NONSINGULAR COSMOLOGY AND PLANCK SCALE PHYSICS

ROBERT H. BRANDENBERGER

Department of Physics
Brown University, Providence, RI 02912, USA

and

Physics Department
University of British Columbia
Vancouver, B.C. V6T 1Z1, CANADA

ABSTRACT

New Planck scale physics may solve the singularity problems of classical general relativity and may lead to interesting consequences for very early Universe cosmology. Two approaches to these questions are reviewed in this article. The first is an effective action approach to including the effects of Planck scale physics in the basic framework of general relativity. It is shown that effective actions with improved singularity properties can be constructed. The second approach is based on superstring theory. A scenario which eliminates the big bang singularity and possibly explains the dimensionality of space-time is reviewed.

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1. Introduction

Through its implications for very early Universe cosmology, Planck scale physics (and specifically string theory) might well have directly observable consequences for the physical world. The aim of this lecture is to explore some possibilities of how this may occur.

To begin, let us list a couple of problems of modern cosmology: the homogeneity and flatness problems, the singularity problem, and the origin of the dimensionality of space-time.

The Universe is observed to be isotropic on large scales to a high accuracy, the best evidence being the near isotropy of the cosmic microwave background (CMB). In standard cosmology there can be no causal explanation for these observations since the radiation was emitted from causally disconnected regions of space. The Universe is also observed to be approximately spatially flat (the density \( \rho \) lies within a factor of 5 from the critical density \( \rho_c \) which a spatially flat Universe would have). This fact is also a mystery in the context of standard cosmology, since \( \rho = \rho_c \) is an unstable fixed point in an adiabatically expanding Universe.

Particle physicists have proposed the inflationary Universe scenario\(^1\) as a solution of the homogeneity and flatness problems. However, closer analysis has shown that standard particle physics models do not yield inflation\(^2\). Inflation requires a fundamental scalar field with a reasonably flat potential (in order to have inflation) and with very small coupling constants (in order that quantum fluctuations present during inflation do not lead to CMB temperature anisotropies in excess of those recently detected\(^3\); see Appendix A). Such potentials are not generic in particle physics models.

The first challenge from cosmologists to Planck scale physics is therefore to provide a generic mechanism for inflation. It may be that Planck scale physics predicts the type of scalar field potentials for which successful inflation results. Another possibility is that Planck scale physics leads to a realization of inflation.
which does not involve scalar fields. A possible scenario for this is suggested in
Section 2. Finally, it may be that Planck scale physics leads to a solution of the
homogeneity and flatness problems which does not require inflation.

Standard cosmology is plagued by an internal inconsistency. It predicts that
the Universe started in a “Big Bang” singularity with infinite curvature and matter
temperature. However, it is known that the physics on which the standard cosmol-
ogical model is built must break down at very high temperature and curvature.
Therefore, the second challenge for Planck scale physics is to find a solution of the
singularity problem. Two very different scenarios in which this may happen are
suggested in Sections 2 and 3.

Finally, Planck scale physics (string theory as a concrete example) allows us to
ask questions about the physical world which cannot be posed in standard physics.
For example, is there a dynamical mechanism which singles out a Universe in
which three space and one time dimensions are observable? One mechanism in the
context of string theory will be reviewed in Section 3.

I will review two very different approaches to Planck scale cosmology. The first
is an attempt to incorporate Planck scale effects on the space-time structure by
writing down an effective action for the space-time metric. It will be shown that a
class of effective actions exists whose solutions have a less singular structure. More
specifically, all homogeneous and isotropic solutions are nonsingular (see Section
2).

In Section 3, I will summarize some aspects of string cosmology and indicate
how in the context of string theory the cosmological singularities can be avoided.
A dynamical mechanism which explains why at most three-spatial dimensions are
large (and thus observable) is suggested.
2. A Nonsingular Universe

2.1. Motivation

Planck scale physics will generate corrections to the Einstein action which determines the dynamics of the space-time metric $g_{\mu\nu}$. This can be seen by considering the effective action obtained by integrating out quantum matter fields in the presence of a dynamical metric, by calculating first order perturbative quantum gravity effects, or by studying the low energy effective action of a Planck scale unified theory such as string theory.

