CLASSIFICATION OF INDECOMPOSABLE INTEGRAL FLOWS ON SIGNED GRAPHS

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Abstract. An indecomposable flow $f$ on a signed graph $\Sigma$ is a nontrivial integral flow that cannot be decomposed into $f = f_1 + f_2$, where $f_1, f_2$ are nontrivial integral flows having the same sign (both $\geq 0$ or both $\leq 0$) at each edge. This paper is to classify indecomposable flows into characteristic vectors of Eulerian cycle-trees — a class of signed graphs having a kind of tree structure in which all cycles can be viewed as vertices of a tree. Moreover, each indecomposable flow other than circuit flows can be further decomposed into a sum of certain half circuit flows having the same sign at each edge. The variety of indecomposable flows of signed graphs is much richer than that of ordinary unsigned graphs.

1. Introduction

A signed graph is an ordinary graph whose each edge is endowed with either a positive sign or a negative sign. The system was formally introduced by Harary [8] who characterized balanced signed graphs up to switching, and was much developed by Zaslavsky [13, 14] who successfully extended most important notions of ordinary graphs to signed graphs, such as circuit, bond, orientation, incidence matrix, Laplacian, and associated matroids, etc. Based on Zaslavsky’s work, Chen and Wang [5] introduced flow and tension lattices of signed graphs and obtained fundamental properties on flows and tensions, including a few characterizations of cuts and bonds.

Now it is natural to ask, inside the flow and tension lattices, how integral flows and tensions are build up from more basic integral flows and tensions. More specifically, what does an integral flow or tension look like if it cannot be decomposed at integer scale but can be possibly decomposed further at fractional scale? The answer is not only interesting but fundamental in nature because if one considers circuit flows to be at atomic level then indecomposable flows are at molecular level.

For ordinary graphs it is easy to see that indecomposable flows are simply the graph circuit flows at integer scale. For signed graphs, however, we shall see that indecomposable flows are much richer than that of unsigned graphs because in addition to circuit flows, the fixed spin (signs on edges) produces a new class of characteristic vectors of so-called directed Eulerian cycle-trees, which are not decomposable at integer scale while decomposable (into signed graph circuit flows) at half-integer scale. The present paper is to present the solution of such a new phenomenon on signed graphs.

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Let $\Sigma = (V, E, \sigma)$ be a signed graph throughout, where $(V, E)$ is an ordinary finite graph with possible loops and multiple edges, $V$ is the vertex set, $E$ is the edge set, and $\sigma : E \to \{-1, 1\}$ is the sign function. Each edge subset $S \subseteq E$ induces a signed subgraph $\Sigma(S) := (V, S, \sigma_S)$, where $\sigma_S$ is the restriction of $\sigma$ to $S$. A cycle of $\Sigma$ is a simple closed path. The sign of a cycle is the product of signs on its edges. A cycle is said to be balanced (unbalanced) if its sign is positive (negative). A signed graph is said to be balanced if all cycles are balanced, and unbalanced if one of its cycles is unbalanced. A connected component of $\Sigma$ is called a balanced (unbalanced) component if it is balanced (unbalanced) as a signed subgraph.

An orientation of a signed graph $\Sigma$ is an assignment that each edge $e$ is assigned two arrows at its end-vertices $u, v$ as follows: (i) if $e$ is a positive edge, the two arrows are in the same direction; (ii) if $e$ is a negative edge, the two arrows are in opposite directions; see Figure 1. We may think of an arrow on an edge $e$ at its one end-vertex $v$ as $+1$ if the arrow points away from $v$ and $-1$ if the arrow points toward $v$. Then there are both $+1$ and $-1$ for a positive loop at its unique end-vertex, and two $+1$’s or two $-1$’s for a negative loop at its unique end-vertex. So an orientation on $\Sigma$ can be considered as a multi-valued function $\varepsilon : V \times E \to \{-1, 0, 1\}$ such that

(i) $\varepsilon(v, e)$ has two values $+1$ and $-1$ if $e$ is a positive loop at its unique end-vertex $v$, and is single-valued otherwise;
(ii) $\varepsilon(v, e) = 0$ if $v$ is not an end-vertex of the edge $e$; and
(iii) $\varepsilon(u, e)\varepsilon(v, e) = -\sigma(e), e = uv$.

A signed graph $\Sigma$ with an orientation $\varepsilon$ is called an oriented signed graph $(\Sigma, \varepsilon)$. We assume that $(\Sigma, \varepsilon)$ is an oriented signed graph throughout the whole paper.

Let $\varepsilon_i$ ($i = 1, 2$) be orientations on signed subgraphs $\Sigma_i$ of $\Sigma$. The coupling of $\varepsilon_1, \varepsilon_2$ is a function $[\varepsilon_1, \varepsilon_2] : E \to \mathbb{Z}$, defined for $e = uv$ by

$$
[\varepsilon_1, \varepsilon_2](e) = \begin{cases} 
1 & \text{if } e \in \Sigma_1 \cap \Sigma_2, \varepsilon_1(v, e) = \varepsilon_2(v, e), \\
-1 & \text{if } e \in \Sigma_1 \cap \Sigma_2, \varepsilon_1(v, e) \neq \varepsilon_2(v, e), \\
0 & \text{otherwise}.
\end{cases} \quad (1.1)
$$

In other words, $[\varepsilon_1, \varepsilon_2](e) = \varepsilon_1(v, e)\varepsilon_2(v, e)$ if $e = uv$.

Let $A$ be an abelian group and be assumed automatically a $\mathbb{Z}$-module. For each edge $e$ and its end-vertices $u, v$, let $\text{End}(e)$ denote the multiset $\{u, v\}$. Associated with $(\Sigma, \varepsilon)$ is the boundary operator $\partial : A^E \to A^V$ defined for $f \in A^E$ and $v \in V$ by

$$(\partial f)(v) = \sum_{e \in E} m_{v,e}f(e) = \sum_{e \in E, u \in \text{End}(e), u = v} \varepsilon(u, e)f(e), \quad (1.2)$$

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{orientation.png}
\caption{Orientations on loops and non-loop edges.}
\end{figure}
where \( m_{v,e} = \varepsilon(v, e) \) if \( e \) is a non-loop, \( m_{v,e} = 2\varepsilon(v, e) \) if \( e \) is a negative loop, and \( m_{v,e} = 0 \) otherwise. A function \( f : E \to A \) is said to be conservative at a vertex \( v \) if \( (\partial f)(v) = 0 \), and is said to be an A-flow of \((\Sigma, \varepsilon)\) if it is conservative at each vertex of \( \Sigma \). The support of a flow \( f \) is the edge subset

\[
\text{supp } f = \{ e \in E \mid f(e) \neq 0 \}. \tag{1.3}
\]

The set of all A-flows forms an abelian group, called the flow group of \((\Sigma, \varepsilon)\) with values in \( A \), denoted \( F(\Sigma, \varepsilon; A) \). We call \( F(\Sigma, \varepsilon) := F(\Sigma, \varepsilon; \mathbb{R}) \) the flow space, and \( Z(\Sigma, \varepsilon) := F(\Sigma, \varepsilon; \mathbb{Z}) \) the flow lattice of \((\Sigma, \varepsilon)\). For further information about flows of signed graphs, see [1, 4, 5, 6, 9]. For notions of ordinary graphs, we refer to the books [2, 3, 7].

A flow is said to be nontrivial if its support is nonempty. A nontrivial integral flow \( f \) is said to be decomposable if \( f \) can be written as

\[
f = f_1 + f_2,
\]

where \( f_i \) are nontrivial integral flows having the same sign (both nonnegative or both nonpositive) at every edge, that is, \( f_1(e)f_2(e) \geq 0 \) for all \( e \in E \). Nontrivial integral flows that are not decomposable are called indecomposable flows. An integral flow \( f \) is said to be elementary if it is indecomposable and there is no nontrivial integral flow \( g \) such that \( \text{supp } g \) is properly contained in \( \text{supp } f \). Compare with Tutte’s definition of elementary chains [11, 12].

