REGULAR MODULES WITH PREPROJECTIVE GABRIEL-ROITER SUBMODULES OVER \( n \)-KRONECKER QUIVERS

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Abstract. Let \( Q \) be a wild \( n \)-Kronecker quiver, i.e., a quiver with two vertices, labeled by 1 and 2, and \( n \geq 3 \) arrows from 2 to 1. The indecomposable regular modules with preprojective Gabriel-Roiter submodules, in particular, those \( \tau^{-i}X \) with \( \dim X = (1, c) \) for \( i \geq 0 \) and some \( 1 \leq c \leq n - 1 \) will be studied. It will be shown that for each \( i \geq 0 \) the irreducible monomorphisms starting with \( \tau^{-i}X \) give rise to a sequence of Gabriel-Roiter inclusions, and moreover, the Gabriel-Roiter measures of those produce a sequence of direct successors. In particular, there are infinitely many GR-segments, i.e., a sequence of Gabriel-Roiter measures closed under direct successors and predecessors. The case \( n = 3 \) will be studied in detail with the help of Fibonacci numbers. It will be proved that for a regular component containing some indecomposable module with dimension vector \((1, 1)\) or \((1, 2)\), the Gabriel-Roiter measures of the indecomposable modules are uniquely determined by their dimension vectors.

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1. Introduction

Let \( \Lambda \) be an artin algebra and \( \text{mod} \Lambda \) the category of finitely generated right \( \Lambda \)-modules. For each \( M \in \text{mod} \Lambda \), we denote by \(|M|\) the length of \( M \). The symbol \( \subset \) is used to denote proper inclusion.

We first recall the original definition of Gabriel-Roiter measure \([15,16]\). Let \( \mathbb{N} = \{1, 2, \ldots \} \) be the set of natural numbers and \( \mathcal{P}(\mathbb{N}) \) be the set of all subsets of \( \mathbb{N} \). A total order on \( \mathcal{P}(\mathbb{N}) \) can be defined as follows: if \( I,J \) are two different subsets of \( \mathbb{N} \), write \( I < J \) if the smallest element in \((I\setminus J) \cup (J\setminus I)\) belongs to \( J \). Also we write \( I \ll J \) provided \( I \subset J \) and for all elements \( a \in I, b \in J\setminus I \), we have \( a < b \). We say that \( J \) starts with \( I \) if \( I = J \) or \( I \ll J \). Thus \( I < J < I' \) with \( I' \) starts with \( I \) implies that \( J \) starts with \( I \).

For each \( M \in \text{mod} \Lambda \), let \( \mu(M) \) be the maximum of the sets \( \{|M_1|, |M_2|, \ldots, |M_t|\} \), where \( M_1 \subset M_2 \subset \ldots \subset M_t \) is a chain of indecomposable submodules of \( M \). We call

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\( \mu(M) \) the **Gabriel-Roiter (GR for short) measure** of \( M \). A subset \( I \) of \( \mathcal{P}(\mathbb{N}) \) is called a GR measure for \( \Lambda \) if there is an indecomposable \( \Lambda \)-module \( M \) with \( \mu(M) = I \). If \( M \) is an indecomposable \( \Lambda \)-module, we call an inclusion \( X \subset M \) with \( X \) indecomposable a **GR inclusion** provided \( \mu(M) = \mu(X) \cup \{ |M| \} \), thus if and only if every proper submodule of \( M \) has Gabriel-Roiter measure at most \( \mu(X) \). In this case, we call \( X \) a **GR submodule** of \( M \). Note that the factor of a GR inclusion is indecomposable.

Using Gabriel-Roiter measure, Ringel obtained a partition of the module category for any artin algebra of infinite representation type \[15, 16\]: there are infinitely many GR measures \( I_i \) and \( I_j \) with \( i \) natural numbers, such that

\[ I_1 < I_2 < I_3 < \ldots < I_i < I_j \]

and such that any other GR measure \( I \) satisfies \( I_i < I < I_j \) for all \( i, j \). The GR measures \( I_i \) (resp. \( I_j \)) are called take-off (resp. landing) measures. Any other GR measure is called a central measure. An indecomposable module \( M \) is called a take-off (resp. central, landing) measure. It was proved in \[15\] that every landing module is preinjective in general sense.

Let \( I, I' \) be two GR measures for \( \Lambda \). We call \( I' \) a **direct successor** of \( I \) if, first, \( I < I' \) and second, there does not exist a GR measure \( I'' \) with \( I < I'' < I' \). The so-called **Successor Lemma** in \[16\] states that any GR measure \( I \) different from \( I_1 \), the maximal one, has a direct successor. However, a GR measure, which is not the minimal one \( I_1 \), may not admit a direct predecessor. A **GR segment** is a sequence of GR measures closed under direct predecessors and direct predecessors. It was conjectured that an artin algebra if of wild type if and only if it has infinitely many GR segments.

The GR measure for path algebras of tame quivers (over an algebraically closed field) were studied in \[3, 4, 5\]. In particular, the connection between GR measure and Auslander-Reiten theory was studied. For example, let \( \delta \) be the minimal positive imaginary root and \( H_1 \) be an indecomposable homogeneous simple module (with dimension vector \( \delta \)), then the sequence of irreducible monomorphisms \( H_1 \to H_2 \to H_3 \to \ldots \) gives a sequence of GR submodules. Moreover, \( \mu(H_{i+1}) \) is the direct successor of \( \mu(H_i) \) for each \( i \geq 1 \). It was also shown in \[5\] that for a tame quiver, there are, but only finitely many, GR measures which have no direct predecessors, and in \[9\] that the number of the GR segments is bounded by \( b + 3 \), where \( b \) is the number of the isomorphism classes of exceptional quasi-simple modules.

So far, not much about the GR measures for wild quivers is known. In \[10\], it was proved for 3-Kronecker quiver that there are uncountable many Gabriel-Roiter measures (modules of infinite length were considered). It was also conjectured there the existence of
the maximal central measure (which should be an infinite sequence of natural numbers). In [6], the wild $n$-Kronecker quivers were studied and infinitely many GR measures admitting no direct predecessors were constructed.

In this paper, the study will be focused on $n$-Kronecker quivers:

$$
\begin{array}{c}
2 \\
\vdots \\
\alpha_n \\
\end{array}
\begin{array}{c}
\alpha_1 \\
\alpha_2 \\
1 \\
\end{array}
$$

with $n \geq 3$. The indecomposable modules whose GR submodules are preprojective will be studied. Similar to the case of homogeneous modules for tame quivers, the following result will be shown:

**Theorem 1.** Let $X$ be an indecomposable module containing a preprojective module as a GR submodule. Then for each $i \geq 0$ and each $j \geq 1$, the irreducible monomorphism $\tau^{-i}X[j] \rightarrow \tau^{-1}X[j + 1]$ is a GR inclusion. Moreover, up to isomorphism, $\tau^{-i}X[j]$ is the unique GR submodule of $\tau^{-i}X[j + 1]$.

Here $\tau$ is the Auslander-Reiten translation and $X[j]$ denotes the regular module with quasi-simple submodule $X$ and quasi-length $j$. Note that $X$ is always quasi-simple under the assumption.

The vectors $(1, c)$ with $1 \leq c \leq n - 1$ are imaginary roots. An indecomposable module (existence by [11]) $X$ with dimension vector $(1, c)$ contains the projective simple module as a GR submodule. Thus the above theorem implies that $\tau^{-i}X[j + 1]$ contains $\tau^{-i}X[j]$, up to isomorphism, as the unique GR submodule. Using some combinatorial studies, it will be seen that the Gabriel-Roiter measures $\mu(\tau^{-i}X[j])$ are namely determined by $i, j$ and the dimension vectors $(1, c)$.

**Theorem 2.** Let $X$ be an indecomposable module with dimension vector $\dim X = (1, c)$ for some $1 \leq c \leq n - 1$ and $M$ an indecomposable module. Let $i \geq 0$ and $j \geq 1$. Then $\mu(M) = \mu(\tau^{-i}X[j])$ if and only if $M \cong \tau^{-i}Y[j]$ for some indecomposable module $Y$ with $\dim Y = (1, c) = \dim X$.

As a consequence of this theorem, we may obtain the following result, which can be used to show that the number of the GR segments for $n$-Kronecker quivers is unbounded:

**Theorem 3.** Let $X$ be an indecomposable module with dimension vector $\dim X = (1, c)$ for some $1 \leq c \leq n - 1$. Then for each $i \geq 0$ and $j \geq 1$, $\mu(\tau^{-i}X[j + 1])$ is the direct successor of $\mu(\tau^{-i}X[j])$. 


