KÄHLER MANIFOLDS WITH NEGATIVE $k$-RICCI CURVATURE

JIANCHUN CHU, MAN-CHUN LEE, AND LUEN-FAI TAM

Abstract. Motivated by the $k$-Ricci curvature introduced by Ni, in this work we consider compact Kähler manifolds with non-positive mixed curvature. As an application, we show that a compact Kähler manifold with non-positive $k$-Ricci curvature where $k$ is an integer in between 1 and the complex dimension $n$ has numerically effective canonical line bundle. If in addition, the $k$-Ricci curvature is quasi-negative, then the manifold is projective with canonical line bundle big and nef. If the $k$-Ricci curvature is negative, then the canonical line bundle is ample. This answers a question of Ni and generalize the result of Wu-Yau. Furthermore, our approach unify the sufficient conditions for the ample canonical line bundle in previous works.

1. Introduction

In this work, we will study compact Kähler manifolds $(M^n, h)$ which are nonpositively (or negatively) curved in the following sense:

\begin{equation}
\alpha |X|^2_h (\text{Ric} + \sqrt{-1} \partial \bar{\partial} \phi)(X, \bar{X}) + \beta R(X, \bar{X}, X, \bar{X}) \leq \lambda |X|^4_h
\end{equation}

for some constants $\alpha, \beta > 0$, $\phi \in C^\infty(M)$ and non-positive function $\lambda(x)$. Here $R$ and Ric denote the curvature tensor and Ricci tensor of $h$ respectively. We prove the following:

Theorem 1.1. Let $(M^n, h)$ be a compact Kähler manifold satisfying (1.1) for some constants $\alpha, \beta > 0$, $\phi \in C^\infty(M)$ and a continuous function $\lambda$.

(a) Suppose $\lambda \leq 0$, then the canonical bundle $K_M$ of $M$ is numerically effective (nef).

(b) Suppose $\lambda < 0$, then $K_M$ is ample. In particular, $M$ is projective and supports a Kähler-Einstein metric with negative Ricci curvature.

(c) Suppose $\lambda$ is quasi-positive, then

$$\int_M c_1(K_M)^n > 0.$$ 

In particular, $K_M$ is big and nef. And $M$ is projective. If in addition $M$ does not contains any rational curve, then $K_M$ is ample.

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Note that there is a compact Kähler manifold with Kähler-Einstein metric with negative Ricci curvature and yet it contains $\mathbb{C}P^1$, see [37]. Hence compact Kähler manifolds satisfying (1.1) with $\lambda$ being quasi-negative may contain rational curves. In this sense, part (c) of the theorem is related to a conjecture of Kobayashi [16] that $K_M$ is ample if $M$ is hyperbolic in the sense that there is no nontrivial holomorphic map from $\mathbb{C}$ into $M$.

If $\alpha = 1, \beta = 0$, (1.1) becomes

$$(\text{Ric} + \sqrt{-1} \partial \overline{\partial} \phi)(X, \overline{X}) \leq \lambda |X|^2_h.$$  

In this case, Theorem 1.1 (a) and the conclusion about bigness in (c) are obviously true. The existence of Kähler-Einstein metric in Theorem 1.1 (b) is the celebrated result of Aubin [1] and Yau [44]. Indeed, the existence of Kähler-Einstein metric when $K_M$ is only big and nef was considered by Tsuji [36] and Tian-Zhang [33]. It was shown that there is a codimension 1 analytic subvariety $S$ of $M$ and a Kähler metric $\omega_{SKE}$ on $M \setminus S$ such that $\text{Ric}(\omega_{SKE}) = -\omega_{SKE}$ holds on $M \setminus S$.

If $\alpha = 0, \beta = 1$, (1.1) means that the holomorphic sectional curvature $H$ is bounded from above by $\lambda$. In this case, Theorem 1.1 (a) was proved by Tosatti-Yang [35] and Wu-Yau [38]. Theorem 1.1 (b) was proved by Wu-Yau [37] under an additional assumption that $M$ is projective. This assumption can be removed by the results in [35, 38]. Theorem 1.1 (c) was proved by Wu-Yau [38]. Since $H(X) \leq 0$ implies that $M$ does not contain any rational curve by the work of Royden [29], therefore, Wu-Yau conclude that $K_M$ is ample if the holomorphic sectional curvature is quasi-negative. This last fact was also proved by Diverio-Trapani [7]. There are also contributions by many other people, see [11, 12, 13, 19, 28, 42] for further details. Hence Theorem 1.1 can be considered as an interpolation of these two cases.

Our study of the condition (1.1) is motivated by the study of $k$-Ricci curvature introduced by Ni [25], which is defined as follows. Let $(M^n, h)$ be a Kähler manifold with Kähler form $\omega$ and curvature tensor $R$. For a point $p \in M$, let $U$ be a $k$-dimensional subspace of $T_p^{1,0}(M)$ and $R|_U$ be the curvature tensor restricted to $U$. Then the $k$-Ricci curvature $\text{Ric}_{k,U}$ on $U$ is defined to be the trace of $R|_U$. That is

$$\text{Ric}_{k,U}(X, \overline{Y}) = \text{tr}_h R(X, \overline{Y}, \cdot, \cdot)$$

for $X, Y \in U$ where the trace is taken with respect to $h|_U$, which is denoted by $h$ again for simplicity. One can see that 1-Ricci curvature coincides with the holomorphic sectional curvature $H$ and $n$-Ricci curvature is the usual Ricci tensor $\text{Ric}$ of $M$. Therefore, one can regard $k$-Ricci curvature as an interpolation between $H$ and $\text{Ric}$ of $M$. Moreover, an example by Hitchin [14] showed that $H$ and $\text{Ric}$ are independent to each other when $n \geq 2$. We say that $k$-Ricci curvature is bounded above by $\lambda$ with $\lambda$ being a constant, denoted by $\text{Ric}_k \leq \lambda$, if $\text{Ric}_{k,U} \leq \lambda |U|$ on any $k$-dimensional subspace $U$ of $T^{1,0}(M)$ at every point. $\text{Ric}_k < \lambda$, $\text{Ric}_k \geq \lambda$ and $\text{Ric}_k > \lambda$ are defined analogously.
There are many results on the structure of compact Kähler manifolds related to Ric_k recently. In [39], Yang proved a conjecture of Yau [45] that a compact Kähler manifold with Ric_1 > 0, i.e. positive holomorphic sectional curvature, must be projective and rationally connected, see also [10]. In [24], Ni proved that this is also true if Ric_k > 0 for some 1 ≤ k ≤ n, which generalizes the result of Yang (for k = 1) and the result of Campana [2] and and Kollár-Miyaoka-Mori [17] (for k = n) on Fano manifolds because of the well-known theorem of Yau [44]: any Fano manifold supports a Kähler metric with positive Ricci curvature. For more recent development, we refer interested readers to [10, 21, 22, 23, 26, 27, 40, 41] and references therein.

Given the great success on the positive side, it is natural to ask what one can say if Ric_k ≤ 0. Results on Ric_n ≤ 0, i.e. the Ricci curvature of (M, h) is negative or more generally Ric + \sqrt{-1} \partial \bar{\partial} u ≤ 0 for some smooth function u, and on k = 1, i.e. holomorphic sectional curvature being nonpositive have been described above. Moreover, [25], Ni also established k-hyperbolicity on Kähler manifolds with Ric_k < 0, in the sense that any holomorphic map from \mathbb{C}^k to a compact Kähler manifold with Ric_k < 0 must be degenerate somewhere. This generalized the work of Royden [29] for k = 1. Ni [25] then asked if a Kähler manifold with negative Ric_k is projective.

When M has non-positive (or negative) k-Ricci curvature, one can show that M will satisfy (1.1) with \alpha = (k - 1), \beta = (n - k), \phi = 0 and \lambda ≤ 0 (or < 0), see Section 2 for more details. This motivates our study on compact Kähler manifolds satisfying (1.1). As a corollary of Theorem 1.1 together with results on Ric_n, Ric_1 we have the following:

**Theorem 1.2.** Suppose (M^n, h) is a compact Kähler manifold with Ric_k(h) ≤ -(k + 1)\sigma for some non-negative function \sigma and integer k with 1 ≤ k ≤ n.

(a) Suppose \sigma ≥ 0, then the canonical bundle K_M of M is numerically effective (nef).

