On Multiquadric Shape Determining Strategies in Image Reconstruction Applications: A Comparative Study

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Abstract. After being introduced to approximate two-dimensional geographical surfaces in 1971, the multivariate radial basis functions (RBFs) have been receiving a great amount of attention from scientists and engineers. Over decades, RBFs have been applied to a wide variety of problems. Approximation, interpolation, classification, prediction, and neural networks are inevitable in nowadays science, engineering, and medicine. Moreover, numerically solving partial differential equations (PDEs) is also a powerful branch of RBFs under the name of the ‘Meshfree/Meshless’ method. Amongst many, the so-called ‘Generalized Multiquadric (GMQ)’ is known as one of the most used forms of RBFs. It is of $(\varepsilon^2 + r^2)^\beta$ form, where $r = \|x - x^0\|_2$ for $x, x^0 \in \mathbb{R}^n$ represents the distance function. The key factor playing a very crucial role for MQ, or other forms of RBFs, is the so-called ‘shape parameter $\varepsilon$ ’ where selecting a good one remains an open problem until now. This paper focuses on measuring the numerical effectiveness of various choices of $\varepsilon$ proposed in literature when used in image reconstruction problems. Condition number of the interpolation matrix, CPU-time and storage, and accuracy are common criteria being utilized. The results of the work shall provide useful information on selecting a ‘suitable and reliable choice of MQ-shape’ for further applications in general.

1. Introduction
Radial Basis Functions (RBFs), $\varphi$, are commonly found as multivariate functions whose values are dependent only on the distance from the origin. This means that $\varphi(x) = \varphi(r) \in \mathbb{R}$ with $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$, in other words, on the distance from a point of a given set $\{x_j\}$, and $\varphi(x - x_j) = \varphi(r_j) \in \mathbb{R}$. Here, $r_j$ is the Euclidean distance defined in $\mathbb{R}^n$ as follows.

$$r_j = \|x - x_j\|_2 = \left[ (x_1^{(1)} - x_j^{(1)})^2 + (x_1^{(2)} - x_j^{(2)})^2 + \cdots + (x_1^{(n)} - x_j^{(n)})^2 \right]^{1/2}$$ (1)

One of the most popular types of RBFs is the Multiquadric (MQ) whose general format is expressed below.

$$\varphi(r, \varepsilon) = \left(\varepsilon^2 + r^2\right)^\beta$$ (2)

Where $\beta = \ldots, -3/2, -1/2, 1/2, 3/2, \ldots$. Over decades, RBFs have been receiving a great amount of attention from scientists and engineers. Some successful applications are those for function

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approximation [1], for solving the regulator equations [2], for classifying weblog dataset [3], for support vector machine classifiers [4], and for numerically solving partial differential equations [5, 6]. While being studied and applied in a wide range of applications, it is known that the shape parameter ($\varepsilon$) plays a crucial role. A number of investigations have been dedicated to searching for the most suitable form and yet this still remains an open and rather ‘hot’ topic. Figure 1 illustrates the clear difference in the surface shape of the MQ-RBF when different shapes are in use. Its impact is also observed, in some of our previous works, to be affected by the physics of the problem under investigation and the numerical scheme used [7-10].

Figure 1. Multiquadrics RBF surface with $\beta = 0.5$ plotted at different shapes; (a) $\varepsilon = 0.10$, (b) $\varepsilon = 1.00$, and (c) $\varepsilon = 2.00$.

Another important area is the amendment of damaged images. This challenge is found in a wide range of types such as repairing damaged historical pictures or their parts, removing inpainting (with scratches, texts, or watermarks embedded), and also noises [11]. Amongst different ideas being proposed over the past decades, those based on RBFs include filtering and restoring missing data [12], image restoring using auxiliary points [13, 14], in which the choice of choosing the shape was still in an ‘ad-hoc’ sense. This all brings great challenge and motivation to this present investigation.

