On solvable subgroups of automorphism groups of right-angled Artin groups

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Abstract

For any right-angled Artin group, we show that its outer automorphism group contains either a finite-index nilpotent subgroup or a nonabelian free subgroup. This is a weak Tits alternative theorem. We find a criterion on the defining graph that determines which case holds. We also consider some examples of solvable subgroups, including one that is not virtually nilpotent and is embedded in a non-obvious way.

1 Introduction and Background

1.1 Introduction

Let $A_{\Gamma}$ be the right-angled Artin group of a finite simplicial graph $\Gamma$ with vertex set $X$, i.e. the group with presentation

$$A_{\Gamma} = \langle X|\{xy = yx| \text{ } x \text{ is adjacent to } y \text{ in } \Gamma\} \rangle.$$ 

In this note we find a combinatorial condition on the graph $\Gamma$ that indicates whether the outer automorphism group $Out A_{\Gamma}$ of $A_{\Gamma}$ contains a nonabelian free group. This extends a result of Gutierrez–Piggott–Ruane [4, Theorem 1.10] which gives a condition for a particular subgroup of $Out A_{\Gamma}$ to be abelian. In fact, our theorem indicates a dichotomy: either $Out A_{\Gamma}$ contains a nonabelian free subgroup or $Out A_{\Gamma}$ is virtually nilpotent. This is a weak Tits alternative theorem. A true Tits alternative theorem would consider all subgroups of $Out A_{\Gamma}$; Charney–Vogtmann [1] recently proved such a theorem for a large class of right-angled Artin groups.

Automorphism groups of right-angled Artin groups are sometimes described as intermediate between automorphism groups of free groups and integer general linear groups, which are extreme examples. This result is
a first attempt to discern the cases where this idea seems reasonable, since Aut \( F_n \) and GL(\( n \), \( \mathbb{Z} \)) both have nonabelian free subgroups for every \( n > 1 \).

To state the theorem, we review some notions on graphs. Recall that the link \( \text{lk}(x) \) of a vertex \( x \in \Gamma \) is the set of vertices adjacent to \( x \), and the star \( \text{st}(x) \) is \( \text{lk}(x) \cup \{x\} \). Domination is a useful relation that was considered by Servatius [6, Section IV]:

**Definition 1.1.** For \( x, y \in \Gamma \), say \( y \) dominates \( x \) if \( \text{lk}(x) \subset \text{st}(y) \); denote this by \( y \geq x \). Say \( x \) and \( y \) are domination equivalent if \( x \leq y \) and \( y \leq x \); denote this by \( x \sim y \).

Next we consider the notion of a separating intersection of links defined by Gutierrez–Piggott–Ruane [4, Definition 1.9].

**Definition 1.2.** The graph \( \Gamma \) has a separating intersection of links if there are two vertices \( x, y \in \Gamma \) such that (1) \( x \) is not adjacent to \( y \) and (2) there is a connected component of \( \Gamma \setminus (\text{lk}(x) \cap \text{lk}(y)) \) not containing \( x \) or \( y \).

Now we state our main result.

**Theorem 1.3.** Consider the following conditions on a graph \( \Gamma \):

1. \( \Gamma \) contains a domination-equivalent pair of vertices.
2. \( \Gamma \) contains a separating intersection of links.

If either condition holds, then \( \text{Out} A_{\Gamma} \) contains a nonabelian free subgroup. If both conditions fail, then \( \text{Out} A_{\Gamma} \) is virtually nilpotent.

**Proof of Theorem 1.3.** If either condition holds, then \( \text{Out} A_{\Gamma} \) contains a nonabelian free subgroup, by Lemmas 2.1 and 2.2 below. If both conditions fail, then Proposition 2.11 below produces a finite-index nilpotent subgroup. \( \square \)

In Definition 2.10 we define a number \( \text{depth}(\Gamma) \) that can be read off of the graph \( \Gamma \). If \( \text{Out} A_{\Gamma} \) is virtually nilpotent, then a certain natural finite-index nilpotent subgroup of \( \text{Out} A_{\Gamma} \) turns out to be nilpotent of class \( \text{depth}(\Gamma) \). Further, every finite-index nilpotent subgroup of \( \text{Out} A_{\Gamma} \) has nilpotence class at least \( \text{depth}(\Gamma) \). See Proposition 2.11 below for details.

