Bending analysis of generalized thermoelastic waves in a multilayered cylinder using theory of dual phase lagging

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Abstract. In this present problem, we construct the analytical model for wave propagation in a generalized thermoelastic multilayered composite hollow cylinder construct of inner and outer viscothermo layer fastened together by linear Elastic materials with voids (LEMV). To uncouple the equation of motion, and heat conduction equations, displacement potential functions are introduced. The frequency equations are derived for longitudinal and flexural modes of vibration and are studied numerically for heat conducting viscothermoelastic material. The computed dimensionless frequency is presented in the form of scattering curves against various physical variables. Adhesive layer LEMV is compared with Carbon Fiber Reinforced Polymer (CFRP). We found that the frequency wave characteristics are more in physical variables in the presence of thermal parameter.

1. Introduction

The thermoelasticity applications are used in various branches of science and technology, so a reasonable consideration has been built since last few centuries. It is acknowledged that the postulate of coupled thermoelasticity deteriorate from physical defect that the thermal signal grow with the boundless speed. In order to the extensive use of materials under high temperature in current technology and with the modern development of polymer and plastic industry, the investigation of viscoelastic materials playing a vital role in solid mechanics.

The extended history of thermoelastic damping investigation established from Zener [1] and other authors Nayfeh and Yuxin Sun [2,3] for dissimilar structures like plate, shaft, cylindrical panel etc. Bland and Christensen [4,5] describes a long analyses of viscothermoelasticity. Viscothermoelastic vibrations in micro-scale beam resonators with linearly varying thickness and advecut analysis of circular micro-plate resonators by using generalized viscothermoelasticity theory of DPL model are briefly described by Grover [6,7]. The generalized thermoelasticity theory with the dual phase lag effect has been developed by Da Yu Tzou [8], Naggar, Abd Alla, Fahmy and Ahmed [9] study about thermal stresses in a rotating non-homogeneous orthotropic hollow cylinder. Guo, Wang and Rogerson [10] analysis of thermoelastic damping in micro-and nanomechanical resonators based on DPL generalized thermoelasticity theory. Sharma, Grover and Sangal [11] statistically studied viscothermoelastic waves. Sharma, Mohinder Pal and Dayal Chand [12] investigated three-dimensional vibration analysis of a piezothermoelastic cylindrical panel. Paul and
Raman[13] discussed wave propagation in a hollow pyroelectric circular cylinder of crystal class 6. Also Cowin and Nunziato[14] introduced mathematical expressions for linear elastic materials with void. Adhucan analysis in circular micro-plate resonators by using generalized viscothermoelasticity theory of DPL model analyzed by Grover [15].

In this present problem we construct the analytical model for wave diffusion in a generalized thermoelastic multilayered composite hollow cylinder construct of inner and outer viscothermoelastic layer fastened together by linear Elastic materials with voids (LEMV). To uncouple the equation of movement, and heat convection equations, displacement potential functions are introduced. The frequency equations are derived for longitudinal and flexural modes of shivering and are analyzed numerically for heat conducting viscothermoelastic material.

2. Problem Formulation

In this segment we assume homogeneous transversely isotropic thermally intendance composite hollow cylinders of boundless length with constant temperature $T_0$ in the undisturbed state originally. Here, the $u_r$, $u_\theta$ and $u_z$ are denotes the radial, tangential and axial displacement components respectively.

\[
c_{11}(u_{rr}^l + u_{rr}^l r^{-1} - u_{rr}^l r^{-2}) - (c_{11} + c_{66})r^{-2}u_{,\theta}^l + r^{-2}c_{66}u_{,\theta\theta}^l + c_{44}u_{,zz}^l + (c_{13} + c_{44})u_{,rz}^l + r^{-1}(c_{12} + c_{66})u_{,rr}^l - \beta_l T_{,rr}^l = \rho_l u_{,tt}^l
\]

\[
(c_{44})(u_{,rr}^l + r^{-1}u_{,r}^l + r^{-2}u_{,\theta\theta}^l) + (c_{44} + c_{13})r^{-1}(u_{,rr}^l + u_{,\theta\theta}^l) + (c_{13} + c_{44})u_{,rz}^l + c_{33}u_{,zz}^l - \beta_3 T_{,zz}^l = \rho_l u_{,tt}^l
\]

