RG flow between $W_3$ minimal models by perturbation and domain wall approaches

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Abstract: We explore the RG flow between neighboring minimal CFT models with $W_3$ symmetry. After computing several classes of OPE structure constants we were able to find the matrices of anomalous dimensions for three classes of RG invariant sets of local fields. Each set from the first class consists of a single primary field, the second one of three primaries, while sets in the third class contain six primary and four secondary fields. We diagonalize their matrices of anomalous dimensions and establish the explicit maps between UV and IR fields (mixing coefficients).

While investigating the three point functions of secondary fields we have encountered an interesting phenomenon, namely violation of holomorphic anti-holomorphic factorization property, something that does not happen in ordinary minimal models with Virasoro symmetry solely.

Furthermore, the perturbation under consideration preserves a non-trivial subgroup of $W$ transformations. We have derived the corresponding conserved current explicitly. We used this current to define a notion of anomalous $W$-weights in perturbed theory: the analog for matrix of anomalous dimensions. For RG invariant sets with primary fields only we have derived a formula for this quantity in terms of structure constants. This allowed us to compute anomalous $W$-weights for the first and second classes explicitly.

The same RG flow we investigate also with the domain wall approach for the second RG invariant class and find complete agreement with the perturbative approach.
# Contents

1 A review of $W_3$ minimal models and RG flow 4  
   1.1 $W_3$ conformal field theories 4  
   1.2 RG flow between $A_2^{(p)}$ and $A_2^{(p-1)}$ 7  

2 Matrix of anomalous dimensions 10  
   2.1 Matrix of anomalous dimension for the first class 10  
   2.2 Matrix of anomalous dimensions for the second class 11  
   2.3 Matrix of anomalous dimensions for the third class 13  

3 Matrix of anomalous $W$-weights 15  
   3.1 Matrix of anomalous $W$-weights for the first class 18  
   3.2 Matrix of anomalous $W$-weights for the second class 18  

4 The RG domain wall 19  
   4.1 Current algebra 20  
   4.2 The IR/UV mixing coefficients through domain wall approach 22  

5 Conclusions and perspectives 23  

A The $\tilde{\phi}$ basis 24  

B Structure constants 25  
   B.1 Three point functions 25  
   B.2 OPE up to level one 32  
   B.3 Structure constants from 4-point functions 34  

C Hypergeometric functions for small $\epsilon$ 40  
   C.1 Small $\epsilon$ hypergeometric functions for the second class 41  
   C.2 Small $\epsilon$ hypergeometric functions for the third class 42  

D Structure constants for small $\epsilon$ 44  

E Representation of $\phi_i^{IR} \phi_j$ in terms of direct product WZNW models in case of second class 46
Introduction

In [1] A. Zamolodchikov investigated the RG flow from minimal model $\mathcal{M}_p$ to $\mathcal{M}_{p-1}$ initiated by the relevant field $\phi_{1,3}$. Using leading order perturbation theory valid for $p \gg 1$, for several classes of local fields he calculated the mixing coefficients specifying the UV - IR map. The next to leading order perturbation was analyzed in [2]. A similar RG trajectory connecting $\mathcal{N} = 1$ super-minimal models $\mathcal{S}\mathcal{M}_p$ to $\mathcal{S}\mathcal{M}_{p-2}$ was found in [3] (see also [4–6]). In this case the RG flow is initiated by the top component of the Neveu–Schwarz super-field $\Phi_{1,3}$.

For us it will be important that such a RG trajectory exists [7] also in the case of minimal models $A^{(p)}_{n-1}$ exhibiting $W_n$ extended conformal symmetry [8]. Namely, under certain slightly relevant perturbation the theory $A^{(p)}_{n-1}$ flows to $A^{(p-1)}_{n-1}$. In contrary to previous cases, this RG trajectory is not much investigated and we hope that in case of $n = 3$ the present work will substantially fill this gap. After computing several classes of OPE structure constants we construct the matrices of anomalous dimensions for three RG invariant classes of local fields and establish the detailed pattern of UV/IR map in these sectors.

To find the matrices of anomalous dimensions we need several classes of OPE structure constants. Some of them were already derived for Toda CFTs in [9]. Even for these known cases the analytic continuation to the minimal models of our interest is quite subtle. This is why we have preferred to derive these and some previously unknown structure constants from scratch. We consider three RG invariant classes and derive corresponding matrices of anomalous dimensions. The first class consists of a single primary, the second one three primaries and the third includes six primary and four level one secondary fields. In all cases we have diagonalized the matrices of anomalous dimensions and identified those specific combinations of UV fields which flow to certain primaries of IR theory.

While investigating the three point functions of secondary fields we have encountered an interesting phenomenon, namely violation of holomorphic anti-holomorphic factorization property, something that never happens in ordinary Virasoro minimal models, see (B.45). This does not contradict general principles since $W$ algebra symmetry generically speaking is not strong enough to reduce correlators of secondary fields to those with primaries only.

We have shown that the perturbation under consideration preserves a subgroup of $W$ transformations and constructed the corresponding conserved current explicitly. Based in this analysis we introduce and investigate the notion of anomalous $W$-weights (3.6) in close analogy with anomalous dimensions. Using our definition we derive an elegant expression for this matrix in terms of OPE structure constants (3.16). This
formula holds when the fields are primary which is not the case for third class.

In many cases the UV/IR mixing coefficients can be computed using a completely
different method by constructing the corresponding RG domain wall. Thus in [10, 11] it
was suggested that there exists an interface that encodes the map from UV observables
to IR. In [11] the RG domain wall was constructed for the \( N = 2 \) super-conformal
models using matrix factorization technique.

A nice algebraic construction of RG domain walls for coset CFT models was pro-
posed in [12]. For the case of Virasoro minimal models it was shown that this domain
wall correctly reproduces perturbative results of [1]. Later the consequences of this
proposal has been carefully tested in various situations (see e.g. [2, 13–15]).

The seminal example of RG flow in [1] has been extended for a very large class of
CFT coset models

\[
\mathcal{T}_{UV} = \frac{\hat{g}_l \times \hat{g}_m}{\hat{g}_{l+m}}, \quad m > l
\]

(0.1)
in [5, 16], where it was argued that under perturbation by a relevant field (denoted by
\( \phi = \phi_{1,1}^{Adj} \)), \( \mathcal{T}_{UV} \) flows to the theory

\[
\mathcal{T}_{IR} = \frac{\hat{g}_l \times \hat{g}_{m-l}}{\hat{g}_m}.
\]

(0.2)

Since \( W_3 \) minimal models \( A_2^{(p)} \) belong to this class:

\[
A_2^{(p)} = \frac{su(3)_{p-3} \times su(3)_{1}}{su(3)_{p-2}}.
\]

(0.3)

It is natural to carry out an alternative analysis of RG trajectory \( A_2^{(p)} \rightarrow A_2^{(p-1)} \) by
explicit construction of Gaiotto’s domain wall. As in [14], the method we applied is
based on the current algebra construction directly and, in this sense, is more general
than the one originally employed in [12]. We have developed an appropriate higher rank
generalization of this technique and applied it for the first non-trivial RG invariant set
of three primaries, demonstrating its consistency with our perturbative result. Unfor-
tunately calculations are rather cumbersome, and we have left the analysis of cases
including descendants for a future work. Such investigation is very desirable, since the
leading order perturbative analysis does not fix the mixing matrix completely due to
inherent degeneracy in conformal dimensions. Another route to attack this problem
would be extension of the notion of anomalous W-weights in case of descendants.

The paper is organized as follows: Section 1 (besides expressions for the structure
constants (1.30), (1.32)–(1.36)) is a review on CFTs with \( W_3 \) symmetry and the RG
flow. In section 2 matrices of anomalous dimensions are derived for all three classes
of RG invariant sets. In section 3 we introduce the notion of anomalous $W$-weights and derive its leading order expression in terms of structure constants. This matrix is constructed explicitly for the first two classes. The RG domain wall is reviewed and explicitly constructed for the second class in section 4. Finally many detailed computation can be found in the extensive appendices in this paper. In particular appendix B contains many important results which are instrumental in deriving the structure constants. In this appendix we meet an intriguing absence of holomorphic anti-holomorphic factorization property in three point functions including descendant fields.

1 A review of $W_3$ minimal models and RG flow

1.1 $W_3$ conformal field theories

In any conformal field theory the energy-momentum tensor has two nonzero components: the holomorphic field $T(z)$ with conformal dimension $(2,0)$ and its anti-holomorphic counterpart $\bar{T}(\bar{z})$ with dimensions $(0,2)$. In conformal theories with extended $W_3$ algebra summery one has in addition the currents $W(z)$ and $\bar{W}(\bar{z})$ with dimensions $(3,0)$ and $(0,3)$ respectively. These fields satisfy the OPE rules

$$T(z)T(0) = \frac{c/2}{z^4} + \frac{2T(0)}{z^2} + \frac{T'(0)}{z} + \cdots,$$  \hspace{1cm} (1.1)

$$T(z)W(0) = \frac{3W(0)}{z^2} + \frac{W'(0)}{z} + \cdots,$$  \hspace{1cm} (1.2)

$$W(z)W(0) = \frac{c/3}{z^6} + \frac{2T(0)}{z^4} + \frac{T'(0)}{z^3} + \frac{1}{z^2} \left( \frac{15c + 66}{10(22 + 5c)} T''(0) + \frac{32}{22 + 5c} \Lambda(0) \right) + \cdots.$$  \hspace{1cm} (1.3)

Here the field $\Lambda(z)$ is defined as

$$\Lambda(z) = :TT:(z) - \frac{3}{10} T''(z),$$  \hspace{1cm} (1.4)

where $::$ is regularization by means of subtraction of all OPE singular terms. Without the second term in (1.4) $\Lambda(z)$ wouldn’t be quasi-primary. The state created by this field

$$\Lambda(0)|0\rangle = \left( L_{-2}^2 - \frac{3}{10} L_{-1}^2 L_{-2} \right) |0\rangle.$$  \hspace{1cm} (1.5)

$\textsuperscript{1}$The corresponding expressions for the anti-holomorphic counterparts look exactly the same: one simply substitutes $z$ by $\bar{z}$. This is why we’ll mainly concentrate on the holomorphic part.
indeed is a Virasoro quasi-primary state i.e. \( L_1 \Lambda(0)|0\rangle = 0 \). We can expand these fields in Laurent series

\[
T(z) = \sum_{n=-\infty}^{+\infty} \frac{L_n}{z^{n+2}}, \quad W(z) = \sum_{n=-\infty}^{+\infty} \frac{W_n}{z^{n+3}}, \quad \Lambda(z) = \sum_{n=-\infty}^{+\infty} \frac{\Lambda_n}{z^{n+4}}.
\]

The OPEs (1.1), (1.2) and (1.3) are equivalent to the \( W_3 \) algebra relations

\[
[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0},
\]

\[
[L_n, W_m] = (2n - m)W_{n+m},
\]

\[
[W_n, W_m] = \alpha(n, m)L_{n+m} + \frac{16(n - m)}{22 + 5c}\Lambda_{n+m} + \frac{c}{360}(n^2 - 4)(n^2 - 1)n\delta_{n+m,0},
\]

where

\[
\alpha(n, m) = (n - m)\left(\frac{1}{15}(n + m + 2)(n + m + 3) - \frac{1}{6}(n + 2)(m + 2)\right),
\]

\[
\Lambda_n = d_n L_n + \sum_{m=-\infty}^{+\infty} :L_m L_{n-m}:.
\]

Here \( : \) means normal ordering (i.e operators with smaller index come first) and

\[
d_{2m} = \frac{1}{5}(1 - m^2), \quad d_{2m-1} = \frac{1}{5}(1 + m)(2 - m).
\]

The central charge of Virasoro algebra in \( A_2 \)-Toda CFT is given by

\[
c = 2 + 24q^2, \quad q = b + \frac{1}{b},
\]

where \( b \) is the (dimensionless) Toda coupling. In what follows it would be convenient to represent the roots, weights and Cartan elements of \( A_2 \) algebra as 3-component vectors (endowed with usual Kronecker scalar product) subject to the condition that sum of components is zero. In this notation the highest weights of fundamental and anti-fundamental representations take the form

\[
\omega_1 = \begin{pmatrix} 2/3 \\ -1/3 \\ -1/3 \end{pmatrix}, \quad \omega_2 = \begin{pmatrix} 1/3 \\ 1/3 \\ -2/3 \end{pmatrix}.
\]

Furthermore the weights of fundamental representation are

\[
h_1 = \begin{pmatrix} 2/3 \\ -1/3 \\ -1/3 \end{pmatrix}; \quad h_2 = \begin{pmatrix} -1/3 \\ 2/3 \\ -1/3 \end{pmatrix}; \quad h_3 = \begin{pmatrix} -1/3 \\ -1/3 \\ 2/3 \end{pmatrix}.
\]
The Weyl vector $e_0$ is half sum of positive roots or, alternatively the sum of highest weights of all fundamental representations

$$e_0 = \omega_1 + \omega_2. \quad (1.15)$$

The conformal (Virasoro) dimensions and $W$-weights of the exponential fields with charge $\alpha$ are given by

$$\Delta(\alpha) = \frac{\alpha \cdot (2Q - \alpha)}{2}, \quad (1.16)$$

$$w(\alpha) = \frac{\sqrt{6}bi}{\sqrt{(3b^2 + 5)(5b^2 + 3)}} \prod_{i=1}^{3} ((\alpha - Q) \cdot h_i), \quad (1.17)$$

where

$$Q = qe_0.$$ 

The conjugate charge $\alpha^*$ is defined through conditions

$$\alpha \cdot \omega_i = \alpha^* \cdot \omega_{3-i}; \quad i = 1, 2. \quad (1.18)$$

If represented as a three component vector, the conjugation amounts to reversing the direction and permuting the first and third components.

To pass from the Toda theory to the minimal models $A^{(p)}_2$, $p = 4, 5, 6, \cdots$, one specifies the parameter $b$ as

$$b = i \sqrt{\frac{p}{p+1}}. \quad (1.19)$$

From (1.12) for the central charge we get

$$c_p = 2 - \frac{24}{p(p+1)}. \quad (1.20)$$

Furthermore, all primary fields of the minimal models are doubly-degenerated. The allowed set of charges is listed below:

$$\alpha [n \quad m] = \frac{i (((n-1)(p+1) + (1-m)p)\omega_1 + ((n'-1)(p+1) + (1-m')p)\omega_2)}{\sqrt{p(p+1)}}, \quad (1.21)$$

where $n, n', m, m'$ are positive integers subject to constraints

$$n + n' \leq p - 1, \quad m + m' \leq p.$$
According to (1.18) the charge of conjugate field is given by
\[ \alpha^* \left[ \begin{array}{c} n \\ m \end{array} \right] = \alpha \left[ \begin{array}{c} n' \\ m' \end{array} \right]. \quad (1.22) \]

In view of (1.16) the conformal dimensions are given explicitly by
\[ \Delta \left[ \begin{array}{c} n \\ m \end{array} \right] = \frac{(p + 1)(n - n') - p(m - m')^2 + 3((p + 1)(n + n') - p(m + m'))^2 - 12}{12p(p + 1)}. \quad (1.23) \]

For later reference let us also present here the explicit expression of corresponding w weights derived from (1.17)
\[ w \left[ \begin{array}{c} n \\ m \end{array} \right] = \sqrt{\frac{2}{3}} \left((p + 1)(n - n') - p(m - m')\right) \times \frac{(p + 1)(n + 2n') - p(m + 2m')((p + 1)(2n + n') - p(2m + m'))}{9p(p + 1)\sqrt{(2p + 5)(2p - 3)}}. \quad (1.24) \]

One can check that the transformation \( \tau \) acting on charges as
\[ \tau \cdot \alpha \left[ \begin{array}{c} n \\ m \end{array} \right] = \alpha \left[ \begin{array}{c} n' \\ m' \end{array} \right], \quad (1.25) \]

leaves both dimensions and \( W_3 \) weights intact. \( \tau \) generates a \( \mathbb{Z}_3 \) group \( \tau^3 = 1 \). Thus the fields with \( \tau \) related charges get naturally identified.

### 1.2 RG flow between \( A_2^{(p)} \) and \( A_2^{(p-1)} \)

In what follows a special role is played by the primary field \( \varphi(x) \) characterized by charge
\[ \alpha \left[ \begin{array}{c} 1 \\ 2 \end{array} \right] = -bc_0. \quad (1.26) \]

This is a relevant field with conformal dimension
\[ \Delta := \Delta \left[ \begin{array}{c} 1 \\ 2 \end{array} \right] = \frac{p - 2}{p + 1} \equiv 1 - \epsilon < 1, \quad \epsilon = \frac{3}{p + 1}. \quad (1.27) \]

Notice also that the \( \varphi \) is \( w \)-neutral:
\[ w \left[ \begin{array}{c} 1 \\ 2 \end{array} \right] = 0. \quad (1.28) \]

Consider the family generated from this field by multiple application of OPE. It appears that \( \varphi \) is the only member of this family (besides the identity operator) which is relevant. This is important since it allows one to construct a consistent perturbed CFT with a single coupling constant:
\[ A = A_{CFT} + g \int \varphi(x)d^2x. \quad (1.29) \]

- 7 -
At large values of \( p \), which is the same as \( \epsilon \ll 1 \), the perturbing field is only slightly relevant and the conformal perturbation theory becomes applicable along a large portion of RG flow. In the case of positive coupling \( g > 0 \) this flow has been investigated in [7] and it was shown that in the infrared our initial theory \( A^{(p)}_2 \) flows to \( A^{(p-1)}_2 \). Our aim in this paper is to investigate this RG trajectory in more details, in particular investigating the UV/IR mixing matrices for some families of fields. One non-trivial part of our work was computation of some structure constants of OPE, which where not available explicitly in literature until now. For the diagonal structure constants we obtained:

\[
C_{\left[ \frac{n}{1} \frac{m}{1} \right]}^{\left[ \frac{n'}{1} \frac{m'}{1} \right]} = -\frac{\sqrt{\gamma (3 - 4 \rho) \gamma (2 - 2 \rho) \gamma (2 - 3 \rho) \gamma (1 - 3 \rho)}}{\gamma (2 - 3 \rho) \gamma (2 - 3 \rho) \gamma (1 - 3 \rho) \gamma (1 - 3 \rho)} \left[ \frac{3F_2}{\Gamma (1 - 3 \rho)} \right]
\]

\[
\times \left[ \frac{\sin (\pi (A_1 - 2 \rho)) \gamma (\rho + A_1) \gamma (\rho + B_1) \sin (\pi A_1) \gamma (2 - 2 \rho + A_1) \gamma (1 + B_1) \sin (\pi A_1) \gamma (2 - 2 \rho + A_1) \gamma (1 + B_1)}{\sin (\pi (A_2 - 2 \rho)) \gamma (\rho + A_2) \gamma (\rho + A_2) \gamma (2 - 2 \rho + A_2) \gamma (1 - A_1) \sin (\pi A_2) \gamma (2 - 2 \rho + A_2) \gamma (1 - A_1)} \right] \left[ \frac{1 - \rho + A_1, 1 - \rho + B_1}{1 + B_1, 2 - 2 \rho + A_1} \right]^2
\]

\[
+ \left[ \frac{\sin (\pi (A_2 - 2 \rho)) \gamma (\rho + A_2) \gamma (\rho + A_2) \gamma (2 - 2 \rho + A_2) \gamma (1 - A_1)}{\sin (\pi A_2) \gamma (2 - 2 \rho + A_2) \gamma (1 - A_1)} \right] \left[ \frac{1 - \rho + A_2, 1 - \rho + B_2}{1 + B_2, 2 - 2 \rho + A_2} \right]^2
\]

\[
+ \left[ \frac{\sin (\pi (A_2 - 2 \rho)) \gamma (\rho - A_2) \gamma (\rho - A_2) \gamma (2 - 2 \rho - A_2) \gamma (1 - B_1)}{\sin (\pi A_2) \gamma (2 - 2 \rho - A_2) \gamma (1 - B_1)} \right] \left[ \frac{1 - \rho + A_3, 1 - \rho + B_3}{1 + B_3, 2 - 2 \rho + A_2} \right]^2,
\]

where

\[
\gamma (x) \equiv \frac{\Gamma (x)}{\Gamma (1 - x)}, \quad \rho \equiv p/(p + 1)
\]

and

\[
A_1 = -n' + m' \rho, \quad A_2 = n + n' - (m + m') \rho, \quad A_3 = -A_2;
\]

\[
B_1 = n - m \rho, \quad B_2 = n' - m' \rho, \quad B_3 = -B_1.
\]

