Non commutative geometry and super Yang-Mills theory

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Abstract

We aim to connect the non commutative geometry “quotient space” viewpoint with the standard super Yang Mills theory approach in the spirit of Connes-Douglas-Schwartz and Douglas-Hull description of application of noncommutative geometry to matrix theory. This will result in a relation between the parameters of a rational foliation of the torus and the dimension of the group $U(N)$. Namely, we will be provided with a prescription which allows to study a noncommutative geometry with rational parameter $p/N$ by means of a $U(N)$ gauge theory on a torus of size $\Sigma/N$ with the boundary conditions given by a system with $p$ units of magnetic flux. The transition to irrational parameter can be obtained by letting $N$ and $p$ tend to infinity with fixed ratio. The precise meaning of the limiting process will presumably allow better clarification.

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1 Foliated torus and the noncommuting $U$, $V$ algebra

We would like to clarify the relation between the so-called non-commuting torus geometry and the picture of a torus foliation in the situation where the slope is irrational (that is, the leaf is infinite). Let us recall the main features of the two approaches.

In the noncommuting torus representation, one generalizes the toroidal geometry by supposing that the two coordinates are not ordinary variables, but satisfy the commutation relations

$$[p, q] = \frac{2\pi i}{N}$$

(1)

A more convenient labeling in order to provide an useful representation for all $N \times N$ matrices is

$$U = e^{ip}$$

(2)

$$V = e^{iq}$$

(3)

so that

$$UV = e^{2\pi i/N} VU$$

(4)

This gives a representation in the same spirit of the Fourier sum expansion:

$$Z = \sum_{n,m=1}^{N} Z_{n,m} (U^n)(V^m)$$

(5)

To any $N \times N$ matrix we thus associate a function of two periodic variables.

Let’s now switch to the foliated torus description. Let us take the relation of equivalence which identifies, over the torus, the points belonging to the same straight line within a line of parallel straight line of fixed slope. The point is slightly tricky (since the torus itself is already born out of an equivalence relation: $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$), so let us be as explicit as possible. We consider the square unit torus (that is, $[0, 1] \times [0, 1]$ with the opposite sides ordinately glued). The correct procedure is to refer to the line bundle in $\mathbb{R}^2$

$$y = ax + b \quad (a \text{ fixed})$$

(6)

and to identify the points belonging to any such leaf. After this, we will proceed to carry on the equivalence which defines the torus.

A leaf of the foliation is parametrized by the value of $b$, but in a very redundant way. Actually two values of the intercept correspond to the same leaf provided that

$$b' = b + an \quad n \in \mathbb{Z}$$

(7)
Now let’s introduce functions of two variables (one of which is an integer): \( F(b, n) \). They actually admit to be interpreted as matrices (infinite, but with discrete entries), if one rewrites them in the form \( F(b, b') \) where \( b \sim b' \) (actually \( b' = b + an \)). For such functions we are about to define a multiplication structure, which, of course, will be inspired by the usual matrix product:

\[
H \equiv F \ast G
\]

\[
H(b, m) = \sum_n F(b, n) G(b + an, m - n)
\]

The matrices of (9) are actually parametrized by \( b \), but if we replace \( b \) with \( b' = b + ap \), this results, at the level of the matrix product, in a relabeling (by positions) of rows and columns of an infinite matrix. In other words, they are actually dependent on the leaf only.

It is straightforward to check the matching of the \( \ast \) definition and of the matrix product and, thus, the equivalence of the two descriptions.

Switching to the rational case and choosing, moreover, \( \theta = 1/N \), let us now give an explicit representation for the matrices \( U, V \) which will turn out very useful in the following. Let’s have as \( U \) the shift matrix

\[
U = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

and as \( V \) the matrix of phases

\[
U = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & e^{2\pi i/N} & 0 & 0 & 0 & 0 \\
0 & 0 & e^{4\pi i/N} & 0 & 0 & 0 \\
0 & 0 & 0 & e^{6\pi i/N} & 0 & 0 \\
0 & 0 & 0 & 0 & e^{8\pi i/N} & 0 \\
0 & 0 & 0 & 0 & 0 & \ddots \\
\end{pmatrix}
\]

Both \( U, V \) are to be interpreted as \( N \times N \) matrices.

It is straightforward to check that (10), (11) satisfy the commutation relation (4).

