INTERNAL ABSOLUTE GEOMETRY I: DESINGULARIZATION

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Abstract. We introduce an axiomatization of Grothendieck sites with additional structure, and we describe sheaves that reconstruct groupoids which are internal to the site structure. This setting applies to various concrete situations, where a Nash blowup of a singular space results in an almost regular foliation. It also potentially applies to various types of moduli spaces. The sheaf can encode candidate holonomy groupoids that desingularize such spaces.

1. Introduction

A singular foliation, i.e. a foliation with leaves of varying dimension, over a smooth manifold, possesses an associated holonomy groupoid. The construction furnishes a groupoid with smooth source fibers, the orbits of which align precisely with the leaves of the foliation. C. Ehresmann initially studied the holonomy groupoid [7]. It was later extended to the regular foliation case by H. E. Winkelnkemper [18] and to singular foliations by J. Pradines [15]. More recent contributions have come from C. Debord [6], G. Skandalis, and I. Androulidakis [1]. Notably, a singular foliation, as defined by Androulidakis-Skandalis, becomes a Debord foliation after a Nash blowup [11].

Various authors have expanded this concept to other settings. For instance, the holonomy groupoid has been constructed for Teichmüller and Riemann moduli spaces when viewed as stacks. This is connected with partially foliated structures based on infinite-dimensional Fréchet manifolds [12]. Generally, even if one’s focus isn’t strictly on smooth systems, the holonomy groupoid can offer a valuable geometric interpretation of structures similar to foliation. For instance, within cognitive systems, the geometry of information spaces is smooth only in a generalized (diffeological) sense [8]. The setting of this work is an ambient category with additional structure and groupoids internal to this category. Some notable instances of specific ambient site structures \((\mathcal{A}, \mathcal{E}, \mathcal{T})\) are as follows:

1. The case of smooth manifolds \(\mathcal{A} = \text{Man} \), or compact smooth manifolds with (embedded) corners, \(\mathcal{A} = \text{Man}_{\text{ec}}\), with \(\mathcal{T}\) the open covers and \(\mathcal{E}\) the (tame) surjective submersions.
2. Complex manifolds, where the ambient category is \(\mathcal{A} = \text{Man}_{\mathbb{C}}\), \(\mathcal{T}\) are the open covers, \(\mathcal{E}\) are surjective holomorphic submersions.
3. Real analytic manifolds, \(\mathcal{A} = \text{Man}_{\mathbb{R}A}\), with \(\mathcal{T}\) the open covers, \(\mathcal{E}\) the surjective real analytic submersions.
4. Algebraic manifolds, \(\mathcal{A} = \text{Man}_{\mathbb{A}lg}\), \(\mathcal{T}\) the étale covers, \(\mathcal{E}\) the surjective étale morphisms.

Other examples may concern moduli spaces, e.g. atlas constructions for orbifold stratified spaces that are modeled by étale Lie groupoids, [4]. The category \(\mathcal{A} = \text{Man}_{\mathbb{F}}\) of Fréchet manifolds fulfills the assumptions 1 through 3 that are stated below, cf.

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[13][Section 9], and the embedding into the category of diffeological spaces is fully faithful, cf. [10, 19]. Let us recall the discussion of a singular foliation from [11]: A subsheaf $\mathcal{F}$ of the sheaf of vector fields $\mathcal{X}$ that is closed with regard to Lie bracket and locally finitely generated as an $\mathcal{O}$-module for the relevant sheaf of functions $\mathcal{O}$. After a single Nash blowup, this results in a Debord foliation: the data $(\tilde{M}, \pi)$ with $\pi: \tilde{M} \to M$ is onto and proper; $\pi_{\text{reg}}, \mathcal{F}: \pi^{-1}(\mathcal{M}_{\text{reg}}, \mathcal{F}) \to \mathcal{M}_{\text{reg}}, \mathcal{F}$ is one-to-one; the pullback $\pi^* \mathcal{F}$ exists and $\pi^* \mathcal{F}|_{\pi^{-1}(\mathcal{M}_{\text{reg}}, \mathcal{F})} \cong \mathcal{F}|_{\mathcal{M}_{\text{reg}}, \mathcal{F}}$.

These examples call for a more integrated approach to the holonomy groupoid and its related geometry. To this end, we introduce the category of virtual manifolds, where an object of this category is a sheaf that reconstructs an atlas which may desingularize a foliation in a particular way; the global object of the sheaf is a groupoid internal to the site structure. These groupoids are not universal integrating groupoids, but may become so, e.g. the $s$-connected component of such a groupoid results in a universal integrating groupoid for the case of a foliation that is defined by an almost injective Lie algebroid over the ambient category of smooth manifolds. In this sense, the atlas described by a given sheaf or virtual manifold prescribes also the particular singularities that are encoded by the original foliation we started from. Another motivation for this study was the idea that it is useful to localize the resulting bicategory of sheaves, which synthesizes the structure of a general atlas, at the regular parts, $\mathcal{M}_{\text{reg}}$.

1.1. The ambient site structure. The axioms of the ambient category concern a structure encoded by the 3-tuple $(\mathcal{A}, \mathcal{T}, \mathcal{E})$, where $\mathcal{A}$ is the ambient category, $\mathcal{T}$ is a Grothendieck site and $\mathcal{E}$ is a suitable collection of admissible epimorphisms. We make the following assumptions:

1. The triple $(\mathcal{A}, \mathcal{T}, \mathcal{E})$ forms a site-structure 2.5.
2. The site $\mathcal{T}$ is subcanonical 2.2 and its covers are local 2.4.
3. Any $\mathcal{E}$-sheaf over a Čech groupoid of an $\mathcal{E}$-cover, together with some bundle projection, is part of a principal bundle. In addition, this property is $\mathcal{T}$-local.
4. There is a fully faithful embedding functor $\Phi$ of $\mathcal{A}$ into the category of diffeological spaces, Diff, with the compatibility properties listed below.

The first three of these conditions suffice to define groupoids internal to $\mathcal{A}$ in the style of Meyer-Zhu [13], as well as study their generalized tensor products. The fourth condition is completely specified in the following subsection below; it is used to construct the generalized sheaves studied in this work and to show that they reconstruct internal groupoids as desired.

