\[ \mathcal{N} = 2 \] strings and the twistorial Calabi-Yau

Andrew Neitzke and Cumrun Vafa

Jefferson Physical Laboratory, Harvard University, Cambridge, MA 02138, USA
neitzke@fas.harvard.edu
vafa@physics.harvard.edu

Abstract

We interpret the A and B model topological strings on \( \mathbb{CP}^{3|4} \) as equivalent to open \( \mathcal{N} = 2 \) string theory on spacetime with signature (2, 2), when covariantized with respect to \( SO(2, 2) \) and supersymmetrized a la Siegel. We propose that instantons ending on Lagrangian branes wrapping \( \mathbb{RP}^{3|4} \) deform the self-dual \( \mathcal{N} = 4 \) Yang-Mills sector to ordinary Yang-Mills by generating a `t Hooft like expansion. We conjecture that the A and B versions are S-dual to each other. We also conjecture that mirror symmetry may explain the recent observations of Witten that twistor transformed \( \mathcal{N} = 4 \) Yang-Mills amplitudes lie on holomorphic curves.
1 Introduction

In a beautiful recent paper [1], Witten provided highly non-trivial evidence that the planar amplitudes of $\mathcal{N} = 4$ supersymmetric $U(N)$ Yang-Mills theory in four dimensions, once transformed to the twistor space $\mathbb{CP}^{3|4}$, are supported on holomorphic curves (see also the followup work [2–4].) This work was based on the spinor formalism for gauge theory amplitudes; see e.g. the early work [5–7] and more recent papers [8, 9]. Since holomorphic curves are traditional hallmarks of string theory, it was attempted in [1] to provide a tentative string theory interpretation of their appearance. This interpretation involved the topological B model on the Calabi-Yau manifold $\mathbb{CP}^{3|4}$, with $N$ D5-branes wrapping the Calabi-Yau. The computation of the amplitude then reduced to a current algebra calculation on the holomorphic curves as suggested long ago in [10]. However, the objects lying on the holomorphic curves were proposed to be closed D1-instantons.

There were a number of puzzles associated with these D1-instantons: i) The amplitude came out as a holomorphic $n$-form on an instanton moduli space of complex dimension $n$; such a form can be integrated only after choosing a contour, but it was not clear a priori which contour to choose; ii) Only the planar gauge theory amplitudes were considered; it was not clear how non-planar contributions could fit in (by this we mean ‘t Hooft diagrams with handles [11]).

It could be that somehow a natural real cycle in the instanton moduli space is identified, or alternatively that all nontrivial choices are somehow equivalent, thus solving problem i). As for problem ii), it could be that this story is just a planar story, and the non-planar diagrams just do not work in such a simple way. In particular, it could be that the non-planar diagrams cannot be decoupled from the gravity sector and consistency of the theory would require a fixed finite rank gauge group (a possibility we will briefly discuss in the context of orientifolding the string realization, although it seems unlikely since there is no sign of any pathology for arbitrary rank $\mathcal{N} = 4$ gauge theory on its own.) However, it would be nice if one could find an alternative mechanism to solve these two puzzles, which would enhance the usefulness of the twistor transform for arbitrary rank classical gauge groups beyond the planar limit. The main content of this paper is an exploration of one possible mechanism. We will see that this possibility leads to some intriguing ideas which, while far from proven,
could merit further investigation. Moreover, some of the ideas we encounter may be relevant even if the original puzzle is solved in other ways.

Here is the mechanism. If somehow the D1-instantons were required to end on the Lagrangian submanifold $\mathbb{RP}^{3|4} \subset \mathbb{CP}^{3|4}$, then the dimension of the moduli space would naturally be reduced by half. Moreover, this could also solve the second problem; the topology of the instantons would be characterized by two parameters instead of one (the number of boundaries on $\mathbb{RP}^{3|4}$ and the number of handles), so one could read off a ‘t Hooft-like expansion. For this ‘t Hooft counting to work out correctly, we would need to have $N$ branes wrapping $\mathbb{RP}^{3|4}$. However, this raises a puzzle: What is the meaning of these Lagrangian branes in the B model? Certainly they cannot be D-branes as that would violate the BRST symmetry of the B model.

The need for Lagrangian branes would suggest an A model interpretation, where we have $N$ Lagrangian D-branes wrapping on $\mathbb{RP}^{3|4}$ and the D1-brane instantons are replaced by worldsheet instantons. However, there would be a dual puzzle in this context [1]: There are too few states on the Lagrangian D-brane to support the fields of $\mathcal{N} = 4$ Yang-Mills\textsuperscript{1}. The moduli space of flat connections on $\mathbb{RP}^{3|4}$ seems to support at most a finite number of states and no continuous fields.

Here we suggest a tentative resolution to these puzzles, starting in the B model. We propose that there could be a gravitational tadpole in the B model which needs to be cancelled, and that this cancellation can be accomplished by the introduction of $N$ Lagrangian branes (not D-branes) wrapped on $\mathbb{RP}^{3|4}$. Moreover, if we assume the existence of these Lagrangian branes (which we call “NS2-branes”) with their natural couplings, the D1-branes can end on them. We interpret these open D1-instantons as the relevant ones for deforming self-dual Yang-Mills to the full Yang-Mills, following the picture of [1]. This would resolve the issue in the B model context.