In Section 2.3 we consider a few other conditions that imply \( \text{Out} A_{\Gamma} \) contains a nonabelian free subgroup. Then we construct examples of graphs \( \Gamma \) with \( \text{Out} A_{\Gamma} \) containing finite-index nilpotent subgroups of arbitrary nilpotence class.

The corollary below follows from Theorem 1.3 by standard arguments.
Corollary 1.4. If $\Gamma$ is a graph such that $\text{Out} A_\Gamma$ has a solvable, finite-index subgroup, then every solvable subgroup of $\text{Out} A_\Gamma$ is virtually nilpotent.

Of course, for $n \geq 3$ the group $\text{GL}(n, \mathbb{Z})$ contains examples of solvable subgroups of infinite index that are not virtually solvable. Given Corollary 1.4, one might conjecture that a solvable subgroup of $\text{Out} A_\Gamma$ that is not virtually nilpotent must be essentially contained in an embedded copy of $\text{GL}(n, \mathbb{Z})$. In Section 3, we produce an example where this is not the case.

1.2 Background

We will use the following four classes of automorphisms. The inversion of $x \in X$ is the automorphism sending $x$ to $x^{-1}$ and fixing $X - \{x\}$. If $\pi$ is an automorphism (a symmetry) of the graph $\Gamma$, then the graphic automorphism of $\pi$ is the automorphism sending $x$ to $\pi(x)$ for each $x \in X$. If $x \in X \cup X^{-1}$ and $Y$ is a connected component of $\Gamma - \text{st}(x)$, the partial conjugation of $Y$ by $x$ is the automorphism sending $y$ to $x^{-1}yx$ for each $y \in Y$ and fixing $X - Y$. Denote this automorphism by $c_{x,Y}$. If $x \in X \cup X^{-1}$ and $y \in X$ are distinct and $x \geq y$, then the transvection of $y$ by $x$ is the automorphism sending $y$ to $yx$ and fixing $X - \{y\}$. Denote this automorphism by $\tau_{x,y}$. Sometimes we will refer to the automorphism just defined as the right transvection, and refer to its conjugate by the inversion in $y$ as the left transvection. The multiplier of a transvection $\tau_{x,y}$ is $x$ and the multiplier of a partial conjugation $c_{y,Y}$ is $y$. Servatius defined these automorphisms and showed that they are well defined in [6, Section IV]. Laurence [5] proved the following, which was a conjecture of Servatius.

Theorem 1.5 (Laurence [5]). The finite set of all transvections, partial conjugations, inversions, and graphic automorphisms is a generating set of $\text{Aut} A_\Gamma$.

Of course the images of these generators form a finite generating set for $\text{Out} A_\Gamma$. Since we are working in $\text{Out} A_\Gamma$, in this paper we will demand that partial conjugations are not inner automorphisms. Specifically, whenever we declare that $C_{y,Y}$ is a partial conjugation with multiplier $y$, we also assume $Y$ and $\Gamma \backslash (\text{st}(y) \cup Y)$ are both nonempty.

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2 Proof of the dichotomy

2.1 Conditions for free subgroups

Lemma 2.1. If $\Gamma$ contains distinct $x, y$ with $x \sim y$, then $\text{Out } A_\Gamma$ contains a nonabelian free subgroup.

Proof. Let $x, y \in \Gamma$ be vertices with $x \sim y$. Let $G$ be the subgroup of $\text{Out } A_\Gamma$ generated by the images of $\tau^2_{x,y}$ and $\tau^2_{y,x}$. The vector space $H_1(A_\Gamma; \mathbb{R})$ has a basis given by the vertex-set of $\Gamma$; note that $G$ leaves the 2-dimensional subspace $V = \langle [x], [y] \rangle$ invariant. Let $A \subset V$ be the set of $a[x] + b[y]$ with $|a| > |b|$ and let $B \subset V$ be the set of such vectors with $|b| > |a|$. It is easy to see that $A$ and $B$ are nonempty, $(\tau^2_{x,y})^*(B) \subset A$ and $(\tau^2_{y,x})^*(A) \subset B$. Then by the well-known Table-Tennis Lemma (see de La Harpe [3, II.B.24]), we see that $G$ is free of rank 2.