The generalized heat equation based on non-classical viscothermoelasticity theory of non dual phase lagging model is given by

\[
K[T_{,rr}^l + r^{-2}T_{,\theta\theta}^l + T_{,zz}^l + \tau_l(T_{,rr}^l + r^{-1}T_{,r}^l + r^{-2}T_{,\theta\theta}^l + T_{,zz}^l),l] - \rho l c_v(T_{,r} + \tau_q T_{,tt}^l) + \beta^\ast T_0( u_{,r}^l + r^{-1}u_{,t}^l + \tau_q u_{,tt}^l) = 0
\]

where $C_v$ is the specific heat capacity, $K$ is the thermal conductivity, $\tau_0$ denotes phase lags of the heat flux and $\tau_T$ is the temperature gradient, $\beta^\ast = \beta^\ast (1 + (\beta_0), \beta_0 = (3\lambda_0 + 2\mu_1)\alpha T/\beta$ and $\beta = (3\lambda + 2\mu)\alpha T$ are the thermal moduli, $\alpha_0$ and $\alpha_1$ denotes mechanical relaxation times.

Solution of the field equation.

To discuss the effect of viscothermoelastic combining on the vibrations of a composite hollow cylinder, We take for a solution of the form \cite{[16],[17]}

\[
u^l(t, r, z) = (\phi^l) e^{ikz + \rho pt} \]

\[
u^l(t, r, z) = \frac{i}{a} W^l e^{ikz + \rho pt} \]

\[
T^l(t, r, z) = \frac{c_{44}}{\beta^* a^2} T^l e^{ikz + \rho pt}
\]

k denote wave number, $p$ denote angular frequency, $\phi^l$ and $W^l$ are denotes displacement potentials, $T^l$ is the temperature change and $a$ denotes geometrical parameter of the hollow cylinder. By introducing the dimensionless quantities such as $x = \frac{r}{a}$, $\zeta = ak$, $\Omega^2 = \frac{\rho a^2}{c_{44}^2}$, $\tilde{c}_{11} = \frac{c_{11}}{c_{44}}$, $\tilde{c}_{13} = \frac{c_{13}}{c_{44}}$, $\tilde{c}_{33} = \frac{c_{33}}{c_{44}}$, $\tilde{c}_{66} = \frac{c_{66}}{c_{44}}$, $\tilde{\beta} = \frac{\beta_l}{\beta_3}$, $\tilde{K}_l = \frac{(c_{44}^2)}{\beta^* T_0 \rho t}$, $\tilde{d} = \rho c_{44}^2 (\frac{\beta_3}{k^2 T_0})$, $\tilde{M} = \frac{\rho a^2}{c_{44}^2}$, $\tilde{M}_1 = \frac{K^2}{\beta^*}, \tilde{M}_2 = \frac{\rho c_{44}^2}{\beta^* (1 + \rho t)}$
Substituting equation (4) in equations (1),(2) and (3) we get

\[
\begin{align*}
[\bar{c}_{11}\nabla^2 - (\zeta^2 - \Omega^2)]\phi^j - \zeta(\bar{c}_{13} + 1)W^j - \bar{\beta}T^j &= 0 \tag{5} \\
\zeta(1 + \bar{c}_{13})\nabla^2 \phi^j + [\nabla^2 + (\Omega^2 - \bar{c}_{33}\zeta^2)]W^j - \zeta T^j &= 0 \tag{6} \\
i\nabla^2 \phi^j - MW^j + \frac{K\nabla^2}{\beta^2\alpha^2}(1 + \tau_t) - M_1 - M_2]T^j &= 0 \tag{7}
\end{align*}
\]

\[
\frac{(\Omega^2 - \zeta^2)}{\bar{c}_{66}} + \nabla^2 \psi = 0 \tag{8}
\]

where \(\nabla^2 = \frac{\partial^2}{\partial x^2} + r^{-1} \frac{\partial}{\partial r} + r^{-2} \frac{\partial^2}{\partial r^2} \).

