Hyperbolic actions and 2nd bounded cohomology of subgroups of $\text{Out}(F_n)$
Part I: Infinite lamination subgroups

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Abstract
In this two part work with sequel [HM17a] we prove that for every finitely generated subgroup $\Gamma < \text{Out}(F_n)$, either $\Gamma$ is virtually abelian or $H^2_b(\Gamma; \mathbb{R})$ contains an embedding of $\ell^1$. The method uses actions on hyperbolic spaces, for purposes of constructing quasimorphisms. Here in Part I, after presenting the general theory, we focus on the case of infinite lamination subgroups $\Gamma$ — those for which the set of all attracting laminations of all elements of $\Gamma$ is infinite — using actions on free splitting complexes of free groups.

1 Introduction
The study of hyperbolic actions — group actions on Gromov hyperbolic spaces — has co-evolved with the study of the 2nd bounded cohomology $H^2_b(\Gamma; \mathbb{R})$ of a group $\Gamma$. This started with Brooks' theorem, using the action of a free group $\Gamma$ of rank $\geq 2$ on its Cayley tree to prove that there is an embedding $\ell^1 \hookrightarrow H^2_b(\Gamma; \mathbb{R})$ [Bro81]. In works to follow, the same conclusion for $H^2_b(\Gamma; \mathbb{R})$ was proved in increasing generality for certain hyperbolic actions that are proper(ly discontinuous) [BS84], [EF97], [Fuj98]. Fujiwara extended the method to work for certain nonproper actions [Fuj00]. Bestvina and Fujiwara [BF02], using hyperbolicity of the curve complex $\mathcal{C}(S)$ of a finite type surface $S$ [MM99], proved an “$H^2_b$-alternative” for subgroups $\Gamma < \text{MCG}(S)$ of the mapping class group: either $\Gamma$ is virtually abelian, or there is an embedding $\ell^1 \hookrightarrow H^2_b(\Gamma; \mathbb{R})$. While the action on $\mathcal{C}(S)$ is not proper, nonetheless Bestvina and Fujiwara distilled enough proper discontinuity to generalize $H^2_b$-methods, by introducing the WPD or “weak proper discontinuity” property; and a really weak but still useful version of

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WPD known as WWPD was introduced by Bestvina, Bromberg, and Fujiwara in [BBF15], for their study of the asymptotic dimension of MCG(S). For later steps of this co-evolution, using WPD to further the study of hyperbolic actions, see [Bow08], [Osi16], and [BHS14].

For subgroups \( \Gamma < \text{Out}(F_n) \), Bestvina and Feighn [BF10] produced enough WPD hyperbolic actions to prove the \( H_2^b \)-alternative if \( \Gamma \) contains a fully irreducible outer automorphism; for another proof of this result by Hamenstädt see [Ham14]. Applying our early subgroup decomposition theory [HM09], the same conclusion follows if \( \Gamma \) is finitely generated and fully irreducible (as defined in Section 4.1). By work of Horbez [Hor16] using different methods, the conclusion follows for all fully irreducible subgroups, without assuming finite generation.

Here is our main result, to be proved over Parts I and II of this work:

**Theorem A.** Finitely generated subgroups of \( \text{Out}(F_n) \) satisfy the \( H_2^b \)-alternative.

Theorem A is proved using Theorems B and C regarding constructions of hyperbolic actions and WWPD elements, and Theorem D regarding WWPD methods that generalize WPD methods of [BF02]. To state these results, we need some definitions.

**WWPD.** Given an action \( G \act X \) on a hyperbolic space, an element \( \gamma \in G \) is loxodromic if its action on the Gromov boundary \( \partial X \) has north-south dynamics, with repeller–attractor pair \( \partial_{\pm} \gamma = (\partial_-, \partial_+) \in \partial X \times \partial X - \Delta \). The action \( G \act X \) is nonelementary if there exists a pair \( \delta, \gamma \in G \) of loxodromic elements which are independent meaning that \( \{ \partial_-, \partial_+ \} \cap \{ \partial_-, \partial_+ \} = \emptyset \). An element \( \gamma \in G \) satisfies WWPD if \( \gamma \) is loxodromic and, under the natural induced action \( G \act \partial X \times \partial X - \Delta \), the orbit \( G \cdot \partial_{\pm} \gamma \) is discrete in \( \partial X \times \partial X - \Delta \). The WPD condition requires in addition that the subgroup of \( G \) that stabilizes \( \partial_{\pm} \gamma \) be virtually cyclic.

**Finite and infinite lamination subgroups.** Behind the results of [BF02] are Thurston’s decomposition theory for elements of \( \text{MCG}(S) \) [FLP+12], Ivanov’s decomposition theory for subgroups of \( \text{MCG}(S) \) [Iva92], and the Masur-Minsky results on hyperbolicity of \( \mathcal{C}(S) \) [MM99]. Given \( \phi \in \text{MCG}(S) \), its lamination set \( \mathcal{L}(\phi) \) is a finite set consisting of one unstable lamination for each pseudo-Anosov piece of the Thurston decomposition of \( \phi \). Each subgroup \( \Gamma < \text{MCG}(S) \) has its associated lamination set \( \mathcal{L}(\Gamma) = \bigcup_{\phi \in \Gamma} \mathcal{L}(\phi) \) which is useful in applying subgroup decomposition theory. For example, \( \Gamma < \text{MCG}(S) \) is virtually abelian if and only if \( \mathcal{L}(\Gamma) \) is finite.

Behind the proof of Theorem A are the decomposition theory for elements of \( \text{Out}(F_n) \) due to Bestvina, Feighn, and Handel [BFH97, BFH00, BFH05, FH11], a decomposition theory for abelian subgroups [FH09] due to Feighn and Handel, our decomposition theory for general finitely generated subgroups, [HM17b]–[HM17f], and our results on hyperbolicity and dynamics of the free splitting complex \( \mathcal{FS}(F_n) \) [HM13, HM14b]. Associated to each \( \phi \in \text{Out}(F_n) \) is its finite set \( \mathcal{L}(\phi) \) of attracting laminations [BFH00]. Each subgroup \( \Gamma < \text{Out}(F_n) \) has its associated lamination set \( \mathcal{L}(\Gamma) = \bigcup_{\phi \in \Gamma} \mathcal{L}(\phi) \). If \( \mathcal{L}(\Gamma) \) is finite then \( \Gamma \) is a finite lamination subgroup, otherwise
it is an infinite lamination subgroup. Every virtually abelian subgroup of $\text{Out}(F_n)$ is a finite lamination subgroup, but the converse does not hold, unlike in $\text{MCG}(S)$.

**Subgroups of $\text{IA}_n(Z/3)$ with (virtually) abelian restrictions.** In [BF02], Bestvina and Fujiwara use Ivanov’s results [Iva92] to reduce to the case of an “irreducible” subgroup $\Gamma < \text{MCG}(S)$ — one having no invariant curve system. We also reduce to special subgroups of $\text{Out}(F_n)$ as stated in the heading just above, a reduction of a somewhat different nature. Recall the finite index, characteristic subgroup

$$\text{IA}_n(Z/3) = \text{IA}(F_n; Z/3) = \ker(\text{Out}(F_n) \rightarrow GL(n, Z/3))$$

In Section 4.1 we review the following important invariance properties of $\text{IA}_n(Z/3)$. First, $\text{IA}_n(Z/3)$ is torsion free ([BT68] and see [Vog02]). Next, if $\phi \in \text{IA}_n(Z/3)$ then the conjugacy class of any element or free factor of $F_n$ which is $\phi$-periodic is fixed by $\phi$ [HM17d]. Finally, every virtually abelian subgroup of $\text{IA}_n(Z/3)$ is abelian [HM17g], which we sometimes emphasize by writing “(virtually) abelian” in the context of a subgroup of $\text{IA}_n(Z/3)$. In various theorems we often restrict to subgroups of $\text{IA}_n(Z/3)$ in order to have these properties, and in applying such theorems we achieve this restriction by replacing a subgroup by its intersection with $\text{IA}_n(Z/3)$.

For any free factor $B < F_n$ with conjugacy class $[B]$ and stabilizer subgroup $\text{Stab}[B] < \text{Out}(F_n)$, there is a natural restriction homomorphism $\text{Stab}[B] \rightarrow \text{Out}(B)$ [HM17c, Fact 1.4]. A subgroup $\Gamma < \text{IA}_n(Z/3)$ has (virtually) abelian restrictions if for any proper free factor $B < F_n$ such that $\Gamma < \text{Stab}[B]$, the restriction map $\Gamma \mapsto \text{Out}(B)$ has (virtually) abelian image. The two versions of this property — with or without “virtually” — are equivalent by [HM17g] (see Corollary 4.1 for details).

Among finitely generated subgroups of $\text{IA}_n(Z/3)$ with (virtually) abelian restrictions, Theorems B and C are concerned, respectively, with infinite lamination and finite lamination subgroups, and their actions on hyperbolic spaces. Theorem B uses the free splitting complex $\mathcal{FS}(F_n)$, hyperbolicity of which is proved in [HM13], and loxodromic elements of which are characterized in [HM14b]; in particular, every element of $\text{Out}(F_n)$ acts elliptically or loxodromically on $\mathcal{FS}(F_n)$.

**Theorem B.** For any infinite lamination subgroup $\Gamma < \text{IA}_n(Z/3)$ with (virtually) abelian restrictions, and any maximal, $\Gamma$-invariant, proper free factor system $\mathcal{A}$, we have:

1. The action $\Gamma \rhd \mathcal{FS}(F_n)$ is nonelementary.
2. There exists a loxodromic $\phi \in [\Gamma, \Gamma]$ which is fully irreducible rel $\mathcal{A}$, and any such $\phi$ is a WWPD element for the action of $\Gamma$ on $\mathcal{FS}(F_n)$.

**Theorem C.** For any finitely generated, finite lamination subgroup $\Gamma < \text{IA}_n(Z/3)$ which is not (virtually) abelian and has (virtually) abelian restrictions, there is a finite index normal subgroup $N < \Gamma$ and there is an action $N \rhd X$ on a hyperbolic space, such that the following hold:
(1) Every element of $N$ acts elliptically or loxodromically on $X$.

(2) The action $N \curvearrowright X$ is nonelementary.

(3) Every element of $[N, N]$ is either elliptic or WWPD with respect to the action $N \curvearrowright X$.

The proof of Theorem C, in which some new hyperbolic actions are constructed, is found in Part II. Here in Part I, Theorem C is used as a black box for purposes of proving Theorem A.

**WWPD methods and Theorem D.**

Reducing Theorem A to Theorems B and C is carried out in Section 4.3, using WWPD methods that are summarized in Theorem D. Here is a brief description of those methods (see also the Remark at the end of Section 2.4, regarding how to avoid Theorem D at the expense of weaker conclusions).

Given a nonelementary action of a group $\Gamma$ on a hyperbolic space, the WPD property comes in an individual version defined for one loxodromic element at a time, and in a global version which simply asserts that all loxodromic elements satisfy the individual version (see Section 2.1). In the special setting of the action $\text{MCG}(S) \curvearrowright \mathcal{C}(S)$, that “local–global” distinction was unimportant because every loxodromic element of the action $\text{MCG}(S) \curvearrowright \mathcal{C}(S)$ — every pseudo-Anosov mapping class — satisfies WPD. For a general hyperbolic action action, the individual and global versions of WPD are connected by a theorem of Osin [Osi16] saying that if a given nonelementary hyperbolic action of $\Gamma$ has a WPD element then $\Gamma$ has a nonelementary hyperbolic action on a possibly different space that satisfies the global WPD property (even better, $\Gamma$ has an acylindrical hyperbolic action).

Although WPD suffices for studying $H^2_b$ properties of subgroups of mapping class groups [BF02], and although it suffices for subgroups of $\text{Out}(F_n)$ containing a fully irreducible element [BF10], we do not know whether it suffices for general subgroups of $\text{Out}(F_n)$. Instead our proof of Theorem B uses the loxodromic elements of the action $\text{Out}(F_n) \curvearrowright \mathcal{F} \mathcal{S}(F_n)$; these form a strictly larger collection than the fully irreducible outer automorphisms, and some of them fail to satisfy WPD [HM14b]. What our proof shows is that sufficiently many of these loxodromics satisfy the weaker WWPD property so that $H^2_b$ methods can be applied.

We study WWPD first as an individual property (Section 2.2), and then as a global property (Section 2.5). The global WWPD property is designed to weaken several different aspects of the global WPD property, retaining enough properties so that the quasimorphism constructions found in [BF02] will still work. First, we do not require a hyperbolic action of $\Gamma$ itself, only a hyperbolic action $N \curvearrowright X$ of a finite index normal subgroup $N \triangleleft \Gamma$. Second, we do not require every loxodromic of $N$ to satisfy WPD or WWPD, only that WWPD hold for all the nontrivial elements of some
Schottky subgroup $F \subset [N,N]$ of the commutator subgroup. The global WWPD property also imposes similar conditions on the actions of $N$ that are obtained by pre-composing the given action of $N$ with inner automorphisms of $\Gamma$. For full details see Definition 2.13.

**Theorem D.** If a group $\Gamma$ satisfies the global WWPD property then $H^2_b(\Gamma; \mathbb{R})$ contains an embedded $\ell^1$.

The proof of Theorem D (see the appendix) follows closely the methods of [Fuj00] and [BF02], combined with an application of Kalužnin-Krasner embeddings into wreath products in order to avoid any special wreath product hypothesis as was needed in [BF02].

**Methods of proof of Theorem B.**

Consider a subgroup $\Gamma < IA_n(\mathbb{Z}/3)$ satisfying the hypotheses of Theorem B, so $\Gamma$ is an infinite lamination subgroup with (virtually) abelian restrictions. Consider also a maximal, proper, $\Gamma$-invariant free factor system $\mathcal{A}$. In Section 4.2, we attack Theorem B using lamination ping-pong methods from [HM17f] to produce $\phi \in \Gamma$ which is irreducible relative to $\mathcal{A}$, has a filling attracting lamination, and is in the commutator subgroup of $\Gamma$. The element $\phi$ acts loxodromically on $\mathcal{F}S(F_n)$ because, as proved in [HM14b], the elements of $\text{Out}(F_n)$ acting loxodromically on $\mathcal{F}S(F_n)$ are precisely those which have a filling attracting lamination (a strictly weaker property than being fully irreducible, which is the criterion for loxodromic action on the free factor complex [BF14]). Since the image of the restriction homomorphism $\Gamma \mapsto \text{Out}(A)$ is abelian, and since $\phi$ is in the commutator subgroup of $\Gamma$, the restriction $\phi \mid \text{Out}(A)$ is trivial.

Theorem B is thus reduced to the following theorem, the reduction argument being carried out in full detail in Section 4.2.

**Theorem E.** Let $n \geq 3$, let $\Gamma$ be a subgroup of $IA_n(\mathbb{Z}/3)$ that preserves a (possibly empty) proper free factor system $\mathcal{F}$, and let $\phi \in \Gamma$ have the following properties:

**Relative irreducibility:** $\phi$ is irreducible rel $\mathcal{F}$;

**Trivial restrictions:** $\phi \mid \text{Out}(A)$ is trivial for each component $[A] \in \mathcal{F}$;

**Filling lamination:** $\phi$ has a filling attracting lamination, equivalently $\phi$ acts loxodromically on $\mathcal{F}S(F_n)$ [HM14b].

Then $\phi$ is a WWPD element for the action of $\Gamma$ on $\mathcal{F}S(F_n)$.

The proof of Theorem E itself is carried out in Section 6. That proof depends on results about well functions in the free splitting complex, developed in [HM14b] in analogy to the well functions of Algom-Kfir in the context of outer space [AK11], and further developed in Section 5. In particular, Section 5.4 describes regularity properties of attracting laminations which we use in place of measure theoretic properties.
of currents as applied in [BF02] and [BF10]. Underlying Section 5.4 is a study of uniform splitting properties of EG strata of relative train track maps carried out in Section 3.2, strengthening splitting properties derived in earlier works.

In Theorems B and E, the free factor system $F$ is empty if and only if the subgroup $H$ is fully irreducible, in which case we recover the result of Bestvina and Feighn [BF10] with a different proof. There might be various ways to relax the assumption that $F$ be empty. The way expressed in Theorem E—allowing $F$ to be nonempty but requiring $H$ to be abelian when restricted to each component of $F$—was chosen because it is sufficient for purposes of proving Theorem B, and because the laminations involved in Theorem E have a particularly simple structure that is easier than general laminations. For a description of that structure see the heading The topological structure of $\Lambda_\phi^\sim$, under Case 2b of Section 6.3.

**Application to the Bridson-Wade Theorem**

As a corollary to Theorem A we prove the following theorem of Bridson-Wade, following the same lines as the analogous proof for mapping class groups found in [BF02].

**Corollary** ([BW11]). If $\Gamma$ is an irreducible lattice in a connected, semisimple Lie group of real rank $\geq 2$ having finite center, then every homomorphism $h: \Gamma \to \text{Out}(F_n)$ has finite image.

**Proof.** By the Margulis-Kahzdan normal subgroup theorem [Zim84, Theorem 8.1.2], $K = \text{Ker}(h)$ is either finite or of finite index in $\Gamma$. Assuming $K$ is finite we derive a contradiction.

If $\Gamma$ is a nonuniform lattice then it contains a solvable subgroup $H$ which does not surject with finite kernel onto an abelian group, so the group $H/H \cap K$ is solvable and not virtually abelian, but it injects to $\text{Out}(F_n)$ contradicting [BFH04, Ali02].

If $\Gamma$ is a uniform lattice then it has a free subgroup of rank $\geq 2$ [Mar91, Theorem 5.6], so the group $\Gamma/K$ is not virtually abelian. Also, $\Gamma$ is finitely generated. But $\Gamma/K$ injects to $\text{Out}(F_n)$ and so, by Theorem A, $H^2_0(\Gamma/K; \mathbb{R})$ contains an embedded $\ell^1$. In particular there is an unbounded quasimorphism $\Gamma/K \to \mathbb{R}$ (see Section 2.1), and by composition we get an unbounded quasimorphism $\Gamma \to \Gamma/K \to \mathbb{R}$, contradicting the theorem of Burger and Monod [BM99, Corollary 1.3].

**Outline of the logic**

To summarize, we shall reduce Theorem A to Theorem C by the following outline:

- Section 4.2: Proof that Theorem E implies Theorem B.
- Section 4.3: Proof that Theorems B, C, D imply Theorem A.
- Section 6: Proof of Theorem E.
- Section 7 (Appendix): Proof of Theorem D.

In Part II [HM17a] will be found the proof of Theorem C.
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2 Weakening weak proper discontinuity

This section contains a review of the WPD and WWPD properties, together with a further development of WWPD culminating with the definition of the global WWPD property and the statement of Theorem D, the proof of which is found in Section 7 (Appendix).

Section 2.1 contains a review of certain basic notions needed throughout the paper. Section 2.2 reviews the WPD property of [BF02] and the WWPD property of [BBF15]; Proposition 2.6 establishes several equivalent formulations of WWPD, and proves various properties of WWPD. In Section 2.3 we review the equivalence relation $g \sim h$ on loxodromic elements of a hyperbolic action, which was crystallized by Bestvina and Fujiwara [BF02], and which plays a central role in proofs that second bounded cohomology contains an embedded $\ell^1$; Lemma 2.10 describes how WWPD and the equivalence relation $g \sim h$ interact with each other. The emphasis in Section 2.3 is to interpret WPD, WWPD, and the relation $g \sim h$ in terms of induced actions on the Gromov boundary of a hyperbolic space.

The rest of this section further develops the theory of WWPD. In Section 2.4 we will explain how to use the existence of an individual WWPD element of a nonelementary hyperbolic group action $\Gamma \curvearrowright X$ to deduce that $H^2_b(\Gamma; \mathbb{R})$ contains an embedded $\ell^1$. In Section 2.5 we formulate a global WWPD property of a hyperbolic group action, we study how this property reduces under passage to finite index subgroups and quotient groups, and we use this property to state Theorem D, whose proof will be found in the appendix.

2.1 Basic concepts.

Metrics. A metric complex is a connected simplicial complex equipped with the geodesic metric making each $k$-simplex isometric to a regular $k$-simplex in $\mathbb{R}^{k+1}$ of edge lengths equal to 1, using barycentric coordinates to define the isometry. A hyperbolic complex $X$ is a Gromov hyperbolic metric complex; equivalently, the 1-skeleton is Gromov hyperbolic. The Gromov boundary is denoted $\partial X$, and we denote $\overline{X} = X \cup \partial X$ with its Gromov topology. The space of two point subsets is denoted $\partial^2 X = \{\xi, \eta \mid \xi \neq \eta \in \partial X\}$ with topology induced by the 2-1 covering map $(\partial X \times \partial X) - \Delta \mapsto \partial^2 X$, equivalently the Hausdorff topology on compact subsets of $\partial X$.

Remark. Using the Rips complex one easily sees that every geodesic metric space is quasi-isometric to some metric complex [Gro87]. Since the spaces we use in this work are all metric complexes, we couch our results in that language.

Actions. An action of a group $\Gamma$ on an object $X$ is given by a homomorphism $\mathcal{A}: \Gamma \mapsto \text{Isom}(X)$ from $\Gamma$ to the self-isomorphism group of $X$ (we use this definition primarily for metric complexes and occasionally for objects in other categories). We
often use “action notation” \( A \): \( \Gamma \curvearrowright X \), usually suppressing \( A \) and writing simply \( \Gamma \curvearrowright X \). For \( g \in \Gamma \) and a subset \( Y \in X \), we use expressions like \( g(Y) \) or \( g \cdot Y \) for \( A(g)(Y) \). Given an action \( \Gamma \curvearrowright X \), the stabilizer of a subset \( Y \subset X \) is denoted \( \text{Stab}(Y) = \text{Stab}(Y; \Gamma) = \{ g \in \Gamma \mid g \cdot Y = Y \} \).

Consider a hyperbolic complex \( X \). Any isometric action \( \Gamma \curvearrowright X \) extends to a topological action \( \Gamma \curvearrowright \overline{X} = X \cup \partial X \). An isometry \( h: X \to X \) is elliptic if each orbit \( \{ h^i \cdot x \} \) is of bounded diameter in \( X \), it is loxodromic if each orbit map \( i \mapsto h^i \cdot x \) is a quasi-isometric embedding \( \mathbb{Z} \to X \), and otherwise it is parabolic. For \( h \) to be loxodromic is equivalent to saying that the extension \( h: \overline{X} \to \overline{X} \) has source-sink or north-south dynamics, meaning that there is a repelling fixed point \( \partial_- h \in \partial S \), an attracting fixed point \( \partial_+ h \in \partial S \), and for each \( x \in S - \{ \partial_- h, \partial_+ h \} \) the sequence \( h^i(x) \) converges to \( \partial_- h \) as \( i \to -\infty \) and to \( \partial_+ h \) as \( i \to +\infty \). As ordered and as unordered pairs we denote these fixed points as \( \partial_\pm h = (\partial_- h, \partial_+ h) \in \partial S \times \partial S - \Delta \) and \( \partial h = \{ \partial_- h, \partial_+ h \} \in \partial^2 S \). Note also that \( h \) is loxodromic if and only if the action of \( h \) on \( S \) has a quasi-axis which is a quasi-isometric embedding \( \gamma: \mathbb{R} \to S \) such that for some \( T > 0 \) called the period we have \( \gamma(t+T) = h(\gamma(t)), t \in \mathbb{R} \); we can always take \( \gamma \) to be a bi-infinite edge path in the 1-skeleton of \( X \). A quasi-axis is determined by its restriction \( \gamma \mid [0, T] \), subject to the requirement that \( h(\gamma(0)) = \gamma(T) \); that restriction, and any of its translates \( \gamma \mid [(k-1)T, kT] \), are called fundamental domains for the action of \( h \) on its quasi-axis.

Two loxodromic isometries \( g, h: X \to X \) are independent if \( \partial h \cap \partial g = \emptyset \). An action \( \Gamma \curvearrowright X \) on a hyperbolic complex is elementary if it does not contain an independent pair of loxodromic elements, and it is elliptic if every element is elliptic.

**Bounded cohomology.** As in other works cited earlier, we study second bounded cohomology of a group via quasimorphisms on the group; see for example the introduction to [BF02]. We recall the precise connection between these two concepts, which depends on basic definitions of cohomology theory and the snake lemma.

Consider a group \( G \) and its cochain complex \( C^*(G; \mathbb{R}) \), where \( C^k(G; \mathbb{R}) \) is the vector space of functions \( f: G^k \to \mathbb{R} \), with the standard coboundary operator \( \delta: C^k(G; \mathbb{R}) \to C^{k+1}(G; \mathbb{R}) \) [Bro82]. We need only the formula for \( \delta: C^1(G; \mathbb{R}) \to C^2(G; \mathbb{R}) \) which is \( \delta(f)(g_1, g_2) = f(g_1) + f(g_2) - f(g_1 g_2) \). The bounded cohomology \( H_b^k(G; \mathbb{R}) \) is the cohomology of the subcomplex of bounded cochains \( C_b^k(G; \mathbb{R}) = \{ f \in C^k(G; \mathbb{R}) \mid f \text{ is bounded} \} \).

The vector space of quasimorphisms of \( G \) is defined to be

\[
QH(G; \mathbb{R}) = \{ f \in C^1(G; \mathbb{R}) \mid \delta f \in C_b^2(\mathbb{R}) \}
\]

The defect of \( f \in QH(G; \mathbb{R}) \) is the real number \( \sup |\delta f(g_1, g_2)| \).

The quotient space \( \overline{QH}(G, \mathbb{R}) \) is defined by modding out \( QH(G; \mathbb{R}) \) by the subspace spanned by those \( f: G \to \mathbb{R} \) which are either actual homomorphisms or
bounded functions. One obtains an exact sequence
\[ 0 \rightarrow \tilde{QH}(G; \mathbb{R}) \rightarrow H^2_b(G; \mathbb{R}) \rightarrow H^2(G; \mathbb{R}) \]
by applying the snake lemma to the short exact sequence of cochain complexes of the form
\[ 1 \mapsto \mathcal{C}^* \rightarrow \mathcal{C}^* \rightarrow \mathcal{C}^* / \mathcal{C}^* \rightarrow 1 \]
Thus to prove that \( H^2_b(G; \mathbb{R}) \) contains an embedded \( \ell^1 \), it is sufficient to prove the same for \( \tilde{QH}(G; \mathbb{R}) \), and that is the method we will use.

### 2.2 WPD and WWPD

In this section we fix a group action \( \Gamma \curvearrowright X \) on a hyperbolic complex with associated Gromov extension \( \Gamma \curvearrowright \overline{X} = X \cup \partial X \).

**Definition 2.1.** Given \( h \in \Gamma \) we say that \( h \) is *individually WPD* with respect to \( \Gamma \) if \( h \) is loxodromic and for every \( x \in X \) and \( R > 0 \) there exists an integer \( M > 0 \) such that any subset \( Z \subset \Gamma \) satisfying the following property is finite:

- For each \( g \in Z \) we have \( d(x, g(x)) < R \) and \( d(h^M(x), gh^M(x)) < R \).

**Definition 2.2** ([BF02] Section 3). The action \( \Gamma \curvearrowright X \) is said to satisfy WPD if \( \Gamma \) is not virtually cyclic, \( \Gamma \) contains at least one loxodromic element, and every loxodromic element of \( \Gamma \) is individually WPD with respect to \( \Gamma \).

In these definitions we usually drop “with respect to \( \Gamma \)” when it is clear from context.

It is shown in the proof of [BF02] Proposition 6 that if \( h \in \Gamma \) satisfies the individual WPD property then its endpoint pair \( \partial_{\pm} h \in \partial X \times \partial X - \Delta \) has discrete orbit under the diagonal action of \( \Gamma \) on \( \partial X \times \partial X - \Delta \) and the stabilizer subgroup of the endpoint pair \( \partial_{\pm} h \) is virtually infinite cyclic. The converse also holds, as is well known to experts, and we make this explicit in Corollary 2.7 below. For purposes of formulating the WWPD property we focus on discreteness:

**Definition 2.3** (Equivariantly discrete fixed set). We say that a loxodromic element \( h \in \Gamma \) has *equivariantly discrete fixed set* if the \( \Gamma \)-orbit of \( \partial h \) in the space \( \partial^2 X \) is a discrete subset, equivalently the \( \Gamma \)-orbit of \( \partial_{\pm} h \) in the space \( \partial X \times \partial X - \Delta \) is a discrete subset.

The WWPD property, formulated by Bestvina, Bromberg, and Fujiwara [BBF15], has several equivalent reformulations given in Proposition 2.6 below. One may think of WWPD as a fragment of the individual WPD property that is designed to allow for arbitrary behavior of the subgroup \( \text{Stab}(\partial_{\pm} h) \), while still capturing the essential discreteness concepts of WPD that are useful for applications to second bounded cohomology and other purposes. The idea is that instead of counting the total number of group elements that move point pairs a small amount, one counts only the number of left cosets of \( \text{Stab}(\partial_{\pm} h) \) represented by such group elements.
Definition 2.4. Given a group action $\Gamma \curvearrowright X$ on a hyperbolic space, an element $h \in \Gamma$ satisfies WWPD with respect to $\Gamma$ if $h$ is loxodromic and for every $x \in X$ and $R > 0$ there exists an integer $M \geq 1$ such that any subset $Z \subseteq \Gamma$ that satisfies the following two properties is finite:

1. For each $g \in Z$ we have $d(x, g(x)) < R$ and $d(h^M(x), gh^M(x)) < R$; and
2. No two elements of $Z$ lie in the same left coset of $\text{Stab}(\partial_\pm h)$.

As with Definition 2.1 and 2.2, we usually drop “with respect to $\Gamma$” if it is clear from context.

Combining Definitions 2.2 and 2.4 we immediately have:

Corollary 2.5. If a loxodromic $h \in \Gamma$ satisfies WPD then it satisfies WWPD. \qed

In the following proposition, item (2) clarifies that when generalizing from WPD to WWPD, what one loses control over is the structure of the group $\text{Stab}(\partial_\pm h)$; what one retains is precisely the discrete topology of the orbit of $\partial_\pm h$. Also, item (3) is how WWPD was originally formulated in [BBF15]. Item (4) is what we will actually apply in the proof of Theorem E; see the heading “Setting up the proof of Theorem E” in Section 6.

Proposition 2.6. Given a loxodromic element $h \in \Gamma$, the following are equivalent:

1. $h$ satisfies WWPD.
2. $h$ has equivariantly discrete fixed set.
3. For any quasi-axis $\ell$ of $h$ there exists $D \geq 0$ such that for any $g \in \Gamma$, if $g \not\in \text{Stab}(\partial_\pm h)$ then the image of a closest point map $g(\ell) \mapsto \ell$ has diameter $\leq D$.
4. In the group $\Gamma$ there is NO infinite sequence $g_1, g_2, g_3, \ldots$ satisfying the following properties:
   
   (a) For all $i \neq j$ the elements $g_i, g_j$ lie in different left cosets of $\text{Stab}(\partial_\pm h)$.
   
   (b) For all (there exists) $x \in X$ there exists $R > 0$ such that for all $M \geq 0$ there exists $I \geq 0$ such that if $0 \leq m \leq M$ and if $i \geq I$ then $d(g_i h^m(x), h^m(x)) < R$.

Furthermore if $h$ satisfies WWPD then:

5. $\text{Stab}(\partial_- h) = \text{Stab}(\partial_+ h) = \text{Stab}(\partial_\pm h)$;
6. If $k \in \Gamma - \text{Stab}(\partial h)$ then $h$ and $khk^{-1}$ are independent.
Remark: The two versions of item (4b) — one with quantifier “for all” and the other with “there exists” — are equivalent, as we shall show in the course of the proof.

Before turning to the proof of Proposition 2.6, we first apply it to prove two corollaries:

**Corollary 2.7.** Given a loxodromic \( h \in \Gamma \), the following are equivalent:

1. \( h \) is individually WPD.
2. The subgroup \( \text{Stab}(\partial \pm h) < \Gamma \) is virtually cyclic, and \( h \) has equivariantly discrete fixed set.

**Proof.** Assuming (1) holds, the proof that \( \text{Stab}(\partial \pm h) < \Gamma \) is virtually cyclic can be found in the proof of [BF02] Proposition 6, which uses only the individual WPD property for \( h \). To verify that \( h \) has equivariantly discrete fixed set one can combine Corollary 2.5 and Proposition 2.6. Thus (2) is proved.

Assuming (2) holds, it follows that \( h \) satisfies WWPD (by Proposition 2.6 (2) \( \implies \) (1)). Using that the subgroup \( \text{Stab}(\partial \pm h) \) is virtually cyclic, its cyclic subgroup \( \langle h \rangle \) has finite index, and since \( h \) is loxodromic the action \( \langle h \rangle \curvearrowright S \) is metrically proper it follows that the action \( \text{Stab}(\partial \pm h) \curvearrowright S \) is metrically proper. Fixing \( x \in S \) and \( R > 0 \), consider for each integer \( N > 0 \) the following set (as in Definitions 2.2 and 2.4):

\[
Z_N = \{ g \in \Gamma \mid d(x, g(x)) < R \text{ and } d(h^N(x), gh^N(x)) < R \}
\]

Since \( h \) satisfies WWPD, there exists \( N \) such that \( Z_N \) intersects only finitely many left cosets of the subgroup \( \text{Stab}(\partial \pm h) \). But since the action of \( \text{Stab}(\partial \pm h) \) is metrically proper, each of its left cosets has finite intersection with \( Z_N \). The set \( Z_N \) is therefore finite, showing that \( h \) satisfies WPD and so (1) is proved.

The next result generalizes Corollary 2.5:

**Corollary 2.8.** Consider a group action \( \Gamma \curvearrowright X \) on a hyperbolic space with image subgroup \( Q = \text{Image}(\Gamma \mapsto \text{Isom}(X)) \), an element \( h \in \Gamma \), and the image \( q \in Q \) of \( h \) under the quotient homomorphism \( \Gamma \mapsto Q \). If \( q \) is a WPD element of the action \( Q \curvearrowright X \) then \( h \) is a WWPD element of the action \( \Gamma \curvearrowright X \). More generally (by Corollary 2.5), if \( q \) is a WWPD element of \( Q \curvearrowright X \) then \( h \) is a WWPD element of \( \Gamma \curvearrowright X \).

**Proof.** Note that \( h \) is loxodromic, that \( \partial \pm h = \partial \pm q \), and that the orbits \( H \cdot \partial \pm h \) and \( Q \cdot \partial \pm q \) are the same subset of \( \partial X \times \partial X - \Delta \). Applying the equivalence of items (1), (2) in Proposition 2.6, and using that \( q \) is WWPD, it follows first that the set \( H \cdot \partial \pm h = Q \cdot \partial \pm q \) is discrete in \( \partial X \times \partial X - \Delta \), and then that \( h \) satisfies WWPD.

We finish this section with:
Proof of Proposition 2.6. In this proof we adapt interval notation \([x, y]\) to denote a choice of geodesic between points \(x, y \in X\) in a geodesic metric space. Usually we use function notation for quasigeodesics, such as \(\gamma: [s, t] \to X\), but we also briefly use the notation \(\overline{xy}\) where \(x = \gamma(s), y = \gamma(t)\).

The equivalence \((2) \iff (3)\) is elementary; here is a sketch with vague constants. Let \(\ell\) be a quasi-axis for \(h\). If \((2)\) fails then there is a sequence \(g_i \in \Gamma - \text{Stab}(\partial \pm h)\) such that \(g_i(\partial \pm h) = g_i(\partial \pm \ell)\) converges to \(\partial \pm h = \partial \pm \ell\). It follows that there are sequences \(x_i, y_i \in \ell\) such that \(x_i \to \partial - h, y_i \to \partial + h\), and \(x_i, y_i\) are within uniformly bounded distance of \(g_i(\ell)\). The image of the closest point projection map \(\pi_i: g_i(\ell) \mapsto \ell\) therefore has points within uniformly bounded distance of \(x_i, y_i\), and since \(d(x_i, y_i) \to +\infty\) it follows that \(\text{diam}(\text{Image}(\pi_i)) \to +\infty\), and so \((3)\) fails. Conversely if \((3)\) fails then there is a sequence \(g_i \in \Gamma - \text{Stab}(\partial \pm h)\) such that, letting \(\pi_i: g_i(\ell) \to \ell\) be a closest point map, the diameter of \(\text{Image}(\pi_i)\) goes to \(+\infty\) with \(i\). After postcomposing each \(g_i\) with an appropriate power of \(h\), \(\text{Image}(\pi_i)\) comes uniformly Hausdorff close to an entire subsegment \([x_i, y_i]\) of \(\ell\) such that \(x_i \to \partial - h\) and \(y_i \to \partial + h\). Furthermore, there is a subsegment \([x_i', y_i']\) of \(g_i(\ell)\) which is uniformly Hausdorff close to \([x_i, y_i]\). It follows that \(\partial - g_i(\ell) \to \partial - h\) and \(\partial + g_i(\ell) \to \partial + h\), proving that \((2)\) fails.

Item \((4)\) asserts that there does not exist an infinite sequence satisfying the conjunction of \((4a)\) and \((4b)\). Let \((4)_\forall\) denote item \((4)\) with the quantifier at the beginning of \((4b)\) specified to be “for all”, and similarly for \((4)_\exists\). Since \((4)\) is a negation, clearly \((4)_\exists \implies (4)_\forall\). It remains to prove the chain of implications \((4)_\forall \implies (1) \implies (2) \implies (4)_\exists\).

For the rest of the proof, let \(L(k, c, \delta) \geq 0\) be a thinness constant for \(k, c\) quasi-geodesic quadrilaterals in a \(\delta\)-hyperbolic space \(X\), with finite or infinite endpoints: for any \(w, x, y, z \in \overline{X}\) and any \(k, c\) quasigeodesics \(\overline{wx}, \overline{xy}, \overline{yz}, \overline{zw}\), the quasigeodesic \(\overline{wz}\) is contained in the \(L\)-neighborhood of \(\overline{wx} \cup \overline{yz} \cup \overline{zw}\). Note for example that \(L(1, 0, \delta) = 2\delta\).

We turn to the proof that \((4)_\forall \implies (1)\). Fix \(\delta\) to be a hyperbolicity constant for \(X\). Suppose that \(h\) fails to satisfy \((1)\), i.e. \(h\) is not WWPD, so there exists \(x \in \overline{X}\) and \(R_0 > 0\), and there exist sequences \(M_i \to +\infty\) in the positive integers and \(g_i\) in \(\Gamma\), such that the elements \(g_i\) all lie in different left cosets of \(\text{Stab}(\partial \pm h)\) (thus satisfying item \((4a)\)), and such that \(d(x, g_i(x)) < R_0\) and \(d(h^{M_i}(x), g_i h^{M_i}(x)) < R_0\). Since \(h\) is loxodromic, the sequence \(m \mapsto h^m(x)\) is a quasi-isometric embedding \(\mathbb{Z} \to X\) and can be interpolated to obtain a \(k, c\) quasi-axis for \(h\) of the form

\[
\cdots [h^{-2}(x), h^{-1}(x)] \ast [h^{-1}(x), h^0(x)] \ast [h^0(x), h^1(x)] \ast [h^1(x), h^2(x)] \ast \cdots
\]

Since \(g_i\) moves the points \(x = h^0(x)\) and \(h^{M_i}(x)\) a distance at most \(R_0\), by an elementary argument in Gromov hyperbolic geometry there exists a constant \(R\) depending only on \(R_0, k, c,\) and \(\delta\) such that all points on this segment between \(x = h^0(x)\) and \(h^{M_i}(x)\) are moved by \(g_i\) a distance at most \(R\) (independent of \(i\)). Thus item \((4b)_\exists\) is satisfied, and so \(h\) fails to satisfy \((4)_\forall\).
We next prove that \((2) \implies (4)\). Assuming that \((4)\) fails, we may pick an infinite sequence of elements \(f_1, f_2, f_3, \ldots\) lying in different left cosets of \(\text{Stab}(\partial_\pm h)\), and we may pick \(x \in X\) and \(R > 0\), so that the following holds: for all \(M \geq 0\) there exists \(I \geq 0\) such that if \(0 \leq m \leq M\) and if \(i \geq I\) then \(d(f_i h^m(x), h^m(x)) < R\). It follows that there exists a growing sequence of positive even integers \(m_i = 2k_i \to +\infty\) such that for each \(i\) the isometry \(f_i\) moves each of the points \(x, h(x), h^2(x), \ldots, h^{2k_i}(x)\) a distance at most \(R\). Let \(g_i = h^{-k_i} f_i h^{k_i}\), and note that \(g_i\) moves each of the points \(h^{-k_i}(x), h^{-k_i+1}(x), \ldots, h^{k_i-1}(x), h^{k_i}(x)\) a distance at most \(R\). Since the sequence \(k \mapsto h^k(x)\) defines a quasi-isometric embedding \(\mathbb{Z} \to X\), it follows that the sequence \((h^{-k_i}(x), h^{k_i}(x))\) converges to \(\partial_\pm h\) in the space of ordered pairs of points in \(X \cup \partial X\), and therefore the sequence \((g(h^{-k_i}(x)), g(h^{k_i}(x)))\) converges to \(\partial_\pm h\), and therefore the sequence \(\partial_\pm (g_i h g_i^{-1}) = g_i(\partial_\pm h)\) converges to \(\partial_\pm h\). We may assume, dropping at most a single term of the sequence, that \(f_i \not\in \text{Stab}(\partial_\pm h)\) for each \(i\), and hence \(g_i \not\in \text{Stab}(\partial_\pm h)\). Since \(g_i(\partial_\pm h) \neq \partial_\pm h\) but \(g_i(\partial_\pm h)\) converges to \(\partial_\pm h\), we have proved that \(h\) does not have equivariantly discrete fixed set and so \((2)\) fails.

