NEW INEQUALITIES FOR INTERPOLATIONAL OPERATOR MEANS

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Abstract. The main goal of this article is to present several refinements and reverses of well known operator inequalities. These inequalities include operator means, operator monotone functions, operator log-convex functions and positive linear maps.

Among many other results, we show that for any \(0 \leq \alpha, \beta \leq 1\),
\[
f(A \nabla^{\alpha} B) \leq f\left((A \nabla^{\alpha} B) \nabla^{\beta} A\right) \leq f\left((A \nabla^{\alpha} B) \nabla^{\beta} B\right)
\]
whenever \(f\) is a non-negative operator log-convex function, \(A, B \in \mathcal{B}(\mathcal{H})\) are positive operators, and \(0 \leq \alpha, \beta \leq 1\). Further, we consider some inequalities of Ando’s type, and prove that if \(\Phi\) is a positive linear map, then
\[
\Phi(A \sharp^{\alpha} B) \leq \Phi\left((A \sharp^{\alpha} B) \sharp^{\beta} A\right) \leq \Phi\left((A \sharp^{\alpha} B) \sharp^{\beta} B\right) \leq \Phi(A) \sharp^{\alpha} \Phi(B).
\]

Many other refinements and reverses are shown by invoking ideas related to the so called interpolational operator means.

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REFERENCES

[1] T. ANDO, Concavity of certain maps on positive definite matrices and applications to Hadamard products, Linear Algebra Appl., 26 (1979), 203–241.
[2] T. ANDO AND F. HIAI, Operator log-convex functions and operator means, Math. Ann., 350 (2011), 611–630.
[3] T. ANDO AND X. ZHAN, Norm inequalities related to operator monotone functions, Math. Ann., 315 (1999), 771–780.
[4] J. S. AUJLA, Some norm inequalities for completely monotone functions, SIAM J. Matrix Anal. Appl., 22(2) (2000), 569–573.
[5] J. S. AUJLA, M. S. RAWLA AND H. L. VASUDEVA, Log-convex matrix functions, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat., 11 (2000), 19–32.
[6] J. C. BOURIN AND M. UCHIYAMA, A matrix subadditivity inequality for \(f(A + B)\) and \(f(A) + f(B)\), Linear Algebra Appl., 423 (2007), 512–518.
[7] J. I. FUJII AND E. KAMEI, Uhlmann’s interpolational method for operator means, Math. Japon., 34 (4) (1989), 541–547.
[8] T. FURUTA, J. MIČÍĆ HOT, J. PEČARIĆ AND Y. SEO, Mond-Pečarić Method in Operator Inequalities, Element, Zagreb, 2005.
[9] I. H. GÜMÜŞ, H. R. MORADI, AND M. SABABHEH, Further subadditive matrix inequalities, Math. Ineq. Appl., in press.
[10] F. HIAI, Matrix Analysis: Matrix Monotone Functions, Matrix Means, and Majorization, Interdisciplinary Information Science, 16 (2010),139–248.
[11] F. KUBO AND T. ANDO, Means of positive linear operators, Math. Ann., 246 (3) (1979), 205–224.
[12] J. LAWSON AND Y. LIM, Monotonic properties of the least squares mean, Math. Ann., 351 (2011), 267–279.
[13] E.-Y. LEE, A matrix reverse Cauchy-Schwarz inequality, Linear Algebra Appl., 430 (2009), 805–810.
[14] Y. Lim, *Convex geometric means*, J. Math. Anal. Appl., **404** (2013), 115–128.

[15] H. R. Moradi, Z. Heydarbeygi and M. Sababheh, *Subadditive inequalities for operators*, Math. Ineq. Appl., **23** (1) (2020), 317–327.