STRONG APPROXIMATION FOR A FAMILY OF NORM VARIETIES

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Abstract. We study strong approximation of the equation

\[ N_{L/k}(x) = \prod_{i=1}^{n} p_i(t) \]

where \( L/k \) is a finite extension of number fields and \( p_i(t) \)'s are non-proportional irreducible polynomials over \( k \). We prove this equation satisfies strong approximation with Brauer-Manin obstruction when \( L \) can be imbedded in \( k[t]/(p_i(t)) \) over \( k \) for all \( 1 \leq i \leq n \). Under Schinzel’s hypothesis, we prove that the same result is true without assuming that \( L \) can be imbedded in \( k[t]/(p_i(t)) \) for all \( 1 \leq i \leq n \) when \( L/k \) is cyclic.

1. Introduction

Weak approximation of the following family of norm varieties

\[ N_{L/k}(x) = q(t) \]

(1.1)

has been studied extensively, where \( L/k \) is a finite extension of number fields and \( q(t) \) is a polynomial over \( k \). So far, there are two approaches to study this problem. The first approach is to study weak approximation of certain torsors over equation (1.1), which depends heavily on the explicit description of the torsors and the descent theory developed in [16], by

- geometric methods (\( k \)-rationality, \( [L:k] \leq 3 \) and \( \deg(q(t)) \leq 6 \); see [17], [52], [18]);
- the circle method (all irreducible factors of \( q(t) \) are at most two distinct linear polynomials; see [33], [14], [36], [48]);
- the sieve method (\( k = \mathbb{Q} \) and \( q(t) \) is irreducible with \( \deg(q(t)) \leq 3 \); see [3], [24], [34]);
- additive combinatorics (\( k = \mathbb{Q} \) and \( q(t) \) is a product of linear polynomials; see [4], [5]).

Another approach is the fibration method which can be traced back to Hasse’s proof about the local-global principle of quadratic forms. Along the same spirit of Hasse’s proof, Colliot-Thélène and Sansuc in [15] proved the equation (1.1) satisfies weak approximation by assuming Schinzel’s hypothesis when \( k = \mathbb{Q} \), \( [L:k] = 2 \) and \( q(t) \) is an irreducible polynomial over \( \mathbb{Q} \). Such a conditional result (under Schinzel’s hypothesis) was largely extended in [19] and [56]. In [30], Harpaz, Skorobogatov and Wittenberg explained how to replace Schinzel’s hypothesis with the recent achievement in additive combinatorics developed by Green, Tao and Ziegler to obtain the unconditional results. In [31], Harpaz and Wittenberg further improved the fibration method such that most of the above mentioned unconditional results can be covered. Moreover, they also proposed a conjecture in [31] which implies that (1.1) satisfies weak approximation with Brauer-Manin obstruction in general.

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For strong approximation of equation (1.1), the first non-trivial example was given by Derenthal and Wei in [25] where \([L:k] = 4\) and \(q(t)\) is an irreducible quadratic polynomial which has a root in \(K\). By using the explicit description of the universal torsor of (1.1) in this special case, they proved that this universal torsor satisfies strong approximation in [25]. Under Schinzel’s hypothesis, the related results for integral points and strong approximation of Châtelet surfaces are studied in [27] and [40].

In [13], Colliot-Thélène and Harari developed the fibration method for studying strong approximation. However, their method can not applied to study the equation (1.1). As pointed out in [13, P.175], there are two difficulties for proving strong approximation of the equation (1.1) by fibration method.

D1: non-trivial elements in Brauer groups of special fibres over rational points are infinite.

D2: the fibers over \(q(t)\) may not split.

In this paper, we will overcome the first difficulty (D1) by using the natural action of torus \(\text{Res}_{L/k}^1(\mathbb{G}_m)\). For any given open set of adelic points of (1.1), one can find a finite subgroup of Brauer group of the generic fiber via this action such that the existence of rational points over almost all fibers of rational points can be tested by using the restriction of this finite group to these special fibers (see Proposition 3.4). It is well known that one can overcome the second difficulty (D2) by using Schinzel’s hypothesis. However, one needs to get rid of obstruction from finite extensions from residue fields of fibers over \(p(t)\) to their algebraic closure in their function fields. This new ingredient forces us to restrict the field extension \(L/k\), for example to be cyclic. Over \(\mathbb{P}^1\), Harpaz and Wittenberg in [31] proposed a conjecture to replace Schinzel hypothesis to solve the similar difficulty as (D2). One can also raise an integral version of Harpaz-Wittenberg conjecture to establish strong approximation of equation (1.1) for any finite field extension \(L/k\).

Notation and terminology are standard. Let \(k\) be a number field, \(\Omega_k\) the set of all primes in \(k\) and \(\infty_k\) the set of all Archimedean primes in \(k\). Write \(v < \infty_k\) for \(v \in \Omega_k \setminus \infty_k\). Let \(\mathfrak{o}_k\) be the ring of integers of \(k\) and \(\mathfrak{o}_{k,S}\) the \(S\)-integers of \(k\) for a finite set \(S\) of \(\Omega_k\) containing \(\infty_k\). For each \(v \in \Omega_k\), the completion of \(k\) at \(v\) is denoted by \(k_v\), the completion of \(\mathfrak{o}_k\) at \(v\) by \(\mathfrak{o}_{k,v}\) and the residue field at \(v\) by \(k(v)\) for \(v < \infty_k\). Write \(\mathfrak{o}_{k,v} = k_v\) for \(v \in \infty_k\) and \(k_\infty = \prod_{v \in \infty_k} k_v\). Let \(A_k\) be the adele ring of \(k\) and \(A_k^f\) the finite adele ring of \(k\). For a finite extension \(L/k\) of number fields and a finite set \(S\) of \(\Omega_k\) containing \(\infty_k\), the set of elements in \(L\) which are integral over all primes not lying above \(S\) is denoted by \(\mathfrak{o}_{L,S}\).

A variety \(X\) over \(k\) is defined to be a reduced separated scheme of finite type over \(k\). We denote \(X_{\bar{k}} = X \times_k \bar{k}\) with \(\bar{k}\) a fixed algebraic closure of \(k\). Let

\[
\text{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m), \quad \text{Br}_1(X) = \ker[\text{Br}(X) \to \text{Br}(X_{\bar{k}})]
\]

and

\[
\text{Br}_a(X) = \ker[\text{Br}(k) \to \text{Br}_1(X)], \quad \text{Br}(X) = \ker[\text{Br}(k) \to \text{Br}(X)].
\]

For any subset \(B\) of \(\text{Br}(X)\), we set

\[
X(A_k)^B = \{(x_v)_{v \in \Omega_k} \in X(A_k) : \sum_{v \in \Omega_k} \text{inv}_v(\xi(x_v)) = 0, \quad \forall \xi \in B\}.
\]
We denote by $\pi_0(X(k_\infty))$ and $\pi_0(X(k_v))$ the set of connected components of $X(k_\infty)$ and $X(k_v)$ for $v \in \infty_k$ with discrete topology respectively. Define $X(A_k)_* = \pi_0(X(k_\infty)) \times X(A_k^{\prime})$ and $X(A_k)_f := E \times X(A_k^{\prime}) \subset X(A_k)_*$ and $X(k)_E := X(k) \cap X(A_k)_E \subset X(A_k)_*$ for any subset $E \subset \pi_0(X(k_\infty))$. Since an element in $\text{Br}(X)$ takes the constant value at each connected component of $X(k_\infty)$, the set $X(A_k)_E$ is well-defined. Write $pr_* : X(A_k) \to X(A_k)_*$ the natural projection map. If $G$ is a connected algebraic group over $k$, then $G(k_\infty)^+$ is denoted as the connected component of identity of Lie group $G(k_\infty)$.

Let $\phi : X \to Y$ be a morphism of schemes. For $y \in Y$, we write $X_y$ for the fiber of $\phi$ over $y$.

Definition 1.1. Let $X$ be a variety over a number field $k$, $B$ a subset of $\text{Br}(X)$ and $S_0$ a finite subset of $\Omega_k$. We say $X$ satisfies strong approximation with respect to $B$ off $S_0$ if $X(k)$ is dense in $pr^{S_0}(X(A_k)_B) \neq \emptyset$ where $pr^{S_0} : X(A_k) \to X(A_k^{S_0})$ the natural projection map and $A_k^{S_0}$ is the adele ring of $k$ without $S_0$-components.

The first main result of this paper is the following theorem (see Corollary 3.6).

**Theorem 1.2.** Let $X$ be the smooth locus of the following affine variety

$$\prod_{i=1}^m N_{L_i/k}(x_i) = c \prod_{j=1}^n p_j(t)^{e_j} \quad \text{with} \quad c \in k^\times$$

where $L_i/k$’s are finite extensions of number fields and $p_j(t)$’s are distinct irreducible monic polynomials over $k$ and $e_j$’s are positive integers. If for each $1 \leq j \leq n$ there is $L_i$ such that either $L_i$ can be imbedded in $k[t]/(p_j(t))$ over $k$ or $[L_i : k] | e_j$ with $1 \leq i \leq m$, then $X$ satisfies strong approximation off $\infty_k$ with respect to $\text{Br}_1(X)$.

Under Schinzel’s hypothesis, one can remove the assumption that for each $1 \leq j \leq n$ there is $L_i$ such that either $L_i$ can be imbedded in $k[t]/(p_j(t))$ over $k$ or $[L_i : k] | e_j$ with $1 \leq i \leq m$ in the above theorem if one of extensions $L_i/k$ for $1 \leq i \leq m$ is cyclic (see Corollary 4.6).

**Theorem 1.3.** Let $X$ be the smooth locus of the following affine variety

$$\prod_{i=1}^m N_{L_i/k}(x_i) = c \prod_{j=1}^n p_j(t)^{e_j} \quad \text{with} \quad c \in k^\times$$

where $L_i/k$’s are finite extensions of number fields and $p_j(t)$’s are distinct irreducible monic polynomials over $k$ and $e_j$’s are positive integers. Assume that one of extensions $L_i/k$ for $1 \leq i \leq m$ is cyclic. Suppose the projection of each connected component of $X(k_\infty)$ to $t$-coordinate is unbounded for all real primes $v \in \infty_k$. If Schinzel’s hypothesis holds, then $X$ satisfies strong approximation off $\infty_k$ with respect to $\text{Br}_1(X)$.

It should be pointed out that the case of Theorem 1.3 is even unknown for weak approximation before. The second named author further applies this idea to study weak approximation in [57].

The paper is organized as follows. In §2, we work out the local necessary condition at $\infty_k$ of strong approximation for quasi-affine varieties. Then we prove Theorem 1.2 and Theorem 1.3 in §3 and §4 respectively. In §5, we further apply our method to show certain results in opposite direction of [31, Corollary 9.10].
2. Necessary conditions at $\infty_k$

For a number field $k$, it is well known that $k$ is discrete and closed in $\mathbb{A}_k$ by the product formula. When $X$ is a quasi-affine variety over $k$, then $X(k)$ is discrete and closed in $X(\mathbb{A}_k)$. Therefore $X(k,\infty)$ is not compact if $X$ satisfies strong approximation off $\infty_k$ in the classical sense. In this section, we extend this necessary condition to strong approximation with Brauer-Manin obstruction when $\text{Br}(X)$ is finite.

**Definition 2.1.** Let $X$ be a variety over a number field $k$ and $B$ be a subgroup of $\text{Br}(X)$. A connected component $D$ of $X(k,\infty)$ is called admissible with respect to $B$ if $D \subset \text{pr}_\infty(X(\mathbb{A}_k)^B)$ where $\text{pr}_\infty : X(\mathbb{A}_k) \to X(k,\infty)$ is the projection map.

Since each element $b \in \text{Br}(X)$ takes a constant value over a connected component $D_v$ of $X(k_v)$ for $v \in \infty_k$, one can simply write $b(D_v)$ for this constant value.

Let $B$ be a subgroup of $\text{Br}(X)$. We define $D \sim_B D'$ for two connected components

$$D = \prod_{v \in \infty_k} D_v \quad \text{and} \quad D' = \prod_{v \in \infty_k} D'_v$$

of $X(k,\infty)$ if

$$\sum_{v \in \infty_k} b(D_v) = \sum_{v \in \infty_k} b(D'_v)$$

for all $b \in B$. This $\sim_B$ provides an equivalent relation among the admissible connected components of $X(k,\infty)$ with respect to $B$.

**Lemma 2.2.** Let $B$ be a finite subgroup of $\text{Br}(X)$. Then two admissible connected components $D$ and $D'$ of $X(k,\infty)$ with respect to $B$ satisfy $D \sim_B D'$ if and only if there is an open compact subset $W \neq \emptyset$ of $X(\mathbb{A}_k^f)$ such that

$$D \times W \subset X(\mathbb{A}_k)^B \quad \text{and} \quad D' \times W \subset X(\mathbb{A}_k)^B.$$ 

**Proof.** Suppose that $D$ and $D'$ are the admissible connected components of $X(k,\infty)$ with respect to $B$ satisfying $D \sim_B D'$. There are open compact subsets $W \neq \emptyset$ and $V \neq \emptyset$ in $X(\mathbb{A}_k^f)$ such that

$$D \times W \subset X(\mathbb{A}_k)^B \quad \text{and} \quad D' \times V \subset X(\mathbb{A}_k)^B$$

respectively. Since $B$ is finite, one can further assume that

$$W = \prod_{v < \infty_k} W_v \quad \text{and} \quad V = \prod_{v < \infty_k} V_v$$

such that each element in $B$ takes a single value over $W_v$ and $V_v$ respectively for each $v < \infty_k$. One can simply denote this single value by $b(W_v)$ and $b(V_v)$ respectively for each $v < \infty_k$. Moreover, one has $b(W_v) = b(V_v) = 0$ for almost all $v < \infty_k$.

