A NEW OVER-PENALIZED WEAK GALERKIN FINITE ELEMENT METHOD. PART II: ELLIPTIC INTERFACE PROBLEMS

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ABSTRACT. We introduce an over-penalized weak Galerkin method for elliptic interface problems with non-homogeneous boundary conditions and discontinuous coefficients, where the method combines a weak Galerkin stabilizer with interior penalty terms. This method employs double-valued functions on interior edges of elements instead of single-valued ones and elements $(P_k, P_k, [P_{k-1}]^2)$. As an advantage of the method, elliptic interface problems with low regularity are approximated well. The over-penalized weak Galerkin method is based on weak functions whose edge part is double-valued on each interior edge sharing by two neighboring elements. Jumps between the edge parts are naturally used to define penalty terms. The over-penalized weak Galerkin method allows to use arbitrary meshes, even for low regularity solutions. These features make the new method more flexible and efficient for solving interface equations. Furthermore, a priori error estimates in energy and $L^2$ norms are derived rigorously, and numerical results confirm the effectiveness of the method.

1. Introduction. Weak Galerkin (WG) finite element method was first introduced by Ye and Wang for solving second order elliptic problems in [20]. Owing to its new features of flexibility in variational formulation and domain geometry, the WG method have been applied to various mathematical and engineering problems in many literatures, for instance, Stokes equation [21], Helmholtz equation [12], Maxwell equation [14], low regular elliptic problems [18], singularly perturbed convection-diffusion-reaction problems [9], and so on. In general, the standard differential operators are replaced by their weak forms, with optional stabilizer term to enforce weak continuity of the approximating functions. We refer to [23] for a comprehensive analysis to the WG methods. Interface problems can be used in modeling fluid mechanics [8], computational electro-magnetics [4], material science [7] and elasticity problems [5] etc. Some methods are mainly designed for a homogeneous dispersive medium, however, a common difficulty exists in treating an inhomogeneous dispersive media. Indeed, solution loses its regularity across an interface. As we know, discontinuity exists in interface problems, resulting in a...
natural manner to mathematically define interface conditions for solution, although physically interface is often complicate. We have to face globally low regularity of solutions in many applications, such as computing electrostatic potential for macromolecule in solvent, simulation of tumor growth and surface plasmon etc.

To strengthen flexibility of weak Galerkin methods for elliptic interface problem with global low regularity, it is desired to design a totally discontinuous type of weak Galerkin finite element methods. Analogous to discontinuous Galerkin (DG) finite element methods, some piecewise polynomials spaces are employed in weak Galerkin finite element methods. It is worthy mentioning that two kinds of elements are commonly employed in WG method. The first type of elements include Raviart-Thomas (RT) element [17, 20] and Brezzi-Douglas-Marini (BDM) element [1, 20]. Meanwhile, with stabilizers provided for weak continuity, the second type of elements are elements \((\mathbb{P}_k(T), \mathbb{P}_k(e), [\mathbb{P}_{k-1}(T)]^d)\) first introduced in [12] for Helmholtz equation and elements \((\mathbb{P}_k(T), \mathbb{P}_{k-1}(e), [\mathbb{P}_{k-1}(T)]^d)\) in [13] for second-order elliptic problems, where \(T\) means any element with dimensions of space \(d\) and \(e \in \partial T\) is any interior edge of the element.

For complex and irregular interfaces, there are a lot of interests in formulating a variational form dependent on body-fitted meshes. Interface problems often exist low global regularity, although the local regularity can be high. Based upon this distinguishing feature, an interior-penalty-stabilized Lagrange multiplier method was proposed in [2] for elliptic interface problems. With a Lagrange multiplier stressed on interface conditions and a discrete weak gradient defined by local RT elements, a weak Galerkin method in [11] has been applied in solving elliptic interface problems, leading to a saddle point system. With elements \((\mathbb{P}_k(T), \mathbb{P}_k(e), [\mathbb{P}_{k-1}(T)]^d)\) being employed, a weak Galerkin method with a parameter-free stabilizer was introduced for the same problem in [15]. The latter has some interesting features including symmetric and positive definite systems, permission of arbitrary meshes, high-order convergence, and good performance on complicated interfaces. Recently, a class of relaxed weak Galerkin finite element methods has been presented for elliptic problems with low regularity in [18], by applying an over-relaxed factor to build up a connection with low regularity of solution and strengthen weak continuity across edges. In [10], Mu introduced a posteriori error estimate of weak Galerkin methods for the second-order elliptic interface problems, in which a promising algorithm was presented for adaptively solving interface problems. Notice that all methods mentioned above apply single-valued edge functions in numerical discretizations.

In [16], we defined weak double-valued edge functions and adopted elements \((P_k(T), P_k(e), RT_k(T))\), for strengthening the flexibility of the WG finite element method and utilizing its numerical solution elementwise belonging to \(H(div)\). In this case, the usual stabilization term does not appear. As an alternative, we introduced penalized terms defined by jumps from double-valued functions on interior edges. More importantly, the penalized terms are commonly applied to interior penalty discontinuous Galerkin finite element methods, which shows a good connection between DG and WG finite element methods. We made an extension of WG finite element method to weak functions, referred to over-penalized weak Galerkin (OPWG) finite element method.

As a motivation, one can employ elements \((P_k(T), P_k(e), [P_{k-1}(T)]^2)\) in the over-penalized weak Galerkin finite element method. Similar as the existing WG methods, a stabilizer more will be used in the new algorithm.
In this paper, we consider the following elliptic interface problem with non-homogeneous boundary and interface conditions
\[ -\nabla \cdot A \nabla u = f, \text{ in } \Omega, \quad (1) \]
\[ u = g, \text{ on } \partial \Omega \setminus \Gamma, \quad (2) \]
\[ [u]_{\Gamma} = \psi, \text{ on } \Gamma, \quad (3) \]
\[ [A \nabla u \cdot n]_{\Gamma} = \phi, \text{ on } \Gamma, \quad (4) \]
where \( \Omega = \Omega_1 \cup \Omega_2 \) is a polygonal or polyhedral domain in \( \mathbb{R}^2 \), \( \Gamma = \Omega_1 \cap \Omega_2 \) and \([A \nabla u \cdot n]_{\Gamma} := A_1 \nabla u \mid_{\Omega_1} \cdot n_1 + A_2 \nabla u \mid_{\Omega_2} \cdot n_2 \). Note that \( n_1 \) and \( n_2 \) are the unit normal vectors exterior to \( \Omega_1 \) and \( \Omega_2 \), respectively. \( \Gamma \) is a polygonal interface with Lipschitz continuity. Set \( f \in L^2(\Omega) \) and let \( A \) be a symmetric and positive definite matrix, whose entries are piecewise constants, i.e. there exist two positive constants \( \lambda_1, \lambda_2 > 0 \) such that
\[ \lambda_1 \xi^t \xi \leq \lambda_2 \xi^t A \xi \leq \lambda_2 \xi^t A \xi, \quad \forall \xi \in \mathbb{R}^2, \]
where \( \xi^t \) is the transpose of vector \( \xi \). To describe over-penalized and stabilized weak Galerkin method, we will use elements \((P_k(T), P_k(e), [P_{k-1}(T)])^2\), while one can take other elements \((P_k(T), P_{k-1}(e), [P_{k-1}(T)])^2\) in our method to retain discrete system of (1)-(4) symmetric and positive definite.

The rest of the paper is organized as follows. In Sect. 2, we will introduce the variational form for elliptic interface problems using the over-penalized and stabilized weak Galerkin method and prove the uniqueness of numerical solution. In Sect. 3, we will give an error equation and show convergence rates in \( H^1 \) energy norm and \( L^2 \) norm in the view of appropriate regular assumptions for the elliptic interface problems. In Sect. 4, we report numerical results to validate our theoretical results.

2. Preliminary. Throughout this paper, we use the standard definitions and notations for the Sobolev spaces. In this section, we concentrate on the preliminaries including some definitions on weak function, weak gradient, approximation spaces and so on.

