Hybrid Angle Control and Almost Global Stability of Non-synchronous Hybrid AC/DC Power Grids

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Abstract—This paper explores the stability of non-synchronous hybrid ac/dc power grids under the hybrid angle control strategy. We formulate detailed dynamical models for the ac grids and transmission lines, interlinking converters, and dc generations and interconnections. Next, we establish the existence and uniqueness of the closed-loop equilibria and demonstrate the global attractiveness of the equilibria, local asymptotic stability of the desired equilibrium point, and instability and zero-Lebesgue-measure region of attraction for other equilibria. The theoretical results are derived under mild, parametric, and unified stability/instability conditions. Further, we conclude the almost theoretic results are derived under mild, parametric, and unified stability/instability conditions. Further, it is envisioned that the ILCs are interconnected via generations and interconnections. Next, we establish the existence of equilibria for the closed-loop dynamics under a verifiable observation, this work explores the stability certificates of the primary and fast frequency controls that are respectively associated with the underlying synchronous machines and power converters in the aggregated COI model. In Subsection III-B, the connection of (1) to the other system dynamics is characterized. We emphasize that, in the sequel, all three-phase quantities are transformed to dq-coordinates aligned with \( \theta_g \) in (1a), hence, the ac impedance and admittance matrices are dynamic and depend on \( \omega_g \).

I. INTRODUCTION

The global paradigm shift toward harvesting energy from renewable sources has recently led to the emergence of hybrid ac/dc power grids. Such systems are typically comprised of several non-synchronous ac power grids that interact with each other through dc/ac interlinking converters (ILCs) that are interconnected by a dc transmission network [1]–[4]. The complex nonlinear dynamics of the hybrid ac/dc power grids with multiple timescales and interactions between the dc network, renewable generations, and ac grids renders the control of interlinking converters a daunting task. It has been recently reported that the grid-forming converter control techniques [5] are viable candidates for controlling the ILCs in hybrid ac/dc power grids [6]. In particular, [6] suggests that the matching control [7], [8] exhibits superior dynamic performance in hybrid ac/dc grids compared to classic control schemes for the interlinking converters, e.g., dual-droop control among others [6]. Inspired by this intriguing observation, this work explores the stability certificates of the hybrid angle control (HAC) [9] for multiple ILCs. We provide detailed linear/nonlinear dynamical models for the ac grids and transmission lines, ILCs, dc generations and interconnections. Next, we prove the existence and uniqueness of equilibria for the closed-loop dynamics under a verifiable assumption. Further, we prove the almost global asymptotic stability (AGAS) of hybrid ac/dc power grids with ILCs under the HAC. Last, we present a numerical verification of the presented results.

II. HYBRID AC/DC GRID MODEL DESCRIPTION

In this section, we describe a dynamical model of the hybrid ac/dc grids. We consider \( n \in \mathbb{Z}_{>0} \) ac grids, \( n \) ILCs, and define \( \mathcal{N}_{ac} \triangleq \{1, \ldots, n\} \) that collects the labels of the ac systems. Further, it is envisioned that the ILCs are interconnected via \( m \in \mathbb{Z}_{>0} \) dc lines; see Figure 1 for the overall configuration.

1) AC grids: we model the ac grids by aggregated dynamic center-of-inertia (COI) models [9], [10], i.e.,

\[
\begin{align}
\theta_g &= \omega_g, \\
\dot{\omega}_g &= \frac{J^{-1}}{m} \left( T_m - D_i \omega_g - D_d (\omega_g - \omega_i) - T_e \right), \\
T_m &= \tau_g^{-1} \left( T_r - \kappa_g (\omega_g - \omega_r) \right),
\end{align}
\]

where \( \theta_g \triangleq (\theta_{g,1}, \ldots, \theta_{g,n}) \in \mathbb{S}^n \) denotes the stacked vector of the absolute phase angles of the ac grids, \( \omega_g \triangleq (\omega_{g,1}, \ldots, \omega_{g,n}) \in \mathbb{R}^n \) denotes the vector of angular frequencies, \( J \triangleq \text{diag} \left( \{J_j\}_{j=1}^n \right) \in \mathbb{R}^{n \times n} \) denotes the diagonal matrix of the moment of inertia constants, \( T_m \triangleq (T_{m,1}, \ldots, T_{m,n}) \in \mathbb{R}^n \) denotes the vector of mechanical torques, \( D_i \triangleq \text{diag} \left( \{D_{ij}\}_{j=1}^n \right) \in \mathbb{R}^{n \times n} \) denotes the diagonal matrix of the aggregated damping constants associated with the friction torques that are proportional to the absolute frequencies, \( D_d \triangleq \text{diag} \left( \{D_{dj}\}_{j=1}^n \right) \in \mathbb{R}^{n \times n} \) denotes the diagonal matrix of the aggregated damping constants associated with the damper windings that are proportional to the frequency deviations, and \( T_e \triangleq (T_{e,1}, \ldots, T_{e,n}) \in \mathbb{R}^n \) collects the reference mechanical torque inputs for the turbines, \( \kappa_g \triangleq \text{diag} \left( \{\kappa_{g,j}\}_{j=1}^n \right) \) is the diagonal matrix of governor proportional control gains, and finally \( \omega_i \triangleq (\omega_{i,1}, \ldots, \omega_{i,n}) \in \mathbb{R}^n \) denotes the nominal frequencies of the ac grids (that are not necessarily identical). The damping terms in (1b) can be seen as a representation of the primary and fast frequency controls that are respectively associated with the underlying synchronous machines and power converters in the aggregated COI model. In Subsection III-B, the connection of (1) to the other system dynamics is characterized. We emphasize that, in the sequel, all three-phase quantities are transformed to dq-coordinates aligned with \( \theta_g \) in (1a), hence, the ac impedance and admittance matrices are dynamic and depend on \( \omega_g \).

