

1. Introduction

The construction of the twistor space of an odd-dimensional Riemannian manifold (as well as the one in the even-dimensional case) can be traced back to R. Penrose [9, 10]. These spaces have been studied by many people, mainly from the point of view of the CR-geometry (see, for example, the literature quoted in [5, 6, 7]).

It is convenient to modify slightly the construction in [8, 11] and to define the twistor space of an odd-dimensional Riemannian manifold \((M, g)\) as the bundle \(\mathcal{C}\) over \(M\) whose fibre at a point \(p \in M\) consists of all pairs \((\varphi, \xi)\) of a skew-symmetric endomorphism \(\varphi\) and a unit vector \(\xi\) of \(T_pM\) such that \((\varphi, \xi, g_p)\) satisfies the (algebraic) identities in the definition of an almost contact metric structure ([5, 6]). The smooth manifold \(\mathcal{C}\) admits two natural partially complex structures \((f\)-structures\) \(\Phi_1\) and \(\Phi_2\), a 1-parameter family of Riemannian metrics \(h_t, t > 0\), compatible with \(\Phi_1\) and \(\Phi_2\), and a globally defined \(h_t\)-unit vector field \(\chi\) such that \((\Phi_\alpha, \chi, h_t), \alpha = 1, 2\), is an almost contact metric structure on \(\mathcal{C}\) (we refer to [8] for general facts about such structures). According to [7] the metrics \(h_t\) on \(\mathcal{C}\) are never Einstein. In this note we consider on \(\mathcal{C}\) a generalization of the Einstein condition adapted for almost contact metric manifolds, namely the so-called eta-Einstein condition. Let us note that the almost contact metric structures \((\Phi_\alpha, \chi, h_t), \alpha = 1, 2\), are never Sasakian (even they are never contact [5, 9]). The eta-Einstein condition for Sasakian manifolds has been recently discussed in [4].

Recall that an almost contact metric structure with contact form \(\eta\) and associated metric \(h\) on an odd-dimensional manifold \(N\) is said to be eta-Einstein if there exist smooth functions \(a\) and \(b\) on \(N\) such that

\[
Ricci_h(X,Y) = ah(X,Y) + b\eta(X)\eta(Y), \quad X,Y \in TN.
\]
It is clear that the functions $a$ and $b$ are uniquely determined - denoting by $s$ and $\xi$ the scalar curvature of $h$ and the $h$-dual vector field of $\eta$, we have $s = a(dim N) + b$, $Ricci(\xi, \xi) = a + b$. Note that, in contrast to the Einstein case, the functions $a$ and $b$ are not constant in the general case.

The main purpose of this paper is to prove the following

**Theorem 1.** Let $(M, g)$ be a Riemannian manifold of odd dimension $n \geq 3$. Then its twistor space $\mathcal{C}$ endowed with the metric $h_t$ and the contact form $\eta_t = h_t(\cdot, \chi)$ is eta-Einstein if and only if $n = 3$, the manifold $(M, g)$ is of positive constant curvature $\nu$ and $t\nu = \frac{1}{2}$; in this case we have $a = \frac{3\nu}{2}$ and $b = -\frac{\nu}{2}$ where the functions $a$ and $b$ on $\mathcal{C}$ are defined by means of (1).

The proof is based on a coordinate-free formula for the Ricci tensor of the metric $h_t$ in terms of the curvature of the base manifold $(M, g)$ obtained in [7].

Suppose that the base manifold $M$ is oriented. Then its twistor space $\mathcal{C}$ is the disjoint union of the open subsets $\mathcal{C}_{\pm}$ consisting of the points $(\varphi, \xi)$ that yield $\pm$ the orientation of $T_pM$ via the decomposition $T_pM = \text{Im} \varphi \oplus \mathbb{R} \xi$ in which the vector space $\text{Im} \varphi$ is oriented by means of the complex structure $\varphi$ on it. These open sets are diffeomorphic by the map $(\varphi, \xi) \rightarrow (\varphi, -\xi)$ which sends $(\Phi_\alpha, \chi, h_t)$ to $(\Phi_\alpha, -\chi, h_t)$. If, in addition, $M$ is of dimension 3, the map $(\varphi, \xi) \rightarrow \xi$ is a diffeomorphism of $\mathcal{C}_{\pm}$ onto the unit tangent bundle $T_1M$ of $M$ which sends the characteristic vector field $\chi$ to the standard characteristic vector field on $T_1M$ (but neither of the structures $\Phi_\alpha$ is going to the standard partially complex structure of $T_1M$); this map sends also the metric $h_t$ to the dilation of the Sasaki metric by $2t$ in the vertical directions. Thus Theorem 1 implies that the Sasaki metric on the unit tangent bundle of the 3-sphere endowed with the +1-curvature metric is eta-Einstein. In fact, by a result of S.Tanno [12], the Sasaki metric on the unit tangent bundle of any unit sphere $S^m$ is eta-Einstein and a suitable modification of this metric gives a homogeneous Einstein metric on $T_1S^m$.

