Quantum Brègman distances and categories

Ryszard Paweł Kostecki\(^1,2\)

\(^1\)Perimeter Institute for Theoretical Physics
31 Caroline St North, N2J 2Y5, Waterloo, Ontario, Canada
\(^2\)Institute for Theoretical Physics, University of Warsaw
Pasteura 5, 02-093 Warszawa, Poland

ryszard.kostecki@fuw.edu.pl
http://www.fuw.edu.pl/~kostecki

October 4, 2017

Abstract

We introduce, and investigate the properties of, the family of quantum Brègman distances, based on embeddings into suitable vector spaces (with the reflexive noncommutative Orlicz spaces over semi-finite \(W^*\)-algebras and noncommutative \(L_p\) spaces over any \(W^*\)-algebras providing two important examples). This allows us to define geometric categories for nonlinear quantum inference theory, with morphisms given by constrained minimisations of quantum Brègman distances.

1 Introduction

In this paper (which provides a further technical development of the ideas in [95]) we discuss information geometric structures on two levels: general, with an information model \(M\) understood as a set (or an object in a category) and an information distance \(D\) understood as a nonsymmetric function (or a functor) on it, and particular, with quantum information models defined as arbitrary dimensional spaces of nonnormalised quantum states (subsets of positive cones of preduals of \(W^*\)-algebras). We consider these quantum geometries as a quite generic setting to develop an approach to foundations of quantum theory, understood as a theory of an intersubjective inductive inference (see [97] for a discussion), however we aim at making the theory as general as possible, expecting operator spaces as a ‘next level to go’ for the (quantum and post-quantum) information geometric theories. Due to consideration of analytic and geometric aspects of information geometry on the equal footing, as two constitutive components for a category-theoretic framework, the approach underlying this text can be considered as a nonlinear follow-up to the approach of Chencov [40, 41, 42, 113], based on replacing markovian morphisms by brègmannian projections.

In what follows, we will first introduce the notation and terminology that we use. Next we will define two different perspectives on (statistical and quantum) information geometry, associated with two different classes of morphisms between information models (resp., coarse grainings and \(D\)-projections) and two different classes of distances (resp., \(f\)-distances and Brègman distances).

For a given \(W^*\)-algebra\(^1\) \(\mathcal{N}\), we define a quantum information model as a subset \(\mathcal{M}(\mathcal{N}) \subseteq L_1(\mathcal{N})^+ \cong \mathcal{N}^+_+\). Its elements would be called (quantum information) states. For a commutative \(W^*\)-algebra \(\mathcal{N}\) the quantum information models \(\mathcal{M}(\mathcal{N})\) turn into statistical

\(^1\)See Appendix for some notions and facts from the theory of operator algebras.
models $\mathcal{M}(A) \subseteq L_1(A)^+$, where $\mathcal{N} \cong L_\infty(A)$. Restriction to normalised states in this case gives $\mathcal{N}_*^+ \cong L_1(A)^+_1$, and $A$ is an mcb-algebra of projections in $\mathcal{N}$.

Given any set $X$, a **distance** is defined as a map $D : X \times X \to [0, \infty]$ such that $D(x, y) = 0 \iff x = y$. A distance is called **bounded** iff $\text{ran}(D) = \mathbb{R}^+$; **symmetric** iff $D(x, y) = D(y, x)$; **metrical** iff it is bounded, symmetric and satisfies [triangle inequality](#):

$$D(x, z) \leq D(x, y) + D(y, z) \ \forall x, y, z \in X. \quad (1)$$

We will use the symbol $d$ instead of $D$ to denote metrical distances. We will use the notion **information distance** to refer to a distance on any set $\mathcal{M}$ that is considered as an information model. A **quantum distance** is defined as a distance on a quantum model $\mathcal{M}(\mathcal{N}) \subseteq \mathcal{N}_*^+$, and it becomes a **statistical distance** if $\mathcal{N}$ is commutative. A **relative entropy** is defined as a map $\mathbf{S} : X \times X \to [-\infty, 0]$ such that $-\mathbf{S}$ is an information distance. This closely follows Wiener’s idea that the «amount of information is the negative of the quantity defined as entropy» [137].

The standard point of departure of commutative (statistical) and noncommutative (quantum) information geometry, as introduced and developed by Chencov [40, 42, 111, 112], is Wald’s [136] unification of Fisher’s and Neyman–Pearson approaches. According to it, the foundation of statistics is decision making: given some evidential data, two information models and their coarse grainings is a subcategory of the category $\mathbf{QMod}^+$ of quantum information models and their coarse grainings is a subcategory of the category $\mathbf{QMod}^+$ of quantum models and positive linear functions.

A function $f : \mathbb{R}^+ \to \mathbb{R}$ is called **operator convex** [98] iff

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \ \forall x, y \in \mathcal{B}(\mathcal{H})^+ \ \forall \lambda \in [0, 1]. \quad (2)$$

If $f : \mathbb{R}^+ \to \mathbb{R}$ is operator convex (hence, continuous on $[0, \infty]$) with $f(0) \leq 0$ and $f(1) = 0$, and if $(\mathcal{H}, \pi, J, \mathcal{H}^3)$ is standard representation of a $W^*$-algebra $\mathcal{N}$, then the $f$-**distance** [92, 121] is defined as a function $D_f : \mathcal{N}_*^+ \times \mathcal{N}_*^+ \to [0, \infty]$ such that

$$D_f(\omega, \phi) := \left\{ \begin{array}{ll}
\langle \xi_\pi(\phi), f(\Delta_{\omega, \phi})\xi_\pi(\phi) \rangle & : \omega \ll \phi \\
+\infty & : \text{otherwise,}
\end{array} \right. \quad (3)$$

where $\xi_\pi(\phi)$ is standard vector representative of $\phi$ in $\mathcal{H}^3$. In the commutative case, this distance was introduced in [45, 110, 6]. By Petz’s theorem [121], if $f$ is bounded from above (hence, operator monotone decreasing), then $D_f$ given by (3) satisfies

$$D_f(\omega, \phi) \geq D_f(T_\omega(\omega), T_\phi(\phi)) \ \forall \omega, \phi \in \mathcal{N}_*^+ \quad (4)$$

---

2The functions that we call ‘(quantum/statistical) distances’ are often called ‘(quantum) information divergences’. However, this causes very unfortunate collision of terms with well established notion of divergence used in differential calculus and differential geometry. Moreover, the term ‘divergence’ was introduced and used by Kullback and Leibler [100] in the context of relative entropy, but in order to refer to an example of what we call a symmetric distance. Rényi [125] proposed to use the term ‘information gain’. Chencov [42] proposed to use the term ‘deviation’, but it seems for us to sound too awkward comparing with a generality and omnipresence of its designate. Eguchi [65] (following Pfanzagl [124]) used the term ‘contrast functional’. We think that it is more reasonable to extend the range of the meaning of term ‘distance’, which is also in agreement with some of the prominent works in the field of information theory, e.g. [37, 50, 119].

3Somewhat similar functionals were considered earlier in [114] under the name “generalised Hellinger integrals”, and with different assumptions on $f$ (it was considered to be a Young function).
for any unital 2-positive function $T$ such that $\text{dom}(T_\ast) = N_+^\ast$ (hence, in particular, for every quantum coarse graining\footnote{By $\text{Mark}_\ast(N_+^\ast)$ we will denote the space of all coarse grainings with a domain $N_+^\ast$.} $T_\ast \in \text{Mark}_\ast(N_+^\ast)$), and the equality is attained iff $T_\ast$ is an isomorphism. In [134] the inequality (4) has been shown to hold for any quantum coarse graining $T_\ast$ such that $\text{dom}(T_\ast) = N_+^\ast$ and for any $f$-distance (without assuming that $\ast$ is bounded from above). In the commutative case the ‘data processing inequality’ (4) was established in [45, 46, 115, 103, 104]. In [47, 48] Csiszár provided a characterisation of the $f$-distances on finite dimensional statistical models by means of (4). The property (4) can be understood as a requirement of compatibility of the quantum distance on a quantum model with the structure of the category $\text{QMod}^\ast$, expressing the requirement that “the coarse graining of information models should always be indicated by nonincreasing of the quantification of relative information content of information states”.

On the other hand, starting from the works of Brègman [29], Chenchov [41], and Hobson [81], there has emerged an alternative approach to statistical inference. Its main idea is to consider the minimisation of information distances as a process of inductive inference [138, 131], with the unique minimiser (whenever it exists) considered as a nonlinear projection onto a codomain model. This way one is led to consideration of a class of nonlinear morphisms of information models that is different from coarse grainings, but admits (as we will see below) also a legitimate information theoretic semantics.

Let $D$ be an information distance on an information model $\mathcal{M}$. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be sub-models of $\mathcal{M}$. We define a $D$-projection from $\mathcal{Q}_1$ to $\mathcal{Q}_2$ as a map

$$\Psi^D_{\mathcal{Q}_2|\mathcal{Q}_1} : \mathcal{Q}_1 \ni \psi \mapsto \arg\inf_{\phi \in \mathcal{Q}_2} \{ D(\phi, \psi) \} \in \mathcal{Q}_2,$$

(5)

whenever the right hand side is a singleton set. We will denote $\Psi^D_{\mathcal{Q}_2} := \Psi^D_{\mathcal{Q}_2|\mathcal{Q}_1}$. From definition of $D$ it follows that $\Psi^D_{\mathcal{Q}_2|\mathcal{Q}_1}(\psi) = \psi \forall \psi \in \mathcal{Q}$, hence $\Psi^D_{\mathcal{Q}_2|\mathcal{Q}_1}$ is an idempotent operation on an arbitrary information model $\mathcal{Q}$. A family of $D$-projections $\{ \Psi^D_{\mathcal{Q}_2|\mathcal{Q}_1} | i \in I, j \in J \}$, where $I$ and $J$ are arbitrary sets, and $\mathcal{Q}_i, \mathcal{Q}_j \subseteq \mathcal{M}$, will be called composable iff $\Psi^D_{\mathcal{Q}_2|\mathcal{Q}_1} = \Psi^D_{\mathcal{Q}_2|\mathcal{Q}_k} \circ \Psi^D_{\mathcal{Q}_k|\mathcal{Q}_j}$ for all $k \in I \cap J$. A category consisting of objects given by information models as objects and composable $D$-projections as arrows will be denoted $\text{Mod}^D$. For objects given by quantum information models we will use the notation $\text{QMod}^D$. A restriction of objects to subsets of $N_+^\ast$ for a given $W^+$-algebra $\mathcal{N}$ defines a category $\text{QMod}^D(\mathcal{N}_+^\ast)$. Note that $\text{QMod}^D$ is not a subcategory of $\text{QMod}^+$, because a composable $D$-projection $\Psi^D_{\mathcal{Q}_2|\mathcal{Q}_1}$ may possess no extension to the full positive cone of $L_1(\mathcal{Q}_1)$\footnote{For any quantum model $\mathcal{M}(\mathcal{N})$ we define $L_1(\mathcal{M}(\mathcal{N}))$ as the smallest space $L_1(\mathcal{C})$, by means of a partial order given by isometric embeddings and for some $W^+$-algebra $\mathcal{C}$, that contains the linear span of $\mathcal{M}(\mathcal{N})$. Note that the existence of infimum in such poset is not guaranteed a priori, and requires to be proven.}. In practice, the requirement of existence and uniqueness of the solution of (5) is achieved by the choice of a distance that is convex and lower semi-continuous, and the choice of models that are convex and closed under suitable mappings determined by $D$. This inseparability of choice of objects and morphisms shows the relevance of category-theoretic perspective for the description of the max-relative-entropic approach to information geometry.

The key feature of this approach is an observation that a class of distances, called Brègman distances, and denoted here as $D_\phi$, admits a generalisation of pythagorean theorem beyond the linear framework of euclidean and Hilbert spaces, providing the additive decomposition of the nonlinear, yet ‘orthogonal’, projection onto a suitably affine class of subsets $K$ of $\mathcal{M}$, \footnote{These sets are required to be affine, but under their local embedding into a suitably chosen linear space that is used to define $D_\phi$ and also determines what ‘orthogonality’ means. See below and Section 3.}

$$D_\phi(\omega, \phi) = D_\phi(\omega, \Psi^D_{\mathcal{K}}(\phi)) + D_\phi(\Psi^D_{\mathcal{K}}(\phi), \phi) \quad \forall (\omega, \phi) \in K \times \mathcal{M}.$$  

(6)

This in turn allows to use geometry for the purpose of a nonlinear nonparametric “data = signal + noise” inference.
In Section 2 we consider a class $\tilde{D}_\Psi$ of two-point nonlinear functionals on vector spaces, known as Brègman functionals [29, 33, 21]. While some of the Brègman functionals are also distances, which allows to consider them as information distances in the case of $L_1(A)$ or $L_1(N)$ vector spaces, this perspective is of limited applicability, especially when infinite dimensional (nonparametric) quantum models are considered. As for now, there is an important gap in the theory of Brègman distances: while very nice results on existence and uniqueness of projections, generalised pythagorean theorem, as well as composability, exist for Brègman functionals on reflexive Banach spaces, the nonreflexivity of Orlicz spaces allowing for an adequate treatment of $D_1$-projections goes hand in hand with the fact that $D_1$ distances (97) are constructed from the most general definition of Brègman distance, based on the right Gâteaux derivative.

In order to investigate the possibilities of bridging this gap, in Section 3 we develop a theory of general abstract Brègman distances, without embedding them into topological, bornological, or differential framework. The key elements of this construction are the Young–Fenchel inequality, dual pairs of coordinate systems\(^7\) and a suitable generalisation of the bijective Legendre transform to the infinite dimensional case. This approach includes the large part of theory of Brègman (and Alber) functionals as a special case. We believe that this study, while currently lacking a conclusive strong theorem, can serve as a good point of departure for a future research on the “optimal” definition of Brègman distance that would unify the reflexive and nonreflexive approaches by balancing better the convex and topological structure\(^8\).

Taking some lessons from this general investigation, in Section 4 we return back to the particular, proposing two definitions of a quantum Brègman distance: more restricted one, based on the embedding of quantum states into the reflexive Banach space (with an explicit example provided by the noncommutative Orlicz spaces $L_{\gamma}(N)$ over semi-finite $W^*$-algebras $N$, which allow us to define a class $D_{\Psi,\gamma}$ of quantum Brègman–Orlicz distances), and more general one, in which the key properties are moved from the proposition to a definition. We end this Section with a construction of a category $\text{QMod}^{D_{\Psi}}$ and its affine subcategory $\text{QAff}^{D_{\Psi}}$, for which the generalised pythagorean theorem holds globally. From this it follows that $D_{\Psi}$-projections are commutative inside $\text{QAff}^{D_{\Psi}}$:

$$
\Psi^{D_{\Psi}}_{\mathcal{Q}_i|\mathcal{Q}_k} \circ \Psi^{D_{\Psi}}_{\mathcal{Q}_k|\mathcal{Q}_j} = \Psi^{D_{\Psi}}_{\mathcal{Q}_i|\mathcal{Q}_j} \circ \Psi^{D_{\Psi}}_{\mathcal{Q}_i|\mathcal{Q}_j} = \Psi^{D_{\Psi}}_{\mathcal{Q}_i|\mathcal{Q}_j} \cap \mathcal{Q}_i|\mathcal{Q}_j, \quad (7)
$$

hence, sequential and parallel updating/learning coincide (see [135, 36] for a discussion of an importance of this feature in the commutative case of $D_1$).

The families of $f$-distances and Brègman distances are widely regarded as two most important classes of information distances (cf. e.g. [48, 51, 52]). This leads to ask about the class of quantum information distances that belong to both families. Amari has recently shown [7] that for the finite dimensional statistical models $L_1(X,\mathcal{U}(X),\mu)^+$ this intersection is characterised by the Liese–Vajda family [103] of $\gamma$-distances. In Section 5 we use the Falcone–Takesaki theory [66] of noncommutative integration and $L_p(N)$ spaces over arbitrary $W^*$-algebras (without a restriction to semi-finite $N$) to construct the canonical family $D_\gamma$ of quantum $\gamma$-distances, which provides a common generalisation of the Jenčová–Ojima family [118, 84] and the Liese–Vajda family of $\gamma$-distances. We prove that the family of $D_\gamma$ belongs to an intersection of quantum $f$-distances $D_f$ and quantum Brègman distances $D_{\Psi}$. Following Amari’s result, we conjecture that this property characterises $\gamma$-distances on $L_1(N)^+ \cong N_1^+$. Similarly to characterisation of quantum $f$-distances by the monotonicity under coarse grainings, and characterisation of Brègman distances by the generalised pythagorean equation, the proof of this conjecture remains an open problem. We discuss also the conditions of exis-

\(^7\) A research on the role of coordinate embeddings (translating between a distance on nonlinear model and a functional on a linear space) for establishing the existence and uniqueness of projections has been initiated by Nagaoka and Amari [116, 9], and our work can be understood as an investigation of the nonsmooth functional analytic foundations for this approach.

\(^8\) Possibly by inducing the latter from the bornology determined by bounded level sets, as in [102].
2 Brègman functionals

At least five different inequivalent general notions of a Brègman functional are present in the literature, each one having its own virtues and flaws (we review them below, to a reasonable extent determined by our later applications). The substantial part of the theory of Brègman functionals is developed for the reflexive Banach spaces. However, this excludes the discussion of the most interesting case of $L_1$ spaces as well as nonreflexive Orlicz spaces, which are naturally related with $D_1$ distances. For that case, there are at least three approaches possible: the general approach based on one-sided Gâteaux derivatives on arbitrary Banach spaces, the measure theoretic approach based on integrals over premeasurable spaces and pointwise composition of gradients over $\mathbb{R}^n$ with $\mathbb{R}^n$-valued measure functions, and the intermediate approach, which can be applied to arbitrary Banach space, but requires its Fréchet differentiability.

A dual pair is defined \[59, 60, 105\] as a triple $(X, X^d, \langle \cdot, \cdot \rangle_{X \times X^d})$, where $X$ and $X^d$ are vector spaces over $\mathbb{K} \in \{ \mathbb{R}, \mathbb{C} \}$, equipped with a bilinearity pairing $\langle \cdot, \cdot \rangle_{X \times X^d} : X \times X^d \to \mathbb{K}$ satisfying \[9\]

$$\langle x, y \rangle_{X \times X^d} = 0 \quad \forall x \in X \Rightarrow y = 0, \quad \langle x, y \rangle_{X \times X^d} = 0 \quad \forall y \in X^d \Rightarrow x = 0. \quad (8)$$

An example of a dual pair is given by a Banach space $X$, $X^d = X^*$, and the dual pairing given by the Banach space duality. The Fenchel subdifferential [67, 109, 31] of a proper $\Psi : X \to [-\infty, +\infty]$ at $x \in \operatorname{efd}(\Psi)$ is a set

$$\partial \Psi(x) := \{ y \in X^d : \Psi(z) - \Psi(x) \geq \text{re} \langle z - x, y \rangle_{X \times X^d} \quad \forall z \in X \}. \quad (9)$$

For $x \in X \setminus \operatorname{efd}(\Psi)$ one defines $\partial \Psi(x) := \emptyset$. The elements of $\partial \Psi(x)$ are called Fenchel subgradients at $x$. The Fenchel dual of $\Psi$ is defined as $\Psi^F : X^d \to [-\infty, +\infty]$ such that

$$\Psi^F(y) := \sup_{x \in X} \{ \text{re} \langle x, y \rangle_{X \times X^d} - \Psi(x) \} \quad \forall y \in X^d. \quad (10)$$

Given $X^{dd}$ such that $(X^d, X^{dd}, \langle \cdot, \cdot \rangle_{X^d \times X^{dd}})$ is a dual pair and $X \subseteq X^{dd}$, one defines $\Psi^{FF} : X \to [-\infty, +\infty]$ by $\Psi^{FF} := (\Psi^F)^F$. The functions $\Psi^F$ and $\Psi^{FF}$ are convex for any $\Psi$, and $\Psi^{FF}|_X \leq \Psi$. If $\operatorname{efd}(\Psi) \neq \emptyset$, then $\Psi^F(x) > -\infty \forall x \in X^d$. If $(X, X^t)$ is a dual pair of locally convex topological vector spaces, equipped with weak-$\star$ and weak topologies, respectively, and $\Psi$ is proper, then $\Psi^F$ is weakly-$\star$ lower semi-continuous, $\Psi^{FF}$ is weakly lower semi-continuous, and $(\Psi^{FF}|_X = \Psi$ holds iff $\Psi$ is weakly lower semi-continuous and convex) \[82, 30\]. A lower semi-continuous convex $\Psi$ on $X$ is proper iff $\Psi^F$ on $X^t$ is proper. If $X$ is a Banach space and $\Psi : X \to [-\infty, +\infty]$ is proper, convex, then it is lower semi-continuous in norm topology of $X$ iff it is lower semi-continuous in weak topology on $X$. In what follows, we will always assume $\operatorname{efd}(\Psi) \neq \emptyset$. If $\Psi : X \to \mathbb{R} \cup \{ +\infty \}$ is convex and $y \in X^d$, then the Young–Fenchel inequality \[139, 67\]

$$\Psi(x) + \Psi^F(y) - \text{re} \langle x, y \rangle_{X \times X^d} \geq 0 \quad (11)$$

holds, with equality iff $y \in \partial \Psi(x)$. If $(X, X^t)$ is a dual pair of locally convex topological vector spaces, and $\Psi$ is proper, convex, and lower semi-continuous, then equality in (11) holds iff $x \in \partial^\star \Psi^F(y)$. There exist various criteria for nonemptiness of Fenchel subdifferential. The

\[\text{We use here the general setting of dual vector spaces, and do not restrict our considerations to locally convex topological vector spaces, because we have in mind the possible future use of convenient vector spaces [72, 99] and stereotype spaces [1].}\]
key role of Fenchel subdifferential $\partial \Psi(x)$ is to characterise minimisers of $\Psi$ at $x$. In particular, if $X$ is a Banach space, $x \in X$, and $\Psi : X \to [\infty, +\infty]$ is proper and convex, then
\[ x_0 \in \arg \inf_{x \in X} \{ \Psi(x) \} \iff 0 \in \partial \Psi(x_0). \] (12)

If $\Psi$ is also lower semi-continuous with respect to norm topology on $X$, then the conditions in (12) are equivalent to $\partial \Psi^F(0) \cap X^* \neq \emptyset$, where $\Psi^F$ is a Fenchel dual with respect to the Banach duality of $X$ and $X^*$.

