Safety-Guarantee Controller Synthesis for Cyber-Physical Systems

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ABSTRACT
The verification and validation of cyber-physical systems is known to be a difficult problem due to the different modeling abstractions used for control components and for software components. A recent trend to address this difficulty is to reduce the need for verification by adopting correct-by-design methodologies. According to the correct-by-design paradigm, one seeks to automatically synthesize a controller that can be refined into code and that enforces temporal specifications on the cyber-physical system. In this paper we consider an instance of this problem where the specifications are given by a fragment of Linear Temporal Logic (LTL) and the physical environment is described by a smooth differential equation. The contribution of this paper is to show that synthesis for cyber-physical systems is viable by considering a fragment of LTL that is expressive enough to describe interesting properties but simple enough to avoid Safra’s construction. We report on two examples illustrating a preliminary implementation of these techniques on the tool PessoaLTL.

1. INTRODUCTION
The correct-by-design, or controller synthesis, paradigm offers a compelling alternative to current system design methodologies relying on extensive testing and/or verification to prove correctness. Intuitively, synthesis is the problem of algorithmically constructing an implementation from a given specification of the desired functionality and performance, and a partial model of the system. Controller synthesis has been studied in various forms in different communities, differing in the form of the model and the specification. For example, in continuous control theory, the partial model is the open loop plant

\[ \dot{x} = f(x, u), \]

and the controller is a feedback function \( u = k(x) \) such that the controlled system \( \dot{x} = f(x, k(x)) \) satisfies certain stability and performance criteria. Similarly, in (discrete) reactive synthesis, the partial implementation is usually an input-enabled, unconstrained automaton, the specification is given as a temporal logic formula capturing the good behaviors of the system, and the controller is an automaton ensuring that its product with the partial implementation only generates good behaviors.

Over the past decades, there has been a convergence of control-theoretic methods with automata-theoretic ones, in order to model hybrid or cyber-physical systems in which discrete components interact with continuous ones. These systems are often complex yet safety-critical, and thus, the application of program synthesis techniques—as opposed to the current practice of design and extensive verification and validation—is likely to have a large impact. However, there are some key technical challenges that have to be overcome in order to apply synthesis to this domain.

First, we have to abstract the underlying continuous state space into discrete parts so that reactive synthesis techniques can be applied. Moreover, such abstractions need to be constructed in such a way that a controller designed for the abstraction can be refined to a controller enforcing the specification on the original continuous model.

Second, the specification language must be expressive enough to capture many properties of interest in the domain. In the reactive synthesis world, linear temporal logic (LTL) [19] (or equivalently, automata over infinite words [29]) is usually considered as a robust and expressive specification formalism. Synthesis algorithms based on deep automata-theoretic constructions [5, 23, 18, 20, 21, 12] are well-known for this formalism. Unfortunately, these algorithms have very high theoretical and practical complexities. Theoretically, the problem is complete for 2EXPTIME. Moreover, Safra’s determinization construction [25], a key step in the algorithms, is extremely difficult to implement, and the best implementations so far can only handle small automata. This has limited the possibility of practical synthesis tools.

In this paper, we present PessoaLTL, an automatic synthesis tool for cyber-physical systems. PessoaLTL takes as input a controlled differential equation modeling the physical components, a specification consisting of two parts: a safety part in safe-LTL and an easily determinizable liveness part, and a parameter \( \varepsilon \) specifying the desired precision, and outputs, if possible, a software controller that ensures that the model together with the controller satisfies the specification up to precision \( \varepsilon \) (in a technical sense). The controller
is refined to Simulink blocks for closed-loop simulation.

We overcome the two challenges mentioned above in the following way. First, we use recent techniques reported in [22, 33, 17] to compute discrete abstractions of the differential equation model of the underlying continuous state space. Second, we use a restricted subset of LTL for our specification language, chosen to be expressive enough to naturally capture many requirements that frequently arise in cyber-physical systems design, and yet enabling controller synthesis without Safra’s construction (or the manipulation of co-Büchi tree automata [12]).

Our choice of the specification formalism is driven by our observation that many specifications for controller synthesis problems in embedded systems and robotics essentially consist of an “involved” safety part (stating that the system should always remain in “safe” states) and a “simple” liveness or guarantee part (stating that eventually a goal state should be reached). For example, a typical requirement in robotic applications is to reach a goal state while avoiding obstacles. A typical problem in control is to force a system to move between different operating points while staying within a desired operational envelope. This occurs, e.g., when we press a button in an elevator requesting that we reach a different floor while maintaining the elevator velocity and acceleration within certain limits for safety as well as comfort reasons. Accordingly, our specification language consists of two parts: a safety part in safe LTL, and a guarantee part given as an until formula. We use the fact that automata for safe LTL can be determinized using the usual subset construction [13] letting us avoid Safra’s construction in the implementation. Moreover, we can symbolically compute maximal strategies for the safety part. In a second step, we can compute the strategy to ensure the guarantee part while ensuring the safety specification. Although our synthesis algorithms are based on enforcing a safety invariant on the product of the system and the automaton constructed from the safe LTL formula, the use of safe LTL directly allows us to write specifications more naturally than if using invariants.

