QNM\#s of branes, BHs and fuzzballs from Quantum SW geometries

Massimo Bianchi, Dario Consoli, Alfredo Grillo, Josè Francisco Morales,
Dipartimento di Fisica, Università di Roma “Tor Vergata” & Sezione INFN Roma2,
Via della ricerca scientifica 1, 00133, Roma, Italy

QNOs govern the linear response to perturbations of BHs, D-branes and fuzzballs and the gravitational wave signals in the ring-down phase of binary mergers. A remarkable connection between QNOs of neutral BHs in 4d and quantum SW geometries describing the dynamics of $\mathcal{N} = 2$ SYM theories has been recently put forward. We extend the gauge/gravity dictionary to a large class of gravity backgrounds including charged and rotating BHs of Einstein-Maxwell theory in $d = 4,5$ dimensions, D3-branes, D1D5 ‘circular’ fuzzballs and smooth horizonless geometries; all related to $\mathcal{N} = 2$ SYM with a single $SU(2)$ gauge group and fundamental matter. We find that photon-spheres, a common feature of all examples, are associated to degenerations of the classical elliptic SW geometry whereby a cycle pinches to zero size. Quantum effects resolve the singular geometry and lead to a spectrum of quantized energies, labeled by the overtone number $n$. We compute the spectrum of QNOs using exact WKB quantization, geodetic motion and numerical simulations and show excellent agreement between the three methods. We explicitly illustrate our findings for the case D3-brane QNOs.

I. INTRODUCTION

Direct detection of gravitational waves (GWs) produced in binary mergers allows to test General Relativity (GR) in extreme (e.g. strong-field) regimes and to discriminate Black Holes (BHs) from fuzzballs or other exotic compact objects (ECOs) in terms of their multipoles [11,12,13], shadows [5,6,7], or tidal effects [8,9].

The GW signal can be decomposed into three main phases: inspiral, merger and ring-down. The latter is dominated by the quasi-normal modes (QNOs) present in the linear response to perturbations. In the case of ECOs in alternative theories of gravity and fuzzballs in string theory [10], the ‘prompt ring-down signal’ decomposes in the early-stage resonant modes, produced around their ‘photon-spheres’ that may differ from the BH ones. At later stages both ECOs and fuzzballs produce a peculiar train of echoes, probing their internal ‘cavity’ and not only their external ‘walls’, with significant deviations from GR [11,12,13].

In the eikonal approximation, the real and imaginary parts of the QNM frequencies or the ‘prompt ring-down modes’ can be expressed as [9,15,20]

$$\omega_{QNM} \approx \omega_c(\ell) - i(2n + 1)\lambda_c, \quad (1)$$

where $\omega_c(\ell)$ is the orbital frequency of unstable ‘circular’ orbits forming the light-ring, while $\lambda_c$ is the Lyapunov exponent governing the chaotic behaviour of nearly critical geodesics around it.

The crucial role played by QNOs in discriminating BHs from fuzzballs or other ECOs motivated renewed effort in their determination with higher and higher accuracy [20,21]. Going beyond the WKB approximation proved to be a hard task even in the simpler GR context. QNOs of Schwarzschild BHs are governed by the celebrated Regge-Wheeler-Zerilli equation [22,23] that can be put in Schrödinger-like (canonical) form

$$\Psi''(z) + Q(z)\Psi(z) = 0 \quad (2)$$

where $Q(z)$ is a rational function with poles associate to horizons or singularities and zeros to turning points in the $\hbar \to 0$ limit. The problem is classically integrable but the wave equation is not exactly solvable and the QNOs are only known numerically, with high precision, though.

The Kerr BH case is even more involved in that the two equations for radial and angular motion, known as Teukolsky equations [24–28], talk to each other via the ‘separation constant’, the ‘wavy’ analogue of Carter constant.

A very efficient approach was later developed by Leaver [29,30] that allowed to determine numerically the QNOs of Kerr or Reissner-Nordstöm BHs by making use of continuous fractions.

More recently a remarkable connection with quantum Seiberg-Witten (SW) curves of $\mathcal{N} = 2$ super-symmetric Yang-Mills (SYM) theory on a Nekrasov-Shatashvili (NS) $\Omega$-background ($\epsilon_1 = \hbar$, $\epsilon_2 = 0$) [31,32] was revealed in [33]. In the NS background, the gauge theory is described by a differential equation of type (2), that is solved (exactly in $\hbar$) by the NS prepotential [37,38]. QNOs of neutral rotating BHs in four dimensions were computed by imposing exact WKB quantization conditions on the quantum phases of the SW curve. Possible extensions to BHs in AdS or in higher dimension were sketched in [36], while finite frequency greybody factor, QNOs and Love numbers of Kerr BHs were determined using irregular 2-d conformal blocks [33].

