Note on the Hurwitz–Lerch Zeta Function of Two Variables

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Abstract: A number of generalized Hurwitz–Lerch zeta functions have been presented and investigated. In this study, by choosing a known extended Hurwitz–Lerch zeta function of two variables, which has been very recently presented, in a systematic way, we propose to establish certain formulas and representations for this extended Hurwitz–Lerch zeta function such as integral representations, generating functions, derivative formulas and recurrence relations. We also point out that the results presented here can be reduced to yield corresponding results for several less generalized Hurwitz–Lerch zeta functions than the extended Hurwitz–Lerch zeta function considered here. For further investigation, among possibly various more generalized Hurwitz–Lerch zeta functions than the one considered here, two more generalized settings are provided.

Keywords: beta function; gamma function; Pochhammer symbol; Hurwitz–Lerch zeta function; Hurwitz–Lerch zeta function of two variables; hypergeometric functions; confluent hypergeometric functions; Appell hypergeometric functions; Humbert hypergeometric functions of two variables; integral representations; generating functions; derivative formulas; recurrence relation

1. Introduction and Preliminaries

The generalized (or Hurwitz) zeta function $\zeta(s, \nu)$ is defined by (see, e.g., [1], pp. 24–27); see also ([2] Chapter XIII)

$$\zeta(s, \nu) = \sum_{n=0}^{\infty} \frac{1}{(n+\nu)^s} \quad (\Re(s) > 1, \ \nu \in \mathbb{C} \setminus \mathbb{Z}^-), \quad (1)$$

which is a generalization of the Riemann zeta function $\zeta(s) := \zeta(s, 1)$ (see, e.g., [1] Section 1.12). Apostol [3] showed that the following analytic continuation formula (see, e.g., ([1] p. 26, Equation (6)), ([4] Equation (5.1))

$$\zeta(1-s, \nu) = \frac{2\Gamma(s)}{(2\pi)^s} \sum_{n=1}^{\infty} \frac{\cos(\pi s/2 - 2\pi \nu n)}{n^s} \quad (0 < \nu \leq 1, \Re(s) > 1), \quad (2)$$

where $\Gamma(s)$ is the familiar Gamma function whose Euler’s integral (see, e.g., [1] pp. 1–24) is

$$\Gamma(s) = \int_{0}^{\infty} e^{-u} u^{s-1} \, du \quad (\Re(s) > 0), \quad (3)$$
and it can be derived from a known transformation formula for the Lerch zeta function

\[ \Phi(x,v,s) = \sum_{n=0}^{\infty} \frac{e^{2\pi inx}}{(n+v)^s}, \quad (\Re(s) > 1, \; v \in \mathbb{C} \setminus \mathbb{Z}, \; x \in \mathbb{R}). \]  

(4)

Note that it is easy to read that \( \Phi(x,v,s) = \zeta(s,v) \) when \( x \in \mathbb{Z} \). The Hurwitz–Lerch zeta function \( \Phi(z,s,v) \) is defined by (see, e.g., [1] p. 27)

\[ \Phi(z,s,v) = \sum_{n=0}^{\infty} \frac{z^n}{(n+v)^s}, \]  

(5)

where \( v \in \mathbb{C} \setminus \mathbb{Z}_0^- \). The \( \Phi(z,s,v) \) in (5) converges for all \( s \in \mathbb{C} \) when \( |z| < 1 \) and for \( \Re(s) > 1 \) when \( |z| = 1 \). Here and elsewhere, let \( \mathbb{N}, \mathbb{Z}, \mathbb{R}, \) and \( \mathbb{C} \) be the sets of positive integers, integers, real numbers, and complex numbers, respectively. Furthermore, let us denote \( \mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N} \) and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \).