We developed PessoaLTL as an extension of Pessoa using both the abstraction algorithms as well as a solver for safety games using BDDs provided by Pessoa. We report preliminary results on the use of PessoaLTL. Drawing inspiration from robotics, we illustrate by two non-trivial examples how embedded control software synthesis problems can be automatically solved. The first example considers the motion planning problem with obstacles and requires a LTL formula comprising both safety as well as guarantee properties. In the second example we consider a more detailed model for the robot by incorporating information about the protocol used to mediate between the sensors and the main processor. Since the main processor may fail to acquire sensor measurements, we consider the requirement of reducing the robot velocity, or even completely stopping the robot, when not enough measurements are acquired. While in the worst case, the complexity of the algorithm is still $2\text{EXPTIME}$ [13], in practice, the subset construction has not been a bottleneck.

Related work We have already mentioned the rich history of reactive synthesis using automata-theoretic techniques. Work on the synthesis problem for cyber-physical systems is quite recent. The use of finite-state abstractions of differential equations and hybrid systems to solve synthesis problems has been pursued by several authors [3, 9, 24, 10, 51, 27]. However, no new novel synthesis algorithms, at the automata level, are proposed in these references.

Most tools for synthesis restrict specifications to state invariants. This is mostly because automata theoretic synthesis algorithms for general LTL properties require a complex determinization step [25] which is hard to implement efficiently [1, 28].

In [14, 32] controller synthesis enforcing temporal requirements on cyber-physical systems is discussed. Although different synthesis algorithms are proposed in these references, both assume a bounded temporal horizon for the satisfaction of the property. The work [14] uses model checking algorithms to find the feasible set of inputs. These inputs are bounded, since it is based on bounded temporal horizon assumptions. The liveness properties with bounded horizon are examples of bounded-safe properties. The fragment of LTL handled by PessoaLTL includes all bounded-safe properties. Furthermore, PessoaLTL also supports guarantee properties that require no restrictions on the time it takes for satisfaction.

In [7, 8], the authors have also restricted attention to specification formalisms which have efficient game solving algorithms, and used such algorithms to synthesize hardware components. Our focus here is embedded and robotics applications, for which our restricted specification language is a good fit. The abstraction of differential equation models for the physical components is an added dimension of complexity in our case.

The synthesis of switching policies for cyber-physical systems is discussed in [6]. Although, the resulting switching policies enforce the desired specifications, the work in [6] assumes that the continuous dynamics in each mode is fixed. In contrast, our algorithms do not assume the a priori existence of different modes with different dynamics.

While our constructions do not introduce any new deep insight into the nature of synthesis, we believe our specification formalism and implemented algorithms represent a practical sweet spot in controller synthesis for cyber-physical systems.

2. BACKGROUND

2.1 Systems

We consider the following notion of system that will be used to model software components as well as the abstraction of physical components.

**Definition 1.** A system

$$S = (X, X_0, U, \rightarrow; Y, H)$$

consists of: a set of states $X$; a set of initial states $X_0 \subseteq X$; a set of inputs $U$; a transition relation $\rightarrow \subseteq X \times U \times X$; a

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1 Available from http://www.cyphylab.ee.ucla.edu/pessoa.
A system is said to be finite when the set of states $X$ is finite. When the set of outputs $Y$ of a system $S$ is equipped with a metric $d: Y \times Y \to \mathbb{R}_{0+}^+$, we say that $S$ is a metric system. Metric systems will be used to formalize finite abstractions of differential equations in Section 2.3.

We write $x \xrightarrow{u} x'$ when $(x, u, x') \in \rightarrow$. For such a transition, state $x'$ is called a $u$-successor, or simply successor, of state $x$. Similarly, $x$ is called a $u$-predecessor, or predecessor, of state $x'$. For technical reasons, we assume that for every $x$ and $u$, there is some $x'$ such that $x \xrightarrow{u} x'$. We denote the set of $u$-successors of a state $x$ by $\text{Post}_u(x)$. A system is said to be deterministic if $(x, u, x') \in \rightarrow$ and $(x, u, x'') \in \rightarrow$ implies $x'' = x'$, or equivalently, if $\text{Post}_u(x)$ is a singleton for each $x \in X$ and $u \in U$.

A run of a system $S$ is an infinite sequence

$$x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} \ldots (1)$$

where $x_0 \in X_0$, and for each $i \geq 0$, we have $x_i \xrightarrow{u_i} x_{i+1}$. The outputs associated with the run (1) is the trace

$$H(x_0)H(x_1)\ldots \in Y^\omega.$$ 

Given an infinite string $z \in Z^\omega$, we will use the notation $z(i)$ to denote the $i$th element in the string $z$ and the notation $z[k]$ to denote the infinite string obtained from $z$ by removing its first $k$ elements, i.e., $z[k](i) = z(i+k)$.

The notion of system in Definition [1] allows for nondeterminism in the sense that for a given state $x \in X$ and input $u \in U$, there may be more than one $u$-successor of $x$. We assume that once the input $u$ is chosen at the state $x$, the exact $u$-successor of $x$ is selected from $\text{Post}_u(x)$ by the environment. We regard this nondeterminism as the adversarial influence of the environment, and consider a two-person game between the controller (player 0) and the non-determinism (player 1).

### 2.2 Controllers

A strategy for the controller (player 0) in a system $S = (X, X_0, U, Y, H)$ is a mapping $\pi_0: (X \times U)^* \times X \to U$ that associates with every non-empty finite sequence of states and inputs ending in $X$, representing the past history of the game, an action. A strategy for player 1 is a mapping $\pi_1: (X \times U)^* \times X \to U$ that associates with every non-empty finite sequence of states and inputs ending in $X$ and after action $u \in U$ has been taken, representing the past history of the game, a successor state $x' \in \text{Post}_u(x)$. A controller strategy $\pi_0$ is memoryless if the strategy depends on the current state only i.e., $\forall x \in X, \forall z, w \in (X \times U)^*$, $\pi_0(z \cdot x) = \pi_0(w \cdot x)$.

