Light propagation around a relativistic vortex flow of dielectric medium

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Abstract

We determine the path of the light around a dielectric vortex described by the relativistic vortex flow of a perfect fluid.

1 Introduction

Recently, Leonhardt and Piwnicki [1] have studied the light propagation in a vortex flow of non-dispersive dielectric medium in order to obtain a so-called optical black hole. However, Visser [2] has shown, in a comment of another paper of Leonhardt and Piwnicki [3], that this vortex geometry has not the sense of black hole such that it is defined in general relativity. Moreover, we emphasize that their vortex flow is simply an ad hoc generalization of the Newtonian case. Despite these points, the analysis of the light propagation in a dielectric vortex is interesting in principle.

In this work, we take up again the question of the light propagation around a dielectric vortex in the case where the relativistic vortex flow is really determined within the framework of relativistic hydrodynamics. It is different from the vortex flow considered by the authors. Then, we establish the qualitative features of the trajectories of the light rays dependent on the values of the angular momentum of the light rays.

The plan of the work is as follows. In section 2, we derive the relativistic vortex flow. For this case, we calculate in section 3 the effective metric for which the light rays define a null congruence of geodesics. In section 4, we obtain the equations determining the path of the light. Then, we discuss the qualitative features of the trajectories of the light rays in section 5. We add some concluding remarks in section 6.

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2 Relativistic vortex flow

We are concerned with a perfect fluid having the energy-momentum tensor

$$T^{\mu\nu} = (\rho c^2 + p)u^\mu u^\nu + p\eta^{\mu\nu}$$  \hspace{1cm} (1)

where $\rho$ is the mass density, $p$ the pressure, $u^\mu$ the four-velocity of the fluid, and $\eta^{\mu\nu}$ is the Minkowskian metric with signature $(- + + +)$. It is conserved $\partial_\mu T^{\mu\nu} = 0$. We assume the existence of a conserved mass density $r$, i.e. $\partial_\mu (ru^\mu) = 0$. We also define the relativistic specific enthalpy $h = (\rho c^2 + p)/r$.

We suppose that the motion of the fluid is isentropic. So, it is not necessary to consider the specific entropy and we postulate merely an equation of state $h(p)$. From the equations of the fluid motion, it is easy to prove that $dh/dp = 1/r$. Moreover, by setting

$$\mu^\tau = hu^\tau,$$  \hspace{1cm} (2)

we obtain the following equation

$$u^\tau \partial_\tau \mu_\sigma = -\partial_\sigma h.$$  \hspace{1cm} (3)

We describe the Minkowskian spacetime in cylindrical coordinates $(x^0, \rho, \varphi, z)$ in which the Minkowskian metric has the form

$$ds^2 = -(dx^0)^2 + d\rho^2 + \rho^2 d\varphi^2 + dz^2,$$  \hspace{1cm} (4)

The symmetries of the vortex flow are given by the three Killing vectors : $\partial_0, \partial_\varphi, \partial_z$, denoted $\xi^{\tau}_{(a)}$ $(a = 1, 2, 3)$, since we assume that the solutions to the equations of the fluid motion are stationary and cylindrically symmetric. In particular, we have $\xi^{\tau}_{(a)} \partial_\tau h = 0$. Taking into account the fact that $\partial_\mu \xi^{(a)\nu} + \partial_\nu \xi^{(a)\mu} = 0$, we obtain from (3) that $u^\sigma \partial_\sigma (\xi^{\tau}_{(a)} \mu_\tau) = 0$. Thus, $\xi^{\tau}_{(a)} \mu_\tau$ are constants of the fluid motion. We limit ourselves to the case $\mu^\rho = 0$ and we put

$$\mu_0 = -\mathcal{E}, \quad \mu_\rho = 0, \quad \mu_\varphi = \mathcal{M} \text{ and } \mu_z = k$$  \hspace{1cm} (5)

where $\mathcal{E} (\mathcal{E} > 0)$, $\mathcal{M}$ and $k$ are arbitrary constants of the fluid motion. By using a Lorentz transformation along $z$, we may take $k = 0$.

