SOME QUESTIONS ON GLOBAL DISTINCTION FOR $\text{SL}_n$

U. K. ANANDAVARDHANAN AND NADIR MATRINGE

ABSTRACT. Let $E/F$ be a quadratic extension of number fields and let $\pi$ be an $\text{SL}_n(\mathbb{A}_F)$-distinguished cuspidal automorphic representation of $\text{SL}_n(\mathbb{A}_E)$. Using an unfolding argument, we prove that an element of the $L$-packet of $\pi$ is distinguished if and only if it is $\psi$-generic for a non-degenerate character $\psi$ of $N_n(\mathbb{A}_E)$ trivial on $N_n(E + \mathbb{A}_F)$, where $N_n$ is the group of unipotent upper triangular matrices of $\text{SL}_n$. We then use this result to analyze the non-vanishing of the period integral on different realizations of a distinguished cuspidal automorphic representation of $\text{SL}_n(\mathbb{A}_E)$ with multiplicity $> 1$, and show that in general some canonical copies of a distinguished representation inside different $L$-packets can have vanishing period. We also construct examples of everywhere locally distinguished representations of $\text{SL}_n(\mathbb{A}_E)$ the $L$-packets of which do not contain any distinguished representation.

1. Introduction

Distinguished representations are the central objects of study in the relative Langlands program. They are defined and studied both locally over a $p$-adic field and globally over a number field. Roughly speaking there is a group $G$, $p$-adic or adelic, and distinction for an irreducible representation, say $\pi$, automorphic in the adelic case, with respect to a subgroup $H$, amounts to the existence of a non-trivial $H$-invariant linear form on the space of $\pi$. In the number field case, the interest is not in any invariant linear form but in a specific invariant linear form, called the $H$-period integral, so distinction with respect to $H$ amounts to the question of whether or not the $H$-period integral is non-vanishing on the space of automorphic forms in $\pi$.

One of the most well understood situations of this phenomenon is the case of a quadratic extension of fields $E/F$, local or global, with $G = \text{GL}_n(E)$ or $\text{GL}_n(\mathbb{A}_E)$ and $H = \text{GL}_n(F)$ or $\text{GL}_n(\mathbb{A}_F)$, where $\mathbb{A}_L$ denotes the adele ring of a number field $L$. Much is known in this situation: distinction is closely related to the theory of the Asai $L$-function, there is multiplicity one for the space of invariant forms in the local picture and as a consequence the period integral is factorizable in the global case, a local-global principle can be formulated which is true under a mild hypothesis, to mention a few important results. If a cuspidal representation is distinguished then each of its local components is distinguished and the local-global question asks whether or not the converse is true; i.e., if each local component of a cuspidal representation $\pi$ is distinguished then is $\pi$ distinguished? For $\text{GL}(n)$ for $E/F$, this is true if $\pi$ has a discrete series component at least at one place.

One specific simplifying aspect of the case of $\text{GL}(n)$ is that there are no non-trivial $L$-packets. However, in the closely related case of $\text{SL}(n)$, which is the group under
consideration in this paper, this is not the case as was shown by Labesse and Langlands in their seminal work on \(L\)-indistinguishability \([LL79]\). Thus, the representation theory of \(SL(n)\) is more complicated but there is a way to understand it via the representation theory of \(GL(n)\). The complications on the one side together with the fact that there is an approach via a well understood picture make it an interesting test case to investigate the various aspects of the general theory of distinguished representations. The initial papers here were for \(SL(2)\) \([AP03, AP06]\) and they brought to light several new features: local multiplicity one can fail even for supercuspidal representations, the period integral is not factorizable in general, the local-global principle can fail even at the level of \(L\)-packets, that is to say there are cuspidal representations \(\pi\) of \(SL_2(\mathcal{A}_E)\) which are locally distinguished everywhere but not only that \(\pi\) is not distinguished in fact no member of its \(L\)-packet is distinguished. The local-global question in this context was investigated further in \([AP13]\) which further clarified parts of \([AP06]\) and in particular \([AP13]\) proved several positive results about the local-global question.

In probing distinction inside an \(L\)-packet, the key finding of \([AP03, AP06]\) was that distinction inside an \(L\)-packet that contains at least one distinguished representation can be characterized in terms of Whittaker models; i.e., distinguished representations in such “distinguished” \(L\)-packets are precisely the ones which admit a Whittaker model with respect to a non-trivial character of \(E/F\) or \(\mathcal{A}_E/(E + \mathcal{A}_F)\) as the case may be. We may also remark here that a crucial role in the global papers on \(SL(2)\) \([AP06, AP13]\) is played by “multiplicity one for \(SL(2)\)”, i.e., a cuspidal representation of \(SL_2(\mathcal{A}_L)\) appears exactly once in the space of cusp forms on \(SL_2(\mathcal{A}_L)\) \([Ram00]\).

With more progress in the \(GL(n)\) theory it was natural to expect more progress in the \(SL(n)\) theory. Thus, with a finer understanding of distinction for \(GL(n)\), came the recent work of the first named author with Dipendra Prasad on \((SL_n(E), SL_n(F))\) over \(p\)-adic fields \([AP18]\) which generalized \([AP03]\) from \(n = 2\) to any \(n\). However, the global \(SL(n)\) case, already quite involved for \(n = 2\) as can be seen from \([AP06, AP13]\), is considerably more difficult for several reasons one of which is that “multiplicity one” is not true for \(SL(n)\) for \(n \geq 3\) as was first shown in the famous work of D. Blasius \([Bla94, Lap99]\).

In this paper, we prove the most basic result about characterizing distinction inside a distinguished \(L\)-packet in terms of Whittaker models, thus generalizing \([AP06, Theorem 4.2]\) from \(n = 2\) to any \(n\). This is the global analogue for cuspidal representations of the corresponding \(p\)-adic result for generic \(L\)-packets in \([AP18]\).

We may mention here in passing that somewhat surprisingly even the finite field analogue of this characterization of distinction in a generic \(L\)-packet turned out to be non-trivial and is settled only fairly recently \([AM18]\). However, if \(\pi\) is a cuspidal representation, then a uniform proof of this characterization for both \(p\)-adic and finite fields can be given in an elementary manner \([AP18, Proposition 4.2 & Remark 4]\). The key point is to prove the result for \(n = 2\) first and then to deduce the general case by an inductive argument which is facilitated by Clifford theory as in \([DP19, Proposition 1]\).

This is the strategy that we follow in this paper as well to get the desired characterization. Thus we create an inductive set up, this time by an unfolding method, and
make use of the base case for \( n = 2 \) which is known by [AP06, Theorem 4.2]. It should be mentioned here that the method that we follow to create this inductive set up is very parallel to that employed in [DP19, Section 5], as was brought to our attention by Dipendra Prasad (see also Remark 3).

As mentioned earlier a cuspidal representation may appear in the space of cusp forms with multiplicity more than 1 for \( \text{SL}(n) \) for \( n \geq 3 \) [Bla94, Lap99]. This phenomenon leads to certain natural questions regarding the non-vanishing of the period integral on different realizations of a given cuspidal representation in the space of cusp forms. We show that there are cuspidal representations \( \pi \) of \( \text{SL}_n(\mathbb{A}_E) \) of multiplicity more than 1 which are \( \text{SL}_n(\mathbb{A}_F) \)-distinguished with realizations in the space of cusp forms on \( \text{SL}_n(\mathbb{A}_E) \) with vanishing \( \text{SL}_n(\mathbb{A}_F) \)-period integral. In fact we do this very generally whenever \( n \) is odd and the cuspidal multiplicity \( m(\pi) \) of \( \pi \) is \( \geq 2 \) (cf. §5.4). One key ingredient in achieving this is the fact that \( (\text{SL}_n(K), \text{SL}_n(k)) \) for odd \( n \) is a Gelfand pair for a quadratic extension \( K/k \) of local fields [Ana05, AP18].

We then consider the canonical realizations inside the L-packets obtained from restricting the cusp forms on \( \text{GL}_n(\mathbb{A}_E) \). We exhibit two types of examples of cuspidal representations of \( \text{SL}_n(\mathbb{A}_E) \) of multiplicity more than 1 which are \( \text{SL}_n(\mathbb{A}_F) \)-distinguished (cf. §5.4, §5.5). In one set of examples, \( F \) is any number field and \( E/F \) is chosen so that the period integral vanishes on some of the \( m(\pi) \) many canonical realizations but not on all the canonical realizations. In the second set of examples, \( F \) is any number field and \( E/F \) is chosen so that the period integral does not vanish in any of the \( m(\pi) \) many canonical realizations inside the L-packets. The key ingredient in all these constructions is the explicit nature of the examples of cuspidal representations of high multiplicity in [Bla94, Lap98, Lap99]. In these examples, we also need to make a crucial use of the main result of this paper (cf. Theorem 3.1). These examples partially answer certain questions (cf. §5.1) posed to us by Raphaël Beuzart-Plessis and Dipendra Prasad.

The methods we employ in order to produce examples of cuspidal distinguished representations of high multiplicity with different canonical realizations admitting or not admitting a non-vanishing period integral can be tweaked to also show that the local-global principle fails at the level of L-packets for \( \text{SL}(n) \) (cf. §5.6). As has already been mentioned such a phenomenon was observed for \( \text{SL}(2) \) as well by an explicit construction in [AP06, Theorem 8.2]. The construction in [AP06] is somewhat involved whereas our analogous examples in §5.6 are conceptually simpler however the methods here are tailor-made for \( n \geq 3 \) and will not give new and easy examples for \( \text{SL}(2) \).

All our constructions of specific examples, of cuspidal representations of \( \text{SL}_n(\mathbb{A}_E) \) of high multiplicity which are \( \text{SL}_n(\mathbb{A}_F) \)-distinguished, that highlight a variety of different phenomena, owe a lot to the specific examples of Blasius of high cuspidal multiplicity making use of the representation theory of the Heisenberg group \( H \) and in particular the fact that different Heisenberg representations are such that their value at any element of the group are conjugate in \( \text{PGL}_n(\mathbb{C}) \) but they are projectively inequivalent [Bla94, Section 1.1]. To give a rough idea, in fact Blasius does this more generally, in [Bla94], high multiplicity examples on \( \text{SL}_n(\mathbb{A}_E) \) are produced by basically transferring this representation theoretic information about Heisenberg groups to Galois
groups of $L/E$ for suitable number fields, via Shafarevich’s theorem, and then to the automorphic side via the strong Artin conjecture which is a theorem in the situation at hand, as $\text{Gal}(L/E) \simeq H$ is nilpotent, due to Arthur-Clozel [AC89, Theorem 7.1]. For our examples, we start with an involution on $H$ and consider the corresponding semi-direct product $H \rtimes \mathbb{Z}/2$, which cuts out extensions $L \supset E \supset F$, and essentially we play with these involutions at our disposal to construct a variety of examples answering several natural question about distinction for the pair $(\text{SL}_n(\mathbb{A}_E), \text{SL}_n(\mathbb{A}_F))$.

