The structure of vortextube segments in fluid turbulence

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Abstract

Geometrical description of the flow fields is an important direction to understand the physics of turbulence. Recently several new analysis approaches addressing the entire field properties have been developed, such as dissipation element analysis for the scalar fields and streamtube segment analysis (J. Fluid Mech. 2010, 648: 183-203) for the velocity vector field. By decomposing into a fundamental structure, i.e. streamtube segments, the velocity field can be understood from the statistics of these relative simple units. Similar idea can be adopted to analyze the vorticity field. The classic concept of vortex tube has been remaining as a topic of essential importance in many aspects. However, the vortex tube structure is not complete to represent the entire turbulent fields, because of its ambiguous definition and small volume portion. This work presents tentatively the vorticitytube segment structure to overcome the existing deficiency. Vorticitytube segments reveal an inherent topology of turbulence vorticity fields. Based on statistics conditioned on different vorticitytube segments, some problems can be newly understood, such as the enstrophy production. Results hereof may also serve for turbulence modeling.

1 Introduction

Strong nonlinearity from convective motions and nonlocality of the flow fields make turbulence prohibitively difficult. The vorticity vector plays an outstanding role in studying turbulence due to both kinematical and dynamical reasons. In turbulent flows, strong interaction of large compression and stretching regions makes local vorticity lumped. Regions of significant vorticity are inclined to be organized into tubes [1][2], named as vortex tubes. Those high vorticity regions reveal important features of turbulence such as intermittency [2] and flow structures [3][4]. Dynamically, vortex stretching causes velocity fluctuations to spread at different scales. This stretching mechanism, or the interaction between vorticity and the strain rate tensor, is believed important to understand in 3D turbulence some most basic and essential features [5]. Mathematically vorticity \( \vec{\omega} \) is defined as the curl of the velocity vector \( \vec{u} \), i.e. \( \vec{\omega} = \nabla \times \vec{u} \). There is abundant literature in discussing the mathematical and
topological features of vorticity and vortices [5]-[7]. Generally it is believed that the length and radius of vortices are of orders of magnitude of the Taylor and dissipative scales, respectively. However, results from larger DNS date analysis suggest there may still be some uncertainty [8]. Ruetsch and Maxey [9] showed that energy dissipation is correlated with vortex tubes. Regions of moderate dissipation tend to surround the tubes, but intensely dissipative regions tend to exist between two or more neighboring vortex tubes. Moffatt described vortex tube as the ‘sinews’ of turbulence [6]. Usually the region of large dissipation does not overlap with the region of large vorticity. Some work focusing on the topological features of the vortices helps to reveal flow structures, such as effects of solids on vortex tubes [10] and the possible symmetrical modes evolved from the interaction of spiral sheets and vortex tubes [11].

Overall the existing work depicts vortex structures in a qualitative and illustrative manner without universal threshold in defining high vorticity. Meanwhile most of the discussion about the vorticity field is limited to local statistics. To understand turbulence, as a field phenomena, the nonlocal and quantitative properties are more essential and informative. Recent years some novel approaches to investigate the turbulent scalar and velocity fields have been developed. Wang and Peters defined the dissipation element structure in turbulent scalar fields [12]. Thus the entire flow field can be partitioned in a space-filling and non-arbitrary way, from which the properties of entire flow fields can be understood from those of relative simple units. To study the velocity vector field, Wang put forward streamtube segment analysis [13] and from the segment structure the characteristic scales of the velocity field can be unambiguously defined. An interesting conclusion is that the skewness of velocity derivatives is naturally a kinematic outcome.

Vorticity and velocity both are among the most important vector field variables in turbulence dynamics. In this paper, some tentative attempts are made to use vorticitytube segment analysis to study the vorticity field by using the similar idea. The vorticitytube segment structure is defined and statistics conditioned on different vorticitytube segments may shed light on a deeper understanding of turbulence dynamics.