If $(X, X^d, [\cdot, \cdot]_{X \times X^d})$ is a dual pair and $\Psi : X \to ]-\infty, +\infty]$ is proper, then:
\[ \text{efd}(\partial \Psi) := \{ x \in \text{efd}(\Psi) \mid \partial \Psi(x) \neq \emptyset \}, \] (13)
\[ \text{efc}(\partial \Psi) := \{ \hat{y} \in X^d \mid \hat{y} \in \partial \Psi(x), x \in \text{efd}(\partial \Psi) \}, \] (14)
\[ (\partial \Psi)^{-1}(x) \ni \hat{y} \mapsto (\partial \Psi)^{-1}(\hat{y}) := \{ x \in X \mid \hat{y} \in \partial \Psi(x) \} \in \wp(X), \] (15)

where $\wp(X)$ denotes a power set of $X$. If $X$ is a Banach space, $X^d = X^*$, and $\Psi$ is proper, convex, and lower semi-continuous in norm topology on $X$, then $\text{int}(\text{efd}(\Psi)) \subseteq \text{efd}(\partial \Psi)$, and $\text{efd}(\partial \Psi)$ is dense in $\text{efd}(\Psi)$.

If $X$ is a vector space over $\mathbb{K}$, $t \in \mathbb{R}$, and $\Psi : X \to ]-\infty, +\infty]$ is proper then the right Gâteaux derivative of $\Psi$ at $x \in X$ in the direction $h \in X$ reads
\[ X \times X \ni (x, h) \mapsto \mathcal{D}^G_+ \Psi(x; h) := \lim_{t \to 0^+} (\Psi(x + th) - \Psi(x))/t \in [0, +\infty]. \] (16)

If $x$ is fixed and (16) exists for all $h \in X$, then $\Psi$ is called Gâteaux differentiable at $x$.

If $\Psi : X \to ]-\infty, +\infty]$ is convex and Gâteaux differentiable at $x$, then $\mathcal{D}^G_+ \Psi(x; \cdot) \in \partial \Psi(x)$.

If $\Psi : X \to ]-\infty, +\infty]$ is convex and continuous at $x$, then $\partial \Psi(x) = \{ \star \}$ iff $\Psi$ is Gâteaux differentiable at $x$. If $\Psi : X \to ]-\infty, +\infty]$ is convex, lower semi-continuous, and Gâteaux differentiable at $x$, then it is continuous at $x$. If $X$ is a Banach space and $\Psi$ is convex and lower semi-continuous, then $\mathcal{D}^G_+ \Psi(x; \cdot)$ is convex on $X$, and continuous on $\text{int}(\text{efd}(\Psi))$, while $\mathcal{D}^G_+ \Psi(\cdot, \cdot)$ is finite and upper semi-continuous on $\text{int}(\text{efd}(\Psi)) \times X$. If $x \in \text{efd}(\Psi)$ and $\mathcal{D}^G_+ \Psi(x; \cdot)$ is continuous at some $h \in X$, then $\partial \Psi(x) \neq \emptyset$. If $X$ is a Banach space and $\Psi$ is Gâteaux differentiable at $x \in X$, then $\mathcal{D}^G_+ \Psi(x; y) := [y, \mathcal{D}^G_+ \Psi]_{X \times X^*}, \forall y \in X$ defines the Gâteaux derivative [73, 74, 75] $(\mathcal{D}^G \Psi)(x) \equiv \mathcal{D}^G_+ \Psi \in X^*$ of $\Psi$ at $x$. A function $\Psi$ is called Gâteaux differentiable iff $\text{int}(\text{efd}(\Psi)) \neq \emptyset$ and $\Psi$ is Gâteaux differentiable for all $x \in \text{int}(\text{efd}(\Psi))$. If $X$ is a Banach space, $\Psi : X \to ]-\infty, +\infty]$ is proper, convex, and lower semi-continuous in norm topology, then: (i) if $\Psi^F$ (with respect to Banach space duality) is strictly convex at all elements of $\text{efd}(\Psi^F)$, then $\Psi$ is Gâteaux differentiable; (ii) if $\Psi^F$ is Gâteaux differentiable at all $x \in X^*$, then $\Psi$ is strictly convex at all elements of $\text{int}(\text{efd}(\Psi^F))$. Given a normed space $X$, a Fréchet derivative of $\Psi : X \to ]-\infty, +\infty]$ at $x \in X$ will be denoted as $\mathcal{D}^F \Psi(x)$. If $\Psi$ is Fréchet differentiable at all $x \in \text{int}(\text{efd}(\Psi))$, then it is also norm continuous and Gâteaux differentiable. For $\dim X < \infty$ these two notions of derivative coincide.

A Banach space $X$ is called: strictly convex [69, 44] iff
\[ \forall x, y \in X \ | x + y | = | x | + | y |, \ x \neq 0 \neq y \Rightarrow \exists \lambda > 0 \ y = \lambda x; \] (17)
Gâteaux differentiable [13, 107] iff $\| \cdot \|$ is Gâteaux differentiable at every $x \in X \setminus \{ 0 \}$; uniformly convex [44] iff
\[ \forall \epsilon_1 > 0 \ \exists \epsilon_2 > 0 \ \forall x, y \in X \ | x | = | y | = 1, \ | x - y | \geq \epsilon_1 \Rightarrow | (x + y) / 2 | \leq 1 - \epsilon_2; \]
uniformly Fréchet differentiable [130] iff
\[ \forall \epsilon_1 > 0 \ \exists \epsilon_2 > 0 \ \forall x, y \in X \ | x | = 1, \ | y | \leq \epsilon_2 \Rightarrow | x + y | + | x - y | \leq 2 + \epsilon_1 \| y \|; \]
**reflexive** [77] iff the map \( j : X \to X^{\ast\ast} \), defined by \( j(x)(y) := y(x) \forall x \in X \forall y \in X^\ast \) is an isometric isomorphism. If \( X \) (resp. \( X^\ast \)) is Gâteaux differentiable, then \( X^\ast \) (resp. \( X \)) is strictly convex [129, 91]. A Banach space \( X \) is uniformly convex (rep. uniformly Fréchet differentiable) iff \( X^\ast \) is uniformly Fréchet differentiable (resp. uniformly convex) [55]. If \( X \) is uniformly convex (resp. uniformly Fréchet differentiable), then it is also strictly convex (resp. Gâteaux differentiable). If \( X \) is uniformly convex or uniformly Fréchet differentiable, then it is reflexive [108, 87, 120, 130]. If \( X \) is Gâteaux differentiable, then there exists a norm-to-weak-* continuous map \( \{ x \in X \mid \|x\|_X = 1 \} \to \{ x \in X^\ast \mid \|x\|_{X^\ast} = 1 \} \) that is uniquely determined by a condition \( [x, \hat{x}]_{X \times X^\ast} = 1 \) [129].

Let \( X \) be a Banach space with a norm \( \| \cdot \| \). In what follows, we will refer to Banach spaces assuming implicitly that they are over \( \mathbb{R} \). For Banach spaces over \( \mathbb{C} \) all definitions and results require to replace \( [\cdot, \cdot]_{X \times X^\ast} \) by re \( [\cdot, \cdot]_{X \times X^\ast} \). A function \( T : X \to \phi(X^\ast) \) is called **locally bounded** at \( x \in X \) iff [132]

\[
\exists \epsilon > 0 \sup \{ \|T(x + \epsilon y)\| \mid y \in X, \|y\| \leq 1 \} < +\infty.
\]

If \( \Psi : X \to ] - \infty, +\infty] \) is proper, then

\[
(\partial \Psi)^{-1}(\hat{y}) = \arg \min_{x \in X} \{ \Psi(x) - [x, \hat{y}]_{X \times X^\ast} \}.
\]

A function \( \Psi : X \to ] - \infty, +\infty] \) is called **coercive** iff \( \lim_{\|x\| \to +\infty} \Psi(x) = +\infty \). A Banach space \( X \) is reflexive iff every proper, convex, coercive function that is lower semi-continuous in norm topology attains its minimum on \( X \). If \( \Psi : X \to ] - \infty, +\infty] \) is proper, convex, lower semi-continuous and \( \Psi^F \) denotes its Fenchel dual with respect to the Banach space duality of \( X \) and \( X^\ast \), then \( \Psi \) is called [126, 21, 26, 28]:

- **essentially Gâteaux differentiable** iff \( (\partial \Psi) \) is locally bounded on \( \text{efd}(\partial \Psi) \) or \( \text{int}(\text{efd}(\Psi)) \neq \emptyset \) and \( \partial \Psi(x) = \{ * \} \forall x \in \text{efd}(\partial \Psi) \);

- **essentially strictly convex** iff \( (\partial \Psi)^{-1} \) is locally bounded on \( \text{efd}((\partial \Psi)^{-1}) \) and \( \Psi \) is strictly convex on every convex subset of \( \text{efd}(\partial \Psi) \);

- **Legendre** iff \( \Psi \) is essentially Gâteaux differentiable and essentially strictly convex;

- **essentially Fréchet differentiable** iff it is essentially Gâteaux differentiable and Fréchet differentiable for all \( x \in \text{int}(\text{efd}(\Psi)) \);

- **Fréchet–Legendre** iff \( \Psi \) and \( \Psi^F \) are essentially Fréchet differentiable.

If \( \Psi \) is continuous and is Gâteaux differentiable at all \( x \in X \) then it is essentially Gâteaux differentiable. If \( \Psi \) is essentially Gâteaux differentiable then \( \text{int}(\text{efd}(\Psi)) \neq \emptyset \) and \( \Psi \) is Gâteaux differentiable on \( \text{int}(\text{efd}(\Psi)) \) [21]. If \( X \) is reflexive, then \( \Psi \) is essentially Gâteaux differentiable (resp. Legendre, Fréchet–Legendre) iff \( \Psi^F \) is essentially strictly convex (resp. Legendre, Fréchet–Legendre). If \( X \) is reflexive and \( \Psi \) is Legendre, then

\[
\mathcal{D}^G \Psi : \text{int}(\text{efd}(\Psi)) \to \text{int}(\text{efd}(\Psi^F))
\]

is bijective, \( (\mathcal{D}^G \Psi)^{-1} = \mathcal{D}^G (\Psi^F) \), and both \( \mathcal{D}^G \Psi \) and \( \mathcal{D}^G (\Psi^F) \) are norm-to-weak continuous and locally bounded on their respective domains [21].

Let \( X \) be a Banach space, and let \( \Psi : X \to ] - \infty, +\infty] \) be proper. Then the **Brègman functional** \( \tilde{D}_\Psi : X \times X \to [0, +\infty] \) can be defined in any of the following **inequivalent** ways (see also [32]):

(B1) for \( \Psi \) convex, with \( \text{efd}(\Psi) \neq \emptyset \) [88, 89, 90, 33, 34]:

\[
\tilde{D}_\Psi : X \times X \ni (x, y) \mapsto \begin{cases} 
\Psi(x) - \Psi(y) - \mathcal{D}^G \Psi(y; x - y) & : y \in \text{efd}(\Psi) \\
+\infty & : \text{otherwise};
\end{cases}
\]

(21)
\( (B_2) \) for \( \Psi \) convex and lower semi-continuous, with \( \text{int}(\text{efd}(\Psi)) \neq \emptyset \) [21]:

\[
\tilde{D}_\Psi : X \times X \ni (x, y) \mapsto \begin{cases}
\Psi(x) - \Psi(y) - [x-y, D^G_y \Psi](x-y) : y \in \text{int}(\text{efd}(\Psi)) \\
\infty : \text{otherwise};
\end{cases}
\]  

\( (B_3) \) for \( \Psi \) convex, lower semi-continuous, and Gâteaux differentiable on \( \text{int}(\text{efd}(\Psi)) \neq \emptyset \) [5]:

\[
\tilde{D}_\Psi : X \times X \ni (x, y) \mapsto \begin{cases}
\Psi(x) - \Psi(y) - [x-y, D^G_y \Psi]_{X \times X} : y \in \text{int}(\text{efd}(\Psi)) \\
\infty : \text{otherwise};
\end{cases}
\]  

\( (B_4) \) for \( \Psi \) convex, lower semi-continuous, and Fréchet differentiable on \( \text{int}(\text{efd}(\Psi)) \neq \emptyset \) [71, 70] (here we generalise the definition given in these papers):

\[
\tilde{D}_\Psi : X \times X \ni (x, y) \mapsto \begin{cases}
\Psi(x) - \Psi(y) - [x-y, D^G_y \Psi]_{X \times X} : y \in \text{int}(\text{efd}(\Psi)) \\
\infty : \text{otherwise};
\end{cases}
\]  

\( (B_5) \) for \( \text{MeFun}(\mathcal{X}, \mathcal{U}(\mathcal{X}); \mathbb{R}^+) \) denoting the space of \( \mathcal{U}(\mathcal{X}) \)-measurable functions \( h : \mathcal{X} \to \mathbb{R}^+ \), \( \tilde{\mu} \) denoting a countably additive finite measure on \( \mathcal{U}(\mathcal{X}) \), \( \tilde{\Psi} : \mathbb{R} \to ]-\infty, +\infty[ \) proper, strictly convex, and differentiable on \( ]0, +\infty[ \) with \( \tilde{\Psi}(0) = \lim_{t \to +\infty} \tilde{\Psi}(t) \) and \( t < 0 \Rightarrow \tilde{\Psi}(t) = +\infty \), \( X \) given by a suitable Banach space of some elements of \( \text{MeFun}(\mathcal{X}, \mathcal{U}(\mathcal{X}); \mathbb{R}^+) \), \( \Psi(x) := \int \tilde{\mu}(x) \tilde{\Psi}(x(x)) \) [86, 48, 49, 50, 53], the map \( \tilde{D}_\Psi : X \times X \to [0, +\infty] \) is defined by:

\[
(x, y) \mapsto \int \tilde{\mu}(x) \left( \tilde{\Psi}(x(x)) - \tilde{\Psi}(y(x)) - (\text{grad}\tilde{\Psi})(y(x)) (x(x) - y(x)) \right).
\]

Some of these definitions are special cases of others, which can be written symbolically as:

\( (B_1) \supseteq (B_2) \supseteq (B_3) \supseteq (B_4) \supseteq (B_5). \)  

(25)

The definitions \( (B_1) \) and \( (B_2) \) are intended to deal with nondifferentiable functions \( \Psi \). In all cases, \( (B_1)-(B_5) \), the convexity of \( \Psi \) implies \( \tilde{D}_\Psi(x, y) \geq 0 \). If \( \Psi \) is strictly convex, \( (B_1) \) is used, and any of the following inequivalent conditions holds,

\[
D^G_+ \Psi(y; x-y) = \sup_{\tilde{z} \in \partial \Psi(y)} \{ [x-y, \tilde{z}]_{X \times X^*} \},
\]

\( (26) \)  

\[
D^G_+ \Psi(y; x-y) = - \sup_{\tilde{z} \in \partial \Psi(y)} \{ [y-x, \tilde{z}]_{X \times X^*} \},
\]

\( (27) \)  

then [88]

\[
\tilde{D}_\Psi(x, y) = 0 \iff x = y \ \forall x, y \in \text{efd}(\Psi).
\]

\( (28) \)

The equation \( (28) \) holds also for \( (B_2)-(B_4) \) under the same conditions as above, if \( \forall x, y \in \text{efd}(\Psi) \) is replaced by \( \forall x, y \in \text{int}(\text{efd}(\Psi)) \). For \( (B_3) \) the strict convexity of \( \Psi \) on \( \text{efd}(\Psi) \) implies that \( D_\Psi(\cdot, y) \) is strictly convex on \( \text{efd}(\Psi) \) [5]. If \( X \) is reflexive and \( (B_2) \) is used, then for \( (x, y) \in \text{int}(\text{efd}(\Psi)) \) [21]:

1) \( \tilde{D}_\Psi(\cdot, y) \) is proper, convex, lower semi-continuous, with \( \text{efd}(\tilde{D}_\Psi(\cdot, y)) = \text{efd}(\Psi) \);

2) \( \tilde{D}_\Psi(x, y) = \Psi(x) - \Psi(y) + \max_{\tilde{z} \in \partial \Psi(y)} \{ [y-x, \tilde{z}]_{X \times X^*} \} \);

3) \( \tilde{D}_\Psi(x, y) = \Psi(x) + \Psi^F(\tilde{z}) - [x, \tilde{z}]_{X \times X^*} \) for all \( \tilde{z} \in \partial \Psi(y) \) such that

\[
[y-x, \tilde{z}]_{X \times X^*} = \max_{\tilde{w} \in \partial \Psi(y)} \{ [y-x, \tilde{w}]_{X \times X^*} \};
\]

(29)
4) if \( \Psi \) is Gâteaux differentiable at \( y \), then
\[
\tilde{D}_\Psi(x, y) = \Psi(x) - \Psi(y) - \left[ x - y, \mathcal{D}_y \Psi \right]_{X \times X^*} = \Psi(x) + \Psi^F(\mathcal{D}_y \Psi) - \left[ x, \mathcal{D}_y \Psi \right]_{X \times X^*}; \tag{30}
\]

5) if \( \Psi \) is essentially strictly convex, then
\[
\tilde{D}_\Psi(x, y) = 0 \iff x = y; \tag{31}
\]

6) if \( \Psi \) is Gâteaux differentiable at \( \text{int}(\text{efd}(\Psi)) \) and essentially strictly convex, then
\[
\tilde{D}_\Psi(x, y) = \tilde{D}_\Psi^F(\mathcal{D}_y \Psi, \mathcal{D}_x \Psi) \forall x \in \text{int}(\text{efd}(\Psi)). \tag{32}
\]

We can conclude that the Brègman functional can be considered a distance if \( \Psi \) is strictly convex, one of the conditions \((\text{26})-(\text{27})\) holds, and \((\text{B}_1)\) is used) or \( \Psi \) is essentially strictly convex, \( X \) is reflexive, and \((\text{B}_2)\) is used).

