EQUIVARIANT COHOMOLOGY AND THE SUPER
RECIPROCAL PLANE OF A HYPERPLANE
ARRANGEMENT

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Abstract. In this paper, we investigate certain graded-commuta-
tive rings which are related to the reciprocal plane compactification
of the coordinate ring of a complement of a hyperplane arrange-
ment. We give a presentation of these rings by generators and
defining relations. Our presentation was recently used by Holler
and I. Kriz [7] to calculate the $\mathbb{Z}$-graded coefficients of localizations
of ordinary $RO((\mathbb{Z}/p)^n)$-graded equivariant cohomology at a given
set of representation spheres. We also give an interpretation of
these rings in terms of superschemes, which can be used to further
illuminate their structure.

1. Introduction

$G$-equivariant generalized homology and cohomology theory for a
compact lie group $G$ is best behaved when the (co)-homology groups are
graded by elements of the real representation ring $RO(G)$. In this case
(see Lewis, May, Steinberger [13] for background), the theory enjoys
many of the properties of non-equivariant (co)-homology, for example,
Spanier-Whitehead duality. Explicit calculations of equivariant coho-
mology groups, however, are much harder than in the non-equivariant
case. A telling example is the case of “ordinary” $G$-equivariant co-
homology theories, defined by Lewis, May and McClure [12]. These
theories satisfy a “dimension axiom” in the sense that the $\mathbb{Z}$-graded
part of their coefficients (i.e. (co)-homology of a point) are zero except
in dimension 0 for all (closed) subgroups of $G$.

However, calculation of the $RO(G)$-graded coefficients of these “or-
dinary” $G$-equivariant cohomology theories has been an open problem
since the 1980s, and these groups carry some deep information. For ex-
ample, for the “constant” $\mathbb{Z}$ Mackey functor coefficients, (which means
that restrictions to subgroups are identities), a partial calculation of
the $RO(G)$-graded coefficients for $G = \mathbb{Z}/8$ was a key ingredient in the
solution by Hill, Hopkins and Ravenel [6] of the Kervaire invariant 1
problem.
The algebraic calculations made in the present paper are relevant to the ordinary $RO(G)$-graded (co)homology theory with constant $\mathbb{Z}/p$ coefficients for $G = (\mathbb{Z}/p)^n$. We denote this theory by $H\mathbb{Z}/p_{(\mathbb{Z}/p)^n}$. In the paper [8], Holler and I. Kriz calculated the “positive” part of these coefficients, meaning the groups

\[(1) \quad H\mathbb{Z}/p^V_{(\mathbb{Z}/p)^n}(\ast)\]

with $V$ an actual (not virtual) representation for $p = 2$. A key ingredient in this calculation was the geometric fixed point ring

\[(2) \quad (\Phi_{(\mathbb{Z}/p)^n} H\mathbb{Z}/p)_*\]

which is the localization of the full $RO((\mathbb{Z}/p)^n)$-graded coefficient ring by inverting the inclusions $S^0 \to S^\alpha$ for all non-trivial irreducible representations $\alpha$ (see Tom Dieck [20] and [13], chapter 11, Def. 9.7).

Holler and I. Kriz [8] calculated the ring (2) for $p = 2$ by hand using a spectral sequence, and commented that the rings seemed to have an unusual algebraic structure, and asked about its geometric significance. They also did not know how to complete the same computation for $p > 2$, where the structure seemed much more complicated.

Answering these algebraic questions is the main purpose of the present paper. Using our main theorem (Theorem 2 below), Holler and I. Kriz [7] then generalized their calculations of the geometric fixed point coefficient ring (2) to $p > 2$, and also answered the following more general question:

What is the structure of the $\mathbb{Z}$-graded coefficient ring $R_S$ of the $((\mathbb{Z}/p)^n)$-fixed point spectrum given by localizing $H\mathbb{Z}/p_{(\mathbb{Z}/p)^n}$ by inverting the maps $S^0 \to S^\alpha$ for a given set $S$ of irreducible $(\mathbb{Z}/p)^n$-representations?

Symbolically, we may write

\[(3) \quad R_S = (\bigwedge_{i=1}^m S^{\infty \alpha_i}) \wedge H\mathbb{Z}/p_{(\mathbb{Z}/p)^n}^\ast\]

where $S = \{\alpha_1, \ldots, \alpha_m\}$.

Then, in particular, the geometric fixed point coefficient ring (2) is equal to $R_S$ where

$S = \{\alpha_1, \ldots, \alpha_{p^n-1}\}$

consists of all non-trivial irreducible representations of $(\mathbb{Z}/p)^n$.

The contribution of the present paper was essential to 1. better understanding the algebraic structure for the case of $\mathbb{Z}/2$, which was necessary for considering the case of an arbitrary set $S$, and 2. finding
a graded-commutative analog, which is relevant for the case of \( \mathbb{Z}/p \)-coefficients for \( p > 2 \) (since in that case, \( \mathbb{Z}/p \)-valued cohomology forms a graded-commutative, and not a commutative ring), which give a candidate for the \( R_S \). The algebraic computations of this present paper then gives sufficient control on the structure of this ring to show that the candidate is correct by a counting argument [7].

What kind of algebra are we talking about? In [8] Theorem 2, Holler and I. Kriz proved that

\[
(\Phi^{(\mathbb{Z}/2)^{n}}H\mathbb{Z}/2)_{\ast} = \mathbb{Z}/2[t_\alpha | \alpha \in (\mathbb{Z}/2)^{n} \setminus \{0\}] / (t_\alpha t_\beta + t_\alpha t_\gamma + t_\beta t_\gamma | \alpha + \beta + \gamma = 0).
\]

Where \( t_\alpha \) are in degree 1. They proved this by counting the dimension of the submodule of homogeneous elements of a given degree and matching it with a spectral sequence. But what do these relations mean?

Consider the affine space

\[
\mathbb{A}^{n}_{\mathbb{F}_2} = \text{Spec}(\mathbb{F}_2[x_1, \ldots, x_n]).
\]

Then the elements \( z_\alpha \), can be identified with non-zero linear combinations of the coordinates \( x_1, \ldots, x_n \) with coefficients in \( \mathbb{F}_2 \). Such linear combinations can, in turn, be identified with equations of hyperplanes through the origin in \( \mathbb{A}^{n}_{\mathbb{F}_2} \). (All possible rational hyperplanes, as it turns out.) If we remove these hyperplanes from \( \mathbb{A}^{n}_{\mathbb{F}_2} \), we obtain an affine variety with coordinate ring

\[
(\prod_{\alpha \in (\mathbb{Z}/2)^{n} \setminus \{0\}} z^{-1}_\alpha)\mathbb{F}_2[x_1, \ldots, x_n].
\]

I showed that the ring (4) is isomorphic to the subring of the ring (5) generated by the elements \( t_\alpha = z^{-1}_\alpha \). This result turned out to be known (for example, [16], Theorem 4). In fact, the affine variety with coordinate ring (4) is known as the reciprocal plane of the hyperplane arrangement \( \{z_\alpha\} \) (see [3]).

The reciprocal plane can, of course, be considered over any field, and the likely reason this significance of the ring (4) was not noticed before is that the focus of the previous work was mostly on characteristic 0: certainly not on the arrangement of all rational hyperplanes over a finite field. This interpretation, then, begged the question as to what happens if we remove just some subset \( S \) of hyperplanes from \( \mathbb{A}^{n}_{\mathbb{F}_2} \)? What is the topological significance of the reciprocal plane in that case? Using the known presentation [16] I rediscovered, Holler and I. Kriz subsequently proved that those rings are isomorphic to the rings (3).
The real story, and the main contribution of the present paper, however, is for $p > 2$. From the point of view of algebraic geometry, there is no difference: As we already mentioned, the reciprocal plane construction is independent of characteristic.