2. Distinguished L-packets of $\text{SL}_n$

Let $E/F$ be a quadratic extension of number fields and let $\mathbb{A}_F$ and $\mathbb{A}_E$ denote the corresponding rings of adeles. Let $\pi$ be a (smooth) cuspidal automorphic representation of $\text{SL}_n(\mathbb{A}_E)$. We say that $\pi$ is $\text{SL}_n(\mathbb{A}_F)$-distinguished if there is a realization $V_\pi$ of $\pi$ in the space $A_0^\infty(\text{SL}_n(E)/\text{SL}_n(\mathbb{A}_E))$ of smooth cusp forms on $\text{SL}_n(\mathbb{A}_E)$ (cf. [Cog04, p. 26]) and a cusp form $\phi$ in $V_\pi$ such that the absolutely convergent period integral

$$
P_{\text{SL}_n(\mathbb{A}_F)}(\phi) = \int_{\text{SL}_n(F)/\text{SL}_n(\mathbb{A}_F)} \phi(h)dh$$

is non-zero. Note that

$$|P_{\text{SL}_n(\mathbb{A}_F)}(\phi)| \leq \text{vol}(\text{SL}_n(F)/\text{SL}_n(\mathbb{A}_F)) ||\phi||_\infty$$

so $P_{\text{SL}_n(\mathbb{A}_F)}$ is in fact continuous for the Fréchet topology on $A_0^\infty(\text{SL}_n(E)/\text{SL}_n(\mathbb{A}_E))$.

It is known that there is a cuspidal automorphic representation $\tilde{\pi}$ of $\text{GL}_n(\mathbb{A}_E)$ such that $\pi$ is isomorphic to a summand of $\text{res}(\tilde{\pi})$, where res is the restrictions of cusp forms (i.e., of functions) on $\text{GL}_n(\mathbb{A}_E)$ to $\text{SL}_n(\mathbb{A}_E)$ (cf. [HS12, Chapter 4]). We denote by $L(\tilde{\pi})$ the set of irreducible summands of $\text{res}(\tilde{\pi})$, and call it the L-packet defined by $\tilde{\pi}$: we emphasize on the fact that the elements of $L(\tilde{\pi})$ are submodules of $A_0^\infty(\text{SL}_n(E)/\text{SL}_n(\mathbb{A}_E))$, and not their isomorphism classes. A cuspidal automorphic representation appears with multiplicity one in $L(\tilde{\pi})$ and moreover for $\tilde{\pi}_1$ and $\tilde{\pi}_2$ two cuspidal automorphic representations of $\text{GL}_n(\mathbb{A}_E)$, the equality $L(\tilde{\pi}_1) = L(\tilde{\pi}_2)$ holds if and only if $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are twists of each other by a Hecke character (see [Bla94, Lap99]).

The aim of section is to characterize cuspidal L-packets of $\text{SL}_n(\mathbb{A}_E)$ containing distinguished representations, whereas in the next section we will characterize distinguished cuspidal representations inside such an L-packet.

The characterization of distinguished L-packets is already there in [AP06], we recall it now.

**Lemma 2.1.** Let $\tilde{\pi}$ be a cuspidal automorphic representation of $\text{GL}_n(\mathbb{A}_E)$ which is is distinguished by $(\text{GL}_n(\mathbb{A}_F), \eta)$ for a Hecke character $\eta$ of $\mathbb{A}_F^\times$, then $L(\tilde{\pi})$ contains a representation of $\text{SL}_n(\mathbb{A}_E)$ on which $P_{\text{SL}_n(\mathbb{A}_F)}$ does not vanish. Conversely, assume that $F$ and $n$ are such that the Grunwald-Wang theorem is applicable and suppose that $\tilde{\pi}$ is a cuspidal automorphic representation of $\text{GL}_n(\mathbb{A}_E)$ such that $L(\tilde{\pi})$ contains a representation of $\text{SL}_n(\mathbb{A}_E)$ on which $P_{\text{SL}_n(\mathbb{A}_F)}$ does not vanish, then $\tilde{\pi}$ is $(\text{GL}_n(\mathbb{A}_F), \eta)$-distinguished for a Hecke character $\eta$ of $\mathbb{A}_F^\times$. 

Proof. The first assertion is a straightforward generalization of [AP06] Proposition 3.2, whereas the second assertion is again an immediate generalization of [AP06] Proposition 3.4 and its proof.

Remark 1. Note that the Grunwald-Wang theorem holds in particular for any $F$ when $n$ is odd and this will be the situation in the later sections of this paper where we will need to apply Lemma 2.1.

Definition 1. We call a cuspidal $L$-packet containing a representation on which $\mathcal{P}_{\text{SL}_n}(A_F)$ does not vanish a distinguished $L$-packet of $\text{SL}_n(A_E)$.

Under the conditions of the Grunwald-Wang theorem one can translate the definition of distinguished $L$-packets into an even more convenient one which does not refer to distinction anymore. By the work of Jacquet and Shalika ([JJS81]) on one hand, and that of Flicker (and Flicker-Zinoviev) on the other hand ([Fli88], [FZ95]), the following result is known.

Theorem 2.2. Denote by $\omega_{E/F}$ the quadratic character attached to $E/F$ by global class field theory, and let $\pi$ be a cuspidal automorphic representation of $\text{GL}_n(A_E)$. Then $\pi$ is conjugate self-dual, i.e., $\pi^\vee \simeq \pi^{\sigma}$ if and only if $\pi$ is either distinguished or $\omega_{E/F}$-distinguished (and in fact not both together).

Proof. Let $\pi_1$, $\pi_2$ and $\pi_3$ be cuspidal automorphic representations of $\text{GL}_n(A_E)$. By the aforementioned references, the partial Rankin-Selberg $L^S(s, \pi_1, \pi_2)$ has a pole at $s = 1$, which is necessary simple, if and only if $\pi_2 \simeq \pi_1^\vee$, whereas the partial Asai $L$-function $L^S_{\text{As}}(s, \pi_3)$ has a pole (necessarily simple) at $s = 1$ if and only if $\pi_3$ is $\text{GL}_n(A_F)$-distinguished. The result now follows from the equality

$$L^S(s, \pi_1, \pi_1^\vee) = L^S_{\text{As}}(s, \pi_1) L^S_{\text{As}}(s, \omega \otimes \pi_1)$$

where $\omega$ is any Hecke character of $A_E^\times$ extending $\omega_{E/F}$. □

In particular it implies that if $L(\pi)$ is a distinguished $L$-packet then $\pi^\vee \simeq \mu \otimes \pi^{\sigma}$ for $\mu$ a Hecke character of $A_E^\times$ which factors through the norm $N_{E/F}$. Indeed if $\pi$ is $\eta$-distinguished for some Hecke character $\eta$ of $A_E^\times$ then extending $\eta$ to a Hecke character of $A_E^\times$ and applying Theorem 2.2 to $\eta^{-1} \otimes \pi$, we get that $\pi^\vee \simeq \mu \otimes \pi^{\sigma}$ for $\mu = \eta^{-1} \circ N_{E/F}$. Conversely, if $\pi^\vee \simeq \mu \otimes \pi^{\sigma}$ for a Hecke character $\mu$ of $A_E^\times$ which factors through the norm $N_{E/F}$, setting $\mu = \eta^{-1} \circ N_{E/F}$, one gets again from Theorem 2.2 that $\pi$ is $\eta$-distinguished or $\eta \omega_{E/F}$-distinguished. The following proposition summarizes this discussion.

Proposition 2.3. Let $\pi$ be a cuspidal automorphic representation of $\text{GL}_n(A_E)$. If $\pi^\vee \simeq \mu \otimes \pi^{\sigma}$ for a Hecke character $\mu$ of $A_E^\times$ which factors through the norm $N_{E/F}$, then $L(\pi)$ is distinguished. Conversely, assume that $F$ and $n$ are such that the Grunwald-Wang theorem is applicable, then if an $L$-packet $L(\pi)$ is distinguished, there is a Hecke $\mu$ character of $A_E^\times$ which factors through the norm $N_{E/F}$ such that $\pi^\vee \simeq \mu \otimes \pi^{\sigma}$.

3. Distinction inside $L$-packets

In this section we primarily solve the problem of characterizing members of distinguished $L$-packets of $\text{SL}_n(A_E)$ on which $\mathcal{P}_{\text{SL}_n}(A_F)$ does not vanish. This is Theorem
and the answer is in terms of Whittaker models. Then we look at the analogous problem where global distinction is replaced by local distinction at every place and in this case we show that if an L-packet contains an everywhere locally distinguished representation, then for every non-degenerate character $\psi$ of $N_n(\mathbb{A}_E)$ which is trivial on $N_n(E + \mathbb{A}_F)$, the $\psi$-generic representation in the L-packet has that property (cf. Proposition 3.6).

3.1. Non-vanishing of the period inside distinguished L-packets. Here we characterize members of distinguished L-packets of $\text{SL}_n(\mathbb{A}_E)$ with non-vanishing $\text{SL}_n(\mathbb{A}_F)$-period. This is done in terms of Whittaker periods.

If $\psi$ is a non-degenerate character of $N_n(\mathbb{A}_E)$ trivial on $N_n(E)$, we say that $\pi$ is $\psi$-generic if there is a realization $V_\pi$ of $\pi$ in $\mathcal{A}_0^\infty(\text{SL}_n(E) \setminus \text{SL}_n(\mathbb{A}_E))$ and a cusp form $\phi$ in $V_\pi$ such that

$$P_{N_n(\mathbb{A}_E),\psi}(\phi) = \int_{N_n(E) \setminus N_n(\mathbb{A}_E)} \phi(n)\psi^{-1}(n)dn$$

is non-zero.

