FINITE-TYPE-DYCK SHIFT SPACES

MARIE-PIERRE BÉAL, MICHEL BLOCKELET, AND CĂTÂLIN DIMA

ABSTRACT. We study some basic properties of sofic-Dyck shifts and finite-type-Dyck shifts. We prove that the class of sofic-Dyck shifts is stable under proper conjugacies. We prove a Decomposition Theorem of a proper conjugacy between edge-Dyck shifts into a sequence of Dyck splittings and amalgamations.

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1. INTRODUCTION

Shifts (or subshifts) of sequences are defined as sets of bi-infinite sequences of symbols over a finite alphabet avoiding a given set of finite factors (or blocks) called forbidden factors [9]. In [2], we defined the notions of sofic-Dyck shifts and finite-type-Dyck shifts which extend the notion of Markov-Dyck shifts introduced by Krieger and Matsumoto (see [6], [3], [4], [10], [11], [8], [7]). Sofic-Dyck shifts are shifts of sequences whose set of forbidden (or allowed) blocks is a visibly pushdown language [2]. Visibly pushdown languages [1] is a strict subclass of context-free languages which is closed by intersection and complementation.

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A sofic-Dyck shift is accepted by a finite-state automaton (or a labelled graph) equipped with a graph semigroup, over an alphabet which is partitioned into three disjoint sets of symbols, the call symbols, the return symbols, and internal symbols (for which no matching constraints are required). Such automata are called Dyck automata. Finite-type Dyck shifts are accepted by Dyck automata which are local (or definite).

In this paper, we study some basic properties of these classes of context-free shifts. We introduce the notion of proper block map. A proper block map \( \Phi \) is a block map between two shifts over two three-type alphabets such that \( \Phi(x) = y \) implies \( y_i \) and \( x_i \) have the same type for any integer \( i \) Roughy speaking, a call (resp. return, internal) symbol is mapped to a call symbol (resp. return, internal). We show that a subshift is a sofic-Dyck shift if and only if it is the proper factor of a finite-type-Dyck shift and that the class of sofic-Dyck shifts is stable under proper conjugacies.

We define two notions of in and out state-splitting map, in and out state-amalgamation map, together with the notions of in-split, out-split, in-amalgamation, and out-amalgamation Dyck automaton. Amalgamation and splitting maps are proper conjugacies. The first notion is the classical notion of state splittings. The second one, called trim splitting, allows one to remove some edges or matchings which are not essential.

We define the notion of edge-Dyck shifts. They play the same role as edge shifts for the sofic-Dyck class. An edge-Dyck shift is defined by a Dyck graph whose edges are partitioned into three types of edges: call edges, return edges and internal edges. We prove a Decomposition Theorem for edge-Dyck shifts. We show that two edge-Dyck shifts are conjugate through a proper conjugacy if and only there is a sequence of splittings and amalgamations allowing to transform one Dyck graph into the other one. We use trim splittings for the final step. Since trim in-splittings do not commute (as classical in-splittings), the result does not allow us to derive a decision process for the proper conjugacy of one-sided edge-Dyck shifts (see [5] or [9]) for the notion of one-sides shifts of sequences.

2. Sofic-Dyck shifts and finite-type-Dyck shifts

2.1. Shifts. We briefly introduce below some basic notions of symbolic dynamics. We refer to [9] [5] for an introduction to this theory. Let \( A \) be a finite alphabet. The shift transformation \( \sigma \) on \( A^\mathbb{Z} \) is defined by

\[
\sigma((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}},
\]

for \((x_i)_{i \in \mathbb{Z}} \in A^\mathbb{Z}\).

A subshift (or shift) of \( A^\mathbb{Z} \) is a closed shift-invariant subset of \( A^\mathbb{Z} \) equipped with the product of the discrete topology. If \( X \) is a shift, a finite word is a block of \( X \) if it appears as a factor of some bi-infinite sequence of \( X \). We denote by \( B(X) \) the set of factors (or blocks) of \( X \) and by \( B_n(X) \) the set of blocks of length \( n \) of \( X \). If \( x = (x_i)_{i \in I} \) is a word and \( i,j \in I \) with \( i \leq j \), we denote by \( x[i,j] \) the factor \( x_i x_{i+1} \cdots x_j \) of \( x \). Let \( F \) be a set of finite words
over the alphabet $A$. We denote by $X_F$ the set of bi-infinite sequences of $A^\mathbb{Z}$ avoiding each word of $F$. The set $X_F$ is a shift and any shift is the set of bi-infinite sequences avoiding each word of some set of finite words. When $F$ can be chosen finite (resp. regular), the shift $X_F$ is called a shift of finite type (resp. sofic). When $F$ can be chosen context-free, the shift $X_F$ is called a context-free shift.

Let $X \subset A^\mathbb{Z}, Y \subset B^\mathbb{Z}$ be subshifts and $m, a$ be nonnegative integers. A map $\Phi : X \rightarrow Y$ is called an $(m,a)$-local map (or an $(m,a)$-block map) if there exists a function $\phi : B^{m+a+1}(X) \rightarrow B$ such that, for all $x \in X$ and any $i \in \mathbb{Z}$, $\Phi(x)_i = \phi(x_{i-m} \cdots x_{i-1}x_ix_{i+1} \cdots x_{i+a})$. A block map is a map which is an $(m,a)$-block map for some nonnegative integers $(m,a)$. The function $\phi$ is called a local function associated to $\Phi$ and $(m,a)$ and $\Phi(X)$ is said to be a factor of $X$.

Let $X$ be a shift over the alphabet $A$, the higher-block shift of order $n$ is the shift denoted $X[n]$ over $B = A^n$ defined as the image of $X$ by an $(m,a)$-block map such that $m + a + 1 = n$ and whose local function is the map $\phi : A^n \rightarrow B$ which is the identity map over $A^n$. This map is a conjugacy. The higher-block shift of order $n$ is also called the shift of sequences of overlapping blocks of lengths $n$ of $X$.

### 2.2. Dyck automata

We define the notion of Dyck automata and sofic-Dyck shift introduced in [2].

We consider an alphabet $A$ which is a disjoint union of three finite sets of letters ($A_c, A_r, A_i$). The sets $A_c$, $A_r$ and $A_i$ are called the call alphabet, the return alphabet, and the internal (or local) alphabet respectively.

Let $A = (Q, E, A)$ be a directed labelled graph (or automaton) on a finite alphabet $A$ with a finite set of vertices $Q$ and a (finite) set of edges $E \subset Q \times B \times Q$. We say that $A$ is deterministic if there is at most one edge with a given label and a given starting state.