Lemma 2.2. If $\Gamma$ contains a separating intersection of links, then $\text{Out } A_\Gamma$ contains a nonabelian free subgroup.

Proof. Let $x, y, z \in \Gamma$ with $y$ not adjacent to $z$ and $\text{lk}(y) \cap \text{lk}(z)$ separating $x$ from both $y$ and $z$. Let $Y$ be the component of $x$ in $\Gamma \setminus \text{lk}(y)$ and let $Z$ be the component of $x$ in $\Gamma \setminus \text{lk}(y)$. The hypotheses imply that $y \notin Z$ and $z \notin Y$. Let $\tilde{G}$ be the subgroup of $\text{Aut } A_\Gamma$ generated by $c_{y,Y}$ and $c_{z,Z}$ and let $G < \text{Out } A_\Gamma$ be its image. Then $\tilde{G}$ fixes $y$ and $z$, and therefore contains no nontrivial inner automorphisms. Therefore the projection $\tilde{G} \to G$ is an isomorphism.

Map $\tilde{G}$ to the free group $F_2 = \langle y, z \rangle$ by sending $\alpha \in \tilde{G}$ to the unique $w \in \langle y, z \rangle$ with $\alpha(x) = w^{-1}xw$. It is easy to see that this map is a homomorphism with $c_{y,Y}$ mapping to $y$ and $c_{z,Z}$ mapping to $z$. Since an inverse homomorphism $F_2 \to \tilde{G}$ is easy to construct, we see that $\tilde{G}$ and $G$ are free of rank 2.

2.2 Conditions for virtual nilpotence

Lemma 2.3. Suppose $x, y$ and $z$ are in $\Gamma$ such that $x$ is not adjacent to $y$, $\text{lk}(x)$ separates $y$ from $z$ and $\text{lk}(y)$ separates $x$ from $z$. Then $\text{lk}(x) \cap \text{lk}(y)$ separates $x$ and $y$ from $z$ and therefore $\Gamma$ contains a separating intersection of links.
Proof. Suppose $\text{lk}(x) \cap \text{lk}(y)$ does not separate both $x$ and $y$ from $z$. Then there is a shortest path from $z$ to $x$ or $y$ through $\Gamma \setminus (\text{lk}(x) \cap \text{lk}(y))$. Starting from $z$, the first time this path hits $\text{lk}(x) \cup \text{lk}(y)$ must also be the last, or else there would be a shorter path. Then the hypotheses imply that the point on the path in $\text{lk}(x) \cup \text{lk}(y)$ must also be in $\text{lk}(x) \cap \text{lk}(y)$, a contradiction. □

**Lemma 2.4.** Suppose $x, y \in \Gamma$ with $x$ not adjacent to $y$, we have $x \geq y$ and $\text{st}(y)$ separates $\Gamma$. Then $\Gamma$ contains a separating intersection of links.

Proof. Let $z$ be in a component of $\Gamma \setminus \text{st}(y)$ not containing $x$. This means $\text{lk}(y)$ separates $z$ from $x$. Since $x \geq y$, we know $\text{lk}(x) \cap \text{lk}(y) = \text{lk}(y)$, so $\text{lk}(x) \cap \text{lk}(y)$ separates $z$ from $x$ and $y$. Therefore $\Gamma$ contains a separating intersection of links. □

**Lemma 2.5.** Suppose $\Gamma$ does not contain a separating intersection of links. Suppose $\alpha$ and $\beta$ are automorphisms that are either partial conjugations or transvections (or one of each) and that $\alpha$ and $\beta$ fix each other’s multipliers. Then $\alpha$ and $\beta$ commute in $\text{Aut } A \Gamma$.

Proof. Let $x$ be the multiplier of $\alpha$ and let $y$ be the multiplier of $\beta$. If $x = y \pm 1$ or $x$ is adjacent to $y$ then $\alpha$ and $\beta$ commute, so assume $x$ and $y$ are distinct and not adjacent. Suppose there is some $z \in \Gamma$ such that neither $\alpha$ nor $\beta$ fixes $z$.