2. Preliminaries

Let $V$ be a real $n$-dimensional vector space with a metric $g$. A partially complex structure on $V$ (or f-structure) of rank $2k$ is an endomorphism $F$ of $V$ of rank $2k$, $0 < 2k \leq n$, satisfying $F^3 + F = 0$. We shall say that such a structure $F$ is compatible with the metric $g$ if the endomorphism $F$ is skew-symmetric with respect to $g$.

Given a compatible partially complex structure $F$, we have the orthogonal decomposition $V = \text{Im} F \oplus \ker F$ and $F$ is a complex structure on the vector space $\text{Im} F$ compatible with the restriction of $g$.

Denote by $F_{2k}(V, g)$ the set of all compatible partially complex structures of rank $2k$ on $(V, g)$. The group $O(V)$ of orthogonal transformations of
$V$ acts transitively on $F_k(V, g)$ by conjugation and $F_k(V, g)$ can be identified with the homogeneous space $O(n)/(U(k) \times O(n-2k))$; in particular, $\dim F_k(V, g) = 2nk - 3k^2 - k$. By the results of [11], the homogeneous manifold $F_k(V, g)$ admits a unique (up to homotety) invariant Kähler-Einstein structure $\phi, J$. It can be described in the following way (see, for example, [7, 5, 6]): Consider $F_k(V, g)$ as a (compact) submanifold of the vector space $so(V)$ of skew-symmetric endomorphisms of $(V, g)$. Then the tangent space of $F_k(V, g)$ at a point $F$ consists of all endomorphisms $Q \in so(V)$ such that $QF^2 + FQF + F^2Q + Q = 0$. Let $G(S, T) = -\frac{1}{2} \text{Trace} ST$ be the standard metric on the space $so(V)$. Then the metric $h$ and the complex structure $J$ of $T_F F_k(V, g)$ are given by:

$$h(P, Q) = 2G(P, Q) - G(FPF, Q), \quad JQ = FQ - QF + FQF^2$$

(the complex structure $J$ coincides with the one defined in [11]). It is not hard to compute (see [11]) that the scalar curvature of $h$ is equal to $\frac{1}{2}(n - k - 1)(2nk - 3k^2 - k)$.

Now suppose that $V$ is of odd dimension $n = 2k + 1$. A (linear) almost contact metric structure on the Euclidean space $(V, g)$ is a pair $(\phi, \xi)$ of an endomorphism $\phi$ and a unit vector $\xi$ of $V$ such that $\phi^2 x = -x + g(x, \xi)\xi$ and $g(\phi x, \phi y) = g(x, y) - g(x, \xi)g(y, \xi)$, $x, y \in V$.

Denote the set of these structures by $C(V, g)$. It is easy to see (cf. e.g. [3]) that if $(\phi, \xi) \in C(V, g)$, then $\phi$ is a compatible partially complex structure of rank $2k$ and $\phi(\xi) = 0$. Conversely, if $\phi$ is a compatible partially complex structure of rank $2k$ and $\phi(\xi) = 0$ for a unit vector $\xi$, then $(\phi, \xi) \in C(V, g)$.

The set $C(V, g)$ is a compact submanifold of $so(V) \times V$; its tangent space at a point $\sigma = (\phi, \xi)$ consists of all pairs $(Q, \phi Q(\xi))$ with $Q \in T_{\sigma} F_k(V, g)$. The group $O(V)$ of orthogonal transformations of $V$ acts transitively on $C(V, g)$ in an obvious way and $C(V, g)$ has the homogeneous representation $C(V, g) = O(2k+1)/(U(k) \times \{1\})$. Thus we have an obvious two-fold covering map

$$C(V, g) = O(2k+1)/(U(k) \times \{1\}) \rightarrow F_k(V, g) = O(2k+1)/(U(k) \times \{1, -1\})$$

In fact this is the projection map $(\phi, \xi) \rightarrow \phi$. We lift the Kähler-Einstein structure of $F_k(V, g)$ to the manifold $C(V, g)$ by means of this map and denote the lifted structure again by $(h, J)$.