An initial state $x_0 \in X_0$, strategy $\pi_0$ for player 0, and $\pi_1$ for player 1 uniquely determine a run:

$$\text{Outcome}(x_0, \pi_0, \pi_1) = x_0 \xrightarrow{u_0} x_1 \ldots \in (X \times U)^\omega$$ (2)

where for $k \geq 0$, we have $u_k = \pi_0(x_0, \ldots, x_k)$, and $x_{k+1} = \pi_1(x_0, \ldots, x_k, u_k)$. Based on (2) we define the infinite state behavior:

$$\text{states}(x_0, \pi_0, \pi_1) = x_0x_1x_2\ldots \in X^\omega$$

and the corresponding outputs as:

$$\text{outputs}(x_0, \pi_0, \pi_1) = H(x_0)H(x_1)H(x_2)\ldots \in Y^\omega.$$ 

For $i \in \{0,1\}$, given an initial state $x$ and a winning objective $\Phi \subseteq Y^\omega$, we say the state $x \in X$ is winning for player-$i$ if there is a player $i$ strategy $\pi_i$ such that, for all player-$(1-i)$ strategies $\pi_{1-i}$, we have $\text{outputs}(x, \pi_0, \pi_1) \in \Phi$. The controller synthesis problem asks, given a system $S$ and an objective $\Phi \subseteq Y^\omega$, to construct a strategy $\pi$ for player 0 such that every initial state $x_0$ is winning for $\Phi$, that is, $\text{outputs}(x_0, \pi, \pi) \in \Phi$ for every $x_0 \in X_0$ and every player 1 strategy $\pi_1$. In that case, $\pi$ is called a controller for $\Phi$, and player 0 is said to enforce $\Phi$.

A strategy-set (for player 0) is a function $\bar{\pi}_0: (X \times U)^* \times X \to 2^U$. A strategy $\pi_0$ for player 0 is compatible with a strategy-set $\pi_0$ if for each $z \in (X \times U)^*$ and $x \in X$, we have $\pi_0(z \cdot x) = \bar{\pi}_0(z \cdot x)$. A strategy-set $\bar{\pi}_0$ for player 0 is winning for a winning objective $\Phi$ if every strategy compatible with $\bar{\pi}_0$ is winning for player 0. A strategy-set $\bar{\pi}_0$ is maximal for $\Phi$ if it is winning for $\Phi$ and every winning strategy of player 0 for $\Phi$ is compatible with $\bar{\pi}$. A strategy-set $\bar{\pi}_0$ is memoryless if it only depends on the final state and not the history of the play. As with strategies, we represent a memoryless strategy-set as a function $\bar{\pi}_0: X \to 2^U$.

As an example, let $Z \subseteq Y$ and consider the property $\Phi$ to be the set of traces $Z^\omega$. This is called a safety game, and player 0 wins this game from $x$ if she has a strategy $\pi_0$ such that for every strategy $\pi_1$ of player 1, $\text{outputs}(x, \pi_0, \pi_1)$ is a trace consisting only of outputs in $Z$ (the game always remains in $Z$). It is known that player 0 has a memoryless maximal strategy in a safety game [34].

For a set $X' \subseteq X$, define $CPre(X') = \{ x \in X | \exists u \in U. Post_u(x) \subseteq X' \}$. The set $CPre(X')$ consists of all states from which player 0 can force a visit to $X'$ in one step, no matter how player 1 resolves the nondeterminism. One can solve a safety game by iterating $CPre$, starting from the set $H^{-1}(Z)$, until a fixpoint is reached [16][44]:

$$\nu x. H^{-1}(Z) \cap CPre(x)$$

Indeed, this algorithm for solving safety games has been implemented in several tools, including Pessoa.

### 2.3 Approximate Alternating Simulation

In order to capture the adversarial intent of the environment, the notion of equivalence and pre-order used in this paper is that of alternating simulation. Moreover, since the results in Section 2.2 are used to relate differential equation models of physical systems to finite abstractions, we consider approximate alternating simulation relations.

**Definition 2.** Let $S_a$ and $S_i$ be metric systems with $Y_a = Y_i$ and let $\epsilon \in \mathbb{R}_{0+}^+$. A relation $R \subseteq X_a \times X_i$ is an $\epsilon$-approximate alternating simulation relation from $S_a$ to $S_i$ if the following three conditions are satisfied:

1. \(\text{Win}(a) \subseteq \text{Win}(i)\) for all player 0 winning objectives \(\Phi\) from \(\text{Win}(a)\) to \(\text{Win}(i)\).
2. \(\text{Win}(a) \subseteq \text{Win}(i)\) for all player 1 winning objectives \(\Phi\) from \(\text{Win}(a)\) to \(\text{Win}(i)\).
3. For any two player 0 winning objectives \(\Phi_0, \Phi_1\) from \(\text{Win}(a)\) to \(\text{Win}(i)\), there exists a \(\epsilon\)-approximate simulation relation between the winning sets of player 0 winning objectives \(\Phi_0\) and \(\Phi_1\) from \(\text{Win}(a)\) to \(\text{Win}(i)\).
1. for every $x_{a0} \in X_{a0}$ there exists $x_{b0} \in X_{b0}$ with $(x_{a0}, x_{b0}) \in R$.
2. for every $(x_a, x_b) \in R$ we have $d(H_a(x_a), H_b(x_b)) \leq \varepsilon$.
3. for every $(x_a, x_b) \in R$ and for every $u_a \in U_a(x_a)$ there exists $u_b \in U_b(x_b)$ such that for every $x'_b \in \text{Post}_{u_a}(x_b)$ there exists $x'_a \in \text{Post}_{u_b}(x_a)$ satisfying $(x'_a, x'_b) \in R$.

