Four-loop QED contributions to the electron $g-2$

Stefano Laporta
Dipartimento di Fisica e Astronomia, Università di Padova, Istituto Nazionale Fisica Nucleare, Sezione di Padova, Via Marzolo 8, I-35131 Padova, Italy
E-mail: stefano.laporta@pd.infn.it

Abstract. The anomalous magnetic moment of the electron is one of the physical quantities measured with the highest precision. Such high precision demands a similar precision in the theoretical evaluations in order to obtain stringent tests of QED. In this talk I will summarize the situation of the theoretical calculations of the contributions to the electron $g-2$; then, I will describe in detail the results of the twenty-year long project of the evaluation of all the 891 mass-independent four-loop QED Feynman diagrams contributing to the electron $g-2$ [11], with the 1100-digits result

\[ a_{e}^{\text{QED}}(4\text{-loop}) = -1.912245764926445574152647167439830054060873390658725345\ldots \left( \frac{\alpha}{\pi} \right)^{4} \]

and high-precision analytical fits. The consequences of this result on the QED tests and the determination of the fine structure constant will be also discussed.

Introduction
In the precision physics of simple systems, the anomalous magnetic moment of the electron has a special place. Its relative insensitivity to QCD and weak effects, put it as an ideal test for QED. We can define the adimensional factor $g$ as the ratio between the electron magnetic moment $\mu$ and the Bohr magneton,

\[ \mu = \frac{g_{e}}{2} \mu_{B} . \]  

According to the Dirac’s theory [1], $g_{e} = 2$. Radiative corrections shift slightly the value of $g_{e}$. Experimentally this was observed for the first time by Kusch and Foley [2]; they measured a value of the anomaly $a_{e}$ slightly different from zero:

\[ a_{e} = \frac{g_{e} - 2}{2} = 0.001 15(4) . \]  

Using Q.E.D. Schwinger [3, 4] was able to calculate the first term of the expansion in the fine structure constant $\alpha = \frac{e^{2}}{\hbar c} \approx \frac{1}{137}$:

\[ a_{e} = \frac{\alpha}{2\pi} = 0.001 161\ldots , \]
Experimental values
The current measurements of $a_e$ are based on the Penning trap method, developed by H. Dehmelt and his group at the University of Washington. For the development of this technique, he won the Nobel prize in Physics in 1989. The anomaly is expressed as the ratio of two frequencies, which can be measured with a very high precision. Their final results were [5]:

$$a_{e^-}^{exp} = 1.159652.188.4(4.3) \times 10^{-12} \text{ (4.3 ppb)} ,$$

$$a_{e^+}^{exp} = 1.159652.187.9(4.3) \times 10^{-12} \text{ (4.3 ppb)} .$$

The most recent value obtained with the same technique by the Harvard group is [6, 7]:

$$a_{e^-}^{exp} = 1.159652.180.73(0.28) \times 10^{-12} \text{ (0.24 ppb)} .$$

Theoretical expression
The standard model contribution to the anomaly can be separated into QED, hadronic and weak contributions.

$$a_e^{SM} = a_e^{QED} + a_e^{weak} + a_e^{hadr} .$$

The QED contribution can be split up in mass-independent and mass-dependent parts:

$$a_e^{QED} = A_1 + A_2 \left( \frac{m_e}{m_\mu} \right) + A_3 \left( \frac{m_e}{m_\tau} \right) .$$

The functions $A^{(i)}$ can be expanded in power series

$$A_i = A_i^{(2)} \left( \frac{\alpha}{\pi} \right) + A_i^{(4)} \left( \frac{\alpha}{\pi} \right)^2 + A_i^{(6)} \left( \frac{\alpha}{\pi} \right)^3 + A_i^{(8)} \left( \frac{\alpha}{\pi} \right)^4 + A_i^{(10)} \left( \frac{\alpha}{\pi} \right)^5 + \ldots .$$

