Note On Elliptic Groups of Prime Orders

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Abstract: Let $E$ be an elliptic curve of rank $\text{rk}(E) \geq 0$, and let $E(\mathbb{F}_p)$ be the elliptic group of order $\#E(\mathbb{F}_p) = n$. The number of primes $p \leq x$ such that $n$ is prime is expected to be $\pi(x, E) = \delta(E)x/\log^2 x + o(x/\log^2 x)$, where $\delta(E) \geq 0$ is the density constant. This note proves a lower bound $\pi(x, E) \gg x/\log^2 x$.

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1 Introduction

The applications of elliptic curves in cryptography demands elliptic groups of certain orders $n = \#E(\mathbb{F}_p)$, and certain factorizations of the integers $n$. The extreme cases have groups of $\mathbb{F}_p$-rational points $E(\mathbb{F}_p)$ of prime orders, and small multiples of large primes.

Conjecture 1.1 (Koblitz). Let $E : f(X,Y) = 0$ be an elliptic curve of discriminant $\Delta \neq 0$ defined over the integers $\mathbb{Z}$ which is not $\mathbb{Q}$-isogenous to a curve with nontrivial $\mathbb{Q}$-torsion and does not have CM. Then,

$$\pi(x, E) = \# \{ p \leq x : p \nmid \Delta \text{ and } \#E(\mathbb{F}_p) \text{ is prime} \} = x \log^2 x \prod_{p \geq 2} \left( 1 - \frac{p^2 - p - 1}{(p - 1)^3(p + 1)} \right) + O \left( \frac{x}{\log^3 x} \right),$$

for large $x \geq 1$.

A new refined version of this conjecture, for any elliptic curve over any number field $K$, was recently proposed. The simpler case for the rational number field is stated below.

Conjecture 1.2 ([30]). Let $E$ be an elliptic curve defined over the rational number field $\mathbb{Q}$, and let $t \geq 1$ be a positive integer. Then, there is an explicit constant $C(E, t) > 0$ such that

$$\pi(x, E, t) = \# \{ p \leq x : p \nmid \Delta \text{ and } \#E(\mathbb{F}_p)/t \text{ is prime} \} = C(E, t) \frac{x}{\log^2 x} + O \left( \frac{x}{\log^3 x} \right),$$

for large $x \geq 1$. 
The product expression appearing in the above formula (1) is basically the average density of prime orders, some additional details are given in Section 7. A result for groups of prime orders generated by primitive points is proved here. Let

\[ \pi(x, E) = \#\{p \leq x : p \nmid \Delta \text{ and } d_{E}^{-1} \cdot \#: E(F_{p}) = \text{prime} \}. \] (3)

The parameter \( t = d_{E} \geq 1 \) is a small integer defined in Section 6.

**Theorem 1.1.** Let \( E : f(X, Y) = 0 \) be an elliptic curve over the rational numbers \( \mathbb{Q} \) of rank \( \text{rk}(E(\mathbb{Q})) > 0 \). Then, as \( x \to \infty \),

\[ \pi(x, E) \geq \delta(d_{E}, E) \frac{x}{\log^{2} x} \left( 1 + O \left( \frac{x}{\log x} \right) \right), \] (4)

where \( \delta(d_{E}, E) \) is the density constant.

The proof of this result is split into several parts. The next sections are intermediate results. The proof of Theorem 1.1 is assembled in the penultimate section, and the last section has examples of elliptic curves with infinitely many elliptic groups \( E(F_{p}) \) of prime orders \( n \).

## 2 Representation of the Characteristic Function

### 2.1 Primitive Points Tests

For a prime \( p \geq 2 \), the group of points on an elliptic curve \( E : y^{2} = f(x) \) is denoted by \( E(F_{p}) \). Several definitions and elementary properties of elliptic curves and the \( n \)-division polynomial \( \psi_{n}(x, y) \) are sketched in Chapter 14.

**Definition 2.1.** The order \( \min\{k \in \mathbb{N} : kP = O\} \) of an elliptic point is denoted by \( \text{ord}_{E}(P) \). A point is a primitive point if and only if \( \text{ord}_{E}(P) = n \).

**Lemma 2.1.** If \( E(F_{p}) \) is a cyclic group, then it contains a primitive point \( P \in E(F_{p}) \).

**Proof.** By hypothesis, \( E(F_{p}) \cong \mathbb{Z}/n\mathbb{Z} \), and the additive group \( \mathbb{Z}/n\mathbb{Z} \) contains \( \varphi(n) \geq 1 \) generators (primitive roots).

More generally, there is map into a cyclic group

\[ E(\mathbb{Q})/E(\mathbb{Q})_{\text{tors}} \rightarrow \mathbb{Z}/m\mathbb{Z}, \] (5)

for some \( m = \#: E(F_{p})/d \), with \( d \geq 1 \); and the same result stated in the Lemma holds in the smaller cyclic group \( \mathbb{Z}/m\mathbb{Z} \subset \mathbb{Z}/n\mathbb{Z} \).

**Lemma 2.2.** Let \( \#: E(F_{p}) = n \) and let \( P \in E(F_{p}) \). Then, \( P \) is a primitive point if and only if \( (n/q)P \neq O \) for all prime divisors \( q \mid n \).

Basically, this is the classical Lucas-Lehmer primitive root test applied to the group of points \( E(F_{p}) \). Another primitive point test intrinsic to elliptic curves is the \( n \)-division polynomial test.
Lemma 2.3. (n-Division primitive point test) Let \( \#E(\mathbb{F}_p) = n \) and let \( P \in E(\mathbb{F}_p) \). Then, \( P \) is a primitive point if and only if \( \psi_{n/q}(P) \neq 0 \) for all prime divisors \( q \mid n \).

The basic proof stems from the division polynomial relation

\[
mP = \mathcal{O} \iff \psi_m(P) = 0,
\]

see [28 Proposition 1.25]. The elliptic primitive point test calculations in the penultimate lemma takes place in the set of integer pairs \( \mathbb{Z} \times \mathbb{Z} \), while the calculations for the \( n \)-division polynomial primitive point test takes place over the set of integers \( \mathbb{Z} \). The elementary properties of the \( n \)-division polynomials are discussed in [28], and the periodic property of the \( n \)-division polynomials appears in [26].

2.2 Additive Elliptic Character

The discrete logarithm function, with respect to the fixed primitive point \( T \), maps the group of points into a cyclic group. The diagram below specifies the basic assignments.

\[
E(\mathbb{F}_p) \quad \rightarrow \quad \mathbb{Z}/n\mathbb{Z}, \\
\mathcal{O} \quad \rightarrow \quad \log_T(\mathcal{O}) = 0, \\
T \quad \rightarrow \quad \log_T(T) = 1.
\]

(7)

In view of these information, an important character on the group of \( E(\mathbb{F}_p) \)-rational points can be specified.

Definition 2.2. A nontrivial additive elliptic character \( \chi \) mod \( n \) on the group \( E(\mathbb{F}_p) \) is defined by

\[
\chi(\mathcal{O}) = e^{i\frac{2\pi}{n}\log_T(\mathcal{O})} = 1,
\]

(8)

and

\[
\chi(mT) = e^{i\frac{2\pi}{n}\log_T(mT)} = e^{i2\pi m/n},
\]

(9)

where \( \log_T(mT) = m \) with \( m \in \mathbb{Z} \).

