LARGE AREA-CONSTRAINED WILLMORE SURFACES IN ASYMPTOTICALLY SCHWARZSCHILD 3-MANIFOLDS

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Abstract. We apply the method of Lyapunov-Schmidt reduction to study large area-constrained Willmore surfaces in Riemannian 3-manifolds asymptotic to Schwarzschild. In particular, we prove that the end of such a manifold is foliated by distinguished area-constrained Willmore spheres. The leaves are the unique area-constrained Willmore spheres with large area, non-negative Hawking mass, and distance to the center of the manifold at least a small multiple of the area radius. Unlike previous related work, we only require that the scalar curvature satisfies mild asymptotic conditions. We also give explicit examples to show that these conditions on the scalar curvature are necessary.

1. Introduction

Let \((M, g)\) be an asymptotically flat Riemannian 3-manifold with non-negative scalar curvature. Such manifolds arise as maximal initial data sets for the Einstein field equations and thus play an important role in general relativity.

Let \(\Sigma \subset M\) be a sphere with unit normal \(\nu\), mean curvature vector \(-H\nu\), area measure \(d\mu\), and area \(|\Sigma|\). The Hawking mass

\[
m_H(\Sigma) = \sqrt{|\Sigma|/16\pi} \left(1 - \frac{1}{16\pi} \int_\Sigma H^2 \, d\mu\right)
\]

of \(\Sigma\) has been used to probe the gravitational field in the domain bounded by \(\Sigma\); see e.g. [19, 13].

R. Geroch [18, p. 115] has noted that the Hawking mass does not increase if \(\Sigma\) flows in direction of the unit normal \(\nu\) at a speed equal to \(H^{-1}\), provided \(H > 0\). Moreover, he has proposed a proof of the positive energy theorem based on evolving by inverse mean curvature flow a small geodesic sphere in \((M, g)\) with Hawking mass close to zero into a large, centered sphere in the asymptotically flat end whose Hawking mass is close to the ADM-mass of \((M, g)\). Expanding upon Geroch’s idea, P. S. Jang and R. Wald [23, p. 43] have sketched a proof of the Riemannian Penrose inequality in the special case where the apparent horizon is connected. These programs have been completed in the paper [20] by G. Huisken and T. Ilmanen, where a suitable, necessarily non-smooth notion of inverse mean curvature flow is developed. H. Bray has proven the Riemannian Penrose inequality with no restriction on the number of boundary components in [4] using a different method.

D. Christodoulou and S.-T. Yau [13] have noted that the Hawking mass of stable constant mean curvature spheres is non-negative. Note that \(m_H(\Sigma) \leq 0\) in flat \(\mathbb{R}^3\) with equality if and only if \(\Sigma\) is a round sphere. The apparent tension between these results is indicative of the potential role of the Hawking mass as a measure of the gravitational field. In this relation, note that stable constant mean curvature surfaces abound in every initial data set. Indeed, as discussed in Appendix K of [7],

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there exist isoperimetric regions of every volume.

To describe our contributions here, we say that \((M, g)\) is \(C^k\)-asymptotic to Schwarzschild with mass \(m > 0\) if there is a non-empty compact set whose complement in \(M\) is diffeomorphic to \(\{ x \in \mathbb{R}^3 : |x| > 1/2 \}\) and such that, in this so-called asymptotically flat, there holds

\[
g = \left(1 + \frac{m}{2|x|}\right)^4 \tilde{g} + \sigma.
\]

Here, \(x\) is the Euclidean position vector and \(\tilde{g}\) is the Euclidean metric on \(\mathbb{R}^3\), while \(\sigma\) is a symmetric two-tensor that satisfies, as \(x \to \infty\) for every multi-index \(J\) with \(|J| \leq k\),

\[
\partial_J \sigma = O(|x|^{-2-|J|}).
\]

Note that \((M, g)\) is modeled upon the initial data of a Schwarzschild black hole given by

\[
\left( \left\{ x \in \mathbb{R}^3 : |x| \geq \frac{m}{2} \right\} \cap \left(1 + \frac{m}{2|x|}\right)^4 \tilde{g} \right).
\]

Given \(r > 1/2\), we define \(B_r \subset M\) to be the compact domain whose boundary corresponds to \(S_r(0)\) in the asymptotically flat chart. We say that a surface \(\Sigma \subset M\) is on-center if it bounds a compact region that contains \(B_1\). If \(\Sigma\) bounds a compact region disjoint from \(B_1\), it will be called outlying.

In pioneering work [21], G. Huisken and S.-T. Yau have shown that an end that is \(C^4\)-asymptotic to Schwarzschild with positive mass is foliated by stable constant mean curvature spheres. This foliation detects fundamental physical quantities associated with the initial data set such as the ADM mass and the Hamiltonian center of mass. Moreover, they have shown that the leaves of the foliation are the only stable constant mean curvature spheres of their respective mean curvature within large classes of competing surfaces. The original characterization of the leaves in [21] has been sharpened by J. Qing and G. Tian in [35], by S. Brendle and the first-named author in [6], and by A. Carlotto, O. Chodosh, and the first-named author in [7]. The optimal uniqueness result for large stable constant mean curvature spheres in asymptotically Schwarzschild initial data sets has recently been obtained by O. Chodosh and the first-named author [11, 10].

The characterization of the leaves of the foliation as the unique solutions of the isoperimetric problem for large volumes has been established by H. Bray in [3] for exact Schwarzschild (1) and by J. Metzger and the first-named author in [16, 17] for initial data asymptotic to Schwarzschild. In fact, these optimal global uniqueness results for large isoperimetric surfaces hold for asymptotically flat manifolds with positive mass, in particular for the examples constructed by A. Carlotto and R. Schoen in [8], as has recently been shown by O. Chodosh, Y. Shi, H. Yu, and the first-named author in [12] and by H. Yu in [39].

A different approach to obtain surfaces that are well-adapted to the ambient geometry is to maximize the Hawking mass under a suitable geometric constraint. Here, fixing the area is a natural choice. Area-constrained critical points of the Hawking mass are also area-constrained critical points of the Willmore energy

\[
\mathcal{W}(\Sigma) = \frac{1}{4} \int_{\Sigma} H^2 \, d\mu.
\]
We refer to such surfaces as area-constrained Willmore surfaces. Note that in e.g. [28], such surfaces are said to be of Willmore type.

Critical points of the Willmore energy, known as Willmore surfaces, satisfy the Euler-Lagrange equation

\[-W = 0\]

where

\[W = \Delta H + (|\hat{h}|^2 + \text{Ric}(\nu, \nu)) H.\]  

Here, \(\Delta\) is the non-positive Laplace-Beltrami operator, \(\hat{h}\) the traceless part of the second fundamental form \(h\), and \(\text{Ric}\) the Ricci curvature of \((M, g)\). Likewise, area-constrained Willmore surfaces satisfy the area-constrained Willmore equation

\[-W = \kappa H,\]

where \(\kappa \in \mathbb{R}\) is a Lagrange multiplier. Note that \(\kappa\) is denoted by \(\lambda\) in [28]. The linearization of the Willmore operator is denoted by \(Q\). It measures how \(-W\) changes along a normal variation of the surface \(\Sigma\). We refer to Appendix A for more details, including a discussion of the notion of stability of such surfaces.

The cross-sections of rotationally symmetric Riemannian manifolds are easily seen to form a foliation by area-constrained Willmore spheres. This observation applies in particular to the spheres of symmetry in the spatial Schwarzschild manifold (1). In [28], T. Lamm, J. Metzger, and F. Schulze have applied a delicate singular perturbation analysis to prove the existence of such a foliation also in the case of small perturbations of the Schwarzschild manifold. To state their result, we define the area radius \(\lambda(\Sigma) > 0\) of a surface \(\Sigma \subset M \setminus K\) by

\[4\pi \lambda(\Sigma)^2 = |\Sigma|\]

and its inner radius \(\rho(\Sigma)\) by

\[\rho(\Sigma) = \sup\{r > 1/2 : B_r \cap \Sigma = \emptyset\}.\]

Moreover, we use \(R\) to denote the scalar curvature of \((M, g)\). Below, we summarize Theorem 1 and Theorem 2 in [28].

**Theorem 1** ([28]). Given \(m > 0\), there is a constant \(\eta > 0\) with the following property. Suppose that \((M, g)\) is \(C^3\)-asymptotic to Schwarzschild with mass \(m > 0\) such that

\[\limsup_{|x| \to \infty} \left(|x|^2 |\sigma| + |x|^3 |D\sigma| + |x|^4 |D^2\sigma| + |x|^5 |D^3\sigma| \right) < \eta\]

and

\[\limsup_{|x| \to \infty} |x|^5 |R| < \eta.\]

There is a compact set \(K \subset M\), a number \(\kappa_0 > 0\), and spheres \(\{\Sigma(\kappa) : \kappa \in (0, \kappa_0)\}\) such that the following hold:

- \(\Sigma(\kappa)\) is a stable area-constrained Willmore sphere that satisfies (3) with parameter \(\kappa\).
- \(M \setminus K\) is smoothly foliated by the family \(\{\Sigma(\kappa) : \kappa \in (0, \kappa_0)\}\).
Moreover, there is a constant $\epsilon_0 > 0$ such that every on-center, strictly mean convex area-constrained Willmore sphere $\Sigma \subset M \setminus K$ with

$$\frac{\lambda(\Sigma)}{\rho(\Sigma)} - 1 < \epsilon_0 \quad \text{and} \quad \int_{\Sigma} |\hat{h}|^2 \, d\mu < \epsilon_0$$

(5)

is part of this foliation.

**Remark 2.** In [28], the uniqueness result is stated in terms of smallness conditions on the rescaled barycenter

$$\frac{1}{\lambda(\Sigma) |\Sigma|} \int_{\Sigma} x \, d\mu$$

and the quotient $\rho(\Sigma)^{-2} \lambda(\Sigma)$. These conditions are implied by (5) and Theorem 1.1 in [15].

The stability of the leaves $\Sigma(\kappa)$ suggests that each contains a maximal amount of Hawking mass given their surface area. Locally, this has been confirmed by the second-named author; see Theorem 1.2 in [24].

**Theorem 3** ([24]). Assumptions as in Theorem 1. Let $\Sigma \subset M \setminus K$ be a closed, on-center sphere

$$\frac{\lambda(\Sigma)}{\rho(\Sigma)} - 1 < \epsilon_0$$

and $|\Sigma| = |\Sigma(\kappa)|$ for some $\kappa \in (0, \kappa_0)$. Then

$$m_H(\Sigma) \leq m_H(\Sigma(\kappa))$$

with equality if and only if $\Sigma = \Sigma(\kappa)$. Moreover, every on-center area-constrained Willmore sphere $\Sigma \subset M \setminus K$ with

$$\frac{\lambda(\Sigma)}{\rho(\Sigma)} - 1 < \epsilon_0 \quad \text{and} \quad \int_{\Sigma} |\hat{h}|^2 \, d\mu < \epsilon_0$$

is part of the foliation from Theorem 1.

**Remark 4.** Unlike in Theorem 1, there is no assumption on the sign of the mean curvature in the uniqueness statement in Theorem 3.

Comparing with the results available for stable constant mean curvature surfaces, the assumptions of Theorem 1 and Theorem 3 are quite restrictive. Yet, there has been no subsequent result which either establishes the existence of a foliation by area-constrained Willmore spheres in a more general setting or characterizes the leaves of such a foliation more globally. Even in exact Schwarzschild initial data (1), our variational understanding of the Willmore energy is limited. In fact, it is not known if an area-constrained maximizer of the Hawking mass exists unless the prescribed area is either very small or an integer multiple of the area of the horizon; see [38, Theorem 1.6 and Remark 1.7]. By contrast, S. Brendle has shown in [5] that the spheres of symmetry are the only closed, embedded constant mean curvature surfaces in exact Schwarzschild (1). Previously, it had been known from the work of H. Bray in [3] that these spheres are the only solutions of the isoperimetric problem for the volume they enclose. Note that such a result fails for the Willmore energy, as one can construct surfaces of arbitrarily large area and Hawking mass by gluing small catenoidal necks between spheres of symmetry that are close to the horizon; see the remark below Corollary 5.4 in
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Consequently, any reasonable characterization of such surfaces can only possibly hold outside a compact set or under a small energy assumption.

In his habilitation thesis [30], P. Laurain conjectures the existence of a foliation by area-constrained Willmore spheres if the metric \(g\) satisfies the so-called Regge-Teitelboim condition (see [36]) and that all area-constrained Willmore surfaces with small energy that enclose a sufficiently large compact set are part of this foliation; cf. [30, Theorem 49 (In progress)]. Note that – being non-linear and of fourth order – the area-constrained Willmore equation (3) poses hard analytical challenges and is not as accessible geometrically as the constant mean curvature equation. What is more, Willmore stability does not appear to be as useful of a condition as the stability of a constant mean curvature surface. For instance, every closed minimal surface is a stable Willmore surface.

In this work, we establish the existence and uniqueness of foliations by area-constrained Willmore spheres in a generality analogous to the optimal results for stable constant mean curvature surfaces in [11, 10]. In summary, we discover optimal conditions on the scalar curvature under which the end of every asymptotically Schwarzschild manifold is foliated by large stable area-constrained Willmore spheres. These surfaces are unique among all large area-constrained Willmore spheres with non-negative Hawking mass whose inner radius is at least a small multiple of the area radius. Our results differ from those in Theorem 1 in that we do not require smallness of the perturbation \(\sigma\) off Schwarzschild or the centering quantity

\[
\frac{\lambda(\Sigma)}{\rho(\Sigma)} - 1
\]

such as (4) or (5), respectively.

More precisely, we first establish the existence of a foliation by area-constrained Willmore spheres assuming that \((M, g)\) is \(C^4\)-asymptotic to Schwarzschild. We also assume that the scalar curvature is asymptotically even and satisfies a certain growth condition.

**Theorem 5.** Let \((M, g)\) be \(C^4\)-asymptotic to Schwarzschild with mass \(m > 0\) and suppose that the scalar curvature \(R\) satisfies

\[
\sum_{i=1}^{3} x^i \partial_i (|x|^2 R) \leq o(|x|^{-2}) \quad \text{and}
\]

\[
R(x) - R(-x) = o(|x|^{-4}).
\]

There exists a compact set \(K \subset M\), a number \(\kappa_0 > 0\), and on-center stable area-constrained Willmore spheres \(\Sigma(\kappa)\), \(\kappa \in (0, \kappa_0)\), satisfying (3) with parameter \(\kappa\) such that \(M \setminus K\) is foliated by the family \(\{\Sigma(\kappa) : \kappa \in (0, \kappa_0)\}\). Moreover, there holds

\[
\lim_{\kappa \to 0} \frac{\lambda(\Sigma(\kappa))}{\rho(\Sigma(\kappa))} = 1.
\]

**Remark 6.** Note that the assumptions of the theorem are satisfied if \((M, g)\) is \(C^4\)-asymptotic to Schwarzschild with \(R = o(|x|^{-4})\). The \(C^4\)-decay gives \(DR = o(|x|^{-5})\) in this case, which implies (6).

**Remark 7.** In Theorem 5 and Theorem 8, it would be sufficient to require appropriate \(C^{3,\alpha}\)-decay of the metric for some \(\alpha \in (0, 1)\). We use the slightly stronger assumption for the sake of readability.
Next, we focus on the geometric characterization of the foliation \( \{ \Sigma(\kappa) : \kappa \in (0, \kappa_0) \} \). Continuing to assume the same asymptotic conditions on the scalar curvature, we show that the leaves of the foliation are the unique large area-constrained Willmore spheres whose inner radius and area radius are comparable and with traceless second fundamental form small in \( L^2 \). The conclusion of Theorem 8 below is illustrated in Figure 1.

**Theorem 8.** Assumptions as in Theorem 5. There exist a small constant \( \epsilon_0 > 0 \) and a compact set \( K \subset M \) which only depend on \( (M, g) \) such that the following holds. For every \( \delta > 0 \), there exists a large constant \( \lambda_0 > 1 \) such that every area-constrained Willmore sphere \( \Sigma \subset M \setminus K \) with

\[
|\Sigma| > 4 \pi \lambda_0^2, \quad \delta \lambda(\Sigma) < \rho(\Sigma), \quad \delta \rho(\Sigma) < \lambda(\Sigma),
\]

and

\[
\int_{\Sigma} |\hat{h}|^2 \, d\mu < \epsilon_0
\]

belongs to the foliation from Theorem 5.

**Remark 9.** The small energy assumption (9) and the assumption that \( \Sigma \) be spherical may be replaced by requiring a lower bound on the Hawking mass and an upper bound on the genus of \( \Sigma \); see Proposition 26.

**Remark 10.** The proof of Theorem 8 shows that there are no outlying area-constrained Willmore spheres satisfying (8) and (9) if the scalar curvature of \( (M, g) \) satisfies (6) but not necessarily (7). The condition (6) has been discovered by O. Chodosh and the first-named author in [10, Theorem 1.4] to be sufficient to rule out sequences of large outlying stable constant mean curvature spheres whose area radius and inner radius are comparable.

Finally, we consider large area-constrained Willmore spheres that are far-outlying in the sense that their inner radius dominates their area radius. In this regime, the contribution of the Schwarzschild metric to the Hawking mass is so weak that a stronger assumption on the scalar curvature is needed to preclude the existence of such surfaces.

**Theorem 11.** Suppose that \( (M, g) \) is \( C^5 \)-asymptotic to Schwarzschild with mass \( m > 0 \) and that its scalar curvature \( R \) satisfies

\[
\sum_{i=1}^3 x^i \partial_i (|x|^2 R) \leq 0.
\]

There exist small constants \( \epsilon_0, \delta_0 > 0 \) and a large constant \( \lambda_0 > 1 \) which only depend on \( (M, g) \) such that the following holds. Every area-constrained Willmore sphere \( \Sigma \subset M \) with

\[
|\Sigma| > 4 \pi \lambda_0^2, \quad \text{and} \quad \int_{\Sigma} |\hat{h}|^2 \, d\mu < \epsilon_0
\]

satisfies

\[
\delta_0 \rho(\Sigma) < \lambda(\Sigma).
\]

**Remark 12.** Condition (10) is stronger than (6) and, for instance, satisfied if the scalar curvature of \( (M, g) \) vanishes. In any case, the assumptions of Theorem 11 are weaker than those discovered in [10]
to be sufficient to rule out far-outlying stable constant mean curvature spheres, where stronger decay of the metric is required and the scalar curvature is assumed to either vanish or to be radially convex. This improvement owes to a conservation law for the Einstein tensor known as the Pohozaev identity. In the generality required here, this law has been observed by R. Schoen, see [37, Proposition 1.4] and [34], and applied by T. Lamm, J. Metzger, and F. Schulze in [28] in a similar context. This identity precisely brings out the contribution of the scalar curvature to the Willmore energy as we explain in Lemma 43. It turns out that the assumptions on the scalar curvature required in Theorem 11 are sufficient to rule out large far-outlying stable constant mean curvature spheres as well. We include a proof of this fact in Appendix E.

The assumptions on the scalar curvature in Theorem 5, Theorem 8, and Theorem 11 are essentially optimal. We show that the growth condition (6) cannot be relaxed to requiring the scalar curvature to be non-negative and even. In fact, these weaker conditions are not sufficient to preclude large area-constrained Willmore spheres – on-center or outlying – that satisfy (8) and (9) but do not belong to the foliation from Theorem 5. In particular, the assumptions conjecturally proposed in [30] are not quite sufficient to conclude that large area-constrained Willmore spheres are unique.

**Theorem 13.** There exist rotationally symmetric metrics \( g_1 \) and \( g_2 \) on \( M = \{ x \in \mathbb{R}^3 : |x| > 1 \} \) both \( C^k \)-asymptotic to Schwarzschild with mass \( m = 2 \) for every \( k \geq 2 \) and with non-negative scalar curvature such that the following holds. There exist sequences of stable area-constrained Willmore spheres \( \{ \Sigma^1_j \}_{j=1}^{\infty} \) and \( \{ \Sigma^2_j \}_{j=1}^{\infty} \) that are on-center in \((M, g_1)\) and outlying in \((M, g_2)\), respectively, such that

\[
\lim_{j \to \infty} |\Sigma^1_j| = \lim_{j \to \infty} |\Sigma^2_j| = \infty, \quad \lim_{j \to \infty} m_H(\Sigma^1_j) = 2, \quad \lim_{j \to \infty} m_H(\Sigma^2_j) = 0,
\]
while, for all \( j \),
\[
\frac{1}{4} < \frac{\rho(\Sigma^1_j)}{\lambda(\Sigma^1_j)} < \frac{7}{8} \quad \text{and} \quad 2\sqrt{2} < \frac{\rho(\Sigma^2_j)}{\lambda(\Sigma^2_j)} < 5.
\]

Conversely, centering of the foliation from Theorem 5 may fail if the assumption that the scalar curvature is asymptotically even is dropped. We refer to the work of C. Cederbaum and C. Nerz [9] for a thorough investigation of various divergent notions of center of mass.

