Holomorphic curves at one point

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Abstract
Let $(M, \omega)$ be a closed symplectic manifold with a compatible almost complex structure $J$. We prove that for $p \in M$ and $E > 0$, if $v : \Sigma \to M$ is a non-constant $J$-holomorphic curve with symplectic area smaller than $E$, then the number of the elements in $v^{-1}(p)$ is bounded, and the bound is independent of $v$. We also provide a uniform Hofer’s energy bound for $J$-holomorphic curves in $M \setminus p$ based on the symplectic area. Using these two results we compactify the moduli space of $J$-holomorphic curves in $M$ by adding holomorphic buildings at the point $p$.

Let $(M, \omega)$ be a closed smooth symplectic manifold of dimension $2N$ and $J$ be a compatible almost complex structure on $M$. For a fixed a point $p \in M$, we assume that $J$ is integrable inside a small neighborhood $U$ of $p$. For a sufficiently small neighborhood $V \subseteq U$ of $p$ there exists a coordinate chart $\varphi : V \to B(0, \epsilon) \subseteq \mathbb{C}^N$ such that $\varphi(p) = 0$, $\varphi^* i = J$, and $\varphi^* \omega_0 = \omega$, where $B(0, \epsilon) := \{z \in \mathbb{C}^N | |z| < \epsilon\}$ and $\omega_0 := \frac{1}{2} \sum_{k=1}^n dz_k \wedge d\bar{z}_k = \sum_{k=1}^n dx_k \wedge dy_k$ is the standard symplectic structure on $\mathbb{C}^N$. We identify $B(0, \epsilon) \setminus 0$ with $W := (-\infty, \log \epsilon) \times S^{2N-1}$ via the map $\psi(z) = (\log |z|, \frac{z}{|z|})$.

Let $\lambda$ be the standard contact 1-form on the unit sphere $S^{2N-1} \subseteq \mathbb{C}^N$, namely

$$\lambda = \sum_{k=1}^N x_k dy_k - y_k dx_k \bigg|_{S^{2N-1}}.$$ 

The corresponding Reeb vector field on $S^{2N-1}$ is denoted by $R$, i.e. $R \in \ker d\lambda$ and $\lambda(R) = 1$. We extend $\lambda$ trivially to $W$ via the pull back of the projection map $\Theta : (-\infty, \log \epsilon) \times S^{2N-1} \to S^{2N-1}$.

Let $(\Sigma, j)$ be a Riemann surface with finitely many punctures and $v : \Sigma \to M \setminus p$ be a $J$-holomorphic curve, i.e. $J(u) \circ Tu = Tu \circ j$.

Definition 1. The Hofer’s energy $E(v)$ of $v$ is defined to be

$$E(v) = E_{\text{symp}}(v) + E_d\lambda(v) + E_\lambda(v),$$

$$E_{\text{symp}}(v) = \int_{\Sigma} v^* \omega,$$

$$E_d\lambda(v) = \int_{v^{-1}(W)} v^* d\lambda.$$
and

\[ E_\lambda(v) = \sup_{\phi \in \mathcal{C}} \int_{v^{-1}(W)} v^*(\phi(r)dr \wedge \lambda) \]

where

\[ \mathcal{C} = \left\{ \phi \in C_\infty^\infty((\infty, \log \epsilon), [0, 1]) \left| \int_{-\infty}^{\log \epsilon} \phi(x)dx = 1 \right\}. \]

It is easy to check that \( v^*\omega, v^*d\lambda \), and \( v^*(\phi(r)dr \wedge \lambda) \) are non-negative multiples of a volume form on \( \Sigma \).

