Asymptotic Analysis for Optimal Dividends in a Dual Risk Model

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Abstract

The dual risk model is a popular model in finance and insurance, which is often used to model the wealth process of a venture capital or high tech company. Optimal dividends have been extensively studied in the literature for a dual risk model. It is well known that the value function of this optimal control problem does not yield closed-form solutions except in some special cases. In this paper, we study the asymptotics of the optimal dividend problem when the parameters of the model go to either zero or infinity. Our results provide insights to the optimal strategies and the optimal values when the parameters are extreme.

1 Introduction

In a classic risk model in the insurance literature, the surplus process increases continuously in time with the constant premium rate and decreases due to the claims that follow a compound Poisson process. In a dual risk model (see, e.g., Avanzi et al. \cite{2}), the opposite happens, that is, the surplus process decreases continuously in time with the constant rate, and increases according to a compound Poisson process. A dual risk model can be used to model wealth of a venture capital, where the running cost is deterministic whereas the revenues are stochastic, see, e.g., \cite{1, 2, 3, 4, 7, 8, 11, 13, 14, 16, 17} etc.

In the pioneering work by Avanzi et al. \cite{2}, they studied the optimal dividend problem in a dual risk model, that is, the optimal dividend strategy for maximizing the expected payments of all the future dividends to the shareholders until the time of the ruin. They proved that the optimal strategy is a barrier strategy, that is, there exists an optimal barrier below which no dividend is paid out and when the wealth process jumps above the barrier, all the surplus is paid out as the dividends to the shareholders immediately and the surplus drops to the level of the barrier.

There have been many related works of Avanzi et al. \cite{2}. In Avanzi et al. \cite{3}, they studied a dividend barrier strategy for a dual risk model whereby dividend decisions are made only periodically, but ruin is still allowed to occur in continuous time. Ng \cite{13} studied a dual model with a threshold dividend strategy, with exponential interclaim times. In another

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related work, Afonso et al. [1] studied the connections between dual and classical risk models and used that to compute various quantities of interest. Cheung and Drekic [8] considered the dividend moments in a dual risk model. They derived integro-differential equations for the moments of the total discounted dividends which can be solved explicitly assuming the jump size distribution has a rational Laplace transform. The expected discounted dividends assuming the profits follow a Phase Type distribution were studied in Rodríguez-Martínez et al. [16]. The Laplace transform of the ruin time, expected discounted dividends for the Sparre-Andersen dual model were derived in Yang and Sendova [17]. In Dimitrova et al. [9], the finite time distribution of the ruin time was studied for the case when the interarrival times are not independent. Bayraktar et al. [5] showed that for all spectrally positive Lévy processes, the optimal strategy is of barrier type and they provided a closed-from solution for the optimal dividend value function in terms of the inverse Laplace transform of scale functions of the Lévy process. In a separate paper [6], they extended their analysis to the case where payment of dividend carries a fixed transaction cost by using double-barrier dividend strategies, i.e., when the dual process hits above the upper barrier, as much dividend is paid as to bring the process to the lower barrier. When there is no transaction cost, the two barriers collapse to one. An iterative approach toward finding optimal barrier and value function is adopted in [10] where they showed via a fixed point theorem that the convergence of an initial guess of the barrier to the optimal one is exponentially fast, as well as the convergence of the value of the initial barrier to the value function. When there is a random delay for the innovations turned to profits, the dual risk model becomes time inhomogeneous and the ruin probabilities and the ruin time distributions are studied in Zhu [18].

Recently, Fahim and Zhu [11] considered the optimal investment on research and development to minimize the ruin probability for a dual risk model. Additional investment on a risky market index and the generalization to a state-dependent dual risk model was also considered in [11].

In a dual risk model, except for the special case including when the probability density function of the jump size in the compound Poisson process is an exponential function or a sum of exponential functions, in general, there is no closed-form formulas for the value function and the no closed-form formula for the optimal barrier.

In this paper, we are going to focus on the asymptotics for the optimal dividend problem in a dual risk model. Even though the general problem does not yield closed-form formulas, the asymptotics are very explicit and intuitive. They also provide useful insights to help us understand better the nature of the optimal dividend problem in the dual risk model. For the optimal dividend problem in the dual risk model, we know that the optimal strategy is a barrier strategy. But in practice, most of the companies pay quarterly dividends and the investors prefer continuous yield rather than the barrier dividend payments for dual risk models. One of the interesting discoveries of our paper is that in many asymptotic regimes, one can find a nearly optimal strategy that pays the continuous dividend yield when the surplus of the company is sufficiently large. The nearly optimal strategy we will present
does not pay any dividend until the surplus reaches a large value and then it starts to pay out dividend continuously. In other words, a start-up company should wait till it becomes a mature company before paying out dividend continuously. Many high-tech companies, after the successful IPO (Initial Public Offering), remain growth companies and do not pay dividends for a very long period of time, until they get more mature, or sometimes in response to big shareholders and activists’ demand. After that, they will start paying dividends continuously with dividend yields being constant, and the dividend yield usually increases modestly and consistently year over year. Therefore, the nearly optimal strategy we will present is perhaps more consistent with practice. It also suggests that the most commonly adopted dividend strategies in the corporate world may not be optimal, but at least, nearly optimal. We will show that in some asymptotic regimes, continuous dividend yield strategy can be nearly optimal, even though not exactly optimal.

We will introduce the model setup in Section 2. The paper has all the main results in Section 3 and all the proofs in Section 4. Before we proceed, let us introduce the standard notations that will be used throughout the rest of the paper. The standard notion $f \sim g$ is used to denote $\frac{f}{g}$ has limit equal to 1. We use the notation $f = O(g)$ to denote $\limsup \frac{|f|}{g} < \infty$. Finally, we use the notation $f = o(1)$ to denote that $f$ has limit equal to 0.

2  Model Setup

In this paper, we are interested in studying the dual risk model, see, e.g., [2], where the surplus or wealth process satisfies the dynamics:

$$dX_t = -\rho dt + dJ_t, \quad X_0 = x > 0, \quad (2.1)$$

where $\rho > 0$ is the running cost of the company and $J_t = \sum_{i=1}^{N_t} Y_i$ is the stream of profits, where $Y_i$ are i.i.d. $\mathbb{R}^+$ valued random variables with common probability density function $p(y), y > 0$ and $N_t$ is a Poisson process with intensity $\lambda > 0$. $Y_i$’s are often known as the innovation sizes or the random future revenues, and $\lambda$ can be interpreted as the innovation rate. In a classic risk model in the insurance literature, the surplus process increases continuously in time with the constant premium rate $\rho$ and decreases due to the claims that follow a compound Poisson process $J_t$, whereas in the dual risk model (2.1), the opposite happens, that is, the surplus process decreases continuously in time with the constant rate $\rho$, and increases according to a compound Poisson process $J_t$.

Let $\tau := \inf\{t > 0 : X_t \leq 0\}$ be the ruin time of the company. Under the assumption that $\lambda \mathbb{E}[Y_1] > \rho$, it is well known that the infinite-horizon ruin probability has the formulas $\mathbb{P}_x(\tau < \infty) = e^{-ax}$, where $x$ is the initial wealth $X_0$ of the company and $\alpha$ is the unique position value that satisfies the equation (see, e.g., [2]):

$$\rho \alpha + \lambda \int_0^\infty [e^{-\alpha y} - 1]p(y)dy = 0. \quad (2.2)$$
Similarly, one can also compute the Laplace transform of the ruin time, $E_x[e^{-\delta \tau}] = e^{-\beta x}$, where $\beta$ is the unique positive value that satisfies the equation (see, e.g., \cite{2}):

$$\beta \rho + \lambda \int_0^\infty [e^{-\beta y} - 1]p(y)dy - \delta = 0. \quad (2.3)$$

In the pioneering work by Avanzi et al. \cite{2}, they studied the optimal dividend problem in the dual risk model (2.1). Let $\delta > 0$ be the discount rate used in the discount factor and $D_t$ be the rate of the dividend payment at time $t$ by the company to the shareholders.

Given a dividend payment strategy $D \in \mathcal{D}$, where the set of admissible dividend payment strategies $\mathcal{D}$ is the collection of all adapted nondecreasing càdlàg processes, $D_t$ is the cumulated dividend until time $t$, and the wealth process is given by

$$dX_t = -\rho dt - dD_t + dJ_t, \quad X_0 = x > 0. \quad (2.4)$$

Avanzi et al. \cite{2} studied the optimal dividend strategy for maximizing the expected payments of all the future dividends to the shareholders until the time of the ruin, that is

$$V(x) := \sup_{D \in \mathcal{D}} E_x \left[ \int_0^\tau e^{-\delta t}dD_t \right], \quad (2.5)$$

with initial wealth $X_0 = x$.

They proved that the optimal strategy is a barrier strategy, that is, there exists an optimal barrier $b > 0$, such that the optimal strategy is as follows. When the wealth process is below $b$, no dividend is paid out. When the wealth process jumps above the barrier $b$ at the stopping time $\tau_b$, $X_{\tau_b} - b$ is paid out as the dividends to the shareholders immediately and the surplus drops to the level $b$. The value function $V(x)$ satisfies the equations:

$$V'(x) = 1, \quad \text{for any } x > b, \quad (2.6)$$

and for any $x < b$,

$$-\rho V'(x) + \lambda \int_0^\infty [V(x + y) - V(x)]p(y)dy - \delta V(x) = 0, \quad (2.7)$$

with $V(0) = 0$. For $x > b$, it is clear that $V(x) = x - b + V(b)$ and hence we can plug this into (2.7) and obtain

$$-\rho V'(x) - \lambda V(x) + \lambda \int_0^{b-x} V(x+y)p(y)dy + \lambda \int_{b-x}^\infty (x+y-b+V(b))p(y)dy - \delta V(x) = 0. \quad (2.8)$$

By the smooth-fit condition, i.e., $V(x)$ is $C^1$ at $x = b$, we have

$$-\rho - \lambda V(b) + \lambda \int_0^\infty yp(y)dy - \delta V(b) = 0, \quad (2.9)$$

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which gives us the value of $V(b)$:

$$V(b) = \frac{\lambda \mathbb{E}[Y_1] - \rho}{\delta}. \quad (2.10)$$

It was assumed in Avanzi et al. [2] that $\rho < \lambda \mathbb{E}[Y_1]$. Under this condition, $\mathbb{P}_{x}(\tau = \infty) > 0$ and $\mathbb{E}_{x}[\tau] = \infty$. They also assume that $\delta > 0$.