As an application of the general discussion, we will study in detail the regular components over the 3-Kronecker quiver, which contains an indecomposable module with dimension vector (1,1) or (1,2). In [18], it was proved that the indecomposable modules in a regular component of any wild hereditary algebra are uniquely determined by their dimensions. Thus in each regular component, only finitely many indecomposable modules have the same length. It may be asked if these modules of the same length have the same GR measure. However, this is not always the case (see Section 5.4 for an example). We can partially answer this question as follows:

**Theorem 4.** Let Q be the 3-Kronecker quiver. Let X be an indecomposable module with dimension vector (1,1) or (1,2) and C a regular component containing X. Then the GR measures of the indecomposable modules in C are uniquely determined by their dimension vectors.

In section 2 some preliminaries, notations and elementary results will be recalled. Section 3 is devoted to a study of the indecomposable regular modules with preprojective GR submodules. In particular, the theorem concerning the coincidence of the irreducible monomorphisms and the GR inclusions will be shown. In Section 4 the indecomposable modules $\tau^{-i}X$ with $i \geq 0$ and $\dim X = (1,c)$ will be studied. In particular, the direct successors of $\tau^{-i}X[j]$ will be described for $i \geq 0$ and $j \geq 1$. The regular components containing an indecomposable module with dimension vector (1,1) or (1,2) over a 3-Kronecker quiver will be studied in detail in Section 5.

2. Preliminaries and known results

2.1. **Representations of n-Kronecker quivers.** We recall some facts of representations of quivers. The best references are [1, 14]. We also refer to [12, 13] for general structures of representations of wild quivers. We also refer to [11] for the relationship between the roots and the indecomposable modules.

Let $Q$ be an $n$-Kronecker quiver with $n \geq 3$ and $k$ an algebraically closed field. A representation of $Q$ over $k$ is simply called a module. The Cartan matrix and the Coxeter matrix are the following:

$$C = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}, \quad \Phi = -C^{-t}C = \begin{pmatrix} n^2 - 1 & n \\ -n & -1 \end{pmatrix}, \quad \Phi^{-1} = \begin{pmatrix} -1 & -n \\ n & n^2 - 1 \end{pmatrix}.$$ 

The dimension vectors can be calculated using $\dim \tau M = (\dim M)\Phi$ if $M$ is not projective and $\dim \tau^{-1}N = (\dim N)\Phi^{-1}$ if $N$ is not injective, where $\tau$ denotes the Auslander-Reiten
translation. The quadratic form \( q((x_1,x_2)) = x_1^2 + x_2^2 - nx_1x_2 \). A vector \((a,b)\) is a real root if \( q((a,b)) = 1 \). The positive real roots are precisely the dimension vectors of the indecomposable preprojective modules and those of the indecomposable preinjective modules. For each positive imaginary root \((a,b)\), i.e., \( q((a,b)) < 0 \), there are infinitely many indecomposable modules with dimension vector \((a,b)\). Note that the dimension vector of an indecomposable module is either a positive real root or a positive imaginary root. The Euler form is \( \langle (x_1,x_2),(y_1,y_2) \rangle = x_1y_1 + x_2y_2 - nx_1y_2 \). For two indecomposable modules \( X \) and \( Y \),

\[
\dim \text{Hom}(X,Y) - \dim \text{Ext}^1(X,Y) = \langle \dim X, \dim Y \rangle.
\]

The Auslander-Reiten quiver of \( Q \) consists of one preprojective component, one preinjective component and infinitely many regular ones. An indecomposable regular module \( X \) is called quasi-simple if the Auslander-Reiten sequence starting with \( X \) has an indecomposable middle term. For each indecomposable regular module \( M \), there is a unique quasi-simple module \( X \) and a unique natural number \( r \geq 1 \) (called quasi-length of \( M \) and denoted by \( \text{ql}(M) = r \)) such that there is a sequence of irreducible monomorphisms \( X = X[1] \rightarrow X[2] \rightarrow \ldots \rightarrow X[r] = M \). In this case, we denote by \( \text{qs}(M) = X \). Dually, there is a unique quasi-simple module \( Y \) (denote by \( \text{qt}(M) \)) with a sequence of irreducible epimorphisms \( M = [r]Y \rightarrow \ldots \rightarrow [2]Y \rightarrow [1]Y = Y \).

### 2.2. Properties of GR measure

We present some known results being used later on. The following proposition was proved in \([15]\):

**Proposition 2.1.** Let \( \Lambda \) be an artin algebra and \( X \) and \( Y_1,Y_2,\ldots,Y_r \) be indecomposable modules. Assume that \( X \xrightarrow{f} \bigoplus_{i=1}^r Y_i \) is a monomorphism.

1. \( \mu(X) \leq \max\{\mu(Y_i)\} \).
2. If \( \max\{\mu(Y_i)\} = \mu(X) \), then \( f \) splits.

We collect some properties of GR inclusions in the following lemma. The proof can be found for example in \([2, 3]\).

**Lemma 2.2.** Let \( \Lambda \) be an artin algebra and \( X \subset M \) a GR inclusion.

1. If all irreducible maps to \( M \) are monomorphisms, then the GR inclusion is an irreducible map.
2. Every non-zero homomorphism \( Y \rightarrow M/X \), which is not an epimorphism, factors through the canonical projection \( M \rightarrow M/X \).
(3) There is an irreducible monomorphism $X \to Y$ with $Y$ indecomposable and an epimorphism $Y \to M$. 

(4) If $Y$ is indecomposable with $\mu(X) < \mu(Y) < \mu(M)$, then $|Y| > |M|$.

2.3. The partition for $n$-Kronecker quivers. Let $Q$ be an $n$-Kronecker quiver. We are going to describe the partition obtained using GR measure for $Q$.

The preprojective component is the following (note that there are actually $n$ arrows from $P_i$ to $P_{i+1}$):

\[
\begin{align*}
P_2 &= (1, n) \\
P_1 &= (0, 1) \\
P_4 &= (n^2 - 1, n^3 - 2n) \\
P_3 &= (n, n^2 - 1) \\
P_5 &= \ldots \\
P_6 &= \ldots \\
\end{align*}
\]

Since every irreducible map in the preprojective component is a monomorphism, $P_i$ is, up to isomorphism, the unique GR submodule of $P_{i+1}$ by Lemma 2.2(1). Similarly, the preinjective component is of the following form:

\[
\begin{align*}
\ldots & Q_5 = (n^3 - 2n, n^2 - 1) \\
Q_4 & \ldots \\
Q_3 & = (n, 1) \\
Q_2 & = (n^2 - 1, n) \\
Q_1 & = \ldots \\
\end{align*}
\]

Let us denote by $I_i$ (resp. $I^i$) the take-off (resp. landing) measures and by $\mathcal{A}(I)$ the set of the representatives (of the isomorphism classes) of indecomposable modules with GR measure $I$.

**Proposition 2.3.** (1) The take-off part contains precisely the simple injective module and the indecomposable preprojective modules.

(2) The landing part contains precisely all non-simple indecomposable preinjective modules.

(3) An indecomposable module is a central module if and only if it is regular.

**Proof.** (1) We show, by induction on $m$, that $\mathcal{A}(I_m) = \{P_m\}$ for each $m \geq 2$. If $m = 2$, the assertion holds by the description of $I_2$, which is the GR measure of a local module with maximal length. Assume that $\mu(M) = I_{m+1}$ for some indecomposable module $M$. Since $M$ is not simple, we may assume that $Y$ is a GR submodule of $M$. Then $\mu(Y) = I_i \leq I_m$ for some $i \leq m$, and thus $Y \cong P_i$ by induction. It follows from Lemma 2.2(3) that there is an epimorphism $P_{i+1} \to M$. In particular $|M| \leq |P_{i+1}|$. If the equality does not hold, then $I_{m+1} = \mu(M) = I_i \cup \{|M|\} > I_i \cup \{|P_{i+1}|, \ldots, |P_m|, |P_{m+1}|\} > I_i \cup \{|P_{i+1}|, \ldots, |P_m|\} = I_m$. 


This is a contradiction because the GR measure \( \mu(P_{m+1}) = I_i \cup \{ |P_{i+1}|, \ldots, |P_m|, |P_{m+1}| \} \) lies between \( I_m \) and \( I_{m+1} \). Therefore, \( |P_{i+1}| = |M| \) and thus \( P_{i+1} \cong M \). Since \( \mu(M) = I_{m+1} \), we have \( i = m \) and thus \( \mathcal{A}(I_{m+1}) = \{ P_{m+1} \} \).

(2) Since there is a short exact sequence \( 0 \to I_{r+1} \to I_r \to I_{r-1} \to 0 \) for each \( r \geq 1 \), \( \mu(I_{r+1}) < \mu(I_r) \) by Proposition 2.1. Because landing modules are preinjective (see [15]), \( \mathcal{A}(I^m) \) contains precisely one isomorphism class \( Q_m \).