3. **The twistor space $(\mathcal{C}, h_t)$ and its Ricci tensor**

First we recall the definition of the twistor space of partially complex structures [11].

Let $(M, g)$ be a Riemannian manifold of dimension $n \geq 3$. Denote by $\pi : \mathcal{F}_k \rightarrow M$ the bundle over $M$ whose fibre at a point $p \in M$ consists of all compatible partially complex structures of rank $2k$ on the Euclidean space $(T_p M, g_p)$. This is the associated bundle

$$O(M) \times_{O(n)} F_k(\mathbb{R}^n)$$
where $O(M)$ denotes the principal bundle of orthonormal frames on $M$.

As it is usual in the twistor theory, the manifold $F_k$ admits two partially complex structures $\Phi_1$ and $\Phi_2$ defined as follows \([11]\): (recall that a partially complex structure on a manifold is an endomorphism $\Phi$ of its tangent bundle having constant rank and such that $\Phi^2 + \Phi = 0$): The Levi-Civita connection on $M$ gives rise to a splitting $V \oplus H$ of the tangent bundle of any bundle associated to $O(M)$ into vertical and horizontal parts. The vertical space $V_f$ of $F_k$ at a point $f \in F_k$ is the tangent space at $f$ of the fibre through this point and $\Phi_1|V_f$ is defined to be the complex structure $J_f$ of the fibre while $\Phi_2|V_f$ is defined as the conjugate complex structure, i.e. $\Phi_2|V_f = -J_f$. The horizontal space $H_f$ is isomorphic via the differential $\pi_* f$ to the tangent space $T_p M$, $p = \pi(f)$, and both $\Phi_1|H_f$, $\Phi_2|H_f$ are defined to be the lift to $H_f$ of the endomorphism $f$ of $T_p M$.

The metrics of $F_k(\mathbb{R}^n)$ and $M$ yield a 1-parameter family $h_t$, $t > 0$, of Riemannian metrics on $F_k$ such that $h_t|V_f$ is $t$ times the metric $h$ of the fibre through $f$, $h_t|H_f = \pi_* g$, and the spaces $V_f$ and $H_f$ are orthogonal. The endomorphisms $\Phi_1$ and $\Phi_2$ are skew-symmetric with respect to the metrics $h_t$ and the projection $\pi : (F_k, h_t) \to (M, g)$ is a Riemannian submersion with totally geodesic fibres (by the Vilms theorem).

Now let $(M, g)$ be a Riemannian manifold of odd dimension $n = 2k + 1$, $k \geq 1$. Slightly modify the twistor construction in \([3, 11]\), we define the twistor space of $(M, g)$ as the bundle $C$ over $M$ whose fibre at a point $p \in M$ consist of all almost contact metric structures on the Euclidean space $(T_p M, g_p)$, i.e.

$$C = O(M) \times O(2k+1) C(\mathbb{R}^{2k+1}).$$

Using the Levi-Civita connection on $M$, we can define on $C$ a 1-parameter family $h_t$, $t > 0$, and two partially complex structures $\Phi_1, \Phi_2$ of rank $k^2 + 3k$, skew-symmetric with respect to $h_t$ in the same way as we did it for the space $F_k$. Define a vector field $\chi$ on $C$ by setting

$$\chi_\sigma = \xi_\sigma^h, \quad \sigma = (\varphi, \xi),$$

where $\xi_\sigma^h$ is the horizontal lift of $\xi$ at the point $\sigma$. Then $(\Phi_\alpha, \chi, h_t)$, $\alpha = 1, 2$, is an almost contact metric structure on $C$. The contact distribution of this structure is obviously $V \oplus \{A \in H : A \perp \chi\}$.