Let \((r, \Theta)\) be the coordinate of \( W := (\infty, \log \epsilon) \times S^{2N-1}, q\) be a puncture of \( \Sigma \), and \( f \) be a biholomorphic map from \((\infty, 0) \times \mathbb{R}/\mathbb{Z}\) to a small open subset of \( \Sigma \) around \( q \). If we choose \((s, t)\) as the coordinate of \((\infty, 0) \times \mathbb{R}/\mathbb{Z}\), then the map \( v \circ f \) can be written as \((r(s, t), \Theta(s, t))\). We say the map \( v \) converges to a Reeb orbit of period \( T > 0 \) around the puncture \( q \), if there exists a map \( \gamma : \mathbb{R}/\mathbb{Z} \rightarrow S^{2N-1} \) such that

\[ \frac{d}{dt}(\gamma(Tt)) = R(\gamma(Tt)), \]

\[ \lim_{s \rightarrow -\infty} \Theta(s, t) = \gamma(Tt), \quad \text{and} \quad \lim_{s \rightarrow -\infty} \frac{r(s, t)}{s} = T. \]

The definition of converging to a Reeb orbit is independent of the choice of \( f \). It is easy to see that in this case \( T = 2k\pi \) for some \( k \in \mathbb{Z}_{>0} \). Every Reeb orbit of \( R \) satisfies the Morse-Bott condition, so the result in [3] applied to this special case gives us

**Theorem 2.** [3] Assume the J-holomorphic curve \( v \) has finite Hofer’s energy, i.e. \( E(v) < +\infty \). Around each puncture of \( \Sigma \), \( v \) converges to either a point in \( M \setminus \{ \} \) or a Reeb orbit.

We say a puncture \( q \) is removable if around \( q \), \( v \) converges to a point in \( M \setminus p \). Otherwise, we say \( q \) is non-removable. The next proposition says that the number of non-removable punctures of \( v \) is bounded by a finite constant independent of \( v \).

**Proposition 3.** Given \( E > 0 \), there exists a number \( N \in \mathbb{N} \) such that for any finitely punctured Riemann surface \( (\Sigma, j) \), and any non-constant J-holomorphic map \( v : \Sigma \rightarrow M \setminus p \) with \( E(v) \leq E \), the number of non-removable punctures of \( v \) is no greater than \( N \).

**Proof.** Suppose to the contrary. Let \( v_n \) be a J-holomorphic curves from a finitely punctured Riemann surface \( \Sigma_n \) to \( M \setminus \{ \} \) with \( E(v_n) \leq E \), such that the number of non-removable punctures of \( v_n \) goes to infinity as \( n \rightarrow \infty \).

Let \( \eta^{q_n} \) be the Reeb orbit of \( R \) to which around the non-removable puncture \( q_n \), \( v_n \) converges. We denote the period of \( \eta^{q_n} \) by \( 2k\pi \), with \( k_{q_n} \in \mathbb{Z}_{>0} \). We can pick \([r_n - 1, r_n + 1] \times S^{2N-1} \) inside \( W = (\infty, \log \epsilon) \times S^{2N-1} \), such that \( v_n^{-1} ([r_n - 1, r_n + 1] \times S^{2N-1}) \) consists of connected components \( A_{l,n} \subset \Sigma_n \) for \( l \in I_n \) with

\[ \partial A_{l,n} = \partial_1 A_{l,n} - \partial_2 A_{l,n}, \]
\[ v_n(\partial_1 A_{i,n}) \subset \{ r_n + 1 \} \times S^{2N-1}, \]
\[ v_n(\partial_2 A_{i,n}) \subset \{ r_n - 1 \} \times S^{2N-1}, \]
and the cardinality \(|I_n|\) of the index set \(I_n\) satisfies \(|I_n| = \sum k_{q_n}\), where the summation is taken over all the punctures \(q_n\) for fixed \(n\). Thus, \(|I_n|\) goes to \(\infty\), as \(n \to \infty\).