Let \( T_h \) be a shape regular and body-fitted partition of \( \Omega \) (see [22]). Denote by \( T_h = T_{1,h} \cup T_{2,h} \) fitted with the interface, where \( T_{1,h} \subset \Omega_1, T_{2,h} \subset \Omega_2 \). Define \( E \) by a set of all edges in \( T_h \), in particular, \( \Gamma_h \) by a subset of \( E \) on \( \Gamma \) and \( E_I = E \setminus (\partial \Omega_1 \cup \Gamma_h) \) by a set of all interior edges except those located on interface. Let \( h_T \) be the diameter of \( T \in T_h \) and define \( h = \max_{T \in T_h} h_T \). Further, denote by \( P_k(T) \) a set of piecewise polynomials in \( T \in T_h \) with degree no more than \( k \). Moreover, \( P_k(e) \) is a set of piecewise polynomials on \( e \in E \) with degree no more than \( k \).

For any weak function \( v = \{v_0, v_b\} \) on a polygon \( K \) with boundary \( \partial K \) satisfying \( v_0 \in L^2(K), \; v_b \in H^\frac{1}{2}(\partial K) \), we define its weak gradient,
\[ (\nabla_w v, q)_K := -(v_0, \nabla \cdot q)_K + \langle v_b, q \cdot n \rangle_{\partial K}, \; \forall q \in H(\text{div}, K), \]
where \( n \) is the outward normal direction to \( \partial K \). Basing on the weak function, a weak Galerkin finite element space is defined by
\[ V_h = \{(v_0, v_b) : v_0 \mid_T \in P_k, T \in T_h; \]
\[ v_b \mid_e \in P_k(e) \times P_k(e), e \in E_I; \]
\[ v_b \mid_e \in P_k(e), e \in \partial \Omega \cup \Gamma, k \geq 1 \}, \]

furthermore,
\[ V_h^0 = \{(v_0, v_b) : v \in V_h, \; v_b = 0 \text{ on } \partial \Omega \setminus \Gamma \}. \]
Here, \( v_h \) is a double-valued function on each interior edge except on the interface and boundary. Set neighboring elements \( T_1 \) and \( T_2 \) share a common edge \( e \), and from the above definition, it is natural to denote the weak jump by

\[
\|v_b\| := v_b|_{\partial T_1 \cap e} - v_b|_{\partial T_2 \cap e}.
\] (5)

Note that, different from the definition of jump in the interior penalty discontinuous Galerkin methods, the present weak jump directly comes from the weak functions rather than a limit passing from interior of neighboring elements to their common edge. Especially, we also define \( \|v_b\| = v_b|_e \), for any \( e \in \partial \Omega \cup \Gamma \).

For each \( v \in V_h \), the local discrete weak gradient \( \nabla_w v \in [P_{k-1}(T)]^2 \) of \( v \) on each element \( T \in \mathcal{T}_h \) is uniquely and piecewise defined by the following equation

\[
(\nabla_w v, q)_T := -(v_0, \nabla \cdot q)_T + (v_b, q \cdot n)_T \quad \forall q \in [P_{k-1}(T)]^2.
\] (6)

Integrating by parts for the first term on the right hand side of the above equation leads to

\[
(\nabla_w v, q)_T = (\nabla v_0, q)_T - (v_0 - v_b, q \cdot n)_T \quad \forall q \in [P_{k-1}(T)]^2.
\] (7)

Assume that \( u \in H^{k+1}(\Omega_i) \), \( i = 1, 2 \), \( k \geq 1 \) be the exact solution of (1)-(4), and let \( T_1 \) and \( T_2 \) be two adjacent elements sharing a common edge \( e \), then we have

(i) if \( e \notin \Gamma \), then \( u|_{\partial T_1 \cap e} = u|_{\partial T_2 \cap e} \); 
(ii) if \( e \in \Gamma \), \( T_1 \subset I_1 \), and \( T_2 \subset I_2 \), then \( u|_{\partial T_1 \cap e} - u|_{\partial T_2 \cap e} = \psi \), in particular, when \( \psi = 0 \), \( u|_{\partial T_1 \cap e} = u|_{\partial T_2 \cap e} \).

For each \( e \in \mathcal{E}_I \), we denote the direction of normal \( n_e \) by:

- on \( e \in \Gamma_h \), \( n_e : T_1 \rightarrow T_2 \), where \( T_1 \in \Omega_1, T_2 \in \Omega_2 \),
- on \( e \in \mathcal{E}_I \), \( n_e : \) any normal direction to \( e \),
- on \( e \in \partial \Omega \cap \mathcal{E} \), \( n_e : \) unit outward normal direction to \( e \).

And the following \( L^2 \) projections are introduced

\[
Q_0 : L^2(T) \rightarrow \mathbb{P}_k(T), \quad \forall T \in \mathcal{T}_h, \\
Q_h : L^2(e) \rightarrow \mathbb{P}_k(e), \quad \forall \partial T \in \mathcal{E}_I, \\
R_h : [L^2(T)]^2 \rightarrow [\mathbb{P}_{k-1}(T)]^2, \quad \forall T \in \mathcal{T}_h.
\]

Here we write \( Q_h := \{Q_0, Q_h\} \).

3. **An over-penalized weak Galerkin Method.** Define a bilinear form

\[
a(w, v) = (A\nabla_w w, \nabla_w v) + s(w, v) + J_0(w, v), \quad \forall w, v \in V_h,
\] (8)

where the stabilizer \( s(w, v) \) and the penalty term \( J_0(w, v) \) are defined by

\[
s(w, v) := \sum_{T \in \mathcal{T}_h} h^{-1}_T \langle Q_h w_0 - w_b, Q_h v_0 - v_b \rangle_{\partial T},
\]

\[
J_0(w, v) := \sum_{e \in \mathcal{E}_I} |e|^{-\beta_0} \langle \|w_b\|, \|v_b\| \rangle_e,
\]

respectively, for \( \beta_0 \geq 1 \). The two forms can ensure uniqueness and stability of the bilinear form in the elements \((\mathbb{P}_k(T), \mathbb{P}_k(e), [\mathbb{P}_{k-1}(T)]^2)\) (see [12]).
Let $e \in \Gamma$ be shared by $T_1$ and $T_2$ with $T_1 \subset \Omega_1$, $T_2 \subset \Omega_2$, and $\psi$, $\phi$ be given in (3)-(4), we define three bilinear forms as follows

$$
\langle \psi, A\nabla_w v \cdot n \rangle_{\Gamma} := \sum_{e \in \Gamma_h} \langle \psi, A\nabla_w (v|_{T_2}) \cdot n_e \rangle_e,
$$

$$
h^{-1} \langle \psi, v_0 - v_b \rangle_{\Gamma} := \sum_{e \in \Gamma_h} h^{-1}_{T_2} \langle \psi, v_0|_{T_2} - v_b \rangle_e,
$$

$$
\langle \phi, v_b \rangle_{\Gamma} := \sum_{e \in \Gamma_h} \langle \phi, v_b \rangle_e.
$$

We present the over-penalized weak Galerkin method as follows.

A numerical approximation for (1)-(4) can be obtained by seeking $u_h = \{u_0, u_b\} \in V_h$ with $u_b = Q_b g$ on $\partial \Omega$ such that for all $v = \{v_0, v_b\} \in V_h^0$

$$
a(u_h, v) = (f, v_0) + \langle \phi, v_b \rangle_{\Gamma} + \langle \psi, A\nabla_w v \cdot n_e \rangle_{\Gamma} - h^{-1} \langle \psi, v_0 - v_b \rangle_{\Gamma}. \quad (9)
$$

It is easy to see that the method is symmetric and positive definite. Then, based on (8), we define by a mesh-dependent norm $\|\cdot\|$. 

**Definition 3.1.** For each $v \in V_h$,

$$
\|v\| := \sqrt{a(v, v)}. \quad (10)
$$

Note that the functional $\|\cdot\|$ is a semi-norm in $V_h$, but a norm in $V_h^0$.

