Anharmonicity in light nuclei near drip lines

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Abstract. Single-particle energy levels of nucleons moving in an anisotropic harmonic oscillator potential along with a spin-orbit and an orbit-orbit interaction have been calculated. The model proposed herein incorporates shapes generated by higher multipolar anisotropies other than the quadruple one. The gaps among the energy levels are strongly dependent on the degree of anisotropy and hence, affect the nucleonic separation energies. For large anisotropies, this characteristic disturbs the occurrences of the usual magic numbers and the sequential ordering of the shells characterized by principal quantum number, $N$, associated with the elementary shell model. For large asymmetries, the states characterized by higher $N$, may come down and lie below the energies of the states with lower $N$ and vice versa. This feature of the model provides a natural explanation of intruder states or the island of inversion observed experimentally. For certain choices of the parameters, the large gaps in single nucleonic separation energies could occur at neutron numbers, 10, 18, 30, and 32 as observed, respectively, in $^{20}$Ne, $^{28}$Ne, $^{52}$Ti, and $^{54}$Ti. For neutron as well as proton number 14, the magnitude of this energy gap depends strongly on the choice of the strength of the spin-orbit and orbit-orbit terms. The model provides a simple understanding of the observed fact that some nuclei with proton or neutron number 14 sometimes exhibit this gap but not others. It also provides an interesting insight of having a low-laying ($1/2^-$) state in $^7$He, and confirms the ground state spin and parity of $^{11}$Be to be ($1/2^+$) and those of $^9$He to be ($1/2^+$), both of which have been contentious. This investigation indicated the density distribution, i.e., the shapes of many of nuclei near drip line to be anisotropic having a symmetry about their body-fixed $z$-axis.

1. Introduction

Experimental evidence for the existence of large energy gaps in the separation energies of neutrons and protons, reminiscence of the energy gaps associated with the magic numbers in elementary shell model, has been reported in many meta-stable nuclei located away from the valley of stable nuclei [1-5]. A few of these nuclei may also have significant large deformation compared to those exhibited by nuclei in the valley of stability. In addition, in some cases, e.g. $^{32-24}$Mg, the energies of the states of unnatural parities or intruder states seem to lie lower in energies than those of the natural parity. For example, the state originating from the $1f$-$2p$ configurations seem to have lower energies than those associated with the $2s$-$1d$ states in $^{32}$Mg. Some of these features have been understood within the framework of shell model calculations with configuration mixing requiring large basis sets [1,2]. However, some of the other features, in particular the abnormal gap in single nucleon separation energies and low-lying or ground state of abnormal parity, e.g., in $^{11}$Be require further investigations.
Since many of these nuclei exhibit characteristics that are associated with spheroidal shape, single particle orbitals in these nuclei are likely to follow this non-spherical shape. Bohr-Mottelson’s collective model [6,7] visualizes the deformed nuclei to be primarily spheroids having a symmetry about the body fixed z-axis. The equipotentials of nucleonic motions in such configuration follow closely, in the first approximation, the spheroidal shapes [8]. For such shapes, the symmetry axis is proportional to \( \exp(\alpha) \), \( \alpha \) being the degree of anharmonicity, and the other two axes are then proportional to \( \exp(-\alpha/2) \). The motion of a nucleon in the potential associated with this shape is no longer isotropic in space but is best described by an anisotropic harmonic potential given by the equation (1) of the next section. In the limit of small anisotropy, this potential leads to the Nilsson potential [9] when a spin-orbit and an orbit-orbit interactions are added to it. For \( \alpha = 0 \), this leads to the Meyer-Jensen [10] isotropic harmonic oscillator. For large anharmonicity, it is, therefore, interesting to study the nucleonic motion and orbitals in a complete anisotropic potential given by the equation (2) incorporating in it the spin-orbit and orbit-orbit interaction and investigate to the extent, the properties of these meta-stable nuclei located near drip lines are compatible with the overall characteristic of these nucleonic orbitals. In the next section, we derive the expressions for energy eigenvalues and eigenfunctions associated with nucleonic motion in an anharmonic oscillator potential along with the spin-orbit and orbit-orbit interaction. Our investigation, as discussed in section 3, indicates that the magic numbers associated with Meyer and Jensen’s [10] isotropic harmonic oscillator model may not be valid for large anharmonicity. The large separation energies of nucleons could also occur at non-magic proton and neutron numbers. Additionally, there could be intruder states at large anharmonic potential, i.e. states originating from higher oscillator shells could have energies that are below those of a lower quantum state of an isotropic oscillator and vice-versa. Many such cases have been observed experimentally.

2. The Theory
2.1. Energy Levels

In investigating nucleonic motion in large anharmonic oscillator mean fields, we consider an anharmonic potential along with a spin-orbit and orbit-orbit interaction which, in the limit of small anharmonicity or deformation, reduces to the successful Nilsson model [9] and in case of zero deformation reproduces the magic number of the nuclear shell model of Meyer and Jensen [10]. This is achieved by considering the following anharmonic oscillator potential symmetric about body-fixed z-axis:

\[
V(r) = (1/2) \frac{1}{2} \frac{\hbar^2}{m} \nabla^2 + V(r) = \frac{1}{2} \frac{1}{2} \frac{\hbar^2}{m} \nabla^2 + V(r)
\]

where \( \alpha \) is the degree of anharmonicity. Thus, the total Hamiltonian for a nucleonic motion is given by

\[
H = H_0 + C (l \cdot s) + D (l \cdot l)
\]

with

\[
H_0 = -\left(\frac{\hbar^2}{2m}\right) \nabla^2 + V(r)
\]

The unit of \( C \) and \( D \) is that of energy, usually MeV, as noted in [9,16,17]. In the limit of small anisotropy, \( V(r) \), represents only quadruple deformation. Thus,

\[
V(r) \xrightarrow[\alpha \rightarrow 0]{} (1/2) \frac{1}{2} \frac{1}{2} \frac{\hbar^2}{m} \nabla^2 \left(1 - 4\sqrt{\pi/5}\right) \alpha Y_{20}(\theta, \phi)
\]

\[
= (1/2) \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{\hbar^2}{m} \nabla^2 \beta Y_{20}(\theta, \phi)
\]

with the Bohr-Mottelson’s deformation parameter, \( \beta = 2\alpha \sqrt{\pi/5} m \omega_0^2 = (4/3) \sqrt{\pi/5} \sigma \).
The potential function (5), when substituted in (2), results into the Nilsson Hamiltonian. Thus, the Nilsson’s Hamiltonian may be considered as a special case of Hamiltonian (2) in the limit of small anharmonicity.

