On the arithmetical rank of certain monomial ideals

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Abstract We determine a new technique which allows the computation of the arithmetical rank of certain monomial ideals.

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Introduction and Preliminaries

Given a commutative ring with identity $R$, the arithmetical rank of an ideal $I$ of $R$, denoted $\text{ara } I$, is defined as the minimum number of elements which generate $I$ up to radical, i.e., generate an ideal which has the same radical as $I$. Determining this number is, in general, a very hard open problem; a trivial lower bound is given by the height of $I$, but this is the actual value of $\text{ara } I$ only in special cases ($I$ is then called a set-theoretic complete intersection). There are, however, techniques which allow us to provide upper bounds. Some results in this direction have been recently proved in [1], [3], [5], [2], and less recently in [1]. These apply especially to the case where $R$ is a polynomial ring over a field, and $I$ is a monomial ideal (i.e., an ideal generated by products of indeterminates) and are essentially based on the following criterion by Schmitt and Vogel (see [14], p. 249).

Lemma 1 Let $P$ be a finite subset of elements of $R$. Let $P_0, \ldots, P_r$ be subsets of $P$ such that

(i) $\bigcup_{i=0}^{r} P_i = P$;

(ii) $P_0$ has exactly one element;

(iii) if $p$ and $p''$ are different elements of $P_i$ ($0 < i \leq r$) there is an integer $i'$ with $0 \leq i' < i$ and an element $p' \in P_{i'}$ such that $pp'' \in (p')$.

Let $0 \leq i \leq r$, and, for any $p \in P_i$, let $e(p) \geq 1$ be an integer. We set $q_i = \sum_{p \in P_i} p^{e(p)}$. We will write $(P)$ for the ideal of $R$ generated by the elements of $P$. Then we get

$\sqrt{(P)} = \sqrt{(q_0, \ldots, q_r)}$.
If, in the construction given in the claim, we take all exponents $e(p)$ to be equal to 1, then $q_0, \ldots, q_r$ are sums of generators; in this paper we present a new method, which gives rise to elements of the same form, but applies under a different assumption. It will enable us to determine the arithmetical rank of certain monomial ideals which could not be treated by the above lemma.

For the determination of the arithmetical rank, every ideal can be replaced by its radical. In the sequel we will therefore throughout consider radical (or reduced) monomial ideals, i.e., ideals generated by squarefree monomials. These are the so-called Stanley-Reisner ideals of simplicial complexes. For the basic notions on this topic we refer to [7], Section 5.

It is well-known that a reduced monomial ideal which is a set-theoretic complete intersection is Cohen-Macaulay; for the Stanley-Reisner ideals of one-dimensional simplicial complexes this condition is independent of the ground field, and is equivalent to the connectedness of the simplicial complex. It is not known whether all reduced monomial ideals which are Cohen-Macaulay are set-theoretic complete intersections. The question is open even for Gorenstein ideals.

The problem of the arithmetical rank of monomial ideals has been intensively studied by several other authors over the past three decades: see [8], [9], [10], [11], [12], [13], [14], [15] and [16].

1 The first result and some applications

Let $R$ be a commutative ring with identity.

**Proposition 1** Let $p_0, p_1, p_2, p_3, p_4 \in R$ be such that $p_0$ divides $p_1 p_2$, $p_1$ divides $p_2 p_3$, and $p_2$ divides $p_3 p_4$. Set $p_1 = p_1 + p_2$ and $p_2 = p_2 + p_3$. Then

$$\sqrt{(p_0, p_1, p_2)} = \sqrt{(p_0, p_1, p_2, p_3, p_4)}.$$

**Proof.** It suffices to show that $p_1, p_2, p_3, p_4 \in \sqrt{(p_0, p_1, p_2)}$. Let $a, b \in R$ be such that $p_3 = ap_0$ and $p_4 = bp_2$. Then

$$\begin{align*}
p_1^3 &= p_1^2 (p_1 + p_2) - p_1^2 p_2 = p_1^2 (p_1 + p_2) - p_1 b p_2 \\
    &= p_1^2 (p_1 + p_2) - b p_1 (p_2 + p_2) + b p_1 p_2 \\
    &= p_1^2 (p_1 + p_2) - b p_1 (p_2 + p_2) + b a p_0 \\
    &= p_1^2 (p_1 + p_2) - b p_1 p_2 + b a p_0 \
\end{align*}$$

This shows that $p_1 \in \sqrt{(p_0, p_1, p_2)}$, whence $p_2 = p_1 - p_1 \in \sqrt{(p_0, p_1, p_2)}$. The claim for $p_2, p_3, p_4$ follows by symmetry. This completes the proof.