The remaining structure constants including the perturbing field \( \varphi (x) \) are much simpler

\[
C_{\left[ \frac{n}{1} \frac{m}{1} \right]}^{\left[ \frac{n}{1} \frac{m}{1} + 1 \right]} = \frac{\sqrt{\gamma (3 - 4 \rho) \gamma (2 - 2 \rho) \gamma (1 - 2 \rho) \gamma (2 - 3 \rho)}}{\gamma (1 - 2 \rho) \gamma (1 - 2 \rho) \gamma (2 - 3 \rho) \gamma (1 - 2 \rho)} \times
\]

\[
\frac{\sin (\pi (A_1 - 2 \rho)) \gamma (\rho + A_1) \gamma (\rho + B_1) \sin (\pi A_1) \gamma (2 - 2 \rho + A_1) \gamma (1 + B_1) \sin (\pi A_1) \gamma (2 - 2 \rho + A_1) \gamma (1 + B_1)}{\sin (\pi (A_2 - 2 \rho)) \gamma (\rho + A_2) \gamma (\rho + A_2) \gamma (2 - 2 \rho + A_2) \gamma (1 - A_1) \sin (\pi A_2) \gamma (2 - 2 \rho + A_2) \gamma (1 - A_1)} \left[ \frac{1 - \rho + A_1, 1 - \rho + B_1}{1 + B_1, 2 - 2 \rho + A_1} \right]^2
\]

\[
+ \left[ \frac{\sin (\pi (A_2 - 2 \rho)) \gamma (\rho + A_2) \gamma (\rho + A_2) \gamma (2 - 2 \rho + A_2) \gamma (1 - A_1)}{\sin (\pi A_2) \gamma (2 - 2 \rho + A_2) \gamma (1 - A_1)} \right] \left[ \frac{1 - \rho + A_2, 1 - \rho + B_2}{1 + B_2, 2 - 2 \rho + A_2} \right]^2
\]

\[
+ \left[ \frac{\sin (\pi (A_2 - 2 \rho)) \gamma (\rho - A_2) \gamma (\rho - A_2) \gamma (2 - 2 \rho - A_2) \gamma (1 - B_1)}{\sin (\pi A_2) \gamma (2 - 2 \rho - A_2) \gamma (1 - B_1)} \right] \left[ \frac{1 - \rho + A_3, 1 - \rho + B_3}{1 + B_3, 2 - 2 \rho + A_3} \right]^2,
\]

For reference let us quote here also the structure constants

\[
C_{\left[ \frac{n}{1} \frac{m}{1} \right]}^{\left[ \frac{n}{1} \frac{m}{1} + 1 \right]} = \sqrt{\gamma (n + n' - (m + m') \rho) \gamma (n + n' - (m + m') \rho)} \left[ \frac{1 - \rho + A_1, 1 - \rho + B_1}{1 + B_1, 2 - 2 \rho + A_1} \right]^2,
\]

\[
C_{\left[ \frac{n}{1} \frac{m}{1} \right]}^{\left[ \frac{n}{1} \frac{m}{1} + 1 \right]} = \sqrt{\gamma (3 - 3 \rho) \gamma (n + (1 - m) \rho) \gamma (n' - m' \rho)} \left[ \frac{1 - \rho + A_2, 1 - \rho + B_2}{1 + B_2, 2 - 2 \rho + A_2} \right]^2,
\]

\[
C_{\left[ \frac{n}{1} \frac{m}{1} \right]}^{\left[ \frac{n}{1} \frac{m}{1} - 1 \right]} = \sqrt{\gamma (3 - 3 \rho) \gamma (n + (1 - m) \rho) \gamma (n + n' - (m + m' - 1) \rho)} \left[ \frac{1 - \rho + A_3, 1 - \rho + B_3}{1 + B_3, 2 - 2 \rho + A_3} \right]^2.
\]
The three point function of the field \( \phi \) corresponds to the special choice \( \alpha = -b(\omega_1 + \omega_2) \) in (1.30). The formula gets simplified drastically. The final result reads (see [7])

\[
C_{\phi} \equiv C_{\left[\frac{1}{2}1\right]} - C_{\left[\frac{1}{2}1\right] \left[\frac{1}{2}1\right]} = \frac{2(4 - 5p)^2}{(3p - 2)(4p - 3)} \frac{\gamma^2(2 - \frac{3\rho}{2})}{\gamma(1 - \frac{p}{2})} \frac{\sqrt{\gamma(4 - 4\rho)\gamma(2 - 2\rho)}}{\gamma(3 - \frac{3\rho}{2})\gamma(3 - 3\rho)}. \quad (1.37)
\]

In the limit when \( p \gg 1 \) we get

\[
C_{\phi} = \frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2} \epsilon - \frac{4\sqrt{2}}{3} \epsilon^2 + O(\epsilon^3), \quad (1.38)
\]

\[
\beta(g) = \epsilon g - \frac{\pi}{2} C_{\phi} g^2 + O(g^3). \quad (1.39)
\]

Thus at

\[
g = g^* = \frac{2\sqrt{2} \epsilon}{3\pi} + O(\epsilon^2) \quad (1.40)
\]

the beta-function vanishes and we get an infrared fixed point. The shift of central charge is given by [1]

\[
c_p - c^* = 12\pi^2 \int_0^{g^*} \beta(g) dg = \frac{16}{9} \epsilon^3 + O(\epsilon^4). \quad (1.41)
\]

On the other hand from (1.20)

\[
c_p - c_{p-1} = \frac{48}{p(p^2 - 1)} = \frac{16}{9} \epsilon^3 + O(\epsilon^4). \quad (1.42)
\]

This strongly supports the identification of IR fixed point with \( A_2^{(p-1)} \) as proposed in [7]. Later we will give many more evidence in supporting and detailing such identification.

It is well known that the slope of beta function at a fixed point is directly related to the anomalous dimension of the perturbing field. In our case

\[
\Delta^* = 1 - \frac{d\beta}{dg} \bigg|_{g=g^*} = 1 + \epsilon + O(\epsilon^2) \quad (1.43)
\]

As expected the perturbing slightly relevant field \( \varphi \) at UV becomes slightly irrelevant at IR. Remind that the \( w \) weight of \( \varphi \) is zero. It is possible to show that this \( w \) weight doesn’t get perturbative corrections at the IR point so that also \( w_{IR} = 0 \). Examining (1.23) and (1.24) we see that the only primary field of \( A_2^{(p-1)} \) with charge

\[
\alpha \left[\frac{2}{2}1\right] = -b^{-1}(\omega_1 + \omega_2) \quad (1.44)
\]

has vanishing \( w \) weight and the desired dimension

\[
\Delta \left[\frac{2}{2}1\right]_{p \rightarrow p-1} = \frac{p+2}{p-1} = 1 + \epsilon + O(\epsilon^2). \quad (1.45)
\]

So that this field should be identified with the perturbing field at the IR fixed point. Another evidence for this conclusion comes from D. Gaiotto’s RG domain-wall approach discussed in section 4.
2 Matrix of anomalous dimensions

According to [1] the matrix of anomalous dimensions in leading order of coupling constant is given by

\[ \Gamma_{\alpha\beta} = \Delta_\alpha \delta_{\alpha\beta} + \pi g C^\alpha_{\beta\alpha} + O(g^2). \]  

(2.1)

In this section for the first three smallest RG invariant classes we are going to construct this matrix explicitly.

2.1 Matrix of anomalous dimension for the first class

It follows from the structure of OPE that each field \( \phi \left[ \frac{n}{n'} \frac{n}{n'} \right] \) by itself is RG invariant. From (1.30) by using following small \( \epsilon \) expansions of generalized hypergeometric functions of unit argument

\[
\begin{align*}
\text{3F2} \left( \frac{\epsilon}{3}, 1 - \frac{\epsilon}{3}(1 + n'), \frac{\epsilon}{3}(1 + n); 1 + \frac{n\epsilon}{3}, \frac{\epsilon}{3}(2 - n'); 1 \right) &= \frac{n' - n - 3}{n' - 2} + O(\epsilon), \\
\text{3F2} \left( \frac{\epsilon}{3}, 1 - \frac{\epsilon}{3}(1 - n - n'), \frac{\epsilon}{3}(1 + n'); 1 + \frac{n'\epsilon}{3}, \frac{\epsilon}{3}(2 + n + n'); 1 \right) &= \frac{n + 2n' + 3}{n + n' + 2} + O(\epsilon), \\
\text{3F2} \left( \frac{\epsilon}{3}, 1 - \frac{\epsilon}{3}(1 + n + n'), \frac{\epsilon}{3}(1 - n); 1 - \frac{n\epsilon}{3}, \frac{\epsilon}{3}(2 - n - n'); 1 \right) &= \frac{2n + n' - 3}{n + n' - 2} + O(\epsilon),
\end{align*}
\]

(2.3)
(2.4)
(2.5)

we obtain

\[
C_{\left[ \frac{n}{n'} \frac{n}{n'} \right]} = \frac{\epsilon^2 (n^2 + nn' + n'^2 - 3)}{27\sqrt{2}} + O(\epsilon^3). \]

(2.6)

From (2.1) and (1.40) at the fixed point for the anomalous dimension we immediately get

\[
\Gamma = \frac{1}{27} \epsilon^2 (n^2 + nn' + n'^2 - 3) + \frac{1}{27} \epsilon^3 (n^2 + nn' + n'^2 - 3) + O(\epsilon^4). \]

(2.7)

From (1.23) it is straightforward to see that

\[
\Gamma = \Delta^{(p-1)} \left[ \frac{n}{n'} \frac{n}{n'} \right] + O(\epsilon^4). \]

(2.8)

So we conclude that the UV field \( \phi^{(p)} \left[ \frac{n}{n'} \frac{n}{n'} \right] \) flows to

\[
\phi^{(p-1)} \left[ \frac{n}{n'} \frac{n}{n'} \right]
\]

(2.9)

in the infrared limit.
2.2 Matrix of anomalous dimensions for the second class

The second class of RG invariant sets is less trivial. Each set contains three primary fields:

\[ \phi_1 := \phi \left[ \frac{n}{n'} \frac{n+1}{n' +1} \right], \quad \phi_2 := \phi \left[ \frac{n}{n'} \frac{n}{n'} - \frac{n}{n'} +1\right], \quad \phi_3 := \phi \left[ \frac{n}{n'} \frac{n-1}{n' +1} \right]. \tag{2.10} \]

According to (1.23) their Virasoro dimensions for small \( \epsilon \) are

\[ \Delta_1 = \frac{1}{3} - \frac{\epsilon}{9} (2n + n' + 1) + O(\epsilon^2), \tag{2.11} \]
\[ \Delta_2 = \frac{1}{3} + \frac{\epsilon}{9} (n + 2n' - 1) + O(\epsilon^2), \tag{2.12} \]
\[ \Delta_3 = \frac{1}{3} + \frac{\epsilon}{9} (n - n' - 1) + O(\epsilon^2). \tag{2.13} \]

The relevant leading order diagonal structure constants are derived from (1.30) and (C.1). Here are the final results:

\[ C_{\left[ \frac{n}{n'} \frac{n+1}{n' +1} \right]}^{\left[ \frac{n}{n'} \frac{n}{n'} \right]} = \frac{2n(n + n' + 3) + 3(n' + 1)}{6\sqrt{2n(n + n')}} + O(\epsilon), \tag{2.14} \]
\[ C_{\left[ \frac{n}{n'} \frac{n}{n'} \right]}^{\left[ \frac{n}{n'} \frac{n}{n'} - \frac{n}{n'} +1\right]} = \frac{n(2n' - 3) + 2n'(n' - 3) + 3}{6\sqrt{2n'(n + n')}} + O(\epsilon), \tag{2.15} \]
\[ C_{\left[ \frac{n}{n'} \frac{n-1}{n' +1} \right]}^{\left[ \frac{n}{n'} \frac{n}{n'} \right]} = \frac{2nn' + 3n - 3n' - 3}{6\sqrt{2nn'}} + O(\epsilon). \tag{2.16} \]

The derivation of off diagonal components is easier. From (1.32) for small \( \epsilon \) we have

\[ C_{\left[ \frac{n}{n'} \frac{n+1}{n' +1} \right]}^{\left[ \frac{n}{n'} \frac{n}{n'} - \frac{n}{n'} +1\right]} = \frac{1}{n + n'} \sqrt{\frac{(n + 1)(n' - 1)((n + n')^2 - 1)}{8nn'}} + O(\epsilon). \tag{2.17} \]

From (1.33) and the symmetry of structure constants\(^2\)

\[ C_\gamma^\alpha = C_\gamma^{\alpha*} = \overline{C}_\gamma^\alpha \tag{2.18} \]

for small \( \epsilon \) we find

\[ C_{\left[ \frac{1}{n} \frac{1}{n'} \right]}^{\left[ \frac{n}{n'} \frac{n+1}{n' +1} \right]} = \frac{1}{n} \sqrt{\frac{(n^2 - 1)(n' + 1)(n + n' + 1)}{8n'(n + n')}} + O(\epsilon). \tag{2.19} \]

Similarly

\[ C_{\left[ \frac{1}{n} \frac{1}{n'} \right]}^{\left[ \frac{n}{n'} \frac{n-1}{n' +1} \right]} = \frac{1}{n'} \sqrt{\frac{(n - 1)(n'^2 - 1)(n + n' - 1)}{8n(n + n')}} + O(\epsilon). \tag{2.20} \]

\(^2\)Remind that \( * \) is the conjugation operation defined in (1.22).
Taking into account these results for the matrix of anomalous dimensions we get

\[\Gamma_{11} = \frac{1}{3} - \frac{\epsilon}{9} \left(1 + 2n + n' - \frac{2n(n + n' + 3) + 3(n + 1)}{n(n + n')}\right) + O(\epsilon^2), \quad (2.21)\]
\[\Gamma_{22} = \frac{1}{3} - \frac{\epsilon}{9} \left(1 - n - 2n' - \frac{n(2n' - 3) + 2(n' - 3)n + 3}{n'(n + n')}\right) + O(\epsilon^2), \quad (2.22)\]
\[\Gamma_{33} = \frac{1}{3} + \frac{\epsilon}{9} \left(1 + n - n' - \frac{3n' + 3}{n} + \frac{3}{n'}\right) + O(\epsilon^2), \quad (2.23)\]
\[\Gamma_{12} = \Gamma_{21} = \frac{\epsilon}{3(n + n')} \sqrt{\frac{(n + 1)(n' - 1) ((n + n')^2 - 1)}{nn'}} + O(\epsilon^2), \quad (2.24)\]
\[\Gamma_{13} = \Gamma_{31} = \frac{\epsilon}{3n} \sqrt{\frac{(n^2 - 1)(n' + 1)(n + n' + 1)}{n'(n + n')}} + O(\epsilon^2), \quad (2.25)\]
\[\Gamma_{23} = \Gamma_{32} = \frac{\epsilon}{3n'} \sqrt{\frac{(n - 1)(n'^2 - 1)(n + n' - 1)}{n(n + n')}} + O(\epsilon^2). \quad (2.26)\]

The eigenvalues of this matrix are

\[\Delta^{IR}_1 = \frac{1}{3} + \frac{\epsilon}{9}(1 - 2n - n') + O(\epsilon), \quad (2.27)\]
\[\Delta^{IR}_2 = \frac{1}{3} + \frac{\epsilon}{9}(1 + n + 2n') + O(\epsilon), \quad (2.28)\]
\[\Delta^{IR}_3 = \frac{1}{3} + \frac{\epsilon}{9}(1 + n - n') + O(\epsilon). \quad (2.29)\]

It is not difficult to identify the primary fields of \(A^{p-1}_2\) which have such dimensions with appropriate accuracy

\[\phi^{IR}_1 := \phi^{(p-1)} \left[ \begin{array}{c} n - 1 \\ n' \end{array} \right], \quad \phi^{IR}_2 := \phi^{(p-1)} \left[ \begin{array}{c} n \\ n' + 1 \end{array} \right], \quad \phi^{IR}_3 := \phi^{(p-1)} \left[ \begin{array}{c} n + 1 \\ n' - 1 \end{array} \right]. \quad (2.30)\]

To specify the combinations of our initial fields \(\phi_i\), \(i = 1, 2, 3\) which flow to (2.30) one should calculate the orthogonal matrix diagonalizing \(\Gamma\)

\[(R^T \Gamma R)_{ij} = \Delta_i^{IR} \delta_{ij}. \quad (2.31)\]

It is straightforward to get convinced that this mixing matrix is given explicitly by

\[R = \left( \begin{array}{ccc} \frac{1}{n(n + n')} \sqrt{(n^2 - 1)((n + n')^2 - 1)} & \frac{1}{n + n'} \sqrt{(n + 1)(n'n + 1) n} & \frac{1}{n} \sqrt{(n - 1)(n'n + 1) n(n + n')} \\ - \frac{1}{n + n'} \sqrt{(n - 1)(n'n + 1) n} & \frac{1}{n} \sqrt{(n + 1)(n'n + 1) n(n + n')} & - \frac{1}{n} \sqrt{(n - 1)(n'n + 1) n(n + n')} \\ - \frac{1}{n} \sqrt{(n + 1)(n'n + 1) n} & \frac{1}{n} \sqrt{(n - 1)(n'n + 1) n(n + n')} & \frac{1}{n + n'} \sqrt{(n^2 - 1)((n + n')^2 - 1)} \end{array} \right). \quad (2.32)\]

Thus we have established that

\[\phi_i^{IR}(x) = \sum_{j=1}^{3} \phi_j(x) R_{ji}. \quad (2.33)\]
2.3 Matrix of anomalous dimensions for the third class

The next class of sets we consider is substantially larger. Each set include 6 primaries and 4 first level descendants:

\[
\phi_1 = \phi \left[ \frac{n}{n'} \frac{n+1}{n'-1} \right], \quad \phi_2 = \phi \left[ \frac{n}{n'} \frac{n+1}{n'-1} \right], \quad \phi_3 = \phi \left[ \frac{n}{n'} \frac{n+1}{n'-2} \right],
\]

\[
\phi_i = \hat{O}_i \phi \left[ \frac{n}{n'} \frac{n}{n'} \right]; \quad \text{for } i = 4, 5, 6, 7,
\]

\[
\phi_8 = \phi \left[ \frac{n}{n'} \frac{n-1}{n'+2} \right], \quad \phi_9 = \phi \left[ \frac{n}{n'} \frac{n-1}{n'-1} \right], \quad \phi_{10} = \phi \left[ \frac{n}{n'} \frac{n-2}{n'+1} \right],
\]

where the operators \( \hat{O}_i \) are defined as

\[
\hat{O}_4 := L_{-1} L_{-1}; \quad \hat{O}_5 := L_{-1} (W_{-1} - \frac{3w_0}{2\Delta_0} L_{-1}); \quad \hat{O}_6 := (W_{-1} - \frac{3w_0}{2\Delta_0} L_{-1}) L_{-1};
\]

\[
\hat{O}_7 := (W_{-1} - \frac{3w_0}{2\Delta_0} L_{-1})(W_{-1} - \frac{3w_0}{2\Delta_0} L_{-1}).
\]

Here \( \Delta_0 \) and \( w_0 \) denote conformal dimension and \( W \)-weight of the field \( \phi \left[ \frac{n}{n'} \frac{n}{n'} \right] \) (cf. (1.24), (1.23)). The specific linear combination of \( L_{-1} \) and \( W_{-1} \) is chosen so that the descendant field

\[
\left( W_{-1} - \frac{3w_0}{2\Delta_0} L_{-1} \right) \phi \left[ \frac{n}{n'} \frac{n}{n'} \right]
\]

is a Virasoro quasi-primary. The conformal dimension of fields \( \phi_i \) are

\[
\Delta_1 = 1 - \frac{\epsilon}{2}(n+1) + O(\epsilon^2), \quad \Delta_2 = 1 - \frac{\epsilon}{2}(n+n'+1) + O(\epsilon^2), \quad \Delta_3 = 1 + \frac{\epsilon}{2}(n' - 1) + O(\epsilon^2),
\]

\[
\Delta_{4,5,6,7} = 1 + O(\epsilon^2), \quad \Delta_8 = 1 - \frac{\epsilon}{2}(n' + 1) + O(\epsilon^2), \quad \Delta_9 = 1 + \frac{\epsilon}{2}(n + n' - 1) + O(\epsilon^2), \quad \Delta_{10} = 1 + \frac{\epsilon}{2}(n - 1) + O(\epsilon^2).
\]

