THE QUANTIZATION FOR MARKOV-TYPE MEASURES ON A CLASS OF RATIO-SPECIFIED GRAPH DIRECTED FRACTALS

MARC KESSEBÖHMER AND SANGUO ZHU

Abstract. We study the asymptotic quantization error of order \( r \) for Markov-type measures \( \mu \) on a class of ratio-specified graph directed fractals. We show that the quantization dimension of \( \mu \) exists and determine its exact value \( s_r \) in terms of spectral radius of a related matrix. We prove that the \( s_r \)-dimensional lower quantization coefficient of \( \mu \) is always positive. Moreover, inspired by Mauldin-Williams’ work on the Hausdorff measure of graph directed fractals, we establish a necessary and sufficient condition for the \( s_r \)-dimensional upper quantization coefficient of \( \mu \) to be finite.

1. Introduction

In this paper, we study the asymptotics of the quantization error for Markov-type measures on a class of ratio-specified graph directed fractals. One of the main mathematical aims of the quantization problem is to study the error in the approximation of a given probability measure with probability measures of finite support. We refer to [3, 4, 6, 7, 15] for more theoretical results and [13, 14] for promising applications of quantization theory. One may see [8, 16] for its deep background in information theory and engineering technology. In the following, let us recall some of the crucial definitions and known results.

1.1. The upper (lower) quantization dimension and quantization coefficient. We set \( D_n := \{ \alpha \subset \mathbb{R}^q : 1 \leq \text{card}(\alpha) \leq n \} \) for \( n \in \mathbb{N} \). Let \( \nu \) be a Borel probability measure on \( \mathbb{R}^q \). For every \( n \geq 1 \), the \( n \)th quantization error for \( \nu \) of order \( r \) is defined by (see [3] for a number of equivalent definitions):

\[
ed_{n,r}(\nu) := \inf_{\alpha \in D_n} \left( \int d(x, \alpha)^r d\nu(x) \right)^{1/r}.
\]

Here \( d(x, \alpha) := \inf_{a \in \alpha} d(x, a) \) and \( d(\cdot, \cdot) \) is the metric induced by an arbitrary norm on \( \mathbb{R}^q \). For \( r \geq 1 \), \( e_{n,r}(\nu) \) agrees with the error in the approximation of \( \nu \) by discrete probability measures supported on at most \( n \) points, in the sense of the Wasserstein \( L_r \)-metric [3].

The convergence rate of \( e_{n,r}(\nu) \) is characterized by the upper and lower quantization dimension of order \( r \) as defined below:

\[
\overline{D}_r(\nu) := \limsup_{n \to \infty} \frac{\log n}{\log e_{n,r}(\nu)}, \quad D_r(\nu) := \liminf_{n \to \infty} \frac{\log n}{\log e_{n,r}(\nu)}.
\]

2000 Mathematics Subject Classification. 28A75, 28A80, 94A15.

Key words and phrases. Markov-type measures, graph directed fractals, quantization dimensions, quantization coefficients.

The second author was supported by the China Scholarship Council, File No. 201308320049.
Theorem The self-similar measure associated with \((f_i)_{i=1}^N\) and a probability vector \((q_i)_{i=1}^N\). Let \(P\) be the unique solution of the equation \(\sum_{i=1}^N q_i \nu \circ f_i^{-1} = \nu\). Let \((f_i)_{i=1}^N\) be a family of contractive similitudes on \(\mathbb{R}^d\) by \(\prod_{i=1}^N\) and a probability vector \((q_i)_{i=1}^N\). Recall that \((f_i)_{i=1}^N\) is said to satisfy the open set condition (OSC), if there exists a non-empty bounded open set \(U\) such that \(\bigcup_{i=1}^N f_i(U) \subset U\) and \(f_i(U) \cap f_j(U) = \emptyset\) for all \(1 \leq i \neq j \leq N\). Graf and Luschgy proved the following result which provides us with significant insight into the study for non-self-similar probability measures:

**Theorem** (Graf/Luschgy [4, 5]). Assume that \((f_i)_{i=1}^N\) satisfies the OSC. Let \(\nu\) be the self-similar measure associated with \((f_i)_{i=1}^N\) and a probability vector \((q_i)_{i=1}^N\). Let \(k_r\) be the unique solution of the equation \(\sum_{i=1}^N (q_i s_i^r) = 1\). Then

\[
D_r(\nu) = k_r, \quad 0 < \overline{Q}_r(\nu) \leq Q_r(\nu) < \infty.
\]

### 1.2. A class of graph directed fractals and Markov-type measures.

In this subsection, we recall the definitions of a class of graph directed fractals and Markov-type measures on these sets. One may see [1, 2, 12] for more details.

Let \(P := (p_{ij})_{N \times N}\) be a row-stochastic matrix, namely, \(p_{ij} \geq 0, 1 \leq i, j \leq N\), and \(\sum_{j=1}^N p_{ij} = 1, 1 \leq i \leq N\). We always assume

\[
\text{card} (\{1 \leq j \leq N : p_{ij} > 0\}) \geq 2 \quad \text{for all} \quad 1 \leq i \leq N.
\]

We will need the following notations. Set

\[
\begin{align*}
\emptyset &:= \text{empty word}, \quad \Omega_0 := \{\emptyset\}, \quad \Omega_1 := \{1, \ldots, N\}; \\
\Omega_k &:= \{\sigma \in \Omega_k^k : p_{\sigma_1 \sigma_2 \ldots \sigma_k} > 0\}, \quad k \geq 2; \\
\Omega^* &:= \bigcup_{k \geq 0} \Omega_k, \quad \Omega_\infty := \{\sigma \in \Omega^k : p_{\sigma_h \sigma_{h+1}} > 0 \, \text{for all} \, h \geq 1\}.
\end{align*}
\]

We denote by \(|\sigma|\) the length of \(\sigma\), namely, \(|\sigma| := k\) for \(\sigma \in \Omega_k\) and \(|\emptyset| := 0\). For every word \(\sigma = (\sigma_1, \ldots, \sigma_n)\) with \(n \geq k\) or \(\sigma \in \Omega_\infty\), we write \(|\sigma| := (\sigma_1, \ldots, \sigma_k)\). If \(\sigma, \omega \in \Omega^*\) and \((\sigma|\sigma|, \omega_1) \in \Omega_{|\sigma|+|\omega|}\) then we define

\[
\sigma \ast \omega = (\sigma_1, \sigma_2, \ldots, \sigma_{|\sigma|}, \omega_1, \ldots, \omega_{|\omega|}) \in \Omega_{|\sigma|+|\omega|}.
\]

Let \(J_i, 1 \leq N\) be non-empty compact subsets of \(\mathbb{R}^d\) with \(J_i = \text{int}(J_i)\) for all \(1 \leq i \leq N\), where \(\overline{B}\) and \(\text{int}(B)\) respectively denote the closure and interior in \(\mathbb{R}^d\) of a set \(B \subset \mathbb{R}^d\). For convenience, we always assume that

\[
\text{diam}(J_i) = 1 \quad \text{for all} \quad 1 \leq i \leq N.
\]
Let \((c_{ij})_{N \times N}\) be a non-negative matrix such that \(c_{ij} > 0\) if and only if \(p_{ij} > 0\). For each pair \(1 \leq i, j \leq N\) with \(p_{ij} > 0\), let \(T_{ij}\) be a contracting similitude on \(\mathbb{R}^q\) of contraction ratio \(c_{ij}\). Assume that, \(T_{ij}(J_j), (i, j) \in \Omega_2\), are non-overlapping subsets of \(J_j\). By [11, Corollary 3.5] (see also [12, Theorem 3]), there exists a unique vector set \((K_i)_{i=1}^N \subset \prod_{i=1}^N J_i\) such that

\[
K_i = \bigcup_{j: (i, j) \in \Omega_2} T_{ij}(K_j), \quad 1 \leq i \leq N.
\]

We call \(K := \bigcup_{i=1}^N K_i\) the recurrent self-similar set associated with the contracting similitudes \(T_{ij}, 1 \leq i, j \leq N\). One can see that \(K\) is also a map-specified MW-fractal which is defined in terms of a directed graph [12].

