CONVERGENCE RATES FOR ITERATIVELY REGULARIZED LANDWEBER ITERATION METHOD FOR NONLINEAR INVERSE PROBLEMS VIA HÖLDER STABILITY ESTIMATES

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Abstract. In this paper, we discuss the local convergence of Iteratively regularized Landweber iteration method for solving nonlinear inverse problems in Banach spaces. Basis of our analysis is based on the assumption that the inverse mapping satisfies the Hölder stability estimate locally. Under certain assumptions, we show the convergence of the Iterative Landweber iterates to the exact solution. As a by-product, under two different conditions, two different convergence rates are obtained.

1. INTRODUCTION

Let \( F : D(F) \subset U \to V : F(u) = v \) be a non-linear forward operator between the Banach spaces \( U \) and \( V \). The classical meaning of an inverse problem is the determination of \( u \), provided \( v \) is given to us. In general, due to the lack of continuous dependence on the data, almost all the inverse problems are ill-posed in nature. Thus, the regularization methods are incorporated to find the stable approximate solutions, i.e. these are used to introduce prior knowledge and make the feasible approximations of ill-posed inverse problems. For further details on inverse problems, see [6] for settings in Hilbert spaces, and [2] for Banach space settings. Variational regularization methods for finding a stable approximate solution are well studied in [4, 6]. Nevertheless, Iterative methods are often a proficient alternative to variational methods, peculiarly for large-scale problems. Among all the iterative methods, Landweber iteration method is one of the well known iteration method. For the convergence of Landweber iteration and its modifications, extensive research has been done in [7]. Especially, for the Hilbert spaces, considerable study has been done in [3]. Using the duality mapping, nonlinear extension of the Landweber iteration is given in [8]. Schrezer in [9], gave the modification of Landweber iteration method named as Iteratively regularized Landweber iteration method. The motivation for this method comes from the Iteratively regularized Gauss-Newton method introduced by Bakushinskii in [10] and issues about its convergence were discussed in [29]. Further, there is an important role of duality mappings in iterative methods (see [14, 15, 17, 30]). In our study the data space \( V \) can be any arbitrary Banach space but the model space \( U \) needs to be uniformly convex and smooth. In the theory of Banach spaces, Bregman distance plays an important role and are more pertinent
to employ rather than Ljapunov functionals to prove the convergence of regularization schemes [16] and hence it is more convenient to derive the convergence rates with the help of Bregman distance.

Conceptually, convergence rates can be derived with two different approaches for non-linear problems. First one is on the basis of source and non-linearity conditions, see for instance, [6, 4, 2, 20] for variational regularization, and [18, 3, 2] for iterative regularization. The second approach relies on the stability estimates which has been derived in [19] for variational regularization methods and in [1] for iterative regularization in Banach spaces. The results regarding the rates of convergence using Hölder stability estimates can be found in [22, 23] and logarithmic stability in [24, 33].

In our analysis, we consider the Iteratively regularized Landweber iteration scheme which is taken from [2]. The motivation for this paper comes from [1] in which the convergence rates for Landweber method have been obtained via Hölder stability estimates. The prime motive of the present work is to study the convergence of the iterates of Iteratively regularized Landweber iteration method provided the inverse mapping satisfies the Hölder stability estimate and hence find the convergence rates. By assuming two different assumptions we obtain two different convergence rates through our analysis. The main novelty of our work is to determine the convergence rates without using the classical approach based on source conditions or the variational inequalities as the smoothness conditions.

The plan of this paper is the following: All the basic results and definitions required in our framework are recapitulated in Section 2. In the third section, the main result on the convergence rates is stated and proved in the Theorem 3.1 along with the necessary assumptions. In addition, a convergence rate is also established in the Theorem 3.2 for the special case of Hölder stability estimate. In section 4, we give the example where our results can be applied. In the end, a few conclusions are made.

2. Preliminaries

Definition 2.1. Convexity modulus of $U$: It is a function $\delta : [0, 2] \to [0, 1]$ defined by

$$\delta_U(\epsilon) = \inf \left\{ \frac{1}{2} \left( 2 - \|u_1 + u_2\| \right) : u_1, u_2 \in S, \|u_1 - u_2\| \geq \epsilon \right\},$$

where $S$ is the boundary of the unit sphere in the Banach space $U$. Further, if $\delta_U(\epsilon) > 0$ for any $\epsilon \in (0, 2]$, then $U$ is uniformly convex.

Definition 2.2. Smoothness modulus of $U$: It is a function $\rho : [0, \infty) \to [0, \infty)$ defined by

$$\rho_U(\tau) = \sup \left\{ \frac{1}{2} \left( \|u_1 + \tau u_2\| + \|u_1 - \tau u_2\| - 2 \right) : u_1, u_2 \in S \right\},$$
where $S$ is the boundary of the unit sphere in Banach space $U$. Further, if \( \lim_{\tau \to 0} \frac{\rho_U(\tau)}{\tau} = 0 \), then $U$ is uniformly smooth.

**Definition 2.3.** A Banach space $U$ is

1. $p$ convex if \( \delta_U(\epsilon) \geq Y\epsilon^p \), where $Y$ is some positive real.
2. $q$ smooth if \( \rho_U(\tau) \leq Z\tau^q \), where $Z$ is some positive real.

### 2.1. Iteratively Regularized Landweber Iteration Method

In Banach spaces, we consider the following version of the iteratively regularized Landweber iteration method given in [2].

\[
J_p(u_{k+1} - u_0) = (1 - \beta_k)J_p(u_k - u_0) - \mu F'(u_k)^* j_p(F(u_k) - v) \tag{2.1}
\]

\[
u_{k+1} = u_0 + J_p^*(J_p(u_{k+1} - u_0)), \quad 0 < \beta_k \leq \beta_{\text{max}} < 1 \tag{2.2}
\]

Here $J_p : U \to U^*$, $J_p^* : U^* \to U$, $j_p : V \to V^*$ are duality mappings and $\mu$ is a positive constant. $u_0$ is the initial guess of the solution and $k \geq 0$.