Since

$$\sum_{v \in \infty_k} b(D_v) = \sum_{v \in \infty_k} b(D'_v) \quad \text{with} \quad D = \prod_{v \in \infty_k} D_v \quad \text{and} \quad D' = \prod_{v \in \infty_k} D'_v$$

...
for all $b \in B$, one concludes that
\[ \sum_{v < \infty_k} b(W_v) = \sum_{v < \infty_k} b(V_v) \]
for all $b \in B$. This implies that $D' \times W \subset X(\mathbb{A}_k)^B$ as desired.

Conversely, one takes an element $(x_v)_{v < \infty_k} \in W$ and obtains $D \sim_B D'$. □

The following proposition gives the necessary condition of strong approximation with Brauer-Manin obstruction for quasi-affine varieties at $\infty_k$ when $\text{Br}(\mathcal{X})$ is finite.

**Proposition 2.3.** Let $X$ be a quasi-affine geometrically integral variety over a number field $k$ with $\dim(X) \geq 1$. Suppose that $B$ is a finite subgroup of $\text{Br}(X)$. If $X$ satisfies strong approximation with respect to $B$ off $\infty_k$, then each equivalent class of admissible connected components of $X(k_\infty)$ with respect to $B$ contains a non-compact one.

**Proof.** If $k$ has a complex prime $v$, then $X(k_v)$ is connected (see [44, Chapter 3, Theorem 3.5]). Since $X$ is quasi-affine, the set $X(k_v)$ is not compact. This implies that every connected component of $X(k_\infty)$ is not compact. Therefore one only needs to consider the case that $k$ is totally real.

Suppose there is an equivalent class $[D]$ of admissible connected components of $X(k_\infty)$ such that every element in this class is compact. Since $D$ is admissible with respect to $B$, there is an open compact subset $W \neq \emptyset$ of $X(\mathbb{A}_k)$ such that $D \times W \subset X(\mathbb{A}_k)^B$. Since $B$ is finite, one can further assume that $W = \prod_{v < \infty_k} W_v$ such that each element in $B$ takes a constant value over $W_v$ for all $v < \infty_k$. Therefore
\[ (X(k_\infty) \times \prod_{v < \infty_k} W_v) \cap X(\mathbb{A}_k)^B = (\bigcup_{C \in [D]} C) \times \prod_{v < \infty_k} W_v. \]

Since the number of connected components of $X(k_\infty)$ is finite by [44, Chapter 3, Theorem 3.6], the set $\bigcup_{C \in [D]} C$ is compact by our assumption. Since $X$ is quasi-affine, one obtains that $X(k)$ is discrete and closed in $X(\mathbb{A}_k)$. This implies that the set
\[ X(k) \cap \left( \bigcup_{C \in [D]} C \times \prod_{v < \infty_k} W_v \right) = \{x_1, \cdots, x_l\} \]
is finite.

Since $X$ is geometrically integral over $k$, there is $v_0 < \infty_k$ such that
\[ V_0 = W_{v_0} \setminus \{x_1, \cdots, x_l\} \neq \emptyset \]
is open and compact in $X(k_{v_0})$ by the Lang-Weil’s estimation (see [45, Theorem 7.7.1]). Consider an open subset
\[ X(k_\infty) \times V_0 \times \prod_{v < \infty_k, v \neq v_0} W_v \]
of $X(\mathbb{A}_k)$. Then
\[ [X(k_\infty) \times V_0 \times \prod_{v < \infty_k, v \neq v_0} W_v] \cap X(\mathbb{A}_k)^B = (\bigcup_{C \in [D]} C) \times V_0 \times \prod_{v < \infty_k, v \neq v_0} W_v \neq \emptyset \]
contains no points in \( X(k) \) any more. This contradicts that \( X \) satisfies strong approximation with respect to \( B \) off \( \infty_k \). \[ \square \]

Example 6.2 in [25] and Example 8.3 in [35] can be explained by Proposition 2.3.

**Example 2.4.** Let \( X \) be a variety over \( \mathbb{Q} \) defined by the equation
\[
x^2 + y^2 = t(t-2)(t-10) \subset \mathbb{A}^3_{\mathbb{Q}}
\]
and the quaternions \( \beta_1 = (t,-1) \) and \( \beta_2 = (t-2,-1) \) are the representative of \( \text{Br}(X) \). Write \( B \) the subgroup of \( \text{Br}(X) \) generated by \( \beta_1 \) and \( \beta_2 \). The set \( X(\mathbb{R}) \) consists of two connected components. One connected component \( D_1 \) is given by \( 0 \leq t \leq 2 \), which is compact. The other one \( D_2 \) is given by \( t \geq 10 \), which is not compact. Since \( \beta_2 \) takes \( \frac{1}{2} \) over \( D_1 \) but takes 0 over \( D_2 \), one gets \( D_1 \not\sim_B D_2 \). The local points
\[
(t_v, x_v, y_v) = \begin{cases} (5, x_5, y_5) & v = 5 \\ (1, 3, 0) & v \neq 5 \end{cases}
\]
evident in [25, Example 6.2] imply that \( D_1 \) is admissible. By Proposition 2.3, one concludes that \( X \) does not satisfy strong approximation with Brauer-Manin obstruction off \( \infty_k \).

**Example 2.5.** Let \( X \) be a Del Pezzo surfaces of degree four defined by the equations
\[
\begin{cases}
x_0(x_0 + x_1) = x_2^2 + (x_2 + x_4)^2 \\
x_0 + x_2)(x_0 + 2x_2) = 2x_1^2 + 3x_3^2
\end{cases}
\]
in \( \mathbb{P}^4 \) and \( U \) be an open subset of \( X \) defined by \( x_4 \neq 0 \). As pointed out in the first step of [35, Example 8.1], the set \( U(\mathbb{R}) \) consists of three connected components \( D_0, D_1 \) and \( D_2 \) and one of them is compact with a rational point. Assume this component is \( D_0 \). Then \( D_0 \) is admissible.

Let \( B \) be the subgroup of \( \text{Br}(U) \) generated by \( \alpha_1 \) and \( \alpha_2 \) in the second step of [35, Example 8.1]. We claim that \( D_0 \not\sim_B D_1 \) and \( D_0 \not\sim_B D_2 \). Indeed, suppose \( D_0 \sim_B D_1 \). Since \( \alpha_1 \) and \( \alpha_2 \) are not constant over \( U(\mathbb{R}) \) by the fourth step in [35, Example 8.1], one obtains that
\[
\alpha_1(D_0) = \alpha_1(D_1) \neq \alpha_1(D_2) \quad \text{and} \quad \alpha_2(D_0) = \alpha_2(D_1) \neq \alpha_2(D_2).
\]
This implies that
\[
(\alpha_1 + \alpha_2)(D_0) = (\alpha_1 + \alpha_2)(D_1) = (\alpha_1 + \alpha_2)(D_2)
\]
which contradicts that \( \alpha_1 + \alpha_2 \) are not constant over \( U(\mathbb{R}) \) in the fifth step in [35, Example 8.1]. Therefore \( U \) does not satisfy strong approximation with Brauer-Manin obstruction off \( \infty \) by the above claim and Proposition 2.3.

To end up this section, we provide a refined version of strong approximation for \( \mathbb{A}^1_k \) which is needed in the next section. There are already several refinements for strong approximation for \( \mathbb{A}^1_k \) implicitly in [8, Proposition 4.6] and [6, Theorem 6.2].

**Definition 2.6.** Let \( v \in \infty_k \) be a real place and \( C_v \) be a subset of \( k_v \). We say
- \( C_v \) is unbounded above if \( \sup\{x \in C_v\} = +\infty \).
- \( C_v \) is unbounded below if \( \inf\{x \in C_v\} = -\infty \).
- \( C_v \) is unbounded if \( C_v \) is either unbounded above or unbounded below.
Proposition 2.7. Let \( C = \prod_{v \in \infty_k} C_v \) be an open connected subset of \( k_{\infty} \). Suppose that there is either a complex \( v \in \infty_k \) with \( C_v = \mathbb{C} \) or a real place \( v \in \infty_k \) such that \( C_v \) is unbounded. If \( W \neq \emptyset \) is an open subset of \( \mathbb{A}^I \), then \( k \cap (C \times W) \neq \emptyset \).

Proof. One only needs to consider the case that \( k \) is totally real and there is \( v_0 \in \infty_k \) such that \( C_{v_0} \) is unbounded. Therefore \( C_{v_0} = (a, +\infty) \) or \(( -\infty, a) \) for \( a \in \mathbb{R} \). Without loss of generality, one can assume that \( W = \prod_{v < \infty_k} W_v \).

When \( k = \mathbb{Q} \), the set \( \mathbb{Q} \cap (C_{v_0} \times W) \neq \emptyset \) by Dirichlet’s prime number theorem with modification on sign if necessary.

Otherwise, there is \( \epsilon \in \mathcal{O}^*_k \) such that
\[
|\epsilon|_{v_0} > 1 \quad \text{and} \quad |\epsilon|_v < 1 \quad \text{for all} \quad v \in \infty_k \setminus \{v_0\}
\]
by [43, 33:8]. Let \( \Sigma \) be a finite subset of \( \Omega_k \) containing \( \infty_k \) such that \( W_v = \mathfrak{o}_{k_v} \) for all \( v \not\in \Sigma \). For each \( v < \infty_k \), one can fix \( \beta_v \in \mathfrak{o}_k \) such that \( \text{ord}_v(\beta_v) > 0 \) and \( \text{ord}_v(\beta_v) = 0 \) for all finite \( w \neq v \) by finiteness of class number of \( \mathfrak{o}_k \). By strong approximation for \( \mathbb{A}^1 \), there is \( a \in k \) such that
\[
a \in k_{v_0} \times \left( \prod_{v \in \infty_k \setminus \{v_0\}} C_v \right) \times W.
\]
Let \( l_v \) be a sufficiently large integer such that \( a + \beta_v^{l_v} \mathfrak{o}_{k_v} \subseteq W_v \) for each \( v \in \Sigma \setminus \infty_k \). Take a sufficiently large positive integer \( N \) such that
\[
b = a + \epsilon^{2N+1} \prod_{v \in \Sigma \setminus \infty_k} \beta_v^{l_v} \in C_v
\]
for all \( v \in \infty_k \setminus \{v_0\} \) and \( b \in (a, +\infty) \) or \(( -\infty, a) \) respectively at \( v_0 \) by replacing \( \epsilon \) with \(-\epsilon \) if necessary. Therefore
\[
b \in k \cap (C \times W)
\]
as desired. \( \square \)

3. Fibration over \( \mathbb{A}^1_k \) with an action of torus

Let \( X \) be a smooth and geometrically integral variety over a number field \( k \). Suppose that \( X \xrightarrow{f} \mathbb{A}^1_k \) is a surjective morphism over \( k \) with geometrically integral generic fiber. There is an open dense subset \( U \) of \( \mathbb{A}^1_k \) over \( k \) such that \( f|_V : V = f^{-1}(U) \rightarrow U \) is smooth with geometrically integral fibers. Write
\[
\mathbb{A}^1_k \setminus U = \{P_1, \cdots, P_n\}
\]
where \( P_1, \cdots, P_n \) are the closed points over \( k \) and \( k_i = k(P_i) \) are the residue fields of \( P_i \) for \( 1 \leq i \leq n \). Let \( D_i = f^{-1}(P_i) \) and \( \{D_{i,j}\}_{j=1}^{g_i} \) be the set of irreducible components of \( D_i \) over \( k_i \) for \( 1 \leq i \leq n \). Then one has the exact sequence with the residue maps
\[
0 \rightarrow \text{Br}(X) \xrightarrow{} \text{Br}(V) \xrightarrow{(\partial_{D_{i,j}})} \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{g_i} H^1(D_{i,j}, \mathbb{Q}/\mathbb{Z})
\]
given by [10, (3.9)]. By the functoriality of residue maps and the Faddeev exact sequence (see [20, §1.1 and §1.2]), one has the following commutative diagram

$$\begin{array}{ccc}
\text{Br}_a(U) & \xrightarrow{\partial_U} & \bigoplus_{i=1}^n H^1(k_i, \mathbb{Q}/\mathbb{Z}) \\
f^* \downarrow & & \downarrow \text{res} \\
\text{Br}_a(V) & \xrightarrow{(\partial_{D_{i,j}})} & \bigoplus_{i=1}^n \bigoplus_{j=1}^{g_i} H^1(D_{i,j}, \mathbb{Q}/\mathbb{Z})
\end{array} \tag{3.1}
$$

Recall that $f$ is split if $D_{i,1}$ is geometrically integral with multiplicity 1 in $D_i$ for all $1 \leq i \leq n$. In this case, one has $\partial_{D_{i,1}}(b) \in H^1(k_i, \mathbb{Q}/\mathbb{Z})$ for any $b \in \text{Br}_1(V)$ with $1 \leq i \leq n$.