It remains to prove that \((1) \implies (2)\). Assuming that \(h\) does not have equivariantly discrete fixed set, there is a sequence \(f_i \in \Gamma\) such that \(f_i(\partial_\pm h) \neq \partial_\pm h\) and such that \(f_i(\partial_\pm h) \to \partial_\pm h\) as \(i \to +\infty\). Notice that for some values of \(i\) we might have \(f_i(\partial_\pm h) = \partial_\pm h\) and for others \(f_i(\partial_\pm h) \neq \partial_\pm h\), although those cannot happen simultaneously. Let \(\gamma: \mathbb{R} \to X\) be a \(k, c\) quasi-axis for \(h\) with period \(T\), let \(L = L(k, c, \delta)\), and let \(\Delta\) be diameter of the fundamental domain \(\gamma[0, T]\). For each \(i\) note that \(f_i \gamma: \mathbb{R} \to X\) is a quasi-axis for \(f_i h_i f_i^{-1}\). After extracting a subsequence of \(f_i\) we may assume that the following three things hold. First, if \(i \neq j\) then \(f_i(\partial_\pm h) \neq f_j(\partial_\pm h)\), and hence \(f_i^{-1} f_j \not\in \text{Stab}(\partial_\pm h)\), and so after deleting at most a single term of the sequence we have \(f_i \not\in \text{Stab}(\partial_\pm h)\) for all \(i\). Second (and after possibly inverting \(h\)) we have \(f_i(\partial_\pm h) \neq \partial_\pm h\) and \(f_i(\partial_\pm h) \neq \partial_\pm h\). Third, there are integer sequences \(a_i \to -\infty\), \(b_i \to +\infty\), and \(c_i < d_i\) such that \(d(\gamma(a_i T), f_i \gamma(c_i T)) \leq L\) and \(d(\gamma(b_i T), f_i \gamma(d_i T)) \leq L + \Delta = L_1\). Letting \(g_i = h^{-a_i} f_i h^{c_i}\), and letting \(B_i = b_i - a_i\) and \(D_i = d_i - c_i\), we have \(d(\gamma(0), g_i(\gamma(0))) \leq L_2\) and \(d(\gamma(B_i T), g_i(\gamma(D_i T))) \leq L_1\), and we also have \(B_i, D_i \to +\infty\). We shall prove that some infinite subsequence of \(g_i\) is pairwise distinct and satisfies Definition 2.4 (1) and (2) and hence \(h\) does not satisfy WWPD.

For proving that Definition 2.4 (1) holds, we claim that \(d(\gamma(D_i T), \gamma(B_i T)) \leq 4L + 2\Delta\), from which we obtain the bound

\[
d\left(\gamma(D_i T), g_i(\gamma(D_i T))\right) \leq \overbrace{d(\gamma(D_i T), \gamma(B_i T)) + d(\gamma(B_i T), g_i(\gamma(D_i T)))}^{\leq 4L + 2\Delta} \leq 5L + 3\Delta.
\]

We prove the claim assuming \(B_i \geq D_i\); the other case is similar. The point \(\gamma(D_i T)\), which lies on the \(k, c\) quasi-geodesic path \(\gamma[0, B_i T]\), is within distance \(L\) of some point.
$P$ in a geodesic $[\gamma(0), \gamma(B_iT)]$. We have
\[
\begin{align*}
d(\gamma(0), \gamma(D_iT)) + d(\gamma(D_iT), \gamma(B_iT)) & \\
& \leq d(\gamma(0), P) + d(P, \gamma(D_iT)) + d(\gamma(D_iT), P) + d(P, \gamma(B_iT)) \\
& = d(\gamma(0), \gamma(B_iT)) + 2L \\
& \leq d(g_i(\gamma(0)), g_i(\gamma(D_iT))) + 2L + 2L_1 \\
& = d(\gamma(0), \gamma(D_iT)) + 4L + 2\Delta
\end{align*}
\]
from which the claim follows.

We next prove, after passing to a subsequence, that for all $i \neq j$ we have
\[
g_i^{-1}g_j \not\in \text{Stab}(\partial_i h)
\]
(*) implying pairwise distinctness of the $g_i$, and implying that $g_i^{-1}g_j \not\in \text{Stab}(\partial \pm h)$ and so Definition 2.4 (2) holds. Fixing $j$ it suffices to prove that (*) holds for all sufficiently large $i$, and so consider those values of $i$ such that the following hold:
\[
g_i^{-1}g_j = h^{-c_i}f_i^{-1}h^{a_i-a_j}f_jh^{c_i} \in \text{Stab}(\partial_i h) \\
f_i^{-1}h^{a_i-a_j}f_j \in \text{Stab}(\partial_i h) \\
h^{a_i-a_j}(f_j(\partial_i h)) = f_i(\partial_i h)
\]
(**)

On the left hand side of (**), since $f_j(\partial_i h) \neq \partial_i h$, and since $a_i - a_j \to -\infty$ as $i \to +\infty$, it follows by source-sink dynamics that $h^{a_i-a_j}(f_j(\partial_i h)) \to \partial_- h$. But on the right hand side we have $f_i(\partial_i h) \to \partial_+ h$. Equation (**) is therefore impossible for all sufficiently large $i$.

This completes the proof of equivalence of items (1), (2), (3), (4)$_\forall$ and (4)$_\exists$.

To prove (5) it suffices by symmetry it suffices to prove that $\text{Stab}(\partial_- h) = \text{Stab}(\partial \pm h)$.

For that it suffices, given $\gamma$ such that $\gamma(\partial_- h) = \partial_- h$, to prove that $\gamma(\partial_+ h) = \partial_+ h$.

If not then $\partial_- (\gamma h \gamma^{-1}) = \gamma(\partial_- h) = \partial_- h$ and $\partial_+ (\gamma h \gamma^{-1}) = \gamma(\partial_+ h) \neq \partial_+ h$. Using source sink dynamics of $h$ acting on $\partial X$ it follows that the sequence $h^n(\partial_+(\gamma h \gamma^{-1})) = \partial_+ ((h^n \gamma)h(h^n \gamma)^{-1})$ converges to $\partial_\pm h$ as $n \to +\infty$, but the terms of that sequence are all distinct from $\partial_\pm h$, contradicting that $h$ has equivariantly discrete fixed points.

To prove (6), consider $k \in \Gamma - \text{Stab}(\partial h)$ and suppose that $khk^{-1}$ is not independent of $h$, so the set $\partial h \cap k(\partial h) = \partial h \cap \partial(khk^{-1})$ consists of a single point. If $\partial_- h = k(\partial_- h)$ or if $\partial_+ h = k(\partial_+ h)$ then either of the elements $k$ or $khk^{-1}$ gives an immediate contradiction to (5). If $\partial_- h = k(\partial_- h) = \partial_+(khk^{-1}) = \partial_-(kh^{-1}k^{-1})$ then $kh^{-1}k^{-1}$ gives a contradiction to (5), and similarly if $\partial_+ h = k(\partial_+ h)$.

\[\square\]

### 2.3 Equivalence $g \sim h$ in a hyperbolic group action

Given two paths $\gamma: [a, b] \to X$ and $\rho: [c, d] \to X$ in a metric space $X$, we say that $\gamma, \rho$ are $L$-Hausdorff close rel endpoints if $d(\gamma(a), \rho(c)) \leq L$, $d(\gamma(b), \rho(d)) \leq L$, and the Hausdorff distance between the sets $\gamma[a, b]$ and $\rho[c, d]$ is $\leq L$.  

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The following relation is defined for loxodromic elements by Bestvina and Fujiwara [BF02]. We trivially extend the relation by convention to nonloxodromic elements, with the effect that if \( g \) is not loxodromic and \( g \sim h \) then \( g = h \).

**Definition 2.9.** Define an equivalence relation \( g \sim h \) on \( \Gamma \) to mean: either \( g = h \); or \( g, h \) are loxodromic, and for some (any) quasi-axes \( \gamma_g \) and \( \gamma_h \) there exists \( L \) such that for any subsegment \( \gamma_g[a, b] \) there is a subsegment \( \gamma_h[c, d] \) and an element \( k \in \Gamma \) such that \( k(\gamma_g[a, b]) \) is \( L \)-Hausdorff close rel endpoints to \( \gamma_h[c, d] \).

For each \( g \in \Gamma \) the orbit of \( \partial_\pm g \) under the action \( \Gamma \acts \partial S \times \partial S - \Delta \) is denoted \( \Gamma \cdot \partial_\pm g \), and its closure in \( \partial S \times \partial S - \Delta \) is denoted \( \Gamma \cdot \overline{\partial_\pm g} \).

**Lemma 2.10.** Given loxodromic \( g, h \in \Gamma \), the following are equivalent:

1. \( g \sim h \)
2. \( \Gamma \cdot \overline{\partial_\pm g} = \Gamma \cdot \overline{\partial_\pm h} \)
3. \( \Gamma \cdot \overline{\partial_\pm g} \cap \Gamma \cdot \overline{\partial_\pm h} \neq \emptyset \)

If in addition \( g \) satisfies WWPD then the following condition is also equivalent:

4. \( \Gamma \cdot \partial_\pm g = \Gamma \cdot \partial_\pm h \).

**Proof.** Evidently (2) \( \implies \) (3). Pick quasi-axes \( \gamma_g \) of \( g \) and \( \gamma_h \) of \( h \).

To prove (1) \( \implies \) (2), assuming \( g \sim h \), for each \( n \) there exists \( \alpha_n, \beta_n \in \Gamma \) such that \( \gamma_g[-n, n] \) is \( L \)-Hausdorff close rel endpoints to some subsegment of \( \alpha_n(\gamma_h) = \gamma_{\alpha_nh\alpha_n^{-1}} \), and it follows that \( \partial_\pm(\alpha_nh\alpha_n^{-1}) \to \partial_\pm g \) as \( n \to +\infty \). This shows that \( \partial_\pm g \in \Gamma \cdot \overline{\partial_\pm h} \), and so the inclusion \( \Gamma \cdot \overline{\partial_\pm g} \subset \Gamma \cdot \overline{\partial_\pm h} \) holds. The opposite inclusion holds by symmetry.

To prove (3) \( \implies \) (1), choosing \( (\xi, \eta) \in \Gamma \cdot \overline{\partial_\pm g} \cap \Gamma \cdot \overline{\partial_\pm h} \), choose sequences \( \alpha_n, \beta_n \) in \( \Gamma \) such that the sequences \( \alpha_n(\partial g) = \partial(\alpha_n g \alpha_n^{-1}) \) and \( \beta_n(\partial g) = \partial(\beta_n h \beta_n^{-1}) \) both approach \( (\xi, \eta) \). Choose an oriented quasigeodesic \( \delta \) with initial end \( \xi \) and terminal end \( \eta \). By precomposing both \( \alpha_n \) and \( \beta_n \) with appropriate powers of \( g \) we may assume that \( \alpha_n(\gamma_g(0)) \) and \( \beta_n(\gamma_h(0)) \) each stay within a uniform distance of \( \delta(0) \).

After passing to subsequences of \( \alpha_n \) and \( \beta_n \) it follows there is a constant \( L \) and sequences \( s_n^-, t_n^- \to -\infty \) and \( s_n^+, t_n^+ \to +\infty \) such that \( \alpha(\gamma_g[s_n^-, s_n^+]) \) and \( \beta(\gamma_h[t_n^-, t_n^+]) \) are each \( L/2 \)-Hausdorff close rel endpoints to \( \delta[-n, +n] \), and so are \( L \) Hausdorff close to each other. Since the paths \( \gamma_g[s_n^-, s_n^+] \) exhaust the quasi-axis \( \gamma_g \), as do the paths \( \gamma_h[t_n^-, t_n^+] \), it follows that \( g \sim h \).

Clearly (4) \( \implies \) (2). For the converse, assuming \( g \) satisfies WWPD then by discreteness of \( \Gamma \cdot \overline{\partial_\pm g} \) it follows that \( \Gamma \cdot \overline{\partial_\pm g} = \Gamma \cdot \partial_\pm g \). Assuming (2) we therefore have \( \Gamma \cdot \overline{\partial_\pm h} = \Gamma \cdot \partial_\pm g \), and so \( \Gamma \cdot \partial_\pm h \subset \Gamma \cdot \partial_\pm g \). By transitivity of the \( \Gamma \)-action on \( \Gamma \cdot \partial_\pm g \) it follows that \( \Gamma \cdot \partial_\pm g = \Gamma \cdot \partial_\pm h \) which is (4). \( \square \)
Remark. Note that if two loxodromics \( g, h \in \Gamma \) satisfy \( \partial_- g = \partial_- h \) then \( g \sim h \) because \( \lim_{i \to +\infty} g^i(\partial_{\pm} h) = \partial_{\pm} g \) and one may apply Lemma 2.10 (3) \( \Rightarrow \) (1); similarly if \( \partial_+ g = \partial_+ h \). On the other hand if \( \partial_- g = \partial_+ h \), or even if \( h = g^{-1} \), one cannot in general conclude anything about whether \( g \sim h \).

2.4 Individualized WWPD methods.

We recall the main outline of [BF02], which gives a method for using a group action \( \Gamma \acts S \) on a hyperbolic complex to compute second bounded cohomology.

[BF02] Theorem 1: If the action has an independent pair of loxodromic elements \( g, h \in \Gamma \) such that \( g \not\sim h \), then \( H^2_b(\Gamma; \mathbb{R}) \) is of uncountably infinite dimension.

[BF02] Proposition 6 (5): If the action \( \Gamma \acts S \) is WPD then \( \Gamma \) has independent loxodromic elements \( g \not\sim h \).

Putting these together one obtains:

[BF02] Theorem 7: If the action \( \Gamma \acts S \) is WPD then \( H^2_b(\Gamma; \mathbb{R}) \) is of uncountably infinite dimension.

An important feature of the proof of [BF02] Proposition 6 (5) is that it works under weaker hypotheses, namely that some loxodromic \( g \in \Gamma \) satisfies the individualized WPD property and that the action \( \Gamma \acts S \) has an independent pair of loxodromic elements. The proof of [BF02] Theorem 7 therefore also works under the same weaker hypotheses.

What follows in Proposition 2.11 is a version of [BF02] Proposition 6 based on the WWPD property for an individual element. Proposition 2.11 also contains some additional details needed for later use.

We first review the basic facts about hyperbolic ping-pong (see e.g. [Mos07, Theorem 6.7]). Consider a geodesic hyperbolic space \( S \). Recall that a group action \( F \acts S \) is Schottky if \( F \) is free of finite rank and for some (any) \( y \in S \) the orbit map \( F \mapsto f \cdot y \) is a quasi-isometric embedding with respect to the word metric on \( F \); in particular each nontrivial element of \( F \) is loxodromic. Recall also the following basic facts:

The Cayley tree of a Schottky group: For any Schottky action \( F \acts S \), and for any free basis of \( F \) with associated Cayley tree \( T \), there exists an \( F \)-equivariant continuous quasi-isometric embedding \( T \mapsto Y \) which has a unique continuous extension to the Gromov bordifications \( T \cup \partial T \mapsto Y \cup \partial Y \).

Hyperbolic ping pong: For any pair of independent loxodromic isometries \( \alpha, \beta \in \text{Isom}(S) \), and any pairwise disjoint neighborhoods \( U_-, U_+, V_-, V_+ \subset \partial S \) of \( \partial_- \alpha, \partial_+ \alpha, \partial_- \beta, \partial_+ \beta \) there exists an integer \( M \geq 1 \) such that for any \( m, n \geq M \) the following hold: \( \alpha^m(\partial S - U_-) \subset U_+ \), \( \beta^n(\partial S - V_-) \subset V_+ \), and \( \alpha^m, \beta^n \) freely generate a Schottky action \( F(\alpha^m, \beta^n) \acts S \).
Proposition 2.11. Let $\gamma \in \Gamma$ satisfy WWPD and suppose that $\Gamma \neq \text{Stab}(\partial \gamma)$. For any $\alpha \in \Gamma - \text{Stab}(\partial \gamma)$ there exists $A > 0$ such that for any $a \geq A$, letting $g_1 = \gamma^a$ and $h_1 = \gamma^a(\alpha \gamma^{-a})^{-a} = \gamma^a \alpha \gamma^{-a} \alpha^{-1}$, we have:

1. $h_1$ is also loxodromic
2. $g_1 \not\sim h_1$ and $g_1 \not\sim h_1^{-1}$
3. $g_1, h_1$ freely generate a Schottky subgroup of $\Gamma$.

Proof. Consider the loxodromic $\delta = \alpha \gamma \alpha^{-1}$. Applying Proposition 2.6 (6), $\delta$ and $\gamma$ are independent. Now we play hyperbolic ping-pong with some WWPD spin. Choose pairwise disjoint neighborhoods $U_-, U_+, V_-, V_+ \subset \partial S \times \partial S - \Delta$ of $\partial_- \gamma$, $\partial_+ \gamma$, $\partial_- \delta$, $\partial_+ \delta$, respectively, where $U_-, U_+$ are chosen so small that $\partial_+ \gamma$ is the only element of the $\Gamma$-orbit of the ordered pair $\partial_+ \gamma$ that is contained in $U_- \times U_+$. Choose $M$ sufficiently large so that for $m, n \geq M$ the conclusions of hyperbolic ping pong hold as stated above, in particular $\gamma^m, \delta^m$ freely generate a Schottky action on $S$. For $a \geq A = 2M$, letting $g_1 = \gamma^a$ and $h_1 = \gamma^a \delta^{-a} = \gamma^a \alpha \gamma^{-a} \alpha^{-1}$, it follows that $h_1$ is loxodromic and that the two elements $g_1, h_1$ freely generate a Schottky subgroup, because any finitely generated subgroup of the Schottky group $\langle \gamma^a, \delta^a \rangle$ is also Schottky.

Having verified conclusions (1), (3) of Proposition 2.11, we turn to (2).

Letting $U'_+ = \gamma^M(U_+) \subset U_+$, we have $\partial_- h_1 \in V_+$, $\partial_+ h_1 \in U'_+$, and $\gamma^{-M}(\partial_\pm h_1) = \partial_\pm (\gamma^{-M} h_1 \gamma^M) \in U_- \times U_+$. But $\gamma^{-M} h_1 \gamma^M$ is independent of $\gamma$ and so $\gamma^{-M}(\partial_\pm h_1) \neq \partial_\pm \gamma$. From our choice of $U_- \times U_+$ and the fact that $\gamma$ satisfies WWPD, by applying Lemma 2.10 (4) it follows that $\gamma \not\sim \gamma^{-M} h_1 \gamma^M$. Since $\gamma$ is invariant under conjugacy and under passage to powers, it follows that $g_1 \not\sim h_1$.

The exact same argument (with $\alpha^{-1}$ in place of $\alpha$, and with the inclusion $\gamma^{-M}(U_-) \subset U_-$ in place of the inclusion $\gamma^M(U_+) \subset U_+$) shows that $g_1 \not\sim h'_1 = \gamma^a \alpha^{-1} \gamma^{-a} \alpha$. But since $h'_1$ is conjugate to $h_1^{-1}$ it follows that $g_1 \not\sim h_1^{-1}$. \qed

Remark. Putting together Proposition 2.11 and [BF02] Theorem 1 we immediately obtain Theorem 2.12 below regarding group actions on hyperbolic spaces possessing WWPD elements. One may sometimes use Theorem 2.12 in place of Theorem D, perhaps at the expense of weaker conclusions. For example, using Theorem 2.12 it follows that a group satisfying the hypotheses of Theorem E and whose action is nonelementary has an embedded $\ell^1$. Also, the reduction arguments of Section 4.3 may be adapted, and combined with Theorem 2.12, to prove a weak version of Theorem A saying that any finitely generated subgroup of $\text{Out}(F_n)$ has a finite index subgroup satisfying the $H^2_b$-alternative.

Theorem 2.12. If $\Gamma \rhd S$ is a hyperbolic action possessing an independent pair of loxodromic elements, and if $\Gamma$ has a WWPD element, then $H^2_b(\Gamma; \mathbb{R})$ contains an embedded $\ell^1$. \qed
2.5 Global WWPD methods: Statement of Theorem D.

In [BF02] Theorem 8, Bestvina and Fujiwara generalize their Theorem 7 by giving hypotheses on a hyperbolic action of a finite index subgroup $N$ of a group $\Gamma$ which are sufficient to determine $H^{2}_b(\Gamma; \mathbb{R})$. The hypotheses require that the subgroup action $N \actson X$ satisfies the global WPD property, and that there is an embedding of $\Gamma$ into a certain wreath product which interacts with the subgroup action in a certain way. Those hypotheses are tailored to the case where $\Gamma$ is a subgroup of a surface mapping class groups, allowing one to apply Ivanov’s subgroup decomposition theory to obtain a natural finite index subgroup of $\Gamma$ acting on a certain subsurface curve complex, and allowing one to use the theorem of [BF02] saying that every element of the mapping class group that acts loxodromically on the curve complex satisfies WPD.

Theorem D (aka Theorem 2.5) generalizes [BF02] Theorem 8 in a few ways. First, WPD can be replaced by WWPD. Second, one does not need any wreath product hypothesis at all: instead, one applies the Kalužnin-Khassner embedding of $\Gamma$ into the wreath product of $N$ by $\Gamma/N$. Third, when we come to apply Theorem D in the setting of $\text{Out}(F_n)$, we will not know that every loxodromic element of the action satisfies WWPD, but we will know that the presence of one WWPD element implies an entire Schottky subgroup’s worth of WWPD elements. The definition of the global WWPD property therefore, by necessity, has a somewhat technical formulation, which is designed to allow us to jump into the middle of the proof [BF02] Theorem 8 where a certain Schottky subgroup of the action begins to play a significant role.

**Definition 2.13.** A group $\Gamma$ is said to satisfy the *global WWPD hypothesis* if there exists a normal subgroup $N \triangleleft \Gamma$ of finite index, an action $N \actson X$ on a hyperbolic complex, and a rank 2 free subgroup $F < N$, such that the following hold:

1. Each element of $N$ acts either loxodromically or elliptically on $X$.

2. The restricted action $F \actson X$ is Schottky and each of its nonidentity elements is WWPD with respect to the action $N \actson X$.

3. For each inner automorphism $i_g: \Gamma \to \Gamma$, letting $N \actson_g X$ denote the composed action $N \overset{i_g}{\to} N \actson X$, the restricted action $F \actson_g X$ satisfies one of the following:
   
   (a) $F \actson_g X$ is Schottky and each of its nonidentity elements is WWPD with respect to the action $N \actson_g X$; or
   
   (b) $F \actson_g X$ is elliptic.

In situations where $N$, its action $N \actson X$, and/or $F$ are specified, we shall adopt phrases like “$\Gamma$ satisfies WWPD with respect to $N$, its action $N \actson X$, and/or $F$”.

In practice we reformulate item (3) by using that $N$ has finite index in order to cut down the inner automorphisms that must be checked to a finite subset. Since $N$ is normal in $\Gamma$, the action of $\Gamma$ on itself by inner automorphisms restricts to an
action of $\Gamma$ on $N$ that we denote $i: \Gamma \to \text{Aut}(N)$. Choose coset representatives $g_\kappa$ of $N$ in $\Gamma$, where $\kappa \in \{1, \ldots, K\}$ and $K = [\Gamma : N]$, and by convention choose $g_1$ to be the identity. Let $i_\kappa \in \text{Aut}(N)$ be the restriction to $N$ of $i_{g_\kappa} \in \text{Inn}(\Gamma)$. We refer to $i_1, \ldots, i_K$ as outer representatives of the action of $\Gamma$ on $N$ (see diagram below). Let $N \acts^\kappa X$ be the composed action $N \acts^\kappa N \acts X$, and let $F \acts^\kappa X$ be its restriction. With this notation, to verify (3) as stated for all the actions $F \acts g X$ it suffices to check only the actions $F \acts^\kappa X$. In other words, (3) is equivalent to:

(3)' For each $\kappa = 1, \ldots, K$ the action $F \acts^\kappa X$ satisfies one of the following:

(a) $F \acts^\kappa X$ is Schottky and each of its nonidentity elements is WWPD with respect to the action $N \acts^\kappa X$; or

(b) $F \acts^\kappa X$ is elliptic.

The terminology of “outer representatives” refers to the fact that in the commutative diagram

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{i} & \text{Aut}(N) \\
\downarrow & & \downarrow \\
\Gamma/N & \xrightarrow{i_\kappa} & \text{Out}(N) \rightarrow \text{Aut}(N)/\text{Inn}(N)
\end{array}
$$

the automorphisms $i_\kappa \in \text{Aut}(N)$ represent all of the elements of the image of the homomorphism $\Gamma/N \to \text{Out}(N)$ (that homomorphism need not be injective, and so there may be some duplication of elements of $\text{Out}(N)$ represented by the list $i_1, \ldots, i_K$, but this is inconsequential). The equivalence of (3) and (3)' holds because if we replace $g_\kappa$ by something else $h = \nu g_\kappa$ in its coset ($\nu \in N$), then $i_\kappa \in \text{Aut}(N)$ is replaced by $i_h = i_\nu \circ i_\kappa \in \text{Aut}(N)$, and so the restricted actions $F \acts^\kappa X$ and $F \acts h X$ are conjugate by an isometry of $X$, namely the action of $\nu \in N$. But each of properties (a) and (b) is invariant under such conjugation.

This completes Definition 2.13.

Remark. We emphasize that although the various actions $N \acts^\kappa X$ are equivalent up to inner automorphisms of $\Gamma$ restricted to its normal subgroup $N$, that does not mean that they are equivalent up to conjugation by isometries of $X$, because $\Gamma$ itself does not act on $X$. Thus, for example, an element of $N$ may be loxodromic with respect to one of the actions $N \acts^\kappa X$ but not with respect to a different one.

The following lemma explains how the global WWPD hypothesis is a “fragment” of the WPD property, and it gives more information we shall use elsewhere.

**Lemma 2.14.** Let $\mathcal{A}: \Gamma \to \text{Isom}(X)$ be an action on a hyperbolic complex, and let $Q = \text{Image}(\mathcal{A})$. If $Q$ satisfies WPD, and if each element of $Q$ is either loxodromic or elliptic, then the given action $\mathcal{A}$ of the group $\Gamma$ on $X$ satisfies the global WWPD hypothesis with respect to the $N = \Gamma$. 

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Proof. Applying [BF02] Proposition 6 (4) to \(Q\) we obtain independent loxodromic elements of \(Q\), and by applying hyperbolic ping-pong we obtain a rank 2 Schottky subgroup \(F' < Q\) with respect to the action \(Q \curvearrowright X\). Under the homomorphism \(\mathcal{A} : \Gamma \to Q\) there is a section \(F' \to \Gamma\) whose image \(F < \Gamma\) is a rank 2 Schottky subgroup with respect to the action \(\Gamma \curvearrowright X\). Since \(Q\) satisfies WPD, applying Corollary 2.8 it follows that each nontrivial element of \(F\) satisfies WWPD with respect to the action \(\Gamma \curvearrowright X\).

Here, restated from the introduction, is our main result regarding the global WWPD hypothesis; see the appendix for the proof.

**Theorem D.** If a group \(\Gamma\) satisfies the global WWPD hypothesis then \(H^2_b(\Gamma; \mathbb{R})\) contains an embedding of \(\ell^1\).

When we apply Theorem D, the free subgroup implicit in the global WWPD hypothesis will be constructed by ping-pong methods.

## 3 Background material on \(\text{Out}(F_n)\)

Section 3.1 reviews basic definitions, terminology, and facts regarding \(\text{Out}(F_n)\), referring to the literature for full details.

Section 3.2 contains Lemma 3.1, a uniform splitting property which will be used for our study of well functions in the proof of Proposition 5.7, and in Part II of this work.

### 3.1 Basic concepts of \(\text{Out}(F_n)\)

In this subsection we briefly review basic terminology and notation and key definitions, focussing on relative train track maps. The original sources for this material include [CV86], [BH92], [BFH00], and [FH11]. See [HM17c, Section 1] for full definitions and citations in a still brief but much more comprehensive overview.

**Marked graphs and topological representatives.** A rank \(n\) marked graph is a finite graph \(G\) without valence 1 vertices equipped with a homotopy equivalence \(R_n \to G\), where \(R_n\) is the base rose of rank \(n\) whose edges are indexed and oriented so as to identify \(F_n \approx \pi_1(R_n)\). The marking induces an isomorphism \(F_n \approx \pi_1(G)\), a deck transformation action \(F_n \curvearrowright \tilde{G}\) on the universal cover, and an equivariant homeomorphism \(\partial F_n \approx \partial \tilde{G}\), all well-defined up to inner automorphism of \(F_n\). A homotopy equivalence \(f : G \to G'\) of marked graphs is always assumed to take vertices to vertices and to be locally injective on each edge, and so \(f\) induces a map of directions (initial germs of oriented edges) and a map of turns (pairs of directions at the same vertex). Given a marked graph \(G\) there is an induced isomorphism between the group of self-homotopy equivalences of \(G\) modulo homotopy and the group \(\text{Out}(F_n)\). A self-homotopy equivalence \(f : G \to G\) that represents \(\phi \in \text{Out}(F_n)\)
in this way is called a topological representative of \( \phi \); in such a case, iterating the induced map on turns, a turn is illegal if some iterate is degenerate, and it is legal otherwise.

**Paths, lines, and circuits.** Following [BFH00] a path in a marked graph \( G \) is a concatenation of edges and partial edges without backtracking, such that a partial edge may occur only incident to an endpoint of the path; in the degenerate case, a trivial path consists of just a vertex. An infinite path in \( G \) can be a line or a ray depending on whether the concatenation is doubly infinite or singly infinite. In some situations we must restrict to paths with endpoints at vertices, e.g. for defining complete splitting of paths; in other situations we must allow for more general paths with arbitrary endpoints, e.g. when taking pre-images of paths. In other situations we will use more general edge-paths in which backtracking is allowed. We will use appropriate language to indicate such situations.

A circuit is a cyclic concatenation of edges without backtracking, i.e. a circle immersion.

Two paths or circuits are equivalent if they differ only by reparameterizaiton and inversion. The space of paths and circuits up to equivalence is denoted \( \partial F \), and is equipped with the weak topology having a basis element for each finite path \( \alpha \), consisting of all paths and circuits having \( \alpha \) as a subpath. The subspace of lines is denoted \( B(G) \subset \partial F \). The abstract space of lines of the group \( F_n \) is the quotient space \( B = B(F_n) = \tilde{B}(F_n)/F_n \) where \( \tilde{B}(F_n) \) is the space of two point subsets of \( \partial F_n \), endowed with the weak topology naturally induced by the topology of \( \partial F_n \). The identification \( \partial F_n \approx \partial \tilde{G} \) induces a canonical homeomorphism \( \mathcal{B}(F_n) \approx B(G) \). The element of \( \mathcal{B}(G) \) corresponding to a given \( \ell \in B(F_n) \) is called the realization of \( \ell \) in \( G \). The natural action \( \mathrm{Aut}(F_n) \curvearrowright \partial F_n \) induces an action \( \mathrm{Out}(F_n) \curvearrowright B \).

**Path maps.** [HM17c, Section 1.1.6]. Consider a homotopy equivalence of possibly distinct marked graphs \( f : G \to G' \) that represents \( \phi \in \mathrm{Out}(F_n) \) in a manner similar to a topological representative: the composition \( R_n \mapsto G \overset{f}{\twoheadrightarrow} G' \mapsto R_n \) defined using marking maps or their homotopy inverses is homotopic to a topological representative of \( \phi \). There is an induced path map \( f_\# : \tilde{B}(G) \to \tilde{B}(G') \), where \( f_\#(\gamma) \) is obtained from the composition \( f \circ \gamma \) by straightening rel endpoints, lifting first to the universal cover when \( \gamma \) is infinite in order to make use of endpoints at infinity. For each line in \( \mathcal{B} \) realized as \( \ell \in \mathcal{B}(G) \) its image \( \phi(\ell) \in \mathcal{B} \) is realized as \( f_\#(\ell) \in \mathcal{B}(G') \).

There is another induced path map \( f_{\#\#} : \tilde{B}(G) \to \tilde{B}(G') \) defined as follows: for any path \( \gamma \in \tilde{B}(G) \), lift to a path \( \tilde{\gamma} \) in the universal cover \( \tilde{G} \), choose any lift \( \tilde{f} : \tilde{G} \to \tilde{G}' \), let \( \tilde{f}_{\#\#}(\tilde{\gamma}) \) be the intersection of all paths in \( \tilde{G}' \) that contain \( \tilde{\gamma} \) as a subpath, and let \( f_{\#\#}(\gamma) \) be the projection to \( G' \) of \( \tilde{f}_{\#\#}(\tilde{\gamma}) \), which is well-defined independent of choices. This path \( f_{\#\#}(\gamma) \) is obtained from \( f_\#(\gamma) \) by truncating a subpath incident to each endpoint of length at most equal to the bounded cancellation constant of \( f \).

Assuming now that \( G = G' \), a finite path \( \gamma \) is a Nielsen path if \( f_\#(\gamma) = \gamma \). If a Nielsen path can not be written as a concatenation of two non-trivial Nielsen paths
then it is *indivisible*.

A decomposition \( \gamma = \gamma_1 \cdot \ldots \cdot \gamma_K \) of a finite path into subpaths is a *splitting* if
\[
 f_{\#}(\gamma) = f_{\#}(\gamma_1) \ldots f_{\#}(\gamma_K)
\]
for all integers \( i \geq 1 \) (the single dot \( \cdot \) in expressions like \( \gamma_1 \cdot \ldots \cdot \gamma_K \) always refers to a splitting). If \( P \) is a set of finite paths and if each \( \gamma_i \in P \) then we say that \( \gamma = \gamma_1 \cdot \ldots \cdot \gamma_K \) is a *splitting of \( \gamma \) with terms in \( P \). This concept of “splitting with terms in \( P \)” is most useful if for each \( \alpha \in P \) the path \( f_{\#}(\alpha) \) splits with terms in \( P \); see for example the definition of a CT.

**Free factor systems and free factor supports.** [BFH00]. Conjugacy classes in \( F_n \) are denoted \([\cdot]\). A *subgroup system* in \( F_n \) is a set \( \mathcal{A} = \{[A_1], \ldots, [A_K]\} \) such that \( A_1, \ldots, A_K < F_n \) are nontrivial finitely generated subgroups [HM17c, Section 1.1.2]. The elements of a subgroup system are called its *components*. The partial ordering of *containment* \( \mathcal{A} \subset \mathcal{A} \) means that for all \( [A] \in \mathcal{A} \) there exists \([A'] \in \mathcal{A} \) such that \( A < A' \).

A subgroup system \( \mathcal{A} \) is a *free factor system* if it has the form \( \mathcal{A} = \{[A_1], \ldots, [A_K]\} \) so that there is a free factorization \( F_n = A_1 \ast \cdots \ast A_K \ast B \). Associated to each subgraph \( H \subset G \) of a marked graph is the free factor system \( [H] = [\pi_1 H] = \{[\pi_1 H_1], \ldots, [\pi_1 H_K]\} \), where \( H_1, \ldots, H_K \) are the noncontractible components of \( H \) and \( [\pi_1 H_k] \) denotes the conjugacy class of the image of the inclusion induced monomorphism \( \pi_1(H_k) \hookrightarrow \pi_1(G) \approx F_n \). Every chain \( \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_L \) of free factor systems can be realized in this manner as a filtration \( G_1 \subset \cdots \subset G_L \subset G \) of subgroups of some marked graph \( G \).

A line \( \ell \in B(F_n) \) is *supported by* or *carried by* a subgroup system \( \mathcal{A} \) if there exists \( \tilde{\ell} \in \tilde{B}(F_n) \) covering \( \ell \) and \( A < F_n \) such that \([A] \in \mathcal{A} \) and such that \( \partial \tilde{\ell} \subset \partial A \); let \( B(\mathcal{A}) = \{\ell \in B(F_n) \mid \ell \text{ is carried by } \mathcal{A}\} \); equivalently, if \( \mathcal{A} \) is realized by a subgraph \( H \subset G \) then the realization of \( \ell \) in \( G \) is contained in \( H \).

The *free factor support* of a subset \( L \subset B(F_n) \), denoted \( F_{\text{supp}}(L) \), is the unique minimal free factor system \( F_{\text{supp}}(L) \) which supports each element of \( L \). Also, given a subgroup system \( \mathcal{A} \) we use the shorthand notation \( F_{\text{supp}}(L; \mathcal{A}) = F_{\text{supp}}(L \cup B(\mathcal{A})) \), and we call this the *free factor support of \( L \) relative to \( \mathcal{A} \).* Also, we say that \( L \) fills relative to \( \mathcal{A} \) if \( F_{\text{supp}}(L; \mathcal{A}) = \{[F_n]\} \).

**Relative train track maps.** [BH92]. A topological representative \( f : G \to G \) is *filtered* if it comes equipped with an \( f \)-invariant filtration of subgraphs \( \emptyset = G_0 \subset G_1 \subset \cdots \subset G_R = G \), with corresponding subgraphs \( H_r = G_k \setminus G_{r-1} \) called *strata*, such that if \( H_r \) has \( m \) edges with corresponding \( m \times m \) transition matrix \( M \) then either \( H_r \) is an irreducible *stratum* meaning that \( M \) is irreducible or \( H_r \) is a zero *stratum* meaning that \( M \) is a zero matrix. Irreducible strata \( H_r \) are further classified by the value of the Perron-Frobenius eigenvalue \( \lambda \geq 1 \) of \( M \): \( H_r \) is an *EG-stratum* if \( \lambda > 1 \); and \( H_r \) is an *NEG-stratum* if \( \lambda = 1 \). An EG stratum is *aperiodic* if \( M \) is Perron-Frobenius (some power of \( M \) is positive).

For \( f : G \to G \) to be a *relative train track* representative requires some conditions to be imposed on its EG strata. The definition and some further properties are found in [BH92] Section 5, including a reformulation of the definition saying that for each
Each initial direction of an edge in $H_r$ maps to some initial direction of an edge in $H_r$.

Paths in $G_r$ whose only turns in $H_r$ are legal turns are called $H_r$-legal paths. An edge in $H_r$, having no turns at all, is by default $H_r$-legal.

([BH92] Lemma 5.8) If $\sigma$ is an $r$-legal path in $G_r$ with endpoints in $H_r$ then $\sigma$ splits with terms in

$$\{\text{edges of } H_r\} \cup \{\text{nontrivial subpaths of } G_{r-1} \text{ with endpoints in } H_r\}$$

and $f_\#(\sigma)$ is $r$-legal.

Paths of the form $f^k_\#(E)$, for $k \geq 0$ and edges $E \subset H_r$, are called $k$-tiles of $H_r$ (note that when $f^k(E)$ is straightened to form $f^k_\#(E)$, no edges of $H_r$ are cancelled).

Every $\phi \in \Out(F_n)$ has a relative train track representative [BH92]. Furthermore, some positive power $\phi^k$ has a relative train track representative $f : G \to G$ which is $EG$-aperiodic, meaning that the transition matrix of each $EG$ stratum is a Perron-Frobenius matrix; one may take $f$ to be the straightened $k^{th}$ power of any relative train track representative of $\phi$ itself, with a refined filtration.

**Attracting Laminations.** [BFH00]. Consider $\phi \in \Out(F_n)$ and an $EG$-aperiodic relative train track representative $f : G \to G$ of some positive power $\phi^i$. Associated to each $EG$ stratum $H_r \subset G$ is its *attracting lamination* $\Lambda_r$, a closed subset of $\mathcal{B}(F_n)$ whose realization in $G$ consists of all lines $\ell$ such that each subpath of $\ell$ is contained in some $k$-tile of $H_r$; equivalently, the lines of $\Lambda_r$ are the weak limits in $\widehat{\mathcal{B}}(G)$ of sequences of $k$-tiles. The lamination $\Lambda_r$ is also characterized as the closure of some birecurrent, nonperiodic line $\ell \in \mathcal{B} \approx \mathcal{B}(G)$ of height $r$ such that for some weak neighborhood $U \subset \mathcal{B}$ of $\ell$ we have $\phi(U) \subset U$ and $\{\phi^i_\#(U) \mid i \geq 0\}$ is a neighborhood basis of $\ell$; each such line $\ell$ is in $\Lambda_r$, $\ell$ is called a *generic leaf* of $\Lambda_r$, and each such neighborhood $U$ is an attracting neighborhood of a generic leaf. The set $\mathcal{L}(\phi) = \{\Lambda_r \mid H_r$ is an $EG$ stratum$\}$ is well-defined independent of the choice of $f : G \to G$, and it is a finite set.

The elements of the set $\mathcal{L}(\phi)$ are distinguished by their free factor supports: if $\Lambda_1 \neq \Lambda_2 \in \mathcal{L}(\phi)$ then $\mathcal{F}_{\supp}(\Lambda_1) \neq \mathcal{F}_{\supp}(\Lambda_2)$. The elements of the two sets $\mathcal{L}(\phi)$ and $\mathcal{L}(\phi^{-1})$ correspond one-to-one according to their free factor supports: the relation $\mathcal{F}_{\supp}(\Lambda^+) = \mathcal{F}_{\supp}(\Lambda^-)$ between $\Lambda^+ \in \mathcal{L}(\phi)$ and $\Lambda^- \in \mathcal{L}(\phi^{-1})$ is a bijection. We will refer to the ordered pair $\Lambda^\pm = (\Lambda^+, \Lambda^-)$ as a (dual) lamination pair of $\phi$, and we write their common support as $\mathcal{F}_{\supp}(\Lambda^\pm)$. The set of all lamination pairs of $\phi$ is denoted $\mathcal{L}^\pm(\phi)$.

Given a lamination pair $\Lambda^\pm \in \mathcal{L}^\pm(\phi)$ which is fixed by $\phi$, a line $\ell \in \mathcal{B}$ is weakly attracted to $\Lambda^+$ (by iteration of $\phi$) if $\phi^k(\ell)$ weakly converges to a generic leaf of $\Lambda^+$. More details of weak attraction are found in Section 4.1.
EG properties of CTs. There is a particularly nice kind of relative train track map called a CT. The necessary and sufficient condition for \( \phi \) to be represented by a CT is that \( \phi \) be rotationless \([FH11,\ \text{Definition 3.13}]\). Every \( \phi \) has a rotationless iterate with uniformly bounded exponent \([FH11,\ \text{Theorem 4.28}]\) so one can often work exclusively with CTs. The defining conditions for \( f \) to be a CT are found in \([FH11,\ \text{Definition 4.7}]\), a sequence of nine conditions with parenthesized names (some mentioned here). Most of our applications will be to a CT \( f : G \to G \) with an \( f \)-invariant filtration \( \emptyset = G_0 \subset G_1 \subset \cdots \subset G_s = G \) satisfying the following:

- (Special Assumption) The top stratum \( H_s \) is EG-aperiodic.