2.3 Divisors Dependent Characteristic Function

A characteristic function for primitive points on elliptic curve is described in the [27, p. 5]; it was used there to derive a primitive point search algorithm.

Lemma 2.4. Let \( E \) be a nonsingular elliptic curve, and let \( E(\mathbb{F}_p) \) be its group of points of cardinality \( n = \#E(\mathbb{F}_p) \). Let \( \chi \) be the additive character of order \( d \) modulo \( d \), and assume \( P = (x_0, y_0) \) is a point on the curve. Then,

\[
\Psi_E(P) = \sum_{d \mid n} \mu(d) \sum_{0 \leq t < d} \chi(tP) = \begin{cases} 
1 & \text{if } \text{ord}_E(P) = n, \\
0 & \text{if } \text{ord}_E(P) \neq n.
\end{cases}
\]

(10)
2.4 Divisors Free Characteristic Function

A new divisors-free representation of the characteristic function of elliptic primitive points is developed here. This representation can overcome some of the limitations of its equivalent in 2.3 in certain applications. The divisors representation of the characteristic function of elliptic primitive points, Lemma 2.4, detects the order \( \text{ord}_E(P) \) of the point \( P \in E(\mathbb{F}_p) \) by means of the divisors of the order \( n = \#E(\mathbb{F}_p) \). In contrast, the divisors-free representation of the characteristic function, Lemma 2.5, detects the order \( \text{ord}_E(P) \) of the point \( P \in E(\mathbb{F}_p) \) by means of the solutions of the equation \( mT - P = \mathcal{O} \), where \( P, T \in E(\mathbb{F}_p) \) are fixed points, \( \mathcal{O} \) is the identity point, and \( m \) is a variable such that \( 0 \leq m \leq n-1 \), and \( \gcd(m,n) = 1 \).

**Lemma 2.5.** Let \( p \geq 2 \) be a prime, and let \( T \) be a primitive point in \( E(\mathbb{F}_p) \). For a nonzero point \( P \in E(\mathbb{F}_p) \) of order \( n \) the following hold: If \( \chi \neq 1 \) is a nonprincipal additive elliptic character of order \( \text{ord}_E \chi = n \), then

\[
\Psi_E(P) = \sum_{\gcd(m,n)=1} \frac{1}{n} \sum_{0 \leq r \leq n-1} \chi((mT - P)r) = \begin{cases} 
1 & \text{if } \text{ord}_E(P) = n, \\
0 & \text{if } \text{ord}_E(P) \neq n,
\end{cases}
\tag{11}
\]

where \( n = \#E(\mathbb{F}_p) \) is the order of the rational group of points.

**Proof.** As the index \( m \geq 1 \) ranges over the integers relatively prime to \( n \), the element \( mT \in E(\mathbb{F}_p) \) ranges over the elliptic primitive points. Ergo, the linear equation

\[
mT - P = \mathcal{O},
\tag{12}
\]

where \( P, T \in E(\mathbb{F}_p) \) are fixed points, \( \mathcal{O} \) is the identity point, and \( m \) is a variable such that \( 0 \leq m \leq n-1 \), and \( \gcd(m,n) = 1 \), has a solution if and only if the fixed point \( P \in E(\mathbb{F}_p) \) is an elliptic primitive point. Next, replace \( \chi(t) = e^{2\pi i t/n} \) to obtain

\[
\Psi_E(P) = \sum_{\gcd(m,n)=1} \frac{1}{n} \sum_{0 \leq r \leq n-1} e^{2\pi \log_T(mT - P)r/n} = \begin{cases} 
1 & \text{if } \text{ord}_E(P) = n, \\
0 & \text{if } \text{ord}_E(P) \neq n,
\end{cases}
\tag{13}
\]

This follows from the geometric series identity \( \sum_{0 \leq k \leq N-1} w^k = (w^N - 1)/(w - 1) \) with \( w \neq 1 \), applied to the inner sum. \qed

3 Primes In Short Intervals

There are many unconditional results for the existence of primes in short intervals \([x, x + y]\) of subsquareroot length \( y \leq x^{1/2} \) for almost all large numbers \( x \geq 1 \). One of the earliest appears to be the Selberg result for \( y = x^{19/77} \) with \( x \leq X \) and \( O \left(X (\log X)^2 \right) \) exceptions, see [24, Theorem 4]. Recent improvements based on different analytic methods are given in [32, 10], and other authors. One of these results, but not the best, has the following claim. This calculation involves the weighted prime indicator function (vonMangoldt), which is defined by

\[
\Lambda(n) = \begin{cases} 
\log n & \text{if } n = p^k, k \geq 1, \\
0 & \text{if } n \neq p^k, k \geq 1,
\end{cases}
\tag{14}
\]

where \( p^k, k \geq 1 \), is a prime power.
Theorem 3.1. (14) Given $\varepsilon > 0$, almost all intervals of the form $[x - x^{1/6+\varepsilon}, x]$ contains primes except for a set of $x \in [X, 2X]$ with measure $O \left( X (\log X)^{-C+1} \right)$, and $C > 1$ constant. Moreover,

$$\sum_{x - y \leq n \leq x} \Lambda(n) > \frac{y}{2}. \quad (15)$$

Proof. The proof is based on zero density theorems, see [10], p. 190. ■

An introduction to zero density theorems and its application to primes in short intervals are given in [16], p. 98, [12], p. 264, et cetera.

The same result works for larger intervals $[x - x^{\theta+\varepsilon}, x]$, where $\theta \in (1/6, 1/2)$ and $(1 - \theta)C < 2$, with exceptions $O \left( X (\log X)^{-C+1} \right), C > 1$.

4 Evaluation Of The Main Term

A lower bound for the main term $M(x)$ in the proof of Theorem 3.1 is evaluated here.

Lemma 4.1. Let $x \geq 1$ be a large number, and let $p \in [x, 2x]$ be prime. Then,

$$\sum_{x \leq p \leq 2x} \frac{1}{4\sqrt{p}} \sum_{p - 2\sqrt{p} \leq n \leq p + 2\sqrt{p}} \frac{\Lambda(n)}{\log n} \cdot \frac{1}{n} \sum_{\gcd(m,n)=1} 1 \gg \frac{x}{2 \log^2 x} \left( 1 + O \left( \frac{x}{\log x} \right) \right). \quad (16)$$

Proof. The main term can be rewritten in the form

$$M(x) = \sum_{x \leq p \leq 2x} \frac{1}{4\sqrt{p}} \sum_{p - 2\sqrt{p} \leq n \leq p + 2\sqrt{p}} \frac{\Lambda(n)}{\log n} \cdot \frac{1}{n} \sum_{\gcd(m,n)=1} 1$$

(17)