Assume that \(P\) is irreducible. Let \(v = (q_i)_{i=1}^N\) be the unique normalized positive left eigenvector of \(P\) with respect to the Perron-Frobenius eigenvalue 1, or equivalently,

\[
\sum_{i=1}^N v_i = 1, \quad v_i > 0, \quad 1 \leq i \leq N; \quad \sum_{i=1}^N v_i p_{ij} = v_j.
\]

We accordingly have a unique vector set \((\nu_i)_{i=1}^N\) of probability measures such that

\[
\nu_i = \sum_{j: (i, j) \in \Omega_2} p_{ij} \nu_j \circ T_{ij}^{-1}
\]

and \(\nu_i\) is supported on \(K_i\) for each \(1 \leq i \leq N\). Hence, we get a Markov-type measure \(\nu := \sum_{i=1}^N p_i \nu_i\) supported on \(K\).

Assuming the irreducibility of the corresponding transition matrices (or strong connectedness of the corresponding graphs), Lindsay has studied the quantization problem for Markov-type measures on map-specified graph directed fractals in [11]; in there he expressed the quantization dimension \(D_r\) in terms of temperature functions and showed that the \(D_r\)-dimensional upper quantization coefficient is finite. Let us note the following facts:

1. the arguments in [11] depend on the invariance properties (1.3) and (1.4); these arguments are not applicable to ratio-specified cases due to the absence of the invariance properties;
2. the interesting cases, where the transition matrices are reducible, have not been explored.

In the present paper, we consider the Markov-type measures \(\mu\) on a class of ratio-specified graph directed fractals \(E\). Mauldin and Williams [12, Theorem 4] have established a necessary and sufficient condition for the Hausdorff measure of a graph directed fractal to be positive and finite. Significantly inspired by this result, we will establish a necessary and sufficient condition for the upper and lower quantization coefficient of \(\mu\) to be both positive and finite, allowing the corresponding transition matrices to be reducible.

Let \(J_i, P = (p_{ij})_{N \times N}\), be given as above. We call \(J_i, 1 \leq i \leq N\), cylinder sets of order one. For each \(1 \leq i \leq N\), let \(J_{ij}, (i, j) \in \Omega_2\), be non-overlapping subsets of \(J_i\) such that \(J_{ij}\) is geometrically similar to \(J_i\) with \(\text{diam}(J_{ij})/\text{diam}(J_i) = c_{ij}\). We call these sets cylinder sets of order two. Assume that cylinder sets of order \(k\) are determined, namely, for each \(\sigma := (i_1, \ldots, i_k) \in \Omega_k\), we have a cylinder set \(J_\sigma\). Let \(J_{\sigma, i_{k+1}}\), \(\sigma \ast i_{k+1} \in \Omega_{k+1}\), be non-overlapping subsets of \(J_\sigma\) such that \(J_{\sigma, i_{k+1}}\) is geometrically similar to \(J_\sigma\) with \(\text{diam}(J_{\sigma, i_{k+1}})/\text{diam}(J_\sigma) = c_{i_{k+1}}\). Inductively,
cylinder sets of order $k$ are determined for all $k \geq 1$. The ratio specified MW-fractal is then given by

$$E := \bigcap_{k \geq 1} \bigcup_{\sigma \in \Omega_k} J_\sigma.$$  

Note that we only fix the contraction ratios $c_{ij}$, $1 \leq i \leq j \leq N$, and we do not fix the similarity mappings, so a ratio-specified MW-fractal typically does not enjoy the invariance property 1.3 of $K$.

Let $(X_i)_{i=1}^{N}$ be an arbitrary probability vector with $\min_{1 \leq i \leq N} X_i > 0$. By Kolmogorov consistency theorem, there exists a unique probability measure $\tilde{\mu}$ on $\Omega_\infty$ such that $\tilde{\mu}([\sigma]) = X_{\sigma_1}p_{\sigma_1\sigma_2}\cdots p_{\sigma_{k-1}\sigma_k}$ for every $k \geq 1$ and $\sigma = (\sigma_1, \ldots, \sigma_k) \in \Omega_k$, where $[\sigma] := \{\omega \in \Omega_\infty : \omega|_{[\sigma]} = \sigma\}$. Let $\pi$ denote the projection from $\Omega_\infty$ to $E$: $\pi(\sigma) = x$, where

$$\{x\} := \bigcap_{k \geq 1} J_{\sigma|_k}, \text{ for } \sigma \in \Omega_\infty.$$  

To overcome the difficulty caused by the absence of invariance properties, we assume the following separation property for $E$: there is some constant $0 < t < 1$ such that for every $\sigma \in \Omega^*$ and distinct $i_1, i_2 \in \Omega_1$ with $p_{\sigma_{i_1}i_2} > 0, l = 1, 2,

$$d(J_{\sigma_{i_1}}, J_{\sigma_{i_2}}) := \inf\{||x - y|| : x \in J_{\sigma_{i_1}}, y \in J_{\sigma_{i_2}}\} \geq t \max\{|J_{\sigma_{i_1}}|, |J_{\sigma_{i_2}}|\}.$$  

Here $|A|$ denotes the diameter of a set $A \subset \mathbb{R}^d$. Under this assumption, $\pi$ is a bijection. We consider the image measure of $\tilde{\mu}$ under the projection $\pi$: $\mu := \tilde{\mu} \circ \pi^{-1}$.

We call $\mu$ a Markov-type measure which satisfies

$$\mu(J_\sigma) = X_{\sigma_1}p_{\sigma_1\sigma_2}\cdots p_{\sigma_k\sigma_k} \quad \text{for } \sigma = (\sigma_1, \ldots, \sigma_k) \in \Omega_k.$$  

As there are infinitely many similitudes corresponding to given contraction ratios $c_{ij}$, $\mu$ generally does not enjoy the invariance property 1.3.

1.3. Statement of main results. Before we state our main result, we need to recall some facts on spectral radius of matrices and some notations on directed graphs.

For $1 \leq i, j \leq N$, we define $a_{ij}(s) := (p_{ij}c_{ij}^{s})^s$. Then we get an $N \times N$ matrix $A(s) = (a_{ij}(s))_{N \times N}$. Let $\psi(s)$ denote the spectral radius of $A(s)$. By [12] Theorem 2], $\psi(s)$ is continuous and strictly decreasing. Note that, by the assumption [12], $\psi(0) \geq 2$; by Perron-Frobenius theorem, we have,

$$\psi(1) \leq \max_{1 \leq i \leq N} \sum_{j=1}^{N} a_{ij}(1) < \max_{1 \leq i \leq N} \sum_{j=1}^{N} p_{ij} = 1.$$  

Intermediate-value theorem implies that there exists a unique number $\xi \in (0, 1)$ such that $\psi(\xi) = 1$. Thus, for every $r > 0$, there exists a unique positive number $s_r$ such that $\psi(s_r/(s_r + r)) = 1$.