(b) Suppose \sigma > 0, then K_M is ample. In particular, M supports a Kähler-Einstein metric negative Ricci curvature and M is projective.

(c) Suppose \sigma is quasi-positive, then

\[ \int_M c_1(K_M)^n > 0. \]

In particular, K_M is big and M is projective. If in addition M does not contains any rational curve, then K_M is ample.

This gives an affirmative answer to Ni’s question. This also generalizes the results by Wu-Yau [38] and Diverio-Trapani [7] on Kähler manifolds with quasi-negative holomorphic sectional curvature. The curvature condition (1.1) gives an unifying approach to all previous known results. Furthermore, the projectivity and simply connectedness of Kähler manifolds satisfying positive counterpart of (1.1) also holds following the argument in [24]. See Section 6 for detailed exposition.
We will use the twisted Kähler-Ricci flow to study the nefness and ampleness of $K_M$. One can also use a twisted version of continuity path in [38] to treat the non-positive case, see Remark 3.2. We use parabolic method here so that the deformation of metrics is clearer. To prove the bigness of $K_M$, we will consider a sequence of Monge-Ampère solution. Since the method is analytic, Theorem 1.1 (a),(b) can be generalized to complete non-compact Kähler manifolds with bounded curvature by modifying Shi’s Ricci flow solution [31] (see also [4]) and its existence time characterization in case of complete non-compact manifolds with bounded curvature [20]. Namely, one can prove that if $(M, h)$ is a complete noncompact Kähler manifold with bounded curvature satisfying (1.1) with $\lambda \leq 0$, then the Kähler-Ricci flow with initial data $h$ will have longtime solution. If in addition, $\lambda < -c < 0$ for some $c > 0$, then $M$ will support a Kähler-Einstein metric with negative Ricci curvature. We leave the details to interested readers.

The paper is organized as follows: In Section 2 we show that $\text{Ric}_k < 0$ implies an interpolation equation between $H$ and Ric by adapting the Royden’s idea [29] to $\text{Ric}_k$. In Section 3, we will prove the nefness and ampleness of the canonical line bundle using the twisted Kähler-Ricci flow. In Section 4, we will consider the quasi-negative case using compactness argument in [38] together with the twisted Monge-Ampère equation. In Section 5, we apply Theorem 1.1 to prove Theorem 1.2. In Section 6, we will discuss some results concerning Kähler manifolds with positive mixed curvature.

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2. ON SYMMETRIC BIHERMITIAN FORMS

In this section we will prove some results in linear algebra on symmetric bihermitian form on a Hermitian vector space. In particular, we will generalize the result of Royden [29] which gives some upper bound on the trace of bisectional curvature in terms of an upper bound on holomorphic sectional curvature.

Let $(V^n, h)$ be a Hermitian vector space with Hermitian metric $h$. Let $S(X, \overline{Y}, Z, \overline{W})$ be a bihermitian form on $V^n$, namely $S(X, \overline{Y}, \cdot, \cdot)$ and $S(\cdot, \cdot, W, \overline{Z})$ are Hermitian forms for fixed $X, Y$ or $W, Z$. Moreover, we assume that the
bihermitian form $S$ satisfies the following symmetry:

\[
\begin{align*}
S(X, Y, Z, W) &= S(Z, Y, X, W); \\
S(X, Y, Z, W) &= S(Y, X, W, Z).
\end{align*}
\]

For a nonzero vector $X \in V$, we define the holomorphic sectional curvature for $S$ as

\[
H^S(X) = \frac{1}{|X|^4} S(X, \overline{X}, X, \overline{X}),
\]

the Ricci tensor for $S$ as

\[
\text{Ric}^S(X, \overline{Y}) = \text{tr}_h(S(X, \overline{Y}, \cdot, ))
\]

and the scalar curvature of $S$ as

\[
S = \text{tr}_h \text{Ric}^S.
\]

For any $k$-dimensional subspace $U$ of $V$, define the $\text{Ric}^S_{k,U}$ to be the Ricci tensor for $S|_U$. Note that $\text{Ric}^S_{k,U}$ depends also on the subspace $U$. We say that $\text{Ric}^S_k \leq \lambda$ for some $\lambda \in \mathbb{R}$ if for any $X \in V$ and any $k$-dimensional subspace $U$ containing $X$, we have $\text{Ric}^S_{k,U}(X, \overline{X}) \leq \lambda|X|^2_h$. The notion $\text{Ric}^S_k \geq \lambda$ is defined analogously. It is easy to see that $\text{Ric}^S_k$ is the holomorphic sectional curvature if $k = 1$ and is the Ricci tensor for $V$ if $k = n$.

First we will derive some relation under the assumption $\text{Ric}^S_k \leq \lambda$. We first show that $\text{Ric}^S_k \leq \lambda$ implies an inequality relating $\text{Ric}^S_1$ and $\text{Ric}^S_n$.

**Lemma 2.1.** Suppose $(V^n, h)$ is a Hermitian vector space and $S$ is a bihermitian form on $V$ satisfying (2.1). Suppose $\text{Ric}^S_k \leq -(k + 1)\sigma$ for some integer $1 \leq k \leq n$ and $\sigma \in \mathbb{R}$. Then for any $X \in V$, we have

\[
(k - 1)|X|^2_h \cdot \text{Ric}^S(X, \overline{X}) + (n - k)S(X, \overline{X}, X, \overline{X}) \leq -(n - 1)(k + 1)\sigma|X|^4_h.
\]

**Proof.** First, we consider the case that $\sigma = 0$. Let $0 \neq X \in V$, by rescaling we may assume $|X| = 1$. If $k = 1$ or $n$, the result holds trivially. Therefore we will assume $2 \leq k \leq n - 1$. Choose an unitary frame \(\{e_i\}_{i=1}^n\) such that $X = e_1$. For notational convenience, we use $S_{i j k l}$ to denote $S(e_i, \overline{e_j}, e_k, \overline{e_l})$. Let $I$ be a subset of $\{2, \ldots, n\}$ such that $|I| = k - 1$. Then there are $C^{n-1}_{k-1}$ different choice of $I$.

By assumption, for each $I$ we have

\[
S_{1111} + \sum_{j \in I} S_{11jj} \leq 0.
\]
For each \( i \neq 1 \), there are \( C_{n-2}^{m-2} \) different \( I \) containing \( i \). Therefore by summing all possible \( I \), we have

\[
0 \geq \sum_I \left( S_{1111} + \sum_{j \in I} S_{11j_1j_2} \right)
\]

\[
= C_{k-1}^{m-1} S_{1111} + C_{k-2}^{m-2} \sum_{i=2}^{n} S_{11i}
\]

\[
= \left( C_{k-1}^{m-1} - C_{k-2}^{m-2} \right) S_{1111} + C_{k-2}^{m-2} S_{11}
\]

\[
= C_{k-2}^{m-2} \left( \frac{n}{k-1} S_{1111} + S_{11} \right).
\]

The result for \( \sigma = 0 \) follows since \( X = e_1 \).

For general \( \sigma \), let \( \tilde{S} = S + \sigma B \) where

\[
B(X, \tilde{Y}, Z, \tilde{W}) = h(X, \tilde{Y})h(Z, \tilde{W}) + h(X, \tilde{W})h(Z, \tilde{Y}).
\]

Then for any \( k \)-dimensional subspace \( U \), one can check

\[
\text{Ric}_{k,U}^{\tilde{S}}(X, \tilde{Y}) = \text{Ric}_{k,U}^{S}(X, \tilde{Y}) + (k+1)\sigma h(X, \tilde{Y}),
\]

while

\[
H^{\tilde{S}}(X) = H^{S}(X) + 2\sigma.
\]

By the assumption, we have \( \text{Ric}_{k}^{\tilde{S}} \leq 0 \) and hence

\[
0 \geq (k-1)h(X, \tilde{X}) \cdot \text{Ric}_{k}^{\tilde{S}}(X, \tilde{X}) + (n-k)H^{\tilde{S}}(X)
\]

\[
= (k-1)h(X, \tilde{X}) \cdot \text{Ric}_{k}^{S}(X, \tilde{X}) + (n-k)H^{S}(X)
\]

\[
+ ((n+1)(k-1) + 2(n-k))\sigma |h(X, \tilde{X})|^2.
\]

This completes the proof of the lemma. \( \square \)

Now we adapt the trick of Royden \[29\] to deduce a relation on bisectional curvature. Motivated by Lemma 2.1, we consider bihermitian form \( S \) such that there is a real \((1,1)\) form \( \rho \) satisfying the following:

\[
\alpha h(X, \tilde{X}) \cdot \rho(X, \tilde{X}) + \beta S(X, \tilde{X}, X, \tilde{X}) \leq \lambda |X|^4.
\]

for all \( X \in V^{1,0} \) for some constants \( \alpha, \beta, \lambda \) where \( \alpha, \beta \) are positive.