**Remark 2.1.** There is another version of the Iteratively regularized Landweber Iteration method which is:

\[
J_p(u_{k+1}) = J_p(u_k) - \mu F'(u_k)^* j_p(F(u_k) - v) + \beta_k J_p(u_0 - u_k) \]

\[
u_{k+1} = J_p^*(J_p(u_{k+1})), \quad 0 < \beta_k \leq \beta_{\text{max}} < \frac{1}{2}
\]

In Hilbert spaces, above method reduces to the method discussed in [3] with $\mu = 1$. The only difference there is that noise in the data is taken into consideration.

**Remark 2.2.** If $\beta_k = 0$ for each $k$ in (2.1), i.e. in Iteratively regularized Landweber iteration, then the method is nothing but Landweber iteration method discussed in [1] with $x_{k+1} = u_{k+1}$ and $u_0 = 0$.

### 2.2. Duality map.

The set valued (generally) mapping $J_p : U \to 2^{U^*}$ of the convex functional $u \to \frac{1}{p}\|u\|^p$ defined by

\[
J_p(u) = \{u^* \in U^* \mid \langle u, u^* \rangle = \|u\|\|u^*\|, \|u^*\| = \|u\|^{p-1}\}
\]

is known as the duality mapping of $U$ with the gauge function $t \to t^{p-1}$.

Now, we discuss the properties of the duality mappings which we need in our analysis in the next theorem. See [2, chapter 2] for the proofs.

**Theorem 2.1.** For $p > 1$, following holds:

1. For every $u \in U$, the set $J_p(u)$ is non empty.
2. The map $J_p(u)$ is one-one provided the Banach space is uniformly smooth.
(3) If a Banach space is uniformly convex and uniformly smooth, then \( J_p(u) \) is one-one and onto and its inverse is \( J_p^{-1} = J_q^* \), with \( J_q^* \) is the duality mapping of \( U^* \), where \( p,q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) and the associated gauge function is \( t \to t^{p-1} \).

(4) Uniform smoothness (uniform convexity) of a Banach space \( U \) is equivalent to the uniform convexity (Uniform smoothness) of the dual space \( U^* \).

2.3. Bregman distance. Let \( U \) be a uniformly smooth Banach space and \( J_p \) is the duality mapping from \( U \) to \( U^* \) with the gauge function \( t \to t^{p-1} \). Then the functional

\[
\Delta_p(u_1, u_2) = \frac{1}{p} ||u_1||^p - \frac{1}{p} ||u_2||^p - \langle J_p(u_2), u_1 - u_2 \rangle, \quad u_1, u_2 \in U
\]

is the Bregman distance of the convex functional \( u \to \frac{1}{p} ||u||^p \) at \( u \in U \).

Next result recapitulates the main facts of Bregman distance and its relationship with the norm. See [2, chapter 2] for the proofs.

**Theorem 2.2.** Let \( U \) be a uniformly convex and uniformly smooth Banach space. Then, for all \( u_1, u_2 \in U \), following result holds:

1. \( \Delta_p(u_1, u_2) \geq 0 \) and \( \Delta_p(u_1, u_2) = 0 \) if and only if \( u_1 = u_2 \).
2. If \( U \) is \( p \)-convex, then we have

\[
\Delta_p(u_1, u_2) \geq \frac{C_p}{p} ||u_1 - u_2||^p
\]

where \( C_p > 0 \) is some constant.

3. If \( U^* \) is \( q \)-smooth, then we have

\[
\Delta_q(u_1^*, u_2^*) \leq \frac{G_q}{q} ||u_1^* - u_2^*||^q \quad \forall \ u_1^*, u_2^* \in U^*
\]

where \( G_q > 0 \) is some constant.

4. Following are equivalent:

   (a) \( \lim_{n \to \infty} ||u_n - u|| = 0 \),

   (b) \( \lim_{n \to \infty} \Delta_p(u_n, u) = 0 \) and

   (c) \( \lim_{n \to \infty} ||u_n|| = ||u|| \) and \( \lim_{n \to \infty} \langle J_p(u_n), u \rangle = \langle J_p(u), u \rangle \).

Next two lemmas are about the convergent sequence of positive terms. For proofs, see [5, chapter 4].

**Lemma 2.1.** If \( \{\beta_k\} \) is a sequence satisfying (2.2) and \( \sum_k \beta_k < \infty \), then following holds:

1. \( \lim_{k \to \infty} \beta_k = 0 \).
2. \( \lim_{k \to \infty} \beta_k^p \to 0 \) for any \( p > 1 \).
3. \( \lim_{k \to \infty} c \beta_k^p \to 0 \) for any \( p \geq 1 \), where \( c > 0 \) is a positive constant.

**Lemma 2.2.** If \( \{\alpha_n\} \) and \( \{\beta_n\} \) are two real convergent sequences, then their sum is also convergent and

\[
\lim_{n \to \infty} (\alpha_n + \beta_n) = \lim_{n \to \infty} \alpha_n + \lim_{n \to \infty} \beta_n
\]
3. Convergence and convergence rates

In the present section, we analyze the convergence and its rate for Iteratively regularized Landweber iteration method. Here, we consider the notation

\[ B = B_\rho^\Delta(u^\dagger) = \{ u \in U : \Delta_\rho(u^\dagger, u) \leq \rho \} \]

where \( \Delta_\rho(u^\dagger, u) = \Delta_\rho(u^\dagger - u_0, u - u_0) \) and \( \rho > 0 \) is some constant. To prove our main result, we need to have certain assumptions accumulated below. The present note is mainly motivated from [1,2]. So, let the following assumption holds:

Assumption 3.1.

1. \( U \) is \( q \)-smooth and \( p \)-convex with \( \frac{1}{p} + \frac{1}{q} = 1 \) with \( p, q > 1 \).
2. \( F \) has a Fréchet derivative \( F'(\cdot) \) and it satisfies the local estimate, i.e., there exist a positive constant \( L \) such that

\[ \| F'(u_1) - F'(u_2) \| \leq L \| u_1 - u_2 \| \quad \forall u_1, u_2 \in B \] (3.1)

3. \( F'(\cdot) \) satisfies the condition, i.e., \( \| F'(u) \| \leq \hat{L} \) for all \( u \in B \) for some positive constant \( \hat{L} \).
4. \( F \) is weakly sequentially closed.
5. Elements in \( B \) satisfy the Hölder stability estimate, i.e.,

\[ \Delta_\rho(u_1, u_2) \leq C_F \| F(u_1) - F(u_2) \|^{\frac{p+1}{p}} \quad u_1, u_2 \in B, \; \epsilon \in (0,1] \] (3.2)

where \( C_F > 0 \) is a constant.
6. \( u_0 \) lies in \( B \), i.e., \( u_0 \in B_\rho^\Delta(u^\dagger) \).
7. Sequence \( \{ \beta_k \} \) satisfies (2.2), \( \sum_k \beta_k < \infty \) and \( \beta_{\text{max}} \) is sufficiently small.
8. \( \mu \) is chosen such that

\[ \mu^{q-1} < \frac{q}{2 \epsilon L^2 G_q} \] (3.3)

9. \( \rho^2 \) satisfies

\[ \rho^2 = \hat{L}^{-p}(LC_F^2) \frac{C_p}{p} \left( \frac{C_p}{p} \right)^{1+\frac{2}{p}} \] (3.4)

Now, we have enough ingredients to give our main result about the convergence and its rates with which we obtain the convergence and the its rates with some additional assumptions on the sequence \( \{ \beta_k \} \).