The following differential equation reduced and simplified as above.

\[
(A^j\nabla^6 + B^j\nabla^4 + C^j\nabla^2 + D^j)(\phi^j, W^j, T^j)^T = 0 \tag{9}
\]

The solution of equation (9) is obtained as\([18],[19],[20]\)

\[
\begin{align*}
\phi^j &= \sum_{j=1}^{3} [A^j_j J_n(\alpha_j^j x) + B^j_j Y_n(\alpha_j^j x)], \\
W^j &= \sum_{j=1}^{3} d^j_j [A^j_j J_n(\alpha_j^j x) + B^j_j Y_n(\alpha_j^j x)], \\
T^j &= \sum_{j=1}^{3} e^j_j [A^j_j J_n(\alpha_j^j x) + B^j_j Y_n(\alpha_j^j x)] \tag{10}
\end{align*}
\]

Here \(J_n\) denotes Bessel function of the first kind of order \(n\) is, \((\alpha_j^j a)^2 > 0 \) (j = 1, 2, 3) are the zeros of the relation

\[
A^j(\alpha_j^j a)^6 + B^j(\alpha_j^j a)^4 + C^j(\alpha_j^j a)^2 + D^j = 0 \tag{11}
\]

The constants \(d^j_j\) and \(e^j_j\) mentioned in the equation (10) derived from the given equation

\[
\zeta(\bar{c}_{13} + 1)d_j + \bar{\beta}e_j = -[\bar{c}_{11}(\alpha_j^j a)^2 - \Omega^2 + \zeta^2]
\]

\[
[(\Omega^2 - \bar{c}_{33}\zeta^2) - (\alpha_j^j a)^2]d_j - \zeta e_j = (\alpha_j^j a)^2(\bar{c}_{13} + 1)\zeta
\]

From equation (8),

\[
\psi^j_n = [A^j_{4n} J_n(\alpha_j^j x) + B^j_{4n} Y_n(\alpha_j^j x)] \tag{12}
\]

where \((\alpha_j^j a)^2 = \Omega^2 - \zeta^2\). If \((\alpha_j^j a)^2 \leq 0\) the Bessel function \(J_n\) is replaced by the Modified Bessel function \(I_n\).
3. Conceptualization of Linearly Elastics Material with Void (LEMV)

The equations of motion for isotropic LEMV materials are given as

\[
\mu^l \nabla^2 \bar{u}^l + (\lambda^l + \mu^l) \nabla \nabla \bar{u}^l = \rho^l \ddot{u}^l_{,tt}
\]

where \( \bar{u}^l \) is the displacement potential, \( \lambda^l = C_{12}, \mu^l = C_{11} - \frac{C_{12}}{2} \) denotes lame's constant, \( \rho^l \) symbolize mass density and \( t \) symbolize time. To solve the above equation we take a solution as follows

\[
\begin{align*}
  u^l &= (u^l_r) \exp(kz + pt) \\
  w^l &= \left( \frac{i}{a} \right) w^l \exp(kz + pt)
\end{align*}
\]

The above solution and dimensionless variables \( x \) and \( \epsilon \), in (13) equation can be reduced as

\[
\begin{vmatrix}
  2\bar{\mu}^l + \bar{\lambda}^l & G_1^l \\
  G_2^l & \bar{\mu}^l \nabla^2_1 + G_4^l
\end{vmatrix} (u^l, w^l) = 0
\]

where

\[
\begin{align*}
  \bar{\lambda}^l &= \frac{\lambda^l}{C_{44}}, \quad \bar{\mu}^l = \frac{\mu^l}{C_{44}} \\
  G_1^l &= (ca)^2 \left( \frac{\rho^l}{\rho_i} \right) - \bar{\mu}^l \epsilon^2, \\
  G_2^l &= (\bar{\lambda}^l + \bar{\mu}^l) \epsilon^2, \\
  G_4^l &= \left( \frac{\rho^l}{\rho_i} \right) (ca)^2 - (\bar{\lambda}^l + 2\bar{\mu}^l) \epsilon^2
\end{align*}
\]

