DEFICIENCY OF \( p \)-CLASS TOWER GROUPS AND MINKOWSKI UNITS

by

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Abstract. — Let \( p \) be a prime. We define the deficiency of a finitely-generated pro-\( p \) group \( G \) to be \( r(G) - d(G) \) where \( d(G) \) is the minimal number of generators of \( G \) and \( r(G) \) is its minimal number of relations. For a number field \( K \), let \( K_\varnothing \) be the maximal unramified \( p \)-extension of \( K \), with Galois group \( G_\varnothing = \text{Gal}(K_\varnothing/K) \). In the 1960s, Shafarevich (and independently Koch) showed that the deficiency of \( G_\varnothing \) satisfies
\[
0 \leq \text{Def}(G_\varnothing) \leq \dim(\Theta_K^p/\Theta_K^q)
\]
relating the deficiency of \( G_\varnothing \) to the \( p \)-rank of the unit group \( \Theta_K^p \) of the ring of integers \( \Theta_K \) of \( K \). In this work, we further explore connections between relations of the group \( G_\varnothing \) and the units in the tower \( K_\varnothing/K \), especially their Galois module structure. In particular, under the assumption that \( K \) does not contain a primitive \( p \)th root of unity, we give an exact formula for \( \text{Def}(G_\varnothing) \) in terms of the number of independent Minkowski units in the tower. The method also allows us to infer more information about the relations of \( G_\varnothing \), such as their depth in the Zassenhaus filtration, which in certain circumstances makes it easier to show that \( G_\varnothing \) is infinite. We illustrate how the techniques can be used to provide evidence for the expectation that the Shafarevich-Koch upper bound is “almost always” sharp.

Let \( p \) be a prime number, and let \( K \) be a number field. For a finite set \( S \) of places of \( K \), let \( K_S \) be the maximal \( p \)-extension of \( K \) unramified outside \( S \) and \( G_S = \text{Gal}(K_S/K) \), its Galois group. Note in particular that \( K_H \) is the maximal pro-\( p \) extension of \( K \) unramified everywhere. We call the extension \( K_\varnothing/K \) the \( p \)-class field tower of \( K \) and the group \( G_\varnothing \) its \( p \)-class tower group. Let
\[
d(G_\varnothing) := \dim H^1(G_\varnothing, \mathbb{Z}/p) \quad \text{and} \quad r(G_\varnothing) := \dim H^2(G_\varnothing, \mathbb{Z}/p)
\]

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be, respectively, the minimal number of generators and relations of \( G_\mathcal{O} \). By class field theory, the maximal abelian quotient of \( G_\mathcal{O} \) is isomorphic to the \( p \)-Sylow subgroup of the class group of \( K \), and is therefore finite. It follows that \( r(G_\mathcal{O}) \geq d(G_\mathcal{O}) \), prompting us to define \( \text{Def}(G_\mathcal{O}) := r(G_\mathcal{O}) - d(G_\mathcal{O}) \) as the \textit{deficiency} of \( G_\mathcal{O} \). By the Burnside Basis Theorem, \( d(G_\mathcal{O}) \) is the \( p \)-rank of the class group of \( K \) and is in particular computable in any given case. By contrast, we do not know an algorithm for computing \( r(G_\mathcal{O}) \). However, thanks to the celebrated work of Shafarevich [35] (and, independently, Koch – see for example [13, Chapter 11]) we know that

\[
0 \leq \text{Def}(G_\mathcal{O}) \leq d(G_\mathcal{O}^\times),
\]

where \( d(G_\mathcal{O}^\times) := d(G_\mathcal{O}^\times/\mathcal{O}_K^\times)^p \) is the \( p \)-rank of the unit group \( G_\mathcal{O}^\times \) of the ring of integers \( \mathcal{O}_K \) of \( K \). We recall that if \( K \) has \( r_1 \) embeddings into \( \mathbb{R} \) and \( r_2 \) pairs of complex conjugate embeddings into \( \mathbb{C} \), then \( d(G_\mathcal{O}^\times) = r_1 + r_2 - 1 + \delta \) where \( \delta \) is 1 or 0 according to whether \( K \) contains a primitive \( p \)th root of unity \( \zeta_p \) or not. The pioneering number-theoretic work of Shafarevich on the above deficiency bound [35] subsequently led to the group-theoretic work of Golod and Shafarevich [7]. This pair of papers gave a criterion for the infinitude of \( G_\mathcal{O} \) and produced the first such examples. More historical details can be found in [34], [28], [13], and [24].

Now let us recall the notion of a Minkowski unit. If \( F/K \) is a Galois extension of number fields, an element \( \varepsilon \in \mathcal{O}_F^\times \) is called a Minkowski unit for \( F/K \) if the subgroup of \( \mathcal{O}_F^\times \) generated by \( \varepsilon \) and all of its Gal\((F/K)\) conjugates is “maximal” in some sense. For the precise notion of maximality that we will use, see below.

In this work, we relate the existence of Minkowski units along the tower \( K_\mathcal{O}/K \) to the deficiency of \( G_\mathcal{O} \), as well as to the depth of its relations. To do this, we introduce a constant \( \lambda \) measuring the free-part of the structure of the units in \( K_\mathcal{O}/K \) (see §2.3). Here free-part means the following: if \( F/K \) is a finite Galois extension in \( K_\mathcal{O}/K \) with Galois group \( G \), we are interested in the \( \mathbb{F}_p[G] \)-structure of \( \mathbb{F}_p \otimes \mathcal{O}_F^\times \), that is the units \( \mathcal{O}_F^\times \) modulo \( p \)th powers. Recall that since \( G \) is a \( p \)-group, the category of \( \mathbb{F}_p[G] \)-modules is not semisimple. When the \( \mathbb{F}_p[G] \)-free part of \( \mathbb{F}_p \otimes \mathcal{O}_F^\times \) is nontrivial, we say the extension \( F/K \) admits a \textit{Minkowski unit} (see §1.3 for further details). It is not difficult to see that the number of independent Minkowski units is non-increasing and stabilizes as we move up the tower \( K_\mathcal{O}/K \) and therefore after a finite number of steps reaches a constant value we denote \( \lambda := \lambda_{K_\mathcal{O}/K} \). We also define \( \beta \) to be

\[
\beta := \begin{cases} 
\frac{d(G_\mathcal{O}^\times \cap \mathcal{O}_K^\times)^p}{d(\mathcal{O}_K^\times)^p} & \zeta_p \in K \\
0 & \text{otherwise}
\end{cases}.
\]

Note that when \( \zeta_p \in K \), if we set \( L = K_\mathcal{O} \cap K(\sqrt[p]{\mathcal{O}_K^\times}) \), then \( [L : K] = p^\beta \). Thus, \( 0 \leq \beta \leq \min(r_1 + r_2, d(G_\mathcal{O})) \). Moreover, we note that \( \beta > 0 \) if and only if \( K(u^{1/p})/K \) is a Galois degree \( p \) unramified extension for some \( u \in \mathcal{O}_K^\times \).

In the much better understood wild case, i.e. when \( S \) contains all the primes of \( K \) above \( p \) (as well as the infinite places when \( p = 2 \)), we can apply the very powerful global duality theorem for \( G_S \) to obtain an exact and easily computable formula for the deficiency of

\footnote{1. We should note that in most of the group theory literature, the deficiency of \( G \) is defined as \( d(G) - r(G) \).}
Theorem A below (see Theorem 2.9) is a first step toward refining our understanding the unramified situation by relating it to the presence of Minkowski units in $K \varnothing / K$.

**Theorem A.** — One has

$$d(\mathcal{O}_K^\times) - \lambda - \beta \leq \text{Def}(G_{\varnothing}) \leq d(\mathcal{O}_K^\times) - \lambda.$$  

In particular, if $\mathcal{O}_K^\times \cap (\mathcal{O}_K^\times)^p = (\mathcal{O}_K^\times)^p$ or if $K$ does not contain a primitive $p$th root of unity, then $\text{Def}(G_{\varnothing}) = d(\mathcal{O}_K^\times) - \lambda$.

We give two proofs of this result (see proof of Theorem 2.9). In our second proof, which is more constructive, we realize $G_{\varnothing}$ as a quotient of some $G_S$, where $S$ is a well-chosen finite set of prime ideals of $\mathcal{O}_K$ coprime to $p$ (see also Notations at the end of this section), and we use the Hochschild-Serre spectral sequence induced by the natural map $G_S \rightarrow G_{\varnothing}$ to produce $d(\mathcal{O}_K^\times) - \lambda - \beta$ distinct elements of $\mathbb{H}^2_p(G_{\varnothing}, \mathbb{Z} / p)$.

**Remark 0.1.** — As mentioned earlier, the non-negativity of $\text{Def}(G_{\varnothing})$ follows from a basic group-theoretical property of $G_{\varnothing}$, namely that its maximal abelian quotient is finite. In other words, one knows that $G_{\varnothing}$ has at least $d(G_{\varnothing})$ relations. For the group $G_{\varnothing}$, one can concretely produce $d(G_{\varnothing})$ relations for $G_{\varnothing}$ as follows. In Figure 1, we show a tower of fields $K \subseteq K' \subseteq L'_1 \subseteq L'_2$ for whose definition, the reader may consult the Notations section at the end of this introduction. For the moment, the key point is that $L'_2/L'_1$ is an elementary abelian $p$-extension of dimension $d(G_{\varnothing})$. By the Chebotarev density theorem, we can choose $d(G_{\varnothing})$ primes of $K$ whose Frobenii form a basis of the elementary $p$-abelian group $\text{Gal}(L'_2/L'_1)$. Letting $S_1$ be the set consisting of these primes, in Section 2.2, we will describe in detail how applying the Gras-Munnier Theorem (Theorem 1.1) and Lemma 1.5 (ii) to the primes in $S_1$ gives us $d(G_{\varnothing})$ distinct elements in $\mathbb{H}^2_p = H^2(G_{\varnothing}, \mathbb{Z} / p)$, which in turn correspond to $d(G_{\varnothing})$ distinct relations in a minimal presentation of $G_{\varnothing}$. We refer to relations constructed in this way as “accessible” via $S_1$.

A key observation we make in this work is that aside from the $d(G_{\varnothing})$ relations accessible via $S_1$, we can construct $d(\mathcal{O}_K^\times) - \lambda - \beta$ additional relations via a modification of this construction using a further set $S_2$ of auxiliary primes whose Frobenii span a Galois group in a more complicated tower of governing fields (Figure 2) described in §2.2. The existence of such primes is tied up with the Galois module structure of units in the Hilbert $p$-class field tower. We refer to the resulting relations as “extra” relations “detected” by
This set of ideas leads to the lower bound for Def(G\(\mathcal{S}\)) in the theorem. The upper bound, on the other hand, is a consequence of a result of Wingberg [38]. When \(\beta = 0\) (which is always the case if \(K\) does not contain a primitive \(p\)th root of unity), these upper and lower bounds coincide, in which case all the relations are either accessible by \(S_1\) or detected by \(S_2\). But when \(\beta > 0\), only \(d(\mathcal{O}_K^\times) - \lambda - \beta\) of the relations are constructible in this way, and there is an open question as to whether there exist (up to) \(\beta\) additional “elusive” relations.

Thanks to the work of Labute [15], Schmidt [31] and others we know that there are special sets \(S\) (finite and tame) for which \(G_S\) is of cohomological dimension 2; however, their methods do not allow \(S\) to be empty. In particular, the question of the computation of the cohomological dimension of \(G_{\mathcal{S}}\) has only been resolved in a few cases, namely when \(G_{\mathcal{S}}\) is known to be finite. A consequence of Theorem A is the following (Theorem 3.11):

**Corollary.** — Let \(K\) be a number field such that

(i) \(K\) contains a primitive \(p\)th root of unity;

(ii) \(\mathcal{O}_K^\times \cap (\mathcal{O}_K^\times)^p = (\mathcal{O}_K^\times)^p\).

Then \(\dim H^3(G_{\mathcal{S}}, \mathbb{F}_p) > 0\). Moreover:

- If \(\dim H^3(G_{\mathcal{S}}, \mathbb{F}_p) = 1\), then \(G_{\mathcal{S}}\) is finite or of cohomological dimension 3;

- If \(\text{Def}(G_{\mathcal{S}}) = 0\) and \(G_{\mathcal{S}}\) is of cohomological dimension 3, then \(G_{\mathcal{S}}\) is a Poincaré duality group.

We also deduce (Corollary 3.4):

**Corollary.** — Let \(K\) be a number field such that

(i) \(K\) contains a primitive \(p\)th root of unity;

(ii) \(\mathcal{O}_K^\times \cap (\mathcal{O}_K^\times)^p = (\mathcal{O}_K^\times)^p\);

(iii) \(\text{Def}(G_{\mathcal{S}}) = 0\).

Then for every open normal subgroup \(H\) of \(G_{\mathcal{S}}\), one has \(\text{Def}(H) = 0\).

Note that in the situation of the Corollary, i.e. when \(\text{Def}(H) = 0\) for all open subgroups of \(G\), \(\lambda\) is maximal all along the tower. When \(\text{Def}(G_{\mathcal{S}}) = 0\) and the tower is finite, \(G_{\mathcal{S}}\) is either cyclic or, when \(p = 2\), a generalized quaternion group. We know of no examples with \(\text{Def}(G_{\mathcal{S}}) = 0\) and \(\#G_{\mathcal{S}} = \infty\). Observe also that Poincaré groups of dimension 3 have deficiency zero.

When \(G_{\mathcal{S}}\) is infinite we suspect \(\text{Def}(G_{\mathcal{S}})\) to be maximal (namely equal to \(d(\mathcal{O}_K^\times)\)) very often in accordance with the heuristics of Liu-Wood [18]. In fact, we elaborate a strategy to investigate maximality of the deficiency by testing for the presence of Minkowski units through computer computation. We further note that if in the first steps of the tower \(K_{\mathcal{S}}/K\) there is some Minkowski unit, preventing us from concluding that \(\text{Def}(G_{\mathcal{S}})\) is maximal, it implies that the group \(G_{\mathcal{S}}\) can be described by relations of high depth in the Zassenhaus filtration. Denote by \((K_n)\) the sequence in \(K_{\mathcal{S}}/K\) where \(K_1 := K\) and \(K_{n+1}\) is the maximal elementary abelian \(p\)-extension of \(K_n\) in \(K_{\mathcal{S}}/K\). Put \(H_n = \text{Gal}(K_n/K)\). Let \(r_{\text{max}} = d(G_{\mathcal{S}}) + d(\mathcal{O}_K^\times)\) be the maximal possible value of \(r(G_{\mathcal{S}})\). To each presentation of a pro-\(p\) group, there is associated a Golod-Shafarevich polynomial; for the basic facts of these polynomials, see §4.1.2. Golod and Shafarevich proved that if this polynomial vanishes on the open unit interval, then the group must be infinite. In §4, we prove the following result (see Theorem 4.12).
Theorem B. — Let $\lambda_n$ be the number of independent $\mathbb{F}_p[H_n]$-Minkowski units in $K_n$. Then $G_{\varnothing}$ can be generated by $d(G_{\varnothing})$ generators and $r_{\text{max}}$ relations $\{\rho_1, \cdots, \rho_{r_{\text{max}}}\}$ such that at least $\lambda_n$ relations are of depth greater than $2^n$. Hence, we can take $1 - d(G_{\varnothing})t + (r_{\text{max}} - \lambda_n)t^2 + \lambda_n t^{2n}$ as a Golod-Shafarevich polynomial for $G_{\varnothing}$.

The more familiar Golod-Shafarevich polynomial in this context is $1 - d(G_{\varnothing})t + r_{\text{max}}t^2$, which is less likely to have a root and thus indicate $\#G_{\varnothing} = \infty$. Also, we will allow the possibility of $n = \infty$, that is there may be fewer than $r_{\text{max}}$ relations.

The imaginary quadratic case is particularly easy to study (here $p = 2$). Indeed when $K$ is an imaginary quadratic field, one has $\text{Def}(G_{\varnothing}) \in \{0, 1\}$. Here we show that, almost always, there is no Minkowski unit in any quadratic extension $F/K$ of $K_{\varnothing}/K$, which implies $\text{Def}(G_{\varnothing}) = 1$. Denote by $\mathcal{F}$ the set of imaginary quadratic fields, and for $X \geq 2$, put

$$\mathcal{F}(X) = \{K \in \mathcal{F}, \ |\text{disc}(K)| \leq X\}, \quad \mathcal{F}_0(X) = \{K \in \mathcal{F}(X), \ \text{Def}(G_{\varnothing}) = 0\}.$$ 

We obtain that almost all the time $\text{Def}(G_{\varnothing}) = 1$ (see Theorem 5.12):

Theorem C. — Let $K$ be imaginary quadratic and $p = 2$. One has

$$\frac{\#\mathcal{F}_0(X)}{\#\mathcal{F}(X)} \leq C \frac{\log \log X}{\log X},$$

where $C$ is an absolute constant. In particular, the proportion of imaginary quadratic fields of discriminant at most $X$ for which $\text{Def}(G_{\varnothing}) = 0$ tends to zero as $X \to \infty$.

Example. — Take $p = 2$. Our method allows us to show that for $K = \mathbb{Q}(\sqrt{-5460})$, the example studied extensively by Boston-Wang [2], one has $\text{Def}(G_{\varnothing}) = r - d = 5 - 4 = 1$.

Notations.

• Let $p$ be a prime number, and let $K$ be a number field.