Let \( W \) be a walk of length \( n \) in \( \Sigma \) and be written as a vertex-edge sequence

\[
W = u_0x_1u_1x_2\ldots u_{n-1}x_nu_n, \tag{1.4}
\]

where each \( x_i \) is an edge with end-vertices \( u_{i-1}, u_i \). The walk \( W \) is said to be closed if the initial vertex \( u_0 \) is the same as the terminal vertex \( u_n \). The sign of \( W \) is the product

\[
\sigma(W) = \prod_{i=1}^{n} \sigma(x_i). \tag{1.5}
\]

The support of \( W \) is the set \( \text{supp } W \) of edges \( x_i \) \((i = 1, \ldots, n)\) with no repetition. We may think of \( W \) as the multiset \( \{ x_1, x_2, \ldots, x_n \} \) (with repetition allowed) of \( n \) edges on \( \text{supp } W \).

A direction of \( W \) is a function \( \varepsilon_W \) with values either 1 or \(-1\), defined for all vertex-edge pairs \((u_{i-1}, x_i)\) and \((u_i, x_i)\), such that

\[
\varepsilon_W(u_{i-1}, x_i)\varepsilon_W(u_i, x_i) = -\sigma(x_i), \quad \varepsilon_W(u_i, x_i) + \varepsilon_W(u_i, x_{i+1}) = 0. \tag{1.6}
\]

Every walk has exactly two opposite directions. A walk \( W \) with a direction \( \varepsilon_W \) is called a directed walk, denoted \((W, \varepsilon_W)\), and is further called a directed closed walk if \( u_0 = u_n \) and

\[
\varepsilon_W(u_0, x_1) + \varepsilon_W(u_n, x_n) = 0.
\]

A directed walk \((W, \varepsilon_W)\) is said to be midway-back avoided, provided that if \( u_\alpha = u_\beta \) with \( 0 \leq \alpha < \beta < n \) in (1.3) then

\[
\varepsilon_W(u_\beta, x_\beta) = \varepsilon_W(u_\alpha, x_{\alpha+1}). \tag{1.7}
\]

Figure 2 demonstrates four possible orientation patterns at a double vertex in a directed midway-back avoided walk.

An Eulerian walk is a balanced closed walk whose directions have the same orientation on repeated edges of each fixed edge. A midway-back avoided walk
with positive sign is necessarily a directed Eulerian walk and has no triple vertices. An Eulerian walk with a direction is called a directed Eulerian walk.

An Eulerian walk $W$ is said to be minimal if there is no Eulerian walk $W'$ such that $W'$ is contained properly in $W$ as edge multisets, and said to be elementary if it is minimal and there is no minimal Eulerian walk $W'$ such that $\text{supp} W'$ is properly contained in $\text{supp} W$ as edge subsets. A minimal Eulerian walk with a direction is called a minimal directed Eulerian walk.

Let $(W, \varepsilon_W)$ be a directed closed walk of length $n$, where $W$ is given in (1.4). The characteristic vector of $(W, \varepsilon_W)$ on $(\Sigma, \varepsilon)$ is a function $f(W, \varepsilon_W) : E \rightarrow \mathbb{Z}$ defined by

$$f(W, \varepsilon_W)(x) = \sum_{x_i \in W, x_i = x} [\varepsilon, \varepsilon_W](x_i). \quad (1.8)$$

By Lemma 2.1, $f(W, \varepsilon_W)$ is an integral flow on $(\Sigma, \varepsilon)$. Whenever $\varepsilon_W = \varepsilon$ on $W$, we simply write $f(W, \varepsilon_W)$ as $f_W$.

Given a real-valued function $f$ on $E$. Let $\varepsilon_f$ be the orientation on $\Sigma$ defined by

$$\varepsilon_f(u, x) = \begin{cases} -\varepsilon(u, x) & \text{if } f(\varepsilon) < 0, \ x = uv, \\ \varepsilon(u, x) & \text{otherwise}. \end{cases} \quad (1.9)$$

It is trivial that $f$ is a flow on $(\Sigma, \varepsilon)$ if and only if the absolute value function $|f|$ is a flow on $(\Sigma, \varepsilon_f)$. Moreover, $|f| = [\varepsilon, \varepsilon_f] \cdot f$.

A cycle-tree of $\Sigma$ is a connected signed subgraph $T$ which can be decomposed into edge-disjoint cycles $C_i$ (called block cycles) and vertex-disjoint simple paths $P_j$ (called block paths), denoted $T = \{ C_i, P_j \}$, satisfying the four conditions:

(i) $\{ C_i \}$ is the collection of all cycles in $T$.

(ii) The intersection of two cycles is either empty or a single vertex (called an intersection vertex).

(iii) Each $P_j$ intersects exactly two cycles and the intersections are exactly the initial and terminal vertices of $P_j$ (also called intersection vertices).

(iv) Each intersection vertex is a cut-point (a vertex whose removal increases the number of connected components of the underline graph as a topological space of 1-dimensional CW complex), also known as a separating vertex [3, p.119].

A cycle-tree is said to be Eulerian if it further satisfies

(v) Parity Condition: Each balanced cycle has even number of intersection vertices, while each unbalanced cycle has odd number of intersection vertices.

We call a block cycle in a cycle-tree to be an end-block cycle if it has exactly one intersection vertex. The name cycle-tree is justified as follows: if one converts each block cycle $C_i$ into a vertex, each common intersection vertex of two block cycles into an edge adjacent with the two vertices converted from the two block cycles,
and keep each block path $P_j$ connecting two vertices converted from the two block cycles connected by $P_j$, then the graph so obtained is indeed a tree.

An orientation $\varepsilon_T$ on a cycle-tree $T$ is called a direction if $(T, \varepsilon_T)$ has neither sink nor source, and for each block cycle $C$, the restriction $(C, \varepsilon_T)$ has either sink or source at each cut-point of $T$ on $C$. We shall see that $T$ admits a direction if and only if $T$ satisfies the Parity Condition, and the direction is unique up to opposite sign. An Eulerian cycle-tree $T$ with a direction $\varepsilon_T$ is called a directed Eulerian cycle-tree $(T, \varepsilon_T)$. For instance, the oriented signed graph given in Figure 3 is an Eulerian cycle-tree with a direction.

![Figure 3. An Eulerian cycle-tree and its direction.](image-url)

Let $T = \{C_i, P_j\}$ be an Eulerian cycle-tree of $\Sigma$. The indicator of $T$ is a function $I_T : E \to \mathbb{Z}$ defined by

$$I_T(e) = \begin{cases} 1 & \text{if } e \text{ is on a block cycle of } T, \\ 2 & \text{if } e \text{ is on a block path of } T, \\ 0 & \text{otherwise.} \end{cases} \quad (1.10)$$

Given a direction $\varepsilon_T$ of $T$. Viewing both $(\Sigma, \varepsilon)$ and $(T, \varepsilon_T)$ as oriented signed subgraphs of $\Sigma$, we have the coupling $[\varepsilon, \varepsilon_T]$. The product function $[\varepsilon, \varepsilon_T] : I_T$ determines a vector in $\mathbb{Z}^E$ and is an integral flow of $(\Sigma, \varepsilon)$ by Theorem 3.5, called the characteristic vector of the directed Eulerian cycle-tree $(T, \varepsilon_T)$ for $(\Sigma, \varepsilon)$.

An Eulerian cycle-tree is called a (signed graph) circuit if it does not contain properly any Eulerian cycle-tree. We shall see that each circuit $C$ must be one of the following three types.

- **Type I**: $C$ consists of a single balanced cycle.
- **Type II**: $C$ consists of two edge-disjoint unbalanced cycles $C_1, C_2$ having a single common vertex, written $C = C_1C_2$.
- **Type III**: $C$ consists of two vertex-disjoint unbalanced cycles $C_1, C_2$, and a simple path $P$ of positive length, such that $C_1 \cap P$ is the initial vertex and $C_2 \cap P$ the terminal vertex of $P$, written $C = C_1PC_2$.

The present definition of circuit seems different from that defined in [13] and that adopted in [5, 6], but they are equivalent. The following characterization of signed graph circuits shows the motivation of the concept.