This B model story is quite similar to the A model picture which we first considered above, and we make the tentative suggestion that the two are in fact related by S-duality. In other words, we propose that the Montonen-Olive duality at the level of topological strings converts the B model into an A model on the same Calabi-Yau. In this language we can

\textsuperscript{1}We would like to thank E. Witten and H. Ooguri for discussions on this point.
see a possible resolution to the problem of the A model not having enough physical states: we now have an extra ingredient, namely the $\mathbb{CP}^{3|4}$ filling branes which are S-dual to the D5-branes of the B model. Borrowing the terminology of type IIB superstrings, we will call these “NS5-branes”.

The fact that $\mathbb{RP}^{3|4}$ features prominently here as the locus where instantons can end suggests that this is the natural arena for the comparison to gauge theory. In other words, we rephrase the observations of [1] as the statement that the amplitudes of $\mathcal{N} = 4$ supersymmetric Yang-Mills on spacetime with signature $++--$, when twistor-transformed to functions on $\mathbb{RP}^{3|4}$, are supported on the real boundaries of holomorphic curves in twistor space $\mathbb{CP}^{3|4}$. Then the holomorphic curves in question would be identified as the boundaries of instantons ending on the Lagrangian branes (either D1-instantons or worldsheets depending on whether we use the A or B model.) In fact, it was already noted in [1] that $\mathbb{RP}^{3|4}$ can arise naturally if one specializes to the signature $++--$. This signature is also natural from the viewpoint of $\mathcal{N} = 2$ strings [12], and as we will explain in the next section, we believe that the topological theory in twistor space $\mathbb{CP}^{3|4}$ is indeed equivalent to $\mathcal{N} = 2$ strings.

It is also natural to ask what role mirror symmetry can play in this story. We have proposed that S-duality exchanges the A and B models on the same twistorial Calabi-Yau. On the other hand, mirror symmetry also exchanges the A and B model, but also changes the Calabi-Yau in question. We propose that mirror symmetry, together with this S-duality, may explain from first principles why the amplitudes of $\mathcal{N} = 4$ Yang-Mills are localized on boundaries of holomorphic curves on twistor space. Mirror symmetry would map the worldsheets of A model to field theory computations of a B model on a different Calabi-Yau, which we conjecture to be the quadric on $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$. The reason for this conjecture is simply that this space is already known to lead to a twistorial formulation of the full $\mathcal{N} = 4$ Yang-Mills theory including all interactions, at least at the level of the classical equations of motion, without requiring any instanton corrections [13], as was re-emphasized in [1].

We have suggested that the natural signature for the $\mathcal{N} = 4$ super Yang-Mills theory in this twistor correspondence is the $++--$ signature, which is precisely the one arising
naturally for $\mathcal{N} = 2$ strings. Now what about the gravitational sector? The gravity version of this theory has been suggested in [1] to be $\mathcal{N} = 4$ conformal supergravity. This can in principle be reconciled with an $\mathcal{N} = 2$ closed string theory, with signature $++--$. This would be viewed as the instanton corrected version of $\mathcal{N} = 4$ self-dual conformal gravity [14]. This also explains the fact that the propagators of the gravity theory for $\mathcal{N} = 2$ strings are expected to be $1/(k^2)^2$ [12] as is the case for conformal gravity. In this context it is amusing to note that a non-perturbative definition of the target space gravity theory of the topological A model has been proposed in [15, 16]; there the target space theory was described as a gravitational quantum foam. This would suggest a gravitational quantum foam picture in the A model on $\mathbb{CP}^{3/4}$; the body of this space, $\mathbb{CP}^3$, projects to the tetrahedral crystal. It would be very interesting to clarify the nature of this gravitational quantum foam for this topological theory with the additional NS5-branes, as it would correspond to the twistor transform of ordinary gravitational foam in $\mathcal{N} = 4$ conformal supergravity in four dimensions.

The organization of this paper is as follows. At the end of this section we summarize how $\mathcal{N} = 2$ strings enter the picture, as the relevant literature seems to be unfamiliar to most readers, and is one of the main motivations of this paper. In Section 2 we briefly review topological A and B model strings with D-branes included. In Section 3 we argue for the existence of extra branes in the A and B models. In section 4 we show how these additional branes affect the twistorial Calabi-Yau and we propose a role for S-duality. In section 5 we discuss the possible role of mirror symmetry in this story. In section 6 we discuss some additional issues.