(3) is straightforward. \( \square \)

3. Regular modules with preprojective GR submodules

Let \( Q \) be an \( n \)-Kronecker quiver with \( n \geq 3 \). Before studying the regular modules whose GR submodules are preprojective, we present some combinatorial descriptions of the indecomposable regular modules with dimension \((a, b)\) such that \( a \leq b \). We write two vectors \((a, b) < (c, d)\) if \( a < c \) and \( b < d \).

Lemma 3.1. Let \( X \) be an indecomposable regular module with dimension vector \( \dim X = (a, b) \) such that \( a \leq b \). Let \( i \geq 1 \) and assume that \( \dim \tau^{-i} X = (c, d) \). Then

1. \((a, b) < (c, d)\).
2. \( c < d \).
3. For each \( r \geq 0 \), \( \sum_{i=0}^{r} \dim \tau^{-i} X < \dim \tau^{-(r+1)} X \).

Proof. We show (3) and (1) and (2) follow similarly. Let \( i = 1 \) and we show \( c - 2a \geq 0 \) and \( d - 2b > 0 \). Since \( n \geq 3 \) and \( b \geq a \), we have \( c - 2a = nb - 3a \geq 0 \). Note that the equality hold only for \( n = 3 \) and \( a = b \). Similarly, \( d - 2b = (n^2 - 1)b - na - 2b = (n^2 - 3)b - na > 0 \). Then the proof follows by induction. \( \square \)

Corollary 3.2. Let \( X \) be a quasi-simple module with dimension vector \( \dim X = (a, b) \) and \( a \leq b \). Consider the following short exact sequence

\[
0 \to \tau^{-i} X[j] \xrightarrow{f} \tau^{-i} X[j+1] \to \tau^{-(i+j)} X \to 0
\]

where \( f \) is an irreducible monomorphism and \( i \geq 0 \), \( j \geq 1 \). Then \( \dim \tau^{-i} X[j] < \dim \tau^{-(i+j)} X \) and thus \( |\tau^{-i} X[j]| < |\tau^{-(i+j)} X| \).

Proof. This follows directly from the lemma with the assumption that \( X \) is quasi-simple. \( \square \)

Let \( \mathcal{B} \) be the set of the isomorphism classes of the indecomposable regular modules whose GR submodules are preprojective. Note that \( \mathcal{B} \) is not empty since it contains all indecomposable modules \( X \) with dimension vector \( \dim X = (1, 1) \). By Proposition 2.3
an indecomposable module $X$ is contained in $B$ if and only if $X$ has no proper regular submodules. In particular, $X \in B$ implies that $X$ is quasi-simple.

**Lemma 3.3.** Let $X \in B$ with dimension vector $\dim X = (a, b)$. Then $a \leq b$.

**Proof.** There is nothing to show if $a = 1$. Assume $a \geq 2$ and let $M$ be an indecomposable regular module with dimension vector $(1, n - 1)$ (note that $M$ exists since $(1, n - 1)$ is an imaginary root). Since there does not exist an epimorphism $M \to X$ and $X$ has no proper regular submodules, we have $\text{Hom}(M, X) = 0$. It follows that $\langle (1, n - 1), (a, b) \rangle \leq 0$ and thus $a - b = a + (n - 1)b - nb \leq 0$. □

**Lemma 3.4.** Let $X \in B$ and $Y$ be a GR submodule of $X$.

1. For each $i \geq 0$, $\tau^{-i}X \in B$.
2. There exists an $m \geq 1$ such that $\tau^iX \notin B$ for any $i \geq m$.
3. If the GR factor $X/Y$ is not simple, then $X/Y \notin B$.
4. If $M$ is a non-simple indecomposable proper factor module of $X$, then $\mu(M) > \mu(X)$.

**Proof.**

1. Since a proper inclusion $M \subset \tau^{-i}X$ with $M$ a regular module induces a proper regular submodule $\tau^iM$ of $X$, $\tau^{-i}X$ has no proper regular submodules and thus $\tau^{-i}X \in B$ for all $i \geq 0$.

2. Without loss of generality, we may assume that $\dim X$ is minimal in the $\tau$-orbit of $X$. Using the Auslander-Reiten formula we have $\text{Hom}(X, \tau X) \cong \mathbb{D}\text{Ext}^1(X, X) \neq 0$. If $\tau X$ has no proper regular submodules, then there is an epimorphism $X \to \tau X$, which contradicts the minimality of $\dim X$. Therefore, $\tau X \notin B$ and thus $\tau^iX \notin B$ for any $i \geq 1$ by (1).

3. Assume that $X/Y$ is not simple and $N$ is a GR submodule of $X/Y$. Then the inclusion $N \to X/Y$ factors through $X$ and thus $N$ is isomorphic to a proper submodule of $X$. Note that $N$ is preprojective since $X \in B$. On the other hand, a GR submodule of a non-simple preinjective module is always a regular one. Therefore, $X/Y$ is regular.

4. We may assume that $M$ is not preinjective by the description of the landing part. If $\mu(M) < \mu(X)$, then $\mu(P_r) < \mu(M) < \mu(X)$, where $P_r$ is a GR submodule of $X$, since $M$ is regular. It follows that $|M| > |X|$ by Lemma 2.2(4), which is a contradiction. □

Let $X \in B$ and $i \geq 1$. Then $\tau^{-i}X \in B$ by above lemma. We are able to determine the GR submodules of $\tau^{-i}X$.

**Lemma 3.5.** Let $X \in B$ and $P_r$ a GR submodule of $X$. 
(1) If $X/P_r$ is regular, then $\tau^{-i}P_r$ is, up to isomorphism, the unique GR submodule of $\tau^{-i}X$ for each $i \geq 0$.

(2) If $X/P_r$ is simple, then $\tau^{-(i-1)}P_{r+1}$ is, up to isomorphism, the unique GR submodule of $\tau^{-i}X$ for each $i \geq 1$.

(3) For all $0 \leq i < j$, $\mu(\tau^{-i}X) > \mu(\tau^{-j}X)$.

Proof. (1) If $X/P_r$ is regular, then the GR inclusion induces a monomorphism $P_{r+2} = \tau^{-1}P_r \rightarrow \tau^{-1}X$ with a regular factor. If there is a monomorphism $P_{r+3} \rightarrow \tau^{-1}X$, then there is a monomorphism $P_{r+1} = \tau P_{r+3} \rightarrow X$. This contradicts $P_r$ is a GR submodule of $X$. Thus $\tau^{-1}P_r$ is a GR submodule of $\tau^{-1}X$. Since the factor is regular, we have $\tau^{-i}P_r$ is a GR submodule of $\tau^{-i}X$ for all $i \geq 1$ by induction.

(2) Assume that $X/P_r$ is simple. Let $\dim P_r = (a, b)$. Then $\dim X = (a+1, b)$. It follows that $\dim \tau^{-1}X = (nb-a-1, (n^2-1)b- n(a+1))$, $\dim \tau^{-1}P_r = (nb-a, (n^2-1)b-na)$ and $\dim P_{r+1} = (b, nb-a)$. Comparing the dimension vectors, we know that $P_{r+1}$ is a GR submodule of $\tau^{-1}X$ (using Lemma 2.2(3)). Note that $(nb-a-1) - b = (n-1)b - a - 1 > 1$. Thus the GR factor $\tau^{-i}X/P_{r+1}$ is not simple. It follows from (1) that $\tau^{-(i-1)}P_{r+1}$ is a GR submodule of $\tau^{-i}X$.

(3) This is straightforward by (1) and (2). □

As a consequence of the last statement of this lemma, we have:

**Corollary 3.6.** There does not exist a minimal central measure.

**Proof.** For the purpose of a contradiction, we assume that $M$ is an indecomposable module such that $\mu(M)$ is the minimal central GR measure. It follows that $M$ is regular by the description of the partition and a GR submodule $N$ of $M$ is preprojective by the minimality of $\mu(M)$. This implies that $\mu(\tau^{-i}M) < \mu(M)$ for each $i \geq 1$, which is a contradiction. □

**Remark** Note that for a tame quiver, the minimal central measure always exists [4, 5]. However, it does not mean that any wild quiver has no minimal central measure. For example, let $Q'$ be the wild quiver with three vertices, labeled by 1, 2, 3, and one arrow from 1 to 2 and two arrows from 2 to 3. Then the GR measure of the indecomposable projective module $P_1$ is $\mu(P_1) = \{1, 3, 4\}$, which is the minimal central measure [7].

