On financial applications of the two-parameter Poisson-Dirichlet distribution

Research Note

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January 7, 2015

Abstract

Capital distribution curve is defined as log-log plot of normalized stock capitalizations ranked in descending order. The curve displays remarkable stability over periods of time.

Theory of exchangeable distributions on set partitions, developed for purposes of mathematical genetics and recently applied in non-parametric Bayesian statistics, provides probabilistic-combinatorial approach for analysis and modeling of the capital distribution curve. Framework of the two-parameter Poisson-Dirichlet distribution contains rich set of methods and tools, such as consistent family of distributions on partitions, limiting distribution in infinite ranked simplex and associated diffusion process, induced by Markov chain of special form.

The purpose of this note is to provide an informal introduction to the framework of exchangeable distributions on partitions in financial context. In particular, it is shown that averaged samples from the Poisson-Dirichlet distribution provide approximation to the capital distribution curves in equity markets. This suggests that the two-parameter model can be employed for modeling evolution of ranked market weights fluctuating in stochastic equilibrium.
1 Introduction

Log-log plot of stock market weights ranked in descending order defines the capital distribution curve. Temporal stability of the shape of this curve is one of the cornerstones of the Stochastic Portfolio Theory (SPT), developed by R. Fernholz and I. Karatzas ([8], [15] and [9]). In contrast to normative theories, such as the MPT and the CAPM, Stochastic Portfolio Theory is descriptive model, in a sense that it is consistent with empirical characteristics and dynamics of equity markets. In particular, the SPT captures tendency of stocks to retain their ranks. The SPT model employs machinery of rank-interacting Brownian particles and semimartingales.

Framework of partition structures, combining combinatorial and probabilistic methods, provides complementary approach for modeling and analysis of the capital distribution curve and can be summarized as follows.

- The market is considered as large combinatorial structure - partition of the set of the invested units of money. Capitalizations of individual stocks provide block or cluster sizes of the partition. It is assumed that all market values are represented by integers, for instance measured in cents.
- Vector of block sizes defines number of compatible partitions of the set. In other words, the market can be represented as a giant Young diagram with vector of capitalizations determining (potentially very large) number of ways the market can be realized combinatorially.
- In principle it is possible to consider arbitrary probability distributions on n-partitions. In the context of mathematical genetics, Kingman [18] noticed (actually attributing this observation to F. Kelly) that there are natural consistency conditions linking distributions for partitions with n and n + 1 elements. He called families of such consistent distributions partition structures.

Partition structures are important for several reasons.

- First of all, partition structure provides a model of random transitions with dynamic dimensions. In other words, at any time number of diffusion components may change due to appearance of a new stock or bankruptcy of existing firm. Also, due to exchangeability, the diffusion components can be considered in descending order, which simplifies modeling of ranked market weights.
- Second, partition structure, with non-trivial limiting distribution, defines asymptotic shape of the corresponding combinatorial structure. In particular, this may provide an explanation of the phenomenon of capital distribution curve stability.

The two-parameter Poisson-Dirichlet model is a remarkable and well studied instance of partition structure. It possesses analytically tractable limiting distribution defined in the simplex with ranked weights.

**PD distribution.** The Dirichlet distribution with m-dimensional vector of parameters $(\alpha_1, \ldots, \alpha_m)$ defines probability for non-negative proportions in a standard simplex. Kingman [17] considered limiting behavior of this distribution with symmetric vector of parameters $(\alpha, \ldots, \alpha)$ such that $\theta = m\alpha = \text{const}$ for $m \to \infty$ and called distribution of ranked components the Poisson-Dirichlet ($PD$) distribution (with one parameter $\theta$). This distribution is defined in the infinite simplex of ranked weights, known as Kingman simplex

$$\nabla = \{ x_1 \geq x_2 \geq ... \mid x_i \geq 0, \sum x_i = 1 \}$$

Size-biased permutation provides an efficient method of sampling from the Dirichlet and the Poisson-Dirichlet distributions. In a framework of population biology Engen [5] suggested modification of the size-biased method, which produced another class of PD distributions. It was called the two-parameter Poisson-Dirichlet distribution by Perman, Pitman and Yor, who rediscovered it in the context of studying of ranked jumps of gamma and stable subordinators (see [20],[24]). Corresponding partition structure was found by Pitman[22]. Monograph by Pitman [23] contains wealth of information on the two-parameter Poisson-Dirichlet model. Chatterjee and Pal [4] showed that limiting behavior of rank-interacting system of Brownian particles is characterized by the $PD(\alpha,0)$ distribution.

Aoki pioneered applications of exchangeable distributions in economics ([1],[2]), in particular using finitary characterization by Garibaldi, Costantini, et al. ([11], also book [14]). Markov chain approach with transitions between partitions was independently developed by Garibaldi, Costantini, et al. [11], [12]. Petrov [21], inspired by works of Kerov, Fulman [10], Borodin and Olshanski [3] constructed a diffusion process preserving the two-parameter Poisson-Dirichlet distribution in the infinite-dimensional ranked simplex.