• We denote by
  
  - $O_K$ the ring of integers of $K$, and by $O_K^\zeta$ the group of units of $O_K$;
  
  - $\mathcal{E}_K = \mathbb{F}_p \otimes O_K^\zeta$, the units modulo the $p$th-powers,
  
  - $K^H$ the Hilbert $p$-class field of $K$,
  
  - $\text{Cl}_K$ the $p$-Sylow subgroup of the class group of $K$.

• Let $\zeta_p \in \mathbb{Q}^{alg}$ be a primitive $p$th root of 1. Put $\delta := \delta_{K,p} := 1$ when $\zeta_p \in K$, 0 otherwise.

• Let $S = \{p_1, \cdots, p_s\}$ be a finite set of prime ideals of $K$. We identify a prime $p \in S$ with the place $v$ it defines.

  - We assume each $p_i$ is tame (prime to $p$) and satisfies $|O_K/p_i| \equiv 1 \pmod{p}$.
  
  - We denote by $\text{RCG}_K(p_1, \cdots, p_s)$ the $p$-Sylow subgroup of the ray class group of $K$ of modulus $p_1 \cdots p_s$. When $S = \emptyset$, one has $\text{RCG}_K(\emptyset) = \text{Cl}_K$.
  
  - Let $K_S$ be the maximal pro-$p$ extension of $K$ unramified outside $S$, put $G_S = G_{K,S} = \text{Gal}(K_S/K)$.

  - By class field theory, one has $G_S^p \simeq \text{RCG}_K(p_1, \cdots, p_s)$.
  
  - Put $V_S := \{x \in K^\times, \ (x) = F \text{ as a fractional ideal of } K; \ x \in (K^\times)^p, \forall v \in S\}$. Then $V_S \supset (K^\times)^p$ and we have the exact sequence:
    
    $$0 \to \mathcal{E}_K/\mathcal{E}_K^p \to V_S/(K^\times)^p \to \text{Cl}_K[p] \to 0.$$ 

• If $M$ is a $\mathbb{Z}$-module, we set $d(M) = \dim_{\mathbb{F}_p}(\mathbb{F}_p \otimes M)$.
For each prime ideal 

- When $G$ is a pro-$p$ group, we denote $d(G) = d(G^{ab})$, where $G^{ab} = G/[G, G]$.
- If $(r_1, r_2)$ is the signature of $K$ and $\delta$ equals 1 or 0 as $K$ contains or does not contain the $p$th roots of unity. By Dirichlet’s Theorem $d(\Theta_K) = r_1 + r_2 - 1 + \delta$.
- From the exact sequence above, $d(V_\sigma/(K^\times)^p) = d(\Theta_K^\sigma) + d(\text{Cl}_K)$.

- Unless otherwise specified, all cohomology groups have $\mathbb{Z}/p\mathbb{Z}$ as their coefficient group.
- Hence $d(G_\sigma) := d(H^1(G_\sigma)) = \dim H^1(G_\sigma, \mathbb{Z}/p\mathbb{Z})$ and $r(G_\sigma) := \dim H^2(G_\sigma, \mathbb{Z}/p\mathbb{Z})$.
- The deficiency (2) $\text{Def}(G_\sigma)$ of $G_\sigma$ is defined to be $r(G_\sigma) - d(G_\sigma)$.

For the computations in this paper we have used the programs GP-PARI [26] and Magma [37] and have assumed the GRH to speed up the computations.

1. Preliminaries

In this section we develop the results we need to detect elements of $H^2(G_\sigma) = \text{III}^2$ as described in the Remark in the Introduction. In particular, Lemma 1.5 shows how one can detect elements of $\text{III}^2$ via ramified extensions of $K_\sigma$; we illustrate our strategy by finding $\text{III}^2$ for the field $\mathbb{Q}(\sqrt{-5460})$ with $p = 2$.

In §1.2, we relate $\text{Def}(G_\sigma)$ to norms of units from number fields in the tower $K_\sigma/K$. In §1.3 we develop the basics of the theory of Minkowski units and show, using the Gras-Munnier Theorem 1.1, that the existence of a Minkowski unit in some number field $F$ in the tower $K_\sigma/K$ follows when $G^ab_F, [p] = G^ab_{F, \sigma}$ for some prime $p$ of $K$.

1.1. Saturated sets, and a spectral sequence. —

1.1.1. Degree-$p$ cyclic extension with prescribed ramification. — Take $p$, $K$ and $S$ as in §“Notations”. The fields of Figure 1 are called governing fields as the existence of a $\mathbb{Z}/p$-extension of $K$ ramified exactly at a given set of primes depends on their Frobenii in these extensions. See Theorem 1.1 below.

For each prime ideal $p \in S$, let us choose a prime ideal $\mathfrak{P}|p$ of $\mathfrak{O}_{L_2}$, and denote by $\sigma_p := \left(\frac{L_2/K'}{\mathfrak{P}}\right)$, the Frobenius at $\mathfrak{P}$ in the governing extension $L_2/K'$.

Using that $L_2$ is formed by taking $p$th roots of elements of $K$ (not $K'$), one can show that $\sigma_p$ depends, up to a nonzero scalar multiple, only on $p$. This serves our purposes. By abuse we also denote by $\sigma_p$ its restriction to $L_1'$. One says that the Frobenii $\sigma_p$, $p \in S$, satisfy a nontrivial relation if

$$\prod_{p \in S} a_p^{\sigma_p} = 1,$$

in $\text{Gal}(L_2/K')$ (or in $\text{Gal}(L_1'/K')$) with the $a_i \in \mathbb{Z}/p\mathbb{Z}$ not all zero. Thus the existence of a nontrivial relation is independent of the ambiguity in the choice of $\sigma_p$.

**Theorem 1.1 (Gras-Munnier [9]).** — Let $S = \{p_1, \cdots, p_t\}$ be a set of tame prime ideals of $K$. One has:

(i) $d(G_S) \neq d(G_\sigma)$, if and only if the $\sigma_p, p \in S$, satisfy a nontrivial relation in $\text{Gal}(L_2'/K')$.

(ii) $|G_S^{ab}| > |G_\sigma^{ab}|$ if and only if the $\sigma_p, p \in S$, satisfy a nontrivial relation in $\text{Gal}(L_1'/K')$.

2. Usually the deficiency of a pro-$p$ group $G$ is the quantity $d(G) - r(G)$, but for clarity we use the opposite
For a generalization of Theorem 1.1, see [8, Chapter V].

**Remark 1.2.** — Let us observe that the Kummer radical of \( L'/K' \) is \( \mathcal{O}_K^\infty(K'^\infty)^p/(K'^\infty)^p \) which is isomorphic to \( \mathcal{O}_K^\infty \) since \( \mathcal{O}_K^\infty \cap (K'^\infty)^p = (\mathcal{O}_K^\infty)^p \) and \([K' : K]\) is coprime to \( p \). For the same reason, \( V_{\mathcal{O}}/(K'^\infty)^p \) is the Kummer radical of \( L'/K' \).

1.1.2. **Saturated sets.** — For \( v \in S \), we denote by \( G_v \) the absolute Galois group of the maximal pro-\( p \) extension \( \overline{K}_v \) of the completion \( K_v \) of \( K \) at \( v \). Let \( \Pi_2^S \) be the kernel of the localization map of \( H^2(G_S, \mathbb{F}_p) \):

\[
\Pi_2^S = \ker\left(H^2(G_S, \mathbb{F}_p) \to \oplus_{v \in S} H^2(G_v, \mathbb{F}_p)\right).
\]

Put \( B_S = (V_S/(K^\infty)^p)^\vee \); then one has (see Theorem 11.3 of [13]) \( \Pi_2^S \to B_S \). When \( S \) contains the places of \( K \) above \( \{p, \infty\} \) this map is an isomorphism and \( \Pi_2^S \) is dual to

\[
\Pi_1^S(\mu_p) := \ker\left(H^1(G_S, \mu_p) \to \oplus_{v \in S} H^1(G_v, \mu_p)\right).
\]

The failures of the isomorphism and duality in the tame case are reasons it is especially challenging.

**Definition 1.3.** — The \( S \) set of places \( K \) is called saturated if \( V_S/(K^\infty)^p = \{1\} \).

As consequence of Theorem 1.1, one has (see [11, Theorem 1.12])

**Theorem 1.4.** — A finite tame set \( S \) is saturated if and only if, the Frobenius \( \sigma_p, p \in S \), span the elementary \( p \)-abelian group \( \text{Gal}(K'/(\sqrt[p]{\mathcal{O}})/K') \).

We recall below the formula of Shafarevich applied in the case where \( S \) is tame (see for example [24, Chapter X, §7, Corollary 10.7.7]):

\[
d(G_S) = |S| - d(\mathcal{O}_K^\infty) + d(V_S/(K^\infty)^p).
\]

Hence when \( S \) is saturated, one has \( d(G_S) = |S| - d(\mathcal{O}_K^\infty) \).

1.1.3. **Spectral sequence.** — Let us start with the natural exact sequence

\[1 \to H_S \to G_S \to G_{\mathcal{O}} \to 1,\]

where the group \( H_S \) is the closed normal subgroup of \( G_S \) generated by the inertia \( \tau_p \in G_S, p \in S \). Set

\[X_S := H_S/[H_S, G_{\mathcal{O}}]H_S^p.\]

Recall as \( G_{\mathcal{O}} \) is a pro-\( p \) group, the compact ring \( \mathbb{F}_p[G_{\mathcal{O}}] \) is local and acts continuously on \( H_S/[H_S, H_S]H_S^p \). We give an easy lemma that can be found in [11] (see Lemmas 1.11 and 1.12).

**Lemma 1.5.** — Let \( S \) be a finite set of tame prime ideals of \( \mathcal{O}_K \).

(i) The \( \mathbb{F}_p[G_{\mathcal{O}}] \)-module \( H_S/[H_S, H_S]H_S^p \) is topologically finitely generated by at most \( |S| \) elements.

(ii) One has the exact sequence

\[1 \to H^1(G_{\mathcal{O}}) \to H^1(G_S) \to X_\mathcal{O}^S \to \Pi_2^\mathcal{O} \to \Pi_2^S.\]

In particular, if \( S \) is such that \( H^1(G_{\mathcal{O}}) \cong H^1(G_S) \), then \( X_\mathcal{O}^S \to \Pi_2^\mathcal{O} \). If moreover \( S \) is saturated then \( X_\mathcal{O}^S \cong \Pi_2^\mathcal{O} \).
To conclude this subsection, let us observe the following: Let \( F_0/K_\emptyset \) be a cyclic extension of degree \( p \) in \( K_\emptyset/K \) such that \( F_0/K \) is Galois. Then \( F_0 \) comes from a finite level: there exists a finite extension \( F/K \) and a cyclic extension \( F_1/F \) of degree \( p \), ramified at some places above \( S \), such that \( F_2 = K_\emptyset F_1 \). The estimate for \( \dim H^2(G_\emptyset) \) can be done by using the previous lemma, typically by seeking the fields \( F_0 \): this is the spirit of the method involving the Hochschild-Serre spectral sequence.

**Example 1.6 (The field \( \mathbb{Q}(\sqrt{-5460}) \)).** — Set \( p = 2 \) and \( K = \mathbb{Q}(\sqrt{-5460}) \). The rational primes in \( \{43, 53, 101, 149, 157\} \) all split in \( K \). Let \( S = \{p_{43}, p_{53}, p_{101}, p_{149}, p_{157}\} \); the first primes above each of these as Magma computes them. We denote the abelian group \( O_p \) and equality (1), and \( p \)ramified at \( S \) for \( \emptyset \), typically by seeking the fields \( F_0 \): this is the spirit of the method involving the Hochschild-Serre spectral sequence.

**Example 1.6 (The field \( \mathbb{Q}(\sqrt{-5460}) \)).** — Set \( p = 2 \) and \( K = \mathbb{Q}(\sqrt{-5460}) \). The rational primes in \( \{43, 53, 101, 149, 157\} \) all split in \( K \). Let \( S = \{p_{43}, p_{53}, p_{101}, p_{149}, p_{157}\} \); the first primes above each of these as Magma computes them. We denote the abelian group \( \text{RCG}_K(\emptyset) = (2, 2, 2, 2) \); \( \text{RCG}_K(\emptyset, p_{43}, p_{53}, p_{101}, p_{149}, p_{157}) = (4, 8, 8, 8) \); \( \text{RCG}_K(\emptyset, p_{43}) = (4, 2, 2, 2) \); \( \text{RCG}_K(\emptyset, p_{43}) = (4, 2, 2, 2) \). As \( F_\emptyset = K_\emptyset \) we have an extension over \( K_\emptyset \) ramified at \( p_{43} \), so \( d(X_S) = 5 \), and by Lemma 1.5 we conclude that \( r(G_\emptyset) = 5 \).

**1.2. Universal norms and relations.** — Put \( \mathcal{O}_{K_\emptyset}^\times = \bigcup_F \mathcal{O}_F^\times \), where \( F/K \) run through the finite Galois extensions in \( K_\emptyset/K \). Recall the following theorem due to Wingberg [38]; see also [24, Theorem 8.8.1, Chapter VIII, §8] where we take \( S = T = \emptyset, \mathfrak{c} \) to be the full class of finite \( p \)-groups and \( A = \mathbb{Z} \). We have written the results there in our notation.

**Theorem 1.7 (Wingberg).** — One has \( \hat{H}^i(G_\emptyset, \mathcal{O}_{K_\emptyset}^\times) \simeq \hat{H}^{3-i}(G_\emptyset, \mathbb{Z})^\vee \).

The exact sequence \( 0 \rightarrow \mathbb{Z} \xrightarrow{x_p} \mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 0 \) gives:
\[
0 \rightarrow H^2(G_\emptyset, \mathbb{Z})/p \rightarrow H^2(G_\emptyset, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^3(G_\emptyset, \mathbb{Z})[p] \rightarrow 0.
\]

Taking the Pontryagin dual, one obtains:
\[
(2) \quad 0 \rightarrow H^3(G_\emptyset, \mathbb{Z})^\vee/p \rightarrow H^2(G_\emptyset, \mathbb{Z}/p\mathbb{Z})^\vee \rightarrow H^2(G_\emptyset, \mathbb{Z})^\vee[p] \rightarrow 0.
\]

By Theorem 1.7:
\[
H^2(G_\emptyset, \mathbb{Z})^\vee \simeq H^1(G_\emptyset, \mathcal{O}_{K_\emptyset}^\times), \text{ and } H^3(G_\emptyset, \mathbb{Z})^\vee \simeq \hat{H}^0(G_\emptyset, \mathcal{O}_{K_\emptyset}^\times).
\]

Recall
\[
\hat{H}^i(G_\emptyset, \mathcal{O}_{K_\emptyset}^\times) \simeq \varprojlim_F \mathcal{O}_F^\times/N_{F/K} \mathcal{O}_F^\times,
\]
where \( F/K \) run through the finite Galois extensions in \( K_\emptyset/K \), \( N_{F/K} \) is the norm in \( F/K \), and \( H^1(G_\emptyset, \mathcal{O}_{K_\emptyset}^\times) \) is the \( p \)-part of \( \text{Cl}_K \) (see for example [24, Lemma 8.8.4, Chapter VIII, §8]).

This observation associated to Theorem 1.7 allows us to prove:
1.3. Minkowski units. —

Corollary 1.8. — One has \( \text{Def}(G_{\mathcal{O}}) = d(\mathcal{O}/N_{K_{\mathcal{O}}/K}\mathcal{O}_{K_{\mathcal{O}}}) \), where \( N_{K_{\mathcal{O}}/K} := \bigcap_{F/K} N_{F/K}\mathcal{O}_F \).

In particular when \([F : K]\) is sufficiently large one has \( \text{Def}(G_{\mathcal{O}}) = d(\mathcal{O}/N_{F/K}\mathcal{O}_F) \).

Proof. — If \( F/K \) is a finite Galois extension in \( K_{\mathcal{O}}/K \) then \( (\mathcal{O}_{K_{\mathcal{O}}})^{[F:K]} \subset N_{F/K}\mathcal{O}_{K_{\mathcal{O}}} \), hence \( \mathcal{O}_K/N_{F/K}\mathcal{O}_F \) is a finite abelian \( p \)-group and \( \lim \mathcal{O}_K/N_{F/K}\mathcal{O}_F \) is an abelian pro-\( p \) group (obviously finitely generated). Then \( \lim \mathcal{O}_K/N_{F/K}\mathcal{O}_F = \lim \mathbb{Z}_p \otimes \mathcal{O}_K \). But as \( \mathbb{Z}_p \) is \( \mathcal{O}_F \)-flat, one gets \( \mathbb{Z}_p \otimes (\mathcal{O}_K/N_{F/K}\mathcal{O}_F) = \mathcal{O}_K/N_{F/K}\mathcal{O}_F \), where \( E_K = \mathbb{Z}_p \otimes \mathcal{O}_K \), and where \( N_{F/K}\mathcal{O}_F \) is the closure of \( N_{F/K}\mathcal{O}_F \) in \( \mathbb{Z}_p \otimes \mathcal{O}_K \). Hence,

\[
\lim \mathcal{O}_K/N_{F/K}\mathcal{O}_F = \mathcal{O}_K/N_{F/K}\mathcal{O}_F.
\]

Thus

\[
\mathbb{F}_p \otimes \lim \mathcal{O}_K/N_{F/K}\mathcal{O}_F \simeq \mathcal{O}_K/N_{F/K}\mathcal{O}_F \simeq \mathcal{O}_K/N_{K_{\mathcal{O}}/K}\mathcal{O}_{K_{\mathcal{O}}}.
\]

The exact sequence (2) becomes

\[
0 \longrightarrow \mathcal{O}_K/N_{K_{\mathcal{O}}/K}\mathcal{O}_{K_{\mathcal{O}}} \longrightarrow H^2(G_{\mathcal{O}}, \mathbb{Z}/p\mathbb{Z})^\vee \longrightarrow \text{Cl}_K[p] \longrightarrow 0,
\]

and computing dimensions gives the result. 

