Phase transition in ferromagnetic Ising model with a cell-board external field.

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Abstract

We show the presence of a first-order phase transition for a ferromagnetic Ising model on $\mathbb{Z}^2$ with a periodical external magnetic field. The external field takes two values $h$ and $-h$, where $h > 0$. The sites associated with positive and negative values of external field form a cell-board configuration with rectangular cells of sides $L_1 \times L_2$ sites, such that the total value of the external field is zero. The phase transition holds if $h < \frac{2J}{L_1} + \frac{2J}{L_2}$, where $J$ is an interaction constant. We prove a first-order phase transition using the reflection positivity (RP) method. We apply a key inequality which is usually referred to as the chessboard estimate.

Keywords: Ising model, periodic external field, Peierls condition, reflection positivity, phase transition.

1 Introduction

In many models of statistical physics the phase transition is a result of spontaneous breaking of the symmetry of a system. The best known model with phase transition is the ferromagnetic Ising model (system) in the absence of a magnetic field. Essentially, this fact has been shown by Peierls [18]. It has become a theorem by Griffiths [15] and Dobrushin [8] (see also [21] and [7]). Peierls’ ideas are referred to as Peierls’ arguments based on Peierls’ condition and Peierls’ transformation connected to a symmetry of the Ising model. Peierls’ condition means that energy required for a droplet formation of one of the phases surrounded by the sides of opposite spin value is proportional to the size of the droplet boundary. For a two dimensional model (on $\mathbb{Z}^2$) the boundary size is the length of the droplet’s boundary. Peierls’ condition is unrelated to the model symmetry. Peierls’ condition is satisfied for the Ising model with a uniform external field, however the symmetry is violated in this case. The second component of Peierls’ argument is constructing Peierls’ transformation such that it does not change the energy of a configuration when a droplet is removed by the transformation. Symmetry of the model is needed for constructing a Peierls transformation.

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Peierls’ arguments show a type of “stability” of the ground states. It means that at a low temperature there exist small perturbations of a ground state which would result in a configuration “close” to the starting ground state.

Unlike Peierls’ argument (specifically Peierls’ transformation), the Pirogov-Sinai theory of phase transitions allows one to find a low-temperature phase diagram of models with no symmetry requirement. In addition, there exist a several more approaches. One such approach, Reflection Positivity (RP), requires showing a type of reflection symmetry. Essentially, it is possible to prove a phase transition with the chessboard inequality obtained from the RP property.

An external field added to the Hamiltonian can change the whole phase diagram. In the case of the ferromagnetic Ising model, any non-zero uniform external field suppresses the phase transition. In some models where the magnetic field is not supposed to be uniform, it is possible to prove phase uniqueness, see for instance, [5], [6]. A random external field can also suppress the phase transition in a planar Ising model (see [1], [2]), even in the case when the total average of the external field is equal to 0.

In this paper we will address the problem of the existence of phase transitions in a planar Ising model where the external field is periodic, forming a cell-board configuration such that total value of the magnetic field is zero. These models were numerically studied by M. Sigelle in [20] (see also [16]). The motivation is coming from image processing where Ising models with non-uniform external fields are used for analyzing segmentation.

In this work we consider the Ising model where the external field takes two values $h$ and $-h$, where $h > 0$. The lattice $\mathbb{Z}^2$ is split into the union of disjointed cells of the same size, and the signs of the external field are alternated similar to a chessboard. Specifically, a cell with one sign of the external field is surrounded by four neighbor cells with opposite value of the external field. We use a specific term for this partition, cell-board partition, to avoid a confusion with the chessboard estimate, to be used later in the paper.

A particular case of this model was studied by Nardi, Olivieri and Zahradnik in [17]. In their work, the cells have infinite horizontal length, while their height equals one.

The paper is organized as follows: In Sect. 2 we define the model we study, and present our main result (Theorem 2.2). The reflection positivity technique, which is the main tool we use for the proofs, will be discussed in Sect. 3 where we will also describe the chessboard estimates. The rigorous proofs of the main result are given in Sect. 4. The proof of the main result using the RP technique follows the standard scheme, see for example [3], Chapters 5, 6. Finally, in Sect. 5 we study a generalization of a phase transition proved in [17], for which we also use the RP approach.

2 Definitions and results

We consider the ferromagnetic Ising model on $\mathbb{Z}^2$ with a periodic external field introduced in [16]. We represent the lattice $\mathbb{Z}^2$ as the union of rectangular cells of the size $L_1 \times L_2$, $L_i \in \mathbb{N}$: for each pair of integers $n, m$ we define

$$C(n, m) = \{(t_1, t_2) \in \mathbb{Z}^2 : \quad nL_1 \leq t_1 < (n+1)L_1, \quad mL_2 \leq t_2 < (m+1)L_2\}.$$  \hspace{1cm} (2.1)
Thus $\mathbb{Z}^2 = \bigcup_{n, m \in \mathbb{Z}} C(n, m)$. Let us define the following subsets $\mathbb{Z}_+$ and $\mathbb{Z}_-$ of $\mathbb{Z}^2$.

$$\mathbb{Z}_+ = \bigcup_{n, m: n + m \text{ is even}} C(n, m), \quad \mathbb{Z}_- = \mathbb{Z}^2 \setminus \mathbb{Z}_+. \quad (2.2)$$

Let us imagine that each site is colored white if it is from $\mathbb{Z}_+$ and black otherwise. Thus, the whole lattice is like a chessboard (see Figure 2(a) where $L_1 = 3$ and $L_2 = 2$).

Let $\Omega = \{-1, +1\}^{\mathbb{Z}^2}$ be the set of all configurations on $\mathbb{Z}^2$. The formal Hamiltonian is defined by

$$H(\sigma) = -J \sum_{\langle t, s \rangle} \sigma(t)\sigma(s) - \sum_s h(s)\sigma(s), \quad (2.3)$$

for any $\sigma \in \Omega$, where $\sigma(t)$ is the spin value of configuration $\sigma$ at the site $t \in \mathbb{Z}^2$, $J > 0$ is the interaction constant, the symbol $\langle t, s \rangle$ denotes nearest neighbors $s, t \in \mathbb{Z}^2$, that is the Euclidean distance between points is one, $|t - s| = 1$, and the external field $h$ is given by

$$h(s) = \begin{cases} h, & \text{if } s \in \mathbb{Z}_+, \\ -h, & \text{if } s \in \mathbb{Z}_-. \end{cases} \quad (2.4)$$

Further, for any subset $\Lambda \subset \mathbb{Z}^2$ and any configuration $\sigma \in \Omega$, we will use the notation $\sigma(\Lambda)$ for the configuration of $\sigma$ restricted to the set of sites $\Lambda$.

We recall the standard definitions of a Gibbs field on the infinite lattice $\mathbb{Z}^2$ and related notations. Let $W$ be a finite subset from $\mathbb{Z}^2$, and let $\Omega_W$ be the set of all configurations on $W$: $\Omega_W = \{-1, 1\}^W$. The Gibbs probability of the configuration $\sigma \in \Omega_W$ with boundary conditions $\omega \in \Omega$, is given by

$$\mu_{\beta, W}(\sigma | \omega) = \frac{1}{Z_W(\beta)} \exp \left( \beta J \sum_{\langle t, s \rangle: t, s \in W} \sigma(t)\sigma(s) + \beta \sum_{t \in W: s \notin W} \sigma(t)\omega(s) + \beta \sum_{s \in W} h(s)\sigma(s) \right), \quad (2.5)$$

where $\beta$ is a positive constant usually interpreted as the inverse temperature, and $Z_W(\beta)$ is the normalizing constant, called the partition function.

A configuration $\tilde{\sigma} \in \Omega$ is a local perturbation of a configuration $\sigma \in \Omega$ if there exists a finite set $V \subset \mathbb{Z}^2$ such that

$$\tilde{\sigma}(t) = \begin{cases} -\sigma(t), & \text{if } t \in V, \\ \sigma(t), & \text{if } t \notin V. \end{cases}$$

A configuration $\sigma \in \Omega$ is called a ground state for the Hamiltonian $H$, if for any local perturbation $\tilde{\sigma}$ of the configuration $\sigma$ the inequality

$$H(\tilde{\sigma}) - H(\sigma) \geq 0,$$

is valid. Following [16] we say that the Peierls’ condition holds true, if there exists a positive constant $c_P > 0$ such that for any local perturbation $\tilde{\sigma}$ of a ground state $\sigma$ the inequality

$$H(\tilde{\sigma}) - H(\sigma) \geq c_P |\partial V|, \quad (2.6)$$

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holds, where $\partial V = \{(t, s) : t \in V, s \notin V\}$ is the boundary of the set $V$. The constant $c_P$ is called the Peierls’ constant in this case.

The following theorem provides the known results from [16] about ground states and Peierls’ condition for our model.