Let $X \in \mathcal{B}$. Lemma 3.1 and Corollary 3.2 give some combinatorial descriptions of the dimension vectors of $\tau^{-i}X$ for $i \geq 0$. We will use these to study the GR submodules of $\tau^{-i}X[j]$ for all $i \geq 0$ and $j \geq 2$. We first recall what a piling submodule is [17].
Definition 3.7. Let $\Lambda$ be an artin algebra and $M$ be an indecomposable $\Lambda$-module. Then an indecomposable submodule $X$ of $M$ is called a piling submodule if $\mu(X) \geq \mu(Y)$ for all submodules $Y$ of $M$ with $|Y| \leq |X|$.

Lemma 3.8 ([17]). Let $\Lambda$ be an artin algebra and $M$ be an indecomposable $\Lambda$-module. Let $X$ be an indecomposable submodule of $M$. Then $X$ is a piling submodule of $M$ if and only if $\mu(M)$ starts with $\mu(X)$ (meaning that $\mu(X) = \mu(M) \cap \{1, 2, 3, \ldots, |X|\}$).

The following result is crucial when calculating the GR submodules of $\tau^{-i}X[j]$ for $X \in \mathcal{B}$ over $n$-Kronecker quivers.

Proposition 3.9. Let $0 \to X \xrightarrow{f} Y \xrightarrow{\pi} Z \to 0$ be an short exact sequence of indecomposable regular modules such that

1. $f$ is an irreducible monomorphism,
2. $Z$ contains a preprojective module as a GR submodule,
3. $|X| < |Z|$.

Then $f$ is a GR inclusion. Moreover, $X$ is, up to isomorphism, the unique GR submodule of $Y$.

Proof. Let $U \xrightarrow{g} Y$ be an indecomposable regular submodule. If the composition $\pi g$ is zero, then the inclusion $g$ factors through $f$ and thus $U$ is isomorphic to a submodule of $X$. If $\pi g$ is not zero, then it is an epimorphism since $Z$ contains no proper regular submodules. In particular, $|U| > |Z|$. Therefore, an indecomposable proper regular submodule of $Y$ is either isomorphic to a submodule of $X$, or with length greater than $|Z|$. Let $V$ be an indecomposable submodule of $M$ such that $|V| \leq |X|$. If $V$ is regular, then $V$ is isomorphic to a submodule of $Y$ by above discussion since $|V| \leq |X| < |Z|$. If $V$ is preprojective, then $\mu(V) < \mu(X)$. It follows that $X$ is a piling submodule of $Y$ and thus $\mu(Y)$ starts with $\mu(X)$ by Lemma 3.8. Let $U$ be a GR submodule of $Y$. Then $U$ is a regular module. For the purpose of a contradiction, we assume that $U \not\cong X$. Then by above discussion, $|U| > |Z| \geq |X|$. Let $U_1 \subset U_2 \subset \ldots \subset U_r = U$ be a GR filtration of $U$. Since $\mu(X) < \mu(U) < \mu(Y)$, we have $\mu(U)$ starts with $\mu(X)$. Therefore, there is an $U_i$ such that $|U_i| = |X|$, and thus $U_i \cong X$. However, $X \xrightarrow{f} Y$ is an irreducible monomorphism implies $U$ is decomposable. This contradiction shows $X$ is the unique, up to isomorphism, GR submodule of $Y$. □

The following theorem is a direct consequence of Lemma 3.3, Corollary 3.2 and Proposition 3.9.
Theorem 3.10. Let $X \in \mathcal{B}$. Then for each $i \geq 0$ and each $j \geq 1$, the irreducible monomorphism $\tau^{-i}X[j] \rightarrow \tau^{-i}X[j + 1]$ is a GR inclusion. Moreover, up to isomorphism, $\tau^{-i}X[j]$ is the unique GR submodule of $\tau^{-i}X[j + 1]$.

Before ending this section, we give a description of the dimension vectors of indecomposable regular modules with the same lengths and trivial Hom-spaces.

Lemma 3.11. Let $X$, $Y$ be indecomposable regular modules with dimension vectors $(a, b)$ and $(r, s)$, respectively. Assume that $|X| = |Y|$, i.e., $a + b = r + s$, and $\text{Hom}(X, Y) = 0$. Then $s \geq b + \frac{q((a, b))}{(n + 1)a - b}$.

Proof. Since $\text{Hom}(X, Y) = 0$, we have
$$\langle (\dim X, \dim Y) \rangle = \dim \text{Hom}(X, Y) - \dim \text{Ext}^1(X, Y) \leq 0.$$ It follows that
$$ar + bs - nas \leq 0.$$ Using $a + b = r + s$, we obtain that $a(a + b - s) + bs - nas \leq 0$. Therefore,
$$(n + 1)a - b)s \geq a(a + b).$$ Assume for a contradiction that $(n + 1)a \leq b$. Since $(a, b)$ is an imaginary root, $\frac{b}{a} \leq \frac{n + \sqrt{n^2 - 4}}{2} < n$. It follows that $n + 1 \leq \frac{b}{a} < n$. This contradiction implies $(n + 1)a > b$ and thus
$$s \geq \frac{a^2 + ab}{(n + 1)a - b} = b + \frac{a^2 - nab + b^2}{(n + 1)a - b} = b + \frac{q((a, b))}{(n + 1)a - b}.$$ The proof is completed. \hfill $\square$

4. INDECOMPOSABLE MODULES $\tau^{-i}X$ WITH $\dim X = (1, c)$

For each natural number $1 \leq c \leq n - 1$, the regular components containing indecomposable modules with dimension vectors $(1, c)$ are of special interests. For example, in section 5, we will see that in case $n = 3$, the dimension vectors of the indecomposable modules in a regular component containing some $X$ with $\dim X = (1, 1)$ or $(1, 2)$ relate to pairs of Fibonacci numbers and the GR measures of the indecomposable modules in such a component are uniquely determined by their dimensions.

Let $\dim X = (1, c)$. It is easily seen (for example in the following lemma) that the GR submodule of $X$ is a projective simple module. Therefore, $\tau^{-i}X \in \mathcal{B}$ for all $i \geq 1$ by Lemma 3.4. It follows that $\tau^{-i}X[j]$ is, up to isomorphism, the unique GR submodule of $\tau^{-i}X[j + 1]$ for each $i \geq 0$ and each $j \geq 1$ (Theorem 3.10). It turns out that the dimension
vector \((1, c)\) and the indexes \(i\) and \(j\) determine the GR measures. Using this we can show that \(\mu(\tau^{-i}X[j+1])\) is a direct successor of \(\mu(\tau^{-i}X[j])\) for every \(i \geq 0\) and \(j \geq 1\).

**Lemma 4.1.** Let \(c\) be a natural number such that \(1 \leq c \leq n-1\). Then the vector \((1, c)\) is an imaginary root. Let \(X\) be an indecomposable module with dimension vector \(\dim X = (1, c)\).

1. \(X\) is a (regular) quasi-simple module.
2. A GR submodule of \(X\) is isomorphic to the projective simple module \(P_1\).
3. If \(c = 1\), \(\tau^{-(i-1)}P_2\) is a GR submodule of \(\tau^{-i}X\) for each \(i \geq 1\). If \(c > 1\), \(\tau^{-i}P_1\) is a GR submodule of \(\tau^{-i}X\) for each \(i \geq 0\).
4. Let \(M\) be an indecomposable module. Then \(\mu(M) = \mu(X)\) if and only if \(\dim M = (1, c) = \dim X\).

**Proof.** It is easily seen that \(q((1, c)) < 0\) and thus \((1, c)\) is an imaginary root. Let \(X\) be indecomposable with \(\dim X = (1, c)\) and \(Y\) a GR submodule of \(X\). Then \(\dim Y = (0, 1)\) or \(\dim Y = (1, r)\) with \(r < c\). If the second case holds, then the GR factor has dimension \((0, c-r)\) which is impossible. Thus \(Y\) is isomorphic to \(P_1\), the projective simple module. In particular, \(X\) is quasi-simple since it has no proper regular submodules. Thus we may describe the GR submodules of \(\tau^{-i}X\) using Lemma 3.3. If \(M\) is an indecomposable module with \(\mu(M) = \mu(X)\), then \(P_1\) is a GR submodule of \(M\) and thus there is an epimorphism \(P_2 \to M\) (Lemma 2.2). In particular, we have \(\dim M = (1, r)\) for some \(r < n\) since \(\dim P_2 = (1, n)\). Therefore, \(\dim M = \dim X\) since \(|M| = |X|\). \(\square\)

**Lemma 4.2.** Let \(X\) be an indecomposable module with dimension \(\dim X = (1, c)\) and \(1 \leq c \leq n-1\). Let \(i \geq 1\) and suppose that \(\dim \tau^{-i}X = (a, b)\). Then \(0 < -q((a, b)) < (n+1)a - b\).