In general, the optimal value $V(x)$ has no closed-form formulas. But in some special cases, e.g., when $Y_i$ are exponentially distributed, the optimal value $V(x)$ and the optimal barrier $b$ can be computed explicitly, see, e.g., Avanzi et al. [2]. From Avanzi et al. [2] et al., when $p(y) = \nu e^{-\nu y}$, we have

$$V(x) = \frac{\lambda}{\nu} \frac{e^{rx} - e^{sx}}{(\rho r + \delta) e^{rb} - (\rho s + \delta) e^{sb}} 1_{[0,b]}(x) + (V(b) + x - b) 1_{(b,\infty)}(x), \quad (2.11)$$

where $r, s$ are the solutions of

$$\rho \xi^2 + (\lambda + \delta - \nu \rho) \xi - \nu \delta = 0, \quad (2.12)$$

and the optimal $b$ is given by

$$b = \frac{1}{r - s} \log \left( \frac{s \rho s + \delta}{r \rho r + \delta} \right). \quad (2.13)$$

Without loss of generality, we can assume that $s > r$ and therefore from (2.12), we have

$$s = -\frac{(\lambda + \delta - \nu \rho) + \sqrt{(\lambda + \delta - \nu \rho)^2 + 4 \rho \nu \delta}}{2 \rho}, \quad r = -\frac{(\lambda + \delta - \nu \rho) - \sqrt{(\lambda + \delta - \nu \rho)^2 + 4 \rho \nu \delta}}{2 \rho}. \quad (2.14)$$

If we assume that

$$\rho \geq \lambda \mathbb{E}[Y_1], \quad (2.15)$$

then $\mathbb{P}_{x}(\tau < \infty) = 1$, i.e., the ruin occurs with probability 1. Intuitively, it says that when you are certain that the company is going to get ruined, the optimal strategy to maximize the dividend payments to shareholders is to give all the surplus of the company to the shareholders immediately. Therefore, for the finite-horizon case, under assumption (2.15), we have the same conclusion. That is, for any time horizon $T > 0$,

$$\sup_{D \in D} \mathbb{E}_{x} \left[ \int_{0}^{\tau \wedge T} e^{-\delta t} dD_{t} \right] = x. \quad (2.16)$$

Notice that the assumption (2.15) is satisfied when the innovation rate $\lambda \to 0$, or when the running cost $\rho \to \infty$. Therefore, these two asymptotics are trivial. We will study instead the $\lambda \to \infty$ asymptotics and the $\rho \to 0$ asymptotics.
Also notice that under the usual condition \( \rho < \lambda \mathbb{E}[Y_1] \), \( \mathbb{P}_x(\tau = \infty) > 0 \) and \( \mathbb{E}_x[\tau] = \infty \). Therefore, if \( \delta = 0 \), then

\[
\sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^\tau dD_t \right] = \infty. \tag{2.17}
\]

That is because we can always choose a constant dividend payment strategy that is \( D_t \equiv \hat{D} \), where \( \hat{D} > 0 \) is a positive constant chosen sufficiently small so that \( \rho + \hat{D} < \lambda \mathbb{E}[Y_1] \). Then, let \( \hat{\tau} \) be the ruin time of this wealth process with \( D_t \equiv \hat{D} \), we have \( \hat{\tau} < \infty \) a.s. and \( \mathbb{E}_x[\hat{\tau}] = \infty \). Then, we have

\[
\sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^\tau dD_t \right] \geq \hat{D} \mathbb{E}_x[\hat{\tau}] = \infty. \tag{2.18}
\]

Therefore, we expect that when \( \delta \to 0 \),

\[
\sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^\tau e^{-\delta t} dD_t \right] \to \infty. \tag{2.19}
\]

To obtain a finer estimate than (2.19) and understand how fast the term in (2.19) goes to infinity, we consider the limit

\[
\lim_{\delta \to 0} \sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^\tau e^{-\delta t} dD_t \right], \tag{2.20}
\]

specifically because of its asymptotic convergence and proper scaling. We will show that the above limit is finite and, therefore, we can determine how fast the value function approaches to infinity as the discount rate \( \delta \to 0 \).

This is also of practical interest because the value function for generally distributed \( Y_i \) does not yield closed-form formula and the asymptotic behavior is particularly useful in the low interest-rate environment because a common choice of discount rate \( \delta \) is by letting \( \delta = r \), where \( r > 0 \) is the risk-free rate.

When \( \delta = 0 \), we have seen already from (2.17) that the value function is \( \infty \). But we can also study the finite-horizon case with a time horizon \( T > 0 \). In the finite-horizon case,

\[
\sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^{T \wedge \tau} dD_t \right] < \infty. \tag{2.21}
\]

But from (2.17), it is clear that

\[
\sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^{T \wedge \tau} dD_t \right] \to \infty, \tag{2.22}
\]

as \( T \to \infty \). So we are interested to study how fast this approaches to \( \infty \) as \( T \to \infty \).

When the discount rate \( \delta \to \infty \), intuitively, it is clear that the company should pay all the surplus as dividends to the shareholders immediately because the cost of carry goes to
infinity. When the time horizon \( T \to 0 \), there is little time to accumulate new wealth and what the company can pay to the shareholders is approximately the initial wealth of the company.

To summarize, in this paper, we will focus on the following asymptotic regimes: (i) Large innovation rate \((\lambda)\) regime; (ii) Small running cost \((\rho)\) regime; (iii) Small discount rate \((\delta)\) regime; (iv) Large time horizon \((T)\) regime; (v) Large discount rate \((\delta)\) regime; (vi) Small time horizon \((T)\) regime.

It is well known that, see, e.g., Avanzi et al.\(^2\), the optimal strategy for the optimal dividend problem

\[
\sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^{\tau} e^{-\delta t} dD_t \right] 
\]

is a barrier strategy. But in practice, shareholders prefer continuous dividend yield and most public companies do not use barrier dividend strategies. We will show that in some asymptotic regimes, continuous dividend yield strategy can be nearly optimal, even though not exactly optimal. More precisely, we define our nearly optimal dividend strategy \(D^{M,\epsilon}\) as follows:

**Definition 1.** The strategy \(D^{M,\epsilon} \in \mathcal{D}\) pays no dividend until \(\tau_M := \inf\{t \geq 0 : X_t \geq M\}\), i.e., the first time that the process jumps above \(M\) before the ruin time. After \(\tau_M\), a constant dividend yield \((1 - \epsilon)(\lambda \mathbb{E}[Y_1] - \rho)\) is paid out to the shareholders.

Let \(\tau_0\) be the ruin time of the process \(X_t = x - \rho t + J_t\) with no dividends. Conditional on \(\tau_M < \tau_0\), a constant dividend yield \((1 - \epsilon)(\lambda \mathbb{E}[Y_1] - \rho)\) is paid out to the shareholders until the ruin time \(\tau'\), which is defined as

\[
\tau' := \inf\{t > 0 : X'_t \leq 0\}, \tag{2.24}
\]

where

\[
dX'_t = -\rho dt - (1 - \epsilon)(\lambda \mathbb{E}[Y_1] - \rho)dt + dJ^M_t, \tag{2.25}
\]

with \(X'_0 = X_{\tau_M}\) and \(J^M_t := J_{\tau_M + t} - J_{\tau_M}\).

Let \(\tau^{M,\epsilon}\) be the ruin time of the process with dividend strategy \(D^{M,\epsilon}\). Then, when \(\tau_M > \tau_0\), we have \(\tau^{M,\epsilon} = \tau_0\) and when \(\tau_M < \tau_0\), we have \(\tau^{M,\epsilon} = \tau_M + \tau'\).

We will show that for small discount rate \(\delta\), large time horizon \(T\), and large innovation rate \(\lambda\) regimes that for sufficiently large \(M\) and sufficiently small \(\epsilon\), \(D^{M,\epsilon}\) is nearly optimal.

### 3 Main Results

We will focus on the following asymptotic regimes: Large innovation rate \((\lambda)\) regime (Section 3.1); Small running cost \((\rho)\) regime (Section 3.2); Small discount rate \((\delta)\) regime (Section 3.3); Large time horizon \((T)\) regime (Section 3.4); Large discount rate \((\delta)\) regime (Section 3.5); Small time horizon \((T)\) regime (Section 3.6). We will use the notation \(V(x; p)\) to emphasize the dependence of the value function \(V\) on the parameter \(p\); for example, for small \(\delta\) regime we use \(V(x; \delta)\).
3.1 Large Innovation Rate ($\lambda$) Regime

Let us consider the large innovation rate ($\lambda$) asymptotics in this section. This corresponds to the regime when the company has a large growth rate and fast expansions. For the moment, let us assume that $Y_i$ are exponentially distributed, say $p(y) = \nu e^{-\nu y}$. Let us recall that for $x \geq b$ we have

$$V(x; \lambda) = x - b + V(b; \lambda).$$

(3.1)

Now as $\lambda \to \infty$, we can easily see from (2.14) that $s \to 0$ and $r \to -\infty$ and by (2.13) $b \to 0$. The asymptotics of the value function is fully determined by

$$V(b; \lambda) \sim \frac{\lambda}{\nu \delta}, \quad \text{as } \lambda \to \infty. \quad (3.2)$$

More generally, we have

$$V(b; \lambda) \sim \frac{\lambda \mathbb{E}[Y_1]}{\delta}, \quad \text{as } \lambda \to \infty. \quad (3.3)$$

The intuition is the following. As the innovation rate $\lambda \to \infty$, the ruin probability will tend to zero. Let us assume that $X_0 = x$. At any given time $t$, after a small time step $\Delta t$, the expected wealth increases to $x + (\lambda \mathbb{E}[Y_1] - \rho) \Delta t$, and then at time $t + \Delta t$, you immediately pay the amount $(\lambda \mathbb{E}[Y_1] - \rho) \Delta t$ to the shareholders and then you restart with wealth $x$ and continue the process. By letting $\Delta t \to 0$, we get

$$V(x; \lambda) \sim (\lambda \mathbb{E}[Y_1] - \rho) \int_0^\infty e^{-\delta t} dt = \frac{\lambda \mathbb{E}[Y_1] - \rho}{\delta} \sim \frac{\lambda \mathbb{E}[Y_1]}{\delta}. \quad (3.4)$$

**Theorem 2.** (i) Assume the discount rate $\delta > 0$. We have

$$\sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^T e^{-\delta t} dD_t \right] = \frac{\lambda \mathbb{E}[Y_1]}{\delta} (1 + o(1)), \quad \text{as the innovation rate } \lambda \to \infty. \quad (3.5)$$

(ii) Assume the discount rate $\delta > 0$, for any time horizon $T > 0$, we have

$$\sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^{T \wedge \tau} e^{-\delta t} dD_t \right] = \frac{\lambda \mathbb{E}[Y_1]}{\delta} (1 - e^{-\delta T})(1 + o(1)), \quad \text{as the innovation rate } \lambda \to \infty. \quad (3.6)$$

(iii) Assume the discount rate $\delta = 0$, for any time horizon $T > 0$, we have

$$\sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^{T \wedge \tau} dD_t \right] = \lambda \mathbb{E}[Y_1] T (1 + o(1)), \quad \text{as the innovation rate } \lambda \to \infty. \quad (3.7)$$
We can see from Theorem 2 that in the large innovation rate regime, i.e., with large \( \lambda \), the maximized expected discounted payment of all the future dividends is large, which is consistent with the intuition that innovation boosts the future value of the company and hence the payout to the shareholders. Indeed Theorem 2 implies that the maximized expected discounted payment of all the future dividends is of the order \( \lambda \), i.e., linear in the innovation rate \( \lambda \).