We have derived the matrix of anomalous dimensions using structure constants given in appendix D. Here is the final result

\[
\Gamma_{11} = 1 - \epsilon \frac{(n^2-5)n'^2+n(n^2-5)n'+n(n+2)}{3(n+1)n'(n'+n)},
\]

\[
\Gamma_{12} = \frac{\epsilon}{5n'} \sqrt{\frac{(n+2)(n'+n+2)(n'^2-1)}{(n+1)(n'+n+1)}},
\]

\[
\Gamma_{13} = \frac{\epsilon}{3(n'+n)} \sqrt{\frac{(n+2)(n'-2)(n'+n+1)(n'+n+2)}{(n+1)(n'-1)(n'+n+1)}},
\]

\[
\Gamma_{14} = \frac{(n-2)^2}{2(n^2+nn'+n'^2-3)} \sqrt{\frac{(n+2)(n'+1)(n'+n+1)}{nn'(n'+n)}},
\]

\[
\Gamma_{15} = \frac{(n-2)(2n'+n)}{2(n^2+nn'+n'^2-3)} \sqrt{\frac{(n^2-2)(n'+1)(n'+n-1)}{3n(n+1)(n'+n+1)}},
\]

\[
\Gamma_{16} = \Gamma_{15},
\]

\[
\Gamma_{17} = \frac{(n-1)(n'+1)(n'+n-1)(2n'+n)}{(n+1)(n^2+nn'+n'^2-3)} \sqrt{\frac{n+2}{n(n-1)n'(n'+n)(n'+n+1)}},
\]

\[

- 13 -
\]
\[ \Gamma_{22} = 1 + \epsilon \frac{n(-n'^3 + 7n' + 2) + n'(n' + 2) - n^3n' + n^2(1-2n'^2)}{3nn'(n' + n + 1)}, \] 
(2.44)

\[ \Gamma_{24} = \frac{\epsilon(n + n - 2)^2}{2(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{(n + 1)(n' + 1)(n' + n + 2)}{nn'(n' + n)}}, \] 
(2.45)

\[ \Gamma_{25} = \frac{\epsilon(n' - n)(n' + n - 2)}{2(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{(n - 1)(n' - 1)(n' + n - 1)(n' + n + 2)}{3nn'(n' + n + 1)}}, \] 
(2.46)

\[ \Gamma_{26} = \Gamma_{25}, \] 
(2.47)

\[ \Gamma_{27} = \epsilon \frac{(n - 1)(n - n')^2(n' - 1)(n' + n - 1)}{6(n' + n + 1)(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{n' + n + 2}{n(n + 1)n'(n' + 1)(n' + n)}}, \] 
(2.48)

\[ \Gamma_{28} = \epsilon \frac{\sqrt{n' - 1)(n' + 2)(n' + n + 2)}{3nn'(n' + n + 1)}}, \] 
(2.49)

\[ \Gamma_{33} = 1 + \epsilon \frac{nn'(n'^2 - 5) + (n' - 2)n'^2 + 2n'^2 - 5}{3nn'(n' + n)}, \] 
(2.50)

\[ \Gamma_{34} = \epsilon \frac{(n' + 2)^2}{2(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{(n + 1)(n' - 2)(n' + n - 1)}{nn'(n' + n)}}, \] 
(2.51)

\[ \Gamma_{35} = \frac{\epsilon(n' + 2)(n' + n - 2)}{2(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{(n - 1)(n' - 2)(n' + n + 1)(n' + n + 1)}{3nn'(n' + n)^2}}, \] 
(2.52)

\[ \Gamma_{36} = \Gamma_{35}, \] 
(2.53)

\[ \Gamma_{37} = \epsilon \frac{(n - 1)(n' + 1)(n' + n + 1)(n' + 2n)}{6(n' - 1)(6 + nn' + n'^2 - 3)} \sqrt{\frac{n'^2 - 2}{n(n + 1)n'(n' + n)(n' + n - 1)}}, \] 
(2.54)

\[ \Gamma_{39} = \epsilon \frac{\sqrt{n' - 2)(n' + 2)(n' + n + 2)}{3nn'(n' + n + 1)}}, \] 
(2.55)

\[ \Gamma_{44} = 1 + \frac{9\epsilon}{2(n^2 + nn' + n'^2 - 3)}, \quad \Gamma_{47} = \frac{9\epsilon}{2(n^2 + nn' + n'^2 - 3)}, \] 
(2.56)

\[ \Gamma_{48} = \frac{\epsilon(n - 2)^2}{2(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{(n - 1)(n' + 2)(n' + n + 1)}{nn'(n' + n)}}, \] 
(2.57)

\[ \Gamma_{49} = \frac{\epsilon(n' + 2)^2}{2(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{(n - 1)(n - 1)(n' + n - 2)}{nn'(n' + n)}}, \] 
(2.58)

\[ \Gamma_{410} = \frac{\epsilon(n + 2)^2}{2(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{(n - 2)(n' + 1)(n' + n + 1)}{nn'(n' + n)}}, \] 
(2.59)

\[ \Gamma_{55} = \Gamma_{44}, \quad \Gamma_{66} = \Gamma_{55}, \] 
(2.60)

\[ \Gamma_{56} = \Gamma_{47}, \] 
(2.61)

\[ \Gamma_{57} = \frac{\epsilon(n' - n)(2n + n')(n + 2n')}{n^2 + nn' + n'^2 - 3} \sqrt{\frac{3}{(n^2 - 1)(n'^2 - 1)(n + n + 1)}}, \] 
(2.62)

\[ \Gamma_{58} = -\frac{\epsilon(n' - 2)(n' + 2n)}{2(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{(n - 1)(n' + 1)(n'^2 + n' - 2)}{3nn'(n' + 1)(n' + n)}}, \] 
(2.63)

\[ \Gamma_{59} = \epsilon \frac{(n' - n)(n' + n + 2)}{2(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{(n + 1)(n' + 1)(n' + n - 2)(n' + n + 1)}{3nn'(n' + n + 1)(n' + n)}}, \] 
(2.64)

\[ \Gamma_{610} = \frac{\epsilon(n + 2)(2n' + n)}{2(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{(n - 2)(n' + 1)(n' + n + 1)}{3nn'(n' + n)}}, \] 
(2.65)

\[ \Gamma_{66} = \Gamma_{55}, \quad \Gamma_{67} = \Gamma_{57}, \quad \Gamma_{68} = \Gamma_{58}, \quad \Gamma_{69} = \Gamma_{59}, \quad \Gamma_{610} = \Gamma_{510}. \]
\[ \Gamma_77 = 1 + \epsilon \frac{3(n-n')^2(n'+2n)^2(2n'+n)^2 + 4(n^2+n'n'^2-3)^3 + 9(n^2+n'n'^2-3)^2}{6(n^2-1)(n'^2-1)(n^2+n'n'^2-3)((n'+n)^2-1)}, \]  
\[ \Gamma_{78} = \epsilon \frac{(n+1)(n'-1)(n'+n-1)(n'+2n)^2}{6(n+1)(n'^2+n'^2-3)} \sqrt{\frac{n'+2}{(n-1)n'n'(n'+n+1)}}, \]  
\[ \Gamma_{79} = \epsilon \frac{(n-n')(n'-n'^2)(n'+n+1)}{6(n'+n-1)(n^2+n'n'^2-3)} \sqrt{\frac{n'+n-2}{(n-1)n'(n'+n)}}, \]  
\[ \Gamma_{710} = \epsilon \frac{(n+1)(n'-1)(n'+n+1)(2n'+n)^2}{6(n-1)(n'+1)(n^2+n'n'^2-3)} \sqrt{\frac{(n-2)(n'+1)(n'+n-1)}{nn'(n'+n)}}, \]  
\[ \Gamma_{88} = 1 - \epsilon \frac{nn'(n'^2-5)+n'(n'+2)+n^2(n'^2-5)}{3n(n'+1)(n'^2-n-1)}, \]  
\[ \Gamma_{810} = \frac{\epsilon}{3(n'+1)} \sqrt{\frac{(n-2)(n'+2)(n'+n)^2}{(n-1)(n'+n)}}, \]  
\[ \Gamma_{99} = 1 + \epsilon \frac{n(n^2-7n'+2)-(n'-2)nn'2n'+n^2n'^2}{3nn'(n'+n-1)} \]  
\[ \Gamma_{910} = \frac{\epsilon}{3n} \sqrt{\frac{(n-2)(n'+n-2)(n'+n-1)}{n+1}}, \]  
\[ \Gamma_{1010} = 1 + \epsilon \frac{(n^2-5)nn'+n(n'-2)n'+(n-2)n}{3(n-1)n'(n'+n)}. \]  

Remind that \( \Gamma \) is symmetric and the matrix elements not listed above are equal to 0. Its eigenvalues are

\[ \Delta_{1R}^1 = 1 + \frac{\epsilon}{3}(1 + n') + O(\epsilon^2), \quad \Delta_{2R}^1 = 1 + \frac{\epsilon}{3}(1 + n + n') + O(\epsilon^2), \quad \Delta_{3R}^1 = 1 + \frac{\epsilon}{3}(1 - n) + O(\epsilon^2), \]
\[ \Delta_{4R}^1,6,7 = 1 + O(\epsilon^2), \]
\[ \Delta_{8R}^1 = 1 + \frac{\epsilon}{3}(1 + n) + O(\epsilon^2), \quad \Delta_{9R}^1 = 1 + \frac{\epsilon}{3}(1 - n - n') + O(\epsilon^2), \quad \Delta_{10R}^1 = 1 + \frac{\epsilon}{3}(1 - n') + O(\epsilon^2). \]

Using (1.23) we see that the \( A_2^{(p)} \) fields given in (2.34) flow to

\[ \phi^{(p-1)} \left[ \begin{array}{c} n-1 \\ n' + 2n' \end{array} \right], \quad \phi^{(p-1)} \left[ \begin{array}{c} n+1 \\ n' + 1n' \end{array} \right], \quad \phi^{(p-1)} \left[ \begin{array}{c} n-2 \\ n' + 1n' \end{array} \right], \]
\[ \phi^{(p-1)} = \hat{O}_i \phi^{(p-1)} \left[ \begin{array}{c} n' \\ n' \end{array} \right], \quad \text{for } i = 4, 5, 6, 7, \]
\[ \phi^{(p-1)} \left[ \begin{array}{c} n+2 \\ n' - 1 \end{array} \right], \quad \phi^{(p-1)} \left[ \begin{array}{c} n-1 \\ n' - 1n' \end{array} \right], \quad \phi^{(p-1)} \left[ \begin{array}{c} n+1 \\ n' - 2n' \end{array} \right] \]

in \( A_2^{(p-1)} \).

## 3 Matrix of anomalous \( W \)-weights

In perturbed theory \( W \) is no longer holomorphic. Using Ward identity for \( W \) current [9] together with (1.28) and null vector condition

\[ W_{-2}\varphi(z_i) = \frac{2}{\Delta + 1} \partial z_i W_{-1}\varphi(z_i) \]
one can verify the following equality

\[
\bar{\partial} \langle W(z) e^{-\int g \varphi d^2 x} \rangle = \sum_{n=0}^{\infty} \sum_{i=1}^{n} \frac{(-g)^n}{n!} \times
\int \bar{\partial} \langle \varphi(x_1) \cdots \left( \frac{1}{(z-x_i)^2} + \frac{2}{(\Delta+1)(z-x_i)} \right) W_{-1} \varphi(x_1) \cdots \varphi(x_n) \rangle d^2 x_1 \cdots d^2 x_n.
\] (3.1)

Next we use

\[
\bar{\partial}(z - x_i)^{-1} = \pi \delta^{(2)}(z - x_i), \quad \text{hence} \quad \bar{\partial}(z - x_i)^{-2} = -\pi \delta'(z - x_i)
\] (3.2)

and evaluate the integral over \(x_i\) in (3.1). The result can be conveniently represented as

\[
\bar{\partial} \langle W(z) e^{-\int g \varphi d^2 x} \rangle = -g \pi \frac{1 - \Delta}{1 + \Delta} \langle (\partial W_{-1} \varphi(z)) e^{-\int g \varphi d^2 x} \rangle,
\] (3.3)

which indicates that the following conservation law is satisfied

\[
\bar{\partial} W(z, \bar{z}) + \pi g \frac{1 - \Delta}{1 + \Delta} \partial W_{-1} \varphi(z, \bar{z}) = 0.
\] (3.4)

This identity can be rewritten as

\[
\bar{\partial}(z^2 W) + \pi g \frac{1 - \Delta}{1 + \Delta} \partial(z^2 W_{-1} \varphi) = 2\pi g \frac{1 - \Delta}{1 + \Delta} z W_{-1} \varphi.
\] (3.5)

Certainly this is not a conservation law (due to non zero right hand side). Still a charge (non conserved) denoted by \(\Omega\) - the analog of \(W\) zero mode in perturbed theory can be defined as

\[
\Omega = \int_{C} \left( z^2 W(z) \frac{dz}{2\pi i} - \pi g \frac{1 - \Delta}{1 + \Delta} z^2 W_{-1} \varphi(z) \frac{d\bar{z}}{2\pi i} \right).
\] (3.6)

Since in 2d QFT it is common to consider radial quantization one can choose \(C\) to be a circle centered at zero. Let \(\Lambda\) be a region bounded by two circles one of large radius \(R\) and the second with a very small radius \(\varepsilon\). Integrating the lhs of equation (3.5) in presence of a field located at zero and applying Stokes’ theorem we find

\[
\left( \int_{C_R} - \int_{C_{\varepsilon}} \right) \left( z^2 W(z) \frac{dz}{2\pi i} - \pi g \frac{1 - \Delta}{1 + \Delta} z^2 W_{-1} \varphi(z) \frac{d\bar{z}}{2\pi i} \right) \phi_\beta(0).
\]

Due to smallness of \(\varepsilon\) the influence of interaction is irrelevant and the contribution of \(C_{\varepsilon}\) is equal to \(w_\beta \phi_\beta\) where \(w_\beta\) is the \(W\)-weight of \(\phi_\beta\) in unperturbed UV theory. Instead the contribution of large contour \(C_R\) is just the variation of \(\phi_\beta\) by the charge \(Q\). Thus we finally arrive at

\[
\hat{Q}(\phi_\alpha) = w_\alpha \phi_\alpha(0) + 2\pi g \frac{\varepsilon}{2 - \varepsilon} \int_{\Lambda} z W_{-1} \varphi(z) \phi_\alpha(0) \frac{dz}{2\pi i}.
\] (3.7)
Consider the OPE

$$W_{-1}\varphi(z)\phi_{\alpha}(0) = \sum_{\beta} z^{\Delta_{\beta}-\Delta-\Delta_{\alpha}-1}b_{\alpha}^{\beta}\phi_{\beta} + \ldots.$$  (3.8)

It is convenient to represent the structure constants $b_{\alpha}^{\beta}$ as

$$b_{\alpha}^{\beta} = \left(1 - \frac{\epsilon}{2}\right)a_{\alpha\beta}C_{\alpha}^{\beta}.$$  (3.9)

It follows from $W$-Ward identities (see (B.16)) that

$$a_{\alpha\beta} = w_{\alpha} - w_{\beta} \quad \text{if} \quad \alpha \neq \beta.$$  (3.10)

The diagonal element $a_{\alpha\alpha}$ can not be determined by Ward identities alone. Still in cases of our interest we are able to determine it by investigating four point correlation functions (see appendix B.3). In (3.7) we pass to radial coordinates $dz\bar{z} = 2i\pi drd\theta$ and the integral over $r$ is evaluated from $[0,R]$ where $R$ is an infrared scale which for convenience we choose $R = 1$. The result is

$$\hat{Q}(\phi_{\alpha}) = w_{\alpha}\phi_{\alpha}(0) + \pi g\epsilon\sum_{\beta} \frac{a_{\alpha\beta}C_{\alpha}^{\beta}}{\Delta_{\beta\alpha} + \epsilon}\phi_{\beta}(0).$$  (3.11)

Let us introduce the renormalized fields (see appendix A for details)

$$\tilde{\phi}_{\alpha} = \phi_{\alpha} + \sum_{\beta} \frac{\pi gC_{\alpha}^{\beta}}{\Delta_{\beta\alpha} + \epsilon}\phi_{\beta}.$$  (3.12)

From (3.11)

$$\hat{Q}(\tilde{\phi}_{\alpha}) = w_{\alpha}\tilde{\phi}_{\alpha} + \pi g\epsilon\sum_{\beta} \frac{a_{\alpha\beta}C_{\alpha}^{\beta}}{\Delta_{\beta\alpha} + \epsilon}\tilde{\phi}_{\beta} + O(g^2) =$$

$$w_{\alpha}\left(\phi_{\alpha} + \frac{\pi gC_{\alpha}^{\beta}}{\Delta_{\beta\alpha} + \epsilon}\phi_{\beta}\right) - \frac{\pi gw_{\alpha}C_{\alpha}^{\beta}}{\Delta_{\beta\alpha} + \epsilon}\phi_{\beta} + \frac{\pi g\epsilon a_{\alpha\beta}C_{\alpha}^{\beta}}{\Delta_{\beta\alpha} + \epsilon}\phi_{\beta} + \frac{\pi gw_{\alpha}C_{\alpha}^{\beta}}{\Delta_{\beta\alpha} + \epsilon}\phi_{\beta} + O(g^2)$$

or

$$\hat{Q}(\tilde{\phi}_{\alpha}) = w_{\alpha}\tilde{\phi}_{\alpha} + \frac{\pi gC_{\alpha}^{\beta}}{\Delta_{\beta\alpha} + \epsilon}(w_{\beta\alpha} + \epsilon a_{\alpha\beta})\tilde{\phi}_{\beta} + O(g^2).$$  (3.13)

Using (3.10) we get

$$\hat{Q}(\tilde{\phi}_{\alpha}) = w_{\alpha}\tilde{\phi}_{\alpha} + \pi g\sum_{\beta} a_{\alpha\beta}C_{\alpha}^{\beta}\tilde{\phi}_{\alpha} + O(g^2).$$  (3.14)

Thus for the matrix element of the $W$-charge $\mathcal{Q}$ we find

$$\mathcal{Q}_{\alpha\beta} = w_{\beta}\delta_{\alpha\beta} + \pi ga_{\alpha\beta}C_{\alpha}^{\beta} + O(g^2).$$  (3.15)

- 17 -
3.1 Matrix of anomalous $W$-weights for the first class

Here each RG invariant set consists of a single field (2.2). Its $W$-weight is

$$w\left[\begin{array}{cc} n & n' \\
\end{array}\right] = \frac{\epsilon^3(n' - n)(2n + n')(n + 2n')}{243 \sqrt{6}} + \frac{\epsilon^4(n' - n)(2n + n')(n + 2n')}{486 \sqrt{6}} + O(\epsilon^5). \hspace{1cm} (3.17)$$

For this case the coefficient $a$ entering in (3.16) is given by eq. (B.54). Taking into account also (2.6) and (1.40) we obtain

$$\Omega = \frac{\epsilon^3(n' - n)(2n + n')(n + 2n')}{243 \sqrt{6}} + \frac{\epsilon^4(n' - n)(2n + n')(n + 2n')}{162 \sqrt{6}} + O(\epsilon^5), \hspace{1cm} (3.18)$$

which indeed coincides with $W$-weight of $\phi(p^{-1})\left[\begin{array}{cc} n & n' \\
\end{array}\right]$ with desired accuracy.

3.2 Matrix of anomalous $W$-weights for the second class

The fields of second class (2.10) in UV limit have the following $W$-dimensions

$$w_1 = \frac{2}{9 \sqrt{6}} - \frac{\epsilon}{9 \sqrt{6}} (2n + n' + 1) + O(\epsilon^2), \hspace{1cm} (3.19)$$

$$w_2 = \frac{2}{9 \sqrt{6}} + \frac{\epsilon}{9 \sqrt{6}} (n + 2n' - 1) + O(\epsilon^2), \hspace{1cm} (3.20)$$

$$w_3 = \frac{2}{9 \sqrt{6}} + \frac{\epsilon}{9 \sqrt{6}} (n - n' - 1) + O(\epsilon^2). \hspace{1cm} (3.21)$$

Comparison with (2.11) ensures that for $\alpha \neq \beta$

$$\frac{w_\alpha - w_\beta}{\Delta_\alpha - \Delta_\beta} = \frac{1}{\sqrt{6}}.$$

In appendix B.3 we show how to derive the value

$$a_{\alpha\alpha} = 1/\sqrt{6}. \hspace{1cm} (3.22)$$

Thus (3.16) becomes

$$\Omega_{\alpha\beta} = w_\beta \delta_{\alpha\beta} + \frac{\pi g}{\sqrt{6}} C^\alpha_\beta + O(g^2). \hspace{1cm} (3.23)$$
Explicitly

\[ \Omega_{11} = \frac{2}{9\sqrt{6}} + \frac{\epsilon}{9\sqrt{6}} \left( \frac{2n(n+n'+3)+3(n'+1)}{n(n+n')} - 2n - n' - 1 \right) + O(\epsilon^2), \]

\[ \Omega_{22} = \frac{2}{9\sqrt{6}} + \frac{\epsilon}{9\sqrt{6}} \left( \frac{n(2n'-3)+2(n'-3)n'+3}{n'(n+n')} + n + 2n' - 1 \right) + O(\epsilon^2), \]

\[ \Omega_{33} = \frac{2}{9\sqrt{6}} + \frac{\epsilon}{9\sqrt{6}} \left( \frac{3n-3(n'+1)}{nn'} + n - n' + 1 \right) + O(\epsilon^2), \]

\[ \Omega_{12} = \frac{\epsilon}{3\sqrt{6(n+n')}} \sqrt{\frac{(n+1)(n'-1)((n+n')^2-1)}{nn'}} + O(\epsilon^2), \]

\[ \Omega_{13} = \frac{\epsilon}{3\sqrt{6n}} \sqrt{\frac{(n^2-1)(n'+1)(n+n'+1)}{n'(n+n')}} + O(\epsilon^2), \]

\[ \Omega_{23} = \frac{\epsilon}{3\sqrt{6n'}} \sqrt{\frac{(n-1)(n'^2-1)(n+n'-1)}{n(n+n')}} + O(\epsilon^2). \]

Here are the eigenvalues of this matrix

\[ w_1^{IR} = \frac{2}{9\sqrt{6}} - \frac{\epsilon(2n+n'-1)}{9\sqrt{6}} + O(\epsilon^2), \]

\[ w_2^{IR} = \frac{2}{9\sqrt{6}} + \frac{\epsilon(n+2n'+1)}{9\sqrt{6}} + O(\epsilon^2), \]

\[ w_3^{IR} = \frac{2}{9\sqrt{6}} + \frac{\epsilon(n-n'+1)}{9\sqrt{6}} + O(\epsilon^2), \]

which coincide with the \( w \) dimensions of the fields (2.30) in \( A_2^{(p-1)} \) as expected.