Suppose $x$ is adjacent to $z$. Then $\alpha$ is a transvection (partial conjugations fix the links of their multipliers) and $x \geq z$. If $y \geq z$, then $y$ is adjacent to $x$, counter to our assumption. If $y \not\geq z$, then $\beta$ is a partial conjugation and $y$ is not adjacent to $z$. This implies that $x$ and $z$ are in the same connected component of $\Gamma \setminus \text{lk}(y)$, meaning that $\beta$ cannot fix $x$ and change $z$. This contradiction implies that $x$ is not adjacent to $z$, and similarly, that $y$ is not adjacent to $z$.

Suppose $\text{lk}(x)$ does not separate $y$ from $z$. Then $x \not\geq z$ and $\alpha$ is a partial conjugation. However, in that case $\alpha$ cannot fix $y$ and change $z$. So $\text{lk}(x)$ separates $y$ from $z$, and similarly $\text{lk}(y)$ separates $x$ from $z$. Then by Lemma 1.10, we have that $\Gamma$ does contain a separating intersection of links, which is a contradiction. From this we deduce that for each $z \in \Gamma$, either $\alpha$ fixes $z$ or $\beta$ fixes $z$. This is enough to deduce that $\alpha$ and $\beta$ commute. □

In the case that $\alpha$ and $\beta$ are partial conjugations, the following lemma is a special case of Theorem 1.10 from Gutierrez–Piggott–Ruane [1].

**Lemma 2.6.** Suppose $\Gamma$ does not contain a separating intersection of links. Let $\alpha$ be a partial conjugation. Suppose that $\beta$ is a transvection fixing the
multiplier of \( \alpha \), or that \( \beta \) is a partial conjugation (not necessarily fixing the multiplier of \( \alpha \)). Then the images of \( \alpha \) and \( \beta \) commute in \( \text{Out} \, A_\Gamma \).

**Proof.** Suppose the multipliers of \( \alpha \) and \( \beta \) are distinct (otherwise \( \alpha \) and \( \beta \) commute). Then possibly by multiplying \( \alpha \) and \( \beta \) by inner automorphisms, we may assume that \( \alpha \) fixes the multiplier of \( \beta \), and if \( \beta \) is a partial conjugation, we may assume that \( \beta \) fixes the multiplier of \( \alpha \). The lemma then follows from Lemma 2.5.

**Lemma 2.7.** Suppose conditions (1) and (2) from Theorem 1.3 both fail. Suppose \( \alpha = \tau_{x,y} \) is a transvection and \( \beta \) is a transvection or partial conjugation with multiplier \( y^\pm 1 \). Let \( \gamma \) be any commutator of \( \alpha \) or \( \alpha^{-1} \) with \( \beta \) or \( \beta^{-1} \). Then \( \gamma \) is a transvection or partial conjugation with multiplier \( x^\pm 1 \) in \( \text{Out} \, A_\Gamma \). Further, if \( \beta \) is a transvection acting on \( z^\pm 1 \), then so is \( \gamma \), and if \( \beta \) is a partial conjugation acting on \( Y \subset X \), then so is \( \gamma \).

**Proof.** If \( \beta \) is a transvection and doesn’t fix \( x \), then \( x \sim y \) and condition (1) holds. If \( \beta \) is a partial conjugation, then up to an inner automorphism we may assume that it fixes \( x \). So assume \( \beta \) fixes \( x \).

We claim that \( x \) is adjacent to \( y \). Let \( z \in \Gamma \) be an element not fixed by \( \beta \). If \( y \) is adjacent to \( z \), then \( \beta \) is a transvection and \( y \geq z \). Since \( x \geq y \), this implies that \( x \geq z \), and that \( x \) is adjacent to \( y \). So suppose \( y \) is not adjacent to \( z \). If \( x \) is not adjacent to \( y \), then \( \text{lk}(y) \) separates \( x \) from \( z \) since \( \beta \) fixes \( x \) but not \( z \). Then by Lemma 2.4, \( \Gamma \) contains a separating intersection of links, contradicting the failure of condition (2). So \( x \) is adjacent to \( y \). Then the lemma follows by a computation.

**Definition 2.8.** A domination chain in \( \Gamma \) is a sequence of distinct vertices \( x_1, \ldots, x_m \) of \( \Gamma \) such that \( x_m \geq x_{m-1} \geq \cdots \geq x_1 \). The length of the domination chain \( x_1, \ldots, x_m \) is \( m - 1 \). The domination depth of \( x \) is the length of the longest domination chain with \( x \) as the dominant member.