The above result can be used for computing the arithmetical rank of some monomial ideals. We first apply it for proving the set-theoretic complete intersection property of the Stanley-Reisner ideals of two simplicial complexes. Recall that if the maximal faces of a simplicial complex on $N$ vertices all have the same cardinality $d$, then its Stanley-Reisner ideal has pure height $N - d$. Any Stanley-Reisner ideal that is Cohen-Macaulay has pure height (see [7], Corollary
5.1.5), so that this holds, in particular, for set-theoretic complete intersections.

**Example 1** In the polynomial ring $R = K[x_1, \ldots, x_5]$, where $K$ is any field, consider the ideal

$$I = (x_1 x_3, x_1 x_4, x_2 x_4, x_2 x_5, x_3 x_5),$$

which is the Stanley-Reisner ideal of the Cohen-Macaulay simplicial complex on the vertex set $\{1, \ldots, 5\}$ whose maximal faces are $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{4, 5\}$, $\{5, 1\}$. It is of pure height 3. Proposition 1 applies to $p_0 = x_1 x_3$, $p_{11} = x_1 x_4$, $p_{12} = x_2 x_5$, $p_{21} = x_2 x_4$, $p_{22} = x_3 x_5$, so that

$$I = \sqrt{(x_1 x_3, x_1 x_4 + x_2 x_5, x_2 x_4 + x_3 x_5)}.$$ 

Hence $\text{ara } I = 3$, and $I$ is a set-theoretic complete intersection. Note that three elements of $R$ generating $I$ up to radical cannot be found by applying Lemma 1 to the set of minimal monomial generators of $I$.

**Example 2** In $R = K[x_1, \ldots, x_6]$ consider the ideal

$$I = (x_1 x_3, x_1 x_4, x_1 x_5, x_2 x_4, x_2 x_5, x_3 x_5, x_3 x_6, x_4 x_6),$$

which is the Stanley-Reisner ideal of the Cohen-Macaulay simplicial complex on the vertex set $\{1, \ldots, 6\}$ whose maximal faces are $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{4, 5\}$, $\{5, 1\}$, $\{5, 6\}$, $\{2, 6\}$. It has pure height 4. It is a set-theoretic complete intersection, since, as we will see,

$$I = \sqrt{(x_1 x_6, x_3 x_5, x_1 x_3 + x_2 x_4 + x_3 x_6, x_1 x_4 + x_2 x_5 + x_4 x_6)}.$$ 

(1)

We only need to prove the inclusion $\subset$. Note that, according to Proposition 1

$$\sqrt{(x_3 x_5, x_3 x_6 + x_2 x_4, x_4 x_6 + x_2 x_5)} = (x_3 x_5, x_3 x_6, x_2 x_4, x_4 x_6, x_2 x_5).$$

Therefore, to prove (1) it suffices to show that $x_1 x_3, x_1 x_4$ belong to the right-hand side of (1). This is true, because, firstly

$$x_1^3 x_3^2 = x_3^3 (\quad x_2 x_4 - x_1 x_3 \quad) x_1 x_6 + x_1 x_2 x_3 x_5 x_3 x_5 + x_2 x_3 x_5 (x_1 x_3 + x_2 x_4 + x_3 x_6) - x_1 x_2 x_3 (x_1 x_4 + x_2 x_5 + x_4 x_6),$$

which proves the claim for $x_1 x_3$, and secondly,

$$x_1^2 x_4^2 = (x_3 x_5 - x_2^2) x_1 x_6 + x_1^2 x_3 x_5 - x_1 x_5 (x_1 x_3 + x_2 x_4 + x_3 x_6) + x_1 x_4 (x_1 x_4 + x_2 x_5 + x_4 x_6).$$