**Characterization of Signed Graph Circuits.** Let $f$ be a nontrivial integral flow of $(\Sigma, \varepsilon)$. Then the following statements are equivalent.

(a) $f$ is elementary.
(b) $f$ is the characteristic vector of a directed circuit.
(c) There exists an elementary directed Eulerian walk \((W, \varepsilon_W)\) such that 
\[
 f = f_{(W, \varepsilon_W)}. 
\]

**Remark.** The characterization of signed graph circuits was obtained by Bouchet \([4, p.283]\) (Corollary 2.3), using Zaslavsky’s definition of circuits \([13]\). As Zaslavsky pointed out himself, the central observation of \([13, p.53]\) is the existence of a matroid over the edge set of a signed graph whose circuits are exactly those of Types I, II, III. Bouchet \([4]\) assumed (without argument) that Zaslavsky’s matroid is the same as the matroid whose circuits are the supports of elementary flows. Indeed, it is trivial to see that the circuits of the former are the circuits of the latter. However, the converse seems not so obvious that no need argument, though it is anticipated. Corollary 3.7 implies that the converse is indeed true. Now it is logically clear and aesthetically complete that the matroid constructed by Zaslavsky \([13]\) for a signed graph is the same matroid whose circuits are the supports of elementary chains (= elementary flows) of the signed graph in the sense of Tutte \([12]\); so are their dual matroids.

**Main Theorem.** (Classification of Indecomposable Integral Flows). Let \(f\) be an integral flow of an oriented signed graph \((\Sigma, \varepsilon)\).

(a) Then \(f\) is indecomposable if and only if there exists an Eulerian cycle-tree \(T\) such that 
\[
 f = [\varepsilon, \varepsilon f] \cdot I_T. 
\]

(b) If \(T\) is an Eulerian cycle-tree other than a circuit, then for each closed walk \(W\) of minimum length that uses all edges of \(T\), there is a decomposition 
\[
 W = C_0P_1C_1 \cdots P_kC_kP_{k+1}, \quad k \geq 1, 
\]
where \(C_i\) are entire end-block cycles of \(T\) and \(C_iP_{i+1}C_{i+1}\) are circuits of Type III with \(C_{k+1} = C_0\), such that 
\[
 I_T = \frac{1}{2} \sum_{i=0}^{k} I_{\Sigma(C_iP_{i+1}C_{i+1})}. 
\]

## 2. Flow Reduction Algorithm

The decomposability of an integral flow \(f\) on \((\Sigma, \varepsilon)\) is equivalent to the decomposability of the flow \(|f|\) on \((\Sigma, \varepsilon_f)\). So without loss of generality, to decompose an integral flow, one only needs to consider nonnegative nontrivial integral flows of \((\Sigma, \varepsilon)\). The following Flow Reduction Algorithm (FRA) finds explicitly a minimal directed Eulerian walk from a given nontrivial integral flow.

Let us first show that the characteristic vector of a directed closed walk is an integral flow.

**Lemma 2.1.** Let \((W, \varepsilon_W)\) be a directed closed walk. Then the function \(f_{(W, \varepsilon_W)}\) defined by \([12, 18]\) is an integral flow of \((\Sigma, \varepsilon)\).

**Proof.** Let \(W = u_0x_1u_1x_2 \ldots x_nu_n\) be the vertex-edge sequence, where each edge \(x_i\) has end-vertices \(u_{i-1}, u_i\), and \(u_0 = u_n\). Fix a vertex \(v \in V\), let \(u_{a_1}, u_{a_2}, \ldots, u_{a_k}\)
be the sequence that $v$ appears in $W$. Since $\varepsilon_W(u_a, x_{a_j}) + \varepsilon_W(u_a, x_{a_j+1}) = 0$, we have

$$\partial f(W, \varepsilon_W)(v) = \sum_{x \in E, u \in \text{End}(x), u = v} \varepsilon(u, x)f(W, \varepsilon_W)(x)$$

$$= \sum_{x \in E, u \in \text{End}(x), u = v, \beta \in \text{supp}E} \varepsilon(u, x)[\varepsilon, \varepsilon_W](x)$$

$$= \sum_{x \in W, u \in \text{End}(x), u = v} \varepsilon_W(u, x)$$

$$= \sum_{j=1}^{k} [\varepsilon_W(u_{a_j}, x_{a_j}) + \varepsilon_W(u_{a_j}, x_{a_j+1})] = 0.$$ 

Hence the function $f(W, \varepsilon_W)$ is an integral flow of $(\Sigma, \varepsilon)$. \hfill \Box

Flow Reduction Algorithm (FRA). Given a nontrivial integral flow on $(\Sigma, \varepsilon)$.

**Step 0.** Choose an edge $x_1$ in supp $f$ with end-vertices $u_0, u_1$. Initiate a half-closed and half-open walk $u_0x_1$. Set $W := u_0x_1$ and $\ell := 1$. Go to Step 1.

**Step 1.** If $u_\ell \notin W$, go to Step 2. If $u_\ell \in W$, say, $u_\ell = u_\beta$ with the greatest index $\beta < \ell$, go to Step 3.

**Step 2.** There exists an edge $x_{\ell+1}$ in supp $f'$ other than $x_\ell$, where $f' := f - f(W, \varepsilon_W)$, having end-vertices $u_\ell, u_{\ell+1}$ such that $\varepsilon_f(u_\ell, x_{\ell+1}) = -\varepsilon_f(u_\ell, x_\ell)$. Set $W := Wu_\ell x_{\ell+1}$ and $\ell := \ell + 1$. Return to Step 1.

**Step 3.** If $u_\beta$ is a double point of $W$, say, $u_\alpha = u_\beta$ with $\alpha < \beta < \ell$, STOP. For the case $\varepsilon_f(u_\ell, x_\ell) = -\varepsilon_f(u_\ell, x_{\ell+1})$, set

$$W := u_\beta x_{\beta+1}u_{\beta+1}\ldots u_{\ell-1}x_{\ell}u_\ell.$$  \hfill (2.1)

For the case $\varepsilon_f(u_\ell, x_\ell) = \varepsilon_f(u_\beta, x_{\beta+1})$, set

$$W := u_\alpha x_{\alpha+1}u_{\alpha+1}\ldots u_{\ell-1}x_{\ell}u_\ell.$$ \hfill (2.2)

Then $(W, \varepsilon_f)$ is a directed Eulerian walk. If $u_\beta$ is a single point of $W$, go to Step 4.

**Step 4.** If there exist double vertices $u_\alpha, u_\gamma$ in $W$ with $\alpha < \beta < \gamma$ such that $u_\alpha = u_\gamma$, STOP. For the case $\varepsilon_f(u_\ell, x_\ell) = -\varepsilon_f(u_\beta, x_{\beta+1})$, set $W$ to be of (2.1). For the case $\varepsilon_f(u_\ell, x_\ell) = \varepsilon_f(u_\beta, x_{\beta+1})$, set

$$W := u_\beta x_{\beta}u_{\beta-1}\ldots u_{\alpha+1}x_{\alpha+1}u_\alpha(u_\gamma)x_{\gamma+1}u_{\gamma+1}\ldots u_{\ell-1}x_{\ell}u_\ell.$$ \hfill (2.3)

Then $(W, \varepsilon_f)$ is a directed Eulerian walk. Otherwise, go to Step 5.

**Step 5.** If $\varepsilon_f(u_\ell, x_\ell) = -\varepsilon_f(u_\beta, x_{\beta+1})$, STOP. Set $W$ to be of (2.1). Then $(W, \varepsilon_f)$ is a directed Eulerian walk. If $\varepsilon_f(u_\ell, x_\ell) = \varepsilon_f(u_\beta, x_{\beta+1})$, return to Step 2.

