Gravitational potential of a planet modeled by a visco-elastis sphere

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Abstract. The paper investigates the gravitational potential of a planet moving in a gravitational field of an attracting center and a satellite. The planet is modeled by a homogeneous isotropic visco-elastic body that in its natural undeformed state has a full sphere shape. The satellite and the attracting center are represented by material points. The planet-satellite system moves relatively to the common center of gravity, which in its turn moves along a Keplerian orbit relatively to a motionless attracting center. Based on the solution of the quasi-static problem of elasticity theory for the current problem, the authors deduced a formula for evaluating the gravitational potential of a planet and also determined the Earth’s gravitational potential, taking into account the Moon and Sun’s tidal effects both in an external point.

Key words: gravitational potential, viscoelastic planet, tides.

1. Introduction

Nowadays one of the most important problems to solve is determining the shape of the Earth geoid (ground reference plane). In order to determine the exact surface of the geoid in any point of the planet one has to carry out a set of measurements on the geoid surface or in a certain point of the Earth surface taking into account the pattern of mass distribution, which is virtually impossible to carry out.

In order to solve this task, GRACE (2002) and GOCE (2009) spacecrafts were launched into near-Earth orbit. These satellites helped to acquire crucial gravimetric and geodesic data so that an advanced model of the geoid was created, exceeding any previous one in accuracy.

However, if we take so-called tidal effects into account, including elastic deformations of the globe due to gravitational forces of the Moon and the Sun, all the results acquired when measuring the Earth geoid shape represent a somewhat average image and carry a certain measure of inaccuracy.

As early as in 1950 Soviet scientist Molodenskiy M.S. [1] suggested using a quasi-geoid surface – a surface close enough to the geoid one, however not requiring to know the earth crust’s inner composition – in order to solve the fundamental problem of determining the real shape of Earth geoid with all the tidal effects taken into account. When comparing the quasi-geoid and the geoid, one
should consider that the difference in high mountains is approximately 2-4 m, in lowland plains – 0.02-0.12 m, and as for the water-plains, there is no difference at all. The quasi-geoid surface is determined by the gravitational potential values. The paper [2] gives an overview of methods for studying the classical theory of tides. Currently, in accordance with the agreements of the IERS (2010) [3], tidal deformations are recorded in the form of small corrections to the coefficients of the geopotential model.

Lunar-solar tides affect the change in the angular velocity of rotation of the Earth, the change in weather and climate [4]. The relevance of the research topic is associated with high-precision prediction of the motion of artificial Earth satellites, with high-precision measurement of the Earth’s gravitational field [5].

The aim of the present work is to calculate gravitational potential of a planet, modeled by a homogeneous isotropic visco-elastic sphere, moving in a gravitational field of an attracting body and a satellite. The satellite is modeled by a material point. The planet-satellite system moves relatively to a common center of gravity, which in its turn moves along a Keplerian orbit relatively to a motionless attracting center. The motion separation method is used for mechanical systems with an infinite number of degrees of freedom [6].

2. Setting up the problem

We shall now consider the movement problem of a planet-satellite system in a gravitational field of an attracting center. We will take as a model a homogeneous isotropic visco-elastic body with \( m \) mass, which in its natural undeformed state occupies the volume \( V = \{ \mathbf{r} \in \mathbb{R}^3, |\mathbf{r}| \leq r_0 \} \) in a 3-dimensional Euclidean space, i.e. its shape is a full sphere with \( r_0 \) radius. The satellite will be modeled as a single mass point \( F \) with mass \( m_2 \), and the attracting center – as a single mass point \( O \) with mass \( m_1 \). The planet-satellite system moves relatively to the common center of gravity \( C_0 \), which in its turn moves along a Keplerian orbit relatively to a motionless attracting center. We assume that \( m_2 \ll m \ll m_1 \) and \( |\mathbf{R}_2| \ll |\mathbf{R}_1| \). Here \( \mathbf{R}_1 = \overrightarrow{OC_0} \) is the radius vector of a mass center of the planet-satellite system, \( \mathbf{R}_2 = \overrightarrow{CF} \), where \( C \) is the planet’s center of mass.

We introduce the following coordinate systems:

1) \( OXYZ \) inertial system with the attracting center being its origin;
2) \( Cx_1x_2x_3 \), moving coordinate system linked to the visco-elastic full sphere;
3) $C'X'Y'Z'$ König's system of axes [7], which moves translationally with its axes parallel to the corresponding axes of the inertial coordinate system (Figure 1).