This research note aims at illustration of applications of partition structures and the two-parameter model for modeling of stochastic evolution of the capital distribution curve. In particular, it is shown in Section 6 that the two-parameter model provides reasonable approximation of capital distribution curves in equity markets. Moreover the model also provides fit for distribution of relative total capitalizations of stock exchanges.

Main results of this paper were presented at the 8th World Congress of the Bachelier Finance Society, 2014. The author is very grateful to Prof. I. Karatzas for useful advice and suggestions.
1.1 Toy model

Let us imagine that there are only two stocks with capitalizations 3 and 2 in the market with capitalization 5. Tickers or names do not play important role and used only to distinguish the stocks. Ten ways in which 5 units of money can form a state with these capitalizations is represented by the following Young tableaux

Since these partitions have the same block sizes it is convenient to use Young diagram, shown on the right, to denote all partitions with the same shape. The 10 partitions above arise by adding a new box:

- in one way to 4 partitions with shape \( \begin{array}{|c|c|} \hline a & b \\ \hline \end{array} \) \( \begin{array}{|c|c|c|} \hline a & b & c \\ \hline \end{array} \) \# = 4,

and

- in two ways to 3 partitions with shape \( \begin{array}{|c|c|c|} \hline a & b \\ \hline \end{array} \) \( \begin{array}{|c|c|c|} \hline a & b & c \\ \hline \end{array} \) \# = 3,

For a Young diagram \( c \) combinatorial formula (1) provides \( \#(c) \) equal to

- number of set partitions with given block sizes,
- which in financial terms is the same as number of ways the market can form a state described by ranked capitalizations.

Exchangeable probability distribution on partitions assigns the same probability to all partitions with the same shape. This framework is useful when one is interested in studying distribution of ranked block sizes (capitalizations) regardless of block labels (tickers). If \( \pi_n(c) \) denotes probability of a partition with \( n \) elements and shape \( c \), then total probability of all partitions with this shape is

\[
p_n(c) = \#(c) \cdot \pi_n(c)
\]

Obviously sum of these probabilities over all shapes (Young diagrams) with \( n \) elements must be 1.

In a context of mathematical genetics Kingman [18] considered family of distributions \( \{p_n\} \) on partitions of \( n = 1, 2, 3, \ldots \) elements and noticed that random sampling induces natural consistency constraints connecting distributions for levels \( n - 1 \) and \( n \). He called distributions \( \{p_n\} \) satisfying such constraints partition structures.

Continuing the example, let’s consider 10 partitions with the shape \( \begin{array}{|c|c|} \hline a & b \\ \hline \end{array} \). In each of these partitions any of 5 boxes can be removed so remaining partitions will have 4 elements. For each partition there are

- 2 ways to get to \( \begin{array}{|c|c|} \hline a & b \\ \hline \end{array} \)

- 3 ways to obtain \( \begin{array}{|c|c|c|} \hline a & b \\ \hline \end{array} \)

Uniform deletion of a box on partitions of 5 elements induces probability distribution on 4-partitions. For example

\[
p_4(\begin{array}{|c|c|} \hline a & b \\ \hline \end{array}) = \frac{3}{5}p_5(\begin{array}{|c|c|} \hline a & b \\ \hline \end{array}) + \frac{1}{5}p_5(\begin{array}{|c|c|} \hline c \\ \hline \end{array})
\]

On the other hand this consistency constraint defines forward conditional probabilities of partition growth, for instance

\[
p(\begin{array}{|c|c|} \hline a & b \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline a & b \\ \hline \end{array}) = \frac{3}{5}p_5(\begin{array}{|c|c|} \hline a & b \\ \hline \end{array})/p_4(\begin{array}{|c|c|} \hline a & b \\ \hline \end{array})
\]

In other words partition structure is a family of distributions on partitions consistent under growth (or Up moves) and recession (or Down moves). This enables considering market dynamics as a process of combinatorial random walks on partitions driven by sequences of up or down transitions.
1.2 Capital distribution curve

Log-log plot of ranked market weights, known as the capital distribution curve displays
  • largest stocks have power law behavior,
  • concavity of the curve and
  • stability over periods of time

For example, figure below shows capital distribution curve of the NASDAQ market on three dates in 2014. As it can be seen from the chart most of market weights had relatively small fluctuations, despite that total capitalization of the NASDAQ market changed significantly during that period of time. Stability of the capital distribution curve suggests certain independence market weights of total market capitalization.

![Figure 1: Capital distribution curves on May 27, Sep 24, Dec 9, 2014, NASDAQ](image)

More detailed chart reveals behavior of weights of top 100 stocks.

![Figure 2: weights of top 100 stocks, NASDAQ](image)

Capital distribution curves in many markets, plus distribution of capitalizations of stock exchanges have shapes similar to one shown at Figure 1, and Section 6 contains examples of fit of these curves by the PD model.

