Principal Eigenvalue and Landscape Function of the Anderson Model on a Large Box

Daniel Sánchez-Mendoza

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Abstract
We state a precise formulation of a conjecture concerning the product of the principal eigenvalue and the sup-norm of the landscape function of the discrete Anderson model restricted to a large box. We first provide the asymptotic of the principal eigenvalue as the size of the box grows, and then use it to give a partial proof of the conjecture. For the one dimensional case, we give a complete proof by means of Green function bounds.

Keywords Anderson model · Principal eigenvalue · Landscape function · Green function

1 Introduction and Results

The landscape function, introduced by Filoche and Mayboroda in [1], has been conjectured to capture the low eigenvalues of the Anderson model operator, discrete or continuous, restricted to a finite large box. We can find this conjecture loosely stated in [2, Equation 1.4] as: If 0 is the minimum of the support of the potential distribution then

$$\lambda_i L_i \approx 1 + \frac{d}{4}, \quad 1 \leq i \ll n^d,$$

where \(\{\lambda_i\}_i\) are the eigenvalues ordered increasingly, \(\{L_i\}_i\) are the local maxima of the landscape function ordered decreasingly, \(d\) is the dimension, and \(n\) is the linear size on the box. Numerical experiments with Bernoulli and Uniform potential distributions support the conjecture (see [3, 4]), but to this moment there is no mathematical proof. In this article we give a precise formulation of the conjecture on the discrete setting for the case \(i = 1\), that is, for the product of the principal (smallest) eigenvalue and the sup-norm of the Landscape function on a large box. We claim such product converges almost surely to an explicit dimensional constant, different from \(1 + \frac{d}{4}\), as the size of the box goes to infinity and give the proof of the lim inf. For the special case \(d = 1\), we also give the proof of the lim sup.
We start with some definitions and notation. Given a finite set $A \subseteq \mathbb{Z}^d$ and a positive potential $W : A \rightarrow [0, \infty)$ we consider the Schrödinger operator

$$\Delta_A + W : \ell^2(A) \rightarrow \ell^2(A),$$

$$(-\Delta_A + W)\phi(x) = \sum_{|y-x|=1} [\phi(y) - \phi(x)] + W(x)\phi(x),$$

where $-\Delta_A$ has Dirichlet boundary conditions. From it, we define its principal eigenvalue and landscape function

$$\lambda_{A,W} := \inf \sigma(-\Delta_A + W), \quad L_{A,W} := (-\Delta_A + W)^{-1} \mathbb{1}_A.$$ 

Notice that $\lambda_{A,W} > 0$ and that $L_{A,W}$ is well defined on $A$, since $-\Delta_A > 0$ and $W \geq 0$.

Let $V = \{V(x)\}_{x \in \mathbb{Z}^d}$ be an independent and identically distributed (i.i.d.) random non-negative potential whose probability measure and expectation we denote $\mathbb{P}$ and $\mathbb{E}$, and define for $n \in \mathbb{N}$ the box $A_n := [-n, n]^d \cap \mathbb{Z}^d$. Our main objectives are the asymptotics of $\lambda_{A_n,V}$ and $\|L_{A_n,V}\|_\infty$ as $n \rightarrow \infty$, where, as customary, the restriction of $V$ to $A_n$ is implicit.

In addition to $V$ being non-negative (i.e., $\mathbb{P}[V(0) \in (-\infty, 0)] = 0$) we will always assume the distribution function $F(t) = \mathbb{P}[V(0) \leq t]$ satisfies one of the following mutually exclusive conditions:

(C1) $0 < F(0) < 1$, (e.g., Bernoulli($p$))

(C2) $F(t) = c t^\eta (1 + o(1))$ as $t \downarrow 0$ for some $c, \eta > 0$. (e.g., Uniform($0, 1$))

We write $n$ instead of $A_n$ whenever convenient, for instance $-\Delta_n = -\Delta_{A_n}$ and $\lambda_{n,V} = \lambda_{A_n,V}$. We denote by $\omega_d$ and $\mu_d$ respectively, the volume of the unit ball in $\mathbb{R}^d$ and the principal eigenvalue of the continuous Laplacian $(-\sum^{d}_{i=1} \partial^2/\partial x_i^2)$ on such ball with Dirichlet boundary conditions.

We now state our conjecture and results. We are always assuming that $V$ is non-negative and satisfies (C1) or (C2). We claim that:

**Conjecture 0**

$$\lim_{n \rightarrow \infty} \lambda_{n,V} \|L_{n,V}\|_\infty = \frac{\mu_d}{2d} \quad \mathbb{P}\text{-a.s.}$$

The heuristic argument behind this conjecture is that both $\lambda_{n,V}$ and $\|L_{n,V}\|_\infty$ are controlled by the largest ball inside of $A_n$ with zero or very low potential. If the radius of such ball is $r$ then, roughly, $\lambda_{n,V}$ is proportional to $r^{-2}$ and $\|L_{n,V}\|_\infty$ is proportional to $r^2$, making the product of “order one” in $r$. The appearance of the continuous constant $\frac{\mu_d}{2d}$ is another instance of the solution of a discrete problem converging to the solution of the corresponding continuous one. The disagreement between the dimensional constants $\frac{\mu_d}{2d}$ and $1 + \frac{d}{4}$ is simply explained by the fact that $1 + \frac{d}{4}$ was “guessed” from the numerical experiments, and the two constants are close to each other. For example, for $d = 1$ we have $1 + \frac{1}{4} = 1.25$ and $\frac{\mu_1}{2} = \frac{\pi^2}{8} \approx 1.23$.

Using the Min–Max Principle and our hypothesis on $V$ it is straightforward to show that $\lambda_{n,V}$ is decreasing in $n$ and converges to 0. Our first result is on the speed of this convergence; to state it we first need to define the deterministic sequences:

$$\epsilon_n := \begin{cases} 0, & \text{(C1)}, \\ (\ln n)^{-2/d}, & \text{(C2)}. \end{cases} \quad y_n := \left( \frac{d \ln n}{\omega_d |\ln F(\epsilon_n)|} \right)^{1/d}.$$ 

**Theorem 1**

$$\lim_{n \rightarrow \infty} y_n^2 \lambda_{n,V} = \mu_d \quad \mathbb{P}\text{-a.s.}$$
The proof of Theorem 1 is given in Sect. 2, and it is divided into the lim sup and lim inf bounds. The lim sup bound follows from the Min–Max Principle and the previously mentioned heuristic of the largest ball with zero or very low potential. The lim inf bound is more involved; it uses a Lifshitz tails result from [5] and the connection between the integrated density of states of the (infinite) Anderson model and the cumulative distribution function of $\lambda_{n,V}$.

Our second result is a partial proof of Conjecture 0, and a complete proof when $d = 1$.

**Theorem 2**

(i) \[ \lim_{n \to \infty} \| \lambda_{n,V} \|_{L_{\infty}} \leq \frac{\mu_d}{2d} \quad \mathbb{P}\text{-a.s.} \]

(ii) If $d = 1$ then \[ \lim_{n \to \infty} \lambda_{n,V} \| L_{n,V} \|_{L_{\infty}} = \frac{\mu_1}{2} \quad \mathbb{P}\text{-a.s.} \]

**Remark** The article [6] has a proof of (ii) in the continuous setting for the (C1) case. Both proofs follow the heuristic of the largest ball with zero potential, but differ on how to obtain a lower bound of $\lambda_{n,V}$ and an upper bound of $L_{n,V}$.

We prove Theorem 2 in Sect. 3 after deriving some general properties of landscape functions. Most notable among these properties is Proposition 9, which states that $\lambda_{A,W} \| L_{A,W} \|_{L_{\infty}}$ is bounded from above and below by two dimensional constants uniformly on $A$ and $W$. This is a consequence of an upper bound of the $\ell^\infty \to \ell^\infty$ norm of the semigroup generated by the Schrödinger operator, which we adapted from the book [7] to the discrete setting. The statement (i) of Theorem 2 follows from domain monotonicity of the landscape function and the asymptotic of $\lambda_{n,V}$ given in Theorem 1, while (ii) is based on the geometric resolvent identity and the restrictions of one dimensional geometry.