The radius vector of a $M$ fixed point of the visco-elastic planet is represented as:

$$\mathbf{R}_M = \mathbf{R}_1 - \frac{m_2}{m + m_2} \mathbf{R}_2 + \Gamma (\mathbf{r} + \mathbf{u}(\mathbf{r}, t)),$$

where $\Gamma$ is the operator for transition from $C'X'Y'Z'$ moving coordinate system to $C'X'Y'Z'$ König’s system of axes, $\mathbf{u}(\mathbf{r}, t)$ is the elastic displacement vector. Based on the motion separation method for mechanical systems with infinite number of degrees of freedom, paper [8] determines a solution for the quasi-static problem of the elasticity theory for the problem under consideration. The above-mentioned solution is given by:

$$\mathbf{u}(\mathbf{r}, t) = E^{-1}(\mathbf{u}_{10}(\mathbf{r}, t) + \mathbf{u}_{11}(\mathbf{r}, t) + \mathbf{u}_{12}(\mathbf{r}, t)),$$

(1)

$$\mathbf{u}_{10}(\mathbf{r}, t) = \rho \left[ \frac{2}{3} \omega^2 [d_1 r^2 + d_2 r_0^2] \mathbf{r} + b_1 \left[ \frac{1}{6} \omega^2 r^2 - \frac{1}{2} (\omega, \mathbf{r})^2 \right] \mathbf{r} + \left[ b_2 r^2 + b_3 r_0^2 \right] \left[ \frac{1}{3} \omega^2 \mathbf{r} - (\omega, \mathbf{r}) \mathbf{r} \right] \right],$$

$$\mathbf{u}_{11}(\mathbf{r}, t) = -\frac{3 \rho f m_1}{Q_1^3} \left[ 1 + 3 \chi \frac{Q_1}{Q_1} \right] \left[ b_1 \left[ \frac{1}{6} \omega^2 r^2 - \frac{1}{2} (\xi_1, \mathbf{r})^2 \right] \mathbf{r} + \left[ b_2 r^2 + b_3 r_0^2 \right] \left[ \frac{1}{3} \mathbf{r} - (\xi_1, \mathbf{r}) \xi_1 \right] \right],$$

$$\mathbf{u}_{12}(\mathbf{r}, t) = -\frac{3 \rho f m_2}{R_2^3} \left[ 1 + 3 \chi \frac{R_2}{R_2} \right] \left[ b_1 \left[ \frac{1}{6} \omega^2 r^2 - \frac{1}{2} (\xi_2, \mathbf{r})^2 \right] \mathbf{r} + \left[ b_2 r^2 + b_3 r_0^2 \right] \left[ \frac{1}{3} \mathbf{r} - (\xi_2, \mathbf{r}) \xi_2 \right] \right] - \frac{3 \rho f m_2}{R_2^3} \left[ b_1 \left( \xi_2, \mathbf{r} \right) \xi_2 \mathbf{r} + \left[ b_2 r^2 + b_3 r_0^2 \right] \left[ \xi_2, \mathbf{r} + (\xi_2, \mathbf{r}) \xi_2 \right] \right].$$

(2)

$$d_1 = -\frac{(1 + v)(1 - 2v)}{10(1 - v)}, \quad d_2 = \frac{(1 - 2v)(3 - v)}{10(1 - v)},$$

$$b_1 = \frac{2(1 + v)}{5v + 7}, \quad b_2 = \frac{-(1 + v)(2 + v)}{5v + 7}, \quad b_3 = \frac{(1 + v)(2v + 3)}{5v + 7}.$$

Here $E, v, \rho$ are Young’s modulus, Poisson’s ratio and average density of the planet respectively (constant values), $\omega$ is the planet’s angular velocity, $\mathbf{Q}_1 = \frac{\partial \mathbf{c}}{\partial \mathbf{c}}$, $\mathbf{Q}_1 = |\mathbf{Q}_1|$, $R_2 = |R_2|$, $\xi_1 = \Gamma^{-1} \mathbf{Q}_1 / Q_1$, $\xi_2 = \Gamma^{-1} R_2 / R_2$.

