First Order Formalism
for Massive Mixed Symmetry Tensor Fields
in Minkowski and (A)dS Spaces

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Abstract

In this paper we extend our recent results (hep-th/0304067) on the first order formulation for the massless mixed symmetry tensor fields to the case of massive fields both in Minkowski as well as in (Anti) de Sitter spaces (including all possible massless and partially massless limits). Main physical results are essentially the same as in hep-th/0211233.

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Introduction

Last times there is a renewed interest in the mixed symmetry high spin tensor fields [1]-[7]. The reason is that such fields naturally appear in a number of physically interesting theories such as superstrings, higher dimensional supergravities and (supersymmetric) high spin theories. One of the technical difficulties (besides purely combinatorial ones) one faces working with such fields is that to get analog of gauge invariant ”field strengths” one has to build expressions with more and more derivatives (or has to work with non-local terms in the equations of motion or the Lagrangians) [8, 9, 10, 11]. The structure of gauge transformation laws is also appears to be rather complicated, moreover, these transformations often turn out to be reducible. All this make the problem of investigation of possible interactions among such fields a very complicated task [12, 13, 14, 15, 16, 17]. Recently [18] we have constructed a few examples of first order formulation for such mixed symmetry fields which turned out to be very similar to the well known tetrad formulation of gravity\(^1\). The Lagrangians obtained have simple and very suggestive form, so that one could hope that such formulation can help in investigations of possible interactions among these fields. Let us note also that first order ”parent” Lagrangians play a very important role in investigations of dualities for such fields [20, 21]. In this paper we construct examples of first order formulation for massive mixed symmetry tensor fields both in flat Minkowski as well as in (Anti) de Sitter spaces. Particles in (A)dS reveal a number of very peculiar features such as unitary forbidden regions (i.e. not all values of mass and cosmological constant are allowed) and appearance of partially massless theories [22]-[30]. Moreover not all massless fields in flat Minkowski space could be deformed into the (A)dS space without introduction of additional fields [31]. In our previous works on this subject [32, 33] we used gauge invariant description of massive particles. Such description being gauge invariant and unitary from the very beginning turns out to be very well suited for the investigation of unitarity, gauge invariance and partial masslessness. In this paper we construct such gauge invariant description of massive particles using a first order formalism for all appropriate massless components.

The paper is organized as follows. In the following section we give simple but very instructive example of massive spin-2 particle. In the next section we give a first order gauge invariant formulation for the simplest mixed symmetry tensor field \(\Phi^{\mu\nu}_{\ a}\) including possible massless and partially massless limits. The last section devoted to analogous construction for the tensor \(R^{\mu\nu}_{\ ab}\) having the symmetries of Riemann tensor.

1 Second rank tensor

Let us start with the simple example of second rank tensor \(h^{\mu\nu}_{\ a}\). Strictly speaking it is not a mixed symmetry tensor, but this case turns out to be very instructive and interesting by itself. To have gauge invariant description of massive spin-2 particle one has to introduce two additional (Goldstone) fields — vector \(A_{\mu}\) and scalar \(\varphi\) ones. To be consistent in what follows we will use first order formalism for all massless components which will serve as building blocks for our massive particles. So we introduce three pairs of fields: \((\omega^{\mu\nu}_{\ ab}, h^{\mu\nu}_{\ a})\),

\(^1\)First order formulation for high spin fields corresponding to symmetric tensors has been given recently in [19]
\((F^{ab}, A_\mu)\) and \((\pi^a, \varphi)\). As in our previous papers on this subject \[32, 33\] our starting point will be just the sum of the massless Lagrangians in flat Minkowski space:

\[
L_0 = L_0(\omega^{ab}_\mu, h^a_\mu) + L_0(F^{ab}, A_\mu) + L_0(\pi^a, \varphi) \tag{1}
\]

\[
L_0(\omega^{ab}_\mu, h^a_\mu) = \frac{1}{2} \{ \mu^\nu \}_a \omega^{ac}_\mu \omega^{bc}_\nu - \frac{1}{2} \{ \mu^\nu a \}_b \omega^{ab}_\mu \partial_\nu h^c_\alpha \]

\[
L_0(F^{ab}, A_\mu) = \frac{1}{4} F^{ab}_\mu F^{\mu}_{ab} - \frac{1}{2} \{ \mu^\nu \}_a F^{ab}_\mu \partial_\nu A_\mu
\]

\[
L_0(\pi^a, \varphi) = -\frac{1}{2} \pi^a_\mu \pi^a_\mu + \{ \mu^a \}_a \pi^a_\mu \partial_\mu \varphi
\]