Next we apply Proposition 1 to a class of ideals which extends Example 1. We will determine the arithmetical rank of these ideals and show that they are not set-theoretic complete intersections except for the ideal studied in that example.
Example 3 Let \( m \geq 2 \) be an integer, and let \( I_m \) be the reduced monomial ideal of \( R = K[x_1, \ldots, x_{3m+3}] \) generated by the following monomials:

\[
r_1 = x_1 x_2, \quad s_n = x_{3n-2} x_{3n+2} \\
t_n = x_{3n+1} x_{3n+3} \\
u_n = x_{3n+1} x_{3n+2} \\
v_n = x_{3n-1} x_{3n+3} \quad (n = 1, \ldots, m).
\]

We prove that

\[
\text{ara} I_m = 2m + 1. \tag{2}
\]

Let \( J_m \) be the ideal of \( R \) generated by the following \( 2m+1 \) elements:

\[
x_1 x_2, \quad s_n + t_n, \quad u_n + v_n \quad (n = 1, \ldots, m).
\]

We first show that

\[
\text{ara} I_m \leq 2m + 1 \tag{3}
\]

by proving that

\[
I_m = \sqrt{J_m}.
\]

It suffices to show that \( I_m \subset \sqrt{J_m} \), i.e., that \( s_n, t_n, u_n, v_n \in \sqrt{J_m} \) for \( 1 \leq n \leq m \).

We proceed by finite induction on \( n \), \( 1 \leq n \leq m \). For \( n = 1 \) the claim follows by applying Proposition 1 to the following elements:

\[
p_0 = r_1 = x_1 x_2, \quad p_1 = s_1 + t_1 = x_1 x_6 + x_4 x_6, \quad p_2 = u_1 + v_1 = x_4 x_5 + x_2 x_6.
\]

Now suppose that \( n > 1 \) and that the claim is true for \( n-1 \). Then, in particular, \( u_{n-1} = x_{3n-2} x_{3n-1} \in \sqrt{J_m} \). Since \( u_{n-1} \) divides \( s_n v_n \), \( t_n \) divides \( u_n v_n \) and \( u_n \) divides \( s_n t_n \),

\[
p_0 = u_{n-1}, \quad p_1 = s_n + t_n, \quad p_2 = u_n + v_n
\]

fulfil the assumption of Proposition 1. It follows that

\[
s_n, t_n, u_n, v_n \in \sqrt{p_0, p_1, p_2} \subset \sqrt{J_m},
\]

which achieves the induction step and proves (3).

We now show the opposite inequality. Let

\[
S = \{x_1\} \cup \{x_{3n+2}, x_{3n+3}, | n = 1, \ldots, m\},
\]

and set \( P = (S) \). Then \( P \) is a prime ideal and \( I_m \subset P \), since \( x_1 \) divides \( r_1 \) and, for all \( n = 1, \ldots, m \), \( x_{3n+2} \) divides \( s_n \) and \( u_n \), and \( x_{3n+3} \) divides \( t_n \) and \( v_n \). Ideal \( P \) is in fact a minimal prime of \( I_m \), because

\[
r_1 \notin (S \setminus \{x_1\}), \text{ and } u_n \notin (S \setminus \{x_{3n+2}\}), \text{ and } t_n \notin (S \setminus \{x_{3n+3}\}) \quad (n = 1, \ldots m).
\]

We have that \( \text{height } P = 2m + 1 \). By Krull’s principal ideal theorem it follows that

\[
2m + 1 \leq \text{ara} I_m,
\]
as required. This completes the proof of (2). Now let
\[ T = \{x_2\} \cup \{x_{3n+1}, x_{3n+2}, | n = 1, \ldots, m - 1\} \cup \{x_{3m+1}\}, \]
and set \( Q = (T) \). Then \( I_m \subset Q \), since
- \( x_2 \) divides \( r_1, v_1 \);
- for \( n = 1, \ldots, m - 1 \), \( x_{3n+2} \) divides \( s_n \) and \( u_n \), and \( x_{3n+1} \) divides \( t_n \);
- for \( n = 2, \ldots, m \), \( x_{3(n-1)+2} = x_{3m-1} \) divides \( v_n \);
- \( x_{3(m-1)+1} = x_{3m-2} \) divides \( s_m \);
- \( x_{3m+1} \) divides \( u_m \) and \( t_m \).