2) AC transmission lines: the lines that couple the ac grid models (1) to the ILCs’ (see Figure 1) are modeled by [9]

\[
i_g = L_g^{-1} \left( v - (R_g - L_g \omega_g \otimes I_2) i_g - v_g \right),
\]

where \( i_g \triangleq (i_{g,1}, \ldots, i_{g,n}) \in \mathbb{R}^{2n} \) denotes line currents in the respective dq-frames that are aligned with the COI angles \( \theta_g \) and \( L_g \triangleq \text{diag} \left( \{L_{g,j} \otimes I_2\}_{j=1}^n \right) \in \mathbb{R}^{2n \times 2n} \) denotes the augmented inductance matrix associated with transmission lines, and \( \otimes \), \( I_2 \), and \( J_2 \) denotes the Kronecker product, 2-D identity matrix and rotation by \( \pi/2 \), respectively. Further, \( v \triangleq (v_1, \ldots, v_n) \in \mathbb{R}^{2n} \) denotes the ac output voltages of
the ILCs, $R_g \triangleq \text{diag} \left( \{R_{g,j} \otimes I_2\}_{j=1}^n \right) \in \mathbb{R}_+^{2n \times 2n}$ is the augmented diagonal resistance matrix of the line impedance, and $L_g \triangleq \text{diag} \left( \{L_{g,j}\}_{j=1}^n \right) \in \mathbb{R}_+^{n \times n}$ is the $n$-D diagonal reduction of $L_g$. Last, $v_{g,j} \triangleq (v_{g,1,j}, \ldots, v_{g,n,j}) \in \mathbb{R}^{2n}$ denotes the dynamic grid voltages.

3) Interlinking converters: the ILCs dynamics in dq-frames aligned with the COI angles in (1a) are given by [5], [6]

\[
\begin{align*}
\dot{\theta}_c &= \omega_c, \\
i_{dc,g} &= \tau_{dc}^{-1} (i_{dc,r} + \kappa_{dc} (v_{dc} - v_{dc,r}) - i_{dc,g}), \\
i_{dc} &= C_{dc}^{-1} \left( B_{dc,n} + i_{dc,g} - G_{dc} v_{dc} - m_{dc}^2 i \right), \\
i &= L^{-1} (m \theta - (R - L \omega_g \otimes J) i - v), \\
v &= C^{-1} (i - (G - C \omega_g \otimes J) v - i_g),
\end{align*}
\]

where $\theta_c \triangleq (\theta_{c,1}, \ldots, \theta_{c,n}) \in \mathbb{S}^n$ denotes the ILCs modulation angles evolving on the $n$-D torus and $\omega_c \triangleq (\omega_{c,1}, \ldots, \omega_{c,n}) \in \mathbb{R}^n$ denotes the converter frequency. The time constants associated with the first-order dc generation models are denoted by $\tau_{dc} \triangleq \text{diag} \left( \{\tau_{dc,j}\}_{j=1}^n \right) \in \mathbb{R}_+^{n \times n}$ and $i_{dc,g} \triangleq (i_{dc,g,1}, \ldots, i_{dc,g,n}) \in \mathbb{R}^n$ denotes the currents flowing out of the dc current sources that are collocated with the ILCs dc-sides, $i_{dc,r} \triangleq (i_{dc,r,1}, \ldots, i_{dc,r,n}) \in \mathbb{R}^n$ denotes the reference currents for the dc sources, and $\kappa_{dc} \triangleq \text{diag} \left( \{\kappa_{dc,j}\}_{j=1}^n \right) \in \mathbb{R}_+^{n \times n}$ denotes the matrix of proportional dc voltage control gains. The dc-link capacitances are denoted by the diagonal matrix $C_{dc} \triangleq \text{diag} \left( \{C_{dc,j}\}_{j=1}^n \right) \in \mathbb{R}_+^{n \times n}$, the signed incidence matrix associated with the directed graph of the dc interconnections is denoted by $B \in \mathbb{R}_+^{n \times m}$, $i_{dc,n} \triangleq (i_{dc,n,1}, \ldots, i_{dc,n,m}) \in \mathbb{R}^m$ collects the dc edge currents, and $G_{dc} \triangleq \text{diag} \left( \{G_{dc,j}\}_{j=1}^n \right) \in \mathbb{R}_+^{n \times n}$ denotes the nodal dc conductances that models the ILCs dc-losses and/or the resistive dc loads. The ILCs modulation signals are captured by $m(\delta) \triangleq (m_1(\delta), \ldots, m_n(\delta)) \in \mathbb{R}^{2n \times 1}$ with $m_j(\delta) = \mu_j r(\delta) \diamond e_j \in \mathbb{R}^{2n}$ where $e_j$ denotes the $j$-th orthonormal basis of $\mathbb{R}^n$, $r(\delta) \triangleq (\cos(\delta), \sin(\delta))$, and $\mu_j \in \mathbb{R}_{[0,1]}$ denotes the $j$-th modulation signal magnitude. Last, $i \triangleq (i_1, \ldots, i_n) \in \mathbb{R}^{2n}$ is the vector of the currents flowing through the ILCs output filters. Furthermore, $L \triangleq \text{diag} \left( \{L_j \otimes I_2\}_{j=1}^n \right) \in \mathbb{R}_+^{2n \times 2n}$ denotes the augmented diagonal matrix of ILCs filter inductances, $R \triangleq \text{diag} \left( \{R_j \otimes I_2\}_{j=1}^n \right) \in \mathbb{R}_+^{2n \times 2n}$ denotes the resistance matrix associated with the filter impedance, and $L \triangleq \text{diag} \left( \{L_{j}\}_{j=1}^n \right) \in \mathbb{R}_+^{n \times n}$ is the reduced version of $L$, $C \triangleq \text{diag} \left( \{C_j \otimes I_2\}_{j=1}^n \right) \in \mathbb{R}_+^{2n \times 2n}$ denotes the augmented diagonal matrix of filter capacitance, $G \triangleq \text{diag} \left( \{G_j \otimes I_2\}_{j=1}^n \right) \in \mathbb{R}_+^{2n \times 2n}$ is the filter conductance, and $C \triangleq \text{diag} \left( \{C_{j}\}_{j=1}^n \right) \in \mathbb{R}_+^{n \times n}$ is the reduced version of $C$.