Here

\[
\{ \mu^\nu \}_a = \delta^\nu_a \delta^\mu_b - \delta^\nu_b \delta^\mu_a
\]

and so on. This Lagrangian is invariant under the following local gauge transformations:

\[
\delta_0 h^a_\mu = \partial_\mu \xi_a + \eta^a_\mu \quad \delta_0 \omega^{ab}_\mu = \partial_\mu \eta^{ab} \quad \delta_0 A_\mu = \partial_\mu \Lambda \tag{2}
\]

Working with the first order formalism it is very convenient to use tetrad formulation of the underlying (Anti) de Sitter space. We denote tetrad as \(e^a_\mu\) (let us stress that it is not a dynamical quantity here, just a background field) and Lorentz covariant derivative as \(D_\mu\).

(Anti) de Sitter space is a constant curvature space with zero torsion, so we have:

\[
D_\mu [\mu^a_\nu] = 0, \quad [D_\mu, D_\nu] v^a = R^{\mu\nu}_{ab} v^b = \kappa (e^a_\mu e^b_\nu - e^b_\mu e^a_\nu) v^b \tag{3}
\]

where \(\kappa = -2\Lambda/(d - 1)(d - 2)\).

Now we replace all the derivatives in the Lagrangian and gauge transformation laws by the covariant ones. Due to noncommutativity of covariant derivatives the Lagrangian becomes not invariant and we get:

\[
\delta_0 L_0 = \kappa (d - 2) \omega^a \xi_a - \kappa (d - 2) h^{ab} \eta_{ab} \tag{4}
\]

So we proceed by adding to the Lagrangian additional low derivatives terms:

\[
L_1 = a_1 \{ \mu^\nu \}_a \omega^{ab}_\mu A_\nu + a_2 \{ \mu^a \}_b F^{ab}_\mu h^b_\mu + a_3 \{ \mu^a \}_a \pi^a A_\mu \tag{5}
\]

as well as corresponding corrections to the gauge transformation laws:

\[
\delta_1 h^a_\mu = \alpha_1 e^a_\mu \Lambda \quad \delta_1 F^{ab}_\mu = \alpha_2 \eta^{ab}_\mu \quad \delta_1 A_\mu = \alpha_3 \xi_\mu \quad \delta_1 \varphi = \alpha_4 \Lambda \tag{6}
\]

In this, the requirement that all variations in that order cancel each other i.e. \(\delta_0 L_1 + \delta_1 L_0 = 0\) gives us the following relations:

\[
\alpha_1 = \frac{2a_1}{d - 2}, \quad \alpha_2 = -2a_1, \quad \alpha_3 = a_1, \quad \alpha_4 = -a_3, \quad a_2 = a_1
\]

At last, we introduce all possible mass-like terms into Lagrangian:

\[
L_2 = b_1 \{ \mu^\nu \}_b h^a_\mu h^b_\nu + b_2 \{ \mu^a \}_a h^a_\mu \varphi + b_3 \varphi^2 \tag{7}
\]
as well as necessary corrections to the gauge transformations:
\[
\delta_2 \omega_{\mu}^{ab} = \beta_1 (e_{\mu}^{a} \xi^{b} - e_{\mu}^{b} \xi^{a}) \quad \delta_2 \pi^{a} = \beta_2 \xi^{a}
\] (8)

Then, if one requires that all new variations \(\delta_0 L_2 + \delta_1 L_1 + \delta_2 L_0\) cancel each other (taking into account the residue of \(\delta_0 L_0\)), then one could express all the parameters in the Lagrangian and the gauge transformation laws in terms of two parameters, say \(a_1\) and \(a_3\):
\[
\beta_1 = -\frac{2b_1}{d-2}, \quad \beta_2 = b_2 = a_1 a_3, \quad b_1 = a_1^2 + \frac{\kappa (d-2)}{2}, \quad b_3 = \frac{d}{d-2} a_1^2
\]

