1,k} \right), \left( \bar{\nu}_{n+1,k+2}, u_{n+1,k+2} \right)]\).

Now we are going to define \( \tilde{J}_2 \) which is the same I curves as \( J_2 \) except that the strength of the shock wave which cross \( \tilde{J}_2 \setminus J_1 \) is coming from the Riemann problem with left state \( \left( \bar{\nu}_{n+1,k}, u_{n+1,k} \right) \) and with right state \( \left( \bar{\nu}_{n+1,k+2}, u_{n+1,k+2} \right) \) when \( \delta_{n+1,k} = \delta_{n+1,k+2} = 0 \). It is natural now to consider the quantities \( F(J_2), V(J_2) \) when \( \delta_{n+1,k} = \delta_{n+1,k+2} = 0 \). Note that if \( \delta_{n+1,k}, \delta_{n+1,k+2} \) does not contribute in solution then the changes in \( F, V, Q \) for the diamond \( D_{n+1,k} \) is same as the homogeneous case done in [23]. We recall the following Lemma issue from [23].

**Lemma 4.10.** If \( \epsilon F(J_1) \) is sufficiently small, then:

\[
F(\tilde{J}_2) \leq F(J_1),
\]

with \( \tilde{J}_2 \) is an immediate successor to \( J_1 \).

**Remark 10.** Note that by induction hypothesis \( F(\tilde{J}_i), F(O^n) \) satisfy (72) and (71) respectively. Since \( J_1 \) coincides with \( \tilde{J}_i \) for some \( i \), we get \( 4\epsilon CF(J_1) \leq \min\{C_0, 1\} \). Therefore, proof of Lemma 4.10 follows from [23, Lemma 5, page-197].
Therefore, we can write
\[ \tilde{F}(J_2) = \tilde{V}(J_2) + K \tilde{Q}(J_2) \leq F(J_1) = V(J_1) + KQ(J_1). \] (73)

By using Lemma 4.3 we can now evaluate the difference of the strength between \( \tilde{J}_2 \) and \( J_2 \) on \( J_2 \setminus J_1 \)
\[ V(J_2) - V(\tilde{J}_2) \leq 2|\delta_{n+1,k+2} - \delta_{n+1,k}|, \]
\[ Q(J_2) - Q(\tilde{J}_2) = \sum_{\beta \notin J_2 \setminus J_1, \beta \text{ approaching } \gamma_2} |\beta||\gamma_1'| + \sum_{\beta \notin J_2 \setminus J_1, \beta \text{ approaching } \beta_2'}|\beta||\beta_2'| \]
\[ - \sum_{\beta \notin J_2 \setminus J_1, \beta \text{ approaching } \gamma_2} |\beta||\tilde{\gamma}_2| - \sum_{\beta \notin J_2 \setminus J_1, \beta \text{ approaching } \tilde{\beta}_2}|\beta||\tilde{\beta}_2| \]
\[ \leq 2|\delta_{n+1,k+2} - \delta_{n+1,k}| \sum_{\beta \in J_1, \notin J_2} |\beta|. \] (74)

Above \( \gamma_2, \beta_2 \) and \( \tilde{\gamma}_2, \tilde{\beta}_2 \) are respectively the shock waves crossing respectively \( J_2 \) and \( \tilde{J}_2 \) on \( J_2 \setminus J_1 \). Hence,
\[ F(J_2) - F(\tilde{J}_2) \leq 2|\delta_{n+1,k+2} - \delta_{n+1,k}| + 2K|\delta_{n+1,k+2} - \delta_{n+1,k}| \sum_{\beta \in J_1, \notin J_2} |\beta|. \] (75)

Using (73) and (75) we deduce that:
\[ F(J_2) \leq F(J_1) + 2|\delta_{n+1,k+2} - \delta_{n+1,k}| + 2K|\delta_{n+1,k+2} - \delta_{n+1,k}| \sum_{\beta \in J_1, \notin J_2} |\beta|. \] (76)