4) DC interconnections: we model the dc lines with RL dynamics that [11], i.e.,

\[
i_{dc,n} = L_{dc}^{-1} (-B^T v_{dc} - R_{dc,i_{dc,n}}),
\]

where $L_{dc} \triangleq \text{diag} \left( \{L_{dc,j}\}_{j=1}^m \right) \in \mathbb{R}_+^{m \times m}$ and $R_{dc} \triangleq \text{diag} \left( \{R_{dc,j}\}_{j=1}^m \right) \in \mathbb{R}_+^{m \times m}$ respectively denote the diagonal inductance and resistance matrices associated with the dc lines. Note that we do not make any assumption on the sparsity of the underlying graph associated with the dc interconnections.

III. HYBRID ANGLE CONTROL AND STABILITY ANALYSIS

In this section, we equip the ILCs with the HAC, formulate the closed-loop dynamics, and present our stability analysis.

A. Hybrid angle control for interlinking converters

We define the frequency of the ILCs in (3a) according to the multi-variable grid-forming HAC [9], i.e.,

\[
\omega_c \triangleq \omega_t + \eta (v_{dc} - v_{dc,r}) - \gamma \sin \left( \frac{\delta - \delta_0}{2} \right),
\]

where $\eta \triangleq \text{diag} \left( \{\eta_j\}_{j=1}^n \right) \in \mathbb{R}_+^{n \times n}$ and $\gamma \triangleq \text{diag} \left( \{\gamma_j\}_{j=1}^n \right) \in \mathbb{R}_+^{n \times n}$ respectively denote the diagonal matrix of the dc and ac gains associated with HAC. Further, for any $z \in \mathbb{R}^n$, $\sin(z) \triangleq (\sin(z_1), \ldots, \sin(z_n))$. Last, $\delta \triangleq \theta_c - \theta_g$ denotes the vector of relative ILC-COI angles and $\delta_0 \triangleq (\delta_{0,1}, \ldots, \delta_{0,n}) \in \mathbb{S}^n$ collects the reference relative angles. We consider that all angular quantities evolve on the boundary of a Möbius strip, i.e., $M \triangleq \mathbb{R} \times [0,\pi]$ where $-\pi \equiv \pi \mod \pi$.[9], hence, $\delta \in M$. The HAC (5) resembles the hybrid control laws, e.g., see [4], [12]. Unlike the classic techniques, e.g., [6], [11], [13], the hybrid strategies unify the dc and ac feedback in a single controller. In particular, HAC encodes trade-off between the dc voltages (that relate to the dc energies) and the ac angles (that relate to the ac power flows) deviations.

B. Closed-loop analysis

In order to combine the models introduced in Section II, we first define the aggregated electrical torque and the voltage associated with (1). Similar to the modeling approach in [9], [14] we define the $j$-th stiff COI voltage (that resembles the synchronous generator electromotive force) as

\[
v_{g,abc,j} \triangleq B_j \omega_{g,j} \left( \sin \theta_{g,j}, \sin (\theta_{g,j} - \frac{2\pi}{3}), \sin (\theta_{g,j} - \frac{4\pi}{3}) \right),
\]
where \( v_{g,\text{abc},j} \) is the three-phase representation of \( v_{g,j} \) in (2) and \( b_j \in \mathbb{R}_{>0} \) is a constant. Note that we can alternatively simplify the frequency-dependent magnitude in (6) to a constant reference \( v_{r,j} \). The implicit assumption in (6) is that \( b_j \triangleq v_{r,j}/\omega_j^*, \) realizes the desired magnitude at the equilibrium frequency \( \omega_j^* \), for the \( j \)-th ac grid. Subsequently, the \( j \)-th electrical torque in (1b) is defined by [9], [14]

\[
T_{e,j} \triangleq \omega_j^* v_{g,\text{abc},j}^\top i_{g,\text{abc},j},
\]

where \( i_{g,\text{abc},j} \) is the three-phase representation of \( i_{g,j} \) in (2).