For \( \#G_{\mathcal{O}} < \infty \) it has been known for a long time that the number of relations of \( G_{\mathcal{O}} \) is related to the norm of the units in the tower. See for example §2 of [28]

As a consequence, one also has

Corollary 1.9. — Let \( F/K \) be a finite Galois extension in \( K_{\mathcal{O}}/K \). Then \( \text{Def}(G_{\mathcal{O}}) \geq d(\mathcal{O}_K/N_{F/K}\mathcal{O}_F) \), and one has equality when \( F \) is sufficiently large.

Proof. — Obvious by using \( \mathcal{O}_K/N_{K_{\mathcal{O}}/K}\mathcal{O}_{K_{\mathcal{O}}} \rightarrow \mathcal{O}_K/N_{F/K}\mathcal{O}_F \). For the equality, use the fact that \( \mathcal{O}_K \) is finite.

When \( p = 2 \), if \(-1\) is not a norm of a unit in a quadratic subextension \( F/K \) of \( K_{\mathcal{O}}/K \), then \( -1 \notin N_{K_{\mathcal{O}}/K}\mathcal{O}_{K_{\mathcal{O}}} \), which implies \( \text{Def}(G_{\mathcal{O}}) \geq 1 \). We will see that this condition appears almost all the time when \( K \) is an imaginary quadratic extension. We close this subsection with a basic fact.

Fact. — For \( S \) a finite set of tame places, \( 0 \leq \text{Def}(G_S) \).

Proof. — We refer to [27], especially Lemma 6.8.6, for the facts we need concerning the homology of profinite groups. From the exact sequence of compact groups

\[
0 \longrightarrow \mathbb{Z}_p \xrightarrow{x_p} \mathbb{Z}_p \longrightarrow \mathbb{Z}/p \longrightarrow 0
\]

we obtain the homology sequence

\[
\cdots \longrightarrow H_2(G_S, \mathbb{Z}/p) \rightarrow H_1(G_S, \mathbb{Z}_p)[p] \rightarrow 0.
\]

As \( H_1(G_S, \mathbb{Z}_p) \simeq G_S^{ab} \), we have \( d(G_S) = d(G_S^{ab}[p]) \leq d(H_2(G_S, \mathbb{Z}/p)) = r(G_S) \).

1.3. Minkowski units. —
1.3.1. — Recall that for a finite group $G$, the ring $\mathbb{F}_p[G]$ is a Frobenius algebra (see for example [3, §62]): every free submodule of an $\mathbb{F}_p[G]$-module $M$ is in direct sum so we may write $M = \mathbb{F}_p[G]^r \oplus N$, where $N$ is torsion (for every element $n \in N$, there exists $0 \neq h \in \mathbb{F}_p[G]$ such that $h \cdot n = 0$), and $t$ is uniquely determined (by Krull-Schmidt Theorem). Observe that if $M^\wedge$ is the Pontryagin dual of $M$, then $t_G(M) = t_G(M^\wedge)$.

**Definition 1.10.** — If $M$ is a finitely generated $\mathbb{F}_p[G]$ module, denote by $t := t_G(M)$, the $\mathbb{F}_p[G]$-rank of the maximal free submodule of $M$.

We record some useful properties. Let $H \subset G$ be a subgroup of $G$.

(i) Recall first that by Mackey’s decomposition theorem, one has the isomorphism of $\mathbb{F}_p[H]$-modules $\text{Res}_H^G \mathbb{F}_p[G] \simeq \mathbb{F}_p[H]^{\otimes[G,H]}$.

(ii) Suppose moreover $H < G$, and denote by $N_H = \sum_{h \in H} h \in \mathbb{F}_p[G]$ the norm map from $H$. For an $\mathbb{F}_p[G]$-module $M$ let $M^H$ denote the invariants. Then one has easily the isomorphism of $\mathbb{F}_p[G]$-modules

\[
\mathbb{F}_p[G/H] \simeq \mathbb{F}_p \otimes_{\mathbb{F}_p[H]} \mathbb{F}_p[G] \simeq \mathbb{F}_p[G]^H
\]

and $N_H(\mathbb{F}_p[G]) = \mathbb{F}_p[G]^H$ so

\[
N_H(\mathbb{F}_p[G]) \simeq \mathbb{F}_p[G/H]
\]

as $\mathbb{F}_p[G/H]$-modules.

1.3.2. — Let $F/K$ be a finite Galois extension of number fields with Galois group $G$.

**Definition 1.11.** — Let $\mathcal{E}_F := \mathbb{F}_p \otimes_{\mathcal{O}_F^\times} \mathcal{O}_F$. We say that $F/K$ has a Minkowski unit (at $p$), if $\mathcal{E}_F$ contains a nontrivial free $\mathbb{F}_p[G]$-submodule. In other word, $F/K$ has a Minkowski unit if $t_G(\mathcal{E}_F) \geq 1$.

Hence the quantity $t_G(\mathcal{E}_F)$ measures *the number* of independent Minkowski units in $F/K$.

If $(p, [G]) = 1$ then $\mathcal{E}_F$ is a semisimple $\mathbb{F}_p[G]$-module. Determining the existence of Minkowski units is more difficult when $(p, |G|) = p$. Indeed, when $G$ is a $p$-group, and $F/K$ is unramified, the presence of a Minkowski unit in $F/K$ is probably a very strong condition as noted by Ozaki [25, Lemma 2] (for general $G_S$, see [10]).

**Theorem 1.12 (Ozaki).** — Let $F/K$ be a finite Galois extension in $K_{G}/K$. Then

\[ t_G(\mathcal{E}_F) \geq r_1 + r_2 - \frac{C}{|\text{Gal}(F/K)| + \text{Cl}_K} \]

where $C \geq 0$ is a constant depending on $\text{Gal}(F/K)$ and on $\text{Cl}_K$. Moreover $C \to \infty$ with $|\text{Gal}(F/K)|$.

Here as usual, $(r_1, r_2)$ is the signature of $K$.

1.3.3. Example. — We want to illustrate the notion of Minkowski units.

**Lemma 1.13.** — Let $F/K$ be a $p$-extension of Galois group $G$. Let $S = \{p_1, \ldots, p_k\}$ be a set of tame primes of $K$ that split completely in $F/K$. If $d(G_{F,S}) = d(G_{F,G})$ then $t_G(\mathcal{V}_{F,G}) \geq k$, and if $|G_{F,S}^{\text{ab}}| = |G_{F,G}^{\text{ab}}|$ then $t_G(\mathcal{E}_F) \geq k$. 

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We now show that the extension between the Frobenii of all the primes above might be free. The fourth and seventh rows of the table suggest that we easily see completely in $H_{\mathfrak{p}}$ assertion follows by (6). For the second assertion, use the second part of Theorem 1.1.

For all prime ideals $\mathfrak{p}_i$ of $\mathcal{O}_F$ we consider the Frobenii $\sigma_{\mathfrak{p}_i}$ in $\text{Gal}(F'(\sqrt[4]{D_i})/F')$. Note that we are no longer in the abelian situation as just before Theorem 1.1. By Theorem 1.1 (i), the hypothesis $d(G_{F,S}) = d(G_{F,D})$ implies the Frobenii $\sigma_{\mathfrak{p}_i}$ in $\text{Gal}(F'(\sqrt[4]{D_i})/F')$ are without nontrivial relation. As each $\mathfrak{p}_i$ splits completely in $F/K$, we have that $\text{Gal}(F'(\sqrt[4]{D_i})/F')$ contains $k$ distinct free $\mathbb{F}_p[G]$-modules, one for each $\mathfrak{p}_i$. The first assertion follows by (6). For the second assertion, use the second part of Theorem 1.1 and (7).

Recall that we denote the abelian group $\prod_{i=1}^{d} \mathbb{Z}/a_i \mathbb{Z}$ by $(a_1, \ldots, a_d)$. Let $p = 2$ and $K = \mathbb{Q}(\sqrt[4]{5 \cdot 13 \cdot 17 \cdot 29})$ and let $H = \mathbb{Q}(\sqrt[4]{5}, \sqrt[4]{13}, \sqrt[4]{17}, \sqrt[4]{29})$ be its Hilbert class field. Here $\text{Cl}_K = (2, 2, 2)$, and $\text{Cl}_H = (4, 4)$. Consider the primes $\ell = 2311$ and $q = 3319$. We easily see $\ell \mathcal{O}_K = l_1 l_2$ and $q \mathcal{O}_K = q_1 q_2$ and these ideals are all principal. In the table below we compute the 2-parts of the ray class groups for $K$ and $H$ of the given conductors. The computations were done with MAGMA (see [37]) and assume the GRH. Note that in the first three rows, the ray class groups are identical. As the principal ideals $l_i$ and $q_i$ split completely in $H/K$, by Lemma 1.13 one sees that $\mathcal{O}_H^\times \otimes \mathbb{F}_2$ has a Minkowski unit over $K$: in other words putting $G = \text{Gal}(H/K) \simeq (\mathbb{Z}/2\mathbb{Z})^5$, one has $t_G(\mathcal{O}_H) \geq 1$. Note $H$ is a degree 16 totally real field so $\dim \mathcal{O}_H = 16$ and $\mathcal{O}_H \simeq G \mathbb{F}_2[G] \oplus M$ where $\dim M = 8$ so $M$ might be free. The fourth and seventh rows of the table suggest that $\mathcal{O}_H^\times \otimes \mathbb{F}_2$ may have two Minkowski unit over $K$. The fifth and sixth rows indicate a relation in the governing extension between the Frobenii of all the primes above $l_i$ and $q_j$ in $\mathcal{O}_H$.

We now show $M$ is not free. Set $K_0 = K$, $K_1 = \mathbb{Q}(\sqrt[4]{5 \cdot 17})$, and $K_2 = \mathbb{Q}(\sqrt[4]{13 \cdot 29})$. Let $F$ be the biquadratic field $K_1 K_2$. Computations show that $\text{Cl}_{K_1} = (2)$, $\text{Cl}_{K_2} = (2)$, and $\text{Cl}_F = (2, 4)$. Denote by $\varepsilon_i$ the fundamental unit of $K_i$, and put

$$e = \# \left( \mathcal{O}_F^\times /\langle -1, \varepsilon_i, i = 0, 1, 2 \rangle \right).$$

Applying the Brauer class formula in the biquadratic extension $F/\mathbb{Q}$, i.e. $|\text{Cl}_F| = 4e|\text{Cl}_{K_1}||\text{Cl}_{K_2}|$, to deduce $e = 1$, and then $\mathcal{O}_F^\times = \langle -1, \varepsilon_i, i = 0, 1, 2 \rangle$.

Let $\sigma$ be a generator of $G = \text{Gal}(F/K)$. Observe that:

| Conductor | $K$ | $H$ |
|-----------|-----|-----|
| 1         | (2, 2, 2) | (4, 4) |
| $l_i, i \in \{ 1, 2 \}$ | (2, 2, 2) | (4, 4) |
| $q_i, i \in \{ 1, 2 \}$ | (2, 2, 2) | (4, 4) |
| $l_1 q_1$ | (2, 2, 2) | (2, 2, 2, 4, 4) |
| $l_1 q_2$ | (2, 2, 2, 2) | (2, 2, 2, 2, 4, 8) |
| $l_2 q_1$ | (2, 2, 2, 2) | (2, 2, 2, 2, 4, 8) |
| $l_2 q_2$ | (2, 2, 2) | (2, 2, 2, 4, 4) |
(i) the norm of $\varepsilon_2$ in $F/K$ is $+1$,
(ii) the norm of $\varepsilon_1$ in $F/K$ is $-1$,
(iii) $\sigma$ acts trivially on $\varepsilon_0$.

Hence, as we will observe in Lemma 5.1, one obtains that $\mathcal{E}_F \simeq \mathbb{F}_2[G'] \oplus \mathbb{F}_2^2$, where $G' = \text{Gal}(F/K)$. Finally since $\mathcal{E}_H \rightarrow \mathcal{E}_F$, and $t_G(\mathcal{E}_F) \geq t_G(\mathcal{E}_H)$ we conclude that $t_G(\mathcal{E}_H) = 1$.

2. Detecting the relations along $K_\ell/K$

As mentioned in the remark in the introduction, we can easily find $d$ elements of $\Pi_{\ell}^2$ by constructing ramified extensions at a low level in the tower $K_\ell/K$. For $(G_n)$ a sequence of open normal subgroups of $G$ with $\bigcap_{n=1}^{\infty} G_n = \{e\}$, let $K_n$ be the fixed field of $G_n$ and set $H_n = \text{Gal}(K_n/K)$. In this section we show that the presence of torsion elements in the $F_p[H_n]$-module $\text{Gal}(K_n \left( \sqrt{\mathcal{O}_{K_n}^\times} \right)/K_n)$ can give rise to more relations.

2.1. First observations. — Let $p$ be a prime number and $K$ be a number field. If $F/K$ is a Galois extension with Galois group $G$, the norm map $N_G$ sends $\mathcal{E}_F$ to $\mathcal{E}_K^\times \cap (\mathcal{E}_F^\times)^p \subseteq \mathcal{E}_F$; denote by $N'_G : \mathcal{E}_F \rightarrow \mathcal{E}_K$ the map from $\mathcal{E}_F$ to $\mathcal{E}_K$ induced by the norm in $F/K$. The commutative diagram:

\[
\begin{array}{ccc}
\mathcal{E}_F & \xrightarrow{N'_G} & \mathcal{E}_K \\
\downarrow & & \downarrow \\
\mathcal{E}_K & \xrightarrow{N_G} & (\mathcal{E}_F^\times)^p \\
\end{array}
\]

implies the following easy lemma:

**Lemma 2.1.** — One has $N'_G(\mathcal{E}_F) \rightarrow N_G(\mathcal{E}_F)$. Moreover, $\mathcal{E}_K^\times \cap (\mathcal{E}_F^\times)^p = (\mathcal{E}_K^\times)^p \implies N'_G(\mathcal{E}_F) \simeq N_G(\mathcal{E}_F)$.

The study of the norm map $N_G$ is "purely algebraic", i.e. it does not involve number theory. Lemma 2.2 below is proved at the beginning of the proof of [25, Lemma 2]). Since that Lemma is stated differently we include a proof that is essentially from [25].

**Lemma 2.2.** — Let $G$ be a finite $p$-group and $M$ an $\mathbb{F}_p[G]$-module. Let $N_G : M \rightarrow M$ be the norm map. Let $m \in M$. Then $N_G(m) = 0$ if and only if $m$ is a torsion element.

**Proof.** — Let $0 \neq m \in M$. Recall that the annihilator $A_m$ of a nontrivial element $m \in M$ is an ideal of $\mathbb{F}_p[G]$.

If the annihilator of $m$ is trivial, then its $\mathbb{F}_p[G]$-span is isomorphic to $\mathbb{F}_p[G]$ and so $N_G(\mathbb{F}_p[G]) = \mathbb{F}_p$ by (5).

Conversely, suppose that $A_m \neq 0$. Then $A_m^G \neq 0$ since $G$ is a $p$-group acting on a nontrivial $\mathbb{F}_p$-vector space. Hence $A_m^G \subseteq (\mathbb{F}_p[G])^G$ which is in turn the one-dimensional vector space $\mathbb{F}_p N_G$. Thus $N_G = A_m^G \subseteq A_m$ so $N_G(m) = 0$. \[\square\]

More generally, one has
**Theorem 2.3.** — Let $F/K$ be a finite $p$-extension with Galois group $G$ and write $N_G(\mathcal{O}_F) \cong F_p^d$. Then $t_G(\mathcal{O}_F) \leq t \leq t_G(\mathcal{O}_F) + d \left( \frac{\mathcal{O}_F^* \cap (\mathcal{O}_F^*)^p}{(\mathcal{O}_F^*)^p} \right)$. In particular if $\mathcal{O}_K^* \cap (\mathcal{O}_F^*)^p = (\mathcal{O}_K^*)^p$, then $t = t_G(\mathcal{O}_F)$.

**Proof.** — Write $\mathcal{O}_F \cong F_p[G]^{t_G(\mathcal{O}_F)} \oplus \mathbb{N}$, where $\mathbb{N}$ is generated by torsion elements as an $F_p[G]$-module. By (5) and Lemma 2.2 one has $N_G(\mathcal{O}_F) \cong F_p^{t_G(\mathcal{O}_F)}$. So by Lemma 2.1 we see $N_G(\mathcal{O}_F) \cong N_G(\mathcal{O}_F) = F_p^{t_G(\mathcal{O}_F)}$, proving the result when $\mathcal{O}_K^* \cap (\mathcal{O}_F^*)^p = (\mathcal{O}_K^*)^p$.