Let \( a_{n,i}, b_{n,i} \) be defined as
\[ a_{n,i} = \left[ \exp(-\sigma_i^A \Delta t) - 1 \right] u_{n+\frac{1}{2},i} \text{ and } b_{n,i} = \frac{q}{m} \frac{1 - \exp(-\sigma_i^A \Delta t)}{\sigma_i^A} \Psi_{n+1,i}. \] (77)

We observe in particular that \( \delta_{n,i} = a_{n,i} + b_{n,i} \). From triangular inequality we get
\[ |a_{n,i+2} - a_{n,i}| \leq \Delta t |\sigma_{i+2} - \sigma_i| |u_{n+\frac{1}{2},i+2}| + \Delta t |\sigma_i| |u_{n+\frac{1}{2},i+2} - u_{n+\frac{1}{2},i}|. \] (78)

Similarly, we obtain
\[ |b_{n,i+2} - b_{n,i}| \leq \frac{q}{m} \left| \frac{1 - \exp(-\sigma_{i+2}^A \Delta t)}{\sigma_{i+2}^A} - \frac{1 - \exp(-\sigma_i^A \Delta t)}{\sigma_i^A} \right| |\Psi_{n+1,i+2}| \]
\[ + \frac{q}{m} \left| \frac{1 - \exp(-\sigma_i^A \Delta t)}{\sigma_i^A} \right| |\Psi_{n+1,i+2} - \Psi_{n+1,i}| \]
\[ \leq \frac{q}{m} |\Psi_{n+1,i+2}| |\sigma_{i+2}^A - \sigma_i^A| \Delta t^2 + \frac{q}{m} |\Psi_{n+1,i+2} - \Psi_{n+1,i}| \Delta t. \] (79)

Combining (76), (78) and (79) and the fact that \( V(J_1) \leq F(J_1) \), we have
\[ F(J_2) - F(J_1) \leq \left( 2|u_{n+\frac{1}{2},k+2}| |\sigma_{k+2} - \sigma_k| \Delta t + 2 |\sigma_k| |u_{n+\frac{1}{2},k+2} - u_{n+\frac{1}{2},k}| \Delta t \right. \]
\[ + 2 \left[ \frac{q}{m} |\Psi_{n+1,k+2}| |\sigma_{k+2}^A - \sigma_k^A| \Delta t^2 + \frac{q}{m} |\Psi_{n+1,k+2} - \Psi_{n+1,k}| \Delta t \right] \right) (1 + K \sum_{\beta \in J_1, \notin J_2} |\beta|). \] (80)
We can now estimate \(|u_{n+\frac{1}{2},k+2} - u_{n+\frac{1}{2},k}|\) in terms of the strength of the shock wave crossing \(J_1\) by using (60) (again we assume that \(J_1\) is an line \(k^+\Delta t, k^-\Delta t\) at infinity)

\[|u_{n+\frac{1}{2},k+2} - u_{n+\frac{1}{2},k}| \leq 4V(J_1) + C'_T \leq 4F(J_1) + C'_T.\]

Now using Lemma 2.1 and (60) we have \(|u_{n+\frac{1}{2},k+2} - u_{n+\frac{1}{2},k}| \leq |u_n^-| + TV(u|_{O_n}) \leq C_T + 4V(J_1) \leq C_T + 4F(J_1).\) Here \(C_T\) is a constant depending only on \(T\). Then it yields from (80) and choice of \(K\):

\[F(J_2) - F(J_1) \leq \left(2C_T \sigma_{k+2} - \sigma_k|\Delta t + 8 |\sigma_{k+2} - \sigma_k|\Delta t F(J_1) + 8 |\sigma_k|\Delta t F(J_1) + 8 |\sigma_k|\Delta t C'_T + \frac{q}{m} |\Psi_{n+1,k+2}| \sigma_{k+2} - \sigma_k|\Delta t + \frac{q}{m} |\Psi_{n+1,k+2} - \Psi_{n+1,k}|\Delta t \right) (1 + 4C\epsilon F(J_1)). \] (81)