Aim of the present investigation is to extend the gauge/gravity dictionary to a large class of gravity backgrounds. The common feature of the examples we consider is the existence of a photon-sphere, that traps light on unstable null orbits eventually leaking out in ringdown modes. We find that QNOs are encoded in differential equations of type (2) with $Q_{SW}(y) = P_4(y)/P_2(y)^2$ given as a ratio of polynomials of degree 4 in the SW variable $y$.

We find that photon-spheres are associated to degeneration of the quantum elliptic SW geometry where a cy-
cle pinches to zero size in the semiclassical eikonal limit \( h \to 0 \). QNMs can be obtained by solving the exact WKB quantization conditions

\[
a_\gamma = \oint \lambda_{SW} = \hbar(n + \frac{1}{2})
\]

with \( \gamma \) the cycle vanishing at the photon-sphere and \( \lambda_{SW} \) the quantum SW differential

\[
\lambda_{SW} = \frac{\hbar}{2\pi} \sqrt{Q_{SW}(y)} dy + \ldots
\]

with dots denoting higher \( h \)-corrections. The singular geometry is resolved by quantum effects leading to a spectrum of quantized energies labeled by the overtone number \( n \).

We restrict ourselves to classical integrable geometries that allow to write down separate, but sometimes intertwined, ordinary differential equations, generalising the famous Teukolsky equations. We find that radial and angular equations can be mapped to \( N = 2 \) theories with a single SU(2) gauge group factor, possibly with hypermultiplets in the fundamental. The dictionary is not one-to-one: a single gravity background can be described in terms of apparently unrelated gauge theories with different numbers of flavours related by modular transformations of the underlying elliptic geometry.

The systems we will consider include Kerr-Newman BHs in \( D = 4 \), CCLP geometries describing charged and rotating solutions of Einstein-Maxwell theory in \( D = 5 \) [44][45], including Myers-Perry [46] and BMPV BHs [47], D3-branes and D1D5 circular fuzzballs [48].

For simplicity, we will only consider neutral scalar perturbations. Metric and vector perturbations lead to similar equations but the derivation is more laborious in general, and may spoil the elegance of the approach \[25\] [28].

One immediate outcome of our analysis is a better knowledge of their QNMs that play a crucial role in (in)stability analyses in these contexts [49][50].

The presentation is organised as follows. We describe the three different approaches to study QNMs: geometic motion, SW exact quantization, numerical methods based on continuous fractions. We illustrate the procedure for D3-branes as working example and present the gauge/gravity dictionary for a large number of BHs and D-brane gravity solutions. Detailed solutions of the various problems will be described in a forthcoming paper.

II. GEODETIC MOTION

Geodetic motion of massless neutral probes is governed by the null Hamiltonian \( \mathcal{H} = \frac{1}{2g^{MN}}P_M P_N = 0 \). If the dynamics is separable, radial and angular motion can be disentangled and effectively described by one dimensional Hamiltonians.

For example, for a D3-brane the metric reads

\[
ds^2 = H(r)^{-\frac{1}{2}}(-dt^2 + dx^2) + H(r)^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2)
\]

where \( x \) are the longitudinal coordinates, \( H(r) = (1 + \frac{L^4}{r^4}) \) and \( d\Omega_5^2 \) denotes the metric of the transverse round \( S^5 \)-sphere. Radial motion is governed by the null Hamiltonian \( \mathcal{H} \sim P_r^2 - Q_{geo}(r) \) with \( P_r \) the radial momentum and

\[
Q_{geo}(r) = \frac{\omega^2 (r^4 + L^4) - J^2 r^2}{r^4}
\]

where \( \omega^2 = E^2 - k^2 \) and \( J \) is the transverse angular momentum. Simple zeros \( r_+ \) of \( Q_{geo}(r) \) are associated to turning points, while double zeros \( r_0 \) define the photon-sphere, where light gets trapped orbiting around forever for specific choice of the frequency \( \omega_c \). The critical equations

\[
Q_{geo}(\omega_c, r_0) = 0
\]

can be solved for \( r_0 \) and \( \omega_c \). Nearly critical geodesics fall with radial velocity \( \dot{r} \approx -2\lambda_c (r - r_c) \), where

\[
\lambda_c = \left( \sqrt{2} \partial_\omega Q_{geo}(r_c, \omega_c) \right)^{-1} \sqrt{\partial_r^2 Q_{geo}(r_c, \omega_c)}
\]

is the Lyapunov exponent that characterises the chaotic behaviour of geodesics around the photon-sphere. For D3-branes one finds

\[
D3: \quad r_c = L , \quad \omega_c \approx \ell + 2 \frac{1}{\sqrt{2L}} , \quad \lambda_c = \frac{1}{2L}
\]

where we used \( J \approx \sqrt{\ell(\ell + 4)} \approx \ell + 2 \) in the large \( \ell \) limit.