Theorem 3.1. Let \( F \) be a non-linear operator between the Banach spaces \( U, V \) and the operator equation \( F(u) = v, v \in V \) has a solution \( u^\dagger \). Let the assumption (3.1) holds. Then all the iterates of iteratively regularized Landweber iteration method (2.1)-(2.2), remain in \( B \) and converge to the solution \( u^\dagger \). Further, we get the following rates under different assumptions.
(1) If \( p < C_p \), then the iterates \( \gamma_k = \Delta_p^{\omega_0}(u^i, u_k) \) satisfies the recursion formula

\[
\gamma_{k+1} \leq -K_2\gamma_k^2 + \alpha_k\gamma_k + K_5\beta_k
\]

for some positive constants \( K_2, K_5 \) (see proof for their actual meaning) and \( \{\alpha_k\} \) is a sequence converges to 1. Further, if \( \{\beta_k\} \) satisfies \( \beta_k \leq C\gamma_k \), then the convergence rate for \( \epsilon \in (0, 1) \) is given by

\[
\Delta_p^{\omega_0}(u^i, u_k) \leq \left( (g_k\rho^2)^{-\frac{k-1}{k+1}} + h_k \right)^{-\frac{1+k}{1-\epsilon}}, \quad k = 1, 2, 3, \ldots
\]

where

\[
g_k = \prod_{i=0}^{k-1} d_i, \text{ for } k \geq 1,
\]

and

\[
h_k = \sum_{j=1}^{k-1} \left( d_jd_{j+1}\ldots d_{k-1} \right)^{-\frac{j}{k-1}} f_{j-1} + f_k, \quad k \geq 2, \text{ and } h_1 = f_0
\]

with \( f_k = \frac{k-1}{k+1} e_k \), \( d_k = \alpha_k + C K_5 \) and \( e_k = \frac{K_5}{\alpha_k} \).

For \( \epsilon = 1 \), we get

\[
\gamma_k \leq \prod_{i=0}^{k-1} (-K_2 + \alpha_i + K_5 C) \rho^2, \quad k = 1, 2, \ldots
\]

(2) we also obtained the rate

\[
\Delta_p^{\omega_0}(u^i, u_k) = \gamma_k = O(\beta_k^{q-1}) \quad \text{as } k \to \infty
\]

provided \( p < C_p \) and

\[
K_5 + \eta\beta_k^{-1} \left[ \alpha_k - \left( \frac{\beta_{k+1}}{\beta_k} \right)^{q-1} \right] \leq 0
\]

for some constants \( \eta, K_5, t = \frac{1}{1+\epsilon} \) and \( \{\alpha_k\} \) is a sequence converging to 1.

**Proof.** Firstly we give the outline of the proof.

(1) We find the upper bound on the difference \( \Delta_p^{\omega_0}(u^i, u_{k+1}) - \Delta_p^{\omega_0}(u^i, u_k) \) for which we further need to find four different estimates.

(2) Then we show that the sequence \( \{\Delta_p^{\omega_0}(u^i, u_k)\} \) is monotonically decreasing and converges to 0.

(3) Thereafter, we find out the convergence rates via two different techniques.

By the definition of Bregmann distance and using (2.1), we can write

\[
\begin{align*}
\Delta_p^{\omega_0}(u^i, u_{k+1}) - \Delta_p^{\omega_0}(u^i, u_k) &= \Delta_p^{\omega_0}(u_k, u_{k+1}) - \mu(j_p(F(u_k) - v), F'(u_k)(u_k - u^i)) \\
&\quad + \beta_k(j_p(u^i - u_0), u^i - u_k) - \beta_k(j_p(u^i - u_0) - J_p(u_k - u_0), u^i - u_k)
\end{align*}
\]

(3.6)
As said above, first we estimate each of the four terms of the right side of above equality individually. For the first term, using lemma (2.63) from [2], (2.4) and (2.1), we get

\[
\Delta^w_p(u_k, u_{k+1}) \leq \frac{G_q}{q} \|J_p(u_{k+1} - u_0) - J_p(u_k - u_0)\|^q
\]

\[
= \frac{G_q}{q} \|\beta_k J_p(u_k - u_0) + \mu F'(u_k)^* j_p(F(u_k) - v)\|^q
\]

(3.7)

Now using the following estimate twice

\[
\|u_1 + u_2\|^p \leq 2^{p-1}(\|u_1\|^p + \|u_2\|^p)
\]

(3.8)

for \(p \geq 1\), see [4, lemma 3.20], in (3.7), we get

\[
\Delta^w_p(u_k, u_{k+1}) \leq 2^{q-1} \frac{G_q}{q} \left( \beta_k^q \|J_p(u_k - u_0)\|^q + \mu q \|F'(u_k)^* j_p(F(u_k) - v)\|^q \right)
\]

\[
= 2^{q-1} \frac{G_q}{q} \left( \beta_k^q \|u_k - u_0\|^p + \mu q \|F'(u_k)^* j_p(F(u_k) - v)\|^q \right)
\]

\[
\leq 2^{q-1} \frac{G_q}{q} \left( 2^{p-1} \beta_k^q (\|u_1 - u_0\|^p + \|u_1 - u_k\|^p) + \mu q \|F'(u_k)^* j_p(F(u_k) - v)\|^q \right)
\]

\[
\leq 2^{q-1} \frac{G_q}{q} \left( 2^{p-1} \beta_k^q (\|u_1 - u_0\|^p + \frac{p}{C_p} \Delta^w_p(u_1, u_k)) + \mu q \|F(u_k) - v\|^p \right)
\]

(3.9)

where the last inequality is obtained by incorporating (2.3) and the condition (3) of assumption (3.1) provided \(u_k\) satisfies estimate (3.2) which we will show later.