**Theorem 2.1.** If

$$h < \frac{2J}{L_1} + \frac{2J}{L_2}, \quad (2.7)$$

then there exist two periodical ground states, namely the constant configurations $\sigma^+ \equiv +1$ and $\sigma^- \equiv -1$. In addition, the Peierls’ condition holds, and the Peierls’ constant $c_P$ is equal to $2J - hL_1L_2/(L_1 + L_2)$. If (2.7) does not hold and

$$h > \frac{2J}{L_1} + \frac{2J}{L_2}, \quad (2.8)$$

then the configuration

$$\sigma_c(t) = \begin{cases} 1, & \text{if } t \in \mathbb{Z}_+, \\ -1, & \text{if } t \in \mathbb{Z}_-. \end{cases} \quad (2.9)$$

is the unique periodical ground state.

Here we use the notion of *periodical configurations*: we say that a configuration $\sigma \in \Omega$ is periodical, if there exists a finite rectangle $A = \{(t_1, t_2) : N' < t_1 \leq N'', M' < t_2 \leq M''\}$, such that for any integers $k_1, k_2 \in \mathbb{Z}$ and any $t \in A$ $\sigma(t) = \sigma(t + (k_1(N'' - N'), k_2(M'' - M')))$. 

**Main result. Phase transition for cell-board model.**

Let $\mathcal{G}_\beta$ denote the set of all Gibbs measures for the Hamiltonian (2.3) with the inverse temperature $\beta$. The next theorem provides the presence of a first-order phase transition in the cell-board model.

**Theorem 2.2.** Let the condition (2.7) holds true, then there exists $\beta_c = \beta_c(L_1, L_2)$, such that for any $\beta > \beta_c$, there exist two distinct measures $\mu^+_\beta$ and $\mu^-_\beta \in \mathcal{G}_\beta$, which satisfy

$$\mu^\pm_\beta(\sigma(t) = \pm 1) > \frac{1}{2}, \quad (2.10)$$

That means $|\mathcal{G}_\beta| > 1$.

This is the main result in the paper. The proof is in Sect. 4. It is based on the reflection positivity machinery. In the next section we introduce this technique.

We conclude this section with an observation about conenction between our model and the well known antiferromagnetic Ising model with constant external field. Remind that the formal Hamiltonian for antiferromagnetic field is

$$H_a(\sigma) = -J_a \sum_{\langle t, s \rangle} \sigma(t)\sigma(s) - h_a \sum_s \sigma(s), \quad (2.11)$$

where the parameter $J_a$ is negative, $J_a < 0$, that creates the antiferromagnetic interactions between nearest spins, the external field $h_a$ is a real constant. Intuitively, the cell-board external field, $\pm h$,
of our model, when $L_1 = L_2 = 1$, should be equivalent to the antiferromagnetic model with the constant external field $h_a = h$. Thus, the known result about phase transition for antiferromagnetic model with external field (see for example [9]) should be the consequence of the phase transition for our model. Indeed, the following result is the consequence of Theorem 2.2.

**Corollary 2.3.** Let $J, h > 0$. If $h < 4J$ and $\beta > \frac{2k}{4J-h}$ for some $k > 0$, then the antiferromagnetic Ising model (2.11) with $J_a = -J$ and $h_a = h$ has two phases.

**Proof.** The transformation $\Psi$ of the configuration space $\Omega$, $\Psi : \Omega \rightarrow \Omega$, 

$$\Psi(\sigma)(t_1, t_2) = \begin{cases} \sigma(t_1, t_2), & \text{if } t_1 + t_2 \text{ even,} \\ -\sigma(t_1, t_2), & \text{otherwise,} \end{cases}$$

is the one-to-one transformation of the $\Omega$. Note that if in (2.11) we choose $J_a = -J$, where $J > 0$, and $h_a = h$ then

$$H_a(\Psi(\sigma)) = J \sum_{(t, s)} \Psi(\sigma)(t)\Psi(\sigma)(s) - h \sum_s \Psi(\sigma)(s)$$

$$= -J \sum_{(t, s)} \sigma(t)\sigma(s) - \sum_s h(s)\sigma(s) = H(\sigma)$$

where

$$h(t) = \begin{cases} h, & \text{if } t = (t_1, t_2), \ t_1 + t_2 \text{ is even,} \\ -h, & \text{otherwise.} \end{cases}$$

It means that the transformation $\Psi$ does not change energy of configurations and provides the direct equivalence of the models. \qed

### 3 Reflection Positivity and Chessboard estimate

In this section we explain the reflection positivity (RP) technique in a way adapted to our model. Surveys about this method can be found in Georgii [13] and Shlosman [19], see also [3, 4, 14, 10, 11, 12]. Here we will mainly use the notation and definitions of Biskup and Kotecký [4] and Biskup [3].

The main consequence of RP is the chessboard estimates, which is used to prove phase coexistence in the models with RP property.

**Reflection Positivity (RP).**

We place the spin system on a two-dimensional torus. Let $T_N = T_N(L_1, L_2)$ be the subset of $\mathbb{Z}^2$ of side $NL_1 \times NL_2$:

$$T_N = \{ t = (t_1, t_2) \in \mathbb{Z}^2 : 0 \leq t_1 < NL_1, 0 \leq t_2 < NL_2 \}.$$ 

Further for a technical reason we will need $N$ a multiple of 4. $T_N$ is composed by $N^2/2$ cells of $\mathbb{Z}_+$ and the same amount of cells of type $\mathbb{Z}_-$. We will consider $T_N$ as a torus: for any integer
Let \( \Omega_N = \{-1,+1\}^{T_N} \) be the set of all configurations on the torus \( T_N \). We consider Hamiltonian with so called periodical boundary conditions: for any \( \sigma \in \Omega_N \)

\[
H_N(\sigma) = -J \sum_{(t,s) \in T_N} \sigma(t)\sigma(s) - \sum_{s \in T_N} h(s)\sigma(s). \tag{3.1}
\]

Gibbs measure is

\[
\mu_{\beta,N}(\sigma) = \frac{1}{Z_N(\beta)} \exp \left( \beta J \sum_{(t,s) \in T_N} \sigma(t)\sigma(s) + \beta \sum_{s \in T_N} h(s)\sigma(s) \right), \tag{3.2}
\]

where \( Z_N(\beta) \) is the corresponding partition function:

\[
Z_N(\beta) = \sum_{\sigma \in \Omega_N} \exp \left( -\beta H_N(\sigma) \right). \tag{3.3}
\]

Now, we define reflection symmetries along lines orthogonal to one of the lattice directions. First, consider the lattice \( \mathbb{Z}^2 \) embedded in \( \mathbb{R}^2 \). We denote by \( \Theta \) the group of all transformations of \( \mathbb{R}^2 \) generated by reflections of \( \mathbb{R}^2 \) with respect to lines orthogonal to one of the lattice directions such that \( \mathbb{Z}^2 \) is invariant for any \( \vartheta \in \Theta \): \( \vartheta \mathbb{Z}^2 = \mathbb{Z}^2 \). Let \( \vartheta_P \) denote the reflection \( \vartheta \) along the line \( P \). The group \( \Theta \) composed by the two distinct subgroups \( \Theta^k \) \((k = 0,1/2)\), generated by reflections \( \vartheta_P \) for which the corresponding line has the form \( P = \{ t = (t_1,t_2) \in \mathbb{R}^2 : t_i = n + k \} \) for \( i = 1 \) or \( 2 \), integer \( n \) and \( k = 0 \) or \( 1/2 \). Reflections from \( \Theta^0 \) we will call the reflections through sites: the corresponding reflection lines pass through the sites of \( \mathbb{Z}^2 \). Reflections from the set \( \Theta^{1/2} \) we will call reflections through bonds: the corresponding reflection lines bisect bonds.

The groups \( \Theta^k \), \( k = 0,1/2 \), generate naturally the reflections of the torus \( T_N \) such that \( \vartheta_P(T_N) = T_N \) (we may think of \( T_N \) as embedded into a continuum torus). For any line \( P \), \( \vartheta_P \in \Theta^k \) splits the torus into two symmetric components, say \( T_N^l \) and \( T_N^r \), the left and the right half, such that \( \vartheta_P(T_N^l) = T_N^r \) and vice versa (see Figure 1). Note that \( T_N^l \cap T_N^r \in P \) for reflections through sites \((k = 0)\) and are disjoint for reflections through bonds \((k = 1/2)\).

