Scaffoldings of totally positive matrices and line insertion

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ABSTRACT

Given a totally positive matrix, can one insert a line (row or column) between two given lines while maintaining total positivity? This question was first posed and solved by Johnson and Smith who gave an algorithm that results in one possible line insertion. In this work, we revisit this problem. First, we show that every totally positive matrix can be associated with a certain vertex-weighted graph in such a way that the entries of the matrix are equal to sums over certain paths in this graph. We call this graph a scaffolding of the matrix. We then use this to give a complete characterization of all possible line insertions as the strongly positive solutions to a given homogeneous system of linear equations.

1. Introduction

The study of totally positive (TP) and totally nonnegative (TN) matrices, i.e. real matrices where each minor is, respectively, positive and nonnegative, has a long history. See the monographs [1] and [2] for extensive classical theory and applications. In the mid 2000’s, deep connections between the theory of TN matrices and the prime ideal theory of the so-called algebra of quantum matrices were discovered (see [3] for a survey). Therefore one might hope to bring tools developed in one field to bear on the other. Indeed Cauchon’s Deleting Derivations Algorithm of [4] developed in the context of quantum algebra was used by Adm et al. [5] to study the rank of TN matrices. In the other direction, the work of Postnikov [6] inspired the “path model” of quantum matrices in [7].

This paper continues the theme by transferring the path model of quantum matrices back to the study of TP matrices to what we will call the scaffolding of a matrix. We demonstrate its utility by addressing the problem of inserting a new row or column into a given TP matrix while maintaining total positivity. This is the Line Insertion Problem. Johnson and Smith [8] first proposed and solved this problem by finding an algorithm that always results in a valid line insertion between any given two rows or columns of a given TP matrix. In contrast, the present work will lead to Theorem 4.6 that gives a characterization of the set of possible line insertions as the set of strongly positive solutions to a homogeneous system of linear equations.
This paper is structured as follows. Section 3 introduces the concept of a scaffolding of a TP matrix $X$ whereby we interpret each entry of $X$ as a sum over paths in a certain grid-like directed graph. We show that every TP matrix has a unique scaffolding and discuss the consequences of the Lindström-Gessel-Viennot Lemma as applied to scaffoldings. We then use these ideas in Section 4 to address the Line Insertion Problem and derive the aforementioned system of equations. Finally, we show these equations do indeed have a strongly positive solution, thereby solving the Line Insertion Problem in a new way.

2. Preliminaries

For a positive integer $k$, let $[k] = \{1, 2, \ldots, k\}$. Unless otherwise noted, the rows and columns of an $m \times n$ matrix are indexed in the usual way by $[m]$ and $[n]$. We may also index rows and columns using other sets but we trust the reader will easily extend the material below to these situations.

If $A$ is an $m \times n$ matrix and $I$ and $J$ are subsets of the rows and columns of $A$ respectively, then $A[I, J]$ denotes the submatrix of $A$ formed by $I$ and $J$. When $|I| = |J|$, the determinant $\det A[I, J]$ is a minor of $A$.

A matrix is positive if all entries are positive. A matrix is totally positive (TP) if every minor of that matrix is positive. Of course every TP matrix is positive but not conversely.

Let $A$ be an $m \times n$ matrix. The notation

$$A[[i \cdots], [j \cdots]]$$

will be shorthand for the submatrix

$$A[[i, i + 1, \ldots, i + k], [j, j + 1, \ldots, j + k]]$$

where $k = \min(m - i, n - j)$. In other words it is the largest contiguous submatrix with $(i, j)$ in the top-left corner. Similarly,

$$A[[\cdots i], [\cdots j]]$$

is the largest contiguous submatrix with $(i, j)$ in the bottom-right corner. We extend this notation to, for $i_0 < i$ and $j_0 < j$, setting

$$A[[i_0, i \cdots], [j_0, j \cdots]] = A[[i_0, i, i + 1, \ldots, i + k], [j_0, j, j + 1, \ldots, j + k]]$$

where $k = \min(m - i, n - j)$. Similarly define

$$A[[\cdots i, i_0], [\cdots j, j_0]]$$

for $i_0 > i$ and $j_0 > j$.

It is easily seen that the transpose of a TP matrix is again TP. If $H_k$ denotes the $k \times k$ matrix with a 1 in the entries $\{(1, k), (2, k - 1), \ldots\}$ and 0 everywhere else, then we also have the following which is immediate from Theorem 1.4.1 in [2].

**Proposition 2.1:** If $A$ is an $m \times n$ TP matrix, then $A^\tau = H_n A^T H_m$ is an $n \times m$ TP matrix.

The matrix $A^\tau$ may be thought of as the reflection of $A$ across the ‘anti-diagonal’ $\{(1, n), (2, n - 1), \ldots\}$ and so we will call the map $A \mapsto A^\tau$ the anti-transpose.
Note that because of the transpose (or anti-transpose) map, it suffices to explain how to insert a row in order to solve the Line Insertion Problem.

It is convenient to use the language of directed graphs to visualize some of the concepts in this paper. We need nothing beyond the most elementary definitions here; however, when we talk about paths in a directed graph, we always will mean directed paths. We also write $P \in G$ to mean the path $P$ is contained in the directed graph $G$. Additional special notation will be given in Notation 3.2.

Finally, an $n$-tuple $v \in \mathbb{R}^n$ is strongly positive if each component of $v$ is positive.