2 Vectorline/vectortube segment structure

Mathematically for a given vector field \( \vec{v} \), the corresponding vector line is defined as

\[
\frac{d\vec{x}}{ds} = \vec{v}.
\]

(1)

If applied to the velocity and vorticity fields, the above equation then determines streamlines and vortex lines, respectively. Along a vortex line the vorticity vector \( \vec{\omega} \) can be expressed as \( \vec{\omega} \hat{n} \), where \( \hat{n} \) is the unit orientation vector. From the N-S equations it yields

\[
\frac{\partial \vec{\omega}}{\partial t} + \vec{u} \cdot \nabla \vec{\omega} = \nu \nabla \cdot \nabla \vec{\omega} + \vec{\omega} \cdot \nabla \vec{u},
\]

(2)

where \( \nu \) is the kinematic viscosity and \( \vec{u} \) is the velocity vector.

Multiply \( \hat{n} \) on both sides of Eq. (2), we have

\[
\frac{\partial \omega}{\partial t} + \hat{n} \frac{\partial \omega}{\partial s} = \nu \hat{n} \cdot \nabla \cdot \vec{\omega} + \omega(\hat{n} \cdot \frac{\partial \vec{u}}{\partial t}),
\]

(3)
where $s$ and $l$ are the curvilinear coordinates along streamlines and vortex lines, respectively.

Multiply $\omega$ to (3), we then have the the governing equation of enstrophy $\omega^2/2$ as

$$\frac{\partial \omega^2}{\partial t} + u \frac{\partial \omega^2}{\partial s} = \nu \tilde{n} \cdot \nabla \omega + P,$$

where the enstrophy production $P$ is

$$P = \omega^2 \tilde{n} \cdot \frac{\partial \tilde{u}}{\partial t}.$$  \hspace{1cm} (5)

$\omega$ is a key parameter in understanding vortex dynamics and will be discussed in the following.

The vorticity equation is a fundamental equation in another form (compared with the N-S equations) in turbulence dynamics. Although the pressure term is absent, Eq. (3) is by no means simpler than the N-S equations. Efforts to study turbulence directly from governing equations need to be made with great discretion because of the inherent nonlinearity and sources of complexity from external boundary effects. For example the famous Kolmogorov theory is not based on the Navier-Stokes equations although some connections have been made [14]. Instead in the following some turbulence properties will be studied from a geometrical viewpoint.

Vectorline segment analysis helps to understand vector fields by studying the properties of vectorline or vectortube segments, a fundamental geometrical structure in turbulence [13]. Taking streamlines as example, from any grid point its streamline can be determined by the local velocity vector, as stated in Eq. (1). Tracing along a streamline in the ascending and descending directions of the velocity magnitude, a local maximum and minimum can finally be reached, respectively, as shown in Fig. 1 (a). The part of the streamline bounded by the two adjacent extrema (one max. and one min.) is defined as the streamline segment with respect to the given grid point. Furthermore, segments can be classified to be either positive or negative, if the velocity magnitude $u$ increases or decreases along the $\tilde{u}$ direction, respectively. Streamtube segments are the volumetric representation of streamline segments. As shown in Fig. 1 (b), from each grid

![Figure 1: Schematic explanation of (a) streamline segments; (b) streamtube segments.](image)
point in a homogeneous point array, streamtubes encompassing streamline segments can be configured in such a way that all the streamtube segments are non-overlapping and space filling. In this sense, the streamtube segment partition provides a space-filling structure and a well defined length scale, i.e. the arclength of segments. The volumetric streamtube segment structure ensures the meaningfulness of field statistics. Although the boundaries of the individual streamtube segments are undetermined, theoretically by volume-weighted average, the final statistical results are independent of the boundary demarcation.

In the following a similar vortexline/vortextube segment structure is put forward to attack the turbulence vorticity field.