Let $M$ be a set of pairs $(p, a, q), (r, b, s)$ of edges of $A$ with $a \in A_c$ and $b \in A_r$. It is called the set of matched edges. We define the graph semigroup $S$ associated to $(A, M)$ as the semigroup generated by the set $E \cup \{x_{pq} | p, q \in Q\} \cup \{0\}$ equipped with the following relations.

\[
\begin{align*}
0s &= s0 = 0 & \text{for } s \in S, \\
x_{pq}x_{qr} &= x_{pr} & \text{for } p, q, r \in Q, \\
x_{pq}x_{rs} &= 0 & \text{for } p, q, r, s \in Q, q \neq r, \\
(p, \ell, q) &= x_{pq} & \text{for } p, q, \ell \in A_i, \\
(p, a, q)x_{qr}(r, b, s) &= x_{ps} & \text{for } ((p, a, q), (r, b, s)) \in M,
\end{align*}
\]
A bi-infinite path is admissible matched with (\((p,a,q),(r,b,s)\) for \(a \in A_c, b \in A_r, ((p,a,q),(r,b,s)) \notin M\),

\[(p,a,q)(r,b,s) = 0,\]

for \(p,q,r,s \in Q, q \neq r, a, b \in A\),

\[x_{pp}(p,a,q) = (p,a,q) x_{qq} \quad \text{for} \quad p,q \in Q, \ a \in A,

\]

\[x_{pq}(r,a,s) = 0 = (r,a,s)x_{tu} \quad \text{for} \quad p,q \in Q, \ a \in A, \ q \neq r, s \neq t.\]

Note that \(X = (x_{pq})_{p,q \in Q}\) satisfies \(X^2 = X\) assuming that \(x_{pq} + x_{qp} = x_{pq}\).

If \(\pi\) is a finite path of \(\mathcal{A}\), we denote by \(f(\pi)\) its image in the graph semigroup \(S\). A finite path \(\pi\) of \(\mathcal{A}\) such that \(f(\pi) \neq 0\) is said to be an admissible path of \((\mathcal{A}, M)\) (or of \(\mathcal{A}\) when \(M\) is understood). A finite word is admissible for \((\mathcal{A}, M)\) if it is the label of some admissible path of \((\mathcal{A}, M)\).

A bi-infinite path is admissible if all its finite factors are admissible. A path \(\pi\) such that \(f(\pi) = x_{pq}\) is called a Dyck path going from \(p\) to \(q\). It is called a prime Dyck path if it has no strict prefix which is also a Dyck path.

The sofic-Dyck shift \(X \subset A^\mathbb{Z}\) is defined as the set of labels of admissible bi-infinite paths of \((\mathcal{A}, M)\). The pair \((\mathcal{A}, M)\) is called a Dyck automaton over \(A\) and the shift is denoted by \(X_{(\mathcal{A}, M)}\).

Note that there may exist in \(\mathcal{A}\) several bi-infinite paths having the same bi-infinite label. We say that the sofic-Dyck shift is accepted (or presented) by the Dyck automaton \((\mathcal{A}, M)\).

The full-Dyck shift over the alphabet \(A = (A_c, A_r, A_i)\), denoted \(X_A\), is the shift of all sequences accepted by the one-state Dyck automaton \((\mathcal{A} = (Q = \{p\}, E, A), M)\) containing each loop \((p,a,p)\) for \(a \in A\), and where each edge \((p,a,p)\) is matched with each edge \((p,b,p)\) when \(a \in A_c, b \in A_r\). Hence \(X_A\) is the set of all sequences over \(A_r \cup A_c \cup A_i\).

For \(m, a\) nonnegative integers, we define the \((m,a)\)-Dyck-De-Bruijn automaton as the Dyck automaton \((\mathcal{A} = (Q = A^m \times A^a, E, A), M)\) containing the edges \(((au, bv), (ub, vc))\) and where each edge \(((au, bv), (ub, vc))\) is matched with \(((a'u', b'v'), b'(u'b', v'c'))\) when \(b \in A_c\) and \(b' \in A_r\). Any \((m,a)\)-Dyck-De-Bruijn automaton over the alphabet \(A = (A_c, A_r, A_i)\) accepts the full-Dyck shift over \(A\). We denote by \(\text{In}(p)\) (resp. \(\text{Out}(p)\) the set of edges coming in \(p\) (resp. going out of \(p\)).

A Dyck automaton is normalized if any label of a finite admissible path is a block of the Dyck shift accepted the automaton. This means that for any finite word \(u\) labeling a finite admissible path \(\pi\) of the automaton there is a bi-infinite path labelled by a word \(x\) containing \(u\) as factor. Note that this does not mean that \(\pi\) itself is extensible to a bi-infinite admissible path. It is shown in [2] that any sofic-Dyck shift is accepted by a normalized Dyck automaton.

Let \((G = (Q, E), M)\) be a Dyck automaton. An edge which belongs to a bi-infinite admissible path is called essential. A pair of edges \((e, f)\), where \(e\) is a call edge and \(f\) a return edge, for which there is a bi-infinite admissible path \((p_i \to p_{i+1})_{i \in \mathbb{Z}}\) such that \(e = (p_0 \to p_1), f = (p_j \to p_{j+1})\) for some \(j > 0\) and \((p_i \to p_{i+1})_{1 \leq i \leq j-1}\) is a Dyck path or is empty, is called an essential
matched pair. A Dyck automaton is essential if each edge is essential and each matched pair of edges is essential.

The following proposition was proved in [2].

**Proposition 1.** The set of blocks of a sofic-Dyck shift is a visibly-pushdown language. Conversely, any factorial extensible visibly-pushdown language is the set of blocks of a sofic-Dyck shift.

Let $(A = (Q, E, A), M)$ and be $(A' = (Q', E', A'), M')$ be two Dyck automata. We define their product as the Dyck automaton $(A, M) \times (A', M') = ((Q \times Q', F, B), N)$ over $B$ where

- $B = (A_c \times A'_c, A_r \times A'_r, A_i \times A'_i)$,
- $((p, p'), (a, a'), (q, q'))$ belongs to $F$ if and only if $(p, a, q)$ belongs to $E$ and $(p', a', q')$ belongs to $E'$.
- $(((p_1, p'_1), (a_1, a'_1), (q_1, q'_1)), ((p_2, p'_2), (a_2, a'_2), (q_2, q'_2)))$ belongs to $N$ if and only if $((p_1, a_1, q_1), (p_2, a_2, q_2))$ belongs to $M$ and $((p'_1, a'_1, q'_1), (p'_2, a'_2, q'_2))$ belongs to $M'$.

**Proposition 2.** The intersection of two sofic-Dyck shifts accepted by $(A, M)$ and $(B, N)$ is a sofic-Dyck shift accepted by the product of $(A, M)$ and $(B, N)$. The product of two local Dyck automata is a local Dyck automaton.

**Proof.** The proof is straightforward. □

**Proposition 3.** Let $X, Y$ be two sofic-Dyck (resp. finite-type-Dyck) shifts accepted by $(A, M)$ and $(B, N)$ respectively, $X \cap Y$ is a sofic-Dyck (resp. finite-type-Dyck) shift accepted by the product $(A, M) \times (B, N)$.

**Proof.** The proof follows from Proposition 2. The first part is also a consequence of the known fact that the intersection of two visibly pushdown languages is a visibly pushdown language (see for instance [1]). □

Let $(A, M)$ be a Dyck automaton over $A = (A_c, A_r, A_i)$. Let $m, a$ be nonnegative integers. We say that $A$ is $(m, a)$-local (resp. $(m, a)$-weak-local if whenever two paths (resp. two admissible paths) of length $m + a$ $(p_i, a_i, p_{i+1})$, $m \leq i \leq a - 1$, $(q_i, a_i, q_{i+1})$, $m \leq i \leq a - 1$, of $A$ have the same label, then $p_0 = q_0$. We say that $A$ (or $(A, M)$) is local if it is $(m, a)$-local for some nonnegative integers $m$ and $a$.

Note that if $A$ is deterministic local if there is a nonnegative integer $m$ such that for each word $x$ of length $m$ of $A$, all paths of $A$ labelled by $x$ end in a same state (depending on $x$).

