FOLLYTONS
AND THE REMOVAL OF EIGENVALUES
FOR FOURTH ORDER DIFFERENTIAL OPERATORS

J. HOPPE, A. LAPTEV AND J. ÖSTENSSON

Abstract. A non-linear functional $Q[u, v]$ is given that governs the loss, respectively gain, of (doubly degenerate) eigenvalues of fourth order differential operators $L = \partial^4 + \partial u \partial + v$ on the line. Apart from factorizing $L$ as $A^*A + E_0$, providing several explicit examples, and deriving various relations between $u$, $v$ and eigenfunctions of $L$, we find $u$ and $v$ such that $L$ is isospectral to the free operator $L_0 = \partial^4$ up to one (multiplicity 2) eigenvalue $E_0 < 0$. Not unexpectedly, this choice of $u$, $v$ leads to exact solutions of the corresponding time-dependent PDE’s.

1. Factorization of the operator $L = \partial^4 + \partial u \partial + v$.

Let us assume that $u$ and $v$ are real-valued functions and $u, v \in S(\mathbb{R})$, where $S(\mathbb{R})$ denotes the Schwarz class of rapidly decaying functions. Let $L$ be a linear fourth order selfadjoint operator in $L^2(\mathbb{R})$

\begin{equation}
L := \partial^4 + \partial u \partial + v
\end{equation}

defined on functions from the Sobolev class $H^4(\mathbb{R})$. This operator is bounded from below and we assume that its lowest eigenvalue $E_0 < 0$ is of double multiplicity and therefore there exist two orthogonal in $L^2(\mathbb{R})$ eigenfunctions $\psi_+$ and $\psi_-$ satisfying the equation

\begin{equation}
L \psi = E_0 \psi.
\end{equation}

As shown in the appendix, the Wronskian

\begin{equation}
W(x) := \psi_+(x) \psi_-'(x) - \psi_-(x) \psi_+'(x)
\end{equation}

is necessarily non-vanishing, $W(x) \neq 0$, $x \in \mathbb{R}$. Let us try to factorize $L - E_0$ as

\begin{equation}
A^*A = \left(-\partial^2 - f \partial + g - f'\right) \left(-\partial^2 + f \partial + g\right),
\end{equation}

1991 Mathematics Subject Classification. Primary 34L40; Secondary 34L30.
with \( f \) and \( g \) real-valued. Clearly, 
\[
\begin{aligned}
\begin{cases}
  f' + f^2 + 2g &= -u \\
  g^2 - (fg + g')' &= v - E_0.
\end{cases}
\end{aligned}
\tag{1.5}
\]

