Freely Generated Vertex Algebras and Non–Linear Lie Conformal Algebras

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Received: 12 December 2003 / Accepted: 22 June 2004
Published online: 11 January 2005 – © Springer-Verlag 2005

Abstract: We introduce the notion of a non–linear Lie conformal superalgebra and prove a PBW theorem for its universal enveloping vertex algebra. We also show that conversely any graded freely generated vertex algebra is the universal enveloping algebra of a unique, up to isomorphism, non–linear Lie conformal superalgebra. This correspondence will be applied in the subsequent work to the problem of classification of finitely generated simple graded vertex algebras.

1. Introduction

After the work of Zamolodchikov [Zam85] it has become clear that the chiral (= vertex) algebra of a conformal field theory gives rise to a Lie algebra with non–linearities in commutation relations. This and the subsequent works in the area clearly demonstrated that the absence of non–linearities, like in the Virasoro algebra, the current algebras and their super analogues, is an exception, rather than a rule.

This is closely related to the fact that in the singular parts of the operator product expansions of generating fields, as a rule, not only linear combinations of these fields and their derivatives occur, but also their normally ordered products. In fact, the absence of terms with normally ordered products is equivalent to the absence of non–linearities in the commutation relations.

The latter case is encoded in the notion of a Lie conformal superalgebra [Kac96], and a complete classification of finite simple Lie conformal superalgebras was given in [FK02] (completing thereby a sequence of works that began with [RS76] and continued in the conformal algebra framework in [Kac97, DK98, FK02]).

A complete classification in the general non–linear case is much harder and is still far away. In the present paper we lay rigorous grounds to the problem by introducing the notion of a non–linear conformal superalgebra.

In order to explain the idea, let us define this notion in the quantum mechanical setting. Let \( g \) be a vector space, endowed with a gradation \( g = \bigoplus_j g_j \), by a discrete subsemigroup
of the additive semigroup of positive real numbers, and with a linear map \([,]\) : \(g \otimes g \to T(g)\), where \(T(g)\) is the tensor algebra over \(g\). We write \(\Delta(a) = j\) if \(a \in g_j\), and we extend the gradation to \(T(g)\) by additivity, i.e. \(\Delta(1) = 0\), \(\Delta(A \otimes B) = \Delta(A) + \Delta(B)\) for \(A, B \in T(g)\), and the bracket \([,]\) \(T(g) \otimes T(g) \to T(g)\) by the Leibniz rule. We write \(\Delta(a) < \Delta\) if \(\Delta(a_j) < \Delta\) for all non-zero homogeneous summands \(a_j\) of \(a\).

\[M_j(g) = \text{span}\left\{A \otimes (b \otimes c - c \otimes b - [b, c]) \otimes D \mid b, c \in g, A, D \in T(g), \Delta(A \otimes b \otimes c \otimes D) \leq j\right\}\]

and let \(M(g) = \bigcup_j M_j(g)\). Note that \(M(g)\) is the two sided ideal of the tensor algebra generated by elements \(a \otimes b - b \otimes a - [a, b]\), where \(a, b \in g\). The map \([,]\) is called a non-linear Lie algebra if it satisfies the following two properties \((a, b, c\) are homogeneous elements of \(g)\):

1. (grading condition) \(\Delta([a, b]) < \Delta(a) + \Delta(b)\),
2. (Jacobi identity) \([a, [b, c]] - [b, [a, c]] - [[a, b], c] \in M_\Delta(g)\),

where \(\Delta < \Delta(a) + \Delta(b) + \Delta(c)\).

Let \(U(g) = T(g)/M(g)\). It is not difficult to show by the usual method (see e.g. [Jac62]) that the PBW theorem holds for the associative algebra \(U(g)\), i.e. the images of all ordered monomials in an ordered basis of \(g\) form a basis of \(U(g)\).

The purpose of the present paper is to introduce the notion of a non-linear Lie conformal (super)algebra \(R\), which encodes the singular part of general OPE of fields, in order to construct the corresponding universal enveloping vertex algebra \(U(R)\), and to establish for \(U(R)\) a PBW theorem.

The main difficulty here, as compared to the Lie (super)algebra case, comes from the fact that \(U(R) = T(R)/M(R)\), where \(M(R)\) is not a two-sided ideal of the associative algebra \(T(R)\). This makes the proof of the PBW theorem much more difficult.

Of course, in the "linear" case these results are well known [Kac96, GMS04, BK03].

Another important result of the paper is a converse theorem, which states that any graded vertex algebra satisfying the PBW theorem is actually the universal enveloping vertex algebra of a non-linear Lie conformal superalgebra.

The special case when the vertex algebra is freely generated by a Virasoro field and primary fields of conformal weight 1 and 3/2 was studied in [DS03].

Here is a brief outline of the contents of the paper. After reviewing in Sect. 2 the definition of a Lie conformal algebra and a vertex algebra, we introduce in Sect. 3 the definition of a non-linear Lie conformal algebra \(R\). We also state the main result of the paper, a PBW theorem for the universal enveloping vertex algebra \(U(R)\), which is proved in Sects. 4 and 5. In Sect. 6 we state and prove the converse statement, namely to every graded freely generated vertex algebra \(V\) we associate a unique, up to isomorphism, non-linear Lie conformal algebra \(R\) such that \(V \simeq U(R)\).

The results of this paper have important applications to the classification problem of conformal vertex algebras. Such applications will be discussed in subsequent work.

2. Definitions of Lie Conformal Algebras and Vertex Algebras

**Definition 2.1.** A Lie conformal superalgebra (see [Kac96]) is a \(\mathbb{Z}/2\mathbb{Z}\)-graded \(\mathbb{C}[T]\)-module \(R = R_0 \oplus R_1\) endowed with a \(\mathbb{C}\)-linear map \(\eta: R \otimes R \to \mathbb{C}[\lambda] \otimes R\) denoted by \(a \otimes b \mapsto [a, \lambda, b]\) and called \(\lambda\)-bracket, satisfying the following axioms:
\[ [Ta \ λ b] = -\lambda [a \ λ b], \quad [a \ λ Tb] = (\lambda + T)[a \ λ b] \quad \text{(sesquilinearity)}, \]

\[ [b \ λ a] = -p(a, b)[a -\lambda - T b] \quad \text{(skewsymmetry)}, \]

\[ [a \ λ [b \ μ c]] - p(a, b)[b \ μ [a \ λ c]] = [[a \ λ b] \ λ \ + \ μ c] \quad \text{(Jacobi identity)}, \]

for \( a, b, c \in R \). Here and further \( p(a, b) = (-1)^{p(a)p(b)} \), where \( p(a) \in \mathbb{Z}/2\mathbb{Z} \) is the parity of \( a \), and \( \otimes \) stands for the tensor product of vector spaces over \( \mathbb{C} \).

In the skewsymmetry relation, \( [a -\lambda - T b] \) means that we have to replace in \( [a \ λ b] \) the indeterminate \( \lambda \) with the operator \((-\lambda - T)\), acting from the left. For brevity, we will often drop the prefix super in superalgebra or superspace.

For a Lie conformal algebra we can define a \( \mathbb{C} \)-bilinear product \( R \otimes R \to R \) for any \( n \in \mathbb{Z}_+ = \{0, 1, 2, \cdots \} \), denoted by \( a \otimes b \mapsto a(n)b \) and given by

\[ [a \ λ b] = \sum_{n \in \mathbb{Z}_+} \lambda^{(n)} a(n)b, \quad (2.1) \]

where we are using the notation: \( \lambda^{(n)} := \frac{\lambda^n}{n!} \).

Vertex algebras can be thought of as a special class of Lie conformal superalgebras.

**Definition 2.2.** A **vertex algebra** is a pair \( (V, |0 \rangle) \), where \( V \) is a \( \mathbb{Z}_2 \)-graded \( \mathbb{C}[T] \)-module, called the **space of states**, and \( |0 \rangle \) is an element of \( V \), called the **vacuum state**, endowed with two parity preserving operations: a \( \lambda \)-bracket \( V \otimes V \to \mathbb{C}[\lambda] \otimes V \) which makes it a Lie conformal superalgebra, and a **normally ordered product** \( V \otimes V \to V \), denoted by \( a \otimes b \mapsto :ab:\ ), which makes it a (not necessarily associative nor commutative) unital differential algebra with unity \( |0 \rangle \) and derivative \( T \). They satisfy the following axioms

1. quasi–associativity

   \[ :ab:c : - :a(bc:) = : \left( \int_0^T d\lambda \ a \right) [b \ λ c] : + \ p(a, b) \ : \left( \int_0^T d\lambda \ b \right) [a \ λ c] :, \]

2. skewsymmetry of the normally ordered product

   \[ :ab : - p(a, b) : ba : = \int_{-T}^0 d\lambda [a \ λ b], \]

3. non-commutative Wick formula

   \[ [a \ λ :bc:] = :a \ λ b c : + p(a, b) : b a \ λ c : + \int_0^\lambda d\mu [a \ λ b] \mu c]. \]

In quasi–associativity, the first (and similarly the second) integral in the right-hand side should be understood as

\[ \sum_{n \geq 0} : (T^{n+1} a)(b_n c) :. \]
An equivalent and more familiar definition of vertex algebras can be given in terms of local fields, [FLM88, FHL93, DL93, Kac96]. A proof of the equivalence between the above definition of a vertex algebra and the one given in [Kac96] can be found in [BK03]. It also follows from [Kac96, Li96] that these definitions are equivalent to the original one of [Bor86].

Remark 2.3. Since $|0\rangle$ is the unit element of the differential algebra $(V, T)$, we have $T|0\rangle = 0$. Moreover, by sesquilinearity of the $\lambda$–bracket, it is easy to prove that the torsion of the $\mathbb{C}[T]$–module $V$ is central with respect to the $\lambda$–bracket; namely, if $P(T)a = 0$ for some $P(T) \in \mathbb{C}[T]\{0\}$, then $[a, b] = 0$ for all $b \in V$. In particular the vacuum element is central: $[a, |0\rangle] = [|0\rangle, a] = 0, \forall a \in V$.

Remark 2.4. Using skewsymmetry (of both the $\lambda$–bracket and the normally ordered product) and non commutative Wick formula, one can prove the right Wick formula [BK03]:

$$ab \lambda c = (e^{T\lambda a}b \lambda c) + \int_0^\lambda d\mu [b \lambda (a \lambda - \mu c)].$$

For a vertex algebra $V$ we can define $n^{th}$ products for every $n \in \mathbb{Z}$ in the following way. Since $V$ is a Lie conformal algebra, all $n^{th}$ products with $n \geq 0$ are already defined by (2.1). For $n \leq -1$ we define $n^{th}$ product as

$$a(n) b = (T^{-(n-1)}a)b.$$ (2.2)

By definition every vertex algebra is a Lie conformal algebra. On the other hand, given a Lie conformal superalgebra $R$ there is a canonical way to construct a vertex algebra which contains $R$ as a Lie conformal subalgebra. This result is stated in the following

**Theorem 2.5.** Let $R$ be a Lie conformal superalgebra with $\lambda$–bracket $[a, b]$. Let $R_{\text{Lie}}$ be $R$ considered as a Lie superalgebra over $\mathbb{C}$ with respect to the Lie bracket:

$$[a, b] = \int_{-T}^0 d\lambda [a, \lambda b], \quad a, b \in R,$$

and let $V = U(R_{\text{Lie}})$ be its universal enveloping superalgebra. Then there exists a unique structure of a vertex superalgebra on $V$ such that the restriction of the $\lambda$–bracket to $R_{\text{Lie}} \times R_{\text{Lie}}$ coincides with the $\lambda$–bracket on $R$ and the restriction of the normally ordered product to $R_{\text{Lie}} \times V$ coincides with the associative product of $U(R_{\text{Lie}})$.

The vertex algebra thus obtained is denoted by $U(R)$ and is called the **universal enveloping vertex algebra** of $R$. For a proof of this theorem, see [Kac96, GMS04], [BK03, Th 7.12], [Primc99].

In the remainder of this section we introduce some definitions which will be used in the sequel.

**Definition 2.6.** Let $R$ be a $(\mathbb{Z}/2\mathbb{Z}$ graded) $\mathbb{C}[T]$–submodule of a vertex algebra $V$. Choose a basis $A = \{a_i, \ i \in I\}$ of $R$ compatible with the $\mathbb{Z}/2\mathbb{Z}$ gradation, where $I$ is an ordered set.

1. One says that $V$ is **strongly generated** by $R$ if the normally ordered products of elements of $A$ span $V$. 

2. One says that $V$ is **freely generated** by $R$ if the set of elements

$$B = \left\{ :a_{i_1} \cdots a_{i_n} : \mid i_1 \leq i_2 \leq \cdots \leq i_n, \quad i_k < i_{k+1} \text{ if } p(a_{i_k}) = 1 \right\}$$

forms a basis of $V$ over $\mathbb{C}$. In this case $V$ is called a freely generated vertex algebra.

The normally ordered product of more than two elements is defined by taking products from right to left. In other words, for elements $a, b, c, \cdots \in V$, we will denote

$$:abc\cdots: = :a(:b(:c\cdots:)\cdots:) :.$$ 

Also, by convention, the empty normal ordered product is $|0\rangle$, and it is included in $B$.

For example, the universal enveloping vertex algebra $U(R)$ of a Lie conformal algebra $R$ is freely generated by $R$, by Theorem 2.5.

From now on, fix an ordered abelian semigroup $\Gamma$ with a zero element 0, such that for every $\Delta \in \Gamma$ there are only finitely many elements $\Delta' \in \Gamma$ such that $\Delta' < \Delta$. The most important example is a discrete subset of $\mathbb{R}_+$ containing 0 and closed under addition.

**Definition 2.7.** By a $\Gamma$–gradation of a vector space $U$ we mean a decomposition

$$U = \bigoplus_{\Delta \in \Gamma \setminus \{0\}} U[\Delta],$$

in a direct sum of subspaces labelled by $\Gamma \setminus \{0\}$. If $a \in U[\Delta]$, we call $\Delta$ the **degree** of $a$, and we will denote it by $\Delta(a)$ or $\Delta_a$. By convention $0 \in U$ is of any degree. The associated $\Gamma$–filtration of $U$ is defined by letting

$$U_{\Delta} = \bigoplus_{\Delta' \leq \Delta} U[\Delta'].$$

Let $T(U)$ be the tensor algebra over $U$, namely

$$T(U) = \mathbb{C} \oplus U \oplus U \otimes U \oplus \cdots.$$

A $\Gamma$–gradation of $U$ extends to a $\Gamma$–gradation of $T(U)$,

$$T(U) = \bigoplus_{\Delta \in \Gamma} T(U)[\Delta],$$

by letting

$$\Delta(1) = 0, \quad \Delta(A \otimes B) = \Delta(A) + \Delta(B), \quad \forall A, b \in T(U).$$

The induced $\Gamma$–filtration is denoted by

$$T_{\Delta}(U) = \bigoplus_{\Delta' \leq \Delta} T(U)[\Delta'].$$

If moreover $U$ is a $\mathbb{C}[T]$–module, we extend the action of $T$ to $T(U)$ by Leibniz rule:

$$T(1) = 0, \quad T(A \otimes B) = T(A) \otimes B + A \otimes T(B), \quad \forall A, B \in T(U).$$
3. Non–linear Lie Conformal Superalgebras and Corresponding Universal Enveloping Vertex Algebras

In the previous section we have seen that, to every Lie conformal algebra \( R \), we can associate a universal enveloping vertex algebra \( U(R) \), freely generated by \( R \). In general, though, it is not true that a freely generated vertex algebra \( V \) is obtained as the universal enveloping vertex algebra of a finite Lie conformal algebra. In fact the generating set \( R \subset V \) needs not be closed under the \( \lambda \)–bracket.

