CONES AND CONVEX BODIES WITH MODULAR FACE LATTICES.

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Dedicated to Claus M. Ringel on the occasion of his 60th birthday.

Abstract. If a convex body $C$ has modular and irreducible face lattice (and is not strictly convex), there is a face-preserving homeomorphism from $C$ to a section of a cone of hermitian matrices over $\mathbb{R}, \mathbb{C},$ or $\mathbb{H}$, or $C$ has dimension 8, 14 or 26.

1. Introduction.

Let $C$ be a convex body in $\mathbb{R}^n$. A subset $F$ of $C$ is a face of $C$ if every open interval in $C$ that contains a point of $F$ is contained in $F$. An extreme point is a 1-point face. If $S$ is any subset of $C$, the face generated by $S$ is the minimal face of $C$ containing $S$. The set $\mathcal{F}(C)$ of all faces of $C$ ordered by inclusion is a lattice, where $F \wedge G$ is the intersection of $F$ and $G$, and $F \vee G$ is the face generated by $F \cup G$. The lattice $\mathcal{L}(C)$ is always algebraic (the chains of faces are finite), atomic (faces are generated by extreme points) and complemented (for every face $F$ there exists a face $G$ such that $F \wedge G = \emptyset$ and $F \vee G = C$). We want to consider convex bodies for which $\mathcal{F}(C)$ is modular, i.e. $F \vee (G \wedge H) = (F \vee G) \wedge H$ whenever $F \leq H$. Modularity is a ‘weak distributivity’ property satisfied by the lattice of normal subgroups of a group and by the lattice of subspaces of a vector space. For algebraic, atomic lattices, modularity is equivalent to the existence of a rank function such that $rk(F) + rk(G) = rk(F \vee G) + rk(F \wedge G)$ for all $F$ and $G$ [4].

Strictly convex bodies and simplices clearly have modular face lattices. No other polytopes have this property [3], but there are beautiful examples of non-polytopal convex bodies in which every pair of extreme points is contained in a proper face and every pair of faces with more than one point meet.

If $\mathcal{L}_1$ and $\mathcal{L}_2$ are lattices, their direct product is given by $(\mathcal{L}_1 \times \mathcal{L}_2, \leq)$, where $(a, b) \leq (c, d)$ if and only if $a \leq c$ and $b \leq d$. It follows that the direct product of two lattices is modular if and only if the factors are modular. A lattice is called irreducible if it is not isomorphic to a direct product of two nontrivial lattices.

If $C_1 \subset \mathbb{R}^m$ and $C_2 \subset \mathbb{R}^n$ are convex bodies, define $C_1 \ast C_2 \subset \mathbb{R}^{m+n+1}$ as the convex hull of a copy of $C_1$ and a copy of $C_2$ placed in general position in the sense that their linear spans are disjoint and have no common directions. So $C_1 \ast C_2$ is well defined up to a linear transformation: it is the convex join of $C_1$ and $C_2$ of largest dimension. For example $C \ast \{pt\}$ is a pyramid with base $C$. Let’s say that a convex body $C$ is $\ast$-decomposable if $C = C_1 \ast C_2$ for two convex bodies $C_1$ and $C_2$. 

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The natural correspondence (up to linear transformation) between convex bodies in \( \mathbb{R}^n \) and closed cones in \( \mathbb{R}^{n+1} \) gives an isomorphism of face lattices in which \( C_1 \ast C_2 \) corresponds to the direct product of the cones, so the results of this paper apply to cones. This project started with the undergraduate thesis of D. Labardini-Fragoso \[9\], who showed that in dimension less than 6 any cone with modular face lattice is strictly convex or is decomposable (this was conjectured by Barker in \[3\]).

**Lemma 1.** A convex body \( C \) is \(*\)-decomposable if and only if its lattice of faces \( \mathcal{L}(C) \) is reducible.

**Proof.** Let \( C = C_1 \ast C_2 \). Observe that each point \( p \) of \( C_1 \ast C_2 \) with \( p \notin C_i \), lies in a unique segment joining a point \( p_1 \) of \( C_1 \) and a point \( p_2 \) of \( C_2 \). For, if a point lies in two segments \( p_1p_2 \) and \( p'_1p'_2 \) then the lines \( p_1p'_1 \) and \( p_2p'_2 \) are parallel or they intersect, contradicting the assumptions on the spans of \( C_1 \) and \( C_2 \). Moreover, if a point \( p \) moves along a straight line in \( C_1 \ast C_2 \) then the corresponding points \( p_1 \) and \( p_2 \) move along straight lines in \( C_1 \) and \( C_2 \): If \( p \) and \( q \) are points in \( C \) and \( x \in pq \) then \( x = tp + (1-t)q = t\lambda p_1 + t(1-\lambda)p_2 + (1-t)\mu q_1 + (1-t)(1-\mu)q_2 \) which can be rewritten as a linear combination of a point in \( p_1q_1 \) and a point in \( p_2q_2 \) with coefficients adding up to 1 so \( x_1 \in p_1q_1 \) and \( x_2 \in p_2q_2 \). Now if \( C'_i \) is a face of \( C_i \) then \( C'_1 \ast C'_2 \) is a face of \( C_1 \ast C_2 \). For, if \( x \in C'_1 \ast C'_2 \) and \( x = \lambda p + (1-\lambda)q \) with \( p, q \in C_1 \ast C_2 \), then \( x_1 \) lies in \( p_1q_1 \) and \( x_2 \) lies in \( p_2q_2 \) so as \( C'_i \) is a face of \( C_i \), \( p_i \) and \( q_i \) lie in \( C'_i \) so \( p \) and \( q \) lie in \( C'_i \ast C'_2 \). Conversely, if \( C' \) is a face of \( C_1 \ast C_2 \) and \( p \in C' \) then \( p_1 \) and \( p_2 \) lie in \( C' \) so \( C' = (C'_1 \cap C_1) \ast (C'_2 \cap C_2) \). It remains to show that \( C' \cap C_1 \) is a face of \( C_1 \). If \( x \in C' \cap C_1 \) and \( x = \lambda p + (1-\lambda)q \) with \( p, q \in C_1 \ast C_2 \) then as \( C'_i \) and \( C_1 \ast C_2 \) are faces of \( C_1 \ast C_2 \), \( p \) and \( q \) lie in \( C' \) and also in \( C_1 \), so \( C' \cap C_1 \) is a face of \( C_1 \). \( C' \cap C_2 \) is a face of \( C_2 \). So \( \mathcal{L}(C_1 \ast C_2) \cong \mathcal{L}(C_1) \times \mathcal{L}(C_2) \).

