Triangle-free graphs with the maximum number of cycles

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Abstract

It is shown that for \( n \geq 141 \), among all triangle-free graphs on \( n \) vertices, the balanced complete bipartite graph \( K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor} \) is the unique triangle-free graph with the maximum number of cycles. Using modified Bessel functions, tight estimates are given for the number of cycles in \( K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor} \). Also, an upper bound for the number of Hamiltonian cycles in a triangle-free graph is given.

1 Introduction

In a recent article [14], the maximum number of cycles in a triangle-free graph was considered. It was asked which triangle-free graphs contain the maximum number of cycles; this question arose from the study of path-finding algorithms [10]. The same authors posed the following conjecture:

Conjecture 1 (Durocher–Gunderson–Li–Skala, 2014 [14]). For each \( n \geq 4 \), the balanced complete bipartite graph \( K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor} \) contains more cycles than any other \( n \)-vertex triangle-free graph.

The authors [14] confirmed Conjecture 1 when \( 4 \leq n \leq 13 \), and made progress toward this conjecture in general. For example, they showed the conjecture to be true when restricted to “nearly regular graphs”, that is, for each positive integer \( k \) and sufficiently large \( n \), \( K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor} \) has more cycles than any other triangle-free graph on \( n \) vertices whose minimum degree and maximum degree differ by at most \( k \).

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In Theorems 5.1 and 5.2 below, it is shown that Conjecture 1 holds true for \( n \geq 141 \). Theorem 3.4 gives a useful estimate for the number of cycles in \( K_{[n/2],[n/2]} \). In Lemma 4.3, an upper bound is given for the number of Hamiltonian cycles in a triangle-free graph.

Even though Conjecture 1 arose from a very specific problem in computing, it can be considered as a significant problem in two aspects of graph theory: counting cycles in graphs, and the structure of triangle-free graphs. In recent decades, bounds have been proved for the maximum number of cycles in various classes of graphs. Some of these classes include complete graphs (e.g., [21]), planar graphs (e.g., [4], [5], [11]), outerplaner graphs and series-parallel graphs (e.g., [13]), graphs with large maximum degree (e.g., [8] for those without a specified odd cycle), graphs with specified minimum degree (see, e.g., [35]) graphs with a specified cyclomatic number or number of edges (e.g., [2], [23], [15], [19]; see also [25, Ch4, Ch10]), cubic graphs (e.g., [3], [12]), graphs with fixed girth (e.g., [27]), \( k \)-connected graphs (e.g., [24]), Hamiltonian graphs (e.g., [28], [32], [35]) Hamiltonian graphs with a fixed number of edges (e.g., [20]), 2-factors of the de Bruijn graph (e.g., [17]), graphs with a cut-vertex (e.g., [35]) complements of trees (e.g., [22], [29], [36]), and random graphs (e.g., [34]). In some cases, the structure of the extremal graphs are also found (e.g., [8], [28]).

In 1973, Erdős, Kleitman, and Rothschild [16] showed that for \( r \geq 3 \), as \( n \to \infty \), the number of \( K_r \)-free graphs on \( n \) vertices is

\[
2^{\left(1 - \frac{1}{r} + o(1)\right)\left(\frac{n}{r}\right)}.
\]

As a consequence, the number of triangle-free graphs is very close to the number of bipartite graphs, and so almost all triangle-free graphs are bipartite. By Mantel’s theorem [26], among graphs on \( n \) vertices, the triangle-free graph with the most number of edges is the balanced complete bipartite graph \( K_{[n/2],[n/2]} \).

Since \( K_{[n/2],[n/2]} \) is the triangle-free graph on \( n \) vertices with the most number of edges, and nearly all triangle-free graphs are bipartite, Conjecture 1 might seem reasonable, even though \( K_{[n/2],[n/2]} \) contains no odd cycles.

## 2 Notation and approximations used

A graph \( G \) is an ordered pair \( G = (V, E) = (V(G), E(G)) \), where \( V \) is a non-empty set and \( E \) is a set of unordered pairs from \( V \). Elements of \( V \) are called vertices and elements of \( E \) are called edges. Under this definition, graphs are simple, that is, there are no loops nor multiple edges.

An edge \( \{x, y\} \in E(G) \) is denoted by simply \( xy \). The neighbourhood of any vertex \( v \in V(G) \) is \( N_G(x) = \{y \in V(G) : xy \in E(G)\} \), and the degree of \( x \) is \( \text{deg}_G(x) = |N(x)| \). When it is clear what \( G \) is, subscripts are deleted, using only \( N(x) \) and \( \text{deg}(x) \). The minimum degree of vertices in a graph \( G \) is denoted by \( \delta(G) \), and the maximum degree is denoted \( \Delta(G) \). If \( Y \subset V(G) \), the subgraph of \( G \) induced by \( Y \) is denoted \( G[Y] \).

A graph \( G = (V, E) \) is called bipartite iff there is a partition \( V = A \cup B \) so that \( E \subset \{\{x, y\} : x \in A, y \in B\} \); if \( E = \{\{x, y\} : x \in A, y \in B\} \), then \( G \) is called the complete
bipartite graph on partite sets \(A\) and \(B\), denoted \(G = K_{|A|,|B|}\). The balanced complete bipartite graph on \(n\) vertices is \(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}\). A cycle on \(m\) vertices is denoted \(C_m\). The complement of a graph \(G\) is denoted \(\bar{G}\). For any graph \(G\), let \(c(G)\) denote the number of cycles in \(G\).

The number \(e\) is the base of the natural logarithm. Stirling’s approximation formula says that as \(n \to \infty\),

\[
\sqrt{2\pi n} \left(\frac{n}{e}\right)^n < n! < \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}}.
\]  

(1)

In 1955, Robbins [30] proved that

\[
\sqrt{2\pi \cdot \sqrt{n}} \left(\frac{n}{e}\right)^n < n! < e \cdot \sqrt{n} \left(\frac{n}{e}\right)^n.
\]  

(2)

Slightly more convenient bounds are used (that are valid for all \(n \geq 1\)):

\[
\sqrt{2\pi \cdot \sqrt{n}} \left(\frac{n}{e}\right)^n < n! < e \cdot \sqrt{n} \left(\frac{n}{e}\right)^n.
\]  

(3)

Two modified Bessel functions (see, e.g., [1]) are used:

\[
I_0(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^{2k}(k!)^2}.
\]  

(4)

In particular, when \(x = 2\) is used in either modified Bessel function, useful approximations are obtained:

\[
2.27958 \leq \sum_{k=0}^{\infty} \frac{1}{(k!)^2} = I_0(2) \leq 2.279586;
\]  

(5)

\[
1.5906 \leq \sum_{i=0}^{\infty} \frac{i}{(i!)^2} = \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} = I_1(2) \leq 1.59064.
\]  

(6)

3 Preliminaries

The following shows that among all bipartite graphs, the balanced one has the most cycles.

