Programming Pulse Driven Quantum Computers

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Abstract: Arrays of weakly–coupled quantum systems can be made to compute by subjecting them to a sequence of electromagnetic pulses of well–defined frequency and length. Such pulsed arrays are true quantum computers: bits can be placed in superpositions of 0 and 1, logical operations take place coherently, and dissipation is required only for error correction. Programming such computers is accomplished by selecting the proper sequence of pulses.

Introduction

A recent paper proposed a technologically feasible quantum computer.\(^1\) This paper contains proofs of the results set forth in that paper. The proposed computers are composed of arrays of weakly coupled quantum systems that are subjected to a sequence of electromagnetic pulses of well-defined frequency and length. Selective driving of resonances induces a parallel, cellular–automaton like logic on the array, a method proposed by Teich et al.\(^2\) for inducing a logic in arrays of quantum dots. In reference 1, it is shown that this method extends to arrays of quantum systems with generic weak couplings, and that the resulting computers, when operated in a quantum–mechanically coherent fashion, are examples of logically reversible computers that dissipate less than \(k_B T\) per logical operation; dissipation is only required for error correction.\(^3\) In fact, the systems are true quantum computers in the sense of Deutsch:\(^8\) bits can be placed in superpositions of 0 and 1, quantum uncertainty can be used to generate random numbers, and states can be created that exhibit purely quantum–mechanical correlations.\(^7\)\(^–\)\(^12\)

In this paper, it is shown how such systems can be programmed. A simple sequence of pulses suffices to realize a universal parallel computer. The highly parallel operation of the system also allows fast and robust error–correction routines. A more complicated sequence of pulses instructs the machine to perform arbitrary unitary transformations on collections of quantum bits.
How it works

For the purposes of exposition, consider a heteropolymer, ABCABCABC..., in which each unit possesses an electron that has a long–lived excited state. For each unit, A, B or C, call the ground state 0, and the excited state 1. Since the excited states are long–lived, the transition frequencies $\omega_A, \omega_B$ and $\omega_C$ between the ground and excited states are well-defined. In the absence of any interaction between the units, it is possible to drive transitions between the ground state of a given unit, B say, and the excited state by shining light at the resonant frequency $\omega_B$ on the polymer.13–14 Let the light be in the form of a $\pi$ pulse, so that $\hbar^{-1} \int \vec{\mu}_B \cdot \hat{e} E(t) dt = \pi$, where $\vec{\mu}_B$ is the induced dipole moment between the ground state and the excited state, $\hat{e}$ is the polarization vector for the light that drives the transition, and $E(t)$ is the magnitude of the pulse envelope at time $t$. If the $\pi$ pulse is long compared with $1/\omega_B$, so that its frequency is well–defined, and if the polymer is oriented, so that each induced dipole moment along the polymer has the same angle with respect to $\hat{e}$, then its effect is to take each B that is in the ground state and put it in the excited state, and to take each B in the excited state and put it in the ground state.

Now suppose that there are local interactions between the units of the polymer, given by interaction Hamiltonians $H_{AB}, H_{BC}, H_{CA}$. Almost any local interaction will do. Consider first the case in which these interaction Hamiltonians are diagonal in the original energy eigenstates for each unit (the effect of off–diagonal terms is considered below). The only effect of such interactions is to shift the energy levels of each unit as a function of the energy levels of its neighbors, so that the resonant frequency $\omega_B$, for instance, takes on a value $\omega_{B1}$ if the A on its left is in its ground state and the C on its right is in its first excited state. If the resonant frequencies for all transitions are different for different values of a unit’s neighbors, then the transitions can be driven selectively: if a $\pi$ pulse with frequency $\omega_{B1}$ is applied to the polymer, then all the the B’s with an A = 0 on the left and a C = 1 on the right will switch from 0 to 1 and from 1 to 0 are. If all transition frequencies are different, these are the only units that will switch. Each unit that undergoes a transition coherently emits or absorbs a photon of the given frequency: no dissipation takes place in the switching process.

Driving transitions selectively by the use of resonant $\pi$ pulses induces a parallel logic on the states of the polymer: a particular resonant pulse updates the states of all units of a given type as a function of its previous state and the states of its neighbours. All units of the given type with the same values for their neighbours are updated in the same way. That is, applying a resonant pulse to the polymer effects the action of a cellular automaton rule on the states of units of the polymer.2,15 The cellular automaton rule is of a particularly simple type: if the neighbours of the unit to be switched take on a specific pair of values, then a permutation that interchanges two states of that unit is induced. Since an arbitrary permutation of $N$ states can be built up of successive interchanges of two states, one can by the proper sequence of pulses realize any cellular automaton rule that permutes first the states of one type of unit as a function of its neighbours, then permutes the states of another type of unit, then another, etc. Any reversible cellular automaton rule in which updating takes place by acting first on one type of unit, then another, then another, must be of this form, a permutation $\pi_{ij}^X$ that induces the permutation $\pi_{ij}$ on the states of all units of type $X$ for all different states $ij$ of the neighbours of a unit.

The system is much more computationally powerful than a simple cellular automaton, since one can change the cellular automaton rule from step to step by changing the sequence of pulses applied. Wolfram15 discussed variable rule cellular automata in the form of coupled ‘Master–Slave’ automata, in which the state of the Master cellular automaton varies the rule of the Slave automaton. The systems discussed here are more general than Wolfram’s example, however, for the simple reason that the ‘Master’ is the programmer of the computer: any sequence of pulses, any program, is allowed.
In fact, as will be shown below, by selecting the sequence of pulses, one can make even the simplest of such systems perform any computation that one desires: pulse driven quantum computers are universal digital computers.

