DETERMINANT OF PSEUDO-LAPLACIANS

TAYEB AISSIOU, LUC HILLAIRET, AND ALEXEY KOKOTOV

Abstract. We derive comparison formulas relating the zeta-regularized
determinant of an arbitrary self-adjoint extension of the Laplace operator
with domain \( C_c^\infty(X \setminus \{P\}) \subset L_2(X) \) to the zeta-regularized
determinant of the Laplace operator on \( X \). Here \( X \) is a compact Riemannian manifold of
dimension 2 or 3; \( P \in X \).

1. Introduction

Let \( X_d \) be a complete Riemannian manifold of dimension \( d \geq 2 \) and let
\( \Delta \) be the (positive) Laplace operator on \( X_d \). Choose a point \( P \in X_d \) and
consider \( \Delta \) as an unbounded symmetric operator in the space \( L_2(X_d) \) with
domain \( C_c^\infty(X_d \setminus \{P\}) \). It is well-known that thus obtained operator is
essentially self-adjoint if and only if \( d \geq 4 \). In case \( d = 2, 3 \) it has deficiency
indices \((1, 1)\) and there exists a one-parameter family \( \Delta_{\alpha,P} \) of its self-adjoint
extensions (called pseudo-laplacians; see [3]). One of these extensions (the
Friedrichs extension \( \Delta_{0,P} \)) coincides with the self-adjoint operator \( \Delta \) on
\( X_d \).

In case \( X_d = \mathbb{R}^d \), \( d = 2, 3 \) the scattering theory for the pair \((\Delta_{\alpha,P}, \Delta)\) was
extensively studied in the literature (see e. g., [1]). The spectral theory of
the operator \( \Delta_{\alpha,P} \) on compact manifolds \( X_d \) \((d = 2, 3)\) was studied in [3],
note also a recent paper [15] devoted to the case, where \( X_d \) is a compact
Riemann surface equipped with Poincaré metric.

The zeta-regularized determinant of Laplacian on a compact Riemannian
manifold was introduced in [11] and since then was studied and used in an
immense number of papers in string theory and geometric analysis, for our
future purposes we mention here the memoir [5], where the determinant of
Laplacian is studied as a functional on the space of smooth Riemannian
metrics on a compact two-dimensional manifold, and the papers [6] and
[13], where the reader may find explicit calculation of the determinant of
Laplacian for three-dimensional flat tori and for the sphere \( S^3 \) (respectively).

The main result of the present paper is a comparison formula relating
det(\( \Delta_{\alpha,P} - \lambda \)) to det(\( \Delta - \lambda \)), for \( \lambda \in \mathbb{C} \setminus (\text{Spectrum}(\Delta) \cup \text{Spectrum}(\Delta_{\alpha,P})) \).

It should be mentioned that in case of two-dimensional manifold the zeta-
regularization of det(\( \Delta_{\alpha,P} - \lambda \)) is not that standard, since the corresponding
operator zeta-function has logarithmic singularity at 0.

It should be also mentioned that in the case when the manifold \( X_d \) is flat
in a vicinity of the point \( P \) we deal with a very special case of the situation
(Laplacian on a manifold with conical singularity) considered in \[10\], \[8\], \[9\] and, via other method, in \[7\]. The general scheme of the present work is close to that of \[7\], although some calculations from \[9\] also appear very useful for us.

Acknowledgements. The work of T. A. was supported by FQRNT. Research of A. K. was supported by NSERC.

2. Pseudo-laplacians, Krein formula and scattering coefficient

Let \(X_d\) be a compact manifold of dimension \(d = 2\) or \(d = 3\); \(P \in X_d\) and \(\alpha \in [0, \pi)\). Following Colin de Verdière \[3\], introduce the set
\[
D(\Delta_\alpha, P) = \{ f \in H^2(X_d \setminus \{P\}) : \exists c \in \mathbb{C} : \text{in a vicinity of } P \text{ one has}
\]
\[
f(x) = c(\sin \alpha \cdot G_d(r) + \cos \alpha) + o(1) \text{ as } r \to 0 \},
\]
where
\[
H^2(X_d \setminus \{P\}) = \{ f \in L^2(X_d) : \exists C \in \mathbb{C} : \Delta f - C\delta_P \in L^2(X_d) \}.
\]
\(r\) is the geodesic distance between \(x\) and \(P\) and
\[
G_d(r) = \begin{cases} 
\frac{1}{2\pi} \log r, & d = 2 \\
-\frac{1}{4\pi r}, & d = 3.
\end{cases}
\]

Then (see \[3\]) the self-adjoint extensions of symmetric operator \(\Delta\) with domain \(C^\infty_c(X_d \setminus \{P\})\) are the operators \(\Delta_{\alpha, P}\) with domains \(D(\Delta_{\alpha, P})\) acting via \(u \mapsto \Delta u\). The extension \(\Delta_{0, P}\) coincides with the Friedrichs extension and is nothing but the self-adjoint Laplacian on \(X_d\).