**Proposition 3.1.** Assume that $f$ is split. Let $B$ be a finite subgroup of $\text{Br}_1(V)$ and $F_i/k_i$ be a finite abelian extension such that $\partial_{D_{i,1}}(B)|F_i = 0$ for all $1 \leq i \leq n$. Let $B_1$ be a finite subgroup of $\text{Br}_1(V)$ such that the image of $B_1$ in $\text{Br}_a(V)$ contains

$$f^*(\partial^{-1}_U(\bigoplus_{i=1}^n H^1(F_i/k_i, \mathbb{Q}/\mathbb{Z}))). \tag{3.9}$$

If $E = \prod_{v \in \infty_k} E_v$ is a connected component of $X(k_\infty)$ such that $f(E_v)$ is unbounded for some $v_0 \in \infty_k$ when $k$ is totally real, then

$$\bigcup_{c \in U(k)} X_c(A_k^{B+B_1}) \text{ is dense in } X(A_k^{(B+B_1)\cap \text{Br}_1(X)})$$

where $X_c$ is the fiber of $f$ over $c \in U(k)$ and $E_c = X_c(k_\infty) \cap E$.

**Proof.** Write $A_k^1 = \text{Spec}(k[t])$ and $P_i = (p_i(t))$ where $p_i(t)$’s are the fixed irreducible polynomials over $k$ for $1 \leq i \leq n$. Let $X'$ be an open subset of $X$ over $k$ such that the restriction $f|_{X'} : X' \rightarrow A_k^1$ is smooth and $f|_{X'}^{-1}(P_i)$ contains the smooth part $D_{i,1}^{sm}$ of $D_{i,1}$ for $1 \leq i \leq n$.

For any open subset

$$W = E \times \prod_{v \in \infty_k} W_v \subset X(A_k) \text{ with } W^{(B+B_1)\cap \text{Br}_1(X)} \neq \emptyset,$$

there is a finite subset $S$ of $\Omega_k$ containing $\infty_k$ such that the following conditions hold.

(a) The morphism $f : X \rightarrow A_k^1$ and the open immersion $X' \hookrightarrow X$ are extended to their integral models $f : \mathcal{X} \rightarrow A_{k,S}^1$ and $\mathcal{X}' \hookrightarrow \mathcal{X}$ over $\mathfrak{O}_{k,S}$ such that the restriction map $\mathcal{X}' \rightarrow A_{k,S}^1$ is smooth.

(b) The morphism $D_i \rightarrow P_i$ is extended to their integral models $\mathcal{D}_i \rightarrow \mathcal{P}_i$ over $\mathfrak{O}_{k,S}$ such that

$$\mathcal{P}_i = \text{Spec}(\mathfrak{O}_{k,S}[t]/(p_i(t))) = \text{Spec}(\mathfrak{O}_{k,S})$$

is smooth over $\mathfrak{O}_{k,S}$ and $\mathcal{D}_i = f^{-1}(\mathcal{P}_i)$ for $1 \leq i \leq n$. Moreover,

$$A_{k,S}^1 \setminus \mathcal{U} = \{P_1, \cdots, P_n\} \text{ and } \mathcal{X} \setminus \mathcal{V} = \{D_1, \cdots, D_n\}$$

(c) The closed immersion $D_{i,1} \hookrightarrow D_i$ is extended to their models $\mathcal{D}_{i,1} \hookrightarrow \mathcal{D}_i$ over $\mathfrak{O}_{k_i,S}$ such that the smooth points over residue fields $\mathcal{D}_{i,1}^{sm}(k(w)) \neq \emptyset$ for all primes $w$ of $k_i$ with $(w \cap k) \not\in S$ for $1 \leq i \leq n$. All field extensions $F_i/k_i/k$ are unramified outside $S$ for $1 \leq i \leq n$. 

(d) All elements in \((B + B_1) \cap \text{Br}_1(X)\) take the trivial value over \(W_v = \mathcal{X}(\mathfrak{o}_{k_v})\) and all elements in \(B + B_1\) take the trivial value over \(\mathcal{V}(\mathfrak{o}_{k_v})\) for all \(v \notin S\).

If there is a complex prime \(v \in \infty_k\), then \(E_v = X(\mathbb{C})\) by [44, Chapter 3, Theorem 3.5] and \(f(E_v) = \mathbb{C}\). In this case, we define \(W_v = E_v \cap V(k_v)\) for all \(v \in \infty_k\). Otherwise, \(k\) is totally real. Then there is a connected component \(C_{v_0}\) of \(V(k_v)\) such that \(f(E_{v_0} \cap C_{v_0})\) is unbounded. In this case, we define

\[
W_v = \begin{cases} 
E_{v_0} \cap C_{v_0} & v = v_0 \\
E_v \cap V(k_v) & v \in \infty_k \setminus \{v_0\}.
\end{cases}
\]

By Harari’s formal lemma (see [11, Theorem 1.4]), one can enlarge \(S\) such that there are \(x_v \in W_v\) for all \(v \in S\) with

\[
\sum_{v \in S} \xi(x_v) = 0
\]

for all \(\xi \in (B + B_1)\). By shrinking \(W_v\) for \(v \in S \setminus \infty_k\), one can assume that each element in \(B + B_1\) takes a single value over \(W_v\). Since \(f\) is split, the set \(f(\prod_{v \in \Omega_k} W_v)\) contains a non-empty open set in \(A_k\) such that \(f(W_v)\) is unbounded when \(k\) is totally real. By Proposition 2.7, there is \(t_0 \in \mathfrak{o}_{k,S}\) such that \(X_{t_0}(A_k) \cap W \neq \emptyset\).

If \(v \notin S\) with \(\text{ord}_v(p_i(t_0)) > 0\) for some \(1 \leq i \leq n\), then \(p_i(t)\) splits in \(\mathfrak{o}_{k_v}\) by (b) and Hensel’s lemma. This implies that there is a prime \(w\) of \(k_i\) above \(v\) such that \((k_i)_w = k_v\). There is \(y_v \in \mathcal{X}_{t_0}'(\mathfrak{o}_{k_v}) \subset \mathcal{X}(\mathfrak{o}_{k_v})\) such that

\[
y_v \mod v \in \mathcal{D}_{i,1}(k(v)) \subset \mathcal{X}_{t_0}'(k(v))
\]

by Hensel’s lemma and (a) and (c) for \(1 \leq i \leq n\). One can extend these \(y_v\) to

\[
(y_v)_{v \in \Omega_k} \in X_{t_0}(A_k) \cap W
\]

by taking any element in \(W_v \cap X_{t_0}(k_v)\) for \(v \in S\) and any element in \(X_{t_0}(\mathfrak{o}_{k_v})\) for the rest of primes \(v\) of \(k\). Then

\[
\sum_{v \in \Omega_k} b(y_v) = \sum_{v \notin S} b(y_v) = \sum_{i=1}^{n} \sum_{v \notin S, \text{ord}_v(p_i(t_0)) > 0} b(y_v)
\]

by (3.2) for all \(b \in (B + B_1)\).

When \(\text{ord}_v(p_i(t_0)) > 0\) for some \(1 \leq i \leq n\) and \(v \notin S\), then \(\text{ord}_v(p_j(t_0)) = 0\) for all \(j \neq i\) by (b). Therefore there is \(\tau_i \in \text{Gal}(F_i/k_i)\) such that

\[
b(y_v) = \partial_{D_{i,1}}(b)(\tau_i) \quad \text{with} \quad v \notin S \quad \text{and} \quad \text{ord}_v(p_i(t_0)) > 0
\]

by [28, Corollary 2.4.3] and the choice of \(y_v\) for all \(b \in B + B_1\). One concludes that

\[
\sum_{v \in \Omega_k} b(y_v) = \sum_{i=1}^{n} \partial_{D_{i,1}}(b)(\tau_i) \in \mathbb{Q}/\mathbb{Z}
\]

(3.3)

with \((\tau_i)_{i=1}^{n} \in \bigoplus_{i=1}^{n} \text{Gal}(F_i/k_i)\) for all \(b \in (B + B_1)\).

Since

\[
\sum_{v \in \Omega_k} f^*(\xi)(y_v) = \sum_{v \in \Omega_k} \xi(t_0) = 0
\]
for all $\xi \in \partial_U^{-1}(\bigoplus_{i=1}^n H^1(F_i/k_i, \mathbb{Q}/\mathbb{Z}))$ by the functoriality of Brauer-Manin pairing and the reciprocity law, one obtains

$$\chi((\tau_i)_{i=1}^n) = 0$$

for all $\chi \in \bigoplus_{i=1}^n H^1(F_i/k_i, \mathbb{Q}/\mathbb{Z})$ by (3.1) and (3.3). This implies that $(\tau_i)_{i=1}^n$ is the trivial element in $\bigoplus_{i=1}^n \text{Gal}(F_i/k_i)$. Therefore

$$(y_v)_{v \in \Omega_k} \in X_{t_0}(A_k)^{B+\mathcal{B}_1} \cap W$$

by (3.3) as desired. \hfill \Box

Remark 3.2. In the proof of Proposition 3.1, the assumption that $f$ is split is needed only for the fact that the set $f(\prod_{v \in \Omega_k} W_v)$ contains a non-empty open set in $A_k$. When $f$ has a section over $k$, this fact is also true. Instead that $f$ is split, the same result still holds when $f$ has a section over $k$.

Recall that a geometrically integral variety $Y$ over a field $k$ is quasi-trivial by [12, Definition 1.1] if the following two conditions hold:

(i) $\bar{k}[Y]^\times /\bar{k}^\times$ is a permutation Galois module;

(ii) $\text{Pic}(Y_{\bar{k}}) = 0$.

Lemma 3.3. Let $U$ be a quasi-trivial variety and $T$ be a torus over a number field $k$. If $V \to U$ is a torsor under $T$, then the Sansuc morphism $\text{Br}_a(V) \to \text{Br}_a(T)$ by [46, Lemma 6.4] is surjective.

Proof. Since $V \to U$ is a torsor under $T$, one gets an exact sequence

$$1 \to \bar{k}[U]^\times /\bar{k}^\times \to \bar{k}[V]^\times /\bar{k}^\times \to \bar{k}[T]^\times /\bar{k}^\times \to \text{Pic}(U_{\bar{k}}) \to \text{Pic}(V_{\bar{k}}) \to \text{Pic}(T_{\bar{k}})$$

by [46, Proposition 6.10]. Since $\text{Pic}(U_{\bar{k}}) = \text{Pic}(T_{\bar{k}}) = 1$, one obtains $\text{Pic}(V_{\bar{k}}) = 1$ and

$$H^2(k, \bar{k}[U]^\times /\bar{k}^\times) \to H^2(k, \bar{k}[V]^\times /\bar{k}^\times) \to H^2(k, \bar{k}[T]^\times /\bar{k}^\times) \to H^3(k, \bar{k}[U]^\times /\bar{k}^\times)$$

(3.4)

by using Galois cohomology for the above short exact sequence. Since $\bar{k}[U]^\times /\bar{k}^\times$ is a permutation $\text{Gal}(\bar{k}/k)$-module, one obtains $H^3(k, \bar{k}[U]^\times /\bar{k}^\times) = 0$ by Shapiro lemma (see [42, Chapter 1, (1.6.4) Proposition]) and [39, Chapter I, Corollary 4.17]. The Hochschild-Serre spectral sequence implies the canonical isomorphisms

$$H^2(k, \bar{k}[V]^\times /\bar{k}^\times) \cong \text{Br}_a(V) \quad \text{and} \quad H^2(k, \bar{k}[T]^\times /\bar{k}^\times) \cong \text{Br}_a(T).$$

The result follows from (3.4). \hfill \Box

Now we further assume that $X$ admits an action of a torus $T$ over $k$. Namely, there is a morphism $T \times_k X \xrightarrow{\lambda} X$ over $k$ satisfying the properties listed in [41, Definition 0.3].

Proposition 3.4. Let $X$ be a smooth variety with an action of a torus $T$ over a number field $k$. Suppose that $X \xrightarrow{f} Y$ is a morphism over $k$ such that $f^{-1}(Z) \xrightarrow{f} Z$ is a torsor under $T$ where $Z$ is an open dense and quasi-trivial sub-variety of $Y$ over $k$. 

If $E$ is a connected component of $X(k_\infty)$ and $W = \prod_{v < \infty} W_v$ is an open compact subset of $X(A_k^2)$, then there is a finite subgroup $B \subset \text{Br}_a(f^{-1}(Z))$ such that
\[(E \times W) \cap X_c(A_k)^B \neq \emptyset \iff (E \times W) \cap X_c(A_k)^{\text{Br}_a(X_c)} \neq \emptyset\]
for all $c \in Z(k)$.