4 The RG domain wall

In [12] D. Gaiotto suggests a candidate for RG domain wall for the general RG flow between theories (0.1) and (0.2). Let us briefly recall his construction. Any conformal interface between two CFTs can be alternatively represented as a conformal boundary condition for the direct product of these two theories (folding trick). This boundary condition encodes the UV/IR map. The construction of appropriate boundary is heavily based on coset representation of \( W \) minimal models (0.3). For the direct product theory we get

\[ T_{UV} \times T_{IR} \sim \frac{\hat{g}_{m-l} \times \hat{g}_l}{\hat{g}_m} \times \frac{\hat{g}_m \times \hat{g}_l}{\hat{g}_{m+l}} \sim \frac{\hat{g}_{m-l} \times \hat{g}_l \times \hat{g}_l}{\hat{g}_{l+m}}. \]

Notice the appearance of two identical factors \( \hat{g}_l \) in the numerator. Hence the resulting theory admits a non trivial \( \mathbb{Z}_2 \) symmetry which intertwines these two factors. Essentially D. Gaiotto conjecture boils down to the statement that boundary of the theory

\[ T_B = \frac{\hat{g}_l \times \hat{g}_l \times \hat{g}_{m-l}}{\hat{g}_{l+m}}, \quad m > l \]
acts as a $\mathbb{Z}_2$ twisting mirror. The basic building block of this boundary state is defined by the property that mirror image of any local field of product theory is identified with its $\mathbb{Z}_2$ counterpart. As in standard Cardy construction these building blocks should be superposed with appropriate coefficients in order to get the “physical” boundary state.

These coefficients have been explicitly identified in [12] in terms of current algebra modular matrix. The final expression can be formally displayed as

$$ |\tilde{B}\rangle = \sum_{s,t} \sqrt{S_{1,t}^{(m-l)} S_{1,s}^{(m+l)}} \sum_d |t,d,d,s;Z_2\rangle , \quad (4.3) $$

where the indices $t$, $d$, $s$ refer to the representations of $\hat{g}_{m-l}$, $\hat{g}_t$, $\hat{g}_{l+m}$ respectively, $S_{1,r}^{(k)}$ are the modular matrices of the $\hat{g}_k$ WZNW model and $|t,d,d,s;Z_2\rangle$ is above mentioned $\mathbb{Z}_2$ image of the Ishibashi state $|t,d,d,s\rangle$ of the product theory.

In what follows we will examine in details the RG flow between $W_3$ minimal models for our second class.

4.1 Current algebra

The WZNW models have extended chiral symmetry generated by Virasoro and Kač-Moody algebras. The Kač–Moody generators can be considered as Laurent coefficients of spin one holomorphic currents. In case of $SU(3)$ WZNW models we deal with the algebra $\hat{A}_2$. It will be convenient for us to choose generators (inherited from standard $sl(3)$ Cartan-Weyl basis) $E_{n}^{ij}$ and $H_{n}^{i}$ with $i,j \in \{1,2,3\}$, $i \neq j$ and $n \in \mathbb{Z}$ is the loop index.

$$ [E_{n}^{ij} E_{m}^{rl}] = \delta^{jr} E_{n+m}^{il} - \delta^{li} E_{n+m}^{rj} + kn \delta_{n+m} \delta^{il} \delta^{jr} , $$

$$ [E_{n}^{ij} H_{m}^{r}] = \delta^{jr} E_{n+m}^{ir} - \delta^{ri} E_{n+m}^{rj} , $$

$$ [H_{n}^{i} H_{m}^{j}] = kn \delta_{n+m} \left( \delta_{ij} - \frac{1}{3} \right) , \quad (4.4) $$

where $k = 1,2,3,...$ referred as the level specifies a model. It is also assumed that

$$ \sum_{i=1}^{3} H_{n}^{i} = 0 . $$

The Virasoro generators can be expressed in terms of current algebra through the Sugawara construction

$$ L_{n} = \frac{1}{2(k+3)} \sum_{l \in \mathbb{Z}} \sum_{i,j=1}^{3} :E_{n-l}^{ij} E_{n}^{ji} : . \quad (4.5) $$
The normal ordering prescription is the standard one: The order of generators should be flipped provided the loop index of first generator is less than that of the second one. Using (4.4) and (4.5) one can get convinced that the Virasoro charge equals to

$$c_k = \frac{8k}{k+3}. \quad (4.6)$$

The primary fields of the theory (denoted as $\phi_\lambda$) are labeled by the highest weights

$$\lambda = (n-1)\omega_1 + (n'-1)\omega_2, \quad (4.7)$$

where $n, n'$ are positive integers and $\omega_1, \omega_2$ are the same as in (1.13). The following formula for conformal dimensions of primary fields can be readily deduced from (4.5)

$$h_k(\lambda) = \frac{\lambda \cdot (\lambda + 2e_0)}{2(k+3)}. \quad (4.8)$$

The corresponding primary states satisfy the conditions

$$H^i_0 |\phi_\lambda\rangle = \lambda^i |\phi_\lambda\rangle, \quad (4.9)$$

$$E^{ij}_0 |\phi_\lambda\rangle = 0 \quad \text{for} \quad i < j, \quad (4.10)$$

$$E^{ij}_n |\phi_\lambda\rangle = H^i_n |\phi_\lambda\rangle = 0 \quad \text{for} \quad n > 0. \quad (4.11)$$

To construct the domain wall we need also the explicit form of the $\hat{A}_2$ modular matrix [17] which in notations specified above takes the form

$$S^{(k)}_{\lambda, n, n', \lambda, m, m'} = \frac{i^3}{(k+3)\sqrt{3}} \sum_{w \in W} \varepsilon(w) \exp \left[ -\frac{2\pi i w(\lambda_{m+1, m'+1}) \cdot \lambda_{n+1, n'+1}}{k+3} \right]. \quad (4.12)$$

Here the sum runs over 6 elements of the Weil group $W$ acting on three component vectors through permutations. More explicitly

$$S^{(k)}_{\lambda, n, n', \lambda, m, m'} = \frac{-i}{\sqrt{3}(k+3)} \left( e_k \left[ 2m'n' + nm' + mn' - mn \right] - e_k \left[ 2m'n + nm' + mn' + 2mn \right] \right)$$

$$+ e_k \left[ -m'n' + nm' + mn' + 2mn \right] - e_k \left[ -m'n - 2mn' + mn' - mn \right]$$

$$-e_k \left[ -m'n' + nm' - 2mn' - mn \right] + e_k \left[ -m'n' - 2mn' - 2mn' - mn \right]. \quad (4.13)$$

with notation

$$e_k [x] := e^{-\frac{2\pi ix}{k+3}}. \quad (4.14)$$

According to [18, 19] the $W_3$ minimal models can be alternatively represented as coset theory

$$\frac{\hat{su}(3)_k \times \hat{su}(3)_1}{\hat{su}(3)_{k+1}}, \quad (4.15)$$
where $\tilde{s}u(3)_k$ stands for level $k$ WZNW model [20, 21] with identification

$$p = k + 3. \quad (4.16)$$

The stress energy tensor of the coset theory (4.15) is given by

$$T_{(su(3)_k \times su(3)_1)/su(3)_{k+1}} = T_{su(3)_k} + T_{su(3)_1} - T_{su(3)_{k+1}}. \quad (4.17)$$

In particular for the central charge we get

$$c_{(su(3)_k \times su(3)_1)/su(3)_{k+1}} = c_{su(3)_k} + c_{su(3)_1} - c_{su(3)_{k+1}}. \quad (4.18)$$

Taking into count (4.6) one easily gets convinced that (4.18) coincides with the central charge of $A_2^{(p)}$ minimal model (1.20) with already mentioned identification (4.16).

### 4.2 The IR/UV mixing coefficients through domain wall approach

For the $W_3$ minimal models (4.1) becomes

$$\tilde{s}u(3)_{k-1} \times \tilde{s}u(3)_1 \sim \tilde{s}u(3)_{k+1} \times \tilde{s}u(3)_1. \quad (4.19)$$

The formula for mixing coefficients dictated by (4.3) is

$$\langle \phi^{IR} \left[ \frac{m_j}{m_j'} \right] \phi^{UV} \left[ \frac{n}{n'} \frac{m_j}{m_j'} \right] | RG \rangle = \frac{\tilde{S}^{(k-1)}_{(k+1)}}{\tilde{S}^{(k+1)}_{(k+1)}} \frac{\tilde{S}^{(k-1)}_{(k+1)}}{\tilde{S}^{(k+1)}_{(k+1)}} \frac{\tilde{S}^{(k-1)}_{(k+1)}}{\tilde{S}^{(k+1)}_{(k+1)}}. \quad (4.20)$$

For the second class we set $\phi^{UV} \left[ \frac{n}{n'} \frac{m_j}{m_j'} \right] \equiv \phi_j$ from (2.10) and $\phi^{IR} \left[ \frac{m_j}{m_j'} \frac{n}{n'} \right] \equiv \phi_i^{IR}$ from (2.30). $|ij\rangle$ is the representative of state $\phi_i^{IR} \phi_j |0 \rangle$ in direct product theory $\tilde{s}u(3)_{k-1} \times \tilde{s}u(3)_1 \times \tilde{s}u(3)_1$. Tilde indicates permutation of the $\tilde{s}u(3)_k$ factors. We will use the notations $E$, $J$, $\tilde{J}$ for current algebra generators of above mentioned three factors respectively.

Let us start with identification of $\phi^{IR} \left[ \frac{n-1}{n} \frac{n'}{n'} \right] \phi^{UV} \left[ \frac{n}{n'} \frac{n+1}{n} \right]$ in triple product theory. Notice that from (1.23) and (4.8) we have

$$\Delta^{IR} \left[ \frac{n-1}{n} \frac{n'}{n'} \right] = h_{k-1} (\lambda_{n-1,n'}) + h_1 (\lambda_{2,1}) - h_k (\lambda_{n,n'}), \quad (4.21)$$

$$\Delta^{UV} \left[ \frac{n}{n'} \frac{n+1}{n'} \right] = h_{k} (\lambda_{n,n'}) + h_1 (\lambda_{2,1}) - h_{k+1} (\lambda_{n+1,n'}). \quad (4.22)$$

Also using (4.7) one can check that

$$\lambda_{n-1,n'} + \lambda_{2,1} + \lambda_{2,1} = \lambda_{n+1,n'}. \quad (4.23)$$
Thus there is a single way to construct a highest weight \( \lambda_{n+1,n'} \) of total current algebra inside the tensor product \( V_{n-1,n'} \otimes V_{2,1} \otimes V_{2,1} \). Namely from each factor one should take the corresponding highest weight state

\[
|11\rangle = |\lambda_{n-1,n'}\rangle |\lambda_{2,1}\rangle |\lambda_{2,1}\rangle.
\]  

(4.24)

Notice that \( \lambda_{n-1,n'} + \lambda_{2,1} = \lambda_{n,n'} \) so the constraint coming from the IR coset is also satisfied.

Identification of the remaining states \( |i,j\rangle \) are more subtle. The details can be found in App.E. Using the results of this appendix and (4.13), (4.20), it is easy to derive UV/IR mixing coefficients explicitly. The result for large \( k \) is

\[
\langle \phi^R_1 \phi_1|RG\rangle = \frac{\sqrt{(n^2 - 1)(n + n')^2 - 1}}{n(n + n')} + O\left(\frac{1}{k}\right),
\]

(4.25)

\[
\langle \phi^R_1 \phi_2|RG\rangle = -\frac{1}{n + n'} \sqrt{\frac{(n-1)(n'-1)}{nn'}} + O\left(\frac{1}{k}\right),
\]

\[
\langle \phi^R_1 \phi_3|RG\rangle = -\frac{1}{n} \sqrt{\frac{(n'+1)(n + n'-1)}{n'(n + n')}} + O\left(\frac{1}{k}\right),
\]

\[
\langle \phi^R_2 \phi_1|RG\rangle = \frac{1}{(n + n') \sqrt{\frac{(n+1)(n'+1)}{n'n}} + O\left(\frac{1}{k}\right)},
\]

\[
\langle \phi^R_2 \phi_2|RG\rangle = \frac{1}{n} \sqrt{\frac{(n+1)(n'+1)}{n(n + n')}} + O\left(\frac{1}{k}\right),
\]

\[
\langle \phi^R_2 \phi_3|RG\rangle = \frac{1}{n} \sqrt{\frac{(n-1)(n + n'+1)}{n(n + n')}} + O\left(\frac{1}{k}\right),
\]

\[
\langle \phi^R_3 \phi_1|RG\rangle = \frac{1}{n} \sqrt{\frac{(n'-1)(n + n'+1)}{n'(n + n')}} + O\left(\frac{1}{k}\right),
\]

\[
\langle \phi^R_3 \phi_2|RG\rangle = \frac{1}{n'} \sqrt{\frac{(n+1)(n + n'-1)}{n(n + n')}} + O\left(\frac{1}{k}\right),
\]

\[
\langle \phi^R_3 \phi_3|RG\rangle = \frac{\sqrt{(n^2 - 1)(n'+2-1)}}{nn'} + O\left(\frac{1}{k}\right).
\]

These is in complete agreement with our perturbative result (2.32). This is a strong evidence confirming that Gaiotto’s RG domain wall conjecture is correct also in the higher rank case.

### 5 Conclusions and perspectives

In this paper we investigated the RG flow between \( A^{(p)}_2 \) and \( A^{(p-1)}_2 \) minimal models initiated by the relevant field \( \phi[\ ] \).
We have identified three classes of RG invariant sets of local fields (2.2), (2.10) and (2.34) and have shown that in IR limit they flow to the classes (2.9), (2.30) and (2.76) respectively. We obtained these results by computing the matrices of anomalies dimensions explicitly using a well known formula by A. Zamolodchikov (2.1). Not all structure constants which enter in this formula are available in literature and their computation was one of the main obstacles we had to overcome. Our final results for matrices of anomalous dimensions are given in (2.7), (2.21) and (2.37).

For the second class we were able to drive the IR/UV mixing coefficients (2.32) by simply finding the eigenvectors of anomalies dimensions matrix $\Gamma$. This result was checked also by constructing the RG domain wall (4.25).

Since in the third class there are secondary fields with identical Virasoro dimensions the spectrum of $\Gamma$ is degenerate. This is why the linear transformation which diagonalizes $\Gamma$ is not defined uniquely. Consequently some of UV/IR mixing coefficients (namely those associated to secondary fields) remain unspecified. It is reasonable to expect that this uncertainty can be cured by application of the RG domain wall method. We hope to address this issue in a future publication. For a different approach to these type of perturbation see [22].

In this paper we introduce the notion of anomalous $W$ weights (see (3.6)) in close analogy with $\Gamma$. The expression (3.23) for this matrix holds for sets with primary fields only. Though our third class includes secondary fields, nevertheless we expect that a similar expression should exist in this case too. Finding the mixing coefficients for this class would be helpful.

While studying the three point functions including secondary fields we have encountered an interesting phenomenon: the violation of holomorphic factorizability.

Finally it would be interesting to find the UV/IR mixing coefficients for general $W_n$ minimal models. This would allow to investigate the large $k$, $n$ limit with fixed $n/k$ ratio, when in addition a holographic description is available [23].

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A The $\tilde{\phi}$ basis

Consider a basis of primary fields satisfying the condition

$$\langle \tilde{\phi}_\alpha(1)\tilde{\phi}_\beta(0) \rangle_\lambda = \delta_{\alpha\beta}.$$
Let us try to find the matrix which connects our initial fields to this new one. So

\[ \tilde{\phi}_\beta = B_{\beta n} \phi_n \quad \text{where} \quad B_{\beta n} = \delta_{\beta n} + \lambda B_{\beta n}^{(1)}. \]

To derive \( B^{(1)} \) let us first consider

\[ \langle \phi_\alpha (1) \phi_\beta (0) \rangle = \langle \phi_\alpha (1) \phi_\beta (0) \rangle - \lambda \int \langle \phi_\alpha (1) \varphi(x) \phi_\beta (0) \rangle d^2 x = \]

\[ = \delta_{\alpha \beta} - \lambda \pi C_\alpha^\beta \frac{\gamma(\Delta_{\alpha \beta} + \epsilon) \gamma(\Delta_{\beta \alpha} + \epsilon)}{\gamma(2 \epsilon)} = \delta_{\alpha \beta} - \lambda \pi C_\alpha^\beta \frac{2 \epsilon}{\epsilon - \Delta_{\alpha \beta}}, \]

where in the last line we have used the fact that \( \Delta_{\alpha \beta} \) is small (of order \( \epsilon \)). Now let us consider

\[ \langle \phi_\alpha (1) \phi_\beta (0) \rangle = \langle \tilde{\phi}_\alpha \tilde{\phi}_\beta \rangle - \lambda B^{(1)}_{\alpha n} \langle \tilde{\phi}_n \tilde{\phi}_\beta \rangle - \lambda B^{(1)}_{\beta n} \langle \tilde{\phi}_\alpha \tilde{\phi}_n \rangle = \delta_{\alpha \beta} - \lambda B^{(1)}_{\beta \alpha} - \lambda B^{(1)}_{\alpha \beta}. \]

By comparing this to (A.1) we obtain

\[ B^{(1)}_{\alpha \beta} + B^{(1)}_{\beta \alpha} = \pi C_\alpha^\beta \frac{2 \epsilon}{(\epsilon + \Delta_{\alpha \beta})(\epsilon + \Delta_{\beta \alpha})} = \pi C_\alpha^\beta \left( \frac{1}{\epsilon + \Delta_{\alpha \beta}} + \frac{1}{\epsilon + \Delta_{\beta \alpha}} \right). \]

Since \( C_\alpha^\beta \) is symmetric the natural solution is \( B^{(1)}_{\alpha \beta} = \frac{\pi C_\alpha^\beta}{(\epsilon + \Delta_{\alpha \beta})} \). Thus we recover the result (3.12).

**B  Structure constants**

**B.1 Three point functions**

In this appendix we will be interested in the three point functions including secondary fields created by the operator \( W_{-1} \). The results of this appendix will be important for the calculation of structure constants of descendant fields used in the main text. In particular we prove here the result (3.10) which is used to calculate the matrix of anomalous \( W \)-weights.