If the manifold $M$ is oriented, then the twistor space $C$ is the disjoint union of the open subsets $C_{\pm}$ consisting of the points $(\varphi, \xi)$ that yield $\pm$ the orientation of $T_p M$ via the decomposition $T_p M = Im \varphi \oplus \mathbb{R} \xi$ in which the vector space $Im \varphi$ is oriented by means of the complex structure $\varphi$ on it. The bundles $C_{\pm}$ are isomorphic by the map $(\varphi, \xi) \to (\varphi, -\xi)$ which preserves the horizontal spaces and sends $(\Phi_\alpha, \chi, h_t)$ to $(\Phi_\alpha, -\chi, h_t)$.

The natural covering map $C(\mathbb{R}^{2k+1}) \to F_k(\mathbb{R}^{2k+1})$ is $O(2k+1)$-equivariant, so it determines a bundle map

$$C \to F_k, \quad (\varphi, \xi) \to \xi,$$
which is a two-fold covering. This map preserves the vertical and horizontal spaces, the metrics and the partially complex structures of $\mathcal{C}$ and $\mathcal{F}_k$. If $M$ is oriented, each of the spaces $\mathcal{C}_+$ and $\mathcal{C}_-$ is isomorphic to $\mathcal{F}_k$.

The curvature of the Riemannian manifold $(\mathcal{F}_k, h_\sharp)$ has been computed in [2] by means of the O'Neill formulas. The computation there immediately gives the curvature of the manifold $(\mathcal{C}, h_\sharp)$ since the map $(\varphi, \xi) \rightarrow \xi$ above is a Riemannian covering. To formulate the corresponding result for the curvature of $(\mathcal{C}, h_\sharp)$, we shall introduce some notations. Denote by $A(TM)$ the bundle of skew-symmetric endomorphisms of $TM$ and consider $\mathcal{C}$ as a submanifold of the bundle $\pi : A(TM) \oplus TM \rightarrow M$. Then the inclusion of $\mathcal{C}$ is fibre-preserving and the horizontal subspace of $T_\sigma \mathcal{C}$ at a point $\sigma = (\varphi, \xi)$ coincides with the horizontal space of $A(TM) \oplus TM$ at that point. The vertical space of $\mathcal{C}$ at $\sigma$, considered as a subspace of the vertical space of $A(TM) \oplus TM$ at $\sigma$, consists of all pairs $(Q, \varphi Q(\xi))$ for which $Q \in A(T_pM)$, $p = \pi(\sigma)$, and satisfies the identity $Q\varphi^2 + \varphi^2 Q + \varphi Q \varphi + Q = 0$.

Further we shall freely make use of the standard isometric identification $A(TM) \cong \Lambda^2 TM$ that assigns to each $a \in A(T_pM)$ the $2$-vector $a^\wedge$ for which $g(aX, Y) = g(a^\wedge, X \wedge Y)$, $X, Y \in T_p M$ (the metric on $\Lambda^2 TM$ is given by $g(X_1 \wedge X_2, X_3 \wedge X_4) = g(X_1, X_3)g(X_2, X_4) - g(X_1, X_4)g(X_2, X_3)$). If $Q \in T_\varphi F_k(T_pM, g_p) \subset A(T_pM)$, the element $Q^\wedge$ of $\Lambda^2 T_pM$ will be also denoted by $Q$. For brevity, denote by $m_{\varphi}$ the image of the tangent space $T_\varphi F_k(T_pM, g_p)$, $p = \pi(\sigma)$, under the identification $A(T_pM) \cong \Lambda^2 T_pM$. Finally, let $\mathcal{R} : \Lambda^2 TM \rightarrow \Lambda^2 TM$ be the curvature operator of $(M, g)$; it is defined by $g(\mathcal{R}(X_1 \wedge X_2), X_3 \wedge X_4) = g([\nabla X_1, X_3] - [\nabla X_1, X_3])X_4)$. Now [7] Proposition 2 implies the following

**Proposition 1.** Let $(M, g)$ be a Riemannian manifold of odd dimension $n = 2k+1$, $k \geq 1$. Then the the Ricci tensor $c_\varphi$ of its twistor space $(\mathcal{C}, h_\sharp)$ is given as follows: For any $E \in T_\sigma \mathcal{C}$, $\sigma = (\varphi, \xi)$, setting $X = \pi_* E$, $(Q, \varphi Q(\xi)) = \nabla E$ (the vertical component of $E$), we have:

$$c_\varphi(E, E) = c_M(X, X) - 2t \text{Trace}(Z \rightarrow (\nabla Z R)(JQ, X)) +$$

$$2t^2 ||\mathcal{R}(JQ)||^2 - 2t ||\iota_X \circ \mathcal{R}|| m_{\varphi} \cdot ||Q||^2_{h, g} + \frac{1}{2} k ||Q||^2_{h, g}$$

where $c_M$ is the Ricci tensor of $(M, g)$, $\iota_X : \Lambda^2 TM \rightarrow TM$ is the interior product and $|| : ||_{h, g}$ is the norm of the metric on the space of linear maps $m_{\varphi} : T_{\pi(\sigma)}M \rightarrow T_{\pi(\sigma)}M$ induced by the metrics $h$ on $m_{\varphi}$ and $g$ on $T_{\pi(\sigma)}M$.