It is worth noting that any such variable rule cellular automaton with \( M \) types of unit, and \( m_i \) states for the \( i \)-th unit, is equivalent in terms of its logical operation to any other such system with the same \( M \) and \( m_i \). In the following exposition of sequences of pulses needed to program such variable rule cellular automata, we will concentrate on automata that are one–dimensional, have two states per site, and have three different types of units, \( A, B, C \), as above. We will show that even such extremely simple systems can be made to perform arbitrary computations. One–dimensional systems with more states per site, or more types of units, or both, are then also computationally universal. We will also prove computational universality for a one–dimensional system with only two types of units, \( A, B \), in which \( B \) has three states, one of which exhibits a fast decay to the ground state. Whether a two–unit, two state reversible variable rule automaton is computationally universal is an open question. Although the exposition here will concentrate on one–dimensional systems, we will also note explicitly when the techniques supplied can be generalized to systems of higher dimension.

**Loading and unloading information**

A simple sequence of pulses allows one to load information onto the polymer. There is one unit on the polymer that can be controlled independently — the unit on the end. If the unit on the end in general has different resonant frequencies from all other units of the same type. Suppose this unit is an \( A \): the resonant frequencies \( \omega^A_{i \text{end}} \) for this unit are functions only of the state \( i \) of the \( B \) on its right. If these resonant frequencies are different from the resonant frequencies \( \omega^A_{ij} \) of the \( A \)'s in the interior of the polymer, then one can switch the end unit from 0 to 1 on its own.

Suppose that all units are initially in their ground state. To load a 1 onto the polymer, apply a \( \pi \) pulse at frequency \( \omega^A_{0 \text{end}} \). This pulse switches the end unit to 1. To move this 1 along the polymer, apply a \( \pi \) pulse with frequency \( \omega^B_{01} \). The only \( B \) that responds to this pulse is the first: it will switch to 1. Now apply a pulse with frequency \( \omega^A_{1\text{end}} \). This pulse switches the \( A \) on the end back to 0. (This act of reversibly restoring a bit to zero using a copy of the bit is called ‘uncopying,’ and is typical of reversible computation schemes.)

To load an arbitrary sequence onto the computer, note first that a sequence of \( \pi \) pulses with frequencies \( \omega^B_{10}, \omega^A_{11}, \omega^A_{01}, \omega^B_{10}, \omega^B_{11} \), swaps information between adjacent \( A \)'s and \( B \)'s, taking whatever information is registered in each \( A \) (except the \( A \) on the end) and exchanging it for the information in its neighbouring \( B \). Adding in the middle of this sequence a pulse with frequency \( \omega^A_{1\text{end}} \) causes the \( A \) on the end to swap information with the \( B \) on its right as well. Similarly, one can exchange information between adjacent \( B \)'s and \( C \)'s and \( C \)'s and \( A \)'s. (No additional pulses to address the end unit is required for these exchanges.)

To load an arbitrary sequence of values, \( a_1b_1c_1 \ldots a_nb_n c_n \) onto the polymer, first load \( b_n \) onto the \( A \) on the end. Swap information between \( A \)'s and \( B \)'s, including the \( A \) on the end. Now load \( a_n \) onto the \( A \) on the end. Swapping the information in the \( B \)'s with the information in the \( C \)'s, then the \( A \)'s with the \( B \)'s, then the \( C \)'s with the \( A \)'s, then the \( B \)'s with the \( C \)'s moves the bit \( b_n \) to the first \( A \) from the end, and \( a_n \) to the first \( C \) from the end. Now load \( b_{n-1} \) onto the \( A \) on the end. Swap information between \( A \)'s and \( B \)'s, etc. Continuing this process loads \( a_1b_1a_2b_2 \ldots a_n b_n 0 \) onto the first \( n \) \( ABC \)'s.

To load on the \( c \)'s as well, note that the sequence of swaps, \( B \)'s with \( C \)'s, \( A \)'s with \( B \)'s, has
the effect of taking the information on the A’s and B’s, and moving each bit one unit to the right, while taking the information on the C’s, and moving each bit two units to the left. Continuing with the sequence of swaps, C’s with A’s, B’s with C’s, then A’s with B’s, C’s with A’s, moves each \( ab \) pair to the \( AB \) one triple, \( ABC \) to the right, while moving each C two triples to the left. The same set of swaps in opposite order undoes the motion, moving each \( ab \) one triple to the left, and each c two triples to the right. It is clear that one can, by the proper sequence of swaps, shift the information in one type of units by any amount with respect to the information in the other types of units (subject to the constraint that the overall ‘center of gravity’ of the array remains fixed, in the sense that sums of the displacements of the information in all types of units remains zero). To load on the c’s, move the string \( a_1 b_1 \ldots a_n b_n \) \( 2n \) units to the right, then add on the c’s at intervals of 3 units, starting with \( c_n \), then with \( c_{n-1} \), etc., moving the a’s and b’s left by one unit for each shift of the c’s right by 2 units. When all the c’s have been loaded, they will be paired with the proper a’s and b’s in the first \( n \) triples.

(It is clear from the discussion of loading information above that for an array with \( M \) different types of units, it is simpler to load information in chunks whose size does not exceed \( M - 1 \) bits, since \( M - 1 \) bits can be translated as a block. Whenever possible, we will use schemes for computation that require chunks of no greater than this size. This practice will prove important when error–correction is introduced.)

This technique clearly works when there are two or more different types of units. The different types of units can have different numbers states, although the maximum amount of information that can be stored and transferred per unit is limited by the type of unit with the smallest number of states. For arrays of more than one dimension, note that the same type of unit will tend to have distinct resonant frequencies if it is on a corner, edge, face, or in the interior of the array. In addition, two of the same type of unit on two different corners (edges, faces) will tend to have distinct resonant frequencies if the type and configuration of their neighbours are different. To load an arbitrary block of bits onto a multi–dimensional array, one starts at a corner, loads an arbitrary string onto an edge, moves it inward one unit onto a face, loads in the next string, moves the two strings a unit further onto the face, and continues until the face contains the first 2-d cross–section of the block. This cross section can be moved into the interior of the block location while the next 2-d section is built up. Etc. Symmetries of the array can interfere with this process.