Ideal \( Q \) is a minimal prime of \( I_m \), because
- \( r_1 \notin (T \setminus \{x_2\}) \);
- \( t_n \notin (T \setminus \{x_{3n+1}\}) \) \( (n = 1, \ldots, m) \);
- \( v_{n+1} \notin (T \setminus \{x_{3n+2}\}) \) \( (n = 1, \ldots, m - 1) \).

We have that height \( Q = 1 + 2(m - 1) + 1 = 2m < 2m + 1 \), so that the height of \( I_m \) is less than \( 2m + 1 \). Hence \( I \) is not a set-theoretic complete intersection.

Let us remark that for \( m = 1 \), ideal \( I_m \) is, up to a change of variables, the same as ideal \( I \) of Example 1.

Finally we present an example of a Stanley-Reisner ideal which is a set-theoretic complete intersection over any field \( K \) with more than two elements, but, unlike in all previously examined cases, the minimal set of elements which generate \( I \) up to radical strictly depends on \( K \).

**Example 4** Let \( I \) be the Stanley-Reisner ideal of \( R = K[x_1, \ldots, x_6] \) associated with the Cohen-Macaulay simplicial complex on the vertex set \( \{1, \ldots, 6\} \) whose maximal faces are \( \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}, \{4, 6\}, \{5, 6\} \). Then
\[ I = (x_1x_3, x_1x_4, x_1x_6, x_2x_4, x_2x_5, x_2x_6, x_3x_5, x_3x_6, x_4x_5x_6), \]
and \( I \) is of pure height 4. We show that ara \( I = 4 \). First we assume that char \( K \neq 2 \). We set
\[ J = (x_1x_4 + x_3x_5, x_1x_4 + x_2x_6 + x_4x_5x_6, x_1x_6 + x_2x_5, x_2x_4 + x_3x_6) \]
and prove that
\[ I = \sqrt{J}. \tag{4} \]

We have that \( x_4^2x_5^2x_6^2 \in J \), since
\[
\begin{align*}
x_4^2x_5^2x_6^2 &= \frac{1}{2}x_6^2 - x_1x_4(x_1x_4 + x_3x_5) + x_4x_5x_6(x_1x_3 + x_2x_6 + x_4x_5x_6) + x_4(x_1x_4 - x_6^2)(x_1x_6 + x_2x_5) - \frac{1}{2}x_5(x_1x_4 + x_3x_5)^2.
\end{align*}
\]
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which implies that $x_4x_5x_6 \in \sqrt{J}$. Moreover, $x_3^2x_5^2 \in \sqrt{J}$ (and hence $x_3x_5 \in \sqrt{J}$), because

$$x_3^2x_5^2 = (x_3x_5 - \frac{1}{2}x_6^2)(x_1x_4 + x_3x_5) - x_4x_5(x_1x_3 + x_2x_6) + \frac{1}{2}x_4x_6(x_1x_6 + x_2x_5) + \frac{1}{2}x_5x_6(x_2x_4 + x_3x_6),$$

where $x_1x_3 + x_2x_6 \in \sqrt{J}$. On the other hand we know that $x_1x_6 + x_2x_5 \in \sqrt{J}$.