We say that $S_a$ is $\varepsilon$-approximately alternatingly simulated by $S_b$, or that $S_b$ $\varepsilon$-approximately alternatingly simulates $S_a$, denoted by $S_a \preceq_{\varepsilon} S_b$, if there exists an $\varepsilon$-approximate alternating simulation relation from $S_a$ to $S_b$.

The results in [22,33] show that for any differential equation model of the physical world, it is possible to construct a finite system $S$ that is $\varepsilon$-approximately alternatingly simulated by the differential equation. Hence, once we synthesize a controller for the finite abstraction, such controller can be refined to a controller enforcing the same specification on the differential equation up to an error of $\varepsilon$. Note that $\varepsilon$ is a design parameter that can be made as small as desired, at the expense of a larger finite abstraction. In the remainder of the paper we will assume that we have already abstracted the differential equation into a finite system. The constructions of such abstractions has been implemented in the freely available tool Pessoa [17].

3. SPECIFICATIONS

3.1 Linear Temporal Logic

We now review the syntax and semantics of linear-temporal logic (LTL) [19].

**Definition 3.** The set of LTL formulae is generated by the following grammar:

$$
\varphi ::= p \; | \; \neg \varphi \; | \; \varphi \lor \psi \; | \; \varphi \land \psi \; | \; \varphi \rightarrow \psi \; | \; \varphi \lor \psi \; | \; \varphi \land \psi \; | \; \varphi \lor \psi \; | \; \varphi \land \psi \; | \; \varphi W \psi \; | \; \varphi U \psi \; | \; \varphi W \psi
$$

where $p$ is chosen from a set $P$ of atomic propositions.

We define shorthands true and false as shorthand for $p \lor \neg p$ and $p \land \neg p$ respectively. We use $\lor \varphi$ and $\neg \varphi$ as shorthands of $(\text{true} \lor \varphi)$ and $(\varphi \lor \text{false})$ respectively.

An LTL formula is in negation normal form (NNF) if negation occurs only before the atomic propositions. It is known that any formula can be put in NNF by applying de Morgan’s laws (for Boolean operations), and the identities $\neg(\varphi \lor \psi) \equiv \neg \varphi \land \neg \psi$, $\neg(\varphi \land \psi) \equiv \neg \varphi \lor \neg \psi$, and $\neg(\varphi W \psi) \equiv \neg \varphi W \neg \psi \lor \neg \varphi U \psi \lor \neg \varphi F \psi$. The length $|\varphi|$ of a formula $\varphi$ is the number of symbols in $\varphi$ and defined by induction on the structure of $\varphi$ in a standard way.

The semantics of LTL formulae is defined over infinite sequences $z \in (2^P)^\omega$:

- $z \models p$ iff $p \in z(0)$;
- $z \models \neg \varphi$ iff $z \not\models \varphi$;
- $z \models \varphi \land \psi$ iff $z \models \varphi$ and $z \models \psi$;
- $z \models \varphi \lor \psi$ iff $z \models \varphi$ or $z \models \psi$;
- $z \models \varphi U \psi$ iff $\exists k \geq 0$ s.t. $z[k] \models \psi$ and $z[j] \models \varphi$ for all $0 \leq j < k$;
- $z \models \varphi W \psi$ iff $\exists k \geq 0$ $z[k] \models \psi$ and $z[j] \models \varphi$ for all $0 \leq j < k$.

If $z \models \varphi$, we say $z$ satisfies $\varphi$. For an LTL formula $\varphi$, the language $L(\varphi)$ of all strings satisfying $\varphi$ is defined by:

$$
L(\varphi) = \{z \in (2^P)^\omega \mid z \models \varphi\}.
$$

Let $S$ be a system where $Y = 2^P$ and thus $H$ maps each state $x \in X$ to the set of atomic propositions that are true at $x$. We say player 0 enforces the LTL formula $\varphi$ if there exists a player 0 strategy $\pi_0$ such that for each player 1 strategy $\pi_1$ and each $x_0 \in X_0$ we have that $\text{outputs}(x_0, \pi_0, \pi_1)$ satisfies $\varphi$.

3.2 Safe-LTL

We now define a subset of LTL formulas that capture all safety properties.

**Definition 4.** The set of safe-LTL formulae is generated by the following grammar:

$$
\varphi ::= p \; | \; \neg p \; | \; \varphi \lor \psi \; | \; \varphi \land \psi \; | \; \varphi W \psi \; | \; \varphi U \psi \; | \; \varphi W \psi
$$

where $p$ ranges over a set $P$ of atomic propositions.

A safe-LTL formula always defines a safety property. Intuitively, a formula $\varphi$ defines a safety property if $z \not\models \varphi$ can be checked by looking at a finite prefix of $z$.