**Proof.** Since the quadratic \(q\) is invariant on the dimension vectors of the indecomposable modules in a \(\tau\)-orbit, we have

\[-q(\dim \tau^{-i}X) = -q((a, b)) = -q((1, c)) = -c^2 + nc - 1 = -(c - \frac{n}{2})^2 + \frac{n^2}{4} - 1.\]

If \(i = 1\), then \((a, b) = \dim \tau^{-1}X = (1, c)\left(\begin{array}{cc} -1 & -n \\ n & n^2 - 1 \end{array}\right) = (nc - 1, (n^2 - 1)c - n)\). Assume for a contradiction that \(-q((a, b)) \geq (n+1)a - b\). Thus \(nc - c^2 - 1 \geq (n+1)(nc - 1) - (n^2 - 1)c + n\). It follows that \(c^2 + c \leq 0\) which is impossible. Thus \(-q((a, b)) < (n+1)a - b\).

Now we assume that \(i \geq 2\). If \(i = 2\), then

\((a, b) = \dim \tau^{-2}X = (nc - 1, (n^2 - 1)c - n)\left(\begin{array}{cc} -1 & -n \\ n & n^2 - 1 \end{array}\right).\)
Thus \( a - n^2 = n^3c - 2nc - 2n^2 + 1 = nc(n^2 - 2) - 2(n^2 - 2) - 3 = (nc - 2)(n^2 - 2) - 3 \geq 0 \) since \( c \geq 1 \) and \( n \geq 3 \). Thus the first coordinate of the dimension vector \( \dim \tau^{-1}X \) is greater than \( n^2 \) for every \( i \geq 2 \) by Lemma 3.11. Since \( -q(\dim \tau^{-1}X) = -q((a, b)) = -(c - \frac{n^2}{2}) + \frac{n^2}{4} - 1 \leq \frac{n^2}{4} \), it is sufficient to show for each \( i \geq 2 \) that \( \frac{n^2}{4} < (n + 1) a - b \). If \( \frac{n^2}{4} \geq (n + 1) a - b \), then \( n + 1 \leq \frac{n^2}{4a} + \frac{b}{a} < \frac{n + \sqrt{n^2 - 4}}{2} + \frac{n^2}{4a} < n + \frac{i^2}{4a} < n + 1 \), since \( a \geq n^2 \). This contradiction implies that \( 0 < -q((a, b)) < (n + 1) a - b \).

**Corollary 4.3.** Let \( X \) be an indecomposable module with dimension \( \dim X = (1, c) \) for some \( 1 \leq c \leq n - 1 \) and \( M = \tau^{-1}X \) for some \( i \geq 1 \) with \( \dim M = (a, b) \). Let \( N \) be an indecomposable regular module with dimension vector \( (r, s) \) such that \( |M| = |N| \). If \( \operatorname{Hom}(M, N) = 0 \), then \( s \geq b \).

**Proof.** We have seen in Lemma 3.11 that \( s \geq b - \frac{q(a, b)}{(n + 1)a - b} \). The statement follows since \( 0 < \frac{q(a, b)}{(n + 1)a - b} < 1 \).

**Lemma 4.4.** Let \( M \) be an indecomposable module with \( \dim M = (a, b) \), \( 1 < a \leq b \). Thus \( \dim \tau^{-1}M = (nb - a, (n^2 - 1)b - na) \). Assume that that \( (a - 1, b + 1) \) is not an imaginary root. Then neither is \( (nb - a - t, (n^2 - 1)b - na + t) \) for any \( 1 \leq t \leq nb - a - 1 \).

**Proof.** Let \( t = 1 \). By assumption \( a \leq b \), we have \( \frac{n^2 - 1}{n^2 - 4} \leq \frac{n + \sqrt{n^2 - 4}}{2} \), since \( (a - 1, b + 1) \) is not an imaginary root, and \( 1 < nb - a < (n^2 - 1)b - na \). Thus we need to show that

\[
\frac{(n^2 - 1)b - na + 1}{nb - a - 1} = n - \frac{b - n - 1}{nb - a - 1} \geq n + \frac{\sqrt{n^2 - 4}}{2}.
\]

Therefore, it is sufficient to show that

\[
\frac{b - n - 1}{nb - a - 1} \leq \frac{a - 1}{b + 1}.
\]

Assume for a contradiction that \( \frac{b - n - 1}{nb - a - 1} > \frac{a - 1}{b + 1} \). Then \( b^2 - nb - b - n - 1 \geq nab - a^2 - a - nb + a + 1 \), and thus \( 0 > q(\dim M) = q((a, b)) = b^2 - nab + a^2 \geq 2 + n \), which is impossible. Since \( \frac{(n^2 - 1)b - na + 1}{nb - a - 1} \geq \frac{n + \sqrt{n^2 - 4}}{2} \), for each \( 1 \leq t \leq nb - a - 1 \), we have

\[
\frac{(n^2 - 1)b - na + t}{nb - a - t} \geq \frac{(n^2 - 1)b - na + 1}{nb - a - 1} \geq \frac{n + \sqrt{n^2 - 4}}{2}.
\]

The proof is completed.

**Corollary 4.5.** Let \( X \) be an indecomposable module with dimension \( \dim X = (1, c) \) with \( 1 \leq c \leq n - 1 \) and \( M = \tau^{-1}X \) for \( i \geq 1 \) with \( \dim M = (a, b) \). Then \( (a - t, b + t) \) is not an imaginary root for any \( 1 \leq t \leq a - 1 \).
Proof. By Lemma [4.3] we need only to show for $i = 1$ that $(a - 1, b + 1) = (nc - 2, (n^2 - 1)c - n + 1)$ is not an imaginary root. It is sufficient to show that
\[
\frac{b + 1}{a - 1} = \frac{(n^2 - 1)c - n + 1}{nc - 2} \geq n > \frac{n + \sqrt{n^2 - 4}}{2}.
\]
If \(\frac{(n^2 - 1)c - n + 1}{nc - 2} < n\), then we have \(n \leq c - 1\) which is impossible. \qed

**Theorem 4.6.** Let $X$ be an indecomposable module with dimension $\dim X = (1, c)$ with $1 \leq c \leq n - 1$ and $M$ an indecomposable module. Let $i \geq 0$ and $j \geq 1$. Then $\mu(M) = \mu(\tau^{-i}X[j])$ if and only if $M \cong \tau^{-i}Y[j]$ for some indecomposable module $Y$ with $\dim Y = (1, c) = \dim X$.

**Proof.** We first assume that $j = 1$. By Lemma [4.1(4)], it is sufficient to consider $i \geq 1$. If $M \cong \tau^{-i}Y$ for some indecomposable module $Y$ with $\dim Y = (1, c) = \dim X$, then the GR measures are obvious the same by Lemma [3.5]. Conversely, since $\mu(M) = \mu(\tau^{-i}X)$, a GR submodule of $M$ is preprojective. In particular, $M$ has no proper regular submodules. Because $|M| = |\tau^{-i}X|$, the vector space $\Hom(\tau^{-i}X, M) = 0$ if $M \not\cong \tau^{-i}X$. Let $\dim \tau^{-i}X = (a, b)$ and $\dim M = (r, s)$. Then we have $s \geq b$ by Corollary [4.3] and thus $(r, s) = (a - t, b + t)$ for some $t \geq 0$. However, Corollary [4.5] implies that $(r, s)$ is not an imaginary root if $t \geq 1$. Therefore, $(r, s) = (a, b)$ and thus $(r, s)\Phi^i = (a, b)\Phi^i = (1, c)$. It follows that $M \cong \tau^{-i}Y$ for some $Y$ with dimension vector $\dim Y = (1, c) = \dim X$.