Finally, combining (1)-(7) yields the overall dynamics, i.e.,

\[
\dot{x} = K^{-1} f(x),
\]

where \( x \triangleq (\delta, i_{\text{dc,n}}, i_{\text{dc,g}}, v_{dc}, i, v, i_g, \omega_g, T_m), \)

\[
K \triangleq \text{diag} (I_n, L_{dc}, \tau_{dc}, C_{dc}, L, C, L_g, J, \tau_g),
\]

and

\[
f(x) \triangleq \begin{pmatrix}
\omega_t + \eta (v_{dc} - v_{dc,r}) - \gamma \sin ((\delta - \delta_0)/2) - \omega_g \\
-B^\top v_{dc} - R_{dc,i_{\text{dc,n}}} \\
i_{\text{dc,r}} - \kappa_{dc} (v_{dc} - v_{dc,r}) - i_{dc,g} \\
B_{dc,i} + i_{dc,g} - G_{dc,i} v_{dc} - m (\delta)^\top i \\
m (\delta) v_{dc} - (R - L \omega_g \otimes J_2) i - v \\
i - (G - C \omega_g \otimes J_2) v - i_g \\
v - (R_g - L \omega_g \otimes J_2) i_g - \psi_i t \\
T_m - D_1 \omega_g - D_2 (\omega_g - \omega_t) + \psi_i^\top i_g \\
T_t - \kappa_g (\omega_g - \omega_t) - T_{min}
\end{pmatrix},
\]

in which the three-phase quantities in (6) and (7) are transformed to the dq-frames aligned with \( \theta_{g,j} \) and written in terms of \( \psi \triangleq (\psi_1, \ldots, \psi_n) \in \mathbb{R}^{2n \times n} \) with \( \psi_1 \triangleq b_j r(0) \otimes e_j \in \mathbb{R}^{2^n} \) (hence \( r(0) \) is used since the dq-frame is aligned with the COI angle). We partition the state vector as \( x \triangleq (\delta, y) \in \mathbb{X} \triangleq \mathbb{X}^n \times \mathbb{R}^{10n+m} \) where \( y \triangleq (i_{dc,n}, i_{dc,g}, v_{dc}, i, v, i_g, \omega_g, T_m) \) and remark that \( f(x) \) is smooth in \( \mathbb{X} \). Last, we define \( D \triangleq D_1 + D_2 \).

**Assumption 1** (Frequency and dc voltage regulation)

Assume that the equilibrium frequency \( \omega_j^* \) and dc voltage \( v_{dc}^* \) of (8) coincide with the respective references \( \omega_t \) and \( v_{dc,r} \).

Assumption 1 implies requirements for frequency and dc voltage balancing across the ac/dc grids. This is met by an appropriate choice of reference-parameter pairs \((T_t, \kappa_g)\) and \((i_{dc,t}, \kappa_{dc})\) in (1c) and (3b), respectively [9]. Note that considering secondary integral-type controllers in (1c) and (3b) also ensures that Assumption 1 holds, but, the integral control hinders the frequency and dc voltage droop mechanisms that are crucial for load-sharing [6]. Thus, the blend of consistent references and adequately tuned proportional controllers is recommended for verification of Assumption 1. Last, Assumption 1 is conceptually similar to a widely recognized assumption in control of power systems that requires the given set-points to be consistent with (feasible) solutions of power flows equations, e.g., see [15] among others.

**Theorem 1** (Existence and uniqueness)

Under Assumption 1, the closed-loop dynamics (8) admits a unique equilibrium set that is described by

\[
\Omega^* \triangleq \{ (\delta^*, y^*) | \delta^*_j, \delta^*_{j+2\pi} + 2\pi, \forall j \in \mathbb{N}_m \},
\]

where \( y^* \) is unique with respect to (w.r.t.) \((\delta_t, v_{dc,r}, \omega_g)\) and \( \Omega^* \) only contains disjoint points that only differ in their angles.

The proof is provided in the Appendix A. Among all the points in \( \Omega^* \), \( x^*_j \triangleq (\theta_j, y^*) \) has a different stability nature (more on this later). Last, we define \( \Omega^*_u \triangleq \Omega^* \setminus x^*_j \).

**Theorem 2** (Decentralized certificates for global attractivity)

The equilibria of system (8) as in (9) are globally attractive if the following decentralized conditions hold for all \( j \in \mathbb{N}_m \):

\[
D_j > D_{\text{min},j} \quad \text{and} \quad \gamma_j > \gamma_{\text{min},j},
\]

where the critical COI damping, i.e., \( D_{\text{min},j} \) is defined by

\[
\frac{(L_j ||v_j^*||)^2}{R_j} + \frac{(C_j ||v_j^*||)^2}{G_j} + \frac{(L_{g,j} ||i_{g,j}^*||)^2}{R_{g,j}} + \frac{1}{2(D_j - D_{\text{min},j})},
\]

and the critical ILC angle damping, i.e., \( \gamma_{\text{min},j} \) is defined by

\[
\eta_j \left(1 + \frac{(\mu_j ||i_j^*||)^2}{R_j} + \frac{\eta_j (\mu_j ||v_{dc,j}^*||)^2}{G_{dc,j}} + \frac{1}{2(D_j - D_{\text{min},j})}\right).
\]

The proof is provided in the Appendix A. Next, we employ the function in (14), and leverage the Lyapunov’s direct method to establish the local asymptotic stability of \( x^*_j \) in \( \Omega^* \).