Let \( \mathcal{F}_P \) \((\mathcal{F}_P) \) be the minimal \( \sigma \)-algebra on \( \Omega_N \) such that all functions \( \sigma(t), t \in T_N^l \) \((T_N^r)\) are measurable. As in [4] we introduce a reflection operator \( \vartheta_P : \Omega_N \to \Omega_N : \vartheta_P(\sigma(s)) := \sigma(\vartheta_P(s)) \) for any spatial reflection \( \vartheta_P : T_N^l \leftrightarrow T_N^r \). The operator \( \vartheta_P \) obeys the following properties:

1. \( \vartheta_P \) is an involution, \( \vartheta_P \circ \vartheta_P = id \);
2. \( \vartheta_P \) is a reflection in the sense that if \( \mathcal{A} \in \mathcal{F}_P \) depends only on configurations in \( \Lambda \subset T_N^l \), then \( \vartheta_P(\mathcal{A}) \in \mathcal{F}_P \) depends only on configurations in \( \vartheta_P(\Lambda) \).

**Definition 3.1.** (Reflection Positivity, see Definition 2.2 of [4]). Let \( \mu \) be a probability measure on \( \Omega_N \), denote \( E_\mu \) the corresponding expectation, and let \( P \) be a reflecting line. We say that \( \mu \) is
reflection positive measure with respect to $\theta_P$ if for any two bounded $F_P$-measurable functions $f$ and $g$

$$E_\mu(f \theta_P(g)) = E_\mu(g \theta_P(f)), \quad (3.4)$$

and

$$E_\mu(f \theta_P(f)) \geq 0, \quad (3.5)$$

where $\theta_P(f)$ is the $F_P$-measurable function $f \circ \theta_P$. A consequence of RP is an inequality like the Cauchy-Schwarz inequality

$$[E_\mu(f \theta g)]^2 \leq E_\mu(f \theta f)E_\mu(g \theta g). \quad (3.6)$$

Chessboard estimates.

Next, we describe the chessboard estimates. To this end we decompose $\mathbb{Z}^2$ (or the torus $\mathbb{T}_N$) into rectangular blocks. Let $P_i, i = 1, 2$, be the set of lines

$$P_i^{(n)} = \{ t = (t_1, t_2) \in \mathbb{R}^2 : t_i = nL_i + (L_i - 1)/2 \} \quad (3.7)$$

on $\mathbb{R}^2$. Note that if $L_i$ is odd then the corresponding reflection $\vartheta_{P_i^{(n)}} \in \Theta^0$, and if $L_i$ is even then the corresponding reflection $\vartheta_{P_i^{(n)}} \in \Theta^{1/2}$. Any such line cuts in half corresponding cells $C(n, m)$, see (2.1). The set of lines $\mathcal{P} = P_1 \cup P_2$ provide the decomposition of $\mathbb{Z}^2$ (correspondingly $\mathbb{T}_N$) into rectangular blocks, see Figure 2(a).

Let $\Lambda$ be the minimal block, obtained with divisions by $\mathcal{P}$, which contains the origin. Further another construction of the block $\Lambda$ will be useful: consider in $\mathbb{R}^2$ the rectangle

$$\tilde{\Lambda} = \left\{ (t_1, t_2) : \left| t_1 + \frac{1}{2} \right| \leq \frac{L_1}{2}, \left| t_2 + \frac{1}{2} \right| \leq \frac{L_2}{2} \right\}. \quad (3.8)$$
then Λ as \( \Lambda = \tilde{\Lambda} \cap \mathbb{Z}^2 \). Note that the block \( \Lambda \) contains \( B_1B_2 \) sites, where

\[
B_i = \begin{cases} 
L_i, & \text{if } L_i \text{ is even,} \\
L_i + 1, & \text{if } L_i \text{ is odd,}
\end{cases}
\]  (3.9)

and \( \mathbb{T}_N \) can be covered by translations of \( \Lambda \),

\[
\mathbb{T}_N = \bigcup_{t \in \tilde{\mathbb{T}}_N} (\Lambda + t),
\]  (3.10)

where \( \tilde{\mathbb{T}}_N = \{t = (t_1, t_2) \in \mathbb{T}_N : t_1 = nL_1, t_2 = mL_2, n, m \in \mathbb{Z} \} \). Note that the neighboring translations of \( \Lambda \) can have a side in common. Any event defined by values \( \sigma(t) \) for \( t \in \Lambda \) is called \( \Lambda \)-event. Let \( \mathcal{F}_\Lambda \) be the set of \( \Lambda \)-events. For each \( s \in \mathbb{T}_N \), the map \( \tau_s : \Omega_N \to \Omega_N \) is the translation by \( s \) defined by \( (\tau_s \sigma)(t) = \sigma(t - s) \). For any \( A \in \mathcal{F}_\Lambda \) and any \( t \in \tilde{\mathbb{T}}_N \) we define the event \( \theta_t(A) \) as follows:

1. If \( t = (nL_1, mL_2) \) has both \( n \) and \( m \) even, then \( \theta_t(A) \) is simply the translation of \( A \) by vector \( t \), i.e., \( \theta_t(A) = \{\sigma \in \Omega_N : \tau_t^{-1}(\sigma) \in A\} \).

2. For the remaining \( t \in \tilde{\mathbb{T}}_N \), we first reflect \( A \) through the midline of \( \Lambda \) in all directions whose component \( n \) or \( m \) of \( t \) is odd, and then translate the result by \( t \) as before.

Thus, \( \theta_t(A) \) will be a cylindrical set of configurations from \( \mathcal{F}_{\Lambda+t} \).

**Theorem 3.2.** (Chessboard estimate) Let \( \mu_{\beta,N} \) a measure on \( \Omega_N \) which is RP with respect to all reflections between the neighboring blocks \( \Lambda + t, t \in \tilde{\mathbb{T}}_N \). Then for any \( \Lambda \)-events \( A_1, \ldots, A_m \) and any distinct sites \( t_1, \ldots, t_m \in \tilde{\mathbb{T}}_N \),
For the proof see, for example, [3], Theorem 5.8 or [4], Theorem 2.4.

The following quantities play the main role in the proof of the phase transition

\[ \mathcal{z}_{\beta,N}(\mathcal{A}) := \mu_{\beta,N} \left( \bigcap_{t \in \tilde{T}_N} \theta_t(\mathcal{A}) \right)^{1/|\tilde{T}_N|}. \]  

(3.12)

The function \( \mathcal{A} \to \mathcal{z}_{\beta,N}(\mathcal{A}) \) is not additive. However, given the \( \sigma \)-additivity of \( \mu \) and using the chessboard estimate, it is easy to prove that it is sub-additive (see [3], Lemma 5.9). That is, for any collection of \( \Lambda \)-events \( \mathcal{A}, \mathcal{A}_1, \mathcal{A}_2, \ldots \) such that \( \mathcal{A} \subset \bigcup_k \mathcal{A}_k \), the inequality

\[ \mathcal{z}_{\beta,N}(\mathcal{A}) \leq \sum_k \mathcal{z}_{\beta,N}(\mathcal{A}_k), \]  

(3.13)

holds. The limiting version of this quantity will be of particular interest for us. Thus, we define

\[ \mathcal{z}_\beta(\mathcal{A}) := \lim_{N \to \infty} \mathcal{z}_{\beta,N}(\mathcal{A}). \]  

(3.14)

### 4 Proof of phase coexistence

Here we prove Theorem 2.2. The proof based on the reflection positivity technique is standard. We follow the ideas of [3]. The proof essentially consists on two steps. First, the easiest step, Proposition 4.1, we apply a known criterion for establish RP property for our model. Second, as usual we construct two measures \( \mu_+^\beta \) and \( \mu_-^\beta \) and prove that the probabilities \( \mu_+^\beta(\sigma(0) = -1) \) and \( \mu_-^\beta(\sigma(0) = +1) \) can be made less than 1/2 for the same \( \beta \). This will prove the phase coexistence. In order to provide this we use chessboard estimate (3.11) for a sort of Peierls’ arguments evaluating the contour probabilities. It is implemented in the proof of Proposition 4.4. However, we need some previous results about \( \mathcal{z}_{\beta,N} \) in (3.12). Proposition 4.2 and Proposition 4.3 give upper bounds of \( \mathcal{z}_{\beta,N} \) for bad block events, which will be defined further.

**Proposition 4.1.** For any \( P \in \mathcal{P} \) (see (3.7)), and all \( \beta \geq 0 \) the Gibbs measure \( \mu_{\beta,N} \) (3.2) on the torus \( T_N \) is reflection positive (RP) with respect to \( \theta_P \).

We prove the proposition later.