A finite-type-Dyck shift is a sofic-Dyck shift presented by a weak local Dyck automaton. As shown in Proposition 4, it is also presented by a local Dyck automaton.

**Proposition 4.** If $X$ is a finite-type-Dyck shift, then there are nonnegative integers $m, a$ such that $X$ is accepted by an $(m, a)$-local Dyck automaton.
Example 1. The full-Dyck shift $X_A$ over $A = (A_c, A_r, A_i)$ is a finite-type-Dyck shift. It is the set of bi-infinite sequences presented by the automaton $(A, M)$ of Figure 1. On the right part of this figure is pictured a Dyck automaton $(A', M)$ over $A' = (A_c, A_r, A_i')$ where the edge labelled by $a$ is matched with the edge labelled by $b$, and the edge labelled by $c$ is matched with the edge labelled by $d$. It accepts a finite-type-Dyck shift which is not a full-Dyck shift.

2.3. Proper block-maps. A block map $\Phi : X_A \to X_{A'}$, where $A = (A_c, A_r, A_i)$ and $A' = (A'_c, A'_r, A'_i)$, is called proper if $\Phi(x)_j \in A'_i$ (resp. $A'_r$, $A'_c$) whenever $x_j \in A_c$ (resp. $A_r$, $A_i$) for any $j \in \mathbb{Z}$.

Proposition 5. A subshift is a sofic-Dyck shift if and only it is the proper factor of a finite-type-Dyck shift.
Proof. Let \( (A = (Q, E), M) \) be a Dyck automaton over \( A = (A_c, A_r, A_i) \) accepting a sofic-Dyck shift \( X \).

Let \( (B = (Q, F, E), N) \) be the \((1,0)\)-local Dyck automaton over \( E = (E_c, E_r, E_i) \) where \( E_c \) (resp. \( E_r, E_i \)) is the set of edges labelled by \( A_c \) (resp. \( A_r, A_i \)). The edges of \( B \) are \((p, e, q)\) for \( e = (p, a, q) \in E \). Two edges \((p, e, q)\) and \((p', e', q')\) are matched in \( N \) if and only if \( e \) and \( e' \) are matched in \( M \).

Let \( \Phi \) be the \((0,0)\)-block map with local function \( \phi(p, a, q) = a \). We have \( \Phi(X(B, N)) = X \). Hence \( X \) is a factor of a finite-type-Dyck shift.

Conversely, suppose that \( X \) is a shift space for which there is a finite-type-Dyck shift \( Y \) and a proper block map \( \Phi \) from \( Y \) onto \( X \). Suppose that \( \Phi \) has memory \( m \) and anticipation \( a \) with a local function \( \phi \). By increasing \( m \) if necessary, we can assume that \( Y \) is accepted by an \((m,a)\)-local Dyck automaton \( (A = (Q, E), M) \) over \( A = (A_c, A_r, A_i) \).

We define the Dyck automaton \( (T = (R, F, B), N) \) (called a transducer) with \( R = Q \times A^m \times A^a \), \( B = (A_c \times A'_c, A_r \times A'_r, A_i \times A'_i) \).

An edge \( ((p, (au, bv)), (b, \phi(auvbc)), (q, (ub, vc))) \) with \( a, b, c \in A \), belongs to \( F \) if and only if \((p, b, q)\) belongs to \( E \) and \( p \) (resp. \( q \)) is the state \( p_0 \) (resp. \( q_0 \)) of any path labelled by \(aubv \) (resp. \(ubvc\)) going from \( p_{-m} \) to \( p_a \) (resp. going from \( q_{-m} \) to \( q_a \)).

A pair \( ((p, (au, bv)), (b, \phi(auvbc)), (q, (ub, vc))) ((r, (dw, ez)), (e, \phi(dwezf)), (s, (we, zf))) \) of \( F \) with \( a, b, c, d, e, f \in A \), belongs to \( N \) if and only if \((p, b, q), (r, e, s)\) belongs to \( M \). The Dyck automaton obtained from \( (T, N) \) by discarding the second component of all labels of edges of \( T \) is an \((m,a)\)-local Dyck automaton accepting \( Y \). The image \( X \) of \( Y \) by \( \Phi \) is accepted by Dyck automaton obtained from \( (T, N) \) by discarding the first component of all labels of edges of \( T \). Hence \( X \) is sofic-Dyck.

Corollary 1. A proper factor of a sofic-Dyck shift is a sofic-Dyck shift.

Proof. Suppose that \( \Phi \) is a proper block map from \( Y \) onto \( X \) and that \( Y \) is a sofic-Dyck shift. By Proposition [5] there is a proper block map \( \Psi \) from \( Z \) onto \( Y \) with \( Z \) a finite-type-Dyck shift. Since \( \Phi \circ \Psi \) is a proper block map, we conclude again by Proposition [5] that \( X \) is sofic-Dyck. □
Proposition 6. If $X$ is a sofic-Dyck (resp. finite-type-Dyck) shift, then the higher-block shifts of $X$ are sofic-Dyck (resp. finite-type-Dyck) shifts.

Proof. The proposition is a consequence of Proposition 5. □

2.4. Dyck state-splitting. In this section, we define two notions of state splitting for Dyck automata and sofic-Dyck shifts: state-splitting and trim state-splitting.

The notion of state-splitting is an extension of the notion of state splitting for sofic shifts. Let $(A = (Q,E,A),M)$ be a Dyck automaton over $A = (A_c,A_r,A_i)$. Let $p \in Q$ and $P$ a partition $(P_1,\ldots,P_k)$ of size $k$ of the edges coming in $p$. We define a Dyck automaton $(A' = (Q',E',A),M')$ by

- $Q' = Q \setminus \{p\} \cup \{p_1,\ldots,p_k\}$,
- $(q,a,r) \in E'$ if $q,r \neq p$ and $(p,a,r) \in E$,
- $(q,a,p_i) \in E'$ for each $1 \leq i \leq k$ such that $(q,a,p) \in P_i$,
- $(p_i,a,r) \in E'$ for each $1 \leq i \leq k$ such that $(p,a,r) \in E$.
- $M'$ is the set of pairs of edges $(q,a,r),(s,b,t)$ where $a \in A_r,b \in A_c$ such that $(\pi(q),a,\pi(r)),(\pi(s),b,\pi(t)) \in M$ where $\pi(q) = q$ for $q \neq p$ and $\pi(p_i) = p$ for $1 \leq i \leq k$.

This automaton $(A',M')$ is called a Dyck in-split of $(A,M)$ and $(A,M)$ is called a Dyck in-amalgamation of $(A',M')$. If the classes of the partition $P$ are singletons, the Dyck in-split graph (resp. map) is called a complete Dyck in-split graph (resp. map). The notions of Dyck out-split and Dyck out-amalgamation are defined similarly.

Note that, if $(A,M)$ is a Dyck amalgamation of $(A',M')$, then the map $\pi : E' \to E$ where $\pi(p,a,q) = (\pi(p),a,\pi(q))$ defines a $(0,0)$-block conjugacy $(A',M')$ onto $(A,M)$.

