THREE-DIMENSIONAL CENTRAL CONFIGURATIONS
IN $\mathbb{H}^3$ AND $\mathbb{S}^3$

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Abstract. We show that each central configuration in the three-dimensional hyperbolic sphere is equivalent to one central configuration on a particular two-dimensional hyperbolic sphere. However, there exist both special and ordinary central configurations in the three-dimensional sphere that are not confined to any two-dimensional sphere.

Key Words: celestial mechanics; curved $N$-body problem; central configurations.

1. Introduction

The curved $N$-body problem is a natural extension of the Newtonian $N$-body problem in $\mathbb{R}^3$ to isotropic, complete, simply connected spaces of constant nonzero curvature, $\mathbb{S}^3$ and $\mathbb{H}^3$. For its history, we refer the readers to [3], where the equations of motion are written in extrinsic coordinates in $\mathbb{R}^4$ for $\mathbb{S}^3$, and the Minkowski space $\mathbb{R}^{3,1}$ for $\mathbb{H}^3$. This approach, different from more traditional ones like [9], led to fruitful results, especially in the study of relative equilibria, which are rigid motions that become fixed points in some rotating coordinates system, [2, 3, 4, 5, 6, 7].

Based on the work of Diacu, especially [3, 5], the authors of [8] proposed to study central configurations. Roughly speaking, central configurations are special arrangements of the point particles and the exact definition will be given later. The central configurations of the Newtonian $N$-body problem, first formulated by Laplace [10], are quite important in the study of the Newtonian $N$-body problem. In [8], the authors have also showed the importance of central configurations for the curved $N$-body problem. For instance, each central configuration gives rise to
a one-parameter family of relative equilibria, and central configurations are the bifurcation points in the topological classification of the curved $N$-body problem.

Some questions about these configurations were also raised in [8]. For example, find all central configurations for $N$ point particles when $N$ is small (the three-particles case has been recently solved and will appear in a forthcoming paper). Another interesting problem is to prove (or disprove) that for generic $N$ point particles, the number of equivalent classes of central configurations is finite. Though these questions are similar to those of the Newtonian $N$-body problem [11, 12], the answers are quite different in general. For example, Moulton’s theorem concerning the collinear central configurations has been generalized to $\mathbb{H}^3$, [8], but it can not be directly generalized to $S^3$.

In this paper, we put into the evidence another difference: each central configuration on $\mathbb{H}^3$ is equivalent to one central configuration on $\mathbb{H}^3_{xyw}$, which will be defined later, whereas in $S^3$ there are central configurations that are not confined to any two-dimensional sphere. In some sense, the number of central configurations in $\mathbb{H}^3$ is smaller than that of $S^3$. When we consider the Wintner-Smale conjecture in $\mathbb{H}^3$ raised in [8] asking whether the number of classes of central configurations in $\mathbb{H}^3$ is finite or not for generic $N$ point particles, we only need to study the problem on $\mathbb{H}^3_{xyw}$.

The paper is organized as follows: in Section 2, we recall the basic setting of the curved $N$-body problem and the corresponding facts about central configurations; in Section 3, we prove the result about central configurations in $\mathbb{H}^3$; in Section 4, we construct a two-parameter family of three-dimensional central configurations in $S^3$.

2. THE CURVED $N$-BODY PROBLEM AND CENTRAL CONFIGURATIONS

2.1. Equations of motion. As done in [3, 5], the equations will be written in $\mathbb{R}^4$ for $S^3$ and in the Minkowski space $\mathbb{R}^3,1$, for $\mathbb{H}^3$. For convenience, we understand the two linear spaces as $\mathbb{R}^4$ endowed with two inner products: for two vectors, $\mathbf{q}_1 = (x_1, y_1, z_1, w_1)^T$ and $\mathbf{q}_2 = (x_2, y_2, z_2, w_2)^T$, the inner products are given by

$$\mathbf{q}_1 \cdot \mathbf{q}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2 + \sigma w_1 w_2,$$