The question we wish to address in this section is whether it is possible to construct a class of effective actions for gravity which have improved singularity properties and which predict inflation, with the constraint that they give the correct low curvature limit.

What follows is a summary of recent work\textsuperscript{4−7} in which we have constructed an effective action for gravity in which all solutions with sufficient symmetry are nonsingular. The theory is a higher derivative modification of the Einstein action, and is obtained by a constructive procedure well motivated in analogy with the analysis of point particle motion in special relativity. The resulting theory is asymptotically free in a sense which will be specified below.

A possible objection to our approach is that near a singularity quantum effects will be important and therefore a classical analysis is doomed to fail. This argument is correct in the usual picture in which at high curvatures there are large fluctuations and space-time becomes more like a “quantum foam.” However, in our theory, at high curvature space-time becomes highly regular and thus a classical analysis of space-time is self-consistent. The property of asymptotic freedom is essential in order to reach this conclusion.

Our aim is to construct a theory with the property that the metric $g_{\mu\nu}$ approaches the de Sitter metric $g^{DS}_{\mu\nu}$, a metric with maximal symmetry which admits a geodesically complete and nonsingular extension, as the curvature $R$ approaches
the Planck value $R_{pl}$. Here, $R$ stands for any curvature invariant. Naturally, from our classical considerations, $R_{pl}$ is a free parameter. However, if our theory is connected with Planck scale physics, we expect $R_{pl}$ to be set by the Planck scale.

Figure 1: Penrose diagrams for collapsing Universe (left) and black hole (right) in Einstein’s theory (top) and in the nonsingular Universe (bottom). C, E, DS and H stand for contracting phase, expanding phase, de Sitter phase and horizon, respectively, and wavy lines indicate singularities.

If successful, the above construction will have some very appealing consequences. Consider, for example, a collapsing spatially homogeneous Universe. According to Einstein’s theory, this Universe will collapse in finite proper time to a final “big crunch” singularity (top left Penrose diagram of Figure 1). In our theory, however, the Universe will approach a de Sitter model as the curvature increases. If the Universe is closed, there will be a de Sitter bounce followed by re-expansion (bottom left Penrose diagram in Figure 1). Similarly, in our theory spherically symmetric vacuum solutions would be nonsingular, i.e., black holes would have no singularities in their centers. The structure of a large black hole would be unchanged compared to what is predicted by Einstein’s theory (top right, Figure 1).
outside and even slightly inside the horizon, since all curvature invariants are small in those regions. However, for $r \to 0$ (where $r$ is the radial Schwarzschild coordinate), the solution changes and approaches a de Sitter solution (bottom right, Figure 1). This would have interesting consequences for the black hole information loss problem.

To motivate our effective action construction, we turn to a well known analogy, point particle motion in the theory of special relativity.

2.2. An Analogy

The transition from the Newtonian theory of point particle motion to the special relativistic theory transforms a theory with no bound on the velocity into one in which there is a limiting velocity, the speed of light $c$ (in the following we use units in which $\hbar = c = 1$). This transition can be obtained by starting with the action of a point particle with world line $x(t)$:

$$
S_{\text{old}} = \int dt \frac{1}{2} \dot{x}^2,
$$

and adding a Lagrange multiplier which couples to $\dot{x}^2$, the quantity to be made finite, and which has a potential $V(\varphi)$:

$$
S_{\text{new}} = \int dt \left[ \frac{1}{2} \dot{x}^2 + \varphi \dot{x}^2 - V(\varphi) \right].
$$