**Theorem 14.** There exists a metric \( g_3 \) on \( \{ x \in \mathbb{R}^3 : |x| > 1 \} \) \( C^k \)-asymptotic to Schwarzschild with mass \( m = 2 \) for every \( k \geq 2 \) with non-negative scalar curvature satisfying (6) such that the following holds. There exists a number \( \kappa_0 > 0 \) and a smooth asymptotic foliation \( \{ \Sigma(\kappa) : \kappa \in (0, \kappa_0) \} \) by on-center stable area-constrained Willmore spheres such that
\[
\limsup_{\kappa \to 0} \frac{\lambda(\Sigma(\kappa))}{\rho(\Sigma(\kappa))} > 1.
\]

Finally, the following result shows that if we relax the growth condition on the scalar curvature only slightly, large far-outlying area-constrained Willmore spheres may exist.

**Theorem 15.** There exists a rotationally symmetric metric
\[
g_4 = (1 + |x|^{-1})^4 \bar{g} + \sigma_4
\]
on \( \{ x \in \mathbb{R}^3 : |x| > 1 \} \) with non-negative scalar curvature and, as \( x \to \infty \) for every multi-index \( J \),
\[
\partial_J \sigma_4 = O(|x|^{-3-|J|})
\]
that has the following property. There exists a sequence of stable area-constrained Willmore spheres \( \{ \Sigma_j \}^\infty_{j=1} \) such that
\[
\lim_{j \to \infty} |\Sigma_j| = \infty, \quad \lim_{j \to \infty} m_H(\Sigma_j) = 0, \quad \text{and} \quad \lim_{j \to \infty} \frac{\rho(\Sigma_j)}{\lambda(\Sigma_j)} = \infty.
\]

The study of large area-constrained Willmore spheres \( \Sigma \) with \( \lambda(\Sigma) \gg \rho(\Sigma) \) is challenging. This is on account of the loss of analytic control in the part of the surface where the area radius \( \lambda(\Sigma) \) is much larger than \( |x| \). In particular, the non-linearities owing to the Schwarzschild background dominate so the method in this paper loses its grip. We remark that large coordinate spheres cease to be mean convex in this regime and exhibit a first-order defect in their Hawking mass; see Remark 44. This suggests that the comparability assumptions on the area radius and inner radius in Theorem 8 are not necessary.

In order to prove Theorem 5 and Theorem 8, we use a strategy modeled upon the Lyapunov-Schmidt reduction developed for stable constant mean curvature spheres in [6, 10]. We will follow by and large the notation of [6, 10] throughout this paper.

By scaling, we may assume that \( m = 2 \), that is,
\[
g = (1 + |x|^{-1})^4 \bar{g} + \sigma.
\]

We use a bar to indicate that a geometric quantity has been computed with respect to the Euclidean background metric \( \bar{g} \). When the Schwarzschild metric
\[
g_S = (1 + |x|^{-1})^4 \bar{g}
\]
with mass $m = 2$ has been used in the computation, we use the subscript $S$.

For every $\xi \in \mathbb{R}^3$ and $\lambda > 1$ large, depending on $|1 - |\xi||^{-1}$, we use the implicit function theorem to perturb the sphere $S_\lambda(\lambda \xi)$ to a surface $\Sigma_{\xi,\lambda}$ with area $4 \pi \lambda^2$ and which satisfies the area-constrained Willmore equation (3) up to a sum of first spherical harmonics. On the one hand, we show that $\Sigma_{\xi,\lambda}$ is an area-constrained Willmore surface if and only if $\xi$ is a critical point of the function $G_\lambda$ given by

$$G_\lambda(\xi) = \begin{cases} 
\lambda^2 \left( \int_{\Sigma_{\xi,\lambda}} H^2 \, d\mu - 16 \pi + 64 \pi \lambda^{-1} \right) & \text{if } |\xi| < 1, \\
\lambda^2 \left( \int_{\Sigma_{\xi,\lambda}} H^2 \, d\mu - 16 \pi \right) & \text{if } |\xi| > 1.
\end{cases}$$

The absence of the term $64 \pi \lambda^{-1}$ in the case $|\xi| > 1$, which would equal $32 \pi \lambda^{-1} m$ without the normalization $m = 2$, owes to the fact that the Hawking mass of an outlying coordinate sphere does not detect the mass of $(M, g)$; see Lemma 43. On the other hand, we show that

$$G_\lambda(\xi) = 64 \pi + \frac{32 \pi}{1 - |\xi|^2} - 48 \pi |\xi|^{-1} \log \frac{1 + |\xi|}{1 - |\xi|} - 128 \pi \log(1 - |\xi|^2) + 2 \lambda \int_{\mathbb{R}^3 \setminus B_\lambda(\lambda \xi)} R \, d\bar{v} + o(1)$$

if $|\xi| < 1$ and

$$G_\lambda(\xi) = 32 \pi \frac{1}{1 - |\xi|^2} - 48 \pi |\xi|^{-1} \log \frac{|\xi| + 1}{|\xi| - 1} - 128 \pi \log(1 - |\xi|^2) - 2 \lambda \int_{B_\lambda(\lambda \xi)} R \, d\bar{v} + o(1)$$

if $|\xi| > 1$. Assuming that the scalar curvature is asymptotically even (7) and satisfies the growth condition (6), we show that $G_\lambda$ has a unique minimum near the origin for every $\lambda > 1$ large. Moreover, the corresponding surfaces $\Sigma_{\xi,\lambda}$ form a foliation. We show that these are in fact the only critical points of $G_\lambda$ if $\lambda > 1$ is large. Theorem 8 follows from this and a compactness argument. The strategy for the proof of Theorem 11 is formally similar. However, a key difference is that the first three terms in (11) become very small when $\xi$ is large. In a more precise analysis, we verify that (11) remains true up to lower-order error terms that decay sufficiently fast as $|\xi| \to \infty$. Finally, we give explicit examples that show that $G_\lambda$ may have other critical points for suitable choices of $\sigma$ that violate the particular assumptions on the scalar curvature.

Unlike large area-constrained Willmore spheres, small area-constrained Willmore spheres in closed manifolds are well-understood. It is shown in the work of T. Lamm and J. Metzger [27] and the work of A. Mondino and T. Riviere [33] that minimizers of the Willmore energy with small prescribed area exist in closed manifolds. Moreover, T. Lamm, J. Metzger, and F. Schulze [29] as well as N. Ikoma, A. Malchiodi, and A. Mondino [22] have shown that a neighborhood of a non-degenerate critical point of the scalar curvature is foliated by small area-constrained Willmore spheres. This extends previous work of T. Lamm and J. Metzger [26] as well as of A. Mondino and P. Laurain [31]. We also mention the work [1] of R. Alessandroni and E. Kuwert on small area-constrained Willmore surfaces with free boundary and the work by A. Mondino [32], where unconstrained Willmore surfaces are studied in a semi-perturbative setting. The method of Lyapunov-Schmidt reduction is used in [22, 1, 32].
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2. Proof of Theorem 5 and Theorem 8

In this section, we assume that \( g \) is a Riemannian metric on \( \mathbb{R}^3 \) such that

\[
g = (1 + |x|^{-1})^4 \bar{g} + \sigma
\]

where \( \sigma \) is a symmetric, covariant two-tensor with, as \( x \to \infty \) for every multi-index \( J \) with \( |J| \leq 4 \),

\[
\partial_J \sigma = O(|x|^{-2-|J|}).
\]

Applying a strategy similar to that in [6, 10], we perform a Lyapunov-Schmidt reduction to analyze area-constrained Willmore spheres \( \Sigma \subset \mathbb{R}^3 \) with large area and comparable area radius \( \lambda(\Sigma) \) and inner radius \( \rho(\Sigma) \).

Given \( \xi \in \mathbb{R}^3 \) and \( \lambda > 0 \), we abbreviate

\[
S_{\xi,\lambda} = S_\lambda(\lambda \xi) = \{ x \in \mathbb{R}^3 : |x - \lambda \xi| = \lambda \}.
\]

Given \( u \in C^\infty(S_{\xi,\lambda}) \), we define the map

\[
\Phi_{\xi,\lambda}^u : S_{\xi,\lambda} \to \mathbb{R}^3 \quad \text{given by} \quad \Phi_{\xi,\lambda}^u(x) = x + u(x) (\lambda^{-1} x - \xi).
\]

We denote by

\[
\Sigma_{\xi,\lambda}(u) = \Phi_{\xi,\lambda}^u(S_{\xi,\lambda})
\]

the Euclidean graph of \( u \) over \( S_{\xi,\lambda} \). Throughout, for instance in (14) below, we tacitly identify functions defined on \( \Sigma_{\xi,\lambda}(u) \) with functions defined on \( S_{\xi,\lambda} \) by precomposition with \( \Phi_{\xi,\lambda}^u \).

Let \( \alpha \in (0, 1) \) and \( \gamma > 1 \). We denote by \( \mathcal{G} \) the space of \( C^{3,\alpha} \)-Riemannian metrics on

\[
\{ y \in \mathbb{R}^3 : \gamma^{-1} \leq |y| \leq \gamma \}
\]

with the \( C^{3,\alpha} \)-topology. Let \( \Lambda_0(S_1(0)) \) and \( \Lambda_1(S_1(0)) \) be the constants and first spherical harmonics viewed as subspaces of \( C^{4,\alpha}(S_1(0)) \), respectively. We use the symbol \( \perp \) to denote the \( L^2(S_1(0)) \)-orthogonal complements of these spaces.

Lemma 16. There exist open neighborhoods \( \mathcal{U} \) of \( \bar{g} \in \mathcal{G} \), \( \mathcal{V} \) of \( 0 \in \Lambda_1(S_1(0)) \perp \), and \( I \) of \( 0 \subset \mathbb{R} \) as well as smooth maps \( u : \mathcal{U} \to \mathcal{V} \) and \( \kappa : \mathcal{U} \to I \) such that the surface \( \Sigma_{0,1}(u(g)) \) has area equal to \( 4 \pi \) and satisfies

\[
\Delta H + (|\hat{h}|^2 + \text{Ric}(\nu, \nu) + \kappa(g)) H \in \Lambda_1(S_1(0)).
\]

Here, all geometric quantities are computed with respect to the surface \( \Sigma_{0,1}(u(g)) \) and the metric \( g \). Moreover, if \( g_0 \in \mathcal{U} \), \( u_0 \in \mathcal{V} \), and \( \kappa_0 \in I \) are such that \( \Sigma_{0,1}(u_0) \) satisfies (14) with respect to \( g_0 \) and has area equal to \( 4 \pi \), then \( u_0 = u(g_0) \) and \( \kappa_0 = \kappa(g_0) \).

Proof. Let \( \Lambda_{0,0}(S_1(0)) \) and \( \Lambda_{1,0}(S_1(0)) \) be the constants and first spherical harmonics viewed as subspaces of \( C^{0,\alpha}(S_1(0)) \), respectively. Note that there are neighborhoods \( \tilde{\mathcal{U}} \) of \( \bar{g} \in \mathcal{G} \) and \( \tilde{\mathcal{V}} \) of
$0 \in \Lambda_1(S_1(0))^\perp$ such that the map
\[ T : \tilde{V} \times \mathbb{R} \times \tilde{U} \to \Lambda_{1,0}(S_1(0))^\perp \times \mathbb{R} \]
given by
\[ T(u, \kappa, g) = \left( \text{proj}_{\Lambda_{1,0}(S_1(0))^\perp} \left[ \Delta H + (|\tilde{h}|^2 + \text{Ric}(\nu, \nu) + \kappa) H \right], |\Sigma| \right), \]
where all geometric quantities are with respect to $\Sigma_{0,1}(u)$ and the metric $g$, is well-defined and smooth. Specifying (51) to $S_1(0)$ in flat $\mathbb{R}^3$, we find
\[ (DT)_{|0,0,0}(u, 0, 0) = \left( -\tilde{\Delta}^2 u - 2 \tilde{\Delta} u, 8 \pi \text{proj}_{\Lambda_0(S_1(0))} u \right). \]
Moreover, there holds
\[ (DT)_{|0,0,0}(\kappa, 0, 0) = (2 \kappa, 0). \]
As discussed in Corollary 33, the kernel of the operator
\[ -\tilde{\Delta}^2 - 2 \tilde{\Delta} : C^{4,\alpha}(S_1(0)) \to C^{0,\alpha}(S_1(0)) \]
is given by $\Lambda_0(S_1(0)) \oplus \Lambda_1(S_1(0))$. It follows from the Fredholm alternative and elliptic regularity that
\[ -\tilde{\Delta}^2 - 2 \tilde{\Delta} : [\Lambda_0(S_1(0)) \oplus \Lambda_1(S_1(0))]^\perp \to [\Lambda_{0,0}(S_1(0)) \oplus \Lambda_{1,0}(S_1(0))]^\perp \]
is an isomorphism. Thus,
\[ (DT)_{|0,0,0}(\cdot, \cdot, 0) : \Lambda_1(S_1(0))^\perp \times \mathbb{R} \to \Lambda_{1,0}(S_1(0))^\perp \times \mathbb{R} \]
is an isomorphism, too. The assertions follow from this and the implicit function theorem. \hfill \Box

We consider the map
\[ \Theta_{\xi,\lambda} : \mathbb{R}^3 \to \mathbb{R}^3 \quad \text{given by} \quad \Theta_{\xi,\lambda}(y) = \lambda (\xi + y). \]
Note that $\Theta_{\xi,\lambda}(S_1(0)) = S_{\xi,\lambda}$. The rescaled metric
\[ g_{\xi,\lambda} = \lambda^{-2} \Theta_{\xi,\lambda}^* g \]
satisfies, as $\lambda \to \infty$,
\[ ||g_{\xi,\lambda} - \bar{g}||_G = O(\lambda^{-1} |1 - |\xi||^{-1}). \]

Let $\delta \in (0, 1/2)$. The following proposition follows from Lemma 16 and scaling. We let $\Lambda_0(S_{\xi,\lambda})$ and $\Lambda_1(S_{\xi,\lambda})$ be the constants and first spherical harmonics viewed as subspaces of $C^{4,\alpha}(S_{\xi,\lambda})$, respectively.

**Proposition 17.** There are constants $\lambda_0 > 1$, $c > 1$, and $\epsilon > 0$ depending on $g$ and $\delta \in (0, 1/2)$ such that for every $\xi \in \mathbb{R}^3$ with $|\xi| < 1 - \delta$ or $|\xi| > 1 + \delta$ and every $\lambda > \lambda_0$ there exist $u_{\xi,\lambda} \in C^\infty(S_{\xi,\lambda})$ and $\kappa_{\xi,\lambda} \in \mathbb{R}$ such that the following hold. The surface
\[ \Sigma_{\xi,\lambda} = \Sigma_{\xi,\lambda}(u_{\xi,\lambda}) \]
has the properties
\[ \circ \Delta H + (|\tilde{h}|^2 + \text{Ric}(\nu, \nu) + \kappa_{\xi,\lambda}) H \in \Lambda_1(S_{\xi,\lambda}) \quad \text{and} \]
\[ \circ |\Sigma_{\xi,\lambda}| = 4 \pi \lambda^2. \]
There holds $u_{\xi, \lambda} \perp \Lambda_1(S_{\xi, \lambda})$ and

$$|u_{\xi, \lambda}| + \lambda |\nabla u_{\xi, \lambda}| + \lambda^2 |\nabla^2 u_{\xi, \lambda}| + \lambda^3 |\nabla^3 u_{\xi, \lambda}| + \lambda^4 |\nabla^4 u_{\xi, \lambda}| < c,$$

$$\lambda^3 |\kappa_{\xi, \lambda}| < c.$$  

Moreover, if $\kappa \in \mathbb{R}$ and $\Sigma_{\xi, \lambda}(u)$ with $u \perp \Lambda_1(S_{\xi, \lambda})$ are such that

- $\Delta H + (|\bar{H}|^2 + \text{Ric}(\nu, \nu) + \kappa) H \in \Lambda_1(S_{\xi, \lambda})$,
- $|\Sigma_{\xi, \lambda}(u)| = 4 \pi \lambda^2$,

and

$$|u| + \lambda |\nabla u| + \lambda^2 |\nabla^2 u| + \lambda^3 |\nabla^3 u| + \lambda^4 |\nabla^4 u| < \epsilon \lambda,$$

$$\lambda^3 |\kappa| < \epsilon \lambda,$$

then $u = u_{\xi, \lambda}$ and $\kappa = \kappa_{\xi, \lambda}$.

**Remark 18.** By the implicit function theorem and scaling, $$(\bar{D}u)|_{(\xi, \lambda)} = O(\lambda^{-1}) \quad \text{and} \quad u'|_{(\xi, \lambda)} = O(\lambda^{-2})$$

where $\bar{D}$ and the dash indicate differentiation with respect to the parameters $\xi$ and $\lambda$, respectively.

We abbreviate $u_{\xi, \lambda}$ by $u$, $\kappa_{\xi, \lambda}$ by $\kappa$, and $\Lambda_{\ell}(S_{\xi, \lambda})$ by $\Lambda_{\ell}$ for $\ell = 0, 1$. To obtain more precise information about the perturbation, we expand $u$ in terms of spherical harmonics.

**Lemma 19.** There holds

$$\text{proj}_{\Lambda_0} u = \begin{cases} 
-2 + O(\lambda^{-1}) & \text{if } |\xi| < 1 - \delta, \\
-2|\xi|^{-1} + O(\lambda^{-1}) & \text{if } |\xi| > 1 + \delta.
\end{cases}$$

**Proof.** On the one hand, we have $|\Sigma_{\xi, \lambda}| = 4 \pi \lambda^2$ by construction. By Lemma 40,

$$|S_{\xi, \lambda}| = \begin{cases} 
4 \pi \lambda^2 + 16 \pi \lambda + O(1) & \text{if } |\xi| < 1 - \delta, \\
4 \pi \lambda^2 + 16 \pi \lambda |\xi|^{-1} + O(1) & \text{if } |\xi| > 1 + \delta.
\end{cases}$$

On the other hand, by the first variation of area formula,

$$|\Sigma_{\xi, \lambda}| - |S_{\xi, \lambda}| = \int_{S_{\xi, \lambda}} H u \, d\mu + O(1) = \int_{S_{\xi, \lambda}} \bar{H} u \, d\bar{\mu} + O(1).$$

In the second equation, we have used Lemma 39 and Lemma 41. Since

$$\frac{1}{4 \pi} \lambda^{-2} \int_{S_{\xi, \lambda}} u \, d\bar{\mu} = \text{proj}_{\Lambda_0} u,$$

the assertion follows. \qed

For the statement of the next lemma, recall the definition of the Legendre polynomials $P_{\ell}$ from Appendix B.

**Lemma 20.** If $|\xi| < 1 - \delta$, there holds

$$\kappa = 4 \lambda^{-3} + O(\lambda^{-4}),$$
\[
W(\Sigma_{\xi,\lambda}) + \kappa H(\Sigma_{\xi,\lambda}) = O(\lambda^{-5}),
\]

\[
u = -2 + 4 \sum_{\ell=2}^{\infty} \frac{|\xi|^{\ell}}{\ell} P_\ell \left(-|\xi|^{-1} g(y, \xi)\right) + O(\lambda^{-1}).
\]

If \(|\xi| > 1 + \delta\), there holds

\[
k = O(\lambda^{-4}),
\]

\[
W(\Sigma_{\xi,\lambda}) + \kappa H(\Sigma_{\xi,\lambda}) = O(\lambda^{-5}),
\]

\[
u = -2|\xi|^{-1} - 4 \sum_{\ell=2}^{\infty} \frac{|\xi|^{-\ell-1}}{\ell + 1} P_\ell \left(-|\xi|^{-1} g(y, \xi)\right) + O(\lambda^{-1}).
\]

**Proof.** It follows from (52) and Lemma 48 that

\[
W(\Sigma_{\xi,\lambda}) - W(S_{\xi,\lambda}) = Q_{S_{\xi,\lambda}} u + O(\lambda^{-5}) = -\bar{\Delta}_S^2 u - 2 \lambda^{-2} \bar{\Delta}_S u + O(\lambda^{-5}).
\]

Likewise, (49), Proposition 17, Lemma 39, and Lemma 41 imply that

\[
H(\Sigma_{\xi,\lambda}) = 2 \lambda^{-1} + O(\lambda^{-2}).
\]

Conversely, by Proposition 17,

\[
W(\Sigma_{\xi,\lambda}) + H(\Sigma_{\xi,\lambda}) \kappa = Y_1 \in \Lambda_1.
\]

Thus,

\[
\bar{\Delta}_S^2 u + 2 \lambda^{-2} \bar{\Delta}_S u - 2 \lambda^{-1} \kappa = W(S_{\xi,\lambda}) - Y_1 + O(\lambda^{-5}).
\]

By Corollary 46, there holds

\[
W(S_{\xi,\lambda}) = \begin{cases}
4 \lambda^{-4} \sum_{\ell=0}^{\infty} (\ell-1)(\ell+1)(\ell+2)|\xi|^{\ell} P_\ell \left(-|\xi|^{-1} g(y, \xi)\right) + O(\lambda^{-5}) & \text{if } |\xi| < 1 - \delta, \\
-4 \lambda^{-4} \sum_{\ell=0}^{\infty} (\ell-1)(\ell+2)|\xi|^{-\ell-1} P_\ell \left(-|\xi|^{-1} g(y, \xi)\right) + O(\lambda^{-5}) & \text{if } |\xi| > 1 + \delta.
\end{cases}
\]

Projecting (16) onto \(\Lambda_1\), we find \(Y_1 = O(\lambda^{-5})\). The assertions follow from Corollary 33.