1.2. Embedding into the category of diffeological spaces. The ambient site structure $(\mathcal{A}, \mathcal{T}, \mathcal{E})$ is assumed to embed into the category of diffeological spaces Diff. The latter category is also endowed with a suitable Grothendieck topology $\mathcal{T}_{\text{D-open}}$ which is comprised of covers that are $D$-open sets. There is also a class of admissible epimorphisms in Diff, the local subductions, which are denoted by $\mathcal{E}_{\text{loc}}$. The second set of assumptions concerns the existence and properties of a fully faithful functor

$$\Phi: \mathcal{A} \to \text{Diff}.$$ 

This embedding functor is assumed to be continuous, by which is meant that it maps coverings to coverings, preserves the pullbacks of coverings and furthermore, $\Phi$ is assumed to preserve finite products (the ones that exist in $\mathcal{A}$). There is a canonical functor $D: \text{Diff} \to \text{Top}$ which equips a diffeological space with the $D$-topology; the largest topology such that all plots are continuous. We point out that $D$ has not as many desirable properties. For example, while it preserves colimits, it fails to preserve products in
general and it is not full. The usual solution for the lack of product-preservation is to co-restrict \( D \) to \( \Delta \)-generated spaces (e.g. [5]), i.e. we consider the functor:
\[
D^\Delta : \text{Diff} \to \text{Top}^\Delta.
\]
Altogether, we assume our ambient category to fit into the functorial extension
\[
\xymatrix{ \mathbb{A} \ar[r]^{\Phi} & \text{Diff} \ar[r]^{D^\Delta} & \text{Top}^\Delta. }
\]
The site structures on \( \mathbb{A} \) and Diff combined with their compatibility allow us to talk about and study sheaves on these categories. Given the assumptions, the category of sheaves on \( \mathbb{A} \) compares well with the category of sheaves on Diff. In the case of the category of finite dimensional smooth manifolds, \( \mathbb{A} = \text{Man} \), the pullback functor furnishes an equivalence of these categories by the Grothendieck-Verdier theorem, cf. [17]. The same cannot be said for Diff in comparison with \( \text{Top}^\Delta \).

2. \( \mathcal{T} \)-local internal Groupoids and Spanoids

Let \( (\mathbb{A}, \mathcal{T}) \) denote the ambient site and \( E = E(\mathcal{T}) \) the maximal collection of \( \mathcal{T} \)-universal epimorphisms that are stable with respect to pullback, cf. [16].

**Definition 2.1.** A site \( \mathcal{T} \) on an ambient category \( \mathbb{A} \) is a collection of families \( \{(U_i \to X)_{i \in I}\}_{X \in \mathbb{A}_0} \) of arrows in \( \mathbb{A}_1 \) called covers between objects of \( \mathbb{A}_0 \) called opens. A site verifies the following conditions:

1. We have that \( \{U' \sim U\} \in \mathcal{T} \), i.e. all isomorphisms are contained in \( \mathcal{T} \).
2. Given \( \{U_i \to U\}_{i \in I} \) collection of covers in \( \mathcal{T} \), then for each \( V \to U \in \mathbb{A}_1 \) we have that \( \{U_i \ast_U V \to V\}_{i \in I} \) is in \( \mathcal{T} \), i.e. \( \mathcal{T} \) is closed with regard to base change.
3. Given \( \{U_i \to U\}_{i \in I} \in \mathcal{T} \) and \( \{U_{ij} \to U_i\}_{j \in J} \) in \( \mathcal{T} \) for each \( i \in I \), we have that the composite \( \{U_{ij} \to U\}_{(i,j) \in I \times J} \) is also in \( \mathcal{T} \), i.e. \( \mathcal{T} \) is closed with regard to intersections.

If each cover consists of a single map, the site is called singleton site.

**Definition 2.2.** A site \( (\mathbb{A}, \mathcal{T}) \) fulfilling the conditions of Proposition 2.3 is called subcanonical.

**Proposition 2.3** ([13], Lem. 2.2). Let \( (\mathbb{A}, \mathcal{T}) \) be a site. The following conditions are equivalent

1. Each cover \( f : U \to X \) in \( \mathcal{T} \) is a coequalizer of some parallel arrows \( g_1, g_2 : Z \to U \).
2. Each cover \( f : U \to X \) in \( \mathcal{T} \) is the coequalizer of \( \text{pr}_1, \text{pr}_2 : U \ast_f U \to U \).
3. For each \( W \in \mathbb{A}_0 \) and \( f : U \to X \) in \( \mathcal{T} \), we have a bijection
   \[
   \text{Mor}_\mathbb{A}(X,W) \sim \{h \in \text{Mor}_\mathbb{A}(U,W) : h \circ \text{pr}_1 = h \circ \text{pr}_2 \text{ in } \text{Mor}_\mathbb{A}(U \ast_f U,W)\},
   \]
   \[
   g \mapsto g \circ f.
   \]
4. All representable functors \( \text{Mor}_\mathbb{A}(\_ ,W) \) on \( \mathbb{A} \) are sheaves.

**Definition 2.4.** Let \( (\mathbb{A}, \mathcal{T}) \) be a site. We say that \( f : Y \to X \in \mathbb{A}_1 \) has property \( (P) \) \( \mathcal{T} \)-locally if there is a cover \( g : U \to X \) such that \( \text{pr}_2 : Y \ast_f U \to U \) has property \( (P) \).

The condition (3) in particular implies that \( \mathcal{T} \) is a sieve-structure, reflecting the alternative formulation of the three axioms to be found in the literature, in terms of the notion of a covering sieve. We can always consider a single map \( (f_i)_{i \in I} : \bigsqcup_{i \in I} U_i \to X \) in lieu of the collection of covers \( \{U_i \to U\}_{i \in I} \).
Definition 2.5. The site $\mathbb{E} = \mathbb{E}(\mathcal{T})$ is the singleton pre-topological site of $\mathcal{T}$-universal epimorphisms, i.e. arrows that are locally sectionable in the sense that there is a covering family $\{c_i: U_i \to X\}$ and lifts $p_i: U_i \to S$ such that $f \circ p_i = c_i$. In addition, $f$ is assumed universal with this property. We call the triple $(\mathcal{A}, \mathcal{T}, \mathbb{E})$ a site structure.