From the constraint equation

$$
\dot{x}^2 = \frac{\partial V}{\partial \varphi},
$$

it follows that $\dot{x}^2$ is limited provided $V(\varphi)$ increases no faster than linearly in $\varphi$ for large $|\varphi|$. The small $\varphi$ asymptotics of $V(\varphi)$ is determined by demanding that
at low velocities the correct Newtonian limit results:

\[ V(\varphi) \sim \varphi^2 \text{ as } |\varphi| \to 0 , \]
\[ V(\varphi) \sim \varphi \text{ as } |\varphi| \to \infty . \] (2.4)

Choosing the simple interpolating potential

\[ V(\varphi) = \frac{2\varphi^2}{1 + 2\varphi} , \] (2.5)

the Lagrange multiplier can be integrated out, resulting in the well-known action

\[ S_{\text{new}} = \frac{1}{2} \int dt \sqrt{1 - \dot{x}^2} \] (2.6)

for point particle motion in special relativity.

2.3. CONSTRUCTION

Our procedure for obtaining a nonsingular Universe theory\(^4\) is based on generalizing the above Lagrange multiplier construction to gravity. Starting from the Einstein action, we can introduce a Lagrange multiplier \(\varphi_1\) coupled to the Ricci scalar \(R\) to obtain a theory with limited \(R\):

\[ S = \int d^4x \sqrt{-g}(R + \varphi_1 R + V_1(\varphi_1)) , \] (2.7)

where the potential \(V_1(\varphi_1)\) satisfies the asymptotic conditions (2.4).

However, this action is insufficient to obtain a nonsingular gravity theory. For example, singular solutions of the Einstein equations with \(R = 0\) are not affected at all. The minimal requirements for a nonsingular theory is that all curvature invariants remain bounded and the space-time manifold is geodesically complete. Implementing the limiting curvature hypothesis\(^9\), these conditions can be reduced to more manageable ones. First, we choose one curvature invariant \(I_1(g_{\mu\nu})\) and
demand that it be explicitly bounded, i.e., $|I_1| < I^{pl}_1$, where $I^{pl}_1$ is the Planck scale value of $I_1$. In a second step, we demand that as $I_1(g_{\mu\nu})$ approaches $I^{pl}_1$, the metric $g_{\mu\nu}$ approach the de Sitter metric $g_{\mu\nu}^{DS}$, a definite nonsingular metric with maximal symmetry. In this case, all curvature invariants are automatically bounded (they approach their de Sitter values), and the space-time can be extended to be geodesically complete.

Our approach is to implement the second step of the above procedure by another Lagrange multiplier construction$^4$. We look for a curvature invariant $I_2(g_{\mu\nu})$ with the property that

$$I_2(g_{\mu\nu}) = 0 \iff g_{\mu\nu} = g_{\mu\nu}^{DS},$$

introduce a second Lagrange multiplier field $\varphi_2$ which couples to $I_2$ and choose a potential $V_2(\varphi_2)$ which forces $I_2$ to zero at large $|\varphi_2|:

$$S = \int d^4x \sqrt{-g} [R + \varphi_1 I_1 + V_1(\varphi_1) + \varphi_2 I_2 + V_2(\varphi_2)],$$

with asymptotic conditions (2.4) for $V_1(\varphi_1)$ and conditions

$$V_2(\varphi_2) \sim \text{const as } |\varphi_2| \to \infty$$

$$V_2(\varphi_2) \sim \varphi_2^2 \text{ as } |\varphi_2| \to 0,$$

for $V_2(\varphi_2)$. The first constraint forces $I_2$ to zero, the second is required in order to obtain the correct low curvature limit.

These general conditions are reasonable, but not sufficient in order to obtain a nonsingular theory. It must still be shown that all solutions are well behaved, i.e., that they asymptotically reach the regions $|\varphi_2| \to \infty$ of phase space (or that they can be controlled in some other way). This must be done for a specific realization of the above general construction.
2.4. Specific Model

At the moment we are only able to find an invariant $I_2$ which singles out de Sitter space by demanding $I_2 = 0$ provided we assume that the metric has special symmetries. The choice

$$I_2 = (4R_{\mu\nu}R^{\mu\nu} - R^2 + C^2)^{1/2},$$  \hspace{1cm} (2.11)

singles out the de Sitter metric among all homogeneous and isotropic metrics (in which case adding $C^2$, the Weyl tensor square, is superfluous), all homogeneous and anisotropic metrics, and all radially symmetric metrics.

