Inter-brane Interactions in Compact Spaces and Brane Inflation

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Abstract. It was pointed out that brane-anti-brane inflation without warped geometry is not viable due to compactification effects (in the simplified scenario where the inflaton is decoupled from the compactification moduli). We show that the inflationary scenario with branes at a small angle in this simplified scenario remains viable. We also point out that brane-anti-brane inflation may still be viable under some special conditions. We also discuss a way to treat potentials in compact spaces that should be useful in the analysis of more realistic brane inflationary scenarios.

PACS numbers: 11.25 Wx, 98.80 Cq, 61.50 Ah, 11.25 Mj

Submitted to: JCAP
1. Introduction

Brane interactions in the brane world have been proposed as the origin of inflation [1, 2, 3], an epoch in the early universe that initiated the radiation-dominated big bang. In brane inflation [4] the inflaton is an open string mode, while the brane interaction comes from the exchange of closed string modes. A particularly simple scenario is the $Dp$-brane-anti-$Dp$-brane system. Although the potential between them seems too steep, it was proposed [5] that the compactification effect in the special case of a hyper-cubic torus will flatten the inflaton potential for enough inflation. Let us call this the $D\bar{D}$ scenario.

Recently, it was pointed out [6, 7] that the Poisson equation for the inflaton potential $\Phi$ in a compactified manifold should include a background term (the so-called “jellium” term in solid state physics). As a result, the slow-roll parameter $\eta$ in the $D\bar{D}$ inflationary scenario (in the simplified version where the stabilization of compactification moduli is independent of the inflaton) becomes

$$\eta \simeq -\frac{2}{d_\perp}$$

(1)

where $d_\perp$ is the number of dimensions perpendicular to the branes, $d_\perp = 9 - p \leq 6$. Since we need at least $N_e = 50$ e-folds of inflation, and $|\eta| < 1/N_e$, the $D\bar{D}$ scenario is not viable as an inflationary model. The impact of the jellium term is clearly important to the analysis of the inflationary properties in some of these scenarios. This was first studied in Ref.[8].

Here we would like to point out that the brane inflationary scenario where branes are at a small angle [9, 10] remains robust because the jellium term is much smaller in this scenario. In the $(n, 1)$, $(n, -1)$ wrapping scenario (which reduces to $2n$ parallel D4-branes after inflation),

$$\eta \simeq -\frac{\theta^2}{4n}$$

(2)

where $\theta$ is the angle between the two branes ($\theta \simeq 1/12$ and $n = 8$ are reasonable values). When the jellium contribution to $|\eta|$ is much less than $1/50$, the slow-roll behavior of the inflaton is dictated by the other terms in the potential, in particular the quartic harmonic term as measured around the antipodal point, as studied in Ref.[10].

We would also like to point out that the $D\bar{D}$ scenario may still be possible under some special conditions. Whether that special condition on the background charge distribution is realized or not depends on the dynamics of moduli stabilization (the dilaton, the complex and Kähler structures of the compactified manifold etc.), an issue that needs better understanding [11, 12, 7].

In section 2, we give a review of the Poisson equation in compact spaces. A simple ansatz of calculating the potential energy between two charges is $q_1\Phi_2$ where $\Phi_2$ is the potential due to the the second charge $q_2$, irrespective of whether the source charges in compact space add to zero or not. We shall justify this ansatz by showing that it is equivalent (up to a constant) to the potential energy between two charges (NS-NS or
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RR) by directly integrating the potential energy density over the compact space. This equivalence is simply illustrated in the 1-dimensional case \[8\]. In higher dimensions, the Green’s function in compact space may be represented in terms of Jacobi theta functions. In section 4, we generalize the method used in solid state physics to show how to obtain the Green’s function numerically. This method should be useful in more realistic brane world models. For flat compact spaces, the Green’s functions obtained by these two methods agree. In section 5, we apply the result to brane inflation in the cosmological context. As pointed out in Ref.\[6, 7\], the slow-roll parameter \(\eta\) is too big for the brane-anti-brane scenario. On the other hand, the impact of the jellium term in the branes-at-small-angle scenario can be made negligibly small. In this sense, the branes-at-small-angle inflationary scenario remains robust. Section 6 contains discussion.

2. Poisson Equation and Potential Energy in Compact Spaces

In non-compact space, the determination of the potential energy (Coulombic or gravitational) between two charges is well known: we treat one of the charge \(q_1\) to be a probe charge, and the potential energy is given by \(q_1\Phi_2\), where \(\Phi_2\) is the potential due to charge \(q_2\) (even if \(|q_1| > |q_2|\)). Here, we argue that this simple ansatz is equally applicable in compact spaces, where \(\Phi_2\) includes the contribution of the jellium term. That is, the potential energy between two charged (Coulombic, Ramond-Ramond, gravitational or NS-NS) objects in a compact space will include the same quadratic component due to the jellium term, irrespective of whether the sum of the source charges in the compact space vanishes or not.

Consider a compact manifold \(M (\partial M = \emptyset)\). The Green’s function is given by the Poisson equation,

\[
\nabla^2 G(r, r') = \delta(r - r') - 1/V
\]

(3)

where \(\delta(r - r')\) is the \(d\)-dimensional \(\delta\) function and \(V\) is the volume of the compactified manifold. Integrating both sides over the \(d\)-dimensional compact space and using Stoke’s theorem, we obtain Gauss’s law. Since a compact space has no boundary, this volume integral over \(\nabla^2 G\) vanishes, as does the integral over the RHS of Eq.(3).