We adopt standard two point function normalization

\[ \langle \phi_1 (z_1) \phi_2 (z_2) \rangle = z_{12}^{-2 \Delta} \quad \text{if} \quad \Delta_1 = \Delta_2 \equiv \Delta. \]

Since in this section we deal with the holomorphic part only for the three point function we will simply assume that

\[ \langle \phi_1 (z_1) \phi_2 (z_2) \phi_3 (z_3) \rangle = z_{12}^{\Delta_1 - \Delta_2} z_{13}^{\Delta_2 - \Delta_3} z_{23}^{\Delta_3 - \Delta_1} \]

and we will restore actual structure constant \( C_{123} \) whenever the “physical” correlator, including both holomorphic and antiholomorphic parts is considered. Remind the OPE of primary fields with \( W \)-current

\[ W(z) \phi_i (z_i) = \frac{w_i}{(z - z_i)^3} \phi_i (z_i) + \frac{1}{(z - z_i)^2} W_{-1} \phi_i (z_i) + \frac{1}{z - z_i} W_{-2} \phi_i (z_i) + \ldots. \]
For brevity we will denote the perturbing field as
\[ \phi_3 = \phi \left[ \frac{1}{2} \frac{1}{2} \right]. \]

This field has two independent null vectors at the second level
\[
\left( W_{-2} - \frac{2}{\Delta_3 + 1} L_{-1} W_{-1} \right) \phi_3 = 0, \tag{B.4}
\]
\[
\left( W_{-1}^2 + \frac{2 \Delta_3 (\Delta_3 + 1) (\Delta_3 + 2)}{(\Delta_3 - 3) (5 \Delta_3 + 1)} L_{-2} - \frac{3 (\Delta_3^2 - 1)}{2 (\Delta_3 - 3) (5 \Delta_3 + 1)} L_{-1}^2 \right) \phi_3 = 0. \tag{B.5}
\]

Observe that due to \( W_3 \) algebra relations
\[
L_1 W_{-1} |\phi_3\rangle = 3 w_3 |\phi_3\rangle = 0, \tag{B.6}
\]
so that \( W_{-1} |\phi_3\rangle \) is Virasoro quasi primary. This is why the three point function takes the form
\[
\langle \phi_1 \phi_2 W_{-1} \phi_3 \rangle = a z_{1z_2}^{\Delta_3+1-\Delta_1-\Delta_2} z_{1z_3}^{\Delta_2-\Delta_1-\Delta_3-1} z_{2z_3}^{\Delta_1-\Delta_2-\Delta_3-1} \tag{B.7}
\]
with \( a \) being a \( z \) independent constant which we will identify below. Notice that for large \( z \) we have
\[
\langle \phi_1(z_1) \phi_2(z_2) \phi_3(z_3) W(z) \rangle \sim \frac{1}{z^6}, \tag{B.8}
\]
which guarantees the vanishing of the integral
\[
0 = \oint_{\text{large contour}} \langle \phi_1(z_1) \phi_2(z_2) \phi_3(z_3) W(z) \rangle (z - z_1)^2 (z - z_2)^2 \frac{dz}{2\pi i} = \tag{B.9}
\]
\[
= \sum_{k=1}^{3} \oint_{z_k} \langle \phi_1(z_1) \phi_2(z_2) \phi_3(z_3) W(z) \rangle (z - z_1)^2 (z - z_2)^2 \frac{dz}{2\pi i}.
\]

Integrals around the poles can be derived by making use of the OPE (B.3). The results are
\[
\oint_{z_1} \langle \phi_1 \phi_2 \phi_3 W(z) \rangle (z - z_1)^2 (z - z_2)^2 \frac{dz}{2\pi i} = w_1 z_{1z_2}^{2} \langle \phi_1 \phi_2 \phi_3 \rangle, \tag{B.10}
\]
\[
\oint_{z_2} \langle \phi_1 \phi_2 \phi_3 W(z) \rangle (z - z_1)^2 (z - z_2)^2 \frac{dz}{2\pi i} = w_2 z_{1z_2}^{2} \langle \phi_1 \phi_2 \phi_3 \rangle, \tag{B.11}
\]
\[
\oint_{z_3} \langle \phi_1 \phi_2 \phi_3 W(z) \rangle (z - z_1)^2 (z - z_2)^2 \frac{dz}{2\pi i} = \tag{B.12}
\]
\[
= 2 (z_{31} z_{32}^{2} + z_{31} z_{32}) \langle \phi_1 \phi_2 W_{-1} \phi_3 \rangle + z_{31} z_{32}^{2} \langle \phi_1 \phi_2 W_{-2} \phi_3 \rangle.
\]
As it is dictated by (B.4)

\[ W_{-2}\phi_3(z_3) = \frac{2}{\Delta_3 + 1} \frac{\partial}{\partial z_3} W_{-1}\phi_3(z_3). \]  

(B.13)

Together with (B.7) we can further simplify (B.12)

\[ \oint_{z_3} \langle \phi_1\phi_2\phi_3 W(z) \rangle (z - z_1)^2(z - z_2)^2 \frac{dz}{2\pi i} = 2a \frac{\Delta_2}{\Delta_3 + 1} \frac{z_{12}^2 \langle \phi_1\phi_2\phi_3 \rangle}{z_{12}^2 z_{23}}. \]  

(B.14)

Now we insert (B.10), (B.11) and (B.14) in (B.9) to get

\[ a = \frac{(\Delta_3 + 1)(w_1 + w_2)}{2(\Delta_1 - \Delta_2)}. \]  

(B.15)

So for the three point function (B.7) when \( \Delta_1 \neq \Delta_2 \) we finally get

\[ \langle \phi_1\phi_2 W_{-1}\phi_3 \rangle = \frac{(\Delta_3 + 1)(w_1 + w_2)}{2(\Delta_1 - \Delta_2)} \frac{z_{12}}{z_{12}^2 z_{23}} \langle \phi_1\phi_2\phi_3 \rangle. \]  

(B.16)

The case \( \Delta_1 = \Delta_2 \) and \( w_1 + w_2 = 0 \) should be treated more carefully. In this case the constant \( a \) can not be derived through Ward identities. We will address this question later in this appendix using bootstrap approach.

To derive \( \Gamma \) for the third class we need structure constants with descendant fields. In particular we need to know \( \langle (W_{-1}\phi_1)(W_{-1}\phi_2)\phi_3 \rangle \). As a first step let us derive \( \langle (W_{-1}\phi_1)\phi_2\phi_3 \rangle \). The technique we use here is the same used for (B.16). From (B.8) we observe that

\[ 0 = \oint_{z_3 \text{large contour}} \langle W(z)\phi_1\phi_2\phi_3 \rangle (z - z_1)(z - z_2)^2(z - z_3) \frac{dz}{2\pi i} = \sum_{k=1}^{3} \oint_{z_k} \langle W(z)\phi_1\phi_2\phi_3 \rangle (z - z_1)(z - z_2)^2(z - z_3) \frac{dz}{2\pi i}. \]  

(B.17)

Inserting the OPE (B.3) and evaluating contour integrals we get

\[ \oint_{z_1} \langle W(z)\phi_1\phi_2\phi_3 \rangle (z - z_1)(z - z_2)^2(z - z_3) \frac{dz}{2\pi i} = w_1(z_{12}^2 + 2z_{12}z_{13})\langle \phi_1\phi_2\phi_3 \rangle + z_{12}^2 z_{13} \langle (W_{-1}\phi_1)\phi_2\phi_3 \rangle, \]  

(B.18)

\[ \oint_{z_2} \langle W(z)\phi_1\phi_2\phi_3 \rangle (z - z_1)(z - z_2)^2(z - z_3) \frac{dz}{2\pi i} = -w_2 z_{12} z_{23} \langle \phi_1\phi_2\phi_3 \rangle, \]  

(B.19)

\[ \oint_{z_3} \langle W(z)\phi_1\phi_2\phi_3 \rangle (z - z_1)(z - z_2)^2(z - z_3) \frac{dz}{2\pi i} = -z_{13} z_{23}^2 \langle \phi_1\phi_2 W_{-1}\phi_3 \rangle. \]  

(B.20)

Inserting these expressions in (B.17) and using (B.16) we obtain

\[ \langle (W_{-1}\phi_1)\phi_2\phi_3 \rangle = \left[ (a + w_2) \frac{z_{12}^2}{z_{12} z_{13}} - w_1 \left( \frac{2}{z_{12}} + \frac{1}{z_{13}} \right) \right] \langle \phi_1\phi_2\phi_3 \rangle. \]  

(B.21)
Similarly
\[ \langle \phi_1(W_{-1}\phi_2)\phi_3 \rangle = \left((a - w_1) \frac{z_{13}}{z_{12}z_{23}} + w_2 \left( \frac{2}{z_{12}} - \frac{1}{z_{23}} \right) \right) \langle \phi_1\phi_2\phi_3 \rangle. \] (B.22)

Another correlator we need is \( \langle (W_{-1}\phi_1)\phi_2W_{-1}\phi_3 \rangle \). Similar to previous cases one has
\[ 0 = \oint_{\text{large contour}} \langle W(z)\phi_1\phi_2W_{-1}\phi_3 \rangle (z - z_1)(z - z_2)^2(z - z_3) \frac{dz}{2\pi i} = \] (B.23)
\[ = \sum_{k=1}^{3} \oint_{z_k} \langle W(z)\phi_1\phi_2W_{-1}\phi_3 \rangle (z - z_1)(z - z_2)^2(z - z_3) \frac{dz}{2\pi i}. \]

On the other hand
\[ W(z)W_{-1}\phi(0)|0\rangle = \sum_{n\in\mathbb{Z}} \frac{W_n}{z^{n+3}} W_{-1}\phi(0)|0\rangle = \sum_{i=-2}^{1} \frac{W_i}{z^{i+3}} W_{-1}\phi(0)|0\rangle + O(z). \] (B.24)

Using \( W_3 \) algebra relations (1.9) this OPE can be rewritten as
\[ W(z)W_{-1}\phi(0)|0\rangle = \left( \frac{2 - c + 32\Delta_3}{(22 + 5c)} \frac{1}{z} + (wW_{-1} + \frac{2 - c + 32\Delta_3}{22 + 5c} L_{-1}) \frac{1}{z} + \right. \] (B.25)
\[ + W_{-2} \frac{1}{2\pi i} \phi(0)|0\rangle + O(z). \]

This OPE allows one to evaluate the contour integral around \( z_3 \) in (B.23) (integrals around \( z_1 \) and \( z_2 \) are standard). Here are the results
\[ \oint_{z_3} \langle W(z)\phi_1\phi_2W_{-1}\phi_3 \rangle (z - z_1)(z - z_2)^2(z - z_3) \frac{dz}{2\pi i} = -z_{13}z_{23}^2 \langle \phi_1\phi_2 W_{-1}^2 W_{-1} \phi_3 \rangle + \] (B.26)
\[ + \left( -\Delta_3 \frac{2 - c + 32\Delta_3}{22 + 5c} (2z_{23} + z_{13}) + \frac{2 - c + 32\Delta_3}{22 + 5c} (2z_{13}z_{23} + z_{23}^2) \frac{\partial}{\partial z_{23}} \right) \langle \phi_1\phi_2\phi_3 \rangle, \]
\[ \oint_{z_2} \langle W(z)\phi_1\phi_2W_{-1}\phi_3 \rangle (z - z_1)(z - z_2)^2(z - z_3) \frac{dz}{2\pi i} = -w_2 z_{12}z_{23} \langle \phi_1\phi_2 W_{-1} W_{-1} \phi_3 \rangle, \] (B.27)
\[ \oint_{z_1} \langle W(z)\phi_1\phi_2W_{-1}\phi_3 \rangle (z - z_1)(z - z_2)^2(z - z_3) \frac{dz}{2\pi i} = \] (B.28)
\[ = w_1 (2z_{12}z_{13} + z_{12}^2) \langle \phi_1\phi_2 W_{-1} W_{-1} \phi_3 \rangle + z_{12}^2 z_{13} \langle (W_{-1}\phi_1)\phi_2 W_{-1} \phi_3 \rangle. \]

It remains to evaluate \( \langle \phi_1\phi_2 W_{-1}^2 \phi_3 \rangle \) using the null vector condition (B.5). According to conformal Ward identity
\[ \langle T(z)\phi_1\phi_2\phi_3 \rangle = \sum_{i=1}^{3} \left( \frac{\Delta_i}{(z - z_i)^2} + \frac{1}{z - z_i} \frac{\partial}{\partial z_i} \right) \langle \phi_1\phi_2\phi_3 \rangle. \] (B.29)

Since
\[ L_{-2}\phi_3(z_3) = \oint_{z_3} T(z)\phi_3(z_3)(z - z_3)^{-1} \frac{dz}{2\pi i} \] (B.30)
we have

\[ \langle \phi_1 \phi_2 L_{-2} \phi_3 \rangle = \sum_{i=1,2} \left( \frac{\Delta_i}{z_i^{\Delta_i}} + \frac{1}{\bar{z}_i} \frac{\partial}{\partial \bar{z}_i} \right) \langle \phi_1 \phi_2 \phi_3 \rangle. \]  

(B.31)

Thus using (B.5) we get

\[ \langle \phi_1 \phi_2 W^2 \phi_3 \rangle = \frac{(\Delta_3 + 1)}{2 (\Delta_3 - 3) (5 \Delta_3 + 1)} \left( 3 (\Delta_3 - 1) \frac{\partial^2}{\partial z_3^2} - 4 \Delta_3 (\Delta_3 + 2) \sum_{i=1,2} \left( \frac{\Delta_i}{z_i^{\Delta_i}} + \frac{1}{\bar{z}_i} \frac{\partial}{\partial \bar{z}_i} \right) \langle \phi_1 \phi_2 \phi_3 \rangle \right). \]  

(B.32)

Incorporating the above results in (B.23) we obtain

\[ \langle (W_{-1} \phi_1) \phi_2 W_{-1} \phi_3 \rangle = \left[ a \left( w_2 \frac{z_{23}}{z_{12} z_{13}} - w_1 \frac{2}{z_{12}} + \frac{1}{z_{13}} \right) \frac{z_{12}}{z_{13} z_{23}} + \frac{z_{23}^2 (\Delta_3 + 1)}{2 z_{12}^2 (\Delta_3 - 3) (5 \Delta_3 + 1)} \left( 3 (\Delta_3 - 1) \frac{\partial^2}{\partial z_3^2} - 4 \Delta_3 (\Delta_3 + 2) \sum_{i=1,2} \left( \frac{\Delta_i}{z_i^{\Delta_i}} + \frac{1}{\bar{z}_i} \frac{\partial}{\partial \bar{z}_i} \right) \right) \right] \langle \phi_1 \phi_2 \phi_3 \rangle. \]  

(B.33)

Now we have all ingredients to evaluate \( \langle (W_{-1} \phi_1)(W_{-1} \phi_2) \phi_3 \rangle \). Indeed, consider the vanishing integral

\[ 0 = \oint_{\text{large contour}} \langle W(z)(W_{-1} \phi_1) \phi_2 W(z) \phi_3 \rangle (z-z_1)^2 (z-z_2)(z-z_3) \frac{dz}{2 \pi i} = \sum_{k=1}^3 \oint_{z_k} \langle W(z)(W_{-1} \phi_1) \phi_2 W(z) \phi_3 \rangle (z-z_1)^2 (z-z_2)^2 (z-z_3) \frac{dz}{2 \pi i}, \]

where the summands are equal to

\[ \oint_{z_1} \langle W(z)(W_{-1} \phi_1) \phi_2 \phi_3 \rangle (z-z_1)^2 (z-z_2)(z-z_3) \frac{dz}{2 \pi i} = \]  

(B.36)

\[ = \frac{2 - c + 32 \Delta_1}{22 + 5c} \left( \Delta_1 (z_{12} + z_{13}) + z_{12} z_{13} \frac{\partial}{\partial z_{1}} \right) \langle \phi_1 \phi_2 \phi_3 \rangle + w_{1} z_{12} z_{13} \langle (W_{-1} \phi_1)(W_{-1} \phi_2) \phi_3 \rangle, \]

\[ \oint_{z_2} \langle W(z)(W_{-1} \phi_1) \phi_2 \phi_3 \rangle (z-z_1)^2 (z-z_2)(z-z_3) \frac{dz}{2 \pi i} = \]  

(B.37)

\[ = w_{2} (z_{12}^2 - 2 z_{23} z_{13}) \langle (W_{-1} \phi_1) \phi_2 \phi_3 \rangle + z_{23} (z_{12}^2) \langle (W_{-1} \phi_1)(W_{-1} \phi_2) \phi_3 \rangle, \]

\[ \oint_{z_3} \langle W(z)(W_{-1} \phi_1) \phi_2 \phi_3 \rangle (z-z_1)^2 (z-z_2)(z-z_3) \frac{dz}{2 \pi i} = - w_{3} z_{13}^2 \langle (W_{-1} \phi_1)(W_{-1} \phi_2)(W_{-1} \phi_3) \rangle. \]  

(B.38)

All three point functions on the right hand sides, besides the one we are interested in,
are already in our disposal. So, from the equality (B.35) we finally get

$$
\langle (W_{-1}\phi_1)(W_{-1}\phi_2)\phi_3 \rangle = \frac{z_{13}^2}{z_{12}^2} \langle (W_{-1}\phi_1)\phi_2(W_{-1}\phi_3) \rangle - \left( w_1 \frac{z_{13}}{z_{23}z_{12}} + w_2 \left( \frac{1}{z_{23}} - \frac{2}{z_{12}} \right) \right) \langle (W_{-1}\phi_1)\phi_2\phi_3 \rangle - \\
2 - c + 32\Delta_1 \left( \frac{22 + 5c}{22 + 5}\Delta_{1}(z_{12} + z_{13}) + z_{12}z_{13} \frac{\partial}{\partial z_{12}} \right) \langle \phi_1\phi_2\phi_3 \rangle.
$$

Actually we are interested in the specific combination

$$
\hat{W}\phi := (W_{-1} - \frac{3w}{2\Delta} L_{-1}) \phi,
$$

where $w$ and $\Delta$ are the $W$ and Virasoro weights of $\phi$. From (B.21) we get

$$
\langle (\hat{W}\phi^*)(z_1)\phi_i(z_2)\phi_3(z_3) \rangle = \frac{(2a + \Delta + 2w) - 3\Delta_1 + 3\Delta_3}{2\Delta} - \frac{w^2}{z_{12}z_{13}} \langle \phi^*\phi_i\phi_3 \rangle.
$$

Remind that star stands for conjugate. We can use (B.39) together with (B.21) and (B.22) to get

$$
\langle (\hat{W}\phi^*)\hat{W}\phi \rangle = \left( \frac{3a(\Delta_3 - 1)w}{\Delta} - \frac{9w^2(-\Delta_3^2 + \Delta_3 + 2\Delta)}{4\Delta^2} - \frac{38(5\Delta + 1) - \Delta(2\Delta_3 + 1)}{125} \right) \langle \phi^*\phi \rangle - \\
\frac{1}{50} \Delta_3 (5\Delta_3 - 1) - \frac{(\Delta + 3)(5\Delta + 1)}{4(\Delta_3 - 3)} + \frac{9(5\Delta + 1) + 25\Delta + 3}{500(5\Delta_3 + 1)} \frac{z_{12}^2}{z_{13}^2} \langle \phi^*\phi \rangle.
$$

where the relation between central charge $c$ and dimension $\Delta_3$

$$
2 - c = \frac{8(1 - \Delta_3)^2}{2 + \Delta_3}
$$

is used. To fix normalization we will need also the two point function of $\hat{W}\phi$, which can be easily derived e.g. considering the three point functions (B.39), (B.21) with insertion of the unit operator instead of $\phi_3$. Technically this amounts to replacing $\Delta_3 \rightarrow 0$ and $a \rightarrow 0$. This results in

$$
\langle (\hat{W}\phi^*)\hat{W}\phi \rangle = \frac{\Delta}{z_{12}^2} \left( \frac{2 - c + 32\Delta}{22 + 5c} - \frac{9w^2}{2\Delta^2} \right) \langle \phi^*\phi \rangle.
$$

As one sees from (2.34) we need also three point functions like

$$
\langle (\hat{W}\phi^*)(\hat{W}\phi)\phi_3 \rangle, \quad \langle \phi^*(\hat{W}\phi)(\hat{W}\phi)\phi_3 \rangle, \quad (\hat{W}\phi^*)(\hat{W}\phi)(\hat{W}\phi)\phi_3.
$$

The behavior of a three point function of these type is somewhat counter intuitive since the usual holomorphic anti-holomorphic factorization fails. From conformal symmetry

$$
\langle \phi^*\phi\hat{W}W_{-1}\phi_3 \rangle = C(a\bar{a}) |z_{12}|^{2(\Delta_3 + 1 - 2\Delta)} |z_{13}z_{23}|^{-2(\Delta_3 + 1)},
$$

$$
\langle \phi^*\phi\hat{W}W_{-1}\phi_3 \rangle = C(a\bar{a}) |z_{12}|^{2(\Delta_3 + 1 - 2\Delta)} |z_{13}z_{23}|^{-2(\Delta_3 + 1)},
$$

$$
\langle (\hat{W}\phi^*)(\hat{W}\phi)\phi_3 \rangle, \quad ((\hat{W}\phi^*)(\hat{W}\phi))(\hat{W}\phi)\phi_3, \quad ((\hat{W}\phi^*)(\hat{W}\phi)^*)(\hat{W}\phi)\phi_3.
$$

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\langle \phi^*\phi\hat{W}W_{-1}\phi_3 \rangle = C(a\bar{a}) |z_{12}|^{2(\Delta_3 + 1 - 2\Delta)} |z_{13}z_{23}|^{-2(\Delta_3 + 1)},
$$

$$
\langle (\hat{W}\phi^*)(\hat{W}\phi)\phi_3 \rangle, \quad ((\hat{W}\phi^*)(\hat{W}\phi))(\hat{W}\phi)\phi_3, \quad ((\hat{W}\phi^*)(\hat{W}\phi)^*)(\hat{W}\phi)\phi_3.
$$

The behavior of a three point function of these type is somewhat counter intuitive since the usual holomorphic anti-holomorphic factorization fails. From conformal symmetry

$$
\langle \phi^*\phi\hat{W}W_{-1}\phi_3 \rangle = C(a\bar{a}) |z_{12}|^{2(\Delta_3 + 1 - 2\Delta)} |z_{13}z_{23}|^{-2(\Delta_3 + 1)},
$$

$$
\langle (\hat{W}\phi^*)(\hat{W}\phi)\phi_3 \rangle, \quad ((\hat{W}\phi^*)(\hat{W}\phi))(\hat{W}\phi)\phi_3, \quad ((\hat{W}\phi^*)(\hat{W}\phi)^*)(\hat{W}\phi)\phi_3.
$$
where the structure constant of primaries $C$ is explicitly factored out and the remaining numerical coefficient is denoted as $\langle a\bar{a} \rangle$. Actual calculation of this quantity shows that

$$\langle a\bar{a} \rangle \neq a\bar{a}. \quad (B.45)$$

From (B.21) and (B.22) we obtain

$$\langle (\hat{W}\hat{\phi}^*)(\hat{W}\phi) \rangle = (4\langle a\bar{a} \rangle \Delta^2 + 12a\Delta_3\Delta w + 9\Delta_3^2w^2) \bigg|_z^z \langle \phi^* \phi \rangle, \quad (B.46)$$

$$\langle \phi^*(\hat{W}
\hat{\phi}) \phi \rangle = (4\langle a\bar{a} \rangle \Delta^2 + 12a\Delta_3\Delta w + 9\Delta_3^2w^2) \bigg|_z^z \langle \phi^* \phi \rangle. \quad (B.47)$$

