The original Weyl-Titchmarsh functions and sectorial Schrödinger L-systems

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Dedicated with great pleasure to Seppo Hassi on the occasion of his 60-th birthday

Abstract. In this paper we study the L-system realizations generated by the original Weyl-Titchmarsh functions \( m_\alpha(z) \) in the case when the minimal symmetric Schrödinger operator in \( L^2(\ell, +\infty) \) is non-negative. We realize functions \((-m_\alpha(z))\) as impedance functions of Schrödinger L-systems and derive necessary and sufficient conditions for \((-m_\alpha(z))\) to fall into sectorial classes \( S^{\beta_1, \beta_2} \) of Stieltjes functions. Moreover, it is shown that the knowledge of the value \( m_\infty(-0) \) and parameter \( \alpha \) allows us to describe the geometric structure of the L-system that realizes \((-m_\alpha(z))\). Conditions when the main and state space operators of the L-system realizing \((-m_\alpha(z))\) have the same or not angle of sectoriality are presented in terms of the parameter \( \alpha \). Example that illustrates the obtained results is presented in the end of the paper.

Contents

1. Introduction
2. Preliminaries
3. Sectorial classes and and their realizations
4. L-systems with Schrödinger operator and their impedance functions
5. Realizations of \(-m_\infty(z)\), \((1/m_\infty(z))\) and \(m_\alpha(z)\).
6. Non-negative Schrödinger operator and sectorial L-systems
7. Example
8. References

1. Introduction

This paper is a part of an ongoing project studying the realizations of the original Weyl-Titchmarsh function \( m_\infty(z) \) and its linear-fractional transformation \( m_\alpha(z) \) associated with a Schrödinger operator in \( L^2(\ell, +\infty) \). In this project the Herglotz-Nevanlinna functions \((-m_\infty(z))\) and \((1/m_\infty(z))\) as well as \((-m_\alpha(z))\) and
(1/\(m_\alpha(z)\)) are being realized as impedance functions of L-systems with a dissipative Schrödinger main operator \(T_h\), (\(\Im h > 0\)). For the sake of brevity we will refer to these L-systems as Schrödinger L-systems for the rest of the manuscript. The formal definition, exposition and discussions of general and Schrödinger L-systems are presented in Sections 2 and 3. We capitalize on the fact that all Schrödinger L-systems \(\Theta_{\mu,h}\) form a two-parametric family whose members are uniquely defined by a real-valued parameter \(\mu\) and a complex boundary value \(h\) of the main dissipative operator.

The focus of this paper is set on the case when the realizing Schrödinger L-systems are based on non-negative symmetric Schrödinger operator with (1, 1) deficiency indices and have accretive state-space operator. It is known (see [2]) that in this case the impedance functions of such L-systems are Stieltjes. Here we study the situation when the realizing Schrödinger L-systems are also sectorial and the Weyl-Titchmarsh functions \((-m_\alpha(z))\) fall into sectorial classes \(S^\beta\) and \(S^{\beta_1,\beta_2}\) of Stieltjes functions that are discussed in details in Section 4. Section 5 provides us with the general realization results (obtained in [2]) for the functions \((-m_\infty(z))\), \((1/m_\infty(z))\), and \((-m_\alpha(z))\). It is shown that \((-m_\infty(z))\), \((1/m_\infty(z))\), and \((-m_\alpha(z))\) can be realized as the impedance function of Schrödinger L-systems \(\Theta_{0,i}, \Theta_{\infty,i}\), and \(\Theta_{\tan,\alpha,i}\), respectively.

The main results of the paper are contained in Section 6. Here we apply the realization theorems from Section 4 to Schrödinger L-systems that are based on non-negative symmetric Schrödinger operator to obtain additional properties. Utilizing the results presented in Section 6, we derive some new features of Schrödinger L-systems \(\Theta_{\tan,\alpha,i}\) whose impedance functions fall into particular sectorial classes \(S^{\beta_1,\beta_2}\) with \(\beta_1\) and \(\beta_2\) explicitly described. The results are given in terms of the parameter \(\alpha\) that appears in the definition of the function \(m_\alpha(z)\). Moreover, the knowledge of the limit value \(m_\infty(-0)\) and the value of \(\alpha\) allows us to find the angle of sectoriality of the main and state-space operators of the realizing L-system. This, in turn, leads to connections to Kato’s problem about sectorial extension of sectorial forms.

The paper is concluded with an example that illustrates main results and concepts. The present work is a further development of the theory of open physical systems conceived by M. Livšic in [13].

2. Preliminaries

For a pair of Hilbert spaces \(\mathcal{H}_1, \mathcal{H}_2\) we denote by \([\mathcal{H}_1, \mathcal{H}_2]\) the set of all bounded linear operators from \(\mathcal{H}_1\) to \(\mathcal{H}_2\). Let \(\hat{A}\) be a closed, densely defined, symmetric operator in a Hilbert space \(\mathcal{H}\) with inner product \((f, g), f, g \in \mathcal{H}\). Any non-symmetric operator \(T\) in \(\mathcal{H}\) such that \(\hat{A} \subset T \subset \hat{A}^\star\) is called a quasi-self-adjoint extension of \(\hat{A}\).

Consider the rigged Hilbert space (see [11], [2]) \(\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-\), where \(\mathcal{H}_+ = \text{Dom}(\hat{A}^\star)\)

\[
(1) \quad (f, g)_+ = (f, g) + (\hat{A}^\star f, \hat{A}^\star g), \quad f, g \in \text{Dom}(\hat{A}^\star).
\]

Let \(\mathcal{R}\) be the Riesz-Berezansky operator \(\mathcal{R}\) (see [11], [2]) which maps \(\mathcal{H}_-\) onto \(\mathcal{H}_+\) such that \((f, g) = (f, \mathcal{R}g)_+ (\forall f \in \mathcal{H}_+, g \in \mathcal{H}_-)\) and \(\|\mathcal{R}g\|_+ = \|g\|_-\). Note that identifying the space conjugate to \(\mathcal{H}_\pm\) with \(\mathcal{H}_\mp\), we get that if \(A \in [\mathcal{H}_+, \mathcal{H}_-]\), then \(A^\star \in [\mathcal{H}_+, \mathcal{H}_-]\). An operator \(A \in [\mathcal{H}_+, \mathcal{H}_-]\) is called a self-adjoint bi-extension of a
symmetric operator $\hat{A}$ if $\mathcal{A} = \mathcal{A}^*$ and $\mathcal{A} \supset \hat{A}$. Let $\mathcal{A}$ be a self-adjoint bi-extension of $\hat{A}$ and let the operator $\hat{A}$ in $\mathcal{H}$ be defined as follows:

$$\text{Dom}(\hat{A}) = \{f \in \mathcal{H}_+: \mathcal{A} f \in \mathcal{H}\}, \quad \hat{A} = \mathcal{A}|\text{Dom}(\hat{A}).$$

The operator $\hat{A}$ is called a quasi-kernel of a self-adjoint bi-extension $\mathcal{A}$ (see \cite[Section 2.1]{2}). A self-adjoint bi-extension $\mathcal{A}$ of a symmetric operator $\hat{A}$ is called t-self-adjoint bi-extension (see \cite[Definition 4.3.1]{2}) if its quasi-kernel $\hat{A}$ is self-adjoint operator in $\mathcal{H}$. An operator $\mathcal{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a quasi-self-adjoint bi-extension of an operator $T$ if $\mathcal{A} \supset T \supset A$ and $\mathcal{A}^* \supset T^* \supset A$. We will be mostly interested in the following type of quasi-self-adjoint bi-extensions. Let $T$ be a quasi-self-adjoint extension of $\hat{A}$ with nonempty resolvent set $\rho(T)$. A quasi-self-adjoint bi-extension $\mathcal{A}$ of an operator $T$ is called (see \cite[Definition 3.3.5]{2}) a $(\ast)$-extension of $T$ if $\text{Re} \mathcal{A}$ is a t-self-adjoint bi-extension of $A$. In what follows we assume that $\hat{A}$ has deficiency indices $(1,1)$. In this case it is known \cite{2} that every quasi-self-adjoint extension $T$ of $\hat{A}$ admits $(\ast)$-extensions. The description of all $(\ast)$-extensions via Riesz-Berezansky operator $\mathcal{R}$ can be found in \cite[Section 4.3]{2}.