3 Topological feature

First of all, some primary topological features need to be investigated from direct numerical simulation (DNS) data analysis. The flow configuration to perform DNS is a homogeneous shear flow as described in [13]. Table 1 shows the characteristic flowing parameters.

| grid points | $d(u_1)/dx$ | viscosity | $k$ | $\varepsilon$ | $\lambda$ | $Re_\lambda$ |
|-------------|-------------|-----------|-----|--------------|----------|------------|
| 2048$^3$    | 1.5         | 0.0009    | 3.5 | 1.16         | 0.160    | 295.0      |

In accordance with the case of streamtube segments, the primary parameters in describing vortextube segments are the arclength $l$ and $\Delta \omega$, the difference of $\omega$ at two extremal points. More specifically

$$\Delta \omega \equiv \omega_e - \omega_s,$$

where the subscripts $e$ and $s$ stand for the ending and starting points, respective, if following the $\vec{\omega}$ vector. It is obvious that for the positive segments, $\Delta \omega$ is positive and vice versa. Fig. 2 shows the probability density function (PDF) of the segment length $l$ both in the linear-linear and log-linear representations. For large segments the length PDF approximately decays exponentially. The length PDF of vortextube segments is similar to that of streamtube segments, which may suggest some relationship of these two vector variables. Fig. 3 is the joint PDF of $l$ and $\Delta \omega$. Compared with the streamtube segment case, it is symmetrical. For streamtube segments, because of the kinematic effect, positive and negative units are inclined to be stretched and compressed, respectively, to lead to the joint PDF unsymmetrical, while for vortextube segments this unsymmetry disappears due to the absence of the kinematic mechanism.

Velocity and vorticity are two interplayed vector field variables, whose relation-ship may be revealed in vectortube segment analysis. As illustrated in Fig. 4 (a), starting from any grid point, there exist one streamtube segment and one vortextube segment. The joint PDF of the lengths of the two different segments is calculated, shown in Fig. 4 (b). Overall these two lengths are positively correlated, i.e., grid points with longer streamline segments are more possible to have longer vortex line segments as well and vice versa.
Figure 2: The marginal PDF of $l$. Figure 3: The joint PDF of $l$ and $\Delta \omega$.

Figure 4: (a) Illustration of the connection of a streamline segment and vortex line segment at a same grid point; (b) The joint PDF of the lengths of these two different segments.
Figure 5: The PDFs of $\Delta u$ conditioned on different $\Delta \omega$.

4Conditional Statistics

The vectortube segment analysis provides a natural partition of flow fields. Turbulent statistics conditioned on different segments are enlightening to understand finer properties of the system to be studied. In the present conditional analysis, vortex tube segments are classified into different groups according to the respective $\Delta \omega$. In studying turbulence dynamics, the enstrophy product term $P$ defined in Eq. (5) plays an outstanding role. $P$ includes the interaction between vorticity, velocity and strain (velocity gradient). It is helpful to understand $P$ in the context of vortex tube segment analysis, by studying the PDF of velocity difference, $\Delta u$, and the strain rate difference, $\Delta a$, conditioned on different magnitudes of $\Delta \omega$, where $a = \frac{\partial \boldsymbol{u} \cdot \hat{n}}{\partial l}$ is the variation of velocity projection $\boldsymbol{u} \cdot \hat{n}$ along the arclength coordinate $l$.

Similar to the definition of $\Delta \omega$ in Eq. 6, $\Delta u = u_e - u_s$ and $\Delta a = a_e - a_s$. Fig. 5 shows the statistics of positive segments. Results for negative segments are almost identical. It can be seen that for segments with small absolute value of $\Delta \omega = \omega_e - \omega_s$, the PDFs of $\Delta u$ are quite symmetrical; while for segments with larger $|\Delta \omega|$, the PDFs are more skewed. More specifically, both for positive and negative segments with large $|\Delta \omega|$, $\Delta u$ is more probably positive. $\Delta u$ is an index reflecting the meaning stretching effect and thus it suggests that on average vortex tube segments with large difference of $\omega$ tend to be elongated, while segments with small $|\Delta \omega|$ at two extremal points statistically are weakly influenced by the strain action.