In this investigation, we shall not be using the potential (5) but consider the full anisotropic potential (1). Thus, the Hamiltonian used herein, is the following:

\[
H = \left(\frac{-\hbar^2}{2m}\right) \nabla^2 + \frac{1}{2} \omega_0^2 \left\{ (x^2 + y^2) e^{\alpha} + z^2 e^{-2\alpha} \right\} + C (l \cdot s) + D (l \cdot l) \tag{6}
\]

The advantage of using (6), instead of Nilsson Hamiltonian, is that (6) encompasses anharmonicity of higher orders instead of the quadruple one used by Nilsson.

In the spirit of Nilsson, we treat \(C (l \cdot s) + D (l \cdot l)\) as perturbing part of the Hamiltonian evaluated using the solution of \(H_0\) as the basis set, subjected to the condition, that

\[
\omega_x + \omega_y + \omega_z = \text{constant} \tag{7}
\]

with

\[
\omega_x^2 = \omega_y^2 = \omega_0^2 e^{\alpha} \quad \text{and} \quad \omega_z^2 = \omega_0^2 e^{-2\alpha} \tag{8}
\]

Thus, the eigensolutions of the unperturbed Hamiltonian are the solutions of the following equation:

\[
H_0 \psi_N = E_N \psi_N \tag{9}
\]

The solution of (9) in body-fixed cylindrical co-ordinate system is well known [11], and are noted below.

\(N\) in (9) refers to major oscillator, or principal quantum number in the limit of isotropic oscillator. In the spirit of the Nilsson model, in this treatment, \(N\) is taken to be a good quantum number although, in principle, the interaction (1) or (5) can mix oscillator levels with a given \(N\) with those of \(2 \pm N\).

In case \(N\) is retained as a good quantum number, the solution of (9) can be obtained in cylindrical coordinate analytically. In that case, (9) can be written as

\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial z^2} + \left( \lambda - \gamma^2 \rho^2 + \gamma z^2 \right) \psi = 0 \tag{10}
\]

with

\[
\lambda = \left(2mE_N/\hbar^2\right), \quad \gamma^2 = \left(m^2 \omega_0^2 e^\alpha/\hbar^2\right), \quad \lambda_z^2 = m^2 \omega_0^2 e^{-2\alpha}/\hbar^2. \tag{11}
\]

Then, the eigenvalues are given by

\[
E_N \equiv E_{Nn_z} = \left[ (N - n_z + 1) e^{\alpha/2} + \left(n_z + \frac{1}{2}\right) e^{-\alpha} \right] \hbar \omega_0 \tag{12}
\]

\[
= \left[ (n_r + 1) e^{\alpha/2} + \left(n_z + \frac{1}{2}\right) e^{-\alpha} \right] \hbar \omega_0 \tag{13}
\]

In (12) and (13), \(N, n_z\) and \(n_r\) are, respectively, the principal quantum number, and the number of oscillator quanta perpendicular and along the axis of symmetry, each being zero or a positive integer. Thus,

\[
N = n_r + n_z \tag{14}
\]

The energy can also be expressed in terms of the \(z\)-projection of the orbital angular momentum, \(\Lambda\) along the body-fixed \(z\)-axis. This leads to the following:

\[
E_N \equiv E_{n'_z, n_z, \Lambda} = \left[ (|\Lambda| - n_z + 1) e^{\alpha/2} + \left(n_z + \frac{1}{2}\right) e^{-\alpha} \right] \hbar \omega_0 \tag{15}
\]
where
\[ N = |\Lambda| + n' + n_z \]
\[ n_r = N - n_z \] (16)

with
\[ n_z = 0, 1, 2, \ldots \]
\[ \Lambda = 0, \pm 1, \pm 2, \ldots \] (17)

and \( n' = 0, 2, 4, \ldots \)

For the case of \( \alpha = 0 \) one obtains the expression of the energy eigen-values for threedimensional isotropic harmonic oscillator, i.e.,
\[ E_N = (N + 3/2) \hbar \omega \]

The orbital angular momentum, \( l \), does not commute with \( H_0 \) but its \( z \)-projection onto the body fixed axis, \( \Lambda \), does. Hence, eigensolutions are classified according to the set of quantum numbers \((N, n_z, \Lambda)\) or equivalently \((N, n_z, \Lambda)\). Thus, the eigensolutions of (11,12) are
\[ \psi_N(\rho, z, \phi) = |N, n_z, \Lambda\rangle = N_0 R(\rho) Z(z) \phi(\phi) \] (18)

with
\[ R_\rho = N_r \exp\left( -\frac{1}{2} \gamma \rho^2 \right) (\sqrt{\gamma} \rho)^{|\Lambda|} L_{\frac{|\Lambda|}{2}}^{\frac{|\Lambda|}{2}}(\lambda \rho^2) \] (19)
\[ Z(z) = N_{n_z} \exp\left( -\frac{1}{2} \lambda z^2 \right) H_{n_z}\left( \sqrt{\lambda} z \right) \] (20)
and
\[ \phi(\phi) = \left( \frac{1}{2\pi} \right)^{1/2} \exp(i\Lambda \phi) \] (21)

where \( N_0, N_r \) and \( N_{n_z} \) are the normalization constants and \( L_{\frac{n'}{2}}^{\frac{|\Lambda|}{2}} \) and \( H_{n_z} \) are, respectively, associate Laguerre polynomial of degree \((n' - |\Lambda|)\) and order \(|\Lambda|\) and Hermite polynomial of order \(n_z\). The normalization constant, \( N_0 \), is given by
\[ N_0 = \left( m \omega_0 / 2 \pi \hbar \right)^{3/4} \left[ \frac{(n'/2)!}{2^{n_z-1} n_z! (n'/2 + |\Lambda|)!} \right]^{1/2} \] (22)

The eigenfunctions of the total Hamiltonian, \( H \), can now be obtained by diagonalization using (18) as a basis set along with the spin wave function \(|s\Sigma\rangle\), \( \Sigma \) being the \( z \)-projection of spin, \( s \), of a nucleon in the body fixed system. However, because of the presence of the spin-orbit interaction in (2), neither \( \Lambda \) nor \( \Sigma \) commutes with \( H \), but \( \Omega \), the projection of \( j = l + s \), onto the body fixed \( z \)-axis does. Hence, the eigenfunctions, \( \chi(N\Omega) \) of \( H \), may be expanded in terms of the basis set \( \psi(Nn_z\Lambda) \) as follows [12,13]:
\[ \chi(N\Omega) = \sum_{n_z\Lambda\Sigma} A(Nn_z\Lambda\Sigma) |Nn_z\Lambda\rangle |s\Sigma\rangle \] (23)

where \( A(Nn_z\Lambda\Sigma) \) are the expansion coefficients, and \( s\Sigma \) is the spin function in the body fixed coordinate system. The normalization condition on \( N\Omega \) leads to
\[ \sum_{Nn_z\Lambda\Sigma} |A(Nn_z\Lambda\Sigma)|^2 = 1 \] (24)
The choice of basis set (23) implies the omission of the admixture of states of different \( N \). Thus, \( N \) is retained as a good quantum number.

