First constraints on the running of non-Gaussianity

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We use data from the WMAP temperature maps to constrain a scale-dependent generalization of the popular ‘local’ model for primordial non-Gaussianity. In the model where the parameter $f_{NL}$ is allowed to run with scale $k$, $f_{NL}(k) = f_{NL}(k/k_{piv})^{n_{fNL}}$, we constrain the running to be $n_{fNL} = 0.30^{+1.2}_{-1.1}$ at 95% confidence, marginalized over the amplitude $f_{NL}$. The constraints depend somewhat on the prior probabilities assigned to the two parameters. In the near future, constraints from a combination of Planck and large-scale structure surveys are expected to improve this limit by about an order of magnitude and usefully constrain classes of inflationary models.

Introduction. Non-Gaussianity in the distribution of primordial density fluctuations provides a unique window into the physics of inflation. The magnitude of primordial non-Gaussianity and its dependence on scale provide information about the dynamics of scalar field(s), their interactions, and the speed of sound during inflation. Constraints on non-Gaussianity have traditionally come from the measurements of the three-point correlation function of the cosmic microwave background (CMB) temperature anisotropies. Upper limits from COBE [1] have been improved by two orders of magnitude by the WMAP experiment [2]. Moreover, clustering of galaxies and galaxy clusters has also been identified as a powerful probe of non-Gaussianity [3], already leading to interesting constraints that are complementary in their information content to the CMB measurements.

So far most attention has been devoted to the ‘local’ model of primordial non-Gaussianity, where the primordial Newtonian potential $\phi(x)$ is modified with a quadratic term: $\phi = \phi_G + f_{NL}(\phi_G^2 - \langle \phi_G^2 \rangle)$, where $\phi_G$ is a Gaussian potential [1]. The parameter $f_{NL}$ is currently constrained to be $32 \pm 21$ by WMAP ([2]; see also [5, 6]) and $28 \pm 23$ by the large-scale structure [7, 9]. Several other non-Gaussian models have been constrained as well (e.g. [10, 11]). However, the ‘running’ with physical scale of these models, which may carry important information about the number of inflationary fields and their interactions [12, 22], has not yet been constrained with current data (except for a very rough estimate of the angular-multipole dependence of $f_{NL}$ [11] and implicit constraints on a braneworld-motivated model [23]). Such constraints have only been forecasted for future experiments [24, 28]. Constraining the running of non-Gaussianity therefore presents a major new opportunity to probe inflationary physics, and is just becoming feasible. In this Letter, we present the first such constraints.

Model. In this work we consider a physically motivated generalization of the local model, where the parameter $f_{NL}$ is promoted to a function of scale $k$. In particular, we seek to constrain the two-parameter power-law subclass of the generalized models [25]

$$f_{NL}(k) = f_{NL}^*(k/k_*)^{n_{fNL}},$$

where $k_*$ is an arbitrary fixed parameter, leaving $f_{NL}^*$ and $n_{fNL}$ as the parameters of interest in this model. Such scaling is expected in inflation when more than one field dominates or when there is self-interaction, and its signatures in the CMB and LSS have been discussed in the literature [24, 25, 29]. The parameter $n_{fNL}$ is often, though certainly not always, expected to be $\lesssim O(1)$ in inflationary models, but in our phenomenological model it is allowed to take any value.

Bispectrum and $f_{NL}^*$ estimator. The primordial bispectrum of the $f_{NL}(k)$ model from Eq. (1) is straightforward to calculate:

$$F(\vec{k}_1, \vec{k}_2, \vec{k}_3) = 2[f_{NL}(k_1)P(k_2)P(k_3) + \text{perm.}],$$

where the full bispectrum is $B(\vec{k}_1, \vec{k}_2, \vec{k}_3) \equiv (2\pi)^3 \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)F(\vec{k}_1, \vec{k}_2, \vec{k}_3)$. Here $P$ is the power spectrum of the primordial curvature perturbations, and $\delta$ is the Dirac delta function.