3. Scaffoldings of totally positive matrices

3.1. From $\Gamma$-Scaffoldings to totally positive matrices

The $\Gamma$-scaffolding of an $m \times n$ TP matrix $X$ is defined using a certain vertex-weighted directed graph. Roughly speaking, this graph is an $m \times n$ grid with extra vertices attached to the right and below, one for each row and column, and with horizontal edges oriented “right to left” and vertical edges ‘top to bottom.’ Figures 1 and 2 are examples of such a graph.

**Figure 1.** The graph $G_{2,3}^\Gamma(T)$ of Example 3.4. Internal vertices are labelled by their weights.

**Figure 2.** The graph $G_{3,3}^\Gamma(T)$. 
Definition 3.1: Let $T = [t_{ij}]$ be an $m \times n$ positive matrix. Define the vertex-weighted directed graph $G^\Gamma_{m,n}(T)$ as follows. The vertex set is the disjoint union $([m] \times [n]) \cup [m] \cup [n]$. The vertices $[m]$ are the row vertices and the vertices $[n]$ are the column vertices.

Next, for each $i \in [m]$, there is a directed edge from row vertex $i$ to the vertex $(i, n)$ and a directed edge from $(i, j)$ to $(i, j - 1)$ for each $j \in [n] \setminus \{1\}$. These directed edges will be called horizontal edges. Also, for each $j \in [n]$ and $i \in [m] \setminus \{1\}$ there is a directed edge from $(i, j)$ to $(i + 1, j)$ and a directed edge from $(m, j)$ to column vertex $j$. These directed edges will be called vertical edges.

Finally, equip $G^\Gamma_{m,n}(T)$ with the function $w : [m] \times [n] \rightarrow \mathbb{R}$ defined by $w(i, j) = t_{ij}$.

In drawings of $G^\Gamma_{m,n}(T)$, we will label the internal vertices by their weight.

Suppose we have the graph $G^\Gamma_{m,n}(T)$ and let $P$ be a path in this graph that starts at row vertex $i$ and ends at column vertex $j$. Notice that $P$ is uniquely determined by the sequence of vertices at which it turns: either proceeding from a horizontal edge to a vertical edge ($\Gamma$-turns), or from a vertical edge to a horizontal edge ($L$-turns). In fact if

$((i,j_1), (i_2,j_1), \ldots, (i_{\ell}, j))$

is this sequence of turns, then it alternates between $\Gamma$-turns and $J$-turns, starting and ending with a $\Gamma$-turn. Paths are crucial in this work so we here set some notation.

Notation 3.2: With respect to the graph $G^\Gamma_{m,n}(T)$,

1. A path $P$ starting at vertex $v$ and ending at vertex $w$ will be denoted $P : v \rightarrow w$.
2. Paths that start at an internal vertex $(a, b)$ and end at a column vertex will always be assumed to begin with a vertical edge.
3. Let $\ell \in [n]$ be a column index. If the path $P$ starts at row vertex $i$, ends at column vertex $j$ and has its first turn at a vertex $(i, j_1)$ for some $j_1 \leq \ell$, then write $P_{\leq \ell} : i \rightarrow j$.
4. Let $P : i \rightarrow j$ be a path with associated sequence of turns

$((i,j_1), (i_2,j_1), \ldots, (i_{\ell}, j))$.

Define the weight of $P$ to be

$w(P) = t_{j_1}^{-1}t_{i_2j_1}^{-1}t_{i_3j_2}^{-1} \cdots t_{i_{\ell}j_{\ell-1}}^{-1}t_{ij}$.

5. There exists a unique path $P : i \rightarrow j$ with exactly one $\Gamma$-turn, and weight $t_{ij}$. We call this path the primary path from $i$ to $j$.

The following definition is crucial to this work.

Definition 3.3: Let $T = [t_{ij}]$ be an $m \times n$ positive matrix. Define the $m \times n$ matrix $X(T) = [x_{ij}]$ by

$x_{ij} = \sum_{P : i \rightarrow j, P \in G^\Gamma_{m,n}(T)} w(P)$.

We say that $T$ is the $\Gamma$-scaffolding of $X(T)$.
Example 3.4: Let

\[ T = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}. \]

Then \( G_{2,3}^\Gamma (T) \) is illustrated in Figure 1 and

\[ X(T) = \begin{bmatrix} 1 + 3 \left( \frac{1}{2} \right)^{-1} & 1 + 1(1)^{-1} & 1 + 1(1)^{-1} \frac{1}{2} \\ 1 & \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 1 \\ 1 & \frac{1}{2} & 1 \end{bmatrix}. \]

Notice that \( X(T) \) is TP.