**Lemma 3.2.** $\|\cdot\|$ defined by (10) is a norm in $V_h^0$.

**Proof.** In order to prove the lemma, we just need to prove that $v \equiv 0$ when $\|v\| = 0$ (see [20]). Let $\|v\| = 0$, by (8), we can obtain

$$
(A\nabla_w v, \nabla_w v) + \sum_{T \in T_h} h^{-1}_T \langle Q_b v_0 - v_b, Q_b v_0 - v_b \rangle_{\partial T} + \sum_{e \in E_T} |e|^{-\frac{1}{2}} \int_e \|v_b\|^2 = 0,
$$

which implies that for each $T \in T_h$, $\nabla_w v = 0$, and on $\partial T$, $Q_b v_0 = v_b$. We also have $\|v_b\| = 0$ on $e \in E_T$. By the definition of discrete weak gradient, we arrive at

$$
0 = (\nabla_w v, q)_T = (\nabla v_0, q)_T - \langle v_0 - v_b, q \cdot n \rangle_{\partial T}
$$

$$
= (\nabla v_0, q)_T - \langle Q_b v_0 - v_b, q \cdot n \rangle_{\partial T}
$$

$$
= (\nabla v_0, q)_T, \forall q \in [P_{k-1}(T)]^2.
$$

Taking $q = \nabla v_0$ on each element, it is obviously $\nabla v_0 = 0$, so $v_0$ is a constant. Since $Q_b v_0 = v_b$, $\|v_b\| = 0$ and $v_b = 0$ on $\partial \Omega$, $v_0 = v_b = 0$ is given.

The aim of following lemma is to derive the uniqueness for numerical solution of (9).

**Lemma 3.3.** The over-penalized weak Galerkin method defined in (9) has a unique solution.

**Proof.** Let $u_1^h, u_2^h$ be two solutions of (9), denote $e = u_1^h - u_2^h$, then

$$
a(e, v) = 0, \forall v \in V_h^0,
$$

with $e \in V_h^0$, choosing $v = e$ in the above equation leads to

$$
\|e\|^2 = a(e, e) = 0,
$$

by the Lemma 3.2, we can have $e \equiv 0$ i.e. $u_1^h \equiv u_2^h$.
Let \( u_h = \{u_0, u_b\} \in V_h \) and \( u \in H^{k+1}(\Omega_i), i = 1, 2 \) be the solutions to (9) and (1)-(4) respectively. The \( L^2 \) projection of \( u \) is \( Q_h u = \{Q_0 u, Q_b u\} \in V_h \). In order to promise \( Q_h u \in V_h \), we give the following definition.

**Definition 3.4.** Let \( T_1 \) and \( T_2 \) be two adjacent elements sharing \( e \in \partial T_1 \cap \partial T_2 \), if \( e \not\in \Gamma \), let \( Q_b u|_e = Q_b(u|_e) \), if \( e \in \Gamma \), let \( Q_b u|_e = Q_b(u|_{\partial T_i \cap e}) \) with \( T_1 \subset T_{1,h}, T_2 \subset T_{2,h} \), and \( Q_b(u|_{\partial T_i \cap e}) = Q_b u|_e - Q_b \psi|_e \).

**Lemma 3.5.** [15] Assume \( Q_h \) and \( R_h \) are projections defined as in the last Section, then we have for each \( \tau \in [P_{k-1}(T)]^2 \)

\[
(\nabla_w(Q_h u), \tau)_T = (R_h(\nabla u), \tau)_T, \quad T \not\subseteq T_{2,h} \text{ or } \partial T \cap \Gamma = \emptyset, \tag{11}
\]

\[
(\nabla_w(Q_h u), \tau)_T = (R_h(\nabla u), \tau)_T + \langle \psi, \tau \cdot n \rangle|_{\partial T \cap \Gamma}, \quad T \subset T_{2,h} \text{ and } \partial T \cap \Gamma \neq \emptyset. \tag{12}
\]

3.1. **Error equation.** Assume that the entries of the coefficient matrix \( A \) are made of piecewise constants with respect to the subdomains \( \Omega_i \) \((i = 1, 2)\). Let \( e_h := Q_h u - u_h = \{Q_0 u - u_0, Q_b u - u_b\} = \{e_0, e_b\} \). Simultaneously, define three formulae \( l_1(u, v), l_2(u, v), l_3(u, v) \) by

\[
l_1(u, v) := \sum_{T \in T_h} (A(\nabla u - R_h \nabla u) \cdot n, v_0 - v_b)|_{\partial T},
\]

\[
l_2(u, v) := \sum_{e \in E_i} (A \nabla u \cdot n_e, [v_b]|_e), \tag{13}
\]

\[
l_3(u, v) := \sum_{T \in T_h} h_T^{-1}(Q_0 u - Q_b(u|_T), v_0 - v_b)|_{\partial T}.
\]

An error equation is given by the following lemma, for analyzing the difference between an \( L^2 \) projection of the exact solution \( u \) of the original problem and numerical solution \( u_h \) from the over-penalized weak Galerkin approximation.

**Lemma 3.6.** Let \( u_h \) and \( u \) be the solutions of (9) and (1)-(4), respectively. Then for each \( v \in V_h^0 \), the error equation is

\[
a(e_h, v) = l_1(u, v) + l_2(u, v) + l_3(u, v). \tag{14}
\]

**Proof.** Testing equation (1) with \( v_0 \) of \( v = \{v_0, v_b\} \) and using Green formula lead to

\[
(f, v_0) = - (\nabla \cdot A \nabla u, v_0) = \sum_{T \in T_h} (A \nabla u, \nabla v_0)|_T - \sum_{T \in T_h} (A \nabla u \cdot n, v_0)|_{\partial T}
\]

\[
= \sum_{T \in T_h} (A \nabla u, \nabla v_0)_T - \sum_{T \in T_h} (A \nabla u \cdot n, v_0 - v_b)|_{\partial T} \tag{15}
\]

\[
- \sum_{T \in T_h} (A \nabla u \cdot n, v_b)|_{\partial T},
\]

then we arrive at

\[
(A \nabla u, \nabla v_0) = (f, v_0) + \sum_{T \in T_h} (A \nabla u \cdot n, v_0 - v_b)|_{\partial T} + \sum_{T \in T_h} (A \nabla u \cdot n, v_b)|_{\partial T}. \tag{16}
\]