Remark 2.6. One can check that the subcanonicity assumption on $\mathcal{T}$ implies that $\mathbb{E} = \mathbb{E}(\mathcal{T})$ is subcanonical.

In what follows we make use of elementwise notation for groupoids and actions, which can be justified by the algorithm described in [13, Section 3]. Denote by $\mathcal{A}\mathcal{S}(\mathbb{E})$ the category of $\mathcal{A}$-internal spans over $\mathbb{E}$ with internal span arrows as the arrows of the category. We introduce next $\mathcal{T}$-local internal groupoids, by which is meant a span that has a multiplication which furnishes a groupoid in a $\mathcal{T}$-local sense.

Definition 2.7. A $\mathcal{T}$-local internal groupoid in $(\mathcal{A}, \mathcal{T}, \mathbb{E})$ consists of the data $(\mathcal{L}, s, r) \in \mathcal{A}\mathcal{S}(\mathbb{E})_0$, $u \in \text{Mor}_{\mathcal{A}}(\mathcal{L}_0, \mathcal{L}_1)$ and $\mathcal{A}$-isomorphism $i: \mathcal{L}_1 \sim \mathcal{L}_0$ so that $u$ and $i$ are compatible with the span-structure:

(1)
$$s \circ i = r, \quad r \circ i = s.$$

(2)
$$u^* s|_{\mathcal{L}_0} = u^* r|_{\mathcal{L}_0} = \text{id}_{\mathcal{L}_0}.$$

Multiplication is an arrow
$$p^* \mathcal{L}_2 = \mathcal{L}_1 s^* r \mathcal{L}_1 \to \mathcal{L}_1$$

such that there is a cover
$$c: D^2 \mathcal{L} \to \mathcal{L}_2$$
in $\mathcal{T}$ with
$$\text{pr}_2^* \mathcal{L}_2 \ast_c D^2 \mathcal{L} \to D^2 \mathcal{L}$$
being compatible with the remaining groupoid structure $(\mathcal{L}_0, \mathcal{L}_1, r, s, u, i)$:

(3)
$$s((p \circ \text{pr}_2)(\gamma_1, \gamma_2)) = s(\gamma_2),$$

(4)
$$r((p \circ \text{pr}_2)(\gamma_1, \gamma_2)) = r(\gamma_1), \quad (\gamma_1, \gamma_2) \in \mathcal{L}_2 \ast_c D^2 \mathcal{L}.$$

(5)
$$(i \circ (p \circ \text{pr}_2))(\gamma_1, \gamma_2) = (p \circ \text{pr}_2)(i(\gamma_1), i(\gamma_2)), \quad (\gamma_1, \gamma_2) \in \mathcal{L}_2 \ast_c D^2 \mathcal{L}.$$

(6)
$$p \circ \text{pr}_2((u \circ r)(\gamma), \gamma) = \gamma = (p \circ \text{pr}_2)(\gamma, (u \circ s)(\gamma)).$$

$$p \circ \text{pr}_2((u \circ r)(\gamma_1, \gamma_2), \gamma_3) = (p \circ \text{pr}_2)(\gamma_1, (p \circ \text{pr}_2)(\gamma_2, \gamma_3)),$$
$$(\gamma_1, \gamma_2) \in \mathcal{L}_2 \ast_c D^2 \mathcal{L},$$
$$(\gamma_2, \gamma_3) \in \mathcal{L}_2 \ast_c D^2 \mathcal{L}.$$

Definition 2.8. An internal groupoid $\mathcal{G}$ is called a spanoid if for any span $\mathcal{S}$ over $\mathbb{E}$ there is at most one span arrow $\mathcal{S} \to \mathcal{G}$. Analogously, a $\mathcal{T}$-local spanoid is a $\mathcal{T}$-local groupoid with this property.
We recall some of the constructions from [6] and how they carry over to our setting; see also [3].

**Remark 2.9.** 1) Let $\mathcal{L} \Rightarrow \mathcal{L}_0$ be a $\mathcal{T}$-local spanoid and $D \to \mathcal{L}_2$ a cover, then there is at most one arrow $p: D \to \mathcal{L}$ such that $s \circ p = s \circ pr_2$, $r \circ p = r \circ pr_1$. Therefore, a maximal cover $D^{\text{max}}_{\text{max}} \to \mathcal{L}_2$ exists. It is understood from now on that we always fix the maximal cover for any given $\mathcal{T}$-local spanoid.

2) Let $\varphi: \mathcal{L} \to \mathcal{L}'$ be a span-arrow for a $\mathcal{T}$-local groupoid $\mathcal{L}$ and a $\mathcal{T}$-local spanoid $\mathcal{L}'$. Then $\varphi$ is promoted to a strict groupoid morphism. The reasoning is that the spanoid property fixes the inverse and multiplication as uniquely defined internal arrows.

**Proposition 2.10.** A ($\mathcal{T}$-local) groupoid is a ($\mathcal{T}$-local) spanoid if and only the only local section of both the source and the range arrow is the unit arrow.

**Proof.** Let $\mathcal{L}$ be a ($\mathcal{T}$-local) spanoid and let $\sigma$ be a local section of $r$ and $s$ for a cover $c_U: U \to \mathcal{L}_0$, i.e. $r \circ \sigma = c_U$, $s \circ \sigma = c_U$. Then $c_{U}'U$ is a span arrow between the spans $\mathcal{L}_0 \leftarrow c_U U \to \mathcal{L}_0$ and $\mathcal{L}_0 \leftarrow U \to \mathcal{L}_0$. By the $\mathcal{T}$-universality of the epimorphisms $E$ in $\mathcal{T}$, we can take $\bigsqcup_i U_i$ for a covering family $\{U_i \to \mathcal{L}_0\}_{i \in I}$ sufficient to cover $\mathcal{L}_0$, so that $c_U$ is in $E$. Since $\mathcal{L}$ is a spanoid by assumption, we have that there is at most one span-arrow $U \to \mathcal{L}$ and hence $\sigma = c_U'.u$. For the other direction, let $\mathcal{L}$ have the property that the only $\mathcal{T}$-local section of $r, s$ is the unit arrow. Let $S$ be a span with arrows $a, b$ to $\mathcal{L}_0$ such that $f_1, f_2: S \Rightarrow S$ are two span arrows with $r \circ f_i = a$, $s \circ f_i = b$. Choose a $\mathcal{T}$-local section $\sigma: U \to S$ of $a$ and define $\nu: U \to \mathcal{L}$ via $x \mapsto (f_1(\sigma(x)))^{-1} \cdot f_2(\sigma(x))$.