Example 3.5: Let \( T = [t_{ij}] \) be a positive \( 3 \times 3 \) matrix. Then \( G_{3,3}^\Gamma (T) \) is illustrated in Figure 2 and \( X(T) = [x_{ij}] \) is the \( 3 \times 3 \) matrix with

\[
\begin{align*}
x_{11} &= t_{11} + t_{12}t_{21}^{-1}t_{21} + t_{12}t_{22}^{-1}t_{31} + t_{13}t_{23}^{-1}t_{21} + t_{13}t_{23}^{-1}t_{22}t_{32}^{-1}t_{31} + t_{13}t_{23}^{-1}t_{33}t_{31}, \\
x_{12} &= t_{12} + t_{13}t_{23}^{-1}t_{22} + t_{13}t_{23}^{-1}t_{32}, \\
x_{21} &= t_{21} + t_{22}t_{32}^{-1}t_{31} + t_{23}t_{33}^{-1}t_{31}, \\
x_{22} &= t_{22} + t_{23}t_{33}^{-1}t_{32}, \\
x_{ij} &= t_{ij}, \quad \text{if } i = 3 \text{ or } j = 3.
\end{align*}
\]

3.2. Minors and \( \Gamma \)-scaffolding

When \( T \) is a positive matrix, it turns out that \( X(T) \) is totally positive. To see why, we need a relationship between minors of \( X(T) \) and the \( \Gamma \)-scaffolding \( T \). This is provided by the well-known Lindström–Gessel–Viennot Lemma.

To explain, fix a \( G_{m,n}^\Gamma (T) \), and let \( I = \{i_1 < i_2 < \cdots < i_k\} \) be a subset of the row vertices and \( J = \{j_1 < \cdots < j_k\} \) a subset of column vertices with \( |I| = |J| \). A path system from \( I \) to \( J \) in \( G_{m,n}^\Gamma (T) \) is a sequence \( \mathcal{P} = (P_1, P_2, \ldots, P_k) \) of paths where \( P_\ell : i_\ell \rightarrow j_\ell \) for each \( \ell \in [k] \). We say that \( \mathcal{P} \) is vertex-disjoint if its paths are mutually vertex-disjoint. Finally, the weight of the path system is the product of the weights of its paths, i.e.

\[ w(\mathcal{P}) = w(P_1)w(P_2)\cdots w(P_k). \]

Note that since \( T \) is positive, so is \( w(\mathcal{P}) \).

We may now state the following special case of the Lindström–Gessel–Viennot Lemma (see [9]).

Lemma 3.6: Let \( T \) be a positive \( m \times n \) matrix and set \( X = X(T) \). If \( I \subseteq [m] \) and \( J \subseteq [n] \) are such that \( |I| = |J| \), then

\[ \det X[I, J] = \sum_{\mathcal{P}} w(\mathcal{P}), \]

where the sum is over all vertex-disjoint path systems from \( I \) to \( J \) in \( G_{m,n}^\Gamma (T) \).
It should be noted that the Lindström-Gessel-Viennot Lemma is usually stated for edge-weighted directed graphs. The graph \( G_{m,n}^\Gamma(T) \) can be modified to this setting by defining the edge-weight of all vertical edges to be 1, the weight of the edge from the row vertex \( i \) to \( (i,n) \) to be \( t_{in} \), and the weight of the edge from \((i,j)\) to \((i,j-1)\) to be \( t_{ij-1}t_{ij}^{-1} \). It is then easy to verify that the edge-weight of a path, being the product of the edge weights, equals the (vertex-) weight of a path as defined in Notation 3.2.

**Example 3.7:** Referring back to Example 3.5 where \( T = [t_{ij}] \) is a \( 3 \times 3 \) positive matrix and \( X = X(T) \), one has, for example, that

\[
\det X([1,2],[1,2]) = t_{11} \cdot t_{22} + t_{11} \cdot t_{23} t_{33}^{-1} t_{32} + t_{12} t_{22}^{-1} t_{21} \cdot t_{23} t_{33}^{-1} t_{32}
\]

\[
\det X([1,2],[1,3]) = t_{11} \cdot t_{23} + t_{12} t_{22}^{-1} t_{21} \cdot t_{23} + t_{12} t_{32}^{-1} t_{31} \cdot t_{23},
\]

and

\[
\det(X) = t_{11} t_{22} t_{33}.
\]

Note that there always exists at least one vertex-disjoint path system from \( I \) to \( J \) in \( G_{m,n}^\Gamma(T) \), namely \( \mathcal{P} = (P_1, \ldots, P_k) \) where each \( P_\ell \) is the primary path from \( i_\ell \) to \( j_\ell \). Call this the **primary path system from \( I \) to \( J \)**. The existence of this path system together with Lemma 3.6 has two immediate consequences. The first keeps our earlier promise.

**Corollary 3.8:** If \( T \) is a positive matrix, then \( X(T) \) is totally positive.

The second corollary is related to certain contiguous minors and will be needed for our work in Section 4. First, notice that if

\[
X[I,J] = X[[i \cdots], [j \cdots]]
\]

is a contiguous submatrix of \( X(T) \), then the primary path system from \( I \) to \( J \) in \( G_{m,n}^\Gamma(T) \) is in fact the **unique** path system from \( I \) to \( J \). It follows that

\[
\det X[[i \cdots], [j \cdots]] = t_{ij} t_{i+1,j+1} \cdots t_{i+k,j+k},
\]

where \( k = \min(m-i, n-j) \).

In Section 4 we will encounter sums of the form

\[
\sum_{P:\ell : i \rightarrow j} w(P).
\]

We may use Lemma 3.6 to write this quantity in terms of minors of \( X(T) \). The paths in this sum may be thought of as exactly those paths from row vertex \( i \) to column vertex \( j \) that are ‘blocked’ by (i.e. are disjoint from) the paths in the primary path system from \([i+1 \cdots]\) to \([\ell+1 \cdots]\). See Figure 3. Given this, the next result follows immediately from Lemma 3.6.