To relate the variational structure of the area-constrained Willmore equation on the families of surfaces \(\{\Sigma_{\xi,\lambda} : |\xi| < 1 - \delta\}\) and \(\{\Sigma_{\xi,\lambda} : |\xi| > 1 + \delta\}\) to a 3-dimensional problem, we introduce the functional

\[
F_\lambda(\Sigma) = \begin{cases}
\lambda^2 \left( \int_{\Sigma} H^2 \, d\mu - 16\pi + 64\pi \lambda^{-1} \right) & \text{if } \Sigma \text{ is on-center}, \\
\lambda^2 \left( \int_{\Sigma} H^2 \, d\mu - 16\pi \right) & \text{if } \Sigma \text{ is outlying},
\end{cases}
\]

for closed, two-sided surfaces \(\Sigma \subset M\). Essentially, \(F_\lambda\) measures the Willmore energy on the relevant scales for on-center and outlying surfaces, respectively. We then define the function

\[
G_\lambda : \{\xi \in \mathbb{R}^3 : |\xi| < 1 - \delta \text{ or } |\xi| > 1 + \delta\} \to \mathbb{R} \quad \text{given by } \quad G_\lambda(\xi) = F_\lambda(\Sigma_{\xi,\lambda}).
\]
Lemma 21. There is $\lambda_0 > 1$ depending on $g$ and $\delta \in (0, 1/2)$ with the following property. Let $\lambda > \lambda_0$. Then $\Sigma_{\xi, \lambda}$ is an area-constrained Willmore sphere if and only if $\xi$ is a critical point of $G_{\lambda}$.

Proof. Fix $\xi \in \mathbb{R}^3$ with either $|\xi| < 1 - \delta$ or $|\xi| > 1 + \delta$.

Let $a \in \mathbb{R}^3$ with $|a| = 1$ and $\epsilon > 0$ be small. Note that the normal speed $f$ of the area-preserving variation

$$\{ \Sigma_{\xi + sa, \lambda} : |s| < \epsilon \}$$

of $\Sigma$ at $s = 0$ is given by

$$f = g \left( \nu, \frac{d}{ds} \bigg|_{s=0} \Phi_{\xi + sa, \lambda} \right) = \lambda \bar{g}(a, \bar{\nu}) + O(1).$$

Assume that $\xi$ is a critical point of $G_{\lambda}$. Using (50), we find

$$\int_{\Sigma_{\xi, \lambda}} W(\Sigma_{\xi, \lambda}) f \, d\mu = 0, \quad \int_{\Sigma_{\xi, \lambda}} H(\Sigma_{\xi, \lambda}) f \, d\mu = 0.$$

In particular,

$$\int_{\Sigma_{\xi, \lambda}} (W(\Sigma_{\xi, \lambda}) + \kappa H(\Sigma_{\xi, \lambda})) (\bar{g}(a, \bar{\nu}) + O(\lambda^{-1})) \, d\mu = 0$$

for every choice of $a \in \mathbb{R}^3$ with $|a| = 1$. Since $W(\Sigma_{\xi, \lambda}) + \kappa H(\Sigma_{\xi, \lambda}) \in \Lambda_1$ by Proposition 17, it follows that $W(\Sigma_{\xi, \lambda}) + \kappa H(\Sigma_{\xi, \lambda}) = 0$ provided $\lambda_0 > 1$ is sufficiently large.

Conversely, if $\Sigma_{\xi, \lambda}$ is an area-constrained Willmore sphere, then

$$\int_{\Sigma_{\xi, \lambda}} W(\Sigma_{\xi, \lambda}) f \, d\mu = -\kappa \int_{\Sigma_{\xi, \lambda}} H(\Sigma_{\xi, \lambda}) f \, d\mu = 0.$$

In conjunction with Lemma 31, we see that $\xi$ is a critical point of $G_{\lambda}$. \hfill $\Box$

In the next step, we compute the asymptotic expansions of $G_{\lambda}$ as $\lambda \to \infty$.

Lemma 22. If $|\xi| < 1 - \delta$, there holds

$$G_{\lambda}(\xi) = 64 \pi + \frac{32 \pi}{1 - |\xi|^2} - 48 \pi |\xi|^{-1} \log \frac{1 + |\xi|}{1 - |\xi|} - 128 \pi \log(1 - |\xi|^2) + 2 \lambda \int_{\mathbb{R}^3 \setminus B_{\lambda}(\lambda \xi)} R \, d\bar{v} + O(\lambda^{-1}).$$

If $|\xi| > 1 + \delta$, there holds

$$G_{\lambda}(\xi) = -\frac{32 \pi}{|\xi|^2 - 1} - 48 \pi |\xi|^{-1} \log \frac{|\xi| + 1}{|\xi| - 1} - 128 \pi \log(1 - |\xi|^2) - 2 \lambda \int_{B_{\lambda}(\lambda \xi)} R \, d\bar{v} + O(\lambda^{-1}).$$

Proof. Using Lemma 31, we compute

$$\int_{\Sigma_{\xi, \lambda}} H(\Sigma_{\xi, \lambda})^2 \, d\mu = \int_{S_{\xi, \lambda}} H^2 \, d\mu - 2 \int_{S_{\xi, \lambda}} W(\Sigma_{\xi, \lambda}) u \, d\mu + \int_{S_{\xi, \lambda}} [u Qu - WH u^2] \, d\mu + O(\lambda^{-3})$$

$$= \int_{S_{\xi, \lambda}} H^2 \, d\mu - 2 \int_{S_{\xi, \lambda}} W(\Sigma_{\xi, \lambda}) u \, d\mu + \int_{S_{\xi, \lambda}} u Qu \, d\mu + O(\lambda^{-3}),$$

where we have abbreviated $H = H(S_{\xi, \lambda})$, $W = W(S_{\xi, \lambda})$, $\Delta = \Delta_{S_{\xi, \lambda}}$, and $Q = Q_{S_{\xi, \lambda}}$. Using (62) and (48), we find that

$$\int_{S_{\xi, \lambda}} u Qu \, d\mu = \int_{S_{\xi, \lambda}} [\Delta u^2 + 2 \lambda^{-2} u \Delta u] \, d\bar{v} + O(\lambda^{-3}).$$
Conversely, Lemma 20 and (16) imply
\[ \int_{S_{\xi,\lambda}} W \mu = \int_{S_{\xi,\lambda}} [(\Delta u)^2 + 2 \lambda^{-2} u \Delta u - 2 \lambda^{-1} \kappa u] \, d\mu + O(\lambda^{-3}). \]

Using Lemma 20 again, we obtain
\[ 2 \lambda^{-1} \kappa \int_{S_{\xi,\lambda}} u \, d\mu = \begin{cases} -64 \pi \lambda^{-2} + O(\lambda^{-3}) & \text{if } |\xi| < 1 - \delta, \\ O(\lambda^{-3}) & \text{if } |\xi| > 1 + \delta. \end{cases} \]

Using Corollary 33, Lemma 20, (54), and Lemma 36 in the case where $|\xi| < 1 - \delta$, we compute
\[ -\int_{S_{\xi,\lambda}} [(\Delta u)^2 + 2 \lambda^{-2} u \Delta u] \, d\mu = -16 \pi \lambda^{-2} \sum_{\ell=2}^{\infty} \frac{(\ell - 1)(\ell + 1)(\ell + 2)}{\ell(2\ell + 1)} |\xi|^{2\ell} \int_{S_{\xi,\lambda}} P_{\ell}^2 \left( -|\xi|^{-1} \tilde{g}(y, \xi) \right) \, d\mu \]
\[ = -64 \pi \lambda^{-2} \sum_{\ell=2}^{\infty} \frac{(\ell - 1)(\ell + 1)(\ell + 2)}{\ell(2\ell + 1)} |\xi|^{2\ell} \]
\[ = -16 \pi \lambda^{-2} \left[ \frac{9}{2} |\xi|^{-1} \log \frac{1 + |\xi|}{1 - |\xi|} + 8 \log(1 - |\xi|^2) + \frac{23 |\xi|^2 - 12 |\xi|^4 - 9}{(1 - |\xi|^2)^2} \right]. \]

Similarly, if $|\xi| > 1 + \delta$, we obtain
\[ -\int_{S_{\xi,\lambda}} [(\Delta u)^2 + 2 \lambda^{-2} u \Delta u] \, d\mu = -64 \pi \lambda^{-2} \sum_{\ell=2}^{\infty} \frac{(\ell - 1)(\ell + 1)(\ell + 2)}{\ell(2\ell + 1)} |\xi|^{-2\ell - 2} \]
\[ = -16 \pi \lambda^{-2} \left[ \frac{9}{2} |\xi|^{-1} \log \frac{|\xi| + 1}{|\xi| - 1} + 8 \log(1 - |\xi|^{-2}) + \frac{3 - |\xi|^2}{(|\xi|^2 - 1)^2} \right]. \]

We have used Lemma 38 in the last equation. The assertions follow from this and Lemma 43. \qed

In order to proceed, we note the following technical result.

**Lemma 23.** There are $c > 0$ and $\lambda_0 > 1$ which only depend on $g$ and $\delta \in (0, 1/2)$ such that
\[ ||G_\lambda||_{C^3(\{\xi \in \mathbb{R}^3: |\xi| \leq 1 - \delta \text{ or } |\xi| \geq 1 + \delta\})} < c \]
for every $\lambda > \lambda_0$.

**Proof.** This estimate follows from a straightforward computation using the regularity properties of the implicit function theorem, (12), and the variational formulae for the Willmore energy in the same way as in the proof of Proposition 6 in [6]. \qed

We now investigate the qualitative behavior of $G_\lambda$ for large values of $\lambda$. In this analysis, the assumptions
\[ \sum_{i=1}^{3} x^i \partial_i (|x|^2 R) \leq o(|x|^{-2}), \]
\[ R(x) - R(-x) = o(|x|^{-4}), \]

where $\sigma = 0$ or 1.
are used. Note that \((19)\) integrates to
\[
R \geq -o(|x|^{-4}).
\]
We remark that the weaker decay
\[
R = O(|x|^{-4})
\]
is implied by \((12)\).

**Lemma 24.** Suppose that the scalar curvature \(R\) of \((\mathbb{R}^3, g)\) satisfies \((19)\) and \((20)\). There exist \(\tau > 0, \delta_0 \in (0, 1/2)\), and \(\lambda_0 > 1\) depending only on \(g\) such that, provided \(\lambda > \lambda_0\),
\[
D^2G_\lambda \geq \tau \text{ Id}
\]
holds on \(\{\xi \in \mathbb{R}^3 : |\xi| < \delta_0\}\). Moreover, given \(\delta \in (0, 1/2)\) and \(\delta_1 \in (0, 1 - \delta)\), there is \(\lambda_1 > \lambda_0\) such that \(G_\lambda\) is strictly increasing in radial directions on \(\{\xi \in \mathbb{R}^3 : \delta_1 < |\xi| < 1 - \delta\}\) provided \(\lambda > \lambda_1\).

**Proof.** We write
\[
G_\lambda = G_1 + G_{\lambda, 2} + O(\lambda^{-1})
\]
where
\[
G_1(\xi) = 64\pi + \frac{32\pi}{1 - |\xi|^2} - 48\pi |\xi|^{-1} \log \frac{1 + |\xi|}{1 - |\xi|} - 128\pi \log(1 - |\xi|^2)
\]
and
\[
G_{\lambda, 2}(\xi) = 2\lambda \int_{\mathbb{R}^3 \setminus B_\lambda(\xi)} R \, d\bar{\nu}.
\]
The \(C^4\)-decay of the metric implies that the family of functions \(\{G_{2, \lambda} : \lambda > \lambda_0\}\) is uniformly bounded in \(C^3(\{\xi \in \mathbb{R}^3 : |\xi| \leq 1 - \delta\}\)) provided \(\lambda_0 > 1\) is sufficiently large. Hence, by Lemma 23 and interpolation, the error term in \((23)\) converges to 0 in \(C^2(\{\xi \in \mathbb{R}^3 : |\xi| \leq 1 - \delta\})\).

We compute
\[
\sum_{i=1}^{3} |\xi|^{-1} \xi^i (\partial_i G_{\lambda, 2})(\xi) = -2\lambda^2 \int_{S_{\xi, \lambda}} |\xi|^{-1} \bar{g}(\xi, \bar{\nu}) R \, d\bar{\mu}.
\]
For ease of notation, we will assume that \(\xi = (0, 0, \xi^3)\) and \(\xi^3 > 0\). Consider the subsets
\[
S_+ = \{x \in S_{\xi, \lambda} : \bar{g}(x, e_3) \leq 0\}, \quad S_- = \{x \in S_{\xi, \lambda} : \bar{g}(x, \bar{\nu}(x)) \geq 0\}, \quad -S_+ = \{-x : x \in S_+\}.
\]
Using \((20)\) and \((21)\), we obtain
\[
-2\lambda^2 \int_{S_{\xi, \lambda}} |\xi|^{-1} \bar{g}(\xi, \bar{\nu}) R \, d\bar{\mu} \geq 2\lambda^2 \int_{S_+} |\xi|^{-1} \bar{g}(\xi, \bar{\nu}) R \, d\bar{\mu} - 2\lambda^2 \int_{S_-} |\xi|^{-1} \bar{g}(\xi, \bar{\nu}) R \, d\bar{\mu} - o(1).
\]
We parametrize almost all of \(S_-\) via
\[
\Psi : (0, \pi/2) \times (0, 2\pi) \to S_- \quad \text{given by} \quad \Psi(\zeta, \varphi) = \lambda (\sin \zeta \sin \varphi, \sin \zeta \cos \varphi, \cos \zeta + \xi_3) .
\]
Likewise, we parametrize almost all of \(-S_+\) via
\[
(0, \arccos(\xi^3)) \times (0, 2\pi) \to -S_+, \quad (\theta, \varphi) \mapsto \lambda (\sin \theta \sin \varphi, \sin \theta \cos \varphi, \cos \theta - \xi_3).
\]
As shown in Figure 2, given $\zeta \in (0, \pi)$, there exists a unique angle $\theta = \theta(\zeta) \in (0, \arccos(\xi_3))$ with $\theta < \zeta$ and a number $t = t(\zeta) > 1$ such that
\[
\sin \theta \sin \varphi, \sin \theta \cos \varphi, \cos \theta - \xi_3 = \sin \zeta \sin \varphi, \sin \zeta \cos \varphi, \cos \zeta + \xi_3.
\]
Moreover,
\[
t = \frac{\sin \zeta}{\sin \theta}.
\]
Since $0 < \theta < \zeta < \pi/2$, we have $\tan \zeta > \tan \theta$. It follows that $t^{-1}$ is increasing on $(0, \pi/2)$. Consequently, $-\log t$ is also increasing on $(0, \pi/2)$ and thus
\[
\dot{t} \sin \theta \cos \varphi \geq t^{-2} \sin \zeta \cos \zeta.
\]
Performing a change of variables and using (25) and (21), we obtain
\[
2 \lambda^2 \int_{S_+} |\xi|^{-1} g(\xi, \bar{\nu}) R d\bar{\mu} - 2 \lambda^2 \int_{S_-} |\xi|^{-1} g(\xi, \bar{\nu}) R d\bar{\mu} \\
\geq 2 \lambda^4 \int_0^{2\pi} \int_0^{\pi/2} [t^{-2} R(t^{-1} \Psi(\zeta, \varphi)) - R(\Psi(\zeta, \varphi))] \sin \zeta \cos \zeta d\zeta d\varphi - o(1).
\]
By (19),
\[
\lambda^4 [t^{-2} R(t^{-1} \Psi(\zeta, \varphi)) - R(\Psi(\zeta, \varphi))] \geq -o(1).
\]
In particular,
\[
\sum_{i=1}^{3} |\xi|^{-1} \xi^i (\partial_t G_{\lambda,2})(\xi) \geq -o(1).
\]
Conversely, it is elementary to check that the function $G_1$ defined in (24) is strictly increasing in radial directions on $\{\xi \in \mathbb{R}^3 : 0 < |\xi| < 1\}$. Given $\delta_1 \in (0, 1 - \delta)$, it follows that $G_\lambda$ is strictly increasing in radial directions on $\{\xi \in \mathbb{R}^3 : \delta_1 < |\xi| < 1 - \delta\}$, provided $\lambda > 1$ is sufficiently large.
It remains to show that $G_\lambda$ is strictly convex near the origin. Again, it is elementary to check that $G_1$ is strictly convex on $\{\xi \in \mathbb{R}^3 : |\xi| < 1\}$. Moreover, given $a \in \mathbb{R}^3$ with $|a| = 1$, we compute

$$(\bar{D}^2 G_\lambda)(\xi, a, a) = -2 \lambda^2 \int_{S_{\xi, \lambda}} \left( \lambda \bar{g}(a, \nu)^2 \bar{D}_\nu R + 3 \bar{g}(a, \nu)^2 R - R \right) d\bar{\mu}.$$  

If $\xi = 0$, the growth condition (19) implies that

$$-\lambda \bar{D}_\nu R \geq 2 R - o(\lambda^{-4}).$$

Combined with (21), this gives

$$(\bar{D}^2 G_\lambda)(0, 0, 0) \geq -o(1) \text{Id}.$$  

Using (12), we conclude that there are $c > 0$ and $\delta_0 \in (0, 1/2)$, both independent of $\lambda$, such that

$$(\bar{D}^2 G_\lambda)(\xi) \geq -o(1) + c |\xi| \text{Id}$$

for every $\xi \in \mathbb{R}^3$ with $|\xi| < \delta_0$, provided $\lambda > 1$ is sufficiently large. The assertions of the lemma follow. $\Box$

**Lemma 25.** Suppose that the scalar curvature $R$ of $(\mathbb{R}^3, g)$ satisfies (19). There is a constant $\lambda_2 > \lambda_1$ depending only on $g$ and $\delta \in (0, 1/2)$ such that $G_\lambda$ is strictly increasing in radial directions on $\{\xi \in \mathbb{R}^3 : 1 + \delta < |\xi| < 1 + \delta^{-1}\}$, provided $\lambda > \lambda_2$.

**Proof.** The argument is almost the same as the proof of Lemma 24 except that we do not need to consider the reflection $-S_{\perp}$. In particular, the assumption (20) is not required. $\square$

**Proof of Theorem 5.** First, we show that for every $\lambda > 1$ sufficiently large, there exists a stable area-constrained Willmore sphere with area radius $\lambda$.