As in [12], we consider the directed graph $G$ associated with the transition matrix $(p_{ij})_{N \times N}$. Namely, $G$ has vertices $1, 2, \ldots, N$; there is an edge from $i$ to $j$ if and only if $p_{ij} > 0$. In the following, we will simply denote by $G = \{1, \ldots, N\}$ both the directed graph and its vertex sets. We also write

$$b_{ij}(s) := (p_{ij}c_{ij}^{s/(s+r)})^{s/(s+r)}, \quad A_{G,s} := (b_{ij}(s))_{N \times N}; \quad \Psi_{G}(s) := \psi(s/(s+r)).$$  

We also refer to an element $(i_1, \ldots, i_k) \in \Omega_k$ as a path in $G$. We call $H \subset G$, with edges inherited from $G$, a subgraph of $G$. A subgraph $H$ of $G$ is called strongly connected if for every pair $i_1, i_2 \in H$, there exists a path $\gamma$ in $H$ which begins at $i_1$ and ends at $i_2$. A strongly connected component of $G$ refers to a maximal strongly connected subgraph. Let $SC(G)$ denote the set of all strongly connected
components of $G$. For $H_1, H_2 \in SC(G)$, we write $H_1 \prec H_2$, if there is a path $\gamma = (i_1, \ldots, i_k)$ in $G$ such that $i_1 \in H_1$ and $i_k \in H_2$. If we have neither $H_1 \prec H_2$ nor $H_2 \prec H_1$, then we say $H_1, H_2$ are incomparable.

For every $H \in SC(G)$, we denote by $A_{H,s}$ the sub-matrix $(b_{ij}(s))_{i,j \in H}$ of $A_G(s)$. Let $\Psi_\mu(s)$ be the spectral radius of $A_{H,s}$ and $s_r(H)$ be the unique positive number satisfying $\Psi_{H}(s_r(H)) = 1$. With assumption (1.2), one can see by pigeon-hole principle that $G$ has at least one strongly connected component $H$ with $\text{card}(H) \geq 2$. As our main result, we will prove

**Theorem 2.1.** Assume that (1.2) and (1.5) are satisfied and let $\mu$ be the Markov-type measure defined in (1.6), and let $s_r$ be the unique positive number satisfying $\Psi_G(s_r) = 1$. Then we have,

$$D_r(\mu) = s_r \text{ and } Q^r(\mu) > 0.$$  

Furthermore, $Q^r(\mu) < \infty$ if and only if $M := \{H \in SC(G) : s_r(H) = s_r\}$ consists only of incomparable elements; otherwise, we have $Q^r(\mu) = \infty$.

At this point, we remark that, although the quantization problem for probability measures and the Hausdorff measure of sets are two substantially different objects, we benefit significantly from some methods previously developed in [12].

2. Notations and preliminary facts

For every $k \geq 2$ and $\sigma = (\sigma_1, \ldots, \sigma_k) \in \Omega_k$, we write

$$\sigma^- := (\sigma_1, \ldots, \sigma_{k-1}); \ p_\sigma := p_{\sigma_1} \sigma_2 \cdots p_{\sigma_{k-1}} \sigma_k; \ c_\sigma := c_{\sigma_1} \sigma_2 \cdots c_{\sigma_{k-1}} \sigma_k.$$  

If $|\sigma| = 1$, we set $\sigma^- = \theta$, where $\theta$ denotes the empty word; we also define $p_\sigma := 1, c_\sigma := 1$ for $\sigma \in \Omega_1 \cup \{\theta\}$. If $\sigma, \omega \in \Omega^*$ satisfy $|\sigma| \leq |\omega|$ and $\sigma = \omega|\sigma|$, then we call $\omega$ a descendant of $\sigma$ and write $\sigma \prec \omega$. Two words $\sigma, \omega \in \Omega^*$ are said to be incomparable if neither $\sigma \prec \omega$, nor $\omega \prec \sigma$. A finite subset $\Gamma$ of $\Omega^*$ is called a finite antichain if any two words in $\Gamma$ are incomparable; a finite antichain $\Gamma$ is said to be maximal, if every $\tau \in \Omega_\infty$ is the descendant of some word $\sigma \in \Gamma$. Set

$$p := \min_{i,j \in \Omega_2} p_{ij}, \ c := \min_{i,j \in \Omega_2} c_{ij}, \ p := \min_{i,j \in \Omega_2} p_{ij}, \ c := \max_{i,j \in \Omega_2} c_{ij}.$$  

For $r > 0$, let $\eta := \frac{p}{\eta^r}$. To study the quantization error $e_{n,r}(\mu)$, we define

$$\Lambda_{j,r} := \{\sigma \in \Omega^* : p_\sigma c^r_\sigma \geq \eta^j > p_\sigma c^r_\sigma\}.$$  

Then $(\Lambda_{j,r})_{j=1}^\infty$ is a sequence of finite maximal antichains. This type of sets were constructed by Graf and Luschgy in their work on the quantization for self-similar measures (cf. [3]). The spirit of these constructions is to seek some kind of uniformity while general measures are not uniform. We define

$$\phi_{j,r}(\mu) := \lim inf_{j \to \infty} \frac{1}{\phi_{j,r}(\mu)} e^r_{j,r}(\mu), \ \overline{P}_r(\mu) := \lim sup_{j \to \infty} \phi_{j,r}(\mu).$$  

**Lemma 2.1.** We have

$$Q^r(\mu) > 0 \iff P^r(\mu) > 0 \text{ and } Q^r(\mu) < \infty \iff \overline{P}_r(\mu) < \infty.$$  

**Proof.** Let $N_1 := \min\{h \in \mathbb{N} : (\overline{P}_r)^h < \eta\}$. For every $\sigma \in \Lambda_{j,r}$, we have, $p_\sigma c^r_\sigma < \eta^j$. Hence, for every $\omega \in \Omega_{N_1}$ with $(\sigma|\sigma|, \omega) \in \Omega_2$, we have

$$p_\sigma \omega c^r_\sigma \omega = (p_\sigma c^r_\sigma) (p_\sigma c^r_\sigma) (p_\sigma c^r_\sigma) \leq (p_\sigma c^r_\sigma) (p_\sigma c^r_\sigma)^N_1 < \eta^{j+1}. \$$
For every \( a \) numbers, we write \( a \) is denoted by \( r \) which is independent of \( \sigma \). For all large \( j \), \( \sigma \).

\begin{equation}
\Gamma(\sigma, h) := \{ \omega \in \Omega_{\sigma \cap h} : \sigma < \omega \}.
\end{equation}

It follows that \( \phi_{j,r} \leq \phi_{j+1,r} \leq N^{N_1} \phi_{j,r} \). This and \([18], \text{Lemma 2.4}\) completes the proof of the lemma. \( \square \)

If the infimum in \((1.1)\) is attained at some \( \alpha \) with \( \text{card}(\alpha) \leq n \), we call \( \alpha \) an \( n \)-optimal set for \( \nu \) of order \( r \). The collection of all \( n \)-optimal sets for \( \nu \) of order \( r \) is denoted by \( C_{\nu,r}(\nu) \). For two sequences \((a_n)_{n=1}^\infty \) and \((b_n)_{n=1}^\infty \) of positive real numbers, we write \( a_n \ll b_n \) if there is some constant \( B \) independent of \( n \) such that \( a_n \leq B \cdot b_n \). If \( a_n \ll b_n \) and \( b_n \ll a_n \) we write \( a_n \asymp b_n \).