Apply this process to a singleton cover $U = \bigsqcup_i U_i \to S$ in $E$, so that

$$r \circ \nu = (a \circ \sigma)(x) = c_U(x), \ x \in U.$$

Hence, $\nu$ is an $E$-section of both $r, s$. Therefore $f_1 \circ \sigma = f_2 \circ \sigma$. Since this works for any singletoned $\mathcal{T}$-local sections of $a$, it follows that $f_1 = f_2$. \hfill $\square$

We record the following strengthening which is relevant for an alternate approach to atlas constructions via sheaves of germs, as will be discussed below.

**Proposition 2.11.** Let $\mathcal{L} \Rightarrow \mathcal{L}_0$ be a $\mathcal{T}$-local groupoid. The following assertions are equivalent:

1) $\mathcal{L}$ is isomorphic as a span to a $\mathcal{T}$-local groupoid $\mathcal{L}' \Rightarrow \mathcal{L}_0$ such that the only $\mathcal{T}$-local section of $\tau, \bar{s}$ is the inclusion of the unit arrow.

2) $\mathcal{L}$ is isomorphic as a span to a $\mathcal{T}$-local spanoid.

In order to prove this statement, we introduce notation that deals with restrictions of $\mathcal{T}$-local groupoids. To this end, let $c_U: V \to \mathcal{L}$ be a cover and $V^+$ all arrows $\gamma \in V$ such that $\gamma^{-1}$ is in $V$. Set $\mathcal{L}(V) := \mathcal{L}_s \ast c_U \ast V \Rightarrow V$, where we denote the range and source arrows by $\tau$ and $\bar{s}$ respectively. Assuming $\mathcal{V}(\mathcal{L}_0) = \mathcal{L}_s \ast u(\mathcal{L} \ast \mathcal{L}_0)_{u, \ast} \mathcal{L}$ set $\mathcal{L}^*(V) := V^+ \ast \tau \mathcal{L}(V \ast \mathcal{L}_0)_{\bar{s}} \ast V^+ \Rightarrow V \ast \mathcal{L}_0 =: \mathcal{L}^*(V)$. This furnishes a $\mathcal{T}$-local groupoid with domain:

$$D^2 \mathcal{L}^+(V) := D^2 \mathcal{L} \ast (\mathcal{L}^+(V) \ast \mathcal{L}^+(V))$$.
Let two $T$-local groupoids $L \ni L_0$, $\tilde{L} \ni \tilde{L}_0$ and let $L_0 \ast \tilde{L}_0 \in \mathcal{L}_0$ be given. Then $L$ is isomorphic to $\tilde{L}$ as spans if there are covers $c: V \to L_0 \ast \tilde{L}_0$ and $\overline{c}: \overline{V} \to L_0 \ast \tilde{L}_0$ as well as an internal isomorphism $\varphi: V \to \overline{V}$ such that $\overline{s} \circ \varphi = s$, $\overline{r} \circ \varphi = r$.

**Proof.** By virtue of Proposition 2.10 we obtain the direction $2) \Rightarrow 1)$. Let $\varphi: \overline{V} \to V$ be an isomorphism between two spans, where $\overline{V} \to L_0 \ast \tilde{L}_0$, $V \to L_0 \ast \tilde{L}_0$ such that $\overline{s} \circ \varphi = s$, $\overline{r} \circ \varphi = r$. We note that the domain $D^2 L$ covers $\{ (\gamma, \gamma^{-1}) : \gamma \in L \}$ in the sense that there is a cover $c_W: W \to L$ with the following properties:

- $W = W^{-1}$,
- $c: (W \times W) \ast L_2 \to D^2 L$ is a cover,
- $c^* p_L: (W \times W) \ast L_2 \to V$ is a cover.

Also, pulling back $r, s$ via $c_W$ induces a $T$-local groupoid structure on $W$:

\[
\begin{array}{ccc}
W & \xrightarrow{c_W} & L_1 \\
\downarrow c_W & & \downarrow r \\
L_0 & \xrightarrow{s} & L_0 \\
\end{array}
\]

By construction, $W$ and $L$ are $T$-locally isomorphic as spans. Let $L_0 \overset{a}{\leftarrow} S \overset{b}{\to} L_0$ be a span with two span arrows $f_1, f_2: S \to W$. Given a singleton cover $c := \bigsqcup_i c_i: U := \bigsqcup_i U_i \to L_0$ and local sections $\sigma := \bigsqcup_i \sigma_i: U \to S$, $c \in \mathcal{E}$, define

$\nu: U \to W$, $x \mapsto f_1(\sigma(x))^{-1} f_2(x)$.

Then $\nu$ is an $\mathcal{E}$-local section for $r$ and $s$, thereby it is the unit arrow and $f_1 \circ \sigma = f_2 \circ \sigma$. Repeating this line of argument for any such constructed $\mathcal{E}$-local section of $a$, we obtain that $f_1 = f_2$. \hfill $\square$

**Definition 2.12.** Let $L \ni L_0$ denote a $T$-local groupoid with domain $D^2 L$. A right $T$-local action $Z \ni L$ is implemented by $a$ where $c_a: D_a \to Z_{\ast q}, L$ is a cover, and its anchor is denoted by $q: Z \to L_0$, such that the following conditions hold:

1. $(q \circ a)(z, \gamma) = s(\gamma)$, $(z, \gamma) \in D_a$.

2. Given that $(z, \gamma_1) \in D_a$, $(\gamma_1, \gamma_2) \in D^2 L$ and if one of $a(\alpha(z, \gamma_1), \gamma_2)$ or $a(z, m(\gamma_1, \gamma_2))$ is defined, then so is the other and they are equal.

3. $(z, id_{q(z)})$ is contained in $D_a$ and $a(z, id_{q(z)}) = z$, for all $z \in Z$.