Thus, reasoning about safety properties on infinite behaviors can be reduced to reasoning about their finite prefixes. First, we recall nondeterministic finite automata as acceptors of languages over finite words. A nondeterministic finite automaton (NFA) is a 5-tuple $A = (Q, \Sigma, \delta, F)$, where $Q$ is a finite set of states, $Q_0 \subseteq Q$ is a set of initial states, $F \subseteq Q$ is a set of final states, $\Sigma$ is an alphabet, and $\delta \subseteq Q \times \Sigma \times Q$ is a set of transitions. An NFA is deterministic, written DFA, if $|Q_0| = 1$ and $\delta$ defines a total function from $Q \times \Sigma$ into $Q$. The unique successor of a state $q \in Q$ under the letter $\sigma \in \Sigma$ in a deterministic automaton is denoted by $\delta(q, \sigma)$. A run of an NFA on a word $\sigma \equiv \sigma_0 \ldots \sigma_{n-1} \in \Sigma^*$ is a sequence $q_0 \xrightarrow{\sigma_0} q_1 \ldots q_{n-1} \xrightarrow{\sigma_{n-1}} q_n$ such that $q_0 \in Q_0$ and for each $0 \leq i \leq n - 1$ we have $(q_i, \sigma_i, q_{i+1}) \in \delta$. A run is accepting if moreover $q_n \in F$, and we say the NFA accepts $\sigma$. The language of an NFA is the set of all words $\sigma \in \Sigma^*$ such that the NFA has an accepting run on $\sigma$.

The set of bad prefixes for a safety formula $\varphi$ is defined by:

$$
\text{Bad}(\varphi) = \{z \in (2^P)^* \mid \forall w \in (2^P)^* z \cdot w \not\models \varphi\}.
$$

That is, a (finite) prefix $z$ is bad if none of its infinite extensions $z \cdot w$ satisfies the formula $\varphi$. The set of finite prefixes is the set of finite prefixes that are sufficient to prove that the computation is unsafe. We say that a set $Z \subseteq \text{Bad}(\varphi)$ is a trap for the safety language $L(\varphi)$ iff every word $w \not\in L(\varphi)$
has at least one prefix z ∈ Z. We denote all the traps for 
\( L(\varphi) \) by \( \text{trap}(L(\varphi)) \).

We say that a nondeterministic automaton \( N_\psi \) is fine for \( \psi \) if there exist \( Z \in \text{trap}(L(\psi)) \) such that \( L(N_\psi) = Z \). Thus, a fine automaton \( N_\psi \) may not accept all the bad prefixes, however it should accept at least one bad prefix of every computation that does not satisfy \( \psi \).

Kupferman and Vardi [13, 11] show that an automaton fine for \( \varphi \) can be constructed from \( \varphi \). The translation is based on the reverse deterministic automaton defined in [30]. In PessoaLTL we implemented the version of Kupferman and Vardi’s algorithm reported in [15] and presented here as Algorithm 1. This algorithm computes \( N_\varphi \) from a safe-LTL formula \( \varphi \). It first computes the set of subformulas \( \text{cl} \) of \( \neg \varphi \) by the procedure \( \text{computeClosure} \). Since each state of the automaton represent whether each of the subformulas is either true or false in that state, the fine automaton can have at most \( 2^{\left| \text{cl} \right|} \) states.

**Proposition 1.** For every safe-LTL formula \( \varphi \), Algorithm 1 constructs a nondeterministic fine automaton for \( \varphi \) with at most \( 2^{\left| \varphi \right|} \) states.

### 4. CONTROLLER SYNTHESIS

In this section, we assume that we have already computed a finite abstraction, in the form of a system \( S \), of the physical components. PessoaLTL accepts a pair of specifications \( (\varphi_S, \varphi_L) \): the first, \( \varphi_S \), is a safe-LTL formula that specifies the safety requirements of the system, and the second, \( \varphi_L \), is a guarantee formula of the form \( \diamond p \) that specifies that the goal \( p \) is eventually reached. We perform controller synthesis in two steps. First, we compute the maximal winning strategy for player 0 for the safe-LTL part of the specification. Second, we compute a controller that ensures the guarantee property using a strategy compatible with the maximal strategy.

#### 4.1 Controller Synthesis for Safe-LTL

For synthesizing a controller for a safe-LTL formula \( \varphi \), we construct a deterministic automaton on finite words that is fine for \( \varphi \). Note that Algorithm 1 may produce an NFA. However, determinization for NFAs over finite words uses the (easier to implement) subset construction. Theoretically, the determinization step adds one more exponential, making the complexity of the construction doubly exponential in the size of \( \varphi \). In our practical examples, this double exponential behavior has not shown up. For example, given the fine automaton for \( p \wedge q \), the subset construction creates the deterministic automaton Figure 1(b).

Given a system \( S = (X, X_0, U, \to, Y, H) \) and a DFA \( D_\varphi = (Q, q_0, Y, \delta, F) \) fine for \( \varphi \), we define the synchronous product \( S \times D_\varphi = (X', X'_0, U', \to', Y', H') \) where

- \( X' = X \times Q; \)
- \( X'_0 = \{(x, q) \mid x \in X_0, q = \delta(q_0, H(x))\}; \)
- \( U' = U; \)
- \( (x, q) \to' \leftarrow' (x', q') \) if \( x \to u x' \) and \( \delta(q, H(x')) = q' \);
- \( Y' = Y; \)
- \( H'(x, q) = H(x) \) for each \( (x, q) \in X' \).