![Figure 3. A Dyck state-splitting of the state 1 into 1’ and 1’'. The alphabet is $A = \{a,b,\{\bar{a},\bar{b}\}\}$. The edges labelled by $a$ (resp. $b$) are matched with the edges labelled by $\bar{a}$ (resp. $\bar{b}$).](image)

The notion of trim state splitting is defined on Dyck automata is the following. Let $(A = (Q,E,A),M)$ be a Dyck automaton over $A = (A_c,A_r,A_i)$. Let $p \in Q$ and $P$ a partition $(P_1,\ldots,P_k)$ of size $k$ of the edges coming in $p$. We define a Dyck automaton $(A' = (Q',E',A),M')$ by

- $Q' = Q \setminus \{p\} \cup \{p_1,\ldots,p_k\}$,
• \((q, a, r) \in E'\) if \(q, r \neq p\) and \((p, a, r) \in E\),
• \((q, a, p_i) \in E'\) for each \(1 \leq i \leq k\) such that \((q, a, p) \in \mathcal{P}_i\),
• \((p_i, a, r) \in E'\) for each \(1 \leq i \leq k\) such that \((p, a, r) \in E\).
• \(M'\) is the set of pairs of edges \((q, a, r), (s, b, t)\) where \(a \in A_r, b \in A_c\) such that \((\pi(q), a, \pi(r)), (\pi(s), b, \pi(t)) \in M\) where \(\pi(q) = q\) for \(q \neq p\) and \(\pi(p_i) = p\) for \(1 \leq i \leq k\).
• Edges \((p_i, a, r)\) which are not essential in \((A', M')\) are removed from \(E'\). Matched pairs \((q, a, r), (p_i, b, t)\) or \((p_i, b, t), (q, a, r)\) which are not essential are removed from \(M'\).

This automaton \((A', M')\) is called a **trim Dyck in-split** of \((A, M)\) and \((A, M)\) is called a **trim Dyck in-amalgamation** of \((A', M')\). Note that a trim Dyck in-split automaton is an essential Dyck automaton. Since removing unessential edges or matched pairs does not affect the bi-infinite admissible paths of a Dyck automaton, the map \(\pi : E' \to E\) defined by \(\pi(p, a, q) = (\pi(p), a, \pi(q))\) defines a \((0, 0)\)-block conjugacy from \((A', M')\) onto \((A, M)\).

**Figure 4.** A trim Dyck state-splitting of the state 1 into 1' and 1''. The alphabet is \(A = \{\{a, b\}, \{\bar{a}, \bar{b}\}, \{i\}\}\). The edges labelled by \(a\) (resp. \(b\)) are matched with the edges labelled by \(\bar{a}\) (resp. \(\bar{b}\)).

**Proposition 7.** The in-split (resp. out-split) of a Dyck automaton accepts the same sofic-Dyck shift. The in-split (resp. out-split) of an \((m,a)\)-local Dyck automaton is an \((m+1,a)\)-local (resp. \((m,a+1)\)-local) Dyck automaton.

2.5. Characterization of finite-type-Dyck shifts. In this section we give an intrinsic characterization of finite-type-Dyck shifts.

Let \(A = (A_c, A_r, A_l)\) be an alphabet. Let \(m, a\) be nonnegative integers, \(F\) be a (finite) set of words of length \(m + a + 1\) over \(A\), and \(G\) be a (finite) set of pairs \((u_m \cdots u_a, v_m \cdots v_a)\) of words of length \(m + a + 1\) such that \(u_0 \in A_c, v_0 \in A_r\). We define a semigroup \(S(F,G)\) generated by the set \(\{(au, ub) \mid u \in A^{m+a}, a, b \in A\} \cup \{x_{u',v'} \mid u', v' \in A^{m+a}\} \cup \{0\}\) with the
following relations.

\[
0s = s0 = 0 \quad \text{for } s \in S,
\]

\[
x_{u',v'}x_{v',w'} = x_{u',w'} \quad \text{for } u', v', w' \in A^{m+a},
\]

\[
x_{u',v'}x_{v',s'} = 0 \quad \text{for } u', v', s' \in A^{m+a}, v \neq r,
\]

\[
(au, ub) = 0 \quad \text{if } aub \in F,
\]

\[
(au, ub) = x_{au,ub} \quad \text{if } aub \notin F, b \in A_i,
\]

\[
(au, ub)x_{ub,cv}(cv, vd) = x_{au,vd} \quad \text{for } b \in A_r, d \in A_c,
\]

\[
((au, ub), (cv, vd)) \notin G, 
\]

\[
((au, ub), (cv, vd)) \in G, 
\]

\[
ub \neq vc, 
\]

\[
au,ub \quad \text{for } u' \neq au, v' \neq ub.
\]

The image \( f(w) \) of a finite word \( w \) of length greater than or equal to \( m + a \) in \( S(F, G) \) is defined as the product of its overlapping blocks of length \( m + a + 1 \) in \( S(F, G) \). A word \( w \in A^{m+a+1} \) is said to be admissible for \( S(F, G) \) if and only if \( f(w) \neq 0 \).

A language \( L \) of finite words over \( A \) is said to be strictly-locally-Dyck testable if and only there are nonnegative integers \( m, a \), finite sets \( F, G \) as above such that, if \( w \) has a length at least \( m + a + 1 \), then \( w \in L \) if and only if \( w \) is admissible for \( S(F, G) \).

**Proposition 8.** A sofic-Dyck shift is a finite-type-Dyck shift if and only if its set of blocks is strictly-locally-Dyck testable.

**Proof.** If \( X \) is a finite-type-Dyck shift over \( A \), it is accepted by an \((m, a)\)-local Dyck automaton \((A, M)\) for some nonnegative integers \( m, a \). Let \( F \) be the set of word of \( A^{m+a+1} \) which are not blocks of \( X \) and \( G \) be the set of pairs of words \((u_m \cdots u_a, v_m \cdots v_a)\) of length \( m + a + 1 \) such that \( u_0 \in A_c, v_0 \in A_r \) and \((p, u_0, q), (r, v_0, s) \notin M \), where \( p \) (resp. \( q, r, s \)) is the shared state \( p_0 \) of any path \((p_i, u_i, p_{i+1})_{-m \leq i \leq a-1}\) (resp. \((p_i, u_i, p_{i+1})_{-m+1 \leq i \leq a-1}, (p_i, v_i, p_{i+1})_{-m \leq i \leq a-1}, (p_i, v_i, p_{i+1})_{-m+1 \leq i \leq a-1}\)). Then \( B(X) \) with the sets \( F, G \) is a strictly-locally-Dyck testable.

Conversely, if there are nonnegative integers \( m, a \) and sets \( F, G \) such that \( B(X) \) is the set of admissible words for \( S(F, G) \) union the factors of length less than \( m + a + 1 \) of these words. Let \((A, M)\) be the Dyck automaton obtained from the \((m, a)\)-Dyck-De-Bruijn automaton by discarding the edges \((au, bv), b, (ub, vc)\) such that \( aub \in F \) and setting that each edge \((au, bv), b, (ub, vc)\) is matched with \((a'u', b', (u'b', v'c'))\) when \( b \in A_c, b' \in A_r \), and \((aubcv, a'u'b'v'c') \notin G \). Then \((A, M)\) is an \((m, a)\)-local Dyck automaton accepting \( X \). Thus \( X \) is a finite-type-Dyck shift. \( \square \)
3. Edge-Dyck shifts

A Dyck graph \( (\mathcal{G} = (Q, E \subset Q \times Q), M) \) is composed of a graph \( \mathcal{G} \), where the edges \( E = (E_c, E_r, E_i) \) are partitioned into three categories: the call edges (denoted \( E_c \)), the return edges \( E_r \), and the internal edges (denoted \( E_i \)). The set \( M \) is a set of pairs \((e, f)\), with \( e \in E_c, f \in E_r \) of matched edges. A finite path in \((\mathcal{G}, M)\) is admissible if it is admissible in the Dyck automaton \((\mathcal{A}, M)\) obtained from \((\mathcal{G}, M)\) by labeling an edge of \( \mathcal{G} \) by itself. An infinite path is admissible if all its finite factor are admissible.