The mass-independent coefficients at 1, 2 and 3 loop are known in analytical form [3,4,8–10]:

$$A_1^{(2)} = \frac{1}{2} ,$$

$$A_1^{(4)} = \frac{197}{144} + \frac{1}{12} \pi^2 - \frac{1}{2} \pi^2 \ln 2 + \frac{3}{4} \zeta(3) = -0.328 \text{ 478 965 579} \ldots ,$$

$$A_1^{(6)} = \frac{83}{72} \pi^2 \zeta(3) - \frac{215}{24} \zeta(5) + \frac{100}{3} \left[ (a_4 + \frac{1}{24} \ln^4 2) - \frac{1}{24} \pi^2 \ln^2 2 \right] - \frac{239}{2160} \pi^4 + \frac{139}{18} \zeta(3) - \frac{298}{9} \pi^2 \ln 2 + \frac{17101}{810} \pi^2 + \frac{28259}{5184} = 1.181 \text{ 241 456} \ldots ,$$

where $\zeta(n) = \sum_{i=1}^{\infty} i^{-n}$, $a_n = \sum_{i=1}^{\infty} 2^{-i} i^{-n}$. In table 1 we list some older theoretical evaluations of the two, three and four loop coefficients. In [11] I have evaluated up to 1100 digits of precision the 4-loop contribution $A_1^{(8)}$, finalizing a twenty-year effort [13–19] begun after the completion of the calculation of $A_1^{(6)}$ [10]. The first digits of the result are

$$A_1^{(8)} = -1.912245764926445574152647167439830054060873390658725345171329848\ldots .$$

A selection of the 891 diagrams contributing to $A_1^{(8)}$ is shown in figure 1.

The full-precision result is shown in table 3. The result (12) is in excellent agreement (0.9σ) with the numerical value

$$A_1^{(8)} \text{ [29]} = -1.91298(84) ,$$

(13)
Table 1. Numerical results of the evaluations of $A^{(4)}_1$, $A^{(6)}_1$, $A^{(8)}_1$ and $A^{(10)}_1$.

| $A^{(4)}_1$ | $A^{(6)}_1$ | $A^{(8)}_1$ | $A^{(10)}_1$ |
|------------|------------|------------|------------|
| -2.973... [12] | 1.499(20) [31] | -1.434(158) [24] | 7.795(330) [29] |
| -0.328 478 965 579 ... [8, 9] | 1.165(20) [32] | -1.593(384) [35] | 0.599(223) [29] |
| 1.1761(42) [33] | 1.7283(35) [25] | -1.944(35) [26] |
| 1.181259(40) [34] | -1.9144(35) [28] |
| 1.181241456587200006... [10] | -1.912245764926445... [11] |

Table 2. Contribution to $a_e$.

| contribution | value in units of $10^{-12}$ |
|--------------|-------------------------------|
| $A^{(2)}_1 (\alpha/\pi)$ | 1 161 409 733.640 ± 0.720 |
| $A^{(4)}_1 (\alpha/\pi)^2$ | -1 772 305.065 ± 0.003 |
| $A^{(6)}_1 (\alpha/\pi)^3$ | 14 804.203 |
| $A^{(8)}_1 (\alpha/\pi)^4$ | -55.667 |
| $A^{(10)}_1 (\alpha/\pi)^5$ | 0.451 ± 0.013 |
| $A^{(4)}_2 (m_e/m_\mu)(\alpha/\pi)^2$ | 2.804 |
| $A^{(6)}_2 (m_e/m_\mu)(\alpha/\pi)^3$ | -0.092 |
| $A^{(8)}_2 (m_e/m_\mu)(\alpha/\pi)^4$ | 0.026 |
| $A^{(10)}_2 (m_e/m_\mu)(\alpha/\pi)^5$ | -0.0002 |
| $A^{(4)}_2 (m_e/m_\tau)(\alpha/\pi)^2$ | 0.010 |
| $A^{(6)}_2 (m_e/m_\tau)(\alpha/\pi)^3$ | -0.0008 |
| $a_e$ (hadronic v.p.) | 1.8490 ± 0.0108 |
| $a_e$ (hadronic v.p.,NLO) | -0.2213 ± 0.0012 |
| $a_e$ (hadronic v.p.,NNLO) | 0.0280 ± 0.0002 |
| $a_e$ (hadronic l-l ) | 0.0370 ± 0.0050 |
| $a_e$ (weak) | 0.03053 ± 0.00023 |