In this section we will introduce the notion of non–linear Lie conformal superalgebra, which is “categorically equivalent” to that of the freely generated vertex algebra: to every non–linear Lie conformal algebra \( R \) we will associate a universal enveloping vertex algebra \( U(R) \), freely generated by \( R \). Conversely, let \( V \) be a vertex algebra freely generated by a free \( \mathbb{C}[T] \)–module \( R \) satisfying a “grading condition”; we will show that \( R \) is a non–linear Lie conformal algebra and \( V \) is isomorphic to the universal enveloping vertex algebra \( U(R) \).

Definition 3.1. A non–linear conformal superalgebra \( R \) is a \( \mathbb{C}[T] \)–module which is \( \Gamma \{0\} \)–graded by \( \mathbb{C}[T] \)–submodules: \( R = \oplus_{\Delta \in \Gamma \{0\}} R[\Delta] \), and is endowed with a \( \lambda \)–bracket, namely a parity preserving \( \mathbb{C} \)–linear map

\[
[\lambda] : R \otimes R \longrightarrow \mathbb{C}[\lambda] \otimes T(R),
\]

such that the following conditions hold

1. sesquilinearity

\[
[T a \lambda b] = -\lambda[a \lambda b],
\]

\[
[a \lambda Tb] = (\lambda + T)[a \lambda b].
\]

2. grading condition

\[
\Delta([a \lambda b]) < \Delta(a) + \Delta(b).
\]

The grading condition can be expressed in terms of the \( \Gamma \)–filtration of \( T(R) \) (see Definition 2.7) in the following way:

\[
[R[\Delta_1] \lambda, R[\Delta_2]] \subset \mathbb{C}[\lambda] \otimes T_\Delta(R), \quad \text{for some } \Delta < \Delta_1 + \Delta_2.
\]

Note also that \( T(R)[0] = \mathbb{C} \) and \( T(R)[\delta] = R[\delta] \), where \( \delta \) is the smallest non–zero element of \( \Gamma \) such that \( R[\delta] \neq 0 \). This will be important for induction arguments.

The following lemma allows us to introduce the normally ordered product on \( T(R) \) and to extend the \( \lambda \)–bracket to the whole tensor algebra \( T(R) \), by using quasi–associativity and the left and right Wick formulas.

Lemma 3.2. Let \( R \) be a non–linear conformal algebra.

A) There exist unique linear maps

\[
N : T(R) \otimes T(R) \longrightarrow T(R),
\]

\[
L_\lambda : T(R) \otimes T(R) \longrightarrow \mathbb{C}[\lambda] \otimes T(R),
\]
such that, for $a, b, c, \ldots \in R$, $A, B, C, \ldots \in T(R)$, the following conditions hold

\[
N(1, A) = N(A, 1) = A, \tag{3.1}
\]

\[
N(a, B) = a \otimes B, \tag{3.2}
\]

\[
N(a \otimes B, C) = N(a, N(B, C)) + N\left( \int_0^T d\lambda \ a \right), L_\lambda(B, C) \right) + p(a, b)N\left( \int_0^T d\lambda \ B \right), L_\lambda(a, C) \right), \tag{3.3}
\]

\[
L_\lambda(1, A) = L_\lambda(A, 1) = 0, \tag{3.4}
\]

\[
L_\lambda(a, b) = [a, L_\lambda(b)], \tag{3.5}
\]

\[
L_\lambda(a \otimes B, C) = N\left( \left( e^{T \partial_\lambda}a \right), L_\lambda(B, C) \right) + p(a, B)N\left( \left( e^{T \partial_\lambda}B \right), L_\lambda(a, C) \right) \tag{3.6}
\]

\[
L_\lambda(a \otimes B, C) = N\left( \left( e^{T \partial_\lambda}a \right), L_\lambda(B, C) \right) + p(a, B)N\left( \left( e^{T \partial_\lambda}B \right), L_\lambda(a, C) \right) \tag{3.7}
\]

B) These linear maps satisfy the following grading conditions ($A, B \in T(R)$):

\[
\Delta(N(A, B)) \leq \Delta(A) + \Delta(B), \quad \Delta(L_\lambda(A, B)) < \Delta(A) + \Delta(B). \tag{3.8}
\]

**Proof.** We want to prove, by induction on $\Delta = \Delta(A) + \Delta(B)$, that $N(A, B)$ and $L_\lambda(A, B)$ exist and are uniquely defined by Eqs. (3.1–3.7) and that they satisfy the grading conditions (3.8). Let us first look at $N(A, B)$. If $A \in C$ or $A \in R$, $N(A, B)$ is defined by conditions (3.1) and (3.2). Suppose then $A = a \otimes A'$, where $a \in R$ and $A' \in R \oplus R^{\otimes 2} \oplus \cdots$. Notice that, in this case, $\Delta(a) < \Delta(A)$, $\Delta(A') < \Delta(A)$. By condition (3.3) we have

\[
N(A, B) = a \otimes N(A', B) + N\left( \int_0^T d\lambda \ a \right), L_\lambda(A', B) \right) + p(a, A')N\left( \int_0^T d\lambda \ A' \right), L_\lambda(a, B) \right), \tag{3.9}
\]

and, by the inductive assumption, each term in the right-hand side is uniquely defined and is in $T_\Delta(R)$. Consider then $L_\lambda(A, B)$. If either $A \in C$ or $B \in C$ we have, by (3.4), $L_\lambda(A, B) = 0$. Moreover, if $A, B \in R$, $L_\lambda(A, B)$ is defined by (3.5). Suppose now $A = a \in R$, $B = b \otimes B'$, with $b \in R$ and $B' \in R \oplus R^{\otimes 2} \cdots$. In this case we have, by (3.6)
\[ L_\lambda(A, B) = N(L_\lambda(a, b), B') + p(a, b)N(b, L_\lambda(a, C)) + \int_0^\lambda d\mu L_\mu(L_\lambda(a, b), B'), \]

and each term in the right-hand side is well defined by induction, and it lies in \( T_{\Delta'}(R) \), for some \( \Delta' < \Delta \). Finally, consider the case \( A = a \otimes A' \), with \( a \in R \) and \( A', B \in R \oplus R^\otimes 2 \oplus \cdots \). In this case we have, by (3.7),

\[ L_\lambda(A, B) = N(e^{T_0}a, L_\lambda(A', B)) + p(a, A')N(e^{T_0}A', L_\lambda(a, B)) \]

and as before each term in the right-hand side is defined by induction and lies in \( T_{\Delta'}(R) \), for some \( \Delta' < \Delta \). \( \Box \)

**Definition 3.3.** Let \( R \) be a non–linear conformal algebra. For \( \Delta \in \Gamma \), we define the subspace \( \mathcal{M}_{\Delta}(R) \subset T(R) \) by

\[ \mathcal{M}_{\Delta}(R) = \text{span}_{\mathbb{C}} \left\{ A \otimes \left( (b \otimes c - p(b, c)c \otimes b) \otimes D \right) \mid b, c \in R, A, D \in T(R), -N\left( \int_0^\lambda d\lambda L_\lambda(b, c), D \right) \right\}. \]

Furthermore, we define

\[ \mathcal{M}(R) = \bigcup_{\Delta \in \Gamma} \mathcal{M}_{\Delta}(R). \]

**Remark 3.4.** By definition we clearly have \( \mathcal{M}_{\Delta}(R) \subset \mathcal{M}(R) \cap T_{\Delta}(R) \). But in general \( \mathcal{M}_{\Delta}(R) \) needs not be equal to \( \mathcal{M}(R) \cap T_{\Delta}(R) \). In particular, if \( \delta \) is the smallest non–zero element of \( \Gamma \) such that \( R[\delta] \neq 0 \), then \( \mathcal{M}_{\delta}(R) = 0 \), since, for \( b, c \in R \), \( \Delta(b) + \Delta(c) \geq 2\delta \), whereas \( \mathcal{M}(R) \cap T_{\delta}(R) \) can be a priori non-zero.

**Definition 3.5.** A non–linear Lie conformal algebra is a non–linear conformal algebra \( R \) such that the \( \lambda \)–bracket \( [\cdot, \cdot]_\lambda \) satisfies the following additional axioms:

1. skewsymmetry
   \[ [a, b]_\lambda = -p(a, b)[b, a]_{-\lambda, -T} \quad \forall a, b \in R, \]

2. Jacobi identity
   \[ L_\lambda(a, L_\mu(b, c)) - p(a, b)L_\mu(b, L_\lambda(a, c)) - L_{\lambda, \mu}(L_\lambda(a, b), c) \in \mathbb{C}[\lambda, \mu] \otimes \mathcal{M}_{\Delta}(R), \]

for every \( a, b, c \in R \) and for some \( \Delta' \in \Gamma \) such that \( \Delta' < \Delta(a) + \Delta(b) + \Delta(c) \).

**Definition 3.6.** A homomorphism of non–linear Lie conformal algebras \( \phi : R \to R' \) is a \( \mathbb{C}[T] \)–module homomorphism preserving the \( \mathbb{Z}/2\mathbb{Z} \) and \( \Gamma \) gradations such that the induced map \( T(R) \to T(R') \) is a homomorphism for the operations \( N \) and \( L_\lambda \) modulo \( \mathcal{M}(R') \) in the following sense:

\[ \phi(N(A, B)) - N'(\phi(A), \phi(B)) \in \mathcal{M}_{\Delta}(R') \quad \text{for} \ \Delta \leq \Delta(A) + \Delta(B), \]

\[ \phi(L_\lambda(A, B)) - L'_{\lambda}(\phi(A), \phi(B)) \in \mathbb{C}[\lambda] \otimes \mathcal{M}_{\Delta}(R') \quad \text{for} \ \Delta < \Delta(A) + \Delta(B). \]
Remark 3.7. Given a $\lambda$–bracket on $R$, consider the map $L : \mathcal{T}(R) \otimes \mathcal{T}(R) \rightarrow \mathcal{T}(R)$ given by

$$L(A, B) = \int_0^\lambda d\lambda L_\lambda(A, B).$$

It is immediate to show that, if $R$ is a non–linear Lie conformal algebra, then $L$ satisfies skewsymmetry

$$L(a, b) = -p(a, b)L(b, a),$$

and the Jacobi identity

$$L(a, L(b, c)) - p(a, b)L(b, L(a, c)) - L(L(a, b), c) \in \mathcal{M}_{\Delta'}(R),$$

for every $a, b, c \in R$ and for some $\Delta' \in \Gamma$ such that $\Delta' < \Delta(a) + \Delta(b) + \Delta(c)$. Thus we obtain a non–linear Lie superalgebra, discussed in the introduction.

Remark 3.8. In the definition of non–linear Lie conformal algebra we could write the skew–symmetry axiom in the weaker form:

$$[a \lambda b] + p(a, b)[b \lambda -T a] \in \mathcal{M}_{\Delta'}(R),$$

for some $\Delta' < \Delta(a) + \Delta(b)$. In this way we would get a similar notion of non–linear Lie conformal algebra, which, as it will be clear from the results in Sect. 6, is not more useful than that given by Definition 3.5, from the point of view of classification of vertex algebras.

We are now ready to state the main results of this paper.

Theorem 3.9. Let $R$ be a non–linear Lie conformal algebra. Consider the vector space $U(R) = \mathcal{T}(R)/\mathcal{M}(R)$, and denote by $\pi : \mathcal{T}(R) \rightarrow U(R)$ the quotient map, and by $\cdot : \cdot$ the image in $U(R)$ of the tensor product of elements of $R$:

$$ab \ldots c := \pi(a \otimes b \otimes \cdots \otimes c), \quad \forall a, b, \ldots, c \in R.$$

Let $A = \{a_i, i \in I\}$ be an ordered basis of $R$ compatible with the $\mathbb{Z}/2\mathbb{Z}$–gradation and the $\Gamma$–gradation. We denote by $B$ the image in $U(R)$ of the collection of all ordered monomials, namely

$$B = \left\{ :a_1 i_1 \ldots a_n i_n : \left| i_1 \leq i_2 \leq \cdots \leq i_n, \quad \text{and } i_k < i_{k+1} \text{ if } p(a_{i_k}) = \overline{1} \right. \right\} \subset U(R).$$

Then

1. $B$ is a basis of the vector space $U(R)$. In particular we have a natural embedding

$$R \rightarrow \pi(R) \subset U(R).$$

2. There is a canonical structure of a vertex algebra on $U(R)$, called the universal enveloping vertex algebra of $R$, such that the vacuum vector $|0\rangle$ is $\pi(1)$, the infinitesimal translation operator $T : U(R) \rightarrow U(R)$ is induced by the action of $T$ on $\mathcal{T}(R)$:

$$T(\pi(A)) = \pi(TA), \quad \forall A \in \mathcal{T}(R),$$

the normally ordered product on $U(R)$ is induced by $N$:

$$\pi(A)\pi(B) : = \pi(N(A, B)), \quad \forall A, B \in \mathcal{T}(R),$$

and the $\lambda$–bracket $[\cdot \lambda : U(R) \otimes U(R) \rightarrow \mathbb{C}[\lambda]U(R)$ is induced by $L_\lambda$:

$$[\pi(A) \lambda \pi(B)] = \pi(L_\lambda(A, B)), \quad \forall A, B \in \mathcal{T}(R).$$

In other words, $U(R)$ is a vertex algebra freely generated by $R$. 

The above notion of a homomorphism of non-linear Lie conformal algebras corresponds to that of homomorphism of the corresponding universal enveloping vertex algebras. Namely we have the following.

**Proposition 3.10.** Let \( \phi : R \to R' \) be a homomorphism of non-linear Lie conformal algebras.

(i) If we extend \( \phi \) to a homomorphism of the associative tensor algebras \( T(R) \to T(R') \), we have \( \phi(M_{\Delta}(R)) \subset M_{\Delta}(R') \).
(ii) \( \phi \) induces a homomorphism of the universal enveloping vertex algebras: \( \phi : U(R) \to U(R') \).

**Proof.** We have

\[
\phi \left( A \otimes \left( b \otimes c \otimes D - p(b, c)c \otimes b \otimes D - N \left( \int_{-T}^{0} d\lambda L_\lambda(b, c) , D \right) \right) \right)
\]

\[
= \phi(A) \otimes \left( \phi(b) \otimes \phi(c) \otimes \phi(D) - p(b, c)\phi(c) \otimes \phi(b) \otimes (D) \right)
\]

\[
- N' \left( \int_{-T}^{0} d\lambda L'_\lambda(\phi(b), \phi(c)) , \phi(D) \right)
\]

\[
- \phi(A) \otimes \left( N \left( \int_{-T}^{0} d\lambda L_\lambda(b, c) , D \right) \right) - N' \left( \int_{-T}^{0} d\lambda \phi(L_\lambda(b, c)) , \phi(D) \right)
\]

\[
- \phi(A) \otimes N' \left( \int_{-T}^{0} d\lambda \left( \phi(L_\lambda(b, c)) - L'_\lambda(\phi(b), \phi(c)) \right) , \phi(D) \right)
\]

All three terms in the right hand side belong to \( M_{\Delta}(R') \) by the assumptions on \( \phi \) and the results in the next section (see, in particular, Corollary 4.5). The second part of the proposition is obvious.