III. WAVE EQUATION

The wave equation for a massless scalar field (or a scalar fluctuation of the metric) in the metric \( g_{MN} \) reads

\[
\Box \Phi = g^{-\frac{1}{2}} \partial_M (g^{\frac{1}{2}} g^{MN} \partial_N) \Phi = 0
\]

For the metric \[9\] using the ansatz

\[
\Phi = e^{-iEt + i\vec{k} \cdot \vec{x}} V_{\vec{k}}^\ell (\hat{\theta})
\]

with \( \nabla_{S^5} J_{\vec{m}}^\ell = -\ell(\ell + 1) J_{\vec{m}}^\ell \), one finds a radial equation in the canonical form \[2\] with

\[
Q(r) = 4\omega^2 (r^4 + L^4) - r^2 (4\ell(\ell + 4) + 15)
\]

with

\[
D3: \quad \Phi(r) \sim e^{i\omega r} \quad \text{as} \quad r \to +\infty
\]

and in-going waves in the deep interior of the photon-sphere (e.g. at the horizon) or vanishing at the centrifugal barrier of a smooth horizonless compact object). For D3-branes, one requires

\[
D3: \quad \Phi(r) \sim e^{i\omega r} \quad \text{as} \quad r \to 0
\]
In the semiclassical limit, where $\omega, \ell$ are large, the equation can be integrated and QNMs follow from the Bohr-Sommerfeld quantization condition
\[
\int_{r_-}^{r_+} \sqrt{Q(r)} \, dr = \pi \left( n + \frac{1}{2} \right)
\] (15)
with $r_\pm$ the inversion points, where $Q(r_\pm) = 0$. The integral \cite{15} can be approximated by expanding $Q(r)$ around its minimum at $r_0$ to quadratic order leading to
\[
\frac{Q(r_c)}{\sqrt{2 \partial_r^2 Q(r_c)}} = -i \left( n + \frac{1}{2} \right)
\] (16)
This equation can be solved by giving a small imaginary part to $\omega$, i.e. writing $\omega = \omega_R + i \omega_I$ and solving perturbatively in $\omega_I$. To linear order in $\omega_I$ one finds
\[
\ell^2 \omega \approx \frac{1}{\sqrt{2L}} \sqrt{(\ell + 2)^2 - \frac{1}{4}} - i \frac{1}{2L} (2n + 1)
\] (17)
in agreement with the geodetic results \cite{1, 9} at large $\ell$.

IV. QNMs FROM QUANTUM SW CURVES

The dynamics of $\mathcal{N} = 2$ gauge theory with fundamental matter in a non-trivial NS-background can be described by the differential equation
\[
\left[ q \gamma^2 P_+(x) y\gamma^2 + P_0(x) + y^{-\gamma} P_-(x) y^{-\gamma} \right] \Psi = 0
\] (18)
where
\[
P_+(x) = \prod_{i=1}^{N_+} (x-m_i), \quad P_-(x) = \prod_{i=N_++1}^{N_+} (x-m_i)
\]
\[
P_0(x) = x^2 + q \delta_{N_+3} x - \hat{u}
\] (19)
with $\hat{u} = u+q(m_1+m_2+m_3 - \frac{h}{2})\delta_{N_+3} - q \delta_{N_+2}$. Here $u = \frac{1}{2}(tr \varphi^2) = a^2 + \ldots$ parametrizes the Coulomb branch, $m_i$ the masses, $q = \Lambda^{2-N_+}$ the gauge coupling. Finally, $x, y$ are operators satisfying the commutation relation $[x, \ln y] = h$ and $\Psi$ an energy eigenstate.