where $\sigma = 1$ for the Euclidean space and $\sigma = -1$ for the Minkowski space. We define the unit sphere $S^3$ and the unit hyperbolic sphere $\mathbb{H}^3$ as

$$S^3 := \{(x, y, z, w)^T \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\} \quad \text{and} \quad \mathbb{H}^3 := \{(x, y, z, w)^T \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 - w^2 = -1, \; w > 0\},$$

respectively. We can merge these two manifolds into

$$\mathbb{M}^3 := \{(x, y, z, w)^T \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + \sigma w^2 = \sigma, \; \text{with} \; w > 0 \; \text{for} \; \sigma = -1\}.$$
Given the positive masses \( m_1, \ldots, m_N \), whose positions are described by the configuration \( \mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N) \in (\mathbb{R}^3)^N \), \( \mathbf{q}_i = (x_i, y_i, z_i, w_i)^T \), \( i = \overline{1,N} \), we define the singularity set
\[
\Delta = \bigcup_{1 \leq i < j \leq N} \{ \mathbf{q} \in (\mathbb{R}^3)^N ; \mathbf{q}_i \cdot \mathbf{q}_j = \pm 1 \}.
\]
Let \( d_{ij} \) be the geodesic distance between the point masses \( m_i \) and \( m_j \), which is computed by
\[
d_{ij}(\mathbf{q}) = \arccos(\mathbf{q}_i \cdot \mathbf{q}_j) \quad \text{for } S^3, \quad d_{ij}(\mathbf{q}) = \arccosh(-\mathbf{q}_i \cdot \mathbf{q}_j) \quad \text{for } H^3.
\]
The force function \( U(\mathbf{q}) \) in \((\mathbb{R}^3)^N \setminus \Delta\) is
\[
U(\mathbf{q}) := \sum_{1 \leq i < j \leq N} m_i m_j \cot d_{ij}(\mathbf{q}),
\]
where \( \cot(x) \) stands for \( \cos(x) \) in \( S^3 \) and \( \cosh(x) \) in \( H^3 \). We also introduce two more notations, which unify the trigonometric and hyperbolic functions,
\[
\sinh(x) = \sin(x) \quad \text{or } \sinh(x), \quad \cosh(x) = \cos(x) \quad \text{or } \cosh(x).
\]
Define the kinetic energy as
\[
T(\dot{\mathbf{q}}) = \sum_{i=1}^{N} m_i (\dot{x}_i^2 + \dot{y}_i^2), \quad \dot{\mathbf{q}} = (\dot{\mathbf{q}}_1, \ldots, \dot{\mathbf{q}}_N).
\]
Then the curved \( N \)-body problem is given by the Lagrange system on \( T((\mathbb{R}^3)^N \setminus \Delta) \), with
\[
L(\mathbf{q}, \dot{\mathbf{q}}) := T(\dot{\mathbf{q}}) + U(\mathbf{q}).
\]
Using variational methods, we obtain the equations of motion in \( S^3 \) and in \( H^3 \), \( \mathbb{S} \). Merged into one, they are
\[
\begin{align*}
\ddot{\mathbf{q}}_i &= \sum_{j=1, j \neq i}^{N} \frac{m_i m_j (\mathbf{q}_j - \mathbf{q}_i)}{\sinh d_{ij}} - \sigma m_i (\dot{\mathbf{q}}_i \cdot \mathbf{q}_i) \mathbf{q}_i, \\
\mathbf{q}_i \cdot \mathbf{q}_i &= \sigma, \quad i = \overline{1,N}.
\end{align*}
\]
The first part of the acceleration access from the gradient of the force function, \( \nabla_{\mathbf{q}_i} U(\mathbf{q}) \), and we will denote it by \( \mathbf{F}_i \). It is the sum of \( \mathbf{F}_{ij} = \frac{m_i m_j (\mathbf{q}_j - \mathbf{q}_i)}{\sinh d_{ij}} \) for \( j \neq i \).