If \( \mathcal{L}(C) \cong \mathcal{L}_1 \ast \mathcal{L}_2 \) then \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are isomorphic to sublattices of \( \mathcal{L}(C) \), so \( \mathcal{L}_i \cong \mathcal{L}(C_i) \) for two faces of \( C \) with \( C_1 \land C_2 = \emptyset \) and \( C_1 \lor C_2 = C \). To show that \( C = C_1 \ast C_2 \) we need to prove that \( \text{span}(C_1) \) and \( \text{span}(C_2) \) are disjoint and have no directions in common. Suppose that \( x \in \text{span}(C_1) \cap \text{span}(C_2) \). Take \( x_1 \in \text{Int}(C_1) \) then the line through \( x \) and \( x_1 \) meets \( \partial C_1 \) at two points \( a_1 \) and \( b_1 \). As \( a_2 \) lies in a proper subface \( C'_2 \) of \( C_2 \), the face generated by \( C_1 \) and \( a_2 \) lies in \( C_1 \lor C'_2 \) which is a proper subface of \( C_1 \lor C_2 \). But the points \( a_1, b_1, a_2, b_2 \) determine a plane quadrilateral whose side \( a_1b_1 \) lies in the interior of \( C_i \) so its diagonals intersect at an interior point \( c \) of \( C_1 \lor C_2 \) so the face generated by \( C_1 \) and \( a_2 \) (which contains \( c \)) must be \( C_1 \lor C_2 \), a contradiction. Now suppose that \( \text{span}(C_1) \) and \( \text{span}(C_2) \) have a common direction \( v \). Take \( x_1 \in \text{Int}(C_1) \) then the line through \( x_1 \) in the direction \( v \) meets \( \partial C_1 \) at two points \( a \) and \( b \). As before \( a_1, b_1, a_2, b_2 \) determine a plane quadrilateral whose diagonals intersect at an interior point \( c \) of \( C_1 \lor C_2 \), but \( c \) lies in the face generated by \( C_1 \) and \( a_2 \) which is a proper face of \( C_1 \lor C_2 \). \( \square \)

Recall that a **projective space** consists of a set \( P \) (the points) and a set \( L \) (the lines) so that (a) Each pair of points is contained in a unique line, (b) If \( a, b, c, d \) are distinct points and the lines \( ab \) and \( cd \) intersect, then the lines \( ac \) and \( bd \) intersect (c) Each line contains at least 3 points and there are at least 2 lines (d) Every chain of subspaces (also called **flats**) has finite length. The maximum length of a chain starting with a point is the **projective dimension** of the space.

The flats of a projective space form an algebraic, atomic, irreducible, modular lattice. Conversely, any lattice with these properties is the lattice of flats of a
projective space, whose points are atoms and whose lines are joins of two atoms \([0,1]\). It is a classic result of Hilbert \([3]\) that a projective space in which Desargues theorem holds is isomorphic to the projective space \(\mathbb{A}P^n\) determined by the linear subspaces of \(\mathbb{A}^{n+1}\), for some division ring \(\mathbb{A}\), and that \(\mathbb{A}P^n\) and \(\mathbb{B}P^n\) are isomorphic if and only if \(\mathbb{A}\) and \(\mathbb{B}\) are isomorphic and \(m = n\). All projective spaces of dimension larger than 2 are desarguesian, but there are many non-desarguesian projective planes.

Examples of convex bodies whose face lattices determine the projective spaces \(\mathbb{A}P^n\), \(\mathbb{B}P^n\), and \(\mathbb{B}P^n\), and the octonionic projective plane arise as sections of some classical cones.

**Example 1.** Let \(F \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}\), let \(H_n(F)\) be the set of Hermitian (self-adjoint) \(n \times n\) matrices with coefficients in \(F\), and let \(C_n(F)\) be the subset of positive-semidefinite matrices (\(A\) is positive-semidefinite if \(\mathbf{v}^T A \mathbf{v} \geq 0\), for all \(\mathbf{v} \in F^n\)). Then \(C_n(F)\) is a real cone whose face lattice is isomorphic to the lattice of subspaces of \(F^n\).

To see this, let \(A, B \in C_n(K)\), and let \(\varphi(B)\) denote the face generated by \(B\). Then \(A \in \varphi(B)\) if and only if \(\ker A \supseteq \ker B\). For, \(A \in \varphi(B) \iff \exists \lambda > 0\) such that \(B - \lambda A \in C_n(K) \iff \exists \lambda > 0\) such that \(\mathbf{w}^T B \mathbf{w} \geq \lambda \mathbf{w}^T A \mathbf{w} \geq 0\) for all \(\mathbf{w} \in F^n \iff \mathbf{w}^T B \mathbf{w} = 0\) implies \(\mathbf{w}^T A \mathbf{w} = 0\) for all \(\mathbf{w} \in F^n \iff \ker A \supseteq \ker B\) (since for \(A \in C_n(K), \mathbf{w}^T A \mathbf{w} = 0\) if and only if \(A \mathbf{w} = 0\)). Therefore \(\varphi(A) \rightarrow (\ker A)^\perp\) defines a bijection \(\nu\) from the set of faces of \(C_n(K)\) to the set of linear subspaces of \(K^n\). To prove that \(\nu\) is an isomorphism of lattices observe that \(\nu((A + B)^\perp) = \nu((A + B)^\perp) = (\ker (A + B))^\perp = (\ker A \cap \ker B)^\perp = (\ker A)^\perp \cup (\ker B)^\perp\), and on the other hand, if \(\varphi(A) \wedge \varphi(B)\) is a non-empty face, then it is generated by a matrix \(C\) with \(\ker C = \text{span}(\ker A \cup \ker B)\), so \(\nu((\varphi(A) \wedge \varphi(B))) = (\ker C)^\perp = (\ker A)^\perp \cap (\ker B)^\perp\).

**Example 2.** Let \(H_3(\mathbb{O})\) be the set of Hermitian \(3 \times 3\) matrices over \(\mathbb{O}\). Then the subset \(C_3(\mathbb{O})\) of all sums of squares of elements in \(H_3(\mathbb{O})\) is a real cone whose face lattice determines an octonionic projective plane.

This can be shown using the nontrivial fact that each matrix in \(H_3(\mathbb{O})\) is diagonalizable by an automorphism of \(H_3(\mathbb{O})\) that leaves the trace invariant \([2]\), so:

(a) A matrix \(A\) in \(H_3(\mathbb{O})\) lies in \(C_3(\mathbb{O})\) if and only if it can be diagonalized to a matrix \(A'\) with non-negative entries, because if \(A\) lies in \(C_3(\mathbb{O})\) then \(A'\) is a sum of squares of matrices in \(H_3(\mathbb{O})\), which have non-negative diagonal entries.