If \( X \) is a Banach space and \( \Psi : X \to ]-\infty, +\infty[ \) is proper, then an **Alber functional** on \( X \) is defined as \([2, 3, 4]\)
\[
W_\Psi : X \times X^* \ni (x, \hat{y}) \mapsto \Psi(x) + \Psi^F(\hat{y}) - \left[ x, \hat{y} \right]_{X \times X^*} \in [0, +\infty]. \tag{33}
\]
The condition \( \Psi \) is Gâteaux differentiable at \( x \) and \( \hat{y} = \mathcal{D}_x \Psi \) is equivalent to \( W_\Psi(x, \hat{y}) = 0 \).

If \( \Psi \) is also convex, lower semi-continuous, and Gâteaux differentiable on \( \text{int}(\text{efd}(\Psi)) \neq \emptyset \), and \( X \) is reflexive, then the Young–Fenchel inequality gives
\[
\Psi(x) + \Psi^F(\mathcal{D}_x \Psi) - \left[ x, \mathcal{D}_x \Psi \right]_{X \times X^*} = 0 \quad \forall x \in \text{int}(\text{efd}(\Psi)) \tag{34}
\]
and
\[
W_\Psi(x, \mathcal{D}_x \Psi) = \Psi(x) + \Psi^F(\mathcal{D}_x \Psi) - \left[ x, \mathcal{D}_x \Psi \right]_{X \times X^*} = \tilde{D}_\Psi(x, y) \tag{35}
\]
for all \( x, y \in \text{int}(\text{efd}(\Psi)) \), with \( \tilde{D}_\Psi \) given by \((\text{B}_3)\). These equations are special cases of \((\text{30})\).

If \( X \) is a Banach space and \((\text{B}_3)\) is used, then for every \( x, y \in X \) and \( z, w \in \text{int}(\text{efd}(\Psi)) \)[\(39, 24, 22\]]
\[
\tilde{D}_\Psi(z, w) + \tilde{D}_\Psi(w, z) = \left[ z - w, \mathcal{D}_z \Psi - \mathcal{D}_w \Psi \right]_{X \times X^*} \tag{36}
\]
\[
\tilde{D}_\Psi(x, w) + \tilde{D}_\Psi(w, z) = \tilde{D}_\Psi(x, z) + \left[ x - w, \mathcal{D}_x \Psi - \mathcal{D}_w \Psi \right]_{X \times X^*} \tag{37}
\]
\[
\tilde{D}_\Psi(x, w) + \tilde{D}_\Psi(y, z) - \tilde{D}_\Psi(x, z) - \tilde{D}_\Psi(y, w) = \left[ x - y, \mathcal{D}_z \Psi - \mathcal{D}_w \Psi \right]_{X \times X^*} \tag{38}
\]
The equation \((\text{37})\) is an instance of a **generalised cosine equation**, while the equation \((\text{38})\) is an instance of a **quadrilateral equation**.

A **Brègman functional projection** \([37, 19, 21, 22]\) from a set \( C_1 \subseteq X \) onto a set \( C_2 \subseteq X \) is the function \( \tilde{\Psi}_{(C_2|C_1)} \) defined by
\[
C_1 \ni y \mapsto \left\{ x \in C_2 \cap \text{efd}(\Psi) \mid \tilde{D}_\Psi(x, y) = \inf_{z \in C_2} \left\{ \tilde{D}_\Psi(z, y) \right\} < +\infty \right\} \in \varphi(C_2). \tag{39}
\]
For \( C_1 = X \) we denote \( \tilde{\Psi}^\Psi_{C_2} := \tilde{\Psi}^\Psi_{(C_2|X)} \). If \( \tilde{\Psi}^\Psi_{C_1}(y) = \{ x \} \), then we will use the notation \( \tilde{\Psi}^\Psi_{C_2}(y) = x \).

The main problems considered in the context of Brègman functional projections are their existence, uniqueness, characterisation, and stability (which means the behaviour of sequences converging to the unique solution of the minimisation problem). Various results, depending on different sets of assumptions, are present in the literature. Here we will present the main existence, uniqueness and characterisation results obtained for the Banach space setting and the measure theoretic setting (which generalise earlier results of \([37, 56, 38, 133, 64, 19]\), obtained for \( \mathbb{R}^n \)).

9
If \((B_3)\) is used, \(\Psi\) is strictly convex on \(efd(\Psi)\), \(C \subseteq X\) is convex, and \(C \cap efd(\Psi) \neq \emptyset\), then \(\widehat{\Psi}_C^\Psi(y)\) contains at most one element. If, in addition, \(X\) is reflexive and \(C\) is nonempty and weakly closed\(^{10}\), then \(\widehat{\Psi}_C^\Psi(y) = \{\ast\} \forall y \in \text{int}(efd(\Psi))\) whenever \((C \cap efd(\Psi))\) is norm bounded or limit \(\lim_{t \downarrow +\infty} \Psi(x) \to +\infty\ \forall x \in C \cap efd(\Psi))\). Moreover, if \(X\) is an arbitrary Banach space, \((B_3)\) is used, \(\Psi\) is strictly convex, \(C \subseteq X\) is nonempty and closed, \(y \in X\), \(x \in C\), then equivalent are:

\[
\begin{align*}
\hat{D}_\Psi(z, x) + \hat{D}_\Psi(z, y) & \leq \hat{D}_\Psi(z, y) \forall z \in C, \\
\left[ z - x, \mathcal{D}_y^\Psi - \mathcal{D}_x^\Psi \right]_{X \times X^*} & \leq 0 \forall z \in C, \\
x & = \widehat{\Psi}_C^\Psi(y).
\end{align*}
\]

If \(X\) and \(\Psi\) are as above, \(K\) is a vector subspace of \(X\), then

\[
\hat{D}_\Psi(x, y) = \hat{D}_\Psi(x, \widehat{\Psi}_K^\Psi(y)) + \hat{D}_\Psi(\widehat{\Psi}_K^\Psi(y), y) \ \forall (x, y) \in K \times X.
\]

Equation \((43)\) is an instance of a generalised pythagorean equation, discovered originally by Chencov \([41, 42]\) in the case of \(D_1\) distance. In \([57, 18]\) an instance of \((43)\) has been established for \(\widehat{\Psi}_C^\Psi(y)\) with \(\hat{D}_\Psi\) defined by \((B_2)\), \(X = \mathbb{R}^n\), \(\Psi\) Legendre but not necessarily lower semi-continuous, and \((P_2)\) with \(C = K + x_0\), where \(K \subseteq X\) is a closed vector subspace and \(x_0 \in \text{int}(efd(\Psi))\). Another instance of a generalised pythagorean equation, independent of \((43)\), was established in a measure theoretic setting of \((B_3)\) in \([53]\).

The composability of Brègman projection holds when they are zone consistent \([37, 38, 19]\): that is, when the projection onto convex set (whenever it exists and is unique) is within a domain of applicability of this projection onto any other convex set (again, with existence and uniqueness). According to \([21]\), if the conditions \((P_2)\) for \(\widehat{\Psi}_C^\Psi(y) = \{\ast\}\) are used with \(\Psi\) Legendre, then \(\widehat{\Psi}_C^\Psi(y)\) is zone consistent (meaning: \(\widehat{\Psi}_C^\Psi(y) \in \text{int}(efd(\Psi))\)) and \(\widehat{\Psi}_C^\Psi(\widehat{\Psi}_C^\Psi(y)) = \widehat{\Psi}_C^\Psi(y)\). According to \([4]\), if the conditions \((P_1)\) are used, then \(\widehat{\Psi}_C^\Psi(x) = \widehat{\Psi}_C^\Psi(\widehat{\Psi}_C^\Psi(x)) = \widehat{\Psi}_K^\Psi(\widehat{\Psi}_C^\Psi(x))\) for nonempty convex closed \(C\), a vector subspace \(K\) of \(X\) with \(C \subseteq K\), and \(x \in \text{int}(efd(\Psi))\).

The Brègman functional \((B_5)\) has been characterised in \([86]\) by means of a generalised pythagorean equation. The Brègman functional \((B_3)\) has been characterised in finite dimensional case of \(X = \mathbb{R}^n\) (for which it coincides with \((B_4)\)) in \([48]\) by a set of conditions which have geometric character, and in \([16]\) by the condition that

\[
\arg \inf_{y \in X} \left\{ \int_\mathcal{X} \tilde{\mu}(x) \hat{D}_\Psi(x(x), y) \right\} = \int_\mathcal{X} \tilde{\mu}(x) x(x)
\]

for some measure space \((\mathcal{X}, \mathcal{B}(\mathcal{X}), \tilde{\mu})\) and \(\tilde{\mu}\)-integrable function \(x : \mathcal{X} \to X\). Generalisation of equation \((45)\) (but not of the associated characterisation) to \((B_4)\) in arbitrary dimension, under some additional conditions, was provided in \([70, 68]\). The equality \((45)\) was proved for the family of Liese–Vajda \(\gamma\)-distances \((85)\) in \([145]\).

\(^{10}\)Note that, by the Hahn–Banach theorem, each norm closed convex set in a reflexive Banach space is weakly closed.
3 Brègman distances

Our main objects of interest are not Brègman functionals, but Brègman distances, considered over information models. While most of research deals only with Brègman functionals on vector spaces as presented in the previous section, we will follow here the idea considered in [9, 140, 84, 8, 7, 95], according to which Brègman distances shall be defined in terms of Brègman functionals on vector spaces composed with (nonlinear) embeddings of statistical or quantum models. Apart from requirement $D_ϕ(ψ, φ) = 0 ⇐⇒ φ = ψ$, this approach stresses that a Brègman distance is an information distance defined by means of some choice of representation of an information model in a linear space, which forms a domain for corresponding Brègman functional. This formulation amounts to expose the dualistic properties of Brègman distance that are responsible for generalised pythagorean theorem. The novel aspect of our work is a systematic treatment of an extension of this approach to infinite dimensional case. The main idea is to introduce a generalisation of a Brègman functional using the Young–Fenchel inequality (11), and to subsequently define a Brègman distance over an arbitrary set $Z$, using this functional together with a pair of (not necessarily linear) embeddings $(ℓ, ℓΩ) : Z \times Z \to X^d$ into a dual pair of vector spaces. The current stage of development of this approach does not provide any strong theorems. Nevertheless, it introduces a valuable structural clarification, and we consider it an important heuristic tool that might help unify various results in the theory of Brègman distances. In particular, we will use it in Section 5 to analyse the properties of a family of quantum $γ$-distances (94).

Given a dual pair $(X, X^d, \left[\cdot, \cdot\right]_{X \times X^d})$ over $𝕂 \in \{𝕂, ℂ\}$ and a convex proper $Ψ : X \to ℝ \cup \{+∞\}$, let us define a generalised Alber functional as a map

$$X \times X^d \ni (x, y) \mapsto W_Ψ(x, y) := Ψ(x) + Ψ^F(y) - ϕ \left[ x, y \right]_{X \times X^d} \in [0, ∞].$$

(46)

By definition and (11),

i) $W_Ψ(x, y)$ is convex in each variable separately,

ii) $W_Ψ(x, y) \geq 0 \quad ∀(x, y) \in X \times X^d$,

iii) $W_Ψ(x, y) = 0 ⇐⇒ (y \in \partial Ψ(x) \text{ and } x \in \mathcal{fr}(\partial Ψ))$.

If $X$ is a Banach space and $X^d = X^*$ with duality given by Banach space duality, then a generalised Alber functional (46) coincides with an Alber functional (33).

For a given dual pair $(X, X^d, \left[\cdot, \cdot\right]_{X \times X^d})$ a dual coordinate system on a set $Z$ is defined as a map

$$(ℓ, ℓΩ) : Z \times Z \ni (ω, φ) \mapsto (ℓ(ω), ℓΩ(φ)) \in X \times X^d.$$

(47)

If $W_Ψ : X \times X^d \to [0, ∞]$ is a generalised Alber functional and $(ℓΨ, ℓΨΩ) : Z \times Z \to X \times X^d$ is a dual coordinate system such that

$$\begin{align*}
∂Ψ(x) &\neq ∅ \quad ∀x \in \mathcal{fr}(∂Ψ) \cap \text{cod}(ℓΨ) \\
ℓΨΩ(ω) &\in ∂Ψ(ℓΨ(ω)) \quad ∀ω \in Z,
\end{align*}$$

(48)

then a Brègman pre-distance is defined as a function

$$D_Ψ : Z \times Z \ni (ω, φ) \mapsto D_Ψ(ω, φ) := W_Ψ(ℓΨ(ω), ℓΨΩ(φ)) \in [0, ∞].$$

(49)

The conditions (48) can be understood either as constraints on allowed dual coordinate systems if $Ψ$ is given, or as constraints on $Ψ$ if $(ℓΨ, ℓΨΩ)$ is given. By definition, $D_Ψ(ω, φ)$ is convex in each variable separately, $D_Ψ(ω, φ) \geq 0 \quad ∀ω, φ \in Z$, and $ω = φ \Rightarrow D_Ψ(ω, φ) = 0 \quad ∀ω \in Z$. This

---

11We have proposed the definition (46) in [95], while being unaware of Alber’s work (which is summarised in Section 2).
weakening of the usual property of distance \((\omega = \phi \iff D(\omega, \phi) = 0)\) is caused by restriction of domain of \(W_\psi\) to \(\text{cod}(f_\psi) \times \text{cod}(f_\psi^0)\). In order to impose an implication in the opposite direction, one would have to impose additional conditions that are not natural at this level of generality (they will be discussed below).

Definition (49) exposes the dualistic and variational structures underlying Brègman distances. However, the standard definition of Brègman distance uses only a single coordinate system instead of a dual pair, exposing geometric properties of Brègman distance and imposing \(D_\psi(x, y) = 0 \iff x = y\) at the price of nontrivial restrictions on the domain of duality and convexity. Usually these restrictions are introduced in order to adapt to presupposed topological and differential framework (e.g. of a reflexive Banach space), which imposes some specific restrictions on Brègman distance (as exemplified by various definitions of Brègman functional in previous section), and requires one to prove that such Brègman distance encodes the Legendre case of the Fenchel duality with the dual variable \(y \in X^d\) given by some suitably defined notion of derivative (e.g. Fréchet, Gâteaux, right Gâteaux), see e.g. [37, 19, 35, 27] for standard examples in commutative case, [123] for an example in the finite dimensional noncommutative case, and [84] for an example in the infinite dimensional noncommutative case.

Our approach is different, because we do not assume any fixed framework for continuity or smoothness, so we can consider general properties of the relationship between explicitly dualistic Brègman distance and its standard (hence, restricted) version, which has both arguments represented on the same space. The transition between these two formulations in the real finite dimensional case is provided by means of bijective Legendre transformation \(L_\psi : \Theta \rightarrow \Xi\), which acts between suitable open subsets \(\Theta \subset \mathbb{R}^n\) and \(\Xi \subset \mathbb{R}^n\), and is given by the gradient,

\[
L_\psi : \Theta \ni \theta \mapsto \eta := \text{grad}\Psi(\theta) \in \Xi. \tag{50}
\]

In the coordinate-dependent form this reads

\[
\eta_i = (L_\psi(\theta))_i := \frac{\partial \Psi(\theta)}{\partial \theta^i}, \quad \theta^i = (L_\psi^{-1}(\eta))^i := \frac{\partial \Psi^F(\eta)}{\partial \eta_i}, \quad \tag{51}
\]

whenever the duality pairing is given by

\[
[\cdot, \cdot]_{\mathbb{R}^n \times \mathbb{R}^n} : \mathbb{R}^n \times \mathbb{R}^n \ni (\theta, \eta) \mapsto \theta \cdot \eta^\top := \sum_{i=1}^n \theta^i \eta_i \in \mathbb{R}. \tag{52}
\]

We will now construct a general framework for conversion between these two forms of the Brègman distance, which is independent of any particular assumptions about continuity or differentiability. The key element in this setting is the (generally, nonlinear) dualiser function. It will provide also an infinite dimensional generalisation of the bijective transformation between the dual coordinate systems that strengthens (48).

The relationship between dual coordinate systems is in the infinite dimensional case is more complicated than just replacing gradient by the Gâteaux derivative. It involves characterisation in terms of subdifferential, and depends on the function \(\Psi\) and on the specific structure of the dual pair \((X, X^d, [\cdot, \cdot]_{X \times X^d})\) of vector spaces. In [95] we have proposed the following generalisation of the Legendre transformation to the case of arbitrary dual pair of vector spaces of arbitrary dimension, which preserves its bijective character without any fixed choice of topological background. The generalisation of (50) is provided by the dualiser, defined as a map \(L_\psi : X \rightarrow X^d\) associated with a convex proper function \(\Psi : X \rightarrow \mathbb{R} \cup \{+\infty\}\) such that there exists a nonempty set \(\Theta_\psi \subseteq \text{efd}(\Psi)\) satisfying:

(i) \(L_\psi\) is a bijection on \(\Theta_\psi\),

(ii) \(\Psi^F(L_\psi(y)) - \Psi(y) = \text{re} \|y, L_\psi(y)\|_{X \times X^d} \quad \forall y \in \Theta_\psi\).
(iii) \( L_\Psi(y) \in \partial \Psi(x) \iff x = y \, \forall x, y \in \text{efd(\partial \Psi)}. \)

If such \( L_\Psi \) exists, then \( \Theta_\Psi \) will be called an **admissible domain** of \( L_\Psi \) and denoted \( \text{add}(L_\Psi) \), while \( \text{adc}(L_\Psi) \equiv \Xi_\Psi := L_\Psi(\Theta_\Psi) \) will be called its **admissible codomain**. The function \( \Psi \) will be called **dualisable** with respect to \( (X, X^d, [\cdot, \cdot]_{X \times X^d}) \) iff there exists at least one dualiser \( L_\Psi \). Each triple \( (\Theta_\Psi, \Xi_\Psi, L_\Psi) \) will be called a **generalised Legendre transformation**. A bijection

\[
L_\Psi : X \ni \Theta_\Psi \mapsto \Xi_\Psi \subseteq X^d,
\]

is a generalisation of (50). A change of domain \( X \) or a change of duality structure \( [\cdot, \cdot]_{X \times X^d} \) on \( X \) changes the available dualisers. Also, there might be several different dualisers for a given quadruple \( (X, X^d, [\cdot, \cdot]_{X \times X^d}), \Psi \). The existence of different dualisers is equivalent to \( \partial \Psi \) being a non-singleton, nonempty, set-valued function.

Given a generalised Legendre transformation \( (\Theta_\Psi, \Xi_\Psi, L_\Psi) \), we can define the **Brègman functional** \( \tilde{D}_\Psi : X \times X \to [0, +\infty] \) associated to a generalised Alber functional \( W_\Psi \) [95].

\[
\tilde{D}_\Psi(x, y) := \begin{cases} W_\Psi(x, L_\Psi(y)) = \Psi(x) - \Psi(y) - \text{re} \left[ x - y, L_\Psi(x) \right]_{X \times X^d} & : y \in \Theta_\Psi \\ +\infty & : \text{otherwise.} \end{cases}
\]

The equality above follows from the property (ii) of \( L_\Psi \). The bounded version of this functional is given by restriction of the domain of (54) to \( \tilde{D}_\Psi : \text{efd}(\Psi) \times \Theta_\Psi \to [0, +\infty[. \) From the property (iii) of \( L_\Psi \) it follows that \( \tilde{D}_\Psi \) satisfies

\[
\tilde{D}_\Psi(x, y) = 0 \iff x = y \, \forall (x, y) \in X \times X,
\]

or for all \((x, y) \in \text{efd}(\Psi) \times \Theta_\Psi\) whenever \( \tilde{D}_\Psi \) is bounded. The equivalence appears here at the price of loss of convexity of \( \tilde{D}_\Psi \) in the second variable (it is a common problem in standard treatments, see e.g. [20]). This is because using the inverse of a dualiser \( L_\Psi \) may not preserve the convexity properties. From \( \Theta \subseteq \text{efd}(\Psi) \) is follows that the definition (54) is a generalisation of (B2). We will call this definition (B\( \tilde{D} \)), and consider it as an alternative to (B2), aimed at preservation of convex and dualistic properties without reducing them to the setting of topological differentiability. From the results discussed in the previous section it follows that (B2) with reflexive \( X \) and Legendre \( \Psi \) is a special case of (B\( \tilde{D} \)). More precisely, if \( X \) is a reflexive Banach space, \( X^d = X^* \), \( \Psi \) is convex, proper, lower semi-continuous, and Legendre, then \( (\Theta_\Psi, \Xi_\Psi, L_\Psi) \) is given by \((\text{int}(\text{efd}(\Psi)), \text{int}(\text{efd}(\Psi^F)), \Xi^G) \) due to (20), and in such case (54) reduces to (23). Properties (54) and (55) follow then from (35), and property 5) in Section 2, respectively.