The vertex algebra \( U(R) \) has the following universality property:

**Proposition 3.11.** Let \( \phi \) be a homomorphism from a non-linear Lie conformal algebra \( R \) to a vertex algebra \( V \), namely a parity preserving \( \mathbb{C}[T] \)-module homomorphism \( \phi : R \to V \) such that, if we extend it to a linear map \( \phi : T(R) \to V \) by letting \( \phi(a_1 \otimes \cdots \otimes a_i) =: \phi(a_1) \cdots \phi(a_i) \), we have

\[
\phi(L_\lambda(a, b)) = [\phi(a), \phi(b)], \quad \forall a, b \in R.
\]

Then:

(a) \( \phi(N(A, B)) =: \phi(A)\phi(B) \), \( \phi(L_\lambda(A, B)) = [\phi(A), \phi(B)] \), \( \forall A, B \in T(R) \).

(b) \( \phi \) extends to a unique vertex algebra homomorphism \( \phi : U(R) \to V \).

**Proof.** (a) is proved by an easy induction on degree of \( T(R) \) and (b) follows from (a). \( \square \)

**Remark 3.12.** Notice that the universal enveloping vertex algebra \( U(R) \) is independent of the choice of the grading of \( R \).
Example 3.13. Suppose \( \hat{R} = R \oplus \mathbb{C}(0) \), where \( \mathbb{C}(0) \) is a torsion submodule, is a Lie conformal algebra. In this case, if we let \( \Gamma = \mathbb{Z}_+ \) and assign degree \( \Delta = 1 \) to every element of \( R \), we make \( R \) into a non-linear Lie conformal algebra. In this case we have a Lie algebra structure on \( R \) defined by the Lie bracket (recall that \( \mathbb{C}(0) \) is in the center of the \( \lambda \)-bracket)

\[
[a, b] = \int_{-T}^{0} d\lambda [a \lambda b], \quad \forall a, b \in R.
\]

Therefore the space \( U(R) \) coincides with the universal enveloping algebra of \( R \) (viewed as a Lie algebra with respect to \( [ \, ] \)), and the first part of Theorem 3.9 is the PBW theorem for ordinary Lie superalgebras. Moreover in this case \( U(R) \) simply coincides with the quotient of the universal enveloping vertex algebra of \( \hat{R} \), defined by Theorem 2.5, by the ideal generated by \( |0\rangle - 1 \). In particular, if \( \hat{R} \) is the Virasoro Lie conformal algebra \( [L \lambda L] = (T + 2\lambda)L + \frac{c}{12}\lambda^3|0\rangle \), then \( U(R) \) is the universal Virasoro vertex algebra with central charge \( c \).

Conversely, suppose \( R \) is a non-linear Lie conformal algebra graded by \( \Gamma = \mathbb{Z}_+ \) and such that every element of \( R \) has degree \( \Delta = 1 \). It follows by the grading condition on \( [ \, ] \) that \( \hat{R} = R \oplus \mathbb{C} \) is a Lie conformal algebra, and therefore Theorem 3.9 holds by the above considerations.

Example 3.14. The first example of a “genuinely” non-linear Lie conformal algebra \( R \) is Zamolodchikov’s \( W_3 \)-algebra [Zam85], which, in our language, is

\[
W_3 = \mathbb{C}[T]L + \mathbb{C}[T]W,
\]

where \( \Delta(L) = 2 \), \( \Delta(W) = 3 \), and the \( \lambda \)-brackets are as follows:

\[
[L \lambda L] = (T + 2\lambda)L + \frac{c}{12}\lambda^3, \quad [L \lambda W] = (T + 3\lambda)W,
\]

\[
[W \lambda W] = (T + 2\lambda)\left(\frac{16}{22 + 5c}(L \otimes L) + \frac{c - 10}{3(22 + 5c)}T^2L + \frac{1}{6}(T + \lambda)L\right) + \frac{c}{360}\lambda^5.
\]

We want to prove that this is the unique, up to isomorphisms, non-linear Lie conformal algebra generated by two elements \( L \) and \( W \), such that \( L \) is a Virasoro element of central charge \( c \) and \( W \) is an even primary element of conformal weight 3. We assume that \( [W \lambda W] \notin \mathbb{C}[\lambda]1 \).

We only need to show that \( [W \lambda W] \) has to be as in (3.10). By simple conformal weight considerations (see Example 6.2) and by skewsymmetry of the \( \lambda \)-bracket, the most general form of \( [W \lambda W] \) is

\[
[W \lambda W] = (T + 2\lambda)\left(\alpha(L \otimes L) + \beta TW + \gamma T^2L + \delta\lambda(T + \lambda)L\right) + \epsilon\lambda^5,
\]

for \( \alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{C} \). We need to find all possible values of these parameters such that the Jacobi identity (3.9) holds. We may assume \( \alpha \neq 0 \), since otherwise \( W_3 \oplus \mathbb{C}(0) \) is a “linear” Lie conformal algebra, and from their classification [DK98] it follows that \( [W \lambda W] \in \mathbb{C}[\lambda]1 \), which is not allowed. There are only two non-trivial Jacobi identities.
We will show how to deal with the first, and we will leave the second to the reader. We need to impose

\[ [W \lambda, [W \mu L]] - [W \mu, [W \lambda L]] \equiv [[W \lambda, W \lambda + \mu], L] \mod \mathbb{C}[[\lambda, \mu]] \otimes \mathcal{M}_7(R). \]  

Using the axioms of non–linear conformal algebra, we can compute every term of Eq. (3.11). The first term of the left-hand side is

\[ (2T + 2\lambda + 3\mu)(T + 2\lambda) \left[ \alpha L \otimes L + \beta TW + \gamma T^2 \lambda L \right] + \delta \lambda (T + \lambda)L \]  

and the second term is obtained by replacing \( \lambda \) by \( \mu \). The right-hand side of (3.11) is

\[ (\lambda - \mu) \left\{ 2\alpha L \otimes TL + 4\alpha TL \otimes L + 4\alpha (\lambda + \mu)L \otimes L + \frac{1}{6} \alpha c(T + \lambda + \mu)^3 L 
+ \alpha \left( \frac{1}{3} (\lambda + \mu)^3 + \frac{2}{3} (\lambda + \mu)^2 T + (\lambda + \mu)T^2 \right) L + \frac{1}{30} \alpha c(\lambda + \mu)^5 
- \beta (\lambda + \mu) (2T + 3\lambda + 3\mu)W 
+ (\gamma (\lambda + \mu)^2 - \delta \lambda \mu)((T + 2\lambda + 2\mu)L + \frac{1}{12} (\lambda + \mu)^3) \right\}. \]  

If \( d \neq 0 \), (3.11) can’t be an exact identity, but only an identity modulo \( \mathcal{M}_4(R) \). Indeed in (3.13) we have a term \( 2\alpha L \otimes TL \), which does not appear in (3.12). But we can replace in (3.13)

\[ TL \otimes L - L \otimes TL \]  

by \( -\frac{1}{6} T^3 L \mod \mathcal{M}_4(R) \).

After this substitution, both sides of (3.11) are linear combinations of \( L \otimes L, L, W, 1 \), with coefficients in \( \mathbb{C}[[\lambda, \mu, T]] \). Hence by Theorem 3.9 all these coefficients are zero. It is then a simple exercise to show that the only solution of (3.11) is given, up to rescaling of \( W \), by choosing the parameters \( \alpha, \beta, \gamma, \delta, \epsilon \) as in (3.10). After this, one checks that the Jacobi identity involving three \( W \) elements is automatic.

A large class of non–linear Lie conformal (super)algebras is obtained by quantum Hamiltonian reduction attached to a simple Lie (super)algebra \( g \) and a nilpotent orbit in \( g \) (see e.g. [FF90, KW04, dBT94]). In particular, the Virasoro algebra and the \( W_3 \)-algebra are obtained by making use of the principal nilpotent orbit of \( g = sl_2 \) and \( sl_3 \) respectively.

The next two sections will be devoted to the proof Theorem 3.9. In the next section we will prove some technical results, and in the following section we will complete the proof of Theorem 3.9.

4. Technical Results

Let \( R \) be a non–linear Lie conformal algebra. For convenience, we introduce the following notations which will be used throughout the following sections (\( A, B, C \cdots \in \mathbb{C} \)):
Freely Generated Vertex Algebras and Non-linear Lie Conformal Algebras

\( T(R) \):

\[
\text{sl}(A, B; \lambda) = L_\lambda(A, B) + p(A, B)L_{-\lambda - T}(B, A),
\]

\[
\text{sn}(A, B, C) = N(A, N(B, C)) - p(A, B)N(B, N(A, C))
\]

\[
- N\left( \int_{-T}^0 d\lambda \ L_\lambda(A, B) \right), C),
\]

\[
W^l(A, B, C; \lambda) = L_\lambda(A, N(B, C)) - p(A, B)N(B, L_\lambda(A, C))
\]

\[
- N\left( \int_{-T}^0 d\mu \ L_\mu\left( L_\lambda(A, B), C \right) \right),
\]

\[
W^r(A, B, C; \lambda) = L_\lambda(N(A, B), C) - p(A, B)\int_{-T}^0 d\mu \ L_\mu\left( L_\lambda(A, C) \right)
\]

\[
Q(A, B, C) = N(N(A, B), C) - N(A, N(B, C)) - p(A, B)N\left( \int_{-T}^0 d\lambda \ L_\lambda(A, B) \right), L_\lambda(A, C)),
\]

\[
J(A, B, C; \lambda, \mu) = L_\lambda(A, L_\mu(B, C)) - p(A, B)L_\mu\left( L_\lambda(A, C) \right)
\]

\[
- L_{\lambda + \mu}(L_\lambda(A, B), C).
\]

Remark 4.1. Using the above notation, we can write in a more concise form all the definitions introduced in the previous section. The maps \( N : T(R) \otimes T(R) \to T(R) \) and \( L_\lambda : T(R) \otimes T(R) \to \mathbb{C}[\lambda] \otimes T(R) \) are defined respectively by the equations

\[
Q(a, B, C) = 0, \quad W^l(a, b, c; \lambda) = 0, \quad W^r(a, b, c; \lambda) = 0,
\]

for all \( a, b \in R \) and \( B, C \in T(R) \). The axioms of non–linear Lie conformal algebras can be written as

\[
\text{sl}(a, b ; \lambda) = 0, \quad J(a, b, c ; \lambda, \mu) \in \mathbb{C}[\lambda, \mu] \otimes \mathcal{M}_\Delta(R),
\]

for all \( a, b, c \in R \) and for some \( \Delta' < \Delta(a) + \Delta(b) + \Delta(c) \). Finally, the subspaces \( \mathcal{M}_\Delta(R) \subset T(R), \Delta \in \Gamma \), are defined as

\[
\mathcal{M}_\Delta(R) = \text{span}_\mathbb{C} \left\{ A \otimes \text{sn}(b,c, D) \mid A, D \in T(R), b, c \in R, \Delta(A) + \Delta(b) + \Delta(c) + \Delta(D) \leq \Delta \right\},
\]

and, by definition, \( \mathcal{M}(R) = \bigcup_{\Delta \in \Gamma} \mathcal{M}_\Delta(R) \).

Lemma 4.2. The endomorphism \( T : T(R) \to T(R) \) is a derivation of the product \( N : T(R) \otimes T(R) \to T(R), \) namely, for \( A, B \in T(R) \),

\[
TN(A, B) = N(TA, B) + N(A, TB).
\]

Furthermore, the \( \lambda \)-bracket \( L_\lambda : T(R) \otimes T(R) \to \mathbb{C}[\lambda] \otimes T(R) \) satisfies sesquilinearity, namely, for \( A, B \in T(R) \),

\[
\text{sl}(A, B; \lambda) = L_\lambda(A, B) + p(A, B)L_{-\lambda - T}(B, A),
\]

\[
\text{sn}(A, B, C) = N(A, N(B, C)) - p(A, B)N(B, N(A, C))
\]

\[
- N\left( \int_{-T}^0 d\lambda \ L_\lambda(A, B) \right), C),
\]

\[
W^l(A, B, C; \lambda) = L_\lambda(A, N(B, C)) - p(A, B)N(B, L_\lambda(A, C))
\]

\[
- N\left( \int_{-T}^0 d\mu \ L_\mu\left( L_\lambda(A, B), C \right) \right),
\]

\[
W^r(A, B, C; \lambda) = L_\lambda(N(A, B), C) - p(A, B)\int_{-T}^0 d\mu \ L_\mu\left( L_\lambda(A, C) \right)
\]

\[
Q(A, B, C) = N(N(A, B), C) - N(A, N(B, C)) - p(A, B)N\left( \int_{-T}^0 d\lambda \ L_\lambda(A, B) \right), L_\lambda(A, C)),
\]

\[
J(A, B, C; \lambda, \mu) = L_\lambda(A, L_\mu(B, C)) - p(A, B)L_\mu\left( L_\lambda(A, C) \right)
\]

\[
- L_{\lambda + \mu}(L_\lambda(A, B), C).
\]
In particular $T$ is a derivation of $L_\lambda$.

**Proof.** We will prove both statements of the lemma by induction on $\Delta = \Delta(A) + \Delta(B)$.

If $\Delta = 0$, we have $A, B \in \mathbb{C}$, and there is nothing to prove. Suppose then $\Delta > 0$. For $A \in \mathbb{C}$, all three Eqs. (4.1), (4.2) and (4.3) are obviously satisfied. We will consider separately two cases:

1. $A = a \in \mathbb{R}$,
2. $A = a \otimes A'$, with $a \in \mathbb{R}, A' \in \mathbb{R} \oplus \mathbb{R} \otimes 2 \oplus \cdots \subset T(R)$.

**Case 1.** For $A \in \mathbb{R}$, Eq. (4.1) follows by the fact that $T$ is a derivation of the tensor product and that $N(a, B) = a \otimes B$. Consider then Eqs. (4.2) and (4.3). For $B \in \mathbb{C}$ they are trivial, and for $B \in \mathbb{R}$ they hold by (3.5). Suppose then $B = b \otimes B'$, with $b \in \mathbb{R}$ and $B' \in \mathbb{R} \oplus \mathbb{R} \otimes 2 \oplus \cdots$. By (3.6) we have

$$L_\lambda(Ta, B) = N(L_\lambda(Ta, b), B') + p(a, b)N(b, L_\lambda(Ta, B'))$$

$$+ \int_0^\lambda d\mu \, L_\mu(L_\lambda(Ta, b), B') = -\lambda L_\lambda(a, B).$$

In the last identity we used the inductive assumption and the grading conditions (3.8).