The various special cases of the Hurwitz–Lerch zeta function (5) including the Riemann zeta function \( \zeta(s) \), the Hurwitz zeta function (1), and the Lerch zeta function (4) have been intensively studied and applied. We choose to take some examples: Adamchik and Srivastava ([5] Proposition 5) studied certain equivalent conditions of a reverse Hilbert-type integral inequality, which, in an example, the generalized zeta function \( \zeta(s,a) \) is shown to be related to a best possible constant factor (see also [7–9]). Recently, a number of generalizations of the Hurwitz–Lerch zeta function (5) have been actively investigated (see, e.g., [10–19] and the references cited therein). Furthermore, very recently, Choi and Parmar [20] have introduced and investigated the following two-variable extension of the Hurwitz–Lerch zeta function (5)

\[ \Phi_{a,b,b',c}(x,y,s,a) = \sum_{k,\ell=0}^{\infty} \frac{(a)_{k+\ell} (b)_{k} (b')_{\ell}}{(c)_{k+\ell} k! \ell!} x^{k} y^{\ell} (k+\ell+a)^s, \]  

(6)

where \( a, b, b' \in \mathbb{C} \) and \( c, a \in \mathbb{C} \setminus \mathbb{Z}_0^- \). The function \( \Phi_{a,b,b',c}(x,y,s,a) \) in (6) converges for all \( s \in \mathbb{C} \) when \( |x| < 1 \) and \( |y| < 1 \), and for \( \Re(s+c-a-b-b') > 1 \) when \( |x| = 1 \) and \( |y| = 1 \). Here \( (\eta)_v \) is the Pochhammer symbol given (for \( \eta, v \in \mathbb{C} \)) by

\[ (\eta)_v := \frac{\Gamma(\eta+v)}{\Gamma(\eta)} \quad (\eta+v \in \mathbb{C} \setminus \mathbb{Z}_0^-) \]

\[ = \begin{cases} 1 & (v = 0), \\ \eta(\eta+1) \cdots (\eta+n-1) & (v = n \in \mathbb{N}). \end{cases} \]  

(7)

From (3) and (7), the following integral formula for the Pochhammer symbol

\[ (s)_v = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-u} u^{\delta+v-1} \, du \quad (\Re(s+v) > 0) \]  

(8)

can be easily obtained.

Here, in a systematic way, we aim to establish certain formulas and representations for the extended Hurwitz–Lerch zeta function of two variables (6) such as integral representations, generating functions, derivative formulas and recurrence relations. We also point out that the results presented here can be reduced to produce corresponding results for several less generalized Hurwitz–Lerch zeta functions than the extended Hurwitz–Lerch zeta function (6). Further, two more generalized settings than (6) are provided.
2. Integral Representations of (6)

We begin by recalling a known integral representation of the extended Hurwitz–Lerch zeta function (6) (see [20] Theorem 1)

\[
\Phi_{a,b,b',c} (x,y,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \int_0^\infty \int_0^\infty u^{x-1} e^{-au} F_1 \left[ a, b, b'; c; xe^{-u}, ye^{-u} \right] \, du,
\]

which converges for \( \min \{ \Re(s), \Re(a) \} > 0 \) when \( |x| \leq 1 \left( x \neq 1 \right) \) and \( |y| \leq 1 \left( y \neq 1 \right) \), and for \( \Re(s) > 1 \) when \( x = 1 \) and \( y = 1 \). Here \( F_1 \) is the Appell hypergeometric function of two variables defined by (see, e.g., ([1] p. 224, Equation (6))); see also ([21] p. 22))

\[
F_1 \left[ a, b, b'; c; x, y \right] = \sum_{m,n=0}^\infty \frac{(a)_{m+n} (b)_m (b')_n}{(c)_{m+n}} \frac{x^m y^n}{m! n!}
\]

\[
\Phi(a,b,b',c) = \frac{1}{\Gamma(s)} \int_0^\infty \int_0^\infty \int_0^\infty u^{x-1} v_1^{y_1-1} v_2^{y_2-1} \exp(-au - v_1 - v_2)
\]

\[
\times F_1 \left[ a; c; e^{-u} (xv_1 + yv_2) \right] \, du \, dv_1 \, dv_2,
\]

where \( a, b, b' \in \mathbb{C}, \ c \in \mathbb{C} \setminus \mathbb{Z}_0^- \) and whose convergence region is \( \max \{ \Re(s), \Re(y) \} < 1 \). Here \( F_{pq} (p, q \in \mathbb{N}_0) \) are the generalized hypergeometric functions (see, e.g., ([1] Chapters II and V); see also ([22] Section 1.5), [21,23–26]).

