THE DISPERSION PROPERTY FOR SCHRÖDINGER EQUATIONS

par

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Résumé. — In this paper we analyze the dispersion property of some models involving Schrödinger equations. First we focus on the discrete case and then we present some results on graphs.

1. Introduction

Let us first recall some classical properties of the solutions of the Schrödinger equation. The solution of the equation
\[
\begin{cases}
  iu_t + u_{xx} = 0, & x \in \mathbb{R}, t \in \mathbb{R}, \\
  u(0, x) = \varphi(x), & x \in \mathbb{R},
\end{cases}
\]
can be obtained by using the Fourier transform as follows
\[
u(t) = S(t) \varphi = (e^{-4\pi^2|\xi|^2 t} \hat{\varphi})^\vee = e^{i|\xi|^2 / 4t} \ast \varphi.
\]
There are two properties that follow from the above representation. The first one is the conservation of the $L^2(\mathbb{R})$ norm:
\[
\|S(t) \varphi\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})}.
\]
The second one is the so-called dispersive property, which shows that the solutions of system (1) decay as time increases:
\[
\|S(t) \varphi\|_{L^\infty(\mathbb{R})} \leq \frac{1}{(4\pi|\xi|^2)^{1/2}} \|\varphi\|_{L^1(\mathbb{R})}.
\]
These simple properties can be used to obtain more refined estimates for the linear semigroup. There are properties of gain on integrability/regularity with respect to the initial data: the Strichartz estimates and the so-called local smoothing property. We state them here in their simplest form. Similar estimates can be written for the

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inhomogenous problem by using the $TT^*$ argument and eventually Christ-Kiselev's argument \[9\]. For any initial data in $L^2(\mathbb{R})$ Strichartz's estimates state that the solutions of (1) satisfy

\[
\|S(t)\varphi\|_{L^q_t(L^r_x(\mathbb{R}))} \lesssim \|\varphi\|_{L^2(\mathbb{R})}.
\]

A scaling argument forces the following admissibility condition for the pairs $(q, r)$:

\[
\frac{2}{q} + \frac{1}{r} = \frac{1}{2}.
\]

The local smoothing property means that the linear semigroup gains locally a one-half space derivative with respect to the initial data:

\[
\sup_{x \in \mathbb{R}} \int_{\mathbb{R}} |(-\Delta)^{1/4}(S(t)\varphi)|^2 dt \lesssim \|\varphi\|_{L^2(\mathbb{R})}.
\]

The above estimates play a crucial role in proving the well-posedness of the non-linear Schrödinger equation. We recall here only one result in this direction \[29\] and the interested reader should consult the classical references \[8, 28, 22, 26\]. For any initial data in $L^2(\mathbb{R})$ and $p \in (0, 4)$ there exists a unique solution $u \in C(\mathbb{R}, L^2(\mathbb{R})) \cap L^q_{\text{loc}}(\mathbb{R}, L^r(\mathbb{R}))$ of the equation

\[
\begin{cases}
iu + uu_x = |u|^pu, & x \in \mathbb{R}, t \in \mathbb{R}, \\
u(0, x) = \varphi(x), & x \in \mathbb{R}.
\end{cases}
\]

In this paper we analyze the dispersion property (3) in the context of discrete equations and of the equations on graphs. Similar estimates in the spirit of (4) and (5) can also be obtained.

2. Discrete equations

Let us consider the discrete Schrödinger equation

\[
\begin{cases}
iu_t + \Delta_d u = 0, & j \in \mathbb{Z}, t \in \mathbb{R}, \\
u(0, j) = \varphi(j), & j \in \mathbb{Z},
\end{cases}
\]

where, for any function $u : \mathbb{Z} \to \mathbb{R}$, the discrete Laplacian $\Delta_d$ acts as follows:

\[
(\Delta_d u)(j) = u(j + 1) - 2u(j) + u(j - 1).
\]

In this paper we will not discuss the possible numerical approximation of the NSE as in \[17, 18, 2, 23\] where there is a parameter $h$, the mesh size, that it is going to zero.