**Corollary 3.9:** Let \( T = [t_{ij}] \) be a positive matrix and set \( X = X(T) \). Let \( i \) be a row vertex and \( j, \ell \in [n] \) column vertices with \( j \leq \ell \). Then

\[
\sum_{P:\ell : i \rightarrow j} w(P) = \frac{\det X[[i, i+1 \cdots], [j, \ell+1 \cdots]]}{\det X[[i+1 \cdots], [\ell+1 \cdots]]}.
\]
Figure 3. A path $P_{\leq \ell}: i \to j$ that is “blocked” by the primary path system (dashed) from $\{i + 1 \cdots \}$ to $\{\ell + 1 \cdots \}$.

It may be worth pointing out that setting $\ell = j$ gives the following formula for the entries of the $\Gamma_1$-scaffolding of $X = X(T)$ in terms of minors of $X$:

$$t_{ij} = \frac{\det X[i \cdots, j \cdots]}{\det X[i + 1 \cdots, [j + 1 \cdots]}.$$

### 3.3. From totally positive matrices to $\Gamma_1$-Scaffoldings

Corollary 3.8 begs the question: does every totally positive matrix $X$ have a $\Gamma_1$-scaffolding, i.e. a positive matrix $T$ with $X = X(T)$? The answer is yes.

One may find the $\Gamma_1$-scaffolding of $X$ using the next procedure. To explain, totally order $[m] \times [n]$ using the reverse lexicographic order $\prec$, that is, $(i, j) \prec (k, \ell)$ if either $i > k$, or $i = k$ and $j > \ell$. If $(i, j) \in [m] \times [n]$, then set $(i, j)^+$ to be the next largest element in this order.

**Algorithm 3.10 (Cauchon’s Algorithm [4]):** Let $X$ be an $m \times n$ TP matrix.

1. **Set** $X^{(m, n)} = X$.
2. **Suppose** $X^{(i, j)} = [x_{k\ell}^{(i, j)}]$ has been defined. If $(i, j) \prec (1, 1)$, then set

$$x_{k\ell}^{(i, j)^+} = \begin{cases} x_{k\ell}^{(i, j)} - x_{kj}^{(i, j)} & x_{ij}^{(i, j)} \dfrac{1}{x_{i\ell}^{(i, j)}} & \text{if } k < i \text{ and } \ell < j, \\ x_{k\ell}^{(i, j)} & \text{otherwise.} & \end{cases}$$

3. **Set** $T = X^{(1,1)}$.

**Example 3.11:** For

$$X = X^{(2,3)} = \begin{bmatrix} 8 & 7 & 1 \\ 2 & 1 & \frac{1}{2} \\ 1 & \frac{1}{2} & 1 \end{bmatrix},$$

there are effectively only two steps in Cauchon’s Algorithm. The first step results in

$$X^{(2,2)} = \begin{bmatrix} 8 - 1 & 7 & 1 \\ 2 - 1 & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 1 \\ 1 & \frac{1}{2} & 1 \end{bmatrix}. $$
Then
\[ X^{(2,1)} = \begin{bmatrix} 7 - 3 \left( \frac{1}{2} \right)^{-1} & 1 & 3 & 1 \\ 1 & 1/2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 1/2 & 1 \end{bmatrix}. \]

The remainder of the steps do not change this matrix and so the output \( T = X^{(2,1)} \). Notice this is the same \( T \) that began Example 3.4.

In general, if \( i = 1 \) or \( j = 1 \), then one has \( X^{(i,j)} = X^{(i,j)} \) and so these steps may be skipped. It follows that for \( m \geq 2 \), \( T = X^{(2,1)} \).

Of course, Cauchon’s Algorithm is not a priori sensible since the \( (i,j) \)-entry of \( X^{(i,j)} \) could have ended up as zero. Fortunately this never happens, and in fact, we have the following.

**Theorem 3.12 ([10, Theorem 3.3]):** If \( X \) is a TP matrix, then every step of Cauchon’s Algorithm produces a positive matrix. Moreover, the entries of \( X^{(i,j)} \) in positions \( \{(1, 1) \succ (1, 2) \succ \cdots \succ (i, j)\} \) form a partial TP matrix, i.e. all minors that are completely determined by these coordinates are positive.

It may be helpful to understand Cauchon’s Algorithm as simply a careful reversal of the process of forming \( X(T) = [x_{ij}] \) from \( T \). Indeed all paths from \( k \) to \( \ell \) other than the primary path contain at least one \( J \)-turn, and therefore a final \( J \)-turn. Thus we can decompose each sum \( x_{k\ell} \) as

\[ x_{k\ell} = t_{k\ell} + \sum_{(i,j) \in [m] \times [n]} \left( \sum_{P} w(P) \right), \]

where each interior sum is over those paths from row vertex \( k \) to column vertex \( \ell \) whose last \( L \)-turn occurs at vertex \( (i,j) \). (Many of these interior sums may be 0.) Then a careful analysis reveals that the \((i,j)\)-step of Cauchon’s Algorithm is deleting from each \( x_{k\ell} \) the set of paths in \( G_{m,n}^{\Gamma} (T) \) from row vertex \( k \) to column vertex \( \ell \) whose last \( J \)-turn occurs at \( (i,j) \).