For every \( k \geq 1 \) and a vector \( w = (w_i)_{i=1}^k \in \mathbb{R}^k \), we define
\[
\overline{w} := \max_{1 \leq i \leq k} w_i, \quad \underline{w} := \min_{1 \leq i \leq k} w_i.
\]

**Lemma 2.2.** For all large \( j \geq 1 \), we have
\begin{equation}
e_{\phi_{j,r}}(\mu) \leq \sum_{\sigma \in \Lambda_{j,r}} p_\sigma e_{\sigma,r}^r.
\end{equation}

**Proof.** For every \( \sigma \in \Lambda_{j,r} \), let \( a_\sigma \) be an arbitrary point in \( J_\sigma \). We have
\[
e_{\phi_{j,r}}(\mu) \leq \sum_{\sigma \in \Lambda_{j,r}} \int_{J_\sigma} d(x, a_\sigma)^r d\mu(x)
\leq \sum_{\sigma \in \Lambda_{j,r}} \mu(J_\sigma)|J_\sigma|^r = \sum_{\sigma \in \Lambda_{j,r}} \chi_{\sigma_1} p_\sigma e_{\sigma,r}^r \leq \overline{\chi} \sum_{\sigma \in \Lambda_{j,r}} p_\sigma e_{\sigma,r}^r.
\]

Using \((1.6)\) and the method in \([17], \text{Lemma 3}\), one can find a constant \( L \geq 1 \), which is independent of \( j \), and a set \( \beta(\sigma) \) with \( \text{card}(\beta(\sigma)) \leq L \) such that
\begin{equation}
e_{\phi_{j,r}}(\mu) \geq \sum_{\sigma \in \Lambda_{j,r}} \int_{J_\sigma} d(x, \beta(\sigma))^r d\mu(x).
\end{equation}

Then by \((1.7)\) and the arguments in \([17], \text{Lemma 4}\), one may find a constant \( D > 0 \) which is independent of \( \sigma \in \Omega^* \), such that
\begin{equation}
\int_{J_\sigma} d(x, \beta(\sigma))^r d\mu(x) \geq D \mu(J_\sigma)|J_\sigma|^r \geq D \overline{\chi} \sum_{\sigma \in \Lambda_{j,r}} p_\sigma e_{\sigma,r}^r.
\end{equation}

By \((2.4)\) and \((2.5)\), we conclude that \( e_{\phi_{j,r}}(\mu) \geq D \sum_{\sigma \in \Lambda_{j,r}} p_\sigma e_{\sigma,r}^r \). \( \square \)

### 3. Proof of Theorem 1.1

We will treat the irreducible and non-irreducible case separately.

#### 3.1. Markov measures with irreducible transition matrix.

**Lemma 3.1.** Assume that \( P = (p_{ij})_{N \times N} \) is irreducible. Then there exist constants \( \delta_i > 0, i = 1, 2 \) such that, for every finite maximal antichain \( \Gamma \subset \Omega^* \),
\[
\delta_1 \leq \sum_{\sigma \in \Gamma} (p_\sigma e_{\sigma,r}^r)^{s_\sigma/(s_\sigma + r)} \leq \delta_2.
\]
Thus, by Kolmogorov consistency theorem, we set $\sigma \xi$. As $\Psi$ Frobenius theorem, 1 is an eigenvalue of $A$, Proposition 3.2.

The lemma follows by setting $\nu_1([\sigma]) := (p_\sigma c_\sigma^{s_r/(s_r+r)})_{\xi_{\sigma}}$. Then one can easily see

$$\sum_{i=1}^N \nu_1([\sigma * i]) = (p_\sigma c_\sigma^{s_r/(s_r+r)})_{\sum_{i=1}^N (p_\sigma c_{}\sigma|_{\xi_{\sigma}})}_{s_r/(s_r+r)} \xi_i = \nu_1([\sigma]) ,$$

$$\sum_{\sigma \in \Omega_2} (p_\sigma c_\sigma^{s_r/(s_r+r)})_{\xi_{\sigma}} = N \sum_{i=1}^N \sum_{j=1}^N (p_\sigma c_{ij})_{\xi_j} = N \sum_{i=1}^N \xi_i = 1.$$

Thus, by Kolmogorov consistency theorem, $\nu_1$ extends a probability measure on $\Omega_\infty$. Since $\Gamma$ is a finite maximal antichain, we have

$$\sum_{\sigma \in \Gamma} (p_\sigma c_\sigma^{s_r/(s_r+r)})_{\xi_{\sigma}} = \nu_1(\Omega_\infty) = 1.$$

As an immediate consequence, we have

$$\xi_1^{-1} \leq \sum_{\sigma \in \Gamma} (p_\sigma c_\sigma^{s_r/(s_r+r)}) \leq \xi^{-1}.$$ 

The lemma follows by setting $\delta_1 := \xi_1^{-1}$ and $\delta_2 := \xi_1^{-1}$. \hfill $\Box$

**Proposition 3.2.** Assume that $P = (p_{ij})_{N \times N}$ is irreducible. Then we have

(3.1) \(0 < Q_r^{s_r}(\mu) \leq Q_r^{s_r}(\mu) < \infty\).

**Proof.** By Lemma 5.1 for $j \geq 1$, we see \(\phi_{j,r}^{s_r/(s_r+r)} = \sum_{\sigma \in \Lambda_{j,r}} (p_\sigma c_\sigma^{s_r/(s_r+r)}) \approx 1\), implying \(\eta^j = \phi_{j,r}^{-s_r/(s_r+r)}\).

This, together with (2.3), leads to

$$e_{\phi_{j,r}}^{\mu} = \sum_{\sigma \in \Lambda_{j,r}} p_\sigma c_\sigma^{s_r/(s_r+r)} \approx \phi_{j,r} \cdot \eta^j = \phi_{j,r} \cdot \phi_{j,r}^{-s_r/(s_r+r)} = e_{\phi_{j,r}}^{\mu}.$$ 

It follows that $\phi_{j,r}^{s_r/(s_r+r)} c_{\phi_{j,r}}^{\mu} \approx 1$. Hence, (3.1) follows by Lemma 2.1. \hfill $\Box$

### 3.2. Markov measures with reducible transition matrix.

For every $H \in \text{SC}(G)$, we write

$$H_\infty := \{ \sigma \in \Omega_\infty : \sigma_i \in H \text{ for } i \geq 1 \} , \quad H^* := \bigcup_{k=1}^\infty H^k ;$$

$$H_k(i) := \{ \sigma \in H^k : \sigma_i = i \} , \quad H^*(i) := \bigcup_{k=1}^\infty H_k(i) ;$$

$$H_\infty(i) := \{ \sigma \in H_\infty : \sigma_1 = i \} ; \quad i \in H .$$

We define finite maximal antichains in $H^*$ or $H^*(i)$ in the same way as we did for those in $\Omega^*$. 