**Unloading information**

There are several ways to get information off the polymer. All involve a certain amount of redundancy, since detection efficiencies for single photons are not very good. The simplest way is to have many copies of the polymer. The same sequence of pulses will induce the same sequence of bits on each copy. To read a bit, one applies a sequence of pulses that moves it to the end. Then one applies two π pulses, with frequencies \( \omega_{0,1} \). If either of these pulses is attenuated, then the bit on the end is a 1; if either is amplified, then the bit is a 0. This method has the disadvantage that all bits must be moved to the end of the polymer to be read.

If the light in the π pulses can be focussed to within a radius of a few wavelengths, information can also be read out in parallel, simply by copying the bit that is to be read out onto all or most units of the same type within a few wavelength neighbourhood, and then seeing whether π pulses aimed at that neighbourhood are attenuated or amplified. (The error–correction schemes described below already require some redundancy.) Other schemes that require less redundancy exist. For example, if the end unit has a fast decay mode (as described below in the section on dissipation), the signal for a bit being a 0 or 1 can be a photon of a different frequency than that of the switching...
pulse. Only a small numbers of such photons in a distinct frequency channel need be present to be detected with high accuracy.

**Computation**

Once information is loaded onto the polymer, a wide variety of schemes can be used to process it in a useful fashion. It is not difficult to find sequences of pulses that realize members of the following class of parallel processing computers.

The polymer is divided up into sections of equal length. By choosing the proper sequence of pulses, and by properly formatting the input information, one can simulate the action of any desired reversible logic circuit on the information within each section. (Since every logical action described up until now is reversible, the entire circuit must be reversible: the logical operation induced by a sequence of $\pi$ pulses can be reversed simply by applying the same sequence in reverse order.) The logic circuit realized is, of course, the same for each section, although the initial information on which the circuit acts can be different from section to section. A second sequence of pulses allows each section to exchange an arbitrary number of bits with the sections to its left and right. Input and output can be obtained from the sections on the end, as above, or from each individual section using focussed light.

By choosing the proper section size and sequence of pulses, one can then realize a string of identical microprocessors of arbitrary reversible circuitry, each communicating with its neighbours. Such a device is obviously computationally universal, in the sense that one can embed in it the operation of a reversible universal Turing machine. A device with the parallel architecture described here, however, is likely to be considerably more useful than a Turing machine for performing actual computations. The number of pulses required to realize such a machine is proportional to the length of the wires, measured in terms of the number of units over which bits must be transported, and number of logic gates in one microprocessor.

We justify the above assertions by giving a recipe for constructing a sequence of pulses that realizes the parallel computer described.

The sequence of $\pi$ pulses with frequencies, $\omega_{10}^C$, $\omega_{11}^C$, $\omega_{11}^B$, $\omega_{10}^C$, $\omega_{11}^C$, induces the operation of a Fredkin gate on each triple $ABC$: a Fredkin gate is a binary gate with three inputs, $X,Y,Z$ and three outputs $X',Y',Z'$ in which $X' = X$, and $Y' = Y$, $Z' = Z$ if $X = 0$; $Y' = Z$, $Z' = Y$ if $X = 1$.\(^4\) (Note that this sequence is closely related to, but simpler than, the set of pulses required to exchange information between the $B$'s and the $C$'s: the only difference is the lack of the pulse with frequency $\omega_{01}^B$). That is, if $X = 0$, all three inputs go through unchanged; if $X = 1$, the second and third input are exchanged: a Fredkin gate effects an exchange of information between two units conditioned on the value of a third. Fredkin gates suffice to give the logical operations $\text{AND}$, $\text{OR}$, $\text{NOT}$ and $\text{FANOUT}$, which in turn form a basis for digital computation.

These two operations, exchange of information between adjacent units, and conditional exchange of information, suffice to create the sort of parallel computer described. The trick is encoding the information in the proper format. Any reversible logic circuit can be constructed from Fredkin gates that operate first on a given triplet of bits, then on another triplet of bits, etc. But the sequence of pulses described in the previous paragraph applies a Fredkin gate to all collections of three bits at once. This extreme parallelism can be overcome as follows.

Each of the processors in the parallel design described above can be described by the same logical circuit. Consider an implementation of this circuit by Fredkin gates: the circuit design consists of wires that move bits to the proper location, and Fredkin gates that then operate on the the proper bits three by three. There are many possible ways to deliver a sequence of pulses that causes the
computer to realize the operation of the desired logic circuit. Here we consider a few of the simplest.

Suppose that the implementation of the circuit in terms of Fredkin gates requires \( N \) bits of input, or in the circuit diagram, \( N \) wires leading into the circuit. The \( k \)th processor will be realized on the \( 2kN \) to \( 2(k + 1)N \) triples \( ABC \). Let the \( N \) bits \( x_1, \ldots, x_N \) that are to be input into the \( k \)th processor be loaded onto the first \( N \) \( A \) units of this section, and let the \( N + 1 \)st triple \( ABC \) contain \( 011 \). Let all the remaining units in the section be set to \( 0 \). Since the only \( B \) and \( C \) units in the entire section that contain a 1 are in the \( N + 1 \)st unit, and since the information in the \( A \)'s, \( B \)'s and \( C \)'s can be shifted relative to each other at will, the 1's in the \( B \) and \( C \) can be used as pointers to move bits around, and to locate triples of bits on which one can operate with Fredkin gates.

For example, to interchange \( x_i \) and \( x_j \), simply shift the information in the \( C \)'s \( N + 1 - i \) triples to the left. The only triple that has \( C = 1 \) is now the \( i \)th triple. Now act on all triples \( ABC \) with a Fredkin gate with \( C \) as the control input that determines whether the other two inputs are to be interchanged. Since the \( i \)th triple is the only one in which \( C = 1 \), this is the only triple in which anything happens. In this triple, \( x_i \) is moved from \( A \) to \( B \). Now shift the information in the \( B \)'s and \( C \)'s \( j - i \) triples to the right. The \( j \)th triple now contains \( ABC = x_jx_i1 \). Act with a Fredkin gate with \( C \) as the control input, as before. Once again, there is only one triple in which \( C = 1 \): the \( j \)th triple, which after the operation of the gate takes the values \( ABC = x_i x_j 1 \). Now shift the information in the \( B \)'s and \( C \)'s \( j - i \) triples to the left and operate with a Fredkin gate again. The \( i \)th triple now contains \( ABC = x_j 01 \). Shifting the information in the \( C \)'s \( N + 1 - i \) triples to the right results in the initial state, but with \( x_i \) and \( x_j \) interchanged.