Pick \(\phi_n(r) \in \mathcal{C}\) to be a function satisfying \(\phi(r) = \frac{1}{\pi}\) for \(r_n - 1 \leq r \leq r_n + 1\). Look at the 2-form \(\Omega_n := \phi_n(r) dr \wedge \lambda + d\lambda\) on \(W\), we know it is non-degenerate over \([r_n-1, r_n+1] \times S^{2N-1}\). Since \(S^{2N-1}\) is compact, by the Gromov’s Monotonicity Theorem we have \(\int_{A_{i,n}} v_n^* \Omega_n > \delta_0 > 0\), for some \(\delta_0\) independent of \(n\). Therefore, we get

\[
E \geq E(v_n) \\
\geq E_{d\lambda}(v_n) + E_{\lambda}(v_n) \\
= \int_{v_n^{-1}(W)} v_n^* d\lambda + \sup_{\phi \in \mathcal{C}} \int_{v_n^{-1}(W)} v_n^*(\phi(r) dr \wedge \lambda) \\
\geq \int_{v_n^{-1}(W)} v_n^* d\lambda + \int_{v_n^{-1}(W)} v_n^*(\phi_n(r) dr \wedge \lambda) \\
\geq \int_{v_n^{-1}([r_n-1, r_n+1] \times S^{2N-1})} v_n^* \Omega_n \to \infty. \\
= \sum_{i \in I_n} \int_{A_{i,n}} v_n^* \Omega_n \\
\geq \sum_{i \in I_n} \delta_0 \\
= \delta_0 |I_n| \to +\infty.
\]

\(\Box\)

**Remark 4.** For each non-removable punctures of \(v\), we can associate a multiplicity which is the multiplicity of the Reeb orbit. The above proof actually shows that the number of non-removable punctures of \(v\) counted with multiplicity is bounded by \(N\).

For a non-constant \(J\)-holomorphic curve \(v\) from a Riemann surface \((S, j)\) to \(M\), we know \(v^{-1}(p)\) is discrete, and hence finite. Let \((\Sigma, j)\) be the punctured Riemann surface defined by \((S \setminus v^{-1}(p), j)\). Now \(v\) can be viewed as a \(J\)-holomorphic curve from \(\Sigma\) to \(M\setminus p\). From the local behavior of a holomorphic map, we know that \(v\) converges to a Reeb orbit along each puncture of \(\Sigma\). From this we can see by the Stokes’ Theorem that \(E(v|_{\Sigma}) < \infty\). We define the Hofer’s energy \(E(v)\) of \(v\) to be \(E(v|_{\Sigma})\). The following proposition says the number of elements in \(v^{-1}(p)\), denoted by \(|v^{-1}(p)|\) is bounded by a finite number depends on the Hofer’s energy \(E(v)\), and independent of \(v\).

**Proposition 5.** Given \(E > 0\), there exists \(N\) such that for any Riemann surface \((S, j)\), and any non-constant \(J\)-holomorphic curve \(v : S \to M\) with \(E(v) \leq E\), we have \(|v^{-1}(p)| \leq N\).
Proof. It follows directly from the Proposition.

Remark 6. The Proposition is also true if we count multiplicity.

The following theorem gives a Hofer’s energy bound by the Symplectic area (compare to 9.2 in [4]).

**Theorem 7.** There exists $C_1 > 0$, such that for any Riemann surface $(S, j)$ and any non-constant $J$-holomorphic curve $v : S \to M$ satisfying $E_{\text{symp}}(v) < +\infty$, we have $E(v) \leq C_1 E_{\text{symp}}(v)$.

**Proof.** We restrict $v$ to the punctured Riemann surface $\Sigma := S \setminus v^{-1}(p)$, so around each puncture $q \in v^{-1}(p)$, $v$ converges to a Reeb orbit. Pick $\varepsilon \in [\frac{1}{2} \log \epsilon, \frac{4}{3} \log \epsilon]$ such that $\varepsilon$ is a regular value of $r \circ v$, where $r : W \to (-\infty, \log \epsilon)$ is the projection map defined by $(r, \Theta) \mapsto r$. Denote $A := v^{-1}((-\infty, \varepsilon] \times S^{2n-1}) \subseteq \Sigma$ and $B_+ := v^{-1}(\{\varepsilon\} \times S^{2n-1})$. Let $\hat{A}$ be the orient blow up of $A$ around all $q$’s in $v^{-1}(p)$, i.e. $\hat{A} = A \sqcup B_-$ with $B_- := \bigsqcup_{q \in v^{-1}(p)} S_1^1$ being the disjoint union of circles parametrized by $v^{-1}(p)$. Hence we have $\partial \hat{A} = B_+ - B_-$. We continuously extend $v$ to $\hat{A}$ by defining $v|_{S_1^1}$ to be the Reeb orbit at negative infinity to which $v$ converges around $q$. Now we have