For now, instead of a full and general definition, we only list certain properties of a CT \( f : G \to G \) which hold under the special assumption above (citations are to \([FH11]\) unless otherwise noted):

1. (Definition 4.7 (Filtration)) There is no \( \phi \)-periodic free factor system strictly contained between the free factor systems \( [\pi_1 G_{s-1}] \) and \( [\pi_1 G_s] = \{ [F_n] \} \).

2. (Corollary 4.19 eg-(i)) There exists (up to reversal) at most one indivisible Nielsen path \( \rho \) of height \( s \). If \( \rho \) exists then it decomposes as \( \rho = \alpha \beta \) where \( \alpha \) and \( \beta \) are \( s \)-legal paths with endpoints at vertices and \((\bar{\alpha}, \beta)\) is an illegal turn in \( H_s \). In particular, \( f(\alpha) = \alpha \gamma \) and \( f(\beta) = \bar{\gamma} \beta \) for some path \( \gamma \). Moreover:
   - ([HM17c, Fact 1.42]) At least one endpoint of \( \rho \) is disjoint from \( G_{s-1} \).
   - (Corollary 4.19 eg-(i)) The initial and terminal directions of \( \rho \) are distinct fixed directions in \( H_s \).

3. ([BH92, Lemma 5.10]) Any assignment \( \ell(E) \) of lengths to the edges \( e \) of \( H_s \) extends to an assignment of lengths to all paths in \( G \) with endpoints at vertices by adding up the lengths of the edges of \( H_s \) in the edge-path description of \( \sigma \). There is an eigenvector assignment of lengths \( \ell(e) \) so that \( \ell(f(e)) = \lambda \ell(e) \) for all edges \( e \) of \( H_s \) where \( \lambda > 1 \) is the Perron-Frobenius eigenvalue of the transition matrix for \( H_s \). If \( \rho = \alpha \beta \) exists as in (2) it follows that \( \ell(\alpha) = \ell(\beta) \) because \( \ell(\alpha) \) and \( \ell(\beta) \) both satisfy \( \lambda L = L + \ell(\gamma) \).

4. ([HM17c, Fact 1.35]) If \( \gamma \) is a path with endpoints at vertices \( H_s \) or a circuit crossing an edge of \( H_s \) then for all sufficiently large \( i \) the path \( f_{\gamma i}(\gamma) \) has a splitting with terms in the set \{edges of \( H_s \}\} \cup \{\text{height } s \text{ indivisible Nielsen paths}\} \cup \{\text{paths in } G_{s-1} \text{ with endpoints in } H_s\} \).

Remark on item (4): This is implicit in \([BFH00]\), e.g. Lemma 4.2.6, Remark 5.1.2, and Step 2 of the proof of Proposition 6.0.4 of \([BFH00]\).

EG strata are further classified into geometric and nongeometric strata, as defined and studied in \([BFH00,\ \text{Sections 5.1, 5.3}]\) and further studied in \([HM17c]\). Very roughly speaking \( H_s \) is geometric if the action of \( f \) on \( G \) may be modeled up to homotopy by a self-homotopy-equivalence of a 2-complex that is obtained by attaching
a surface $S$ to $G_{s-1}$, such that on $S$ the map is pseudo-Anosov. For now we need only the following facts:

(5) ([HM17c, Fact 2.3]) $H_s$ is geometric if and only if there is a closed height $s$ individual Nielsen path.

(6) ([HM17c, Section 2.4]) Geometricity is well-defined for a lamination pair $\Lambda^\pm \in \mathcal{L}(\phi)$ independent of the choice of a CT representative of a rotationless power of $\phi$. To be precise, if $f : G \to G$ and $f' : G' \to G'$ are CT representatives of rotationless positive or negative powers of $\phi$, with EG strata $H_i \subset G$, $H'_j \subset G'$ corresponding to either $\Lambda^+$ or $\Lambda^-$ depending on the sign of the power, then $H_i$, $H'_j$ are either both geometric or both nongeometric.

### 3.2 A uniform splitting lemma

In this section we consider a rotationless $\phi \in \text{Out}(F_n)$ represented by a CT $f : G \to G$ with an EG stratum $H_s$ and a corresponding attracting/repelling lamination pair $\Lambda^\pm_s$ each realized in $G$. If there exists an indivisible Nielsen path of height $s$ (which must be unique up to reversal), denote that path as $\rho$; otherwise, ignore $\rho$.

Lemma 3.1 will uniformize the splitting property that was reviewed in Section 3.1, under item (4) of the heading “EG properties of CTs”. That property said that for each finite path $\sigma$ in $G$ of height $\leq s$ having endpoints at vertices of $H_s$, some iterate $f^d_\#(\sigma)$ splits into terms each of which is either an edge of $H_s$, an indivisible Nielsen path of height $s$, or a path in $G_{s-1}$ with endpoints at vertices of $H_s$. The exponent $d$ needed for this splitting is unbounded in general, as one sees for example by letting $\sigma$ be a leaf segment of $\Lambda^-_s$ that crosses a large number of edges of $H_s$. The following lemma says that such examples are the only reason for the splitting exponent $d$ to be unbounded.

For any path $\sigma \subset G$ of height $\leq s$, let $|\sigma|_s$ denote the number of times that $\sigma$ crosses edges of $H_s$. Also, let $\ell^-_s(\sigma)$ denote the maximum of $|\tau|_s$ over all paths $\tau$ in $G$ such that $\tau$ is a subpath both of $\sigma$ and of a generic leaf of $\Lambda^-_s$.

**Lemma 3.1.** With notation as above, for all $L > 0$ there is a positive integer $d$ so that if $\sigma \subset G$ is a circuit or a finite path with endpoints at vertices of $H_s$, and if $\ell^-_s(\sigma) < L$, then $f^d_\#(\sigma)$ splits into terms each of which is an edge of $H_s$, an indivisible Nielsen path of height $s$, or a subpath of $G_{s-1}$ with endpoints at vertices; equivalently, $f^d_\#(\sigma)$ splits into terms each of which is $s$-legal or an indivisible Nielsen path of height $s$.

**Proof.** We give the argument when $\sigma$ is a finite path with endpoints at vertices of $H_s$; the argument for circuits is almost the same.

Arguing by contradiction, if the lemma fails then there exists $L > 0$ so that for all even integers $2i$ there is a finite path $\sigma_i \subset G$ with endpoints at vertices and with $\ell^-_s(\sigma_i) < L_1$ such that $f^{2i}_\#(\sigma_i)$ does not split into terms each of which is either $s$-legal or is an indivisible Nielsen path of height $s$. Let $\beta_i = f^{i}_\#(\sigma_i)$, and let $\beta_i = \beta_{i,1} \cdots \beta_{i,n}$ be
a maximal splitting of $\beta_i$ into subpaths with endpoints at vertices. There is a splitting $\sigma_i = \sigma_{i,1} \cdot \ldots \cdot \sigma_{i,n}$ with $f^i_#(\sigma_{i,j}) = \beta_{i,j}$; the paths $\sigma_{i,j}$ need not have endpoints at vertices.

If $|\beta_{i,j}|_s$ is bounded independently of $i$ and $j$ then by [HM17c, Lemma 1.54] there is a positive integer $d$ independent of $i, j$ such that each $f^i_#(\beta_{i,j})$ splits into subpaths that are either $s$-legal or indivisible Nielsen paths of height $s$. The same is then true for $f^i_#(\beta_{i,j})$ if $i \geq d$, in contradiction to the fact that $f^{2i}_#(\sigma_i) = f^i_#(\beta_{i,1}) \cdot \ldots \cdot f^i_#(\beta_{i,n})$ has no such splitting.

After passing to a subsequence we may therefore choose for each $i$ an integer $j(i) \in \{1, \ldots, J_i\}$ such that $|\beta_{i,j(i)}|_s \to \infty$ as $i \to \infty$. Since $\beta_{i,j(i)}$ does not split at any interior vertex, there is a uniform bound to the number of edges of $H_s$ in an $s$-legal subpath $\beta_{i,j(i)}$. It follows that the number of illegal turns of height $s$ in $\beta_{i,j(i)}$ goes to infinity with $i$. This in turn implies that $|\sigma_{i,j(i)}|_s$ goes to infinity with $i$, because $\sigma_{i,j(i)}$ has at least as many height $s$ illegal turns as $\beta_{i,j(i)}$.

It follows that some weak limit of the sequence $\sigma_{i,j(i)}$ is a line of height $s$, contained in $G_s$ and containing at least one edge of $H_s$. Consider a height $s$ line $\ell$ which is a weak limit of a subsequence of $\sigma_{i,j(i)}$. If $H_s$ is non-geometric then by the weak attraction theory of [HM17e], the line $\ell$ has one of two options: either the weak closure of $\ell$ contains $\Lambda^-$; or $\ell$ is weakly attracted to $\Lambda^+$ [HM17e, Lemma 2.18]. If $H_s$ is geometric — and so $\rho$ exists and is closed — then there is a third option [HM17e, Lemma 2.19], namely that $\ell$ is a bi-infinite iterate of $\rho$ or $\bar{\rho}$.

Since no $\sigma_i$ contains a subpath of $\Lambda^-$ that crosses $L$ edges of $H_s$, neither does $\ell$, and so the weak closure of $\ell$ does not contain $\Lambda^-$, ruling out the first option. Suppose $\ell$ is weakly attracted to $\Lambda^+$. There exists $m > 0$ such that $f^m_#(\ell)$, and hence $f^m_#(\sigma_{i,j(i)})$ for arbitrarily large $i$, contains an $s$-legal subpath that crosses $> C$ edges of $H_s$, for any choice of $C$. By [BFH00, Lemma 4.2.2], by choosing $C$ to be sufficiently large we may conclude that $f^m_#(\sigma_{i,j(i)})$ splits at an interior vertex. But then for $i \geq m$ it follows that $\beta_{i,j(i)} = f^{i-m}_#(f^m_#(\sigma_{i,j(i)}))$ splits at an interior vertex, contradicting maximality of the splitting of $\beta_i$, and thus ruling out the second option for $\ell$. This concludes the proof if $H_s$ is non-geometric.

Assuming now that $H_s$ is geometric, it remains to show that the third option can be avoided, and hence by the previous paragraph the desired contradiction is achieved. That is, if $H_s$ is geometric we show that some weak limit of a subsequence of $\sigma_{i,j(i)}$ contains at least one edge of $H_s$ and is not a bi-infinite iterate of the closed path $\rho$ or $\bar{\rho}$. This may be done by setting up an application of [HM17e, Lemma 1.11], but it is just as simple to give a direct proof. Lift $\sigma_{i,j(i)}$ to the universal cover of $G$ and write it as an edge path $\tilde{\sigma}_{i,j(i)} = \tilde{E}_{i,1} \tilde{E}_{i,2} \ldots \tilde{E}_{i,M_s} \subset \tilde{G}$; the first and last terms are allowed to be partial edges. Let $b$ equal twice the number of edges in $\rho_s$. Given $m \in \{2 + b, \ldots, M_s - b - 1\}$, we say that $\tilde{E}_{im}$ is well covered if there is a periodic line $\tilde{\rho}_{im} \subset \tilde{G}$ that projects to $\rho$ or to $\bar{\rho}$ and that contains $\tilde{E}_{i,m-b} \ldots \tilde{E}_{im} \ldots \tilde{E}_{i,m+b}$ as a subpath. Since the intersection of distinct periodic lines cannot contain two fundamental domains of both lines, $\tilde{\rho}_{im}$ is unique if it exists. Moreover, if both $\tilde{E}_{im}$
and $\tilde{E}_{i,m+1}$ are well covered then $\tilde{\rho}_{im} = \tilde{\rho}_{i,m+1}$. It follows that if $\tilde{E}_{im}$ is well covered then we can inductively move forward and backward past other well covered edges of $\tilde{\sigma}_{i,j(i)}$ all in the same lift of $\rho$, until either encountering an edge that is not well covered, or encountering initial and terminal subsegments of $\tilde{\sigma}_{i,j(i)}$ of uniform length. After passing to a subsequence, one of the following is therefore satisfied:

1. There exists a sequence of integers $K_i$ such that $2 + b < K_i < M_i - b - 1$, and $K_i \to \infty$, and $M_i - K_i \to \infty$, and such that $\tilde{E}_{iK_i} \subset \tilde{H}_s$ is not well covered.

2. $\sigma_{i,j(i)} = \alpha_i \rho^p \beta_i$ where the number of edges crossed by $\alpha_i$ and $\beta_i$ is bounded independently of $i$ and $|p_i| \to \infty$.

If (1) holds then the existence of a subsequential weak limit of $\sigma_{i,j(i)}$ that crosses an edge of $H_s$ and is not a bi-infinite iterate of $\rho$ or $\bar{\rho}$ follows immediately. If (2) holds then $\beta_{i,j(i)} = f^p_{\#}(\beta_{i,j(i)})$ decomposes as $\mu_i \rho^u \nu_i$ where $|q_i| \to \infty$ and the number of illegal turns of height $s$ in $\mu_i$ and $\nu_i$ is uniformly bounded. But then for any $p \geq 0$ the number of edges of $\rho^u$ that are cancelled when $f^p_{\#}(\mu_i) \rho^u f^p_{\#}(\nu_i)$ is tightened to $f^p_{\#}(\beta_{i,j(i)})$ is bounded independently of $i$ and $p$ and therefore $\beta_{i,j(i)}$ can be split at an interior vertex, which is a contradiction.

4 Reducing Theorem A to Theorems C, D, E

4.1 Weak attraction and subgroup decomposition theory

In this section we review some terminology and notation regarding proofs and applications of lamination ping-pong, particularly from our papers on subgroup classification theory [HM17c, HM17e, HM17f] and from Ghosh’s work on free subgroups of $\text{Out}(F_n)$ [Gho16].

The group $\text{IA}_n(\mathbb{Z}/3)$. Recall the finite index normal subgroup $\text{IA}_n(\mathbb{Z}/3) < \text{Out}(F_n)$ defined by

$$\text{IA}_n(\mathbb{Z}/3) = \text{Ker}(\text{Out}(F_n) \to \text{Aut}(H_1(F_n; \mathbb{Z}/3) \approx GL(n, \mathbb{Z}/3))$$

This subgroup has several important invariance properties:

[BT68] (and see [Vog02]) $\text{IA}_n(\mathbb{Z}/3)$ is torsion free.

[HM17d, Theorem 3.1] For every $\psi \in \text{IA}_n(\mathbb{Z}/3)$ and every free factor $A < F_n$, if its conjugacy class $[A]$ is $\psi$-periodic then it is fixed by $\psi$.

[HM17d, Theorem 4.1] For every $\psi \in \text{IA}_n(\mathbb{Z}/3)$ and every $\gamma \in F_n$, if its conjugacy class $[\gamma]$ is $\psi$-periodic then it is fixed by $\psi$.

[HM17g] Every virtually abelian subgroup of $\text{IA}_n(\mathbb{Z}/3)$ is abelian.
As said in the introduction, we sometimes emphasize the last result, from [HM17g], by using the terminology “(virtually) abelian” in the context of a subgroup of $IA_n(\mathbb{Z}/3)$. For example, here is a consequence of [HM17g] which justifies that terminology for the property of virtually abelian restrictions:

**Corollary 4.1.** A subgroup $\Gamma < IA_n(\mathbb{Z}/3)$ has virtually abelian restrictions if and only if it has abelian restrictions.

**Proof.** For any free factor $A < F_n$ the inclusion $A \hookrightarrow F_n$ induces an injection $H_1(A; \mathbb{Z}/3) \hookrightarrow H_1(F_n; \mathbb{Z}/3)$ which is equivariant with respect to the natural actions $\text{Stab}[A] \hookrightarrow \text{Out}(F_n) \overset{\sim}{\hookrightarrow} H_1(F_n; \mathbb{Z}/3)$ and $\text{Stab}[A] \hookrightarrow \text{Out}(A) \overset{\sim}{\hookrightarrow} H_1(A; \mathbb{Z}/3)$. It follows that the image of $\Gamma$ under the induced homomorphism $\Gamma \hookrightarrow \text{Out}(A)$ is contained in $IA_A(\mathbb{Z}/3)$, and so by [HM17g] that image is virtually abelian if and only if it is abelian.

Relative full irreducibility. An extension of free factor systems is simply an instance $\mathcal{F} \sqsubset \mathcal{F}'$ of the usual partial ordering such that $\mathcal{F} \neq \mathcal{F}'$. Given an extension $\mathcal{F} \sqsubset \mathcal{F}'$, its co-edge number is the minimum, over all marked graphs $G$ and subgraphs $H \subset H' \subset G$ realizing $\mathcal{F} \sqsubset \mathcal{F}'$, of the number of edges of $H' \setminus H$. If the co-edge number equals 1 then $\mathcal{F} \sqsubset \mathcal{F}'$ is a one-edge extension, and this holds if and only if the free factor systems $\mathcal{F}$ and $\mathcal{F}'$ are related in one of two ways: either two components of $\mathcal{F}$ having ranks $i, j$ are replaced by a single component of $\mathcal{F}'$ of rank $i + j$ and containing the given two components of $\mathcal{F}$; or a single component of $\mathcal{F}$ having rank $i$ is replaced by a single component of $\mathcal{F}'$ of rank $i + 1$ and containing the given component of $\mathcal{F}$. Otherwise $\mathcal{F} \sqsubset \mathcal{F}'$ is a multi-edge extension. If $\mathcal{F}' = \{[F_n]\}$ then we drop $\mathcal{F}'$ from the terminology and speak of the co-edge number of $\mathcal{F}$.

For any subgroup $\mathcal{H} < \text{Out}(F_n)$ and any $\mathcal{H}$-invariant extension $\mathcal{F} \sqsubset \mathcal{F}'$, recall that $\mathcal{H}$ is fully irreducible relative to the extension if there does not exist a free factor system contained strictly between $\mathcal{F}$ and $\mathcal{F}'$ which is invariant under a finite index subgroup of $\mathcal{H}$. If in addition $\mathcal{F}' = \{[F_n]\}$ then it is dropped from the terminology and we say that $\mathcal{H}$ is fully irreducible rel $\mathcal{F}$. When $\mathcal{H} = \langle \phi \rangle$ is cyclic then we extend this terminology to $\phi$, and so $\phi$ is fully irreducible relative to a $\phi$-invariant extension $\mathcal{F} \sqsubset \mathcal{F}'$ if and only if no free factor system contained strictly between $\mathcal{F}$ and $\mathcal{F}'$ is $\phi$-periodic. In the context of a subgroup or element of $IA_n(\mathbb{Z}/3)$ we may drop the adverb “fully”, because in $IA_n(\mathbb{Z}/3)$ periodic free factor systems are fixed [HM17d, Theorem 3.1].

Weak attraction; nonattracting subgroup systems [HM17e, Section 1]. Consider a subgroup system $\mathcal{A}$ in $F_n$. We say that $\mathcal{A}$ is a vertex group system if there exists a minimal action $F_n \acts T$ on an $\mathbb{R}$-tree $T$ with trivial arc stabilizers such that $\mathcal{A}$ is the set of conjugacy classes of nontrivial point stabilizers of the action $F_n \acts T$ [HM17c, Section 3].

Consider $\phi \in \text{Out}(F_n)$ and a lamination pair $\Lambda_{\phi}^\pm \in \mathcal{L}^\pm(\phi)$ which is fixed by $\phi$. Earlier we defined weak attraction of a line to the lamination $\Lambda_{\phi}^+$ under iteration
of \( \phi \). A conjugacy class is weakly attracted to \( \Lambda_\phi^+ \) if the periodic line representing that conjugacy class is weakly attracted. The nonattracting subgroup system of \( \Lambda_\phi^+ \) is denoted \( A_{na}(\Lambda_\phi^+) \), it is concretely described in terms of any CT representative of a rotationless power of \( \phi \) in [HM17e, Definitions 1.2], and it is more abstractly characterized by item (2) of the following compilation of results:

Fact 4.2. For any rotationless \( \phi \in \text{Out}(F_n) \) and \( \Lambda_\phi^\pm \in L^\pm(\phi) \) we have:

1. \( A_{na}(\Lambda_\phi^\pm) \) is a vertex group system, and it is a free factor system if and only if \( \Lambda_\phi^\pm \) is nongeometric ([HM17c, Section 3], [HM17e, Proposition 1.4]).

2. \( A_{na}(\Lambda_\phi^\pm) \) is the unique vertex group system such that: a conjugacy class \( c \) is carried by \( A_{na}(\Lambda_\phi^+) \) \iff \( c \) is not weakly attracted to \( \Lambda_\phi^- \) by iteration of \( \phi \), \iff \( c \) is not weakly attracted to \( \Lambda_\phi^- \) by iteration of \( \phi^{-1} \) [HM17e, Corollaries 1.9 and 1.10].

3. If \( \mathcal{F} \) is a proper, \( \phi \)-invariant free factor system, and if there exists a lamination pair \( \Lambda_\phi^+ \in L^\pm(\phi) \) not carried by \( \mathcal{F} \), then:
   
   (a) \( \mathcal{F} \) has co-edge number \( \geq 2 \).
   
   (b) If \( \phi \) is fully irreducible rel \( \mathcal{F} \) then \( \Lambda_\phi^\pm \) is the unique lamination pair in \( L^\pm(\phi) \) not carried by \( \mathcal{F} \), and furthermore:
      
      i. \( \Lambda_\phi^\pm \) fills relative to \( \mathcal{F} \), meaning that \( \mathcal{F}_{\text{supp}}(\Lambda_\phi^\pm; \mathcal{F}) = \{ [F_n] \} \).
      
      ii. If \( \Lambda_\phi^\pm \) is nongeometric then \( A_{na}(\Lambda_\phi^\pm) = \mathcal{F} \);
      
      iii. If \( \Lambda_\phi^\pm \) is geometric then there exists a primitive cyclic subgroup \( C \subset F_n \) such that \( A_{na}(\Lambda_\phi^\pm) = \mathcal{F} \cup \{ [C] \} \) and \( \mathcal{F}_{\text{supp}}(\mathcal{F}, [C]) = \{ [F_n] \} \); in particular \( [C] \not\in \mathcal{F} \).

Proof. Only item (3) needs some justification.

We prove (3a) by contradiction, so suppose that \( \mathcal{F} \) has co-edge number 1. Let \( f : G \to G \) be a relative train track representative of \( \phi \) in which \( \mathcal{F} \) is represented by a filtration element \( G_r \). There are two cases to consider, regarding how \( G \) is obtained topologically from \( G_r \): by attaching either an arc or a “sewing needle” to \( G_r \).

In the first case \( G \) is obtained by attaching an arc \( \alpha \) to \( G_r \), identifying each endpoint of \( \alpha \) with a vertex of \( G_r \). Let \( f' : G \to G \) be obtained from \( G \) by tightening the restriction \( f \mid \alpha \). Applying [BFH00, Corollary 3.2.2], the path \( f'(\alpha) \) crosses \( \alpha \) at most once, and so \( f' \) has no EG stratum above \( G_r \). Since \( f' \mid G_r = f \mid G_r \), the map \( f' \) is also a relative train track representative of \( \phi \), because the definition of a relative train track map imposes no condition on non-EG strata. Applying [BFH00, Lemma 3.1.10], the highest stratum of \( f' \) crossed by leaves of \( \Lambda_\phi^+ \) is EG and so all such leaves are contained in \( G_r \), implying that \( \Lambda_\phi^+ \) is supported by \( \mathcal{F} \), a contradiction.

In the second case \( G \) is obtained from \( G_r \) by attaching a “sewing needle”, consisting of a vertex \( v \) disjoint from \( G_r \), an arc \( \alpha \) with both endpoints identified to \( v \), and
another arc $\alpha'$ with one endpoint identified to $v$ and opposite endpoint identified to a vertex on $G_r$. Let $f_1: G_1 \to G_1$ be the topological representative obtained from $f: G \to G$ by first collapsing $\alpha'$ to a point to get $G_1$ and then tightening the induced map on $\alpha$. The analysis of the first case now applies to this map $f_1$, reaching the same contradiction.

We prove (3b). For item (i), the free factor system $F_{\text{supp}}(\Lambda_\phi^\pm, \mathcal{F})$ equals $\{[F_n]\}$ because it contains $\mathcal{F}$ and is $\phi$-invariant, and $\phi$ is fully irreducible rel $\mathcal{F}$. It also follows that $H_s$ is the top stratum. For uniqueness of $\Lambda_\phi^\pm$, if $\phi$ had a different lamination pair not supported by $\mathcal{F}$ then there would exist a $\phi$-invariant free factor system strictly between $\mathcal{F}$ and $\{[F_n]\}$ (by [HM17c, Fact 1.55]), contradicting that $\phi$ is fully irreducible rel $\mathcal{F}$. The further conclusions of (3b) then follow from [HM17e, Definition 1.2, “Remark: The case of a top stratum”].

The conclusions of the following “mutual weak attraction” fact match key hypotheses of lamination ping-pong results, for example [HM17f, Proposition 1.3], and Ghosh’s theorem [Gho16, Theorem 7.3] discussed below, enabling us to apply these results below in Propositions 4.7 and 4.9.

**Fact 4.3.** Given $\phi, \psi \in \text{Out}(F_n)$ and lamination pairs $\Lambda_\phi^\pm \in \mathcal{L}^\pm(\phi)$ and $\Lambda_\psi^\pm \in \mathcal{L}^\pm(\psi)$, suppose that the following hypotheses hold:

1. $\{\Lambda_\psi^-, \Lambda_\psi^+\} \cap \{\Lambda_\phi^-, \Lambda_\phi^+\} = \emptyset$;
2. No generic leaves of $\Lambda_\psi^-$ or $\Lambda_\phi^+$ are carried by $A_{na}(\Lambda_\phi^\pm)$, and similarly with $\phi, \psi$ switched;

It follows that generic leaves of $\Lambda_\phi^+$ and of $\Lambda_\psi^-$ are weakly attracted to $\Lambda_\phi^+$ by iteration of $\psi$ and to $\Lambda_\psi^-$ by iteration of $\psi^{-1}$, and similarly with $\phi, \psi$ switched.

**Proof.** By symmetry we need only check that a generic leaf $\ell_\phi^+$ of $\Lambda_\phi^+$ is weakly attracted to $\Lambda_\phi^+$ by iteration of $\psi$, which we do by applying [HM17e, Theorem H].

Consider a generic leaf $\ell_\psi^-$ of $\Lambda_\psi^-$. If $\ell_\psi^-$ is a leaf of $\Lambda_\phi^+$ then by combining [BFH00, Lemma 3.1.15] with birecurrence of $\ell_\psi^-$ it follows that one of two cases holds. In the first case, $\ell_\psi^-$ is a generic leaf of $\Lambda_\phi^+$ and so $\Lambda_\phi^- = \Lambda_\phi^+$ which contradicts (1). In the second case, choosing any CT representative $f: G \to G$ of a rotationless power of $\phi$ with core filtration element $G_r$ and EG stratum $H_r$ corresponding to $\Lambda_\phi^\pm$, the realization in $G$ of $\ell_\psi^-$ is contained in $G_{r-1}$ and so $\ell_\psi^-$ is carried by $A_{na}(\Lambda_\phi^\pm)$, which contradicts (2).

We conclude that $\ell_\psi^-$ is not a leaf of $\Lambda_\phi^+$ and hence there exists a weak neighborhood $V_\psi^-$ of $\ell_\psi^-$ such that $\ell_\phi^+ \notin V_\psi^-$. Also, from (2) we conclude that $\ell_\phi^+$ is not carried by $A_{na}(\Lambda_\phi^\pm)$. Applying [HM17e, Theorem H (2)] it follows that for every weak neighborhood $V_\psi^+$ of $\ell_\psi^+$ there exists $m$ such that $\psi^m(\ell_\phi^+) \in V_\psi^+$, that is, $\ell_\phi^+$ is weakly attracted to $\ell_\psi^+$ by iteration of $\psi$. 

\qed
Lamination ping-pong. The next result is a pared down version of [HM17f, Proposition 1.7], omitting several conclusions of that proposition labelled (2±), (3±), (4±) that we do not need here.

Proposition 4.4 (Proposition 1.7 of [HM17f]). Given a free factor system $F$, rotationless $\phi, \psi \in \text{Out}(F_n)$ both leaving $F$ invariant, and lamination pairs $\Lambda_\phi^± \in \mathcal{L}^±(\phi)$, $\Lambda_\psi^± \in \mathcal{L}^±(\psi)$ each having a generic leaf fixed by $\phi^±, \psi^±$ (resp.) with fixed orientation, assume that the following hypotheses hold:

(a) $F \subseteq A_n \Lambda_\phi^±$ and $F \subseteq A_n \Lambda_\psi^±$.

(i) Generic leaves of $\Lambda_\phi^+$ are weakly attracted to $\Lambda_\phi^+$ under iteration by $\phi$.

(ii) Generic leaves of $\Lambda_\phi^-$ are weakly attracted to $\Lambda_\phi^-$ under iteration by $\phi^{-1}$.

(iii) Generic leaves of $\Lambda_\psi^+$ are weakly attracted to $\Lambda_\psi^+$ under iteration by $\psi$.

(iv) Generic leaves of $\Lambda_\psi^-$ are weakly attracted to $\Lambda_\psi^-$ under iteration by $\psi^{-1}$.

Under these hypotheses, there exists an integer $M$, such that for any $m, n \geq M$ the outer automorphism $\xi = \psi^m \phi^n$ has an invariant attracting lamination $\Lambda_\xi^+ \in \mathcal{L}(\xi)$ and an invariant repelling lamination $\Lambda_\xi^- \in \mathcal{L}(\xi^{-1})$ such that each is nongeometric if $\Lambda_\phi^+$ and $\Lambda_\psi^+$ are nongeometric, and the following hold:

(1) $F$ is carried by both $A_n \Lambda_\xi^+$ and $A_n \Lambda_\xi^-$, and so neither $\Lambda_\xi^+$ nor $\Lambda_\xi^-$ is carried by $F$. Also, both $A_n \Lambda_\xi^+$ and $A_n \Lambda_\xi^-$ are carried by $A_n \Lambda_\phi^±$ and by $A_n \Lambda_\psi^±$.

(5+) For any weak neighborhood $U^+ \subseteq B$ of a generic leaf of $\Lambda_\xi^+$ there exists an integer $M^+$ such that if $m, n \geq M^+$ then a generic leaf of $\Lambda_\xi^+$ is in $U^+$.

(5-) For any weak neighborhood $U^- \subseteq B$ of a generic leaf of $\Lambda_\xi^-$ there exists an integer $M^-$ such that if $m, n \geq M^-$ then a generic leaf of $\Lambda_\xi^-$ is in $U^-$.

(6) $\Lambda_\xi^±$ is a dual lamination pair of $\xi$ under either of the following conditions:

— Both pairs $\Lambda_\phi^±$ and $\Lambda_\psi^±$ are nongeometric; or

— Both laminations $\Lambda_\phi^+$ and $\Lambda_\psi^-$ are geometric. □

Subgroup decomposition theory. We review [HM17f, Theorem I] which is the main result of subgroup decomposition theory. This theorem and its proof will be applied in Proposition 4.6 to follow.

Consider $\phi, \psi \in \text{Out}(F_n)$ and lamination pairs $\Lambda_\phi^± \in \mathcal{L}^±(\phi)$, $\Lambda_\psi^± \in \mathcal{L}^±(\psi)$. The Independence Theorem [HM14b, Theorem 1.2] says that the sets $\{\Lambda_\phi^+, \Lambda_\phi^−\}$ and $\{\Lambda_\psi^+, \Lambda_\psi^−\}$ are either equal or disjoint. Furthermore $\text{Stab}(\Lambda_\phi^±) = \text{Stab}(\Lambda_\psi^±) = \text{Stab}(\Lambda_\phi^−)$, in other words if $\theta \in \text{Out}(F_n)$ stabilizes either of $\Lambda_\phi^+$ or $\Lambda_\phi^-$ then $\theta$ stabilizes both $\Lambda_\phi^+$ and $\Lambda_\phi^-$ [HM14b, Corollary 1.3].

Given $\phi \in \text{Out}(F_n)$ and a $\phi$-invariant free factor system $F$ define

$$\mathcal{L}(\phi; F) = \{\Lambda \in \mathcal{L}(\phi) \mid \Lambda \text{ is not carried by } F\}$$
and let \( \mathcal{L}^\pm(\phi; \mathcal{F}) \subset \mathcal{L}^\pm(\phi) \) be similarly defined. Given \( \Gamma < \text{Out}(F_n) \) and an \( \Gamma \)-invariant free factor system \( \mathcal{F} \), let \( \mathcal{L}(\Gamma; \mathcal{F}) = \cup_{\phi \in \Gamma} \mathcal{L}(\phi; \mathcal{F}) \). Since free factor support is well-defined for a dual lamination pair, the set \( \mathcal{L}(\Gamma; \mathcal{F}) \) decomposes as a disjoint union of dual lamination pairs.

Following [HM17f, Definition 1.2], we say that \( \Gamma \) is geometric above \( \mathcal{F} \) if every element of \( \mathcal{L}(\Gamma; \mathcal{F}) \) is geometric.

**Theorem 4.5** ([HM17f, Theorem 1]). Let \( \mathcal{H} < IA_n(\mathbb{Z}/3) \) be a subgroup and \( \mathcal{F} \subset \{[F_n]\} \) an \( \mathcal{H} \)-invariant multi-edge extension of free factor systems such that \( \mathcal{H} \) is irreducible relative to \( \mathcal{F} \). If either \( \mathcal{H} \) is finitely generated or \( \mathcal{L}(\mathcal{H}, \mathcal{F}) \neq \emptyset \) then there exists \( \phi \in \mathcal{H} \) which is fully irreducible relative to \( \mathcal{F} \). Moreover, for any \( \theta \in \mathcal{H} \) and \( \Lambda^-_\theta \in \mathcal{L}(\theta^{-1}, \mathcal{F}) \), if either \( \mathcal{H} \) is geometric above \( \mathcal{F} \) or \( \Lambda^-_\theta \) is non-geometric then for any weak neighborhood \( U \subset B \) of a generic leaf of \( \Lambda^-_\theta \) we may choose \( \phi \) so that generic leaves of \( \Lambda^-_\phi \), the unique element of \( \mathcal{L}(\phi^{-1}, \mathcal{F}) \), are contained in \( U \).

**Laminations with infinite orbit.** The following result will be used in Section 4.2. It is an application of [HM17f, Theorem 1] and of the lamination ping-pong tools underlying its proof.

**Proposition 4.6.** Consider a subgroup \( \Gamma < IA_n(\mathbb{Z}/3) \) and an \( \Gamma \)-invariant free factor system \( \mathcal{F} \) such that \( \Gamma \) is irreducible relative to \( \mathcal{F} \). If \( \mathcal{L}(\Gamma; \mathcal{F}) \) contains more than one dual lamination pair then for each \( \Lambda \in \mathcal{L}(\Gamma; \mathcal{F}) \), the stabilizer of \( \Lambda \) in \( \Gamma \) has infinite index, equivalently the \( \Gamma \)-orbit of \( \Lambda \) is infinite.

**Proof.** We begin with:

**Claim 1:** For any \( \alpha, \beta \in \Gamma \) and lamination pairs \( \Lambda^\pm_{\alpha} \in \mathcal{L}^\pm(\alpha; \mathcal{F}) \), \( \Lambda^\pm_{\beta} \in \mathcal{L}^\pm(\beta; \mathcal{F}) \), if \( \alpha \) is fully irreducible rel \( \mathcal{F} \), and if \( \{\Lambda^-_{\alpha}, \Lambda^+_{\alpha}\} \neq \{\Lambda^-_{\beta}, \Lambda^+_{\beta}\} \), then neither of \( \Lambda^\pm_{\beta} \) is a sublamination of either of \( \Lambda^\pm_{\alpha} \), equivalently if \( \gamma \) is a generic leaf of one of \( \Lambda^-_{\beta} \) or \( \Lambda^+_{\beta} \) then \( \gamma \) is a leaf of neither \( \Lambda^-_{\alpha} \) nor \( \Lambda^+_{\alpha} \).

To prove this, note first that the sets \( \{\Lambda^-_{\alpha}, \Lambda^+_{\alpha}\} \) and \( \{\Lambda^-_{\beta}, \Lambda^+_{\beta}\} \) are disjoint (by the Independence Theorem [HM14b, Theorem 1.2]), so \( \gamma \) is a generic leaf of neither \( \Lambda^+_{\beta} \) nor \( \Lambda^-_{\beta} \). Also, by applying [BFH00, Lemma 3.1.15] to a relative train track representative of \( \alpha \) in which \( \mathcal{F} \) is realized by a filtration element, it follows that every birecurrent, nongeneric leaf of \( \Lambda^\pm_{\alpha} \) is carried by \( \mathcal{F} \) and so no such leaf equals \( \gamma \), completing the proof of Claim 1.

Since \( \mathcal{L}(\Gamma; \mathcal{F}) \neq \emptyset \) it follows by Fact 4.2 (3) that \( \mathcal{F} \subset \{[F_n]\} \) is a multi-edge extension. Applying Theorem 4.5, choose \( \eta \in \Gamma \) which is fully irreducible rel \( \mathcal{F} \). The set \( \mathcal{L}^\pm(\eta; \mathcal{F}) \) consists of a unique lamination pair \( \Lambda^\pm_{\eta} \).

Let \( \Theta_\mathcal{L} \) be the set of pairs \( (\theta, \Lambda^\pm_{\theta}) \) where \( \theta \in \Gamma \), \( \Lambda^\pm_{\theta} \in \mathcal{L}(\theta; \mathcal{F}) \) and the pairs \( \{\Lambda^-_{\theta}, \Lambda^+_{\theta}\} \) and \( \{\Lambda^-_{\eta}, \Lambda^+_{\eta}\} \) are unequal and hence disjoint. By hypothesis, \( \Theta_\mathcal{L} \neq \emptyset \). For each \( (\theta, \Lambda^\pm_{\theta}) \in \Theta_\mathcal{L} \), it follows by Claim 1 that generic leaves \( \ell^\pm_{\theta} \) of \( \Lambda^\pm_{\theta} \) are neither
leaves of $\Lambda_\eta^-$ nor leaves of $\Lambda_\eta^+$. Since $\Lambda_\eta^-$ and $\Lambda_\eta^+$ are weakly closed, there are weak neighborhoods $U_\eta^- \subset \mathcal{B}$ of $\ell_\eta^-$ disjoint from $\Lambda_\eta^-$ and $\Lambda_\eta^+$.

If $\Lambda \in \mathcal{L}(\Gamma; \mathcal{F})$ is neither $\Lambda_\eta^-$ nor $\Lambda_\eta^+$, choose $(\theta, \Lambda_\mu^\pm) \in \Theta_\mathcal{L}$ such that $\Lambda = \Lambda_\theta^\pm$. From Claim 1, $\Lambda$ is not a sublamination of either of $\Lambda_\eta^\pm$. Applying [BFH00, Proposition 6.0.8], $\Lambda$ is weakly attracted to $\Lambda_\eta^+$ under iteration of $\xi$. The $\eta$-orbit of $\Lambda$ is therefore infinite, and so its $\Gamma$-orbit is infinite. This completes the proof if $\Lambda \not\in \{\Lambda_\eta^-, \Lambda_\eta^+\}$.

The same argument applies with $\eta$ replaced by any other element of $\Gamma$ that is irreducible rel $\mathcal{F}$. We are therefore reduced to

**Claim 2:** There exists $(\mu, \Lambda_\mu^\pm) \in \Theta_\mathcal{L}$ such that $\mu$ is irreducible rel $\mathcal{F}$.

As a first case, suppose that there exists $(\theta, \Lambda_\theta^\pm) \in \Theta_\mathcal{L}$ with non-geometric $\Lambda_\theta^\pm$. By Theorem 4.5 there exists $\mu \in \Gamma$ which is fully irreducible rel $\mathcal{F}$ and such that $\Lambda_\mu^-$ is contained in $U_\theta^-$; in particular, $\Lambda_\mu^- \not\in \{\Lambda_\eta^-, \Lambda_\eta^+\}$ and we have verified claim 2. We are now reduced to the case that each $\Lambda_\mu^\pm$ is geometric. If $\Lambda_\mu^\pm$ is geometric then the same argument applies without change.

We may therefore assume that $\Lambda_\mu^\pm$ is non-geometric and that each $\Lambda_\theta^\pm$ is geometric.

Choose $(\mu, \Lambda_\mu^\pm) \in \Theta_\mathcal{L}$ so that $\mathcal{A}_{na}\Lambda_\mu^\pm$ is minimal with respect to $\subset$. In other words, if $(\theta, \Lambda_\theta^\pm) \in \Theta_\mathcal{L}$ and $\mathcal{A}_{na}\Lambda_\theta^\pm \sqsubset \mathcal{A}_{na}\Lambda_\mu^\pm$ then $\mathcal{A}_{na}\Lambda_\theta^\pm = \mathcal{A}_{na}\Lambda_\mu^\pm$. Such a $(\mu, \Lambda_\mu^\pm)$ exists because there is a bound to the length of a chain of proper inclusions of non-attracting subgroup systems [HM17c, Proposition 3.2]. We will complete the proof by showing that $\mu$ is irreducible rel $\mathcal{F}$. Since $\Lambda_\mu^\pm$ is geometric, it suffices by [HM17f, Lemma 2.3 (3)] to show that $\text{Stab}_\Gamma(\mathcal{A}_{na}\Lambda_\mu^\pm)$ has finite index in $\Gamma$.