It follows that the intermediate matrices \( X^{(i,j)} = [x^{(i,j)}_{k\ell}] \) in Cauchon’s Algorithm can be formed from the final output \( T = [t_{ij}] \) by \( x^{(i,j)}_{k\ell} = \sum_{P: k \to \ell} w(P) \) where the sum is over all paths whose \( L \)-turns occur only at vertices greater than or equal to \((i,j)\) in the reverse lexicographic order. Hence we can state the following.

**Corollary 3.13:** If \( X \) is a TP matrix and Cauchon’s Algorithm applied to \( X \) results in the positive matrix \( T \), then \( T \) is the \( \Gamma \)-scaffolding of \( X \).

### 3.4. L-scaffoldings.

Recall that if \( X \) is a TP matrix, then so is its anti-transpose \( X^\tau \). Thus \( X^\tau \) has a \( \Gamma \)-scaffolding, say \( S \) which may be found by Cauchon’s Algorithm. The matrix \( T = S^\tau \) will be called the \( L \)-scaffolding of \( X \).

We may easily modify the concepts of the previous sections to essentially avoid the intermediate use of the anti-transpose. One defines, for example, the graph \( G_{m,n}^{\Gamma} (T) \) and a “ \( L \)-version” of Cauchon’s Algorithm which runs from \((1, 1)\) to \((m, n)\) lexicographically.
Figure 4. The graph $G_{2,3}(T)$ of Example 3.13.

Example 3.14: Once again, consider the TP matrix

$$X = \begin{bmatrix} 8 & 7 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Running the ‘L-version’ of Cauchon’s Algorithm produces

$$T = \begin{bmatrix} 8 & 7 & 1 \\ 2 & 1 & 16 \\ 1 & 1 & 6 \end{bmatrix}$$

The graph $G_{m,n}(T)$ is in Figure 4. Notice, for example, that

$$\sum_{P: 2 \to 3} w(P) = \frac{6}{7} + \frac{1}{16} \left( \frac{7}{2} \right)^{-1} 1 + 1(8)^{-1}1 = 1 = x_{23}.$$ 

4. Line insertion in totally positive matrices

4.1. Bordering

In this section, we reduce the TP line insertion problem to that of finding a strongly positive solution to a certain homogeneous system of linear equations. The key steps in our reduction use the idea of bordering a TP matrix, i.e. adding a row or column to the outside while retaining total positivity. That one may do this at all is well-known and easy to show. The novelty here is that we use scaffolding to characterize all possible borderings.

Let $X$ be a TP matrix and suppose we wish to append a new row above $X$. Do this as follows:

1. Find the $\Gamma$-scaffolding $T$ of $X$.
2. Append above $T$ a new row with only positive entries to form $T'$. Give this new row the index 0.
3. Then the matrix $X(T')$ is TP and contains $X$ in rows 1, 2, \ldots, $m$. 

This method works since in $G_{m+1,n}(T')$, no path from a row vertex $i > 0$ to column vertex $j$ turns in row 0. On the other hand, every such path corresponds to a path in $G_{m,n}(T)$. Hence if $X' = X(T')$, then $X$ is the submatrix of $X'$ formed by rows 1, 2, ..., $m$. Conversely, suppose $X'$ is an $(m + 1) \times n$ TP matrix with rows indexed by 0, 1, ..., $n$ such that rows 1 to $m$ form $X$. When applying Cauchon’s Algorithm to $X'$, the steps involving only entries in rows 1 to $m$ are identical to the application of Cauchon’s Algorithm to $X$. Hence the $\Gamma$-scaffolding $T'$ of $X'$ contains the $\Gamma$-scaffolding $T$ of $X$ in rows 1 to $m$ together with a positive row 0.

Next, to add a column to the left of $X$, we need only append a strongly positive column to the left of $T'$. On the other hand, to append a row beneath $X$ or to the right of $X$, we proceed similarly but using the $L$-scaffolding of $X$ instead.

We now carefully analyse the output of this bordering technique in the case that we are adding a row above $X$. Suppose $T'$ has been formed by appending $[r_1 \ r_2 \ \ldots \ r_n]$ above the $\Gamma$-scaffolding $T = [t_{ij}]$ of $X$ to form row 0. Let $x_{0j}$ be the $j$th entry in the 0th row of $X(T')$. By definition, $x_{0j}$ is the sum over all paths from row vertex 0 to column vertex $j$ in $G_{m+1,n}(T')$. Each path begins with a $\Gamma$-turn at $(0, \ell)$ for some $j \leq \ell \leq n$ and (recalling Notation 3.2) we conclude

$$x_{0j} = \sum_{\ell=j}^{n} \sum_{P: (0, \ell) \to j} w(P)r_{\ell}.$$  \hfill (1)

The coefficient of $r_\ell$ in Equation (1) may be written using minors of $X$. The key is to notice that if $P: (0, \ell) \to j$ is a path in $G_{m+1,1}(T')$, then $t_{1\ell}w(P)$ is the weight of a path $Q: 1 \to j$. Moreover, $Q$ contains no turns in any columns from $\ell + 1$ to $n$. See Figure 5. Conversely, any path $Q: 1 \to j$ with no turns in columns $\ell + 1$ to $n$ arises in this way.

Therefore,

$$\sum_{P: (0, \ell) \to j} w(P) = t_{1\ell}^{-1} \left( \sum_{P: (0, \ell) \to j} t_{1\ell}w(P) \right) = t_{1\ell}^{-1} \left( \sum_{Q\leq\ell: 1 \to j} w(Q) \right).$$

Applying Corollary 3.6 gives us the following.