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1 Data source is [http://www.google.com/finance#stockscreener](http://www.google.com/finance#stockscreener)
1.3 Poisson-Dirichlet distribution

The two-parameter Poisson-Dirichlet distribution is analytically tractable exchangeable probability distribution. Log-log plot of samples from the Poisson-Dirichlet law displays

- largest components have power law behavior,
- concavity of the curve and
- stability

The infinite-dimensional Poisson-Dirichlet distribution generalizes symmetric finite-dimensional Dirichlet distribution. Moreover, both distributions can be represented by normalization of sequences of iid random variables \((y_1, y_2, \ldots)\) by sum \(s = \sum y_j\) as \((y_1/s, y_2/s, \ldots)\).

*The Poisson-Dirichlet law is a family of distributions with property of independence proportions \(y_i/s\) of sum \(s\).*

This together with results of Section 6 suggest that \(PD\) model can be used for capital distribution curve modeling.

Figure below illustrates fit of NASDAQ market weights by averages of samples from the two-parameter distribution. Estimation of parameters is by least squares method.

![Figure 3: NASDAQ fit by \(PD(0.60, 55)\), (data as of Dec 9, 2014)](image)

![Figure 4: 20 sample paths of \(PD(0.60, 55)\)](image)
1.4 The broken-stick model

The broken-stick is a simple model illustrating how uniform partition produces inequality patterns. MacArthur [19] proposed this model for explanation of relative species abundances in closed environment.

Let’s assume that stick of unit length represents some finite resource, such as territory, available food, water reservoir, etc., which must be shared between species. The resource is broken at random by throwing uniformly \( n - 1 \) cutting points on this stick and breaking it into \( n \) pieces. Length of each piece represents share, which is taken by some class of species. While on average length of each piece will be \( 1/n \), ranked lengths of pieces display interesting behavior.

For instance, if stick is broken just into two pieces, then length of smaller piece is never larger than 50% and since cut point is uniformly distributed it is easy to see that smaller stick on average represents 25% of length, while larger one takes 75%. In general it can be shown that after breaking stick into \( n \) pieces expected length of the \( k \)-th largest piece is given by

\[
 x_k = \frac{1}{n} \sum_{j=k}^{n} \frac{1}{j}
\]

In case of 3 pieces expected proportions ranked in descending order are 61.1%, 27.8% and 11.1%. It can be checked by straightforward simulation that dropping 4 points at uniform on unit interval produces on average following ranked lengths of 5 subintervals

\[
(46\%, 26\%, 16\%, 9\%, 4\%)
\]

Obviously, sampled proportions will fluctuate around these expected lengths. For larger values of \( n \) ranked expected proportions start to decay rapidly and it is more convenient to display them on a log-log plot.

![Figure 5: Expected proportions for \( n = 10, 25, 50 \)](image)

These examples illustrate that asymmetry in ranked proportions appears with completely uniform distribution of resource.
2 Exchangeable partitions

2.1 Descriptions of set partitions

Galaxies, stars, companies, people form clusters and sizes of these clusters are rarely uniform. In finance and economics stocks and companies can be considered from the following point of view.

- Stock market comprise \( k \) stocks with total capitalization \( n \) of units of money. If \( n_i \) denotes value of \( i \)-th largest company (by capitalization), then market weights are given by \( x_i = n_i/n \) and vector \( \mathbf{x} = (x_1, x_2, ...) \) represents capital distribution curve.

New unit of money can join any of the stocks thus increasing capitalization of particular stock to \( n_i + 1 \) and total capitalization to \( n + 1 \), or unit of money can leave a stock decreasing corresponding stock and market capitalizations by 1. Also there is a possibility that a new stock will be issued during the IPO, which leads to increase of number of clusters to \( k + 1 \).

- In the same way companies assets/values may experience increase or decrease. Also there is a possibility that a new company enters the market.

- In mutual funds industry, money coming to the market join existing funds proportionally to their size, but there is always opportunity that new fund emerges.

Process of clustering can represented by partitions of a set. For instance, the set of three letters 'a', 'b' and 'c' can be partitioned as shown below in the left column, with corresponding Young diagrams in the right column representing partition classes:

\[
\begin{align*}
\{a, b, c\} & \quad \begin{array}{c}
\square \\
\square \\
\square
\end{array} \\
\{a, b\}, \{c\} & \quad \begin{array}{c}
\square \\
\square
\end{array} \\
\{a, c\}, \{b\} & \quad \begin{array}{c}
\square
\end{array} \\
\{b, c\}, \{a\} & \quad \begin{array}{c}
\square
\end{array} \\
\{a\}, \{b\}, \{c\} & \quad \begin{array}{c}
\square
\end{array}
\end{align*}
\]

If in set partition, represented by clusters/blocks, cluster labels are not important and order of items inside of each cluster is irrelevant, then such partition called exchangeable. Such partitions have the same shape and completely described by vector of their block sizes. Partitions with the same shape belong to the same exchangeable class (or partition class). For the example with three companies, the set \( \{a, b, c\} \) has 5 partitions and 3 exchangeable classes, represented by Young diagrams in the right column.