We choose the action\textsuperscript{4.5}

$$S = \int d^4x \sqrt{-g} \left[ R + \varphi_1 R - (\varphi_2 + \frac{3}{\sqrt{2}} \varphi_1) I_2^{1/2} + V_1(\varphi_1) + V_2(\varphi_2) \right]$$  \hspace{1cm} (2.12)

with

$$V_1(\varphi_1) = 12 H_0^2 \frac{\varphi_1^2}{1 + \varphi_1} \left( 1 - \frac{\ln(1 + \varphi_1)}{1 + \varphi_1} \right)$$  \hspace{1cm} (2.13)

$$V_2(\varphi_2) = -2\sqrt{3} H_0^2 \frac{\varphi_2^2}{1 + \varphi_2^2}. \hspace{1cm} (2.14)$$

The general equations of motion resulting from this action are quite messy. However, when restricted to homogeneous and isotropic metrics of the form

$$ds^2 = dt^2 - a(t)^2 (dx^2 + dy^2 + dz^2),$$  \hspace{1cm} (2.15)

the equations are fairly simple. With $H = \dot{a}/a$, the two $\varphi_1$ and $\varphi_2$ constraint
equations are
\[ H^2 = \frac{1}{12} V'_1 \]  
(2.16)

\[ \dot{H} = -\frac{1}{2\sqrt{3}} V'_2, \]  
(2.17)

and the dynamical \( g_{00} \) equation becomes
\[ 3(1 - 2\varphi_1)H^2 + \frac{1}{2}(V_1 + V_2) = \sqrt{3}H(\dot{\varphi}_2 + 3H\varphi_2). \]  
(2.18)

The phase space of all vacuum configurations is the half plane \( \{(\varphi_1 \geq 0, \varphi_2)\} \). Equations (2.16) and (2.17) can be used to express \( H \) and \( \dot{H} \) in terms of \( \varphi_1 \) and \( \varphi_2 \). The remaining dynamical equation (2.18) can then be recast as
\[ \frac{d\varphi_2}{d\varphi_1} = -\frac{V''_1}{4V_2} \left[ -\sqrt{3}\varphi_2 + (1 - 2\varphi_1) - \frac{2}{V'_1}(V_1 + V_2) \right]. \]  
(2.19)

The solutions can be studied analytically in the asymptotic regions and numerically throughout the entire phase space.

The resulting phase diagram of vacuum solutions is sketched in Fig. 2 (for numerical results, see Ref. 5). The point \( (\varphi_1, \varphi_2) = (0, 0) \) corresponds to Minkowski space-time \( M^4 \), the regions \( |\varphi_2| \to \infty \) to de Sitter space. As shown, all solutions either are periodic about \( M^4 \) or else they asymptotically approach de Sitter space. Hence, all solutions are nonsingular. This conclusion remains unchanged if we add spatial curvature to the model.
One of the most interesting properties of our theory is asymptotic freedom\(^5\), i.e., the coupling between matter and gravity goes to zero at high curvatures. It is easy to add matter (e.g., dust or radiation) to our model by taking the combined action

$$S = S_g + S_m, \quad (2.20)$$

where \(S_g\) is the gravity action previously discussed, and \(S_m\) is the usual matter action in an external background space-time metric.

We find\(^5\) that in the asymptotic de Sitter regions, the trajectories of the solutions in the \((\phi_1, \phi_2)\) plane are unchanged by adding matter. This applies, for example, in a phase of de Sitter contraction when the matter energy density is increasing exponentially but does not affect the metric. The physical reason for asymptotic freedom is obvious: in the asymptotic regions of phase space, the space-time curvature approaches its maximal value and thus cannot be changed even by adding an arbitrary high matter energy density.
Naturally, the phase space trajectories near \((\varphi_1, \varphi_2) = (0, 0)\) are strongly effected by adding matter. In particular, \(M^4\) ceases to be a stable fixed point of the evolution equations.