Consider the eigenfunctions \(u_\lambda (r)\) of the Laplacian operator on the compact manifold with eigenvalues \(\lambda \leq 0\)

\[
\nabla^2 u_\lambda (r) = \lambda u_\lambda (r).
\]

(4)

The eigenfunctions satisfy the completeness relation:

\[
\sum_\lambda u_\lambda (r) u_\lambda (r') = \delta(r - r')
\]

(5)

Integrating both sides of Eq.(11) over \(M\), we obtain \((dv = d^d r \sqrt{-g})\),

\[
\int_M dv u_\lambda(r) = 0 \quad \lambda \neq 0
\]

(6)
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In a compact space there is a normalizable zero mode of the Laplacian, \( \nabla^2 u_0 (r) = 0 \), where \( u_0 (r) = 1/\sqrt{V} \). The Green’s function can be written as:

\[
G (r, r') = \sum_{\lambda \neq 0} \frac{u_\lambda (r) u_\lambda (r')}{\lambda} \tag{7}
\]

Note that the zero mode is absent in \( G(r, r') \) in Eq.(7), thus avoiding an obvious divergence. Using Eq.(6) and (7), we have

\[
\int_M dv\ G(r, r') = 0 \tag{8}
\]

Let \( \Phi \) be the static potential due to a given set of charges \( q_i \) (at \( r_i \)), \( i = 1, 2, ..., \)

\[
\Phi (r) = \sum_i q_i G(r, r_i) \tag{9}
\]

Using Eq.(9) and (8), the total static energy due to this set of charges \( q_k \) is

\[
V(r_k) = \frac{1}{2} \int_M dv (\nabla \Phi)^2 = -\int_M dv \frac{1}{2} \Phi \nabla^2 \Phi \tag{10}
\]

\[
= -\frac{1}{2} \sum_i q_i \Phi (r_i) + \sum_i q_i \int_M dv \Phi (r) \]

Using Eq.(8) and Eq.(9), we see that the last term in Eq.(10) vanishes. So, irrespective of whether or not the net charge vanishes,

\[
V(r_k) = -\sum_{i>j} q_i G(r_i, r_j)q_j + \text{constant} \tag{11}
\]

where the constant comes from the \( r_i \)-independent self-interaction terms. When applied to the RR interaction, where \( q_i = \mu_i \), we must have \( \sum_i \mu_i = 0 \) in our compact space. For the NS-NS interaction, \( q_i \) are the brane tensions \( T_i \), \( \sum_i T_i \neq 0 \) and \( V_{NS-NS} (r_i) \) has the opposite sign since same sign charges attract. Putting them together we obtain, up to a constant

\[
V(r_k) = +\sum_{i>j} (T_i T_j - \mu_i \mu_j) G(r_i, r_j) \tag{12}
\]

where the presence of the jellium term is encoded in \( G(r_i, r_j) \). In the two brane potential, the “Coulomb” \( (1/r^{d-2}) \) term and the jellium \( (r^2) \) term have the same effective strength, namely \( T_i T_j - \mu_i \mu_j \). This effective strength is very small for branes at a small angle. Since the constant term in the potential is linear in the brane tension \( T \), \( V''/V \) can be made arbitrarily small.

2.1. One-dimensional Example

This well-known example (given in Ref.[8]) illustrates the above result. Consider a circle of circumference \( L \). For \( 0 \leq y \leq L \),

\[
G(y) = -\frac{y^2}{2L} + \frac{|y|}{2} - \frac{L}{12} \tag{13}
\]
where the constant term may be determined by the $\zeta$-function regularization of the divergent Green’s function $\hat{G}$ of the 1-dimensional lattice,

$$2\hat{G}(y) = |y| + \sum_{n=1}^{\infty} |y + nL| + |y - nL|$$

(14)

Notice that $G(y)$ satisfies Eq.$(\mathbf{13})$.

Let us put a charge $q_1 = q$ at $y_1 = 0$ and a second charge $q_2 = -q$ at $y_2 = y$, where $0 \leq y < L$ so that the total charge in the circle is zero. Using Eq.$(\mathbf{11})$, one finds

$$V(y) = q^2 G(y) + \frac{q^2 L}{12} = q^2 \left( \frac{y}{2} - \frac{y^2}{2L} - \frac{L}{12} \right) + \frac{q^2 L}{12}$$

(15)

where the first term is the linear potential and the quadratic term is due to the jellium effect. The last term is the self-energy term and is independent of $y$. As we put the charges on top of each other, $V(0) = 0$, as expected.

We may also calculate the potential at $x$ due to these two charges ($+q$ and $-q$):

$$\phi_1(x) = q \frac{x}{L}(L - y) - qy + \phi_0 \quad 0 \leq x \leq y$$

$$\phi_2(x) = -q \frac{xy}{L} + \phi_0 \quad y \leq x \leq L$$

(16)

Note that $\phi(x)$ is continuous and piecewise linear in $x$, i.e., $\phi_1(0) = \phi_2(L)$ and $\phi_1(y) = \phi_2(y)$. Now we can integrate over the circle the energy density $((\nabla\phi)^2/2)$ to obtain the energy as a function of the relative position $y$ of the charges. This exactly reproduces $V(y)$ in Eq.$(\mathbf{15})$. In short, $\phi(x, y)$ is the potential for a probe charge at $x$, while $V(y)$ is what we are interested in.