**Lemma 3.1** ([14]). For \(n \geq 4\), among all bipartite graphs on \(n\) vertices, \(K_{\lfloor n/2 \rfloor,\lceil n/2 \rceil}\) has the greatest number of cycles; that is, \(K_{\lfloor n/2 \rfloor,\lceil n/2 \rceil}\) is the unique cycle-maximal bipartite graph on \(n\) vertices.

So, to settle Conjecture [1], it is then sufficient to prove that a cycle-maximal triangle-free graph is bipartite. To this end, the following result is essential:
**Theorem 3.2** (Andrásfai, 1964 [6]). Any triangle-free graph $G$ on $n$ vertices with $\delta(G) > 2n/5$ is bipartite.

See also [7] for an English proof of Theorem 3.2 and related results. Theorem 3.2 is sharp because of $C_5$ (or a blow-up of $C_5$).

**Lemma 3.3** ([14]). For $n \geq 4$, the number of cycles in the balanced complete bipartite graph is

$$c(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) = \sum_{k=2}^{\lfloor n/2 \rfloor} \frac{\lfloor n/2 \rfloor! \lfloor n/2 \rfloor!}{2^k \lfloor n/2 \rfloor!}.$$

(7)

The following form for the number of cycles in $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ gives a way to estimate the right hand side of (7) in Lemma 3.3:

**Theorem 3.4.** For $n \geq 12$,

$$c(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) \geq \frac{n/2! \lfloor n/2 \rfloor!}{2 \lfloor n/2 \rfloor!} \cdot \begin{cases} I_0(2) & \text{if } n \text{ is even} \\ I_1(2) & \text{if } n \text{ is odd.} \end{cases}$$

(8)

and as $n \to \infty$,

$$c(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) = (1 + o(1)) \begin{cases} I_0(2) \pi \left(\frac{n}{2e}\right)^n & \text{if } n \text{ is even} \\ I_1(2) \pi \left(\frac{n}{2e}\right)^n & \text{if } n \text{ is odd.} \end{cases}$$

(9)

**Proof:** Using (2), the proof that (9) follows from (8) is elementary, and so is omitted.

By Lemma 3.3, write

$$c(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) = \sum_{k=2}^{\lfloor n/2 \rfloor} \frac{\lfloor n/2 \rfloor! \lfloor n/2 \rfloor!}{2^k \lfloor n/2 \rfloor!} \cdot \begin{cases} I_0(2) & \text{if } n \text{ is even} \\ I_1(2) & \text{if } n \text{ is odd.} \end{cases}$$

(12)

Case 1 ($n$ even): Suppose that for $\ell \geq 2$, $n = 2\ell$, and set

$$a_\ell = \sum_{k=2}^{\ell} \frac{\ell}{k((\ell-k)!)^2} = \sum_{i=0}^{\ell-2} \frac{\ell}{(\ell-i)(i)!^2}.$$ 

Then equation (11) becomes

$$c(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) = \frac{n/2! \lfloor n/2 \rfloor!}{2 \lfloor n/2 \rfloor!} \cdot a_\ell.$$ 

(12)
Claim: For $\ell \geq 6$, $a_{\ell+1} \leq a_{\ell}$.

Proof of Claim:

$$a_{\ell} - a_{\ell+1} = \sum_{i=0}^{\ell-2} \left( \frac{\ell}{\ell - i} - \frac{\ell + 1}{\ell + 1 - i} \right) \frac{1}{(i!)^2} - \frac{\ell + 1}{2((\ell - 1)!)^2}$$

$$= \sum_{i=0}^{\ell-2} \left( \frac{i}{(\ell + 1)(\ell - i)} \right) \frac{1}{(i!)^2} - \frac{\ell + 1}{2((\ell - 1)!)^2}$$

$$= \sum_{i=2}^{\ell-2} \frac{i}{(\ell + 1 - i)(\ell - i)} \cdot \frac{1}{(i!)^2} - \frac{1}{\ell(\ell - 1)} - \frac{\ell + 1}{2((\ell - 1)!)^2}$$

$$\geq 0 + \frac{2((\ell - 1)!)^2 - (\ell + 1)\ell(\ell - 1)}{2((\ell - 1)!)^2}$$

$$\geq 0 \quad \text{(for $\ell \geq 6$),}$$

finishing the proof of the claim.

Since the sequence $\{a_{\ell}\}$ is non-increasing and bounded below (by 0, e.g.), $\lim_{\ell \to \infty} a_{\ell}$ exists. To find this limit, first apply partial fractions:

$$a_{\ell} = \sum_{i=0}^{\ell-2} \frac{\ell}{(\ell - i)(i!)^2} = \sum_{i=0}^{\ell-2} \frac{1}{(i!)^2} + \sum_{i=0}^{\ell-2} \frac{i}{(\ell - i)(i!)^2}.$$ 

Put $b_{\ell} = \sum_{i=0}^{\ell-2} \frac{1}{(i!)^2}$ and $c_{\ell} = \sum_{i=0}^{\ell-2} \frac{i}{(\ell - i)(i!)^2}$. Then

$$c_{\ell} = \sum_{i=0}^{\ell-2} \frac{i}{(\ell - i)(i!)^2}$$

$$= \sum_{i=0}^{3} \frac{i}{(\ell - i)(i!)^2} + \sum_{i=4}^{\ell-2} \frac{i}{(\ell - i)(i!)^2}$$

$$\leq \frac{3}{\ell - 3} + \frac{1}{\ell} \sum_{i=4}^{\ell-2} \frac{1}{i!}$$

$$\leq \frac{3}{\ell - 3} + \frac{e}{\ell},$$

and therefore, $\lim_{\ell \to \infty} c_{\ell} = 0$. Thus,

$$\lim_{\ell \to \infty} a_{\ell} = \lim_{\ell \to \infty} (b_{\ell} + c_{\ell}) = \lim_{\ell \to \infty} b_{\ell}$$
\[ \sum_{i=0}^{\infty} \frac{1}{(i!)^2} = I_0(2) \tag{by (5)}. \]