2.2. Central configurations.

Definition 1. A configuration \( \mathbf{q} \in (\mathbb{R}^3)^N \setminus \Delta \) is called a central configuration if there is some constant \( \lambda \) such that
\[
\nabla_{\mathbf{q}_i} U(\mathbf{q}) = \lambda \nabla_{\mathbf{q}_i} I(\mathbf{q}), \quad i = \overline{1,N},
\]
where \( \nabla \) is the gradient operator in \( \mathbb{H}^3 \), \( I(\mathbf{q}) = \sum_{i=1}^{N} m_i (x_i^2 + y_i^2) \), and the explicit form of \( \nabla_{\mathbf{q}_i} I \) is
Theorem 1. Each central configuration in \( H^3 \) is equivalent to some central configuration on \( H^2_{xyz} \).

Proof. We first show that all central configurations in \( H^3 \) must lie on a two-dimensional hyperbolic sphere. Then we show that there is some action \( \chi \in \)

\[
(2) \quad 2m_i \begin{bmatrix} x_i(w_i^2 + z_i^2) \\ y_i(w_i^2 + z_i^2) \\ -z_i(x_i^2 + y_i^2) \\ -w_i(x_i^2 + y_i^2) \end{bmatrix} \text{ in } T(S^3)^N \quad \text{ and } \quad 2m_i \begin{bmatrix} x_i(w_i^2 - z_i^2) \\ y_i(w_i^2 - z_i^2) \\ z_i(x_i^2 + y_i^2) \\ w_i(x_i^2 + y_i^2) \end{bmatrix} \text{ in } T(H^3)^N.
\]

Since the two functions \( U \) and \( I \) are both invariant under the group action of \( SO(2) \times SO(1,1) \) (in the case of \( H^3 \)), it is easy to check that a central configuration remains a central configuration after an \( SO(2) \times SO(2) \) action (or an \( SO(2) \times SO(1,1) \) action), \([8]\). Two central configurations are said to be equivalent if one can be transformed to the other by these group actions. When we say a central configuration, we mean a class of central configurations as defined by the equivalence relation.

A central configuration with \( \lambda = 0 \) is called a special central configuration, which only occurs in \( S^3 \), \([3]\). Otherwise, it is called an ordinary central configuration. A central configuration lying on a geodesic is called a geodesic central configuration. A central configuration lying on a two-dimensional sphere is called an \( S^2 \) central configuration, a central configuration lying on a two-dimensional hyperbolic sphere is called an \( H^2 \) central configuration. All the other central configurations are called three-dimensional central configurations.

Here, a two-dimensional sphere (hyperbolic sphere) means a sphere (hyperbolic sphere) isometric to the unit sphere (hyperbolic sphere) in \( R^3 \left( R^{3,1} \right) \). It is the non-empty intersection of \( M^3 \) with a three-dimensional linear subspace: \( \{ (x, y, z, w)^T \in R^4 | ax + by + cz + dw = 0 \} \), \([1]\). We begin with the following result.

Proposition 1. Let \( V = \{ (x, y, z, w)^T \in R^4 | cz + \sigma dw = 0 \} \). If a configuration \( q = (q_1, \cdots, q_N) \) lies on the two-dimensional sphere (hyperbolic sphere): \( V \cap M^3 \), then \( \nabla q_i I \) is in \( V \) for \( i = 1, N \).

Proof. By the explicit form of \( \nabla q_i I \), equation (2), we get

\[
\nabla q_i I \cdot (0, 0, c, d)^T = -\sigma 2m_i (x_i^2 + y_i^2)(cz_i + \sigma dw_i) = 0.
\]

This equation completes the proof. \( \square \)

3. Central configurations on \( H^3 \)

Let us define \( H^2_{xyz} := \{ (x, y, z, w)^T \in R^4 | z = 0 \} \cap H^3 \). We can prove the following result.

Theorem 1. Each central configuration in \( H^3 \) is equivalent to some central configuration on \( H^2_{xyz} \).