The following confluent form of the Appell hypergeometric function \( F_1 \) is recalled (see, e.g., [1] p. 225, Equation (21)); see also ([21] p. 22 et seq.)

\[
\Phi_2(b,c; x, y) = \sum_{m,n=0}^\infty \frac{(b)_m (b')_n}{(c)_{m+n}} \frac{x^m y^n}{m! n!} \left( |x| < \infty, |y| < \infty \right).
\]

We provide further integral representations of the extended Hurwitz–Lerch zeta function (6), asserted in the following theorem.

**Theorem 1.** Each of the following integral representations holds.

\[
\Phi_{a,b,b',c} (x,y,s,a) = \frac{1}{\Gamma(s)} \frac{\Gamma(a)\Gamma(b)\Gamma(b')}{\Gamma(A(a,b,b',c))} \int_0^\infty \int_0^\infty \int_0^\infty u^{x-1} v_1^{y_1-1} v_2^{y_2-1} \exp(-au - v_1 - v_2)
\]

\[
\times F_1 \left[ a; c; e^{-u} (xv_1 + yv_2) \right] \, du \, dv_1 \, dv_2,
\]

where \( \min \{ \Re(s), \Re(a), \Re(b), \Re(b') \} > 0 \);

\[
\Phi_{a,b,b',c} (x,y,s,a) = \frac{1}{\Gamma(s)} \frac{\Gamma(a)\Gamma(b)\Gamma(b')}{\Gamma(A(a,b,b',c))} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty u^{x-1} v_1^{y_1-1} v_2^{y_2-1} v_3^{y_3-1}
\]

\[
\times \exp(-au - v_1 - v_2 - v_3) F_1 \left[ a; c; e^{-u} (xv_1 + yv_2 + v_3) \right] \, du \, dv_1 \, dv_2 \, dv_3,
\]

where \( \min \{ \Re(s), \Re(a), \Re(a), \Re(b), \Re(b') \} > 0 \);

\[
\Phi_{a,b,b',c} (x,y,s,a) = \frac{1}{\Gamma(s)} \frac{\Gamma(a)}{\Gamma(A(a,b,b',c))} \int_0^\infty \left( u^{x-1} \exp(-au - v) \times \Phi_2 \left[ b, b'; c; xe^{-u} v, ye^{-u} v \right] \right) \, du \, dv,
\]
where \( \min \{ \Re(s), \Re(a), \Re(b) \} > 0 \) and \( \max \{ \Re(x), \Re(y) \} < 1 \);

\[
\Phi_{a,b,b',c}(x, y, s, a) = \frac{\Gamma(c)}{\Gamma(s) \Gamma(a) \Gamma(b) \Gamma(b') \Gamma(c - b - b')}
\times \int_0^\infty \int_0^1 \int_0^1 \left( u^{s-1} e^{-u \xi} \eta^{b-1} (1 - \xi)^{c-b-1}
\times (1 - \eta)^{c-b-b'-1} \exp \left[ -au - v + xe^{-u \xi} + ye^{-v (1 - \xi)} \right] \right) d\xi \, d\eta \, du \, dv,
\]  

(15)

where \( \min \{ \Re(s), \Re(a), \Re(b) \} > 0 \) and \( \Re(c - b) > \Re(b') > 0 \);

\[
\Phi_{a,b,b',c}(x, y, s, a) = \frac{\Gamma(d)}{\Gamma(s) \Gamma(a) \Gamma(b) \Gamma(b') \Gamma(c - d - b')}
\times \int_0^\infty \int_0^1 \int_0^1 \left( u^{d-1} v^{d-1} \xi^{d-1} \eta^{b-1} (1 - \xi)^{c-d-b-1}
\times (1 - \eta)^{c-d-b'-1} \exp \left[ -au - v + xe^{-u \xi} + ye^{-v (1 - \xi)} \right] \right) d\xi \, d\eta \, du \, dv,
\]  

(16)

where \( \min \{ \Re(s), \Re(a), \Re(b) \} > 0 \) and \( \Re(d - b) > \Re(b') > 0 \);

\[
\Phi_{a,b,b',c}(x, y, s, a) = \frac{\Gamma(c)}{\Gamma(s) \Gamma(a) \Gamma(d) \Gamma(b) \Gamma(b') \Gamma(c - d - b')}
\times \int_0^\infty \int_0^1 \int_0^1 \left( u^{s-1} v^{d-1} \xi^{d-1} \eta^{b-1} (1 - \xi)^{c-d-b-1}
\times (1 - \eta)^{c-d-b'-1} \exp \left[ -au - v + xe^{-u \xi} + ye^{-v (1 - \xi)} \right] \right) d\xi \, d\eta \, du \, dv,
\]  