In the case of the discrete equation (7) the solutions decay as follows:

**Theorem 1.** — (\[16, 25\]) For any $\varphi \in l^1(\mathbb{Z})$ the solution of system (7) satisfies

\[
\|u(t)\|_{l^\infty(\mathbb{Z})} \lesssim (|t| + 1)^{-1/3}\|\varphi\|_{L^1(\mathbb{Z})}, \quad \forall t \in \mathbb{R}.
\]

Similar to the case of the real line we can write the solution of system (7) as a convolution:

\[
u(t, j) = \sum_{k \in \mathbb{Z}} K_t(j - k)\varphi(k),\]
where

\[ K_t(j) = \int_{-\pi}^{\pi} e^{-4it \sin^2(\xi/2)} e^{i j \xi} d\xi. \]

Using a classical result for oscillatory integrals, the Van der Corput Lemma [23, Ch. 1], we have that

\[ |K_t(j)| \lesssim (1 + |t|)^{-1/3}, \quad \forall j \in \mathbb{Z}. \]

This is a consequence of the fact that the second and the third derivatives of the phase function \( \psi(\xi) = 4\sin^2(\xi/2) \) do not vanish simultaneously at the same point: \( |\psi''| + |\psi'''| > C > 0 \). Observe that \( \|K_t\|_{L^p(\mathbb{Z})} \leq C \) for some positive constant, so, for any \( p \in [2, \infty] \),

\[ \|K_t\|_{L^p(\mathbb{Z})} \lesssim (|t| + 1)^{-1/3}, \quad \forall t \in \mathbb{R}. \]

This implies a decay property for \( u(t) \) in the \( L^p(\mathbb{Z}) \)-norm similar to the one in [8]:

\[ \|u(t)\|_{L^p(\mathbb{Z})} \lesssim (|t| + 1)^{-1/p} \|\psi\|_{L^1(\mathbb{Z})}. \]

However, in [24], the authors show that this decay property can be improved as follows

\[ \|u(t)\|_{L^p(\mathbb{Z})} \lesssim (|t| + 1)^{-\alpha_p} \|\psi\|_{L^1(\mathbb{Z})}, \]

where

\[ \alpha_p = \begin{cases} \frac{p-2}{2p}, & p \in [2, 4), \\ \frac{p-1}{3p}, & p \in (4, \infty]. \end{cases} \]

Discrete equations on positive integers can also be considered

\[ \begin{cases} iu_t(t, j) + (\Delta_x u)(t, j) = 0, & j \geq 1, t \neq 0, \\ u(0, j) = \psi(j), & j \geq 1. \end{cases} \]

One may also impose Dirichlet or Neumann boundary conditions at \( j = 1: u(t, 0) = 0 \) or \( u(t, 0) = u(t, 1) \), respectively. The solutions of these systems have similar decay.

In a joint paper with D. Stan [15], we considered the case of two discrete Schrödinger equations coupled at \( j = 0 \):

\[ \begin{cases} iu_t(j) + b_1^{-1}(\Delta_x u)(j) = 0, & j \leq -1, \\ iv_t(j) + b_2^{-1}(\Delta_x v)(j) = 0, & j \geq 1, \\ u(t, 0) = v(t, 0), & t > 0, \\ b_1^{-2}(u(t, -1) - u(t, 0)) = b_2^{-2}(v(t, 0) - v(t, 1)), & t > 0, \\ u(0, j) = \varphi(j), & j \leq -1, \\ v(0, j) = \varphi(j), & j \geq 1. \end{cases} \]

In the particular case when \( b_1 = b_2 \) the solution of system (10) can be written in terms of the solutions of some Dirichlet and Neumann problems. Thus, it decays with the same power \( (|t| + 1)^{-1/3} \). For \( b_1 \neq b_2 \) there are some difficulties. In matrix formulation system (10) can be written as \( iU_t + AU = 0 \) where \( U = (u(-j), u(j))_{j \neq 0} \).
and

\[ A = \begin{pmatrix}
... & ... & 0 & 0 & 0 & 0 \\
0 & b_1^{-2} & -2b_1^{-2} & b_1^{-2} & 0 & 0 \\
0 & 0 & b_1^{-2} & -b_1^{-2} - \frac{1}{b_1^2 + b_2^2} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{b_1^2 + b_2^2} & -b_2^{-2} & 0 \\
0 & 0 & 0 & 0 & b_2^{-2} & 0 \\
0 & 0 & 0 & 0 & 0 & ... \\
\end{pmatrix}. \]