One can view \cite{18} as an ordinary differential equation of second order in $y$ by setting $\hat{x} \equiv hy \partial_y$ or as a difference equation in $x$ by setting $y = e^{-h\partial_x}$. For instance, using $P(x)y = yP(x + h)$ to bring all the dependence on $y$ to the left, one can write \cite{18} as
\[
0 = \left[ A(y) \hat{x}^2 + B(y) \hat{x} + C(y) \right] \Psi(y)
\] (20)
\[
= \left[ q y^2 P_+(\hat{x} + \frac{h}{2}) + y P_0(\hat{x}) + P_-(\hat{x} - \frac{h}{2}) \right] \Psi(y)
\]
with $A(y), B(y), C(y)$ some polynomials of order at most two. For example, for $(N_+, N_-) = (1, 2)$, one finds
\[
A(y) = 1+y, \quad B = qy(y+1)-m_2-m_3-h
\]
\[
C(y) = qy^2(\frac{m_1}{2}-m_1)-y(\hat{u} + h\hat{q}) + (m_2+\frac{h}{2})(m_3+\frac{h}{2})
\] (21)
Massive fundamentals can be decoupled by sending $m \to \infty$ and $q \to 0$ keeping $q = -q \text{m} \text{ fixed}$. Bringing equation \cite{20} to canonical form one finds
\[
Q_{SW}(y) = \frac{4CA-B^2+2h(yB-AB')+h^2A^2}{4h^2y^2A^2}
\] (22)
Alternatively, viewing \cite{18} as a difference equation one can write the $h$-deformed SW equation \cite{54, 55}
\[
qM(x)W(x-h) + P_0(x)W(x) + 1 = 0
\] (23)
with $M(x) = P_+(x - \frac{h}{2})P_-(x - \frac{h}{2})$ and
\[
W(x) = \frac{1}{P_-(x + \frac{h}{2})} \Psi(x + h)
\] (24)
Eq. \cite{23} can be recursively solved order by order in $q$, $W(x) = -\frac{1}{P_0(x)} + \ldots$. The quantum period $a(u, q, h)$ can therefore be written as a sum over residues
\[
a(u, q, h) = \oint \lambda = 2\pi i \sum_{s=0}^{\infty} \text{Res} \sqrt{\alpha} \beta^s \lambda_s(x)
\] (25)
of the $h$-deformed Seiberg-Witten differential $\lambda_{SW}$
\[
2\pi i \lambda_{SW}(x) = -x d\ln W(x)
\] (26)
The $a_D$ period is computed in terms of NS prepotential $F(a, q, h)$ via the identification
\[
2\pi i a_D(a, q, h) = \partial_a F(a, q, h)
\] (27)
The prepotential can be determined by inverting $a(u)$ given in \cite{25} for $u(a) = a^2 + \ldots$ order by order in $q$, using the quantum version of the Matone relation $u(a, q, h) = q \lambda_{SW}(a, q, h)$ \cite{55} and adding the $q$-independent one-loop term.

The QNM frequencies are obtained by imposing the exact WKB conditions on the vanishing cycle and using the gauge/gravity dictionary following from
\[
Q(y) = Q(z) \frac{z'(y)^2}{2z'(y)} - \frac{3}{4} \left( \frac{z''(y)^2}{z'(y)} \right)^2
\] (28)
Using this dictionary, the WKB conditions translate into an equation for the frequencies $\omega$ that can be solved numerically.

V. THE D3-BRANE QUASI-NORMAL MODES

The radial wave equation for a scalar field (such as the dilaton) on a D3-brane background can be mapped to the quantum SW curve of pure $SU(2)$ gauge theory or equivalently to the Mathieu equation. The characteristic functions are
\[
Q_{SW}(y) = \frac{4y^2q + y(h^2 - 4u) + 4}{4h^2y^3}
\]
\[
Q_{Mathieu}(z) = \alpha - 2\beta \cos(2z)
\] (29)
and gauge/gravity/Mathieu parameters are related by
\[
\alpha = \frac{4u}{h^2} = (\ell + 2)^2, \quad \beta = \frac{4\sqrt{7}}{h^2} = \omega^2 L^2
\]
\[
r = \frac{h_{\omega} L^2}{2} \sqrt{y} = L e^{iz}
\] (30)
The general solution to Mathieu equation reads
\[ \Psi(z) = c_1 \text{me}(\alpha, \beta, z) + c_2 \text{me}(\alpha, \beta, -z) \]  
(31)
with \( \text{me}(\alpha, \beta, z) \) the exponential Mathieu function which is quasi-periodic \( \text{me}(\alpha, \beta, z+\pi) = e^{i\pi \nu} \text{me}(\alpha, \beta, z) \), with \( \nu(\alpha, \beta) \) the Floquet exponent. The latter can be related to the quantum \( \alpha \)-period at weak coupling and to the quantum \( a_D \)-period at strong coupling [57]. In the weak coupling limit, one finds [42]
\[ a(u, q) = \frac{\hbar}{2} \nu(\alpha, \beta) = \frac{\hbar}{2\pi i} \log \left[ \frac{\text{me}(\alpha, \beta, z)}{\text{me}(\alpha, \beta, z/2)} \right] \]  
(32)
that matches [25] after using the the weak coupling (\( \beta \ll \alpha \)) expansion of the Mathieu function
\[ \text{me}(\alpha, \beta, z) = e^{iz\nu} \left[ 1 - \frac{\beta}{4} \left( \frac{e^{2iz}}{\nu+1} - \frac{e^{-2iz}}{\nu-1} \right) + \ldots \right] \]  
(33)
and of its energy eigenvalue
\[ \alpha = \nu^2 + \frac{\beta^2}{2(\nu^2 - 1)} + \frac{(5\nu^2 + 7)\beta^4}{32(\nu^2 - 1)^2(\nu^2 - 4)} + \ldots \]  
(34)
The photon-sphere [0] corresponds to the opposite limit \( \alpha \approx \pm 2\beta \) (\( u \approx \pm 2\sqrt{q} \)) where the gauge theory is strongly coupled and the \( a_D \)-cycle degenerates. In this limit, the energy eigenvalue is given by [55]
\[ \alpha = -2\beta + 2s \sqrt{\beta - \frac{1}{8}(1 + s^2)} - \frac{s(s^2 + 3)}{2^7 \sqrt{\beta}} - \frac{5s^4 + 34s^2 + 9}{2^{12} \beta} + \ldots \]  
(35)
with \( s = 2\nu \) and the Floquet exponent \( \nu \) is now related to the \( a_D \)-quantum period [57]. The WKB quantization conditions translate then to \( s = n \) with \( n \) the overtone. QNM frequencies are obtained by plugging [30] into [35] and solving numerically for \( \omega \) as a function of \( \ell \) and \( n \).