Let \((X, X, [\cdot, \cdot]_{X \times X^d})\) be a dual pair, let \( \Psi : X \to \mathbb{R} \cup \{+\infty\} \) be a convex proper function, let \((\Theta_\Psi, \Xi_\Psi, L_\Psi)\) be a generalised Legendre transformation, let \( Z \) be a set, and let \((\ell_\Psi, \ell^G_\Psi) : Z \times Z \to X \times X^d \) be a dual coordinate system such that \( \text{cod}(\ell^G_\Psi) \subseteq \Xi_\Psi \). Then we define the **dualistic Brègman distance** on \( Z \) as a function \( D_\Psi : Z \times Z \to [0, \infty] \) such that

\[
D_\Psi(\omega, \phi) := W_\Psi(\ell_\Psi(\omega), \ell^G_\Psi(\phi)) = \tilde{D}_\Psi(\ell_\Psi(\omega), L_\Psi^{-1} \circ \ell^G_\Psi(\phi)) = \Psi(\ell_\Psi(\omega)) - \Psi(L_\Psi^{-1} \circ \ell^G_\Psi(\phi)) - \text{re} \left[ \ell_\Psi(\omega) - L_\Psi^{-1} \circ \ell^G_\Psi(\phi), \ell^G_\Psi(\phi) \right]_{X \times X^d}.
\]

Note that it is possible to weaken the above definition by weakening the condition (iii) of definition of \( L_\Psi \) by replacing \( \text{efd}(\partial \Psi) \) and \( L_\Psi(y) \) by \( \text{efd}(\partial \Psi) \cap \text{cod}(\ell_\Psi) \) and \( L_\Psi(y) \cap \text{cod}(\ell^G_\Psi) \) respectively. Both definitions imply

\[
D_\Psi(\omega, \phi) = 0 \iff \omega = \phi \, \forall \omega, \phi \in Z.
\]

It follows that a single Brègman pre-distance (49) may have several different representations in terms of dualistic Brègman distances, depending on the choice of the dualiser \( L_\Psi \) (56), corresponding to the choice of the generalised Legendre transformation \( (\Theta_\Psi, \Xi_\Psi, L_\Psi) \). If \( D_{\Psi, L_1} \)
and \( D_{\Psi, L_2} \) are two Brègman functionals defined from a single generalised Alber functional \( W_{\Psi} \) by two dualisers \( L_1 \) and \( L_2 \) of \( \Psi \), then they are equal to each other on \( V \subseteq \text{add}(L_1) \cap \text{add}(L_2) \) if there exists a dualiser \( L_3 \) of \( \Psi \) such that \( \text{add}(L_3) = V \). Every choice of a triple \((\Theta_{\Psi}, \Xi_{\Psi}, L_{\Psi})\) that turns Brègman pre-distance to a dualistic Brègman distance can be considered as a localisation of the former.

Especially interesting case of the dualistic Brègman distance (56) is when the equality
\[
\ell^0_{\Psi} = L_{\Psi} \circ \ell_{\Psi}
\]
holds for all elements of \( Z \). Relation (58) is a special case of (48) and allows to rewrite (56) as
\[
D_{\Psi}(\omega, \phi) = D_{\Psi}(\ell_{\Psi}(\omega), \ell_{\Psi}(\phi)) = \Psi(\ell_{\Psi}(\omega)) - \Psi(\ell_{\Psi}(\phi)) - \text{re} \left[ \ell_{\Psi}(\omega) - \ell_{\Psi}(\phi), L_{\Psi} \circ \ell_{\Psi}(\phi) \right]_{X \times X^d},
\]
which does not depend on \( \ell^0_{\Psi} \). Functional of the form (59) will be called a standard Brègman distance. If \( \ell_{\Psi} \) and \( \ell^0_{\Psi} \) are bijections on \( \Theta_{\Psi} \) and \( \Xi_{\Psi} \), respectively, so the diagram
\[
\begin{array}{ccc}
\Theta_{\Psi} & \xrightarrow{L_{\Psi}} & \Xi_{\Psi} \\
\downarrow{\ell_{\Psi}^{-1}} & & \downarrow{\ell^0_{\Psi}^{-1}} \\
Z & \xrightarrow{\ell_{\Psi}} & Z
\end{array}
\]
commutes, then we will call an associated distance (59) a strong Brègman distance. In particular, if \( X = X^d = \mathbb{R}^n \) with duality given by (52), \( \Psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is convex and proper, \( L_{\Psi} \) is given by the Legendre transformation (50), \( D_{\Psi} \) is given by a functional introduced originally by Brègman in [29],
\[
D_{\Psi}(x, y) = \Psi(x) - \Psi(y) - [x - y, \text{grad} \Psi(y)]_{\mathbb{R}^n \times \mathbb{R}^n},
\]
\( Z = \mathcal{M}(A) \subseteq L_1(A)^+ \) for some mcP-algebra \( A \) or \( Z = \mathcal{M}(N) \subseteq L_1(N)^+ \) for some \( W^* \)-algebra \( N \), \( \dim Z =: n < \infty \), while \( (\ell_{\Psi}, \ell^0_{\Psi}) \) satisfies (58) by means of
\[
\ell^0_{\Psi} = \text{grad} \Psi(\ell_{\Psi}(\cdot)),
\]
so the generalised Legendre transformation is determined by such \((\Theta_{\Psi}, \Xi_{\Psi})\) that \( \text{cod}(\ell_{\Psi}) \subseteq \Xi_{\Psi} \), then the associated standard Brègman distance reads
\[
D_{\Psi}(\omega, \phi) = \Psi(\ell_{\Psi}(\omega)) - \Psi(\ell_{\Psi}(\phi)) - \sum_{i=1}^n (\ell_{\Psi}(\omega) - \ell_{\Psi}(\phi))^t (\text{grad} \Psi(\ell_{\Psi}(\phi)))_i.
\]
If \( A \) is represented in terms of a measureable space \((\mathcal{X}, \mathcal{U}(\mathcal{X}), \mathcal{U}^0(\mathcal{X}))\) and if \( \phi_1 \) and \( \phi_2 \) and densities in \( \text{MeFun}(\mathcal{X}, \mathcal{U}(\mathcal{X}); \mathbb{R}^n) \) with respect to a fixed measure \( \tilde{\mu} \) on \((\mathcal{X}, \mathcal{U}(\mathcal{X}))\) such that \( \mathcal{U}^0(\mathcal{X}) = \mathcal{U}^1(\mathcal{X}) \), then we can identify with the elements of \( L_1(\mathcal{X}, \mathcal{U}(\mathcal{X}), \bar{\mu}; \mathbb{R}^n) \), and if
\[
\Psi(\ell_{\Psi}(\phi_i)) = \int_{\mathcal{X}} \tilde{\mu}(\chi) \tilde{\Psi}(\phi_i(\chi)),
\]
then (63) takes the form (B3), with domain of \( \tilde{\Psi} \) generalised from \( \mathbb{R}^+ \) to \((\mathbb{R}^+)^n \). If \( \mathcal{H} = \mathcal{B}(\mathcal{H}) \) with \( \dim \mathcal{H} < \infty \), and \( \phi_1, \phi_2 \in \mathcal{S}_1(\mathcal{H})_0^+ \), then a condition analogous to (64) reads (cf. [79, 58])
\[
\Psi(\ell_{\Psi}(\phi_i)) = \text{tr}_{\mathcal{H}}(\tilde{\Psi}(\phi_i)),
\]
where \( \tilde{\Psi} : \mathbb{R} \to \(-\infty, +\infty[ \) is proper, operator strictly convex function, differentiable on \([0, +\infty[ \) with \( \tilde{\Psi}(0) = \lim_{t \to +\infty} \tilde{\Psi}(t) \) and \( t < 0 \Rightarrow \tilde{\Psi}(t) = +\infty \), and it is applied to density operator \( \phi_i \) in terms of functional calculus on its spectrum.
Note that the relations (62), (58), and (48) quite specifically correspond to three sectors of the information geometry theory: finite dimensional, infinite dimensional with good duality properties, and generally infinite dimensional.

From the definitions (49) and (46) it follows that every dualistic Brègman distance \( D_\Psi \) with its corresponding dual coordinate system \((\ell_\Psi, \ell^0_\Psi)\) satisfies the quadrilateral equation

\[
D_\Psi(z_1, z_2) + D_\Psi(z_4, z_3) - D_\Psi(z_1, z_3) - D_\Psi(z_4, z_2) = \text{re} \left[ \ell_\Psi(z_1) - \ell_\Psi(z_4), \ell^0_\Psi(z_3) - \ell^0_\Psi(z_2) \right]_{X \times X^d},
\]

(66)

and the generalised cosine equation

\[
D_\Psi(z_1, z_2) + D_\Psi(z_2, z_3) - D_\Psi(z_1, z_3) = \text{re} \left[ \ell_\Psi(z_1) - \ell_\Psi(z_2), \ell^0_\Psi(z_3) - \ell^0_\Psi(z_2) \right]_{X \times X^d},
\]

(67)

for all \( z_1, z_2, z_3, z_4 \in Z \) (cf. [141]). From the definition (54) of bounded Brègman functional \( \bar{D}_\Psi \) it follows that \( \bar{D}_\Psi \) satisfies the generalised cosine equation that generalises (37),

\[
\bar{D}_\Psi(x_1, x_3) + \bar{D}_\Psi(x_2, x_3) - \bar{D}_\Psi(x_1, x_2) = \text{re} \left[ x_1 - x_2, L_\Psi(x_3) - L_\Psi(x_2) \right]_{X \times X^d}
\]

(68)

\[\forall x_1, x_2, x_3 \in \text{add}(L_\Psi) \cap \text{efd}(\Psi), \]

and it also satisfies the corresponding generalisation of the quadrilateral relation (38). From (68) it follows that for any given \( x, y, \bar{y} \in \text{add}(L_\Psi) \cap \text{efd}(\Psi) \), the generalised orthogonality decomposition

\[
\bar{D}_\Psi(x, \bar{y}) + \bar{D}_\Psi(\bar{y}, y) = \bar{D}_\Psi(x, y) \quad \forall x \in \text{add}(L_\Psi) \cap \text{efd}(\Psi)
\]

(69)

is equivalent with the orthogonality condition,

\[
\text{re} \left[ x - \bar{y}, L_\Psi(y) - L_\Psi(\bar{y}) \right]_{X \times X^d} = 0.
\]

(70)

Moreover, the equivalence holds also if \( = \) is replaced by \( \ge \) in (69) and \( = \) is replaced by \( \le \) in (70). As we will see below, under suitable assumptions that guarantee the existence and uniqueness of solution of the corresponding variational problem, the generalised orthogonal decomposition can be turned into a theorem stating the existence and uniqueness of generalised additive decomposition of information distance under projection onto subspace (submodel), known as generalised pythagorean theorem (or equation).

Let \( y \in \text{add}(L_\Psi) \cap \text{efd}(\Psi) \), let \( C \subseteq \text{add}(L_\Psi) \cap \text{efd}(\Psi) \) be nonempty, convex, and containing at least one element \( z \) such that \( \bar{D}_\Psi(z, y) < \infty \), let \( x \in C \). In such case the Brègman functional projection (39) of \( y \) using \( \bar{D}_\Psi \) will be denoted

\[
\bar{y} \in \bar{\Psi}^{\Psi}_C(y) = \arg \inf_{x \in C} \{ \bar{D}_\Psi(x, y) \}.
\]

(71)

The main problem with this definition is that in general case \( \bar{\Psi}^{\Psi}_C(y) \) might not exist or might be nonunique. The existence and uniqueness can follow from various assumptions. In particular, if \( X \) is a locally convex space, \( C \) is weakly compact, and \( \bar{D}_\Psi \) is weakly lower semi-continuous, then the existence can be guaranteed by means of Bauer’s theorem [17]. On the other hand, if \( X \) is a reflexive Banach space, \( C \) is closed, \( \bar{D}_\Psi \) is lower semi-continuous, strictly convex, and Gâteaux differentiable at \( y \), with \( \text{int}(\text{efd}(\bar{D}_\Psi)) \neq \emptyset \), \( C \cap \text{efd}(\bar{D}_\Psi) \neq \emptyset \) and \( y \in \text{int}(\text{efd}(\bar{D}_\Psi)) \), then \( \bar{\Psi}^{\Psi}_C(y) \) is at most a singleton [27]. The conjunction of these two conditions is sufficient to guarantee the existence and uniqueness of \( \bar{\Psi}^{\Psi}_C(y) \). Unfortunately, we know neither the sufficient conditions for existence that would not require lower semi-continuity, nor the sufficient conditions for uniqueness that would not require Gâteaux differentiability.

If there exists a unique Brègman functional projection \( \bar{y} = \bar{\Psi}^{\Psi}_C(y) \) for \( y \in \text{add}(L_\Psi) \cap \text{efd}(\Psi) \), such that \( (y, \bar{y}) \) satisfies the orthogonality condition (70), then \( \bar{y} = \bar{\Psi}^{\Psi}_C(y) \) is called orthogonal. Property (69) generalises in such case the additive decompositions of norm under linear projections on closed convex subsets in the Hilbert space to the class of nonlinear
projections onto convex subsets \( C \) in the linear space \( X \). Note that the ‘orthogonality’ of projection is understood in the sense of the bilinear duality pairing \( \langle \cdot, \cdot \rangle_{X \times X^d} \), while the nonlinearity of projection \( \Psi_C^\Phi \) corresponds to the nonlinear dualiser \( \Psi_\Phi \). In particular, if \( \bar{D}_\Psi \) is given by \( (B_3) \), then condition \( (70) \) turns to equality in \( (41) \), so the orthogonality condition \( (70) \) satisfied by \( \bar{y} = \Psi_\Phi^\Phi(y) \) turns to generalised pythagorean equation \( (43) \).

Given a dualistic Brègman distance \( D_\Psi \) on \( Z \) and \( K_1, K_2 \subseteq Z \), we define a dualistic Brègman projection as a map

\[
\Psi_{D_\Psi}^{K_2|K_1} : K_1 \ni \phi \mapsto \arg \inf_{\omega \in K_2} \{ D_\Psi(\omega, \phi) \} \subseteq \psi(K_2),
\]

with \( \Psi_{D_\Psi}^{K_2|K_1} \). If \( \ell_\Psi \times \ell_\Psi^0 \) is bijective on \( K_2 \times K_1 \), then the existence (resp., uniqueness) of \( \Psi_{D_\Psi}^{K_2|K_1}(\phi) \) follows from the existence (resp., uniqueness) of \( \bar\Psi_{D_\Psi}^{K_2|K_1}(\ell_\Psi^0(K_1)) \). The generalised cosine equation \( (68) \) and the above discussion leads us to call a dualistic Brègman projection \( \Psi_{D_\Psi}^{K_2} \) orthogonal iff it is a singleton and satisfies

\[
\text{re} \left[ \ell_\Psi(\phi) - \ell_\Psi(\Psi_{K_2}^{D_\Psi}(\psi)), \ell_\Psi(\psi) - \ell_\Psi^0(\Psi_{K_2}^{D_\Psi}(\psi)) \right]_{X \times X^d} = 0 \quad \forall \phi \in K,
\]

which is equivalent the generalised pythagorean equation

\[
D_\Psi(\phi, \Psi_{K_2}^{D_\Psi}(\psi)) + D_\Psi(\Psi_{K_2}^{D_\Psi}(\psi), \psi) = D_\Psi(\phi, \psi) \quad \forall \phi \in K.
\]

The problem of characterisation of orthogonal \( \Psi_{D_\Psi}^{K_2} \) for a given \( D_\Psi \) and \( K \) remains open.

Let us summarise the insights gained in last two sections. There are few different candidates for the general notion of a Brègman distance on a general Banach space:

\begin{itemize}
  \item \((BD_1)\): the Brègman functional \( \bar{D}_\Psi \) defined by \( (B_1) \) under additional assumptions that \( \Psi \) is strictly convex on efd(\( \Psi \)) and that one of the equations \( (26)-(27) \) holds;
  \item \((BD_2)\): the Brègman functional \( \bar{D}_\Psi \) defined by \( (B_D) \), with duality given by Banach space duality;
  \item \((BD_3)\): the dualistic Brègman distance \( (56) \), which is defined as a special case of \( (B_D) \), but its domain is shifted to the space \( Z \), which in turn can be an arbitrary subset of a Banach space;
  \item \((BD_4)\): the Brègman functional \( \bar{D}_\Psi \) defined by \( (B_2) \) for reflexive \( X \) and \( \Psi \) essentially strictly convex on int(efd(\( \Psi \))) \( \neq \emptyset \);
  \item \((BD_5)\): defined as \( (BD_4) \), but with an additional assumption of essential Gâteaux differentiability on int(efd(\( \Psi \))). This is a special case of both \( (BD_1) \) and \( (BD_2) \). The essential Gâteaux differentiability implies Gâteaux differentiability on int(efd(\( \Psi \))) \( \neq \emptyset \), so this definition assumes, a posteriori, \( D_\Psi \) defined by \( (B_3) \).
\end{itemize}

In principle, there are four main properties that one would expect from a general notion of the Brègman distance:

- it should be a distance;
- it should possess well defined existence and uniqueness properties for the Brègman projections onto a well defined class of subsets;
- it should allow for generalised pythagorean and cosine theorems;
- it should allow for composable projections.
All above candidates satisfy the first condition. The second and fourth conditions can be guaranteed at the level of (BD_3). The third condition requires either to strengthen (B2) in (BD_5) with an additional assumption of strict convexity of \( \Psi \), in order to use (P1), or to use (BD_3) with an additional orthogonality condition (73). However, the condition (73) is abstract and we do not know what are necessary and sufficient topological/convex conditions for it to hold. On the other hand, using (BD_5) as a Brègman distance restricts the underlying Banach space to be reflexive.

4 Quantum Brègman distances

The discussion in the last two sections provides us a fast track to the construction and investigation of the properties of quantum Brègman distances.

**Definition 4.1.** Let \( X \) be a reflexive Banach space, \( \Psi : X \to ]-\infty, +\infty[ \) be convex, lower semi-continuous, and Legendre, let \( \ell : U \to X \), for \( U \subseteq N_+^* \) and \( U \cap N_+^* \neq \emptyset \), be a map that is bijective on its codomain, \( \ell(U) \subseteq \text{int(efd}(\Psi)) \), and let \( \tilde{D}_\Psi \) be a Brègman functional defined by (23), i.e. (B_3). Then the reflexive quantum Brègman distance on \( V \subseteq U \cap N_+^* \) reads

\[
D_\Psi(\phi, \psi) := \tilde{D}_\Psi(\ell(\phi), \ell(\psi)) \quad \forall (\phi, \psi) \in V \times V.
\]  

(75)

**Proposition 4.2.** \( D_\Psi \) given by (75) is a standard Brègman distance in the sense of (59). Furthermore, if \( C \subseteq V \subseteq N_+^* \) is nonempty, \( \ell(C) \subseteq \text{int(efd}(\Psi)) \subseteq X \) is convex, \( C \) is closed in the topology induced by \( \ell^{-1} \) from the weak topology of \( X \), and \( \psi \in V \), then:

i) \[
P_C^{D_\Psi}(\psi) := \arg\inf_{\phi \in C} \{ D_\Psi(\phi, \psi) \} = \{ \ast \}.
\]

(76)

ii) if \( \ell(C) \) is a vector subspace of \( X \), then the generalised pythagorean equation holds:

\[
D_\Psi(\omega, \psi) = D_\Psi(\omega, \Psi_C^{D_\Psi}(\psi)) + D_\Psi(\Psi_C^{D_\Psi}(\psi), \psi) \quad \forall (\omega, \psi) \in C \times V.
\]

(77)

**Proof.** Follows directly from (P2).

\[\square\]

**Remark 4.3.** Note that in the above proposition \( C \) does not have to be convex (resp.: closed, affine) in \( N_+^* \) this what matters is only whether it becomes convex (resp.: closed, affine) under the coordinate system \( \ell \). We will use the terminology \( \ell \)-convex (resp.: \( \ell \)-closed, \( \ell \)-affine) to refer to this property.