**Corollary 1.** If $(M, g)$ is of constant curvature $\nu$, then the Ricci tensor $c_\varphi$ of $(\mathcal{C}, h_\sharp)$ is given by

$$c_\varphi(E, E) = 2k\nu(1 - t\nu)||X||^2_{h, g} + t^{\nu^2} ||\varphi X||^2_{h, g} +$$

$$\frac{1}{2}k(1 + 2t^2 \nu^2) ||Q||^2_{h, g} + t^2 \nu^2 h(\varphi \circ Q \circ \varphi, Q)$$

where $\sigma = (\varphi, \xi) \in \mathcal{C}$, $E \in T_\sigma \mathcal{C}$, $X = \pi_* E$ and $(Q, \varphi Q(\xi)) = \nabla E$. 


4. Proof of the Theorem

Suppose that \((\mathcal{C}, h_t, \chi)\) is eta-Einstein. Then, by Proposition 1, there exist smooth functions \(a\) and \(b\) on \(\mathcal{C}\) such that for every point \(\sigma = (\varphi, \xi) \in \mathcal{C}\), every \(h\)-orthonormal basis \(\{U_\alpha\}\) of \(T_p F_k(T_p, g_p)\), \(p = \pi(\sigma)\), and every \(X \in T_p M\), \(Q \in T_p F_k(T_p, g_p)\) the following two equations are satisfied

\[
(2) \quad c_M(X, X) - 2t \sum_{a=1}^{k^2+k} ||R(U_\alpha)X||^2 = a(\sigma)||X||^2 + b(\sigma)g(X, \xi)^2
\]

\[
(3) \quad 2t^2||\mathcal{R}(Q)||^2 + \frac{1}{2}k||Q||^2_h = a(\sigma)t||Q||^2_h
\]

Lemma 1. The functions \(a\) and \(b\) on \(\mathcal{C}\) descend to smooth functions \(\bar{a}\) and \(\bar{b}\) on \(M\).

Proof. We have

\[
(4) \quad c_t(E', E'') = ah_t(E', E'') + bh_t(E', \chi)h_t(E'', \chi)
\]

for every \(E', E'' \in T\mathcal{C}\).

Take a point \(\sigma = (\varphi, \xi) \in \mathcal{C}\) and set \(p = \pi(\sigma)\). Let \(e_1, ..., e_{2k+1}\) be an orthonormal basis of \(T_p M\) with \(e_{2k+1} = \xi\) and let \(V_1, ..., V_{k^2+k}\) be a \(h_t\)-orthonormal basis of the vertical space \(\mathcal{V}_\sigma\) of \(\mathcal{C}\). Denote by \(\{A_i\}\) the \(\mathcal{h}_t\)-orthonormal basis \(\{e_1^h, ..., e_{2k+1}^h, V_1, ..., V_{k^2+k}\}\) of \(T_p \mathcal{C}\). Denote by \(D\) the Levi-Civita connection of the metric \(h_t\) on \(\mathcal{C}\). Then a standard application of the differential Bianchi identity gives:

\[
\sum_i A_m(c_t(A_i, A_i)) = \sum_{i,j} h_t((D_{A_m} R_t)(A_i, A_j, A_i), A_j) = 2 \sum_i A_i(c_t(A_i, A_m))
\]

where \(R_t\) is the curvature tensor of the metric \(h_t\) on \(\mathcal{C}\). For every \(l = 1, ..., 2k+1\) and \(\beta = 1, ..., k^2+k\), we have \(c_t(e_l^h, V_\beta) = 0\) by (4). Therefore

\[
(k^2+3k+1)V_\beta(a) + V_\beta(b) = \sum_i V_\beta(c_t(A_i, A_i)) = 2 \sum_{\gamma=1}^{k^2+k} V_\gamma(c_t(V_\gamma, V_\beta)) = 2V_\beta(a).
\]

It follows that the function \((k^2+3k-1)a + b\) is constant on the fibers of \(\mathcal{C}\). Denote by \(s\) the scalar curvature of \((M, g)\). Then (2) and (3) imply that

\[
s(p) + t^{-1}(k^2+k^2) = (2k^2+4k+1)a(\sigma) + b(\sigma).
\]

Thus, the function \((2k^2 + 4k + 1)a + b\) is also constant on the fibers of \(\mathcal{C}\). This proves the lemma.