The QNM wave function is obtained by imposing the boundary conditions [13] and [14] at \( z = \pm i\infty \). For \( n \) odd, the result can be written in terms of the Mathieu function \( M(3)(iz, \beta) \) [58]
\[ \psi(z) \sim \frac{e^{2i\sqrt{\beta} \cos z}}{(\cos z + 1)^{1/2}} \sum_{m=0}^{\infty} \frac{(-)^m a_m(\alpha, \beta)}{4i \sqrt{\beta} (\cos z + 1)^m} \]  
(36)
with \( z \) purely imaginary and the coefficients \( D_m \) determined by the recursion relation with \( D_{-1} = 0, D_0 = 1 \)
\[ (m+1)D_{m+1} + [(m+\frac{1}{2})^2 + 8i\sqrt{\beta} (m+\frac{1}{2}) + 2\beta - \alpha]D_m + 8i\sqrt{\beta} m(m-\frac{1}{2})D_{m-1} = 0 \]  
(37)

\[ \text{VI. NUMERICAL COMPUTATIONS} \]
To test the results obtained from geodetic motion and SW quantization one can solve the differential equation numerically using the method of continuous fractions introduced by Leaver [29]. We start from the ansatz
\[ \Phi(z) = e^{i\omega z}(z-z_-)^{\sigma_+} (z-z_+)^{\sigma_-} \sum_{n=0}^{\infty} C_n (\frac{z-z_+}{z-z_-})^n \]  
(38)
where \( z_+ > z_- \). The constants \( \sigma_+, \omega \) are determined by requiring that the ansatz solves the differential equation near \( z_+ \) and infinity, and imposing that only outgoing waves are present at infinity. On the other hand \( \sigma_- \) is fixed by requiring that the recursion involve only three terms. One finds
\[ \omega^2 = \frac{P_4(\nu)(z_+)}{4!} , \quad \sigma_+ (\sigma_+-1) + \frac{P_4(z_+)}{\Delta^2} = 0 \]  
\[ \sigma_- = -\sigma_+-i\omega \Delta + \frac{iP_4''(z_+)}{12\omega} \]  
(39)
with \( \Delta = z_+ - z_- \) and prime denoting derivatives wrt \( z \). There are two solutions for \( \sigma_+ \), generating two infinite towers of QNMs. Plugging the ansatz into the wave equation one finds the recurrence
\[ \alpha_n c_{n+1} + \beta_n c_n + \gamma_n c_{n-1} = 0 \]  
(40)
with \( c_{-1} = 0 \) and
\[ \alpha_n = (n+\sigma_+)(n+1+\sigma_+) + \frac{P_4(z_+)}{\Delta^2} \]  
\[ \beta_n = 2(n+\sigma_+)(\Delta\nu-n+\sigma_-) + \frac{P_4'(z_+)}{\Delta} - \frac{2P_4(z_+)}{\Delta^2} \]  
\[ \gamma_n = 2\Delta \nu (\sigma_-+\sigma_+) + (n-\sigma_-)(n-\sigma_--1)(n-\sigma_-) + \Delta^2 \nu^2 + \frac{P_4'(z_+)}{\Delta} + \frac{1}{2} \frac{P_4''(z_+)}{\Delta^2} \]  
(41)
The QNM frequencies \( \omega_n \) associated to the overtone \( n \) can be obtained by truncating the recursion to level \( N \) and solving numerically the equation
\[ \beta_n = \frac{\alpha_n-1}{\beta_{n-1}} \frac{\gamma_n}{\beta_{n-2}} + \frac{\alpha_n \gamma_{n+1}}{\beta_{n+1}} \frac{\sigma_{n+1} \gamma_{n+2}}{\beta_{n+2}} \]  
(42)
viewed as an equation for \( \omega_n \). For D3-brane, the differential equation can be put in the canonical form with two regular singular points and an irregular one by setting (with \( L=\hbar=1 \))
\[ r = 1 + 2y + 2\sqrt{y(1+y)} \]  
(43)
leading to
\[ Q(y) = \frac{4y(y+1) \left[ 8\omega^2(1+8y(y+1))-(\ell+2)^2+1 \right]+3}{16y^2(y+1)^2} \]  
(44)
which is the characteristic function of the \( \mathcal{N} = 2 \) SYM with \( SU(2) \) gauge group coupled to \( (N_+, N_-) = (1,2) \) flavours and parameters
\[ u = (\ell + 2)^2 + 2i\omega - 2\omega^2 , \quad q = 8i \omega \]  
\[ m_1 = m_2 = 0 , \quad m_3 = \frac{1}{2} \]  
(45)
There are two solutions around this point corresponding to \( \sigma_+ = \frac{1}{2} \) and \( \sigma_- = \frac{3}{2} \). Comparing with the results coming from geodetic motion, and the large \( q \) expansion of the Mathieu equation, one finds that the two choices reproduce QNM frequencies with even and odd overtones \( n \)'s respectively.