(3)

The function $\mathbf{u}_{10}(\mathbf{r}, t)$ characterizes the planet’s deformations due to its rotation and describes its compression along the axis of rotation. The $\mathbf{u}_{11}(\mathbf{r}, t)$ and $\mathbf{u}_{12}(\mathbf{r}, t)$ functions in formula (1) determine the tidal deformation of the planet on the sides of the attracting center and the satellite respectively. The summands containing $\chi$ factor in expressions for $\mathbf{u}_{11}(\mathbf{r}, t)$ and $\mathbf{u}_{12}(\mathbf{r}, t)$ characterize the lag time of the given tidal deformations due to viscous friction forces. Since the impact of dissipative forces is quite small within the timespans comparable to the planet’s spin period, we will not take these dissipative forces into account and assume $\chi = 0$.

The $\omega, \mathbf{Q}_1, R_2, \xi_1, \xi_2$ values in the (1) expression for the $\mathbf{u}(\mathbf{r}, t)$ viscous displacement vector are prescribed time functions. According to the motion separation method, time dependence of the given values corresponds to the unperturbed problem, when $C_0$ mass center of the planet-satellite system moves along a Keplerian elliptic orbit relatively to $O$ attracting center and $C, F$ points move according to the classic two-body problem. Moreover, the planet rotates uniformly around the axis which is oriented inalterably in the $OXY$ inertial coordinate system.

To make the problem simpler we will assume that $C, F$ points’ movement is within the $OXY$ plane. In the $OXY$ inertial coordinate system the $R_1, R_2$ vector coordinates are given by:
\[ R_i = R_i(\cos\psi_i, \sin\psi_i, 0), R_i = \frac{a_i(1 - e_i^2)}{(1 + e_i \cos\theta_i)}, \psi_i = g_i + \vartheta_i (i = 1,2), \tag{4} \]

\( a_1 \) is the major semi-axis, \( e_1 \) is the eccentricity, \( g_1 \) stands for the longitude of the pericenter, \( \vartheta_1 \) is the true anomaly of the \( C_0 \) point’s orbit relatively to the \( O \) motionless center; \( a_2 \) is the major semi-axis, \( e_2 \) is the eccentricity, \( g_2 \) stands for the longitude of the perihelion, \( \vartheta_2 \) is the true anomaly of the \( F \) point’s orbit relatively to the \( C \) point. The \( a_1, e_1, g_1, a_2, e_2 \) values are constant. The true anomalies time functions:

\[ \dot{\vartheta}_i = \frac{(1 + e_i \cos\theta_i)^2}{(1 - e_i^2)^{3/2}} n_i, n_i = \frac{2\pi}{T_i} (i = 1,2). \tag{5} \]

The respective rotation periods are denoted by \( T_i \) in (5) formula.

The operator for transition from \( Cx_1x_2x_3 \) moving coordinate system to \( CX’Y’Z’ \) König’s system of axes can be represented as a product of three orthogonal matrices:

\[ \Gamma = \Gamma_3(\psi)\Gamma_1(\theta)\Gamma_3(\varphi), \]

\[ \Gamma_3(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \Gamma_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}. \]

Here \( \psi, \theta, \varphi \) are Euler angles [7]. The \( p, q, s \) – components of the \( \mathbf{w} \) angular velocity vector in the \( Cx_1x_2x_3 \) moving coordinate system are connected to the Euler angles via Euler kinematic equations:

\[ p = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi, q = \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi, s = \dot{\psi}\cos\theta + \dot{\varphi}. \tag{6} \]

If we point the \( Cx_3 \) axis along the \( \mathbf{w} \) vector, the first two components of the vector will be equal to zero. Therefore the (6) system gives us: \( \dot{\psi} = 0, \dot{\theta} = 0, \dot{\varphi} = \omega, \) i.e., the \( \psi, \theta \) Euler angles are constant and the \( \varphi \) angle is linearly dependent to time. We can consider \( \psi_0 = 0 \) without loss of generality. Thus the \( \Gamma \) operator and its inverse one are given by the following equations:

\[ \Gamma = \Gamma_1(\theta_0)\Gamma_3(\varphi), \Gamma^{-1} = \Gamma_3(-\varphi)\Gamma_1(-\theta_0), \theta_0 = \theta(0), \varphi = \omega t + \varphi(0). \]

The \( \mathbf{r} \) radius vector of the \( M \) arbitrary point of the planet in \( Cx_1x_2x_3 \) moving coordinate system can be defined via \( r, \lambda, \mu \) spherical coordinates (Figure 2):

\[ \mathbf{r} = r(\cos\lambda\cos\mu; \sin\lambda\cos\mu; \sin\mu). \]