**Proof.** Let
\[\text{St}(W_v) = \{g \in T(k_v) : g \cdot W_v = W_v\}\]
for $v < \infty_k$. Since $W_v$ is an open compact subset of $X(k_v)$ with $W_v = X(\mathfrak{o}_{k_v})$ for almost all $v$ for an integral model $X$ of $X$, the group $\text{St}(W_v)$ is an open subgroup of $T(k_v)$ such that $\text{St}(W_v) = T(\mathfrak{o}_{k_v})$ for almost all $v$ for an integral model $\mathcal{T}$ of $T$. Then
\[\text{St}(W) = T^+(k_\infty) \times \prod_{v < \infty_k} \text{St}(W_v)\]
is an open subgroup of $T(A_k)$, where $T^+(k_\infty)$ is the connected Lie subgroup $T(k_\infty)$. Let
\[B_1 = \{\xi \in Br_1(T) : \sum_{v \in \Omega_k} \xi(\sigma_v) = 0 \text{ for all } (\sigma_v)_{v \in \Omega_k} \in \text{St}(W)\}.

By [23, Theorem 3.19], one has the following commutative diagram of short exact sequences
\[
\begin{array}{ccccccc}
0 & \longrightarrow & T(A_k)/T(k) & \longrightarrow & \text{Hom}(\text{Br}_a(T), \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \text{III}^1(T) & \longrightarrow & 0 \\
& \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \longrightarrow & T(A_k)/(T(k) \cdot \text{St}(W)) & \longrightarrow & \text{Hom}(B_1/\text{Br}(k), \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \text{III}^1(T) & \longrightarrow & 0
\end{array}
\] (3.5)
and $B_1/\text{Br}(k)$ is finite by [44, Theorem 6.15 and Theorem 8.1].

Let $B$ be a finite subgroup of $\text{Br}_a(f^{-1}(Z))$ such that $\lambda(B) = B_1/\text{Br}(k)$ where $\lambda$ is the Sansuc morphism in Lemma 5.2. If
\[(E \times W) \cap X_c(A_k)^B \neq \emptyset\]
for $c \in Z(k)$, there is $(x_v)_{v \in \Omega_k} \in (E \times W) \cap X_c(A_k)$ which can be viewed
\[(x_v)_{v \in \Omega_k} \in \text{Hom}(\text{Br}_a(X_c), \mathbb{Q}/\mathbb{Z})\]
such that $(x_v)_{v \in \Omega_k}|_B = 0$.

By [46, Lemma 6.8], there is a canonical isomorphism $\lambda_c : \text{Br}_a(X_c) \cong \text{Br}_a(T)$ such that the following diagram commutes
\[
\begin{array}{ccc}
\text{Br}_a(f^{-1}(Z)) & \xrightarrow{i_c^*} & \text{Br}_a(X_c) \\
\lambda \downarrow & & \lambda_c \downarrow \\
\text{Br}_a(T) & & 
\end{array}
\] (3.6)
where $i_c : X_c \hookrightarrow f^{-1}(Z)$ is the closed immersion. Since
\[(x_v)_{v \in \Omega_k} \circ \lambda_c^{-1} \in \text{Hom}(\text{Br}_a(T), \mathbb{Q}/\mathbb{Z})\]
with
\[(x_v)_{v \in \Omega_k} \circ \lambda_c^{-1}(B_1/\text{Br}(k)) = (x_v)_{v \in \Omega_k} \circ \lambda_c^{-1} \circ \lambda(B) = i_c^*(B)((x_v)_{v \in \Omega_k}) = 0\]
by (3.6), there are \( t \in T(k) \) and \((s_v)_{v \in \Omega_k} \in \text{St}(W)\) such that
\[
t \cdot (s_v)_{v \in \Omega_k} = (x_v)_{v \in \Omega_k} \circ \lambda_c^{-1} \in \text{Hom}(\text{Br}_a(T), \mathbb{Q}/\mathbb{Z})
\] (3.7)
by (3.5). Consider
\[
(s_v^{-1} \cdot x_v)_{v \in \Omega_k} \in (E \times W) \cap X_c(A_k).
\]
Then
\[
b((s_v^{-1} \cdot x_v)_{v \in \Omega_k}) = \sum_{v \in \Omega_k} \lambda_c(b)(s_v^{-1}) + \sum_{v \in \Omega_k} b(x_v) = - \sum_{v \in \Omega_k} \lambda_c(b)(t \cdot s_v) + \sum_{v \in \Omega_k} b(x_v) = 0
\]
for all \( b \in \text{Br}_a(X_c) \) by (3.7) and [9, Proposition 2.9]. Namely,
\[
(s_v^{-1} \cdot x_v)_{v \in \Omega_k} \in (E \times W) \cap X_c(A_k)^{\text{Br}_a(X_c)} \neq \emptyset
\]
as desired. \( \square \)

Combining Proposition 3.1 and Proposition 3.4, one obtains the main result of this section.

**Theorem 3.5.** Let \( X \xrightarrow{f} A_k^1 \) be a surjective morphism over a number field \( k \). Assume that \( X \) admits an action of a torus \( T \) over \( k \) such that the generic fiber of \( f \) is a torsor under \( T \). Suppose that each equivalent class of admissible connected components of \( X(k_{\infty}) \) contains a connected component \( E = \prod_{v \in \infty_k} E_v \) such that \( f(E_v) \) is unbounded for some \( v \in \infty_k \) when \( k \) is totally real. If \( f \) is split or has a section over \( k \), then \( X \) satisfies strong approximation off \( \infty_k \) with respect to \( \text{Br}_1(X) \).

**Proof.** For any open subset \( X(k_{\infty}) \times \prod_{v < \infty_k} W_v \) of \( X(A_k) \) with
\[
(X(k_{\infty}) \times \prod_{v < \infty_k} W_v) \cap X(A_k)^{\text{Br}_1(X)} \neq \emptyset,
\]
there is a connected component \( E \) of \( X(k_{\infty}) \) such that
\[
(E \times \prod_{v < \infty_k} W_v) \cap X(A_k)^{\text{Br}_1(X)} \neq \emptyset
\]
and \( f(E) \) is unbounded. Since \( X \) admits an action of a torus \( T \) over \( k \) such that the generic fiber of \( f \) is a torsor under \( T \), there is an open dense subset \( U \) of \( A_k^1 \) over \( k \) with \( V = f^{-1}(U) \) such that \( f|_V : V \to U \) is a torsor under \( T \). By Proposition 3.4, there is a finite subgroup \( B \subset \text{Br}_1(V) \) such that
\[
(E \times \prod_{v < \infty_k} W_v) \cap X_c(A_k)^B \neq \emptyset \iff (E \times \prod_{v < \infty_k} W_v) \cap X_c(A_k)^{\text{Br}_a(X_c)} \neq \emptyset
\]
for all \( c \in U(k) \). Let \( B_1 \) be a finite subgroup of \( \text{Br}_1(V) \) as defined in Proposition 3.1 for \( B \). Then there is \( c_0 \in U(k) \) such that
\[
(E \times \prod_{v < \infty_k} W_v) \cap X_{c_0}(A_k)^{B + B_1} \neq \emptyset
\]
by Proposition 3.1 and Remark 3.2. Therefore
\[ (E \times \prod_{v < \infty_k} W_v) \cap X_{c_0}(A_k)_{Br_1(X)} \neq \emptyset. \]
This implies that \( X_{c_0} \) is a trivial torsor by [46, Corollary 8.7] and
\[ X_{c_0}(k) \cap (E \times W) \neq \emptyset \]
by [23, Theorem 3.19].

**Corollary 3.6.** Let \( X \) be the smooth locus of the following affine variety
\[ \prod_{i=1}^m N_{L_i/k}(x_i) = c \prod_{j=1}^n p_j(t)^{e_j} \quad \text{with} \quad c \in k^\times \]
where \( L_i/k \)'s are finite extensions of number fields and \( p_j(t) \)'s are distinct irreducible monic polynomials over \( k \) and \( e_j \)'s are positive integers. If for each \( 1 \leq j \leq n \) there is \( L_i \) such that either \( L_i \) can be imbedded in \( k[t]/(p_j(t)) \) over \( k \) or \( [L_i : k] \mid e_j \) with \( 1 \leq i \leq m \), then \( X \) satisfies strong approximation off \( \infty_k \) with respect to \( Br_1(X) \).

**Proof.** Consider the following homomorphism of tori
\[ \phi : \prod_{i=1}^m \text{Res}_{L_i/k}(G_m) \rightarrow G_m; \quad (x_1, \ldots, x_m) \mapsto \prod_{i=1}^m N_{L_i/k}(x_i) \]
and \( T = \ker \phi \). Then the surjective morphism
\[ f : X \rightarrow A_k^1; \quad (x,t) \mapsto t \]
with the action of \( T \)
\[ T \times X \rightarrow X; \quad (\alpha, (x,t)) \mapsto (\alpha \cdot x, t). \]
For each \( 1 \leq j \leq n \), there is \( L_i \) such that either \( L_i \) can be imbedded in \( k[t]/(p_j(t)) \) over \( k \) or \( [L_i : k] \mid e_j \) with \( 1 \leq i \leq m \). This implies that each \( p_j(t)^{e_j} \) can be written as a norm from \( L_i \) to \( k \) with the variable \( t \) for \( 1 \leq j \leq n \). Since
\[ X(A_k)_{Br_1(X)} \neq \emptyset \quad \Rightarrow \quad c \in N_{L/k}(L^\times) \]
by [46, Theorem 5.1], one concludes that \( f \) has a section over \( k \).

If there are a real prime \( v \in \infty_k \) and \( L_i \) with \( 1 \leq i \leq m \) such that \( L_i \) has a real prime above \( v \), then \( X(k_v) \) is connected by inspecting the equation of \( X \). Therefore one only needs to consider that \( k \) is totally real and all primes of \( L_i \) above \( \infty_k \) are complex for all \( 1 \leq i \leq m \). In this case, one can list all real roots of \( p_i(t)^{e_i} \) with the odd integers \( e_i \)
\[ -\infty < \alpha_1 < \alpha_2 < \cdots < \alpha_s < \infty. \]
Since the sign of \( \prod_{i=1}^n p_i(t)^{e_i} \) will change when \( t \) crosses such a root, one can concludes that
\[ f(E_v) \cap f(E'_v) = \emptyset \]
where \( E_v \) and \( E'_v \) are two different connected components of \( X(k_v) \) for \( v \in \infty_k \). On the other hand, the morphism \( f \) has a section. This implies that \( X(k_v) \) is connected and \( f(X(k_v)) = k_v \). \qed
Example 3.7. Let $a \in k^\times \setminus (k^\times)^2$. For any positive integer $d$, there is an irreducible polynomial $f(t)$ over $k$ with $\deg(f) = 2d$ such that the affine Châtelet surface 

$$x^2 - ay^2 = f(t)$$

satisfies strong approximation off $\infty_k$.

Proof. Let $L = k(\sqrt{a})$ and $Q^{ab}$ be the maximal abelian extension of $\mathbb{Q}$. Since

$$\text{Gal}(Q^{ab}/\mathbb{Q}) = \prod_{p \text{ prime}} \mathbb{Z}_p^\times,$$

one concludes that $\text{Gal}(k \cdot Q^{ab}/k)$ is an open subgroup of $\text{Gal}(Q^{ab}/\mathbb{Q})$ with a finite index. This implies that there is a Galois extension $K/k$ containing $L$ with $[K : k] = 2d$. Let $K = k(\theta)$ and $f(x)$ be an irreducible monic polynomial of $\theta$ over $k$. Write $X$ a variety defined by the equation

$$x^2 - ay^2 = f(t).$$

Since $\overline{k}[X]^\times = \overline{k}^\times$ by [24, Proposition 2], one has

$$\text{Br}_a(X) = H^1(k, \text{Pic}(X_{\overline{k}}))$$

by [21, Lemma 2.1] and [39, Chapter 1, Corollary 4.21]. Let $U$ be an open subset of $X$ defined by $f(t) \neq 0$. Then $\text{Pic}(U_{\overline{k}}) = 1$. Moreover, one has the short exact sequence

$$1 \to \overline{k}[U]^\times / \overline{k}^\times \xrightarrow{\text{div}} \text{Div}_{X_{\overline{k}}/U_{\overline{k}}}(X_{\overline{k}}) \to \text{Pic}(X_{\overline{k}}) \to 1.$$ 

This implies that $\text{Pic}(X_{\overline{k}})$ is a free abelian group of rank $2d - 1$. By the inflation-restriction sequence, one further has

$$\text{Br}_a(X) = H^1(k, \text{Pic}(X_{\overline{k}})) = H^1(K/k, \text{Pic}(X_K)).$$

Since $\text{Pic}(U_K) = 1$, one has the short exact sequence

$$1 \to K[U]^\times / K^\times \to \text{Div}_{X_K/U_K}(X_K) \to \text{Pic}(X_K) \to 1.$$ 