Let \(p \in M\) and let \(e_1, e_2, ..., e_n\) be an orthonormal basis of \(T_p M\), \(n = 2k+1\). Set \(e_{ij} = e_i \wedge e_j\). Assume that \(n = 2k+1 \geq 5\).
Lemma 2. For any \( X \in T_pM \) we have
\[
c_M(X, X) - t \sum_{p=1}^{k} \sum_{j=2p+1}^{\infty} (\|R(e_{2p-1,j})X\|^2 + \|R(e_{2p,j})X\|^2) = \frac{a(p)}{\sqrt{2}} (e_{2p-1,2q-1} - e_{2p,2q}) + \frac{a(p)}{\sqrt{2}} (e_{2p-1,2q} + e_{2p,2q-1})
\]
where \( p = 1, \ldots, k-1, q = p+1, \ldots, n, r = 1, \ldots, k \). Then \( \{U_\alpha\} = \{A_{pq}', A_{pq}'', B_{rs}', B_{rs}''\} \) is a h-orthonormal basis of \( T_{x}F_k(T_pM, g_p) \) (such that \( \mathcal{J} A_{pq}' = A_{pq}'', \mathcal{J} B_{rs}' = B_{rs}'' \)). Writing (2) for this basis we get an identity which involves the basis \( e = (e_1, \ldots, e_n) \); in view of Lemma 1, its right-hand side depends on the choice of the vector \( e_n \) and does not depend on the particular choice of \( \varphi \). We denote by \( (2)_{e} \) the identity we obtain in this way. Set \( e' = (e_1, e_2, \ldots, e_n) \). Then the identity \( (2)_{e} - (2)_{e'} \) reads as
\[
g(R(e_{13})X, R(e_{24})X) - g(R(e_{14})X, R(e_{23})X) + \ldots +
g(R(e_{1,2k-1})X, R(e_{2,2k})X) - g(R(e_{1,2k})X, R(e_{2,2k-1})X) = 0.
\]
Now we apply (3) for the bases \( e = (e_1, \ldots, e_4, \ldots, e_n) \) and \( e'' = (e_1, \ldots, -e_4, \ldots, e_n) \). Then the identity \( (3)_{e} - (3)_{e''} \) gives
\[
g(R(e_{13})X, R(e_{24})X) - g(R(e_{14})X, R(e_{23})X) = 0.
\]
It follows that \( g(R(e_{2p-1,2q-1})X, R(e_{2p,2q})X) = g(R(e_{2p-1,2q})X, R(e_{2p,2q-1})X) \) for \( p = 1, 2, \ldots, k-1, q = p + 1, p + 2, \ldots, n \). The latter identity, \( (2)_{e} \) and Lemma 1 imply Lemma 2.

The proofs of the next two lemmas go in the same lines as the proofs of Lemmas 6 and 7 in [7] (in view of Lemmas 1 and 2 above) and will be omitted.

Lemma 3. If \( i, j, l, m \in \{1, \ldots, n\} \) are four different indexes, then
\[
g(R(e_{ij})X, R(e_{lm})X) = 0
\]
for any \( X \in T_pM \).

Lemma 4. For any \( i \neq j, l \neq m \) and \( X \in T_pM \), we have
\[
\|R(e_{ij})X\| = \|R(e_{lm})X\|.
\]
Proof of the theorem in the case \( n = 2k + 1 \geq 5 \).