Since \( a_\ell \) is non-increasing for \( \ell \geq 6 \), for \( n \geq 12 \),

\[ c(K_{[n/2],[n/2]}) \geq \frac{[n/2]![n/2]!}{2^{[n/2]}} \cdot I_0(2), \]

which proves the even case of (8). By (5), as \( n \to \infty \),

\[ c(K_{[n/2],[n/2]}) = (1 + o(1)) \frac{[n/2]![n/2]!}{2^{[n/2]}} \cdot I_0(2), \]

and by Stirling’s approximation (1), the proof of the even case of (10) is complete.

Case 2 (\( n \) odd): Suppose that for \( \ell \geq 6 \), \( n = 2\ell + 1 \). The proof follows the even case, and so is only outlined. Put

\[ a_\ell = \sum_{k=2}^{\ell} \frac{\ell}{k(\ell-k)!(\ell+1-k)!} = \sum_{i=0}^{\ell-2} \frac{\ell}{(\ell-i)!((i+1)!}. \]

Claim: For \( \ell \geq 4 \), \( a_{\ell+1} \leq a_\ell \).

Proof of claim: Letting \( \ell \geq 4 \),

\[ a_\ell - a_{\ell+1} = \sum_{i=0}^{\ell-2} \frac{i}{(\ell+1-i)(\ell-i)} \cdot \frac{1}{i!(i+1)!} - \frac{\ell + 1}{2(\ell-1)!}. \]

\[ = \sum_{i=2}^{\ell-2} \frac{i}{(\ell+1-i)(\ell-i)} \cdot \frac{1}{i!(i+1)!} - \frac{1}{2(\ell-1)!} - \frac{\ell + 1}{2(\ell-1)!}. \]

\[ \geq 0 + \frac{(\ell-2)!(\ell-1)! - (\ell + 1)}{2(\ell-1)!}. \]

\[ \geq 0, \]

finishing the proof of the claim.

Therefore, \( \lim_{\ell \to \infty} a_\ell \) exists. To find this limit, write

\[ a_\ell = \sum_{i=0}^{\ell-2} \frac{1}{i!(i+1)!} + \sum_{i=0}^{\ell-2} \frac{i}{(\ell-i)i!(i+1)!}. \]

Letting \( b_\ell = \sum_{i=0}^{\ell-2} \frac{1}{i!(i+1)!} \) and \( c_\ell = \sum_{i=0}^{\ell-2} \frac{i}{(\ell-i)i!(i+1)!} \), observe that

\[ c_\ell = \sum_{i=0}^{\ell-2} \frac{i}{(\ell-i)i!(i+1)!} + \sum_{i=0}^{\ell-2} \frac{i}{(\ell-i)i!(i+1)!} \leq \frac{3}{\ell - 3} + \frac{e}{\ell}, \]

\[ \sum_{i=0}^{\infty} \frac{1}{i!(i+1)!} = I_0(2) \tag{by (5)}. \]
and so \( \lim_{\ell \to \infty} c_\ell = 0 \). Thus,
\[
\lim_{\ell \to \infty} a_\ell = \lim_{\ell \to \infty} b_\ell = \sum_{i=0}^{\infty} \frac{1}{i!(i+1)!} = \sum_{i=0}^{\infty} \frac{i+1}{((i+1)!)^2} = \sum_{i=0}^{\infty} \frac{i}{(i!)^2}.
\]

which, by (6), is equal to \( I_1(2) \). Then again
\[
c(K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}) \geq \frac{|n/2|! [n/2]!}{2^{n/2}} \cdot I_1(2)
\]
\[
= \frac{\ell!(\ell+1)!}{2\ell} \cdot I_1(2)
\]
\[
= \frac{(\ell!)^2}{2\ell}(\ell+1) \cdot I_1(2)
\]
\[
= (1 + o(1)) \pi \left( \frac{\ell}{e} \right)^{2\ell} (\ell + 1) \cdot I_1(2)
\]
\[
> (1 + o(1)) \pi \left( \frac{\ell}{e} \right)^{2\ell} (\ell - 1) \cdot I_1(2)
\]
\[
= 1 + o(1)) \pi \left( \frac{n-1}{2e} \right)^n \left( \frac{n-1}{2} \right) \cdot I_1(2)
\]
\[
= (1 + o(1)) \pi e \left( \frac{n-1}{2e} \right)^n \cdot I_1(2)
\]
\[
= (1 + o(1)) \pi e \left( \frac{n-1}{n} \right)^n \left( \frac{n}{2e} \right)^n \cdot I_1(2)
\]
\[
= (1 + o(1)) \pi \left( \frac{n}{2e} \right)^n \cdot I_1(2),
\]
and as \( n \to \infty \),
\[
c(K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}) = (1 + o(1)) \pi \left( \frac{n}{2e} \right)^n \cdot I_1(2).
\]

This completes the proof for odd \( n \), and so the proof of the lemma. \( \square \)

**Lemma 3.5.** Let \( H \) be a triangle-free graph on 6 vertices with \( x, y \in V(H) \). Then there are at most 9 different \( x-y \) paths.

**Proof:** Consider two cases.

Case 1: \( H \) contains no copy of \( C_5 \). Then \( H \) contains no odd cycle, and so is bipartite. Without loss of generality, add edges to \( H \) to make \( H \) a complete bipartite graph. There are only four different complete bipartite graphs on six vertices, namely \( \overline{K}_6, K_{1,5}, K_{2,4}, \) and \( K_{3,3} \). By inspection, in any of these, the maximum number of paths between any two vertices is at most 9.
Case 2: $H$ contains a copy of $C_5$. Suppose that $x_1, x_2, x_3, x_4, x_5, x_1$ forms a cycle $C$, and that $x_6$ is the remaining vertex. Then $x_6$ is adjacent to at most two vertices of $C$. If $x_6$ is adjacent to fewer than two vertices of $C$, add an extra edge or two so that $x_6$ is adjacent to precisely two vertices of $C$; without loss, suppose that $x_6$ is adjacent to $x_1$ and $x_3$. Then the maximum number of paths between any two vertices is 4 (for example, between $x_2$ and $x_6$).