The Euler phi function $\varphi(n) = \#\{1 \leq m < n : \gcd(m,n) = 1\}$, and it has the value $\varphi(n)/n = 1 - 1/n$ at a prime argument $n$. Thus, the expression

$$\frac{\Lambda(n) \varphi(n)}{\log n} n = \begin{cases} \frac{\Lambda(n)}{\log n} \left( 1 - \frac{1}{n} \right) & \text{if } n = p^m, m \geq 1, \\ 0 & \text{if } n = p^m, m \geq 1, \end{cases} \quad (18)$$

is supported on the set of prim powers $n = p^m, m \geq 1$. Consequently,

$$M(x) = \sum_{x \leq p \leq 2x} \frac{1}{4\sqrt{p}} \sum_{p - 2\sqrt{p} \leq n \leq p + 2\sqrt{p}} \frac{\Lambda(n)}{\log n} \left( 1 - \frac{1}{n} \right)$$

(19)

since $1 - 1/n \geq 1/2$. Now observe that as the prime $p \in [x, 2x]$ varies, the number of intervals $[p - 2\sqrt{p}, p + 2\sqrt{p}]$ is the same as the number of primes in the interval $[x, 2x]$, namely,

$$\pi(2x) - \pi(x) = \frac{x}{\log x} \left( 1 + O \left( \frac{1}{\log x} \right) \right) > \frac{x}{2 \log x} \quad (20)$$
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for large $x \geq 1$. By Theorem 3.1, the finite sum over the short interval satisfies

$$\sum_{p-2\sqrt{p} \leq n \leq p+2\sqrt{p}} \Lambda(n) > \frac{4\sqrt{p}}{2},$$

with $O(x \log^{-C} x)$ exceptions $p \in [x, 2x]$, where $1 < C < 4$. Take $C > 2$, then, the number of exceptions is small in comparison to the number of intervals $\pi(2x) - \pi(x) > x/2 \log x$ for large $x \geq 1$. Hence, an application of this Theorem yields

$$M(x) \geq \frac{1}{2} \sum_{x \leq p \leq 2x} \frac{1}{4\sqrt{p}} \cdot \frac{1}{\log p} \sum_{p-2\sqrt{p} \leq n \leq p+2\sqrt{p}} \Lambda(n)$$

$$\geq \frac{1}{2} \sum_{x \leq p \leq 2x} \frac{1}{4\sqrt{p}} \cdot \frac{1}{\log p} \cdot \left(\frac{4\sqrt{p}}{2}\right) + O\left(\frac{x}{\log x}\right)$$

$$\geq \frac{1}{4} \frac{1}{\log x} \sum_{x \leq p \leq 2x} 1 + O\left(\frac{x}{\log C x}\right)$$

$$\geq \frac{1}{4} \frac{1}{\log x} \cdot \frac{x}{\log x} \left(1 + O\left(\frac{1}{\log x}\right)\right),$$

where all the errors terms are absorbed into one term. ■

**Remark 4.1.** The exceptional intervals $[p-2\sqrt{p}, p+2\sqrt{p}]$, with $p \in [x, 2x]$, contain fewer primes, that is,

$$\sum_{p-2\sqrt{p} \leq n \leq p+2\sqrt{p}} \Lambda(n) = o(\sqrt{p}).$$

This shortfall is accounted for in the correction term

$$C(x) = \sum_{x \leq p \leq 2x} \frac{1}{4\sqrt{p}} \cdot \frac{1}{\log p} \left(1 - \frac{1}{p}\right) \sum_{p-2\sqrt{p} \leq n \leq p+2\sqrt{p}} \Lambda(n)$$

$$= o\left(\sum_{x \leq p \leq 2x} \frac{1}{4\sqrt{p}} \cdot \frac{1}{\log^2 p} \cdot \sqrt{p}\right)$$

$$= O\left(\frac{x}{\log C x}\right),$$

where $C > 2$, in the previous calculation.

### 5 Estimate For The Error Term

The analysis of an upper bound for the error term $E(x)$, which occurs in the proof of Theorem 1.1, is split into two parts. The first part in Lemma 5.1 is an estimate for the triple inner sum. And the final upper bound is assembled in Lemma 5.2.

**Lemma 5.1.** Let $E$ be a nonsingular elliptic curve over rational number, let $P \in E(\mathbb{Q})$ be a point of infinite order. Let $x \geq 1$ be a large number. For each prime $p \geq 3$, fix a primitive point $T$, and suppose that $P \in E(\mathbb{F}_p)$ is not a primitive point for all primes $p \geq 2$, then
\[
\sum_{p^{-2\sqrt{p}} \leq n \leq p+2\sqrt{p}} \frac{\Lambda(n)}{\log n} \cdot \frac{1}{n} \sum_{\gcd(m,n)=1} \sum_{1 \leq r < n} \chi((mT - P)r) \leq 2.
\] (25)

**Proof.** Let \(\log_T : E(\mathbb{F}_p) \rightarrow \mathbb{Z}_n\) be the discrete logarithm function with respect to the fixed primitive point \(T\), defined by \(\log_T(mT) = m\), \(\log_T(P) = k\), and \(\log_T(O) = 0\). Then, the nontrivial additive character evaluates to
\[
\chi(rmT) = e^{\frac{2\pi i}{n} \log_T(rmT)} = e^{2\pi rm/n},
\] (26)
and
\[
\chi(-rP) = e^{\frac{2\pi i}{n} \log_T(-rP)} = e^{-2\pi rk/n},
\] (27)
respectively. To derive a sharp upper bound, rearrange the inner double sum as a product
\[
T(p) = \sum_{p^{-2\sqrt{p}} \leq n \leq p+2\sqrt{p}} \frac{\Lambda(n)}{\log n} \cdot \frac{1}{n} \sum_{\gcd(m,n)=1} \sum_{1 \leq r < n} \chi((mT - P)r)
\]
\[
= \sum_{p^{-2\sqrt{p}} \leq n \leq p+2\sqrt{p}} \frac{\Lambda(n)}{\log n} \cdot \frac{1}{n} \left( \sum_{1 \leq r < n} \chi(-rP) \sum_{\gcd(m,n)=1} \chi(rmT) \right)
\] (28)
\[
= \sum_{p^{-2\sqrt{p}} \leq n \leq p+2\sqrt{p}} \frac{\Lambda(n)}{\log n} \cdot \frac{1}{n} \left( \sum_{1 \leq r < n} e^{-2\pi rk/n} \right) \left( \sum_{\gcd(m,n)=1} e^{2\pi rm/n} \right).
\]
The hypothesis \(mT - P \neq O\) for \(m \geq 1\) such that \(\gcd(m, n) = 1\) implies that the two inner sums in (28) are complete geometric series, except for the terms for \(m = 0\) and \(r = 0\) respectively. Moreover, since \(n \geq 2\) is a prime, the two geometric sums have the exact evaluations
\[
\sum_{0 < r \leq n-1} e^{-2\pi rk/n} = -1 \quad \text{and} \quad \sum_{\gcd(m,n)=1} e^{2\pi rm/n} = -1
\] (29)
for any \(1 \leq k < n\), and \(1 \leq r < n\) respectively. Therefore, it reduces to
\[
T(p) = \sum_{p^{-2\sqrt{p}} \leq n \leq p+2\sqrt{p}} \frac{\Lambda(n)}{\log n} \cdot \frac{1}{n}(-1)(-1)
\]
\[
\leq \frac{1}{\log p} \sum_{p^{-2\sqrt{p}} \leq n \leq p+2\sqrt{p}} \frac{\Lambda(n)}{n}
\]
\[
\leq \frac{1}{\log p} \sum_{n \leq p+2\sqrt{p}} \frac{\Lambda(n)}{n}
\]
\[
\leq 2,
\] (30)
refer to [23 Theorem 2.7] for additional details. \(\blacksquare\)