Recall that a linear operator $T$ in a Hilbert space $\mathcal{H}$ is called accretive \cite{17} if $\text{Re}(T f, f) \geq 0$ for all $f \in \text{Dom}(T)$. We call an accretive operator $T$ $\beta$-sectorial \cite{17} if there exists a value of $\beta \in (0, \pi/2)$ such that

$$\text{(cot} \beta)|\text{Im}(T f, f)| \leq \text{Re}(T f, f), \quad f \in \text{Dom}(T).$$

We say that the angle of sectoriality $\beta$ is exact for a $\beta$-sectorial operator $T$ if

$$\tan \beta = \sup_{f \in \text{Dom}(T)} \frac{|\text{Im}(T f, f)|}{\text{Re}(T f, f)}.$$  

An accretive operator is called extremal accretive if it is not $\beta$-sectorial for any $\beta \in (0, \pi/2)$. A $(\ast)$-extension $\mathcal{A}$ of $T$ is called accretive if $\text{Re}(\mathcal{A} f, f) \geq 0$ for all $f \in \mathcal{H}_+$. This is equivalent to that the real part $\text{Re} \mathcal{A} = (\mathcal{A} + \mathcal{A}^*)/2$ is a nonnegative t-self-adjoint bi-extension of $\hat{A}$.

The following definition is a “lite” version of the definition of L-system given for a scattering L-system with one-dimensional input-output space. It is tailored for the case when the symmetric operator of an L-system has deficiency indices $(1,1)$. The general definition of an L-system can be found in \cite[Definition 6.3.4]{2} (see also \cite{16} for a non-canonical version).

**Definition 1.** An array

$$(3) \quad \Theta = \left( \begin{array}{ccc} \mathcal{A} & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & \mathcal{C} \end{array} \right)$$

is called an L-system if:

1. $T$ is a dissipative ($\text{Im}(T f, f) \geq 0$, $f \in \text{Dom}(T)$) quasi-self-adjoint extension of a symmetric operator $\hat{A}$ with deficiency indices $(1,1)$;
2. $\mathcal{A}$ is a $(\ast)$-extension of $T$;
3. $\text{Im} \mathcal{A} = KK^*$, where $K \in [\mathcal{C}, \mathcal{H}_-]$ and $K^* \in [\mathcal{H}_+, \mathcal{C}]$.

Operators $T$ and $\mathcal{A}$ are called a main and state-space operators respectively of the system $\Theta$, and $K$ is a channel operator. It is easy to see that the operator $\mathcal{A}$ of the system (3) is such that $\text{Im} \mathcal{A} = (\cdot, \chi)\chi$, $\chi \in \mathcal{H}_-$ and pick $Kc = c \cdot \chi$, $c \in \mathcal{C}$ (see \cite{2}). A system $\Theta$ in (3) is called minimal if the operator $\hat{A}$ is a prime operator in $\mathcal{H}$, i.e., there exists no non-trivial reducing invariant subspace of $\mathcal{H}$ on which it induces
a self-adjoint operator. Minimal L-systems of the form (3) with one-dimensional input-output space were also considered in [6].

We associate with an L-system $\Theta$ the function

$$W_\Theta(z) = I - 2iK^*(A - zI)^{-1}K, \quad z \in \rho(T),$$

which is called the transfer function of the L-system $\Theta$. We also consider the function

$$V_\Theta(z) = K^*(\text{Re} A - zI)^{-1}K,$$

that is called the impedance function of an L-system $\Theta$ of the form (3). The transfer function $W_\Theta(z)$ of the L-system $\Theta$ and function $V_\Theta(z)$ of the form (5) are connected by the following relations valid for $\text{Im} z \neq 0$, $z \in \rho(T)$,

$$V_\Theta(z) = i[W_\Theta(z) + I]^{-1}[W_\Theta(z) - I],$$

$$W_\Theta(z) = (I + iV_\Theta(z))^{-1}(I - iV_\Theta(z)).$$

An L-system $\Theta$ of the form (3) is called an accretive L-system (9), (14) if its state-space operator $A$ is accretive, that is $\text{Re}(Af, f) \geq 0$ for all $f \in \mathcal{H}_+$. An accretive L-system is called sectorial if the operator $A$ is sectorial, i.e., satisfies (2) for some $\beta \in (0, \pi/2)$ and all $f \in \mathcal{H}_+$.

### 3. Sectorial classes and their realizations

A scalar function $V(z)$ is called the Herglotz-Nevanlinna function if it is holomorphic on $\mathbb{C} \setminus \mathbb{R}$, symmetric with respect to the real axis, i.e., $V(z)^* = V(\bar{z})$, $z \in \mathbb{C} \setminus \mathbb{R}$, and if it satisfies the positivity condition $\text{Im} V(z) \geq 0$, $z \in \mathbb{C}_+$. The class of all Herglotz-Nevanlinna functions, that can be realized as impedance functions of L-systems, and connections with Weyl-Titchmarsh functions can be found in [2], [6], [13], [15] and references therein. The following definition can be found in [16]. A scalar Herglotz-Nevanlinna function $V(z)$ is a Stieltjes function if it is holomorphic in $\text{Ext}(0, +\infty)$ and

$$\frac{\text{Im}[zV(z)]}{\text{Im} z} \geq 0.$$ 

It is known [16] that a Stieltjes function $V(z)$ admits the following integral representation

$$V(z) = \gamma + \int_0^\infty \frac{dG(t)}{t - z},$$

where $\gamma \geq 0$ and $G(t)$ is a non-decreasing on $[0, +\infty)$ function such that $\int_0^\infty \frac{dG(t)}{t + \epsilon} < \infty$. We are going to focus on the class $S_0(R)$ (see [2], [14], [2]) of scalar Stieltjes functions such that the measure $G(t)$ in representation (7) is of unbounded variation. It was shown in [2] (see also [3]) that such a function $V(z)$ can be realized as the impedance function of an accretive L-system $\Theta$ of the form (3) with a densely defined symmetric operator if and only if it belongs to the class $S_0(R)$.

Now we are going to consider sectorial subclasses of scalar Stieltjes functions introduced in [1]. Let $\beta \in (0, \pi/2)$. Sectorial subclasses $S^\beta$ of Stieltjes functions
are defined as follows: a scalar Stieltjes function $V(z)$ belongs to $S^\beta$ if

$$K_\beta = \sum_{k,l=1}^n \left[ \frac{z_k V(z_k) - \bar{z}_l V(\bar{z}_l)}{z_k - \bar{z}_l} - (\cot \beta) V(\bar{z}_l) V(z_k) \right] h_k \bar{h}_l \geq 0,$$

for an arbitrary sequences of complex numbers $\{z_k\}$, $(\text{Im } z_k > 0)$ and $\{h_k\}$, $(k = 1, ..., n)$. For $0 < \beta_1 < \beta_2 < \frac{\pi}{2}$, we have

$$S^{\beta_1} \subset S^{\beta_2} \subset S,$$

where $S$ denotes the class of all Stieltjes functions (which corresponds to the case $\beta = \frac{\pi}{2}$). Let $\Theta$ be a minimal L-system of the form (1) with a densely defined non-negative symmetric operator $\hat{A}$. Then (see [2]) the impedance function $V_\Theta(z)$ defined by (1) belongs to the class $S^\beta$ if and only if the operator $A$ of the L-system $\Theta$ is $\beta$-sectorial.