For the case of two masses on a circle: $m_1$ at $y_1 = 0$ and $m_2$ at $y_2 = y$, the potential at $x$ ($0 \leq x < L$) due to these 2 masses is $\phi(x) = m_1 G(x) + m_2 G(x - y)$, where there is a jellium contribution in each Green’s function. From this,

$$\nabla\phi_1(x) = -\frac{m_1 + m_2}{L} x + \frac{m_2 y}{L} + \frac{m_1 - m_2}{2} \quad 0 \leq x \leq y$$

$$\nabla\phi_2(x) = -\frac{m_1 + m_2}{L} x + \frac{m_2 y}{L} + \frac{m_1 + m_2}{2} \quad y \leq x \leq L$$

(17)

and, as expected, we have

$$V(y) = -\frac{1}{2} \int dx (\nabla\phi)^2 = m_1 m_2 G(y) - \frac{(m_1^2 + m_2^2)L}{24}$$

(18)

where the last constant term is the self-energy contribution.

### 3. Green’s Function in Arbitrary Dimension

When two branes are moving in a compactified space, we need to calculate the interbrane potential. This requires finding $\Phi(\mathbf{r}) = -G(\mathbf{r})$ that satisfies Eq.$(\mathbf{3})$. Our main goal is to determine the Green’s function around the antipodal point (the point farthest
from the source). To simplify the problem, consider a square, flat $d$-dimensional torus of volume $L^d$. Using the identity
\[
\frac{1}{L^2 \lambda} = - \int_0^\infty e^{L^2 \lambda s} ds \quad (\lambda < 0)
\]
and the eigensolutions
\[
u_n(x) = \sqrt{L^{-d}} e^{2\pi i n x / L}, \quad \lambda_n = - \frac{4\pi^2 n^2}{L^2}
\]
we can regularize
\[
G(x) = - \sum_{n \in \mathbb{Z}^d} (1 - \delta_{n,0}) \frac{e^{2\pi i n x / L}}{4\pi^2 n^2 L^{d-2}}
\]
\[
= L^{2-d} \int_0^\infty \left(1 - \prod_{j=1}^d \sum_{n_j \in \mathbb{Z}} e^{2\pi i n_j x_j / L - 4\pi^2 n_j^2 s_j}\right) ds
\]
\[
= L^{2-d} \int_0^\infty \left[1 - \prod_{j=1}^d \theta_3\left(\frac{\pi x_j}{L}, e^{-4\pi^2 s_j}\right)\right] ds
\]
Where $\theta_3$ is a Jacobi theta function \[19\]
\[
\theta_3\left(\frac{b}{2}, e^{-a}\right) = \sum_n e^{-an^2 + ibn}
\]
The rapid convergence of the above expression makes the flat $d$-torus especially simple to numerically simulate.

When one considers compactification on a manifold ($T^2$, $K3$, $CY_3$) with a non-trivial metric, another numerical method suggests itself. Developed by Ewald and others \[13\], this method is suited to spaces that may be represented as a periodic lattice (perhaps modulo some additional discrete symmetries). To illustrate the method, we proceed with a simple example, a square, flat torus. Let us again write $\Phi$ as in Eq.(7):
\[
\Phi(r) = \frac{1}{V} \sum_{k_j \neq 0} \frac{\exp(\mathbf{i k}_j \cdot \mathbf{r})}{k_j^2}
\]
where $\mathbf{k}_j \in \frac{2\pi}{L} \mathbb{Z}^d$ is a reciprocal lattice vector and $k_j$ its magnitude. This sum diverges for $d \geq 2$, since the number of terms grows as $k_j^{d-1}$ (i.e., with the surface area of a $d$-dimensional ball). Alternatively, one may treat the torus as an infinite lattice and sum over the source and its images at the lattice points:
\[
\Phi(r) = \sum_{r_j} \frac{\alpha}{(r - r_j)^{d-2}}
\]
where
\[
\alpha = \frac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{d/2}}, \quad d \neq 2
\]
As we saw in the one-dimensional case, this will lead to a divergence and the need to regularize it. The answer depends on how the lattice points are summed and how the regularization is carried out.
Ewald’s method is to add and subtract a diffuse charge distribution around each lattice point. The added charge makes the monopole moment of each cell zero, aiding the convergence of the real lattice sum. This additional charge (minus a jellium term) is convergently subtracted away in reciprocal space, since a reciprocal space sum converges for non-singular charge distribution. The final result is independent of the diffuse charge distribution used to regulate the sums. Details on how to implement this method in two, three and four dimensions are included in the appendix. The numerical data generated by the Ewald method may be fit to an expression for the potential near the source of the form:

\[
\Phi^{(d)}(r) = \frac{\alpha}{r^{d-2}} + \frac{r^2}{2dV} + \Phi^{(d)}_{\text{harmonic}}(r) + \text{constant}, \quad d > 2
\]

\[
= -\frac{1}{2\pi} \ln \left(\frac{r}{L}\right) + \frac{r^2}{4V} + \Phi^{(2)}_{\text{harmonic}}(r) + \text{constant}, \quad d = 2
\]

Notice that although individual terms in \(\Phi^{(d)}(r)\) are not periodic, their sum is. The constant is fixed by requiring \(\Phi^{(d)}(r)\) to be independent of the added and subtracted charge. The harmonic piece satisfies the Laplace equation (when restricted to the cell containing the origin):

\[
\nabla^2 \Phi^{(d)}_{\text{harm}} = 0 \quad (27)
\]