Assuming that $\text{Stab}_\Gamma(\mathcal{A}_{na}\Lambda_\mu^\pm)$ has infinite index we will arrive at a contradiction by applying Proposition 4.4. Passing to a power of $\mu$ we may assume that $\Lambda_\mu^\pm$ have generic leaves $\ell_\mu^\pm$ that are fixed by $\mu$ with fixed orientations, which is one of the hypotheses of Proposition 4.4. Applying [HM17f, Lemma 2.1] there exists $\zeta \in \Gamma$ such that the following hold. None of the lines $\zeta(\ell_\mu^+), \zeta(\ell_\mu^-), \zeta^{-1}(\ell_\mu^+), \zeta^{-1}(\ell_\mu^-)$ is carried by $\mathcal{A}_{na}\Lambda_\mu^\pm$ ([HM17f, Lemma 2.1 (1)]), and hence by [HM17e, Corollary 2.17 (Theorem H)] each of these lines is weakly attracted to $\Lambda_\mu^\pm$ under iteration by $\mu$. Letting $\beta = \zeta(\mu^{-1})^{-1}$ with geometric lamination pair $\Lambda_\beta^\pm = \zeta(\Lambda_\mu^\pm) \in \mathcal{L}^\pm(\beta; \mathcal{F})$, it follows that $\mu$ and $\beta$ with their lamination pairs $\Lambda_\mu^\pm$ and $\Lambda_\beta^\pm$ satisfy hypotheses (i)–(iv) of Proposition 4.4. Also, $\mathcal{A}_{na}\Lambda_\beta^\pm = \zeta(\mathcal{A}_{na}\Lambda_\mu^\pm) \neq \mathcal{A}_{na}\Lambda_\mu^\pm$ [HM17f, Lemma 2.1 (4)]. Hypothesis (a) of Proposition 4.4 is that $\mathcal{F} \sqsubset \mathcal{A}_{na}\Lambda_\mu^\pm$ and $\mathcal{F} \sqsubset \mathcal{A}_{na}\Lambda_\beta^\pm$, which holds because $\Lambda_\mu^\pm, \Lambda_\beta^\pm$ are not supported by $\mathcal{F}$. We may now apply Proposition 4.4 with $U^+ = U_\beta^+$ and $U^- = U_\mu^-$.

Applying the conclusions of Proposition 4.4, for some $m, n \geq 1$ we have: an outer automorphism $\nu = \beta^m \mu^n$; a $\nu$-invariant attracting lamination $\Lambda_\nu^+ \in \mathcal{L}(\nu)$; and a $\nu$-invariant repelling lamination $\Lambda_\nu^- \in \mathcal{L}(\nu)$. By Conclusion (1), $\Lambda^-_\nu, \Lambda^+_\nu \in \mathcal{L}(\Gamma; \mathcal{F})$. By Conclusion (5), generic leaves of $\Lambda_\nu^-$ are contained in $U^-$, and generic leaves of $\Lambda_\nu^+$ are contained in $U^+$. Since $U^-$ and $U^+$ are disjoint from $\Lambda_\nu^\pm$, neither $\Lambda_\nu^-$ nor $\Lambda_\nu^+$ is in $\{\Lambda_\eta^-, \Lambda_\eta^+\}$ and so $\Lambda^-_\nu$ and $\Lambda^+_\nu$ are geometric. By Conclusion (6), $\Lambda_\nu^\pm$ is
a dual lamination pair for \( \nu \), and so \( A_{\mu}^\pm \Lambda_{\mu}^\pm \) is a well-defined vertex group system. By Conclusion (1), \( A_{\mu}^\pm \Lambda_{\mu}^\pm \subseteq A_{\nu}^\pm \Lambda_{\nu}^\pm \) and \( A_{\mu}^\pm \Lambda_{\mu}^\pm \subseteq A_{\beta}^\pm \Lambda_{\beta}^\pm \) and \( (\nu, \Lambda_{\mu}^\pm) \in \Theta_{\mathcal{L}} \). The containment \( A_{\mu}^\pm \Lambda_{\mu}^\pm \subseteq A_{\nu}^\pm \Lambda_{\nu}^\pm \) is therefore proper, in contradiction to our choice of \( \mu \). 

**Free subgroups.** We will need the following Proposition 4.7 which, inside certain subgroups of \( \text{Out}(F_n) \), produces useful free subgroups. In the nongeometric case this is a consequence of a theorem of Ghosh [Gho16]. The geometric case combines tools from our subgroup decomposition theory in [HM17f] with a result of Farb and Mosher [FM02] that produces useful free subgroups of mapping class groups.

**Proposition 4.7.** Given \( \phi, \psi \in \text{Out}(F_n) \) and a proper free factor system \( \mathcal{F} \) that is preserved by \( \phi \) and \( \psi \), and given lamination pairs \( \Lambda_{\phi}^\pm \in \mathcal{L}^\pm(\phi) \) and \( \Lambda_{\psi}^\pm \in \mathcal{L}^\pm(\phi) \) that fill \( F_n \), suppose that the following hold:

1. \( \phi, \psi \) are both fully irreducible rel \( \mathcal{F} \).
2. \( \{\Lambda_{\phi}^-, \Lambda_{\phi}^+\} \cap \{\Lambda_{\psi}^-, \Lambda_{\psi}^+\} = \emptyset \);
3. Either both of the lamination pairs \( \Lambda_{\phi}^\pm, \Lambda_{\psi}^\pm \) are nongeometric, or the group \( \langle \phi, \psi \rangle \) is geometric above \( \mathcal{F} \).

Then there exists \( M \geq 1 \) such that for any integers \( m, n \geq M \) the outer automorphisms \( \phi^m \) and \( \psi^n \) freely generate a rank two free subgroup \( \langle \phi^m, \phi^n \rangle \subseteq \text{Out}(F_n) \) such that any nontrivial \( \xi \in \langle \phi^m, \psi^n \rangle \) is fully irreducible rel \( \mathcal{F} \) and has a lamination pair \( \Lambda_{\xi}^\pm \) that fills rel \( \mathcal{F} \), and if both of \( \Lambda_{\phi}^\pm, \Lambda_{\psi}^\pm \) are nongeometric then \( \Lambda_{\xi}^\pm \) is nongeometric.

**Remark.** In a more restricted context contained in the proof of Proposition 4.9, we shall prove a stronger conclusion saying that, after further increasing \( M \), each pair \( \Lambda_{\xi}^\pm \) fills \( F_n \) in the absolute sense.

**Proof.** The result breaks naturally into two cases.

**Case 1: \( \Lambda_{\phi}^\pm, \Lambda_{\psi}^\pm \) are both nongeometric.** The conclusion in this case exactly matches the conclusion of [Gho16, Theorem 7.3]. Our work here is therefore just to verify the hypothesis of that theorem, which is that \( (\phi, \Lambda_{\phi}^\pm) \) and \( (\psi, \Lambda_{\psi}^\pm) \) are independent rel \( \mathcal{F} \) [Gho16, Definition 7.2]. We verify each of the six clauses of independence rel \( \mathcal{F} \).

**Independence rel \( \mathcal{F} \), clause (1):** Neither of \( \Lambda_{\phi}^\pm, \Lambda_{\psi}^\pm \) is carried by \( \mathcal{F} \).

**Independence rel \( \mathcal{F} \), clause (2):** \( \{\Lambda_{\phi}^\pm\} \cup \{\Lambda_{\psi}^\pm\} \) fill rel \( \mathcal{F} \).

These hold because each pair \( \Lambda_{\phi}^\pm \) and \( \Lambda_{\psi}^\pm \) individually fills.
Independence rel $\mathcal{F}$, clauses (3,4): Generic leaves of $\Lambda_\phi^\pm$ are weakly attracted to $\Lambda^-_\psi$ by iteration of $\psi^{-1}$ and to $\Lambda^+_\psi$ by iteration of $\psi$, and similarly with $\phi, \psi$ switched.

This follows from Fact 4.3 after checking its hypotheses. Hypothesis (1) of Fact 4.3 is identical to Proposition 4.7 (2). Hypothesis (2) of Fact 4.3 follows from Fact 4.2 (3) using that a filling generic leaf $\ell$ of an attracting lamination cannot be carried by a component of a proper free factor system, nor by the conjugacy class of a cyclic subgroup because $\ell$ is nonperiodic.

Independence rel $\mathcal{F}$, clause (5): The free factor systems $A_{na} \Lambda^-_\phi = A_{na} \Lambda^-_\psi$ are mutually malnormal rel $\mathcal{F}$.

The meaning of mutual malnormality of two free factor systems $A_1$ and $A_2$ rel $\mathcal{F}$ is that for each subgroup $C < F_n$, if $[C] \subseteq A_1$ and $[C] \subseteq A_2$ then $[C] \subseteq \mathcal{F}$. In the current situation is is obvious from the fact that $A_{na} \Lambda^\pm_\phi = A_{na} \Lambda^\pm_\psi = \mathcal{F}$ (Fact 4.2 (3)).

Independence rel $\mathcal{F}$, clause (6): Both lamination pairs $\Lambda^\pm_\phi, \Lambda^\pm_\psi$ are nongeometric.

This holds by assumption of Case 1, and so Case 1 is complete.

**Case 2: $\Lambda^\pm_\phi$ and $\Lambda^\pm_\psi$ are not both nongeometric.** Denote $\Gamma = \langle \phi, \psi \rangle < \text{Out}(F_n)$. Combining hypothesis (3) of the proposition with the hypothesis of Case 2 it follows that $\Gamma$ is geometric above $\mathcal{F}$. Also, the group $\Gamma$ is fully irreducible rel $\mathcal{F}$, because it contains an element which is fully irreducible rel $\mathcal{F}$, namely $\phi$. The hypotheses of [HM17f, Theorem J] for the subgroup $\Gamma$ are therefore satisfied.

From the conclusions of [HM17f, Theorem J] we obtain a compact surface $S$ with nonempty boundary and an injection $\pi_1(S) \hookrightarrow F_n$ whose image is its own normalizer, such that $\Gamma$ stabilizes the subgroup system $[\pi_1(S)]$. It follows from [HM17c, Fact 1.4] that there is a well-defined homomorphism $\Gamma \to \text{Out}(\pi_1(S))$ which assigns to each $\xi \in \Gamma$ the element of $\text{Out}(\pi_1(S))$ that is represented by the element of $\text{Aut}(\pi_1(S))$ obtained by choosing an automorphism $\Xi \in \text{Aut}(F_n)$ that represents $\xi$ and preserves $\pi_1(S)$, and restricting $\Xi$ to $\pi_1(S)$. Also, the image of this homomorphism is contained in the natural $\text{MCG}(S)$ subgroup of $\text{Out}(\pi_1(S))$ thereby giving a homomorphism $dj^\#: \Gamma \to \text{MCG}(S)$. Also, $dj^\#(\xi)$ is pseudo-Anosov if and only if $\xi$ is fully irreducible rel $\mathcal{F}$. Also, the induced map $dj_B: \mathcal{B}(\pi_1(S)) \to \mathcal{B}(F_n) = \mathcal{B}$ induces a bijection between the following two sets: the set of all geodesic laminations on $S$ which are unstable laminations of pseudo-Anosov elements of $dj^\#(\Gamma)$ (here we pick a hyperbolic structure on $S$ with totally geodesic boundary); and the set of all attracting laminations not supported by $\mathcal{F}$ of elements of $\Gamma$ that are fully irreducible rel $\mathcal{F}$.

Consider $dj^\#(\phi), dj^\#(\psi) \in dj^\#(\Gamma) < \text{MCG}(S)$. Their lamination pairs $\Lambda^\pm_\phi, \Lambda^\pm_\psi$ form four distinct closed subsets of $\mathcal{B}$, and therefore the unstable/stable geodesic laminations pairs $\Lambda^u_\phi, \Lambda^s_\phi, \Lambda^u_\psi, \Lambda^s_\psi$ form four distinct laminations on $S$. The hypotheses of [FM02, Theorem 1.4] are therefore satisfied, the conclusions of which give
the existence of $M \geq 1$ such that for all $m, n \geq M$ the mapping classes $dj^#(\phi^m)$, $dj^#(\psi^n)$ freely generate a rank 2 subgroup of $dj^#(\Gamma)$ such that any nontrivial element is pseudo-Anosov. It follows that $\phi^m, \psi^n$ freely generate a rank 2 subgroup of $\Gamma$ such that any nontrivial element $\xi$ is fully irreducible rel $F$, and has a unique attracting/repelling lamination pair $\Lambda^{\pm}_\xi$ not supported by $F$ which maps via $dj_M$ to the unique unstable/stable lamination pair $\Lambda^{un}_\xi, \Lambda^{st}_\xi$ of $dj^#(\xi)$ in $S$. Since $\xi$ is fully irreducible rel $F$, this pair $\Lambda^{\pm}_\xi$ fills rel $F$. Case 2 is now complete. \hfill \Box

4.2 Proof that Theorem E implies Theorem B.

Throughout this section we fix the following notation, taken from the hypotheses of Theorem B:

(1)$_\Gamma\quad \Gamma < IA_n(\mathbb{Z}/3)$ is an infinite lamination group with (virtually) abelian restrictions.

(2)$_\Gamma\quad \mathcal{A}$ is a maximal, $\Gamma$-invariant, proper free factor system of $F_n$.

Without further reference we make use of the fact [HM17d, Theorem 3.1] that if $\eta \in IA_n(\mathbb{Z}/3)$ then every free factor conjugacy class that is $\eta$-periodic is fixed by $\eta$. Since $\Gamma$ actually has abelian restrictions (Corollary 4.1), we have:

(3)$_\Gamma\quad$ For each component $[A]$ of $\mathcal{A}$,

- $[A]$ is fixed by $\Gamma$.
- The group $\Gamma_A = \text{Image}(\Gamma \mapsto \text{Out}(A))$ is abelian.

We will need the following minor extension of [FH09, Lemma 4.4].

**Lemma 4.8.** Each virtually abelian subgroup $A$ of $\text{Out}(F_n)$ is a finite lamination subgroup. Furthermore if $A < IA_n(\mathbb{Z}/3)$ then each element of $\mathcal{L}(A)$ is $A$-invariant.

**Proof.** By [FH09, Corollary 3.14] there is a finite index abelian subgroup $A' < A$ that is generated by rotationless elements. Applying [FH09, Lemma 4.4] it follows that $\mathcal{L}(A')$ is a finite collection of $A'$-invariant laminations. Since each $\phi \in A'$ has a power $\phi^k \in A, k \neq 0$, and since $\mathcal{L}(\phi) = \mathcal{L}(\phi^k)$, it follows that $\mathcal{L}(A') = \mathcal{L}(A)$ and so $\mathcal{L}(A)$ is finite.

Suppose now that $A < IA_n(\mathbb{Z}/3)$, that $\Lambda \in \mathcal{L}(A)$, and that $\psi \in A$; we prove that $\psi(\Lambda) = \Lambda$. Choose $\phi \in A$ so that $\Lambda \in \mathcal{L}(\phi) = \{\Lambda_1, \ldots, \Lambda_m\}$. Since $A$ is virtually abelian there exists $k \geq 1$ such that $\psi^k$ commutes with $\phi^k$. It follows that $\psi^k$ permutes $\mathcal{L}(\phi)$. The free factor supports of the $\Lambda_i$’s are distinct [BFH00, Lemma 3.2.4] and are permuted by $\psi^k$. Since $\psi \in IA_n(\mathbb{Z}/3)$, it preserves the free factor support of each $\Lambda_i$ and so $\psi$ preserves each $\Lambda_i$. \hfill \Box
Returning now to the context of the subgroup $\Gamma$ in property (1) above, by Lemma 4.8 it follows that $\Gamma$ is not virtually abelian (and hence that hypothesis is not needed in the statement of Theorem B, as it was in Theorem C).

Applying Lemma 4.8 together with (3) we have:

(4) For each proper free factor $A < F_n$ such that $\Gamma < \text{Stab}[A]$, the image of the homomorphism $\Gamma \hookrightarrow \text{Stab}[A] \twoheadrightarrow \text{Out}(A)$ is a finite lamination group. In particular, for each component $[A]$ of $\mathcal{A}$ the subgroup $\Gamma_A < \text{Out}(A)$ is a finite lamination group.

Given $\xi \in \Gamma$ consider the set $L(\xi; \mathcal{A})$ consisting of those laminations in $L(\xi)$ that are not carried by $\mathcal{A}$. Let $L(\Gamma; \mathcal{A}) = \cup_{\xi \in \Gamma} L(\xi; \mathcal{A})$, which is an infinite set, because $L(\Gamma)$ is infinite by hypothesis, but by (4) only finitely many elements of $L(\Gamma)$ are carried by $\mathcal{A}$. We have:

(5) $\mathcal{A} \sqsubseteq \{F_n\}$ is a multi-edge extension (by Fact 4.2 (3a)), and every element of $L(\Gamma; \mathcal{A})$ has infinite orbit under $\Gamma$ (by Proposition 4.6).

For each $\Lambda \in L(\Gamma; \mathcal{A})$ consider the free factor system $\mathcal{F}_{\text{supp}}(\Gamma \cdot \Lambda)$, that is, the smallest free factor system carrying every lamination in the $\Gamma$-orbit of $\Lambda$. Clearly $\mathcal{F}_{\text{supp}}(\Gamma \cdot \Lambda)$ is $\Gamma$-invariant. By (1) we have $\Gamma < I\text{A}_n(\mathbb{Z}/3)$, and so each component of $\mathcal{F}_{\text{supp}}(\Gamma \cdot \Lambda)$ is $\Gamma$ invariant. By item (5), one of the finitely many components of $\mathcal{F}_{\text{supp}}(\Gamma \cdot \Lambda)$ supports infinitely many elements of $L(\Gamma; \mathcal{A})$ and so the restriction of $\Gamma$ to that component is not a finite lamination group; by (4) that component must be $\{[F_n]\}$. This shows:

(6) The $\Gamma$-orbit of each element of $L(\Gamma; \mathcal{A})$ fills $F_n$.

Using this we next verify most of conclusion (2) of Theorem B:

(7) There exists $\eta \in [\Gamma, \Gamma]$ such that $\eta$ is fully irreducible rel $\mathcal{A}$ and such that $\eta$ acts loxodromically on $\mathcal{F}_S(F_n)$.

We prove (7) in two cases depending on whether $\Gamma$ is geometric above $\mathcal{A}$ (see just before Theorem 4.5 to recall the definition).

**Case 1: $\Gamma$ is not geometric above $\mathcal{A}$.** Applying [HM17f, Proposition 2.2 (1)], we obtain $\eta \in \Gamma$ and a non-geometric lamination pair $\Lambda^\pm_\eta \in L(\eta; \mathcal{A})$ such that $A_{\text{na}}(\Lambda^\pm_\eta) = \mathcal{A}$. We may assume in addition that $\eta$ is chosen so that $\mathcal{F}_{\text{supp}}(\Lambda^\pm_\eta)$ is maximal with respect to $\sqsubseteq$.

We claim that the free factor system $\mathcal{F}_{\text{supp}}(\Lambda^\pm_\eta)$ is $\Gamma$-invariant. If not, then there exists $\zeta \in \Gamma$ such that $\mathcal{F}_{\text{supp}}(\Lambda^\pm_\eta) \neq \zeta(\mathcal{F}_{\text{supp}}(\Lambda^\pm_\eta))$. Since $\mathcal{F}_{\text{supp}}(\Lambda^\pm_\eta) = \mathcal{F}_{\text{supp}}(\Lambda^\pm_\eta)$, and similarly with $\zeta$ applied, it follows that $\{\Lambda^+, \Lambda^-\} \cap \{\zeta(\Lambda^+_\eta), \zeta(\Lambda^-_\eta)\} = \emptyset$. Applying [HM17f, Inductive Step of Proposition 2.4 (2)] it follows that if the integer $m > 0$ is sufficiently large then $\eta' = \zeta \eta m \zeta^{-1} \eta^{-m}$ has a nongeometric lamination pair
$\Lambda_\eta^\pm$ such that $A_{na}(\Lambda_\eta^\pm) = A$ and such that $F_{supp}(\Lambda_\eta^\pm)$ is strictly contained in $F_{supp}(\Lambda_\eta'^\pm)$, contradicting maximality and therefore proving the claim.

Applying $\Gamma$-invariance of $F_{supp}(\Lambda_\eta^\pm)$, for each $\theta \in \Gamma$ we have $F_{supp}(\theta \cdot \Lambda_\eta^\pm) = \theta(F_{supp}(\Lambda_\eta^\pm)) = F_{supp}(\Lambda_\eta^\pm)$ and so $F_{supp}(\Lambda_\eta^\pm) = F_{supp}(\Gamma \cdot \Lambda_\eta^\pm) = \{[F_n]\}$ where the latter equation follows from (6)$_\Gamma$. Thus $\Lambda_\eta^\pm$ fills $F_n$ and so $\eta$ acts loxodromically on $FS(F_n)$, by [HM14b].

If $\eta$ were not fully irreducible rel $A$, that is if there existed an $\eta$-invariant free factor system $A'$ contained strictly between $A$ and $\{[F_n]\} = F_{supp}(\Lambda_\eta^\pm)$, then any conjugacy class carried by $A'$ but not by $A$ would not be weakly attracted to $\Lambda_\eta^\pm$, contradicting Fact 4.2 (2).

Lastly, it remains to arrange that $\eta \in [\Gamma, \Gamma]$ (if it is not already true). Applying (5)$_\Gamma$ we may choose $\zeta \in \Gamma$ such that $\{\zeta(\Lambda_\eta^\pm), \zeta(\Lambda_\eta^-)\} \cap \{\Lambda_\eta^+, \Lambda_\eta^-\} = \emptyset$. Applying [HM17f, Inductive Step of Proposition 2.4] again, if the integer $m > 0$ is sufficiently large then $\zeta^m \cdot \zeta^{-1} \eta^{-m} \in [\Gamma, \Gamma]$ satisfies all the portions of (7)$_\Gamma$ already established for $\eta$.

**Case 2: $\Gamma$ is geometric above $A$.** The proof is similar to Case 1 but cites different results from [HM17f]. Applying [HM17f, Proposition 2.2 (b)], there exists $\eta \in \Gamma$ which is fully irreducible rel $A$ and which has a geometric lamination pair $\Lambda_\eta^\pm \in \mathcal{L}(\eta; A)$ whose nonattracting subgroup system $A_{na}(\Lambda_\eta^\pm)$ is $\Gamma$-invariant, that is, $\text{Stab}(A_{na}(\Lambda_\eta^\pm)) = \Gamma$. Also, the nonattracting subgroup system has the form $A_{na}(\Lambda_\eta^\pm) = A \cup \{[C]\}$ for some rank 1 subgroup $C < F_n$ [HM17f, Proposition 2.2 (b)(iii)]. Note that $[C] \not\in A$, for otherwise it would follow that $A_{na}(\Lambda_\eta^\pm) = A$ is a free factor system, contradicting Fact 4.2 (1).

We will need that there is no vertex group system $A'$ strictly contained between $A$ and $A \cup \{[C]\}$, for suppose that $A \subset A' \subset A \cup \{[C]\}$. Using that vertex group systems are malnormal [HM17c, Lemma 3.1], it follows that $A \subset A'$. If $A' = A$ we are done. Otherwise, consider any component $[C'] \in A' - A$. From malnormality it follows that $[C'] \subset [C]$, and so up to conjugacy we may assume that $C < C'$ and hence $C, C'$ each have rank 1. By [HM17c, Proposition 3.2] it follows that $C' = C$ and hence $A' = A \cup \{[C]\}$.

By applying [HM17f, Proposition 2.3 (3)(b)] we conclude that $F_{supp}(\Lambda_\eta^\pm)$ is $\Gamma$-invariant. Using the same argument as in Case 1, it follows that $\Lambda_\eta^\pm$ fills $F_n$, and so $\eta$ acts loxodromically on $FS(F_n)$ [HM14b].

To arrange that $\eta$ is in $[\Gamma, \Gamma]$, as before choose $\zeta \in \Gamma$ such that $\{\zeta(\Lambda_\eta^\pm), \zeta(\Lambda_\eta^-)\} \cap \{\Lambda_\eta^+, \Lambda_\eta^-\} = \emptyset$. Applying [HM17f, Induction Step of Proposition 2.2], if the integer $m > 0$ is sufficiently large then $\eta' = \zeta^m \cdot \zeta^{-1} \eta^{-m} \in [\Gamma, \Gamma]$ has a geometric lamination pair $\Lambda_{\eta'}^\pm \in \mathcal{L}(\eta'; A)$ whose nonattracting subgroup system satisfies the containment relations

$$A \subset A_{na}(\Lambda_{\eta'}^\pm) \subset A_{na}(\Lambda_\eta^\pm) = A \cup \{[C]\}$$

As shown above, one of these containment relations is an equation, and it cannot be the first because $\Lambda_{\eta'}^\pm$ is geometric and so $A_{na}(\Lambda_\eta^\pm)$ is not a free factor system. Thus $A_{na}(\Lambda_{\eta'}^\pm) = A_{na}(\Lambda_\eta^\pm)$ and so $\text{Stab}(A_{na}(\Lambda_\eta^\pm)) = \Gamma$. Applying [HM17f, Lemma 2.3 (3)]
it follows that \( \eta' \) is fully irreducible rel \( A \) and that \( \mathcal{F}_{\text{supp}}(\Lambda_{\eta'}^+) \) is \( \Gamma \)-invariant, and so again by the same argument as in Case 1 it follows that \( \eta' \) acts loxodromically on \( \mathcal{F}S(F_n) \).

To complete the proof of Theorem B there are just two more properties to verify:

(8)\(_\Gamma\) Every \( \eta \) as in (7)\(_\Gamma\) — that is, every \( \eta \in [\Gamma, \Gamma] \) that is loxodromic and fully irreducible rel \( A \) — is a WWPD element for the action \( \Gamma \curvearrowright \mathcal{F}S(F_n) \).

(9)\(_\Gamma\) The action \( \Gamma \curvearrowright \mathcal{F}S(F_n) \) is nonelementary.

To prove item (8)\(_\Gamma\), note that for each component \([A]\) of \( A \) the restriction of \( \eta \) to \( \text{Out}(A) \) is in the commutator subgroup of the abelian group \( \Gamma_A \) and hence is trivial. Applying Theorem E it follows that \( \eta \) is WWPD.

To prove item (9)\(_\Gamma\), take \( \eta \) as in (8)\(_\Gamma\), let \( \Lambda\pm_{\eta} \) be its filling lamination pair, and apply (5)\(_\Gamma\) to conclude that the \( \Gamma \) orbit of the pair \( \Lambda_{\eta}^+ \) is infinite. In particular, for some \( \delta \) the pair \( \Lambda_{\eta}^+ \) is disjoint from the pair \( \delta(\Lambda_{\eta}^+ - 1) \) which also fills. Thus \( \eta^\delta = \delta \eta \delta^{-1} \) is also loxodromic. Furthermore \( \eta, \eta^\delta \) form an independent pair of loxodromics, because by [HM14b] the set of attracting (repelling) points of loxodromic elements of \( \Gamma \) in the Gromov boundary \( \partial \mathcal{F}S(F_n) \) corresponds bijectively and \( \Gamma \)-equivariantly to the set of repelling (attracting) laminations \( \Lambda \in \mathcal{L}(\Gamma) \) such that \( \Lambda \) fills \( F_n \).

### 4.3 Proof that Theorems B, C, D imply Theorem A

Given a finitely generated subgroup \( G < \text{Out}(F_n) \) which is not virtually abelian, to prove Theorem A it suffices to verify the global WWPD hypothesis for the group \( G \) (Definition 2.13), for then we can apply Theorem D from which we obtain an embedding \( \ell^1 \hookrightarrow H^2_b(G; \mathbb{R}) \). It remains to show how to apply Theorems B and C to verify the global WWPD hypothesis for \( G \).

**Setup:** Throughout this section we denote the following objects satisfying various properties:

- Denote \( G_0 = G \cap \text{IA}_n(\mathbb{Z}/3) \), a finite index normal subgroup of \( G \). It follows that \( G_0 \) is not abelian.

- Choose \( F_r < F_n \) to be a free factor of rank \( r \neq n \), with corresponding restriction homomorphism \( \pi: \text{Stab}[F_r] \to \text{Out}(F_r) \), such that the following two properties hold and the rank \( r \) is minimal with respect to these properties:
  - \( G_0 < \text{Stab}[F_r] \),
  - the group \( \Gamma = \pi(G_0) < \text{Out}(F_r) \) is not virtually abelian.

By restriction we have a surjective homomorphism \( \pi: G_0 \to \Gamma \). Since \( G_0 \) is contained in \( \text{IA}_n(\mathbb{Z}/3) \cap \text{Stab}[F_r] \), and since the subgroup \( \pi(\text{IA}_n(\mathbb{Z}/3) \cap \text{Stab}[F_r]) < \text{Out}(F_r) \) is contained in \( \text{IA}_r(\mathbb{Z}/3) \), and since virtually abelian subgroups of \( \text{IA}_r(\mathbb{Z}/3) \) are abelian, [HM17g] we have
• \( \Gamma < \text{IA}_r(\mathbb{Z}/3) \).

By choice of the rank \( r \) we have:

• \( \Gamma \) has virtually abelian restrictions.

To prove this, if \( B < F_r \) is a proper free factor such that \( \Gamma < \text{Stab}_{\text{Out}(F_r)}[B] \) then by malnormality of \( F_r \) it follows also that \( G_0 < \text{Stab}_{\text{Out}(F_r)}[B] \), and by minimality of \( r \) it follows that \( \text{Image}(\Gamma \mapsto \text{Out}(B)) = \text{Image}(G_0 \mapsto \text{Out}(B)) \) is virtually abelian. Combining the last two bullet points with Corollary 4.1 we have:

• \( \Gamma \) has abelian restrictions.

With this setup, henceforth we work primarily in \( \text{Out}(F_r) \), thinking of \( G \) and its subgroups somewhat abstractly rather than as subgroups of \( \text{Out}(F_n) \). In particular, our strategy for verifying the global WWPD hypothesis of \( G \) is to work with a hyperbolic action of (a certain subgroup of) \( G_0 \) that factors through a hyperbolic action of (a certain subgroup of) \( \Gamma < \text{Out}(F_r) \), the latter action being obtained by applying Theorem B or C. As a notational side effect, many standard notations should be interpreted in \( F_r \), for example the attracting lamination notation \( \mathcal{L}(\cdot) \).

The proof now breaks into two cases, depending on whether \( \Gamma \) is an infinite lamination subgroup of \( \text{Out}(F_r) \).

**Case 1:** \( \Gamma < \text{Out}(F_r) \) is a finite lamination subgroup. Applying Theorem C, we obtain the following objects: a nontrivial free factor \( A < F_r \) with \( \Gamma \)-invariant conjugacy class, a homomorphism \( \rho: \Gamma \mapsto \text{Aut}(A) \) with image \( \hat{\mathcal{H}} = \rho(\Gamma) < \text{Aut}(A) \), a finite index subgroup \( N' < \hat{\mathcal{H}} \), and an action \( N' \acts X \) on a hyperbolic space. Also from Theorem C we have the following:

(a)' Every element of \( N' \) acts elliptically or loxodromically on \( X \);

(b)' The action \( N' \acts X \) is nonelementary;

(c)' Every element of \( [N', N'] \) is either elliptic or WWPD with respect to the action \( N' \acts X \).

Let \( N < G_0 < G \) be a finite index normal subgroup of \( G \) such that \( \rho \circ \pi(N) < N' \); for instance, we can take \( N \) to be the intersection of all the \( G \)-conjugates of \( (\rho \circ \pi)^{-1}(N') \).

By composing the homomorphism \( N \xrightarrow{\rho \circ \pi} N' \) with the action \( N' \acts X \) we obtain an action \( N \acts X \), and clearly properties (a)'—(c)' for \( N' \) imply the same properties for \( N \):

(a) every element of \( N \) acts elliptically or loxodromically on \( X \);

(b) the action \( N \acts X \) is nonelementary;

(c) every element of \( [N, N] \) is either elliptic or WWPD with respect to the action \( N \acts X \).
We shall produce a certain rank 2 subgroup $E < [N,N]$ and for this subgroup we shall verify Definition 2.13 items (1), (2) and (3)', which proves the global WWPD hypothesis for $G$. Definition 2.13 (1) automatically holds because it is the same as property (a).

Let $i_1, \ldots, i_K \in \text{Aut}(N)$ be outer representatives of the inner action of $G$ on $N$, with $i_1 = \text{Id}_N$. Let $N \actson X$ denote the composed action $N \overset{i_\kappa}{\actson} X$; so $N \actson_1 X$ is another notation for the given action $N \actson X$. We also use $\actson_\kappa$ to denote restriction of the action $N \actson X$ to subgroups of $N$. Since $[N,N]$ is a characteristic subgroup of $N$, it follows that item (c) above holds not just for the given action $N \actson_1 X$ but for each of the composed actions $N \actson_\kappa X$. That is,

(d) For every $a \in [N,N]$ and every $\kappa = 1, \ldots, K$, the element $a$ is either elliptic or WWPD with respect to the action $N \actson_\kappa X$.

Apply (b) to obtain an independent pair of loxodromic elements in $N$, and then apply hyperbolic ping pong to this pair of elements to obtain a rank 2 Schottky subgroup $E_0 < N$. The commutator subgroup $[E_0, E_0] < E_0$ is free of infinite rank and hence the restricted action $[E_0, E_0]$ is nonelementary. Since $[E_0, E_0] < [N,N]$, we may pick an independent pair of loxodromic elements in $[N,N]$, and then apply hyperbolic ping pong to obtain a rank 2 Schottky subgroup $E_1 < [N,N]$. Every nonidentity element of $E_1$ is loxodromic, and by applying (c), every nonidentity element of $E_1$ satisfies WWPD, thus every rank 2 subgroup $E \subset E_1$ satisfies Definition 2.13 (2).

To complete the verification of the WWPD hypothesis it remains to describe a rank 2 subgroup $E < E_1$ that satisfies Definition 2.13 (3)', and by applying (d) it suffices to show that $E$ satisfies the following:

- Each restricted action $E \actson_\kappa X$ is either Schottky or elliptic, for $\kappa = 1, \ldots, K$.

We do this with an argument of Bestvina and Fujiwara extracted from the proof of [BF02, Theorem 8], as follows. Assume by induction that for some $\kappa = 1, \ldots, K - 1$ we have a rank 2 subgroup $E_\kappa \subset E_1$ such that for each $i = 1, \ldots, \kappa$ the action $E_\kappa \actson_i X$ is either Schottky or elliptic, and hence the restriction of the action $\actson_i$ to any rank 2 subgroup of $E_\kappa$ is either Schottky or elliptic. We break into cases depending on the nature of the action $E_\kappa \actson_{\kappa + 1} X$. If that action is elliptic then, taking $E_{\kappa + 1} = E_\kappa$, the induction is complete. If that action is nonelementary then, picking independent loxodromic elements and using hyperbolic ping-pong, we obtain a rank 2 subgroup $E_{\kappa + 1} < E_\kappa$ for which the action $E_{\kappa + 1} \actson_{\kappa + 1} X$ is Schottky, and the induction is complete. In the remaining case the action $E_\kappa \actson_{\kappa + 1} X$ is nonelliptic and elementary, and so there exists a point $\xi \in \partial X$ which is either an attractor or repeller for every loxodromic element of the action $E_\kappa \actson_{\kappa + 1} X$. By combining the inclusion $E_\kappa \subset [N,N]$ with property (d) and Proposition 2.6 (5), it follows that there exists $\eta \neq \xi \in \partial X$ such that $\{\xi, \eta\}$ is the attractor repeller pair for every loxodromic element in $E_\kappa$. In this situation the translation number homomorphism $\tau: E_\kappa \to \mathbb{R}$
is defined by picking a base point \( x \in X \), and for each \( \theta \in E_\kappa \) setting

\[
\tau(\theta) = \begin{cases} 
\lim_{n \to \infty} d(x, \theta^n(x))/n & \text{if } \xi \text{ is the attractor for } \theta \\
-\lim_{n \to \infty} d(x, \theta^n(x))/n & \text{if } \eta \text{ is the attractor for } \theta \\
0 & \text{otherwise}
\end{cases}
\]

Since \( \text{Image}(\tau) \) is abelian, \( \text{Ker}(\tau) \) is an infinite rank subgroup of \( E_\kappa \). Taking \( E_{\kappa+1} < \text{Ker}(\tau) \) to be any rank 2 subgroup, the action \( E_{\kappa+1} \curvearrowright_{\kappa+1} X \) is elliptic and the induction is complete.

**Case 2: \( \Gamma < \text{Out}(F_r) \) is an infinite lamination subgroup.** Letting \( A \) be any maximal, proper, \( \Gamma \)-invariant free factor system in \( F_r \), the hypotheses of Theorem B are satisfied, and from its conclusions we obtain the following:

(a) The action \( \Gamma \curvearrowright \mathcal{FS}(F_r) \) is nonelementary;

(b) There exists a loxodromic element of \([\Gamma, \Gamma]\) which is fully irreducible rel \( A \), and any such element satisfies WWPD for the action \( \Gamma \curvearrowright \mathcal{FS}(F_r) \).

We may apply all of the numbered properties (1)\(_\Gamma\) - (9)\(_\Gamma\) from Section 4.2, each of which was proved starting only with the assumption that \( \Gamma \) and \( A \) satisfy the hypotheses of Theorem B.

We also adopt all the notation from the setup at the beginning of Section 4.3, but to simplify notation and highlight parallels with Case 1 we use the notation

\[
N = G_0
\]

We have an action \( N \curvearrowright \mathcal{FS}(F_r) \), given by the composition \( N \xrightarrow{\pi} \Gamma \curvearrowright \mathcal{FS}(F_r) \). Evidently the following properties corresponding to (a) and (b) are satisfied:

(a)\(_N\) The action \( N \curvearrowright \mathcal{FS}(F_r) \) is nonelementary;

(b)\(_N\) There exists a loxodromic \( \phi \in [N, N] \) such that \( \pi(\phi) \in \text{Out}(F_r) \) is fully irreducible rel \( A \), and any such \( \phi \) satisfies WWPD for the action \( N \curvearrowright \mathcal{FS}(F_r) \).

The proof in Case 2 will parallel Case 1 for a brief while, before diverging. Choose \( i_1, \ldots, i_K \in \text{Aut}(N) \) to be outer representatives of the inner action of \( G \) on \( N \), with \( i_1 = \text{Id}_N \). Let \( N_{\kappa} \mathcal{FS}(F_r) \) denote the composed action \( N \xrightarrow{i} N \curvearrowright \mathcal{FS}(F_r) \), and use the same action symbol \( \curvearrowright_{\kappa} \) for restrictions of \( N \curvearrowright \mathcal{FS}(F_r) \) to subgroups of \( N \). Definition 2.13 (1) is simply property (a)\(_N\) above. Thus to verify the global WWPD hypothesis for \( G \) we must produce a rank 2 subgroup \( E < [N, N] \) and use it to verify the following parts of Definition 2.13:

(2) The restricted action \( E \curvearrowright \mathcal{FS}(F_r) \) is Schottky and its nontrivial elements all satisfy WWPD with respect to the action \( N \curvearrowright \mathcal{FS}(F_r) \).
(3') For each $\kappa = 1, \ldots, K$, the action $E \acts_{\kappa} FS(F_r)$ is either elliptic, or it is Schottky and its nontrivial elements all satisfy WWPD with respect to the action $N \acts_{\kappa} FS(F_r)$.

Where Cases 1 and 2 diverge is that we do not know that every loxodromic $\phi \in [N, N]$ is a WWPD element for the action $N \acts FS(F_r)$; we know this only for those $\phi$ such that $\pi(\phi) \in \Out(F_r)$ is fully irreducible rel $A$. Thus, to verify the global WWPD hypothesis for $G$ using the action $N \acts FS(F_r)$ there is still quite a bit of intricate work to do involving subgroup decomposition theory, lamination ping-pong, and Ghosh’s theorem.

Define $\pi_{\kappa} : N \to \Gamma$ to be the composition $N \xrightarrow{i_{\kappa}} N \xrightarrow{\pi} \Gamma$. The action $N \acts_{\kappa} FS(F_r)$ is thus the same as the composed action $N \xrightarrow{\pi_{\kappa}} \Gamma \acts FS(F_r)$. For each component $[A]$ of $\mathcal{A}$, since the restriction of $\Gamma$ to $\Out(A)$ is abelian, it follows that the restriction of $[\Gamma, \Gamma]$ to $\Out(A)$ is trivial. Since the characteristic subgroup $[N, N]$ is preserved by the isomorphism $i_{\kappa}$, we have $\pi_{\kappa}[N, N] = \pi[N, N] < [\Gamma, \Gamma]$ for $\kappa = 1, \ldots, K$, and so

(c)$_N$ For each component $[A]$ of $\mathcal{A}$ and each $\kappa = 1, \ldots, K$, the restriction of $\pi_{\kappa}[N, N]$ to $\Out(A)$ is trivial.

Consider $a \in N$, and denote its images in $\Gamma$ as $\alpha_\kappa = \pi_\kappa(a) \in \Gamma$, $1 \leq \kappa \leq K$. We say that $\kappa$ is a PG index for $a$ if $\mathcal{L}(\alpha_\kappa; \mathcal{A}) = \emptyset$; otherwise $\kappa$ is an EG index for $a$. Assuming that $\kappa$ is an EG index for $a$, we say that $\kappa$ is a non-geometric index of $a$ if some element of $\mathcal{L}(\alpha_\kappa; \mathcal{A})$ is non-geometric; otherwise $\kappa$ is a geometric index for $a$. Let $t_1 \geq 0$ be the maximum number of non-geometric indices that occurs for any element of $N$, and let $\mathcal{M}'$ be the set of all $a \in N$ having $t_1$ non-geometric indices; note that $t_1 = 0$ if and only if the subgroup $\Gamma < \Out(F_r)$ is geometric above $\mathcal{A}$. Let $t_2 \geq t_1$ be the maximum number of EG indices that occur for some $a \in \mathcal{M}'$, and let $\mathcal{M}$ be the set of all $a \in \mathcal{M}'$ having $t_2$ EG indices. There exists $a \in N$ and $\kappa \in \{1, \ldots, K\}$ such that $\kappa$ is an EG index for $a$, because by property (a) some element of $\Gamma$ is loxodromic and so has a filling lamination. The set $\mathcal{M}$ is therefore nonempty and $t_2 \geq 1$. After permuting the $\kappa$'s if necessary we may assume that the following subset of $\mathcal{M}$ is nonempty:

$$\mathcal{M}_0 = \{a \in N \mid \kappa \text{ is a non-geometric index of } a \text{ for } 1 \leq \kappa \leq t_1, \text{ and } \kappa \text{ is a geometric index of } a \text{ for } t_1 < \kappa \leq t_2\}$$

That is, $a \in \mathcal{M}_0$ if and only if: $\mathcal{L}(\alpha_\kappa; \mathcal{A})$ has a non-geometric lamination for each $1 \leq \kappa \leq t_1$; $\mathcal{L}(\alpha_\kappa; \mathcal{A})$ is a nonempty set of geometric laminations for $t_1 < \kappa \leq t_2$; and $\mathcal{L}(\alpha_\kappa; \mathcal{A})$ is empty for $t_2 < \kappa \leq K$.