Let $0 \leq \beta_1 < \frac{\pi}{2}$, $0 < \beta_2 < \frac{\pi}{2}$, and $\beta_1 \leq \beta_2$. We say that a scalar Stieltjes function $V(z)$ belongs to the class $S^{\beta_1,\beta_2}$ if

$$\tan \beta_1 = \lim_{x \to -\infty} V(x), \quad \tan \beta_2 = \lim_{x \to 0} V(x).$$

The following connection between the classes $S^\beta$ and $S^{\beta_1,\beta_2}$ can be found in [2]. Let $\Theta$ be an L-system of the form (1) with a densely defined non-negative symmetric operator $\hat{A}$ with deficiency numbers $(1, 1)$. Let also $A$ be an $\beta$-sectorial ($\ast$)-extension of $T$. Then the impedance function $V_\Theta(z)$ defined by (1) belongs to the class $S^{\beta_1,\beta_2}$, $\tan \beta_2 \leq \tan \beta$. Moreover, the main operator $T$ is $(\beta_2 - \beta_1)$-sectorial with the exact angle of sectoriality $(\beta_2 - \beta_1)$. In the case when $\beta$ is the exact angle of sectoriality of the operator $T$ we have that $V_\Theta(z) \in S^{0,\beta}$ (see [2]). It also follows that under this set of assumptions, the impedance function $V_\Theta(z)$ is such that $\gamma = 0$ in representation (1).

Now let $\Theta$ be an L-system of the form (1), where $A$ is a ($\ast$)-extension of $T$ and $\hat{A}$ is a closed densely defined non-negative symmetric operator with deficiency numbers $(1, 1)$. It was proved in [2] that if the impedance function $V_\Theta(z)$ belongs to the class $S^{\beta_1,\beta_2}$ and $\beta_2 \neq \pi/2$, then $A$ is $\beta$-sectorial, where

$$\tan \beta = \tan \beta_2 + 2\sqrt{\tan \beta_1 (\tan \beta_2 - \tan \beta_1)}.$$

Under the above set of conditions on L-system $\Theta$, it is shown in [2] that $A$ is $\beta$-sectorial ($\ast$)-extension of an $\beta$-sectorial operator $T$ with the exact angle $\beta \in (0, \pi/2)$ if and only if $V_\Theta(z) \in S^{0,\beta}$. Moreover, the angle $\beta$ can be found via the formula

$$\tan \beta = \int_0^\infty \frac{dG(t)}{t},$$

where $G(t)$ is the measure from integral representation (1) of $V_\Theta(z)$.

4. L-systems with Schr"{o}dinger operator and their impedance functions

Let $H = L_2[\ell, +\infty)$, $\ell \geq 0$, and $l(y) = -y'' + q(x)y$, where $q$ is a real locally summable on $[\ell, +\infty)$ function. Suppose that the symmetric operator

$$\begin{cases}
Ay = -y'' + q(x)y \\
y(\ell) = y'(\ell) = 0
\end{cases}$$

was defined. Under these assumptions the main operator $T$ is $S^{0,\beta}$-sectorial. Moreover, the angle $\beta$ can be found via the formula

$$\tan \beta = \int_0^\infty \frac{dG(t)}{t},$$

where $G(t)$ is the measure from integral representation (1) of $V_\Theta(z)$.
has deficiency indices (1,1). Let $D^*$ be the set of functions locally absolutely continuous together with their first derivatives such that $l(y) \in L_2[\ell, +\infty)$. Consider $\mathcal{H}_+ = \text{Dom}(A^*) = D^*$ with the scalar product

$$(y, z)_+ = \int_{\ell}^{\infty} \left( y(x) \bar{z(x)} + l(y) \overline{l(z)} \right) dx, \quad y, \ z \in D^*.$$ 

Let $\mathcal{H}_+ \subset L_2[\ell, +\infty) \subset \mathcal{H}_-$ be the corresponding triplet of Hilbert spaces. Consider the operators

$$
\begin{cases}
T_h y = l(y) = -y'' + q(x)y \\
\frac{hy(\ell) - y'(\ell)}{\mu,h} = 0
\end{cases},
\begin{cases}
T_h^* y = l(y) = -y'' + q(x)y \\
\frac{hy(\ell) - y'(\ell)}{\mu,h} = 0
\end{cases},
$$

where $\text{Im} \ h > 0$. Let $\bar{A}$ be a symmetric operator of the form (13) with deficiency indices (1,1), generated by the differential operation $l(y) = -y'' + q(x)y$. Let also $\varphi_k(x,\lambda)(k=1,2)$ be the solutions of the following Cauchy problems:

$$
\begin{cases}
l(\varphi_1) = \lambda \varphi_1 \\
\varphi_2(\ell,\lambda) = 0 \\
\varphi_1'(\ell,\lambda) = 1
\end{cases},
\begin{cases}
l(\varphi_2) = \lambda \varphi_2 \\
\varphi_2(\ell,\lambda) = -1 \\
\varphi_2'(\ell,\lambda) = 0
\end{cases}.
$$

It is well known [20], [18] that there exists a function $m_\infty(\lambda)$ introduced by H. Weyl [27], [28] for which

$$\varphi(x,\lambda) = \varphi_2(x,\lambda) + m_\infty(\lambda)\varphi_1(x,\lambda)$$

belongs to $L_2[\ell, +\infty)$. The function $m_\infty(\lambda)$ is not a Herglotz-Nevanlinna function but $(-m_\infty(\lambda))$ and $(1/m_\infty(\lambda))$ are.

Now we shall construct an $L$-system based on a non-self-adjoint Schrödinger operator $T_h$ with $\text{Im} \ h > 0$. It was shown in [3], [2] that the set of all $(\ast)$-extensions of a non-self-adjoint Schrödinger operator $T_h$ of the form (13) in $L_2[\ell, +\infty)$ can be represented in the form

$$
\begin{align*}
\mathbb{A}_{\mu,h} y &= -y'' + q(x)y - \frac{1}{\mu-h} \left[ y'(\ell) - hy(\ell) \right] \left[ \mu \delta(x-\ell) + \delta'(x-\ell) \right], \\
\mathbb{A}_{\mu,h}^* y &= -y'' + q(x)y - \frac{1}{\mu-h} \left[ y'(\ell) - hy(\ell) \right] \left[ \mu \delta(x-\ell) + \delta'(x-\ell) \right].
\end{align*}
$$

Moreover, the formulas (14) establish a one-to-one correspondence between the set of all $(\ast)$-extensions of a Schrödinger operator $T_h$ of the form (13) and all real numbers $\mu \in [-\infty, +\infty)$. One can easily check that the $(\ast)$-extension $\mathbb{A}$ in (13) of the non-self-adjoint dissipative Schrödinger operator $T_h$, $(\text{Im} \ h > 0)$ of the form (13) satisfies the condition

$$\text{Im} \ \mathbb{A}_{\mu,h} = \frac{\mathbb{A}_{\mu,h} - \mathbb{A}_{\mu,h}^*}{2i} = (\cdot, g_{\mu,h}) g_{\mu,h},$$

where

$$\begin{align*}
g_{\mu,h} &= \frac{(\text{Im} \ h)^2}{|\mu-h|} \left[ \mu \delta(x-\ell) + \delta'(x-\ell) \right] \\
\text{and} \quad \delta(x-\ell), \delta'(x-\ell) \quad \text{are the delta-function and its derivative at the point } \ell, \text{ respectively. Furthermore,}
\end{align*}$$

$$(y, g_{\mu,h}) = \frac{(\text{Im} \ h)^2}{|\mu-h|} \left[ \mu y(\ell) - y'(\ell) \right].$$
where \( y \in \mathcal{H}_+, \ g \in \mathcal{H}_-, \) and \( \mathcal{H}_+ \subset L_2[\ell, +\infty) \subset \mathcal{H}_- \) is the triplet of Hilbert spaces discussed above.