For a hypercubic torus (with \(V = L^d\)) it has the general form

\[
\Phi^{(d)}_{\text{harm}}(r) = \frac{C^{(d)}_4}{r^{d-2}} + A^{(d)}_4 h^{(d)}_4(r) + A^{(d)}_6 h^{(d)}_6(r) + \ldots \quad (28)
\]

where the \(h^{(d)}_n\) are polynomials of order \(n\) with coefficients determined by Eq. (27). For example, for a hyper-cubic lattice with coordinates \(x_i\) measured from the source (podal point) and \(r = (x_1, x_2, \ldots, x_d)\)

\[
h^{(d)}_4(r) = \frac{1}{V^{d-1}} \left[ \sum_{i=1}^{d} x_i^4 - \frac{6}{d-1} \sum_{i \neq j} x_i^2 x_j^2 \right] \quad (29)
\]

\[
h^{(d)}_6(r) = \frac{1}{V^{d-1}} \left[ \sum_{i \neq j \neq k} x_i^2 x_j^2 x_k^2 + \frac{(d-2)(d-1)}{180} \sum_{i} x_i^6 - \frac{(d-2)}{12} \sum_{i \neq j} x_i^4 x_j^2 \right]
\]

In two dimensions, terms of order \(4m + 2\) are not present in the harmonic piece. Beyond the sixth order terms in the hypercubic case, and for \(d > 2\) for a rectangular lattice, there is more than one undetermined coefficient \(A^{(d)}_{n,i}\) at each order. A two-dimensional rectangular lattice has only one parameter at any even order. At a given order \(n\), the number of parameters for a hypercubic lattice reaches a maximum and becomes independent of dimension for \(d \geq \frac{n}{2}\).

There is an expression similar to Eq. (26) for the potential near the antipodal point. This is the expression that is suitable for applications in brane inflation. With coordinates \(z_i\) now measured from the antipodal point \((z = r - (L/2, L/2, \ldots))\), \(\Phi\) has the form

\[
\Phi^{(d)}_{\text{antipodal}}(z) = \frac{C^{(d)}_a}{V^{d-1}} + \frac{1}{2dV} \sum_{i=1}^{d} z_i^2 + B^{(d)}_4 h^{(d)}_4(z) + B^{(d)}_6 h^{(d)}_6(z) + \ldots \quad (30)
\]
Table 1. Constant, fourth order and sixth order coefficients in potential near source.

| $d_{\perp}$ | 2   | 3   | 4   |
|------------|-----|-----|-----|
| $C_s$      | -0.21 | -0.21 | -0.17 |
| $A_4$      | 0.12  | 0.44  | 0.34 |
| $A_6$      | 0.00  | 0.0072 | 3.05 |

Table 2. Constant and coefficients of the fourth order and sixth order terms in the potential near the antipodal point.

| $d_{\perp}$ | 2   | 3   | 4   |
|-------------|-----|-----|-----|
| $C_a$       | -0.055 | -0.064 | -0.070 |
| $B_4$       | -0.62 | -0.53 | -2.20 |
| $B_6$       | 0.00  | 0.0024 | -101.5 |

The results for the coefficients in Eq.(26) and Eq.(30) were obtained by the method described below, and are summarized in Tables 1 and 2. The lattice spacing has been set to one. At least four terms in $\Phi_{\text{harm}}$ were kept in each dimension, but the accuracy of the numerical values could be improved by keeping more. In general, the convergence of Eq.(26) is somewhat better than that of Eq.(30).

It is easy to check that the integral of $\Phi$ over a unit cell is zero, that is, $\Phi$ satisfies Eq.(8). In solid state physics, the Madelung constant is found by considering the potential due to both the positive and negative ions. Since the negative ions are found at the antipodal point in a simple cubic lattice, the Madelung constant is given by $\alpha_m = C_s - C_a$. In three dimensions, our value for the Madelung constant of a simple cubic lattice agrees with Ref.[15]. The constants $C_a$ in Table 2 agree with Eq.(21) evaluated at the antipodal point, and are four times those listed in Ref.[14].

For rectangular torus, there will be quadratic harmonic terms of the form $z_i^2 - z_j^2$. Their impact on inflation is discussed in Ref.[10]. The hypercubic way to sum the lattice generates only the harmonic terms. The numerical values of $B_4$ is at least a factor of 3 smaller than that given in Ref.[10]. This weakens the potential and improves the inflationary scenario.

4. Application to Brane Inflationary Scenarios

Let us consider a few brane inflationary scenarios, where the moduli stabilization effects are ignored. In order to find the potential of the inflaton as seen by a 4D observer we need to calculate the low-energy effective action for the brane system. The interaction between the branes due to the exchange of closed strings depends on their separation, so we will decompose the coordinates of the two (stacks of) branes into the center-of-mass
and the relative separation.

Assuming that the branes wrap $n$ and $m$ times the volume $V_{\parallel}$, the low-energy effective action is obtained by expanding the DBI action:

$$S_{\text{eff}} = n\tau_p \int d^{p+1}\xi_1 \sqrt{1 + \partial_\mu y_1 \partial^\mu y_1} + m\tau_p \int d^{p+1}\xi_2 \sqrt{1 + \partial_\mu y_2 \partial^\mu y_2}$$

$$\simeq (m + n) \ell_{\parallel} \tau_p + n\tau_p \int d^{p+1}\xi_1 \frac{1}{2} \partial_\mu y_1 \partial^\mu y_1 + m\tau_p \int d^{p+1}\xi_2 \frac{1}{2} \partial_\mu y_2 \partial^\mu y_2$$

(31)

where $\tau_p$ is the brane tension

$$\tau_p = M_s^{p+1}/(2\pi)^p g_s$$

(32)

where $M_s$ is the string scale and $g_s$ the string coupling. The coordinates of the brane in the transverse directions are expressed as:

$$y_1 = y_{CM} + \frac{m}{m + n} y_r$$

(33)

$$y_2 = y_{CM} - \frac{n}{m + n} y_r$$

(34)

and substituting these into the expression for $S_{\text{eff}}$ we obtain:

$$S_{\text{eff}} = \tau_p \frac{mn}{2(m + n)} \int d^{p-3}\xi \int d^4\xi \partial_\mu y_r \partial^\mu y_r = \int d^4\xi \frac{1}{2} \partial_\mu \psi \partial^\mu \psi$$

(35)

The relationship between $y_r$ and $\psi$ is given by:

$$\psi = y_r \sqrt{\frac{mn}{(m + n)^2} \tau_p V_{\parallel}}$$

(36)

where $V_{\parallel} = \int d^{p-3}\xi = \ell^{p-3}_{\parallel}$.