**Lemma 3.3** (cf. [12]). We have \(s_r = \max_{H \in \text{SC}(G)} s_r(H)\).
Proof. By the factor theorem [12, pp. 813], we have
\[
0 = \det(I - A_{G,s_r}) = \prod_{H \in \text{SC}(G)} \det(I - A_{H,s_r}).
\]

Since \( A_{G,s_r} \) is non-negative, by the definition of \( s_r \), 1 is an eigenvalue of \( A_{G,s_r} \).

For every \( H \in \text{SC}(G) \), we know that \( A_{H,s_r}(H) \) is irreducible since \( H \) is strongly
connected. Thus, there exists an eigenvalue \( \lambda_H > 0 \) which equals the spectral
radius \( \Psi_H(s_r) \) of \( A_{H,s_r} \), namely, \( \lambda_H \) is the Perron root of \( A_{H,s_r} \).

It follows that \( 1 = \max_{H \in \text{SC}(G)} \lambda_H \) and there exists an \( H \in \text{SC}(G) \) such that
\( \lambda_H = 1 \). If \( \lambda_H = 1 \), then \( s_r = s_r(H) \); otherwise, we have \( \lambda_H = \Psi_H(s_r) < 1 \). By [12 Theorem 2], \( \Psi_H(s) \)
is strictly decreasing with respect to \( s \). Thus, \( s_r > s_r(H) \). Combining the above
analysis, the lemma follows. \( \square \)

Remark. The factor theorem is an easy consequence of the following facts. For
the non-negative reducible matrix \( A_{G,s} \), there exists some permutation matrix \( T \)
(which is necessarily orthogonal) such that \( T A_{G,s} T^{-1} \) is a block upper triangular
matrix, where the blocks on the diagonal are either irreducible matrices or 1 \( \times \) 1
null matrices; thus the set of the eigenvalues of \( A_{G,s} \) are exactly the union of those
of all the blocks on the diagonal.

Further, the non-zero blocks on the diagonal are the images of the maximal irre-
ducible sub-matrices of \( A_{G,s} \) corresponding to the strongly connected compo-
nents under symmetric permutations which are orthogonal and preserve eigenvalues.
Therefore, if we denote by \( m \) (possibly zero) the number of 1 \( \times \) 1 null matrices
on the diagonal of \( T A_{G,s} T^{-1} \), then
\[
\det(\lambda I - A_{G,s}) = \lambda^m \prod_{H \in \text{SC}(G)} \det(\lambda I - A_{H,s}).
\]

Lemma 3.4. There exists constant \( M_1, M_2 > 0 \) such that
\[
M_1 \leq \sum_{\sigma \in \Gamma} (p_{\sigma c_\sigma^T})^{s_r(H)}/(s_r(H)+r) \leq M_2.
\]

for every finite maximal antichain in \( H^* \) or \( H^*(i), i \in H \), with \( H \in \text{SC}(G) \).

Proof. By Lemma 3.1 for each \( H \), one can choose a constant \( M_H \) such that the second
inequality in (3.3) holds with \( M_H \) in place of \( M_2 \). Set \( M_2 := \max\{M_H : H \in \text{SC}(G)\} \). Then for every \( H \in \text{SC}(G) \) and every finite maximal antichain \( \Gamma \) in
\( H^* \), we have
\[
\sum_{\sigma \in \Gamma} (p_{\sigma c_\sigma^T})^{s_r(H)}/(s_r(H)+r) \leq M_2
\]

Since every finite maximal antichain \( \Gamma(i) \) in \( H^*(i) \) is contained in a finite maximal
antichain in \( H^* \), we conclude that the second inequality of (3.3) also holds for such
a \( \Gamma(i) \). Next we need to choose \( M_1 \) such that the first inequality also holds for such
a \( \Gamma(i) \).

Let \( H \in \text{SC}(G) \) and \( i \in H \). We denote by \( \zeta \) the unique normalized positive right
eigenvector of \( A_{H,s_r}(H) \) with respect to the Perron-Frobenius eigenvalue 1. We
consider the measure \( \nu_{1,i} \) on \( H_\infty(i) \) satisfying
\[
\nu_{1,i}([\sigma]) := (p_{\sigma c_\sigma^T})^{s_r(H)}/(s_r(H)+r) \zeta_{|\sigma|}, \sigma \in H^*(i);
\]
\[
\nu_{1,i}(H_\infty(i)) = \sum_{j=1}^N (p_{ij} c_i^T)^{s_r(H)}/(s_r(H)+r) \zeta_j = \zeta_i,
\]
Thus, for large $k$, we have
\[ \xi^{-1} \zeta_i \leq \sum_{\sigma \in \Gamma(i)} (p_\sigma c^\sigma_\sigma)^{s_r(H)/(s_r(H)+r)} \leq \xi^{-1} \zeta_i. \]

Let $m_H := \min\{\xi^{-1} \zeta_i : i \in H\}$ and $M_1 := \min\{m_H : H \in SC(G)\}$. Then (3.3) holds for every finite maximal antichain $\Gamma(i)$ in $H^*(i)$ and finite maximal antichain in $H^*$. This completes the proof of the lemma.

**Lemma 3.5.** Let $H \in SC(G)$ and $i \in H$. Then $Q^{\mu_r(H)}(\mu(|J_i)|) > 0$.

**Proof.** For each $k \geq 1$, we write
\[ \Lambda_{k,r}(i) := \{ \sigma \in \Omega^* : \sigma_1 = i, p_\sigma c^\sigma_\sigma \geq \eta^k > p_\sigma c^\sigma_0 \}, \]
\[ \Lambda_{k,r}(i,H^c) := \{ \sigma \in \Lambda_{k,r}(i) : \sigma_h \notin H \text{ for some } h \}. \]

For $\sigma \in \Lambda_{k,r}(i,H^c)$, we write $h(\sigma) := \min\{h : \sigma_h \notin H\}$. Then, since $H$ is a strongly connected component, we deduce that $\sigma_l \notin H$ for all $l \geq h(\sigma)$. Note that $A_{h,s_r(H)}$ is irreducible and that $\Lambda_{k,r}(i) \setminus \Lambda_{k,r}(i,H^c)$ is a maximal finite antichain in $H^*(i)$. Hence, by Lemma 3.4 we have
\[ \sum_{\sigma \in \Lambda_{k,r}(i) \setminus \Lambda_{k,r}(i,H^c)} (p_\sigma c^\sigma_\sigma)^{s_r(H)/(s_r(H)+r)} \geq M_1 > 0. \]

Let $\phi_{k,r}(i)$ denote the cardinality of $\Lambda_{k,r}(i)$. As we did for (2.3), one can show
\[ e^{r}_{\phi_{k,r}(i),r}(\mu(|J_i|)) \gg \sum_{\sigma \in \Lambda_{k,r}(i)} p_\sigma c^\sigma_\sigma. \]

By H"older’s inequality for exponent less than one, we have
\[ e^{r}_{\phi_{k,r}(i),r}(\mu(|J_i|)) \gg \left( \sum_{\sigma \in \Lambda_{k,r}(i)} (p_\sigma c^\sigma_\sigma)^{s_r(H)/(s_r(H)+r)} \right)^{s_r(H)/(s_r(H)+r)} \cdot \phi_{k,r}(i)^{-r/s_r(H)} \geq \left( \sum_{\sigma \in \Lambda_{k,r}(i) \setminus \Lambda_{k,r}(i,H^c)} (p_\sigma c^\sigma_\sigma)^{s_r(H)/(s_r(H)+r)} \right)^{s_r(H)/(s_r(H)+r)} \cdot \phi_{k,r}(i)^{-r/s_r(H)} \gg \phi_{k,r}(i)^{-r/s_r(H)}. \]

This and Lemma 2.1 yields that $Q^{\mu_r(H)}(\mu(|J_i|)) > 0$. The lemma follows. \qed

**Proposition 3.6.** For any Markov-type measure $\mu$ as defined in (1.6) we have $D_r(\mu) = s_r$ and $Q^{\mu_r}(\mu) > 0$.