Now we assume that $j > 1$. If $M \cong \tau^{-i}Y[j]$ for some indecomposable module $Y$ with $\dim Y = (1, c) = \dim X$, then $\tau^{-i}X$ and $\tau^{-i}Y$ have the same dimension vector and isomorphic GR submodules for each $i \geq 0$. Thus $\mu(\tau^{-i}Y[j]) = \mu(\tau^{-i}X[j])$ since the irreducible monomorphisms are GR inclusions (Theorem [3.10]). Conversely, if $\mu(M) = \mu(\tau^{-i}X[j])$, then $\mu(M) = \mu(\tau^{-i}X[j - 1]) \cup \{\text{dim}\}$ since $\tau^{-i}X[j - 1]$ is a GR submodule of $\tau^{-i}X[j]$ (Theorem [3.10]). In particular, if $N$ is a GR submodule of $M$, then $\mu(N) = \mu(\tau^{-i}X[j - 1])$. By induction on $i + j$, we have $N \cong \tau^{-i}Y[j - 1]$ for some indecomposable module $Y$ with $\dim Y = (1, c) = \dim X$. It follows that $\dim \tau^{-i}Y[j] = \dim \tau^{-i}X[j]$ and thus $|\tau^{-i}Y[j]| = |\tau^{-i}X[j]| = |M|$. Note that there is an epimorphism $\tau^{-i}Y[j] \rightarrow M$ since $N \cong \tau^{-i}Y[j - 1]$ is a GR submodule of $M$ (Lemma [2.2(3)]). Therefore, $M \cong \tau^{-i}Y[j]$. \qed

This theorem concludes that the GR measures $\mu(\tau^{-i}X[j])$ are determined by the indexes $i$ and $j$ and the dimension vector $(1, c)$. Using this result, we can describe the direct successors of $\mu(\tau^{-i}X[j])$ for all $i \geq 0$ and $j \geq 1$, where $\dim X = (1, c)$.

**Theorem 4.7.** Let $X$ be an indecomposable module with dimension vector $\dim X = (1, c)$ for some $1 \leq c \leq n - 1$. Then for each $i \geq 0$ and $j \geq 1$, $\mu(\tau^{-i}X[j + 1])$ is the direct successor of $\mu(\tau^{-i}X[j])$. 
Proof. For the purpose of a contradiction, we assume that $M$ is an indecomposable module such that $\mu(\tau^{-1}X[j]) < \mu(M) < \mu(\tau^{-1}X[j+1])$. It follows that $\mu(M) = \mu(\tau^{-1}X[j]) \cup \{m_1, m_2, \ldots, m_t\}$ and $m_1 > |\tau^{-1}X[j+1]|$. Let $N \subset N'$ be indecomposable modules in a GR filtration of $M$ with $\mu(N) = \mu(\tau^{-1}X[j])$ and $|N'| = m_1$. Then $N \cong \tau^{-i}Y[j]$ for some indecomposable $Y$ with $\dim Y = \dim X = (1, c)$ by Theorem 4.7. Since $N$ is a GR submodule of $N'$, there is an epimorphism $\tau^{-i}Y[j] \to N'$. It follows that $|N'| = m_1 \leq |\tau^{-i}Y[j+1]| = |\tau^{-1}X[j+1]|$. This is a contradiction. \qed

Recall that a GR-segment is a sequence of Gabriel-Roiter measures, which is closed under direct predecessors and direct successors. We have proved in [9] that a tame quiver has only finitely many GR-segments and conjectured that a wild quiver has infinitely many GR-segments. It was already constructed in [6] for $n$-Kronecker quivers infinitely many GR measures of regular modules, which admit no direct predecessors. Since each GR measure that does not admit a direct predecessor produce a GR segment by taking direct successors, we have already infinitely many GR segments for $n$-Kronecker quivers. Now Theorem 4.7 actually gives a new series of (infinitely many) GR segments.

**Theorem 4.8.** There are infinitely many GR-segments of $n$-Kronecker quiver.

Proof. Fix some $1 \leq c \leq n-1$ and an indecomposable module $X$ with $\dim X = (1, c)$. Let $i \geq 0$. Starting with $\mu(\tau^{-i}X)$, we obtains a sequence of GR measures by taking direct successors

$$\mu(\tau^{-i}X) < \mu(\tau^{-i}X[2]) < \mu(\tau^{-i}X[3]) < \ldots$$

by Theorem 4.7. We may also take direct predecessors of $\mu(\tau^{-i}X)$. It is easily seen that $\mu(\tau^{-j}X[s])$ never appears in this sequence for any $j > i$ and $s \geq 1$. It follows that \({\mu(\tau^{-i}X[r])}_{r \geq 1}\) and \({\mu(\tau^{-j}X[s])}_{s \geq 1}\) are in different GR segments for all $0 < i \neq j$. Thus there are infinitely many GR segments. \qed

5. 3-Kronecker quiver

It was proved in [18] that indecomposable modules in a regular component of the Auslander-Reiten quiver of a wild hereditary algebra are uniquely determined by their dimension vectors. Thus given a regular component of the Auslander-Reiten quiver of a wild quiver, there are only finitely many indecomposable modules with the same length. Since length is an invariant of GR measure, it is interesting to know if these indecomposable modules with the same length have the same GR measure. However, this is not always the case (see, for example, Section 5.3).
Now we consider a fixed 3-Kronecker quiver. This quiver is of special interests because it relates to Fibonacci numbers. Let $C$ be a regular component which contains an indecomposable module with dimension vector $(1, 1)$ or $(1, 2)$. We show that the Gabriel-Roiter measures of the indecomposable modules in $C$ are uniquely determined by their dimension vectors.

5.1. Fibonacci numbers and dimension vectors. We denote by $F_i$ the Fibonacci numbers, which are defined inductively: $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Thus we have the sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots$$

With the help of Fibonacci numbers, we may describe the dimension vectors of indecomposable modules as follows:

**Lemma 5.1.** Let $M$ be a non-projective indecomposable module with dimension vector $(a, b)$.

1. If $\tau^i M$ exists for $i > 0$, then its dimension vector is $(F_{4i+2}a - F_{4i}b, F_{4i}a - F_{4i-2}b)$.
2. If $\tau^{-i} M$ exists for $i > 0$, then its dimension vector is $(F_{4i}b - F_{4i-2}a, F_{4i+2}b - F_{4i}a)$.

**Proof.** We show (1) and (2) follows similarly. We use induction on $i$. This is clear for $i = 1$.

Assume that $\dim \tau^i M = (F_{4i+2}a - F_{4i}b, F_{4i}a - F_{4i-2}b)$. Then

$$\dim \tau^{i+1} M = (8(F_{4i+2}a - F_{4i}b) - 3(F_{4i}a - F_{4i-2}b), 3(F_{4i+2}a - F_{4i}b) - (F_{4i}a - F_{4i-2}b))$$

Since for each $n \geq 2$, $F_{n+2} = 3F_n - F_{n-2}$, i.e., $(F_{n+2}, F_n) = (F_n, F_{n-2}) \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$, we have

$$\begin{pmatrix} F_{n+6} \\ F_{n+4} \end{pmatrix} = (F_{n+4}, F_{n+2}) \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix} = (F_{n+2}, F_n) \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix} = (F_{n+2}, F_n) \begin{pmatrix} 8 & 3 \\ -3 & -1 \end{pmatrix}$$

Therefore, $\dim \tau^{i+1} M = (F_{4(i+1)+2}a - F_{4(i+1)}b, F_{4(i+1)}a - F_{4(i+1)-2}b)$. \qed
5.2. Regular components containing an indecomposable module with dimension vector \((1, 1)\) or \((1, 2)\). First of all, we are able to describe the regular components such that a \(\tau\)-orbit contains two different indecomposable modules with the same length. It turns out that up to a scalar such a component is exactly the one that we have mentioned above. The following result was shown in \([8]\) using Fibonacci numbers:

**Proposition 5.2.** Let \(M\) be an indecomposable regular module such that \(|M| = |\tau^i M|\) for some \(i \geq 1\). Then the \(\tau\)-orbit contains an indecomposable module with dimension vector \((m, m)\) or \((m, 2m)\) for some \(m \geq 1\).

The regular components containing some indecomposable module \(X\) with dimension vector \((1, 1)\) or \((1, 2)\) are of special interests. On one hand, the dimension vectors of the indecomposable modules in such a component strongly relate to pairs of Fibonacci numbers. On the other hand, the indecomposable modules \(\tau^i X\) (resp. \(\tau^{-i} X\)) have no proper regular factors (resp. regular submodules) for any \(i \geq 0\).

In the following, we always denote by \(X\) an indecomposable module with dimension vector \((1, 1)\) in a regular component \(C\). We are going to describe some properties of the dimension vectors of the indecomposable modules in \(C\).

**Remark** All properties to be presented also hold similarly for a regular component containing an indecomposable module with dimension vector \((1, 2)\).