2. Interpolation with RBFs
It starts with considering the interpolation of a multivariate function $f : \Omega \rightarrow \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$, from a set of sample values $\left\{ f(x_j) \right\}_{j=1}^N$ on a discrete set $X = \left\{ x_j \right\}_{j=1}^N \subset \Omega$. Such multivariate functions can be efficiently reconstructed if they are approximated by linear combinations of univariate interpolation functions with the Euclidean norm $\| \cdot \|_2$. This can be achieved by using translates $\Phi(x - x_j)$ of a single continuous real-valued function $\Phi$ defined on $\mathbb{R}$, and by letting $\Phi$ be radially symmetric; i.e.,

$$\Phi(x) := \varphi(\| x \|_2)$$

with a continuous function $\varphi$ on $\mathbb{R}_+$. In the mathematical literature, $\varphi$ is often called a radial basis function with centres $\{x_j\}_{j=1}^N$ and $\Phi$ is the associated kernel.

Interpolants $F$ to $f$ can be constructed as

$$F(x) = \sum_{j=1}^N a_j \varphi(\| x - x_j \|_2)$$

with real coefficients, $\{a_j\}_{j=1}^N$ which can be determined by the following interpolating condition.

$$F(x_i) = f(x_i), \quad \text{for all } i = 1, 2, ..., N.$$
\[ \sum_{j=1}^{N} \alpha_{ij} \varphi_i(\|x_j - x_i\|) = f(x_i) = F(x_i) \quad 1 \leq i \leq N, \]  

(6)

Hence, what comes next is a system of linear equations with \( \{\alpha_{ij}\}_{j=1}^{N} \) being the unknowns. The symmetric \( N \times N \) matrix expressed below.

\[
\Phi = \begin{bmatrix}
\varphi_{11} & \cdots & \varphi_{1N} \\
\vdots & \ddots & \vdots \\
\varphi_{N1} & \cdots & \varphi_{NN}
\end{bmatrix},
\begin{bmatrix}
\varphi_1(\|x_j - x_i\|) \\
\vdots \\
\varphi_N(\|x_j - x_i\|)
\end{bmatrix}
\]

(7)

where \( \varphi_j = \varphi(\|x_i - x_j\|) \), can be non-singular in only some cases [15]. One way to ensure the solvability of the system is to add a polynomial term to the interpolants, yielding:

\[
F(x) = f(x) = \sum_{j=1}^{N} \alpha_{ij} \varphi_i(\|x_j - x_i\|) + \sum_{k=1}^{n} \beta_k p_k(x)
\]

(8)

for \( 1 \leq i \leq N \), and together with the following constraints

\[
\sum_{j=1}^{N} \alpha_{ij} p_k(x_j) = 0, \quad 1 \leq k \leq n
\]

(9)

Where \( \{p_k\}_{k=1}^{n} \) is a basis for \( P_{m-1} \), the set of polynomials in two variables of degree \( \leq m - 1 \).

Let

\[
p^T = \begin{bmatrix}
p_1(x_1) & \cdots & p_i(x_i) & \cdots & p_n(x_n)
\end{bmatrix}
\]

(10)

, the interpolation conditions

\[
F(x_i) = \sum_{j=1}^{N} \alpha_{ij} \varphi_i(\|x_j - x_i\|) + \sum_{k=1}^{n} \beta_k p_k(x_i),
\]

(11)

for \( 1 \leq i, j \leq N \) and \( 1 \leq k \leq n \) can be reached. Therefore, the final system can be represented by the following matrix form.

\[
\begin{bmatrix}
\Phi & P \\
P^T & 0
\end{bmatrix}\begin{bmatrix}
a \\
b
\end{bmatrix} = \begin{bmatrix}
f \\
0
\end{bmatrix},
\]

(12)

or

\[
\Phi A = F,
\]

(13)

where \( \Phi = \begin{bmatrix} \Phi & P \\ P^T & 0 \end{bmatrix}, A = \begin{bmatrix} a & b \end{bmatrix}^T, F = \begin{bmatrix} f & 0 \end{bmatrix}^T, a = [a_1, \ldots, a_n]^T, b = [b_1, \ldots, b_n]^T \) and

\[
f = [f(x_1), \ldots, f(x_N)]^T. \]

Once the coefficient matrix is obtained, the solution calculation process can then be proceeded, explained in section 4.1.