4.1. The D̅D scenario

The D̅D potential is given in Ref.[5] [10], with $d_\perp = 9 - p$ :

$$V (y) = 2\tau_p \ell_{\parallel}^{p-3} - \frac{\kappa^2 \beta \tau_p^2 \ell_{\parallel}^{p-3}}{y^{d_{\perp}-2}}$$

(37)

where

$$\kappa^2 = 8\pi G_{10} = \frac{g_s^2 (2\pi)^7}{2M_s^8}$$

(38)

and $\beta = 2\alpha$ given in Eq.(25). Measured relative to the antipodal point, the position of the anti-brane is given by $z$, which is simply $y$ shifted by half the lattice size. The important pieces of the potential for the slow-roll condition on inflation are the constant term and the quadratic piece due to the jellium term :

$$V (z) = 2\tau_p \ell_{\parallel}^{p-3} - z^2 \frac{\kappa^2 \beta \tau_p^2 \ell_{\parallel}^{p-3}}{d_{\perp} V_{\perp}}$$

(39)

where $V = V_{\perp}$. As pointed out in Ref.[6] [7], the relevant slow-roll parameter $\eta$ is given by:

$$\eta = M_p^2 \frac{V''}{V} \simeq -\frac{2}{d_{\perp}}$$

(40)
where the derivative is taken with respect to the scalar field $\psi$ that appears in the low-energy effective theory, Eq. (36). Since $d_\perp \leq 6$, the slow-roll condition is never satisfied in this case, that is, the branes will collide far too early for any significant inflation to take place.

A priori, it is still possible that the stabilization dynamics of the extra dimensions has some unusual features that suppress $\eta$ and realize the condition required for the viability of the $D \bar{D}$ inflationary scenario. One such possibility utilizes a warped geometry to suppress the inter-brane attractive potential [7].

Here, let us consider

$$\nabla^2 \Phi = \sum_i q_i \delta (r - r_i) - F(r - (L/2, L/2, ..))$$

$$F(z) = \frac{\sum_i q_i}{V} + f(z)$$

where $f(z)$ is multi-periodic and consistency requires it to satisfy

$$\int_M dv f(z) = 0$$

It is not hard to imagine that $f(z)$ originates from the stabilization of the moduli in the extra dimensions. As one can easily see, $D \bar{D}$ inflationary scenario is viable if $F(z)$ vanishes at the antipodal point. To suppress the quadratic term in the inflaton potential, we need to decrease the value of $F(z)$ at the antipodal point so that

$$|\eta| \simeq |2VF(0)/d_\perp| < 1/N_e$$

Suppose, for a torus, measured with respect to the antipodal point,

$$F(z) = \frac{1 - \Pi \cos(k_j z_j)}{V} \simeq \frac{\sum (k_j z_j)^2}{V}$$

so that $F(0) = 0$. This implies that the inflaton potential does not have an anharmonic quadratic term around the antipodal point. Such a $D \bar{D}$ scenario will be able to give enough inflation to render the model viable. (In fact, one may choose $F(z)$ to reduce the contribution of the quartic term to $\eta$ as well.) Among other factors, the form of $F(z)$ depends on the dynamics of moduli stabilization, an issue that is not fully understood [11, 12, 7]. Generically, we should consider $F(0) \ll 1/V$ as a fine-tuning.

Since $\eta$ is generically around 1 for the $D \bar{D}$ system, we need a fine-tuning of 1 in 100 on $F(z)$ to suppress $\eta$. In more realistic constructions of string models, it will be very interesting to see how such a condition can be satisfied.

**4.2. Branes at a Small Angle**

Let us consider the simple scenario [9] where $p = 4$, $d_\perp = 4$, but with different wrappings of the branes. The $X^4, X^5$ torus, has sides $\ell_\parallel$ and $u\ell_\parallel$ and the branes wrap one-cycles on the torus, as shown in Figure 1. These branes are separated in the $X^6, X^7, X^8, X^9$ directions by a distance $y$. 
The Planck mass is given by
\[ M_P^2 = \frac{M_S^8 \ell^2 \parallel u V_\perp}{g_s^2 \pi (2\pi)^6} \] (45)

We start with the example shown in Figure 1 where the wrapping numbers are (1, 1) and (1, −1). We shall consider small \( \theta \), so the angle between the branes is \( \theta \simeq \tan \theta = 2u \).

The constant piece of the potential is given by:
\[ V_0 = \tau_p \ell \parallel \left( 2\sqrt{1 + u^2} - 2 \right) \simeq \tau_p \ell \parallel u^2 \] (46)

and the full potential is:
\[ V(y) = \tau_4 \ell \parallel u^2 - \frac{Q y^2}{2d_\perp V_\perp} - \frac{\beta Q}{2y^2} + \text{harmonic} \] (47)

where \( Q \) includes the contributions from both the NS-NS and RR sectors.