Similarly we have

$$\begin{align*}
L_\lambda(Ta, B) &= N(L_\lambda(Ta, b), B') + p(a, b)N(b, L_\lambda(Ta, B')) \\
&+ \int_0^\lambda d\mu \, L_\mu(L_\lambda(Ta, b), B') \\
&+ N(L_\lambda(a, b), T B') + p(a, b)N(b, L_\lambda(a, T B')) \\
&+ \int_0^\lambda d\mu \, L_\mu(L_\lambda(a, b), T B') = (\lambda + T) L_\lambda(a, B).
\end{align*}$$

**Case 2.** By Eq. (3.3), the inductive assumption and the grading conditions (3.8) we have

$$TN(A, B) = T \left[ N(a, N(A', B)) + N(\int_0^T d\lambda a), L_\lambda(A', B) \right]$$

$$+ p(a, A')N\left( \int_0^T d\lambda A', L_\lambda(a, B) \right) = N(TA, B) + N(A, TB),$$

which proves (4.1). Similarly by Eq. (3.7) we prove (4.2) and (4.3):

$$\begin{align*}
L_\lambda(Ta, B) &= L_\lambda(Ta \otimes A', B) + L_\lambda(a \otimes TA', B) \\
&= N\left( e^{T\partial_\lambda} Ta, L_\lambda(A', B) \right) + p(a, A')N\left( e^{T\partial_\lambda} A', L_\lambda(Ta, B) \right) \\
&+ p(a, A') \int_0^\lambda d\mu \, L_\mu(A', L_{\lambda-\mu}(Ta, B)) \\
&+ N\left( e^{T\partial_\lambda} a, L_\lambda(TA', B) \right) + p(a, A')N\left( e^{T\partial_\lambda} TA', L_\lambda(a, B) \right) \\
&+ p(a, A') \int_0^\lambda d\mu \, L_\mu(TA', L_{\lambda-\mu}(a, B)) = -\lambda L_\lambda(A, B),
\end{align*}$$

$$L_\lambda(TA, B) = -\lambda L_\lambda(A, B),$$

$$L_\lambda(A, TB) = (\lambda + T)L_\lambda(A, B).$$
and
\[ L_\lambda(A, T B) = N(e^{T_\lambda}a), L_\lambda(A', T B) + p(a, A') N(e^{T_\lambda}A'), L_\lambda(a, T B) \]
\[ + p(a, A') \int_0^\lambda d\mu L_\mu(A', L_{\lambda-\mu}(a, T B)) = (\lambda + T)L_\lambda(A, B). \]

In the above equations we have used the obvious commutation relation \( e^{T_\lambda} = (\lambda + T)e^{T_\lambda} \). This completes the proof of the lemma. \( \square \)

As an immediate consequence of the above lemma, we get the following

**Corollary 4.3.** The subspaces \( \mathcal{M}_A(R) \subset \mathcal{T}(R) \) are invariant under the action of \( T \). In particular \( T_\mathcal{M}(R) \subset \mathcal{M}(R) \).

The following lemma follows by a straightforward but quite lengthy computation, which we omit.

**Lemma 4.4.** The following equations hold for all \( a, b, \cdots \in R \) and \( A, B, \cdots \in \mathcal{T}(R) \):

\[ N(sn(a, b, c), D) = sn(a, b, N(C, D)) + \int_0^{T_\lambda} \frac{d\lambda}{\lambda} sn(a, b, L_\lambda(C, D)) \]
\[ - Q\left( \left( \int_{-T}^0 d\lambda L_\lambda(a, b), C, D \right) \right) \]
\[ - p(a, C) p(b, C) N\left( \int_0^T d\lambda \int_0^{T-\lambda} d\mu C, J(a, b, D; \lambda, \mu) \right), \]

\[ L_\lambda(a, sn(b, c, D)) = sn(L_\lambda(a, b), c, D) - p(b, c)sn(L_\lambda(a, c), b, D) \]
\[ + p(a, b) p(a, c)sn(b, c, L_\lambda(a, D)) \]
\[ + \int_0^\lambda d\mu \left[ W^i(L_\lambda(a, b), c, D; \mu) - p(b, c)W^i(L_\lambda(a, c), b, D; \mu) \right] \]
\[ - W^i(a, \int_{-T}^0 d\mu L_\mu(b, c), D; \lambda) \]
\[ + N\left( \left( \int_{-T}^\lambda d\mu \text{sl}(L_\lambda(a, b), c; \mu) - p(b, c)\text{sl}(L_\lambda(a, c), b; \mu) \right) \right), D \]
\[ - N\left( \left. \int_{-T}^\lambda d\mu \text{sl}(a, L_\mu(b, c); \lambda) \right) \right), D \]
\[ - p(a, b) p(a, c) N\left( \left. \int_{-T}^\lambda d\mu J(b, c, a; -\mu - T, \mu - \lambda) \right) \right), D \]
\[ + \int_0^\lambda d\nu \int_0^\lambda d\mu L_\nu(\text{sl}(L_\lambda(a, b), c; \mu) - p(b, c)\text{sl}(L_\lambda(a, c), b; \mu), D) \]
\[ - \int_0^\lambda d\nu \int_0^\lambda d\mu L_\nu(\text{sl}(a, L_{\mu-\lambda}(b, c); \lambda), D) \]
\[ - p(a, b) p(a, c) \int_0^\lambda d\nu \int_0^\lambda d\mu L_\nu(J(b, c, a; \mu - \lambda, v - \mu), D), \]
\[ L_\lambda(\text{sn}(a, b, C), D) = \text{sn}\left( \begin{pmatrix} e^{T_0}a \\ e^{T_0}b \end{pmatrix}, L_\lambda(C, D) \right) \]

\[ -W^r\left( \int_{-T}^0 d\mu L_\mu(a, b), C, D; \lambda \right) \]

\[ -p(a, C)p(b, C)N\left( \int_0^T d\mu e^{T_0}C, J(a, b, D; \mu, \lambda) \right) \]

\[ -p(a, C)p(b, C) \int_0^\lambda d\mu \int_0^{\lambda-\mu} d\nu L_\nu(C, J(a, b, D; \mu-v, \lambda-\mu)), \]

\[ Q(N(a, A'), B, C) = N(a, Q(A', B, C)) + p(a, A') \int_0^{T_{A'}} d\lambda Q(A', L_\lambda(a, B), C) \]

\[ -\int_{-T}^0 d\lambda N(a, W^d(A', B, C; \lambda)) \]

\[ p(a, A') \int_{-T}^{T_{A'}} d\lambda N(A', W^d(a, B, C; \lambda)) \]

\[ + p(a, A')p(a, B) \int_0^{T_0} d\lambda Q(A', B, L_\lambda(a, C)) \]

\[ + p(A', B) \int_{-T}^{T_0} d\lambda \text{sn}(a, B, L_\lambda(A', C)) \]

\[ + p(a, A')p(a, B) \int_0^{T_0} d\lambda \text{sn}(a, B, L_\lambda(A', C)) \]

\[ - \int_{-T}^0 d\lambda \int_{-\lambda}^0 d\mu N(a, J(A', B, C; \lambda, \mu)) \]

\[ - p(a, A') \int_{-T}^{T_{A'}} d\lambda \int_0^{T_{A'}-\lambda} d\mu N(A', J(a, b, C; \lambda, \mu)), \]

\[ W^d(a, N(b, B', C; \lambda)) = p(a, b)N(b, W^d(a, B', C; \lambda)) - Q(L_\lambda(a, b), B', C) \]

\[ + \int_{-T}^0 d\mu W^d(L_\lambda(a, b), B', C; \mu) \]

\[ + p(b, B')W^d(a, \left( \int_0^T d\mu B' \right), L_\mu(b, C; \lambda)) \]

\[ - \int_{-T}^0 d\mu W^r(L_\lambda(a, b), B', C; \mu) \]

\[ + p(a, b)N\left( \int_0^T d\mu b \right), J(a, B', C; \lambda, \mu)) \]

\[ + p(a, B')p(b, B')N\left( \int_0^T d\mu B' \right), J(a, b, C; \lambda, \mu)) \]

\[ + \int_{-T}^0 d\mu \int_0^{\lambda-\mu} d\nu J(L_\lambda(a, b), B', C; \mu, \nu), \]
\[ W^f(N(a, A'), B, C; \lambda) = N\left( e^{T_{\lambda}a}, W^f(A', B, C; \lambda) \right) \]

\[ + p(a, A') N\left( e^{T_{\lambda}A'}, W^f(a, B, C; \lambda) \right) \]

\[ - p(a, A') Q\left( e^{T_{\lambda}A'}, L_{\lambda}(a, B), C \right) \]

\[ + p(A', B) \text{sn}( e^{T_{\lambda}a}, B, L_{\lambda}(A', C) ) \]

\[ + p(a, A') p(a, B) \text{sn}( e^{T_{\lambda}A'}, B, L_{\lambda}(a, C) ) \]

\[ + p(a, A') \int_0^\lambda d\mu L_{\mu}(A', W^f(a, B, C; \lambda - \mu)) \]

\[ - p(a, A') \int_0^\lambda d\mu W^f\left( e^{T_{\lambda}A'}, L_{\lambda}(a, B), C; \mu \right) \]

\[ + p(a, A') \int_0^\lambda d\mu W^f(A', L_{\lambda-\mu}(a, B), C; \mu) \]

\[ + p(a, A') p(a, B) \int_0^\lambda d\mu W^f(A', B, L_{\lambda-\mu}(a, C); \mu) \]

\[ + p(a, A') \int_0^\lambda d\mu \int_0^{\lambda-\mu} dv J(A', L_{\lambda-\mu}(a, B), C; \mu, v), \]

\[ W^r(N(a, A'), B, C; \lambda) = N\left( e^{T_{\lambda}a}, W^r(A', B, C; \lambda) \right) \]

\[ + p(a, A') p(a, B) Q\left( e^{T_{\lambda}a}, \left( e^{T_{\lambda}A'}, B, L_{\lambda}(a, C) \right) \right) \]

\[ + p(A', B) \text{sn}( e^{T_{\lambda}a}, B, L_{\lambda}(A', C) ) \]

\[ + p(a, A') p(a, B) \text{sn}( e^{T_{\lambda}A'}, e^{T_{\lambda}B}, L_{\lambda}(a, C) ) \]

\[ + p(a, A') W^r\left( \int_0^T d\mu A', L_{\mu}(a, B), C; \lambda \right) \]

\[ - p(a, B) p(A', B) \int_0^\lambda d\mu W^d(B, \left( e^{T_{\lambda}a}, L_{\lambda-\mu}(A', C); \mu \right) \]

\[ - p(a, A') p(a, B) p(A', B) \int_0^\lambda d\mu W^d(B, \left( e^{T_{\lambda}A'}, L_{\lambda-\mu}(a, C); \mu \right) \]

\[ + p(a, A') p(a, B) \int_0^\lambda d\mu W^d\left( B, \left( e^{T_{\lambda}A'}, L_{\lambda-\mu}(a, C); \mu \right) \right) \]

\[ - N\left( \int_0^T d\mu e^{T_{\lambda}a}, \left( e^{T_{\lambda}A'}, B, L_{\lambda}(a, C) \right) \right) \]

\[ - p(a, A') N\left( \int_0^T d\mu e^{T_{\lambda}A'}, B, L_{\lambda}(a, C); \mu, \lambda - \mu \right) \]

\[ - p(a, A') p(a, B) \int_0^\lambda d\mu N\left( e^{T_{\lambda}A'}, B, L_{\lambda}(a, C) \right) \]

\[ - p(a, B) p(A', B) \int_0^\lambda d\mu N\left( e^{T_{\lambda}A'}, B, L_{\lambda}(a, C) \right) \]

\[ + p(A', B) \int_0^T d\mu \int_0^\lambda dv L_{\nu}(\text{sl}(a, B; \mu), L_{\lambda-\nu}(A', C)) \]

\[ + p(a, A') p(a, B) \int_0^T d\mu \int_0^\lambda dv L_{\nu}(\text{sl}(A', B; \mu), L_{\lambda-\nu}(a, C)). \]
\[ J(a, b, N(c, D); \lambda, \mu) = W^i(a, L_{\mu}(b, c), D; \lambda) - p(a, b)W^i(b, L_{\lambda}(a, c), D; \mu) \]
\[ -W^i(L_{\lambda}(a, b), c, D; \lambda + \mu) \tag{4.11} \]
\[ + N(J(a, b; c, \lambda, \mu), D) \]
\[ + p(a, c)p(b, c)N(c, J(a, b; D, \lambda, \mu)) \]
\[ + \int_0^\mu dv J(a, L_{\mu}(b, c), D; \lambda, v) \]
\[ - p(a, b) \int_0^\lambda dv J(b, L_{\lambda}(a, c), D; \mu, v) \]
\[ + \int_0^{\lambda + \mu} dv L_{\nu}(J(a, b; c, \lambda, \mu), D), \]

\[ J(A, N(b, B'), D; \lambda, \mu) = W^i(A, b, L_{\mu} + T_{b'}(B', D); \lambda) \]
\[ + p(b, B')W^i(A, B', L_{\mu} + T_{b'}(b, D); \lambda) \]
\[ - L_{\lambda + \mu}(W^i(A, b, B'; \lambda), D) - W^i(L_{\lambda}(A, b), B', D; \lambda + \mu) \]
\[ + p(A, b)N(b, J(A, B'; D; \lambda, \mu + T_{b'})) \]
\[ + p(A, B')p(b, B')N(B', J(A, b; D; \lambda, \mu + T_{b'})) \] \tag{4.12}
\[ + \int_0^\lambda dv J(L_{\lambda}(A, b), B', D; \nu, \lambda + \mu - \nu) \]
\[ + p(b, B') \int_0^\mu dv J(A, B', L_{\mu}(b, D); \lambda, \nu) \]
\[ + p(A, B')p(b, B') \int_0^\mu dv L_{\nu}(B', J(A, b; D; \lambda, \mu - \nu)), \]

\[ J(N(a, A'), B; C; \lambda, \mu) = - p(a, B)p(A', B)J(B, N(a, A'), C; \mu, \lambda) \]
\[ - L_{\lambda + \mu}(sl(N(a, A'), B; \lambda), C), \tag{4.13} \]

\[ sn(a, N(b, B'), C) = sn(a, b, N(B', C)) + p(a, b)N(b, sn(a, B', C)) \]
\[ - p(a, b)p(a, B')N\left( \int_0^T d\lambda b \right), W^i(B', a; C; \lambda) \]
\[ - \int_{-T_0 - T_{b'}}^{-T_{b'} - T_{b'}} d\lambda Q(L_{\lambda}(a, b), B', C) \]
\[ + p(a, b) \int_{T_{b}}^{T_{b'}} d\lambda sn(L_{\lambda}(b, a), B', C) \]
\[ - p(a, b)p(a, B')N\left( \int_0^T d\lambda b \right), N(sl(B', a; \lambda), C) \]
\[ + sn(a, \left( \int_0^T d\lambda b \right), L_{\lambda}(B', C)) \]
\[ + p(b, B')sn(a, \left( \int_0^T d\lambda B' \right), L_{\lambda}(b, C)) \]
\[ - p(a, b)N\left( \int_0^T d\lambda \int_0^\lambda d\mu b \right), L_{\mu}(sl(a, B'; \mu - \lambda), C), \]
Corollary 4.5. For every $\Delta_1, \Delta_2, \Delta_3 \in \Gamma$ there exists $\Delta' \in \Gamma$ such that the following inclusions hold

\[
N(T_{\Delta_1}(R), M_{\Delta_2}(R)) \subset M_{\Delta_1 + \Delta_2}(R),
\]

\[
N(M_{\Delta_1}(R), T_{\Delta_2}(R)) \subset M_{\Delta_1 + \Delta_2}(R),
\]

\[
L_{\lambda}(T_{\Delta_1}(R), M_{\Delta_2}(R)) \subset \mathbb{C}[\lambda] \otimes M_{\Delta'}(R), \Delta' < \Delta_1 + \Delta_2,
\]

\[
L_{\lambda}(M_{\Delta_1}(R), T_{\Delta_2}(R)) \subset \mathbb{C}[\lambda] \otimes M_{\Delta'}(R), \Delta' < \Delta_1 + \Delta_2,
\]

\[
Q(T_{\Delta_1}(R), T_{\Delta_2}(R), T_{\Delta_3}(R)) \subset M_{\Delta_1 + \Delta_2 + \Delta_3}(R),
\]

\[
W(T_{\Delta_1}(R), T_{\Delta_2}(R), T_{\Delta_3}(R); \lambda) \subset \mathbb{C}[\lambda] \otimes M_{\Delta'}(R), \Delta' < \Delta_1 + \Delta_2 + \Delta_3,
\]

\[
W'(T_{\Delta_1}(R), T_{\Delta_2}(R), T_{\Delta_3}(R); \lambda) \subset \mathbb{C}[\lambda] \otimes M_{\Delta'}(R), \Delta' < \Delta_1 + \Delta_2 + \Delta_3,
\]
Proof.