The diagonalization of \( H \) requires the determination of non-vanishing matrix elements of the operators \( l.s \) and \( l^2 \). Following [13], the diagonal matrix elements of these operators are given by

\[
\langle Nn\Lambda | l^2 | Nn\Lambda \rangle = \Lambda^2 + 2n_z N_r + 2n_z + N_r \tag{25}
\]

\[
\langle Nn\Lambda | l^2 | Nn\Lambda \rangle = \Lambda \Sigma \tag{26}
\]

The off-diagonal matrix elements are given by

\[
\langle n_z + 2\Lambda \Sigma | l^2 | n_z \Lambda \Sigma \rangle = \left[ (N - \Lambda) (N_r + \Lambda) (n_z + 1) (n_z + 2) \right]^{\frac{1}{2}} \tag{27}
\]

\[
\langle n_z - 2\Lambda \Sigma | l^2 | n_z \Lambda \Sigma \rangle = \left[ (N_r + \Lambda + 2) (N_r - \Lambda + 2) n_z (n_z - 1) \right]^{\frac{1}{2}} \tag{28}
\]

\[
\langle n_z + 1\Lambda \pm 1\ Sigma | l.s | n_z \Lambda \Sigma \rangle = \frac{1}{2} \left[ (N_r \mp \Lambda) (n_z + 1) \right]^{\frac{1}{2}} \tag{29}
\]

\[
\langle n_z + 1\Lambda \pm 1\ Sigma | l.s | n_z \Lambda \Sigma \rangle = \frac{1}{2} \left[ (N_r \pm \Lambda + 2) n_z \right]^{\frac{1}{2}} \tag{30}
\]

Instead of \( C \) and \( D \) in (2), the results are presented in terms of Nilsson parameters, \( \kappa \) and \( \mu \) which are related to \( C \) and \( D \) as

\[
\kappa = -\left( \frac{1}{2} \right) \left( C/\hbar\omega_0 \right) \quad \text{and} \quad \mu = \left( 2D/C \right) \tag{31}
\]

The expression for eigenvalues of \( H \) after diagonalization can be written in a closed form as

\[
E(N\Omega\nu) = \left\{ (N - n_z + 1) e^{(1/2)\alpha} + (n_z + \frac{1}{2}) e^{-\alpha} \right\} \hbar\omega_0 + \kappa\hbar\omega_0 R(N\Omega\nu) \tag{32}
\]

In (28), \( R(N\Omega\nu) \) is the matrix elements of the operator \( R = -2l.s - \mu l^2 \) and the index \( \nu \) is introduced to label the states for a particular set of \( N\Omega \). The eigenvalues and eigenfunctions are obtained by the Jacobi diagonalization procedure.

2.2. Equilibrium Energy

Nuclei investigated herein are radioactive and not stable and as such, there is no equilibrium deformation in general. Nevertheless, it is interesting to investigate the equilibrium energy of the potential energy surface corresponding to the absolute minimum in energy as a function of \( \alpha \) and the corresponding anharmonicity, \( \alpha_{eq} \), and investigate whether in the limit of main shell closure, one can recover the magic numbers.

Since

\[
E^{NO}_{\alpha} = \left\{ (N - n_z + 1) e^{\frac{1}{2}\alpha} (n_z + 1/2) e^{-\alpha} \right\} \hbar\omega_0 + \text{terms independent of } \alpha \tag{33}
\]

we have for equilibrium condition,

\[
\frac{\partial E}{\partial \alpha} = \sum_{N,n_z} \left\{ (N - n_z + 1) \frac{1}{2} e^{\frac{1}{2}\alpha} - (n_z + 1/2) e^{-\alpha} \right\} \hbar\omega_0 = 0 \tag{34}
\]

or,

\[
\alpha_{eq} = 2 \left[ \frac{\sum_{N,n_z} (2n_z + 1)}{\sum_{N,n_z} (N - n_z + 1)} \right] = \frac{2}{3} \ln \left[ \frac{\sum_{N,n_z} (2n_z + 1)}{\sum_{N,n_z} (N - 3n_z) \sum_{N,n_z} (2n_z + 1)} \right] \tag{35}
\]
Now for the shell closure

$$\sum_{N,n_z} (N - 3n_z) = 0$$

(36)

and hence, for shell closure associated with $N$, $\alpha_{eq} = 0$. Thus, if the shells are filled in proper sequence of $N$, one recovers the major shell closure of the elementary shell model, associated with $N$ at the equilibrium deformation.

The above observation along with the fact that many properties of stable isotopes of these radioactive nuclei are amenable to collective model description [16, 17, 53] allow us to choose values of the oscillator constant, parameters $C$ and $D$ to be similar to those used in [16, 17, 53]. The most of the experimental data is, then, discussed in terms of the anharmonicity parameter. The calculation of absolute band head energies will require a knowledge of the moment of inertia [16,17].

3. Results and Discussion

For actual determination of the eigenvalues and eigenfunctions, one needs to specify the oscillator quantum, $\hbar \omega_0$, the strengths of the spin-orbit, $C$, and of the orbit-orbit interactions, $D$, which may vary from shell to shell or from nuclei to nuclei. Energy levels, moments and transition rates of many light and medium-light nuclei have been extensively studied rather successfully within the framework of the Nilsson [9,14] and the Coriolis coupling model [16,17], for nuclei in the valley of stability. It is, therefore, reasonable to examine, whether many of these meta-stable nuclei near drip lines could exhibit shapes commensurate with higher multipolar anisotropy.

In figure 1 and figure 2, we present the eigenvalues as a function of anisotropic parameter, $\alpha$, for two sets of $\kappa$ and $\mu$ which are representative of the values used in the studies by Nilsson [9], Nilsson and Mottelson [14] and Malik and Scholz [16,17]. $\kappa$ and $\mu$ are related to $C$ and $D$ by (27). $\hbar \omega_0$ is taken to be 41 MeV/$\sqrt{A}$ as used in those investigations. Nuclei studied in those investigations lie in the valley of stability exhibiting relatively small anisotropy, $\alpha$, and have only quadrupole deformation, $\beta$, which is usually less than 0.25. This corresponds to $\alpha = 0.16$ (in the limit of small anharmonicity, $(\alpha = (1/2) \sqrt{5/\pi \beta}) \approx 0.633 \beta$ and hence, $V(r)$ reduces to the Nilsson potential.

