\[ N = 1 \] geometries for M–theory and type IIA strings with fluxes

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ABSTRACT

We derive a set of necessary and sufficient conditions for obtaining \[ N = 1 \] backgrounds of M–theory and type IIA strings in the presence of fluxes. Our metrics are warped products of four–dimensional Minkowski space–time with a curved internal manifold. We classify the different solutions for irreducible internal manifolds as well as for manifolds with \( S^1 \) isometries by employing the formalism of group structures and intrinsic torsion. We provide examples within these various classes along with general techniques for their construction. In particular, we generalize the Hitchin flow equations so that one can explicitly build irreducible 7–manifolds with 4–form flux. We also show how several of the examples found in the literature fit in our framework and suggest possible generalizations.
1 Introduction

The need to connect ordinary four-dimensional physics with string theory or M-theory motivates the study of all types of solutions which can be described as a (possibly warped) product of four-dimensional Minkowski space-time with an internal 6- or 7-manifold. Although one would finally need to completely break supersymmetry, retaining some control on the effective theory suggests to look for solutions leading to $\mathcal{N} = 1$ supersymmetry in four-dimensions.

The simplest such scenario consists in a setup where all fields but the metric are vanishing. There, it is known that the resulting internal space must fall in Berger’s classification of special holonomy manifolds. Being more specific, in M-theory one uses 7-manifolds of $G_2$ holonomy, whereas in the context of the heterotic string theories one needs Calabi-Yau three-folds, whose holonomy is $SU(3)$. Although these solutions may be interesting phenomenologically, one can also consider the more general case where other fields besides the metric acquire non-vanishing expectation values. This is actually very natural in string theory, where it is known that D-branes couple to the various tensor fields appearing in the theory. Therefore, an obvious extension consists in the analysis of string- and M-theory vacua in the presence of non-trivial fluxes, i.e. non-vanishing expectation values for the tensor fields. Moreover, in recent times, the inclusion of fluxes has provided new insight in addressing the moduli problem (see for instance [1] and references therein) and in constructing potentials leading to $dS$ vacua.

Although the concept of holonomy is no longer a very useful tool for classifying these types of solutions, it can be shown that an analogous rôle is now played by group structures. It was already noted in the ‘80’s that the requirement of supersymmetric solutions implies the existence of tensor structures given by bilinears in the supersymmetry parameters [2]. One outstanding example is given by the Kähler structure $J_{m}^{n} = i \eta^{\dagger} \gamma_{m}^{n} \eta$ of Calabi-Yau manifolds where $\eta$ is the supersymmetry parameter. This fact was reconsidered in [3], where a more precise connection with the group structure of the solution was remarked. Indeed, if such tensors are globally defined, they imply a reduction of the structure group of the tangent bundle. As a consequence, the supersymmetry requirement on the solutions can be reinterpreted as a restriction on the possible group structures. As a final outcome, it is quite important to know and to classify group structures not only as a way to extract general information on the solutions but also in order to find techniques for their construction.

So far, the main effort in studying flux compactifications has been devoted to type IIB [4, 5, 6] and heterotic theories [7, 8, 9]. In these cases, it was shown that the internal manifold is no longer Calabi-Yau, but retains the property of being a complex manifold. On the other hand, not so much has been done for the type IIA theory, despite its prominent
role in connection with intersecting brane–world scenarios (see [10, 11, 12] and references therein). However, type IIA solutions can be obtained from circle reductions of M–theory and as such they appear indirectly in the analysis of all possible supersymmetric solutions of M–theory [13, 14]. Moreover, always in the context of M–theory, a classification of the solutions we seek in terms of $G_2$ structures has been given in [15, 16] and [18, 19, 20] analyzed the $SU(3)$ structures of six–dimensional manifolds that can be used for type IIA compactifications with two–form flux.

Despite these very general classifications, no explicit examples nor guideline for their construction were given in these papers. For this reason, we would like to take a more concrete approach and analyze once more M–theory in the presence of fluxes. In particular, by performing a systematic analysis of the various classes of group structures allowed by supersymmetry, we will finally be able to produce explicit examples. In doing so, we are not only going to discuss the purely eleven–dimensional backgrounds, but also the various possible IIA reductions.

An important mathematical result in this vein is that a spin seven–manifold admits always an $SU(2)$ structure [21]. Hence, it would seem natural at first sight to classify the supersymmetric solutions of M–theory according to their $SU(2)$ structure. However, this turns out to give a very complicated rewriting of the susy conditions that makes further analysis quite cumbersome. For this reason we choose to consider an intermediate setup and analyze $SU(3)$ structures. We will see that this analysis is fine enough to capture the main properties of the supersymmetric solutions and contains the guidelines for the construction of explicit examples.

By using this strategy, we find the necessary and sufficient conditions for obtaining supersymmetric solutions, in terms of restrictions on the 4–form flux and the intrinsic torsion of the internal manifold. We then use this result to study the two main classes of solutions: irreducible 7–manifolds and manifolds with an $S^1$ isometry.

Regarding the first class, we find two interesting results. First, it is possible to construct a generalization of the Hitchin flow equations [22]. In the same way as the Hitchin construction yields $G_2$–holonomy manifolds fibering half–flat 6–manifolds over an interval $I \subseteq \mathbb{R}$ [30], we construct 7–manifolds with the appropriate $SU(3)$ structure starting from special–hermitian manifolds. Second, by analyzing a more general setup in which the einbein over $I$ depends on the $M_6$ coordinates, we are able to recover the Fayyazuddin–Smith solution [23, 24, 25, 26].

Subsequently, we consider 7–manifolds with an $S^1$ isometry, where we can further distinguish two classes of solutions. These arise because of the presence of a vector $v$ in the definition of an $SU(3)$–structure in 7 dimensions. Then, one can distinguish between reduc-

\footnote{Solutions with a 3–dimensional Minkowski space–time were analyzed in [17].}
tions to type IIA where \( v \) is respectively proportional or orthogonal to the Killing vector describing the \( S^1 \) isometry along which we reduce M–theory. In the first instance we find that the type IIA theory can be described in terms of \( SU(3) \) classes in six dimensions. More importantly, it contains only the NS 3–form flux and therefore it gives rise to the known results of \([8, 9]\), where the common sector of type I, type II and heterotic string theories was analyzed. Conversely, when the isometry is “orthogonal” to \( v \), we can further refine the analysis according to the type of fluxes one obtains in 10 dimensions. An interesting result is that without 4–form flux and appropriately chosen 2–form and 3–form fluxes, one can use conformal Calabi–Yau manifolds to compactify type IIA string theory to four dimensions. As a final application we provide a technique to build vacua of type IIA with all fluxes and dilaton turned on starting from \( T^2 \) fibrations over \( K3 \) manifolds.

The plan of the paper is the following. After this introduction, in section 2 we review \( SU(3) \) and \( SU(2) \) structures for 6– and 7–dimensional manifolds and we describe how one can classify the various possibilities in terms of the irreducible modules of the intrinsic torsion. In section 3 we recall the conditions to obtain supersymmetric solutions of M–theory with non–vanishing 4–form flux and express these in terms of \( SU(3) \) structures, providing a set of necessary and sufficient conditions the flux and the internal manifold should obey. The construction of explicit examples and the description of general techniques to obtain them starts in section 4, where we analyze irreducible 7–manifolds. We show how to obtain them as fibration of 6–manifolds on real intervals giving generalizations to the Hitchin construction of \( G2 \)–holonomy manifolds and recovering the Fayyazuddin–Smith solution of M5–branes wrapped on holomorphic 2-cycles of the internal manifold. Finally, in section 5 we discuss the type IIA reduction for 7–manifolds admitting isometries, making contact with known results and discussing new possibilities arising from turning on all possible fluxes.

**Note added:** While this paper was under completion we were informed of the work by Behrndt and Jeschek \([27]\) which has some overlap with our section 4.1 and 5.1 and discusses also the superpotentials for M–theory with 4–form flux. A refined discussion of type II theories with NS–fluxes and the relation with mirror symmetry appeared in \([28]\).

2 Group structures and torsion classes

The existence of a \( G \)–structure on a \( d \)–dimensional Riemannian manifold implies that the structure group of the frame bundle can be reduced to \( G \subset O(d) \) (if the manifold is spin then \( G \subset Spin(d) \)). An alternative and sometimes more convenient way to define \( G \)–structures is via \( G \)–invariant tensors (spinors). A non–vanishing, globally defined tensor (spinor) \( \eta \) is \( G \)–invariant if it is invariant under \( G \)–rotations of the orthonormal frame. Since \( \eta \) is globally
defined, by considering the set of frames for which \( \eta \) takes the same form, one can see that the structure group of the frame bundle reduces to \( G \) or a subgroup thereof. Thus the existence of \( \eta \) implies a \( G \)–structure.

Typically, the converse is also true. Tensors of a given type, relative to an orthonormal frame, form a vector space, or module, for a given representation of \( O(d) \). If the structure group of the frame bundle is reduced to \( G \subset O(d) \), this module can be decomposed into irreducible modules of \( G \). If there are tensors admitting invariant components under \( G \), the corresponding vector bundle must be trivial, and thus it will admit a globally defined non–vanishing section \( \eta \).

The existence of a \( G \)–structure does not a priori put any constraints on the possible holonomy groups. In particular, the failure of the holonomy of the Levi-Civita connection to reduce to \( G \subset GL(n) \) is measured by the intrinsic torsion and this latter can be used to describe the \( G \)–structure. Given some \( G \)–invariant form \( \eta \) defining a \( G \)–structure, the derivative of \( \eta \) with respect to the Levi–Civita connection, \( \nabla \eta \), can be decomposed into \( G \)–modules. The different types of \( G \)–structures are then specified by which of these modules are present, if any. One first uses the fact that there is no obstruction to find some connection \( \nabla^{(T)} \) so that \( \nabla^{(T)} \eta = 0 \) \[29\]. Then \( \nabla^{(T)} - \nabla \) is a tensor which has values in \( \Lambda^1 \otimes \Lambda^2 \). Since \( \Lambda^2 \cong so(d) = g \oplus g^{\perp} \) where \( g^{\perp} \) is the orthogonal complement of the Lie algebra \( g \) in \( so(d) \), and \( \eta \) is invariant with respect to \( g \), we conclude that \( \nabla \eta = (\nabla - \nabla^{(T)}) \eta \) can be identified with an element \( \tau \) of \( \Lambda^1 \otimes g^{\perp} \). Furthermore, this element is a function only of the particular \( G \)–structure, independent of the choice of \( \nabla^{(T)} \) and it is in one-to-one correspondence with the intrinsic torsion. Explicitly, for a \( p \)–form \( \eta \)

\[
\nabla_m \eta_{n_1...n_p} = -p \tau^{[m} \eta_{q|n_1...n_2...n_p]}, \tag{2.1}
\]

where \( \tau \in \Lambda^1 \otimes g^{\perp} \), \( m \) is the one–form index and \( n, q \) label the two–form \( g^{\perp} \subset \Lambda^2 \).

