Addendum to Supersymmetric Dyonic Black Holes in Kaluza-Klein Theory

Mirjam Cveti\v{c} * and Donam Youm†

Physics Department
University of Pennsylvania, Philadelphia PA 19104-6396
(March 1995)

Abstract

We complete the study of 4-dimensional (4-d), static, spherically symmetric, supersymmetric black holes (BH’s) in Abelian (4 + n)-d Kaluza-Klein theory, by showing that for such solutions n electric charges $\vec{Q} \equiv (Q_1, ..., Q_n)$ and n magnetic charges $\vec{P} \equiv (P_1, ..., P_n)$ are subject to the constraint $\vec{P} \cdot \vec{Q} = 0$. All such solutions can be obtained by performing the SO(n) rotations, which do not affect the 4-d space-time metric and the volume of the internal space, on the supersymmetric $U(1)_M \times U(1)_E$ BH’s, i.e., supersymmetric BH’s with a diagonal internal metric.

In Ref. [1], 4-dimensional (4-d), static, spherically symmetric, supersymmetric black holes (BH’s) in Abelian (4 + n)-d Kaluza-Klein (KK) theory were obtained with a diagonal internal metric Ansatz. Such BH’s correspond to $U(1)_M \times U(1)_E$ configurations, i.e., they have at most one electric charge $Q$ and one magnetic charge $P$ which necessarily arise from different $U(1)$ groups. Here, we complete the study of 4-d static, spherically symmetric, supersymmetric BH’s in Abelian KK theories by addressing the corresponding solutions with a general non-diagonal internal metric Ansatz. We show that n electric charges $\vec{Q} \equiv (Q_1, ..., Q_n)$ and n magnetic charges $\vec{P} \equiv (P_1, ..., P_n)$ of such BH’s are subject to the constraint $\vec{P} \cdot \vec{Q} = 0$. All such solutions can be obtained by performing the global SO(n) rotations [2] on the supersymmetric $U(1)_M \times U(1)_E$ BH solutions.

Throughout we use the notation specified in Ref. [1].

As discussed in Section 4.1 of Ref. [1], the supersymmetric BH configurations are solutions of the Killing spinor equations, which correspond to the vanishing of supersymmetric
transformations on the dimensionally reduced \((4+n)\)-d gravitino, i.e., \(\delta \psi^m_\mu = \delta \psi^m_\mu = 0\). Note, \(\psi^m_\mu\) and \(\psi^m_\mu\) \((m = 1, \ldots, 2^l \tilde{\phi})\), \(\mu = 0, \ldots, 3\), \(\tilde{\mu} = 4, \ldots, (n + 3)\) are the corresponding 4-d gravitino(s) and dilatino(s), obtained by the dimensional reduction of \((4+n)\)-d gravitino(s). (See Eqs. (3.6) and (3.7) of Ref. \[1\] for the explicit form of the corresponding supersymmetry transformations.)

For spherically symmetric configurations with the 4-d metric Ansatz given by Eq. (4.2) of Ref. \[1\] and, however, now with a general non-diagonal internal metric Ansatz, one can rewrite the \(t\)-component \(\delta \psi^m_t = 0\) of the 4-d gravitino Killing spinor equation, analogous to Eq. (4.9) in Ref. \[1\], as:

\[
R \left( \partial_r \lambda - \frac{1}{\alpha} \lambda \partial_r \varphi \right) \gamma^{\alpha \beta} \varepsilon - \sum_{a=4}^{n+3} \tilde{Q}^a (\gamma^{35} \otimes \gamma^\tilde{a}) \varepsilon = 0 ,
\]

where \(\tilde{Q}^a \equiv e^{-\frac{1}{4} \varphi} (\Phi^{-1})^\tilde{a} \tilde{Q}^\tilde{a}\). Note, \(\alpha = \sqrt{\frac{n+2}{n}}\), \(\Phi^\tilde{a}_\tilde{a} (a, \tilde{a} = 4, \ldots, n + 3)\) is the \(n\)-bein of the unimodu lar part of the internal metric, \(\varphi\) is the dilaton (the volume of the internal space), and \(\lambda\) and \(R\) are the \((t, t)\) and \((\theta, \theta)\) components of the 4-d metric \(g_{\mu \nu}\), respectively. Here, \(\tilde{Q}^a\) are electric charges obtained by integrating the Maxwell’s equation \(\partial_r (e^{\varphi} R \lambda \Phi^\tilde{a}_\tilde{a} \Phi^\tilde{b}_\tilde{b} F^\lambda_{\tilde{a} \tilde{b}}) = 0\).