We will prove, by induction on $\Delta$, the following conditions hold for all $\lambda, \mu \in \mathbb{C}$. Let then $\Delta' < \Delta_1 + \Delta_2 + \Delta_3$.  

1. $A = a \in R$, 
2. $A = a \otimes A'$, with $a \in R$ and $A' \in R \otimes R \otimes \cdots$.

Case 1. For $B \in \mathbb{C}$ or $B \in R$ condition (4.28) is trivially satisfied, and for $A \in R$ it holds thanks to Eq. (3.3). Let then $B = b \otimes B'$, with $b \in R$ and $B' \in R \otimes R \otimes \cdots$. In this case $W^l(A, B, C; \lambda)$ is given by Eq. (4.15), and, by the inductive assumption, every term in the right-hand side belongs to $\mathcal{M}_{\Delta'}(R)$, for some $\Delta' < \Delta$. 

Proof of (4.28). For $A \in \mathbb{C}$ condition (4.28) is trivially satisfied, and for $A \in R$ it holds thanks to Eq. (3.3). Let then $A = a \otimes A'$, with $a \in R$ and $A' \in R \otimes R \otimes \cdots$. In this case $Q(A, B, C)$ is given by Eq. (4.7), and every term in the right-hand side belongs to $\mathcal{M}_{\Delta'}(R)$ by the inductive assumption.

Proof of (4.29). For $A \in \mathbb{C}$ there is nothing to prove. We will consider separately two cases:

1. $A = a \in R$, 
2. $A = a \otimes A'$, with $a \in R$ and $A' \in R \otimes R \otimes \cdots$. 

Case 1. For $B \in \mathbb{C}$ or $B \in R$ condition (4.29) is trivially satisfied, thanks to Eq. (3.6). Let then $B = b \otimes B'$, with $b \in R$ and $B' \in R \otimes R \otimes \cdots$. In this case $W^l(A, B, C; \lambda)$ is given by Eq. (4.8), and, by the inductive assumption, every term in the right-hand side lies in $\mathcal{M}_{\Delta'}(R)$, for some $\Delta' < \Delta$. 

Proof of (4.30). For $A \in \mathbb{C}$ condition (4.30) is trivially satisfied, and for $A \in R$ it holds thanks to Eq. (3.6). Let then $A = a \otimes A'$, with $a \in R$ and $A' \in R \otimes R \otimes \cdots$. In this case $J(A, B, C)$ is given by Eq. (4.7), and every term in the right-hand side belongs to $\mathcal{M}_{\Delta'}(R)$ by the inductive assumption.
Case 2. In this case $W'(A, B, C; \lambda)$ is given by Eq. (4.10), and again, by induction, every term of the right-hand side lies in $\mathcal{M}_{\Delta'}(R)$, for some $\Delta' < \Delta$.

Proof of (4.30). For $A \in \mathbb{C}$ or $A \in R$, condition (4.30) holds trivially, thanks to Eq. (3.7). Suppose then $A = a \otimes A'$, with $a \in R$ and $A' \in R \oplus R^{\otimes 2} \oplus \cdots$. In this case $W'(A, B, C; \lambda)$ is given by Eq. (4.10), and every term in the right-hand side lies in $\mathcal{M}_{\Delta'}(R)$, for some $\Delta' < \Delta$.

Proof of (4.31). If either $A \in \mathbb{C}$, or $B \in \mathbb{C}$, or $C \in \mathbb{C}$, $J(A, B, C; \lambda, \mu) = 0$, so that (4.31) holds. Moreover, if $A, B, C \in R$, condition (4.31) holds by the definition of non–linear Lie conformal algebra. We will consider separately the following three cases:

1. $A = a \in R$, $B = b \in R$, $C = c \otimes D$, with $c \in R$ and $D \in R \oplus R^{\otimes 2} \oplus \cdots$.
2. $B = b \otimes B'$, with $b \in R$ and $A, B', C \in R \oplus R^{\otimes 2} \oplus \cdots$.
3. $A = a \otimes A'$, with $a \in R$ and $A', B, C \in R \oplus R^{\otimes 2} \oplus \cdots$.

In the first case $J(A, B, C; \lambda, \mu)$ is expressed by Eq. (4.11), and, by the inductive assumption, every term in the right-hand side lies in $\mathcal{M}_{\Delta'}(R)$, for some $\Delta' < \Delta$. In the second case $J(A, B, C; \lambda, \mu)$ is given by Eq. (4.12), and it lies by induction in $\mathcal{M}_{\Delta'}(R)$, for some $\Delta' < \Delta$. Finally, the third case reduces to the second case, thanks to Eq. (4.13) and the inductive assumption.

Proof of (4.32). For $A \in \mathbb{C}$ condition (4.32) is obvious, and for $A \in R$ it holds by definition of $\mathcal{M}_{\Delta}(R)$. Let then $A = a \otimes B$, with $a \in R$ and $B \in R \oplus R^{\otimes 2} \oplus \cdots$. By (3.3) we have

$$N(A, E) = a \otimes N(B, C) + N\left(\int_0^T d\lambda \ a, L_\lambda(B, E)\right) + p(a, b)N\left(\int_0^T d\lambda \ B, L_\lambda(a, E)\right),$$

and each term in the right-hand side is in $\mathcal{M}_{\Delta}(R)$ by the inductive assumption.

Proof of (4.33). We will consider separately the following two cases:

1. $E = a \otimes F$, with $a \in R$ and $F \in \mathcal{M}_{\Delta_F}$, where $\Delta_a + \Delta_F = \Delta_E$.
2. $E = \text{sn}(a, b, C)$, with $a, b \in R$, $C \in \mathcal{T}(R)$, and $\Delta_a + \Delta_b + \Delta_C = \Delta_E$.

In the first case we have, by (3.3),

$$N(E, D) = a \otimes N(F, D) + N\left(\int_0^T d\lambda \ a, L_\lambda(F, D)\right) + p(a, F)N\left(\int_0^T d\lambda \ F, L_\lambda(a, D)\right),$$

and each term in the right-hand side is in $\mathcal{M}_{\Delta}(R)$ by the inductive assumption. In the second case $N(E, D)$ is expressed by Eq. (4.4), and again each term in the right-hand side lies in $\mathcal{M}_{\Delta}(R)$.

Proof of (4.34). For $A \in \mathbb{C}$ condition (4.34) is obvious. We will consider separately two cases:
1. \( A = a \in R \),
2. \( A = a \otimes B \), with \( a \in R \) and \( B \in R \oplus R^{\otimes 2} \oplus \cdots \).

Consider the second case first. We have, by (3.7),
\[
L_\lambda(A, E) = N\left((e^{Tb}a), L_\lambda(B, E)\right) + p(a, B)N\left((e^{Tb}B), L_\lambda(a, E)\right) + p(a, B) \int_0^\lambda d\mu(L_\mu(B, L_\lambda - \mu(a, E)),
\]
and, by induction, each term in the right-hand side lies in \( M_{\Delta'}(R) \), for some \( \Delta' < \Delta \).

Suppose then \( A = a \in R \). As before, there are two different situations to consider:
1. \( E = b \otimes F \), with \( b \in R \), \( F \in M_{\Delta'} \), and \( \Delta b + \Delta F = \Delta E \).
2. \( E = \text{sn}(b, c, D) \), with \( b, c \in R \), \( D \in T(R) \), and \( \Delta b + \Delta c + \Delta D = \Delta E \).

In the first case we have, by (3.6),
\[
L_\lambda(a, b \otimes F) = N(L_\lambda(a, b), F) + p(a, b)N(b, L_\lambda(a, F)) + \int_0^\lambda d\mu(L_\mu(a, b), F),
\]
and each term in the right-hand side lies by induction in \( M_{\Delta'}(R) \), for some \( \Delta' < \Delta \). Finally, if \( E = \text{sn}(b, c, D) \), \( L_\lambda(a, E) \) is expressed by Eq. (4.5), and again each term in the right-hand side lies in \( M_{\Delta'}(R) \), for some \( \Delta' < \Delta \).

Proof of (4.35). Consider separately two cases:
1. \( E = a \otimes F \), with \( a \in R \), \( F \in M_{\Delta'} \), and \( \Delta a + \Delta F = \Delta E \).
2. \( E = \text{sn}(a, b, C) \), with \( a, b \in R \), \( C \in T(R) \), and \( \Delta a + \Delta b + \Delta C = \Delta E \).

In the first case we have, by (3.7),
\[
L_\lambda(E, D) = N\left((e^{Tb}a), L_\lambda(F, D)\right) + p(a, B)N\left((e^{Tb}B), L_\lambda(a, D)\right) + p(a, B) \int_0^\lambda d\mu(L_\mu(F, L_\lambda - \mu(a, D)),
\]
and, by the inductive assumption, each term in the right-hand side is in \( M_{\Delta'}(R) \), for some \( \Delta' < \Delta \). In the second case \( L_\lambda(E, D) \) is given by Eq. (4.6). All the terms of the right-hand side are in \( M_{\Delta'}(R) \), for some \( \Delta' < \Delta \), thanks to the inductive assumption and condition (4.31).

Proof of (4.36). For \( A \in C \) or \( B \in C \), condition (4.36) holds trivially. Moreover, for \( A, B \in R \), (4.36) holds by definition of \( M_{\Delta}(R) \). Suppose first \( A = a \in R \), \( B = b \otimes B' \), with \( b \in R \) and \( B' \in R \oplus R^{\otimes 2} \oplus \cdots \). In this case \( \text{sn}(A, B, C) \) is given by Eq. (4.14), and it belongs to \( M_{\Delta}(R) \) by induction. We are left to consider the case \( A = a \otimes A' \), with \( a \in R \) and \( A' \), \( B \in R \oplus R^{\otimes 2} \oplus \cdots \). In this case \( \text{sn}(A, B, C) \) is given by Eq. (4.15). Every term in the right-hand side lies in \( M_{\Delta}(R) \) thanks to the inductive assumption and the above result.
Proof of (4.37). If either $A \in \mathbb{C}$, or $B \in \mathbb{C}$, or $A, B \in \mathbb{R}$, condition (4.37) is trivial. Suppose $A = a \in \mathbb{R}$ and $B = b \otimes B'$, with $b \in \mathbb{R}$ and $B' \in \mathbb{R} \oplus \mathbb{R}^2 \oplus \cdots$. In this case $\text{sl}(A, B; \lambda)$ is given by Eq. (4.16), and it lies by induction in $\mathcal{M}_{\Delta'}(\mathbb{R})$, for some $\Delta' < \Delta$. Finally, consider the case $A = a \otimes A'$, with $a \in \mathbb{R}$ and $A' \in \mathbb{R} \oplus \mathbb{R}^2 \oplus \cdots$. In this case $\text{sl}(A, B; \lambda)$ is given by Eq. (4.17), and again it lies in $\mathcal{M}_{\Delta'}(\mathbb{R})$, for some $\Delta' < \Delta$, by the inductive assumption and by condition (4.29).

Corollary 4.6. The map $L : T(\mathbb{R}) \otimes T(\mathbb{R}) \to T(\mathbb{R})$ defined in Remark 3.7 satisfies skewsymmetry:

$$L(A, B) + p(A, B)L(B, A) \in \mathcal{M}_{\Delta'}(\mathbb{R}),$$

for every $A, B \in T(\mathbb{R})$ and for some $\Delta' < \Delta(A) + \Delta(B)$, and Jacobi identity,

$$L(A, L(B, C)) - p(A, B)L(B, L(A, C)) - L(L(A, B), C) \in \mathcal{M}_{\Delta'}(\mathbb{R}),$$

for every $A, B, C \in T(\mathbb{R})$ and for some $\Delta' < \Delta(A) + \Delta(B) + \Delta(C)$.

Proof. The corollary follows immediately from Corollary 4.5 and the following identities:

$$L(A, B) = -p(A, B)L(B, A) + \int_{-\lambda}^{0} d\lambda \text{sl}(A, B; \lambda),$$

$$L(A, L(B, C)) = p(A, B)L(B, L(A, C)) + L(L(A, B), C) + \int_{-\lambda}^{0} d\lambda \int_{-\mu}^{0} d\mu \text{J}(A, B, C; \lambda, \mu).$$

5. Proof of Theorem 3.9

In this section we will use Corollary 4.5 to prove Theorem 3.9. Notice that by (4.18)–(4.21), the products $N : T(\mathbb{R}) \otimes T(\mathbb{R}) \to T(\mathbb{R})$ and $L_{\lambda} : T(\mathbb{R}) \otimes T(\mathbb{R}) \to \mathbb{C}[\lambda] \otimes T(\mathbb{R})$ induce products on the quotient space $U(\mathbb{R}) = T(\mathbb{R})/\mathcal{M}(\mathbb{R})$, which we denote by : and $[\lambda]$ respectively. Moreover, by (4.22)–(4.27), the space $U(\mathbb{R})$, with normally ordered product $:$ and $\lambda$–bracket $[,]$, satisfies all the axioms of vertex algebra. This proves the second part of Theorem 3.9.

We are left to prove the first part of Theorem 3.9, which provides a PBW basis for $U(\mathbb{R})$, and therefore guarantees that $U(\mathbb{R}) \neq 0$. Let us denote by $\bar{B}$ the collection of 1 and all ordered monomials in $T(\mathbb{R})$:

$$\bar{B} = \left\{ a_{i_1} \otimes \cdots \otimes a_{i_n} \mid i_1 \leq \cdots \leq i_n, n \in \mathbb{Z}_+, i_k < i_{k+1} \text{ if } p(a_{i_k}) = \bar{1} \right\}.$$  

We also denote by $\bar{B}[\Delta]$ the collection of ordered monomials of degree $\Delta$, namely $\bar{B}[\Delta] = \bar{B} \cap T(\mathbb{R})[\Delta]$, and by $\bar{B}_\Delta$ the corresponding filtration, namely $\bar{B}_\Delta = \bar{B} \cap T_{\Delta}(\mathbb{R})$. By definition $B = \pi(\bar{B})$. 
Lemma 5.1. Any element \( E \in T_\Delta(R) \) can be decomposed as
\[
E = P + M,
\]
(5.1)
where \( P \in \text{span}_C \tilde{B}_\Delta \) and \( M \in \mathcal{M}_\Delta(R) \). Equivalently
\[
T_\Delta(R)/\mathcal{M}_\Delta(R) = \text{span}_C \pi(\tilde{B}_\Delta), \quad \forall \Delta \in \Gamma,
\]
and therefore \( U(R) = \text{span}_C B \).