The search for supersymmetric solutions of string and supergravity theories demands the existence of spinors which annihilate all the supersymmetry transformations. In geometrical terms, such spinors are parallel with respect to a generalized connection which include the Levi–Civita connection and the fluxes contributions:

\[
\nabla^{(T')} \eta = 0. \tag{2.2}
\]

This gives us the possibility of understanding whether a certain solution preserves supersymmetry or not by analyzing its group structure in terms of the intrinsic torsion. Indeed one needs its group–structure to be contained in those allowed by \[2.2\]

\[
\nabla^{(T)} \subseteq \nabla^{(T')} . \tag{2.3}
\]
It is therefore very important to express supersymmetry conditions as constraints on the intrinsic torsion and at the same time to classify the possible group–structures of the candidate solutions in terms of the irreducible components of the same intrinsic torsion. This is still not enough to satisfy the equations of motion, unless one requires maximal supersymmetry. As we will see later, only in certain favorable cases one can translate the extra conditions coming from such a requirement in terms of torsion classes. We will always try to achieve this, so that specifying the group structure is everything one needs in order to completely satisfy all the conditions.

So far we assumed that the supersymmetry parameter is a spinor and that in order to fulfill the supersymmetry conditions one has to use spin manifolds. Actually, in certain cases a weaker requirement can guarantee the existence of (locally) supersymmetric solutions. There are cases where a Spin\(_c\)–structure is enough. If this happens\(^2\), one can still use supersymmetry parameters to build tensors which, in general, are not globally defined but can be used to define a local group structure. Though this case looses interest from a mathematical point of view, it can still be very valuable for classifying and constructing locally supersymmetric solutions.

### 2.1 Static SU(3)–structures

Let us discuss first the case \( G = SU(3) \) for \( d = 6 \) and \( d = 7 \). The six–dimensional case is well known \[30\]. For \( d = 6 \), the generic structure group is \( SO(6) \simeq SU(4) \) and the decomposition of \( SU(4) \) irrepses under \( SU(3) \) gives

\[
\begin{align*}
4 & \rightarrow 1 + 3, \\
6 & \rightarrow 3 + 3, \\
10 & \rightarrow 1 + 3 + 6, \\
15 & \rightarrow 1 + 3 + 3 + 8. \\
\end{align*}
\tag{2.4}
\]

This implies the well–known fact that an \( SU(3) \) structure in six dimensions is specified by an almost complex structure \( J \) (and its associated 2–form) and an invariant complex 3–form \( \Psi \), which is of \( (3,0)\)–type with respect to \( J \). These are the \( SU(3) \) singlets of the corresponding 15 and 10 representations of \( SU(4) \). In addition, they satisfy the following compatibility relations

\[
\Psi \wedge J = 0, \quad \Psi \wedge \overline{\Psi} = -\frac{4i}{3} J \wedge J \wedge J. \tag{2.5}
\]

It is also worth noting that (2.4) implies the existence of two invariant spinors \( \eta_{\pm} \). From such spinors (that can be normalized to 1) one can build the invariant tensors \( J \) and \( \Psi \) by contractions with two and three gamma matrices respectively. Then the compatibility

\(^2\)We thank D. Martelli for explaining this to us.
relations (2.5) follow from the properties of gamma matrices and rearrangements using Fierz identities.

The different $SU(3)$ structures are then classified by the decomposition of the torsion $\tau$ into five complex modules

$$\tau \to (\mathbf{3} + \mathbf{\overline{3}}) \times (\mathbf{1} + \mathbf{3} + \mathbf{\overline{3}}) = (\mathbf{1} + \mathbf{1}) + (\mathbf{8} + \mathbf{\overline{8}}) + (\mathbf{6} + \mathbf{\overline{6}}) + (\mathbf{3} + \mathbf{\overline{3}}) + (\mathbf{3} + \mathbf{\overline{3}})$$

(2.6)

and these are completely determined by $dJ$ and $d\Psi$ in the following way

$$dJ = \frac{3}{4} i \left( \mathcal{W}_1 \overline{\Psi} - \overline{\mathcal{W}_1} \Psi \right) + \mathcal{W}_3 + J \wedge \mathcal{W}_4,$$

(2.7)

$$d\Psi = \mathcal{W}_1 J \wedge J + J \wedge \mathcal{W}_2 + \Psi \wedge \mathcal{W}_5,$$

(2.8)

where $J \wedge \mathcal{W}_3 = J \wedge J \wedge \mathcal{W}_2 = 0$ and $\Psi \wedge \mathcal{W}_3 = 0$. The fact that the $(2,2)$ piece of $d\Psi$ defines the same class as the $(0,3)$ piece of $dJ$ is a consequence of the first relation in (2.5).

Operatively one can obtain the various classes by proper contractions of $J$ and $\Psi$ with $dJ$ and $d\Psi$. For example,

$$\mathcal{W}_1 = -i \frac{3}{32} \overline{\Psi} \lrcorner \, dJ = \frac{1}{12} (J \wedge J) \lrcorner \, d\Psi,$$

(2.9)

$$\mathcal{W}_4 = \frac{1}{2} J \lrcorner \, dJ,$$

$$\mathcal{W}_5 = \frac{1}{4} \overline{\Psi} \lrcorner \, d\Psi.$$

We remind that a choice of $J$ and $\Psi$ fixes also the metric on the six–dimensional manifold and its orientation. Moreover, one can choose an orthonormal basis of $T^*$ such that $^3 J = e^{12} + e^{34} + e^{56}$ and $\Psi = (e^1 + i e^2) \wedge (e^3 + i e^4) \wedge (e^5 + i e^6)$.

The description of the seven dimensional case is a simple extension of the above. The decomposition of $SO(7)$ to $SU(3)$ gives

$$7 \rightarrow 1 + 3 + \mathbf{\overline{3}},$$

$$21 \rightarrow 1 + 2 \cdot 3 + 2 \cdot \mathbf{\overline{3}} + 8,$$

$$35 \rightarrow 3 \cdot 1 + 2 \cdot 3 + 2 \cdot \mathbf{\overline{3}} + 6 + \mathbf{6} + \mathbf{8}.$$ (2.10)

Therefore, the only difference with the six–dimensional case is the existence of a globally defined vector $^4 v$ (the extra singlet $3$–form is then $J \wedge v$) An $SU(3)$ structure in $d = 7$ is then described by a triplet $v, J, \Psi$, satisfying the compatibility relations (2.5) and, in addition,

$$v \lrcorner \, J = 0, \quad v \lrcorner \, \Psi = 0.$$ (2.11)

$^3$To avoid cluttering we use the notation $e^{i_1\cdots i_n} \equiv e^{i_1} \wedge \cdots \wedge e^{i_n}$.

$^4$We will use the same symbol for both the one-form and the dual vector. The precise identification should be clear from the context.
Again, to the 2–form $J$ one can associate a $(1,1)$–tensor, which now satisfies $J^a_b J^c_b = -\delta^c_a + v^a v^c$. One can then decompose the horizontal part of the forms according to their type with respect to this tensor. The vector $v$ allows also for the definition of an almost–product structure which, if integrable, implies that the metric of the seven–dimensional space can be written as $ds_7^2(x,t) = ds_6^2(x,t) + v \otimes v$ with $v = e^{\phi(x)} dt$.

In seven dimensions we have $T^*(M_7) \otimes SU(3)^\perp \sim (1 + 3 + \overline{3}) \otimes (1 + 2 \cdot 3 + 2 \cdot \overline{3})$ and therefore the decomposition of the torsion gives a total of 14 classes

$$\tau \rightarrow \begin{array}{c} 5 \cdot 1 + 4 \cdot (3 + \overline{3}) + 2 \cdot (6 + \overline{6}) + 4 \cdot 8, \\
R, C_{1,2} + V_{1,2}, W_{1,2} + S_{1,2} + A_{1,2}, T. \end{array}$$

(2.12)

Notice that $C_{1,2}$ and $T$ are complex.

Also in this case they can be read from the exterior differentials of the forms defining the structure:

$$dv = RJ + \overline{W}_{1} \cdot \Psi + W_{1} \cdot \overline{\Psi} + A_1 + v \wedge V_{1},$$

$$dJ = \frac{2i}{3}(C_1 \Psi - \overline{C}_1 \overline{\Psi}) + J \wedge V_2 + S_1$$

$$+ v \wedge \left[ \frac{1}{3}(C_2 + \overline{C}_2) J + \overline{W}_2 \cdot \Psi + W_2 \cdot \overline{\Psi} + A_2 \right],$$

$$d\Psi = C_1 J \wedge J + J \wedge T + \Psi \wedge V_3 + v \wedge (C_2 \Psi - 2J \wedge W_2 + S_2).$$

(2.13)

(2.14)

(2.15)

### 2.2 Static $SU(2)$–structures

The definition of $SU(2)$–structures in six and seven dimensions is a bit more involved, but can be obtained from the previous one by further decomposing the $SU(3)$ representations in terms of $SU(2)$ ones.

Using that $3 \rightarrow 1 + 2$, $6 \rightarrow 1 + 2 + 3$ and $8 \rightarrow 1 + 2 + 2 + 3$, it follows that an $SU(2)$ structure in six dimensions is specified by an invariant complex 1–form $w$, one invariant 2–form $J$, and one invariant complex 2–form $K$. All the extra singlets in the $SU(2)$ decomposition can be written in terms of these three objects. Compatibility of these forms now imposes that

$$K \wedge K = 0, \quad J \wedge K = 0, \quad K \wedge \overline{K} = 2J \wedge J,$$

(2.16)
as well as

$$w \wedge K = \overline{w} \wedge K = 0, \quad w \wedge J = \overline{w} \wedge J = 0.$$

(2.17)

In what follows we will also often further decompose real and imaginary parts as

$$K \equiv J_2 + i J_3, \quad w \equiv w_1 + i w_2,$$

(2.18)
and define $J_1 \equiv J$.

Locally one can introduce a frame such that:

$$w = e^5 + i e^6, \quad J = e^{12} + e^{34}, \quad K = (e^1 + i e^2) \wedge (e^3 + i e^4).$$

(2.19)

Notice that $J$ can be thought as an almost complex structure in the 4–dimensional part of the tangent bundle spanned by \{e\}, $i = 1, \ldots, 4$, with respect to which $J$ is of $(1, 1)$ and $K$ of $(2, 0)$ type. In the same way, the triplet of two–forms $J_i$ induce a triplet of almost–complex structures satisfying $J_i J_j = - \delta_{ij} + \epsilon_{ijk} J_k$.