3. Shape parameter determining strategies

As mentioned earlier, one crucial factor affecting the method’s performance is the shape parameter. Choices available in literature can be categorised into three forms; constant/fixed, variable, and iterative-based. This work concerns only those in variable forms. Table 1 contains popular strategies proposed and documented in the literature with their graphs illustrated in Figure 2. These forms are under our investigation.
Table 1. MQ-RBF parameter selection strategies.

| MQ-Shape Strategy | Proposed by (Year) | Mathematical Formula (for \(j = 1, 2, \ldots, N\)) |
|-------------------|--------------------|-------------------------------------------------|
| Shape-1           | Nojavan et.al. (2017) [16] | \(e_j = \left( c_{\text{min}} + \frac{c_{\text{max}} - c_{\text{min}}}{N-1} (j-1) \right)^{1/2} \) |
| Shape-2           | Nojavan et.al. (2017) [16] | \(e_j = \left( c_{\text{min}} + \frac{c_{\text{max}} - c_{\text{min}}}{N-1} \right)^{1/4} (j-1) \) |
| Shape-3           | Kansa (1990) [17] | \(e_j = 2 \left( c_{\text{min}}^{2} + \left( \frac{c_{\text{max}} - c_{\text{min}}}{c_{\text{min}}} \right)^{N-2} \right)^{1/2} \) |
| Shape-4           | Nojavan et.al. (2017) [16] | \(e_j = \left( c_{\text{min}} + \frac{c_{\text{max}} - c_{\text{min}}}{N-1} \right) \exp \left( -j \right) \) |
| Shape-5           | Xiang et.al. (2012) [18] | \(e_j = c_{\text{min}} + (c_{\text{max}} - c_{\text{min}}) \sin(j) \) |
| Shape-6           | Kansa and Carlson (1992) [19] | \(e_j = c_{\text{min}} + \left( \frac{c_{\text{max}} - c_{\text{min}}}{N-1} \right) j \) |
| Shape-7           | Sara and Sturgill (2009) [20] | \(e_j = c_{\text{min}} + (c_{\text{max}} - c_{\text{min}}) \times \text{rand} \left( 1, N \right) \) |
| Shape-8           | Hardy (1971) [21] | \(e = 0.815d, \quad d = \frac{1}{N} \sum_{i=1}^{N} d_i, \quad d_i \) is the distance between the \(i\)-th data point and its nearest neighbour. |
| Shape-9           | Franke (1979) [22] | \(e = 1.25 \frac{D}{\sqrt{N}}, \quad D \) is the smallest of the diameter of the circle containing all data points. |
| Shape-10          | Carlson et.al. (1991) [23] | \(e = \frac{1}{1 + 120V'}, \quad V = \frac{N}{2} \sum_{i=1}^{N} \left( \frac{z_i}{1} \right)^2 \) where \(z_i = \phi(\overline{x}_i, \overline{y}_i)\) and \(\phi(\overline{x}, \overline{y})\) are bivariate quadratic polynomial, \(\overline{x}_i = \frac{x_i - x_{\text{min}}}{x_{\text{max}} - x_{\text{min}}} \) \(\overline{y}_i = \frac{y_i - y_{\text{min}}}{y_{\text{max}} - y_{\text{min}}} \) \(\overline{z}_i = \frac{z_i - z_{\text{min}}}{z_{\text{max}} - z_{\text{min}}} \). |

4. Numerical algorithm and setups

4.1 The algorithm

**Step 1.** The original image is damaged and the pixels are grouped into two non-intersecting sets; the set of corrupted pixels (\(\Xi\), containing \(N\) elements), and the rest (\(\Xi\), containing \(N\) elements).