1.1 Historical background

The string worldsheet can be viewed as a theory of 2d gravity coupled to matter. The corresponding 2d gravity theories can be classified by the number of supersymmetries $\mathcal{N}$. Bosonic strings and superstrings correspond to $\mathcal{N} = 0$ and $\mathcal{N} = 1$ worldsheet supersymmetry respectively, and the heterotic string is where left-movers and right-movers have $(\mathcal{N}_L, \mathcal{N}_R) = (0, 1)$. The case with 2 worldsheet supersymmetries (known as $\mathcal{N} = 2$ string theory) has been studied far less. It was first discussed in [17,18]. Its target space meaning was clarified in [12], where it was shown that the closed string sector leads to self-dual gravity, and in [19, 20].
where it was shown that the heterotic or open string sector corresponds to self-dual Yang-Mills, in four dimensions with signature $++--$. Moreover, it was shown that for this string theory the infinite tower of string oscillations are absent from the physical spectrum (they are BRST trivial). This is what one would ordinarily expect from a non-critical string theory, such as $c \leq 1$ bosonic strings. Indeed, in a sense, for $\mathcal{N} = 2$ strings the “non-critical” theories are equivalent to “critical” theories (the analog of the gap $1 < c < 25$ for bosonic string is absent for $\mathcal{N} = 2$ strings).

On the other hand, topological strings on Calabi-Yau 3-folds, or more precisely $\mathcal{N} = 2$ topological strings, were introduced in [21]. The construction of the $\mathcal{N} = 2$ topological string was modeled after bosonic string theory, namely, a twisted $\mathcal{N} = 2$ worldsheet supersymmetry was used to construct a bosonic-string-like BRST complex. This $\mathcal{N} = 2$ topological string has two versions [22, 23], known as the A and B models, exchanged by mirror symmetry. It was shown in [24] that the open topological strings in the A version lead to ordinary Chern-Simons theory on Lagrangian submanifolds of a Calabi-Yau space, while the B version leads to holomorphic Chern-Simons theory on the full Calabi-Yau. The corresponding closed string theory in the B model was studied in [25], leading to a gravity theory quantizing complex structure of Calabi-Yau threefolds. In any case, the target space description always involves only finitely many fields.

Since the $\mathcal{N} = 2$ string also involves only finitely many target space fields, it was natural to expect [25] that the $\mathcal{N} = 2$ string is also related to some kind of topological string; this was indeed shown to be the case in [26]. The theory in question is the $\mathcal{N} = 4$ topological string theory, so named because it is connected to a twisted version of $\mathcal{N} = 4$ worldsheet supersymmetry. Moreover, this formulation leads to vanishing theorems [26] for perturbative $\mathcal{N} = 2$ string amplitudes for $n > 3$ points on flat space at all loops. It is also useful computationally, leading in certain cases to computations of amplitudes at all loops [27].

Thus both the $\mathcal{N} = 2$ and the $\mathcal{N} = 4$ topological strings emerged as string theories for which string oscillations are absent from the physical spectrum. It was suggested in [25] (pages 140-141) that these two theories are indeed also related to each other. The motivation for this was that in the open $\mathcal{N} = 2$ topological string one finds holomorphic Chern-Simons theory on a three complex dimensional Calabi-Yau space (topological B model), while on...
the other hand $\mathcal{N} = 2$ strings, via the twistor transform, map self-dual Yang-Mills geometry to holomorphic bundles in twistor space [28] which is also of complex dimension 3. However, there was an obstacle to realizing this idea: the twistor space is $\mathbb{CP}^3$ which is not a Calabi-Yau threefold.

A few ideas emerged to resolve this puzzle [29–31]: $\mathcal{N} = 2$ strings naturally lead to a theory in 2 complex dimensions [12], with Lorentz group $U(1,1)$. Although this is not necessarily a problem, it was suggested in [29] that one could try to recover Lorentz invariance by “integrating” over the choices of $U(1,1) \subset SO(2,2)$, which leads to bringing in the twistor sphere as part of the physical space. However, it was found that this was not possible unless one also added fermions and supersymmetrized the theory to $\mathcal{N} = 4$ in the target space. This echoes the fact that $\mathbb{CP}^3$ is not a Calabi-Yau manifold but $\mathbb{CP}^3|_4$ is, as was recently exploited in [1]; thus there is a possibility to unify the two topological strings, as also discussed by [31, 32]. We believe that the results of [1] should be viewed as a step toward such a unification.

Together with the recent observations in [33] that $\mathcal{N} = 2$ topological strings are equivalent to non-critical bosonic strings and superstrings, we can view the equivalence of $\mathcal{N} = 2$ strings (which can also be viewed as “non-critical”) with $\mathcal{N} = 2$ topological strings as an extension of the principle that all non-critical strings are unified by topological strings$^2$.

## 2 Brief review of A and B models

In this section we provide a brief review of A and B model strings on Calabi-Yau manifolds. The reader can consult [35] for a more thorough review. To make the discussion simple we first discuss the case which has been most studied, namely the case where the Calabi-Yau is an ordinary (bosonic) manifold.