In this, the whole Lagrangian will be invariant under all gauge transformations provided:
\[
4(d-1)a_1^2 - (d-2)a_3^2 = -2\kappa (d-1)(d-2)
\] (9)

Now, having in our disposal complete description of general massive particle in (A)dS space we can investigate all possible massless or partially massless limits. Recall that parameter \(\kappa\) is proportional to the cosmological constant \(\Lambda\) so it could be positive as well as negative. As a result one get two possible special cases. In the de Sitter space \((\kappa < 0)\) one can set \(a_3 = 0\). In this, scalar component \((\pi^a, \varphi)\) completely decouples from the system, while the rest fields with the Lagrangian:
\[
L = L_0(\omega_{\mu}^{ab}, h_{\mu}^a) + m \left\{ \mu^\nu \right\} \omega_{\mu}^{ab} A_\nu + m \left\{ \mu \right\} F^{ab} h_{\mu}^b
\] (10)

where \(m^2 = -\kappa (d-2)/2\), which is invariant under the following gauge transformations:
\[
\delta h_{\mu}^a = D_{\mu} \xi^a + e_{\mu b} \eta^b + \frac{2m}{d-2} e_{\mu}^a \Lambda \quad \delta \omega_{\mu}^{ab} = D_{\mu} \eta^{ab}
\]
\[
\delta A_\mu = D_\mu \Lambda + m e_{\mu a} \xi^a \quad \delta F^{ab} = -2m \eta^{ab}
\] (11)
describe the rather well known partially massless spin-2 particle. Note, that in this particular limit there are no any explicit mass-like terms in the Lagrangian.

On the other side, in the Anti de Sitter space one can set \(a_1 = 0\). In this, the whole system also decompose onto two subsystems. One of them is just a usual massless spin-2 particle with the Lagrangian:
\[
L = L_0(\omega_{\mu}^{ab}, h_{\mu}^a) + \frac{\kappa (d-2)}{2} \left\{ \mu^\nu \right\} h_{\mu}^a h_{\nu}^b
\] (12)
and corresponding gauge transformations:
\[
\delta h_{\mu}^a = D_{\mu} \xi^a + e_{\mu b} \eta^b \quad \delta \omega_{\mu}^{ab} = D_{\mu} \eta^{ab} - \kappa (e_{\mu}^a \xi^b - e_{\mu}^b \xi^a)
\] (13)

The other subsystem with the Lagrangian:
\[
L = L_0(F^{ab}, A_\mu) + L_0(\pi^a, \varphi) + m \left\{ \mu \right\} \pi^a A_\mu
\] (14)
where \(m^2 = 2\kappa (d-1)\), which is invariant under
\[
\delta A_\mu = D_\mu \Lambda \quad \delta \varphi = -m \Lambda
\]
describes massive vector particle (in a gauge invariant first order formalism). Note that in such description there is no explicit mass term for the vector field \(A_\mu\), but if one solves (algebraic) equation of motion for the field \(\pi^a\) and puts the result back into the Lagrangian one obtains appropriate term.
2 \( \Phi_{[\mu\nu],\alpha} \) tensor

In this section we consider a truly mixed symmetry tensor \( \Phi_{[\mu\nu],\alpha} \). In \cite{33} we have shown that gauge invariant description of corresponding massive particle requires introduction of three additional Goldstone fields: second rank tensor \( h_{\mu}^{\ a} \), two form \( B_{[\mu\nu]} \) and vector \( A_\mu \) ones. To construct appropriate first order formalism we introduce four pairs of fields: \( (\Omega_{\mu}^{abc}, \Phi_{\mu}^{a}) \), \( (\omega_{\mu}^{ab}, h_{\mu}^{\ a}) \), \( (C^{abc}, B_{\mu\nu}) \) and \( (F^{ab}, A_\mu) \). We start with the sum of massless first order Lagrangians in flat Minkowski space:

\[
\mathcal{L} = \mathcal{L}_0(\Omega_{\mu}^{abc}, \Phi_{\mu}^{a}) + \mathcal{L}_0(\omega_{\mu}^{ab}, h_{\mu}^{\ a}) + \mathcal{L}_0(C^{abc}, B_{\mu\nu}) + \mathcal{L}_0(F^{ab}, A_\mu) \tag{15}
\]