Constraining the running parameter $n_{fNL}$ seems difficult because of the apparent requirement to find an estimator for a parameter in an exponent. To avoid this, we resort to an indirect approach where, for a fixed value of $n_{fNL}$, we estimate the parameter $f_{NL}^*$ using modifications of the well-known KSW estimator [30], which is known to be nearly optimal [31, 32]. We then iterate over the values of the running $n_{fNL}$ to obtain the full likelihood $L(f_{NL}^*, n_{fNL})$.

The theoretical expectation for the bispectrum of the temperature anisotropies in the cosmic microwave background can be explicitly evaluated, starting from the definition of the generalized non-Gaussian local model in Eq. (1) to account for the running $n_{fNL}$:

$$B^{\text{theory}}_{\ell_1 \ell_2 \ell_3} (f_{NL}^*, n_{fNL}) = 2f_{NL}^* I_{\ell_1 \ell_2 \ell_3} \times \int_0^\infty r^2 dr \left( \alpha_{\ell_1}(n_{fNL}, r)\beta_{\ell_2}(r)\beta_{\ell_3}(r) + \text{perm.} \right) \text{(3)}$$
\[ I_{\ell_1 \ell_2 \ell_3} \text{ is the Gaunt integral and} \]
\[ \alpha_\ell(r) = \frac{2}{\pi} \frac{1}{k_{\text{NL}}^{\text{piv}}} \int k^{2+n_{\text{NL}}} t_\ell(k) j_\ell(kr)dk \]
\[ \beta_\ell(r) = \frac{2}{\pi} \int k^3 P_k(k) t_\ell(k) j_\ell(kr)dk. \]

Here, \( t_\ell \) is the radiation transfer function, which can be calculated using CAMB. Following KSW, we can define new, filtered maps \( A(\tilde{\mathbf{n}}, r) \) and \( B(\tilde{\mathbf{n}}, r) \),
\[ A(\tilde{\mathbf{n}}, r) = \sum_{\ell, m} \alpha_\ell(n_{\text{NL}}, r) \frac{b_\ell}{C_\ell} a_{\ell m} Y_{\ell m}(\tilde{\mathbf{n}}), \]
\[ B(\tilde{\mathbf{n}}, r) = \sum_{\ell, m} \beta_\ell(r) \frac{b_\ell}{C_\ell} a_{\ell m} Y_{\ell m}(\tilde{\mathbf{n}}). \]

Then, we write down the skewness \( S(n_{\text{NL}}) \):
\[ S(n_{\text{NL}}) = \int r^2 dr \int d^2 \tilde{\mathbf{n}} A(\tilde{\mathbf{n}}, r) B^2(\tilde{\mathbf{n}}, r), \]

which requires \( n_{\text{NL}} \) as input (through \( A \)), and does not require \textit{a priori} knowledge of \( f_{\text{NL}} \).

The observed CMB bispectrum is defined as \( B_{\text{obs}}^{\text{obs}} = \langle a_{\ell_1 m_1} a_{\ell_2 m_2} a_{\ell_3 m_3} \rangle \), and \( S(n_{\text{NL}}) \) therefore reduces to
\[ S = \sum_{\ell_1 \leq \ell_2 \leq \ell_3} \frac{B_{\ell_1 \ell_2 \ell_3}^{\text{obs}} \tilde{B}_{\ell_1 \ell_2 \ell_3}^{\text{theory}}(n_{\text{NL}} = 1)}{C_{\ell_1} C_{\ell_2} C_{\ell_3}}, \]

where \( \tilde{B}_{\ell_1 \ell_2 \ell_3}^{\text{theory}} = b_{\ell_1} b_{\ell_2} b_{\ell_3} P_{\text{nl}}^{\text{theory}}(\tilde{\mathbf{n}}) \), and \( b_\ell \) is the beam transfer function.