$$E_d\lambda(v) = \int_{\hat{A}} v^* d\lambda + \int_{\{\Sigma \setminus A\} \cap v^{-1}(W)} v^* d\lambda$$

$$\leq \int_{B_+} v^* \lambda - \int_{B_-} v^* \lambda + \int_{\{\Sigma \setminus A\} \cap v^{-1}(W)} v^* d\lambda$$

$$\leq \int_{B_+} v^* \lambda + \int_{\{\Sigma \setminus A\} \cap v^{-1}(W)} v^* d\lambda. \quad (1)$$

Given $\phi \in C = \left\{ \phi \in C^\infty((-\infty, \log \epsilon), [0, 1]) \left| \int_{-\infty}^{r} \phi(x) dx = 1 \right. \right\}$, we define $\Phi(r) := \int_{-\infty}^{r} \phi(x) dx$, so we get

$$\int_{v^{-1}(W)} v^* (\phi(r) dr \wedge \lambda)$$

$$= \int_{\hat{A}} v^* (\phi(r) dr \wedge \lambda) + \int_{\{\Sigma \setminus A\} \cap v^{-1}(W)} v^* (\phi(r) dr \wedge \lambda)$$

$$= \int_{\hat{A}} v^* d(\Phi(r) \lambda) - \int_{\hat{A}} v^* (\Phi(r) d\lambda) + \int_{\{\Sigma \setminus A\} \cap v^{-1}(W)} v^* (\phi(r) dr \wedge \lambda)$$

$$\leq \int_{\hat{A}} v^* d(\Phi(r) \lambda) + \int_{\{\Sigma \setminus A\} \cap v^{-1}(W)} v^* (dr \wedge \lambda)$$

$$= \int_{B_+} v^* (\Phi(r) \lambda) - \int_{B_-} v^* (\Phi(r) \lambda) + \int_{\{\Sigma \setminus A\} \cap v^{-1}(W)} v^* (dr \wedge \lambda)$$

$$\leq \int_{B_+} v^* \lambda + \int_{\{\Sigma \setminus A\} \cap v^{-1}(W)} v^* (dr \wedge \lambda). \quad (2)$$
We choose a smooth function $\tau$ defined by $\tau(r) = \frac{\log \varepsilon - r}{\log \varepsilon}$ for $\varepsilon \leq r \leq \log \varepsilon$. Since $\tau(\varepsilon) = 1$ and $\tau(0) = 0$, by Stokes' Theorem we have

$$\int_{B_n} v^*\lambda = \int_{(\Sigma \setminus A) \cap v^{-1}(W)} v^*d(\tau(r)\lambda). \quad (3)$$

Since the symplectic form $\omega$ is non-degenerate, on $(\Sigma \setminus A) \cap v^{-1}(W)$ we have $\nu^*\omega \leq C(\varepsilon) \nu^*\omega$ and $\nu^*d(\tau(r)\lambda) \leq C(\varepsilon) \nu^*\omega$. Therefore from (1), (2), and (3) we get

$$E(v) \leq C_1(\varepsilon) E_{\text{symp}}(v).$$

Proposition 5 and Theorem 7 imply the following theorem

**Theorem 8.** Given $E > 0$ there exists $N \in \mathbb{N}$, such that for any Riemann surface $(S, j)$ and any non-constant $J$-holomorphic curve $v : \Sigma \to M$ with symplectic area $E_{\text{symp}}(v) \leq E$, we have $|v^{-1}(p)| \leq N$.