Assume that \((C, h, \chi)\) is eta-Einstein. Let \( p \in M \) and let \( e_1, \ldots, e_n \), \( n = 2k + 1 \), be an orthonormal basis of \( T_p M \). Set \( \varphi = e_{12} + \ldots + e_{2k-1,2k} \) and \( \xi = e_{2k+1} \). Set also

\[
Q_1 = e_{13} - e_{24}, \quad Q_2 = e_{1,2k+1}.
\]

Then \( Q_1, Q_2 \in T_\varphi F_k(T_p M, g_p) \) and according to (3) and Lemma 1

\[
||R(Q_1)||^2 = ||R(Q_2)||^2 = t^{-2} \left[ \bar{a}(p) t - \frac{1}{2} k \right].
\]

On the other hand, by Lemmas 3 and 4

\[
||R(Q_1)X||^2 = 2||R(e_{13})X||^2 = 2||R(e_{1,2k+1})X||^2 = 2||R(Q_2)X||^2
\]

for any \( X \in T_p M \). Therefore \( ||R(Q_1)||^2 = 2||R(Q_2)||^2 \) and we see that \( R(Q_1) = R(Q_2) = 0 \). Then

\[
||R(e_{13})|| = 2^{-1/2} ||R(Q_1)|| = 0 \quad \text{and} \quad \bar{a}(p) t - \frac{1}{2} k = 0.
\]

The first of these identities implies \( R = 0 \). Now taking \( X = e_1 \) in (2) we see that \( \bar{a}(p) = 0 \) which contradicts to the second identity above.

Proof of the theorem in the case \( n = 3 \).

Let \( c_M \) be the Ricci tensor of \((M, g)\) and \( \rho : T_p M \to T_p M \) the symmetric operator corresponding to \( c_M \). Denote by \( s \) the scalar curvature of \((M, g)\). It is well-known that the curvature operator of a 3-dimensional Riemannian manifold is given by

\[
R(X \wedge Y) = -\frac{s}{2} X \wedge Y + \rho(X) \wedge Y + X \wedge \rho(Y)
\]

for \( X, Y \in TM \) (see e.g. [2, Sec. 1 G]). Let \( p \in M \) and put

\[
\lambda = \frac{1}{2t^2} \left[ a(p) t - \frac{1}{2} k \right].
\]

Let \( e_1, e_2, e_3 \) be an arbitrary orthonormal basis of \( T_p M \). Consider the point \( \sigma = (\varphi, \xi) \in C \) with \( \varphi = e_{12}, \xi = e_3 \). Then \( e_{13}, e_{23} \in T_\varphi F_1(T_p M, g_p) \) and by (3) we have

\[
||R(e_{13})||^2 = ||R(e_{23})||^2 = 2\lambda \quad \text{and} \quad g(\mathcal{R}(e_{13}), \mathcal{R}(e_{23})) = 0.
\]

It follows that either \( \lambda = 0 \) or the operator \( T = (1/\sqrt{2\lambda})\mathcal{R} \) is orthogonal. If \( \lambda = 0 \), then \( \mathcal{R} = 0 \) and identity (2) implies \( \bar{a}(p) = 0 \). This together with \( \lambda = 0 \) gives \( k = 0 \), a contradiction. Thus, the operator \( T \) is orthogonal. Since this operator is also symmetric, its square is equal to \( Id \). Therefore the eigenvalues of \( T \) are +1 or -1. Suppose that both +1 and -1 are eigenvalues of the operator \( T \) and denote by \( \alpha \) and \( \beta \) the dimensions of the corresponding eigenspaces. Then

\[
\frac{s}{2} = \text{Trace} \mathcal{R} = (\alpha - \beta) \sqrt{2\lambda}.
\]
Further, by (2) and (3), we have that
\[ s - 8t\lambda = 3\bar{a} + \bar{b}, \]
therefore
\[ \lambda = \frac{2ts - 3 - 2\bar{b}t}{28t^2}. \] (10)

Set \( c_{ij} = c_M(e_i, e_j). \) Now take the point \( \sigma = (e_{13}, e_2) \in C \) and apply (2) with \( U_1 = \frac{1}{\sqrt{2}}e_{12}, \) \( U_2 = \frac{1}{\sqrt{2}}e_{23}, \) \( X = e_1. \) This gives
\[ c_{11} - t(||R(e_{12})e_1||^2 + ||R(e_{23})e_1||^2) = \bar{a}(p). \] (11)

Similarly, considering the point \( (e_{23}, e_1) \in C \) and applying (2), we get
\[ c_{11} - t(||R(e_{12})e_1||^2 + ||R(e_{13})e_1||^2) = \bar{a}(p) + b(p). \]
Hence, \( \bar{b}(p) = t(||R(e_{12})e_1||^2 - ||R(e_{13})e_1||^2), \) which, in view of (8), implies
\[ \bar{b}(p) = [c_{12}^2 - \left( \frac{s}{2} - c_{22} \right)^2]. \]