**Lemma 2.2.** Suppose \((V^n, h)\) is a Hermitian vector space and \( S \) is a bi-Hermitian form on \( V \) satisfying \eqref{eq:2.1} and \eqref{eq:2.7}. If \( g \) is another Hermitian metric on \( V \), then we have:

\[
2g^{i\bar{j}}g^{k\bar{l}}S_{ijkl} \leq \frac{1}{\beta} \left( \lambda (tr_g h)^2 - \alpha tr_g h \cdot tr_g \rho \right) + \sum_{i=1}^{n} S(E_i, E_{\bar{i}}, E_i, E_{\bar{i}})
\]

\[
\leq \frac{\lambda}{\beta} \left( (tr_g h)^2 + |h|_g^2 \right) - \frac{\alpha}{\beta} tr_g h \cdot tr_g \rho - \frac{\alpha}{\beta} \langle \omega_h, \rho \rangle_g.
\]
where \( \{ E_i \}_{i=1}^n \) is an unitary frame with respect to \( g \) so that \( h \) is diagonal. Here \( S_{ijkl} = S(E_i, E_j, E_k, E_l) \) and \( \omega_h \) is the Kähler form for \( h \).

Proof. We follow closely the argument of Royden [29]. Let \( \{ E_i \}_{i=1}^n \) be a frame such that \( g(E_i, E_j) = \delta_{ij} \) and \( h(E_i, E_j) = \tau_i \delta_{ij} \). Let \( \eta_A = \sum_{i=1}^n \varepsilon_i^A E_i \) where \( (\varepsilon_i^A) \in \mathbb{Z}_4^n \) with \( \mathbb{Z}_4 \) being the finite group consisting of 4-th root of unity.

Note that for any \( A \in \mathbb{Z}_4^n \), we have

\[
\sum_{A \in \mathbb{Z}_4^n} \varepsilon_i^A \varepsilon_j^A = \begin{cases} 0 & \text{for all } i \neq j, \\ n & \text{for } i = j. \end{cases}
\]

Also since \( \sum_{A \in \mathbb{Z}_4^n} \varepsilon_i^A \varepsilon_j^A \rho(E_i, E_j) = 0 \) for all \( i \neq j \), by symmetry we obtain

\[
\sum_{A \in \mathbb{Z}_4^n} \rho(\eta_A, \bar{\eta}_A) = \sum_{A \in \mathbb{Z}_4^n} \sum_{i,j=1}^n \varepsilon_i^A \varepsilon_j^A \rho(E_i, E_j)
= 4^n \sum_{i=1}^n \rho(E_i, \bar{E}_i)
= 4^n \text{tr}_g \rho.
\]

Similarly,

\[
\sum_{A \in \mathbb{Z}_4^n} S(\eta_A, \bar{\eta}_A, \eta_A, \bar{\eta}_A) = \sum_{A \in \mathbb{Z}_4^n} \sum_{i,j,\gamma,\delta=1}^n \varepsilon_i^A \varepsilon_j^A \varepsilon_{\gamma}^A \varepsilon_{\delta}^A S(E_i, E_j, E_\gamma, E_\delta)
= 4^n \sum_{i \neq j} S(E_i, \bar{E}_j, E_j, \bar{E}_i) + S(E_i, \bar{E}_i, E_j, \bar{E}_j)
+ 4^n \sum_{i=1}^n S(E_i, \bar{E}_i, E_i, \bar{E}_i)
= 4^n g^{ij} g^{kl} (S_{ijkl} + S_{iklj}) - 4^n \sum_{i=1}^n S(E_i, \bar{E}_i, E_i, \bar{E}_i)
= 4^n \left( 2g^{ij} g^{kl} S_{ijkl} - \sum_{i=1}^n S(E_i, \bar{E}_i, E_i, \bar{E}_i) \right).
\]
And,
\begin{align}
\sum_{A \in \mathbb{Z}_4^n} |\eta_A|^4 &= \sum_{A \in \mathbb{Z}_4^n} \sum_{i,j,\gamma,\delta=1}^n \varepsilon_i^A \varepsilon_j^A \varepsilon^A_{\gamma} \varepsilon^A_{\delta} h(E_i, E_j) h(E_{\gamma}, E_{\delta}) \\
&= 4^n \sum_{i \neq j} h(E_i, E_j) h(E_j, E_i) + h(E_i, E_i) h(E_j, E_j) \\
&= 4^n \left( \sum_{i \neq j} \tau_i \tau_j + \sum_i \tau_i^2 \right) \\
&= 4^n (\text{tr}_g h)^2.
\end{align}

(2.11)

Apply (2.7) for each \( \eta_A \) and sum over \( A \in \mathbb{Z}_4^n \), we conclude that
\[ \lambda (\text{tr}_g h)^2 \geq \alpha \text{tr}_g h \cdot \text{tr}_g \rho + \beta \left( 2 g^{ij} g^{kl} S_{ijkl} - \sum_{i=1}^n S(E_i, E_i, E_i, E_i) \right). \]

From this the first inequality in the lemma follows. The second follows from (2.7).

If we choose \( g = h \), then we have the following relation on \( \text{Ric}^S \) and \( S \) under the assumption \( \text{Ric}_k^S \leq -(k+1)\sigma \).

**Lemma 2.3.** Suppose \((V^n, h)\) is a Hermitian vector space and \( S \) is a bi-Hermitian form on \( V \) satisfying (2.1) and \( \text{Ric}_k^S \leq -(k+1)\sigma \) for some \( \sigma \in \mathbb{R} \) and \( n \geq k > 1 \), then we have
\[ (nk + n - k - 2) S h + n \text{Ric}^S \leq -n(n+1)(n-1)(k+1)\sigma h. \]

**Proof.** As in the proof of Lemma 2.1, we may assume \( \sigma = 0 \) by considering \( S + \sigma B \) instead of \( S \). Then \( S \) now satisfies the assumption of Lemma 2.2 with \( \alpha = k - 1 \), \( \rho = \text{Ric}^S \), \( \beta = n - k \) and \( \lambda = 0 \). Hence Lemma 2.2 implies that for \( g = h \), we have
\[ (nk + n - 2k) S \leq (n - k) \sum_{i=1}^n S_{ii\bar{i}} \]

for any unitary frame \( \{E_i\}_{i=1}^n \) of \( h \).

For any \( X \in V \) with \( |X| = 1 \), we choose an unitary frame \( \{E_i\}_{i=1}^n \) such that \( X = E_1 \). For each \( i \neq 1 \), let \( I \) be a subset of \( \{1, \ldots, n\} \) such that \( |I| = k-2 \geq 0 \) and excludes 1 and \( i \). There are \( C_{k-2}^{n-2} \) different choice of \( I \). By assumption on \( \text{Ric}_k^S \), we have
\[ S_{ii\bar{i}} \leq -S_{1i\bar{i}} - \sum_{j \in I} S_{ij\bar{j}} \]

(2.13)
We first assume $k > 2$ so that $I \neq \emptyset$. By summing up over all possibility of $I$, we have for $i \neq 1$ that
\begin{equation}
C_{k-2}^{m-2}S_{i\bar{i}i} \leq -C_{k-2}^{m-2}S_{1\bar{1}i} - \sum_{j \in I} \sum_{I} S_{i\bar{j}j}
= -C_{k-2}^{m-2}S_{1\bar{1}i} - C_{k-3}^{m-3} \sum_{j \neq 1, i} S_{i\bar{j}j}.
\end{equation}

If $k = 2$, then the inequality follows from the definition of $\text{Ric}_2^S \leq 0$.