4 Counting types of cycles

Lemma 4.1. There exists $n_0 \in \mathbb{Z}^+$ so that for every even integer $n \geq n_0$, if $G$ is a triangle-free graph on $n$ vertices, and $x_1x_2 \in E(G)$, then the number of cycles containing the edge $x_1x_2$ is at most $10n^{n-1}/(2e)^n$.

Proof: Let $G$ be a triangle-free graph on $n$ vertices, and let $x_1x_2 \in E(G)$. For each $k = 4, \ldots, n$, let $c_k$ denote the number of cycles of length $k$ that contain the edge $x_1x_2$. The goal is to give an upper bound for $\sum_{k=4}^n c_k$.

Let $2 \leq i \leq \frac{n-4}{2}$, an upper bound on $c_{2i} + c_{2i+1}$ is first calculated; to do so, count all possible cycles of the form $x_1, x_2, \ldots, x_{2i}$ or $x_1, x_2, \ldots, x_{2i+1}$. For each $j > 1$, there are at most $d_j = |N(x_j)\{x_1, \ldots, x_{j-1}\}|$ ways to choose an $x_{j+1}$. Note that $N(x_j) \cap N(x_{j+1}) = \emptyset$, since otherwise a triangle is formed with $x_j$ and $x_{j+1}$. Also,

$$|(N(x_j)\{x_1, \ldots, x_{j-1}\}) \cup (N(x_{j+1})\{x_1, \ldots, x_j\})| \leq |V(G)\{x_1, \ldots, x_j\}| = n - j.$$ 

Therefore,

$$d_j + d_{j+1} \leq |N(x_j)\{x_1, \ldots, x_{j-1}\}| + |N(x_{j+1})\{x_1, \ldots, x_j\}|$$

$$= |((N(x_j)\{x_1, \ldots, x_{j-1}\}) \cup (N(x_{j+1})\{x_1, \ldots, x_j\}))|$$

$$\leq n - j,$$

and thus

$$d_j d_{j+1} \leq \left\lfloor \frac{n-j}{2} \right\rfloor \cdot \left\lfloor \frac{n-j}{2} \right\rfloor \cdot n-j.$$ (13)

Using (13), the number of ways to choose vertices $x_3, x_4, \ldots, x_{2i}$ so that $x_1, x_2, x_3, x_4, \ldots, x_{2i}$ form a path is at most

$$\prod_{j=2}^{2i-1} d_j = \prod_{j=1}^{i-1} (d_{2j}d_{2j+1}) \leq \prod_{j=1}^{i-1} \left( \left\lfloor \frac{n-2j}{2} \right\rfloor \cdot \left\lfloor \frac{n-2j}{2} \right\rfloor \cdot n-j \right)$$

$$= \prod_{j=1}^{i-1} \left( \frac{n-2j}{2} \right)^2.$$ (14)

If there is an edge $x_2x_1 \in E(G)$, there is one cycle $x_1, x_2, \ldots, x_{2i}$ of length $2i$, and no cycles of the form $x_1, x_2, \ldots, x_{2i+1}$ because otherwise, $x_1, x_2, x_{2i+1}$ form a triangle. So, in total, there is exactly one cycle that contains the path $x_1, x_2, \ldots, x_{2i}$ and has length $2i$ or $2i + 1$. 

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If there is no edge \(x_{2i}x_1\), there is no cycle \(x_1, \ldots, x_{2i}\) and at most \(n - 2i\) cycles of the form \(x_1, \ldots, x_{2i}x_{2i+1}\). In any case, there are at most \(n - 2i\) cycles containing the path \(x_1, \ldots, x_{2i}\).

By these observations and inequality (14),

\[
c_{2i} + c_{2i+1} \leq (n - 2i) \prod_{j=1}^{i-1} \left( \frac{n - 2j}{2} \right)^2. \tag{15}
\]

To evaluate \(\sum_{k=4}^{n} c_k\), separate the sum into two parts:

\[
\sum_{k=4}^{n-5} c_k = \sum_{i=2}^{(n-6)/2} (c_{2i} + c_{2i+1}) \\
\leq \sum_{i=2}^{(n-6)/2} \left((n - 2i) \prod_{j=1}^{i-1} \left( \frac{n - 2j}{2} \right)^2\right) \tag{by (15)} \\
= \sum_{i=2}^{(n-6)/2} (n - 2i) \left( \frac{n-2}{2}! \right)^2 \left( \frac{2j}{(n-2i)!} \right)^2 \\
= \left( \frac{n-2}{2}! \right)^2 \sum_{j=3}^{n-4} \frac{2j}{(j!)^2} \\
= \left( \frac{n-2}{2}! \right)^2 \left( \sum_{j=1}^{n-4} \frac{2j}{(j!)^2} - \frac{2}{(1!)^2} - \frac{2 \cdot 2}{(2!)^2} \right) \\
\leq \left( \frac{n-2}{2}! \right)^2 \left( 2 \cdot (1.591) - 3 \right) \tag{by (6)} \\
< 0.19 \left( \frac{n-2}{2}! \right)^2. \tag{16}
\]

To count \(\sum_{k=n-4}^{n} c_k\), note that by (13), there are at most

\[
\prod_{i=2}^{n-5} d_i \leq \prod_{j=1}^{n-6} \left( \frac{n - 2j}{2} \right)^2
\]

ways to choose a path \(x_1, x_2, \ldots, x_{n-4}\), and by Lemma 3.5, there are at most 9 paths that connect \(x_{n-4}\) and \(x_1\) in the graph \(G\{x_1, \ldots, x_{n-5}\}\); that is, there are at most 9 ways to complete the path \(x_1, x_2, \ldots, x_{n-4}\) to a cycle. Therefore,

\[
\sum_{k=n-4}^{n} c_k \leq 9 \prod_{j=1}^{n-6} \left( \frac{n - 2j}{2} \right)^2 = 9 \cdot \left( \frac{n-2}{2}! \right)^2 = 9 \cdot \left( \left( \frac{n-2}{2}! \right)^2 \right). \tag{17}
\]