It was also shown in \( [2] \) that the quasi-kernel \( \hat{A}_\xi \) of \( \text{Re} \, A_{\mu, h} \) is given by

\[
\begin{cases}
\hat{A}_\xi y = -y'' + q(x)y, & \text{where } \xi = \frac{\mu \text{Re} \, h - |h|^2}{\mu - \text{Re} \, h}, \\
y'(\ell) = \xi g(\ell)
\end{cases}
\]

Let \( E = \mathbb{C}, \ K_{\mu, h}c = cg_{\mu, h}, \ (c \in \mathbb{C}) \). It is clear that

\[
K_{\mu, h}^* y = (y, g_{\mu, h}), \ \ y \in \mathcal{H}_+,
\]

and \( \text{Im} \, A_{\mu, h} = K_{\mu, h}^* K_{\mu, h} \). Therefore, the array

\[
\Theta_{\mu, h} = \begin{pmatrix} A_{\mu, h} & K_{\mu, h} \\ \mathcal{H}_+ \subset L_2[\ell, +\infty) \subset \mathcal{H}_- & 1 \end{pmatrix}.
\]

is an \( L \)-system with the main operator \( T_h \), (Im \( h > 0 \)) of the form \( [3] \), the state-space operator \( A_{\mu, h} \) of the form \( [4] \), and with the channel operator \( K_{\mu, h} \) of the form \( [5] \). It was established in \([3, 4]\) that the transfer and impedance functions of \( \Theta_{\mu, h} \) are

\[
W_{\Theta_{\mu, h}}(z) = \frac{\mu - \text{Re} \, h \ m_\infty(z) + i \hbar}{\mu - \text{Re} \, h \ m_\infty(z) + i \hbar},
\]

and

\[
V_{\Theta_{\mu, h}}(z) = \frac{(m_\infty(z) + \mu) \text{Im} \, h}{(\mu - \text{Re} \, h) m_\infty(z) + \mu \text{Re} \, h - |h|^2}.
\]

It was shown in \([2, 3] \) Section 10.2) that if the parameters \( \mu \) and \( \xi \) are related via \([16]\), then the two \( L \)-systems \( \Theta_{\mu, h} \) and \( \Theta_{\xi, h} \) of the form \([18]\) have the following property

\[
W_{\Theta_{\mu, h}}(z) = -W_{\Theta_{\xi, h}}(z), \ V_{\Theta_{\mu, h}}(z) = -\frac{1}{V_{\Theta_{\xi, h}}(z)}, \ \text{where } \xi = \frac{\mu \text{Re} \, h - |h|^2}{\mu - \text{Re} \, h}.
\]

5. Realizations of \( -m_\infty(z) \), \( 1/m_\infty(z) \) and \( m_\omega(z) \).

It is known \([15, 20]\) that the original Weyl-Titchmarsh function \( m_\infty(z) \) has a property that \( (-m_\infty(z)) \) is a Herglotz-Nevanlinna function. The question whether \( (-m_\infty(z)) \) can be realized as the impedance function of a Schrödinger \( L \)-system is answered in the following theorem that was proved in \([7]\).

**Theorem 2** \([7]\). Let \( \hat{A} \) be a symmetric Schrödinger operator of the form \([12]\) with deficiency indices \( (1, 1) \) and locally summable potential in \( \mathcal{H} = L_2[\ell, \infty) \). If \( m_\infty(z) \) is the Weyl-Titchmarsh function of \( \hat{A} \), then the Herglotz-Nevanlinna function \( (-m_\infty(z)) \) can be realized as the impedance function of a Schrödinger \( L \)-system \( \Theta_{\mu, h} \) of the form \([18]\) with \( \mu = 0 \) and \( h = i \).

Conversely, let \( \Theta_{\mu, h} \) be a Schrödinger \( L \)-system of the form \([18]\) with the symmetric operator \( \hat{A} \) such that \( V_{\Theta_{\mu, h}}(z) = -m_\infty(z) \), for all \( z \in \mathbb{C}_\pm \) and \( \mu \in \mathbb{R} \cup \{\infty\} \). Then the parameters \( \mu \) and \( h \) defining \( \Theta_{\mu, h} \) are such that \( \mu = 0 \) and \( h = i \).

A similar result for the function \( 1/m_\infty(z) \) was also proved in \([6]\).

**Theorem 3** \([7]\). Let \( \hat{A} \) be a symmetric Schrödinger operator of the form \([12]\) with deficiency indices \( (1, 1) \) and locally summable potential in \( \mathcal{H} = L_2[\ell, \infty) \). If \( m_\infty(z) \) is the Weyl-Titchmarsh function of \( \hat{A} \), then the Herglotz-Nevanlinna function \( (1/m_\infty(z)) \) can be realized as the impedance function of a Schrödinger \( L \)-system \( \Theta_{\mu, h} \) of the form \([18]\) with \( \mu = \infty \) and \( h = i \).
Conversely, let \( \Theta_{\mu,h} \) be a Schrödinger \( L \)-system of the form (18) with the symmetric operator \( \hat{A} \) such that \( V_{\Theta_{\mu,h}}(z) = \frac{1}{m_{\infty}(z)} \), for all \( z \in \mathbb{C}_\pm \) and \( \mu \in \mathbb{R} \cup \{ \infty \} \). Then the parameters \( \mu \) and \( h \) defining \( \Theta_{\mu,h} \) are such that \( \mu = \infty \) and \( h = i \).

We note that both \( L \)-systems \( \Theta_{0,i} \) and \( \Theta_{\infty,i} \) obtained in Theorems 3 and 4 share the same main operator

\[
(22) \quad \begin{cases}
T_1 y = -y'' + q(x)y \\
y'(\ell) = i y(\ell)
\end{cases}
\]

Now we recall the definition of Weyl-Titchmarsh functions \( m_\alpha(z) \). Let \( \hat{A} \) be a symmetric operator of the form (12) with deficiency indices \((1,1)\), generated by the differential operation \( l(y) = -y'' + q(x)y \). Let also \( \varphi_\alpha(x,z) \) and \( \theta_\alpha(x,z) \) be the solutions of the following Cauchy problems:

\[
\begin{aligned}
&\left\{ \begin{array}{l}
l(\varphi_\alpha) = z\varphi_\alpha \\
\varphi_\alpha(\ell, z) = \sin \alpha \\
\varphi'_\alpha(\ell, z) = -\cos \alpha
\end{array} \right. \\
&\left\{ \begin{array}{l}
l(\theta_\alpha) = z\theta_\alpha \\
\theta_\alpha(\ell, z) = \cos \alpha \\
\theta'_\alpha(\ell, z) = \sin \alpha
\end{array} \right.
\end{aligned}
\]

It is known [12, 20, 21] that there exists an analytic in \( \mathbb{C}_\pm \) function \( m_\alpha(z) \) for which

\[
(23) \quad \psi(x, z) = \theta_\alpha(x, z) + m_\alpha(z)\varphi_\alpha(x, z)
\]

belongs to \( L_2[\ell, +\infty) \). It is easy to see that if \( \alpha = \pi \), then \( m_\pi(z) = m_\infty(z) \). The functions \( m_\alpha(z) \) and \( m_\infty(z) \) are connected (see [12, 21]) by

\[
(24) \quad m_\alpha(z) = \frac{\sin \alpha + m_\infty(z) \cos \alpha}{\cos \alpha - m_\infty(z) \sin \alpha}.
\]

We know [20, 21] that for any real \( \alpha \) the function \( -m_\alpha(z) \) is a Herglotz-Nevanlinna function. Also, modifying (24) slightly we obtain

\[
(25) \quad -m_\alpha(z) = \frac{\sin \alpha + m_\infty(z) \cos \alpha}{-\cos \alpha + m_\infty(z) \sin \alpha} = \frac{\cos \alpha + \frac{1}{m_\infty(z)} \sin \alpha}{\sin \alpha - \frac{1}{m_\infty(z)} \cos \alpha}.
\]

The following realization theorem (see [7]) for Herglotz-Nevanlinna functions \( -m_\alpha(z) \) is similar to Theorem 3.