The action is called an $\mathcal{E}$-sheaf if $q \in \mathcal{E}$. The action is abbreviated as $a(z, \gamma) = z \cdot \gamma$. Denote by $\overline{Y}$ the arrow $(pr_1, a): D_a \to Z_{\ast q}Z$. The action is called principal if there is a covering family $\{ c_i: V_i \to Z \}_{i \in I}$ so that $\overline{c_i}: V_i \ast q \ni L \to D_a$ is a cover, induced by $c_i$, with the property that the arrow

$\overline{c_i}^* Y: D_a \ni V \ast \overline{c_i} (V_i \ast q, V_i) \to V_i \ast q V_i$

furnishes an isomorphism for each $i \in I$.

The $\mathcal{E}$-sheaves and principal actions are also being studied in [13, 19] in order to construct the bicategory of internal groupoids, $A \mathcal{G}_{hi}$. The aim here is to extend these constructions in order to compose $T$-local bibundle correspondences. Let us fix some useful notation.
Notation 2.13. We call $\text{Sh}(E)_L$ the category of $E$-sheaves, which consists of right actions by $L$ where the anchor of the action is contained in $E$ and the arrows are given by the obvious equivariant maps. Dually, $\mathcal{L}$ denotes the corresponding category of left actions that are sheaves.

By $\mathcal{A}_L$ we denote the category of right actions and dually, by $\mathcal{A}_L$ the category of left actions. Given an admissible epimorphism $p: X \to Z \in E$, denote by $\mathcal{C}(p)$ the so-called Čech groupoid, given by the data

$$\mathcal{C}(p) = X, \mathcal{C}(p) = X \ast_p X, \hat{r}(x_1, x_2) = x_1, \hat{s}(x_1, x_2) = x_2 \text{ for } p(x_1) = p(x_2);$$

the inverse $i(x_1, x_2) = (x_2, x_1)$ and multiplication $(x_1, x_2) \cdot (x_2, x_3) = (x_1, x_3)$ for $p(x_1) = p(x_2) = p(x_3)$.

Remark 2.14. We recall the definition of orbit spaces [13, Lem. 5.3]. Let $Z \rhd G$ be a given right action of a $\mathcal{T}$-local internal groupoid on $Z$, then the orbit-space is the coequalizer:

$$Z_q *_{\mathcal{T}} G \xrightarrow{\pi_2} Z \xrightarrow{\pi_\mathcal{T}} Z/G.$$

If the action is a principal $G$ bundle, then $q$ is equivalent to $\pi_\mathcal{T}$ and $\pi_\mathcal{T} \in E$.

Definition 2.15. a) A $\mathcal{T}$-local bibundle equivalence or Morita equivalence between $\mathcal{T}$-local groupoids $\mathcal{L}(1)$ and $\mathcal{L}(0)$ is given by the following data and relations:

$$\mathcal{L}(1) \rhd Z_f \rhd \mathcal{L}(0)$$

Notation: $f: \mathcal{L}(1) \rhd Z_f \rhd \mathcal{L}(0)$.

1. The actions are commuting $\mathcal{T}$-principal actions.
2. Denoting by $\{(V_i \to Z_f)_i\}_{i \in I}$ the defining covering family, $q_f$ induces an isomorphism $V_i/\mathcal{L}(0) \xrightarrow{\sim} q_f(V_i)$ and $p_f$ induces an isomorphism $V_i/\mathcal{L}(1) \xrightarrow{\sim} p_f(V_i)$ for each $i \in I$.
3. $(p_f, q_f)$ furnish an arrow $Z_f \to O_1 \times O_0$ contained in $E$.

b) We denote the equivalence class of $\mathcal{T}$-local Morita equivalences by $[\mathcal{L}(1) \rhd Z_f \rhd \mathcal{L}(0)] =: f: \mathcal{L}(1) \to \mathcal{L}(0)$ and we call this a Morita isomorphism, where the equivalence relation is defined as the span-isomorphisms that intertwine the actions. A representative of the class is called a span for $f$.

c) A bi-bundle correspondence between $\mathcal{L}(1)$ and $\mathcal{L}(0)$ is given by the same data, but with the variation on the relations that the right action is principal, $q_f \in E$ and $p_f$ induces the isomorphisms $V_i/\mathcal{L}(1) \xrightarrow{\sim} p_f(V_i)$, $i \in I$.

Lemma 2.16. A $\mathcal{T}$-local Morita isomorphism $f$ between $\mathcal{T}$-local spanoids is entirely determined by a span for $f$, up to span isomorphism.

Proof. Let $V_i \to Z_f$ be a cover that is part of a covering family with the property that $V_i/\mathcal{L}(0) \xrightarrow{\sim} p_f(V_i)$ is an isomorphism induced by $p_f$. Given a span $O_1 \xleftarrow{b} S \xrightarrow{a} O_0$ and arrows $g_0, g_1: S \to V_i$ such that $q_f \circ g_0 = a = q_f \circ g_1$ and $p_f \circ g_0 = b = q_f \circ g_1$. Define $h: S \to \mathcal{L}(0)$, $x \mapsto \gamma$ via $g_0(x) = g_1(x) \cdot \gamma$ and $\tilde{h}: S \to \mathcal{L}(0)$ via $x \mapsto a(x)$. Then $s_0 \circ h = s_0 \circ \tilde{h} = a$ and $r_0 \circ h = r_0 \circ \tilde{h} = a$. If $\mathcal{L}(0)$ is a $\mathcal{T}$-local spanoid, there is at most one span arrow between $O_0 \xleftarrow{b} S \xrightarrow{a} O_0$ and $O_0 \xrightarrow{r_0} \mathcal{L}(0) \xrightarrow{r_0} O_0$, hence $h = \tilde{h}$ and $g_0 = g_1$. The assertion follows. \qed
The main result of this section is the composition theorem for $\mathcal{T}$-local bibundle correspondences. We make use of the assumptions (1) through (3) as stated in Section 1.1.