By noting that the ‘difference’ between $N_G(\mathcal{O}_F)$ and $N_G(\mathcal{O}_F)$ is exactly $\frac{N_G(\mathcal{O}_F) \cap (\mathcal{O}_F^*)^p}{(\mathcal{O}_F^*)^p}$ which has $p$-rank at most $d \left( \frac{\mathcal{O}_F^* \cap (\mathcal{O}_F^*)^p}{(\mathcal{O}_F^*)^p} \right)$, we obtain the general case. $\square$

**2.2. Exhibiting relations via the Hochschild-Serre spectral sequence.** — In this subsection, we flesh out the details of the process described in Remark 0.1 for explicitly exhibiting $d(G_{\mathcal{O}})$ relations in a minimal presentation of $G_{\mathcal{O}}$. It would be helpful to refer to Figure 1 from the introduction. Put $K' = K(\zeta_p)$. Let $S = S_1 \cup S_2$ be a set of tame prime ideals of $\mathcal{O}_F$ such that:

- $S_1$ is a minimal set whose Frobenii generate the $d(G_{\mathcal{O}})$-dimensional $F_p$-vector space $\text{Gal} \left( \overline{K'}(\sqrt[p]{\mathcal{O}_K})/\overline{K}'(\sqrt[p]{\mathcal{O}_K}) \right)$, and

- $S_2$ is a minimal set whose Frobenii generate the $F_p$-vector space $\text{Gal} \left( \overline{K'}(\sqrt[p]{\mathcal{O}_K})/\overline{K}' \right)$ of dimension $r_1 + r_2 - 1 + \delta$.

Recall the Frobenii above are well-defined up to nonzero scalar multiples in the Galois groups, which are vector spaces over $F_p$. This ambiguity does not affect their spanning properties. One has

**Lemma 2.4.** — The set $S$ is saturated, in particular $\Pi_3^S = \{0\}$. Moreover $d(G_S) = d(G_{\mathcal{O}})$, and $r(G_{\mathcal{O}}) = d(X_S)$.

**Proof.** — That $S$ is saturated follows immediately from Theorem 1.4. As there is no dependence relation between the Frobenii (the set $S$ is minimal), Theorem 1.1 implies $d(G_S) = d(G_{\mathcal{O}})$. That $r(G_{\mathcal{O}}) = d(X_S)$ follows from the second part with Lemma 1.5. $\square$

**Lemma 2.5.** — Write $(a_1, \ldots, a_d)$ for the $p$-part of $\text{RCG}_K(\mathcal{O})$ and let $S_1 = \{p_1, \ldots, p_d\}$ as above. Then $\text{RCG}_K(p_1, \ldots, p_d) \rightarrow (pa_1, \ldots, pa_d)$.

**Proof.** — This is a consequence of Theorem 1.1. As the primes of $S_1$ split completely in the governing extension $\text{Gal}(\overline{K'}(\sqrt[p]{\mathcal{O}_K})/K')$, for each prime ideal $p \in S_1$ we have $\#G_{(p)} \neq \#G_{(p)}^{ab}$. We conclude by noting that $d(G_{S_1}) = d(G_{\mathcal{O}})$. $\square$

Lemma 2.5 implies the existence of $d$ independent degree-$p$ cyclic extensions $F_i$ of $K_{\mathcal{O}}$, each totally ramified at $p_i$, $i = 1, \ldots, d$, and on which $G_{\mathcal{O}}$ acts trivially, implying that $d(X_S) \geq d$. These additional relations are accessible via the set $S_2$.

**2.3. Proof of Theorem A.** — Let $(G_n)$ be a sequence of open normal subgroups of $G_{\mathcal{O}}$ such that $G_n \subset G_{n+1}$ and $\cap_n G_n = \{e\}$. Put $H_n := G_{\mathcal{O}}/G_n$, $K_n := K_{\mathcal{O}}^{G_n}$, and write $\mathcal{O}_n := F_p[H_n]^{\mathbb{N}} \oplus \mathbb{N}$ where $\mathbb{N}$ is torsion as an $F_p[H_n]$-module.

**Lemma 2.6.** — The sequence $(t_n)$ is nonincreasing.
Proof. — Recall from (4) that the norm map from $H_{n+1,n} := \text{Gal}(K_{n+1}/K_n)$ on $F_p[H_{n+1}]$ induces the following $F_p[H_n]$-isomorphisms:

$$F_p[H_n] \cong F_p[H_{n+1}H_{n-1,n} \cong F_p[H_{n+1},n].$$

The norm map $N_{H_{n+1,n}}$ of $K_{n+1}/K_n$ induces a morphism from $\mathcal{E}_{K_{n+1}}$ to $\mathcal{E}_{K_{n+1}}$ which allows us to obtain

$$\mathbb{F}_p[H_n]^{t_{n+1}} \xrightarrow{\mathcal{E}_{K_n}^x} \mathcal{E}_{K_n}^x \cap (\mathcal{E}_{K_{n+1}}^x)^p \leftarrow \mathcal{E}_{K_n},$$

which implies $t_n \geq t_{n+1}$. \hfill $\Box$

Definition 2.7. — Set $\lambda := \lambda_{K/G} = \lim_n t_n$. We call this the Minkowski-rank of the units along $K_{\mathbb{Q}}/K$.

One easily sees that $\lambda$ does not depend on the sequence $(G_n)$.

Let us write $p^\beta := [K'(\sqrt{\mathcal{O}_{K}^x}) \cap K'K_{\mathbb{Q}} : K'] = [\mathcal{O}_{K}^x \cap (\mathcal{O}_{K_{\mathbb{Q}}}^x)^p : (\mathcal{O}_{K}^x)^p]$. Obviously, $\beta \leq \min(d(\mathcal{O}_{K}^x), d(G_{\mathbb{Q}}))$.

Proposition 2.8. — One has: $\delta = 0 \implies \beta = 0$.

Proof. — Let $\Delta = \text{Gal}(K'/K)$ be the Galois group of $K'/K$; by hypothesis $\Delta$ is of order coprime to $p$. As $\Delta$ acts trivially on $\mathcal{O}_{K}^x$, by Kummer duality the action of $\Delta$ over $\text{Gal}(K'(\sqrt{\mathcal{O}_{K}^x})/K')$ is given by the cyclotomic character; in particular, there is no non-trivial subspace of $\text{Gal}(K'(\sqrt{\mathcal{O}_{K}^x})/K')$ on which $\Delta$ acts trivially. But $\Delta$ acts trivially on $\text{Gal}(K'K_{\mathbb{Q}}/K')$; the result holds. \hfill $\Box$

Theorem 2.9. — We have the estimates:

$$d(\mathcal{O}_{K}^x) - \lambda - \beta \leq \text{Def}(G_{\mathbb{Q}}) \leq d(\mathcal{O}_{K}^x) - \lambda.$$

In particular,

- if $\mathcal{O}_{K}^x \cap (\mathcal{O}_{K_{\mathbb{Q}}}^x)^p = (\mathcal{O}_{K}^x)^p$ or if $\delta = 0$, then $\text{Def}(G_{\mathbb{Q}}) = d(\mathcal{O}_{K}^x) - \lambda$.
- if $\lambda = d(\mathcal{O}_{K}^x)$ then $\text{Def}(G_{\mathbb{Q}}) = 0$.

Proof. — We keep the notations of the beginning of the section.

We give two proofs for the lower bound. The first one is ‘algebraic’ while the second is number-theoretic and is more ‘explicit’ in how we determine the existence of the relations.

We first establish the upper bound. Denote by $N_{H_{n}}$ the norm map for the extension $K_{n}/K$. Observe that by Corollary 1.9, $\text{Def}(G_{\mathbb{Q}}) = d(\mathcal{E}_{K}) - d(N_{H_{n}}(\mathcal{E}_{K_{n}}))$ for $n \gg 0$. Take $n$ sufficiently large such that $t_n = \lambda$. One has $d(N_{H_{n}}(\mathcal{E}_{K_{n}})) \geq \lambda$ (see Theorem 2.3), implying that $d(N'_{H_{n}}(\mathcal{E}_{K_{n}})) \geq \lambda$. Hence one gets:

$$\text{Def}(G_{\mathbb{Q}}) \leq d(\mathcal{O}_{K}^x) - \lambda.$$

Below are the two proofs of the lower bound.

- First proof:

Observe that $\beta = d\left(\frac{\mathcal{O}_{K}^x \cap (\mathcal{O}_{K_{n}}^x)^p}{(\mathcal{O}_{K}^x)^p}\right)$ since $n \gg 0$. By Theorem 2.3 one also has $d(N'_{H_{n}}(\mathcal{E}_{K_{n}})) \leq \lambda + \beta$, and then by Corollary 1.8 we get

$$\text{Def}(G_{\mathbb{Q}}) \geq d(\mathcal{O}_{K}^x) - \lambda - \beta.$$
Here we show $d(\mathcal{O}_K^\times) - \lambda - \beta \leq \text{Def}(G_{\mathcal{O}})$ using saturated sets and the Hochschild-Serre exact sequence.

First assume that $\zeta_p \in K$ i.e. $\delta = 1$. Choose $n \gg 0$, and write $\mathcal{O}_K = F_p[H_n] \oplus N_n$, where $N_n$ is a $H_n$-torsion $F_p[H_n]$-module.

![Diagram of fields](image)

**Figure 2.**

Put $\mathcal{O}_K' := \mathcal{O}_K/\mathcal{O}_K^{(p)} \to \mathcal{O}_K$. Hence $[L_4 : K_n] = \#(\mathcal{O}_K')$. Observe that $\text{Gal}(L_4/K_n) \simeq F_p^t$, where $t = d(\mathcal{O}_K^\times) - \beta$.

We will find a set of primes of $S = \{p_1, \ldots, p_{1 - \lambda}\}$ in $K$ such that:

- $p_i$ splits completely in $K_n/K$,
- Their Frobenii span a $(t - \lambda)$-dimensional space in $\text{Gal}(L_2/L_1) \simeq \text{Gal}(L_4/K_n)$,
- For each $i$, let $b_{ij}$ be the primes above $p_i$ in $K_n$. There is a dependence relation on the Frobenii of the $b_{ij}$ in $\text{Gal}(L_5/K_n)$. By Gras-Munnier (Theorem 1.1) this implies the existence of a $\mathbb{Z}/p$-extension $R_i/K_n$ ramified only at (these primes above) $p_i$.

Let $R_i$ be the Galois closure over $K$ of $R_i$. As the $p$-group $\text{Gal}(K_n/K)$ must act on the $F_p$-vector space $\text{Gal}(R_i/K_n)$ with a fixed point, by iteration we may assume $R_i/K$ is Galois. The $\mathbb{Z}/p$-extension $R_iK_{\mathcal{O}}$ is ramified only at $p_i$ and gives an element of $\Pi_{\mathcal{O}}^2$. We have produced $t - \lambda$ elements of $\Pi_{\mathcal{O}}^2$ in addition to the $d$ elements of $\mathcal{O}_K^\times$. 

---
we get by choosing primes \( \{ q_1, \ldots, q_d \} \) of \( K \) whose Frobenii form a basis of \( \text{Gal}(L_3/L_2) \).

This gives the lower bound. We now construct \( S \).

As \( L_2/K \) is abelian (\( \zeta_p \in K \)), \( H_n = \text{Gal}(K_n/K) \) acts trivially on \( \text{Gal}(L_2/L_1) \) (and thus on \( \text{Gal}(L_4/K_n) \) as well).

After taking the Kummer dual of \( \mathcal{E}_{K_n} \), one obtains \( \text{Gal}(L_5/K_n) \cong \mathbb{F}_p[G] \oplus M \), where \( M_n = N_n^* \) is a \( H_n \)-torsion \( \mathbb{F}_p[H_n] \)-module. The natural surjection \( \pi : \text{Gal}(L_5/K_n) \to \text{Gal}(L_4/K_n) \) induces, upon taking \( H_n \)-coinvariants, the map

\[
\text{Gal}(L_5/K_n)_{H_n} \cong \mathbb{F}_p \oplus (M_n)_{H_n} \xrightarrow{\pi} \text{Gal}(L_4/K_n) \cong \text{Gal}(L_2/L_1) \cong \mathbb{F}_p^d.
\]

Thus

\[d(\pi((M_n)_{H_n})) \geq t - \lambda = d(\mathcal{E}_K^\times) - \beta - \lambda.\]

Take (at least) \( t - \lambda \) elements \( x_i \) in \( \text{Gal}(L_5/K_n) \), such that their image under the projection \( \pi \) forms a basis of \( \pi((M_n)_{H_n}) \subset \text{Gal}(L_4/K_n) \cong \text{Gal}(L_2/L_1) \cong \mathbb{F}_p^d \). We choose \( p_i \) to split completely in \( K_n/K \) and have Frobenius \( \pi(x_i) \in \text{Gal}(L_2/L_1) \), so clearly \( p_i \) satisfies the first two points above. We have chosen \( p_i \) so that the primes above it in \( K_n \) have Frobenii generating a \( \mathbb{F}_p[H_n] \)-torsion module in \( \text{Gal}(L_5/K_n) \). This settles the third point and the case \( \delta = 1 \).

Suppose now that \( \delta = 0 \). Replace every field \( E \) above by \( E' := E(\zeta_p) \). The key fact is this: by Proposition 2.8, one has \( d(\text{Gal}(L_2/L_1')) = d(\mathcal{E}_K^\times) \) so \( L_2' \cap K_n = K \). The rest of the proof is word for word the same from this point on.

The last result follows since \( \text{Def}(G_G) \geq 0 \).

**Remark 2.10.** — Observe that

(i) the inequality \( \text{Def}(G_G) \leq d(\mathcal{E}_K^\times) - \lambda \) comes from universal norms of units and is Wingberg’s result (Theorem 1.7);

(ii) the group \( G_G \) has at least \( \lambda \) fewer relations than the maximal possible number,

\[\dim V_G/(K^\times)^p.\]

**Corollary 2.11.** — Suppose \( K_G/K \) is finite. Then \( \lambda < d(\mathcal{E}_K^\times) - d^2/4 + d \).

**Proof.** — By the Theorem of Golod-Shafarevich one has \( \text{Def}(G_G) > d^2/4 - d \); then apply Theorem 2.9.

**2.4. Remarks.** —

2.4.1. When \( G_G \) is abelian. — • Consider first the case where \( G_G \) is cyclic. Clearly \( d(G) = r(G) = 1 \) so \( \text{Def}(G_G) = 0 \). By Theorem 2.3, we get

\[\lambda = t_{G_G}(\mathcal{E}_{K_G}) \geq d(\mathcal{E}_K^\times) - \beta \geq d(\mathcal{E}_K^\times) - 1,\]

due to the fact that \( \beta \leq 1 \). In particular, this situation forces \( K_G \) to have a Minkowski unit provided \( K \) is neither \( \mathbb{Q} \) nor imaginary quadratic. We can recover this fact by using the well-known following result: as \( G_G \) is cyclic, every element of \( \mathcal{E}_K^\times \) is the norm of an element of \( \mathcal{E}_{K_G}^\times \). Note this last argument applies in the quadratic imaginary case as well.

As an example, take the imaginary quadratic number field \( K = \mathbb{Q}(\sqrt{-q-\ell}) \), with \( -q \equiv \ell \equiv 1 \pmod{4} \). Here, \( p = 2 \), \( G_G \) is cyclic, and \( \mathcal{E}_K^\times \cap \mathcal{E}_{K_G}^\times = \mathcal{E}_K^{\times^2} \). We find \( \lambda = 1 \), and finally that \( \mathcal{E}_{K_G} \cong \mathbb{F}_2[G_G] \).
Observe that if $G_{\mathcal{O}} \simeq \mathbb{Z}/2\mathbb{Z}$, then the fundamental unit of the biquadratic extension $\mathbb{K}(\sqrt{7})$ is exactly the fundamental unit of the quadratic field $\mathbb{Q}(\sqrt{7})$ and then is of norm $-1$: hence, as we will see later, in this case we find again $\delta_{K_{\mathcal{O}}} \simeq F_2[G_{\mathcal{O}}]$.

- Take $p = 2$, and $K$ such that $G_{\mathcal{O}} \simeq (\mathbb{Z}/2\mathbb{Z})^2$. Here $d(G) = 2$ and $r(G) = 3$ so $\text{Def}(G_{\mathcal{O}}) = 1$, implying the existence of an extra relation. By Theorem 2.3, we get

$$\lambda = t_{G_{\mathcal{O}}}(\delta_{K_{\mathcal{O}}}) \geq d(\theta_{K}^x) - 2,$$

due to the fact that $\beta \leq 2$.

Let us be more precise: Kisilevsky in [12] showed that if $G_{\mathcal{O}} \simeq (\mathbb{Z}/2\mathbb{Z})^2$, then for every quadratic subextension $F_i/K$ in $G_{\mathcal{O}}/K$, one has $(\theta_{K}^x : N_{F_i/K} \theta_{F_i}^x) = 2$. We prove

**Proposition 2.12.** — Let $K/q$ be a quadratic extension such that $G_{\mathcal{O}} \simeq (\mathbb{Z}/2\mathbb{Z})^2$. Then the extra relation is detected at one of the three quadratic subextensions $F/K$ in $G_{\mathcal{O}}/K$.