The *modus operandi* is clear: the \( N \) bits in the \( A \)'s are the sheep, and the two 1’s in the \( B \) and the \( C \) are the shepherds. By the method of the previous paragraph, one can move to adjacent triples groups of three bits on which one desires to act with a Fredkin gate. To operate with a Fredkin gate on the three bits \( x, y, z \) in the \( i \)th, \( i + 1 \)st and \( i + 2 \)nd \( A \)'s, one performs the following sequence of operations. First, shift the information in the \( C \)'s \( N - i \) units to the left, and the information in the \( B \)'s \( N - i - 1 \) units to the left. The three triples now read \( x00 y01 z10 \): the \( i + 1 \)st triple is the only one that has \( C = 1 \), and the \( i + 2 \)nd triple is the only one that has \( B = 1 \). Operate on all triples with a Fredkin gate with \( C \) as the control input: the only triple affected is the \( i + 1 \)st, which now has \( ABC = 0y1 \). Shift the information in the \( C \)'s 2 units to the right: the \( C = 1 \) unit is now in the \( i + 3 \)rd triple, one to the right of the three triples under consideration, which read \( x00 0y0 z10 \). Now operate on all triples with a Fredkin gate with \( B \) as the control input: the only unit affected is the \( i + 2 \)nd, which now has \( ABC = 01z \). Shift the \( B \)'s to the left by one triple, and the \( C \)'s to the left by two triples. The three triples now read \( xyz 011 000 \). Now operate on all triples with a Fredkin gate with \( A \) as the control. The only triple that can be affected is the \( i \)th: all other triples that have \( A = 1 \) have \( B = C = 0 \). The new values of \( x, y, z \) can now be moved back to their original positions simply by undoing the reversible set of operations that brought them together.

By the above operations, one can move bits anywhere in the section, act with Fredkin gates on any three bits that one desires, move bits again, act on three more bits, etc. This method allows one to translate the design for any logic circuit composed of Fredkin gates into a sequence of pulses that realizes that logic circuit on the bits of information within a section. In each section, the same logical circuit is realized. After the operation of the circuit has been completed, information can be exchanged between sections very simply. First, identify the bits that need to be transferred to the section on the right. By moving the \( C \) control unit to the triples in which those bits reside, each of those bits can be transferred from the \( A \) unit to the \( B \) unit. Now move the information in the \( B \)'s \( 2N \) triples to the right, and transfer the information in the desired bits from the \( B \)'s back to the \( A \)'s. The transfer of information from one section to the next is complete. An analogous procedure
allows the transfer of information to the section on the left. The total number of pulses required to realize a particular circuit design is proportional both to the number of gates and the length of the wires in the design.

The following is an even simpler method for inducing parallel computing. Its drawback is that it takes up more space and requires more pulses to realize than the previous method.

Suppose that as before one has the circuit design for the processors in the parallel computer, and that each processor takes \( N \) bits of input. Let \( m = \lceil N/3 \rceil \), the smallest integer greater than \( N/3 \). When loading information onto the polymer, place the first \( m \) bits of information on the \( A \)'s in the section in which the processor is to operate, at intervals of \( m \), so that the first \( A \) in the section contains the first bit, the \( m+1 \)th \( A \) contains the second bit, the \( 2m+1 \)th bit contains the third bit, etc., all other \( A \)'s in the section taking the value 0. Now place the next \( m \) bits on the \( B \)'s at intervals of \( m+1 \), and the remaining bits on the \( C \)'s at intervals of \( m+2 \). It can be seen immediately that in each section, there is at most one triple \( ABC \) in which information is stored in adjacent units, and that this is the only triple \( ABC \) in which there can be more than a single unit that takes the value 1. If the pulses that effect a Fredkin gate are applied to the polymer, then this triple is the only triple of bits in each section whose values can change, since a Fredkin gate changes the values of its inputs only if more than one of those bits takes the value 1. For each Fredkin gate in the circuit diagram for the processor, the exchange operations described above can be used to bring together in the same triple the proper inputs to the gate, and the pulses that effect a Fredkin gate can then be applied. Any reversible logic circuit on the \( N \) input bits can be realized in this fashion. The exchange operations can then be used to transmit information to the neighbouring processors. The total number of pulses required to enact the circuit is proportional to the number of Fredkin gates, and length of the wires, measured in terms of the number of units over which information must be moved.

Both of the methods for inducing parallel computation described above are easily adapted to providing parallel computation in more than one dimension.

**Quantum computation**

The resulting computer is not only a universal digital computer, but a true quantum computer. Bits can be placed in superpositions of 0 and 1 by the simple expedient of applying pulses at the proper resonant frequencies, but of length different from that required to fully switch the bit. For example, if in loading information on the polymer, as in the section above, instead of applying a \( \pi \) pulse, one applies a \( \pi/2 \) pulse of frequency \( \omega_0^{A_{end}} \) of length \( T_1 \), the effect is to put the \( A \) unit on the end in the state, \( 1/\sqrt{2}(|0\rangle + e^{-i\phi_1}|1\rangle) \), where \( \phi_1 = \pi/2 + \omega_0^{A_{end}}T_1 \). Applied at a time \( T_2 \) later, a \( \pi \) pulse of frequency \( \omega_0^{B_{end}} \) and length \( T_3 \) then puts the first two units in the state, \( 1/\sqrt{2}(|00\rangle + e^{-i\phi_2}|11\rangle) \), where \( \phi_2 = 3\pi/2 + \omega_0^{A_{end}}(T_1 + T_2) + (\omega_1^{A_{end}} + \omega_0^{B_{end}})T_3 \).