### 2.5. Connection with Dilaton Gravity

The low energy effective actions for the space-time metric in 4 dimensions which come from string theory are only known perturbatively. They contain higher derivative terms, but not if the exact same form as the ones used in our construction. The connection between our limiting curvature construction and string theory-motivated effective actions is more apparent in two space-time dimensions\(^6\).\(^7\).

The most general renormalizable Lagrangian for string-induced dilaton gravity is

\[
\mathcal{L} = \sqrt{-g}[D(\varphi)R + G(\varphi)(\nabla \varphi)^2 + H(\varphi)], \tag{2.21}
\]

where \(\varphi(x, t)\) is the dilaton. In two space-time dimensions, the kinetic term for \(\varphi\) can be eliminated, resulting in a Lagrangian (in terms of rescaled fields) of the form

\[
\mathcal{L} = \sqrt{-g}[D(\varphi)R + V(\varphi)]. \tag{2.22}
\]

We can now apply the limiting curvature construction to find classes of potentials for which the theory has nonsingular black hole\(^6\) and cosmological\(^7\) solutions. In the following, we discuss the nonsingular two-dimensional black hole.

To simplify the algebra, the dilaton is redefined such that

\[
D(\varphi) = \frac{1}{\varphi}. \tag{2.23}
\]

The most general static metric can be written as

\[
ds^2 = f(r)dt^2 - g(r)dr^2 \tag{2.24}
\]
and the gauge choice

\[ g(r) = f(r)^{-1} \]  

(2.25)
is always possible. The variational equations are

\[ f' = -V(\varphi)\frac{\varphi^2}{\varphi'} , \]  

(2.26)

\[ \left( \frac{\varphi'}{\varphi^2} \right)' = 0 \]  

(2.27)

and

\[ \varphi^{-2}R = \frac{\partial V}{\partial \varphi} , \]  

(2.28)

where a prime denotes the derivative with respect to \( r \).

Equation (2.27) can be integrated to find (after rescaling \( r \))

\[ \varphi = \frac{1}{Ar} . \]  

(2.29)

To give the correct large \( r \) behavior for the metric, we need to impose that

\[ f(r) \to 1 - \frac{2m}{r} \quad \text{as} \quad r \to \infty . \]  

(2.30)

From (2.26) this leads to the asymptotic condition

\[ V(\varphi) \to 2mA^3\varphi^2 \quad \text{as} \quad \varphi \to 0 . \]  

(2.31)
The limiting curvature hypothesis requires that \( R \) be bounded as \( \varphi \to \infty \). From (2.28) this implies

\[ V(\varphi) \to \frac{2}{\ell^2\varphi} \quad \text{as} \quad \varphi \to \infty , \]  

(2.32)

where \( \ell \) is a constant which determines the limiting curvature. As an interpolating potential we can choose

\[ V(\varphi) = \frac{2mA^3\varphi^2}{1 + mA^3\ell^2\varphi^3} , \]  

(2.33)

which allows (2.26) to be integrated explicitly\(^6\) to obtain \( f(r) \).
The resulting metric coefficient $f(r)$ describes a nonsingular black hole with a single horizon at $r \simeq 2m$. The metric is indistinguishable from the usual Schwarzschild metric until far inside of the horizon, where our $f(r)$ remains regular and obtains vanishing derivative at $r = 0$, which allows for a geodesically complete extension of the manifold.