An edge-Dyck shift is the set of admissible bi-infinite paths of a Dyck graph. The edge-Dyck shift defined by the Dyck graph \((\mathcal{G}, M)\) is denoted by \( X_{(\mathcal{G}, M)} \).

A Dyck graph is essential if each edge and each matched pair of edges are essential.

The following proposition shows that each finite-type Dyck shifts are properly conjugate to an edge-Dyck shift.

**Proposition 9.** Each finite-type-Dyck shift is properly conjugate to a finite-type edge-Dyck shift.

**Proof.** Let \((\mathcal{A} = (Q, E, A), M)\) be an \((m, a)\)-local Dyck automaton accepting the finite-type-Dyck shift \( X \) over \( A = (A_c, A_r, A_i) \). Without loss of generality, we may assume that for any pair of states \( p, q \), there is at most one edge going from \( p \) to \( q \). Otherwise we build the automaton \((\mathcal{A}' = (Q \times E, E', A), M')\) where the states are pairs \((p, e)\) where \( p \in Q \) and \( e \) is an edge coming in \( p \). There is an edge \(((p, e), a, (q, f))\) in \( E' \) whenever \( f = (p, a, q) \in E \). The pair of the two edges \(((p, e), a, (q, f))\) and \(((r, g), b, (s, g))\) belongs to \( M' \) whenever the pair \((p, a, q)\), \((r, b, s)\) belongs to \( M \). The Dyck-shift \( X \) is still accepted by \((\mathcal{A}', M')\).

Let \( E = (E_c, E_r, E_i) \) be the alphabet of call edges, return edges and internal edges of \( E \). Let \( N \) be the set of pairs \((p, q), (r, s)\) such that there is \( a \in A_c \) and \( b \in A_r \) and \((p, a, q), (r, b, s) \in M \). The Dyck graph \((\mathcal{G} = (Q, E), N)\) over the alphabet \( E \) defines an edge-Dyck shift \( Y \). There is a proper \((0,0)\)-local block map \( \Phi \) from \( Y \) to \( X \) defined by \( \phi(p, a, q) = a \). The \((m, a)\)-block map \( \Psi : X \to Y \) defined by \( \psi(auvb) = (p, b, q) \), where \( p \) (resp. \( q \)) is the state \( p_0 \) (resp. \( q_0 \)) of any path labelled by \( auvb \) (resp. \( ubvc \)) going from \( p_{-m} \) to \( p_a \) (resp. going from \( q_{-m} \) to \( q_a \)), is the inverse of \( \Phi \). \qed

The notion of Dyck splittings of Dyck graphs is a particular case of Dyck splittings of Dyck automata.

Let \((\mathcal{G} = (Q, E), M)\) be a Dyck automaton over \( E = (E_c, E_r, E_i) \). Let \( p \in Q \) and \( \mathcal{P} \) a partition \((\mathcal{P}_1, \ldots, \mathcal{P}_k)\) of size \( k \) of the edges coming in \( p \). We define a Dyck automaton \((\mathcal{G}' = (Q', E', M'))\) by

- \( Q' = Q \setminus \{p\} \cup \{p_1, \ldots, p_k\} \),
- \((q, r) \in E'_c\) (resp. \( E'_r, E'_i\)) if \( q, r \neq p \) and \((p, r) \in E_c\),
- \((q, p_i) \in E'\) for each \( 1 \leq i \leq k \) such that \((q, p) \in \mathcal{P}_i\),
- \((p_i, r) \in E'\) for each \( 1 \leq i \leq k \) such that \((p, r) \in E\).
\( M' \) is the set of pairs of edges \((q, r), (s, t)\) such that \((\pi(q), a, \pi(r)), (\pi(s), b, \pi(t)) \in M\) where \(\pi(q) = q\) for \(q \neq p\) and \(\pi(p_i) = p\) for \(1 \leq i \leq k\).

This Dyck graph \( (G', M') \) is called a Dyck in-split graph of \((A, M)\) and \((A, M)\) is called a Dyck in-amalgamation graph of \((A', M')\).

\[ \begin{array}{c}
3 \\
\alpha \\
1 \\
\alpha \\
5 \\
\beta \\
4 \\
\beta \\
\end{array} \quad \begin{array}{c}
3 \\
\alpha \\
1 \\
\alpha \\
5 \\
\beta \\
4 \\
\beta \\
\end{array} \]

FIGURE 5. A trim Dyck in-amalgamation of \((G', M')\) to \((G, M)\). The blue edges are call edges, the red edges are return edges and the black edges are internal edges. In \(G'\) (on the left) the pairs of matched edges are \((5, 1')(1', 3)\) and \((4, 1')(1'', 2)\). In \(G\) (on the right), the pairs of matched edges are \((5, 1)(1, 3)\) and \((4, 1)(1, 2)\). Call edges are labelled in \(\Sigma\) and return edges are labelled in \(\Psi(\Sigma)\), where \(\Sigma\) is a finite alphabet and \(\Sigma\) is a disjoint copy of \(\Sigma\). A call edge \((p, \alpha, q)\) and a return edge \((r, u, s)\) are matched if and only if \(\alpha \in u\). Note that \((5, 1)(1, 2)\) and \((4, 1)(1, 3)\) are not admissible and thus the amalgamation defines a proper conjugacy between the admissible bi-infinite paths of \(G'\) and \(G\).

The following proposition shows that Dyck in-amalgamation maps commute. A similar result holds for Dyck out-amalgamation maps.

**Proposition 10.** Let \(X_{(G_i, M_i)}\) be an edge-Dyck shift. Let \(\Phi\) (resp. \(\Psi\) be a Dyck in-amalgamation transforming \((G_1, M_1)\) to \((G_2, M_2)\) (resp. \((G_3, M_3)\). Then there is a Dyck graph \((G_4, M_4)\) and a Dyck in-amalgamation \(\Omega\) (resp. \(\Theta\)) from \(X_{(G_2, M_2)}\) (resp. \(X_{(G_3, M_3)}\)) to \(X_{(G_4, M_4)}\) such that \(\Phi \circ \Omega = \Psi \circ \Theta\).

**Proof.** Suppose that \(X_i = X_{(G_i, M_i)}\) where \(G_i = (Q_i, E_i)\) for \(i = 1, 2, 3\). Let us assume that the states \(p_1, \ldots, p_k\) of \(Q_1\) are amalgamated to a state \(p\) of \(Q_2\) and that the vertices \(q_1, \ldots, q_\ell\) of \(Q_1\) are amalgamated to a vertex \(q\) of \(Q_3\).