**Proposition 4.4.** Let \( D_{\Psi_{1,U_1}} \) and \( D_{\Psi_{2,U_2}} \) be two reflexive quantum Brègman distances, with \( U_1 \subseteq N_+^* \supseteq U_2 \), \( U_1 \cap U_2 \neq \emptyset \), \( \Psi_1 : X_1 \to ]-\infty, +\infty[ \), \( \Psi_2 : X_2 \to ]-\infty, +\infty[ \), \( \tilde{\ell}_1 : U_1 \to X_1 \), \( \tilde{\ell}_2 : U_2 \to X_2 \) satisfying: \( D_{\Psi_{1,U_1}}(u_1, u_2) = D_{\Psi_{2,U_2}}(u_1 \cap u_2), \psi \circ \tilde{\ell}_1 = \tilde{\ell}_2, \tilde{\ell}^{-1} \circ \tilde{\ell}_2 = \tilde{\ell}_1, \Psi_1 \circ \tilde{\ell}^{-1} = \Psi_2, \Psi_2 \circ \psi = \Psi_1 \), where \( \psi : X_1 \to X_2 \) is an isometric isomorphism. Then \( \Psi_C^{D_{\Psi_{1,U_1}}} \) (resp., \( \Psi_C^{D_{\Psi_{2,U_2}}} \)) is zone consistent over the class of nonempty sets \( C_i \subseteq U_1 \cap U_2 \), \( i \in I \) (some index set), if \( \ell_1(C_1) \) (resp., \( \ell_2(C_1) \)) are convex and \( C_i \) are closed in the topology induced by \( \tilde{\ell}_1^{-1} \) (resp., \( \tilde{\ell}_2^{-1} \)) from the weak topology of \( X_1 \) (resp., \( X_2 \)). Furthermore, in such case the projections \( \Psi_C^{D_{\Psi_{1,U_1}}} \) and \( \Psi_C^{D_{\Psi_{2,U_2}}} \) agree.

**Proof.** Straightforward from (P2), the fact that isometric isomorphism preserves the norm topology, and that for a convex set in a reflexive Banach space the weak and norm topological closures coincide.

\[\square\]

A family \( \{ D_{\Psi_{j,U_j}} \mid j \in J \} \) satisfying the above conditions for some index set \( J \) will be called a reflexive Brègman system.
Remark 4.5. We would like to be able to strengthen this result, generalising a definition of a Brègman system to a larger class of quantum Brègman distances, which are also strong Brègman distances in terms of (60), but are not restricted to reflexive spaces $X$. In such case the interplay between the domains of definition of $\ell_\Psi$ and $L_\Psi$ becomes even more apparent, because (in general) instead of an isometric isomorphism $\iota$ we would need a weaker map, satisfying the compatibility conditions with respect to $\ell_j$ and $\Psi_j$, $j \in \{1, 2\}$, and guaranteeing the transferability of existence and uniqueness results (as well as of composability/zone consistency and pythagoreanity/affinity criteria).

On the other hand, some properties of (B1) applied to $X = \Phi_1(\mathcal{H}) = \mathcal{B}(\mathcal{H})$, for $\dim \mathcal{H} < \infty$ were analysed by Petz in [123] (see also [58]). Furthermore, Jenčová [85] defined a nonreflexive noncommutative Orlicz space

$$L_{Y_\phi}(N) := \{ x \in N_{sa} \mid \exists \lambda > 0 \; Y_\phi(\lambda x) < \infty \}$$

where

$$Y_\phi : N_{sa} \ni h \mapsto \frac{1}{2} \left( \phi^h(1) + \bar{\phi}^{-h} \right) - 1 \in \mathbb{R}^+,$$

$$\phi^h(1) := \inf_{\omega \in N_{\nu}} \{ D_1[1]_{N_{\nu}}(\omega, \phi) + \omega(h) - \omega(1) \},$$

$$|x|_{Y_\phi} := \inf \{ \lambda > 0 \mid Y_\phi(\lambda^{-1} x) \leq 1 \}.$$  

Given the family of open subsets $U(\phi) := \{ x \in L_{Y_\phi}(N) \mid |x|_{Y_\phi} < 1 \}$ for $\phi \in N_{0+}$, the maps $w_\phi^{-1} : U(\phi) \ni h \mapsto \bar{\phi}^{-h} \in N_{0+}$ are diffeomorphisms that form a smooth atlas $\{(w_\phi^{-1}(U(\phi)), w_\phi) \mid \phi \in N_{0+}\}$ that turn $N_{0+}$ into a smooth manifold. The conditions for the local existence and uniqueness of $\Psi_C^{D_1}$ are already incorporated in this construction, the pythagorean theorem (as follows from the earlier results of Araki [10, 11] and Donald [63]) holds for the affine subsets of the positive cones of $N_{sa}$ and $L_{Y_\phi}(N)$ as well, while the (smoothness of) local composability of $D_1$-projections is precisely what underlines the proof that the above system of maps makes a (smooth) atlas. Thus, with a suitable Orlicz space construction one can Brègman-localise a globally defined distance, even maintaining the compatibility conditions in a resulting Brègman system to be given by isometric isomorphisms, despite the lack of reflexivity. This leads us to propose the following definition.

Definition 4.6. Let $X$ be a topological vector space, $\Psi : X \to [-\infty, +\infty]$ be proper, convex, lower semi-continuous, with $\text{efd}(\Psi) \neq \emptyset$, let $D_\Psi$ be defined by (21), i.e. (B1), let $\bar{\ell} : U \to \text{efd}(\Psi)$, for $U \subseteq N_*$ and $U \cap N_{0+}^* \neq \emptyset$, be bijective on its codomain. Then a quantum Brègman distance on $V \subseteq U \cap N_{0+}^*$ is defined as (75), whenever:

(i) $D(\phi, \psi) = 0 \iff \phi = \psi \forall \phi, \psi \in V$,  

(ii) for any $\psi \in V$ and $\bar{\ell}$-convex $\bar{\ell}$-closed $C \subseteq V$ (76) holds,  

(iii) $\Psi_{\Psi_{C_i}}^{D_\Psi}$ are composable over $\bar{\ell}$-convex $\bar{\ell}$-closed sets $C_i \subseteq V$,  

(iv) if $C \subseteq V$ in (ii) is also $\bar{\ell}$-affine, then (77) holds.

A family $\{D_{\Psi_j, U_j} \mid j \in J\}$ of quantum Brègman distances will be called a Brègman system iff for every $i, j \in J$ and for every nonempty $C \subseteq U_i \cap U_j$ it holds that $C$: is $\bar{\ell}_i$-convex iff it is $\bar{\ell}_j$-convex, is $\bar{\ell}_i$-closed iff it is $\bar{\ell}_j$-closed, and is $\bar{\ell}_i$-affine iff it is $\bar{\ell}_j$-affine.

Definition 4.7. If $D_\Psi$ is a quantum Brègman distance over $U \subseteq N_{0+}^*$, defined using a coordinate system $\bar{\ell}$, then the category of $\bar{\ell}$-convex $\bar{\ell}$-closed subsets of $U$ with $D_\Psi$-projections as arrows will be denoted as $\mathbf{QMod}_{D_\Psi}^U(U)$. A full subcategory of $\mathbf{QMod}_{D_\Psi}^U(U)$, with the class of
objects restricted to ℓ-affine subsets will be denoted QAff$^{D_y}(U)$. Admitting all Brégnan systems (over all preaulas of W*-algebras)\(^{12}\) that contain $D_y$ determines a category QMod$^{D_y}$. Its restriction to ℓ-affine subsets gives a “pythagorean” category QAff$^{D_y}$.

The category QMod$^{D_y}$ is an instance of QMod$^D$ defined abstractly in Section 1. By construction, all morphisms in QAff$^{D_y}$ satisfy the generalised pythagorean equation (77), so one can think of QAff$^{D_y}$ as the subcategory of “ideal” signal+noise ‘orthogonally decomposing’ figures. The construction of these two categories does not require to have a single Brégnan distance defined over all positive cones of all noncommutative $L_1$ spaces. What is important here is an availability of the coordinate systems ℓ$_i$ that are used to define the ‘local’ instances of a Brégnan distance as well as to establish the sheaf compatibility between these instances.

Now we will provide an example implementing the above abstract definitions with some nontrivial and novel content. Let $X$ be a real locally convex topological vector space, and $X^t$ its topological dual. A function $\Upsilon : X \to [0, \infty]$ that is convex, lower semi-continuous, satisfying $\Upsilon \not= +\infty$, $\Upsilon \not= 0$, $\Upsilon(0) = 0$, $\Upsilon(\lambda x) = 0 \forall \lambda > 0 \Rightarrow x = 0$, $x \not= 0 \Rightarrow \lim_{\lambda \to +\infty} \Upsilon(\lambda x) = +\infty$, is called a Young function. A Young–Brégnan–Orlicz dual of a Young function $\Upsilon$ is defined as [25]

$$\Upsilon^Y : X^t \ni y \mapsto \Upsilon^Y(y) := \sup_{x \in X} \{\|x, y\|_{X \times X^t} - \Upsilon(x)\}. \quad (82)$$

If $\{y \in X^t \mid \|x, y\| = 0 \forall x \in efd(\Upsilon)\} = 0$ then $\Upsilon^Y$ is also a Young function [85]. A Young function $\Upsilon$ is said to satisfy a global $\Delta_2$ condition iff [25] $\exists \lambda > 0 \forall x \geq 0 \Upsilon(\lambda x) \leq \lambda \Upsilon(x)$.

Let $L_\Upsilon(N)$ be a noncommutative Orlicz space defined as $L_\Upsilon(N) := \overline{M_{\Upsilon^Y}}$, using a Young function $\Upsilon : N^{sa} \to [0, \infty]$:

$$N^\Upsilon := \{x \in N^{sa} \mid \lim_{\lambda \to +0} \Upsilon(\lambda x) = 0\} = \{x \in N^{sa} \mid \exists \lambda > 0 \Upsilon(\lambda x) < \infty\},$$

$$|\cdot|_\Upsilon : N^\Upsilon \ni x \mapsto \inf\{\lambda > 0 \mid \Upsilon(\lambda^{-1} x) \leq 1\} \in \mathbb{R}^+.$$

If $\Upsilon$ satisfies a global $\Delta_2$ condition, then ($L_\Upsilon(N)$)$^* \cong L_{\Upsilon^Y}(N)$, where $\Upsilon^Y$ is calculated with respect to the Banach duality between $N^{sa}$ and $N^{sa*}$. If also $\Upsilon^Y$ satisfies global $\Delta_2$ conditions and $N$ is semi-finite, then $L_\Upsilon(N)$ and $L_{\Upsilon^Y}(N)$ are reflexive [43, 61, 127]\(^{13}\). In such case $L_\Upsilon(N)$ is isometrically isomorphic to [101]

$$L_\Upsilon(N, \tau) := \{x \in M(N, \tau) \mid \tau(\Upsilon(|x|)) < \infty\}, \quad (83)$$

where $\tau$ is an arbitrary faithful normal semi-finite trace on $N$, while $\Upsilon : [0, \infty] \to [0, \infty]$ is an Orlicz function (that is: convex, continuous, nondecreasing, satisfying $\Upsilon(0) = 0$ and $\lim_{\lambda \to +\infty} \Upsilon = +\infty$) determined by $\Upsilon(x) = \tau(\Upsilon(|x|)) \forall x \in N^\Upsilon$. In [14] it was shown that for any two semi-finite traces $\tau_1, \tau_2 \in \mathcal{W}_0(N)$ the spaces $L_\Upsilon(N, \tau_1)$ and $L_\Upsilon(N, \tau_2)$ are isometrically isomorphic, provided that $\Upsilon$ satisfies a global $\Delta_2$ condition.

**Definition 4.8.** Let $N$ be a semi-finite $W^*$-algebra, let $\Upsilon : N^{sa} \to [0, \infty]$ be a Young function satisfying a global $\Delta_2$ condition, and let $\Psi : L_\Upsilon(N) \to (-\infty, +\infty]$ be convex, lower semi-continuous, and Legendre. Then a quantum Brégnan–Orlicz distance is defined as

$$D_\Psi,\Upsilon : N^\Upsilon \times N^\Upsilon \ni (\phi, \psi) \mapsto D_\Psi(\ell_\Upsilon(\phi), \ell_\Upsilon(\psi)) \in [0, \infty], \quad (84)$$

where $\ell_\Upsilon : N^\Upsilon \ni \phi \mapsto \Upsilon(\Upsilon(\Upsilon^{-1}(\Upsilon^{-1}(\phi))) \in L_\Upsilon(N, \tau)$, and $\Upsilon$ is defined as (B3) for $X = L_\Upsilon(N, \tau)$.

\(^{12}\)Which means that one assumes that $U_i \cap U_j \subseteq (N_i)_i^+$, and $N_{ij}$ varies over all $W^*$-algebras, provided ($\ell_\Upsilon, \ell_\Upsilon$) are well defined.

\(^{13}\)The statement of the sufficient condition for reflexivity in Corollary 4.3 of [127] is missing the requirement of the global $\Delta_2$ condition for $\Upsilon^Y$. For example, $\Upsilon(x) = (1 + |x|) \log(1 + |x|) - |x|$ satisfies global $\Delta_2$ condition, but its YBO dual, $\Upsilon^Y(x) = e^{|x|} - |x| - 1$, does not.
A direct calculation shows that \( D \) does not depend on the choice of \( \tau \). By construction, \( D_{\Psi, \chi} \) is a reflexive quantum Bréгman distance (in particular, continuous nondecreasing under ‘information loss’).

By imposing the condition of monotonicity under coarse graining on the dualistic Bréгman systems \( \Psi \) or the corresponding dual coordinate systems \((\ell_\Psi, \ell_\Phi)\). Such families of information distances are of special interest, because they satisfy two main information theoretic constraints: existence of orthogonal decomposition under 

\[
D(\gamma, \phi) := \begin{cases} 
\int \frac{1}{(1-\gamma)} \left( \gamma \mu_\omega + (1-\gamma) \nu_\phi - \nu_\phi \left( \frac{\mu_\omega}{\nu_\phi} \right)^\gamma \right) & : \gamma \in [0,1], \mu_\omega \ll \nu_\phi \\
\int_{+\infty} \left( \gamma \mu_\omega + (1-\gamma) \nu_\phi - \nu_\phi \left( \frac{\mu_\omega}{\nu_\phi} \right)^\gamma \right) & : \gamma \in \{0,1\}, \mu_\omega \ll \nu_\phi \\
\text{otherwise,} 
\end{cases}
\]

where the right limit, \( \tilde{\gamma} \to^+ \gamma \), is considered for \( \gamma = 0 \), while the left limit, \( \tilde{\gamma} \to^- \gamma \), is considered for \( \gamma = 1 \). Here \( \mu_\omega \) and \( \nu_\phi \) are finite positive measures corresponding to the positive integrals \( \omega \) and \( \phi \). It follows directly that \( D_\gamma \) satisfies

i) \( \nu \ll \mu \ll \nu \Rightarrow D_\gamma(\mu, \nu) = D_{1-\gamma}(\mu, \nu) \forall \gamma \in [0,1], \)

ii) \( D_\gamma(\lambda \mu, \lambda \nu) = \lambda D_\gamma(\mu, \nu) \forall \lambda \in ]0,\infty[. \)

A direct calculation shows that \( D_\gamma \) is an \( f \)-distance with

\[
f_\gamma(t) = \begin{cases} 
\frac{1}{\gamma} + \frac{1}{1-\gamma} t - \frac{1}{\gamma (1-\gamma)} t^\gamma & : \gamma \in [0,1] \\
t \log t - (t - 1) & : \gamma = 1 \\
- \log t + (t - 1) & : \gamma = 0. 
\end{cases}
\]

All above properties hold for the domain of \( \gamma \) extended from \( [0,1] \) to \( \mathbb{R} \) with the conditions satisfied for \( \gamma \in [0,1] \) extending to \( \gamma \in \mathbb{R} \setminus \{0,1\} \). Nevertheless, we will consider this extension separately.

The Liese–Vajda \( \gamma \)-distances are generalised Bréгman distances for \( \gamma \in [0,1] \) (see below), while for \( \gamma \in \{0,1\} \) and \( \text{dim}(L_1(\mathcal{A})) =: n < \infty \) they are standard Bréгman distances in the sense of \( (B_3) \) and \( (63) \) with \( X = \mathbb{R}^n \) and \( \Psi_{\gamma=1}(x) = \sum_{i=1}^n (x_i \log(x_i) - x_i + 1) \).

Amarí [7] has shown that the Liese–Vajda \( \gamma \)-distances with \( \gamma \in \mathbb{R} \) can be characterised in the finite dimensional case as a unique (up to a multiplicative constant) class of standard Bréгman distances that are monotone under coarse grainings.\(^{14}\) Csizsárk [48] (see also [119, 78])

\(^{14}\)The assumption of decomposability used in Amari’s proof is a discrete version of \( (64) \), so, together with \( (62) \), it amounts to a choice of a specific dual coordinate system.
has shown that under restriction to \( L_1(A)^+ \), the uniqueness result is stronger, characterising the pair \( \{ D_1|_{L_1(A)^+}, D_0|_{L_1(A)^+} \} \). So far no corresponding characterisation results for the noncommutative case are known.\(^{15}\)

Consider the \( \gamma \)-embedding functions on \( \mathcal{N}_\gamma^+ \) valued in \( L_{1/\gamma}(\mathcal{N})^+ \) spaces:

\[
\ell_\gamma : \mathcal{N}_\gamma^+ \ni \omega \mapsto \ell_\gamma(\omega) := \gamma^{-1} \omega^\gamma \in L_{1/\gamma}(\mathcal{N}),
\]

(87)

with \( \gamma \in [0, 1] \). These functions arise as restrictions of

\[
\tilde{\ell}_\gamma : \mathcal{N}_\gamma \ni \omega \mapsto \tilde{\ell}_\gamma(\omega) := \gamma^{-1} u|\omega|^\gamma \in L_{1/\gamma}(\mathcal{N}),
\]

(88)

which are bijections due to uniqueness of the polar decomposition \( \omega = |\omega|(\cdot, u) \). In particular, \( \tilde{\ell}_{1/2} \) maps bijectively \( \mathcal{N}_\gamma \) onto Hilbert space \( L_2(\mathcal{N}) \). The special case of the function (87) was introduced by Nagaoka and Amari \([116]\) in commutative finite dimensional setting,

\[
\ell_\gamma : \mathcal{M}(\mathcal{X}, \mathcal{U}(\mathcal{X}), \tilde{\mu}) \ni p(x) \mapsto \ell_\gamma(p(x)) := \left\{ \begin{array}{ll}
\frac{1}{\gamma} p(x)^\gamma & : \gamma \in [0, 1], \\
\log p(x) & : \gamma = 0
\end{array} \right\} \in L_{1/\gamma}(\mathcal{X}, \mathcal{U}(\mathcal{X}), \tilde{\mu})^+.
\]

(89)

Since then it became a standard tool of information geometry theory. However, the Nagaoka–Amari formulation (89), as well as its noncommutative generalisations \([79, 9]\)\(^{16}\),

\[
\ell_\gamma : \mathcal{G}_1(\mathcal{H})^+ \cong \mathfrak{B}(\mathcal{H})^+ \ni \rho \mapsto \ell_\gamma(\rho) := \left\{ \begin{array}{ll}
\frac{1}{\gamma} \rho^\gamma & : \gamma \in [0, 1], \\
\log \rho & : \gamma = 0
\end{array} \right\} \in L_{1/\gamma}(\mathfrak{B}(\mathcal{H}), \text{tr}_\rho)^+,
\]

(90)

and \([84]\)

\[
\ell_\gamma^\psi : \mathcal{N}_\gamma^+ \ni \omega \mapsto \ell_\gamma^\psi(\omega) := \gamma^{-1} \Delta_{\omega, \psi}^\gamma \in L_{1/\gamma}(\mathcal{N}, \psi) \text{ for } \gamma \in [0, 1],
\]

(91)

use \( \gamma \)-powers of densities (Radon–Nikodým quotients) with respect to a fixed reference measure \( \tilde{\mu} \), trace \( \text{tr}_\mu \), or weight \( \psi \in W_0(\mathcal{N}) \), respectively. (For a semi-finite \( \mathcal{N} \) the embeddings (89) and (91) are the special cases of \( \ell_\gamma^\psi \) used in (84).) This restricts the generality of formulation. An important attempt to solve this problem in the commutative case was made by Zhu \([142, 145]\), who considered the spaces of measures constructed through an equivalence relation based on \( \gamma \)-powers of Radon–Nikodým quotients, but without fixing any particular reference measure (hence, without passing to densities). However, his work remained unfinished and widely unknown, and it covered only the finite measures. The embeddings (87) solve these problems in the noncommutative case.