\[
\mathcal{L}_0(\Omega_{\mu}^{abc}, \Phi_{\mu}^{a}) = -\frac{3}{4} \{ \omega_{\mu}^{ab} \Omega_{\mu}^{a} \} + \frac{1}{4} \Omega_{\mu}^{abc} \partial_{\nu} \Phi_{\nu}^{d} \]

\[
\mathcal{L}_0(C^{abc}, B_{\mu\nu}) = -\frac{1}{6} C_{abc}^{2} + \frac{1}{6} \{ \omega_{\mu}^{ab} \} C^{abc} \partial_{\nu} B_{\nu a} \]

where \( \mathcal{L}_0(\omega_{\mu}^{ab}, h_{\mu}^{\ a}) \) and \( \mathcal{L}_0(F^{ab}, A_\mu) \) are the same as in the previous section. This Lagrangian is invariant under the following set of local gauge transformations:

\[
\delta_{0} \Phi_{\mu}^{a} = \partial_{\mu} \zeta^{a} + \eta^{a} \delta_{\mu}, \quad \delta_{0} \Omega_{\mu}^{abc} = \partial_{\mu} \eta^{abc} \quad \delta_{0} h_{\mu}^{\ a} = \partial_{\mu} \eta^{ab} \quad \delta_{0} B_{\mu\nu} = \partial_{\mu} \zeta^{\nu} - \partial_{\nu} \zeta^{\mu} \quad \delta_{0} A_\mu = \partial_{\mu} \Lambda \tag{16}
\]

Now we replace all the derivatives in the Lagrangian and gauge transformation laws by the Lorentz covariant ones (with the same notations and conventions for the description of the (Anti) de Sitter space as in the previous section). As usual, the Lagrangians becomes to be non invariant under the gauge transformations and we get:

\[
\delta_{0} \mathcal{L} = 3\kappa(d-3)[\frac{1}{2} \Phi^{abc} \eta_{abc} - \Omega^{ab} z_{ab}] + \kappa(d-2)[\omega^{a} \zeta^{a} - h^{ab} \eta_{ab}] \tag{17}
\]

Now we add to the Lagrangian all possible low derivatives terms which could be written as forms with equal number of "world" and "local" indices:

\[
\mathcal{L}_1 = a_1 \{ \omega_{\mu}^{ab} \} \Omega_{\mu}^{abc} h_{\nu}^{\ c} + a_2 \{ \omega_{\mu}^{ab} \} \Omega_{\mu}^{abc} \Phi_{\nu}^{c} + a_3 \{ \omega_{\mu}^{ab} \} \Omega_{\mu}^{abc} B_{\nu a} + a_4 \{ \omega_{\mu}^{ab} \} C^{abc} \Phi_{\mu}^{c} + a_5 \{ \omega_{\mu}^{ab} \} \omega_{\mu}^{ab} F^{ab} h_{\nu}^{\ b} + a_6 \{ \omega_{\mu}^{ab} \} F^{ab} B_{\mu\nu} \tag{18}
\]

as well as corresponding terms to the gauge transformation laws:

\[
\delta_{1} \Phi_{\mu\nu} = \alpha_1 (e_{\mu}^{a} \zeta^{b} - e_{\mu}^{b} \zeta^{a}) + \alpha_2 (e_{\mu}^{a} \zeta^{b} - e_{\mu}^{b} \zeta^{a}) \quad \delta_{1} \Omega_{\mu}^{abc} = \alpha_3 e_{\mu}^{a} \eta^{bc} \]

\[
\delta_{1} h_{\mu}^{\ a} = \alpha_4 \zeta^{\mu} + \alpha_5 e_{\mu}^{a} \Lambda \quad \delta_{1} \omega_{\mu}^{ab} = \alpha_6 e_{\mu}^{a} \eta^{ab} \tag{19}
\]

\[
\delta_{1} B_{\mu\nu} = \alpha_7 \zeta_{\mu\nu} \quad \delta_{1} C^{abc} = \alpha_8 \eta^{abc} \quad \delta_{1} F^{ab} = \alpha_9 \eta^{ab} \quad \delta_{1} A_\mu = \alpha_{10} \zeta_\mu + \alpha_{11} \zeta_\mu
\]