In fact, by the proper sequence of pulses, it is possible not only to create any quantum state of \( N \) bits, but to effect any unitary transformation desired on those \( N \) bits. The proof is by induction: The inductive assumption is that it is possible to perform any unitary transformation on the space spanned by vectors \( |1\rangle, \ldots, |k\rangle \), where each \( |i\rangle \) is a member of the set, \( \{|00\ldots0\rangle, |10\ldots0\rangle, \ldots, |11\ldots1\rangle\} \). This is clearly true for \( k = 2 \), since it is possible by applying a resonant pulse of the proper intensity and length to effect any desired unitary transformation between the states \( |00\ldots0\rangle \) and \( |10\ldots0\rangle \), and since it is possible by the logical operations described above to arrange a sequence of pulses whose only effect is to produce some desired permutation of the states \( \{|00\ldots0\rangle, |10\ldots0\rangle, \ldots, |11\ldots1\rangle\} \). To derive this result, we use methods developed by Bennett. The desired permutation, \( \Pi \) can be accomplished by some reversible logical circuit, that
to derive this result, we use methods developed by Bennett. The desired permutation, \( \Pi \) can be accomplished by some reversible logical circuit, that takes as input the state \( |i, i\rangle \), and gives as output the state \( |e^{i\phi}i, j\rangle \). One of the two copies of \( i \) can now be used to reversibly ‘uncopy’ the other, leaving \( |\Pi(i), j\rangle \). A second circuit can perform the inverse transformation, \( \Pi^{-1}(i) \), giving the state \( |i', i\rangle \). Some 'junk' information that tells how the computation was performed, and is a phase whose value can be manipulated arbitrarily by varying the length and intensity of the \( \pi \) pulses used to effect the computation. The pulses can always be delivered in such a way that \( \phi = 0 \), the value which we will assume from this point on. The ‘junk’ can be cleaned up by making a copy of \( |\Pi(i), i\rangle \). A second circuit can perform the inverse transformation, \( \Pi^{-1}(i) \), giving the state \( |i', i\rangle \). Some more 'junk.' One of the two copies of \( i \) can now be used to reversibly ‘uncopy’ the other, leaving \( |\Pi(i), i\rangle \). For example, each \( \beta \) that decays quickly to a long–lived state, this fast decay can be exploited to provide error correction. Error correction is a logically irreversible process, and requires dissipation if errors are not to accumulate. In addition, by giving each bit a quantum ‘twist’ when loading it on the computer (for example, by applying a \( \pi/2 \) pulse or a \( 3\pi/2 \) pulse at random), information could be encoded and stored in such a way that only the person who knows by how much each bit has been rotated could read the information. All others who try to read it will get no information, and will leave a signature of their attempt to read it in the process, by randomizing the states of the bits.

**Dissipation and error correction**

Errors in switching and storing bits are inevitable. It is clear that without a method for error correction, the computer described here will not function. Error correction is a logically irreversible process, and requires dissipation if errors are not to accumulate. If in addition to a long–lived excited state, any of the units possesses an excited state that decays quickly to a long–lived state, this fast decay can be exploited to provide error correction. For example, each \( B \) could have an additional excited state, 2, that decays to the ground state, 0, in an amount of time short compared with the time in between pulses. Any \( B \) in a long–lived state, 1, e.g., can be restored to the ground state conditioned on the state of its neighbours by applying pulses with the resonant frequency \( \omega^B_{ij}(12) \) of the transition between the states 1 and 2 given that its neighbours \( A \) and \( C \) are in the states \( i \) and \( j \). The pulse need not have a well–defined length, provided that it is long enough to drive the transition efficiently.
If just one type of unit has a fast decay of the sort described, then one can realize not only any reversible cellular automaton rule that updates first one type of unit, then another, but any irreversible cellular automaton rule as well. The scheme described above that allows the construction of one-dimensional arrays of arbitrary parallel-processing reversible microprocessors then allows one to produce one-dimensional arrays of arbitrary irreversible microprocessors, each one of which can contain arbitrary error-correcting circuitry. Many error-correcting schemes are possible, using check sums and parity bits multiplexing, etc. A particularly simple and robust scheme is given below. For each logically irreversible operation accomplished, a photon is emitted incoherently to the environment. In contrast to the switching of bits using $\pi$ pulses, in which photons are emitted and absorbed coherently, the switching of bits using fast decays is inherently dissipative. The amount of dissipation depends on what is done with the incoherently emitted photons. If the photon is absorbed and its energy thermalized, then considerably more than $k_B T$ is dissipated; if the energy of the photon is put to work, dissipation can be brought down to close to $k_B T$.

Such a computer can function reliably in the face of a small error rate in principle. Error correction for the method of computation proposed here takes the place of gain and signal restoration in conventional circuits. Whether such a computer can actually be made to function reliably in the face of a finite error rate depends crucially on whether the error correction routine suffices to correct the number of errors generated in the course of the computational cycle, in between error correction cycles.

Suppose that the probability of error per unit per computational cycle is $\epsilon$. Suppose that all bits come in $2k + 1$ redundant copies, and that after each cycle, error correction is performed in parallel by having the copies vote amongst each other as to their proper value, and all copies are restored to that value: there exist quick routines for performing this operation, that are insensitive to errors generated during their execution. The error rate per cycle is reduced to $\eta \approx (2\epsilon)^k$ by this process. For a computation that uses $b$ bits over $c$ cycles to have a probability no greater than $f$ for the failure of a single bit, we must have $b(1 - \eta)^{bc} \geq b - f$, which implies that $\eta \leq 1/cb^2$. For example, suppose that the error rate per bit per cycle is a quarter of a percent, $\epsilon = .0025$. To have a computation involving $10^{12}$ bits over $10^{20}$ steps have a probability of less than 1% of getting a bit wrong requires that each bit have 47 redundant copies. Although such computers have much higher error rates and require much more error correction than conventional computers, because of their high bit density and massively parallel operation, error correction can be carried out without too great a sacrifice in space or time.

**Robust error correction schemes**

As noted, a wide variety of error correction schemes are possible. Here we present several schemes that are robust: they correct errors quickly and efficiently, even if errors are committed during their execution.