Every exchangeable partition of \( n \) elements into \( k \) clusters (blocks) can be described in two ways.

1. For the first order description, since labeling of clusters is not important, it is convenient to consider cluster sizes arranged in descending order

\[ n_1 \geq n_2 \geq \ldots \geq n_k , \]

where \( n_i \) denotes size of \( i \)-th largest cluster, hence \( n = n_1 + \cdots + n_k \)

In population biology terminology it is called a frequency vector:

\[ \mathbf{n} = [n_1, n_2, \ldots n_k] \]

Obviously Young diagrams correspond to this description.

2. For the second description, let \( c_1 \) denote number of clusters of size one, \( c_2 \) denote number of clusters of size two, etc. If \( c_j \) denote number of clusters with \( j \) items then total number of items is

\[ n = c_1 + 2 \cdot c_2 + \cdots + m \cdot c_m \]

and number of clusters is given by \( k = c_1 + \cdots + c_m \) where \( m \) is the size of the largest cluster. To distinguish partition vector \( \mathbf{c} \) from frequency vector \( \{\} \)-notation is used

\[ \mathbf{c} = \{c_1, c_2, \ldots \} \]

Let \( \mathbf{c} \vdash (n, k) \) denote that vector \( \mathbf{c} \) describes a partition of \( n \) elements into \( k \) clusters.
2.2 Size of exchangeable class

Every exchangeable class contains set partitions, represented by the same partition vector $c = \{c_1, c_2, c_3, \ldots\}$. Number of partitions in a class, described by vector $c$, is given by

$$\#(c) = \frac{n!}{\prod c_j!^{c_j}}$$  \hspace{1cm} (1)

Indeed, by multinomial formula number of partitions with $c_1$ one-element subsets, $c_2$ two-element subsets, etc.
is

$$\frac{n!}{1! \cdots 1! \cdots 2! \cdots \cdots}$$

which must be divided by $\prod c_j!$ since permutations of blocks of the same size play no role.

For instance, the 4 element set $\{a, b, c, d\}$ has 15 partitions and 5 exchangeable classes:

| $\{a, b, c, d\}$ | $n = [4]$ | $c = \{0, 0, 0, 1\}$ | $\#(c) = \frac{4!}{(4!)^1} = 1$ |
|------------------|----------|---------------------|-----------------------------|
| $\{a, b\}, \{c\}$ | $n = [3, 1]$ | $c = \{1, 0, 1, 0\}$ | $\#(c) = \frac{4!}{(1!)^1(3!)^1} = 4$ |
| $\{a, c\}, \{b\}$ | $n = [2, 2]$ | $c = \{0, 2, 0, 0\}$ | $\#(c) = \frac{4!}{2!(2!)^2} = 3$ |
| $\{a, d\}, \{b, c\}$ | $n = [2, 1, 1]$ | $c = \{2, 1, 0, 0\}$ | $\#(c) = \frac{4!}{2!(1!)^2(2!)^1} = 6$ |
| $\{a\}, \{b\}, \{c\}, \{d\}$ | $n = [1, 1, 1, 1]$ | $c = \{4, 0, 0, 0\}$ | $\#(c) = \frac{4!}{4!(1!)^4} = 1$ |

Interestingly that partition $n = [2, 1, 1]$ can be realized in 6 ways and uniform partition $n = [2, 2]$ only in 3 ways.

2.3 Partition structure

If all partitions from the class with partition vector $c$ are considered to be equivalent then they should have the same probability.

If $\pi_n(c)$ denotes probability of an element from the partition class $c$ with $\#(c)$ elements then probability of that exchangeable class is

$$p_n(c) = \#(c) \cdot \pi_n(c)$$

Obviously these probabilities should satisfy

$$\sum_{c \parallel n} p_n(c) = 1$$  \hspace{1cm} (2)

Here $c \parallel n$ denotes that summation runs over all classes of partitions of $n$ elements.

**Partition structure.** In general, it is not enough to assign probability measures over partition classes for all values of $n$. Kingman [18] noticed that besides (2) there are consistency conditions linking exchangeable probability measures $p_{n-1}$ and $p_n$ and called such consistent sequences of $\{p_i\}$ *partition structures.*
2.4 Ewens-Pitman Sampling Formulae

A finite dimensional counterpart of one-parameter stick-breaking model (6) has been proposed by Ewens in the context of population biology. Given partition vector \( \mathbf{c} \) Ewen’s Sampling Formula assigns probability as

\[
p(\mathbf{c}) = \frac{\theta^k}{\theta + k} \prod \frac{n!}{c_j! c_j}
\]

Pitman studied size-biased representation of the two-parameter model (7) and obtained corresponding extension of Ewen’s sampling formula in [22]. The two-parameter Pitman’s Sampling Formula (PSF) gives probability for partition class \( \mathbf{c} \)

\[
p(\mathbf{c}) = \frac{\theta^{[k,\alpha]}}{\theta + k} \prod \frac{n!}{c_j! c_j} \prod \left( \frac{1 - \alpha}{n} \right)^{c_i}
\]

where \( \theta^{[k,\alpha]} = \theta(\theta + \alpha) \cdots (\theta + \alpha(k - 1)) \), which shows that for \( \alpha = 0 \) the formula converges to (3).