Take $\pi \in L(\tilde{\pi})$, then the Whittaker period $P_{N_n(\mathbb{A}_E),\psi}$ does not vanish on $\pi$ for some non-degenerate character $\psi$ of $N_n(\mathbb{A}_E)$ trivial on $N_n(E)$. Moreover it is a well-known consequence of local multiplicity one for Whittaker models of $\text{GL}_n$ that for fixed $\psi$ there is only one element of $L(\tilde{\pi})$ on which $P_{N_n(\mathbb{A}_E),\psi}$ does not vanish. Indeed, the representation $\tilde{\pi}$ of $\text{GL}_n(\mathbb{A}_E)$ is generic with respect to any non-degenerate character of $N(\mathbb{A}_E)/N(E)$ and observe that multiplicity one for local Whittaker models of $\text{GL}_n$ implies multiplicity one inside an L-packet [Lap99, Remark 2.1 (3)].

We denote by $P_n$ the mirabolic subgroup of $\text{GL}_n$ (matrices with last row $(0,\ldots,0,1)$), by $U_n$ its unipotent radical, and by $P_{n}^1$ its intersection with $\text{SL}_n$, and thus $P_{n}^1 = \text{SL}_{n-1}.U_n$. We set $N_n$ to be the unipotent radical of the Borel subgroup of upper triangular matrices in $\text{GL}_n$. We denote by $Q_n$ the proper parabolic subgroup of $\text{SL}_n$ containing $P_{n}^1$. We now characterize distinction inside distinguished L-packets of $\text{SL}_n(\mathbb{A}_E)$, thus generalizing [AP06, Theorem 4.2].

**Theorem 3.1.** Let $L(\tilde{\pi})$ be a distinguished L-packet of $\text{SL}_n(\mathbb{A}_E)$. Then the period integral $P_{\text{SL}_n(\mathbb{A}_F)}$ does not vanish on $\pi \in L(\tilde{\pi})$ if and only if there exists a non-degenerate character $\psi$ of $N_n(\mathbb{A}_E)$ trivial on $N_n(E + \mathbb{A}_F)$ such that $P_{N_n(\mathbb{A}_E),\psi}$ does not vanish on $\pi$.

Before we move on to the proof of Theorem 3.1, let us state a simple but very useful consequence of it.

**Corollary 3.2.** Let $\pi$ be a cuspidal automorphic $\text{SL}_n(\mathbb{A}_F)$-distinguished representation of $\text{SL}_n(\mathbb{A}_E)$ and let $L(\tilde{\pi}')$ be a distinguished L-packet of $\text{SL}_n(\mathbb{A}_E)$ containing an isomorphic copy of $\pi$. Then the period $P_{\text{SL}_n(\mathbb{A}_F)}$ does not vanish on the unique representation in $L(\tilde{\pi}')$ isomorphic to $\pi$.

**Proof.** Call $\pi'$ the isomorphic copy of $\pi$ in $L(\tilde{\pi}')$. Thanks to Theorem 3.1 $\pi$ is $\psi$-generic for $\psi$ a distinguished non-degenerate character of $N_n(\mathbb{A}_E)$ trivial on $N_n(E + \mathbb{A}_F)$ and therefore $\pi'$ is locally $\psi_v$-generic for every place $v$ of $F$. By Theorem 3.1 again, the $\psi$-generic cuspidal representation $\pi''$ in $L(\tilde{\pi}')$ is also $\text{SL}_n(\mathbb{A}_F)$-distinguished. But thanks to multiplicity one of local Whittaker models, two locally $\psi$-generic cuspidal representations.
representations in the same L-packet are equal, hence $\pi' = \pi''$, and we deduce that $\mathcal{P}_{\text{SL}_n}(\mathbb{A}_F)$ does not vanish on $\pi'$.

The rest of this section is devoted to the proof of Theorem 3.1; we first prove one direction. The following lemma is a generalization of [AP06, Lemma 4.3], but the proof in [AP06, Lemma 4.3] does not generalize to this case. For $n \geq 3$, we set $R_n = \{\text{diag}(x, I_{n-2}, x^{-1}), x \in \mathbb{G}_n\}$, so $Q_n$ is the semi-direct product $P_n^1.R_n$.

**Lemma 3.3.** Take $n \geq 3$. Let $\phi$ be a cusp form on $\text{SL}_n(\mathbb{A}_E)$ such that

$$\int_{\text{SL}_n(F)\backslash \text{SL}_n(\mathbb{A}_F)} \phi(h)dh \neq 0,$$

then there is $h_0 \in \text{SL}_n(\mathbb{A}_F)$ (and in fact in $R_n(\mathbb{A}_F)$) such that

$$\int_{P_n(F)\backslash P_n(\mathbb{A}_F)} \phi(hh_0)dh \neq 0,$$

where this integral is absolutely convergent.

**Proof.** According to [SV17, Section 18.2], there is $s \in \mathbb{C}$ such that for $\Re(s)$ large enough, the integral $\int_{Q_n(F)\backslash Q_n(\mathbb{A}_F)} \phi(p)\delta^s_{Q_n}(p)dp$ is absolutely convergent. Moreover it has meromorphic continuation, and there is a meromorphic function $r(s)$ with $r(0) = 0$ such that $r(s)\int_{Q_n(F)\backslash Q_n(\mathbb{A}_F)} \phi(h)\delta^s_{Q_n}(h)dh$ tends to $\int_{\text{SL}_n(F)\backslash \text{SL}_n(\mathbb{A}_F)} \phi(h)dh \neq 0$ when $s \to 0$. In particular there is an $s \in \mathbb{R}$ large enough in the realm of absolute convergence such that

$$0 \neq \int_{Q_n(F)\backslash Q_n(\mathbb{A}_F)} \phi(p)\delta^s_{Q_n}(p)dp = \int_{P_n(F)\backslash P_n(\mathbb{A}_F)} \int_{R_n(F)\backslash R_n(\mathbb{A}_F)} \phi(pa)\delta^s_{Q_n}(a)dpa$$

hence there is an $a \in R_n(\mathbb{A}_F)$ such that $\delta^s_{Q_n}(a)\int_{P_n(F)\backslash P_n(\mathbb{A}_F)} \phi(pa)dpa \neq 0$ and the result follows. \hfill $\square$

**Remark 2.** A result similar to Lemma 3.3 is [DP19, Proposition 8] where it is proved via unfolding an Eisenstein series $E(h, s)$ on $\text{SL}_n(\mathbb{A}_F)$ and using that

$$\text{Res}_{s=1} \left( \int_{\text{SL}_n(F)\backslash \text{SL}_n(\mathbb{A}_F)} \phi(h)E(h, s)dh \right) = \mathcal{P}_{\text{SL}_n(\mathbb{A}_F)}(\phi),$$

a trick that [DP19] attributes to [AGR93]. A straightforward adaptation of the proof of [DP19, Proposition 8] can also be used to prove Lemma 3.3. Though our proof here looks much shorter where we appeal to [SV17, Section 18.2], however the core of [SV17, Proposition 18.2.1] is the equality (18.6) and what follows in *loc. cit.*, and it relies on the exact same considerations on Eisenstein series as in [DP19, Proposition 8]. Hence the proof above is in fact essentially the same as that of [DP19, Proposition 8] but the main part of the argument is contained in the statement of [SV17, Section 18.2]. Note that [SV17, Section 18.2] is done in general for any semisimple group.

Now we can prove one implication of Theorem 3.1. We set

$$N_{k,n} = U_k \ldots U_n < N_n = N_{2,n}.$$ 

For $\psi_{k,n}$ a character of $N_{k,n}(\mathbb{A}_E)$ and $\phi$ a cusp form on $\text{SL}_n(\mathbb{A}_E)$, we set

$$\phi_{\psi_{k,n}}(x) = \int_{N_{k,n}(E)\backslash N_{k,n}(\mathbb{A}_E)} \phi(nx)\psi_{k,n}^{-1}(n)dn.$$
for \( x \in \text{SL}_n(\mathbb{A}_E) \). When \( k = 2 \) and \( \psi_{2,n} \) is non-degenerate, we write \( \phi_{\psi_{2,n}} = W_{\phi,\psi_{2,n}} \).

The reader familiar with it will recognize what is often called the unfolding method in the following proof (see [JS90, Section 6] for a famous and difficult instance of this technique).

**Proposition 3.4.** Let \( \phi \) be a cusp form on \( \text{SL}_n(\mathbb{A}_E) \) such that

\[
\int_{\text{SL}_{n-1}(F)} \phi(h) dh \neq 0,
\]

then there is a non-degenerate character \( \psi \) of \( N_n(\mathbb{A}_E)/N_n(E + \mathbb{A}_F) \) such that \( W_{\phi,\psi} \) does not vanish on \( \text{SL}_n(\mathbb{A}_E) \). In particular thanks to Lemma 3.3 if \( \pi \) is an \( \text{SL}_n(\mathbb{A}_F) \)-distinguished cuspidal automorphic representation of \( \text{SL}_n(\mathbb{A}_E) \), then it is \( \psi \)-generic for a non-degenerate character \( \psi \) of \( N_n(\mathbb{A}_E)/N_n(E + \mathbb{A}_F) \).