We tried to illustrate Theorem 2(ii) when $V(0) \overset{d}{=} \text{Bernoulli}(p)$ by plotting $\lambda_{n,V} \| L_{n,V} \|_{L_{\infty}}$ v.s. $n$ for a single realization of the potential. However, the plot does not show any kind of accumulation up to $n = 10^5$, suggesting the convergence is very slow. Instead, we draw the empirical distribution of $\lambda_{n,V} \| L_{n,V} \|_{L_{\infty}} - \frac{\mu_1}{2}$ from $10^5$ realizations for $n = 10^2, 10^3, 10^4, 10^5$. These are given in Fig. 1, from which we can see that the empirical distribution concentrates towards 0 as $n$ increases.

In the proofs that follow, $C$ or $C(d)$ is a finite positive constant that may only depend on the dimension and can change from line to line. By $a_t \sim b_t$ we mean $\lim_{t \to \infty} \frac{a_t}{b_t} = 1$.

**2 Principal Eigenvalue (Proof of Theorem 1)**

**2.1 The Lim Sup Bound**

The goal of this subsection is to show

\[ \lim_{n \to \infty} y_n^2 \lambda_{n,V} \leq \mu_d \quad \mathbb{P}\text{-a.s.} \quad (1) \]

As usual, getting a good upper bound on $\lambda_{n,V}$ just requires choosing a good test function and applying the Min–Max Principle.

Define $Y_n$ to be the radius of the largest open euclidean ball contained in $\Lambda_n$ in which $V$ is uniformly bounded by $\epsilon_n$, that is,

\[ Y_n := \max \left\{ r \in \mathbb{N} \mid \exists x \in \Lambda_n \text{ such that } B(x,r) \cap \mathbb{Z}^d \subseteq \Lambda_n \cap V^{-1}([0,\epsilon_n]) \right\}, \]

where $B(x,r) = \{ x' \in \mathbb{R}^d \mid |x - x'| < r \}$. Also, define $x_n \in \Lambda_n$ to be the center of a ball at which the maximum is attained (it may not be unique); and let $\phi \in \ell^2(B(x_n,Y_n) \cap \mathbb{Z}^d)$ be...
Fig. 1 Empirical distribution of $\lambda_{n,V}$ for $d=1$ and $V(0) \overset{d}{=} \text{Bernoulli}(0.3)$ computed from $10^5$ samples. The empirical mean ($m$) and empirical standard deviation ($s$) are shown in red and blue respectively.

the normalized eigenvector of $-\Delta_{B(x_n,Y_n)}$ associated to $\lambda_{B(x_n,Y_n)\cap \mathbb{Z}^d,0}$, extended to $\Lambda_n$ by 0. Then, by the Min–Max Principle, we have

$$\lambda_{n,V} \leq \langle \phi, (-\Delta_n + V) \phi \rangle_{L^2(\Lambda_n)}$$

$$= \langle \phi, (-\Delta_{B(x_n,Y_n)\cap \mathbb{Z}^d} + V) \phi \rangle_{L^2(B(x_n,Y_n)\cap \mathbb{Z}^d)}$$

$$\leq \lambda_{B(x_n,Y_n)\cap \mathbb{Z}^d,0} + \epsilon_n.$$ 

To conclude the proof of (1) we use that the event $\{\lim_{n \to \infty} Y_n/y_n = 1\}$ has probability one (this will be proven in Proposition 3), together with translation invariance and the limit

$$\lim_{r \to \infty} r^2 \lambda_{B(0,r)\cap \mathbb{Z}^d,0} = \mu_d,$$

to obtain

$$\lim_{n \to \infty} y_n^2 \lambda_{n,V} \leq \lim_{n \to \infty} y_n^2 (\lambda_{B(x_n,Y_n)\cap \mathbb{Z}^d,0} + \epsilon_n)$$

$$= \lim_{n \to \infty} y_n^2 \frac{\lambda_{B(x_n,Y_n)\cap \mathbb{Z}^d,0} + \epsilon_n}{Y_n^2} = \mu_d \mathbb{P}\text{-a.s.}$$

The limit $\lim_{r \to \infty} r^2 \lambda_{B(0,r)\cap \mathbb{Z}^d,0} = \mu_d$ is a consequence of the discrete Laplacian converging to the continuous one, or random walk converging to Brownian motion. A proof following the latter approach can be found in [8, Proposition 8.4.2], where an extra factor $d$ appears as a result of the probabilistic normalization of the Laplacian.

**Proposition 3** $Y_n \sim_{n} Y_n \mathbb{P}\text{-a.s.}$
Proof If $Y_n < y_n(1 - \delta)^{1/d}$ for some $0 < \delta < 1$, then the inscribed ball of each of the
\[
\left(\frac{2n}{y_n(1 - \delta)^{1/d}}\right)^d (1 + o(1)) \text{ disjoint cubes of side length } \left[2y_n(1 - \delta)^{1/d}\right]
\]
that make up $\Lambda_n$ contains a point $x$ with $V(x) > \epsilon_n$. Approximating the number of points in such balls by
\[
\#(B(0, r) \cap \mathbb{Z}^d) \sim_r \text{Vol}(B(0, r)) = \omega_d r^d,
\]
we obtain for large $n$
\[
\mathbb{P}\left[ Y_n < y_n(1 - \delta)^{1/d}\right] \leq \left(1 - F(\epsilon_n)^{\omega_d y_n^d (1-\delta)(1+o(1))}\right)^{\frac{n^d}{y_n^d (1-\delta)(1+o(1))}}
\]
\[
= \left(1 - \frac{1}{n^d (1-\delta)(1+o(1))}\right)^{\frac{n^d \omega_d |\ln F(\epsilon_n)|}{d n (1-\delta)(1+o(1))}}
\]
\[
\leq \exp\left(-\frac{n^d \omega_d |\ln F(\epsilon_n)|}{2d n (1-\delta)}\right),
\]
which is summable. Therefore, the Borel–Cantelli Lemma and sending $\delta \to 0$ give
\[
1 \leq \lim_{n \to \infty} y_n^{-1} Y_n \quad \mathbb{P}\text{-a.s.}
\]
We show the lim sup bound, first on an exponential sub-sequence, and then we extend it to the whole sequence. The extending argument requires a monotone sequence of random variables, which $Y_n$ may fail to be if (C2) holds. For this reason we introduce
\[
Y_{n,n'} := \max\{r \in \mathbb{N} \mid \exists x \in \Lambda_n \text{ such that } B(x, r) \cap \mathbb{Z}^d \subseteq \Lambda_n \cap V^{-1}([0, \epsilon_{n'})\}\}
\]
which is increasing on $n$, decreasing on $n'$ and satisfies $Y_{n,n} = Y_n$. Since for $\delta > 0$ and large $m$ we have
\[
\mathbb{P}\left[ Y_{\lfloor e^m \rfloor, \lfloor e^m \rfloor} > y_{\lfloor e^m \rfloor} (1 + \delta)^{1/d}\right]
\]
\[
\leq \sum_{x \in \Lambda_{\lfloor e^m \rfloor + 1}} \mathbb{P}\left[ B(x, y_{\lfloor e^m \rfloor} (1 + \delta)^{1/d}) \cap \mathbb{Z}^d \subseteq V^{-1}([0, \epsilon_{\lfloor e^m \rfloor}])\right]
\]
\[
= \#\Lambda_{\lfloor e^m \rfloor + 1} F(\epsilon_{\lfloor e^m \rfloor})^{\omega_d y_{\lfloor e^m \rfloor}^d (1+\delta)(1+o(1))}
\]
\[
= \#\Lambda_{\lfloor e^m \rfloor + 1} \frac{1}{\lfloor e^m \rfloor d (1+\delta)(1+o(1))}
\]
\[
\leq C(d) e^{-md\delta/2},
\]
the Borel–Cantelli Lemma and the limit $\delta \to 0$ give
\[
\lim_{m \to \infty} \frac{1}{y_{\lfloor e^m \rfloor}} y_{\lfloor e^m \rfloor} Y_{\lfloor e^m \rfloor, \lfloor e^m \rfloor} \leq 1 \quad \mathbb{P}\text{-a.s.}
\]
For $n \in \mathbb{N}$ define $m(n) \in \mathbb{N}$ by $\lfloor e^{m(n)} \rfloor \leq n < \lfloor e^{m(n)+1} \rfloor$. Since $y_n \sim_n Y_{\lfloor e^{m(n)} \rfloor}$ and $Y_n \leq Y_{\lfloor e^{m(n)+1} \rfloor, \lfloor e^{m(n)} \rfloor}$, we conclude
\[
\lim_{n \to \infty} y_n^{-1} Y_n \leq \lim_{n \to \infty} y_{n}^{-1} Y_{\lfloor e^{m(n)+1} \rfloor, \lfloor e^{m(n)} \rfloor}
\]
\[
= \lim_{n \to \infty} y_n^{-1} Y_{\lfloor e^{m(n)} \rfloor, \lfloor e^{m(n)} \rfloor} \leq 1 \quad \mathbb{P}\text{-a.s.}
\]
\[\square\]
2.2 The Lim Inf Bound