The optimal strategy is always the barrier strategy. But in practice, shareholders prefer continuous dividend yield. Consider a dividend strategy \( D^{M, \epsilon} \) from Definition 1 that pays continuous dividend yield \((1 - \epsilon)(\lambda\mathbb{E}[Y_1] - \rho)\) after the surplus process hits above \( M \). We will show that the \( D^{M, \epsilon} \) strategy is nearly optimal:

**Proposition 3.** Given any one of the cases in Theorem 2, for any \( \varepsilon > 0 \), there exist some \( M', \epsilon' > 0 \) such that for any \( M > M' \) and \( 0 < \epsilon < \epsilon' \), the \( D^{M, \epsilon} \) strategy is \( \varepsilon \lambda \)-optimal, i.e., for any \( \delta \geq 0 \) and \( T \in (0, \infty) \) or \( \delta > 0 \) and \( T = \infty \), there exist some \( M', \epsilon' > 0 \) such that for any \( M > M' \) and \( 0 < \epsilon < \epsilon' \),

\[
\left| \sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^{T \wedge \tau} e^{-\delta t} dD_t \right] - \mathbb{E}_x \left[ \int_0^{T \wedge \tau} e^{-\delta t} dD^{M, \epsilon}_t \right] \right| \leq \varepsilon \lambda,
\]

for any sufficiently large \( \lambda > 0 \).

We can see from Proposition 3 that \( D^{M, \epsilon} \) strategy (defined in Definition 1) is nearly optimal in the large innovation rate (\( \lambda \)) regime, that is, the company does not pay out any dividend until the surplus jumps above a high threshold \( M \), and afterwards, a constant dividend yield \((1 - \epsilon)(\lambda\mathbb{E}[Y_1] - \rho)\) is paid out to the shareholders. Managerial implication is that when the company has significant innovation power, it should focus on accumulating wealth by continuing to innovate and deferring dividend payments to the shareholders until the company has reached a considerably large scale.

**Remark 4.** It is also possible to have a discrete dividend strategy that is nearly optimal, see Remark 14.

### 3.2 Small Running Cost (\( \rho \)) Regime

Let us consider the small running cost (\( \rho \)) asymptotics in this section. We will see that when the running cost \( \rho = 0 \), for any \( x > 0 \),

\[
V(x; \rho) = x + \frac{\lambda\mathbb{E}[Y_1]}{\delta}.
\]  

(3.8)

The intuition is that since when the running cost \( \rho \to 0 \), the probability of ruin is negligible, and if the discount rate \( \delta > 0 \), it is optimal to give almost all the surplus as dividends to the shareholders immediately rather than holding it. Then, each time the surplus increases, we also pay excess surplus as dividend. More precisely, for \( \epsilon > 0 \), let \( D^\epsilon \) be the strategy
which pays dividend \(x - \epsilon\) at the beginning and any surplus above \(\epsilon\) thereafter. At time 0, with initial wealth \(x\), you give \(x - \epsilon\) dollars as amount of dividends to the shareholders. Then at time of the \(n\)th jump of the process \(J, \tau^{(n)}\), the wealth grows to \(Y_n + \epsilon\) and then you give \(Y_n\) as dividend. Therefore, when the running cost \(\rho = 0\),

\[
V(x; \rho) = x + \int_0^\infty e^{-\delta t} \lambda E[Y_1] dt = x + \frac{\lambda E[Y_1]}{\delta}. \tag{3.9}
\]

To have a more rigorous proof, notice that for \(x > b\), \(V(x; \rho) = x - b + V(b; \rho)\). Recall from (2.10) that \(V(b; \rho) = \frac{\lambda E[Y_1] - \rho}{\delta}\). Hence when the running cost \(\rho = 0\), we have (3.9). In fact, the optimal strategy for the finite horizon case should be the same.

**Theorem 5.** Assume that the running cost \(\rho = 0\).

(i) For any discount rate \(\delta > 0\), we have

\[
\sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^T e^{-\delta t} dD_t \right] = x + \frac{\lambda E[Y_1]}{\delta}. \tag{3.10}
\]

(ii) For any discount rate \(\delta > 0\) and time horizon \(T > 0\), we have

\[
\sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^{T \wedge \tau} e^{-\delta t} dD_t \right] = x + \frac{\lambda E[Y_1]}{\delta} (1 - e^{-\delta T}). \tag{3.11}
\]

(iii) When the discount rate \(\delta = 0\), for any time horizon \(T > 0\), we have

\[
\sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^{T \wedge \tau} dD_t \right] = x + \lambda E[Y_1] T. \tag{3.12}
\]

**Proposition 6.** Assume that the running cost \(\rho = 0\). Given any of the cases in Theorem 5, for any \(\epsilon > 0\), there exists an \(\epsilon > 0\) such that the strategy \(D^\epsilon\) is \(\epsilon\)-optimal, i.e., for any \(\epsilon > 0\) and for any \(\delta \geq 0\), \(T \in (0, \infty)\) or \(\delta > 0\), \(T = \infty\), there exists an \(\epsilon > 0\) such that

\[
\left| \sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^{T \wedge \tau} e^{-\delta t} dD_t \right] - \mathbb{E}_x \left[ \int_0^{T \wedge \tau} e^{-\delta t} dD_t^\epsilon \right] \right| \leq \epsilon.
\]

We can see from Proposition 6 that \(D^\epsilon\) is nearly optimal, that is, to pay dividend \(x - \epsilon\) at the beginning and any surplus above \(\epsilon\) thereafter. The intuition is that when the running cost \(\rho\) is small, so is the ruin risk. Since the future value of the payment is always smaller than the present value due to the discount rate, it is nearly optimal to pay as much dividend as possible to the shareholders at time zero.

In practice, the running cost \(\rho\) should always be positive, even though it can be small. Therefore, the discussions for the \(\rho = 0\) case in Theorem 5 and Proposition 6 serve as a first-order approximation when \(\rho > 0\) is small, instead of describing a real world scenario.
Indeed, for the special case when \( p(y) = \nu e^{-\nu y} \), we can even find the second-order approximation as the running cost \( \rho \to 0 \). Let us recall that for \( x > b \), \( V(x; \rho) = x - b + V(b; \rho) \), where \( b \) is given by (2.13). From (2.14), it is easy that see that \( r \sim \frac{-(\lambda + \delta)}{\rho} \) and \( s \sim \frac{\nu \delta}{\lambda + \delta} \). This implies that

\[
b \sim \frac{\rho}{-(\lambda + \delta)} \log \left( \frac{\nu \delta - \rho}{\lambda + \delta} \frac{\delta}{\lambda - \lambda} \right) \sim \frac{\rho \log \rho}{\lambda + \delta},
\]

as the running cost \( \rho \to 0 \). Hence, we conclude that for any \( x > 0 \),

\[
V(x; \rho) \sim x + \frac{\lambda}{\nu \delta} + \frac{\rho \log \rho}{\lambda + \delta},
\]

as the running cost \( \rho \to 0 \).

### 3.3 Small Discount Rate (\( \delta \)) Regime

Let us consider the small discount rate (\( \delta \)) asymptotics in this section. This is practically relevant when the interest rate is low, which is a new environment, e.g., after the 2008 financial crisis in the United States. Recall that \( \lambda \mathbb{E}[Y_1] > \rho \) so that \( \mathbb{P}_x(\tau = \infty) > 0 \) and \( \mathbb{E}_x[\tau] = \infty \). Therefore, by considering a constant dividend yield strategy, it easily follows that

\[
V(x; \delta) = \sup_{D \in D} \mathbb{E}_x \left[ \int_0^\tau e^{-\delta t} dD_t \right] \to \infty, \quad \text{as } \delta \to 0.
\]

We are interested to see how fast it goes to \( \infty \) as \( \delta \to 0 \).

To get some intuitions, let us first consider the case when \( p(y) = \nu e^{-\nu y} \) so that there are explicit formulas for the optimal value function and the optimal barrier. Let us recall that for \( x \leq b \),

\[
V(x; \delta) = \frac{\lambda}{\nu} \frac{e^{rx} - e^{sx}}{(\rho r + \delta) e^{rb} - (\rho s + \delta) e^{sb}},
\]

where \( r, s \) are given by (2.14) and the optimal \( b \) is given by (2.13). Therefore, as \( \delta \to 0 \), we have \( s \to 0 \) and \( r \to \nu - \frac{\lambda}{\rho} \) and \( b \to \infty \).

By the definition of the optimal \( b \), we have

\[
\frac{r}{s} (\rho r + \delta) e^{br} = (\rho s + \delta) e^{bs},
\]

This implies that for \( x \leq b \)

\[
V(x; \delta) = \frac{\lambda}{\nu} \frac{e^{rx} - e^{sx}}{1 - \frac{r}{s} (\rho r + \delta) e^{br}}.
\]

Notice that

\[
e^{br} = e^{r v\delta} \log \left( \frac{s^2(v - r)}{r^2(v - s)} \right) \sim \frac{\lambda}{\nu} \frac{\delta^2}{\nu(v - \frac{\lambda}{\rho})^2},
\]

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as \( \delta \to 0 \). Therefore, we have
\[
V(x; \delta) \sim \frac{\lambda(1 - e^{(\nu - \frac{\lambda}{\rho})x})}{\nu(\rho\nu - \lambda)} \left( \nu - \frac{\lambda}{\rho} \right)^2 \frac{1}{s},
\]
(3.20)
as \( \delta \to 0 \).

Note that \( s \sim \frac{\nu}{\lambda - \nu \rho} \delta \) as \( \delta \to 0 \). Therefore, we conclude that
\[
\lim_{\delta \to 0} \delta V(x; \delta) = \lambda - \nu \rho \left( 1 - e^{(\nu - \frac{\lambda}{\rho})x} \right).
\]
(3.21)

For generally distributed \( Y_i \), we provide some heuristic arguments. Notice that for the optimization problem
\[
V(x; \delta) = \sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^\tau e^{-\delta t} dD_t \right],
\]
(3.22)
the optimal strategy is a barrier strategy, that is,
\[
V'(x; \delta) = 1, \quad \text{for any } x > b,
\]
(3.23)
and for any \( x < b \),
\[
- \rho V'(x; \delta) + \lambda \int_0^\infty [V(x + y; \delta) - V(x; \delta)] p(y) dy - \delta V(x; \delta) = 0,
\]
(3.24)
with \( V(0; \delta) = 0 \).

