Density distribution of point-contact current

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Abstract. The density distribution of point-contact current is theoretically investigated. Finite current is introduced to a semi-infinite body from the outside. The following assumptions are made: the contact area is circular, magnetic effects are ignored, the contact area is equipotential, the potential equals zero at an infinite distance from the contact area, and the electric permittivity and resistivity are constant. The density distribution of point-contact current for a non-steady state is obtained in an integral form and the derivation process is discussed. The results show that the current density distribution for the non-steady state agrees with that for the steady state. Further, the results suggest that the analytical solution of the density distribution of point-contact current for the steady state is applicable to thermal analysis as well.

1. Introduction
The density distribution of point-contact current is significant for the design of various processes and devices such as resistance welding and mechanical relays. Melting by current input has been investigated considering the heat input only from the surface [1, 2]. However, heat is also generated inside a body because there exists a current density distribution. It is important to evaluate the effects of the current density distribution. Greenwood [3] investigated the steady-state heat generation inside a body; in this state, the heat generation at each point of the body equals the heat loss by diffusion at that point. Kubono [4] investigated heat generation for a non-steady state. In both these researches [3, 4], the same current density distribution was employed; however, the method for obtaining this distribution was not clarified. In the present research, the current distribution for the simplest boundary conditions is theoretically derived.

2. Theory
Figure 1 shows the simplest model that has an axially symmetric semi-infinite conductive body. The dark part represents the contact area. V [V] is the potential at the contact area; a [m] is the contact radius; and I [A] is the amount of current introduced in the body. The following assumptions are made here: (1) magnetic effects are ignored; (2) the contact area is equipotential V; (3) potential equals zero at an infinite distance from the contact area; (4) electric permittivity ε [N/V²] is constant; and (5) resistivity ρ [Ω m] is constant.
The basic equations for this model are Gauss’ law without charge $\text{div}\mathbf{E}(\mathbf{r}) = 0$ and the relationship between potential and electric field $\mathbf{E}(\mathbf{r}) = -\text{grad} V(\mathbf{r})$. By combining these relations, the Poisson equation is obtained as follows:

$$\Delta V(\mathbf{r}) = 0 \tag{1}$$

The potential distribution must satisfy the Poisson equation and boundary conditions described in the assumptions. In order to obtain the density distribution for point-contact current, an imaginary charge distribution in the contact area is assumed. The charge distribution is represented by equation (2).

$$c(r) = \frac{4Ve}{\pi} \frac{1}{\sqrt{a^2 - r^2}} \tag{2}$$

An integral pass is introduced as shown in Figure 2. $r$, $x$, and $s$ are the length of OR, OX, and O$_x$X, respectively. $x$ is expressed by $r$ and $s$ as $x = \sqrt{s^2 + r^2\sin^2\phi}$. $\phi$ is the angle between OR and RX. The charge in the black part is $c(x)|s + r\cos\phi|d\phi$ because the length of RX is $|s + r\cos\phi|$. The potential change at R due to the charge is $c(x)|s + r\cos\phi|d\phi(4\pi\varepsilon|s + r\cos\phi|)^{-1}$. The potential at R is obtained by integrating the potential change over the contact area where the range of $s$ and $\phi$ are $[-\sqrt{a^2 - r^2\sin^2\phi}, \sqrt{a^2 - r^2\sin^2\phi}]$ and $[0, \pi]$, respectively. The potential is obtained as follows:

$$V(r) = \int_{0}^{\pi} \int_{-\sqrt{a^2 - r^2\sin^2\phi}}^{\sqrt{a^2 - r^2\sin^2\phi}} c(x)|s + r\cos\phi|d\phi \frac{d\phi}{4\pi\varepsilon|s + r\cos\phi|}$$

$$= V$$

Figure 1. Model: Current through a semi-infinite bulk with voltage applied across contact area.

Figure 2. Integration path for deriving potential distribution in contact area.
It means that the potential based on equation (2) in the contact area satisfies the boundary condition for this model. It is clear that the potential based on equation (2) at an infinite distance from the origin is equal to zero.

In addition, the electric field based on equation (2) satisfies Gauss’ law inside the semi-infinite body since the electric field based on Coulomb’ law satisfies Gauss’ law. Therefore, the potential distribution based on equation (2) is the solution for the given system owing to the uniqueness of the Poisson equation. The electric field distribution is obtained on the basis of equation (2) because the electric field distribution has a one-to-one relationship with the potential distribution. Then, the current density distribution is obtained by Ohm’s law given by equation (3).

\[ i(r) = \frac{1}{\rho} E(r) \quad (3) \]

The current density distribution can be obtained as follows in a normalized vector form, on the basis of the steps described above. This equation is originally derived.

\[ i(\tilde{r}, \tilde{z}) = \frac{1}{\pi} \int_0^1 \int_0^{\pi/2} \frac{1}{\sqrt{1 - s^2} (\tilde{r}^2 - \tilde{z}^2 + s^2 + 2\tilde{r}s \cos \theta)^{3/2}} (\tilde{r} - s \cos \theta) d\theta ds, \quad (4) \]

where \( \tilde{r} := r/a, \tilde{z} := z/a, \tilde{i} := i/(2V/\pi \rho a) \).

### 3. Results and discussion

Figure 3 shows the results of equation (4). The current density distribution \(|i(\tilde{r}, \tilde{z})|\) has a singular point at \((\tilde{r}, \tilde{z}) = (1,0)\), which is the edge of the contact area. \(|i(\tilde{r}, \tilde{z})|\) is close to \(|i(\tilde{r}, \tilde{z})| = 1/(\tilde{r}^2 + \tilde{z}^2)\) and is sufficiently far from the origin \((0,0)\). In some special cases, the analytic solutions of \(|i(\tilde{r}, \tilde{z})|\) are obtained as follows:

\[ |i(\tilde{r}, 0)| = \begin{cases} 
\frac{1}{\sqrt{1 - \tilde{r}^2}} & (\tilde{r} \leq 1) \\
\frac{1}{\tilde{r} \sqrt{\tilde{r}^2 - 1}} & (\tilde{r} > 1)
\end{cases} \quad (5) \]

\[ |i(0, \tilde{z})| = \frac{1}{1 + \tilde{z}^2} \quad (6) \]

Equation (4) agrees with the current density distribution suggested in literature [3].

**Figure 3.** Current density distribution inside bulk.
\[ i(\xi, \eta) = \frac{I}{2\pi a^2 \sqrt{(1 + \xi^2)(\eta^2 + \xi^2)}} \]  

(7)

where \( \tilde{r} = [(1 + \xi^2)(1 - \eta^2)]^{1/2}, \tilde{z} = \xi \eta. \)

This agreement suggests that the current density distribution obtained under steady-state conditions is applicable for the non-steady state under the assumptions introduced in this research. As shown in Figure 3, the current density distribution has a singular point at the edge of the contact area. In general, if the heat input has an infinite value at some point, the temperature has an infinite value at that point. It is suggested that since the current density distribution is obtained under simple assumptions and temperature should have a finite value, it is necessary to treat the singular point appropriately for thermal analysis.

4. Conclusion
The density distribution of point-contact current is theoretically investigated under simple assumptions. The density distribution is obtained in an integral form and shows great agreement with the current density distribution suggested by Greenwood [3]. In particular, for special cases, the two distributions shows analytical agreement for \( \tilde{z} = 0 \) and \( \tilde{r} = 0 \). This result suggests the applicability of the current density distribution obtained by Greenwood for the thermal analysis of a system.

References
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