By summing up $i$, we obtain
\begin{equation}
\sum_{i=1}^{n} S_{i\bar{i}i} \leq 2S_{1\bar{1}1} - \sum_{i=1}^{n} S_{1\bar{1}i} - \frac{k-2}{n-2} \sum_{2 \leq i, j \leq n; i \neq j} S_{i\bar{j}j}
= 2S_{1\bar{1}1} - \sum_{i=1}^{n} S_{1\bar{1}i} - \frac{k-2}{n-2} \left( S - \sum_{i=1}^{n} S_{i\bar{i}i} + 2S_{1\bar{1}1} - 2 \sum_{i=1}^{n} S_{1\bar{1}i} \right)
= -\frac{k-2}{n-2} S + \frac{k-2}{n-2} \sum_{i=1}^{n} S_{i\bar{i}i} + \frac{2(n-k)}{n-2} S_{1\bar{1}1} + \frac{2k-n-2}{n-2} S_{1\bar{1}}.
\end{equation}

By applying Lemma 2.1 again, we deduce that
\begin{equation}
(n-k) \sum_{i=1}^{n} S_{i\bar{i}i} \leq -(k-2)S + 2(n-k)S_{1\bar{1}1} + (2k-n-2)S_{1\bar{1}}
\leq -(k-2)S - nS_{1\bar{1}}.
\end{equation}

Substitute it back to (2.12), we have
\begin{equation}
(nk + n - k - 2)S + nS_{XX} \leq 0
\end{equation}
since $X = E_1$. This completes the proof as $X$ is arbitrary unit vector. 

As an immediate consequence, we have the following rigidity result.

**Corollary 2.1.** Suppose $(M, g)$ is a compact Kähler manifold such that its curvature tensor $\text{Rm}(g)$ satisfies $\text{Ric}_k \leq -(k+1)\sigma$ for some $\sigma \in \mathbb{R}$ and integer $k$ with $1 \leq k < n$. If $S_g = -n(n+1)\sigma$, then $g$ has constant holomorphic sectional curvature.

**Proof.** By Lemma 2.3 we have $\text{Ric}_g \leq -(n+1)\sigma$. Since $S = -n(n+1)\sigma$, we conclude that $\text{Ric}_g \equiv -(n+1)\sigma$ and hence Lemma 2.1 implies $H_g \leq -2\sigma$. As we have
\begin{equation}
S = \frac{n(n+1)}{2} \int_{Z \in T^{1,0}_p M, |Z| = 1} H(Z)d\theta(Z)
\end{equation}
by the Berger’s lemma. We conclude that $H = -2\sigma$. □

Remark 2.1. Using the same argument, it is easy to see that under $\text{Ric}_k^S \geq (k + 1)\sigma$, Lemma 2.1 2.2 2.3 and Corollary 2.1 are still true with $\leq$ replaced by $\geq$.

3. Canonical line bundle under non-positive curvature

In this section, we are going to prove Theorem 1.1 (a), (b). Consider a compact Kähler manifold $(M, h)$ with curvature $R$ satisfying (2.7) for $\rho = \text{Ric}_h + \sqrt{-1} \partial \bar{\partial} \phi$. That is:

$$\alpha h(X, \bar{X}) \cdot (\text{Ric}(h) + \sqrt{-1} \partial \bar{\partial} \phi)(X, \bar{X}) + \beta R^h(X, \bar{X}, X, \bar{X}) \leq \lambda |X|^4$$

for some $\alpha, \beta > 0$ and for some function $\phi \in C^\infty(M)$ and non-positive function $\lambda$. We will first show that $K_M$ is nef by making use of a twisted Kähler-Ricci flow which is the one parameter family of Kähler metrics $g(t)$ with Kähler forms $\omega(t)$ satisfying

$$\left\{ \begin{array}{l}
\partial_t \omega(t) = -\text{Ric}(\omega(t)) - \eta; \\
\omega(0) = \omega_h.
\end{array} \right.$$  

where $\eta$ is a smooth closed real $(1,1)$ form on $M$. Equation (3.2) is equivalent to the following Monge-Ampère type flow:

$$\left\{ \begin{array}{l}
\partial_t \varphi = \log \frac{(\omega_h - t\text{Ric}(\omega_h) - t\eta + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\omega_h^n} \\
\varphi(0) = 0,
\end{array} \right.$$

in the sense that if $\varphi$ satisfies (3.3) on $M \times [0, T]$ so that

$$\omega_h - t\text{Ric}(\omega_h) - t\eta + \sqrt{-1} \partial \bar{\partial} \varphi > 0$$

then $\omega(t) = \omega_h - t\text{Ric}(\omega_h) - t\eta + \sqrt{-1} \partial \bar{\partial} \varphi$ will satisfy (3.2). Moreover if $\omega(t)$ satisfies (3.2), then

$$\varphi(t) = \int_0^t \log \frac{\omega(s)^n}{\omega_h^n} ds,$$

satisfies (3.3).

Since $M$ is closed, the twisted Kähler-Ricci flow $\omega(t)$ admits a short time solution, for example see [9]. We will denote $g(t)$ to be the Kähler metric associated to $\omega(t)$. In this section, we will estimate the existence time of the flow $g(t)$ under the assumption (2.7). We need the following fact, which states that if the solution $g(t)$ to (3.2) is uniformly equivalent to a fixed metric $h$ on $M \times [0, T_0)$, then we have higher order regularity of $g(t)$ and hence the solution can be extended beyond $T_0$, see [5, 8, 18, 30, 34]:

Lemma 3.1. Let $g(t)$ be a smooth solution to (3.3) on $M \times [0, T_0)$. Suppose there is a positive constant $C > 0$ such that

$$C^{-1} h \leq g(t) \leq C h$$
on $M \times [0, T_0)$. Then there is $\varepsilon > 0$ such that $g(t)$ can be extended to $M \times [0, T_0 + \varepsilon)$ which satisfies (3.3).

We have the following useful formulas.

**Lemma 3.2.**

\[
\begin{cases}
  \left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) \phi = -\text{tr}_g (\text{Ric}(h) + \eta); \\
  \left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) (t\dot{\phi} - \phi - nt) = -\text{tr}_g h,
\end{cases}
\]

where $\dot{\phi} = \partial_t \phi$.

**Proof.** By differentiating $\dot{\phi}$ with respect to time, we have

\[
\frac{\partial}{\partial t} \dot{\phi} = \text{tr}_g \left( -\text{Ric}(h) - \eta + \sqrt{-1} \partial \bar{\partial} \phi \right) = \Delta_{g(t)} \dot{\phi} - \text{tr}_g (\text{Ric}(h) + \eta).
\]

This proved the first equation. For the second equation,

\[
\left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) (t\dot{\phi}) = \text{tr}_g (t(-\text{Ric}(h) + \eta)) + \dot{\phi} = \text{tr}_g (g - h) + \left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) \phi.
\]

This proved the second equation. \qed

Along the Kähler-Ricci flow, it is well-known that the scalar curvature is bounded from below. We have the following analogy for the twisted Kähler-Ricci flow.

**Lemma 3.3.** Let $g(t), t \in [0, T)$ be a solution to the twisted Kähler-Ricci flow on $M$. Then the scalar curvature $S(g(t))$ satisfies

\[
S(g(t)) + \text{tr}_g \eta \geq -\frac{n}{t + \sigma}
\]
on $M \times [0, T)$ where $\sigma > 0$ so that $\inf_M (S(g(0)) + \text{tr}_h \eta) \geq -n\sigma^{-1}$. In particular,

\[
\sup_M \log \frac{\det g}{\det h} = \sup_M \dot{\phi}(\cdot, t) \leq n \log \left( \frac{t + \sigma}{\sigma} \right).
\]

**Proof.**

\[
\partial_t (S + \text{tr}_g \eta) = -g^{il} g^{kj} (R_{ij} + \eta_{ij}) \partial_i g_{kl} + g^{ij} \partial_i R_{ij}
\]

\[
= |\text{Ric} + \eta|^2 - g^{ij} \partial_i \partial_j \left( g^{kl} \partial_l g_{ki} \right)
\]

\[
= |\text{Ric} + \eta|^2 + \Delta_{g(t)} (S + \text{tr}_g \eta).
\]

The lower bound of $S + \text{tr}_g \eta$ follows from maximum principle. The upper bound of $\dot{\phi}$ follows from the fact that $\partial_t \dot{\phi} = -S - \text{tr}_g \eta$ and $\dot{\phi}(0) = 0$. \qed
The lemma gives an upper bound of \((\det g)/(\det h)\). Next, we want to estimate the upper bound of \(\operatorname{tr}_g h\). Combining these two bounds, one can obtain \(C^0\) estimates along the twisted Kähler-Ricci flow. This in turns will give an estimate on the existence time.