**Proof.** We do an induction on \( n \), the case \( n = 2 \) being part of the proof of [AP06, Theorem 4.2]. Hence we suppose \( n \geq 3 \). By hypothesis we have

\[
\int_{\text{SL}_{n-1}(F) \backslash \text{SL}_{n-1}(\mathbb{A}_F)} \int_{U_{n}(F) \backslash U_{n}(\mathbb{A}_F)} \phi(uh)dudh \neq 0.
\]

Set

\[
\phi^{U_{n,F}}(x) = \int_{U_{n}(F) \backslash U_{n}(\mathbb{A}_F)} \phi(ux) du
\]

for \( x \in \text{SL}_{n-1}(\mathbb{A}_F) \). By Poisson formula for \( (F \backslash \mathbb{A}_F)^{n-1} \subset (E \backslash \mathbb{A}_E)^{n-1} \), we have

\[
\phi^{U_{n,F}}(x) = \sum_{\psi_{n,n} \in U_n(\mathbb{A}_E)/U_n(E + \mathbb{A}_F)} \phi_{\psi_{n,n}}(x),
\]

which is in turn equal to

\[
\sum_{\psi_{n,n} \in U_n(\mathbb{A}_E)/U_n(E + \mathbb{A}_F) \backslash \{1\} } \phi_{\psi_{n,n}}(x)
\]

by cuspidality of \( \phi \). The convergence of the series is absolute (and can be shown to be uniform for \( x \) in compact subsets of \( \text{SL}_{n-1}(\mathbb{A}_F) \) but we will not use it). For fixed non-degenerate \( \psi_{n,n}^0 \) of \( U_{n}(\mathbb{A}_E)/U_{n}(E + \mathbb{A}_F) \), one has

\[
\phi^{U_{n,F}}(x) = \sum_{\psi_{n,n} \in U_n(\mathbb{A}_E)/U_n(E + \mathbb{A}_F)} \phi_{\psi_{n,n}}(x) = \sum_{\gamma \in P^1_{n-1}(F) \backslash \text{SL}_{n-1}(F)} \phi_{\psi_{n,n}}(\gamma x)
\]

because as \( n \geq 3 \), the group \( \text{SL}_{n-1}(F) \) acts transitively on the set of non-trivial characters of \( \text{U}_{n}(\mathbb{A}_E) \) trivial on \( U_{n}(E + \mathbb{A}_F) \), and the stabilizer of \( \psi_{n,n}^0 \) is \( P^1_{n-1}(F) \). Hence

\[
0 \neq \int_{\text{SL}_{n-1}(F) \backslash \text{SL}_{n-1}(\mathbb{A}_F)} \int_{U_{n}(F) \backslash U_{n}(\mathbb{A}_F)} \phi(uh)dudh = \int_{P^1_{n-1}(F) \backslash \text{SL}_{n-1}(\mathbb{A}_F)} \phi_{\psi_{n,n}}(h) dh
\]

where the right hand side is absolutely convergent (by Fubini). Now

\[
\int_{P^1_{n-1}(F) \backslash \text{SL}_{n-1}(\mathbb{A}_F)} \phi_{\psi_{n,n}}(h) dh = \int_{P^1_{n-1}(\mathbb{A}_F) \backslash \text{SL}_{n-1}(\mathbb{A}_F)} \int_{P^1_{n-1}(F) \backslash P^1_{n-1}(\mathbb{A}_F)} \phi_{\psi_{n,n}}(hx) dh dx,
\]

and this implies that

\[
\int_{P^1_{n-1}(F) \backslash P^1_{n-1}(\mathbb{A}_F)} \phi_{\psi_{n,n}}(hh_0) dh \neq 0
\]
for some $h_0 \in \text{SL}_{n-1}(\mathbb{A}_F)$. The function $\phi_0 = (\rho(h_0)\phi)_{\psi_{0,n}} = \rho(h_0)\phi_{\psi_{0,n}}$ is a cusp form on $\text{SL}_{n-1}(\mathbb{A}_F)$, and we can apply our induction hypothesis to it, to conclude that $W_{\phi_0,\psi'}$ is non-zero on $\text{SL}_{n-1}(\mathbb{A}_E)$ for some non-degenerate character $\psi'$ of $N_{n-1}(\mathbb{A}_E)$ trivial on $N_{n-1}(\mathbb{A}_F + E)$. Setting $\psi = \psi' \otimes \psi_{0,n} : n'.u \mapsto \psi'(n')\psi_{0,n}(u)$, one checks that by definition:

$$W_{\phi_0,\psi'}(x) = W_{\rho(h_0)\phi,\psi}(x) = W_{\phi,\psi}(xh_0)$$

for $x \in \text{SL}_{n-1}(\mathbb{A}_E)$. The result follows.

**Remark 3.** As mentioned in §1 our strategy in proving Proposition 3.4 is to have an inductive set up to reduce the proof to the case of $n = 2$. In the finite field case as well as in the $p$-adic field case such an inductive machinery can be set up via Clifford theory [DP19, Proposition 1] and this is carried out in [AP18, Proposition 4.2 & Remark 4]. A similar approach in the number field case can be carried out as well by making use of the global analogue of [DP19, Proposition 1] which is [DP19, Proposition 6]. In fact [DP19, Proposition 6] is stated more generally and our inductive set up would follow by taking $H = \text{SL}_{n-1}(\mathbb{A}_F)$ and $A = \frac{U_n(\mathbb{A}_F)\mathbb{A}_F}{U_n(E + \mathbb{A}_F)}$, in the notations of [DP19, Proposition 6].

To end the proof of Theorem 3.1 it now suffices to prove the following implication, which is part of the proof of [AP06, Theorem 4.2], and which we repeat.

**Lemma 3.5.** Let $L(\overline{\pi})$ be a distinguished $L$-packet of $\text{SL}_n(\mathbb{A}_F)$. If $\pi \in L(\overline{\pi})$ is $\psi$-generic with respect to a non-degenerate character $\psi$ of $N_{n}(\mathbb{A}_E)$ trivial on $N_{n}(E + \mathbb{A}_F)$, then $\mathcal{P}_{\text{SL}_n(\mathbb{A}_F)}$ does not vanish on $\pi$.

**Proof.** By definition there is $\pi' \in L(\overline{\pi})$ such that $\mathcal{P}_{\text{SL}_n(\mathbb{A}_F)}$ does not vanish on it. By Proposition 3.4, the representation $\pi'$ is $\psi'$-generic for a non-degenerate character $\psi'$ of $N_{n}(\mathbb{A}_E)$ trivial on $N_{n}(E + \mathbb{A}_F)$. Denoting by $T_n$ the diagonal torus of $\text{GL}_n$, there is $t \in T_n(F)$ such that $\psi = \psi'^t$ where $\psi'^t(n) = \psi'(t^{-1}nt)$. Now the representation $\pi'^t$ given by $\pi'^t(g) = \pi'(t^{-1}gt)$ appears in $L(\overline{\pi})$ and is $\psi$-generic. We deduce that $\pi = \pi'^t$, and the result follows since $t \in \text{GL}_n(F)$. 

**Remark 4.** We cease the occasion to fill a small gap in the literature, which uses the ideas of this paper: namely the unfolding of the Asai $L$-function. The proofs given in [Fli88, p. 303] and [Zha14, p. 558] are a bit quick. Here we add the details to the proof of [Flicker, 2 Proposition, p. 303]. The transition between the second and third line of the equality there relies on the following step: take $\phi$ a cusp form on $\text{GL}_n(\mathbb{A}_F)$, then

$$\int_{\text{N}_n(F) \setminus \text{N}_n(\mathbb{A}_F)} \phi(n)dn = \sum_{\gamma \in \text{N}_n(F) \setminus \text{P}_n(F)} W_{\phi,\psi}(\gamma),$$

where both the ”integrals” are absolutely convergent and $\psi$ is a non-degenerate character of $N_n(\mathbb{A}_F)$ trivial on $N_n(E + \mathbb{A}_F)$. We use the same notations as in Proposition 3.4, and denote by $\psi_{n,n}'$ the restriction of $\psi$ to $U_n(\mathbb{A}_E)$.

Let us write

$$\int_{\text{N}_n(F) \setminus \text{N}_n(\mathbb{A}_F)} \phi(n)dn = \int_{\text{N}_{n-1}(F) \setminus \text{N}_{n-1}(\mathbb{A}_F)} \phi_{U_n,F}(n)dn.$$
By induction applied to the cusp form $\phi^{U_n,F}$ on $\text{GL}_{n-1}(\mathbb{A}_F)$, we have

$$\int_{N_{n-1}(F)\backslash N_{n-1}(\mathbb{A}_F)} \phi^{U_n,F}(n)dn = \sum_{\gamma' \in N_{n-1}(F)\backslash P_{n-1}(F)} \phi^{U_n,F}(\gamma').$$

Now replace $\phi^{U_n,F}(\gamma')$ by $\sum_{\gamma \in P_{n-1}(F)\backslash U_n(F)} \int_{P_{n-1}(F)\backslash U_n(F)} \phi^{\psi}_{n,n}(\gamma \gamma')$ this time (still by Poisson formula and because $P_n(F)$ also acts transitively on the set of non-trivial characters of $U_n(\mathbb{A}_F)$), the stabilizer of $\psi^0_{n,n}$ being $P_{n-1}(F)\backslash U_n(F))$. We get

$$\int_{N_n(F)\backslash N_n(\mathbb{A}_F)} \phi(n)dn = \sum_{\gamma' \in N_{n-1}(F)\backslash P_{n-1}(F)} \sum_{\gamma \in P_{n-1}(F)\backslash U_n(F)} W_{\phi_{n,n},\psi_{n,n}}(\gamma \gamma')dn$$

$$= \sum_{\gamma' \in N_{n-1}(F)\backslash P_{n-1}(F)} \sum_{\gamma \in P_{n-1}(F)\backslash U_n(F)} W_{\phi,\psi}(\gamma \gamma')$$

$$= \sum_{\gamma \in N_{n-1}(F)\backslash U_n(F)} W_{\phi,\psi}(\gamma),$$

which is what we wanted.

3.2. **Locally distinguished representations inside $L$-packets.** Here we prove a result of a similar flavor concerning the cuspidal automorphic representations of $\text{SL}_n(\mathbb{A}_E)$ which are $\text{SL}_n(F_v)$-distinguished at every place $v$ of $F$ in the $L$-packet determined by a cuspidal automorphic representations of $\text{GL}_n(\mathbb{A}_E)$ which is $\text{SL}_n(F_v)$-distinguished at every place $v$ of $F$. Our main tool is the local analogue of Theorem 3.1 proved in [AP03] and [AP18].

**Proposition 3.6.** Let $\pi$ be a cuspidal automorphic representations of $\text{GL}_n(\mathbb{A}_E)$ and suppose that $\pi_v$ is $\text{SL}_n(F_v)$-distinguished for every place $v$ of $F$. Then there is a cuspidal automorphic representation $\pi \in L(\pi)$ such that $\pi_v$ is $\text{SL}_n(F_v)$-distinguished for every place $v$ of $F$. In fact all $\psi$-generic representations in $L(\pi)$ for a non-degenerate character $\psi$ of $N_n(\mathbb{A}_E)$ trivial on $N_n(E + \mathbb{A}_F)$ are locally distinguished at every place $v$ of $F$.