As \( \delta \to 0 \), the optimal \( b \to \infty \). Therefore, for fixed \( x \), we have \( x/b \to 0 \) and \( V(x; \delta) \)
roughly satisfies the equation:
\[
- \rho V'(x; \delta) + \lambda \int_0^\infty [V(x + y; \delta) - V(x; \delta)] p(y) dy = 0,
\]
(3.25)
with \( V(0; \delta) = 0 \), which yields that
\[
V(x; \delta) \sim c(1 - e^{-\alpha x}),
\]
(3.26)
for some positive constant \( c \), where \( \alpha \) is the unique positive solution to the equation:
\[
\rho \alpha + \lambda \int_0^\infty [e^{-\alpha y} - 1] p(y) dy = 0.
\]
(3.27)

Next, let us determine the positive constant \( c \). Recall that by (2.10), for any \( x > b \),
\[
V(x; \delta) = \frac{\lambda \mathbb{E}[Y_1] - \rho}{\delta} + (x - b) + o(1).
\]
(3.28)
This implies that for \( x \) fixed and large, \( V(x; \delta) \delta \sim \lambda \mathbb{E}[Y_1] - \rho \) as \( \delta \to 0 \). Hence, we have
\[
c = \frac{\lambda \mathbb{E}[Y_1] - \rho}{\delta}
\]
and
\[
\lim_{\delta \to 0} \delta V(x; \delta) = (\lambda \mathbb{E}[Y_1] - \rho)(1 - e^{-\alpha x}).
\]
(3.29)

Indeed, we can prove it rigorously:

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Theorem 7. We have the following asymptotic result for small discount rate $\delta$:

$$\sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^T e^{-\delta t} dD_t \right] = \frac{\lambda \mathbb{E}[Y_1] - \rho}{\delta} (1 - e^{-\alpha x}) (1 + o(1)), \quad \text{as the discount rate } \delta \to 0.$$  

(3.30)

We can see from Theorem 7 that in the low interest rate environment, i.e., the small discount rate $\delta$, the maximized expected discounted payment of all the future dividends is large, which is consistent with the conventional wisdom that low interest rate environment boosts the asset values. Indeed Theorem 7 implies that the maximized expected discounted payment of all the future dividends is of the order $1/\delta$, i.e., the reciprocal of the discount rate.

Proposition 8. For any $\varepsilon > 0$, let $\delta > 0$ be such that

$$\left| \delta \sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^T e^{-\delta t} dD_t \right] - (\lambda \mathbb{E}[Y_1] - \rho)(1 - e^{-\alpha x}) \right| \leq \varepsilon.$$  

(3.31)

Then, there exist some $M', \epsilon' > 0$ such that for any $M > M'$ and $0 < \epsilon < \epsilon'$, the $D^{M,\epsilon}$ strategy is an $\frac{\delta}{\varepsilon}$-optimal strategy, i.e.,

$$\left| \delta \mathbb{E}_x \left[ \int_0^T e^{-\delta t} dD^{M,\epsilon}_t \right] - (\lambda \mathbb{E}[Y_1] - \rho)(1 - e^{-\alpha x}) \right| \leq \varepsilon.$$  

(3.32)

We can see from Proposition 8 that $D^{M,\epsilon}$ strategy (defined in Definition 1) is nearly optimal, that is, the company does not pay out any dividend until the surplus jumps above a high threshold $M$, and afterwards, a constant dividend yield $(1 - \epsilon)(\lambda \mathbb{E}[Y_1] - \rho)$ is paid out to the shareholders. Managerial implication is that in the low interest rate environment, the company should focus on accumulating wealth and deferring dividend payments until the company has reached a considerably large scale.

3.4 Large Time Horizon ($T$) Regime

Let us consider the large time horizon ($T$) asymptotics in this section. Denote the value function as

$$V(x; T) := \sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^{T \land \tau} e^{-\delta t} dD_t \right].$$

(3.33)

Let us differentiate two cases: $\delta > 0$ and $\delta = 0$.

When $\delta > 0$, intuitively, it is clear that the finite-horizon value function will converge to the infinite-horizon value function as time horizon $T \to \infty$:

$$V(x; T) = \sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^{T \land \tau} e^{-\delta t} dD_t \right] \to \sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^\tau e^{-\delta t} dD_t \right].$$

(3.34)

Indeed, we can give an upper bound on the speed of the convergence and show rigorously the following result:
Theorem 9. Assume $\delta > 0$, we have

$$\left| V(x; T) - \sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^T e^{-\delta t} dD_t \right] \right| \leq (x + \lambda \mathbb{E}[Y_1] T) e^{-\delta T} + \frac{\lambda \mathbb{E}[Y_1]}{\delta} e^{-\delta T}. \quad (3.35)$$

Next, let us consider the $\delta = 0$ case, i.e.,

$$V(x; T) = \sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^{T \land \tau} dD_t \right]. \quad (3.36)$$

Then, we expect that

$$\lim_{T \to \infty} \frac{V(x; T)}{T} = (\lambda \mathbb{E}[Y_1] - \rho)(1 - e^{-\alpha x}). \quad (3.37)$$

We shall show this result rigorously later. Before we proceed, let us give some heuristic arguments and gain some intuition behind this result. Notice that when the discount rate $\delta = 0$, the present value of the future dividends does not decay as the payment time evolves. Moreover, $\lambda \mathbb{E}[Y_1] > \rho$, so there is an upward drift and if the company holds the wealth rather than pay the dividends, there will be less chance that the company is going to get ruined. On the other hand, we have already seen that the value of the dividends do not decay over time because $\delta = 0$, therefore, the optimal strategy is not to pay any dividends until the time of the maturity $T$ if by that time the company is not ruined. As the time horizon $T \to \infty$, the probability that the company is not ruined is the ultimate survival probability, that is, $1 - e^{-\alpha x}$. On the other hand, ignoring ruin probability, the wealth of the company right before time $T$ is given by (2.1). By taking expectation, we obtain

$$\mathbb{E}_x[X_t] = x - \rho t + \mathbb{E}[J_t], \quad t < T. \quad (3.38)$$

By monotone convergence theorem,

$$\mathbb{E}_x[X_{T-}] = x - \rho T + \mathbb{E}[J_T] = x + (\lambda \mathbb{E}[Y_1] - \rho)T. \quad (3.39)$$

where $X_{T-}$ is the left limit of the process $X_t$ at $t \uparrow T$. So for large $T$, on average, $\lambda \mathbb{E}[Y_1] - \rho$ is the rate of the growth of the company. Therefore, we expect to get

$$\lim_{T \to \infty} \frac{V(x; T)}{T} = (\lambda \mathbb{E}[Y_1] - \rho)(1 - e^{-\alpha x}). \quad (3.40)$$

The rigorous result is as follows.

Theorem 10. As the time horizon $T \to \infty$, we have

$$\sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^{T \land \tau} dD_t \right] = (\lambda \mathbb{E}[Y_1] - \rho) \left[ (1 - e^{-\alpha x}) \int_0^\infty \frac{e^{-\alpha x} x}{\rho - \lambda \int_0^\infty e^{-\alpha y} p(y) dy} + o(1) \right]. \quad (3.41)$$
The optimal strategy when $\delta = 0$ is to withhold any dividend until $T \land \tau$. But in practice, that is not very realistic. Indeed, we will show that the $D^{M,\epsilon}$ strategy that pays continuous dividend yield after the surplus reaches above the level $M$ is nearly optimal:

**Proposition 11.** For any $\epsilon > 0$, let $T > 0$ be such that

$$\left| \frac{1}{T} \sup_{D \in D} \mathbb{E}_x \left[ \int_0^{T \land \tau} e^{-\delta t} dD_t \right] - (\lambda \mathbb{E}[Y_1] - \rho)(1 - e^{-\alpha x}) \right| \leq \epsilon. \quad (3.42)$$

Then, there exist some $M', \epsilon' > 0$ such that for any $M > M'$ and $0 < \epsilon < \epsilon'$, the $D^{M,\epsilon}$ strategy is an $\epsilon T$-optimal strategy, i.e.,

$$\left| \frac{1}{T} \mathbb{E}_x \left[ \int_0^{T \land \tau} e^{-\delta t} dD_t^{M,\epsilon} \right] - (\lambda \mathbb{E}[Y_1] - \rho)(1 - e^{-\alpha x}) \right| \leq \epsilon. \quad (3.43)$$

We can see from Proposition 11 that $D^{M,\epsilon}$ strategy (defined in Definition 1) is nearly optimal in the zero discount rate $\delta = 0$ and large time horizon ($T$) regime, that is, the company does not pay out any dividend until the surplus jumps above a high threshold $M$, and afterwards, a constant dividend yield $(1 - \epsilon)(\lambda \mathbb{E}[Y_1] - \rho)$ is paid out to the shareholders. Managerial implication is that in the low rate environment (small $\delta$) with a long-term view (large $T$), the company should focus on accumulating wealth and deferring dividend payments until the company has reached a considerably large scale. This is consistent with our conclusion from Proposition 8.

### 3.5 Large Discount Rate ($\delta$) Regime

Let us consider the large discount rate ($\delta$) asymptotics in this section. When the discount rate $\delta \to \infty$, intuitively, it becomes clear that if the company waits, the present value of the future dividends will be virtually zero because of the extreme discount factor. Therefore, it is easy to see that the optimal strategy is to give the surplus as the dividends to the shareholders sooner rather than later. Intuitively, one might guess that all the surplus should be given to the shareholders as the dividends at time zero and thus

$$\sup_{D \in D} \mathbb{E}_x \left[ \int_0^{T \land \tau} e^{-\delta t} dD_t \right] \sim x, \quad \text{as the discount rate } \delta \to \infty. \quad (3.44)$$

We will show later that this indeed is true. Moreover, we can obtain the second order approximations as the discount rate $\delta \to \infty$ when $Y_i$ are exponentially distributed.

Let us consider the special case when $p(y) = \nu e^{-\nu y}$. Then by (2.10), for any $x > b$, we have

$$V(x; \delta) = x - b + \frac{\lambda \mathbb{E}[Y_1] - \rho}{\delta} = x - b + \frac{\lambda - \rho}{\delta}. \quad (3.45)$$
The optimal barrier $b$ is given by (2.13). Then from (2.14), it is easy to see that $r \sim -\frac{\delta}{\rho}$ as $\delta \rightarrow \infty$ and $s \sim \nu$ as $\delta \rightarrow \infty$. Furthermore, we can compute that

$$
\rho r + \delta = \frac{-\left(\lambda + \delta - \nu \rho\right)}{2} - \frac{\sqrt{(\lambda + \delta - \nu \rho)^2 + 4 \rho \nu \delta}}{2} + \delta 
$$

as the discount rate $\delta \rightarrow \infty$. Therefore, plugging into (2.13), we get

$$
b \sim \frac{\rho \log(\frac{\lambda}{\nu \rho})}{\delta}, \text{ as the discount rate } \delta \rightarrow \infty.
$$

(3.47)

Hence, we conclude that for any $x > 0$,

$$
V(x; \delta) = x - b + \frac{\lambda}{\delta} - \rho \sim x + \frac{1}{\delta} \left(\frac{\lambda}{\nu} - \rho \log \left(\frac{\lambda}{\nu\rho}\right)\right),
$$

as the discount rate $\delta \rightarrow \infty$. We can indeed prove the first order approximation for generally distributed $Y_i$ rigorously and have the following result:

**Theorem 12.**

$$
\sup_{D \in D} \mathbb{E}_x \left[ \int_0^T e^{-\delta t} dD_t \right] = x(1 + o(1)), \text{ as the discount rate } \delta \rightarrow \infty.
$$

(3.49)

When the discount rate $\delta$ goes to infinity, intuitively, it is clear that the company should pay all the surplus as dividends to the shareholders immediately because the cost of carry goes to infinity. Indeed, we can see from Theorem 12 that a nearly optimal strategy is to pay out the amount of the initial wealth $x$ at time zero. Therefore in the high discount rate (e.g., interest rate) environment, to maximize the shareholders value, the company should issue the dividend payment immediately instead of focusing on the growth.