Now $T^*(\mathcal{M}_6) \otimes SU(2) = (2 \cdot 1 + 2 \cdot 2) \otimes (4 \cdot 1 + 4 \cdot 2)$ and therefore the decomposition of the torsion gives a total of 20 classes

$$\tau \rightarrow \begin{array}{c}
16 \cdot 1 \\
16 \cdot 2 \\
8 \cdot 3 \\
S_{1, \ldots, 8} \\
V_{1, \ldots, 8} \\
T_{1, \ldots, 4}.
\end{array}$$

(2.20)

As usual, one can define these 20 classes from the exterior differentials on the forms defining the $SU(2)$ structure

$$dw = S_1 K + w \wedge V_1 + S_2 J + \overline{w} \wedge V_2 + S_3 w \wedge \overline{w} + T_1 + S_4 \overline{K},$$

$$dJ = S_5 (K \wedge w) + S_6 (K \wedge \overline{w}) + \frac{1}{2} (S_7 + S_8) J \wedge w + J \wedge V_4,$$

$$+ w \wedge \overline{w} \wedge V_5 + w \wedge T_2 + c.c$$

(2.21)

Here the torsion components satisfy the following consistency relations

$$J \wedge T_i = K \wedge T_i = 0, \quad K \wedge V_i = 0, \quad J \wedge J \wedge V_i = 0.$$  

(2.22)

The $SU(2)$ structures in seven dimensions are now straightforward to obtain. One has simply an extra globally defined vector $v$. To keep the notation compact we denote \{v, w_1, w_2\} collectively by $v^i$, with $i = 1, 2, 3$. The intrinsic torsions of the $SU(2)$ structure are then

$$dv^i = C^{ij} J^j + S^{ij} \epsilon_{jkl} v^k \wedge v^l + W^i \wedge v^j + T^i,$$

$$dJ^i = C^{ijk} J^j \wedge v^k + \epsilon_{ijk} J^j \wedge W^k + V^{ij} \wedge v^k \wedge v^j \epsilon_{jkl} + T^{ij} v^j,$$

(2.23)

where

$$C^{ijk} = \delta^{ijk} \hat{C}^k + \epsilon^{ijm} \hat{C}^m, \quad J^1 \wedge V^1 = J^2 \wedge V^2 = J^3 \wedge V^3,$$

(2.24)

due to the consistency conditions (2.16), to which one has to add $v \wedge J^i = 0$. The number of independent classes is 30 singlets, 15 doublets along with their conjugates and 30 triplets of $SU(2)$, exactly as expected from $T^*(\mathcal{M}_7) \otimes SU(2)^{\perp}$. 

8
3 Intrinsic torsion classes for M–theory with fluxes

In this section we derive the necessary and sufficient conditions for obtaining supersymmetric solutions of M–theory with fluxes. In doing so, we will show how to make contact with [16], where necessary conditions were found and discussed in terms of $G_2$–structures.

The gravitino variation of eleven–dimensional supergravity in the presence of a nontrivial 4–form flux $G = dC$ and with vanishing gravitino background values reads

$$\delta \Psi_A = \left\{ D_A[\omega] + \frac{1}{144} G_{BCDE} \left( \Gamma^{BCDE}_A - 8 \Gamma^{CDE} \eta^B_A \right) \right\} \epsilon,$$

where $\epsilon$ is a Majorana spinor in eleven dimensions and we have denoted flat indices by using letters from the beginning of the alphabet and curved ones using letters from the middle of the alphabet.

In what follows we consider warped compactifications to four–dimensional Minkowski space–time

$$ds_{11}^2 = e^{2\Delta} \eta_{\mu\nu} dx^\mu dx^\nu + ds_7^2,$$

where the warp factor depends only on the internal coordinates, $\Delta = \Delta(y^m)$. Now greek letters will be used for the 4d part and small latin letters for the internal manifold. Poincaré invariance of the four–dimensional part of the solution allows a non-zero four–form flux $G$ only on the internal manifold and depending only on the internal coordinates. We are not going to discuss modifications due to a non–trivial cosmological constant in space–time, since this is a simple extension [16] [15].

The decomposition for the eleven–dimensional $\gamma$–matrices is the standard one

$$\Gamma^a = \gamma^a \otimes I, \quad \Gamma^a = \gamma^{(5)} \otimes \gamma^a,$$

where $\gamma^{(5)} = i\gamma^1 \gamma^2 \gamma^3 \gamma^4$ is the four–dimensional chirality operator. A useful choice for these matrices is given by the Majorana representation. In this representation the $\gamma$–matrices are either purely real ($\gamma^a$) or purely imaginary ($\gamma^{(5)}$ and $\gamma^a$) and the Majorana condition on $\epsilon$ reduces to the reality constraint $\epsilon^* = \epsilon$. One can therefore split the supersymmetry parameter as

$$\epsilon = \psi_+ \otimes \eta_+ + \psi_- \otimes \eta_-$$

where $\eta_\pm$ depend only on the internal coordinates and the $\pm$ label refers to the chirality of the four–dimensional part. The Majorana constraint on $\epsilon$ then requires $(\psi_\pm)^* = \psi_\mp$ and $(\eta_\pm)^* = \eta_\mp$.

The gravitino variation [3.1] leads to the following supersymmetry constraints on the internal spinors:

$$\left[ \pm \frac{1}{2} (\partial_c \Delta) \gamma^c + \frac{1}{144} G_{bcde} \gamma^{bcde} \right] \eta_\pm = 0,$$

where $\epsilon$ is a Majorana spinor in eleven dimensions and we have denoted flat indices by using letters from the beginning of the alphabet and curved ones using letters from the middle of the alphabet.
from the space–time part $\alpha = 0, \ldots, 3$ and
\[
D_\alpha [\omega] \eta_\pm = \mp \frac{1}{144} (G_{bcde} \gamma^{bcde} a - 8 G_{abcd} \gamma^{bcd}) \eta_\pm \\
= \pm \left( \frac{i}{12} (*G)_{abc} \gamma^{bc} + \frac{1}{18} G_{abcd} \gamma^{bcd} \right) \eta_\pm
\]
(3.6)

from the internal part $a = 4, \ldots, 10$. We have defined $(*G)_{abc} \equiv \frac{1}{4!} \epsilon_{abcdefg} G_{defg}$.

The existence of $\eta_\pm$ implies the definition of an $SU(3)$ structure since they are in one to one correspondence with the two singlets of the decomposition of the fundamental representation of $Spin(7)$. In order to discuss the $SU(3)$ structure as in section 2.1, one first needs to normalize them properly. This normalization can be read from (3.6) by considering the contraction of that equation with $\eta_\pm^\dagger$. Defining $\Xi \equiv \eta_\pm^\dagger \eta_\pm = \eta_\pm^\dagger - \eta_\pm^\dagger \eta_\pm$, one obtains
\[
d (e^{-\Delta} \Xi) = 0,
\]
which implies that a good normalization is given by $\Xi = e^\Delta$.

The three tensors defining the $SU(3)$ structure are then obtained from
\[
v_a = e^{-\Delta} \eta_\pm^\dagger \gamma_a \eta_+ , \\
J_{ab} = -i e^{-\Delta} \eta_\pm^\dagger \gamma_{ab} \eta_+ , \\
\Psi_{abc} = -i e^{-\Delta} \eta_\pm^\dagger \gamma_{abc} \eta_+ .
\]
(3.8)

By using the gamma–matrices relations, one can indeed check that such tensors (and the corresponding forms) satisfy the requirements of section 2.1. Moreover, it can be shown that these are the only independent contractions of the gamma–matrices with the $\eta_\pm$ spinors since
\[
\eta_\pm^\dagger \gamma^{[n]} \eta_+ = 0 , \quad \text{for } n = 0, 1, 2 ,
\]
(3.9)
and the rest follow from duality relations.

We are now ready to interpret the supersymmetry conditions (3.5) and (3.6) as conditions on the $SU(3)$ structure of the internal manifold defined by (3.8). In order to do so, one considers the contraction of these two equations with the full basis of tensor–spinors constructed from $\eta_\pm^\dagger \gamma^{[n]}$. Then, using some gamma–algebra and the definitions of the $SU(3)$ structure tensors, the independent conditions will be summarized as constraints on the torsion classes and on the allowed fluxes. The last ingredient one needs is the decomposition

\footnote{It can be shown that $\eta_\pm^\dagger \eta_+$ vanishes because $\eta_\pm$ are orthogonal. If this does not happen then one has just one independent supersymmetry parameter and non–trivial 4–form flux necessarily curves the space–time.}
of the four–form flux in terms of irreducible $SU(3)$ representations. Since it is a four–form it decomposes as $35 \rightarrow 3 \cdot 1 \oplus (3 \oplus \overline{3}) \oplus (6 \oplus \overline{6}) \oplus 8$ and one can therefore write

$$G = -\frac{Q}{6} J \wedge J + J \wedge A + \psi_+ \wedge V + v \wedge (c_1 \psi_+ + c_2 \psi_+ + J \wedge W + U). \quad (3.10)$$

Here the first term was normalized in a way that the singlet $Q$ corresponds to the same one shown in [16] in the $G_2$ decomposition of $G$ for supersymmetric configurations. The same thing can be done for the dual, which then reads\(^6\)

$$* G = \frac{Q}{3} J \wedge v + v \wedge A - J \wedge (W \cdot J) + S + c'_1 \psi_+ + c'_2 \psi_+ + v \wedge (V \cdot \psi_+). \quad (3.11)$$

We emphasize that $*$ denotes the 7-dimensional Hodge duality operator.

Since $A$, $S$, $V$, $U$ and $W$ are horizontal with respect to $v$, one can decompose them according to their type with respect to $J$. For instance $A$ is a primitive (1,1)–form, $S$ is a primitive (2,1)–form plus its complex conjugate and so on. For future reference, we also give the definition of some components of the flux as contractions with the structure tensors

$$Q = -\frac{1}{2} (J \wedge J) \cdot G, \quad W = \frac{1}{2} (J \wedge v) \cdot G, \quad A = J \cdot G + \frac{2}{3} Q J - 2 v \wedge W, \quad U = v \cdot G - J \wedge W. \quad (3.12)$$

The first condition following from supersymmetry, namely (3.5), does not contain derivatives of the spinors and therefore will be realized as simple constraints on the flux and an equation for the warp factor. The independent conditions on the flux are

$$\Psi \cdot G = 0 = \nabla \cdot G, \quad (3.13)$$

which remove two singlets and one vector from (3.10) and (3.11). The other independent conditions can be written as an equation for the warp factor

$$d\Delta = -\frac{1}{3} Q v + \sigma, \quad (3.14)$$

and a further relation between $\sigma$ and the remaining one–form in the fluxes:

$$\sigma = \frac{2}{3} W \cdot J. \quad (3.15)$$

It should be noted that there are no constraints at all concerning the primitive (1,1)–form $A$ and the (2,1)–form $S$.

\(^6\)To be consistent with [16] and in order to compare results, we had to choose the volume form as $\frac{1}{6} J \wedge J \wedge J \wedge v$, so that for instance $*J = -\frac{1}{3} J \wedge J \wedge v$. 

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After some tedious calculations one can also obtain differential conditions on the $SU(3)$ structures by using (3.16). The resulting conditions can be summarized in the following concise expressions\footnote{The relations and the above conditions on the flux are in agreement with [10], but for a typo in their eq. (3.19) where the r.h.s. should have a factor of 2. The conditions on the flux also give naturally $Q_{ab}v^b = 0$ and not the weaker condition $Q_{ab}v^a v^b = 0$, but, as noted also by A. Tomasiello, using the expression for $Q_{ab}$ in terms of its irreducible components and the other conditions on the flux one can show that they are equivalent.}:

\begin{align*}
    dv &= 2v \wedge d\Delta, \quad (3.16) \\
    dJ &= -4J \wedge d\Delta - 2*G, \quad (3.17) \\
    d\Psi &= 3\Psi \wedge d\Delta. \quad (3.18)
\end{align*}

From (3.17) we conclude that $J$ is a generalized calibration (c.f. section 4.1.2) of the 7-dimensional internal space.