The equation (15) can be reduced as

\[
(\nabla^4_1 + P^l \nabla^2_1 + Q^l)(u^l, w^l) = 0
\]

where

\[
\begin{align*}
  P^l &= \frac{[(\bar{\lambda}^l + 2\bar{\mu}^l)G_4^l + \bar{\mu}^l G_1^l + (G_2^l)^2]}{(\bar{\lambda}^l + 2\bar{\mu}^l)\bar{\mu}^l} \\
  Q^l &= \frac{[G_1^l G_4^l]}{(\bar{\lambda}^l + 2\bar{\mu}^l)\bar{\mu}^l}
\end{align*}
\]

The solution of equation (16) are as follows

\[
\begin{align*}
  u^l &= \sum_{j=1}^{3} [A^l_j J_n(\alpha^l_j x) + B^l_j Y_n(\alpha^l_j x)] \\
  w^l &= \sum_{j=1}^{3} d^l_j [A^l_j J_n(\alpha^l_j x) + B^l_j Y_n(\alpha^l_j x)]
\end{align*}
\]

where \((\alpha^l_j)^2\) is the nonzero roots of \((\alpha^l_j)^4 + P^l (\alpha^l_j)^2 - Q^l = 0\), and \(d^l_j\) is arbitrary constant. Its obtained from

\[
d^l_j = \frac{[-(\bar{\lambda}^l + 2\bar{\mu}^l)(\alpha^l_j)^2 + G_4^l]}{G_2^l}
\]
4. Interface Boundary Conditions and Frequency Equations

The frequency equations of the problem can be enumerated applying the given boundary and conjoin conditions:

(i) On the smooth inner and outer surface

\[ \sigma_{rr}^l = \sigma_{rz}^l = T^l = 0 \quad \text{with} \quad l = 1, 3. \]  \hspace{1cm} (17)

(ii) At the conjoin between (outer-center and center-inner) hollow cylinder

\[ \sigma_{rr}^l = \sigma_{rz}^l = \sigma_{r\beta}^l, \quad \phi^l = \phi, \quad W^l = W_0, \quad T^l = 0, \quad \text{with} \quad l = 1, 2, 3, 4, 5. \]  \hspace{1cm} (18)

swapping the solutions in the boundary conjoin conditions, we obtained the following determinant form frequency equations. It is denoted as follows,

\[ |(N_{i,j})| = 0, \quad (i, j = 1, 2, 3, \ldots, 26) \]  \hspace{1cm} (19)

The nonzero elements at \( x_1 = \frac{a_2}{a_1} \), by varying \( j = 1, 2, 3 \) and \( k = 1, 2 \) are

\[ N(1, j) = 2\epsilon_{66} \left( \frac{a_j}{x_0} \right) J_1(\alpha_j x_0) + \left[ -\epsilon_{11}(\alpha_j)^2 + \epsilon_{13} \zeta d_j + \beta e_i \right] J_0(\alpha_j x_0), \]

\[ N(2, j) = (\zeta + d_j)(\alpha_j)J_1(\alpha_j x_0), \]

\[ N(3, j) = \frac{\epsilon_j}{x_0} J_0(\alpha_j x_0) - (\alpha_j)J_1(\alpha_j x_0), \]

At \( x_1 = \frac{a_1}{a_1} \),

\[ N(4, j) = 2\epsilon_{66} \left( \frac{a_j}{x_1} \right) J_1(\alpha_j x_1) + \left[ -\epsilon_{11}(\alpha_j)^2 + \epsilon_{13} \zeta d_j + \beta e_i \right] J_0(\alpha_j x_1), \]

\[ N(4, k + 6) = 2[\mu(\alpha_j)^2]J_1(\alpha_j x_1) + \left[ -\epsilon_{11}(\alpha_j)^2 + \epsilon_{13} \zeta d_j + \beta e_i \right] J_0(\alpha_j^2 x_1), \]

\[ N(5, j) = (\zeta + d_j)(\alpha_j^2)J_1(\alpha_j^2 x_1), \]

\[ N(5, k + 6) = -\mu(-\zeta + d_j)(\alpha_j^2)J_1(\alpha_j x_1), \]

\[ N(6, j) = -\zeta J_0(\alpha_j x_1), \]

\[ N(6, k + 6) = -\zeta J_0(\alpha_j x_1), \]

\[ N(7, j) = d_j J_0(\alpha_j x_1), \]

\[ N(7, k + 6) = -d_j J_0(\alpha_j x_1), \]

\[ N(8, j) = \frac{\epsilon_j}{x_1} J_0(\alpha_j x_1) - (\alpha_j)J_1((\alpha_j x_1)) \]

and the another elements are enumerated by the same from the above relations. The frequency equation acquired up above are valid for different solid middle and outer hollow materials of CdSe and various thickness of layers.