Note that in this case we can’t add to the Lagrangian any explicit mass-like terms, because the only possible term \( \{ \omega_{\mu}^{ab} \} h_{\mu}^{\ a} h_{\nu}^{\ b} \) is forbidden by \( \Lambda \)-symmetry. Nevertheless it turned out possible to achieve full gauge invariance of the Lagrangian by appropriate choice of the parameters. Once again all the parameters could be expressed in terms of two ones (we choose \( a_1 \) and \( a_3 \))

\[
\alpha_1 = \frac{2a_1}{3(d-3)}, \quad \alpha_2 = \frac{4a_3}{d-3}, \quad \alpha_3 = 2\alpha_1, \quad \alpha_4 = 4a_1, \quad \alpha_5 = \frac{2a_5}{d-2}, \quad \alpha_6 = -2a_1
\]
\[ \alpha_7 = -4a_3, \quad \alpha_8 = 6a_3, \quad \alpha_9 = -2a_5, \quad \alpha_{10} = a_5, \quad \alpha_{11} = 4a_7 \]

\[ a_2 = a_1, \quad a_4 = a_3, \quad a_6 = a_5, \quad a_5 = 2 \sqrt{\frac{d-2}{d-3}} a_3, \quad a_7 = \sqrt{\frac{d-2}{d-3}} a_1 \]

In this, the following constraint must be satisfied:

\[ a_1^2 - 3a_3^2 = \frac{3}{8} \kappa (d - 3) \quad (20) \]

In the gauge invariant description of massive particles we used to work the massless limit is just the limit where all Goldstone fields completely decouple from the main gauge field. For the case at hand it means that one has to set \( a_1 = 0 \) and \( a_3 = 0 \) simultaneously. But the last relation clearly shows that it is possible in the flat Minkowski space \( \kappa = 0 \) only. So there is no truly massless limit for the field \( \Phi_{\mu \nu}^a \) in (A)dS space. Instead there exist two partially massless limits depending on the sign of the cosmological term. In the Anti de Sitter space one can set \( a_3 = 0 \). In this, the whole system decouples onto two disconnected subsystems. One of them with the Lagrangian:

\[ L = L_0(\Omega_{\mu}^{abc}, \Phi_{\mu \nu}^a) + L_0(\omega_{\mu}^{ab}, h_{\mu}^a) + m \{^\mu_{\mu}^\nu\} \Omega_{\mu}^{abc} h_{\nu}^c + m \{^\mu_{\mu}^\nu\} \omega_{\mu}^{ab} \Phi_{\nu \alpha}^c \quad (21) \]

where \( m^2 = \frac{3}{8} \kappa (d - 3) \) is invariant under the following set of gauge transformations:

\[ \delta \Phi_{\mu \nu}^a = D_{\mu} z_{\nu}^a - D_{\nu} z_{\mu}^a + \eta_{\mu \nu}^a + \frac{2m}{3(d-3)} (e_{\mu}^a \zeta_{\nu} - e_{\nu}^a \zeta_{\mu}) \]

\[ \delta \Omega_{\mu}^{abc} = D_{\mu} \eta^{abc} + \frac{4m}{3(d-3)} e_{\mu}^{[a} \eta^{bc]} \quad (22) \]

\[ \delta h_{\mu}^a = D_{\mu} \zeta^a + \eta_{\mu}^a + 4m z_{\mu}^a \quad \delta \omega_{\mu}^{ab} = D_{\mu} \eta_{\mu}^{ab} - 2m \eta_{\mu}^{ab} \]

and describes partially massless theory \([31]\). The rest fields with the Lagrangian:

\[ L = L_0(C^{\mu \nu}, B_{\mu \nu}) + L_0(F_{\mu \nu}, A_{\mu}) + \frac{M}{4} \{^\mu_{\mu}^\nu\} F_{\mu \nu} B_{\mu \nu} \quad (23) \]

where \( M^2 = 6 \kappa (d - 2) \) and gauge transformations:

\[ \delta B_{\mu \nu} = D_{\mu} \zeta_{\nu} - D_{\nu} \zeta_{\mu} \quad \delta A_{\mu} = D_{\mu} \Lambda + M \zeta_{\mu} \quad (24) \]

gives a gauge invariant description of partially massless antisymmetric tensor in (A)dS space. Note again the absence of explicit mass term.