Using the expression for the dual flux (3.11), the equation for the warp factor (3.14) and the relation (3.15) one can further rewrite (3.16)–(3.18) completely in terms of the $G$–flux components. In detail, it is easy to check that

\begin{align*}
    dv &= 2v \wedge \sigma, \quad (3.19) \\
    dJ &= \frac{2}{3}Qv \wedge J - 2S - 2v \wedge A - J \wedge \sigma, \quad (3.20) \\
    d\Psi &= -Qv \wedge v + 3\Psi \wedge \sigma. \quad (3.21)
\end{align*}

A comparison with (2.13)–(2.15) now yields

\[ R = C_1 = W_1 = W_2 = A_1 = T = S_1 = 0, \quad (3.22) \]

and the following identifications

\begin{align*}
    C_2 &= \overline{C}_2 = Q, \quad V_1 = \frac{2}{3}V_3 = \sigma, \quad (3.23) \\
    A_2 &= -A, \quad S_1 = -2S.
\end{align*}

It should also be noted that consistency requires some differential constraints on these torsion components:

\begin{align*}
    3d\sigma &= dQ \wedge v + 2Qv \wedge \sigma, \quad (3.24) \\
    v \wedge \Psi \wedge (dQ - 2Q\sigma) &= 0, \quad (3.25) \\
    3d\sigma \wedge J - 6dS + 6v \wedge dA &= -4Qv \wedge S + 6v \wedge \sigma \wedge A - 6S \wedge \sigma. \quad (3.26)
\end{align*}
Supersymmetric backgrounds for M–theory with fluxes can now be constructed by choosing seven–dimensional manifolds with an $SU(3)$ structure whose torsion classes are of the form (3.23). Of course, one has to check the equations of motion for the four–form field $G$ and the metric, and also the Bianchi identity for $G$. However, as proven in [13], if one solves the supersymmetry conditions, the Bianchi identity for $G$ and its equation of motion, the Einstein equation is identically satisfied. Let us then analyze these conditions.

The equation of motion for the four–form can be written in terms of differential forms as

$$d (\ast_{11} G) = G \wedge G, \quad (3.27)$$

where the Hodge dual is taken with respect to the full eleven–dimensional metric. Since we are considering configurations admitting only a non–vanishing expectation value for $G$ in the internal space, one can rewrite (3.27) as

$$d (\ast e^{4\Delta} G) = 0, \quad (3.28)$$

where now the Hodge dual is taken only with respect to the internal space and the term $G \wedge G$ is vanishing because it is an 8–form in 7–dimensions. It is then straightforward to check that such an equation is identically satisfied for our backgrounds [31], as (3.20) implies that $\ast e^{4\Delta} G = -\frac{1}{2} d (e^{4\Delta} J)$. This finally means that the complete equations of motion are satisfied once the supersymmetry conditions (3.19)–(3.21) are fulfilled along with the source-free Bianchi identity $dG = 0$. We should mention that we consider for the moment the five–brane sources.

In conclusion, in order to get a complete set of necessary and sufficient conditions for supersymmetric solutions of M–theory with fluxes, one just needs to understand the 4–form Bianchi identity. Assuming that there are no sources, one can obtain differential conditions on the flux components from the $SU(3)$ decomposition of the $G$–flux by requiring that $dG = 0$. These conditions read

$$\frac{dQ}{6} \wedge J \wedge J + \frac{2}{9} Q^2 J \wedge J \wedge v - \frac{4}{3} Q v \wedge J \wedge A - \frac{Q}{3} J \wedge J \wedge \sigma + 2 S \wedge A + 2 v \wedge A \wedge A$$

$$+ J \wedge \sigma \wedge A - 3 v \wedge \sigma \wedge J \wedge W - 2 v \wedge S \wedge W + v \wedge J \wedge dW + d \ast S = 0. \quad (3.29)$$

As it is obvious, this is not a simple restriction on the torsion classes, but it imposes rather non–trivial differential relations. We will see in the construction of explicit examples that such relation can result in reasonable restrictions on the fluxes and the geometry and that it can be identically satisfied in certain cases. Of course, a non-vanishing $dG$ will not be disastrous, provided that it can be interpreted as due to the presence of 5–brane sources.
4 Irreducible 7–manifolds

The first type of solutions we want to discuss are 7–manifolds $\mathcal{M}_7$ which do not admit any isometries and therefore we will call them irreducible. In detail, we are going to focus on fibered products of 6–manifolds $\mathcal{M}_6$ with an interval $I \subset \mathbb{R}$. Moreover, we assume $\mathcal{M}_6$ allows for an $SU(3)$ structure $\{\hat{J}, \hat{\Psi}\}$ which will also be fibered over $I$. This means that there is a naturally induced $SU(3)$–structure on the 7–dimensional manifold given by $\{v = e^{q\phi}dt, J = \hat{J}(t), \Psi = \hat{\Psi}(t)\}$, where $t$ is the variable parameterizing $I$.

For the sake of clarity we discuss the case $q = 0$ separately than the general case; as we will see this distinction is actually physically relevant, in the sense that the two classes of solutions we obtain admit different physical interpretations. Moreover, the case $q = 0$ provides a direct generalization of the Hitchin construction of $G_2$–holonomy manifolds.

4.1 Case I: $q = 0$

In this case, the metric of the 7–dimensional spaces reads

$$ds^2_7(y, t) = ds^2_6(y, t) + dt^2. \quad (4.1)$$

As previously assumed, at any given $t$, the 6–dimensional manifold has an $SU(3)$ structure. This cannot be generic if we require a supersymmetric solution of M–theory. In detail, its structure should follow from the restriction of (3.14)–(3.15) for a given $t$. In order to understand this, it is useful to split the 7–dimensional differential $d$ as $d = \hat{d} + dt \frac{\partial}{\partial t}$, $\hat{d}$ being the 6–dimensional one. Using this fact and the above definition of the 7–dimensional $SU(3)$ structure we can finally provide the 6–dimensional structure and a set of differential equations in $t$ which describe the fibration of this structure over $t$. Before giving these conditions explicitly let us note that since $dv = 0$ by construction, the vector component of the flux has to vanish: $\sigma = 0$.

The conditions on $\mathcal{M}_6$ read

$$\begin{cases}
\hat{d}J = -2S, \\
\hat{d}\Psi = 0,
\end{cases} \quad (4.2)$$

and this means that $\mathcal{M}_6$ is a special–hermitian manifold, i.e. it is a complex, non–Kähler manifold with $\hat{d}J \neq 0$ but $\hat{d}(J \wedge J) = 0$. Given such type of manifolds, one can build a 7–manifold that can be used in a flux solution of M–theory, by solving the following first order differential equations in $t$:

$$\begin{cases}
\frac{\partial J}{\partial t} = \frac{2}{3} Q J - 2A, \\
\frac{\partial \Psi}{\partial t} = Q \Psi.
\end{cases} \quad (4.3)$$
This construction mimics the Hitchin construction of $G_2$–holonomy manifolds \[22\] as presented in \[30\]. In that case the base is a half–flat manifold\(^8\) and the flow equations are \(\partial_t \psi_+ = \tilde{d} J, \partial_t (J \wedge J) = -2 \tilde{d} \psi_-\), where the split \(\Psi = \psi_+ + i \psi_-\) is used. It should also be noted that (4.3) preserve the compatibility relations \(\tilde{\Psi} \wedge \tilde{J} = 0\) and \(\tilde{\Psi} \wedge \tilde{\Psi} = -\frac{4i}{3} \tilde{J} \wedge \tilde{J} \wedge \tilde{J}\).

It is tempting to conjecture that for the common class of 6–dimensional base spaces given by special–hermitian manifolds the two pictures are related. The idea would be that by turning on an appropriate flux on 3–cycles of \(M_6\) one could obtain a deformation of its fibration, described by the set of equations just presented. We present a very simple example along these lines in the next subsection, but it would be interesting to make this idea more precise.

Given this general construction, let us now present some explicit examples in order to illustrate better the procedure one has to follow. We stress that even though we specify the flux by solving the conditions rather than trying to obtain solutions for a given flux, the procedure is general.

4.1.1 Calabi–Yau base

As a first instance of special hermitian 6–manifolds we analyze the case \(S = 0\), i.e. \(M_6\) is Calabi–Yau (\(\tilde{d} \tilde{J} = 0 = \tilde{d} \tilde{\Psi}\)). In order to further simplify the setup, we also demand that the flux depends only on \(t\) and therefore \(\tilde{d} Q = 0\). From the first flow equation follows \(\tilde{d} A = 0\) and the Bianchi identity (3.29) for \(G\) becomes

\[-\frac{1}{6} \left( \dot{Q} + \frac{4}{3} Q^2 \right) J \wedge J + \frac{2}{3} Q J \wedge A - 2 A \wedge A + J \wedge \dot{A} = 0,\]

which can be easily solved by setting \(A = 0\). In this case one has to satisfy \(\dot{Q} + \frac{4}{3} Q^2 = 0\), which specifies

\(Q(t) = \frac{3q_0}{3 + 4q_0 t},\)

where \(q_0\) will be associated with the “charge” of the solution. With this information, the warp factor is also completely determined by (3.14)

\(e^{\Delta(t)} = \left( 1 + \frac{4q_0}{3} t \right)^{-\frac{1}{4}},\)

where we chose \(\Delta(0) = 0\) for simplicity.

The \(SU(3)\) structure in 7 dimensions is now easily fixed by using (4.3). All in all, the \(t\)–dependence results in an overall factor in front of the 6–dimensional structure:

\(J(t) = \left( 1 + \frac{4q_0}{3} t \right)^{\frac{1}{2}} \tilde{J},\)

\(^8\)Recall that a 6-dimensional manifold with an \(SU(3)\) structure is called half-flat when \(d \psi_+ = 0\) and \(J \wedge dJ = 0\).
\[ \Psi(t) = \left(1 + \frac{4q_0}{3} t\right)^{\frac{4}{3}} \hat{\Psi}. \] (4.8)

The full metric of 11–dimensional supergravity is then given by

\[ ds_{11}^2(x,y,t) = \frac{1}{\sqrt{1 + \frac{4q_0}{3} t}} \eta_{\mu\nu} dx^\mu dx^\nu + \sqrt{1 + \frac{4q_0}{3} t} \, ds_{CY}^2(y) + dt^2 \] (4.9)

where \( ds_{CY}^2(y) \) is a 6-dimensional Calabi–Yau metric.

The flux is completely determined by its singlet component \( Q \) and it reads

\[ G = -\frac{Q}{6} J \wedge J = -\frac{q_0}{2} \hat{J} \wedge \hat{J}. \] (4.10)

One can therefore interpret this solution as a deformation of the direct product of Calabi–Yau manifolds with \( I \), which corresponds to \( q_0 = 0 \), when a non–trivial flux of strength \( q_0 \) is turned on the 4–cycles of the Calabi–Yau.

Such a solution can be employed as a genuine heterotic–M–theory background once the space–time fields satisfy appropriate conditions at the boundaries of \( I \). The metric (4.9) corresponds to an isotropic deformation of the CY metric over the interval \( I \). It would be very interesting to construct anisotropic solutions by allowing for a non–zero \( A \).

4.1.2 M5–branes in special–hermitian manifolds

We now discuss generic flux configurations based on special–hermitian manifolds with M5–branes; as a concrete application we analyze the case of M5–branes wrapping 2–cycles of the Iwasawa manifold.