A controller enforcing \( \varphi \) on \( S \) can be constructed by synthesizing a controller on the synchronous product \( S \times D_\varphi \) enforcing the specification that the system always remains
Algorithm 1 ConstructFineAutomaton(\(\psi\))

\[
\psi' := \text{NFA}((\neg \psi)); \text{cl} := \text{computeClosure}(\psi')
\]

\[
F := \{0\}, Q := \{\emptyset\}; X := \{0\}, Q_0 := \{\}, \delta = \{\}
\]

while \(X \neq \emptyset\) do

\[
s := \text{Dequeue}(X)
\]

foreach \(\sigma \in \Sigma\)

\[
s' = \{\}
\]

foreach \(\phi \in \text{cl}\) do

switch \(\phi\) begin

\[
case p \in q \text{ or } p = \neg q \text{ for } q \in Y;
\]

if \(p\) is satisfied by \(\sigma\), then \(s' := s' \cup \{p\}\)

\[
case \phi = \phi_1 \lor \phi_2 \text{ :}
\]

if \(\phi_1 \in s'\) or \(\phi_2 \in s'\) then \(s' := s' \cup \{\phi\}\)

\[
case \phi = \phi_1 \land \phi_2 \text{ :}
\]

if \(\phi_1 \in s'\) and \(\phi_2 \in s'\) then \(s' := s' \cup \{\phi\}\)

\[
case \phi = \neg \phi_1 \text{ :}
\]

if \(\phi_1 \in s\) then \(s' := s' \cup \{\phi\}\)

\[
case \phi = \phi_1 \lor \phi_2 \text{ :}
\]

if \(\phi_2 \in s'\) or \((\phi_1 \in s'\) and \(\phi \in s\)) then \(s' := s' \cup \{\phi\}\)

end switch

if \(\neg \phi \in s'\) then \(Q_0 := Q_0 \cup \{s'\}\)

\[
\delta := \delta \cup \{(s', \sigma, s)\}
\]

\[
X := X \cup \{s'\}, Q := Q \cup \{s'\}
\]

end for

end while

return \(A_{\psi}^{\text{fine}} = (Q, Q_0, 2^P, \delta, F)\)

in the states \(X \times (Q \setminus F)\), i.e., that player 0 ensures that no word in the language of \(D_\psi\) is seen. This is a safety game where player 0 keeps the states into an invariant set \((X \times (Q \setminus F))\), and can be solved using existing methods by iterating a symbolic controllable-predecessor operator \[34\]. Moreover, it is well-known that player 0 has memoryless maximal winning strategies in this game.

**Theorem 1.** Let \(S = (X, X_0, U, \to, Y, H)\) be system and let \(D_\psi = (Q, Q_0, Y, \delta, F)\) be a deterministic finite automaton fine for the safe-LTL formula \(\psi\). For any initial state \(x \in X_0\), player 0 has a winning strategy for the safe-LTL formula \(\psi\), if player 0 has a memoryless winning strategy from the unique \(x_0 \in X_0\) to stay in \(X \times (Q \setminus F)\) states in system \(S \times D_\psi\). Moreover, player 0 has a maximal winning strategy in \(S \times D_\psi\).

Thus, the algorithm to construct a maximal memoryless controller for a system \(S\) and a safe-LTL property \(\psi\) proceeds as follows. First, we construct an NFA \(N_\psi\) fine for \(\psi\). Second, we use the subset construction to determinize \(N_\psi\) into a DFA \(D_\psi\). Third, we construct the synchronous product of \(S\) with \(D_\psi\). Finally, we solve the safety game on \(S \times D_\psi\) for the winning set \(X \times (Q \setminus F)\) and construct a maximal memoryless winning strategy.

### 4.2 Controller Synthesis for the Guarantee Part

Let \(S \times D_\psi = (X, X_0, U, \to, Y, H)\) be the synchronous product of a system and a DFA fine for the safe-LTL \(\psi\), and let \(\pi\) be a maximal memoryless winning strategy for player 0

which ensures that all runs of the system stay in the states \(X \times (Q \setminus F)\).

We define the restriction of \(S \times D_\psi\) modulo \(\pi\) to be the system \((X, X_0, U, \to, Y, H)\) where \(x \to u x'\) if \(x \to u x'\) and \(u \in \pi(x)\). That is, we restrict the actions available at a state to only those allowed by the maximal strategy \(\pi\).

We now consider constructing a controller for the guarantee part \(\Diamond p\). We solve this by constructing a winning strategy in the reachability game on the product \(S \times D_\psi\) modulo \(\pi\), the maximal memoryless winning strategy for the safety game. Again, the solution to the reachability game is constructed by iterating a symbolic controllable predecessor operator \[34\].

The resulting strategy ensures that the guarantee part \(\Diamond p\) is enforced by player 0 (by construction in the reachability game), while always maintaining the safety part (by ensuring that the strategy is compatible with \(\pi\)). Together, the controller enforces the specification \(\varphi_S \land \varphi_L\).

While the current implementation of PESSOALTL only handles guarantee properties of the form \(\Diamond p\) (or some syntactic sugar, e.g., properties of the form \(p_1 \cup p_2\) using the identity \(p_1 \cup p_2 \equiv p_1 \wp_2 \lor \Diamond p_2\)), notice that all we need is that a deterministic generator for the liveness part of the specification is efficiently computable. For example, it is easy to extend the algorithm when the liveness part of the specification is a Büchi requirements \(\Box \Diamond p\), or more generally, from the fragments described in \[2\].