This implies the exact sequence

$$H^1(K/k, \text{Div}_{X_K/U_K}(X_K)) \to H^1(K/k, \text{Pic}(X_K)) \to H^2(K/k, K[U]^\times / K^\times)$$

by Galois cohomology. Since $\text{Div}_{X_K/U_K}(X_K)$ is a permutation Gal$(K/k)$-module, one has

$$H^1(K/k, \text{Div}_{X_K/U_K}(X_K)) = 0$$

by Shapiro’s lemma (see [42, (1.6.3) Proposition]). Since

$$K[U]^\times / K^\times \cong \mathbb{Z}(x - \sqrt{ay}) \oplus \bigoplus_{\sigma \in \text{Gal}(K/k)} \mathbb{Z}(t - \sigma \theta),$$

one concludes $H^1(K/L, K[U]^\times / K^\times) = 0$. Therefore one obtains the following inflation-restriction sequence

$$1 \to H^2(L/k, (K[U]^\times / K^\times)^{\text{Gal}(K/L)}) \to H^2(K/k, K[U]^\times / K^\times) \to H^2(K/L, K[U]^\times / K^\times)^{\text{Gal}(L/k)}.$$ 

Since

$$H^2(K/L, K[U]^\times / K^\times) = H^2(K/L, \mathbb{Z}(x - \sqrt{ay})) = H^1(K/L, \mathbb{Q}/\mathbb{Z}(x - \sqrt{ay}))$$

by Shapiro’s lemma, one has
\[ H^2(K/L, K[U]^\times/K^\times)^{\text{Gal}(L/k)} = \text{Hom}(\text{Gal}(K/L), \mathbb{Q}/\mathbb{Z}(x - \sqrt{ay}))^{\text{Gal}(L/k)} = 0. \]
Note
\[ (K[U]^\times/K^\times)^{\text{Gal}(K/L)} \cong \mathbb{Z}(x - \sqrt{ay}) \oplus \mathbb{Z} f_1(x) \oplus \mathbb{Z} f_2(x) \]
where
\[ f_1(t) = \prod_{\sigma \in \text{Gal}(K/L)} (t - \sigma \theta) \quad \text{and} \quad f_2(t) = \prod_{\sigma \in \text{Gal}(K/k) \setminus \text{Gal}(K/L)} (t - \sigma \theta) \]
satisfying \( \tau f_1(t) = f_2(t) \) for the non-trivial element \( \tau \in \text{Gal}(L/k) \). Since
\[ \tau(x - \sqrt{ay}) = f_1(t) \cdot f_2(t) \cdot (x - \sqrt{ay})^{-1}, \]
one concludes that
\[ H^2(L/k, (K[U]^\times/K^\times)^{\text{Gal}(K/L)}) = \hat{H}^0(L/k, (K[U]^\times/K^\times)^{\text{Gal}(K/L)}) = 0. \]
Therefore \( \text{Br}_a(X) = 0 \). Since \( f(t) \) is a monic polynomial, each connected component of \( X(k_\infty) \) is not compact. The result follows from Corollary 3.6. \( \square \)

The local-global principle for integral points of Diophantine equations in [55] can be reduced to study strong approximation of
\[ q(x_1, \cdots, x_n) = p(t) \quad (3.8) \]
where \( q(x_1, \cdots, x_n) \) is a non-degenerated quadratic form and \( p(t) \) is a non-zero polynomial over a number field \( k \). Indeed, strong approximation has been established in [22] and [58] for \( n \geq 3 \). The remaining \( n = 2 \) case is an affine Châtelet surface and [55, Theorem 4] can be recovered from the following example when the equation (3.8) is smooth.

**Example 3.8.** Let \( X \) be the smooth variety defined by (3.8) with \( n = 2 \) over \( k \). If \( q(x_1, x_2) = 0 \) has a non-trivial solution over \( k \) and \( p(t) \equiv 0 \mod v \) is soluble for almost all primes \( v \) of \( k \), then \( X \) satisfies strong approximation off \( k_\infty \).

**Proof.** Since \( q(x_1, x_2) = 0 \) has a non-trivial solution over \( k \), one can assume that
\[ q(x_1, x_2) = x_1 x_2. \]
Applying Corollary 3.6 for \( m = 2 \) and \( L_1 = L_2 = k \), one concludes that \( X \) satisfies strong approximation off \( k_\infty \) with respect to \( \text{Br}_1(X) \).

Let \( U \) be an open subset of \( X \) defined by \( x_1 \neq 0 \). Then \( U \cong \mathbb{G}_m \times_k \mathbb{A}^1_k \) over \( k \). This implies that \( \bar{k}[U]^\times/\bar{k}^\times \) is free of rank 1 with a generator \( x_1 \). Since \( p(t) \equiv 0 \mod v \) is soluble for almost all primes \( v \) of \( k \), then \( p(t) \) is not a constant. Therefore \( x_1 \notin \bar{k}[X]^\times \) and \( \bar{k}[X]^\times = \bar{k}^\times \). One concludes that
\[ \text{Br}_a(X) = H^1(k, \text{Pic}(X_\bar{k})) \quad (3.9) \]
by [21, Lemma 2.1] and [39, Chapter 1, Corollary 4.21] and the short exact sequence
\[ 1 \to \bar{k}[U]^\times/\bar{k}^\times \xrightarrow{\text{div}} \text{Div}_{X_\bar{k}\setminus U_\bar{k}}(X_\bar{k}) \to \text{Pic}(X_\bar{k}) \to 1 \quad (3.10) \]
of $\text{Gal}(\bar{k}/k)$-modules. Since $X$ is smooth, one can write

$$p(t) = c \prod_{j=1}^{n} p_j(t) \quad \text{and} \quad K_j = k[t]/(p_j(t))$$

with $c \in k^\times$ and $1 \leq j \leq n$, where $p_j(t)$’s are distinct irreducible monic polynomials over $k$. Then

$$\text{Div}_{X_k \backslash U_k}(X_k) \cong \bigoplus_{j=1}^{n} \text{Ind}_{\text{Gal}(\bar{k}/k)}^{\text{Gal}(\bar{k}/K_j)} \mathbb{Z}$$

as $\text{Gal}(\bar{k}/k)$-modules. Since

$$H^1(k, \text{Div}_{X_k \backslash U_k}(X_k)) = 0 \quad \text{and} \quad H^2(k, \text{Div}_{X_k \backslash U_k}(X_k)) = \bigoplus_{j=1}^{n} H^2(K_j, \mathbb{Z})$$

by Shapiro’s lemma (see [42, (1.6.3) Proposition]) and

$$0 = H^1(k, \text{Div}_{X_k \backslash U_k}(X_k)) \to H^1(k, \text{Pic}(X_k)) \to H^2(k, \bar{k}[U]^{\times}/\bar{k}^{\times}) \to H^2(k, \text{Div}_{X_k \backslash U_k}(X_k))$$

by applying Galois cohomology to (3.10), one obtains that

$$\text{Br}_a(X) \cong \ker(H^1(k, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\text{res}} \bigoplus_{j=1}^{n} H^1(K_j, \mathbb{Q}/\mathbb{Z}))$$

by (3.9) and the identifications

$$H^2(k, \mathbb{Z}) \cong H^1(k, \mathbb{Q}/\mathbb{Z}) \quad \text{and} \quad H^2(K_j, \mathbb{Z}) \cong H^1(K_j, \mathbb{Q}/\mathbb{Z})$$

with $1 \leq j \leq n$.

Suppose $\text{Br}_a(X)$ is not trivial. There is

$$\psi \in \ker[\text{Hom}_{\text{cts}}(\text{Gal}(\bar{k}/k), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\text{res}} \bigoplus_{j=1}^{n} \text{Hom}_{\text{cts}}(\text{Gal}(\bar{k}/K_j), \mathbb{Q}/\mathbb{Z})]$$

such that $\ker(\psi) = \text{Gal}(\bar{k}/L)$ where $L/k$ is a finite abelian extension with $\sigma \in \text{Gal}(L/k)$ and $\psi(\sigma) \neq 0$. By the Chebotarev density theorem, there are infinitely many primes $p$ of $k$ such that $Frob_{L/k}(p) = \sigma$. Since $p(t) \equiv 0 \pmod{v}$ is soluble for almost all primes $v$ of $k$, there is a prime $v_0$ of $k$ such that

$$Frob_{L/k}(v_0) = \sigma \quad \text{and} \quad (K_j)_w = k_{v_0}$$

for some $1 \leq j \leq n$ and some prime $w$ of $K_j$ above $v_0$. This implies that

$$Frob_{L/K_j}(w)|_{L/k} = \sigma \quad \text{and} \quad \psi(\sigma) = 0$$

by (3.11), which contradicts the choice of $\psi$. Therefore $\text{Br}_a(X)$ is trivial and $X$ satisfies strong approximation off $\infty_k$.

It should be pointed out that the Brauer groups of certain affine Châtelet surfaces are computed with some algorithm for Brauer-Manin set in [1].
4. Application of Schinzel hypothesis to non-split fibrations

We keep the same notations as those in the previous section if not explained. More precisely, we assume that $X$ is a smooth and geometrically integral variety over a number field $k$. Let $X \xrightarrow{f} \mathbb{A}^1_k$ be a surjective morphism over $k$ with geometrically integral generic fiber. The significant difference is that we do not assume that $f$ is split in this section. Then the map on the adelic points $X(\mathbb{A}_k) \to \mathbb{A}_k$ induced by $f$ is not necessarily open. This prevents us from applying strong approximation for $\mathbb{A}_k^1$ to find a rational point over $\mathbb{A}_k^1$.

**Example 4.1.** Let $X$ be a smooth and geometrically integral variety defined by $x^2 + y^2 = t$ over $\mathbb{Q}$ and consider the fibration $X \xrightarrow{f} \mathbb{A}^1_\mathbb{Q}$ by sending $(x, y, t) \mapsto t$. For the prime $p \equiv 3 \bmod 4$ and $\text{ord}_p(t) \equiv 1 \bmod 2$, the equation $x^2 + y^2 = t$ has no solution over $\mathbb{Z}_p$. There are infinitely many primes $p$ such that the map $f : X(\mathbb{Z}_p) \to \mathbb{Z}_p$ is not surjective. Therefore the map $X(\mathbb{A}_\mathbb{Q}) \to \mathbb{A}_\mathbb{Q}$ induced by $f$ is not open.

To overcome (D2) in §1, one can apply Schinzel’s hypothesis. The hypothesis $(H_1)$ in [20, pp. 71] is a consequence of Schinzel’s hypothesis which is due to Serre by [20, Prop. 4.1]. One can further refine the hypothesis $(H_1)$ to the following hypothesis with the signs.

**Hypothesis** $(H_s)$. Let $p_1(t), \ldots, p_n(t)$ be irreducible polynomials over a number field $k$ and $\epsilon_v \in \{\pm 1\}$ for each real prime $v \in \infty_k$. Then there is a finite subset $S$ of $\Omega_k$ containing $\infty_k$ satisfying the following property:

For any $\lambda_v \in k_v$ with $v \in S \setminus \infty_k$, there is $\lambda \in \mathfrak{o}_{k,S}$ close to $\lambda_v$ for $v \in S \setminus \infty_k$ with arbitrarily large $\epsilon_v \lambda$ at all real primes $v$ of $k$ such that for each $1 \leq i \leq n$, $p_i(\lambda)$ is a unit of $k_w$ for all primes $w \notin S$ except one prime $w_i$, where $p_i(\lambda)$ is a uniformizing parameter.

**Proposition 4.2.** Schinzel hypothesis over $\mathbb{Q}$ implies $(H_s)$ over a number field $k$.

**Proof.** Fix $a \in k$ such that $\epsilon_v a > 0$ for all real primes $v \in \infty_k$ and write $P_i(t) = p_i(at)$ with $1 \leq i \leq n$. Let $S$ be a finite set of primes of $k$ satisfying [20, Hypothesis $(H_1)$] for $P_1(t), \ldots, P_n(t)$ and $a \in \mathfrak{o}_{k,S}$. For any given $\lambda_v \in k_v$ with $v \in S \setminus \infty_k$, there is $t_0 \in \mathfrak{o}_{k,S}$ close to $a^{-1}\lambda_v$ for $v \in S \setminus \infty_k$ and arbitrarily large at all real primes $v \in \infty_k$ such that for each $1 \leq i \leq n$, $P_i(t_0)$ is a unit of $k_w$ for all primes $w \notin S$ except one prime $w_i$, where $P_i(t_0)$ is a uniformizing parameter by [20, Prop.4.1]. Then one can choose $\lambda = at_0$ as desired. \hfill $\Box$

Let $U$ be an open dense subset of $\mathbb{A}_k^1$ over $k$ such that $f|_V : V = f^{-1}(U) \to U$ is smooth with geometrically integral fibers. Write

$$\mathbb{A}_k^1 \setminus U = \{P_1, \ldots, P_n\}$$

where $P_1, \ldots, P_n$ are the closed points over $k$ and $k_i = k(P_i)$ are the residue fields of $P_i$ for $1 \leq i \leq n$. Let $D_i = f^{-1}(P_i)$ and $\{D_{i,j}\}_{j=1}^{g_i}$ be the set of irreducible components of $D_i$ for $1 \leq i \leq n$. For any $b \in \text{Br}_1(V)$, one has

$$\partial_{D_{i,j}}(b) \in H^1(L_{i,j}, \mathbb{Q}/\mathbb{Z})$$

where $L_{i,j}$ is the algebraic closure of $k_i$ inside the function field $k_i(D_{i,j})$ for $1 \leq j \leq g_i$ with $1 \leq i \leq n$. 
The following result is an analogue of Proposition 3.1 without assuming that all fibers of $f$ are split under Schinzel’s hypothesis.