Let us first assume that the vertices \(p_1, \ldots, p_k\) and \(q_1, \ldots, q_\ell\) are all distinct. We define \((G_4, M_4)\) as the in-amalgamation of \((G_2, M_2)\) obtained by amalgamating the vertices \(p, q_1, \ldots, q_\ell\) to a vertex \(q\). It is also equal to the in-amalgamation of \(X_3\) obtained by amalgamating the vertices \(q, p_1, \ldots, p_k\) to a vertex \(p\). Let us now assume that \(p_1 = q_1, \ldots, p_n = q_n\) for some integer \(1 \leq n \leq \min(k, \ell)\). This implies the following properties:

- for any \(1 \leq j \leq k, 1 \leq j' \leq \ell\), one has \((p_j, r) \in E_{1,c}\) (resp. \(E_{1,r}, E_{1,i}\)) if and only if \((p_{j'}, r) \in E_{1,c}\) (resp. \(E_{1,r}, E_{1,i}\)) for \(1 \leq j, j' \leq k\).
Figure 6. The commutation of Dyck in-amalgamation maps. If $X(G_2, M_2)$ and $X(G_3, M_3)$ are edge-Dyck shifts which are Dyck in-amalgamations of $X(G_1, M_1)$, then there is an edge-Dyck shift $X(G_4, M_4)$ which is a common Dyck in-amalgamation shift of $X(G_2, M_2)$ and $X(G_3, M_3)$.

- $(r, p_j) \in E_1$ implies $(r, q_{j'}) \notin E_1$ for any $j \neq j'$.
- $(r, s)(p_j, t) \in M_1$ if and only if $(r, s)(q_{j'}, t) \in M_1$.
- $(p_j, t)(r, s) \in M_1$ if and only if $(q_{j'}, q)(r, s) \in M_1$ for $1 \leq j \leq k$, $1 \leq j' \leq \ell$.

We define $(G_4, M_4)$ as the Dyck in-amalgamation of $(G_2, M_2)$ obtained by the map $\Omega$ amalgamating the vertices $p, q_{n+1}, \ldots, q_\ell$ to the state $p$. It is equal to the Dyck in-amalgamation of $(G_3, M_3)$ obtained by the map $\Theta$ amalgamating the states $q, p_{n+1}, \ldots, p_\ell$ to a vertex $p$. The maps $\Omega$ and $\Theta$ are Dyck in-amalgamations and $\Phi \circ \Omega = \Psi \circ \Theta$.

Unfortunately, two trim Dyck in-amalgamations do not commute in general as is shown in the following example.

Example 2. Let $(G, M)$ be the Dyck graph of Figure 7. Two trim in-amalgamations are possible. States 1 and 2 are in-amalgamated in the Dyck graph $(G_1, M_1)$ on the right of Figure 8 and states 2 and 3 are in-amalgamated in the Dyck graph $(G_2, M_2)$ on the left of Figure 8 through a trim in-amalgamation.

The Dyck graphs $(G_1, M_1)$ and $(G_2, M_2)$ cannot be (trim) in-amalgamated to the Dyck graph $(G_3, M_3)$ of Figure 9. For instance the states $(1, 2)$ and...
3 of $G_1$ cannot be in-amalgamated since the bijection between bi-infinite admissible paths of $(G_1, M_1)$ and $(G_3, M_3)$ would be lost. Indeed, the path

$\cdots 9 \rightarrow 9 \overset{\alpha}{\rightarrow} 4 \rightarrow 5 \rightarrow (1,2,3) \overset{\alpha}{\rightarrow} 7 \rightarrow 8 \cdots$

is an admissible path of $(G_3, M_3)$ which is not the image of an admissible path of $(G_1, M_1)$ by the $(0,0)$-block amalgamation map.

3.1. Decomposition Theorem. The Decomposition Theorem for shifts of infinite words states that any conjugacy between shifts of finite type can be decomposed into a finite sequence of splittings and amalgamations (see for instance [5]). In this section, we prove a similar result for proper conjugacies between edge-Dyck shifts. As for shifts of finite type, the crucial lemma will show that the memory of a block map can be reduced using proper splittings.

Theorem 1. Let $(G, M)$, $(H, N)$ be two Dyck graphs such that $X_{(G,M)}$ and $X_{(H,N)}$ are properly conjugate. Then there are sequences of Dyck graphs $(G_i, M_i)_{1 \leq i \leq k}$, $(H_j, N_j)_{1 \leq j \leq r}$, and Dyck (or trim Dyck) in-splittings $\Psi_i : (G_i, M_i) \rightarrow (G_{i+1}, M_{i+1})$, $\Delta_j : (H_j, N_j) \rightarrow (H_{j+1}, N_{j+1})$, for $1 \leq i \leq k - 1$, $1 \leq j \leq k' - 1$ such that $(G_1, M_1) = (G, M)$, $(H_1, N_1) = (H, N)$, and
\((G_k, M_k) = (H_k', N_k')\), up to renaming of the states.

\((G, M) \xrightarrow{\psi_1} \ldots \xrightarrow{\psi_k} (G_k, M_k) = (H_k', N_k') \triangleleft \Delta_{k'} \ldots \triangleleft \Delta_1 (H, N)\)

We also obtain the following corollary.

**Corollary 2.** Any proper conjugacy between edge-Dyck shifts can be decomposed as a composition of Dyck in-splitting maps and Dyck in-amalgamation maps.

**Lemma 1.** Let \(\Phi : X_{(G, M)} \rightarrow X_{(H, N)}\) be a proper \((m, a)\)-block conjugacy between two edge-Dyck shifts with \(m \geq 1\). Then there is a Dyck in-split graph \((\tilde{G}, \tilde{M})\) of \((G, M)\) and a Dyck in-splitting map \(\Psi_1\) from \(X_{(G, M)}\) to \(X_{(\tilde{G}, \tilde{M})}\), and a proper \((m-1, a)\)-block conjugacy \(\tilde{\Phi}\) from \(X_{(\tilde{G}, \tilde{M})}\) onto \(X_{(H, N)}\) such that \(\Phi = \tilde{\Phi} \circ \Psi_1\).

**Proof.** Let \(E = (E_c, E_r, E_i)\) (resp. \(F = (F_c, F_r, F_i)\)) the edges of \(X = X_{(G, M)}\) (resp. \(Y = X_{(H, N)}\)). Let \(\phi : B_{m+a}(X) \rightarrow F\) be the local function defining \(\Phi\). Let \(X_{(\tilde{G}, \tilde{M})}\) be the edge-Dyck graph obtained by splitting each state \(p\) into states \((p, e)\), where \(e \in \text{In}(p)\), according to the trivial partition of the edges \(\text{In}(p)\) coming in \(p\) where each class is a singleton. Hence we perform a full Dyck in-splitting. There is an edge \(((q, f), (p, e))\) in \(\tilde{G}\) if and only if \(f = (r, q)\) and \(e = (q, p)\) for some \(r \in Q\).

Let \(\Psi_1 : X_{(G, M)} \rightarrow X_{(\tilde{G}, \tilde{M})}\) be the \((1,0)\)-block conjugacy defined by \(\psi_1(f e) = ((q, f)(p, e))\).

Let \(\tilde{\Phi} : X_{(\tilde{G}, \tilde{M})} \rightarrow Y\) be the \((m-1, a)\)-block map defined, for any block of \(\tilde{E}^{m-1+a}\), by

\[
\tilde{\phi}(\tilde{e}_{-m+1} \cdot \tilde{e}_0 \cdot \tilde{e}_a) = \phi(e_{-m} \cdot e_0 \cdot e_a),
\]

where \(\tilde{e}_i = ((p_{i-1}, e_{i-1}), (p_i, e_i))\). We have \(\Phi = \tilde{\Phi} \circ \Psi_1\). \(\square\)

A similar result holds for reducing the anticipation of \(\Phi\).