**Proof.** By [AP03] Section 3] and [AP18] Section 4], we know that if $v$ is a non-split finite place of $F$, the $\text{SL}_n(F_v)$-distinguished representations occurring in $L(\pi_v)$ (the set of irreducible components of $(\pi_v)_{|\text{SL}_n(E_v)}$ are exactly the $\psi_v$-generic ones, for $\psi_v$ a non-degenerate character of $N_n(E_v)$ trivial on $N_n(F_v)$. On the other hand, if $v$ is split (finite or not), this result is still true and easy to prove. Finally if $E_v/F_v = C/\mathbb{R}$, then the result is again true because $L(\pi_v) = \{(\pi_v)_{|\text{SL}_n(E_v)}\}$. In particular fixing a global non-degenerate character $\psi$ of $N_n(\mathbb{A}_E)$ trivial on $N_n(E + \mathbb{A}_F)$, we see that for each place $v$ of $F$, the $\psi_v$-generic representation $\pi_v'$ in $L(\pi_v)$ is $\text{SL}_n(F_v)$-distinguished. Now consider $\pi$ the $\psi$-generic representation in $L(\pi)$, then $\pi_v'$ and $\pi_v$ are both $\psi_v$-generic in $L(\pi_v)$ and hence they are isomorphic. If $v$ is a non-degenerate character of $N_n(\mathbb{A}_E)$ trivial $N_n(E + \mathbb{A}_F)$, then the $\psi$-generic representation of $L(\pi)$ is locally distinguished.

This result will be used in Section 5.6 to exhibit everywhere locally distinguished representations of $\text{SL}_n(\mathbb{A}_E)$ appearing in no distinguished $L$-packet.

**Remark 5.** One could ask conversely if an element of $L(\pi)$ as above which is locally distinguished everywhere is $\psi$-generic for a non-degenerate character $\psi$ of $N_n(\mathbb{A}_E)$
trivial $\eta_n (E + \mathbb{A}_F)$. We do not know whether or not this is true, however note that an affirmative answer to this question together with Theorem 3.1 would imply that inside a distinguished L-packet, the locally distinguished members are exactly those on which $P_{\text{SL}_n (\mathbb{A}_F)}$ does not vanish. Such a local-global principle is known to be true for SL(2) [AP13, Theorem 1.2]. In proving [AP13, Theorem 1.2], the key input there is a characterization of the fibers of the Asai lift from GL$_2 (\mathbb{A}_E)$ to GL$_4 (\mathbb{A}_F)$ which is [AP13, Theorem 6.5].

4. Higher multiplicity for $\text{SL}_n$

We now suppose $n \geq 3$ and recall consequences of the works of Blasius, Lapid and Hiraga-Saito [Bla94, Lap98, Lap99, HS05, HS12]. This section contains no original result.

4.1. Different notions of multiplicity. Let $\pi$ be a cuspidal automorphic representation $\pi$ of $\text{SL}_n (\mathbb{A}_E)$. We set

$$m(\pi) = \dim_{\text{SL}_n (\mathbb{A}_E)} (\text{Hom}(\pi, \mathbb{A}_0^\infty (\text{SL}_n (E) \backslash \text{SL}_n (\mathbb{A}_E))))$$

and call it the multiplicity of $\pi$ in the cuspidal spectrum of $\text{SL}_n (\mathbb{A}_E)$. There are several other notions of multiplicity for $\pi$, both on the automorphic side and on the Galois parameter side of the putative global Langlands correspondence. We shall need to pass from one to another and we explain the process in this paragraph. We follow [Lap98, p. 293] and [Lap99, p. 162]. First we consider the automorphic side. Thus, let $\pi$ and $\pi'$ be two cuspidal representations of $\text{GL}_n (\mathbb{A}_E)$. We write:

(i) $\tilde{\pi} \sim_s \tilde{\pi}'$ if $\tilde{\pi} \simeq \tilde{\pi}' \otimes \eta$ for a Hecke character $\eta$ of $\mathbb{A}_E^\times$,

(ii) $\tilde{\pi} \sim_{\text{cw}} \tilde{\pi}'$ if $\tilde{\pi}_v \simeq \tilde{\pi}'_v \otimes \eta_v$ for a character $\eta_v$ of $E_v^\times$ at each place $v$ of $E$

(iii) $\tilde{\pi} \sim_w \tilde{\pi}'$ if $\tilde{\pi}_v \simeq \tilde{\pi}'_v \otimes \eta_v$ for a character $\eta_v$ of $E_v^\times$ for almost places $v$ of $E$.

One denotes by $M(L(\tilde{\pi}))$ the number of $\sim_s$ equivalence classes in the $\sim_{\text{cw}}$ equivalence class of $\tilde{\pi}$, and by $\mathcal{M}(L(\tilde{\pi}))$ the number of $\sim_s$ equivalence classes in the $\sim_w$ equivalence class of $\tilde{\pi}$. It was expected by Labesse and Langlands that if $\pi$ is a cuspidal automorphic representation of $\text{SL}_n (\mathbb{A}_E)$ contained in $L(\tilde{\pi})$, then its multiplicity $m(\pi)$ inside the cuspidal automorphic spectrum is equal to $M(L(\tilde{\pi}))$ so that in particular $M(L(\tilde{\pi}))$ is finite [LL79]. This was proved for $\text{SL}_2 (\mathbb{A}_E)$ in [LL79] and in general for $\text{SL}_n (\mathbb{A}_E)$ by Hiraga and Saito [HS12, Theorem 1.6].

On the other hand the multiplicity $\mathcal{M}(L(\tilde{\pi}))$, which is conjectured to be finite and bounded by a function of $n$ in [Lap99, Conjecture 1], is certainly at least equal to $M(L(\tilde{\pi}))$ by definition, and related to a similar multiplicity on the “Galois parameter side”. To this end we introduce equivalence relations $\sim_s$ and $\sim_w$ on the set of representations of a group $G$. Let $\phi$ and $\phi'$ be two morphisms from $G$ to $\text{GL}_n (\mathbb{C})$, we write:

(i) $\phi \sim_s \phi'$ if there is $x \in \text{PGL}_n (\mathbb{C})$ such that $\phi'(g) = x^{-1} \phi(g)x$ for all $g \in G$, in which case we say that $\phi$ and $\phi'$ are strongly equivalent.

(ii) $\phi \sim_w \phi'$ if for all $g \in G$, there is $x_g \in \text{PGL}_n (\mathbb{C})$ such that $\phi'(g) = x_g^{-1} \phi(g)x_g \in \text{PGL}_n (\mathbb{C})$, in which case we say that $\phi$ and $\phi'$ are weakly equivalent.
We denote by \( \mathcal{M}(\phi) \) the number of \( \sim_s \) equivalence classes in the \( \sim_w \) equivalence class of \( \phi \). One of the main achievements of [Lap98, Lap99] is the following result (cf. [Lap98, Theorem 6] and [Lap99, Theorem 2]).

**Theorem 4.1.** Let \( L \) be a Galois extension of \( E \) with respective Weil groups \( W_L \) and \( W_E \) such that \( \text{Gal}(L/E) \) is nilpotent, and let \( \chi \) be a Hecke character of \( \mathbb{A}_E^\times \) such that \( \phi = \text{Ind}_{W_L}^{W_E}(\chi) \) is irreducible. Denote by \( \tilde{\pi} = \tilde{\pi}(\phi) \) the cuspidal automorphic representation of \( \text{GL}_n(\mathbb{A}_E) \) associated to \( \text{Ind}_{W_L}^{W_E}(\chi) \) by [AC89]. Then \( \mathcal{M}(\phi) = \mathcal{M}(L(\tilde{\pi})) \).

**Remark 6.** In the proof of this result Lapid invokes the Chebotarev density theorem to argue that for such representations, the relations \( \sim_s \) and \( \sim_w \) are compatible on the Galois parameter side and the automorphic side, and shows that if \( \tilde{\pi}' \sim_w \tilde{\pi} \) (i.e., almost everywhere a twist of \( \tilde{\pi} \)) for \( \tilde{\pi} \) as in the statement of Theorem 4.1 then \( \tilde{\pi}' \) is of Galois type, i.e., there exists a Galois representation \( \phi' \), necessarily unique, of \( W_E \) with Satake parameters equal to those of \( \tilde{\pi}' \) at almost every place of \( E \). We shall use these facts as well in what follows.

**Remark 7.** In particular suppose that \( \tilde{\pi} \) and \( \mathcal{M}(\phi) \) as in the statement of Theorem 4.1 and suppose moreover that the weak equivalence class of \( \tilde{\pi} \) (its \( \sim_w \) class) is the same as its \( \sim_{cw} \) class, then for any \( \pi \in L(\tilde{\pi}) \), we have:

\[
m(\pi) = M(L(\tilde{\pi})) = \mathcal{M}(L(\tilde{\pi})) = \mathcal{M}(\phi).
\]

Note that the middle equality can in general be a strict inequality, see for example [Bla94, Proposition 2.5].

### 4.2. Examples of higher cuspidal multiplicity due to Blasius

In this section we recall the first fundamental construction, due to D. Blasius ([Bla94]), of representations appearing with a multiplicity greater than one in the cuspidal spectrum of \( \text{SL}_n(\mathbb{A}_E) \). In view of the more recent results of Lapid and Hiraga-Saito recalled in Section 4.1 we give a slightly more modern treatment of the construction of Blasius, however following its exact same lines. For \( p \) a fixed prime number, we denote by \( H_p \) the Heisenberg subgroup of \( \text{GL}_3(\mathbb{F}_p) \) of upper triangular unipotent matrices with order \( p^3 \). Blasius considers finite products of Heisenberg groups

\[
H_{p_i} = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z}/p_i \right\},
\]

where for our purpose we restrict a finite number of odd primes \( p_i \) possibly equal for \( i \neq j \). For each index \( i \), we denote by \( Z_i \) the center of \( H_{p_i} \), and by \( L_i \) the Lagrangian subgroup of \( H_{p_i} \) given by \( a = 0 \). We then set \( H = \prod_i H_{p_i}, L = \prod_i L_i \) and \( Z = \prod_i Z_i \).