\[ Q = \frac{M_s^2}{2\pi \sin \theta} \left( 1 - \frac{\sin^2 \theta}{2} - \cos \theta \right) \simeq M_s^2 \theta^3 / 16\pi \simeq M_s^2 u^3 / 2\pi \] (48)

where the \( (1 - \sin^2 \theta / 2) \) term comes from the NS-NS sector while the \( - \cos \theta \) term comes from the RR sector. The \( \sin \theta \) in the denominator comes from the length of the brane along the \( \ell \parallel \) direction. \( y \) measures the relative positions of the branes. With the above expressions for the potential and the Planck mass, we evaluate the potential \( V(z) \) at the antipodal point. The relevant slow-roll parameter \( \eta \) there becomes:
\[ \eta = \frac{M_P^2 V''}{V} = -\frac{M_S^8 \ell^2 \parallel u V_\perp}{g_s^2 \pi (2\pi)^6} \frac{Q}{d_\perp V_\perp \ell \parallel u^2} \left( \frac{\partial y}{\partial \psi} \right)^2 = -\frac{4u^2}{d_\perp} = -\frac{\theta^2}{4} \] (49)

For \( u \simeq 1/M_s \ell \parallel \simeq \alpha_{\text{GUT}} \simeq 1/25 \) (a reasonable choice, where \( \alpha_{\text{GUT}} \) is the standard model coupling at the GUT scale), the slow-roll condition is easily satisfied, and it is possible to obtain more than 60 e-foldings of inflation since \( N_e \simeq 1/\eta \).

A few comments are in order. As the jellium term contribution to \( \eta \) is very small, the number of e-foldings is dictated by the first non-zero harmonic term. Here it is the quartic term with strength \( B_4 \) given in Table 2. As we pointed out earlier, the \( B_4 \) obtained here are at least a factor of three smaller than those used in Ref. [10]. Thus the inter-brane potential is weaker than Ref. [10] uses, and so it is much easier to get sufficient inflation than they indicate.
The Madelung term $C_a$ will shift the vacuum energy of the inflaton potential. As is clear from Eq.(47) and Table 2, this shift is positive, that is, it increases the vacuum energy term in the inflaton potential. Its contribution tends to decrease the magnitude of $\eta$.

After collision, the two branes at a small angle reduce to two parallel branes (horizontal in Fig. (1)) with zero vacuum energy. In an orientifold, the tension and RR charge of these two branes are canceled by the presence of orientifold planes. This implies that during inflation, the branes at angle will have a non-zero force with the remaining branes and orientifold planes. For example, the interaction between the $(1,-1)$ D4-brane and a $(1,0)$ D4-brane is proportional to

$$Q_{(1,0)} = -\frac{M_s^2}{2\pi \sin \theta} \left( 1 - \frac{\sin^2(\theta/2)}{2} - \cos(\theta/2) \right) \simeq \frac{Q}{16}$$

and its interaction with orientifold planes is also suppressed by the same factor of 16, but with opposite sign. These contribute a small correction to the interaction between the $(1,-1)$ and the $(1,1)$ branes. One can always place the $(1,1)$-brane, the $(1,-1)$-brane, and the $(1,0)$-branes at initial positions that are favorable to inflation.

### 4.3. Other Branes-at-a-Small-Angle Inflationary Scenarios

Next we consider branes wrapping the long dimension of the torus more than once. Suppose one branes has $(n,1)$ wrapping and the other has $(n,-1)$ wrapping. After collision, we are left with $2n$ parallel branes. In this case the angle between the branes is $\theta = 2u/n$ and the constant piece of the potential is:

$$V_0 = 2\tau_p \ell_{\parallel} \left( \sqrt{n^2 + u^2} - n \right) \simeq \tau_p \ell_{\parallel} \frac{u^2}{n}$$

The branes now intersect in $n^2$ points, so the charge $Q$ is given by:

$$Q = n^2 \frac{M_s^2 u^3}{2\pi n^3} = \frac{M_s^2 u^3}{2\pi n}$$

The relationship between $\psi$ and $y$ becomes:

$$\psi = y \sqrt{\tau_p \ell_{\parallel} \frac{n}{2}}$$

and the slow-roll parameter $\eta$ becomes:

$$\eta \simeq -\frac{u^2}{n}$$

For small $u$ and large $n$, the slow-roll condition is easily satisfied, and it is possible to obtain many more than 60 e-foldings of inflation since $N_e \sim 1/|\eta|$. The other slow-roll parameter $\epsilon = 0$ at the antipodal point. In the actual case, the slow-roll parameters are dictated by the quartic harmonic term where $\epsilon$ is negligibly small. For a reasonable choice $n = 8$, we see that the number of e-foldings that can be obtained for inflation is further improved. In fact, the quadratic term is negligible in this case and the inflaton rolling is dictated by the quartic harmonic term discussed in Ref. [10].
We may also consider branes with dimensionalities other than $p = 4$. In the $p = 6$, $d_\perp = 2$ case the D6-branes span the $X^4, X^5$ torus and wrap different cycles of the $X^6, X^7$ torus. They are localized in the $X^8, X^9$ torus and the interaction potential is logarithmic. The potential and the Planck mass are given by:

$$M_P^2 = \frac{M_S^8 V_{45} \ell_\parallel^3 \theta V_\perp}{g_s^2 \pi (2\pi)^6}$$
$$V(y) = \frac{\tau_6 V_{45} \ell_\parallel \tan^2 \theta}{4} - \frac{Qy^2}{2d_\perp V_\perp} - \beta Q \log(M_s y)$$

where $Q \simeq M_4^4 \theta^3 / 16\pi$. Again, in the small angle approximation, the relevant slow-roll parameter becomes:

$$\eta \simeq -\left(\frac{(2\pi)^2 \theta^2}{(M_s r_\parallel)^2}\right)$$

The slow-roll parameter is again small in this case, and the end of the slow-roll is determined by the attractive logarithmic potential. The region yielding enough slow-roll is reasonably large and there is no need to fine-tune the initial conditions.