The results coming from geodetic motion, SW techniques and numerical methods are plotted in figure 1. We find excellent agreement between the three methods even for small \( \ell \), where the semi-classical geodetic approximation is not fully justified.

VII. EXAMPLES

Kerr-Newman BH: The wave equation is separable into two ordinary differential equations describing radial and angular motion and depending on the frequency \( \omega \), the separation constant \( K \), the mass \( M \), angular momentum variable \( a = J/M \), the charge \( Q \) and an orbital mode \( m_\phi \). Both radial and angular equations match that of \( SU(2) \) gauge theory with \( (N_+, N_-) = (1, 2) \) fundamental parameters \( \ell_1, \ell_2 \). Introducing the variables \( z = r^2 \) and \( \xi = \cos^2 \theta \) the radial and angular wave equations can be mapped to \( SU(2) \) gauge theory with \( (N_+, N_-) = (0, 2) \) flavours. The gauge/gravity dictionary for the radial wave equation reads

\[
\frac{\omega}{\sqrt{z_+ + z_-}} = -1 \quad \frac{\ell_1 \ell_2}{4} - (Q + \ell_1 \ell_2)^2 \quad (49)
\]

while the dictionary for the angular part \( (y = -\xi) \) reads

\[
\frac{\omega}{\sqrt{z_+ + z_-}} = -1 \quad \frac{\ell_1 \ell_2}{4} - (Q + \ell_1 \ell_2)^2 \quad (50)
\]

D1D5 fuzzball: Finally we consider a circular D1D5 profile with \( a \) the radius and \( Q_1 = Q_5 = L^2 \). The wave equation now separates into equations of type \( (2) \) with \( Q \)'s functions matching that of \( SU(2) \) gauge theory with \( (N_+, N_-) = (0, 2) \) fundamental hypermultiplets. The gauge/gravity dictionary for the radial variables is

\[
q = \frac{(\omega^2 - P_y^2)}{4} a^2 , \quad u = \frac{K^2 + 1 - (2L^2 - a^2)(\omega^2 - P_y^2)}{4}
\]

while for the angular ones, using \( y = -\cos^2 \theta \), one finds

\[
q = \frac{a^2}{4} (\omega^2 - P_y^2) , \quad u = \frac{K^2 + 1}{4} , \quad m_{1,2} = \frac{m_\phi \pm m_\psi}{2}
\]

Acknowledgments.

We would like to thank A. Aldi, C. Argento, G. Bonelli, V. Cardoso, G. Di Russo, D. Fioravanti, M. Firrota, F. Fucito, A. Grassi, T. Hikeda, M. Marinó, P. Pani, G. Raposo, R. Savelli, and Y. Zenkevich for interesting discussions and valuable suggestions.
[1] M. Bianchi, D. Consoli, A. Grillo, J. F. Morales, P. Pani, and G. Raposo, “Distinguishing fuzzballs from black holes through their multipolar structure,” *Phys. Rev. Lett.* **125** no. 22, (2020) 221601, [arXiv:2007.01743 [hep-th]].

[2] M. Bianchi, D. Consoli, A. Grillo, J. F. Morales, P. Pani, and G. Raposo, “The multipolar structure of fuzzballs,” *JHEP* **01** (2021) 003, [arXiv:2008.01445 [hep-th]].

[3] I. Bena and D. R. Mayerson, “Multipole Ratios: A New Window into Black Holes,” *Phys. Rev. Lett.* **125** no. 22, (Nov, 2020) 221602, [arXiv:2006.10750 [hep-th]].

[4] I. Bena and D. R. Mayerson, “Black Holes Lessons from Multipole Ratios,” *JHEP* **03** (2021) 114, [arXiv:2007.09152 [hep-th]].