By using Lemma 3.5, the definition of discrete weak gradient (7) and Green formula, we have the following equations:
(i) for $T \not\subseteq \Omega_2$ or $\partial T \cap \Gamma = \emptyset$,
\[
(\nabla_w(Q_h u), \nabla_w v)_T = (AR_h \nabla u, \nabla_w v)_T \\
= (AR_h \nabla u, \nabla_v 0)_T - (AR_h \nabla u \cdot n, v_0 - v_b)_{\partial T} \\
= (\nabla u, \nabla_v 0)_T - (AR_h \nabla u \cdot n, v_0 - v_b)_{\partial T},
\]
(ii) for $T \subset \Omega_2$ and $\partial T \cap \Gamma \neq \emptyset$,
\[
(\nabla_w(Q_h u), \nabla_w v)_T \\
= (AR_h \nabla u, \nabla_w v)_T + \langle \psi, A \nabla_w \cdot n \rangle_{\partial T \cap \Gamma} \\
= (AR_h \nabla u, \nabla_v 0)_T - (AR_h \nabla u \cdot n, v_0 - v_b)_{\partial T} + \langle \psi, A \nabla_w \cdot n \rangle_{\partial T \cap \Gamma} \\
= (\nabla u, \nabla_v 0)_T - (AR_h \nabla u \cdot n, v_0 - v_b)_{\partial T} + \langle \psi, A \nabla_w \cdot n \rangle_{\partial T \cap \Gamma}.
\]
Therefore, we get
\[
(\nabla_w Q_h u, \nabla_w v) = (\nabla u, \nabla_v 0) - \sum_{T \in T_h} (AR_h \nabla u \cdot n, v_0 - v_b)_{\partial T} \\
+ \langle \psi, A \nabla_w v \cdot n \rangle_{\Gamma}.
\]
Substituting (16) into (18) yields
\[
(\nabla_w(Q_h u), \nabla_w v) \\
= (f, v_0) + \sum_{T \in T_h} (\nabla u \cdot n, v_0 - v_b)_{\partial T} + \sum_{T \in T_h} (\nabla u \cdot n, v_0 - v_b)_{\partial T} \\
- \sum_{T \in T_h} (AR_h \nabla u \cdot n, v_0 - v_b)_{\partial T} + \langle \psi, A \nabla_w v \cdot n \rangle_{\Gamma}.
\]
If $e = \partial T_1 \cap \partial T_2$, $v_1^e$ and $v_2^e$ are defined on $\partial T_1$, $\partial T_2$, respectively, then it holds that for all $e \not\subseteq \Gamma_h$
\[
\langle \nabla u \cdot n_1, v_1^e \rangle_{\partial T_1 \cap e} + \langle \nabla u \cdot n_2, v_2^e \rangle_{\partial T_2 \cap e} = \langle \nabla u \cdot n_e, [v_b]_e \rangle,
\]
and for all $e \in \Gamma_h$
\[
\langle \nabla u \cdot n_1, v_1^e \rangle_{\partial T_1 \cap e} + \langle \nabla u \cdot n_2, v_2^e \rangle_{\partial T_2 \cap e} = \langle \nabla u \cdot n_1 + \nabla u \cdot n_2, v_b \rangle_e \\
= \langle \phi, v_b \rangle_e.
\]
Thus the term $\sum_{T \in T_h} (\nabla u \cdot n, v_b)_{\partial T}$ in (19) can be rewritten as
\[
\sum_{T \in T_h} (\nabla u \cdot n, v_b)_{\partial T} = \sum_{e \in \mathcal{E}_I} (\nabla u \cdot n_e, [v_b]_e) + \langle \phi, v_b \rangle_{\Gamma}.
\]
Plugging (20) into (19) implies
\[
(\nabla_w(Q_h u), \nabla_w v) = (f, v_0) + \sum_{T \in T_h} (\nabla u - R_h \nabla u \cdot n, v_0 - v_b)_{\partial T} \\
+ \sum_{e \in \mathcal{E}_I} (\nabla u \cdot n_e, [v_b]_e)_{\partial T} + \langle \phi, v_b \rangle_{\Gamma} + \langle \psi, A \nabla_w v \cdot n \rangle_{\Gamma}.
\]
By the definition of penalty term $J_0(u, v) = \sum_{e \in \mathcal{E}_I} |e|^{-\beta_0} \int_{\Gamma} \|v_b\| ds$ and $\|Q_h u\| = 0$ on the interior edges, we have
\[
J_0(Q_h u, v) = \sum_{e \in \mathcal{E}_I} |e|^{-\beta_0} \int_{\Gamma} \|Q_h u\| \|v_b\| ds = 0.
\]
When $T \notin \Omega_2$ or $\partial T \cap \Gamma = \emptyset$,
\[
\langle Q_0u - Q_b(u|_T), v_0 - v_b \rangle_{\partial T} = \langle Q_0u - Q_b(u|_T), v_0 - v_b \rangle_{\partial T}.
\]

When $T \subset \Omega_2$ and $\partial T \cap \Gamma \neq \emptyset$, the definition of $Q_b$ leads to
\[
\langle Q_0u - Q_b(u|_T), v_0 - v_b \rangle_{\partial T} = \langle Q_0u - Q_b(u|_T), v_0 - v_b \rangle_{\partial T} - \langle Q_b \psi, v_0 - v_b \rangle_{\partial T \cap \Gamma}.
\]

With the two cases above analyzed, the stabilizer can be written as
\[
s(Q_h u, v) = \sum_{T \in T_h} h_T^{-1} \langle Q_0u - Q_b(u|_T), v_0 - v_b \rangle_{\partial T} - h^{-1} \langle \psi, v_0 - v_b \rangle_{\Gamma}.
\]

Adding the stabilizer $s(Q_h u, v)$ and the penalty term $J_0(Q_h u, v)$ to the both sides of (21) results in
\[
a(Q_h u, v) = (A \nabla_w(Q_h u), \nabla_w v) + s(Q_h u, v) + J_0(Q_h u, v)
\]
\[
= (f, v_0) + \sum_{T \in T_h} \langle A(\nabla u - R_h \nabla u) \cdot n, v_0 - v_b \rangle_{\partial T} + \sum_{e \in E_h} \langle A \nabla u \cdot n_e, \|v_b\|_e \rangle
\]
\[
+ \sum_{T \in T_h} h_T^{-1} \langle Q_0u - Q_b(u|_T), v_0 - v_b \rangle_{\partial T} - h^{-1} \langle \psi, v_0 - v_b \rangle_{\Gamma}
\]
\[
+ \langle \phi, v_b \rangle_{\Gamma} + \langle \psi, A \nabla_w v \cdot n \rangle_{\Gamma}
\]
\[
= (f, v_0) + l_1(u, v) + l_2(u, v) + l_3(u, v) - h^{-1} \langle \psi, v_0 - v_b \rangle_{\Gamma}
\]
\[
+ \langle \psi, A \nabla_w v \cdot n \rangle_{\Gamma} + \langle \phi, v_b \rangle_{\Gamma},
\]
where $l_1(u, v), l_2(u, v), l_3(u, v)$ are defined by (13). Furthermore, the above equation can be rewritten as
\[
a(Q_h u, v) = (f, v_0) + l_1(u, v) + l_2(u, v) + l_3(u, v) - h^{-1} \langle \psi, v_0 - v_b \rangle_{\Gamma}
\]
\[
+ \langle \psi, A \nabla_w v \cdot n \rangle_{\Gamma} + \langle \phi, v_b \rangle_{\Gamma}.
\]

Substituting (9) from (23) completes the proof of Lemma.

3.2. Error estimate. In this section, thanks to the error equation (14), a priori estimates are given as follows.

**Lemma 3.7.** (Trace inequality [22]) Assume the partition $T_h$ is shape regular, then for each $T \in T_h$, there exists a constant $C$ such that
\[
\|\varphi\|_{H^1(T)}^2 \leq C(\delta_T^{-1}\|\varphi\|_T^2 + h_T\|\nabla \varphi\|_{T}^2), \quad \forall \varphi \in H^1(T)
\]
where $C$ is independent of $\varphi$ and $h_T$.

**Definition 3.8.** The discrete $H^1$ norm is defined by :
\[
\|v\|_{H^1_{T_h}}^2 = \sum_{T \in T_h} (\|\nabla v_0\|_T^2 + h_T^{-1}\|Q_b v_0 - v_b\|^2_{\partial T}) + \sum_{e \in E_h} |e|^{-\delta_0} \|v_b\|_e^2, \quad \forall v \in V_h.
\]

By [13], it is easily verified that $\|\cdot\|_{1,T}$ is equivalent with $\|\cdot\|$. Since the elliptic interface problem has local regularity of $u \in H^{k+1}(\Omega_i), \quad i = 1, 2$, we define a norm to describe the property,
\[
\|u\|_{H^{k+1}(\Omega_1 \cup \Omega_2)}^2 = \|u\|_{H^{k+1}(\Omega_1)}^2 + \|u\|_{H^{k+1}(\Omega_2)}^2.
\]
Lemma 3.9. Assume the partition $T_h$ is shape regular, then for $u \in H^{k+1}(\Omega_i)$, $k \geq 1$, $i = 1, 2$ and $v = \{v_0, v_b\} \in V_h$, then the following estimates holds
\[
|l_1(u, v)| \leq C h^k ||u||_{H^{k+1}(\Omega_i \cup \Omega_2)} ||v||, \\
|l_2(u, v)| \leq C h^{(\beta_0 - 1)/2} ||u||_{H^{k+1}(\Omega_i \cup \Omega_2)} ||v||, \tag{25}
|l_3(u, v)| \leq C h^k ||u||_{H^{k+1}(\Omega_i \cup \Omega_2)} ||v||.
\]