We now define \( F = F(n_{\text{NL}}) \), the Fisher matrix for \( f_{\text{NL}} \), equivalent to the cumulative signal-to-noise squared of the theoretical bispectrum for \( f_{\text{NL}} = 1 \)
\[ F(n_{\text{NL}}) = \sum_{\ell_1 \leq \ell_2 \leq \ell_3} \frac{\left( \tilde{B}_{\ell_1 \ell_2 \ell_3}^{\text{theory}}(n_{\text{NL}} = 1) \right)^2}{C_{\ell_1} C_{\ell_2} C_{\ell_3}}. \]

The theoretical expectation for \( B_{\ell_1 \ell_2 \ell_3} \propto f_{\text{NL}}^2 \), so the cubic KSW estimator for \( f_{\text{NL}} \) is:
\[ \hat{f}_{\text{NL}} = \frac{S}{F}. \]

We used HEALPix, by way of HealPy, to do the forwards and backwards spherical harmonic transforms required to obtain the \( A \) and \( B \) maps.

\textbf{Cut-sky maps.} Equation (\ref{eq:cut-sky}) works well for a full-sky map, but a sky cut introduces a spurious non-Gaussian signal. To account for the masking of the CMB sky, we make the substitution \( S \rightarrow S_{\text{cut}} = S/f_{\text{sky}} + S_{\text{linear}} \). \( S_{\text{linear}} \) is an addition to skewness from Eq. (\ref{eq:skewness}), calibrated to account for partial-sky observations:
\[ S_{\text{linear}} = -\frac{1}{f_{\text{sky}}} \int r^2 dr \int d^2 \tilde{\mathbf{n}} [A(\tilde{\mathbf{n}}, r) \langle B_{\text{sim}}^3(\tilde{\mathbf{n}}, r) \rangle_{\text{MC}} + 2B(\tilde{\mathbf{n}}, r) \langle A_{\text{sim}}(\tilde{\mathbf{n}}, r) B_{\text{sim}}(\tilde{\mathbf{n}}, r) \rangle_{\text{MC}}]. \]

The subscripted filtered maps \( A_{\text{sim}} \) and \( B_{\text{sim}} \) are created from Python-produced Monte Carlo realizations of the cut CMB sky; the brackets \( \langle \rangle_{\text{MC}} \) indicate an average over all 300 Monte-Python maps. The simulated maps were produced using the prescription laid out in Appendix A of the WMAP5 paper; the only difference (aside from using the WMAP7 cosmological model) is that we used a uniform weighting for the maps, rather than the slightly more complicated weighting given there, since it only gives a marginal improvement in estimating \( f_{\text{NL}} \).

\textbf{Likelihood Evaluation.} To find the likelihood, we first find a \( \chi^2 \) statistic for \( f_{\text{NL}} \), given a value of \( n_{\text{NL}} \). Taking the angular-averaged bispectrum \( B_{\ell_1 \ell_2 \ell_3} \) as our observables, we have:
\[ \chi^2(f_{\text{NL}}, n_{\text{NL}}) = \sum_{\ell_1 \ell_2 \ell_3} \frac{(B_{\ell_1 \ell_2 \ell_3}^{\text{obs}} - f_{\text{NL}} \tilde{B}_{\ell_1 \ell_2 \ell_3}^{\text{theory}}(n_{\text{NL}} = 1))^2}{C_{\ell_1} C_{\ell_2} C_{\ell_3}}. \]

Again, this works because the theoretical expectation for \( B_{\ell_1 \ell_2 \ell_3} \propto f_{\text{NL}} \). Using Eqs. (\ref{eq:fisher}) and (\ref{eq:chi2}), we can rewrite \( \chi^2 \) as:
\[ \chi^2(f_{\text{NL}}, n_{\text{NL}}) = F \left( f_{\text{NL}} - \frac{S}{F} \right)^2 + \chi_0^2 - \frac{S^2}{F}. \]

where \( \chi_0^2 = \sum_{\ell_1 \ell_2 \ell_3} (B_{\ell_1 \ell_2 \ell_3}^{\text{obs}})^2 / (C_{\ell_1} C_{\ell_2} C_{\ell_3}) \) is the goodness-of-fit parameter for the data with respect to the \( f_{\text{NL}} = 0 \) case. Note that the numerator of \( \chi_0 \) is an observed quantity, and the denominator is based solely on the theoretical prediction for the power spectrum (as well as a few noise and beam parameters of WMAP).