On the other hand, in the de Sitter space one can set \( a_1 = 0 \). Then one also obtains two decoupled subsystems. This time our mixed symmetry tensor \( \Phi_{\mu \nu}^a \) combines with the two form \( B_{\mu \nu} \) and gives us another example of partially massless theory with the Lagrangian:

\[ L = L_0(\Omega_{\mu}^{abc}, \Phi_{\mu \nu}^a) + L_0(C^{\mu \nu}, B_{\mu \nu}) + m \{^\mu_{\mu}^\nu\} \Omega_{\mu}^{abc} B_{\nu \alpha} + m \{^\mu_{\mu}^\nu\} C^{\mu \nu} \Phi_{\mu \nu}^c \quad (25) \]

where \( m^2 = -\kappa (d - 3)/8 \) and the following set of gauge transformations:

\[ \delta \Phi_{\mu \nu}^a = D_{\mu} z_{\nu}^a - D_{\nu} z_{\mu}^a + \eta_{\mu \nu}^a + \frac{4m}{d-3} (e_{\mu}^a \zeta_{\nu} - e_{\nu}^a \zeta_{\mu}) \quad \delta \Omega_{\mu}^{abc} = D_{\mu} \eta^{abc} \]

\[ \delta B_{\mu \nu} = D_{\mu} \zeta_{\nu} - D_{\nu} \zeta_{\mu} - 4m z_{\mu \nu} \quad \delta C^{\mu \nu} = 6m \eta^{abc} \quad (26) \]

In this, the rest fields \( (h_{\mu}^a, A_{\mu}) \) gives exactly the same partially massless spin-2 theory as in the previous section.
3 \textit{R}_{[\mu
u],[\alpha\beta]} \textit{tensor}

The results of the previous section could be easily generalized to the case of the mixed tensors with arbitrary number of "world" indices and the only "local" one \( \Phi_{[\mu_1\ldots\mu_a]}^a \). We will not proceed along this line here. Instead, in this section we consider more interesting field \( \textit{R}_{[\mu
u],[\alpha\beta]} \) having the symmetry of Riemann tensor. As we have shown in [33] for gauge invariant description of appropriate massive particle one needs two additional Goldstone fields: \( \Phi_{\mu
u}^a \) and \( h_\mu^a \). So to construct first order form of such description we introduce three pairs of fields: \( (\Sigma_{\mu
u}^{abc}, R_{\mu
u}^{ab}) \), \( (\Omega_{\mu}^{abc}, \Phi_{\mu}^a) \) and \( (\omega_{\mu}^{ab}, h_\mu^a) \). The sum of flat space massless Lagrangians:

\[
\mathcal{L}_0 = \mathcal{L}_0(\Sigma_{\mu
u}^{abc}, R_{\mu
u}^{ab}) + \mathcal{L}_0(\Omega_{\mu}^{abc}, \Phi_{\mu}^a) + \mathcal{L}_0(\omega_{\mu}^{ab}, h_\mu^a)
\]

where \( \mathcal{L}_0(\Omega_{\mu}^{abc}, \Phi_{\mu}^a) \) and \( \mathcal{L}_0(\omega_{\mu}^{ab}, h_\mu^a) \) are the same as before is invariant under the following gauge transformations:

\[
\begin{align*}
\delta_0 R_{\mu
u}^{ab} &= \partial_\mu \chi_{\nu}^{ab} - \partial_\nu \chi_{\mu}^{ab} + \psi_{\mu,\nu}^{ab} - \psi_{\nu,\mu}^{ab} \\
\delta_0 \Phi_{\mu}^a &= \partial_\mu z_{\mu}^a - \partial_\nu z_{\nu}^a + \eta_{\mu}^a \\
\delta_0 h_\mu^a &= \partial_\mu \xi_{\mu} + \eta_{\mu} \\
\delta_0 \omega_{\mu}^{ab} &= \partial_\mu \eta_{\mu}^{ab}
\end{align*}
\]