(17)

where \( \min \{ \Re(s), \Re(a), \Re(b) \} > 0 \) and \( \Re(c - d) > \Re(b') > 0 \);

\[
\Phi_{a,b,b',c}(x, y, s, a) = \frac{\Gamma(c)}{\Gamma(s) \Gamma(a) \Gamma(d) \Gamma(d_1) \Gamma(b_2) \Gamma(c - \delta - d - b')}
\times \int_0^\infty \int_0^1 \int_0^1 \left( u^{s-1} v^{d-1} \xi^{d-1} \eta^{b-1} (1 - \xi)^{c-d-b-1}
\times (1 - \eta)^{c-d-b'-1} \exp \left[ -au - v + xe^{-u \xi} + ye^{-v (1 - \xi)} \right] \right) d\xi \, d\eta \, du \, dv,
\]  

(18)

where \( \min \{ \Re(s), \Re(a), \Re(b) \} > 0 \) and \( \Re(c - d) > \Re(b') > 0 \);

\[
\Phi_{a,b,b',c}(x, y, s, a) = \frac{\Gamma(c)}{\Gamma(s) \Gamma(a) \Gamma(d_1) \Gamma(d_2) \Gamma(c - \delta_1 - \delta_2)
\times \int_0^\infty \int_0^1 \int_0^1 \left( u^{s-1} v^{d-1} \xi^{d-1} \eta^{b-1} (1 - \xi)^{c-d-b-1}
\times (1 - \eta)^{c-d_1-\delta_1-\delta_2-1} \exp \left[ -au - v + xe^{-u \xi} + ye^{-v (1 - \xi)} \right] \right) d\xi \, d\eta \, du \, dv,
\]  

(19)
where \( \min \{ \Re(s), \Re(a), \Re(b), \Re(c) \} > 0 \) and \( \Re(c - \delta_1) > \Re(\delta_2) > 0 \);

\[
\Phi_{a,b,b';c}(x,y,s,a) = \frac{\Gamma(c)}{\Gamma(s) \Gamma(a) \Gamma(b) \Gamma(c - a - b)} \times \frac{1}{\Gamma(\delta_1) \Gamma(\delta_2) \Gamma(c - \delta_1 - \delta_2)} \times \int \int \int \left( e^{-au} u^{a-1} \delta_1^{\delta_1-1} (1 - \delta_1)(1 - \eta)^{c-\delta_1-\delta_2-1} \right) \exp \left[ -au + x e^{-u} \eta^{c-b} + ye^{-u} (1 - \xi) \eta \right] \times \Phi_2 \left[ b - \delta_1; b' - \delta_2; c; xe^{-u} (1 - \xi)(1 - \eta), ye^{-u} (1 - \eta) \right] \, dx \, dy \, du \, dv,
\]

where \( \min \{ \Re(s), \Re(a), \Re(b), \Re(b') \} > 0 \) and \( \Re(c - b - b') > 0 \);

\[
\Phi_{a,b,b';c}(x,y,s,a) = \frac{\Gamma(c)}{\Gamma(s) \Gamma(a) \Gamma(c - a)} \times \int \int \left( e^{-au} u^{a-1} \delta_1^{\delta_1-1} (1 - \delta_1)(1 - \eta)^{c-a-1} (1 - \eta)^{c-a-1} (1 - vxe^{-u})^{-b} (1 - vye^{-u})^{-b'} \right) \, dv \, du,
\]

where \( \min \{ \Re(s), \Re(a), \Re(b) \} > 0 \) and \( \Re(c - a) > 0 \);

\[
\Phi_{a,b,b';c}(x,y,s,a) = \frac{\Gamma(c)}{\Gamma(s) \Gamma(b) \Gamma(c - b)} \int \int \left( e^{-au} u^{a-1} \delta_1^{\delta_1-1} (1 - \eta)^{c-b-1} (1 - \eta)^{c-b-1} \right) \times \left[ (1 - xe^{-u})^{c-a-b} (1 - vxe^{-u})^{a-c} \right] F_1(a,b';c-b;ye^{-u}) \, dv \, du,
\]

where \( \min \{ \Re(s), \Re(a), \Re(b) \} > 0 \) and \( \Re(c - b) > 0 \).