**Step 2.** For each \(x_i = (x_i, y_i) \in \Xi, \quad i = 1, 2, \ldots, N\), and choose \(n = 3\).

2.1. Choose the shape determining strategy shown in Table 1, and select \(\beta \in \{0.5, -0.5\}\).

2.2. Construct the interpolation matrix \(\phi_{N\times3}(N+3)^{-1}\).
2.3 Solve the linear system $[\varphi^{(N+3)\times(N+3)}]A^{(N+3)\times(1)} = [F]^{(N+3)\times(1)}$, (equation (13)), to get the coefficient matrix $A$.

**Step 3.** For each $\hat{x}_j = (x_j, y_j) \in \hat{\Xi}$, $j = 1, 2, ..., \hat{N}$.

Construct the interpolation matrix $\hat{\varphi}_{\hat{N}(N+3)}$ similarly to Step 2.2.

$$\hat{\varphi}_{\hat{N}(N+3)} = \hat{\varphi}(\|\hat{x}_j - x\|, c_j)$$ (14)

**Step 4.** Obtain the matrix containing reconstructed pixels, $\hat{F}_{\hat{N} \times 1}$, by the following calculation.

$$\hat{F}_{\hat{N} \times 1} = \hat{\varphi}_{\hat{N}(N+3)}^{(N+3)\times(1)}A$$ (15)

**Step 5.** Solution validation using equation (16), see below.

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**Figure 2.** Multiquadrics’ shape determining strategies plotted in 2D using $c_{\min} = 1/\sqrt{N}$, $c_{\max} = 3/\sqrt{N}$, and $N = 30$. 

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4.2 Performances' effectiveness criteria

Three criteria are briefly provided, on which the overall performance of the work is judged upon.

4.2.1 Accuracy. Results obtained for each case are evaluated using Mean K-order Absolute (MKE) Error, $S^{[k]}$, defined by the following expression, [14].

$$S^{[k]} = \frac{1}{N} \sum_{j=1}^{N} |\Omega_j(x,y) - \hat{\Omega}_j(x,y)|$$  (16)

Where $N$ is the number of the damaged pixels, $\Omega_j$ and $\hat{\Omega}_j$ are the original (undamaged) and reconstructed images respectively.

4.2.2 CPU-time (and storage). With less amount of time and storage required for the computational process, a method would be more desirable. As all numerical cases under this work were carried out using MATLAB programming platform, the so-called ‘tic-toc’ command was utilized (The ‘tic’ function records the current time, and the ‘toc’ function uses the recorded value to calculate the elapsed time.)

4.2.3 Condition number. The system can be solvable if the collocation matrix, $\phi$, has an inverse and this can be indicated by the means of its condition number ($\text{Cond}_{\phi}(-)$) expressed as:

$$\text{Cond}_{\phi}(\phi) = \left\| \phi \right\| \left\| \phi^{-1} \right\|$$  (17)

Two of the crucial factors affecting the condition number for an RBF-interpolation matrix are the number of interpolating nodes, and the shape parameter ($\ell$). The trending behaviour of this number is therefore observed throughout this experiment, providing information on the solvability of the collocation matrix for future uses.

4.3 Salt-and-Pepper impulse noise.

A well-known type of damaging image is called ‘Salt-and-Pepper’ where the targeted pixels are converted into either 0 or 255, equation (18).

$$\Omega_j(x,y) \in (0,255) \text{ corrupted to } \hat{\Omega}_j(x,y) = \begin{cases} 0 & \text{'Pepper noise'} \\ 255 & \text{'Salt noise'} \end{cases}$$  (18)

For some $j = 1, 2, ..., N + M$. Typically, the Salt-and Pepper noise can be resulted from a defect of the camera sensor, software failure, or hardware failure in image capturing or transmission. Figure 3 displays a damaged picture of the famous Lena corrupted with 40% of Salt-and-Pepper impulse noise.