We start with the replacement of all derivatives in the Lagrangian and gauge transformation laws by the covariant ones. As a result Lagrangian \( \mathcal{L}_0 \) is not invariant now, instead we have:

\[
\delta_0 \mathcal{L}_0 = -6\kappa(d-4)[2\Sigma_{\mu}^{a} \chi_{\nu,ab} - \Sigma_{a}^{\mu} \chi_{\nu}^{ab} + R_{\mu,ab}^{a} \psi_{d,abc}^{b} + 2R_{\mu}^{ab} \psi_{ab}^{b}] + 3\kappa(d-3)[\frac{1}{2} \Phi_{\mu}^{abc} \eta_{abc} - \Omega_{\mu}^{abc} \zeta_{abc}] + \kappa(d-2)[\omega_{\mu}^{abc} \eta_{abc} + h_{\mu}^{ab} \eta_{ab}]
\]

To proceed we add all possible low derivative terms to the Lagrangian (by possible we mean those that could be written as forms with equal number of "world" and "local" indices):

\[
\mathcal{L}_1 = a_1 \{ \frac{\mu\nu\alpha\beta}{abcd} \} \Sigma_{\mu
u}^{abc} \Phi_{\alpha\beta}^{d} + a_2 \{ \frac{\mu\nu\alpha}{abc} \} R_{\mu\nu}^{ab} \Omega_{\mu}^{bcd} + a_3 \{ \frac{\mu\nu}{ab} \} \Omega_{\mu}^{abc} h_{\mu}^{c} + a_4 \{ \frac{\mu\nu\alpha}{abc} \} \omega_{\mu}^{ab} \Phi_{\nu\alpha}^{c}
\]

as well as corresponding terms to the gauge transformations:

\[
\begin{align*}
\delta_1 R_{\mu\nu}^{ab} &= \alpha_1 e_{[\mu}^{[a} z_{\nu]}^{b]} \\
\delta_1 \Phi_{\mu}^{a} &= \alpha_3 (\chi_{\mu}^{a} - \chi_{\nu}^{a}) + \alpha_4 e_{[\mu}^{[a} \xi_{\nu]} \\
\delta_1 \Omega_{\mu}^{abc} &= \alpha_5 \psi_{\mu}^{abc} + \alpha_6 e_{[\mu}^{[a} \eta_{\nu]} \\
\delta_1 h_{\mu}^{a} &= \alpha_7 z_{\mu}^{a} \\
\delta_1 \omega_{\mu}^{ab} &= \alpha_8 \eta_{\mu}^{ab}
\end{align*}
\]

At this order the gauge invariance \( \delta_0 \mathcal{L}_1 + \delta_1 \mathcal{L}_0 = 0 \) gives a number of relations among the parameters:

\[
\alpha_1 = -\frac{2a_1}{4}, \quad \alpha_2 = \frac{a_1}{4}, \quad \alpha_3 = 4a_1, \quad \alpha_5 = -8a_1
\]
\[ \alpha_6 = \frac{4a_3}{3(d-3)}, \quad \alpha_7 = 4a_3, \quad \alpha_8 = -2a_3, \quad a_2 = -3a_1, \quad a_4 = a_3 \]

but to achieve complete invariance we have to introduce a number of mass-like terms in the Lagrangian (this time they do exist):

\[ \mathcal{L}_2 = b_1 \left\{ \mu^{\alpha\beta} \right\}_{abcd} R_{\mu\nu}^{ab} R_{\alpha\beta}^{cd} + b_2 \left\{ \mu^{\alpha\nu} \right\}_{ab} R_{\mu\nu}^{ab} h_{\alpha}^{c} + b_3 \left\{ \mu^{\nu} \right\}_{ab} h_{\nu}^{a} h_{\nu}^{b} \]  

(33)

as well as necessary corrections to the gauge transformations:

\[ \delta_2 \Sigma_{\mu\nu}^{abc} = \beta_1 \epsilon_{[\mu}^{[a} \chi_{\nu]}^{bc]} + \beta_2 \epsilon_{[\mu}^{[a} \epsilon_{\nu]}^{b} \zeta_{c]} \quad \delta_2 \omega_{\mu}^{ab} = \beta_3 \chi_{\mu}^{ab} + \beta_4 \epsilon_{\mu}^{[a} \xi^{b]} \]  

(34)