It will be interesting to work out the situation of other brane inflationary scenarios [16, 17, 18].

5. Discussions

In the simplified scenario discussed here, the $D\bar{D}$ inflationary scenario is not viable. On the other hand, the branes at a small angle scenario remains a viable model for inflation. Cosmic strings are generically produced towards the end of the brane inflationary epoch. Using the temperature fluctuation in the cosmic microwave background radiation to fix the superstring scale, the cosmic string tension arising from the brane recombination in the branes at a small angle inflation happens to be much bigger than that in the $D\bar{D}$ scenario [10, 20]. If branes-at-a-small-angle scenario is preferred, one consequence is that the cosmic string tension will be on the high side, up to values just below the present experimental bounds. This enhances the hope to test the brane inflationary scenario via the search of signatures of cosmic strings.

As is clear from the analysis, the brane inflationary scenario depends on the dynamics of moduli stabilization. Presumably compactification moduli are stabilized by some string dynamics, or effective potential. The minimum of such an effective potential measures the cosmological constant. So understanding the moduli stabilization problem implies some understanding of the cosmological constant problem, or in a less ambitious framework, moduli stabilization must accommodate the smallness of the observed dark energy [12]. Hopefully, brane inflation in string theory allows us to address this important issue.

We thank Cliff Burgess, Nick Jones, Shamit Kachru, Juan Maldacena, Fernando Quevedo and Gary Shiu for many valuable discussions. This material is based upon work supported by the National Science Foundation under Grant No. PHY-0098631.
Inter-brane Interactions

Appendix A. Ewald’s Method

Here we will detail Ewald’s technique and extend it to suit our purpose. Consider the \(d\)-dimensional torus as an infinite lattice and let \(\Phi(r)\) satisfy

\[
\nabla^2 \Phi(r) = \sum_j -\delta(\Delta r_j) + \frac{1}{V}
\]

(A.1)

\[
\Phi(r) = \Phi_1(r) + \Phi_2(r) + \text{constant}
\]

\[
\nabla^2 \Phi_1(r) = \sum_j -P(\Delta r_j) \exp(-\epsilon^2 \Delta r_j^2) + 1/V
\]

\[
\nabla^2 \Phi_2(r) = \sum_j [-\delta(\Delta r_j) + P(\Delta r_j) \exp(-\epsilon^2 \Delta r_j^2)]
\]

Where \(\Delta r_j = r - r_j\), and \(r_j\) is the \(j\)th lattice vector. Here \(P(\Delta r_j) \exp(-\epsilon^2 \Delta r_j^2)\) can be thought of as a charge distribution inserted at the \(j\)th lattice site, normalized so that

\[
\int_{\mathbb{R}^d} P(r) \exp(-\epsilon^2 r^2) dv = 1
\]

(A.2)

The charge distribution may extend outside the unit cell, but integrating one distribution over all space is identical to integrating all distributions over the unit cell. The effect of adding and subtracting a diffuse charge distribution is to regularize the summation over the lattice. The full potential \(\Phi(r)\) is the sum of \(\Phi_2(r)\) (the real-space sum) and \(\Phi_1(r)\) (the reciprocal lattice sum). For a good choice of \(P(r)\) the sums are separately convergent, and together reproduce \(\Phi(r)\) for the original point-charge distribution. Clearly, \(\Phi(r)\) should be independent of \(P(r)\) and \(\epsilon\), which are chosen to enhance the convergence of the sums. There is a range of choices of \(\epsilon\) (for our calculation \(18 \leq \epsilon \leq 24\)) for which the results of Eq.(A.1) are independent of \(\epsilon\). Other crucial conditions are listed and demonstrated in the next section. This procedure is well tested since it is widely used in solid state physics to compute the Madelung constant.

Appendix A.1. Three dimensions

In three dimensions, a simple choice is to add and subtract a Gaussian charge distribution at each lattice site. This gives \(P(r) = \epsilon^3 \pi^{-3/2}\). Then we have

\[
\Phi_1^{(3)}(r) = \frac{1}{V} \sum_{k_j \neq 0} \frac{\exp(-k_j^2/4\epsilon^2) \exp(i k_j \cdot r)}{k_j^2}
\]

(A.3)

\[
\Phi_2^{(3)}(r) = \frac{1}{4\pi} \sum_{\Delta r_j} \frac{1 - \text{erf}(\epsilon\Delta r_j)}{\Delta r_j}
\]

where \(k_j\) is the \(j\)th reciprocal lattice vector. Note that the following necessary conditions are satisfied:

1) The terms in \(\Phi_1\) converge faster than \(k_j^{-3}\).
2) \(\Phi_1\) reduces to Eq.(23) when \(\epsilon \to \infty\).
3) The terms in \(\Phi_2\) converge faster than \(\Delta r_j^{-3}\).
4) $\Phi_2$ goes to zero as $\epsilon \to \infty$.
5) $\Phi_2$ reduces to $\sum_j \Delta r_j^{-1}$ as $\Delta r_j \to 0$.

Any other choice for $P(r)$ must lead to potentials that satisfy these same limits for the method to work.

In principle there is also an arbitrary constant of integration, but the requirement that $\Phi$ be independent of $\epsilon$ yields

$$\Phi^{(3)}(r) = \Phi_1^{(3)}(r) + \Phi_2^{(3)}(r) - \frac{1}{4V\epsilon^2}$$  (A.4)

The $\epsilon$-dependent term comes from the missing zero mode in $\Phi_1(r)$, as can be seen by differentiating with respect to $\epsilon$. The constant vanishes as $\epsilon \to \infty$, so that Eq.(A.4) recovers Eq.(23). This also satisfies Eq.(8). In practice, a suitable choice of $\epsilon$ increases the convergence of the calculation. In condensed matter a jellium is always used to make the lattice neutral, but a more general term can be easily incorporated into this technique by including the appropriate Fourier expansion in $\Phi_1$.