In order to apply the chessboard inequality we describe bad block events we deal with. For each \( \Lambda \)-block configuration \( \sigma_\Lambda \in \{-1,+1\}^\Lambda \) we define the event

\[ \mathcal{B}(\sigma_\Lambda) = \{ \sigma \in \Omega_N : \sigma(\Lambda) = \sigma_\Lambda \}. \]  

(4.1)

Let \( \sigma_\Lambda^+ \) and \( \sigma_\Lambda^- \) be the constant \( \Lambda \)-block configurations with all spins plus and all spins minus, respectively. For a given configuration \( \sigma \in \Omega_N \), we say that the block \( \Lambda \) is good if all spins of \( \sigma \) take the same value on \( \Lambda \) (\( \sigma \in \mathcal{B}(\sigma_\Lambda^+) \cup \mathcal{B}(\sigma_\Lambda^-) \)), and bad otherwise. Let \( R(\Lambda) \) be the set of all non-constant
\( \Lambda \)-block configurations, \( R(\Lambda) = \{ -1, +1 \}^\Lambda \setminus \{ \sigma_\Lambda^+, \sigma_\Lambda^- \} \). Remember that the size of the block \( \Lambda \) is equal to \( B_1B_2 \) sites, and \( B_i \geq 2 \) as defined in (3.9). This implies that \( |R(\Lambda)| = 2^{B_1B_2} - 2 \geq 14 \).

Let \( \mathcal{R} \) denote the event that the block \( \Lambda \) is bad,

\[
\mathcal{R} = \bigcup_{\sigma_\Lambda \in R(\Lambda)} B(\sigma_\Lambda).
\]

(4.2)

Further we show that bad \( \Lambda \)-blocks have small probability when \( \beta \) is large.

Depending on parity of \( L_1 \) and \( L_2 \) we will need another blocks construction. We will call the new blocks \( \Lambda^* \)-blocks. Consider the four lines \( P_1^{(-1)}, P_1^{(0)}, P_2^{(-1)}, P_2^{(0)} \in \mathcal{P} \) defined by (3.7), then we define the following double \( \Lambda \)-blocks:

\[
\begin{align*}
\Lambda_{1,h} &= \Lambda \cup \vartheta_{P_2^{(0)}}(\Lambda), \\
\Lambda_{2,h} &= \Lambda \cup \vartheta_{P_2^{(-1)}}(\Lambda), \\
\Lambda_{1,v} &= \Lambda \cup \vartheta_{P_1^{(0)}}(\Lambda), \\
\Lambda_{2,v} &= \Lambda \cup \vartheta_{P_1^{(-1)}}(\Lambda),
\end{align*}
\]

(4.3)

with covering

\[
\begin{align*}
\mathbb{T}_N &= \bigcup_{t \in \mathbb{T}_N^h} (\Lambda_{1,h} + t) = \bigcup_{t \in \mathbb{T}_N^h} (\Lambda_{2,h} + t) = \bigcup_{t \in \mathbb{T}_N^v} (\Lambda_{1,v} + t) = \bigcup_{t \in \mathbb{T}_N^v} (\Lambda_{2,v} + t)
\end{align*}
\]

(4.4)

where

\[
\begin{align*}
\mathbb{T}_N^h &= \{ t = (t_1, t_2) \in \mathbb{T}_N : t_1 = nL_1, t_2 = 2mL_2, \ n, m \in \mathbb{Z} \}, \\
\mathbb{T}_N^v &= \{ t = (t_1, t_2) \in \mathbb{T}_N : t_1 = 2nL_1, t_2 = mL_2, \ n, m \in \mathbb{Z} \}.
\end{align*}
\]

Defining the chessboard inequality for such block divisions we need to consider \( N \) a multiple of 4. In the same way, see (4.1), (4.2), we define the sets of non-constant \( \Lambda^* \)-block configurations \( R(\Lambda_{i,k}) \), events \( B(\sigma_{\Lambda_{i,k}}) \) and events that \( \Lambda^* \)-block is bad, \( \mathcal{R}_{i,k} \), with corresponding \( \mathfrak{g}_{\beta,N} \) values, see (3.12): \( \mathfrak{g}_{\beta,N}(\mathcal{R}_{i,k}) \) \( (i = 1, 2, k = h, v) \). Note, by symmetry

\[
\begin{align*}
\mathfrak{g}_{\beta,N}(\mathcal{R}_{1,h}) &= \mathfrak{g}_{\beta,N}(\mathcal{R}_{2,h}) = : \mathfrak{g}_{\beta,N}(\mathcal{R}_h), \\
\mathfrak{g}_{\beta,N}(\mathcal{R}_{1,v}) &= \mathfrak{g}_{\beta,N}(\mathcal{R}_{2,v}) = : \mathfrak{g}_{\beta,N}(\mathcal{R}_v).
\end{align*}
\]

(4.5)

**Proposition 4.2.** If the condition (2.7) holds true, then for any \( \Lambda \)-block configuration \( \sigma_\Lambda \in R(\Lambda) \), the inequality

\[
\mathfrak{g}_{\beta,N}(B(\sigma_\Lambda)) \leq \exp \left( -\beta \left( 2J - \frac{hL_1L_2}{L_1 + L_2} \right) \right),
\]

(4.6)

holds for any \( N \) even. The same inequality (4.6) holds true for any \( \Lambda^* \)-block configuration \( \sigma_{\Lambda_{i,k}} \in R(\Lambda_{i,k}) \) with fixed \( i \) and \( k \) \( (i = 1, 2, k = h, v) \), when \( N \) is multiple of 4.

Observe that the term inside brackets is the Peierls’ constant defined in Theorem 2.1 We will prove this technical proposition later.

In the next proposition we show that \( \mathfrak{g}_{\beta,N}(\mathcal{R}), \mathfrak{g}_{\beta,N}(\mathcal{R}_h) \) and \( \mathfrak{g}_{\beta,N}(\mathcal{R}_v) \) defined by (3.12) can be made small.
Proposition 4.3. Let the condition (2.7) holds true, then for any small $\varepsilon > 0$ and any $N$ multiple of 4, there exists $\beta_\varepsilon > 0$ such that if $\beta > \beta_\varepsilon$, then $\max(\tilde{z}_{\beta,N}(R), \tilde{z}_{\beta,N}(R_h), \tilde{z}_{\beta,N}(R_v)) < \varepsilon$.

Proof. By sub-additivity of $\tilde{z}_{\beta,N}$, see (3.13), using Proposition 4.2 we obtain

\[
\tilde{z}_{\beta,N}(R) \leq \sum_{\sigma \in R(\Lambda)} \tilde{z}_{\beta,N}(B(\sigma)) \\
\leq \sum_{\sigma \in R(\Lambda)} \exp \left(-\beta \left(2J - h \frac{L_1 L_2}{L_1 + L_2}\right)\right) \\
= |R(\Lambda)| \exp \left(-\beta \left(2J - h \frac{L_1 L_2}{L_1 + L_2}\right)\right).
\]

Note that inequality (4.6) holds true for all $\Lambda^*$-events in (4.3), when $N$ is multiple of 4. Then, the inequalities (4.7) are holds true for bad $\Lambda^*$-events too. Let denote $R := \max(|R(\Lambda_1,h)|, |R(\Lambda_1,v)|)$ then

\[
\max(\tilde{z}_{\beta,N}(R), \tilde{z}_{\beta,N}(R_1,h), \tilde{z}_{\beta,N}(R_1,v)) \leq R \exp \left(-\beta \left(2J - h \frac{L_1 L_2}{L_1 + L_2}\right)\right).
\]

Therefore, for each $\varepsilon$ we can take

\[
\beta_\varepsilon = \frac{1}{2J - h \frac{L_1 L_2}{L_1 + L_2}} \ln \left(\varepsilon^{-1} R\right).
\]

Now, we can state the main proposition. It will be used for construction of two measures $\mu^+_{\beta}$ and $\mu^-_{\beta}$ in order to prove the phase transition.

Proposition 4.4. Let the condition (2.7) holds true. There exist a constant $c > 1$ and $\beta_c$, such that for any $s,t \in \mathbb{T}_N$, the following inequality holds

\[
\mu_{\beta,N}(\sigma(s) = +1, \sigma(t) = -1) \leq 4c^*_{\beta,N},
\]

for any $\beta > \beta_c$, where

\[
3^*_{\beta,N} := \max(\tilde{z}_{\beta,N}(R)^{1/2}, \tilde{z}_{\beta,N}(R_h)^{1/8}, \tilde{z}_{\beta,N}(R_v)^{1/8}).
\]

We will prove the proposition later. Here we note that the constant $c$ appears from the combinatorial argument: there exists a constant $c > 1$ such that the number of contours builded from $n \Lambda$-blocks can be estimated by $c^n$.

Finally, we can apply Proposition 4.3 and Proposition 4.4 to complete the proof of phase coexistence.
Proof of Theorem 2.2.