Proof. We first show that all central configurations in \( H^3 \) must lie on a two-dimensional hyperbolic sphere. Then we show that there is some action \( \chi \in \)
SO(2) \times SO(1, 1) which transforms that hyperbolic sphere to $\mathbb{H}^2_{xyw}$. Thus by the definition of equivalent central configurations, each central configuration in $\mathbb{H}^3$ is equivalent to some central configuration on $\mathbb{H}^2_{xyw}$.

Consider the two-dimensional hyperbolic sphere: $\mathbb{H}^2_{\phi} := \{(x, y, z, w)^T \in \mathbb{R}^4| \cosh \phi z - \sinh \phi w = 0\} \cap \mathbb{H}^3$. The intersection is not empty, since the linear subspace and $\mathbb{H}^3$ share the point $(0, 0, \sin \phi, \cosh \phi)^T$. We show that each central configuration will be confined to only one such two-dimensional hyperbolic sphere.

Assume that this is not the case. Suppose that there is a central configuration $\mathbf{q} = (q_1, \cdots, q_N)$ with $q_i \in \mathbb{H}^2_{\phi_i}$, $\phi_1 \geq \phi_i$ for $i \neq 1$ and there is at least one $i$ such that $\phi_1 > \phi_i$. Then $\mathbf{q}_i$ can be written as $(x_i, y_i, \rho_i \sinh \phi_i, \rho_i \cosh \phi_i)^T$ with $\rho_i > 0$ since $w_i = \rho_i \cosh \phi_i > 0$. By Proposition 1, $\nabla_{\mathbf{q}_i} \mathbf{I}$ is in the linear subspace $\{(x, y, z, w)^T \in \mathbb{R}^4| \cosh \phi_1 z - \sinh \phi_1 w = 0\}$. In order to have a central configuration, $\nabla_{\mathbf{q}_i} \mathbf{U}$ must be in the linear subspace, i.e.,

$$\nabla_{\mathbf{q}_i} \mathbf{U} \cdot (0, 0, \cosh \phi_1, \sinh \phi_1)^T = \mathbf{F}_{1z} \cosh \phi_1 - \mathbf{F}_{1w} \sinh \phi_1 = 0,$$

where $\mathbf{F}_{1z}$ and $\mathbf{F}_{1w}$ stand for the $z$-coordinate and $w$-coordinate of $\mathbf{F}_1$, respectively. However, using the explicit form of $\mathbf{F}_1$, we get

$$\mathbf{F}_{1z} \cosh \phi_1 - \mathbf{F}_{1w} \sinh \phi_1$$

$$= \sum_{i=2}^{N} m_i m_1 \left( \frac{z_i - \cosh d_{i1} z_1}{\sinh^3 d_{i1}} \cosh \phi_1 - \frac{w_i - \cosh d_{i1} w_1}{\sinh^3 d_{i1}} \sinh \phi_1 \right)$$

$$= \sum_{i=2}^{N} m_i m_1 \rho_i \sinh \phi_i \cosh \phi_1 \cosh \phi_i \sinh \phi_1 - \cosh d_{i1} (z_1 \cosh \phi_1 - w_1 \sinh \phi_1)$$

$$= \sum_{i=2}^{N} m_i m_1 \rho_i \sinh(\phi_i - \phi_1) \frac{\sinh^3 d_{i1}}{\sinh^3 d_{i1}} < 0,$$

since $\phi_i \leq \phi_1$ for $i \neq 1$ and there is at least one $i$ such that $\phi_i < \phi_1$.

Thus any central configuration must lie on only one such hyperbolic sphere, say $\mathbb{H}^2_{\phi}$. Let

$$\chi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \cosh \phi & -\sinh \phi \\ -\sinh \phi & \cosh \phi \end{bmatrix} \in SO(2) \times SO(1, 1).$$