To see this, we decompose

$$G_\lambda(\xi) = G_1(\xi) + 2 \lambda \int_{\mathbb{R}^3 \setminus B_\lambda(\xi)} R d\bar{\nu} + O(\lambda^{-1})$$

as in (23). Note that $G_1(0) = 0$ and $\lim_{|\xi| \to 1} G_1(\xi) = \infty$. Using (22), we find

$$2 \lambda \int_{\mathbb{R}^3 \setminus B_\lambda(0)} R d\bar{\nu} = O(1).$$

Conversely, (21) implies, as $\lambda \to \infty$,

$$2 \lambda \int_{\mathbb{R}^3 \setminus B_\lambda(\xi)} R d\bar{\nu} \geq -2 \lambda \int_{\mathbb{R}^3 \setminus B_\lambda(1-|\xi|)(0)} o(|x|^{-4}) d\bar{\nu} \geq -o(1 - |\xi|^{-1}).$$

It follows that there is a number $z \in (1/2, 1)$ such that

$$G_\lambda(0) < G_\lambda(\xi)$$

for every $\xi$ with $|\xi| = z$ and every sufficiently large $\lambda > 1$. Consequently, there is a local minimum $\xi(\lambda)$ of $G_\lambda$ with

$$|\xi(\lambda)| < z.$$  

Lemma 21 shows that $\Sigma(\lambda) = \Sigma_{\xi(\lambda), \lambda}$ is a stable area-constrained Willmore surface. We claim that

$$\xi(\lambda) = o(1).$$
Otherwise, there exists a suitable sequence \( \{ \lambda_j \}_{j=1}^{\infty} \) with \( \lim_{j \to \infty} \lambda_j = \infty \) such that

- \( \lim_{j \to \infty} \xi(\lambda_j) = \xi_0 \) with \( \xi_0 \in \{ \xi \in \mathbb{R}^3 : 0 < |\xi| < 1 \} \) and
- \( (\bar{DG}_{\lambda_j})_{|\xi(\lambda_j)} = 0 \).

This is incompatible with Lemma 24. Lemma 20 implies that \( \kappa = \kappa(\Sigma(\lambda)) \) is decreasing and approaches 0 as \( \lambda \to \infty \). By Proposition 49, the surfaces \( \{ \Sigma(\lambda) : \lambda > \lambda_0 \} \) form a smooth foliation. This finishes the proof of Theorem 5.

**Proof of Theorem 8.** Suppose, for a contradiction, that there is a sequence of area-constrained Willmore sphere \( \{ \Sigma_j \}_{j=0}^{\infty} \) with

\[
\liminf_{j \to \infty} |\Sigma_j| = \infty, \quad \limsup_{j \to \infty} \int_{\Sigma_j} \hat{h}^2 \, d\mu < \epsilon_0, \quad 0 < \liminf_{j \to \infty} \frac{\rho(\Sigma_j)}{\lambda(\Sigma_j)} \leq \limsup_{j \to \infty} \frac{\rho(\Sigma_j)}{\lambda(\Sigma_j)} < \infty,
\]

and \( \Sigma_j \neq \Sigma(\lambda_j) \) where \( \lambda_j = \lambda(\Sigma_j) \). For \( \epsilon_0 > 0 \) small enough, Lemma 4.2 in [24] implies the uniform, scale invariant curvature estimate

\[
||h||_{L^\infty(\Sigma_j)} = O(\lambda_j^{-1})
\]

with corresponding higher-order estimates as well as the estimate

\[
\kappa(\Sigma_j) = O(\lambda_j^{-3}).
\]

Passing to a subsequence, if necessary, the rescaled surfaces \( \tilde{\Sigma}_j = \lambda_j^{-1} \Sigma_j \) converge smoothly to a Euclidean Willmore surface \( \tilde{\Sigma} \subset \mathbb{R}^3 \) satisfying

\[
\int_{\tilde{\Sigma}} \hat{h}^2 \, d\tilde{\mu} < \epsilon_0, \quad |\tilde{\Sigma}| = 4\pi, \quad 1 - \frac{1}{4\pi} \int_{\tilde{\Sigma}} y \, d\tilde{\mu} = \xi_0
\]

where \( |\xi_0| \neq 1 \). Now, the gap theorem [25, Theorem 2.7] for Euclidean Willmore surfaces due to E. Kuwert and R. Schätzle implies that \( \tilde{\Sigma} = S_1(\xi_0) \). It follows that \( \Sigma_j \) is a perturbation of a coordinate sphere for large \( j \). By Proposition 17, it is captured in our Lyapunov-Schmidt reduction in the sense that \( \Sigma_j = \Sigma_{\xi_j, \lambda_j} \) where \( \xi_j \) is a critical point of \( G_{\lambda_j} \) and \( \lim_{j \to \infty} \xi_j = \xi_0 \). If \( |\xi_0| < 1 \), Lemma 24 implies that \( \xi_0 = 0 \). However, since \( G_{\lambda_j} \) is strictly convex near the origin, it follows that \( \xi_j = \xi(\lambda_j) \), a contradiction. If \( |\xi_0| > 1 \), we use Lemma 25 instead to obtain a contradiction in a similar way.

**Proposition 26.** Assumptions as in Theorem 5. There is no sequence \( \{ \Sigma_j \}_{j=1}^{\infty} \) of connected, closed area-constrained Willmore surfaces with

\[
\liminf_{j \to \infty} |\Sigma_j| = \infty, \quad \liminf_{j \to \infty} m_H(\Sigma_j) > -\infty, \quad \limsup_{j \to \infty} \text{genus}(\Sigma_j) < \infty
\]

\[
0 < \liminf_{j \to \infty} \frac{\rho(\Sigma_j)}{\lambda(\Sigma_j)} \leq \limsup_{j \to \infty} \frac{\rho(\Sigma_j)}{\lambda(\Sigma_j)} < \infty
\]

that are not part of the foliation from Theorem 5.

**Proof.** According to Theorem 8, it suffices to verify that \( \Sigma_j \) is a sphere for every \( j \) sufficiently large and that

\[
\limsup_{j \to \infty} \int_{\Sigma_j} \hat{h}^2 \, d\mu = 0. \tag{26}
\]
Let $K$ denote the Gauss curvature of $\Sigma_j$. Using the Gauss equation in the form
\[ 4K = 2R - 4 \text{Rc}(\nu,\nu) + H^2 - 2|\hat{h}|^2 \]
and the Gauss-Bonnet theorem
\[ \int_{\Sigma_j} K \, d\mu = 4\pi \left(1 - \text{genus}(\Sigma_j)\right), \]
we find that
\[ \int_{\Sigma_j} H^2 \, d\mu = 16\pi \left(1 - \text{genus}(\Sigma_j)\right) + 2 \int_{\Sigma_j} |\hat{h}|^2 \, d\mu + 2 \int_{\Sigma_j} (2 \text{Ric}(\nu,\nu) - R) \, d\mu. \]
In particular,
\[ \int_{\Sigma_j} |\hat{h}|^2 \, d\mu = O(1). \]

As in Lemma 41, we compute that
\[ \hat{\hat{h}} = (1 + |x|)^2 \hat{h} + O(|x|^{-2}|h|) + O(|x|^{-3}) \]
and consequently
\[ \int_{\Sigma_j} |\hat{h}|^2 \, d\bar{\mu} = \int_{\Sigma_j} |\hat{h}|^2 \, d\mu + O(\lambda(\Sigma_j)^{-2}). \]

Now, using the Gauss equation for the Euclidean surface $\Sigma_j \subset \mathbb{R}^3$ and the Gauss-Bonnet theorem, we find that
\[ 16\pi - \int_{\Sigma_j} \vec{H}^2 \, d\bar{\mu} = O(\lambda(\Sigma_j)^{-1}). \]

According to the result [2, Theorem 1.2] of E. Kuwert and M. Bauer, for any fixed genus, there exists an embedded surface which attains the infimum of the Euclidean Willmore energy. Since the round spheres are the only compact surfaces with Euclidean Willmore energy equal to $4\pi$, it follows that $\text{genus}(\Sigma_j) = 0$ for $j$ large. Thus, (26) follows directly from (28).

\[ \square \]

3. Proof of Theorem 11

In this section, we assume that $g$ is a Riemannian metric on $\mathbb{R}^3$ such that
\[ g = (1 + |x|^{-1})^4 \bar{g} + \sigma \]
where $\sigma$ is a symmetric, covariant two-tensor with, as $x \to \infty$ for every multi-index $J$ with $|J| \leq 5$,
\[ \partial_J \sigma = O(|x|^{-2-|J|}). \]
Let $|\xi| > 2$ and $\lambda > \lambda_0$ for some $\lambda_0 > 1$ large. As in Section 2, given $\ell \in \{0, 1, 2, \ldots \}$, we use $\Lambda_\ell$ to denote the space of the $\ell$-th spherical harmonics on $S_{\xi, \lambda} = S_{\lambda}(\lambda \xi)$. Likewise, we define $\Lambda_{>2}$, $\Lambda_{>1}$, and $\Lambda_{>0}$ to be the orthogonal complements of $\Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2$, $\Lambda_0 \oplus \Lambda_1$, and $\Lambda_0$ in $C^{4,\alpha}(S_{\xi, \lambda})$, respectively. We suppress the dependence on $\xi$ and $\lambda$ to relax the notation.

We recall the definition of the rescaled metric $g_{\xi, \lambda}$ in (15). Note that

$$||g_{\xi, \lambda} - \bar{g}||_g = O(\lambda^{-1} |\xi|^{-1})$$

Consequently, Lemma 16 leads to the following proposition.

**Proposition 27.** There are constants $\lambda_0 > 1$, $c > 1$, and $\epsilon > 0$ depending on $g$ such that for every $|\xi| > 2$ and $\lambda > \lambda_0$ there exist $u_{\xi, \lambda} \in C^\infty(S_{\xi, \lambda})$ and $\kappa_{\xi, \lambda} \in \mathbb{R}$ such that the following hold. The surface $\Sigma_{\xi, \lambda} = \Sigma_{\xi, \lambda}(u_{\xi, \lambda})$ has the properties

- $W(\Sigma_{\xi, \lambda}) + \kappa_{\xi, \lambda} H(\Sigma_{\xi, \lambda}) \in \Lambda_1$,
- $|\Sigma_{\xi, \lambda}| = 4\pi \lambda^2$.

There holds $u_{\xi, \lambda} \perp \Lambda_1$ and

$$|u_{\xi, \lambda}| + \lambda |\nabla u_{\xi, \lambda}| + \lambda^2 |\nabla^2 u_{\xi, \lambda}| + \lambda^3 |\nabla^3 u_{\xi, \lambda}| + \lambda^4 |\nabla^4 u_{\xi, \lambda}| < c |\xi|^{-1},$$

$$\lambda^3 |\kappa_{\xi, \lambda}| < c |\xi|^{-1}.$$  \hspace{1cm} (30)

Moreover, if $\kappa \in \mathbb{R}$ and $\Sigma_{\xi, \lambda}(u)$ with $u \perp \Lambda_1(S_{\xi, \lambda})$ are such that

- $\Delta H + (|\tilde{h}|^2 + \text{Ric}(\nu, \nu) + \kappa) H \in \Lambda_1$,
- $|\Sigma_{\xi, \lambda}(u)| = 4\pi \lambda^2$,

and

$$|u| + \lambda |\nabla u| + \lambda^2 |\nabla^2 u| + \lambda^3 |\nabla^3 u| + \lambda^4 |\nabla^4 u| < \epsilon \lambda,$$

$$\lambda^3 |\kappa| < \epsilon \lambda,$$

then $u = u_{\xi, \lambda}$ and $\kappa = \kappa_{\xi, \lambda}$.

As in the previous section, we abbreviate $u = u_{\xi, \lambda}$ and $\kappa = \kappa_{\xi, \lambda}$. Lemma 20 suggests that a stronger estimate for $u$ than (30) might hold. However, more care is required since error terms of order $O(\lambda^{-1})$ may be larger than any inverse power of $|\xi|$.

First, we introduce some notation. We define

$$\phi(x) = 1 + |x|^{-1}$$

to be the conformal factor of the Schwarzschild metric with mass $m = 2$. As in [10], we use a bar underneath a quantity to indicate evaluation at $\lambda \xi$. If the quantity includes derivatives, these are taken first before we evaluate. For instance, we have

$$\tilde{\sigma} = \sigma(\lambda \xi), \quad \tilde{D}\sigma = (\tilde{D}\sigma)(\lambda \xi).$$

We note that the decay assumptions and Taylor’s theorem imply for example that

$$\sigma(\lambda \nu + \lambda \xi) = \sigma + \lambda D\nu \sigma + O(\lambda^{-2} |\xi|^{-4}).$$
We now adapt Lemma 20 to the current setting. In the statement of the following lemma, we let

\[ Y_{ij}^2 = \frac{1}{2} \left[ 3 \bar{g}(\bar{\nu}, e_i) g(\bar{\nu}, e_j) - \delta_{ij} \right] \in \Lambda_2 \]

where \( i, j = 1, 2, 3 \) and \( \{e_1, e_2, e_3\} \) denotes the standard basis of \( \mathbb{R}^3 \).

**Lemma 28.** There holds

\[ \kappa = O(\lambda^{-4} \, |\xi|^{-4}), \]

\[ W(\Sigma_{\xi,\lambda}) + \kappa H(\Sigma_{\xi,\lambda}) = O(\lambda^{-5} \, |\xi|^{-3}), \]

\[ u = -2|\xi|^{-1} + O(\lambda^{-1} \, |\xi|^{-2}) + O(|\xi|^{-3}). \]

More precisely,

\[ \text{proj}_{\Lambda_2} u = -\frac{1}{3} \sum_{i,j=1}^{3} \left[ 4 |\xi|^{-5} \xi^i \xi^j + \lambda \phi^{-6} \sigma(e_i, e_j) \right] Y_{ij}^2 + O(|\xi|^{-4}), \]

\[ \text{proj}_{\Lambda_{>2}} u = O(|\xi|^{-4}) + O(\lambda^{-1} \, |\xi|^{-3}). \]

These identities may be differentiated once with respect to \( \xi \).

**Proof.** From Corollary 46 we obtain

\[ \text{proj}_{\Lambda_0} W(S_{\xi,\lambda}) = O(\lambda^{-5} \, |\xi|^{-4}), \]

\[ \text{proj}_{\Lambda_1} W(S_{\xi,\lambda}) = O(\lambda^{-5} \, |\xi|^{-3}), \]

\[ \text{proj}_{\Lambda_2} W(S_{\xi,\lambda}) = O(\lambda^{-4} \, |\xi|^{-3}) + O(\lambda^{-5} \, |\xi|^{-2}), \]

\[ \text{proj}_{\Lambda_{>2}} W(S_{\xi,\lambda}) = O(\lambda^{-4} \, |\xi|^{-4}) + O(\lambda^{-5} \, |\xi|^{-3}). \]

We consider the family of surfaces

\[ \{ \Phi_{\xi,\lambda}^{tu} : \xi, \lambda \} \]

where \( \Phi_{\xi,\lambda}^{tu} : S_{\xi,\lambda} \rightarrow M \) is as in (13). The initial velocity of this variation with respect to the metric \( g \) is given by

\[ w = u \, g(\bar{\nu}, \nu). \]

Note that

\[ g(\bar{\nu}, \nu) = \phi^2 + O(\lambda^{-2} \, |\xi|^{-2}). \]

By (30) and Taylor’s theorem,

\[ Q_{S_{\xi,\lambda}} w = W(S_{\xi,\lambda}) - W(\Sigma_{\xi,\lambda}) + O(\lambda^{-5} \, |\xi|^{-2}). \]

Using (62), we find that

\[ Q_{S_{\xi,\lambda}} w = \Delta_{S_{\xi,\lambda}}^2 w + 2 \lambda^{-2} \phi^{-4} \Delta_{S_{\xi,\lambda}} w + O(\lambda^{-5} \, |\xi|^{-2}) \]

\[ = \phi^{-6} (\Delta_{S_{\xi,\lambda}}^3 u + 2 \lambda^{-2} \Delta_{S_{\xi,\lambda}} u) + O(\lambda^{-5} \, |\xi|^{-2}). \]

By Proposition 27, we have

\[ W(\Sigma_{\xi,\lambda}) + \kappa H(\Sigma_{\xi,\lambda}) = Y_1 \in \Lambda_1. \]
Moreover, (49), Proposition 27, Lemma 39, and Lemma 41 imply that

\[ \text{proj}_{\Lambda_0} H(\Sigma_{\xi,\lambda}) = 2\lambda^{-1} + O(\lambda^{-2}|\xi|^{-1}). \]

Thus, projecting (33) onto \( \Lambda_0 \), we find

\[ \kappa = O(\lambda^{-4}|\xi|^{-2}). \]

Next, we observe that

\[ \text{proj}_{\Lambda_{>0}} H(\Sigma_{\xi,\lambda}) = O(\lambda^{-2}|\xi|^{-1}). \]

Projecting (33) onto \( \Lambda_1 \), we conclude that

\[ Y_1 = O(\lambda^{-5}|\xi|^{-2}). \]

Similarly, we obtain

\[ \text{proj}_{\Lambda_{>1}} u = O(\lambda^{-1}|\xi|^{-2}) + O(|\xi|^{-3}). \]

Finally, Lemma 40, (30), and the formula for the first variation of area imply

\[ \int_{S_{\xi,\lambda}} H(S_{\xi,\lambda}) \, w \, d\mu = -16\pi \lambda|\xi|^{-1} + O(|\xi|^{-2}). \]

Using the improved estimates (34) for \( \text{proj}_{\Lambda_{>1}} u \) and arguing as in the proof of Lemma 19, we obtain

\[ \text{proj}_{\Lambda_0} u = -2|\xi|^{-1} + O(\lambda^{-1}|\xi|^{-2}). \]

To remove this scaling effect of the perturbation, we define

\[ \tilde{u} = u + 2|\xi|^{-1}, \quad \tilde{\lambda} = \lambda - 2|\xi|^{-1} \quad \text{and} \quad \tilde{\xi} = \lambda (\lambda - 2|\xi|^{-1})^{-1} \xi \]

and note that \( \lambda \xi = \tilde{\lambda} \tilde{\xi} \). Then, \( \Sigma_{\xi,\lambda} \) = \( \Phi_{\xi,\lambda}^\tilde{u} (S_{\xi,\lambda}) \) and there holds

\[ \tilde{u} = O(\lambda^{-1}|\xi|^{-2}) + O(|\xi|^{-3}). \]

We now repeat the above argument to obtain an improved estimate for \( \tilde{u} \). As before, we consider the family of surfaces

\[ \{\Phi_{\xi,\lambda}^t(S_{\xi,\lambda}) : t \in [0,1]\}. \]

The initial velocity of this family with respect to the metric \( g \) is given by

\[ \tilde{w} = \tilde{u} g(\tilde{\nu},\nu). \]

Note that (36) implies the improved family

\[ Q_{S_{\xi,\lambda}} \tilde{w} = W(S_{\xi,\lambda}) - W(\Sigma_{\xi,\lambda}) + O(\lambda^{-6}|\xi|^{-4}) + O(\lambda^{-5}|\xi|^{-6}). \]

Revisiting equation (62) and recalling (32), we find

\[ Q_{S_{\xi,\lambda}} \tilde{w} = \Delta^2_{S_{\xi,\lambda}} \tilde{w} + 2\tilde{\lambda}^{-2} \phi^{-4} \Delta_{S_{\xi,\lambda}} \tilde{w} + O(\lambda^{-6}|\xi|^{-4}) + O(\lambda^{-5}|\xi|^{-6}) \]

\[ = \phi^{-6} (\Delta^2_{S_{\xi,\lambda}} \tilde{u} + 2\tilde{\lambda}^{-2} \Delta_{S_{\xi,\lambda}} \tilde{u}) + O(\lambda^{-6}|\xi|^{-4}) + O(\lambda^{-5}|\xi|^{-6}). \]

Arguing as before, this improved estimate yields

\[ \kappa = O(\lambda^{-4}|\xi|^{-4}), \quad Y_1 = O(\lambda^{-5}|\xi|^{-3}) \quad \text{and} \quad \text{proj}_{\Lambda_{>2}} \tilde{u} = O(|\xi|^{-4}) + O(\lambda^{-1}|\xi|^{-3}). \]
Finally, we project (37) onto $\Lambda_2$. Recalling Corollary 46, we find
\begin{align}
\Delta_{\tilde{\Sigma}}^2 \text{proj}_{\Lambda_2} \tilde{u} + 2 \tilde{\lambda}^{-2} \Delta_{\tilde{\Sigma}} \text{proj}_{\Lambda_2} \tilde{u} \\
= \text{proj}_{\Lambda_2} \left( \phi^6 W(S_{\tilde{\xi}, \tilde{\lambda}}) \right) + O(\lambda^{-6} |\xi|^{-4}) + O(\lambda^{-5} |\xi|^{-6}) \\
= -32 \tilde{\lambda}^{-4} |\xi|^{-3} P_2 \left( -|\xi|^{-1} \bar{g}(\bar{\nu}, \xi) \right) - 4 \tilde{\lambda}^{-3} \phi^{-4} (3 \sigma(\bar{\nu}, \bar{\nu}) - \bar{\pi} \sigma) + O(\lambda^{-4} |\xi|^{-4})
\end{align}
where $P_2$ is the second Legendre polynomial defined by (53). To conclude, we use Corollary 33 and the estimate
\begin{equation}
\tilde{\lambda} = \phi^{-2} \lambda + O(\lambda^{-1} |\xi|^{-2}),
\end{equation}
which is immediate from the definition (35). □

We recall from (18) that the function $G_\lambda : \{ \xi \in \mathbb{R}^3 : |\xi| > 2 \} \to \mathbb{R}$ is given by
\begin{equation}
G_\lambda(\xi) = \lambda^2 \left( \int_{\Sigma_{\xi, \lambda}} H^2 \, d\mu - 16 \pi \right).
\end{equation}
Using the improved estimates for $u$ obtained in Lemma 28, we now show that the expansion in Lemma 22 holds with better error control. This is the key step in the proof of Theorem 11.