The most general quantum distance that has been shown so far to be a standard Brégnan distance that is monotone under coarse grainings is the \textit{Jenčová–Ojima \( \gamma \)-distance} \([83, 84, 118]\),

\[
D_\gamma(\omega, \phi) := \left\{ \begin{array}{ll}
\frac{\gamma \omega(1) + (1-\gamma)\phi(1) - \| \Delta_{\omega, \psi}^\gamma - \Delta_{\phi, \psi}^\gamma \|_\psi }{\gamma(1-\gamma)} & : \omega \ll \phi \\
+\infty & : \text{otherwise}
\end{array} \right\}
\]

(92)

where \( \gamma \in ]0, 1[, \psi \in W_0(\mathcal{N}) \) is an arbitrary reference functional, \( \| \cdot \|_\psi \) is the Banach space duality pairing between the Araki–Masuda noncommutative \( L_{1/\gamma}(\mathcal{N}, \psi) \) and \( L_{1/(1-\gamma)}(\mathcal{N}, \psi) \) spaces (see \([12]\) or \([96]\)). However, (92) is not a canonical noncommutative generalisation of (85). The construction of (92) is dependent on the choice of fixed reference weight \( \psi \), while (85) does not depend on any additional measure. (Nevertheless, the values taken by (92) are independent of the choice of \( \psi \).) Using the Falcone–Takesaki theory we can make the reference-independent approach valid in all cases, including the infinite dimensional noncommutative one \([94, 95]\).\(^{15}\)

\(^{15}\)However, one should note Donald’s \([62]\) characterisation of Donald’s distance, which coincides with \( D_1|_{\mathcal{N}_1^+} \) at least for injective \( W^\ast \)-algebras, as well as Petz’s \([122]\) characterisation of \( D_1|_{\mathcal{N}_1^+} \) for injective \( W^\ast \)-algebras (however, see \([15]\) for a discussion of a mistake in Petz’s proof).

\(^{16}\)The condition \( \gamma \in [0, 1] \) in (89) and (90) can be replaced by \( \gamma \in \mathbb{R} \setminus \{0\} \). However, for \( \gamma \in \mathbb{R} \setminus [0, 1] \) the codomain of \( \ell_\gamma \) is no longer given by the \( L_{1/\gamma} \) space, so for some purposes we will consider this case separately.
Proposition 5.5. Whenever required, the family (94) can be extended to the range \(\gamma \in \mathbb{R}\) with the condition \(\gamma \in [0,1]\) replaced by \(\gamma \in \mathbb{R} \setminus \{0,1\}\), using the fact that (140) is well defined for any \(\gamma > 0\), and defining \(D_\gamma(\phi, \omega)\) for \(\gamma < 0\) as \(D_{1-\gamma}(\omega, \phi)\).

Proposition 5.3. A quantum \(\gamma\)-distance (94) for \(\gamma \in [0,1]\) is an \(\mathfrak{f}\)-distance on \(N^*_\gamma\) with \(\mathfrak{f}\) given by (86).

Proof. Applying (86) for \(\gamma \in [0,1]\) to (3) for \(\omega \ll \phi\) and using identity (140), we obtain

\[
D_\gamma(\omega, \phi) = \left\langle \xi_\pi(\phi), \left( \frac{1}{\gamma} + \frac{1}{1-\gamma} \Delta_{\omega,\phi} - \frac{1}{\gamma(1-\gamma)} \Delta^\gamma_{\omega,\phi} \right) \xi_\pi(\phi) \right\rangle_H
= \frac{1}{\gamma} \phi(\mathbb{I}) + \frac{1}{1-\gamma} \omega(\mathbb{I}) - \frac{1}{\gamma(1-\gamma)} \int \omega^\gamma \phi^{1-\gamma} = D(\omega, \phi). \tag{95}
\]

We have also used the identity \(\Delta^{1/2}_{\omega,\phi,\pi}(\phi) = \text{supp}(\phi)\xi_\pi(\phi)\), which holds for any \(\phi, \omega \in N^*_\gamma\). Using \(I_0(t) = \lim_{\gamma \to +0} f_\gamma(t)\) and \(f_1(t) = \lim_{\gamma \to -1} f_\gamma(t)\) in (86), we obtain \(D_\gamma(\omega, \phi) = D_\gamma(\omega, \phi)\) also for \(\gamma \in \{0,1\}\). \(\square\)

Corollary 5.4. From the above proof it follows that, for \(\gamma \in \{0,1\}\), (94) can be written explicitly as

\[
D_0(\omega, \phi) = \langle \xi_\pi(\phi), (\log(\Delta_{\omega,\phi})) \rangle_H = (\omega - \phi) \langle \mathbb{I}, \log(\Delta_{\omega,\phi})\rangle_H \tag{96}
\]

and

\[
D_1(\omega, \phi) = \langle \xi_\pi(\phi), (\Delta_{\omega,\phi}, \log(\Delta_{\omega,\phi}) - 1) \rangle_H = (\phi - \omega) \langle \mathbb{I}, \log(\Delta_{\omega,\phi})\rangle_H \tag{97}
\]

Hence,

\[
\phi \ll \omega \ll \phi \Rightarrow D_\gamma(\omega, \phi) = D_{1-\gamma}(\phi, \omega) \quad \forall \gamma \in [0,1], \tag{98}
\]

\[
D_0(\omega, \phi) = D_0(\phi, \omega) \iff \gamma = 0.5. \tag{99}
\]

Proposition 5.5. If \(\gamma \in [0,1]\), then quantum \(\gamma\)-distance (94) is a dualistic Brègman distance (56), a standard Brègman distance (59), and a reflexive quantum Brègman distance (75), with a dual coordinate system \((\ell_\gamma, \ell_1-\gamma)\) given by (87), with a convex proper function

\[
\Psi_\gamma : \mathbb{L}_{1/\gamma}(\mathbb{N}) \ni x \mapsto \Psi_\gamma(x) := \frac{1}{1-\gamma} \int (\gamma x)^{1/\gamma} = \frac{\|\gamma x\|^{1/\gamma}}{1-\gamma} \in [0, +\infty[. \tag{100}
\]
with a dualiser
\[ L_{\psi_\gamma} := \tilde{\ell}_{1-\gamma} \circ \tilde{\ell}^{-1}_\gamma : L_{1/\gamma}(\mathcal{N}) \ni \frac{1}{\gamma} u|\phi|^\gamma \mapsto \frac{1}{1-\gamma} u|\phi|^{1-\gamma} \in L_{1/(1-\gamma)}(\mathcal{N}), \tag{101} \]
and with a Brègman functional, in the sense of \((B_D)\) and \((B_4)\),
\[ D_{\gamma}(x, y) = \Psi_\gamma(x) + \Psi_{1-\gamma}(L_{\psi_\gamma}(y)) - \text{re} \left[ x, L_{\psi_\gamma}(y) \right]_{L_{1/\gamma}(\mathcal{N}) \times L_{1/(1-\gamma)}(\mathcal{N})}. \tag{102} \]

Proof. Our method of proof will be based on the approach of [84] (which in turn used some of the ideas introduced in [76]).

The embeddings \( \ell_\gamma \) defined by (87) allow to construct the real valued functional on \( \mathcal{N}_+^\gamma \) using the duality (135),
\[ \mathcal{N}_+^\gamma \times \mathcal{N}_+^\gamma \ni (\omega, \phi) \mapsto \int \ell_\gamma(\omega)\ell_1(\gamma)(\phi) = \left[ \ell_\gamma(\omega), \ell_1(\gamma)(\phi) \right]_{L_{1/\gamma}(\mathcal{N}) \times L_{1/(1-\gamma)}(\mathcal{N})} \in \mathbb{R}. \tag{103} \]
In these terms, \( D_\gamma \) defined in (94) for \( \gamma \in [0, 1] \) is equal to
\[ D_\gamma(\omega, \phi) = \int \left( \frac{\omega}{1-\gamma} + \frac{\phi}{\gamma} - \ell_\gamma(\omega)\ell_1(\gamma)(\phi) \right) = \frac{\omega(1)}{1-\gamma} + \frac{\phi(1)}{\gamma} - \left[ \ell_\gamma(\omega), \ell_1(\gamma)(\phi) \right]_\gamma, \tag{104} \]
where we have simplified the notation by \([\cdot, \cdot]_\gamma := [\cdot, \cdot]_{L_{1/\gamma}(\mathcal{N}) \times L_{1/(1-\gamma)}(\mathcal{N})}\).

We begin by proving that that a function \( L_{\psi_\gamma} \) is a homeomorphism in the corresponding norm topologies. Its bijectivity follows from the bijectivity of \( \ell_\gamma \). For \( \phi \in \mathcal{N}_+ \) denote its unique polar decomposition as \( |\phi| (\cdot, u) \). From (134) it follows that
\[ \|u|\phi|^{1/\gamma}\|_{1/\gamma} = (|\phi|)_{\gamma}, \tag{105} \]
so
\[ \|\gamma x\|^{1/\gamma} := \|(1 - \gamma)L_{\psi_\gamma}(x)\|_{1/(1-\gamma)}^{1/(1-\gamma)} = \int |\phi| \mathbb{I} = \int |\phi|^{1/(1-\gamma)} \supp(\phi) = \int u|\phi| |\phi||^{1-\gamma}u^* = \int u|\phi|^{\gamma} (u|\phi|^{1-\gamma})^* = \gamma(1 - \gamma) \left[ x, (L_{\psi_\gamma}(x))^\gamma \right]_{\gamma}. \tag{106} \]

For a Banach space \( X \) let \( v_x/|x| \) denote a unique point on a unit sphere in \( X^\ast \) such that
\[ \left[ x, v_x/|x| \right]_{X \times X^\ast} = \|x\|_{X^\ast}. \tag{107} \]
According to [54], if \( X \) is uniformly convex and \( \|\cdot\|_X \) is Fréchet differentiable, then a map
\[ F_v : \begin{cases} \quad X \setminus \{0\} \ni x \mapsto [x, v_x/|x|] \in X^\ast \setminus \{0\} \\ X \ni 0 \mapsto 0 \in X^\ast \end{cases} \tag{108} \]
is a homeomorphism in the norm topologies of \( X \) and \( X^\ast \). The function
\[ v_\gamma(x) := \|\gamma x\|^{1-1/\gamma} (1 - \gamma) (L_{\psi_\gamma}(x))^\gamma \]satisfies
\[ \left[ x, v_\gamma(x) \right]_{\gamma} = \|\gamma x\|^{1-1/\gamma} (1 - \gamma) \left[ x, (L_{\psi_\gamma}(x))^\gamma \right]_{\gamma} = \|\gamma x\|^{1-1/\gamma} (1 - \gamma) \|\gamma x\|^{1/\gamma} \gamma^{-1} (1 - \gamma)^{-1} = \|x\|_{1/\gamma}. \tag{110} \]
hence \( v_\gamma(x) = v_{x/\|x\|} \) for \( X = L_{1/\gamma}(\mathcal{N}) \). From (106) it follows that \( L_{\Psi_\gamma}(x) \) is continuous at 0.

From uniform convexity and uniform Fréchet differentiability of \( L_{1/\gamma}(\mathcal{N}) \) for \( \gamma \in ]0,1[ \) it follows that for \( x \in L_{1/\gamma}(\mathcal{N}) \setminus \{0\} \) the function \( F_{v_\gamma} \) reads

\[
F_{v_\gamma}(x) = \|x\|_{1/\gamma}^{1/\gamma} v_\gamma(x) = (1 - \gamma)^{1 - 1/\gamma} \|x\|_{1/\gamma}^{2 - 1/\gamma} (L_{\Psi_\gamma}(x))^*,
\]

which implies that \( L_{\Psi_\gamma} \) is also a homeomorphism.

Next, we will prove that \( \Psi_\gamma \) is Fréchet differentiable, with

\[
(\mathcal{D}^F_x \Psi_\gamma)(y) = \text{re} \left[ [y, L_{\Psi_\gamma}(x)] \right]_{\gamma} \quad \forall x \in L_{1/\gamma}(\mathcal{N})
\]

and

\[
\Psi_\gamma(x) + \Psi_{1-\gamma}(L_{\Psi_\gamma}(x)) - \text{re} \left[ [x, L_{\Psi_\gamma}(x)] \right]_{\gamma} = 0 \quad \forall x \in L_{1/\gamma}(\mathcal{N}).
\]

If a Banach space \( X \) is Gâteaux differentiable except \( 0 \in X \), then

\[
\left[ [y, \mathcal{D}^G_x \|\cdot\|_{\mathcal{N}}] \right]_{X} = \text{re} \left[ [y, v_{x/\|x\|}] \right]_{X}.
\]

From the uniform Fréchet differentiability of \( L_{1/\gamma}(\mathcal{N}) \) it follows that \( \|\|_{1/\gamma} \) is Fréchet differentiable at any \( x \in L_{1/\gamma}(\mathcal{N}) \setminus \{0\} \), and

\[
(\mathcal{D}^F_x \|1/\gamma\|)(y) = \text{re} \left[ [y, v_{1/\gamma}] \right]_{\gamma} \quad \forall y \in L_{1/\gamma}(\mathcal{N}),
\]

so

\[
(\mathcal{D}^F_x \Psi_\gamma)(y) = \left( \mathcal{D}^F \left( \frac{1 - \gamma}{1 - \gamma} \|\gamma x\|_{1/\gamma}^{1/\gamma} \right) \right)(y) = \left( \frac{1 - \gamma}{1 - \gamma} \|\gamma x\|_{1/\gamma}^{1/\gamma - 1} \mathcal{D}^F \|x\|_{1/\gamma} \right)(y)
\]

\[
= \text{re} \left[ [y, \frac{1 - \gamma}{1 - \gamma} \|\gamma x\|_{1/\gamma}^{1/\gamma - 1} \|\gamma x\|_{1/\gamma}^{1/\gamma - 1} (1 - \gamma)(L_{\Psi_\gamma}(x))^* \right]_{\gamma}
\]

\[
= \text{re} \left[ [y, L_{\Psi_\gamma}(x)] \right]_{\gamma}.
\]

The function \( \|\gamma x\|_{1/\gamma}^{1/\gamma} \) is also Fréchet differentiable at \( x = 0 \), which implies

\[
(\mathcal{D}^F_x \Psi_\gamma)(y) = 0 = \text{re} \left[ [y, L_{\Psi_\gamma}(0)] \right]_{\gamma}.
\]

This gives (112). The equation (113) follows as straightforward calculation. Note that (113) is just \( D_{\Psi_\gamma}(x, x) = 0 \) for \( D_{\Psi_\gamma} \) given by (102). From the fact that (102) satisfies (B_4), it follows that \( D_{\Psi_\gamma}(x, y) \geq 0 \). Moreover, from Fréchet differentiability and continuity of \( \Psi_{1-\gamma} \) on all \( L_{1/(1-\gamma)}(\mathcal{N}) \) and reflexivity of \( L_{1/\gamma}(\mathcal{N}) \) spaces it follows that \( \Psi_\gamma \) is essentially strictly convex, hence, due to (31), \( D_{\Psi_\gamma}(x, y) = 0 \iff x = y \). This implies that the equation (113) is a unique solution of the variational problem

\[
\Psi_{1-\gamma}(L_{\Psi_\gamma}(x)) = \sup_{y \in L_{1/\gamma}(\mathcal{N})} \left\{ \text{re} \left[ [y, L_{\Psi_\gamma}(x)] \right]_{\gamma} - \Psi_\gamma(y) \right\},
\]

because

\[
y \neq x \ \Rightarrow \ \Psi_\gamma(y) + \Psi_{1-\gamma}(L_{\Psi_\gamma}(x)) - \text{re} \left[ [y, L_{\Psi_\gamma}(x)] \right]_{\gamma} > 0,
\]

\[
\Psi_{1-\gamma}(L_{\Psi_\gamma}(x)) > \text{re} \left[ [y, L_{\Psi_\gamma}(x)] \right]_{\gamma} - \Psi_\gamma(y).
\]

Comparing (118) with (10), we see that

\[
\Psi_{1-\gamma} = \Psi_\gamma^F,
\]

with respect to the duality \([\cdot, \cdot]_\gamma\).
If $X$ is a Banach space and $f : X \to \mathbb{R}$ is norm continuous and convex function, then $f$ is Gâteaux differentiable iff $\partial f(x) = \{ * \} \ \forall x \in X$. The norm continuity and Fréchet differentiability of $\Psi_\gamma$ on $L_{1/\gamma}(N)$ implies that

$$\partial \Psi_\gamma(x) = \{ * \} = \mathcal{D}_x^F \Psi_\gamma,$$

so

$$L_{\Psi_\gamma}(y) \in \partial \Psi_\gamma(x) \iff x = y \ \forall x, y \in \text{efd}(\partial \Psi_\gamma).$$

Hence, $(L_{1/\gamma}(N), L_{1/(1-\gamma)}(N), L_{\Psi_\gamma})$ is a generalised Legendre transform, and $D_\gamma(\omega, \phi)$ is a dualistic Brègman distance of the form

$$D_\gamma(\omega, \phi) = \Psi_\gamma(\ell_\gamma(\omega)) + \Psi_{1-\gamma}(\ell_{1-\gamma}(\phi)) - \| \ell_\gamma(\omega), \ell_{1-\gamma}(\phi) \|_{L_{1/\gamma}(N) \times L_{1/(1-\gamma)}(N)}$$

with $\Psi_\gamma(\ell_\gamma(\omega)) = \frac{1}{1-\gamma}(\| \).$

**Proposition 5.6.** If $\gamma \in [0, 1[$, then $D_\gamma(\omega, \phi)$ satisfies the generalised cosine equation

$$D_\gamma(\omega, \phi) + D_\gamma(\phi, \psi) = D_\gamma(\omega, \psi) + \int (\ell_\gamma(\omega) - \ell_\gamma(\phi)) (\ell_{1-\gamma}(\psi) - \ell_{1-\gamma}(\phi)).$$

In finite dimensional setting (125) holds also for $\gamma \in \{0, 1\}$, with $\ell_\gamma$ given by (90).

**Proof.** Straightforward calculation based on equations (104) and (67).

**Corollary 5.7.** The equation (125) is an instance of the ‘standard’ generalised cosine equation (68) applied to $D_\gamma$, given by (102), while the equation (98) follows from the ‘representation-index duality’ equation\footnote{The finite dimensional commutative version of the equation (126), with a dualiser given by gradient, was discussed in [140].}

$$\bar{D}_\gamma(x, y) = \bar{D}_{\psi_{1-\gamma}}(L_{\Psi_\gamma}(y), L_{\Psi_\gamma}(x)),$$

where $x, y \in L_{1/\gamma}(N)$. For $\gamma = 1/2$ the $L_{1/\gamma}(N)$ space becomes a Hilbert space $\mathcal{H}$, the generalised Brègman functional $D_{\Psi_\gamma}$ becomes the norm distance on it,

$$D_{\Psi_{1/2}}(x, y) = \| x - y \|_{\mathcal{H}}^2 / 2,$$

so the generalised cosine equation for $D_{\Psi_\gamma}$ turns to the cosine equation in Hilbert space $\mathcal{H}$,

$$\| x - y \|^2_{\mathcal{H}} + \| y - z \|^2_{\mathcal{H}} = \| x - z \|^2_{\mathcal{H}} + 2 \langle x - y, z - y \rangle_{\mathcal{H}}.$$

**Remark 5.8.** From the fact that (94) is an f-distance it follows that it has the following properties [92, 121, 117, 84]:

1) $D_\gamma(\omega, \phi) \geq D_\gamma(T_\star(\omega), T_\star(\phi))$,

2) $D_\gamma$ is jointly convex on $\mathcal{N}_+^\star \times \mathcal{N}_+^\star$,

3) for $\gamma \in [0, 1[$, $D_\gamma$ is lower semi-continuous on $\mathcal{N}_+^\star \times \mathcal{N}_+^\star$, endowed with the product of norm topologies, while for $\gamma \in \{0, 1\}$ it is also lower semi-continuous on $\mathcal{N}_+^\star \times \mathcal{N}_+^\star$ endowed with the product of weak-* topologies.