**Theorem 4 ([7])**. Let \( \hat{A} \) be a symmetric Schrödinger operator of the form (12) with deficiency indices \((1,1)\) and locally summable potential in \( \mathcal{H} = L^2[\ell, \infty) \). If \( m_\alpha(z) \) is the function of \( \hat{A} \) described in (24), then the Herglotz-Nevanlinna function \( (-m_\alpha(z)) \) can be realized as the impedance function of a Schrödinger \( L \)-system \( \Theta_{\mu,h} \) of the form (18) with

\[
(26) \quad \mu = \tan \alpha \quad \text{and} \quad h = i.
\]

Conversely, let \( \Theta_{\mu,h} \) be a Schrödinger \( L \)-system of the form (18) with the symmetric operator \( \hat{A} \) such that

\[
V_{\Theta_{\mu,h}}(z) = -m_\alpha(z),
\]

for all \( z \in \mathbb{C}_\pm \) and \( \mu \in \mathbb{R} \cup \{ \infty \} \). Then the parameters \( \mu \) and \( h \) defining \( \Theta_{\mu,h} \) are given by (24), i.e., \( \mu = \tan \alpha \) and \( h = i \).
We note that when \( \alpha = \pi \) we obtain \( \mu_\alpha = 0, m_\alpha(z) = m_\infty(z) \), and the realizing Schrödinger L-system \( \Theta_{0,i} \) is thoroughly described in [24, Section 5]. If \( \alpha = \pi/2 \), then we get \( \mu_\alpha = \infty, -m_\alpha(z) = 1/m_\infty(z) \), and the realizing Schrödinger L-system is \( \Theta_{\infty,i} \) (see [24, Section 5]). Assuming that \( \alpha \in (0, \pi] \) and neither \( \alpha = \pi \) nor \( \alpha = \pi/2 \) we give the description of a Schrödinger L-system \( \Theta_{\mu,i} \) realizing \( -m_\alpha(z) \) as follows.

\[
\Theta_{\tan,\alpha,i} = \begin{pmatrix}
A_{\tan,\alpha,i} & K_{\tan,\alpha,i} & 1 \\
\mathcal{H}_+ \subset L_2[\ell, \infty) & \mathcal{H}_- & \mathcal{C}
\end{pmatrix},
\]

where

\[
A_{\tan,\alpha,i} y = l(y) - \frac{1}{\tan \alpha} \left[ y'(\ell) - iy(\ell) \right] [(\tan \alpha)\delta(x - \ell) + \delta'(x - \ell)],
\]
\[
A_{\tan,\alpha,i}^* y = l(y) - \frac{1}{\tan \alpha + i} \left[ y'(\ell) + iy(\ell) \right] [(\tan \alpha)\delta(x - \ell) + \delta'(x - \ell)],
\]

\[
K_{\tan,\alpha,i} c = c g_{\tan,\alpha,i}, (c \in \mathbb{C}) \text{ and }
\]
\[
\Theta_{\tan,\alpha,i} = (\tan \alpha)\delta(x - \ell) + \delta'(x - \ell).
\]

Also,

\[
V_{\Theta_{\tan,\alpha,i}}(z) = -m_\alpha(z)
\]
\[
W_{\Theta_{\tan,\alpha,i}}(z) = \frac{\tan \alpha - i}{\tan \alpha + i} \frac{m_\infty(z) - i}{m_\infty(z) + i} = (e^{2\alpha i}) \frac{m_\infty(z) - i}{m_\infty(z) + i}
\]

The realization theorem for Herglotz-Nevanlinna functions \( 1/m_\alpha(z) \) is similar to Theorem [3] and can be found in [3].

6. Non-negative Schrödinger operator and sectorial L-systems

Now let us assume that \( \hat{A} \) is a non-negative (i.e., \( (\hat{A}f, f) \geq 0 \) for all \( f \in \text{Dom}(\hat{A}) \)) symmetric operator of the form [23] with deficiency indices (1,1), generated by the differential operation \( l(y) = -y' + q(x)y \). The following theorem takes place.

**Theorem 5 ([23], [24], [25]).** Let \( \hat{A} \) be a non-negative symmetric Schrödinger operator of the form [23] with deficiency indices (1,1) and locally summable potential in \( H = L^2[\ell, \infty) \). Consider operator \( T_h \) of the form \[24\]. Then

1. operator \( \hat{A} \) has more than one non-negative self-adjoint extension, i.e., the Friedrichs extension \( A_F \) and the Krein-von Neumann extension \( A_K \) do not coincide, if and only if \( m_\infty(-0) < \infty \);
2. operator \( T_h \), \( (h = \hbar) \) coincides with the Krein-von Neumann extension \( A_K \) if and only if \( h = -m_\infty(-0) \);
3. operator \( T_h \) is accretive if and only if
\[
\text{Re } h \geq -m_\infty(-0);
\]
4. operator \( T_h \), \( (h \neq \hbar) \) is \( \beta \)-sectorial if and only if \( \text{Re } h > -m_\infty(-0) \) holds;
5. operator \( T_h \), \( (h \neq \hbar) \) is accretive but not \( \beta \)-sectorial for any \( \beta \in (0, \frac{\pi}{2}) \) if and only if \( \text{Re } h = -m_\infty(-0) \);
6. If \( T_h \), \( (\text{Im } h > 0) \) is \( \beta \)-sectorial, then the exact angle \( \beta \) can be calculated via
\[
\tan \beta = \frac{\text{Im } h}{\text{Re } h + m_\infty(-0)}.
\]
For the remainder of this paper we assume that $m_{\infty}(-0) < \infty$. Then according to Theorem 1 above (see also [3, 23, 24]) we have the existence of the operator $T_h$, $(\Im h > 0)$ that is accretive and/or sectorial. It was shown in [2] that if $T_h$ $(\Im h > 0)$ is an accretive Schrödinger operator of the form (13), then for all real $\mu$ satisfying the following inequality

$$\mu \geq \frac{(\Im h)^2}{m_{\infty}(-0) + \Re h} + \Re h,$$

formulas (14) define the set of all accretive ($\ast$)-extensions $A_{\mu,h}$ of the operator $T_h$. Moreover, an accretive ($\ast$)-extensions $A_{\mu,h}$ of a sectorial operator $T_h$ with exact angle of sectoriality $\beta \in (0, \pi/2)$ also preserves the same exact angle of sectoriality if and only if $\mu = +\infty$ in (14) (see [3, Theorem 3]). Also, $A_{\mu,h}$ is accretive but not $\beta$-sectorial for any $\beta \in (0, \pi/2)$ ($\ast$)-extension of $T_h$ if and only if in (14)

$$\mu = \frac{(\Im h)^2}{m_{\infty}(-0) + \Re h} + \Re h,$$

(see [3, Theorem 4]). An accretive operator $T_h$ has a unique accretive ($\ast$)-extension $A_{\infty,h}$ if and only if $\Re h = -m_{\infty}(-0)$. In this case this unique ($\ast$)-extension has the form

$$A_{\infty,h}y = -y'' + q(x)y + [hy(\ell) - y'(')]\delta(x-\ell),$$

$$A_{\infty,h}^{\ast}y = -y'' + q(x)y + [hy(\ell) - y'(\ell)]\delta(x-\ell).$$

Now we are going to turn to functions $m_{\alpha}(z)$ described by (24)-(24) and associated with the non-negative operator $\hat{A}$ above. We need to see how the parameter $\alpha$ in the definition of $m_{\alpha}(z)$ affects the L-system realizing $(-m_{\alpha}(z))$. This question was answered in [5, Theorem 6.3]. It tells us that if the non-negative symmetric Schrödinger operator is such that $m_{\infty}(-0) \geq 0$, then the L-system $\Theta_{\tan,\alpha,i}$ of the form (24) realizing the function $(-m_{\alpha}(z))$ is accretive if and only if

$$\tan \alpha \geq \frac{1}{m_{\infty}(-0)}.$$

Note that if $m_{\infty}(-0) = 0$ in (26), then $\alpha = \pi/2$ and $-m_{\alpha}(z) = 1/m_{\alpha}(z)$. Also, from [7, Theorem 6.2] we know that if $m_{\infty}(-0) \geq 0$, then $1/m_{\infty}(z)$ is realized by an accretive system $\Theta_{\infty,i}$.