The numerical potential generated by summing Eq.(A.4) over the lattice can be fit to a potential of the form given in Eq.(26)

$$\Phi^{(3)}(r) = \frac{1}{4\pi r} + \frac{r^2}{6V} + \frac{C^{(3)}}{V^2} + A_4^{(3)}h_4^{(3)}(x_i) + A_6^{(3)}h_6^{(3)}(x_i) + \cdots$$  (A.5)

where the quadratic piece comes from the jellium, and the constant is related to the Madelung energy of the lattice. The data generated by the Ewald method can also be used to fit the potential near the antipodal point. We expect this expansion to be harmonic, except for the $r^2$ term. With coordinates now measured from the center of each cube, this is

$$\Phi^{(3)}_{\text{antipodal}}(z) = \frac{C^{(3)}}{V^2} + \frac{1}{6V} \sum_{i=1}^{3} z_i^2 + B_4^{(3)}h_4^{(3)}(z_i) + B_6^{(3)}h_6^{(3)}(z_i) + \cdots$$  (A.6)

Of course, if the expression near the podal point is known exactly, Eq(A.6) can be obtained algebraically using $z_1 = x_1 - \frac{L}{2}$, etc. Generally, though, the higher order coefficients $A_{10}, A_{12}, \ldots$ will not be small, so it is best to use a numerical fit.

Appendix A.2. Four dimensions

To implement the same procedure in dimensions other than three, we first need to choose a suitable form for $P(r)$. We will need the Laplacian in $d$ dimensions:

$$\nabla^2 \varphi(r) = \frac{1}{r^{d-1}} \frac{\partial}{\partial r} \left[ r^{d-1} \frac{\partial}{\partial r} \varphi \right]$$  (A.7)

The choice for $P(r)$ is guided by the behavior of $\Phi_1$ and $\Phi_2$ in the limits mentioned below Eq.(A.3). In particular, each should converge at least faster than $\frac{1}{r^{d-2}}$ or $\frac{1}{k^{d-2}}$ since the number of points in the sums over the lattice grow as the $(d-1)$ power of $r$ or $k$. Also, $\Phi_2(r)$ should reduce to the usual Coulomb potential as $r \to 0$. Taking $\epsilon \to \infty$, 

which amounts to eliminating the added charge distributions, recovers Eq.(23). In four dimensions, a simple choice is [13]:

$$\Phi^{(4)}(\mathbf{r}) = \frac{1}{4\pi^2} \sum_{\Delta r_j} \exp(-\epsilon^2 \Delta r_j^2) \left( \frac{1}{\Delta r_j^2} + ae^2 + be^4 \Delta r_j^2 + ... \right)$$

(A.8)

The coefficients \((a, b, ..)\) can be adjusted to give a single term in the charge distribution \(P(r) \propto r^n\). The choice of how many higher powers of \(r\) to include in the above equation is a matter of taste and simply changes \(n\). Consider \(a = \frac{1}{4}, b = 0\). Then

$$P^{(4)}(r) = \frac{\epsilon^6 r^2}{2\pi^2}$$

(A.9)

$$\Phi_1^{(4)}(\mathbf{r}) = \frac{1}{V} \sum_{k_j \neq 0} \frac{\exp(-k_j^2/4\epsilon^2) \exp(ik_j \cdot \mathbf{r})}{k_j^2} \left( 1 - \frac{k_j^2}{8\epsilon^2} \right)$$

$$\Phi^{(4)}(\mathbf{r}) = \Phi_1^{(4)}(\mathbf{r}) + \Phi_2^{(4)}(\mathbf{r}) - \frac{3}{8V\epsilon^2}$$

The constant in the last expression above guarantees that \(\Phi\) is independent of \(\epsilon\).

**Appendix A.3. Two dimensions**

In 2 dimensions, a useful form for the real-space part is

$$\Phi_2^{(2)}(\mathbf{r}) = \frac{1}{2\pi} \sum_{\Delta r_j} \int_{\Delta r_j}^{\infty} \exp(-\epsilon^2 r'^2) \left( \frac{1}{r'} + ae^2 r' + be^4 r'^3 + ... \right) dr'$$

(A.10)

To evaluate \(\Phi_2\) in the Ewald sum, note that some convenient expressions for the exponential integral are:

$$\int_{t}^{\infty} \frac{\exp(-u)}{u} du = -\gamma_E - \log(t) - \sum_{n=1}^{\infty} \frac{(-1)^nt^n}{n!}$$

(A.11)

$$= \exp(-t) \frac{1}{1 + t - \frac{1}{3+t} - \frac{1}{8+t} + ...}$$

(A.12)

For the simplest choice \(a = b = 0\), we have

$$P^{(2)}(r) = \frac{\epsilon^2}{\pi}$$

(A.13)

$$\Phi_1^{(2)}(\mathbf{r}) = \frac{1}{V} \sum_{k_j \neq 0} \frac{\exp(-k_j^2/4\epsilon^2) \exp(ik_j \cdot \mathbf{r})}{k_j^2}$$

$$\Phi^{(2)}(\mathbf{r}) = \Phi_1^{(2)}(\mathbf{r}) + \Phi_2^{(2)}(\mathbf{r}) - \frac{1}{4V\epsilon^2}$$

Extending the techniques from the previous three sections, we now have a well-defined way to evaluate the potential numerically for general dimension, charge configuration, and metric.

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