Given $a \in \mathcal{M}_0$ with $\alpha_\kappa = \pi_\kappa(a)$, an assignment of lamination pairs for $a$ is a function

$$\kappa \mapsto \Lambda^\pm_{\alpha_\kappa} \in \mathcal{L}(\alpha_\kappa; \mathcal{A}) \text{ defined for } 1 \leq \kappa \leq t_2,$$

denoted in shorthand as $(\Lambda^\pm_{\alpha_\kappa})$. We say that this assignment $(\Lambda^\pm_{\alpha_\kappa})$ is $\mathcal{M}_0$-consistent if $\Lambda^\pm_{\alpha_\kappa}$ is non-geometric for each $1 \leq \kappa \leq t_1$. 44
Proposition 4.9.

(1) There exists $a \in \mathcal{M}_0$ with an $\mathcal{M}_0$-consistent assignment $(\Lambda^\pm_{\alpha_\kappa})$ of filling lamination pairs, such that $\alpha_\kappa = \pi_\kappa(a) \in \Gamma$ is fully irreducible rel $\mathcal{A}$ for each $1 \leq \kappa \leq t_2$.

(2) Either $t_1 = 0$ or $t_2 = t_1 \geq 1$. More precisely: either each $\Lambda^\pm_{\alpha_\kappa}$ is non-geometric for each $\kappa$; or $\Gamma$ is geometric above $\mathcal{A}$ and therefore $\Lambda^\pm_{\alpha_\kappa}$ is geometric for each $\kappa$.

Before proving Proposition 4.9, we apply it to the construction of a rank 2 subgroup $E < \Gamma$ by property (5). Consider the set $\mathcal{C}$ of closed subsets of $\mathcal{B}(F_r)$, on which $\text{Out}(F_r)$ acts naturally. Each lamination $\Lambda^-_{\alpha_\kappa}, \Lambda^+_{\alpha_\kappa}$ for $1 \leq \kappa \leq t_2$ is an element of $\mathcal{C}$. By [HM14b, Corollary 1.3] we have equality of stabilizer subgroups $\text{Stab}_\Gamma(\Lambda^-_{\alpha_\kappa}) = \text{Stab}_\Gamma(\Lambda^+_{\alpha_\kappa})$, and by property (5) this subgroup has infinite index in $\Gamma$. The subgroup $\text{Stab}_\Gamma(\Lambda^-_{\alpha_\kappa}, \Lambda^+_{\alpha_\kappa})$, which contains $\text{Stab}_\Gamma(\Lambda^-_{\alpha_\kappa}) = \text{Stab}_\Gamma(\Lambda^+_{\alpha_\kappa})$ with index at most 2, therefore also has infinite index in $\Gamma$.

For each $1 \leq \kappa \leq t_2$, consider the composed action $N \xrightarrow{\pi_\kappa} \Gamma < \text{Out}(F_r) \cong C_r$, denoted as $N \curvearrowright C_r$. The Second Sublemma of Lemma 2.1 of [HM17f] applied to the actions $N \curvearrowright C_r$ produces an element $b \in N$ such that for all $1 \leq \kappa \leq t_2$, letting $\beta_\kappa := \pi_\kappa(b)$, we have

$$\beta_\kappa(\{\Lambda^+_{\alpha_\kappa}, \Lambda^-_{\alpha_\kappa}\}) \neq \{\Lambda^+_{\alpha_\kappa}, \Lambda^-_{\alpha_\kappa}\}$$

and therefore by the Independence Theorem [HM14b, Theorem 1.2] we have

$$(*) \quad \beta_\kappa(\{\Lambda^+_{\alpha_\kappa}, \Lambda^-_{\alpha_\kappa}\}) \cap \{\Lambda^+_{\alpha_\kappa}, \Lambda^-_{\alpha_\kappa}\} = \emptyset$$

Let $c = bab^{-1} \in N$, let $\gamma_\kappa = \pi_\kappa(c) = \beta_\kappa(\alpha_\kappa)^{-1}$, and let $\Lambda^\pm_{\gamma_\kappa} = \beta_\kappa(\Lambda^\pm_{\alpha_\kappa}) \in \mathcal{L}^\pm(\gamma_\kappa; \mathcal{A})$. Then $\gamma_\kappa$ is fully irreducible rel $\mathcal{A}$ and $\{\Lambda^+_{\alpha_\kappa}, \Lambda^-_{\alpha_\kappa}\} \cap \{\Lambda^+_{\gamma_\kappa}, \Lambda^-_{\gamma_\kappa}\} = \emptyset$ for all $1 \leq \kappa \leq t_2$; moreover, $(\Lambda^\pm_{\gamma_\kappa})$ is an $\mathcal{M}_0$-consistent assignment of filling lamination pairs for $c$.

We now apply Proposition 4.7 to the pair of elements $\alpha_\kappa$ and $\gamma_\kappa$ for each $1 \leq \kappa \leq t_2$, noting that we have already verified hypotheses (1) and (2) of Proposition 4.7 for that pair, and that hypothesis (3) follows from $\mathcal{M}_0$-consistency of each of the assignments $(\Lambda^\pm_{\alpha_\kappa})$ and $(\Lambda^\pm_{\gamma_\kappa})$ and from Proposition 4.9 (2). From the conclusion of Proposition 4.7, for each $1 \leq \kappa \leq t_2$ there exists $M_\kappa$ so that for each $m \geq M_\kappa$ the subgroup $E'_\kappa$ of $\Gamma$ generated by $\alpha^m_\kappa$ and $\gamma^m_\kappa$ is free of rank 2, each non-trivial element $\xi_\kappa \in E'_\kappa$ is irreducible rel $\mathcal{A}$, each $\xi_\kappa$ has a unique lamination pair $\Lambda^\pm_{\xi_\kappa}$ that fills rel $\mathcal{A}$, and $\Lambda^\pm_{\xi_\kappa}$ is nongeometric for $1 \leq \kappa \leq t_1$ and geometric for $t_1 < \kappa \leq t_2$. Increasing each $M_\kappa$ to $M = \max_\kappa M_\kappa$, we obtain a subgroup $E'$ of $\Gamma$ freely generated by $\alpha^m$ and $c^m$ such that that $E'_\kappa = \pi_\kappa(E')$, and such that for each nontrivial $x \in E'$, letting $\xi_\kappa = \pi_\kappa(x)$, the assignment $(\Lambda^\pm_{\xi_\kappa})$ is $\mathcal{M}_0$-consistent.

By [HM14b, Theorems 1.1 and 1.2], the actions of $\alpha_\kappa$ and $\gamma_\kappa$ on $\mathcal{F}S(F_r)$ are loxodromic and independent. By hyperbolic ping-pong (see the beginning of Section 2.4),
after a further increase of $M$, we may also assume that the action of each $E'_\kappa$ on $FS(F_r)$ is Schottky. In particular, each non-trivial $\xi_\kappa \in E'_\kappa$ has a filling lamination pair, and that pair must be $\Lambda_{\xi_\kappa}^\pm$ since all other lamination pairs for $\xi_\kappa$ are supported by $A$. It follows that for each $1 \leq \kappa \leq t_2$ the restricted action $E' \acteq \kappa \mathcal{FS}(F_r)$ is Schottky and each non-trivial element of $\pi_\kappa(E')$ is irreducible rel $A$. Moreover each non-trivial element of $E'$ is contained in $M_0$.

Choose a rank 2 free subgroup $E < [E' \cap N, E' \cap N] < [N, N]$. Note that each restricted action $E \acteq \kappa \mathcal{FS}(F_r)$ is still Schottky for each $1 \leq \kappa \leq t_2$, and each non-trivial element of $E$ is contained in $M_0$. Applying property (c), for each component $[A]$ of $A$ the restriction of each $\pi_\kappa(E)$ to $\text{Out}(A)$ is trivial. Theorem E therefore implies that $E \acteq \kappa \mathcal{FS}(F_r)$ is a WWPD Schottky group for each $1 \leq \kappa \leq t_2$. Since each non-trivial element of $E$ is contained in $M_0$, the actions $E \acteq \kappa \mathcal{FS}(F_r)$ must be elliptic for $\kappa > t_2$. This completes the verification of (2) and (3)' using $E$, thus finishing Case 2 subject to the proof of Proposition 4.9.

**Proof of Proposition 4.9.** We start with two lemmas needed to set up the ping-pong proof. Most of the following lemma is cited from [HM17f, Lemma 2.3 (3)].

**Lemma 4.10.** Suppose that $\Gamma < IA_r(\mathbb{Z}/3)$ is irreducible relative to a free factor system $A$, and suppose that $\phi \in \Gamma$ is rotationless and has a geometric lamination pair $\Lambda_{\phi}^\pm \in \mathcal{L}(\phi; A)$ such that $\mathcal{A}_{\text{na}}(\Lambda_{\phi}^\pm)$ is $\Gamma$-invariant. Then

1. $\phi$ is irreducible rel $A$ [HM17f, Lemma 2.3 (3)(a)].
2. The free factor support of $\Lambda_{\phi}^\pm$ is $\Gamma$-invariant [HM17f, Lemma 2.3 (3)(b)].
3. $\mathcal{A}_{\text{na}}(\Lambda_{\phi}^\pm) = \mathcal{A} \cup \{[C]\}$ where $C$ is a maximal infinite cyclic subgroup of $F_r$ [HM17f, Lemma 2.3 (3)(c)].
4. $\Gamma$ is geometric above $A$.

**Proof.** To prove (4), if $\Gamma$ is not geometric above $A$ then by [HM17f, Proposition 2.4] there exists a rotationless $\theta \in \Gamma$ that is irreducible rel $A$ and there exists $\Lambda_{\theta}^\pm \in \mathcal{L}(\theta; A)$ such that $\mathcal{A}_{\text{na}}(\Lambda_{\theta}^\pm) = \mathcal{A}$. But this contradicts the fact that $[C]$ is $\theta$-invariant and not carried by the free factor system $A$.

Ping-pong arguments in groups can start with one player — that is, one group element — producing a second player by conjugating the first. The conjugating element is chosen carefully, depending on the desired outcome. The proof of Proposition 4.9 is a ping-pong game in the group $N$, carried out inductively: given $a_i \in M_0$, after choosing a conjugator $b_i \in N$ we produce the second player $c_i = b_i a_i b_i^{-1} \in M_0$ and then define $a_{i+1} = c_i^m a_i^n$ for sufficiently large $m, n$. We then iterate this until the resulting $a_i$ satisfies the conclusions of Proposition 4.9. Each step of this iteration can also be viewed as $t_2$ simultaneous ping-pong games in the quotient group $\Gamma = \pi(N)$,
with $t_2$ first players $\alpha_\kappa (=\pi_\kappa(a_1))$, requiring a very careful and consistent choice of $t_2$ conjugating elements $\beta_\kappa (=\pi_\kappa(b_1))$. The following lemma describes the choice of conjugating element $b_1$. It is a straightforward generalization of [HM17f, Lemma 2.1], which is used to choose the conjugating maps for the proof of [HM17f, Theorem 1].

**Lemma 4.11.** Suppose that $a \in \mathcal{M}_0$ and $\alpha_\kappa = \pi_\kappa(a) \in \Gamma$, $1 \leq \kappa \leq K$. Let $\Lambda_\pm^\kappa \in \mathcal{L}(\alpha_\kappa;\mathcal{A})$ be an $\mathcal{M}_0$-consistent assignment of lamination pairs, with generic leaves $\ell^\pm_{\alpha_\kappa}$, respectively. There exists $b \in N$ such that $\beta_\kappa = \pi_\kappa(b)$ satisfies the following properties for all $1 \leq \kappa \leq t_2$.

1. None of the lines $\beta_\kappa(\ell^+_\alpha), \beta_\kappa(\ell^-_\alpha), \beta_\kappa^{-1}(\ell^+_\alpha), \beta_\kappa^{-1}(\ell^-_\alpha)$ is carried by $\mathcal{A}_{na}\Lambda^\pm_{\alpha_\kappa}$.
2. $\beta_\kappa\{\Lambda^+_\alpha,\Lambda^-\alpha\} \cap \{\Lambda^+_\alpha,\Lambda^-\alpha\} = \emptyset$.
3. If $\mathcal{A}_{na}\Lambda^\pm_{\alpha_\kappa}$ is not $\Gamma$-invariant then $\beta_\kappa(\mathcal{A}_{na}\Lambda^\pm_{\alpha_\kappa}) \neq \mathcal{A}_{na}\Lambda^\pm_{\alpha_\kappa}$.
4. If $\mathcal{F}_{supp}\Lambda^\pm_{\alpha_\kappa}$ is not $\Gamma$-invariant then $\beta_\kappa(\mathcal{F}_{supp}\Lambda^\pm_{\alpha_\kappa}) \neq \mathcal{F}_{supp}\Lambda^\pm_{\alpha_\kappa}$.

**Proof.** Following the proof of [HM17f, Lemma 2.1], we shall show the following for all $1 \leq \kappa \leq t_2$:

1. There is a finite index subgroup $\Gamma_\kappa < \Gamma$ such that for any $\beta \in \Gamma_\kappa$ none of the lines $\beta(\ell^+_\alpha), \beta(\ell^-_\alpha), \beta^{-1}(\ell^+_\alpha), \beta^{-1}(\ell^-_\alpha)$, is carried by $\mathcal{A}_{na}\Lambda^\pm_{\alpha_\kappa}$.
2. $\text{Stab}_\Gamma\{\Lambda^+_\alpha,\Lambda^-\alpha\}$ has infinite index.
3. If $\mathcal{A}_{na}\Lambda^\pm_{\alpha_\kappa}$ is not $\Gamma$-invariant then $\text{Stab}_\Gamma(\mathcal{A}_{na}\Lambda^\pm_{\alpha_\kappa})$ has infinite index.
4. If $\mathcal{F}_{supp}\Lambda^\pm_{\alpha_\kappa}$ is not $\Gamma$-invariant then $\text{Stab}_\Gamma(\mathcal{F}_{supp}\Lambda^\pm_{\alpha_\kappa})$ has infinite index.

Item (a) follows from the First Sublemma in the proof of [HM17f, Lemma 2.1]. Item (b) follows from property (5)$_\Gamma$. Item (c) follows from [HM17f, Lemma 2.3]. Item (d) follows because if $\text{Stab}_\Gamma(\mathcal{F}_{supp}\Lambda^\pm_{\alpha_\kappa})$ has finite index then for each $\theta \in \Gamma$ some power of $\theta$ fixes the free factor system $\mathcal{F}_{supp}\Lambda^\pm_{\alpha_\kappa}$, but since $\theta \in IA_n(\mathbb{Z}/3)$ it follows that $\theta$ itself fixes $\mathcal{F}_{supp}\Lambda^\pm_{\alpha_\kappa}$, by [HM17d, Theorem 3.1].

By pulling back the infinite index subgroups in (b)$_\kappa$, (c)$_\kappa$, and (d)$_\kappa$ under the homomorphism $\pi_\kappa: N \to \Gamma$, and letting $\kappa = 1, \ldots, t_2$ vary, we obtain a finite collection of infinite index subgroups of $N$. Applying the Second Sublemma in the proof of [HM17f, Lemma 2.1], there is an infinite subset of $N$ any two elements of which lie in distinct left cosets of each subgroup in this collection. By the pigeonhole principle, this subset must contain two elements $b_1 \neq b_2$ lying in the same left coset of the finite index subgroup

$$\bigcap_{\kappa=1}^{t_2} j_\kappa^{-1}(\Gamma_\kappa)$$

The element $b = b_1^{-1}b_2$ satisfies the conclusions of the lemma. □
Proof of Proposition 4.9. The construction is inductive, using the fact that $M_0 \neq \emptyset$ to choose an arbitrary $a(1) \in M_0$ and an $M_0$-consistent assignment of lamination pairs $(\Lambda_{\pi_\kappa(a(1))}^\pm)$, and producing a sequence $a(2), a(3), \ldots \in M_0$ with $M_0$-consistent assignments $(\Lambda_{\pi_\kappa(a(i))}^\pm)$. The first part of the proof is a description of a single step in the induction. The remainder of the proof explains how the induction eventually terminates in an element of $M_0$ satisfying conclusions (1) and (2) of the proposition.

Assuming that we have produced $a_i \in M_0$, we describe how to produce $a_{i+1}$. Let $\alpha_{i+1} := \pi_\kappa(a_i)$ for $1 \leq \kappa \leq t_2$. Let $(\Lambda_{\alpha_{i+1}}^\pm)$ be the corresponding $M_0$-consistent assignment of lamination pairs. For notational convenience we suppress the subscript “i” for now, writing $a, \alpha = \pi_\kappa(a)$, and $(\Lambda_{\alpha}^\pm)$.

Choose $b \in N$ as in Lemma 4.11, the conclusions of which we shall apply below. Let $\beta_\kappa = \pi_\kappa(b)$, let $c = bab^{-1}$, let $\gamma_\kappa = \pi_\kappa(c) = \beta_\kappa \alpha \beta_\kappa^{-1}$, and let $\Lambda_{\gamma_\kappa}^\pm = \beta_\kappa(\Lambda_{\alpha_{i+1}}^\pm) \in \mathcal{L}(\gamma_\kappa; \mathcal{A})$. Fixing $m, n \geq 1$ subject to lower bounds to be given, consider $x = c^m a^n$ and for each $1 \leq \kappa \leq t_2$ consider $\xi_\kappa = \pi_\kappa(x) = \gamma_\kappa^m \alpha_{i+1}^\kappa$.

We want to simultaneously apply Proposition 4.4 (aka [HM17f, Proposition 1.7]) to $\alpha_\kappa, \Lambda_{\alpha_{i+1}}^\pm, \gamma_{\kappa}$ and $\Lambda_{\gamma_{\kappa}}^\pm$ for $1 \leq \kappa \leq t_2$, and so we must check its hypotheses. After replacing $a$, and hence $c$, with an iterate of itself, we may assume for $1 \leq \kappa \leq t_2$ that $\Lambda_{\alpha_{i+1}}^\pm$ and $\Lambda_{\gamma_{\kappa}}^\pm$ have generic leaves $\ell_{\alpha_{i+1}}^\pm$ and $\ell_{\gamma_{\kappa}}^\pm = \beta_\kappa(\ell_{\alpha_{i+1}}^\pm)$ that are fixed by $\phi$ and $\psi$ respectively, with fixed orientation. Note that if $1 \leq \kappa \leq t_1$ then the pair $\Lambda_{\gamma_{\kappa}}^\pm$ is nongeometric because it is the $\beta_\kappa$ image of the nongeometric pair $\Lambda_{\alpha_{i+1}}^\pm$. It remains to check hypotheses (a) and (i)–(iv) of Proposition 4.4.

Hypothesis (a) of Proposition 4.4 says that $\mathcal{A} \sqsubseteq \mathcal{A}_{\alpha_{i+1}} \Lambda_{\alpha_{i+1}}^\pm$ and $\mathcal{A} \sqsubseteq \mathcal{A}_{\alpha_{i+1}} \Lambda_{\gamma_{\kappa}}^\pm = \beta_\kappa \mathcal{A}_{\alpha_{i+1}} \Lambda_{\alpha_{i+1}}^\pm$. This follows from the assumption that $\Lambda_{\alpha_{i+1}}^\pm$, and hence also $\Lambda_{\gamma_{\kappa}}^\pm$, is not carried by $\mathcal{A}$: every iterate under $\alpha$ and $\gamma$ of every conjugacy class carried by $\mathcal{A}$ is also carried by $\mathcal{A}$, and so such iterates cannot weakly converge to a lamination that is not carried by $\mathcal{A}$.

Hypotheses (i)–(iv) of Proposition 4.4 follow from the conclusions of Fact 4.3, whose two hypotheses we verify by applying the conclusions of Lemma 4.11. Hypothesis (1) of Fact 4.3 is just Lemma 4.11 (2). Hypothesis (2) of Fact 4.3 requires that: neither of the lines $\ell_{\alpha_{i+1}}^\pm = \beta_\kappa(\ell_{\alpha_{i+1}}^\pm)$ should be carried by $\mathcal{A}_{\alpha_{i+1}} \Lambda_{\alpha_{i+1}}^\pm$; also neither of the lines $\ell_{\gamma_{\kappa}}^\pm$ should be carried by $\mathcal{A}_{\alpha_{i+1}} \Lambda_{\gamma_{\kappa}}^\pm = \beta_\kappa(\mathcal{A}_{\alpha_{i+1}} \Lambda_{\alpha_{i+1}}^\pm)$ which is equivalent to saying that neither of the lines $\beta_\kappa^{-1}(\ell_{\alpha_{i+1}}^\pm)$ is carried by $\mathcal{A}_{\alpha_{i+1}} \Lambda_{\alpha_{i+1}}^\pm$; but these altogether follow from Lemma 4.11 (1).

We may now apply the conclusions of Proposition 4.4 for each $1 \leq \kappa \leq t_2$, producing threshold constants $M^1_\kappa$, and if $m, n \geq M^1_\kappa$ producing laminations $\Lambda^+_\kappa \in \mathcal{L}(\xi_{\kappa}; \mathcal{A})$ and $\Lambda^-_{\kappa} \in \mathcal{L}(\xi_{\kappa}^{-1}; \mathcal{A})$.

Define a simultaneous threshold constant $M^1 = \max_{1 \leq \kappa \leq t_2} M^1_\kappa$. Fix $m, n \geq M^1$, and so for each $1 \leq \kappa \leq t_2$ we have a lamination pair $\Lambda^\pm_{\xi_{\kappa}} \in \mathcal{L}(\xi_{\kappa})$.

In the case that $1 \leq \kappa \leq t_1$, since $\Lambda_{\alpha_{i+1}}^\pm$ and $\Lambda_{\gamma_{\kappa}}^\pm$ are both nongeometric, Proposition 4.4 lets us conclude that $\Lambda^+_\xi$ and $\Lambda^-_{\xi}$ are both nongeometric. In the case that $t_1 < \kappa \leq t_2$, the laminations $\Lambda^+_\xi$ and $\Lambda^-_{\xi}$ are both geometric by definition of $M_0$. In either case Proposition 4.4 (6) lets us conclude that $\Lambda^\pm_{\xi}$ form a dual lamination pair.
Thus we have shown that \( (\Lambda^\pm_{\xi}) \) is an \( \mathcal{M}_0 \)-consistent assignment of lamination pairs for \( x \).

Next we show for each \( 1 \leq \kappa \leq t_2 \) that

\((*)\) \( \mathcal{A}_{na}\Lambda_{\xi,\kappa} \sqsubset \mathcal{A}_{na}\Lambda_{\alpha,\kappa} \). Furthermore, this containment is strict if \( \mathcal{A}_{na}\Lambda_{\alpha,\kappa} \) is not \( \Gamma \)-invariant.

The containment \( \mathcal{A}_{na}\Lambda_{\xi,\kappa} \sqsubset \mathcal{A}_{na}\Lambda_{\alpha,\kappa} \) follows from Proposition 4.4 (1) which moreover gives the containment \( \mathcal{A}_{na}\Lambda_{\xi,\kappa} \sqsubset \mathcal{A}_{na}\Lambda_{I,\kappa} \). For the “furthermore” part, if \( \mathcal{A}_{na}\Lambda_{\alpha,\kappa} \) is not \( \mathcal{H} \)-invariant then, by Lemma 4.11 (3), we have

\[
\mathcal{A}_{na}\Lambda_{\alpha,\kappa} = \beta_{\kappa}(\mathcal{A}_{na}\Lambda^\pm_{\alpha,\kappa}) \not\sqsubset \mathcal{A}_{na}\Lambda^\pm_{\alpha,\kappa},
\]

Arguing by contradiction, if furthermore \( \mathcal{A}_{na}\Lambda_{\xi,\kappa} = \mathcal{A}_{na}\Lambda_{\alpha,\kappa} \) then we have

\[
\mathcal{A}_{na}\Lambda_{\alpha,\kappa} = \beta_{\kappa}^{-1}(\mathcal{A}_{na}\Lambda_{\alpha,\kappa}) \sqsubset \beta_{\kappa}^{-1}(\mathcal{A}_{na}\Lambda_{\alpha,\kappa}) = \beta_{\kappa}^{-1}(\mathcal{A}_{na}\Lambda_{\alpha,\kappa})
\]

which must be a strict containment, and by iteration we have an infinite sequence of strict containments

\[
\mathcal{A}_{na}\Lambda_{\alpha,\kappa} \sqsubset \beta_{\kappa}^{-1}(\mathcal{A}_{na}\Lambda_{\alpha,\kappa}) \sqsubset \beta_{\kappa}^{-1}(\mathcal{A}_{na}\Lambda_{\alpha,\kappa}) \sqsubset \cdots
\]

But this contradicts [HM17c, Proposition 3.2] which says any decreasing sequence of vertex groups systems is eventually constant.

Restoring the “i” subscript, and so \( a = a_i \) and \( c = c_i \), now define \( a_{i+1} = x = e_i a_i^n \), so \( \xi(\alpha,\kappa) = \pi_\kappa(a_{i+1}) \) and \( \Lambda^\pm_{\alpha_{i+1,\kappa}} = \Lambda^\pm_{\xi(\alpha,\kappa)} \). This completes the inductive construction of \( a_{i+1} \) and its \( \mathcal{M}_0 \)-consistent assignment of lamination pairs \((\Lambda^\pm_{\alpha_{i+1,\kappa}})\). We have a containment \( \mathcal{A}_{na}\Lambda^\pm_{\alpha_{i,\kappa}} \sqsubset \mathcal{A}_{na}\Lambda^\pm_{\alpha_{i+1,\kappa}} \) which is strict if \( \mathcal{A}_{na}\Lambda^\pm_{\alpha_i,\kappa} \) is not \( \Gamma \)-invariant.

Consider the whole sequence of containments

\[
\mathcal{A}_{na}\Lambda^\pm_{\alpha_{i,\kappa}} \sqsubset \mathcal{A}_{na}\Lambda^\pm_{\alpha_{i+1,\kappa}} \sqsubset \mathcal{A}_{na}\Lambda^\pm_{\alpha_{i+2,\kappa}} \sqsubset \cdots
\]

Another application of [HM17c, Proposition 3.2] shows this sequence to be eventually constant, and again applying (*) it follows that \( \mathcal{A}_{na}\Lambda^\pm_{\alpha_i,\kappa} \) is eventually \( \Gamma \)-invariant. It follows that there exists \( I \geq 0 \) independent of \( \kappa \) such that for \( i \geq I \) each \( \mathcal{A}_{na}\Lambda^\pm_{\alpha_i,\kappa} \) is \( \Gamma \)-invariant and is independent of \( i \).

We now complete the proof of Proposition 4.9.

Conclusion (2) of Proposition 4.9 follows by observing that if \( t_2 > t_1 \) then, by Lemma 4.10 combined with \( \Gamma \)-invariance of \( \mathcal{A}_{na}\Lambda^\pm_{\alpha_i,\kappa} \) for each \( i \geq I \) and each \( t_1 < \kappa \leq t_2 \), the entire group \( \Gamma \) is geometric above \( \mathcal{A} \), from which it follows that \( t_1 = 0 \).

Conclusion (1) will be proved in two cases.

**Case 1:** \( t_2 > t_1 = 0 \). As just shown, in this case \( \Gamma \) is geometric above \( \mathcal{A} \) and \( \mathcal{A}_{na}\Lambda^\pm_{\alpha_i,\kappa} \) is \( \Gamma \)-invariant for all \( i \geq I \) and all \( 1 \leq \kappa \leq t_2 \). Applying Lemma 4.10, it then follows that \( \alpha_{i,\kappa} \) is irreducible rel \( \mathcal{A} \) and \( \Lambda^\pm_{\alpha_i,\kappa} \) has \( \Gamma \)-invariant free factor support.
and so is filling by property (6)\(r\). This proves (1) and so completes the proof of Proposition 4.9 in Case 1.

Case 2: \(t_2 = t_1\). In this case for each \(1 \leq \kappa \leq t_2\) we know that \(\Lambda^\pm_{\alpha_i,\kappa}\) is nongeometric for all \(i\), and hence \(A_{na}\Lambda^\pm_{\alpha_i,\kappa}\) is a proper free factor system. Combining this with the fact that \(A \sqsubset A_{na}\Lambda^\pm_{\alpha_i,\kappa}\), that \(\Gamma\) is irreducible rel \(A\), and that \(A_{na}\Lambda^\pm_{\alpha_i,\kappa}\) is \(\Gamma\)-invariant for \(i \geq I\), it follows that \(A_{na}\Lambda^\pm_{\alpha_i,\kappa} = A\) for \(i \geq I\).

Considering now each \(1 \leq \kappa \leq t_2\) separately, after imposing stricter threshold constants we wish to prove by induction on \(j \geq I\) that:

\((\#)\) We have containments \(F^\supp(\Lambda^\pm_{\alpha_I,\kappa}) \sqsubseteq F^\supp(\Lambda^\pm_{\alpha_{I+1},\kappa}) \sqsubseteq \cdots \sqsubseteq F^\supp(\Lambda^\pm_{\alpha_j,\kappa})\)

\((\#\#)\) For \(I \leq i < j\) the containment \(F^\supp(\Lambda^\pm_{\alpha_i,\kappa}) \sqsubseteq F^\supp(\Lambda^\pm_{\alpha_{i+1},\kappa})\) is proper if and only if \(F^\supp(\Lambda^\pm_{\alpha_i,\kappa})\) is not \(\Gamma\)-invariant.

Assuming that this has been proved for a particular \(j\), and returning temporarily to the earlier notation where the subscript \(j\) has been dropped, we shall apply the [HM17f, Inductive Step of Proposition 2.4], the hypotheses of which are already known to be true: the pair \(\Lambda^\pm_{\alpha}\) is nongeometric and its nonattracting subgroup system equals \(A\); by Lemma 4.11 (2) we have \(\{\Lambda^-_{\alpha}, \Lambda^+_{\alpha}\} \cap \{\Lambda^-_{\alpha}, \Lambda^+_{\alpha}\} = \emptyset\); and there are generic leaves of \(\Lambda^\pm_{\alpha}\) that are fixed by \(\alpha_{\kappa}\) with fixed orientations. From the conclusions of [HM17f, Inductive Step of Proposition 2.4] we obtain a threshold constant \(M^2_{\kappa}\) such that if \(m, n \geq M^2_{\kappa}\) then \(F^\supp(\Lambda_{\alpha}) \sqsubseteq F^\supp(\Lambda_{\kappa})\) with proper inclusion if and only if \(F^\supp(\Lambda_{\alpha})\) is not \(\Gamma\)-invariant.

Letting \(M^2 = \max_{1 \leq \kappa \leq t_2} M^2_{\kappa}\), and requiring that \(m, n \geq M^2\), for all \(1 \leq \kappa \leq t_2\) we have completed the inductive verification of (\#) and (\#\#) for all \(j \geq I\).

Since the length of a proper chain of free factor systems of \(F_{\kappa}\) is uniformly bounded, there exists \(j \geq I\) such that (\#) and (\#\#) are satisfied and such that the free factor support of \(\Lambda^\pm_{\alpha_i,\kappa}\) is \(\Gamma\)-invariant, and so by (6)\(r\) the laminating pair \(\Lambda^\pm_{\alpha_j,\kappa}\) fills \(F_{\kappa}\). Since \(A_{na}\Lambda^\pm_{\alpha_j,\kappa}\), it follows that \(\alpha_{j,\kappa}\) is irreducible rel \(A\). This being true for each \(1 \leq \kappa \leq t_2\), we have proved conclusion (1) of Proposition 4.9 in Case 2, and so the proof of the proposition is complete.

\[\square\]

5 Well functions and weak tiling functions

In this section and the next we are concerned with the proof of Theorem E. For this section we assume that \(n \geq 3\) and we shall pursue a further study of the “well functions” that were defined in [HM14b, Section 4.4] in the setting of outer automorphisms \(\phi\) possessing a filling lamination pair \(\Lambda^\pm_{\phi}\), and that generalize the well functions defined in [AK11] in the setting of fully irreducible outer automorphisms.

The idea behind a well function is that for any conjugacy class \(c\) that is weakly attracted to \(\Lambda^\pm_{\phi}\) by iteration of \(\phi_{w(c)}^{\pm 1}\), there is an iterate \(\phi_{w(c)}^{w(c)}\) with coarsely well-defined exponent \(w(c)\) such that the conjugacy class \(\phi_{w(c)}^{w(c)}\) is not a good approximation
of either $\Lambda^-_\phi$ or $\Lambda^+_\phi$. This contrasts with being outside of the well: for exponents $l \gg w(c)$ the class $\phi^l(c)$ is a good approximation of $\Lambda^+_\phi$ but a bad approximation of $\Lambda^-_\phi$; whereas for exponents $l \ll w(c)$ the class $\phi^l(c)$ is a good approximation of $\Lambda^-_\phi$ but a bad approximation of $\Lambda^+_\phi$. The vicinity of the integer $w(c)$ within the whole of the integers $\mathbb{Z}$ therefore defines a “well” in which approximations to $\Lambda^+_\phi$ and $\Lambda^-_\phi$ are simultaneously at their worst.

Our main result in this section is Proposition 5.7, which allows one to coarsely evaluate the well function $w(c)$ using a CT representative $f : G \to G$ of $\phi$, by studying how the circuit in $G$ representing $c$ can be “weakly tiled” using natural collections of paths in $G$ (see Definition 5.4). As an application we prove Corollaries 5.10 and 5.11 which describe strong combinatorial regularity properties of weak tilings. These regularity properties of weak tilings are what we shall use to verify WWPD in Section 6, in lieu of measure theoretic regularity properties of currents used to verify WPD in the arguments of [BF02] and [BF10].

5.1 Well functions on intermediate conjugacy classes

We review well functions and their properties from [HM14b, Lemma 4.14], although see Remark 5.3 to understand some differences of formulation.

**Definition 5.1.** Given a free factor system $F$ and a conjugacy class $c$ of $F_n$, we say that $c$ is **filling relative to** $F$ if there is no proper free factor system that carries both $c$ and $F$. If $c$ is not carried by $F$ nor filling relative to $F$ then $c$ is said to be **intermediate** (relative to $F$).

We note some properties relating this definition to an outer automorphism $\phi \in \text{Out}(F_n)$ that preserves $F$, the first of which is obvious:

- A conjugacy class $c$ is intermediate rel $F$ if and only if $\phi(c)$ is intermediate rel $F$.

- If $\phi$ is fully irreducible rel $F$ and if $\Lambda^\pm$ is a lamination pair for $\phi$ that is not carried by $F$ and hence fills rel $F$, then a conjugacy class $c$ is intermediate rel $F$ if and only if $c$ is weakly attracted to $\Lambda^\pm$ and is not filling relative to $\phi$.

To see why the second item is true, recall [HM17e, Theorem F] which says that $c$ is weakly attracted to $\Lambda^+$ under iteration by $\phi$ if and only if $c$ is weakly attracted to $\Lambda^-$ under iteration by $\phi^{-1}$ if and only if $c$ is not carried by the nonattracting subgroup system $A_{na}(\Lambda^\pm)$. If $\Lambda^\pm$ is non-geometric then $F = A_{na}(\Lambda^\pm)$ by [HM17e, Remark: The case of a top stratum] and so “not carried by $F$” is equivalent to “weakly attracted to $\Lambda^\pm$”. If $\Lambda^\pm$ is geometric then there are conjugacy classes that are not carried by $F$ and are not weakly attracted to $\Lambda^\pm$ but each of these fills relative to $F$; each such conjugacy class is represented by a multiple of the peripheral curve in a geometric model ([HM17c, Proposition 2.18]).
For most of this section we focus on the following objects whose notations we fix:

**Notation A:**
- \( F \) is a proper free factor system in \( F_n \), possibly trivial.
- \( \phi, \phi^{-1} \in \text{Out}(F_n) \) are rotationless, and they fix \( F \) and are irreducible relative to \( F \).
- \( \Lambda^\pm \in \mathcal{L}^\pm(\phi) \) is a lamination pair that fills \( F_n \).

With this notation we also have the following property:
- \( F \) has co-edge number \( \geq 2 \) (Fact 4.2 (3a)).

**Definition 5.2.** Consider the space of lines \( B \) with its weak topology. For any attracting neighborhood \( U \subset B \) of a generic leaf of \( \Lambda^+ \), and any conjugacy class \( c \) that is weakly attracted to \( \Lambda^+ \) under iteration by \( \phi \), there exists a maximal integer \( w_{\phi,U}(c) \) such that \( c \in \phi^{w_{\phi,U}(c)}(U) \) or equivalently \( \phi^{-w_{\phi,U}(c)}(c) \in U \). We refer to \( w_{\phi,U} \) as the well function of \( \phi \) with respect to \( U \). It is immediate from the definition that

\[
w_{\phi,U}(\phi^m(c)) = w_{\phi,U}(c) + m
\]

for all \( m \).

The two key properties of well functions are as follows:

**Coarse well-definedness of wells:** For any two \( U, V \subset B \) attracting neighborhoods of a generic leaf of \( \Lambda^+ \) there exists \( K \geq 0 \) so that \( \phi^K(V) \subset U \) and \( \phi^K(U) \subset V \). It follows that

\[
|w_{\phi,U}(c) - w_{\phi,V}(c)| \leq K
\]

for all \( c \) that are weakly attracted to \( \Lambda^\pm \). In other words the function \( w_{\phi,U}(\cdot) \) is coarsely well-defined independent of \( U \).

**Coarse additive inverse property of wells:** For any attracting neighborhoods \( U^\pm \) of generic leaves of \( \Lambda^\pm \), respectively, there exists \( L \geq 0 \) so that for all intermediate conjugacy classes \( c \) we have

\[
|w_{\phi,U^+}(c) + w_{\phi^{-1},U^-}(c)| \leq L
\]

In other words, the well functions \( w_{\phi,U^+}(\cdot) \) and \( w_{\phi^{-1},U^-}(\cdot) \) are coarse additive inverses of each other.

The coarse well-definedness property is obvious. The coarse additive inverse property is proved in [HM14b, Lemma 4.14(1)] (which is in turn based on [HM17f, Proposition 3.1]) for a particular choice of \( U^\pm \), and so it holds for all choices. Note that coarse well-definedness does not require the hypothesis of “intermediate”; that hypothesis need only be brought in for results, like the coarse additive inverse property, that ultimately depend on [HM17f, Proposition 3.1].
Henceforth, when attracting neighborhoods of generic leaves of $\Lambda^+$ and of $\Lambda^-$ are chosen, implicitly or explicitly, we write the corresponding well functions of $\phi$ and $\phi^{-1}$ as $w_{\phi}(\cdot)$ and $w_{\phi^{-1}}(\cdot)$, suppressing the dependence on the choice of attracting neighborhoods.

Remark 5.3. Our presentation here of well functions differs from the presentation in [HM14b] version 1 in a few regards. One difference is that well functions were defined there only for particular choices of $U^-, U^+$; the coarse well-definedness property lets us use any choices. Another difference is that $w_{\phi}$ as defined here equals the additive inverse of $w_{\phi^{-1}}$ as it was defined in Lemma 4.14 of [HM14b] version 1; by the coarse additive inverse property, this changes the definition of $w_{\phi}$ by a constant depending only on the choice of $U^-, U^+$.

5.2 Weak tiling functions on intermediate conjugacy classes

Following up on Notation A, for much of this section we shall also focus on these additional objects whose notations we also fix:

Notation B:
- $f: G \to G$ is a CT representative of $\phi$ with core filtration element $G_r$ representing the free factor system $F$.
- The attracting lamination $\Lambda^+$ corresponds to the top stratum $H_s$ of $G$, an EG stratum.
- $H^+_s$ is the union of $H_s$ with the zero strata, if any, that it envelops, those strata being the contractible components of $G_{s-1}$ [FH11, Definition 2.18].
- $G_u = G - H^+_s$ is the maximal filtration element that deformation retracts to $G_r$.
- $\rho_s$ is an indivisible Nielsen path of height $s$ if one exists (and see items (2) and (5) under the heading “EG properties of CTs” in Section 3.1).

With this notation we have the following property:
- If $\rho_s$ exists and is closed then the conjugacy class $c$ represented by $\rho_s$ fills relative to $F$ and so is not intermediate [HM17c, Lemma 2.5].

Our immediate goal, formulated in Proposition 5.7, is to give quantitative bounds on well functions, expressed in terms of the CT chosen in Notation B, and more specifically in terms of tiles as defined and studied in [BFH00, Section 3].

Earlier we recalled that a $k$-tile of the stratum $H_s$ is a path of the form $f^k_{\#}(E)$ where $E$ is an edge of $H_s$ and $k \geq 0$. Tiles satisfy a self-similarity property saying that for any integers $l > k > 0$, if the difference $l - k$ is sufficiently large then every $l$-tile contains every $k$-tile as a subpath [BFH00, Lemma 3.1.8 (3)].

Given a path or circuit $\sigma$ in $G$, a $k$-tiling of $\sigma$ is a splitting of $\sigma$ each term of which is either a $k$-tile or a subpath of $G_{s-1}$ with endpoints on $H_s$. Every generic leaf $\gamma^+$ of $\Lambda^+$ has a $k$-tiling for all $k \geq 1$ [BFH00, Lemma 3.1.10 (3)].

Given a line $\gamma$ in $G$, an exhaustion by tiles is an increasing family of finite subpaths $\gamma_1 \subset \gamma_2 \subset \cdots$ whose union is all of $\gamma$ such that each $\gamma_i$ is a $k_i$-tile for some sequence
Every generic leaf $\gamma^+$ of $\Lambda^+$ has an exhaustion by tiles [BFH00, Lemma 3.1.10 (4)].

**Remark:** Given a generic leaf $\gamma^+$ with an exhaustion by tiles $\gamma_1 \subset \gamma_2 \subset \cdots$, those tiles define a neighborhood basis $V_1 \supset V_2 \supset \cdots$ of $\gamma^+$ where $V_k \subset B$ is the set of lines having $\gamma_k$ as a subpath. Once an attracting neighborhood $U^+ \subset B$ of $\ell$ is chosen, the sequence $\phi^i(U^+)$ also forms neighborhood basis of $\ell$. It follows that for any sequence of conjugacy classes $c_i$, the sequence of values of a well function $w_{\phi}(c_i)$ correlates with the value of the maximum integer $k_i \geq 0$ such that $c_i$ contains a $k_i$ tile. Namely, $w_{\phi}(c_i) \to +\infty$ if and only if $k_i \to +\infty$. This gives the first hint to the quasi-comparability of well functions and of the “weak tiling functions” to be introduced in Definition 5.6 below.