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Adding equations (16) and (17),

\[
\sum_{k=4}^{n} c_k \leq 0.19 \left( \left( \frac{n-2}{2} \right)! \right)^2 + \frac{9}{4} \left( \left( \frac{n-2}{2} \right)! \right)^2 = 2.44 \left( \left( \frac{n-2}{2} \right)! \right)^2 .
\]  (18)

By Stirling’s approximation, as \( n \to \infty \),

\[
2.44 \left( \left( \frac{n-2}{2} \right)! \right)^2 = (1 + o(1))2.44 \frac{n^{n-1}}{(2e)^n} \cdot \pi(n-2)
\]

\[
= (1 + o(1))2.44\pi \frac{n^{n-1}}{(2e)^n} \left( \frac{n-2}{n} \right)^{-n-1}
\]

\[
= (1 + o(1))2.44\pi \frac{n^{n-1}}{(2e)^n} \cdot 4e^2 \cdot \frac{1}{e^2}
\]

\[
= (1 + o(1))9.76\pi \frac{n^{n-1}}{(2e)^n}
\]

\[
< 10\pi \frac{n^{n-1}}{(2e)^n}
\]  (for \( n \) suff. large)

completing the proof of the lemma. \( \square \)

**Lemma 4.2.** There exists \( n_0 \in \mathbb{Z}^+ \) so that for every odd integer \( n \geq n_0 \), if \( G \) is a triangle-free graph on \( n \) vertices, and \( x_1 x_2 \in E(G) \) with \( \deg_G(x_2) \leq \frac{2}{5}n \), then the number of cycles containing the edge \( x_1 x_2 \) is at most \( 7.81\pi \frac{n^{n-1}}{(2e)^n} \).

**Proof:** The proof is similar to that of Lemma 4.1. Let \( G \) be a triangle-free graph on \( n \) vertices, and let \( x_1 x_2 \in E(G) \), where \( \deg(x_2) \leq \frac{2}{5}n \). For each \( k = 4, \ldots , n \), let \( c_k \) denote the number of cycles of length \( k \) that contain the edge \( x_1 x_2 \).

For \( 3 \leq i \leq \frac{n-5}{2} \), an upper bound on \( c_{2i-1} + c_{2i} \) is first calculated; to do so, count all possible cycles of the form \( x_1, x_2, \ldots , x_{2i-1} \) or \( x_1, x_2, \ldots , x_{2i} \). As in Lemma 4.1, for each \( j > 1 \), there are at most \( d_j = |N(x_j) \setminus \{x_1, \ldots , x_{j-1}\}| \) ways to choose an \( x_{j+1} \), and

\[
d_j d_{j+1} \leq \left[ \frac{n-j}{2} \right] \cdot \left[ \frac{n-j}{2} \right].
\]  (19)

Using (19) and the fact that \( d_2 \leq \frac{2}{5}n \), the number of ways to choose vertices \( x_3, x_4, \ldots , x_{2i-1} \) so that \( x_1, x_2, x_3, x_4, \ldots , x_{2i-1} \) form a path is at most

\[
\prod_{j=2}^{2i-2} d_j = d_2 \prod_{j=3}^{2i-2} d_j \leq \frac{2}{5} n \prod_{j=1}^{i-2} (d_{2j+1}d_{2j+2}) \leq \frac{2}{5} n \prod_{j=1}^{i-2} \left( \left[ \frac{n-2j-1}{2} \right] \cdot \left[ \frac{n-2j-1}{2} \right] \right)
\]

\[
= \frac{2}{5} n \prod_{j=1}^{i-2} \left( \frac{n-2j-1}{2} \right)^2 .
\]  (20)
If $x_{2i-1}x_1 \in E(G)$, there is one cycle of length $2i - 1$ and no cycles of length $2i$; if there is no such edge, there are no cycles of length $2i - 1$ and at most $n - 2i - 1$ cycles of length $2i + 1$. By these observations and (20),

$$c_{2i-1} + c_{2i} \leq (n - 2i - 1) \frac{2}{5} n \prod_{j=1}^{i-2} \left( \frac{n - 2j - 1}{2} \right)^2. \quad (21)$$

To evaluate $\sum_{k=4}^{n} c_k$, separate the sum into three parts:

$$\sum_{k=4}^{n} c_k = c_4 + \sum_{k=5}^{n-5} c_k + \sum_{k=n-4}^{n} c_k.$$

First,

$$c_4 \leq d_2 d_3 < n \cdot n = n^2. \quad (22)$$

Next,

$$\sum_{k=5}^{n-5} c_k = \sum_{i=3}^{(n-5)/2} (c_{2i-1} + c_{2i})$$

$$\leq \sum_{i=3}^{(n-5)/2} \left[ (n - 2i - 1) \frac{2}{5} n \prod_{j=1}^{i-2} \left( \frac{n - 2j - 1}{2} \right)^2 \right] \quad (\text{by (21)})$$

$$= \frac{2}{5} n \sum_{i=3}^{(n-5)/2} \left[ (n - 2i - 1) \prod_{j=1}^{i-2} \left( \frac{n - 2j - 1}{2} \right)^2 \right]$$

$$= \frac{2}{5} n \sum_{i=3}^{(n-5)/2} (n - 2i - 1) \left( \frac{(n-3)!}{(n-2i-1)!} \right)^2$$

$$= \frac{2}{5} n \left( \frac{(n-3)!}{2} \right)^2 \sum_{j=3}^{n-5} \frac{2j}{(j!)^2}$$

$$= \frac{2}{5} n \left( \frac{(n-3)!}{2} \right)^2 \left( \sum_{j=1}^{n-5} \frac{2j}{(j!)^2} - \frac{2}{(1!)^2} - \frac{2}{(2!)^2} \right)$$

$$\leq \frac{2}{5} n \left( \frac{(n-3)!}{2} \right)^2 (3.19 - 3) \quad (\text{by (6)})$$

$$= 0.076n \left( \frac{n-3}{2} \right)^2. \quad (23)$$
To count $\sum_{k=n-4}^{n} c_k$, note that by (22), there are at most
\[
\prod_{i=2}^{n-5} d_i = d_2 \cdot \prod_{j=1}^{(n-7)/2} d_{2j+1}d_{2j+2} \leq \frac{2}{5} n \prod_{j=1}^{n-7} \left( \frac{n - 2j - 1}{2} \right)^2
\]
ways to choose a path $x_1, x_2, \ldots, x_{n-4}$, and by Lemma 3.5, there are at most 9 ways to complete to a cycle (by paths that connect $x_{n-4}$ and $x_1$) in the graph $G\{x_1, \ldots, x_{n-5}\}$.