We can treat the remaining cases of (B.43) similarly. For example to calculate

$$\langle (\hat{W}\hat{\phi}^*)(\hat{W}\phi) \rangle$$

one must “square” (B.41), keeping in mind (B.45). To conclude in this way we can express the structure constants involving descendant fields from the third class (2.34) in terms of structure constants of primary fields and the constants $a = \bar{a}$ and $\langle a\bar{a} \rangle$. For brevity let us introduce the notations

$$\phi_0 := \phi \left[ \begin{array}{c} n \\ n' \end{array} \right]$$

and

$$\Delta_0 = \epsilon^2 h_0; \quad h_0 = \frac{1}{27} \left( n^2 + nn' + n'^2 - 3 \right), \quad (B.48)$$

$$w_0 = \epsilon^2 s_0, \quad s_0 = -\frac{((n-n')(2n+n')(n+2n'))}{243\sqrt{6}}, \quad (B.49)$$

$$\Delta_i = 1 + \epsilon h_i, \quad w_i = \epsilon s_i \quad \text{for} \quad i = 1, 2, 3, 8, 9, 10. \quad (B.50)$$

For effective structure constants needed when computing the matrix of anomalous dimensions in third class we have obtained

$$C^4_i = C^0_i \frac{(h_i+1)^2}{2h_0}, \quad (B.51)$$

$$C^6_i = C^5_i = C^0_i \frac{(h_i+1)(3s_0(h_i+1)-s_0h_0(2h_i+1))}{h_0\sqrt{\frac{1}{4}h_0^2(12h_0+1)-36s_0^2}}, \quad (B.52)$$

$$C^7_i = C^0_i \frac{3(s_0h_0(2h_i+1)-3s_0(h_i+1))^2}{54s_0^2h_0+12h_0^2+h_0^3}, \quad (B.53)$$

where the explicit form of $h_i$ is determined from (2.36). For the structure constants $C^j_i$ with $i, j \in \{4, 5, 6, 7\}$ we have used the explicit expressions (to be derived later on):

$$a = \frac{\epsilon((n'^{-n})(2n+n')(n+2n'))}{6\sqrt{6}(n^2+nn'+n'^2-3)} + O(\epsilon^2), \quad (B.54)$$

$$\langle a\bar{a} \rangle = \frac{\epsilon^2(4(n^2+n'n'^2-3)+9)}{27} + O(\epsilon^3). \quad (B.55)$$

- 31 -
To avoid confusion let us emphasize that this result is valid only in current case i.e. when both fields are descendants of $\phi_0$. Here are the final expressions

\[
C_4' = C_0' \left( \frac{729\epsilon^{-2}}{4(n^2+n'+n'^2-3)^2} + O \left( \frac{1}{\epsilon} \right) \right), \quad (B.56)
\]

\[
C_5' = C_6' = C_0' (1 + O(\epsilon^{-1})), \quad (B.57)
\]

\[
C_7' = C_4' + O(\epsilon), \quad (B.58)
\]

\[
C_5 = C_6 = C_4 + O(\epsilon), \quad (B.59)
\]

\[
C_7 = -C_0', \left( \frac{81\sqrt{3}\epsilon^{-2}(n-n')(2n+n')(n+2n')} {2\sqrt{(n+1)(n^2-1)((n+n')^2-1)(n^2+nn'+n'^2-3)^2}} + O \left( \frac{1}{\epsilon} \right) \right), \quad (B.60)
\]

\[
C_7 = C_0' \left( \frac{27(4n^2+nn'+n'^2-3)^3+9(n^2+nn'+n'^2-3)^2+3(n-n')^2(2n+n')^2(2n+2n')^2} {4\epsilon^2(n^2-1)((n+n')^2-1)(n^2+nn'+n'^2-3)^2} + O \left( \frac{1}{\epsilon} \right) \right). \quad (B.61)
\]

### B.2 OPE up to level one

In order to identify $a$ and $\langle a\bar{a}\rangle$ from four point correlation function we need the next to leading order terms of OPE expansion explicitly. It follows from OPE

\[
T(z)\phi(0) = \frac{\Delta}{z^2}\phi(0) + \frac{1}{z}(\partial\phi)(0) + ...
\]

and (B.3) \cite{8, 24} that

\[
[L_n, \phi(z)] = z^n ((n+1)\Delta + z\partial_z) \phi(z) , \quad (B.63)
\]

\[
[W_n, \phi(z)] = z^n \left( \frac{(n+2)(n+1)}{2} w + (n+2)zW_{-1} + z^2W_{-2} \right) \phi(z). \quad (B.64)
\]

For brevity let us denote

\[
\phi_{-\omega_1} = \phi \left[ \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right] \equiv \phi_1.
\]

This is a doubly degenerated primary field with independent null vectors

\[
\left( W_{-1} - \frac{3w_1}{2\Delta_1} L_{-1} \right) \phi_1 = 0, \quad (B.65)
\]

\[
\left( W_{-2} - \frac{12w_1}{\Delta_1(5\Delta_1 + 1)} L_{-1}^2 + \frac{6w_1(\Delta_1 + 1)}{\Delta_1(5\Delta_1 + 1)} L_{-2} \right) \phi_1 = 0. \quad (B.66)
\]

From general arguments one has\footnote{We have suppressed the contribution of unit operator. For our proposes it is sufficient to concentrate on $W$-block $[\varphi]$ only.}

\[
\phi_1(z)\phi_1(0) = z^{\Delta - 2\Delta_1} (\varphi(0) + z (\alpha L_{-1} + \beta W_{-1}) \varphi(0) + ...), \quad (B.67)
\]
where as usual \( \varphi \equiv \phi \left[ \frac{1}{2} \right] \). We can act on This OPE with \( L_1 \) and find that \( \alpha = 1/2 \). Similarly to find \( \beta \) we act on (B.67) with \( W_1 \)
\[
[W_1, \phi^*_1(z)] \phi_1(0) = \beta z^{\Delta - 2 \Delta_1 + 1} [W_1, W_{-1}] \varphi(0) = \\
= \beta z^{\Delta - 2 \Delta_1 + 1} \left( -\frac{1}{5} \Delta + \frac{32}{22 + 5c} \left( \frac{1}{5} \Delta + \Delta^2 \right) \right) \varphi(0),
\]
where in second row the commutation relation (1.9) is used. On the other hand using (B.64) and (B.65) for the lhs of (B.68) we have
\[
[W_1, \phi^*_1(z)] \phi_1(0) = - \left( 3w_1 z + z^2 \frac{9w_1}{2\Delta} \partial_z \right) \phi^*_1(z) \phi_1(0) + z^{3(W_-2 \phi^*_1(z))} \phi_1(0). \tag{B.69}
\]

From null vector condition (B.66)
\[
(W_-2 \phi^*_1(z)) \phi_1(0) = - \frac{12w_1}{\Delta_1(5\Delta_1 + 1)} \partial^2 \phi^*_1(z) \phi_1(0) + \frac{6w_1(\Delta_1 + 1)}{\Delta_1(5\Delta_1 + 1)} (L_-2 \phi^*_1(z)) \phi_1(0). \tag{B.70}
\]

The second summand of this expression can be refurnished as
\[
(L_-2 \phi^*_1(z)(\phi_1)(0) = \oint_{\mathbb{R}} T(\zeta)(\zeta - z)^{-1} \phi^*_1(z) \phi_1(0) \frac{d\zeta}{2\pi i} = \tag{B.71}
\]
\[
\oint_{\mathbb{R}} z^{\Delta - 2 \Delta_1} (\zeta - 1)^{-1} T(\zeta) \varphi(0) \frac{d\zeta}{2\pi i} - \oint_{\mathbb{R}} (\zeta - 1)^{-1} \phi^*_1(z) \left( \frac{\Delta_1}{\zeta^2} \phi_1(0) + \frac{1}{\zeta} (\partial \phi_1)(0) \right) \frac{d\zeta}{2\pi i},
\]
where besides the definition of \( L_-2 \) in first term of the second row OPE (B.67) is applied. Using \( T \varphi \) OPE and performing \( \zeta \) integration around 0 and \( z \) one can easily get convinced that it is of order \( z^{\Delta - 2 \Delta_1} \). This term can be ignored since we are interested in more singular terms of order \( z^{\Delta - 2 \Delta_1 - 2} \) (notice the factor \( z^3 \) in front of second term in (B.69)). The remaining integral around zero in (B.71) is of desired order:
\[
- \oint_{0} (\zeta - 1)^{-1} \phi^*_1(z) \left( \frac{\Delta_1}{\zeta^2} \phi_1(0) + \frac{1}{\zeta} (\partial \phi_1)(0) \right) \frac{d\zeta}{2\pi i} = \tag{B.72}
\]
\[
\oint_{0} z^{\Delta - 2 \Delta_1} (\zeta - 1)^{-1} T(\zeta)(\zeta - z)^{-1} (\zeta - 1)^{-1} \phi^*_1(z) \phi_1(0) = \frac{1}{z^2} \phi^*_1(0) \phi_1(0) (3\Delta_1 - \Delta) z^{\Delta - 2 \Delta_1 - 2} \varphi(0).
\]

Combining (B.72), (B.71) and (B.70) we will get
\[
(W_-2 \phi^*_1(z)) \phi_1(0) = - \frac{12w_1(\Delta - 2 \Delta_1)(\Delta - 2 \Delta_1 - 1) + 6w_1(\Delta_1 + 1)(3\Delta_1 - \Delta)}{\Delta_1(5\Delta_1 + 1)} z^{\Delta - 2 \Delta_1} \varphi(0) + ... \tag{B.73}
\]

Now we can find \( \beta \) by inserting this in (B.69) and comparing the result with (B.68)
\[
\beta = - \frac{3(5c + 22)(13\Delta_1 - 8\Delta + 1)}{2\Delta_1(5\Delta_1 + 1)(c - 32\Delta - 2)} w_1. \tag{B.74}
\]

Taking into account that
\[
\Delta_1 = \frac{p - 3}{3(p + 1)}, \quad w_1 = \frac{(p - 3)}{9(p + 1)} \sqrt{\frac{2(2p - 3)}{3(2p + 5)}}, \tag{B.75}
\]
and expressing \( c \) and \( \Delta \) in terms of \( p \) we finally get
\[
\beta = - \frac{\sqrt{(2p - 3)(2p + 5)}}{\sqrt{6(2p - 1)}}. \tag{B.76}
\]
The parameters are given by \( i \) where
\[
\binom{3}{\phi_{\alpha_2}(\infty)\phi_{b\omega_2}(1)\phi_{-b\omega_1}(x)\phi_{\alpha_1}(0)} = (x\bar{x})^{b(\alpha_1-h_1)}((1-x)(1-\bar{x}))^{\frac{b_b}{2}} G(x, \bar{x}) .
\]
(B.77)
The function \( G \) can be expressed in terms of the generalized hypergeometric function \( _3F_2 \) as follows:
\[
G(x, \bar{x}) = \sum_{i=1}^{3} s_i |F_i^{(s)}(x)|^2 ,
\]
(B.78)
where
\[
F_i^{(s)}(x) = _3F_2 \left( \begin{array}{c} A_1; A_2; A_3 \\ B_1; B_2 \end{array} \bigg| x \right) ,
\]
(B.79)
\[
F_2^{(s)}(x) = x^{1-B_1} _3F_2 \left( \begin{array}{c} 1 - B_1 + A_1; 1 - B_1 + A_2; 1 - B_1 + A_3 \\ 2 - B_1; 1 - B_1 + B_2 \end{array} \bigg| x \right) ,
\]
(B.80)
\[
F_3^{(s)}(x) = x^{1-B_2} _3F_2 \left( \begin{array}{c} 1 - B_2 + A_1; 1 - B_2 + A_2; 1 - B_2 + A_3 \\ 2 - B_2; 1 - B_2 + B_1 \end{array} \bigg| x \right) .
\]
(B.81)
The parameters are given by \( i = 1, 2, 3 \):
\[
A_i = \frac{b(\kappa - 2b)}{3} + b (\alpha_2 - Q) \cdot h_i + b (\alpha_1 - Q) \cdot h_1 ,
\]
\[
B_1 = 1 + b (\alpha_1 - Q) \cdot (h_1 - h_2) ,
\]
\[
B_2 = 1 + b (\alpha_1 - Q) \cdot (h_1 - h_3) .
\]
(B.82)
We’ll call eq. (B.78) the s-channel representation of the correlation function since \( x \sim 0 \) behavior (in particular the single valuednes) is quite transparent. The CFT blocks \( F_i^{(s)} \), for \( i = 1, 2, 3 \) correspond to intermediate primaries \( \phi_{\alpha_1-b\omega_1}, \phi_{\alpha_1+b\omega_1-b\omega_2} \) and \( \phi_{\alpha_1+b\omega_2} \) respectively, as can be easily seen by examining dimensions.

Using the transformation formula under \( x \to 1/x \)

\[
\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(e)\Gamma(f)} \binom{a; b; c}{e; f} = _3F_2 \left( \begin{array}{c} a; 1 - e + a; 1 - f + a \\ e; f \end{array} \bigg| \frac{1}{x} \right)
\]
(B.83)
\[
= \frac{\Gamma(a)\Gamma(b-a)\Gamma(c-a)}{\Gamma(e)\Gamma(f-a)} x^{-a} _3F_2 \left( \begin{array}{c} a; 1 - e + a; 1 - f + a \\ 1 - b + a; 1 - c + a \end{array} \bigg| \frac{1}{x} \right)
\]
\[
+ \frac{\Gamma(b)\Gamma(a-b)\Gamma(c-b)}{\Gamma(e)\Gamma(f-b)} x^{-b} _3F_2 \left( \begin{array}{c} b; 1 - e + b; 1 - f + b \\ 1 - a + b; 1 - c + b \end{array} \bigg| \frac{1}{x} \right)
\]
\[
+ \frac{\Gamma(c)\Gamma(a-c)\Gamma(b-c)}{\Gamma(e)\Gamma(f-c)} x^{-c} _3F_2 \left( \begin{array}{c} c; 1 - e + c; 1 - f + c \\ 1 - a + c; 1 - b + c \end{array} \bigg| \frac{1}{x} \right)
\]
eq. (B.78) alternatively can be represented as (u-channel representation)

\[ G(x, \bar{x}) = \sum_{i=1}^{3} u_i |F_i^{(u)}(x)|^2, \]  

(B.84)

where

\[ F_1^{(u)}(x) = x^{-A_1} F_2 \left( A_1; 1 - B_1 + A_1; 1 - B_2 + A_1 \left| \frac{1}{x} \right. \right), \]  

\[ F_2^{(u)}(x) = x^{-A_2} F_2 \left( A_2; 1 - B_1 + A_2; 1 - B_2 + A_2 \left| \frac{1}{x} \right. \right), \]  

\[ F_3^{(u)}(x) = x^{-A_3} F_2 \left( A_3; 1 - B_1 + A_3; 1 - B_2 + A_3 \left| \frac{1}{x} \right. \right). \]  

(B.85)

(B.86)

(B.87)

Consideration similar to the s-channel case ensures that the CFT blocks \( F_i^{(u)} \), for \( i = 1, 2, 3 \) correspond to u-channel intermediate primaries \( \phi_{\alpha_2 - b\omega_1} \), \( \phi_{\alpha_3 + b\omega_1 - b\omega_2} \) and \( \phi_{\alpha_2 + b\omega_2} \). The absence of cross terms in (B.84) is guaranteed provided \( s_1, s_2, s_3 \) are related as [9]

\[ s_2 : s_1 = \frac{\gamma(B_1)\gamma(B_2)}{\gamma(2 - B_1)\gamma(1 - B_1 + B_2)} \prod_{i=1}^{3} \frac{\gamma(1 - B_1 + A_i)}{\gamma(A_i)}, \]

\[ s_3 : s_1 = \frac{\gamma(B_1)\gamma(B_2)}{\gamma(2 - B_2)\gamma(1 - B_2 + B_1)} \prod_{i=1}^{3} \frac{\gamma(1 - B_2 + A_i)}{\gamma(A_i)}. \]  

(B.88)

Thus such choice ensures single valuedness of correlation function around \( x \sim \infty \) too. The single valuedness around remaining singularity at \( x \sim 1 \) now is automatically guaranteed, since a small cycle surrounding \( x = 1 \) is equivalent to the difference of cycles around \( x = \infty \) and \( x = 0 \). Using (B.83) one can show that the coefficients \( u_i \) are related to \( s_1 \):

\[ u_1 : s_1 = \frac{\gamma(A_2 - A_1) \gamma(A_3 - A_1) \gamma(B_1) \gamma(B_2)}{\gamma(A_2) \gamma(A_3) \gamma(B_1 - A_1) \gamma(B_2 - A_1)}, \]

\[ u_2 : s_1 = \frac{\gamma(A_1 - A_2) \gamma(A_3 - A_2) \gamma(B_1) \gamma(B_2)}{\gamma(A_1) \gamma(A_3) \gamma(B_1 - A_2) \gamma(B_2 - A_2)}, \]  

\[ u_3 : s_1 = \frac{\gamma(A_1 - A_3) \gamma(A_2 - A_3) \gamma(B_1) \gamma(B_2)}{\gamma(A_1) \gamma(A_2) \gamma(B_1 - A_3) \gamma(B_2 - A_3)}. \]  

(B.89)

It is not surprising that the correlation function is fixed up to an overall constant factor, since we have note imposed any field normalization condition yet.

To calculate the OPE structure constants of our interest, let us start with the choice

\[ \kappa = -b; \quad \alpha_2 = \alpha_1^*. \]  

(B.90)
For later use let us emphasize that in this case the parameters of hypergeometric functions become related as
\[ B_1 = 1 + A_2 - A_3, \quad B_2 = 1 + A_1 - A_3. \quad \text{(B.91)} \]

The t-channel (i.e. \( x \sim 1 \)) in this case gets contribution of the unit operator which emerges in OPE
\[ \phi_{-b\omega}(z)\phi_{-b\omega}(1) \sim |z - 1|^{-4\Delta(-b\omega)}[I] + |z - 1|^2\Delta(-b\omega)\Delta(-b\omega)C_{-b\omega, -b\omega}[\phi_{-b\omega}]. \quad \text{(B.92)} \]

Then imposing canonical unit-normalization condition on two-point functions we can remove the remaining ambiguity thus completely fixing the 4-point correlation function.

Behavior of the hypergeometric function around \( x \sim 1 \), \( \arg(1-x) < \pi \) is given by the formula
\[ \Gamma(a)\Gamma(b)\Gamma(c)\Gamma(e)\Gamma(f) \frac{3F_2 \left( \begin{array}{c} a \ b \ c \\ e \ f \end{array} \right| x \right)}{\Gamma(s)} = \sum_{n=0}^{\infty} g_n(0)(1-x)^n + \sum_{n=0}^{\infty} g_n(s)(1-x)^{n+s}, \quad \text{(B.93)} \]

where \( s = e + f - a - b - c \). The general formulae for the coefficients could be found in [25]. For our purposes we need only the coefficients \( g_0(0) \), \( g_1(0) \) and \( g_0(s) \) (using obvious symmetry we have exchanged the roles of \( b \) and \( c \) compared to [25] for later convenience):

\[ g_0(s) = \Gamma(-s), \]
\[ g_0(0) = \frac{\Gamma(a)\Gamma(c)\Gamma(s)}{\Gamma(a+s)\Gamma(c+s)} 3F_2 \left( \begin{array}{c} e-b \ f-b \ s \\ a+s \ c+s \end{array} \right| 1 \right), \quad \text{(B.94)} \]
\[ g_1(0) = \frac{\Gamma(a+1)\Gamma(c+1)\Gamma(s-1)}{\Gamma(a+s)\Gamma(c+s)} 3F_2 \left( \begin{array}{c} e-b \ f-b \ s-1 \\ a+s \ c+s \end{array} \right| 1 \right). \]

The hypergeometric function of argument 1 should be understood as the result of analytic continuation from a region of parameters, where the hypergeometric series converges. It is well known, that
\[ \frac{3F_2 \left( \begin{array}{c} a \ b \ c \\ e \ f \end{array} \right| x \right)}{\Gamma(s)} = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(e)_n(f)_n n!} x^n \quad \text{(B.95)} \]
converges at \( x = 1 \) provided \( e + f - a - b - c > 0 \). If we apply (B.93) to investigate \( x \sim 1 \) limit of the correlation function (B.78), one generically gets hypergeometric
functions of unit argument which do not meet the convergence condition. Fortunately this difficulty can be overcome performing one more transformation using the identity

\[ \binom{a}{e; a + s} = \frac{\Gamma(f)\Gamma(-a - b - c + e + f)}{\Gamma(f - a)\Gamma(-b - c + e + f)} \binom{a - e - b}{e; a + s} \]  

(B.96)

for the cases (B.79), (B.80) and the identity (obtained from the previous one by reshuffling \( e \) with \( f \))

\[ \binom{a}{e; a + s} = \frac{\Gamma(e)\Gamma(-a - b - c + e + f)}{\Gamma(e - a)\Gamma(-b - c + e + f)} \binom{a - f - b}{f; a + s} \]  

(B.97)

for the case (B.80). These formulae can be derived combining Lemma 1 of [25] with obvious symmetry properties of the hypergeometric function with respect to its parameters. Interestingly in all three cases one arrives at the unit argument hypergeometric functions with \( s = 1 + b^2 = 1/(p + 1) > 0 \), so that convergence criteria is fulfilled.