**Proposition 4.3.** Let $B$ be a finite subgroup of $Br_1(V)$. Suppose that $E = \prod_{v \in \infty_k} E_v$ is a connected component of $X(k_\infty)$ such that $f(E_v)$’s are unbounded for all real primes $v$ of $k$. Assume that

(i) Each $D_i$ contains an irreducible component $D_{i,1}$ of multiplicity 1 such that the algebraic closure $L_i$ of $k_i$ inside $k_i(D_{i,1})$ is cyclic over $k_i$ with $1 \leq i \leq n$.

(ii) Hypothesis $(H_s)$ holds.

Let $M_i/L_i$ be a finite abelian extension such that $\partial_{D_{i,1}}(B)|_{M_i} = 0$ and $M_i/k_i$ is Galois for $1 \leq i \leq n$. Let $B_1$ be a finite subgroup of $Br_1(V)$ such that the image of $B_1$ in $Br_1(V)$ contains $f^*(\partial_U^{-1}(\bigoplus_{i=1}^{n} H^1(M_i/k_i, \mathbb{Q}/\mathbb{Z})))$

by the commutative diagram (3.1)

Then

$$\bigcup_{c \in U(k)} X_c(A_k)_{E_c}^{B+B_1}$$

is dense in

$$X(A_k)^{(B+B_1)\cap Br_1(X)}$$

where $X_c$ is the fiber of $f$ over $c \in U(k)$ and $E_c = E \cap X_c(k_\infty)$.

**Proof.** Write $A^1_k = \text{Spec}(k[t])$ and $P_i = (p_i(t))$ where $p_i(t)$’s are the fixed irreducible polynomials over $k$. Let $X'$ be an open subset of $X$ over $k$ such that the restriction $f|_{X'} : X' \to A^1_k$ is smooth and $f|_{X'}^{-1}(P_i)$ contains the smooth part $D_{i,1}^{sm}$ of $D_{i,1}$ for $1 \leq i \leq n$.

For any open subset

$$W = E \times \prod_{v < \infty_k} W_v \subset X(A_k)$$

with $W^{Br_1(X)} \neq \emptyset$,

there is a finite subset $S_0$ of $\Omega_k$ containing $\infty_k$ such that the following conditions hold.

(a) The morphism $f : X \to A^1_k$, the open immersions $U \hookrightarrow A^1_k$, $V \hookrightarrow X$ and $X' \hookrightarrow X$ are extended to their integral models over $\mathfrak{o}_{k,S_0}$ with the following commutative diagram

such that $f(V(\mathfrak{o}_{k_v})) = U(\mathfrak{o}_{k_v})$ for $v \not\in S_0$ and the restriction $f|_{X'} : X' \to A^1_{\mathfrak{o}_{k,S_0}}$ is smooth.

(b) The morphism $D_i \to P_i$ is extended to their integral models $D_i \to P_i$ over $\mathfrak{o}_{k,S_0}$ such that

$$\mathcal{P}_i = \text{Spec}(\mathfrak{o}_{k,S_0}[t]/(p_i(t))) = \text{Spec}(\mathfrak{o}_{k_i,S_0})$$

is smooth over $\mathfrak{o}_{k,S_0}$ and $D_i = f^{-1}(\mathcal{P}_i)$ for $1 \leq i \leq n$. Moreover,

$$A^1_{\mathfrak{o}_{k,S_0}} \setminus U = \{\mathcal{P}_1, \cdots, \mathcal{P}_n\}$$

and $X \setminus V = \{D_1, \cdots, D_n\}$.
(c) The closed immersion $D_{i,1} \hookrightarrow D_i$ is extended to their models $D_{i,1} \hookrightarrow D_i$ over $\mathfrak{o}_{k,S_0}$ such that the smooth points over residue fields $D_{i,1}^\infty(k(w)) \neq \emptyset$ for all primes $w$ in $L_i$ with $(w \cap k) \not\subseteq S_0$ for $1 \leq i \leq n$. All field extensions $M_i/L_i/k_i/\mathfrak{k}$ are unramified outside $S_0$ for $1 \leq i \leq n$.

(d) All elements in $(B + B_1) \cap Br(X)$ take the trivial value over $W_v = \mathcal{X}(\mathfrak{o}_{k_v})$ and all elements in $B + B_1$ take the trivial value over $\mathcal{V}(\mathfrak{o}_{k_v})$ for all $v \not\subseteq S_0$.

By the Chebotarev density theorem, there is a finite set $T_i$ of primes of degree 1 over $k$ in $L_i$ with $|Gal(M_i/k_i)|$ elements such that

1. for each $\sigma \in Gal(M_i/L_i)$, there are exactly $|Gal(L_i/k_i)|$ primes $q \in T_i$ with sufficiently large cardinalities of the residue fields and $Frob(q) = \sigma$ for $1 \leq i \leq n$;

2. the set $S_i = \{q \cap k : q \in T_i\}$ has $|Gal(M_i/k_i)|$ primes with $S \cap S_i = \emptyset$ for $1 \leq i \leq n$ and $S_i \cap S_j = \emptyset$ for $i \neq j$.

Since all primes in $T_i$ are of degree 1 over $k$, one concludes that $(L_i)_q = k_p$ for each $q \in T_i$ with $p = q \cap k \in S_i$ for $1 \leq i \leq n$. Moreover, the scheme $D_{i,1} \times_{\mathfrak{o}_{k,S}} \mathfrak{o}_{k_p}$ splits into geometrically irreducible components $\{\sigma D_{i,p} : \sigma \in Gal(L_i/k_i)\}$ over $\mathfrak{o}_{k_p}$ by (b) and (c), where $D_{i,p}$ is a fixed geometrically irreducible component of $D_{i,1} \times_{\mathfrak{o}_{k,S}} \mathfrak{o}_{k_p}$ for $p \in S_i$ with $1 \leq i \leq n$. Since the polynomial $p_i(t)$ has a root over $\mathfrak{o}_{k_p}$ for each $p \in S_i$ by (b) and (c), one obtains $t_p \in \mathfrak{o}_{k_p}$ with $\text{ord}_p(p_i(t_p)) = 1$ such that there is $x_{p} \in \mathcal{X}'_{p}(\mathfrak{o}_{k_p}) \subset \mathcal{X}(\mathfrak{o}_{k_p})$ satisfying

$$x_{p} \mod p \in D_{i,p}^\infty(k(p)) \subset \mathcal{X}'_{p}(k(p))$$

by (1), the Lang-Weil’s estimation (see [45, Theorem 7.7.1]), (a) and Hensel’s lemma for $1 \leq i \leq n$.

For any real prime $v \in \infty_k$, there is a connected component $C_v$ of $V(k_v)$ such that $f(E_v \cap C_v)$ is unbounded. We define

$$W_v = \begin{cases} E_v \cap C_v & \text{if } v \text{ real prime in } \infty_k \\ E_v \cap V(k_v) & \text{if } v \text{ complex prime in } \infty_k. \end{cases}$$

By Harari’s formal lemma (see [11, Theorem 1.4]), there is a finite set $S$ of primes of $k$ containing $\bigcup_{i=1}^n S_i$ with

$$ \begin{cases} x_{p} \in \mathcal{X}'_{p}(\mathfrak{o}_{k_p}) & \text{as above for } p \in \bigcup_{i=1}^n S_i \\ x_{v} \in W_v & \text{for } v \in S \setminus \bigcup_{i=1}^n S_i \end{cases}$$

such that

$$ \sum_{v \in S} \xi(x_v) = 0 \tag{4.1} $$

for all $\xi \in (B + B_1)$. By shrinking $W_v$ for $v \in S \setminus \infty_k$, one can assume that each element in $B + B_1$ takes a single value over $W_v$ and $f(W_v)$ is open in $k_v$.

By (ii), there is $t_0 \in \mathfrak{o}_{k,S}$ close to $f(x_v)$ for $v \in S \setminus \infty_k$ and $t_0 \in f(W_v)$ for $v \in \infty_k$ such that $\text{ord}_v(p_i(t_0)) = 0$ for all $v \not\subseteq S$ except one prime $v_i$ of $k$ with $\text{ord}_{v_i}(p_i(t_0)) = 1$ for $1 \leq i \leq n$. Write

$$ p_i(t) = \prod_{j=1}^g q_j(t) $$
where \( q_j(t) \) are irreducible polynomials over \( \mathfrak{o}_{k_{v_i}} \) for \( 1 \leq j \leq g \). There are \( g \) primes \( w_1, \ldots, w_g \) of \( k_i \) above \( v_i \) corresponding to irreducible polynomials \( q_1(t), \ldots, q_g(t) \) respectively. Since \( \text{ord}_{v_i}(p_i(t_0)) = 1 \), there is \( 1 \leq j_0 \leq g \) such that

\[
\text{ord}_{v_i}(q_{j_0}(t_0)) = 1 \quad \text{and} \quad \text{ord}_{v_i}(q_j(t_0)) = 0 \quad \text{for all} \quad j \neq j_0.
\] (4.2)

Then \( q_{j_0}(t) \) is of degree 1 by (b) and Hensel’s lemma and \( w_{j_0} \) is a prime of degree 1 over \( v_i \).

Let \( \alpha_{i,v} \) be the image of \( t \) in \( k[t]/(p_i(t)) = k_i \). Then

\[
\text{Cores}_{k_i/k}(\chi, t - \alpha_i) \in Br_1(U) \quad \text{with} \quad f^*(\text{Cores}_{k_i/k}(\chi, t - \alpha_i)) \in B_1
\]

for any \( \chi \in H^1(L_i/k_i, \mathbb{Q}/\mathbb{Z}) \) by [20, §1.2]. Since

\[
0 = \sum_{v \in \Omega_k} \text{Cores}_{k_i/k}(\chi, t_0 - \alpha_i)_v = \sum_{v \in S} \text{Cores}_{k_i/k}(\chi, t_0 - \alpha_i)_v + \sum_{v \not\in S} \text{Cores}_{k_i/k}(\chi, t_0 - \alpha_i)_v
\]

by the reciprocity law, (b) and (c), one concludes that

\[
\sum_{j=1}^g (\chi, t_0 - \alpha_i)_w_j = - \sum_{v \not\in S} \text{Cores}_{k_i/k}(\chi, t_v - \alpha_i)_v = - \sum_{v \not\in S} f^*(\text{Cores}_{k_i/k}(\chi, t - \alpha_i))(x_v) = 0
\]

by (4.1) and the functoriality of Brauer-Manin pairing. By (4.2), one obtains that

\[
(\chi, t_0 - \alpha_i)_{w_{j_0}} = 0 \quad \text{for all} \quad \chi \in H^1(L_i/k_i, \mathbb{Q}/\mathbb{Z}).
\]

Since \( L_i/k_i \) is abelian, one concludes that \( w_{j_0} \) splits completely in \( L_i/k_i \). There is

\[
y_{v_i} \in X'_{t_0}(\mathfrak{o}_{k_{v_i}}) \subset X(\mathfrak{o}_{k_{v_i}})
\]

such that \( y_{v_i} \bmod v_i \in D^{\text{sm}}_{i,v_i}(k(v_i)) \subset X'_{t_0}(k(v_i)) \)

by Hensel’s lemma and (a) and (c) for \( 1 \leq i \leq n \). One can extend these \( y_{v_i} \) to

\[
(y_{v_i})_{v \in \Omega_k} \in X_{t_0}(\mathbb{A}_k) \cap W
\]

by taking any element in \( W_v \cap X_{t_0}(k_v) \) for \( v \in S \) and any element in \( X_{t_0}(\mathfrak{o}_{k_v}) \) for the rest of primes \( v \) of \( k \). Then

\[
\sum_{v \in \Omega_k} b(y_v) = \sum_{v \not\in S} b(y_v) = \sum_{i=1}^n b(y_{v_i})
\]

by (4.1) and (d) for all \( b \in (B + B_1) \). By [28, Corollary 2.4.3] and the choice of \( y_{v_i} \), there is

\[
\tau_i \in \text{Gal}(M_i/k_i) \quad \text{such that} \quad b(y_{v_i}) = \partial_{D, i, 1}(b)(\tau_i)
\]

for all \( b \in B + B_1 \). Therefore

\[
\sum_{v \in \Omega_k} b(y_v) = \sum_{i=1}^n \partial_{D, i, 1}(b)(\tau_i) \in \mathbb{Q}/\mathbb{Z}
\] (4.3)

with \( (\tau_i)_{i=1}^n \in \bigoplus_{i=1}^n \text{Gal}(M_i/k_i) \) for all \( b \in (B + B_1) \). Since

\[
\sum_{v \in \Omega_k} f^*(\xi)(y_v) = \sum_{v \in \Omega_k} \xi(t_0) = 0
\]
for all $\xi \in \partial_U^{-1}(\bigoplus_{i=1}^n H^1(M_i/k_i, \mathbb{Q}/\mathbb{Z}))$ by the functoriality of Brauer-Manin pairing and the reciprocity law, one obtains

$$\chi((\tau_i)_{i=1}^n) = 0$$

for all $\chi \in \bigoplus_{i=1}^n H^1(M_i/k_i, \mathbb{Q}/\mathbb{Z})$ by (3.1) and (4.3). This implies that $(\tau_i)_{i=1}^n \in \bigoplus_{i=1}^n[\text{Gal}(M_i/k_i), \text{Gal}(M_i/k_i)]$. Write

$$[\text{Gal}(L_i/k_i), \text{Gal}(M_i/L_i)] = \prod_{\gamma \in \text{Gal}(L_i/k_i)} (\gamma \circ \sigma_\gamma)\sigma_\gamma^{-1} : \sigma_\gamma \in \text{Gal}(M_i/L_i)$$

for $1 \leq i \leq n$. Then

$$[\text{Gal}(L_i/k_i), \text{Gal}(M_i/L_i)] \subseteq [\text{Gal}(M_i/k_i), \text{Gal}(M_i/k_i)] \subseteq \text{Gal}(M_i/L_i).$$