**Lemma 5.2.** Let \(E\) be a nonsingular elliptic curve over rational number, let \(P \in E(\mathbb{Q})\) be a point of infinite order. For each large prime \(p \geq 3\), fix a primitive point \(T\), and suppose that \(P \in E(\mathbb{F}_p)\) is not a primitive point for all primes \(p \geq 2\), then
\[
\sum_{x \leq p \leq 2x, p^{-2\sqrt{p}} \leq n \leq p+2\sqrt{p}} \frac{\Lambda(n)}{\log n} \cdot \frac{1}{n} \sum_{\gcd(m,n)=1} \sum_{1 \leq r < n} \chi((mT - P)r) = O(x^{1/2}).
\] (31)
Proof. By assumption $P \in E(\mathbb{F}_p)$ is not a primitive point. Hence, the linear equation $mT - P = 0$ has no solution $m \in \{ m : \gcd(m, n) = 1 \}$. This implies that the discrete logarithm $\log_T (mT - P) \neq 0$, and $\sum_{1 \leq r < n} \chi((mT - P)r) = -1$. This in turns yields

$$|E(x)|$$

$$= \left| \sum_{x \leq p \leq 2x} \frac{1}{4\sqrt{p}} \sum_{p-2\sqrt{p} \leq n \leq p+2\sqrt{p}} \frac{\Lambda(n)}{\log n} \cdot \frac{1}{n} \sum_{\gcd(m, n) = 1} \sum_{1 \leq r < n} \chi((mT - P)r) \right|$$

$$\leq \sum_{x \leq p \leq 2x} \frac{1}{4\sqrt{p}} \sum_{p-2\sqrt{p} \leq n \leq p+2\sqrt{p}} \frac{\Lambda(n)}{\log n} \cdot \frac{1}{n} \sum_{\gcd(m, n) = 1} 1$$

$$\ll \frac{x}{\log^2 x} + O\left( \frac{x}{\log^3 x} \right).$$

Here, the inequality $(1/n) \sum_{\gcd(m, n) = 1} 1 = \varphi(n)/n \leq 1/2$ for all integers $n \geq 1$ was used in the second line. Hence, there is a sharper nontrivial upper bound for the error term. To derive a sharper upper bound, take absolute value, and apply Lemma 5.1 to the inner triple sum to obtain this:

$$|E(x)|$$

$$\leq \sum_{x \leq p \leq 2x} \frac{1}{4\sqrt{p}} \sum_{p-2\sqrt{p} \leq n \leq p+2\sqrt{p}} \frac{\Lambda(n)}{\log n} \cdot \frac{1}{n} \sum_{\gcd(m, n) = 1} \sum_{1 \leq r < n} \chi((mT - P)r)$$

$$\leq \sum_{x \leq p \leq 2x} \frac{1}{4\sqrt{p}} \cdot 2 \log n$$

$$\ll \frac{x}{\log^2 x} + O\left( \frac{x}{\log^3 x} \right).$$

The last line uses the trivial estimate $\sum_{x \leq p \leq 2x} 1 \leq x$. 

6 Elliptic Divisors

The divisor $\text{div}(f) = \gcd(f(\mathbb{Z}))$ of a polynomial $f(x)$ is the greatest common divisor of all its values over the integers, confer [7, p. 395]. Basically, the same concept extents to the setting of elliptic groups of prime orders, but it is significantly more complex.

Definition 6.1. Let $\mathcal{O}_K$ be the ring of integers of a quadratic numbers field $K$. The elliptic divisor is an integer $d_E \geq 1$ defined by

$$d_E = \gcd \left( \{ \#E(\mathbb{F}_p) : p \geq 2 \text{ and } p \text{ splits in } \mathcal{O}_K \} \right).$$

(34)

Considerable works, [8], [22], [11], [14], have gone into determining the elliptic divisors for certain classes of elliptic curves.
Theorem 6.1. ([14, Proposition 1]) The divisor of an elliptic curve $E : y^2 = x^3 + ax + b$ over the rational numbers $\mathbb{Q}$ with complex multiplication by $\mathbb{Q}(\sqrt{D})$ and conductor $N$ satisfies $d_E|24$. The complete list, with $c,m \in \mathbb{Z} - \{0\}$, is the following.

| $D$ | $(a,b)$ | $d_E$ | $D$ | $(a,b)$ | $d_E$ |
|-----|---------|------|-----|---------|------|
| $-3$ | $(0,m)$ | $1$  | $-7$ | $(-140c^2, -784c^3)$ | $4$  |
| $-3$ | $(0,m^2); (0, -27m^2)$ | $3$  | $-8$ | $(-30c^2, -56c^3)$ | $2$  |
| $-3$ | $(0,m^4)$ | $4$  | $-11$ | $(-1056c^2, -13552c^3)$ | $1$  |
| $-3$ | $(0,c^6); (0, 27c^6)$ | $12$ | $-19$ | $(-608c^2, -5776c^3)$ | $1$  |
| $-4$ | $(m,0)$ | $2$  | $-43$ | $(-13760c^2, -621264c^3)$ | $1$  |
| $-4$ | $(m^2,0); (-m^2,0)$ | $4$  | $-67$ | $(-117920c^2, -15585808c^3)$ | $1$  |
| $-4$ | $(-c^4,0); (4c^4,0)$ | $8$  | $-163$ | $(-34790720c^2, -78984748304c^3)$ | $1$  |

7 Densities Expressions

The product expression appearing in Conjecture 1.1, id est,

$$P_0 = \prod_{p \geq 2} \left(1 - \frac{p^2 - p - 1}{(p-1)^3(p+1)}\right) \approx 0.505166168239435774,$$

is the basic the average density of prime orders, it was proved in [17], and very recently other proofs are given in [1, Theorem 1], [15, 21], et alii. The actual density has a slight dependence on the elliptic curve $E$ and the point $P$. The determination of the dependence is classified into several cases depending on the torsion groups $E(\mathbb{Q})_{\text{tors}}$, and other parameters.

Lemma 7.1. ([30, Proposition 4.2]). Let $E : f(x,y) = 0$ be a Serre curve over the rational numbers. Let $D$ be the discriminant of the numbers field $\mathbb{Q}(\sqrt{\Delta})$, where $\Delta$ is the discriminant of any Weierstrass model of $E$ over $\mathbb{Q}$. If $d_E = 1$, then

$$\delta(1,E) = \begin{cases} \left(1 + \prod_{q|D} \frac{1}{q^2 - 2q^2 - q + 3}\right) \prod_{p \geq 2} \left(1 - \frac{p^2 - p - 1}{(p-1)^3(p+1)}\right) & \text{if } D \equiv 1 \text{ mod } 4; \\ \prod_{p \geq 2} \left(1 - \frac{p^2 - p - 1}{(p-1)^3(p+1)}\right) & \text{if } D \equiv 0 \text{ mod } 4. \end{cases}$$

8 Elliptic Brun Constant

Assuming the generalized Riemann hypothesis, the upper bound

$$\pi(x,E,t) \ll \frac{x}{\log^2 x}$$

was proved in [3] and [31]. In addition, the unconditional upper bound

$$\pi(x,E,t) \ll \frac{x}{(\log x)(\log \log x)}$$
for elliptic curves with complex multiplication was proved in [31 Proposition 7]. Later, the same upper bound for elliptic curves without complex multiplication was proved in [31 Theorem 1.3]. An improved version for any elliptic curve over the rational numbers is proved here.