Now once we established a criteria for an L-system realizing $(-m_{\alpha}(z))$ to be accretive, we can look into more of its properties. There are two choices for an accretive L-system $\Theta_{\tan,\alpha,i}$: it is either (1) accretive sectorial or (2) accretive extremal. In the case (1) we have that $A_{\tan,\alpha,i}$ of the form (28) is $\beta_1$-sectorial with some angle of sectoriality $\beta_1$ that can only exceed the exact angle of sectoriality $\beta$ of $T_i$. In the case (2) the state-space operator $A_{\tan,\alpha,i}$ is extremal (not sectorial for any $\beta \in (0, \pi/2)$) and is a ($\ast$)-extension of $T_i$ that itself can be either $\beta$-sectorial or extremal. These possibilities were described in details in [5, Theorem 6.4]. In particular, it was shown that for the accretive L-system $\Theta_{\tan,\alpha,i}$ realizing the function $(-m_{\alpha}(z))$ the following is true:

1. If $m_{\infty}(-0) = 0$, then there is only one accretive L-system $\Theta_{\infty,i}$ realizing $(-m_{\alpha}(z))$. This L-system is extremal and its main operator $T_i$ is extremal as well.
2. If $m_{\infty}(-0) > 0$, then $T_i$ is $\beta$-sectorial for $\beta \in (0, \pi/2)$ and
   a. If $\tan \alpha = 1/m_{\infty}(-0)$, then $\Theta_{\tan,\alpha,i}$ is extremal;
above describes the dependence of the properties of realizing (−\(m_\alpha(z)\)) extremal L-systems on the value of \(\mu\) and hence \(\alpha\). The bold part of the real line depicts values of \(\mu = \tan \alpha\) that produce accretive L-systems \(\Theta_{\mu,i}\).

Additional analytic properties of the functions (−\(m_\alpha(z)\))1 were described in [3, Theorem 6.5]. It was proved there that under the current set of assumptions we have:

1. the function \(1/m_\infty(z)\) is Stieltjes if and only if \(m_\infty(-0) \geq 0\);
2. the function \((-m_\infty(z))\) is never Stieltjes;
3. the function \((-m_\alpha(z))\) given by (2) is Stieltjes if and only if

\[
0 < \frac{1}{m_\infty(-0)} \leq \tan \alpha.
\]

Now we are going to turn to the case when our realizing L-system \(\Theta_{\mu,i}\) is accretive sectorial. To begin with let \(\Theta\) be an L-system of the form (0), where \(\mathcal{A}\) is a (\(*\))-extension (1) of the accretive Schrödinger operator \(T_h\). Here we summarize and list some known facts about possible accretivity and sectoriality of \(\Theta\).

- The operator \(\mathcal{A}_{\mu,h}\) of \(\Theta_{\mu,h}\) is accretive if and only if (2) holds (see [2]).
- According to Theorem 2 if an accretive operator \(T_h\), \((\text{Im} \, h > 0)\), is \(\beta\)-sectorial, then (2) holds. Conversely, if \(h\), \((\text{Im} \, h > 0)\), is such that \(\text{Re} \, h > -m_\infty(-0)\), then operator \(T_h\) of the form (0) is \(\beta\)-sectorial and \(\beta\) is determined by (2).
- \(T_h\) is accretive but not \(\beta\)-sectorial for any \(\beta \in (0, \pi/2)\) if and only if \(\text{Re} \, h = -m_\infty(-0)\).
- If \(\Theta_{\mu,h}\) is such that \(\mu = +\infty\), then \(V_{\Theta_{\mu,h}}(z)\) belongs to the class \(S^{0,\beta}\). In the case when \(\mu \neq +\infty\) we have \(V_{\Theta_{\mu,h}}(z) \in S^{\beta_1,\beta_2}\) (see [3]).
- The operator \(\mathcal{A}_{\mu,h}\) is a \(\beta\)-sectorial \((\ast)\)-extension of \(T_h\) (with the same angle of sectoriality) if and only if \(\mu = +\infty\) in (8) (see [2], [8]).
- If \(T_h\) is \(\beta\)-sectorial with the exact angle of sectoriality \(\beta\), then it admits only one \(\beta\)-sectorial \((\ast)\)-extension \(\mathcal{A}_{\mu,h}\) with the same angle of sectoriality \(\beta\). Consequently, \(\mu = +\infty\) and \(\mathcal{A}_{\mu,h} = \mathcal{A}_{\infty,h}\) has the form (3).
- A \((\ast)\)-extension \(\mathcal{A}_{\mu,h}\) of \(T_h\) is accretive but not \(\beta\)-sectorial for any \(\beta \in (0, \pi/2)\) if and only if the value of \(\mu\) in (4) is given by (5).

Note that it follows from the above that any \(\beta\)-sectorial operator \(T_h\) with the exact angle of sectoriality \(\beta \in (0, \pi/2)\) admits only one accretive \((\ast)\)-extension \(\mathcal{A}_{\mu,h}\) that

1It will be shown in an upcoming paper that if \(m_\infty(-0) \geq 0\), then the function \((-m_\infty(z))\) is actually inverse Stieltjes.
is not $\beta$-sectorial for any $\beta \in (0, \pi/2)$. This extension takes form (14) with $\mu$ given by (34).

Now let us consider a function $(-m_\alpha(z))$ and Schrödinger $L$-system $\Theta_{\tan \alpha, i}$ of the form (27) that realizes it. According to [7, Theorem 6.4-6.5] this $L$-system $\Theta_{\tan \alpha, i}$ is sectorial if and only if

$$\tan \alpha > \frac{1}{m_\infty(-0)}. \quad (37)$$

If we assume that $L$-system $\Theta_{\tan \alpha, i}$ is $\beta$-sectorial, then its impedance function $V_{\Theta_{\tan \alpha, i}}(z) = -m_\alpha(z)$ belongs to certain sectorial classes discussed in Section 4. Namely, $(-m_\alpha(z)) \in S^\beta$. The following theorem provides more refined properties of $(-m_\alpha(z))$ for this case.

**Theorem 6.** Let $\Theta_{\tan \alpha, i}$ be the accretive $L$-system of the form (27) realizing the function $(-m_\alpha(z))$ associated with the non-negative operator $\hat{A}$. Let also $h_{\tan \alpha, i}$ be a $\beta$-sectorial $(\ast)$-extension of $T_i$ defined by (22). Then the function $(-m_\alpha(z))$ belongs to the class $S^{\beta_1, \beta_2}$, $\tan \beta_2 \leq \tan \beta_1$, $\tan \beta_1 = \cot \alpha$, and

$$\tan \beta_2 = \frac{\tan \alpha + m_\infty(-0)}{(\tan \alpha)m_\infty(-0) - 1}. \quad (39)$$

Moreover, the operator $T_i$ is $(\beta_2 - \beta_1)$-sectorial with the exact angle of sectoriality $(\beta_2 - \beta_1)$.

**Proof.** It is given that $\Theta_{\tan \alpha, i}$ is $\beta$-sectorial and hence (37) holds. For further convenience we re-write $(-m_\alpha(z))$ as

$$-m_\alpha(z) = \frac{\sin \alpha + m_\infty(z)\cos \alpha}{-\cos \alpha + m_\infty(z)\sin \alpha} = \frac{\tan \alpha + m_\infty(z)}{(\tan \alpha)m_\infty(z) - 1}. \quad (40)$$

Since under our assumption $\Theta_{\tan \alpha, i}$ is $\beta$-sectorial, then its impedance function $V_{\Theta_{\tan \alpha, i}}(z) = -m_\alpha(z)$ belongs to certain sectorial classes discussed in Section 4. Namely, $-m_\alpha(z) \in S^\beta$ and $-m_\alpha(z) \in S^{\beta_1, \beta_2}$. In order to describe $\beta_1$ we take into account (see [2, Section 10.3]) that $\lim_{x \to -\infty} m_\infty(x) = +\infty$ to obtain

$$\tan \beta_1 = \lim_{x \to -\infty} (-m_\alpha(x)) = \frac{\tan \alpha + m_\infty(-\infty)}{(\tan \alpha)m_\infty(-\infty) - 1} = \frac{\tan \alpha + m_\infty(-\infty) + 1}{\tan \alpha - m_\infty(-\infty)} = \frac{1}{\tan \alpha} = \cot \alpha.$$  

In order to get $\beta_2$ we simply pass to the limit in (40). The above confirms (38) and (27). In order to show the rest, we apply [2, Theorem 9.8.4]. This theorem states that if $\hat{A}$ is a $\beta$-sectorial $(\ast)$-extension of a main operator $T$ of an $L$-system $\Theta$, then the impedance function $V_\Theta(z)$ belongs to the class $S^{\beta_1, \beta_2}$, $\tan \beta_2 \leq \tan \beta_1$, and $T$ is $(\beta_2 - \beta_1)$-sectorial with the exact angle of sectoriality $(\beta_2 - \beta_1)$. It can also be checked directly that formulas (38) and (27) (under condition (27)) imply $0 < \beta_2 - \beta_1 < \pi/2$ and hence the definition of $(\beta_2 - \beta_1)$-sectoriality applies correctly. \qed
Now we state and prove the following.