The most of the nuclei considered in this study may have substantial multipolar anharmonicity and hence, we present here energy levels generated by the Hamiltonian (6) as a function of alpha. In figure 1 and figure 2 we present two sets of energy-level diagrams using the Hamiltonian (6). The energy-level diagrams exhibited in figure 1 and figure 2 have been calculated using $(\mu = 0.035$ and $\kappa = 0.0626)$ and $(\mu = 0.614$ and $\kappa = 0.0626)$, respectively. The lowest energy levels originating from the $(1s_{1/2})$ parentage have not been shown in either of the figures, since the energy levels originating from $(1s_{1/2})$ remains well separated from other energy levels. For large values of $\alpha$, the anisotropic parameter, the eigenstates are strongly mixed, an example of which is given by (36) and hence, the asymptotic quantum numbers are not convenient to characterize these states. They have, therefore, been identified by simple numbers. Thus, the parentage of the groups of state (2 and 3), (4), (5, 6 and 8), (9), (7 and 10), (17, 14, 12 and 11), (15 and 18), (16, 20, and 22) and (19) etc. at $\alpha = 0$ are, respectively, $(1p_{3/2})$, $(1p_{1/2})$, $(1d_{5/2})$, $(2s_{1/2})$, $(1d_{3/2})$, $(1f_{7/2})$, $(2p_{1/2})$ states of an isotropic oscillator. The parentage, thus, refers to spherical symmetric states in the limit of $\alpha = 0$.

A comparison between these two figures clearly indicates that the relative locations of the energy levels among shells at $\alpha = 0$, i.e. in the limit of spherical symmetry, depends strongly on the choice of $\kappa$ and $\mu$. For example, the energy gap between $1d_{5/2}$ and $2s_{1/2}$ at $\alpha = 0$ in figure 1 is considerably less than that in figure 2. Thus, for the parameters chosen to generate energy levels of figure 2, neutron number $N$ or proton number $Z = 14$ could be considered a close sub-shell but not for the parameter set related to figure 1. Similarly, in figure 2 the energy
Figure 1. Energy level diagram as a function of anisotropy, $\alpha$, for $\mu = 0.035$, and $\kappa = 0.0626$ for major shells, $N = 1, 2, 3, 4$ and $5$. Energy levels for $N = 0$ is not plotted.

gap at $\alpha = 0$ between the $(1f_{7/2})$ shell and the bunch of states $(2p_{3/2}, 1f_{5/2}, 2p_{1/2})$ is quite large, making $N$ or $Z = 28$ a good closed shell, but this is not the case for energy levels shown in figure 1.

Another important observation is that both for positive and negative $\alpha$ and $\alpha \neq 0$, there are large energy gaps among the energy levels which are comparable to those existing for $\alpha = 0$. The exact location and magnitude of these energy gaps depend critically on the choice of $\alpha$, $\mu$ and $\kappa$.

Recently, Hamamoto and Mottelson [20] studied the factors leading to the dominance of prolate over the oblate shape using the Hamiltonian (2) with $C = D = 0$ and showed that under certain circumstances, this dominance of prolate shapes over the oblate ones could occur for small $\alpha$. Hence, we shall be discussing the intruder states, energy gaps, etc. referring to positive
Figure 2. Energy level diagram as a function of anisotropy, $\alpha$, for $\mu = 0.614$, and $\kappa = 0.0626$ for major shells, $N = 1, 2, 3, 4$ and $5$. Energy levels for $N = 0$ is not shown.

$\alpha$ but many of these discussed features are also pertinent for negative $\alpha$.

3.1. Disturbed shell ordering, island of inversion and intruder states
As noted earlier, for closed major shells characterized by the principal quantum number $N$, the equilibrium deformation is zero, provided shells are filled in proper sequence with respect to $N$ serially. Both figure 1 and figure 2 indicate that with increasing $\alpha$, some energy levels associated with a higher $N$ could come down in energy and lie below the levels associated with a lower $N$ both for positive and negative values of $\alpha$. These are noted as intruder states. Thus, as alpha increases, the natural shell ordering associated with those of the spherical symmetry is disturbed.
and one could have intruder states, or island of inversion, as observed in some experiments. In addition, there are many cases where the energy gaps between two levels for $\alpha \neq 0$ are of about the same order, as the energy gaps associated with the Meyer and Jensen’s magic numbers. For example, the eigenvalues as a function of $\alpha$, marked with numbers 8 and 17 in figures 1 and 2 are generated by states 8 and 17 in column 5 of table 1 and have eigenvectors noted in column 4. In the limit of spherical symmetry, i.e., $\alpha = 0$, they are generated from $1d_{5/2}$ and $1f_{7/2}$ states, and noted as of ‘parentage of $1d_{5/2}$ and $1f_{7/2}$’, respectively.

Similarly, the state 31 of parentage $(1p_{1/2})$ comes down in energy for large $\alpha$ and lies lower in energy compared to levels originating from lower $N$. Hence, for large positive values of alpha, one expects many intruder states at low energies, in light nuclei. These intruder states have been established in many nuclei, e.g. $^{32-34} Mg$ [15], $^{31-33} Na$ and $^{30-32} Ne$ [21,22].

For $\alpha$ greater than approximately 0.2, energies of some states of the $(1f_{7/2})$ parentage which have negative parity, lie lower in energies compared to some states of the parentage $(2s-1d)$ which have positive parity in both figure 1 and figure 2. Experimentally, the presence of these intruder states have been observed in $^{28} Ne$ [5], $^{29} Ne$[21] and $^{32} Mg$ [24]. A striking example of an intruder state is the ground state of $^{11} Be$ [23-25] which is $(1/2^+)$. In isotropic harmonic oscillator model, one expects this to be $(1/2^-)$. However, in our model, the energy of the state number 8 of the $(1d_{5/2})$ parentage lies lower than that of the state number 4 of the $(1p_{1/2})$ parentage for $\alpha$ approximately between 0.4 and 0.35 in figure 1 and figure 2, respectively. In case the anharmonicity of $^{11} Be$ is about $\alpha = 0.35$ in figure 1 and $\alpha = 0.35$ in figure 2, its ground state spin and parity should $(1/2^+)$, but the first excited state could be $(1/2^-)$ . This is indeed, the case in $^{11} Be$ experimentally [44]. In this model, the stripping spectroscopic factors (SF) to this state is expected to be similar to those in the Nilsson or the Coriolis Coupling model and hence, is expected to be weak [47, 48], since its parentage is $(1d_{5/2})$. This is in agreement with the measurement [44].