First, we examine the simplest possible form of error correction. If information is stored redundantly in triplicate, so that each $ABC$ is supposed to contain the same bit of information, a simple form of error correction is provided by applying a sequence of pulses that restores $ABC$ to 111 if at least two of the units are 1, and to 000 if at least two of the units are 0.

Suppose that each $B$ unit has an excited state, call it 2, that decays to the ground state 0 in an amount of time comparable to the length of the $\pi$ pulses used to do switching. To restore $B$ to 0 if $A$ and $C$ are both 0, one simply applies a pulse at the resonant frequency $\omega_{20} B(12)$ of $B$’s transition from 1 to 2, given that $A = 0, C = 0$. As noted above, the pulse need not be of any specific length, as long as it is considerably longer than the lifetime of the state 2. This pulse has the following
effect: If $B$ is initially in the state 0, it remains in that state. If $B$ is initially in the state 1, then it is excited to the state 2, and then decays to 0. The net effect is to reset $B$ to 0 provided that $A$ and $C$ are both 0. To restore $B$ to 1 if both $A$ and $C$ are 1, first apply a pulse at the resonant frequency $\omega_{11}^B(12)$ of $B$’s transition from 1 to 2, given that $A = 1$, $C = 1$. This pulse insures that if $A$ and $C$ are 1, $B$ is set to 0. Then apply a $\pi$ pulse of frequency $\omega_{11}^B(01)$ to take $B$ from 0 to 1. The net effect is to reset $B$ to 1 provided both $A$ and $C$ are 1. If $A$ and $C$ have different values, the sequence of pulses has no effect on $B$. A simple sequence of pulses will interchange the bits in $B$ and $C$. If one interchanges $B$ with $C$, and performs the resetting procedure of the previous paragraph, then interchanges $A$ with $B$ and resets $B$ once again, the required error correction is accomplished.

What is desired is a method of correcting errors by the method of voting, but to have the voting take place over an arbitrary number of copies. One way to do this would be to design a circuit that performs the voting, and then realize it by the method given above. Such a circuit might take a large number of pulses to realize, however, and would moreover be vulnerable to errors committed in the course of its operation. We have designed some quick and dirty methods that perform error correction massively in parallel. The basic idea is to store all bits with $n$-fold redundancy, in blocks, to have each bit in the block vote with its neighbours in threes, as above, then to scramble up the bits in the block and to perform the voting again. If the error rate per computational step is $\epsilon$, and if no errors are made in the voting, then after a single vote, the rate will be $\epsilon^2$, after two votes, $\epsilon^4$, etc.: the the effective error rate rapidly drops to zero. If during voting, there is a probability of error per unit of $\theta$, then the fraction of units in each block with the ‘wrong’ value eventually converges to $\theta$. The process described is insensitive to errors committed during its execution.

There are several ways to realize this particular method of error correction. In the methods for inducing computing described above, we have some bits stored in the $A$’s, some in the $B$’s, and some in the $C$’s: Suppose that each bit is stored with $n$-fold redundancy, with $n$ triples of blank space between each redundantly registered bit. When error correction begins, then, the data is stored in blocks of $n$ triples, and in each block, all the $A$’s are supposed to be the same, with all the $B$’s and $C$’s equal to zero, or all the $B$’s are supposed to be the same, with all the $A$’s and $C$’s equal to zero, etc. Between the blocks of data, there are blocks $n$ triples long in which all units are 0. Both of the computational methods described above can be made to work with this formatting of the data.

The error correction routine restores first the $A$’s to a common value, then the $B$’s, then the $C$’s. The method is simple: To restore the $A$’s, first supply a sequence of pulses that transforms $ABC = 100$ to $ABC = 111$, and $ABC = 111$ to $ABC = 100$, leaving all other values for $ABC$ unchanged. Such a sequence is easy to devise. This transformation has the effect of making each unit in the block in which information is stored in $A$’s take on the value of the $A$ in the triple, while leaving the blocks in which information is stored in $B$’s or $C$’s unchanged. One can now make the different $A$’s in the block vote three by three amongst themselves. First, shift the information in the $B$’s in one direction by some number of triples $\leq n$, and shift the $C$’s to the opposite direction by some independently chosen number $\leq n$; then restore an $A$ to one if its neighbours are one, and to zero if its neighbours are zero. By differing the shifts, one can make each $A$ in the block vote with any $A$ to its left and any $A$ to its right. After the voting is done once, one can shift the $B$’s and the $C$’s back again, and perform the inverse transformation within each triple, mapping 111 to 100, and 001 to 111, leaving other values fixed.

The only weakness in this voting scheme occurs at the ends of blocks in which all the $A$’s are supposed to be equal to 1: the last $A$ on each block doesn’t get to vote. As a result, without some further measure, blocks in which all the $A$’s are supposed to be equal to 1 will tend to be eaten away from the ends. This problem is easily remedied by scrambling up the redundant bits in each
block. There is a wide variety of ways to induce this scrambling, one of the simplest of which is the following.

The redundant bits in blocks where information is stored in the $A$'s can be scrambled amongst each other by inducing the following interaction between that block and a block in which the $C$'s are supposed to be equal to $1$, (for example, the block of $C = 1$ that is used to shepherd bits around in the first computing scheme above). First, shift the information in the blocks relative to each other so that the $m$ triples on the right of the block in which information is stored in the $C$'s overlaps the first $m$ triples on the left of the block in which information is stored in the $A$'s, where $m \leq n/2$. Enact a Fredkin gate with $C$ as the control bit. Since the $C$'s are by and large equal to $1$, the effect is to interchange the information in the $m A$'s of the overlapping blocks with the $0$'s in the $m B$'s. Shift the information in the $B$'s and $C$'s $m$ triples to the right, and act with a Fredkin gate with control $C$ again. Then shift the information in the $B$'s and the $C$'s $m$ triples to the left, and act with a Fredkin gate with control $C$ again. The effect of these actions is as follows: the only place in which anything happens is in the first $2m$ triples of the blocks where information is stored in the $A$'s, as in all other triples where $C = 1$, $A = B = 0$. In such blocks, the effect is to interchange the information that was in the first $m A$'s of the blocks with the information in the second $m A$'s of the blocks. Similarly, it is possible to interchange the information in the last $m A$'s of the blocks with the information in the second to last $m A$'s of the blocks. By varying $m$, one can scramble up the $A$'s in any way that one chooses. Similarly, one can use a block of $A = 1$ to scramble up the bits in blocks in which information is stored in $B$'s or $C$'s. There are other, more involved scrambling procedures that do not require blocks of $C = 1$ to scramble up the bits in blocks in which information is stored in the $B$'s or $C$'s.