### 3.6 Small Time Horizon ($T$) Regime

Let us consider the small time horizon ($T$) asymptotics in this section. When the time horizon $T \rightarrow 0$, there is little time for the company to accumulate new wealth and thus all what the company can pay to the shareholders is roughly $x$. Within the small time interval $[0, T]$, for sufficiently small, the ruin probability is zero because ruin will not occur until after the time $\frac{x}{\rho}$. So on this short time interval, there is no ruin risk. For small time horizon $T$, it is easy to compute that $\mathbb{E}_x[X_T] = x + (\lambda \mathbb{E}[Y_i] - \rho)T$. For any dividend,
the difference between paying the dividend at time 0 and at the time $T$ is only a discount factor at most $e^{-\delta T}$. Since there is no ruin risk, the initial wealth $x$ should be given upfront as the dividends to shareholders. $(\lambda \mathbb{E}[Y_1] - \rho)T$ is of order $O(T)$, and thus the impact of the discount factor on this amount of new wealth is of order $O(T^2)$ which is negligible. Therefore, we have the following result:

**Theorem 13.** Assume that $\lambda \mathbb{E}[Y_1] > \rho$. As the time horizon $T \to 0$, we have

\[
\sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^{\tau \wedge T} e^{-\delta t} dD_t \right] = x + (\lambda \mathbb{E}[Y_1] - \rho)T + O(T^2). \tag{3.50}
\]

When the time horizon $T$ goes to zero, there is little time to accumulate new wealth and what the company can pay to the shareholders is approximately the initial wealth $x$ of the company. Indeed, we can see from Theorem 13 that a nearly optimal strategy is to pay out the amount of the initial wealth $x$ at time zero. Therefore with a short-term view (i.e., small time horizon $T$), to maximize the shareholders value, the company should issue the dividend payment immediately instead of focusing on the growth.

### 4 Proofs of the Main Results

#### 4.1 Large Innovation Rate ($\lambda$) Regime

**Proof of Theorem 2 and Proposition 3.**

(i) First, let us prove the upper bound. Notice that the optimal strategy is the barrier strategy with

\[
V(x; \lambda) = x - b + V(b; \lambda), \quad \text{for } x > b, \tag{4.1}
\]

and $V'(x; \lambda) > 1$ for $x < b$ and $V(0; \lambda) = 0$. Therefore, it is easy to see that for $x < b$, we have $V(x; \lambda) \leq x - b + V(b; \lambda)$. On the other hand by (2.10), $V(b; \lambda) = \frac{\lambda \mathbb{E}[Y_1] - \rho}{\delta}$. Therefore, for any $x$,

\[
V(x; \lambda) \leq x - b + \frac{\lambda \mathbb{E}[Y_1] - \rho}{\delta} \leq x + \frac{\lambda \mathbb{E}[Y_1] - \rho}{\delta}. \tag{4.2}
\]

This gives us the upper bound.

Next, let us prove the lower bound. For any $M > 0$ and $\epsilon > 0$, let us recall that the definition of the dividend strategy $D^{M,\epsilon}$ in Definition 11 no dividend is paid out until the first time that the process jumps above $M$ and then it pays dividend with continuous rate...
(1 − ε)\(\lambda E[Y_1] - \rho\), and also recall the definitions of \(\tau_M, X_\tau, \tau^\varepsilon, \tau^{M,\varepsilon}\), and \(\tau_0\). Then,

\[
\sup_{D \in \mathcal{D}} E_x \left[ \int_0^\tau e^{-\delta t} dD_t \right] \geq E_x \left[ \int_0^{\tau^{M,\varepsilon}} e^{-\delta t} dD_t^{M,\varepsilon} \right]
\]

\[
\geq E_x \left[ e^{-\delta \tau_M 1_{\tau_M < \tau_0}} \right] \frac{(1 - \varepsilon) \lambda E[Y_1] - \rho}{\delta} \left( 1 - E \left[ e^{-\delta \tau^\varepsilon} 1_{X_\tau = M} \right] \right)
\]

\[
= E_x \left[ e^{-\delta \tau_M 1_{\tau_M < \tau_0}} \right] \frac{(1 - \varepsilon) \lambda E[Y_1] - \rho}{\delta} \left( 1 - e^{-\beta^{\varepsilon,\lambda} M} \right),
\]

where \(\beta^{\varepsilon,\lambda}\) is the unique positive value that satisfies the equation

\[
\rho \beta^{\varepsilon,\lambda} + (1 - \varepsilon) \lambda E[Y_1] - \rho \beta^{\varepsilon,\lambda} + \lambda \int_0^\infty \left[ e^{-\beta^{\varepsilon,\lambda} y} - 1 \right] p(y) dy - \delta = 0,
\]

(4.4)

It is easy to check that \(\beta^{\varepsilon,\lambda} \to \beta^{\varepsilon,\infty}\), where \(\beta^{\varepsilon,\infty}\) is the unique positive value that satisfies:

\[
(1 - \varepsilon) E[Y_1] \beta^{\varepsilon,\infty} + \int_0^\infty \left[ e^{-\beta^{\varepsilon,\infty} y} - 1 \right] p(y) dy = 0.
\]

(4.5)

Since \(\tau_M \to 0\) when \(\lambda \to \infty\), it follows from bounded convergence theorem that

\[
\lim_{\lambda \to \infty} E_x \left[ e^{-\delta \tau_M 1_{\tau_M < \tau_0}} \right] = 1.
\]

(4.6)

Therefore,

\[
\liminf_{\lambda \to \infty} \frac{1}{\lambda} \sup_{D \in \mathcal{D}} E_x \left[ \int_0^\tau e^{-\delta t} dD_t \right] \geq (1 - \varepsilon) \frac{E[Y_1]}{\delta} \left( 1 - e^{-\beta^{\varepsilon,\infty} M} \right).
\]

(4.7)

Finally, letting \(M \to \infty\) first and then \(\varepsilon \to 0\), we get

\[
\liminf_{\lambda \to \infty} \frac{1}{\lambda} \sup_{D \in \mathcal{D}} E_x \left[ \int_0^\tau e^{-\delta t} dD_t \right] \geq \frac{E[Y_1]}{\delta}.
\]

(4.8)

(ii) Assume \(\delta > 0\) and \(T > 0\). Let us first prove the upper bound. Let

\[
V(x, t; \lambda) := \sup_{D \in \mathcal{D}} E_x \left[ \int_t^{T \wedge \tau} e^{-\delta t} dD_t \right].
\]

(4.9)

Then, \(V(x, t)\) satisfies the equation:

\[
\max \left\{ \frac{\partial V}{\partial t} - \rho \frac{\partial V}{\partial x} + \lambda \int_0^\infty \left[ V(x + y, t; \lambda) - V(x, t; \lambda) \right] p(y) dy - \delta V, - \frac{\partial V}{\partial x} + 1 \right\} = 0,
\]

(4.10)
with terminal condition $V(x, T; \lambda) = x$. Now define the function $U_1(x, t) := x + \frac{\lambda E[Y_1] - \rho}{\delta} (1 - e^{-\delta(T-t)})$ and consider an arbitrage dividend strategy $D \in \mathcal{D}$. Recall that $dX_t = \rho dt + dJ_t - dD_t$. Thus, by change of variable formula for processes of bounded variation (see, e.g., [15, Theorem II.31]), we obtain

$$U_1(x, 0) = \mathbb{E}_x [e^{-\delta(T\wedge\tau)} U_1(X_{T\wedge\tau}, T \wedge \tau) + \mathbb{E}_x \left[ \int_0^{T\wedge\tau} e^{-\delta s} \left( \frac{\partial U_1}{\partial x}(X_s, s) + \frac{\partial U_1}{\partial t}(X_s, s) \right) ds \right]$$

(4.11)

$- \lambda \int_0^\infty [U_1(X_s + y, s) - U_1(X_s, s)] p(y) dy + \delta U_1(X_s, s) ds \right]$

$+ \mathbb{E}_x \left[ \int_0^{T\wedge\tau} e^{-\delta s} \frac{\partial U_1}{\partial x}(X_s, s) dD_s \right]$

$- \mathbb{E}_x \left[ \sum_{s \leq T\wedge\tau} e^{-\delta s} \left( U_1(X_{s+}, s) - U_1(X_s, s) + \frac{\partial U_1}{\partial x}(X_s, s) \Delta D_s \right) \right]$.

By direct calculation, the Riemann integral in the above and the term

$$\mathbb{E}_x [e^{-\delta(T\wedge\tau)} U_1(X_{T\wedge\tau}, T \wedge \tau)]$$

are non-negative. Also for the last term, we have

$$U_1(X_{s+}, s) - U_1(X_s, s) - \frac{\partial U_1}{\partial x}(X_s, s) \Delta D_s = 0.$$

Therefore, we can write

$$U_1(x, 0) \geq \mathbb{E}_x \left[ \int_0^{T\wedge\tau} e^{-\delta s} dD_s \right]$$

Taking supremum over $D \in \mathcal{D}$ gives us the desired upper bound.