Since

$$\begin{bmatrix} \cosh \phi & -\sinh \phi \\ -\sinh \phi & \cosh \phi \end{bmatrix} \begin{bmatrix} \rho_i \sinh \phi \\ \rho_i \cosh \phi \end{bmatrix} = \begin{bmatrix} \rho_i \\ \rho_i \cosh \phi \end{bmatrix}$$

$$\chi(\mathbb{H}^2_{\phi}) = \mathbb{H}^2_{xyw}. \text{ This calculation completes the proof.} \qquad \Box$$
To offer more insight into this result, we provide a heuristic argument. Recall that the Poincaré ball model of $\mathbb{H}^3$ is
\[ (\bar{x}^2 + \bar{y}^2 + \bar{z}^2 < 1, \quad ds^2 = \frac{4(d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2)}{(1 - \bar{x}^2 - \bar{y}^2 - \bar{z}^2)^2} ). \]
In this model, a two-dimensional hyperbolic sphere is the intersection of the three-dimensional ball with a two-dimensional Euclidean sphere that orthogonally intersects the boundary of the ball. The hyperbolic spheres $\mathbb{H}^2_\phi$ defined above are those that intersect the \( \bar{z} \)-axis orthogonally, \([1]\). For example, $\mathbb{H}^2_{xyw}$ in this model is the disk in the plane $\bar{z} = 0$. Now suppose that $q_i \in \mathbb{H}^2_{\phi_i}$ and $\phi_1 > \phi_2$. Proposition \([1]\) implies that $\nabla_{q_i} I \in T_{q_i} \mathbb{H}^2_{\phi_1}$, as showed in Figure \([1]\). However, $F_{12}$ points towards the lower hyperbolic sphere $\mathbb{H}^2_{\phi_2}$. Thus $\nabla_{q_i} I$ and $\nabla_{q_i} U$ cannot be collinear.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{A configuration in the Poincaré ball model.}
\end{figure}

**Remark 1.** Recall that there are central configurations not in a plane (called spatial central configurations) in the Newtonian $N$-body problem, such as the regular tetrahedron for any given four masses. However, those spatial configurations do not lead to rigid motions. Thus if we defined central configurations in $\mathbb{R}^3$ as those that lead to rigid motions, there would be no spatial ones.

4. **Central configurations in $S^3$**

Apparently, the compactness of $S^3$ makes the set of central configuration in it richer than in $\mathbb{H}^3$. With computations similar to the ones we performed in $\mathbb{H}^3$, we can get the following necessary conditions for central configurations in $S^3$,
\[ \sum_{j=1, j \neq i}^{N} m_i m_j \frac{\rho_j \sin(\phi_j - \phi_i)}{\sin^3 d_{ij}} = 0, \quad i = 1, N. \]
These equations, however, do not rule out the existence of three-dimensional central configurations. For example, we have the pentatope special central configuration of five equal masses, \[8\],

\[
\begin{align*}
x_1 &= 1, & y_1 &= 0, & z_1 &= 0, & w_1 &= 0, \\
x_2 &= -1/4, & y_2 &= \sqrt{15}/4, & z_2 &= 0, & w_2 &= 0, \\
x_3 &= -1/4, & y_3 &= -\sqrt{5}/(4\sqrt{3}), & z_3 &= \sqrt{5}/\sqrt{6}, & w_3 &= 0, \\
x_4 &= -1/4, & y_4 &= -\sqrt{5}/(4\sqrt{3}), & z_4 &= -\sqrt{5}/(2\sqrt{6}), & w_4 &= \sqrt{5}/(2\sqrt{2}), \\
x_5 &= -1/4, & y_5 &= -\sqrt{5}/(4\sqrt{3}), & z_5 &= -\sqrt{5}/(2\sqrt{6}), & w_5 &= -\sqrt{5}/(2\sqrt{2}).
\end{align*}
\]