Since the matrix is not diagonal we cannot use the Fourier transform. We use the explicit form of the resolvent \( R(\lambda) = (A - \lambda I)^{-1} \) and spectral calculus to write the following limiting absorption principle for \( \varphi \in L^1(\mathbb{R}) \)

\[ e^{itA}\varphi = \frac{1}{2it\pi} \int_{\sigma(A)} e^{it\omega} (R^+ (\omega) - R^- (\omega))\varphi d\omega, \]

where \( R^\pm = \lim_{\epsilon \to 0} R(\omega \pm i\epsilon) \). Using this representation we obtain that the solutions of system (10) also decay as \( (|t| + 1)^{-1/3} \). To prove the decay property it is enough to show that for any \( a \in (0, 1] \) there is a positive constant \( C(a) \) such that the following

\[ \int_0^\pi |e^{it(2\cos \theta + 2z \arcsin(a \sin \frac{\theta}{2})})e^{it\theta \sin \theta d\theta}| \leq C(a)(|t| + 1)^{-1/3} \]

holds for any real numbers \( y, z \) and \( t \). The main difficulty comes from the fact that we want an estimate that should be uniform with respect to the parameters \( y \) and \( z \). To obtain that we use previous improvements of the Van der Corput lemma given in [19] and new ones. Full details are given in [15].

Many questions can arise regarding the problems discussed above. Using matrices, the discrete equations can be rewritten as follows:

\[ iU_t + AU = 0. \]

We can assume that matrix \( A \) has a few non-identical vanishing diagonals. One of the questions that arises here is the following: what properties of matrix \( A \) guarantee the decay of the solutions of system (12)? We admit that also other types of decay may appear \( t^{-1/4}, t^{-\alpha} \log(t), \) etc... In the case when \( A \) is a diagonal matrix we can use Fourier analysis to reduce the question to the study of the roots of some trigonometric polynomials. It will be interesting to see how resolvent estimates similar to those in [7] used in the continuous case can be obtained and used in the matrix case. The analysis of system (11) has been mainly done in the context of the \( l^3(\mathbb{Z}) - l^\infty(\mathbb{Z}) \) decay of the solutions. It is still possible to refine the analysis to obtain better estimates as those in [24].

We have mainly analyzed the particular case of two discrete equations coupled at one point, but more general coupling conditions can be used or more coupled equations can be considered. We recall here [12] where some discrete models on trees have been analyzed.
3. Equation on graphs

In this section we present some results concerning the Schrödinger equation on graphs. The results obtained until now concern trees, i.e., graphs without cycles, having the last generation of edges formed from infinite strips. For a graph $\Gamma = (V, E)$ we consider the following equation

\begin{equation}
\begin{aligned}
&iu_t(t, x) + \Delta_\Gamma u(t, x) = 0, \quad x \in \Gamma, t \neq 0, \\
&u(0, x) = u_0(x), \quad x \in \Gamma.
\end{aligned}
\end{equation}

For the precise definition of operator $\Delta_\Gamma$ we refer to [14]. Denoting $u = (u_e)_{e \in E}$ system (13) means that on each edge $e \in E$ of the graph we have a Schrödinger equation for $u_e$ and at any vertex $v \in V$ we couple the equations on the edges that enter into $v$ by assuming for example, Kirchhoff’s law: continuity and sum of the normal derivatives to be zero. Other coupling conditions can be imposed: $\delta$ or $\delta'$ coupling as in [1] or those given in [20].

The first result regarding the dispersion property on trees has been obtained in [14] where the case of regular trees has been considered. The regular trees are special trees where all the edges in the same generation have the same length and all the vertices in the same generation have the same number of children. In this case we can make averages of the solutions defined in the same generation and reduce the analysis of dispersion on the tree to the case of the Schrödinger equation on the real line with a piecewise constant coefficient $\sigma$, $iu_t + (\sigma u_x)_x = 0$. When the coefficient $\sigma$ is given by a finite number of piecewise constant functions we have the following result.

**Theorem 2 ([3]).** — Consider a partition of the real axis $-\infty = x_0 < x_1 < \cdots < x_{n+1} = \infty$ and a step function $\sigma(x) = \sigma_i$ for $x \in (x_i, x_{i+1})$, where $\sigma_i$ are positive numbers. The solution $u$ of the Schrödinger equation

\begin{equation}
\begin{aligned}
&iu_t(t, x) + (\sigma(x)u_x)_x(t, x) = 0, \quad \text{for } x \in \mathbb{R}, t \neq 0, \\
&u(0, x) = \varphi(x), \quad x \in \mathbb{R},
\end{aligned}
\end{equation}

satisfies the dispersion inequality

$$
\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq C|t|^{-1/2}\|\varphi\|_{L^1(\mathbb{R})}, \quad t \neq 0.
$$

Using this result and an inductive argument on the generations of the tree, in [14] the dispersion property for the solutions of system (13) has been proved.