In algebraic topology, however, when we are dealing with characteristic $p \neq 2$, coefficient rings become *graded-commutative*, i.e.

$$xy = (-1)^{|x||y|}yx$$

where $|x|$ denotes the degree of $x$. So to solve the structure of the rings (2), (3) for $p > 2$, it was necessary to discover the appropriate *graded-commutative analogue* of the reciprocal plane, and to prove structure results analogous to [16]. This is the main result of the present paper.

Very briefly, we consider the ring

$$\mathbb{F}_p[x_1, \ldots, x_n] \otimes \Lambda \mathbb{F}_p[dx_1, \ldots, dx_n]$$

where $\Lambda$ denotes the exterior algebra. In this ring, invert a set of linear combinations $z_\alpha$ of the elements $x_\alpha$. The right ring turns out to be the subring generated by $t_\alpha = z_\alpha^{-1}$ and $u_\alpha = z_\alpha^{-1} dx_\alpha$. Topologically, the element $t_\alpha$ has degree 2 and the element $u_\alpha$ has degree 1, corresponding to the fact that we are dealing with complex, not real, representations for $p > 2$. I determine the structure of these subrings in a way analogous to (but more complicated than) the commutative case. Holler and I. Kriz [7] then used my structure theorems to prove that these rings are isomorphic to the rings (3) for $p > 2$. This is the main topological application of the results of the present paper. The very striking geometric interpretation of the reciprocal planes begs the question what is the appropriate analogue of this interpretation in the graded-commutative case. The Spec of a graded-commutative ring is a *superscheme* (for a survey, see [21]). In section 6, I develop the superscheme analog of some of the known geometric structures associated with the reciprocal plane, which correspond to my algebraic generalization to graded-commutative rings. (Again, the algebraic geometry side of the story is independent of characteristic).

The present paper is organized as follows: In the next section, I give precise statements of the algebraic results of this paper, which amount to finding a presentation of the rings in question, in Theorem 1 in the commutative case and Theorem 3 in the graded-commutative case. Essentially, the proof is by describing an explicit algorithm of reducing a given relation to the relations in my presentation, which I do not think was known before. In Section 3, as a warm-up, I give an
explicit proof of Theorem 1 which can be generalized to the graded-commutative case. In Section 4, I use this method to prove Theorem 3. In Section 5, I prove that the relation ideals $I, K$ are also generated by the relation polynomials $P_L, P_L.S$ where the $L$’s are restricted to “minimal” relations. The commutative case is particularly simple. This was also proved in [16] in the commutative case by less explicit methods. In Section 6, I discuss the geometric interpretation, including the construction of the superscheme corresponding to the graded-commutative case (Theorem 3).

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2. Statement of the results

Following Terao [19], consider an $n$-dimensional affine space $\mathbb{A}^n_F$ over a field $F$. Let $z_1, \ldots, z_m$ be non-zero linear combinations of the coordinates $x_1, \ldots, x_n$ with coefficients in $F$. We can think of the $z_i$’s as equations of hyperplanes in $\mathbb{A}^n_F$. Then the coordinates $t_i = z_i^{-1}$ define a morphism of affine varieties

$$\pi : \mathbb{A}^n_F \setminus Z(z_1 \ldots z_m) \to \mathbb{A}^m_F$$

where $Z(I) = Z(I)$ is the set of zeros of an ideal $I$. The morphism $\pi$ is an embedding if the $z_i$’s linearly span the $x_i$’s. Consider the Zariski closure of $\text{Im}(\pi)$. As we shall see, this variety is a cone, so we can speak of the corresponding projective variety. This construction, called the reciprocal plane, has been studied extensively (see [16, 14, 9, 15, 18, 17, 10, 11]). For a survey, see [3].

To understand this construction better, we must describe it algebraically, which will also bring us closer to the motivation of the present paper. Let

$$R = z_1^{-1} \ldots z_m^{-1} F[x_1, \ldots, x_n] = F[x_1, \ldots, x_n][z_1^{-1}, \ldots, z_n^{-1}]$$

Then we have a homomorphism of rings

$$h : F[t_1, \ldots, t_m] \to R$$

with $h(t_i) = z_i^{-1}$ (which is, of course, not onto). Consider the ideal $I = \text{Ker}(h)$. Denote $\mathcal{A} = \{z_1, \ldots, z_m\}$, and put

$$R_{A,\mathbb{A}^n_F} = F[t_1, \ldots, t_m]/I.$$ 

Then $\text{Spec}(R_{A,\mathbb{A}^n_F})$ is, by definition, the Zariski closure of $\text{Im}(\pi)$. Also by the homomorphism theorem, $R_{A,\mathbb{A}^n_F}$ is a subring of $R$. Observe that
$I$ is a prime ideal (therefore a radical) since $R$ is an integral domain, and hence so are its subrings. Further, if the $z_i$’s generate the $x_j$’s, then

$$R = (t_1 \cdots t_m)^{-1} R_{A,A^n_2}.$$  

Thus, in particular, in this case $\pi$ is an open embedding of the hyperplane arrangement complement into the Zariski closure of its image.

The ideal $I$ is non-zero when there are linear dependencies among the hyperplane equations $z_i$. Suppose, then,

$$L = a_1z_{i_1} + \cdots + a_kz_{i_k} = 0 \in F[x_1, \ldots, x_n]$$

where $a_1, \ldots, a_k \in F$ are not 0, and

$$1 \leq i_1 < \cdots < i_k \leq m.$$  

So, in $R$, we have $\frac{a_1}{t_{i_1}} + \cdots + \frac{a_k}{t_{i_k}} = 0$ where $k > 1$ (where, in the rest of this paper, we indentify $t_j = z_j^{-1}$). Thus,

$$\frac{a_1 t_{i_2} \cdots t_{i_k} + \cdots + a_j t_{i_1} \cdots \hat{t}_{i_j} \cdots t_{i_k} + \cdots + a_k t_{i_1} \cdots t_{i_{k-1}}}{t_{i_1} \cdots t_{i_k}} = 0 \in R,$$

where the hat means an omitted term.

Hence, the numerator $P_L$ of the left hand side of (7) is in $I$.

**Theorem 1.** ([16], [3], (5.3)) Let $Z$ be the set of all linear relations $L$ among the hyperplane equations $z_i$. Then

$$I = (P_L(t_1, \ldots, t_m) | L \in Z),$$

or in other words,

$$R_{A,A^n_2} = F[t_1, \ldots, t_m] / (P_L(t_1, \ldots, t_m) | L \in Z).$$

**Corollary 2.** ([7, 8]) For $p = 2$, the $\mathbb{Z}$-graded coefficient ring (6) of the constant $\mathbb{Z}/2$-Mackey functor ordinary $(\mathbb{Z}/2)^n$-equivariant cohomology spectrum with the inclusion $S^0 \to S^{\alpha_i}$ inverted where $\alpha_i$ are real irreducible representations corresponding to the hyperplanes $z_i$ is

$$R_S = R_{A,A^n_2}.$$  

This was proved in [8] using a direct method for the case of all $2^n - 1$ rational hyperplanes through the origin in $A^n_2$. The authors of [8] asked about the algebraic interpretation of this ring. I found the above interpretation and proved Theorem 1 by describing an explicit algorithm for reducing relations. Using this and the commutative case of Theorem 8 below then led to the proof of the general case of Corollary 2 in [7].
Both Theorem 1 and the commutative case of Theorem 8 turned out to be known ([16], Theorem 4). The reason the connection with [8], and Corollary 2 of [7] were not noticed before is probably that the focus of [16] and other previous work was not on the case of all hyperplanes over a finite field.

While the work of [16] (and hence Theorem 1 and the commutative case of Theorem 8) work over any field, in algebraic topology, those calculations are not relevant in characteristic \( p > 2 \), where the relevant rings are graded-commutative. In fact, [8] was written entirely in characteristic 2 because the graded-commutative analog of the ring \( R_{A,K_F}^p \) was not known. I found this ring algebraically by looking for a “graded-commutative” analog of the geometric structures described above, and my reduction algorithm. I then proved a graded-commutative analog of Theorem 1, (and the corresponding part of Theorem 8), which is the main result of the present paper.