The \(\theta\)-component \(\delta \psi^m_\theta = 0\) of the 4-d gravitino Killing spinor equation, analogous to Eq. (4.10), supplemented by the constraint Eq. (5.2) in Ref. \[1\], which is always satisfied by the Killing spinors of spherically symmetric configurations, assumes the following form:

\[
\left[ 2 \sqrt{R} - \sqrt{\lambda} \left( \partial_r R - \frac{1}{\alpha} R \partial_r \varphi \right) \right] \gamma^{13} \varepsilon - \sum_{a=4}^{n+3} \tilde{P}^a (\gamma^{25} \otimes \gamma^\tilde{a}) \varepsilon = 0 ,
\]

where \(\tilde{P}^a = e^{-\frac{1}{4} \varphi} \Phi^\tilde{a}_\tilde{a} \tilde{P}^\tilde{a}\). Here, \(\tilde{P}^\tilde{a}\) are magnetic charges satisfying the Maxwell’s equation \(\partial_\theta (e^{\varphi} R \lambda \sin \theta \Phi^\tilde{a}_\tilde{a} \Phi^\tilde{b}_\tilde{b} F^\lambda_{\tilde{a} \tilde{b}}) = 0\) with \(F^\lambda_{\tilde{a} \tilde{b}} = P^\lambda \sin \theta\).

Eqs. (1) and (2) can be satisfied if and only if the lower two-component spinors \(\varepsilon^m_u\) and the upper two-component spinors \(\varepsilon^m_u\) \(((\varepsilon^m)^T \equiv (\varepsilon^m_u, \varepsilon^m_\ell))\) satisfy the following constraint:

\[
\eta_Q \sum_{a=4}^{n+3} \tilde{Q}^a (\gamma^\tilde{a})^m \varepsilon^m_u = \varepsilon^m_u = i \eta_P \sum_{a=4}^{n+3} \tilde{P}^a (\gamma^\tilde{a})^m \varepsilon^m_\ell , \quad \eta_Q, P = \pm 1 ,
\]

where \(i \equiv \sqrt{-1}\). Here, \(\tilde{Q}^a \equiv \tilde{Q}^a / \left[ R \lambda^{-1/2} (\partial_r \lambda - \frac{1}{\alpha} \lambda \partial_r \varphi) \right]\) and \(\tilde{P}^a \equiv \tilde{P}^a / \left[ 2 \sqrt{R} - \sqrt{\lambda} (\partial_r R - \frac{1}{\alpha} R \partial_r \varphi) \right]\) satisfy the constraint:

\[
\sum_{a=4}^{n+3} (\tilde{Q}^a)^2 = \sum_{a=4}^{n+3} (\tilde{P}^a)^2 = 1 .
\]

Multiplying the left-most hand side of Eq. (3) by \(\sum_{a=4}^{n+3} \tilde{P}^a (\gamma^\tilde{a})\) and the right-most hand side by \(\sum_{b=4}^{n+3} \tilde{Q}^b (\gamma^\tilde{b})\), and summing the two resultant equations, along with the identity \(\{\gamma^\tilde{a}, \gamma^\tilde{b}\} = -2 \delta^{\tilde{a}\tilde{b}}\), one has the result:

\[
\sum_{a=4}^{n+3} \tilde{P}^a \tilde{Q}^a = 0 ,
\]

(5)
or equivalently:

\[ \sum_{\tilde{\pi}=4}^{n+3} P_{\tilde{\pi}} Q_{\tilde{\pi}} = \vec{\mathcal{P}} \cdot \vec{\mathcal{Q}} = 0. \] (6)

Therefore, supersymmetric BH’s have constrained charge configurations with the electric charge vector \( \vec{Q} \) and the magnetic charge vector \( \vec{P} \) orthogonal to one another.

New supersymmetric BH solutions can be generated by performing the global \( SO(n) \) rotations on known supersymmetric BH solutions. Namely, the effective 4-d KK Lagrangian density (see Eq. (3.4) of Ref. [1]) is invariant under the global \( SO(n) \) transformations [2]:

\[ \Phi_{\tilde{\lambda}}^{\tilde{a}} \rightarrow U_{\tilde{\pi}}^{\tilde{a}} \Phi_{\tilde{\lambda}}^{\tilde{a}} , \quad A_{\tilde{\lambda}}^{\tilde{a}} \rightarrow U_{\tilde{\pi}}^{\tilde{a}} A_{\tilde{\lambda}}^{\tilde{a}} , \] (7)

where \( U \) is an \( n \times n \) matrix of the \( SO(n) \) transformations. Transformations (7) affect the unimodular part of the internal metric and the gauge fields, however, they leave the 4-d space-time metric \( g_{\mu\nu} \) and the dilaton \( \varphi \) intact. In addition, since the \( SO(n) \) transformations are also the symmetry of the Killing spinor equations, the transformed solutions remain supersymmetric, i.e., they also satisfy the Killing spinor equations.

The \( SO(n) \) transformations (7) on the \( U(1)_M \times U(1)_E \) supersymmetric BH’s generate solutions with \( n \) electric \( \vec{Q} \) and \( n \) magnetic \( \vec{P} \) charges which are subject to the constraint \( \vec{P} \cdot \vec{Q} = 0 \) [2]. Therefore, those are all the 4-d static, spherically symmetric, supersymmetric BH’s in Abelian KK theories.

ACKNOWLEDGMENTS

The work is supported by U.S. DOE Grant No. DOE-EY-76-02-3071, and the NATO collaborative research grant CGR 940870.
REFERENCES

[1] M. Cvetič and D. Youm, Nucl. Phys. B\textbf{438}, 182 (1995), hep-th \# 9409119.
[2] M. Cvetič and D. Youm, UPR-645-T preprint (1994), hep-th \# 9502099.