Now let \( E \) be our number field. Since \( H \) is a product of \( p \)-groups it is solvable, and therefore by the well-known result of Shafarevich in inverse Galois theory, there is a Galois extension \( L/E \) such that \( \text{Gal}(L/E) = H \). Now take for each \( i \) a non-trivial character \( \chi_i \) of \( Z_i \) and extend \( \chi_i \) to a character \( \tilde{\chi}_i \) of \( L_i \) by

\[
\tilde{\chi}_i \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \chi_i \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
Now set $\chi = \otimes_i \chi_i$ the corresponding character of $Z$, and call it a regular character of $Z$ (meaning all the $\chi_i$ are non-trivial), and $\tilde{\chi} = \otimes_i \tilde{\chi}_i$ to be the corresponding character of $L = \text{Gal}(L/L_E)$ (for $L_E$ an extension of $E$). This character can be seen as a Hecke character of the Weil group $W_{L_E}$ (which is trivial on $W_L$). The induced representation $I_{\chi} = \text{Ind}_{W_{L_E}}^{W_L} (\tilde{\chi})$ is an irreducible representation of $H$ of dimension $n = \prod_i p_i$ and when $\chi$ varies, the representations $I_{\chi}$ are non-isomorphic and describe all the irreducible representations of $H$, their number being equal to

$$m(n) = \prod_i (p_i - 1).$$

We then set $\tilde{\pi}_\chi$ to be the cuspidal automorphic representation of $\text{GL}_n(A_E)$ attached to $I_{\chi}$ in [AC89]. By Theorem 4.1 we obtain the following result from Section 1.1 of [Bla94].

**Proposition 4.2.** In the situation above, let $\pi \in \mathcal{A}_E^\psi (\text{SL}_n(E) \backslash \text{SL}_n(A_E))$ be an irreducible summand of $\tilde{\pi}_\chi$. Then $\mathcal{M}(L(\tilde{\pi}_\chi)) = m(n)$.

**Proof.** According to Theorem 4.1 it is sufficient to check that the conjugacy class of $I_{\chi}(w)$ in $\text{PGL}_n(C)$ is independent of $\chi$ for any $w \in W_E$ but that the $I_{\chi}$’s are inequivalent projective representations. This is done in [Bla94, Section 1.1].

We are however looking for information on $m(\pi)$ rather than $\mathcal{M}(L(\tilde{\pi}_\chi))$. Therefore we follow Blasius again to put us in a situation where $\mathcal{M}(L(\tilde{\pi}_\chi)) = \mathcal{M}(L(\tilde{\pi}_{\chi'}))$ in order to apply Remark 7. To this end we select $L$ as in the proof of [Bla94, Proposition 2.1], such that at all the places in $L$ lying above $p$ for each $p$ dividing $|H|$, $L$ is unramified.

Then in such a situation, by [Bla94, Proposition 2.1, (2)], we deduce that two representations $\tilde{\pi}_\chi$ and $\tilde{\pi}_{\chi'}$, for regular characters $\chi$ and $\chi'$ of $Z$, are not only weakly equivalent (which we already know from [Bla94, Section 1.1] and Section 4.1), but they are in fact in the same $\sim_{ew}$-class, i.e., they are twists of each other at every place of $E$. Finally, by Remark 6 if $\tilde{\pi}$ is a cuspidal automorphic representation of $\text{GL}_n(A_E)$ weakly equivalent to $\tilde{\pi}_{\chi'}$, it is of Galois type with Galois parameter say $\phi$. Because for every $w \in W_E$, the conjugacy class of $I_{\chi}(w)$ in $\text{GL}_n(C)$ is equal to that of $\phi(w)$, we deduce that $I_{\chi}$ and $\phi$ have the same kernel, and are thus in fact both irreducible representations of $H$. This implies that $\phi$ is itself of the form $I_{\chi'}$ for a regular character $\chi'$ of $Z$, in particular the $\sim_{ew}$ class of $\pi$ is equal to its $\sim_{ew}$ class. In view of Remark 7 the outcome of this discussion is the following result, which also follows from the proof of [Bla94, Proposition 3.3].

**Proposition 4.3.** Let $E$ be a number field and let $L$ be an extension of $E$ such that $\text{Gal}(L/E) \simeq H$ and such that $L$ is unramified at every place of $L$ lying over a prime divisor of the cardinality $n = |H|$. Let $\chi$ be a regular character of $Z$ and let $\pi \in \mathcal{A}_E^\psi (\text{SL}_n(E) \backslash \text{SL}_n(A_E))$ be an irreducible summand of $\tilde{\pi}_\chi$. Then $m(\pi) = m(n)$ and the $L$-packets containing a copy of $\pi$ are those of the form $L(\tilde{\pi}_{\chi'})$ for a regular character $\chi'$ of $Z$ and they are all different.

**Remark 8.** Such extensions $L$ of $E$ exist in abundance by Shafarevich’s theorem in inverse Galois theory.
Remark 9. In [Bla94], Blasius had conjectured that two L-packets, say \( L(\tilde{\pi}) \) and \( L(\tilde{\pi}') \), would be isomorphic if \( \tilde{\pi} \) and \( \tilde{\pi}' \) are locally isomorphic at every place up to a character twist [Bla94, Conjecture on p. 239]. This conjecture was later proved by Hiraga-Saito in 2005 [HS05]. Lacking the truth of the conjecture at that point in time, [Bla94] resorted to a trick using complex conjugation. Note that reading out the precise multiplicity \( m(\pi) \) is an immediate consequence of this result.

5. Three questions

In this section we attempt to answer a number of natural and important questions. We thank Raphaël Beuzart-Plessis and Dipendra Prasad for posing the first two of these questions to us in the context of this paper. We then consider one more question which in the case of \( SL(2) \) was answered by an explicit construction in [AP06, Theorem 8.2].

5.1. Questions. We formulate three natural questions for each of which we provide answers in the later subsections.

Question 1. Is there a cuspidal representation \( \pi \) of \( SL_n(A_E) \) distinguished with respect to \( SL_n(A_F) \) with one realization \( \pi_1 \) in \( A_\infty^0(SL_n(E) \backslash SL_n(A_E)) \) with non-vanishing period and another realization \( \pi_2 \) in \( A_\infty^0(SL_n(E) \backslash SL_n(A_E)) \) with vanishing period?

Remark 10. We shall see in Section 5.2 that the answer to the above question is Yes, in particular implying that there are cuspidal automorphic representations of \( SL_n(A_E) \) for \( n \geq 3 \) which are locally distinguished, but with a realization in the space of smooth cusp forms on which \( P_{SL_n(A_F)} \) vanishes.

Question 2. Consider the natural decomposition of \( A_\infty^0(SL_n(E) \backslash SL_n(A_E)) \) into L-packets. Let \( \pi_1 \) and \( \pi_2 \) be two irreducible submodules of \( A_\infty^0(SL_n(E) \backslash SL_n(A_E)) \) such that \( \pi_1 \simeq \pi_2 \) but which belong to two different L-packets \( L(\tilde{\pi}_1) \neq L(\tilde{\pi}_2) \). If \( P_{SL_n(A_F)} \) does not vanish on \( \pi_1 \), then is it true that it does not vanish on \( \pi_2 \)?

Remark 11. We shall see in Section 5.5 that the answer is No in general. This implies a refinement of Remark 10 which is that for \( n \geq 3 \), there are cuspidal automorphic representations of \( SL_n(A_E) \) which are locally distinguished, but with at least one canonical realization in the space of smooth cusp forms on which \( P_{SL_n(A_F)} \) vanishes.

The following question arises immediately after the above remark.

Question 3. For \( n \geq 3 \), are there cuspidal automorphic representations of \( SL_n(A_E) \) which are locally distinguished at every place of \( F \), but not globally? In fact is it even possible to construct such a representation which belongs to no distinguished L-packet?

We shall see in Section 5.6 that such representations do exist. Note that though Questions 1 and 2 are not meaningful for \( SL_2(A_E) \) according to Ramakrishnan’s multiplicity 1 result [Ram00], the issues addressed by Remarks 10 and 11 as well as Question 3 make sense for \( n = 2 \). In this case they are all answered in [AP06]. In fact it is sufficient to answer Question 3 for \( n = 2 \), and this is done by [AP06].
8.2], the proof of which is quite involved: there are indeed cuspidal automorphic representations of $SL_2(A_E)$ which are locally distinguished at every place of $F$ but not globally. We shall provide easier examples of this type in Section 5.5 for $n \geq 3$.

5.2. **Answer to Question 1.** To answer Question 1 in the affirmative suppose we have a cuspidal representation $\pi$ of $SL_n(A_E)$ of multiplicity $m(\pi) \geq 2$ which is $SL_n(A_F)$-distinguished. Suppose also that we are in a situation where

$$\dim \text{Hom}_{SL_n}(\pi, C) = 1,$$

where the linear forms considered are continuous with respect to the Fréchet topology. That is to say the period integral $P_{\text{SL}_n}(A_F)$ is, up to multiplication by scalars, the only $SL_n(A_F)$-invariant linear form on the space of $\pi$. Observe that the above assumption is equivalent to having local multiplicity one for $SL_n(F_v)$-invariant forms for each place $v$ of $F$. Now let $U_1$ and $U_2$ be two linearly independent elements of

$$\text{Hom}_{SL_n}(\pi, A_0^\infty(SL_n(E) \backslash SL_n(A_E)))$$

with $P_{\text{SL}_n}(A_F)$ non-vanishing on $\text{Im}(U_1)$. We set

$$\Lambda_i = P_{\text{SL}_n}(A_F) \circ U_i.$$

If $\Lambda_2 = 0$ then we are done. If not then both $\Lambda_1$ and $\Lambda_2$ are non-trivial elements of $\text{Hom}_{SL_n}(\pi, C)$ and therefore there is $s \in C$ such that $\Lambda_1 - s\Lambda_2 = 0$. We conclude that $P_{\text{SL}_n}(A_F)$ vanishes on $\text{Im}(U)$ for $U = U_1 - sU_2$, which is indeed a realization of $\pi$ as a submodule of $A_0^\infty(SL_n(E) \backslash SL_n(A_E))$. So an $SL_n(A_F)$-distinguished cuspidal automorphic representation of $SL_n(A_E)$ can have a realization in $A_0^\infty(SL_n(E) \backslash SL_n(A_E))$ on which $P_{\text{SL}_n}(A_F)$ vanishes, thus answering Question 1.

It only remains to construct cuspidal representations of $SL_n(A_E)$ satisfying the above two assumptions which we do in the next subsections.