In this subsection we show that

$$\mu_d \leq \lim_{n \to \infty} \frac{\lambda_n^2}{n^2} \ln \lambda_n, \quad \text{P-a.s.}$$  \hspace{1cm} (2)

The main input for this is a result from [5] on the Lifshitz tail of the integrated density of states. We recall the integrated density of states of the Anderson model is a deterministic distribution function given by the \( \text{P-a.s.} \) limit

$$I(t) := \lim_{n \to \infty} \frac{1}{\#\Lambda_n} \#\{\lambda \in \sigma(-\Delta_n + V) \mid \lambda \leq t\}, \quad t \in \mathbb{R},$$

where the eigenvalues are counted with multiplicities. The central hypothesis of [5] is a scaling assumption on the cumulant-generating function of \( V(0) \), \( H(t) := \ln \mathbb{E}[e^{-t V(0)}] \), which we will check in the following proposition. To state it, we first need to define

$$1 \leq t \mapsto \alpha(t) := \begin{cases} t^{1/(d+2)}, & \text{C1}, \\ (\frac{t}{\ln t})^{1/(d+2)}, & \text{C2}, \end{cases} \quad \tilde{H} := \begin{cases} \ln F(0), & \text{C1}, \\ \frac{2n}{d+2}, & \text{C2}. \end{cases}$$

**Proposition 4** (Scaling assumption of [5]) For any compact \( K \subset (0, \infty) \) we have

$$\lim_{t \to \infty} t^{\alpha^d(t)} \frac{\alpha^d(t)}{t} H\left( \frac{t}{\alpha^d(t)} y \right) = -\tilde{H}$$

uniformly on \( y \in K \).

**Proof** First assume (C1). In this case \( \frac{\alpha^d(t)}{t} = 1 \) and \( \frac{t}{\alpha^d(t)} = t^{2/(d+2)} \). Since for \( t > 0 \) we have

$$\ln F(0) \leq H(t) = \ln \left( \mathbb{E}\left[ e^{-t V(0)} 1_{V(0) \leq \frac{1}{\sqrt{t}}} \right] + \mathbb{E}\left[ e^{-t V(0)} 1_{V(0) > \frac{1}{\sqrt{t}}} \right] \right) \leq \ln \left( F\left( \frac{1}{\sqrt{t}} \right) + e^{-\sqrt{t}} \right),$$

we conclude that

$$\sup_{y \in K} \left| \frac{\alpha^d(t)}{t} H\left( \frac{t}{\alpha^d(t)} y \right) - \ln F(0) \right| \leq \sup_{y \in K} \ln \left( \frac{1}{t^{1/(d+2)} \sqrt{y}} \right) + e^{-t^{1/(d+2)} \sqrt{y}} - \ln F(0)$$

$$= \ln \left( F\left( \frac{1}{t^{1/(d+2)} \sqrt{\min K}} \right) + e^{-t^{1/(d+2)} \sqrt{\min K}} \right) - \ln F(0) \xrightarrow{t \to \infty} 0.$$

Now assume (C2). In this case \( \frac{\alpha^d(t)}{t} = \frac{1}{\ln t} \) and \( \frac{t}{\alpha^d(t)} = t^{2/(d+2)} (\ln t)^{d/(d+2)} \). We introduce a parameter \( 0 < \delta < 1 \) and observe that

$$H(t) = \ln \left( \mathbb{E}\left[ e^{-t V(0)} 1_{V(0) \leq t^{-\delta}} \right] + \mathbb{E}\left[ e^{-t V(0)} 1_{V(0) > t^{-\delta}} \right] \right) \leq \ln \left( F\left( t^{-\delta} \right) + e^{-t^{1-\delta}} \right), \quad t > 0,$$
which implies
\[
\lim_{t \to \infty} \sup_{y \in K} \frac{\alpha^{d+2}(t)}{t} H \left( \frac{t}{\alpha^{d}(t)} y \right) \\
\leq \lim_{t \to \infty} \sup_{y \in K} \frac{1}{\ln t} \ln \left( F \left( \left[ \frac{t}{\alpha^{d}(t)} \right]^{-\delta} \right) \right) + \exp \left( - \left[ \frac{t}{\alpha^{d}(t)} \right]^{-1} \right) \\
= \lim_{t \to \infty} \frac{1}{\ln t} \ln \left( F \left( \left[ \frac{t}{\alpha^{d}(t)} \min K \right]^{-\delta} \right) \right) + \exp \left( - \left[ \frac{t}{\alpha^{d}(t)} \min K \right]^{-1} \right) \\
= \frac{-2\delta\eta}{d+2} - 2\eta.
\]

For the \( \lim_{t \to \infty} \inf_{y \in K} \) we use
\[
H(t) = \ln \left( \mathbb{E} \left[ e^{-tV(0)} \mathbb{1}_{V(0) \leq t^{-1}} \right] + \mathbb{E} \left[ e^{-tV(0)} \mathbb{1}_{V(0) > t^{-1}} \right] \right) \\
\geq \ln \left( e^{-1} F(t^{-1}) \right), \quad t > 0,
\]
to obtain
\[
\lim_{t \to \infty} \inf_{y \in K} \frac{\alpha^{d+2}(t)}{t} H \left( \frac{t}{\alpha^{d}(t)} y \right) \geq \lim_{t \to \infty} \inf_{y \in K} \frac{1}{\ln t} \ln \left( e^{-1} F \left( \left[ \frac{t}{\alpha^{d}(t)} \right]^{-1} \right) \right) \\
\geq \lim_{t \to \infty} \frac{1}{\ln t} \ln \left( e^{-1} F \left( \left[ \frac{t}{\alpha^{d}(t)} \max K \right]^{-1} \right) \right) \\
= \frac{-2\eta}{d+2}.
\]

\[\square\]

Having checked the scaling assumption on \( H \), we now have the Lifshitz tail result:

**Theorem 5** [5, Theorem 1.3] Define the constant
\[
\chi := \inf_{g \in H^1(\mathbb{R}^d), \|g\|_2 = 1} \left( \|\nabla g\|^2_2 + \tilde{H} \text{Vol(supp } g) \right),
\]
then
\[
\lim_{t \downarrow 0} \frac{\ln I(t)}{t\alpha^{-1}(t^{-1/2})} = -2d^{d/2} \left( \frac{\chi}{d+2} \right)^{(d+2)/2}.
\]

**Remark** The function \( t \mapsto \alpha(t) \) is eventually increasing so \( \alpha^{-1}(t) \) is well defined for large \( t \). The original statement from [5] is far more general; our conditions on \( V \) make \( H \) fall into, what is there called, the \((\gamma = 0)\)-class.