In the proof of Lemma 29, we use the notation
\begin{equation}
\hat{\sigma} = \sigma - \frac{1}{3} (\bar{\pi} \sigma) \bar{g}
\end{equation}

**Lemma 29.** There holds
\begin{equation}
G_\lambda(\xi) = -\frac{128 \pi}{15} |\xi|^{-6} - 2 \lambda \int_{B_\lambda(\lambda \xi)} R \, d\tilde{v} + O(\lambda^{-1} |\xi|^{-6}) + O(|\xi|^{-7}).
\end{equation}
This identity may be differentiated once with respect to $\xi$.

**Proof.** We recall from the proof of Lemma 28 that $\Sigma_{\xi, \lambda} = \Phi_{\xi, \lambda}^\mu(S_{\tilde{\xi}, \tilde{\lambda}})$, where
\begin{equation}
\tilde{u} = u + 2 |\xi|^{-1}, \quad \tilde{\lambda} = \lambda - 2 |\xi|^{-1} \quad \text{and} \quad \tilde{\xi} = \lambda (\lambda - 2 |\xi|^{-1})^{-1} \xi.
\end{equation}
As in the proof of Lemma 28, we consider the family of surfaces
\begin{equation}
\{ \Phi_{\xi, \lambda}^t(S_{\tilde{\xi}, \tilde{\lambda}}) : t \in [0, 1] \}
\end{equation}
connecting $S_{\tilde{\xi}, \tilde{\lambda}}$ and $\Sigma_{\xi, \lambda}$. Recall the functional $F_\lambda$ defined in (17). We compute the Taylor expansion of the function
\begin{equation}
[0, 1] \to \mathbb{R}^3, \quad t \mapsto F_\lambda(\Phi_{\xi, \lambda}^t(S_{\tilde{\xi}, \tilde{\lambda}}))
\end{equation}
at $t = 0$. To this end, we abbreviate $W = W(S_{\tilde{\xi}, \tilde{\lambda}})$ and $Q = Q S_{\tilde{\xi}, \tilde{\lambda}}$. Since the initial velocity is given by $\phi^2 \tilde{u}$, we find, using Lemma 31, Lemma 28, as well as (31), that
\begin{align}
F_\lambda(\Sigma_{\xi, \lambda}) &= F_\lambda(S_{\tilde{\xi}, \tilde{\lambda}}) - 2 \lambda^2 \int_{S_{\tilde{\xi}, \tilde{\lambda}}} \phi^2 W \, d\mu + \lambda^2 \int_{S_{\tilde{\xi}, \tilde{\lambda}}} \phi^4 \left[ \tilde{u} Q \tilde{u} - W H \tilde{u}^2 \right] \, d\mu + O(\lambda^{-1} |\xi|^{-6}) \\
&= F_\lambda(S_{\tilde{\xi}, \tilde{\lambda}}) - 2 \lambda^2 \int_{S_{\tilde{\xi}, \tilde{\lambda}}} \phi^2 W \, d\mu + \lambda^2 \int_{S_{\tilde{\xi}, \tilde{\lambda}}} \phi^4 \tilde{u} Q \tilde{u} \, d\mu + O(\lambda^{-1} |\xi|^{-6}) \\
&= F_\lambda(S_{\tilde{\xi}, \tilde{\lambda}}) - 2 \lambda^2 \int_{S_{\tilde{\xi}, \tilde{\lambda}}} \text{proj}_{\Lambda_2}(\phi^6 W) \, d\tilde{\mu} + \lambda^2 \int_{S_{\tilde{\xi}, \tilde{\lambda}}} \phi^4 \tilde{u} Q \tilde{u} \, d\mu + O(\lambda^{-1} |\xi|^{-6}).
\end{align}
Revisiting equation (62) again and using (38), we find
\[
\lambda^2 \int_{S_{\xi,\lambda}} \phi^4 \ddot{u} \dot{Q} \dd u + \mu = \lambda^2 \int_{S_{\xi,\lambda}} \left[ (\Delta \ddot{u})^2 + 2 \ddot{u} \Delta \ddot{u} \right] \dd \mu + O(\lambda^{-1} |\xi|^{-6})
\]
\[
= 2 \lambda^2 \int_{S_{\xi,\lambda}} \text{proj}_{\lambda^2} (\phi^6 W) \text{proj}_{\lambda^2} \dd \dd \mu + O(\lambda^{-1} |\xi|^{-6})
\]
\[
= 24 \lambda^2 \tilde{\lambda}^{-4} \int_{S_{\xi,\lambda}} (\text{proj}_{\lambda^2} \dd \mu)^2 \dd \mu + O(\lambda^{-1} |\xi|^{-6}).
\]
It follows that
\[
F_\lambda(S_{\xi,\lambda}) = F_\lambda(S_{\xi,\lambda}) - 24 \lambda^2 \tilde{\lambda}^{-4} \int_{S_{\xi,\lambda}} (\text{proj}_{\lambda^2} \dd \mu)^2 \dd \mu + O(\lambda^{-1} |\xi|^{-6}).
\]
For ease of notation, we assume that $\xi$ is a multiple of $e_3$. Using Lemma 28 and Lemma 35, we compute
\[
- 24 \lambda^2 \tilde{\lambda}^{-4} \int_{S_{\xi,\lambda}} (\text{proj}_{\lambda^2} \dd \mu)^2 \dd \mu
\]
\[
= -\frac{8}{3} \lambda^2 \tilde{\lambda}^{-2} \int_{S_{\xi,\lambda}} \left( 16 |\xi|^{-6} Y_2^{3,3} Y_2^{3,3} + 8 \lambda |\xi|^{-3} \phi^{-6} \sigma(e_i, e_j) Y_2^{2,3} Y_2^{ij} \right.
\]
\[
+ \lambda^2 \phi^{-12} \sum_{i,j,k,l=1}^{3} \sigma(e_i, e_j) \sigma(e_k, e_l) Y_2^{ij} Y_2^{kl} \left. \right) \dd \mu
\]
\[
= -\frac{512 \pi}{15} \lambda^2 \tilde{\lambda}^{-2} |\xi|^{-6} - \frac{128 \pi}{15} \lambda^3 \tilde{\lambda}^{-2} |\xi|^{-3} (3 |\xi|^{-2} \sigma(\xi, \xi) - \bar{\sigma} \bar{\sigma}) - \frac{48 \pi}{15} \lambda^4 \tilde{\lambda}^{-2} \phi^{-12} |\hat{\sigma}|^2
\]
\[
+ O(\lambda^{-1} |\xi|^{-6})
\]
\[
= -\frac{512 \pi}{15} |\xi|^{-6} - \frac{128 \pi}{15} \lambda |\xi|^{-3} (3 |\xi|^{-2} \sigma(\xi, \xi) - \bar{\sigma} \bar{\sigma}) - \frac{48 \pi}{15} \lambda^2 \phi^{-8} |\hat{\sigma}|^2
\]
\[
+ O(\lambda^{-1} |\xi|^{-6}).
\]
In the last equation, we have used (39). Conversely, Lemma 43 and Taylor’s theorem give that
\[
F_\lambda(S_{\xi,\lambda}) = \lambda^2 \tilde{\lambda}^{-2} F_\lambda(S_{\xi,\lambda})
\]
\[
\int_{B_\lambda(\xi)} R \dd \bar{v}
\]
\[
+ \frac{48 \pi}{15} \lambda \phi^{-8} |\hat{\sigma}|^2 + \frac{128 \pi}{15} \lambda |\xi|^{-3} (3 |\xi|^{-2} \sigma(\xi, \xi) - \bar{\sigma} \bar{\sigma}) + O(\lambda^{-1} |\xi|^{-6}) + O(|\xi|^{-7}).
\]
Finally, using (29) and (39), we find
\[
\lambda^2 \tilde{\lambda}^{-1} \phi^4 \int_{B_\lambda(\xi)} R \dd \bar{v} = \lambda \phi^6 \int_{B_{\lambda}(-\lambda \xi)} R \dd \bar{v} + O(\lambda^{-1} |\xi|^{-6})
\]
\[
\int_{B_\lambda(\xi)} R \dd \bar{v} + O(\lambda^{-1} |\xi|^{-6}).
\]
Assembling (40), (41), (42), and (43), the assertion follows. $\square$
Proof of Theorem 11. Suppose, for a contradiction, that there exists a sequence of outlying area-constrained Willmore spheres \( \{\Sigma_j\} \) with
\[
\lim_{j \to \infty} |\Sigma_j| = \infty, \quad \limsup_{j \to \infty} \int_{\Sigma_j} |\hat{h}|^2 \, d\mu < \epsilon_0, \quad \lim_{j \to \infty} \rho(\Sigma_j) = \infty.
\]
As in the proof of Theorem 8, we may assume that
\[
\Sigma_j = \Sigma_{\xi_j, \lambda_j}
\]
for suitable \( \xi_j \in \mathbb{R}^3 \) and \( \lambda_j \in (\lambda_0, \infty) \) where
\[
\lim_{j \to \infty} |\xi_j| = \infty \quad \text{and} \quad \lim_{j \to \infty} \lambda_j = \infty.
\]
Arguing as in the proof of Lemma 25, but this time using the exact growth condition
\[
\sum_{i=1}^{3} x^i \partial_i(||x|^2 R)(x) \leq 0,
\]
we find that
\[
\sum_{i=1}^{3} \xi^i_j \partial_i \left( -2 \lambda_j \int_{B_\lambda(\lambda_j \xi_j)} R \, d\bar{v} \right) \geq 0.
\]
In conjunction with Lemma 29, this gives
\[
\sum_{i=1}^{3} \xi^i_j \partial_i (\partial_\lambda G)(\xi_j) \geq \frac{256 \pi}{5} |\xi_j|^{-6} + O(|\lambda_j^{-1} \xi_j|^{-6}) + O(|\xi_j|^{-7}).
\]
In particular,
\[
\sum_{i=1}^{3} \xi^i_j (\partial_\lambda G)(\xi_j) > 0.
\]
This is incompatible with Lemma 21. \( \square \)

4. Proof of Theorems 13, 14 and 15

We first prove Theorem 13 and Theorem 15. To this end, we adapt a construction from [10, §3 and §6]. The metrics in this construction are rotationally symmetric. Their scalar curvature has a pulse.

We briefly recall some steps from [10].

Given a function \( S : (0, \infty) \to (-\infty, 0] \) with
\[
S^{(\ell)} = O(s^{-4-\ell})
\]
for every integer \( \ell \geq 0 \), we define the function \( \Psi : (0, \infty) \to (-\infty, 0] \) by
\[
\Psi(s) = s^{-1} \int_{s}^{\infty} (t-s) \, t \, S(t) \, dt.
\]
Note that
\[
\Psi^{(\ell)} = O(s^{-2-\ell})
\]
for every integer \( \ell \geq 0 \). The metric
\[
g = (1 + |x|^{-1} + \Psi(|x|))^4 \bar{g}
\]
on $\mathbb{R}^3 \setminus \{0\}$ is $C^k$-asymptotic to Schwarzschild with mass $m = 2$ for every $k \geq 2$. Its scalar curvature $R$ is given by

\begin{equation}
R(x) = -8 \left(1 + O(|x|^{-1})\right) S(|x|).
\end{equation}

In particular, $R \geq 0$ outside a compact set.

Proof of Theorem 13. First, as shown in Figure 3, we construct a metric $g_2$ which admits large outlying area-constrained Willmore spheres $\{\Sigma_j\}_{j=1}^{\infty}$ with

$$2\sqrt{2} < \frac{p(\Sigma_j)}{\lambda(\Sigma_j)} < 5 \quad \text{and} \quad m_H(\Sigma_j) > -o(1).$$

Let $\chi \in C^\infty(\mathbb{R})$ be such that $\chi(t) > 0$ for all $t \in (3, 4)$ and supp $\chi \subset [3, 4]$. Let

$$S(s) = -B \sum_{k=0}^{\infty} 10^{-4k} \chi(10^{-k}s).$$

The constant $B > 0$ will be chosen (large) later. Let $j \geq 1$ be a large integer and $\lambda_j = 10^j$. Recall the definition of $G_1$ in (18). $G_{\lambda_j}$ is rotationally symmetric on $\{\xi \in \mathbb{R}^3 : |\xi| > 2\}$. As in the proof of Lemma 24, we have

$$G_{\lambda_j} = G_1 + G_{2, \lambda_j} + o(1).$$

This expansion may be differentiated twice. Here, $G_1$ is strictly increasing in radial directions and independent of both $\lambda$ and $B$, while

$$\sum_{i=1}^{3} \xi_i \left(\partial_i G_{2, \lambda_j}\right)(\xi) = -2 \lambda_j^2 \int_{S_{\xi, \lambda_j}} \bar{g}(\xi, \bar{v}) \, d\bar{\mu} = -16 B \int_{S_1(\xi)} \bar{g}(\xi, \bar{v}) \, d\bar{\mu} + o(1).$$

The integral on the right hand side vanishes if $|\xi| = 5$ and is negative if $|\xi| = 2\sqrt{2}$. Thus, using that $G_{\lambda_j}$ is rotationally symmetric and that $G_1$ is strictly increasing in radial directions, we may increase $B > 0$ appropriately so $G_{\lambda_j}$ attains a local minimum at some $\xi_j \in \mathbb{R}^3$ with

$$2 \sqrt{2} < |\xi_j| < 5$$

for every sufficiently large $j$.

Next, we construct a metric $g_1$ which admits large on-center area-constrained Willmore spheres that are not part of the foliation from Theorem 5. This time, we choose a smooth function $\chi$ such that $\chi(t) > 0$ for all $t \in (9/8, 11/8)$ and supp $\chi \subset [9/8, 11/8]$. We write

$$G_{\lambda_j} = G_1 + G_{2, \lambda_j} + o(1).$$

As before, $G_1$ is strictly increasing in radial directions and independent of both $\lambda$ and $B$, while

$$\sum_{i=1}^{3} \xi_i \left(\partial_i G_{2, \lambda_j}\right)(\xi) = -16 B \int_{S_1(\xi)} \bar{g}(\xi, \bar{v}) \, d\bar{\mu} + o(1).$$

The integral on the right hand side is negative if $|\xi| = 1/4$ and positive if $|\xi| = 7/8$. Thus, we may again increase $B$ appropriately such that for every $j$ large, $G_{\lambda_j}$ attains a local minimum $\xi_j \in \mathbb{R}^3$ with

$$1/4 < |\xi_j| < 7/8.$$
Figure 3. An illustration of the construction for the proof of Theorem 13. The scalar curvature is positive in the shaded region and vanishes elsewhere. On the left, the surface $\Sigma_{\xi,\lambda_j}$ with $|\xi| = 5$ is shown. It does not overlap with the shaded region. In particular, the radial derivative of $G_{2,\lambda_j}$ vanishes. On the right, the surface corresponds to the choice $|\xi| = 2\sqrt{2}$. If this surface is moved upwards, the overlap with the shaded region increases. The radial derivative of $G_{2,\lambda_j}$ is negative.

This completes the proof of Theorem 13. 

Proof of Theorem 15. To construct $g_4$, we choose a suitable function $\Psi$ in (44) which satisfies

$$(46) \quad \Psi^{(\ell)} = O(s^{-3-\ell}).$$

In particular, $\Psi$ decays one order faster than the perturbations used to construct $g_1$ and $g_2$. Due to the fast decay of the Schwarzschild contribution, this perturbation will still be strong enough to admit large far-outlying area-constrained Willmore spheres with Hawking masses bounded from below. More precisely, we choose $\chi \in C^\infty(\mathbb{R})$ with $\chi(t) > 0$ for all $t \in (4,6)$, supp $\chi \subset [4,6]$, and $\chi^{(5)}(5) = 1$. Let

$$S(s) = -\sum_{k=0}^{\infty} 10^{-5k} \chi(10^{-k}s)$$

and note that $S^{(\ell)} = O(s^{-5-\ell})$ for every integer $\ell \geq 0$. This ensures that (46) holds. Now, let $j \geq 1$ be large, $\lambda_j = 10^j$, and $\xi_t = t10^j a$ where $t \in [3,7]$ and $a \in \mathbb{R}^3$ is such that $|a| = 1$. From (45), we
Figure 4. An illustration of the construction for the proof of Theorem 14. The odd part of the scalar curvature is positive in the shaded region and negative in the hatched region. On the left and right, the surfaces $\Sigma_{\xi,\lambda_j}$ corresponding to the choices $\xi = 0$ and $\xi = t e_1$ for some small $t > 0$, respectively, are shown. The latter has larger overlap with the shaded region, causing the derivative of $G_{2,\lambda_j}$ in direction $e_1$ to be negative at $\xi = 0$.

see that

$$R = O(|x|^{-5}).$$

Using Taylor’s theorem, we find that

$$-2\lambda \int_{B_{\lambda_j}(\lambda_j, \xi_t)} R \, d\bar{v} = -\frac{8\pi}{3} \lambda_j^4 R + O(\lambda^{-1} |\xi_t|^{-6}) + O(|\xi_t|^{-7}).$$

Lemma 29 implies that

$$G_{\lambda_j}(\xi_t) = -\frac{128 \pi}{15} |\xi_t|^{-6} - \frac{8\pi}{3} \lambda_j^4 R + O(\lambda^{-1} |\xi_t|^{-6}) + O(|\xi_t|^{-7})$$

$$= -\frac{128 \pi}{15} t^{-6} 10^{-6j} - \frac{64 \pi}{3} \chi(t) 10^{-6j} + O(10^{-7j}).$$

The derivative of the quantity on the right hand side with respect to $t$ is positive if $t = 7$ and negative if $t = 5$ provided $j$ is large. Since $G_{\lambda_j}$ is rotationally symmetric, it follows that, for every $j$ large, there is a number $t_j \in [5, 7]$ with $(\partial G_{\lambda_j})(\xi_j) = 0$ where $\xi_j = t_j 10^j a$. In particular, $\{\Sigma_{\xi_j,\lambda_j}\}_{j=1}^\infty$ is a sequence of far-outlying stable area-constrained Willmore spheres with diverging area and Hawking mass bounded from below.

□

**Remark 30.** The proofs of Theorem 13 and Theorem 15 follow the proofs of Theorem 1.3 and 1.8 in [10] closely. Note that the analysis of the function $G_\lambda$ differs from the analysis of the reduced area functional in [10]. The construction of the metric $g_1$ has features not considered in [10].
Proof of Theorem 14. We construct a metric \( g_3 \) on \( \mathbb{R}^3 \setminus \{0\} \) that admits a foliation by area-constrained Willmore spheres whose leaves do not center around the origin; see Figure 4. To this end, we let \( \chi : \mathbb{R}^3 \to [0, \infty) \) be a standard bump function

\[
\chi(y) = \begin{cases} 
 e^{-(1-|y|^2)^{-1}} & \text{if } |y| < 1, \\
 0 & \text{if } |y| \geq 1.
\end{cases}
\]

If \( y \in B_1(0) \), we compute

\[
(\bar{\Delta} \chi)(y) = 2 \chi(y) \frac{4|y|^2 + |y|^4 - 3}{(1 - |y|^2)^4}.
\]

In particular, \( \chi \) is strictly subharmonic on \( \{ y \in \mathbb{R}^3 : \sqrt{3}/2 < |y| < 1 \} \). Let

\[
\psi(x) = \sum_{k=0}^{\infty} 10^{-2k} \chi \left( 2 \cdot 10^{-k} (x - 10^k e_1) \right)
\]

and define the conformally flat metric

\[
g_3 = [1 + |x|^{-1} - \epsilon (|x|^{-2} + \delta \psi(x))]^4 \bar{g}
\]

on \( \mathbb{R}^3 \setminus \{0\} \) where \( \epsilon, \delta > 0 \) will be chosen (small) later. Note that \( g_3 \) is \( C^k \)-asymptotic to Schwarzschild with mass \( m = 2 \) for every \( k \geq 2 \). The scalar curvature of \( g_3 \) is given by

\[
R = 8 \epsilon \Delta(|x|^{-2} + \delta \psi(x)) + O(|x|^{-5})
\]

\[
= 8 \epsilon \left( 2 |x|^{-4} + 4 \delta \sum_{k=0}^{\infty} 10^{-4k} (\bar{\Delta} \chi) \left( 2 \cdot 10^{-k} (x - 10^k e_1) \right) \right) + O(|x|^{-5}).
\]

In particular, \( R \geq 0 \) outside a bounded set, provided \( \delta > 0 \) is sufficiently small. A similar computation shows that, taking \( \delta > 0 \) smaller if necessary,

\[
\sum_{i=1}^{3} x^i \partial_i (|x|^2 R) \leq 0
\]

outside a bounded set.