First of all, we use the following symmetry of the torus measure. Let $\Lambda_s$ be the block containing the site $s \in \mathbb{T}_N$, thus

$$\mu_{\beta,N}(\sigma : \sigma(\Lambda_s) \equiv +1) = \mu_{\beta,N}(\sigma : \sigma(\Lambda_s) \equiv -1) = \frac{1 - \mu_{\beta,N}(\mathcal{R})}{2},$$

(4.12)

for any $s \in \mathbb{T}_N$. We check this equality by taking some configuration $\sigma \in \Omega_N$, such that $\sigma(\Lambda) \equiv +1$. Now consider the following two transformations. First, we apply on $\sigma$ the reflection operator $\theta_P$, defined as in Section 3, for which $P$ is the line bisecting the set of bonds $\langle u,v \rangle$, where $u = (0,k) = v + (1,0)$ and the bonds $\langle u',v' \rangle$, with $u' = (NL_1/2,k) = v' + (1,0)$ (recall that $N$ is even). That is, $\omega := \theta_P(\sigma) \in \Omega_N$ takes the value $\omega(t_1,t_2) = \sigma(-t_1 - 1,t_2)$, for all $t = (t_1,t_2) \in \mathbb{T}_N$. Second, we obtain $\sigma' = -\omega$, flipping all the spin values. In other words, $\sigma'(t) = -\sigma(-t_1 - 1,t_2)$, for all $t \in \mathbb{T}_N$. Clearly, $\sigma'(\Lambda) \equiv -1$ and given $h(t_1,t_2) = -h(-t_1 - 1,t_2)$, the Hamiltonians are equal $H_N(\sigma) = H_N(\sigma')$.

Using the chessboard estimate (3.11) and (3.12), we obtain the inequality

$$\mu_{\beta,N}(\mathcal{R}) \leq \mu_{\beta,N}(\bigcap_{t \in \mathbb{T}_N} \theta_t(\mathcal{R}))^{1/|\mathbb{T}_N|} = \delta_{\beta,N}(\mathcal{R}),$$

(4.13)

then, from (4.12)

$$\mu_{\beta,N}(\sigma : \sigma(\Lambda_s) \equiv +1) \geq \frac{1 - \delta_{\beta,N}(\mathcal{R})}{2}.$$  

(4.14)

Let $t \in \mathbb{T}_N$ such that $t = s + (NL_1/2,0)$, and define

$$\mu_{\beta,N}^{\pm}() := \mu_{\beta,N}(\cdot | \sigma(t) = \pm 1).$$ 

(4.15)

By (4.14) and Proposition 4.4 we have

$$\mu_{\beta,N}^{+}(\sigma(s) = -1) \leq \frac{\mu_{\beta,N}(\sigma(s) = -1, \sigma(t) = +1)}{\mu_{\beta,N}(\sigma(\Lambda_t) \equiv +1)} \leq \frac{8c_3^{\ast}_{\beta,N}}{1 - \delta_{\beta,N}(\mathcal{R})},$$

(4.16)

and

$$\mu_{\beta,N}^{-}(\sigma(s) = +1) \leq \frac{8c_3^{\ast}_{\beta,N}}{1 - \delta_{\beta,N}(\mathcal{R}).}$$

(4.17)

When $N$ is multiple of 4 and $N \nearrow \infty$ we extract from the sequences $(\mu_{\beta,N}^{+})$ and $(\mu_{\beta,N}^{-})$, two converging subsequences, $\mu_{\beta}^{+}$ and $\mu_{\beta}^{-}$, which are infinite-volume Gibbs measures. By (4.16) and (4.17), the inequalities (2.10) are satisfied if $16c_3^{\ast}_{\beta,N} + \delta_{\beta}(\mathcal{R}) < 1$. By Proposition 4.3 there exists $\beta_c$, such that this last inequality holds true for any $\beta > \beta_c$.

That proves the main result of the paper.
Remaining proofs

Proof of Proposition 4.1.

The proof is the application of the known criteria for a measure to be reflection positive. Fix a line $P \in \mathcal{P}$ of the reflection and let $\theta_P$ be the corresponding reflection operator. The criteria claims that the measure $\mu_{\beta,N}$ is reflection positive, if its Hamiltonian can be represented in the form

$$-H_N = A + \theta_P(A) + \sum_{\alpha} C_\alpha \theta_P(C_\alpha), \quad (4.18)$$

where $A, C_\alpha$ are $\mathcal{F}_P$-measurable functions. Then for all $\beta \geq 0$ the torus Gibbs measure $\mu_{\beta,N}$, is RP with respect to $\theta_P$ (see Definition 3.1). The criteria can be found for example in Theorem 2.1 of Shlosman [19] or Corollary 5.4 of Biskup [3].

Indeed, there are two possibilities for $P \in \mathcal{P}$: $P$ passes trough sites of $\mathbb{T}_N$ or not. In the case of $P$ passing through sites of $\mathbb{T}_N$ choose

$$A = J \sum_{\langle t,s \rangle \in \mathbb{T}_N \setminus P} \sigma(t) \sigma(s) + \frac{J}{2} \sum_{\langle t,s \rangle \in P} \sigma(t) \sigma(s) + \sum_{s \in \mathbb{T}_N \setminus P} h_s \sigma(s) + \frac{1}{2} \sum_{s \in P} h_s \sigma(s),$$

then

$$-H_N(\sigma) = A + \theta_P(A),$$

here functions $C_\alpha \equiv 0$. In the case of reflections through bonds choose

$$A = J \sum_{\langle t,s \rangle \in \mathbb{T}_N} \sigma(t) \sigma(s) + \sum_{s \in \mathbb{T}_N} h_s \sigma(s),$$

then

$$-H_N(\sigma) = A + \theta_P(A) + J \sum_{t \in \mathbb{T}_N : |t-P|=1/2} \sigma(t) \theta_P(\sigma(t)).$$

That proves the proposition.

Proof of Proposition 4.2.

Let $\sigma_{\Lambda,N} := \cap_{t \in \mathbb{Z}_N} \theta_t(\mathcal{B}(\sigma_\Lambda))$ be the configuration on $\mathbb{T}_N$, obtained by reflecting a fixed block configuration $\sigma_\Lambda$. The proof of the proposition 4.2 will be based on the estimation (see Lemma 4.5 and Lemma 4.6) of the right-hand side of the following inequality

$$\delta_{\beta,\Sigma}(\mathcal{B}(\sigma_\Lambda)) = \frac{\exp(-\beta H_N(\sigma_{\Lambda,N}))}{Z_N(\beta)} \leq \exp(-\beta [H_N(\sigma_{\Lambda,N}) - H_N(\sigma^+)]) \quad (4.19)$$

In order to formulate the next lemma we need introduce some notations. Consider the configuration $\sigma_{\Lambda,N}$ introduced above. For any $\sigma_\Lambda$ the configuration $\sigma_{\Lambda,N}$ has the following periodicity property: for any $t \in \mathbb{T}_N$ and $t' \in \mathbb{T}_N$ we have

$$\sigma_{\Lambda,N}(t) = \sigma_{\Lambda,N}(t + 2t). \quad (4.20)$$
It means that there exists some “minimal” sub-configuration such that the configuration \( \sigma_{\Lambda,N} \) can be obtained by its translations. Indeed, using the rectangle \( \tilde{\Lambda} \) defined by (3.8) let us define

\[
\bar{\Lambda}^{(2)} := \{ \tilde{\Lambda} \cup \{ \tilde{\Lambda} + (0, L_2) \} \cup \{ \tilde{\Lambda} + (L_1, 0) \} \cup \{ \tilde{\Lambda} + (L_1, L_2) \} \} + \left( \frac{1}{4}, \frac{1}{4} \right),
\]

\[
T_{\Lambda} := T_N \cap \bar{\Lambda}^{(2)}.
\]

(4.21)

See Figure 3 for illustration.

Remark that \( (T_{\Lambda} + 2t_1) \cap (T_{\Lambda} + 2t_2) = \emptyset \), if \( t_1 \neq t_2 \), with \( t_1, t_2 \in \mathbb{T}_N \). For any \( t \in \mathbb{T}_N \), we have \( h(s) = h(s + 2t) \), as well as,

\[
\sigma_{\Lambda,N}(T_{\Lambda}) = \sigma_{\Lambda,N}(T_{\Lambda} + 2t) \text{ and } T_N = \bigcup_{t \in \mathbb{T}_N} (T_{\Lambda} + 2t).
\]

(4.22)

Adding for \( T_{\Lambda} \) periodic relation we denote by \( \mathbb{T}^{(2)} \) the corresponding torus. Let us define the torus Hamiltonian: for any \( \sigma \in \{-1, +1\}^{\mathbb{T}^{(2)}} \)

\[
H_2(\sigma) = -J \sum_{\langle t,s \rangle \in \mathbb{T}^{(2)}} \sigma(t)\sigma(s) - \sum_{s \in \mathbb{T}^{(2)}} h(s)\sigma(s).
\]

(4.23)

**Lemma 4.5.** For any block configuration \( \sigma_{\Lambda} \in \{-1, +1\}^{\Lambda} \), denote \( \sigma_{\Lambda,2} := \sigma_{\Lambda,N}(T_{\Lambda}) \), then

\[
H_N(\sigma_{\Lambda,N}) = \left( \frac{N}{2} \right)^2 H_2(\sigma_{\Lambda,2}).
\]

(4.24)