2.6. Discussion

We have shown that a class of higher derivative extensions of the Einstein theory exist for which many interesting solutions are nonsingular. Our class of models is very special. Most higher derivative theories of gravity have, in fact, much worse singularity properties than the Einstein theory. What is special about our class of theories is that they are obtained using a well motivated Lagrange multiplier construction which implements the limiting curvature hypothesis. We have shown that

i) all homogeneous and isotropic solutions are nonsingular\textsuperscript{4,5}

ii) the two-dimensional black holes are nonsingular\textsuperscript{6}

iii) nonsingular two-dimensional cosmologies exist\textsuperscript{7}.

We also have evidence that four-dimensional black holes and anisotropic homogeneous cosmologies are nonsingular\textsuperscript{10}.

By construction, all solutions are de Sitter at high curvature. Thus, the theories automatically have a period of inflation (driven by the gravity sector in analogy to Starobinsky inflation\textsuperscript{11}) in the early Universe.

A very important property of our theories is asymptotic freedom. This means that the coupling between matter and gravity goes to zero at high curvature, and might lead to an automatic suppression mechanism for scalar fluctuations.

In two space-time dimensions, there is a close connection between dilaton gravity and our construction. In four dimensions, the connection between fundamental physics and our class of effective actions remains to be explored.
3. Aspects of String Cosmology

3.1. Motivation

In the previous section we studied effective actions for the space-time metric which might arise in the intermediate energy regime of a fundamental theory such as string theory. However, it is also of interest to explore the predictions of string theory which depend specifically on the “stringy” aspects of the theory and which are lost in any field theory limit. It is to a description of a few of the string-specific cosmological aspects to which we turn in this section.

3.2. Implications of Target Space Duality

Target space duality\textsuperscript{12} is a symmetry specific to string theory. As a simple example, consider a superstring background in which all spatial dimensions are toroidally compactified with equal radii. Let $R$ denote the radius of the torus.

The spectrum of string states is spanned by oscillatory modes which have energies independent of $R$, by momentum modes whose energies $E_n$ (with integer $n$) are

$$E_n = \frac{n}{R}, \quad (3.1)$$

and by winding modes with energies $E'_m$ (m integer)

$$E'_m = mR. \quad (3.2)$$

Target space duality is a symmetry between two superstring theories, one on a background with radius $R$, the other on a background of radius $1/R$, under which winding and momentum modes are interchanged.

Target space duality has interesting consequences for string cosmology\textsuperscript{13}. Consider a background with adiabatically changing $R(t)$. While $R(t) \gg 1$, most of the
energy in thermal equilibrium resides in the momentum modes. The position eigenstates $|x>$ are defined as in quantum field theory in terms of the Fourier transform of the momentum eigenstates $|p>$

$$|x> = \sum_p e^{ixp}|p>.$$  \hspace{1cm} (3.3)

However, for $R(t) \ll 1$, most of the energy flows into winding modes, and it takes much less energy to measure the “dual distance” $|\tilde{x}>$ than $|x>$, where

$$|\tilde{x}> = \sum_w e^{i\tilde{x}w}|w>.$$ \hspace{1cm} (3.4)

is defined in terms of the winding modes $|w>$. 

We conclude that target space duality in string theory leads to a minimum physical length in string cosmology. As $R(t)$ decreases below 1, the measured length starts to increase again. This could lead to a bouncing or oscillating cosmology$^{13}$. 

It is well known that for strings in thermal equilibrium there is a maximal temperature, the Hagedorn temperature$^{14}$. Target space duality implies that in thermal equilibrium the temperature in an adiabatically varying string background begins to decrease once $R(t)$ falls below 1:

$$T\left(\frac{1}{R}\right) = T(R).$$ \hspace{1cm} (3.5)

Thus, the $T(R)$ curve in string cosmology is nonsingular and very different from its behavior in standard cosmology. For further discussions of the thermodynamics of strings see, e.g., Refs. 15 and 16 and references therein.
3.3. Strings and Space-Time Dimensionality

Computations\(^{13}\) using the microcanonical ensemble show that for all spatial directions compactified at large total energy \(E\), the entropy \(S\) is proportional to \(E\):

\[
S = \beta_H E ,
\]

with \(\beta_H\) denoting the inverse of the Hagedorn temperature \(T_H\). Thus, the \(E(R)\) curve in string cosmology is very different from the corresponding curve in standard cosmology.