3. Solving the problem

Using the viscous displacement vector (1), the gravitational potential of the planet can be evaluated with the formula:

\[ \Pi(\mathbf{R}, t) = -f \iiint_V \frac{\rho dv}{|\Gamma(\mathbf{r} + \mathbf{u}(\mathbf{r}, t)) - \mathbf{R}|}. \tag{7} \]

Here \( f \) is the universal gravitational constant, \( \mathbf{R} = \overline{CK} \) is the radius vector of the \( K \) point, the potential is determined, \( \mathbf{r} \) – radius-vector of the \( dv \) differential of volume with \( \rho \) density. The \( \mathbf{R} \) vector in formula (7) is given in the \( CX’Y’Z’ \) coordinate system. The integral is evaluated by the \( V \) full sphere of \( r_0 \) radius.

It is assumed that:

\[ |\mathbf{R}| > r_0, |\mathbf{u}(\mathbf{r}, t)| < |\mathbf{r} - \Gamma^{-1}\mathbf{R}|. \tag{8} \]

Let \( \mathbf{R} \) be the radius vector of some external point. We denote this vector as \( \mathbf{a} \) plotted in the \( Cx_1x_2x_3 \) moving coordinate system:

\[ \mathbf{a} = \Gamma^{-1}\mathbf{R} = \mathbf{R}e_R, e_R = (\cos\lambda\cos\mu, \sin\lambda\cos\mu, \sin\mu), R = r_0 + h, h > 0. \tag{9} \]
We shall now transform the expression in the right of equation (7) taking equation (8) into account while limiting ourselves to the first-order members by \(|u|/|r - \Gamma^{-1}R|\), and acquire the following expression for the potential:

\[
\Pi(R, t) = -f \iiint_{V} \frac{\rho dv}{|r - \Gamma^{-1}R|} + f \iiint_{V} \frac{(r - \Gamma^{-1}R, u)}{|r - \Gamma^{-1}R|^3} \rho dv. \tag{10}
\]

Utilizing the solution of the quasi-static problem of elasticity theory for the viscous displacement vector (1)-(2), we now calculate the triple integrals over the domain \(V\) in the formula (10). As a result the gravitational potential of Earth, modeled by a visco-elastic full sphere, with tidal forces of the Sun and the Moon taken into consideration is represented by formula:

\[
\Pi = -\frac{f m}{R} \Phi(\nu) \left\{ \frac{(\omega, \omega)}{R^3} - \frac{3(\omega, a)^2}{R^5} \right\} + \frac{9f^2 m^2 m_1 R_0}{140\pi E}\Phi(\nu) \left\{ \frac{(\xi_1, \xi_1)}{R^3} - \frac{3(\xi_1, a)^2}{R^5} \right\} + \frac{9f^2 m^2 m_2 R_0}{140\pi E^3} \Phi(\nu) \left\{ \frac{(\xi_2, \xi_2)}{R^3} - \frac{3(\xi_2, a)^2}{R^5} \right\} \tag{11}
\]

Here:

\(\omega = (0, 0, \omega)\), \(\xi_1 = (1 + h_{12} \cos \psi_{12}) \eta_1 - h_{12} \eta_2\), \(\xi_2 = \eta_2, \Phi(\nu) = (1 + \nu)(9\nu + 13)(5\nu + 7)^{-1}\),

\(\eta_1 = \nu^{-1}(\cos(g_1 + \vartheta_1), \sin(g_1 + \vartheta_1), 0), \eta_2 = \nu^{-1}(\cos(g_2 + \vartheta_2), \sin(g_2 + \vartheta_2), 0)\),

\(\nu^{-1} = \nu_3(-\varphi)\nu_1(-\vartheta_0), \vartheta_0 = 0.409280\ \text{rad}, \varphi = \omega t + \varphi(0), \psi_{12} = g_1 + \vartheta_1 - g_2 - \vartheta_2\),

\[h_{12} = k_4(1 + e_1 \cos \vartheta_1), k_4 = \frac{m_2 a_2 (1 - e_2^2)}{(m + m_2) a_1 (1 - e_1^2)}, h_{12} \ll 1,\]

\[R_1 = \frac{a_1 (1 - e_1^2)}{(1 + e_1 \cos \vartheta_1)}, R_2 = \frac{a_2 (1 - e_2^2)}{(1 + e_2 \cos \vartheta_2)}, Q_1 = \frac{a_2 (1 - e_2^2)}{(1 + e_1 \cos \vartheta_1)} (1 - h_{12} \cdot \cos \psi_{12}). \tag{12}\]