**Theorem 2.17.** Let $g: \mathcal{L}(2) \circ Z_g \circ \mathcal{L}(1)$ and $f: \mathcal{L}(1) \circ Z_f \circ \mathcal{L}(0)$ be $\mathcal{T}$-local bibundle correspondences. There is a $\mathcal{T}$-local subgroupoid cover $\mathcal{H}_1 \to \mathcal{L}(1)$, as well as covers $c_g: V_1 \to Z_g$, $c_f: V_2 \to Z_f$ so that

$$g|_{\mathcal{L}(2), V_1, \mathcal{H}_1}: \mathcal{L}(2) \circ V_1 \circ \mathcal{H}_1, \quad f|_{\mathcal{H}_1, V_1, \mathcal{L}(0)}: \mathcal{H}_1 \circ V_2 \circ \mathcal{L}(0)$$

become composable in the sense that the product

$$V_1 \circ_{\mathcal{H}_1} V_2 := V_1 c_g q_g \circ_{c_f p_f} V_2 / \mathcal{H}_1$$

exists as an object in $\mathcal{A}_0$ and carries the canonical actions $\mathcal{L}(2) \circ V_1 \circ_{\mathcal{H}_1} V_2 \circ \mathcal{L}(0)$ that furnish a $\mathcal{T}$-local bibundle correspondence.

**Notation:** We denote the product by $Z_f \circ_{\text{loc}} Z_g$, with underlying span $Z_g \circ_{\text{loc}} Z_f$.

**Proof.** For a given admissible epimorphism $p: X \to Z \in \mathcal{E}$, consider $Z$ as a 0-groupoid, $X$ an equivalence, then we have the induced functor

$$\text{Sh}(\mathcal{E})_Z \to \text{Sh}(\mathcal{E})_{\text{C}(p)}.$$ 

The application of this functor, combined with a straightforward adaption of the argument of [13, proof of Prop. 7.6], making appropriate use of the slightly stronger assumption 3, furnishes the functor

$$\text{Sh}(\mathcal{E})_{\mathcal{L}(1)} \to \mathcal{A}, \quad W \mapsto Z \circ_{\text{loc}} W.$$ 

Here $\mathcal{L}(0) \circ Z_g \circ \mathcal{L}(1)$ denotes the given $\mathcal{T}$-local bibundle correspondence.

In other words, there exist subgroupoid covers $V_1, V_2$, that fit into the above diagram, in such a way that the local composition exists when constructed out of the induced coequalizer diagrams.

3. **Virtual Manifolds and Groupoid Reconstruction**

The sheaves are constructed using a pseudo-group generated by $\mathcal{T}$-local Morita isomorphisms, see also [6]. We make use of the embedding functor into the category of diffeological spaces - using assumption 4 of Section 1.1 - in order to specify a path holonomy diffeology on the quotient of the pseudo-group using the final diffeology generated by the natural projection. The main result is that the sheafification produces a category of $\mathcal{E}$-sheaves which globalize to groupoids that are internal to the site structure. Conditions on the path holonomy diffeology correspond to certain properties of the underlying foliation, e.g. in the special case of the ambient category of smooth manifolds, projectivity and involutivity of a foliation are reflected by properties of the associated path-holonomy diffeology.
**Definition 3.1.** A generalized atlas $\mathcal{U}_X := \{\mathcal{L}_i \Rightarrow U_i\}_{i \in I}$ consists of $\mathcal{T}$-local groupoids adapted to the covering family $\{\varphi_i: U_i \to X\}_{i \in I}$ such that

1. For each $i \in I$ the $\mathcal{T}$-local groupoid $\mathcal{L}_i$ is a $\mathcal{T}$-local spanoid.
2. For each $i, j \in I$ there are $\mathcal{T}$-local subgroupoids $\mathcal{H}^j_i \to \mathcal{L}_i$, $\mathcal{H}^j_j \to \mathcal{L}_j$, where the arrows are covers, and span-isomorphisms $\varphi_{ij}: \mathcal{H}^j_i \cong \mathcal{H}^i_j$.

**Remark 3.2.** The $\mathcal{T}$-local spanoid property yields that $\varphi_{ij}$ are uniquely determined, i.e. there is a maximal cover $\mathcal{D}(\varphi_{ij}) \to \mathcal{L}_i$ such that it factors through $U_i \ast_U U_j \to \mathcal{D}(\varphi_{ij})$; called the domain of $\varphi_{ij}$.

We describe operations on a $\mathcal{T}$-local Morita isomorphism $f: \mathcal{L}(1) \to \mathcal{L}(0)$ with a given span $\mathcal{O}_1 \xrightarrow{p_f} Z_f \xrightarrow{q_f} \mathcal{O}_0$ for $f$.

- **Identity:** $\operatorname{id}_{\mathcal{L}(0)}: \mathcal{L}(0) \to \mathcal{L}(0)$, $Z_{\operatorname{id}_{\mathcal{L}(0)}} = \mathcal{L}(0)$, $p_{\operatorname{id}_{\mathcal{L}(0)}} = s(0)$, $q_{\operatorname{id}_{\mathcal{L}(0)}} = r(0)$, with actions given by right and left multiplication.
- **Inversion:** $Z_{f^{-1}} = Z_f$, $p_{f^{-1}} = q_f$, $q_{f^{-1}} = p_f$. The right $\mathcal{T}$-local action $Z_{f^{-1}} \circ \mathcal{L}(1)$ (respectively $\mathcal{L}(0) \circ Z_{f^{-1}}$) is implemented by $\alpha_1(z, \gamma_1) = \gamma_1^{-1} \cdot z$ (respectively $\alpha_0(\gamma_0, z) = z \cdot \gamma_0^{-1}$). Then $f^{-1}: \mathcal{L}(0) \to \mathcal{L}(1)$ is a $\mathcal{T}$-local Morita isomorphism with span $\mathcal{O}_0 \xleftarrow{q_{f^{-1}}} Z_{f^{-1}} \xrightarrow{p_{f^{-1}}} \mathcal{O}_1$ for $f^{-1}$.
- **Restriction:** Let $\mathcal{H}_0 \to \mathcal{L}(0)$, $\mathcal{H}_1 \to \mathcal{L}(1)$ be $\mathcal{T}$-local subgroupoid covers and a cover $V \to Z_f$ such that $p_f(V)$ is the units of $\mathcal{H}_0$ and $q_f(V)$ is the units of $\mathcal{H}_1$. The restriction of $f$, denoted by $f|_{\mathcal{H}_1,V,\mathcal{H}_0} : \mathcal{H}_0 \to \mathcal{H}_0$, admits as span the restriction $q_f(V) \xrightarrow{q_f} V \xrightarrow{p_f} p_f(V)$ with $\mathcal{T}$-local action induced by $\mathcal{L}(1)$ and $\mathcal{L}(0)$.
- **Composition:** Explained by Theorem 2.17.