For the graded-commutative case, consider

\[
\Omega = F[x_1, \ldots, x_n] \otimes \Lambda[dx_1, \ldots, dx_n]
\]

where \( \Lambda \) denotes the exterior algebra over the field \( F \). Then the non-zero \( F \)-linear combinations \( z_i \) of the \( x_i \)'s are in the center of \( \Omega \). Now consider

\[
T = z_1^{-1} \cdots z_m^{-1} \Omega \supset \Omega.
\]

This is the graded-commutative analog of the ring \( R \). We are interested in the subring \( T_{A,K_F} \) of \( T \) generated by \( z_1^{-1}, \ldots, z_m^{-1}dz_1, \ldots, z_m^{-1}dz_m \).

Put \( t_i = z_i^{-1} \) and \( u_i = z_i^{-1}dz_i \). Then we have a canonical homomorphism of rings

\[
\psi : \Xi = F[t_1, \ldots, t_m] \otimes \Lambda[u_1, \ldots, u_m] \to T.
\]

Let \( K = \ker(\psi) \). Note that \( I \subsetneq K \). Thus, we have

\[
T_{A,K_F} = \Xi / K.
\]

We want to find the generators of the ideal \( K \). If \( L \) is again the left hand side of (6), then

\[
dL = a_{i_1}dz_{i_1} + \cdots + a_{i_k}dz_{i_k} = 0 \in T.
\]

If we multiply

\[
P_L = a_{i_1}t_{i_2} \cdots t_{i_k} + a_{i_2}t_{i_1}t_{i_2} \cdots t_{i_k} + \cdots + a_{i_k}t_{i_1} \cdots t_{i_{k-1}} + a_{i_k}t_{i_1} \cdots t_{i_k} - 1
\]

by \( dz_{j_1} \cdots dz_{j_l} \), where

\[
S = \{ j_1 < \cdots < j_l \} \subseteq \{ i_1, \ldots, i_k \}.
\]
some monomial summands can be expressed in terms of the $u_j$'s. If a monomial summand does not contain $t_j$ but does contain $dz_j$, then use $dL = a_i dz_i + \cdots + a_k dz_k$ to eliminate $dz_j$. Explicitly, let

$$P_{L,S} = P_L dz_{j_1} \cdots dz_{j_l}$$

$$- \sum_{s=1}^l t_{j_s} \cdots \hat{t}_{j_s} \cdots t_{j_k} dz_{j_1} \cdots \hat{dz}_{j_s} dL \cdots dz_{j_l}.$$  

We have $P_{L,S} \in \Xi$. Note that, by definition, $P_{L,\emptyset} = P_L$. Our main result is

**Theorem 3.** Let $\mathcal{Y}$ be the set of all pairs $(L,S)$ where $L$ is a linear relation among hyperplanes equations as in (6), and $S$ is a subset of the index set as in (9). Then

$$K = (P_{L,S} \mid (L,S) \in \mathcal{Y}).$$

In other words,

$$T_{\mathcal{A},\mathcal{A}_p^R} = \Xi/(P_{L,S} \mid (L,S) \in \mathcal{Y}).$$

This algebraic Theorem, along with Theorem 8 below was used in [7] to prove the following result:

**Corollary 4.** ([7]) For $p > 2$, the $\mathbb{Z}$-graded coefficient ring (6) of the constant $\mathbb{Z}/p$-Mackey functor ordinary $(\mathbb{Z}/p)^n$-equivariant cohomology spectrum with inclusions $S^0 \to S^{\alpha_i}$ inverted where $\alpha_i$ are complex irreducible representations corresponding to the hyperplanes $z_i$ is

$$R_S = T_{\mathcal{A},\mathcal{A}_p^R}.$$  

Since the commutative algebra methods of [16] are at present not available for graded-commutative rings, our proof of Theorem 3 is elementary and in fact is an elaboration of my algorithm used to prove Theorem 1. As a warm-up, I also include my original elementary proof of Theorem 1, which I then generalized to the graded-commutative case. Some brief notes on the interpretation of the graded-commutative result in algebraic geometry are given in Section 6 below.

**Example:** Let $L = z_1 + z_2 + z_3 = 0 \in \Omega$. Then we have

$$P_L = P_{L,\emptyset} = \frac{z_1 + z_2 + z_3}{z_1 z_2 z_3} = t_2 t_3 + t_1 t_3 + t_1 t_2.$$  

Now to compute $P_{L,(2)}$, write

$$(10) \quad P_L dz_2 = t_2 t_3 dz_2 + t_1 t_2 dz_2 + t_1 t_3 dz_2 = u_2 t_3 + t_1 u_2 + t_1 t_3 dz_2.$$  

Now use

$$(11) \quad dL = dz_1 + dz_2 + dz_3 = 0$$
to express $dz_2 = -dz_1 - dz_3$, which we use to conclude
\[ t_2 t_3 dz_2 = -t_1 t_3 (dz_1 + dz_3) = u_1 t_3 + u_3 t_1. \]
Substituting this into (10) gives the relation
\[ P_{L,\{2\}} = u_2 (t_1 + t_3) - u_1 t_3 - u_3 t_1. \]
To calculate $P_{L,\{1,2\}}$, we start with the expression
\[ p_L dz_2 = t_2 t_3 dz_1 + t_1 t_2 dz_1 dz_2 + t_1 t_3 dz_1 dz_3 =
\[ = t_2 t_3 dz_1 dz_2 + u_1 u_2 + t_1 t_3 dz_1 dz_3. \]
Using (11) again, we get
\[ t_2 t_3 dz_1 dz_2 = t_2 t_3 (-dz_2 - dz_3) dz_1 = t_2 t_3 dz_2 dz_3 = u_2 u_3. \]
and
\[ t_1 t_3 dz_1 dz_2 = t_1 t_3 dz_1 (-dz_1 - dz_3) = u_3 u_1. \]
Thus, we obtain the relation
\[ P_{L,\{1,2\}} = u_1 u_2 + u_2 u_3 + u_3 u_1. \]
The reader should keep in mind that the above derivation of examples of the relations $P_{L,S}$ is used simply to explain our definition of these relations. Nevertheless, they illustrate the fact that $P_{L,S}$ is a relation in $t_1^{-1} \ldots t_m^{-1} \Xi$ which is contained in $\Xi$, and thus is valid in $\Xi$.

3. The commutative case

The purpose of this section is to prove Theorem 1.

**Lemma 5.** The relation ideal $I$ of Theorem 1 satisfies
\[ I = \{ q = p_1 L_1 + \cdots + p_N L_N \mid p_i \in t_1^{-1} \ldots t_m^{-1} F[t_1, \ldots, t_m], q \in F[t_1, \ldots, t_m] \}. \]