On the other hand, Talmi and Unna in 1960 [34] pointed out that within the framework of strong configuration interaction, the energy of the $2s_{1/2}$ state could lie in the neighborhood of the energies of $(1p_{3/2})$ states making the ground state of $^{11} Be$ to be $(1/2^+)$. However, the SF to this state should be that of a single particle shell model state, i.e., large, contradicting the observation [46].

3.2. Unusual energy gaps and the ground state, spin and parity of $^9 He$

3.2.1. $^7 He$, $^8 He$ and $^9 He$ Skaza et al. [18] and the authors of a few other publications prior to that, have suggested that the neutron configuration of $^8 He$ could have a semi-closed $(1p_{3/2})^4$ shell, i.e. the separation between the energies of $(1p_{3/2})$ and $(1p_{1/2})$ of the isotropic shell model is significantly large compared to those of the neighboring isotopes. It has been difficult to generate this gap in shell model calculations with configuration mixture. In our model, for $\alpha$ less than about +0.35, this is the case for both sets of parameters as shown in figures 1 and 2. More interestingly, Skaza et al. claims that the first excited state of $^7 He$ is $(1/2^-)$ having an excitation energy of about 1 MeV (this is not reported in some other experiments) and notes: “The reproduction of the first excited state below 1 MeV would be a challenge for the most sophisticated nuclear theories.”

Within the context of this investigation, however, one expects a low-lying $(1/2^-)$ state in $^7 He$. In the Bohr-Mottelson’s strong coupling theory, the level spectra of $^7 He$ is expected to exhibit rotational structure when the rotational particle coupling (RPC) is omitted. Thus, low-lying spectrum would compose of bands based on state 2, with $\Omega = 3/2$ and state 4 having $\Omega = 1/2$. Although these two levels are well separated, the RPC introduces an interaction between them. As a result the band based on the $\Omega = 1/2$ state is decoupled [19]. The energy of the $K = 1/2^-$ band originating from and $(1p_{1/2})$ configuration in the lowest order perturbation theory is given.
by

\[
E_k(I) = E_k(0) + A \left[ I (I + 1) + (-I)^{I+1/2} a (I + 1/2) \sigma (K_{1/2}) \right]
\]  

(37)

where \(E_k(0)\) is the bandhead energy and \(A = \hbar / 2 \mathcal{I} \) and \(I, a \) and \(I \) are, respectively, the moment of inertia, decoupling parameter and spin of the state. The decoupling parameter, \(a\), is given by

\[
a = \sum_j (-1)^{j-1/2} (j + 1/2) |A (N \Omega \nu)|^2
\]  

(38)

\(a\) in this case is positive, since the main contribution to (35) is the band originating from \((1p_{1/2})\) configuration for which \(j = 1/2\) and \(\Omega = 1/2\). Thus, the contribution of the second term in (34) to the energy is negative for \(I = 1/2\). This will lower the energy of the \((1/2)^-\) state based on \(\Omega = 1/2\). Thus, in this model, the first excited state of \((1/2)^-\) could easily be close to the \((3/2)^-\) ground state, thereby supporting the observation of Skaza et al. [18]. The large SF to this state observed in [18] is expected in this model [47, 48], thereby supporting that the spin and parity of this state to be \((1/2)^-\).

Recent attempt [54] to combine, no core shell model (NCSM), the Hamiltonian of which does not allow for continuum states, with the latter state generated by the resonating group theory (RGT) using \(R\)-matrix method to generate this low lying \((1/2)^-\) state, met with limited success.

Chen et al.’s [43] experiment provides some indication that the ground state of \(^9\text{He}\) could be \((1/2)^+\) which could be the case, if energy levels in the shell model calculation using the WBT interaction of Warburton and Brown [44] is shifted arbitrarily by about 4 MeV. The key component of this \((1/2)^+\) state is the unperturbed \((2s_{1/2})\) shell model state which under some strong interaction mixing, could come down in energy as noted by Talmi and Unna [34]. However, in the stripping reaction \(^2\text{H} (^8\text{He}, p)^9\text{He}\), Golovkov et al. [45], found the SF for stripping to this state to be very small, less than 0.1, whereas, the expected SF to this state based primarily on the \((2s_{1/2})\) configuration should be large, as noted by Barker [46], and Golovkov et al. [45].

In our model, if the anaharmocity lies between about 0.35 and 0.45, as shown in both figures, the ground state of \(^9\text{He}\) could be \((1/2)^+\) of the \(9d_{5/2}\) parentage. This being a \(\Omega = 1/2\) band originating from \((1d_{5/2})\) configuration should have very small SF [47,48] as reported in [45]. The approximate rule for SF in the Coriolis Coupling model which mixes bands of the Nilsson model is that the SF to states with \(\Omega = j\) (i.e., if this is the dominant band) is large, and to other states, small. Thus, the model can account for the observed \((1/2)^+\) ground state of \(^9\text{He}\) with small SF. The expected \((5/2)^+\) state based on this band could be around 5 MeV, since this is also decoupled. A \((3/2)^+\) state in this region with small SF should also exist. Indeed, the stripping experimental of Golovkov et al. [45] reports a state with large width, possibly \((5/2)^+\) around 5 MeV excitation energy. The resolution of this broad state was not good enough to separate other states, but in [47] Rogachev et al. noted the possibility of a \((3/2)^+\) state in this region.

The first two exited states of the band \(\Omega = 1/2\) of the parentage \(1p_{1/2}\), are \((1/2)^-\) and \((3/2)^-\) which are expected to have small SF as reported in [45]. Thus, the model provides strong evidence of \((1/2)^+\) being the ground state of \(^9\text{He}\) and a qualitative understanding of its low-lying spectrum.

3.2.2. Single nucleonic separation energies at \(N = 10, 16, 18\)  In [4, 5], the authors report that nucleon numbers 10, 16 and 18 may form closed shells. A close look at the levels shown in both figure 1 and figure 2 indicates the presence of a substantial energy gap between state 8 and state 4, both of the \((1d_{5/2})\) parentage for \(\alpha\) approximately equal to 0.35 to 0.4, respectively. Because of this energy separation, neutron number, \(N = 10\) could appear as a closed shell.
Similarly, there is a substantial energy gap between the state 10 and the next bunch of states around $\alpha = 0.45$ in figure 1, which would make $N = 16$ appear as a closed shell. In the same figure, state 31 lies below state 14 in energy, for $\alpha \approx 0.52$, which would allow $N = 18$ to appear as a closed shell.