**Proof.** Using (8) in the series definition of \( F_1 \) in (9), we get

\[
F_1 \left[ a, b, b'; c; xe^{-u}, ye^{-u} \right] = \frac{1}{\Gamma(b) \Gamma(b')} \int \int \left( e^{-v_1 - v_2} v_1^{b-1} v_2^{b'-1} \right) \times \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}}{(c)_{m+n}} \frac{(xv_1 e^{-u})^m}{m!} \frac{(yv_2 e^{-u})^n}{n!} \, dv_1 \, dv_2.
\]

Applying the following identity (see, e.g., [26] p. 52)

\[
\sum_{m,n=0}^{\infty} f(m + n) \frac{x^m y^n}{m! n!} = \sum_{N=0}^{\infty} f(N) \frac{(x + y)^N}{N!}
\]
to the double series in the right side of (24), we have an integral representation of $F_1$

$$
F_1 [a, b, b'; c; xe^{-u}, ye^{-u}] = \frac{1}{\Gamma (b) \Gamma (b')} \times \int_0^\infty \int_0^\infty e^{-v_1 - v_2} v_1^{b-1} v_2^{b'-1} F_1 [a; c; xv_1 e^{-u} + yv_2 e^{-u}] \, dv_1 \, dv_2.
$$

(26)

Finally, using (26) in (9), we get (12).

Similarly, using (8), we have an integral representation of

$$
_{1}F_{1}[a; c; x] = \frac{1}{\Gamma (a)} \int_0^\infty e^{-u} u^{a-1} _{1}F_{1}[-; c; xu] \, du \quad (\Re (a) > 0).
$$

(27)

Applying (27) in (12), we get (13).

Using the following known integral formula (see [21] p. 282, Equation (27))

$$
F_1 [a, b, b'; c; x, y] = \frac{1}{\Gamma (a)} \int_0^\infty u^{a-1} e^{-u} \Phi_2 [b, b'; c; xu, yu] \, du,
$$

(28)

where $\max \{\Re (x), \Re (y)\} < 1$ and $\Re (a) > 0$, in the integrand of (9) yields (14).

From the known integral representation of $\Phi_2$ (see [27] Equation (4.2)) in (14), we obtain (15).

Similarly, applying the known integral representations ([27] Equation (4.10)–Equation (4.14)) of $\Phi_2$ to (14), respectively, yields (16)–(20).

Applying known integral representations for $F_1$ (see, e.g., ([23] p. 76, Equation (1)); see also ([28] Equation (3.2)) and (see, e.g., ([23] p. 77, Equation (4)); see also ([28] Equation (3.1)) to (9), respectively, we obtain (21) and (22).

Using a known integral representation for $F_1$ (see [28] Equation (3.3)) in (9), we get (23).

3. Generating Functions of (6)

Certain generating functions for the extended Hurwitz–Lerch zeta function (6) are given in the following theorem.

**Theorem 2.** The following two formulas hold true:

$$
\sum_{m=0}^\infty \binom{\lambda + m - 1}{m} \Phi_{a+m,b,b';x} (x, y, s, \alpha) u^m = (1 - u)^{-\lambda} \Phi_{a,b,b';x} \left( \frac{x}{1-u}, \frac{y}{1-u}, s, \alpha \right)
$$

and

$$
\sum_{m=0}^\infty \binom{\lambda + m - 1}{m} \Phi_{a,b,a+m,x} (x, y, s, \alpha) u^m = (1 - u)^{-\lambda} \Phi_{a,b,a,x} \left( \frac{x}{1-u}, \frac{y}{1-u}, s, \alpha \right).
$$

(29)

(30)

**Proof.** We begin by recalling the generalized binomial theorem

$$
(1 - y)^{-\lambda} = \sum_{m=0}^\infty (-1)^m \binom{-\lambda}{m} y^m
$$

$$
= \sum_{m=0}^\infty \binom{\lambda + m - 1}{m} y^m,
$$

where $\lambda \in \mathbb{C}$ and $|y| < 1$. 