2 Towards M-theory

We start the analysis by considering IIA theory on a torus and using Seiberg’s view of matrix theory [4], that is, we replace the lightlike compactification with a compactification along a spacelike circle of shrinking radius. The paper [4] discusses a weakly coupled IIA theory
as above with a background 2-form potential; let its value be $\theta$. We take a square torus, which gives immediately $\tau = i$. It will be convenient (for T duality purposes) to choose as parameters of the system

- $\tau \equiv i$
- $P := \theta + ia$

where $a$ is the area of the torus.

Let’s do now a T-duality transformation along one cycle of the torus, let’s say, the “1” 1-cycle. Such a transformation interchanges $\tau$ and $P$; that is, we obtain a new torus with parameters

$$P' = \tau$$

$$\tau' = P$$

that is

$$P' = i \Rightarrow a' = 1, \quad \theta' = 0$$

$$\tau' = \theta + ia$$

If the area gets small, the dual torus becomes very elongated (a sharp angle and the other nearly flat).

What happens to a D0-brane under this duality? We will obtain a D1-brane oriented along the “1” direction of the new torus. (See figure 1.) Now suppose we have a fundamental string whose ends are on the D1-brane but which is wound $n$ times along the “2” 1-cycle. If we try to impose a constraint of minimal length (provided the winding number $n$ is fixed) and if we imagine to “open up” the torus and refer to a tiling of the plane made by its copies (see figure 2), we realize that the two points where the fundamental string is attached to the D-string are separated by a distance $n\theta$ along a straight line almost perpendicular to the D-string. In other words, they are equivalent points in the sense of the foliated torus construction described before.

If we consider, at this stage, the field operators which create and annihilate such strings, they are fields whose arguments are two points belonging to the D-string and related by the equivalence relation of the foliated torus. In other words, it is now natural to interpret them as objects which live in the noncommutative algebra of the foliated torus.

### 3 The rational foliation case

Let us follow in this section the approach of [4] outlined in the previous pages, but with the additional assumption that the value $\theta$ of the background is rational:

$$\theta \equiv \frac{p}{N} \quad p, \ N \text{ relatively prime}$$

3
The purpose of the present work is to connect the parameters of a super Yang-Mills theory on a torus (in particular the value of \(N\)) with the geometrical viewpoint of the noncommutative foliation. The first step will be to just assume \(\theta = 1/N\). We will later handle the general case of \([11]\).

Let us fix once more the notations (see figure 3).

\[
\begin{align*}
\tau &= i \\
\theta &= \theta + i\Sigma h \\
P &= i \\
\end{align*}
\]

Let the horizontal axis be called \(\sigma\), \(0 \leq \sigma \leq \Sigma\).

We have already seen that if we have a single D0-brane and make a T-duality transformation in the \(\sigma\) direction, we obtain a single D1-brane oriented along the \(\sigma\) direction. Moreover, strings compelled by “minimal length constraints” dynamically connect points \(\sigma\) and \(\sigma + \frac{\Sigma}{N}\) where \(w \in \mathbb{Z}\), thus defining (nonlocal) fields

\[\phi(\sigma, w)\] (18)

To symmetrize the aspect of the above field, one might Fourier transform with respect to the periodic coordinate \(\sigma\), thus obtaining a function of two integers:

\[\tilde{\phi}(k, w) \quad k, w \in \mathbb{Z}\] (19)

The energy of such a mode is

\[E(k, w) = \left(\frac{1}{\Sigma^2} \left[k^2 + w^2\right]\right)^{\frac{1}{2}}\] (20)

Let us now imagine of “folding” the circle according to the equivalence relation induced by the foliation (see figure 4). The small circle has now length \(\frac{\Sigma}{N}\) and its periodic coordinate \(x\) satisfies \(0 \leq x \leq \frac{\Sigma}{N}\).

We should also notice that the choice of \(p\) in the numerator only corresponds to a relabeling of the \(N\) sectors, which are rearranged in a permuted order. (See figure 5). This is a consequence of the (trivial) fact that, given \(p, N\) relatively prime, no one of the numbers

\[p, 2p, 3p, \ldots (N - 1)p\]

is a multiple of \(N\) and thus

\[\{0, (p)_{\text{mod } N}, (2p)_{\text{mod } N}, \ldots\}\]

is a set of \(N\) integer numbers belonging to \([0, N]\) and all different. Thus choosing \(p/N\) instead of \(1/N\) results only in a relabeling of the equivalence relation: the “arches” are superimposed in a different order.
We are now going to split the winding and the momentum modes into a part which is an integer multiple of $N$ and the remainder (see figure 6). First we define an appropriate splitting of the integer $k$ of equation (19):

$$k = KN + q \quad K, q \in \mathbb{Z} \quad 0 \leq q < N \quad (21)$$

We simultaneously replace the fields $\phi$ of eq. (18) with matrix valued fields

$$\phi_{ab}(x, W) \quad a, b = 1, ... N \quad (22)$$

where the index $a$ (resp. $b$) tells us in which of the $N$ intervals (of length $\frac{\Sigma}{N}$ each) the string begins (resp. ends) and the integer $W$ is the winding number along the big circle of length $\Sigma$.