It is clear from Step 3 that the multiplicity of each vertex in the closed walk $W$ (obtained by FRA) is at most two. So $W$ has only possible double vertices and possible double edges. At each double vertex of $W$, say $u_\alpha = u_\beta$ with $\alpha < \beta$, Step 5 implies

$$\varepsilon(u_\beta, x_\beta) = \varepsilon_W(u_\alpha, x_{\alpha+1}) = -\varepsilon_W(u_\alpha, x_\alpha) = -\varepsilon_W(u_\beta, x_{\beta+1});$$ \hfill (2.4)

see Figure 2. It is possible that $(u_\beta, x_{\beta+1}) = (u_\alpha, x_\alpha)$; if so, the repeated edges $x_\alpha, x_{\beta+1}$ have the same orientation in $(W, \varepsilon_W)$. This means that $(W, \varepsilon_W)$ is a directed Eulerian walk.
In Step 2, both functions \( f_W, f' \) are not conservative at \( u_\ell \). In fact, if \( u_\ell \neq u_0 \), then \( \partial f_W(u_\ell) (u_\ell) = \varepsilon_f(u_\ell, x_\ell) \) and \( \partial f'(u_\ell) = -\varepsilon_f(u_\ell, x_\ell) \); if \( u_\ell = u_0 \), then \( \partial f'(u_\ell)(u_\ell) = 2\varepsilon_f(u_\ell, x_\ell) = 2\varepsilon_f(u_0, x_1) \). Hence \( \partial f_W(u_\ell) \neq 0 \). This means that there exists an edge \( x_{\ell+1} \) in \( \text{supp} f' \) at \( u_\ell \) such that \( \varepsilon_f(u_\ell, x_{\ell+1}) = \varepsilon_f(u_\ell, x_\ell) \). Thus the length of \( W \) is increased by one. Since the multiset \( (E, |f|) \) is finite, FRA stops with a directed closed walk \((W, \varepsilon_f)\). Moreover, Step 4 implies that each double vertex in \( W \) (obtained by FRA) is a cut-point of \( \Sigma(W) \).

**Lemma 2.2.** Let \( W \) be a directed walk. Then FRA finds no directed closed walk along \( W \) if and only if FRA finds no directed closed walk along \( W^{-1} \).

**Proof.** This seems to be quite trivial. In fact, let FRA find a directed closed walk along \( W \). Then \( W \) contains one of the three patterns of closed walks: (a), (b) and (c) in Figure 4, where \( \alpha < \beta < \gamma < \delta \), \( \varepsilon_W(u_\beta, x_\beta) = -\varepsilon_W(u_\alpha, x_{\alpha+1}) \) in (a), \( \varepsilon_W(u_\gamma, x_\gamma) = -\varepsilon_W(u_{\alpha}, x_{\alpha+1}) \) in (b), and \( \varepsilon_W(u_\delta, x_\delta) = -\varepsilon_W(u_{\alpha}, x_{\alpha+1}) \) in (c). The reversions of patterns (a), (b) and (c), as subwalks in \( W^{-1} \), have the same patterns as (a), (b) and (c) respectively. The subwalks from \( u_\alpha \) to \( u_\beta \) in (a), (b), (c) may contain some double vertices and double edges; so do the subwalks from \( u_\beta \) to \( u_\gamma \) in (b) and (c); so does the subwalk from \( u_\gamma \) to \( u_\delta \) in (c).

Note that when FRA is applied to \( W \), the algorithm may stop and find a directed closed walk before it reaches \( u_\beta \) in (a), or before it reaches \( u_\gamma \) in (b), or before it reaches \( u_\delta \) in (c). If so, when FRA is applied to \( W^{-1} \), the algorithm stops and finds a directed closed walk along \( W^{-1} \) before it reaches \( u_{\beta-1} \), or \( u_{\gamma-1} \), or \( u_{\delta-1} \). If not, when FRA is applied to \( W^{-1} \), the algorithm stops and finds a directed closed walk when it reaches \( u_{\alpha-1} \). This means that FRA finds a directed closed walk along \( W^{-1} \).

![Figure 4. Three patterns that FRA stops.](image)

Conversely, let FRA find a directed closed walk along \( W^{-1} \). Then FRA finds a directed closed walk along \( (W^{-1})^{-1} \), which is exactly the walk \( W \). We have seen that FRA finds a directed closed walk along \( W \) if and only if FRA finds a directed closed walk along \( W^{-1} \). \( \square \)

**Lemma 2.3.** Let \((W, \varepsilon_W)\) be a directed midway-back avoided walk. Then

(a) \( W \) has no triple vertices, that is, the multiplicity of each vertex and of each edge in \( W \) is at most two.

(b) \((W, \varepsilon_W)\) is a directed Eulerian walk.
Proof: Write $W = u_0x_1u_1x_2\cdots u_{n-1}x_nu_n$.

(a) Suppose there is a vertex appeared three times in $W$, say, $u_\alpha = u_\beta = u_\gamma$ with $\alpha < \beta < \gamma$; see Figure 5. Since $(W, \varepsilon_W)$ is midway-back avoided, we have

$$
\varepsilon_W(u_\beta, x_\beta) = \varepsilon_W(u_\alpha, x_{\alpha+1}),
$$

$$
\varepsilon_W(u_\gamma, x_\gamma) = \varepsilon_W(u_\alpha, x_{\alpha+1}),
$$

$$
\varepsilon_W(u_\beta, x_\gamma) = \varepsilon_W(u_\beta, x_{\beta+1}).
$$

Then

$$
\varepsilon_W(u_\gamma, x_\gamma) = -\varepsilon_W(u_\beta, x_\beta) = -\varepsilon_W(u_\alpha, x_{\alpha+1}) = -\varepsilon_W(u_\gamma, x_\gamma),
$$

which is a contradiction.

(b) Let $u_\alpha = u_\beta$ with $\alpha < \beta$ and let $x_{\beta+1}(= x_\alpha)$ be a repeated edge. Then $u_{\beta+1} = u_{\alpha-1}$. Suppose $\varepsilon_W(u_\beta, x_{\beta+1}) = -\varepsilon_W(u_\alpha, x_\alpha)$. Then

$$
\varepsilon_W(u_{\beta+1}, x_{\beta+1}) = -\varepsilon_W(u_{\alpha-1}, x_\alpha).
$$

This means that $(W, \varepsilon_W)$ is midway-back at $u_{\alpha-1}$, which is a contradiction. So $\varepsilon_W(u_\beta, x_{\beta+1}) = \varepsilon_W(u_\alpha, x_\alpha)$. This means that $\varepsilon_W$ has the same orientation on repeated edges. Thus $(W, \varepsilon_W)$ is a directed Eulerian walk. \hfill $\square$

Lemma 2.4. Let $(W, \varepsilon_f)$ be a directed closed walk found by FRA. Then

(a) $(W, \varepsilon_f)$ is midway-back avoided.

(b) Each double vertex in $W$ is a cut-point of $\Sigma(W)$.

Proof. (a) The directed walk $(W, \varepsilon_f)$ satisfies (2.4). By definition $(W, \varepsilon_f)$ is midway-back avoided.

(b) Assume that FRA stops at time $\ell$ and finds a directed closed walk $(W, \varepsilon_f)$, but did not stop before $\ell$. The two forms (2.1) and (2.2) of $W$ are the same kind, having indices increasing. However, the form (2.3) of $W$ is special; its indices from $u_\beta$ to $u_\alpha$ are decreasing. We may reduce the form (2.4) of $W$ to the form whose indices are increasing as follows.

Consider the directed walk $(W', \varepsilon_f)$, where $W' = W_1W_2$,

$$
W_1 = u_\alpha x_{\alpha+1}u_{\alpha+1}\cdots u_\beta x_{\beta+1}u_{\beta+1}\cdots u_{\gamma-1}x_\gamma u_\gamma,
$$

$$
W_2 = u_\gamma x_{\gamma+1}u_{\gamma+1}\cdots u_{\ell-1}x_\ell u_\ell, \quad u_\gamma = u_\alpha, \ u_\ell = u_\beta.
$$

Applying FRA to $W_1W_2$, the algorithm cannot stop before $\ell$, but stops at time $\ell$ and finds the directed closed walk $W$. Of course, FRA finds no directed closed walk along $W_1$. Writing $W_1^{-1}$ in increasing-order of indices and applying FRA to $W_1^{-1}$, by Lemma 2.2 the algorithm finds no directed closed walk along $W_1^{-1}$. Now
applying FRA to \( W_{l+1}^{-1}W_2 \), the algorithm cannot stop before \( l \), but stops at time \( l \) and finds the same directed closed walk \( W \), having indices increasing.