### 5. CONTROLLER REFINEMENT

The discussion so far has focused on the synthesis of strategies enforcing LTL formulas over the finite abstraction \(S\) of a physical system. The natural next step is to refine the controller synthesized for \(S\) to a controller enforcing the specification on the differential equation model of the physical system. Typical controller implementations are done on digital platforms, hence it is convenient to assume a periodic\(^2\) execution of the controller implementation with period \(\tau\). Moreover, a time discretized version of the differential equation:

\[
\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \quad (3)
\]

modeling the physical system being controlled can be described by the system \(S_\tau = (X_\tau, X_{r0}, U_\tau, \to, Y_\tau, H_\tau)\) consisting of:

- \(X_\tau = \mathbb{R}^n\);
- \(X_{r0} = X\);
- \(U_\tau = \mathbb{R}^m\);
- \(x \to u\ x'\) if there exists a solution \(\xi\) of \(\dot{\xi} = f(\xi(t), u)\) for the constant input \(u\) satisfying \(\xi(0) = x\) and \(\xi(\tau) = x'\).
- \(Y_\tau = X_\tau\);
- \(H_\tau(x) = x\) for any \(x \in X_\tau\).

\(^2\)There are also considerable advantages to consider non-periodic implementations as in \[2\], however such approaches are outside the scope of this paper.
The results in [22, 33] guarantee the existence of a finite system \( S \) and of an \( \varepsilon \)-approximate alternating simulation relation \( R \) from \( S \) to \( S_r \). Note that while \( S_r \) is deterministic, the abstraction process introduces nondeterminism in \( S \). Nevertheless, the existence of the relation \( R \) guarantees that any controller synthesized for \( S \) can be refined to a controller for \( S_r \). A formal description of the refined controller can be found in [26]. Here, we provide an informal description which we believe to be more informative. Any state \( x_r \in X_r \) of the system \( S_r \) is related by \( R \) to a state \( x \in X \) in the finite abstraction \( S \). If the strategy \( \pi_0 \) dictates that the input \( u \in U \) should be used at the state \( x \), then by using a constant input curve of duration \( \tau \) and value \( u \) in \( S_r \), we are guaranteed to reach a state \( x' \in X_r \) that is \( R \) related to a state \( x' \in \text{Post}_u(x) \). Hence, the refined controller consists in a loop performing the following steps:

1. Acquire the current state from sensors/estimators;
2. Identify the state in \( S \) that is related by \( R \) to the current state;
3. Compute the input \( u \) given by the strategy \( \pi_0 \);
4. Send the value \( u \) to the actuators and keep it constant for \( \tau \) units of time;
5. Loop to step 1.

This refined controller enforces the specification on \( S_r \) up to an error \( \varepsilon \) as stated in the next result.

**Proposition 2.** Let \( S_r \) be the time discretization of a differential equation governing the physical system to be controlled and let \( \varphi \) be a LTL formula whose predicates correspond to subsets of \( Y_r \). Consider the finite abstraction \( S \) of \( S_r \) and let \( R \) be the \( \varepsilon \)-approximate alternating simulation relation from \( S \) to \( S_r \). For any strategy \( \pi_0 \) enforcing \( \varphi \) on \( S \), the strategy \( \pi_{r0} \) obtained by refining \( \pi_0 \) on \( S_r \) up to an error of \( \varepsilon \), that is, for any environment strategy \( \pi \) for \( S \) we have \( d(y(i), y_r(i)) \leq \varepsilon \) for every \( i \in \mathbb{N} \), for the unique \( y \in \text{outputs}(x, \pi_0, \pi_1) \), the unique \( y_r \in \text{outputs}(x_r, \pi_{r0}) \), and for any \((x, x_r) \in R \).

6. CASE STUDY: ROBOT CONTROLLER

We consider a nonholonomic robot described by the following differential equations:

\[
\dot{x} = v \cos \theta, \quad \dot{y} = v \sin \theta, \quad \dot{\theta} = \omega
\]

where \((x, y)\) denotes the robot position and \( \theta \) its orientation. The inputs are \( v \) and \( \omega \) and correspond to the linear and angular velocity of the robot, respectively. Using PESSOA we compute a finite abstraction \( S \) of the differential equation model of the robot. This abstraction is approximately alternatingly simulated by the differential equation model with a precision of \( \varepsilon = 0.1 \). In this abstraction the input \( v \) is restricted to take values in the set \( \{0, 0.2, 0.4\} \) while the input \( \omega \) is restricted to take values in the set \( \{-0.2, 0, 0.2\} \).

![Figure 2: Closed-loop diagram in Simulink showing the automatically synthesized controller.](image)

**6.1 Reachability with Obstacle Avoidance**

For every obstacle (see the blue sets in Figure 3) we construct a predicate \( \text{obstacle}_i \), \( i \in \{1, 2, 3\} \), that is true whenever the robot is inside the set defined by the obstacle. Similarly, we define the predicate \( \text{target} \) describing the target set represented by the red set in Figure 3. The objective of reaching the target set, if possible, while avoiding the obstacles is naturally expressed by the safe-LTL formula:

\[
\psi = (\neg(\text{obstacle}_1 \lor \text{obstacle}_2 \lor \text{obstacle}_3)) W \text{target}.
\]

Note that \( \varphi \) does not require the target set to be reached. Such requirement can be prescribed by using instead the LTL formula:

\[
\varphi = (\neg(\text{obstacle}_1 \lor \text{obstacle}_2 \lor \text{obstacle}_3)) U \text{target}.
\]

Since \( \varphi \) can be decomposed as:

\[
\varphi = \psi \land \Diamond \text{target}
\]

we first solve the safety problem specified by \( \psi \) and then we solve the reachability problem specified by \( \Diamond \text{target} \). The synthesized controller is automatically refined to a Simulink block in PESSOA, see Figure 2 in order to simulate the closed-loop behavior. In Figure 3 we show the trajectory followed by the robot, and in Figure 4 we show the inputs used to steer the robot. The yellow line represents the translational velocity input while the magenta line represents the angular velocity input.