Now we set \(J_1 = \tilde{J}_j\) and \(J_2 = \tilde{J}_{j+1}\). Note that by our assumption, \(F(\tilde{J}_j)\) satisfies (72) and \(F(O^n)\) satisfies (71). Due to the choice of \(\epsilon\) and the fact \(m\Delta x \leq 2L + 2(n + 1)\Delta x \leq 2L + \frac{2T}{\lambda} = L_1\), we have

\[1 + KF(\tilde{J}_j) \leq 1 + 4C\epsilon \left[e^{\lambda A L_1/2} F(O^n) + B \frac{e^{\lambda A L_1 - 1}}{A}\right] \leq 1 + 4C\epsilon \left[e^{\lambda A L_1} e^{A_1 n\Delta t} F(O^n) + e^{\lambda A L_1} B_1 \left(e^{A_1 n\Delta t} - 1 + B \frac{e^{\lambda A L_1 - 1}}{A}\right)\right] \leq 2. \] (82)

Hence, we have

\[F(\tilde{J}_{j+1}) - F(\tilde{J}_j) \leq \left(2C_T \sigma_{k+2} - \sigma_k|\Delta t + 8 |\sigma_{k+2} - \sigma_k|\Delta t F(\tilde{J}_j) + 8 |\sigma_k|\Delta t F(\tilde{J}_j) + 8 |\sigma_k|\Delta t C'_T + \frac{q}{m} |\Psi_{n+1,k+2}| \sigma_{k+2} - \sigma_k|\Delta t + \frac{q}{m} |\Psi_{n+1,k+2} - \Psi_{n+1,k}|\Delta t \right) (1 + KF(\tilde{J}_j)) \leq 8(C_T + C'_T) |\sigma| \Delta t + 32 |\sigma| \Delta t F(\tilde{J}_j) + \frac{8q}{m} |\Psi| \Delta t + |\Psi| \Delta t. \] (83)

By choice of \(A, B\) as in (69) we have using the recurrence hypothesis (72) at the level \(j\)

\[F(\tilde{J}_{j+1}) \leq (1 + A\Delta t) F(\tilde{J}_j) + B\Delta t \leq (1 + A\Delta t) e^{\lambda A j \Delta x} F(O^n) + B\Delta t \sum_{i=1}^{j} (1 + A\Delta t)^i + B\Delta t \leq e^{\lambda A (j+1) \Delta x} F(O^n) + B\Delta t \sum_{i=0}^{j} (1 + A\Delta t)^i.\]

This completes proof of Claim 4.9. \(\square\)
From (81) with $J_1 = \bar{J}_k, J_2 = \bar{J}_{k+1}$, we obtain

$$F(\bar{J}_{k+1}) - F(\bar{J}_k)$$

$$\leq \left( 2 \left| u_{n+\frac{1}{2},k+2} \right| \left| \sigma_{k+2} - \sigma_k \right| \Delta t + 2 \left| u_{n+\frac{1}{2},k+2} - u_{n+\frac{1}{2},k} \right| \Delta t$$

$$+ 2 \left( \frac{q}{m} \right) \left| \Psi_{n+1,k+2,2} \right| \left| \sigma_{k+2} - \sigma_k \right| \Delta t^2 + \frac{q}{m} \left| \Psi_{n+1,k+2} - \Psi_{n+1,k+1} \right| \Delta t \right)^2$$

$$(1 + K \sum_{\beta \in J_1, \delta J_1 \setminus J_2} |\beta|).$$

(84)

Using Claim 4.9, (82) and the fact that $\left| u_{n+\frac{1}{2},k+2} \right| \leq C_T + 4F(J_1)$ we obtain

$$F(\bar{J}_{k+1}) - F(\bar{J}_k)$$

$$\leq 4C_T \left| \sigma_{k+2} - \sigma_k \right| \Delta t + 16 \left( e^{\lambda \lambda L_1} F(O^n) + B \frac{e^{A \lambda L_1}}{A} \right) \left| \sigma_{k+2} - \sigma_k \right| \Delta t$$

$$+ 4 \left| \sigma_{\Delta} \right| \left( 4F(O^n) + C_T \right) \Delta t + \frac{4q}{m} \left| \Psi_{n+1} \right| \Delta t^2$$

$$+ \frac{4q}{m} \left| \Psi_{n+1,k+2} - \Psi_{n+1,k+1} \right| \Delta t.$$  

(85)