Kerov[16] proposed that formula (4) can be obtained via model of random allocation with conditionally independent variates.

2.5 Chinese Restaurant Process

Chinese Restaurant Process provides probabilistic dynamics of partitions, ensuring that probabilities remain exchangeable. Zabell [25] explains the metaphor as "on any given evening in Berkley a large number of people go to some Chinese restaurant in the downtown area. As each person arrives, he looks in the window of each restaurant to decide whether or not to go inside. His chances of going in increases with the number of people already seen inside... But there’s some probability that he goes to an empty restaurant...”

More formally, it is assumed that there are infinite number of tables (restaurants) and first customer always sits at first unoccupied table, say table 1. Customer \( n + 1 \) observes occupied \( k \) tables and

- joins table with \( n_i \) people with probability
  \[
p_i = \frac{n_i - \alpha}{n + \theta}
\]

- joins new, unoccupied table with probability
  \[
p^* = \frac{\theta + \alpha k}{n + \theta}
\]

It is important that this process provides exchangeable probability on partitions.

For instance partition \( \mathbf{n} = [2, 1, 1] \) can migrate to following states

\[
\begin{align*}
2 & \xrightarrow{\frac{\alpha}{\theta + 4}} 1 \\
1 & \xrightarrow{\frac{\alpha}{\theta + 4}} 1 \\
1 & \xrightarrow{\frac{\alpha}{\theta + 4}} 1 \\
\theta + 3\alpha & \xrightarrow{\frac{\alpha}{\theta + 4}}
\end{align*}
\]
3 Diffusions as random walks on partitions

Following results and methods of Kingman and Kerov, diffusion process in ordered infinite simplex was developed by Petrov, Olshanski, Borodin, et al. ([3],[21],[10],[7] and monograph [6]). Independently, Markov chain induced by down-up transitions was studied by Costantini, Garibaldi et. al. ([13], [12]).

The idea of approximating a two-parameter diffusion process is relatively simple. Let’s fix some large \( n \) and starting with some partition let’s consider Down and Up jumps, where

- down move randomly deletes one cell in Young diagram
- up move acts according to the Chinese Restaurant Process

For instance, figure below shows possible transitions between partitions of 4 and corresponding DU-chain:

![Diagram of Down-Up transitions](image)

(a) **Down-chain**  
(b) **Up-chain**  
(c) **DU-chain**

Figure 6: Example of Down-Up transitions

D- and U- operators preserve probability structures given by Ewens-Pitman Sampling formulae, hence obtained Markov chain also preserves this distribution. Figure below illustrates approximation of diffusion process in Kingman simplex with parameters \( \alpha = 0.3, \theta = 5 \)

![Diffusion sample paths on partitions](image)

Figure 7: Diffusion sample paths on partitions, top five \( x_i(t) \)
4 Dirichlet distribution and size-biased sampling

4.1 Dirichlet distribution

The broken stick model is a particular case of more general Dirichlet distribution, which is a probability measure over vectors of proportions. Its density function is parametrized by vector \( \alpha = (\alpha_1, \ldots, \alpha_m) \)

\[
p(x) = \frac{\Gamma(\sum \alpha_i)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_m)} x_1^{\alpha_1-1} \cdots x_m^{\alpha_m-1}
\]

The distribution is defined for vectors of proportions from the simplex

\[\Delta_m = \{ x \in \mathbb{R}^m | \sum_{i=1}^m x_i = 1, x_i \geq 0 \}\]

It is convenient to denote that vector \( x \in \Delta_m \) has \( m \)-dimensional Dirichlet distribution described by vector of parameters \( \alpha \) as

\[x \sim D_m(\alpha)\]

An alternative and equivalent definition of the Dirichlet distribution, which makes proofs of its properties almost immediate is given by normalization of vector of gamma variables. For \( m \) independent gamma distributed variables \( y_i \sim \mathcal{G}(\alpha_i) \) vector of proportions defined by

\[x = \left( \frac{y_1}{\sum y_i}, \ldots, \frac{y_m}{\sum y_i} \right)\]

has Dirichlet distribution \( x \sim D_m(\alpha) \).

Gamma distribution possess convolution property

\[\text{for } y_1 \sim \mathcal{G}(\alpha_1), y_2 \sim \mathcal{G}(\alpha_2) \quad y_1 + y_2 \sim \mathcal{G}(\alpha_1 + \alpha_2)\]

This property together with Lukacs characterization, saying that it is unique distribution which possesses independence of \( y_1/y_2 \) from \( y_1 + y_2 \), simplifies proofs of the properties below.

Properties. Let \( \theta = \sum_{i=1}^m \alpha_i \) denote sum of parameters. If \( y_0 = \sum_{j=1}^m y_j \) denotes sum of independent gamma variables \( y_i \sim \mathcal{G}(\alpha_i) \), then this sum has gamma distribution \( y_0 \sim \mathcal{G}(\theta) \) by convolution property.