**Proof.** — Suppose first that $K/q$ is an imaginary quadratic extension. By Kisilevsky’s result $-1$ is not a norm of any unit in each of the three subextensions $F_i/K$ of $G_{\mathcal{O}}/K$. Let us choose $F := F_1$ such that $F \neq K(\sqrt{-1})$; put $G = \text{Gal}(F/K)$. By using $N_G(\theta_{F}^x) \subset \{\pm 1\}$, it is then easy to see that, modulo $\theta_{F}^{x^2}$, $-1$ is not a norm of any unit in $F/K$ implying that the norm map $N_G : \delta_{F} \to \frac{\theta_{K}^x}{\theta_{F}^x}$ is not onto.

Recall that $\frac{\theta_{K}^x}{\theta_{F}^x} \in \delta_{F}$. As dim$_F \delta_{F} = 2$, the only possibilities for the $F_2[G]$-module $\delta_{F}$ are $F_2$ and $F_2[G]$. As the norm map is onto in the latter case we see $\delta_{F} \simeq F_2$ and then $t_G(\delta_{F}) = 0$ so $\lambda = 0$: the extra relation is detected by the quadratic extension $F/K$.

We now settle the case where $K/q$ is real quadratic extension. Denote by $\varepsilon$ the positive fundamental unit of $K$. By Kisilevsky’s result, one knows that for every quadratic subextension $F_i/K$, $-1$ or $\varepsilon$ is not a norm of any unit in $F_i/K$. Take one such quadratic extension $F/K$, and put $G = \text{Gal}(F/K)$.

Suppose that $-1$ is not a norm of from $F$ to $K$ of any unit but $-1 \in N_G(\theta_{F}^x) \theta_{F}^{x^2}$. First, $N_G(\theta_{F}^x) \subset \{1, \pm \varepsilon\}$ modulo squares. The equations $-1 = z^2$ and $-1 = \varepsilon z^2$ have no solutions with $z \in \theta_{F}^x$ for sign reasons. Hence the only possible solution is that $-1 = -\varepsilon z^2$, and then, necessarily $F = K(\sqrt{\varepsilon})$.

Suppose now that $\varepsilon$ is not a norm of any unit in $F/K$. As before, if we test the condition $\varepsilon \in N_G(\theta_{F}^x) \theta_{F}^{x^2}$, we see the equations $\varepsilon = -z^2$, and $\varepsilon = -\varepsilon z^2$ have no solution for sign reasons. Suppose that $\varepsilon = \varepsilon z^2$ for some odd integer $a$ with $\varepsilon a \in N_G(\theta_{F}^x)$. As $N_G(\varepsilon) = \varepsilon^2$, it is easy to see this implies $\varepsilon \in N_G(\theta_{F}^x)$, which contradicting our assumption. Thus $a$ is even. We conclude that $\varepsilon \in \theta_{F}^{x^2}$, i.e. $F = K(\sqrt{\varepsilon})$.

Hence, in any quadratic subextension $F/K$ in $G_{\mathcal{O}}/K$ such that $F \neq K(\sqrt{\varepsilon})$, one has that the map $N_G : \delta_{F} \to \frac{\theta_{K}^x}{\theta_{F}^x}$ is not onto, and the result follows as in the imaginary case.

**2.4.2. Remarks on $\beta$.** — Let $d \in \mathbb{Z}_{>0}$ square free, and let $K = \mathbb{Q}(\sqrt{d})$ be a real quadratic field with fundamental unit $\varepsilon$. One wants to test if $\varepsilon \in \theta_{K}^{x^2}$. This will imply $\beta > 0$.

Obviously, we assume $N_{K/q}\varepsilon = 1$. Let us write $\delta(\varepsilon) = 2 + \text{Tr}_{K/q}(\varepsilon) \in \mathbb{Z}$ where $\text{Tr}_{K/q}$ is the trace map; this quantity has been introduced by Kubota in [14].

Recall that, up to square of $\mathbb{Q}$, the quantity $\delta(\varepsilon)$ divides $\text{disc}(K)$ (see [14, Hilfssatz 8]). Moreover by [14, Hilfssatz 11], given a real biquadratic extension $F/q$ containing $K$, then $\sqrt{\varepsilon} \in F$ if and only if, $\sqrt{\delta(\varepsilon)} \in F$. In particular as easy consequence we get:
Lemma 2.13. — Let $K = \mathbb{Q}(\sqrt{d})$ be a real quadratic field. Let $\varepsilon$ be the fundamental unit of $K$ that we suppose of norm +1. Then $K(\sqrt{\varepsilon})/K$ is unramified if and only if, $K(\sqrt{\delta(\varepsilon)})/K$ is unramified.

In particular, when $d = 3 \pmod{4}$, then $\mathcal{O}_K^\times \cap \mathcal{O}_{K_{\frak{p}}}^2 = \mathcal{O}_K^{\times 2}$ if and only if the 2-valuation $v_2(\delta(\varepsilon))$ of $\delta(\varepsilon)$ is odd.

Take $K = \mathbb{Q}(\sqrt{d})$ as in the following table; let us write $d = d_0 \cdot q$, where $d_0$ is fixed and where $q$ is some varying prime number. For $X \geq 1$, put $A(X) = \# \{ q \equiv 3 \pmod{4}, q \leq X \}$, and $B(X) = \{ q \in A(X), v_2(\delta(\varepsilon)) = 1 \pmod{2} \}$, where $\varepsilon$ is the fundamental unit of the quadratic field $K$. We compute the proportion $\# B(X)/\# A(X)$: that is the proportion of quadratic real fields $K = \mathbb{Q}(\sqrt{d})$ for which $\mathcal{O}_K^\times \cap \mathcal{O}_{K_{\frak{p}}}^2 = \mathcal{O}_K^{\times 2}$.

| $d$ | $q \leq 10^5$ | $10^6$ | $10^7$ |
|-----|----------------|--------|--------|
| $3 \cdot 5 \cdot 7 \cdot 9$ | 2604/4806 $\approx$ .5418 | .5428 | .5416 |
| $3 \cdot 5 \cdot 11 \cdot 9$ | .5420 | .5401 | .5421 |
| $5 \cdot 7 \cdot 11 \cdot 9$ | .6271 | .6252 | .6251 |
| $3 \cdot 7 \cdot 13 \cdot 9$ | .6317 | .6267 | .6249 |
| $3 \cdot 17 \cdot 7 \cdot 3$ | .4621 | .4593 | .4784 |
| $3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot q$ | .5980 | .5945 | .5936 |

We can do more computations: the next table indicates the proportion of quadratic real fields $K = \mathbb{Q}(\sqrt{d})$ for which $\mathcal{O}_K^\times \cap \mathcal{O}_{K_{\frak{p}}}^2 = \mathcal{O}_K^{\times 2}$:

| $d \equiv 3 \pmod{4}$ | $\leq 10^5$ | $10^6$ | $10^7$ | $10^8$ |
|-----------------------|-----------|--------|--------|--------|
| $7170$ | .6948 | .6885 | .6659 |

3. Consequences

Throughout this section, we explore consequences of the previous results, including:

— How $\lambda$ and deficiency change as we move up the tower $K_{\frak{p}}/K$;
— That Def($G_{\frak{p}}$) = 0 implies the same for open subgroups of $G_{\frak{p}}$ when $\delta = 1$;
— The rapid growth of $\lambda$ as we move up a $p$-adic analytic quotient tower of $G_{\frak{p}}$. The Tame Fontaine-Mazur conjecture predicts that infinite $p$-adic analytic quotient of $G_{\frak{p}}$ do not exist; thus, proving $\lambda$ cannot grow rapidly would lend support to the Fontaine-Mazur conjecture;
— Some results in the direction of better understanding the cohomological dimension of $G_{\frak{p}}$;
— A computable test for maximality of Def($G_{\frak{p}}$).

3.1. Conserving the deficiency along the tower. — Let $F$ be a number field in the tower $K_{\frak{p}}/K$ and recall that $F_{\frak{p}} = K_{\frak{p}}$. We denote by $\lambda_{F_{\frak{p}}/F}$ the asymptotic Minkowski rank in $F_{\frak{p}}/F$.

Proposition 3.1. — One has $\lambda_{F_{\frak{p}}/F} \geq [F : K] \lambda_{K_{\frak{p}}/K}$.

Proof. — Let $L \supset F \supset K$ in $K_{\frak{p}}/K$ be a large enough number field so that $\lambda_{L/K} = \lambda_{K_{\frak{p}}/K}$ and $\lambda_{L/F} = \lambda_{F_{\frak{p}}/F}$. Set $G = \text{Gal}(L/K)$ and $H = \text{Gal}(L/F)$. Then $\delta_L = \mathbb{F}_p[G]^{\lambda_{K_{\frak{p}}/K}} \oplus N$, where $N$ is $G$-torsion. The result follows by noting that $\mathbb{F}_p[G] \cong H \mathbb{F}_p[H]^{[F : K]}$ (see §1.3.1). \qed
Corollary 3.2. — For every number field $F$ in $K_{\mathcal{O}}/K$, we have
\[ \text{Def}(G_{F,\mathcal{O}}) \leq d(\mathcal{O}_F^\times) - [F : K]\lambda_{K_{\mathcal{O}}/K}. \]

Proof. — This follows immediately from Theorem 2.9 and Proposition 3.1. □

Remark 3.3. — The above Corollary is very close to what one could expect from strictly group-theoretic considerations. Namely, from equations (5.2) and (5.4) of [13] one deduces that for an open subgroup $H$ of a pro-$p$ group $G$, one has
\[ \text{Def}(H) + 1 \leq (G : H)(\text{Def}(G) + 1). \]

3.2. When $\text{Def}(G_{\mathcal{O}}) = 0$. —

Corollary 3.4. — Let $K$ be a number field containing $\zeta_p$. Suppose that $\mathcal{O}_{K_{\mathcal{O}}}^\times \cap (\mathcal{O}_{K_{\mathcal{O}}}^\times)^p = (\mathcal{O}_K^\times)^p$, and that $\text{Def}(G_{\mathcal{O}}) = 0$. Then, for every finite extension $F/K$ in $K_{\mathcal{O}}/K$, one has
\[ \text{Def}(F, G_{\mathcal{O}}) \leq [F : K]\lambda_{K_{\mathcal{O}}/K} = [F : K]d(\mathcal{O}_K^\times) = d(\mathcal{O}_F^\times). \]

Proof. — Applying Theorem 2.9, we see $\lambda = d(\mathcal{O}_K^\times)$ and is maximal and hence constant in the tower $K_{\mathcal{O}}/K$, relative to the base field $K$. By Proposition 3.1 we see
\[ \lambda_{F_{\mathcal{O}}/F} \geq [F : K]\lambda = [F : K]d(\mathcal{O}_K^\times) = d(\mathcal{O}_F^\times). \]
The result follows by Theorem 2.9. □

Corollary 3.5. — Let $G$ be a pro-$2$ group such that:

(i) $\text{Def}(G) = 0$,

(ii) there exists a normal open subgroup $H$ of $G$ such that $r(H) \neq d(H)$.

Then $G$ cannot be realized as the $2$-tower of an imaginary quadratic field $K$ of discriminant $\text{disc}_K = 1(\text{mod } 4)$ nor $\text{disc}_K = 0(\text{mod } 8)$.

Proof. — The discriminant hypotheses imply $-1 \notin \mathcal{O}_{K_{\mathcal{O}}}^\times$. The result follows from Corollary 3.4. □

The condition that the number of Minkowski units is maximal is very strong\(^3\):

Proposition 3.6. — Let $G$ be a finite $p$-group such that $\text{Def}(H) = 0$ for every subgroup $H$ of $G$. Then $G$ is cyclic, or the generalized quaternion group $Q_{2^n} = \langle x, y \mid x^{2^n-1} = 1, x^{2^{n-1}} = y^2, yxy^{-1} = x^{-1} \rangle$, $n \geq 3$.

Proof. — In this case every abelian subgroup $H$ of $G$ is of deficiency 0, forcing $H$ to be cyclic. Then, $G$ is cyclic or the generalized quaternion group $Q_{2^n}$ of order $2^n$ (see for example [32, Theorem 9.7.3]). For the converse, obviously cyclic groups $G$ satisfy $\text{Def}(H) = 0$ for every subgroup $H$ of $G$. Concerning $Q_{2^n}$, recall that its subgroups are cyclic or isomorphic to $Q_{2^{n-1}}$, and that the Schur multiplier of the generalized quaternion groups $Q_{2^n}$ are trivial (or in other words that $\text{Def}(Q_{2^n}) = 0$). □

Remark 3.7. — Take $p = 2$, and let $K$ be an imaginary quadratic field. Recall that $\text{Def}(G_{\mathcal{O}}) \in \{0, 1\}$. We suspect that when $G_{\mathcal{O}}$ is infinite then $\text{Def}(G_{\mathcal{O}})$ is maximal. In fact, if it is not the case, then by Theorem 3.4, we get that $r(H) = d(H)$ for every open distinguished subgroup $H$ of $G_{\mathcal{O}}$.

3. We thank Ozaki for bringing this result to our attention.
Remark 3.8. — Observe that Poincaré pro-$p$ groups of dimension 3 satisfy condition of Corollary 3.4, see for example [24, Chapter III, §7].

We close this subsection with an explicit, albeit contrived, example with $p = 2$.

Example. — Let $K = \mathbb{Q}(\sqrt{-3}, 5, 53)$. An easy MAGMA computation gives that the class group of $K$ is $(2, 2)$ and its 2-Hilbert Class Field tower has order 8. Some straightforward computations show this group has at least three cyclic subgroups of order 4, hence it is the quaternion group of order 8. Here $\mathcal{O}_K^\times = \{1, -1\}$, and as the discriminant of $K$ is prime to 4, $i^2 = -1 \notin \mathcal{O}_K^\times$ so $\mathcal{O}_K^\times \cap \mathcal{O}_K^\times = \{1\}$ and $\beta = 0$. Then Theorem 2.9 gives $\text{Def}(G_{\mathcal{O}}) = 1 - \lambda$. But it is well-known the quaternion group has deficiency 0 so $\lambda = 1$. There is a Minkowski unit in this (short) tower.

3.3. In the context of the Fontaine-Mazur conjecture. — The conjecture of Fontaine-Mazur [5, Conjecture 5a] asserts that every analytic quotient of $G_{\mathcal{O}}$ must be finite. By class field theory, one knows that every infinite analytic quotient of $G_{\mathcal{O}}$ must be of analytic dimension at least 3 (see [21, Proposition 2.12]).

One knows that $G_{\mathcal{O}}$ is not $p$-analytic when the $p$-rank of $d(\text{Cl}_K)$ of the class group $\text{Cl}_K$ of $K$ is large compared to $[K : \mathbb{Q}]$. See A.3.11 of [17]. Alternatively, this is (literally!) an exercise on page 78 of [33].

Suppose $G := G_{\mathcal{O}}$ is infinite and analytic. One knows that every infinite analytic pro-$p$ group contains an open uniform subgroup. To simplify, assume $G$ is uniform. Denote by $(G_n)$ the $p$-central descending series of $G$ (it is also the Frattini series), and let $K_n = K_{G_n}$. Put $H_n = \text{Gal}(K_n/K)$; recall that $\#H_n = p^{\lambda_n}$, where $d = d(G_{\mathcal{O}})$ is also the dimension of $G_{\mathcal{O}}$ as analytic group. For $n \geq 1$, denote by $\lambda_n$ the Minkowski-rank of the units along $K_{G_n}/K_n$.

The hypothesis of Corollary 3.9 below is, assuming the Fontaine-Mazur Conjecture, never satisfied. We include the Corollary to indicate a possible strategy to prove $G_{\mathcal{O}}$ is not analytic, namely show the number of Minkowski units does not grow so rapidly in the tower.

Corollary 3.9. — Let $G_{\mathcal{O}}$ be pro-$p$ analytic of dimension $d$. Then for $m$ large,

$$(r_1 + r_2)[K_m : K] - 1 + \delta - \frac{d(d - 1)}{2} \leq \lambda_m \leq (r_1 + r_2)[K_m : K] - 1 + \delta - \frac{d(d - 3)}{2}$$

Proof. — Theorem 2.9 here, Theorem 4.35 of [4], and the assumption that $G_{\mathcal{O}}$ is uniform imply $\text{Def}(G_n)$ is constant and equal to $\text{Def}(G_{\mathcal{O}}) = \frac{d(d - 3)}{2}$. As remarked in the introduction, $\beta_m \leq d(G_m) = d(G) = d$. We immediately see $\lambda_m \sim (r_1 + r_2)[K_m : K]$, proving the main terms in the estimates.

We now prove the more refined estimates. Let us choose $n \gg m \gg 0$ such that:

$$\mathcal{O}_{K_n}^\times = \mathbb{Z}_p[H_{n,m}]^\times \oplus H_{n,m} \cap \mathbb{Z}_p[H_{n,m}] \cap \mathbb{Z}_p[H_{n,m}]$$

where $\lambda = \lambda_{K_{G_n}/K}$, $\lambda_m = \lambda_{K_{G_n}/K_m}$, $H_{n,m} = \text{Gal}(K_n/K_m)$, and $N_{n,m}$ and $N_n$ are torsion modules over $\mathbb{Z}_p[H_{n,m}]$ and $\mathbb{Z}_p[H_n]$ respectively.