**Proof.** Consider the diagram
\[ \begin{array}{ccc}
F[t_1, \ldots, t_m] & \xrightarrow{\pi} & R \\
\downarrow & & \downarrow \\
F[t_1, t_1^{-1}, \ldots, t_m, t_m^{-1}] & \xrightarrow{\pi} & F[z_1, \ldots, z_m]
\end{array} \]
We know $\ker(\pi) = (L_1, \ldots, L_N) \subseteq F[z_1, \ldots, z_m]$. Therefore we know that
\[ \ker(\tilde{\pi}) = z_1^{-1}, \ldots, z_m^{-1} \ker(\pi) = (L_1, \ldots, L_N) \subseteq z_1^{-1} \ldots z_m^{-1} F[z_1, \ldots, z_m] = t_1^{-1} \ldots t_m^{-1} F[t_1, \ldots, t_m] \]
by exactness of localization.
Proof of Theorem 1. Let $J$ be the ideal in $I$ which is generated by all the $P_L$’s. Then we want $J$ to equal $I$. We shall perform induction on $m$ (the number of the variables $z_j$). If $m = 0$, then $R_{A, A_x^p} = F$ and there is nothing to prove. Let

$$(14) \quad q = p_1 L_1 + \cdots + p_N L_N \in I$$

be as in Lemma 5. We want to show that $q \in J$. We may assume $L_1, \ldots , L_N$ are in reduced row echelon form where the order of columns corresponds to the order of variables $z_m, \ldots z_1$. Let $A' = \{z_1, \ldots , z_{m-1}\}$ and define an ideal $I' \subset F[t_1, \ldots , t_{m-1}]$ by

$$R_{A', A_x^p} = F[t_1, \ldots , t_{m-1}]/I'.$$

If the first pivot of our RREF is not in the first column, there is no relation $L_1, \ldots , L_N$ with $a_1 \neq 0$ involving $z_m = t_m$. In this case, by construction, we have a homomorphism of rings

$$R_{A', A_x^p} \to R_{A, A_x^p}.$$

Now, we may write each $p_1, \ldots , p_N$ as a Laurent polynomial in the variable $t_m$. Since $L_1, \ldots , L_N$ do not involve $z_m$, by Lemma 5, the coefficients $q_i$ of $q$ at $t_m^i$ are in $F[t_1, \ldots , t_{m-1}]$, and are 0 for $i < 0$. Thus, by Lemma 5, each $q_i$ maps to 0 in $R_{A', A_x^p}$, and thus, by the induction hypothesis, is an $F[t_1, \ldots , t_{m-1}]$-linear combination of the elements $P_L$ where $L$ runs through all linear relations among $z_1, \ldots , z_{m-1}$. Thus, $q \in J$ and we are done. Thus, assume that the RREF of the relations $L_1, \ldots , L_N$, as described above, has a pivot in the first column, corresponding to $z_m$.

Now write $p_i \in F[t_1, \ldots , t_{m-1}, t_1^{-1}, \ldots , t_{m-1}^{-1}][t_m, t_m^{-1}]$. Denote by

$$p_{i,k} \in t_1^{-1} \cdots t_{m-1}^{-1} F[t_1, \ldots , t_{m-1}]$$

the coefficients of every power $t_m^k$, $k \in \mathbb{Z}$.

Without loss of generality,

$$(15) \quad L_1 = a_1 z_m + \ldots ,$$

$a_1 \neq 0 \in F$

has more than two non-zero terms. Otherwise, the number of nonzero terms in $L$ would be exactly 2 (since we did not allow $z_m = 0$). But if the number of non-zero terms in $L_1$ is exactly 2, then $z_m$ is a non-zero multiple of some $z_i$, $i < m$. Therefore, if we put $A' = \{z_1, \ldots , z_{m-1}\}$, $R_{A, A_x^p} = R_{A', A_x^p}$, and our statement follows from the induction hypothesis.
Suppose, therefore, that (15) has at least 3 non-zero terms. Now consider the highest \( k \in \mathbb{Z} \) such that \( p_{i,k} \neq 0 \) for some \( i \). Let \( L'_1 \) be the linear combination obtained from \( L_1 \) by omitting the \( a_1 z_m \) summand.

**Case 1:** \( k > 0 \). Then

\[
\tilde{q} = p_{1,k} L'_1 + p_{2,k} L_2 + \cdots + p_{N,k} L_N \in F[t_1, \ldots, t_{m-1}].
\]

Note: we do not, of course, claim that (16) is a relation among the chosen \( z_j \)'s. However, since the relations are in reduced row echelon form, and \( L_1 \) has at least three terms, the relation \( L'_1 \) only introduces a linear relation among at least two of the non-pivot variables of the relationns \( L_2, \ldots, L_N \). Therefore, there exist some non-zero linear combinations \( z_2, \ldots, z_m \) of some other parameters \( y_1, \ldots, y_{m'} \), which satisfy the relations \( L'_1, L_2, \ldots, L_N \). Since our induction is only on the number \( m \) of the hyperplanes, the induction hypothesis applies. By Lemma 5, \( \tilde{q} \in I' \). By the induction hypothesis, (16) is a linear combination

\[
w_1 P_{L'_1} + w_2 P_{L_2} + \cdots + w_N P_{L_N}, w_i \in F[t_1, \ldots, t_{m-1}].
\]

Then subtracting

\[
w_1 t_m^{k-1} P_{L_1} + w_2 t_m^k P_{L_2} + \cdots + w_N t_m^k P_{L_N}
\]

from (14), we obtain an element \( q' \in F[t_1, \ldots, t_m] \) which differs from \( L \) by an element of \( J \), and for which the number \( k \in \mathbb{Z} \) is lower. Thus, we are reduced to:

**Case 2:** \( k \leq 0 \). Then consider the lowest \( l \leq 0 \) for which there exists an \( i \) with \( p_{i,l} \neq 0 \). Then, \( p_{i,l} = 0 \), since \( p_{1,l} z_m^{-l+1} \) has nothing to cancel out against in (14) (since all the other powers of \( z_m \) are \( \leq -l \). Thus, since \( -l + 1 > 0 \), this contradicts \( q \in F[t_1, \ldots, t_m] \).

Thus,

\[
q' = p_{2,l} L_2 + \cdots + p_{N,l} L_N \in F[t_1, \ldots, t_m],
\]

and thus, by Lemma 5, we have \( q' \in I' \). Thus, by the induction hypothesis applied to \( p_{i,l} \), \( q' \) is a linear combination

\[
w_2 P_{L_2} + \cdots + w_N P_{L_N}, w_2, \ldots, w_N \in F[t_1, \ldots, t_{m-1}].
\]

If \( k = l = 0 \), we are done. If \( l < 0 \), we must have

\[-p_{1,l+1} = a_1^{-1}(p_{2,l} L_2 + \cdots + p_{N,l} L_N)\]

for cancellation, so

\[
(p_{2,l} L_2 + \cdots + p_{N,l} L_N) z_m^{-l} +
\]

\[
p_{1,l+1} L_1 z_m^{-l-1} + a_1^{-1}(p_{2,l} L_2 + \cdots + p_{N,l} L_N) L'_1 z_m^{-l-1}
\]

\[
= 0 \in t_1^{l+1} \cdots t_m^{l+1} F[t_1 \ldots t_m]
\]

can be subtracted from (14), thus increasing \( l \) without violating \( k \leq 0 \). Thus, by repeating this process, we are done.
4. The odd case

In this section, we prove Theorem 3.

Lemma 6. The relation ideal $K$ of Theorem 3 is given by

$$K = \{ q = p_1 L_1 + \cdots + p_N L_N + r_1 dL_1 + \cdots + r_N dL_N \mid q \in \Xi, p_i, r_i \in \Xi[t_1^{-1} \cdots t_m^{-1}] \}.$$ 

Proof. Denote $Y = F[z_1 \ldots z_m] \otimes \Lambda[dz_1 \ldots dz_m]$. The analog of diagram (13) is

$$\begin{array}{ccc}
\Xi & \xrightarrow{\psi} & T = z_1^{-1} \ldots z_m^{-1} \Omega \\
\Xi[t_1^{-1} \ldots t_m^{-1}] & \xrightarrow{=} & z_1^{-1} \ldots z_m^{-1} Y \\
\end{array}$$

where $\overline{\psi}$ is the canonical map. Then

$$\text{Ker}(\overline{\psi}) = (L_1, \ldots, L_N, dL_1, \ldots, dL_N).$$

Now certainly $\text{Ker}(\psi) \supseteq z_1^{-1} \ldots z_m^{-1}(L_1, \ldots, L_N, dL_1, \ldots, dL_N)$. To prove the converse, note that exactness of localization works the same here as in the commutative case. If $\psi(x) = 0$, $x \in z_1^{-1} \ldots z_m^{-1}Y$, then $(z_1, \ldots, z_m)^N \psi(x) = \psi((z_1, \ldots, z_m)^N x) = 0$. Without loss of generality (by increasing $N$ if necessary), we may then also assume $x \in Y$, $\tilde{\psi}(x) = 0$. \qed