Arguing as in the proof of Lemma 24, we may choose \( \epsilon > 0 \) small such that \( G_{\lambda} \) is strictly convex in \( \{ \xi \in \mathbb{R}^3 : |\xi| < 1/2 \} \) and strictly radially increasing near \( \{ \xi \in \mathbb{R}^3 : |\xi| = 1/2 \} \) provided \( \lambda > 1 \) is sufficiently large. It follows that \( G_{\lambda} \) has a unique critical point \( \xi(\lambda) \) with \( |\xi(\lambda)| < 1/2 \). Let

\[
\lambda_j = \frac{9}{16} 10^j.
\]

As in (23), we consider the decomposition

\[
G_{\lambda_j} = G_1 + G_{2,\lambda_j} + o(1).
\]

Note that \( (\bar{D} G_1)(0) = 0 \). Using that \( \chi \) is strictly subharmonic on \( \{ y \in \mathbb{R}^3 : \sqrt{3}/2 < |y| < 1 \} \) and that odd functions integrate to zero on a sphere, we obtain

\[
(\partial_t G_{2,\lambda_j})(0) = -4 \left( \frac{9}{16} \right)^4 \epsilon \delta \int_{S_1(0)} (9 x - 16 e_1) \bar{g}(e_1, \bar{\nu}) (\bar{\Delta} \chi) d\bar{\mu} + o(1) = -c \epsilon \delta + o(1)
\]
where \(c > 0\) is independent of \(j\). By Lemma 23, the \(C^2\)-norm of \(G_\lambda\) is bounded. It follows that there is \(z \in (0, 1/2)\) with
\[
|\xi(\lambda_j)| \geq z
\]
provided \(j\) is sufficiently large.

To conclude, note that the same argument as in the proof of Proposition 49 shows that the family of surfaces \(\{\Sigma_{\lambda, \xi(\lambda)} : \lambda > \lambda_0\}\) forms a smooth asymptotic foliation of \(\mathbb{R}^3\). □

A. THE WILLMORE ENERGY

In this section, we provide some background material on the Willmore energy. We refer to [28, §3] for proofs and further information.

Let \((M, g)\) be a Riemannian 3-manifold without boundary. Let \(\Sigma \subset M\) be a closed, two-sided surface with unit normal \(\nu\). The Willmore energy of \(\Sigma\) is the quantity
\[
\mathcal{W}(\Sigma) = \frac{1}{4} \int_\Sigma H^2 \, d\mu,
\]
where \(H\) is the mean curvature scalar computed as the divergence of \(\nu\) along \(\Sigma\).

Let \(\epsilon > 0\) and \(U \in C^\infty(\Sigma \times (-\epsilon, \epsilon))\) with \(U(\cdot, 0) = 0\). Decreasing \(\epsilon > 0\) if necessary, we obtain a smooth variation \(\{\Sigma_s : s \in (-\epsilon, \epsilon)\}\) of embedded surfaces \(\Sigma_s = \Phi_s(\Sigma_s)\) where
\[
\Phi_s : \Sigma \to M \quad \text{is given by} \quad \Phi_s(x) = \exp_x(U(x, s)\nu(x)).
\]
We denote the initial velocity and initial acceleration of the variation by
\[
u(x) = \hat{U}(x, 0) \quad \text{and} \quad v(x) = \hat{\ddot{U}}(x, 0).
\]

In the following lemma, we recall the formulae for the first and the second variation of the Willmore energy (47). To this end, we recall that
\[
L f = -\Delta f - (|h|^2 + \text{Ric}(\nu, \nu)) f
\]
denotes the linearization of the mean curvature operator. In particular,
\[
Lu = \frac{d}{ds} \bigg|_{s=0} (H(\Sigma_s) \circ \Phi_s).
\]

Lemma 31 ([28, §3]). There holds
\[
\frac{d}{ds} \bigg|_{s=0} \int_{\Sigma_s} H^2 \, d\mu = -2 \int_{\Sigma} W u \, d\mu
\]
where
\[
W = \Delta H + (|\dot{h}|^2 + \text{Ric}(\nu, \nu)) H.
\]
Moreover,
\[
\frac{d^2}{ds^2} \bigg|_{s=0} \int_{\Sigma_s} H^2 \, d\mu = -2 \int_{\Sigma} \left[ u Qu + H W u^2 + W v \right] \, d\mu
\]
where
\[
Qu = L(Lu) + \frac{1}{2} H^2 Lu + 2 H g(\hat{h}, \nabla^2 u) + 2 H \text{Ric}(\nu, \nabla u) + 2 \hat{h}(\nabla H, \nabla u)
\]
\[
+ u \left[ |\nabla H|^2 + 2 \text{Ric}(\nu, \nabla H) + H \Delta H + 2 g(\hat{h}, \nabla^2 H) + 2 H^2 |\hat{h}|^2
\right.
\]
\[
+ 2 H g(\text{Ric}, \hat{h}) - H (D_\nu \text{Ric})(\nu, \nu)
\]
\]

is the linearization of the Willmore operator.

The operator \( Q \) measures how \( W \) changes along a normal variation of \( \Sigma \). More precisely,
\[
Q u = -\frac{d}{d s}\bigg|_{s=0} (W(\Sigma_s) \circ \Phi_s).
\]

Surfaces that are critical for the Willmore energy are called Willmore surfaces. The corresponding Euler-Lagrange equation is
\[-W = 0.
\]

Surfaces that are critical for the Willmore energy among so-called area-preserving variations are called area-constrained Willmore surfaces. They satisfy the area-constrained Willmore equation
\[-W = \kappa H
\]

where \( \kappa \in \mathbb{R} \) is a Lagrange parameter. A variation \( \{\Sigma_s : |s| < \epsilon\} \) is called area-preserving if \( |\Sigma_s| = |\Sigma| \) for all \( s \in (-\epsilon, \epsilon) \).

An area-constrained Willmore surface \( \Sigma \) is stable if it passes the second derivative test for the Willmore energy among all area-preserving variations. Note that this is always satisfied if \( \Sigma \) is a minimal surface. If \( \Sigma \) is not a minimal surface, we define \( u^\perp = u + s H \) where
\[
s = -\left( \int_{\Sigma} H u \, d\mu \right)^{-1} \int_{\Sigma} H^2 d\mu
\]
is chosen such that
\[
\int_{\Sigma} u^\perp H \, d\mu = 0.
\]
It can be seen that \( \Sigma \) is stable if and only if
\[
\kappa \int_{\Sigma} u^\perp L u^\perp \, d\mu \leq \int_{\Sigma} u^\perp Q u^\perp \, d\mu
\]
for every \( u \in C^\infty(\Sigma) \); see [28, (41) on p. 16].

\textbf{B. Spherical harmonics and Legendre polynomials}

In this section, we collect some standard facts about the Laplace operator on the unit sphere.

\textbf{Lemma 32.} The eigenvalues of the operator
\[-\tilde{\Delta} : H^2(S_1(0)) \to L^2(S_1(0))
\]
are given by
\[
\{ \ell (\ell + 1) : \ell = 0, 1, 2, \ldots \}.
\]
We denote the eigenspace corresponding to the eigenvalue $\ell(\ell + 1)$ by
$$\Lambda_\ell(S_1(0)) = \{ f \in C^\infty(S_1(0)) : -\Delta f = \ell(\ell + 1)f \}.$$ 
Recall that these eigenspaces are finite dimensional and that
$$L^2(S_1(0)) = \bigoplus_{\ell=0}^\infty \Lambda_\ell(S_1(0)).$$

**Corollary 33.** The eigenvalues of the operator
$$\bar{\Delta}^2 + 2\bar{\Delta} : H^4(S_1(0)) \to L^2(S_1(0))$$
are given by
$$\{(\ell - 1)\ell(\ell + 1)(\ell + 2) : \ell = 0, 1, 2, \ldots \}.$$

**Lemma 34.** There holds
\begin{align*}
\Lambda_0(S_1(0)) &= \text{span}\{1\}, \\
\Lambda_1(S_1(0)) &= \text{span}\{y^1, y^2, y^3\}, \\
\Lambda_2(S_1(0)) &= \text{span}\{Y_{11}^2, Y_{22}^2, Y_{12}^2, Y_{13}^2, Y_{23}^2\}.
\end{align*}

Here $y^1, y^2, y^3$ are the coordinate functions and
$$Y_{ij}^2 = \frac{1}{2} (3 y^i y^j - \delta_{ij}).$$

We also record the following useful orthogonality relations.

**Lemma 35.** Let $i, j, k, \ell \in \{1, 2, 3\}$. There holds
\begin{align*}
\int_{S_1(0)} y^i y^j \, d\bar{\mu} &= \frac{4\pi}{3} \delta_{ij}, \\
\int_{S_1(0)} y^i y^j y^k y^\ell \, d\bar{\mu} &= \frac{4\pi}{15} (\delta_{ij} \delta_{k\ell} + \delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}), \\
\int_{S_1(0)} Y_{ij}^2 Y_{k\ell}^2 \, d\bar{\mu} &= \frac{\pi}{5} (3 \delta_{ik} \delta_{j\ell} + 3 \delta_{i\ell} \delta_{jk} - 2 \delta_{ij} \delta_{k\ell}), \\
\int_{B_1(0)} y^i y^j \, d\bar{v} &= \frac{4\pi}{15} \delta_{ij}.
\end{align*}

The Legendre polynomials $P_0, P_1, P_2, \ldots$ may be defined via a generating function. More precisely, given $s \in [0, 1]$ and $t \in [0, 1)$, there holds
\begin{equation}
(1 - 2 s t + t^2)^{-\frac{1}{2}} = \sum_{\ell=0}^\infty P_\ell(s) t^\ell. \tag{53}
\end{equation}

A proof of the following lemma can be found in [14, §8].

**Lemma 36.** Let $i \in \{1, 2, 3\}$ and $\ell, \ell_1, \ell_2 \in \{0, 1, 2, \ldots\}$. There holds
\begin{equation}
P_\ell(-y^i) \in \Lambda_\ell(S_1(0)). \tag{54}
\end{equation}
Moreover,
\[ \int_{S_1(0)} P_{\ell_1}(-y^i) P_{\ell_2}(-y^i) \, d\bar{\mu} = \frac{4\pi}{2\ell + 1} \delta_{\ell_1 \ell_2}. \]

The next lemma extends the calculation of inverse powers of \(|y + \xi|\) in [6, p. 668].

**Lemma 37.** Let \( \xi \in \mathbb{R}^3 \) with \(|\xi| \neq 1\) and \( k \in \{0, 1, 2, 3\} \). There holds, for all \( y \in S_1(0) \)
\[ |y + \xi|^{-2k - 1} = \begin{cases} \sum_{\ell=0}^{\infty} a_{k,\ell}(\xi) |\xi|^\ell P_\ell(-|\xi|^{-1} \tilde{g}(y, \xi)) & \text{if } |\xi| < 1, \\ \sum_{\ell=0}^{\infty} \tilde{a}_{k,\ell}(\xi) |\xi|^{-\ell-1} P_\ell(-|\xi|^{-1} \tilde{g}(y, \xi)) & \text{if } |\xi| > 1. \end{cases} \]
Here,
\[ a_{0,\ell}(\xi) = 1, \]
\[ a_{1,\ell}(\xi) = (2\ell + 1) \frac{1}{1 - |\xi|^2}, \]
\[ a_{2,\ell}(\xi) = (2\ell + 1) \frac{(2\ell + 3) - (2\ell - 1) |\xi|^2}{3(1 - |\xi|^2)^3}, \]
\[ a_{3,\ell}(\xi) = (2\ell + 1) \frac{(2\ell + 3)(2\ell + 5) - 2(2\ell - 3)(2\ell - 5) |\xi|^2 + (2\ell - 3)(2\ell - 1)|\xi|^4}{15(1 - |\xi|^2)^5}, \]
and
\[ \tilde{a}_{k,\ell}(\xi) = (-1)^k |\xi|^{-2k} a_{k,\ell}(|\xi|^{-2} \xi). \]

**Proof.** If \( k = 0 \), the expansions follow from (53) with \( s = -|\xi|^{-1} \tilde{g}(y, \xi) \) and choice of
\[ t = \begin{cases} |\xi| & \text{if } |\xi| < 1, \\ |\xi|^{-1} & \text{if } |\xi| > 1. \end{cases} \]
The asserted formula follows from the recursive relation
\[ |y + \xi|^{-2k - 1} = \frac{1}{1 - |\xi|^2} \left( \frac{2}{2k - 1} \sum_{i=1}^{3} \xi^i \partial_i(|y + \xi|^{-2k+1}) + |y + \xi|^{-2k+1} \right), \]
where the partial derivative is with respect to \( \xi \). \( \square \)

**Lemma 38.** Let \( t \in (-1, 1) \). There holds
\[ \log(1 + t) = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1}}{\ell} t^\ell \quad \text{and} \quad \frac{1}{(1 - t^2)^2} = \sum_{\ell=0}^{\infty} \frac{1}{1 + \ell} t^{2\ell}. \]

**C. Some geometric expansions**

In this section, we compute several geometric expansions needed in this paper. Recall that \( S_{\xi,\lambda} = S_\lambda(\lambda \xi) \). We distinguish between the cases \(|\xi| < 1 - \delta\) and \(|\xi| > 1 + \delta\) where \( \delta \in (0, 1/2) \). We will assume that \( \lambda > \lambda_0 \) where \( \lambda_0 > 1 \) is large. We note that, for every \( x \in S_{\xi,\lambda} \),
\[ \delta \leq \lambda^{-1} |x| \leq 2 - \delta \quad \text{if} \quad |\xi| < 1 - \delta \]
and
\[ |\xi| - 1 \leq \lambda^{-1} |x| \leq |\xi| + 1 \quad \text{if} \quad |\xi| > 1 + \delta. \]

Throughout, we assume that \( g \) is a Riemannian metric on \( \mathbb{R}^3 \) such that
\[ g = (1 + |x|^{-1})^4 \tilde{g} + \sigma \]
where \( \sigma \) is a symmetric, covariant two-tensor with, as \( x \to \infty \) for every multi-index \( J \) with \( |J| \leq 4 \),
\[ \partial_J \sigma = O(|x|^{-2-|J|}). \]
(55)

We point out additional assumptions on \( g \) where required.

The estimates below depend on \( \lambda_0 > 1, \delta \in (0, 1/2) \), and \( g \). They are otherwise independent of \( \xi \) and \( \lambda \). For outlying surfaces, where \( |\xi| > 1 + \delta \), we recall that a bar underneath a quantity indicates evaluation at \( \lambda \xi \), possibly after taking derivatives. For example,
\[ \sigma(\lambda \tilde{\nu} + \lambda \xi) = \sigma + \lambda D_\nu \sigma + O(\lambda^{-2} |\xi|^{-4}) \]
is shorthand for
\[ \sigma(\lambda \tilde{\nu} + \lambda \xi) = \sigma(\lambda \xi) + \lambda (D_\nu \sigma)(\lambda \xi) + O(\lambda^{-2} |\xi|^{-4}). \]

When stating that an error term such as \( \mathcal{E} = O(\lambda^{-\ell_1} |\xi|^{-\ell_2}) \) may be differentiated with respect to \( \xi \) with \( \ell_1, \ell_2 \in \mathbb{Z} \), we mean that
\[ \bar{D}\mathcal{E} = O(\lambda^{-\ell_1} |\xi|^{-\ell_2-1}), \]
where differentiation is with respect to \( \xi \).

For the statements below, recall that
\[ \phi(x) = 1 + |x|^{-1} \]
denotes the conformal factor of the Schwarzschild metric.

**Lemma 39.** We have, for \( i, j = 1, 2, 3 \),
\[ (\text{Ric}_S)(e_i, e_j) = 2 \phi^{-2} |x|^{-3} (\delta_{ij} - 3 |x|^{-2} x^i x^j). \]
Moreover, there holds
\[ \nu_S(S_{\xi, \lambda}) = \phi^{-2} \tilde{\nu} \quad \text{and} \quad H_S(S_{\xi, \lambda}) = 2 \phi^{-2} \lambda^{-1} - 4 \phi^{-3} |x|^{-3} \tilde{g}(x, \tilde{\nu}). \]

A more precise version of the expansion in the following lemma was computed in [6, p. 670].

**Lemma 40 ([6]).** There holds
\[ |S_{\xi, \lambda}| = \begin{cases} 4 \pi \lambda^2 + 16 \pi \lambda + O(1) & \text{if } |\xi| < 1 - \delta, \\ 4 \pi \lambda^2 + 16 \pi \lambda |\xi|^{-1} + O(|\xi|^{-2}) & \text{if } |\xi| > 1 + \delta. \end{cases} \]

We need a more precise expansion of the Willmore energy of \( S_{\xi, \lambda} \) than that computed in [11].
To this end, we first compute the dependence of certain geometric quantities on the perturbation \( \sigma \) away from \( g_S \).
Lemma 41. Let \( \{e_1, e_2\} \) be a local Euclidean orthonormal frame for \( TS_{\xi,\lambda} \). There holds
\[
\nu - \nu_S = -\frac{1}{2} \phi^{-6} \sigma(\bar{\nu}, \bar{\nu}) \bar{\nu} - \phi^{-6} \sum_{\alpha=1}^{2} \sigma(\bar{\nu}, e_\alpha) e_\alpha + O(\lambda^{-4} (1 + |\xi|)^{-4}),
\]
\[
\hat{h}_\alpha^\beta = -\frac{1}{2} \lambda^{-1} \phi^{-6} \left[ 2 \sigma(e_\alpha, e_\beta) - (\bar{\tau} \sigma - \sigma(\bar{\nu}, \bar{\nu})) \delta_\alpha^\beta \right]
\]
\[
- \frac{1}{2} \left[ (\bar{D}_e e_\alpha)(\bar{\nu}, e_\beta) + (\bar{D}_e e_\beta)(\bar{\nu}, e_\alpha) - (\bar{D}_\nu e_\alpha)(e_\alpha, e_\beta) \right] + \frac{1}{4} \left[ (\bar{D}_\nu e_\alpha)(\bar{\nu}, \bar{\nu}) - 2 (\bar{D}_\nu \bar{\tau} \sigma)(\bar{\nu}, \bar{\nu}) - (\bar{D}_\nu e_\alpha)(\bar{\nu}, \bar{\nu}) \right] \delta_\alpha^\beta + O(\lambda^{-4} (1 + |\xi|)^{-4}),
\]
\[
H - H_S = \lambda^{-1} \phi^{-6} \left[ 2 \sigma(\bar{\nu}, \bar{\nu}) - \bar{\tau} \sigma \right] + \frac{1}{2} \left[ (\bar{D}_\nu \bar{\tau} \sigma + (\bar{D}_\nu \sigma)(\bar{\nu}, \bar{\nu}) - 2 (\bar{\text{div}} \sigma)(\bar{\nu}) \right] \right.
\]
\[
\left. + O(\lambda^{-4} (1 + |\xi|)^{-4}), \right. 
\]
\[
\Delta(H - H_S) = 4 \lambda^{-3} \phi^{-10} \left[ \bar{\tau} \sigma - 3 \sigma(\bar{\nu}, \bar{\nu}) \right] + Y_1 + Y_3 + O(\lambda^{-5} (1 + |\xi|)^{-4})
\]
where \( \alpha, \beta = 1, 2 \). Here, \( Y_1 \) and \( Y_3 \) are respectively first and third spherical harmonics with
\[
Y_1 = O(\lambda^{-5} (1 + |\xi|)^{-3}) \quad \text{and} \quad Y_3 = O(\lambda^{-5} (1 + |\xi|)^{-3}).
\]
If \( g \) satisfies (55) for every multi-index \( J \) with \( |J| \leq 5 \), these identities may be differentiated once with respect to \( \xi \).

Proof. Given \( t \in [0, 1] \), we define the family of metrics \( g_t = g_S + t \sigma \) such that \( g_0 = g_S \) and \( g_1 = g \). The identities can be obtained upon linearizing the respective quantities at \( t = 0 \).