Let \(R(x, y; \lambda)\) be the resolvent kernel of the self-adjoint Laplacian \(\Delta\) on \(X_d\).

Following \[3\] define the scattering coefficient \(F(\lambda; P)\) via
\[
- R(x, P; \lambda) = G_d(r) + F(\lambda; P) + o(1)
\]
as \(x \to P\). (Notice that in \[3\] the resolvent is defined as \((\lambda - \Delta)^{-1}\), whereas for us it is \((\Delta - \lambda)^{-1}\). This results in the minus sign in \((2)\).

As it was already mentioned the deficiency indices of the symmetric operator \(\Delta\) with domain \(C^\infty_c(X_d \setminus \{P\})\) are \((1, 1)\), therefore, one has the following Krein formula (see, e. g., \[1\], p. 357) for the resolvent kernel, \(R_\alpha(x, y; \lambda)\), of the self-adjoint extension \(\Delta_{\alpha, P}\):
\[
R_\alpha(x, y; \lambda) = R(x, y; \lambda) + k(\lambda; P)R(x, P; \lambda)R(P, y; \lambda)
\]
with some \(k(\lambda; P) \in \mathbb{C}\).

The following Lemma relates \(k(\lambda; P)\) to the scattering coefficient \(F(\lambda; P)\).

**Lemma 1.** One has the relation
\[
k(\lambda; P) = \frac{\sin \alpha}{F(\lambda; P) \sin \alpha - \cos \alpha}.
\]
Proof. Send $x \to P$ in (3), observing that $R_{\alpha}(\cdot, y; \lambda)$ belongs to $D_{\alpha,P}$, make use of (1) and (2), and then compare the coefficients near $G_{d}(r)$ and the constant terms in the asymptotical expansions at the left and at the right. □

It follows in particular from the Krein formula that the difference of the resolvents $(\Delta_{\alpha,P} - \lambda)^{-1} - (\Delta - \lambda)^{-1}$ is a rank one operator. The following simple Lemma is the key observation of the present work.

Lemma 2. One has the relation

$$\text{Tr} \left( (\Delta_{\alpha,P} - \lambda)^{-1} - (\Delta - \lambda)^{-1} \right) = \frac{F'_{\lambda}(\lambda; P) \sin \alpha}{\cos \alpha - F(\lambda; P) \sin \alpha}.$$  

Proof. One has

$$-F'_{\lambda}(\lambda; P) = \left. \frac{\partial R(y, P; \lambda)}{\partial \lambda} \right|_{y=P} = \lim_{\mu \to \lambda} \frac{R(y, P; \mu) - R(y, P; \lambda)}{\mu - \lambda}.$$

Using resolvent identity we rewrite the last expression as

$$\lim_{\mu \to \lambda} \int_{X_d} R(y, z; \mu) R(P, z; \lambda) dz \bigg|_{y=P} = \int_{X_d} [R(P, z; \lambda)]^{2} dz.$$

From (3) it follows that

$$[R(P, z; \lambda)]^{2} = \frac{1}{k(\lambda; P)} (R_{\alpha,P}(x, z; \lambda) - R(x, z; \lambda)) \bigg|_{x=z}. $$

This implies

$$-F'_{\lambda}(\lambda; P) = \frac{1}{k(\lambda; P)} \text{Tr} \left( (\Delta_{\alpha,P} - \lambda)^{-1} - (\Delta - \lambda)^{-1} \right)$$

which, together with Lemma 1, imply (5). □

Introduce the domain

$$\Omega_{\alpha,P} = \mathbb{C} \setminus \{ \lambda - it, \lambda \in \text{Spectrum}(\Delta) \cup \text{Spectrum}(\Delta_{\alpha,P}); t \in (-\infty, 0] \}.$$ 

Then in $\Omega_{\alpha,P}$ one can introduce the function

$$\tilde{\xi}(\lambda) = -\frac{1}{2\pi i} \log(\cos \alpha - F(\lambda; P) \sin \alpha)$$

(It should be noted that the function $\xi = \Re(\tilde{\xi})$ is the spectral shift function of $\Delta$ and $\Delta_{\alpha,P}$.) One can rewrite (5) as

$$\text{Tr} \left( (\Delta_{\alpha,P} - \lambda)^{-1} - (\Delta - \lambda)^{-1} \right) = 2\pi i \tilde{\xi}'(\lambda).$$

3. Operator zeta-function of $\Delta_{\alpha,P}$

Denote by $\zeta(s, A)$ the zeta-function

$$\zeta(s, A) = \sum_{\mu_k \in \text{Spectrum}(A)} \frac{1}{\mu_k^s}$$

of the operator $A$. (We assume that the spectrum of $A$ is discrete and does not contain 0.)
Take any \( \tilde{\lambda} \) from \( \mathbb{C} \setminus (\text{Spectrum } (\Delta_{\alpha,P}) \cup \text{Spectrum } (\Delta)) \). From the results of [3] it follows that the function \( \zeta(s, \Delta_{\alpha,P} - \tilde{\lambda}) \) is defined for sufficiently large \( \Re s \). It is well-known that \( \zeta(s, \Delta - \tilde{\lambda}) \) is meromorphic in \( \mathbb{C} \).