**Remark 5.9.** The family (94) provides a canonical infinite dimensional noncommutative generalisation of the family (85) of Liese–Vajda $\gamma$-distances, and generalises the family (92) of Jenčová–Ojima $\gamma$-distances in terms of canonical noncommutative $L_{1/\gamma}(N)$ spaces. These properties, considered together with Propositions 5.3 and 5.5 suggest a quantum analogue of Amari’s [7] characterisation of the Liese–Vajda $\gamma$-distances. Amari’s characterisation holds for $\gamma \in \mathbb{R}$. On the other hand, Hasegawa [80] proved that (86), when extended with the range of $\gamma$ to $\mathbb{R}$, is operator convex only for $\gamma \in [-1, 2]$. This leads us to:
Conjecture 5.10. The family $D_\gamma$ defined by (94) for $\gamma \in [-1,2]$ is the unique (up to a multiplicative constant) family of quantum distances $D$ on $N^+_*$ that satisfies the conditions: (strong version):

1) $D(\omega, \phi) \geq D(T_*(\omega), T_*(\phi)) \quad \forall \omega, \phi \in N^+_* \quad \forall T_* \in \text{Mark}_*(N^+_*)$,

2) $D$ is representable in the form (59),

3) $\exists C \subseteq N^+_* \quad \forall (\phi, \psi) \in K \times C$

$$\mathcal{P}^D_K(\psi) = \{\ast\} \implies D(\phi, \psi) = D(\phi, \mathcal{P}^D_K(\psi)) + D(\mathcal{P}^D_K(\psi), \psi),$$

(129)

for every $K \subseteq C \subseteq N^+_*$ such that $\ell_\psi(K)$ is affine, where $\ell_\psi$ is as in (59), and $\mathcal{P}^D_K(\psi) := \text{arg inf}_{\phi \in K} \{D(\phi, \psi)\}$.

(weak version):

1) $D_\gamma$ belongs to the class $D_1$ given by (3),

2) $D_\gamma$ is a quantum Bregman distance.

(both versions): Moreover, under restriction from $N^+_*$ to $N^+_1$, the above conditions are satisfied only by $D_\gamma$ for $\gamma \in \{0,1\}$.\footnote{In [123] Petz claims without proof this uniqueness property for $D_1$ for normalised states.}

Now let us consider the projections $\mathcal{P}^D_C(\psi)$ given by (94) for $\gamma \in \{0,1\}$. The following results were obtained first by Jenčová [84] for the $\gamma$-distance (92) and its corresponding dualistic Bregman functional.

Proposition 5.11. 1) if $y \in L_{1/\gamma}(N)$ and $K \subseteq L_{1/\gamma}(N)$ is nonempty, weakly closed, convex, then:

i) $\mathcal{P}^\psi_K(y) := \text{arg inf}_{x \in K} \{D_\psi, (x, y)\} = \{\ast\}$,

iii) $\bar{D}_\psi, (x, y) \geq \bar{D}_\psi, (x, \mathcal{P}^\psi_K(y)) + \bar{D}_\psi, (\mathcal{P}^\psi_K(y), y) \quad \forall x \in K$, (130) and, equivalently,

$$\text{re} \left[ \left[ x - \mathcal{P}^\psi_K(y), L_\psi, (y) - L_\psi, (\mathcal{P}^\psi_K(y)) \right] \right]_{L_{1/\gamma}(N) \times L_{1/(1-\gamma)}(\mathcal{N})} \leq 0 \quad \forall x \in K. \quad (131)$$

iv) the equality in (130) and (131) holds if $K$ is additionally a vector subspace of $L_{1/\gamma}(N)$,

2) if $\psi \in N^+_*$ and $C \subseteq N^+_*$ is nonempty, $\ell_\gamma(C) \subseteq L_{1/\gamma}(N)$ is convex, and $C$ is closed in the topology induced by $\ell_\gamma^{-1}$ from the weak topology of $L_{1/\gamma}(N)$, then

i) $\mathcal{P}^D_C(\psi) := \text{arg inf}_{\phi \in C} \{D_\gamma(\phi, \psi)\} = \{\ast\}$,

iii) if $\ell_\gamma(C)$ is a vector subspace of $L_{1/\gamma}(N)$, then the generalised pythagorean equation holds:

$$D_\gamma(\omega, \psi) = D_\gamma(\omega, \mathcal{P}^D_C(\psi)) + D_\gamma(\mathcal{P}^D_C(\psi), \psi) \quad \forall \omega \in C. \quad (132)$$

Proof. Because $\bar{D}_\psi, \bar{D}_\gamma, \bar{D}_1$ given by (102) is a Bregman functional in the sense of (B1), the theorems (P1) on existence, uniqueness and properties of Bregman projections for definitions (B3) and (B4) provided in Section 2 apply also in this case. The corresponding results for $D_\gamma$ follow from the fact that it is a reflexive quantum Bregman distance, so the Proposition 4.2 applies. More specifically, this can be obtained by an extension of $D_\gamma$ to $\bar{D}_\gamma$, defined on the whole space
$\mathcal{N}_*$ by replacing the term $[\ell_1(\omega), \ell_{1-\gamma}(\phi)]_\gamma$ in (104) by re $[\tilde{\ell}_1(\omega), \tilde{\ell}_{1-\gamma}(\phi)]_\gamma$. Because $\tilde{\ell}_1$ are homeomorphisms (hence, bijections) between Banach spaces $\mathcal{N}_*$ and $L_{1,\gamma}(\mathcal{N})$, the theorems on existence, uniqueness, and pythagorean theorem for projections for $D_{\Psi}$ on $L_{1,\gamma}(\mathcal{N})$ can be translated in terms of topology induced by $\tilde{\ell}_{1-\gamma}^{-1}$ on $\mathcal{N}_*$, turning them into the corresponding theorems on projections for $\tilde{D}_{\gamma}$. The results for $D_{\gamma}$ follow then by the restriction of domain of $\tilde{D}_{\gamma}$ to $\mathcal{N}_*^+$.  

Most of the conditions for (P$_1$) were already verified: $L_{1,\gamma}(\mathcal{N})$ is reflexive, $\Psi_{\gamma}$ is lower semi-continuous, Gâteaux differentiable, essentially Gâteaux differentiable and essentially strictly convex on $\text{efd}(\Psi_{\gamma}) = L_{1,\gamma}(\mathcal{N})$. The strict convexity of $\Psi_{\gamma}$ follows from Gâteaux differentiability of $\Psi_{1-\gamma}$. Finally,  

$$\lim_{\|x\|_{1,\gamma} \to +\infty} \|x\|_{1,\gamma}^{\gamma - 1} = \frac{\gamma - 1}{1 - \gamma} \lim_{\|x\|_{1,\gamma} \to +\infty} \|x\|_{1,\gamma}^{-\gamma} = +\infty \ \forall x \in K. \quad (133)$$

\[\square\]

**Remark 5.12.** Jenčová [84] proved also that, under the same assumptions as in 1) and 2) above, respectively:

1.ii) $y \mapsto \bar{\mathfrak{P}}_K^{\Psi_{\gamma}}(y)$ is a continuous function from $L_{1,\gamma}(\mathcal{N}, \phi)$ with its norm topology to $K$ with the relative weak topology,

2.ii) $\psi \mapsto \mathfrak{P}^{D_{\gamma}}_{\mathcal{N}}(\psi)$ is a continuous function from $\mathcal{N}_*^+$ with the topology induced by $\tilde{\ell}_{1-\gamma}^{-1}$ from the norm topology of $L_{1,\gamma}(\mathcal{N}, \phi)$ to $C$ with the relative topology induced by $\tilde{\ell}_{1-\gamma}^{-1}$ from the weak topology of $L_{1,\gamma}(\mathcal{N}, \phi)$.

By the isometric isomorphism of the Araki–Masuda $L_p(\mathcal{N}, \phi)$ spaces and the Falcone–Takesaki $L_p(\mathcal{N})$ spaces, these results hold also in our case. They provide a topological specification of the stability of the behaviour of $D_{\gamma}$ projection under the change of initial state $\psi$.

**Acknowledgments.** I would like to thank: P.Gibilisco, A.Jenčová, W.Kamiński, and S.L.Woronowicz for discussions, D.Pavlov and D.E.Sherman for correspondence, as well as L.Hardy, R.Kunjwal, and J.Lewandowski for hosting. This research was supported in part by Perimeter Institute for Theoretical Physics as well as by Polish National Science Centre (NCN grant N202 343640). Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research and Innovation.

**Appendix: Noncommutative $L_p(\mathcal{N})$ spaces**

In this work we use the terminology and notation which is in agreement with the exposition of noncommutative integration theory given in [96]. We refer to that text, and references in it, for the detailed discussion. Here we will just recall briefly few key notions and facts used in the current paper.

A $W^*$-algebra is defined as such $C^*$-algebra that has a Banach predual. If a predual of $C^*$-algebra exists then it is unique. Given a $W^*$-algebra $\mathcal{N}$, we will denote its predual by $\mathcal{N}_*-\mathcal{N}$. Moreover, $\mathcal{N}_*^+ := \{ \phi \in \mathcal{N}_* \mid \phi(x^*x) \geq 0 \ \forall x \in \mathcal{N} \}$, $\mathcal{N}_*^{\gamma,0} := \{ \phi \in \mathcal{N}_*^+ \mid \omega(x^*x) = 0 \Rightarrow x = 0 \ \forall x \in \mathcal{N} \}$, $\mathcal{N}_*^{\gamma} := \{ \phi \in \mathcal{N}_*^+ \mid \| \phi \| = 1 \}$, $\mathcal{N}_*^{\text{sa}} := \{ x \in \mathcal{N} \mid x^* = x \}$, $\mathcal{N}_*^+ := \{ x \in \mathcal{N} \mid \exists y \in \mathcal{N} : x = y^*y \}$. A space of all semi-finite normal weights on $\mathcal{N}$ is denoted $W_0(\mathcal{N})$. We will call a boolean algebra $\mathcal{A}$ an mcb-algebra iff it is Dedekind–MacNeille complete and allows for a semi-finite strictly positive countably additive measure. Every commutative $W^*$-algebra $\mathcal{N}$ is isometrically isomorphic and $*$-isomorphic to $L_{\infty}(\mathcal{A})$ space, where $\mathcal{A}$ is an mcb-algebra constructed as the lattice of projections of $\mathcal{N}$ (in such case $L_{\infty}(\mathcal{A})_* \cong L_1(\mathcal{A})$ is an
isometric isomorphism and a Riesz isomorphism). More generally, there holds an equivalence between the categories of: commutative $W^*$-algebras with $*$-homomorphisms, mcb-algebras with order continuous boolean homomorphisms, and localisable measure spaces with complete morphisms. Let $\tau \in \mathcal{W}_0(\mathcal{N})$ be a semi-finite trace. A closed densely defined linear operator $x : \text{dom}(x) \to \mathcal{H}$, with $\text{dom}(x) \subseteq \mathcal{H}$, is called $\tau$-measurable iff $\exists \lambda > 0 \; \tau(P^{[\lambda]}[x]) < \infty$, where $P^{[\lambda]}$ is a spectral measure of $|x|$. The space of all $\tau$-measurable operators affiliated with the GNS representation $\pi_\tau(\mathcal{N})$ will be denoted by $\mathcal{M}(\mathcal{N}, \tau)$. For $x, y \in \mathcal{M}(\mathcal{N}, \tau)$ the algebraic sum $x + y$ and algebraic product $xy$ may not be closed, hence in general they do not belong to $\mathcal{M}(\mathcal{N}, \tau)$. However, their closures (denoted with the abuse of notation by the same symbol) belong to $\mathcal{M}(\mathcal{N}, \tau)$.

Falcone and Takesaki [66] have constructed a family of noncommutative $L_p(\mathcal{N})$ spaces that are canonically associated to every $W^*$-algebra, including also those that do not admit faithful normal semi-finite traces. For a detailed review of this construction, see [96]. Here we will need only several facts about them. Its key feature is a construction of a semi-finite von Neumann algebra $\tilde{\mathcal{N}}$ and a faithful normal semi-finite trace $\tilde{\tau} : \tilde{\mathcal{N}} \to [0, \infty]$ that are uniquely defined for any $W^*$-algebra $\mathcal{N}$, with no dependence of an additional weight or state on $\mathcal{N}$. Using these objects, a topological $*$-algebra $\mathcal{M}(\tilde{\mathcal{N}}, \tilde{\tau})$ of $\tilde{\tau}$-measurable operators is defined. It is equipped with a graded function $\text{grd} : \mathcal{M}(\tilde{\mathcal{N}}, \tilde{\tau}) \to \mathbb{C}$ satisfying $\text{grd}(x^*) = (\text{grd}(x))^*$, $\text{grd}(|x|) = \text{re}(\text{grd}(x)) = \frac{1}{2}(\text{grd}(x) + \text{grd}(x^*))$, $\text{grd}(xy) = \text{grd}(x) + \text{grd}(y)$, $\text{re}^{(\text{grd}(x))} > 0$ implies $|x|^{1/\text{re}(\text{grd}(x))} \in \mathcal{N}_r^+$, where $\mathcal{N}_r$ is the closure of $xy$. The Falcone–Takesaki canonical integral $\mathcal{M}(\tilde{\mathcal{N}}, \tilde{\tau}) \to \mathbb{C}$ satisfies $\int : L_1(\tilde{\mathcal{N}}) \ni \phi \mapsto \int \phi = \phi(\mathbb{I}) = \in \mathbb{C}$. The spaces $L_p(\tilde{\mathcal{N}})$ for $p = \in \mathbb{C} \setminus \{0\}$ are defined as the spaces of $\tilde{\tau}$-measurable operators of grade $1/p$ affiliated with $\tilde{\mathcal{N}}$. The norms $\mathcal{M}(\tilde{\mathcal{N}}, \tilde{\tau})$ for $p, q \leq 1$ read

$$\|x\|_p : L_p(\mathcal{N}) \ni x \mapsto \|x\|_p := \left( \right)^{1/\text{re}(p)} \in \mathbb{R}^+,$$

and turn $L_p(\mathcal{N})$ into Banach spaces, with their Banach duals given by $L_q(\mathcal{N})$ spaces with $\frac{1}{p} + \frac{1}{q} = 1$. The space $L_\infty(\mathcal{N})$ is defined as $\mathcal{N}$, and an isometric isomorphism $\mathcal{N} \cong L_1(\mathcal{N})$ holds. The Banach space duality between $L_p(\mathcal{N})$ and $L_q(\mathcal{N})$ for $1/p + 1/q = 1$ and $p \in \{ \lambda \in \mathbb{C} | \text{re}^{(\lambda)} > 0 \}$ reads

$$L_p(\mathcal{N}) \times L_q(\mathcal{N}) \ni (x,y) \mapsto [x,y]_{\mathcal{N}} := \int xy \in \mathbb{C}.$$  

The space $L_2(\mathcal{N})$ is a Hilbert space with respect to the inner product

$$L_2(\mathcal{N}) \times L_2(\mathcal{N}) \ni (x_1,x_2) \mapsto \langle x_1,x_2 \rangle_{L_2(\mathcal{N})} := \int x_1^*x_2 \in \mathbb{C}.$$  

If $\{x_i\}_{i=1}^n \subseteq \mathcal{M}(\tilde{\mathcal{N}}, \tilde{\tau})$, $\sum_{i=1}^n \text{grd}(x_i) = r \leq 1$ and $\text{re}(\text{grd}(x_i)) \geq 0 \forall i \in \{1, \ldots, n\}$, then the noncommutative analogue of the Rogers–Hölder inequality holds [93],

$$\|x_1 \cdots x_n\|_r^{1/r} \leq \|x_1\|_{\text{re}(\text{grd}(x_1))} \cdots \|x_n\|_{\text{re}(\text{grd}(x_n))}.$$  

The stronger condition $\sum_{i=1}^n \text{grd}(x_i) = 1$ implies that $x_1 \cdots x_n \in L_1(\mathcal{N})$, and in such case

$$\int x_1 \cdots x_n = \int x_n x_1 \cdots x_{n-1}.$$  

This suggests to use the symbolic notation $y = x_\phi^{\text{grd}(y)} = x_\phi^\gamma$ with $(x, \phi) \in \mathcal{N} \times \mathcal{W}_0(\mathcal{N})$ for a generic element $y$ of the space $L_{1/\gamma}(\mathcal{N})$ with $\text{re}(\gamma) \in [0, 1[$, with boundary cases given by $x \in L_\infty(\mathcal{N}) = \mathcal{N}$ and $\phi \in L_1(\mathcal{N}) \cong \mathcal{N}_r$. For the negative powers of weights, $\phi^{-p}$ for $p > 0$, there are no corresponding $L_{-p}(\mathcal{N})$ spaces. However, as shown in [128], the right
and left multiplications, \( \mathcal{R}(\phi^{-p}) \) and \( \mathcal{L}(\phi^{-p}) \), for \( \phi \in \mathcal{W}_0(\mathcal{N}) \) are well defined\(^{19}\) and satisfy \( \mathcal{R}(\phi^{-p}) = (\mathcal{R}(\phi^p))^{-1} \), \( \mathcal{L}(\phi^{-p}) = (\mathcal{L}(\phi^p))^{-1} \), \( \mathcal{R}(\phi^{-p})\mathcal{R}(\phi^p) = I \), as well as
\[
\Delta^{1/p}_{\phi, \psi} = \mathcal{R}(\phi^{-1/p})\mathcal{L}(\psi^{1/p}),
\]
where \( \psi \in \mathcal{W}(\mathcal{N}) \). This gives
\[
\int \psi^\gamma \phi^{1-\gamma} = \int \psi^\gamma \phi^{-\gamma} \phi = \int (\mathcal{R}(\phi^{-\gamma})\mathcal{L}(\psi^\gamma))\mathcal{I} \phi = \langle \xi_\pi(\phi), \Delta^{\gamma}_{\psi, \phi} \xi_\pi(\phi) \rangle_{\mathcal{H}}
\]
for any standard representation \((\mathcal{H}, \pi, J, \mathcal{H}^2)\). The equation (140) holds also when \( \phi, \psi \in \mathcal{N}_s^+ \) and \( \psi \ll \phi \), because in such case \( \phi \) is faithful on \( \mathcal{N}_{\text{supp}(\phi)} \) and this algebra contains the support of \( \phi \).