latest result of a really impressive pluridecennial effort [20–29], and with the independent value

$$A^{(8)}_1 ( [30] ) = -1.87(12).$$

At 5-loop level there is only the numerical evaluation by the Kinoshita’s group

$$A^{(10)}_1 ( [29] ) = 6.599(223).$$

Concerning the mass-dependent part $A_2(r)$, the contributions are known (expanded in series for small $r$, or numerically) up to 5 loops [29, 36–43]. The numerical values of QED, hadronic and weak contributions [44–47] are listed in table 2.

It is evident that the electron anomaly is dominated by the QED mass-independent contributions. QED mass-dependent, hadronic and weak effects contribute only to a few $10^{-12}$.

Comparison of theory and experiment

The total theoretical prediction is therefore

$$a_e^{th} = 1 159 652 182.032(13)(12)(720) \times 10^{-12},$$
Table 3. First 1100 digits of $A^{(8)}_1$.

\[ -1.9122457649264455741526471674398092700053058936451373727056376514912865391 \\
95280425705273823043455767897045562729039341299998720177867277992211178472035964519108666275979087436811151514797227432164273431913795686 \\
07405057837894606718574352831424838025192249407274285893646350814040225655043094423125563568812086620654014322497759430040292887867617 \\
489893615108870969970126358533757878631720526381901134757449363128848675182068332873878473703831510506274180315305147894055369778 \\
36946278642361843177850881153745669503230434970361342658864995311858788117437455386432448863846055844418823724176467608719442803073403017443 \\
05573459611710508589611449950621260612469604731184092974723040023464696351735842584175982240973757107376707404651592112026281111377253 \\
02154457210148112115984697088442532797972048420144612282945151605123661786594592050091733031713200865767212345450034910470728024487250 \\
616044261325492069000431915198230047488181949311038495378289940622967586763852659873979313261227975755067701142904987976208505785592 \ldots \]

where the latest measurement of the fine structure constant [48, 49] has been used,

\[ \alpha^{-1} = 137.035 \, 998 \, 996(85) \, (0.62 \text{ ppb}) \, . \]

In equation (16) the first error comes from $A^{(10)}_1$, the second one from the hadronic and electroweak corrections, the last one from $\alpha$, respectively.

One sees that the experimental error of the value of $\alpha$ is dominating; as a consequence, one can assume the validity of the theory and, inverting the theoretical expression of $\alpha_e$, one uses the experimental measurement (6) to determine a more precise value of $\alpha$

\[ \alpha^{-1}(\alpha_e) = 137.035 \, 999 \, 1491(15)(14)(330) \, (0.25 \text{ ppb}) ; \]

the errors come from $A^{(10)}_1$, hadronic and electroweak corrections, and $\alpha^{exp}_e$, respectively.

Figure 1. Typical 4-loop vertex diagrams

Factorial grow of complexity

There is a gap of 39 years between two- and three-loop results, and 19 years to complete the 4-loop calculation. Surely much time. Considered the precision needed, at sub-p.p.b. level, calculations on the lattice are out of consideration. Only the perturbative calculations
Table 4. Number of diagrams, number of self-mass families, typical number of integrals in the contribution to $\alpha^{-2}$ of one diagram and number of dimensions of MonteCarlo integrands.

| loop | vertex diagrams | self-mass families | integrals | MC dim |
|------|-----------------|--------------------|-----------|--------|
| 1    | 1               | 1                  | 14        | 1      |
| 2    | 7               | 3                  | $\sim 200$ | 3      |
| 3    | 72              | 15                 | $\sim 4000$ | 5      |
| 4    | 891             | 104                | $\sim 200000$ | 7      |
| 5    | 12672           | $<\approx 915$    | $\sim 10^7$ | 9      |
| 6    | 202770          |                    |           | 11     |

can attain such a precision. But there is a problem: a factorial growth of the size and the complexity as the number of loops increases, as it can seen in table 4. Whichever approach is used (numerical/analytical) one needs to deal with this rapid growth of complexity.