The relation with $w$ is given by

$$w = NW + (a - b) \quad (23)$$

We can now rewrite the energy (20) in terms of the capital variables:

$$E^2 = \frac{1}{\Sigma^2} \left( (NK + q)^2 + (NW + (a-b))^2 \right) \quad (24)$$

We now wish to reproduce the energy spectrum (24) with a description on a small torus of size $\Sigma/N$. On such torus we will have two directions, $x$ and $y$ respectively. The description (22) of the matrix fields can yield a description in one more direction if, as usual, we wish to interpret the winding mode as a Kaluza-Klein momentum in an additional direction; that is, Fourier transformation with respect to $W$ gives

$$\tilde{\phi}_{ab}(x, y) \quad (25)$$

Of course, one might also Fourier transform (22) with respect to $x$:

$$\tilde{\phi}_{ab}(K, W) \quad (26)$$

In order to reproduce the spectrum (24) we will introduce a background $U(N)$ Yang-Mills field. We will also have to introduce non trivial boundary conditions for the fields when transporting them around the $x$ axis. The first couple of condition we impose are

$$\phi_{a,b}(x + \frac{\Sigma}{N}, y) = \phi_{a+1,b+1}(x, y) \quad (27)$$

$$\phi_{ab}(x, y + \frac{\Sigma}{N}) = \phi_{ab}(x, y) \quad (28)$$

that is, if we move along $x$ direction making a complete turn on the small circle we get a unit shift of both indexes (of the “begin” and “end” sectors on the big circle), while the $y$ direction is associated to the “big” winding number. In matrix language

$$\phi(x + \frac{\Sigma}{N}, y) = U^\dagger \phi(x, y)U \quad (29)$$
φ(x, y + \frac{\Sigma}{N}) = φ(x, y) \quad (30)

where U is the \(N \times N\) shift matrix

\[
U = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \quad (31)
\]

Along the \(y\) direction we wish to introduce Wilson loops, in order to mimic the fractional contribution to the momentum which we encountered in eq. (24):

\[
W(x) = \exp \left( i \oint A_y(x, y) \, dy \right) \quad (32)
\]

We assume \(A_x = 0\) and \(A_y\) independent of \(y\):

\[
W(x) = \exp \left( i \frac{\Sigma}{N} A_y(x) \right) \quad (33)
\]

We put in a background vector potential:

\[
W(x) = \exp \left( i \frac{x}{\Sigma} V \right) \quad (34)
\]

where the matrix \(V\) is

\[
V = \begin{pmatrix}
\ddots & 0 & 0 & 0 & 0 & 0 \\
0 & e^{-2\pi i/N} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & e^{2\pi i/N} & 0 & 0 \\
0 & 0 & 0 & 0 & e^{4\pi i/N} & 0 \\
0 & 0 & 0 & 0 & 0 & \ddots
\end{pmatrix} \quad (35)
\]

Notice that the background satisfies the same conditions of \(\phi\) if we make a complete turn on the small circle:

\[
Γ(x + \frac{\Sigma}{N}) = U^\dagger Γ(x) U \quad (36)
\]

From (33) and (34) it follows immediately for the vector potential

\[
A_y = \frac{x}{\Sigma^2} N I + \frac{1}{\Sigma} \begin{pmatrix}
\ddots & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \ddots
\end{pmatrix} \quad (37)
\]
where $I$ is the $N \times N$ identity matrix.

The vector potential can thus be split in an “abelian” and in a “non abelian” part. One should notice how the form of the abelian gauge field corresponds to a unit of abelian magnetic flux through the torus.

Now let’s consider the free part of the action for the scalars

$$\mathcal{L} = \int dx \, dy \, Tr(\dot{\phi}^2 - (\nabla \phi)^2)$$  \hspace{1cm} (38)

where the covariant derivative is defined as usual

$$\nabla \phi = \partial \phi + i [A, \phi]$$  \hspace{1cm} (39)

Let us consider separately the spectra of $\nabla_x$ and $\nabla_y$.