The proof of the following lemma coincides verbatim with the proof of Proposition 5.9 from [7].

**Lemma 3.** Suppose that the function \( \hat{\xi}'(\lambda) \) from [7] is \( O(|\lambda|^{-1}) \) as \( \lambda \to -\infty \). Let \( -C \) be a sufficiently large negative number and let \( c_{\lambda,\epsilon} \) be a contour encircling the cut \( c_{\lambda} \) which starts from \( -\infty + 0i \), follows the real line till \( -C \) and then goes to \( \tilde{\lambda} \) remaining in \( \Omega_{\alpha,P} \). Assume that \( \text{dist}(z, c_{\lambda}) = \epsilon \) for any \( z \in c_{\lambda,\epsilon} \). Let also

\[
\zeta_2(s) = \int_{c_{\lambda,\epsilon}} (\lambda - \tilde{\lambda})^{-s} \hat{\xi}'(\lambda)d\lambda,
\]

where the integral at the right hand side is taken over the part \( c_{\lambda,\epsilon} \) of the contour \( c_{\lambda,\epsilon} \) lying in the half-plane \( \{ \lambda : \Re \lambda > -C \} \). Let

\[
\hat{\zeta}_2(s) = \lim_{\epsilon \to 0} \zeta_2(s) = 2i \sin(\pi s) \int_{-C}^{\tilde{\lambda}} (\lambda - \tilde{\lambda})^{-s} \hat{\xi}'(\lambda)d\lambda,
\]

where \( (\lambda - \tilde{\lambda})^{-s} = e^{-is\pi} \lim_{\lambda \to c_{\lambda}} (\lambda - \tilde{\lambda})^{-s} \). Then the function

\[
R(s, \tilde{\lambda}) = \zeta(s, \Delta_{\alpha,P} - \tilde{\lambda})) - \zeta(s, \Delta - \tilde{\lambda}) - 2i \sin(\pi s) \int_{-\infty}^{-C} |\lambda|^{-s} \hat{\xi}'(\lambda)d\lambda - \hat{\zeta}_2(s)
\]

can be analytically continued to \( \Re s > -1 \) with \( R(0, \tilde{\lambda}) = R'_s(0, \tilde{\lambda}) = 0 \).

For completeness we give a sketch of proof. Using (7), one has for sufficiently large \( \Re s \)

\[
\zeta(s, \Delta_{\alpha,P} - \tilde{\lambda}) - \zeta(s, \Delta - \tilde{\lambda}) = \frac{1}{2\pi i} \int_{c_{\lambda,\epsilon}} (\lambda - \tilde{\lambda})^{-s} \text{Tr}((\Delta_{\alpha,P} - \lambda)^{-1} - (\Delta - \lambda)^{-1})d\lambda =
\]

\[
= \int_{c_{\lambda,\epsilon}} (\lambda - \tilde{\lambda})^{-s} \hat{\xi}'(\lambda)d\lambda = \zeta_1(s) + \zeta_2(s),
\]

where

\[
\zeta_1(s) = \left\{ \int_{-\infty + ic}^{-C + ic} - \int_{-\infty - ic}^{-C - ic} \right\} (\lambda - \tilde{\lambda})^{-s} \hat{\xi}'(\lambda)d\lambda.
\]

It is easy to show (see Lemma 5.8 in [7]) that in the limit \( \epsilon \to 0 \) \( \zeta_1(s) \) gives

\[
2i \sin(\pi s) \int_{-\infty}^{-C} |\lambda|^{-s} \hat{\xi}'(\lambda)d\lambda + 2i \sin(\pi s) \int_{-\infty}^{-C} |\lambda|^{-s} \hat{\xi}'(\lambda)\rho(s, \tilde{\lambda}/\lambda)d\lambda,
\]

where \( \rho(s, z) = (1 + z)^{-s} - 1 \) and

\[
\rho(s, \tilde{\lambda}/\lambda) = O(|\lambda|^{-1})
\]
as \( \lambda \to -\infty \). Using the assumption on the asymptotics of \( \tilde{\xi}(\lambda) \) as \( \lambda \to -\infty \) and the obvious relation \( \rho(0,z) = 0 \) one can see that the last term in (9) can be analytically continued to \( \Re s > -1 \) and vanishes together with its first derivative w. r. t. \( s \) at \( s = 0 \). Denoting it by \( R(s, \tilde{\lambda}) \) one gets the Lemma.

□

As it is stated in the introduction the main object we are to study in the present paper is the zeta-regularized determinant of the operator \( \Delta_{\alpha,P} - \lambda \).

Let us remind the reader that the usual definition of the zeta-regularized determinant of an operator \( A \)

\[
\det A = \exp \left( -\zeta'(0, A) \right)
\]

requires analyticity of \( \zeta(s, A) \) at \( s = 0 \).