(b) All the idempotent matrices in \(H_3(\mathbb{O})\) lie in \(C_3(\mathbb{O})\) as they squares \((A = A^2)\). The idempotent matrices of trace 1 correspond to the extreme rays of \(C_3(\mathbb{O})\) since they can’t be written as non-negative combinations of other idempotent matrices.

(c) Each face of \(C_3(\mathbb{O})\) is generated by an idempotent matrix, because in any cone all the positive linear combinations of the same set of vectors generate the same face, so a diagonal matrix with non-negative entries generates the same face as a matrix with only zeros and ones.

(d) Any two idempotent matrices of trace 1 lie in a face generated by an idempotent matrix of trace 2, because they can be put simultaneously in the form

\[
\begin{bmatrix}
  a & x & 0 \\
  \mathbf{\tau} & b & 0 \\
  0 & 0 & 0 \\
\end{bmatrix},
\]

and these lie in the face generated by

\[
\begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 0 \\
\end{bmatrix}.
\]

(e) \(A \in C_3(\mathbb{O})\) is an idempotent of trace 1 if and only if \(I - A\) is an idempotent with trace 2. If \(A\) and \(B\) are idempotents of trace 1, then \(A\) lies in the face generated by \(I - B\) if and only if \(B\) lies in the face generated by \(I - A\). This duality and (d)
show that any two faces generated by idempotent matrices of trace 2 meet in a face generated by an idempotent matrix of trace 1.

2. Face lattices defining projective spaces.

If \( C \) is a convex body whose face lattice is modular and irreducible and \( C \) is not strictly convex, the set of extreme points of \( C \) is a projective space with flats determined by the faces of \( C \). We would like to know which projective spaces arise in this way, and what convex bodies give rise to them.

By Blaschke selection theorem \([2]\), the space of all compact, convex subsets of a convex body in \( \mathbb{R}^n \), with the Hausdorff metric, is compact. So the subspace formed by the compact convex subsets of the boundary is closed, but the subspace formed by the faces is not closed in general.

**Lemma 2.** If \( C \) is a convex body whose face lattice \( F \) is modular, then the set \( \mathcal{F}_h \) of faces of rank \( h \), with the Hausdorff metric, is compact for each \( h \).

**Proof.** Let \( F_i \) be a sequence of faces of rank \( h \). Then \( F_i \) has a subsequence \( F_{i_j} \) that is convergent in \( C \), and its limit is a compact convex set \( K \) contained in \( \partial C \), so \( K \) generates a proper face \( F \) of \( C \) of some rank \( h' \). We claim that \( h' = h \) and \( K = F \).

If the rank of \( F \) was less than \( h \), there would be a face \( F^c \) of \( C \) of rank \( n - h \) with \( F^c \cap F = \emptyset \). As \( F \) and \( F^c \) are two disjoint compact sets in \( \mathbb{R}^n \), there exists \( \epsilon > 0 \) such that the \( \varepsilon \)-neighborhoods of \( F \) and \( F^c \) in \( \mathbb{R}^n \) are disjoint. But as \( F_{i_j} \to K \subset F \) in the Hausdorff metric, then for sufficiently large \( j \), \( F_{i_j} \) is contained in the \( \varepsilon \)-neighborhood of \( F \), therefore \( F_{i_j} \cap F^c = \emptyset \), but these 2 faces have ranks that add up to \( n \), so they should meet, a contradiction.

If the rank of \( F \) is \( h \) and \( K \neq F \), there is an extreme point \( p \in F - K \), and there is a face \( F' \) of rank \( n - h \) that meets \( F \) only at \( p \), so \( F' \cap K = \emptyset \) and the previous argument gives a contradiction.

To show that the rank of \( F \) cannot be larger than \( h \), proceed inductively on \( n - h \). As a limit of proper faces is contained in a proper face, the claim holds if \( n - h = 1 \). Given a sequence \( F_i \) of faces of rank \( h \), let \( F \) be a face generated by the limit of a convergent subsequence \( F_{i_j} \). If \( p \) is an extreme point of \( C \) not in \( F \), then for sufficiently large \( j \), \( p \notin F_{i_j} \) (otherwise \( p \) would be in \( F \)). Let \( G_{i_j} \) be a face of rank \( h + 1 \) containing \( F_{i_j} \) and \( p \). Now we can assume inductively that the limit of a convergent subsequence of \( G_{i_j} \) generates a face of rank \( h + 1 \). This face contains \( F \) properly (because it contains \( p \)) so \( h' < h + 1 \). \( \square \)

Now recall that a topological projective space is a projective space in which the sets of flats of each rank are given nontrivial topologies that make the join and meet operations \( \vee \) and \( \wedge \) continuous, when restricted to pairs of flats of fixed ranks whose join or meet have a fixed rank \([5]\).

**Lemma 3.** If \( C \) is a convex body whose face lattice is modular and irreducible then \( C \) is strictly convex or the set of extreme points \( E(C) \) is a topological projective space which is compact and connected.

**Proof.** A natural topology for the set of flats is given by the Hausdorff distance between the faces. By lemma \([2]\) \( E = F_0 \) is compact. As the lattice is irreducible and has more than 2 points, the 1-flats have more than 2 points, so (as they are topological spheres) they are connected. Now every pair of points in \( E \) is contained
in one of these spheres, so \( \mathcal{E} \) is connected (one can actually show that each \( \mathcal{F}_h \) is connected).

It remains to show that \( \lor \) and \( \land \) are continuous on the preimages of each \( \mathcal{F}_h \). Suppose \( A_i \to A', B_i \to B \) where \( A_i \land B_i \) and \( A \land B \) are faces corresponding to \( h \) flats. We need to show that \( A_i \land B_i \to A \land B \). By lemma \( \ref{lemma:interior} \) \( C_i = A_i \land B_i \) has convergent subsequences and the limit of a convergent subsequence \( C_{i_\alpha} \) is a face \( C_{\alpha} \) corresponding to an \( h \) flat. As \( C_{i_\alpha} \) is contained in \( A_{i_\alpha} \) and \( B_{i_\alpha} \), \( C_{\alpha} \) is contained in \( A \land B \). But \( C_{\alpha} \) and \( A \land B \) are both faces corresponding to \( h \) flats, so \( C_{\alpha} = A \land B \). Similarly, if \( A_i \to A', B_i \to B' \) and \( A_i \lor B_i \), \( A \lor B \) are faces corresponding to \( h \) flats, the limit of each convergent subsequence of \( D_i = A_i \lor B_i \) is a face \( D \) corresponding to an \( h \) flat. As \( D_i \) contains \( A_i \) and \( B_i \), \( D \) contains \( A \lor B \), and as both faces correspond to \( h \) flats they must be equal. \( \square \)

Let \( C \) be any convex body. Denote by \( \mathcal{B}(C) \subset C \) the set of baricenters of faces of \( C \) and let \( b : \mathcal{F}(C) \to \mathcal{B}(C) \) the function that assigns to each face its baricenter.