Using the (12) equations we now calculate all the acquired coefficients and scalar products and then deduce time-dependent expression for the gravitational potential in an external point:

\[
\Pi = -\frac{f m}{R} \left\{ 1 + k_3 \left[ 1/3 - \sin^2 \vartheta_1 \right] - k_2 (1 + e_1 \cos \vartheta_1)^3 \left[ 1/3 - (\eta_1, e_R)^2 \right] \right\} + \left\{ (1 - 5(\eta_1, e_R)^2) \cos \psi_{12} + 2(\eta_1, e_R)(\eta_2, e_R) h_{12} \right\} - k_3 (1 + e_2 \cos \vartheta_2)^3 \left[ 1/3 - (\eta_2, e_R)^2 \right], \tag{13}\]

where

\[k_1 = \frac{9m r_0 a_0^2}{140\pi E R^2} \Phi(\nu), k_2 = \frac{27f m m_1 R_0}{140\pi E R^2} \frac{\Phi(\nu)}{a_1^3 (1 - e_1^2)^3}, k_3 = \frac{27f m m_2 R_0}{140\pi E R^2} \frac{\Phi(\nu)}{a_2^3 (1 - e_2^2)^3}, \tag{14}\]

\((\eta_1, e_R) = \cos \mu \cos \vartheta_0 \sin(g_1 + \vartheta_1) \sin(\varphi + \lambda) + \cos \mu \cos(g_1 + \vartheta_1) \cos(\varphi + \lambda) - \sin \vartheta_0 \sin \mu \sin(g_1 + \vartheta_1), i = 1, 2.\)
4. Plotting the graphs

We shall now consider the following dimensionless functions

\[ \Pi_1 = \left( \Pi - \Pi_0 \right) / \Pi_0, \]

where \( \Pi_0 = -f m/R \) undisturbed gravitational potential for an external point.

From (13) follows:

\[ \Pi_1 = k_1 \left[ \frac{1}{3} - \sin^2 \mu \right] - k_2 (1 + e_1 \cos \vartheta_1)^3 H_1 - k_3 (1 + e_2 \cos \vartheta_2)^3 H_2, \]

(15)

\[ H_1 = \frac{1}{3} - (\eta_1, \nu_R)^2 + \left[ \frac{1}{3} - 5 (\eta_1, \nu_R)^2 \right] \cos \vartheta_1 + 2 (\eta_1, \nu_R) (\eta_2, \nu_R) \mu, \]

\[ H_2 = \frac{1}{3} - (\eta_2, \nu_R)^2. \]

We have to make a transition from dimensional time \( t \) to the dimensionless variable \( \tau \) equal to the number of Earth’s spins:

\[ \tau = \frac{\varphi}{2 \pi} = \frac{\omega t + \varphi(0)}{2 \pi}. \]

Then \( \varphi = 2 \pi \tau \). With the derivative of \( \tau \) denoted by a character stroke we deduce:

\[ \dot{\varphi}_i = \frac{2 \pi}{\omega} \dot{\varphi}_i = \frac{2 \pi (1 + e_i \cos \vartheta_i)^2 n_i}{(1 - e_i^2)^{3/2}} \frac{n_i}{\omega} = \frac{2 \pi (1 + e_i \cos \vartheta_i)^2 T_{rot}}{T_1}, \]

\[ (i = 1, 2), \]

\[ g_2 = \frac{2 \pi}{\omega} \dot{g}_2 = \frac{2 \pi T_{rot}}{T_3}. \]

Here \( T_{rot} \) – spin period of Earth, \( T_1 \) – rotation period of the barycenter of the “Earth-Moon” system around the Sun; \( T_2 \) – the Moon’s rotation period around Earth, \( T_3 \) rotation period of the Moon’s major semi-axis.