Since the operator zeta-function \( \zeta(s, \Delta - \tilde{\lambda}) \) is regular at \( s = 0 \) (in fact, it is true in case of \( \Delta \) being an arbitrary elliptic differential operator on any compact manifold) and the function \( \tilde{\xi}(s) \) is entire, Lemma 3 shows that the behavior of the function \( \zeta(s, \Delta_{\alpha,P} - \tilde{\lambda}) \) at \( s = 0 \) is determined by the properties of the analytic continuation of the term

\[
2i \sin(\pi s) \int_{-\infty}^{-C} |\lambda|^{-s} \tilde{\xi}'(\lambda) d\lambda
\]

in (8). These properties in their turn are determined by the asymptotical behavior of the function \( \tilde{\xi}'(\lambda) \) as \( \lambda \to -\infty \).

It turns out that the latter behavior depends on dimension \( d \). In particular, in the next section we will find out that in case \( d = 2 \) the function \( \zeta(s, \Delta_{\alpha,P} - \tilde{\lambda}) \) is not regular at \( s = 0 \), therefore, in order to define \( \det(\Delta_{\alpha,P} - \tilde{\lambda}) \) one has to use a modified version of (10).

4. DETERMINANT OF PSEUDO-LAPLACIAN ON TWO-DIMENSIONAL COMPACT MANIFOLD

Let \( X \) be a two-dimensional Riemannian manifold, then introducing isothermal local coordinates \((x, y)\) and setting \( z = x + iy \), one can write the area element on \( X \) as

\[
\rho^{-2}(z) |dz|^2
\]

The following estimate of the resolvent kernel, \( R(z', z; \lambda) \), of the Laplacian on \( X \) was found by J. Fay (see [5]; Theorem 2.7 on page 38 and the formula preceding Corollary 2.8 on page 39; notice that Fay works with negative Laplacian, so one has to take care of signs when using his formulas).

Lemma 4. (J. Fay) The following equality holds true

\[
- R(z, z'; \lambda) = G_2(r) + \frac{1}{2\pi} \left[ \gamma + \log \frac{\sqrt{|\lambda|} + 1}{2} \right.
\]

\[
- \frac{1}{2(|\lambda| + 1)} \left( \frac{4}{3} \rho^2(z) \frac{\partial^2}{\partial z \partial \bar{z}} \rho(z) + \hat{R}(z', z; \lambda) \right),
\]
where $\hat{R}(z', z; \lambda)$ is continuous for $z'$ near $z$,

$$\hat{R}(z, z; \lambda) = O(|\lambda|^{-2})$$

uniformly w. r. t. $z \in X$ as $\lambda \to -\infty$; $r = \text{dist}(z, z')$, $\gamma$ is the Euler constant.

Using (12), we immediately get the following asymptotics of the scattering coefficient $F(\lambda, P)$ as $\lambda \to -\infty$:

(13) $$F(\lambda, P) = \frac{1}{4\pi} \log(|\lambda| + 1) + \frac{\gamma - \log 2}{2\pi} - \frac{1}{4\pi(|\lambda| + 1)} \left[ 1 + \frac{4}{3} \rho^2(z) \partial^2_{zz} \rho(z) \right]_{z = z(P)} + O(|\lambda|^{-2}).$$

Remark 1. It is easy to check that the expression $\rho^2(z) \partial^2_{zz} \rho(z)$ is independent of the choice of conformal local parameter $z$ near $P$.

Now from (6) and (13) it follows that

$$2\pi i \hat{\xi}'(\lambda) = -\frac{1}{4\pi(|\lambda| + 1)} - \frac{b}{(|\lambda| + 1)^2} + O(|\lambda|^{-3})$$

where $a = \frac{1}{2\pi}(\gamma - \log 2)$ and $b = \frac{1}{4\pi}(1 + \frac{4}{3} \rho^2(z) \partial^2_{zz} \rho(z))$. This implies that for $-\infty < \lambda \leq -C$ one has

(14) $$2\pi i \hat{\xi}'(\lambda) = \frac{1}{|\lambda|(|\log |\lambda| - 4\pi \cot \alpha + 4\pi a)|} + f(\lambda),$$

with $f(\lambda) = O(|\lambda|^{-2})$ as $\lambda \to -\infty$. Now knowing (14), one can study the behaviour of the term (11) in (8). We have

(15) $$2i \sin(\pi s) \int_{-\infty}^{-C} |\lambda|^{-s-1} \hat{\xi}'(\lambda) d\lambda = - \frac{\sin(\pi s)}{\pi} \int_{-\infty}^{-C} |\lambda|^{-s-1} \frac{d\lambda}{(|\log |\lambda| - 4\pi \cot \alpha + 4\pi a)|} + \sin(\pi s) \int_{-\infty}^{-C} |\lambda|^{-s} f(\lambda) d\lambda.$$