5.3. **Local Gelfand pair.** First we claim that if we take $n$ to be an odd integer it is automatic that

$$\dim \text{Hom}_{SL_n}(\pi, C) = 1.$$

This is because then we have

$$\dim \text{Hom}_{SL_n(F_v)}(\pi_{F_v}, C) = 1$$

at each place $v$ of $F$. For a place of $F$ that splits in $E$ this is obvious by Schur’s lemma whereas for a finite place of $F$ that remains inert this statement is [Ana05, Theorem 1.1] (for a slightly different proof see [APT18, Proposition 3.3 (1)]). The remaining case, that of $(SL_n(C), SL_n(R))$, is in fact easier to prove by similar considerations as in [Ana05, APT18] once we have multiplicity one for $(GL_n(C), GL_n(R))$ which is known thanks to [AG09, Theorem 8.2.5]; we state the result as a proposition since it is not stated as such in the literature.

**Proposition 5.1.** Let $\pi$ be an irreducible admissible smooth Fréchet representation (as in [AG09]) of $SL_n(C)$ and suppose that $n$ is odd. Then

$$\dim \text{Hom}_{SL_n(R)}(\pi, C) \leq 1.$$
Proof. Any such \(\pi\) appears in the restriction of an irreducible admissible smooth Fréchet representation \(\tilde{\pi}\) of \(\text{GL}_n(\mathbb{C})\) which is unique up to a character twist. By [AG09 Theorem 8.2.5], we know that \(\dim \text{Hom}_{\text{GL}_n(\mathbb{R})}(\tilde{\pi}, \mathbb{C}) = 1\), and therefore arguing as in the proof of [API18 Proposition 3.3 (1)], we see that \(\dim \text{Hom}_{\text{SL}_n(\mathbb{R})}(\tilde{\pi}, \mathbb{C})\) is the number of characters \(\alpha \in \mathbb{R}^\times\) for which \(\tilde{\pi}\) is \(\alpha\)-distinguished. We may also assume that \(\tilde{\pi}\) is \(\text{GL}_n(\mathbb{R})\)-distinguished if \(\pi\) is \(\text{SL}_n(\mathbb{R})\)-distinguished and then since \(n\) is odd, for central character reasons, there is no such non-trivial character. Thus,

\[
\dim \text{Hom}_{\text{SL}_n(\mathbb{R})}(\tilde{\pi}, \mathbb{C}) \leq \dim \text{Hom}_{\text{GL}_n(\mathbb{R})}(\tilde{\pi}, \mathbb{C}) \leq 1.
\]

We remark in passing that the first inequality above is in fact an equality as \(\tilde{\pi}\) restricts irreducibly to \(\text{SL}_n(\mathbb{C})\) but we do not need this. \(\square\)

5.4. Distinguished cuspidal representations of higher multiplicity. Now we need to construct cuspidal representations \(\pi\) of \(\text{SL}_n(\mathbb{A}_E)\) which are \(\text{SL}_n(\mathbb{A}_F)\)-distinguished with \(m(\pi) \geq 2\) for odd \(n\) (and this will complete answering Question 1).

Let us explain our general recipe for this, using the examples of Blasius in §4.2. We take \(n \geq 3\) odd and write it as \(n = \prod_i p_i\). We set \(H = \prod_H H_{p_i}\) as before and take an involution \(\sigma\) of the group \(H\). Associated to this involution is the semi-direct product

\[G = H \rtimes \mathbb{Z}/2\]

where \(\mathbb{Z}/2\) acts on \(H\) via \(\sigma\). Now let \(F\) be any number field and let \(L\) be an extension of \(F\) such that \(\text{Gal}(L/F) \simeq G\). In fact we choose \(L\) in such a way that \(L/F\) is unramified at each place of \(F\) lying above any \(p\) dividing \(n\). Note that all these can be done by Shafarevich’s theorem since \(G\) is solvable. Let \(E\) be the fixed field of \(H\) so that

\[\text{Gal}(L/E) \simeq H \& \text{Gal}(E/F) = \langle \sigma \rangle.\]

Take an irreducible representation \(\rho\) of \(H\). It identifies with \(I_{\chi_{\rho}}\) for \(\chi_{\rho}\) a regular character of \(Z\) and we set \(\tilde{\pi}(\rho) = \tilde{\pi}_{\chi_{\rho}}\) (cf. §4.2). In particular, because \(L/E\) is unramified at places of \(E\) lying above the prime divisors of \(n\), if \(\pi \in \text{L}(\tilde{\pi}(\rho))\), we obtain \(m(\tilde{\pi}(\rho)) = m(\pi) = m(n)\) thanks to Proposition 4.3. In this situation, we have the following very useful result due to the rigidity of the representation theory of Heisenberg groups, which we will apply in order to produce examples answering Question 2.

**Proposition 5.2.** In the situation described above, take an irreducible representation \(\rho\) of \(H\) and denote by \(c_{\rho}\) its central character. The \(L\)-packet \(\text{L}(\tilde{\pi}(\rho))\) is distinguished if and only if \(c_{\rho}(z^r) = c_{\rho}(z^{-1})\) for all \(z \in \mathbb{Z}\).

Proof. By Proposition 4.3 \(\text{L}(\tilde{\pi}(\rho))\) is distinguished if and only if \((\tilde{\pi}(\rho)^{\vee})^r \simeq \mu \otimes \tilde{\pi}(\rho)\) for a Hecke character \(\mu\) factoring through \(N_{E/F}\). This is equivalent to \(\tilde{\pi}((\rho^{\vee})^r) \simeq \mu \otimes \tilde{\pi}(\rho)\). However as the \(L\)-packets determined by different irreducible representations are different thanks to Proposition 4.3, we easily deduce that \(\text{L}(\tilde{\pi}(\rho))\) is distinguished if and only if \(\rho^r \simeq \rho^{\vee}\). The result now follows from the fact that \(\rho\) is determined by its central character. \(\square\)

In view of Corollary 3.2, a consequence of Proposition 5.2 is the following.

**Corollary 5.3.** In the situation of Proposition 5.2, let \(\rho\) be an irreducible representation of \(H\) such that \(c_{\rho}^r = c_{\rho}^{-1}\), and \(\pi \in \text{L}(\tilde{\pi}(\rho))\) such that \(\mathcal{P}_{\text{SL}_n(\mathbb{A}_F)}\) does not vanish on \(\pi\). Then the
canonical copies of \( \pi \) on which \( \mathcal{P}_{\mathrm{SL}_n(A_F)} \) does not vanish are those contained in the \( L \)-packets of the form \( L(\tilde{\pi}(\rho')) \) with \( \rho' \) an irreducible representation of \( H \) such that \( c_{\rho'}^\pi = c_{\rho'}^{-1} \).

5.5. **Examples for Question 2**. We first give two examples for which we answer Question 2. In the first one, all the canonical copies of the considered distinguished representation have a non-vanishing period, whereas in the second example only some of the canonical copies of the considered distinguished representation have a non-vanishing period and some others do not have a non-vanishing period.

For the first set of examples, the group \( H \) is as in Section 5.4 and the involution that we consider on it, for \( a, b \) and \( c \) in \( \prod_i \mathbb{Z}/p_i \), is given by

\[
\sigma : \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & a & -c \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}.
\]

In this case because the associated involution acts as the inversion on \( Z \), Proposition 5.2 tells us that all \( L \)-packets \( L(\tilde{\pi}(\rho)) \) are distinguished when \( \rho \) varies in the set of irreducible classes of representations of \( H \), and that if one fixes a representation \( \pi \) in one \( L \)-packet on which \( \mathcal{P}_{\mathrm{SL}_n(A_F)} \) does not vanish, then it does not vanish on any of the \( m(n) \) canonical copies of \( \pi \).

For the second set of examples, we consider \( H \) as above (of odd cardinality \( n \)) and \( H' = H \times H \) (which is in fact a special type of \( H \)) endowed with the switching involution

\[
\sigma : (x, y) \mapsto (y, x).
\]

In this case Proposition 5.2 tells us that the distinguished \( L \)-packets of \( \mathrm{SL}_{n^2}(A_E) \) of the form \( L(\tilde{\pi}(\rho')) \) are the \( m(n) \) ones such that \( \chi_{\rho'} \) is of the form \( \chi \otimes \chi^{-1} \) with \( \chi \) regular, whereas the others are not. Then again by Corollary 5.3 we conclude that if \( \pi \) is a fixed distinguished representation of \( \mathrm{SL}_{n^2}(A_E) \) appearing in one of the \( m(n)^2 \) many \( L \)-packets above, then the period \( \mathcal{P}_{\mathrm{SL}_{n^2}(A_F)} \) does not vanish on the \( m(n) \) canonical copies inside the distinguished \( m(n) \) many distinguished \( L \)-packets, and does vanish on the \( m(n)^2 - m(n) \) remaining ones.

5.6. **Examples for Question 3**. Now we give a set of examples answering Question 3, using Proposition 5.6. For simplicity we take \( H = H_p \) for \( p \) an odd prime (i.e., \( n = p \)). Let \( \sigma \) be an involution of \( H \) such that \( z^\sigma = z \) for all \( z \in \mathbb{Z} \). Thus, we may take the trivial involution or the involution of \( H \) given by

\[
\sigma : \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -a & c \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}.
\]

Since \( z^\sigma = z \) for all \( z \in \mathbb{Z} \), Proposition 5.2 implies that no \( L \)-packet of the form \( \tilde{\pi}(\rho) \) for \( \rho \) an irreducible representation of \( H \) is distinguished because, as \( |\mathbb{Z}| \) is odd, the only character of \( \mathbb{Z} \) of order \( \leq 2 \) is trivial.

It remains to prove that if we fix \( \rho \) as above, and set \( \tilde{\pi} = \tilde{\pi}(\rho) \), then \( L(\tilde{\pi}) \) contains an automorphic representation \( \pi \) such that \( \pi_v \) is \( \mathrm{SL}_p(F_v) \)-distinguished for every place \( v \) of \( F \). This is equivalent to showing that \( \tilde{\pi}_v \) is \( (\mathrm{GL}_p(F_v), \gamma_v) \)-distinguished for some
character $\gamma_v$ of $F_v^\times$, which is what we do. Recall that by [Bla94 Proposition 2.1],
\[ \tilde{\pi}_v^\times \simeq \tilde{\pi}_v^\vee \otimes \eta_v \]
at each place $v$ for a character $\eta_v$ of $E_v^\times$.