Therefore by summing (85) that we combine with (60), it gives

$$F(O^{n+1}) - F(O^n)$$

$$\leq 4C_T TV(\sigma_{\Delta}) \Delta t + 4 \left( e^{\lambda \lambda L_1} F(O^n) + B \frac{e^{A \lambda L_1}}{A} \right) TV(\sigma_{\Delta}) \Delta t$$

$$+ 4 \left| \sigma_{\Delta} \right| \left( 4F(O^n) + C_T \right) \Delta t + \frac{4q}{m} \left| \Psi_{n+1} \right| \Delta t^2 + \frac{4q}{m} TV(\Psi_{n+1} \Delta) \Delta t.$$  

Let $A_1, B_1 > 0$ be as in (67), (68), then we get

$$A_1 \geq 4e^{\lambda \lambda L_1} TV(\sigma_{\Delta}) + 16 \left| \sigma_{\Delta} \right| \Delta t,$$

$$B_1 \geq 4C_T TV(\sigma_{\Delta}) + 4B \frac{e^{A \lambda L_1}}{A} - 1 TV(\sigma_{\Delta}) + 4C_T \left| \sigma_{\Delta} \right| \Delta t + \frac{4q}{m} \left| \Psi_{n+1} \right| \Delta t^2 \Delta t.$$  

Hence,

$$F(O^{n+1}) \leq F(O^n)(1 + A_1 \Delta t) + B_1 \Delta t.$$  

This completes the proof of Lemma 4.7.

Once we obtain (71), subsequently, we have

$$F(O^n) \leq e^{A_1 n \Delta t} F(O_1) + B_1 \frac{e^{A_1 n \Delta t} - 1}{A_1}$$

$$\leq e^{A_1 T} F(O_1) + B_1 \frac{e^{A_1 T} - 1}{A_1}$$

for $1 \leq n \leq \frac{T}{\Delta t}$.  

(86)

Remark 4.11. From (86), we have $\left\| (\sigma_{\Delta}, u_{\Delta}) \right\|_{L^\infty(\mathbb{R} \times [0,T])} \leq C(T, \delta_0, u_0)$. By (27), (28) we get then we have uniform bound of $\gamma_n, \xi_n$ depending only on $T, \delta_0$ and $u_0$. Subsequently, we obtain $\left| \Psi_{n,i+2} - \Psi_{n,i} \right| \leq \tilde{C} \Delta x$ and $\left| u_{n+1,i} - u_{n+\frac{1}{2},i} \right| \leq C^* \Delta t.$

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For uniform lower bound of approximate density \( \varrho^\Delta \), we have the following lemma.

**Lemma 4.12.** Let \( \bar{\varrho}_0 > 0 \) be satisfying the following

\[
\sup_{x \in \mathbb{R}} s(\varrho_0(x), u_0(x)) - \inf_{x \in \mathbb{R}} r(\varrho_0(x), u_0(x)) + 2C^* T < \frac{\sqrt{\gamma}}{\epsilon_0}
\]

where \( C^* \) is as in Remark 4.11. Then there exists \( \underline{\varrho} > 0 \) such that \( \varrho^\Delta(x,t) \geq \underline{\varrho} \) for a.e. \((x,t) \in \mathbb{R} \times [0,T]\).

**Proof.** Since in the first step of approximation, we solve Riemann data for homogeneous problem (20)–(21), by Lemma 3.1

\[
\inf_{j \in \mathbb{Z}} r(\varrho_{n,j}, u_{n,j}) \leq r(\varrho_{n+\frac{1}{2},j}, u_{n+\frac{1}{2},j})
\]

\[
\leq s(\varrho_{n+\frac{1}{2},j}, u_{n+\frac{1}{2},j}) \leq \sup_{j \in \mathbb{Z}} s(\varrho_{n,j}(x), u_{n,j}(x)),
\]