**Lemma 8.1.** For any large number \( x \geq 1 \) and any elliptic curves \( E : f(X, Y) = 0 \) of discriminant \( \Delta \neq 0 \),

\[
\pi(x, E) \leq \frac{12x}{\log^2 x}. \tag{39}
\]

**Proof.** The number of such elliptic primitive primes has the asymptotic formula

\[
\pi(x, E) = \sum_{p \leq x, \text{ord}_E(P) = n \text{ prime}} 1 \leq \sum_{p \leq x} \frac{1}{4 \sqrt{p}} \sum_{p - 2 \sqrt{p} \leq n \leq p + 2 \sqrt{p}} \frac{\Lambda(n)}{\log n} \cdot \Psi_E(P), \tag{40}
\]

where \( \Psi_E(P) \) is the characteristic function of primitive points \( P \in E(\mathbb{Q}) \). This is obtained from the summation of the elliptic primitive primes density function over the interval \([1, x] \), see [54].

Since \( \Psi_E(P) = 0, 1 \), the previous equation has the upper bound

\[
\pi(x, E) \leq \sum_{p \leq x} \frac{1}{4 \sqrt{p}} \sum_{p - 2 \sqrt{p} \leq q \leq p + 2 \sqrt{p}} 1\leq \sum_{p \leq x} \frac{1}{4 \sqrt{p}} \sum_{p - 2 \sqrt{p} \leq q \leq p + 2 \sqrt{p}} 1, \tag{41}
\]

where \( q \) ranges over the primes in the short interval \([p - 2 \sqrt{p}, p + 2 \sqrt{p}]\). The inner sum is estimated using either the explicit formula or Brun-Titchmarsh theorem. The later result states that the number of primes \( p \) in the short interval \([x, x + 4 \sqrt{x}]\) satisfies the inequality

\[
\pi(x + 4 \sqrt{x}) - \pi(x) \leq \frac{3 \cdot 4 \sqrt{x}}{\log x}, \tag{41}
\]

see [12] p. 167], [23, Theorem 3.9], and [29] p. 83], and similar references. Replacing (41) into (41) yields

\[
\sum_{p \leq x} \frac{1}{4 \sqrt{p}} \sum_{p - 2 \sqrt{p} \leq q \leq p + 2 \sqrt{p}} 1 \leq \sum_{p \leq x} \frac{1}{4 \sqrt{p}} \left( \frac{3 \cdot 4 \sqrt{p}}{\log p} \right) \leq 3 \sum_{p \leq x} \frac{1}{\log p} \leq \frac{12}{\log^2 x}. \tag{42}
\]

The last inequality follows by partial summation and the prime number theorem \( \pi(x) = x/\log x + O(x/\log^2 x) \).
This result facilitates the calculations of a new collection of constants associated with elliptic curves. The best known results have established the sum
\[
\sum_{p \leq x} p^{-1} \ll \log \log \log x,
\]
but does not establish the convergence of this series, see [3] and [31]. The convergence of this series is proved below.

**Corollary 8.1.** For any elliptic curve \( E \), the elliptic Brun constant
\[
\sum_{p \geq 2} \frac{1}{p} < \infty
\]
converges.

*Proof.* Use the prime counting measure \( \pi(x, E, t) \leq 6x/\log^2 x + O(x/\log^3 x) \) in Theorem 1.1 to evaluate the infinite sum
\[
\sum_{p \geq 2} \frac{1}{p} = \int_{2}^{\infty} \frac{1}{z} d\pi(z, E, t) = O(1) + \int_{2}^{\infty} \frac{\pi(z, E, t)}{z^2} dz < \infty
\]
as claimed. ■

**Corollary 8.2.** For any elliptic curve \( E \), the elliptic Brun constant
\[
\sum_{p \geq 2} \frac{1}{p} < \infty
\]
converges.

*Proof.* Use the prime counting measure \( \pi(x, E) \leq 6x/\log^2 x + O(x/\log^3 x) \) in Lemma 8.1 to evaluate the infinite sum
\[
\sum_{p \geq 2} \frac{1}{p} = \int_{2}^{\infty} \frac{1}{t} d\pi(t, E) = O(1) + \int_{2}^{\infty} \frac{\pi(t, E)}{t^2} dt < \infty
\]
as claimed. ■

The elliptic Brun constants and the numerical data for the prime orders of a few elliptic curves were compiled. The last example shows the highest density of prime orders. Accordingly, it has the largest constant.
Example 8.1. The nonsingular Bachet elliptic curve $E : y^2 = x^3 + 2$ over the rational numbers has complex multiplication by $\mathbb{Z}[\rho]$, and nonzero rank $\text{rk}(E) = 1$. The data for $p \leq 1000$ with prime orders $n$ are listed on the Table 8.1, and the elliptic Brun constant is

$$\sum_{\substack{p \geq 2 \\#E(\mathbb{F}_p) = \text{prime}}} \frac{1}{p} = .520067922 \ldots$$  (48)

Example 8.2. The nonsingular elliptic curve $E : y^2 = x^3 + 6x - 2$ over the rational numbers has no complex multiplication and zero rank $\text{rk}(E) = 0$. The data for $p \leq 1000$ with prime orders $n$ are listed on the Table 8.2, and the elliptic Brun constant is

$$\sum_{\substack{p \geq 2 \\#E(\mathbb{F}_p) = \text{prime}}} \frac{1}{p} = .186641187 \ldots$$  (49)

Example 8.3. The nonsingular elliptic curve $E : y^2 = x^3 - x$ over the rational numbers has complex multiplication by $\mathbb{Z}[\rho]$, and nonzero rank $\text{rk}(E) = 0$. The data for $p \leq 1000$ with prime orders $n/4$ are listed on the Table 8.3, and the elliptic Brun constant is

$$\sum_{\substack{p \geq 2 \\#E(\mathbb{F}_p)/4 = \text{prime}}} \frac{1}{p} = .549568584 \ldots$$  (50)

Example 8.4. The nonsingular elliptic curve $E : y^2 = x^3 - x$ over the rational numbers has complex multiplication by $\mathbb{Z}[\rho]$, and nonzero rank $\text{rk}(E) = 0$. The data for $p \leq 1000$ with prime orders $n/8$ are listed on the Table 8.4, and the elliptic Brun constant is

$$\sum_{\substack{p \geq 2 \\#E(\mathbb{F}_p)/8 = \text{prime}}} \frac{1}{p} = .2067391731 \ldots$$  (51)
Table 3: Prime Orders \(n/4 = \#E(\mathbb{F}_p)/4\) modulo \(p\) for \(y^2 = x^3 - x\).