3.2.3. $Z = 14$ subshell closure and near degeneracy of $d_{3/2}$ and $s_{1/2}$ orbit The experiment of Fridmann et al. [2] establishes that $d_{3/2}$ and $s_{1/2}$ proton orbits are nearly degenerate in $^{42}Si$, $^{43}P$ and $^{44}S$ and are well separated from $1d - 5/2$ orbitals making $Z = 14$ a closed sub-shell. The level scheme plotted in figure 2 indicates that for positive $\alpha$, the state 7 with $\Omega = 3/2$ originating from the $1d_{3/2}$ state and the state 9 with $\Omega = 1/2$ level originating from the $2s_{1/2}$ state are nearly degenerate, and there is a sharp energy gap between these two states and the state 5 with $\Omega = 5/2$ originating from the $1d_{5/2}$ state. For small values of $\alpha$, i.e. $\alpha \leq 0.2$, this energy difference between the set of states 7 and 9 and the state 5 is significant for $Z = 14$, which will make $Z = 14$ to appear as closed shell. Interestingly, in figure 2, for $\alpha$ between 0.2 to 0.4, $\Omega = 1/2$ level originating from the $1d_{7/2}$ state, i.e. the state 17, comes down in energy and state 5 with $\Omega = 5/2$ level originating from the $1d_{5/2}$ moves up in energy, thereby creating a large gap for $Z = 14$, making $Z = 14$ to appear also as a closed shell. This shell closure, however, depends strongly on the choice for $\kappa$ and $\mu$. For example, it disappears in figure 1 which is compatible with the situation reported in [22].

3.2.4. $N = 30, 32$ shell closure In their experiment, Janssens et al. [3] raises the possibility that $N = 30$ as well as 32 neutrons in $^{52,54}Ti$ may exhibit shell closure. In figure 2, for $\alpha$ slightly greater than 0.5, there is a substantial energy gap between state 20 (or state 5) and the next bunch of levels, which would allow $N = 30$ to behave like a closed shell.

At a slightly higher alpha, the gap between state 56 and state 5 may make $N = 32$ appear also as a closed shell.

4. State Vectors and Degeneracies

In table 1 (continued in table 2), the coefficients $A(Nn_\Lambda \Sigma)$ of the state vectors, $\vert N\Omega \nu \Lambda \rangle$, corresponding to eigenvalues plotted in figure 1 are listed for $N = 1$ to 5. For $N = 0$, and $\Omega = 1/2$, the state vector is a pure $\vert 000+ \rangle$ configuration with $A(000+) = 1$ and is not listed. The eigenvalues associated this state are also not shown in figure 1. Columns 1 through 4 in table 1 list, respectively, $N$, $\Omega$, the coefficient $A(Nn_\Lambda \Sigma)$ and the basis state. The last column lists the number assigned to the corresponding state in column 4. The base vector $A(Nn_\Lambda \Sigma)$ is then obtained by matrix multiplication of the columns 3 and 4. For example, the base vector $\vert 3 1/2 19 \rangle$ for $N = 3$, $\Omega = 1/2$ and $\nu = 19$ is given by

$$\vert 3 1/2 19 \rangle = 0.447\vert 330+ \rangle - 0.365\vert 310+ \rangle + 0.730\vert 301- \rangle - 0.365\vert 321- \rangle$$  \hspace{1cm} (39)

It is, thus, strongly an admixture of four states of Hamiltonian $H_0$ with $N = 3$. All states are normalized to one. For example, for the above state

$$(0.447)^2 + (-0.365)^2 + (0.730)^2 + (0.365)^2 = 1$$ \hspace{1cm} (40)

In the absence of $\ell s$ and $\ell^2$ terms the states with the same $N$ and $n_\Lambda$ are degenerate. For example, the eigenvalues of $H_0$ for $N = 1$, $n_3 = 0$, $\Lambda \pm 1$ are twofold degenerate. So are the eigenvalues with $N = 2$, $n_3 = 1$, $\Lambda \pm 1$. This degeneracy is removed by the $\ell s$ and $\ell^2$ term. However, the matrix elements of $(\ell s + \ell^2)$ term are independent of $\alpha$ and hence, the energy shift between these degenerate states remains constant for all $\alpha$. This is the case, for example, between states (2 and 4), and (5, 9 and 7) in figure 1. For small shift, such groups of states...
remain nearly degenerate and the degree of splitting can be controlled by the coefficients $C$ and $D$.

There are many indications pointing towards collective properties present in these nuclei. Hamada et al. [52] noted that the spectroscopic factor in $^{7}\text{Li}(\alpha,p)^{10}\text{Be}$ is indicative of the presence of a ground state band in $^{10}\text{Be}$. Measurement of reduced large electric quadrupole transition, B(E2) in $^{32}\text{Mg}$, led Motobashi et al. [39] to suggest this nuclei to be strongly deformed. Level scheme of $^{32}\text{Mg}$ which has presence of the intruder states from $(1f-2p)$ shell, near its ground state, is indicative of an anisotropy of $\alpha \approx 0.25$ both in figure 1 and figure 2. Similarly, Tripathi et al. [21, 26] have raised the possibility of band-structure in $^{32}\text{Na}$. More significantly, the anti-symmetrized molecular dynamic calculation of Horiuchi et al. [50] indicates strong anisotropic density distribution in lithium and boron isotopes. Calculations using algebraic cluster model indicate the same for beryllium isotopes including $^{11}\text{Be}$. Noting that the density distributions, in the first approximation follows the potential-energy surface, these investigations [50, 51] support strongly anisotropic nucleon orbitals in these nuclei.

Various many-body approaches have been applied and actively being pursued to study the properties of these nuclei. Extensive studies have been published within the context of Hartree Fock orbitals with the Skyrme effective interaction [27] to understand the difference between neutron and proton separation energies, $S_n$ and $S_p$, respectively, among neighboring isotopes and isotones with various degrees of success. Green function Monte Carlo (GFMC) [28] method has been quite successful in studying low lying spectra and some electromagnetic properties of some of $(1s)$, $(1p)$ and $(1d)$ nuclei to limited extent. Properties of some nuclei have been extensively studied within the context of no core shell model (NCSM) and its variations [29-38], also with limited success. All these many-body approaches are based on spherical symmetric basis sets. Some calculations using significantly modified versions of generator coordinate method [40 – 42] with Skyrme interaction to accommodate the collective aspects of some of these nuclei have been initiated. These are summarized in the recent article of Görgen [39] who has suggested investigating properties of these nuclei within the framework of the unified theory of Bohr and Mottelson [6, 7]. This investigation is a step in that direction for large anisotropic density distribution.