Next, we come to the estimation of the second term on the right side of (3.6).

\[
-\mu \langle j_p(F(u_k) - v), F'(u_k)(u_k - u_1) \rangle
\]

\[
= -\mu \langle j_p(F(u_k) - v), F(u_k) - v + \mu \langle j_p(F(u_k) - v), F(u_k) - v - F'(u_k)(u_k - u_1) \rangle
\]

\[
= -\mu \|F(u_k) - v\|^p + \mu \|j_p(F(u_k) - v), F(u_k) - v - F'(u_k)(u_k - u_1)\|
\]

By employing fundamental theorem of calculus for \(F'(:)\) and (3.1), we get,

\[
-\mu \langle j_p(F(u_k) - y), F'(u_k)(u_k - u_1) \rangle
\]

\[
\leq -\mu \|F(u_k) - v\|^p + \frac{\mu L}{2} \|F(u_k) - v\|^{p-1} \|u_k - u_1\|^2
\]

Using (2.3) and then (3.2), we get

\[
-\mu \langle j_p(F(u_k) - v), F'(u_k)(u_k - u_1) \rangle
\]

\[
\leq -\mu \|F(u_k) - v\|^p + \frac{p}{2} \|F(u_k) - v\| \|u_k - u_1\|^2
\]

(3.10)

Next, we give the estimation of the third term of the right side of (3.6).

\[
\beta_k \langle J_p(u^1 - u_0), u^1 - u_k \rangle \leq \beta_k \langle J_p(u^1 - u_0), u^1 - u_k \rangle
\]

\[
\leq \beta_k \|J_p(u^1 - u_0)\| \|u^1 - u_k\|
\]

\[
= \beta_k \|u^1 - u_0\|^{p-1} \|u^1 - u_k\|
\]
Using Young’s Inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ for $a = ||u^\dagger - u_0||^{p-1}$, $b = ||u^\dagger - u_k||$, $r = \frac{p}{p-1}$ and $s = p$ and (2.3), yields

$$\beta_k(J_p(u^\dagger - u_0), u^\dagger - u_k) \leq \beta_k \left( \frac{p-1}{p} ||u^\dagger - u_0||^p + \frac{1}{p} ||u^\dagger - u_k||^p \right)$$

$$\leq \beta_k \left( \frac{p-1}{p} ||u^\dagger - u_0||^p + \frac{1}{c_p} \Delta_p^{u_0}(u^\dagger, u_k) \right) \quad (3.11)$$

For the fourth term of (3.6), we have

$$-\beta_k(J_p(u^\dagger - u_0) - J_p(u_k - u_0), u^\dagger - u_k) = -\beta_k \Delta_p^{u_0}(u^\dagger, u_k) - \beta_k \Delta_p^{u_0}(u^\dagger, u_k)$$

$$\leq -\beta_k \Delta_p^{u_0}(u^\dagger, u_k) \quad (3.12)$$

where the last inequality holds because of (1) of theorem (2.2). Combining all the estimates (3.9) – (3.12) in (3.6) and use the notation $\gamma_k = \Delta_p^{u_0}(u^\dagger, u_k)$, we get

$$\gamma_{k+1} - \gamma_k \leq 2^{q-1} G_q \left( 2^{p-1} G_q \beta_k \|u^\dagger - u_0\|^p + \frac{p}{c_p} \gamma_k \right) + \mu^q \hat{L} \|F(u_k) - v\|^p - \mu \|F(u_k) - v\|^p$$

$$+ \frac{\mu}{2} L C^2_F \left( \frac{p}{C_p} \right)^{2/p} \|F(u_k) - v\|^{p+\epsilon} + \beta_k \left( \frac{p-1}{p} \|u^\dagger - u_0\|^p \right) + \beta_k \left( \frac{1}{c_p} \gamma_k \right) - \beta_k \gamma_k$$

$$= \left( 2^{q-1} G_q \mu^q \hat{L} \|u^\dagger - u_0\|^p \right) + \left( 2^{p+q-2} G_q \beta_k \|u^\dagger - u_0\|^p \right) \|u^\dagger - u_0\|^p$$

$$+ \frac{\mu}{2} L C^2_F \left( \frac{p}{C_p} \right)^{2/p} \|F(u_k) - v\|^{p+\epsilon} + \left( \frac{\beta_k}{C_p} - \beta_k + 2^{p+q-2} G_q \|u^\dagger - u_0\|^p \right) \gamma_k \quad (3.13)$$

Let us assume that $u_k \in B$, then after using (3.3) in the first term of the right side of above equation, we get

$$\gamma_{k+1} - \gamma_k \leq -\frac{\mu}{2} \|F(u_k) - v\|^p + \left( 2^{p+q-2} G_q \beta_k \|u^\dagger - u_0\|^p \right) \|u^\dagger - u_0\|^p$$

$$+ \frac{\mu}{2} L C^2_F \left( \frac{p}{C_p} \right)^{2/p} \|F(u_k) - v\|^{p+\epsilon} + \left( \frac{\beta_k}{C_p} - \beta_k + \frac{2^{p+q-2} G_q}{C_p} \gamma_k \right) \gamma_k \quad (3.14)$$

From mean value inequality, point 3 of assumption 3.1, (2.3) and (4.4), we get

$$\|F(u_k) - v\| = \|F(u_k) - F(u^\dagger)\| \leq \hat{L} ||u_k - u^\dagger|| \leq \hat{L} \left( \frac{p}{C_p} \right)^{1/p} \Delta_p^{u_0}(u^\dagger, u_k)^{1/p}$$

$$\leq \hat{L} \left( \frac{p}{C_p} \right)^{1/p} \|u_0\| \leq \left( \frac{C_p}{p} \right)^{2/p} (L C^2_F)^{\frac{1}{p}}$$