For simplicity’s sake, let us denote the left side of (29) by $\mathcal{L}$. Then, by using (6), we have

$$\mathcal{L} = \sum_{m=0}^{\infty} \frac{(\lambda)_m}{m!} \sum_{k, \ell=0}^{\infty} \frac{(\lambda + m)_{k+\ell} (b)_k (b')_{\ell}}{(c)_{k+\ell} k! \ell!} \frac{x^k y^{\ell}}{(k + \ell + a)^m} u^m.$$

In view of $(\lambda)_m (\lambda + m)_{k+\ell} = (\lambda)_{m+k+\ell} (\lambda + k + \ell)_m$, changing the order of summations and using (31), we obtain

$$\mathcal{L} = \sum_{k, \ell=0}^{\infty} \frac{(\lambda)_{k+\ell} (b)_k (b')_{\ell}}{(c)_{k+\ell} k! \ell!} \frac{x^k y^{\ell}}{(k + \ell + a)^m} \sum_{m=0}^{\infty} \frac{(\lambda + k + \ell)_m}{m!} u^m$$

$$= (1 - u)^{-\lambda} \sum_{k, \ell=0}^{\infty} \frac{(\lambda)_{k+\ell} (b)_k (b')_{\ell}}{(c)_{k+\ell} k! \ell!} \frac{x^k y^{\ell}}{(k + \ell + a)^m} (1 - u)^{-k-\ell}$$

which, using the definition (6), leads to the right side of (29). Similarly, we can obtain (30).

We can also prove the generating relations here by using some known generating relations for $F_1$ (see [29] Equation (2.1)) and ([22] Equations (1.2)–(1.3)) in (9). The details of the proof are omitted here. □

4. Derivative Formulas of (6)

Certain derivative formulas for the extended Hurwitz–Lerch zeta function (6) are established in the following theorem.

**Theorem 3.** Each of the following derivative formulas holds true for $m, n \in \mathbb{N}_0$:

$$\frac{\partial^m}{\partial x^m} \Phi_{a,b,b',c} (x, y, s, \alpha) = \frac{(a)_m (b)_m}{(c)_m} \Phi_{a+m,b+m,b',c+m} (x, y, s, \alpha + m),$$

$$\frac{\partial^n}{\partial y^n} \Phi_{a,b,b',c} (x, y, s, \alpha) = \frac{(a)_n (b')_n}{(c)_n} \Phi_{a+n,b,b'+n,c+n} (x, y, s, \alpha + n),$$

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} \Phi_{a,b,b',c} (x, y, s, \alpha) = \frac{(a)_{m+n} (b)_m (b')_n}{(c)_{m+n}} \Phi_{a+m+n,b+m,b'+n,c+n+m} (x, y, s, \alpha + m + n).$$

**Proof.** Differentiating the series definition (6) with respect to the variable $x$ under the double summations, which is valid under the conditions in (6), we have

$$\frac{\partial}{\partial x} \Phi_{a,b,b',c} (x, y, s, \alpha) = \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \frac{(a)_{k+\ell} (b)_k (b')_{\ell}}{(c)_{k+\ell} (k-1)! \ell!} \frac{x^{k-1} y^{\ell}}{(k + \ell + a)^m}.$$

Putting $k - 1 = k'$ in (35) and cancelling the prime on $k$, we obtain

$$\frac{\partial}{\partial x} \Phi_{a,b,b',c} (x, y, s, \alpha) = \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \frac{(a)_{k+1+\ell} (b)_{k+1} (b')_{\ell}}{(c)_{k+1+\ell} k! \ell!} \frac{x^k y^{\ell}}{(k + \ell + 1 + a)^m}.$$

Using $(\lambda)_{k+1} = \lambda (\lambda + 1)_k$ ($k \in \mathbb{N}_0$) in (36), we get

$$\frac{\partial}{\partial x} \Phi_{a,b,b',c} (x, y, s, \alpha) = \frac{ab}{c} \Phi_{a+1,b+1,b'+1} (x, y, s, \alpha + 1).$$

Differentiating the right side of (37), successively, $m - 1$ times, with respect to the variable $x$, we have (32). Similarly, we can obtain (33) and (34). The details are omitted. □
5. Recurrence Relations of (6)

Wang [30] presented a number of recurrence relations for $F_1$, which are chosen to give some recurrence relations for the extended Hurwitz–Lerch zeta function (6), asserted in Theorem 4.