[5] I. Bah, I. Bena, P. Heidmann, Y. Li, and D. R. Mayerson, “Gravitational Footprints of Black Holes and Their Microstate Geometries,” [arXiv:2104.10686 [hep-th]].

[6] M. Bianchi, A. Grillo, and J. F. Morales, “Chaos at the rim of black hole and fuzzball shadows,” *JHEP* **05** (2020) 078, [arXiv:2002.05574 [hep-th]].

[7] F. Bacchini, D. R. Mayerson, B. Ripperda, J. Davelaar, H. Olivares, T. Hertog, and B. Vercnocke, “Fuzzball Shadows: Emergent Horizons from Microstructure,” [arXiv:2103.12075 [hep-th]].

[8] E. J. Martinec and N. P. Warner, “The Harder They Fall, the Bigger They Become: Tidal Trapping of Strings through Microstate Geometries,” *JHEP* **04** (2021) 259, [arXiv:2009.07847 [hep-th]].

[9] I. Bena, A. Houppe, and N. P. Warner, “Delaying the Inevitable: Tidal Disruption in Microstate Geometries,” *JHEP* **02** (2021) 103, [arXiv:2006.13939 [hep-th]].

[10] S. D. Mathur, “The Information paradox: A Pedagogical introduction,” *Class. Quant. Grav.* **26** (2009) 224001, [arXiv:0909.1038 [hep-th]].

[11] V. Cardoso and P. Pani, “Tests for the existence of black holes through gravitational wave echoes,” *Nature Astron.* **1** no. 9, (2017) 586–591, [arXiv:1709.01525 [gr-qc]].

[12] M. R. Correia and V. Cardoso, “Characterization of echoes: A Dyson-series representation of individual pulses,” *Phys. Rev. D* **97** no. 8, (2018) 084030, [arXiv:1802.07735 [gr-qc]].

[13] V. Cardoso, E. Franzin, and P. Pani, “Is the gravitational-wave ringdown a probe of the event horizon?,” *Phys. Rev. Lett.* **116** no. 17, (2016) 171101, [arXiv:1602.07309 [gr-qc]]. [Erratum: *Phys. Rev. Lett.* **117** no. 8, (2016) 089902].

[14] D. R. Mayerson, “Fuzzballs and Observations,” *Gen. Rel. Grav.* **52** no. 12, (2020) 115, [arXiv:2010.09736 [hep-th]].

[15] V. Cardoso, A. S. Miranda, E. Berti, H. Witek, and V. T. Zanchin, “Geodesic stability, Lyapunov exponents and quasi-normal modes,” *Phys. Rev. D* **79** (2009), 064016, [arXiv:0812.1806 [hep-th]].

[16] B. Mashhoon, “Stability of charged rotating black holes in the eikonal approximation,” *Phys. Rev. D* **31** no. 2, (1985) 290–292, [arXiv:hep-th/9808099].

[17] B. F. Schutz and C. M. Will, “BLACK HOLE NORMAL MODES: A SEMIANALYTIC APPROACH,” *Astrophys. J. Lett.* **291** (1985) L33–L36.

[18] S. Iyer and C. M. Will, “Black Hole Normal Modes: A WKB Approach. 1. Foundations and Application of a Higher Order WKB Analysis of Potential Barrier Scattering,” *Phys. Rev. D* **35** (1987) 3621.

[19] M. Bianchi, D. Consoli, A. Grillo, and J. F. Morales, “Light rings of five-dimensional geometries,” *JHEP* **03** (2021) 210, [arXiv:2011.04334 [hep-th]].

[20] T. Ikeda, M. Bianchi, D. Consoli, A. Grillo, J. F. Morales, P. Pani, and G. Raposo, “Black-hole microstate spectroscopy: ringdown, quasi-normal modes, and echoes,” [arXiv:2103.10960 [gr-qc]].

[21] I. Bena, F. Eperon, P. Heidmann, and N. P. Warner, “The Great Escape: Tunneling out of Microstate Geometries,” *JHEP* **04** (2021) 112, [arXiv:2005.11323 [hep-th]].

[22] T. Regge and J. A. Wheeler, “Stability of a Schwarzschild singularity,” *Phys. Rev.* **108** (1957) 1063–1069.

[23] F. J. Zerilli, “Effective potential for even parity Regge-Wheeler gravitational perturbation equations,” *Phys. Rev. Lett.* **24** (1970) 737–738.

[24] S. A. Teukolsky, “Rotating black holes - separable wave equations for gravitational and electromagnetic perturbations,” *Phys. Rev. Lett.* **29** (1972) 1114–1118.