Then by Proposition 3.1, we see $\lambda_m = [K_m : K] \lambda + \lambda_{m,n}^{\text{new}}$, the quantity $\lambda_{m,n}^{\text{new}}$ corresponds to the $\mathbb{Z}_p[H_n]$-free part in $N_n$. Hence, by Theorem 2.9, one has

$$\text{Def}(G_m) \geq d(\mathcal{O}_{K_m}^\times) - \lambda_m - \beta_m \geq d(\mathcal{O}_{K_m}^\times) - [K_m : K] \lambda - \lambda_{m,n}^{\text{new}} - d(G_m).$$

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After noting that \( d(O_{K_m}^\times) = (r_1 + r_2)[K_m : K] - 1 + \delta \), we get
\[
(8) \quad \text{Def}(G_m) \geq (r_1 + r_2 - \lambda)[K_m : K] - 1 + \delta - d - \lambda_{m,n}^{\text{new}}.
\]
But \( \text{Def}(G_m) = \text{Def}(G) = \frac{d(d-3)}{2} \). Hence (8) becomes
\[
(9) \quad \lambda_{m,n}^{\text{new}} \geq (r_1 + r_2 - \lambda)[K_m : K] - 1 + \delta - \frac{d(d-1)}{2},
\]
and
\[
(10) \quad \lambda_m \geq (r_1 + r_2)[K_m : K] - 1 + \delta - \frac{d(d-1)}{2},
\]
proving the first inequality. The upper bound follows as \( \text{Def}(G_m) \leq d(O_{K_m}^\times) - \lambda_m \) so
\[
(11) \quad \lambda_m \leq (r_1 + r_2)[K_m : K] - 1 + \delta - \frac{d(d-3)}{2}.
\]
\[\square\]

### 3.4. On the cohomological dimension of \( G_\varnothing \).

Since the works of Labute [15], Labute-Mináč [16] and Schmidt [31], etc. one knows that in certain cases the groups \( G_S \), for \( S \) tame, are of cohomological dimension 2. In all the examples of these papers \( S \neq \varnothing \). The question of the computation of cohomological dimension of \( G_\varnothing \) is still an open problem (one can find partial negative answers in [20]). To prove Theorem 3.11, we need the following lemma due to Schmidt [29, Proposition 1].

**Lemma 3.10.** — (Schmidt) Let \( G \) be an infinite pro-\( p \) group such that for a fixed constant \( n \geq 0 \) and every open subgroup \( H \) of \( G \), one has
\[
-\chi_3(H) + n := -1 - \text{Def}(H) + \dim H^3(H) + n \geq [G : H](-1 - \text{Def}(G) + \dim H^3(G)) \quad := -[G : H]\chi_3(G).
\]

Then \( \text{cd}(G) \leq 3 \).

**Theorem 3.11.** — Let \( K \) be a number field such that

(i) \( K \) contains a primitive \( p \)th root of unity;

(ii) \( O_K^\times \cap (O_K^\times)^p = (O_K^\times)^p \).

Then \( \dim H^3(G_\varnothing) > 0 \). Moreover:

- If \( \dim H^3(G_\varnothing) = 1 \), then \( G_\varnothing \) is finite or of cohomological dimension 3;

- If \( \text{Def}(G_\varnothing) = 0 \), and if \( G_\varnothing \) is of cohomological dimension 3, then \( G_\varnothing \) is a Poincaré duality group.

**Proof.** — As \( O_K^\times \cap (O_K^\times)^p = (O_K^\times)^p \) and \( \delta = 1 \), one has, by Theorem 2.9,
\[
\text{Def}(G_\varnothing) = d(O_K^\times) - \lambda_{K_\varnothing/K} = r_1 + r_2 - \lambda_{K_\varnothing/K}.
\]

Let \( H \) be an open normal subgroup of \( G_\varnothing \) and set \( F = K^H_\varnothing \). Proposition 3.1 implies \( \lambda_{F_\varnothing/F} \geq \lambda_{K_\varnothing/K}[G_\varnothing : H] \), so Theorem 2.9 implies \( \text{Def}(H) \leq [G_\varnothing : H](r_1 + r_2 - \lambda_{K_\varnothing/K}) \).

Recalling that \( \chi_2 \) is the Euler characteristic truncated at second cohomology,
\[
\chi_2(H) = 1 + \text{Def}(H) \leq 1 + [G_\varnothing : H](r_1 + r_2 - \lambda_{K_\varnothing/K}),
\]
so \( \chi_2(H) \) cannot be equal to \([G_\varnothing : H]\chi_2(G_\varnothing) = [G_\varnothing : H](1 + r_1 + r_2 - \lambda_{K_{/K}})\), a necessary condition, by Theorem 5.4 of [13], for \( G_\varnothing \) to be of cohomological dimension 2. Hence \( G_\varnothing \) is not of cohomological 2 so \( \dim H^3(G_\varnothing) > 0 \).

Now suppose \( G_\varnothing \) is infinite and \( \dim H^3(G_\varnothing) = 1 \). By Theorem 2.9 and Proposition 3.1, one has

\[
\left[ -1 - \text{Def}(H) + \dim H^3(H) \right] + 1 = \lambda_{\varnothing/L} - d(\varnothing) + \dim H^3(H) \\
\geq [G_\varnothing : H](\lambda_{\varnothing/L} - (r_1 + r_2)) \\
= [G_\varnothing : H](-1 - \text{Def}(G_\varnothing) + \dim H^3(G_\varnothing))
\]

where the last equality follows from the proof of the Principal Ideal Theorem: Namely, for a group \( G \) let \( G_1 \) be the trivial group. As the transfer map is the dual of the corestriction map, it follows from the proof of the Principal Ideal Theorem: Namely, for a group \( G \) let \( G_1 \) be the trivial group.

Finally, to check that our group is a Poincaré group, following [24, Chapter III, §7], we verify that \( D_i(Z/pZ) := \lim \nabla H^i(U) = 0 \) for \( i = 0, 1, 2 \), where the limit is taken over open subgroups \( U \) of \( G_\varnothing \) and the transition maps are dual to the corestriction. As \( G_\varnothing \) is assumed to be of finite cohomological dimension, then \( G_\varnothing \) is infinite and thus \( D_0(Z/pZ) = 0 \). Moreover that

\[
D_1(Z/pZ) = \lim_{\nabla U} U_{ab}/p = 0
\]

follows from the proof of the Principal Ideal Theorem: Namely, for a group \( G \) let \( G' \) be its (closed) commutator subgroup and let \( G'' \) be the (closed) commutator subgroup of \( G' \). The key part of the proof of the Principal Ideal Theorem is that the transfer map

\[
\text{Ver}: G_1/G' \rightarrow G'/G''
\]

is the zero map. As the transfer map is the dual of the corestriction map, \( D_1(Z/pZ) = 0 \).

We now show \( D_2(Z/pZ) = 0 \). Let \( U \subset G_\varnothing \) be open. Taking the \( U \)-cohomology of the short exact sequence of trivial \( U \)-modules

\[
0 \to Z/pZ \to Q/Z \xrightarrow{p} Q/Z \to 0
\]

gives

\[
H^1(U, Q/Z) \xrightarrow{p} H^1(U, Q/Z) \to H^2(U, Z/pZ)
\]

which becomes \( \{U^{ab}\} / p \to H^2(U, Z/pZ) \). As \( \dim \{U^{ab}\} / p = \dim H^1(U) \) and \( \text{Def}(U) = 0 \), this injection is an isomorphism and \( H^2(U, Z/pZ) = \{U^{ab}\} / p \simeq H^1(U, Z/pZ) \). Since the duals of the two corestriction maps are induced by the transfer, \( D_1(Z/pZ) = 0 \) \( \implies \) 

\( D_2(Z/pZ) = 0 \).

**Remark 3.12.** — The first part of Theorem 3.11 extends the following observation that can be deduced from the relationship between Galois cohomology and étale cohomology. We use the the formalism of étale cohomology as in [23]. Suppose \( \text{Def}(G_\varnothing) \) is maximal. Then the étale version of the Hochschild-Serre spectral sequence with [30, Theorem 3.4] shows that \( H^i(G_\varnothing) \simeq H^i_\varnothing(\text{Spec } \varnothing_{/K}) \) for \( i = 1, 2 \). More, if \( G_\varnothing \) has cohomological dimension 2, then \( G_\varnothing \) is infinite: by [30, Lemma 3.7] and the Hochschild-Serre spectral sequence we also get \( G_\varnothing \simeq H^3_\varnothing(\text{Spec } \varnothing_{/K}) \simeq \mu_{K, p} \), where here \( \mu_{K, p} = \langle \zeta_p \rangle \cap K \) (by [30, Theorem 3.4]). Hence \( \delta \) must be zero.

**3.5. On the maximality of \( \text{Def}(G_\varnothing) \).** —
3.5.1. Detecting maximality. — The strategy of the Hochschild-Serre spectral sequence allows us to prove:

**Theorem 3.13.** — Suppose there exist two linearly disjoint unramified (and nontrivial) \( \mathbb{Z}/p \)-extensions \( F_1/K \) and \( F_2/K \) such that \( t_{G_{\mathcal{O}}}({\mathcal{O}}_{F_i}) = 0, i = 1, 2 \), where \( G_i = \text{Gal}(F_i/K) \). Then \( \text{Def}(G_{\mathcal{O}}) = d(\mathcal{O}_K^{\infty}) \) is maximal. Only one such extension \( F_i/K \) is sufficient if \( F_i \subset K' \bigl( \sqrt[3]{\mathcal{O}_K^{\infty}} \bigr) \), which is the case when \( \delta = 0 \).

**Proof.** — First note that Lemma 2.6 and the fact that \( t_{G_i}(\mathcal{O}_{F_i}) = 0 \) implies \( \lambda = 0 \). If \( \delta = 0 \), Proposition 2.8 implies \( \beta = 0 \) so \( \text{Def}(G_{\mathcal{O}}) \) is maximal by Theorem 2.9.

We now address the \( \delta = 1 \) case. First suppose \( F_1 \subset K(\sqrt[3]{\mathcal{O}_K^{\infty}}) \). Then one can choose \( d(\mathcal{O}_K^{\infty}) \) primes \( p \) of \( K \) that split completely in \( F_1 \) and whose Frobenii form a basis of \( \text{Gal}(K(\sqrt[3]{\mathcal{O}_K^{\infty}})/K) \). Since \( t_{G_1}(\mathcal{O}_{F_1}) = 0 \), we see that for each \( p \in S_2 \) there is a \( \mathbb{Z}/p \mathbb{Z} \)-extension of \( F_1 \), and hence of \( K_{\mathcal{O}} \), ramified only at \( ( \text{the primes above} ) \) \( p \). Each of these elements gives rise to a relation of \( G_{\mathcal{O}} \). As usual, one gets the rest of the relations 'for free' by choosing primes that split completely in \( K(\sqrt[3]{\mathcal{O}_K^{\infty}})/K \) but form a basis of \( \text{Gal} \left( K(\sqrt[3]{\mathcal{O}_{K,K}})/K(\sqrt[3]{\mathcal{O}_K^{\infty}}) \right) \). For such primes \( p \) there is always an abelian extension of \( K \) ramified only at \( p \), also giving rise to a relation of \( G_{\mathcal{O}} \).

We study the remaining case, namely when \( F_1, F_2 \subset K(\sqrt[3]{\mathcal{O}_K^{\infty}}) \). Choose a prime \( q_1 \) of \( K \) such that its Frobenius generates \( \text{Gal}(F_1/K) \) and \( q_1 \) splits in \( F_2 \). Choose \( q_2 \) similarly. Then, as before, when we allow ramification at \( q_1 \) we obtain a ramified extension over \( F_2 \) and when we allow ramification at \( q_2 \) we obtain a ramified extension over \( F_1 \). Set \( S_2 = \{ q_1, q_2 \} \) and augment \( S_2 \) to include primes that split completely in \( F_1, F_2 \) and whose Frobenii, along with those of \( q_1 \) and \( q_2 \), form a basis of \( \text{Gal}(K(\sqrt[3]{\mathcal{O}_K^{\infty}})/K) \). For each of these primes we get a ramified extension over \( F_1 \) for \( i = 1, 2 \). Each of these primes gives rise to a relation of \( G_{\mathcal{O}} \) and along with the 'free relations' we get \( \text{Def}(G_{\mathcal{O}}) = d(\mathcal{O}_K^{\infty}) \).

3.5.2. Road to a Conjecture. — Given \( K_{\mathcal{O}}/K \), the condition \( \lambda > 0 \) should be seen as follows: for every finite Galois extension \( F/K \) in \( K_{\mathcal{O}}/K \) with Galois group \( G \), there exists at least \( \lambda \) Minkowski units, i.e., \( t_G(\mathcal{O}_F) \geq \lambda \). Regarding relations, we ask:

**Question 3.14.** — When \( K_{\mathcal{O}}/K \) is sufficiently large compared to \([ K : \mathbb{Q} ] \), in particular when \( K_{\mathcal{O}}/K \) is infinite, do we have \( \text{Def}(G_{\mathcal{O}}) \geq d(\mathcal{O}_K^{\infty}) - \beta \)? In particular, when \( \delta = 0 \) and \( G_{\mathcal{O}} \) is infinite, do we have \( \text{Def}(G_{\mathcal{O}}) = d(\mathcal{O}_K^{\infty}) \)?

In fact, even when \( \delta = 1 \), we suspect this to be the case when \( d(G_{\mathcal{O}}) \) is large compared to \([ K : \mathbb{Q} ] \). For a number field \( K \), put \( \alpha_K = 3 + 2\sqrt{r_1 + r_2 + 2} \). Note \( \alpha_K \) is small relative to \( d(\mathcal{O}_K^{\infty}) = r_1 + r_2 - 1 + \delta \).

**Proposition 3.15.** — Assume \( \delta = 1 \) and suppose \( d(G_{\mathcal{O}}) \geq \alpha_K \). Then for every cyclic degree \( p \)-extension \( F/K \) in \( K(\sqrt[3]{\mathcal{O}_K^{\infty}})/K \), there exists an infinite Galois extension \( K^T/K \) in \( K_{\mathcal{O}}/K \) such that \( F \subset K^T \).

**Proof.** — Choose a prime ideal \( p \subset \mathcal{O}_K \) whose Frobenius in \( \text{Gal}(K(\sqrt[3]{\mathcal{O}_K^{\infty}})/K) \) generates \( \text{Gal}(F/K) \). Put \( T = \{ p \} \), and consider \( K^T \) the maximal pro-\( p \) extension of \( K \) unramified everywhere and where \( p \) splits completely. Since

\[
\text{d}(\text{Gal}(K^T/K)) \geq d(G_{\mathcal{O}}) - 1 \geq 2 + 2\sqrt{d(\mathcal{O}_K^{\infty}) + 2},
\]

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the pro-p extension $K^T/K$ is infinite (see for example [19, Théorème 2.1]). And as $p$ splits completely in $K^T/K$, we conclude that $F \subseteq K^T$. □

Proposition 3.15 shows the following: Suppose $d(G_{\mathcal{O}}) \geq \alpha_K$. Then every unit in $\mathcal{O}_K^x \cap (\mathcal{O}_K^2)^p$ gives an extra relation (following the proof of Theorem 2.9) except if there is a Minkowski unit in some infinite extension. In other words: when $\delta = 1$, one has $\text{Def}(G_{\mathcal{O}}) < r_1 + r_2$ if and only if there exists a prime ideal $p \in \mathcal{O}_K$ with nontrivial Frobenius in $\text{Gal}(K(\sqrt[p]{\mathcal{O}_K^2})/K)$ such that $K^{(p)}/K$ has a Minkowski unit. Here, as $K^{(p)}/K$ is infinite, having a Minkowski unit means that for every finite Galois quotient $L/K$ of $K^{(p)}/K$ of Galois group $G$, one has $h_G(\mathcal{O}_L) > 0$.

**Question 3.16.** Assume $\delta = 1$. When $d(G_{\mathcal{O}}) \geq \alpha_K$, do we have $\text{Def}(G_{\mathcal{O}}) = d(\mathcal{O}_K^x)$?

### 4. On the depth of the relations

It is easier to use software for computations along the Frattini tower of $K_{\mathcal{O}}/K$ than along the Zassenhaus tower. In this section we show the existence of Minkowski units deep in the Frattini tower imply that some of the relations of $G_{\mathcal{O}}$ are very deep. This makes it 'more likely' that one can prove $G_{\mathcal{O}}$ is infinite using the Golod-Shafarevich series. We also prove a converse, namely the existence of very deep relations implies the existence of Minkowski units along the Frattini tower.

#### 4.1. On the Zassenhaus filtration.

**4.1.1. Basic properties.** We refer to Lazard [17, Appendice A3]. Given a finitely presented prop-p group $G$, let us take a minimal presentation of $G$

$$1 \rightarrow R \rightarrow F \xrightarrow{\varphi} G \rightarrow 1,$$

where $F$ is a free prop-p group on $d$ generators; here $d = d(G)$. Let $I = \ker(F_p[F] \rightarrow F_p)$ be the augmentation ideal of $F_p[F]$, and for $n \geq 1$ consider $F^n = \{x \in F, x - 1 \in I^n\}$. The sequence $(F^n)$ of open subgroups of $F$ is the Zassenhaus filtration of $F$. The depth $\omega$ of $x \in F$ is defined as being $\omega(x) = \max\{n, x - 1 \in I^n\}$, with the convention that $\omega(1) = \infty$; the function $\omega$ is a valuation following terminology of Lazard. Hence $F^n = \{g \in F, \omega(g) \geq n\}$. This allows us to define a depth $\omega_G$ on $G$ as follows: $\omega_G(x) = \max\{\omega(g), g \in F, \varphi(g) = x\}$. Put $G^n = \{x \in G, \omega_G(x) \geq n\}$. Observe that $G^n = F^n R/R \simeq F^n/(F^n \cap R)$; the sequence $(G^n)$ is the Zassenhaus series of $G$, it corresponds to the filtration arising from the augmentation ideal $I_G$ of $F[G]$, see [17, Appendice A3, Theorem 3.5]. One has the following property. If $\pi : G' \rightarrow G$ is surjective, then $\omega_G$ is the restriction of $\omega_{G'}$; in other word, $\omega_{G'}(y) = \max\{\omega_{G'}(y), y \in G', \pi(y) = x\}$.