For the moment let us concentrate on the second simpler summand of expansion (B.93), which behaves as \((1 - x)^{-s}\) times something analytic at \( x \sim 1 \). The main contribution of this part in (B.78) is equal to (remind that \( B_{1,2} \) are related to \( A_{1,2,3} \) via (B.91))

\[ C = \Gamma(3\rho - 2)^2 \times \left( \frac{s_1\Gamma^2(B_1)\Gamma^2(B_2)}{\prod_{i=1}^3\Gamma^2(A_i)} + \frac{s_2\Gamma^2(2 - B_1)\Gamma^2(1 - B_1 + B_2)}{\prod_{i=1}^3\Gamma^2(1 - B_1 + A_i)} + \frac{s_3\Gamma^2(2 - B_2)\Gamma^2(1 - B_2 + B_1)}{\prod_{i=1}^3\Gamma^2(1 - B_2 + A_i)} \right). \]  

(B.98)

Notice also that the exponent of prefactor \(|1 - x|^{\frac{b\rho}{3}}\), relating the correlator (B.77) to (B.78), in our case is equal to \( \frac{b\rho}{3} = -\frac{b^2}{3} = -4\Delta(-b\omega_1) \), which exactly is the exponent expected for the contribution of unit operator (see (B.67)). Under this circumstances the 4-point correlator factorizes as

\[ \langle \phi_{\alpha_1}(\infty)\phi_{-b\omega_2}(1)\phi_{-b\omega_1}(x)\phi_{\alpha_1}(0) \rangle \sim \langle \phi_{\alpha_1}(\infty)\phi_{\alpha_1}(0) \rangle \langle \phi_{-b\omega_1}(x)\phi_{-b\omega_2}(1) \rangle. \]  

(B.99)

Thus standard unit normalization for 2-point functions is compatible with (B.98) if \( C \equiv 1 \). Combining this condition with (B.88) we fix

\[ s_1 = \frac{\gamma(A_1)\gamma(A_2)\gamma(3 - 3A_3)\gamma(A_3)}{\gamma(A_1 - A_3 + 1)\gamma(A_2 - A_3 + 1)}, \]  

(B.100)

\[ s_2 = \frac{\gamma(2A_3 - A_2)\gamma(A_1 + A_3 - A_2)\gamma(A_3)\gamma(3 - 3A_3)}{\gamma(A_1 - A_2 + 1)\gamma(A_3 - A_2 + 1)}, \]  

(B.101)

\[ s_3 = \frac{\gamma(2A_3 - A_1)\gamma(A_2 + A_3 - A_1)\gamma(A_3)\gamma(3 - 3A_3)}{\gamma(A_2 - A_1 + 1)\gamma(A_3 - A_1 + 1)}. \]  

(B.102)
Using (B.89) we can easily get also the $u$-channel constants

$$u_1 = \frac{\gamma(A_1 - A_2 + A_3) \gamma(A_1) \gamma(A_3)}{\gamma(A_1 - A_2 + 1) \gamma(A_1 - A_3 + 1)} \gamma(3 - 3A_3),$$

$$u_2 = \frac{\gamma(A_2 - A_1 + A_3) \gamma(A_2) \gamma(A_3)}{\gamma(A_2 - A_1 + 1) \gamma(A_2 - A_3 + 1)} \gamma(3 - 3A_3),$$

$$u_3 = \frac{\gamma(2A_3 - A_1) \gamma(2A_3 - A_2) \gamma(A_3)}{\gamma(A_3 - A_1 + 1) \gamma(A_3 - A_2 + 1)} \gamma(3 - 3A_3).$$

Remind that the $s$- and $t$-channel intermediate fields are already identified hence the coefficients $s_i, b_i$ are just the (squared) OPE structure constants

$$s_1 = [C_{-b\omega_1, \alpha_1}]^2, \quad s_2 = [C_{-b\omega_1, \alpha_1 + b\omega_2}]^2, \quad s_3 = [C_{-b\omega_1, \alpha_1}]^2$$

and

$$u_1 = [C_{-b\omega_1, \alpha_1 + b\omega_2}]^2, \quad u_2 = [C_{-b\omega_1, \alpha_1 + b\omega_2}]^2, \quad u_3 = [C_{-b\omega_1, \alpha_1 + b\omega_2}]^2.$$

Using (B.82) and (B.91) one can check that both (B.106) and (B.107) consistently lead to the formulae (1.34), (1.35) and (1.36).

To find structure constants including the perturbing field $\phi_{-b(\omega_1 + \omega_2)}$, it is necessary to investigate the $t$-channel contributions coming from the first summand of the formula (B.93). Besides the leading terms we shall keep also the subleading contributions, since we are also interested in the structure constants of certain first level descendents fields.

$$C_{-b(\omega_1 + \omega_2)} C_{-b(\omega_1 + \omega_2), \alpha_1, \alpha_1} = \frac{\sin(\pi A_3) \gamma^2(A_3) \gamma(2 - 3A_3) \gamma(3 - 3A_3)}{\sin(3\pi A_3)}$$

$$\times \left( \frac{\sin(\pi(A_1 - 3A_3)) \gamma(A_1) \gamma(A_2) F_1^2}{\sin(\pi(A_1 - A_3)) \gamma(A_1 - 3A_3) + 2} \frac{\gamma(A_2 - A_3 + 1)}{\gamma(A_2 - A_3 + 2)} + \frac{\sin(\pi(A_1 - A_3)) \gamma(A_1 - A_2) \gamma(A_2 + A_3 - A_1) F_2^2}{\sin(\pi(A_1 - A_3)) \gamma(A_1 - A_2 - A_3 + 1)} \frac{\gamma(A_1 - A_2) \gamma(A_2 + A_3 - A_1 F_3^2)}{\gamma(A_1 - A_2) \gamma(A_2 + A_3 - A_1 F_3^2)} \right),$$

where

$$F_1 = 3F_2 \left( \begin{array}{c} A_1; 1 + A_2 - 2A_3; 1 - A_3 \\ 2 + A_1 - 3A_3; 1 + A_2 - A_3 \\ 1 \end{array} \right),$$

$$F_2 = 3F_2 \left( \begin{array}{c} 1 - A_2; 1 - A_3; A_1 - A_2 + A_3 \\ 2 + A_1 - A_2 - 2A_3; 1 - A_2 + A_3 \\ 1 \end{array} \right),$$

$$F_3 = 3F_2 \left( \begin{array}{c} 1 - A_3; 1 - A_1 + A_2 - A_3; -A_1 + 2A_3 \\ 1 - A_1 + A_2; 2 - A_1 - A_3 \\ 1 \end{array} \right).$$

The structure constant $C_{-b(\omega_1 + \omega_2)}$ is easily calculated using the special case $\alpha_1 = -b\omega_2$ of the first equality in (B.106):

$$C_{-b(\omega_1 + \omega_2)} = \frac{\sqrt{\gamma(2 - 3\rho) \gamma(3 - 3\rho)}}{\gamma(3 - 4\rho) \gamma(2 - 2\rho)}.$$
Thus, in principle, (B.108) determines the diagonal structure constants of type

\[ C_{-b(\omega_1 + \omega_2), \alpha, \alpha^*}. \]

Unfortunately, in general case the generalized hypergeometric functions with unit arguments in (1.30) can not be expressed via gamma-functions and we do not know if there is a way to simplify further these expressions. Fortunately in large \( p \) limit, which is considered in this paper the (rather subtle) calculations lead to concise algebraic expressions. Notice also, that in an important case \( \alpha = -b(\omega_1 + \omega_2) \), above mentioned generalized hypergeometric functions of unit argument can be represented in terms of gamma-functions using so called Watson’s sum:

\[
\begin{align*}
3F_2 \left( \begin{array}{c} a; b; c \\ \frac{1}{2}(a + b + 1); 2c \end{array} \right) &= \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} + c \right) \Gamma \left( \frac{1}{2}(a + b + 1) \right) \Gamma \left( c + \frac{1}{2}(1 - a - b) \right)}{\Gamma \left( \frac{1}{2}(a + 1) \right) \Gamma \left( \frac{1}{2}(b + 1) \right) \Gamma \left( c + \frac{1}{2}(1 - a) \right) \Gamma \left( c + \frac{3}{2}(1 - b) \right)}. \quad (B.111)
\end{align*}
\]

Calculation is rather lengthy, but the final expression has a nice factorized form

\[
C_{-b(\omega_1 + \omega_2), -b(\omega_1 + \omega_2)} = \frac{2(4 - 5\rho)^2 \gamma \left( \frac{5}{2} \right) \gamma \left( \frac{5}{2} \right)}{(3\rho - 2)(4\rho - 3)\gamma(3 - 3\rho)\gamma \left( \frac{3\rho}{2} - 1 \right)^2}. \quad (B.112)
\]

Generalization of this formula for \([A_{n-1}^{(p)}]\) and \([D_n^{(p)}]\) minimal models has been found long time ago by Fateev and Lukyanov \([7]\) by completely different method, using Coulomb gas integrals.

To conclude this appendix let us briefly demonstrate how to calculate non-diagonal structure constants including the perturbing field \( \phi_{-b\omega_1 + \omega_2} \). It follows from the OPE structure that there are two essentially different cases to investigate: \( \alpha_2 = \alpha^*_1 - b(\omega_1 + \omega_2) \) and \( \alpha_2 = \alpha^*_2 - 2b\omega_1 + b\omega_2 \). Other admissible choices are related to these two via obvious symmetry of the structure constants under permutation of fields and the symmetry under particular Weyl reflection \( \omega_1 \leftrightarrow \omega_2 \). Specifying parameters from (B.77)-(B.82) and (B.88) one can see that the 4-point correlation functions get simplified drastically:

\[
\begin{align*}
\langle \phi_{\alpha_1^* - b(\omega_1 + \omega_2)} \phi_{-b\omega_2} (1) \phi_{-b\omega_1} (x) \phi_{\alpha_1} (0) \rangle &= s_1 \langle x | 2b(\alpha_1 - h_1) | 1 - x | -\frac{2b^2}{4} \rangle, \quad (B.113)
\langle \phi_{\alpha_2^* - 2b\omega_1 + 2\omega_2} \phi_{-b\omega_2} (1) \phi_{-b\omega_1} (x) \phi_{\alpha_1} (0) \rangle &= s_2 \langle x | 2 + 2b^2 + 2b(\alpha_1 - h_2) | 1 - x | -\frac{2b^2}{4} \rangle. \quad (B.114)
\end{align*}
\]

Comparing s-and t-channels of above correlators with respective OPE’s and using known structure constants (1.34)-(1.36), one obtains (1.32) and (1.33).

We have seen that investigation of the leading singularities in s-, t- and u-channels allows to find several structure constants. Similarly, investigating sub-leading singularities one can compute also some descendant structure constants. In particular case of our interest

\[
\phi_1 = \phi_0^*, \quad \phi_2 = \phi_0, \quad \phi_0 := \phi \left[ \frac{n}{n'}, \frac{n}{n'} \right]. \quad (B.115)
\]
the constants $a$ and $\langle a\bar{a} \rangle$ defined in (B.7) and (B.44) can be evaluated as follows.

The OPE (B.92) supplemented also by the subleading term becomes

$$
\phi_{-b\omega_1}(z)\phi_{-b\omega_2}(1) \sim |z - 1|^{-4\Delta(-b\omega_1)}|1| + |z - 1|^{2\Delta(-b\omega_2) - 4\Delta(-b\omega_1)C_{-b\omega_1, -b\omega_2}} \times (1 + (z - 1) (\frac{1}{2}L_{-1} + \beta W_{-1}) + \ldots) (1 + (\bar{z} - 1) (\frac{1}{2}\bar{L}_{-1} + \beta \bar{W}_{-1}) + \ldots) \phi_{-b\omega_0},
$$

where $\beta$ is given in (B.76). Now we can apply this OPE inside the correlation function (B.77). Comparing appropriate powers of $z$ and $\bar{z}$ on both sides we arrive at the expressions (B.54) and (B.55) for $a$ and $\langle a\bar{a} \rangle$ respectively.

Similarly for the constants $a_{\alpha\alpha}$ needed for investigation of anomalous $W$-weights for second class we get the result (3.22).

C Hypergeometric functions for small $\epsilon$

For our purposes we need explicit expressions for the diagonal structure constants (1.30) in small $\epsilon$ limit. The parameters of generalized hypergeometric functions at this limit approach to integer values, which makes their investigation rather subtle. In the next subsections we display the results of analysis explicitly. To illustrate our method we outline the details of procedure on one specific example

$$
\binom{3}{2} F_2 \left( \begin{array}{c} 1 - (1 + n')x; \, x; \, -1 + (2 + n)x \\ (n + 1)x; \, (2 - n')x \end{array} \right| 1 \right) \tag{C.1}
$$

where

$$
x = \frac{\epsilon}{3}
$$

is supposed to be small. Let us examine the expansion (B.95) of our hypergeometric function carefully. The sum of first two terms behaves as

$$
1 + \frac{(1 - (n' + 1)x)((n + 2)x - 1)}{(n + 1)(2 - n')x} = \frac{1}{(n + 1)(n' - 2)x} + O(1). \tag{C.2}
$$

Since $x$ is small, we can choose a large integer $M$ such that

$$
1 \ll M \ll \frac{1}{x}. \tag{C.3}
$$

Then the terms with $i > M$ can be approximated as

$$
\frac{(1 - (1 + n')x)i(n + 1)(2 - n')x}{(n + 1)x_i((2 - n')x)_i!} = i^{-x-1} \left( \frac{\Gamma((n + 1)x)\Gamma(-(n' - 2)x)}{\Gamma((n + 2)x - 1)\Gamma(1 - (n' + 1)x)} + O \left( \frac{1}{i} \right) \right) \tag{C.4}
$$

$$
= i^{-x-1} \left( \frac{n + 2}{(n + 1)(n' - 2)} + O \left( \frac{1}{i} \right) \right),
$$

where on second line we have suppressed $O(x)$ correction since, due to (C.3) it is much smaller than $O(1/i)$. Instead, for $i + 1$'th term with $i < M$ we take small $x$ limit

$$
\frac{(1 - (1 + n')x)i(n + 1)(2 - n')x}{(n + 1)x_i((2 - n')x)_i(i + 1)!} = \frac{n + 2}{(n + 1)(n' - 2)i} + O(x). \tag{C.5}
$$
Notice that the result is almost the same as (C.4) the only difference being that now the factor \( i^{-x} \) is missing. Let us estimate how the sum will be affected if we put this factor by hand. In fact
\[
\sum_{i=1}^{M-1} \left( \frac{1}{i} - \frac{1}{i^{1+x}} \right) = \sum_{i=1}^{M-1} \left( \frac{x \log i}{i} + O(x^2) \right) < (M - 1)x,
\]
which due to our choice of \( M \) (C.3) is small. By definition of Riemann zeta function
\[
\sum_{i=1}^{\infty} i^{-(x+1)} = \zeta(x + 1) = \frac{1}{x} + O(1).
\]
Thus
\[
\sum_{i=2}^{\infty} \frac{(1 - (1 + n')x)_i (x)_i (-1 + (2 + n)x)_i}{((n + 1)x)_i ((2 - n')x)_i i!} = \frac{n + 2}{(n + 1)(n' - 2)x} + O(1).
\]
Adding contribution of the first two terms (C.2) we find
\[
\begin{align*}
\binom{n + n' + 1}{n' n' - 1} \bigg|_{\left[n \atop \frac{1}{2} \atop n' \atop n' - 1\right]} \approx & \frac{n + 3}{(n + 1)(n' - 2)x} + O(1) \quad \text{(C.8)}
\end{align*}
\]
\[
\begin{align*}
\binom{n + n' + 1}{n' n' - 1} \bigg|_{\left[n \atop 2 \atop n' \atop n' - 1\right]} \approx & \frac{n + 3}{(n + 1)(n' - 2)x} + O(1) \quad \text{(C.11)}
\end{align*}
\]
\[
\begin{align*}
\binom{n + n' + 1}{n' n' - 1} \bigg|_{\left[n \atop 2 \atop n' \atop n' - 1\right]} \approx & \frac{n + 3}{(n + 1)(n' - 2)x} + O(1) \quad \text{(C.9)}
\end{align*}
\]
\[
\begin{align*}
\binom{n + n' + 1}{n' n' - 1} \bigg|_{\left[n \atop 2 \atop n' \atop n' - 1\right]} \approx & \frac{n + 3}{(n + 1)(n' - 2)x} + O(1) \quad \text{(C.12)}
\end{align*}
\]

C.1 Small \( \epsilon \) hypergeometric functions for the second class

Here we will be interested in the small \( \epsilon \) behavior of the hypergeometric functions needed for calculation of diagonal structure constants (1.30) in case of the second invariant class (2.10). For calculation of
\[
\binom{n + n' + 1}{n' n' - 1} \bigg|_{\left[n \atop 2 \atop n' \atop n' - 1\right]} \approx \frac{n + 3}{(n + 1)(n' - 2)x} + O(1).
\]

For
\[
\binom{n + n' + 1}{n' n' - 1} \bigg|_{\left[n \atop 2 \atop n' \atop n' - 1\right]} \approx \frac{n + 3}{(n + 1)(n' - 2)x} + O(1).
\]
we need
\[ 3F_2 \left( \begin{array}{c} x; \frac{-n'x; (1 + n)x}{1 + nx \frac{-1 + (3 - n')x}{1} } \end{array} \right) \approx 1, \]  \hspace{1cm} (C.13)
\[ 3F_2 \left( \begin{array}{c} x; \frac{2 + (n + n' - 2)x; 1 + n'x}{2 + (n' - 1)x \frac{1 + (n + n' + 1)x}{1} } \end{array} \right) \approx 2, \]  \hspace{1cm} (C.14)
\[ 3F_2 \left( \begin{array}{c} x; \frac{-(n + n')x; (1 - n)x}{1 - nx \frac{-1 - (n + n' - 3)x}{1} } \end{array} \right) \approx 1. \]  \hspace{1cm} (C.15)

For
\[ C_2 \left( \begin{array}{c} \frac{n}{n'} \frac{n-1}{n'+1} \\ \frac{1}{2} \frac{n}{n'} \frac{n-1}{n'+1} \end{array} \right) \]
we need
\[ 3F_2 \left( \begin{array}{c} x; \frac{2 - (n' + 2)x; 1 + nx}{2 + (n - 1)x \frac{1 + (1 - n')x}{1} } \end{array} \right) \approx 2, \]  \hspace{1cm} (C.16)
\[ 3F_2 \left( \begin{array}{c} x; \frac{1 + (n + n' - 1)x; (n' + 2)x - 1}{(n' + 1)x \frac{(n' + 2)x + 1}{(n' + 1)x} \frac{1}{1} } \end{array} \right) \approx -\frac{n' + 3}{(n' + 1)(n' + n' + 2)x}, \]  \hspace{1cm} (C.17)
\[ 3F_2 \left( \begin{array}{c} x; \frac{1 - x(n + n' + 1); (2 - n)x - 1}{(1 - n)x \frac{(2 - n - n'x}{1} \frac{1}{1} } \end{array} \right) \approx \frac{n - 3}{(n - 1)(n + n' - 2)x}. \]  \hspace{1cm} (C.18)

C.2 Small $\epsilon$ hypergeometric functions for the third class

Here we list the small $\epsilon$ formulae for those hypergeometric functions which enter in expressions of the structure constants relevant for the third class (2.34).