A bit of history

The first numerical result for the three-loop coefficient $A_3^{(6)}$ was obtained by Levine and Wright [31] in 1971 (MonteCarlo integration, 15% precision), followed 3 years after by a 2% result obtained by the group of Toichiro Kinoshita [32]. He and his collaborators kept to tirelessly increase the precision.

MonteCarlo integration of Feynman parametric integrals is straightforward, but suffers some drawbacks, in particular the difficulty to increase the precision (one digits more requires $10^2 \times$ the computation time), the enormous size of integrands (in the generation of integrand the polynomial is squared for each loop), and the strong numerical cancellations which may force to use (slow) multiple precision in some zones of the domain of integration. A complete analytical calculation does not have these problems.

I joined Ettore Remiddi in Bologna in 1989 as doctoral student. He was involved in the analytical calculation of the three-loop coefficient for a long time. In that time there were 3 families of diagrams (12 diagrams) still not known in analytical form. It took seven years to complete the analytical calculations.

One of the problems was that the methods used were highly diagram-specific, being combinations of dispersion relations and hyperspherical techniques. That is, one changes diagram, for example by moving a photon to another position on the same electron line and every analytical structure changes, and all must be done from scratch, again.

In 1994 I realized that an attack to the four-loop calculation with this method was almost impossible.

Thanks to a visit in Bologna of David Broadhurst, the relatively new technique of integration by parts in loop variables developed by Chetyrkin and Tkachov [50, 51] in 1981 came into help.

By operating in continuous $n$-dimensions, it allows to find linear relations between Feynman integrals with different power of denominators. The idea was to reduce expressions composed of a large number of complicated Feynman integrals into a (hopefully small) number of irreducible (called ‘master’) integrals. This makes a strong simplification (integrals on the boundary surfaces are zero by definition), but there is a problem.

The integration by parts provides an many linear relations among different integrals, with no indication on how to combine them. In literature the relations were combined with methods specific to the considered topology, an approach quite unsatisfactory if one wants to apply it to arbitrary diagrams.

Therefore, I studied the problem and devised an algorithm [13,14] able to solve automatically
Table 5. Number of diagrams, typical number of integrals in a system of identities, number of irreducible master integrals.

| loop | diagrams size of the system | master integrals |
|------|-----------------------------|------------------|
| 1    | 1                           | $\sim 10$        | 1                |
| 2    | 7                           | $\sim 10^3$      | 3                |
| 3    | 72                          | $\sim 10^5$      | 17               |
| 4    | 891                         | $\sim 10^7$      | 334              |

these systems of identities in order to reduce everything to the minimum number of integrals. The algorithm is general, and works for every topology, not only diagrams for the electron $g$-2.

Here is the effect of the application of the algorithm to electron $g$-2 At the same time (always in [13, 14]) I developed and presented a method of calculation of master integrals based on the solution of difference equations able to provide results with very high precision (i.e. thousands of digits).

In the late nineties there were no way of perform all the needed operations (symbolic, exact or floating point) within a single program. So I wrote from scratch a comprehensive tool, SYS, (30000 lines in C), containing all the needed parts, like a simplified algebraic manipulator, arithmetic routines for operation on polynomials and rational functions, an arbitrary precision solver of systems of difference and differential equations, etc, etc....

The development of SYS and the calculation of simpler 4-loop diagrams was done on desktop computers. But the whole calculation needed much more computer power. The key intervention was Thomas Gehrmann of the ITP of Zurich which in 2007 allowed me to use the cluster ZBOX2 of ITP and subsequently the Schrödinger supercomputer of the University of Zurich (now both decommissioned). The main part of the calculations was done there. Using SYS I was able to calculate all the 334 master integrals, with a precision of 1100 digits.