We will need a less restrictive kind of tiling, allowing terms that are Nielsen paths of height $s$.

**Definition 5.4 (Weak tilings).** A weak $k$-tiling of a path or circuit $\sigma \subset G$ is a splitting of $\sigma$ each term of which is either a $k$-tile, or an indivisible Nielsen path of height $s$, or a maximal subpath of $\sigma$ in $G_{s-1}$. Weak $k$-tilings are most useful when $k$ is large in the following sense: the weak tiling threshold constant $k_0 \geq 0$ is defined to be the minimum integer such that for all $k \geq k_0$ the endpoints of each $k$-tile are periodic.

**Examples of weak tilings:** The following lemma gives general methods for constructing weak tilings. Lemma 3.1 gives another method, with tighter quantitative control, for producing circuits with weak 0-tilings; that lemma plays a key role in the proof of Proposition 5.7.

**Lemma 5.5.** Following Notations A and B, we have:

1. For each circuit or finite path $\sigma \subset G$ with endpoints, if any, at vertices there exists $m_{\sigma} \geq 0$ such that $f_{\#}^{m_{\sigma}}(\sigma)$ has a weak 0-tiling.

2. For all $k \geq 0$, if $\sigma = \mu_1 \cdots \mu_I$ is a weak $k$-tiling then $f_{\#}(\sigma) = f_{\#}(\mu_1) \cdots f_{\#}(\mu_I)$ is a weak $(k + 1)$-tiling.

3. Suppose that $f_{\#}(\sigma) = \nu_1 \cdots \nu_I$ is a weak $(k + 1)$-tiling for some $k \geq k_0$ and that any endpoint of $f_{\#}(\sigma)$ that is contained in $G_{s-1}$ is fixed by $f$ (and hence is contained in $G_u$). Then there is weak $k$-tiling $\sigma = \mu_1 \cdots \mu_I$ such that $f_{\#}(\mu_i) = \nu_i$ for all $i$.

**Proof.** Item (1) follows from [BFH00, Lemma 4.2.6]. Item (2) follows from the fact that if $\mu_i$ is a path in $G_{s-1}$, an indivisible Nielsen path of height $s$, or a $k$-tile, then $f_{\#}(\mu_i)$ is a path in $G_{s-1}$, an indivisible Nielsen path of height $s$ or a $(k + 1)$-tile respectively.

For (3) we define $\mu_i$ given $\nu_i$ as follows. If $\nu_i$ is an indivisible Nielsen path of height $s$, let $\mu_i = \nu_i$. Suppose next that $\nu_i$ is a maximal subpath of $G_{s-1}$. An endpoint
of \(\nu_i\) is either an endpoint of \(f_\#(\sigma)\) or is contained in \(H_s\). In the former case it is fixed by hypothesis and in the latter case it is a principal vertex by [FH11, Remark 4.9] and hence fixed because \(f\) is rotationless. Since the components of \(G_{s-1} - G_u\) are zero strata, and since zero strata contain no fixed points, it follows that \(\nu_i \subset G_u\) and that there is a path \(\mu_i \subset G_u\) with the same endpoints as \(\nu_i\) and satisfying \(f_\#(\mu_i) = \nu_i\). Suppose finally that \(\nu_i\) is a \((k + 1)\)-tile with endpoints say \(y_1\) and \(y_2\). For \(i = 1, 2\), let \(x_i\) be the unique vertex in the \(f\)-orbit of \(y_i\) such that \(f(x_i) = y_i\). There exists a \(k\)-tile \(\mu_i\) such that \(f_\#(\mu_i) = \nu_i\). Since \(k \geq k_0\), the endpoints of \(\mu_i\) are periodic and map to \(y_1\) and \(y_2\) and so must be \(x_1\) and \(x_2\). To complete the proof we need only verify that \(\mu_1 \cdots \mu_I\) is a splitting and that adjacent terms are not both contained in \(G_{s-1}\). If \(\mu_i\) is an indivisible Nielsen path of height \(s\) or a \(k\)-tile then \(f\) maps the initial and terminal directions of \(\mu_i\) to the initial and terminal directions of \(f_\#(\mu_i)\), all of which are contained in \(H_s\). If \(\mu_i \subset G_u\) then the initial and terminal directions of \(f_\#(\mu_i)\) are contained in \(G_u\). Since \(\nu_1 \cdots \nu_m\) is a splitting and since adjacent terms are not both contained in \(G_u\), the same is true for \(\mu_1 \cdots \mu_I\).

**Definition 5.6 (Weak tiling functions).** Following notations A and B, we can now define the integer valued weak tiling function \(\tau_f\) on the set of intermediate conjugacy classes. Consider an intermediate conjugacy class \(c\) that is represented by a circuit \(\sigma \subset G\). If \(c\) is such that \(\sigma\) has a weak \(k_0\)-tiling, then choose the maximal \(k \geq k_0\) such that \(\sigma\) has a weak \(k\)-tiling and define \(\tau_f(c) = k - k_0\). By Lemma 5.5 we can extend this to arbitrary \(c\), by defining \(\tau_f(c)\) to be the maximal integer such that \(f_{\#}^{-\tau_f(c)}(\sigma)\) has a weak \(k_0\)-tiling.

### 5.3 Coarse equivalence of well functions and weak tiling functions.

The following proposition is the main result of this section. It states that the weak tiling function determined by \(f : G \to G\) is coarsely equivalent to any well function associated to \(\Lambda^+\). As corollaries we will state some strong regularity properties of weak \(k\)-tilings (see Corollaries 5.10 and 5.11).

**Proposition 5.7.** Following notations A and B, let \(w_\phi = w_{\phi,U}\) be a well function for \(\Lambda^+\) defined with respect to some attracting neighborhood \(U\) of a generic leaf of \(\Lambda^+\), let \(k_0\) be the weak tiling threshold for \(f : G \to G\) (Definition 5.4), and let \(\tau_f\) be the associated weak tiling function (Definition 5.6). Then there is a constant \(N\) so that for all intermediate \(c\) we have

\[|w_\phi(c) - \tau_f(c)| \leq N\]

The proof of the proposition is given after the statement and proof of the following lemma and a corollary thereof. This lemma is a relative version of [AK11, Proposition 3.8].
Lemma 5.8. Suppose that the lamination pair $\Lambda^\pm$ is geometric and consider $\rho_s$, the indivisible Nielsen path of height $s$. There exists an integer $M \geq 1$ so that if $\sigma \subset G$ is a circuit representing an intermediate conjugacy class then $\sigma$ does not contain a subpath of the form $\rho_s^M$ or $\rho_s^{-M}$.

Proof. Let $c_0$ be the conjugacy class of the circuit in $G$ determined by the closed path $\rho_s$. By [HM17c, Proposition 2.18] the free factor support of $c_0$ and $F$ is the same as the free factor support of $\Lambda^\pm$ and $\mathcal{F}$ and hence is $\{[F_n]\}$. We may therefore choose conjugacy classes $c_1, \ldots, c_k$ carried by $\mathcal{F}$ whose free factor support equals $\mathcal{F}$, and therefore $\{c_0, c_1, \ldots, c_k\}$ fills. By Whitehead’s theorem [Whi36] there is a marked rose $R$ such that the Whitehead graph for $\{c_0, \ldots, c_k\}$ is connected and has no cut vertices—this is the graph with a vertex $v_e$ for every directed edge $e$ in $R$ and an edge $v_e-v_{e'}$ whenever $ee'$ is a subpath of the circuit in $R$ representing one of $c_0, \ldots, c_k$ or their inverses. Let $\tau \subset R$ be a closed path determining the circuit in $R$ that corresponds to $c_0$. It follows that if $\sigma' \subset R$ is any circuit that contains $\tau^2$ as a subpath then the Whitehead graph for $\{[\sigma'], c_1, \ldots, c_k\}$ contains the Whitehead graph for $\{c_0, \ldots, c_k\}$ and so is also connected and has no cut points. A second application of Whitehead’s theorem shows that $\{[\sigma'], c_1, \ldots, c_k\}$ is filling. To complete the proof, apply the bounded cancellation lemma and choose $M$ so large that if $\sigma \subset G$ contains a subpath of the form $\rho_s^M$ then the realization of $[\sigma]$ in $R$ contains a subpath of the form $\tau^2$. \qed

Corollary 5.9. For any circuit $\sigma \subset G$ representing an intermediate conjugacy class $c$ the following hold:

1. If $\sigma$ has a weak $k$-tiling then some term of that tiling is a $k$-tile.

2. For all $k \geq 0$ there exists $L \geq 1$ so that if $\sigma$ has a weak $k$-tiling and if $\sigma'$ is a subpath of $\sigma$ that crosses $L$ edges of $H_s$ then $\sigma'$ contains a $k$-tile.

Proof. We prove (1) by contradiction, assuming that no term (of the weak $k$-tiling of $\sigma$) is a $k$-tile. If $\rho_s$ does not exist, or if it exists and no term is a copy of $\rho_s$ or $\bar{\rho}_s$, then $\sigma$ is contained in $G_r$ and so $c$ is supported by $\mathcal{F} = [G_r]$, contradicting that $c$ is intermediate. We may therefore assume that $\rho_s$ exists and that at least one term is a copy of $\rho_s$ or $\bar{\rho}_s$. At least one endpoint $x$ of $\rho_s$ is contained in the interior of $H_s$. If $\rho_s$ is not closed then $\sigma$ has a term incident to $x$ that is not a copy of $\rho_s$ or $\bar{\rho}_s$, but that term must contain an edge of $H_s$ and so can only be a $k$-tile, contradicting the assumption. If $\rho_s$ is closed then by a similar argument $\sigma$ can only be an iterate of $\rho_s$ or $\bar{\rho}_s$, and so $c$ fills relative to $\mathcal{F}$ (see Notation B), also contradicting that $c$ is intermediate.

For item (2) suppose that $\sigma = \sigma_1 \ldots \sigma_m$ is a weak $k$-tiling. Let $L_0$ be the maximal number of $H_s$-edges in a $k$-tile. If $\rho_S$ exists let $L_s$ be the number of $H_s$ edges in $\rho_s$, otherwise $L_s = 0$. If $\Lambda^\pm$ is nongeometric and $\rho_s$ does not exist then the conclusion is evident with $L = 2L_0$. If $\Lambda^\pm$ is non-geometric and $\rho_s$ exists then $\rho_s$ has distinct
endpoints, so if $2 \leq i \leq m - 1$ and if $\sigma_i = \rho_s^\pm$ then either $\sigma_{i-1}$ or $\sigma_{i+1}$ is $k$-tile; the conclusion holds with $L = L_s + 2L_0$. If $\Lambda^\pm$ is geometric then $\rho_s$ exists and is closed and, letting $M$ be the constant of Lemma 5.8, at most $M$ consecutive terms of any cyclic permutation of the given weak $k$-tiling of $\sigma$ can be copies of $\rho_s$ or $\bar{\rho}_s$. In all cases the conclusion follows with $L = \max\{2, M\} \cdot L_s + 2L_0$. \hfill \Box

Proof of Proposition 5.7. We simplify notation slightly by writing $w$ and $\tau$ for the functions $w_\phi$ and $\tau_f$, suppressing the dependence on $\phi$ and $f$. Let $\Lambda^- \in \mathcal{L}(\phi^{-1})$ denote the dual repelling lamination of $\Lambda^+.$

Since $w(\phi^m(c)) = w(c) + m$ and $\tau(\phi^m(c)) = \tau(c) + m$ for all $m \in \mathbb{Z}$ it suffices to show that there are uniform upper and lower bounds to $\tau(c)$ as $c$ varies over all intermediate conjugacy classes with $w(c) = 0.$

An upper bound is easy to find, using properties of $\Lambda^+.$ Choose a generic leaf $\gamma^+ \subset G$. Since $U$ is a weak neighborhood of $\gamma^+$ there is subpath $\delta \subset G$ of $\gamma^+$ such that every line in $G$ that contains $\delta$ as a subpath is contained in $U$. By [BFH00, Lemma 3.1.1 (4)] there exists $k_3 > 0$ and a $k_3$-tile $\kappa$ that contains $\delta$ as a subpath. By [BFH00, Lemma 3.1.8 (4)] there exists $l > \max\{k_0, k_3\}$ such that every $l$-tile contains every $k_3$-tile as a subpath. It follows that every $l$-tile contains $\delta$ as a subpath. Corollary 5.9 (1) implies that any circuit that represents an intermediate $c$ and that has a weak $l$-tiling must contain an $l$-tile, and therefore that circuit determine a bi-infinite line contained in $U$. To put it another way, if $\tau(c) \geq l - k_0$ then $w(c) \geq 0$. It follows that $w(c) < 0 \implies \tau(c) < l - k_0$ and that $w(c) \leq 0 \implies \tau(c) \leq l - k_0$, and so $l - k_0$ is the desired upper bound to $\tau(c)$ when $w(c) = 0.$

It remains to find the lower bound, using properties of $\Lambda^-$. We show that there exists $d > 0$ so that if $w(c) = 0$ and if $\sigma \subset G$ is the circuit representing $c$ then $f^d_\#(\sigma)$ has a weak 0-tiling, for once that is done it follows that $-(d + k_0)$ is a lower bound for $\tau(c)$.

Let $\zeta$ be the line in $G$ that wraps bi-infinately around $\sigma$.

Claim (\ast): There exists $L > 0$, independent of $c$, so that no common subpath of $\zeta$ and a generic leaf of $\Lambda^- \subset G$ crosses $L$ edges of $H_s$.

The conclusion of Claim (\ast) implies that no common subpath of $\sigma$ and a generic leaf of $\Lambda^- \subset G$ crosses $L$ edges of $H_s$, which is equivalent to the inequality $\ell^\pm_\sigma(\sigma) < L$ in the hypothesis of Lemma 3.1, and so we may apply the conclusion of that lemma to obtain $d$, depending only on $L$ and so independent of $c$, such that $f^d_\#(\sigma)$ has a weak 0-tiling.

To prove Claim (\ast), let $g : G' \to G'$ be a CT representing $\phi^{-1}$ with top stratum $H'_s$, corresponding to $\Lambda^-$ and with core filtration element $G'_s \subset G'_{s-1}$ representing $\mathcal{F}$. Let $\gamma^-$ be a generic leaf for $\Lambda^-$ realized in $G'$. Let $\sigma' \subset G'$ be the circuit in $G'$ representing $c$, and let $\zeta'$ be the line in $G'$ wrapping bi-infinately around $\sigma'$. We shall show:

Claim (\ast\ast): There exists $L' > 0$, independent of $c$, so that no common subpath of $\zeta'$ and $\gamma^-$ in $G'$ crosses $L'$ edges of $H'_s$. 

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This suffices to prove Claim (\(*\)), for the following reasons. In \(\tilde{G}\) let \(\tilde{H}_s\) be the total lift of \(H_s\), and let \(\tilde{G}_{s-1} = \tilde{G} \setminus \tilde{H}_s\) be the total lift of \(G_{s-1}\); in \(\tilde{G}'\) define \(\tilde{H}'_r\) and \(\tilde{G}'_{s'-1}\) similarly. Let \(T\) be the simplicial \(F_n\)-tree obtained from the universal cover \(\tilde{G}\) by collapsing each component of \(\tilde{G}_{s-1}\) to a point, and let \(T'\) be similarly obtained from \(\tilde{G}'\) and \(\tilde{G}'_{s'-1}\). Using that \([G'_{s'-1}] = [G'_r] = [F_r] = [G_{s-1}]\) it follows that there are equivariant quasi-isometries between \(T\) and \(T'\) in either direction, and that these are equivariant coarse inverses to each other [GL07, Theorem 3.8]. For any lifts \(\tilde{\zeta}, \tilde{\gamma} \in \tilde{B}\), using subscripts to represent their realizations as lines in \(\tilde{G}\) and \(\tilde{G}'\) and the projections to lines in \(T\) and \(T'_r\) it follows that the number of \(\tilde{H}_s\) edges of \(\tilde{\zeta} \cap \tilde{\gamma}\) equals the edge path length of \(\tilde{\zeta}_T \cap \tilde{\gamma}_T\), which is quasi-comparable to the edge path length of \(\tilde{\zeta}'_T \cap \tilde{\gamma}'_T\), which equals the number of \(H'_s\) edges of \(\tilde{\zeta}'_G \cap \tilde{\gamma}'_G\). The existence of the uniform bound \(L'\) in Claim (\(*'\)) therefore implies the existence of the uniform bound \(L\) in Claim (\(*\)).

Choose an attracting neighborhood \(U^- \subset B\) of \(\gamma^-\) and let \(w' := w_{\phi^{-1}}U^-\) be the corresponding well function. By the coarse additive inverse property of well functions we have a uniform upper bound \(|w'(c) + w(c)| < W|\) independent of \(c\). The set \(\phi^{-W}(U^-) \subset B\) is evidently also an attracting neighborhood of \(\gamma^-\). Using an exhaustion by tiles of \(\gamma^-\), there exists an integer \(l \geq 1\) and an \(l\)-tile \(\delta\) such that every line having \(\delta\) as a subpath is contained in \(\phi^{-W}(U^-)\). Using the self-similarity property of tiles there exists an integer \(m > l\) such that every \(m\)-tile has \(\delta\) as a subpath. Since we are restricting our attention to those \(c\) with \(w(c) = 0\), we obtain a uniform upper bound \(w'(c) < W\). It follows that \(c \notin \phi^{-W}(U^-)\), and so \(\zeta'\) does not contain \(\delta\) as a subpath. We may therefore choose \(L'\) to be twice the maximal number of \(H'_s\)-edges in an \(m\)-tile: since \(\gamma^-\) has an \(m\)-tiling, every subpath of \(\gamma^-\) that contains \(L'\) edges of \(H'_s\) contains an \(m\)-tile and so also contains \(\delta\), and that subpath of \(\gamma^-\) is therefore not a subpath of \(\zeta'\). This completes the proof of Claim (\(*'\)), of Claim (\(*\)), and of the lemma.

\[\square\]

5.4 Regularity properties of weak tilings.

We continue to follow notations A and B.

A general circuit \(\sigma\) in \(G\) can have highly irregular behavior with respect to tiles and weak tilings. For example, choose \(\alpha\) to be a finite path in \(G\) having at least two distinct illegal turns, and choose \(j \geq 2\) so that each \(k\)-tile with \(k \geq j\) has more \(H_s\) edges than \(\alpha\) has. It follows that if \(k \geq j\) and if \(\sigma\) has a weak \(k\)-tiling then \(\sigma\) does not contain \(\alpha\) as a subpath. On the other hand, one can easily construct \(\sigma\) so as to contain \(\alpha\) as a subpath and to also contain some \(k\)-tile with arbitrarily large \(k\). This kind of irregularity — a large gap between the greatest \(j\) for which \(\sigma\) has a weak \(j\)-tiling and the greatest \(k\) for which \(\sigma\) contains a \(k\)-tile — is ruled out by the following corollary for those circuits representing intermediate conjugacy classes. As an application of the corollary, the circuit \(\sigma\) just constructed must fill rel \(F\) if \(k\) is sufficiently large.
Corollary 5.10. Following notations $A$ and $B$, and letting $k_0$ be as in Definition 5.4, there is a positive constant $M$ so that for all $k \geq k_0 + M$, every circuit $\sigma \subset G$ that realizes an intermediate conjugacy class and that contains a $k$-tile has a weak $k - M$ tiling.

Proof. If the corollary fails then there are sequences $k_i \geq k_0$ and $l_i \geq k_i$ such that $l_i - k_i \rightarrow +\infty$, and there are circuits $\sigma_i \subset G$ representing intermediate conjugacy classes, such that $\sigma_i$ contains an $l_i$-tile but has no weak $k_i$-tiling. Assuming this, we argue to a contradiction.

The main step in the proof is to reduce to the case that $k_i = k_0$ for all $i$. More precisely, we claim that there exist a sequence $l'_i \rightarrow +\infty$ and circuits $\alpha_i \subset G$ representing intermediate conjugacy classes such that $\alpha_i$ contains an $l'_i$-tile but has no weak $k_0$-tiling. To prove the claim, decompose $\sigma_i = \nu_i\nu'_i$ as a concatenation of two subpaths where $\nu_i$ is an $l_i$-tile. Let $y_i^-$ and $y_i^+$ be the initial and terminal endpoints of $\nu_i$ respectively and let $x_i^-$ and $x_i^+$ be the unique periodic points satisfying $f_{k_i-k_0}(x_i^-) = y_i^-$ and $f_{k_i-k_0}(x_i^+) = y_i^+$. There are unique paths $\mu_i$ connecting $x_i^-$ to $x_i^+$ and $\mu'_i$ connecting $x_i^-$ to $x_i^+$ such that $f_{k_i-k_0}^{k_i-k_0} = \nu_i$ and $f_{k_i-k_0}^{k_i-k_0} = \nu'_i$. As argued in the proof of Lemma 5.5, $\mu_i$ is an $(l_i + k_0 - k_i)$-tile; it follows that $\mu_i$ has an $l$ tiling for all $l = 1, \ldots, l_i + k_0 - k_i$ [BFH00, 3.1.8]. The circuit $\alpha_i$ obtained by tightening $\mu_i\mu'_i$ satisfies $f_{k_i-k_0}^{k_i-k_0} = \sigma_i$ and so has no weak $k_0$-tiling by Lemma 5.5 and by the assumption that $\sigma_i$ has no weak $k_i$-tiling. Since $\mu_i$ is $s$-legal and the initial [resp. terminal] directions of $f_{k_i-k_0}^{k_i-k_0} = \nu_i$ and $f_{k_i-k_0}^{k_i-k_0} = \nu'_i$ are distinct, there is a uniform upper bound to the number of $H_s$-edges in the maximal common initial [resp. terminal] subpath of $\mu_i$ and $\mu'_i$ [BFH00, Lemma 4.2.2]. Thus only finitely many edges of $H_s$ are cancelled when $\mu_i\mu'_i$ is tightened to $\alpha_i$. We may therefore choose $l''_i < l_i$ so that $l_i - l''_i$ is independent of $i$ and so that at least one of the terms in the $(l''_i + k_0 - k_i)$-tiling of the $(l_i + k_0 - k_i)$-tile $\mu_i$ remains after cancellation and so occurs in $\alpha_i$. This verifies the claim with $l'_i = l''_i + k_0 - k_i$.

Choose an attracting neighborhood $U$ associated to $\Lambda^+$ and let $w_\phi$ be the corresponding well function. From the claim we have $\tau_f[\alpha_i] < 0$. Proposition 5.7 therefore implies that $w_\phi[\alpha_i]$ has a uniform upper bound, and hence that there exists a positive integer $N$ so that for each $i$, $\alpha_i \not\in f_N(U)$. This contradicts the fact that $\Lambda^+$ is a weak limit of the $\alpha_i$’s which follows from the assumption that $l'_i \rightarrow \infty$. \hfill \Box

Corollary 5.11. Following notations $A$ and $B$, let $w_\phi$ be a well function for $\Lambda^+$ defined with respect to some attracting neighborhood of $\Lambda^+$, and let $k_0$ be as in Definition 5.4. Then there exists $K > 0$ so that for all $A \geq k_0$ there exist $L > 0$ so that if $\alpha, \beta$ are circuits in $G$ that represent intermediate conjugacy classes and that have a common subpath that crosses at least $L$ edges of $H_s$ and if $w_\phi(\alpha) \leq A$ then $w_\phi(\beta) \leq A + K$.

Proof. By Proposition 5.7 and Corollary 5.10 there exists an even integer $K$ so that so that the following are satisfied for all circuits $\alpha, \beta \subset G$ representing intermediate conjugacy classes and all $A \geq k_0$. 59
(1) If \( w_\phi([\alpha]) \leq A \) then \( \alpha \) does not contain any \( (A + K/2) \)-tiles.

(2) If \( w_\phi([\beta]) > A + K \) then \( \beta \) has a weak \( (A + K/2) \)-tiling.

It therefore suffices to show that there exists \( L > 0 \) so that if \( \beta \) has a weak \( (A + K/2) \)-tiling then every subpath of \( \beta \) that crosses at least \( L \) edges of \( H_s \) contains an \( (A + K/2) \)-tile. The existence of \( L \) follows from Lemma 5.9 (2).

\[ \square \]

6 Proof of Theorem E

After repeating the theorem for convenience, we shall then review some basic facts and notations regarding the free splitting complex \( \mathcal{FS}(F_n) \).

**Theorem E.** Suppose that \( n \geq 3 \), that \( \mathcal{H} \) is a subgroup of \( \text{IA}_n(\mathbb{Z}/3) \) that preserves a (possibly empty) proper free factor system \( \mathcal{F} \) and that \( \phi \in \mathcal{H} \) has the following properties:

Relative irreducibility: \( \phi \) is irreducible rel \( \mathcal{F} \);

Trivial restrictions: \( \phi \mid \text{Out}(A) \) is trivial for each component \([A] \in \mathcal{F}\);

Filling lamination: There exists a filling attracting lamination \( \Lambda_+^\phi \in \mathcal{L}(\phi) \), and hence \( \phi \) acts loxodromically on \( \mathcal{FS}(F_n) \) \cite{HM14b}.

Then \( \phi \) satisfies WWPD with respect to the action of \( \mathcal{H} \) on \( \mathcal{FS}(F_n) \).

Henceforth we fix \( \mathcal{F}, \phi, \) and \( \Lambda_+^\phi \in \mathcal{L}(\phi) \) as in the theorem and we let \( \Lambda^-_\phi \in \mathcal{L}(\phi^{-1}) \) be the dual repelling lamination of \( \mathcal{L}(\phi) \), and so \( \Lambda^-_\phi \) also fills. The existence of a filling lamination forces \( \mathcal{F} \) to have co-edge number \( \geq 2 \) (Fact 4.2 (3a)).

6.1 Free splitting complex and spine of relative outer space.

We first set some notation regarding the free splitting complex \( \mathcal{FS}(F_n) \). A free splitting is a minimal, simplicial action of \( F_n \rtimes T \) on a simplicial tree \( T \) having trivial edge stabilizers; usually we suppress the action from the notation and write simply \( T \) for a free splitting. The conjugacy classes of vertex stabilizers of \( T \) forms a free factor system \( \mathcal{F}(T) \). Two free splittings are equivalent if there is an \( F_n \)-equivariant homeomorphism \( S \mapsto T \), in which case \( \mathcal{F}(S) = \mathcal{F}(T) \). On each free splitting we have the natural simplicial structure whose vertices are the points of valence \( \geq 3 \), and we have the natural geodesic metric which assigns length 1 to each natural edge. Fix a set of representatives \( \mathcal{S} \) of equivalence classes of free splittings.

The free splitting complex \( \mathcal{FS}(F_n) \) has a \( k \)-simplex for each \( T \in \mathcal{S} \) such that the set of natural edges has \( k + 1 \) orbits under the action of \( F_n \). The simplex corresponding to \( S \in \mathcal{S} \) has a face corresponding to \( T \in \mathcal{S} \) if and only if there is a collapse map \( S \mapsto T \).
meaning an $F_n$-equivariant map whose point pre-images are connected. Alternatively, we may regard $\mathcal{S}$ as the vertex set of the first barycentric subdivision $\mathcal{FS}'(F_n)$; the 1-skeleton of $\mathcal{FS}'(F_n)$ has a directed edge from $S \in \mathcal{S}$ to $T \in \mathcal{S}$ if and only if there is a collapse map $S \mapsto T$; and $\mathcal{FS}'(F_n)$ is characterized as the directed simplicial complex generated by its 1-skeleton, with a $k$-simplex for every directed path of length $k$.

A map $f: S \to T$ between $S, T \in \mathcal{S}$ is a function which is simplicial with respect to some simplicial structures on $S$ and $T$ that refine the natural structures. Given $\Phi \in \text{Aut}(F_n)$, a map $f$ is $\Phi$-twisted equivariant if $f(\gamma \cdot x) = \Phi(\gamma) \cdot f(x)$; the case $\Phi = \text{Id}$ yields ordinary equivariance. If not otherwise specified, maps are assumed to be equivariant in the ordinary sense.

The canonical right action of $\text{Out}(F_n)$ on $\mathcal{S}$ is defined as follows: for each $T \in \mathcal{S}$ and $\theta \in \text{Out}(F_n)$ the image $T^\theta \in \mathcal{S}$ is well-defined up to equivalence by precomposing the given action $F_n \rhd T$ with an automorphism $\Theta \in \text{Aut}(F_n)$ representing $\theta$. This extends to a simplicial right action of $\text{Out}(F_n)$ on $\mathcal{FS}'(F_n)$, and to a continuous action on the Gromov bordification $\overline{\mathcal{FS}}(F_n) = \mathcal{FS}(F_n) \cup \partial \mathcal{FS}(F_n)$.

For readability, expressions of the right action $T^\theta$ will sometimes be rewritten $T \cdot \theta$.

Many relations between free splittings and conjugacy classes (of group elements or subgroups) satisfy an equivariance property with respect to each $\theta \in \text{Out}(F_n)$ acting from the right on free splittings and $\theta^{-1}$ acting from the left on conjugacy classes. For example, a conjugacy class $c$ is elliptic with respect to $S \in \mathcal{S}$ if and only if $\theta^{-1}(c)$ is elliptic with respect to $S^\theta$. The same format holds for relations between free splittings and lines, and the following example of this relation will play a key role in the proof of Theorem E:

**Theorem 6.1.** [HM14b, Theorem 1.2] The set of all $\tau \in \partial \mathcal{FS}(F_n)$ which are attracting or repelling points for elements of $\text{Out}(F_n)$ acting loxodromically on $\mathcal{FS}(F_n)$ corresponds bijectively with the set of filling laminations $\Lambda$ for elements of $\text{Out}(F_n)$. This correspondence is equivariant in the following manner: for all corresponding pairs $\tau \leftrightarrow \Lambda$ and all $\theta \in \text{Out}(F_n)$, $\tau^\theta \leftrightarrow \theta^{-1}(\Lambda)$ is also a corresponding pair.

Let $\mathcal{S}_F$ denote those $T \in \mathcal{S}$ such that $\mathcal{F}(T) = \mathcal{F}$. The flag subcomplex of $\mathcal{FS}'(F_n)$ spanned by $\mathcal{S}_F$ is denoted $\mathcal{K}_F$ and is called the spine of the outer space of $F_n$ rel $\mathcal{F}$; for example if $\mathcal{F} = \emptyset$ then $\mathcal{K}_F$ is the ordinary spine of the outer space of $F_n$. The action of any $\theta \in \text{Out}(F_n)$ takes $\mathcal{K}_F$ to $\mathcal{K}_{\theta^{-1}(\mathcal{F})}$, and hence we have an equation of stabilizer subgroups $\text{Stab}(\mathcal{F}) = \text{Stab}(\mathcal{K}_F)$. The complex $\mathcal{K}_F$ is contractible, being a restricted deformation space in the sense of [GL07, Theorem 6.1]).

Given $S \in \mathcal{S}_F$ and a conjugacy class $c$ of $F_n$ which is not carried by $\mathcal{F}$ and hence is loxodromic on $S$, we say that $c$ is $S$-intermediate or that $c$ does not fill $S$ if there exists an $F_n$-orbit of edges which misses the axes of $c$. Note that an $S$-intermediate conjugacy class is intermediate in the sense of Definition 5.1. Note also that an $S$-intermediate conjugacy class exists for each $S \in \mathcal{S}_F$, as follows. Since $\mathcal{F}$ has co-edge number $\geq 2$, there exists a finite arc $\alpha \in S$ whose endpoints $v, w \in S$ have nontrivial stabilizer, such that $\alpha$ misses the $F_n$-orbit of some edge $E \subset S$. Since edges of $S$
have trivial stabilizers, one easily produces $\gamma \in F_n$ acting loxodromically on $S$ with axis equal to a union of translates of $\alpha$ which therefore misses the orbit of $E$; the conjugacy class of $\gamma$ is therefore $S$-intermediate.

The well function on the spine of relative outer space. Henceforth, as we did in Section 5, we shall fix attracting neighborhoods of $\Lambda^-$ and of $\Lambda^+$ with respect to which the well functions $w_{\phi^{-1}}$ and $w_{\phi}$ are defined.

Consider $S \in \mathcal{S}_F$ and a pair of $S$-intermediate conjugacy classes $c_1,c_2$, so for $i = 1,2$ the axis of $c_i$ in $S$ misses the orbit of the interior of some natural edge $E_i \subset S$. If $E_1, E_2$ have the same orbits then $c_1,c_2$ are supported by the same proper free factor system — namely, the conjugacy classes of stabilizers of components of $S \setminus (F_n \cdot E_1) = S \setminus (F_n \cdot E_2)$ — and so we may apply [HM14b, Lemma 4.14 (3)] to conclude that $|w_{\phi^{-1}}(c_1) - w_{\phi^{-1}}(c_2)|$ is uniformly bounded independent of $c_1,c_2$. If on the other hand $E_1, E_2$ have different orbits then there are one-edge free splittings $S_1,S_2$ connected by an edge such that $c_i$ is elliptic in $S_i$ — namely, $S_i$ is obtained from $S$ by collapsing to a point each component of $S \setminus F_n \cdot E_i$ — and so we may apply [HM14b, Lemma 4.20] to get the same conclusion. Define a well function

$$W_{\phi}(S) = w_{\phi^{-1}}(c)$$

for all $S \in \mathcal{S}_F$, by simply choosing $c$ to be any $S$-intermediate conjugacy class. Note that $W_{\phi}(S)$ is coarsely well-defined, meaning that changing the choice of $c$ changes the value of $W_{\phi}(S) \in \mathbb{Z}$ by at most a constant independent of $S$ and $c$. Note also that $W_{\phi}(S)$ is a Lipschitz function on the spine of relative outer space: if $S,T \in \mathcal{S}_F$ and if there is a collapse map $S \mapsto T$ then any $T$-intermediate conjugacy class $c$ is also $S$-intermediate and so $|W_{\phi}(S) - W_{\phi}(T)|$ is uniformly bounded. We record this as:

**Lemma 6.2.** $W_{\phi}(S) = w_{\phi^{-1}}(c): \mathcal{S}_F \rightarrow \mathbb{Z}$ is coarsely well-defined and Lipschitz.

**Remark 6.3.** Here we have defined $W_{\phi}(S)$ for free splittings $S \in \mathcal{S}_F$ where $\mathcal{F} = \mathcal{F}(S)$ is a multi-edge free splitting, whereas in [HM14b, Definition 4.15] we instead defined $W_{\phi}(T)$ for one-edge free splittings $T$. While the domains of these two well functions are at opposite extremes in some sense, they are nonetheless related: if there is a collapse map $S \mapsto T$ then $|W_{\phi}(T) - W_{\phi}(S)|$ is uniformly bounded. This is an easy consequence of [HM14b, Definition 4.15] and of the properties laid out in Section 5.1 regarding well functions on intermediate conjugacy classes, namely coarse well-definedness and the coarse additive inverse property; see also Remark 5.3. One could therefore unify the two definitions to obtain a coarsely well-defined, Lipschitz well function $W_{\phi}$ on the free splitting complex of $F_n$ relative to $\mathcal{F}$ in the sense of [HM14a].

If $c$ is $S$-intermediate and $\xi \in \mathcal{H}$ then $\xi^{-1}(c)$ is $S^{\xi}$-intermediate. Thus

$$W_{\phi}(S^{\xi}) = w_{\phi^{-1}}(\xi^{-1}(c))$$
for any $S$-intermediate $c$. In the special case that $\xi = \phi^m$ we have

$$W_\phi(S^{\phi^m}) = w_{\phi^{-1}}(\phi^{-m}(c)) = w_{\phi^{-1}}(c) + m = W_\phi(S) + m$$

(This accounts for our defining $W_\phi$ in terms of $w_{\phi^{-1}}$ and not $w_\phi$; if we had done the latter, we would have $W_\phi(S^{\phi^m}) = W_\phi(S) - m$.)

### 6.2 Setting up the proof of Theorem E.

We may assume without loss of generality that $\phi$ and $\phi^{-1}$ are rotationless. The action of $\phi$ on $FS(F_n)$ is loxodromic by [HM14b, Theorem 1.1].

For proving Theorem E by contradiction, we assume that the action of $\phi$ is not WWPD. From the equivalence of items (1) and (4) of Proposition 2.6, it follows that there exists a sequence $\theta_i \in H$ satisfying the following:

**Distinct Coset Property:** $\theta_i^{-1}\theta_j \notin \text{Stab}(\partial_+\phi)$ for $i \neq j$.

**Long Cylinder Property:** For each $S_0 \in S_F$ there exists $R > 0$ such that for any $K \geq 0$ there exists $I \geq 0$ such that if $i \geq I$ then

$$d(S_0 \cdot \phi^k\theta_i, S_0 \cdot \phi^k) < R \text{ for all } 0 \leq k \leq K.$$ Setting $k = 0$ and applying Lemma 6.2, after further increasing $R$ the following inequality also holds for all $i$:

$$|W_\phi(S_0 \cdot \theta_i)| < R$$

Therefore, for any $K \geq 0$ there exists $I \geq 0$ such that if $i \geq I$ then

$$|W_\phi(S_0 \cdot \phi^k\theta_i) - k| < R \text{ for all } 0 \leq k \leq K$$

The latter inequality uses that $W_\phi(S_0 \cdot \phi^k) = W_\phi(S_0) + k$.

**Henceforth** we fix an initial choice of the sequence $\Theta = (\theta_1, \theta_2, \ldots)$ satisfying the Distinct Coset Property and the Long Cylinder Property, although notice that these two properties each continue to hold after passing to an arbitrary subsequence of $\Theta$, which we will do as the proof proceeds. We also fix a choice of $S_0$ and a corresponding choice of $R$, although in later parts of the proof we may impose further constraints on the choice of $S_0$ (the Long Cylinder Property lets us change $S_0$ at the expense of increasing $R$).

**Case analysis and outline of the proof:** Let $C_\phi$ be the set whose elements are infinite sequences $(c_1, c_2, \ldots)$ of intermediate conjugacy classes such that for some $S \in S_F$ — and hence for every $S \in S_F$ (Lemma 6.5 (2)) — the length $L_S(c_i)$ of $c_i$
in $S$ is bounded independently of $i$, whereas the length $L_S(\theta_i^{-1}(c_i))$ is not bounded independently of $i$.

The proof of Theorem E will break into Case 1 where $\mathcal{C}_\Theta \neq \emptyset$, and Case 2 where $\mathcal{C}_\Theta = \emptyset$; the geometric meaning of this case analysis is explained in Lemma 6.4. Case 2 will break into two further subcases, based on an analysis of the images of a generic leaf of $\Lambda^-$ under the outer automorphisms $\theta_i^{-1}$.

In Cases 1 and 2(a), using neither the *Distinct Coset Property*, nor the hypothesis on triviality of the restrictions of $\phi$, after passing to a subsequence of $(\theta_1, \theta_2, \ldots)$ we will obtain contradictions to regularity properties of well functions that were derived in Section 5. Having eliminated cases 1 and 2(a) we will be able to pass to a further subsequence of $(\theta_1, \theta_2, \ldots)$ satisfying very strong properties. In the final case 2(b) those properties will be combined with the hypothesis on triviality of restrictions of $\phi$, allowing us to pass to one final subsequence for which the images $\theta_i^{-1}(\Lambda^-)$ of the repelling lamination $\Lambda^-$ are all the same. By applying results of [HM14b] it will follows that the images $\theta_i(\partial_\phi) \in \partial \mathcal{F}S(F_n)$ of the repelling fixed point are all the same, as are the images $\theta_i(\partial_+ \phi) \in \partial \mathcal{F}S(F_n)$ of the attracting fixed point, from which we will obtain a contradiction to the *Distinct Coset Property*.

For each $S \in \mathcal{S}_\mathcal{F}$, define $\mathcal{C}_\Theta(S)$ to be set of all sequences $(c_1, c_2, \ldots)$ in $\mathcal{C}_\Theta$ such that each $c_i$ is $S$-intermediate.

**Lemma 6.4.** The following are equivalent.

1. $\mathcal{C}_\Theta = \emptyset$.
2. $\mathcal{C}_\Theta(S) = \emptyset$ for some $S \in \mathcal{S}_\mathcal{F}$.
3. For all $S \in \mathcal{S}_\mathcal{F}$, the outer automorphisms $\theta_i^{-1}$ are represented by $\theta_i^{-1}$-twisted equivariant maps $\bar{h}_i : S \rightarrow S$ with uniformly bounded Lipschitz constants, uniformly bounded cancellation constants, and uniformly bounded quasi-isometry constants.

Underlying the proof of Lemma 6.4 are Lemmas 6.5 and 6.6. The first of these includes results of Forester and of Guirardel-Levitt, together with a uniformity clause.

**Lemma 6.5.** For all $S, T \in \mathcal{S}_\mathcal{F}$ with their natural geodesic metrics, we have:

1. [For02, Theorem 1.1] There exists an $F_n$-equivariant quasi-isometry $S \mapsto T$.
2. [GL07, Theorem 3.8 (7)] There exists $k \geq 1$ depending only on $S, T$ such that for all conjugacy classes $c$ in $F_n$ we have $\frac{1}{k} \cdot L_T(c) \leq L_S(c) \leq k \cdot L_T(c)$.
3. For every $\ell \geq 1$ there exists $\ell' \geq 1, c' \geq 0$ such that for every $\Phi \in \text{Aut}(F_n)$, each $\ell$-Lipschitz $\Phi$-twisted equivariant map $f : S \mapsto T$ is an $(\ell', c')$-quasi-isometry.
Proof. The twisted version of (3) follows from the untwisted version by precomposing the action homomorphism $F_n \to \text{Aut}(S)$ with an appropriate automorphism. To prove the untwisted version, the given map $f$ factors as a product of an initial collapse map—which collapses to a point each edge whose $f$-image is a vertex of $T$—followed by a Stallings fold sequence. Since $S$ and $T$ have their natural geodesic metrics, the number of folds in the sequence is bounded by a constant depending only on the Lipschitz constant $k$ and the rank $n$. It therefore suffices to consider separately the cases that $f$ is either a collapse map or a single fold, because a composition of sequence of quasi-isometries has QI constants depending only on the sequence of QI constants of the factors of the composition.