Instead of discussing these non-linear differential equations directly, let us express \( f, g, u \) and \( v \) in terms of the functions \( \psi_+, \psi_- \). Straightforwardly, one finds that since \( \psi_+ \) and \( \psi_- \) are eigenfunctions of \( A^*A \) with eigenvalue 0, we have \( A\psi_+ = A\psi_- = 0 \), which implies 
\[
\begin{aligned}
\begin{cases}
  fW &= W' \\
  -gW &= \psi_+'' \psi_- - \psi_+'' \psi_- =: W_{12},
\end{cases}
\end{aligned}
\tag{1.6}
\]
while \((L - E_0) \psi_+ = (L - E_0) \psi_- = 0 \) implies 
\[
\begin{aligned}
\begin{cases}
  uW &= 2W_{12} - W'' + \epsilon \\
  (v - E_0)W &= uW_{12} + W_{12}'' - W_{23},
\end{cases}
\end{aligned}
\tag{1.7}
\]
where \( \epsilon \) is an integration constant and 
\[
W_{23} := \psi_+'' \psi_-'' - \psi_+'' \psi_- 
\tag{1.8}
\]
is expressible in terms of \( W \) and \( W_{12} \) via 
\[
WW_{23} = W_{12}' W' - W_{12} W'' + W_{12}''.
\tag{1.9}
\]
Equations (1.6) say that 
\[
f = \frac{W'}{W}, \quad g = -\frac{W_{12}}{W}.
\tag{1.10}
\]
Since \( uW + W'' - 2W_{12} \) vanishes at infinity, \( \epsilon \) has to be 0, and one finds, using equations (1.7)-(1.9), that 
\[
u = \frac{2W_{12} - W''}{W},
\tag{1.11}
\]
\[
v - E_0 = \frac{W_{12}^2}{W^2} + \left(\frac{W_{12}'}{W}\right)'.
\tag{1.12}
\]
Note that 
\[
\tilde{L} := AA^* + E_0 = L + 4 \partial f' \partial + 2f g' - f f'' + f'''
\tag{1.13}
\]
will be isospectral to \( L \), apart from \( E_0 \), which has been removed. To see why \( E_0 \) is not an eigenvalue of \( \tilde{L} \), let us for simplicity assume that \( u, v \in C^\infty_0(\mathbb{R}) \), say that supp \( u \), supp \( v \subset (-c, c) \). Then, 
\[
\psi_+(x) = \alpha_1 e^{-\kappa x} \cos(\kappa x) + \beta_1 e^{-\kappa x} \sin(\kappa x), \quad x > c,
\]
\[
\psi_-(x) = \alpha_2 e^{-\kappa x} \cos(\kappa x) + \beta_2 e^{-\kappa x} \sin(\kappa x), \quad x < c,
\]
where $E_0 = -4\kappa^4$, $k > 0$. This implies
\[
W(x) = \kappa e^{-2\kappa x} (\alpha_1 \beta_2 - \beta_1 \alpha_2), \quad x > c.
\]
\[
W_{12}(x) = 2\kappa^3 e^{-2\kappa x} (\alpha_1 \beta_2 - \beta_1 \alpha_2), \quad x > c.
\]
(Note that the bracket does not vanish, since $\psi_+$ and $\psi_-$ are linearly independent.) This (and a similar investigation at the other end) implies that
\[
f(x) = \mp 2\kappa, \quad g(x) = -2\kappa^2, \quad \text{for } \pm x > c.
\]
Since $\tilde{L}\psi = E_0 \psi$ implies $A^* \psi = 0$, we obtain
\[
\psi'' - 2\kappa \psi' + 2\kappa^2 \psi = 0, \quad x > c.
\]
It clearly follows that $\psi$ cannot be in $L^2(\mathbb{R})$ unless it vanishes identically.

Before giving some explicit examples, let us make some comments concerning the problem of actually finding $f$ and $g$, or $\psi_+$ and $\psi_-$, when $u$ and $v$ are given. Instead of solving the non-linear system (1.5), or the spectral problem (1.2), one may also try to solve the Hirota-type equation which follows from (1.11), (1.12)
\[
(1.14) \quad 4(v - E_0) = \left(\frac{W''}{W} + u\right)^2 + 2 \left(\frac{W'''}{W} + u' + u \frac{W'}{W}\right),
\]
and which for $u \equiv 0$ reads
\[
4(v - E_0) W^2 = 2(W''' W - W'' W') + W''^2.
\]
Once $W (\neq 0)$ is obtained, $f$ and $g$ can be given by the equations (1.10). With $f$ and $g$ defined in this way [2], equation (1.5) is satisfied and the factorization (1.4) is valid.

Note also the following: the functions $\psi_+$ and $\psi_-$ are solutions of $A \psi = 0$, i.e.
\[
-\psi'' + f \psi' + g \psi = 0.
\]
By writing
\[
\psi_\pm = \sqrt{W} \phi_\pm,
\]
one finds that $\phi_+ \phi'_- - \phi'_+ \phi_- = 1$ and that $\phi_\pm$ are (oscillating) solutions of the equation in Liouville form
\[
-\phi'' + \left(g + \frac{3}{4} \left(\frac{W'}{W}\right)^2 - \frac{1}{2} \frac{W''}{W}\right) \phi = 0,
\]
i.e. associated to a second order self-adjoint differential operator.
2. Addition and removal of eigenvalues.

Although adding and removing eigenvalues may be thought to be a procedure that can be read both ways (symmetrically), the steps involved are actually quite different in both cases (in particular, it is not yet clear, which conditions on $u$ and $v$ allow for the addition of a doubly degenerate eigenvalue below the spectrum of $\partial^4 + \partial u \partial + v$). Let us therefore 'summarize' them separately, in both cases starting from a given operator

$$L_n := \partial^4 + \partial u_n \partial + v_n, \quad n \in \mathbb{N},$$

and the equation (1.14) with $u, v$ replaced by $u_n, v_n$. This equation shall be referred to as (1.14)$_n$.