![Figure 3. Sample of 40%-corrupted pixels in Salt-and-Pepper manner from the original Lena image.](image-url)
5. Numerical experiments and results

A numerical process following the steps listed in section 4.1 was conducted for each test case on the two well-known greyscale images; Lena and Plane, shown in Figure 4. To cover all subcases involved, more than one hundred simulations were performed on a machine with Windows 10 (64bit), Intel Corei7-11800H CPU (2.30GHz, 26MB L3 4.60 GHz), 32 GB DDR4 3200MHz. Simulations were organised regarding to types of images, percentages of damaging, shape strategies, and the values of $\beta$, where all the criteria stated in section 4.2 were being carefully recorded. The choices of $c_{\min} = 1/\sqrt{N}$, and $c_{\max} = 3/\sqrt{N}$, were adopted following suggestion by [18].

![Figure 4. Original images used; (a) 88×128–pixel Lena image, and (b) 128×128–pixel Plane image.](image)

The whole experiment comprised two phases based on the levels of damage. In Phase-1, 20% and 40% of damaging were considered and all ten strategies were numerically tested out for both $\beta = 0.5$ and $\beta = -0.5$. Table 2 and Table 3 provides respectively the errors measured by $S^{(1)}$ and $S^{(2)}$ for Lena and Plane image. From both tables, some similarities can be observed. Shape-4 was clearly noted to worst perform for both image types and for both $\beta$’s. Information in both tables also indicated that Shape-1, Shape-3, Shape-5, Shape-6, Shape-7, and Shape-10 performed equally well. Amongst all these, however, both image types and levels of damaging revealed that Shape-2, Shape-8, and Shape-9 produced comparatively best accuracy and for this reason only these three shape strategies were selected for further studies in Phase-2. The resulting images reconstructed by all the shape types were depicted in Figure 5 and Figure 6 for Lena and Plane images respectively.

| Shape Type | $S^{(1)}$ | $S^{(2)}$ | $S^{(1)}$ | $S^{(2)}$ |
|------------|----------|----------|----------|----------|
| Shape-1    | 6.48E+00 | 1.33E+01 | 3.68E+02 | 7.39E+00 |
| Shape-2    | 7.29E+00 | 6.87E+00 | 1.29E+02 | 8.87E+00 |
| Shape-3    | 6.66E+00 | 2.47E+01 | 1.03E+03 | 7.54E+00 |
| Shape-4    | 1.30E+16 | 1.55E+17 | 2.46E+34 | 6.42E+17 |
| Shape-5    | 6.67E+00 | 2.60E+01 | 1.12E+03 | 7.54E+00 |
Shape-6 | 6.66E+00 | 2.42E+01 | 1.25E+02 | 9.94E+02 | 2.54E+01 | 1.67E+02 | 1.08E+03
Shape-7 | 6.66E+00 | 2.44E+01 | 1.25E+02 | 1.00E+03 | 2.55E+01 | 1.67E+02 | 1.09E+03
Shape-8 | 6.01E+00 | 6.37E+00 | 1.01E+02 | 1.14E+02 | 7.53E+00 | 1.49E+02 | 1.67E+02
Shape-9 | 6.67E+00 | 6.29E+00 | 1.22E+02 | 1.09E+02 | 8.59E+00 | 1.68E+02 | 1.83E+03
Shape-10 | 6.69E+00 | 3.34E+01 | 1.26E+02 | 1.09E+02 | 3.37E+00 | 1.68E+02 | 1.83E+03

Table 3. Plane image’s errors produced by all shape types at two levels of damaging.