If component \( y_i \) is separated and others are lumped together, then vector \( (y_i, \sum_{j \neq i} y_j) \) has independent components with gamma distributions with parameters \( \alpha_i \) and \( \theta - \alpha_i \) correspondingly. From normalization by \( y_0 \) it follows that marginally \( x_i = y_i/y_0 \) have beta distributions

\[x_i \sim \mathcal{B}(\alpha_i, \theta - \alpha_i)\]

Let \( x_{[-i]} \) denote that \( i \)-th component is removed from the vector \( x \)

\[x_{[-i]} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)\]

Since in a vector \( x \in \Delta_m \) sum of components is 1, in a vector \( x_{[-i]} \) sum of components becomes \( 1 - x_i \) and therefore normalized vector \( x_{[-i]}/(1 - x_i) \) belongs to \( \Delta_{m-1} \) simplex.

The Dirichlet distribution possesses an important property of neutrality that is independence of \( x_i \) and \( x_{[-i]}/(1 - x_i) \). Moreover, if original vector \( x \sim D_m(\alpha) \), then

\[\frac{x_{[-i]}}{1 - x_i} \sim D_{m-1}(\alpha_{[-i]})\]

This property is trivial for \( m = 2 \). For arbitrary dimension, let’s consider vector with \( m \) independent gamma variates \( y_i \) and corresponding Dirichlet distributed vector with components \( x_j = y_j/y_o \). Removing \( i \)-th component from both vectors and normalization of the second one yields for \( j \neq i \)

\[x_j \cdot \frac{1}{1 - x_i} = x_j \cdot \frac{1}{1 - y_i/y_o} = \frac{y_j}{y_o} - \frac{y_i}{y_o} - \frac{y_i}{\sum_{k \neq i} y_k}\]

Independence (neutrality) follows from Lucacs property.
**Beta stick-breaking.** These properties suggest a *stick-breaking* method of sampling from the Dirichlet distribution. Let’s imagine that a stick of unit length is broken by the following step-by-step procedure. By marginal property the first component \( x_1 = z_1 \) can be modeled as

\[ z_1 \sim B(\alpha_1, \theta - \alpha_1) \]

By neutrality property the first piece can be broken off. The remaining components have \( D_{m-1}(\alpha_{[-1]}) \) distribution. The procedure is repeated for remaining part of stick with length \( 1 - x_1 \) with

\[ z_2 \sim B(\alpha_2, \theta - \alpha_1 - \alpha_2) \]

producing new piece \( x_2 = z_2(1 - z_1) \) and residual with length \( (1 - z_2)(1 - z_1) = 1 - x_1 - x_2 \) etc.

In general by stage \( k \)

\[ z_k \sim B(\alpha_i, \theta - \sum_{j=1}^{k-1} \alpha_j) \]

\[ x_k = z_k \prod_{j=1}^{k-1} (1 - y_j) = z_k \left( 1 - \sum_{j=1}^{k-1} x_j \right) \]

For \( k = m \) the last component is simply a remainder. In the case of symmetric Dirichlet distribution with \( \alpha_i = \alpha \) breaking rule simplifies to

\[ z_k \sim B(\alpha, \theta - ka) \]

\[ x_k = z_k (1 - x_1 - \cdots - x_{k-1}) \]

### 4.2 Size-biased sampling

Stick-breaking method from the previous paragraph samples proportions from the Dirichlet distribution component by component. In many applications as well as in the development of the Poisson-Dirichlet distribution it is more important to study proportions ranked in descending order

\[ x_{(1)} \geq x_{(2)} \geq \ldots \]

so it would be more convenient to have a procedure which gives as output order statistics from distribution. In general this is not easy, however it is possible to devise a simulation strategy which provides samples in proportions of the appearance in the real distribution - which is given by *size-biased sampling.*

Let’s suppose that given a Dirichlet distributed vector \( \mathbf{x} \sim D_m(\alpha) \) one of its components is chosen at random, such that \( x_j \) is the probability of choosing \( j \)-th component. Alternatively it can be visualized as dropping a point at uniform over stick of length one, which is divided by proportions \( x_1, x_2, \ldots \) and choosing proportion/piece at which the point falls on. Value of this proportion is called *size-biased sample* and obviously chances to choose largest piece are highest, etc. Once proportion is chosen, it is set apart and the procedure is repeated with normalized residual. The outcome is the *size-biased permutation* of vector \( \mathbf{x} \) where components are randomly interchanged, with bias towards ordered case.