It is generally believed that vorticities are more stretched than compressed [5][15], which has been widely used in theoretical analysis and modeling. For instance a stretched spiral vortex model by Lundgren [15] can reasonably predict the some features of the energy spectrum. From the present result, it seems that the it will be more meaningful to relate $\Delta \omega$ at the extremal points of vortextube
segments with the stretching effect, other than considering the magnitude of \( \omega \) itself. Segments with large \( \omega \) may also have small different of \( \omega \).

The above conditional statistics are also helpful to understand the enstrophy production and related topics. In statistically stable turbulent flow field, the production term in Eq. (4) need to be positive to sustain enstrophy. By observing the PDF of the angle between eigenvectors of the strain rate tensor and \( \vec{\omega} \), Tsi-nober [5] argued the positiveness of the enstrophy production. One of the most basis phenomena and distinctive features of 3D turbulence is the predominant vortex stretching, which is manifested in positive net enstrophy production.

The enstrophy production \( P \) in Eq. (5) can be written as

\[
\omega^2 \vec{n} \cdot \frac{\partial \vec{u}}{\partial l} = \omega^2 \frac{\partial (\vec{u} \cdot \vec{n})}{\partial l} - \omega^2 \vec{u} \cdot \frac{\partial \vec{n}}{\partial l}. \tag{7}
\]

Because of randomness of segments' tangential direction in turbulence, the last orientation term can be considered as fluctuation and on average negligibly small. Therefore the remaining part can be well explained by the conditional statistics results from Fig. 5. For segments with small \( |\Delta \omega| \), the symmetry of PDF in Fig. 5 lead to small \( \omega^2 \frac{\partial (\vec{u} \cdot \vec{n})}{\partial l} \), while the main contribution to the positiveness of \( P \) is from segments with large \( |\Delta \omega| \). Overall, the net product must be positive. Furthermore, a more general conclusion hereof is for any positive integer \( n \),

\[
\omega^n \frac{\partial (\vec{u} \cdot \vec{n})}{\partial l} > 0. \tag{8}
\]

A counterpart to check the case with \( n = 1 \) in the Cartesian coordinate system is \( |\omega|a \), where \( i = 1, 2, 3 \) correspond to three Cartesian directions. The calculated results are 4.91, 3.18 and 3.79 in \( x, y \) and \( z \), respectively.

The vorticity-strain relation is also checked in reactive turbulent flows. The PDF of the orientation angle \( \theta = \tan^{-1}(\frac{\omega \times S \cdot \omega}{\omega \cdot S \cdot \omega}) \) has been calculated [16], where \( S \) is the strain rate tensor. It is found that this PDF reaches maximum at \( \cos(\theta) = 1 \), a consistent outcome with the statistical results in Fig. 5.

5 Conclusions

Vorticity and velocity vectors are two most important vector field variables in turbulence. Some geometrical properties have been studied using streamtube segment analysis. Similarly, in the present work the vortextube segment analysis has been conducted to address part of the kinematic and dynamics features of the vorticity field. The primary parameters in characterizing vortex tube segments are the arclength \( l \) and \( \Delta \omega \), difference of two extremal \( \omega \). The length PDF approximately decays exponentially in the large \( l \) range, which is similar to the streamtube segment case. However, the joint PDF of \( l \) and \( \Delta \omega \) is symmetrical, different from the streamtube segment case.

Conditional statistics have been studied according the the different magnitudes of \( \Delta \omega \) of vortextube segments. For segments with large \( |\Delta \omega| \), \( \Delta \omega \) is more probably positive. The index \( \Delta \omega \) reflects the meaning stretching effect and thus it can be concluded that that on average vortextube segments with large \( \omega \) difference
tend to be elongated. This property can explain the positiveness of the enstrophy production in turbulence. The parameter defined in (8) is a generalization of the enstrophy production case.

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