Therefore,
\[
\sum_{k=n-4}^{n} c_k \leq 9 \cdot \frac{2}{5} n \prod_{j=1}^{n-7} \left( \frac{n - 2j - 1}{2} \right)^2 = 9 \cdot \frac{2}{5} n \cdot \left( \frac{(n-3)!}{(2!)^2} \right)^2 = \frac{9}{10} n \left( \frac{(n-3)!}{2!} \right)^2.
\]

(24)

Adding (22), (23), and (24), as $n \to \infty$,
\[
\sum_{k=4}^{n} c_k \leq n^2 + 0.076n \left( \frac{n-3}{2} \right)!^2 + \frac{9}{10} n \left( \frac{(n-3)!}{2!} \right)^2
\]
\[
= n^2 + 0.976n \left( \frac{n-3}{2} \right)!^2
\]
\[
= n^2 + (1 + o(1))0.976n(n-3)\pi \left( \frac{n-3}{2e} \right)^{n-3}
\]
\[
= (1 + o(1))0.976\pi n \cdot \frac{n^{n-2}}{(2e)^n} \left( \frac{n - 3}{n} \right)^{n-2} (2e)^3
\]
\[
= (1 + o(1))0.976\pi \cdot \frac{n^{n-1}}{(2e)^n} e^3 (2e)^3
\]
\[
= (1 + o(1))7.808\pi \cdot \frac{n^{n-1}}{(2e)^n}
\]
\[
< 7.81\pi \frac{n^{n-1}}{(2e)^n} \quad \text{(for n suff. large),}
\]
completing the proof.

Lemma 4.3. Let $H$ be a triangle-free graph on $k$ vertices. Then $H$ has at most $e^2 \left( \frac{k}{2e} \right)^k$ hamiltonian cycles.

Proof: Let $x_1$ be the first vertex of a hamiltonian cycle. For each $i \geq 1$, there are at most $d_i = |N(x_i)\{x_1, \ldots, x_i\}|$ ways to choose a vertex $x_{i+1}$. Note that $N(x_i) \cap N(x_{i+1}) = \emptyset$ because if the intersection contains some vertex $v$, then $v$, $x_i$, and $x_{i+1}$ form a triangle. Also,
\[
|N(x_i)\{x_1, \ldots, x_i\} \cup N(x_{i+1})\{x_1, \ldots, x_{i+1}\}| \leq |V(H)\{x_1, \ldots, x_{i+1}\}| = k - i.
\]
Therefore,
\[ d_i + d_{i+1} = |N(x_i)\backslash \{x_1, \ldots, x_i\}| + |N(x_{i+1})\backslash \{x_1, \ldots, x_{i+1}\}| = |N(x_i)\backslash \{x_1, \ldots, x_i\} \cup N(x_{i+1})\backslash \{x_1, \ldots, x_{i+1}\}| \leq k - i, \]
and thus \( d_i d_{i+1} \leq \left\lfloor \frac{k-i}{2} \right\rfloor \cdot \left\lceil \frac{k-i}{2} \right\rceil \).

When \( k \) is odd, the number of hamiltonian cycles is at most
\[
\prod_{i=1}^{k-1} d_i = \prod_{j=1}^{k-1} d_{2j-1} d_{2j} \leq \prod_{j=1}^{k-1} \left[ \frac{k-2j+1}{2}, \frac{k-2j+1}{2} \right] = \prod_{j=1}^{k-1} \left( \frac{k-2j+1}{2} \right)^2 = \left( \frac{k-1}{2} \right)!^2
\]
and by (2), this number is at most
\[
\left( \frac{k-1}{2} \right)!^2 e^{k-3} \leq k \left( \frac{k}{2} \right)^k e^{k-3} \leq k \left( \frac{k}{2} \right)^k e^{k-3} = k \left( \frac{k}{2} \right)^k e^{k-3}
\]
completing the proof for odd \( k \).

When \( k \) is even, similarly obtain
\[
\prod_{i=1}^{k-1} d_i = \left( \prod_{j=1}^{k-1} d_{2j-1} d_{2j} \right) d_{k-1} \leq \left( \prod_{j=1}^{k-1} \left[ \frac{k-2j+1}{2}, \frac{k-2j+1}{2} \right] \right) \cdot 1
\]
\[
= \prod_{j=1}^{k-1} \left( \frac{k-2j}{2} \right) \cdot \left( \frac{k-2j+2}{2} \right) = k \left( \frac{k-2}{2} \right) \left( \frac{k-2}{2} \right)! \leq k \left( \frac{k-2}{2} \right) e^{k-2} \leq k \left( \frac{k-2}{2} \right) e^{k-2} \leq k \left( \frac{k-2}{2} \right) e^{k-2}
\]
completing the proof for even \( k \), and hence for the lemma.

\[ \square \]

5 Main theorems

In Theorem 5.1 Conjecture 1 is proved for sufficiently large \( n \). Then in Theorem 5.2 a lower bound on such \( n \) is given.

**Theorem 5.1.** There exists \( n_0 \in \mathbb{Z}^+ \) so that for any \( n \geq n_0 \), the triangle-free graph on \( n \) vertices with the largest number of cycles is \( K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil} \).
Proof: Let $G$ be a triangle-free graph on $n$ vertices. It is first shown that if $G$ contains a vertex of small degree, then $G$ has far fewer cycles than does $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

Let $x \in V(G)$, and assume that $\deg(x) \leq \frac{2}{5}n$. Cycles in $G$ are counted according to whether or not they contain $x$.