Proof. It suffices to prove (5.1) for monomials, namely for \( E = a_{j_1} \otimes \cdots \otimes a_{j_n} \in T(R)[\Delta] \), where \( \Delta \in \Gamma \) and \( a_{j_k} \in A \). Let us define the number of inversions of \( E \) as
\[
d(E) = \# \left\{ (p, q) \mid 1 \leq p < q \leq n \text{ and either } a_{j_p} \text{ is even and } j_p > j_q \right. \right. \nonumber
\]
\[
\left. \left. \text{or } a_{j_p} \text{ is odd and } j_p \geq j_q \right\} \right\}.
\]
Let \( d = d(E) \). We will prove that \( E \) decomposes as in (5.1) by induction on the pair \((\Delta, d)\), ordered lexicographically. If \( d = 0 \), we have \( E \in \tilde{B}[\Delta] \), so there is nothing to prove. Suppose that \( d \geq 1 \) and let \( p \in \{1, \ldots, n-1\} \) be such that \( j_p = j_{p+1} \), or \( j_p = j_{p+1} \) for \( a_{j_p} \) odd. In the first case we have, by definition of \( M_\Delta(R) \),
\[
E \equiv p(a_{j_p}, a_{j_{p+1}}) a_{j_1} \otimes \cdots \otimes a_{j_{p-1}} \otimes a_{j_p} \otimes a_{j_{p+2}} \otimes \cdots \otimes a_{j_n} \mod M_\Delta(R).
\]
The first term in the right-hand side has degree \( \Delta \) and \( d - 1 \) inversions, while, by the grading conditions (3.8), the second term lies in \( T_\Delta(R) \), for some \( \Delta' < \Delta \). Therefore, by the inductive assumption, both terms admit a decomposition (5.1). Consider now the second case, namely \( a_{j_p} \) is odd and \( j_p = j_{p+1} \). By definition of \( M_\Delta(R) \) we have
\[
E \equiv \frac{1}{2} a_{j_1} \otimes \cdots \otimes N \left( \left( \int_{-T}^0 d\lambda \right)^a_{j_p} \otimes a_{j_{p+1}} \right) \cdots \otimes a_{j_n} \mod M_\Delta(R),
\]
and again, by the inductive assumption, the right-hand side admits a decomposition (5.1).

Lemma 5.2. Let \( \tilde{U} \) be a vector space with basis \( \tilde{B} \):
\[
\tilde{U} = \bigoplus_{A \in \tilde{B}} CA.
\]
There exists a unique linear map
\[
\sigma : T(R) \rightarrow \tilde{U}
\]
such that
1. \( \sigma(A) = A, \quad \forall A \in \tilde{B} \),
2. \( \mathcal{M}(R) \subset \ker \sigma \).

Proof. We want to prove that there is a unique collection of linear maps \( \sigma_\Delta : T_\Delta(R) \rightarrow \tilde{U} \), for \( \Delta \in \Gamma \), such that
(1) \(\sigma_0(1) = 1\), \(\sigma_{\Delta}|_{T_{\Delta}(R)} = \sigma_{\Delta'}\), if \(\Delta' \leq \Delta\),
(2) \(\sigma_{\Delta}(A) = A\), if \(A \in B[\Delta]\),
(3) \(M_{\Delta}(R) \subset \ker \sigma_{\Delta}\).

This obviously proves the lemma. Indeed for any such sequence we can define the map \(\sigma : T(R) \to \hat{U}\) by
\[
\sigma|_{T_{\Delta}(R)} = \sigma_{\Delta}, \quad \forall \Delta \in \Gamma,
\]
and conversely, given a map \(\sigma : T(R) \to \hat{U}\) satisfying the assumptions of the lemma, we can define such a sequence of maps \(\sigma_{\Delta} : T_{\Delta}(R) \to \hat{U}\) by Eq. (5.2).

The condition \(\sigma_0(1) = 1\) defines completely \(\sigma_0\). Notice that, since \(M_0(R) = 0\), \(\sigma_0\) satisfies all the required conditions. Let then \(\Delta > 0\), and suppose by induction that \(\sigma_{\Delta'}\) is uniquely defined and it satisfies all conditions (1)–(3) for every \(\Delta' < \Delta\).

**Uniqueness of \(\sigma_{\Delta}\).** Given a monomial of degree \(\Delta\), \(E = a_{j_1} \otimes \cdots \otimes a_{j_n} \in T(R)[\Delta]\), we will show that \(\sigma_{\Delta}(E)\) is uniquely defined by induction on the number of inversions \(d\) of \(E\), defined above. For \(d = 0\) we have \(E \in B_{\Delta}\), so it must be \(\sigma_{\Delta}(E) = E\) by condition (2). Let then \(d \geq 1\) and let \((j_p, j_{p+1})\) be the “most left inversion”, namely \(p \in \{1, \ldots, n\}\) is the smallest integer such that either \(j_p > j_{p+1}\) or \(p(a_{j_p}) = 1\) and \(j_p = j_{p+1}\). By condition (3) and the definition of \(M_{\Delta}(R)\) we then have, if \(j_p > j_{p+1}\),
\[
\sigma_{\Delta}(E) = p(a_{j_p}, a_{j_{p+1}})\sigma_{\Delta}(a_{j_1} \otimes \cdots \otimes a_{j_{p+1}} \otimes a_{j_p} \otimes \cdots \otimes a_{j_n})
\]
\[
+ \sigma_{\Delta}(a_{j_1} \otimes \cdots \otimes N(L(a_{j_p}, a_{j_{p+1}}), \cdots \otimes a_{j_n}))
\]
while, if \(p(a_{j_p}) = 1\) and \(j_p = j_{p+1}\),
\[
\sigma_{\Delta}(E) = \frac{1}{2}\sigma_{\Delta}(a_{j_1} \otimes \cdots \otimes N(L(a_{j_p}, a_{j_{p+1}}), \cdots \otimes a_{j_n})�\]
Notice that \((a_{j_1} \otimes \cdots \otimes a_{j_{p+1}} \otimes a_{j_p} \otimes \cdots \otimes a_{j_n})\) has degree \(\Delta\) and \(d - 1\) inversions, so the first term in the right-hand side of (5.3) is uniquely defined by the inductive assumption. Moreover, by the grading conditions (3.8), \(a_{j_1} \otimes \cdots \otimes N(L(a_{j_p}, a_{j_{p+1}}), \cdots \otimes a_{j_n})\) lies in \(T_{\Delta}(R)\), for some \(\Delta' < \Delta\), so the second term in the right-hand side of (5.3) and the right-hand side of (5.4) are uniquely defined by condition (1) and the inductive assumption.

**Existence of \(\sigma_{\Delta}\).** The above prescription defines uniquely a linear map \(\sigma_{\Delta} : T_{\Delta}(R) \to \hat{U}\), which by construction satisfies conditions (1) and (2). We are left to show that condition (3) holds. By definition of \(M_{\Delta}(R)\), it suffices to prove that, for every monomial of degree \(\Delta\), \(E = a_{j_1} \otimes \cdots \otimes a_{j_n} \in T(R)[\Delta]\), and for every \(q = 1, \ldots, n\), we have
\[
\sigma_{\Delta}(a_{j_1} \otimes \cdots \otimes a_{j_{q-1}} \otimes \text{sn}(a_{j_q}, a_{j_{q+1}}, a_{j_{q+2}} \otimes \cdots \otimes a_{j_n})) = 0.
\]
By skewsymmetry of the \(\lambda\)-bracket we can assume, without loss of generality, that \(j_q \geq j_{q+1}\). Moreover, if \(p(a_{j_q}) = 0\), we have \(\int_{-T}^{0} d\lambda [a_{j_q}, a_{j_q}] = 0\), so that when \(j_q = j_{q+1}\), Eq. (5.5) is obvious. In other words, we can assume that \((j_q, j_{q+1})\) is an “inversion” of \(E\). In particular \(d = d(E) \geq 1\). We will prove condition (5.5) by induction on \(d\). Let \((j_p, j_{p+1})\) be the “most left inversion” of \(E\). For \(p = q\) Eq. (5.5) holds by construction. We will consider separately the cases \(p < q = 2\) and \(p = q - 1\).
Case 1. Assume \( p \leq q - 2 \). For simplicity we rewrite

\[
E = A \otimes c \otimes b \otimes D \otimes f \otimes e \otimes H,
\]
where \( A = a_{j_1} \otimes \cdots \otimes a_{j_{p-1}}, c = a_{j_p}, b = a_{j_{p+1}}, D = a_{j_{p+2}} \otimes \cdots \otimes a_{j_{q-1}}, f = a_{j_q}, e = a_{j_{q+1}}, H = a_{j_{q+2}} \otimes \cdots \otimes a_{j_n} \). The left-hand side of Eq. (5.5) then takes the form

\[
\sigma_\Delta \left( A \otimes c \otimes b \otimes D \otimes f \otimes e \otimes H \right) - p(e, f) \sigma_\Delta \left( A \otimes c \otimes b \otimes D \otimes e \otimes f \otimes H \right) - \sigma_\Delta \left( A \otimes c \otimes b \otimes D \otimes N(L(c, b), D \otimes e \otimes f \otimes H) \right).
\]  

(5.6)

By definition of \( \sigma_\Delta \), the first term of (5.6) can be written as

\[
p(b, c) \sigma_\Delta \left( A \otimes b \otimes c \otimes D \otimes f \otimes e \otimes H \right) + \sigma_\Delta \left( A \otimes N(L(c, b), D \otimes f \otimes e \otimes H) \right).
\]  

(5.7)

If \( e \neq f \), \( A \otimes c \otimes b \otimes D \otimes e \otimes f \otimes H \) has degree \( \Delta \) and \( d - 1 \) inversions, so that we can use the inductive assumption to rewrite the second term of (5.6) as

\[
p(e, f) p(b, c) \sigma_\Delta \left( A \otimes b \otimes c \otimes D \otimes e \otimes f \otimes H \right) - p(e, f) \sigma_\Delta \left( A \otimes N(L(c, b), D \otimes e \otimes f \otimes H) \right).
\]  

(5.8)

By the grading conditions (3.8), \( A \otimes c \otimes b \otimes D \otimes N(L(f, e), H) \in T_{\Delta'}(R) \), for some \( \Delta' < \Delta \). We can thus use the fact that \( \sigma_\Delta \big|_{T_{\Delta'}(R)} = \sigma_{\Delta'} \) and the inductive assumption to rewrite the third term of (5.6) as

\[
- p(b, c) \sigma_\Delta \left( A \otimes b \otimes c \otimes D \otimes N(L(f, e), H) \right) - \sigma_\Delta \left( A \otimes N(L(c, b), D \otimes N(L(f, e), H)) \right).
\]  

(5.9)

Combining (5.7), (5.8) and (5.9) we can rewrite (5.6) as

\[
p(b, c) \sigma_\Delta \left( A \otimes b \otimes c \otimes D \otimes sn(f, e, H) \right) + \sigma_\Delta \left( A \otimes N(L(c, b), D \otimes sn(f, e, H)) \right).
\]  

(5.10)

The first term of (5.10) appears only for \( b \neq c \), and in this case it is zero by the inductive assumption, since \( A \otimes b \otimes c \otimes D \otimes f \otimes e \otimes H \) has degree \( \Delta \) and \( d - 1 \) inversions. Consider the second term of (5.10). By the grading conditions (3.8) and by Corollary 4.5, the argument of \( \sigma_\Delta \) lies in \( M_{\Delta'}(R) \), for some \( \Delta' < \Delta \). It follows by induction that also the second term of (5.10) is zero. We thus proved, as we wanted, that (5.6) is zero.

Case 2. We are left to consider the case \( p = q - 1 \). For simplicity we write

\[
E = A \otimes c \otimes b \otimes a \otimes D,
\]
where \( A = a_{j_1} \otimes \cdots \otimes a_{j_{p-1}}, c = a_{j_p}, b = a_{j_{p+1}}, a = a_{j_{p+2}}, D = a_{j_{p+3}} \otimes \cdots \otimes a_{j_n} \). The left-hand side of Eq. (5.5) then takes the form

\[
\sigma_\Delta \left( A \otimes c \otimes b \otimes a \otimes D \right) - p(a, b) \sigma_\Delta \left( A \otimes c \otimes a \otimes b \otimes D \right) - \sigma_\Delta \left( A \otimes c \otimes N(L(b, a), D) \right).
\]  

(5.11)
After some manipulations based on the inductive assumption, similar to the ones used above, we can rewrite (5.11) as

\[
\sigma(A \otimes \left\{ N(L(c, b), a \otimes D) - p(a, b)p(a, c)a \otimes N(L(c, b), D) \right\}) (5.12)
\]

\[
+ \sigma(A \otimes \left\{ p(b, c)b \otimes N(L(c, a), D) - p(a, b)p(b, c)b \otimes N(L(c, a), b \otimes D) \right\})
\]

\[
+ \sigma(A \otimes \left\{ p(a, c)p(b, c)N(L(b, a), c \otimes D)) - c \otimes N(L(b, a), D) \right\})
\].

It follows by Corollary 4.5 that there exists \(\Delta' < \Delta\) such that

\[
A \otimes a \otimes N(L(c, b), D) - p(a, b)p(a, c)A \otimes N(L(c, b), a \otimes D) \equiv A \otimes N(L(a, L(c, b)), D) \mod \mathcal{M}(R),
\]

\[
A \otimes b \otimes N(L(c, a), D) - p(a, b)p(b, c)A \otimes N(L(c, a), b \otimes D) \equiv A \otimes N(L(b, L(c, a)), D) \mod \mathcal{M}(R),
\]

\[
A \otimes c \otimes N(L(b, a), D) - p(b, c)p(a, c)A \otimes N(L(b, a), c \otimes D) \equiv A \otimes N(L(c, L(b, a)), D) \mod \mathcal{M}(R).
\]

We can thus use the inductive assumption to rewrite (5.12) as

\[
\sigma \left( p(b, c)A \otimes N(L(b, L(c, a)), D) - p(a, b)p(a, c)A \otimes N(L(a, L(c, b)), D) \right. \]

\[
- A \otimes N(L(c, L(b, a)), D) \right) (5.13)
\].

By Corollary 4.6, the argument of \(\sigma\) in (5.13) lies in \(\mathcal{M}(R)\) for some \(\Delta' < \Delta\) and therefore (5.13) is zero by the inductive assumption. This concludes the proof of the lemma. \(\square\)

**Lemma 5.3.** If \(\sigma\) is as in Lemma 5.2, the induced map

\[
\hat{\sigma} : U(R) = T(R)/\mathcal{M}(R) \xrightarrow{\sim} \hat{U}
\]

is an isomorphism of vector spaces (namely \(\mathcal{M}(R) = \ker \sigma\)).

**Proof.** By definition the map \(\hat{\sigma} : U(R) \rightarrow \hat{U}\) is surjective. On the other hand, we have a natural map \(\hat{\pi} : \hat{U} \rightarrow U(R)\) which maps every basis element \(A = a_i \otimes \cdots \otimes a_n \in \hat{B}\) to the corresponding \(\hat{\pi}(A) = : a_i \ldots a_n : \in \hat{B} \subset U(R)\). By Lemma 5.1 this map is also surjective. The composition map

\[
\hat{U} \xrightarrow{\hat{\pi}} U(R) \xrightarrow{\hat{\sigma}} \hat{U}
\]

is the identity map (by definition of \(\hat{\pi}\) and \(\hat{\sigma}\)). This of course implies that both \(\hat{\pi}\) and \(\hat{\sigma}\) are isomorphisms of vector spaces. \(\square\)

The last lemma implies that \(\hat{B}\) is a basis of the space \(U(R)\), thus concluding the proof of Theorem 3.9.
6. Pre–Graded Freely Generated Vertex Algebras and Corresponding Non–Linear Lie Conformal Algebras

Let $V$ be a vertex algebra strongly generated by a free $\mathbb{C}[T]$–module $R = \mathbb{C}[T] \otimes \bar{V}$.