[25] O. J. C. Dias, M. Godazgar, and J. E. Santos, “Linear Mode Stability of the Kerr-Newman Black Hole and Its Quasinormal Modes,” *Phys. Rev. Lett.* **114** no. 15, (2015) 151101, [arXiv:1501.04625 [gr-qc]].

[26] P. Pani, E. Berti, and L. Gualtieri, “Gravitoelectromagnetic Perturbations of Kerr-Newman Black Holes: Stability and Isospinarity in the Slow-Rotation Limit,” *Phys. Rev. Lett.* **110** no. 24, (2013) 241103, [arXiv:1304.1160 [gr-qc]].

[27] P. Pani, E. Berti, and L. Gualtieri, “Scalar, Electromagnetic and Gravitational Perturbations of Kerr-Newman Black Holes in the Slow-Rotation Limit,” *Phys. Rev. D* **88** (2013) 064048, [arXiv:1307.7315 [gr-qc]].

[28] Z. Mark, H. Yang, A. Zimmerman, and Y. Chen, “Quasinormal modes of weakly charged Kerr-Newman spacetimes,” *Phys. Rev. D* **91** no. 4, (2015) 044025, [arXiv:1409.5800 [gr-qc]].

[29] E. W. Leaver, “An Analytic representation for the quasi normal modes of Kerr black holes,” *Proc. Roy. Soc. Lond. A* **402** (1985) 285–298.

[30] E. W. Leaver, “Quasinormal modes of Reissner-Nordstrom black holes,” *Phys. Rev. D* **41** (1990) 2986–2997.

[31] N. Seiberg and E. Witten, “Electric–magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory,” *Nucl. Phys. B* **426** (1994) 19–52, [arXiv:hep-th/9407087]. [Erratum: *Nucl.Phys.B* **430**, 485–486 (1994)].

[32] N. Seiberg and E. Witten, “Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD,” *Nucl. Phys. B* **431** (1994) 484–550, [arXiv:hep-th/9408099].

[33] M. Matone, “Instantons and recursion relations in N=2 SUSY gauge theory,” *Phys. Lett. B* **357** (1995) 342–348, [arXiv:hep-th/9506102].
N. A. Nekrasov and S. L. Shatashvili, “Quantization of Integrable Systems and Four Dimensional Gauge Theories,” in 16th International Congress on Mathematical Physics. 8, 2009. arXiv:0908.4052 [hep-th]

L. F. Alday, D. Gaiotto, and Y. Tachikawa, “Liouville Correlation Functions from Four-dimensional Gauge Theories,” Lett. Math. Phys. 91 (2010) 167–197 arXiv:0906.3219 [hep-th]

G. Aminov, A. Grassi, and Y. Hatsuda, “Black Hole Quasinormal Modes and Seiberg-Witten Theory,” arXiv:2006.06111 [hep-th]

A. Mironov and A. Morozov, “Nekrasov Functions and Exact Bohr-Zommerfeld Integrals,” JHEP 04 (2010) 040 arXiv:0910.5670 [hep-th]

Y. Zenkevich, “Nekrasov prepotential with fundamental matter from the quantum spin chain,” Phys. Lett. B 701 (2011) 630–639 arXiv:1103.4843 [math-ph]

J.-E. Bourgine and D. Fioravanti, “Quantum integrability of $N = 2$ 4d gauge theories,” JHEP 08 (2018) 125 arXiv:1711.07935 [hep-th]

D. Fioravanti and D. Gregori, “Integrability and cycles of deformed $N = 2$ gauge theory,” Phys. Lett. B 804 (2020) 135376 arXiv:1908.08030 [hep-th]

A. Grassi and M. Marino, “A Solvable Deformation of Quantum Mechanics,” SIGMA 15 (2019) 025, arXiv:1806.01407 [hep-th]

A. Grassi, J. Gu, and M. Mariño, “Non-perturbative approaches to the quantum Seiberg-Witten curve,” JHEP 07 (2020) 106 arXiv:1908.07065 [hep-th]

G. Bonelli, C. Iossa, D. P. Lichtig, and A. Tanzini, “Exact solution of Kerr black hole perturbations via CFT$_2$ and instanton counting,” arXiv:2105.04483 [hep-th]

Z. W. Chong, M. Cvetic, H. Lu, and C. N. Pope, “General non-extremal rotating black holes in minimal five-dimensional gauged supergravity,” Phys. Rev. Lett. 95 (Oct, 2005) 161301 arXiv:hep-th/0506029

Z. W. Chong, M. Cvetic, H. Lu, and C. N. Pope, “Five-dimensional gauged supergravity black holes with independent rotation parameters,” Phys. Rev. D 72 (Aug, 2005) 041901(R) arXiv:hep-th/0505112

R. C. Myers and M. J. Perry, “Black Holes in Higher Dimensional Space-Times,” Annals Phys. 172 (1986)