Using above equation, we have the estimate

$$-\frac{\mu}{2} \|F(u_k) - v\|^p + \frac{\mu}{2} L C^2_F \left( \frac{p}{C_p} \right)^{2/p} \|F(u_k) - v\|^{p+\epsilon}$$

$$= \|F(u_k) - v\|^p \left[ -\frac{\mu}{2} + \frac{\mu}{2} L C^2_F \left( \frac{p}{C_p} \right)^{2/p} \|F(u_k) - v\|^\epsilon \right] \leq 0 \quad (3.15)$$
Thus, (3.14) and (3.15) implies
\[ \gamma_{k+1} - \gamma_k \leq \left( 2^{p+q-2} \frac{G_a q}{q} \beta_k^p + \beta_k \frac{p-1}{p} \right) \|u^1 - u_0\|^p + \left( \frac{\beta_k}{C_p} - \beta_k + 2^{p+q-2} \frac{\beta_q G_p}{q C_p} \right) \gamma_k \] (3.16)

Because of the assumption \( u_0 \in B \) and (2.3), (3.16) can also be written as
\[ \gamma_{k+1} - \gamma_k \leq \left[ \left( \frac{p}{C_p} \right) \left( 2^{p+q-2} \frac{G_a q}{q} \beta_k^p + \beta_k \frac{p-1}{p} \right) + \left( \frac{\beta_k}{C_p} - \beta_k + 2^{p+q-2} \frac{\beta_q G_p}{q C_p} \right) \right] \rho^2 \]
\[ = \left[ 2^{p-1} \frac{G_a q}{q} \left( \frac{p}{C_p} \right) - \beta_k \left( 1 - \frac{p}{C_p} \right) \right] \rho^2 \] (3.17)

Since \( p < C_p \) and \( \beta_k \) can be sufficiently small, we choose \( \beta_{\text{max}} \) sufficiently small so that right side of (3.17) becomes non positive. The validity of above statement is shown below. From (3.17), we have
\[ \gamma_{k+1} - \gamma_k \leq 0 \implies \beta_k \leq q^{-\frac{1}{\frac{p}{C_p} - 1}} \left( \frac{G_a q}{q} \right) \rho^2 \]

Therefore, by taking \( \beta_k \)'s which satisfies the above inequality, we have
\[ \gamma_{k+1} - \gamma_k \leq 0 \implies \gamma_{k+1} \leq \gamma_k \leq \rho^2 \]

Thus \( u_{k+1} \in B \). Therefore, it is clear that the sequence \( \{\gamma_k\} \) is a monotonically decreasing sequence bounded below, which means that the limit of the sequence exists. Next, we show that the sequence \( \{\gamma_k\} \) converges to 0. Putting (3.15) in (3.13) yields
\[ \gamma_{k+1} - \gamma_k \leq \left( 2^{q-1} \frac{G_a q}{q} \mu^q \tilde{L} \frac{q}{2} - \frac{p}{2} \right) \|F(u_k) - v\|^p + \left( 2^{p+q-2} \frac{G_a q}{q} \beta_k^p + \beta_k \frac{p-1}{p} \right) \|u^1 - u_0\|^p \]
\[ + \left( \frac{\beta_k}{C_p} - \beta_k + 2^{p+q-2} \frac{\beta_q G_p}{q C_p} \right) \gamma_k \]

We rewrite above equation as
\[ \gamma_{k+1} - \gamma_k \leq -K_1 \|F(u_k) - v\|^p + \left( 2^{p+q-2} \frac{G_a q}{q} \beta_k^p + \beta_k \frac{p-1}{p} \right) \|u^1 - u_0\|^p \]
\[ + \left( \frac{\beta_k}{C_p} - \beta_k + 2^{p+q-2} \frac{\beta_q G_p}{q C_p} \right) \gamma_k \] (3.18)

where \( K_1 = -2^{q-1} \frac{G_a q}{q} \mu^q \tilde{L} \frac{q}{2} + \frac{p}{2} > 0 \) by assumption (3.3). Let \( \lim_{k \to \infty} \gamma_k = a \). Taking limit \( k \to \infty \) and then incorporating lemmas (2.1), (2.2), \( \sum_{k \to \infty} \beta_k < \infty \) and (3.2), we get
\[ a - a \leq -K_1 \lim_{k \to \infty} \|F(u_k) - v\|^p + 0 \leq -K_1 \frac{1}{(C_F)^{\frac{2}{\alpha+1}}} \lim_{k \to \infty} \gamma_k^{\frac{2}{\alpha+1}} = -K_2 \lim_{k \to \infty} \gamma_k^{\frac{2}{\alpha+1}} \]
(3.19)

where \( K_2 = \frac{K_1}{(C_F)^{\frac{2}{\alpha+1}}} \) is a new positive constant. Now, using the continuity of the function \( x \to x^\delta \) for any \( \delta > 1 \), (3.19) implies
\[ 0 \leq -K_2 a^{\frac{2}{\alpha+1}} \implies a^{\frac{2}{\alpha+1}} \leq 0 \] (3.20)
But as \( \gamma_k \geq 0 \), we must have \( a \geq 0 \) and thus (3.20) implies \( a = 0 \). Hence, by (4) of theorem 2.2, \( u_k \rightarrow u^1 \). Next, we find the recursion formula satisfied by the sequence \( \{\gamma_k\} \). Using (3.2) and \( u_0 \in B \) in (3.18) yields

\[
\gamma_{k+1} \leq -K_2 \gamma_k^{\frac{2}{1+\gamma_k}} + \left( 2^{p+q-2} \frac{Gq}{q} \beta_k^{q} + \beta_k \frac{p-1}{p} \right) \frac{p}{C_p} \rho^2 + \left( 1 + \frac{\beta_k}{C_p} - \beta_k + 2^{p+q-2} \frac{Gq}{q} \beta_k \frac{p}{C_p} \right) \gamma_k
\]

\[
= -K_2 \gamma_k^{\frac{2}{1+\gamma_k}} + \alpha_k \gamma_k + K_3 \beta_k + K_4 \beta_k \leq -K_2 \gamma_k^{\frac{2}{1+\gamma_k}} + \alpha_k \gamma_k + K_5 \beta_k \tag{3.21}
\]

where

\[
K_3 = 2^{p+q-2} \frac{Gq}{q} \frac{p}{C_p} \rho^2, \quad K_4 = \frac{p-1}{C_p} \rho^2, \quad \alpha_k = 1 + \frac{\beta_k}{C_p} - \beta_k + 2^{p+q-2} \frac{Gq}{q} \beta_k \frac{p}{C_p}
\]

