Permutation symmetry for the tomographic probability distribution of a system of identical particles

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Abstract

The symmetry properties under permutation of tomograms representing the states of a system of identical particles are studied. Starting from the action of the permutation group on the density matrix we define its action on the tomographic probability distribution. Explicit calculations are performed in the case of the two-dimensional harmonic oscillator.

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1 Introduction

In the conventional approach to quantum mechanics, that is the wave function description [1], the peculiarities of the quantum behaviour of a system of identical particles are encoded in the symmetry properties of the wave functions. The description in terms of density operators [4] also implies, for the density matrices associated to the states of identical particles, definite symmetry properties. Nevertheless both the descriptions are very different from the approach of classical statistical mechanics in terms of probability distributions defined on the phase space of the system. The wish to render the quantum picture closer to the classical one gave rise to hidden variables theories [3] and to the introduction of the Wigner quasi-distribution function on phase space [4] obeying to the Moyal equation [5]. The same aim of describing quantum states in terms of a classical distribution of probability determined then the introduction of another class of quasi-distributions for the quantum states [6, 7, 8]. In 1969 Cahill and Glauber [9] considered a set of s-ordered quasi-distribution functions that further generalized the quasi-probability distribution functions previously introduced by Wigner, Husimi, Glauber and Sudarshan. They also showed that for each quantum state one can find some probability distribution which is determined by the density operator of the given state. But it was not clear whether or not such probability distribution determined uniquely the density operator, that is, whether or not the map was invertible. Only recently it was realized that quantum mechanics could be described completely in terms of this kind of probability distributions suitably defined for a random variable [10]. In [10] a consistent scheme has been proposed, the so-called probability representation, that results completely equivalent, in the sense of invertibility, to the ordinary formulation of quantum mechanics: the quantum states are described by a tomographic distribution of probability, or tomogram, also known as marginal distribution function (MDF), and the evolution of the system is described by an integro-differential equation for the MDF of generalized Fokker-Planck type. The probability representation of quantum mechanics uses as a mathematical tool the symplectic tomography map [11] of density operators onto quadrature probability distributions. For a general approach to tomograms and quasi-distributions of quantum states see also [12, 13] and, with the inclusion of spin, [14, 15, 16, 17].

In the case of identical particles it is well known that they obey either Bose or Fermi statistics. As a result the wave function describing the system must be symmetrized or antisymmetrized respectively. Because the states of a physical system belong to a vector space and are, consequently, linearly superposable, it results quite easy to implement the two types of statistics in the ordinary quantum mechanics considering, for example, the wave functions of the system as a basis for a representation of the permutation group. On the other hand, in the probability representation of quantum states one needs to know what is the behaviour of the tomographic probability distributions for systems of identical particles. The aim of our article is the formulation of the symmetry properties of the tomograms for systems of identical particles.
In the following we use the relation between the MDF and the density matrix to construct the action of the permutation group on the marginal distribution functions. Because the set of the MDFs is not a vector space (see for example [18] where a superposition principle was formulated for both the sets of density matrices and tomograms) we obtain a realization, and not a representation, of the permutation group on the MDFs. In this way we can introduce the completely symmetrized and antisymmetrized marginal distribution function.

The paper is organized as follows. In section 2 we review the realization of the permutation group on the set of density matrices and in section 3 we derive the corresponding realization on the set of MDFs. For the sake of clarity we work explicitly on the case of two identical particles and then we extend the results to $n$ particles in section 4. In section 5 we describe a simple application and finally the concluding remarks.

## 2 The permutation group of the density matrix

Identical particles obey either Bose or Fermi statistics. This property implies two specific behaviours of the wave function of two identical particles: it must be symmetric in the case of Bose-particles, antisymmetric for Fermi-particles. These properties are more precisely described by means of the permutation group representation theory. To have a model let us concentrate on the system of two one dimensional particles whose positions are $x_1$ and $x_2$ respectively. The corresponding wave function, $\psi(x_1, x_2)$, can be decomposed into the sum of a symmetric and an antisymmetric function:

$$\psi(x_1, x_2) = \frac{1}{2}(\psi(x_1, x_2) + \psi(x_2, x_1)) + \frac{1}{2}(\psi(x_1, x_2) - \psi(x_2, x_1)) = \psi_+(x_1, x_2) + \psi_-(x_1, x_2)$$  \hspace{1cm} (2.1)