Without loss of generality we may assume that \((W, \varepsilon_W)\) (obtained by FRA) has the form
\[
W = u_0 x_1 u_1 x_2 \ldots u_{l-1} x_l u_l, \quad u_0 = u_l.
\] (2.5)

Suppose \( W \) has a double vertex \( u \) that is not a cut-point of \( \Sigma(W) \), say, \( u = u_\delta = u_\eta \) with \( \delta < \eta \). Then there exist vertices \( u_\mu, u_\nu \) in \( W \) such that \( u_\mu = u_\nu \), where \( \delta < \mu < \eta \) and either \( \eta < \nu \) or \( \nu < \delta \). With the indices modulo \( l \), the closed walk \( W \) can be written as the form (see Figure 6)
\[
W = u_\delta x_\delta+1 u_\delta+1 \ldots x_\mu u_\mu x_\mu+1 \ldots x_\eta u_\eta x_\eta+1 \ldots x_\nu u_\nu x_\nu+1 \ldots u_\delta-1 x_\delta u_\delta.
\]

Consider the case \( \delta < \mu < \eta < \nu \). If \( \nu < l \), FRA stops in Step 4 at time \( \nu \) and finds the directed closed walk
\[
u_\mu x_\mu u_\mu-1 \ldots u_\delta+1 x_\delta+1 u_\delta (u_\eta) x_\eta+1 u_\eta+1 \ldots u_\nu-1 x_\nu u_\nu
\]
in Figure 6 this is a contradiction. If \( \nu = l \), then \( u_\mu x_{\nu+1} u_{\nu+1} = u_0 x_1 u_1 \), FRA stops in Step 4 at time \( \eta \) and finds the directed closed walk
\[
u_\nu x_{\nu+1} u_{\nu+1} \ldots u_\delta-1 x_\delta u_\delta (u_\eta) x_\eta u_\eta-1 \ldots u_{\mu+1} x_{\mu+1} u_\mu
\]
in Figure 6 this is a contradiction. For the case \( \nu < \delta < \mu < \eta \), it is analogous to the case \( \delta < \mu < \eta < \nu \). □

![Figure 6. A double vertex that is not a cut-point.](image)

**Theorem 2.5.** Let \((W, \varepsilon_W)\) be a directed closed walk such that

(i) \((W, \varepsilon_W)\) is a directed midway-back avoided walk;

(ii) each double vertex in \( W \) is a cut-point of \( \Sigma(W) \).

Then \( \Sigma(W) \) is an Eulerian cycle-tree, the restriction of \( \varepsilon_W \) to \( \Sigma(W) \) is a direction on the cycle-tree, and \( W \) uses each edge of block cycles once and each edge of block paths twice, crossing from one block to the other block at each cut-point.

**Proof.** Lemma[23] implies that \( W \) has no triple vertices, that is, \( W \) has only possible double vertices and double edges. Since each double vertex in \( W \) is a cut-point of \( \Sigma(W) \), then each connected component of the signed subgraph induced by the double edges of \( W \) is a simple path (of possible zero length), called a double-edge path of \( W \). Remove the internal part of each double-edge path of positive length from \( W \), we obtain an Eulerian graph whose vertex degrees are either 2 or 4. The Eulerian graph can be decomposed into edge-disjoint cycles, called block cycles.
Each double-edge path (of possible zero length) connects exactly two block cycles. Since \( \Sigma(W) \) is connected and each double vertex in \( W \) is a cut-point of \( \Sigma(W) \), it follows that \( \Sigma(W) \) is a cycle-tree.

It is clear that \( W \) uses each edge of block cycles once and each edge of block paths twice, and crosses from one block to the other block at each cut-point. Since \((W, \varepsilon_W)\) is midway-back avoided, it follows that the restriction of \( \varepsilon_W \) to \( \Sigma(W) \) is a direction on the cycle-tree. Thus \( \Sigma(W) \) is an Eulerian cycle-tree by Lemma 3.3.

**Corollary 2.6.** Let \((W, \varepsilon_f)\) be a directed closed walk found by FRA. Then \( \Sigma(W) \) is an Eulerian cycle-tree with direction \( \varepsilon_f \), and \( W \) uses each edge of block cycles once and each edge of block paths twice, crossing from one block to the other block at each cut-point.

**Proof.** It follows from Lemma 2.4 and Theorem 2.5.

**Theorem 2.7** (Flow Reduction Theorem). Let \( f \) be a nontrivial integral flow of \((\Sigma, \varepsilon)\). Then there exist minimal directed Eulerian walks \((W_i, \varepsilon_f)\) and Eulerian cycle-trees \( T_i = \Sigma(W_i) \) such that

\[
f = \sum f(W_i, \varepsilon_f) = \sum [\varepsilon, \varepsilon_f] \cdot I_{T_i},
\]

where \( f(W_i, \varepsilon_f) \) are given by (1.8) and \( I_{T_i} \) by (1.10).

Furthermore, if \( f \) is indecomposable, then there exists a minimal directed Eulerian walk \((W, \varepsilon_f)\) and an Eulerian cycle-tree \( T = \Sigma(W) \) such that

\[
f = f(W, \varepsilon_f) = [\varepsilon, \varepsilon_f] \cdot I_T.
\]

**Proof.** Consider the nonnegative integral flow \(|f|\) of \((\Sigma, \varepsilon_f)\), where \( \varepsilon_f \) is defined by (1.9). Let \( \Sigma(f) \) denote the signed subgraph induced by the edge subset \( \text{supp} f \). Then FRA finds a directed Eulerian walk \((W_1, \varepsilon_f)\) on the oriented signed graph \((\Sigma(f), \varepsilon_f)\), such that \( f_{W_1} \leq |f| \), where \( f_{W_1} \) is given by (1.8) with \( \varepsilon = \varepsilon_f \). Corollary 2.6 implies that \( \Sigma(W_1) \) is an Eulerian cycle-tree \( T_1 \), and Theorem 3.6 implies that \( W_1 \) is a minimal Eulerian walk. Then Theorem 3.5 implies \( f_{W_1} \equiv I_{T_1} \), the indicator function of \( T_1 \) defined by (1.10).

If \( f_1 := |f| - f_{W_1} \neq 0 \), then FRA finds a minimal directed Eulerian walk \((W_2, \varepsilon_f)\) on \((\Sigma(f_1), \varepsilon_f)\), such that \( f_{W_2} \leq |f| - f_{W_1} \) and \( f_{W_2} = I_{T_2} \), where \( T_2 \) is the Eulerian cycle-tree \( \Sigma(W_2) \). Likewise, if \( f_2 := |f| - f_{W_1} - f_{W_2} \neq 0 \), then FRA finds a minimal directed Eulerian walk \((W_3, \varepsilon_f)\) on \((\Sigma(f_2), \varepsilon_f)\), such that \( f_{W_3} \leq |f| - f_{W_1} - f_{W_2} \) and \( f_{W_3} = I_{T_3} \), where \( T_3 \) is the Eulerian cycle-tree \( \Sigma(W_3) \). Continue this procedure, we obtain minimal directed Eulerian walks

\[(W_1, \varepsilon_f), \ (W_2, \varepsilon_f), \ldots, \ (W_k, \varepsilon_f)\]

on \((\Sigma(f), \varepsilon_f)\), such that \(|f| = \sum_{i=1}^k f_{W_i} = \sum_{i=1}^k I_{T_i}\), where \( T_i \) are the Eulerian cycle-trees \( \Sigma(W_i) \) and \( f_{W_i} = I_{T_i} \). Note that

\[
f = [\varepsilon, \varepsilon_f] \cdot |f|, \quad f(W_i, \varepsilon_f) = [\varepsilon, \varepsilon_f] \cdot f_{W_i}.
\]

We obtain \( f = \sum_{i=1}^k f(W_i, \varepsilon_f) = \sum_{i=1}^k [\varepsilon, \varepsilon_f] \cdot I_{T_i} \).