**Proof.** Let \( \mathbb{T}^{(2)} = \{ t = (t_1, t_2) \in T_N : t_1 = 2nL_1, t_2 = 2mL_2 \} \). Note that the number of sites in \( \mathbb{T}^{(2)} \) is equal to \( (N/2)^2 \). Therefore, for the external field part in the Hamiltonian we have

\[
\sum_{s \in \mathbb{T}_N} h(s)\sigma_{\Lambda,N}(s) = \sum_{t \in \mathbb{T}^{(2)}} \left( \sum_{s \in T_{\Lambda} + t} h(s)\sigma_{\Lambda,N}(s) \right) = \left( \frac{N}{2} \right)^2 \sum_{s \in \mathbb{T}^{(2)}} h(s)\sigma_{\Lambda,2}(s).
\]

(4.25)
Furthermore, for any $\sigma \in \Omega_N$ we can write
\[
\sum_{\langle t,s \rangle \in \mathbb{T}_N} \sigma(t)\sigma(s) = 
= \sum_{t \in \mathbb{T}_2} \left( \sum_{\langle t,s \rangle \in \mathbb{T}_N + t} \sigma(t)\sigma(s) + \sum_{\langle t,s \rangle : t \in (\mathbb{T}_N + t) \cup (2L_1,0)} \sigma(t)\sigma(s) + \sum_{\langle t,s \rangle : s \in (\mathbb{T}_N + t + (0,2L_2))} \sigma(t)\sigma(s) \right).
\]

Using this representation for neighbors interacting part of the hamiltonian we apply it for configuration $\sigma_{\Lambda,N}$ and remembering the periodicity condition (4.22) we obtain
\[
\sum_{\langle t,s \rangle \in \mathbb{T}_N} \sigma_{\Lambda,N}(t)\sigma_{\Lambda,N}(s) = \sum_{t \in \mathbb{T}_2} \left( \sum_{\langle t,s \rangle \in \mathbb{T}_N^{(2)}} \sigma_{\Lambda,2}(t)\sigma_{\Lambda,2}(s) \right)
= \left( \frac{N}{2} \right)^2 \sum_{\langle t,s \rangle \in \mathbb{T}_N^{(2)}} \sigma_{\Lambda,2}(t)\sigma_{\Lambda,2}(s).
\] (4.26)

The lemma follows from (4.25) and (4.26).

Since, the number of sites in $\mathbb{T}_N$ is equal to $N^2$, using Lemma 4.5 we obtain from (4.19): for any $\sigma_{\Lambda} \in R(\Lambda)$
\[
\mathfrak{J}_{\beta,N}(\mathcal{B}(\sigma_{\Lambda})) \leq \exp \left( -\frac{\beta}{4} \left[ H_2(\sigma_{\Lambda,2}) - H_2(\sigma_{\Lambda,2}^+) \right] \right).
\] (4.27)

Finally the Proposition 4.2 readily follows from the next lemma. The proof of this lemma is essentially repeat the arguments of Lemma 3.3 from [16], but we provide the proof for completeness of reading.

**Lemma 4.6.** If $h$ satisfies (2.7) then for all block configuration $\sigma_{\Lambda} \in R(\Lambda)$, its extended configuration $\sigma_{\Lambda,2}$ on torus $\mathbb{T}^{(2)}$ satisfies,
\[
H_2(\sigma_{\Lambda,2}) - H_2(\sigma_{\Lambda,2}^+) \geq 4 \left( 2J - h\frac{L_1L_2}{L_1 + L_2} \right). \quad (4.28)
\]

**Proof.** The proof is taking from [16]. For brevity of notations we omit the index $\Lambda$ everywhere, we think $\sigma$ as a configuration on $\mathbb{T}^{(2)}$. Let for each $V \subset \mathbb{T}^{(2)}$ the configuration $\sigma^V$ being the perturbation of $\sigma$, that is
\[
\sigma^V(t) = \begin{cases} 
-\sigma(t), & \text{if } t \in V, \\
\sigma(t), & \text{if } t \notin V.
\end{cases} \quad (4.29)
\]

Let $\sigma_\pm$ be a perturbation of $\sigma^+$. Note that,
\[
H_2(\sigma_\pm) - H_2(\sigma^+) = 2J|\partial V| + 2 \sum_{s \in V} h_s. \quad (4.30)
\]

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We will prove the inequality \((4.28)\) for all \(V \neq \emptyset\) and \(V \neq \mathbb{T}^{(2)}\). We decompose \(V\) into the sets of horizontal and vertical lines of sites, that we denote \(S_V\) and \(T_V\), respectively. Thus,

\[
\sum_{s \in V} h_s = \sum_{S \in S_V} \sum_{s \in S} h_s = \sum_{T \in T_V} \sum_{s \in T} h_s. \tag{4.31}
\]

Furthermore, since each horizontal line of \(\mathbb{T}^{(2)}\) is composed by exactly one black and one white segments, then for \(S \in S_V\) and for \(T \in T_V\), the following estimates hold

\[
\left| \sum_{s \in S} h_s \right| = h |S \cap \mathbb{Z}_+| - |S \cap \mathbb{Z}_-| \leq h L_1, \tag{4.32}
\]
\[
\left| \sum_{s \in T} h_s \right| = h |T \cap \mathbb{Z}_+| - |T \cap \mathbb{Z}_-| \leq h L_2.
\]

This implies the following inequalities

\[
2 \sum_{s \in S} \sum_{s \in S} h_s \geq -2h L_1 |S_V| \geq -h L_1 |\partial^v V|, \tag{4.33}
\]
\[
2 \sum_{T \in T_V} \sum_{s \in T} h_s \geq -2h L_2 |T_V| \geq -h L_2 |\partial^h V|,
\]

where \(\partial^h V\) (\(\partial^v V\)) is the set of horizontal (vertical) bonds from \(\partial V\). Clearly at each horizontal line \(S \in S_V\) we must be at least 2 vertical bonds from \(\partial V\), then the last inequalities in \((4.33)\) hold.

Finally, by \((4.31)\) and \((4.33)\),

\[
H_2(\sigma^V_+) - H_2(\sigma^+) = 2J |\partial V| + \frac{L_1}{L_1 + L_2} \left( 2 \sum_{s \in V} h_s \right) + \frac{L_2}{L_1 + L_2} \left( 2 \sum_{s \in V} h_s \right)
\geq 2J |\partial V| - h \frac{L_1 L_2}{L_1 + L_2} |\partial^h V| - h \frac{L_1 L_2}{L_1 + L_2} |\partial^v V| \tag{4.34}
\]
\[
= \left( 2J - h \frac{L_1 L_2}{L_1 + L_2} \right) |\partial V|.
\]

Clearly for each \(V \subset \mathbb{T}^{(2)}\) in study, \(|\partial V| \geq 4\).

Therefore, applying this lemma to \((4.27)\) we complete the proof of Proposition \(4.2\). Note that all steps of the proof can be repeated for the \(\Lambda^*\)-block configuration.

**Proof of Proposition \(4.4\)**

**Proof.** Denote \(\Omega_N^* = \{ \sigma \in \Omega_N : \sigma(s) = +1 \text{ and } \sigma(t) = -1 \}\). Then for each \(\sigma \in \Omega_N^*\) we define the set

\[
I^+(\sigma) = \{ u \in \mathbb{T}_N : \sigma(u) = +1 \}, \tag{4.35}
\]
and let $I^+_s(\sigma) \subseteq I^+(\sigma)$ be its maximal connected component containing the site $s$. Clearly $I^+_s(\sigma) \neq \mathbb{T}_N$. Sites $s$ and $t$ are separated by a usual, for Ising model, Peierls’ contour $\gamma$ in $\mathbb{T}_N$, which lies on the dual graph $\mathbb{T}_N^*: = \mathbb{T}_N + (1/2,1/2)$. Let $\gamma_+(\sigma) := \gamma(\sigma : s, +)$ be the external contour for $I^+_s(\sigma)$.