**Lemma 4.** (a) If \( \mathcal{F}(C) \) is compact then \( b \) is a homeomorphism.

(b) If \( \mathcal{E}(C) \) is compact then a sequence of faces \( F_i \) converges to a face \( F \) if and only if \( \mathcal{E}(F_i) \) converges to \( \mathcal{E}(F) \).

**Proof.** (a) The function that assigns to each compact convex set in \( \mathbb{R}^n \) its baricenter is continuous, so \( b : \mathcal{F}(C) \to \mathcal{B}(C) \) is a continuous bijective map from a compact Hausdorff space to a metric space.

(b) The Hausdorff distance between two compact convex sets is bounded above by the Hausdorff distance between their sets of extreme points.

If \( F_i \) converges to \( F \) but \( \mathcal{E}(F_i) \) doesn’t converge to \( \mathcal{E}(F) \) then there is a subsequence \( \mathcal{E}(F_{i_j}) \) that stays at distance at least \( \varepsilon > 0 \) from \( \mathcal{E}(F) \). For each \( i_j \) there is an extreme point \( p_{i_j} \in F_{i_j} \) whose distance from \( \mathcal{E}(F) \) is larger than \( \varepsilon \), or an extreme point \( q_i \in F \) whose distance from \( \mathcal{E}(F_{i_j}) \) is larger than \( \varepsilon \). If there is a convergent subsequence \( p_{i_k} \to p \in F \) then \( p \) is at distance at least \( \varepsilon \) from \( \mathcal{E}(F) \), so \( p \) can’t be an extreme point of \( C \).

If there is a convergent subsequence \( q_{i_k} \to q \in F \), take \( p'_{i_k} \in F_{i_k} \) with \( p'_{i_k} \to q \). Eventually \( |p'_{i_k} - q_{i_k}| < \frac{\varepsilon}{2} \) so the distance from \( p'_{i_k} \) to \( \mathcal{E}(F_{i_k}) \) is at least \( \frac{\varepsilon}{2} \), so \( p'_{i_k} \) is the center of a straight interval \( I_{i_k} \) of length \( \varepsilon \) contained in \( F_{i_k} \). A convergent subsequence of these intervals yields a straight interval centered at \( q \) and contained in \( F \), so \( q \) can’t be an extreme point of \( C \), contradicting the compactness of \( \mathcal{E}(C) \). \( \square \)

**Lemma 5.** If \( C \) and \( C' \) are convex bodies with \( \mathcal{F}(C) \) and \( \mathcal{F}(C') \) compact, then any continuous “face preserving” map \( \varphi : \mathcal{E}(C) \to \mathcal{E}(C') \) extends naturally to a continuous map \( \varphi : C \to C' \).

**Proof.** \( \varphi \) determines a function \( \Psi : \mathcal{F}(C) \to \mathcal{F}(C') \). \( \Psi \) is continuous because by lemma \( \ref{lemma:interior} \) \( F_i \to F \) implies \( \mathcal{E}(F_i) \to \mathcal{E}(F) \), then uniform continuity of \( \varphi \) on \( \mathcal{E}(C) \) implies that \( \varphi(\mathcal{E}(F_i)) \to \varphi(\mathcal{E}(F)) \) so by definition \( \mathcal{E}(\Psi(F_i)) \to \mathcal{E}(\Psi(F)) \) and so \( \Psi(F_i) \to \Psi(F) \). So \( \varphi \) can be extended to a continuous function \( \varphi : \mathcal{B}(C) \to \mathcal{B}(C') \) as \( b \circ \Psi \circ \imath^{-1} \) (recall that \( \mathcal{E}(C) \subset \mathcal{B}(C) \)). Now we can extend \( \varphi \) to the interiors of the faces of \( C \) defining it linearly on rays, as follows.

For each point \( a \in C \), let \( F(a) \) be the unique face of \( C \) containing \( a \) in its interior and let \( b(a) \) be the baricenter of \( F(a) \). Although \( F(a) \) and \( b(a) \) are not continuous functions of \( a \) on all of \( C \), they are continuous on the union of the interiors of the faces corresponding to \( h \)-flats for each \( h \). If \( a \neq b(a) \) let \( p(a) \) be the
projection of $a$ to $\partial F(a)$ from $b(a)$ and let $\lambda(a) = \frac{|a-b(a)|}{|p(a)-b(a)|}$ (or 0 if $a = b(a)$) so $a = (1 - \lambda(a))b(a) + \lambda(a)p(a)$. Define $\varphi(a) = (1 - \lambda(a))\varphi(b(a)) + \lambda(a)\varphi(p(a))$.

Assume inductively that $\varphi$ is continuous on the union of $B(C)$ and the faces of $C$ of dimension less than $d$ (this set is closed because the limit of faces of dimension less than $d$ has dimension less than $d$). and let’s show that for each sequence of points $a_i$ in the interiors of faces of dimension $d$, $a_i \rightarrow a$ implies $\varphi(a_i) \rightarrow \varphi(a)$. We may assume that the $a_i$ are not baricenters, so $p(a_i)$ is well defined.

Case 1. $F(a_i) \rightarrow F(a)$ then $b(a_i) \rightarrow b(a)$ by the continuity of $b$ on faces.

If $b(a) \neq a$ then $p(a_i) \rightarrow p(a)$ and $\lambda(a_i) \rightarrow \lambda(a)$ so $\varphi(a_i) = (1 - \lambda(a_i))\varphi(b(a_i)) + \lambda(a_i)\varphi(p(a_i)) \rightarrow (1 - \lambda(a))\varphi(b(a)) + \lambda(a)\varphi(p(a)) = \varphi(a)$.

If $b(a) = a$ then $\lim b(a_i) = \lim a_i$ but $p(a_i)$ may not converge, so consider a convergent subsequence $p(a_{i_j})$: If $\lim p(a_{i_j}) \neq \lim b(a_{i_j}) = \lim a_{i_j}$ then $\lambda(a_{i_j}) = 0$ so $\varphi(a_{i_j}) = \varphi(b(a_{i_j})) + \lambda(a_{i_j})(\varphi(p(a_{i_j})) - \varphi(b(a_{i_j}))) \rightarrow \varphi(b(a)) + 0 = \varphi(a)$. If $\lim p(a_{i_j}) = \lim b(a_{i_j})$ then $\varphi(p(a_{i_j})) = \varphi(b(a_{i_j}))$ (by continuity of $\varphi$ in the baricenters and faces of lower dimension) and as $\varphi(a_{i_j})$ lies between them, $\lim \varphi(a_{i_j}) = \varphi(b \circ F(a_{i_j})) = \varphi(b(a)) = \varphi(a)$.