For large \(R \gg 1\), most of the energy in a gas of strings in thermal equilibrium will flow into momentum modes, and the thermodynamics will approach that of an ideal gas of radiation for which

\[
E(R) \sim \frac{1}{R} .
\]

By duality, for small \(R\)

\[
E(R) \sim R .
\]

If, however, for some reason the string gas falls out of equilibrium, the \(E(R)\) curve will look very different. Starting at \(R = 1\) with a temperature approximately equal to \(T_H\), a large fraction of the energy will reside in winding modes. If these winding modes cannot annihilate, thermal equilibrium will be lost, and the energy in winding modes will increase linearly in \(R\), and thus for large \(R\):

\[
E(R) \sim R .
\]

Newtonian intuition tells us that out of equilibrium winding modes with an energy relation (3.9) will prevent the background space from expanding\(^{13}\). The
equation of state corresponding to a gas of straight strings is

\[ p = -\frac{1}{N} \rho \quad (3.10) \]

where \( p \) and \( \rho \) denote pressure and energy density, respectively, and \( N \) is the number of spatial dimensions. According to standard general relativity, an equation of state with negative pressure will lead to more rapid expansion of the background. It turns out that the Newtonian intuition is the correct one and that general relativity gives the wrong answer\(^{17}\). At high densities, the specific stringy effects – in particular target space duality – become crucial.

The Einstein action violates duality. In order to restore duality, it is necessary to include the dilaton in the effective action for the string background. The action for dilaton gravity is

\[
S = \int d^{N+1}x \sqrt{-g}e^{-2\phi}[R + 4(D\phi)^2] \quad (3.11)
\]

where \( \phi \) is the dilaton. It is convenient to use new fields \( \varphi \) and \( \lambda \) defined by

\[
a(t) = e^\lambda t \quad (3.12)
\]

and

\[
\varphi = 2\phi - N\lambda. \quad (3.13)
\]

The action (3.11) has the duality symmetry

\[
\lambda \rightarrow -\lambda, \quad \varphi \rightarrow \varphi. \quad (3.14)
\]

The variational equations of motion derived from (3.11) for a homogeneous
and isotropic model are\textsuperscript{17,18}

\begin{align}
\dot{\varphi}^2 &= e^{\varphi} E + N \dot{\lambda}^2 \\
\ddot{\lambda} - \dot{\varphi} \dot{\lambda} &= \frac{1}{2} e^{\varphi} P \\
\ddot{\varphi} &= \frac{1}{2} e^{\varphi} E + N \dot{\lambda}^2,
\end{align}

(3.15)

where $P$ and $E$ are total pressure and energy, respectively. For a winding mode-dominated equation of state (and neglecting friction terms) the equation of motion for $\lambda(t)$ becomes

\begin{equation}
\ddot{\lambda} = -\frac{1}{2N} e^{\varphi} E(\lambda),
\end{equation}

(3.16)

which corresponds to motion in a confining potential. Hence, winding modes prevent the background toroidal dimensions from expanding.

These considerations may be used to put forward the conjecture\textsuperscript{13} that string cosmology will single out three as the maximum number of spatial dimensions which can be large ($R \gg 1$ in Planck units). The argument proceeds as follows. Space can, starting from an initial state with $R \sim 1$ in all directions, only expand if thermal equilibrium is maintained, which in turn is only possible if the winding modes can annihilate. This can only happen in at most three spatial dimensions (in a higher number the probability for intersection of the world sheets of two strings is zero). In the critical dimension for strings, $N = 3$, the evolution of a string gas has been studied extensively in the context of the cosmic string theory (see e.g., Refs. 19 and 20 for recent reviews). The winding modes do, indeed, annihilate, leaving behind a string network with about one winding mode passing through each Hubble volume. Thus, in string cosmology only three spatial dimensions will become large whereas the others will be confined to Planck size by winding modes.
4. Conclusions

Planck scale physics may have many observational consequences and may help cosmologists solve some of the deep puzzles concerning the origin of inflation, the absence of space-time singularities and the dimensionality of space-time.