Second, we recall the conformal killing operator
\[
\mathcal{D}Z = \mathcal{L}Z \sigma - \frac{1}{3} \text{tr}(\mathcal{L}Z \sigma) \sigma
\]
where \( Z \) is a vector field.

Lemma 42. Let \( Z = \phi^{-2} (x - \lambda \xi) \). If \( |\xi| < 1 - \delta \), there holds, in \( \mathbb{R}^3 \setminus B_\lambda(\lambda \xi) \),
\[
\mathcal{D}_S Z = O(\lambda^{-1} |x|^{-1}) \quad \text{and} \quad \mathcal{D}_S Z - \mathcal{D} Z = O(\lambda^{-1} |x|^{-2}).
\]
If \( |\xi| > 1 - \delta \), there holds, in \( B_\lambda(\lambda \xi) \),
\[
\mathcal{D}_S Z = O(\lambda^{-2} |\xi|^{-2}) \quad \text{and} \quad \mathcal{D}_S Z - \mathcal{D} Z = O(\lambda^{-3} |\xi|^{-3}).
\]

Proof. Note that
\[
(\mathcal{D}Z)(e_i, e_j) = g(D_{e_i} Z, e_j) + g(D_{e_j} Z, e_i) - \frac{2}{3} (\text{div} Z) g(e_i, e_j).
\]
We compute
\[
\bar{D}_{e_i} Z = \phi^{-2} \lambda^{-1} e_i + 2 \phi^{-1} |x|^{-3} \bar{g}(x, e_i) Z
\]
and
\[
\text{div} Z = 3 \phi^{-2} \lambda^{-1} + 2 \phi^{-1} |x|^{-3} \bar{g}(x, Z).
\]
Moreover, we have
\[
\begin{align*}
&\circ \quad D_S Z - \bar{D}Z = O(|x|^{-2} |Z|), \\
&\circ \quad DZ - D_S Z = O(|x|^{-3} |Z|), \\
&\circ \quad \text{div}_S Z - \text{div} Z = O(|x|^{-2} |Z|), \quad \text{and} \\
&\circ \quad \text{div} Z - \text{div}_S Z = O(|x|^{-3} |Z|).
\end{align*}
\]

Finally, note that \( Z = O(\lambda^{-1} |x|) \) in \( \mathbb{R}^3 \setminus B_{\lambda}(\lambda \xi) \) if \(|\xi| < 1 - \delta \) and that \( Z = O(1) \) in \( B_{\lambda}(\lambda \xi) \) if \(|\xi| > 1 + \delta \).

The assertion follows from these estimates. \( \square \)

For Lemma 43 below, recall that
\[ \hat{\sigma} = \sigma - \frac{1}{3} (\text{tr} \, \sigma) \tilde{g}. \]

**Lemma 43.** If \(|\xi| < 1 - \delta \), there holds
\[
\int_{S_{\xi,\lambda}} H^2 \, d\mu = 16 \pi - 64 \pi \lambda^{-1} + 8 \pi \lambda^{-2} \left[ 10 - 6 |\xi|^2 \right] \left[ (1 - |\xi|^2)^2 + 3 |\xi|^{-1} \log \frac{1 + |\xi|}{1 - |\xi|} \right] + 2 \lambda^{-1} \int_{\mathbb{R}^3\setminus B_{\lambda}(\lambda \xi)} R \, d\nu + O(\lambda^{-3}).
\]
If \(|\xi| > 1 + \delta \), there holds
\[
\int_{S_{\xi,\lambda}} H^2 \, d\mu = 16 \pi + 8 \pi \lambda^{-2} \left[ 10 - 6 |\xi|^2 \right] \left[ (1 - |\xi|^2)^2 + 3 |\xi|^{-1} \log \frac{|\xi| + 1}{|\xi| - 1} \right] - 2 \lambda^{-1} \phi^4 \int_{B_{\lambda}(\lambda \xi)} R \, d\nu + \frac{48 \pi}{15} \phi^{-8} |\hat{\sigma}|^2 + \frac{128 \pi}{15} \lambda^{-1} |\xi|^{-3} (3 |\xi|^{-2} \sigma(\xi, \xi) - \text{tr} \, \sigma) + O(\lambda^{-3} |\xi|^{-6}).
\]
Both expressions may be differentiated twice with respect to \( \xi \).

**Proof.** We first compute the Schwarzschild contribution. Note that
\[
(57) \quad \hat{g}(x, \bar{\nu}) = \lambda (1 - |\xi|^2) + \lambda^{-1} |x|^2.
\]

In the case where \(|\xi| < 1 - \delta \), we use Lemma 39 to compute
\[
\int_{S_{\xi,\lambda}} H_S^2 \, d\mu_S = \int_{S_{\xi,\lambda}} \left[ 4 \lambda^{-2} - 16 \lambda^{-1} (|x|^{-3} - |x|^{-4}) \hat{g}(x, \bar{\nu}) + 16 |x|^{-6} \hat{g}(x, \bar{\nu})^2 \right] \, d\mu + O(\lambda^{-3})
\]
\[
= 16 \pi - 64 \pi \lambda^{-1} + 16 \pi \lambda^{-2} \left[ 5 - 3 |\xi|^2 \right] \left[ (1 - |\xi|^2)^2 + 24 \pi \lambda^{-2} |\xi|^{-1} \log \frac{1 + |\xi|}{1 - |\xi|} \right] + O(\lambda^{-3}).
\]

In the case where \(|\xi| > 1 + \delta \), we compute that
\[
\int_{S_{\xi,\lambda}} H_S^2 \, d\mu_S = 16 \pi + 16 \pi \lambda^{-2} \left[ 5 - 3 |\xi|^2 \right] \left[ (1 - |\xi|^2)^2 + 24 \pi \lambda^{-2} |\xi|^{-1} \log \frac{|\xi| + 1}{|\xi| - 1} \right] + O(\lambda^{-3} |\xi|^{-6}).
\]

To compute the contribution from the perturbation \( \sigma \) off Schwarzschild, we start from the identity
\[
(58) \quad \int_{S_{\xi,\lambda}} H^2 \, d\mu = 16 \pi + 2 \int_{S_{\xi,\lambda}} |\hat{h}|^2 \, d\mu + 2 \int_{S_{\xi,\lambda}} (2 \, \text{Ric}(\nu, \nu) - R) \, d\mu,
\]
Conversely, if \(|\xi| < 1 - \delta\) and \(\sigma \neq 0\), using Lemma 41, we estimate
\[
\int_{S_{\xi,\lambda}} |h|^2 d\mu = O(\lambda^{-4}).
\]

If \(|\xi| > 1 + \delta\), using Lemma 41, Taylor expansion, and cancellations due to symmetry, we find that
\[
\int_{S_{\xi,\lambda}} |h|^2 d\mu = \phi^{-8} \lambda^{-2} \int_{S_{\xi,\lambda}} \left[ \frac{1}{2} (\bar{\tau} s_{\xi,\lambda} \sigma) \right] d\mu + O(\lambda^{-4} |\xi|^{-6})
\]
\[
= \phi^{-8} \lambda^{-2} \int_{S_{\xi,\lambda}} \left[ \sigma^2 - \frac{1}{2} (\bar{\tau} \sigma)^2 + \frac{1}{2} \sigma(\bar{\nu}, \bar{\nu}) \sigma(\bar{\nu}, \bar{\nu}) + \sigma(\bar{\nu}, \bar{\nu}) \bar{\tau} \sigma + 2 \sum_{i=1}^{3} \sigma(\bar{\nu}, e_i) \sigma(\bar{\nu}, e_i) \right] d\mu + O(\lambda^{-4} |\xi|^{-6})
\]
\[
= \frac{24}{15} \phi^{-8} |\phi|^2 + O(\lambda^{-4} |\xi|^{-6}).
\]

In the last step, we have used Lemma 35.

To compute the second integral in (58), first recall that the Einstein tensor
\[
E = \text{Ric} - \frac{1}{2} R g
\]
is divergence free. If \(|\xi| < 1 - \delta\), this leads to the following form of the Pohozaev identity
\[
\int_{S_{\xi,\lambda}} E(Z, \nu) d\mu = \int_{S_{\xi,\lambda}} E(Z, \nu) d\mu - \int_{B_r(0) \setminus B_\lambda(\lambda \xi)} \left[ \frac{1}{2} g(E, \mathcal{D}Z) - \frac{1}{6} (\text{div} Z) R \right] dv,
\]
valid for every vector field \(Z\) and every \(r > 2 \lambda\). Similarly, if \(|\xi| > 1 + \delta\), we have
\[
\int_{S_{\xi,\lambda}} E(Z, \nu) d\mu = \int_{B_\lambda(\lambda \xi)} \left[ \frac{1}{2} g(E, \mathcal{D}Z) - \frac{1}{6} (\text{div} Z) R \right] dv.
\]

We refer to [28, §6.4] for a discussion of the Pohozaev identity and a related application.

Let
\[
Z = \phi^{-2} \lambda^{-1} (x - \lambda \xi)
\]
and note that \(Z = \nu_S\) on \(S_{\xi,\lambda}\). Consequently,
\[
\int_{S_{\xi,\lambda}} E_S(\nu_S, \nu_S) d\mu_S = \int_{S_{\xi,\lambda}} E_S(\nu_S, Z) d\mu_S
\]
and
\[
\int_{S_{\xi,\lambda}} E(\nu, \nu) d\mu = \int_{S_{\xi,\lambda}} E(\nu, Z) d\mu + \int_{S_{\xi,\lambda}} E(\nu, \nu - \nu_S) d\mu.
\]

If \(|\xi| < 1 - \delta\), we let \(r \to \infty\) in (59) and obtain, using Lemma 39, Lemma 42, and that \(R_S = 0\),
\[
\int_{S_{\xi,\lambda}} E_S(\nu_S, Z) d\mu_S = - \frac{16 \pi \lambda^{-1}}{2} \int_{\mathbb{R}^3 \setminus B_\lambda(\lambda \xi)} g_S(E_S, \mathcal{D}_S Z) dv_S
\]
\[
= - \frac{16 \pi \lambda^{-1}}{2} \int_{\mathbb{R}^3 \setminus B_\lambda(\lambda \xi)} g(E, \mathcal{D} Z) dv + O(\lambda^{-3}).
\]
Likewise, we have
\[\int_{S_{\xi,\lambda}} E(\nu, Z) \, d\mu = -16 \pi \lambda^{-1} - \frac{1}{2} \int_{\mathbb{R}^3 \setminus B_\lambda(\lambda \xi)} g(E, DZ) \, dv + \frac{1}{6} \int_{\mathbb{R}^3 \setminus B_\lambda(\lambda \xi)} (\text{div} Z) \, R \, dv.\]

Finally, we use the coarse estimates
\[\int_{\mathbb{R}^3 \setminus B_\lambda(\lambda \xi)} (\text{div} Z) \, R \, dv = 3 \lambda^{-1} \int_{\mathbb{R}^3 \setminus B_\lambda(\lambda \xi)} R \, dv + O(\lambda^{-3})\]
and
\[\int_{S_{\xi,\lambda}} E(\nu, \nu - \nu_S) \, d\mu = O(\lambda^{-3}).\]

If \(|\xi| > 1 + \delta\), we use (60), Lemma 42, and \(R_S = 0\) to obtain that
\[\int_{S_{\xi,\lambda}} E_S(\nu_S, Z) \, d\mu_S = \frac{1}{2} \int_{B_\lambda(\lambda \xi)} g_S(E_S, D_S Z) \, dv_S = \frac{1}{2} \int_{B_\lambda(\lambda \xi)} g(E, DZ) \, dv + O(\lambda^{-3} |\xi|^{-6}).\]

Likewise, we have
\[\int_{S_{\xi,\lambda}} E_S(\nu, Z) \, d\mu = \frac{1}{2} \int_{B_\lambda(\lambda \xi)} g(E, DZ) \, dv - \frac{1}{6} \int_{B_\lambda(\lambda \xi)} (\text{div} Z) \, R \, dv.\]

Note that
\[-\frac{1}{6} \int_{B_\lambda(\lambda \xi)} (\text{div} Z) \, R \, dv = -\frac{1}{2} \varphi^4 \lambda^{-1} \int_{B_\lambda(\lambda \xi)} R \, dv + O(\lambda^{-3} |\xi|^{-6}).\]

Moreover, using that \(R = O(|x|^{-4})\), we have
\[\int_{S_{\xi,\lambda}} E(\nu - \nu_S, \nu) \, d\mu = \int_{S_{\xi,\lambda}} \text{Ric}(\nu - \nu_S, \nu) \, d\mu + O(\lambda^{-3} |\xi|^{-6}).\]

Finally, using Lemma 39 and the expansion for \(\nu - \nu_S\) from Lemma 41, we obtain
\[\int_{S_{\xi,\lambda}} \text{Ric}(\nu - \nu_S, \nu) \, d\mu \]
\[= -\int_{S_{\xi,\lambda}} |x|^{-3} \left[ \sigma(\bar{\nu}, \bar{\nu}) \left(1 + 3 |x|^{-2} \bar{g}(x, \bar{\nu})^2\right) - 6 |x|^{-2} \sigma(x, \bar{v}) g(x, \bar{\nu}) \right] \, d\bar{\mu} + O(\lambda^{-3} |\xi|^{-6})\]
\[= -\int_{S_{\xi,\lambda}} \lambda^{-3} |\xi|^{-3} \left[ \sigma(\bar{\nu}, \bar{\nu}) \left(1 + 3 |\xi|^{-2} \bar{g}(\xi, \bar{\nu})^2\right) - 6 |\xi|^{-2} \sigma(\xi, \bar{\nu}) \bar{g}(\xi, \bar{\nu}) \right] \, d\bar{\mu} + O(\lambda^{-3} |\xi|^{-6})\]
\[= \frac{32 \pi}{15} \lambda^{-1} |\xi|^{-3} \left[ 3 |\xi|^{-2} \sigma(\xi, \xi) - \text{tr} \, \sigma \right] + O(\lambda^{-3} |\xi|^{-6}).\]

We have used Lemma 35 in the third equality. The assertion follows from these estimates.

\[\square\]

**Remark 44.** Let \(\{S_j\}_{j=1}^\infty\) be a sequence of coordinate spheres \(S_j = S_{\lambda_j}(\lambda_j \xi_j)\) with \(\lambda_j > 1\) and \(\xi_j \in \mathbb{R}^3\) that are slowly divergent in the sense that \(\lim_{j \to \infty} \rho_j = \infty\) and \(\lim_{j \to \infty} \lambda_j^{-1} \rho_j = 0\) where \(\rho_j = \rho(S_j)\). If the spheres are on-center, we compute
\[\int_{S_j} H_S^2 \, d\mu_S = 16 \pi - 32 \pi \lambda_j^{-1} (2 - \rho_j^{-1}) + 8 \pi \rho_j^{-2} + O(\lambda_j^{-2} \log \lambda_j) + O(\lambda_j^{-1} \rho_j^{-2}) + O(\rho_j^{-3}).\]
Next, using the transformation of the Laplacian under a conformal change of the metric, we find
\[ \int_{S_j} H_S^2 \, d\mu_S = 16 \pi + 32 \pi \lambda_j^{-1} \rho_j^{-1} + 8 \pi \rho_j^{-2} + O(\lambda_j^{-2} \log \lambda_j) + O(\lambda_j^{-1} \rho_j^{-2}) + O(\rho_j^{-3}). \]

Using Lemma 39, we compute that, in either case,
\[ \min_{x \in S_j} (\phi^2 H_S) = 2 \lambda_j^{-1} - 4 \rho_j^{-2} + O(\rho_j^{-3}). \]

Thus, if \( \rho_j^2 = o(\lambda_j) \), it follows that \( \min_{x \in S_j} H_S < 0 \) and \( m_H(S_j) < 0 \) for all \( j \) large.

Next, we express the Willmore operator \(-W(S_{\xi,\lambda})\) in terms of spherical harmonics.

**Lemma 45.** There holds
\[ W(S_{\xi,\lambda}) = \frac{1}{2} \, \phi^{-8} \left( -9 \lambda^{-3} |x|^{-1} + (3 |\xi|^2 - 7) \lambda^{-1} |x|^{-3} - 3 (1 - |\xi|^2) (7 |\xi|^2 + 5) \lambda |x|^{-5} \right) \]
\[ + 15 (1 - |\xi|^2)^3 \lambda^3 |x|^{-7} \]
\[ + 4 \phi^{-10} \lambda^{-3} [\bar{tr} \sigma - 3 \sigma(\bar{\nu}, \bar{\nu})] + Y_1 + Y_3 + O(\lambda^{-5} (1 + |\xi|)^{-4}). \]

Here \( Y_1 \) is a first spherical harmonic and \( Y_3 \) is a third spherical harmonic. They satisfy
\[ Y_1 = O(\lambda^{-5} (1 + |\xi|)^{-3}) \quad \text{and} \quad Y_3 = O(\lambda^{-5} (1 + |\xi|)^{-3}). \]

If \( g \) satisfies (55) for every multi-index \( J \) with \( |J| \leq 5 \), this identity may be differentiated once with respect to \( \xi \).

**Proof.** Using Lemma 39 and (57), we find
\[ H \, \text{Ric}(\nu, \nu) = \phi^{-8} \left( -3 \lambda^{-3} |x|^{-1} + 2 (3 |\xi|^2 - 1) \lambda^{-1} |x|^{-3} - 3 (1 - |\xi|^2)^2 \lambda |x|^{-5} \right) \]
\[ + O(\lambda^{-5} (1 + |\xi|)^{-4}). \]

Next, using the transformation of the Laplacian under a conformal change of the metric, we find
\[ \Delta_{S_{\xi,\lambda}} H = \phi^{-4} \Delta_{S_{\xi,\lambda}} H_S + \phi^{-4} \Delta_{S_{\xi,\lambda}} (H - H_S) + O(\lambda^{-6} (1 + |\xi|)^{-4}). \]

Let \( \psi : \mathbb{R}^3 \to \mathbb{R} \) be given by \( \psi(x) = 2 \phi^{-3} \lambda^{-1} - 2 \phi^{-3} \lambda (1 - |\xi|^2) |x|^{-3} \).

By Lemma 39 and (57), we have \( \psi(x) = H_S(x) \) for all \( x \in S_{\xi,\lambda} \). Consequently,
\[ \bar{\Delta}_{S_{\xi,\lambda}} H_S = \bar{\Delta}_{\mathbb{R}^3} \psi - \bar{D}_{\rho,\rho} \psi - 2 \lambda^{-1} \bar{D}_{\rho} \psi. \]

We compute
\[ \partial_i \psi = 6 \phi^{-4} \lambda^{-1} |x|^{-3} x^i + 6 \phi^{-4} \lambda (1 - |\xi|^2) |x|^{-5} x^i \]
and
\[ \partial_i \partial_j \psi = 6 \phi^{-4} \lambda^{-1} |x|^{-3} (\delta_{ij} - 3 |x|^{-2} x^i x^j) + 6 \phi^{-4} \lambda (1 - |\xi|^2) |x|^{-5} (\delta_{ij} - 5 |x|^{-2} x^i x^j) \]
\[ + O(\lambda^{-5} (1 + |\xi|)^{-4}). \]

Using this and (57), we obtain
\[ \bar{\Delta}_{\mathbb{R}^3} \psi = -12 \phi^{-4} \lambda (1 - |\xi|^2) |x|^{-5} + O(\lambda^{-5} (1 + |\xi|)^{-4}), \]
\[
\begin{align*}
\tilde{D}_{\psi,\nu}^2 \psi &= -\frac{9}{2} \phi^{-4} \lambda^{-3} |x|^{-1} - \frac{1}{2} \phi^{-4} \lambda^{-1} (21 - 33 |\xi|^2) |x|^{-3} - \frac{1}{2} \phi^{-4} \lambda (1 - |\xi|^2) (27 - 39 |\xi|^2) |x|^{-5} \\
&\quad - \frac{15}{2} \phi^{-4} \lambda^3 (1 - |\xi|^2) |x|^{-7} + O(\lambda^{-5} (1 + |\xi|)^{-4}), \\
\tilde{D}_{\nu} \psi &= 3 \phi^{-4} \lambda^{-2} |x|^{-1} + 6 \phi^{-4} (1 - |\xi|^2) |x|^{-3} + 3 \phi^{-4} \lambda^2 (1 - |\xi|^2)^2 |x|^{-5}.
\end{align*}
\]

The assertion follows from this and Lemma 41.

The following corollary is an immediate consequence of Lemma 37 and Lemma 45.