**Removal of eigenvalues:**

1. Solve (1.14)$_n$ (with $E_0 \to E_0^{(n)} = -4\kappa^4_n$) for $W_n := W$ (→ 0 at infinity) and define $W^{(n)}_{12}$ as $\frac{1}{2} (W_n u_n + W''_n)$, as is natural in accordance with equation (1.11)$_n$. Alternatively, if $\psi_{\pm}^{(n)}$ are known, calculate $W_n$ and $W_{12}^{(n)}$ via their definitions, i.e. as

$$W_n = \psi_+^{(n)} \psi_-^{(n)} - \psi_-^{(n)} \psi_+^{(n)}$$
$$W_{12}^{(n)} = \psi_+^{(n)} \psi_-^{(n)} - \psi_-^{(n)} \psi_+^{(n)}.$$  

2. Define $f_n$ and $g_n$ according to (1.10)$_n$, thus solving the system (1.5), and obtaining the factorization

$$L_n = A^*_n A_n - 4\kappa^4_n.$$

3. The operator

$$\tilde{L}_n = A_n A^*_n - 4\kappa^4_n =: L_{n-1}$$

will then be isospectral to $L_n$ apart from the lowest eigenvalue $E_0^{(n)} = -4\kappa^4_n$ (of multiplicity 2), which has been removed.

**Addition of eigenvalues:**

1. Solve (1.14)$_n$ (with $E_0 \to E_0^{(n+1)} = -4\kappa^4_{n+1}$) for $\hat{W}_{n+1} := W \sim e^{\pm 2\kappa_{n+1} x}$, as $x \to \pm \infty$, i.e. $\hat{W}_{n+1}$ diverging at infinity and non-vanishing for finite $x$. (As mentioned above, conditions on $u_n, v_n$ ensuring the existence of $\hat{W}_{n+1}$ are still unclear.)

2. Define $W_{n+1} := \frac{1}{W_{n+1}}$, which will then solve the (more complicated looking) equation

$$\begin{align*}
40 \frac{W'^4}{W^4} - 2 \frac{W'''}{W} + 14 \frac{W'' W'}{W^2} + 13 \frac{W''^2}{W^2} - 64 \frac{W'''^2}{W^3} + \\
2u'' + u^2 - 2u' \frac{W'}{W} + 2u \left( \frac{W'^2}{W^2} - 2 \left( \frac{W'}{W} \right)' \right) = 16\kappa^4 + 4v
\end{align*}$$

(2.1)
(with \( u, v \to u_n, v_n \) and \( \kappa \to \kappa_n+1 \)). In fact, (2.1) is equivalent to
\[
-2f''' + 6ff'' + 7f'^2 - 8f'f^2 + f^4 + 2u \left( f^2 - 2f' \right) - 2uf + u^2 + 2u'' = 4v + 16\kappa^4
\]
(via \( f = \frac{W'}{W_{n+1}} = f_{n+1}, \ u, v \to u_n, v_n \) and \( \kappa \to \kappa_{n+1} \)) that arises in the factorization of \( L_{n+1} \).

3. Write
\[
L_n = A_{n+1}A_n^* - 4\kappa_{n+1}^4
\]
(implying \( 2g_{n+1} := 3f_{n+1}' - f_{n+1}^2 - u_n \)).

4. Then,
\[
L_{n+1} := A_{n+1}A_n^* - 4\kappa_{n+1}^4,
\]
will be isospectral to \( L_n \) apart from having one additional (doubly degenerate) eigenvalue \( E_0^{(n+1)} \) below the spectrum of \( L_n \).