Why a precision so high? From a practical point of view 10-20 digits would be enough. This precision was necessary to reconstruct the analytical expressions from the numerical values. If successful the existence of an analytical fit increases enormously the reliability of the result.

There exists the algorithm PSLQ [52] which does this, provided that the unknown expression is a linear combination of known constants with rational coefficients. Usually it needs values with very high precision. For example, the fit of some specific master integrals required values with a precision of 2400, 4800 or 9600 digits.

The numerical fit

The result of the analytical fit for $A_1^{(8)}$ can be written as follows:

$$A_1^{(8)} = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U,$$

(17)
\[ T = \frac{1243127611}{130636800} + \frac{30180451}{25920} \zeta(2) + \frac{255842141}{2721600} \zeta(3) - \frac{8873}{3} \zeta(2) \ln 2 + \frac{6768227}{2160} \zeta(4) + \frac{19063}{360} \zeta(2) \ln^2 2 
+ \frac{12097}{90} \zeta(a + \frac{1}{2} \ln^2 2) - \frac{2862587}{6480} \zeta(5) - \frac{12729007}{64800} \zeta(3) \zeta(2) - \frac{221581}{2160} \zeta(4) \ln 2 
+ \frac{9656}{27} \zeta(a + \frac{1}{2} \ln^2 2) - \frac{12729007}{120 \ln^2 2} + \frac{191490067}{46656} \zeta(6) + \frac{10358551}{33200} \zeta(3)^2 - \frac{40136}{271} a_6 + \frac{26404}{27} \zeta(3) \ln^3 2 
+ \frac{700706}{27} \zeta(a) \zeta(2) - \frac{26404}{27} a_5 \ln 2 + \frac{26404}{27} \zeta(5) \ln 2 - \frac{63779}{50} \zeta(3) (2) \ln 2 - \frac{40723}{135} \zeta(4) \ln^2 2 + \frac{753201}{81} \zeta(3)^2 \ln^3 2 
+ \frac{2700}{7211262673} \zeta(2) \ln^2 2 + \frac{7577}{1620} \ln^6 2 + \frac{43545}{19584} \zeta(7) + \frac{193536}{116506} \zeta(1) \zeta(3) + \frac{85933}{63} a_4 \zeta(3) - \frac{75283}{18} \zeta(2) 
+ \frac{233012}{189} a_5 \ln 2 + \frac{432}{1705273} \zeta(3)^2 \ln 2 - \frac{80937}{2} \zeta(a) \zeta(2) + \frac{3024}{189} a_4 \zeta(2) \ln 2 + \frac{80937}{2} \zeta(5) \ln^2 2 - \frac{3995099}{6048} \zeta(3)^2 \ln^2 2 
- \frac{189}{189} a_5 \ln^2 2 + \frac{432}{1705273} \zeta(4) \ln^2 2 + \frac{602033}{4536} \zeta(3) \ln^4 2 - \frac{1650461}{11340} \zeta(2) \ln^5 2 + \frac{52177}{15870} \ln^7 2 
, \quad (18) \]

\[ V_a = \frac{14101}{480} \zeta(3) \zeta(2) C_l_2 (\frac{\pi}{3}) + \frac{494}{27} \operatorname{Im} H_{0,0,0,1,1,1,1} (e^{i \pi}) + \frac{20812}{297} C_l_6 (\frac{\pi}{3}) \]

\[ V_b = \frac{33487}{60} \operatorname{Re} H_{0,0,0,1,1,1,1} (e^{i \pi}) + \frac{13487}{60} \zeta(3) \zeta(2) C_l_2 (\frac{\pi}{3}) + \frac{136781}{360} \zeta(3)^2 \ln 2 + \frac{651}{4} \operatorname{Re} H_{0,0,0,1,1,1,1} (e^{i \pi}) \]