Next, let us prove the lower bound. For any $0 < \epsilon < 1$, consider the strategy $D^{M, \epsilon}$ from Definition [14] that pays out dividend at constant rate $(1 - \epsilon)(\lambda E[Y_1] - \rho)$ after the surplus hits above $M$. Then,

$$\sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^{T\wedge\tau} e^{-\delta t} dD_t \right]$$

(4.12)

$$\geq \mathbb{E}_x [e^{-\delta \tau_M 1_{\tau_M < \tau_0 \wedge \tau}} (1 - \epsilon)(\lambda E[Y_1] - \rho) \mathbb{E} \left[ \int_0^{(T-\tau_M)\wedge\tau^t} e^{-\delta t} dt \right] X_0^t = M, \tau_M < \tau_0 \wedge T]$$

$$= \mathbb{E}_x [e^{-\delta \tau_M 1_{\tau_M < \tau_0 \wedge \tau}} (1 - \epsilon)(\lambda E[Y_1] - \rho)$$

$$\cdot \frac{1}{\delta} \left( 1 - \mathbb{E} \left[ e^{-\delta ((T-\tau_M)\wedge\tau^t)} \left| X_0^t = M, \tau_M < \tau_0 \wedge T \right] \right),$$

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which implies that
\[
\liminf_{\lambda \to \infty} \frac{1}{\lambda} \sup_{D \in D} E_x \left[ \int_0^{T \wedge \tau} e^{-\delta t} dD_t \right] \geq (1 - \epsilon) \frac{E[Y_1]}{\delta} \left( 1 - E[e^{-\delta(T \wedge \tau^\epsilon)}|X_0^\epsilon = M] \right). \tag{4.13}
\]

Now, by first letting $M \to \infty$ and then $\epsilon \to 0$, and following the same arguments as in (i), we get the desired lower bound.

(iii) Assume $\delta = 0$ and $T > 0$. The upper bound is similar as in (ii) by using the function $U_1(x,t) := x + (\lambda E[Y_1] - \rho)(T - t)$. We can show that
\[
\sup_{D \in D} E_x \left[ \int_0^{T \wedge \tau} e^{-\delta t} dD_t \right] \leq x + (\lambda E[Y_1] - \rho)T. \tag{4.14}
\]

The lower bound can be obtained similarly as in (ii). We can show that
\[
\liminf_{\lambda \to \infty} \frac{1}{\lambda} \sup_{D \in D} E_x \left[ \int_0^{T \wedge \tau} e^{-\delta t} dD_t \right] \geq (1 - \epsilon) \frac{E[Y_1]}{\delta} T. \tag{4.15}
\]

Now, letting $M \to \infty$, we have $\tau^\epsilon \to \infty$ in probability and by bounded convergence theorem, we have
\[
\liminf_{\lambda \to \infty} \frac{1}{\lambda} \sup_{D \in D} E_x \left[ \int_0^{T \wedge \tau} e^{-\delta t} dD_t \right] \geq (1 - \epsilon) \frac{E[Y_1]}{\delta} T. \tag{4.16}
\]

Since it holds for any $\epsilon > 0$, we get the desired lower bound. \qed

Remark 14. Indeed, we can also have a nearly optimal discrete dividend strategy for the large $\lambda$ regime as an alternative to the continuous dividend strategy $D^\epsilon$ defined in the proof of Theorem \[. Let us consider a particular strategy $D^*$ which is a barrier strategy with barrier $x > 0$, the same as the initial surplus. Then,

\[
\sup_{D \in D} E_x \left[ \int_0^\tau e^{-\delta t} dD_t \right] \geq \left( \sum_{n=1}^{\infty} \mathbb{I}_{[\tau_n^x(n) > 0]} e^{-\tau_{n+1}^x(n)} \right) \geq 1 \mathbb{I}_{Y_1 > \rho \Delta t}(Y_1 - \rho \Delta t) \tag{4.17}
\]

where $\tau_{n+1}^x(n)$ is the $n$-th time that the process jumps above the threshold $x$ and $\tau_{n+1}^x(0) := 0$ and $Y_n$ are i.i.d. distributed as $Y_1$ as before. Essentially, we provide a lower bound by counting the events that there is a jump occur before $\Delta t$ after the process starts at $x$ and that jump size is greater than $\rho \Delta t$ which guarantees that the process jumps above the threshold $x$ and pays the dividend. Therefore, we have
\[
\sup_{D \in D} E_x \left[ \int_0^\tau e^{-\delta t} dD_t \right] \geq \mathbb{E}[1_{Y_1 > \rho \Delta t}(Y_1 - \rho \Delta t)] \sum_{n=1}^{\infty} \left( \mathbb{E} \left[ e^{-\tau_{n+1}^x(1)} \mathbb{I}_{N[0,\Delta t] \geq 1_{\tau_{n+1}^x(1)}} \right] \right)^n. \tag{4.18}
\]
It is easy to compute that

\[ E_x \left[ e^{-\tau(1)} 1_{N[0,\Delta t] \geq 1} \right] = \int_0^{\Delta t} e^{-\delta s} \lambda e^{-\delta s} ds \]

\[ = \frac{\lambda}{\delta + \lambda} \left[ 1 - e^{-(\lambda + \delta)\Delta t} \right], \]

which implies that

\[ \sup_{D \in D} \mathbb{E}_x \left[ \int_0^\tau e^{-\delta t} dD_t \right] \geq \mathbb{E} [1_{Y_1 > \rho \Delta t} (Y_1 - \rho \Delta t)] \frac{\lambda}{\delta + \lambda} \left[ 1 - e^{-(\lambda + \delta)\Delta t} \right]. \]

Therefore, for any \( \Delta t > 0 \), we have

\[ \liminf_{\lambda \to \infty} \sup_{D \in D} \mathbb{E}_x \left[ \int_0^\tau e^{-\delta t} dD_t \right] \geq \frac{1}{\delta} \mathbb{E} [1_{Y_1 > \rho \Delta t} (Y_1 - \rho \Delta t)]. \]

By letting \( \Delta t \to 0 \), we get the desired lower bound.

### 4.2 Small Running Cost (\( \rho \)) Regime

**Proof of Theorem 5 and Proposition 6.** (i) By our assumption, \( \rho = 0 \). Let \( v(x) = x + \frac{\lambda \mathbb{E}[Y_1]}{\delta} \). Then, we can compute that

\[ \max \left\{ \lambda \int_0^\infty [v(x + y) - v(x)] p(y) dy - \delta v(x), 1 - v'(x) \right\} = \max \{-\delta x, 0\} = 0. \]

Therefore \( v(x) = x + \frac{\lambda \mathbb{E}[Y_1]}{\delta} \) is a classical solution of the above problem. Classical verification theorem such as [12, Theorem 8.4.1] shows that \( v(x) \geq V(x; \rho) \) for \( \rho = 0 \). Thus, it is sufficient to show that there is a sequence of strategies \( D^\epsilon \in D \) such that

\[ \limsup_{\epsilon \to 0} \mathbb{E}_x \left[ \int_0^\tau e^{-\delta t} dD^\epsilon_t \right] = v(x). \]

Recall the strategy \( D^\epsilon \) which pays dividend \( x - \epsilon \) at the beginning and any surplus above \( \epsilon \) thereafter. Then, the ruin never occurs and

\[ \mathbb{E}_x \left[ \int_0^\tau e^{-\delta t} dD^\epsilon_t \right] = x - \epsilon + \mathbb{E}_x \left[ \sum_{n=1}^\infty e^{-\delta \tau(n)} Y_n \right] = x - \epsilon + \mathbb{E}[Y_1] \sum_{n=1}^\infty \mathbb{E}_x \left[ e^{-n\delta \tau(1)} \right]
\]

\[ = x - \epsilon + \mathbb{E}[Y_1] \frac{\lambda \mathbb{E}[Y_1]}{\delta}, \]

where for \( n \geq 1 \), \( \tau(n) \) is the time of the \( n \)th jump of the process \( J \). Thus, [3.10] holds.
(ii) For \( \delta > 0 \) and finite \( T > 0 \), let \( v(t, x) = x + \frac{\lambda \mathbb{E}[Y_1]}{\delta}\left(1 - e^{-\delta(T-t)}\right) \). Then \( v(T, x) = x \) and
\[
\max \left\{ \frac{\partial v}{\partial t}(t, x) + \lambda \int_0^\infty [v(t, x + y) - v(t, x)]p(y)dy - \delta v(t, x), 1 - \frac{\partial}{\partial x}v(t, x) \right\} = 0.
\]
Therefore \( v(t, x) = x + \frac{\lambda \mathbb{E}[Y_1]}{\delta}\left(1 - e^{-\delta(T-t)}\right) \) is a classical solution. Therefore, one can repeat the above argument for function \( v(t, x) \) to obtain (3.11).

(iii) When \( \delta = 0 \), the function \( v(t, x) = x + \lambda \mathbb{E}[Y_1](T-t) \) should be used to obtain (3.12).

4.3 Small Discount Rate (\( \delta \)) Regime

Proof of Theorem \( 7 \) and Proposition \( 8 \) (i) Let us first prove the upper bound.

Let us recall that
\[
\min \left\{ \delta V(x; \delta) + \rho V'(x; \delta) - \lambda \int_0^\infty [V(x + y; \delta) - V(x; \delta)]p(y)dy, V'(x; \delta) - 1 \right\} = 0, 
\]
with \( V(0) = 0 \).

Let us define \( U(x) := \delta V(x; \delta) \) so that \( U \) satisfies
\[
\min \left\{ \delta U(x) + \rho U'(x) - \lambda \int_0^\infty [U(x + y) - U(x)]p(y)dy, U'(x) - \delta \right\} = 0, 
\]
with \( U(0) = 0 \).

We want to show that for \( \delta > 0 \) sufficiently small,
\[
U(x) \leq U_1(x) := (\lambda \mathbb{E}[Y_1] - \rho) \left(1 - e^{-\alpha(\delta)x}\right) + \delta x, 
\]
where \( \alpha(\eta) \) is the largest positive root of
\[
F(\alpha) = \alpha \rho - \lambda \int_0^\infty [e^{-\alpha y} - 1]p(y)dy + \eta = 0. 
\]
If \( \eta > 0 \), \( \alpha(\eta) \) exists. For \( \eta < 0 \) sufficiently small, \( F(\alpha) = 0 \) has at least one positive root. In addition, \( \alpha(\delta) \) is a continuous function.