The constant \( \chi \) can be explicitly computed by means of the Faber–Krahn inequality:

**Proposition 6** \( \chi = (d+2) \left( \frac{\tilde{H}\omega_d}{2} \right)^{2/(d+2)} \left( \frac{\mu_d}{d} \right)^{d/(d+2)} \).
Proof  Starting from $\chi = \inf_{g \in H^1(\mathbb{R}^d), \|g\|_2 = 1} \left( \|\nabla g\|_2^2 + D \text{Vol}(\text{supp } g) \right)$ we see that we only need to consider the finite volume case. Hence

$$\chi = \inf_{A \subseteq \mathbb{R}^d, \text{Vol}(A) < \infty} \inf_{g \in H^1(\mathbb{R}^d), \|g\|_2 = 1, \text{supp } g = A} \left( \|\nabla g\|_2^2 + \tilde{H} \text{Vol}(A) \right) = \inf_{A \subseteq \mathbb{R}^d, \text{Vol}(A) < \infty} \left( \mu(A) + \tilde{H} \text{Vol}(A) \right),$$

where $\mu(A)$ is the principal eigenvalue of the continuous Laplacian $(-\sum_{i=1}^d \partial^2 / \partial x_i^2)$ defined on $A$ with Dirichlet boundary conditions. The Faber-Krahn inequality states that over all domains of a given volume the one with the lowest principal eigenvalue is the ball, therefore, using $\mu(B(0,r)) = \mu_d / r^2$ and $\text{Vol}(B(0,r)) = \omega_d r^d$ we obtain

$$\chi = \inf_{0<r<\infty} \left( \frac{\mu_d}{r^2} + \tilde{H} \omega_d r^d \right).$$

Evaluating at the only critical point $r = \left( \frac{2\mu_d}{\tilde{H} \omega_d} \right)^{1/(d+2)}$ finishes the proof. \[\square\]

We now exploit the connection between $\chi$ and the distribution of $\lambda_n,V$. This is a classic argument that can be found, for instance, in [9, Equation 4.46]. We present here a slightly modified version. Let $n \in \mathbb{N}$ and define a new potential

$$V'(x) := \begin{cases} \infty, & x \in \Gamma, \\ V(x), & x \in \mathbb{Z}^d \setminus \Gamma, \end{cases}$$

$$\Gamma := \{ x = (x_1, \ldots, x_d) \in \mathbb{Z}^d \mid x_i \in (2n+2)\mathbb{Z} \text{ for some } i = 1, \ldots, d \}.$$ 

Clearly $V \leq V'$ so for any $k \in \mathbb{N}$ and $r \in \mathbb{R}$ we have

$$\frac{\# \{ \lambda \in \sigma(-\Delta(2n+2)k + V) \mid \lambda \leq t \}}{\# \Lambda_{(2n+2)k}} \geq \frac{\# \{ \lambda \in \sigma(-\Delta(2n+2)k + V') \mid \lambda \leq t \}}{\# \Lambda_{(2n+2)k}},$$

where $-\Delta(2n+2)k + V'$ has (by definition) Dirichlet boundary conditions at $\Gamma$. These Dirichlet boundary conditions at $\Gamma$ imply that $-\Delta(2n+2)k + V'$ is a direct sum of $(2k)^d$ independent terms, all equal in distribution to $-\Delta_n + V$. Therefore, by taking the limit $k \to \infty$ on the above inequality and applying the Law of Large Numbers, we obtain

$$I(t) \geq \left( \lim_{k \to \infty} \frac{(2k)^d}{\# \Lambda_{(2n+2)k}} \right) \mathbb{E} \left[ \# \{ \lambda \in \sigma(-\Delta_n + V) \mid \lambda \leq t \} \right] \geq \left( \frac{1}{2n+2} \right)^d \mathbb{P} \left[ \lambda_{n,V} \leq t \right].$$

From the previous inequality, Theorem 5 and Proposition 6 we have

$$\mathbb{P} \left[ \lambda_{n,V} \leq t \right] \leq C(d)n^d I(t) \leq C(d)n^d \exp \left[ -f(1/t)(1 + o(1)) \right] \quad \text{as } t \downarrow 0, \quad (3)$$

where we have introduced $f(t) := \frac{\tilde{H} \omega_d \mu_d^{d/2} \alpha^{-1}(t^{1/2})}{t}$. To finish the proof we need the asymptotic of $f^{-1}(t)$ as $t \to \infty$:

**Proposition 7** (i) For (C1), $f^{-1}(t) = \frac{1}{\mu_d} \left( \frac{t}{\omega_d |\ln F(0)|} \right)^{2/d}$.
(ii) For (C2), \( f^{-1}(t) \sim_{t} \frac{1}{\mu_{d}} \left( \frac{d t}{2\eta \omega_{d} \ln t} \right)^{2/d} \).

**Proof** For (C1) there is nothing to prove since \( f(t) = \omega_{d} |\ln F(0)| \mu_{d}^{d/2} t^{d/2} \).

For (C2) we have

\[
\begin{align*}
f(t) &= \frac{k\alpha^{-1}(t^{1/2})}{t} = k t^{d/2} \ln \alpha^{-1}(t^{1/2}),
\end{align*}
\]

with all the constants collected in \( k = \frac{2\eta \omega_{d} \mu_{d}^{d/2}}{d+2} \). Since \( \alpha \) is eventually increasing and has infinite limit, the same is true for \( f \), in particular \( f^{-1}(t) \) exists for large \( t \). By solving for the \( \alpha^{-1} \) term in the first equality above, applying \( \alpha \) and simplifying some exponents we arrive at

\[
\begin{align*}
f^{-1}(t) &= \left( \frac{t}{k \ln \left[ tf^{-1}(t)/k \right]} \right)\frac{2}{d}.
\end{align*}
\]

In order to deal with the \( \ln \left[ tf^{-1}(t) \right] \) term above, we multiplying the equality by \( t \) and then take the logarithm to obtain

\[
\ln \left[ tf^{-1}(t) \right] = \frac{d+2}{d} \ln t - \frac{2}{d} \ln \left[ k \ln \left[ tf^{-1}(t)/k \right] \right].
\]

Since \( tf^{-1}(t) \xrightarrow[t \to \infty]{} \infty \), the last equality implies \( \ln \left[ tf^{-1}(t) \right] \sim_{t} \frac{d+2}{d} \ln t \) and therefore

\[
\begin{align*}
f^{-1}(t) &\sim_{t} \left( \frac{d t}{(d+2)k \ln t} \right)\frac{2}{d} = \frac{1}{\mu_{d}} \left( \frac{d t}{2\eta \omega_{d} \ln t} \right)^{2/d}.
\end{align*}
\]

\[\square\]

Going back to (3) with \( n = |e^{m}| \) and \( t = 1/f^{-1}(1+\delta)dm \) for some \( m \in \mathbb{N} \) and \( \delta > 0 \), we see that

\[
\begin{align*}
P \left[ \lambda_{[e^{m}],V} f^{-1}((1+\delta)dm) \leq 1 \right] &\leq C(d) \left( |e^{m}| \right)^{d} \exp \left[ -(1+\delta)dm(1+o(1)) \right] \\
&\leq C(d) e^{-md\delta/2},
\end{align*}
\]

which is summable over \( m \in \mathbb{N} \). Therefore, by the Borel–Cantelli Lemma we have

\[
1 \leq \lim_{m \to \infty} \lambda_{[e^{m}],V} f^{-1}((1+\delta)dm) = (1+\delta)^{2/d} \lim_{m \to \infty} \lambda_{[e^{m}],V} f^{-1}(dm) \quad \mathbb{P}\text{-a.s.}
\]

As in the proof of Proposition 3, we define \( m(n) \in \mathbb{N} \) by \( \left[ e^{m(n)} \right] \leq n < \left[ e^{m(n)+1} \right] \), so that \( \ln n \sim_{n} (m(n) + 1) \). Since \( n \mapsto \lambda_{n,V} \) is monotone decreasing we have

\[
\lim_{n \to \infty} \lambda_{n,V} f^{-1}(d \ln n) \geq \lim_{n \to \infty} \lambda_{[e^{m(n)+1}],V} f^{-1}(d \ln n) = \lim_{n \to \infty} \lambda_{[e^{m(n)+1}],V} f^{-1}(d(m(n) + 1)) \geq (1+\delta)^{-2/d} \quad \mathbb{P}\text{-a.s.}
\]

The proof of (2) is finished by sending \( \delta \to 0 \) and noticing that Proposition 7 implies \( f^{-1}(d \ln n) \sim_{n} \frac{\lambda_{n,V}}{\mu_{d}} \) for both (C1) and (C2).
3 Landscape Function

We start this section by deriving some general properties of landscape functions.