| \(p\) | 5  | 7  | 11 | 19 | 43 | 67 | 163 | 211 | 283 | 331 |
|---|---|---|---|---|---|---|---|---|---|---|
| \(n\) | 2  | 2  | 3  | 5  | 11 | 17 | 41 | 53 | 71 | 83 |
| \(p\) | 523 | 547 | 691 | 787 | 907 |   |   |   |   |   |
| \(n\) | 131 | 137 | 173 | 197 | 227 |   |   |   |   |   |

Table 4: Prime Orders \(n/8 = \#E(\mathbb{F}_p)/8\) modulo \(p\) for \(y^2 = x^3 - x\).

| \(p\) | 17 | 23 | 29 | 37 | 53 | 101 | 103 | 109 | 149 | 151 |
|---|---|---|---|---|---|---|---|---|---|---|
| \(n\) | 2  | 3  | 5  | 5  | 5  | 13 | 13 | 13 | 17 | 19 |
| \(p\) | 157 | 277 | 293 | 317 | 389 | 487 | 541 | 631 | 661 | 701 |
| \(n\) | 17 | 37 | 37 | 41 | 37 | 53 | 61 | 73 | 79 | 89 |
| \(p\) | 757 | 773 | 797 | 821 | 823 | 829 | 853 |   |   |   |
| \(n\) | 97 | 101 | 97 | 109 | 103 | 97 | 101 |   |   |   |

**Example 8.5.** The nonsingular Bachet elliptic curve \(E : y^2 = x^3 + 1\) over the rational numbers has complex multiplication by \(\mathbb{Z}[\rho]\), and nonzero rank \(\text{rk}(E) = 1\). The data for \(p \leq 1000\) with prime orders \(n/12\) are listed on the Table 8.5 and the elliptic Brun constant is

\[
\sum_{p \geq 2} \frac{1}{p \#E(\mathbb{F}_p)/12=\text{prime}} = .5495685884\ldots
\]

Table 5: Prime Orders \(n/12 = \#E(\mathbb{F}_p)/12\) modulo \(p\) for \(y^2 = x^3 + 1\).

| \(p\) | 31 | 43 | 59 | 67 | 73 | 79 | 97 | 103 | 131 | 139 |
|---|---|---|---|---|---|---|---|---|---|---|
| \(n\) | 3  | 3  | 5  | 7  | 7  | 7  | 7  | 7  | 11 | 13 |
| \(p\) | 151 | 163 | 181 | 199 | 227 | 241 | 337 | 367 | 379 | 409 |
| \(n\) | 13 | 13 | 13 | 19 | 19 | 19 | 31 | 31 | 31 | 31 |
| \(p\) | 421 | 443 | 463 | 487 | 491 | 523 | 563 | 709 | 751 | 787 |
| \(n\) | 37 | 37 | 37 | 37 | 41 | 43 | 47 | 61 | 67 | 61 |
| \(p\) | 823 | 829 | 859 | 883 | 907 | 947 | 967 | 991 |   |   |
| \(n\) | 73 | 73 | 67 | 73 | 79 | 79 | 79 | 79 |   |   |
9 Prime Orders $n$

The characteristic function for primitive points in the group of points $E(\mathbb{F}_p)$ of an elliptic curve $E : f(X,Y) = 0$ has the representation

$$\Psi_E(P) = \begin{cases} 1 & \text{if } \text{ord}_E(P) = n, \\ 0 & \text{if } \text{ord}_E(P) \neq n. \end{cases}$$

(Theorem 3.2). Let $\langle P \rangle = E(\mathbb{F}_p)$ for at least one large prime $p \leq x_0$, and let $x \geq x_0 \geq 1$ be a large number. Suppose that $P \notin E(\mathbb{Q})_{\text{tors}}$ is not a primitive point of prime order $\text{ord}_E(P) = n$ in $E(\mathbb{F}_p)$ for all primes $p \geq x$. Then, the sum of the elliptic primes measure over the short interval $[x,2x]$ vanishes. Indeed,

$$0 = \sum_{x \leq p \leq 2x} \frac{1}{4\sqrt{p}} \sum_{p-2\sqrt{p} \leq n \leq p+2\sqrt{p}} \Lambda(n) \cdot \Psi_E(P) = \sum_{x \leq p \leq 2x} \frac{1}{4\sqrt{p}} \sum_{p-2\sqrt{p} \leq n \leq p+2\sqrt{p}} \frac{\Lambda(n)}{\log n} \cdot \Psi_E(P).$$

Replacing the characteristic function, Lemma 2.5, and expanding the nonexistence equation (55) yield

$$0 = \sum_{x \leq p \leq 2x} \frac{1}{4\sqrt{p}} \sum_{p-2\sqrt{p} \leq n \leq p+2\sqrt{p}} \Lambda(n) \cdot \Psi_E(P)$$

$$= \delta(d_E,E) \sum_{x \leq p \leq 2x} \frac{1}{4\sqrt{p}} \sum_{p-2\sqrt{p} \leq n \leq p+2\sqrt{p}} \Lambda(n) \cdot \frac{1}{\log n} \sum_{\gcd(m,n)=1} 1 \sum_{1 \leq r \leq n-1} \chi((mT-P)r)$$

$$= \delta(d_E,E)M(x) + E(x),$$

where $\delta(d_E,E) \geq 0$ is a constant depending on both the fixed elliptic curve $E : f(X,Y) = 0$ and the integer divisor $d_E$.

The main term $M(x)$ is determined by a finite sum over the principal character $\chi = 1$, and the error term $E(x)$ is determined by a finite sum over the nontrivial multiplicative characters $\chi \neq 1$. 

Note On Elliptic Groups of Prime Orders

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Applying Lemma 4.1 to the main term, and Lemma 5.2 to the error term yield
\[
\sum_{x \leq p \leq 2x} \frac{1}{4\sqrt{p}} \sum_{p-2\sqrt{p} \leq n \leq p+2\sqrt{p}} \frac{\Lambda(n)}{\log n} \cdot \Psi_E(P) = \delta(d_E, E)M(x) + E(x) \quad (57)
\]
\[
\approx \delta(d_E, E) \frac{x}{\log^2 x} \left( 1 + O\left( \frac{x}{\log x} \right) \right) + O\left( x^{1/2} \right)
\]
\[
\approx \delta(d_E, E) \frac{x}{\log^2 x} \left( 1 + O\left( \frac{x}{\log x} \right) \right).
\]
But, if \( \delta(d_E, E) > 0 \), the expression
\[
\sum_{x \leq p \leq 2x} \frac{1}{4\sqrt{p}} \sum_{p-2\sqrt{p} \leq n \leq p+2\sqrt{p}} \frac{\Lambda(n)}{\log n} \cdot \Psi_E(P) \approx \delta(d_E, E) \frac{x}{\log^2 x} \left( 1 + O\left( \frac{x}{\log x} \right) \right)
\]
\[
> 0,
\]
contradicts the hypothesis \((55)\) for all large numbers \( x \geq x_0 \). Ergo, there are infinitely many primes \( p \geq x \) such that a fixed elliptic curve of rank \( \text{rk}(E) > 0 \) with a primitive point \( P \) of infinite order, for which the corresponding groups \( E(\mathbb{F}_p) \) have prime orders. Lastly, the number of such elliptic primitive primes has the asymptotic formula
\[
\pi(x, E) = \sum_{\text{ord}_E(P) = n \text{ prime}} \frac{1}{\log n}
= \sum_{p \leq x} \frac{1}{4\sqrt{p}} \sum_{p-2\sqrt{p} \leq n \leq p+2\sqrt{p}} \frac{\Lambda(n)}{\log n} \cdot \Psi_E(P)
\geq \delta(d_E, E) \frac{x}{\log^2 x} \left( 1 + O\left( \frac{x}{\log x} \right) \right),
\]
which is obtained from the summation of the elliptic primitive primes density function over the interval \([1, x]\).

\[\blacklozenge\]

10 Examples Of Elliptic Curves

The densities of several elliptic curves have been computed by several authors. Extensive calculations for some specific densities are given in [30].