The first integral in the right hand side of (15) appeared in (9), p. 15, where it was observed that it can be easily rewritten through the function

$$\text{Ei}(z) = - \int_{-z}^{\infty} e^{-y} \frac{dy}{y} = \gamma + \log(-z) + \sum_{k=1}^{\infty} \frac{z^k}{k \cdot k!}$$

which leads to the representation

(16) $$\frac{\sin(\pi s)}{\pi} \int_{-\infty}^{-C} |\lambda|^{-s-1} \frac{d\lambda}{(|\log |\lambda| - 4\pi \cot \alpha + 4\pi a)|} = - \frac{\sin(\pi s)}{\pi} e^{-s \kappa} [\gamma + \log(s(\log C - \kappa)) + e(s)].$$
where $e(s)$ is an entire function such that $e(0) = 0$; $\kappa = 4\pi \cot \alpha - 4\pi a$.

From this we conclude that

\[
\frac{\sin(\pi s)}{\pi} \int_{-\infty}^{-C} |\lambda|^{-s-1} \frac{d\lambda}{(\log |\lambda| - 4\pi \cot \alpha + 4\pi a)} = -s \log s + g(s)
\]

where $g(s)$ is differentiable at $s = 0$.

Now (8) and (17) justify the following definition.

**Definition 1.** Let $\Delta_{\alpha,P}$ be the pseudo-laplacian on a two-dimensional compact Riemannian manifold. Then the zeta-regularized determinant of the operator $\Delta_{\alpha,P} - \tilde{\lambda}$ with $\tilde{\lambda} \in \mathbb{C} \setminus \text{Spectrum}(\Delta_{\alpha,P})$ is defined as

\[
(18) \quad \det(\Delta_{\alpha,P} - \tilde{\lambda}) = \exp \left\{ -\frac{d}{ds} \left[ \zeta(s, \Delta_{\alpha,P} - \tilde{\lambda}) + s \log s \right] \bigg|_{s=0} \right\}
\]

We are ready to get our main result: the formula relating $\det(\Delta_{\alpha,P} - \tilde{\lambda})$ to $\det(\Delta - \tilde{\lambda})$.

From (8, 11) it follows that

\[
\frac{d}{ds} \left[ \zeta(s, \Delta_{\alpha,P} - \tilde{\lambda}) + s \log s - \zeta(s, \Delta - \tilde{\lambda}) \right] \bigg|_{s=0} =
\]

\[
\frac{d}{ds} \hat{\zeta}_2(s) \bigg|_{s=0} + \int_{-\infty}^{-C} f(\lambda) d\lambda +
\]

\[
- \frac{d}{ds} \left\{ \frac{\sin \pi s}{\pi} e^{-s\kappa} [\gamma + \log(s(\log C - \kappa)) + e(s)] + s \log s \right\} \bigg|_{s=0} =
\]

\[
2\pi i \left( \zeta(\tilde{\lambda}) - \zeta(-C) \right) + \int_{-\infty}^{-C} f(\lambda) d\lambda - \gamma - \log(\log C - \kappa) =
\]

\[
(19) \quad 2\pi i \hat{\xi}(\tilde{\lambda}) - \gamma +
\]

\[
\int_{-\infty}^{-C} f(\lambda) d\lambda - 2\pi i \hat{\xi}(-C) - \log(\log C - 4\pi \cot \alpha + 2\gamma - \log 4).
\]

Notice that the expression in the second line of (19) should not depend on $C$, so one can send $C$ to $+\infty$ there. Together with (13) this gives

\[
(20) \quad \frac{d}{ds} \left[ \zeta(s, \Delta_{\alpha,P} - \tilde{\lambda}) + s \log s - \zeta(s, \Delta - \tilde{\lambda}) \right] \bigg|_{s=0} =
\]

\[
2\pi i \hat{\xi}(\tilde{\lambda}) - \gamma + \log(\sin \alpha/(4\pi)) - i\pi
\]

which implies the comparison formula for the determinants stated in the following theorem.

**Theorem 1.** Let $\tilde{\lambda}$ do not belong to the union of spectra of $\Delta$ and $\Delta_{\alpha,P}$ and let the zeta-regularized determinant of $\Delta_{\alpha,P}$ be defined as in (18). Then one has the relation

\[
(21) \quad \det(\Delta_{\alpha,P} - \tilde{\lambda}) = -4\pi e^{\gamma} (\cot \alpha - F(\tilde{\lambda}, P)) \det(\Delta - \tilde{\lambda}).
\]
Observe now that 0 is the simple eigenvalue of $\Delta$ and, therefore, it follows from Theorem 2 in [3] that 0 does not belong to the spectrum of the operator $\Delta_{\alpha,P}$ and that $\Delta_{\alpha,P}$ has one strictly negative simple eigenvalue. Thus, the determinant in the left hand side of (21) is well defined for $\tilde{\lambda} = 0$, whereas the determinant at the right hand side has the asymptotics
\begin{equation}
\det(\Delta - \tilde{\lambda}) \sim (-\tilde{\lambda})\det^* \Delta
\end{equation}
as $\tilde{\lambda} \to 0^-$. Here $\det^* \Delta$ is the modified determinant of an operator with zero mode.