and the last term in (3.21) is written because \( \beta_k < 1 \) where \( K_5 = K_3 + K_4 \). We can easily see that \( \alpha_k \rightarrow 1 \). So, (3.21) is the required recurrence relation. Now, for the particular case, i.e., if \( \{\beta_k\} \) satisfies the condition \( \beta_k \leq C \gamma_k \) for some \( C > 0 \), then we find the rate of convergence motivated by the theorem 4.5 in [1]. With the given condition \( \beta_k \leq C \gamma_k \), (3.21) can be written as

\[
\gamma_{k+1} \leq -K_2 \gamma_k^{\frac{2}{1+\gamma_k}} + \alpha_k \gamma_k + K_5 \beta_k \leq -K_2 \gamma_k^{\frac{2}{1+\gamma_k}} + d_k \gamma_k
\]

\[
= d_k \gamma_k \left( 1 - e_k \frac{1}{\gamma_k^{1+\gamma_k}} \right) \tag{3.22}
\]

where \( d_k = \alpha_k + CK_5 \) and \( e_k = \frac{K_5}{d_k} \) for every \( k \). Let \( t = \frac{1}{1+\gamma_k} \). Then, above yields

\[
(\gamma_{k+1})^{-t} \geq (d_k \gamma_k)^{-t} \left( 1 - e_k \gamma_k^t \right)^{-t} \tag{3.23}
\]

Using the estimate

\[
(1 - y)^{-t} \geq 1 + ty \quad \forall \ y \in (0, 1) \tag{3.24}
\]

in equation (3.23) for \( k \geq 1 \), we get

\[
(\gamma_{k+1})^{-t} \geq (d_k \gamma_k)^{-t} + f_k
\]

where \( f_k = t e_k d_k^{-t} \). Thus, we get

\[
\Delta_p^{u_0}(u^t, u_k) \leq \left( (g_k \rho^2)^{-\frac{1}{1+\gamma_k}} + h_k \right)^{-\frac{1}{1+\gamma_k}}, \quad k = 1, 2, 3, \ldots
\]

where

\[
g_k = \prod_{i=0}^{k-1} d_i, \text{ for } k \geq 1
\]

and

\[
h_k = \sum_{j=1}^{k-1} \left( d_j d_{j+1} \ldots d_{k-1} \right)^{-\frac{1}{1+\gamma_k}} f_{j-1} + f_{k-1}, \quad k \geq 2, \text{ and } h_1 = f_0
\]
For $\epsilon = 1$, (3.21) with $\beta_k \leq C\gamma_k$ implies

$$
\gamma_k \leq \prod_{i=0}^{k-1}(-K_2 + \alpha_i + K_5C)p^2, \quad k = 1, 2, \ldots
$$

So, the first part of the proof is completed now. Now, we proceed for the second part. Equation (3.21) implies

$$
\gamma_{k+1} \leq -K_2\gamma_k + \alpha_k\gamma_k + K_5\beta_k^q
$$

(3.25)

and the last term in (3.25) is written because $\beta_k < 1$. Now, let us define $\eta_k = \frac{\gamma_k}{\beta_k^{q-1}}$. Then from (3.25), we get

$$
\eta_{k+1} \leq \left(\frac{\beta_k}{\beta_{k+1}}\right)^{q-1}\left[-K_2\eta_k^{1+t}\beta_k^{(q-1)t} + \alpha_k\eta_k + K_5\beta_k\right]
$$

(3.26)

where $t = \frac{1}{1+\epsilon}$. For the uniform boundedness of $\{\eta_k\}$ by some $\eta$, sufficient condition is

$$
\alpha_k\eta + K_5\beta_k \leq \eta\left(\frac{\beta_{k+1}}{\beta_k}\right)^{q-1}
$$

Above can be further written as

$$
K_5 + \eta\beta^{-1}_k\left[\alpha_k - \left(\frac{\beta_{k+1}}{\beta_k}\right)^{q-1}\right] \leq 0
$$

(3.27)

Above equation holds good for sufficiently small $\beta_{\text{max}}$. Thus

$$
\Delta_{\alpha}u_0^q(u^\dagger, u_k) = \gamma_k = O(\beta_k^{q-1}) \quad \text{as } k \to \infty
$$

(3.28)

□

Next result is for the crucial case when $\epsilon = 0$ in (3.2).

**Theorem 3.2.** Let the assumptions (1)-(7) and (9) of assumption 3.1 holds with $\epsilon = 0$ in (5), $\mu$ satisfies

$$
\mu^{q-1} < \frac{q}{2^{q-1}C_qL^q}\left[1 - \frac{1}{2}LC_F^2\left(\frac{p}{C_p}\right)^{2/p}\right]
$$

(3.29)

Further, let

$$
M_1 = C_F^{-2p}\left[2^{q-1}\frac{G_q}{q}\mu^q\bar{L}^q - \mu + \frac{\mu}{2}LC_F^2\left(\frac{p}{C_p}\right)^{2/p}\right]
$$

Then, all the iterates of iteratively regularized Landweber method remain in $B$ and converges to the solution $u^\dagger$. Further, we get the following two rates under two different assumptions.

(1) If $p < C_p$, then the iterates $\gamma_k = \Delta_{\alpha}u_0^q(u^\dagger, u_k)$ satisfies the recursion formula

$$
\gamma_{k+1} \leq -M_1\gamma_k^{2} + \alpha_k\gamma_k + K_5\beta_k
$$

(3.30)

Further, if $\{\beta_k\}$ satisfies $\beta_k \leq C\gamma_k$, then the convergence rate is given by

$$
\Delta_{\alpha}u_0^q(u^\dagger, u_k) \leq \left((g_k\rho^2)^{-1} + h_k\right)^{-1}, \quad k = 1, 2, 3, \ldots
$$
where
\[ g_k = \prod_{i=0}^{k-1} d_i, \quad \text{for } k \geq 1 \]
and
\[ h_k = \sum_{j=1}^{k-1} (d_j d_{j+1} \ldots d_{k-1})^{-1} f_{j-1} + f_{k-1}, \quad k \geq 2, \quad \text{and } h_1 = f_0 \]
where the constants used have similar meaning as defined in theorem 3.1.

(2) A similar rate like (3.28) can be obtained provided (3.27) holds.