**Example 10.1.** The nonsingular Bachet elliptic curve \( E : y^2 = x^3 + 2 \) over the rational numbers has complex multiplication by \( \mathbb{Z}[\rho] \), and nonzero rank \( \text{rk}(E) = 1 \). It is listed as 1728.n4 in [20]. The numerical data shows that \#\( E(\mathbb{F}_p) = n \) is prime for at least one prime, see Table 8.1. Hence, by Theorem 1.1, the corresponding group of \( \mathbb{F}_p \)-rational points \#\( E(\mathbb{F}_p) \) has prime orders \( n = \# E(\mathbb{F}_p) \) for infinitely many primes \( p \geq 3 \).

Since \( \Delta = -2^6 \cdot 3^3 \), the discriminant of the quadratic field \( \mathbb{Q}(\sqrt{\Delta}) \) is \( D = -3 \). Moreover, the integer divisor \( d_E = 1 \) since \#\( E(\mathbb{F}_p) \) is prime for at least one prime. Thus, applying Lemma 7.1 gives the natural density
\[
\delta(1, E) = \frac{10}{9} P_0 \approx 0.5612957424882619712979385 \ldots,
\]

(60)
In the table below, we see the data for the elliptic curve $y^2 = x^3 + 2$.

| Invariant            | Value                                                                 |
|----------------------|----------------------------------------------------------------------|
| Discriminant         | $\Delta = -16(4a^3 + 27b^2) = -1728$                                 |
| Conductor            | $N = 1728$                                                            |
| $j$-Invariant $j(E)$ | $(-48b^3/\Delta = 0$                                                |
| Rank                 | $\text{rk}(E) = 1$                                                   |
| Special $L$-Value    | $L(E, 1) \approx 2.82785747365$                                     |
| Regulator            | $R = .754576$                                                        |
| Real Period          | $\Omega = 5.24411510858$                                            |
| Torsion Group        | $E(\mathbb{Q})_{\text{tors}} \approx \{O\}$                         |
| Integral Points      | $E(\mathbb{Z}) = \{O, (-1, 1); (-1, 1)\}$                           |
| Rational Group       | $E(\mathbb{Q}) = \mathbb{Z}$                                        |
| Endomorphisms Group  | $\text{End}(E) = \mathbb{Z}[(1 + \sqrt{-3})/2], \text{CM}$          |
| Integer Divisor      | $d_E = 1$                                                            |

Table 6: Data for $y^2 = x^3 + 2$.

The predicted number of elliptic primes $p \nmid 6N$ such that $\#E(\mathbb{F}_p)$ is prime has the asymptotic formula

$$\pi(x, E) = \delta(1, E) \int_2^x \frac{1}{\log(t + 1)} \frac{dt}{t}. \quad (61)$$

A lower bound for the counting function is

$$\pi(x, E) \geq \delta(1, E) \frac{x}{\log^3 x} \left(1 + O\left(\frac{1}{\log x}\right)\right), \quad (62)$$

see Theorem [1,1].

The associated weight $k = 2$ cusp form, and $L$-function are

$$f(s) = \sum_{n \geq 1} a_n q^n = q - q^7 - 5q^{13} + 7q^{19} + \cdots, \quad (63)$$

and

$$L(s) = \sum_{n \geq 1} \frac{a_n}{n^s}$$

$$= \prod_{p \mid N} \left(1 - \frac{a_p}{p^s}\right)^{-1} \prod_{p \nmid N} \left(1 - \frac{a_p}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1}$$

$$= 1 - \frac{1}{7^s} - \frac{5}{13^s} + \frac{7}{19^s} + \cdots, \quad (64)$$

where $q = e^{i2\pi}$, respectively. The coefficients are generated using $a_p = p + 1 - \#E(\mathbb{F}_p)$, and the formulas

1. $a_{pq} = a_pa_q$ if $\gcd(p, q) = 1$;
2. $a_{pn+1} = a_{pn}a_p - pa_{pn-1}$ if $n \geq 2$. 


The corresponding functional equation is
\[ \Lambda(s) = \left( \frac{\sqrt{N}}{2\pi} \right)^s \Gamma(s)L(s) \quad \text{and} \quad \Lambda(s) = \Lambda(2-s), \] (65)
where \( N = 1728 \), see [18, p. 80].

**Example 10.2.** The nonsingular elliptic curve \( E : y^2 = x^3 + 6x - 2 \) over the rational numbers has no complex multiplication and zero rank, it is listed as 1728.w1 in [20]. The numerical data shows that \( \#E(\mathbb{F}_p) = n \) is prime for at least one prime, see Table 8.2. Hence, by Theorem 1.1, the corresponding group of \( \mathbb{F}_p \)-rational points \( \#E(\mathbb{F}_p) \) has prime orders \( n = \#E(\mathbb{F}_p) \) for infinitely many primes \( p \geq 3 \).

Since \( \Delta = -2^6 \cdot 3^5 \), the discriminant of the quadratic field \( \mathbb{Q}(\sqrt{\Delta}) \) is \( D = -3 \). Moreover, the integer divisor \( d_E = 1 \) since \( \#E(\mathbb{F}_p) \) is prime for at least one prime. Thus, applying Lemma 7.1 gives the natural density
\[ \delta(1,E) = \frac{10}{9} P_0 \approx 0.5612957424882619712979385 \ldots, \] (66)
The predicted number of elliptic primes \( p \nmid 6N \) such that \( \#E(\mathbb{F}_p) \) is prime has the asymptotic formula
\[ \pi(x,E) = \delta(1,E) \int_2^x \frac{1}{\log(t+1)} \frac{dt}{t}. \] (67)

A table for the prime counting function \( \pi(1,E) \), for \( 2 \times 10^7 \leq x \leq 10^9 \), and other information on this elliptic curve appears in [30].

A lower bound for the counting function is
\[ \pi(x,E) \geq \delta(1,E) \frac{x}{\log^3 x} \left( 1 + O \left( \frac{1}{\log x} \right) \right), \] (68)
see Theorem 1.1.