**Definition 3.3.** Let $\mathcal{U} := \mathcal{U}_X$ be a given generalized atlas and assume that $\mathcal{U}$ is stable with regard to restriction. Then the pseudogroup $\Psi_{\mathcal{U}}$ is defined to consist of local Morita isomorphisms between elements of $\mathcal{U}$ such that the identity is contained in $\Psi_{\mathcal{U}}$ and $\Psi_{\mathcal{U}}$ is stable with regard to inversion, local composition and restriction.

We introduce an equivalence relation on a pseudogroup and pass to the quotient; we do this, after utilization of the embedding $\Phi$, inside the category of diffeological spaces. Let $f: \mathcal{L}(1) \to \mathcal{L}(0)$ and $g: \mathcal{L}(1) \to \mathcal{L}(0)$ be $\mathcal{T}$-local Morita isomorphisms. Consider the corresponding spans $\mathcal{O}_1 \xrightarrow{p_f} Z_f \xrightarrow{q_f} \mathcal{O}_0$, $\mathcal{O}_1 \xrightarrow{p_g} Z_g \xrightarrow{q_g} \mathcal{O}_0$. By continuity, the functor $\Phi: (\Lambda, \mathcal{T}, \mathcal{E}) \to (\text{Diff}, \mathcal{D}\text{-open}, \mathcal{E}_{\text{loc}})$ maps the given spans to spans over $\mathcal{E}_{\text{loc}}$ (local subductions). Introduce germs, where $[f]_{z_f} = [g]_{z_g}$ if there are $D$-open neighborhoods $V_{z_f}$, $V_{z_g}$ in $Z_f$ and $Z_g$ respectively and $\mathcal{E}_{\text{loc}}$ span isomorphisms $\varphi: V_{z_f} \cong V_{z_g}$ such that $\varphi(z_f) = z_g$. Denote the underlying equivalence relation by $\sim$. Let us fix the notation $\Phi_{\Psi_{\mathcal{U}}} := \Phi_{\Psi_{\mathcal{U}}} \Phi$ for the pseudo-group formed out of the spans over $\mathcal{E}_{\text{loc}}$ that are images under the functor $\Phi$ of the corresponding spans over $\mathcal{E}$. Note that the span-generators of the pseudogroup $\Psi_{\mathcal{U}}$ form a cocycle. To this end, consider the span isomorphisms $\varphi_{ij}: \mathcal{H}^j_i \cong \mathcal{H}^j_j$, $\varphi_{jk}: \mathcal{H}^j_k \cong \mathcal{H}^j_j$ and $\varphi_{ik}: \mathcal{H}^i_k \cong \mathcal{H}^i_i$ between $\mathcal{T}$-local subgroupoids with given covers $\mathcal{H}^j_i \to \mathcal{L}(i)$, $\mathcal{H}^j_i \to \mathcal{L}(i)$, $\mathcal{H}^j_j \to \mathcal{L}(j)$, $\mathcal{H}^j_j \to \mathcal{L}(j)$, $\mathcal{H}^j_k \to \mathcal{L}(k)$, $\mathcal{H}^i_k \to \mathcal{L}(k)$. 

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**Internal Absolute Geometry 9**

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Then there are span isomorphisms completing the diagram:

Since the elements of $\Phi_{\mathcal{U}}$ for a given $\mathcal{U} = \mathcal{U}_X$, are uniquely determined by their representing spans up to span-isomorphism, we can form the quotient $\Phi_{\mathcal{U}}/\sim$ and endow it with the following so-called path holonomy diffeology $\mathcal{P}(\Phi_{\mathcal{U}})$. As preparation, let us recall some facts regarding diffeology, cf [9].

**Definition 3.4.** The set $\mathcal{P}(X)$ consisting of maps $\chi : \mathcal{O}_X \to X$ with $\mathcal{O}_X \subset \mathbb{R}^n$ open for some $n \in \mathbb{N}$ is called a *diffeology* on $X$ and its elements are called *plots*, if the following conditions hold:

1. The constant functions $\mathbb{R}^n \to x \subset X$ are contained in $\mathcal{P}(X)$ for each $x \in X$, $n \in \mathbb{N}$.
2. If $\chi : \mathcal{O}_X \to X$ is such that for each $y \in \mathcal{O}_X$ there is an open subset $V \subset \mathcal{O}_X$ such that $\chi|_V \in \mathcal{P}(X)$, then $\chi \in \mathcal{P}(X)$.
3. For each smooth $f : V \to \mathcal{O}_X$, $V \subset \mathbb{R}^n$ open, the composition with a given plot $\chi$, $\chi \circ f$ is contained in $\mathcal{P}(X)$.

We endow the quotient $\Phi_{\mathcal{U}}/\sim$ with the quotient diffeology $\mathcal{P}(\Phi_{\mathcal{U}})$ which is the final diffeology induced by the projection $q : \Phi_{\mathcal{U}} \to \Phi_{\mathcal{U}}/\sim$. The D-topology of this diffeology coincides with the quotient topology, when the pseudo-group is endowed with its canonical D-topology, cf. [2, 5, 9].

**Proposition 3.5.** Let $\mathcal{U}$ be some generalized atlas over $X \subset \mathcal{A}_0$.

1. The quotient map $q : \Phi_{\mathcal{U}} \to \Phi_{\mathcal{U}}/\sim$ is a local subduction.
2. Any two final diffeologies over (maximal) generalized atlases $\mathcal{U}$ and $\mathcal{U}'$ over $X$ have the same path-holonomy diffeology.