For the structure constant
\[ C_1 \left( \begin{array}{c} \frac{n}{n'} \frac{n+2}{n'+1} \\ \frac{1}{2} \frac{n}{n'} \frac{n+2}{n'+1} \end{array} \right) \]
we need the small $\epsilon$ expansions
\[ 3F_2 \left( \begin{array}{c} x; \frac{-(n + 3)x - 2}{(n + 2)x - 1; (3 - n')x - 1 \frac{1}{1} } \end{array} \right) \approx \frac{3n(n' - 1) + 9n' - 6}{(n + 2)(n' - 3)}, \]  \hspace{1cm} (C.19)
\[ 3F_2 \left( \begin{array}{c} x; \frac{x(n + n'); n'x + 1}{(n' - 1)x + 2; x(n + n' + 3) - 1 \frac{1}{1} } \end{array} \right) \approx \frac{3}{n + n' + 3}, \]  \hspace{1cm} (C.20)
\[ 3F_2 \left( \begin{array}{c} x; \frac{2 - x(n + n' + 2); 2 - (n + 1)x}{3 - (n + 2)x; 1 - (n + n' - 1)x \frac{1}{1} } \end{array} \right) \approx 3. \]  \hspace{1cm} (C.21)

For
\[ C_2 \left( \begin{array}{c} \frac{n}{n'} \frac{n+1}{n'+1} \\ \frac{1}{2} \frac{n}{n'} \frac{n+1}{n'+1} \end{array} \right) \]
we need
\[ 3F_2 \left( \begin{array}{c} x; \frac{2 - (n' + 2)x; (n + 2)x - 1}{(n + 1)x \frac{1 - n'x + 1}{1} \frac{1}{1} } \end{array} \right) \approx -\frac{3}{1 + n}, \]  \hspace{1cm} (C.22)
\[ 3F_2 \left( \begin{array}{c} x; \frac{x(n + n' + 1); (n' + 2)x - 1}{(n' + 1)x \frac{x(n + n' + 4) - 2}{1} \frac{1}{1} } \end{array} \right) \approx \frac{3((n + 4)n' + n + n'^2 + 2)}{2(n' + 1)(n + n' + 4)}, \]  \hspace{1cm} (C.23)
\[ 3F_2 \left( \begin{array}{c} x; \frac{3 - x(n + n' + 3); 1 - nx}{2 - (n + 1)x \frac{x(n + n')}{2} \frac{1}{1} } \end{array} \right) \approx \frac{3}{2}. \]  \hspace{1cm} (C.23)
For
\[ C\left[ \frac{n}{n'} n+1 \frac{n'}{n'} n+2 \right]_{\frac{3}{2}, \frac{n}{n'} n+2} := C^3 \]
we need

\begin{align}
3F2 \left( \frac{x; (1-n')x - 1; (n + 2)x - 1}{(n + 1)x; (4-n')x - 2} \right) & \approx \frac{3}{2} \left( 1 + \frac{n + 2}{(n + 1)(n' - 4)} \right), \\
C & \quad (C.24) \\
3F2 \left( \frac{x; x(n + n' - 2) + 2; (n' - 1)x + 2}{(n' - 2)x + 3; x(n + n' + 1) + 1} \right) & \approx 3, \\
C & \quad (C.25) \\
3F2 \left( \frac{x; -(n + n')x; 1 - nx}{2 - (n + 1)x; (3 - n - n')x - 1} \right) & \approx -\frac{3}{n + n' - 3}. \\
C & \quad (C.26)
\end{align}

For
\[ C\left[ \frac{n}{n'} n-1 \frac{n'}{n'} n+2 \right]_{\frac{3}{2}, \frac{n}{n'} n+2} := C^8 \]
we need

\begin{align}
3F2 \left( \frac{x; 3 - (n' + 3)x; nx + 1}{(n - 1)x + 2; 2 - n'x} \right) & \approx \frac{3}{2}, \\
C & \quad (C.27) \\
3F2 \left( \frac{x; x(n + n'); (n' + 3)x - 2}{(n' + 2)x - 1; x(n + n' + 3) - 1} \right) & \approx \frac{3(n(n' + 3) + n'(n' + 4) + 2)}{(n' + 2)(n + n' + 3)}, \\
C & \quad (C.28) \\
3F2 \left( \frac{x; 2 - x(n + n' + 2); (2 - n)x - 1}{(1 - n)x; 1 - (n + n' - 1)x} \right) & \approx \frac{3}{n - 1}. \\
C & \quad (C.29)
\end{align}

For
\[ C\left[ \frac{n}{n'} n-1 \frac{n'}{n'} n-1 \right]_{\frac{3}{2}, \frac{n}{n'} n-1} := C^9 \]
we need

\begin{align}
3F2 \left( \frac{x; -n'x; nx + 1}{(n - 1)x + 2; 3 - n'x - 1} \right) & \approx -\frac{3}{n'-3}, \\
C & \quad (C.29) \\
3F2 \left( \frac{x; x(n + n' - 3) + 3; n'x + 1}{(n' - 1)x + 2; (n + n')x + 2} \right) & \approx \frac{3}{2}, \\
C & \quad (C.30) \\
3F2 \left( \frac{x; (1-n-n')x - 1; (2-n)x - 1}{(1-n)x; 4 - (n - n'x) - 2} \right) & \approx \frac{3}{2} \left( 1 + \frac{n - 2}{(n - 1)(n + n' - 4)} \right). \\
C & \quad (C.31)
\end{align}

For
\[ C\left[ \frac{n}{n'} n-2 \frac{n'}{n'} n+1 \right]_{\frac{3}{2}, \frac{n}{n'} n+1} := C^{10} \]
we need

\begin{align}
3F2 \left( \frac{x; 2 - (n' + 2)x; (n - 1)x + 2}{(n - 2)x + 3; (1 - n'x + 1} \right) & \approx 3, \\
C & \quad (C.31) \\
3F2 \left( \frac{x; x(n + n' - 2) + 2; (n' + 2)x - 1}{(n' + 1)x; (n + n' + 1)x + 1} \right) & \approx -\frac{3}{n' + 1}, \\
C & \quad (C.32) \\
3F2 \left( \frac{x; -(n + n')x; (3 - n)x - 2}{2 - n)x - 1; (3 - n - n')x - 1} \right) & \approx \frac{3(n(n' + 4) - 9n' + 6)}{(n - 2)(n + n' - 3)}. \\
C & \quad (C.33)
\end{align}
D Structure constants for small $\epsilon$

Here we list the structure constants needed for computation of anomalous dimensions in the third class. Let us start with the diagonal ones. From (1.30) using the results of C.2 for small $\epsilon$ we get

$$C_1^1 \approx \frac{2(n+3)n'^2 + 2n(n+3)n' - n(n+2)}{2\sqrt{2}(n+1)n'(n+n')} ,$$  \hspace{1cm} (D.1)

$$C_2^2 \approx \frac{n^2(2n' + 1) + 2n(n'(n' + 4) + 1) + n'(n' + 2)}{2\sqrt{2}nn'(n + n' + 1)} ,$$  \hspace{1cm} (D.2)

$$C_3^3 \approx \frac{2n^2(n' - 3) + 2n(n' - 3)n' + (n' - 2)n'}{2\sqrt{2}n(n' - 1)(n + n')} ,$$  \hspace{1cm} (D.3)

$$C_4^4 \approx C_5^5 = C_6^6 = \frac{27}{4\sqrt{2}(n^2 + nn' + n'^2 - 3)} ,$$  \hspace{1cm} (D.4)

$$C_7^7 \approx \frac{4(n^2 + nn' + n'^2 - 3)^3 + 9(n^2 + nn' + n'^2 - 3)^2 + 3(n-n')^2(2n+n')^2(2n+n')^2}{4\sqrt{2}(n^2 - 1)(n^2 - 1)(n+n') - 1}(n^2 + nn' + n'^2 - 3) ,$$  \hspace{1cm} (D.5)

$$C_8^8 \approx \frac{2n^2(n' + 3) + 2nn'(n' + 3) - n'(n'+3)}{2\sqrt{2}n(n'+1)(n+n')} ,$$  \hspace{1cm} (D.6)

$$C_9^9 \approx \frac{n^2(2n' - 1) + 2n((n'-4)n'+1) - (n'-2)n'}{2\sqrt{2}nn'(n + n' - 1)} ,$$  \hspace{1cm} (D.7)

$$C_{10}^{10} \approx \frac{2(n-3)n'^2 + 2(n-3)nn' + (n-2)n}{2\sqrt{2}(n-1)n'(n + n')} .$$  \hspace{1cm} (D.8)

The off diagonal structure constants can be obtained from (1.32) and (1.33). Below we list the nonzero structure constants only.

$$C_2^1 \approx \frac{1}{2n'} \sqrt{\frac{(n+2)(n'^2 - 1)(n+n'+2)}{2(n+1)(n+n'+1)}} ,$$  \hspace{1cm} (D.9)

$$C_3^1 \approx \frac{1}{2(n+n')} \sqrt{\frac{(n+2)(n'-2)((n+n')^2 - 1)}{2(n+1)(n'-1)}} ,$$  \hspace{1cm} (D.10)

$$C_4^1 \approx \frac{3(n-2)^2}{4(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{(n+2)(n'-1)(n+n'+1)}{2nn'(n+n')}} ,$$  \hspace{1cm} (D.11)

$$C_5^1 = C_6^1 \approx \frac{(n-2)(n+2n')}{4(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{3(n^2 + n-2)(n'+1)(n+n'-1)}{2n(n+1)n'(n+n')}} ,$$  \hspace{1cm} (D.12)

$$C_7^1 \approx \frac{(n-1)(n'+1)(n+n'-1)(n+2n')^2}{4(n+1)(n'-1)(n+n'+1)(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{(n+2)(n'-1)(n+n'+1)}{2nn'(n+n')}} .$$  \hspace{1cm} (D.13)
\[ C_4^4 \approx \frac{3(n' + 2)^3}{4(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{(n' - 2)(n^2 + nn' + n' - 1)}{2nn'(n + n')}} \],

\[ C_5^3 = C_6^3 \approx -\frac{(n' + 2)(2n + n')}{4(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{3(n - 1)(n' - 2)(n' + 1)(n + n' + 1)}{2n(n' - 1)n'n'(n + n')}} \],

\[ C_7^3 \approx \frac{(n - 1)(n' + 1)(n + n' + 1)(2n + n')^2}{4(n + 1)(n' - 1)(n + n' - 1)(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{(n' - 2)(n^2 + nn' + n' - 1)}{2nn'(n + n')}} \],

\[ C_9^4 \approx \frac{1}{2n} \sqrt{\frac{(n^2 - 1)(n' - 2)(n + n' - 2)}{2(n' - 1)(n + n' - 1)}} \],

\[ C_7^4 \approx \frac{27}{4\sqrt{2}(n^2 + nn' + n'^2 - 3)} \],

\[ C_8^4 \approx \frac{3(n' - 2)^2}{4(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{(n - 1)(n' + 2)(n + n' + 1)}{2nn'(n + n')}} \],

\[ C_9^4 \approx \frac{3(n + n' + 2)^2}{4(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{(n - 1)(n' - 2)(n + n' - 2)}{2nn'(n + n')}} \],

\[ C_{10}^4 \approx \frac{3(n + 2)^2}{4(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{(n' - 2)(n' + 1)(n + n' - 1)}{2n' + n + n' - 1}} \],

\[ C_5^5 \approx \frac{27}{4\sqrt{2}(n^2 + nn' + n'^2 - 3)} \],

\[ C_7^5 \approx \frac{3(n' - n)(2n + n')(n + 2n')}{2(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{3}{2(n^2 - 1)(n^2 - 1)((n + n')^2 - 1)}} \],

\[ C_8^5 \approx -\frac{(n' - 2)(2n + n')}{4(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{3(n + 1)(n^2 + n' - 2)(n + n' - 1)}{2nn'(n' + 1)(n + n')}} \],

\[ C_9^5 \approx \frac{(n' - n)(n + n' + 2)}{4(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{3(n + 1)(n' + 1)(n + n' - 2)(n + n' + 1)}{2nn'(n + n' + 1)(n + n')}} \],

\[ C_{10}^5 \approx \frac{(n + 2)(n + 2n')}{4(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{3(n - 2)(n + 1)(n' - 1)(n + n' + 1)}{2(n - 1)nn'(n + n')}} \],

\[ (D.14) \]

\[ (D.15) \]

\[ (D.16) \]

\[ (D.17) \]

\[ (D.18) \]

\[ (D.19) \]

\[ (D.20) \]

\[ (D.21) \]

\[ (D.22) \]

\[ (D.23) \]

\[ (D.24) \]

\[ (D.25) \]

\[ (D.26) \]
Next consider

\[
C^6 \approx \frac{3(n'-n)(2n+n')(n+2n')}{2(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{3}{2(n^2 - 1)(n'^2 - 1)((n+n')^2 - 1)}}, \quad \text{(D.27)}
\]

\[
C^6 \approx -\frac{(n'-2)(2n+n')}{4(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{3(n+1)(n'^2 + n' - 2)(n+n')}{2nn'(n+n')}}, \quad \text{(D.28)}
\]

\[
C^6 \approx \frac{(n'-n)(n+n'+2)}{4(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{3(n+1)(n'+1)(n+n'-2)(n+n'+1)}{2nn'(n+n')}}, \quad \text{(D.29)}
\]

\[
C^6_{10} \approx \frac{(n+2)(n+2n')}{4(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{3(n-2)(n+1)(n'+1)(n+n'+1)}{2(1-n)n'n(n+n')}}, \quad \text{(D.30)}
\]

\[
C^7 \approx \frac{(n+1)(n'-1)(n+n'-1)(2n+n')}{4(n-1)(n'+1)(n+n' +1)(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{(n-1)(n'+2)(n+n'+1)}{2nn'(n+n')}}, \quad \text{(D.31)}
\]

\[
C^7 \approx \frac{(n+1)(n'+1)(n-n')^2(n+n'+1)}{4(n-1)(n'-1)(n+n'-1)(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{(n-1)(n'-1)(n+n'-2)}{2nn'(n+n')}}, \quad \text{(D.32)}
\]

\[
C^7_{10} \approx \frac{(n+1)(n'-1)(n+n'+1)(n+2n')}{4(n-1)(n'+1)(n+n'-1)(n^2 + nn' + n'^2 - 3)} \sqrt{\frac{(n-2)(n'+1)(n+n'-1)}{2nn'(n+n')}}, \quad \text{(D.33)}
\]

\[
C^8_{10} \approx \frac{1}{2\sqrt{2(n+n')}} \sqrt{\frac{(n-2)(n'+2)((n+n')^2 - 1)}{(n-1)(n'+1)}}, \quad \text{(D.34)}
\]

\[
C^9_{10} \approx \frac{1}{2\sqrt{2n'}} \sqrt{\frac{(n-2)(n'^2 - 1)(n+n'-2)}{(n-1)(n+n'-1)}}. \quad \text{(D.35)}
\]

\[E\quad \text{Representation of } \phi^I_R \phi_j \text{ in terms of direct product WZNW models in case of second class}\]

We have already shown in the main text (see eq. (4.24)) that

\[|11\rangle = |\lambda_{n-1,n'}\rangle |\lambda_{2,1}\rangle |\lambda_{2,1}\rangle.\]

Next consider \(\phi^I_R \begin{bmatrix} n-1 & n \\ n' & n' \end{bmatrix} \phi^{UV} \begin{bmatrix} n & n \\ n' & n'-1 \end{bmatrix} :\]

\[
\Delta^I_R \begin{bmatrix} n-1 & n \\ n' & n' \end{bmatrix} = h_{k-1}(\lambda_{n-1,n'}) + h_1(\lambda_{2,1}) - h_k(\lambda_{n,n'}), \quad \text{(E.1)}
\]

\[
\Delta^{UV} \begin{bmatrix} n & n \\ n' & n'-1 \end{bmatrix} = h_k(\lambda_{n,n'}) + h_1(\lambda_{2,1}) - h_{k+1}(\lambda_{n,n'-1}). \quad \text{(E.2)}
\]
It is easy to check that
\[
\lambda_{n-1,n'} + \lambda_{2,1} + \lambda_{2,1} + (-1, 0, 1)^T = \lambda_{n,n'+1},
\]
\[
\lambda_{n-1,n'} + \lambda_{2,1} = \lambda_{n,n'}.
\]  

(E.3)  

(E.4)

The second equality implies that one should consider only such states which belong to the irreducible representation of the combined current \( K = E + J \) with highest weight \( \lambda_{n-1,n'} + \lambda_{2,1} \). Taking into account also the constraint coming from the first equality one is lead to the ansatz

\[
|12\rangle = \left( a_1 K_{31} + a_2 \tilde{J}_{31} + a_3 K_{32} K_{21} + a_4 K_{32} \tilde{J}_{21} \right) |\lambda_{n-1,n'}\rangle |\lambda_{2,1}\rangle |\lambda_{2,1}\rangle. 
\]  

(E.5)

Now we can impose the condition that \(|12\rangle\) should be a highest weight state of total current

\[
(K_{ij} + \tilde{J}_{ij}) |12\rangle = 0 \quad \text{with} \quad i < j.
\]  

(E.6)

This fixes the coefficients \( a \) up to an overall constant which we will choose such that \( a_1 = 1 \). Here is the result

\[
|12\rangle = \left( (E + J)_{31} - \frac{(1-n-n')(1-n')}{n'} \tilde{J}_{31} - \frac{1}{n'} (E + J)_{32} (E + J)_{21} - \frac{1}{n} (E + J)_{32} \tilde{J}_{21} \right) |\lambda_{n-1,n'}\rangle |\lambda_{2,1}\rangle |\lambda_{2,1}\rangle.
\]  

(E.7)

The other states are obtained in a similar fashion. The results are listed below.

\[
|13\rangle = \left( (E + J)_{21} + (1-n) \tilde{J}_{21} \right) |\lambda_{n-1,n'}\rangle |\lambda_{2,1}\rangle |\lambda_{2,1}\rangle,
\]  

(E.8)

\[
|21\rangle = \left( E_{31} - \frac{n'(n+n')}{1+n'} J_{31} - \frac{1}{1+n'} E_{32} E_{21} + \frac{n+n'}{n'+1} E_{32} J_{21} \right) |\lambda_{n,n'+1}\rangle |\lambda_{2,1}\rangle,
\]  

(E.9)

\[
|22\rangle = \left( (E + J)_{31} - \frac{(1-n-n')(1-n')}{n'} \tilde{J}_{31} - \frac{1}{n'} (E + J)_{32} (E + J)_{21} - \frac{1}{n} (E + J)_{32} \tilde{J}_{21} \right) \times \left( E_{31} - \frac{n'(n+n')}{1+n'} J_{31} - \frac{1}{1+n'} E_{32} E_{21} + \frac{n+n'}{n'+1} E_{32} J_{21} \right) |\lambda_{n,n'+1}\rangle |\lambda_{2,1}\rangle |\lambda_{2,1}\rangle,
\]  

(E.10)

\[
|23\rangle = \left( (E + J)_{21} + (1-n) \tilde{J}_{21} \right) \times \left( E_{31} - \frac{n'(n+n')}{1+n'} J_{31} - \frac{1}{1+n'} E_{32} E_{21} + \frac{n+n'}{n'+1} E_{32} J_{21} \right) |\lambda_{n,n'+1}\rangle |\lambda_{2,1}\rangle |\lambda_{2,1}\rangle,
\]  

(E.11)

\[
|31\rangle = \left( E_{21} - n \tilde{J}_{21} \right) |\lambda_{n+1,n'-1}\rangle |\lambda_{2,1}\rangle |\lambda_{2,1}\rangle,
\]  

(E.12)

\[
|32\rangle = \left( (E + J)_{31} - \frac{(1-n-n')(1-n')}{n'} \tilde{J}_{31} - \frac{1}{n'} (E + J)_{32} (E + J)_{21} - \frac{1}{n} (E + J)_{32} \tilde{J}_{21} \right) \times \left( E_{21} - n \tilde{J}_{21} \right) |\lambda_{n+1,n'-1}\rangle |\lambda_{2,1}\rangle |\lambda_{2,1}\rangle,
\]  

(E.13)

\[
|33\rangle = \left( (E + J)_{21} + (1-n) \tilde{J}_{21} \right) \left( E_{21} - n \tilde{J}_{21} \right) |\lambda_{n+1,n'-1}\rangle |\lambda_{2,1}\rangle |\lambda_{2,1}\rangle.
\]  

(E.14)
Having above explicit expressions we have computed ratio of scalar products $\langle \tilde{i}j|ij \rangle$ and $\langle ij|i\rangle$.

From (4.24) we simply get

$$\langle \tilde{1}1|11 \rangle \langle 11|11 \rangle = 1.$$  \hspace{1cm} (E.15)

From (E.7)

$$\langle \tilde{1}2|12 \rangle = (E + \tilde{J})_{13} - \frac{(1-n-n')(1-n')}{n'} J_{13} - \frac{1}{n'} (E + \tilde{J})_{12} (E + \tilde{J})_{23} - \frac{1-n-n'}{n'} (E + \tilde{J})_{23} J_{12} \langle \lambda_{n-1,n'} \rangle \langle \lambda_{2,1} \rangle \langle \tilde{\lambda}_{2,1} \rangle.$$  \hspace{1cm} (E.16)

so

$$\langle \tilde{1}2|12 \rangle \langle 12|12 \rangle = \frac{1}{1 - n - n'}.$$  \hspace{1cm} (E.17)

Let us list the other cases

- From (E.8) we get

$$\langle \tilde{1}3|13 \rangle \langle 13|13 \rangle = \frac{1}{1 - n}.$$  \hspace{1cm} (E.18)

- From (E.9) we get

$$\langle \tilde{2}1|21 \rangle \langle 21|21 \rangle = \frac{1}{n + n' + 1}.$$  \hspace{1cm} (E.19)

- From (E.10) we get

$$\langle \tilde{2}2|22 \rangle \langle 22|22 \rangle = 1.$$  \hspace{1cm} (E.20)

- From (E.11) we get

$$\langle \tilde{2}3|23 \rangle \langle 23|23 \rangle = \frac{1}{n' + 1}.$$  \hspace{1cm} (E.21)

- From (E.12) we get

$$\langle \tilde{3}1|31 \rangle \langle 31|31 \rangle = \frac{1}{n + 1}.$$  \hspace{1cm} (E.22)
• From (E.13) we got
\[
\frac{\langle \tilde{3}2|32 \rangle}{\langle 32|32 \rangle} = \frac{1}{1-n'}.
\] (E.23)
• From (E.14) we get
\[
\frac{\langle \tilde{3}3|33 \rangle}{\langle 33|33 \rangle} = 1.
\] (E.24)

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