If \( f \) is indecomposable, by definition we must have \( k = 1 \).
3. Characterizations of Eulerian Cycle-trees

This section is to establish properties satisfied by Eulerian cycle-trees such as the Existence and Uniqueness of Direction, the Minimality, and the Half-integer Scale Decomposition. We shall see the equivalence of indecomposable flows, minimal Eulerian walks, and Eulerian cycle-trees. The byproduct is the equivalence of circuits, elementary flows, and elementary Eulerian walks, and the classification of circuits.

Lemma 3.1. Let $W = u_0x_1u_1x_2 \ldots u_{n-1}x_nu_n$ be a walk with a direction $\varepsilon_W$. Then

$$\varepsilon_W(u_n, x_n) = -\sigma(W)\varepsilon_W(u_0, x_1).$$

In particular, if $(W, \varepsilon_W)$ is a directed closed walk, then $\sigma(W) = 1$.

Proof. The direction $\varepsilon_W$ must be constructed inductively as follows:

$$\varepsilon_W(u_i, x_i) = -\sigma(x_i)\varepsilon_W(u_{i-1}, x_i),$$

$$\varepsilon_W(u_i, x_{i+1}) = -\varepsilon_W(u_i, x_i), \quad 1 \leq i \leq n.$$  

Then $\varepsilon_W(u_n, x_n)$ is determined by $\varepsilon_W(u_0, x_1)$ as $\varepsilon_W(u_n, x_n) = -\sigma(W)\varepsilon_W(u_0, x_1)$.

If $(W, \varepsilon_W)$ is a directed closed walk, then $\varepsilon_W(u_n, x_n) = -\varepsilon_W(u_0, x_1)$. It is clear that $\sigma(W) = 1$. □

Let $T = \{C_i, P_j\}$ be a cycle-tree throughout. We choose a block cycle $C_0$ and write

$$C_0 = u_0x_1u_1x_2 \ldots u_{i-1}x_iu_i, \quad u_i = u_0. \quad (3.1)$$

If $T$ has two or more block cycles, we require $C_0$ to an end-block cycle, having $u_0$ as its unique intersection vertex. Let $P$ be the block path (of possible zero length) from the vertex $u_0$ on $C_0$ to a vertex $w_0$ on another block cycle $C_1$. We write

$$P = v_0y_1v_1y_2 \ldots v_{m-1}y_mv_m, \quad v_m = w_0. \quad (3.2)$$

Remove the cycle $C_0$ and the internal part of the path $P$, we obtain a cycle-tree

$$T_1 = T\setminus(C_0 \cup P), \quad (3.3)$$

which has one fewer block cycle than $T$. Choose an edge $z_1$ on $C_1$ incident with $w_0$ and switch the sign of $z_1$, we obtain a cycle-tree $T_1'$. If $T$ is Eulerian, so is $T_1'$, for the block cycle $C_1$ has one fewer intersection vertex in $T_1'$ than in $T$. This procedure will be recalled in the proof of the following Lemma 3.2 and Theorem 3.3.

Lemma 3.2. Let $T$ be a cycle-tree. Then there exists a closed walk $W$ on $T$ that uses each edge of block cycles once and each edge of block paths twice, and crosses from one block to the other block at each cut-point.

Moreover, each such $W$ is a closed walk of minimum length that uses all edges of $T$, and vice versa.

Proof. If $T$ has only one block cycle, then $T$ is a cycle $C_0$ and can be written as a closed walk in $3.1$. If $T$ has two or more block cycles, then by induction there is a closed walk $W_1$ on $T_1$ in $3.3$ such that $W_1$ crosses from one block to the other block at each intersection vertex. Then $W = C_0P\sigma W_1P^{-1}$ is the required closed walk on $T$; see Figure 7. The property of minimum length is trivial. □

Theorem 3.3 (Existence and Uniqueness of Direction on Eulerian Cycle-Tree). Let $T$ be a cycle-tree. Then $T$ satisfies the Parity Condition if and only if there exists a unique direction $\varepsilon_T$ on $T$ up to opposite sign.
Proof. “⇒”: We proceed by induction on the number of block cycles of \( T \). When \( T \) has only one block cycle, then \( T \) is the cycle itself, and the cycle has to be balanced. It is clear that a balanced cycle has a unique direction up to opposite sign.

Assume that \( T \) has two or more block cycles. Then \( T_1 \) in Figure 7 is a cycle-tree having one fewer block cycle than \( T \). Switch the sign of the edge \( z_1 \) in \( T_1 \), we obtain an Eulerian cycle-tree \( T'_1 \). By induction there exists a unique direction \( \varepsilon_{T'_1} \) (up to opposite sign) on \( T'_1 \). Let us switch the sign of \( z_1 \) in \( T'_1 \) back to the sign of \( z_1 \) in \( T_1 \) and define an orientation \( \varepsilon_{T_1} \) on \( T_1 \) by setting \( \varepsilon_{T_1} = \varepsilon_{T'_1} \) for all vertex-edge pairs except

\[
\varepsilon_{T_1}(w_0, z_1) = -\varepsilon_{T'_1}(w_0, z_1).
\]

Then \((C_1, \varepsilon_{T_1})\) has either a sink or a source at \( w_0 \). Let \( \varepsilon_P \) be a direction on \( P \) such that \( \varepsilon_P(v_m, y_m) = -\varepsilon_{T_1}(w_0, z_1) \), and \( \varepsilon_{C_0} \) be a direction on \( C_0 \) such that \( \varepsilon_{C_0}(u_1, x_1) = -\varepsilon_P(v_0, y_1) \). Then the joint orientation \( \varepsilon_{C_0} \vee \varepsilon_P \vee \varepsilon_{T_1} \) gives rise to a direction \( \varepsilon_T \) on \( T \); see Figure 7.

Let \( \varepsilon'_T \) be an arbitrary direction on \( T \). Then \( \varepsilon'_T \) induces directions \( \varepsilon'_{C_0}, \varepsilon'_P, \varepsilon'_{T_1} \) on \( C_0, P, T_1 \) respectively, where

\[
\varepsilon'_{T_1}(w_0, z_1) = -\varepsilon'_T(w_0, z_1),
\]

\[
\varepsilon'_P(v_m, y_m) = -\varepsilon_T(w_0, z_n),
\]

\[
\varepsilon'_{C_0}(u_1, x_1) = -\varepsilon_P(v_0, y_1).
\]

Then by induction we have that \( \varepsilon'_{T_1} = \pm \varepsilon_{T_1} \). Thus \( \varepsilon'_P = \pm \varepsilon_P \) and \( \varepsilon'_{C_0} = \pm \varepsilon_{C_0} \). Therefore \( \varepsilon'_T = \pm \varepsilon_T \); see Figure 7. This shows that the direction \( \varepsilon_T \) is unique up to opposite sign.

“⇐”: We also proceed by induction on the number of block cycles. Let \( \varepsilon_T \) be a direction on \( T \), and let \( W \) be a closed walk on \( T \) that uses each edge of block cycles once and each edge of block paths twice. Then \((W, \varepsilon_T)\) is a directed Eulerian walk. It is trivially true when \( T \) has only one block cycle, for the cycle has zero number of intersection vertices and is balanced by Lemma 3.1.

Assume that \( T \) has two or more block cycles. Let \((W, \varepsilon_W)\) be the restriction of \((W, \varepsilon_W)\) to \( T_1 \) in Figure 7. Then \((W, \varepsilon_W)\) is a directed walk, having either a sink or a source at \( u_0 \). Switch the sign of the edge \( z_1 \) in \( T \) and its orientation at \( w_0 \). We obtain a directed Eulerian walk \((W'_1, \varepsilon'_{W_1})\) on \( T'_1 \) that uses each edge of block cycles once and each edge of block paths twice. By induction all block cycles of \( T'_1 \) satisfy the Parity Condition. Thus all block cycles of \( T_1 \) other than \( C_1 \) satisfies the Parity Condition. Let us switch the sign of \( z_1 \) back to the sign of \( z_1 \) in \( T \). Since \( C_1 \) has one fewer intersection vertex in \( T \) than that in \( T'_1 \), we see that \( C_1 \) satisfies the Parity Condition in \( T \). Since \((C_0, \varepsilon_T)\) has either a sink or a source at \( u_0 \), it forces that \( C_0 \) is unbalanced. Hence \( C_0 \) also satisfies the Parity Condition. □

**Lemma 3.4.** Let \( W \) be a minimal Eulerian walk with a direction \( \varepsilon_W \). Then
(a) \((W, \varepsilon_W)\) is midway-back avoided.
(b) Each double vertex in \(W\) is a cut-point of \(\Sigma(W)\).