**Proof.** We show (1), the proof of (2) goes along similar lines; see also [2] for a similar setting. Consider an element $\mathcal{L}(U) \rightrightarrows U$ in $\mathcal{U}_X$ and let $\mathcal{H}_U \to \mathcal{L}(U)$ denote the defining cocycle subgroupoid, up to a fixed span isomorphism using Lemma 2.16. Fix an element $w \in \mathcal{H}_U$, $q(w) = z$ and take a plot $\chi : \mathcal{O}_X \to \Phi_{\mathcal{U}}/\sim$ and $x \in \mathcal{O}_X$ with $\chi(x) = z$. There is a connected open neighborhood $V$ of $x$ and a plot $\chi' : V \to \Phi_{\mathcal{U}}$ so that $\chi|_V = q \circ \chi'$. Then $w' = \chi'(x)$ is in some $\mathcal{H}_{U'}$ and shrinking $U'$ if necessary, there is a span isomorphism $\varphi : \mathcal{H}_U \to \mathcal{H}_{U'}$, with $q(w') = q(w) = z$ and $\varphi \circ \chi' : V \to \mathcal{H}_U$ furnishes a lifting. \hfill \Box

Given $[f]_z \in \Phi_{\mathcal{U}}/\sim$ and let $f : \mathcal{L}(1) \to \mathcal{L}(0)$ be a representative with a given span $\mathcal{O}_1 \leftarrow Z_f \rightarrow \mathcal{O}_0$ and $z \in Z_f$. By Lemma 2.16 the map $Z_f \to \Phi_{\mathcal{U}}/\sim$, $w \mapsto [f]_w$ is $D$-locally injective. Also, define the map $X \to \Phi_{\mathcal{U}}/\sim$, $x \mapsto [id_{\mathcal{L}}]_x$ where $\mathcal{L}$ is some element of $U$ and $x$ is contained in $\mathcal{L}_0$.

**Theorem 3.6.** The groupoid $\mathcal{G} := \Phi_{\mathcal{U}}/\sim$ with units $X$ for a given generalized atlas $\mathcal{U} := \mathcal{U}_X$ with $X \subset \mathcal{A}$ is a spanoid internal to the site structure $(\mathcal{A}, \mathcal{T}, \mathcal{E})$. 

Proof. The unit inclusion \( u: X \to \mathcal{G} \) which maps to the identities of the corresponding maximal spanoids. The source \( s: \mathcal{G} \to X \) maps \( [\mathcal{L}(1) \circ Z_f \circ \mathcal{L}(2)]_z \to q_f(z) \) and \( r: \mathcal{G} \to X \) maps \( [\mathcal{L}(1) \circ Z_f \circ \mathcal{L}(2)]_z \to p_f(z) \). The composition is defined by

\[
[\mathcal{L}(2) \circ Z_g \circ \mathcal{L}(1)]_t \cdot [\mathcal{L}(1) \circ Z_f \circ \mathcal{L}(0)]_z = [\mathcal{L}(2) \circ Z_g \circ \mathcal{L}(0)]_{(t,z)}.
\]

The inverse is defined as

\[
i: \mathcal{G} \to \mathcal{G}, \quad [\mathcal{L}(1) \circ Z_f \circ \mathcal{L}(0)]^{-1} = [\mathcal{L}(0) \circ Z_f^{-1} \circ \mathcal{L}(1)]
\]

and this furnishes the groupoid structure on \( \mathcal{G} \). Let \( U \to X \) be a cover in \( \mathcal{T} \). Define \( \nu: U \to \mathcal{G} \) to be a \( \mathcal{T} \)-local section of both \( r \) and \( s \). Take an element \( \mathcal{L} \supset U \) with source / range denoted by \( \overrightarrow{s}, \overrightarrow{r} \) of \( U \). Denote by \( f: \mathcal{L} \to \mathcal{L} \) a \( \mathcal{T} \)-local Morita isomorphism implemented by the span \( U \xleftarrow{p_f} \mathcal{L} \xrightarrow{q_f} U \) such that the image of \( \nu \) is a subset of \( \{ [f]_z : z \in Z_f \} \). Then there is a unique arrow \( \overrightarrow{\nu} \) in \( \mathcal{A} \) such that \( \nu(x) = [f]_{\overrightarrow{\nu}(x)}, \ x \in U \) and \( p_f \circ \overrightarrow{\nu} = q_f \circ \overrightarrow{\nu} = \text{id}_U \). Consider the arrow \( \gamma \to \gamma \cdot \overrightarrow{\nu(\gamma)} \) denoted \( \varphi: \mathcal{L} \to Z_f \). The multiplication sign here refers to the canonical \( \mathcal{T} \)-local action; in particular \( \varphi \) induces an isomorphism between spans in \( \mathcal{A} \). This isomorphism is mapped via \( \Phi \) so that there is a \( D \)-local neighborhood \( W \) of \( U \) in \( \mathcal{L} \) such that

\[
[id_{\mathcal{L}}]_w = [f]_{\varphi(w)}, \ w \in W.
\]

If \( x \in U \), we get that \( [id_{\mathcal{L}}]_x = [f]_{\varphi(x)} = [f]_{\overrightarrow{\nu}(x)} = \nu(x) \). If \( \mathcal{G} \) is \( \mathcal{A} \)-internal this will then furnish the spanoid property of \( \mathcal{G} \) by Proposition 2.10, since we just showed that \( \nu \) is the inclusion of units. We are left to check that \( \nu \) is an admissible monomorphism, or dually that \( \nu^\text{op}: \mathcal{G} \to X \) is an admissible epimorphism, i.e. a universally locally sectionable arrow. To this end, let us consider the decomposition \( \mathcal{G} = \bigsqcup_{x \in X} \mathcal{G}_x \) into the sections and define the canonical action of \( \mathcal{G} \) on itself with anchor given by the range \( r \), as defined above. Let \( (c_i: U_i \to X)_{i \in I} \) be the covering family that defines the generalized atlas \( \mathcal{U} \). Then there is a lifting \( f: U_i \to \mathcal{G} \) which is defined by taking a possibly larger \( D \)-open neighborhood \( U \) and mapping it into a \( \mathcal{G}_x \) via \( \nu \), for a given \( x \in U_i \subset U \) so that \( r \circ f = c_i \). This property of \( r \) is invariant under a change of base by Prop. 3.5 (2). Hence \( r \) is an admissible epimorphism.

\[
\square
\]

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