First of all, the spectrum of $\nabla_x \equiv \partial_x$ has to be evaluated on a circle of length $\frac{\Sigma}{N}$. But actually we imposed a condition which allows $\phi$ to return to its value only after $N$ turns, so the system behaves as the circle was effectively of size $\Sigma$. The spectrum will be then

$$\frac{i}{\Sigma} [KN + p] \quad p \in \mathbb{Z}, \quad 0 \leq p \leq N - 1$$  \hspace{1cm} (40)

with the same decomposition of eq. (21).

Second, let’s turn our attention to $\nabla_y = \partial_y + i[A_y, \cdot]$. The derivative piece has the usual spectrum on the circle

$$\frac{i N W}{\Sigma} \quad W \in \mathbb{Z}$$  \hspace{1cm} (41)

since $\phi(y + \frac{N}{\Sigma}) = \phi(y)$. The commutator (for which the “abelian” part of the vector potential can be dropped) is rewritten

$$[A_y, \phi]_{a, b} = \frac{1}{\Sigma} (a - b) \phi_{a, b}$$  \hspace{1cm} (42)

and the spectrum of $\nabla_y$ is

$$\frac{i}{\Sigma} [NW + (a - b)]$$  \hspace{1cm} (43)

Since the equation of motion of $\phi$ is

$$\ddot{\phi} = (\nabla_x^2 + \nabla_y^2) \phi$$  \hspace{1cm} (44)

the spectrum will be

$$E^2 = \left[ \left( \frac{KN + p}{\Sigma} \right)^2 + \left( \frac{WN + (a - b)}{\Sigma} \right)^2 \right]$$  \hspace{1cm} (45)

as required by eq. (24).
4 Interaction terms

To isolate our main point, we will refer to the quartic scalar interaction term. What happens
is of course more general and could actually be imagined from the remarks in section 3; nevertheless, we want to proceed to an explicit exhibition of how the star product works as
a substitute of matrix multiplication and to discuss the role of the numerator of the fraction
\( \theta \equiv p/N \).

In the 2+1 dimensional Connes-Douglas-Schwartz theory over the “big” torus, the form
of the quartic terms is

\[
\int_0^\Sigma dX dY \left[ \phi^i \ast \phi^j - \phi^j \ast \phi^i \right]^2
\]  

(46)

where \( X, Y \) are coordinates on the “large” torus and the star product is defined as

\[
F \ast G = F(x, y) e^{i\theta (\overleftarrow{\partial_x} \overleftarrow{\partial_y} - \overleftarrow{\partial_y} \overleftarrow{\partial_x})} G(x, y)
\]  

(47)

For the moment we will assume \( \theta \equiv 1/N \).

To evaluate this, we go to the Douglas-Hull representation by Fourier transforming with
respect to \( y \):

\[
F \ast G = \sum_{n,m} F(x, n) e^{in\theta} e^{i\theta (\overleftarrow{\partial_x} \overleftarrow{\partial_y} - \overleftarrow{\partial_y} \overleftarrow{\partial_x})} \tilde{G}(x, m) e^{im\theta}
\]  

(48)

We replace \( \overleftarrow{\partial_y} \) by \( in \) and \( \overleftarrow{\partial_y} \) by \( im \):

\[
F \ast G = \sum_{n,m} F(x, n) e^{-m\theta/2} e^{n\theta/2} G(x, n) e^{i(n+m)y} =
\]  

\[
= \sum F(x - \frac{m\theta}{2}, n) G(x + \frac{n\theta}{2}, m) e^{i(n+m)y}
\]  

(49)

We now recall the intuitive picture of \( F(x, n) \) as a “string” attached at \( x \) and with length
\( n\theta \) along the \( x \) axis (see figure 7). If we regard the strings on a given leaf (that is, the ones
whose endpoints belong to that leaf, i.e. are equivalent with respect to the leaf induced
relation of equivalence) as matrices (since the endpoints of \( F \) and \( G \) coincide), then the
Fourier transform of \( F \ast G \) is exactly the matrix product:

\[
\sum_n F(x - \frac{k-n}{2}\theta, n) G(x + \frac{n\theta}{2}, (k-n))
\]  

(50)

(remember \( m = k - n \)).