From the standard asymptotics
\[-R(x, y; \lambda) = \frac{1}{\text{Vol}(X)} \frac{1}{\lambda} + G_2(r) + O(1)\]
as $\lambda \to 0$ and $x \to y$ one gets the asymptotics
\begin{equation}
F(\lambda, P) = \frac{1}{\text{Vol}(X)} \frac{1}{\lambda} + O(1)
\end{equation}
as $\lambda \to 0$. Now sending $\tilde{\lambda} \to 0^-$ in (21) and using (22) and (23) we get the following corollary of the Theorem 1.

**Corollary 1.** The following relation holds true
\begin{equation}
\det \Delta_{\alpha,P} = -\frac{4\pi e^\gamma}{\text{Vol}(X)} \det^* \Delta.
\end{equation}

5. **Determinant of pseudo-laplacian on three-dimensional manifolds**

Let $X$ be a three-dimensional compact Riemannian manifold. We start with the Lemma describing the asymptotical behavior of the scattering coefficient as $\lambda \to -\infty$.

**Lemma 5.** One has the asymptotics
\begin{equation}
F(\lambda; P) = \frac{1}{4\pi} \sqrt{-\lambda} + c_1(P) \frac{1}{\sqrt{-\lambda}} + O(|\lambda|^{-1})
\end{equation}
as $\lambda \to -\infty$

**Proof.** Consider Minakshisundaram-Pleijel asymptotic expansion ([12])
\begin{equation}
H(x, P; t) = (4\pi t)^{-3/2} e^{-d(x,P)^2/(4t)} \sum_{k=0}^{\infty} u_k(x, P) t^k
\end{equation}
for the heat kernel in a small vicinity of $P$, here $d(x, P)$ is the geodesic distance from $x$ to $P$, functions $u_k(\cdot, P)$ are smooth in a vicinity of $P$, the equality is understood in the sense of asymptotic expansions. We will make use of the standard relation
\begin{equation}
R(x, y; \lambda) = \int_0^{+\infty} H(x, y; t) e^{\lambda t} dt.
\end{equation}
Let us first truncate the sum (26) at some fixed \( k = N + 1 \) so that the remainder, \( r_n \), is \( O(t^N) \). Defining
\[
\tilde{R}_N(x, P; -\lambda) := \int_{0}^{\infty} r_n(t, x, P)e^{t\lambda} dt,
\]
we see that
\[
\tilde{R}_N(x, P; \lambda) = O(|\lambda|^{-(N+1)})
\]
as \( \lambda \to -\infty \) uniformly w. r. t. \( x \) belonging to a small vicinity of \( P \).

Now, for each \( 0 \leq k \leq N + 1 \) we have to address the following quantity
\[
R_k(x, P; \lambda) := u_k(x, y) \frac{2}{(4\pi)^{3/2}} \int_{0}^{\infty} t^{k-3/2} e^{-\frac{d(x, P)^2}{4t}} e^{t\lambda} dt.
\]
According to identity (36) below one has
\[
R_0(x, P; \lambda) = \frac{1}{4\pi d(x, P)} - \frac{1}{4\pi} \sqrt{-\lambda} + o(1),
\]
as \( d(x, P) \to 0 \). For \( k \geq 1 \) one has
\[
R_k(x, P; \lambda) = \frac{u_k(x, P)}{(4\pi)^{3/2}} 2^{3/2-k} \left( \frac{d(x, P)}{\sqrt{-\lambda}} \right)^{k-1/2} K_{k-\frac{3}{2}}(d(x, P)\sqrt{-\lambda}) =
\]
\[
- c_k(P) \frac{1}{(\sqrt{-\lambda})^{2k-1}} + o(1)
\]
as \( d(x, P) \to 0 \) (see [2], p. 146, f-la 29). Now (25) follows from (27), (28) and (29). □

Now from Lemma 5 it follows that
\[
2\pi i \hat{\xi}'(\lambda) = -\frac{1}{2\lambda} + O(|\lambda|^{-3/2})
\]
as \( \lambda \to -\infty \), therefore, one can rewrite (11) as
\[
\sin(\pi s) \frac{1}{\pi} \left\{ \int_{-\infty}^{-C} |\lambda|^{-s}(2\pi i \hat{\xi}'(\lambda) + \frac{1}{2\lambda})d\lambda + \frac{C^{-s}}{2s} \right\}
\]
which is obviously analytic in \( \Re s > -\frac{1}{2} \). Thus, it follows from (8) that the function \( \zeta(s, \Delta_{\alpha,P} - \tilde{\lambda}) \) is regular at \( s = 0 \) and one can introduce the usual zeta-regularization
\[
\det(\Delta_{\alpha,P} - \tilde{\lambda}) = \exp\{-\zeta'(0, \Delta_{\alpha,P} - \tilde{\lambda})\}
\]
of \( \det(\Delta_{\alpha,P} - \tilde{\lambda}) \).