The number of cycles not containing $x$: Any cycle in $G \setminus x$ is a hamiltonian cycle for some subgraph, and so the number of cycles in $G$ not containing $x$ is loosely bounded above by

$$
\sum_{Y \subseteq V(G) \setminus x} \left( \text{number of ham. cycles in } G[Y] \right)
$$

(26)

$$
\leq \sum_{k=4}^{n-1} \binom{n-1}{k} e^2 \left( \frac{k}{2e} \right)^k
$$

(by Lemma 4.3)

$$
< e^2 \sum_{k=4}^{n-1} \binom{n-1}{k} \left( \frac{n-1}{2e} \right)^k
$$

$$
< e^2 \left( 1 + \frac{n-1}{2e} \right)^{n-1}
$$

$$
= e^2 \left( \frac{n+2e-1}{2e} \right)^{n-1}
$$

$$
= e^2 \left( \frac{n}{2e} \right)^{n-1} \left( 1 + \frac{2e-1}{n} \right)^{n-1}
$$

$$
< e^2 \left( \frac{n}{2e} \right)^{n-1} e^{2e-1}
$$

$$
= \frac{2e^{2e+2}}{n} \left( \frac{n}{2e} \right)^n.
$$

(27)

The number of cycles containing $x$: Each cycle $C$ containing $x$ has exactly two edges (in $C$) incident with $x$, and so the number of cycles containing $x$ is

$$
\frac{1}{2} \sum_{y \in N(x)} \left( \text{number of cycles containing } xy \right).
$$

(28)

By Lemma 4.1, for even $n$, the expression (28) is at most

$$
\frac{1}{2} \cdot \frac{2}{5} \cdot \frac{10\pi}{(2e)^n} n^{n-1} = 2\pi \left( \frac{n}{2e} \right)^n.
$$

In this case, for $n$ sufficiently large, the total number of cycles in $G$ is at most

$$
2\pi \left( \frac{n}{2e} \right)^n + \frac{2e^{2e+2}}{n} \left( \frac{n}{2e} \right)^n = \left( 2\pi + \frac{2e^{2e+2}}{n} \right) \left( \frac{n}{2e} \right)^n \leq 2.01\pi \left( \frac{n}{2e} \right)^n.
$$
However, by (8), the number of cycles in $K_{[n/2],[n/2]}$ is (for $n$ even) at least $2.27958 \pi \left(\frac{n}{2e}\right)^n$.

Let $n$ be odd; then by Lemma 1.2 the expression (28) is at most

$$\frac{1}{2} \cdot \frac{2}{5} n \cdot 7.81 \pi \frac{n^{n-1}}{(2e)^n} = 1.562 \pi \left(\frac{n}{2e}\right)^n.$$  \hspace{1cm} (29)

Thus, for odd $n$ sufficiently large, by (29) and (27) the total number of cycles in $G$ is at most

$$1.562 \pi \left(\frac{n}{2e}\right)^n + \frac{2e^{2e+2}}{n} \left(\frac{n}{2e}\right)^n \leq 1.57 \pi \left(\frac{n}{2e}\right)^n.$$  

By (8) in Theorem 3.3, the number of cycles in $K_{[n/2],[n/2]}$ for $n$ odd is at least $1.5906 \pi \left(\frac{n}{2e}\right)^n$.

In both the even and odd case, if $G$ contains a vertex of degree at most $\frac{2}{5} n$, then $G$ has far fewer cycles than does $K_{[n/2],[n/2]}$.

So assume that $\delta(G) > \frac{2}{5} n$. Then by Theorem 3.2, $G$ is bipartite. By Lemma 3.1, the number of cycles in $G$ is maximized by $K_{[n/2],[n/2]}$. $\square$

**Theorem 5.2.** The statement of Theorem 3.1 with $n_0 = 141$ is true.

**Proof:** To show that $n_0$ works, further estimations on $c(K_{[n/2],[n/2]})$ are needed for $n \geq 141$. Both when $n$ is even and when $n$ is odd, (12) holds (but the expression for $a_\ell$ changes). Since each (one for odd, one for even) sequence of $a_\ell$s are non-increasing for $n \geq 140$,

$$c(K_{[n/2],[n/2]}) \leq \frac{[n/2]! [n/2]!}{2 [n/2]} \cdot \begin{cases} a_{71} & \text{for } n \text{ even} \\ a_{70} & \text{for } n \text{ odd} \end{cases}$$

$$\leq \frac{[n/2]! [n/2]!}{2 [n/2]} \begin{cases} 2.302786 & \text{for } n \text{ even} \\ 1.60067 & \text{for } n \text{ odd} \end{cases}. \hspace{1cm} \ldots \hspace{1cm} (30)$$

(The values of $a_{70}$ and $a_{71}$ were calculated by computer.) With these estimates in hand, now Theorem 3.1 is proved with $n_0 = 141$. Let $G$ be a triangle-free graph on $n \geq 141$ vertices. Without loss of generality, assume that there is a vertex of degree at most $\frac{2}{5} n$ (since otherwise, the theorem is proved by Theorem 3.2 and Lemma 3.1). In the following calculations, bounds given in (2) and Theorem 3.4 are used freely.

Case 1: Let $n \geq 141$ be odd. By (25) from the proof of Lemma 1.2, the number of cycles passing through an edge $xy$ in $G$ is at most $n^2 + 0.976n \left(\left(\frac{n-3}{2}\right)!\right)^2$. Then the number of cycles in $G$ is bounded by

$$c(G) \leq 1 \cdot \frac{2}{5} \cdot n \cdot \left[n^2 + 0.976n \left(\left(\frac{n-3}{2}\right)!\right)^2 + \frac{2e^{2e+2}}{n} \left(\frac{n}{2e}\right)^n\right]$$

$$= \frac{n-1}{n-1} \cdot n^{n+1} \cdot I_1(2) \cdot \left(n \cdot \frac{2}{5} \cdot n^2 + 0.976n \left(\left(\frac{n-3}{2}\right)!\right)^2 \right) + \frac{n}{n} \cdot \pi \left(\frac{n}{2e}\right)^n \left(\frac{2e^{2e+2}}{n\pi I_1(2)}\right)$$

15
Returning to the case when \( n \) graph with maximum degree \( \Delta \) on \( n \), Schelp [8] showed that for a fixed \( k > 0 \), for \( A \) few natural extensions of Theorem 5.1 and Conjecture 1 can be considered. For example, 6 Concluding remarks