Notice that a given leaf corresponds to a point on a small circle. It was actually to the
purpose of removing (this particular kind of) non locality that the small circle has been introduced.
Similarly, expressions like
\[ \int (F * G)(H * K) \, dx \, dy \] (51)
translate in matrix language
\[ \int_0^\infty Tr (FG)(HK) \] (52)
Thus, going back to the action term (46) we obtain
\[ Tr \int_0^\infty [\phi^i, \phi^j]^2 \, dx \, dy \] (53)
that is, our usual Yang-Mills terms.

Let us now discuss what happens if \( \theta = p/N \), \( p \) and \( N \) relatively prime. We have already discussed how this amounts to a “relabeling” of the sectors of the circle by means of a permutation of their indices. However, it is important to point out what happens to the “boundary condition” \( U \) and to the “Wilson loop” \( V \) matrices.

The first point to realize is that, whatever the permutation of sectors may be, the “periodic” boundary conditions will not change. This is because, no matter what happens to the interactions, the free evolution will move along the interval in the original order; the free part of the lagrangian on the big circle is certainly local.

What happens to the Wilson loop? If we permute the intervals
\[
\begin{align*}
1 & \rightarrow 1 \\
2 & \rightarrow (1 + p)_{\text{mod } N} \\
3 & \rightarrow (1 + 2p)_{\text{mod } N} \\
4 & \rightarrow (1 + 3p)_{\text{mod } N} \\
\ldots & \quad \ldots \\
\end{align*}
\] (54)
we will replace eq. (34) with
\[ W(x) = \exp \left( ip \frac{x}{\Sigma} \right) V^p \] (55)
The non abelian part of the Wilson loop
\[ V = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \xi & 0 & 0 & 0 & 0 \\
0 & 0 & \xi^2 & 0 & 0 & 0 \\
0 & 0 & 0 & \xi^3 & 0 & 0 \\
0 & 0 & 0 & 0 & \xi^4 & 0 \\
0 & 0 & 0 & 0 & 0 & \ddots \\
\end{pmatrix} \quad \xi = e^{2\pi i/N} \] (56)
is replaced with

$$V^p = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \xi^{(1+p)\text{mod } N-1} & 0 & 0 & 0 \\
0 & 0 & \xi^{(1+2p)\text{mod } N-1} & 0 & 0 \\
0 & 0 & 0 & \xi^{(1+3p)\text{mod } N-1} & 0 \\
0 & 0 & 0 & 0 & \ddots \\
0 & 0 & 0 & 0 & 0 & \ddots
\end{pmatrix}$$  \hspace{1cm} (57)

The abelian vector potential is now $p$ times larger, and so is the flux.

Also the commutation relation is accordingly modified:

$$UW = WU \xi^p$$  \hspace{1cm} (58)

(the rewriting in terms of the Wilson loop is allowed since the abelian part gives no contribution).

To summarize, the relations in the general case are

$$\theta = \frac{p}{N}$$

\[
\begin{align*}
\phi(x + \frac{\Sigma}{N}, y) &= U^\dagger \phi(x, y) U \\
\phi(x, y + \frac{\Sigma}{N}) &= \phi(y)
\end{align*}
\]

$$W = V^p$$

$$UW = WU \xi^p$$

Moreover, the abelian part of the vector potential carries now $p$ units of magnetic flux.

## 5 Coupling constant rescaling

It might be interesting to derive the behaviour of the coupling constant $g_{YM}$ along the process of “folding” the big circle into the small one. We know (cfr. for example [5]) that the Yang-Mills coupling for a 2+1 dim gauge theory describing compactification on a 2-torus is

$$g_{YM}^2 = \frac{1}{\sum L_3^3}$$  \hspace{1cm} (60)

This is the coupling constant of the non commutative geometry on the “big” torus. The transition to the small torus yields a factor of $1/N$:

$$\tilde{g}_{YM}^2 = \frac{1}{N\sum L_3^3}$$  \hspace{1cm} (61)

Thus $g^2 N$ is kept fixed during the passage to the “small” torus.
6 Conclusions

To summarize, we have now a prescription for the link between a super Yang Mills theory with large N and a noncommutative geometry with $\theta$ “almost irrational”.

- Take $\theta$ and approximate it with an irreducible fraction $\frac{p}{N}$

\[ \Downarrow \]

- Build a $U(N)$ gauge theory on a torus of size $\frac{\Sigma}{N}$ and choose as boundary conditions those which are explicit in eq. (59) with $p$ units of magnetic flux.

Thus we find that abelian gauge theory on a noncommutative torus is equivalent to an appropriate limit of non abelian gauge theory on a rescaled commutative torus.

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