Moreover, differentiating (8) with respect to \( s \) at \( s = 0 \) similarly to (19) we get
\[
\frac{d}{ds} \left[ \zeta(s, \Delta_{\alpha,P} - \tilde{\lambda}) - \zeta(s, \Delta - \tilde{\lambda}) \right] \bigg|_{s=0} =
\]
\[
2\pi i(\tilde{\xi}(\tilde{\lambda}) - \tilde{\xi}(-C)) + \int_{-\infty}^{-C} (2\pi i \tilde{\xi}'(\lambda) + \frac{1}{2\lambda}) d\lambda - \frac{1}{2} \log C =
\]
which reduces after sending \(-C \to -\infty\) to
\[
2\pi i \tilde{\xi}(\tilde{\lambda}) + \log \sin \alpha - \log(4\pi) + i\pi = -\log(\cot \alpha - F(\tilde{\lambda}; P)) - \log(4\pi) + i\pi
\]
which implies the following theorem.

**Theorem 2.** Let \(\Delta_{\alpha,P}\) be the pseudo-laplacian on \(X\) and \(\tilde{\lambda} \in \mathbb{C}\backslash (\text{Spectrum}(\Delta) \cup \text{Spectrum}(\Delta_{\alpha,P}))\). Then

\[
\det(\Delta_{\alpha,P} - \tilde{\lambda}) = -4\pi(\cot \alpha - F(\tilde{\lambda}; P)) \det(\Delta - \tilde{\lambda}).
\]

Sending \(\tilde{\lambda} \to 0\) and noticing that relation (23) holds also in case \(d = 3\) we get the following corollary.

**Corollary 2.**

\[
\det\Delta_{\alpha,P} = -\frac{4\pi}{\text{Vol}(X)} \det^* \Delta.
\]

In what follows we consider two examples of three-dimensional compact Riemannian manifolds for which there exist explicit expressions for the resolvent kernels: a flat torus and the round (unit) 3d-sphere. These manifolds are homogeneous, so, as it is shown in [3], the scattering coefficient \(F(\lambda, P)\) is \(P\)-independent.

**Example 1: Round 3d-sphere.**

**Lemma 6.** Let \(X = S^3\) with usual round metric. Then there is the following explicit expression for scattering coefficient

\[
F(\lambda) = \frac{1}{4\pi} \coth \left( \frac{\pi}{\sqrt{-\lambda - 1}} \right) \cdot \sqrt{-\lambda - 1}
\]

and, therefore, one has the following asymptotics as \(\lambda \to -\infty\)

\[
F(\lambda) = \frac{1}{4\pi} \sqrt{\vert \lambda \vert - 1} + O(\vert \lambda \vert^{-\infty}).
\]

**Remark 2.** The possibility of finding an explicit expression for \(F(\lambda)\) for \(S^3\) was mentioned in [3]. However we failed to find (34) in the literature.

**Proof.** We will make use the well-known identity (see, e.g., [2], p. 146, f-la 28):

\[
\int_0^{+\infty} e^{\lambda t} t^{-3/2} e^{d^2 / 4t} dt = 2\frac{\sqrt{\pi}}{d} e^{-d\sqrt{-\lambda}};
\]

for \(\lambda < 0\) and \(d \in \mathbb{R}\) and the following explicit formula for the operator kernel \(e^{-tH(x, y; t)}\) of the operator \(e^{-t(\Delta + 1)}\), where \(\Delta\) is the (positive) Laplacian on \(S^3\) (see [3], (2.29)):

\[
e^{-tH(x, y; t)} = -\frac{1}{2\pi \sin d(x, y)} \frac{1}{\partial} \Theta(z, t).
\]
Here $d(x, y)$ is the geodesic distance between $x, y \in S^3$ and

$$
\Theta(z, t) = \frac{1}{\sqrt{4\pi t}} \sum_{k=-\infty}^{+\infty} e^{-(z+2k\pi)^2/4t}
$$

is the theta-function.

Denoting $d(x, y)$ by $\theta$ and using (37) and (36), one gets

$$
R(x, y; \lambda - 1) = \int_0^{+\infty} e^{\lambda t} e^{-t} H(x, y; t) dt =
$$

$$
\frac{1}{4\pi} \left( - \sum_{k<0} e^{(\theta+2k\pi)\sqrt{-\lambda}} + \sum_{k\geq 0} e^{-(\theta+2k\pi)\sqrt{-\lambda}} \right) =
$$

$$
\frac{1}{4\pi} \frac{1}{\sin \theta} \left( - e^{-2\pi\sqrt{-\lambda} \sqrt{-\lambda}} e^{\theta \sqrt{-\lambda}} + e^{-\theta \sqrt{-\lambda}} \right) =
$$

$$
\frac{1}{4\pi} \frac{1}{\sin \theta} \left( - e^{-2\pi\sqrt{-\lambda} \sqrt{-\lambda}} e^{\theta \sqrt{-\lambda}} + e^{-\theta \sqrt{-\lambda}} \right)
$$