Density function of the first size-biased pick in the permuted vector can be found by the following argument. Proportion \( \bar{x} \) may be picked with probability \( \bar{x} \) as a first component of a vector \( \bar{x}, x_2, \ldots, x_m \) or as a second component of a vector \( x_1, \bar{x}, x_3, \ldots, x_m \), etc. In each of these particular cases finding unconditional probability amounts to marginalization over Dirichlet density, which yields beta densities \( B_{\alpha, \theta - \alpha_1}(\bar{x}) \) and therefore total probability is

\[ p(\bar{x}) = \sum_{i=1}^{m} \bar{x} \cdot B_{\alpha, \theta - \alpha_1}(\bar{x}) \]

In case of symmetric Dirichlet distribution density of the size-biased pick simplifies to

\[ p(\bar{x}) = \bar{x} \sum_{i=1}^{m} \frac{\alpha \Gamma(\alpha m)}{\alpha \Gamma(\alpha) \Gamma(\theta - \alpha)} \bar{x}^{\alpha - 1} (1 - \bar{x})^{\theta - \alpha - 1} = \frac{\Gamma(\theta + 1)}{\Gamma(\alpha + 1) \Gamma(\theta - \alpha)} \bar{x}^{\alpha - 1} (1 - \bar{x})^{\theta - \alpha - 1} \]
In other words, first SBP $\tilde{x}_1 = \tilde{y}_1$ has beta distribution with shifted parameter

$$\tilde{x}_1 \sim B(1 + \alpha, \theta - \alpha)$$

After breaking off the first SBP and applying the procedure over and over again it can be shown that piece to be broken off from the remaining part of the stick has beta distribution

$$\tilde{z}_k \sim B(1 + \alpha, \theta - \alpha k) \quad (5)$$

with corresponding proportions

$$\tilde{x}_k = \tilde{z}_k (1 - \tilde{x}_1 - \cdots - \tilde{x}_{k-1})$$

Since $\theta = \alpha m$ simulation terminates at stage $k = m - 1$ with $\tilde{x}_m$ is length of the remainder.

Samples obtained this way will have tendencies for larger proportions appearing first, followed by smaller proportions. Obviously, after ranking of proportions both methods (standard and size-biased ones) produce identically distributed sequences, since SBP only randomly permutes components of symmetric Dirichlet vector.

At first it is not clear what is the purpose of the SBP, since components of ordered Dirichlet distributed vector can be sampled by standard procedures and then ranked. However, as it will be shown below, size-biased simulation allows sampling from directly not accessible cases.

5 The Poisson-Dirichlet distribution

5.1 One-parameter family

For symmetric $m$-dimensional Dirichlet distribution $D_m(\alpha)$ let’s consider limiting case, where dimensionality $m \to \infty$ such that $\theta = \alpha m$, which means that while dimension goes to infinity the total charge $\theta$ remains the same and individual parameters $\alpha = \frac{\theta}{m} \to 0$. In this case direct application of the standard stick-breaking method is impossible since we will have to sample from $B(\varepsilon, \theta - \varepsilon)$ where $\varepsilon$ is infinitesimally small. However, for any $m$, it is possible to consider size-biased stick-breaking with

$$\tilde{z}_k \sim B(1 + \frac{\theta}{m}, \theta - k \cdot \frac{\theta}{m})$$

which gives for $m \to \infty$

$$\tilde{z}_k \sim B(1, \theta) \quad (6)$$

$$\tilde{x}_k = \tilde{y}_k (1 - \tilde{x}_1 - \cdots - \tilde{x}_{k-1})$$

Ranked values of size-biased sequence $\{\tilde{x}_k\}$ have one-parameter Poisson-Dirichlet distribution $\mathcal{PD}(\theta)$

$$\tilde{x}_1 \geq \tilde{x}_2 \geq \ldots$$

It is important to not that in contrast to $(5)$ sequences of $\tilde{z}_k$ and $\tilde{x}_k$ are infinite, since ultimately they correspond to the infinite dimensional Dirichlet distribution.

An alternative way of sampling consists in consideration of jumps of gamma subordinator in time interval $[0, \theta]$ and normalization of ranked jumps in this interval.

5.2 Two-parameter family

Engen[5] noticed that valid size-biased stick-breaking model $(5)$ holds for negative values of parameter in the range $\alpha \in (-1, 0]$, which after relabeling $(-\alpha) \mapsto \alpha$ leads to the following sampling method

$$\tilde{z}_k \sim B(1 - \alpha, \theta + \alpha k) \quad (7)$$

where as before corresponding partitions of unit interval are given by

$$\tilde{x}_k = \tilde{y}_k (1 - \tilde{x}_1 - \cdots - \tilde{x}_{k-1})$$

The two-parameter Poisson-Dirichlet distribution is defined as distribution of ranked values of sequence $\{\tilde{x}_k\}$

$$\tilde{x}_1 \geq \tilde{x}_2 \geq \ldots$$

Obviously in the setting of $(7)$ range of parameter $\alpha$ is $-\theta < \alpha < 1$. For values $\alpha < 0$ such that $\theta = ma$ sequence $\tilde{x}_k$ eventually stops and corresponds to the Dirichlet distribution. When $0 \leq \alpha < 1$ as well as in one-parameter case sequence $\tilde{x}_k$ never stops and thus for this range of parameter $\alpha$ this model is infinite-dimensional.
6 Market capitalizations and $\mathcal{PD}(\alpha, \theta)$

Plots in this section illustrate results of fit of capital distribution curves for several international stock exchanges by averages of the Poisson-Dirichlet law. Blue circles correspond to ranked relative capitalizations and red lines represent averages of several samples from the two-parameter Poisson-Dirichlet distribution. Optimal fitting is by least-squares method.