Our starting point is arbitrary fibrations of special–hermitian manifolds \( M_6 \) over the interval \( I \), with fluxes satisfying (4.2) and (4.3)

\[ S = -\frac{1}{2} \hat{d} J \] (4.11)
\[ A = \frac{1}{3} Q J - \frac{1}{2} \hat{J}. \] (4.12)

Notice that this choice implies identically the constraint \( \hat{J} \wedge \hat{J} \wedge A = 0 \). In addition, the exterior differential of \( G \) reads

\[ dG = -\frac{1}{6} \left( \dot{Q} + \frac{4}{3} Q^2 \right) J \wedge J \wedge dt + \frac{4}{9} Q J \wedge A \wedge dt - 2 A \wedge A \wedge dt + J \wedge \ddot{A} \wedge dt + \hat{d}(\ast S), \] (4.13)

which will not vanish in general. Hence, most of these solutions will be interpreted as M5–branes wrapping 2–cycles in \( M_7 \).
Before we discuss wrapped M5-branes it is useful to review first some facts about generalized calibrations. Recall that a generalized calibration of degree $k$ in a manifold $\mathcal{M}$ is a $k$-form $\phi$ such that its pull-back on any $k$-dimensional submanifold $\Sigma$ of $\mathcal{M}$ is less than or equal to the corresponding volume form, i.e.

$$\phi^* \leq \text{vol}_\Sigma.$$  \hfill (4.14)

As opposed however to standard calibrations, $\phi$ does not have to be a closed form. A submanifold $\mathcal{N}$ of $\mathcal{M}$ is called calibrated with respect to $\phi$ if $\phi^* = \text{vol}_\mathcal{N}$.

The relevance of generalized calibrations to branes in curved backgrounds with fluxes is due to the fact that submanifolds calibrated by $\phi$ minimize functionals of the form $E(\Sigma) = \int_\Sigma (\text{vol}_\Sigma - A)$, with $A$ a $k$-form such that $d\phi = dA$. These functionals correspond precisely to the energy\(^9\) of a $p$-brane wrapping a $k$-dimensional submanifold of spacetime in the presence of background fluxes; the first part is the standard worldvolume contribution, while the second originates from the Wess-Zumino term in the brane action which couples the brane with the appropriate spacetime form–field. The calibrated submanifold minimizes $E(\Sigma)$ among the homology class of $\Sigma$. Notice however that unlike the case of standard calibrations, the cycles corresponding to generalized calibrations may also be topologically trivial. For more details on generalized calibrations consult [33, 34, 35, 36].

In general, branes wrapping calibrated submanifolds are BPS and they preserve some supersymmetry. The relevant conditions on the fluxes are precisely of the form $d\phi = dA$, where $\phi$ is a purely geometrical object, and they determine the corresponding generalized calibration. For example, in the situation described above we are interested in M5–branes wrapping a 2-dimensional submanifold of $\mathcal{M}_7$ in the presence of 4-form flux $G$. Since the M5–branes couple magnetically to $G$, the generalized calibration satisfies

$$d\phi = *_{11}G,$$  \hfill (4.15)

which, compared to the condition (5.17) coming from supersymmetry, implies that

$$\phi = -\frac{1}{2} e^{4\Delta} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge J.$$  \hfill (4.16)

This is expected since the full calibrating form should consist of the volume form of the flat part of the brane\(^{10}\) along with a (generalized) calibration of degree 2 given by $J$. The fact that $J$ is indeed a generalized calibration is just a fiber-wise application of the fact that

---

\(^9\)When $k < p$ we should actually talk about the energy density of the wrapped brane since the energy of an infinitely extended brane is infinite.

\(^{10}\)Notice that due to the warp factor that volume form of the flat 4-dimensional part is actually $e^{4\Delta} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$. 

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\( \hat{J} \) is a generalized calibration on any almost-hermitian (and consequently on any special-hermitian) manifold \cite{36}. Notice that the above calibrating form is the most general possible, but in actual examples one can consider its restriction to a specific configuration of calibrated cycles.

It is important to mention that a priori the flux \( G \) may have nothing to do with the 5–branes, i.e. can be thought of as a background 4-form flux. If we want however to interpret our solution as a configuration of wrapped M5–branes with no other background fluxes, we have to identify \( G \) with the flux due to the branes. Then we obtain a further relation between the calibration and the flux, this time coming from the Bianchi identity with magnetic sources

\[
dG = \ast_{11}J_6, \tag{4.17}
\]

with \( J_6 \) a 6-form specifying where the 5–branes are located in the transverse 5–dimensional space and how they are oriented.

We present now a concrete example of this type based on the Iwasawa manifold \cite{37,38}. This manifold is a \( T^2 \) fibration over \( T^2 \times T^2 \) and its 6–dimensional orthonormal frame \( e^i, i = 1, \ldots, 6 \) satisfies

\[
\begin{align*}
  de^i &= 0, \quad i = 1, \ldots, 4, \\
  de^5 &= e^{13} - e^{24}, \\
  de^6 &= e^{14} + e^{23}.
\end{align*} \tag{4.18}
\]

In terms of complex coordinates we have

\[
dz = e^1 + ie^2, \quad dv = e^3 + ie^4, \quad -du + zdv = e^5 + ie^6. \tag{4.19}
\]

Given \( \hat{J} \) the canonical choice of complex structure on the Iwasawa manifold, i.e.

\[
\hat{J} \equiv e^{12} + e^{34} + e^{56} = \frac{i}{2} (dz \wedge \bar{dz} + dv \wedge \bar{dv} + (-du + zdv) \wedge (-d\bar{u} + \bar{z}d\bar{v})) \tag{4.20},
\]

it can be checked that

\[
\hat{d}\hat{J} = (e^{136} - e^{246} - e^{145} - e^{235}). \tag{4.21}
\]

We are therefore interested in fibrations of such a manifold with a real interval \( I \). The uplift of the complex structure \( \hat{J} \) to the full 7–dimensional space determines also the metric of such space and can be rather general.

As a working assumption, we discuss the case of simple size deformations:

\[
J = e^{2a(t)}e^{12} + e^{2b(t)}e^{34} + e^{2c(t)}e^{56}, \tag{4.22}
\]

with \( a(t), b(t), c(t) \) being arbitrary for the moment. Since the general solution of \cite{43} for \( \Psi \) takes the form

\[
\Psi(t) = e^{\int_{t_0}^t Q(t')dt'} \hat{\Psi} \tag{4.23}
\]

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and \( \tilde{\Psi} \wedge \tilde{\Psi} = -\frac{4}{3} \tilde{J} \wedge \tilde{J} \wedge \tilde{J} \), we obtain a relation between the singlet component of the flux and the size deformations of the metric

\[
Q(t) = \dot{a}(t) + \dot{b}(t) + \dot{c}(t).
\]  

(4.24)

In this way the conditions on the base \( \text{(4.2)} \) and the second equation in \( \text{(4.3)} \) are satisfied by construction. In order to fulfill the first flow equation \( \text{(4.3)} \), we can fix the \( A \) component of the flux as

\[
A = \frac{1}{3} e^{2a(t)} (-2 \dot{a} + \dot{b} + \dot{c}) e^{12} + \frac{1}{3} e^{2b(t)} (-2 \dot{b} + \dot{c} + \dot{a}) e^{34} + \frac{1}{3} e^{2c(t)} (-2 \dot{c} + \dot{a} + \dot{b}) e^{56}.
\]  

(4.25)

Furthermore \( S \) can be computed from \( \text{(4.2)} \):

\[
S = -\frac{1}{2} e^{2c} (e^{136} - e^{246} - e^{145} - e^{235}).
\]  

(4.26)

In order to have a full solution we need to solve the Bianchi identity for \( G \). This reads

\[
dG = \frac{1}{3} e^{2a+2b} (-2 \ddot{a} - 2 \ddot{b} + \ddot{c} - 4 \dot{a}^2 - 4 \dot{b}^2 + 2 \dot{b} \dot{c} + 2 \dot{a} (-4 \dot{b} + \dot{c})) e^{1234} \wedge \nu
\]

\[
+ \frac{1}{3} e^{2b+2c} (-2 \ddot{b} - 2 \ddot{c} + \ddot{a} - 4 \dot{b}^2 - 4 \dot{c}^2 + 2 \dot{c} \dot{a} + 2 \dot{b} (-4 \dot{c} + \dot{a})) e^{3456} \wedge \nu
\]

\[
+ \frac{1}{3} e^{2c+2a} (-2 \ddot{c} - 2 \ddot{a} + \ddot{b} - 4 \dot{c}^2 - 4 \dot{a}^2 + 2 \dot{a} \dot{b} + 2 \dot{c} (-4 \dot{a} + \dot{b})) e^{5612} \wedge \nu - 2 e^{2c} e^{1234} \wedge \nu,
\]  

(4.27)

where we have used \( \text{(4.24)} \). The magnetic 6-form current is

\[
\mathcal{J}_6 = e^{45} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge (\rho_{12}(t) e^{12} + \rho_{34}(t) e^{34} + \rho_{56}(t) e^{56})
\]  

(4.28)

where

\[
\rho_{12}(t) = \frac{1}{3} e^{2b+2c} (-2 \ddot{b} - 2 \ddot{c} + \ddot{a} - 4 \dot{b}^2 - 4 \dot{c}^2 + 2 \dot{c} \dot{a} + 2 \dot{b} (-4 \dot{c} + \dot{a}))
\]

\[
\rho_{34}(t) = \frac{1}{3} e^{2c+2a} (-2 \ddot{c} - 2 \ddot{a} + \ddot{b} - 4 \dot{c}^2 - 4 \dot{a}^2 + 2 \dot{a} \dot{b} + 2 \dot{c} (-4 \dot{a} + \dot{b}))
\]  

(4.29)

\[
\rho_{56}(t) = \frac{1}{3} e^{2a+2b} (-2 \ddot{a} - 2 \ddot{b} + \ddot{c} - 4 \dot{a}^2 - 4 \dot{b}^2 + 2 \dot{b} \dot{c} + 2 \dot{a} (-4 \dot{b} + \dot{c})) - 2 e^{2c}.
\]

One can first look for solutions with no sources, i.e. \( dG = 0 \) or equivalently \( \rho_{12}(t) = \rho_{34}(t) = \rho_{56}(t) = 0 \). Analyzing however the resulting system of differential equations seems quite difficult in practice and we postpone this for future work. Hence, we are left with the possibility of interpreting the non-zero piece of \( dG \) as due to M5–branes wrapping 2–cycles in the internal 7-dimensional manifold. Since this manifold is a fibration of the Iwasawa manifold over an interval, we can actually imagine that our solution corresponds to a fibration of a configuration of wrapped M5–branes on calibrated 2-cycles of the Iwasawa.
Since \( \hat{J} \) is a generalized calibration on the Iwasawa, we conclude that branes wrapping the holomorphic 2-cycles \( u = v = 0 \) and \( z = u = 0 \) are BPS. We can obtain such an interpretation of our example if we demand \( \varrho_{56}(t) = 0 \). A simple way to solve this equation is to set \( a = b = 0 \). Then \( c(t) = -\log(\sqrt{6t}) \) and \( \varrho_{12}(t) = \varrho_{34}(t) = -1/3t^4 \) give the density of wrapped M5–branes in the transverse space. Notice that the 5–branes are distributed uniformly on the Iwasawa manifold and the only non-trivial part of the solution comes from the fibration over \( I \). Finally, to specify completely the solution we present the 11-dimensional metric:

\[
ds_{11}(x,t) = t^{2/3} \eta_{\mu\nu} dx^\mu dx^\nu + \left( dzd\bar{z} + dv\bar{v} + \frac{1}{6t^2}(-du + zd\bar{v})(-d\bar{u} + zdv) \right) + dt^2. \tag{4.30}
\]

Notice that since all holomorphic 2–cycles in a complex manifold are calibrated, one can seek more solutions. However, our purpose here was to illustrate the general technique and to discuss a simple example.