It is useful to note here that in general, for a commutative ring $R$, and a multiplicative set $S \subseteq R$

$$S^{-1}(\Lambda_R[u_1, \ldots, u_m]) = \Lambda_{S^{-1}R}[u_1, \ldots, u_m].$$

Proof of Theorem 3: The reader is encouraged to follow along the corresponding steps of the proof of Theorem 1, which we shall mimic. Let

$$M \subseteq K$$

be the ideal generated by the elements $P_{L,S}$. Again, we prove the statement by induction by $m$, the number of hyperplanes $z_j$. For $m = 0$, again $T_{A,\Lambda_p^n} = F$, so there is nothing to prove. For $m > 0$, again, put

$$A' = \{z_1, \ldots, z_{m-1}\},$$

$$\Xi_0 = F[t_1, \ldots, t_{m-1}] \otimes \Lambda[u_1, \ldots, u_{m-1}],$$

$$T_{A',\Lambda_p^n} = \Xi_0/K'.$$
First we again put \( L_1, \ldots, L_N \) in reduced row echelon form so that the columns correspond to \( z_m, z_{m-1}, \ldots, z_1 \). Then let

\[
(21) \quad w = p_1 L_1 + \cdots + p_N L_N + q_1 dL_1 + \cdots + q_N dL_N \in \Xi
\]

and \( p_j, q_j \in \Xi[t_1^{-1}, \ldots, t_m^{-1}] \). Again, we may assume that the first column (corresponding to \( z_m \)) has a pivot: Otherwise, consider again the canonical homomorphism of rings

\[
T_{A', \mathcal{A}'_k} \rightarrow T_{A, \mathcal{A}_k}.
\]

Again, each coefficients \( w_{i,0} \) and \( w_{i,1} \) of \( w \) at \( t_m^i u_m \), respectively are in \( \Xi' \) by Lemma 6, and hence are linear combinations of \( P_{L,S} \) where \( L \) are the relations among \( z_1, \ldots, z_{m-1} \) by the induction hypothesis. Thus \( w \in M \) and we are done. Again, we may also assume that \( L_1 \) has at least 3 terms: 1 term is excluded by \( z_1 \neq 0 \), and 2 terms would give \( T_{A, \mathcal{A}_k} = R_{A', \mathcal{A}'_k} \), and our statement would follow from the induction hypothesis.

Then

\[
p_i = \sum_{k \in \mathbb{Z}} \left( p_{i,k} t_m^k + \overline{p_{i,k}} t_m^k u_m \right)
\]

and

\[
q_i = \sum_{k \in \mathbb{Z}} \left( q_{i,k} t_m^k + \overline{q_{i,k}} t_m^k u_m \right)
\]

with \( p_{i,k}, \overline{p_{i,k}}, q_{i,k}, \overline{q_{i,k}} \in \Xi_0[t_1^{-1}, \ldots, t_{m-1}^{-1}] \). Let \( L_1' \) be \( L_1 \) with the \( z_m \) term removed. Consider the highest \( k \) for which at least one of the polynomials \( p_{i,k}, \overline{p_{i,k}}, q_{i,k}, \overline{q_{i,k}} \) is non-zero. Let us distinguish two cases. (Note that \( dz_m = t_m^{-1} u_m \), so the omitted terms have lower total power of \( t_m \)).

**Case 1:**

Suppose \( k > 0 \). By maximality of \( k \),

\[
(22) \quad p_{1,k} L_1' + p_{2,k} L_2 + \cdots + p_{N,k} L_N + q_{1,k} dL_1' + \cdots + q_{N,k} dL_N \in \Xi_0.
\]

By the induction hypothesis this is a linear combination of \( P_{L_1',S}, P_{L_i,S}, i > 1 \). Similarly for

\[
(23) \quad \overline{p_{1,k}} L_1' + \overline{p_{2,k}} L_2 + \cdots + \overline{p_{N,k}} L_N + \overline{q_{1,k}} dL_1' + \overline{q_{2,k}} dL_2 + \cdots + \overline{q_{N,k}} dL_N \in \Xi_0.
\]

(This time, we are also using \( u_m^2 = 0 \).) Subtracting the corresponding linear combinations of \( P_{L_1,S}, P_{L_i,S} \) from

\[
p_{1,k} L_1 + \cdots + p_{N,k} L_N + q_{1,k} dL_1 + \cdots + q_{N,k} dL_N
\]

or

\[
\overline{p_{1,k}} L_1 + \cdots + \overline{p_{N,k}} + \overline{q_{1,k}} dL_1 + \cdots + \overline{q_{N,k}} dL_N,
\]

we decrease \( k \). Thus, again, we are reduced to
Case 2:
Suppose \( k \leq 0 \). Then consider the lowest \( \ell \) for which at least one of the polynomials \( p_{i,k}, q_{i,k}, \sigma_{i,k}, \tau_{i,k} \) is non-zero. Let

\[
\begin{align*}
\omega &= p_{1,\ell}L_1 + \cdots + p_{N,\ell}L_N \\
\rho &= q_{1,\ell}dL_1 + \cdots + q_{N,\ell}dL_N \\
\sigma &= \sigma_{1,\ell}L_1 + \cdots + \sigma_{N,\ell}L_N \\
\tau &= \tau_{1,\ell}dL_1 + \cdots + \tau_{N,\ell}dL_N
\end{align*}
\]

and denote by \( \omega', \rho', \sigma', \tau' \) the linear combinations obtained by replacing \( L_1, dL_1 \) by \( L'_1, dL'_1 \) in (24). Then by minimality of \( \ell \),

\[
\omega + \rho + \sigma + \tau = \omega' + \rho' + \sigma' + \tau'.
\]

(The extra terms on the left hand side of (25) have nothing to cancel against so their sum must be 0.) Therefore, if \( k = \ell = 0 \), the right hand side of (25) (which is equal to (21)) is in \( \Xi_0 + u_m\Xi_0 \), and the statement follows from the induction hypothesis and the lemma following this proof.

If \( \ell < 0 \), for cancellation, we must have

\[
a_1 p_{1,\ell+1} = -\rho' - \omega'
\]

\[
a_1 (q_{1,\ell+1}u_m + \tau_{1,\ell+1}u_m) = -\tau' - \sigma',
\]

using the convention (15) for the definition of \( 0 \neq a_1 \in F \). Thus, adding to (21)

\[
0 = -(\omega + \rho + \sigma + \tau)z_m^{-\ell} - (p_{1,\ell+1} + q_{1,\ell+1}u_m + \tau_{1,\ell+1}u_m)z_m^{-\ell-1}L_1 - a_1^{-1}(\rho + \omega + \sigma + \tau)z_m^{-\ell-1}L'_1,
\]

which is a \( t_1^{-1} \ldots t_m^{-1} \Xi \)-linear combination of the elements \( L_i \) and \( dL_i \), increases \( \ell \), without increasing \( k \).

\( \square \)

Lemma 7. Assuming the statement of Theorem 3 holds with \( m \) replaced by \( m - 1 \), and \( L_1, \ldots, L_M \) are in reduced row echelon form in the order of columns \( z_m, z_{m-1}, \ldots, z_1 \) with a pivot in the first column and assume \( L_1 \) has at least 3 non-zero terms. Then, we have

\[
(\Xi_0 + u_m\Xi_0) \cap \text{Ker}(\psi) \subseteq M + (\Xi_0 \cap \text{Ker}(\psi))
\]

(see (20)).