For a finite set $A \subseteq \mathbb{Z}^d$ and a positive potential $W : A \rightarrow [0, \infty)$ we introduce the Green function with 0 as spectral parameter

$$G_{A,W}(x, y):= \begin{cases} \{\delta_x, (-\Delta_A + W)^{-1}\delta_y\}_{l^2(A)}, & (x, y) \in A \times A, \\ 0, & (x, y) \in (\mathbb{Z}^d \times \mathbb{Z}^d) \setminus (A \times A). \end{cases}$$

This function is known to be symmetric, non-negative, and decreasing on the potential $W$. It is also known to satisfy the geometric resolvent identity:

$$G_{A,W}(x, y) = G_{A',W}(x, y) + \sum_{(i,j) \in \partial A'} G_{A',W}(x, i) G_{A,W}(j, y), \quad A' \subseteq A,$$

where $\partial A':=\{(i, j) \in A \times (\mathbb{Z}^d \setminus A) \mid |i - j| = 1\}$ is the boundary of $A'$ (see [10, Section 5.3]). By extending the definition of $L_{A,W}$ to $L_{A,W}(x):=\sum_{y \in \mathbb{Z}^d} G_{A,W}(x, y)$ for all $x \in \mathbb{Z}^d$, the previously stated properties of $G_{A,W}$ translate into non-negativity, potential monotonicity and domain monotonicity of landscape functions:

- $L_{A,W} \geq 0$ and $L_{A,W}(x) = 0$ if $x \in \mathbb{Z}^d \setminus A$.
- If $0 \leq W' \leq W$ then $L_{A,W} \leq L_{A,W'}$.
- If $A' \subseteq A$ then

$$L_{A,W}(x) = L_{A',W}(x) + \sum_{(i,j) \in \partial A'} G_{A',W}(x, i) L_{A,W}(j) \geq L_{A',W}(x). \quad (4)$$

Our last general property is that $\lambda_{A,W} \|L_{A,W}\|_\infty$ is bounded from above and below by two positive constants uniformly on $A$ and $W$. This is based on the following upper bound of the $\ell^\infty \rightarrow \ell^\infty$ norm of the semigroup, which can be found, for the continuous setting, in [7, Chapter 3, Theorem 1.2]. We could not find a proof in the literature for the discrete case, so we provide one in Appendix A.

**Theorem 8** For a finite $A \subseteq \mathbb{Z}^d$ and $W : A \rightarrow [0, \infty)$ we have

$$\|\exp (-t[\Delta_A + W]) 1_A\|_\infty \leq C(d) \left(1 + \left[\lambda_{A,W}t\right]^{d/2}\right) e^{-\lambda_{A,W}t}, \quad t \geq 0.$$ 

As an an immediate corollary we obtain:

**Proposition 9** For a finite $A \subseteq \mathbb{Z}^d$ and $W : A \rightarrow [0, \infty)$ we have

$$1 \leq \lambda_{A,W} \|L_{A,W}\|_\infty \leq C(d).$$

**Remark** The bound 1 is sharp. It is attained when $A$ is a single point of $\mathbb{Z}^d$.

**Proof** For the upper bound we use Theorem 8 and the substitution $u = \lambda_{A,W}t$:

$$\|L_{A,W}\|_\infty = \left\|\int_0^\infty \exp (-t[\Delta_A + W]) 1_A \, dt\right\|_\infty \leq C(d) \int_0^\infty \left(1 + \left[\lambda_{A,W}t\right]^{d/2}\right) \exp (-\lambda_{A,W}t) \, dt = \frac{C(d)}{\lambda_{A,W}} \int_0^\infty \left(1 + u^{d/2}\right) e^{-u} \, du = \frac{C(d)}{\lambda_{A,W}}.$$
For the lower bound we just need to notice that the positivity of $G_{A,W}$ implies
\[ \| (-\Delta_A + W)^{-1} \|_{\ell^\infty(A) \to \ell^\infty(A)} = \| (-\Delta_A + W)^{-1} \mathbb{1}_A \|_\infty = \| L_{A,W} \|_\infty, \]
and therefore, denoting $\phi \in \ell^2(A)$ an eigenvector of $-\Delta_A + W$ associated to $\lambda_{A,W}$, we obtain
\[ \frac{1}{\lambda_{A,W}} = \frac{\| (-\Delta_A + W)^{-1}\phi \|_\infty}{\| \phi \|_\infty} \leq \| (-\Delta_A + W)^{-1} \|_{\ell^\infty(A) \to \ell^\infty(A)} = \| L_{A,W} \|_\infty. \]
\[ \square \]

3.1 Proof of Theorem 2(i)

We start with the asymptotic of the sup-norm of the landscape function on balls with 0 potential.

**Proposition 10** \[ \| L_{B(0,r) \cap \mathbb{Z}^d,0} \|_\infty \sim_r \frac{r^2}{2d}. \]

**Proof** Let $r > 0$ and consider the function $\phi_r(x) := \frac{r^2 - |x|^2}{2d}$ defined on $\mathbb{Z}^d$. Clearly $-\Delta \phi_r(x) = 1$ for all $x \in \mathbb{Z}^d$ and therefore $L_{B(0,r) \cap \mathbb{Z}^d,0} - \phi_r$ is harmonic in $B(0,r) \cap \mathbb{Z}^d$. By the Maximum Principle (see [8, Theorem 6.2.1]) we have
\[ \left| \| L_{B(0,r) \cap \mathbb{Z}^d,0} \|_\infty - \frac{r^2}{2d} \right| = \left| \sup_{x \in B(0,r) \cap \mathbb{Z}^d} L_{B(0,r) \cap \mathbb{Z}^d,0}(x) - \sup_{x \in B(0,r) \cap \mathbb{Z}^d} \phi_r(x) \right| \leq \sup_{x \in B(0,r) \cap \mathbb{Z}^d} \left| L_{B(0,r) \cap \mathbb{Z}^d,0}(x) - \phi_r(x) \right| \leq \sup_{x \in \partial^+ [B(0,r) \cap \mathbb{Z}^d]} \left| \phi_r(x) \right| \leq C(d) r (1 + o(1)) \]
where $\partial^+ A := \{ x \in \mathbb{Z}^d \setminus A \mid \exists y \in A \text{ such that } |x - y| = 1 \}$ is the outer boundary of $A \subseteq \mathbb{Z}^d$. Dividing by $\frac{r^2}{2d}$ and taking the limit $r \to \infty$ give the proposition. \[ \square \]

Recall the definitions of $Y_n$ and $x_n$ from Sect. 2.1. From domain monotonicity of landscape functions we have
\[ L_{n,V} \geq L_{B(x_n,Y_n) \cap \mathbb{Z}^d,V}. \]
For (C1), $V$ is identically 0 in $B(x_n,Y_n) \cap \mathbb{Z}^d$ so Theorem 1, translation invariance and Propositions 3, 10 give
\[ \lim_{n \to \infty} \lambda_{n,V} \| L_{n,V} \|_\infty \geq \lim_{n \to \infty} \lambda_{n,V} \| L_{B(x_n,Y_n) \cap \mathbb{Z}^d,0} \|_\infty = \lim_{n \to \infty} \frac{\mu_d}{\mu_d Y_n^2} = \frac{\mu_d}{2d} \text{ P-a.s.} \]
For (C2), we use the second resolvent identity, domain monotonicity of the eigenvalue, and Propositions 3, 9, 10 to obtain
\[ \lambda_{n,V} \| L_{B(x_n,Y_n) \cap \mathbb{Z}^d,0} - L_{B(x_n,Y_n) \cap \mathbb{Z}^d,V} \|_\infty \]
Proposition 11

Let \( a \) the values of the potential.