Proof. Putting \( \epsilon = 0 \) in (3.13), we get
\[
\gamma_{k+1} - \gamma_k \leq \left( 2^{q-1} \frac{G_q}{q} \mu^q \bar{L}^q - \mu + \frac{\mu}{2} LC^2 \left( \frac{p}{C_p} \right)^{2/p} \right) \| F(u_k) - v \|^p 
+ \left( 2^{p+q-2} \frac{G_q}{q} \beta_k^q + \beta_k \frac{p-1}{p} \right) \| u^1 - u_0 \|^p + \left( \frac{\beta_k}{C_p} - \beta_k + 2^{p+q-2} \beta_k \frac{G_q}{q} \frac{p}{C_p} \right) \gamma_k
\]
After using the estimate (3.29) and proceeding in the similar way as done in the part (1) of theorem (3.1), we reach up to the estimate
\[
\gamma_{k+1} \leq -M_1 \gamma_k^2 + \alpha_k \gamma_k + K_5 \beta_k
\]
Thus again if \( \beta_k \leq C \gamma_k \), we can derive the above-said convergence rate. For the second part, we get an equation alike equation (3.26) with \( K_2 = M_1 \) and \( t = 1 \). Rest part of the proof follows similarly.

4. Example: Electrical Impedance Tomography

Our results can be applied directly on the Calderóns inverse problem which is the mathematical bedrock of EIT, i.e. Electrical Impedance tomography also discussed in [1, Example 5]. Uhlman in [27] has recently studied the EIT and Calderóns problem. Further, see [30, 28, 11, 21] for the results about uniqueness. In [26, 31], result regarding the Lipschitz-type stability estimates were given for the Calderóns inverse conductivity problem provided the a-priori information about the conductivity is known, i.e. it is piecewise constant with a bounded number of unknown values. The difference between the two results is that in [26], real valued case is discussed and in [32], complex valued case is discussed. So, we recall definitions and results discussed there. The problem considered there is the determination of \( v \in H^1(\Omega) \) where \( u \) satisfies
\[
\begin{align*}
\text{div}(\gamma \nabla v) &= 0, \quad \text{in } \Omega \\
u &= g, \quad \text{on } \partial \Omega
\end{align*}
\]
where \( g \in H^{1/2}(\partial \Omega) \), \( \Omega \subset \mathbb{R}^n \) is a bounded domain having smooth boundary and \( \gamma \) is the positive and bounded function representing the electrical conductivity of \( \Omega \). The inverse problem
associated with impedance tomography is the determination of electrical conductivity \( \gamma \) from the information of \( \Lambda_\gamma \), i.e. the Dirichlet to Neumann map which is defined as

\[
\Lambda_\gamma : H^{1/2}(\partial \Omega) \rightarrow H^{-1/2}(\partial \Omega) : g \rightarrow \left( \frac{\partial u}{\partial \nu} \right)_{\partial \Omega}
\]

where the vector \( \nu \) is the outward normal to \( \partial \Omega \).

The operator \( F \) associated with the inverse problem is defined by

\[
F : X \subset L^\infty_\nu(\Omega) \rightarrow L(H^{1/2}(\partial \Omega), H^{-1/2}(\partial \Omega)) : F(\gamma) = \Lambda_\gamma
\]

(4.1)

Further, \( F' \), the Fréchet derivative of \( F \) at \( \gamma = \gamma' \) is given by

\[
F'(\gamma') : U \subset L^\infty(\Omega) \rightarrow L(H^{1/2}(\partial \Omega), H^{-1/2}(\partial \Omega)) : \delta \gamma \rightarrow F'(\gamma')(\delta \gamma)
\]

(4.2)

where \( F'(\gamma')(\delta \gamma) \) is defined by the sesquilinear form

\[
\langle F'(\gamma')(\delta \gamma)g_1, g_2 \rangle = \int_{\Omega} \delta \gamma \nabla v_1 \cdot \nabla v_2 dx, \quad g_1, g_2 \in H^{1/2}(\partial \Omega)
\]

(4.3)

where \( v_1 \) and \( v_2 \) are the solutions of

\[
\begin{cases}
\text{div}(\gamma' \nabla v_1) = 0 = \text{div}(\gamma' \nabla v_2), & \text{in } \Omega \\
v_1 = g_1, & v_2 = g_2 \quad \text{on } \partial \Omega
\end{cases}
\]

Under the assumption that \( \gamma \in L^\infty(\Omega) \), for the case \( n = 2 \), uniqueness of the solution of inverse problem is discussed in [21] and for \( n \geq 3 \) it is discussed in [25] provided \( \gamma \) is in the Sobolev space \( W^{3/2,\infty}(\Omega) \).

Next, we recall the definitions and assumptions proposed in the Section-2 of [26] for our analysis. For \( u \in \mathbb{R}^n \), set \( u = (u', u_n) \), where \( u' \in \mathbb{R}^{n-1}, u_n \in \mathbb{R}, n \geq 2 \). Further, for every \( u \in \mathbb{R}^n \), let \( B_R(u), B'_R(u') \) and \( Q_R(u) \) signifies the open ball having center \( u \) and radius \( R \) in \( \mathbb{R}^n \), ball in \( \mathbb{R}^{n-1} \) centered at \( u' \) of radius \( R \) and the cylinder \( B'_R(u') \times (u_n - R, u_n + R) \) respectively.

**Definition 4.1.** For a bounded domain \( \Omega \subset \mathbb{R}^n \), the boundary \( \partial \Omega \) is of Lipschitz class with constants \( R_0, L > 0 \), if for every \( A \in \partial \Omega \), a rigid transformation of coordinates exists with \( A = 0 \) and

\[
\Omega \cap Q_{R_0}(0) = \{(u', u_n) \in Q_{R_0}(0) : u_n > \psi(u') \}
\]

where the function \( \psi \) is Lipschitz continuous on \( B'_{R_0}(0) \) with \( \psi(0) = 0 \) and

\[
\|\psi\|_{C^{0,1}(B'_{R_0}(0))} \leq LR_0
\]

**Definition 4.2.** For a bounded domain \( \Omega \subset \mathbb{R}^n \), given \( \beta, 0 < \beta < 1 \), the boundary \( \partial \Omega \) is of \( C^{1,\beta} \) class with constants \( R_0, L > 0 \), if for any \( A \in \partial \Omega \), a rigid transformation of coordinates exists with \( A = 0 \) and

\[
\Omega \cap Q_{R_0}(0) = \{(u', u_n) \in Q_{R_0}(0) : u_n > \psi(u') \}
\]

where \( \psi \) is \( C^{1,\beta} \) function on \( B'_{R_0}(0) \) with \( B'_{R_0}(0) \) with \( \psi(0) = |\nabla \psi(0)| = 0 \) and

\[
\|\psi\|_{C^{1,\beta}(B'_{R_0}(0))} \leq LR_0
\]
Next, we recall the assumption assumed in [26] for the main result on Lipschitz stability estimate. Also, we presume that the Lebesgue measure of $\Omega$ is denoted by $|\Omega|$.