**Corollary 46.** If \(|\xi| < 1 - \delta\), there holds

\[
W(S_{\xi,\lambda}) = 4 \lambda^{-4} \sum_{\ell=0}^{\infty} (\ell - 1) (\ell + 1) (\ell + 2) |\xi|^\ell P_\ell (-|\xi|^{-1} \tilde{g}(\nu, \xi)) + O(\lambda^{-5}).
\]

If \(|\xi| > 1 + \delta\), there holds

\[
W(S_{\xi,\lambda}) = -4 \lambda^{-4} \sum_{\ell=0}^{\infty} (\ell - 1) \ell (\ell + 2) |\xi|^{-\ell-1} P_\ell (-|\xi|^{-1} \tilde{g}(\nu, \xi))
\]

\[\text{(61)}
\]

\[
- 4 \lambda^{-3} \phi^{-10} (3 \sigma(\tilde{\nu}, \tilde{\nu}) - \tilde{\nabla} \sigma) + Y_1 + Y_3 + O(\lambda^{-5} |\xi|^{-4}).
\]

Here, \(Y_1\) and \(Y_3\) are, respectively, first and third spherical harmonics with

\[
Y_1 = O(\lambda^{-5} |\xi|^{-3}) \quad \text{and} \quad Y_3 = O(\lambda^{-5} |\xi|^{-3}).
\]

If \(g\) satisfies (55) for every multi-index \(J\) with \(|J| \leq 5\), then (61) may be differentiated once with respect to \(\xi\).

**Remark 47.** Note that

\[
3 \sigma(\tilde{\nu}, \tilde{\nu}) - \tilde{\nabla} \sigma = \sum_{i,j=1}^3 \sigma(e_i, e_j) (3 \tilde{g}(\tilde{\nu}, e_i) \tilde{g}(\tilde{\nu}, e_j) - \delta_{ij}) \in \Lambda_2(S_{\xi,\lambda}).
\]

In the next lemma, we specify the formula for the linearization of the Willmore operator (51) to a sphere.

**Lemma 48.** For every \(u \in C^\infty(S_{\xi,\lambda})\) there holds

\[
Q_{S_{\xi,\lambda}} u = L(Lu) + \frac{1}{2} H^2 Lu + (\nabla^2 u) * O(\lambda^{-4} (1 + |\xi|)^{-2}) + (\nabla u) * O(\lambda^{-4} (1 + |\xi|)^{-3})
\]

\[\text{+} u * O(\lambda^{-5} (1 + |\xi|)^{-3}).
\]

If \(g\) satisfies (55) for every multi-index \(J\) with \(|J| \leq 5\), (62) may be differentiated once with respect to \(\xi\).

**Proof.** This follows from (51), the decay of the metric, Lemma 41, the estimates

\[
\nabla H = O(\lambda^{-3} (1 + |\xi|)^{-2}), \quad \nabla^2 H = O(\lambda^{-4} (1 + |\xi|)^{-2}),
\]

and the estimate

\[
\Delta H = W + O(\lambda^{-4} (1 + |\xi|)^{-3}) = O(\lambda^{-4} (1 + |\xi|)^{-3}).
\]

We have used Corollary 46 in the last equation. □
D. The foliation property

Recall from the proof of Theorem 5 that $\Sigma(\lambda)$ is the sphere

$$\Sigma(\lambda) = \Sigma_{\xi(\lambda),\lambda} = \Sigma_{\xi(\lambda),\lambda}(u_{\xi(\lambda),\lambda})$$

where $\xi(\lambda) \in \mathbb{R}^3$ is the unique local minimum near the origin of the function $G_\lambda$ defined in (18). In particular, by Lemma 21, $\Sigma(\lambda)$ is a stable area-constrained Willmore surface. Moreover, we have seen in the proof of Theorem 5 that, as $\lambda \to \infty$,

$$\xi(\lambda) = o(1). \tag{63}$$

By Proposition 17 and Remark 18, we have

$$u_{\xi(\lambda),\lambda} = O(1), \quad (\bar{D}u)_{|\xi(\lambda),\lambda} = O(\lambda^{-1}), \quad u'_{|\xi(\lambda),\lambda} = O(\lambda^{-2}), \tag{64}$$

where we recall that $\bar{D}$ and the dash indicate differentiation with respect to the parameters $\xi$ and $\lambda$, respectively.

We now verify that the family of spheres $\{\Sigma(\lambda) : \lambda > \lambda_0\}$ forms a smooth foliation, provided $\lambda_0 > 1$ is sufficiently large.

**Proposition 49.** Suppose that $(M,g)$ is $C^4$-asymptotic to Schwarzschild with mass $m > 0$ and that the scalar curvature $R$ satisfies (19) and (20). Then the family $\{\Sigma(\lambda) : \lambda > \lambda_0\}$ of stable area-constrained Willmore spheres is a smooth foliation of the complement of a compact subset of $M$ provided $\lambda_0 > 1$ is sufficiently large.

**Proof.** By Lemma 24, there are constants $\tau > 0$ and $\delta_0 > 0$ such that

$$\bar{D}^2 G_\lambda \geq \tau \text{ Id} \tag{65}$$

on $\{\xi \in \mathbb{R}^3 : |\xi| < \delta_0\}$ provided $\lambda > \lambda_0$ and $\lambda_0 > 1$ is sufficiently large. In particular, by the implicit function theorem, the dependence of $\xi(\lambda)$ on $\lambda$ is smooth. It follows that the map

$$\Psi : S_1(0) \times (\lambda_0, \infty) \to M \quad \text{given by} \quad \Psi(y, \lambda) = \Phi_{u_{\xi(\lambda),\lambda}}^\lambda (\lambda y + \xi)$$

is smooth. Using (63) and (64), we find that $\Sigma(\lambda)$ encloses every given compact set, provided $\lambda > 1$ is sufficiently large.

We claim that

$$\xi'(\lambda) = o(\lambda^{-1}). \tag{66}$$

To see this, let $a \in \mathbb{R}^3$. Differentiating the identity $(\bar{D}G_\lambda)_{|\xi(\lambda)}(a) = 0$ with respect to $\lambda$, we obtain

$$(\bar{D}^2 G_\lambda)_{|\xi(\lambda)}(a, \xi'(\lambda)) + (\bar{D}G_\lambda')_{|\xi(\lambda)}(a) = 0. \tag{67}$$

The argument presented in the proof of Lemma 24 also shows that we may differentiate the error terms in Lemma 22 with respect to $\lambda$. Applying Lemma 22 and using (63), we thus find

$$(\bar{D}G_\lambda')_{|\xi(\lambda)}(a) = -4\lambda \int_{S_{\xi(\lambda),\lambda}} \bar{g}(a, \bar{\nu}) R \, d\bar{\mu} - 2\lambda^2 \int_{S_{\xi(\lambda),\lambda}} g(a, \bar{\nu}) (\bar{D}_\bar{\nu} R) \, d\bar{\mu} + o(\lambda^{-1}).$$
Since \((M, g)\) is \(C^4\)-asymptotic to Schwarzschild, we obtain from (20) that

\[
\sum_{i=1}^{3} \left[ x^i (\partial_i R)(x) + x^i (\partial_i R)(-x) \right] = o(|x|^{-4}).
\]

Indeed, if (68) failed, integration along radial lines would yield that (20) must be violated, too. From this, we find that

\[
(DG'_{\lambda})_{\xi(\lambda)}(a) = \xi(\lambda) O(\lambda^{-1}) + o(\lambda^{-1}) = o(\lambda^{-1}).
\]

Choosing \(a = \xi'(\lambda)\) and using (67) as well as (65), we obtain the asserted estimate (66).

Note that \(\Psi(\cdot, \lambda)\) parametrizes \(\Sigma(\lambda)\) and that \(\bar{\nu} = y + O(\lambda^{-1})\). Using (64) and (66), we compute that

\[
\bar{g}(\Psi', y) = 1 + \bar{g}(\xi(\lambda), y) + \lambda \bar{g}(\xi'(\lambda), y) + (\bar{D}_{\xi(\lambda)} u)_{(\xi(\lambda), \lambda)} + u'_{(\xi(\lambda), \lambda)} = 1 + o(1).
\]

In particular, \(\bar{g}(\Psi', \bar{\nu}) > 0\). This finishes the proof. \(\square\)

E. Remark on far-outlying stable constant mean curvature surfaces

In this section, we show that the assumptions of Theorem 11 are sufficient to preclude large far-outlying stable constant mean curvature spheres in \((M, g)\) as well. In the statement of the following result, \(\text{vol}(\Sigma)\) denotes the volume of the compact domain bounded by \(\Sigma\).

**Theorem 50.** Suppose that \((M, g)\) is \(C^5\)-asymptotic to Schwarzschild with mass \(m > 0\) and that its scalar curvature \(R\) satisfies

\[
\sum_{i=1}^{3} x^i (\partial_i |x|^2 R) \leq 0.
\]

There is no sequence \(\{\Sigma_j\}_{j=1}^{\infty}\) of outlying stable constant mean curvature spheres \(\Sigma_j \subset M\) with

\[
\lim_{j \to \infty} \text{vol}(\Sigma_j) = \infty \quad \text{and} \quad \lim_{j \to \infty} \rho(\Sigma_j) H(\Sigma_j) = \infty.
\]

**Remark 51.** The hypotheses of Theorem 50 are weaker than those of Corollary 1.7 in [10]. First, we only require \(C^5\)-decay of the metric, while \(C^7\)-decay of the metric is assumed in [10]. Second, the growth condition (69) is weaker than the radial convexity assumption

\[
\sum_{i,j=1}^{3} x^i x^j \partial_i \partial_j R \geq 0
\]

in [10].

The proof of Theorem 50 is a small variation of the proof of Corollary 1.7 in [10]. The asserted improvement is obtained by first taking the radial derivative of the area functional of a coordinate sphere and then estimating the resulting terms, rather than first estimating the area functional and then taking the radial derivative. This approach brings out the contribution of the scalar curvature in a more precise way. We only point out the necessary modifications of the proof.

We recall that

\[
\phi(x) = 1 + |x|^{-1}
\]
denotes the conformal factor of the Schwarzschild metric with mass \( m = 2 \). Moreover, continuing the notation introduced on p. 21, we use a bar underneath a quantity to indicate evaluation at \( \lambda \xi \).

Proof of Theorem 50. As in the proof of Theorem 11, there exists a constant \( \lambda_0 > 1 \) which only depends on \( (M, g) \) such that for every \( \lambda > \lambda_0 \) and \( \xi \in \mathbb{R}^3 \) with \( |\xi| > 2 \), there is a surface \( \Sigma_{\xi, \lambda} \) with the following properties:

- \( \Sigma_{\xi, \lambda} \) is a perturbation of the sphere \( S_{\tilde{\lambda}, \tilde{\lambda}} \), where \( \tilde{\lambda} = \lambda \xi \).

Moreover, we have

\[
\tilde{\lambda} = \lambda \phi^{-2} + O(\lambda^{-1} |\xi|^{-2}).
\]

- The mean curvature of \( \Sigma_{\xi, \lambda} \) is constant up to first spherical harmonics.

- There holds

\[
\text{vol}(\Sigma_{\xi, \lambda}) = \frac{4}{3} \pi \lambda^3.
\]

- \( \Sigma_{\xi, \lambda} \) has constant mean curvature if and only if \( \xi \) is a critical point of the function \( A_{\lambda} : \{ \xi \in \mathbb{R}^3 : |\xi| > 2 \} \to \mathbb{R}^3 \) given by \( A_{\lambda}(\xi) = |\Sigma_{\xi, \lambda}| \).

We note that \( \tilde{\lambda} \) is denoted by \( r \) in [10].

In [10, p. 182], it was shown that

\[
A_{\lambda}(\xi) = 4 \pi \lambda^2 - \frac{2 \pi}{15} \lambda^4 R - \frac{\pi}{105} \lambda^6 \Delta R - \frac{8 \pi}{35} |\xi|^{-6} + O(\lambda^{-1} |\xi|^{-6}) + O(|\xi|^{-7}).
\]

This identity may be differentiated once with respect to \( \xi \). In the derivation of this identity and its differentiability in [10, §4], \( C^6 \)-decay of the metric rather than \( C^5 \)-decay is used to analyze the contribution of the term

\[
\frac{1}{2} \int_{S_{\tilde{\lambda}, \tilde{\lambda}}} [\tilde{\text{tr}} \sigma - \sigma(\tilde{\nu}, \tilde{\nu})] d\tilde{\mu} - \tilde{\lambda}^{-1} \int_{B_{\lambda}(\lambda \xi)} \tilde{\text{tr}} \sigma d\tilde{v}
\]

to (71). In [10, §4.1 and §4.2], (72) was computed to be

\[
-\frac{2 \pi}{15} \lambda^4 R - \frac{\pi}{105} \lambda^6 \Delta R - \frac{8 \pi}{35} \lambda |\xi|^{-3} [\tilde{\text{tr}} \sigma - 3|\xi|^{-2} \sigma(\xi, \xi) + \lambda \tilde{D}_{\xi} \tilde{\text{tr}} \sigma] + O(\lambda^{-1} |\xi|^{-6}) + O(|\xi|^{-7}).
\]

It follows that

\[
A_{\lambda}(\xi) = 4 \pi \lambda^2 - \frac{8 \pi}{35} |\xi|^{-6} + \frac{8 \pi}{15} \lambda |\xi|^{-3} (\tilde{\text{tr}} \sigma - 3|\xi|^{-2} \sigma(\xi, \xi) + \lambda \tilde{D}_{\xi} \tilde{\text{tr}} \sigma) + \frac{1}{2} \int_{S_{\tilde{\lambda}, \tilde{\lambda}}} [\tilde{\text{tr}} \sigma - \sigma(\tilde{\nu}, \tilde{\nu})] d\tilde{\mu} - \tilde{\lambda}^{-1} \int_{B_{\lambda}(\lambda \xi)} \tilde{\text{tr}} \sigma d\tilde{v} + O(\lambda^{-1} |\xi|^{-6}) + O(|\xi|^{-7}).
\]

This expansion may be differentiated once with respect to \( \xi \) provided \( (M, g) \) is \( C^5 \)-asymptotic to Schwarzschild. We proceed by computing the radial derivative of \( A_{\lambda} \). Using Taylor’s theorem and
cancellations due to symmetry, we find
\[ \sum_{i=1}^{3} \xi^i \partial_i \left( \frac{8 \pi}{15} \lambda |\xi|^{-3} (\tilde{t} \tilde{r} \sigma - 3 |\xi|^{-2} \sigma(\xi, \xi) + \lambda \tilde{D}_\xi \tilde{t} \tilde{r} \sigma) \right) \]
(73)
\[ = -2 \int_{B_\lambda(\lambda \xi)} \left[ |x|^{-3} \tilde{t} \tilde{r} \sigma - 3 |x|^{-5} \sigma(x, x) + |x|^{-3} \tilde{D}_x \tilde{t} \tilde{r} \sigma \right] \tilde{g}(\lambda \xi - x, \xi) \, d\tilde{v} \]
\[ + O(\lambda^{-1} |\xi|^{-6}). \]

Next, we compute
\[ \sum_{i=1}^{3} \xi^i \partial_i \left( \frac{1}{2} \int_{S_{\tilde{t} \tilde{r} \lambda}} [\tilde{t} \tilde{r} \sigma - \sigma(\tilde{v}, \tilde{v})] \, d\tilde{\mu} - \tilde{\lambda}^{-1} \int_{B_\lambda(\lambda \xi)} \tilde{t} \tilde{r} \sigma \, d\tilde{v} \right) \]
(74)
\[ = \frac{1}{2} \lambda \int_{S_{\tilde{t} \tilde{r} \lambda}} \left[ \tilde{D}_\xi \tilde{t} \tilde{r} \sigma - \tilde{D}_\sigma \sigma(\tilde{v}, \tilde{v}) - 2 \tilde{\lambda}^{-1} \tilde{t} \tilde{r} \sigma \tilde{g}(\xi, \tilde{v}) \right] \, d\tilde{\mu} \]
\[ + \frac{1}{2} \xi^i \partial_i \tilde{\lambda} \left( \int_{S_{\tilde{t} \tilde{r} \lambda}} \left[ \tilde{D}_\sigma \tilde{t} \tilde{r} \sigma - \tilde{D}_\sigma \sigma(\tilde{v}, \tilde{v}) - 2 \tilde{\lambda}^{-1} \sigma(\tilde{v}, \tilde{v}) \right] \, d\tilde{\mu} + 2 \tilde{\lambda}^{-2} \int_{B_\lambda(\lambda \xi)} \tilde{t} \tilde{r} \sigma \, d\tilde{v} \right). \]

From (70), we find that
\[ \sum_{i=1}^{3} \xi^i \partial_i \tilde{\lambda} = 2 |\xi|^{-1} + O(\lambda^{-1} |\xi|^{-2}). \]

Using cancellations due to symmetry, we compute, using Taylor’s theorem to expand all terms up to second derivatives of \( \sigma \) and Lemma 35, that the last line of (74) equals
\[ - \frac{16 \pi}{15} \lambda^3 |\xi|^{-1} (\text{div div} \, \sigma - \tilde{\Delta} \tilde{t} \tilde{r} \sigma) + O(\lambda^{-1} |\xi|^{-6}) \]
(75)
\[ = 4 \int_{B_\lambda(\lambda \xi)} (\text{div div} \, \sigma - \tilde{\Delta} \tilde{t} \tilde{r} \sigma) (|\xi|^{-1} - |x|^{-1}) \tilde{g}(\xi, \lambda \xi - x) \, d\tilde{v} + O(\lambda^{-1} |\xi|^{-6}). \]

Here, we have also used that
\[ |\xi|^{-1} = |x|^{-1} - \lambda^{-2} |\xi|^{-3} \tilde{g}(\xi, \lambda \xi - x) + O(\lambda^{-1} |\xi|^{-3}). \]

Finally, using \( \lambda \xi = \tilde{\lambda} \bar{\xi} \), we can argue exactly as in [10, §2.1] to show that the second line of (74) equals
\[ \frac{1}{2} \int_{B_\lambda(\lambda \xi)} (\text{div div} \, \sigma - \tilde{\Delta} \tilde{t} \tilde{r} \sigma) \tilde{g}(\xi, \lambda \xi - x) \, d\tilde{\mu} \]
(76)
\[ = - \frac{2 \pi}{15} \tilde{\lambda}^5 \tilde{D}_\xi (\text{div div} \, \sigma - \tilde{\Delta} \tilde{t} \tilde{r} \sigma) + O(\lambda^{-1} |\xi|^{-6}) \]
\[ = - \frac{2 \pi}{15} \phi^{-8} \lambda^5 \tilde{D}_\xi (\text{div div} \, \sigma - \tilde{\Delta} \tilde{t} \tilde{r} \sigma) + O(\lambda^{-1} |\xi|^{-6}) \]
\[ = \frac{1}{2} \phi^{-8} \int_{B_\lambda(\lambda \xi)} (\text{div div} \, \sigma - \tilde{\Delta} \tilde{t} \tilde{r} \sigma) \tilde{g}(\xi, \lambda \xi - x) \, d\tilde{\mu} + O(\lambda^{-1} |\xi|^{-6}). \]

In the first and third equality, we have used Taylor’s theorem to expand the integrand up to fourth derivatives of \( \sigma \), the \( C^5 \)-decay of the metric, cancellations due to symmetry, and Lemma 35. In the second equality, we have used (70). According to [10, §4.9], there holds
\[ R = \phi^{-8} (\text{div div} \, \sigma - \tilde{\Delta} \tilde{t} \tilde{r} \sigma) - 4 \left[ |x|^{-3} \tilde{t} \tilde{r} \sigma - 3 |x|^{-5} \sigma(x, x) + |x|^{-3} \tilde{D}_x \tilde{t} \tilde{r} \sigma \right] + O(\lambda^{-1} |\xi|^{-6}) \]
while

$$\bar{\phi}^{-8} = \phi^8 + 8 (|x|^{-1} - |\xi|^{-1}) + O(\lambda^{-1} |\xi|^{-2}).$$

Combing this with (73), (74), (75), and (76), we conclude that

$$\sum_{i=1}^{3} \xi^i (\partial_i A_\lambda)(\xi) = \frac{48 \pi}{35} |\xi|^{-6} + \frac{1}{2} \int_{B_\lambda(\xi)} \bar{g}(\xi, \lambda \xi - x) R \, d\bar{\mu} + O(\lambda^{-1} |\xi|^{-6}) + O(|\xi|^{-7}).$$

In [10, §2.2], it has been shown that this integral is non-negative provided that (69) holds. In particular,

$$\sum_{i=1}^{3} \xi^i (\partial_i A_\lambda)(\xi) > 0$$

provided both $\xi \in \mathbb{R}^3$ and $\lambda > 1$ are large. We may now conclude the proof as in [10].

□

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