Proof. With Cauchy-Schwarz inequality, trace inequality (24) and approximation properties of $L^2$ projection $R_h$, we bound the first term
\[
|l_1(u, v) = | \sum_{T \in T_h} \langle A(\nabla u - R_h \nabla u) \cdot n, v_0 - v_b \rangle_{\partial T} | \leq C \sum_{T \in T_h} ||A(\nabla u - R_h \nabla u)||_{\partial T} ||v_0 - v_b||_{\partial T} \leq C \left( \sum_{T \in T_h} ||A(\nabla u - R_h \nabla u)||^2_{\partial T} + h_T^2 ||\nabla (A(\nabla u - R_h \nabla u))||^2_T \right)^{1/2} \cdot \left( \sum_{T \in T_h} h_T^{-1} ||v_0 - v_b||^2_{\partial T} \right)^{1/2} \leq C h^k ||u||_{H^{k+1}(\Omega_i \cup \Omega_2)} ||v||.
\]

Next, we estimate the term $l_2(u, v)$. By the equivalence of $|| \cdot ||_{1,h}$ and $|| \cdot ||$, and Sobolev embedding inequality, the following estimate holds
\[
|l_2(u, v)| = \left| \sum_{e \in E_I} \langle A\nabla u \cdot n_e, [v_0]_e \rangle \right| \leq \left( \sum_{e \in E_I} ||A\nabla u \cdot n_e||^2_e \right)^{1/2} \left( \sum_{e \in E_I} ||[v_0]_e||^2_e \right)^{1/2} \leq C \sum_{T \in T_h} (h_T^{-1} ||\nabla u||^2_T + h_T ||\nabla^2 u||^2_T)^{1/2} \cdot \frac{h_T^{\beta_0/2}}{h_T} ||v|| \leq C h^{(\beta_0 - 1)/2} ||u||_{H^{k+1}(\Omega_i \cup \Omega_2)} ||v||.
\]

Finally, we derive the estimate for the term $l_3(u, v)$. Note that $l_3(u, v) = s(Q_h u, v)$. With the definition of $Q_h$ and the trace inequality (24), we have
\[
|l_3(u, v)| \leq \left( \sum_{T \in T_h} h_T^{-1} ||Q_0 u - u||^2_{\partial T} \right)^{1/2} \left( \sum_{T \in T_h} h_T^{-1} ||v_0 - v_b||^2_{\partial T} \right)^{1/2} \leq C \left( \sum_{T \in T_h} (h_T^{-2} ||Q_0 u - u||^2_T + ||\nabla (Q_0 u - u)||^2_T) \right)^{1/2} ||v|| \leq C h^k ||u||_{H^{k+1}(\Omega_i \cup \Omega_2)} ||v||,
\]

which completes the proof.

3.3. Convergence Theorem. Basing on the estimates of $l_1(u, v)$, $l_2(u, v)$, and $l_3(u, v)$, the convergence result in the energy norm can be given directly from (25).

Theorem 3.10. For the partition $T_h$ being shape regular, let $u_h \in V_h$ and $u \in H^{k+1}(\Omega_i)$, $k \geq 1$, $i = 1, 2$ be the solutions of (9) and (1)-(4). Then there exists a constant $C$ independent of $h$ such that
\[
||e_h|| \leq C (h^k + h^{(\beta_0 - 1)/2}) ||u||_{H^{k+1}(\Omega_1 \cup \Omega_2)}, \tag{26}
\]
where $\beta_0 \geq 1$. In particular, the convergence rates are optimal if $\beta_0 \geq 2k + 1$. 

In order to show an error estimate in $L^2$ norm of (9), we apply the Nitsche’s trick. Consider the following auxiliary problem of (1)-(4) (see [3] and [18]), i.e. to find $w \in H^1_0(\Omega)$ satisfy

$$
\begin{align*}
- \nabla \cdot (A\nabla w) &= e_0, \quad \text{in } \Omega, \\
\|w\|_\Gamma &= 0, \quad \text{on } \Gamma, \\
\|A\nabla w\|_\Gamma &= 0, \quad \text{on } \Gamma.
\end{align*}
$$

(27)

By [3], the solution $w \in H^1(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2)$ of the auxiliary problem satisfies a local $H^2$ regularity, i.e. there exists a constant $C$ such that

$$
\|w\|_{H^2(\Omega_1 \cup \Omega_2)} \leq C\|e_0\|. 
$$

(28)

**Theorem 3.11.** For the partition $T_h$ being shape regular, let $u_h \in V_h$ and $u \in H^{k+1}(\Omega_i)$, $k \geq 1, \ i = 1, 2$ be the solutions of (9) and (1)-(4). Then there exists a constant $C$ independent of $h$ such that

$$
\|e_0\| \leq C(h^{k+1} + h^{\beta_0+1/2} + h^{k+(\beta_0-1)/2} + h^{\beta_0-1})\|u\|_{H^{k+1}(\Omega_1 \cup \Omega_2)}, 
$$

(29)

where $\beta_0 \geq 1$. In particular, the convergence rates are optimal if $\beta_0 \geq 2k+1$.

**Proof.** Substituting $u$, $v$ with $w$, $e_h$ in (23), respectively, leads to

$$
a(Q_h w, e_h) = (e_0, e_h) + l_1(w, e_h) + l_2(w, e_h) + l_3(w, e_h).$$

Since $a(\cdot, \cdot)$ is symmetry, applying error equation (14) to (27), we have

$$
\|e_0\|^2 = a(Q_h w, e_h) - (l_1(w, e_h) + l_2(w, e_h) + l_3(w, e_h)) = l_1(u, Q_h w) + l_2(u, Q_h w) + l_3(u, Q_h w) - (l_1(w, e_h) + l_2(w, e_h) + l_3(w, e_h)).
$$

Since the jump $\{Q_h w\} = 0$ on each interior boundary, so the second term satisfies

$$
l_2(u, Q_h w) = \sum_{e \in E_i} \langle A\nabla u \cdot n_e, \{Q_h w\} \rangle_e = 0.
$$

The above equations imply

$$
\|e_0\|^2 = l_1(u, Q_h w) + l_3(u, Q_h w) - (l_1(w, e_h) + l_2(w, e_h) + l_3(w, e_h)).
$$

(30)

To the end, by Theorem 6.4 in [13], Cauchy-Schwarz inequality, trace inequality (24) and Theorem 3.10, we can get

$$
\begin{align*}
l_1(u, Q_h w) &\leq Ch^{k+1}\|u\|_{H^{k+1}(\Omega_1 \cup \Omega_2)}\|e_0\|, \\
l_3(u, Q_h w) &\leq Ch^{k+1}\|u\|_{H^{k+1}(\Omega_1 \cup \Omega_2)}\|e_0\|, \\
l_1(w, e_h) &\leq Ch\|w\|_{H^2(\Omega_1 \cup \Omega_2)}\|e_h\| \\
&\leq Ch^{k+1} + h^{(\beta_0+1)/2}\|u\|_{H^{k+1}(\Omega_1 \cup \Omega_2)}\|e_0\|, \\
l_2(w, e_h) &\leq Ch^{(\beta_0-1)/2}\|w\|_{H^2(\Omega_1 \cup \Omega_2)}\|e_h\| \\
&\leq Ch^{(\beta_0-1)/2}(h^{k} + h^{(\beta_0-1)/2})\|u\|_{H^{k+1}(\Omega_1 \cup \Omega_2)}\|e_0\|, \\
l_3(w, e_h) &\leq Ch\|w\|_{H^2(\Omega_1 \cup \Omega_2)}\|e_h\| \\
&\leq Ch^{k+1} + h^{(\beta_0+1)/2}\|u\|_{H^{k+1}(\Omega_1 \cup \Omega_2)}\|e_0\|.
\end{align*}
$$

Combining the estimates above with (30), the inequality (29) follows. \qed
4. Numerical results. In order to demonstrate effectiveness and reliability of the over-penalized weak Galerkin finite element method for elliptic interface problems, we present some numerical examples with a Lipschitz continuous interface ([6]). The partitions are generated by GMSH software, and MATLAB is used as a software development environment. The errors are estimated with respect to the energy norm and the $L^2$ norm.