This completes the proof of the theorem for \( n \geq 141 \).

\[
\begin{align*}
\leq c(K_{[n/2],[n/2]}) \cdot \left( 10^{-10} + \frac{8}{5} \cdot (0.976) \left( \frac{n^2}{n^2 - 1} \right) + \frac{2c^{2e+2}}{n} \right) \cdot \frac{1}{I_1(2)} \cdot 6.
\end{align*}
\]

Case 2: Let \( n \) be even and \( n \geq 142 \). Then by (18), the proof of Theorem 5.1 and by the result in Case 1,

\[
\begin{align*}
c(G) \leq & \left( \frac{4}{3} \cdot \frac{2.44}{I_0(2)} + 6 \cdot \frac{1.60067 \cdot \frac{[n/2]! [n/2]!}{2 [n/2]} \cdot I_0(2)}{I_0(2) [n/2]! [n/2]!} \right) c(K_{[n/2],[n/2]}),
\end{align*}
\]

\[
\begin{align*}
& \leq c(K_{[n/2],[n/2]}),
\end{align*}
\]

\[
\begin{align*}
& \leq c(K_{[n/2],[n/2]}),
\end{align*}
\]

\[
\begin{align*}
& \leq c(K_{[n/2],[n/2]}).
\end{align*}
\]

Returning to the case when \( n \) is odd,

\[
\begin{align*}
c(G) \leq & \left( \frac{10^{-10} + \frac{8}{5} \cdot (0.976) \left( \frac{n^2}{n^2 - 1} \right)}{I_1(2)} + \frac{2.302786 \cdot \frac{[n/2]! [n/2]!}{2 [n/2]} \cdot I_1(2)}{I_1(2) [n/2]! [n/2]!} \right) c(K_{[n/2],[n/2]}),
\end{align*}
\]

\[
\begin{align*}
& \leq c(K_{[n/2],[n/2]}),
\end{align*}
\]

\[
\begin{align*}
& \leq c(K_{[n/2],[n/2]}),
\end{align*}
\]

\[
\begin{align*}
& < c(K_{[n/2],[n/2]}).
\end{align*}
\]

This completes the proof of the theorem for \( n \geq 141 \).

6 Concluding remarks

A few natural extensions of Theorem 5.1 and Conjecture 1 can be considered. For example, for \( k > 1 \), what \( C_{2k+1} \)-free graphs have the most number of cycles? It is well known (see, e.g., [9] p. 150], [18], or [33]) that for \( n \) large enough, the unique \( C_{2k+1} \)-free \( n \)-vertex graph with the maximum number of edges is \( K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor} \). Also, Balister, Bollobás, Riordan, and Schelp [8] showed that for a fixed \( n \) and \( k \) and \( \frac{n}{2} < \Delta < n - k \), any maximal \( C_{2k+1} \)-free graph with maximum degree \( \Delta \) on \( n \) vertices is the complete bipartite graph \( K_{\Delta,n-\Delta} \). Since

\[
\begin{align*}
& \leq c(K_{[n/2],[n/2]}).
\end{align*}
\]
any $C_{2k+1}$-free graph with a maximum number of cycles is also edge-maximal, it might then seem reasonable to pose the following:

**Conjecture 2.** For any $k > 1$, if an $n$-vertex graph $C_{2k+1}$-free graph has the maximum number of cycles, then $G = \overline{K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}}$.

In support of Conjecture 2 by duplicating the proofs in this paper, it can be shown that the order of magnitude for edges in a $C_{2k+1}$-free graph is correct:

**Theorem 6.1.** For any $k \geq 2$, there exists a constant $\alpha_{2k+1} \geq 1$ so that if $G$ is a $C_{2k+1}$-free graph on $n$ vertices with the maximum number of cycles, then $c(G) \leq \alpha_{2k+1} \cdot c(\overline{K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}})$.

With considerably more work, it seems feasible that by refining methods in this paper, when $k = 2$, the constant $\alpha_5$ can be reduced to 1. However, for $k > 2$, our proof of Theorem 6.1 yields a constant $\alpha_{2k+1}$ that grows exponentially in $k$, so it seems that new techniques are required to settle Conjecture 2 for general $k$. In any case, for $k > 2$, to prove the general structure of a cycle-maximal $C_{2k+1}$-free graph seems beyond our reach at this moment.

One might entertain other questions related to Conjecture 1. For example:

**Question 1.** What is the maximum number of cycles in a graph on $n$ vertices with girth at least $g$?

The case $g = 3$ is trivial and this paper addresses this question for $g = 4$; however, there seems to be little known for $g \geq 5$. Another question that might be interesting is:

**Question 2.** For $k \geq 4$ what is the maximum number of cycles in a $K_k$-free graph on $n$ vertices? Could it be that the cycle-maximal $K_k$-free graphs are indeed Turán graphs?

A type of stability result also follows from the techniques given in this paper. Theorem 5.1 shows that among all triangle-free graphs with $n$ vertices and $m = \left\lfloor \frac{n^2}{4} \right\rfloor$ edges, $\overline{K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}}$ has the most number of cycles. Let $\ell = o(n)$, and set $m = \left\lfloor \frac{n^2}{4} \right\rfloor - \ell$. If $G$ has $n$ vertices and $m$ edges, and has the most number of cycles among all triangle-free $n$-vertex graphs with $m$ edges, then same argument as in the proof of Theorem 5.1 implies that $G$ is bipartite. By the maximality of the number of cycles, one can show that $G$ is a subgraph of $\overline{K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}}$.

For $14 \leq n \leq 140$, Conjecture 1 remains open. With a bit more care, it appears that with the techniques in this paper, one might be able to prove Conjecture 1 for the even $n$ to $n \geq 100$ or so, but the techniques used here do not seem to leave much room for the odd $n$. Skala [31] has suggested that Lemma 3.5 might be proved for graphs with slightly more vertices, and such an improvement might yield modest improvements for the bound on $n$ for which Theorem 5.1 holds.

Finally, during preparation of this paper, Alex Scott (Oxford) has informed us that he and his students have been working on Conjecture 1, and he reports that they had some success using the Regularity Lemma (which would give a value for $n_o$ much larger than 141), and that they are working on a different proof of a more general result.
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