This concludes the proof of Theorem 2(i).

\begin{proof}

By translation invariance and isotropy of the Laplacian, it is enough to show the first

which implies

\[
\lim_{n \to \infty} \lambda_n, V \left\| L_n, V \right\|_\infty \geq \lim_{n \to \infty} \lambda_n, V \left\| L_B(x_n, Y_n) \cap \mathbb{Z}^d, V \right\|_\infty
\]

This concludes the proof of Theorem 2(ii).

3.2 Proof of Theorem 2(ii)

We assume from this point on that \( d = 1 \). We set \([a, b] := [a, b] \cap \mathbb{Z}\) for any \( a, b \in \mathbb{Z}, a \leq b \).

This proof is based on the following deterministic bound of the Green function in terms of the values of the potential.

Proposition 11

Let \( a, b \in \mathbb{Z}, a \leq b \) and \( W : [a, b] \to [0, \infty) \). For any \( x \in [a, b] \) we have

\[
G_{[a, b], W}(a, x) \leq \left( \sum_{j=0}^{x-a} (x - a + 1 - j) W(x - j) \right)^{-1},
\]

\[
G_{[a, b], W}(x, b) \leq \left( \sum_{j=0}^{b-x} (b - x + 1 - j) W(x + j) \right)^{-1}.
\]

\begin{proof}

By translation invariance and isotropy of the Laplacian, it is enough to show the first inequality assuming that \( a = 1, b \geq 1 \), \( W : [1, b] \to [0, \infty) \).

Fix some \( x \in [1, b] \). By potential monotonicity and Cramer’s rule we have

\[
G_{[1, b], W}(1, x) \leq G_{[1, b], W_{[1, x]}}(1, x) = \frac{\det \left( \left[-\Delta_{[1, b]} + W_{[1, x]} \right]_{1 \to \delta_x} \right)}{\det \left( \left[-\Delta_{[1, b]} + W_{[1, x]} \right] \right)},
\]

where \([-\Delta_{[1, b]} + W_{[1, x]}]_{1 \to \delta_x}\) is the matrix obtained by replacing the first column of \(-\Delta_{[1, b]} + W_{[1, x]}\) by \( \delta_x \). Computing the determinant on the numerator from its first column we see that

\[
\det \left( \left[-\Delta_{[1, b]} + W_{[1, x]} \right]_{1 \to \delta_x} \right) = (-1)^{x+1} \det \left( \frac{T}{-\Delta_{[1, b-x]}} \right) = (-1)^{x+1} \det(T) \det(-\Delta_{[1, b-x]}) = b - x + 1,
\]

since \( T \) is a lower triangular square matrix of size \( x - 1 \) with \( (-1) \) on all the diagonal, and \( \det(-\Delta_{[1, k]}) = k + 1 \) for all \( k \in \mathbb{N} \). Here we have used that the determinant of an empty matrix is 1.

Consider \( \det(-\Delta_{[1, b]} + W_{[1, x]}) \) as a polynomial in \( (W(j))_{j=1}^x \). It is clear that it does not contain squares, or greater powers, of any \( W(j) \). Moreover, a straightforward computation

\(\square\)
shows that the coefficient of $W(j_1)W(j_2)\cdots W(j_{k-1})W(j_k)$, with $1 \leq j_1 < j_2 < \cdots j_{k-1} < j_k \leq x$ and $1 \leq k \leq x$, is

$$
\det(-\Delta_{[1,j_1-1]}) \det(-\Delta_{[j_1+1,j_2-1]}) \cdots \det(-\Delta_{[j_{k-1}+1,j_k-1]}) \det(-\Delta_{[j_k+1,b]}) = j_1(j_2 - j_1) \cdots (j_k - j_{k-1})(b - j_k + 1).
$$

The remaining coefficient (the constant one) is $\det(-\Delta_{[1,b]}) = b + 1$, which means all coefficients of $\det(-\Delta_{[1,b]} + W\mathcal{I}_{[1,x]})$ are positive. Therefore, by keeping only the linear terms, we obtain

$$
G_{[1,b],W}(1,x) \leq \frac{b - x + 1}{\det(-\Delta_{[1,b]} + W\mathcal{I}_{[1,x]})} \leq \frac{b - x + 1}{\sum_{j=1}^{x} j(b - j + 1)W(j)} \leq \left( \sum_{j=1}^{x} jW(j) \right)^{-1} = \left( \sum_{j=0}^{x-1} (x - j)W(x - j) \right)^{-1}.
$$

With the previous proposition in mind we define for $\delta > 0$ and $x \in \mathbb{Z}^d$

$$
Z^+_{\delta}(x) := \min \left\{ n \in \mathbb{N} \mid \sum_{j=1}^{n} (n + 1 - j)V(x + j) > \delta^{-1} \right\},
$$

$$
Z^-_{\delta}(x) := \min \left\{ n \in \mathbb{N} \mid \sum_{j=1}^{n} (n + 1 - j)V(x - j) > \delta^{-1} \right\},
$$

$$
A_{\delta}(x) := \|x - Z^-_{\delta}(x), x + Z^+_{\delta}(x)\|.
$$

Notice that $V(x)$ is not included in the definition of $Z^\pm_{\delta}(x)$ and therefore $Z^+_{\delta}(x)$ and $Z^-_{\delta}(x)$ are independent for all $x \in \mathbb{Z}$. Moreover, $Z^\pm_{\delta}(x)$ is equal in distribution to $Z^+_{\delta}(0)$ for all $x \in \mathbb{Z}$.

It follows from (4), the definitions above, potential monotonicity, and Propositions 9, 11 that

$$
\lambda_{n,V} \|L_{n,V}\|_\infty \leq \lambda_{n,V} \max_{x \in \Lambda_n} \left[ L_{A_{\delta}(x),0}(x) + 2\delta \|L_{n,V}\|_\infty \right] \leq \lambda_{n,V} \max_{x \in \Lambda_n} \|L_{A_{\delta}(x),0}\|_\infty + 2\delta C.
$$

By domain monotonicity and translation invariance, the last maximum above is attained at the $x \in \Lambda_n$ that also maximises $\#A_{\delta}(x) = Z^+_{\delta}(x) + Z^-_{\delta}(x) + 1$. Moreover, $V$ being i.i.d. implies

$$
P\left[ \lim_{n \to \infty} \max_{x \in \Lambda_n} Z^+_{\delta}(x) + Z^-_{\delta}(x) = \infty \right] = 1,
$$

and therefore Theorem 1 and Proposition 10 give

$$
\lim_{n \to \infty} \lambda_{n,V} \|L_{n,V}\|_\infty \leq \frac{\mu_1}{2} \lim_{n \to \infty} \left( \frac{\max_{x \in \Lambda_n} Z^+_{\delta}(x) + Z^-_{\delta}(x)}{2y_n} \right)^2 + 2\delta C \text{ P-a.s.}
$$

The proof of Theorem 2(ii) is finished with the next proposition followed by the limit $\delta \to 0$. 

\begin{flushright}
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Proposition 12  For all $\delta > 0$, $\lim_{n \to \infty} \frac{1}{2y_n} \max_{x \in \Lambda_n} [Z_\delta^+(x) + Z_\delta^-(x)] \leq 1$ a.s.

Proof  We will prove this over an exponential subsequence by means of the Borel–Cantelli Lemma; the extension to the whole sequence is done as in the proof of Proposition 3 using the monotonicity of $n \mapsto \max_{x \in \Lambda_n} [Z_\delta^+(x) + Z_\delta^-(x)]$.