Let

$$\bar{A} = \{\bar{a}_i, \ i \in \bar{I}\},$$

be an ordered basis of $\bar{V}$, and extend it to an ordered basis of $R$,

$$A = \{a_i, \ i \in I\},$$

where $I = \bar{I} \times \mathbb{Z}_+$ and, if $i = (\bar{i}, n)$, then $a_i = T^n\bar{a}_{\bar{i}}$. The corresponding collection of ordered monomials of $V$ (see Definition 2.6) is denoted by $B$. Recall that $V$ is said to be freely generated by $R$ if $B$ is a basis of $V$. We will denote by $\pi$ the natural quotient map $\pi : T(R) \rightarrow V$, given by $\pi(a \otimes b \otimes \cdots \otimes c) = :ab\cdots c:$.

We will assume that the generating space $\bar{V}$ is graded by $\Gamma \setminus \{0\}$, and that the basis $\bar{A}$ of $\bar{V}$ is compatible with the $\Gamma \setminus \{0\}$–gradation. The $\Gamma$–gradation can be extended to $R = \mathbb{C}[T] \otimes \bar{V} \subset V$ by saying that $T$ has zero degree, and to the whole tensor algebra $T(R)$ by additivity of the tensor product (as defined in Sect. 2). The corresponding $\Gamma$–filtration of $T(R)$ naturally induces a $\Gamma$–filtration on the vertex algebra $V$ (cf. [Li02]):

$$V_{\Delta} = \pi(T_\Delta(R)), \quad \Delta \in \Gamma.$$

**Definition 6.1.** The vertex algebra $V$, strongly generated by a free $\mathbb{C}[T]$–module $\mathbb{C}[T] \otimes \bar{V}$, where $\bar{V} = \oplus_{\Lambda \in \Gamma \setminus \{0\}} \bar{V}[\Lambda]$, is said to be pre–graded by $\Gamma$ if the $\lambda$–bracket satisfies the grading condition

$$[\bar{V}[\Delta_1] ; \bar{V}[\Delta_2]] \subset \mathbb{C}[\lambda] \otimes V_{\Delta'}, \quad \text{for some } \Delta' < \Delta_1 + \Delta_2.$$

**Example 6.2.** The most important examples of pre–graded vertex algebras are provided by graded vertex algebras. Recall that a vertex algebra $V$ is called graded if there exists a diagonalizable operator $L_0$ on $V$ with discrete non–negative spectrum $\Gamma$ and $0$th eigenspace $\mathbb{C}[0]$, such that for all $a \in V$,

$$[L_0, Y(a, z)] = z\partial_z Y(a, z) + Y(L_0a, z).$$

The eigenvalue $w(a)$ of an eigenvector $a$ of $L_0$ is called its conformal weight. Let $V = \oplus_{w \in \Gamma} V(w)$ be the eigenspace decomposition. Recall that conformal weights satisfy the following rules (see e.g. [Kac96]):

$$w(Ta) = w(a) + 1, \quad w(a_{(n)}b) = w(a) + w(b) - n - 1.$$

Assume now that $V$ is strongly generated by a free $\mathbb{C}[T]$–submodule $R = \mathbb{C}[T] \otimes \bar{V}$, where $\bar{V}$ is invariant under the action of $L_0$. Consider the decomposition of $\bar{V}$ in a direct sum of eigenspaces of $L_0$: $\bar{V} = \oplus_{w \in \Gamma \setminus \{0\}} \bar{V}[\Lambda]$, where $\bar{V}[\Lambda] = \bar{V} \cap V(\Delta)$. This is a $\Gamma \setminus \{0\}$–gradation of $\bar{V}$, which induces, as explained above, a $\Gamma$–gradation on $T(R)$ and hence a $\Gamma$–filtration, $V_{\Delta}, \Delta \in \Gamma$, on the vertex algebra $V$. Notice that such $\Gamma$–filtration is not induced by the gradation given by the conformal weights. Indeed the filtered space $V_{\Delta}$ are preserved by $T$, while, by (6.2), $T$ increases the conformal weight by 1. On the other hand, we have

$$\bigoplus_{w \leq \Delta} V(w) \subset V_{\Delta}.$$
The above $\Gamma \setminus \{0\}$–gradation of $\bar{V}$ is a pre–gradation of $V$. Indeed (6.2) and (6.3), imply

$$[\bar{V}[\Delta_1], \bar{V}[\Delta_2]] \subset \mathbb{C}[\lambda] \otimes \bigoplus_{w \leq \Delta_1 + \Delta_2 - 1} V(w) \subset \mathbb{C}[\lambda] \otimes V_{\Delta_1 + \Delta_2 - 1},$$

so that the grading condition in Definition 6.1 holds.

**Remark 6.3.** Notice that if $R = \mathbb{C}[T] \otimes \bar{V}$ is a non–linear Lie conformal algebra graded by $\Gamma \setminus \{0\}$, then by Theorem 3.9 the universal enveloping vertex algebra $U(R)$ is a pre–graded freely generated vertex algebra.

We want to prove a converse statement to Theorem 3.9.

**Theorem 6.4.** Let $V$ be a pre–graded vertex algebra freely generated by a free $\mathbb{C}[T]$–submodule $R = \mathbb{C}[T] \otimes \bar{V}$. Then $R$ has a structure of the non–linear Lie conformal algebra

$$L_\lambda : R \otimes R \rightarrow \mathbb{C}[\lambda] \otimes T(R),$$

compatible with the Lie conformal algebra structure of $V$, in the sense that

$$\pi(L_\lambda(a, b)) = [a \lambda b], \forall a, b \in R,$$

so that $V$ is canonically isomorphic to the universal enveloping vertex algebra $U(R)$.

Any two such structures of a non–linear Lie conformal algebra on $R$ are isomorphic in the sense of Definition 3.6.

The proof will be achieved as a result of 8 lemmas (the first two of which can be derived from [Li02]).

**Lemma 6.5.** The vertex algebra structure on $V$ satisfies the following grading conditions:

$$: V_{\Delta_1} V_{\Delta_2} : \subset V_{\Delta_1 + \Delta_2},$$

$$[V_{\Delta_1} \lambda V_{\Delta_2}] \subset \mathbb{C}[\lambda] \otimes V_{\Delta'}, \text{ for some } \Delta' < \Delta_1 + \Delta_2.$$

**Proof.** By definition $V_{\Delta}$ is spanned by elements $\pi(A)$, where $A \in T_{\Delta}(R)$. We thus want to prove, by induction on $\Delta = \Delta_1 + \Delta_2$, that, for $A \in T_{\Delta_1}(R)$ and $B \in T_{\Delta_2}(R)$, we have

$$: \pi(A) \pi(B) : \in V_{\Delta},$$

$$[\pi(A) \lambda \pi(B)] \in \mathbb{C}[\lambda] \otimes V_{\Delta'} \text{ for some } \Delta' < \Delta. \quad (6.4)$$

If $A \in \mathbb{C}$ or $B \in \mathbb{C}$, both conditions are obvious. Suppose then $A, B \in R \oplus R^{\otimes 2} \oplus \cdots$. If $A \in R$, we have $: \pi(A) \pi(B) : = \pi(A \otimes B)$, so that condition (6.4) follows immediately by the definition of the $\Gamma$–filtration on $V$. Let then $A = a \otimes A'$, with $a \in R$ and $A', B \in R \oplus R^{\otimes 2} \oplus \cdots$. Notice that $\Delta(a) + \Delta(A') = \Delta(A) + \Delta(a), \Delta(A') > 0$. By quasi–associativity of the normally ordered product we then have

$$: \pi(A) \pi(B) : = a(\pi(A') \pi(B) :) + : (\int_0^T d\lambda a) \pi(A') \pi(B) :) +$$

$$+ p(a, A') : (\int_0^T d\lambda \pi(A')) [a \lambda \pi(B) :].$$
and each term in the right-hand side belongs to $V_\Delta$ by the inductive assumption. We are left to prove condition (6.5). If $A, B \in \mathcal{R}$, (6.5) holds by sesquilinearity and by the assumption of the grading condition on $V$. Suppose now $A = a \in \mathcal{R}$ and $B = b \otimes B'$, with $b \in \mathcal{R}$ and $B' \in \mathcal{R} \oplus \mathcal{R}^\otimes 2 \oplus \cdots$. In this case we can use the left Wick formula to get

$$[[\pi(A) \lambda \pi(B)]] = [a \lambda b] \pi(B') + \int_0^\lambda d\mu [a \lambda \mu \pi(B')]$$

and each term in the right-hand side belongs to $V/\Delta_1'$, for some $\Delta_1'$. Finally, let $A = a \otimes A'$, where $a \in \mathcal{R}$ and $A', B \in \mathcal{R} \oplus \mathcal{R}^\otimes 2 \oplus \cdots$. We then have by the right Wick formula

$$[[\pi(A) \lambda \pi(B)]] = \left( e^{i\theta} \pi(A') \lambda \pi(B) \right) + p(a, A') \int_0^\lambda d\mu [a \lambda \mu \pi(B')]$$

and again, by the inductive assumption, each term in the right-hand side belongs to $V_{\Delta'}$ for some $\Delta' < \Delta$. 

As in Sect. 5, we denote by $\tilde{B}$ the collection of ordered monomials in $\mathcal{T}(\mathcal{R})$:

$$\tilde{B} = \left\{ a_{i_1} \otimes \cdots \otimes a_{i_n} \mid i_1 \leq \cdots \leq i_n, \; n \in \mathbb{Z}_+, \; \forall (i_q \leq i_{q+1} \text{ if } p(a_{i_q}) = 1) \right\} \subset \mathcal{T}(\mathcal{R})$$

so that the basis $B$ of $V$ is the image of $\tilde{B}$ under the quotient map $\pi$:

$$B = \left\{ :a_{i_1} \cdots a_{i_n} : \mid i_1 \leq \cdots \leq i_n, \; n \in \mathbb{Z}_+, \; \forall (i_q \leq i_{q+1} \text{ if } p(a_{i_q}) = 1) \right\} \subset V$$

The basis $B$ of $V$ induces an embedding $\rho : V \hookrightarrow \mathcal{T}(\mathcal{R})$, defined by

$$\rho( :a_{i_1} \cdots a_{i_n} :) = a_{i_1} \otimes \cdots \otimes a_{i_n}, \; \forall :a_{i_1} \cdots a_{i_n} : \in \tilde{B}.$$  

Moreover, the basis $B$ induces a primary $\Gamma$–gradation on $V$,

$$V = \bigoplus_{\Delta \in \Gamma} V[\Delta]^{(1)}$$

defined by assigning to every basis element $:a_{i_1} \cdots a_{i_n} : \in \tilde{B}$ degree

$\Delta( :a_{i_1} \cdots a_{i_n} :) = \Delta(a_{i_1}) + \cdots + \Delta(a_{i_n}).$

**Lemma 6.6.** The $\Gamma$–filtration (6.1) of $V$ is induced by the primary $\Gamma$–gradation (6.6).
Proof. We want to prove that for every $\Delta \in \Gamma$,

$$V_\Delta = \bigoplus_{\Delta \leq \Delta} V[\Delta']^{(1)}.$$

By definition of the $\Gamma$–filtration of $V$

$$V_\Delta = \pi(T_\Delta(R)) = \text{span}_C \left\{ a_{j_1} \cdots a_{j_n} : \Delta(a_{j_1}) + \cdots + \Delta(a_{j_n}) \leq \Delta \right\},$$

while, by definition of the primary $\Gamma$–gradation of $V$, we have

$$\bigoplus_{\Delta \leq \Delta} V[\Delta']^{(1)} = \text{span}_C \left\{ a_{j_1} \cdots a_{j_n} : \Delta(a_{j_1}) + \cdots + \Delta(a_{j_n}) \leq \Delta, i_1 \leq \cdots \leq i_n, i_q < i_{q+1} \text{ if } p(a_{i_q}) = 1 \right\}.$$

Therefore we obviously have the inclusion $\bigoplus_{\Delta \leq \Delta} V[\Delta']^{(1)} \subseteq V_\Delta$. We are left to show that every (non-necessarily ordered) monomial $A = a_{j_1} \cdots a_{j_n}$ can be written as a linear combination of ordered monomials of primary degree $\Delta'$ $\leq \Delta$. We will prove this statement by induction on $(\Delta, d)$, where $d$ is the number of inversions of $A$:

$$d = \# \left\{ (p, q) \mid 1 \leq p < q \leq n \text{ and either } j_p > j_q \text{ or } j_p = j_q \text{ and } p(a_{j_q}) = 1 \right\}.$$

If $d = 0$, then $A \in V[\Delta]^{(1)}$ and the statement is trivial. Suppose then $d \geq 1$, and let $(j_q, j_{q+1})$ be an inversion of $A$. We consider the case $j_q > j_{q+1}$. The case $j_q = j_{q+1}$ and $p(a_{j_q}) = 1$ is similar. By skew–symmetry of the normally ordered product, we have

$$A = p(a_{j_q}, a_{j_{q+1}}) : a_{j_1} \cdots a_{j_{q+1}} a_{j_q} \cdots a_{j_n} : + a_{j_1} \cdots a_{j_{q-1}} \left( \int_{-T}^{0} d\lambda \ [a_{j_q}, a_{j_{q+1}}] a_{j_{q+2}} \cdots a_{j_n} : \right).$$

The first term in the right-hand side of (6.7) is a monomial of $V_\Delta$ with $d-1$ inversions. Moreover, by Lemma 6.5, the second term in the right-hand side of (6.7) belongs to $V_{\Delta'}$ for some $\Delta' < \Delta$. We can thus use the inductive assumption to conclude, as we wanted, that $A \in \bigoplus_{\Delta \leq \Delta} V[\Delta']^{(1)}$. \hfill $\Box$

The embedding $\rho : V \hookrightarrow T(R)$ introduced above does not commute with the action of $T$. The main goal in the following will be to replace $\rho$ with another embedding $\rho_T : V \hookrightarrow T(R)$ which commutes with the action of $T$.

We introduce the following collection of elements of $T(R)$:

$$\tilde{B}_T = \left\{ T^{k_1}(\tilde{a}_{i_1} \otimes a_{i_2} \otimes \cdots \otimes a_{i_n}) : \begin{array}{l} i_1 = (i_1, k_1) \leq i_2 \leq \cdots \leq i_n \leq \Delta, \\
i_q < i_{q+1} \text{ if } p(a_{i_q}) = 1 \end{array} \right\} \subset T(R),$$

and we denote by $B_T$ the corresponding image via the quotient map $\pi$, namely

$$B_T = \left\{ T^{k_1}(\tilde{a}_{i_1}, a_{i_2} \cdots a_{i_n}) : \begin{array}{l} i_1 = (i_1, k_1) \leq i_2 \leq \cdots \leq i_n \leq \Delta, \\
i_q < i_{q+1} \text{ if } p(a_{i_q}) = 1 \end{array} \right\} \subset V.$$
Lemma 6.7. Consider the elements: \( a_{i_1} \ldots a_{i_n} : \in B \) and \( T^{k_1} : \alpha_{i_1} a_{i_2} \ldots a_{i_n} : \in B_T \) ordered lexicographically by the indices \((\Delta, i_1 = (i_1, k_1), i_2 = (i_2, k_2), \ldots, i_n = (i_n, k_n))\), where \( \Delta = \Delta(a_{i_1}) + \cdots + \Delta(a_{i_n}) \in \Gamma \).

(a) Every element \( T^{k_1} : \alpha_{i_1} a_{i_2} \ldots a_{i_n} : \in B_T \) can be written as
\[ T^{k_1} : \alpha_{i_1} a_{i_2} \ldots a_{i_n} : = : a_{i_1} \ldots a_{i_n} : + M, \]
where \( M \) is a linear combination of elements of \( B \) smaller than \( : a_{i_1} \ldots a_{i_n} : \).