Case 2. $F(a_i) \rightarrow F(a)$, then for any convergent subsequence $F(a_{i_j})$ with limit $F \neq F(a)$, $a$ lies in $F$ and so must lie in $\partial F$, so $|a_i - p(a_{i_j})| \rightarrow 0$ and $\lambda(a_{i_j}) \rightarrow 1$, so $\varphi(a_{i_j}) = \lim (1 - \lambda(a_{i_j}))\varphi(b(a_{i_j})) + \lambda(a_{i_j})\varphi(p(a_{i_j})) = \lim \varphi(p(a_{i_j})) = \varphi(a)$ (by continuity of $\varphi$ on the faces of lower dimension).

**Theorem 1.** Let $C$ be a convex body whose face lattice defines a $n$-dimensional projective space.

If $n = 2$, then $C$ has dimension 5, 8, 14 or 26.

If $n > 2$ (or the space is desarguesian) there is a face-preserving homeomorphism from $C$ to a section of a cone of Hermitian matrices over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

**Proof.** First consider the case $n = 2$. All the lines of a topological projective plane $\mathcal{P}$ are homeomorphic because if $l$ is a line and $p$ is a point not in $l$ then the projection $\phi: \mathcal{P} - p \rightarrow l$, $\phi(x) = (x \lor p) \land l$ is continuous and its restriction to each projective line not containing $p$ is one to one. If the projective lines are topological spheres then a famous result of Adams [5, p.1278], shows that their dimension must be $d = 0, 1, 2, 4$ or 8.

To compute the dimension of $C$ take 3 faces of rank 1, $F_0$, $F_1$ and $F_2$ so that $F_1$ and $F_2$ meet at a point $p$ not in $F_0$. The projection $\phi: \mathcal{E}(C) - \{p\} \rightarrow \partial F_0$ extends to a continuous map $\phi: \mathcal{P} = \partial \partial F_0 \rightarrow \mathcal{E}$ whose restriction to each face is one to one (see proof of lemma 4). Now $U = \bigcup \{IntF, F \, face \, of \, C, p \notin F\}$ is an open subset of $\partial F$ and the function $\Phi: U \rightarrow \partial F_0 \times (\partial F_1 - \{p\}) \times (\partial F_2 - \{p\})$ defined as $\Phi(x) = (\phi(x), \partial F_0(x) \land \partial F_1, \partial F_0 \land \partial F_2)$ is continuous and bijective, so $U$ has the same dimension as $F \times \partial F \times \partial F$, which is $3d + 1$, therefore $C$ has dimension $3d + 2$. Note that the discrepancy between the dimensions of the union of the boundaries of the faces $(2d)$ and the union of the faces $(3d + 1)$ arises because the boundaries of the faces overlap (as the lines in a projective plane do) but the interiors of the faces are disjoint. When $n > 2$, there is a similar homeomorphism from an open subset of $\partial C$ and a product $\partial F_0 \times (\partial F_1 - \{p\}) \times (\partial F_2 - \{p\}) \times ... \times (\partial F_n - \{p\})$ where $F_0$ is a face of rank $r - 1$ and $F_1, F_2, ..., F_n$ are faces of rank 1. So $\dim(C) = \dim(F_0) + nd + 1$ and it follows by induction that $\dim C = \frac{n(n-1)}{2}d + n - 1$.

Now assume that the projective space determined by $\mathcal{E}(C)$ is desarguesian. Every topological desarguesian projective space is isomorphic to a projective space over a
topological division ring $A$ (defined on a line minus a point) and the isomorphism is a homeomorphism [5, p.1261]. By a classic result of Pontrin [5, p.1263] the only locally compact, connected division rings are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. Therefore $\mathcal{E}(C)$ is isomorphic to $\mathbb{R}P^r$, $\mathbb{C}P^r$ or $\mathbb{H}P^r$. Therefore $\mathcal{E}(C)$ is isomorphic to $\mathcal{E}(C')$ where $C'$ is a section of a cone of Hermitian matrices, and so by [5] there is a face-preserving homeomorphism from $C$ to $C'$.

3. Face lattices defining affine spaces.

Now let us consider closed (but not necessarily compact) convex sets in $\mathbb{R}^n$ whose faces meet as the subspaces of an affine space. An abstract affine plane consists of a set of points and a set of lines so that 1) there are at least 2 points and 2 lines 2) every pair of points is contained in a line and a point not contained in it, there is a unique line containing the point and parallel to the line. The axioms of an abstract affine space are not so simple, but it is enough to know that if $P$ is a projective space then the complement of a maximal flat of $P$ is an affine space, and any affine space $A$ can be embedded in a projective space in this fashion, by attaching to $A$ a point at infinity for each parallelism class of affine lines.

Observe that if a closed convex set $C$ in $\mathbb{R}^n$ is non-compact, it contains a ray (half of a euclidean line) and if $C$ contains a ray then it contains all the parallel rays starting at points of $C$ (we say that $C$ contains an infinite direction). So if $C$ contains a line, $C$ is the product of that line and a closed convex set $C'$ of lower dimension and the face lattice of $C$ and $C'$ are isomorphic. So from now on we will assume that $C$ doesn't contain lines.

It is easy to see that the faces of a polytope cannot determine an affine space (the faces of rank $i$ would have dimension $i$, two parallel faces of rank 1 generate a face of rank 2 with at least 4 vertices, but the sides of a polygon don’t define an affine plane).

Example 3. Let $C$ be a convex body in $\mathbb{R}^n$ whose faces determine a projective space. Take a cone over $C$ and slice it with a hyperplane parallel to a support hyperplane containing a maximal face. The result is a closed non-compact convex set $C'$ in $\mathbb{R}^n$ whose faces determine an affine space. In particular, the cones of Hermitian matrices have non-compact sections whose face lattice determines a real, complex or quaternionic affine space or an octonionic affine plane.