**Corollary 1** (Local asymptotic stability of \( x^*_j \))

Consider the closed-loop system (8) and the equilibrium point \( x^*_j = (\theta_j, y^*) \), then \( x^*_j \) is locally asymptotically stable if the stability conditions (10) are satisfied for all \( j \in \mathbb{N}_m \).

The proof is provided in the Appendix A.

**Corollary 2** (Instability and region of attraction of \( \Omega^*_u \))

Consider the closed-loop system (8), if the conditions (10) are satisfied for all \( j \in \mathbb{N}_m \) then all equilibria in \( \Omega^*_u \) are unstable with zero-Lebesgue-measure region of attractions.

Proof is skipped due to the lack of space but it follows from a standard albeit lengthy Schur complement analysis as in [9]. More precisely, the Jacobian of (8) admits at least one eigenvalue with positive real part when evaluated over \( \Omega^*_u \). Subsequently, by invoking the results of [14]–[16] one can show that the union of the regions of attraction of the equilibria in \( \Omega^*_u \) is a zero-Lebesgue-measure set. Intuitively speaking, \( \Omega^*_u \) contains the saddle points of the LaSalle function (14) since \( \mathcal{H}(\dot{x}) \) is globally convex and \( S(\delta) \) attains its local maxima on \( \Omega^*_u \); see [9]. Theorem 3 below combines the results of Theorem 2, Corollaries 1, and 2.

**Theorem 3** (Main result: AGAS)

The closed-loop system (8) is almost globally asymptotically stable with respect to the equilibrium \( x^*_j \) if the unified stability-instability conditions (10) are satisfied for all \( j \in \mathbb{N}_m \).

**Remark 1** (Features, conditions, and implementations)

First, HAC (5) provides two degrees of freedom for an optimal frequency tuning. Further, HAC integrates the complementary benefits of purely ac or dc-based grid-forming frequency control laws, i.e., the enhanced performance and robustness; see [5], [17] for a comparison. In addition, the AGAS result is obtained without requiring an assumption on...
the connectivity/sparsity of the dc interconnections. Second, conditions (10) are fully decentralized (i.e., they do not require non-local parameters) and confirm that the stability certificate of HAC for a single converter system is fully scalable; see [9]. Third, the damping requirement for the ILCs i.e., \( \gamma_i > \gamma_{\text{min}, j} \) does not require large physical damping but is met by an appropriate choice of the control parameters \( \gamma_i \) and \( \eta_i \). Next, the COIs damping requirements i.e., \( D_j > D_{\text{min}, j} \) is a reoccurring theme in related works; see [9] for details. Forth, conditions in (10) does not rely on the control of the dc energy sources. Thus, such generation units can be distributed within the dc network. Finally, the reader is referred to [9] for discussions on the implementation of the HAC (5). Note that under the dc power flow assumption and when reducing the ILCs’ filters to resistive-inductive elements, HAC is approximated by \( \omega_e \approx \omega_i + \eta_i (v_{dc} - v_{dc}) - \gamma \sin ((p - p_i)/2) \), where \( p \) and \( p_i \) respectively denote the active power flows between the ILCs and COIs and the associated references.

IV. NUMERICAL VERIFICATION

In this section, the qualitative behavior of the closed-loop dynamics (8) is presented. We consider two ILCs that interconnect two ac grids via a dc transmission line, i.e., we set \( n = 2 \) and \( m = 1 \) in (8); see Figure 1 for an illustration. The model and control parameters of the first subsystem are: \( \omega_{1,1} \approx 314 \text{ [rad/s]} \) (i.e., 50 Hz), \( \eta_1 = 0.01 \), \( v_{dc,1} = 3.168 \times 10^3 \text{ [V]} \), \( \gamma_i = 10^6 \), \( \delta_{1,1} = -0.1 \) [rad], \( \delta_{1,1} = 0.001 \) [rad], \( \delta_{1,1} = 0 \) [rad], \( \delta_{1,1} = 0.001 \) [rad], \( \delta_{1,1} = 0.001 \) [rad], \( \delta_{1,1} = 10^3 \), \( C_{dc,i} = 0.005 \) [F], \( G_{dc,i} = 1 \) [S], \( \delta_{1,1} = 2.59 \), \( \delta_{1,1} = 816.4 \text{ [V]} \), \( J_1 = 500 \text{ [s]} \), \( D_{1,1} = 1 \times 10^3 \), \( \delta_{g,1} = 5 \) [s], \( \gamma_{1,1} = 0.001 \) [rad], \( \mu_1 = 0.25 \), \( C_1 = 0.002 \) [F], \( b_1 = 2.59 \), \( v_{dc,1} = 5 \times 10^3 \), \( b_2 = 2.16 \). Figure 2 illustrates the convergence of state pairs \( (\delta_{1,1}, \omega_{2,1}, \delta_{1,1}, \omega_{2,1}) \) and \( (\omega_{2,1} - \omega_{1,1}, \omega_{2,1} - \omega_{1,1}) \) starting from random initial conditions in \( \mathbb{R} \). Since \( \delta_{1,1} = 0.1 \), the first ILC absorbs power from the first ac grid. In contrast, the second ILC injects power into the second ac grid, since \( \delta_{1,1} = 0.1 \). The power transfer over the dc interconnection is realized by \( v_{dc,1} \rightarrow v_{dc,1} \rightarrow v_{dc,2} \rightarrow v_{dc,2} \); presentation is skipped due to lack of space. It is noteworthy, that even if the dc voltage references are not selected appropriately the droop mechanism of the HAC [9] shifts the equilibrium dc voltages such that the power transfer is realized. Further, this behavior is achieved while coupling ac grids with significantly different angular frequencies. Figure 2 also qualitatively highlights the region of attraction of the stable equilibrium point characterized by \( \delta_{1} = 0 \) and instability of other angle equilibria in \( \mathbb{M}^2 \). We close by remarking that the presented results are preliminary and numerical verification of HAC performance in hybrid AC/DC grids requires further in-depth investigations.