In this, by adjusting the values of all parameters:

\[ \beta_1 = -\frac{8b_1}{3(d-4)}, \quad \beta_2 = \frac{2b_2}{3(d-3)(d-4)}, \quad \beta_3 = 4b_2, \quad \beta_4 = -\frac{2b_3}{d-2} \]

\[ b_1 = -3a_1^2 - \frac{3}{3\kappa(d-4)}, \quad b_2 = -4a_1 a_3, \quad b_3 = -\frac{4(d-2)}{3(d-3)} a_3^2 + 3\frac{d-2}{2\kappa} \]

we obtain gauge invariant description of massive \( R_{\mu\nu}^{ab} \) field in the (A)dS space, provided:

\[ 24(d-3)a_1^2 - 8(d-4)a_3^2 = -3\kappa(d-3)(d-4) \]  

(35)

All these results look very similar to the ones obtained in the previous section, but there is an essential difference which could be traced to different number of fields and the presence of explicit mass-like terms in the Lagrangian. This time we also have two special limits depending on the sign of cosmological constant. But now in the Anti de Sitter space by setting \( a_1 = 0 \) one obtains truly massless theory for the tensor \( R_{\mu\nu}^{ab} \) with rather simple Lagrangian:

\[ \mathcal{L} = \mathcal{L}_0(\Sigma_{\mu\nu}^{abc}, R_{\mu\nu}^{ab}) - \frac{3}{8\kappa(d-4)} \left\{ \mu^{\alpha\beta} \right\}_{abcd} R_{\mu\nu}^{ab} R_{\alpha\beta}^{cd} \]  

(36)

which is invariant under the following gauge transformations:

\[ \delta R_{\mu\nu}^{ab} = D_{\mu} \chi_{\nu}^{ab} - D_{\nu} \chi_{\mu}^{ab} + \psi_{\mu,\nu}^{ab} - \psi_{\nu,\mu}^{ab} \quad \delta \Sigma_{\mu\nu}^{abc} = D_{\mu} \psi_{\nu}^{abc} - D_{\nu} \psi_{\mu}^{abc} + \kappa \epsilon_{[\mu}^{[a} \chi_{\nu]}^{bc]} \]  

(37)

At the same time the rest fields completely decouples and describe exactly the same partially massless theory as in the previous section.

On the other hand, in the de Sitter space one can set \( a_3 = 0 \). This gives us one more example of partially massless theory the Lagrangian being:

\[ \mathcal{L} = \mathcal{L}_0(\Sigma_{\mu\nu}^{abc}, R_{\mu\nu}^{ab}) + \mathcal{L}_0(\Omega_{\mu}^{abc}, \Phi_{\mu\nu}^{a}) + m \left\{ \mu^{\alpha,\beta} \right\}_{abcd} \Sigma_{\mu\nu}^{abc} \Phi_{\alpha\beta}^{d} - 3m \left\{ \mu^{\alpha} \right\}_{abc} R_{\mu\nu}^{ad} \Omega_{\alpha}^{bcd} \]  

(38)

with \( m^2 = \kappa(d-4)/8 \), which is invariant under:

\[ \delta R_{\mu\nu}^{ab} = D_{\mu} \chi_{\nu}^{ab} - D_{\nu} \chi_{\mu}^{ab} + \psi_{\mu,\nu}^{ab} - \psi_{\nu,\mu}^{ab} - \frac{2m}{d-4} \epsilon_{[\mu}^{[a} z_{\nu]}^{b]} \]

\[ \delta \Sigma_{\mu\nu}^{abc} = D_{\mu} \psi_{\nu}^{abc} - D_{\nu} \psi_{\mu}^{abc} + \frac{m}{d-4} \epsilon_{[\mu}^{[a} n_{\nu]}^{bc]} \]

\[ \delta \Phi_{\mu\nu}^{a} = D_{\mu} z_{\nu}^{a} - D_{\nu} z_{\mu}^{a} + n_{\mu}^{a} + 4m(\chi_{\mu,\nu}^{a} - \chi_{\nu,\mu}^{a}) \]

\[ \delta \Omega_{\mu}^{abc} = D_{\mu} n_{abc} - 8m \psi_{\mu}^{abc} \]  

(39)

One more time note that in the partially massless limit we obtain relatively simple Lagrangian without any explicit mass-like terms.
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