This decomposition may be related to the irreducible representation of the permutation group $G = (\hat{E}, \hat{p}_{12})$, where $\hat{E}$ is the identity of the group and $\hat{p}_{12}$ represents the operation of permutation of the coordinates $x_1$ and $x_2$ i.e.

$$\hat{E}\psi(x_1, x_2) = \psi(x_1, x_2); \quad \hat{p}_{12}\psi(x_1, x_2) = \psi(x_2, x_1).$$  \hspace{1cm} (2.2)

The table of characters of the representations of the permutation group of two elements has the form

$$\begin{array}{cc}
\hat{E} & \hat{p}_{12} \\
1 & 1 \\
1 & -1 \\
\end{array}$$  \hspace{1cm} (2.3)

It means that the decomposition (2.1) is connected with the irreducible representations (2.2) through the following formulas:

$$\hat{E}\psi_\pm(x_1, x_2) = +1\psi_\pm(x_1, x_2)$$
\[ \hat{p}_{12} \psi_{\pm}(x_1, x_2) = \pm 1 \psi_{\pm}(x_1, x_2). \] (2.4)

Thus the two functions \( \psi_{\pm} \) realize a basis of the one-dimensional representation of the permutation group. We say that identical particles with symmetric wave function are described by means of the symmetric representation of \( G \), while identical particles with antisymmetric wave function by means of the antisymmetric one.

The density matrix \( \rho(x_1, x_2, x'_1, x'_2) \) of a pure state of two particles with wave function \( \psi(x_1, x_2) \) has the form
\[ \rho(x_1, x_2, x'_1, x'_2) = \psi(x_1, x_2) \psi^*(x'_1, x'_2) \] (2.5)

thus we can extend the action of the permutation group to the set of density matrices because the coordinates \( x_1, x_2 \) and \( x'_1, x'_2 \) can be permuted independently. This means that the group \( \tilde{G} = G \otimes G \) which is the direct product of the permutation groups of two elements is the one related to the symmetry properties of the density matrix. \( \tilde{G} \) contains four elements:
\[ \hat{I} = \hat{E} \otimes \hat{E}, \quad \hat{P}_{12} = \hat{p}_{12} \otimes \hat{E}, \quad \hat{P}'_{12} = \hat{E} \otimes \hat{p}_{12}, \quad \hat{\Pi}_{12} = \hat{p}_{12} \otimes \hat{p}_{12}. \] (2.6)

The action of the four elements on the density matrix (2.5) is defined as follows:
\[ \hat{I} \rho(x_1, x_2, x'_1, x'_2) = \rho(x_1, x_2, x'_1, x'_2) \] (2.7)
\[ \hat{P}_{12} \rho(x_1, x_2, x'_1, x'_2) = \rho(x_2, x_1, x'_1, x'_2) \] (2.8)
\[ \hat{P}'_{12} \rho(x_1, x_2, x'_1, x'_2) = \rho(x_1, x_2, x'_2, x'_1) \] (2.9)
\[ \hat{\Pi}_{12} \rho(x_1, x_2, x'_1, x'_2) = \rho(x_2, x_1, x'_2, x'_1). \] (2.10)

Using the multiplication table of the group \( \tilde{G} \):

\[ \begin{array}{c|cccc}
\hat{I} & \hat{P}_{12} & \hat{P}'_{12} & \hat{\Pi}_{12} \\
\hline
\hat{I} & \hat{I} & \hat{P}_{12} & \hat{P}'_{12} & \hat{\Pi}_{12} \\
\hat{P}_{12} & \hat{P}_{12} & \hat{I} & \hat{\Pi}_{12} & \hat{P}'_{12} \\
\hat{P}'_{12} & \hat{P}'_{12} & \hat{\Pi}_{12} & \hat{I} & \hat{P}_{12} \\
\hat{\Pi}_{12} & \hat{\Pi}_{12} & \hat{P}'_{12} & \hat{P}_{12} & \hat{I} \\
\end{array} \]

we may construct a realization through the action of \( \tilde{G} \) on the following distributions:
\[ \rho_+ = \frac{1}{4} \left( \hat{I} \rho + \hat{\Pi} \rho + \hat{P}_{12} \rho + \hat{P}'_{12} \rho \right) \] (2.11)
\[ \rho_- = \frac{1}{4} \left( \hat{I} \rho + \hat{\Pi} \rho - \hat{P}_{12} \rho - \hat{P}'_{12} \rho \right) \] (2.12)
\[ \rho_1 = \frac{1}{4} \left( \hat{I} \rho - \hat{\Pi} \rho + \hat{P}_{12} \rho - \hat{P}'_{12} \rho \right) \] (2.13)
\[ \rho_2 = \frac{1}{4} \left( \hat{I} \rho - \hat{\Pi} \rho - \hat{P}_{12} \rho + \hat{P}'_{12} \rho \right). \] (2.14)