5. Numerical results and investigations

The purpose of the present study is demonstrating the various applications of composite materials with different adhesive core materials. To carry the numerical calculation we choose the material CdSe, its physical data as follows.
Figure 1. variation of thermoelastic adducent against wave number in presences of viscous effect.

Figure 2. variation of thermoelastic adducent against wave number in presences of without viscous effect.

c_{11} = 7.41 \times 10^4 \text{N}m^{-2}; c_{12} = 4.52 \times 10^4 \text{N}m^{-2}; c_{13} = 3.93 \times 10^4 \text{N}m^{-2}; c_{33} = 8.36 \times 10^4 \text{N}m^{-2}; c_{44} = 1 : 32 \times 10^4 \text{N}m^{-2}; \beta_1 = 0.621 \times 10^6 \text{NK}^{-1}m^{-2}; \beta_3 = 0.551 \times 10^6 \text{NK}^{-1}m^{-2}; \rho = 5504 \text{kgm}^{-3}; T_0 = 298K;

The variations of thermoelastic damping against dimensionless wave number has been shown in Fig:1 and Fig:2. The measured thermoelastic adducent with varying wave number (k) for a viscothermoelastic multilayered composite hollow cylinder using the postulate of DPL model of shivering modes for core/LEMV/Core shown in Fig:1. Its noted the measured adducent factor of shivering modes are originally increase to the high value for particular wave number, then oscillatory in the remaining value of wave number. The measured thermoelastic adducent with differing wave number (k) for a thermoelastic multilayered composite hollow cylinder using the postulate of the DPL model of shivering modes for core/LEMV/Core shown in Fig:2. Its noted that the measured adducent factor of shivering modes originally increase to the high value for the particular wave number, then reduced on the remaining value of wave number.

The variations of thermoelastic adducent against thickness has been shown in Fig:3 and Fig:4. The scaled thermoelastic adducent with differing thickness (h) for a viscothermoelastic multilayered composite hollow cylinder using the postulate of the DPL model of shivering modes...
for core/LEMV/Core in shown Fig:3. Its noted that the measured adducnet factor of shivering modes originally increase to the high value for particular thickness, and then reduced in the remaining level of thickness. The measured thermoelastic adducnet with differing thickness \((h)\) for a thermoelastic multilayered composite hollow cylinder using the postulate of the DPL model of shivering modes for core/LEMV/Core shown in Fig:4. Its noted that the measured adducnet factor of shivering modes originally increases to the high value for particular thickness, and then lessen in the remaining values of thickness. Its perceived that a shivering of the multilayered composite hollow cylinder has a high value in comparison to the core/lemv/core layers with a symmetric behavior.

6. Conclusions
The frequency equation is derived for vibrating waves of viscothermoelastic multilayered hollow cylinders with isotropic LEMV bonding layers. The constitutive equations of a linear viscothermoelastic material was formulated by the equations of motion of the composite hollow cylinder. The frequency equation an include the interaction between the composite hollow
cylinder and affixed layer are obtained for CdSe/LEMV/CdSe with no stress boundary conditions. Based on numerical calculations, its carried out for single layered viscothermoelastic hollow cylinder with LEMV and CFRP cores and the analyses of non dimensional thermal adducent against various physical parameters are represented as the scattering curves. Its observed that from the graphical representation a shivering of the multilayered composite hollow cylinder has a higher value in comparison to the CdSe/LEMV/CdSe layers with a symmetric behavior. The present article is applicable to a broad range in geophysics and gyroscopic sensor applications and its industry.

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