### 3. A NON-LINEAR FUNCTIONAL \( Q \) AND A SYSTEM OF PDE’s ASSOCIATED WITH THE OPERATOR \( L \).

As observed 100 years ago [4], the operator \( L = \partial^4 + \partial u \partial + v \) has a unique 4’th root in the form \( L^{1/4} := \partial + \sum_{k=1}^{\infty} l_k(x) \partial^{-k} \). Define \( M \) to be the positive (differential operator) part of any integer power of \( L^{1/4} \). Then it is well known, that
\[
L_t = [L, M],
\]
where \( L_t \) is the operator defined by \( L_t \varphi = \partial u_t \partial + v_t \varphi \), consistently defines evolution equations (for \( u = u(x,t), v = v(x,t) \)) that have infinitely many conserved quantities (i.e. functionals of \( u \) and \( v \), and their spatial derivatives, that do not depend on \( t \)). We shall make use of this by letting
\[
M := 8 \left( L^{3/4} \right)_+ = 8 \partial^3 + 6u \partial + 3u',
\]
and focusing on the quantity
\[
Q[u, v] := \int_\mathbb{R} \left( 48v^2 + \frac{5}{4}u^4 - 12u^2v - 40uv'' - 13u'v' + 9u'v' \right) \, dx.
\]
This quantity does not change when \( u \) and \( v \) evolve according to
\[
\begin{align*}
  u_t &= 10u''' + 6u'u' - 24v' \\
  v_t &= 3(u'''' + u'')u' - 8v''' - 6u'v'.
\end{align*}
\]
Formula (1.13) for \( \tilde{L} = \partial^4 + \partial \tilde{u} \partial + \tilde{v} \) implies that
\[
\begin{align*}
  \tilde{u} - u &= 4f' \\
  \tilde{v} - v &= 2f'g' - f'' + f'''.
\end{align*}
\]
By using the asymptotic properties of $f$ and $g$ ($f \to \mp 2\kappa$, $g \to -2\kappa^2$, as $x \to \pm \infty$), one can show that

$$\delta Q := Q[\tilde{u}, \tilde{v}] - Q[u, v] = -32\kappa^7 \frac{2^9}{7}. \tag{3.4}$$

(making $\frac{7}{2^9} \sqrt{2} \delta Q = -2 \left(4\kappa^4\right)^{7/4}$). This result is similar to that for Schrödinger operators \cite{1} and reflects the loss of a doubly degenerate eigenvalue $E_0 = -4\kappa^4$, when going from $L$ to $\tilde{L}$. The constant in the right hand side of \eqref{3.4} is related to the semiclassical constant appearing in the trace formula for a fourth order differential operator considered in \cite{3}.

The proof of \eqref{3.4}, just as the derivation of \eqref{3.1}, involves very lengthy calculations. When deriving \eqref{3.4} one uses \eqref{1.5} and \eqref{3.3} to write the expression for $\delta Q$ as an integral of terms involving only the functions $f$ and $g$, and their spatial derivatives. The crucial step is to note that the integrand is a pure derivative of $x$, i.e. $\delta Q = \int Q'/dx$ for some function $Q$, which makes it possible to evaluate the integral solely from the limits of $f$ and $g$ at infinity. Thus, to compute $\delta Q$, one selects the terms in $Q$ which are free of derivatives, as those are the only ones that contribute. The terms in $Q$ still containing derivatives, for instance the ones quadratic in $g$ and linear in $f$,

$$\int \left( (96 - 48) g^2 f''' - 2 \cdot 96 f g' g'' - 8 \cdot 12 g'' f' \cdot 2g - 4 \cdot 40 f''' g^2 
+ 160 g'' g' f - 16 \cdot 26 f'' g' g - 16 \cdot 13 g^2 f' \right) dx,$$

give zero.

4. SOME EXAMPLES.

**Example 1.** The operator

$$L = \partial^4 - 5 \partial^2 + \partial \frac{12}{\cosh^2 x} \partial - \frac{6}{\cosh^2 x} = A^*A - 4$$

with

$$A = -\partial^2 - 3 \tanh x \partial - 2$$

has 2 linearly independent eigenfunctions with eigenvalue $E_0 = -4$,

$$\psi_+(x) = \frac{1}{\cosh^2 x}, \quad \psi_-(x) = \frac{\sinh x}{\cosh^2 x}.$$

One can easily check that $A \psi_\pm = 0$ and that $u, v$ are reflectionless, as

$$\tilde{L} = AA^* - 4 = \partial^4 - 5 \partial^2$$
(note that $\psi_+$ and $\psi_-$ have different fall-off behaviour at $\infty$ and that $W(x) = \cosh^{-3} x$).