**Lemma 2.** Let \(\Phi : X_{(G, M)} \rightarrow X_{(H, N)}\) be a proper \((0,0)\)-block conjugacy between two edge-Dyck shifts such that \(\Phi^{-1}\) is an \((m, a)\)-block map with \(m \geq 1\). Then there is a Dyck in-split graph \((\tilde{G}, \tilde{M})\) of \((G, M)\), a Dyck in-split graph \((\tilde{H}, \tilde{N})\) of \((H, N)\), a Dyck in-splitting map \(\Psi_1\) from \(X_{(G, M)}\) to \(X_{(\tilde{G}, \tilde{M})}\), a Dyck in-splitting map \(\Psi_2\) from \(X_{(H, N)}\) to \(X_{(\tilde{H}, \tilde{N})}\), and a \((0,0)\)-block conjugacy \(\tilde{\Phi}\) from \(X_{(\tilde{G}, \tilde{M})}\) onto \(X_{(\tilde{H}, \tilde{N})}\) such that \(\Phi = \Psi_2^{-1} \circ \tilde{\Phi} \circ \Psi_1\) and \(\tilde{\Phi}^{-1}\) is a proper \((m-1, a)\)-block map. This makes the following diagram commute.

\[
\begin{array}{ccc}
X_{(G, M)} & \xrightarrow{\Phi} & X_{(H, N)} \\
\Psi_1 \downarrow & & \downarrow \Psi_2 \\
X_{(\tilde{G}, \tilde{M})} & \xrightarrow{\tilde{\Phi}} & X_{(\tilde{H}, \tilde{N})}
\end{array}
\]
Proof. Let $E = (E_c, E_r, E_i)$ (resp. $F = (F_c, F_r, F_i)$) the edges of $X = X_{(G, M)}$ (resp. $Y = X_{(H, N)}$). Let $\phi : B_{m+1}(X) \to F$ be the local function defining $\Phi$. Let $X_{(\tilde{G}, \tilde{M})}$ be the edge-Dyck graph obtained by a Dyck in-splitting of each state $p$ into states $(p, [e])$, where $e \in \text{In}(p)$, according to the partition of the edges coming in $p$ such that two edges $e, e'$ belong to the same class if and only if $\phi(e) = \phi(e')$ where $\phi$ is the local function associated to $\Phi$. Let $[e]$ denotes the class of $e$ in this partition. There is an edge $((q, [f]), (p, [e]))$ in $\tilde{G}$ if and only if $f = (r, q), e = (q, p)$ for some $r \in Q$ and $fe$ is admissible. Let $\Psi_1 : X \to X_{(\tilde{G}, \tilde{M})}$ be the $(1, 0)$-block map defined by $\psi_1(fe) = ((q, [f])(p, [e]))$.

Let $X_{(\tilde{H}, \tilde{N})}$ be the edge-Dyck graph obtained by a Dyck in-splitting of each state $p$ into states $(p, e)$, where $e \in \text{In}(p)$, according to the trivial partition of the edges $\text{In}(p)$ coming in $p$ where each class is a singleton. There is an edge $((q, f), (p, e))$ in $\tilde{H}$ if and only if $f = (r, q), e = (q, p)$ for some $r \in Q$, and $fe$ is admissible. We denote by $\Psi_2$ the proper in-splitting map from $Y$ to $X_{(\tilde{H}, \tilde{N})}$.

We define a $(0, 0)$-block map $\Phi$ from $X_{(\tilde{G}, \tilde{M})}$ onto $X_{(\tilde{H}, \tilde{N})}$ by $\Phi([e]) = \phi(e)$. It is consistent by definition of the partition of $\text{In}(p)$ if $e$ ends in $p$. We have $\Phi = \Psi_2^{-1} \circ \Phi \circ \Psi_1$. It remains to check that $\Phi^{-1} = \Psi_1 \circ \Phi^{-1} \circ \Psi_2^{-1}$ is an $(m - 1, a)$-block map. That is, we must show that for any word $x$ in $X_{(\tilde{H}, \tilde{N})}$, the coordinate of index 0 of $\Phi^{-1}(x)$ is determined by its block $u = [x_{-m}x_{m+1}][x_{m+1}x_{m+2}] \cdots [x_{a-1}x_a]$ of length $m + a$. But this follows from the observation that the block $u$ determines $\Psi_2^{-1}(x)_{-m}$ and therefore the block $\Psi_2^{-1}(x)_{[-m, a]}$ of length $m + a + 1$. Hence, if $x' = (\Phi^{-1} \circ \Psi_2^{-1})(x)$, $x'_0$ is determined by $u$ since $\Phi^{-1}$ is an $(m, a)$-block map. Furthermore, since the block $u$ determines the block $\Psi_2^{-1}(x)_{[-m, a]}$ of length $m + a + 1$, it determines $\phi(x'_{-1})$ and thus $\Psi_1(x'_0)$ which depends only on $x'_0, \phi(x'_{-1})$. □

A similar result holds for reducing the anticipation of $\Psi_{-1}$.

**Lemma 3.** Let $\Delta : X_{(G, M)} \to X_{(H, N)}$ be a proper conjugacy between edge-Dyck shifts defined by Dyck graphs $(G, M)$ and $(H, N)$. Let us assume that $\Delta$ and $\Delta^{-1}$ are $(0, 0)$-block maps. Then there is a Dyck graph $(G', M')$ (resp. $(H', N')$) obtained by trim in-splittings of $(G, M)$ (resp. $(H, N)$) such that $(G', M')$ and $(H', N')$ are equal, up to a renaming of the states.

**Proof.** Let $G = (Q, E)$ and $H = (R, F)$. Let $(G' = (Q', E'), M')$ be the Dyck graph whose states are $(e, f)$ where $ef$ is a path of $E$ and edges are $((e, f), (f, g))$ if $efg$ is a path of $G$. The edge $((e, f), (f, g))$ is a call (resp. return, internal) edge if and only if $f$ is a call (resp. return, internal) edge. A return edge $((e, f), (f, g))$ is matched with a call edge $((r, s), (s, t))$ if and only if $f$ is matched with $s$. We define $(H' = (R', F'), N')$ similarly for $(H, N)$.

The Dyck graph $(G', M')$ (resp. $(H', N')$) is obtained from $(G, M)$ (resp. $(H, N)$) by trim in-splittings. Each state $p$ is split into the states $(p, e)$ for each edge $e$ coming in $p$ with a trim in-splitting and each state $(p, e)$
is split into \((e,f)\) for each edge going out of \(p\) (or \((p,e)\) with a trim out-splitting. Since the splittings trim, for each edge \(f\) of \((\mathcal{G}',M')\) there is a bi-infinite admissible path extending \(f\) in \((\mathcal{G}',M')\) (resp. \((\mathcal{H}',N')\)). The \((0,0)\)-block proper conjugacy \(\Delta' : X(\mathcal{G}',M') \rightarrow X(\mathcal{H}',N')\) defined by \(\delta'((e,f)(f,g)) = (\delta(e),\delta(f))\delta(f),\delta(g)\) has an inverse which is also a \((0,0)\)-block map.