Since $\text{Gal}(L_i/k_i)$ is cyclic, one has

$$H^2(L_i/k_i, \mathbb{Q}/\mathbb{Z}) = H^3(L_i/k_i, \mathbb{Z}) = H^1(L_i/k_i, \mathbb{Z}) = 0.$$

This implies the restriction map

$$H^1(M_i/k_i, \mathbb{Q}/\mathbb{Z}) \to H^1(M_i/L_i, \mathbb{Q}/\mathbb{Z})^{\text{Gal}(L_i/k_i)}$$

is surjective for $1 \leq i \leq n$. Therefore the natural map

$$\text{Gal}(M_i/L_i)/[\text{Gal}(L_i/k_i), \text{Gal}(M_i/L_i)] \to \text{Gal}(M_i/k_i)/[\text{Gal}(M_i/k_i), \text{Gal}(M_i/k_i)]$$

is injective by the duality. One concludes that

$$[\text{Gal}(L_i/k_i), \text{Gal}(M_i/L_i)] = [\text{Gal}(M_i/k_i), \text{Gal}(M_i/k_i)]$$

for $1 \leq i \leq n$. Then

$$\tau_i = \prod_{\gamma_\iota \in \text{Gal}(L_i/k_i)} (\gamma_i \circ \sigma_{\gamma_i})\sigma_{\gamma_i}^{-1} \text{ with } \sigma_{\gamma_i} \in \text{Gal}(M_i/L_i)$$

(4.4)

for $1 \leq i \leq n$.

For each pair $(\gamma_i, \sigma_{\gamma_i}) \in \text{Gal}(L_i/k_i) \times \text{Gal}(M_i/L_i)$ in (4.4), there is $q \in T_i$ such that

$$\sigma_{\gamma_i}^{-1} = \text{Frob}(q) \quad \text{and} \quad (L_i)_q = k_p \text{ with } p = q \cap k$$

for $1 \leq i \leq n$. Since $\text{ord}_p(p_i(t_0)) = 1$, there is

$$y'_p \in \mathcal{X}'_{t_0}(\mathcal{O}_{k_p}) \subset \mathcal{X}_{t_0}(\mathcal{O}_{k_p}) \text{ such that } y'_p \text{ mod } p \in \gamma_i D_{i,p}^{sm}(k(p)) \subset \mathcal{X}'_{t_0}(k(p))$$

by Hensel’s lemma and (a) and (c) for $1 \leq i \leq n$. Define

$$z_v = \begin{cases} y'_p & \text{if } v = p \text{ from a pair } (\gamma_i, \sigma_{\gamma_i}) \text{ in } (4.4) \text{ with } 1 \leq i \leq n \\ y_v & \text{otherwise.} \end{cases}$$

Then $(z_v)_{v \in \Omega_k} \in \mathcal{X}_{t_0}(\mathcal{A}_k) \cap W$ by (d). For $p$ from a pair $(\gamma_i, \sigma_{\gamma_i})$ from (4.4), one has

$$b(y'_p) = e(p)\partial_{D_i}(b)(\text{Frob}(y'_p)) = \text{ord}_p(p_i(t_0))\partial_{D_i}(b)(\gamma_i \circ \sigma_{\gamma_i}^{-1}) = -\partial_{D_i}(b)(\gamma_i \circ \sigma_{\gamma_i})$$

(4.5)

and

$$b(y_p) = e(p)\partial_{D_i}(b)(\text{Frob}(y_p)) = \text{ord}_p(p_i(t_0))\partial_{D_i}(b)(\sigma_{\gamma_i}^{-1}) = -\partial_{D_i}(b)(\sigma_{\gamma_i})$$

(4.6)
for all \( b \in (B + B_1) \) by [28, Corollary 2.4.3 and p.244-245] with \( 1 \leq i \leq n \). Therefore

\[
\sum_{v \in \Omega_k} b(z_v) = \sum_{v \in \Omega_k} b(y_v) - \sum_{i=1}^{n} \sum_{\gamma_i \in \text{Gal}(L_i/k_i)} b(y_{i}) + \sum_{i=1}^{n} \sum_{\gamma_i \in \text{Gal}(L_i/k_i)} b(y'_{i})
\]

\[
= \sum_{i=1}^{n} \partial_{D_i}(b)(\tau_i) + \sum_{i=1}^{n} \sum_{\gamma_i \in \Gamma_i} \partial_{D_i}(b)(\sigma_{\gamma_i}) - \sum_{i=1}^{n} \sum_{\gamma_i \in \text{Gal}(L_i/k_i)} \partial_{D_i}(b)(\gamma_i \circ \sigma_{\gamma_i}) = 0
\]

for all \( b \in (B + B_1) \) by (4.4), (4.5) and (4.6). This implies that

\[
(z_v)_{v \in \Omega_k} \in (X_{b_0}(\mathbb{A}_k)^{B+B_1} \cap W) \neq \emptyset
\]
as desired. \( \square \)

**Remark 4.4.** The assumption that \( L_i/k_i \) is cyclic for \( 1 \leq i \leq n \) is needed only for proving the equality

\[
[\text{Gal}(M_i/k_i), \text{Gal}(M_i/k_i)] = \{ \prod_{\gamma \in \text{Gal}(L_i/k_i)} (\gamma \circ \sigma_{\gamma})\sigma_{\gamma}^{-1} : \sigma_{\gamma} \in \text{Gal}(M_i/L_i) \}
\]
in the proof of Proposition 4.3. This equality is equivalent to that the inflation map

\[
H^2(L_i/k_i, \mathbb{Q}/\mathbb{Z}) \to H^2(M_i/k_i, \mathbb{Q}/\mathbb{Z})
\]
is injective. In this case, the assumption that \( L_i/k_i \) is abelian will be enough for Proposition 4.3.

The main result of this section is the following theorem.

**Theorem 4.5.** Let \( X \xrightarrow{f} \mathbb{A}_k^1 \) be a surjective morphism over a number field \( k \). Suppose that \( X \) admits an action of a torus \( T \) over \( k \) such that \( f^{-1}(U) \xrightarrow{f} U \) is a torsor under \( T \) where \( U \) is an open dense subset of \( \mathbb{A}_k^1 \) over \( k \). Write \( \mathbb{A}_k^1 \setminus U = \{ P_1, \ldots, P_n \} \) where \( P_1, \ldots, P_n \) are the closed points over \( k \) and \( k_i = k(P_i) \) are the residue fields of \( P_i \), for \( 1 \leq i \leq n \). Assume that

i) Each \( f^{-1}(P_i) \) contains an irreducible component of multiplicity 1 such that the algebraic closure \( L_i \) of \( k_i \) inside the function field of this component is cyclic over \( k_i \), with \( 1 \leq i \leq n \).

ii) Each equivalent class of admissible connected components of \( X(k(\infty)) \) contains a connected component \( E = \prod_{v \in \infty_k} E_v \) such that all \( f(E_v) \)'s are unbounded over all real primes \( v \).

iii) Hypothesis \( (H_v) \) holds.

Then \( X \) satisfies strong approximation off \( \infty_k \) with respect to \( \text{Br}_a(X) \).

**Proof.** It follows from the same proof of Theorem 3.5 by replacing Proposition 3.1 with Proposition 4.3. \( \square \)

It should be pointed out that weak approximation for varieties satisfying Theorem 4.5 is not known before (see [57]). Applying the above theorem to the norm equation in (1.1), one gets the following result which improves [27, Theorem 2] and [40, Theorem 1.1].
Corollary 4.6. Let $X$ be the smooth part of the following equation

$$
\prod_{i=1}^{m} N_{L_i/k}(x_i) = c \prod_{j=1}^{n} p_j(t)^{e_j} \quad \text{with} \quad c \in k^\times
$$

where $L_i/k$'s are finite extensions of number fields and $p_j(t)$'s are distinct irreducible monic polynomials over $k$ and $e_j$'s are positive integers. Assume that one of extensions $L_i/k$ for $1 \leq i \leq m$ is cyclic. Suppose there is a connected component $E = \prod_{v \in \infty k} E_v$ in each equivalent class of admissible connected components of $X(k_{\infty})$ such that the projection of $t$-coordinate of $E_v$ is unbounded for all real primes $v \in \infty k$. If Schinzel’s hypothesis holds, then $X$ satisfies strong approximation off $\infty k$ with respect to $\text{Br}_1(X)$.

Proof. It follows from the same proof of Corollary 3.6 by replacing Theorem 3.5 with Theorem 4.5. □

5. Fibration over $\mathbb{P}^1$ with an action of torus

In this section, we apply Proposition 3.4 to study the fibration over $\mathbb{P}^1$ with an action of torus. Harpaz and Wittenberg proposed a conjectures in [31, Conjecture 9.1 and 9.2] related with

- a set of distinct irreducible polynomials $\{p_1(t), \cdots, p_n(t)\}$ over $k$ with $k_i = k[t]/(p_i(t))$ and $a_i \in k_i$ representing the class of $t$ for $1 \leq i \leq n$;
- a set of finite extensions $\{L_1/k_1, \cdots, L_n/k_n\}$ and $b_i \in k_i^\times$ for $1 \leq i \leq n$.

Consider the morphism

$$
\Phi : (\mathbb{A}^2_k \setminus \{(0, 0)\}) \times_k \prod_{i=1}^{n} (\text{Res}_{L_i/k}(\mathbb{A}^1_{L_i}) \setminus F_i) \longrightarrow \prod_{i=1}^{n} \text{Res}_{k_i/k}(\mathbb{A}^1_{k_i})
$$

given by

$$(\lambda, \mu, x_1, \cdots, x_n) \mapsto (b_i(\lambda - a_i \mu) - N_{L_i/k_i}(x_i))_{1 \leq i \leq n}$$

where $F_i$ is the singular locus of the variety of $\text{Res}_{L_i/k}(\mathbb{A}^1_{L_i}) \setminus \text{Res}_{L_i/k}(\mathbb{G}_{m,L_i})$. Define

$$W = \Phi^{-1}(0, \cdots, 0) \quad (5.1)$$

in [31, §9.2.2.] and show that [31, Conjecture 9.1] holds if $W$ satisfies strong approximation off any finite prime (see [31, Corollary 9.10]). Contrarily, we study strong approximation property of $W$ by using [31, Conjecture 9.1 and 9.2].

Lemma 5.1. Let $W$ be defined as (5.1). Then

$$\bar{k}[W]^\times = \bar{k}^\times, \quad \text{Pic}(W_\bar{k}) = 0 \quad \text{and} \quad \text{Br}_1(W) = \text{Br}(k)$$

where $\bar{k}$ is an algebraic closure of $k$.

Proof. Consider the variety $W_1$ over $\bar{k}$ defined by the following equations

$$
\prod_{j=1}^{d_i} \sigma_i^{(j)}(b_i)(\lambda - \sigma_i(a_i)\mu) = \sigma_i(x_i, \sigma_i) \quad (5.2)
$$
where $\sigma_i$ runs over all embeddings from $k_i/k$ to $\bar{k}/k$ and $d_i = [L_i : k_i]$ for $1 \leq i \leq n$. For any fixed $\sigma_i$ with $1 \leq i \leq n$, the variety defined by a single equation of (5.2) is isomorphic to $\mathbb{A}_k^{d_i+1}$. This implies
\[
(\bar{k}[z_{i,\sigma_i}^{(1)}, \ldots, z_{i,\sigma_i}^{(d_i)}], \lambda, \mu)/(\prod_{j=1}^{d_i} z_{i,\sigma_i}^{(j)} - \sigma_i(b_j)(\lambda - \sigma_i(a_i)\mu))^\times = \bar{k}^\times.
\] (5.3)

We claim that $\bar{k}[W_1]^\times = \bar{k}^\times$. Indeed, suppose $u \in (\bar{k}[W]^\times \setminus \bar{k}^\times)$, there is $\sigma_{i_0}$ with $1 \leq i_0 \leq n$ such that the specialization of $u$ with $z_{i,\tau}^{(j)} \in \bar{k}$ for all $\tau \neq \sigma_{i_0}$ and $1 \leq j \leq d_i$ is not a constant. This contradicts to (5.3) and the claim follows. Since $W_\bar{k}$ is obtained by removing a closed subset of $W_1$ of codimension bigger than 1, one concludes that $\bar{k}[W]^\times = \bar{k}^\times$.