**Theorem 4.1:** Let $X$ be an $m \times n$ TP matrix with $\Gamma$-scaffolding $T$. Suppose $X'$ is an $(m + 1) \times n$ TP matrix obtained from $X$ by adding the new row $[x_{01} \ x_{02} \ \ldots \ x_{0n}]$ above the first.

Figure 5. Example of a $P: (0, \ell) \to j$ (solid) and the corresponding $Q: 1 \to j$ (dashed) in $G_{m+1,n}(T')$ where $w(Q) = t_{1\ell}w(P)$. 


Then there exists a strongly positive \( r = [ r_1 \cdots r_n ] \) such that \( T' = \begin{bmatrix} T \end{bmatrix} \) is the \( \Gamma \)-scaffolding of \( X' \) and for all \( j \in [n] \),

\[
x_{0j} = \sum_{\ell=j}^{n} \sum_{P:(0,\ell) \rightarrow (j,\ell) \in G_{m+1,n}(T')} w(P) r_{\ell}
= \sum_{\ell=j}^{n} \frac{\det X([12\ldots],[j,\ell+1\ldots])}{\det X([12\ldots],[\ell,\ell+1\ldots])} r_{\ell}.
\tag{2}
\]

Conversely, if we take positive real numbers \( r_1, r_2, \ldots, r_n \) and define \( x_{0j} \) as in Equation (2), then the matrix \( X' \) obtained from \( X \) by adding

\[
\begin{bmatrix}
x_{01} & x_{02} & \cdots & x_{0n}
\end{bmatrix}
\]

above the first row is totally positive.

**Example 4.2:** Let \( T = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \) so that \( X(T) = \begin{bmatrix} 4 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \). If we add the row \( [ r_1 \ r_2 \ r_3 ] = [1 \ 2 \ 2] \) above \( T \) to get \( T' = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \), then one may count path weights in \( G_{3,3}^{\Gamma}(T') \) to obtain

\[
X(T') = \begin{bmatrix} 15 & 6 & 2 \\ 4 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}.
\]

Notice that

\[
x_{01} = \frac{\det X([1,2],[1,2])}{\det X([1,2],[1,2])} r_1 + \frac{\det X([1,2],[1,3])}{\det X([1,2],[2,3])} r_2 + \frac{\det X([1,1],[1])}{\det X([1],[3])} r_3
= 1 + \left( \frac{3}{1} \right) 2 + \left( \frac{4}{1} \right) 2
= 15,
\]

which agrees with the \( j = 1 \) case of Equation (2).

Similar results can be obtained for the addition of a row or column to the other sides of a TP matrix. For later use, we record this result for the addition of a row below \( X \).

**Theorem 4.3:** Let \( X \) be an \( m \times n \) TP matrix with \( L \)-scaffolding \( T \). Suppose \( X' \) is an \( (m+1) \times n \) TP matrix obtained from \( X \) by adding \( \begin{bmatrix} X_{m+1,1} & X_{m+1,2} & \cdots & X_{m+1,n} \end{bmatrix} \) beneath the \( m \)th row. Then there exists a strongly positive \( q = [ q_1 \cdots q_n ] \) such that \( T' = \begin{bmatrix} T \\ q \end{bmatrix} \) is the \( L \)-scaffolding.
of \( X' \) and for all \( j \in [n] \),

\[
x_{m+1,j} = \sum_{i=1}^{j} \sum_{P \in \text{GL}_{m+1,n}(T')} w(P)q_i
\]

\[
= \sum_{i=1}^{j} \det X[[\cdots m-1, m], [\cdots i-1, j]] q_i.
\]

Conversely, if we take positive real numbers \( q_1, q_2, \ldots, q_n \) and define \( x_{m+1,j} \) as in Equation (3), then the matrix \( X' \) obtained from \( X \) by adding

\[
\begin{bmatrix}
x_{m+1,1} & x_{m+1,2} & \cdots & x_{m+1,n}
\end{bmatrix}
\]

below the \( m \)th row is totally positive.

4.2. Row insertion

Let \( X \) be an \( m \times n \) TP matrix and suppose we wish to insert a new row between rows \( k \) and \( k+1 \) of \( X \) while maintaining total positivity. We will index this new row by \( k' \). We use the bordering results above to show that the possible inserted rows correspond to strongly positive solutions to a homogeneous system of \( 2n \) linear equations in \( 3n \) unknowns. The result will be Theorem 4.6 but let us derive these equations before stating the theorem.

We begin by finding the first \( n \) equations. Let \( X_1 \) be the submatrix of \( X \) consisting of the first \( k \) rows of \( X \) and let \( X_2 \) be the submatrix of \( X \) consisting of rows \( k+1 \) through \( m \) of \( X \). Obviously \( X_1 \) and \( X_2 \) are themselves totally positive.