Let $\mathcal{M}_g(M, J, p, n, E)$ be the space of equivalence classes of $(v, \Sigma, j, \tilde{z})$ such that $(\Sigma, j)$ is a smooth Riemann surface of genus $g$; $\tilde{z} = (z_1, z_2, ..., z_n)$ are $n \in \mathbb{N}$ distinct points in $\Sigma$; $v : \Sigma \to M$ is a $J$-holomorphic curve satisfying $v^{-1}(p) = \{z_1, z_2, ..., z_n\}$; $E_{\text{symp}}(v) \leq E$; if $v$ is a constant map, we require $v \neq p$ and $g \geq 2$. $(v, \Sigma, j, \tilde{z})$ and $(v', \Sigma', j', \tilde{z}')$ are equivalent if there exists a diffeomorphism $\sigma : \Sigma \to \Sigma'$ satisfying $\sigma^* j' = j$, $\sigma(z_i) = z_i'$ for $i = 1, 2, ..., n$, and $v = v' \circ \sigma$.

Let $\mathcal{M}_g^{\text{SFT}}(M \setminus p, J, n, E)$ be the space of equivalent classes of $(v, \Sigma, j, \tilde{z})$ such that $(\Sigma, j)$ is a smooth Riemann surface of genus $g$; $\tilde{z} = (z_1, z_2, ..., z_n)$ are $n$ distinct points (called punctures) in $\Sigma$; $v : (\Sigma \setminus \{z_1, z_2, ..., z_n\}, j) \to (M \setminus p, J)$ is a $J$-holomorphic curve such that around each puncture $z_i$ for $i = 1, 2, ..., n$, $v$ converges to a Reeb orbit of $S^{2n-1}$; $E(v) \leq E$; if $v$ is constant, we require $g \geq 2$. $(v, \Sigma, j, \tilde{z})$ and $(v', \Sigma', j', \tilde{z}')$ are equivalent if there exists a diffeomorphism $\sigma : \Sigma \to \Sigma'$ satisfying $\sigma^* j' = j$, $\sigma(z_i) = z_i'$ for $i = 1, 2, ..., n$, and $v = v' \circ \sigma$.

Proposition 5 and Proposition 7 imply

**Corollary 9.** Given $E_1 > 0$, there exists $N_1 \in \mathbb{N}$, such that for $n_1 > N_1$, $\mathcal{M}_g(M, J, p, n_1, E_1) = \emptyset$. Given $E_2 > 0$, there exists $N_2 \in \mathbb{N}$, such that for $n_2 > N_2$, $\mathcal{M}_g^{\text{SFT}}(M \setminus p, J, n_2, E_2) = \emptyset$.

There exists an obvious map $\pi$ from

$$\mathcal{M}_g^{\text{SFT}}(M \setminus p, J, n) := \bigcup_{E \geq 0} \mathcal{M}_g^{\text{SFT}}(M \setminus p, J, n, E)$$

to

$$\mathcal{M}_g(M, J, p, n) := \bigcup_{E \geq 0} \mathcal{M}_g(M, J, p, n, E)$$

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by $\pi([v, \Sigma, j, \vec{z}]) = [v, \Sigma, j, \vec{z}]$. $\pi$ is a well defined bijective map by the Removable of Singularity Theorem and the local behavior of holomorphic maps.

We equip $\mathcal{M}_g(M, p, n, E)$ with the standard Gromov’s topology, and equip $\mathcal{M}^SFT_g(M\setminus p, J, n, E)$ with the Symplectic Field Theory topology introduced in [4]. From [4] we know that $\pi$ is a homeomorphism.

To compactify $\mathcal{M}_g(M, p, n, E)$, we look at the moduli space of curves with point-wise constraints, denoted by $\mathcal{M}^*_g(M, p, n, E)$, and defined by replacing the requirement $\”v\”^{-1}(p) = \{z_1, z_2, ..., z_n\}$ in the definition of $\mathcal{M}_g(M, p, n, E)$ by the requirement $\”v\”(z_i) = p$ for $i = 1, 2, ..., n$. There exists an obvious inclusion map

$$i : \mathcal{M}_g(M, p, n, E) \to \mathcal{M}^*_g(M, p, n, E).$$

We denote by $\mathcal{M}^*_g(M, p, n, E)$ the standard Gromov’s compactification of $\mathcal{M}^*_g(M, p, n, E)$. We define $\mathcal{M}_g(M, p, n, E)$ to be the closure of the subset $i(M_g(M, p, n, E))$ in $\mathcal{M}^*_g(M, p, n, E)$, and hence $\mathcal{M}_g(M, p, n, E)$ is compact.