### 4.1.3 Generalization of Strong Kähler Torsion (SKT) manifolds

For the case of zero flux in the singlet component of \( G \), the special hermitian manifolds one has to use as base of the full 7–dimensional spaces have to satisfy an interesting relation. Since \( Q = 0 \), from (4.2) and (4.3) it follows that \( \hat{d}A = \hat{S} \). Using the flow equations and the fact that \( J \wedge S = 0 \) we find that the source-free Bianchi identity becomes

\[
\hat{d} \ast \hat{d}J = \frac{1}{2} \frac{d^2}{dt^2} (J \wedge J) \wedge v. \tag{4.31}
\]

A simple way to solve this equation is to assume a \( t\)–dependence of the form \( J(t) = e^{\frac{m}{2t}\hat{t}} \hat{J} \). Then we get an equation for the 2–form \( \hat{J} \) that reads

\[
(\hat{d}\hat{d}\hat{J} - m^2)\hat{J} = 0, \tag{4.32}
\]

where we have used \( \ast(J \wedge J \wedge v) = -2J \). Being complex, this manifold has to satisfy a generalization of the SKT condition \( \partial\bar{\partial}\hat{J} = 0 \). It would be interesting to find explicit examples of 6–dimensional manifolds with almost complex structures satisfying this equation.

### 4.2 Case II: \( q \neq 0 \)

Let us now generalize the previous discussion for non–zero \( \phi = \phi(y,t) \). We will show that the Fayyazuddin-Smith solution [23, 24, 25, 26] falls in this class.

We assume an ansatz for the 7–dimensional metric of the following form

\[
ds^2(y,t) = e^{p_0 \phi} ds^2_0(y,t) + e^{2\phi} dt^2. \tag{4.33}
\]

20
From (3.14) and the condition (3.16) on $v = e^\phi dt$ we can specify the warp factor dependence on the coordinates of the 6–dimensional base $y$ and the fiber $t$

$$\sigma \equiv \tilde{d}\Delta = -\frac{1}{2} \tilde{d}\phi, \quad \dot{\Delta} = -\frac{1}{3} Q e^\phi. \quad (4.34)$$

This dependence is further restricted by the consistency condition (3.21) which in the case at hand becomes

$$\tilde{d}Q = -Q \tilde{d}\phi. \quad (4.35)$$

From this latter we can see that $\tilde{d}\Delta = \tilde{d}\phi = 0$ and that therefore the warp factor splits as $\Delta(x, t) = \Delta_1(x) + \Delta_2(t)$.

Now, denoting with a hat the 2– and 3–forms of the $SU(3)$ structure on the 6–dimensional base

$$J = e^{\rho \phi} \hat{J}, \quad \Psi = e^{\frac{3}{2} \rho \phi} \hat{\Psi}, \quad (4.36)$$

by using (3.19)–(3.21) we obtain once again conditions on the geometry of the base space

$$\begin{cases} 
\tilde{d}\hat{J} = -2 e^{-\frac{1}{2} \phi} S \\
\tilde{d}\hat{\Psi} = \frac{3}{4} \tilde{d}\phi \wedge \hat{\Psi}
\end{cases} \quad (4.37)$$

and the flow equations describing the uplift to seven dimensions

$$\begin{cases} 
\frac{\partial \hat{J}}{\partial t} = \frac{2}{3} e^\phi Q \hat{J} - 2 e^\frac{\phi}{2} A - \frac{1}{2} \dot{\phi} \hat{J} \\
\frac{\partial \hat{\Psi}}{\partial t} = Q e^\phi \hat{\Psi} - \frac{3}{4} \dot{\phi} \hat{\Psi}
\end{cases} \quad (4.38)$$

The conditions (4.37) on the $SU(3)$ structure mean that we deal with the so-called balanced manifolds, which are complex but not Kähler. Notice that we have chosen $p = 1/2$ in order to make the equation for $\tilde{d}\hat{J}$ simpler. If we had kept an arbitrary $p$, we would have a non-zero $\mathcal{W}_4$ class and the manifold would be conformally balanced. This is due to the fact that under conformal transformations of the 6-dimensional metric, the combination $3\mathcal{W}_4 + 2\mathcal{W}_5$ remains constant.

It is straightforward now to verify that the Fayyazudin-Smith solution, as described for example in [24], falls in the above class by identifying the metric and warp factors as

$$H_1 = e^\Delta, \quad H_2 = e^\phi, \quad 2G_{MN}dz^Mdz^N = e^{\frac{\phi}{2}} ds_6^2. \quad (4.39)$$

An interesting consistency condition follows from the fact that $\hat{J}^3 = \sqrt{g_6}d^6y$, where $g_6(y, t)$ the metric tensor corresponding to $ds_6^2(y, t)$. This condition, which reads

$$\frac{\partial}{\partial t} \log \sqrt{g_6} = -6 \left( \frac{1}{4} \dot{\phi} + \dot{\Delta} \right), \quad (4.40)$$

is useful in establishing the relation of the above solution to [23, 24, 26].
5 Type IIA reduction

We now turn to the discussion of 7–manifolds with isometries, so that we can reduce the solution to type IIA strings. As we already saw in the introduction, one has to distinguish two cases according to the relation between the vector describing the isometry of the solution and the vector $v$ defining the $SU(3)$ structure in 7 dimensions. First we analyze the case of $v$ being proportional to the isometry and then discuss the more complicated situation arising from a reduction in a direction orthogonal to $v$.

5.1 Case I: reduction to type IIA along $v$

When the 7–manifold has an isometry, we can find a set of coordinates so that this isometry is described by a constant Killing vector field $\partial/\partial z$. The $SU(3)$ structure in 7 dimensions is described also by a globally defined vector field $v$ and we are now going to discuss the possibility of such a vector being proportional to the Killing vector $\partial/\partial z$. Since we need $v$ to be globally defined, the dual one–form must be defined as

$$v = e^{\beta \phi} dz,$$

where $dz$ is the differential associated to the isometry. As opposed to a generic IIA reduction of a 7–manifold with isometries, one cannot introduce a non–trivial gauge potential in (5.1) and hence the RR 2-form flux in type IIA is zero: $F = 0$.

The fact that $v$ is globally defined means that we have an almost product structure and the metric of $M_7$ takes the form

$$ds_7^2(y) = e^{-2\alpha \phi(y)} ds_6^2(y) + v(y) \otimes v(y),$$

where there is no $z$–dependence since the corresponding vector field is Killing and the $\alpha$ and $\beta$ parameters have been introduced in order to maintain the freedom of choice of the 10–dimensional frame. The string frame is obtained by setting $\beta = 2\alpha = 2/3$. From (5.2), one can see that the 7–dimensional $SU(3)$ structure naturally induces an $SU(3)$ structure in 6 dimensions

$$\tilde{J} = e^{2\alpha \phi} J, \quad \tilde{\Psi} = e^{3\alpha \phi} \Psi.$$

This implies that we can discuss the resulting type IIA compactifications in the presence of fluxes in terms of such structures and their intrinsic torsion classes.

The computation of the $SU(3)$–torsion classes for the 6–dimensional manifolds follows from the application of the above definitions (5.1), (5.2) and (5.3) to the conditions (3.19)–(3.21). The equation on $dv$ will not reduce to constraints on the 6–dimensional torsion, but
it will further constrain the possible solution. From the explicit computation \( dv = \beta d\phi \wedge v \) we conclude that
\[
\sigma = -\frac{\beta}{2} d\phi. \quad (5.4)
\]

It can also be noted that even though we set to zero the gauge field from first principles, its introduction would have resulted again in finding \( F = 0 \) as the analysis of \( dv \) shows. Other constraints on the fluxes as well as the definition of the 6–dimensional torsion can be obtained now by the evaluation of \( dJ \) and \( d\Psi \) and their comparison with \([5.20]–[5.21]\). The \( SU(3) \) singlet and adjoint components of the 11–dimensional flux are vanishing, i.e. \( A = 0 \) and \( Q = 0 \). The latter implies that we can identify the warp factor with the 10–dimensional dilaton \( \Delta = -\frac{\beta}{2} \phi \).

The differentials of the 6–dimensional forms read
\[
\begin{align*}
    d\tilde{J} &= -2e^{2\alpha \phi} S + \left( 2\alpha + \frac{\beta}{2} \right) d\phi \wedge \tilde{J} \quad (5.5) \\
    d\tilde{\Psi} &= 3 \left( \alpha + \frac{\beta}{2} \right) d\phi \wedge \tilde{\Psi}. \quad (5.6)
\end{align*}
\]

From these we can read eventually the relevant intrinsic torsion classes
\[
\mathcal{W}_1 = 0, \quad \mathcal{W}_2 = 0, \quad \mathcal{W}_3 = -2e^{2\alpha \phi} S, \quad \mathcal{W}_4 = \left( 2\alpha + \frac{\beta}{2} \right) d\phi, \quad \mathcal{W}_5 = -3 \left( \alpha + \frac{\beta}{2} \right) d\phi. \quad (5.7)
\]

Let us now comment on the solution. The original configuration in M–theory involved a warped solution starting from a 4–form flux given by
\[
G = v \wedge (J \wedge W + S). \quad (5.8)
\]

The reduction to type IIA along \( v \) gives generically a complex manifold with torsion. In the string frame, where \( 2\alpha = \beta \), the torsion components satisfy a constraint given by \( 2\mathcal{W}_4 + \mathcal{W}_5 = 0 \), with both \( \mathcal{W}_4 \) and \( \mathcal{W}_5 \) being exact and proportional to the exterior derivative of the dilaton. \( \mathcal{W}_3 \) remains free and related to the flux. This result is precisely the same as that found in [8, 9] where the “common sector” of type I/II and heterotic theory was analyzed. Indeed \([5.8]\) shows that the only flux present in the reduced theory is the NS–NS 3–form flux. Moreover, one can simply realize that \( H^{(3,0)} = H^{(0,3)} = 0 \) because of the constraint \([3.13]\).