Since indecomposable modules in \(C\) are uniquely determined by their dimension vectors, we use the dimension vectors to denote the indecomposable modules. The following is a part of the regular component \(C\):

\[
\begin{array}{c}
\begin{pmatrix}
275 & 110 \\
273 & 105 \\
272 & 104 \\
34 & 13 \\
\end{pmatrix} & \begin{pmatrix}
55 & 55 \\
42 & 21 \\
40 & 16 \\
5 & 2 \\
\end{pmatrix} & \begin{pmatrix}
110 & 275 \\
21 & 42 \\
16 & 40 \\
2 & 5 \\
\end{pmatrix} & \begin{pmatrix}
272 & 104 \\
39 & 15 \\
36 & 6 \\
13 & 34 \\
\end{pmatrix} \\
\end{array}
\]

**Lemma 5.3.** Let \(M\) be an indecomposable module.

1. If \(\dim M = (m, m), m \geq 1\), then \(\dim \tau^i M = (mF_{4i+1}, mF_{4i-1})\) and \(\dim \tau^{-i} M = (mF_{4i-1}, mF_{4i+1})\), for each \(i > 0\).
(2) If $\dim M = (m, 2m)$, $m \geq 1$, then $\dim \tau^{i+1}M = (mF_{4i+3}, mF_{4i+1})$ and $\dim \tau^{-i}M = (mF_{4i+1}, mF_{4i+3})$ for each $i \geq 0$.

Proof. These are direct consequences of Lemma 5.1

We define inductively a sequence of indecomposable modules in $C$. Let $X_1 = X$. Assume that $X_n$ is already defined. If $n$ is odd, then $X_{n+1}$ is the unique indecomposable module with an irreducible epimorphism $X_{n+1} \to X_n$; if $n$ is even, then $X_{n+1}$ is the unique indecomposable module with an irreducible monomorphism $X_n \to X_{n+1}$. Thus

$\begin{align*}
X_1 &= \begin{pmatrix} 1 & 1 \end{pmatrix} \\
X_2 &= \begin{pmatrix} 6 & 3 \end{pmatrix} \\
X_3 &= \begin{pmatrix} 8 & 8 \end{pmatrix} \\
X_4 &= \begin{pmatrix} 42 & 21 \end{pmatrix} \\
X_5 &= \begin{pmatrix} 55 & 55 \end{pmatrix} \\
\vdots
\end{align*}$

Note that the quasi-length of $X_n$ is $\text{ql} (M_n) = n$.

**Lemma 5.4.** The dimension vector of $X_n$ is

$$\dim X_n = \begin{cases} 
F_{2n}(1, 1), & n \text{ is odd;} \\
F_{2n}(2, 1), & n \text{ is even.}
\end{cases}$$

Proof. It is not difficult to see that for each $n \geq 2$ the dimension vector of $X_n$ is the following:

$$\dim X_n = \begin{cases} 
\sum_{i=1}^{n-1} \dim \tau^iX + \sum_{i=1}^{n-1} \dim \tau^{-i}X + (1, 1), & n \text{ is odd;} \\
\sum_{i=1}^{\frac{n}{2}} \dim \tau^iX + \sum_{i=1}^{\frac{n}{2}-1} \dim \tau^{-i}X + (1, 1), & n \text{ is even.}
\end{cases}$$

Thus if $n$ is odd, then

$$\dim X_n = (\sum_{i=1}^{n-1} F_{4i+1}, \sum_{i=1}^{n-1} F_{4i-1}) + (\sum_{i=1}^{n-1} F_{4i-1}, \sum_{i=1}^{n-1} F_{4i+1}) + (1, 1)$$

$$= (\sum_{i=1}^{n} F_{2i+1}, \sum_{i=1}^{n} F_{2i-1})$$

It follows similarly for $n$ even.

**Corollary 5.5.** Let $M$ be an indecomposable module in $C$ with quasi-length $n$. If $n$ is odd, then the dimension vector of $M$ is $F_{2n}(F_{4i+1}, F_{4i-1})$, $F_{2n}(1, 1)$ or $F_{2n}(F_{4i-1}, F_{4i+1})$. If $n$ is even, then $\dim M = F_{2n}(F_{4i+3}, F_{4i+1})$ or $F_{2n}(F_{4i+1}, F_{4i+3})$.

Proof. Since the quasi-length $\text{ql} (M) = n$, $M$ and $X_n$ defined above are in the same $\tau$-orbit. Thus $M \cong \tau^iX_n$ for some integer $i \in \mathbb{Z}$.

□
5.3. **Indecomposable modules with the same length.** Now we will show that in the regular component $C$, two indecomposable modules with the same length are in the same $\tau$-orbit. Thus we may describe their dimension vectors using the properties we have seen before. For an indecomposable regular module $M$ with quasi-length $\text{ql}(M) = n$, we denote by $\text{qs}(M)$ the unique quasi-simple module $X$ such that $M = X[n]$ and by $\text{qt}(M)$ the unique quasi-simple module $Y$ such that $M = [n]Y$.

**Lemma 5.6.** Let $M$ and $N$ be two indecomposable modules in $C$ with $|M| = |N|$. Then $\text{ql}(M) = \text{ql}(N)$. Thus either there is an indecomposable module $U$ with dimension vector $F_{2n}(1,1)$ such that $M \cong \tau iU$ and $N \cong \tau^{-i}U$ for some $i$, or there is an indecomposable module $V$ with dimension vectors $F_{2n}(2,1)$ and $\dim \tau^{-1}V = F_{2n}(1,2)$ such that $M \cong \tau^iV$ and $N \cong \tau^{-i}(\tau^{-1}V)$ for some $i$, where $n = \text{ql}(M) = \text{ql}(N)$.

**Proof.** The proof depends on a detailed calculation of the dimension vectors. Let $\text{qs}(M) = M_1$, $\text{qt}(M) = M_2$ and $M_i = \tau^{m_i}X$. Similarly, let $\text{qs}(N) = N_1$, $\text{qt}(N) = N_2$ and $N_i = \tau^{n_i}X$. Without loss of generality, we may assume that $m_2 \geq n_2$. It is obvious that $M \cong N$, provided the equality holds.

We first assume that $m_2 > n_2 \geq 0$. Then

$$\dim N \leq \sum_{i=0}^{n_1} \dim \tau^iX < \dim \tau^{n_1+1}X$$

by Lemma 3.3. It follows that $m_1 \leq n_1$. But this implies $\dim M < \dim N$, a contradiction.

Now we assume that $m_2 \geq 0 > n_2$. Obviously, we have $m_1 \geq |n_2|$. If $n_1 \leq 0$, then $m_1 = |n_2|$. Otherwise, $m_1 > |n_2|$ and

$$\dim N \leq \sum_{i=0}^{n_2} \dim \tau^{-i}X < \dim \tau^{-|n_2|+1}X \leq \dim \tau^{-m_1}X,$$

and thus $|N| < |\tau^{-m_1}X| = |\tau^{m_1}X| \leq |M|$, a contradiction. Since $m_1 = |n_2|$, we have $m_2 = |n_1|$ and thus $\text{ql}(M) = \text{ql}(N)$. If $n_1 > 0$, we have two possibilities $n_1 < m_2$ and $n_1 \geq m_2$. In the first case, we have $|n_2| = m_1$. Otherwise, $|n_2| < m_1$ and thus $\sum_{i=0}^{n_2} \dim \tau^{-i}X < \dim \tau^{-m_1}X$ and $\sum_{i=1}^{n_1} \dim \tau^iX < \dim \tau^{m_2}X$. It follows that $|N| < |M|$, which is a contradiction. (Note that here we need $m_1 \neq m_2$, i.e., $M$ is not quasi-simple. If $M$ is quasi-simple, we can discuss similarly.) In the second case, we have $|\sum_{i=n_1+1}^{m_1} \tau^iX| = |\sum_{i=-n_2}^{m_2-1} \dim \tau^iX|$. Then the discussion for the first case applies. The other possibilities follow similarly. The proof of the other statements is straightforward. \qed
Let us denote indecomposable modules in \( C \) by their dimension vectors. Given an odd number \( n \geq 1 \). There is a short exact sequence

\[
0 \to F_{2n}(1, 1) \xrightarrow{f} F_{2n+2}(1, 2) \to \tau^{-\frac{n+1}{2}}(1, 1) \to 0.
\]

Thus we have short exact sequences

\[
0 \to \tau^{-i}F_{2n}(1, 1) \xrightarrow{f_i} \tau^{-i}F_{2n+2}(1, 2) \to \tau^{-(i+\frac{n+1}{2})}(1, 1) \to 0
\]

where \( f_i \) are irreducible monomorphisms.

**Lemma 5.7.** Let \( n \geq 1 \) be odd. Then \( \dim \tau^{-i}F_{2n}(1, 1) < \dim \tau^{-(i+\frac{n+1}{2})}(1, 1) \) for each \( i \geq 0 \). Therefore, \( \tau^{-i}F_{2n}(1, 1) \xrightarrow{f_i} \tau^{-i}F_{2n+2}(1, 2) \) is a GR inclusion.