**Definition 2.9.** A domination chain \( x_m \geq \cdots \geq x_1 \) is star-separation preserving if \( \Gamma \setminus \text{st}(x_1) \) has two components \( Y_1 \) and \( Y_2 \) such that \( Y_i \not\subset \text{st}(x_m) \) for \( i = 1, 2 \). The star-separation depth of \( x \in \Gamma \) is

\[
1 + \max_{x = x_m \geq \cdots \geq x_1} \text{length}(x_m \geq \cdots \geq x_1)
\]

where the maximum is taken over all star-separation-preserving domination chains.
Definition 2.10. The depth depth(x) of a vertex x ∈ Γ is maximum of the domination depth of x and the star-separation depth of x. The depth depth(Γ) of Γ is the maximum depth of its vertices.

Proposition 2.11. The subgroup N of Out AΓ generated by transvections and partial conjugations is finite index in Out AΓ. If the conditions from Theorem 2.3 both fail, then N is nilpotent of class depth(Γ). Further, every finite-index nilpotent subgroup of Out AΓ has nilpotence class at least depth(Γ).

Proof. Let S be the finite subset of Out AΓ consisting of the identity, the images of transvections (both right and left) and partial conjugations, and their inverses. Let N be the subgroup generated by S and let P be the finite subgroup generated by images of inversions and graphic automorphisms in Out AΓ. Note that P normalizes N (since conjugation by P leaves S invariant). By Laurence’s theorem (Theorem 1.5), Out AΓ = PN and therefore N < Out AΓ. By a classical group isomorphism theorem, Out AΓ/N ∼= P/(P ∩ N) and therefore N is finite-index in Out AΓ. (In fact, the failure of condition (1) implies that N ∩ P = 1 and Out AΓ ∼= P × N, as can be seen from the presentation for Aut AΓ in Day [2, Theorem 2.7].)

Let k = depth(Γ). Let S0 = {1}, and let Si be the union of {1} with the set of transvections τx,y with depth(x) − depth(y) ≥ k − i + 1 (and left transvections satisfying the same condition) and partial conjugations cy,y with depth(y) ≥ k − i + 1 for i = 1, . . . , k. The Si are nested and S is S1.

Let α ∈ Si and β ∈ Sj, for 1 ≤ i, j ≤ k. By Lemmas 2.5, 2.6 and 2.7, we see that if [α, β] is nontrivial, then i + j > k + 1 and [α, β] is a member of S1+k−k−1. Since i, j ≤ k, we have that i + j − k - 1 < i, j. This is enough to deduce that N is nilpotent of class at most k.

Select xk ∈ Γ with depth(xk) = k. By definition, there is a domination chain xk ≥ · · · ≥ x1 in Γ, such that either x1 dominates a vertex x0, or Γ \ st(x1) has two components Y1, Y2 with Yi ⊈ st(xk) for i = 1, 2 (depending on whether depth(xk) is the domination depth or the star-separation depth, respectively). In the first of these cases, let α1 denote the transvection τx1,x0, and in the second of these cases, let α1 denote the partial conjugation c1,y1, for i = 2, . . . , k, let αi be the transvection τx1,x1−1. Then by Lemma 2.7, the element

\[
[\cdots[[α1,α2],α3],\ldots,α_k] \in \text{Out } AΓ
\]

is either a transvection τy,x0 or a partial conjugation c0,y, where y = xk±1 and Y = Y1 \ st(y). If it is τy,x0, it is obviously nontrivial in Out AΓ. If it is c0,y, it is nontrivial in Out AΓ since there is an element of Y1 \ st(y) that
is conjugated and an element of $Y_2 \setminus \text{st}(y)$ that is not conjugated. So the nilpotence class of $N$ equals $\text{depth}(\Gamma)$.