Proof. We want to rephrase the Lemma to say

\[
\Xi_0 + (P_{L,S}) \supseteq u_m\Xi_0 \cap \psi^{-1}(\psi\Xi_0).
\]

This is possible because if \( a \in \Xi_0 \) and \( b \in u_m\Xi_0 \), and \( \psi(a + b) = 0 \) then \( -\psi(a) = \psi(b) \in \psi\Xi_0 \). Then \( b \in u_m\Xi_0 \cap \psi^{-1}(\psi\Xi_0) \) so if (27) holds, then
$b \in \Xi_0 + (P_{L,S})$. In other words, $b = c+$ linear combinations of $P_{L,S}$ with $c \in \Xi_0$. Then $a + b = a + c+$ linear combination of $P_{L,S}$, $a + c \in \Xi_0$, $\psi(a + c) = 0$ (since $\psi(P_{L,S}) = 0$). Thus, proving the statement (26) is reduced to proving the statement (27).

To prove (27), let $R = \{p \in \Xi_0 | \psi(um) \in \psi(\Xi_0)\}$ (Note that $u_mR = u_m\Xi_0 \cap \psi^{-1}(\psi(\Xi_0))$.)

Let $Y_0 = F[z_1 \ldots z_{m-1}] \otimes \Lambda[z_1 \ldots dz_{m-1}]$.

Put $Q = \{p \in z_1^{-1} \ldots z_{m-1}^{-1}Y_0 | \psi(pu_m) \in \psi(z_1^{-1} \ldots z_{m-1}^{-1}Y_0)\}$. We have

$z_1^{-1} \ldots z_{m-1}^{-1}Y = t_1^{-1} \ldots t_{m-1}^{-1}\Xi$

and

(28) $z_1^{-1} \ldots z_{m-1}^{-1}Y_0 = t_1^{-1} \ldots t_{m-1}^{-1}\Xi_0$.

Then $R \subseteq Q \cap \Xi_0$. So

(29) $\psi(\psi(p)) = \psi(p)\psi(um) = \psi(p)\frac{\psi(dL_1')}{\psi(L_1')} \in \psi(L_1')^{-1}T$.

Note that $\psi(L_1')$ is a non-zero linear combination of the $x_i$'s which is not an $F$-multiple by any of the $z_j$'s by our assumption on $L_1$. Then $u_mY_0$ goes to $(\psi(L_1'))^{-1}\Omega$ by $\psi$ and $u_m(z_1^{-1} \ldots z_{m-1}^{-1}Y_0)$ goes to $T$ by $\psi$.

Let $\overline{Q} = \{p \in Y_0 | \psi(pu_m) \in \psi(Y_0)\}$. So $Q = z_1^{-1} \ldots z_{m-1}^{-1}\overline{Q}$. By (29), $\overline{Q}$ is the ideal of all $p \in Y_0$ such that

(30) $\psi(L_1')\psi(p)\psi(dL_1') \in T$.

But since $L_1'$ is not an $F$-multiple of any of the $z_j$'s, by (19), (30) is equivalent to

$\tilde{\psi}(L_1')\tilde{\psi}(p)\tilde{\psi}(dL_1') \in \Omega$.

Also, $\psi(L_1') \in \Omega$ is a regular element: let

$\mu = \frac{\tilde{\psi}(p)\tilde{\psi}(dL_1')}{\tilde{\psi}(L_1')} \in \Omega$.

We may now assume that the $z_j$'s linearly span the $x_i$'s (since otherwise we could replace the $x_i$'s by the span of $z_j$'s), so

$\tilde{\psi}|_{Y_0} : Y_0 \rightarrow \Omega$

is onto. Let $\psi(\tilde{\mu}) = \mu$, $\tilde{\mu} \in Y_0$. Thus, by (18), we have

$pdL_1' \in \tilde{\mu}L_1' + (L_2, \ldots, L_N, dL_2, \ldots, dL_N)$.

Now note that in $Y_0$, $L_1'$ is not a linear combination of $L_2, \ldots, L_n$, (since $z_m \neq 0$), so by basic properties of polynomial and exterior algebras, writing

$Y_0 = F[L_1', \gamma_2, \ldots, \gamma_{m-1}] \otimes \Lambda_F[dL_1', d\gamma_2, \ldots, d\gamma_{m-1}]$,
we see that $dL'_1|\bar{\mu} \in Y_0$. Also, the kernel of multiplication by $dL'_1$ is $(dL'_1)$, so
\[ p \in (L'_1, L_2, \ldots, L_N, dL'_1, dL_2, \ldots, dL_N) \subseteq Y_0. \]
Thus, we proved
\[ Q = (L'_1, L_2, \ldots, L_N, dL'_1, dL_2, \ldots, dL_N) \subseteq Y_0. \]
By exactness of localization,
\[ Q = (L'_1, L_2, \ldots, L_N, dL'_1, dL_2, \ldots, dL_N) \subseteq z^{-1}_1 \ldots z^{-1}_{m-1} Y_0. \]
Now we need to prove
\[ u_m(Q \cap \Xi_0) \subseteq \Xi_0 + M \]
(see (20)). Let $v \in Q \cap \Xi_0$. Then we have
\[ v \in p_1 L'_1 + p_2 dL'_1 + (L_2, \ldots, L_N, dL_2, \ldots, dL_N) \subseteq z^{-1}_1 \ldots z^{-1}_{m-1} Y_0 \]
with $p_1, p_2 \in \Xi_0[t^{-1}_1, \ldots, t^{-1}_{m-1}]$. Let
\[ L'_1 = \sum_{j=2}^{k} a_j z_{i_j} \]
for $1 \leq i_2 < \ldots, i_k < m$.

Note that $L'_1$ is linearly independent over $F$ of $L_2, \ldots, L_N$ (since $z_m \neq 0$). Now subtract a linear combination of $L_2, \ldots, L_N$ from (31) so that the right hand side $\tilde{L}_1'$ has the fewest possible non-zero terms. Without loss of generality, thus, $\tilde{L}_1' = L'_1$ and $z_{i_2}, \ldots, z_{i_k}, L_2, \ldots, L_N$ are linearly independent over $F$. Then, choosing a basis for $F\{z_1, \ldots, z_{m-1}\}$ containing $\{z_{i_2}, \ldots, z_{i_k}, L_2, \ldots, L_N\}$ and writing elements of $\Xi_0$ using this basis, we see that the elements $p_1 L'_1, p_2 dL'_1$ must be in $\Xi_0$, if we absorb any monomials from
\[ (L_2, \ldots, L_N, dL_2, \ldots, dL_N) \]
containing $dL'_1$ into $p_2 dL'_1$ and any remaining monomials from (32) containing $L'_1$ into $p_1 L'_1$. This means that (recalling that $z_{i_2}, \ldots, z_{i_k}$ are the summands of $L'_1$ with non-zero coefficients), then $p_1$ is a $\Xi_0$-multiple of $t_{i_2} \ldots t_{i_k}$, and $p_2$ is a $\Xi_0$-multiple of $t_{i_2} \ldots t_{i_k} d z_{j_2} \ldots d z_{j_l}$ for some $\{j_2 < \cdots < j_l\} \subset \{i_2 < \cdots < i_k\}$.

Now $u_m t_{i_2} \ldots t_{i_k} L'_1$ can be eliminated by $P_{L_1, \{m\}}$. For
\[ S = \{ j_1 < \cdots < j_l \} \subseteq \{ i_2, \ldots, i_k \}, \]
\[ u_m t_{i_2} \ldots t_{i_k} d z_{j_1} \ldots d z_{j_l} dL'_1 \]
can be eliminated by \( u_m P_{L_1, S \cup \{m\}} \). This is because

\[
u_m P_{L_1, S \cup \{m\}} = \nu_m P_{L_1} dz_{j_1} \ldots dz_{j_l} dz_m \]

\[
- \sum_{q=1}^{l} u_m t_2 \ldots \hat{t}_{j_q} \ldots t_{i_k} dz_{j_1} \ldots dL_1 \hat{dz}_{j_q} \ldots dz_{j_l} dz_m \\
- u_m t_2 \ldots t_{j_k} dz_{j_1} \ldots dz_{j_l} dL_1' \\
= -u_m t_2 \ldots t_{j_k} dz_{j_1} \ldots dz_{j_l} dL_1'
\]

\[\square\]

Some concrete examples of the eliminations we used in the conclusion of the proof of Lemma 7 are shown below.