Assume (C1). For all $t > 0$ we have

$$E \left[ e^{-t V(0)} \right] = E \left[ e^{-t V(0)} 1_{V(0) \leq \frac{1}{\sqrt{t}}} \right] + E \left[ e^{-t V(0)} 1_{V(0) > \frac{1}{\sqrt{t}}} \right] \leq F \left( \frac{1}{\sqrt{t}} \right) + e^{-\sqrt{t}}.$$ 

With this, we use the exponential Markov inequality and independence to obtain

$$\mathbb{P} \left[ Z_\delta^+(0) > n \right] = \mathbb{P} \left[ \sum_{j=1}^{n} (n+1-j) V(x+j) \leq \delta^{-1} \right] \leq e^{t/\delta} \prod_{j=1}^{n} \mathbb{E} \left[ \exp(-tj V(0)) \right] \leq e^{t/\delta} \left( F \left( \frac{1}{\sqrt{t}} \right) + e^{-\sqrt{t}} \right)^n, \quad n \in \mathbb{N}.$$ 

Now we proceed with the distribution of $Z_\delta^+(0) + Z_\delta^-(0)$ as

$$\mathbb{P} \left[ Z_\delta^+(0) + Z_\delta^-(0) > n \right]$$ 

$$= \mathbb{P} \left[ Z_\delta^+(0) > n-1 \right] + \sum_{j=1}^{n-1} \mathbb{P} \left[ Z_\delta^+(0) = j \right] \mathbb{P} \left[ Z_\delta^-(0) > n-j \right]$$ 

$$\leq 2 \mathbb{P} \left[ Z_\delta^+(0) > n-1 \right] + \sum_{j=2}^{n-1} \mathbb{P} \left[ Z_\delta^+(0) > j-1 \right] \mathbb{P} \left[ Z_\delta^-(0) > n-j \right]$$ 

$$\leq 2e^{t/\delta} \left( F \left( \frac{1}{\sqrt{t}} \right) + e^{-\sqrt{t}} \right)^{n-1} + (n-2)e^{2t/\delta} \left( F \left( \frac{1}{\sqrt{t}} \right) + e^{-\sqrt{t}} \right)^{n-1}$$ 

$$\leq ne^{2t/\delta} \left( F \left( \frac{1}{\sqrt{t}} \right) + e^{-\sqrt{t}} \right)^{n-1}.$$ 

For any $\epsilon > 0$ define $t(\epsilon)$ by $\ln \left( F \left( \frac{1}{\sqrt{t(\epsilon)}} \right) + e^{-\sqrt{t(\epsilon)}} \right) \leq \frac{\ln F(0)}{1+\epsilon}$ so that

$$\mathbb{P} \left[ \max_{x \in \Lambda_n} [Z_\delta^+(x) + Z_\delta^-(x)] > \left( 1 + \epsilon \right)^2 2y_n \right]$$ 

$$\leq Cn \mathbb{P} \left[ Z_\delta^+(0) + Z_\delta^-(0) > \left( 1 + \epsilon \right)^2 2y_n \right]$$ 

$$\leq Ce^{2t(\epsilon)/\delta} n \exp \left[ (1 + \epsilon)^2 2y_n \frac{\ln F(0)}{1+\epsilon} (1+o(1)) \right]$$ 

$$= Ce^{2t(\epsilon)/\delta} n^{-\epsilon(1+o(1))},$$

which is summable over the exponential subsequence $n = [e^m], m \in \mathbb{N}$.

Assume (C2). We follow the same steps as for (C1) above. To bound the Laplace transform of $V(0)$ we consider the function $f(t) := a[F(t)]^{1/\delta}$ for some $a > 0$. From (C2) follows that there exists $t_0 \in (0, \infty)$ such that $F(t) \leq 2 c t^{1/3}$ for all $t \in [0, t_0]$. Therefore, by choosing
a := (t_0^{-1} + (2c)^{1/n})^{-1} we obtain

\[ 0 \leq f(t) \leq \begin{cases} a(2c)^{1/n}t \leq t, & t \in [0, t_0], \\ a \leq t_0 \leq t, & t \in (t_0, \infty). \end{cases} \]

Moreover, since \( P[f(V(0)) \leq t] = \left( \frac{t}{a} \right)^y \) for \( t \in [0, a] \), we have

\[
\mathbb{E}[\exp(-tV(0))] \leq \mathbb{E}[\exp(-tf(V(0)))] = \frac{\eta}{(a)^y} \int_0^a e^{-ty} y^{\eta-1} \, dy
\]

\[ \leq \frac{\eta}{(a)^y} \int_0^\infty e^{-ty} y^{\eta-1} \, dy = \frac{\eta \Gamma(y)}{(at)^y}, \quad t > 0. \]

The exponential Markov inequality at \( t = n \eta \delta \), independence, and the Stirling bound \((n/e)^n \leq n!\) lead us to

\[
P[Z_\delta^+(0) > n] \leq e^{\epsilon/\delta} \prod_{j=1}^n \mathbb{E}[\exp(-t j V(0))] \leq \left( \frac{\eta \Gamma(y)}{(at)^y} \right)^n e^{\epsilon/\delta} (n!)^n
\]

\[ \leq \left( \frac{\eta^{1-y} \Gamma(y) e^{2\eta}}{a^n \delta} \right)^n n^{-2\eta n} =: K_\delta^n n^{-2\eta n}, \quad n \in \mathbb{N}, \]

from which follows

\[
P[Z_\delta^+(0) + Z_\delta^-(0) > n]
\]

\[ \leq 2P[Z_\delta^+(0) > n - 1] + \sum_{j=2}^{n-1} P[Z_\delta^+(0) > j - 1] P[Z_\delta^-(0) > n - j]
\]

\[ \leq 2K_\delta^{n-1}(n - 1)^{-2\eta(n-1)} + K_\delta^{n-1} \sum_{j=2}^{n-1} (j - 1)^{-2\eta(j-1)} (n - j)^{-2\eta(n-j)}. \]

The function \([2, n - 1] \ni j \mapsto (j - 1)^{-(j-1)} (n - j)^{-(n-j)}\) attains its unique maximum at \( j = (n + 1)/2 \), therefore

\[
P[Z_\delta^+(0) + Z_\delta^-(0) > n]
\]

\[ \leq 2K_\delta^{n-1}(n - 1)^{-2\eta(n-1)} + (4^{\eta} K_\delta)^{n-1}(n - 2)(n - 1)^{-2\eta(n-1)}
\]

\[ \leq (4^{\eta} K_\delta)^{n-1} n(n - 1)^{-2\eta(n-1)}. \]

Finally, for any \( \epsilon > 0 \) we have

\[
P\left[ \max_{x \in \Lambda_n} [Z_\delta^+(x) + Z_\delta^-(x)] > [(1 + \epsilon)2y_n] \right]
\]

\[ \leq Cn P[Z_\delta^+(0) + Z_\delta^-(0) > [(1 + \epsilon)2y_n]]
\]

\[ = Cn \exp\left[ -(1 + \epsilon)4\eta y_n (\ln y_n)(1 + o(1)) \right]
\]

\[ = Cn \exp\left[ -(1 + \epsilon)4\eta y_n (\ln \ln n)(1 + o(1)) \right]
\]

\[ = Cn^{-\epsilon(1+o(1))}, \]

which is summable over the exponential subsequence \( n = [e^m], m \in \mathbb{N} \).
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## Appendix A Proof of Theorem 8

Let \((X_t)_{t \geq 0}\) be a continuous time simple symmetric random walk on \(\mathbb{Z}^d\) with jump intensity 1, and let \(P_x, E_x\) be the associated probability measure and expectation conditioned on \(X_0 = x\). We remark that \((X_t)_{t \geq 0}\) is the Markov process of generator \(-\Delta/(2d)\) on \(\ell^2(\mathbb{Z}^d)\).

For a finite \(A \subseteq \mathbb{Z}^d\) and \(W : A \to [0, \infty)\), the Feynman-Kac formula lets us write the semigroup generated by \(-\Delta_A + W\) acting on as

\[
\exp \left( -\frac{t}{2d} [\Delta - A + W] \right) \phi(x) = E_x \left[ \phi(X_t) \exp \left( -\int_0^t \frac{W(X_s)}{2d} \, ds \right) 1_{t < \tau_A} \right],
\]

where \(\tau_A := \inf\{t \geq 0 \mid X_t \notin A\}\) is the exit time of \(A\). To simplify notation we set \(\lambda = \frac{\Delta_A W}{2d}\) and \(K_t = \exp \left( -\frac{t}{2d} [\Delta - A + W]\right)\). Depending on \(\lambda\) we distinguish two cases.