Denote by $(G_n)$ the Frattini filtration of $G$. Recall the well-known relationship between these two filtrations of $G$:

**Lemma 4.1.** One has $G_n \subseteq G^{2n-1}$.

We say a few words about the reverse inclusions. Let $H$ be an open normal subgroup of $G$. Since the groups $(G^n)$ form a basis of neighborhoods of 1, let $a(H)$ be the smallest integer such that $G^{a(H)} \subseteq H$. We want to give some estimates on $a(H)$ in some special cases.
**Definition 4.2.** — For a pro-$p$ group $\Gamma$, denote by $I_\Gamma := \ker(F_p[\Gamma] \to F_p)$, the augmentation ideal of $F_p[\Gamma]$; and denote by $k(\Gamma)$ the smallest integer such that $I^{k(\Gamma)}_\Gamma = \{0\}$, where we allow $k(\Gamma) = \infty$.

**Proposition 4.3.** — Let $1 \to H \to G \to G/H \to 1$ be an exact sequence of pro-$p$ groups. Then $I_H = \ker(F_p[G] \to F_p[G/H])$.

**Proof.** — See for example [13, Chapter 7, §7.6, Theorem 7.6]. □

Following Koch’s book [13, Chapter 7, §7.4], we give some estimates for $a(H)$.

**Proposition 4.4.** — One has:

(i) $a(H) \leq k(G/H) \leq |G/H|$. 
(ii) If $\Gamma' \vartriangleleft \Gamma$ are two finite $p$-groups, then $k(\Gamma) \leq k(\Gamma') k(\Gamma'')$. 
(iii) $k(G/G_2) = p$.

**Proof.** — (i) Take $k$ such that $I^k_{G/H} = \{0\}$. Then by Proposition 4.3 one has $I^k_G \subset I_H$, which implies $G^k \subset H$, and then $a(H) \leq k$. In particular, $a(H) \leq k(G/H)$. For the second part of the inequality, observe that: for every finite $p$-group $\Gamma$, one has $I^1_G = \{0\}$ (see the proof of Lemma 7.4 of [13, Chapter 7, §7.4]), showing that $k(\Gamma) \leq |\Gamma|$. 
(ii) By Proposition 4.3, one has $I^{k(\Gamma')}_{\Gamma'} \subset I^{k(\Gamma'')}_{\Gamma''}$, and then $I^{k(\Gamma)}_{\Gamma} = \{0\}$. 
(iii) This follows as $G/G_2$ is $p$-elementary abelian. □

For every integer $n \geq 1$, put $a_n := a(G_{n+1})$. Observe that $a_1 = 1$.

**Proposition 4.5.** — For $n \geq 2$, one has $a_n \leq p^n$. Therefore $G_n \subset G^{2^{n-1}} \subset G_{(n-1)\log(2)/\log(p)}$.

**Proof.** — That $a_n \leq p^n$ follows from Proposition 4.4 and the fact that $G_n/G_{n+1}$ is elementary $p$-abelian. The second part follows from the first. □

**4.1.2. The Golod-Shafarevich polynomial.** — Consider a minimal presentation of a finitely generated pro-$p$ group $G$:

$$1 \to R \to F \xrightarrow{\varphi} G \to 1.$$ 

Suppose that $R/F[R,R]$ is generated as an $F_p[F]$-module by the family $\mathcal{F} = \{\rho_i\}$ of elements $\rho_i \in F$. For $k \geq 2$, put $r_k = |\{\rho_i, \omega(\rho_i) = k\}|$; here we assume the $r_i$’s finite.

**Definition 4.6.** — The series $P(t) = 1 - dt + \sum_{k \geq 2} r_k t^k$ is a Golod-Shafarevich series associated to the presentation $\mathcal{F}$ of $G$.

The theorem of Golod-Shafarevich implies the following: if for some $t_0 \in (0, 1)$ one has $P(t_0) = 0$, then $G$ is infinite (see [36] or [1]). Observe that when no information on the depth of the $\rho_i$ is available, then one may take $1 - dt + rt^2$ as Golod-Shafarevich series for $G$, where $r = d(H^2(G))$.

**Remark 4.7.** — When $G = G_\mathcal{F}$, the $p$-rank of $G_n$ corresponds to the $p$-rank of the class group of $K_n$, where $K_n = K_\mathcal{F}$. Hence by Class Field Theory and with the help of a software package, in a certain sense it is easier to test if an element of $G$ is in $G_n$ than if it is in $G^n$.

**4.2. Minkowski units and the Golod-Shafarevich polynomial of $G_\mathcal{F}$.** —
4.2.1. The principle. — Let \( S \) be a finite saturated set of tame places of \( K \) as in Lemma 1.5, i.e. such that \( H^1(G_\varnothing) \approx H^1(G_S) \) and \( |S| = d(V_{K,\varnothing}) \). Put \( d = d(G_\varnothing) \). Let \( F \) be the free pro-\( p \)-group on \( d \) generators \( x_1, \ldots, x_d \). Consider now the minimal presentations of \( G_\varnothing \) and \( G_S \) induced by \( F \), and the following diagram

\[
1 \rightarrow R_\varnothing \rightarrow F \rightarrow G_S \rightarrow 1
\]

where \( \varnothing \) is induced by \( i \). The Hochschild-Serre spectral sequences induce the following isomorphisms

\[
R_\varnothing/R_\varnothing^p[R_\varnothing, F] \xrightarrow{\cong} H^2(G_\varnothing)^{\wedge} \xrightarrow{\cong} H_S/H_S^p[G_\varnothing, H_S]
\]

where \( \varnothing \) is induced by \( \varnothing' \). Using \( \varnothing \) we will study the depth of the relations of \( G \): indeed, \( R_\varnothing/R_\varnothing^p[R_\varnothing, F] \) and \( H_S/H_S^p[G_\varnothing, H_S] \) inherit the Zassenhaus valuation from \( R_\varnothing \) and \( H_S \), and then on the Zassenhaus valuation on \( F \). Therefore an element of depth \( k \) in \( R_\varnothing/R_\varnothing^p[R_\varnothing, F] \) corresponds to one element of depth \( k \) in \( H_S/H_S^p[G_\varnothing, H_S] \).

4.2.2. Minkowski element. — Here we extend the notion of Minkowski unit to the notion of Minkowski element. Set \( \mathcal{U}_K = V_{K,\varnothing}/(K^\times)^p \).

Definition 4.8. — Let \( L/K \) be a Galois extension with Galois group \( G \). We denote by \( \lambda_{L/K} := t_G(\mathcal{U}_L) \) the \( \mathbb{F}_p[G] \)-rank of \( \mathcal{U}_L \). One says that \( L/K \) has a Minkowski element if \( \lambda_{L/K} \geq 1 \).

Lemma 4.9. — One has \( \lambda_{L/K} \geq \lambda_{L/K} \), so the existence of a Minkowski unit implies that of a Minkowski element.

Proof. — This follows immediately from the exact sequence

\[
1 \rightarrow \mathcal{E}_L \rightarrow \mathcal{U}_L \rightarrow \text{Cl}[p] \rightarrow 1.
\]
Suppose now that $G_{\mathcal{O}}$ is infinite and, except for finitely many Galois extensions $L/K$ in $K_{\mathcal{O}}/K$, one has $\lambda'_{L/K} = d - 1 + r_1 + r_2$. Then $d(\mathcal{Y}_L) \geq |G/H|(d - 1 + r_1 + r_2)$ and 
\[d(\text{Cl}_L) \geq 1 - \delta + |G/H|(d - 1) \geq |G/H|(d - 1),\]

implying 
\[-\chi_1(H) + 1 \geq -|G/H|\chi_1(G_{\mathcal{O}}).
By [29, Proposition 1], the Galois group $G_{\mathcal{O}}$ must be free pro-$p$, which is impossible. \square

The converse of Lemma 1.13 follows easily from the Chebotarev density theorem:

**Proposition 4.11.** — Let $L/K$ be a finite $p$-extension of Galois group $G$.

(i) If $t_G(\mathcal{O}_{L,\mathcal{O}}) \geq k$, then there exist infinitely many sets $S = \{p_1, \ldots, p_k\}$ of tame primes of $K$ such that $\#C^{ab}_{L,S} = \#G^{ab}_{L,\mathcal{O}}$.

(ii) If $t_G(\mathcal{Y}_L) \geq k$, then there exist infinitely many sets $S = \{p_1, \ldots, p_k\}$ of tame primes of $K$ such that $d(G_{L,S}) = d(G_{L,\mathcal{O}})$.

From the computational view point, we will now consider the sequence of fields $(K_n)$ in $K_{\mathcal{O}}/K$ induced by the Frattini filtration $(G_n)$: in other word, $K_n = K^{G_n}_G$. Put $H_n = \text{Gal}(K_n/K)$, and denote by $\lambda'_n := \lambda'_K$ the $\mathbb{F}_p[H_n]$-free rank of $\mathcal{Y}_{K_n}$.

Put $d := d(G_{\mathcal{O}})$, and $r_{\max} := d + d(\mathcal{O}^\times_K)$.

**Theorem 4.12.** — Take $n \geq 2$. Then $G_{\mathcal{O}}$ can be generated by $d(G_{\mathcal{O}})$ generators and $r_{\max}$ relations $\{\rho_1, \ldots, \rho_{r_{\max}}\}$ such that at least $\lambda'_n$ relations are of depth greater than $2^n$.

**Proof.** — We are assuming that the $\mathbb{F}_p[H_n]$-module $\mathcal{Y}_{K_n}$ is isomorphic to $\mathbb{F}_p[H_n]^{\lambda'_n} \oplus N$ where $N$ is torsion. Using Chebotarev’s theorem, choose a set $S' = \{p_1, \ldots, p_{\lambda'_n}\}$ of primes of $K$ such that

— Each $p_i$ splits completely from $K$ to $K_n$,
— The Frobenius at a prime $\mathfrak{F}_{ij}$ of $K_n$ above $p_i$ in $\text{Gal}(K_n'(\sqrt[n]{\mathcal{Y}_{K_n}})/K_n)$ lies in the $i$th copy of $\mathbb{F}_p[H_n] \subset \mathcal{Y}_{K_n}$ and generates that copy of $\mathbb{F}_p[H_n]$ under the action of $H_n$.

We claim $d(G_{K_{m,S'}}) = d(G_{K_m,\mathcal{O}})$. Indeed, there are $|H_n|$ primes $\mathfrak{F}_{ij}$ of $K_n$ above $p_i$ and they have independent Frobenii in $\text{Gal}(K_n'(\sqrt[n]{\mathcal{Y}_{K_n}})/K_n)$ by choice, even as we take the union over $i$ from 1 to $\lambda'$. Gras-Munnier (Theorem 1.1) gives the equality. In fact, it gives more: $d(G_{K_{m,S'}}) = d(G_{K_{m,\mathcal{O}}})$ for all $m < n$. If this were false for $m_0 < n$, there would exist a $\mathbb{Z}/p\mathbb{Z}$-extension of $L/K_{m_0}$ ramified at primes (above those) of $S'$. Thus $\text{Lk}_{K_n}/K_n$ would be a $\mathbb{Z}/p\mathbb{Z}$-extension ramified only at primes (above those) of $S'$ contradicting the result for $n$. We have shown that the $p$-Frattini towers of $G_{S'}$ and $G_{\mathcal{O}}$ agree at the first $n$ levels. Thus the generators $\tau_{p_i}$ of the tame inertia groups all have depth $2^n$ in $G_{S'}$.

We have 
\[0 \to \mathfrak{I}_{S'}^2 \to H^2(G_{S'}) \xrightarrow{res} \bigoplus_{p_i \in S'} H^2(G_{p_i}) \]
and $\dim \mathfrak{I}_{S'}^2 = \dim \mathfrak{B}_{S'} = r_{\max} - \lambda'_n$. We can say nothing about the depth of the relations coming from $\mathfrak{I}_{S'}^2$, so we assume they have minimal depth two. The local relations are of the form $[\sigma_{p_i}, \tau_{p_i}]_{p_i}^{N(p_i)-1}$ and are easily seen to have depth at least $2^n + 1$. As 
\[G_{S'}/\langle \tau_{p_1}, \ldots, \tau_{p_{\lambda'_n}} \rangle \cong G_{\mathcal{O}}\]
and taking this quotient trivializes the local relations, the theorem follows. \square

**Remark 4.13.** — If $K_n$ corresponds to the Zassenhaus filtration, we get that the $\lambda'_n$ relations are of depth at least $2n$. 

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Corollary 4.14. — Assuming the hypothesis of Theorem 4.12, we may take $1 - dt + (r_{\text{max}} - \lambda_n')t^2 + \lambda_n't^2n$ as a Golod-Shafarevich polynomial for $G_{\mathcal{O}}$.

Example 4.15. — Let us return to the field $K = \mathbb{Q}(\sqrt{5} \cdot 13 \cdot 17 \cdot 19)$. Take $H = K_2$, $G = \text{Gal}(H/K)$. As seen earlier, $t_G(\mathcal{F}_G) \geq 1$. Indeed, the existence of a Minkowski element follows from that of a Minkowski unit. Here a Golod-Shafarevich polynomial of $\alpha$ of elements $\lambda$. —

Corollary 4.16. — If $[K : \mathbb{Q}]$ is large compared to $d$, then $G_{\mathcal{O}}$ has some deep relations. More precisely, one may take $1 - dt + (r_{\text{max}} - \alpha)t^2 + at^4$ as Golod-Shafarevich polynomial for $G_{\mathcal{O}}$, for some integer $a > 0$.

Proof. — By Theorem 1.12 applied to $G = \mathbb{G}_2^\mathcal{O}/G_{\mathcal{O}}[G_{\mathcal{O}}, G_{\mathcal{O}}] = G_{\mathcal{O}}/G_{\mathcal{O}}^2$, one observes that $t_G(\mathcal{O}_K) \geq a$, if $[K : \mathbb{Q}]$ is large as compared to a certain quantity depending on $G$. Now apply Corollary 4.14 since $\lambda'_n \geq t_G(\mathcal{O}_K)$. □

4.2.3. The converse. — Theorem 4.12 shows that the presence of Minkowski elements in the tower implies the existence of very deep relations in $G_{\mathcal{O}}$. Here we show the converse, that the existence of very deep relations implies the presence of Minkowski elements. For $n \geq 1$, recall that $F$ is a free pro-$p$ group on $d$ generators, $F^n$ and $F_m$ are the Zassenhaus and Frattini filtrations, and $a_n$ is the smallest integer such that $F^{a_n} \subset F_{n+1}$. Recall from Lemma 4.1 that $F_n \subset F^{a_n-1}$. See Section 4.1.1. Put $H_n = G_{\mathcal{O}}/G_{\mathcal{O}}$ and $K_n = K_{G_{\mathcal{O}}}^\mathcal{O}$.

Theorem 4.17. — Suppose that all the relations of $G_{\mathcal{O}}$ are of depth at least $a_n$. Then

(i) if $\zeta_p \in K$, $\lambda_{K_n/K} \geq r_1 + r_2$;

(ii) if $\zeta_p \not\in K$, $\lambda_{K_n/K} = r_1 + r_2 - 1 + d$.

Proof. — Since all the relations of $G_{\mathcal{O}}$ have depth $a_n$, we see that $G_{\mathcal{O}}/G_{\mathcal{O}}^{a_n} \simeq F/F^{a_n}$ has maximal Zassenhaus filtration for the first $a_n$ steps. Thus for any set $S$ satisfying $d(G_S) = d(G_{\mathcal{O}})$ we have

$$F/F^{a_n} \simeq G_{\mathcal{O}}/G_{\mathcal{O}}^{a_n} \simeq G_S/G_S^{a_n}$$

and since $F^{a_n} \subset F_{n+1}$, we also have

$$F/F_{n+1} \simeq G_{\mathcal{O}}/G_{\mathcal{O},n+1} \simeq G_S/G_{S,n+1}$$

so all relations of $G_{\mathcal{O}}$ have depth at least $n + 1$ in the Frattini filtration.