We need the following parabolic Schwarz Lemma which is a parabolic version of the Schwarz Lemma by Yau \cite{43}.

**Lemma 3.4.** Let \(g(t)\) be a solution to the twisted Kähler-Ricci flow \(\eqref{3.2}\), then

\[
\left(\frac{\partial}{\partial t} - \Delta_{g(t)} \right) \log \operatorname{tr}_g h \leq \frac{1}{\operatorname{tr}_g h} g^{ij} g^{kl} R_{ijkl}(h) + \frac{1}{\operatorname{tr}_g h} g^{ij} h_{ij} \eta_{kl}.
\]

**Proof.** The proof is similar to the parabolic Schwarz Lemma in Kähler-Ricci flow \cite{32}. By Yau’s Schwarz Lemma \cite{43}, for two Kähler metrics \(g\) and \(h\) we have

\[
\Delta_g \log \operatorname{tr}_g h \geq \frac{1}{\operatorname{tr}_g h} \left( R^{ij}(g) h_{ij} - g^{ij} R_{ijkl}(h) \right).
\]

Applying \(\eqref{3.10}\) with \(g = g(t)\), we conclude that

\[
\left(\frac{\partial}{\partial t} - \Delta_{g(t)} \right) \log \operatorname{tr}_g h \leq \frac{1}{\operatorname{tr}_g h} \left( g^{ij} g^{kl} R_{ijkl}(h) - R^{ij}(g) h_{ij} \right) + \frac{1}{\operatorname{tr}_g h} h_{ij} \partial_i g^{ij}
\]

\[
= \frac{1}{\operatorname{tr}_g h} g^{ij} g^{kl} R_{ijkl}(h) + \frac{1}{\operatorname{tr}_g h} g^{ij} h_{ij} \eta_{kl}.
\]

We are now ready to prove Theorem \(\ref{1.1}(a)\).

**Proof of Theorem \(\ref{1.1}(a)\).** For the Kähler metric \(h\), let

\[
S = \inf \{ s \in \mathbb{R} : \exists f \in C^\infty(M), \text{Ric}(h) < s \omega_h + \sqrt{-1} \partial \bar{\partial} f \}.
\]

We claim that \(S \leq 0\). If the claim is true, then for any \(\varepsilon > 0\) we can find smooth function \(f\) so that

\[
-\text{Ric}(h) - \sqrt{-1} \partial \bar{\partial} f \geq -\varepsilon \omega_h.
\]

Since the first Chern class of the canonical line bundle \(K_M\) is represented by \(-\text{Ric}(h)\), we see that the canonical bundle is nef.

Suppose on the contrary that \(S > 0\). For \(\mu > S\) to be chosen later, we can find \(v \in C^\infty(M)\) such that

\[
\text{Ric}(h) + \sqrt{-1} \partial \bar{\partial} \phi = \rho < \mu \omega_h + \sqrt{-1} \partial \bar{\partial} v.
\]

Let \(g(t)\) be the twisted Kähler-Ricci flow with \(\eta = \sqrt{-1} \partial \bar{\partial} u\) and \(g(0) = h\) where \(u = \frac{\mu}{2\rho} v\). We want to show that the maximal existence time \(T_{\text{max}} > S^{-1}\).
If this is true, then \( (3.4) \) implies that

\[
\omega(t) = \omega_h - t \text{Ric}(h) - t \sqrt{-1} \partial \bar{\partial} u + \sqrt{-1} \partial \bar{\partial} \varphi
\]

is a Kähler metric for some \( t > S^{-1} \) where \( \varphi \) is the solution to \( (3.3) \). But this contradicts the definition of \( S \). See also \([33, 36]\) for the existence time characterization of the Kähler-Ricci flow.

On \( M \times [0, T_{\text{max}}) \), we are going to estimate \( \text{tr} g(h) \). Let \( E_i \) be an unitary frame with respect to \( g(t) \) which diagonalizes \( h \) at a point. By Lemma \( 3.4 \) and Lemma \( 2.2 \) with \( g = g(t) \) and \( \rho = \text{Ric}(h) + \sqrt{-1} \partial \bar{\partial} \varphi \), we have

\[
(3.13) \quad \left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) \log \Lambda \leq \frac{1}{\Lambda} g^{ij} g^{kl} R_{ijkl}(h) + \frac{1}{\text{tr}_g h} g^{ij} g^{kl} h_{ij} u_{kl}
\]

\[
\leq \frac{\lambda}{2\beta} \Lambda + \frac{\lambda}{2\beta} |h|^2 + \frac{1}{\Lambda} \langle \omega_h, \sqrt{-1} \partial \bar{\partial} u \rangle - \frac{\alpha}{2\beta} \text{tr}_g \rho - \frac{\alpha}{2\beta} \langle \rho, \omega_h \rangle
\]

\[
\leq \frac{\lambda}{2\beta} \Lambda + \frac{\lambda}{2\beta} |h|^2 - \frac{\alpha}{\beta} \text{tr}_g \rho + \frac{1}{\Lambda} \langle \omega_h, \sqrt{-1} \partial \bar{\partial} u \rangle + \frac{\alpha}{2\beta} \langle \rho, \omega_h \rangle
\]

\[
= \frac{\lambda}{2\beta} \Lambda + \frac{\lambda}{2\beta} |h|^2 - \frac{\alpha}{\beta} \text{tr}_g \rho + \frac{1}{\Lambda} \langle \omega_h, \sqrt{-1} \partial \bar{\partial} u \rangle + \frac{\alpha}{2\beta} \sum_{i=1}^{n} \rho(E_i, \bar{E}_i) (\Lambda - h(E_i, \bar{E}_i))
\]

\[
\leq \frac{\lambda}{2\beta} \Lambda + \frac{\lambda}{2\beta} |h|^2 - \frac{\alpha}{\beta} \text{tr}_g \rho + \frac{1}{\Lambda} \langle \omega_h, \sqrt{-1} \partial \bar{\partial} u \rangle + \frac{\alpha}{2\beta} \sum_{i=1}^{n} (\mu h(E_i, \bar{E}_i) + (\sqrt{-1} \partial \bar{\partial} v)(E_i, \bar{E}_i)) (\Lambda - h(E_i, \bar{E}_i))
\]

\[
= \left( \frac{\lambda + \alpha \mu}{2\beta} \right) \Lambda + \left( \frac{\lambda - \alpha \mu}{2\beta} \right) |h|^2 + \frac{\alpha}{2\beta} \Delta_{g(t)} v - \frac{\alpha}{\beta} \text{tr}_g \rho.
\]

Here we have used the fact that:

\[
\frac{\alpha}{2\beta} \sum_{i=1}^{n} (\sqrt{-1} \partial \bar{\partial} v)(E_i, \bar{E}_i) h(E_i, \bar{E}_i) = \frac{\alpha}{2\beta} \langle \sqrt{-1} \partial \bar{\partial} v, \omega_h \rangle = \langle \sqrt{-1} \partial \bar{\partial} u, \omega_h \rangle.
\]

Since \( \lambda \leq 0 \) and \( n|h|^2 \geq \Lambda^2 \), we have

\[
\left( \frac{\lambda + \alpha \mu}{2\beta} \right) \Lambda + \left( \frac{\lambda - \alpha \mu}{2\beta} \right) |h|^2 \leq \frac{\alpha \mu (n-1)}{2n\beta} \Lambda.
\]
Together with Lemma 3.2, we deduce that
\[
\left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) \log \Lambda \leq B \Lambda + \frac{\alpha}{2\beta} \Delta_{g(t)} v - \frac{\alpha}{\beta} \text{tr}_g \rho
\]
\[
= - B \left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) w + \frac{\alpha}{2\beta} \Delta_{g(t)} v
\]
\[
+ \frac{\alpha}{\beta} \left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) (\dot{\phi} + \phi - u)
\]
\[
= \left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) \left[ -B w - \frac{\alpha}{2\beta} v + \frac{\alpha}{\beta} (\dot{\phi} + \phi - u) \right],
\]
where \( w = t\dot{\phi} - \phi - nt \) and \( B = \frac{\alpha \mu (n-1)}{2n \beta} \). Hence
\[
\log \Lambda \leq C_1 + \left( \frac{\alpha}{\beta} - Bt \right) \dot{\phi} + B \phi + Bnt
\]
where \( C_1 \) depends only on \( \alpha, \beta, \sup_{M} |v|, \sup_{M} |\phi| \) and the upper bound of \( \mu \). Combining with Lemma 3.3, we conclude that there exists a constant \( C_2 > 1 \) such that for all \( t < \min \{ T_{\max}, \frac{2n}{(n-1)\mu} \} \),
\[
C_2^{-1} h \leq g(t) \leq C_2 h.
\]
By Lemma 3.1, we conclude that \( T_{\max} \geq \frac{2n}{(n-1)\mu} \). Hence \( T_{\max} > S^{-1} \) if we choose \( \mu \) sufficiently close to \( S \). This completes the proof of the theorem. \( \square \)

Remark 3.1. In case \( \lambda > 0 \), the method of proof of the theorem will give an upper bound for \( S \) above, which will tend to zero as \( \lambda \to 0 \).