**Case 1:** \(\lambda \leq \frac{1}{d}\) 

Let \(\overline{B}_\infty(x,r) := \{y \in \mathbb{R}^d \mid \|x - y\|_\infty \leq r\}\). For \(t \geq 1, x \in A\) and \(r = 2t \sqrt{\frac{x}{\lambda d}}\) we decompose \(K_{t/\lambda} \mathbb{1}_A(x)\) as

\[
K_{t/\lambda} \mathbb{1}_A(x) = K_{t/\lambda} \mathbb{1}_{A \cap \overline{B}_\infty(x,r)}(x) + K_{t/\lambda} \mathbb{1}_{A \setminus \overline{B}_\infty(x,r)}(x)
\]

and bound each term as follows.

For the first term use that \(t \mapsto K_t\) is a semigroup to obtain

\[
K_{t/\lambda} \mathbb{1}_{A \cap \overline{B}_\infty(x,r)}(x) = \left\langle \delta_x, K_{t/\lambda} \mathbb{1}_{A \cap \overline{B}_\infty(x,r)} \right\rangle_{\ell^2(A)}
\]

\[
= \left\langle K_{t/\lambda} \delta_x, K_{\lambda(t-1)/\lambda} \mathbb{1}_{A \cap \overline{B}_\infty(x,r)} \right\rangle_{\ell^2(A)}
\]

\[
\leq \left\| K_{t/\lambda} \delta_x \right\|_{\ell^2(A)} \left\| \mathbb{1}_{A \cap \overline{B}_\infty(x,r)} \right\|_{\ell^2(A)} \left\| K_{\lambda(t-1)/\lambda} \right\|_{\ell^2(A) \to \ell^2(A)}
\]

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We can estimate $P_0 \left[ X_{2/\lambda} = 0 \right]$ using the characteristic function of $X_s$, which is $\phi_s(\theta) = \exp \left( -s + \frac{s}{d} \sum_{i=1}^{d} \cos(\theta_i) \right)$, by means of

$$P_0 \left[ X_s = 0 \right] = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \phi_s(\theta) \, d\theta = \frac{e^{-s}}{(2\pi)^d} \left[ \int_{[-\pi,\pi]} \exp \left( \frac{s}{d} \cos(\theta_1) \right) \, d\theta_1 \right]^d.$$ 

Laplace’s method applied to the integral inside the square brackets yields

$$P_0 \left[ X_s = 0 \right] \sim \frac{1}{(2\pi)^d} \left[ \int_{[-\pi,\pi]} \exp \left( \frac{s}{d} \cos(\theta_1) \right) \, d\theta_1 \right]^d \rightarrow e^{-\frac{s}{d}},$$

and therefore $P_0 \left[ X_{2/\lambda} = 0 \right] \leq C(d) \lambda^{d/2}$.

For the second term we use [11, Lemma 4.6] which states that

$$P[S_n > y] \leq e^{-y^2/(2n)}, \quad y > 0,$$

where $S_n$ is a discrete time simple symmetric random walk on $\mathbb{Z}$ starting at 0, and $P$ is its probability measure. Recalling that the first component of $X_t$, which we denote $X_t^1$, is a continuous time simple symmetric random walk on $\mathbb{Z}$ with jump intensity $1/d$ we have

$$K_{t/\lambda} \mathbb{1}_{A \setminus B_\infty(x,r)}(x) \leq P_0 \left[ X_{t/\lambda} \notin B_\infty(0, r) \right] \leq 2dP_0 \left[ X_{t/\lambda}^1 > r \right] \leq 2d \sum_{n \geq 0} e^{-t/(\lambda d)} \left( \frac{t}{\lambda d} \right)^n P[S_n > r].$$

We split the series at $n = \frac{2et}{\lambda d}$ and bound the two terms separately:

$$\sum_{n \leq \frac{2et}{\lambda d}} \frac{e^{-t/(\lambda d)}}{n!} \left( \frac{t}{\lambda d} \right)^n P[S_n > r] \leq \sum_{n \leq \frac{2et}{\lambda d}} \frac{e^{-t/(\lambda d)}}{n!} \left( \frac{t}{\lambda d} \right)^n e^{-r^2/(2n)} \leq \exp \left( -\frac{r^2 \lambda d}{4e} \right) \sum_{n \leq \frac{2et}{\lambda d}} \frac{e^{-t/(\lambda d)}}{n!} \left( \frac{t}{\lambda d} \right)^n \leq e^{-t},$$

$$\sum_{n \geq \frac{2et}{\lambda d}} \frac{e^{-t/(\lambda d)}}{n!} \left( \frac{t}{\lambda d} \right)^n P[S_n > r] \leq e^{-t/(\lambda d)} \sum_{n \geq \frac{2et}{\lambda d}} \frac{1}{n!} \left( \frac{t}{\lambda d} \right)^n \leq e^{-t/(\lambda d)} \sum_{n \geq 0} \frac{1}{n!} \left( \frac{n}{2e} \right)^n \leq Ce^{-t}.$$ 

With this we have shown $\| K_{t/\lambda} \mathbb{1}_A \|_\infty \leq C(d) \left( 1 + r^{d/2} \right) e^{-t}$ for $t \geq 1$. Since $K_{t/\lambda} \mathbb{1}_A(x)$ is always bounded by 1 we can add $(\inf_{0 \leq t \leq 1} \left( 1 + r^{d/2} \right) e^{-t})^{-1}$ to $C(d)$, if necessary, to have

$$\| K_{t/\lambda} \mathbb{1}_A \|_\infty \leq C(d) \left( 1 + r^{d/2} \right) e^{-t}, \quad t \geq 0.$$ 

Replacing $t$ by $2d\lambda t$ and gives the desired bound.

**Case 2:** $\lambda \geq \frac{1}{d}$. 
This case follows from the heat kernel bound

\[ P_0 [X_t = y] \leq C(d) \exp \left[ -t - \| y \|_1 \ln \left( \frac{\| y \|_1}{e t} \right) \right], \quad \| y \|_1 \geq e t, \]

taken from [11, Theorem 5.17]. We proceed as before but now we use \( \overline{B_1}(x, r) := \{ y \in \mathbb{R}^d \mid \| x - y \|_1 \leq r \} \). For \( t \geq 0 \) and \( r = \lambda t d e^2 \) we have

\[
K_t \mathbb{1}_A = K_t \mathbb{1}_{A \cap \overline{B_1}(x, r)} + K_t \mathbb{1}_{A \setminus \overline{B_1}(x, r)}(x)
\]

\[
\leq \left( \delta_x, K_t \mathbb{1}_{A \cap \overline{B_1}(x, r)} \right)_{L^2(A)} + P_0 \left[ X_t \notin \overline{B_1}(0, r) \right]
\]

\[
\leq C(d) r^{d/2} e^{-\lambda t} + P_0 \left[ X_t \notin \overline{B_1}(0, r) \right].
\]

Clearly \( r \geq e t \), so we can apply the heat kernel bound to obtain

\[
P_0 \left[ X_t \notin \overline{B_1}(0, r) \right] \leq C(d) \sum_{y \in \mathbb{Z}^d, \| y \|_1 > r} \exp \left[ -t - \| y \|_1 \ln (e d) \right]
\]

\[
\leq C(d) e^{-r} \sum_{y \in \mathbb{Z}^d, \| y \|_1 > r} \exp \left[ -\| y \|_1 \right]
\]

\[
= C(d) e^{-r} \leq C(d) e^{-\lambda t},
\]

and therefore \( \| K_t \mathbb{1}_A \|_\infty \leq C(d) (1 + [\lambda t]^{d/2}) e^{-\lambda t} \). Replacing \( t \) by \( 2dt \) and gives the desired bound.

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