5. Conclusion

This investigation establishes that many anomalous properties of nuclei away from the valley of stability could be the consequences of large anisotropy which embodies a mixture of quadruple with higher multipolar shapes in many of these nuclei. For large anisotropy, the energy levels associated with the higher principal quantum number $N$ of isotopic harmonic oscillator, in general, come down in energy and lie lower than those associated with lower $N$. This accounts for many intruder states observed in these nuclei, e.g., in $^{11}\text{Be}$, and results in creating the islands of inversion noted in $^{32}\text{Mg}$ and other isotopes. The presence of these intruder states seriously disturb the ordering of shells and the magic numbers associated with the Meyer-Jensen shell model. However, these observed data are compatible with the broad characteristics of our model.

The model provides a qualitative explanation of $^{11}\text{Be}$ having $(1/2^+)$ as its ground state and the possible existence of a $(1/2^-)^-\text{He}$ around 1 MeV excitation energy. The model further establishes that $(1/2^+)$ could be the ground state, and parity of $^{9}\text{He}$ and commensurate with the scanty information of its low lying level scheme. The large $B(E2)$ observed in $^{32}\text{Mg}$ yields, in the Nilsson limit, an anisotropy of $\alpha \approx 0.5$. However, this needs further investigation, since for such a large value of $\alpha$, the Nilsson limit may not be valid. At this deformation, the intruder states of $(1f-2p)$ parentage have about the same energies as those of the $(1d)$ parentage in this model, as observed.

The model also provides an understanding of having 14 as a closed shell in some but not in other nuclei and the possibility of having $N = 30$ and 32 behaving like closed shells. It also
Table 1. List of basevectors $\psi(Nn_{z}\Lambda\Sigma)$ and the coefficients $A(Nn_{z}\Lambda\Sigma)$ along with the state number $\nu$. Continued in Table 2.

| $N$ | $\Omega$ | $A(Nn_{z}\Lambda\Sigma)$ | $\psi$ | $\nu$ |
|-----|----------|---------------------------|-------|------|
| 0   | 1/2      | 1.000                     |       | 000+ |
| 1   | 3/2      | 1.000                     |       | 110+ |
|     | 1/2      | 0.817 0.577               |       | 101+ |
|     |          | -0.577 0.817              |       | 101- |
| 2   | 5/2      | 1.000                     |       | 202+ |
|     | 3/2      | 0.894 0.477               |       | 211+ |
|     |          | -0.477 0.894              |       | 202- |
|     | 1/2      | 0.632 0.477 0.632         |       | 220+ |
|     |          | -0.577 0.817 0.000        |       | 200+ |
|     |          | -0.516 -0.365 0.775       |       | 211- |
| 3   | 7/2      | 1.000                     |       | 303+ |
|     | 5/2      | 0.926 0.378               |       | 312+ |
|     |          | -0.378 0.926              |       | 303- |
|     | 3/2      | 0.756 0.378 0.535         |       | 321+ |
|     |          | -0.447 0.894 0.000        |       | 301+ |
|     |          | -0.478 -0.239 0.845       |       | 312- |
|     | 1/2      | 0.478 0.586 0.293 0.586   |       | 330+ |
|     |          | -0.632 0.516 0.516 -0.258 |       | 310+ |
|     |          | 0.447 -0.365 0.730 -0.365 |       | 301- |
|     |          | -0.414 -0.507 0.338 0.676 |       | 321- |
| 4   | 9/2      | 1.000                     |       | 404+ |
|     | 7/2      | 0.943 0.333               |       | 413+ |
|     |          | -0.333 0.943              |       | 404- |
|     | 5/2      | 0.817 0.333 0.471         |       | 422+ |
|     |          | -0.377 0.926 0.000        |       | 402+ |
|     |          | -0.436 -0.178 0.882       |       | 413- |
|     | 3/2      | 0.617 0.535 0.218 0.535   |       | 431+ |
|     |          | -0.586 0.676 0.414 -0.169 |       | 411+ |
|     |          | 0.293 -0.338 0.828 -0.338 |       | 402- |
|     |          | -0.436 -0.378 0.309 0.756 |       | 422- |
|     | 1/2      | 0.356 0.617 0.218 0.436 0.504 | 440+ | 31 |
|     |          | -0.586 0.169 0.478 0.478 -0.414 | 420+ | 32 |
|     |          | 0.447 -0.516 0.730 0.000 0.000 | 400+ | 33 |
|     |          | 0.478 -0.138 -0.390 0.586 -0.507 | 411- | 34 |
|     |          | -0.318 -0.552 -0.195 0.488 0.563 | 431- | 35 |

raises the possibility of these nuclei to have anisotropic shapes.

This investigation, thus, established that the nucleonic density distribution or shapes of most of the nuclei near drip lines are anisotropic about their body-fixed $z$-axis.

It would be interesting to understand the underlying cause of this from a many-body standpoint. Nuclei at the valley of stability, exhibit increasing quadrupole deformation as one moves away from the magic number up to the middle of the shell. A key cause of this is the interaction among nucleon forming the core with extra-core nucleons leading to strong configuration admixtures. To some extend this must also be the case here. However, the degree of admixture may be very intricate making it difficult to generate these shapes from spherical symmetric band states. Another consideration is to note that many light nuclei exhibit...
Table 2. Continuation of Table 1

| N  | Ω  | A(NnΛΣ) | ψ  | v  |
|----|----|---------|----|----|
| 5  | 11/2 | 1.000  |    |    |
| 9/2 | 0.953 | 0.301  | 505+ | 36 |
|    | 0.302 | 0.953  | 514+ | 37 |
| 7/2 | 0.853 | 0.302  | 523+ | 39 |
|    | 0.333 | 0.943  | 503+ | 40 |
| 5/2 | 0.696 | 0.492  | 505+ | 39 |
|    | 0.302 | 0.953  | 514+ | 41 |
| 3/2 | 0.493 | 0.603  | 532+ | 42 |
|    | 0.302 | 0.953  | 512+ | 43 |
| 1/2 | 0.263 | 0.588  | 532+ | 44 |
|    | 0.504 | 0.113  | 501+ | 45 |
|    | 0.535 | 0.478  | 512+ | 46 |
|    | 0.302 | 0.953  | 512+ | 47 |
|    | 0.390 | 0.239  | 512+ | 48 |
|    | 0.436 | 0.098  | 512+ | 49 |
|    | 0.504 | 0.478  | 512+ | 50 |
|    | 0.390 | 0.239  | 512+ | 51 |

clustering structure, e.g., \(^8\)Be, \(^6\)Li, \(^7\)Li, etc. exhibit alpha-alpha, alpha-deuteron, alpha-tritron, etc. configuration, respectively. These are not spherically symmetric. The relative strength of triplet and singlet two-nucleon tensor force is a factor in that.