Next we want to prove that \( K_M \) is ample if \( \lambda < 0 \).

Proof of Theorem 1.1 (b). By part (a) of the theorem, the canonical line bundle \( K_M \) is nef. Therefore, for all \( \varepsilon > 0 \), we can find \( v \in C^\infty(M) \) such that
\[
\text{Ric}(h) + \sqrt{-1} \partial \bar{\partial} \phi = \rho \leq \varepsilon \omega_h + \sqrt{-1} \partial \bar{\partial} v.
\]
Choose \( \varepsilon > 0 \) small enough so that
\[
B = \frac{(n+1)\lambda + (n-1)\varepsilon \alpha}{2\beta n} \leq -\sigma.
\]
for some constant \( \sigma > 0 \).

Let \( g(t) \) be the twisted Kähler-Ricci flow with \( \eta = \sqrt{-1} \partial \bar{\partial} u \) where \( u = \frac{\alpha}{2\beta} v \). By Theorem 1.1 (a) and the existence time estimates in [33], the flow exists for all time. Let
\[
F = \log \Lambda + \left( 1 + \frac{\alpha}{\beta} \right) u - \frac{\alpha}{\beta} (\dot{\phi} + \phi)
\]
By Lemma 3.2 and the computation in (3.14), we have
\[
\left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) F \leq -\sigma \Lambda
\]
for some $\sigma > 0$. Now let
\[ G = F + \left(1 + \frac{\alpha n}{\beta}\right) \log t. \]
Then
\[ \left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) G \leq -\sigma \Lambda + \left(1 + \frac{\alpha n}{\beta}\right) t^{-1}. \]
Since $G \to -\infty$ as $t \to 0$, for any $T_0 > 0$ suppose
\[ G(x_0, t_0) = \sup_{M \times [0, T_0]} G. \]
Then $t_0 > 0$. At $(x_0, t_0)$,
\[ t_0 \Lambda(x_0, t_0) \leq C_1 \]
for some $C_1 > 0$ depending only on $\alpha, \beta, \sigma$ and $n$. Hence by AM-GM inequality,
\[ (3.17) \quad \sup_{M \times [0, T_0]} G(x, t) \leq G(x_0, t_0) \]
for some constant $C_3 > 0$ independent of $t$. Hence we have
\[ \log \Lambda + \log t \leq -\frac{\alpha n}{\beta} \log t + \frac{\alpha}{\beta} (\dot{\varphi} + \phi) - \left(1 + \frac{\alpha}{\beta}\right) u \]
\[ \leq -\frac{\alpha n}{\beta} \log t + \frac{\alpha}{\beta} \cdot n \log \left(\frac{t + \tau}{\tau}\right) + C_3 \]
\[ \leq C_4 \]
for some constant $C_4$ independent of $t$ where $\tau$ is a constant such that $\inf_M (S(g(0) + \Delta_{g(0)} u) > -n\tau^{-1}$. Here we have used Lemma 3.3. Therefore,
\[ \frac{g(t)}{t} \geq C_4 h. \]
for some $C_4 > 0$ for $t$ large enough. By the potential expression of $g(t)$ from (3.1),
\[ \omega(t) = \omega_h - t\text{R}ic(h) - t\bar{\partial}\partial u + \sqrt{-1}\bar{\partial}\partial \varphi. \]
This implies that for $t$ large enough,
\[ -\text{R}ic(h) - \sqrt{-1}\bar{\partial}\partial f \geq \varepsilon \omega_h \]
for some $\varepsilon > 0$ and $f(t) \in C^\infty(M)$. By the result of Aubin [1] and Yau [44], $M$ supports a Kähler-Einstein metric with negative scalar curvature. In particular, the canonical line bundle $K_M$ is ample and $M$ is projective. $\square$

**Remark 3.2.** Following the idea using twisting Kähler-Ricci flow, one can also consider its elliptic counterpart to treat the non-positive and negative case. For instances, we can consider the continuity path
\[ (t\omega_h - \text{R}ic(h) + \sqrt{-1}\bar{\partial}\partial(f + u_t))^n = e^{u_t}\omega^n_h \]
from large $t$ and estimate the minimal of $t$ such that $u_t$ exists.

### 4. Quasi-negative case

In this section, we will prove Theorem 1.1 (c). We consider compact Kähler manifolds which satisfies (3.1) for some quasi-negative function $\lambda$. In contrast with the deformation path in the proof of parts (a), (b) and its elliptic counterpart in Remark 3.2, we need to choose a sequence of twisting function in addition. We follow closely the arguments in [38]. Our main contribution is the following:

**Lemma 4.1.** Let $(M^n, h)$ be a compact Kähler manifold satisfying (3.1) for some quasi-negative function $\lambda$. Then
\[ \int_M c_1(K_M)^n > 0. \]

**Proof.** It is equivalent to prove that:
\[ \int_M (-\text{R}ic(h))^n > 0. \]
Since $K_M$ is nef by Theorem 1.1 (a), for all $1 \geq \varepsilon > 0$ there exists $u_\varepsilon \in C^\infty(M)$ such that
\[ -\text{R}ic(h) + \varepsilon \omega_h + \sqrt{-1}\bar{\partial}\partial u_\varepsilon > 0. \]
By Yau’s Theorem [44] Theorem 4, p.383], we can find $v_\varepsilon \in C^\infty(M)$ such that
\[ \begin{aligned} (-\text{R}ic(h) + \varepsilon \omega_h + \sqrt{-1}\bar{\partial}\partial(u_\varepsilon + v_\varepsilon))^n &= \exp\left(v_\varepsilon + u_\varepsilon + \frac{\partial}{\partial\theta}(\phi + u_\varepsilon)\right)\omega^n_h; \\
-\text{R}ic(h) + \varepsilon \omega_h + \sqrt{-1}\bar{\partial}\partial(u_\varepsilon + v_\varepsilon) &> 0. \end{aligned} \]
For notational convenience, we denote
\[ f_\varepsilon =: v_\varepsilon + u_\varepsilon + \alpha \frac{\beta}{2} (\phi + u_\varepsilon); \quad \omega_\varepsilon =: -\text{Ric}(h) + \varepsilon \omega_h + \sqrt{-1} \partial \bar{\partial} (u_\varepsilon + v_\varepsilon) \]
so that
\[ \omega_\varepsilon^n = \exp(f_\varepsilon) \omega_h^n; \quad -\text{Ric}(\omega_\varepsilon) + \text{Ric}(\omega_h) = \sqrt{-1} \partial \bar{\partial} f_\varepsilon. \]
By Stokes’ theorem,
\[ \lim_{\varepsilon \to 0} \int_M \omega_\varepsilon^n = \lim_{\varepsilon \to 0} \int_M (-\text{Ric}(h) + \varepsilon \omega_h)^n = \int_M (-\text{Ric}(h))^n. \]
Therefore, it suffices to estimate the lower bound of \( \int_M \omega_\varepsilon^n \) for some \( \varepsilon_i \to 0 \).