These topological models are defined by topological twisting of the $\mathcal{N} = 2$ supersymmetry

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$^2$It is amusing to note that the dimension of the critical topological string is correlated with the upper limit of the central charge of non-critical strings. However, depending on how the twisting is done there are two choices for the $\hat{c}$ of the corresponding $\mathcal{N} = 2$ topological theory [34]. In particular, the $c = 1$ non-critical bosonic string gets mapped to $\hat{c} = 3$ and $\hat{c} = -1$, which are precisely the cases of most interest in topological strings (the latter being explored here and in [1]).
on the worldsheet of the supersymmetric sigma model with Calabi-Yau target space. In the closed string case, this twisting just amounts to shifting the spins of all worldsheet fields by $J \rightarrow J + \alpha/2$, where choosing $\alpha$ to be the $U(1)_R$ or $U(1)_A$ charge defines the A or B model respectively. After this twist one of the two supercharges becomes scalar and can therefore be used as a BRST operator. Correlation functions of BRST-invariant operators are then independent of worldsheet position, and depend on the background in a very restricted way — namely they depend only on the Kähler moduli (A model) or complex moduli (B model).

Most importantly for our purposes, the fermionic BRST symmetry acting on the sigma model field space implies that correlation functions are localized on BRST-invariant field configurations. In the A model these are holomorphic maps of the worldsheet into the target (worldsheet instantons), while in the B model they are simply the constant maps. In this sense the B model reduces to classical geometry of points while the A model admits more complicated corrections from holomorphic curves.

### 2.1 Open string case

The A and B models also admit BRST-invariant boundary conditions for open strings. In the A model such a boundary condition is obtained by restricting the endpoints of the string to lie on a Lagrangian subspace $Y$ of the Calabi-Yau manifold $X$. In the B model the endpoints are coupled instead to a holomorphic bundle, supported on some holomorphic cycle inside the Calabi-Yau.

The corresponding open string field theories were described in [24]. Let us first discuss the A model. In this case the BRST cohomology on the worldsheet gets identified with the ordinary de Rham cohomology on $Y$. Therefore the target space fields can be packaged together as

$$A = A_0 + A_1 + \cdots + A_m.$$  \hfill (2.1)

Here $A_k$ is a $k$-form on $Y$, and $m$ is the real dimension of $Y$.

The target space action consists of two parts. One part is the Chern-Simons theory on $Y$, with action

$$S_{CS} = \frac{1}{g_s} \int_Y \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).$$  \hfill (2.2)
This part of the action can be considered as coming from disc diagrams in which the worldsheet has degenerated to a Feynman diagram; these configurations lie at the boundary of field space and were called “virtual instantons” in [24]. In this Chern-Simons theory the one-form part of $A$, $A_1$, is viewed as a $U(N)$ connection (if we have $N$ branes), while the other $A_k$ can be viewed as the usual ghost fields which arise in covariant quantization of Chern-Simons [36, 37].

The second contribution to the action comes from honest holomorphic maps $(\Sigma, \partial \Sigma) \rightarrow (X, Y)$. Each such instanton modifies the action by adding a Wilson loop term, giving

$$S_{\text{inst}} = \sum_i C_i e^{-t_i} \text{Tr} \text{Pexp} \oint_{\gamma_i} A, \quad (2.3)$$

where $t_i$ is the area of the $i$-th instanton and $\gamma_i$ is its boundary inside $Y$. The coefficients $C_i$ will depend on the details of the map. The full target space field theory can thus be viewed as the Chern-Simons theory coupled to a background source of Wilson lines, determined by the worldsheet instantons. Thus the full action is given by

$$S = S_{\text{CS}} + S_{\text{inst}}. \quad (2.4)$$

The story is generally similar in the B model, with one important difference. In this model, as we described above, the only BRST invariant field configurations are the constant maps. Therefore there are no worldsheet instanton corrections; the full open string theory for any Calabi-Yau target space is reduced to a holomorphic version of Chern-Simons, with action

$$S = \frac{1}{g_s} \int_X \Omega \wedge \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \quad (2.5)$$

For the supermanifold case the same formulations should work, although there are important technical issues to be overcome, as has been recently discussed in the context of the B model in $\mathbb{C}P^{3/4}$ [1].

3 Extra branes for A and B models?

In this section we discuss the possible existence of extra branes in the A and B models. We will be mainly focusing on the case of a bosonic Calabi-Yau 3-fold, but it should be possible to generalize our discussion to supermanifolds such as $\mathbb{C}P^{3/4}$. 
Let us start with the B model. This model has D-branes of all even (Euclidean) dimensions. In particular we have D1-branes. It has been known that D1-branes play a privileged role among the branes of the B model \([33, 38]\): they are a source for the holomorphic 3-form. In other words, if we have a 3-cycle \(C_3\) surrounding \(N\) D1-branes, the period of the holomorphic 3-form \(\Omega\) over this 3-cycle is deformed by the branes,

\[
\Delta \int_{C_3} \Omega = N g_s. \tag{3.1}
\]

It is natural to introduce the 3-form

\[
C = \frac{\Omega}{g_s}. \tag{3.2}
\]

Then (3.1) can be written as

\[
\Delta \int_{C_3} C = N. \tag{3.3}
\]

There is a better way to write this relation. Let us introduce a 2-form field \(B\) under which the D1-brane is charged\(^3\). Then the above relation can be summarized by saying that the action takes the form

\[
S = S_{KS} + \int C \wedge dB, \tag{3.4}
\]

where \(S_{KS}\) is the Kodaira-Spencer action \([25]\). One can easily check that the addition of this term leads, upon variation of the Kodaira-Spencer field, to (3.3).