Now suppose that $N''$ is a nilpotent, finite-index subgroup of $\text{Out} \ A_\Gamma$. Then $N''$ intersects $N$ in a finite index subgroup $N'$. Each of the $\alpha_1, \ldots, \alpha_k$ from the previous paragraph is of infinite order. Since $N'$ is finite index in $N$, the intersection $N' \cap \langle \alpha_i \rangle$ is finite index in $\langle \alpha_i \rangle$ for each $i$. In particular, each $N' \cap \langle \alpha_i \rangle$ is nontrivial. So we have $a_1, \ldots, a_k \in \mathbb{Z}$ with $\alpha_i^{a_i} \in N'$ for each $i$. Then by the same reasoning as in the previous paragraph, we see that

$$\left[ \cdots \left[ \alpha_1^{a_1}, \alpha_2^{a_2} \right], \alpha_3^{a_3} \right], \ldots, \alpha_k^{a_k} \right].$$

is nontrivial. From this, we see the nilpotence class of $N'$ is also $\text{depth}(\Gamma)$, and the nilpotence class of $N''$ is at least $\text{depth}(\Gamma)$.

The following needs no further proof.

**Corollary 2.12.** The group $\text{Out} \ A_\Gamma$ is virtually abelian if and only if both conditions from Theorem 1.3 fail and $\text{depth}(\Gamma) \leq 1$.

**Remark 2.13.** It has long been known that $\text{Out} \ A_\Gamma$ is finite if and only if $\Gamma$ contains no pair of vertices $x, y$ with $x \geq y$ and $\Gamma$ contains no vertex $x$ with $\Gamma - \text{st}(x)$ disconnected. This is an easy corollary of Theorem 1.5.

### 2.3 Examples

**Corollary 2.14.** The group $\text{Out} \ A_\Gamma$ has a nonabelian free subgroup if any of the following conditions on $\Gamma$ hold:

- $\Gamma$ is disconnected.
- $\Gamma$ contains a cut-vertex that breaks $\Gamma$ into three or more components.
- $\Gamma$ contains non-adjacent vertices $x$ and $y$ with $x \geq y$ and $\text{st}(y)$ separating $\Gamma$.
- $\Gamma$ contains pairwise non-adjacent vertices $x, y$ and $z$ with $x \geq y \geq z$.

**Proof.** In each case we find a domination-equivalent pair of vertices or a separating intersection of links in $\Gamma$, and Theorem 1.3 implies the corollary. The final condition is a special case of the second to last condition, which implies $\Gamma$ has a separating intersection of links by Lemma 2.4.

Now suppose that $\Gamma$ is disconnected. If $\Gamma$ is edgeless, then any two vertices are domination equivalent. Otherwise some component of $\Gamma$ has at
least two vertices. If each component of $\Gamma$ is a complete graph, then any two vertices in the same component are domination equivalent. So we have some component of $\Gamma$ that contains two nonadjacent vertices. Then $\Gamma$ contains a separating intersection of links (for $x$ and $y$ not adjacent, $\text{lk}(x) \cap \text{lk}(y)$ separates $x$ and $y$ from any vertex in another component).

Now suppose $\Gamma$ contains a cut-vertex $z$ that breaks $\Gamma$ into at least three components. Without loss of generality we assume $\Gamma$ is connected. If the valence of $z$ is less than 2, then $\Gamma \setminus \{z\}$ has only one component. If for each pair of distinct $x, y \in \text{lk}(z)$, either $x$ is adjacent to $y$ or $\text{lk}(x) \cap \text{lk}(y)$ contains two or more elements, then $\Gamma \setminus \{z\}$ has only one component. Therefore $\Gamma$ contains distinct, non-adjacent vertices $x$ and $y$ with $\text{lk}(x) \cap \text{lk}(y) = \{z\}$.

Then $\Gamma \setminus (\text{lk}(x) \cap \text{lk}(y))$ has at least three components and $\Gamma$ has a separating intersection of links.

**Proposition 2.15.** For each $k \geq 0$, there is a graph $\Gamma_k$ such that $\text{Out} A_{\Gamma_k}$ contains a finite-index subgroup of nilpotence class $k$.

**Proof.** For each $k$, we will construct a graph $\Gamma_k$ with $\text{depth}(\Gamma_k) = k$ and such that $\Gamma_k$ satisfies the hypotheses of Proposition 2.11. We can take $\Gamma_0$ to be the graph with one vertex.

Now fix $k > 0$. For the vertex set of $\Gamma_k$, we will take a set of $2k+2$ vertices labeled as $x_0, \ldots, x_k, y_0, \ldots, y_k$. Take the induced subgraph on $\{x_i\}_i$ to be the complete graph on $k$ vertices, and similarly for $\{y_i\}_i$. Further, connect $x_i$ to $y_j$ by an edge if $i + j > k$. These are the only edges of $\Gamma_k$.