If a place $v$ of $F$ splits in $E$ as $(v_1, v_2)$ then the above condition implies $\tilde{\pi}_v$ is of the form $(\tau, \tau^\vee \otimes \nu)$ which is distinguished for the character $\nu$ of $F_v^\times$.

Now let $v$ be such that it does not split in $E$. We set $B_p(E_v)$ the upper triangular Borel subgroup of $GL_p(E_v)$.

We write as before $\tilde{\pi} = \tilde{\pi}(\rho)$ for $\rho$ an irreducible representation of $H$. We denote by $L$ and $L'$ the first and the second Lagrangian subgroups of $H$ given by $a = 0$ and $b = 0$ respectively (cf. [4.2]). By the proof of [Bla94 Proposition 2.1] the local Galois group of $H_v$ is an abelian subgroup of $H$, hence either trivial, equal to $Z$, $L$ or $L'$. We recall that $\rho = Ind_H^L(\tilde{\chi})$ where

\[ \tilde{\chi} \left( \begin{array}{ccc} 1 & 0 & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{array} \right) = \chi(c) \]

for $\chi$ a non-trivial character of $Z/p$. We fix $\mu$ a non-trivial character $Z/p$ and set

\[ \tilde{\mu} \left( \begin{array}{ccc} 1 & 0 & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{array} \right) = \mu(b). \]

Similarly we set

\[ \tilde{\chi}' \left( \begin{array}{ccc} 1 & a & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) = \chi(c) \]

and

\[ \tilde{\mu}' \left( \begin{array}{ccc} 1 & a & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) = \mu(a). \]

Clearly if $H_v$ is trivial or equal to $Z$, then $\rho|_{H_v}$ is a sum of copies of the same character, hence $\tilde{\pi}_v$ is of the form
\[ Ps(\alpha, \ldots, \alpha) = Ind_{B_p(E_v)}^{GL_p(E_v)}(\alpha \otimes \cdots \otimes \alpha) \]

where the induction is normalized, hence $\alpha|_{F_v^\times}$ distinguished by, for example, [Mat11 Theorem 5.2]. Now we consider the case $H_v = L$. Then by Mackey theory,
\[ \rho|_L = \tilde{\chi}(\oplus_{k=0}^{p-1} \tilde{\mu}^k). \]

Thus the corresponding principal series is of the form
\[ (1) \quad \tilde{\pi}_v = Ps(\alpha, \alpha \beta, \alpha \beta^{-1}, \ldots, \alpha \beta^{(p-1)/2}, \alpha \beta^{-(p-1)/2}). \]

If $\sigma$ is the trivial involution we trivially have $\beta = \beta^\sigma$ so (1) takes the form
\[ \tilde{\pi}_v = \alpha \otimes Ps(1, \beta, \beta^{-\sigma}, \ldots, \beta^{(p-1)/2}, (\beta^{(p-1)/2})^{-\sigma}), \]

which is distinguished by [Mat11 Theorem 5.2].
If $\sigma$ is the non-trivial involution such that $z^\sigma = z$ for $z \in \mathbb{Z}$ then note that $\sigma$ fixes $\tilde{\chi}$ whereas it sends $\tilde{\mu}$ to its inverse. We set $\mu_k = \alpha \beta^k$ for $k = 1, \ldots, (p - 1)/2$, so that (1) takes the form

$$\tilde{\pi}_\nu = \text{Ps}(\alpha, \mu_1, \mu_1^\sigma, \ldots, \mu_{(p-1)/2}, \mu_{(p-1)/2}^\sigma).$$

Now because for $k = 1, \ldots, (p - 1)/2$, one has $\alpha^2 = \mu_k \mu_k^\sigma$ and hence $a^{2|_{F_\nu^\times}} = \mu_k^{2|_{F_\nu^\times}}$, but as both characters in this equality have odd order $p$ we deduce that $\alpha_{|_{F_\nu^\times}} = \mu_k{_{|F_\nu^\times}}$. So

$$\tilde{\pi}_\nu = \alpha \otimes \text{Ps}(1, \alpha^{-1} \mu_1, \alpha^{-1} \mu_1^\sigma, \ldots, \alpha^{-1} \mu_{(p-1)/2}, \alpha^{-1} \mu_{(p-1)/2}^\sigma)$$

and all the characters appearing in the principal series have trivial restriction to $F_\nu^\times$, we deduce again from [Mat11, Theorem 5.2] that $\tilde{\pi}_\nu$ is $\alpha_{|{F_\nu^\times}}$-distinguished.

Finally when $H_\nu = L'$ then

$$\rho_{|L'} = \tilde{\chi}'(\oplus_{k=0}^{p-1} \tilde{\mu}^k)$$

and a completely similar argument proves that $\tilde{\pi}_\nu$ is distinguished by a character.

We apply Proposition 3.6 to conclude that $L(\tilde{\pi})$ does not contain any distinguished representation but it contains cuspidal representations which are everywhere locally distinguished.

Acknowledgements

The authors thank Raphaël Beuzart-Plessis and Yiannis Sakellaridis for useful comments and explanations. The content of §5.4 & 5.5 grew out of a discussion with Beuzart-Plessis. The authors would like to especially thank Dipendra Prasad for his questions and comments over several e-mail conversations; his guidance in general has played a significant role in the writing of this paper.

References

[AG09] A. Aizenbud and D. Gourevitch, Harish-Chandra descent, Gelfand pairs, and an Archimedean analog of Jacquet-Rallis’s theorem, with an appendix by the authors and Eitan Sayag. Duke Math. J. 149 (2009), no. 3, 509–567. MR 2553879

[AP03] U. K. Anandavardhanan and D. Prasad, Distinguished representations for SL(2), Math. Res. Letters 10 (2003), 867–878. MR 2025061

[Ana05] U. K. Anandavardhanan, Distinguished non-Archimedean representations, Algebra and Number Theory, 183–192, Hindustan Book Agency, Delhi, 2005. MR 2193352

[AP06] U. K. Anandavardhanan and D. Prasad, On the SL(2) period integral, Amer. J. Math. 128 (2006), no. 6, 1429–1453. MR 2275907

[AP13] U. K. Anandavardhanan and D. Prasad, A local-global question in automorphic forms, Compos. Math. 149 (2013), no. 6, 959–995. MR 3077658

[AP18] U. K. Anandavardhanan and D. Prasad, Distinguished representations for SL(n), Math. Res. Lett., 25 (2018), no. 6, 1695–1717. MR 3934841

[AM18] U. K. Anandavardhanan and N. Matringe, Test vectors for finite periods and base change, Preprint, 2018.

[AC89] J. Arthur and L. Clozel, Simple algebras, base change, and the advanced theory of the trace formula, Annals of Mathematics Studies, 120. Princeton University Press, Princeton, NJ, 1989. xiv+230 pp. MR 1007299

[AGR93] A. Ash, D. Ginzburg and S. Rallis, Vanishing periods of cusp forms over modular symbols, Math. Ann. 296 (1993), no. 4, 709–723. MR 1233493

[Bla94] D. Blasius, On multiplicities for SL(n), Israel J. Math. 88 (1994), 237–251. MR 1303497
[Cog04] J. Cogdell, Lectures on L-functions, converse theorems, and functoriality for GL_n, Lectures on automorphic L-functions, 1–96, Fields Inst. Monogr., 20, Amer. Math. Soc., Providence, RI, 2004. MR 2071506

[DP19] S. Dijols and D. Prasad, Symplectic models for unitary groups, Trans. Amer. Math. Soc. (2019). DOI:https://doi.org/10.1090/tran/7651.

[Fli88] Yuval Z. Flicker, Twisted tensors and Euler products, Bull. Soc. Math. France 116 (1988), no. 3, 295–313. MR 984899

[FZ95] Yuval Z. Flicker and D. Zinoviev, On poles of twisted tensor L-functions, Proc. Japan Acad. Ser. A Math. Sci. 71 (1995), no. 6, 114–116. MR 1344660

[HS05] K. Hiraga and H. Saito, On restriction of admissible representations, Algebra and Number Theory, 299–326, Hindustan Book Agency, Delhi, 2005. MR 2193361

[HS12] K. Hiraga and H. Saito, On L-packets for inner forms of SL_n, Mem. Amer. Math. Soc. 215 (2012), no. 1013, vi+97 pp. ISBN: 978-0-8218-5364-1. MR 2918491

[JS81] H. Jacquet and J. Shalika, On Euler products and the classification of automorphic representations I, Amer. J. Math. 103 (1981), no. 3, 499–558. MR 0618323

[JS90] H. Jacquet and J. Shalika, Exterior square L-functions, Automorphic forms, Shimura varieties, and L-functions, Vol. II (Ann Arbor, MI, 1988), 143–226, Perspect. Math., 11, Academic Press, Boston, MA, 1990. MR 1044830

[LL79] J.-P. Labesse and R. P. Langlands, L-indistinguishability for SL(2), Canad. J. Math. 31 (1979), no. 4, 726–785. MR 0540902

[Lap98] E. M. Lapid, A note on the global Langlands conjecture, Doc. Math. 3 (1998), 285–296. MR 1661109

[Lap99] E. M. Lapid, Some Results on Multiplicities for SL(n), Israel J. Math. 112 (1999), 157–186. MR 1714998

[Mat11] N. Matringe, Distinguished generic representations of GL(n) over p-adic fields, Int. Math. Res. Not. IMRN 2011, no. 1, 74–95. MR 2755483

[Ram00] D. Ramakrishnan, Modularity of the Rankin-Selberg L-series, and multiplicity one for SL(2), Ann. of Math. (2) 152 (2000), no. 1, 45–111. MR 1792292

[SV17] Y. Sakellaridis and A. Venkatesh, Periods and harmonic analysis on spherical varieties, Astérisque No. 396 (2017), viii+360 pp. ISBN: 978-2-85629-871-8. MR 3764130

[Zha14] W. Zhang, Automorphic period and the central value of Rankin-Selberg L-function, J. Amer. Math. Soc. 27 (2014), no. 2, 541–612. MR 3164988

Department of Mathematics, Indian Institute of Technology Bombay, Mumbai - 400076, India. E-mail address: anand@math.iitb.ac.in

Laboratoire Mathématiques et Applications, Université de Poitiers, France. E-mail address: matringe@math.univ-poitiers.fr