So far we have been discussing the B model. In the A model context the same kind of relation holds (and has been related to the large \(N\) duality between Chern-Simons and closed topological strings \([40]\)): a D2-brane wrapped over a Lagrangian 3-cycle \(Y\) changes the integral of the Kähler form \(k\) over a 2-cycle surrounding \(Y\). In this case we introduce a gauge potential \(\tilde{C}\) which couples to the D2 brane, and a normalized Kähler form \(\tilde{B} = k/g_s\), and then the action includes

\[
S = \ldots + \int \tilde{B} \wedge d\tilde{C}. \tag{3.5}
\]

Note that this is exactly the same as (3.4) with the identifications

\[
B \leftrightarrow \tilde{B}, \tag{3.6}
\]

\[
C \leftrightarrow \tilde{C}. \tag{3.7}
\]

\(^3\)The need for such a field in the gravity sector of the twistorial Calabi-Yau was suggested in \([39]\).
We will later propose that this fact is not accidental but rather a consequence of S-duality, in the specific example of $\mathbb{CP}^{3|4}$.

It is natural to ask whether topological D-branes can end on other branes. Given that this does happen in the superstring, one might expect that the same thing also happens in the topological string context. In particular, the computation of F-terms for superstring compactifications including branes ending on branes should be captured by some topological theory. For D1-branes, the only way this can happen (consistent with holomorphy, which is a consequence of the BRST symmetry of the B model) is for the branes to end on Lagrangian submanifolds. What is the nature of these Lagrangian submanifolds? They certainly cannot be D-branes, since this would violate the BRST symmetry. The Lagrangian submanifold must be supporting some new kind of brane of the B model; let us call this hypothetical brane an “NS2-brane.” If such a brane really exists in the B model, then it should naturally couple to the gravity sector. Since the gravity sector of the B model is rather poor in fields, the only natural guess for the coupling is

$$c \int_Y \Omega.$$  \hspace{1cm} (3.8)

Later we will identify the constant $c$ with $1/g_s$.

Now can a D1-brane end on this NS-brane? As is familiar from superstring theory, charge conservation requires that if branes $Z$ and $Z'$ can end on brane $Y$, the action must contain a Chern-Simons term of the form

$$S_{CS} = \int A \wedge F \wedge F'.$$  \hspace{1cm} (3.9)

Here $F$ and $F'$ denote the field strengths coupled to $Z$ and $Z'$, while $A$ is the corresponding gauge potential for the field coupled to $Y$. In the case at hand $Y$ is our Lagrangian NS-brane, $A$ is the 3-form $C$, and $F = dB$ is the 3-form field strength coupled to the D1. Hence $F'$ should be a closed 0-form, i.e. a constant function. So if D1-branes can end on NS2-branes, we would expect a term in the action of the form $\int C \wedge dB$; but as we have argued above, there is just such a term in the action. This is consistent with the assumption that these Lagrangian NS2-branes exist and that D1-branes can end on them.
4 Tadpoles on the twistorial Calabi-Yau and S-duality

In this section we discuss possible gravitational tadpoles for the B model on $\mathbb{CP}^{3|4}$ with space-filling D5-branes. We argue how such tadpoles can arise and propose possible cancellation mechanisms. We then discuss the relation between this picture and the twistor transform of [1]. Finally we introduce the notion of S-duality for the topological string, and suggest that this B model could in fact be S-dual to an A model on $\mathbb{CP}^{3|4}$.

So, consider a B model with $N$ D5-branes wrapped over $\mathbb{CP}^{3|4}$. In analogy with type I superstrings in 10 dimensions, we know that there could very well be gravitational tadpoles. A comprehensive study of such tadpoles would be important for this theory. Here we describe some possible tadpoles that can exist. One possibility would be a “bulk” tadpole — a gravitational mode which couples to the total brane charge, much as in the type I superstring. Another possibility would be an “induced” tadpole which gives rise to a lower-dimensional brane.

Let us first discuss the possibility of a bulk tadpole. In the case of bosonic Calabi-Yau 3-folds, the only canonical choice would be a brane generated term of the form

$$\int \Omega \wedge \overline{\Omega}. \quad (4.1)$$

Even though $\overline{\Omega}$ is not a field of the B model and depends on anti-holomorphic moduli, one could nevertheless countenance the existence of such a term because of the holomorphic anomaly [25]. If such a term is indeed generated, the only known mechanism in the superstring context to cancel it is the introduction of orientifold planes. Indeed, it is known that orientifold planes introduce brane charge also in topological string theories [41]. In particular, the orientifold action which fixes the full Calabi-Yau gives a brane charge of $\pm 8 = \pm 2^3$ which leads to the introduction of 8 branes and an $SO(8)$ gauge group in the bulk. It is conceivable that a similar mechanism is at work for Calabi-Yaus which are supermanifolds. If so, this would give a fixed rank gauge group. It would be somewhat surprising, however, since $\mathcal{N} = 4$ super Yang-Mills seems to be well-behaved and non-anomalous independent of the chosen gauge group.\footnote{Also, for the branes of [1] the orientifold action presumably would fix just the brane, which is actually not quite space-filling; it has only 4 real fermionic directions. In such a case the analog of $\int \Omega \wedge \overline{\Omega}$ is not clear.}
For now let us assume either that there are no “bulk” tadpoles, or at least that defining a decoupled gauge theory sector only requires cancellations of lower dimensional “induced” tadpoles. Now we consider these. As we have reviewed, on the worldvolume of D5 branes we have the action

\[
\frac{1}{g_s} \int_{\mathbb{CP}^{3|4}} \Omega \wedge CS(A) \quad (4.2)
\]