**Example 2.** The operator

$$L = \partial^4 + 16 \partial \frac{1}{\cosh^2 x} \partial + \frac{40}{\cosh^4 x} - \frac{88}{\cosh^2 x} = A^* A - 64$$

with

$$A = -\partial^2 - 4 \tanh x \partial - 8 + \frac{2}{\cosh^2 x}$$

has 2 linearly independent eigenfunctions with eigenvalue $E_0 = -64$,

$$\psi_+(x) = \frac{\cos 2x}{\cosh^2 x}, \quad \psi_-(x) = \frac{\sin 2x}{\cosh^2 x}.$$  

One easily verifies that $A \psi_{\pm} = 0$, and that

$$\tilde{L} = AA^* - 4 = \partial^4 - \frac{40}{\cosh^2 x}.$$  

A computation gives that

$$Q = \frac{2^{10}}{7} \cdot 4 \cdot 687, \quad \tilde{Q} = 2^{10} \cdot 100, \quad \delta Q = -\frac{2^{21}}{7} \left( = -32 (\kappa = 2) \frac{2^9}{7} \right).$$

**Example 3.** The operator

$$L = \partial^4 + \left( 45 \Psi^4 - 40 \Psi^2 \right) = A^* A - 4$$

with

$$W = \Psi^2 := \frac{1}{\cosh^2 x}, \quad W_{12} = 2 \Psi^2 - 3 \Psi^4$$

and

$$A = -\partial^2 - 2 \tanh x \partial - 2 + 3 \Psi^2$$

has a doubly degenerate eigenvalue $E_0 = -4$. One easily verifies, that

$$\tilde{L} = \partial^4 - 8 \partial \Psi^2 \partial + 25 \Psi^4 - 16 \Psi^2.$$  

**Example 4.** The operator

$$L = \partial^4 - \partial^2 + 4 \partial \frac{1}{\cosh^2 x} \partial + \frac{6}{\cosh^2 x} - \frac{8}{\cosh^4 x} = A^* A$$

with

$$A = -\partial^2 - \tanh x \partial - \frac{1}{\cosh^2 x} = \partial (-\partial - \tanh x)$$

has a unique ground-state $E_0 = 0$ with eigenfunction

$$\psi(x) = \frac{1}{\cosh x}.$$  

The second solution of $A \psi = 0$ is $\psi = \tanh x \not\in L^2(\mathbb{R})$. One easily verifies, that

$$\tilde{L} = \partial^4 - \partial^2.$$
Example 5. For any \( k > 0 \), the operator
\[
L = \partial^4 + \partial u \partial + v
\]
with
\[
\begin{cases}
u(x) = 2 \left( 1 + \frac{2}{k} \right) \Psi^2 \left( \frac{x}{k} \right) \\
u(x) = -4 \left( 1 + \frac{1}{k} - \frac{1}{k^2} \right) \Psi^2 \left( \frac{x}{k} \right) + \left( 1 - \frac{1}{k} \right) \left( 1 + \frac{2}{k} + \frac{6}{k^2} \right) \Psi^4 \left( \frac{x}{k} \right)
\end{cases}
\]
where
\[
\Psi(x) := \frac{1}{\cosh x},
\]
has a doubly degenerate ground-state, \( E_0 = -4 \), with eigenfunctions
\[
\psi^{(k)}_\pm(x) = e^{\pm ix} \left( \frac{1}{\cosh \frac{x}{k}} \right)^k.
\]

5. Follytons.

In order to find \( u \) and \( v \) such that \( L = \mathcal{A}^* \mathcal{A} + E_0 \) is 'conjugate' to the free operator \( \tilde{L} = \partial^4 =: \mathcal{L}_0 \) one has to solve (1.5) with \( u = v = 0 \). Eliminating \( g \) and writing \( E_0 = -4k^4 \) one obtains the ODE
\[
2 f''' + 6 f'' f' + 7 f^2 + 8 f' f^2 + f^4 = 16 k^4.
\]
One may reduce the order by taking \( f \) as the independent variable, and \( F(f) := f' \) as the dependent one, yielding
\[
2 \left( F''' F + F''^2 \right) + 6 F' F' + 7 F^2 + 8 F'' + f^4 = 16 k^4,
\]
but both forms seem(ed) to be too difficult to solve. By using (1.14), however, it takes the form
\[
16k^4 W^2 = 2 (W''' W - W'' W') + W''^2,
\]
in which it is easier to see the solution \[2\]
\[
\hat{W} = \text{const} \cdot \left( \sqrt{2} + \cosh (2 \kappa x) \right).
\]
Correspondingly,
\[
\hat{f} := \frac{\hat{W}''}{\hat{W}} = 2 \kappa \frac{\sinh (2 \kappa x)}{\sqrt{2} + \cosh (2 \kappa x)}.
\]
As interchanging \( \mathcal{A}^* \) and \( \mathcal{A} \) (as far as \( f \) is concerned) only changes the sign of \( \hat{f} \),
\[
f(x) = -2 \kappa \frac{\sinh (2 \kappa x)}{\sqrt{2} + \cosh (2 \kappa x)}.
\]
The Wronskian of the two ground-states \( \psi_{\pm} \) (of \( L = \partial^4 + \partial u \partial + v \), conjugate to \( L_0 = \partial^4 \)) is simply the inverse of \( \hat{W} \), i.e. (choosing the constant in \( \hat{W} \) to be 1),