**Theorem 7.** Let $\Theta_{\tan,\alpha,i}$ be an accretive $L$-system of the form (27) that realizes $(-m_\alpha(z))$, where $A_{\tan,\alpha,i}$ is a $(\ast)$-extension of a $\theta$-sectorial operator $T_i$ with exact angle of sectoriality $\theta$. Let also $\alpha_\ast \in \left( \arctan \left( \frac{1}{m_\infty(-0)} \right), \frac{\pi}{2} \right)$ be a fixed value that defines $A_{\tan,\alpha,i}$ via (14), and $(-m_\alpha(z)) \in S_{\beta_1,\beta_2}$. Then a $(\ast)$-extension $A_{\tan,\alpha,i}$ of $T_i$ is $\beta$-sectorial for any $\alpha \in [\alpha_\ast, \pi/2)$ with

$$\tan \beta = \tan \beta_1 + 2\sqrt{\tan \beta_1 \tan \beta_2}, \quad \tan \beta > \tan \theta.$$ 

Moreover, if $\alpha = \pi/2$, then

$$\beta = \beta_2 - \beta_1 = \theta = \arctan \left( \frac{1}{m_\infty(-0)} \right).$$

**Proof.** We note first that the conditions of our theorem imply the following $\tan \alpha_\ast \in \left( \frac{1}{m_\infty(-0)}, +\infty \right)$. Thus, according to [7, Theorem 6.4] part 2(c), a $(\ast)$-extension $A_{\tan,\alpha,i}$ is $\beta$-sectorial for some $\beta \in (0, \pi/2)$. Then we can apply Theorem 6 to confirm that $(-m_\alpha(z)) \in S_{\beta_1,\beta_2}$, where $\beta_1$ and $\beta_2$ are described by (38) and (39). The first part of formula (41) follows from [8, Theorem 8] applied to the $L$-system $\Theta_{\tan,\alpha,i}$ with $\mu = \tan \alpha$ (see also [2, Theorem 9.8.7]). Note that since $A_{\tan,\alpha,i}$ is a $\beta$-sectorial extension of a $\theta$-sectorial operator $T_i$, then $\tan \beta \geq \tan \theta$ with equality possible only when $\mu = \tan \alpha = \infty$ (see [2, 3]). Since we chose $\alpha \in [\alpha_\ast, \pi/2)$, then $\tan \alpha \neq \infty$ and hence $\tan \beta > \tan \theta$ that confirms the second part of formula (41).

If we assume that $\alpha = \pi/2$, then our function $(-m_\alpha(z)) = 1/m_\infty(z)$ is realized with $\Theta_{\infty,i}$ (see Theorem 6) that preserves the angle of sectoriality of its main operator $T_i$ (see [2, Theorem 6.4] and Figure 4). Therefore, $\beta = \theta$. If we combine this fact with $(-m_\alpha(z)) \in S_{\beta_1,\beta_2}$ and apply Theorem 6 we get that $\beta = \beta_2 - \beta_1$. Finally, since $T_i$ is $\theta$-sectorial, formula (32) yields $\tan \theta = \frac{1}{m_\infty(-0)}$. □

**Figure 2.** Angle of sectoriality $\beta$. Here $\alpha_0 = \arctan \left( \frac{1}{m_\infty(-0)} \right)$. 


Note that Theorem 1 provides us with a value $\beta$ which serves as a universal angle of sectoriality for the entire indexed family of $(\ast)$-extensions $\tilde{A}_{\tan \alpha,i}$ of the form (2) as depicted on Figure 4. It is clearly shown on the figure that if $\alpha = \pi/2$, then $\tan \beta = \tan \theta$.

7. Example

We conclude this paper with a simple illustration. Consider the differential expression with the Bessel potential

$$l_\nu = \frac{d^2}{dx^2} + \frac{\nu^2 - 1/4}{x^2}, \quad x \in [1, \infty)$$

of order $\nu > 0$ in the Hilbert space $\mathcal{H} = L^2[1, \infty)$. The minimal symmetric operator

\begin{equation}
\begin{cases}
\hat{A}y = -y'' + \frac{\nu^2 - 1/4}{x^2}y \\
y(1) = y'(1) = 0
\end{cases}
\end{equation}

generated by this expression and boundary conditions has defect numbers $(1,1)$. Let $\nu = 3/2$. It is known \cite{4} that in this case

$$m_\infty(z) = -\frac{iz - \frac{3}{2}\sqrt{z} - \frac{3}{2}i}{\sqrt{z} + i} - \frac{1}{2} = \frac{\sqrt{z} - iz + i}{\sqrt{z} + i} = 1 - \frac{iz}{\sqrt{z} + i}$$

and $m_\infty(-0) = 1$. The minimal symmetric operator then becomes

\begin{equation}
\begin{cases}
\hat{A}y = -y'' + \frac{3}{2\pi}y \\
y(1) = y'(1) = 0.
\end{cases}
\end{equation}

The main operator $T_i$ of the form (13) is written for $h = i$ as

\begin{equation}
\begin{cases}
T_i y = -y'' + \frac{2}{\pi}y \\
y'(1) = iy(1)
\end{cases}
\end{equation}

will be shared by all the family of L-systems realizing functions $(-m_\alpha(z))$ described by (23)–(24). This operator is accretive and $\beta$-sectorial since $\operatorname{Re} h = 0 > -m_\infty(-0) = -1$ with the exact angle of sectoriality given by (see (24))

\begin{equation}
\tan \beta = \frac{\operatorname{Im} h}{\operatorname{Re} h + m_\infty(-0)} = \frac{1}{0 + 1} = 1 \text{ or } \beta = \frac{\pi}{4}.
\end{equation}

A family of L-systems $\Theta_{\tan \alpha,i}$ of the form (27) that realizes functions $(-m_\alpha(z))$ described by (23)–(24) as

\begin{equation}
-m_\alpha(z) = \frac{(\sqrt{z} - iz + i)\cos \alpha + (\sqrt{z} + i)\sin \alpha}{(\sqrt{z} - iz + i)\sin \alpha - (\sqrt{z} + i)\cos \alpha},
\end{equation}

was constructed in \cite{7}. According to \cite[Theorem 6.3]{7} the L-systems $\Theta_{\tan \alpha,i}$ in (27) are accretive if

$$1 = \frac{1}{m_\infty(-0)} \leq \tan \alpha < +\infty.$$

Using part (2c) of \cite[Theorem 6.4]{7}, we get that the realizing L-system $\Theta_{\tan \alpha,i}$ in (27) preserves the angle of sectoriality and becomes $\frac{1}{4}$-sectorial if $\mu = \tan \alpha = +\infty$ or $\alpha = \pi/2$. Therefore the L-system

\begin{equation}
\Theta_{\infty,i} = \left( \mathcal{H}_+ \subset L_2[1, +\infty) \subset \mathcal{H}_- \right) + \frac{\mathcal{H}_\infty,i}{1 + \mathbb{C}},
\end{equation}