**Example 1:** \( z_m + z_1 + z_2 = L_1, m > 2 \), then \( L_1' = z_1 + z_2 \), so \( u_m t_1 t_2 (z_1 + z_2) = t_m dz_m t_1 t_2 (z_1 + z_2) = u_m (t_1 + t_2) = u_m t_1 + u_m t_2 \). And \( P_1, m = u_m t_1 + u_m t_2 - t_1 u_2 - t_2 u_1 \).

**Example 2:** \( S = 0 \) so

\[ u_m t_1 t_2 dL_1' = u_m u_1 t_2 + u_m u_2 t_1. \]

This is eliminated by

\[ u_m P_{L_1 \cup \{m\}} = -u_m t u_2 - u_m t u_1 \]

**Example 3:** \( S = 1 \) so

\[ u_m t_1 t_2 dz_1 (dz_1 + dz_2) = u_m u_1 u_2 \]

is eliminated by

\[ u_m P_{L_1, \{1, m\}} = u_m u_1 u_2. \]

### 5. Minimality

Let

\[ L = a_1 z_{i_1} + \cdots + a_k z_{i_k} \]

where

\[ 1 \leq i_1 < \cdots < i_k \leq m, \]

\[ a_i \neq 0 \in F. \]

Then put

(33) \[ |L| := \{i_1, \ldots, i_k\}. \]
Call \( L \) \textit{minimal} if there do not exist relations \( L_1, L_2 \) such that
\[
L_1 + L_2 = L
\]
\[
|L_1|, |L_2| \subsetneq |L|.
\]
Define shuffle permutations as follows: for sets of natural numbers
\[
S_1 = \{i_1 < \cdots < i_k\}
\]
\[
S_2 = \{j_1 < \cdots < j_l\}
\]
\( S_1 \cap S_2 = \emptyset \),
denote by \( \sigma_{S_1,S_2} \) the permutation which puts the sequence
\[
(i_1, \ldots, i_k, j_1, \ldots, j_l)
\]
in increasing order. Also define for \( S = \{i_1 < \cdots < i_k\} \):
\[
t_S := t_{i_1} \cdots t_{i_k}
\]
\[
u_S := u_{i_1} \cdots u_{i_k}
\]
\[
dz_S := dz_{i_1} \cdots dz_{i_k}
\]

\textbf{Theorem 8.}
\[
I = (P_L|L \text{ is a minimal relation})
\]
\[
K = (P_{L,S}|L \text{ is a minimal relation and } S \subseteq |L|)
\]

(For the case of \( I \), see [16], Theorem 4.)

Let \( L \) be as in (33), \( S \subseteq |L| \). Put
\[
Q_{L,S} := t_{|L|} dL dz_S.
\]
So obviously, \( Q_{L,S} \in K \).

\textbf{Lemma 9.} \( Q_{L,S} \in (P_{L,T}|T \subseteq |L|) \)

\textit{Proof.} If \( S \neq \emptyset \), let \( i \in S \). Then \( Q_{L,S} = u_i P_{L,S} \). On the other hand,
\[
Q_{L,\emptyset} = u_i P_{L,\emptyset} - t_{i_1} P_{L,\{i_1\}}.
\]
In the first summand the surviving term is the term of \( P_{L,\emptyset} \) which omits \( t_{i_1} \). In the second summand the surviving terms are the “error terms” of the summand of \( P_L \) which omits \( t_{i_1} \). All remaining terms cancel. \( \Box \)

\textit{Proof of Theorem 8:} Even case : Suppose we know
\[
L_1 + L_2 = L
\]
\[
|L_1|, |L_2| \subsetneq |L|.
\]
Then
\[
P_L = t_{L\setminus L_1} P_{L_1} + t_{L\setminus L_2} P_{L_2}.
\]
Odd case: If $L$ is not minimal we know $L_1 + L_2 = L$ and $|L_1|, |L_2| \subseteq |L|$. Based on the even case, the first guess for $P_{L,S}$ could be

$$P_{L_1,S_1} u_{S \setminus S_1} t_{|L| \setminus (|L_1| \cup S)} \text{sign} (\sigma_{S,S \setminus S_1})$$

$$+ P_{L_2,S_2} u_{S \setminus S_2} t_{|L| \setminus (|L_2| \cup S)} \text{sign} (\sigma_{S_2,S \setminus S_2}).$$

The terms that match are those when we omit $t_i$ from $t_L$ with $i \in |L| \setminus S$ or $i \in |L_1| \cap |L_2| \cap S$. The terms which do not match are for $i \in (|L_1| \cap S) \setminus |L_2|$ or $(|L_2| \cap S) \setminus |L_1|$. For $i \in (|L_1| \cap S) \setminus |L_2|$, the term missing in our first guess is

$$u_{S \setminus S_2 \setminus \{i\}} Q_{L_2,S_2} t_{|L| \setminus (|L_2| \cup S)} \text{sign} (\sigma_{S \setminus S_2 \setminus \{i\}, S_2}) \text{sign} (\sigma_{S \setminus S_2 \setminus \{i\}, S_2}).$$

Symmetrically for $|L_2| \cap S \setminus |L_1|$. Thus we have

$$P_{L,S} = P_{L_1,S_1} u_{S \setminus S_1} t_{|L| \setminus (|L_1| \cup S)} \text{sign} (\sigma_{S,S \setminus S_1})$$

$$+ P_{L_2,S_2} u_{S \setminus S_2} t_{|L| \setminus (|L_2| \cup S)} \text{sign} (\sigma_{S_2,S \setminus S_2})$$

$$+ \sum_{i \in S \setminus S_2} u_{S \setminus S_2 \setminus \{i\}} Q_{L_2,S_2} \text{sign} (\sigma_{S \setminus S_2 \setminus \{i\}, S_2})$$

$$\sum_{i \in S \setminus S_1} u_{S \setminus S_1 \setminus \{i\}} Q_{L_1,S_1} \text{sign} (\sigma_{S \setminus S_1 \setminus \{i\}, S_1}).$$

Use Lemma 9. \qed

6. THE GEOMETRIC INTERPRETATION

Since the well known paper by W. Fulton and R. MacPherson [4], compactifications of configuration spaces, and complements of hyperplane arrangements [2], became an important topic of algebraic geometry. For a good survey, see [3]. Our geometric interpretation is related to a compactification known as the reciprocal plane [3], Section 5.1, and its super analog.

Let us assume the $z_i$’s linearly span the vector space $\mathbb{A}^n_F$ (otherwise, we can replace $x_1, \ldots, x_n$ by a basis of the span of $z_1, \ldots, z_m$). Denote

$$\mathcal{A} = \{z_1, \ldots, z_m\}, \mathcal{A}_S = \{z_i | i \in S\}.$$ 

Let $R_{A,\mathbb{A}^n_F} = F[t_1, \ldots, t_m]/I$ (see Theorem 1). We can then similarly write $R_{A,W}$ where $A$ is a set of vectors spanning the dual of an $F$-vector space $W$. A stratification of $\text{Spec}(R_{A,\mathbb{A}^n_F})$ can be described as follows. Recall that we have a canonical embedding

$$\mathbb{A}^n_F \setminus Z(z_1 \ldots z_m) \subseteq \text{Spec}(R_{A,\mathbb{A}^n_F}).$$
Call a vector subspace $V \subseteq \mathbb{A}^n_F$ special if $V = Z(A_S)$ for some $S \subseteq \{1, \ldots, m\}$. (Note: $S$ can be empty.) Put also $S_V = \{i \in \{1, \ldots, m\} | V \subseteq Z(z_i)\}$.