\[
W(x) = \frac{1}{\sqrt{2} + \cosh(2 \kappa x)} =: \chi(2 \kappa x).
\]

The function \( g \) is given by

\[
g = \frac{1}{2} (3 f' - f^2) = -2 \kappa^2 \left( 1 + \sqrt{2} W - 2 W^2 \right).
\]

Insertion into equation (1.5) yields the reflectionless 'potentials'

\[
\begin{align*}
\left\{ 
\begin{array}{ll}
u_{\kappa} & = 16 \kappa^2 \left( \sqrt{2} W - W^2 \right) \\
u_{\kappa} & = 16 \kappa^4 \left( \sqrt{2} W - 12 W^2 + 16 \sqrt{2} W^3 - 8 W^4 \right)
\end{array}
\right.
\end{align*}
\]

with \( L = \partial^4 + \partial u \partial + v \) having exactly one doubly degenerate negative eigenvalue \(-4 \kappa^4\). While in most other examples we scaled \( \kappa \) to be equal to 1 it is, in this case (due to the appearance of \( 2 \kappa \) in \( W \)) easiest to choose \( \kappa = \frac{1}{2} \), i.e. to take

\[
\begin{align*}
\left\{ 
\begin{array}{ll}
u & = 4 \left( \sqrt{2} \chi - \chi^2 \right) \\
u & = \left( \sqrt{2} \chi - 12 \chi^2 + 16 \sqrt{2} \chi^3 - 8 \chi^4 \right)
\end{array}
\right.
\end{align*}
\]

and, when needed, use formulas like

\[
\begin{align*}
\chi'' & = \chi \left( 1 - 3 \sqrt{2} \chi + 2 \chi^2 \right) \\
\chi'^2 & = \chi^2 \left( 1 - 2 \sqrt{2} \chi + \chi^2 \right) \\
\chi''' & = \chi' \left( 1 - 6 \sqrt{2} \chi + 6 \chi^2 \right) \\
\chi'''' & = \chi \left( 1 - 15 \sqrt{2} \chi + 80 \chi^2 - 60 \sqrt{2} \chi^3 + 24 \chi^4 \right).
\end{align*}
\]

(Note that redefining \( \chi \) by a factor of \(-\sqrt{2}\) would make all the coefficients positive (integers)). These formulas are useful when checking that \( u(x+4 t) \) and \( v(x+4 t) \), with \( u \) and \( v \) given by \( (5.2) \), are exact solutions of the nonlinear system of PDE’s \( (5.2) \) (just as \( u_{\kappa}(x + 16 \kappa^2 t), v_{\kappa}(x + 16 \kappa^2 t) \)).

**Appendix A.** \( W \neq 0 \).

We shall prove here that the Wronskian type function defined in (1.3) never equals zero.

**Theorem A.1.** Let \( \psi_{\pm} \) be two orthonormal eigenfunctions of the operator (1.1) corresponding to the lowest eigenvalue \( E_0 \) of double multiplicity. Then

\[
(W[\psi_+, \psi_-])(x) := \psi_+(x) \psi'_-(x) - \psi_-(x) \psi'_+(x) \neq 0, \quad x \in \mathbb{R}.
\]

In order to prove this result we need a simple auxiliary statement.
Lemma A.1. Let $E_0$ be the lowest eigenvalue of the operator $L$ and let $\psi \in L^2(\mathbb{R})$ be a solution of the equation (1.2) satisfying $\psi(x_0) = \psi'(x_0) = 0$ for some $x_0 \in \mathbb{R}$. Then $\psi(x) \equiv 0$.