If $f: S \to T$ is a collapse map, let $\sigma \subset S$ be the union of collapsed natural edges. Each component of $\sigma$ contains at most one natural vertex in each orbit, because $S, T$ have the same vertex stabilizers. The diameters of the components of $\sigma$, in the natural metric on $S$, are therefore uniformly bounded by a constant depending only on the rank $n$. Given $x, y \in S$, the segment $xy$ decomposes into an alternating concatenation of segments of uncollapsed edges and collapsed segments of length at most $D$, and so $d_T(f(x), f(y)) \geq \frac{1}{1+D}d(x, y) - 2D$; since $f$ is 1-Lipschitz, this gives uniform quasi-isometry constants.

If $f: S \to T$ is a fold map, then there are oriented natural edges $E_0, E_1$ with the same initial vertex $v$ and initial segments $e_i \subset E_i$ with terminal points $w_i$ such that $f$ is a quotient map that identifies $\gamma \cdot e_0$ and $\gamma \cdot e_1$ for all $\gamma \in \gamma$. The points $w_0, w_1$ are in different orbits and at most one has nontrivial stabilizer, because $S, T$ have the same vertex stabilizers. It follows that if $x, y \in S$ satisfy $f(x) = f(y) \in T$ then the segment $xy$ has length at most 4. We may assume that if $e_i \neq E_i$ then $e_i$ has length $\frac{1}{2}$ and so on each edge of $S$ the map $f$ stretches length by a factor between 1 and 2. It follows that $f$ has uniform quasi-isometry constants independent of rank.

Suppose that $G$ is a marked graph and that $H$ is a subgraph whose non-contractible components represent $\mathcal{F}$. In this context we refer to the edges of $G \setminus H$ as top edges; since $\mathcal{F}$ has co-edge number $\geq 2$, there are at least two top edges. We obtain a free splitting $S \in S_{\mathcal{F}}$ from the universal cover $\tilde{G}$ by collapsing to a point each component of the full pre-image $\tilde{H}$ of $H$. We say that $S$ is determined by or represented by the graph pair $(G, H)$. (In [HM14b] we required that $H$ be a natural subgraph of $G$ and have only non-contractible components. In the current context it is more useful to drop these requirements.) Every $S \in S_{\mathcal{F}}$ is represented by at least one graph pair $(G, H)$. If $S$ is determined by $(G, H)$ then a conjugacy class is $S$-intermediate if and only its representative in $G$ crosses some but not all top edges.

Let $S_F \subset S_{\mathcal{F}}$ be the set of those $S \in S_{\mathcal{F}}$ such that one of two possibilities holds: either every vertex of $S$ has non-trivial stabilizer; or all vertex stabilizers of $S$ are trivial and there is only one orbit of vertices. Equivalently $S$ is realized by a graph pair $(G, H)$ such that: either $H$ is nonempty contains all the vertices of $G$ and has no contractible components; or $H$ is empty and $G$ is a rose. The first possibility applies when $\mathcal{F}$ is nonempty, the second when $\mathcal{F}$ is empty.
Lemma 6.6. For each $S \in S_F^*$, for each graph pair $(G, H)$ representing $S$ as above, and for each $\theta \in \mathcal{H}$, there exists a homotopy equivalence $h : (G, H) \to (G, H)$ representing $\theta^{-1}$ so that for each top edge $E$ there is a closed path $\tau_E$ forming a circuit and satisfying the following properties.

1. $\tau_E$ crosses $E$ but not all top edges.
2. No top edges are cancelled when $h_\gamma(\tau_E)$ is tightened to $(h_\gamma)_\#(\tau_E)$.
3. Each $\tau_E$ crosses at most two top edges, counted with multiplicity.

Proof. As a first case, assume that $H$ contains every vertex in $G$ and that each component of $H$ is non-contractible. Choose a homotopy equivalence $h : (G, H) \to (G, H)$ representing $\theta$ that restricts to an immersion on each edge and preserves each component of $H$. Let $C_1$ and $C_2$ be the components of $H$ that contain the initial and terminal endpoints of $E$ respectively and let $\mu \subset C_1$ and $\nu \subset C_2$ be the maximal initial and terminal subpaths of $(h)_\#(E)$ that are contained in $H$. If $C_1 \neq C_2$ then we choose $\tau_E = E\beta_2 E\beta_1$ where $\beta_1 \subset C_1$ and $\beta_2 \subset C_2$ are any closed paths. Top edges are not cancelled when $h(E\beta_2 E\beta_1)$ is tightened to $(h)_\#(E\beta_2 E\beta_1)$ because $\nu(h)_\#(\beta_2) \bar{\mu}$ and $\bar{\mu}(h)_\#(\beta_1) \mu$ do not tighten to trivial paths. If $C_1 = C_2$ then we choose $\tau_E = E\beta_1$ where $\beta_1 \subset C_1$ is a closed path such that $\nu(h)_\#(\beta_1) \bar{\mu}$ does not tighten to the trivial path. This completes the proof of (2); (1) and (3) are clear from the construction because $G \setminus H$ has at least two edges.

Suppose now that $H = \emptyset$ and that $G$ is a rose. Choose a homotopy equivalence $h : G \to G$ representing $\theta$ that restricts to an immersion on each edge, that fixes the unique vertex and that has at least two gates at that vertex. The easy case is that for each edge $E$, either the ends of $E$ are contained in distinct gates or the ends of $E$ are contained in a single gate and there is an edge $E'$ with neither end in that gate. In the former case we take $\tau_{E_1} = E_1$ and in the latter $\tau_{E_1} = E_1 E'$.

The hard case is that there is a gate $\alpha$ such that every edge has at least one end in $\alpha$ and some edge has both ends in $\alpha$. We induct on the sum of the lengths $|h(E)|$ of the paths $h(E)$ over all edges $E$ that have both ends in $\alpha$. Enumerate and orient the edges of $G$ as

$$E_1, \ldots, E_K, E_{K+1}, \ldots, E_L, \quad 1 \leq K < L$$

so that $E_1, \ldots, E_K$ have both ends in $\alpha$ and $E_{K+1}, \ldots, E_L$ have only their initial end in $\alpha$.

Let $\beta$ be the complement of $\alpha$, consisting of the terminal ends of $E_{K+1}, \ldots, E_L$. There is an oriented edge $\eta$ so that the edge path $h(E_j)$ begins with $\eta$ [resp. ends with $\bar{\eta}$] if and only $E_j$ has initial end in $\alpha$ [resp. terminal end in $\alpha$]. Post-compose $h$ with a map that drags the unique vertex of $G$ across $\eta$, producing a new map $h' : G \to G$ still representing $\theta^{-1}$. Then the edge path $h'(E_j)$ is obtained from $h(E_j)$ by first removing the initial $\eta$ and then either removing the terminal $\bar{\eta}$ if $j \leq K$ or adding $\eta$ to the terminal end if $j > K$.
Replace \( h : G \to G \) by \( h' : G' \to G' \). If we are now in one of the easy cases then we are done. Otherwise there is a gate \( \alpha' \) such that every edge has at least one end in \( \alpha' \) and some edge has both ends in \( \alpha' \). All ends in the set \( \beta \) (the terminal ends of \( E_{K+1}, \ldots, E_L \)) are mapped by \( h' \) to the direction \( \tilde{\eta} \), and no other end is mapped by \( h' \) to \( \tilde{\eta} \), so \( \beta \) is a single gate of \( h' \) and no edge has both ends in the gate \( \beta \). In particular, \( \beta \neq \alpha' \). Furthermore, \( \alpha' \) must contain all the initial ends of \( E_{K+1} \ldots E_L \). So none of \( E_{K+1} \ldots, E_L \) have both ends in \( \alpha' \). Thus, the set of edges with both ends in \( \alpha' \) is a subset of \( E_1, \ldots, E_{K} \). Since \( |h'(E_k)| = |h(E_k)| - 2 \) for all \( 1 \leq k \leq K \), we are done by induction. \( \square \)

**Proof of Lemma 6.4.** (3) \( \implies \) (1) \( \implies \) (2) is obvious so it suffices to show that (2) \( \implies \) (3). Suppose then that \( \mathcal{C}_\Theta(S) = \emptyset \). Once we produce the maps \( \bar{h}_i \) with uniformly bounded Lipschitz constants, the uniform bounds on cancellation constants and quasi-isometry constants follow from [BFH97, Lemma 3.1] and Lemma 6.5 respectively.

Choose \( S' \in S^*_\mathcal{F} \) that is obtained from \( S \) by collapsing a forest. If a conjugacy class is \( S'-\)intermediate then it is \( S\)-intermediate. Thus \( \mathcal{C}_\Theta(S') = \emptyset \). Since the equivariant collapse map \( S \implies S' \) is 1-Lipschitz, it is a quasi-isometry with uniform constants by Lemma 6.5 and hence has an equivariant coarse inverse \( S' \implies S \) with uniform constants. It therefore suffices to verify (3) for \( S' \).

Let \( (G, H) \) be a graph pair representing \( S' \) as in the definition of \( S^*_\mathcal{F} \). For each \( \theta_i \) in the sequence \( \Theta \) and for each top edge \( E \) of \( G \setminus H \), let \( h_i : (G, H) \to (G, H) \) be the homotopy equivalence representing \( \theta_i^{-1} \) and \( \tau_{E,i} \subset G \) the circuit obtained by applying Lemma 6.6. Lifting \( h_i \) to the universal cover and collapsing the total lift of \( H \), the map \( h_i \) induces a \( \theta_i^{-1}\)-twisted equivariant map \( \bar{h}_i : S' \to S' \). The conjugacy class \( c_i \) determined by \( \tau_{E,i} \) is \( S'\)-intermediate and satisfies \( L_{S'}(c_i) \leq 2 \). Since \( (c_1, c_2, \ldots) \) is not an element of \( \mathcal{C}_\Theta(S') \) it must be that \( L_{S'}(\theta_i^{-1}c_i) = L_{S'}([h_i]_{\#}(\tau_{E,i})) \) is uniformly bounded. Lemma 6.6 (2) implies that there is a uniform bound to the number of top edges crossed by \( h_i(E) \) for each top edge \( E \). Equivalently, the Lipschitz constants are uniformly bounded. \( \square \)

### 6.3 The case analysis of Theorem E.

We begin with:

**Case 1:** \( \mathcal{C}_\Theta \neq \emptyset \)

It is convenient in this case to choose \( S_0 \in S^*_\mathcal{F} \). Let \( (G_0, H_0) \) be a graph pair representing \( S_0 \). For each \( \theta_i \) in the sequence \( \Theta \) and for each top edge \( E \) of \( G_0 \setminus H_0 \), let \( h_i : (G_0, H_0) \to (G_0, H_0) \) be the homotopy equivalence representing \( \theta_i^{-1} \) and \( \tau_{E,i} \subset G_0 \) the circuit obtained by applying Lemma 6.6.

**Lemma 6.7.** Suppose that \( (c_1, c_2, \ldots) \) is an element of \( \mathcal{C}_\Theta \). Then there are arbitrarily large \( i \) and arbitrarily large \( M \) and a top edge \( E \) of \( G_0 \) such that \( h_i(E) \) and the circuit representing \( \theta_i^{-1}(c_i) \) have a common subpath that crosses \( M \) top edges.
Proof. Decompose the circuit \( \sigma_i \subset G_0 \) representing \( c_i \) into a concatenation of a uniformly bounded number of subpaths, each of which is either contained in \( H \) or is a top edge. The circuit \((h_i)\#(\sigma_i)\) representing \( \theta_i^{-1}(c_i) \) therefore decomposes into a concatenation of a uniformly bounded number of subpaths, each of which is either contained in \( H \) or contained in the \( h_i \)-image of some top edge. Since \((c_1,c_2\ldots) \in \mathcal{C}_0\), the number of top edges crossed by the latter subpaths is unbounded.

For each \( m \geq 0 \), let \( S_m = S_0^{m} \). By Lemma 6.4 we can choose an element \((c_1,c_2,\ldots) \in \mathcal{C}(S_m)\). From the fact that \( c_i \) is \( S_m \)-intermediate, it follows that \( \theta_i^{-1}(c_i) \) is \( S_m^\theta \)-intermediate. Thus

\[
W_\phi(S_m^\theta) = w_{\phi^{-1}}(\theta_i^{-1}(c_i))
\]

for all \( i \). Similarly, Lemma 6.6(1) implies that

\[
W_\phi(S_0^\theta) = w_{\phi^{-1}}((h_i)\#(\tau_{E,i}))
\]

where \([((h_i)\#(\tau_{E,i}))] \) is the conjugacy class of the circuit formed by \((h_i)\#(\tau_{E,i})\). By Lemma 6.7, we can choose \( i \) and a top edge \( E \) of \( G_0 \) so that the circuit representing \( \theta_i^{-1}(c_i) \) in \( G_0 \) contains a subpath of \( h_i(E) \) that crosses at least \( m \) top edges. Lemma 6.6(2) then implies that the circuit representing \( \theta_i^{-1}(c_i) \) in \( G_0 \) and the circuit \((h_i)\#(\tau_{E,i})\) have a common subpath that crosses at least \( m \) top edges. Since there is a uniform bound for \( w_{\phi^{-1}}([h_i\#(\tau_{E,i})]) = W_\phi(S_0^\theta) \), Corollary 5.11 implies that there is a uniform bound for \( w_{\phi^{-1}}(\theta_i^{-1}(c_i)) = W_\phi(S_m^\theta) \). This contradicts the assumption that \(|W_\phi(S_m^\theta) - m| < \mathcal{R}\) for all sufficiently large \( i \) and so completes the proof in case 1.

Case 2: \( \mathcal{C} = \emptyset \)

Let \( f : G \to G \) be a CT representing \( \phi^{-1} \) with top stratum \( G_s \) and with \( \mathcal{F} \) realized by a core filtration element \( G_r \). Let \( H_z \) be the union of \( H_s \) with the zero strata, if any, that it envelops [FH11, Definition 2.18] and let \( G_u = G - H_z \). Each zero stratum enveloped by \( H_s \) is a contractible component of \( G_{s-1} \). Since \( \phi \mid [G_r] \) is trivial and \( \phi \) is irreducible rel \([G_r] \subset [G_s], f \mid G_r \) is the identity and \( G_u \) deformation retracts to \( G_r \). The edges of \( G_u - G_r \) are non-fixed NEG and so are linear.

We now constrain \( S_0 \in \mathcal{S} \) to be the free splitting determined by the graph pair \((G,G_{s-1})\). By Lemma 6.4, there exist \( \theta_i^{-1} \) twisted equivariant maps \( \bar{h}_i : S \to S \) with uniformly bounded Lipschitz constants, cancellation constants, and quasi-isometry constants. Letting \( h_i : (G,G_{s-1}) \to (G,G_{s-1}) \) be homotopy equivalences corresponding to \( \bar{h}_i : S \to S \), it follows that:

1. For each top edge \( E_j \) the number of top edges in \( h_i(E_j) \) is uniformly bounded.
2. There is a uniform constant \( C \) so that if \( \sigma = \sigma_1\sigma_2 \subset G \) is a decomposition into subpaths, then for all \( i \) at most \( C \) pairs of top edges are cancelled when \((h_i)\#(\sigma_1)(h_i)\#(\sigma_2)\) is tightened to \((h_i)\#(\sigma)\).
(3) For all $L$ there exists $L'$ independent of $i$ so that if $\sigma \subset G$ crosses $\geq L'$ top edges then $(h_i)\#(\sigma)$ crosses $\geq L$ top edges.

(4) For all $L'$ there exists $L$ independent of $i$ so that for any path $\sigma \subset G$, if $(h_i)\#(\sigma)$ crosses $\geq L$ top edges then $\sigma$ crosses $\geq L'$ top edges.

As a consequence of (2) we have the following consequence for the path maps $(h_i)^\#_0$ (see Section 3.1 or [HM17c, Section 1.1.6] for a more comprehensive review):

(5) For all $i$ and all paths $\sigma \subset G$, $(h_i)\#_0(\sigma)$ contains the path obtained from $(h_i)\#(\sigma)$ by removing the maximal initial and terminal subpaths with exactly $C$ top edges [HM17c, Lemma 1.6].

After passing to a subsequence of the $\theta_i$'s (and the corresponding subsequences of $n_i$'s and $h_i$'s), we may assume by (1):

(6) For each top edge $E_j$ the sequence of top edges crossed by $h_i(E_j)$ is independent of $i$.

Given a top edge $E_j$, write $h_i(E_j) = \alpha_0\epsilon_1\alpha_1 \ldots \epsilon_k\alpha_k$ where $\epsilon_1, \ldots, \epsilon_k$ are the top edges that are crossed. After passing to a further subsequence of the $\theta_i$'s we may assume that for all $0 \leq l \leq k$, the subpath $\alpha_i$ is either always trivial or never trivial, independent of $i$. We may therefore subdivide $E_j$ into subpaths called edgelets and modify $h_i$, without changing $h_i(E_j)$, so that:

(7) For all $i$ and $j$, $h_i$ maps each edgelet in $E_j$ to either a top edge or to a non-trivial path in $G_{s-1}$.

Recall that $f : G \to G$ represents $\phi^{-1}$, not $\phi$, and that tiles are defined with respect to $f$. As $k$ goes to infinity, the number of top edges in a $k$-tile goes to infinity. We may therefore choose a fixed integer $K \geq 1$ so that for each $i, j$, the $K$-tile $\omega_j = f^K_\#(E_j)$ has the property that its $(h_i)\#-$image $\sigma_{ij} := (h_i)\#(\omega_j)$ crosses at least $C$ top edges.

There may be some cancellation of top edges when $h_i(\omega_j)$ is tightened to $\sigma_j$. After passing to a further subsequence of the $\theta_i$'s, we may assume that this cancellation is independent of $i$; in particular, for each $K$-tile $\omega_j$, the sequence of top edges crossed by $\sigma_{ij}$ is independent of $i$. Decompose $\sigma_{ij}$ into subpaths $\sigma_{ij} = \sigma^-_{ij}\sigma^+_{ij}$ where $\sigma^-_{ij}$ is the shortest initial subpath of $\sigma_{ij}$ that completely contains $C$ top edges and that terminates at the midpoint of a top edge. Note that the endpoints of $\sigma^-_{ij}$ are independent of $i$. For each $K$-tile $\omega_j$ we define its prefix–suffix decomposition $\omega_j = \omega^-_j\omega^+_j$ where $\omega^-_j$ is the shortest initial subpath satisfying $(h_i)\#(\omega^-_j) = \sigma^-_{ij}$. The common endpoint of $\omega^-_j$ and $\omega^+_j$, which we refer to as the midpoint of $\omega_j$, is contained in the interior of an edgelet and maps to the common endpoint of $\sigma^-_{ij}$ and $\sigma^+_{ij}$. Since the cancellation of top edges when $h_i(\omega_j)$ is tightened to $\sigma_j$ is independent of $i$ we have:

(8) For each $K$-tile $\omega_j$, its prefix–suffix decomposition $\omega_j = \omega^-_j\omega^+_j$, its midpoint and the $h_i$-image of its midpoint are independent of $i$. 69
A generic leaf $\gamma^-$ of $\Lambda^\phi_-$ has a 0-tiling; i.e. a splitting whose terms are either edges in $H_s$ or paths in $G_{s-1}$. After increasing $K$ if necessary, we may assume that $f^K(H_j) \subset G_u$ for each zero stratum $H_j$ enveloped by $H_s$. Applying $f^K$ to the above 0-splitting, we have a splitting of $\gamma^-$ whose terms are either $K$-tiles or paths in $G_u$. We may therefore write $\gamma^-$ as an alternating concatenation of the form

$$\gamma^- = \cdots \omega_{j_{t-1}} \rho_t \omega_{j_t} \rho_{t+1} \omega_{j_{t+1}} \cdots$$

where each $\omega_j$ is a $K$-tile and each $\rho_t$ is either trivial or contained in $G_u$. After taking the prefix–suffix decomposition of each $K$-tile and then collecting terms we obtain a new decomposition of $\gamma^-$ as follows:

\[
\begin{align*}
\gamma^- &= \cdots \omega_{j_{t-1}}^+ \omega_{j_{t-1}}^- \rho_t \omega_{j_t}^+ \omega_{j_t}^- \rho_{t+1} \omega_{j_{t+1}}^+ \omega_{j_{t+1}}^- \cdots \\
&= \cdots \mu_{t-1} \mu_t \mu_{t+1} \mu_{t+2} \cdots
\end{align*}
\]

Our choice of $C$ guarantees that the above decomposition is a “universal 1-splitting” in the sense that

\[
(h_i)(\gamma^-) = \ldots (h_i)_{\#}(\mu_{t-1}) (h_i)_{\#}(\mu_0) (h_i)_{\#}(\mu_t) \ldots
\]

is a decomposition into subpaths for all $h_i$. By (8),

(9) For each $\mu_t$, the $h_i$-image of the endpoints of $\mu_t$ are independent of $i$.

The proof now divides into two subcases.

Case 2a: There exists $p < q$ so that for infinitely many values of $i$, the path $(h_i)_{\#}(\mu_p \ldots \mu_q)$ is not a subpath of $\gamma^-$. 

After passing to a subsequence of the $\theta_i$’s we may assume that the assumption of Case 2a holds for all values of $i$. Choose $p' < p$ and $q' > q$ so that $(h_i)_{\#}(\mu_{p'} \ldots \mu_{p-1})$ and $(h_i)_{\#}(\mu_{q+1} \ldots \mu_{q'})$ cross at least $C$ top edges. By (5)

\[
(h_i)_{\#}(\mu_{p'} \ldots \mu_{q'}) \supset (h_i)_{\#}(\mu_p \ldots \mu_q)
\]

for all $i$. By [BFH00, Lemma 3.1.8] there exists $K'$ such that

\[
f^K'_{\#}(E_j) \supset \mu_{p'} \ldots \mu_{q'}
\]

for each $j$. From [HM17c, Lemma 1.6-(3)] it therefore follows that

\[
(h_i)_{\#}(f_{\#}^{K'}(E_j)) \supset (h_i)_{\#}(\mu_{p'} \ldots \mu_{q'})
\]

We conclude that:
There is a sequence $L$ is sufficiently large are therefore reduced to bounding the number of edges in a maximal subpath.

Case 2b: For any path $\sigma_0 \subset G$ that crosses some but not all edges of $G - G_{s-1}$. Then the circuit

$$\beta_{i,m} := (h_i)_{\#} f_m^\#(\alpha) \subset G$$

determines a conjugacy class in $F_n$ that is represented by a circuit in $G \cdot \phi^m \theta_i$ which crosses some but not all top edges of $(G, G_{s-1}) \cdot \phi^m \theta_i$. It follows that

$$w_{\phi^{-1}}[\beta_{i,m}] = W(S_0 \cdot \phi^m \theta_i)$$

and hence (after further increasing $R$) for each $m$ we have $|w_{\phi^{-1}}[\beta_{i,m}] - m| < R$ if $i$ is sufficiently large. It follows that:

(##) There is a sequence $L_m \to +\infty$ such that for each $m$, if $i$ is sufficiently large then there exists a common subpath of $\gamma^-$ and $\beta_{i,m}$ that crosses $L_m$ top edges.

For reasons as above we have

$$w_{\phi^{-1}}[f_m^\#(\alpha)] = W(S_0^m)$$

Thus $w_{\phi^{-1}}[f_m^\#(\alpha)]$ goes to infinity with $m$. By Proposition 5.7, for all sufficiently large $m$, $f_m^\#(\alpha)$ has a weak $K'$-tiling. By Lemma 5.9 (2) there exists $L' > 0$ so that for all sufficiently large $m$, every subpath $\sigma \subset f_m^\#(\alpha)$ that contains at least $L'$ top edges contains a $K'$-tile. Choose $L > 0$ so that if $\sigma$ is any path in $G$ and if $\zeta$ is a subpath of $(h_i)_{\#}(\sigma)$ that crosses at least $L$ top edges then there is a subpath $\sigma_0$ of $\sigma$ that crosses at least $L'$ top edges such that $(h_i)_{\#}(\sigma_0)$ is a subpath of $\zeta$; this follows by applying item (4) above, with our current $\zeta$ in the role of the $\sigma$ of item (4). Thus any subpath of $\beta_{i,m}$ that crosses at least $L$ top edges contains a subpath of the form $(h_i)_{\#}(\sigma_0)$ where $\sigma_0$ is a subpath of $f_m^\#(\alpha)$ that contains at least $L'$ top edges, and hence $\sigma_0$ contains a $K'$-tile. Combining this with conclusion (##), it follows that no subpath of $\beta_{i,m}$ that crosses at least $L$ edges occurs as a subpath of $\gamma^-$. This contradicts (##), completing the proof in Case 2a.

Case 2b: For any $p < q$, the path $(h_i)_{\#}(\mu_p \ldots \mu_q)$ is a subpath of $\gamma^-$ for all sufficiently large $i$.

Our first claim is that

(*) For any given $p$, as $i$ varies, the expression $(h_i)_{\#}(\mu_p)$ takes on only finitely many values.

We can see this as follows. The number of top edges in $(h_i)_{\#}(\mu_p)$ is uniformly bounded so the sequence of top edges in $(h_i)_{\#}(\mu_p)$ takes on only finitely many values. We are therefore reduced to bounding the number of edges in a maximal subpath $\nu$ of $(h_i)_{\#}(\mu_p)$ in $G_{s-1}$. Since $\gamma^-$ is birecurrent, there exists $q > p$ such that $\mu_p = \mu_q$.
There is a copy $\nu'$ of $\nu$ in $(h_i)\#(\mu_p \ldots \mu_q)$ that is separated from $\nu$ in $(h_i)\#(\mu_p \ldots \mu_q)$ by a uniformly bounded number of top edges. Assuming without loss that $(h_i)\#(\mu_p \ldots \mu_q)$ is a subpath of $\gamma^-$, we have that $\nu$ and $\nu'$ are separated in $\gamma^-$ by a uniformly bounded number, say $P$, of top edges. Choose $k$ so that every $k$-tile contains at least $P + 1$ edges. Since $\gamma^-$ decomposes into $k$-tiles and subpaths in $G_{s-1}$, at least one of $\nu$ and $\nu'$ must be contained in a $k$-tile. This gives the uniform bound on the number of edges in $\nu$ and so completes the proof of the first claim.

**The topological structure of $\Lambda^{-}_{\phi}$.** The fact that $G_u$ is the union of a filtration element of fixed edges $G_r$ and some NEG-linear edges $G_u \setminus G_r$ implies that $\Lambda^{-}_{\phi}$ has a particularly simple topological structure. If there were no linear edges, then $\Lambda^{-}_{\phi}$ would be minimal, meaning every leaf would be dense. With only linear edges, the structure is still quite simple: for example, the only nondense leaves are the finitely many periodic lines corresponding to circuits around which the linear edges of $G_u \setminus G_r$ twist. We now describe a decomposition of generic leaves of $\Lambda^{-}_{\phi}$ which reflects this structure.

The generic leaf $\gamma^-$ decomposes as a concatenation of subpaths in $H_s$ and maximal subpaths in $G_{s-1}$. We further divide the latter into subpaths in zero strata enveloped by $H_s$ called *middle pieces* (referred to in [FH11] as ‘taken connecting paths’) and subpaths in $G_u$ called *bottom pieces*. There are only finitely many middle pieces and the $f_{\#}$-image of a middle piece is either another middle piece or a bottom piece. Each tile also divides into subpaths in $H$, bottom pieces and middle pieces. A bottom piece is *primitive* if it occurs in a 1-tile or is the $f_{\#}$-image of a middle piece. There are only finitely many primitive bottom pieces; we denote them with the symbol $\delta$. Every bottom piece of a $k$-tile can be written as $f_{\#}^j(\delta)$ for some primitive bottom piece $\delta$ and some $j \geq 0$. This is obvious for $k = 1$ and it follows by the obvious induction argument on $k$ using the fact that each $k$-tile is $s$-legal.

Recall that $f \mid G_r$ is the identity and that each edge of $G_u \setminus G_r$ is NEG-linear. Since the initial vertices of edges of $G_u - G_r$ have valence one in $G_u$, paths in $G_u$ can only cross edges of $G_u - G_r$ in their first or last edges. A primitive bottom piece $\delta$ therefore has one of three types:

**Constant Type:** $\delta$ is contained in $G_r$;

**Once-Linear Type:** Either the initial or terminal edge of $\delta$, but not both, is contained in $G_u - G_r$;

**Twice-Linear Type:** The initial and terminal edges of $\delta$ are contained in $G_u - G_r$.

If $\delta$ has constant type then $f_{\#}^j(\delta)$ is independent of $j$. If $\delta$ has once linear type then there is an edge $E$ of $G_u - G_r$, a closed path $u$ in $G_r$ such that $f_{\#}^j(E) = E u^j$, and a path $\beta$ in $G_r$, such that

$$\delta = E \beta, \quad \text{and} \quad f_{\#}^j(\delta) = E [u^j \beta]$$

or $\delta = \beta E$, and $f_{\#}^j(\delta) = [\beta u^j] E$.
If $\delta$ has twice linear type then there are edges $E, E'$ in $G_u - G_r$, closed paths $u, u'$ in $G_r$ such that $f^{j}_#(E) = Eu^j$ and $f^{j}_#(E') = Eu'^j$, and a path $\beta$ in $G_r$, such that

$$\delta = E, \beta, E'$$

and $f^{j}_#(\delta) = E[u^j \beta u'^j]E'$

Recall that for each term $\mu_t$ of the decomposition $\gamma^- = \cdots \mu_{t-1} \mu_t \mu_{t+1} \mu_{t+2} \cdots$ there is a decomposition $\mu_t = \omega^+_{j+1} \rho_t \omega_{j-1}^-$ where $\omega^+_{j+1}$ is a $K$-tile suffix, $\rho_t$ is either trivial or a maximal subpath in $G_u$, and $\omega^-_{j-1}$ is a $K$-tile prefix. Since every non-trivial subpath of $\gamma^-$ occurs in a tile, each non-trivial $\rho_t$ can be written as $f^{j}_#(\delta)$ for some primitive bottom piece $\delta$ and some $j \geq 0$.

To each $K$-tile suffix $\omega^+$, each primitive bottom piece $\delta$, and each $K$-tile prefix $\omega^-$ we associate the set of paths

$$\mathcal{M}(\omega^+, \delta, \omega^-) = \{ \mu_t \mid \omega^+ = \omega^+_{j+1}, \text{ and } f^{j}_#(\delta) = \rho_t \text{ for some } j \geq 0, \text{ and } \omega^- = \omega^-_{j-1} \}$$

Also, for each $K$-tile suffix $\omega^+$ and $K$-tile prefix $\omega^-$ we define

$$\mathcal{M}(\omega^+, \omega^-) = \begin{cases} \{\omega^+ \omega^-\} = \{\mu_t\} & \text{if } \omega^+ = \omega^+_{j+1} \text{ and } \omega^- = \omega^-_{j-1} \text{ for some } t \\ \emptyset & \text{otherwise} \end{cases}$$

We say that the set $\mathcal{M}(\omega^+, \delta, \omega^-)$ has constant, once-linear or twice-linear type if $\delta$ does. The set $\{\mu_t : t \in \mathbb{Z}\}$ is the (not necessarily disjoint) union of the sets $\mathcal{M}(\omega^+, \delta, \omega^-)$ and the sets $\mathcal{M}(\omega^+, \omega^-)$ as $\omega^+$, $\omega^-$, and $\delta$ vary. Note that there are finitely many of the sets $\mathcal{M}(\omega^+, \delta, \omega^-)$ and $\mathcal{M}(\omega^+, \omega^-)$. Note also that each set $\mathcal{M}(\omega^+, \omega^-)$ has finite cardinality whereas the sets $\mathcal{M}(\omega^+, \delta, \omega^-)$ may have infinite cardinality.

Our second claim is that:

(**) Each $\mathcal{M}(\omega^+, \delta, \omega^-)$ contains a finite subset $\mathcal{M}_0(\omega^+, \delta, \omega^-)$ so that for any $i$ and $j$, if $(h_i)_#(\mu_t) = (h_j)_#(\mu_t)$ for all $\mu_t \in \mathcal{M}_0(\omega^+, \delta, \omega^-)$ then $(h_i)_#(\mu_t) = (h_j)_#(\mu_t)$ for all $\mu_t \in \mathcal{M}(\omega^+, \delta, \omega^-)$.

Before proving this claim we use it to complete the proof of Theorem E, by arriving at a contradiction that settles Case 2b. From claim (**) it follows that for any finite subset $M \subset \{\mu_t : t \in \mathbb{Z}\}$ we can pass to a subsequence so that $(h_i)_#(\mu_t) = (h_j)_#(\mu_t)$ for all $i, j$ and for all $\mu_t \in M$. We may therefore assume that $(h_i)_#(\mu_t) = (h_j)_#(\mu_t)$ for all $i, j$ and for all $\mu_t$ in each $\mathcal{M}(\omega^+, \omega^-)$ and in each $\mathcal{M}_0(\omega^+, \delta, \omega^-)$. From claim (**) it then follows that $(h_i)_#(\mu_t) = (h_j)_#(\mu_t)$ for all $i, j$ and all $\mu_t$. This in turn implies that $(h_i)_#(\gamma^-) = (h_j)_#(\gamma^-)$ for all $i, j$, and hence taking weak closures that $\theta_i^{-1}(\Lambda^-) = \theta_j^{-1}(\Lambda^-)$. It follows by [HM14b, Theorem 1.2] that $\theta_i(\partial_+ \phi) = \theta_j(\partial_+ \phi)$ and that $\theta_i(\partial_+ \phi) = \theta_j(\partial_+ \phi)$, and so $\theta_i^{-1}\theta_j(\partial_+ \phi) = \partial_+ \phi$, in contradiction to the Distinct Coset Property saying that $\theta_i^{-1}\theta_j \notin \text{Stab}(\partial_+ \phi)$.

It remains to verify claim (***). This is obvious if $\mathcal{M}(\omega^+, \delta, \omega^-)$ is finite, in particular if it has constant type and therefore has cardinality at most one. For notational
simplicity we let \( M = \mathcal{M}(\omega^+, \delta, \omega^-) \), we assume \( M \) is infinite, and we denote the desired finite subset by \( M_0 = \mathcal{M}_0(\omega^+, \delta, \omega^-) \). There exists an infinite sequence of integers \( 0 \leq j_1 < j_2 < \ldots \) so that

\[
M = \{ \tau_l := \omega^+ f^{j_l}_\#(\delta) \omega^- \mid l \geq 1 \}
\]

For all \( l \geq 1 \), the paths \( \tau_l \) have the same initial vertex \( v_- \) and the same terminal vertex \( v_+ \). The images \( v'_l := h_i(v_-) \) and \( v'_l := h_i(v_+) \) are independent of \( i \) by (9). We include \( \tau_l = \omega^+ f^{j_l}_\#(\delta) \omega^- \) in \( M_0 \), and then applying (\*) we pass to a subsequence of the \( \theta_i \)'s so that \( h_i(\tau_1) \) is independent of \( i \).

For each \( l \geq 2 \) consider the closed edge path \( \tau_1 \bar{\tau}_l \) based at \( v_- \), with nontrivial path homotopy class denoted \( [\tau_1 \bar{\tau}_l] \in \pi_1(G, v_-) \). Let \( \gamma_l \) denote the closed path obtained by straightening \( \tau_1 \bar{\tau}_l \), and so \( \gamma_l \) is the unique closed path based at \( v_- \) that represents \( [\tau_1 \bar{\tau}_l] \). Let \( D = \{ [\tau_1 \bar{\tau}_l] \mid l \geq 2 \} \subset \pi_1(G, v_-) \), an infinite set of nontrivial elements of \( \pi_1(G, v_-) \).

Let \( \Upsilon_i : \pi_1(G, v_-) \to \pi_1(G, v'_l) \) be the isomorphism induced by \( (h_i)_\# \). Under the identification \( \pi_1(G, v_-) \approx F_n \), we obtain automorphisms

\[
\Upsilon_i^{-1} \Upsilon_1 \in \text{Aut}(\pi_1(G, v_-)) \approx \text{Aut}(F_n)
\]

Denote their fixed subgroups by

\[
B_i = \text{Fix}(\Upsilon_i^{-1} \Upsilon_1) = \text{Fix}(\Upsilon_i^{-1} \Upsilon_i) < F_n
\]

For each \( l \geq 2 \), it follows that the path \( (h_i)_\#(\tau_1) \) is independent of \( i \) if and only if \( \Upsilon_i [\tau_1 \bar{\tau}_l] \in \pi_1(G, v'_l) \) is independent of \( i \) if and only if \( [\tau_1 \bar{\tau}_l] \in B_i \) for all \( i \).

Each \( B_i \) has rank \( \leq n \) by the solution to the Scott conjecture [BH92, Theorem 6.1], and each \( B_i \) is primitive meaning that if \( B_i \) contains a nonzero power of an element then it contains that element.

It therefore suffices to show that there exists a finite subset \( D_0 \subset D \) such that for any subgroup \( B < F_n \) that is a primitive and has rank \( \leq n \), if \( D_0 \subset B \) then \( D \subset B \). Once that is shown then we finish claim (**) by defining \( M_0 = \{ \tau_l \mid [\tau_1 \bar{\tau}_l] \in D_0 \} \).

For the rest of the proof we fix \( B < F_n \) to be primitive and of rank \( \leq n \); we shall define the subset \( D_0 \subset D \) independent of \( B \).

If the set \( M \) is of once-linear type then either \( \delta = E \beta \) or \( \delta = \beta \bar{E} \), and \( f^{j_l}_\#(E) = E u^j \). The two cases are symmetric so we assume that \( \delta = E \beta \). Thus

\[
\tau_l = \omega^+ E [u^j / \beta] \omega^-
\]

\[
\gamma_l = (\omega^+ E) u^{j_l - j_i} (\omega^+ E)^{-1}
\]

It follows that as \( l \) varies, the \( [\tau_1 \bar{\tau}_l] \)'s are all contained in a cyclic subgroup of \( \pi_1(G, v_-) \). Any primitive subgroup that contains a single nontrivial element of this cyclic subgroup contains all of \( D \), so taking \( D_0 \) to be any nonempty, finite subset of \( D \) we are done in the once-linear case.
We are now reduced to the case that $M$ is of twice-linear type. In this case we have $\delta = E_\beta \overline{E}_\tau$, $f'_\#(E) = E\hat{u}$, and $f''_\#(E') = E'\hat{u}'$, and so

$$\tau_l = \omega_+ E [u^j \beta \overline{u}^j] \overline{E} \omega^-$$

$$\gamma_l = (\omega_+ E) [u^j \beta (u')^{j-j_1} \overline{\beta} u^{-j_1}] (\omega_+ E)^{-1}$$

If the number of edges cancelled when $u^j \beta (u')^{j-j_1} \overline{\beta} u^{-j_1}$ is tightened is not bounded independently of $l$ then there is a closed path $\alpha$ such that $u'$, $\beta$ and $u$ are all iterates of $\alpha$. The path $u^j \beta (u')^{j-j_1} \overline{\beta} u^{-j_1}$ therefore tightens to an iterate of $\alpha$ and so each $\gamma_l$ has the form $(\omega_+ E) \alpha^* (\omega_+ E)^{-1}$. Each $[\tau_{l \overline{n}}]$ is therefore contained in the cyclic subgroup of $\pi_1(G, v_-)$ generated by $(\omega_+ E) \alpha (\omega_+ E)^{-1}$, and the proof concludes as in the once linear case with $D_0$ any nonempty, finite subset of $D$.

It remains to consider the case that the cancellation of $u^j \beta (u')^{j-j_1} \overline{\beta} u^{-j_1}$ is uniformly bounded. In this case, there are paths $\sigma_1, \sigma_2, \sigma_3$, root-free closed paths $\alpha_1, \alpha_2$, an integer $l_0 \geq 2$, and increasing sequences of positive integers $\{s(l)\}$ and $\{t(l)\}$ defined for $l \geq l_0$, such that if $l \geq l_0$ then $\gamma_l$ can be written in the form

$$\gamma_l = \sigma_1 \alpha_1^{s(l)} \sigma_2 \alpha_2^{-t(l)} \sigma_3$$

and such that the following hold:

- $u'$ is an iterate of $\alpha_1$, and $u$ is an iterate of $\alpha_2$;
- The path $\sigma_1$ is obtained by straightening the concatenation of $\omega_+ E u^j \beta$ followed by some initial subpath of $(u')^*$;
- The path $\sigma_2$ is obtained by straightening the concatenation of some terminal subpath of $(u')^*$, followed by the path $\overline{\beta}$, followed by some initial subpath of $(u^{-1})^*$;
- The path $\sigma_3$ is obtained by straightening the concatenation of some terminal subpath of $(u^{-1})^*$ followed by $E \overline{\omega}^+$.
- The maximal common initial subpath of $\sigma_2$ and $\sigma_1$ is a proper initial subpath of $\alpha_1$; we denote this subpath $\sigma'_2$.

Let

$$D_0 = \{ [\tau_{l \overline{n}}] \mid 2 \leq l \leq l_0 + 2n - 1 \} \subset \pi_1(G, v_-)$$

Assuming that $D_0 \subset B$ we complete the proof by showing that the path $\sigma_1 \alpha_1^s \sigma_2 \alpha_2^t \sigma_3$ represents an element of $B$ for all $s, t \in \mathbb{Z}$.