**Proof.** (a) Suppose \((W, \varepsilon_W)\) is not midway-back avoided. Then \(W\) can be written as
\[
W = u_0 x_1 u_1 \ldots u_{\alpha-1} x_\alpha u_\alpha \ldots u_{\beta-1} x_\beta u_\beta \ldots u_{\ell-1} x_\ell u_\ell,
\]
where \(0 \leq \alpha < \beta < \ell\), \(u_\alpha = u_\beta\), and \(\varepsilon_W(u_\beta, x_\beta) = -\varepsilon_W(u_\alpha, x_\alpha+1)\). Then \((W', \varepsilon_W)\) is a directed closed walk contained properly in \((W, \varepsilon_W)\) as multisets, where
\[
W' = u_\alpha x_\alpha+1 u_{\alpha+1} \ldots u_{\beta-1} x_\beta u_\beta.
\]
This is a contradiction.

(b) Suppose there is a double vertex in \(W\) that is not a cut-point of \(\Sigma(W)\). Then we can write \(W\) as
\[
W = u_\delta x_\delta+1 u_\delta+1 \ldots x_\mu u_\mu \ldots x_\eta u_\eta \ldots x_\nu u_\nu \ldots u_{\delta-1} x_\delta u_\delta,
\]
where \(\delta < \mu < \eta < \nu\), \(u_\delta = u_\eta\) and \(u_\mu = u_\nu\); see Figure 6. Since \((W, \varepsilon_W)\) is midway-back avoided, we have
\[
\varepsilon_W(u_\eta, x_\eta) = \varepsilon_W(x_\delta, x_{\delta+1}), \quad \varepsilon_W(u_\nu, x_\nu) = \varepsilon_W(u_\mu, x_{\mu+1}). \tag{3.4}
\]
Since \((W, \varepsilon_W)\) is directed, we must have
\[
\varepsilon_W(u_\delta, x_\delta) = \varepsilon_W(u_\eta, x_{\eta+1}), \quad \varepsilon_W(u_\mu, x_\mu) = \varepsilon_W(u_\nu, x_{\nu+1}). \tag{3.5}
\]
Thus (3.4) and (3.5) imply that \((W_1, \varepsilon_W)\) and \((W_2, \varepsilon_W)\) are directed closed walks and are contained properly in \((W, \varepsilon_W)\) as multisets, where
\[
W_1 = u_\delta x_{\delta+1} u_{\delta+1} \ldots u_{\mu-1} x_\mu u_\mu x_\nu u_\nu u_{\nu-1} \ldots u_{\eta+1} x_{\eta+1} u_\eta,
W_2 = u_\eta x_\eta u_{\eta-1} \ldots u_{\mu+1} x_{\mu+1} u_\mu x_\nu+1 u_{\nu+1} \ldots u_{\delta-1} x_\delta u_\delta.
\]
This is a contradiction. \(\square\)

**Theorem 3.5** (Characterization of Minimal Eulerian Walk). Let \(W\) be a minimal Eulerian walk with a direction \(\varepsilon_W\). Then \(\Sigma(W)\) is an Eulerian cycle-tree \(T\), \(W\) uses each edge of block cycles once and each edge of block paths twice of \(T\), and \(\varepsilon_W\) induces a direction \(\varepsilon_T\) on \(T\). Moreover,
\[
f(W, \varepsilon_W) = [\varepsilon, \varepsilon_T] \cdot I_T. \tag{3.6}
\]

**Proof.** It follows from Lemma 3.4 and Theorem 2.5 that \(\Sigma(W)\) is an Eulerian cycle-tree. Since \(\varepsilon_W(u, x) = \varepsilon_T(u, x)\) for vertex-edge pairs \((u, x)\) on \(T\), the identity (3.6) follows from definitions (1.8) and (1.10). \(\square\)

**Theorem 3.6** (Minimality of Eulerian Cycle-Tree). Let \(T\) be an Eulerian cycle-tree with a direction \(\varepsilon_T\). Then \(T\) is minimal in the sense that if \(T_1\) is an Eulerian cycle-tree contained in \(T\) and block paths of \(T_1\) are block paths of \(T\) then \(T_1 = T\).

Moreover, if \(W\) is a closed walk on \(T\) that uses each edge of block cycles once and each edge of block paths twice, then \(W\) is a minimal Eulerian walk. In particular, each closed walk found by FRA is a minimal Eulerian walk.

**Proof.** Suppose there is an Eulerian cycle-tree \(T_1\) contained properly in \(T\), such that block paths of \(T_1\) are block paths of \(T\). Then there exists an edge \(e \in T \backslash T_1\), incident with a vertex \(v\) on a block cycle \(C\) of \(T_1\). The vertex \(v\) must be an intersection vertex in \(T\) but not an intersection vertex in \(T_1\). Let \(e_1, e_2\) be two edges (maybe be an identical loop) on \(C\), incident with \(v\). Then \(\varepsilon_T(v, e_1) = -\varepsilon_T(v, e_2)\) in \(T_1\) and \(\varepsilon_T(v, e_1) = \varepsilon_T(v, e_2)\) in \(T\). This is a contradiction.
Let $W$ be a required closed walk on $T$. It is clear that $(W, \varepsilon_T)$ is a directed Eulerian walk by Lemma 3.2 and by definition of $\varepsilon_T$. Let $W_1$ be a minimal Eulerian walk on $T$, contained properly in $W$ as multisets. Then $T_1 = \Sigma(W_1)$ is contained in $T$ and is an Eulerian cycle-tree by Theorem 3.5. It is clear that block cycles of $T_1$ are block cycles of $T$. Note that edges of block paths of $T$ are double edges in $W$, edges of block paths of $T_1$ are double edges in $W_1$, and double edges in $W_1$ must be double edges in $W$. It follows that block paths of $T_1$ are block paths of $T$. Thus $T_1 = T$ by the first part of the theorem. Therefore $M(W_1) = M(W)$. \[\square\]

**Corollary 3.7** (Characterization and Classification of Circuits). Let $(W, \varepsilon_W)$ be a minimal directed Eulerian walk. Then the following statements are equivalent.

1. $(W, \varepsilon_W)$ is elementary.
2. $f_{(W, \varepsilon_W)}$ is elementary.
3. $\Sigma(W)$ is a circuit.

Moreover, circuits are classified into Types I, II, III.

**Proof.** (a) $\Leftrightarrow$ (b): Assume $(W, \varepsilon_W)$ is not elementary, that is, there exists a minimal directed Eulerian walk $(W_1, \varepsilon_{W_1})$ such that $\text{supp} W_1 \subsetneq \text{supp} W$. Since $\text{supp} W_1 = f_{(W_1, \varepsilon_{W_1})}$ and $\text{supp} W = \Sigma f_{(W, \varepsilon_W)}$, then $\Sigma f_{(W_1, \varepsilon_{W_1})} \subsetneq \Sigma f_{(W, \varepsilon_W)}$. This means that $f_{(W_1, \varepsilon_{W_1})}$ is not elementary.

Conversely, assume $f_{(W, \varepsilon_W)}$ is not elementary, that is, there is a flow $g$ such that $\text{supp} g \subsetneq \text{supp} f_{(W, \varepsilon_W)}$. We may require $\Sigma(\text{supp} g)$ to be connected. By Lemma 2.1 there exists a directed closed walk $(W_1, \varepsilon_g)$ on $\Sigma(\text{supp} g)$ such that $g = f_{(W_1, \varepsilon_g)}$. Since $\text{supp} W_1 = \Sigma g$ and $\text{supp} f_{(W_1, \varepsilon_W)} = \Sigma W$, then $\text{supp} W_1 \subsetneq \Sigma W$. This means that $(W_1, \varepsilon_W)$ is not elementary.