$\mathbb{R}P^n$ can be seen as the space of lines through the origin in $\mathbb{R}^{n+1}$ or as the quotient of the unit sphere $S^n$ (or the sphere at infinity of $\mathbb{R}^{n+1}$) by the action of the antipodal map. Identifying $\mathbb{R}^n$ with a hyperplane of $\mathbb{R}^{n+1}$ that doesn’t contain the origin gives an embedding of $\mathbb{R}^n$ as a dense open subset of $\mathbb{R}P^n$. The remaining points of $\mathbb{R}P^n$ correspond to lines through the origin in $\mathbb{R}^{n+1}$ that don’t meet the hyperplane, i.e., parallelism classes of lines in $\mathbb{R}^n$ (or pairs of antipodal points in the sphere at infinity). Define a set in $\mathbb{R}P^n$ to be convex if it is the image of a convex set in $\mathbb{R}^n$ under one of these embeddings. As convex sets in $\mathbb{R}P^n$ correspond to convex cones based at the origin of $\mathbb{R}^{n+1}$, convexity in $\mathbb{R}P^n$ doesn’t depend on the particular embedding, and a convex set in $\mathbb{R}P^n$ has the usual properties of a convex set in $\mathbb{R}^n$.

Now if $C$ is a closed convex set in $\mathbb{R}^n$ that doesn’t contain lines, its closure $\overline{C}$ is a convex set in $\mathbb{R}P^n$. The faces of $\overline{C}$ are the closures of faces of $C$ and their intersections with the sphere at infinity modulo the antipodal map.
Lemma 6. In a closed convex set in $\mathbb{R}^n$ that has semi-modular face lattice, two faces of rank 1 can share at most one direction, and it corresponds to a ray.

Proof. We are considering convex bodies without lines. Suppose that two rank 1 faces $F_1$ and $F_2$ have a common direction, i.e., there are segments of parallel euclidean lines $l_1$ and $l_2$ lying in $F_1$ and $F_2$. We may assume that $l_i$ goes through an interior point of $F_i$, so $l_i$ meets $\partial F_i$ in one or two extreme points. If $l_1$ or $l_2$ has two extreme points then there is a convex quadrilateral with sides in $l_1$ and $l_2$ with 3 extreme points as vertices. The interior of the quadrilateral lies in the interior of the rank 2 face generated by the 3 extreme points, but the intersection of its diagonals lies in the rank 1 face generated by 2 extreme points, a contradiction. So $l_i$ meets $\partial F_i$ in only one point and so $F_i$ contains a ray $l'_i$. If $F_1$ and $F_2$ have two common directions, there is a common direction which meets $F_1$ in 2 points, giving the same contradiction. \qed

Lemma 7. If the faces of a closed convex set $C$ in $\mathbb{R}^n$ define an affine space, then each face representing a line contains a unique ray, and faces representing parallel lines contain parallel rays.

Proof. We are assuming again that $C$ doesn’t contain lines, so the points of the affine space correspond to the extreme points of $C$. Each affine line is represented by the boundary of a convex set of dimension at least 2, which is connected, so set of extreme points in $C$ is connected.

Let’s first show that the faces representing affine lines cannot be compact. Suppose that $C$ has a compact face $F$. Let $p$ and $q$ be two extreme points in $F$ and let $q_i$ be a sequence of extreme points not in $F$ that converge to $q$. Let $F_i$ be the face generated by $p$ and $q_i$. If $F_i$ is non-compact, it contains a ray $r_i$ through $p$. So $F_i$ contains the “parallelogram” determined by the interval $pq_i$ and the ray $r_i$. An infinite sequence of $l_i$’s would have a subsequence converging to a ray $l$ through $q_i$, so the parallelogram determined by the interval $pq$ and the ray $l$ would be contained in $\partial C$, so it would have to be contained in a face of $C$, which would have to be $F$ because it contains $p$ and $q$. This contradicts the assumption that $F$ is compact and shows that if $q_i$ is sufficiently close to $q$, the face generated by $p$ and $q_i$ is compact. Now take a face $F'$ that doesn’t meet $F$ (i.e., $F$ and $F'$ represent parallel affine lines). As $F$ is compact and $F'$ is closed in $\mathbb{R}^n$, there is an $\varepsilon$ neighborhood of $F$ that doesn’t intersect $F'$. By the previous argument there is a point $q_i$ not in $F$ so that the face $F_i$ generated by $p$ and $q_i$ is contained in the $\varepsilon$ neighborhood of $F$. So $F_i$ doesn’t meet $F'$, but $F$ was supposed to be the only face containing $p$ and disjoint from $F'$.

This proves that $F$ is non-compact, so it contains rays. Let’s show that two faces representing parallel affine lines contain parallel rays. Let $p$ be an extreme point outside $F$, so $F$ and $p$ generate a face $H$ representing an affine plane. There are extreme points $p_0, p_1, p_2, \ldots$ in $F$ so that the sequence of intervals $p_0p_i$ converges to a ray $l_+\subset F$ (because $F$ is closed). The sequence of intervals $p_0p_i$ lie in $\partial H$ and converge to a ray $m_+$ parallel to $l_+$ and containing $p$ so (as $H$ is closed) $m_+$ is contained in a face $G$ of $\partial H$ representing an affine line. As two faces that contain parallel rays cannot meet at a single point, $G$ doesn’t meet $F$ so (as $F$ and $G$ are contained in $H$) $G$ represents the affine line parallel to $F$ through $p$. If $F$ has nonparallel rays, one can construct as before two nonparallel rays $l_+$ and $l'_+$ in $F$ and faces $G$ and $G'$ through $p$ and containing rays $m_+$ and $m'_+$. The uniqueness of
parallel affine lines implies that \( G = G' \), so \( F \) and \( G \) have more than one common direction, contradicting the previous lemma.

This shows that if the faces of a closed convex set \( C \) define an affine space, \( C \) is non-compact. One can show that if the faces of a closed convex set \( C \) (containing no lines) define a projective space, \( C \) must be compact. For this, one has to give a topology to the space of closed convex sets in \( \mathbb{R}^n \) that makes it locally compact, show that this makes \( E(C) \) into a locally compact projective space, and observe that these spaces are necessarily compact.

**Theorem 2.** If \( C \) is a closed convex set in \( \mathbb{R}^n \) whose faces determine an affine space, there is a projective transformation in \( \mathbb{R}P^n \) taking \( C \) to a compact convex set in \( \mathbb{R}^n \) whose faces determine a projective space. If the space is desarguesian, there is a face-preserving homeomorphism from \( C \) to a non-compact slice of a cone of Hermitian matrices.

**Proof.** We need to show that if the face lattice of \( C \) determines an affine space, the face lattice of \( \overline{C} \subseteq \mathbb{R}P^n \) determines its projective completion. The faces of \( C \) representing affine lines are non-compact, and two of them share an infinite direction in \( \mathbb{R}^n \) if and only if they represent parallel affine lines. The closure \( \overline{C} \subseteq \mathbb{R}P^n \) contains one point at infinity for each infinite direction in \( C \), so \( \overline{C} \) contains an extreme point at infinity for each class of faces of \( C \) representing parallel lines. This corresponds precisely with the definition of the projective completion of the affine space. Now the result for \( C \) follows by applying theoremII to \( \overline{C} \).