**Proof.** Let $s > s_r$. By [12, Theorem 2], we have $\Psi_G(s) < \Psi_G(s_r) = 1$. Let $u = (u_i)_{i=1}^N$ be the column vector with $u_i = 1$ for all $1 \leq i \leq N$. We choose $t$ such that $\Psi_G(s) < t < 1$. By Gelfand’s formula, we have
\[ \lim_{k \to \infty} \| A_{G,s}^k u \|_1^{-1/k} = \Psi_G(s) < t < 1. \]

Thus, for large $k$, we have that $\| A_{G,s}^k u \|_1 < t^k$. It follows that
\[ \sum_{\sigma \in \Omega_k} (p_\sigma r^\sigma_\sigma)^{s/(s+r)} = \| A_{G,s}^{k-1} u \|_1 < t^{k-1}. \]

Let $A_{j,r}$ be as defined in (2.1). It is immediate to see that, there exist two constants $A_1, A_2 > 0$ such that
\[ A_1 j \leq l_{ij} \leq l_{2j} < A_2 j. \]
Thus, by (2.1), for all large \( j \), we deduce
\[
\sum_{\sigma \in \Lambda_{j,r}} (p_{\sigma} c_{\sigma})^{\varphi/(s+r)} \leq \sum_{k=l_{j}}^{l_{j}} \sum_{\sigma \in \Omega_{k}} (p_{\sigma} c_{\sigma})^{\varphi/(s+r)} \leq l^{\mu_{l_{j}-1}} \frac{1}{1-t} < 1 \quad \text{for large } j.
\]

Thus, by (2.1), for all large \( j \), we have \( \phi_{j,r} \leq \eta^{-s(j+1)/(s+r)} \). Also, by (2.3)
\[
e_{\phi_{j,r}}(\mu) \leq \sum_{\sigma \in \Lambda_{j,r}} (p_{\sigma} c_{\sigma})^{\varphi/(s+r)} (p_{\sigma} c_{\sigma})^{\varphi/(s+r)} \leq \chi^{p_{\sigma}/(s+r)} \leq \chi^{1/(s+r)}.
\]

Thus, by Lemma 2.1 we have, \( \overline{D}_{r}(\mu) \leq s \). Since \( s > s_{r} \) was chosen arbitrarily, we obtain that \( \overline{D}_{r}(\mu) \leq s_{r} \).

Let \( H \in \mathcal{M} \). Then we have \( s_{r}(H) = s_{r} \). We take an arbitrary vertex \( i_{0} \in H \) and consider the conditional probability measure \( \mu_{i_{0}} := \mu(\cdot|J_{i_{0}}) \). By Lemma 3.5 we have, \( \overline{Q}_{r}^{s_{r}}(\mu_{i_{0}}) > 0 \). Hence,
\[
\overline{Q}_{r}^{s_{r}}(\mu) \geq \mu(J_{i_{0}}) \overline{Q}_{r}^{s_{r}}(\mu_{i_{0}}) \geq \chi \overline{Q}_{r}^{s_{r}}(\mu_{i_{0}}) > 0.
\]

In particular, by [3, Proposition 11.3], we have, \( \overline{D}_{r}(\mu) \geq s_{r} \). Combining this and the first part of the proof, we conclude that \( D_{r}(\mu) \) exists and equals \( s_{r} \). This completes the proof of the theorem. \( \square \)

Define the set \( F := G \setminus \bigcup_{H \in \mathcal{M}} H \) which is possibly empty. Whenever \( F \neq \emptyset \), there corresponds a sub-matrix \( A_{F,s_{r}} \) of \( A_{G,s_{r}} \). We write
\[
F_{0} := \{ \emptyset \}, \quad F_{k} := \{ \sigma \in \Omega_{k} : \sigma_{h} \in F, 1 \leq h \leq k \}, \quad F^{*} := \bigcup_{k=0}^{\infty} F_{k}.
\]

**Lemma 3.7.** There exists a constant \( t \in (0,1) \) such that for \( n \in \mathbb{N} \) large
\[
\sum_{\sigma \in F_{n}} (p_{\sigma} c_{\sigma})^{s_{r}/(s+r)} \ll t^{n}.
\]

**Proof.** Let \( s_{r}(F) \) denote the unique number with \( \Psi_{F}(s_{r}(F)) = 1 \). Then by Lemma 3.3 and the definition of \( F \), we deduce
\[
s_{r}(F) = \max\{ s_{r}(H) : SC(G) \ni H \subset F \} < s_{r}.
\]

According to [12, Theorem 3], \( \Psi_{F}(s) \) is strictly decreasing with respect to \( s \). Thus, \( \Psi_{F}(s_{r}) < 1 \) and we may choose some \( s > 0 \) such that \( \Psi_{F}(s_{r}) < t < 1 \). Following the proof of Proposition 3.6 one can see that, there exists a constant \( k_{0} \in \mathbb{N} \) such that for all \( k \geq k_{0} \) we have
\[
\sum_{\sigma \in F_{k}} (p_{\sigma} c_{\sigma})^{s_{r}/(s+r)} \leq t^{k-1}, \quad k \geq k_{0}.
\]

From this the lemma follows. \( \square \)

**Proposition 3.8.** If \( \mathcal{M} \) consists of pairwise incomparable elements then \( \overline{Q}_{r}^{s_{r}}(\mu) < \infty \).

**Proof.** First note that in this situation we have that any \( \omega \in \Omega^{*} \) has the form \( \omega = \nu' \ast \tau \ast \nu'' \) where \( \nu', \nu'' \in F^{*} \) and \( \tau \in H^{*} \) for some \( H \in \mathcal{M} \). Further, for
Assume that there are two elements \( \nu', \nu'' \in F^* \), we have that

\[
H^*_{j,r}(\nu', \nu'') := \{ \tau \in H^* : \nu' * \tau * \nu'' \in \Lambda_{j,r} \} \subset \bigcup_{k=1}^{M_3} \Gamma_k^{H,j}(\nu', \nu''),
\]

where \( \Gamma_k^{H,j}(\nu', \nu'') \) is some antichain for each \( k = 1, \ldots, M_3 \). With this notation and using Lemmata 5.3 and 5.5 and the definition of \( t \) and \( M_2 \) therein we estimate

\[
\sum_{\omega \in \Lambda_{j,r}} (p_\omega c_{\rho}^{\prime})(s_r/(s_r+r))
= \sum_{H \in M} \sum_{\nu', \nu'' \in F^*} \sum_{\tau \in H^*_{j,r}(\nu', \nu'')} \left( p_{\nu'*\tau_1} c_{\nu'*\tau_1}^{\prime} p_{\nu''} c_{\nu''}^{\prime} c_{\nu'*\tau_1}^{\prime} \right)^{s_r/(s_r+r)}
\leq \sum_{H \in M} \sum_{\nu', \nu'' \in F^*} \left( \sum_{\rho \in \Omega_{2} \cup \{ \theta \}} (p_{\rho} c_{\rho}^{\prime})^{s_r/(s_r+r)} \right)^2 \sum_{k=1}^{M_3} \sum_{\nu', \nu''} \left( p_{\tau} c_{\rho}^{\prime} \right)^{s_r/(s_r+r)}
\leq \text{card}(M) \left( \sum_{\nu', \nu''} (p_{\nu''} c_{\nu''}^{\prime})^{s_r/(s_r+r)} \right)^2 (N^2 + 1)^2 M_3 M_2 \ll \left( \sum_{\nu} \eta \right)^2 \ll 1.
\]