Since $B < \pi_1(G, v_-)$ has rank $\leq n$, there is a finite core graph $X$ of rank $\leq n$ equipped with a basepoint $x$ and an immersion $Q : (X, x) \to (G, v_-)$ such that a closed path in $G$ based at $v_-$ lifts to a closed path in $X$ based at $x$ if and only if it represents an element of $B$. Equivalently, the induced homomorphism $Q_* : \pi_1(X, x) \to \pi_1(G, v_-)$
is an injection with image $B$. The path $\gamma_{l_0 + 2n - 1}$ representing $[a_{l_0} b_{l_0 + 2n - 1}] \in D_0$ lifts to a closed path $\tilde{\gamma}_{l_0 + 2n - 1}$ in $X$ based at $x$. We have the following forms for a decomposition of $\gamma_{l_0 + 2n - 1}$ and the corresponding decomposition of $\tilde{\gamma}_{l_0 + 2n - 1}$:

$$\gamma_{l_0 + 2n - 1} = \sigma_1 \alpha_1 \alpha_2 \ldots \alpha_s \sigma_2 \alpha_2^{-1} \alpha_3^{-1} \ldots \alpha_2^{-1} \sigma_3$$

$$s(l_0 + 2n - 1) \text{ times}$$

$$\tilde{\gamma}_{l_0 + 2n - 1} = \tilde{\sigma}_1 \tilde{\alpha}_1 \tilde{\alpha}_2 \ldots \tilde{\alpha}_1 s(l_0 + 2n - 1) \tilde{\sigma}_2 \tilde{\alpha}_2^{-1} \tilde{\alpha}_2 \ldots \tilde{\alpha}_2^{-1} \tilde{\sigma}_3$$

$$t(l_0 + 2n - 1) \text{ times}$$

In $X$ denote the following vertices, where $1 \leq i < s(l_0 + 2n - 1) - 1$:

$$y_0 = \text{the terminal vertex of } \tilde{\sigma}_1 = \text{the initial vertex of } \tilde{\alpha}_1$$

$$y_i = \text{the terminal vertex of } \tilde{\alpha}_1 = \text{the initial vertex of } \tilde{\alpha}_1$$

$$\tilde{\sigma}_2 = \text{the unique lift of } \sigma_2 \text{ with initial vertex } y_i$$

$$z_i = \text{the terminal vertex of } \tilde{\sigma}_2$$

Also, for $l_0 \leq l \leq l_0 + 2n - 2$, the initial and terminal vertices of $\tilde{\sigma}_2 s(l)$ are denoted $\eta_l = y_{s(l)}$ and let $\zeta_l = z_{s(l)}$.

The vertex $\zeta_{l_0}$ has valence $\geq 3$ in $X$, because both of the paths $\gamma_{l_0}$ and $\gamma_{l_0 + 1}$ lift to paths in $X$ starting at $x$; these two lifts have maximal common initial subpath $\tilde{\sigma}_1 \tilde{\alpha}_1 \ldots \tilde{\alpha}_1 s(l_0) \tilde{\sigma}_2 s(l_0)$; this subpath is proper in each of $\gamma_{l_0}$ and $\gamma_{l_0 + 1}$, and the terminal endpoint of this subpath is $\zeta_{l_0}$; this follows by maximality of $\sigma_2$. Similarly each of the vertices $\zeta_{l_0 + 1}, \ldots, \zeta_{l_0 + 2n - 2}$ has valence $\geq 3$ in $X$.

Since $X$ has at most $2n - 2$ distinct vertices of valence $\geq 3$, it must be that there is repetition amongst the vertices $\zeta_{l_0}, \ldots, \zeta_{l_0 + 2n - 2}$. Let $a < b$ be indices such that $\zeta_a = \zeta_b$. It follows that $y_{s(a)} = \eta_a = \eta_b = y_{s(b)}$. Since there is at most one lift of $\alpha_1$ to $X$ that terminates at each $y_i$, it must be that $y_l = y_{s(l)} - s(l_0)$. This implies that some iterate of $\alpha_1$ lifts to a closed path in $X$ based at $y_0$ and hence that some straightened iterate of $\sigma_1 \alpha_1 \tilde{\sigma}_1$ lifts to a closed path in $X$ based at $x$. Since $B$ is primitive, the path $\sigma_1 \alpha_1 \tilde{\sigma}_1$ itself lifts to a closed path based at $x$ and so determines an element of $B$.

The symmetric argument shows that $\tilde{\sigma}_3 \alpha_2 \sigma_3$ determines an element of $B$ and hence that each path

$$\sigma_1 \alpha_1^s \sigma_2 \alpha_2^t \sigma_3 = [(\sigma_1 \alpha_1^s \sigma_2 \alpha_2^t \sigma_3)(\sigma_1 \alpha_1^s \sigma_2 \alpha_2^t \sigma_3)]$$

determines an element of $B$ for each $s, t \in \mathbb{Z}$ as needed to complete the proof. \qed

7 Appendix: Proof of Theorem D

This proof hews closely to the proof of [BF02] Theorem 8. The one innovation is to use the Kaloujnin–Krasner embedding to avoid the wreath product hypothesis in the statement of [BF02] Theorem 8. This results in many slight technical differences from
[BF02], and so we have written the proof to be primarily self-contained, but with a few references to [BF02] and to [Fuj98]. We begin by setting up the notation of the Kaloujnin–Krasner embedding which is used throughout the proof. Substantial use of algebraic properties of the embedding comes towards the end of the proof.

Fix a group $\Gamma$, a finite index normal subgroup $N \triangleleft \Gamma$, an action $N \actson X$ on a hyperbolic complex $X$, and a rank 2 free subgroup $F \triangleleft N$, with respect to which the global WWPD hypothesis of Definition 2.13 holds: each element of $N$ is either loxodromic or elliptic; the restricted action $F \actson X$ is a Schottky group whose nontrivial elements are all WWPD elements of the action $N \actson X$; and for each inner automorphism $i: G \to G$, the restriction to $F$ of the composed action $N \circledast N \triangleright N \actson X$ is either elliptic or is a Schottky group whose nontrivial elements are all WWPD elements of the composed action. We shall prove that $H_2^b(\Gamma; \mathbb{R})$ contains an embedded $\ell^n$.

We may assume that $F \triangleleft [N, N]$, because there is a natural inclusion of commutator subgroups $[F, F] \triangleleft [N, N]$, and because the global WWPD hypothesis is retained when $F$ is replaced by any of its rank 2 subgroups, so we may replace $F$ with a rank 2 subgroup of $[F, F]$. This assumption will be used only at the very last sentence of the proof.

**Step 1: The Kaloujin–Krasner embedding.** This is an embedding of $\Gamma$ into a certain wreath product constructed from $N \triangleleft \Gamma$. Consider the quotient group $Q = \Gamma/N = \{B = Ng \mid g \in \Gamma\}$. Elements of the wreath product $N^Q \rtimes \text{Sym}(Q)$ are denoted as ordered pairs $(\rho, \phi) \in N^Q \times \text{Sym}(Q)$, the group operation being $(\rho, \phi) \cdot (\sigma, \psi) = (\rho \cdot (\sigma \circ \phi), \psi \phi)$; note the reversal of order in the second coordinate. The embedding $\theta: \Gamma \hookrightarrow N^Q \rtimes \text{Sym}(Q)$ is given by the following formula.¹ For each coset $B \in \Gamma/N = Q$ choose a representative $g_B \in B$, and so $B = Ng_B$; in particular, choose $g_N$ to be the identity. For each $\mu \in \Gamma$ define $\theta(\mu) = (\rho_\mu, \phi_\mu) \in N^Q \rtimes \text{Sym}(Q)$ where for each $B \in Q$ we have:

$$
\rho_\mu(B) = g_B \mu g_B^{-1} \in N
$$

$$
\phi_\mu(B) = B\mu \in Q
$$

One may easily check this to be a homomorphism. To see that it is injective, note first that $\phi_\mu$ is trivial if and only if $B = B\mu$ for all $B \in \Gamma/N$ if and only if $N\nu = N\nu\mu$ for all $\nu \in \Gamma$ if and only if $\mu \in N$ (we shall need this below). Next, if $\mu \in N$ and if $\rho_\mu(B)$ is trivial for all $B \in Q$, then using $B = N$ it follows that $\rho_\mu(N) = g_N \mu g_N^{-1} = g_N \mu g_N^{-1} = \mu$ is trivial.

This completes Step 1. □

We next set up some notation that will be used for the rest of the proof of Theorem 2.5. In particular we match the notation regarding the Kaloujin–Krasner embedding with our notation for outer representatives of the action of $\Gamma$ on $N$.

¹See e.g. [Wel76], proofs of Proposition 12.1 and Theorem 11.1
Under the natural injection \( N^Q \hookrightarrow N^Q \rtimes \text{Sym}(Q) \) given by \( \rho \mapsto (\rho, \text{Id}) \), we identify \( N^Q \) with its image. Under this identification, note that \( \theta(\Gamma) \cap N^Q = \theta(N) \) because, as just seen, \( \phi_\mu \) is trivial if and only if \( \mu \in N \).

Fix an enumeration of \( Q = \Gamma/N \) as \( N = B_1, \ldots, B_K \), and abbreviate \( g_{B_\kappa} \) to \( g_\kappa \). For each \( \kappa = 1, \ldots, K \) consider the map \( i_\kappa : N \rightarrow N \) defined by the following composition:

\[
N \xrightarrow{\theta} \theta(N) \rightarrow \theta(\Gamma) \cap N^Q \xrightarrow{\rho \mapsto \rho(B_\kappa)} N
\]

Tracing through the definitions, and using that if \( \mu \in N \) then \( B_\mu = B_1 \), it follows that \( i_\kappa : N \rightarrow N \) is the outer representative of the action of \( \Gamma \) on \( N \) determined by \( g_\kappa : \mu \mapsto g_\kappa \mu g_\kappa^{-1} \).

In particular, since \( N = B_1 \) is represented by \( g_1 = \text{Id} \) the automorphism \( i_1 \) is the identity and the \( N \rtimes_1 X \) is the given action \( N \rtimes X \); we therefore often drop the subscript 1 from the action \( N \rtimes_1 X \).

Since each action \( F \rtimes_\kappa X \) is either Schottky or elliptic, and since \( F \rtimes_1 X \) is Schottky, after re-indexing there exists \( K_1 \geq 1 \) such that \( F \rtimes_\kappa X \) is Schottky when \( 1 \leq \kappa \leq K_1 \) and it is elliptic when \( K_1 < \kappa \leq K \).

Consider the equivalence relation \( \sim \) on elements of the action \( N \rtimes X \), as defined in Section 2.3. We let \( \sim_\kappa \) denote the analogous equivalence relation on the elements of each action \( N \rtimes_\kappa X \), which is given by

\[
\mu \sim_\kappa \nu \iff i_\kappa(\mu) \sim i_\kappa(\nu)
\]

**Step 2:** A good free basis for \( F \). We may assume, after replacing \( F \) with a rank 2 free subgroup, that there is a free basis \( F = \langle g_1, h_1 \rangle \) such that for each \( \kappa = 1, \ldots, K \) we have

1. \( g_1 \not\sim_\kappa h_1 \).
2. \( g_1 \not\sim_\kappa h_1^{-1} \).

To see why, start with an arbitrary free basis \( F = \langle g_1, h_1 \rangle \), and suppose by induction that we have \( k \in \{1, \ldots, K\} \) such that (1) and (2) hold for \( 1 \leq \kappa < k \) (which is vacuously true when \( k = 1 \)). If \( k > K_1 \) then \( F \rtimes_\kappa X \) is elliptic and so (1) and (2) hold as well for \( \kappa = k \). We may therefore assume that \( \kappa \leq K_1 \). The subgroup \( \langle g_1 \rangle \) is a maximal cyclic subgroup of \( F \), since \( g_1 \) is a free basis element. It follows that for each of the actions \( F \rtimes_\kappa X \), \( 1 \leq \kappa \leq k \), the subgroup of \( F \) that stabilizes the fixed point pair of the loxodromic isometry \( g_1 \rtimes_\kappa X \) is equal to \( \langle g_1 \rangle \), and therefore \( h_1 \) does not stabilize the fixed point pair of \( g_1 \) for any of these actions. Applying Proposition 2.11 we obtain for each \( 1 \leq \kappa \leq k \) a constant \( A_\kappa \).
such that if \( a \geq A_\kappa \) then \( g_1 \) and \( h'_1 = g_1^a h_1 g_1^{-a} h_1^{-1} \) satisfy properties (1) and (2). Taking \( a \geq \max \{ A_\kappa \mid 1 \leq \kappa \leq K_1 \} \) and replacing \( h_1 \) with \( h'_1 \), those properties hold simultaneously for all \( \kappa \leq K_1 \). Having made this replacement, the WWPD hypothesis for \( \Gamma \) with respect to \( N \curvearrowright X \) and \( F \) remains true, completing Step 2.

We shall often use the above “maximization trick” to obtain constant bounds independent of \( \kappa \).

**Some notations:** Let \( T \) denote the Cayley tree of the free group \( F = \langle g_1, h_1 \rangle \). For each \( \kappa \leq K_1 \) we have a continuous, \( K_\kappa, C_\kappa \) quasi-isometric embedding \( \xi_\kappa: T \to X \) which takes vertices to vertices and edges to edge paths and which is \( F \)-equivariant with respect to the action \( N \curvearrowright \kappa X \). The quasi-isometry constants \( K_\kappa \geq 1, C_\kappa \geq 0 \) of the maps \( \xi_\kappa \) at first depend on \( \kappa \), but applying the maximization trick we have:

(3) There exists \( K \geq 1, C \geq 0 \) independent of \( \kappa \) such that \( \xi_\kappa: T \to X \) is a \( K, C \) quasi-isometric embedding for \( \kappa \leq K_1 \).

Fix \( \delta \) to be a hyperbolicity constant for \( X \). Given a nontrivial \( g \in F \), let \( A_T(g) \) denote its axis in the tree \( T \), along which the translation distance of \( g \) denoted \( L_g \) equals the cyclically reduced word length of \( g \). For each \( \kappa \leq K_1 \), let \( A_\kappa(g) = \xi_\kappa(A_T(g)) \) which is a \( K, C \)-quasi-axis of \( g \) with respect to the Schottky action \( F \curvearrowright \kappa X \). By item (3), the quasi-axes \( A_\kappa(g) \) all have uniform quasi-isometry constants \( K \geq 1, C \geq 0 \), independent of \( \kappa \leq K_1 \) and \( g \in F \).

Again we often drop the subscript \( \kappa = 1 \), and so \( A(g) = A_1(g) \) is the \( K, C \)-quasi-axis with respect to the given action \( N \curvearrowright X = N \curvearrowright_1 X \).

**Step 3: A sequence of inequivalencies.** There exists a sequence of nontrivial elements

\[
f_1, f_2, f_3, \ldots \in F < [N, N]
\]

such that the following inequivalence properties hold:

(4) For each \( i \geq 1 \) and each \( 1 \leq \kappa \leq K \) we have \( f_i \not\sim i_\kappa(f_i^{-1}) \).

(5) For each \( i > j \geq 1 \) and each \( 1 \leq \kappa \leq K \) we have \( f_i \not\sim i_\kappa(f_j^\pm) \).

If \( \kappa > K_1 \) then \( i_\kappa(f_i^{-1}) \) and \( i_\kappa(f_j^\pm) \) are both elliptic and are therefore distinct from the loxodromic element \( f_i \), hence (4) and (5) both follow from Definition 2.9. We may therefore assume that \( \kappa \leq K_1 \). In that case the justifications we give will follow [BF02] Proposition 2, Claim 3 together with lines of argument found in [BF02] Theorem 8, on pages 81–82 (here we hew more closely to the notation of [BF02]).

The elements \( f_j \) are given by formulas

\[
f_j = g_i^{-s_j} h_i^{-t_j} g_1^{m_j} h_1^{n_j} g_i^{k_j} h_1^{-l_j}
\]
with a rapidly growing sequence of integer exponents

\[
m_1 \ll n_1 \ll k_1 \ll l_1 \ll s_1 \ll t_1 \ll n_2 \ll k_2 \ll l_2 \ll s_2 \ll t_2 \ll n_3 \ll \ldots
\]

which are chosen by the following protocol. There exists a sequence of functions

\[
M_1, \ N_1, \ K_1, \ L_1, \ S_1, \ T_1,
\]

\[
M_2, \ N_2, \ K_2, \ L_2, \ S_2, \ T_2,
\]

\[
M_3, \ N_3, \ \ldots
\]

with the following properties:

- The 1\textsuperscript{st} function \(M_1\) has 0 arguments (and so is a constant), the 2\textsuperscript{nd} function \(N_1\) has 1 argument, the 3\textsuperscript{rd} has 2 arguments, and so on, the \(p\textsuperscript{th}\) having \(p - 1\) arguments.

- If \(m_1 \geq M_1, n_1 \geq N_1(m_1), k_1 \geq K_1(m_1, n_1), l_1 \geq L_1(m_1, n_1, k_1), \) and so on, the \(p\textsuperscript{th}\) exponent being bounded below by the \(p\textsuperscript{th}\) function applied to the previous \(p - 1\) exponents, then properties (4), (5) hold.

At first these functions are chosen to depend on \(\kappa \leq K_1\), obtaining a sequence of functions denoted \(M_1^\kappa, N_1^\kappa, \ldots, M_2^\kappa, N_2^\kappa, \ldots\). But then, by the maximization trick, one replaces the collection of constants \(M_1^\kappa\) with the single constant \(M_1 = \max_{\kappa \leq K_1}\{M_1^\kappa\}\); and one replaces the collection of 1-variable functions \(N_1^\kappa(m_1)\) with the single 1-variable function \(N_1(m_1) = \max_{\kappa \leq K_1}\{N_1^\kappa(m_1)\}\); and so on.

First we prove (5). The quasi-isometric embedding \(\xi_1 : T \to X\) restricts to a map \(A_T(f_1) \xrightarrow{\xi_1} A_1(f_1)\) taking an \(h_1^{-l_1}\) subsegment of \(A_T(f_1)\) to a subsegment \(\alpha\) of \(A_1(f_1)\) (see the diagram below). The constant \(l_1\) is chosen so large that if \(f_1 \sim i_\kappa(f_j)\) then some translate of \(\alpha\) stays close rel endpoints to a subsegment of \(A_\kappa(f_j)\) which contains the image under \(T \xrightarrow{\xi_\kappa} X\) of an entire fundamental domain for \(A_\kappa(f_j)\) labelled \(\underbrace{g_1^{-s_1} h_1^{-l_1} g_1 m_1 n_1 h_1^{-1} g_1 k_1^{-1}}_{(m_1 \text{ reps})} \xrightarrow{\kappa} \beta_1 \). Under that translate, the \(g_1 m_1 \) subsegment of this fundamental domain is taken by \(\xi_\kappa\) to a subsegment \(\beta\) of \(A_\kappa(f_j)\), and the \(h_1^{n_1}\) subsegment is taken by \(\xi_\kappa\) to a subsegment \(\gamma\) of \(A_\kappa(f_j)\), as shown in the the following diagram. In this diagram the symbol \(\xrightarrow{\kappa}\) is an abbreviation for the map \(\xi_\kappa:\)

\[
\ldots g_1^{-s_1} h_1^{-l_1} g_1 m_1 n_1 h_1^{-1} g_1 h_1^{-1} h_1^{-1} h_1^{-1} h_1^{-1} h_1^{-1} h_1^{-1} h_1^{-1} h_1^{-1} g_1^{-s_1} h_1^{-l_1} g_1 m_1 n_1 h_1^{-1} g_1 k_1^{-1} \ldots
\]

It follows that some translate of a subsegment of \(\alpha\) stays close rel endpoints to \(\beta \gamma\), which implies in turn that there exist integers \(\xi, \eta > 0\) such that some translate of \(\beta\)
stays close rel endpoints to a subsegment of $\alpha$ labelled by $h^{\xi}_1$, and some translate of $\gamma$ stays close rel endpoints to a subsegment of $\alpha$ labelled by $h^{\eta}_1$. Since $m_j < n_j$ it follows that $\xi < \eta$, and therefore some translate of $\beta$ stays close rel endpoints to a subsegment of $\gamma$. By taking $m_j$ sufficiently large this contradicts $g_1 \not\sim h_1$, item (1) above. Similarly if $f_i \sim i_\kappa(f^{-1}_j)$, equivalently if $f^{-1}_i \sim i_\kappa(f_j)$, then after inverting the first line in the above diagram, obtaining the segment $\alpha^{-1}$ labelled by $l_i$ repetitions of $h_1$, one sees that some translate of a subsegment $\alpha^{-1}$ stays close rel endpoints to $\beta \gamma$, which leads similarly to a contradiction of $g_1 \not\sim h_1$.

Next we turn to the proof of (4). The quasi-axis $A_1(f_i)$ can be subdivided into alternating subsegments labelled $\alpha_1$ and $\beta_1$ which are the images under the map $A_T(f_i) \xrightarrow{\xi \alpha} A_1(f_i)$ of alternating subsegments of $A_T(f_i)$ labelled $h_1^{t_i}$ and $g_1^{m_i} h_1^{n_i} h_1^{-l_i} g_1^{-s_i}$. Similarly, $A_\kappa(f^{-1}_i)$ can be subdivided into alternating subsegments labelled $\alpha_\kappa$ and $\beta_\kappa$, the images under $A_T(f^{-1}_i) \xrightarrow{\xi \alpha} A_\kappa(f^{-1}_i)$ of alternating subsegments of $A_T(f^{-1}_i)$ labelled $h_1^{t_i}$ and $g_1^{s_i} h_1^{l_i} h_1^{-n_i} g_1^{-m_i}$. Note that each $\alpha_1 \beta_1$ subsegment of $A_1(f_i)$ is a fundamental domain, and similarly for each $\alpha_\kappa \beta_\kappa$ subsegment of $A_\kappa(f_i)$. This situation is depicted in the following diagram:

$$
\begin{align*}
A_1(f_i) & : \cdots h_1^{-t_i} g_1^{m_i} h_1^{n_i} g_1^{k_i} h_1^{-l_i} g_1^{-s_i} h_1^{-t_i} \cdots \\
A_\kappa(f^{-1}_i) & : \cdots h_1^{t_i} g_1^{s_i} h_1^{-m_i} g_1^{l_i} h_1^{-n_i} g_1^{-l_i} h_1^{t_i} \cdots
\end{align*}
$$

For sufficiently large choice of $t_i \gg s_i, l_i, k_i, m_i, n_i$ it follows that $\alpha_1$ is much longer than $\beta_\kappa$ and $\alpha_\kappa$ is much longer than $\beta_1$.

Arguing by contradiction, suppose that $f_i \sim i_\kappa(f^{-1}_i)$ for arbitrarily large choices of $t_i$. Thus, there are subsegments of $A_1(f_i)$ and of $A_\kappa(f^{-1}_i)$, each containing arbitrarily many fundamental domains, that are close rel endpoints to each other. This is depicted in the above diagram, but the alignment of the fundamental domains is not yet clear; by studying various cases of that alignment we shall arrive in each case at a contradiction. First, it immediately follows that $h_1 \sim i_\kappa(h_1^{-1})$, because by using that $\alpha_1$ is much longer than $\beta_\kappa$ and that $\alpha_\kappa$ is much longer than $\beta_1$ it follows that some subsegment of $\alpha_1$ labelled by a large power of $h_1^{-1}$ is close after translation to some subsegment of $\alpha_\kappa$ labelled by a large power of $h_1$.

We now consider four very similar cases of alignment.

**Case 1:** The terminal $g_1^{-m_i/2}$ subsegment of $\beta_\kappa$ cannot be close rel endpoints to a subsegment of $\alpha_1$: otherwise, since $h_1^{-1} \sim i_\kappa(h_1)$ it would follow that the terminal $g_1^{-m_i/2}$ segment of $\beta_\kappa$ is after translation close rel endpoints to a subsegment of $\alpha_\kappa$ labelled by some large power of $h_1$, which for sufficiently large $m_i$ contradicts that $g_1^{-1} \not\sim h_1$, item (2).

**Case 2:** The initial $g_1^{s_i/2}$ subsegment of $\beta_\kappa$ cannot be close rel endpoints to a subsegment of $\alpha_1$: otherwise for sufficiently large $s_i$ we contradict that $h_1 \not\sim g_1$.

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Case 3: The terminal $g_1^{-s_i/2}$ subsegment of $\beta_1$ cannot be close rel endpoints to a subsegment of $\alpha_\kappa$: otherwise for sufficiently large $s_i$ we contradict that $g_1^{-1} \not\sim_1 h_1^{-1}$.

Case 4: The initial $g_1^{m_i}$ subsegment of $\beta_1$ cannot be close rel endpoints to a subsegment of $\alpha_\kappa$: otherwise for sufficiently large $m_i$ we contradict that $h_1^{-1} \not\sim_1 g_1$.

As a consequence of Cases 1–4 and the fact that $\alpha_1$ is much longer than $\beta_\kappa$ and $\alpha_\kappa$ is much longer than $\beta_1$, it follows that $\alpha_1$ and $\alpha_\kappa$ are somewhat close rel endpoints: the distance between their left endpoints has an upper bound comparable to the lengths of the terminal $g_1^{-m_i/2}$ segment of $\beta_\kappa$ and the terminal $g_1^{-s_i/2}$ segment of $\beta_1$, respectively; and similarly for their right endpoints. But now, using that $s_i \gg m_i, n_i$, it follows that some long subsegment of the $g_1^{-s_i}$ segment of $\beta_1$ is close rel endpoints to the entire $h_1^{-n_i}$ subsegment of $\beta_\kappa$, which for sufficiently large $n_i$ contradicts that $g_1 \not\sim_1 h_1$.

This completes Step 3.

\[ \square \]

Step 4: Some quasimorphisms on $N$. We cite Fujiwara [Fuj98] for a method which associates quasimorphisms to loxodromic elements of hyperbolic group actions. Then we cite a result of [BF02] giving some cyclic subgroups on which these quasimorphisms are unbounded and others on which they are bounded. In later steps these results will be applied to the elements $f_1, f_2, \ldots$ constructed in Step 3 and the cyclic subgroups they generate.

Recall that for each $\kappa \leq K_1$ where the action $F \not\sim_\kappa X$ is Schottky, and for each nontrivial $g \in F$, the quasi-axis $A_\kappa(g)$ is a $K, C$-quasigeodesic path in the $\delta$-hyperbolic space $X$. Using the Morse property of quasigeodesics, we fix a constant $B = B(\delta, K, C)$ such that for any $K, C$ quasigeodesic segment $\alpha$ between points $p, q$ in a $\delta$-hyperbolic space, the Hausdorff distance between $\alpha$ and any geodesic $[p, q]$ is at most $B$. We shall apply this where $\alpha$ is a subsegment of any of the quasi-axes $A_\kappa(g)$.

Henceforth let $W \in (3B, 3B+1)$ be the unique integer, as in [BF02] and in [Fuj98].

Given an edge path $\eta = e_1 \cdots e_K$ in $X$ let $|\cdot| = K$ denote its length and $\overline{\eta} = \bar{e}_K \cdots \bar{e}_1$ its reversal. Two subpaths $\eta' = e_{i_1} \cdots e_{j_1}$ and $\eta'' = e_{i_2} \cdots e_{j_2}$ are said to overlap if $\{j_1, \ldots, j_2\} \cap \{i_1, \ldots, i_2\} \neq \emptyset$.

Consider any edge path $w$ in $X$ such that $|w| > W$. Given another edge path $\alpha$, define a copy of $w$ in $\alpha$ to be a subpath of $\alpha$ which is a translate of $w$ by an element of $N$, and define

$$|\alpha|_w = \text{the maximal number of nonoverlapping copies of } w \text{ in } \alpha$$

Then, given vertices $x, y \in X$, define

\[
(*) \quad c_w(x, y) = d(x, y) - \inf_{\alpha} (|\alpha| - W |\alpha|_w) \\
= \sup_{\alpha} (W |\alpha|_w - (|\alpha| - d(x, y)))
\]

where $\alpha$ varies over all edge paths in $X$ from $x$ to $y$. 

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Remark. If $\alpha$ is a geodesic then the parenthetical quantity inside the supremum of $(\ast)$ is equal to $W |\alpha|_{\omega}$ which hearkens back to the original method of Brooks in [Bro81] computing $H^2_{\delta}$ for a free group, where $X$ is the Cayley tree of a free group and $\alpha$ is a geodesic in that tree. When $X$ is a hyperbolic metric space, the method of Fujiwara in [Fuj98] allows $\alpha$ to travel through a sequence of nonoverlapping “wormholes”—copies of $w$—in an attempt to optimize the parenthetical quantities of $(\ast)$. With this allowance a path $\alpha$ which achieves the optimum need no longer be a geodesic, but at least it is a quasigeodesic (see (6a) below).

Pick once and for all a base point $x_0 \in X$. For each edge path $w$ in $X$ define a function $h_w : N \to \mathbb{R}$ as follows:

\[(\ast\ast) \quad h_w(\gamma) = c_w(x_0, \gamma \cdot x_0) - c_{\pi w}(x_0, \gamma \cdot x_0)\]

(6) Assuming $|w| \geq 2W$:

(a) Any edge path $\alpha$ in $X$ which realizes the optimum in $(\ast)$ is a quasigeodesic with constants $\ell = 2$, $c \geq 0$ depending only on $\delta, K, C$.

(b) For each $\gamma \in N$ we have

\[|h_w(\gamma)| \leq H(\gamma)\]

where the constant $H(\gamma)$ is independent of $w$.

(c) The function $h_w$ is a quasimorphism with defect bounded above by a constant $D$ depending only on $\delta, K, C$.

Item (6a) follows from [Fuj98, Lemma 3.3] which says that $\alpha$ is a quasigeodesic with multiplicative constant $\frac{|w|}{|w|-W} \leq 2 = \ell$ and additive constant $\frac{2W}{|w|-W} \leq 4W < 12B+4 = c$. For item (6b), if the edge path $\alpha$ from $x_0$ to $\gamma \cdot x_0$ realizes the optimum in the first term $c_w(x_0, \gamma \cdot x_0)$ of $h_w(\gamma)$ then by (6a) it follows that

\[0 \leq c_w(x_0, \gamma \cdot x_0) \leq |\alpha| \leq \ell d(x_0, \gamma \cdot x_0) + c\]

and the same bound holds for the second term $c_{\pi w}(x_0, \gamma \cdot x_0)$. Item (6c) follows from [Fuj98, Proposition 3.10] which gives a formula for an upper bound to the defect involving only the quantities $\delta, W$, $\ell = 2$, and $c = 12B + 4$ (the latter two numbers $\ell$ and $c$ being upper bounds for the quantities $\frac{|w|}{|w|-W}$ and $\frac{2W}{|w|-W}$ as we have just shown).

Next define a quasimorphism $h(f) : N \to \mathbb{R}$ for each nontrivial $f \in F$ as follows. First, for each line $L \subset T$ pick a base vertex $x_T(L) \in L$, chosen so that $\xi(x_T(L))$ is a point of $\xi(L)$ closest to $x_0$. Next, for each nontrivial $f \in F$ define $x(f) = \xi(x_T(A_T(f))) \in A(f)$, and notice that $x(f) = x(f_d)$ for all integers $d \neq 0$. Then define $w(f)$ to be an oriented geodesic path in $X$ with initial vertex $x(f)$ and terminal vertex $f \cdot x(f)$. Finally define

\[h(f) = h_{w(f)} : N \to \mathbb{R}\]
Although it need not be true in general that \(w(f) \geq 2W\) as needed to apply (6), as long as \(f\) is loxodromic the inequality \(|w(f^d)| \geq 2W\) is true for sufficiently large integers \(d > 0\), which leads to:

(7) There is an integer \(d(f) > 0\) defined for each loxodromic \(f \in N\), and an integer \(d(f, f') > 0\) defined for each pair \(f, f' \in N\) such that \(f\) is loxodromic, such that the following hold:

(a) If \(f \neq f^{-1}\) then for all integers \(d \geq d(f)\) the values of \(h(f^d)\) are non-negative and unbounded on positive elements of the cyclic group \(\langle f \rangle\).

(b) If \(f \neq f^{\pm 1}\) then for all integers \(d \geq d(f, f')\) the value of \(h(f^d)\) is zero on each element of the cyclic group \(\langle f' \rangle\).

(c) If \(f \neq f'^{-1}\) then for all integers \(d \geq d(f, f')\) the values of \(h(f^d)\) are non-negative on positive elements of the cyclic group \(\langle f' \rangle\).

Item (7a), and item (7b) when \(f'\) is loxodromic, are proved in [BF02] Proposition 5.

Before continuing, here are a few remarks. First we note that [BF02] Proposition 5 has a hypothesis that \(f\) be cyclically reduced, which guarantees that if one chooses \(x_0 \in X\) to be the image under \(\xi\) of the identity vertex in the Cayley tree of \(F\) then \(A(f)\) passes through \(x_0\); but the proof goes through without change using instead that \(A(f)\) passes through \(x(f)\). We note also that [BF02] Proposition 5 has an implicit hypothesis \(|w(f^d)| > W \geq 3B\), which we arrange by requiring \(d\) to be sufficiently large. Finally, we note that item (7b) is a consequence of item (7c) (proved in detail below) applied once as stated and once with \(f'\) replaced by \(f'^{-1}\).

To prove items (7b) and (7c) when \(f'\) is not loxodromic, by the WWPD hypothesis the element \(f'\) must be elliptic meaning that the diameter of the set \(\{f'^i(x_0)\}_{i \in \mathbb{Z}}\) is bounded by some constant \(\Delta\). If \(d\) is sufficiently large then \(w(f^d)\) has length greater than \(\ell \Delta + c\), which by (6a) is an upper bound for the length of any path that realizes the optimum for either term of the expression \(h(f^d)(f'^i) = c_w(f^d)(x_0, f'^i(x_0)) - c_{\pi f^d}(x_0, f'^i(x_0))\). It follows that \(h(f^d)\) is zero on \(\langle f' \rangle\).

Remark. We do not know how to prove (7b) if one were to allow \(f'\) to be parabolic (c.f. the final paragraph of Section 3 of [BF02]).

Here is a proof of item (7c) when \(f'\) is loxodromic, which is similar to elements of the proof of [BF02] Proposition 5. If (7c) fails then there exist integer sequences \(d_i \to +\infty\) and \(e_i > 0\) such that the following quantity is negative for all \(i\):

\[ h(f^{d_i})(f'^{e_i}) = c_{w(f^{d_i})}(f'^{e_i}) - c_{\pi f^{d_i}}(f'^{e_i}) \]

It follows that \(c_{\pi(f^{d_i})}(f'^{e_i}) = c_{w(f^{d_i})}(f'^{-e_i})\) is positive for all \(i\). The geodesic path \(w(f^{d_i})\) has the same endpoints as the \(K, C\)-quasigeodesic subsegment \(\alpha_i \subset A(f)\) with initial endpoint \(x(f)\) and terminal endpoint \(f^{d_i} \cdot x(f)\), and therefore \(w(f^{d_i})\) and \(\alpha_i\) are uniformly Hausdorff close rel endpoints. By (6a) there is an \(\ell, c\) quasigeodesic path \(\beta_i\) from \(x_0\) to \(f'^{-e_i} \cdot x_0\) realizing the optimum in the definition of the number
and since that number is positive it follows that some subsegment of $\beta_i$ is a translate of $w(f^{d_i})$ and therefore is uniformly Hausdorff close rel endpoints to a translate of the $K,C$-quasigeodesic segment $\alpha_i$. Recalling that $x(f')$ is a point of $A(f')$ closest to $x_0$, letting $[x_0, x(f')]$ denote a geodesic segment in $X$ with the indicated endpoints, and letting $\gamma_i \subset A(f'^{-1})$ denote the $K,C$-quasigeodesic subsegment with initial point $x(f')$ and terminal point $f'^{-e_i} \cdot x(f')$, the following path is also a uniform quasigeodesic from $x_0$ to $f'^{-e_i} \cdot x_0$ and so stays uniformly Hausdorff close rel endpoints to $\beta_i$:

$$[x_0, x(f')] \ast \gamma_i \ast (f'^{-e_i} \cdot [x(f'), x_0])$$

It follows that some subsegment of the above path is uniformly Hausdorff close endpoints to a translate of $\alpha_i$. Noting that $\alpha_i$ is a concatenation of $d_i$ consecutive fundamental domains of the axis $A(f)$, noting also that the prefix $[x_0, x(f')]$ is independent of $i$, and noting that the suffix $f'^{-e_i} [x(f'), x_0]$ has length independent of $i$, it follows that some subsegment of $\gamma_i$ is uniformly Hausdorff close to a translate of a subpath $\alpha_i' \subset \alpha_i$ consisting of a concatenation of $d_i - d_0$ consecutive fundamental domains of $A(f)$ where $d_0$ is constant. Since $d_i \to +\infty$ it follows that $\text{Length}(\alpha_i') \to +\infty$, from which it follows that $f \sim f'^{-1}$, a contradiction.

This completes Step 4.

Step 5: Quasimorphisms on $\Gamma$. We now define a sequence of quasimorphisms on $\Gamma$ with uniformly bounded defects, denoted

$$h_1, h_2, h_3, \ldots : \Gamma \to \mathbb{R}$$

We shall use this sequence in Step 6 to prove Theorem 2.5.

By combining (4), (5), (6) and (7), we may choose for each $i \geq 1$ an integer $d_i \geq 1$ such that the following hold:

(8) The function $h(f_i^{d_i}) : N \to \mathbb{R}$ is a quasimorphism with defect bounded by a constant $D \geq 0$ independent of $i$ (by (6c)).

(9) The quasimorphism $h(f_i^{d_i})$ is non-negative and unbounded on positive elements of the cyclic group $\langle f_i \rangle$ (by (4) with $\kappa = 1$, and by (7a)).

(10) If $2 \leq \kappa \leq K$ then the quasimorphism $h(f_i^{d_i})$ is non-negative on positive elements of the cyclic group $\langle i_{\kappa}(f_i) \rangle$ (by (4) with $2 \leq \kappa \leq K$, and by (7c)).

(11) If $i > j$ and $1 \leq \kappa \leq K$ then the quasimorphism $h(f_i^{d_i})$ is zero on the cyclic group $\langle i_{\kappa}(f_j) \rangle$ (by (5) and (7b)).

Note that the constant $d_i$ is made independent of $\kappa$, and in (11) of $j$, by the maximization trick.
Define functions $h'_i: N^Q \rtimes \text{Sym}(Q) \to \mathbb{R}$ and $h_i: \Gamma \to \mathbb{R}$ as follows:

$$h'_i(\rho, \phi) = \sum_{\kappa=1}^{K} h(f_{i_\kappa}^d)(\rho(B_\kappa)), \quad \text{for } \rho \in N^Q, \phi \in \text{Sym}(Q)$$

$$h_i(\mu) = h'_i(\theta(\mu)) = h'_i(\rho_\mu, \phi_\mu), \quad \text{for } \mu \in \Gamma$$

$$= \sum_{\kappa=1}^{K} h(f_{i_\kappa}^d)(\rho_\mu(B_\kappa))$$

$$= \sum_{\kappa=1}^{K} h(f_{i_\kappa}^d)(i_\kappa(\mu)) \quad \text{in the special case } \mu \in N.$$

Note that $h_i(\mu)$ has no dependence on $\phi_\mu$, it depends on $\rho_\mu$ alone. We show:

(12) Each $h_i: \Gamma \to \mathbb{R}$ is a quasimorphism with defect $\leq KD$.

For each $\mu, \nu \in \Gamma$ we have

$$h_i(\mu \cdot \nu) = h'_i(\theta(\mu \cdot \nu)) = h'_i(\theta(\mu) \cdot \theta(\nu)) = h'_i((\rho_\mu, \phi_\mu) \cdot (\rho_\nu, \phi_\nu))$$

$$= h'_i(\rho_\mu \cdot (\rho_\nu \circ \phi_\mu), \phi_\nu \phi_\mu)$$

$$= \sum_{\kappa=1}^{K} h(f_{i_\kappa}^d)(\rho_\mu(B_\kappa) \cdot \rho_\nu(\phi_\mu(B_\kappa))) \quad \text{(recall that } \rho_\mu(B_\kappa), \rho_\mu(\phi_\mu(B_\kappa)) \in N)$$

$$= \sum_{\kappa=1}^{K} h(f_{i_\kappa}^d)(\rho_\mu(B_\kappa)) + \sum_{\kappa=1}^{K} h(f_{i_\kappa}^a)(\rho_\nu(\phi_\mu(B_\kappa)))$$

where the symbol $KD=$ means that the difference of the two sides has absolute value bounded by $KD$; this holds because of (8). Since $\phi_\mu \in \text{Sym}(Q)$ permutes the set of cosets $Q = \{B_1, \ldots, B_K\}$ of $N$ in $\Gamma$, by rewriting the second term on the right hand side of the last equation we obtain:

$$= \sum_{\kappa=1}^{K} h(f_{i_\kappa}^a)(\rho_\mu(B_\kappa)) + \sum_{\kappa=1}^{K} h(f_{i_\kappa}^a)(\rho_\nu(\phi_\mu(B_\kappa)))$$

$$= h_i(\mu) + h_i(\nu)$$

This completes Step 5.

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**Step 6: The proof of Theorem D.** To embed $l^1 \hookrightarrow H^2_b(\Gamma; \mathbb{R})$, consider a sequence of real numbers $(t) = (t_1, t_2, t_3, \ldots)$ with $\sum |t_i| < \infty$. Define the function $h_{(t)}: \Gamma \to \mathbb{R}$ by

$$h_{(t)} = t_1h_1 + t_2h_2 + t_3h_3 + \cdots$$

By (12), each $h_i$ is a quasimorphism with defect $\leq KD$, and it follows that $h_{(t)}$ is a quasimorphism with defect $\leq KD \sum |t_i|$.
The map \((t) \mapsto h(t)\) evidently defines a linear map from the vector space \(\ell^1\) to the vector space of quasimorphisms \(\Gamma \mapsto \mathbb{R}\), and so it remains to show that if \((t)\) is nonzero in \(\ell^1\) then \(h(t)\) is unbounded and has unbounded difference with every homomorphism \(\Gamma \mapsto \mathbb{R}\). Letting \(j \geq 1\) be the least integer such that \(t_j \neq 0\) we have:

\[
h(t)(f_j^d) = \sum_{i<j} t_i h_i(f_j^d) + t_j h_j(f_j^d) + \sum_{i>j} t_i h_i(f_j^d)
\]

\[
= t_j \sum_{\kappa=1}^K h(f_j^d_i)(i_\kappa(f_j^d)) + \sum_{i>j} t_i \left( \sum_{\kappa=1}^K h(f_i^d)(i_\kappa(f_j^d)) \right)
\]

\[
= t_j \left( h(f_j^d_i)(f_j^d) + \sum_{\kappa=2}^K h(f_j^d_i)(i_\kappa(f_j^d)) \right)
\]

Regarding the quantity in the big parentheses on the last line, as \(d > 0\) varies the summand \(h(f_j^d_i)(f_j^d)\) is non-negative and unbounded by (9), and the other summands with \(2 \leq \kappa \leq K\) are non-negative by (10), so the whole quantity is unbounded. This shows that \(h(t)(f_j^d)\) is unbounded. Also, since \(f_j^d \in F < [N, N] < [\Gamma, \Gamma]\), the value of every homomorphism \(\Gamma \mapsto \mathbb{R}\) on \(f_j^d\) is zero, and so \(h(t)\) has unbounded difference with every such homomorphism.

\(\square\)

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