We now solve the equations of the fluid motion. From (2), we get $h^2 = -\mu_\tau \mu^\tau$. Hence by assumptions (3), we find

$$h(\rho) = \sqrt{\mathcal{E}^2 - \frac{\mathcal{M}^2}{\rho^2}}$$  \hspace{1cm} (6)

which is well defined for $\rho > \rho_c$ where $\rho_c = |\mathcal{M}/\mathcal{E}|$. We notice that $h(\rho_c) = 0$. By combining (2) and (3), we get the components of the four-velocity of the fluid

$$u^0 = \frac{1}{\sqrt{1 - \rho_c^2/\rho^2}}, \quad u^\rho = 0, \quad u^\varphi = \pm \frac{\rho_c}{\rho \sqrt{1 - \rho_c^2/\rho^2}} \text{ and } u^z = 0 \text{ for } \rho > \rho_c$$  \hspace{1cm} (7)

in which we choose the sign $+$ henceforth. Solution (7) to the equations of the fluid motion gives a model of vortex flow which is only considered for $\rho > \rho_c$, i.e. outside the core region of the vortex flow. This vortex geometry is usually used in relativistic hydrodynamics [4] and in the relativistic theory of superfluid [5].
3 Light propagation in a non-dispersive medium

The electromagnetic properties of an isotropic non-dispersive dielectric medium are characterized by the permittivity $\epsilon$ and the permeability $\mu$. A relativistic model of such a medium has been found a long time ago [6, 7]. The Maxwell equations are written with the aid of a tensor $\tilde{g}_{\mu\nu}$ defined by

$$\tilde{g}_{\alpha\beta} = \eta_{\alpha\beta} - \left(-1 + \frac{1}{n^2}\right)u_\alpha u_\beta$$

where $n$ is the refractive index, $n^2 = \epsilon \mu c^2$, and $u^\mu$ the four-velocity of the medium. At the approximation of geometrical optics in this medium, the vector of propagation of light is $k_\mu = \partial_\mu S$ where $S$ is the phase and it satisfies the eikonal equation [8]

$$\tilde{g}^{\alpha\beta}k_\alpha k_\beta = 0$$

where $\tilde{g}^{\alpha\beta}$ is the inverse matrix of $\tilde{g}_{\mu\nu}$, having from (8) the expression

$$\tilde{g}^{\alpha\beta} = \eta^{\alpha\beta} - \left(-1 + n^2\right)u^\alpha u^\beta.$$ (10)

Equation (9) for the propagation of light in this dielectric medium is equivalent to the propagation of light in the effective metric $\tilde{g}_{\mu\nu}$. We therefore define the components $\tilde{k}^\mu$ of the vector of propagation by

$$\tilde{k}^\mu = \tilde{g}^{\mu\nu}k_\nu$$ (11)

and we take $k_\mu = k_\mu$. Since $\tilde{k}_\mu = \partial_\mu S$, equation (9) implies that the vector of propagation $\tilde{k}^\mu$ defines a null congruence of geodesics for the effective metric $\tilde{g}_{\mu\nu}$. We have thereby

$$\tilde{k}^\mu \tilde{\nabla}_\mu \tilde{k}_\nu = 0$$ (12)

where $\tilde{\nabla}$ is the metric connection associated to $\tilde{g}_{\mu\nu}$.

We suppose that the dielectric medium consists of a perfect fluid. In the case of the vortex flow given by (7), we can calculate the components of the effective metric

$$\tilde{g}_{00} = -\frac{1 - n^2 \rho_c^2/\rho^2}{n^2 (1 - \rho_c^2/\rho^2)}, \quad \tilde{g}_{0\varphi} = -\frac{\rho_c (1 - 1/n^2)}{1 - \rho_c^2/\rho^2}, \quad \tilde{g}_{\varphi\varphi} = \frac{\rho^2 (1 - \rho_c^2/n^2)}{1 - \rho_c^2/\rho^2},$$

$$\tilde{g}_{\rho\rho} = 1 \text{ and } \tilde{g}_{zz} = 1$$

and these of the inverse metric

$$\tilde{g}^{00} = -\frac{n^2 - \rho_c^2/\rho^2}{1 - \rho_c^2/\rho^2}, \quad \tilde{g}^{0\varphi} = -\frac{\rho_c (-1 + n^2)}{\rho^2 (1 - \rho_c^2/\rho^2)}, \quad \tilde{g}^{\varphi\varphi} = \frac{1 - n^2 \rho_c^2/\rho^2}{\rho^2 (1 - \rho_c^2/\rho^2)},$$

$$\tilde{g}^{\rho\rho} = 1 \text{ and } \tilde{g}^{zz} = 1.$$ (14)

Expressions (13) and (14) are well defined for $\rho > \rho_c$, i.e. outside the core region of the vortex flow.