for all \( j \in \mathbb{Z} \). In the second step \( \varrho \)-variable remains same and \( u \)-variable is changed by \( \delta_{n,j} = u_{n+\frac{1}{2},j} - u_{n,j} \). By Remark 4.11, \( |\delta_{n,j}| \leq C^* \Delta t \). Therefore,

\[
\inf_{x \in \mathbb{R}} r(\varrho_{n,j}(x), u_{n,j}(x)) - C^* \Delta t \leq r(\varrho_{n+1,j}(x), u_{n+1,j}(x))
\]

\[
\leq s(\varrho_{n+1,j}(x), u_{n+1,j}) \leq \sup_{x \in \mathbb{R}} s(\varrho_{n,j}(x), u_{n,j}(x)) + C^* \Delta t.
\]

Hence, we have

\[
\inf_{x \in \mathbb{R}} r(\varrho_0(x), u_0(x)) - C^* n \Delta t \leq r(\varrho_{n,j}(x), u_{n,j}(x))
\]

\[
\leq s(\varrho_{n,j}(x), u_{n,j}) \leq \sup_{x \in \mathbb{R}} s(\varrho_0(x), u_0(x)) + C^* n \Delta t,
\]

equivalently,

\[
s(\varrho_{n,j}(x), u_{n,j}) - r(\varrho_{n,j+2}(x), u_{n,j+2})
\]

\[
\geq \inf_{x \in \mathbb{R}} r(\varrho_0(x), u_0(x)) - \sup_{x \in \mathbb{R}} s(\varrho_0(x), u_0(x)) - 2C^* T
\]

\[
> - \frac{\sqrt{\gamma}}{\epsilon_0}.
\]

Since the bound in (87) does not depend on \( n \) and \( \Delta t \), we get a uniform lower bound \( \underline{\varrho} > 0 \) such that \( \varrho_{n+\frac{1}{2},j+1} \geq \underline{\varrho} \) for all \( n \geq 0, j \in \mathbb{Z} \). \( \square \)

### 4.3 Proof of Theorem 1.1

Note that from (86) and Remark 9 we have a uniform BV bound in \( n \) of the sequence of functions \((\varrho^\Delta(\cdot, \cdot), u^\Delta(\cdot, \cdot))\). We deduce solving the different Riemann problems that for any \( t \in [0,T] \) there exists \( C_{1,T} \) such that for any sequence \( \theta \):

\[
\| (\varrho^\Delta(\cdot, \cdot), u^\Delta(\cdot, \cdot)) \|_{TV(\mathbb{R})} \leq C_{1,T} \text{ for } 0 < t \leq T.
\]

Similarly from (64), (65), (86) and the resolution of the Riemann problem there exists \( C_{2,T} > 0 \) such that for any \( t \in [0,T] \) we have:

\[
\| (\varrho^\Delta(\cdot, \cdot), u^\Delta(\cdot, \cdot)) \|_{L^\infty(\mathbb{R})} \leq C_{2,T}.
\]
This last estimate implies that there exists $C_{3,T} > 0$ such that:
\[
\max(\|\lambda_1(\rho^{\Delta,\theta})\|_{L^\infty([0,T] \times \mathbb{R})}, \|\lambda_2(\rho^{\Delta,\theta})\|_{L^\infty([0,T] \times \mathbb{R})}) \leq C_{3,T}
\]  
(90)

We can now fix the CFL condition on $\lambda$, indeed we wish to solve each Riemann problem at any time $t_n = n\Delta t \leq T$ such that there is no interaction between the different Riemann problem. It suffices to choose $\lambda$ such that:
\[
\lambda < \frac{1}{C_{3,T}}.
\]  
(91)

Now the rest of the proof is similar to the one given in [27], in particular $(\rho^{\Delta,\theta}, u^{\Delta,\theta}, \Psi^{\Delta,\theta})$ converges weakly when $\Delta x$ goes to 0 to a weak solution $(\rho, u, \Psi)$ of the Euler Poisson system provided that the sequence $\theta$ is suitably chosen (indeed the convergence is true for almost every sequence $\theta$ for the uniform probability measured $d\nu$, product of the uniform measures $dm_j = \frac{1}{2}da_j$ on each factor $(-1,1)$).

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