References

[1] Akbarov S.S., 2016, Envelopes and refinements in categories, with applications to functional analysis, Dissert. Math. 513, 1. ↑ 5.
[2] Alber Ya.I., 1996, Metric and generalized projection operators in Banach spaces: properties and applications, in: Kartsatos A.G. (ed.), Theory and applications of nonlinear operators of accretive and monotone type, Dekker, New York, p.15. ↑ 9.
[3] Alber Ya.I., 1998, Generalized projections, decompositions, and the pythagorean-type theorem in Banach spaces, Appl. Math. Lett. 11, 115. ↑ 9.
[4] Alber Ya.I., Butnariu D., 1997, Mathematical Society , Providence, p.1. ↑ 9, 10.
[5] Alber Ya.I., Butnariu D., 1997, Convergence of Bregman projection methods for solving consistent convex feasibility problems in reflexive Banach spaces, J. Optim. Theor. Appl. 92, 33. ↑ 8, 10.
[6] Ali S.M., Silvey S.D., 1966, A general class of coefficients of divergence of one distribution from another, J. Roy. Stat. Soc. B 28, 131. ↑ 2.
[7] Amari S.-i., 2009, \( \alpha \)-divergence is unique, belonging to both \( f \)-divergence and Bregman divergence classes, IEEE Trans. Inf. Theor. 55, 4925. ↑ 4, 11, 20, 25.
[8] Amari S.-i., 2009, Information geometry and its applications: convex function and dually flat manifold, in: Nielsen F. (ed.), Emerging trends in visual computing, LNCS 5416, Springer, Berlin, p.11. ↑ 11.
[9] Amari S.-i., Nagaoka H., 1993, Jōhō kika no hōhō, Iwanami Shoten, Tōkyō (Engl. transl. rev. ed.; 2000, Methods of information geometry, American Mathematical Society, Providence). ↑ 4, 11, 21.
[10] Araki H., 1973, Relative hamiltonian for faithful normal states of von Neumann algebra, Publ. Res. Inst. Math. Sci. Kyōto Univ. 9, 165. dx.doi.org/10.2977/prims/1195192744. ↑ 18.
[11] Araki H., 1977, Relative entropy for states of von Neumann algebras II, Publ. Res. Inst. Math. Sci. Kyōto Univ. 13, 173. dx.doi.org/10.2977/prims/1195190105. ↑ 18.
[12] Araki H., Masuda T., 1982, Positive cones and \( L_p \)-spaces for von Neumann algebras, Publ. Res. Inst. Math. Sci. Kyōto Univ. 18, 339. dx.doi.org/10.2977/prims/1195183577. ↑ 21.
[13] Ascoli G., 1932, Sugli spazi lineari metrici e loro varietà lineari, Ann. Mat. Pura Appl. 10, 33, 203. ↑ 6.
[14] Ayupov Sh.A., Chilin V.I., Abdullaev R.Z., 2012, Orlicz spaces associated with a semi-finite von Neumann algebra, Comment. Math. Univ. Caroliniae 53, 519. arXiv:1108.3267. ↑ 19.

\(^{19}\)More precisely, let the adjective ‘strong’ refer to the topological closure of some algebraic operation in \( \mathcal{M}(\mathcal{N}, \mathcal{T}) \). For any \( \lambda \geq 0, t > 0, \phi \in \mathcal{N}_0^+ \), the map \( \mathcal{R}(\phi^t) : L_{1/\lambda}(\mathcal{N}) \to L_{1/(\lambda+t)}(\mathcal{N}) \), defined as a strong composition with \( \phi^t \) from right, is everywhere defined, bounded, and injective with dense range. Moreover, the maps \( \mathcal{R}(\phi^t)^{-1} \) and \( \mathcal{R}(\phi^{-t}) \) have the same range and agree (from this it follows that they are equal). The map \( \mathcal{R}(\phi^{-t}) \) is closed, and is understood as a strong product, defined only when the closure is \( \mathcal{T} \)-measurable. The same holds for \( \mathcal{R} \) replaced by \( \mathcal{L} \). If \( \phi \in \mathcal{N}_0^+ \) is replaced by \( \phi \in \mathcal{W}_0(\mathcal{N}) \), then all those properties hold except that \( \mathcal{R}(\phi^t) \) and \( \mathcal{L}(\phi^t) \) are no longer everywhere defined or bounded.
[15] Baez J.C., Fritz T., 2014, A bayesian characterization of relative entropy, Theor. Appl. Cat. 29, 421. ↑ 21.
[16] Banerjee A., Guo X., Wang H., 2005, On the optimality of conditional expectation as a Bregman predictor, IEEE Trans. Inf. Theor. 51, 2664. ↑ 10.
[17] Bauer H., 1958, Minimalstellen von Funktionen und Extremalpunkte, Arch. Math. 9, 389. ↑ 15.
[18] Bauschke H.H., 2003, Duality for Bregman projections onto translated cones and affine subspaces, J. Approx. Theor. 121, 1. ↑ 10.
[19] Bauschke H.H., Borwein J.M., 1997, Legendre functions and the method of random Bregman projections, J. Conv. Anal. 4, 27. www.emis.de/journals/JCA/vol_4_no_1/j86.ps.gz. ↑ 9, 10, 12.
[20] Bauschke H.H., Borwein J.M., 2001, Joint and separate convexity of the Bregman distance, Stud. Comp. Math. 8, 23. people.ok.ubc.ca/bauschke/Research/c04.ps. ↑ 13.
[21] Bauschke H.H., Borwein J.M., Combettes P.L., 2001, Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces, Commun. Cont. Math. 3, 615. ↑ 4, 7, 8, 9, 10.
[22] Bauschke H.H., Borwein J.M., Combettes P.L., 2003, Bregman monotone optimization algorithms, Soc. Industr. Appl. Math. J. Contr. Optim. 42, 596. ↑ 9, 10.
[23] Bauschke H.H., Combettes P.L., 2003, Constructions of best Bregman approximations in reflexive Banach spaces, Proc. Amer. Math. Soc. 131, 3757. ↑ 10.
[24] Bauschke H.H., Lewis A.S., 2000, Dykstra’s algorithm with Bregman projections: a convergence proof, Optimization 48, 409. ↑ 9.
[25] Birnbaum Z.W., Orlicz W., 1930, Über Approximation im Mittel, Studia Math. 2, 197. matwbn.icm.edu.pl/ksiazki/sm/sm2/sm2117.pdf. ↑ 5, 19.
[26] Bauschke H.H., Borwein J.M., 2001, Joint and separate convexity of the Bregman distance, Stud. Comp. Math. 8, 23. people.ok.ubc.ca/bauschke/Research/c04.ps. ↑ 13.
[27] Bauschke H.H., Borwein J.M., Combettes P.L., 2001, Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces, Commun. Cont. Math. 3, 615. ↑ 4, 7, 8, 9, 10.
[28] Bauschke H.H., Borwein J.M., Combettes P.L., 2003, Bregman monotone optimization algorithms, Soc. Industr. Appl. Math. J. Contr. Optim. 42, 596. ↑ 9, 10.
[29] Bauschke H.H., Combettes P.L., 2003, Constructions of best Bregman approximations in reflexive Banach spaces, Proc. Amer. Math. Soc. 131, 3757. ↑ 10.
[30] Bauschke H.H., Lewis A.S., 2000, Dykstra’s algorithm with Bregman projections: a convergence proof, Optimization 48, 409. ↑ 9.
[31] Birnbaum Z.W., Orlicz W., 1930, Über Approximation im Mittel, Studia Math. 2, 197. matwbn.icm.edu.pl/ksiazki/sm/sm2/sm2117.pdf. ↑ 5, 19.
[32] Bauschke H.H., Borwein J.M., 2001, Joint and separate convexity of the Bregman distance, Stud. Comp. Math. 8, 23. people.ok.ubc.ca/bauschke/Research/c04.ps. ↑ 13.
[33] Bauschke H.H., Borwein J.M., Combettes P.L., 2001, Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces, Commun. Cont. Math. 3, 615. ↑ 4, 7, 8, 9, 10.
[34] Bauschke H.H., Borwein J.M., Combettes P.L., 2001, Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces, Commun. Cont. Math. 3, 615. ↑ 4, 7, 8, 9, 10.
[40] Chencov N.N., 1965, *Kategorii matematicheskoi statistiki*, Dokl. AN SSSR 164, 511. ↑ 1, 2.
[41] Chencov N.N., 1968, Nesimmetrichnoe rasstoyanie mezdu raspredeleniyami veroyatnostei, entropiya i teorema Pifagora, Mat. Zametki 4, 323. (Engl. transl. 1968, *Nonsymmetrical distance between probability distributions, entropy and the theorem of Pythagoras*, Math. Notes 4, 686.). ↑ 1, 3, 10.
[42] Chencov N.N., 1972, *Statisticheskie reshayushhie pravila i optimal’nye vyvody*, Nauka, Moskva (Engl. transl.: 1982, *Statistical decision rules and optimal inference*, American Mathematical Society, Providence). ↑ 1, 2, 10.
[43] Chilin V.I., Krygin A.W., Sukochev F.A., 1992, Uniform and local uniform convexity of symmetric spaces of measurable operators, Math. Proc. Cambridge Phil. Soc. 111, 355. ↑ 19.
[44] Clarkson J.A., 1936, Uniformly convex spaces, Trans. Amer. Math. Soc. 40, 396. ↑ 6.
[45] Csizsár I., 1963, *Eine informationstheoretische Ungleichung und ihre Anwendung auf den Beweis der Ergodizität von Markoffschen Ketten*, Magyar Tud. Akad. Mat. Kutató Int. Közl. 8, 85. ↑ 2, 3.
[46] Csizsár I., 1967, *Information-type measures of difference of probability distributions and indirect observation*, Stud. Sci. Math. Hungar. 2, 229. ↑ 3.
[47] Csizsár I., 1978, *Information measures: a critical survey*, in: Koženšík J. (ed.), Transactions of the seventh Prague conference on information theory, statistical decision functions, random processes, and of the 1974 european meeting of statisticians held at Prague, from August 18 to 23, 1974, Vol.B, Springer, Berlin, p.73. ↑ 3.
[48] Csizsár I., 1990, Uniform convexity in factor and conjugate spaces, *Entropy* 10, 261. www.mdpi.com/1099-4300/10/3/261/pdf. ↑ 4.
[49] Csizsár I., Matúš F., 2008, Axiomatic characterizations of information measures, *Entropy* 10, 261.
[50] Della Pietra S., Della Pietra V., Lafferty J., 2002, *Duality and auxiliary functions for Bregman programming*. J. Optim. Theor. Appl. 51, 421. ↑ 9.
[51] Dieudonné J.A., 1940, The geometry of Banach spaces. Smoothness, Trans. Amer. Math. Soc. 110, 284. ↑ 23.
[52] Dhillon I.S., Tropp J.A., 2007, *Matrix nearness with Bregman divergences*, Soc. Industr. Appl. Math. J. Matrix Anal. Appl. 29, 1120. ↑ 14, 18.
[53] Dodds P.G., Dodds T.K., de Pagter B., 1993, *Non-commutative Köthe duality*, Trans. Amer. Math. Soc. 339, 717. www.ams.org/journals/tran/1993-339-02/S0002-9947-1993-1113694-3/.
[54] Donald M.J., 1986, *On the relative entropy*, Commun. Math. Phys. 105, 13.
[55] Donald M.J., 1990, *Relative hamiltonians which are not bounded from above*, J. Funct. Anal. 91, 143. ↑ 18.
[64] Eckstein J., 1993, *Nonlinear proximal point algorithms using Bregman functions, with applications to convex programming*, Math. Oper. Res. 18, 202. ↑ 9.

[65] Eguchi S., 1983, *Second order efficiency of minimum contrast estimators in a curved exponential family*, Ann. Statist. 11, 793. euclid:1176346246. ↑ 2.

[66] Falcone A.J., Takesaki M., 2001, *The non-commutative flow of weights on a von Neumann algebra*, J. Funct. Anal. 182, 170. www.math.ucla.edu/~nt/papers/QFlow-Final.tex.pdf. ↑ 4, 28.

[67] Fenchel W., 1949, *On conjugate convex functions*, Canadian J. Math. 1, 73. cms.math.ca/cjm/v1/cjm1949v01.0073-0077.pdf. ↑ 5.

[68] Fischer A., 2010, *Quantization and clustering with Bregman divergences*, J. Multivar. Anal. 101, 2207. ↑ 10.

[69] Fréchet M., 1925, *Les espaces abstraits topologiquement affines*, Acta Math. 47, 25. ↑ 6.

[70] Frigyik B.A., Szristava S., Gupta M.R., 2008, *Functional Bregman divergence and bayesian estimation of distributions*, IEEE Trans. Inf. Theor. 54, 5130. arXivcs/0611123. ↑ 8, 10.

[71] Frigyik B.A., Szristava S., Gupta M.R., 2008, *An introduction to functional derivatives*, Technical report 2008-0001, Dept. Electr. Eng. Univ. Washington, Seattle. www.ee.washington.edu/research/guptalab/publications/functionalDerivativesIntroduction.pdf. ↑ 8.

[72] Frölicher A., Kriegl A., 1988, *Linear spaces and differentiation theory*, Wiley, Chisterter. ↑ 5.

[73] Gâteaux M.R., 1922, *Sur les fonctionnelles continues et les fonctionelles analytiques*, Bull. Soc. Math. France 47, 70. ↑ 6.

[74] Hasegawa H., 1993, *Über lineare Gleichungssysteme in linearen Räumen*, J. Reine Angew. Math. 157, 325. ↑ 6.

[75] Hahn H., 1927, *An introduction to functional derivatives*, Bull. Soc. Math. France 50, 1. ↑ 6.

[76] Hobson A., 1969, *Dual geometry of the Wigner–Yanase–Dyson information content*, Rend. Acad. Sci. Paris 114, 73. ↑ 7.

[77] Hörmander L., 1955, *Sur la fonction d’appui des ensembles convexes dans un espace localement convexe*, Ark. Matem. 3, 181. ↑ 5.

[78] Hasegawa H., 1993, *A new theorem of information theory*, IEEE Information Theory Society, Washington, p.566. ↑ 20.

[79] Hasegawa H., 1993, *α-divergence of the non-commutative information geometry*, Rep. Math. Phys. 33, 87. ↑ 14, 21.

[80] Hasegawa H., 2003, *Dual geometry of the Wigner–Yanase–Dyson information content*, Inf. Dim. Anal. Quant. Prob. Relat. Top. 6, 413. ↑ 25.

[81] Harremoës P., Tishby N., 2007, *The information bottleneck revisited or how to choose a good distortion measure*, in: *Proceedings of the 2007 IEEE International Symposium on Information Theory*, IEEE Information Theory Society, Washington, p.566. ↑ 20.

[82] Jacobs K.R., 1971, *Linear constraints to ada compression, pattern classification and cluster analysis*, Math. Oper. Res. 22, 326. ↑ 7, 8.

[83] Jones L.K., Byrne C.L., 1990, *Functional Bregman divergence and bayesian estimation of distributions*, IEEE Trans. Inf. Theor. 36, 23. ↑ 8, 10.

[84] Kakutani S., 1939, *Weak topology and regularity of Banach spaces*, Proc. Imp. Acad. Tôkyô 15, 169. ↑ 7.

[85] Jenčová A., 2005, *Affine connections, duality and divergences for a von Neumann algebra*, arXiv:math-ph/0311004. ↑ 21.

[86] Jenčová A., 2003, *Quantum information geometry and non-commutative estimation of distributions*, IEEE Trans. Inf. Theory 54, 5130. ArXivcs/0611123. ↑ 8, 10.

[87] Jenčová A., 2006, *Construction of a nonparametric quantum information manifold*, J. Funct. Anal. 239, 1. ArXiv:math-ph/0511065. ↑ 18, 19.

[88] Jones L.K., Byrne C.L., 1990, *General entropy criteria for inverse problems, with applications to ada compression, pattern classification and cluster analysis*, IEEE Trans. Inf. Theory 36, 23. ↑ 8, 10.

[89] Kakutani S., 1939, *Weak topology and regularity of Banach spaces*, Proc. Imp. Acad. Tôkyô 15, 169. ↑ 7.

[90] Kiwiel K.C., 1997, *Proximal minimization methods with generalized Bregman functions*, Soc. Industr. Appl. Math. J. Contr. Optim. 35, 1142. ↑ 7.

[91] Kiwiel K.C., 1998, *Generalized Bregman projections in convex feasibility problems*, J. Optim.
processing theorem of information theory, IEEE Trans. Inf. Theor. 43, 1288. ↑ 2, 20.
[120] Pettis B.J., 1939, A proof that every uniformly convex space is reflexive, Duke Math. J. 5, 249. ↑ 7.
[121] Petz D., 1985, Quasi-entropies for states of a von Neumann algebra, Publ. Res. Inst. Math. Sci. Kyōto Univ. 21, 787. www.math.bme.hu/~petz/pdf/26quasi.pdf. ↑ 2, 25.
[122] Petz D., 1994, On entropy functionals of states of operator algebras, Acta Math. Hungar. 64, 333. www.renyi.hu/~petz/pdf/59.pdf. ↑ 21.
[123] Petz D., 2007, Bregman divergence as relative operator entropy, Acta Math. Hungar. 116, 127. www.renyi.hu/~petz/pdf/112bregman.pdf. ↑ 12, 18, 26.
[124] Pflanzagl J., 1973, Asymptotic expansions related to minimum contrast estimators, Ann. Statist. 1, 993. ↑ 2.
[125] Rényi A., 1962, Wahrscheinlichkeitsrechnung: mit einem Anhang über Informationstheorie, Deutscher Verlag der Wissenschaften, Berlin (Engl. transl. 1970, Probability theory, North-Holland, Amsterdam). ↑ 2.
[126] Rockafellar R.T., 1970, Convex analysis, Princeton University Press, Princeton. www.math.washington.edu/~rtr/papers/rtr-ConvexAnalysis.djvu. ↑ 7.
[127] Sadegh G., 2001, Non-commutative Orlicz spaces associated to a modular on τ-measurable operators, J. Math. Anal. App. 395, 705. ↑ 19.
[128] Sherman D.E., 2001, The application of modular algebras to relative tensor products and noncommutative $L^p$ modules, Ph.D. thesis, University of California, Los Angeles. ↑ 28.
[129] Shmul’yan V.L., 1939, O nekotorykh geometricheskikh svoi˘ıstvakh edinichno ˘ı sfery prostranstva tipa (B), Matem. Sb. N.S. 6, 77, 90. ↑ 7.
[130] Shmul’yan V.L., 1940, Sur les topologies différentes dans l’espace de Banach, Dokl. Akad. Nauk SSSR 23, 331. ↑ 6, 7.
[131] Shore J.E., Johnson R.W., 1980, Axiomatic derivation of the principle of maximum entropy and the principle of minimum cross-entropy, IEEE Trans. Inf. Theory 26, 26. ↑ 3.
[132] Simons S., 1998, Minimax and monotonicity, LNM 1693, Springer, Berlin. ↑ 7.
[133] Teboulle M., 1992, Entropic proximal mappings with applications to nonlinear programming, Math. Oper. Res. 17, 670. ↑ 9.
[134] Tomamichel M., Colbeck R., Renner R., 2009, A fully quantum asymptotic equipartition property, IEEE Trans. Inf. Theor. 55, 5840. arXiv:0811.1221. ↑ 3.
[135] Tribus M., Rossi R., 1973, On the Kullback information measure as a basis for information theory: comments on a proposal by Hobson and Chang, J. Stat. Phys. 9, 331. ↑ 4.
[136] Wald A., 1939, Contributions to the theory of statistical estimation and testing hypothesis, Math. Statist. 10, 299. ↑ 2.
[137] Wiener N., 1948, Cybernetics or control and communication in the animal and the machine, MIT Press, Cambridge. ↑ 2.
[138] Williams P.M., 1980, Bayesian conditionalisation and the principle of minimum information, Brit. J. Phil. Sci. 31, 131. ↑ 3.
[139] Young W.H., 1912, On classes of summable functions and their Fourier series, Proc. Royal Soc. London Ser. A. 87, 225. ↑ 5.
[140] Zhang J., 2004, Divergence function, duality, and convex analysis, Neural Comp. 16, 159. www.lsa.umich.edu/psych/junz/Neural Comp 2004.pdf. ↑ 11, 25.
[141] Zhang J., 2004, Dual scaling of comparision and reference stimuli in multi-dimensional psychological space, J. Math. Psych. 48, 409. ↑ 15.
[142] Zhu H., 1998, Generalized Lebesgue spaces and application to statistics, Santa Fe Inst. Tech. Rep. 98-06-44, Santa Fe. www.santafe.edu/media/workingpapers/98-06-044.ps. ↑ 21.
[143] Zhu H., Rohwer R., 1995, Information geometric measurements of generalisation, Tech. Rep. Neural Comp. Res. Group 4350, Aston. eprints.aston.ac.uk/507/1/NCRG_95_005.pdf. ↑ 20.
[144] Zhu H., Rohwer R., 1997, Measurements of generalisation based on information geometry, in: Ellacott S.W. et al. (eds.), Mathematics of neural networks: models, algorithms and applications, Kluwer, Dordrecht, p.394. eprints.aston.ac.uk/514/1/NCRG_95_012.pdf. ↑ 20.
[145] Zhu H., Rohwer R., 1998, Information geometry, bayesian inference, ideal estimates and error decomposition, Santa Fe Inst. Tech. Rep. 98-06-45, Santa Fe. omega.albany.edu:8008/ignorance/zhn98.pdf. ↑ 10, 20, 21.