Since $\text{Pic}(U_\bar{k}) = \text{Pic}(\ker(\psi)_k) = 0$, one obtains that $\text{Pic}(f^{-1}(U)_\bar{k}) = 0$ and
\[
1 \to \bar{k}[U]^\times \to \bar{k}[f^{-1}(U)]^\times \to \bar{k}[\ker(\psi)]^\times \to 1
\]
by [46, Prop.6.10]. Therefore
\[
\text{Pic}(W_\bar{k}) = \text{coker}(\bar{k}[f^{-1}(U)]^\times \xrightarrow{\text{div}} \text{Div}_{W_\bar{k} \setminus f^{-1}(U)_\bar{k}} W_\bar{k})
\]
and $\{z_{i,\sigma_i}^{(j)}\}$ is a basis of the free abelian group $\bar{k}[f^{-1}(U)]^\times / \bar{k}^\times$ where $\sigma_i$ runs over all embeddings from $k_i/k$ to $\bar{k}/k$ and $1 \leq j \leq d_i$ for $1 \leq i \leq n$. Since $W \to \mathbb{P}^1$ is faithful flat, one gets that $\text{Div}_{W_\bar{k} \setminus f^{-1}(U)_\bar{k}} W_\bar{k}$ is a free abelian group with a basis $\{f^{-1}(\lambda - \sigma_i(a_i)\mu)\}$ where $\sigma_i$ runs over all embeddings from $k_i/k$ to $\bar{k}/k$ with $1 \leq i \leq n$. By the equation (5.2), one concludes that $\text{Pic}(W_\bar{k}) = 0$. By the Hochschild-Serre spectral sequence (see [46, Lemma 6.3 (i)]), one has $\text{Br}_1(W) = \text{Br}(k)$.

The following result is due to Sansuc [47, Theorem 4.1], which provides the explicit version of Waldschmidt’s result in [53] and [54].

**Lemma 5.2.** Let $m$ be an integral ideal of a number field $k$. Suppose $\{v_1, \ldots, v_s\}$ is a set of finite primes of $k$ above a set of primes $\{p_1, \ldots, p_s\}$ of $\mathbb{Q}$ such that each $p_i$ splits completely in Galois closure of $k/\mathbb{Q}$ with $v_i \not\mid m$ for $1 \leq i \leq s$. If $s > [k : \mathbb{Q}]^2 - [k : \mathbb{Q}] + 1$, then the set
\[
\{x \in \mathfrak{o}_{k,S_0}^	imes : x \equiv 1 \mod m, \ x \text{ totally positive at real primes}\}
\]
is dense in the connected component of 1 in $(k \otimes_{\mathbb{Q}} \mathbb{R})^\times$ with $S_0 = \{v_1, \ldots, v_s\} \cup \infty_k$.

One can apply Lemma 5.2 to established strong approximation with Brauer-Manin obstruction for tori with the property of approximation at archimedean primes.

**Theorem 5.3.** Let $T$ be a torus over a number field $k$. If $S$ is a finite subset of $\Omega_k$, there is a finite subset $S_0$ of $\Omega_k$ with $S_0 \cap S = \emptyset$ and $|S_0|$ independent of $S$ such that $T$ satisfies strong approximation with respect to $\text{Br}_1(T)$ off $S_0$.

**Proof.** We first prove the case that $T$ is a quasi-trivial torus. Without loss of generality, we assume that $T = \text{Res}_{K/k}(G_m)$ where $K/k$ is a finite extension. Choose a set of primes $\{p_1, \ldots, p_s\}$ of $\mathbb{Q}$ with $s > [K : \mathbb{Q}]^2 - [K : \mathbb{Q}] + 1$ such that each $p_i$ splits completely in Galois closure of
$K/\mathbb{Q}$ and any prime in $S$ is not above $p_i$ for $1 \leq i \leq s$. Let $S_0$ be a set of all primes of $k$ above $p_i$ for $1 \leq i \leq s$.

For any open subset $\prod_{v \in S_0} T(k_v) \times \prod_{v \not\in S_0} U_v$ in $T(A_k)$ with

$$(\prod_{v \in S_0} T(k_v) \times \prod_{v \not\in S_0} U_v) \cap T(A_k)^{Br_1(T)} \neq \emptyset,$$

there is $x \in T(k) = K^\times$ such that

$$(x \cdot T^+(k_\infty)) \cap (\prod_{v \in S_0} T(k_v) \times \prod_{v \not\in S_0} U_v) \neq \emptyset$$

by [23, Theorem 3.19], where

$$T^+(k_\infty) = \prod_{v \in \infty_k} T^+(k_v)$$

is the connected component of 1 in $T(k_\infty) = (K \otimes_k k_\infty)^\times$.

Let $S_1$ be a finite subset of $\Omega_k$ with $\infty_k \subseteq S_1$ and $S_1 \cap S_0 = \emptyset$ such that

$$U_v = (\mathfrak{O}_K \times_{\mathfrak{O}_k} \mathfrak{O}_{k_v})^\times = \prod_{w \mid v} \mathfrak{O}_{K_w}^\times$$

for any $v \not\in S_1 \cup S_0$. Let $\mathfrak{m}$ be an integral ideal of $\mathfrak{O}_K$ consisting of the primes above $S_1 \setminus \infty_k$ such that

$$x(1 + \mathfrak{m}) \subseteq U_v$$

for all $v \in S_1 \setminus \infty_k$. Let $\tilde{S}_0$ be the set of primes of $K$ above $S_0$ and $\infty_k$. There is $y \in \mathfrak{O}_{K,\tilde{S}_0}^\times$ with $y \equiv 1 \mod \mathfrak{m}$ such that

$$y \in ((x^{-1} \cdot \prod_{v \in \infty_k} U_v) \cap T^+(k_\infty)) \subseteq T^+(k_\infty)$$

by applying Lemma 5.2. This implies that

$$xy \in T(k) \cap (\prod_{v \in \tilde{S}_0} k_v^\times \times \prod_{v \not\in \tilde{S}_0} U_v)$$

as desired.

For a general torus $T$, one consider a flasque resolution of $T$ in sense of [12, Proposition-Definition 3.1]

$$1 \to T_1 \to T_0 \to T \to 1$$

where $T_0$ is a quasi-trivial torus over $k$ and $T_1$ is a flasque torus over $k$. Since $T_0$ satisfies strong approximation with respect to $Br_1(T_0)$ off $S_0$ as above, one concludes that $T$ satisfies strong approximation with respect to $Br_1(T)$ off $S_0$ by the decent relation in [7, Theorem 5.1].

\[\square\]

Corollary 5.4. Let $G$ be a connected linear algebraic group over a number field $k$. If $S$ is a finite subset of $\Omega_k$, there is a finite subset $S_0$ of $\Omega_k$ with $S_0 \cap S = \emptyset$ and $|S_0|$ independent of $S$ such that $G$ satisfies strong approximation with respect to $Br_1(G)$ off $S_0$. 
Proof. Since

\[ 1 \to G^u \to G \to G^{red} \to 1 \]

where $G^u$ is the unipotent radical of $G$, one can assume $G = G^{red}$ by [9, Lemma 2.1] and [7, Proposition 4.1]. By [12, Proposition-Definition 3.1] and [7, Theorem 5.1], one can further assume that $G$ is reductive and quasi-trivial. Then one has the following short exact sequence of algebraic groups

\[ 1 \to [G,G] \to G \to T \to 1 \]

where $[G,G]$ is semi-simple and simply connected and $T$ is quasi-trivial. Enlarge $S_0$ such that each simple factor of $[G,G]$ is isotropic over $S_0$ with $S_0 \cap S = \emptyset$. Since $[G,G]$ satisfies strong approximation off $S_0$ by [44, Theorem 7.12], one obtains the desired result by [7, Proposition 4.1], [9, Lemma 2.1] and [51, Theorem 5.1.1(e)].

The main result of this section is the following result.

**Theorem 5.5.** Let $X$ be a smooth variety with an action of a torus $T$ over a number field $k$. Suppose that $X \xrightarrow{f} \mathbb{P}^1$ is a morphism over $k$ such that all fibres of $f$ contain an irreducible component of multiplicity 1. Assume that there is an open dense subset $U \subset \mathbb{P}^1$ over $k$ such that $f^{-1}(U) \xrightarrow{f} U$ is a torsor under $T$. Write $\mathbb{P}^1 \setminus U = \{P_1, \ldots, P_n\}$.

1. If Conjecture 9.1 in [31] is true for $\{P_1, \ldots, P_n\}$, then for any finite subset $S$ of $\Omega_k$ there is a finite subset $S_0$ of $\Omega_k$ with $S \cap S_0 = \emptyset$ and $|S_0|$ independent of $S$ such that $X$ satisfies strong approximation with respect to $\text{Br}_1(X)$ off $S_0$.

2. If Conjecture 9.1 in [31] is true for $\{P_1, \ldots, P_n\}$, then $T(k_\infty)^+ \cdot f^{-1}(U)(k)$ is dense in $X(\mathbb{A}_k)^{\text{Br}_1(X)}$ where $T(k_\infty)^+$ is the connected component of identity of Lie group $T(k_\infty)$.

3. If Conjecture 9.2 in [31] is true for $\{P_1, \ldots, P_n\}$, then $f^{-1}(U)(k)$ is dense in $X(\mathbb{A}_k)^{\text{Br}_1(X)}$.

**Proof.** By Theorem 5.3, there exists a finite subset $S_0$ of $\Omega_k$ with $S \cap S_0 = \emptyset$ such that $T(k_\infty)^+$ can be approximated by the elements in $T(k) \cdot \prod_{v \in S_0} T(k_v)$. The statement (2) gives

\[ X(\mathbb{A}_k)^{\text{Br}_1(X)} = T(k_\infty)^+ \cdot f^{-1}(U)(k) \subset \left( \prod_{v \in S_0} T(k_v) \right) \cdot X(k) \subset X(\mathbb{A}_k). \]

This implies that the statement (1) holds. We only need to show (2) and (3).

Let $E$ be a connected component of $X(k_\infty)$ and $W$ be an open compact subset of $X(\mathbb{A}_k^f)$ such that

\[ (E \times W) \cap X(\mathbb{A}_k)^{\text{Br}_1(X)} \neq \emptyset. \]

By Proposition 3.4, there is a finite subgroup $B \subset \text{Br}_1(f^{-1}(U))$ such that

\[ (E \times W) \cap X_c(\mathbb{A}_k)^B \neq \emptyset \iff (E \times W) \cap X_c(\mathbb{A}_k)^{\text{Br}_1(X_c)} \neq \emptyset \]

for all $c \in U(k)$.

For (2), there is $u \in U(k)$ such that

\[ X_u(\mathbb{A}_k)^B \cap (E \times W) \neq \emptyset \]

by [31, Theorem 9.17]. Therefore

\[ (E \times W) \cap X_u(\mathbb{A}_k)^{\text{Br}_1(X_u)} \neq \emptyset. \]
This implies that $X_u$ is a trivial torsor under $T$ over $k$ by [51, Theorem 5.2.1]. Moreover, there are $g_{\infty} \in T(k_{\infty})^+$ and $x \in X_u(k) \subset f^{-1}(U)(k)$ such that $g_{\infty} \cdot x \in W$ by [29, Theorem].

For (3), one also can apply [31, Theorem 9.17]. The properness of $f$ is only used to check that a connected component of $f^{-1}(U)(\mathbb{R})$ maps onto a connected component of $U(\mathbb{R})$ in [31, pp. 281, line 12]. This is true in our situation by Hilbert 90. Then there is $u \in U(k)$ such that

$$X_u(\mathbb{A}_k) \cap (E \times W) \neq \emptyset.$$ 

By the same argument as above (2), one concludes

$$[(f^{-1}(U)(k) \cap (E \times W)] \supset [X_u(k) \cap (E \times W)] \neq \emptyset$$

as desired.

\begin{proof}
Consider the surjective homomorphism of tori

$$\psi : \mathbb{G}_m \times \prod_{i=1}^{n} \text{Res}_{L_i/k} \mathbb{G}_m \to \prod_{i=1}^{n} \text{Res}_{k_i/k} \mathbb{G}_m; \ (\alpha, \alpha_1, \ldots, \alpha_n) \mapsto (\alpha^{-1} \cdot N_{L_i/k} \alpha_i)_{1 \leq i \leq n}$$

where $N_{L_i/k_i}$ is the norm map for $1 \leq i \leq n$. Then $\ker(\psi)$ acts on $W$ by

$$(\alpha, \alpha_1, \ldots, \alpha_n) \circ (\lambda, \mu, x_1, \ldots, x_n) = (\alpha \cdot \lambda, \alpha \cdot \mu, \alpha_1 \cdot x_1, \ldots, \alpha_n \cdot x_n).$$

Let $f : W \to \mathbb{P}^1$ be the fibration given by $(\lambda, \mu, x_1, \ldots, x_n) \mapsto [\lambda, \mu]$ and

$$U = \mathbb{P}^1 \setminus \{(p_1(t), \ldots, p_n(t))\}$$

where $p_i(t)$ with $1 \leq i \leq n$ are the irreducible polynomials with $\lambda = t \mu$. Then $f^{-1}(U) \to U$ is a torsor under $\ker(\psi)$. The results follow from Theorem 5.5 and Lemma 5.1.

\end{proof}

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