However, all known three-dimensional central configurations are special central configurations (i.e., \(\lambda = 0\)). We will further construct a two-parameter family of ordinary three-dimensional central configurations of five masses. Suppose that the masses are \(m_1 = m_2 = m, m_3 = m_4 = m_5 = 1\), and their positions are given by

\[
\begin{align*}
x_1 &= 0, & y_1 &= 0, & z_1 &= \cos \theta, & w_1 &= \sin \theta, \\
x_2 &= 0, & y_2 &= 0, & z_2 &= \cos \theta, & w_2 &= -\sin \theta, \\
x_3 &= r, & y_3 &= 0, & z_3 &= c, & w_3 &= 0, \\
x_4 &= r \cos \frac{2\pi}{3}, & y_4 &= r \sin \frac{2\pi}{3}, & z_4 &= c, & w_4 &= 0, \\
x_5 &= r \cos \frac{4\pi}{3}, & y_5 &= r \sin \frac{4\pi}{3}, & z_5 &= c, & w_5 &= 0,
\end{align*}
\]

where \(c \in (-1, 1) \setminus \{0\}\), \(r > 0\), \(r^2 + c^2 = 1\) and \(\theta \in (0, \pi) \setminus \{\pi/2\}\). Such configurations depend on two parameters, \(c\) and \(\theta\), and we denote them by \(q(c, \theta)\). It is easy to see that these configurations are not confined to any two-dimensional sphere. In Figure 2, we illustrate such a configuration in a \(R^3\) hyperplane by the stereographic projection of \(S^3\) from \((0, 0, 1, 0)\) onto the corresponding equatorial \(R^3\) hyperplane, i.e.,

\[
\tilde{x} = \frac{x}{1 - z}, \quad \tilde{y} = \frac{y}{1 - z}, \quad \tilde{w} = \frac{w}{1 - z}.
\]

**Proposition 2.** For any \((c, \theta) \in (-1, 0) \times (0, \pi/2)\), and \((c, \theta) \in (0, 1) \times (\pi/2, \pi)\), the configurations \(q(c, \theta)\) constructed above are central configurations if

\[
(3) \quad m = -\frac{3c \sin^3 2\theta}{2 \cos \theta (1 - c^2 \cos^2 \theta)^{3/2}}.
\]

Generally, they are ordinary central configurations.
Proof. We check that the central configuration equations \( \nabla q_i U = \lambda \nabla q_i I, i = 1, \cdots, 5 \), are satisfied. The function \( U \) can be written as \( U = U_1 + U_2 \), where

\[
U_1 = \cot d_{34} + \cot d_{45} + \cot d_{35}, \quad U_2 = m^2 \cot d_{12} + m \sum_{i=3}^{5} (\cot d_{1i} + \cot d_{2i}).
\]

Note that the three equal masses \( m_3, m_4, \) and \( m_5 \) form an ordinary central configuration themselves, i.e., \( \nabla q_i U_1 = \lambda_1 \nabla q_i I \), for \( i = 3, 4, 5 \), \( \lambda_1 = \frac{-3}{2 \sin^2 d_{45}} \). Note that \( \nabla q_1 I = \nabla q_2 I = 0 \) by equation (2). Thus \( \nabla q_i U = \lambda \nabla q_i I \) is satisfied if and only if there is some constant \( \lambda_2 \) such that

\[
\nabla q_3 U_2 = \lambda_2 \nabla q_3 I, \quad \lambda_2 = -m \cos \theta c \sin^3 d_{13}.
\]

By symmetry, we only need to check \( \nabla q_3 U_2 = \lambda_2 \nabla q_3 I, \) and \( F_1 = F_2 = 0 \).

Note that \( d_{13} = d_{23} = d_{14} = d_{24} = d_{15} = d_{25}, d_{34} = d_{45} = d_{35} \), and

\[
\cos d_{12} = \cos 2\theta, \quad \cos d_{13} = c \cos \theta, \quad \cos d_{34} = \frac{3}{2} c^2 - \frac{1}{2}.
\]

Some straightforward computation shows

\[
\nabla q_3 U_2 = F_{31} + F_{32} = \frac{m (q_1 - \cos d_{13} q_3)}{\sin^3 d_{13}} + \frac{m (q_2 - \cos d_{23} q_3)}{\sin^3 d_{23}}
\]

\[
= \frac{m}{\sin^3 d_{13}} (q_1 + q_2 - 2 \cos d_{13} q_3) = \frac{m}{\sin^3 d_{13}} \left( (0, 0, 2 \cos \theta, 0)^T - 2c \cos \theta (r, 0, c, 0)^T \right)
\]

\[
= \frac{-2mr \cos \theta}{\sin^3 d_{13}} (c, 0, -r, 0)^T.
\]