If we assume that \( v \) itself is Killing, i.e. \( \beta = 0 \), then the resulting 6–manifold is special–hermitian. The solution will not show a warp factor and the only torsion class different from zero will be \( \mathcal{W}_3 \), originating from the primitive part of the flux.
5.2 Case II: reduction to type IIA along $\tau \perp v$

Under the general assumption that the 7–dimensional manifold $M_7$ used in the solution of M–theory with 4–form flux has a Killing isometry, we have a metric ansatz of the form

$$ds^2_7(y) = e^{-2\alpha\phi(y)} ds^2_6(y) + \tau(y) \otimes \tau(y)$$  \hspace{1cm} (5.9)

where $\tau(y) = e^{2\beta\phi(y)} (dz + A(y))$ is a 1–form describing a non–trivial $U(1)$ fibration over the 6–dimensional manifold $M_6$ parameterized by $x$. We will now discuss the case that $\tau$ is not globally defined, so that one can assume $\tau \perp v$. Performing the reduction to type IIA in this way implies that $v$ is inherited by the 6–dimensional manifold so that its group structure is at least reduced to $SO(5)$. On the other hand, we know that the requirement of preserving some supersymmetry in type IIA imposes the existence of an $SU(3)$ structure for the internal manifold. Hence, we finally expect a 6–dimensional $SU(2)$ structure.

Since there is a $U(1)$–worth of $SU(2)$ embeddings in $SU(3)$, one can reconstruct the full $SU(3)$ in different ways. Obviously, this degeneracy does not change the physics, and therefore we arbitrarily fix the extra phase and define

$$v = e^{-\alpha\phi} \tilde{v},$$  \hspace{1cm} (5.10)
$$J = e^{-2\alpha\phi} \tilde{J} + e^{-\alpha\phi} \tilde{w} \wedge \tau,$$  \hspace{1cm} (5.11)
$$\Psi = e^{-2\alpha\phi} \tilde{K} \wedge (e^{-\alpha\phi} \tilde{w} + i \tau).$$  \hspace{1cm} (5.12)

Here the 6–dimensional forms $\tilde{v}, \tilde{w}, \tilde{J}, \tilde{K}$ characterize the $SU(2)$ structure (the complex 1–form of subsection 1.1 is given by $w = \tilde{v} + i \tilde{w}$). We also have

$$d\tau = \beta d\phi \wedge \tau + e^{\beta\phi} F,$$  \hspace{1cm} (5.13)

where $F$ is the 2–form field strength of the type IIA gauge field.

Besides the metric assumption (5.9) and the definition of the $SU(2)$ structure given above, we do not impose extra constraints on the fluxes for the moment. The usual computation of the intrinsic torsion leads to the 6–dimensional $SU(2)$ and $SU(3)$ structures coming from the supersymmetry constraints on $dv, dJ$ and $d\Psi$. After some tedious but straightforward computations, one obtains the conditions on the torsion classes

$$d\tilde{v} = (\alpha d\phi - 2\sigma) \wedge \tilde{v},$$  \hspace{1cm} (5.14)
$$d\tilde{w} = (\alpha - \beta) d\phi - \sigma + 2 \frac{Q}{3} e^{-\alpha\phi} \tilde{v} \wedge \tilde{w} - 2 e^{\alpha\phi} (\tau \wedge S) + 2 \tilde{v} \wedge (\tau \wedge A),$$  \hspace{1cm} (5.15)
$$d\tilde{J} = e^{(\alpha + \beta)\phi} F \wedge \tilde{w} + (2\alpha d\phi - \sigma + 2 \frac{Q}{3} e^{-\alpha\phi} \tilde{v}) \wedge \tilde{J} - 2 e^{2\alpha\phi} S|_h - 2 e^{\alpha\phi} \tilde{v} \wedge A|_h, (5.16)$$
$$d\tilde{K} = (2\alpha - \beta) d\phi - 3\sigma + Q \tilde{v} e^{-\alpha\phi} \wedge \tilde{K}.$$  \hspace{1cm} (5.17)
In the above formulas $S|_h$ and $A|_h$ mean the horizontal part of $S$ and $A$ with respect to $\tau$. In addition the following compatibility constraint arises

$$\left(-2\beta d\phi - \frac{2}{3}Q\bar{v}e^{-\alpha\phi}\right) \wedge \tilde{K} \wedge \bar{w} + ie^{(\alpha+\beta)\phi} \tilde{K} \wedge F - 2e^{\alpha\phi} \tilde{K} \wedge (\tau \lhd S) + 2\tilde{K} \wedge \bar{v} \wedge (\tau \lhd A) = 0.$$  

(5.18)

Though this seems quite a cumbersome expression, we will see later how it can be used to extract useful physical information.

### 5.2.1 10–dimensional vacua with 2–form flux

A purely geometrical solution of M–theory with at least one isometry can be reduced to a type IIA background with non–trivial dilaton and 2–form flux. Though these solutions have been already analyzed in terms of $SU(3)$–structures in \cite{30, 18, 19}, we will analyze them once again in terms of $SU(2)$ structures in order to fit them in our framework.

Once all the 4–form flux components are turned off, the intrinsic torsions of the 6–dimensional $SU(2)$ structure read

$$d\tilde{v} = \alpha d\phi \wedge \bar{v},$$

$$d\tilde{w} = (\alpha - \beta) d\phi \wedge \bar{w},$$

$$d\tilde{J} = e^{(\alpha+\beta)\phi} F \wedge \bar{w} + 2\alpha d\phi \wedge \bar{J},$$

$$d\tilde{K} = (2\alpha - \beta) d\phi \wedge \bar{K},$$

(5.19)

and the consistency constraint (5.18) simplifies to

$$2\beta d\phi \wedge \tilde{K} \wedge \bar{w} = ie^{(\alpha+\beta)\phi} \tilde{K} \wedge F.$$  

(5.20)

In order to compare the above expressions with the known results given in terms of $SU(3)$ structures, we have to introduce an almost complex structure and the associated (3,0)–form in 6–dimensions. In doing so one has to face a $U(1)$ ambiguity following from the embedding of $SU(2) \subset SU(3)$. Since such ambiguity should not result in a physical difference, we will simply fix it in a convenient way and choose

$$J = J^2 + \bar{w} \wedge \bar{v},$$

$$\Omega = (J^3 + iJ^1) \wedge (\bar{w} + i\bar{v}),$$

(5.21)

(5.22)

where $\tilde{J} = J^1$ and $\tilde{K} = J^2 + iJ^3$. Some of the torsion classes follow now straightforwardly from $dJ$:

$$\mathcal{W}_1 = 0, \quad \mathcal{W}_3 = 0, \quad \mathcal{W}_4 = (2\alpha - \beta) d\phi.$$  

(5.23)
Before proceeding with the rest of the torsion classes, we note that the constraint \((5.20)\) implies \(\tilde{v} \cdot d\phi = 0\) and \(\tilde{v} \cdot F = 0\). This, together with \((5.21)\), constrains the possible 2–form field strength, whose general form is given by

\[
F = 2\beta e^{-(\alpha+\beta)\phi} \left( -\frac{1}{2} (\tilde{w} \cdot d\phi) J^1 - (d\phi \cdot J^1) \wedge \tilde{w} \right) + F_0. \tag{5.24}
\]

Here \(F_0\) denotes the primitive part of \(F\) with respect to the three 4–dimensional almost complex structures, i.e. \(F_0 \wedge J^i = 0\) for \(i = 1, 2, 3\). It should be noted that \(F\) is primitive with respect to the \(SU(3)\) structure defined by \((5.21)–(5.22)\), i.e. \(J \cdot F = 0\) or equivalently \(J \wedge J \wedge F = 0\). From this expression one can also obtain

\[
F \cdot \text{Im}\Omega = -2 \beta e^{-(\alpha+\beta)\phi} d\phi, \tag{5.25}
\]

which is the generalized monopole equation noticed in [19].

The remaining torsion classes are now determined by computing \(d\Omega\):

\[
W_2 = -e^{(\alpha+\beta)} F - \beta (d\phi \cdot \text{Im}\Omega), \quad W_5 = -(3\alpha - \beta) d\phi. \tag{5.26}
\]

One can verify that the expression for \(W_2\) reduces to the primitive \((1, 1)\) piece of the 2-form flux with respect to \(J\) using \((5.24)\). In order to do so one first writes \(F\) as a sum of irreducible components

\[
F = m J + F_0^{(1,1)} + (F^{(2,0)} + F^{(0,2)}). \tag{5.27}
\]

From the general solution \((5.24)\) one gets \(m = 0\) and \((F^{(2,0)} + F^{(0,2)}) = \frac{1}{2} (F \cdot \Omega_-) \cdot \Omega_- = -\beta e^{-(\alpha+\beta)\phi} (d\phi \cdot \Omega_-)\). Finally, by using this in \((5.26)\), one gets the expected result: \(W_2 = -e^{(\alpha+\beta)\phi} F_0^{(1,1)}\).

A background satisfying the above relations can be constructed from a deformation of \(T^4 \times R^2\) by functions of the dilaton

\[
d\tilde{s}_6^2 = \sum_{i=1}^{4} dy_i^2 + e^{-2\alpha \phi} dy_5^2 + e^{2\alpha \phi} dy_6^2, \tag{5.28}
\]

where we assume that we are in the string frame, i.e. \(2\alpha = \beta\). This metric satisfies \((5.19)\) with the definitions

\[
\tilde{v} = e^{\alpha \phi} dy_6, \quad \tilde{w} = e^{-\alpha \phi} dy_5, \quad \tilde{J} = dy_1 \wedge dy_2 + dy_3 \wedge dy_4, \quad F = 2\alpha \tilde{J}, \tag{5.29}
\]

and by choosing the dilaton to be a logarithmic function of \(y_5\). A simple extension is given by adding more warpings in the metric depending on the dilaton

\[
d\tilde{s}_6^2 = e^{-2\alpha \phi} (dy_5^2 + dy_4^2 + dy_3^2) + e^{2\alpha \phi} (dy_1^2 + dy_2^2 + dy_6^2). \tag{5.30}
\]
In this case the flux is not given simply in terms of $\tilde{J}$, but we get

$$\tilde{J} = e^{2\alpha \phi} dy_1 \wedge dy_2 + e^{-2\alpha \phi} dy_3 \wedge dy_4, \quad F = 4\alpha dy_3 \wedge dy_4. \tag{5.31}$$

One can check that the full 10–dimensional IIA metric corresponding to (5.31) has 7–dimensional Poincaré invariance and therefore it can be interpreted as a configuration of smeared D6–branes.

A twisted version of the solution just presented is shown in [5]. This is obtained by T-dualizing three times a type IIB solution given by $T^6/\mathbb{Z}_2$ with 3–form fluxes. The type IIA metric in the string frame is

$$d\tilde{s}_6^2 = e^{-2\phi/3} \left[ (dx_1 + 2x_2 \, dx_3)^2 + (dy_1 + 2y_2 \, dx_3)^2 + dy_3^2 \right] + e^{2\phi/3} \left[ dx_2^2 + dx_3^2 + dy_2^2 \right], \tag{5.32}$$

and there is a non–trivial 2–form flux

$$F = 2 (dx_1 + 2x_2 \, dx_3) \wedge dy_2 + 2 (dy_1 + 2y_2 \, dx_3) \wedge dx_2. \tag{5.33}$$

In the presentation of [5], the dilaton was neglected, though it should be a non–trivial function of $x_2$, $x_3$ and $y_2$. This means that one cannot fit this solution in the scheme presented above, unless the dilaton is determined. Moreover, neglecting the dilaton implies that the 5–form flux of the original type IIB solution was neglected and, accordingly, all of its contributions in the dualization procedure.