Example 1. Consider a circle interface problem [24]. Given a square domain $[-1,1] \times [-1,1]$ and a circle interface $r^2 = x^2 + y^2 = 1/4$. Define coefficient matrix $A$ by a diagonal one with $A_{11} = b$ and $A_{22} = 2$, in the subregion $r \leq 0.5$ and $r > 0.5$ respectively. The exact solutions in two subdomains are chosen as $u(x, y) = -[1/4(1 - 1/(8b) - 1/b) + (r^4/2 + r^2)]/b, \quad r > 0.5,$ $v(x, y) = -(x^2 + y^2 - 1), \quad r \leq 0.5,$

where $b = 10$ is taken such that the interface jump is constant, and the subdivision is a uniform triangulation.

Table 1. Example 1 - piecewise linear elements (k=1)

| Mesh | $\beta_0 = 1$ | $\beta_0 = 2$ |
|------|--------------|--------------|
|      | $\|e_h\|_0$ | Order | $\|e_0\|_0$ | Order | $\|e_h\|_1$ | Order | $\|e_0\|_1$ | Order |
| Level 1 | 3.1511e+0 | 2.8316e+0 | 3.1452e+0 | 1.1782e+0 | 1.1782e+0 | 1.0549e+0 | 6.7356e-1 | 0.8067 |
| Level 2 | 1.6608e+0 | 0.9240 | 3.4495e+0 | -0.2848 | 1.6225e+0 | 0.9549 | 6.7356e-1 | 0.8067 |
| Level 3 | 9.6287e-1 | 0.7865 | 3.7937e+0 | -0.1372 | 8.5141e-1 | 0.9303 | 3.5846e-1 | 0.9100 |
| Level 4 | 5.8043e-1 | 0.2223 | 4.0839e+0 | -0.0365 | 2.4081e-1 | 0.9122 | 9.3619e-2 | 0.9795 |

Table 2. Example 1 - piecewise linear elements (k=1)

| Mesh | $\beta_0 = 3$ | $\beta_0 = 4$ |
|------|--------------|--------------|
|      | $\|e_h\|_0$ | Order | $\|e_0\|_0$ | Order | $\|e_h\|_1$ | Order | $\|e_0\|_1$ | Order |
| Level 1 | 3.1387e+0 | 6.5647e-1 | 3.1321e+0 | 5.0657e-1 | 3.1321e+0 | 5.0657e-1 | 3.1321e+0 | 5.0657e-1 |
| Level 2 | 1.5825e+0 | 0.9880 | 2.0026e+0 | 0.9880 | 2.0026e+0 | 0.9880 | 2.0026e+0 | 0.9880 |
| Level 3 | 7.8972e-1 | 1.0028 | 5.5939e-2 | 1.0028 | 5.5939e-2 | 1.0028 | 5.5939e-2 | 1.0028 |
| Level 4 | 3.9408e-1 | 1.0028 | 1.4490e-2 | 1.0028 | 1.4490e-2 | 1.0028 | 1.4490e-2 | 1.0028 |
| Level 5 | 1.9690e-1 | 1.0010 | 3.7056e-3 | 1.0010 | 3.7056e-3 | 1.0010 | 3.7056e-3 | 1.0010 |

Table 3. Example 1 - piecewise quadratic elements (k=2)

| Mesh | $\beta_0 = 2$ | $\beta_0 = 3$ |
|------|--------------|--------------|
|      | $\|e_h\|_0$ | Order | $\|e_0\|_0$ | Order | $\|e_h\|_1$ | Order | $\|e_0\|_1$ | Order |
| Level 1 | 4.1681e-1 | 8.9233e-1 | 3.7338e-1 | 3.1242e-1 | 3.7338e-1 | 3.1242e-1 |
| Level 2 | 3.8786e-1 | 1.0004 | 6.0156e-1 | 0.9298 | 1.9606e-1 | 0.9298 | 1.9606e-1 | 0.9298 |
| Level 3 | 3.1811e-1 | 0.2860 | 3.3979e-1 | 0.8241 | 7.1110e-2 | 1.4627 | 3.1687e-2 | 1.7924 |
| Level 4 | 2.2423e-1 | 0.5045 | 1.7908e-1 | 0.9183 | 1.9708e-2 | 1.8513 | 8.4505e-3 | 1.9068 |
| Level 5 | 1.3977e-1 | 0.6819 | 9.2408e-2 | 0.9603 | 4.9833e-3 | 1.9836 | 2.1778e-3 | 1.9561 |

For piecewise linear elements ($P_1, P_1, [P_0]^2$), Tables 1-2 illustrate numerical errors and rates in the energy and $L^2$ norms with $\beta_0 = 1, 2, 3, 4$. It can be seen that the convergence rate is optimal with $\beta_0 = 3$, and numerical solution and error are
Figure 1. Example 1: piecewise linear elements. Left: numerical solution, Right: error

Table 4. Example 1 - piecewise quadratic elements (k=2)

| Mesh  | $\beta_0 = 4$ |  | $\beta_0 = 5$ |  |
|-------|---------------|---|---------------|---|
|       | $\|e_0\|$ | Order | $\|e_0\|$ | Order | $\|e_h\|$ | Order | $\|e_h\|$ | Order |
| Level 1 | 3.2065e-1  | 1.1501e-1 | 2.8633e-1 | 5.0175e-2 | 2.0367 | 3.1447 |
| Level 2 | 8.8722e-2  | 1.8536  | 2.4276  | 6.8564e-2 | 2.0622 | 3.1447 |
| Level 3 | 1.8873e-2  | 2.2330  | 2.7692  | 1.6740e-2 | 2.0342 | 3.2357 |
| Level 4 | 4.3171e-3  | 2.1282  | 4.2411e-3 | 4.1755e-3 | 2.0033 | 3.0872 |
| Level 5 | 1.0521e-3  | 2.0367  | 5.6286e-5 | 1.0435e-3 | 2.0005 | 2.9454 |

Figure 2. Example 1: piecewise quadratic elements. Left: numerical solution, Right: error

plotted in Fig. 1 in this case. From Fig. 1, we observe that the solution of OPWG is convergent, while it permits discontinuity on edges.

Then, by piecewise quadratic elements ($P_2, P_2, [P_1]^2$), Tables 3-4 give numerical errors and rates in the energy and $L^2$ norms with $\beta_0 = 2, 3, 4, 5$. It is observed from the tables that the convergence rates arrive at the optimal ones as $\beta_0$ increases gradually. In the case $\beta_0 = 5$, numerical solution reaches the optimal convergence orders in the energy and $L^2$ norms, respectively, which verify our theory. We also
show the profiles of numerical solution and error in Fig. 2, which produce less error and smaller jumps along the edges and the interface as a comparison to Fig. 1.

**Example 2.** Consider the polygon interface problem with low regularity to illustrating the capability of OPWG. The example was considered in [19]. Let the domain \( \Omega = (0, 1)^2 \) with \( \Omega_1 = [0.2, 0.8] \times [0.3, 0.7] \) and \( \Omega_2 = \Omega / \Omega_1 \). The piecewise diffusive coefficients are taken as \( A_{11} = 80 \), and \( A_{22} = 1 \), and the exact solutions are chosen as follows:

\[
\begin{align*}
  u(x,y) &= (x^2 + y^2)^{1.5} + \sin(x + y), & \text{if} & \ (x,y) \in \Omega_1, \\
  v(x,y) &= 1 + [(x - 0.5)^2 + (y - 0.5)^2]^{0.5}, & \text{if} & \ (x,y) \in \Omega_2.
\end{align*}
\]

Note that the solution \( u \) has a singularity at \((0.5, 0.5)\). The related interface conditions can be obtained from the exact solutions. Let each subdivision be a uniform triangulation.