(a) $\Leftrightarrow$ (c): If $(W, \varepsilon_W)$ is not a circuit, that is, there exists an Eulerian cycle-tree $T_1$ contained properly in $\Sigma(W)$. Let $\varepsilon_{T_1}$ be a direction on $T_1$, and $W_1$ be a closed walk that uses each edge of block cycles once and each edge of block paths twice of $T_1$. Then $(W_1, \varepsilon_{T_1})$ is a minimal directed Eulerian walk and $\text{supp} W_1 \subsetneq \Sigma W$. This means that $\Sigma(W)$ is not a circuit.

Conversely, if $\Sigma(W)$ is not a circuit, that is, there exists an Eulerian cycle-tree $T_1$ contained properly in $\Sigma(W)$. Let $\varepsilon_{T_1}$ be a direction on $T_1$, and $W_1$ be a closed walk that uses each edge of block cycles once and each edge of block paths twice of $T_1$. Then $(W_1, \varepsilon_{T_1})$ is a minimal directed Eulerian walk and $\text{supp} W_1 \subsetneq \Sigma W$. This means that $(W, \varepsilon_W)$ is not elementary.

Now let $T$ be an Eulerian cycle-tree and be further a circuit. Then $T$ contains at most one block path (of possible zero length). Otherwise, suppose there are two or more block paths in $T$, then one block path together with its two block cycles form an Eulerian cycle-tree, which is properly contain in $T$; this means that $T$ is not a circuit, a contradiction. If there is no block path in $T$, then $T$ must be a single balanced cycle, which is a circuit of Type I. If $T$ contains exactly one block path, the length of the block path is either zero or positive.

In the case of zero length for the block path, $T$ consists of two block cycles having a common vertex, which is a circuit of Type II. In the case of positive length for the block path, $T$ consists of two block cycles and the block path connecting them, which is a circuit of Type III. \[\square\]

**Theorem 3.8** (Half-integer Scale Decomposition). Let $T$ be an Eulerian cycle-tree with a direction $\varepsilon_T$. Let $W$ be a closed walk on $T$ that uses each edge of block cycles once and each edge of block paths twice. If $T$ is not a circuit, then $W$ can be divided
where \( P \) is a block cycle of zero length. Note that \( T \) has exactly two block paths, then \( T \) has the form in Figure 8. Since \( W \) crosses each cut-point from one block to the other block, then \( W \) can be written as \( W = C_0 P_1 C_1 P_2 \cdots P_k C_k P_{k+1}, \quad k \geq 1 \), \hspace{1cm} (3.7)
satisfying the following four conditions:

(i) \( \{C_i\} \) is the collection of all end-block cycles of \( T \) and \( P_i \) are simple open paths of positive lengths.

(ii) Each edge of non-end-block cycles appears in exactly one of the paths \( P_i \), and each edge of block paths appears in exactly two of the paths \( P_i \).

(iii) Each \( (C_i P_{i+1} C_{i+1}, \varepsilon_T) \) \((0 \leq i \leq k)\) is a directed circuit of Type III with \( C_{k+1} = C_0 \).

(iv) Half-integer scale decomposition

\[
I_T = \frac{1}{2} \sum_{i=0}^{k} I_{\Sigma(C_i P_{i+1} C_{i+1})}. \hspace{1cm} (3.8)
\]

**Proof.** We proceed by induction on the number of block paths in \( T \), including those of zero length. Note that \( T \) is a circuit if there is none or exactly one block path. When \( T \) has exactly two block paths, then \( T \) has the form in Figure 8. Since \( W \)

![Figure 8. An Eulerian cycle-tree with two block paths.](image)

can be written as \( W = C_0 P_1 C_1 P_2 \), where \( P_1 = PQ_1 Q, \ P_2 = Q^{-1} Q_2^{-1} P^{-1} \). Then \( C_0 P_1 C_1 P_2 C_0 \) are circuits of Type III. We thus have the decomposition

\[
I_T = \frac{1}{2} I_{\Sigma(C_0 P_1 C_1)} + \frac{1}{2} I_{\Sigma(C_1 P_2 C_0)}. \]

When \( T \) has three or more block paths (of possible zero length), choose an end-block cycle \( C \) and a block path \( P \) (of possible zero length) having its initial vertex \( u \) on \( C \) and its terminal vertex \( v \) on another block cycle \( C' \). Since \( T \) has at least three block paths, the cycle \( C' \) cannot be a loop; so all edges of \( C' \) are not loops. Choose an edge \( x \) on \( C' \) at \( v \), change the sign of \( x \), and remove the cycle \( C \) and the internal part of \( P \) from \( T \). We obtain an Eulerian cycle-tree \( T' \); see Figures 9 and 10. Then \( W \) can be written as \( W = C' P W' P^{-1} \), where \( W' \) is a closed walk on \( T' \) that uses each edge of block cycles once and each edge of block paths twice. Thus \( (W', \varepsilon_{T'}) \) is a minimal Eulerian walk, where \( \varepsilon_{T'} \) is a direction of \( T' \) and \( \varepsilon_{T'} = \varepsilon_T \) except \( \varepsilon_{T'}(v, x) = -\varepsilon_T(v, x) \). By induction \( W' \) can be written as

\[
W' = C'_0 P'_1 C'_1 P'_2 \cdots P'_k C'_{k+1},
\]
satisfying the conditions (i)–(iv). There are two cases: \( C' \) is either an end-block cycle of \( T' \), or \( C' \) is not an end-block cycle of \( T' \).

In the case that \( C' \) is an end-block cycle of \( T' \), we may assume \( C'_k = C' \), having its unique intersection vertex at \( w \) in \( T' \). Let us write \( C' \) as a closed path \( C'_k = P'Q' \), where \( P' \) is a path from \( v \) to \( w \) on \( C' \) and \( Q' \) is the other path from \( w \) to \( v \) on \( C' \).
Note that $P'_k$ is a path whose terminal vertex is $w$, and $P'_{k+1}$ is a path whose initial vertex is $w$; see Figure 9. Set $C_i = C'_i \ (0 \leq i \leq k-1)$, $P_i = P'_i \ (1 \leq i \leq k-1)$, and $P_k = P'_k Q' P'^{-1}$, $C_k = C$, $P_{k+1} = P P' P'_{k+1}$.

Then $W = C_0 P_1 C_1 P_2 \cdots P_k C_k P_{k+1}$ is a closed walk on $T$, satisfying the conditions (i)–(iv); see Figure 9.

In the case that $C'$ is not an end-block cycle of $T'$, we may assume that $P'_{k+1}$ contains the vertex $v$ and the edge $x$. Let us write $P'_{k+1} = P' Q'$, where $P'$ is a path whose terminal vertex is $v$ and $Q'$ is a path whose initial vertex is $v$; see Figure 10. Set $C_i = C'_i \ (0 \leq i \leq k)$, $P_i = P'_i \ (1 \leq i \leq k)$, and $P_{k+1} = P' P'^{-1}$, $C_{k+1} = C$, $P_{k+2} = P Q'$.

Then $W = C_0 P_1 C_1 P_2 \cdots P_{k+1} C_{k+1} P_{k+2}$ is a closed walk on $T$, satisfying the conditions (i)–(iv) with the direction $\varepsilon_T$; see Figure 10. \hfill \Box

**Problem.** An Eulerian cycle-tree is said to be *bridgeless* if it does not contain block paths of positive length. The indicator function of a bridgeless Eulerian cycle-tree has constant value 1 on its support. It should be interesting to consider integral flows $f$ such that $\Sigma(f)$ is connected and has no bridges; we may call such integral flows as *bridgeless flows*. A bridgeless flow $f$ is said to be *bridgeless decomposable* if there exist nontrivial bridgeless flows $f_1, f_2$ such that $f = f_1 + f_2$, where $f_i$ have the same sign, that is, $f_1 \cdot f_2 \geq 0$. It is particularly wanted to classify *bridgeless indecomposable flows*, that is, the integral flows that are not bridgeless decomposable.

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