The three vectors $\partial_0$, $\partial_\varphi$ and $\partial_z$ define again three Killing vectors $\hat{\xi}_\alpha^{\mu}$ of the effective metric $\tilde{g}_{\mu\nu}$, i.e. $\tilde{\nabla}_\alpha \hat{\xi}_{(\alpha)}^{\mu} + \tilde{\nabla}_\beta \hat{\xi}_{(\beta)}^{\mu} = 0$. Taking into account the geodesic equations (12),
we find that $\tilde{k}^\mu \bar{\nabla}_\mu (\tilde{\xi}^\nu (a) \tilde{k}_\nu) = 0$ and so $\tilde{\xi}^\mu (a) \tilde{k}_\mu$ are constants of the light motion and thus we put

$$\tilde{k}_0 = -E \quad \tilde{k}_\varphi = L \quad \text{and} \quad \tilde{k}_z = C$$

where $E \ (E > 0), L$ and $C$ are arbitrary constants, $E$ being the energy and $L$ the angular momentum of the light rays.

### 4 Equations of the path of the light

We parametrize the path of the light $x^\mu (\lambda)$ with the affine parameter $\lambda$ so that we have

$$\tilde{k}^\mu = \frac{dx^\mu}{d\lambda},\quad (16)$$

obeying the geodesic equations (12) for $\rho > \rho_c$. We limit ourselves to consider trajectories of the light rays in the plane orthogonal to the vortex flow. We can put $z(\lambda) = 0$ having $C = 0$. We do not write down the geodesic equations (12) but we know that these are differential equations of second order. To determine uniquely the path of light, we must know at $\lambda = \lambda_0$ the quantities

$$(x^0)_0, \quad \left(\frac{dx^0}{d\lambda}\right)_0, \quad (\rho)_0, \quad \left(\frac{d\rho}{d\lambda}\right)_0, \quad (\varphi)_0 \quad \text{and} \quad \left(\frac{d\varphi}{d\lambda}\right)_0.$$  

However, there exists the constraint (9) on these initial data and consequently there is five independent initial data.

We consider the characterization of the solutions to the geodesic equations (12) by the constants (15). According to (11), we have the differential equations

$$\frac{dx^0}{d\lambda} = -\tilde{g}^{00} E + \tilde{g}^{0\varphi} L \quad \text{and} \quad \frac{d\varphi}{d\lambda} = \tilde{g}^{\varphi\varphi} L - \tilde{g}^{0\varphi} E.$$  

The unknown function $\rho(\lambda)$ is determined from equation (13) which can be written

$$\left(\frac{d\rho}{d\lambda}\right)^2 = -\tilde{g}^{00} E^2 + 2\tilde{g}^{0\varphi} EL - \tilde{g}^{\varphi\varphi} L^2.$$  

In this case, the path of light is characterized by the five quantities

$$(x^0)_0, \quad (\varphi)_0, \quad (\rho)_0, \quad E \quad \text{and} \quad L.$$  

Now we point out that the affine parameter $\lambda$ is only defined up to a constant factor, therefore we can put $E = 1$ by a choice of the affine parameter.

We now turn to analyse the differential equation (18) for $\rho > \rho_c$. By setting

$$x = \frac{\rho}{\rho_c}, \quad \tau = \frac{\lambda}{\rho_c} \quad \text{and} \quad \gamma = \frac{\rho_c}{L},$$  