In order to use Wu-Yau’s method \[38\], we will derive a differential inequality involving of \( \Lambda = \text{tr}_g h \) where \( g \) is the Kähler metric associated to the Kähler form \( \omega_\varepsilon \). Direct computations show that:
\[ -\Delta_g \log \Lambda \leq \frac{1}{\Lambda} g^{ij} g^{kl} R(h)_{ijkl} - \frac{1}{\Lambda} \langle \text{Ric}(g), h \rangle. \]
Here and below the inner product is taken with respect to \( g \). By Yau’s Schwarz Lemma \[43\] and the computation as in (3.13) using (4.2) and Lemma 2.2, for \( \rho = \text{Ric}(h) + \sqrt{-1} \partial \bar{\partial} \phi \), we have in a unitary frame \( E_i \) with respect to \( g \) so that \( \rho \) is diagonal at a point,
\[ \frac{1}{\Lambda} g^{ij} g^{kl} R(h)_{ijkl} \leq \frac{\lambda + \alpha \varepsilon}{2\beta} \Lambda + \frac{\lambda}{2\beta \Lambda} |h|^2 - \frac{\alpha}{\beta} \text{tr}_g \rho + \frac{\alpha}{2\beta} \Lambda \sum_{i=1}^n \rho(E_i, \bar{E}_i)(\Lambda - h(E_i, \bar{E}_i)) \]
\[ = \frac{\lambda}{2\beta} \Lambda + \frac{\lambda}{2\beta \Lambda} |h|^2 - \frac{\alpha}{\beta} \text{tr}_g \rho + \frac{\alpha}{2\beta} \Lambda \sum_{i=1}^n \rho(E_i, \bar{E}_i)(\Lambda - h(E_i, \bar{E}_i)) \]
\[ \leq \frac{\lambda}{2\beta} \Lambda + \frac{\lambda}{2\beta \Lambda} |h|^2 - \frac{\alpha}{\beta} \text{tr}_g \rho \]
\[ + \frac{\alpha}{2\beta \Lambda} \sum_{i=1}^n (\varepsilon \omega_h + \sqrt{-1} \partial \bar{\partial} (u_\varepsilon + \phi))(E_i, \bar{E}_i)(\Lambda - h(E_i, \bar{E}_i)) \]
\[ = \frac{\lambda + \alpha \varepsilon}{2\beta} \Lambda + \frac{\lambda - \alpha \varepsilon}{2\beta \Lambda} |h|^2 - \frac{\alpha}{\beta} \text{tr}_g \text{Ric}(h) + \frac{\alpha}{2\beta} \Delta_g (u_\varepsilon - \phi) \]
\[ - \frac{\alpha}{2\beta \Lambda} (\sqrt{-1} \partial \bar{\partial} (u_\varepsilon + \phi), \omega_h). \]
On the other hand,
\[ -\frac{1}{\Lambda} \langle \text{Ric}(g), h \rangle = -\frac{1}{\Lambda} \langle \text{Ric}(h) - \sqrt{-1} \partial \bar{\partial} f_\varepsilon, \omega_h \rangle \]
\[ = \frac{\alpha}{2\beta \Lambda} (\sqrt{-1} \partial \bar{\partial} (u_\varepsilon + \phi), \omega_h) + \frac{1}{\Lambda} \langle \omega_\varepsilon - \varepsilon \omega_h, \omega_h \rangle \]
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because \( \sqrt{-1} \partial \bar{\partial} f_\varepsilon = \frac{\alpha}{2\beta} \sqrt{-1} \partial \bar{\partial} (u_\varepsilon + \phi) + \omega_\varepsilon - \varepsilon \omega_h + \text{Ric}(h) \). By (4.6)–(4.8), we conclude that

\[
-\Delta_g \log \Lambda \leq \left( \frac{\lambda + \varepsilon \alpha}{2\beta} \right) \Lambda + \left( \frac{\lambda - \varepsilon \alpha}{2\beta \Lambda} \right) |h|^2 + \frac{\alpha}{2\beta} \Delta_g (u_\varepsilon - \phi)
\]

\[
-\frac{\alpha}{\beta} \text{tr}_g \text{Ric}(h) + \frac{1}{\Lambda} \langle \omega_h, \omega_\varepsilon - \varepsilon \omega_h \rangle
\]

(4.9)

Since \( \lambda \leq 0 \), by rewriting \( -\text{Ric}(h) = \omega_\varepsilon - \varepsilon \omega_h - \sqrt{-1} \partial \bar{\partial} (u_\varepsilon + v_\varepsilon) \), we can see that the function \( F = -\log \Lambda - \frac{\alpha}{2\beta} (u_\varepsilon - \phi) + \frac{\alpha}{\beta} (u_\varepsilon + v_\varepsilon) \) satisfies

\[
\Delta_g F \leq \left( 1 + \frac{n\alpha}{\beta} \right) + \left( \frac{\lambda}{2\beta} \right) \Lambda
\]

\[
\leq \left( 1 + \frac{n\alpha}{\beta} \right) + \left( \frac{\lambda}{2\beta} \right) \exp \left( -\frac{\max_M f_\varepsilon}{n} \right).
\]

(4.10)

Here we have used AM-GM inequality at the last inequality. From this, one can proceed as in the proof of [38, Theorem 2]. We sketch the arguments for the convenience of the readers. First one can estimate \( \sup_M f_\varepsilon \) as follows. Since \( M \) is compact, for all \( 1 \geq \varepsilon > 0 \), there is \( x_0 \in M \) such that \( f_\varepsilon(x_0) = \max_M f_\varepsilon \).

At \( x_0 \), we have \( \sqrt{-1} \partial \bar{\partial} (u_\varepsilon + v_\varepsilon) \leq -\frac{\alpha}{2\beta} \sqrt{-1} \partial \bar{\partial} (\phi + u_\varepsilon) \)

\[
\leq -\frac{\alpha}{2\beta} \sqrt{-1} \partial \bar{\partial} \phi + \frac{\alpha}{2\beta} (-\text{Ric}(h) + \varepsilon \omega_h)
\]

\[
\leq C_0(\alpha, \beta, \phi, n, h) \omega_h
\]

(4.11)

where we have used (4.12) on the last inequality. By substituting it back to the Monge-Ampère equation (4.3), we conclude that

\[
\sup_M f_\varepsilon \leq C_1(\alpha, \beta, \phi, n, h),
\]

(4.12)

which is independent of \( \varepsilon \). On the other hand, from (4.3) and (4.2), we see that \( C_1 \omega_h + \sqrt{-1} \partial \bar{\partial} f_\varepsilon > 0 \) for some sufficiently large \( C_1 \) independent of \( \varepsilon \). By [38, Lemma 7], we can find \( \varepsilon_i \to 0 \) such that the sequence \( \{ \exp(1 + \sup_M f_\varepsilon - f_\varepsilon_i) \}_{i=1}^\infty \) converges to \( \exp(e^w) \) for some function \( w \) almost everywhere. Since \( (1 + \sup_M f_\varepsilon - f_\varepsilon_i) \geq 1 \), by Lebesgue dominated convergence theorem, and by integrating (4.10)

\[
\exp \left( -\frac{\max_M f_\varepsilon}{n} \right) \leq \left( 1 + \frac{n\alpha}{\beta} \right) \frac{\int_M \omega^n_\varepsilon}{\int_M \omega^n_{\varepsilon_i}}
\]

\[
\to \left( 1 + \frac{n\alpha}{\beta} \right) \frac{\int_M \exp(-e^w) \omega^n_h}{\int_M \exp(-e^w) \omega^n_h}
\]
and so
\[ \sup_M f_{\varepsilon_i} \geq -C_3 \]
for some $C_3 > 0$ independent of $i$. Together with the upper bound (4.12), passing to a subsequence $f_{\varepsilon_i} \to -e^w + c$ for some constant $c$. This implies that
\[ \int_M \omega_{\varepsilon_i}^n \to \int_M \exp(-e^w + c)\omega^n_h > 0. \]
This completes the proof of the lemma.

Proof of Theorem 1.1 (c). By Lemma 4.1 and the fact that $K_M$ is nef, as pointed out by Diverio-Trapani [7], it follows from [6, Theorem 0.5] that $M$ is Moishezon. Since $M$ is Kähler, $M$ is projective by Moishezon’s Theorem. If in addition that $M$ does not contain any rational curve, then $K_M$ is ample by the proof of [37, Lemma 5], see also [7].

5. Kähler manifolds with non-positive $Ric_k$ and $Ric^+$

Now we are in position to apply Theorem 1.1 to prove Theorem 1.2 by using the interpolation Lemma 2.1.

Proof of Theorem 1.2. If $k = 1$ or $n$, the result is well-known. It suffices to consider $1 < k < n$. By Lemma 2.1, the curvature of $g$ satisfies (3.1) with $\alpha = k - 1$, $\beta = n - k$ and $\lambda = -(n - 1)(k + 1)\sigma$. The result follows from Theorem 1.1.