V. CONCLUSIONS AND OUTLOOK

In this work, we presented a dynamical modelling of hybrid ac/dc grids and derived fully decentralized conditions for the existence, uniqueness, and global stability of the closed-loop equilibria. Our future work includes: 1) the stability analysis of the hybrid AC/DC grids under HAC while incorporating nonlinear constant power sources/loads, 2) revisiting and extending the analysis by considering the port-Hamiltonian representation, 3) deriving stability certificates for the systems that incorporate high-fidelity dc energy source models, e.g., wind generators, 4) stability analysis when decomposing the ac grid models into distributed generators, and 5) an extensive numerical verification of the control performance.

APPENDIX A: PROOF OF THE TECHNICAL RESULTS

Proof of Theorem 1. Setting the right-hand side (RHS) of (8) to zero, by Assumption (1), angle dynamics (8) at the equilibrium, i.e.,

\[
\omega_i + \eta_i (v_{dc} - v_{dc}) - \gamma \sin ((\delta_i - \delta_i)/2) - \omega_g = 0, \quad (11)
\]

reduces to \( \sin ((\delta_i - \delta_i)/2) = 0 \). This implies that the elements of the angle equilibrium \( \delta_i^* \), i.e., \( \delta_j^* \in \{\delta_i, j, \delta_i, j + 2\pi\} \) for all \( j \in \mathcal{N}_{ac} \). Further, Assumption 1 implies the existence of dc voltage and frequency equilibria, thus, their respective dynamics in (8) vanish at the equilibrium. Hence, \( i_{ac}^* = -R^{-1}B^T v_{dc,r} \), \( i_{dc,r}^* = i_{dc,r} \), and \( T_r^* = T_r \) that follow from the dc edge, dc generation, and torque dynamics in (8) at the equilibrium, respectively. Next, the ILC’s filter and transmission dynamics can be written as \( F y^* = h \) where \( F \neq 0 \)

\[
\begin{pmatrix}
-(R - L \omega_g \otimes J_2) & -I_{2n} \\
I_{2n} & -(G - C \omega_g \otimes J_2) \\
0 & -I_{2n}
\end{pmatrix} = 0
\]

\[
\begin{pmatrix}
0 \\
-I_{2n} \\
-(R - \omega_g \otimes J_2)
\end{pmatrix} = 0
\]

\[y^* = (i^*, v^*, i_{ac}^*)\] and \( h = (m(\delta^*) v_{dc}^* - 0, \psi \omega_g^*)\). Note that as in [9], the symmetric part of \( F, i.e., (1/2) (F + F^T) \neq 0 \) that means \( F \) is invertible and \( y^* \) is unique. Thus, the \( y^* \) in (9) is uniquely given by \( y^* = (f_{dc}^*, i_{dc}^*, v^*, i_{ac}^*, v^*, \omega_g^*, T_m^*) \) that completes the proof.

Proof of Theorem 2. Define the error coordinates \( \hat{x} \in \mathbb{R} \) w.r.t. \( x^* = (\delta, y^*) \) (as defined in Subsection (III-B)), i.e.,

\[
\hat{x} \triangleq \tilde{\delta} (\hat{\delta}, \hat{y}) \triangleq (\delta - \delta_i, i_{ac}^* - i_{ac}, i_{ac}^* - i_{dc}, v_{dc} - v_{dc}^*) \quad i - i^*, v - v^*, \hat{\delta} - \hat{\delta}^*, \omega_g - \omega_g^*, T_m - T_m^*)
\]

(12)

Subsequently, the translation of the closed-loop dynamics (8) to the coordinates (12) results in the error dynamics, i.e.,

\[
\hat{x} = K^{-1} f(\hat{x})
\]