Let us introduce the following notations:

\[ k_5 = \frac{2 \pi T_{rot}}{T_1 (1 - e_1^2)^{3/2}}, \]

\[ k_6 = \frac{2 \pi T_{rot}}{T_2 (1 - e_2^2)^{3/2}}, \]

\[ k_7 = \frac{2 \pi T_{rot}}{T_3}. \]

(17)

Thus from (16) it follows:

\[ \varphi_1 = k_5 (1 + e_1 \cos \vartheta_1)^2, \varphi_2 = k_6 (1 + e_2 \cos \vartheta_2)^2, g_2 = k_7 \tau + g_2(0). \]

(18)
We utilize the following parameter values included in the formulae (14), (17) [9]:

\[ r_0 = 6.378 \cdot 10^6 \text{ m}, h = 3 \cdot 10^5 \text{ m}; E = 1.2 \cdot 10^{11} \text{ kg/(m \cdot s^2)}, \nu = 0.2, \]
\[ T_{rot} = 23,934.19 \text{ hours}, \omega = 2\pi/T_{rot} = 7.2922 \cdot 10^{-5} \text{ s}^{-1}, f = 6.672 \cdot 10^{-11} \text{ m}^3/(\text{kg} \cdot \text{s}^2), \]
\[ m_1 = 1.98911 \cdot 10^{30} \text{ kg}, m = 5.9736 \cdot 10^{24} \text{ kg}, m_2 = 7.349 \cdot 10^{22} \text{ kg}, a_1 = 1.4959787 \cdot 10^{11} \text{ m}, \]
\[ e_1 = 0.01671022, a_2 = 3.844 \cdot 10^8 \text{ m}, e_2 = 0.054900, \theta_0 = 23.45^\circ = 0.409280 \text{ rad}, \]
\[ T_1 = 365.26 \text{ d}, T_2 = 27.321661 \text{ d}, T_3 = 8.85 \text{ years}. \]

We point the \( OX \) axis of the inertial coordinate system along the radius vector of the barycenter orbit’s perihelion for the «Earth-Moon» system. Thus \( g_1 = 0 \).

Dimensionless coefficients (12), (14), (17) are:

\[ k_1 = 1.719813 \cdot 10^{-3}; k_2 = 3.849354 \cdot 10^{-8}; k_3 = 8.451862 \cdot 10^{-8}; \]
\[ k_4 = 3.114226 \cdot 10^{-5}; k_5 = 0.017162; k_6 = 0.230382; k_7 = 1.943723 \cdot 10^{-3}. \]

We introduce the function

\[ \Pi_{11} = \Pi_1 \cdot 10^4 + \Pi_{10}, \]

where \( \Pi_{10} \) is constant.

Figure 3 demonstrates the graph of function \( \Pi_{11} = \Pi_{11}(\tau) \), that describes the gravitational potential in an external point within 30 days, with Earth modeled by a homogeneous isotropic visco-elastic full sphere. In the initial moment of time the Sun, Earth and the Moon were set in the right line with the Sun and the Moon being on opposite sides of Earth \( (g_1 = 0, g_2(0) = 0, \theta_1(0) = 0, \theta_2(0) = 0) \). The position of the point in the moving coordinate system is determined by the parameters: \( R = 6.678 \cdot 10^6 \text{ m}, \lambda = 0, \mu = 0 \). In this case \( \Pi_{10} = -5.732 \). The oscillation amplitude depends on the lunar phases.

\[ \Pi_{11} \cdot 10^4 \]

\[ \begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c} \hline \pi & 0 & 5 & 10 & 15 & 20 & 25 & 30 \\ \hline \pi_{11} \cdot 10^4 & 16.42 & 12.93 & 9.44 & 5.96 & 2.47 \\ \hline \end{array} \]

**Figure 3.** Gravitational potential in an external point \( R = 6.678 \cdot 10^6 \text{ m}, \lambda = 0, \mu = 0 \) during 30 days. \( \Pi_{11} = \Pi_1 \cdot 10^4 - 5.732 \).

Figures 4, 5 show dependency graphs for the size of the tidal bulge on the number of Earth’s spins at an external point at 30° and 60° latitudes respectively. Initial conditions in (18) are as follows:

\[ g_2(0) = 0, \theta_1(0) = 0, \theta_2(0) = 0. \]
Figure 4. Gravitational potential in an external point $R = 6.678 \times 10^6 m, \lambda = 0, \mu = 30^\circ$ during 30 days. $P_{11} = P_1 \cdot 10^4 - 1.432$.

Figure 5. Gravitational potential in an external point $R = 6.678 \times 10^6 m, \lambda = 0, \mu = 60^\circ$ during 30 days. $P_{11} = P_1 \cdot 10^4 + 7.166$.

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