**Assumption 4.1.**

1. $\Omega$ is a subset of $\mathbb{R}^n$ satisfying $|\Omega| \leq A|B_{R_0}(0)|$ and $\partial \Omega$ is of Lipschitz class with constants $R_0$ and $L$.
2. The conductivity $\gamma$ is of the form
   \[
   \gamma(u) = \sum_{j=1}^{N} \gamma_j \chi_{D_j}(u)
   \]
i.e., it is a function which is piecewise constant and further satisfies the ellipticity condition
   \[
   S^{-1} \leq \gamma \leq S
   \]
   where $S$ is some constant, $D_j$ are open subsets of $\mathbb{R}^n$ which are already known and $\gamma_j$, $\forall$ $j$ from 1 to $N$ are unknown real numbers.
3. This assumption is about the open sets $D_j$.
   (a) Let $D_j$, $\forall$ $j$ from 1 to $N$ are pairwise disjoint and connected open sets such that
   \[
   \bigcup_{j=1}^{N} \overline{D_j} = \overline{\Omega}.
   \]
   This condition implies that diameter of $\Omega$ has a lower bound.
   (b) $\partial D_j$, $\forall$ $j$ from 1 to $N$ are of Lipschitz class with constants $R_0$ and $L$. This condition implies that for every $j$ from 1 to $N$, diameter of $D_j$ is bounded below.
   (c) Atleast one region, say $D_1$, such that $\partial D_1 \cap \partial \Omega \supseteq \Sigma_1$, where $\Sigma_1$ is of $C^{1,\beta}$ class with constants $R_0$ and $L$ and is also open.
   (d) For each $i$ from 2 to $N$, there exist $j_1, j_2, \ldots, j_M \in \{1, 2, \ldots, N\}$ such that $D_{j_1} = D_1$, $D_{j_M} = D_i$ and, for every $k$ from 1 to $M$, assume $\partial D_{j_k} \cap \partial D_{j_k-1}$ contains $\Sigma_k$, i.e. a portion of $C^{1,\alpha}$ class with constants $R_0$ and $L$ and is also open.

Based on above assumptions and definitions, the following Lipschitz estimate is established in [26].

**Theorem 4.1.** Let $\gamma_1, \gamma_2$ be two real functions which are piecewise constant satisfying (2) of assumption 4.1, $\Omega$ satisfy (1) of assumption (4.1) and for all $j$ from 1 to $N$, $D_j$ satisfy (3) of assumption 4.1. Then we have
\[
\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{L(H^{1/2}(\partial \Omega), H^{-1/2}(\partial \Omega))}
\]
(4.4)
where $C = C(n, R_0, L, \Lambda, S, N)$ is a constant depending upon a-priori data.

Now, we verify that our assumptions of the theorem 3.1 are satisfied. Before that, observe that the Banach space $L^\infty(\Omega)$ is not uniformly convex, so defining the pre-image space as
\[
U = \text{span}\{\chi_{D_1}, \chi_{D_2}, \ldots, \chi_{D_N}\}
\]
(4.5)
fitted with $L^p$ norm where $p > 1$. 

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(1) With the help of basis \(\{\chi_{D_1}, \chi_{D_2}, \ldots, \chi_{D_N}\}\), one can show the Lipschitz continuity of \(F\) and \(DF\). In particular, if \(\gamma\) and \(\gamma'\) satisfy (2) of assumption 4.1 and \(\Omega\) satisfy (1) of assumption 4.1, we have

\[
\|F(\gamma) - F(\gamma')\|_{L(H^{1/2}(\Omega),H^{-1/2}(\Omega))} \leq L \|\gamma - \gamma'\|_{L^p(\Omega)},
\]

\[
\|F'(\gamma) - F'(\gamma')\|_{L(X,L(H^{1/2}(\Omega),H^{-1/2}(\Omega)))} \leq \hat{L},
\]

\[
\|F'(\gamma) - F'(\gamma')\|_{L(H^{1/2}(\Omega),H^{-1/2}(\Omega))} \leq C \|\gamma - \gamma'\|_{L^p(\Omega)},
\]

where \(L, \hat{L}, C\) depend on \(\Omega, N\) and ellipticity constant respectively.

(2) As the notion of weak and strong topology is equivalent for the finite dimensional spaces, \(F\) defined in (4.1) is weakly sequentially closed.

(3) \(p < C_p\) (Still to check??)

Further, assume that \(\gamma = F(\gamma^\dag)\) where \(\gamma^\dag \in X\), \(\Omega\) satisfies (1) of assumption 4.1 and \(U\) is defined by (4.5). Then assumptions 2, 3, 4 of assumption (3.1) are satisfied and (3.2) holds with \(u_0 = 0\) see [1, subsection 5.3]. Hence, iteratively regularized landweber iteration method converges in accordance with the theorem 3.1.

5. Conclusion

We discussed the Iteratively regularized Landweber iteration scheme in Banach spaces and we obtained the rates for convergence under different assumptions. Our results cannot be applied to the Hilbert spaces as we needed strong condition \(p < C_p\) in our analysis. Because in case of Hilbert spaces \(\Delta_2(u_1,u_2) = \frac{1}{2}\|u_1 - u_2\|^2\). So, in equation (2.3), \(C_2 \leq 1 < p\). Here, we are able to give sublinear convergence rates under some additional assumptions. This paper is the first advancement to find the explicit reconstructions by employing the Hölder stability estimate in inverse problems after the reconstructions in [1] and hence based on the obtained convergence rates we can say that Iteratively regularized Landweber iteration scheme is a worthy scheme for the regularization of ill-posed inverse problems in Banach spaces subject to the concerned assumptions.

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