One can also ask whether these branes induce any gravitational backreaction, which would be present even if \( A = 0 \). In particular, since the topological string at the disc level computes superpotential terms for superstring compactifications on the Calabi-Yau \cite{25}, and since we expect that superpotential to vanish when the spin connection \( \omega \) is identified with the gauge connection \( A \) \cite{42}, one would expect that the full disc contribution is

\[
\frac{1}{g_s} \int_{\mathbb{CP}^{3|4}} \Omega \wedge [CS(A) - CS(\omega)] , \quad (4.3)
\]

where \( \omega \) denotes the \((0,1)\) part of the spin connection. The existence of such additional gravitational Chern-Simons terms is also natural from the perspective of ordinary Chern-Simons theory \cite{43}; there such a gravitational term is needed to ensure topological invariance is not spoiled quantum mechanically. However, in that case the gravitational Chern-Simons arises with a coefficient given by the central charge of the WZW model, which in the \( U(N)_k \) case would have been \( kN^2/(N+k) \). In view of this it would be crucial to check in the string theory under consideration what is the precise coefficient of the above term. Let us assume that this term is indeed generated at the level of the disc amplitude as in \((4.3)\).

To write \((4.3)\) in a more convenient form, note that \( CS(\omega) \) should be related to a Poincare dual \((3|4)\) cycle. While a full theory of “superhomology” is apparently lacking, there is one natural candidate for this cycle. Namely, choosing an antiholomorphic involution on \( \mathbb{CP}^{3|4} \), the fixed locus gives an \( \mathbb{RP}^{3|4} \), which on general grounds should be a special Lagrangian cycle and in particular should be nontrivial in homology. Let us assume that this candidate cycle is indeed dual to \( CS(\omega) \) (with coefficient 1, say, though the exact coefficient is inessential.) Then we would have\(^5\)

\[
- \frac{1}{g_s} \int_{\mathbb{CP}^{3|4}} \Omega \wedge CS(\omega) = - \frac{1}{g_s} \int_{\mathbb{RP}^{3|4}} \Omega . \quad (4.4)
\]

\(^5\)We are glossing over some formidable mathematical subtleties here. As usually defined, the holomorphic Chern-Simons action is not gauge invariant and really makes sense only modulo the periods of \( \Omega \) — see e.g. \cite{44}. Nevertheless the question of whether or not there is a tadpole should have a well-defined answer.
Recall that $\Omega$ is the Kodaira-Spencer field which describes the gravitational sector of the B model [25]. Thus (4.4) can be interpreted as a gravitational tadpole created by a single D5-brane. Letting $C = \Omega/g_s$, for $N$ D5-branes we have

$$-N \int_{\mathbb{RP}^{3|4}} C. \tag{4.5}$$

So the only way this B model with D5-branes can be consistent is if it has an additional term in the action involving $C$. But as we have argued in the previous section, there is a candidate: namely, we proposed that the B model could include an NS2-brane coupled to $C$. Moreover, we have argued that, if NS2-branes do exist, D1-branes can end on them. We are thus led to introduce $N$ of these NS2-branes wrapping the Lagrangian cycle $\mathbb{RP}^{3|4}$, which would cancel the tadpole (4.5).

### 4.1 Twistor transform of $\mathcal{N} = 4$ super Yang-Mills revisited

After our discussion above, we can now revisit the D1-brane instantons studied in [1]. Since D1-branes can end on the NS2-branes, we obtain a natural real cycle inside each moduli space of D1-branes. The geometrical picture is that the closed Riemann surfaces considered in [1] should be considered as doubled versions of the open D1-branes ending on $\mathbb{RP}^{3|4}$. The key point is that not every closed Riemann surface is obtained in this way. To characterize the ones which are obtained by doubling, note that the antiholomorphic involution which acts on $\mathbb{CP}^{3|4}$ with fixed locus $\mathbb{RP}^{3|4}$ also acts naturally on the instanton moduli space. The instantons obtained by doubling are exactly the fixed points of this involution; these form a real cycle in the moduli space as needed.