Therefore, \( \chi^2_0 \) does not depend on \( f_{\text{NL}} \) or \( n_{\text{NL}} \) at all. We can use the definition of \( \hat{f}_{\text{NL}} \) in Eq. (\ref{eq:chi2}) to rewrite the expression for \( \chi^2 \) as follows:
\[ \chi^2(f_{\text{NL}}, n_{\text{NL}}) = F \left( f_{\text{NL}} - \hat{f}_{\text{NL}} \right)^2 + \chi_0 - (\hat{f}_{\text{NL}})^2 F. \]

For a fixed value of \( n_{\text{NL}} \), the \( \chi^2 \) is, as expected, minimized in \( f_{\text{NL}} \) when \( f_{\text{NL}} = \hat{f}_{\text{NL}} \), and one obtains
\[ \chi^2_{\text{min}}(n_{\text{NL}}) = \chi_0 - (\hat{f}_{\text{NL}})^2 F. \]

A more interesting task is to calculate the constraints when \( n_{\text{NL}} \) is allowed to vary. With an expression for \( \chi^2(f_{\text{NL}}, n_{\text{NL}}) \) in hand, we can write an expression for the likelihood, \( \mathcal{L}(n_{\text{NL}}, f_{\text{NL}}) \propto \exp(-\chi^2/2) \) (dropping the constant term with \( \chi_0 \))
\[ \mathcal{L}(n_{\text{NL}}, f_{\text{NL}}) \propto \exp \left[ -\frac{F (f_{\text{NL}} - \hat{f}_{\text{NL}})^2}{2} \right] \exp \left[ -\frac{(\hat{f}_{\text{NL}})^2 F}{2} \right]. \]

To marginalize over \( f_{\text{NL}} \) is also straightforward
\[ \mathcal{L}(n_{\text{NL}}) = \int \mathcal{L}(n_{\text{NL}}, f_{\text{NL}}) df_{\text{NL}} \propto \frac{1}{\sqrt{F}} \exp \left[ -\frac{(\hat{f}_{\text{NL}})^2 F}{2} \right]. \]
WMAP7 constraints on $n_{\text{fNL}}$. Figure 1 shows the likelihood $\mathcal{L}$ in the $n_{\text{fNL}} - f_{\text{NL}}^*$ plane, as well as the likelihood for $n_{\text{fNL}}$ alone, calculated from the WMAP7 temperature maps. We used a weighted and masked combination of the WMAP V and W band maps with the monopole and dipole subtracted, as recommended by the WMAP team. To extract full information from WMAP maps, we used multipoles out to $\ell_{\text{max}} = 800$ for the sums in Eqs. (6), (7) and (10). We did not find a significant improvement between $\ell_{\text{max}} = 700$ and $\ell_{\text{max}} = 800$; we chose the higher value to be conservative in our analysis.

The quantity $\chi^2$ is independent of our choice for $k_{\text{piv}}$, but the likelihood itself is not, since $F$ is inversely proportional to $k_{\text{piv}}^{2n_{\text{fNL}}}$. The true pivot scale favored by the data is the value of $k_{\text{piv}}$ for which the errors in $f_{\text{NL}}^*$ are uncorrelated with the errors in $n_{\text{fNL}}$. We find this scale by using the likelihood to calculate the covariance matrix $\mathbf{C}$ between $f_{\text{NL}}^*$ and $n_{\text{fNL}}$:

$$\mathbf{C}_{p_i,p_j} = \langle (p_i - \bar{p}_i)(p_j - \bar{p}_j) \rangle.$$

We can easily find the pivot value $k_{\text{piv}}$ that diagonalizes the covariance matrix $\mathbf{C}$ (see e.g. Ref. [26])

$$k_{\text{piv}} = k_\ast \exp \left( - \frac{C_{f_{\text{NL}}^*,n_{\text{fNL}}}}{f_{\text{NL}}^* C_{f_{\text{NL}}^*,n_{\text{fNL}}}} \right),$$