(13)
where \( \hat{f}(\hat{x}) \equiv f(\hat{x} + x^*_{\gamma}) \) and is given by
\[
\begin{pmatrix}
\eta \hat{v}_{dc} - \gamma \sin(\hat{\delta}/2) - \omega_g \\
-B^T \hat{v}_{dc} - R_{dc} \hat{v}_{dc,n} \\
-m(\hat{\delta}) \hat{v}_{dc} - \hat{m}(\hat{\delta})^T i^* - (m(\delta))^T \hat{\delta} \\
\end{pmatrix}
\]
where we exploited the fact that \( f(x^*) = 0 \) and \( \hat{m}(\hat{\delta}) \equiv m(\delta) - m(\delta^*) \) denotes the vector of the trigonometric modulation errors. Consider the LaSalle function candidate:
\[
V(\hat{x}) \equiv S(\hat{\delta}) + H(\hat{\gamma}) = 2 \sum_{j \in \mathbb{N}_u} \lambda_j \left( 1 - \cos \left( \frac{\hat{\delta}}{2} \right) \right) + \frac{1}{2} (\hat{y}^T P \hat{y}),
\]
where for all \( j \in \mathbb{N}_u \), \( \lambda_j \in \mathbb{R}_>0 \) is a free parameter and \( P =: \text{diag}(L_{dc,dc^{(k)}}, C, L_{eg}, J, T_{dc^{(k)}}) \) > 0 (with the well-defined model and control parameters). Note that \( V(\hat{x}) > 0 \) for all \( \hat{x} \neq 0 \) (modulo \( 4\pi \)). For notational convenience we collect all \( \lambda_j \) in \( \Lambda \equiv \text{diag}(\lambda_j)_{j=1}^n \). We evaluate the time derivative of \( V(\hat{x}) \) along the solutions of (13), that is,
\[
\dot{V}(\hat{x}) = \sin(\hat{\delta}/2)^T (\lambda \hat{v}_{dc} - \lambda \gamma \sin(\hat{\delta}/2) - \omega_g) - \hat{\delta} \hat{v}_{dc,n} G_{dc} \hat{v}_{dc} - \hat{m}(\hat{\delta})^T i^* - (m(\delta))^T \hat{\delta} \hat{v}_{dc,n} G_{dc} \hat{v}_{dc} - \hat{m}(\hat{\delta})^T i^* - (m(\delta))^T \hat{\delta} \hat{v}_{dc,n} G_{dc} \hat{v}_{dc},
\]
where due to the quadratic structure of \( H(\hat{\gamma}) \) in (15), consider that \( \left( \hat{\delta}, \hat{\gamma} \right) \approx \left( \delta^*, \gamma^* \right) \) that, compared to \( \hat{x} \) in (12), replaces \( \hat{\delta} \) with its nonlinear counterpart \( \sin(\hat{\delta}/2) \) and resuffles the elements of \( \hat{x} \) as
\[
\hat{x}_1 \equiv \left( \sin \left( \frac{\hat{\delta}}{2} \right), \hat{v}_{dc,1}, \omega_1, \ldots, \sin \left( \frac{\hat{\delta}}{2} \right), \hat{v}_{dc,n}, \omega_n \right),
\]
\[
\hat{x}_2 \equiv \left( \hat{\delta} \hat{v}_{dc,n} G_{dc} \hat{v}_{dc}, \hat{m}(\hat{\delta})^T i^* - (m(\delta))^T \hat{\delta} \hat{v}_{dc,n} G_{dc} \hat{v}_{dc} \right).
\]
Hence, the RHS of (24) takes a quadratic form in \( \hat{x} \), i.e.,
\[
\dot{V}(\hat{x}) \leq -\hat{x}^T \mathbf{P} \hat{x} = -\left( \mathbf{x}_1^T Q \mathbf{x}_1 + 2 \mathbf{x}_2^T Q \mathbf{x}_2 \right),
\]
where \( \mathbf{Q} \equiv \text{diag}((Q_{1,j})_{j=1}^n) \) with \( Q_{1,j} \equiv \mathbf{Q}_{2,j} \equiv \mathbf{Q}_{2,1} \equiv \mathbf{Q}_{2,n} \equiv \mathbf{Q}_{2,1} \equiv \mathbf{Q}_{2,n} \equiv \mathbf{Q}_{2,1} \equiv \mathbf{Q}_{2,n} \) and \( \mathbf{Q}_{2,j} \equiv \mathbf{Q}_{2,1} \equiv \mathbf{Q}_{2,n} \equiv \mathbf{Q}_{2,1} \equiv \mathbf{Q}_{2,n} \) for all \( j \in \mathbb{N}_u \). This set of parameters directly implies the positive definiteness of \( \mathbf{Q} \). Next, standard Schur complement analysis yields that \( Q_{1,j} \equiv \text{positive definite if and only if (10) is satisfied}. \)
Since \( \dot{V}(\hat{x}) \leq 0 \) for any \( \hat{x}(0) \in \mathbb{X} \), then the c-sublevel sets of \( \mathbf{Q} \), i.e., \( \mathbb{X} \subset \{ x \in \mathbb{X} : \mathbf{Q}(\hat{x}) \leq c \} \) with \( c \equiv \mathbf{Q}(\hat{x}(0)) \), is a forward invariant and compact due to the boundedness of \( \hat{\delta} \) in \( \mathbb{M}_n \) (that is the union of \( n \) compact M"obius strip boundaries) and the radial unboundedness of \( \mathbf{H}(\hat{\gamma}) \). Hence, by invoking the LaSalle's invariance principle, the solutions of (13) globally converge to the largest invariant set \( \mathcal{M} \subset \Omega \).
\[-\hat{v}_d^T m(\hat{\delta})^T \hat{x}^* \leq \hat{v}_d^T \varphi_1 \hat{v}_dc + \sin(\hat{\delta}/2)^T \varphi_2 \sin(\hat{\delta}/2), \varphi_1 \triangleq \text{diag}\left(\left\{\epsilon_{1,j} \mu_j ||i^*_j||\right\}_{j=1}^n\right), \text{ and } \varphi_2 \triangleq \text{diag}\left(\left\{\epsilon_{1,j}^2\right\}_{j=1}^n\right). \quad (18)\]