**Proof.** If \( i = 0 \),

\[
\dim \tau^{-\frac{n+1}{2}} X = (F_{2n+1}, F_{2n+3}) > (F_{2n}, F_{2n}).
\]

Now assume that \( i \geq 1 \). Then we need to show

\[ (F_{2n}F_{4i-1}, F_{2n}F_{4i+1}) < (F_{4i(n+\frac{1}{2})}, F_{4i(n+\frac{1}{2})+1}). \]

Since \( F_iF_s + F_{s-1}F_s = F_{s+1} \), we get \( F_{2n}F_{4i-1} < F_{2n+4i} < F_{2n+4i+1} \) and \( F_{2n}F_{4i+1} < F_{2n+4i+3} \). The second statement follows by Proposition 3.9. \( \square \)

Similarly, let \( n \geq 2 \) be an even number. Then there is a short exact sequence

\[
0 \to F_{2n}(1, 2) \xrightarrow{f_i} \tau^{-1}F_{2n+2}(1, 1) \to \tau^{-\left(\frac{n}{2}+1\right)}(1, 1) \to 0.
\]

Thus we have short exact sequences

\[
0 \to \tau^{-i}F_{2n}(1, 2) \xrightarrow{f_i} \tau^{-i}F_{2n+2}(1, 1) \to \tau^{-\left(i+\frac{n}{2}+1\right)}(1, 1) \to 0
\]

where \( f_i \) are irreducible monomorphisms. As above, the following result can be easily shown:

**Lemma 5.8.** Let \( n \geq 2 \) be even. Then \( \dim \tau^{-i}F_{2n}(1, 2) < \dim \tau^{-\left(i+\frac{n}{2}+1\right)}(1, 1) \) for each \( i \geq 0 \). Therefore, \( \tau^{-i}F_{2n}(1, 2) \xrightarrow{f_i} \tau^{-\left(i+\frac{n}{2}+1\right)}F_{2n+2}(1, 1) \) is a GR inclusion.

**Theorem 5.9.** Let \( X \) be an indecomposable module with dimension vector \((1, 1)\) and \( C \) a regular component containing \( X \). Then the GR measures of the indecomposable modules in \( C \) are uniquely determined by their dimension vectors.

**Proof.** By previous discussion, it is sufficient to consider the following cases:

1. Since for an odd number \( n \geq 1 \) and each \( i \geq 0 \), \( \tau^{-i}F_{2n}(1, 1) \xrightarrow{f_i} \tau^{-i}F_{2n+2}(1, 2) \) is a GR inclusion (Lemma 5.7), we need to show that the length of a GR submodule of \( \tau^iF_{2n+2}(2, 1) \) does not equal to \( |\tau^{-i}F_{2n}(1, 1)| \).
(2) Since for an even number \( n \geq 2 \) and each \( i \geq 0 \), \( \tau^{-i}F_{2n}(1, 2) \xrightarrow{f_i} \tau^{-(i+1)}F_{2n+2}(1, 1) \) is a GR inclusion (Lemma 5.8), we need to show that the length of a GR submodule of \( \tau^{i+1}F_{2n+2}(1, 1) \) does not equal to \( |\tau^{-i}F_{2n}(1, 2)| \).

We show (1) and (2) follows similarly. A GR submodule \( Y \) of \( \tau^{i}F_{2n+2}(2, 1) \) is obviously a regular module. Assume that \( \dim Y = (a, b) \). Then \( \dim \tau^{-1}Y = (3b - a, 8b - 3a) \). Let \( M \) be the unique indecomposable module with an irreducible monomorphism \( Y \rightarrow M \). Then there is an epimorphism \( M \rightarrow \tau^{i}F_{2n+2}(2, 1) \) (Lemma 2.2(3)). Note that \( \dim M \leq (3b, 9b - 3a) \).

Assume that \( |Y| = |\tau^{-1}F_{2n}(1, 1)| \). Then we have

\[
a + b = F_{2n}(F_{4i-1} + F_{4i+1}) \quad \text{and} \quad 3b \geq F_{2n+2}F_{4i+3}.
\]

The second inequality follows because \( \dim \tau^{i}F_{2n+2}(2, 1) = F_{2n+2}(F_{4i+3}, F_{4i+1}) \). Therefore,

\[
a + b \leq \frac{F_{2n}(F_{4i-1} + F_{4i+1})}{F_{2n+2}F_{4i+3}}
\]

and thus

\[
a \leq \frac{3F_{2n} (F_{4i-1} + F_{4i+1})}{F_{4i+3}} - 1.
\]

For the purpose of a contradiction, we show that the right hand side is smaller than \( \frac{3 - \sqrt{5}}{2} = 2 - \varphi \) where \( \varphi = \frac{1 + \sqrt{5}}{2} \) is the golden ratio. If this is the case, then \( (a, b) \) is not a root. Thus there does not exist an indecomposable module with dimension vector \( (a, b) \) and we obtain a contradiction.

We simply write \( A = \frac{(F_{4i-1}+F_{4i+1})}{F_{4i+3}} \). It is sufficient to show

\[
\frac{F_{2n}}{F_{2n+2}} A < \frac{3 - \varphi}{3}.
\]

Using \( F_m = \frac{\varphi^m - (1-\varphi)^m}{\sqrt{5}} \), we may easily obtain that \( \varphi F_m = F_{m+1} - (1 - \varphi)^m \). Thus

\[
(1 + \varphi)F_m = \varphi^2 F_m = \varphi F_{m+1} - (1 - \varphi)^m \varphi = F_{m+2} - (1 - \varphi)^{m+1} - (1 - \varphi)^m \varphi = F_{m+2} - (1 - \varphi)^m (1 - \varphi + \varphi) = F_{m+2} - (1 - \varphi)^m.
\]

Replacing \( m \) by \( 2n \), we get \( \frac{F_{2n}}{F_{2n+2}} < \frac{F_{2n}}{(1+\varphi)F_{2n}} = \frac{1}{1 + \varphi} \). Thus it is sufficient to show that

\[
\frac{1}{(1 + \varphi)} A < \frac{3 - \varphi}{3}.
\]
Note that \((3-\varphi)(1+\varphi) = \frac{2+\varphi}{3} > 1\). However,
\[
A = \frac{(F_{4i-1} + F_{4i+1})}{F_{4i+3}} < \frac{(F_{4i+2} + F_{4i+1})}{F_{4i+3}} = 1.
\]

The proof is finished. \(\Box\)

**Remark** The theorem can be generalized for regular components over \(n\)-Kronecker quivers, which contains an indecomposable module with dimension vector \((1, 1)\) or \((1, n - 1)\).

### 5.4. A counter example.

In the following example, we will see that non-isomorphic indecomposable modules in a regular component may have the same GR measure for some wild quiver.

**Example** Let \(k\) be an algebraically closed field and \(Q = (Q_0, Q_1)\) be a tame quiver of type \(\tilde{A}_n\) with \(n \geq 3\) an odd number and with sink-source orientation, i.e., a vertex in \(Q_0\) is either a sink or a source. Without loss of generality, we may certainly assume that the vertices in \(Q_0\) are labeled by \(\{a_1, a_2, \ldots, a_{n+1}\}\) and there is an arrow \(a_1 \to a_2\). This means that \(a_1\) is a source. Let \(\overline{Q}\) be the one point extension of \(Q\) with respect to the indecomposable projective module \(P_{a_1}\). More precisely, \(\overline{Q}_0 = Q_0 \cup \{a_0\}\) and \(\overline{Q}_1 = Q_1 \cup \{a_0 \to a_1\}\). For example, if \(n = 3\), then \(\overline{Q}\) is the following:

\[
\begin{array}{c}
a_0 \\
\downarrow \\
a_1 \\
\downarrow \\
a_2 \\
\downarrow \\
a_3 \\
\downarrow \\
\downarrow \\
a_4 \\
\end{array}
\]

We know from the structure of the Auslander-Reiten quiver of \(Q\) that there are two exceptional regular tubes, each of which contains precisely \(\frac{n+1}{2}\) non-isomorphic indecomposable modules of length 2 as quasi-simple modules. Let \(M\) be one of those with dimension vector \((\dim M)_{a_i} = \begin{cases} 1, & i = 1, 2; \\ 0, & \text{otherwise.} \end{cases}\)

Then as \(kQ\)-modules, \(M, \tau_Q M, \ldots, \tau_Q^{ \frac{n+1}{2}} M\) are pairwise non-isomorphic quasi-simple regular modules in a regular tube. It is not difficult to see that \(\tau_Q^i M = \tau_Q^{i-1} M\) for every \(0 \leq i \leq \frac{n-1}{2}\) and \(\tau_Q^{\frac{n+1}{2}} M\) is an indecomposable module with length 3. Thus \(\tau_Q^i M\) are quasi-simple regular modules with length 2 for all \(0 \leq i \leq \frac{n-1}{2}\). Obviously, they all have the same GR measure \(\{1, 2\}\).
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