(19)
equation (18) becomes
\[(dx/d\tau)^2 = x^2/(x^2 - 1)^2 \left[ (n^2 - 1/x^2) \gamma^2 - \frac{2(-1 + n^2)}{x^2} \gamma - \left(1 - \frac{n^2}{x^2}\right) \frac{1}{x^2} \right], \quad (20)\]
valid for \(x > 1\) and \(\gamma \neq 0\). We can rewrite (20) in the form
\[(dx/d\tau)^2 = \frac{n^2}{x^2 - 1} \left[ (\gamma - \gamma_1(x)) (\gamma - \gamma_2(x))\right] \quad (21)\]
where
\[\gamma_1(x) = \frac{n - x}{x(nx - 1)} \quad \text{and} \quad \gamma_2(x) = \frac{n + x}{x(nx + 1)}. \quad (22)\]
By virtue of (17), the polar angle \(\varphi(\tau)\) is then determined by
\[\frac{d\varphi}{d\tau} = \frac{1}{x^2 - 1} \left[ \left(1 - \frac{n^2}{x^2}\right) \frac{1}{\gamma} - 1 + n^2 \right] \quad (23)\]
when \(x(\tau)\) is known.
In the case \(\gamma = 1\), we can easily find the general solution to equations (21) and (23)
\[x(\tau) = \sqrt{1 + n^2(\tau_0 - \tau)^2} \quad \text{and} \quad \varphi(\tau) = n \arctan n(\tau_0 - \tau) + \varphi_0. \quad (24)\]
In the case \(\gamma \neq 1\), expressions (21) and (23) have a divergence at \(x = 1\) and therefore we have
\[\lim_{x \to 1} \left(\frac{dx}{d\tau}\right) = \infty \quad \text{and} \quad \lim_{x \to 1} \left(\frac{d\varphi}{d\tau}\right) = \infty. \]
Since the effective metric (13) is singular at \(x = 1\), we cannot discuss the path of the light in terms of \(\tau\) at \(x = 1\). In any way, the affine parameter \(\lambda\) has no physical meaning since \(\tilde{g}_{\mu\nu}\) is an auxiliary metric. Also, it is convenient to express \(x\) in function of \(\varphi\) for \(x > 1\) and we obtain thereby
\[\lim_{x \to 1} \left(\frac{dx}{d\varphi}\right) = 0. \]
This means that the trajectories of the light rays arriving at \(x = 1\) are tangent at the circle \(x = 1\). Then, we can match them with the similar trajectories outgoing at the same position.
In consequence, we would directly determine \(x(\varphi)\) by the following differential equation
\[\left(\frac{dx}{d\varphi}\right)^2 = \frac{(x^2 - 1)(n^2x^2 - 1)}{(1 - n^2)^2[\gamma - \gamma_0(x)]^2} \left[(\gamma - \gamma_1(x))(\gamma - \gamma_2(x))\right] \quad (25)\]
where
\[\gamma_0(x) = \frac{x^2 - n^2}{x^2(1 - n^2)}, \quad (26)\]
deduced from (21) and (23).
5 Qualitative features of the trajectories

We are now in a position to discuss the qualitative features of the trajectories of the light rays. Already for $\gamma = 1$, we have the particular solution (24) which can be rewritten

$$x(\varphi) = \sqrt{1 + \left(\frac{\tan \varphi - \varphi_0}{n}\right)^2}.$$  \hspace{1cm} (27)

We see that the path of light described by trajectory (27) arrives at $x = 1$ from infinity then goes back at the infinity.

A solution to equation (25) exists when its right member is positive. This fact depends on the value of $\gamma$ with respect to the roots $\gamma_1(x)$ and $\gamma_2(x)$. The turning points occur when $dx/d\varphi = 0$. The derivative $dx/d\varphi$ changes of sign there except if the derivative of the right member with respect to $\varphi$ vanishes also at this point. In this latter case, we have a limiting circle. Moreover, as proved in the previous section, we see directly from equation (25) that the trajectories of the light rays passing at $x = 1$ are tangent at the circle $x = 1$.

In order to discuss graphically the form of the trajectories dependent on the values of $\gamma$, we must draw the curves $\gamma_1$ and $\gamma_2$. We note that $\gamma_1(1) = 1$ and $\gamma_2(1) = 1$ and $\gamma_1(\infty) = 0$ and $\gamma_2(\infty) = 0$. The derivatives of $\gamma_1$ and $\gamma_1$ with respect to $x$ are

$$\frac{d\gamma_1}{dx} = \frac{n(1 - 2n + x^2)}{x^2(nx - 1)^2} \quad \text{and} \quad \frac{d\gamma_2}{dx} = -\frac{n(1 + 2n + x^2)}{x^2(nx + 1)^2}. \hspace{1cm} (28)$$

The derivative of the function $\gamma_2$ is always negative for $x > 1$ and so $\gamma_2$ is monotonically decreasing from 1 to 0. On the contrary, the form of the curve of $\gamma_1$ depends on whether $n < 1$ or $n > 1$.