| Table 5. Example 2 - piecewise linear elements (k=1) |
|-----------------------------------------------------|
| \( \beta_0 = 1 \) | \( \beta_0 = 2 \) |
| \( \| \varepsilon_0 \| \) | Order | \( \| \varepsilon_0 \| \) | Order | \( \| \varepsilon_0 \| \) | Order | \( \| \varepsilon_0 \| \) | Order |
| Level 1 | 4.788e+1 | 2.029e+1 | 3.815e+1 | 2.029e+1 |
| Level 2 | 4.063e+1 | 0.2367 | 2.546e+1 | -0.3268 |
| Level 3 | 3.894e+1 | 0.0615 | 2.874e+1 | -0.1726 |
| Level 4 | 3.868e+1 | 0.0097 | 3.008e+1 | -0.0679 |
| Level 5 | 3.871e+1 | -0.0011 | 3.080e+1 | -0.0342 |

| Table 6. Example 2 - piecewise linear elements (k=1) |
|-----------------------------------------------------|
| \( \beta_0 = 3 \) | \( \beta_0 = 4 \) |
| \( \| \varepsilon_0 \| \) | Order | \( \| \varepsilon_0 \| \) | Order | \( \| \varepsilon_0 \| \) | Order | \( \| \varepsilon_0 \| \) | Order |
| Level 1 | 3.409e+1 | 2.670e+0 | 3.210e+1 | 2.305e+0 |
| Level 2 | 1.827e+1 | 0.8997 | 7.422e-1 | 1.8470 |
| Level 3 | 8.958e+0 | 1.0283 | 1.967e-1 | 1.9158 |
| Level 4 | 4.146e+0 | 1.1112 | 5.008e-2 | 1.9736 |
| Level 5 | 1.983e+0 | 1.0638 | 1.263e-2 | 1.9870 |

| Table 7. Example 2 - piecewise quadratic elements (k=2) |
|-----------------------------------------------------|
| \( \beta_0 = 2 \) | \( \beta_0 = 3 \) |
| \( \| \varepsilon_0 \| \) | Order | \( \| \varepsilon_0 \| \) | Order | \( \| \varepsilon_0 \| \) | Order | \( \| \varepsilon_0 \| \) | Order |
| Level 1 | 2.130e+1 | 4.121e+0 | 1.318e+1 | 8.413e-1 |
| Level 2 | 1.896e+1 | 0.1716 | 2.754e+0 | 0.5815 |
| Level 3 | 1.636e+1 | 0.2301 | 1.588e+0 | 0.7847 |
| Level 4 | 1.393e+1 | 0.2290 | 8.532e-1 | 0.9058 |
| Level 5 | 1.126e+1 | 0.3088 | 4.404e-1 | 0.9518 |

Analogously, with piecewise linear elements \( (P_1, P_1, [P_0]^2) \), Tables 5-6 give the numerical errors in the energy and \( L^2 \) norms as well as convergence orders with \( \beta_0 = 1, 2, 3, 4 \). The optimal convergence order can be observed in the case \( \beta_0 = 3 \). In Fig. 3, profiles of numerical solution and error are plotted for linear elements employed. With the singularity of exact solution, from Tables 7-8, although piecewise quadratic elements \( (P_2, P_2, [P_1]^2) \) have convergence as magnitude of \( \beta_0 \) increases, they appear to reduce convergence order due to low regularity of solution.
Figure 3. Example 2: piecewise linear elements, with $\beta_0 = 3$, Level = 5. Left: numerical solution, Right: error

Table 8. Example 2 - piecewise quadratic elements (k=2)

| Mesh | $\beta_0 = 4$ | Order | $\beta_0 = 5$ | Order |
|------|---------------|-------|---------------|-------|
|      | $\|e_h\|$ |       | $\|e_0\|$ |       | $\|e_h\|$ |       | $\|e_0\|$ |       |
| Level 1 | 7.7496e+0 | 1.7252e-1 | 3.2231e+0 | 7.4422e-2 | 7.4422e-2 | 3.0798 |
| Level 2 | 2.1684e+0 | 1.8374 | 2.6829 | 4.0304e-1 | 2.9994 | 8.020e-3 | 3.0798 |
| Level 3 | 3.2927e-1 | 2.7192 | 3.6813e-3 | 2.8674 | 8.216e-2 | 2.2899 | 1.2434e-3 | 3.0798 |
| Level 4 | 4.7524e-2 | 2.7925 | 5.0460e-4 | 2.8670 | 2.421e-2 | 1.7669 | 3.6017e-4 | 1.7875 |
| Level 5 | 1.0420e-2 | 2.1893 | 1.5632e-4 | 1.6851 | 9.0684e-3 | 1.4171 | 1.6761e-4 | 1.1035 |

Figure 4. Example 2: piecewise quadratic elements, with $\beta_0 = 5$, Level = 5. Left: numerical solution, Right: error

Example 3. Consider an elliptic interface problem in [24], and the domain is divided by concave and convex curves, where the interface $\Gamma$ is represented by a polar equation in parametric form

$$r = 1/2 + \sin(5\theta)/7.$$ 

The outer polygon is $[-1, 1] \times [-1, 1]$, and set the diagonal coefficients $A_{11} = 10$ and $A_{22} = 1$, which are defined in the inner and outer subdomains of interface $\Gamma$. 
respectively, the exact solutions are given by

\[ u(x, y) = 0.1(x^2 + y^2)^2 - 0.01 \ln(2\sqrt{x^2 + y^2}), \quad \text{outer subdomain of } \Gamma, \]
\[ v(x, y) = e^{x^2 + y^2}, \quad \text{inner subdomain of } \Gamma. \]

The related interface conditions can be obtained from the exact solutions.

Table 9. Example 3 - piecewise linear elements (k=1)

| Mesh       | \( \beta_0 = 1 \) | \( \beta_0 = 2 \) |
|------------|-------------------|-------------------|
|           | \( \|e_h\| \) | \( \|e_0\| \) | \( \|e_h\| \) | \( \|e_0\| \) | \( \|e_h\| \) | \( \|e_0\| \) |
| Level 1   | 2.1520e+0 | 2.9567e+0 | 2.0507e+0 | 5.9099e-1 |
| Level 2   | 1.5717e+0 | 0.4534 | 3.2200e+0 | -0.1231 | 0.6823 | 3.4432e-1 | 0.7794 |
| Level 3   | 1.2871e+0 | 0.2882 | 3.4488e+0 | -0.0990 | 0.7564 | 1.7768e-1 | 0.9564 |
| Level 4   | 1.1948e+0 | 0.1074 | 3.5461e+0 | -0.0401 | 0.4442e-1 | 0.7679 | 8.9088e-2 | 0.9960 |
| Level 5   | 1.1645e+0 | 0.0370 | 3.5926e+0 | -0.0187 | 2.5595e-1 | 0.9940 | 4.4926e-2 | 0.9876 |

Table 10. Example 3 - piecewise linear elements (k=1)

| Mesh       | \( \beta_0 = 3 \) | \( \beta_0 = 4 \) |
|------------|-------------------|-------------------|
|           | \( \|e_h\| \) | \( \|e_0\| \) | \( \|e_h\| \) | \( \|e_0\| \) | \( \|e_h\| \) | \( \|e_0\| \) |
| Level 1   | 1.9345e+0 | 1.9955e-1 | 1.8922e+0 | 1.389e-1 |
| Level 2   | 1.0109e+0 | 6.4142e-2 | 1.6374 | 9.8873e-1 | 0.9364 | 4.0633e-2 | 1.8242 |
| Level 3   | 4.9094e-1 | 1.6236e-2 | 1.9821 | 4.8750e-1 | 1.0202 | 9.6646e-3 | 2.0719 |
| Level 4   | 2.4285e-1 | 4.9744e-3 | 1.9945 | 2.4244e-1 | 1.0078 | 2.3708e-3 | 2.0273 |
| Level 5   | 1.2177e-1 | 1.0372e-3 | 1.9738 | 1.2172e-1 | 0.9940 | 5.9750e-4 | 1.9883 |