Denote $\Gamma_{s,t} = \{ \gamma_+(\sigma) : \sigma \in \Omega_N^* \}$ the set of all possible contours defined below. For each $\gamma_+ \in \Gamma_{s,t}$ we denote $\text{Int}(\gamma_+) \subset \mathbb{T}_N$ the interior of $\gamma_+$. Each contour $\gamma_+ \in \Gamma_{s,t}$ determines a $\Lambda$-contour composed by $\Lambda$-blocks as follows

$$
\tilde{\gamma}_\text{blocks} = \left\{ t \in \mathbb{T}_N : (\tilde{\Lambda} + t) \cap \gamma_+ \cap \text{Int}(\gamma_+) \neq \emptyset \right\},
$$

$$
\gamma_\text{blocks} = \cup_{t \in \tilde{\gamma}_\text{blocks}} (\Lambda + t),
$$

where $\tilde{\Lambda}$ is defined by (3.8). However, each $\gamma_\text{blocks}$ may come from many different contours of $\Gamma_{s,t}$. We denote $\gamma_\text{blocks}^{-1} = \{ \gamma_+ \in \Gamma_{s,t} : \gamma_+ \text{ defines } \gamma_\text{blocks} \}$. For any $\gamma_+ \in \Gamma_{s,t}$ we define

$$
\sigma_{\gamma_+} = \left\{ \sigma \in \Omega_N^* : \gamma_+ = \gamma(\sigma : s, +) \text{ is a contour} \right\}.
$$

Let us fix some $\gamma_+ \in \Gamma$ and some configuration $\sigma \in \sigma_{\gamma_+}$. Note that if $L_1$ and $L_2$ are both odds (any two neighbor $\Lambda$-blocks have nonempty intersection), then for any $\sigma \in \sigma_{\gamma_+}$ and any $t \in \tilde{\gamma}_\text{blocks}$ the block configuration $\sigma(\Lambda + t)$ is the bad configuration, $\sigma(\Lambda + t) \in \mathcal{R}$. In this case we have

$$
\mu_{\beta,N}(\sigma(s) = +1, \sigma(t) = -1) = \mu_{\beta,N}\left( \bigcup_{\gamma_+ \in \Gamma} \sigma_{\gamma_+} \right) = \mu_{\beta,N}\left( \bigcup_{\gamma_+ \in \gamma_\text{blocks}^{-1}} \bigcup_{\gamma_+ \in \gamma_\text{blocks}} \sigma_{\gamma_+} \right)
$$

$$
\leq \sum_{\gamma_+ \in \gamma_\text{blocks}^{-1}} \mu_{\beta,N}\left( \bigcup_{\gamma_+ \in \gamma_\text{blocks}} \sigma_{\gamma_+} \right) \leq \sum_{\gamma_+ \in \gamma_\text{blocks}^{-1}} \mu_{\beta,N}\left( \bigcap_{t \in \tilde{\gamma}_\text{blocks}} \theta_t(\mathcal{R}) \right) \leq \sum_{\gamma_+ \in \gamma_\text{blocks}} \delta_{\beta,N}(\mathcal{R}) |\gamma_\text{blocks}|,
$$

where $|\gamma_\text{blocks}|$ denotes the number of points in $\tilde{\gamma}_\text{blocks}$ and where we used the chessboard estimates in the last inequality. By standard arguments (see, Lemma 6.6 of Biskup [3]), the number of $\Lambda$-contours of fixed length $n$ is bounded by $c^n$ for some constant $c > 1$. Finally

$$
\mu_{\beta,N}(\sigma(s) = +1, \sigma(t) = -1) \leq \sum_{n \geq 1} c^n \delta_{\beta,N}(\mathcal{R})^n = \frac{c \delta_{\beta,N}(\mathcal{R})}{1 - c \delta_{\beta,N}(\mathcal{R})} \leq 2c \delta_{\beta,N}(\mathcal{R}),
$$

where for last inequality we chose $\beta$ such that inequality $c \delta_{\beta,N}(\mathcal{R}) < 1/2$ holds: it is possible thanks the Proposition 4.3.

Consider now the case when sides $L_1, L_2$ are even. In this case some of configurations $\sigma(\Lambda + t)$, where $t \in \tilde{\gamma}_\text{blocks}$ can be the good block configurations. It occurs when the contour $\gamma_+$ passes between two neighbors $\Lambda$-blocks, one of which is inside good $\Lambda$-block (with $\sigma_+^\Lambda$ configuration) and other outside block with respect to $\gamma_+$ (with negative spins on the sites neighbors to the inside block). We should estimate the contribution of such good blocks. Let $\mathcal{A}(\sigma_+^\Lambda)$ denote the event described above, remember that it depends also of the corresponding neighbor $\Lambda$-block configuration that belongs to the outside of the $\gamma_+$. Now for any $\gamma_+ \in \gamma_\text{blocks}^{-1}$ we define

$$
\sigma_{\gamma_+}^{n,m} = \left\{ \sigma \in \Omega_N^* : \gamma_+ \text{ is the contour and there exists exactly } m \text{ good and } n \text{ bad } \Lambda \text{-block configurations into } \sigma(\gamma_\text{blocks}) \right\}.
$$
Note that $|\gamma_{\text{blocks}}| = n + m$ and $\tilde{\gamma}_{\text{blocks}} = \tilde{\gamma}_{\text{good}}^{\text{blocks}} \cup \tilde{\gamma}_{\text{bad}}^{\text{blocks}}$, with $|\tilde{\gamma}_{\text{good}}^{\text{blocks}}| = m$ and $|\tilde{\gamma}_{\text{bad}}^{\text{blocks}}| = n$. In the same way we obtain

$$\mu_{\beta,N}(\sigma(s) = +1, \sigma(t) = -1) = \mu_{\beta,N}\left( \bigcup_{\gamma_{+} \in \Gamma} \{ \sigma_{\gamma_{+}} \} \right)$$

$$= \mu_{\beta,N}\left( \bigcup_{\gamma_{+} \in \tilde{\gamma}_{\text{blocks}}^{m,n}} \left( \bigcup_{n,m} \bigcup_{\gamma_{+} \in \tilde{\gamma}_{\text{blocks}}^{n,m}} \sigma^{n,m}_{\gamma_{+}} \right) \right) \leq \sum_{\gamma_{+} \in \tilde{\gamma}_{\text{blocks}}^{m,n}} \mu_{\beta,N}\left( \bigcap_{t \in \tilde{\gamma}_{\text{blocks}}^{m,n}} A(\sigma_{t+\Lambda}) \bigcap_{t \in \tilde{\gamma}_{\text{blocks}}^{m,n}} \theta_{t}(\mathcal{R}) \right)$$

$$\leq \sum_{\gamma_{+} \in \tilde{\gamma}_{\text{blocks}}^{m,n}} \sum_{n,m} \mu_{\beta,N}\left( \bigcap_{t \in \tilde{\gamma}_{\text{blocks}}^{m,n}} A(\sigma_{t+\Lambda}) \bigcap_{t \in \tilde{\gamma}_{\text{blocks}}^{m,n}} \theta_{t}(\mathcal{R}) \right) \bigcap_{t \in \tilde{\gamma}_{\text{blocks}}^{m,n}} \theta_{t}(\mathcal{R})^{1/2} \delta_{\beta,N}(\mathcal{R}) |\tilde{\gamma}_{\text{blocks}}^{m,n}|^{1/2}, \quad (4.42)$$

where in the last inequality we used the Cauchy-Schwarz and chessboard inequalities. Let us fix some block contour $\tilde{\gamma}_{\text{blocks}}^{m,n}$ with corresponding $\tilde{\gamma}_{\text{good}}^{\text{blocks}}$ and $\tilde{\gamma}_{\text{bad}}^{\text{blocks}}$, and now we want estimate the contribution of such good block in (4.42). For this end denote $e$ the vector $e = (L_{1}/2 + 1/2, L_{2}/2 + 1/2)$ and denote

$$V := \{ [\mathbf{t} - e, \mathbf{t} - e + (0, L_{2})], \mathbf{t} \in \tilde{T}_{N} \}$$

$$H := \{ [\mathbf{t} - e, \mathbf{t} - e + (L_{1}, 0)], \mathbf{t} \in \tilde{T}_{N} \}$$

the set of vertical (V) and horizontal (H) intervals separating the blocks. Note that a fixed contour $\tilde{\gamma}_{\text{blocks}}^{m,n}$ uniquely determines the set $E_{v}^{\text{blocks}} \subset V$ and $E_{h}^{\text{blocks}} \subset H$ of intervals which are the part of the contour $\gamma_{+}$ separating the good blocks from the outside with respect of $\gamma_{+}$. With any interval we associate the point $\mathbf{t}$ from $\tilde{T}_{N}$ which used in definition of the interval, and let $\tilde{E}_{v}$ and $\tilde{E}_{h}$ be such corresponding points from $\tilde{T}_{N}$ for the sets $E_{v}$ and $E_{h}$, respectively. Moreover, we provide one more division: $E_{v} = E_{v}^{1} \cup E_{v}^{2}$, where $E_{v}^{1}(E_{v}^{2})$ is the subset of intervals for which the corresponding points $\mathbf{t} = (t_{1}, t_{2})$ such that $t_{2}/L_{2}$ is odd (even). Similarly, $E_{h} = E_{h}^{1} \cup E_{h}^{2}$, where $E_{h}^{1}(E_{h}^{2})$ is the subset of intervals for which the corresponding points $\mathbf{t}$ such that $t_{1}/L_{1}$ is odd (even).