Let us show that \( U \leq U_1 \). Consider an arbitrary admissible dividend strategy \( D_t \), increasing càdlàg function with \( D_0 = 0 \). By change of variable formula for processes of
bounded variation (see, e.g., [15, Theorem II.31]), we have

\[
U_1(x) = \mathbb{E}_x\left[e^{-\delta(t \land \tau)}U_1(X_{t \land \tau})\right] + \mathbb{E}_x\left[\int_0^{t \land \tau} e^{-\delta s}\left(\rho U'_1(X_s) - \lambda \int_0^\infty [U_1(X_s + y) - U_1(X_s)]p(y)dy + \delta U_1(X_s)\right)ds\right]
\]

\[
+ \mathbb{E}_x\left[\int_0^{t \land \tau} e^{-\delta s} U'_1(X_s)dD^c_s\right] - \mathbb{E}_x\left[\sum_{s \leq t \land \tau} e^{-\delta s} \left[U_1(X_{s+}) - U_1(X_s) - U'_1(X_s)\Delta D_s\right]\right]
\]

where \(D^c_s\) is the continuous part of \(D_s\) (see, e.g., [15, Pg. 70]). We recall the definition of \(U_1(x)\) in (4.25), where \(\alpha(\delta)\) is defined in (4.26). Direct calculations shows that

\[
\delta U_1 + \rho U'_1 - \lambda \int_0^\infty (U_1(x + y) - U_1(x))p(y)dy = \delta x^2 \geq 0,
\]

(4.28)

\[
U'_1(x) \geq \delta.
\]

(4.29)

Thus

\[
U_1(x) \geq \mathbb{E}_x\left[e^{-\delta(t \land \tau)}U_1(X_{t \land \tau})\right] + \delta \mathbb{E}_x\left[\int_0^{t \land \tau} e^{-\delta s} dD_s\right].
\]

(4.30)

Notice that here we used \(U'_1 \geq \delta\), \(\Delta D_s \geq 0\), \(X_{s+} \leq X_s\) and

\[
\sum_{s \leq t \land \tau} \left[U_1(X_{s+}) - U_1(X_s) - U'_1(X_s)\Delta D_s\right] e^{-\delta s} \leq -\delta \sum_{s \leq t \land \tau} e^{-\delta s} \Delta D_s.
\]

(4.31)

By sending \(t \uparrow +\infty\), we obtain

\[
U_1(x) \geq \lim_{t \to \infty} \mathbb{E}_x\left[e^{-\delta(t \land \tau)}U_1(X_{t \land \tau})\right] + \delta \mathbb{E}_x\left[\int_0^{t \land \tau} e^{-\delta t} dD_t\right].
\]

(4.32)

Note that \(\lim_{t \to \infty} \mathbb{E}_x\left[e^{-\delta(t \land \tau)}U_1(X_{t \land \tau})\right] = 0\). Thus, by taking supremum over \(D \in \mathcal{D}\), we get

\[
U_1(x) \geq \delta \sup_{D \in \mathcal{D}} \mathbb{E}_x\left[\int_0^{\infty} e^{-\delta t} dD_t\right] = \delta V(x; \delta) = U(x).
\]

(4.33)

From \(U(x) \leq U_1(x)\), it follows that

\[
\limsup_{\delta \to 0} \delta V(x; \delta) \leq (\lambda \mathbb{E}[Y_1] - \rho)(1 - e^{-\alpha x}).
\]

(4.34)

(ii) Now let us consider the lower bound (the Proposition 8 will also follow). For any \(M > 0\) and \(\epsilon > 0\), let us recall that the definition of the dividend strategy \(D^{M,\epsilon}\) in Definition
no dividend is paid out until the first time that the process jumps above $M$ and then it pays dividend with continuous rate $(1 - \epsilon)(\lambda\mathbb{E}[Y_1] - \rho)$, and also recall the definitions of $\tau_M$, $X_t^\epsilon$, $\tau^\epsilon$, $\tau^{M,\epsilon}$, and $\tau_0$. Therefore, we have

$$\sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^\tau e^{-bt} dD_t \right] \geq \mathbb{E}_x \left[ \int_0^{\tau^{M,\epsilon}} e^{-bt} dD_t^{M,\epsilon} \right] \geq \mathbb{E}_x \left[ e^{-\delta \tau_M} 1_{\tau_M < \tau_0} \right] \mathbb{E} \left[ \int_0^{\tau^\epsilon} e^{-bt}(1 - \epsilon)(\lambda\mathbb{E}[Y_1] - \rho)dt \bigg| X_0^\epsilon = M \right] = \mathbb{E}_x \left[ e^{-\delta \tau_M} 1_{\tau_M < \tau_0} (1 - \epsilon)(\lambda\mathbb{E}[Y_1] - \rho) \right] \left( 1 - \mathbb{E} \left[ e^{-\delta \tau^\epsilon} \big| X_0^\epsilon = M \right] \right),$$

where the second inequality above uses the simple fact that $X_{\tau_M} \geq M$ and a lower bound is given by starting $X_0^\epsilon = M$ rather than $X_0^\epsilon = X_{\tau_M}^\epsilon$. Notice that

$$\lim_{\delta \to 0} \mathbb{E} \left[ e^{-\delta \tau^\epsilon} | X_0^\epsilon = M \right] = \mathbb{P}(\tau^\epsilon < \infty | X_0^\epsilon = M),$$

and $u(x) := \mathbb{P}(\tau^\epsilon < \infty | X_0^\epsilon = x)$ satisfies the equation

$$- \rho u'(x) - (1 - \epsilon)(\lambda\mathbb{E}[Y_1] - \rho)u(x) + \lambda \int_0^\infty [u(x + y) - u(x)]p(y)dy = 0,$$

with the boundary condition $u(0) = 1$, which implies that

$$\mathbb{P}(\tau^\epsilon < \infty | X_0^\epsilon = M) = e^{-\alpha^\epsilon M},$$

where $\alpha^\epsilon$ is the unique positive value that satisfies the equation:

$$\rho \alpha^\epsilon - (1 - \epsilon)(\lambda\mathbb{E}[Y_1] - \rho)\alpha^\epsilon + \lambda \int_0^\infty [e^{-\alpha^\epsilon y} - 1]p(y)dy = 0.$$

Moreover, we have

$$\lim_{\delta \to 0} \mathbb{E}_x \left[ e^{-\delta \tau_M} 1_{\tau_M < \tau_0} \right] = \mathbb{P}_x(\tau_M < \tau).$$

Hence, we have

$$\lim_{\delta \to 0} \delta \sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^\tau e^{-bt} dD_t \right] \geq \mathbb{P}_x(\tau_M < \tau) (1 - \epsilon)(\lambda\mathbb{E}[Y_1] - \rho) \left( 1 - e^{-\alpha^\epsilon M} \right).$$

Since it holds for any $M > x$, we can let $M \to \infty$ and it follows that

$$\lim_{\delta \to 0} \delta \sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^\tau e^{-bt} dD_t \right] \geq \mathbb{P}_x(\tau = \infty) (1 - \epsilon)(\lambda\mathbb{E}[Y_1] - \rho).$$

Finally, notice that $\mathbb{P}_x(\tau = \infty) = 1 - e^{-\alpha x}$ and it holds for any $\epsilon > 0$ and we can let $\epsilon \to 0$. That yields the desired lower bound. \qed
4.4 Large Time Horizon \((T)\) Regime

**Proof of Theorem 9.** For any \(D \in \mathcal{D}\),

\[
\left| \mathbb{E}_x \left[ \int_0^{T \wedge \tau_0} e^{-\delta t} dD_t \right] \right| - \mathbb{E}_x \left[ \int_0^\tau e^{-\delta t} dD_t \right] = \mathbb{E}_x \left[ \int_0^{T \wedge \tau_0} e^{-\delta t} dD_t \right]
\]

\[
\leq \mathbb{E}_x \left[ \int_0^\infty e^{-\delta t} dD_t \right].
\]

The total expected discounted dividends for \(\rho \geq 0\) is bounded above by the case when \(\rho = 0\). In the case when \(\rho = 0\), there is no ruin risk and any surplus should be paid out immediately to the shareholders. Therefore, for any \(D \in \mathcal{D}\), an upper bound of \(\mathbb{E}_x \left[ \int_0^\infty e^{-\delta t} dD_t \right]\) is given as follows. If no dividend is paid before time \(T\) then the expected value of the surplus at time \(T\) is \(x + \lambda \mathbb{E}[Y_1 | T]\) when \(\rho = 0\) and that the surplus is paid out at time \(T\), after which, any surplus is paid out immediately so that for any \(D \in \mathcal{D}\),

\[
\mathbb{E}_x \left[ \int_0^\infty e^{-\delta t} dD_t \right] \leq (x + \lambda \mathbb{E}[Y_1 | T]) e^{-\delta T} + \lambda \mathbb{E}[Y_1] \int_T^\infty e^{-\delta t} dt,
\]

which yields the desired result. \(\square\)

Now let us turn to the proof of Theorem 10 which would be directly implied by Lemma 15 and Lemma 16. First, let us prove that the optimal strategy is not to pay any dividend until the maturity \(T\) if the company is not ruined by then and all the surplus at the maturity is given to the shareholders as the dividends.

**Lemma 15.**

\[
\sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^{T \wedge \tau_0} dD_t \right] = \mathbb{E}_x \left[ X_0^{T \wedge \tau_0} \right],
\]

where \(X_0^t := x - \rho t + J_t\) and \(\tau_0\) is the ruin time of process \(X^0\).

**Proof.** Let \(D\) be an arbitrary admissible dividend strategy such that \(X_{T \wedge \tau} = 0\), where \(\tau\) is the ruin time of process

\[
dX_t = -\rho dt + dJ_t - dD_t, \quad X_0 = x.
\]

Next, define the dividend strategy \(\tilde{D}\) by \(\tilde{D}_t = 0\) for \(t < T \wedge \tau_0\) and \(\tilde{D}_{T \wedge \tau_0} = X_{T \wedge \tau_0}^0\). For \(t < T\), we have \(X_t = X_{T \wedge \tau}^0 - D_t\). Then,

\[
\mathbb{E}_x \left[ \int_0^{T \wedge \tau} dD_t \right] = \mathbb{E}_x \left[ X_{T \wedge \tau}^0 \right] - \mathbb{E}_x \left[ X_{T \wedge \tau} \right] = \mathbb{E}_x \left[ X_{T \wedge \tau}^0 \right].
\]
On the other hand, since \( \lambda \mathbb{E}[Y_1] - \rho > 0 \), \( X^0 \) is a submartingale and thus, \( \mathbb{E}_x \left[ X^0_{T \wedge \tau} \right] < \mathbb{E}_x [X^0_{T \wedge \tau_0}] \). Here we used the fact that \( \tau_0 \geq \tau \). This implies that
\[
\mathbb{E}_x \left[ \int_0^{T \wedge \tau} dD_t \right] \leq \mathbb{E}_x \left[ \tau_0 \cdot 1_{\tau_0 < \infty} \right].
\]

The above lemma asserts that when there is no discounting, paying surplus as dividend at the terminal time is an optimal strategy. This allows us to focus only on the terminal time dividend payment strategy. When the dividend is paid at terminal time, the expected dividend is equal to \( \mathbb{E}_x \left[ \tau_0 \cdot 1_{\tau_0 < \infty} \right] \), which we will estimate in the next step as \( T \to \infty \).

**Lemma 16.**
\[
\mathbb{E}_x \left[ X^0_{T \wedge \tau_0} \right] = (\lambda \mathbb{E}[Y_1] - \rho) \left[ (1 - e^{-\alpha x})T + \frac{e^{-\alpha x}}{\rho - \lambda \int_0^\infty e^{-\alpha y} p(y) dy} + o(1) \right],
\]
where \( \alpha(\theta) > 0 \) is the unique solution to the equation:
\[
\rho (1 + \lambda) \int_0^\infty e^{-\alpha(\theta) x} - 1 [e^{-\alpha(\theta) y} - 1] p(y) dy - \theta = 0.
\]

Before we proceed to the proof of Lemma 16, let us first state and prove a technical lemma.