We first address the case $\delta = 1$. Consider the $p$-elementary abelian extensions $K(\sqrt{V_{K,n}/K})$ and $K_2/K$, the latter being the maximal unramified $p$-elementary abelian extension of $K$. By Kummer theory each is formed by adjoining to $K$ the $p$th roots of elements $\alpha \in K$. Since $K_2/K$ is everywhere unramified, $(\alpha)$ is the $p$th power of an ideal, that is $\alpha \in V_{K,n}$ so $K(\sqrt{V_{K,n}/K}) \supseteq K_2$ and $d(G(K(\sqrt{V_{K,n}/K})/K_2)) = r_1 + r_2$. Note $K_n \cap K(\sqrt{V_{K,n}/K}) = K_2$ as the intersection is both unramified over $K$ and $p$-elementary abelian over $K$. Let $S := \{p_1, \cdots, p_{r+1} \}$ consist of primes that split completely from $K$.
to $K_2$ to $K_n$ and whose Frobenii form a basis of $\text{Gal}(\sqrt[N]{V_{K,\emptyset}})/K_2$.

\[
\begin{array}{c}
K_n \\
\downarrow \\
K(\sqrt[N]{V_{K,\emptyset}}) \\
\downarrow \\
K_2 \\
\downarrow \\
K
\end{array}
\]

By the above discussion

(13) \[ F/F_{n+1} \cong G_{\emptyset}/G_{\emptyset,n+1} \cong G_S/G_{S,n+1}. \]

This will imply $\lambda_{K_n/K} \geq r_1 + r_2$. Indeed, above each $p_i$ there are $[K_n : K]$ primes $\mathfrak{P}_{ij}$ in $K_n$ upon which $\text{Gal}(K_n/K)$ acts transitively. If for some $i$ the Frobenii of the $\mathfrak{P}_{ij}$ did not generate a distinct copy of $F_p[G_{n}]$ in $\text{Gal}(K_n(\sqrt[N]{V_{K,\emptyset}})/K_n)$, then there would be a dependence relation among them and by Gras-Munnier we would have $d(G_{K_n,S}) > d(G_{K_n,\emptyset})$, contradicting (13). Thus $\lambda_n \geq r_1 + r_2$ completing the proof in the $\delta = 1$ case.

We now consider the case $\delta = 0$. As usual, the key fact is that $K_n' \cap K'(\sqrt[N]{V_{K,\emptyset}}) = K'$ (following the proof of Proposition 2.8) so $d(G(K_n'(\sqrt[N]{V_{K,\emptyset}})/K_n')) = r_1 + r_2 - 1 + d$.

\[
\begin{array}{c}
K_n' \\
\downarrow \\
K'(\sqrt[N]{V_{K,\emptyset}}) \\
\downarrow \\
K'
\end{array}
\]

We choose $S := \{p_1, \cdots , p_{r_1+r_2-1+d}\}$ to consist of primes of $K$ that split completely from $K$ to $K'$ to $K_n'$ and whose Frobenii form a basis of $\text{Gal}(\sqrt[N]{V_{K,\emptyset}})/K')$. We complete the proof exactly as in the $\delta = 1$ case.

\textbf{Corollary 4.18.} — If all the relations of $G_{\emptyset}$ are of depth at least $p^2$ then $K_2$ has a Minkowski element.

\textbf{Proof.} — This follows immediately from Proposition 4.5 and Theorem 4.17.

\textbf{4.3. The alternative.} — There is another way by which we can obtain Theorem 2.9 in the context of Golod-Shafarevich series $P(t)$. Indeed, such a series for a pro-$p$ group $G$ approximates the Hilbert $H_G$ series of the Zassenhaus filtration of $G$. In particular the Golod-Shafarevich Theorem is a consequence of this inequality: if there is some $t_0 \in [0, 1[$ such that $P(t_0) < 0$ then necessarily $H_G(t_0)$ diverges, implying the infiniteness of $G$.

Retain the notations of Section §2.3, and fix $n > 0$. Apply Corollary 4.14 to $K_n/K$ by taking $1 - dt + (r_{\max} - \lambda)t^2 + \lambda t^{2n}$ as a Golod-Shafarevich polynomial for $G_{\emptyset}$. Now, as $n$ can be arbitrarily large, we see that $1 - dt + (r_{\max} - \lambda)t^2$ is a Golod-Shafarevich polynomial for $G_{\emptyset}$.

Of course, the question of determining $\lambda$ when it is nonzero seems a hard problem, except in the case where at the beginning of the tower, we see $\lambda = 0$. Here is an explicit alternative.
Corollary 4.19. — Let $n \in \mathbb{Z}_{>1}$. One has:

(i) if $t_{\text{H}}(\mathfrak{d}_K) = 0$ and $\beta = 0$, then $\text{Def}(G_{\varnothing}) = r_1 + r_2 - 1 + \delta$;

(ii) if $t_{\text{H}}(\mathfrak{d}_K) = \lambda_n > 0$, then one may take $1 - dt_1(r_{\text{max}} - \lambda_n) + \lambda_n t^{2n}$ as a Golod-Shafarevich polynomial for $G_{\varnothing}$.

Remark 4.20. — The condition $\beta = 0$ can be relaxed as noted in Theorem 3.13.

5. The case of imaginary quadratic fields

In this section, we take $p = 2$ and let $K := \mathbb{Q}(\sqrt{D})$ be an imaginary quadratic field of discriminant $D < -7$. Since the unit rank of $K$ is 1, we have $\text{Def}(G_{\varnothing}) \in \{0, 1\}$. In this simplest of all non-trivial situations, we will discuss the deficiency of $G_{\varnothing}$ and explore the extent to which we can detect relations using the machinery and notation set up in Section 2.2.

5.1. The frame. — Let $d = (\text{Cl}_K)$ be the 2-rank of the class group of $K := \mathbb{Q}(\sqrt{D})$. By Gauss’s genus theory, we know that $D$ admits a unique (up to reordering) factorization into $d + 1$ integers, each of which is a “prime fundamental discriminant” – meaning it is the discriminant of a quadratic field in which a single prime ramiﬁes. For an odd prime $q$, we define $q^* := (-1)^{(q-1)/2}q$. The prime discriminants are then $q^*$ as $q$ ranges over all odd primes, as well as $-4$ and $\pm 8$. We write $D = q_1^* \cdots q_d^*$, with the convention that if $D$ is even, then $q_{d+1}^* \in \{-4, -8, 8\}$. Put $q_0^* = -1$ and for each $i$ in the range $0 \leq i \leq d$, put

$$K_i := K(\sqrt{q_0^*, \cdots, \sqrt{q_i^*, q_{i+1}^*, \cdots, \sqrt{q_d^*}}}),$$

where

$$q_d^* = \begin{cases} q_d^* & \text{if } D \text{ is odd} \\ q_d^* & \text{if } q_{d+1}^* = \pm 8 \\ 2 & \text{if } q_{d+1}^* = -4. \end{cases}$$

Also define $L' := K(\sqrt{q_0^*, \sqrt{q_1^*, \cdots, \sqrt{q_{d-1}^*, \sqrt{q_d^*}})}$. A direct computation shows that the number field $L'$ is the governing field $K(\sqrt{\varnothing})$ (see Section 2.2). Choose prime numbers $p_0, \cdots, p_d$ that split in $K$ and such that for each $i$ in the range $0 \leq i \leq d$, the Frobenii of the $p_j, j \neq i$ in $L'/Q$ generate the Galois group of the quadratic extension $L'/K_i$. Fix a prime $p_i|p_i$ of $K$ and put $S_2 = \{p_0\}$, $S_1 = \{p_1, \cdots, p_d\}$, and $S = S_1 \cup S_2$. Observe that the primes $p_1, \cdots, p_d$ all are congruent to 1 mod 4 and that $p_0 \equiv 3 \text{ mod } 4$.

As the 2-part of the class group of $K$ has $d$ generators, Lemma 2.5 shows the existence of $d$ independent quadratic extensions $F_i$ above $K_{\varnothing}$, totally ramiﬁed at $p_i$, $i = 1, \cdots, d$, so $d(X_S) \geq d$. This puts us in the situation where the extra relations are detectable by the set $S_2$. Now, by studying the Galois module structure of units in imaginary biquadratic number ﬁelds, we can specify conditions under which $\text{Def}(G_{\varnothing}) = 1$; see Theorem 5.3 below.

Lemma 5.1. — Let $K_0/Q$ be a real quadratic ﬁeld; $G_0 = \text{Gal}(K_0/Q)$. Then $\mathfrak{e}_{K_0}$ is $\mathbb{F}_2[G_0]$-free if and only if, the norm of the fundamental unit $\varepsilon$ is $-1$. More precisely, as

an $\mathbb{F}_2[G_0]$-module, $\mathfrak{e}_{K} \simeq \begin{cases} \mathbb{F}_2 \oplus \mathbb{F}_2 & N(\varepsilon) = 1 \\ \mathbb{F}_2[G_0] & N(\varepsilon) = -1. \end{cases}$
Proof. — If the norm of ε is +1, then modulo $(\mathcal{O}_K^\times)^2$, we get $\varepsilon^T = \varepsilon$. If the norm of ε is -1, then $\mathcal{O}_K^\times$ is generated by $\varepsilon(\mathcal{O}_K^\times)^2$ as $G$-module, and $\langle \varepsilon(\mathcal{O}_K^\times)^2 \rangle$ is $\mathbb{F}_2[G_0]$-free. □

Recall this well-known result:

**Lemma 5.2.** — Let $F/\mathbb{Q}$ be an imaginary biquadratic field. Let $K_0$ be the real quadratic subfield, and let $\varepsilon$ be the fundamental unit of $K_0$. Then, $|\mathcal{O}_F^\times/\langle \mu_F, \varepsilon \rangle| = 1$ or 2. In particular, if $F/K_0$ is ramified at some odd prime, then $\mathcal{O}_F^\times = \langle \mu_F, \varepsilon \rangle$.

### 5.2. Main result

**Theorem 5.3.** — Let $K$ be an imaginary quadratic field of discriminant $D$. Assume that we can write $D = D_1D_2$, where $D_1 > 0$ and $D_2$ are fundamental discriminants, such that:

(i) the norm of the fundamental unit of $\mathbb{Q}(\sqrt{D_1})$ is +1,

(ii) some odd prime number divides $D_2$.

Then $\text{Def}(G_\sigma) = 1$, and the extra relation is detected by the quadratic extension $K(\sqrt{D_1})/K$.

**Proof.** — Put $F := K(\sqrt{D_1})$. As $D_1$ and $D_2$ are fundamental discriminants, then $F/K$ is unramified. By assumption (ii) and Lemma 5.2, $\mathcal{O}_F^\times = \langle \varepsilon, -1 \rangle$, where $\varepsilon$ is the fundamental unit of $\mathbb{Q}(\sqrt{D_1})$. By assumption (i) and Lemma 5.1, $\mathcal{O}_F$ is not $\mathbb{F}_2[G]$-free, where $G = \text{Gal}(F/K)$: in other words $t_G(\mathcal{O}_F) = 0$. Now we can conclude by Theorem 3.13 (here $\sqrt{-1} \notin F$). □

**Remark 5.4.** — To elaborate further, observe that $p_0$ splits in $F/K$. Indeed, by the choice of $p_0$ we have, for $i = 1, \ldots, d - 1$, $(\frac{q_i^*}{p_0}) = (\frac{\mathfrak{p}_i^*}{p_0}) = 1$. Let us study two cases.

(a) Suppose first that $q_d^* = q_d^+$. Then by recalling that $(\frac{D}{p_0}) = 1$, one also gets $(\frac{q_{d+1}^*}{p_0}) = 1$, and then $(\frac{D_1}{p_0}) = 1$ (here $D_1$ is the product of some of the $q_i^*$).

(b) Suppose now that $q_d^* = 2$. Since $p_0 \equiv 3 \mod 4$ and $D = q_1^* \cdots q_{d-1}^*$, we have $(\frac{q_d^*}{p_0}) = -1$. By assumption, there exists an odd prime $p$ that divides $D_2$. We may choose $p = q_d$ (before fixing $p_0$). Then, $D_1$ is the product of various $q_i^*$, for $i = 1, \ldots, d - 1$ so $(\frac{D_1}{p_0}) = 1$.

As $p_0$ splits completely in $F/K$, we see $\prod_{p|p_0} \mathfrak{B}_p/\mathfrak{B}_p^2$ is $\mathbb{F}_2[G]$-free of rank 1. But as $t_G(\mathcal{O}_F) = 0$, the subgroup $I_{p_0}$ of $\text{RCG}_F(p_0)$ generated by the ramification at $p_0$ is not trivial. Put $I := I_{p_0}/I_{p_0}^2$. By Nakayama’s lemma, the coinvariants $I_G$ are also not trivial, hence there exists at least one quadratic extension $F_1/F$, Galois over $K$, totally ramified at some $\mathfrak{P}|p_0$, such that $G$ acts trivially on $\text{Gal}(F_1/F)$. The compositum $F_1K_{\sigma}/K_\sigma$ is ramified at $p_0$ and produces a $(d + 1)$st relation. This is the formalism of example 1.6.

**Corollary 5.5.** — Let $K$ be an imaginary quadratic field of discriminant $D$. Suppose $D$ is divisible by at least two odd primes $p_1, p_2$ such that $p_1 \equiv p_2 \equiv 3 \mod 4$. Then $\text{Def}(G_\sigma) = 1$.

**Proof.** — If there is another odd prime $q$ that divides $D$, take $D_1 = p_1p_2$. If $K = \mathbb{Q}(\sqrt{-p_1p_2})$ (resp. $\mathbb{Q}(\sqrt{-2p_1p_2})$), take $D_1 = 4p_1$ (resp. $D_1 = 8p_1$). □
Example 5.6 (Martinet [22]). — Take $K = \mathbb{Q}(\sqrt{-21})$. Then, by Odlyzko bounds the 2-tower $K_{\mathcal{O}}/K$ is finite, and it is not hard to see $G_{\mathcal{O}} \simeq (\mathbb{Z}/2\mathbb{Z})^2$, and $\text{Def}(G_{\mathcal{O}}) = 1$.

Example 5.7 (See Example 1.6). — Take $K = \mathbb{Q}(\sqrt{-5460})$, $D_1 = 21$ and $D_2 = -260$. We then get an extra relation coming from the extension $K(\sqrt{21})/K$, and $\text{Def}(G_{\mathcal{O}}) = 1$.

Corollary 5.8. — Suppose $k \geq 2$, and $p_1, \ldots, p_k$ are $k$ distinct odd primes, exactly one of which, say $p_1$, is $\equiv 3 \pmod{4}$. For the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-2p_1 \cdots p_k})$ with discriminant $D = -8p_1 \cdots p_k$, we have: $\text{Def}(G_{\mathcal{O}}) = 1$.

Proof. — Take $D_1 = 8p_1$.

Example 5.9. — Take $K = \mathbb{Q}(\sqrt{-p_1p_2})$, with primes $p_1, p_2$ such that $p_1 \equiv 1 \pmod{4}$ and $p_2 \equiv 3 \pmod{4}$. Here the hypotheses of Theorem 5.3 do not apply and $r = d = 1$ so $\text{Def}(G_{\mathcal{O}}) = 0$.

Example 5.10. — The hypotheses of Theorem 5.3 do not apply for $K = \mathbb{Q}(\sqrt{-130})$. As noted by Martinet [22], in that case, $G_{\mathcal{O}}$ is the quaternion group so $r = d = 2$.

Example 5.11. — Take $K = \mathbb{Q}(\sqrt{-5 \cdot 13 \cdot 41})$. Here $r = d + 1 = 3$; indeed the norm of the fundamental unit of $\mathbb{Q}(\sqrt{5 \cdot 41})$ is $+1$.

5.3. $\text{Def}(G_{\mathcal{O}})$ is maximal almost all the time. — We easily deduce from Theorem 5.3 that the presence of a Minkowski unit in a quadratic unramified extension $F/K$ is rare, with the consequence that, generically, the deficiency of $G_{\mathcal{O}}$ is maximal. Let us say more precisely what we mean by the term “generically” here. Denote by $\mathcal{F}$ the set of imaginary quadratic fields. For $X \geq 2$, put

\[ \mathcal{F}(X) = \{ K \in \mathcal{F}, |\text{disc}(K)| \leq X \}, \]

and

\[ \mathcal{F}_0(X) = \{ K \in \mathcal{F}(X), \text{Def}(G_{\mathcal{O}}) = 0 \}. \]

Theorem 5.12. — There is an absolute constant $C > 0$ such that for all $X$,

\[ \frac{\# \mathcal{F}_0(X)}{\# \mathcal{F}(X)} \leq C \frac{\log \log X}{\sqrt{\log X}}. \]

In particular, when ordered by absolute value of the discriminant, the proportion of imaginary quadratic fields for which $\text{Def}(G_{\mathcal{O}}) = 0$, tends to zero when $X \to \infty$.

Proof. — We use the number theory analytic tools of [20, Theorem 4.6] due to Fouvry. Let $K$ be an imaginary quadratic field. For $X \geq 2$, put

\[ B(X) = \{ K \in \mathcal{F}(X), \exists 2 \text{ distincts odd primes } p \equiv q \equiv 3 \pmod{4}, pq \mid \text{disc}(K) \}. \]

By Corollary 5.5, for every $K \in B(X)$ one has $\text{Def}(G_{\mathcal{O}}) = 1$. Hence $\mathcal{F}_0(X)$ is in the complementary $C(X)$ of $B(X)$.

Note by $A_i(X)$ the set of square-free integers $n \leq X$ having exactly $i$ prime factors $\equiv 3 \pmod{4}$, put $A(X) = A_0(X) \cup A_1(X)$. Clearly, $|C(X)| = O(|A(X)|)$.

In the proof of Theorem 4.6 of [20], it is shown that uniformly in $X \geq 2$, one has $|A_0(X)| = O\left(\frac{X}{\sqrt{\log X}}\right)$ and $|A_1(X)| = O\left(\frac{\log \log X}{\sqrt{\log X}}\right)$. Thus $|C(X)| = O\left(\frac{X \log \log X}{\sqrt{\log X}}\right)$.

We conclude by noting that $|\mathcal{F}(X)| = \frac{3}{32}X + O(\sqrt{\log X})$ (see for example [6, §4]).
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