It is easy to see that such an action is given by:
Incidentally we note that \( \rho_+ \) and \( \rho_- \) are also obtained as:

\[
\rho_+(x_1, x_2, x'_1, x'_2) = \psi_+(x_1, x_2) \psi_+^*(x'_1, x'_2) = \\
\frac{1}{4} \left\{ \psi(x_1, x_2) \psi^*(x'_1, x'_2) + \psi(x_2, x_1) \psi^*(x'_1, x'_2) + \psi(x_1, x_2) \psi^*(x'_2, x'_1) + \psi(x_2, x_1) \psi^*(x'_2, x'_1) \right\},
\]

and

\[
\rho_-(x_1, x_2, x'_1, x'_2) = \psi_-(x_1, x_2) \psi_-^*(x'_1, x'_2) = \\
\frac{1}{4} \left\{ \psi(x_1, x_2) \psi^*(x'_1, x'_2) - \psi(x_2, x_1) \psi^*(x'_1, x'_2) - \psi(x_1, x_2) \psi^*(x'_2, x'_1) + \psi(x_2, x_1) \psi^*(x'_2, x'_1) \right\}
\]

that is \( \rho_+ \) and \( \rho_- \) are the usual density matrices respectively associated to the usual symmetric and antisymmetric wave functions. (Here and in the whole paper everything is obtained for pure states but because of the linearity of the mixtures of pure states we can easily generalise to the density matrices of arbitrary mixed states of two particles.)

Thus, what we have done so far is to extend the action of the permutation group (which we know for the space of the wave functions) to the space of density matrices (Eqs. (2.7)-(2.10)) by means of Eq. (2.5) which links in a simple way the density matrices to the wave functions. The natural generalization of this procedure, that is the definition of the action of the permutation group on the space of tomograms through the expression of tomograms in terms of density matrices, is not so simple since such a relation is an integral transform. This implies that the density matrices and the MDFs depend on different variables and, as we will see in the next section, it is not clear what is the action of the permutation group on the new variables. We follow then a different approach which we illustrate preliminarily for the space of the density matrices. What we are looking for is an expression of the symmetrized and antisymmetrized density matrices in integral form and, more precisely, we search for a realization of the permutation group in terms of integral kernels. Starting from Eqs. (2.7), (2.10) it is easy to realize that

\[
\rho_\pm = \frac{1}{4} \left\{ \rho(x_1, x_2, x'_1, x'_2) + \rho(x_1, x_2, x'_2, x'_1) \pm \rho(x_2, x_1, x'_1, x'_2) + \rho(x_2, x_1, x'_2, x'_1) \right\}
\]

\[
= \frac{1}{4} \int dy_1 dy_2 dy'_1 dy'_2 \left\{ \delta_{y_1,y_2} \delta_{y'_1,y'_2} \delta_{y'_2,y'_1} + \delta_{y_1,y_2} \delta_{y'_2,y'_1} \delta_{y'_1,y'_2} + \delta_{y_1,y_2} \delta_{y'_2,y'_1} \delta_{y'_1,y'_2} + \delta_{y_1,y_2} \delta_{y'_2,y'_1} \delta_{y'_1,y'_2} \right\} \rho(y_1, y_2; y'_1, y'_2),
\]

with \( \delta_{y_i,x_j} = \delta(y_i - x_j) \), that is

\[
\rho_\pm(x_1, x_2; x'_1, x'_2) = \int K_\pm(x_1, x_2, x'_1, x'_2; y_1, y_2, y'_1, y'_2) \rho(y_1, y_2; y'_1, y'_2) dy_1 dy_2 dy'_1 dy'_2,
\]
with the kernel
\[ K_{\pm}(x_1, x_2, x'_1, x'_2; y_1, y_2, y'_1, y'_2) = \frac{1}{4} \left\{ \delta_{y_1, x_1} \delta_{y_2, x_2} \delta_{y'_1, x'_1} \delta_{y'_2, x'_2} + \delta_{y_1, x_2} \delta_{y_2, x_1} \delta_{y'_1, x'_2} \delta_{y'_2, x'_1} \right\} \]

(2.19)