**Lemma 17.** Under the condition that \( \rho < \lambda \mathbb{E}[Y_1] \), we have
\[
\mathbb{E}_x \left[ \tau_0 \cdot 1_{\tau_0 < \infty} \right] = \frac{x}{\rho - \lambda \int_0^\infty e^{-\alpha y} p(y) dy}.
\]

**Proof of Lemma 17.** When \( \mathbb{P}_x (\tau_0 = \infty) > 0 \), first, we compute
\[
v(x) := \mathbb{E}_x \left[ e^{-\theta \tau_0} 1_{\tau_0 < \infty} \right],
\]
which satisfies the equation:
\[
- \rho v'(x) + \lambda \int_0^\infty [v(x) - v(x)] p(y) dy - \theta v(x) = 0,
\]
with the boundary condition \( V(0) = 1 \).

It is easy to see that \( v(x) = e^{-\alpha(\theta) x} \), where \( \alpha(\theta) > 0 \) is the unique solution to the equation:
\[
\rho \alpha(\theta) + \lambda \int_0^\infty (e^{-\alpha(\theta) y} - 1) p(y) dy - \theta = 0.
\]

Differentiating with respect to \( \theta \), we get
\[
\rho \alpha'(\theta) - \lambda \alpha'(\theta) \int_0^\infty e^{-\alpha(\theta) y} p(y) dy - 1 = 0.
\]
On the other hand,

\[ E_x[\tau_01_{\tau_0<\infty}] = \alpha'(0)x e^{-\alpha(0)x}. \]  \hspace{1cm} (4.52)

By letting \( \theta = 0 \) in (4.51), we conclude that

\[ E_x[\tau_01_{\tau_0<\infty}] = \frac{e^{-\alpha x}}{\rho - \lambda \int_0^{\infty} e^{-\alpha y}p(y)dy}, \]  \hspace{1cm} (4.53)

where \( \alpha > 0 \) is the unique solution to the equation:

\[ \rho \alpha + \lambda \int_0^{\infty} \left(e^{-\alpha y} - 1\right)p(y)dy = 0. \]  \hspace{1cm} (4.54)

Hence,

\[ E_x[\tau_0|\tau_0 < \infty] = \frac{x}{\rho - \lambda \int_0^{\infty} e^{-\alpha y}p(y)dy}. \]  \hspace{1cm} (4.55)

**Remark 18.** Note that in Lemma 17, \( E_x[\tau_01_{\tau_0<\infty}] < \infty \) requires that the condition

\[ \rho - \lambda \int_0^{\infty} e^{-\alpha y}p(y)dy > 0 \]  \hspace{1cm} (4.56)

is satisfied. This can be easily checked as follows. Recall that \( \alpha \) is the unique positive solution to the equation

\[ \rho \alpha + \lambda \int_0^{\infty} \left(e^{-\alpha y} - 1\right)p(y)dy = 0. \]  \hspace{1cm} (4.57)

Substituting (4.57) into (4.56), the condition (4.56) is equivalent to:

\[ \int_0^{\infty} \left(1 - e^{-\alpha y} - e^{-\alpha y} \alpha y\right)p(y)dy > 0. \]  \hspace{1cm} (4.58)

Let \( F(x) := 1 - e^{-x} - e^{-\alpha x} \) for \( x \geq 0 \). It is easy to compute that \( F(0) = 0 \) and \( F'(x) = e^{-x}x > 0 \), which implies that \( F(x) > 0 \) for any \( x > 0 \) and hence (4.56) holds.

**Remark 19.** In Lemma 17, for the case when \( Y_i \) are exponentially distributed, say \( p(y) = \nu e^{-\nu y} \) for some \( \nu > 0 \), then, we can compute that \( \alpha = \frac{\lambda}{\rho} - \nu \) and

\[ E_x[\tau_0|\tau_0 < \infty] = \frac{x}{\rho(1 - \frac{\rho \nu}{\lambda})}. \]  \hspace{1cm} (4.59)

Now we are ready to prove Lemma 16.
Proof of Lemma 16. By change of variable formula for processes of bounded variation (see, e.g., [15, Theorem II.31]), we can compute that
\[ E_x \left[ X_{T \wedge \tau_0} \right] = x + (\lambda E[Y_1] - \rho)E_x[T \wedge \tau_0], \]  
(4.60)

Note that
\[ E_x[T \wedge \tau_0] = E_x[\tau_0 \cdot 1_{\tau_0 < T}] + TP_x(\tau_0 > T). \]  
(4.61)

By Lemma 17, we have
\[ E_x[\tau_0 \cdot 1_{\tau_0 < \infty}] = \frac{x}{\rho - \lambda \int_0^\infty e^{-\alpha y}yp(y)dy}. \]  
(4.62)

Moreover,
\[ \lim_{T \to \infty} P_x(\tau_0 > T) = P_x(\tau_0 = \infty) = 1 - e^{-\alpha x}. \]  
(4.63)

Hence, we proved that
\[ \lim_{T \to \infty} \frac{1}{T} E_x[X_{T \wedge \tau_0}] = (\lambda E[Y_1] - \rho)(1 - e^{-\alpha x}). \]  
(4.64)

Finally, notice that \( E_x[\tau_0 \cdot 1_{\tau_0 < \infty}] < \infty \). Therefore,
\[ E_x[\tau_0 \cdot 1_{\tau_0 < \infty}] = \int_0^\infty P_x(t \leq \tau_0 < \infty)dt < \infty, \]  
(4.65)

which implies that
\[ P_x(\tau_0 \geq T) - P_x(\tau_0 = \infty) = P_x(T \leq \tau_0 < \infty) = o \left( T^{-1} \right), \]  
(4.66)

as \( T \to \infty \). Hence, we conclude that
\[ E_x[X_{T \wedge \tau_0}] = (\lambda E[Y_1] - \rho) \left[ (1 - e^{-\alpha x})T + \frac{e^{-\alpha x}x}{\rho - \lambda \int_0^\infty e^{-\alpha y}yp(y)dy} + o(1) \right], \]  
(4.67)

as \( T \to \infty \). \( \square \)

Proof of Proposition 11. For any \( M > 0 \) and \( \epsilon > 0 \), let us recall the definitions of \( \tau_M, X_t^\epsilon, \tau^\epsilon, \tau^{M,\epsilon} \), and \( \tau_0 \). Following the similar arguments as in the proof of Theorem [17] and Proposition [8] we have
\[
\sup_{D \in D} \mathbb{E}_x \left[ \int_0^{\tau \wedge T} dD_t \right] \geq \mathbb{E}_x \left[ \int_0^{\tau^{M,\epsilon} \wedge T} dD_t^{M,\epsilon} \right] \\
\geq \mathbb{P}_x(\tau_M < \tau_0 \wedge T)(1 - \epsilon)(\lambda E[Y_1] - \rho)\mathbb{E}[\tau^\epsilon \wedge (T - \tau_M)|X_0^\epsilon = M, \tau_M < \tau_0 \wedge T] \\
= \mathbb{P}_x(\tau_M < \tau_0 \wedge T)(1 - \epsilon)(\lambda E[Y_1] - \rho) \\
\cdot \left( \mathbb{E}[(T - \tau_M)1_{\tau^\epsilon = \infty}|X_0^\epsilon = M, \tau_M < \tau_0 \wedge T] \\
+ \mathbb{E}[\tau^\epsilon \wedge (T - \tau_M)1_{\tau^\epsilon < \infty}|X_0^\epsilon = M, \tau_M < \tau_0 \wedge T] \right).
\]
Therefore, we have
\[
\liminf_{T \to \infty} \frac{1}{T} \sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^{\tau \wedge T} dD_t \right] \geq \mathbb{P}_x(\tau_M < \tau_0)(1 - \epsilon)(\lambda \mathbb{E}[Y_1] - \rho)\mathbb{P}(\tau^\epsilon = \infty)\mathbb{E}_x[M]. \tag{4.68}
\]
Hence, we conclude that
\[
\lim_{\epsilon \to 0} \lim_{M \to \infty} \liminf_{T \to \infty} \frac{1}{T} \sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^{\tau \wedge T} dD_t \right] \geq \mathbb{P}_x(\tau_0 = \infty)(\lambda \mathbb{E}[Y_1] - \rho) = (1 - e^{-\alpha x})(\lambda \mathbb{E}[Y_1] - \rho). \tag{4.69}
\]

\section{4.5 Large Discount Rate ($\delta$) Regime}

\begin{proof}[Proof of Theorem 12] When the initial surplus is paid out completely at time 0, this strategy gives value $x$. Therefore, this gives us a lower bound. Next, let us prove the upper bound. Notice that the optimal strategy is the barrier strategy with the optimal barrier $b$ and for any $x$
\[
V(x; \delta) \leq x - b + \frac{\lambda \mathbb{E}[Y_1] - \rho}{\delta} \leq x + \frac{\lambda \mathbb{E}[Y_1] - \rho}{\delta}. \tag{4.70}
\]
This gives us the upper bound. \end{proof}

\section{4.6 Small Time Horizon ($T$) Regime}

\begin{proof}[Proof of Theorem 13] Let us first prove the upper bound. It is clear that
\[
\sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^{\tau \wedge T} e^{-\delta t} dD_t \right] \leq \sup_{D \in \mathcal{D}} \mathbb{E}_x \left[ \int_0^{\tau \wedge T} dD_t \right] = \mathbb{E}_x[X_{\tau \wedge T}] = \mathbb{E}_x[X_T], \tag{4.71}
\]
for sufficiently small $T > 0$, since when there is no discount factor, it is never optimal to pay dividend and if no dividend is paid out, then the ruin time $\tau \geq \frac{x}{\rho} > T$ for $T$ sufficiently small. We can easily compute that
\[
\mathbb{E}_x[X_T] = x + (\lambda \mathbb{E}[Y_1] - \rho)T. \tag{4.72}
\]
This gives us the upper bound. Now let us turn to the proof of the lower bound. Let us consider a dividend strategy $D'$ such that $x - \epsilon$ is paid out at time 0 and then the remaining surplus is $\epsilon$ and no dividend is paid out. For any $T$ that is sufficiently small so that $T < \frac{x}{\rho}$,
then, ruin will not occur before time $T$, i.e., $\tau > T$. By considering this particular strategy, we have

$$\sup_{D \in D} \mathbb{E}_x \left[ \int_0^{\tau \wedge T} e^{-\delta t} dD_t \right] \geq \mathbb{E}_x \left[ \int_0^{\tau \wedge T} e^{-\delta t} dD_t \right] = x - \epsilon + e^{-\delta T} \left[ \epsilon + \left( \lambda \mathbb{E}[Y_1] - \rho \right) T \right] = x - \delta T \epsilon + \epsilon O(T^2) + \left( \lambda \mathbb{E}[Y_1] - \rho \right) T + O(T^2),$$

which holds for $T < \frac{\epsilon}{\rho}$. Take $\epsilon = 2 \rho T$ for example will give us the desired lower bound. □

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