The case of the star-shaped tree has been considered in [3] not only with the classical Kirchhoff’s coupling, but also with the $\delta$ and $\delta'$ coupling. The advantage of the star-shaped tree is that explicit formulas can be easily obtained for the resolvent $(\Delta_\Gamma - \lambda I)^{-1}$ and then for the solutions of system (13). Classical results for oscillatory integrals allow us to obtain the dispersion property more easily.

The case of general trees with Kirchhoff’s coupling has been considered in [5]. In contrast to the case of the star-shaped tree, the resolvent is written in an implicit way by using results for almost periodic functions. The same problem with $\delta$-coupling has
been treated in [4]. We emphasize that in the particular case of a tree with all internal vertices having degree two this type of coupling corresponds to the Schrödinger equation on the real line where the Laplacian has been perturbed by a sum of Dirac deltas:

\[ H_\alpha = -\Delta + \sum_{j=1}^{p} \alpha_j \delta(x - x_j). \]

Under some technical assumptions on the strengths \( \{\alpha_j\}_{j=1}^{p} \) and on the lengths of the intervals \( \{x_{j+1} - x_j\}_{j=1}^{p-1} \) that exclude the existence of resonances ([21, 10, 4]) it has been proved that

\[ \left\| e^{-itH_\alpha} Pu_0 \right\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{|t|}} \| u_0 \|_{L^1(\mathbb{R})}, \quad \forall t \neq 0, \]

where \( P \) is the projection on the continuous part of the spectrum of \( H_\alpha \). A similar analysis can be performed on a tree with delta coupling conditions at the vertices. The interested reader can check all the details in [4].

The analysis offered so far concerns Kirchhoff’s coupling or the \( \delta \)-coupling. The \( \delta' \)-coupling has been only considered in the case of a star-shaped tree [1]. In the case of metric graphs, there are various types of coupling conditions that introduce a self-adjoint version of the Laplace operator \( \Delta(A, B) \). In [20] it was proved that for the coupling

\[ A(v)u(v) + B(v)u'(v) = 0, \]

the following conditions are necessary and sufficient: \((A(v), B(v))\) has maximal rank, i.e. \( d(v) \), and \( A(v)B(v)^T = B(v)A(v)^T \) for all vertices \( v \) of the graph \( \Gamma \). As far as we know up to now, the dispersion property has not been proved under general coupling conditions not even on star shaped trees.

One of the problems that emerges from the previous analysis is how to obtain some kind of dispersion when the graphs have cycles. We recall that in the case of a compact manifold without boundary [6] it was proved that the dispersion property holds for small intervals of time depending on the range of the eigenfunctions taken by the initial data. In the case of one-dimensional torus \( T^1 \) this was already known [19, Th. 5.3]:

\[ \left\| \sum_{|k| \leq N} a_k e^{i(tk^2 + kx)} \right\|_{L^\infty(T^1)} \lesssim |t|^{-1/2} \left\| \sum_{k} a_k e^{ikx} \right\|_{L^1(T^1)}, \quad \forall |t| \leq N^{-1}. \]

The existence of an argument based on oscillatory integrals that allows us to obtain the dispersion for \( T^1 \) on small time intervals gives us hope that in the case of graphs with cycles similar results could be obtained. In fact, for very particular structures as those in [11] the analysis performed in [13] shows that the dispersion holds for small intervals of time. The full understanding of the dispersion phenomena when a periodic structure is combined with infinite lines, i.e. a graph with few infinite external edges that contains a cycle, remains to be investigated. In this context we recall the case of a cylinder that was considered in [27].
Another model that can be considered on a graph structure is Dirac’s equation:

\[ iu_t = H u \]

where

\[ H = \begin{bmatrix} -i \partial_x & -1 \\ -1 & i \partial_x \end{bmatrix} . \]

As far as we know the dispersion property for this equation on graphs/trees has not been considered previously.

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