### 5.2.2 10–dimensional vacua with 2–form and 3–form flux

Before dealing with examples of the most general configuration of fluxes, we want to show solutions of type IIA string theory in the presence of 2–form and 3–form flux. From the 11–dimensional point of view this means that the 4–form flux $G$ satisfies $G \wedge \tau = 0$. Using the general expression for $G$ we conclude that the $SU(3)$ singlet and adjoint components are vanishing and that the vector and double symmetric tensor are not arbitrary. Instead

$$Q = 0, \quad A = 0, \quad \sigma = \lambda \tilde{w}, \quad S = \tilde{w} \wedge X, \tag{5.34}$$

where $X$ should be a primitive 2–form with respect to $J^i, i = 1, 2, 3$.

We can analyze this case as an extension of the previous one, by describing the 6–dimensional intrinsic torsions in terms of the ones in (5.19) incorporating of course the new flux contributions. In this way one gets

$$d\tilde{v} = d\tilde{v}_{old} - 2\lambda \tilde{w} \wedge \tilde{v},$$

$$d\tilde{w} = d\tilde{w}_{old},$$

$$d\tilde{J} = d\tilde{J}_{old} - (\lambda \tilde{w} \wedge \tilde{J} + 2e^{2\alpha \phi} X \wedge \tilde{w}),$$

$$d\tilde{K} = d\tilde{K}_{old} - 3\lambda \tilde{w} \wedge \tilde{K}, \tag{5.35}$$

27
where \( \tilde{v}_{\text{old}}, \tilde{w}_{\text{old}}, \tilde{J}_{\text{old}}, \tilde{K}_{\text{old}} \) are the differentials in (5.19). We note now that even though the constraint (5.18) remains the same as (5.20), the precise form of the solution for the 2–form flux changes because \( dJ^1 \) and \( dJ^2 \) are different. The new solution is

\[
F = 2\beta e^{-(\alpha+\beta)\phi} \left( -\frac{1}{2} (\tilde{w} \lrcorner d\phi) J^1 - (d\phi \lrcorner J^1) \wedge \tilde{w} \right) - 2\lambda e^{-(\alpha+\beta)\phi} J^1 + F_0.
\]

(5.36)

Notice the appearance of the extra term proportional to \( \lambda \) which changes the generalized monopole equation (5.25) to

\[
F \lrcorner \text{Im}\Omega = -2\beta e^{-(\alpha+\beta)\phi} d\phi - 4 e^{-(\alpha+\beta)\phi} \lambda \tilde{w}.
\]

(5.37)

As we did in the previous case, we will now interpret the above constraints in terms of \( SU(3) \) structures determined by a \( J \) and \( \Omega \) defined as in (5.21)–(5.22). Although this will not change the mathematical description we just presented, it will help us improve our intuition about the possible solutions. The torsion conditions on the almost complex structure are obviously the same as before, except for the addition of an extra piece depending on the 3–form flux:

\[
dJ = dJ_{\text{old}} - 3\lambda J^2 \wedge \tilde{w} = dJ_{\text{old}} + J \wedge (-3\lambda \tilde{w}).
\]

(5.38)

Hence, the torsion classes determined by \( J \) are now

\[
W_1 = 0, \quad W_3 = 0, \quad W_4 = (2\alpha - \beta) d\phi - 3\lambda \tilde{w}.
\]

(5.39)

The same applies to \( d\Omega \), which reads

\[
d\Omega = d\Omega_{\text{old}} - 5i \lambda J^3 \wedge \tilde{w} \wedge \tilde{v} + 3\lambda J^1 \wedge \tilde{w} \wedge \tilde{v} + 2e^{2\alpha\phi} X \wedge \tilde{w} \wedge \tilde{v}.
\]

(5.40)

The remaining torsion classes are more easily understood if one puts the extra terms in irreducible form

\[
d\Omega = d\Omega_{\text{old}} + \Omega \wedge \frac{5}{2} \lambda (\tilde{w} - i \tilde{v}) + 2 J \wedge (e^{2\alpha\phi} X - \lambda J^1).
\]

(5.41)

From this, one concludes that

\[
W_2 = \left( -e^{(\alpha+\beta)\phi} F - \beta (d\phi \lrcorner \Omega_-) - 2\lambda J^1 \right) + 2e^{\alpha\phi} X,
\]

\[
W_5 = -(3\alpha - \beta) d\phi + 5\lambda \tilde{w}.
\]

(5.42)

The \( W_2 \) class contains again an extra piece depending on \( \lambda \), but inspection of (5.36) shows that this is exactly canceled by the \( \lambda \)-dependent term that appears there. This means that once again we are left with the primitive \((1,1)\) piece of \( F \). Finally, we can describe this class explicitly in terms of primitive \((1,1)\) forms

\[
W_2 = -F_0^{(1,1)} + 2e^{\alpha\phi} X.
\]

(5.43)
So far the torsion classes given above show that the manifolds one has to use in order to find solutions of type IIA strings in the presence of fluxes have to be non–complex. However, it should be noted that for certain choices of the 2–form flux one can obtain an integrable complex structure. Indeed, by choosing the fluxes so that \( F^{(1,1)}_0 = 2e^{a\phi}X \), the classes \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \) are vanishing and therefore the resulting 6–dimensional manifold is complex. Even more interesting is that by selecting \( \beta d\phi = \lambda \tilde{w} \) one can satisfy the extra condition

\[
3\mathcal{W}_1 + 2\mathcal{W}_2 = 0.
\] (5.44)

This condition describes manifolds which are conformal rescalings of Calabi–Yau spaces [30]. Hence, for these particular combinations of 2– and 3–form fluxes we can obtain relatively simple type IIA solutions with full metric in the string frame

\[
ds_{10}^2 = e^{2\phi} \eta_{\mu\nu} dx^\mu dx^{\nu} + e^{-2\phi} ds_{\text{CY}}^2(y),
\] (5.45)

where \( ds_{\text{CY}}^2(y) \) is a 6–dimensional CY metric. One should be careful however to check that the corresponding \( SU(2) \) structure has also the required intrinsic torsion and that the Bianchi identity for the 3–form flux is satisfied. In these cases, the two–form flux satisfies again a generalized monopole equation like (5.25), but with a different coefficient

\[
F_{\perp} \text{Im}\Omega = -4e^{-\phi} d\phi.
\] (5.46)

5.2.3 10–dimensional vacua with general fluxes

In the previous sections we specified the general results of (5.14)–(5.17) for solutions with only 2–form flux \( F \neq 0 \) or for solutions with both 2–form and 3–form fluxes \( F \neq 0, H \neq 0 \). We are going now to present a way to construct solutions where all the 10–dimensional fluxes are turned on.

Our strategy is partially inspired by [39]. The starting point is a 4–dimensional space which admits a triplet of Kähler structures \( dJ = 0 = dK \). This can be a hyper–Kähler space or a Calabi–Yau 2–fold, i.e. \( K3 \). We then construct a 7–dimensional space by taking the simple product with three circles \( K_3 \times S^1 \times S^1 \times S^1 \). The first non–trivial ingredient now added in order to introduce some flux is the twist of the metric on two of the above circles \( K_3 \times S^1_T \times S^1_T \times S^1 \). Explicitly, one makes a non–trivial fibration of two of such circles on the base space adding to their standard einbein an extra 1–form valued on \( K3 \). However, this manifold does not allow yet for a warp–factor different from zero.

In order to achieve this, one can conformally rescale the 6–dimensional space given by the \( K3 \) and the two circles fibered over it. The function which is used in such rescaling will
give the warp factor and should be chosen to depend only on the coordinate of the extra $S^1$. The resulting 7–dimensional metric is

$$ds_7^2 = e^{-2\Delta(y_3)} \left[ ds_{K3}^2 + (dy_1 + \beta_1)^2 + (dy_2 + \beta_2)^2 \right] + dy_3^2,$$

(5.47)

where $y_i$ parameterize the three circles and $\beta_i$ are 1–forms valued on $K3$. This becomes a full solution of M–theory by using the 4–form flux

$$G = e^{-2\Delta(y_3)} \left[ (dy_1 + \beta_1) \wedge dy_3 \wedge \omega_1 - (dy_2 + \beta_2) \wedge dy_3 \wedge \omega_2 \right] + \frac{1}{2} \Delta' e^{-4\Delta} \left[ J \wedge J + 2J \wedge (dy_1 + \beta_1) \wedge (dy_2 + \beta_2) \right],$$

(5.48)

where $J$ is now the Kähler form and $\omega_i = d\beta_i$ are harmonic $(1,1)$–forms on $K3$. This flux is not closed for a generic choice of $\Delta$, but this will then be fixed by the specific setup one uses in order to build the solution. In the case of no source contributions to the 4–form Bianchi identity, i.e. $dG = 0$, one fixes the $\Delta$ dependence as

$$e^\Delta = c_2 \left( c_1 + 4y_3 \right)^{-1/4},$$

(5.49)

with $c_1$, $c_2$ real integration constants.

Using one of the twisted circles as 11th coordinate, the reduction to type IIA gives a solution with all fluxes turned on. This is actually a forced reduction since translations along $y_3$ are not isometries of the metric (5.47). Solutions of this type arise as T–duals of type IIB compactifications with 3–form fluxes on $K3 \times T^2$, as shown in [6]. We notice however that from [6] it is not clear how the terms proportional to $\Delta'$ can be recovered since the warp factor is neglected. It is natural to expect that they arise as contributions from the dualization of the 5–form flux of the type IIB solution which depends on the derivative of the dilaton.

Another type of solution, based on twisted tori was presented in [5]. This solution was obtained as T–dual of $\mathbb{T}^6/\mathbb{Z}_2$, which is a solution of the type IIB theory in the presence of 3–form fluxes. The resulting manifold is a nilmanifold [38] and it should be a consistent background when all the fluxes of type IIA are turned on. Its metric in the string frame in 10 dimensions is given by $\mathbb{T}^3 \times \mathbb{T}^3$, where one of the two tori is twisted

$$ds_6^2 = e^{-2\phi/3} \left( dx_1 + 2x_2 dx_3 \right)^2 + e^{2\phi/3} \left( dx_2^2 + dx_3^2 + \sum_{i=1}^3 dy_i^2 \right).$$

(5.50)

For simplicity we set to 1 the radii of the various tori. The fluxes are

$$F = 2 \, dx_2 \wedge dy_3,$$

(5.51)

$$H = 2 \, dy_1 \wedge dy_2 \wedge dx_3,$$

(5.52)

$$G_{IIA} = 2 \left( dx_1 + 2x_2 dx_3 \right) \wedge dy_1 \wedge dy_2 \wedge dy_3.$$

(5.53)
The four–flux in 11 dimensions can be reconstructed from (5.51)–(5.53) as

\[ G = 2 (dx_1 + 2x_2 \, dx_3) \wedge dy_1 \wedge dy_2 \wedge dy_3 + \tau \wedge dy_1 \wedge dy_2 \wedge dx_3, \]

where \( \tau = dz + 2x_2 \, dy_3 \) and \( z \) is the uplift circle coordinate. The flux (5.54) can also be written as \( G = v \wedge U \) and therefore the only non–trivial component of the flux is given by the \((2,1) + (1,2)\)–form \( U \) or its dual \( S \). Also in this case we can apply the same comments as above concerning the fact that the solution was discussed in [5] by using the \( e^{-2\phi} \sim 1 \) approximation.

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