\[-\hat{v}_d^T m(\hat{\delta})^T \hat{v}_dc \leq \hat{v}_d^T \varphi_1 \hat{v}_dc + \sin(\hat{\delta}/2)^T \varphi_2 \sin(\hat{\delta}/2), \varphi_1 \triangleq \text{diag}\left(\left\{\epsilon_{2,j}^2 \otimes I_2\right\}_{j=1}^n\right), \text{ and } \varphi_4 \triangleq \text{diag}\left(\left\{\mu_j v_{dc,j}/\epsilon_{2,j}\right\}_{j=1}^n\right). \quad (19)\]

\[-i^T \Lambda \varphi_1 \Lambda^T \Lambda \varphi_1 + \lambda^2 \varphi_2 \Lambda \varphi_2, \varphi_1 \triangleq \text{diag}\left(\left\{\epsilon_{2,j} \otimes I_2\right\}_{j=1}^n\right), \lambda \triangleq \text{diag}\left(\left\{|i^*_j|^2/2\epsilon_{2,j}\right\}_{j=1}^n\right). \quad (20)\]

\[-i^T C \varphi_1 C^T \varphi_1 \Lambda \varphi_1 + \lambda^2 \varphi_2 C \varphi_2, \varphi_1 \triangleq \text{diag}\left(\left\{\epsilon_{4,j} \otimes I_2\right\}_{j=1}^n\right), \lambda \triangleq \text{diag}\left(\left\{|i^*_j|^2/2\epsilon_{4,j}\right\}_{j=1}^n\right). \quad (21)\]

\[-i^T \Lambda \varphi_1 \Lambda^T \Lambda \varphi_1 + \lambda^2 \varphi_2 \Lambda \varphi_2, \varphi_1 \triangleq \text{diag}\left(\left\{\epsilon_{6,j} \otimes I_2\right\}_{j=1}^n\right), \lambda \triangleq \text{diag}\left(\left\{|i^*_j|^2/2\epsilon_{6,j}\right\}_{j=1}^n\right). \quad (22)\]

\[
Q_{11,j} \triangleq \left(\begin{array}{cc}
\lambda_j \gamma_j - \frac{1}{\epsilon_{1,j}^2} - \left(\frac{\mu_j i^*_j}{\epsilon_{2,j}}\right)^2 & -\frac{\lambda_j \eta_j}{2} \\
-\frac{\lambda_j \eta_j}{2} & G_{dc,j} - \left(\epsilon_{1,j} \mu_j ||i^*_j||\right)^2
\end{array}\right) - \left(\begin{array}{c}
\frac{\lambda_j}{2} \\
0
\end{array}\right)
- \left(\begin{array}{c}
D_j - \frac{L_j ||i^*_j||^2}{2\epsilon_{3,j}} - \left(\frac{C_j ||i^*_j||^2}{2\epsilon_{4,j}}\right) - \left(\frac{L_{eg,j} ||i^*_j||^2}{2\epsilon_{5,j}}\right)
\end{array}\right). \quad (23)\]

\[
\{\hat{x} \in \mathbb{X} : \hat{V}(\hat{x}) = 0\}. \text{ Under the conditions (10), } Q > 0 \text{ in (25). Thus, } \hat{V}(\hat{x}) = 0 \text{ iff } \hat{x} = 0. \text{ Finally, } \hat{x} = 0 \text{ characterizes a set that is identical to } \Omega^* \text{ in (9), i.e., } \Omega = \Omega^*. \]

**Proof of Corollary 1.** Consider the coordinates (12) that are written w.r.t. $x^*_i$ and dynamics (13). Note that $V(\hat{x})$ vanishes at the origin and $V(\hat{x}) > 0$ otherwise in $\mathbb{X}$. By Theorem 3, if (10) is satisfied, $V(\hat{x}) < 0$ in a sufficiently small open neighborhood of $x^*_i$ (that excludes any other equilibria in $\Omega^*$). The existence of such an open neighborhood is guaranteed since all equilibria in $\Omega^*$ are disjoint. Consider a sufficiently small $\epsilon$-sublevel set of $V(\hat{x})$ i.e., $\mathbb{L}_\epsilon \triangleq \{\hat{x} \in \mathbb{X} : V(\hat{x}) < \epsilon \in \mathbb{R} > 0\}$ such that it excludes all the equilibria in $\Omega^*$ except $x^*_i$. Note that for sufficiently small $\epsilon$, $V(\hat{x}) \leq 0$ for all $\hat{x} \in \mathbb{L}_\epsilon$. Thus, $\mathbb{L}_\epsilon$ is positively invariant w.r.t. (13). Last, applying the Lyapunov’s direct method concludes the asymptotic stability of $x^*_i \in \Omega^*$.

**APPENDIX B: MATHEMATICAL IDENTITIES**

**Lemma 1** (Algebraic and trigonometric identities)

For $u, v \in \mathbb{R}^2$, $\epsilon \in \mathbb{R}_{>0}$ and $\vartheta, \varrho \in \mathbb{S}^1$ the followings hold

\[
\pm u^T w \leq \epsilon^2 ||u||^2 + (1/2\epsilon)^2 ||u||^2, \quad (26)
\]

\[
\cos(\vartheta \pm \varrho) = \cos(\vartheta) \cos(\varrho) \mp \sin(\vartheta) \sin(\varrho), \quad (27)
\]

\[
\sin^2(\vartheta/2) = (1 - \cos \vartheta)/2. \quad (28)
\]

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