Now Proposition 4 applies to $p_0 = x_3x_5$, $p_{11} = x_1x_3$, $p_{12} = x_2x_6$, $p_{21} = x_1x_6$, $p_{22} = x_2x_5$, whence $x_1x_3, x_2x_6, x_1x_6, x_2x_5 \in \sqrt{J}$. Note that $x_3x_5 \in \sqrt{J}$ also implies that $x_1x_4 \in \sqrt{J}$. Finally, from $x_2x_6, x_2x_4 + x_3x_6 \in \sqrt{J}$ we deduce, by Lemma 1 that $x_2x_4, x_3x_6 \in \sqrt{J}$. We have thus proven that $I \subset \sqrt{J}$; since the opposite inclusion is obvious, (4) follows. Now assume that $\text{char} \ K = 2$, and $K$ has more than two elements. Let $t \in K \setminus \{0, 1\}$. This time we set

$$J = (x_1x_4 + x_3x_5, x_1x_3 + x_2x_6 + x_4x_5x_6, x_1x_6 + tx_2x_5, x_2x_4 + x_3x_6).$$

We show that

$$I = \sqrt{J}.$$  

As above, the claim will follow once we have proven that $x_4x_5x_6, x_3x_5 \in \sqrt{J}$.

In fact we have:

$$x_2^2x_5^3x_6^4 = \frac{1}{t + 1}[(x_6^4 - tx_1x_2x_3 - tx_2^2x_6)(x_1x_4 + x_3x_5) + (t + 1)x_6(x_4x_5x_6 - x_1x_3)(x_1x_3 + x_2x_6 + x_4x_5x_6) + (x_1x_3^2 + x_2x_3x_6 - x_4x_6^3)(x_1x_6 + tx_2x_5) + (tx_1x_3 + tx_1x_2x_6 - x_5x_6^3)(x_2x_4 + x_3x_6)],$$

and

$$x_3^2x_5^2 = \frac{1}{t + 1}[(t + 1)x_3x_5 - x_6^2)(x_1x_4 + x_3x_5) - (t + 1)x_4x_5(x_1x_3 + x_2x_6) + x_4x_6(x_1x_6 + tx_2x_5) + x_5x_6(x_2x_4 + x_3x_6)].$$

In [6], Example 4, we determined a set of four polynomials which generate $I$ up to radical over any algebraically closed field, and, in particular, are independent of the characteristic. They, however, unlike those presented in this example, are not formed by linear combinations of the minimal monomial generators.

2 The second result

The ring considered in this section is a polynomial ring $R = K[x_1, \ldots, x_N]$, where $K$ is an algebraically closed field. We will determine the arithmetical rank of certain ideals generated by monomials. We prove a result, based on combinatorial considerations, which generalizes both Lemma 1 and Proposition 4 for this class of ideals and which will allow us to prove the set-theoretic intersection property in various examples.
Proposition 2 Let $G \subset R$ be a set of monomials. Suppose that there are subsets $S_0, \ldots, S_r$ of $G$ such that

(i) $\bigcup_{i=0}^r S_i = G$;

(ii) $S_0$ has exactly one element;

(iii) the following recursive procedure can always be performed and always comes to an end regardless of the choice of the indeterminate $z$ and the index $j$ at each step.

0. Set $T = S_0$.

1. Pick an indeterminate $z$ dividing the only element of $T$.

2. Cancel all monomials divisible by $z$ in every $S_i$.

3. If no element of $G$ is left, then end. Else pick an index $j$ such that there is exactly one element left in $S_j$ and set $T = S_j$.

4. Go to 1.

For all $i = 0, \ldots, r$ we set $q_i = \sum_{\mu \in S_i} \mu$. Then we get

$\sqrt{(G)} = \sqrt{(q_0, \ldots, q_r)}$.