Acknowledgments

The authors are thankful to particularly Prof. Lee Sobotka for many illuminating discussion, to Mr. A. Abu-Nada for the preparation and submission of the manuscript and to Mrs. Luanne Bartholomew for carefully reading the article.

References

[1] Brown B A 2001 Prog. Part. Nucl. Phys. 40 1
[2] Fridmann J et al 2006 Phys. Rev. C 74 034313
[3] Janssens R V F et al. 2002 Phys. Lett. B 546 55
[4] Padgett S W et al. 2005 Phys. Rev. C 72 064330
[5] Belleznik M et al. 2005 Phys. Rev. C 72 054316
[6] Bohr A 1952 Kgl. Danske Videnskab. Selskab. Mat.-fys. Medd. 26 No 14
[7] Bohr A and Mottelson B R 1953 Kgl. Danske Videnskab. Selskab. Mat.-fys. Medd. 29 No 16
[8] Moszkowski S A 1957 Handbuch der Physik XXXIX 411
[9] Nilsson S G 1955 Kgl. Danske Videnskab. Selskab. Mat.-fys. Medd. 29 No 16
[10] Meyer M G and Jensen J H D 1951 Elementary Theory of Nuclear Shell Structure (John Wiley, New York)
[11] Pauling L and Wilson E B 1958 Introduction to Quantum Mechanics (McGraw-Hill Book Co, New York).
[12] Rassey A 1958 Phys. Rev. 109 949
[13] Majumder A 1983 M.S. Thesis, Southern Illinois University at Carbondale, unpublished.
[14] Mottelson B R and Nilsson S G 1959 Kgl. Danske Videnskab. Selskab. Mat.-fys. Skr. 1 No 8
[15] Motobayashi T et al. 1995 Phys. Lett. B 346 9
[16] Malik F B and Scholz W 1996 Phys. Rev. 150919
[17] Malik F B and Scholz W 1967 *Nuclear Structure* edited by Hossain A, Harun-ar-Rashid and Islam M (North Holland Publishing Co.).
[18] Skaza F et al. 2006 *Phys. Rev.* C 73 044301
[19] Bohr A and Mottelson B R 1955 *Kgl. Danske Videnskab Selskab Mat.-fys. Medd.* 30 No 1.
[20] Hamamoto I and Mottelson B R 2009 *Phys. Rev.* C 79 No 034317.
[21] Tripathi V et al. 2005 *Phys. Rev. Lett.* 94 162501
[22] Strongman M J et al. 2009 *Phys. Rev.* C 80 021302 (R)
[23] Warburton E K, Becker I A, Millener D J and Brown B A 1987 *Brookhaven National Laboratory’s report, BNL-4D89D*.
[24] Warburton E K, Becker J A and Brown B A 1990, *Phys. Rev.* C 41 1147
[25] Willkinson D H and Alburger D E 1959 *Phys. Rev.* 113 563
[26] Tripathi V et al. 2008 *Phys. Rev.* C 77 034310
[27] Skyrme T R H 1959 *Nucl. Phys.* A 9 615, 635
[28] Pieper S et al. 2001 *Phys. Rev.* C 64 014001
[29] Navaratiil P and Barrett B R 1998 *Phys. Rev.* C 57 3117
[30] Navaratiil P, Vary J P and Barrett B R 2000 *Phys. Rev.* C 62 05434
[31] Al-Khalid J and Arai K 2006 *Phys. Rev.* C 74 034312
[32] Grigorenko L V and Zhukov M V 2008 *Phys. Rev.* C 77 034611
[33] Korsheninnkov A A, Danilin B V and Zhukov M V 1993 *Nucl. Phys.* A 559 208
[34] Talmi I and Unna I 1960 *Phys. Rev. Lett.* 4 469
[35] Brown B A and Wildenthal B H 1988 *Annual. Rev. Nucl. Part. Sci.* 38 1988
[36] Fukunishi N, Ohtsuka T and Sebe T 1992 *Phys. Lett. B* 296 279
[37] Utsuno Y et al. 2001 *Phys. Rev.* C 70 044307
[38] Utsuno Y et al. 2001 *Phys. Rev.* C 64 011301(R)
[39] Görgen A J 2010 *Phys. G. Nucl. Part. Phys.* 37 103101
[40] Bender M et al. 2004 *Phys. Rev.* C 69 031303
[41] Bender M, Bonche P and Heenen P H 2004 *Phys. Rev.* C 74 024312
[42] Bender M and Heienen P H 2004 *Phys. Rev. C* 78 024309
[43] Chen L, Blank B, Brown B A, Chartier M, Galonsky A, Hanso P G and Thoennessen H 2001 *Phys. Lett. B* 505 21
[44] Warburton E K and Brown B A 1992 *Phys. Rev.* C 46 923
[45] Golovkov M S, it al. 2007 *Phys. Rev.* C 76 021605(R)
[46] Barker F C 2004 *Nucl. Phys.* A 741 42
[47] Knight R H, Scholz W and Malik F B 1978 *Physics of Medium Light Nuclei* 158 Edited by Blasi P and Ricci R A (Editria Composition, Bologna).
[48] Malik F B 1986 *Int’l conf. on Nucl. Structure and Symmetries* Edited by. Meyer R and Paar V P 1069 (World Scientific)
[49] Rogachev G V, it al. 2003 *Phys. Rev.* C 61 041603(R)
[50] Horiiuchi H and Kanada-En’yo Y 1995 *Nucl. Phys.* A 588 121c
[51] de la Peña I H, Hess P O, Léval G and Algara A 2001 *J. Phys. G. Nucl. Part. Phys.* 27 2019
[52] Hamada S, Yasue M, Kubono S, Tanaka M and Petersen R T 1994 *Phys. Rev. C* 49 3192
[53] Cohen S and Kurath D 1965 *Nucl. Phys.* A 73 1
[54] Baroni S, Navrátil P and Quaglioni S 2013 *Phys. Rev. Lett.* 110 022505