Similarly, we have also
\[ \bar{b}(p) = [c_{23}^2 - \left( \frac{s}{2} - c_{33} \right)^2] \quad \text{and} \quad \bar{b}(p) = [c_{13}^2 - \left( \frac{s}{2} - c_{11} \right)^2]. \]

Adding the last three equalities and setting \( \mu = c_{12}^2 + c_{23}^2 + c_{13}^2, \) we obtain
\[ 3\bar{b}(p) = t[\mu - (\frac{s}{2} - c_{11})^2 - (\frac{s}{2} - c_{22})^2 - (\frac{s}{2} - c_{33})^2]. \] (12)

Next, an obvious application of (2) for the point \( (e_{12}, e_3) \in C, \) gives
\[ c_{11} - t(||R(e_{13})e_1||^2 + ||R(e_{23})e_1||^2) = \bar{a}(p). \]
From the latter identity and (11) we see that
\[ ||R(e_{12})e_1|| = ||R(e_{13})e_1||. \]
This and (3) imply
\[ -\left( \frac{s}{2} + c_{11} + c_{22} \right)^2 = -\left( \frac{s}{2} + c_{11} + c_{33} \right)^2 \]
which gives \( c_{11}(c_{22} - c_{33}) = 0. \) Similarly, \( c_{22}(c_{33} - c_{11}) = 0 \) and \( c_{33}(c_{11} - c_{22}) = 0. \) It follows that either \( c_{11} = c_{22} = c_{33} \) or two of the numbers \( c_{11}, c_{22}, c_{33} \) are equal to 0 and the third one is different from zero.

1). Suppose \( c_{11} = c_{22} = c_{33}. \) Then, by (12),
\[ 3\bar{b}(p) = t\mu - \frac{ts^2}{12}. \]
Now it follows from (2) and (10) that \( ts \) satisfies the equation
\[ (7 - \frac{m}{9})x^2 - 4mx + 6m + \frac{4mt^2\mu}{3} = 0 \] (13)
where \( m = (\alpha - \beta)^2. \) This fact implies
\[ 4m^2 \geq (7 - \frac{m}{9})(6m + \frac{4mt^2\mu}{3}) \geq (7 - \frac{m}{9})6m. \]
Thus we see that $m \geq 9$. On the other hand $|\alpha - \beta| < \dim \Lambda^2 T_pM = 3$, so $m < 9$, a contradiction.

2). Assume that $c_{11} = c_{22} = 0$. In this case, according to (12), we have

$$3\bar{b}(p) = t\mu - \frac{3ts^2}{4}$$

and, in view of (9) and (10), $ts$ satisfies the equation

$$6x^2 - 4mx + 6m + \frac{4mt^2\mu}{3} = 0.$$ 

This implies $m \geq 9$ and we come again to a contradiction.

It follows that either $T = Id$ or $T = -Id$ on the whole space $\Lambda^2 T_pM$. Therefore the sectional curvature of $M$ at each point $p$ is constant and the classical Schur theorem implies that $M$ is of constant curvature, say $\nu$. Moreover $ts$ satisfies equation (13) with $m = 9$ and $\mu = 0$. Thus $ts = 3$, i.e. $t\nu = \frac{1}{2}$.

Conversely, suppose that $M$ is a 3-dimensional Riemannian manifold of positive constant curvature $\nu$ and take $t = \frac{1}{2\nu}$. Let $\sigma = (\varphi, \xi) \in C, E \in T\sigma C, X = \pi_E E$ and $\mathcal{V}E = (Q, \varphi Q(\xi))$. Since $\dim M = 3$, we have $\varphi \circ Q \circ \varphi = 0$ and Corollary 1 gives

$$c_t(E, E) = (2\nu - t\nu^2)||X||^2 + \frac{1}{2t}(1 + 2t^2\nu^2)||Q||^2_{h_t} - t\nu^2 g(X, \xi)^2 =$$

$$\frac{3\nu}{2}||E||^2_{h_t} - \frac{\nu}{2}g(X, \xi)^2.$$ 

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Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str. Bl. 8, 1113 Sofia, Bulgaria

E-mail address: jtd@math.bas.bg