Corollary 5.1. Let $(M^n, h)$ be a compact Kähler manifold. Suppose $Ric_k \leq 0$ for $1 \leq k \leq n$. Then the Kähler-Ricci flow with initial data $h$ has long time solution.

Proof. The Kähler-Ricci flow has long time solution is equivalent to the fact that $K_M$ is nef by [33].

The condition (2.7) is also related to curvature $Ric^+$ introduced in [24] which is defined to be
\[ Ric^+(X, \overline{X}) = Ric(X, \overline{X}) + \frac{R(X, \overline{X}, X, \overline{X})}{|X|^2}. \]
and is equivalent to the left hand side of (1.1) with $\alpha = \beta = 1$ and $\phi = 0$. It was proved in [24, Proposition 6.2] that a compact Kähler manifold with $Ric^+ < 0$ has no nontrivial holomorphic vector field. By Theorem 1.1 we have the following stronger results.

Corollary 5.2. Suppose $(M^n, g)$ is a compact Kähler manifold with $Ric^+ \leq -(n+2)\sigma$ for some nonnegative function $\sigma$, then the canonical line bundle $K_M$ is nef. If $\sigma > 0$ on $M$, then $K_M$ is ample. If in addition $\sigma$ is positive at some point, then $M$ is projective with $K_M$ big and nef.
6. Kähler manifolds with positive mixed curvature

In [24], it was shown that compact Kähler manifolds with $\text{Ric}^+ > 0$ or $\text{Ric}_k > 0$ are simply connected and projective. Following the argument in [24, Theorem 2.7], we show that Kähler manifolds satisfying the positive counterpart of (1.1) are also simply connected and projective.

**Theorem 6.1.** Suppose $(M, g)$ is a compact Kähler manifold with
\begin{equation}
\alpha g(X, \bar{X})\text{Ric}(X, \bar{X}) + \beta R(X, \bar{X}, X, \bar{X}) > 0
\end{equation}
for some $\alpha, \beta > 0$, then $h^{p,0} = 0$ for all $1 \leq p \leq n$. In particular, $M$ is simply connected and projective.

We first show that the Hodge numbers vanish. This will follow from a slight modification of argument in [24, Section 6].

**Proposition 6.1.** Suppose $(M, g)$ is a compact Kähler manifold with
\begin{equation}
\alpha g(X, \bar{X})\text{Ric}(X, \bar{X}) + \beta R(X, \bar{X}, X, \bar{X}) > 0
\end{equation}
for some $\alpha, \beta > 0$, then $h^{p,0} = 0$ for all $1 \leq p \leq n$.

**Proof.** The first part of proof follows similarly as in that of [24, Theorem 2.2]. Assuming the existence of a nonzero holomorphic $(p, 0)$-form $\phi$, we may conclude that at the point $x_0$ where the maximum of the comass $||\phi||_0$ is attained,
\begin{equation}
0 \geq \sum_{i=1}^{p} R_{v_i \bar{v}_i}
\end{equation}
for any $v \in T^{1,0}M$, for some choice of unitary frame $\left\{\frac{\partial}{\partial z_l}\right\}_{l=1}^n$. Denote $\Sigma = \text{span}\left\{\frac{\partial}{\partial z_l} : l = 1, \ldots, p\right\}$. In particular, (6.3) implies $\sum_{i=1}^{p} R_{v_i} \leq 0$ and $\text{Ric}(x_0, \Sigma)(v, \bar{v}) \leq 0$ for all $v \in \Sigma$.

On the other hand, assumption (6.2) implies
\begin{equation}
0 < \int_{Z \in \Sigma, |Z|=1} \alpha \text{Ric}(Z, \bar{Z}) + \beta H(Z) \, d\theta(Z)
\end{equation}
\begin{equation}
= \frac{\alpha}{p} \sum_{i=1}^{p} R_{v_i} + \frac{2\beta}{p(p+1)} S_p(x_0, \Sigma).
\end{equation}
Since the right hand side is non-positive, this is impossible. This completes the proof. \hfill \square

The next ingredient is the compactness of positively curved Kähler manifold. This was done by the second variational argument in the proof of Bonnet-Meyer theorem.

**Proposition 6.2.** Let $(M^n, g)$ be a complete Kähler manifold with
\begin{equation}
\alpha |X|^2_g \text{Ric}(X, \bar{X}) + \beta R(X, \bar{X}, X, \bar{X}) \geq \lambda |X|^4_g,
\end{equation}
for some $\alpha, \beta, \lambda > 0$. Then $(M, g)$ is a compact manifold with
\[ \text{diam}(M, g) \leq \pi \sqrt{\frac{\alpha(2n - 1) + \beta}{\lambda}}. \]

**Proof.** For any $p, q \in M$, let $\gamma : [0, \ell] \to M$ be a minimizing geodesic connecting $p$ and $q$. It suffices to show
\[ \ell \leq \pi \sqrt{\frac{\alpha(2n - 1) + \beta}{\lambda}}. \]

Let $\{e_i\}_{i=1}^{2n}$ be an orthonormal parallel vector fields along $\gamma$ with
\[ e_{2n-1} = J\gamma', \quad e_{2n} = \gamma', \]
where $J$ is the complex structure of $(M, g)$. For $1 \leq i \leq 2n - 1$, define
\[ V_i(t) = \sin\left(\frac{\pi t}{\ell}\right) e_i(t), \quad \phi_i(t, s) = \exp_{\gamma(t)}(sV_i(t)), \quad L_i(s) = \text{length}(\phi_i(\cdot, s)). \]

Since $\phi_i(t, 0) = \gamma(t)$ and $\gamma$ is minimizing, then $L_i$ has a minimum point at 0. Using second variation formula of arc length, it is clear that
\[
0 \leq \left. \frac{d^2}{ds^2} \right|_{s=0} L_i(s)
= \int_0^\ell \left( \left| \nabla V_i \right|^2 g - R(V_i, \gamma', \gamma', V_i) \right) dt
= \int_0^\ell \left( \left( \frac{\pi}{\ell} \right)^2 \cos^2 \left( \frac{\pi t}{\ell} \right) - \sin^2 \left( \frac{\pi t}{\ell} \right) R(e_i, \gamma', \gamma', e_i) \right) dt.
\]

Applying (6.5) to $X = 1/\sqrt{2}(\gamma' - \sqrt{-1}J\gamma')$, we see that
\[ \alpha \text{Ric}(\gamma', \gamma') + \beta R(J\gamma', \gamma', \gamma', J\gamma') \geq \lambda, \]
which implies
\[ \alpha \sum_{i=1}^{2n-1} R(e_i, \gamma', \gamma', e_i) + \beta R(e_{2n-1}, \gamma', \gamma', e_{2n-1}) \geq \lambda. \]
Recalling $\alpha, \beta > 0$, we compute
\begin{equation}
0 \leq \left. \frac{d^2}{ds^2} \right|_{s=0} \left( \alpha \sum_{i=1}^{2n-1} L_i(s) + \beta L_{2n-1}(s) \right) \\
= \int_0^\ell \left( \alpha (2n - 1) + \beta \right) \left( \frac{\pi}{\ell} \right)^2 \cos^2 \left( \frac{\pi t}{\ell} \right) dt \\
- \int_0^\ell \sin^2 \left( \frac{\pi t}{\ell} \right) \left( \alpha \sum_{i=1}^{2n-1} R(e_i, \gamma', \gamma', e_i) + \beta R(e_{2n-1}, \gamma', \gamma', e_{2n-1}) \right) dt \\
\leq \int_0^\ell \left( \alpha (2n - 1) + \beta \right) \left( \frac{\pi}{\ell} \right)^2 \cos^2 \left( \frac{\pi t}{\ell} \right) - \lambda \sin^2 \left( \frac{\pi t}{\ell} \right) dt \\
= \frac{\ell}{2} \left( \alpha (2n - 1) + \beta \right) \left( \frac{\pi}{\ell} \right)^2 - \lambda,
\end{equation}
as required. This completes the proof. \qed

**Proof of Theorem 6.7** By Proposition 6.1, $h^{p,0} = 0$ for $1 \leq p \leq n$. Hence $M$ is projective. By Proposition 6.2, the universal cover $\tilde{M}$ is also compact with zero Hodge numbers $\tilde{h}^{p,0}$ for $1 \leq p \leq n$ and is projective. Then one can conclude that $M$ is simply connected by comparing the Euler characteristic numbers of $M$ and $\tilde{M}$ using [15, Lemma 1]. \qed

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