Using equation (2), we obtain \( \nabla q_3 I = 2rc(c, 0, -r, 0)^T \). Thus we can write that

\[
\nabla q_3 U_2 = \lambda_2 \nabla q_3 I, \quad \lambda_2 = -\frac{m \cos \theta}{c \sin^3 d_{13}}.
\]
By direct computation, we obtain
\[
F_1 = F_{12} + \sum_{j=3}^{5} F_{1j} = \frac{m^2}{\sin^2 2\theta} (q_2 - \cos 2\theta q_1) + \sum_{i=3}^{5} \frac{m}{\sin^3 d_{13}} (q_i - \cos d_{13} q_1)
\]
\[
= \frac{m^2}{\sin^2 2\theta} (q_2 - \cos 2\theta q_1) + \frac{m}{\sin^3 d_{13}} \left( \sum_{i=3}^{5} q_i - 3c \cos \theta q_1 \right)
\]
\[
= m \sin \theta \left( \frac{2m \cos \theta}{\sin^3 2\theta} + \frac{3c}{\sin^3 d_{13}} \right) (0, 0, \sin \theta, -\cos \theta)^T.
\]

Thus \(F_1 = 0\) if and only if \(m = -\frac{3c|\sin^3 2\theta|}{2 \cos \theta (1 - c^2 \cos^2 \theta)^{3/2}}\). Since we need positive masses, \(c \cos \theta\) needs to be negative.

We have thus obtained a two-parameter family of central configurations \(q(c, \theta)\) for any \((c, \theta) \in (-1, 0) \times (0, \frac{\pi}{2})\), and \((c, \theta) \in (0, 1) \times (\frac{\pi}{2}, \pi)\). The central configuration equations \(\nabla_q U = \lambda(c, \theta) \nabla_q I, i = 1, \ldots, 5\), are satisfied, and the constant is
\[
\lambda(c, \theta) = \lambda_1 + \lambda_2 = \frac{-3}{2 \sin^3 d_{34}} - \frac{m \cos \theta}{c \sin^3 d_{13}} = \frac{-3}{2 \sin^3 d_{34}} + \frac{3|\sin^3 2\theta|}{2 \sin^6 d_{13}}
\]
\[
= \frac{3}{2} \left( \frac{-8}{3 \sqrt{3}(1 + 3c^2)^{3/2}(1 - c^2)^{3/2}} + \frac{|\sin^3 2\theta|}{(1 - c^2 \cos^2 \theta)^{3/2}} \right),
\]
which is zero on a one-dimensional manifold. Factually, it is homeomorphic to two open unit intervals. Thus generally, \(q(c, \theta)\) are ordinary central configurations.

This remark completes the proof. \(\square\)

Moreover, if \((c, \theta) \in (-1, 0) \times (0, \frac{\pi}{2})\), then the masses \(m_3, m_4, m_5\) are contained in the unit ball, \(x^2 + y^2 + w^2 \leq 1\), and the masses \(m_1, m_2\) are outside, see Figure 2. This happens because
\[
\bar{w}_1 = \frac{w_1}{1 - z_1} = \frac{\sin \theta}{1 - \cos \theta} > 1, \quad \bar{x}_3^2 + \bar{y}_3^2 = \left( \frac{x_3}{1 - z_3} \right)^2 + \left( \frac{y_3}{1 - z_3} \right)^2 = \frac{1 + c}{(1 - c)} < 1.
\]

Similarly, if \((c, \theta) \in (0, 1) \times (\frac{\pi}{2}, \pi)\), then masses \(m_3, m_4, m_5\) are outside, but the masses \(m_1, m_2\) are inside the ball.

Obviously, we can still obtain central configurations if we substitute the equilateral triangle by a regular \(n\)-gon with equal masses, and generally they are ordinary ones.

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