---

4. **Projective planes and the case \( d = 1 \).**

The face lattice of a convex body \( C \) (not a triangle) determines a projective plane if every pair of extreme points is contained in a proper face and every pair of faces with more than one point meet. By theoremI this projective plane is compact and connected, so for some \( d \in \{1, 2, 4, 8\} \), all the faces of \( C \) have dimension \( d + 1 \) and \( C \) has dimension \( n = 3d + 2 \).

**Lemma 8.** A \( d + 1 \) dimensional subspace of \( \mathbb{R}^n \) is the span of a face of \( C \) if and only if it meets all the spans of faces of \( C \).

**Proof.** Let \( S \) be an affine subspace that intersects \( \text{span}(F) \) for every \( F \in F_1 \), the set of faces of rank 1. Then \( \{ F \in F_1 \mid \dim(S \cap \text{span}(F)) \geq i \} \) is closed in \( F_1 \) for each \( i \).

Case 1. \( d = 1 \). We claim that if \( S \) is not the span of a face then \( S \) cannot intersect \( \text{span}(F) \) in more than one point. For, if \( S \cap \text{span}(F) \) contains a line, then \( \text{span}(S \cup F) \) is 3 dimensional. Take an extreme point \( p \notin \text{span}(S \cup F) \) and let \( F_1 \) and \( F_2 \) be 2 faces containing \( p \), and meeting \( F \) at points \( p_1 \) and \( p_2 \) not in \( S \). If \( p'_1 \) and \( p'_2 \) are points in \( S \cap \text{span}(F_1) \) and \( S \cap \text{span}(F_2) \) respectively, then \( p, p_1 \) and \( p_2 \) are not aligned (otherwise \( p \) would be in the span of \( S \cup \text{span}(F) \)) and so the span of \( p, p_1 \) and \( p_2 \), which is \( \text{span}(F_1) \), is contained in \( \text{span}(p \cup S \cup F) \). But \( \text{span}(p \cup S \cup F) \) is 4 dimensional, so it cannot contain the 3 faces \( F, F_1 \) and \( F_2 \) because if it did, it would contain each face that meets \( F, F_1 \) and \( F_2 \) at 3 different points, but every face is a limit of such faces, so it would contain all the faces of \( C \), but \( C \) has dimension 5. This shows that \( S \) intersects each \( \text{span}(F) \) at exactly one point, and so \( S \) contains at most one extreme point of \( C \). The function \( I : F_1 \to S \) that maps each face \( F_i \) to the point of intersection of \( \text{span}(F) \) with \( S \) is continuous, and
as the spans of faces meet only at extreme points, \( I \) only fails to be injective on the faces containing the extreme point in \( S \) (if any). But \( F_1 \) is a 2-dimensional closed surface in which the faces that contain an extreme point form a closed curve, and there are no continuous maps from a closed surface to the plane that fail to be injective only along a curve.

Case 2. \( S \) doesn’t contain extreme points of some face \( F \). Choose \( F \) that minimizes the dimension of the subspace \( S \cap \text{span}(F) \). Then for every \( F' \) in a neighborhood of \( F \), \( S \cap \text{span}(F) \) is a subspace of minimal dimension and with no extreme points. If \( S' \) is the orthogonal complement of \( S \cap \text{span}(F) \) in \( S \) then \( S' \) intersects \( \text{span}(F) \) in one point for all \( F' \) in a smaller neighborhood \( V \) of \( F \). Then the function \( I : V \to S' \) that maps \( F' \) to \( S' \cap \text{span}(F) \) is continuous, and it is injective as the spans of faces only meet at extreme points. But an injective map between manifolds can only exist when the domain has dimension no larger than the target so \( 2d \leq \dim S' \leq \dim S \leq d+1 \), so \( d = 1 \) and we are in case 1.

Case 3. \( S \) contains extreme points of each face \( F \). As \( C \) is convex, either \( S \cap C \subset \partial C \) or \( \partial S \cap C = S \cap \partial C \). In the first case \( S \cap C \) is contained in a face \( F_1 \) of \( C \) and so either \( S \cap C = F_1 \) (so \( F_1 \subset S \)) or there is an extreme point \( p \) of \( F_1 \) not contained in \( S \), but then a face \( F_2 \) that meets \( F_1 \) at \( p \) doesn’t meet \( F_1 \cap S \subset S \cap C \) so \( S \) doesn’t contain extreme points of \( F_2 \).

Let \( p \) be an extreme point not in \( S \), and consider the set \( \mathcal{F}_1^p \) of faces of rank 1 containing \( p \). If \( F \) and \( F' \) are distinct faces in \( \mathcal{F}_1^p \), \( S \cap F \) and \( S \cap F' \) are disjoint. Choose \( F \) so that \( S \cap F \) has minimal dimension, then for all \( F' \) in some neighborhood \( V \) of \( F \), \( S \cap F' \) has the same dimension and the map \( I_B : \mathcal{F}_1^p \cap V \to S \cap \partial C \) that sends \( F' \) to the baricenter of \( S \cap F' \) is continuous and injective. As \( I_B \) is a map between manifolds, \( d = \dim \mathcal{F}_1^p \leq \dim S \cap \partial C \leq d \) and so by domain invariance the image of \( I_B \) is an open subset of \( S \cap \partial C = \partial S(S \cap C) \). This implies that for each \( F' \in \mathcal{F}_1^p \cap V, S \cap F' \) consists of one point (if a face of a convex set has more than 1 point, its baricenter is arbitrarily close to points in the boundary that are not baricenters of other faces, namely, the points in the face) and so, by hypothesis, \( S \cap F' \) is an extreme point of \( C \).

So part of the boundary of \( S \cap C \) in \( S \) is strictly convex, therefore the line segment joining two extreme points in it lies in \( \text{Int}_S(S \cap C) \), but that line segment lies in the face of \( C \) containing the 2 extreme points, so it must lie in \( S \cap \partial C = \partial S(S \cap C) \), a contradiction.

**Lemma 9.** The boundaries of the faces of rank 1 of \( C \) are semi-algebraic sets. If \( d = 1 \), they are conic sections.

**Proof.** By lemma [2] the set \( S \) of spans of faces of \( C \) is the same as the set of \( d + 1 \)-dimensional subspaces of \( \mathbb{R}^n \) that intersect every element of \( S \). The set of all \( d + 1 \)-dimensional affine subspaces of \( \mathbb{R}^n \) forms a real algebraic variety and the condition that the subspaces meet a fixed subspace is algebraic, so (by the finite descending chain condition) there is a finite family of spans \( S_1, S_2, ..., S_m \in S \) such that \( S \in S \) if it intersects these \( S_i \)’s (see [1]).