\[ W_b = \zeta(2) \left( -\frac{28276}{25} C_l_2 (\frac{\pi}{3})^2 + 104 \right) \left( 4 \operatorname{Re} H_{0,0,0,1,1,1,1} (i) + 4 \operatorname{Im} H_{0,0,1,1,1,1} (i) C_l_2 (\frac{\pi}{3}) - 2 C_l_4 (\frac{\pi}{3}) + \zeta(2) \ln 2 \right) \]

\[ E_a = \pi \left( -\frac{24858503}{691200} D_3 + \frac{250076961}{18662400} C_5 \right) + \frac{457593}{7776} f_0 (2, 0, 0, 1) + \frac{483913}{7776} f_0 (2, 0, 0, 1) + \frac{270433}{10935} f_0 (0, 2, 0) \]

\[ E_b = \zeta(2) \left( -\frac{2541575}{82944} f_1 (0, 0, 2) - \frac{556445}{6912} f_1 (0, 1, 1) + \frac{54515}{972} f_1 (0, 2, 0) - \frac{75145}{20736} f_1 (1, 0, 1) \right) \]

\[ -\frac{4715}{1458} \zeta(2) f_0 (1, 0, 1) \]
\[ U = -\frac{541}{300} C_{810} - \frac{629}{60} C_{816} + \frac{49}{3} C_{81c} - \frac{327}{160} C_{83a} + \frac{49}{36} C_{83b} + \frac{37}{6} C_{83c} . \]  

(24)

The integrals \( C_{B3j} \) are the \( \epsilon^0 \) coefficients of the \( \epsilon \)-expansion of six master integrals. In the above expressions \( \zeta(n) = \sum_{i=1}^{\infty} i^{-n} \), \( a_n = \sum_{i=1}^{\infty} 2^{-i} i^{-n} \), \( b_0 = H_{0,0,0,0,1,1}(\frac{1}{2}) \), \( b_7 = H_{0,0,0,0,0,0,1,1}(\frac{3}{2}) \), \( d_7 = H_{0,0,0,0,1,0,1,0}(1) \), \( \mathrm{Cl}_n(\theta) = \text{ImLi}_n(e^{i\theta}) \). \( H_{1,2,\ldots}(x) \) are the harmonic polylogarithms. The integrals \( f_j \) are defined as follows:

\[
f_m(i,j,k) = \int_1^9 ds \, D_1(s) \text{Re} \left( \sqrt{3^{m-1}} D_m(s) \left( s - \frac{9}{5} \right) \ln^i (9-s) \ln^j (s-1) \ln^k (s) \right) ,
\]

(25)

\[
D_m(s) = \frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}} K \left( m-1 - (2m-3) \frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3} \right) ;
\]

(26)

\( K(x) \) is the complete elliptic integral of the first kind. The constants \( B_3 \) and \( C_3 \), defined in [18], admit the hypergeometric representations:

\[
B_3 = \frac{4\pi^2}{3} \left( \frac{\Gamma^2(\frac{3}{5}) \Gamma(\frac{2}{3})}{\Gamma^2(\frac{2}{5}) \Gamma(\frac{1}{3})} \right)^{-1} F_3 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} : 1 \right) + \frac{\Gamma^2(\frac{3}{5}) \Gamma(-\frac{1}{3})}{\Gamma^2(\frac{2}{5}) \Gamma(-\frac{1}{3})} F_3 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} : 1 \right) ,
\]

(27)

\[
C_3 = \frac{486\pi^2}{1925} \gamma F_6 \left( \frac{7}{4}, \frac{7}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} : 1 \right) .
\]

(28)

Conclusions

- 1100-digits value of \( C_4 \) allows a successful analytical fit with (relatively) small coefficients; the ability to fit analytical expression to the numerical value guarantees that all digits computed are correct;
- the error from the 4-loop coefficient is now eliminated;
- experimental and theoretical values of \( a_\epsilon \) agree at level of 1.6\( \sigma \);
- the ultimate limit of the precision of the theoretical value is the error in the hadronic contribution \( \approx 10^{-14} \);
- the remaining QED error comes from the 5-loop coefficient.

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