In particular, the contributions considered in [1] with genus $g$ get mapped in our picture to D1-instantons with no handles and with $b = g + 1$ boundaries ending on $\mathbb{RP}^{3|4}$. This is exactly the same topology as the corresponding $\mathcal{N} = 4$ ‘t Hooft diagram the instanton is computing! This is also consistent with the fact that each boundary of the instanton contributes a factor of $N$, coming from the degeneracy of the Lagrangian branes on which it ends, as would be required for the corresponding gauge theory diagram. Furthermore, this counting suggests the existence of a constant background field $g_{D1}$ coupled to the curvature.
of the $D1$ brane, which we identify with the square of the Yang-Mills coupling constant:

$$g_{YM}^2 = g_{D1}. \quad (4.6)$$

It is natural to ask whether $g_{D1}$ is the same as the B-model topological string coupling $g_s$. This would be reasonable in view of the fact that the tree level amplitude includes a $1/g_s$, which in the Yang-Mills theory should have corresponded to a $1/g_{YM}^2 = 1/g_{D1}$.

At first it seems however that this agreement is spurious, because a $D1$-brane instanton of degree $d$ should get an additional weight $e^{-A/d/g_s}$, where $A$ is the area of the degree 1 curve. This extra weighting can actually be absorbed using the condition given in [1] for instanton contributions to planar Yang-Mills amplitudes:

$$d = q + l - 1. \quad (4.7)$$

Here $q$ is the number of ingoing negative helicity gluons in the Yang-Mills scattering process, and $l$ is the number of loops in Yang-Mills which for planar diagrams is the same as the $l = b - 1$ where $b$ is the number of boundaries. In principle we can consider an instanton with an arbitrary number of handles $h$, and $b$ boundaries lying on $\mathbb{RP}^3|4$. We propose that the generalization of this formula to arbitrary $h$ is given by

$$d = q + 2h + b - 2 \quad (4.8)$$

Now we can see that we can absorb the factor $e^{-A/d/g_s}$ in two rescalings: we rescale

$$g_{D1} \rightarrow e^{A/g_s} g_{D1}, \quad (4.9)$$

and also rescale the vertex operators $V_-$ for all negative helicity gluons by

$$V_- \rightarrow e^{A/g_s} V_- \quad (4.10)$$

This triviality of the Kähler class is reminiscent of [45], where it was shown that the Kähler form is indeed decoupled from the BRST invariant observables in certain A models on supermanifolds.

Thus it is plausible that we can indeed set $g_s = g_{D1}$. At any rate, for the rest of the paper we need not assume this.
Note that there could be a new puzzle associated with the $N$ additional NS2-branes wrapping $\mathbb{RP}^{3|4}$. If these branes carry additional dynamical fields, it will ruin the perfect match with the field multiplet of $\mathcal{N} = 4$ super Yang-Mills, which was obtained in [1] just from the fields on the D5-branes. Clearly, a $U(N)$ gauge field lives on the NS2-branes (due to the fact that D1-branes can end on them). So we expect a $U(N)$ theory with no dynamical fields. The natural candidate is the Chern-Simons theory living on the branes; on $\mathbb{RP}^{3|4}$ the space of classical solutions of this theory is purely discrete.

4.2 S-duality and the A model

The D1 brane instantons we have been discussing are quite similar to the fundamental strings of the A model. In particular, the fact that they end on Lagrangian branes is reminiscent of fundamental strings ending on the D-branes of the A model. Furthermore, we have argued above that consistency requires these branes to support Chern-Simons gauge theory. It is thus natural to suggest that there is some dual description in which the A model picture is the right one. In this picture the “NS2-branes” get mapped to D2-branes and D1-branes get mapped to fundamental strings.

Independently, it is natural to ask how the Montonen-Olive duality $g_{YM} \leftrightarrow 1/g_{YM}$ acts on this picture. This question exists even if we have no tadpoles to cancel and the D1 instantons end up being closed Riemann surfaces. A priori there could be two natural answers. One possibility is that the picture is mapped back to itself. This we find unlikely, given the fact that the D1 string coupling constant is identified with $g_{YM}^2$, and as in type IIB duality we would have expected the duality to map the D1 string to the fundamental string.

We thus conjecture that the S-dual description of this model is in terms of A model topological strings, with $N$ D-branes wrapping $\mathbb{RP}^{3|4}$. However, this A model is novel in that it should also have $N$ additional “NS5-branes.” A correct quantization of the full system including these extra NS5-branes could then avoid the A model no-go theorem [1].

Actually, the fact that the A model can be related to D-instantons has already been encountered [46, 47]. In particular, the worldsheet instantons of the A model capture degeneracies of wrapped D2 branes, which can be viewed as Euclidean D1 brane instantons. This has also been extended to a relation between the degeneracy of D2 branes ending on branes
and A model worldsheets ending on Lagrangian D-branes [48]. If we want to interpret these computations completely within the topological string we would need an S-duality like the one we are proposing.

It would be natural to try to define the amplitudes involving instantons ending on branes directly from the A model perspective. By now there is a huge machinery for such computations for bosonic toric Calabi-Yau threefolds. The Calabi-Yau space $\mathbb{CP}^{3|4}$ is also toric, albeit super, and one would expect that suitable localization techniques could also work here.