Now for \( (x_1, x_2, ..., x_m) \in S_1 \times S_2 \times S_m \), the subspace \( \text{span}(x_1, ..., x_m) \) has dimension at least \( d + 1 \) (otherwise it would be contained in two subspaces of dimension \( d + 1 \) that meet each \( S_i \), so they would both be in \( S \), but two spans can only meet in 1 point). So \( \text{span}(x_1, ..., x_m) \) lies in \( S \) if and only if its dimension is \( d + 1 \), and this happens if and only if some determinants (given by polynomials on \( x_1, ..., x_m \) ) vanish. Therefore the set \( X = \{(x_1, x_2, ..., x_m) \in S_1 \times S_2 \times S_m \mid \text{span}(x_1, ..., x_m) \in S \} \) is
real algebraic, as is the set \( X^p \) formed by the elements of \( X \) that contain a fixed point \( p \). If \( F_1 \) is the face in \( S_1 \) and \( p \) is an extreme point of \( C \) outside \( F_1 \) then \( \partial F_1 \) consists of the intersections of \( S_1 \) with the elements of \( S \) containing \( p \). So \( \partial F_1 \) is the one to one projection of the algebraic set \( X^p \) to \( S_1 \), so \( \partial F_1 \) is at least semi-algebraic.

Now assume \( d = 1 \) so \( n = 5 \). Every projective plane has 7 points and 6 lines so that each line contains 3 points as in figure 1, so \( C \) has 7 extreme points and 6 faces intersecting in that way. The 7 points are in general position in \( \mathbb{R}^5 \) because as each face of \( C \) is spanned by 3 points, the span of any 6 of those points contains the span of 3 faces, which is all of \( \mathbb{R}^5 \). Therefore we may assume (by applying a projective transformation) that the 7 points are \( p_0 = (0, 0, 0, 0, 0), p_1 = (1, 0, 0, 0, 0), ..., p_5 = (0, 0, 0, 1), p_6 = (1, 1, 1, 1, 1) \). Let \( S_1 \) be the plane spanned by the face \( F_i \). A plane \( S \) that intersects \( S_1, S_2 \) and \( S_3 \) has a parametrization \( (x, y, z, v, w) = r(a, b, 0, 0, 0) + s(0, 0, c, d, 0) + t(e, e, e, e, f) \) with \( r + s + t = 1 \). \( S \) intersects \( S_4, S_5 \) and \( S_6 \) only if three systems of linear equations in \( r, s, t \) represented by the following matrices have nontrivial solutions:

\[
\begin{vmatrix}
a & 0 & e \\
0 & d & e \\
b - 1 & c - 1 & 2e + f - 1
\end{vmatrix}
\begin{vmatrix}
b & 0 & e \\
0 & c & e \\
a - 1 & d - 1 & 2e + f - 1
\end{vmatrix}
\begin{vmatrix}
b & 0 & e - f \\
0 & d & e - f \\
a - 1 & c - 1 & 2e - f - 1
\end{vmatrix}
\]

As the determinants of these matrices are linear functions on the variables \( e \) and \( f \), they vanish simultaneously if and only if the matrix of this new system has determinant 0:

\[
\det
\begin{vmatrix}
-ac + 2ad - bd + a + d & ad & -ad \\
-ac + 2bc - bd + b + c & bc & -bc \\
-ad - bc + 2bd + b + d & ad + bc - bd - b - d & -bd
\end{vmatrix}
= 0
\]

This determinant factors as the product of a linear and a quadratic function of \( a \) and \( b \) (with coefficients in \( c \) and \( d \)). Since the boundary of the face \( F_1 \) is formed by the intersections of \( S_1 \) with the planes that meet all \( S_i \)’s and go through a fixed point in the boundary of \( F_2 \) (this corresponds to fixing \( c \) and \( d \)), the boundary of \( F_1 \) is contained in the union of a line and a conic. As the boundary of \( F_1 \) is strictly convex, it must be the conic.
Theorem 3. All convex bodies in $\mathbb{R}^5$ with modular and irreducible face lattice are projectively equivalent.

Proof. Let $C$ and $C'$ be two such bodies. Take extreme points $p_0, p_1, \ldots, p_6$ and faces $F_1, \ldots, F_6$ of $C$ as in figure 1. Pick an extreme point $p'_0$ in $C'$ and two faces $F'_1$ and $F'_2$ of $C'$ intersecting at $p'_0$. Let $S_i$ be the span of $F_i$. As the faces of $C$ and $C'$ are conics, there are linear transformations from $S_1$ to $S'_1$ taking $F_1$ to $F'_1$ and from $P_2$ to $P'_2$ taking $F_2$ to $F'_2$. Together, they define a linear transformation $l$ from $\text{span}(F_1 \cup F_2)$ to $\text{span}(F'_1 \cup F'_2)$. Let $p'_i = l(p_i)$ for $i = 1, \ldots, 4$. The faces $F_4, F_5, F_6$ are generated by unique pairs of $p_i$'s with $i \leq 4$. Let $F'_4, F'_5, F'_6$ be the faces generated by the corresponding pairs of $p'_i$'s. Finally, let $p'_5 = S'_4 \cap S'_5$, let $F'_3$ be the face generated by $p'_4$ and $p'_5$ and let $p'_6 = S'_5 \cap S'_6$. The linear transformation $l$ can be extended to a projective transformation $\rho$ in $\mathbb{R}^5$ that takes $p_5$ to $p'_5$ and $p_6$ to $p'_6$. As $\rho$ sends each $p_i$ to $p'_i$, it sends each $S_i$ to $S'_i$, so it sends each plane in $\mathbb{R}^5$ intersecting every $S_i$ to a plane intersecting every $S'_i$. Since by construction $\rho$ takes those planes that meet $\partial F_1$ and $\partial F_2$ to planes that meet $\partial F'_1$ and $\partial F'_2$, Lemma 5 implies that $\rho$ maps spans of faces of $C$ to spans of faces of $C'$ and therefore it maps faces to faces. \hfill $\Box$

Question 1: Are all the convex bodies whose face lattices determine classical projective spaces projectively equivalent to sections of cones of hermitian matrices?

Question 2: Can two convex bodies of the same dimension define non isomorphic projective planes (so they are not related by a face-preserving homeomorphism)?

In dimensions 8 and 14 this is equivalent to ask if the projective planes are always desarguesian. In dimension 26 there might be enough space for non-equivalent non-desarguesian examples.

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