**Theorem 4.** Let $n \in \mathbb{N}_0$. Then the following recurrence relations are satisfied:

$$
\Phi_{a+n,b,b',c} (x, y, s, \alpha) = \Phi_{a,b,b',c} (x, y, s, \alpha) + \frac{b x}{c} \sum_{m=1}^{n} \Phi_{a+m,b+1,b'+c+1} (x, y, s, \alpha) + \frac{b'y}{c} \sum_{m=1}^{n} \Phi_{a+m,b+1,b'+c+1} (x, y, s, \alpha);
$$

(38)

$$
\Phi_{a-n,b,b',c} (x, y, s, \alpha) = \Phi_{a,b,b',c} (x, y, s, \alpha) - \frac{b x}{c} \sum_{m=0}^{n-1} \Phi_{a-m,b+1,b'+c+1} (x, y, s, \alpha);
$$

(39)

$$
\Phi_{a,b-n,b',c} (x, y, s, \alpha) = \sum_{m=0}^{n} \left( \frac{n}{m} \right) \frac{(a)_m}{(c)_m} (-y)^m \Phi_{a+m,b,b',c+m} (x, y, s, \alpha + m).
$$

(40)

Here the involved empty sum in each identity is assumed to be nil.

**Proof.** Using the following recursion relation for $F_1$ ([30] Theorem 1, Equation (1)):

$$
F_1 [a+n,b,b';c;x,y] = F_1 [a,b,b';c;x,y] + \frac{b x}{c} \sum_{m=1}^{n} F_1 [a+m,b+1,b'+c+1; x,y] + \frac{b'y}{c} \sum_{m=1}^{n} F_1 [a+m,b+1; c+1; x,y]
$$

(41)

and ([30] Theorem 1, Equation (2)) in (9), respectively, we obtain (38) and (39).

Using the recurrence relation for $F_1$ in ([30] Theorem 4, Equation (12)) in (9), we find (40). □

Using six known recurrence relations for $F_1$ in (12), we establish six recurrence relations for the extended Hurwitz–Lerch zeta function (6), which are asserted in Theorem 5.

**Theorem 5.** The following recurrence relations hold true:

$$
a \Phi_{a+1,b,b',c} (x, y, s, \alpha) = (c - a) \Phi_{a-1,b,b',c} (x, y, s, \alpha) + (2a - c) \Phi_{a,b,b',c} (x, y, s, \alpha) + bz \Phi_{a,b+1,b',c} (x, y, s, \alpha + 1) + b'y \Phi_{a,b,b'+1,c} (x, y, s, \alpha + 1);
$$

(42)

$$
c (c - 1) \Phi_{a,b,b',c-1} (x, y, s, \alpha) = c (c - 1) \Phi_{a,b,b',c} (x, y, s, \alpha) + cbz \Phi_{a,b+1,b',c} (x, y, s, \alpha + 1) + cb'y \Phi_{a,b,b'+1,c} (x, y, s, \alpha + 1) + (a - c) bz \Phi_{a,b+1,b',c+1} (x, y, s, \alpha + 1) + (a - c) b'y \Phi_{a,b,b'+1,c+1} (x, y, s, \alpha + 1);
$$

(43)

$$
(1 + a - c) \Phi_{a,b,b',c} (x, y, s, \alpha) = a \Phi_{a+1,b,b',c} (x, y, s, \alpha) + (1 - c) \Phi_{a,b,b',c-1} (x, y, s, \alpha);
$$

(44)

$$
c \Phi_{a,b,b',c} (x, y, s, \alpha) = c \Phi_{a-1,b,b',c} (x, y, s, \alpha) + bz \Phi_{a,b+1,b',c+1} (x, y, s, \alpha + 1) + b'y \Phi_{a,b,b'+1,c+1} (x, y, s, \alpha + 1);
$$