![Figure 3: Trajectory followed by the robot.](image)
6.2 Fault tolerance

We consider the same robot as in the previous case study. We assume that the communication between the several sensors onboard of the robot with the microprocessor running the control code is governed by a protocol that reports if communication is successful or not. There are several reasons for unsuccessful communication such as the fact that the communication medium is shared among several subsystems and sensor failures. We now consider a specification detailing how the robot should operate in case of sensor failures:

The main microprocessor may fail to receive sensor measurements more than once. In such case the controller should have a strategy to protect the robot from either leaving the desired working area or hitting the obstacles. One possible way of encoding this objective as a safety property is to require that if sensor measurements are not received two or more times during three consecutive control cycles, the robot should stop and remain at its current location. In order to formalize this property we extend the model of the robot so as to incorporate the previously used input as part of the state. Consider now the predicate \( \text{stop} \), which is true (resp. false) when the input \( v \) is equal to (resp. different from) zero, and the predicate \( \text{fails}_{3,2} \), which is true when 2 or more sensor measurements were not received during 3 consecutive control cycles. Since in LTL we cannot refer to the past, we encode \( \text{fails}_{3,2} \) by making reference to the future as follows:

\[
\text{fails}_{3,2} = (f \land \Box f) \lor (\Box f \land \Box \Box f) \lor (f \land \Box \Box f).
\]

In the preceding formula \( f \) is the predicate that becomes true every time that the microprocessor fails to receive sensor measurements. The final formula can then be obtained as:

\[
\Box(\text{fails}_{3,2} \rightarrow \bigcirc \bigcirc \bigcirc \text{stop}). \tag{4}
\]

Figure 6 shows the fine-automaton with respect to the previous property. In Figure 6 we show the inputs generated by the controller when the predicate \( f \) evolves according to:

\[fff \neg f \neg f \neg f \neg f \neg f \neg f \neg f \neg f.\]

The yellow line represents the translational velocity input \( (v) \) while the magenta line represents the angular velocity input \( (\omega) \). Note that whenever the protocol returns two consecutive failures \( (f \text{ is true twice}) \), the input \( v \) generated by the controller at the next control cycle is zero. Figure 7 shows the closed-loop evolution of \( \theta, x, y, u \) and \( v \) for the given fault-sequence. The colors of these state variables are cyan, yellow, magenta, red and green respectively. We can easily develop more sophisticated fault tolerance requirements. Let \( \text{slow} \) denote the predicate that holds true when \( v = 0.2 \), corresponding to half of the maximum velocity. We could, e.g., require that when the sensor measurements are not received one in three control cycles, the robot should reduce its translational speed to \( v = 0.2 \). Such specification can be written as:

\[
\Box(\text{fails}_{3,1} \rightarrow \bigcirc \bigcirc \bigcirc \text{slow}) \tag{5}
\]

where \( \text{fails}_{3,1} \) captures one sensor failure in three control cycles:

\[
(f \land \Box f \land \Box \Box f) \lor (\neg f \land \Box f \land \Box \Box f) \lor (f \land \Box \Box f). \]

By conjoining (4) with (5), we would obtain a more detailed requirement asking for the robot to slow down when one
measurement fails in the three consecutive control cycles, and to stop when two measurements fail.

Table 8 show the time and space complexity of fine automata for formula \( \varphi = \Box(fail_{n,k} \rightarrow \Box^n stop) \) where, \( k \) is the number of faults in \( n \) consecutive readings. The length column denotes the length of \( nnf(\neg \varphi) \). \( \Box^n \varphi \) is a shorthand of \( n \)-consecutive \( \Box \) applied to \( \varphi \).

| Parameters | length | Time(s) | NFA | DFA |
|------------|--------|---------|-----|-----|
| \( n=3, k=2 \) | 10     | 0.714   | 245 | 10  |
| \( n=3, k=1 \) | 10     | 1.096   | 253 | 10  |
| \( n=4, k=1 \) | 13     | 12.690  | 1045| 15  |
| \( n=5, k=1 \) | 16     | 110.026 | 2717| 21  |
| \( n=6, k=1 \) | 19     | 1957.450| 7933| 28  |

To illustrate the mode switching problem in the context of the mobile robot example, we consider the scenario to be specified by a remote operator that instructs the robot to move to one of two locations described by the predicates:

\[
\text{goal}_1 = \{(x, y, \theta) \in \mathbb{R}^3 \mid 4.4 \leq x \leq 4.6 \land 1 \leq y \leq 1.6\}
\]

\[
\text{goal}_2 = \{(x, y, \theta) \in \mathbb{R}^3 \mid 4.6 \leq x \leq 5.0 \land 1 \leq y \leq 1.6\}
\]

The formulas defining the scenarios are the predicates \( \text{scen}_1 \) and \( \neg \text{scen}_1 \) whose truth value can be dynamically changed by the robot operator according to the location where he wants the robot to go. The fine automaton for the resulting specification (Figure 9) was constructed in <1 seconds and has 4 \( \text{DFA} \) states.

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