Then $x_k \geq y_0, y_k \geq x_0$, and for $0 \leq i < j \leq k$ we have $x_j \geq x_i$ and $y_j \geq y_i$. Since $k > 0$, these are the only pairs which satisfy the domination relation. In particular, there are no domination-equivalent pairs. There are no vertices whose stars separate $\Gamma$, so the star-separation depth of all vertices is trivial. We compute all depths as equal to domination depths, and find $\text{depth}(x_i) = \text{depth}(y_i) = i$ for $i = 0, \ldots, k$. Therefore $\text{depth}(\Gamma_k) = k$.

The only non-adjacent pairs of vertices are $(x_i, y_j)$ and $(x_j, y_i)$ for $i + j \leq k$. For such $i, j$, every element of $\Gamma_k$ is adjacent to either $x_i$ or $y_j$. In particular, every element of $\Gamma_k \setminus (\text{lk}(x_i) \cap \text{lk}(y_j))$ has a path of length one to either $x_i$ or $y_j$ (and similarly for $x_j$ and $y_i$). Therefore $\Gamma_k$ does not contain a separating intersection of links. \qed

### 3 A non-nilpotent solvable subgroup

Whenever $Y \subset X$ is a clique with $x \sim y$ for all $x, y \in Y$, the transvections of elements of $Y$ acting on each other generate an embedded copy of $\text{SL}(|Y|, \mathbb{Z})$.
inside Out $A_{Γ}$. When we have such a copy of $\text{SL}(n, \mathbb{Z})$, say it is canonically embedded. Of course one can find non-virtually-nilpotent solvable subgroups of Out $A_{Γ}$ inside canonically embedded copies of $\text{SL}(n, \mathbb{Z})$ for $n \geq 3$. Given Corollary 1.4, one might conjecture that when $G < \text{Out} A_{Γ}$ is solvable but not virtually nilpotent, there is $H < \text{Out} A_{Γ}$ a canonically embedded copy of $\text{SL}(n, \mathbb{Z})$, such that $H \cap G$ is not virtually nilpotent. However the following example is not of this type.

**Proposition 3.1.** Let $Γ$ be the graph on three vertices $\{a, b, c\}$ with a single edge from $a$ to $b$. Let $G$ be the subgroup of $\text{Out} A_{Γ}$ generated by the images of the elements $\{τ_{a,c}, τ_{b,c}, τ_{a,b}τ_{b,a}\}$. Then $G$ is a solvable group and is not virtually nilpotent.

The intersection of $G$ with the unique canonically embedded copy of $\text{SL}(n, \mathbb{Z})$ in $\text{Out} A_{Γ}$ is not virtually nilpotent.

**Proof.** It is apparent that $G$ does not contain any inner automorphisms, so $G$ is isomorphic to the subgroup of Aut $A_{Γ}$ generated by these generators. Let $α = τ_{a,c}$, $β = τ_{b,c}$, and let $γ = τ_{a,b}τ_{b,a}$. Since $a$ commutes with $b$, we know that $α$ commutes with $β$. A computation shows that $γαγ^{-1} = α^2β$ and $γβγ^{-1} = αβ$. It is easy to see that $⟨α, β⟩ ∩ ⟨γ⟩ = 1$. From this we can see that $G$ is the semidirect product $\mathbb{Z} ⋉ \mathbb{Z}^2$, where $\mathbb{Z}$ acts on $\mathbb{Z}^2$ by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. So $G$ is solvable.

On the other hand, for all $k > 0$, the centralizer of $γ^k$ in $G$ is $⟨γ⟩$. Let $H$ be a finite index subgroup of $G$. Then $H$ contains a positive power of $γ$ and an element of $G$ outside of $⟨γ⟩$. So $H$ has trivial center and is therefore not nilpotent. In fact $G$ is isomorphic to a lattice in the 3-dimensional Lie group sol; see Thurston [7, Example 3.8.9] for explanation.

The only canonically embedded copy of any $\text{SL}(n, \mathbb{Z})$ in $\text{Out} A_{Γ}$ is generated by $τ_{a,b}$ and $τ_{b,a}$. However, the intersection of $G$ with this subgroup is a copy of $\mathbb{Z}$. $\square$

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