Denote by $\mathcal{M}^{SFT}_g(M\setminus p, J, n, E)$ the Symplectic Field Theory compactification of $\mathcal{M}^{SFT}_g(M\setminus p, J, n)$ by adding holomorphic buildings (see [4]).

**Theorem 10.** [4] $\mathcal{M}^{SFT}_g(M\setminus p, J, n, E)$ is compact.

Denote $\mathcal{M}_g(M, J, p, n) := \bigcup_{E \geq 0} \mathcal{M}_g(M, J, p, n, E)$ and $\mathcal{M}^{SFT}_g(M\setminus p, J, n) := \bigcup_{E \geq 0} \mathcal{M}^{SFT}_g(M\setminus p, J, n, E)$. The map $\pi$ extends continuously to a surjective map from $\mathcal{M}^{SFT}_g(M\setminus p, J, n, E)$ to $\mathcal{M}_g(M, J, p, n)$. Actually, for $[v] \in \mathcal{M}^{SFT}_g(M\setminus p, J, n)$, $\pi$ is just the forgetful map defined by forgetting the all the negative levels, if any, of the holomorphic building $[v]$.

Define $\mathcal{M}_g(M, J, p, n, E)^{SFT}$ to be the closure of the subset $\pi^{-1}(\mathcal{M}_g(M, J, p, n, E))$ in $\mathcal{M}^{SFT}_g(M\setminus p, J, n)$. Theorem 11 implies that there exists $E_2 < \infty$ such that $\mathcal{M}_g(M, J, p, n, E)^{SFT} \subseteq \mathcal{M}^{SFT}_g(M\setminus p, J, n, E_2)$, and hence

**Corollary 11.** $\mathcal{M}_g(M, J, p, n, E)^{SFT}$ is compact.

Now $\pi$ restricts to a map $\pi : \mathcal{M}_g(M, J, p, n, E)^{SFT} \to \mathcal{M}_g(M, J, p, n, E)$. Since for any $[v] \in \mathcal{M}_g(M, J, p, n, E)$, $\pi^{-1}([v])$ consists of exactly one point, $\mathcal{M}_g(M, J, p, n, E)^{SFT}$ can be viewed as another compactification of $\mathcal{M}_g(M, J, p, n, E)$. If we take a further quotient which makes the $n$ punctures unordered, we get two corresponding spaces and denote them by $\mathcal{M}_g(M, J, p, n, E)^{SFT}_2$ and $\mathcal{M}_g(M, J, p, n, E)_2$. Let $\mathcal{M}_g(M, J, E)$ be the standard moduli space of $J$-holomorphic curves of genus $g$ in $M$, with symplectic area no greater than $E$, and let $\mathcal{M}_g(M, J, E)$ be the Gromov’s compactification of $\mathcal{M}_g(M, J, E)$. We have an obvious homeomorphism $h$ from $\bigcup_n \mathcal{M}_g(M, J, p, n, E)^{SFT}_2$ to $\mathcal{M}_g(M, J, E)$, and a continuos map.
\[ \pi' = h \circ \pi \text{ from } \overline{\mathcal{M}_g(M,J,E)} \subset \bigcup_n \mathcal{M}_g(M,J,p,n,E)_{SFT} \text{ to } \mathcal{M}_g(M,J,E). \]

The Corollary 9 implies

**Corollary 12.** \( \overline{\mathcal{M}_g(M,J,E)} \) is compact.

This means that \( \overline{\mathcal{M}_g(M,J,E)} \) serves as a compactification of \( \mathcal{M}_g(M,J,E) \).

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