Let us show that \((\mathcal{G}',M')\) and \((\mathcal{H}',N')\) are equal, up to a renaming of the states. We define a renaming of the states \(\rho : Q' \rightarrow R'\) as follows. If \((e,f)\) is a state of \(\mathcal{G}'\), we set \(\rho(e,f) = (\delta(e),\delta(f))\). If \(((e,f)(f,g))\) is an edge of \(\mathcal{E}'\), there is a bi-infinite admissible path \(\pi = ((e_i,e_{i+1})\mid e_{i+2})\mid i \in \mathbb{Z}\) of \(\mathcal{G}'\) such that \((e_0,e_1) = (e,f)\) and \((e_1,e_2) = (f,g)\). This image of \(\pi\) by \(\Delta'\) is the admissible path \(((\delta(e_i),\delta(e_{i+1}))(\delta(e_{i+1}),\delta(e_{i+2}))\mid i \in \mathbb{Z}\). Hence \((\rho(e,f),\rho(f,g))\) is an edge of \(\mathcal{F}'\). Inverting the roles played by \((\mathcal{G}',M')\) and \((\mathcal{H}',N')\), we obtain that \(\rho\) is a graph isomorphism from \(\mathcal{G}'\) into \(\mathcal{H}'\).

Furthermore, since \((\mathcal{G}',M')\) and \((\mathcal{H}',N')\) are essential, for each pair \((e',f')\), \((r',s')\) of matched edges of \((\mathcal{G}',M')\), there is a bi-infinite admissible path \(\pi\) such that \(\pi = z(e',f')w(r',s')z'\), where \(w\) is a Dyck path. The image of \(\pi\) by \(\Delta'\) is the bi-infinite admissible path \(\pi = \delta'(z)\delta'(e',f')\delta'(w)\delta'(r',s')\delta'(z')\) of \((\mathcal{H}',N')\). Since \(\delta'(w)\) is a Dyck path of \((\mathcal{H}',N')\). Hence \(\delta'(e',f'),\delta'(r',s')\) belongs to \(N'\). We get \(M' \subseteq N'\) and symmetrically \(N' \subseteq M'\). Thus \((\mathcal{G}',M') = (\mathcal{H}',N')\).

Proof of Theorem 1. Let \(\Phi : X(\mathcal{G},M) \rightarrow X(\mathcal{H},N)\) be a proper \((m,a)\)-block conjugacy between two edge-Dyck shifts such that \(\Phi^{-1}\) is an \((m',a')\)-block map where \(m,a,m',a'\) are nonnegative integers.

By Lemma 1 and Lemma 2 there are sequences of Dyck graphs \((\mathcal{G}_i,\mathcal{M}_i)\) for \(1 \leq i \leq k\), and Dyck in-splittings \(\Psi_1 : (\mathcal{G}_1,\mathcal{M}_1) \rightarrow (\mathcal{G}_{i+1},\mathcal{M}_{i+1})\), \(\Delta_j : (\mathcal{H}_j,\mathcal{N}_j) \rightarrow (\mathcal{H}_{j+1},\mathcal{N}_{j+1})\), for \(1 \leq i \leq k-1, 1 \leq j \leq k' - 1\) such that \((\mathcal{G}_1,\mathcal{M}_1) = (\mathcal{G},\mathcal{M}), (\mathcal{H}_1,\mathcal{N}_1) = (\mathcal{H},\mathcal{N})\), and a proper conjugacy \(\Delta: X(\mathcal{G}_k,\mathcal{M}_k) \rightarrow X(\mathcal{H}_{k'},\mathcal{N}_{k'})\) such that \(\Delta^{-1}\) is \((0,0)\)-block maps, with

\[\Phi = \Delta_1^{-1} \circ \Delta_2^{-1} \circ \cdots \circ \Delta_{k'}^{-1} \circ \Delta \circ \Psi_k \circ \Psi_2 \circ \Psi_1.\]

By Lemma 3 there are trim Dyck in-splittings \(\Psi_j(k+1) \leq j \leq n\) splitting \((\mathcal{G}_{j-1},\mathcal{M}_{j-1})\) into an essential Dyck graph \((\mathcal{G}_j,\mathcal{M}_j)\) and trim Dyck in-splittings \((\Delta_j(k')+1) \leq j' \leq n'\) splitting \((\mathcal{H}_{j'-1},\mathcal{N}_{j'-1})\) into an essential Dyck graph \((\mathcal{H}_{j'},\mathcal{N}_{j'})\), and a proper conjugacy \(\Delta': (\mathcal{G}_n,\mathcal{M}_n) \rightarrow (\mathcal{H}_{n'},\mathcal{N}_{n'})\) such that \(\Phi = \Delta_1^{-1} \circ \Delta_2^{-1} \circ \cdots \circ \Delta_{n'}^{-1} \circ \Delta \circ \Psi_n \circ \Psi_2 \circ \Psi_1\) and \(\Delta\) is a renaming map.
The Decomposition Theorem is illustrated in the following diagram where $\Delta$ is a renaming map.

\[
\begin{array}{c}
\xymatrix{
X((G,M)) \ar[r]^\Phi \ar[d]_{\Psi_1} & X((H,N)) \ar[d]_{\Delta_1} \\
\vdots & \vdots \\
X((G_n,N_n)) \ar[d]_{\Psi_n} & \\
X((H_{n'},N_{n'})) \ar[r]_{\Delta} & \\
}
\end{array}
\]

REFERENCES

[1] R. Alur and P. Madhusudan. Visibly pushdown languages. In Proceedings of the 36th Annual ACM Symposium on Theory of Computing, pages 202–211 (electronic), New York, 2004. ACM.

[2] M.-P. Béal, M. Blockelet, and C. Dima. Sofic-Dyck shifts. CoRR, http://arxiv.org/1305.7413, 2013.

[3] K. Inoue. The zeta function, periodic points and entropies of the Motzkin shift. CoRR, math/0602100, 2006.

[4] K. Inoue and W. Krieger. Subshifts from sofic shifts and Dyck shifts, zeta functions and topological entropy. CoRR, 2010.

[5] B. P. Kitchens. Symbolic Dynamics. Universitext. Springer-Verlag, Berlin, 1998. One-sided, two-sided and countable state Markov shifts.

[6] W. Krieger. On $g$-functions for subshifts. In Dynamics & stochastics, volume 48 of IMS Lecture Notes Monogr. Ser., pages 306–316. Inst. Math. Statist., Beachwood, OH, 2006.

[7] W. Krieger. On subshift presentations. CoRR, 2012.

[8] W. Krieger and K. Matsumoto. A notion of synchronization of symbolic dynamics and a class of $C^*$-algebras. CoRR, http://arxiv.org/abs/1105.3249, 2011.

[9] D. Lind and B. Marcus. An Introduction to Symbolic Dynamics and Coding. Cambridge University Press, Cambridge, 1995.

[10] K. Matsumoto. A certain synchronizing property of subshifts and flow equivalence. CoRR, http://arxiv.org/abs/1105.3249, 2011.

[11] K. Matsumoto. A notion of synchronization of symbolic dynamics and a class of $C^*$-algebras. CoRR, http://arxiv.org/abs/1105.4393, 2011.

Université Paris-Est, Laboratoire d’informatique Gaspard-Monge, UMR 8049 CNRS
E-mail address: beal@univ-mlv.fr

Université Paris-Est, Laboratoire d’Algorithmmique, Complexité et Logique
E-mail address: Michel.blockelet@univ-mlv.fr

Université Paris-Est, Laboratoire d’Algorithmmique, Complexité et Logique
E-mail address: catalin.dima@u-pec.fr