(b) Every element: \( : a_{i_1} \ldots a_{i_n} : \in B \) can be written as
\[ : a_{i_1} \ldots a_{i_n} : = T^{k_1} : \alpha_{i_1} a_{i_2} \ldots a_{i_n} : + N, \]
where \( N \) is a linear combination of elements of \( B_T \) smaller than \( T^{k_1} : \alpha_{i_1} a_{i_2} \ldots a_{i_n} : \).

Proof. (a) Since \( T \) is a derivation of the normally ordered product, we have
\[ T^{k_1} : \alpha_{i_1} a_{i_2} \ldots a_{i_n} : = : a_{i_1} \ldots a_{i_n} : + \sum_{l_1 \geq 1, l_2, \ldots, l_n \geq 0} \left( \begin{array}{c} k_1 \\ l_1, \ldots, l_n \end{array} \right) : a_{i_1, l_1} a_{i_2, l_2+i_1} \cdots a_{i_n, k_n+l_n} :. \]
We just need to prove that every monomial \( : a_{i_1, l_1} a_{i_2, l_2+i_1} \cdots a_{i_n, k_n+l_n} : \) is a linear combination of ordered monomials smaller than \( : a_{i_1} \ldots a_{i_n} : \). In general the monomial
\[ : a_{i_1, l_1} a_{i_2, l_2+i_1} \cdots a_{i_n, k_n+l_n} : \] is not ordered. But thanks to skew–symmetry of the normally ordered product and to Lemma 6.5, we can rewrite it (up to a sign) as the sum of an ordered monomial \( : a_{\sigma_1} \cdots a_{\sigma_n} : \) with the same (reordered) indices, and an element \( R \in V_{\Delta'}, \) with \( \Delta' < \Delta \). Notice that the reordered monomial \( : a_{\sigma_1} \cdots a_{\sigma_n} : \) is smaller than \( : a_{i_1} \ldots a_{i_n} : \), since \( \Delta(\sigma_1) + \cdots + \Delta(\sigma_n) = \Delta \) and \( \sigma_1 = (i_1, l_1) < i_1 \). Moreover, by Lemma 6.6, \( R \) is linear combination of ordered monomials with primary degree \( \Delta' < \Delta \), hence smaller than \( : a_{i_1} \cdots a_{i_n} : \). This proves the first part of the lemma.

(b) As before, we can decompose
\[ : a_{i_1} \ldots a_{i_n} : = T^{k_1} : \alpha_{i_1} a_{i_2} \ldots a_{i_n} : - \sum_{l_1 \geq 1, l_2, \ldots, l_n \geq 0} \left( \begin{array}{c} k_1 \\ l_1, \ldots, l_n \end{array} \right) : a_{i_1, l_1} a_{i_2, l_2+i_1} \cdots a_{i_n, k_n+l_n} :, \]
and we want to prove that every monomial \( : a_{i_1, l_1} a_{i_2, l_2+i_1} \cdots a_{i_n, k_n+l_n} : \) is a linear combination of elements of \( B_T \) smaller than \( T^{k_1} : \alpha_{i_1} a_{i_2} \ldots a_{i_n} : \). By the above argument, we can decompose
\[ : a_{i_1, l_1} a_{i_2, l_2+i_1} \cdots a_{i_n, k_n+l_n} : = : a_{\sigma_1} a_{\sigma_2} \cdots a_{\sigma_n} : + R, \]
where \( a_{\sigma_1} a_{\sigma_2} \cdots a_{\sigma_n} : \in B \) is the ordered monomial obtained by reordering the indices of \( : a_{i_1, l_1} a_{i_2, l_2+i_1} \cdots a_{i_n, k_n+l_n} : \), and \( R \in V_{\Delta'}, \) for some \( \Delta' < \Delta \). Since \( (i_1, l_1) < i_1 \leq (i_q, k_q + l_q), \) \( \forall q = 2, \ldots, n \), it follows by induction on \( \Delta, i_1, \ldots, i_n \) that \( : a_{\sigma_1} \cdots a_{\sigma_n} : \) is a linear combination of elements of \( B_T \) smaller than \( T^{k_1} : \alpha_{i_1} a_{i_2} \ldots a_{i_n} : \). By Lemma 6.6 \( R \) is a linear combination of ordered monomials \( : a_{\tau_1} \cdots a_{\tau_p} : \in B \), with
\( \Delta(a_{\tau_1}) + \cdots + \Delta(a_{\tau_p}) \leq \Delta. \) We thus conclude, by induction on \( (\Delta, i_1, \ldots, i_n) \), that each such ordered monomial is a linear combination of elements of \( \mathcal{B}_T \) smaller than \( T^{k_1} : \tilde{a}_{i_1}a_{i_2} \cdots a_{i_n} : \). \( \square \)

**Lemma 6.8.** \( \mathcal{B}_T \) is a basis of \( V \).

**Proof.** By Lemma 6.7(b), \( \mathcal{B}_T \) spans \( V \). Suppose by contradiction that there is a relation of linear dependence among elements of \( \mathcal{B}_T \), and let \( T^{k_1} : \tilde{a}_{i_1}a_{i_2} \cdots a_{i_n} : \) be the largest element of \( \mathcal{B}_T \) (with respect to the ordering defined in Lemma 6.7) with a non zero coefficient \( c \). By Lemma 6.7(a), we can rewrite such a relation as a relation of linear dependence among elements of \( \mathcal{B} \), which is not identically zero since the coefficient of \( :a(\tilde{a}_{i_1},k_1)a_{i_2} \cdots a_{i_n}: \) is \( c \neq 0 \). This clearly contradicts the fact that \( \mathcal{B} \) is a basis of \( V \). \( \square \)

The basis \( \mathcal{B}_T \) of \( V \) induces a new embedding \( \rho_T : V \hookrightarrow T(R) \), defined by

\[
\rho_T(T^{k_1} : \tilde{a}_{i_1}a_{i_2} \cdots a_{i_n} :) = T^{k_1}(\tilde{a}_{i_1} \otimes a_{i_2} \otimes \cdots \otimes a_{i_n}).
\]

for every basis element \( T^{k_1} : \tilde{a}_{i_1}a_{i_2} \cdots a_{i_n} : \in \mathcal{B}_T \). Notice that \( \rho_T \) commutes with the action of \( T \). Moreover, \( \mathcal{B}_T \) induces a secondary \( \Gamma \)–gradation on \( V \),

\[
V = \bigoplus_{\Delta \in \Gamma} V[\Delta]^{(2)}, \tag{6.8}
\]

defined by assigning to every basis element \( T^{k_1} : \tilde{a}_{i_1}a_{i_2} \cdots a_{i_n} : \in \mathcal{B}_T \) degree,

\[
\Delta(T^{k_1} : \tilde{a}_{i_1}a_{i_2} \cdots a_{i_n} :) = \Delta(a_{i_1}) + \cdots + \Delta(a_{i_n}).
\]

**Lemma 6.9.** (a) The \( \Gamma \)–filtration (6.1) of \( V \) is induced by the secondary \( \Gamma \)–gradation (6.8).

(b) The embedding \( \rho_T \) preserves the filtration: \( \rho_T(V[\Delta]) \subset T(\Gamma)([\Delta]) \), \( \forall \Delta \in \Gamma \).

**Proof.** The first part of the lemma is an obvious corollary of Lemma 6.7. The second part follows by the first part and the obvious inclusion \( \rho_T(V[\Delta]^{(2)}) \subset T(R)[\Delta] \), \( \forall \Delta \in \Gamma \). \( \square \)

**Lemma 6.10.** (a) \( R \) is a non–linear skew–symmetric conformal algebra with \( \lambda \)–bracket

\[
L_\lambda : R \otimes R \longrightarrow \mathbb{C}[\lambda] \otimes T(R),
\]

given by

\[
L_\lambda(a, b) = \rho_T([a, b]), \quad \forall a, b \in R.
\]

(b) Such a \( \lambda \)–bracket is compatible with \( [\, \, ] \), namely

\[
\pi(L_\lambda(A, B)) = [\pi(A) \lambda \pi(B)],
\]
\[
\pi(N(A, B)) = :\pi(A)\pi(B) :, \quad \forall A, B \in T(R).
\]

Here \( L_\lambda \) and \( N \) are defined on \( T(R) \) thanks to Lemma 3.2.
Proof. (a) Since $\rho_T$ commutes with the action of $T$, it immediately follows that $L_\lambda$ satisfies sesquilinearity and skew–symmetry. The grading condition for $L_\lambda$ follows by the assumptions on $V$ and Lemma 6.9(b). Indeed

$$L_\lambda([R[|\Delta_1], R[|\Delta_2]]) = \rho_T([R[|\Delta_1], R[|\Delta_2]]) \subset \mathbb{C}[|\lambda] \otimes \rho_T(V_\Delta) \subset \mathbb{C}[|\lambda] \otimes T_\Delta(R),$$

for some $\Delta < \Delta_1 + \Delta_2$.

(b) Notice that, by definition of $\pi$ and $\rho_T$, we have $\pi \circ \rho_T = \mathbf{1}_V$. Hence $\pi(L_\lambda(a, b)) = [a \lambda b], \forall a, b \in R$. Part (b) then follows by the definition of $L_\lambda$ and $N$ on $T(R)$, and by an easy induction argument. □

Notice that in the proof of Lemma 5.1 we did not use the assumption that $R$ was a non–linear Lie conformal algebra. As an immediate consequence we get the following

**Lemma 6.11.** Every element $A \in T_\Delta(R)$ decomposes as

$$A = B + M,$$

with $B \in \text{span}_\mathbb{C} \tilde{B}_\Delta$ and $M \in \mathcal{M}_\Delta(R)$ (we are using the notation introduced in Sect. 5).

**Lemma 6.12.** The $\lambda$–bracket $L_\lambda : R \otimes R \to \mathbb{C}[|\lambda] \otimes T(R)$ satisfies the Jacobi identity (3.9).

**Proof.** We need to show that for elements $a, b, c$ of $R$ we have

$$J(a, b, c; \lambda, \mu) = L_\lambda(a, L_\lambda(b, c)) - p(a, b)L_\mu(b, L_\lambda(a, c))$$

$$-L_{\lambda+\mu}(L_\lambda(a, b), c) \in \mathcal{M}_\Delta(R)$$

for some $\Delta \in \Gamma$ such that $\Delta < \Delta(a) + \Delta(b) + \Delta(c)$. By the grading condition (3.8) on $L_\lambda$, we have

$$J(a, b, c; \lambda, \mu) \in T_\Delta(R),$$

for some $\Delta < \Delta(a) + \Delta(b) + \Delta(c)$. Therefore, by Lemma 6.11, we have

$$J(a, b, c; \lambda, \mu) = J_B + J_M,$$

(6.9)

where $J_B \in \text{span}_\mathbb{C} \tilde{B}_\Delta$ and $J_M \in \mathcal{M}_\Delta(R)$. By Lemma 6.10(b) $L_\lambda$ is compatible with $[\lambda]$. Since $V$ is a vertex algebra, we have

$$\pi(J(a, b, c; \lambda, \mu)) = [a \lambda [b \mu c]] - p(a, b)[b \mu [a \lambda c]]$$

$$-[[a \lambda b] \lambda+\mu c] = 0.$$  

(6.10)

Moreover, since $\mathcal{M}(R) \subset \text{Ker} \pi$, we also have

$$\pi(J_M) = 0.$$  

(6.11)

We thus get, from (6.9),(6.10) and (6.11), that $\pi(J_B) = 0$. On the other hand, since $V$ is freely generated over $\tilde{V}$, we have that $\pi : \text{span}_\mathbb{C} \tilde{B} \to V$ is an isomorphism of vector spaces. We thus conclude, as we wanted, that $J_B = 0$, hence $J(a, b, c; \lambda, \mu) \in \mathcal{M}_\Delta(R)$. □
The above lemmas prove the first part of Theorem 6.4. We are left to prove the second part, namely uniqueness of the structure of non–linear Lie conformal algebra on $R$.

**Proof of Theorem 6.4, second part.** Suppose $L_\lambda, L'_\lambda$ are two structures of non–linear Lie conformal algebra on $R$, such that the corresponding enveloping vertex algebras are both isomorphic to $V$. Let us denote by $\mathcal{M}_\lambda(R)$ (respectively $\mathcal{M}'_\lambda(R)$) the subspace introduced in Definition 3.3 corresponding to $L_\lambda$ (resp. $L'_\lambda$). By assumption

$$V_\Delta \cong T_\Delta(R)/\mathcal{M}_\Delta(R) \cong T_\Delta(R)/\mathcal{M}'_\Delta(R),$$

hence $\mathcal{M}'_\lambda(R) = \mathcal{M}_\lambda(R)$. Let $\pi_\Delta : T_\Delta(R) \to T_\Delta(R)/\mathcal{M}_\Delta(R) \cong V_\Delta$ be the natural quotient map. Since the vertex algebra structures of $V, T(R)/\mathcal{M}(R)$ and $T(R)/\mathcal{M}'(R)$ are compatible, we must have

$$\pi_\Delta(N(A, B)) = \pi_\Delta(N'(A, B)), \quad \text{with } \Delta \leq \Delta(A) + \Delta(B),$$

$$\pi_\Delta(L_\lambda(A, B)) = \pi_\Delta(L'_\lambda(A, B)), \quad \text{with } \Delta < \Delta(A) + \Delta(B).$$

It follows that

$$N'(A, B) - N(A, B) \in \mathcal{M}_\Delta(R), \quad \text{with } \Delta \leq \Delta(A) + \Delta(B),$$

$$L'_\lambda(A, B) - L_\lambda(A, B) \in C[\lambda] \otimes \mathcal{M}_\Delta(R), \quad \text{with } \Delta < \Delta(A) + \Delta(B),$$

namely the identity map on $R$ gives a non–linear Lie conformal algebra isomorphism between $(R, L_\lambda)$ and $(R, L'_\lambda)$. \hfill $\Box$

**Remark 6.13.** Let $V$ be any simple graded vertex algebra strongly generated by a $C[T]$–submodule $R$. Arguments similar to the ones used in the proof of Theorem 6.4 show that the vertex algebra structure of $V$ induces on $R$ a structure of a non–linear conformal algebra. We have to consider separately two cases.

1. If $R$ is a non–linear Lie conformal algebra, then it is not hard to show that $V$ is the simple graded quotient of the universal enveloping vertex algebra $U(R)$ by a unique maximal ideal. In this case we say that the vertex algebra $V$ is non–degenerate. We thus conclude that non–degenerate vertex algebras are classified by non–linear Lie conformal algebras. Their classification will be studied in a subsequent paper.

2. If on the contrary $R$ is a non–linear conformal algebra, but Jacobi identity (3.9) does not hold, we say that the vertex algebra $V$ is degenerate. In this case one can still describe $V$ as a quotient of the universal enveloping vertex algebra $U(R)$ of $R$. The main difference is that in general $U(R)$ is not freely generated by $R$, namely the PBW theorem fails. For this reason, the study (and classification) of degenerate vertex algebras is more complicated than in the non–degenerate case. Their structure theory will be developed in subsequent work.

**Acknowledgement.** We would like to thank M. Artin, B. Bakalov, A. D’andrea and P. Etingof for useful discussions. This research was conducted by A. De Sole for the Clay Mathematics Institute. The paper was partially supported by the NSF grant DMS0201017.

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Communicated by L. Takhtajan