A lot of work needs to be done before these issues are properly understood. I have outlined two ways to address some of these questions. The first investigation was based on classical physics and attempted to analyze what can be said about the origin of inflation and about singularities from an effective action approach to gravity. We constructed a class of higher derivative gravity actions without singular cosmological solutions (i.e., no singular homogeneous and isotropic solutions) and which automatically give rise to inflation.

The second approach was an exploration of some of the cosmological consequences of target space duality in string theory. A nonsingular cosmological scenario was proposed which might even explain why only three-spatial dimensions are large.

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Conditions for Successful Inflation

There are two sets of constraints on scalar field-driven inflationary Universe models. Firstly, the equation of state of the scalar field must be compatible with inflation. Secondly, the induced quantum fluctuations must be consistent with the limits from recent CMB anisotropy experiments.

In order for the equation of state of a scalar field $\varphi$ (assumed to be spatially homogeneous for simplicity) to be consistent with inflation, it is sufficient to require that the energy density be dominated by the potential $V(\varphi)$ and that $\varphi$ is slowly rolling, i.e., that the acceleration term $\ddot{\varphi}$ in the equation of motion

$$\ddot{\varphi} + 3H\dot{\varphi} = -V'(\varphi) \quad (A.1)$$

is negligible. In the above, $H$ is the Hubble expansion rate and a prime denotes the derivative with respect to $\varphi$.

The condition for the energy density $\rho$ to be potential dominated is that

$$\dot{\varphi}^2 \ll V(\varphi). \quad (A.2)$$

In order for this condition to be maintained over many Hubble expansion times $\Delta t = H^{-1}$, it is sufficient to require that $\ddot{\varphi}$ in (A.1) is negligible. A necessary condition for this to be the case is

$$\left| \frac{V''}{24\pi GV} \right| \ll 1, \quad (A.3)$$

$G$ being Newton’s constant. This condition is obtained by taking Equation (A.1) without the $\ddot{\varphi}$ term (i.e., in the slow rolling approximation), solving for $\dot{\varphi}$, differentiating to obtain $\ddot{\varphi}$, and requiring that the result be smaller in absolute magnitude than $V'(\varphi)$. The validity of (A.2) is assumed in the derivation.
Quantum fluctuations produced during inflation generate energy density fluctuations on scales relevant to cosmology (see e.g., Ref. 21 for a recent review). In particular, CMB anisotropies are induced. In order for these anisotropies not to exceed the recent observational results\textsuperscript{21}, the following condition on the scalar field potential must be satisfied\textsuperscript{3}

$$\frac{V(\varphi_i)}{(\Delta \varphi)^4} < 10^{-6}.$$  \hspace{1cm} (A.4)

Here, $\varphi_i$ is the value of the inflaton field at the time when perturbations on present day Hubble radius scale are produced, and $\Delta \varphi$ is the change in $\varphi$ between $\varphi_i$ and the value of $\varphi$ at the end of inflation, at which point it is assumed that the potential $V$ vanishes (see Figure 3).

In the context of standard particle physics, it is hard to satisfy all three conditions (A.2), (A.3) and, in particular, (A.4). For example, in a model of chaotic inflation with potential $V(\varphi) = \lambda \varphi^4$, the condition (A.4) implies

$$\lambda < 10^{-8}.$$  \hspace{1cm} (A.5)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{The potential energy density $V(\varphi)$ in models of new inflation (left) and chaotic inflation (right). $\varphi$ is denoted by $F$, and $\Delta \varphi$ by $DF$.}
\end{figure}

If the correct resolution of the homogeneity and flatness problems is based on some fundamental principle rather than on special choices of parameters in some rather ad hoc model, then new physics seems to be required.
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