where $k_\ast$ is the (arbitrary) pivot used initially, and $f_{\text{NL}}^*$ is the corresponding value used in $\mathbf{C}$. Despite the fact that $k_\ast$ and $f_{\text{NL}}^*$ show up in the expression, $k_{\text{piv}}$ does not depend on them: it is a fixed number telling us roughly where the experiment has greatest power (and where normalization and running of $f_{\text{NL}}(k)$ are precisely uncorrelated). We find that $k_{\text{piv}}^{\text{WMAP7}} \approx 0.064 \, \text{h} \, \text{Mpc}^{-1}$. The 68%, 95%, and 99% constraints on $f_{\text{NL}}^*$ and $n_{\text{fNL}}$ are shown at the left panel of Figure 1 assuming flat priors on $f_{\text{NL}}$ and $n_{\text{fNL}}$ and $k_\ast = k_{\text{piv}}^{\text{WMAP7}} \approx 0.064 \, \text{h} \, \text{Mpc}^{-1}$.

**Dependence on the prior.** As with most present-day cosmological measurements, the precise constraints depend on the prior probability on the parameters we are constraining. Even for a simple flat prior on $f_{\text{NL}}$ and $n_{\text{fNL}}$, the actual effective prior depends on the a priori chosen pivot in wavenumber $k_\ast$. For example, a flat prior on $(f_{\text{NL}}^*)^{(1)} = f_{\text{NL}}(k_{\ast 1})$ defined at some pivot $k_{\ast 1}$ corresponds to a non-flat prior on some $(f_{\text{NL}}^*)^{(2)} = f_{\text{NL}}(k_{\ast 2})$ defined at some other pivot $k_{\ast 2}$, since $(f_{\text{NL}}^*)^{(2)} \equiv (f_{\text{NL}}^*)^{(1)}(k_{\ast 2}/k_{\ast 1})^{n_{\text{fNL}}}$. If we assume some alternate pivot $k_{\ast 2}$ but hold the flat prior in $f_{\text{NL}}^*$, the contours in the $n_{\text{fNL}} - f_{\text{NL}}^*$ plane (left panel of Fig. 1) are stretched vertically by a factor of $(k_{\ast 2}/0.064 \, \text{h} \, \text{Mpc}^{-1})^{n_{\text{fNL}}}$. We have experimented with different k-pivot values for a flat prior on $f_{\text{NL}}^*$ and $n_{\text{fNL}}$. We have also investigated other possibilities, such as the prior that assigns equal weight to each decade in $f_{\text{NL}}^*$ above 0.1 (so uniform in $\log(f_{\text{NL}}^*)$, but cut off at the arguably lowest-ever observable value of $|f_{\text{NL}}^*| = 0.1$ so that the total integrated likelihood is finite). We present the two aforementioned examples, showing constraints on $n_{\text{fNL}}$ marginalized over $f_{\text{NL}}^*$ in the right panel of Fig. 1. In the end, we decide to quote results for the flat prior and the uncorrelated $k_{\text{piv}}$ value from Eq. (19), which most closely follows priors to both non-Gaussian and other cosmological parameters applied in the literature.

Putting it all together, we can get the estimate for $n_{\text{fNL}}$ from the WMAP7 data for a flat prior on $f_{\text{NL}}^*$ at the pivot $k_{\text{piv}}$ from Eq. (19). The 68% (95%) confidence interval is

$$n_{\text{fNL}} = 0.30^{+0.78}_{-0.61} \, (1.9).$$

The current constraints are therefore fully consistent with no running, as Fig. 1 clearly indicates. Figure 2
shows the constraints in the $f_{NL}(k)$ plane together with a few representative models allowed by the data.

**Conclusions.** We have presented the first constraints on the scale-dependence of (any form of) non-Gaussianity using the WMAP7 data. The constraints are compatible with zero running, $n_{NL} = 0$, with very mild ($<1$-sigma) preference for a positive value of $n_{NL}$. We will learn more soon: the Planck data and the data from upcoming large-scale structure surveys should be able to improve constraints on the running of non-Gaussianity by about an order of magnitude [24, 27, 28], thus shedding important new light on the physics of inflation.

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