Proof. It suffices to show that $\sqrt{(G)} \subset \sqrt{(q_0, \ldots, q_r)}$. We proceed by induction on $r \geq 0$. For $r = 0$ we have that $(G) = (S_0) = (q_0)$, so that the claim is trivially true. Now suppose that $r > 0$ and that the claim is true for all smaller $r$. According to Hilbert’s Nullstellensatz, it suffices to show that, whenever all $q_i$ vanish at some $x \in K^N$, the same is true for all $\mu \in G$. In the sequel, as long as this does not cause any ambiguity, we will denote a polynomial and its value at $x$ by the same symbol. So assume that $q_i = 0$ for all $i = 0, \ldots, r$. From $q_0 = 0$ we deduce that one of the indeterminates dividing the only element of $S_0$, say the indeterminate $z$, vanishes. Then all $\mu \in G$ that are divisible by $z$ vanish. Let $\tilde{G}$ be the set of $\mu \in G$ that are not divisible by $z$. We have to show that all $\mu \in \tilde{G}$ vanish. If $\tilde{G} = \emptyset$, then there is nothing to be proven. Otherwise, for all $i = 1, \ldots, r$, set $\tilde{S}_i = S_i \cap \tilde{G}$. By assumption we have that $|\tilde{S}_j| = 1$ for some index $j$: up to a change of indices we may assume that $j = 1$. Then $\tilde{G}$ and its subsets $\tilde{S}_1, \ldots, \tilde{S}_r$ fulfill the assumption of the proposition. For all $i = 1, \ldots, r$ we set $\tilde{q}_i = \sum_{\mu \in \tilde{S}_i} \mu$. Then by induction $\sqrt{(\tilde{G})} = \sqrt{(\tilde{q}_1, \ldots, \tilde{q}_r)}$. Since, by assumption, all $\tilde{q}_i$ vanish, this implies that all $\mu \in \tilde{G}$ vanish, as required. This completes the proof.

Remark 1 (i) Note that in the above recursive procedure, the only element left in $S_j$ at step 3 is cancelled as soon as step 2 is performed. We could therefore cancel it right away.
(ii) In the proof of Proposition 2, we interpret the recursive procedure in terms of the vanishing of monomials: cancelling an indeterminate is the same as supposing that this indeterminate vanishes; cancelling a monomial means concluding that it vanishes. Finishing the procedure means cancelling all monomials of $G$, i.e., concluding that they all vanish.

**Remark 2** Proposition 2 generalizes Lemma 1 for ideals generated by monomials over an algebraically closed field. In fact, if $P$ is a set of monomials and $R_0, \ldots, R_r$ are as in the assumption of Lemma 1 and we set $G = P$ and $S_i = P_i$ for all $i = 0, \ldots, r$, then the assumption of Proposition 2 is fulfilled, as we are going to show next. Let $z$ be any indeterminate dividing the only element of $S_0$. As in the proof of Proposition 2 for all $i = 1, \ldots, r$, let $S_i$ denote the set of monomials in $S_i$ which are not divisible by $z$. Let $j$ be the smallest index $j > 0$ such that $S_j$ is not empty. Then the product of each two distinct monomials of $S_j$ is divisible by a monomial of some $S_i, i < j$, which is divisible by $z$. Hence all but possibly one of the monomials of $S_j$ are divisible by $z$. Therefore $|S_j| \leq 1$. We conclude by finite descending induction.

**Remark 3** If the elements $p_0, p_1, p_{12}, p_{21}, p_{22}$ fulfilling the assumption of Proposition 1 are monomials, then it can be easily verified that the assumption of Proposition 2 is fulfilled by $S_0 = \{p_0\}$, $S_1 = \{p_1, p_{12}\}$ and $S_2 = \{p_{21}, p_{22}\}$.

**Example 5** In the ring $R = K[x_1, \ldots, x_6]$ consider the ideal $I$ generated by the following set of squarefree monomials:

$$G = \{x_1x_3, x_1x_4, x_1x_5, x_2x_4, x_2x_5, x_2x_6, x_3x_5, x_3x_6, x_4x_6\}.$$

We verify that the following subsets of $G$ fulfil the assumption of Proposition 2:

- $S_0 = \{x_3x_6\}$
- $S_1 = \{x_1x_4, x_2x_5\}$
- $S_2 = \{x_1x_3, x_2x_4, x_3x_5\}$
- $S_3 = \{x_1x_5, x_2x_6, x_4x_6\}$

0. Since $x_3x_6$ is the only element of $S_0$, we cancel it.
1. First pick $x_3$, and cancel $x_1x_3, x_3x_5$. Then $x_2x_4$ is the only element left in $S_2$. We cancel it.

For the remaining part of the procedure, we have two possible subcases.