\(\mathcal{H}_+ \subset L_2[1, +\infty) \subset \mathcal{H}_-\)
where
\[ A_{\infty,i} y = -y'' + \frac{2}{x^2} y - [y'(1) - iy(1)] \delta(x - 1), \]
(48)
\[ A_{\infty,i}^* y = -y'' + \frac{2}{x^2} y - [y'(1) + iy(1)] \delta(x - 1), \]
\[ K_{\infty,i} c = cg_{\infty,i} \quad (c \in \mathbb{C}) \] and \( y_{\infty,i} = \delta(x - 1) \), realizes the function \(-m_\varpi(z) = 1/m_\infty(z)\). Also,
\[ V_{\Theta_{\infty,i}}(z) = -m_\varpi(z) = \frac{1}{m_\infty(z)} = \frac{\sqrt{z} + i}{\sqrt{z} - iz + i} \]
(49)
\[ W_{\Theta_{\infty,i}}(z) = (-e^{\pi i})^{m_\infty(z) - i} \frac{m_\infty(z) + i}{m_\infty(z) + i} = \frac{(1 - i)\sqrt{z} - iz + 1 + i}{(1 + i)\sqrt{z} - iz - 1 + i}. \]

This \( \Theta_{\infty,i} \) is clearly accretive according to [4] Theorem 6.2 which is also independently confirmed by direct evaluation

\[ (\text{Re} A_{\infty,i} y, y) = \|y''(x)\|_{L^2}^2 + 2\|y(x)/x\|_{L^2}^2 \geq 0. \]

Moreover, according to [5] Theorem 6.4 (see also [6] Theorem 9.8.7) the \( \Theta_{\infty,i} \) is \( \frac{\pi}{4} \)-sectorial. Taking into account that \((\text{Im} A_{\infty,i} y, y) = |y(1)|^2\), (see formula \((5)\)) we obtain inequality \((4)\) with \( \beta = \frac{\pi}{4} \), that is \((\text{Re} A_{\infty,i} y, y) \geq |(\text{Im} A_{\infty,i} y, y)|\), or
\[ \|y''(x)\|_{L^2}^2 + 2\|y(x)/x\|_{L^2}^2 \geq |y(1)|^2. \]

In addition, we have shown that the \( \beta \)-sectorial form \((T_{\beta}, y, y)\) defined on \( \text{Dom}(T_{\beta}) \) can be extended to the \( \beta \)-sectorial form \((A_{\infty,i} y, y)\) defined on \( \mathcal{H}_+ = \text{Dom}(A^+) \) (see \((13)-(14)\)) having the exact (for both forms) angle of sectoriality \( \beta = \pi/4 \). A general problem of extending sectorial sesquilinear forms to sectorial ones was mentioned by T. Kato in [7]. It can be easily seen that function \(-m_\varpi(z)\) in \((10)\) belongs to a sectorial class \( S^{0,\pi/4} \) of Stieltjes functions.

References

[1] D. Alpay, E. Tsekanovskii, Interpolation theory in sectorial Stieltjes classes and explicit system solutions, Lin. Alg. Appl., 314, (2000), 91–136.
[2] Yu. Arlinskiǐ, S. Belyi, E. Tsekanovskii Conservative Realizations of Herglotz-Nevanlinna functions, Operator Theory: Advances and Applications, Vol. 217, Birkhäuser Verlag, 2011.
[3] Yu. Arlinskiǐ, E. Tsekanovskii, M. Krein's research on semi-bounded operators, its contemporary developments, and applications, Oper. Theory Adv. Appl., vol. 190, (2009), 65–112.
[4] Yu. Arlinskiǐ, E. Tsekanovskii, Linear systems with Schrödinger operators and their transfer functions, Oper. Theory Adv. Appl., 149, 2004, 47–77.
[5] S. Belyi, Sectorial Stieltjes functions and their realizations by \( L \)-systems with Schrödinger operator, Mathematische Nachrichten, vol. 285, no. 14-15, (2012), pp. 1729-1740.
[6] S. Belyi, K. A. Makarov, E. Tsekanovskii, Conservative \( L \)-systems and the Livšic function. Methods of Functional Analysis and Topology, 21, no. 2, (2015), 104–133.
[7] S. Belyi, E. Tsekanovskii, On realization of the original Weyl-Titchmarsh functions by Schrödinger \( L \)-systems, Complex Analysis and Operator Theory, 15 (11), (2021).
[8] S. Belyi, E. Tsekanovskii, On Sectorial \( L \)-systems with Schrödinger operator, Differential Equations, Mathematical Physics, and Applications. Selim Grigorievich Krein Centennial, CONM, vol. 734, American Mathematical Society, Providence, RI (2019), 59-76.
[9] S. Belyi, E. Tsekanovskii, Stieltjes like functions and inverse problems for systems with Schrödinger operator. Operators and Matrices, vol. 2, No.2, (2008), 265–296.
[10] S. Belyi, S. Hassi, H.S.V. de Snoo, E. Tsekanovskii, A general realization theorem for matrix-valued Herglotz-Nevanlinna functions, Linear Algebra and Applications. vol. 419, (2006), 331–358.
[11] Yu. Berezansky, *Expansion in eigenfunctions of self-adjoint operators*, vol. 17, Transl. Math. Monographs, AMS, Providence, 1968.

[12] A.A. Danielyan, B.M. Levitan, *On the asymptotic behaviour of the Titchmarsh-Weyl m-function*, Izv. Akad. Nauk SSSR Ser. Mat., Vol. 54, Issue 3, (1990), 469–479.

[13] V. Derkach, M.M. Malamud, E. Tsekanovskii, *Sectorial Extensions of Positive Operators*. (Russian), Ukrainian Mat.J. 41, No.2, (1989), pp. 151–158.

[14] I. Dovženko and E. Tsekanovskii, *Classes of Stieltjes operator-functions and their conservative realizations*, Dokl. Akad. Nauk SSSR, 311 no. 1 (1990), 18–22.

[15] F. Gesztesy, E. Tsekanovskii, *On Matrix-Valued Herglotz Functions*. Math. Nachr. 218, (2000), 61–128.

[16] I.S. Kac, M.G. Krein, *R-functions – analytic functions mapping the upper halfplane into itself*, Amer. Math. Soc. Transl., Vol. 2, 103, 1-18, 1974.

[17] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, 1966.

[18] B.M. Levitan, *Inverse Sturm-Liouville problems*. Translated from the Russian by O. Ermlov. VSP, Zeist, (1987)

[19] M.S. Livšic, *Operators, oscillations, waves*. Moscow, Nauka, (1966)

[20] M.A. Naimark, *Linear Differential Operators II*, F. Ungar Publ., New York, 1968.

[21] E.C. Titchmarsh, *Eigenfunction Expansions Associated with Second-Order Differential Equations*. Part I, 2nd ed., Oxford University Press, Oxford, (1962).

[22] E. Tsekanovskii, *Accretive extensions and problems on Stieltjes operator-valued functions relations*, Operator Theory: Adv. and Appl., 59, (1992), 328–347.

[23] E. Tsekanovskii, *Characteristic function and sectorial boundary value problems*. Research on geometry and math. analysis, Proceedings of Mathematical Institute, Novosibirsk, 7, 180–194 (1987)

[24] E. Tsekanovskii, *Friedrichs and Krein extensions of positive operators and holomorphic contraction semigroups*. Funct. Anal. Appl. 15, 308–309 (1981).

[25] E. Tsekanovskii, *Non-self-adjoint accretive extensions of positive operators and theorems of Friedrichs-Krein-Phillips*. Funct. Anal. Appl. 14, 156–157 (1980).

[26] E. Tsekanovskii, Yu.I. Šmuljan, *The theory of bi-extensions of operators on rigged Hilbert spaces*. Unbounded operator colligations and characteristic functions, Russ. Math. Surv., 32, (1977), 73–131.

[27] H. Weyl, *Über gewöhnliche lineare Differentialgleichungen mit singularen Stellen und ihre Eigenfunktionen*, (German), Göttinger Nachrichten, 37–64 (1907).

[28] H. Weyl, *Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen*. (Math. Ann., 68, no. 2, (1910), 220–269.

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