Proof. Indeed, the function

$$\tilde{\psi}(x) = \begin{cases} \psi(x), & \text{if } x \leq x_0, \\ -\psi(x), & \text{if } x \geq x_0, \end{cases}$$

is linear independent with $\psi$. Since $\tilde{\psi}(x_0) = \tilde{\psi}'(x_0) = 0$ we obtain $(L\tilde{\psi}, \tilde{\psi}) = E_0\|\tilde{\psi}\|^2$. Then $\tilde{\psi}$ is also an eigenfunction of the operator $L$ with the eigenvalue $E_0$. Consider now the linear combination

$$\psi_1(x) = \tilde{\psi}''(x_0)\psi(x) - \tilde{\psi}'(x_0)\tilde{\psi}(x).$$

Obviously $\psi_1(x_0) = \psi_1'(x_0) = \psi_1''(x_0) = 0$, $\psi_1 \in L^2(\mathbb{R})$ and $\psi_1$ satisfies the fourth order differential equation $L\psi_1 = E_0\psi_1$. Being overdetermined, $\psi_1 \equiv 0$ which also implies $\psi \equiv 0$. □

Remark. In Lemma A.1 the conditions $\psi(x_0) = \psi'(x_0) = 0$ split the problem for the operator $L$ in $L^2(\mathbb{R})$ into two Dirichlet boundary value problems on semiaxes $L^2((x_0, \infty))$ and $L^2((-\infty, x_0))$. Therefore, the lowest eigenvalue moves up.

Proof of Theorem A.1

a. Let $\psi_\pm$, be two orthonormal eigenfunctions corresponding to the lowest eigenvalue $E_0$ of the operator $L$. The functions $\psi_+$ and $\psi_-$ cannot vanish at the same point. Indeed, assume that they do. Then there is a point $x_0$ such that $\psi_+(x_0) = \psi_-(x_0) = 0$. If in addition we assume that $\psi_+(x_0) = 0$, then by Lemma A.1, $\psi_+ \equiv 0$ and we obtain a contradiction. Therefore we can assume that $\psi'_\pm(x_0) \neq 0$. Introduce a new function

$$\psi_2(x) = \psi'_-(x_0)\psi_+(x) - \psi'_+(x_0)\psi_-(x).$$

It is a non-trivial eigenfunction of the equation (1.2) satisfying $\psi_2(x_0) = \psi_2'(x_0) = 0$. By using Lemma A.1 again we find that $\psi_2 \equiv 0$ which cannot be true because $\psi_+$ and $\psi_-$ are linear independent.

b. Consider now the following pair of complex functions

$$\Psi_\pm(x) = \psi_+(x) \pm i\psi_-(x) =: \psi(x)e^{\pm i\phi(x)}.$$

By using a. we observe that $\psi$ never vanishes, $\psi(x) \neq 0$, $x \in \mathbb{R}$. Besides

$$W[\Psi_+, \Psi_-] = (\psi_+ + i\psi_-)(\psi_+ - i\psi_-)' - (\psi_+ + i\psi_-)'(\psi_+ - i\psi_-)$$

$$= -2iW[\psi_+, \psi_-] = -2i\phi'\psi^2.$$
Thus, in order to prove Theorem A.1 it remains to prove that $\phi' \neq 0$. Assume that there is $x_0$ such that $\phi'(x_0) = 0$ and consider

$$\Phi(x) = e^{-i\phi(x_0)}\Psi_+(x) - e^{i\phi(x_0)}\Psi_-(x).$$

Clearly $\Phi(x_0) = \Phi'(x_0) = 0$ and by using Lemma A.1 we obtain $\Phi \equiv 0$ which contradicts the linear independency of the functions $\Psi_\pm$.

The proof is complete. □

Acknowledgments. The authors would like to thank H. Kalf, E. Langmann, A. Pushnitski and O. Safronov for useful discussions, as well as the ESF European programme SPECT and the EU Network: “Analysis & Quantum” for partial support.

REFERENCES

[1] R. Benguria, M. Loss, A simple proof of a theorem of Laptev and Weidl, Mathematical research letters, 7 (2000), 195–203.
[2] J. Hoppe, Factorization of higher order operators, manuscript.
[3] J. Östensson, Trace formulae for fourth order differential operators, KTH, PhD-thesis (to appear).
[4] I. Schur, Über vertauschbare lineare Differentialausdrücke, Sitzungsbericht d. Berliner Mathematischen Gesellschaft (1905), 2-8.

E-mail address: hoppe@math.kth.se, laptev@math.kth.se, jorgen@math.kth.se