(38)

$$
\frac{1}{4\pi} \frac{1}{\sin \theta} \left( - e^{-2\pi\sqrt{-\lambda} \sqrt{-\lambda}} e^{\theta \sqrt{-\lambda}} + e^{-\theta \sqrt{-\lambda}} \right) =
$$

as $\theta \to 0$, which implies the Lemma. □

Example 2: Flat 3d-tori. Let $\{A, B, C\}$ be a basis of $\mathbb{R}^3$ and let $T^3$ be the quotient of $\mathbb{R}^3$ by the lattice $\{mA + nB + lC : (m, n, l) \in \mathbb{Z}^3\}$ provided with the usual flat metric.

Notice that the free resolvent kernel in $R^3$ is

$$
e^{-\sqrt{-\lambda} ||x-y||}$$

and, therefore,

(39)

$$R(x, y; \lambda) = \frac{e^{-\sqrt{-\lambda} ||x-y||}}{4\pi ||x-y||} + \frac{1}{4\pi} \sum_{(m, n, l) \in \mathbb{Z}^3 \setminus \{(0,0,0)\}} \frac{e^{-\sqrt{-\lambda} ||x-y+mA+nB+lC||}}{||x-y+mA+nB+lC||}.$$

From (39) it follows that

$$F(\lambda) = \frac{1}{4\pi} \sqrt{-\lambda} - \frac{1}{4\pi} \sum_{(m, n, l) \in \mathbb{Z}^3 \setminus \{(0,0,0)\}} \frac{e^{-\sqrt{-\lambda} ||mA+nB+lC||}}{||mA+nB+lC||} =
$$

$$\frac{1}{4\pi} \sqrt{-\lambda} + O(|\lambda|^{-\infty})$$

as $\lambda \to -\infty$.

Remark 3. It should be noted that explicit expressions for $\text{det}^* \Delta$ in case $X = S^3$ and $X = T^3$ are given in [13] and [6].
References

[1] Albeverio S., Gesztesy F., Hoegh-Krohn R., Holden H., Solvable models in quantum mechanics, AMS 2005
[2] Erdélyi, A. and Bateman, H. Tables of integral transforms, volume 2, McGraw-Hill, New York, 1954
[3] Yves Colin De Verdiere, Pseudo-laplacians. I, Annales de l’institut Fourier, tome 32, N3 (1982), 275–286
[4] J. Cheeger, M. Taylor, On the diffraction of waves by conical singularities. I, Communications on Pure and Applied Mathematics, Volume 35 (1982), Issue 3, 275-331
[5] Fay, John D., Kernel functions, analytic torsion, and moduli spaces, Memoirs of the AMS 464 (1992)
[6] Furutani K., de Gosson S., Determinant of Laplacians on Heisenberg manifolds, J. Geom. Phys. 48 (2003), pp. 438-479
[7] L. Hillairet, A. Kokotov, Krein formula and S-matrix for euclidean surfaces with conical singularities, J. of Geom. Anal, 2012, to appear, arXiv:1011.5034v1
[8] Klaus Kirsten, Paul Loya, Jinsung Park, Exotic expansions and pathological properties of ζ-functions on conic manifolds, J. Geom. Anal., 18(2009), 835-888
[9] Klaus Kirsten, Paul Loya, Jinsung Park, The very unusual properties of the resolvent, heat kernel, and zeta-function for the operator \(-\frac{d^2}{dr^2} - 1/(4r^2)\), Journal of mathematical physics, 47(2006)
[10] Loya P., McDonald P., Park J., Zeta regularized determinants for conic manifolds, Journal of Functional Analysis (2007), 242, N1, 195-229
[11] Ray, D. B.; Singer, I. M., Analytic torsion for complex manifolds., Ann. of Math., 98 (1973), 154-177
[12] Minakshisundaram, S.; Pleijel, A., Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds, Canadian Journal of Mathematics 1 (1949): 242-256
[13] Kumagai H., The determinant of the laplacian on the n-sphere, Acta Arithmetica, XCL.3 (1999)
[14] Ray D. B., Singer I. M., Analytic torsion for complex manifolds. Ann. of Math., Vol 98 (1973), N1, 154-177
[15] Ueberschaer H., The trace formula for a point scatterer on a compact hyperbolic surface, arXiv:1109.4329v2
E-mail address: aissiou@math.mcgill.ca

Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Blvd. West, Montreal, Quebec H3G 1M8 Canada

E-mail address: Luc.Hillairet@math.univ-nantes.fr

UMR CNRS 6629-Université de Nantes, 2 rue de la Houssinière, BP 92 208, F-44 322 Nantes Cedex 3, France

E-mail address: alexey@mathstat.concordia.ca

Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Blvd. West, Montreal, Quebec H3G 1M8 Canada