Combining this with Lemma 5.11 for \( j \geq 1 \), we get

\[
\phi_{j,r} \eta^{s_r/(s_r+r)} = \sum_{\sigma \in \Lambda_{j,r}} (p_\sigma c_{\rho}^{\prime})^{s_r/(s_r+r)} \ll 1
\]

and hence \( \eta \ll \phi_{j,r}^{-s_r/(s_r+r)} \). This, together with (2.3), leads to

\[
e_{\phi_{j,r}}(\mu) = \sum_{\sigma \in \Lambda_{j,r}} \sigma_{\rho} c_{\rho}^{\prime} \approx \phi_{j,r} \cdot \eta \ll \phi_{j,r} \cdot \phi_{j,r}^{-s_r/(s_r+r)} = \phi_{j,r}^{-s_r/(s_r+r)}.
\]

It follows that \( \phi_{j,r}^{s_r/(s_r+r)} e_{\phi_{j,r}}(\mu) \ll 1 \). Hence, the assertion follows by Lemma 2.23.

\[\square\]

In order to estimate the quantization error from below, we need an auxiliary measure of Mauldin-Williams-type. One may see [12, p. 823] for more details.

Assume that, there are two elements \( H_1, H_2 \in M \) such that \( H_1 \prec H_2 \), i.e., there exists a path \( \gamma = (i_1, \ldots, i_h) \) satisfying

\[
i_1 \in H_1, \quad i_2, \ldots, i_{h-1} \notin H_1 \cup H_2, \quad i_h \in H_2.
\]

Let \( v = (v_i)_{i=0}^{m_{2}+1} \) be the positive normalized right eigenvector of \( A_{H_2, s_r} \) with respect to the Perron-Frobenius eigenvector 1. Set

\[
E_q := \{ \tau \in H^q : \tau_q = 1 \}, \quad \tilde{E} := \{ i_2, \ldots, i_{h-1} \},
\]

\[
F_{q} := \{ \tau \ast \tilde{\gamma} * \rho : \tau \in E_q, \rho \in H_{2}^N, \rho_1 = i_h \}.
\]

For every \( \tau \in E_q \) and \( \rho \in H_{2}^*(i_h) \), we define

\[
\nu_q([\tau \ast \tilde{\gamma} * \rho]) = (p_{\tau} c_{\rho}^{\prime} p_{\gamma} c_{\rho}^{\prime} c_{\rho}^{\prime} \rho v_{\rho,\rho}^{\prime} c_{\rho}^{\prime} \rho v_{\rho,\rho}^{\prime})^{s_r/(s_r+r)} v_{\rho,\rho}^{\prime},
\]

where \( [\tau \ast \tilde{\gamma} * \rho] := \{ \tau \ast \tilde{\gamma} * \omega : \omega \in (H_{2})_{\infty}, \omega|_{1} = \rho \} \). By (3.8), we have

\[
\sum_{\tau \in H_2} \nu_q([\tau \ast \tilde{\gamma} * \rho]) = \sum_{\rho \in H_2} (p_{\tau} c_{\rho}^{\prime} p_{\gamma} c_{\rho}^{\prime} c_{\rho}^{\prime} \rho v_{\rho,\rho}^{\prime} c_{\rho}^{\prime} \rho v_{\rho,\rho}^{\prime})^{s_r/(s_r+r)} v_{\rho,\rho}^{\prime} = (p_{\tau} c_{\rho}^{\prime} p_{\gamma} c_{\rho}^{\prime} c_{\rho}^{\prime} \rho v_{\rho,\rho}^{\prime} c_{\rho}^{\prime} \rho v_{\rho,\rho}^{\prime})^{s_r/(s_r+r)} v_{\rho,\rho}^{\prime}.
\]
Thus, by Kolmogorov consistency theorem, we get a unique measure $\nu_q$ on $F_q$.

**Proposition 3.9.** Assume that there are two comparable elements in $\mathcal{M}$. Then we have, $\bigotimes Q^r (\mu) = \infty$.

**Proof.** Assume that, there are two elements $H_1, H_2 \in \mathcal{M}$ such that $H_1 \prec H_2$. Without loss of generality, as in [12, 18], we assume that $H_1 = \{1, \ldots, m_1\}, i_1 = 1$ and $H_2 = \{m_0 + 1, \ldots, m_0 + m_2\}, i_2 = m_0 + 1$; and (3.7) holds. As above, let \( \tilde{\gamma} := (i_2, \ldots, i_{h-1}) \). For all large $k$, there exist some words $\sigma \in \Lambda_{k, r}$ taking the form $\sigma = \tau * \tilde{\gamma} * \rho$ with $\tau \in E_q$ and $\rho \in H_2, \rho_1 = m_0 + 1$. For every $q \leq l_{1k} - 1 - h$ and $\tau \in E_q$, we have, $p_{\tau} c_{\tau}^{r} c_{\gamma}^{r} \geq \eta^{-k}, \text{otherwise}$, $\min_{\tau \in \Lambda_{k, r}} |\sigma|$ would be strictly less than $l_{1k}$, contradicting the definition of $l_{1k}$. This implies that $\Lambda_{k, r}$ includes some subset $F_q$ of $F_q^*$ such that $\{\rho : \tau * \tilde{\gamma} * \rho \in F_q^*\}$ forms a finite maximal antichain in $H_2^*(m_2 + 1) := \{\sigma \in H_2^* : \sigma_1 = m_1 + 1\}$. For each $\sigma = \tau * \tilde{\gamma} * \rho \in F_q^*$, we have

\[
(p_{\tau} c_{\tau}^{r} c_{\gamma}^{r} \rho)_{v/\nu} = \left( p_{\tau} c_{\gamma}^{r} p_{\gamma} c_{\gamma}^{r} \rho \right)_{v/\nu} \geq \left( p_{\tau} c_{\gamma}^{r} c_{\gamma}^{r} \rho \right)_{v/\nu} \geq \eta^{2k} \theta_{v/\nu} \geq \eta^{2k} \theta_{v/\nu} \mu^{-1}.
\]