The associated weight \( k = 2 \) cusp form, and \( L \)-function are
\[ f(s) = \sum_{n \geq 1} a_n q^n = q + 2q^5 + q^7 + 2q^{11} - q^{13} - 6q^{17} + 5q^{19} + \cdots, \] (69)
and
\[ L(s) = \sum_{n \geq 1} \frac{a_n}{n^s} \]
\[ = \prod_{p|N} \left( 1 - \frac{a_p}{p^s} \right)^{-1} \prod_{p \nmid N} \left( 1 - \frac{a_p}{p^s} + \frac{1}{p^{2s}-1} \right)^{-1} \]
\[ = 1 + \frac{2}{5^s} + \frac{1}{7^s} + \frac{2}{11^s} - \frac{1}{13^s} - \frac{6}{17^s} + \frac{5}{19^s} + \cdots, \] (70)
where \( q = e^{i2\pi} \), respectively. The coefficients are generated using \( a_p = p + 1 - \#E(\mathbb{F}_p) \), and the formulas
1. \( a_{pq} = a_p a_q \) if \( \gcd(p,q) = 1 \);
### Table 7: Data for $y^2 = x^3 + 6x - 2$.

| Invariant             | Value                                                                 |
|-----------------------|----------------------------------------------------------------------|
| Discriminant          | $\Delta = -16(4a^3 + 27b^2) = -2^6 \cdot 3^5$                       |
| Conductor             | $N = 2^6 \cdot 3^3$                                                  |
| j-Invariant           | $j(E) = (-48b)^3 / \Delta = 2^9 \cdot 3$                            |
| Rank                  | $\text{rk}(E) = 0$                                                  |
| Special $L$-Value     | $L(E, 1) \approx 2.24402797314$                                     |
| Regulator             | $R = 1$                                                             |
| Real Period           | $\Omega = 2.2440797314$                                             |
| Torsion Group         | $E(\mathbb{Q})_{\text{tors}} = \{O\}$                               |
| Integral Points       | $E(\mathbb{Z}) = \{O\}$                                            |
| Rational Group        | $E(\mathbb{Q}) = \{O\}$                                            |
| Endomorphisms Group   | $\text{End}(E) = \mathbb{Z}$, nonCM                                 |
| Integer Divisor       | $d_E = 1$                                                           |

2. $a_{p^{n+1}} = a_p a_p^p - pa_p^{p-1}$ if $n \geq 2$.

The corresponding functional equation is

$$\Lambda(s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L(s) \quad \text{and} \quad \Lambda(s) = \Lambda(2 - s),$$  \hspace{1cm} (71)

where $N = 1728$, see [18, p. 80].

**Example 10.3.** The nonsingular elliptic curve $E : y^2 = x^3 - x$ over the rational numbers has complex multiplication by $\mathbb{Z}[i]$, and zero rank, it is listed as 32.a3 in [20]. The numerical data shows that $\# E(\mathbb{F}_p)/8 = n$ is prime for at least one prime, see (??). Hence, by Theorem 1.1, the corresponding group of $\mathbb{F}_p$-rational points $\# E(\mathbb{F}_p)$ has prime orders $n = \# E(\mathbb{F}_p)$ for infinitely many primes $p \geq 3$.

Since $\Delta = -2^6$, the discriminant of the quadratic field $\mathbb{Q}(\sqrt{\Delta})$ is $D = -4$. Moreover, the integer divisor $d_E = 8$ since $\# E(\mathbb{F}_p)/8$ is prime for at least one prime, see Table 8.4. Thus, Lemma 7.1 is not applicable. The natural density

$$\delta(8, E) = \frac{1}{2} \prod_{p \geq 3} \left(1 - \chi(p) \frac{p^2 - p - 1}{(p - \chi(p))(p - 1)^2}\right) \approx 0.5336675447 \ldots,$$  \hspace{1cm} (72)

where $\chi(n) = (-1)^{(n-1)/2}$, is computed in [30, Lemma 7.1]. The predicted number of elliptic primes $p \nmid 6N$ such that $\# E(\mathbb{F}_p)$ is prime has the asymptotic formula

$$\pi(x, E) = \delta(8, E) \int_9^x \frac{1}{\log(t + 1) - \log 8 \log t} \, dt.$$  \hspace{1cm} (73)

A table for the prime counting function $\pi(x, E)$, for $2 \times 10^7 \leq x \leq 10^9$, and other information on this elliptic curve appears in [30].
A lower bound for the counting function is

$$
\pi(x, E) \geq \delta(8, E) \frac{x}{\log^3 x} \left( 1 + O \left( \frac{1}{\log x} \right) \right),
$$

see Theorem 1.1.

The associated weight $k = 2$ cusp form, and $L$-function are

$$
f(s) = \sum_{n \geq 1} a_n q^n = q + 2q^5 + q^7 + 2q^{11} - q^{13} - 6q^{17} + 5q^{19} + \cdots,
$$

and

$$
L(s) = \sum_{n \geq 1} \frac{a_n}{n^s} = \prod_{p \mid N} \left( 1 - \frac{a_p}{p^s} \right)^{-1} \prod_{p \nmid N} \left( 1 - \frac{a_p}{p^s} + \frac{1}{p^{2s-1}} \right)^{-1}
$$

$$
= 1 + \frac{2}{5^s} + \frac{1}{7^s} + \frac{2}{11^s} - \frac{1}{13^s} - \frac{6}{17^s} + \frac{5}{19^s} + \cdots,
$$

where $q = e^{i2\pi}$, respectively. The coefficients are generated using $a_p = p + 1 - \#E(\mathbb{F}_p)$, and the formulas

1. $a_{pq} = a_p a_q$ if $\gcd(p, q) = 1$;
2. $a_{p^{n+1}} = a_p a_p - p a_{p^{n-1}}$ if $n \geq 2$.

The corresponding functional equation is

$$
\Lambda(s) = \left( \frac{\sqrt{N}}{2\pi} \right)^s \Gamma(s) L(s) \quad \text{and} \quad \Lambda(s) = \Lambda(2 - s),
$$

where $N = 1728$, see [18, p. 80].
11 Exercises

Problem 1. Assume random elliptic curve $E : f(x, y) = 0$ has no CM, $P$ is a point of infinite order. Is the integer divisor $d_E < \infty$ bounded? This parameter is bounded for CM elliptic curves, in fact $d_E < 24$, reference: [14, Proposition 1].

Problem 2. Assume the elliptic curve $E : f(x, y) = 0$ has no CM, $P$ is a point of infinite order, and the density $\delta(1, E) > 0$. What is the least prime $p \nmid 6N$ such that the integer divisor $d_E = 1$? Reference: [5, Corollary 1.2].

Problem 3. Fix an elliptic curve $E : y^2 = x^3 + ax + b$ of rank $\text{rk}(E) > 0$, and CM; and $P$ is a point of infinite order. Let the integer divisor $d_E = 4$. Assume the densities $\delta(1, E) > 0$, $\delta(2, E) > 0$, and $\delta(4, E) > 0$ are defined. What is the arithmetic relationship between the densities?

Problem 4. Fix an elliptic curve $E : y^2 = x^3 + ax + b$ of rank $\text{rk}(E) > 0$, and CM; and $P$ is a point of infinite order. Does $\#E(\mathbb{F}_p) = n$ prime for at least one large prime $p \geq 2$ implies that the integer divisor $d_E = 1$?
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