(Note [3] that the sets of $i$’s for which the $z_i$’s are linearly independent are the independent sets of a matroid. Then the sets $S_V$ are precisely what is called the flats of this matroid.) For a scheme $X$, denote by $|X|$ the underlying topological space.

**Theorem 10.** ([16], Remark 6) For $V \subseteq \mathbb{A}^n_F$ special, there is a canonical embedding

\[(35) \quad \text{Spec}(R_{A_{SV}}, \mathbb{A}^n_F/V) \rightarrow \text{Spec}(R_{A_A^n}).\]

Composing (35) with $\mathbb{A}^n_F/V \bigcup_{i \in S} Z(z_i) \subseteq \text{Spec}(R_{A_{SV}}, \mathbb{A}^n_F/V)$, (see (34)), induces a decomposition of sets (not topological spaces),

\[(36) \quad |\text{Spec}(R_{A_A^n})| = \prod_{V \subseteq \mathbb{A}^n_F \text{ special}} |(\mathbb{A}^n_F/V) \bigcup_{i \in S_V} Z(z_i)|.\]

**Proof.** We have

\[R_{A_A^n}/(t_i | i \notin S_V) = R_{A_{SV}},\]

which gives the maps (35). (The point is that there is no linear relation between the $z_i$’s in which all but one term would have $i \in S_V$. Thus, all the relations $P_L$ where $L$ contains a term not in $S_V$ are in $(t_i | i \notin S_V)$.)

To prove (36), first note that the images of the inclusions of the components of the right hand side of (36) are clearly disjoint since they correspond to imposing relations $t_i$ with $i \notin S_V$ for some special vector subspace $V$, and inverting all other $t_i$’s. Thus, our task is to show that the canonical map from the right hand side to the left hand side of (36) is onto. To this end, let $Q \in \text{Spec}(R_{A_A^n})$ and let

\[S = \{j \in \{1, \ldots, m\} | Q \in (t_j)\}.\]

Let

\[V = \bigcap_{j \in S} Z(z_j).\]

We want to prove that $S = S_V$. The fact that $S \subseteq S_V$ is automatic. Suppose $j \in S_V \setminus S$. Then $z_j = a_1 z_{j_1} + \cdots + a_k z_{j_k}$ with $j_1 < \cdots < j_k \in S$, $a_1, \ldots, a_k \neq 0 \in F$. Let

\[L = z_j - a_1 z_{j_1} - \cdots - a_k z_{j_k}.\]
Theorem 10 suggests that \( \text{Spec}(R_{A,A^n_F}) \) should have a compactification where on the right hand side of (36) we replace each
\[
\left( \mathbb{A}^n_F/V \right) \setminus \bigcup_{i \in S_V} Z(z_i)
\]
with the corresponding affine space \( (\mathbb{A}^n_F/V) \). In fact, there is such a compactification \( X_{A^n_F,A} \) and it can be described as the Zariski closure of the image of the embedding
\[
\mathbb{A}^n_F \setminus Z(z_1 \ldots z_m) \xrightarrow{(z_1, \ldots, z_m)} \prod_{i=1}^m \mathbb{P}^1_F.
\]
In the terminology of [3], this is an example of what is called a toric compactification. It was also studied, from a different point of view, in [1]. Note that while (37) resembles superficially the formula for the De Concini-Procesi wonderful compactification [2], (37) is in fact quite different. While the wonderful compactification uses projections to (typically) higher-dimensional projective spaces, (37) uses inclusions of the affine coordinates \( z_i \) into \( \mathbb{P}^1_F \).

The projective variety \( X_{A^n_F,A} \) is covered by a system of affine open sets, closed under intersection,
\[
U_{V,T} = \text{Spec} \prod_{j \in T} z_j^{-1} F[t_i, z_j | i \notin S_V, j \in S_V]/(P_L)_{t_{S_V \cap |L|}}
\]
where \( V \) runs through special subspaces of \( \mathbb{A}^n_F \), \( L \) runs through all linear relations among the \( z_i \)'s, and \( T \) is any subset of \( S_V \). The following fact follows from the definitions:

**Lemma 11.** We have
\[
U_{V,T} \cap U_{V',T'} = U_{W,T \cup T' \cup (S_V - S_{V'}) \cup (S_{V'} - S_V)}
\]
where
\[
V + V' \subseteq W = \bigcap_{i \in S_V \cap S_{V'}} Z(z_i)
\]
so
\[
S_V \cap S_{V'} = S_W.
\]
It follows from Theorem 10 that \(|U_{V,T}|\) are open subsets covering \(X_{\mathbb{A}^n_F, A}\). To show the affine schemes \(U_{V,T}\) are reduced (their coordinate rings have no nilpotent elements), we have the following generalization of Theorem 1:

**Theorem 12.** Let \(V\) be a special subspace of \(\mathbb{A}^n_F\). The kernel of the homomorphism of rings

\[
F[t_i, z_j | i \notin S_V, j \in S_V] \to \prod_{i \notin S_V} z_i^{-1} F[z_1, \ldots, z_m]/(Z_V)
\]

given by \(t_i \mapsto z_i^{-1}\), where \(Z_V\) is the set of all linear relations among the \(z_i\)’s, \(i \in S_V\), is

\[
\left(\frac{P_L}{t_{S_V\cap[L]}}\right).
\]

**Proof.** Note that by the proof of Theorem 10, any linear relation among the \(z_i\)’s which involves a \(z_i\) for \(i \notin S_V\) involves at least two of them. Therefore, we can repeat the induction in Section 3 with \(\{1, \ldots, m\}\) replaced by \(\{1, \ldots, m\} \setminus S_V\). \(\square\)

We also have a similar analog of Theorem 3:

**Theorem 13.** Let \(V\) be a special subspace of \(\mathbb{A}^n_F\). The kernel of the homomorphism of rings

\[
F[t_i, z_j | i \notin S_V, j \in S_V] \otimes \Lambda[u_i, dz_j | i \notin S_V, j \in S_V]
\]

\[
\prod_{i \notin S_V} z_i^{-1} F[z_1, \ldots, z_m] \otimes \Lambda[dz_i, \ldots, dz_m]/(Y_V)
\]

given by \(t_i \mapsto z_i^{-1}\), \(u_i \mapsto z_i^{-1} dz_i\), where \(Y_V = Z_V \cup \{dL | L \in Z_V\}\), is

\[
\left(\frac{P_{L,S}}{t_{S_V\cap[L]}}\right)
\]

where \(L\) runs through the linear relations among the \(z_i\)’s and \(S \subseteq |L|\).

Accordingly, we have a superscheme analog \(\tilde{X}_{\tilde{A}^n_F, A}\) of \(X_{\mathbb{A}^n_F, A}\). Here by a superscheme, we mean a locally ringed space by \(\mathbb{Z}/2\)-graded commutative rings which is locally isomorphic to \(\text{Spec}\) of a \(\mathbb{Z}/2\)-graded commutative ring (see e.g. [21]). \(\tilde{X}_{\tilde{A}^n_F, A}\) is covered by super-affine open subsets

\[
\tilde{U}_{V,T} = \text{Spec} \prod_{j \in T} z_j^{-1} F[t_i, z_j | i \notin S_V, j \in S_V]
\]

\[
\otimes \Lambda[u_i, dz_j | i \notin S_V, j \in S_V]/\left(\frac{P_{L,S}}{t_{S_V\cap[L]}}\right).
\]
We clearly have 

$$|\tilde{U}_{V,T}| = |U_{V,T}|$$

and for $|U_{V',T'}| \subseteq |U_{V,T}|$, $\tilde{U}_{V',T'}$ is a complement of the zero set of an (even) principal ideal in $\tilde{U}_{V,T}$. Therefore, $\tilde{X}_{\tilde{A}^n}^{U_{V,T}}$ can be defined as the colimit of the $\tilde{U}_{V,T}$’s in the category of superschemes.

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