Using this facts, we deduce

\[
\sum_{\sigma \in \Lambda_{k, r}} (p_{\sigma} c_{\sigma}^{r})_{v/\nu} \geq \sum_{q=1}^{l_{1k} - 1 - h} (p_{\sigma} c_{\sigma}^{r})_{v/\nu} \geq \eta^{2k} \theta_{v/\nu} \sum_{q=1}^{l_{1k} - 1 - h} \nu_q (F_q) = \theta_{v/\nu} \sum_{q=1}^{l_{1k} - 1 - h} \nu_q (F_q) = Q_k,
\]

where $v = (v_i)_{i=1}^{m_2}$ is the positive eigenvector in the definition of the measures $\nu_q$ and $\theta := \max_{1 \leq i \leq m_2} v_i$. Note that $s_r (H_1) = \nu_q$. Let $w = (w_i)$ be a positive left eigenvector $A_{H_1, \nu}^*$ with respect to the Perron-Frobenius eigenvector 1. Then we have $w A_{H_1, \nu}^* w = w$ for all $h \geq 1$. Let $A_{H_1, \nu}^* = (c_{ij})_{m_1 \times m_1}$. We have

\[
\sum_{\tau \in E_q} (p_{\tau} c_{\tau}^{r})_{v/\nu} = \sum_{i=1}^{m_1} c_{ij} \geq \theta_{v/\nu} w_1.
\]

This implies that $Q_k \to \infty$. Thus, by (2.3) and Hölder’s inequality, we have

\[
c_{\tau, \nu}^{r} (\mu) \geq D \left( \sum_{\sigma \in \Lambda_{k, r}} (p_{\sigma} c_{\sigma}^{r})_{v/\nu} \right)^{(s_r + 1)/s_r} \phi_{k, r}^{s_r - r/s_r} \geq Q_k^{(s_r + 1)/s_r} \phi_{k, r}^{s_r - r/s_r}.
\]

Hence, by Lemma 2.1 it follows that $\bigotimes Q^r (\mu) = \infty$. The proposition follows. \( \Box \)

**Proof of Theorem 1.1.** For the proof of Theorem 1.1 we just have to combine Proposition 3.8 and 3.9. \( \Box \)

Next, we construct two examples illustrating Theorem 1.1.

**Example 3.10.** Let $Q = (q_{ij})_{2 \times 2}, T = (t_{ij})_{3 \times 3}$ be two positive matrices, i.e., $q_{ij} > 0, 1 \leq i, j \leq 2$ and $t_{ij} > 0, 1 \leq i, j \leq 3$. We define

\[
P = \begin{pmatrix} Q_{2 \times 2} & 0 \\ 0 & T_{3 \times 3} \end{pmatrix}.
\]
Then \( P \) is a reducible matrix. Let \( \mu \) be the Markov-type measure associated with \( P \). Let \( H_1 := \{1,2\} \) and \( H_2 := \{3,4\} \). Clearly, \( \mathcal{M} = \{H_1, H_2\} \) and \( H_1, H_2 \) are incomparable. Thus, by Theorem 3.11, \( 0 < Q_r^\mu(\mu) \leq Q_r^\mu(\mu) < \infty \).

**Example 3.11.** Let the transition matrix be given by

\[
\begin{pmatrix}
1/4 & 1/4 & 1/2 & 0 \\
1/4 & 1/4 & 1/2 & 0 \\
0 & 0 & 1/2 & 1/2 \\
0 & 0 & 1/2 & 1/2 \\
\end{pmatrix}
\]

Fix \( r > 0 \) we set \( s := r/(2r + 1) \) implying \( 2 \left(4^{-1}4^{-r}\right)^{(s+r)} = 1 \). Let \( c_{ij} = 1/8 \) for \( 1 \leq i \leq 2, \ 1 \leq j \leq 3, \) and \( c_{33} = c_{43} = c_{44} = 2^{-1/s} \). With \( H_1 := \{1,2\} \) and \( H_2 := \{3,4\} \) we have

\[
\begin{align*}
A_{H_1,s} = A_{H_2,s} &= \begin{pmatrix}
(2^{-(2+2r)})^{s/(s+r)} & (2^{-(2+2r)})^{s/(s+r)} \\
(2^{-(2+2r)})^{s/(s+r)} & (2^{-(2+2r)})^{s/(s+r)} \\
\end{pmatrix} \\
\end{align*}
\]

Clearly \( A_{H_1,s}, A_{H_2,s} \) are irreducible row-stochastic matrices. Hence, \( s_r = s = s_r(H_i), \ i = 1,2. \) Since \( H_1 \prec H_2 \), by Theorem 3.11 we conclude that \( Q_{<r}^\mu(\mu) = \infty \).

**References**

[1] M. F. Barnsley, J. H. Elton, and D. P. Hardin, Recurrent iterated function systems. *Constr. Approx.* 5 (1989), 3–31.

[2] G. A. Edgar and R. D. Mauldin, Multifractal decompositions of digraph recursive fractals. *Proc. London Math. Soc.* (3) 65 (1992), 604–628.

[3] S. Graf and H. Luschgy, Foundations of quantization for probability distributions. *Lecture Notes in Math.* vol. 1730, Springer, 2000.

[4] S. Graf and H. Luschgy, Asymptotics of the quantization error for self-similar probabilities. *Real. Anal. Exchange* 26 (2001), 795–810.

[5] S. Graf and H. Luschgy, The quantization dimension of self-similar probabilities. *Math. Nachr.* 241(1) (2002), 103–109.

[6] S. Graf and H. Luschgy, Quantization for probability measures with respect to the geometric mean error. *Math. Proc. Camb. Phil. Soc.* 136 (2004), 687–717.

[7] S. Graf, H. Luschgy and G. Pagès, The local quantization behavior of absolutely continuous probabilities. *Ann. Probab.* 40 (2012), 1795–1828.

[8] R. Gray and D. Neuhoff, Quantization. *IEEE Trans. Inform. Theory* 44 (1998), 2325–2383.

[9] J. E. Hutchinson, Fractals and self-similarity. *Indiana Univ. Math. J.* 30 (1981), 713–747.

[10] W. Kreitmeier, Optimal quantization for dyadic homogeneous Cantor distributions. *Math. Nachr.* 281 (2008), 1307–1327.

[11] L. J. Lindsay, *Quantization dimension for probability distributions*. Ph.D. Thesis, University of North Texas, 2001.

[12] R. D. Mauldin and S. C. Williams, Hausdorff dimension Graph-directed constructions. *Trans. AMS. Math.* 309 (1988), 811–829.

[13] G. Pagès, A space quantization method for numerical integration. *J. Comput. Appl. Math.* 89 (1998), 1–38.

[14] G. Pagès, J. Printems, Functional quantization for numerics with an application to option pricing. *Monte Carlo Methods Appl.* 11 (2005), 407–446.

[15] K. Pötzelberger, The quantization dimension of distributions. *Math. Proc. Camb. Phil. Soc.* 131 (2001) 507–519.

[16] P. L. Zador, Development and evaluation of procedures for quantizing multivariate distributions. Ph.D. Thesis, Stanford University, 1964.
[17] S. Zhu, Quantization dimension of probability measures supported on Cantor-like sets. *J. Math. Anal. Appl.* **338** (2008), 742–750.
[18] S. Zhu, On the upper and lower quantization coefficient for probability measures on multiscale Moran sets. *Chaos, Solitons & Fractals* **45** (2012), 1437–1443.

Marc Kesseböhmer, Fachbereich 3 – Mathematik und Informatik, Universität Bremen, Bibliothekstr. 1, Bremen 28359, Germany
*E-mail address*: mhk@math.uni-bremen.de

Sanguo Zhu, School of Mathematics and Physics, Jiangsu University of Technology, Changzhou 213001, China
*E-mail address*: sgzhu@jsut.edu.cn (Corresponding author)