Combining such a scrambling procedure with the procedure of voting by threes is an effective method for correcting errors quickly as long as each bit has a sufficiently large number of redundant copies that the probability of more than a third of them taking on the wrong value in the course of the computation is small. If the probability of error within a block of $A$'s after the computational cycle and before the error correction cycle was $\epsilon$, and if the probability of error per unit during the error–correction cycle is $\theta$, then the probability of error after one round of voting by threes is $\approx 2\epsilon^2 + \theta$. More precisely, if the number of incorrect $A$'s in the block was initially $p$, the probability that an $A$ in the block takes on the correct value after voting once by threes is $\approx 1 - p^2/n^2(2 - p/n) - \theta$. One can now repeat the process, having each $A$ vote with the copies of a different pair of $A$'s in the same block. The probability of error is now reduced to $\approx 2(2\epsilon^2 + \theta)^2 + \theta$. Etc. As long as the number of redundant bits is sufficiently large, this method will rapidly restore all but a fraction $\theta$ of the bits in a block to a common value. Here, ‘sufficiently large,’ depends on how many times the voting by triples takes place. Because it takes place in parallel fashion, each voting requires only a short sequence of pulses to realize. For only a small number of votes, five, say, it suffices that the number of incorrect $A$'s in the block, $p$, never gets larger than $n/3$ in the course of the computation.

The above method, if error–free, leaves unaffected blocks in which only the $B$'s or only the $C$'s contain data. If the error rate is $\theta$ over a single vote and scramble of the $A$'s, then the error–correcting routine for the $A$'s will introduce an error rate of $\approx \ell\theta$ per unit of the $B$'s and $C$'s, if carried out over $\ell$ votes. The same method can now be used to restore first the $B$'s, then the $C$'s to their proper values. The resulting error correction technique is efficient and robust. It can easily be generalized to higher dimensions, and to more types of units with more states.

_Destruction of quantum coherence_

Note that when a photon is emitted incoherently, the quantum coherence of the bit from which
it was emitted, and of any other bits correlated with that bit, is destroyed. Incoherent processes and the generation of errors intrinsically limit the number of steps over which the computer can function in a purely quantum-mechanical fashion.

Errors

There are many potential sources of error in the operation of these pulsed quantum computers. The primary difficulty in the proposed scheme is the delivery of effective $\pi$ pulses. Microwave technology can give complete inversion with error rates of a fraction of a percent in NMR systems. Optical systems are at present harder to invert, since bands suffer considerable homogeneous and inhomogeneous broadening. As noted above, a fraction of a percent error per bit per pulse can be tolerated; but a few percent is probably too much. Techniques such as pulse shaping$^{23}$ or iterative excitation schemes$^{24}$ enhance $\pi$ pulse effectiveness and selectivity. If optical systems with sufficiently narrow bands can be found, and if the systems can be well-oriented, so that the coupling with the pulses is uniform, then the rapid advance of laser technology promises soon to reach a level at which $\pi$ pulses can be delivered at optical frequencies.

In addition to the technological problem of supplying accurate $\pi$ pulses, the following fundamental physical effects can cause substantial errors:

Effect of off–diagonal terms in interaction Hamiltonians. These terms have a number of effects. The simplest is to induce unwanted switching of individual units, with a probability of error per unit per pulse of $(\delta \omega_{off}/\omega)^2$ whenever a unit or its neighbour is switched. Here $\hbar \delta \omega_{off}$ is the characteristic size of the relevant off–diagonal term in the interaction Hamiltonian. Off–diagonal interactions also induce the propagation of excitons along the polymer: this process implies that a localized excited state has an intrinsic finite lifetime equal to the inverse of the bandwidth for the propagation of the exciton associated with that state.$^{25–26}$ For the polymer $ABCABC\ldots$, the bandwidth associated with the propagation of an excited state of $A$ can be calculated either by a decomposition in terms of Bloch states, or by perturbation theory, and is proportional to $\delta \omega_{off}^A \delta \omega_{off}^B \delta \omega_{off}^C / (\omega_A - \omega_B)(\omega_A - \omega_C)$, where $\hbar \delta \omega_{off}^A$, e.g., is the size of the term in $H_{AB}$ that induces propagation of excitation from $A$ to $B$. For a polymer of the form $12\ldots M12\ldots M\ldots$, the characteristic bandwidth goes as $\delta \omega_{off}^M / \Delta \Omega M^{-1}$, where $\Delta \Omega$ is the typical size of the difference between the resonant frequencies of different types of units. For the computer to function, the inverse of the exciton propagation bandwidth must be much longer than the characteristic switching time. If the off–diagonal terms are of the same size as the on–diagonal terms, on average, then for the computer to function, the overall interaction between units must be weak, and $M$, the number of different kinds of units in the polymer, must be at least three. Small off-diagonal terms and a relatively large number of different types of units are essential for the successful operation of the computer.

Quantum–electrodynamic effects. The probability of spontaneous emission from a single unit is assumed to be small. In the absence of interactions, the spontaneous decay rate for a unit with resonant frequency $\omega$ is $4\omega^3 \mu^2 / 3\hbar c^3$. If the lifetime of an optical excited state is to be as long as milliseconds, the induced dipole moment $\mu$ must be suppressed by symmetry considerations. Interactions between different units of the same type can give rise to quantum–electrodynamic effects such as super–radiance, and the coherent emission of a photon by one unit and coherent reabsorption by another.$^{27}$ Fortunately, the states that are being used for computation, in which each unit is in a well–defined excited or ground state, are exactly those that do not give enhanced probabilities for these processes. In the process of switching, however, and when bits are in superpositions of $|0\rangle$ and $|1\rangle$, super–radiant emission gives an enhancement of the spontaneous emission rate by a factor of $n$, 

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where \( n \) is the number of units of the same type within a wavelength of the light used. Since the switching time is short compared to the lifetime, super–radiant emission is not a problem. (Though super–radiance can shorten the lifetime of quantum superpositions of logical states.)