The insertion of a new row between rows \( k \) and \( k+1 \) of \( X \) is simultaneously the addition of a row beneath \( X_1 \) and the addition of a row above \( X_2 \). If \( r = [r_1 \cdots r_n] \) is a strongly positive row we add above the \( \Gamma \)-scaffolding of \( X_2 \) and \( q = [q_1 \cdots q_n] \) is a strongly positive row we add below the \( J \)-scaffolding of \( X_1 \), then by Theorems 4.1 and 4.3, it is necessary that for all \( j \in [n] \),

\[
\sum_{\ell=j}^{n} \frac{\det X_2[[k+1 k+2 \cdots], [j \ell + 1 \cdots]]}{\det X_2[[k+1 k+2 \cdots], [\ell \ell + 1 \cdots]]} r_\ell = \sum_{i=1}^{j} \frac{\det X_1[[\cdots k-1 k], [\cdots i-1 j]]}{\det X_1[[\cdots k-1 k], [\cdots i-1 i]]} q_i.
\]

Since all minors in the above equation are equal to the corresponding minor in \( X \), we may write these equations as

\[
\sum_{\ell=j}^{n} \frac{\det X[[k+1 k+2 \cdots], [j \ell + 1 \cdots]]}{\det X[[k+1 k+2 \cdots], [\ell \ell + 1 \cdots]]} r_\ell = \sum_{i=1}^{j} \frac{\det X[[\cdots k-1 k], [\cdots i-1 j]]}{\det X[[\cdots k-1 k], [\cdots i-1 i]]} q_i.
\]
Example 4.4: We will keep a running example and show how to insert a row after the second row in

\[ X = \begin{bmatrix} 6 & 3 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \]

While one may readily write down the Equations (4) using \( X \), but let us briefly follow their derivation described above. First we have

\[ X_1 = \begin{bmatrix} 6 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}. \]

Using Cauchon's Algorithms, we obtain their J-scaffoldings and \( \Gamma \)-scaffoldings respectively and then the graphs \( G_{1,3}^\Gamma(T'_2) \) and \( G_{2,3}^J(T'_1) \). See Figure 6. Now Equations (4) come from requiring that the sum over paths from the row vertices \( 2' \) to column vertices \( j \) are the same for each \( j \). This yields the equations

\[ r_1 + r_2 + r_3 = q_1 \quad (5) \]
\[ r_2 + r_3 = \frac{2}{3} q_1 + q_2 \quad (6) \]
\[ r_3 = \frac{1}{3} q_1 + q_2 + q_3. \quad (7) \]

While a strongly positive solution to Equations (4) is necessary, it is not sufficient. Indeed, if we do take a strongly positive solution and form \( X' \) by inserting the row \( k' \) so determined, we are not guaranteed that minors of \( X' \) involving row \( k' \) and rows in both \( X_1 \) and \( X_2 \) are positive.

The question then is which strongly positive solutions of the Equations (4) guarantee total positivity of \( X' \)?
Consider the application of Cauchon’s Algorithm to $X'$ up through step $(k + 1, 1)$. The resulting matrix $(X')^{(k+1,1)}$ has the block form

$$(X')^{(k+1,1)} = \begin{bmatrix} \hat{X} \\ r \\ T_2 \end{bmatrix}$$

where note that $\hat{X}$ also equals the first $k$ rows of $X^{(k+1,1)}$. Equivalently, $\hat{X}$ is the TP matrix whose $\Gamma$-scaffolding consists of the first $k$ rows of the $\Gamma$-scaffolding of $X$.

Since $X_2$ is TP, we already know that $T_2$ is positive. By Theorem 3.11, it follows that $X'$ is TP if and only if

$$\begin{bmatrix} \hat{X} \\ r \end{bmatrix}$$

is TP.

Now, since $X$ is TP so is $\hat{X}$, again by Theorem 3.11. Therefore by Theorem 4.3,

$$\begin{bmatrix} \hat{X} \\ r \end{bmatrix}$$

is TP if and only if there is a strongly positive $s = [s_1 \cdots s_n]$ with

$$r_j = \sum_{i=1}^{j} \frac{\det\hat{X}[\cdot\cdot\cdot \cdot k-1, k], \cdot\cdot\cdot i-1, j]}{\det\hat{X}[\cdot\cdot\cdot k-1, k], \cdot\cdot\cdot i-1, i]} s_i. \tag{8}$$

**Example 4.5:** We continue Example 4.4. For the second set of equations we first find $\hat{X}$ which the reader may verify to be

$$\hat{X} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$ 

By Equations (8), we seek a strongly positive solution to Equations (5)–(7) such that there are positive $s_1, s_2, s_3$ with

$$r_1 = s_1$$

$$r_2 = s_1 + s_2$$

$$r_3 = s_1 + 2s_2 + s_3.$$ 

For example, one such solution is

$$r_1 = 1, \quad r_2 = 2, \quad r_3 = 6, \quad q_1 = 9, \quad q_2 = 2, \quad q_3 = 1, \quad s_1 = 1, \quad s_2 = 1, \quad s_3 = 3,$$

which yields the TP matrix

$$X' = \begin{bmatrix} 6 & 3 & 1 \\ 3 & 2 & 1 \\ 9 & 8 & 6 \\ 1 & 1 & 1 \end{bmatrix}.$$
Our discussion has led us to the following result.

**Theorem 4.6:** Let $X$ be an $m \times n$ TP matrix. Let $\hat{X}$ be the TP matrix whose $\Gamma$-scaffolding consists of the first $k$ rows of the $\Gamma$-scaffolding of $X$.