(45)

$$
ac \Phi_{a+1,b,b',c} (x, y, s, \alpha) = ac \Phi_{a,b,b',c} (x, y, s, \alpha) + cbz \Phi_{a,b+1,b',c} (x, y, s, \alpha + 1) + cb'y \Phi_{a,b,b'+1,c} (x, y, s, \alpha + 1) + (a - c) bz \Phi_{a,b+1,b',c+1} (x, y, s, \alpha + 1) + (a - c) b'y \Phi_{a,b,b'+1,c+1} (x, y, s, \alpha + 1);
$$

(46)
\[(c - 1) \Phi_{a, b, b', c'} (x, y, s, \alpha) = (a - 1) \Phi_{a, b, b', c'} (x, y, s, \alpha) + b z \Phi_{a, b + 1, b', c'} (x, y, s, \alpha + 1)
+ b' y \Phi_{a, b, b' + 1, c'} (x, y, s, \alpha + 1) + (c - a) \Phi_{a - 1, b, b', c'} (x, y, s, \alpha + 1).
\] (47)

**Proof.** Using the following known recurrence relation for \(1_F_1\) (see [25] p. 19, Equation (2.2.1))

\[(c - a) \Phi_{a, b, b', c'} (x, y, s, \alpha) = (2a - c + x) \Phi_{a, b, b', c'} (x, y, s, \alpha) - a \Phi_{a, b, b', c'} (x, y, s, \alpha) = 0
\]
in (12), we obtain (42). Similarly, using the recurrence relations ([25] p. 19, Equations (2.2.2)-(2.2.6)) for \(1_F_1\), respectively, we get (43)-(47).

**6. Symmetries and Conclusions**

We can find some interesting identities from symmetries involved in certain definitions and formulas. From (6) and (10), we demonstrate the follow symmetric relations:

\[\Phi_{a, b, b', c'} (x, y, s, \alpha) = \Phi_{a, b, b', c'} (y, x, s, \alpha)\]

and

\[F_1 [a, b, b', c; x, y] = F_1 [a, b', b; c; y, x].\]

Further, in view of the symmetric relation (48), each integral representation in Theorem 1 may yield another integral representation. For example, from (14) and (48), we have

\[\Phi_{a, b, b', c'} (x, y, s, \alpha) = \frac{1}{\Gamma(a) \Gamma(b)} \int_0^\infty \int_0^\infty \exp(-au - v) \times \Phi_2 [b', b; c; y e^{-u v}, x e^{-u v}] \, du \, dv,
\]

where \(\min\{\Re(s), \Re(\alpha), \Re(a)\} > 0\) and \(\max\{\Re(x), \Re(y)\} < 1\).

The extended Hurwitz–Lerch zeta function of two variables in (6) may be further generalized in various ways. Here we introduce two extensions, one of which is due to parametric increase and the other of which is due to variable addition:

\[\Phi_{p, q, k}^{(n)} (x, y, s, \alpha) = \sum_{u, v = 0}^{\infty} \prod_{j=1}^n (a_j)_{u+j} \prod_{j=1}^m (b_j)_{u+j} \prod_{j=1}^k (c_j)_{v+j} x^u y^v \frac{u! v! (u + v + \alpha)^s}{u! v! (u + v + \alpha)^s}
\]

and

\[\Phi_{n}^{(n)} [a, b_1, \ldots, b_n; c; x_1, \ldots, x_n; s, \alpha] = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{\prod_{j=1}^n (a_j)_{m_1 + \cdots + m_n} (b_j)_{m_1 + \cdots + m_n} x_1^{m_1} \cdots x_n^{m_n}}{(m_1 + \cdots + m_n + \alpha)^s}.
\]

Here, for convergence, the parameters and variables in (51) and (52) would be suitably restricted. Obviously,

\[\Phi_{1; 3, 1; 0; 0}^{(2)} (a; b; b'; c; x, y, s, \alpha) = \Phi_{a, b, b', c} (x, y, s, \alpha)\]

and

\[\Phi_{1; 3, 1; 0; 0}^{(2)} (a; b; b'; c; x, y, s, \alpha) = \Phi_{a, b, b', c} (x, y, s, \alpha).\]
The extended Hurwitz–Lerch zeta function (6) can be specialized to yield several known generalizations of the Hurwitz–Lerch zeta function (5) (see, e.g., [20]). Thus, the results presented here can yield corresponding identities regarding several reduced cases of the extended Hurwitz–Lerch zeta function (6), which are still generalizations of the Hurwitz–Lerch zeta function (5).

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