Therefore, any $\mathbf{t} \in \tilde{E}_{v} \cup \tilde{E}_{h}$ can be treated as a new $\Lambda^{*}$-block, as defined in (4.3), with corresponding positions for covering $\tilde{T}_{N}$ as in (4.4). Thus, for a given $\tilde{\gamma}_{\text{blocks}}^{m,n}$ with corresponding $\tilde{\gamma}_{\text{good}}^{\text{blocks}} \cup \tilde{\gamma}_{\text{bad}}^{\text{blocks}}$

$$\bigcap_{t \in \tilde{\gamma}_{\text{blocks}}^{m,n}} A(\sigma_{t+\Lambda}) \subset \bigcap_{t \in E_{h}^{1}} \theta_{t}(\mathcal{R}_{1,h}) \bigcap_{t \in E_{h}^{2}} \theta_{t}(\mathcal{R}_{2,h}) \bigcap_{t \in E_{v}^{1}} \theta_{t}(\mathcal{R}_{1,v}) \bigcap_{t \in E_{v}^{2}} \theta_{t}(\mathcal{R}_{2,v}),$$

and by Cauchy-Schwarz inequalities we obtain

$$\leq \sum_{\gamma_{+} \in \tilde{\gamma}_{\text{blocks}}^{m,n}} \sum_{n,m} \mu_{\beta,N}\left( \bigcap_{t \in E_{h}^{1}} \theta_{t}(\mathcal{R}_{1,h}) \right) \bigcap_{t \in E_{h}^{2}} \theta_{t}(\mathcal{R}_{2,h})^{1/8} \mu_{\beta,N}\left( \bigcap_{t \in E_{v}^{1}} \theta_{t}(\mathcal{R}_{1,v}) \right) \bigcap_{t \in E_{v}^{2}} \theta_{t}(\mathcal{R}_{2,v})^{1/8} \delta_{\beta,N}(\mathcal{R}) |\tilde{\gamma}_{\text{blocks}}^{m,n}|^{1/2}, \quad (4.43)$$
Finally, by (4.5) and denoting
\[ z^*_{\beta,N} := \max\{z_{\beta,N}(\mathcal{R}_h)^{1/8}, z_{\beta,N}(\mathcal{R}_v)^{1/8}, z_{\beta,N}(\mathcal{R})^{1/2}\}, \]
see (4.11), then using chessboard estimate and Proposition 4.3 we obtain
\[ (4.43) \leq \sum_{\gamma_{\text{blocks}}} \sum_{n,m} (z^*_{\beta,N})^{s_{\gamma_{\text{blocks}}}} \leq \sum_{n \geq 1} c^n (2z^*_{\beta,N})^n \leq 4c_3 z^*_{\beta,N} \] (4.44)
where the last inequality holds for such \( \beta \) when \( 2c_3 z^*_{\beta,N} < 1/2 \). That finishes the proof of the proposition.

5 2D Ising model where external field is alternating on 1D sublattices

In [17] it was studied a phase diagram of the 2D Ising model with alternating external field on 1D sublattices. In this section we prove the result of [17] about coexistence of two phases by using RP in a way similar to the considerations in previous sections. In fact we prove a more general result of the coexistence, including the coexistence result of [17]. The model in [17] is as following. The external field is

\[ h_s = \begin{cases} h, & \text{if } s \in \Lambda_1, \\ -h, & \text{if } s \in \Lambda_2, \end{cases} \] (5.1)

where
\[ \Lambda_1 = \{ t = (t_1, t_2) \in \mathbb{Z}^2 : t_2 \text{ odd} \}, \]
\[ \Lambda_2 = \{ t = (t_1, t_2) \in \mathbb{Z}^2 : t_2 \text{ even} \}, \] (5.2)

and the Hamiltonian is defined by (2.3).

Thus this model is an “extreme” case of the cell-board model. Indeed, the external field in (5.1) can be obtained letting \( L_1 = \infty \) and \( L_2 = 1 \). In [17], it is proved that a phase transition in this model holds true for \( \beta \) sufficiently large and \( h \) satisfying the inequality
\[ h < 2J - ke^{-\beta J}, \] (5.3)
where \( k \) being a suitable positive constant. We propose a more general model, with \( L_2 \geq 1 \), for which we use the reflection positivity techniques to prove phase transition.

**Theorem 5.1.** Let \( L \in \mathbb{N} \). For each \( n \) integer define
\[ \Lambda(n) = \{ (t_1, t_2) \in \mathbb{Z}^2 : nL \leq t_2 < (n + 1)L \}, \] (5.4)
and
\[ \Lambda_1 = \bigcup_{n:n \text{ even}} \Lambda(n), \quad \text{and} \quad \Lambda_2 = \bigcup_{n:n \text{ odd}} \Lambda(n), \] (5.5)
Consider the Ising model defined by the Hamiltonian as \((2.3)\) with external field given by \((5.1)\) and \((5.5)\). If \(h < \frac{2}{J/L}\), then there exists a suitable positive constant \(k = k(L)\), such that for any \(\beta > k/(2J - hL)\), there exist two distinct measures \(\mu^+_\beta\) and \(\mu^-_\beta \in \mathcal{G}_\beta\), which satisfy

\[
\mu^+_\beta(\sigma(t) = 1) > \frac{1}{2} \quad \text{and} \quad \mu^-_\beta(\sigma(t) = -1) > \frac{1}{2}.
\]

**Proof.** We follow ideas of Section 3. First, we construct a torus \(T_N\) by taking a subset \(T_N\) of \(\mathbb{Z}^2\) of size \(N \times NL\):

\[
T_N = \{t = (t_1, t_2) \in \mathbb{Z}^2 : 0 \leq t_1 < N, 0 \leq t_2 < NL\},
\]

where \(N\) is multiple of 4. Thus, \(T_N\) is the factor-group \(\mathbb{Z}/(N\mathbb{Z}) \times \mathbb{Z}/(NL\mathbb{Z})\). We consider the corresponding Hamiltonian with periodical boundary condition, as defined in \((3.1)\). To apply the chessboard estimate we define the \(\Lambda\)-blocks by translations of the block \(\Lambda\), which is the smallest red rectangle containing the origin in Figure 2(b). We construct \(\Lambda\) considering in \(\mathbb{R}^2\) the rectangle

\[
\tilde{\Lambda} = \{(t_1, t_2) : |t_1 + \frac{1}{2}| \leq \frac{1}{2}, |t_2 + \frac{1}{2}| \leq \frac{L}{2}\}.
\]

Then \(\Lambda = \tilde{\Lambda} \cap \mathbb{Z}^2\). And \(\tilde{T}_N = \{t = (t_1, t_2) \in T_N : t_1 = n, t_2 = mL\}\).

This particular construction of the torus \(\tilde{T}_N\) allows us to prove a version of Proposition 4.2. Furthermore, we use the same ideas of bad blocks (see \((4.1)\) for the definition of \(R(\Lambda)\) and \((4.2)\) for the event \(\mathcal{R}\)).

**Proposition 5.2.** If \(h < 2J/L\), then for any \(\beta > 0\) and any block configuration \(\sigma_\Lambda \in R(\Lambda)\), the inequality

\[
\mathcal{Z}_\beta(\mathcal{B}(\sigma_\Lambda)) \leq \exp \{-\beta (2J - hL)\},
\]

holds.

**Proof.** The proof is similar to the proof of Proposition 4.2. We use the periodicity property defined in \((4.20)\), and the decomposition of \(T_N\) in sub-tori \(T^{(2)}\), where the sub-torus \(T^{(2)}\) has its size \(2 \times 2L\) sites. Thus, we apply Lemma 4.5 and a version of Lemma 4.6. In this case, we can prove that for all block configuration \(\sigma_\Lambda \in R(\Lambda)\), its extended configuration \(\sigma_{\Lambda,2}\) on torus \(T^{(2)}\) satisfies,

\[
H_2(\sigma_{\Lambda,2}) - H_2(\sigma^+_{\Lambda,2}) \geq H_2(\sigma_c) - H_2(\sigma^+_{\Lambda,2}) \geq 4 (2J - hL),
\]

where \(\sigma_c \in \{-1, +1\}^{T^{(2)}}\) is cell-board configuration, defined in \((2.9)\). The idea is to use that any \(\sigma_{\Lambda,2}\) has constant configuration (the same spin value) in both sites of each horizontal line of the torus \(T^{(2)}\). Then, it is clear that we must compare the energy of cell-board configuration with the ground states on \(T^{(2)}\). \(\square\)
Repeating considerations with the conditional measures (4.15) as in the proof of Theorem 2.2. We use Proposition 5.2 to obtain Proposition 4.4, then by (4.14) and the version of Proposition 4.3 (with $h < 2J/L$), there exists $\beta_c$ such that for any $\beta > \beta_c$, we obtain (5.6):

$$\mu^+_{\beta} (\sigma(s) = -1) < \frac{1}{2} \quad \text{and} \quad \mu^-_{\beta} (\sigma(s) = +1) < \frac{1}{2}. $$

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