Any insertion of a new row between rows $k$ and $k+1$ of $X$ that maintains total positivity corresponds to a strongly positive solution

$$
\begin{bmatrix}
r_1 & \cdots & r_n & q_1 & \cdots & q_n & s_1 & \cdots & s_n
\end{bmatrix}
$$

to the following system of linear equations:

$$
\sum_{\ell=j}^{n} \frac{\det X[[k+1, k+2, \ldots], j \ell + \cdots]}{\det X[[k+1, k+2, \ldots], \ell \ell + \cdots]} r_\ell = \sum_{i=1}^{j} \frac{\det X[[\cdots k-1, k], \cdots i - 1, j]}{\det X[[\cdots k-1, k], \cdots i - 1, i]} q_i
$$

(9)

$$
r_j = \sum_{i=1}^{j} \frac{\det \hat{X}[[\cdots k-1, k], \cdots i - 1, j]}{\det \hat{X}[[\cdots k-1, k], \cdots i - 1, i]} s_i,
$$

(10)

where $j$ runs from 1 to $n$ in both (9) and (10).

If $[x_{k'1} \cdots x_{k'n}]$ is the row inserted, then $x_{k'j}$ equals the common value in the $j$th equation of (9).

That there exist strongly positive solutions to the system of equations in Theorem 4.6 may already be inferred by the main result of [8]. However, in the spirit of independence and in keeping with the theme of the present work, we provide an alternative approach.

**Theorem 4.7:** There always exist strongly positive solutions to the system of equations in the statement of Theorem 4.6.

**Proof:** For any choice of positive $s_1, \ldots, s_n$, each $r_j$ is positive. So we only need to show that there always exists an appropriate choice of positive $s_1, \ldots, s_n$ so that each $q_1, \ldots, q_n$ is positive.

To do this, we show that for each $j$, the expression for $q_j$ as a linear combination of $s_1, \ldots, s_n$ is such that the coefficient of $s_n$ is positive. So for any choice of positive $s_1, \ldots, s_{n-1}$, we then just choose $s_n$ large enough to make $q_j > 0$.

To find the coefficient of $s_n$ in the expression for $q_j$, we set $s_1 = \cdots = s_{n-1} = 0$ and, for convenience, $s_n = x_{k+1,n} > 0$. From Equations (10) we have $r_1 = \cdots = r_{n-1} = 0$ and $r_n = s_n = x_{k+1,n}$.

On the other hand, let $\bar{X} = \begin{bmatrix} X_1 \end{bmatrix}$, where, as above, $X_1$ consists of the first $k$ rows of $X$ and $x_{k+1} = (k+1)$st row of $X$. Let $\bar{T} = [\bar{t}_{ij}]$ be the L -scaffolding of $\bar{X}$. We prove by induction on $j$ that with the above choice of $s_1, \ldots, s_n$, one obtains

$$
q_j = \bar{t}_{k+1,j}.
$$

In other words, in the expression for $q_j$ as a linear combination of $s_1, \ldots, s_n$, the coefficient of $s_n$ is $\frac{\bar{t}_{k+1,j}}{x_{k+1,n}} > 0$, which will complete the proof.
To implement our strategy, we use the first form of the equation in Theorem 4.3 to replace the right sides in Equations (9). We obtain

\[
\frac{x_{k+1,j}}{x_{k+1,n}} r_n = \sum_{i=1}^{j} \sum_{P \in G_{k+1,n}^1(T) : P \rightarrow j} w(P) q_i,
\]

and since \( r_n = x_{k+1,n} \),

\[
x_{k+1,j} = \sum_{i=1}^{j} \sum_{P \in G_{k+1,n}^1(T) : P \rightarrow j} w(P) q_i.
\]

For \( j = 1 \), this reduces to simply \( x_{k+1,1} = q_1 \). Since in the \( J \)-scaffolding of \( X \), one has \( x_{k+1,1} = \tilde{t}_{k+1,1} \), our assertion holds in this case.

Now suppose \( q_i = \tilde{t}_{k+1,i} \) for \( 1 \leq i \leq j - 1 \). Then if \( P : (k+1, i) \rightarrow j \) is a path in \( G_{k+1,n}^1(T) \) for some \( 1 \leq i \leq j - 1 \), then \( w(P) q_i = \tilde{t}_{k+1,i} w(P) \) is the weight of a path \( Q : k+1 \rightarrow j \), and conversely every such path arises in this way. Thus

\[
\sum_{i=1}^{j-1} \sum_{P \in G_{k+1,n}^1(T) : P \rightarrow j} w(P) q_i = \sum_{i=1}^{j} \sum_{P \in G_{k+1,n}^1(T) : P \rightarrow j} \tilde{t}_{k+1,i} w(P)
\]

is precisely the sum of the weights of all paths from \( k+1 \) to \( i \) except the primary path from \( k+1 \) to \( i \). Since \( x_{k+1,i} \) is the sum of the weights of all paths from \( k+1 \) to \( i \), it follows that we must have \( q_j = \tilde{t}_{k+1,j} \). As explained above, this suffices to complete the proof. \( \blacksquare \)

One easily extracts an algorithm from the above proof: First Choose any positive \( s_1, \ldots, s_{n-1} \), then write each \( q_j \) in terms of \( s_1, \ldots, s_n \) using Equations (9) and (10), and finally choose \( s_n \) large enough to ensure \( q_j > 0 \).

We hope the concept of scaffoldings will prove a fruitful method for other TP completion problems. For example, many of the main results of [11] may also be recovered using this approach, and in a sequel paper, we will address the echelon completion problem described in [12].

**Note**

1. We resolve any ambiguity between these labels by explicitly stating the type (row or column) of vertex we mean.

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