The first example shows that the Poisson-Dirichlet distribution provides fit for ranked normalized capitalizations of world stock exchanges. Data source is http://www.world-exchanges.org/statistics/monthly-reports, data are as of end of November, 2014.

Figure 8: NYSE, NASDAQ, Japan, EURONEXT, Hong Kong, ... ($\alpha = 0.44, \theta = 18$)

Figures below illustrate results of modeling of capital distribution curve in major stock exchanges. Data source is http://www.google.com/finance#stockscreener, data are as of December 9, 2014. Possible explanation of these results is provided by the hypothesis that the two-parameter model approximates underlying partition structure in corresponding markets. This suggests that stochastic evolution of ranked market weights can be modeled by the two-parameter diffusion process or its modification.

Figure 9: Australia ($\alpha = 0.45, \theta = 18$)
Figure 10: Belgium ($\alpha = 0.73, \theta = 19$)

Figure 11: Brazil ($\alpha = 0.1, \theta = 50$)

Figure 12: Canada (w/o CNOOC) ($\alpha = 0.75, \theta = 40$)
Figure 13: China (Shanghai) \((\alpha = 0.57, \theta = 20)\)

Figure 14: France \((\alpha = 0.35, \theta = 20)\)

Figure 15: Germany \((\alpha = 0.2, \theta = 34)\)
Figure 16: Hong Kong ($\alpha = 0.47, \theta = 40$)

Figure 17: India ($\alpha = 0.43, \theta = 37$)

Figure 18: Israel ($\alpha = 0.65, \theta = 45$)
Figure 19: Korea ($\alpha = 0.55, \theta = 45$)

Figure 20: Italy ($\alpha = 0.3, \theta = 15$)

Figure 21: Japan ($\alpha = 0.48, \theta = 95$)
Figure 22: Russia ($\alpha = 0.2, \theta = 15$)

Figure 23: Singapore ($\alpha = 0.51, \theta = 20$)

Figure 24: Spain ($\alpha = 0.02, \theta = 16$)
Figure 25: Switzerland ($\alpha = 0.4, \theta = 9$)

Figure 26: Taiwan ($\alpha = 0.5, \theta = 50$)

Figure 27: Turkey ($\alpha = 0.24, \theta = 26$)
Figure 28: United Kingdom ($\alpha = 0.65, \theta = 15$)

Figure 29: United States ($\alpha = 0.60, \theta = 55$)

Figure 30: United States ($\alpha = 0.28, \theta = 255$)
6.1 S&P 500 drawdowns

Figure in this section illustrates another application of the two-parameter Poisson-Dirichlet distribution providing fit to normalized relative daily drawdowns of S&P 500.

![Figure 31: S&P 500 relative daily drawdowns, from 1950 to 2014; $\theta = 80, \alpha = 0.24$](image)

7 Summary

The central idea of the proposed approach is probabilistic-combinatorial and can be summarized as follows:

- **combinatorics**: Stock market is considered as large combinatorial structure - partition of the set of all invested units of money. Stock capitalizations, represented by integers, define partitions of the market value. Number of ways this state can be realized combinatorially is given by the formula (1).

- **probability**: Partition structure for each level $n \geq 1$ defines exchangeable probability for all partitions with $n$ elements, such that distribution on level $n$ is consistent with distribution on partitions with $n + 1$ elements.

  It turns out that the framework of partition structures defines up ($n \rightarrow n + 1$) and down ($n \rightarrow n - 1$) conditional probabilities of transitions. This means that market dynamics can be modeled as random walk on exchangeable set partitions.

- **The two-parameter Poisson-Dirichlet distribution**, defined in infinite ranked simplex has corresponding partition structure, given by the formula (4). Stick-breaking construction provides size-biased method of sampling from the two-parameter model. Associated diffusion process, induced by down/up Markov chains has $\mathcal{PD}(\alpha, \theta)$ as unique equilibrium distribution.

Results of Section 6 suggest the hypothesis that the two-parameter model approximates stationary distribution of capital distribution curve as well as corresponding underlying partition structure. It is proposed that vector of the ranked market weights (capital distribution curve) fluctuates in stochastic equilibrium, which can be modeled by means of the two-parameter diffusion process, or by combinatorial random walks on partitions.

This research note aims at informal discussion of financial applications of partition structures. More detailed analysis of exchangeable distributions and questions

- whether slight asymmetry of probabilities of D- and U- transitions is consistent with observed asymmetry of losses and gains;
- whether jumps in stock prices appear naturally within the proposed model;
- correlation structure of weights in the $\mathcal{PD}(\alpha, \theta)$ model;
- pricing applications;
- analysis of distribution of mutual funds, etc.

are left for a paper to follow.
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