| Shape Type | 20%-damaged | 40%-damaged |
|------------|-------------|-------------|
|            | β = 0.5     | β = -0.5    |
| Shpae-1    | 2.64E+00    | 5.73E+00    |
| Shpae-2    | 2.44E+00    | 5.73E+00    |
| Shpae-3    | 2.74E+00    | 5.73E+00    |
| Shpae-4    | 1.88E+17    | 5.73E+00    |
| Shpae-5    | 2.75E+00    | 5.73E+00    |
| Shpae-6    | 2.74E+00    | 5.73E+00    |
| Shpae-7    | 2.74E+00    | 5.73E+00    |
| Shpae-8    | 2.21E+00    | 5.73E+00    |
| Shpae-9    | 2.19E+00    | 5.73E+00    |
| Shpae-10   | 2.76E+00    | 5.73E+00    |

After Phase-1 has been done, three shape determining strategies were further investigated; Shape-2, Shape-8, and Shape-9. More severe damaging levels were tackled; 45% up to 95%. Figure 7 shows repaired Lena images produced using Shape-2, 8, and -9, after undergoing 75% and 85% of damage.
Figure 6. Lena images; (a) original, (b) 20%-damaged, and reconstructed using $\beta = -0.5$ with; (c) Shape-1, (d) Shape-2, (e) Shape-3, (f) Shape-4, (g) Shape-5, (h) Shape-6, (i) Shape-7, (j) Shape-8, (k) Shape-9, and (l) Shape-10.

Figure 8 provides the same information but for the Plane image. Alongside with $s^{(1)}$ and $s^{(2)}$, CPU-time (and implicitly indicating the storage too), and the condition number of the interpolation matrix were also monitored throughout all damaging levels. While Figure 9(b) does not present any significance in CPU-time (measurement taken from Step 2 to Step 5 in section 4.1), noticeable differences can be seen in terms of condition number displayed in Figure 9(a). Based on this, it can be seen that for both $\beta's$, Shape-8 produced the lowest sensitivity indicating future better and more reliable application. This impression was further confirmed by results illustrated in Figure 9(c) for both Lena and Plane images. Note that other similar results with similar trends are not shown here.

Figure 7. (Top row) 75%-damaged Lena image with its corresponding reconstructed ones produced with shape-2,-8,-9 with $\beta = 0.5$ and $\beta = -0.5$ respectively, and (bottom row) 85%-damaged Lena image with its corresponding reconstructed ones produced with shape-2,-8,-9 with $\beta = 0.5$ and $\beta = -0.5$ respectively.
Figure 8. (Top row) 75%-damaged Plane image with its corresponding reconstructed ones produced with shape-2,-8,-9 with $\beta = 0.5$ and $\beta = -0.5$ respectively, and (bottom row) 85%-damaged Plane image with its corresponding reconstructed ones produced with shape-2,-8,-9 with $\beta = 0.5$ and $\beta = -0.5$ respectively.

Figure 9. Overall performances of the three chosen shape strategies (Shape-2, Shape-8, and Shape-9) in terms of; (a) Condition number of Lena image, (b) CPU-time spent of Plane image, and (c) $S^{(2)}$ obtained by using $\beta = 0.5$. 
6. Conclusions
Emerging as an alternative scheme of data interpolation and function approximation, multiquadric radial basis function (MQ-RBF), \((\varepsilon^2 + r^2)\beta, \beta = -0.5, 0.5\), is well-known amongst scientists and engineers. While its use spreads across a wide range of applications, the issue regarding a suitable value of shape parameter remains challenging. This work gathers ten variable-shape selection strategies for MQ-RBF ever being proposed in the literature. These were used to tackle the problem of damaged image reconstruction, done in a global manner, where Salt-and-Pepper noise was imposed. To observe the full potential of the schemes, normalization is not in use in any domain direction. Accuracy based on mean-square error, CUP-time, and condition number of the interpolation matrix were three criteria adopted for results validation. Overall performance measured in this work indicates that the form proposed in 1971 by Hardy [21] is capable of producing comparatively satisfactory results where the only input is the distance between the data point and its nearest neighbour. It is also found in this work that those containing the pre-determined boundary \(c_{\text{min}}\) and \(c_{\text{max}}\) could lead to desirable yields for both \(\beta = -0.5, 0.5\) if good values are input and this remains one of our future studies along this path.

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