**Nonlocal interactions.** Coherent switching will not work unless the shift in a unit’s resonant frequency induced by nearby units of the same type is too small to throw the unit out of resonance. The dipole–dipole couplings of reference 2 fall off as \( 1/r^3 \). For such a long range coupling, many different types of units are required, and the result of a resonant pulse is to realize a cellular automaton rule with a neighbourhood of radius larger than one.

None of the purely physical effects gives error rates that are insurmountable. If the \( \pi \) pulses are long compared to the inverse frequency shifts due to interaction, if the unperturbed resonant frequencies differ substantially between the different types of unit, and if the off–diagonal terms in the interaction Hamiltonians are small compared with the resonant frequencies and their differences, then this computing scheme will work in principle.

Although putting them together in a working package may prove difficult, precisely timed monochromatic laser pulses, well–oriented polymers, accurately fabricated semi–conductor arrays, and fast, sensitive photodetectors are all available in today’s technology. Continuously tunable titanium–sapphire lasers, or diode–pumped YAG lasers tuned by side–band modulation can currently supply frequency–stable picosecond pulses at nanosecond intervals with an integrated intensity that varies by a fraction of a percent. Currently available electro–optical shutters could be used to generate the proper pulse sequence at a nanosecond clock rate. Photodetectors equipped with photomultipliers and acoustic–optical filters can reliably detect tens of photons (or fewer) within a wavelength band a few nanometers wide. Although arrays of quantum dots created by X-ray lithography are not yet of sufficiently uniform quality, arrays of quantum dots and lines that have been created using interference techniques might be sufficiently uniform to realize the proposed scheme.

**Numbers**

The range of speed of operation of such a pulsed quantum computer within acceptable error rates is determined by the frequency of light used to drive transitions, and by the strength and character of the interactions between units. For square-wave pulses, the intrinsic probability of error per unit per pulse due to indiscriminate transition–driving is \( (1/T\delta\omega_{on})^2 \), where \( T \) is the pulse length and \( \delta\omega_{on} \) is resonant frequency shift induced by on-diagonal terms in the interaction Hamiltonian,\(^{13}\) (this error can be reduced significantly by using shaped pulses\(^{26}\)) while the probability of error per unit per pulse due to off–diagonal terms in the interaction Hamiltonians is \( (\delta\omega_{off}/\omega)^2 \). The decay of localized excitations due to exciton propagation gives a lifetime proportional to \( \Delta\omega^{M-1}/\delta\omega_{off}^M \), where \( M \) is the number of different types of units.

Suppose that the excited states have transition frequencies corresponding to light in the visible range, say \( \omega = 10^{15} \text{ sec}^{-1} \). (Many electronic excited states in molecular systems and quantum dots are in the visible or near–visible range. Visible light is a good range in which to operate, because accurate lasers exist for these frequencies, and because the systems can operate at room temperature.) In the absence of off–diagonal terms in the interaction Hamiltonians, the frequency shifts due to interaction do not need to be small compared to \( \omega \), and to obtain an intrinsic error rate of less than \( 10^{-6} \) per unit per pulse, the pulse length could be as short as \( 10^{-12} \) seconds, and as long as a few thousands of the intrinsic lifetimes of the excited states (assuming that a few thousand pulses are required for error correction). The clock rate of such a computer could be varied to synchronize its input and output with conventional electronic devices. In the presence of off–diagonal terms of
the same magnitude $\delta\omega_{off} \approx \delta\omega_{on} \sim \delta\omega$ as the on–diagonal terms, to obtain an intrinsic error rate of $10^{-6}$ per unit per pulse, one must have $\delta\omega = 10^{12}$ sec$^{-1}$, and a minimum pulse length of $10^{-9}$ seconds. If the computer has three different types of units, the intrinsic exciton lifetime from the local coupling alone is on the order of $10^{-6}$ seconds. Actual exciton lifetime will be shorter as a result of coupling to other modes. The more different kinds of units, the more freedom one has to lengthen the clock cycle.

If the units in the quantum computer are nuclear spins in an intense magnetic field, with dipole–dipole interactions, then the pulses will have frequencies in the microwave or radiofrequency region, and the computers will have clock rates from microseconds to milliseconds.

Conclusion

Computers composed of arrays of pulsed, weakly–coupled quantum systems are physically feasible, and may be realizable with current technology. The units in the array could be quantum dots, nuclear spins, localized electronic states in a polymer, or any multistate quantum system that can be made to interact locally with its neighbours, and can be compelled to switch between states using resonant pulses of light. The exposition here has concentrated on one-dimensional systems with two or three states, but more dimensions, more types of unit, and more states per unit provide higher densities of information storage and a wider range of possibilities for information processing, as long as the different transitions can still be driven discriminately.

The small size, high clock speeds and massively parallel operation of these pulsed quantum computers, if realized, would make them good devices for simulating large, homogeneous systems such as lattice gases or fluid flows. But such systems are capable of more than digital computation. When operated coherently, the devices described here are true quantum computers, combining digital and quantum analog capacities, and could be used to create and manipulate complicated many–bit quantum states. Many questions remain: What are the best physical realizations of such systems? (The answer may be different according to whether the devices are to be used for fast, parallel computing, or for generating novel quantum states.) How can they best be programmed? How can noise be suppressed and errors corrected? How can their peculiarly quantum features be exploited? What are the properties of higher dimensional arrays? The device proposed here, as with all devices in the next generation of nanoscale information processing, cannot be built and made to function without addressing fundamental questions in the physics of computation.

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