Piecewise linear elements (\( \mathbb{P}_1, \mathbb{P}_1, [\mathbb{P}_0]^2 \)) give the numerical errors in the energy and \( L^2 \) norms in Tables 9-10 with \( \beta_0 = 1, 2, 3, 4 \) taken, respectively. They confirm the theory well again. With piecewise linear elements (\( \mathbb{P}_1, \mathbb{P}_1, [\mathbb{P}_0]^2 \)), the convergence rates are optimal if \( \beta_0 \geq 3 \). One can observe profiles of numerical solution and error in Fig. 5. With piecewise quadratic elements (\( \mathbb{P}_2, \mathbb{P}_2, [\mathbb{P}_1]^2 \)), numerical solutions converge to the exact solutions and the convergence rates are shown in Table 11. Due to low regularity on the concave points of the interface, the error plots on
Table 11. Example 3 - piecewise quadratic elements (k=2)

| Mesh | \( \beta_0 = 4 \) | \( \beta_0 = 5 \) |
|------|-----------------|-----------------|
|      | \( \|e_h\| \) | Order | \( \|e_h\| \) | Order | \( \|e_h\| \) | Order | \( \|e_h\| \) | Order |
| Level 1 | 2.0263e-1 | 2.1517e-2 | 1.2596e-1 | 7.5252e-3 |
| Level 2 | 4.6943e-2 | 2.1377 | 4.0745e-3 | 2.4007 | 3.1587e-2 | 1.9955 | 1.0519e-3 | 2.8387 |
| Level 3 | 8.9009e-3 | 2.3709 | 6.1530e-4 | 2.7272 | 8.6377e-3 | 1.9744 | 2.0217e-4 | 2.3793 |
| Level 4 | 2.2145e-3 | 2.0069 | 1.1133e-4 | 2.4690 | 2.1738e-3 | 1.8865 | 5.8895e-5 | 1.7793 |
| Level 5 | 6.0543e-4 | 1.8709 | 2.6078e-5 | 2.0913 | 6.0306e-4 | 1.8498 | 1.9505e-5 | 1.5943 |

Figure 6. Example 3: piecewise quadratic elements, with \( \beta_0 = 5 \), Level = 5. Left: numerical solution, Right: error

geometric singularity can be observed in Fig. 6, resulting in a limitation for other high order elements.

Example 4. Consider an elliptic interface problem with the domain \([-2, 2] \times [-2, 2]\) is divided into two subregions by a heart-shaped interface, where the interface \( \Gamma \) is represented by an implicit form

\[
(x^2 + y^2 - 1)^3 - x^2 y^3 = 0.
\]

For simplicity of implementation, we map linearly the domain into \([0, 1] \times [0, 1]\), and set the diffusion coefficients \( A_1 = 1 \) and \( A_2 = 1000 \), which are defined in the outer and inner subdomains \( \Omega_1 \) and \( \Omega_2 \), respectively. The exact solutions are given by

\[
\begin{align*}
  u(x, y) &= 5e^{-x^2-y^2}, \quad \text{outer subdomain of } \Gamma, \\
  v(x, y) &= e^x \cos y, \quad \text{inner subdomain of } \Gamma.
\end{align*}
\]

The related interface conditions can be obtained from the exact solutions. Note that the interface includes double cusped points, resulting in locally refined grids employed for many methods.

We test the robustness of our method in shape regular meshes (see Fig. 7), without being locally refined, to calculate the convergence rates. From Table 12, it is observed that the method has optimal convergence rates in the energy norm and in the \( L^2 \) norm, amazingly appearing superconvergence behavior in the energy norm as the ratio between the coefficients up to 1 : 1000.
5. Conclusion. In this paper, we give the OPWG method including the stabilizer of weak functions and the penalty term of jumps on weak edge functions for solving second-order elliptic interface problems. By a suitable mathematical setting of the double-valued weak functions on the interior edges, the jumps can be treated analogously as discontinuous Galerkin methods. We have analyzed the optimal parameter $\beta_0$ and assured the convergence rates optimal in the energy and $L^2$ norms. Convergence and flexibility of the OPWG method result in good approximations to locally low regularity problems, while the theory has been verified by numerical examples. In the near future, we will apply the over-penalized weak Galerkin finite element method to parabolic problems with moving interfaces.

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REFERENCES

[1] F. Brezzi, J. Douglas Jr. and L. D. Marini, Two families of mixed finite elements for second order elliptic problems, Numer. Math., 47 (1985), 217–235.
[2] E. Burman and P. Hansbo, Interior-penalty-stabilized Lagrange multiplier methods for the finite-element solution of elliptic interface problems, IMA J. Numer. Anal., 30 (2010), 870–885.
[3] Z. Chen and J. Zou, Finite element methods and their convergence for elliptic and parabolic interface problems, Numer. Math., 79 (1998), 175–202.
[4] G. R. Hadley, High-accuracy finite-difference equations for dielectric waveguide analysis II: Dielectric corners, J. Lightwave Technol., 20 (2002), 1219–1231.
[5] S. Hou, Z. Lin, L. Wang and W. Wang, A numerical method for solving elasticity equations with interfaces, Commun. Comput. Phys., 12 (2012), 595–612.
[6] S. Hou, W. Wang and L. Wang, Numerical method for solving matrix coefficient elliptic equation with sharp-edged interfaces, J. Comput. Phys., 229 (2010), 7162–7179.
[7] T. Y. Hou, Z. Li and S. Osher, Hybrid method for moving interface problems with application to the Hele-Shaw flow, J. Comput. Phys., 134 (1997), 236–252.
[8] A. T. Layton, Using integral equations and the immersed interface method to solve immersed boundary problems with stiff forces, Comput. & Fluids, 38 (2009), 266–272.
[9] R. Lin, X. Ye, S. Zhang and P. Zhu, A weak Galerkin finite element method for singular perturbed convection-diffusion-reaction problems, SIAM J. Numer. Anal., 56 (2018), 1482–1497.
[10] L. Mu, Weak Galerkin based a posteriori error estimates for second order elliptic interface problems on polygonal meshes, J. Comput. Appl. Math., 361 (2019), 413–425.
[11] L. Mu, J. Wang, G. Wei, X. Ye and S. Zhao, Weak Galerkin methods for second order elliptic interface problems, J. Comput. Phys., 250 (2013), 106–125.
[12] L. Mu, J. Wang and X. Ye, A new weak Galerkin finite element method for the Helmholtz equation, IMA J. Numer. Anal., 35 (2015), 1228–1255.
[13] L. Mu, J. Wang and X. Ye, A weak Galerkin finite element method with polynomial reduction, J. Comput. Appl. Math., 285 (2015), 45–58.
[14] L. Mu, J. Wang, X. Ye and S. Zhang, A weak Galerkin finite element method for the Maxwell equations, J. Sci. Comput., 65 (2015), 363–386.
[15] L. Mu, J. Wang, X. Ye and S. Zhao, A new weak Galerkin finite element method for elliptic interface problems, J. Comput. Phys., 325 (2016), 157–173.
[16] W. Qi and L. Song, Weak Galerkin method with implicit $\theta$-schemes for second-order parabolic problems, Appl. Math. Comput., 366 (2020), 11pp.
[17] P.-A. Raviart and J. M. Thomas, A mixed finite element method for 2nd order elliptic problems, in Mathematical Aspects of Finite Element Methods, Lecture Notes in Math., 606, Springer, Berlin, 1977.
[18] L. Song, K. Liu and S. Zhao, A weak Galerkin method with an over-relaxed stabilization for low regularity elliptic problems, J. Sci. Comput., 71 (2017), 195–218.
[19] L. Song, S. Zhao and K. Liu, A relaxed weak Galerkin method for elliptic interface problems with low regularity, Appl. Numer. Math., 128 (2018), 65–80.
[20] J. Wang and X. Ye, A weak Galerkin finite element method for second-order elliptic problems, J. Comput. Appl. Math., 241 (2013), 103–115.
[21] J. Wang and X. Ye, A weak Galerkin finite element method for the stokes equations, Adv. Comput. Math., 42 (2016), 155–174.
[22] J. Wang and X. Ye, A weak Galerkin mixed finite element method for second order elliptic problems, Math. Comp., 83 (2014), 2101–2126.
[23] J. Wang and X. Ye, The basics of weak Galerkin finite element methods, preprint, arXiv:1901.10035.
[24] Y. C. Zhou and G. W. Wei, On the fictitious-domain and interpolation formulations of the matched interface and boundary (MIB) method, J. Comput. Phys., 219 (2006), 228–246.

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