1.1. Pick $x_2$ and cancel $x_2x_5, x_2x_6$. Then $x_1x_4$ is the only element left in $S_1$, and we cancel it. Then we must pick $x_1$ or $x_4$; in the former case we cancel $x_1x_5$, and there is only $x_4x_6$ left in $S_3$; in the latter case the roles of $x_1x_5$ and $x_4x_6$ are interchanged. Hence, in any case, all monomials are cancelled.

1.2. Pick $x_4$ and cancel $x_1x_4, x_4x_6$. Then $x_2x_5$ is the only element left in $S_1$, and we cancel it. Then we must pick $x_2$ or $x_5$; but the first choice takes us back to the previous subcase. In the second case, we cancel $x_1x_5$, and there is only $x_2x_6$ left in $S_3$; we cancel it, and thus all monomials have been cancelled.

2. Then pick $x_6$, and cancel $x_2x_6, x_4x_6$. Then $x_1x_5$ is the only element left in
Therefore, we again have two possible subcases. In the second case, we cancel the Stanley-Reisner ideal associated with an 
N-gon is always Gorenstein. For N = 4 this ideal is generated by \(x_1x_3\) and \(x_2x_4\) and is therefore a complete intersection. We are not able to say whether the Stanley-Reisner ideal associated with an N-gon is a set-theoretic complete intersection for \(N \geq 7\).

In addition to Example 5, we present two more examples of simplicial complexes on \(N = 6\) vertices whose maximal faces have cardinality \(d = 2\) and whose Stanley-Reisner ideals are set-theoretic complete intersections. The combinatorial verifications are left to the reader.

Example 6 Let \(R = K[x_1, \ldots, x_6]\). The Stanley-Reisner ideal of the Cohen-Macaulay simplicial complex on the vertex set \(\{1, \ldots, 6\}\) whose maximal faces are \(\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 1\}, \{2, 5\}\) is

\[I = (x_1x_3, x_1x_4, x_1x_5, x_2x_4, x_2x_6, x_3x_5, x_3x_6, x_4x_6).\]

From Proposition 2, we can deduce that

\[I = \sqrt{(x_1x_4, x_3x_6, x_1x_3 + x_1x_5 + x_2x_4, x_2x_6 + x_3x_5 + x_4x_6)}.\]

Example 7 The Stanley-Reisner ideal of the Cohen-Macaulay simplicial complex on the vertex set \(\{1, \ldots, 6\}\) whose maximal faces are \(\{1, 2\}, \{2, 3\}, \{3, 4\}, \ldots\)
From Proposition 2 we can deduce that

\[ I = \sqrt{(x_1 x_4, x_3 x_6, x_1 x_3 + x_2 x_4 + x_1 x_2 x_5, x_3 x_5 + x_4 x_6 + x_1 x_2 x_6 + x_1 x_5 x_6 + x_2 x_5 x_6)} \]

Many more examples could be provided. We only give one for \( N = 8, d = 2 \).

**Example 8** Let \( R = K[x_1, \ldots, x_8] \). The Stanley-Reisner ideal of the Cohen-Macaulay simplicial complex on the vertex set \( \{1, \ldots, 8\} \) whose maximal faces are \( \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}, \{5, 6\}, \{6, 7\}, \{7, 8\}, \{8, 4\} \) is

\[ I = (x_1 x_3, x_1 x_4, x_1 x_6, x_1 x_7, x_1 x_8, \\
   x_2 x_4, x_2 x_5, x_2 x_6, x_2 x_7, x_2 x_8, \\
   x_3 x_5, x_3 x_6, x_3 x_7, x_3 x_8, x_4 x_6, x_4 x_7, x_5 x_7, x_5 x_8, x_6 x_8). \]

From Proposition 2 we can deduce that

\[ I = \sqrt{(x_3 x_6, x_1 x_8 + x_2 x_7, x_1 x_3 + x_2 x_8 + x_3 x_7, x_1 x_7 + x_2 x_6 + x_6 x_8, \\
   x_1 x_4 + x_2 x_5 + x_3 x_6 + x_4 x_7 + x_5 x_8, x_1 x_6 + x_2 x_4 + x_3 x_5 + x_4 x_6 + x_5 x_7)} \]

Since \( I \) is an ideal of pure height 6, we conclude that it is a set-theoretic complete intersection.

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