## 5 Mirror symmetry

So far we have discussed a possible stringy twistorial formulation of $\mathcal{N} = 4$ supersymmetric $U(N)$ Yang-Mills coupled to $\mathcal{N} = 4$ conformal supergravity; in fact we proposed that there are two such formulations, both as A and B model topological string on $\mathbb{CP}^{3|4}$, in each case with $N$ space-filling branes and $N$ branes wrapping $\mathbb{RP}^{3|4}$. We have seen that instantons do contribute to various amplitudes in the twistor space. In the A-model case, the instantons are worldsheet instantons. As is well known, the main point of mirror symmetry is to compute precisely such worldsheet instanton contributions. Namely, in the mirror (not S-dual) B model description the instantons disappear; the field theory in the B model captures the worldsheet instantons of the A model. This strategy has been successfully applied to Calabi-Yau threefolds in the presence of Lagrangian branes, starting with [49] and culminating in a recent work [33]. Therefore, we would expect this strategy to work here as well.\footnote{For a recent discussion of mirror symmetry for $\mathcal{N} = 2$ strings see [50].}

How does mirror symmetry work for Calabi-Yau supermanifolds? In fact Calabi-Yau supermanifolds were first studied precisely for a better understanding of mirror symmetry [45]! Namely, the puzzle that rigid Calabi-Yaus cannot have ordinary Calabi-Yau mirrors (as the mirror would not have any Kähler classes) was resolved in [45] by extending the space of bosonic Calabi-Yaus to include Calabi-Yau supermanifolds. In particular, examples were found where the mirror of a rigid Calabi-Yau is a Calabi-Yau supermanifold, realized as a complete intersection in a product of projective supermanifolds $\prod_i \mathbb{CP}^{n_i|m_i}$. In such cases...
the superdimension of the manifold and the mirror agree (which is required because the superdimension determines the $\hat{c}$ in the $U(1)$ current algebra), but the individual bosonic and fermionic dimensions need not be equal. In particular it was found in [45] that a Calabi-Yau of dimension $(n|0)$ can be mirror to a Calabi-Yau with dimension $(n + d|d)$.

The techniques to derive mirror geometry from the linear sigma model description have been found in [51] in the context of bosonic Calabi-Yaus. To treat $\mathbb{CP}^{3|4}$ one would need to extend these techniques to the case of Calabi-Yau supermanifolds. This should be possible, given that $\mathbb{CP}^{3|4}$ does admit a simple linear sigma model realization. Here we would like to make a conjecture about what the mirror is. Whatever it is, it will carry the holomorphic version of Chern-Simons. Moreover, via a twistor transform, it should realize Yang-Mills perturbation theory directly from holomorphic Chern-Simons, since that theory has no corrections from worldsheet instantons. Therefore the full classical Yang-Mills should be realized completely in a classical way after the twistor transform. There is only one such formulation presently known: classical solutions of the full Yang-Mills can be identified with holomorphic data on the quadric in $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$ [13]. We thus conjecture that the mirror of $\mathbb{CP}^{3|4}$ is a quadric in $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$. First of all, note that this quadric is a Calabi-Yau supermanifold. This is already a non-trivial test. In fact, it was already conjectured in [1] that the B model on this quadric could reproduce $\mathcal{N} = 4$ super Yang-Mills. Furthermore, both manifolds have the same superdimension, $-1$, as required by mirror symmetry.

We are currently investigating the validity of this mirror conjecture. If the conjecture is proven it would be an important step toward explaining, more or less from first principles, the fact that the twistor transformed amplitudes lie on holomorphic curves; it is because the A model on $\mathbb{CP}^{3|4}$ is the mirror of a B model on twistor space whose perturbation theory is the same as that of ordinary $\mathcal{N} = 4$ super Yang-Mills.

6 Conclusion

In this paper we have presented some ideas which could lead to a different understanding of the twistor transform of $\mathcal{N} = 4$ super Yang-Mills presented in [1]. Many of our ideas are clearly rather speculative; they should be viewed as broadening the possibilities for a consistent stringy realization of the beautiful ideas of the twistor transform [28, 52].
Quite independent of the application to $\mathcal{N} = 4$ super Yang-Mills, it would be very desirable to understand whether the new NS-branes we proposed in the A and B models indeed exist, and whether there is indeed an S-duality exchanging D-branes with NS-branes. These questions would make sense even if it turns out that there are no gravitational tadpoles. In particular, our S-duality conjecture relating the A and B model on the same Calabi-Yau and the mirror conjecture we made in the previous section could have a broader range of validity.

More optimistically, if our picture turns out to be correct, it would be important to understand carefully how the physical states of $\mathcal{N} = 4$ super Yang-Mills can be encoded in the A model with extra NS5-branes — in other words, what is the precise origin in this picture of the Dolbeault cohomology which appears in the B model? In the A model the computation of $\mathcal{N} = 4$ super Yang-Mills amplitudes should amount to some version of worldsheet perturbation theory, and the issue is to determine the proper vertex operators to insert. It should then be possible to calculate $\mathcal{N} = 4$ super Yang-Mills amplitudes in the A model; in particular one should be able to obtain the maximally helicity-violating amplitudes, as was done in [1] for the B model. One would imagine that the effects of the NS5-branes can be taken into account by a suitable definition of the measure on the moduli space of open A model instantons. This would be a crucial test of our proposed A model picture.

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