5.1 Case $n < 1$

The function $\gamma_1$ blows up for $x = 1/n$. The derivative of the function $\gamma_1$ is always positive. For $x > 1/n$, we have $\gamma_2(x) > \gamma_1(x)$ and for $1 < x < 1/n$, we have $\gamma_2(x) < \gamma_1(x)$. We remark that the component $\tilde{g}_{00}$ of the effective metric (13) vanishes for $\rho = \rho_c/n$ and changes of sign.

The solution to equation (25) exists when $\gamma > \gamma_1(x)$ or $\gamma < \gamma_1(x)$ for $x > 1/n$ and when $\gamma_2(x) < \gamma < \gamma_1(x)$ for $1 < x < 1/n$ since the factor in front of $\gamma^2$ in (25) is presently negative. We see graphically on Fig. 1 that for all $\gamma, \gamma \neq 1$, there is a turning point. We have quasi-hyperbolic trajectories. The case $\gamma = 1$ is the particular solution (27) already studied.

5.2 Case $n > 1$

The derivative of the function $\gamma_1$ vanishes at $x = x_l$ with $x_l = n + \sqrt{n^2 - 1}$. We denote $\gamma_l = \gamma_1(x_l)$ which has the expression

$$\gamma_l = -\frac{1}{(n + \sqrt{n^2 - 1})^2}. \hspace{1cm} (29)$$
Figure 1: Curves $\gamma_1(x)$ and $\gamma_2(x)$ for $n = 0.8$
satisfying $-1 < \gamma_1(x_l) < 0$. We have $\gamma_2(x) > \gamma_1(x)$.

The solution to equation (25) exists when $\gamma > \gamma_2(x)$ or $\gamma < \gamma_1(x)$. We see graphically on Fig. 2 that when $\gamma > 1$ or $\gamma < \gamma_l$, the light reaches tangentially to the circle $x = 1$ and goes back at the infinity. We point out the existence of a limiting circle at $x = x_l$ and thus for $\gamma = \gamma_l$ the corresponding light ray spirals around this limiting circle. For $\gamma_l < \gamma < 1$, the form of the curves $\gamma_1$ and $\gamma_2$ shows that there are two turning points. We have either quasi-hyperbolic trajectories or quasi-elliptic trajectories, oscillating between $x = 1$ and $x_\gamma$ defined by $\gamma_1(x_\gamma) = \gamma$.

6 Conclusion

In the present work, we have considered that the dielectric vortex is the relativistic vortex flow of a perfect fluid. The core region defined by $\rho < \rho_c$ is not generally considered. Outside the core region, it is possible to calculate an effective metric for which the light rays define a null congruence of geodesics. By discussing the general features of the trajectories of the light rays, we have proved that the light rays remain external to the core region. The trajectories arriving at $\rho = \rho_c$ are tangent at the circle $\rho = \rho_c$. However, we have merely an idealized model of dielectric vortex and, since the effective metric is singular at $\rho = \rho_c$, we cannot determine the behaviour of the light within geometrical optics in the
neighbourhood of $\rho = \rho_c$.

References

[1] Leonhardt, U and Piwnicki, P (1999) *Phys. Rev. A* **60**, 4301.

[2] Visser, M (2000) [arXiv:gr-qc/0002011](http://arxiv.org/abs/gr-qc/0002011).

[3] Leonhardt, U and Piwnicki, P (2000) *Phys. Rev. Lett.* **84**, 822.

[4] Boisseau, B (2000) *Phys. Rev. D* **61**, 083504.

[5] Carter, B and Langlois, D (1995) *Phys. Rev. D* **52**, 4640.

[6] Gordon, W (1923) *Ann. Phys. (Leipzig)* **72**, 421.

[7] Pham Mau Quan (1956) *C. R. Acad. Sci. (Paris)* **242**, 465.

[8] Tourrenc, Ph (1981) *J. Phys. Colloq.* **8**, 441.