Defining the permutation operator as
\[ P_{j_1, j_2, \ldots, j_n}(x_1, x_2, \ldots, x_n) = f(x_{j_1}, x_{j_2}, \ldots, x_{j_n}) \]
we can write the kernel as:
\[ K_{\pm}(x, x'|y, y'_2) = \frac{1}{(2!)^2} (I \pm P_{12}) \otimes (I \pm P'_{12}) \delta_{y_1, x_1} \delta_{y_2, x_2} \delta_{y'_1, x'_1} \delta_{y'_2, x'_2} \]

(2.21)
where \((x, x'|y, y'_2) \equiv (x_1, x_2, x'_1, x'_2; y_1, y_2, y'_1, y'_2)\) and we assume that \(P, P'\) act on \(x, x'\) respectively. Analogously we may define the kernels \(K_1, K_2\) associated to the density matrices \(\rho_1, \rho_2\). In this way we obtain a realization of the permutation group in terms of integral kernels. This approach is absolutely equivalent to the one previously described but it is the one we need to implement the symmetry properties in the MDF framework.

3 The permutation group of the MDF

We are now in the position to extend the previous procedure to the tomographic probability distribution, but we first need a brief review of the theory.

The MDF of a random variable \(X\) is defined in [1] as the Fourier transform of the quantum characteristic function \(\chi(k) = \langle e^{i k \hat{X}} \rangle\):
\[ w(X, t) = \frac{1}{2\pi} \int \frac{dk e^{-i k \hat{X}}}{e^{i k \hat{X}}} \]

(3.1)
where \(\hat{X}\) is the operator associated to \(X\), and, for each observable \(\hat{O}\), \(\langle \hat{O} \rangle = Tr(\hat{\rho} \hat{O})\) with \(\hat{\rho}\) the time-dependent density operator. The MDF so defined is positive and normalized to unity, provided \(\hat{X}\) is an observable [1].

In Ref. [19] it is shown that by taking
\[ X = \mu q + \nu p, \]
with \(q\) and \(p\) two conjugate variables and \(\mu, \nu\) real parameters labelling different reference frames in phase space, \(w(X, \mu, \nu, t)\) is normalized with respect to the \(X\) variable and there exists an invertible relation among the MDF and the density matrix. The variable \(X\) represents the position coordinate taking values in an ensemble of reference frames. Equation (3.1) may be rewritten in the more convenient form [20]
\[ w(X, \mu, \nu, t) = \frac{1}{2\pi |\nu| \hbar} \int \rho(Z', Z''; t) \exp \left[ -i \frac{Z' - Z''}{\nu \hbar} \left( X - \frac{Z' + Z''}{2} \right) \right] dZ' dZ'' \]

(3.3)
and its inverse is represented by

$$\rho(X, X', t) = \frac{1}{2\pi} \int w(Y, \mu, X - X') \exp \left[ \frac{i}{\hbar} \left( Y - \mu \frac{X + X'}{2} \right) \right] d\mu dY \quad (3.4)$$

It is important to note that, for (3.3) to be invertible, it is necessary that $X$ be a coordinate variable taking values in an ensemble of phase spaces; in other words, the specific choice $\mu = 1, \nu = 0$ or any other fixing of the parameters $\mu$ and $\nu$ would not allow to reconstruct the density matrix. Hence, the MDF contains the same amount of information on a quantum state as the density matrix, only if Eq. (3.2) is assumed.

For Hamiltonians of the form

$$H = \frac{p^2}{2m} + V(q) \quad (3.5)$$

an evolution equation governing the time dependence of the MDF is available (see ref. [20] for a simple derivation)

$$\partial_t w(X, \mu, \nu, t) = \left\{ \frac{\mu}{m} \partial_{\nu} + \frac{i}{\hbar} \left[ V(\partial_X)^{-1} \partial_{\mu} - \frac{i\nu \hbar}{2} \partial_X \right] \right\} w(X, \mu, \nu, t) \quad (3.6)$$

where the inverse derivative is defined as

$$(\partial_X)^{-1} \int f(Z) e^{g(Z)X} dZ = \int \frac{f(Z)}{g(Z)} e^{g(Z)X} dZ. \quad (3.7)$$

The evolution equation, of generalized Fokker-Plank type, plays the rôle of the Schrödinger equation in the alternative scheme we are outlining. Its classical limit is easily seen to be

$$\dot{w}(X, \mu, \nu, t) = \left\{ \frac{\mu}{m} \frac{\partial}{\partial \nu} + \nu V' \left( - \left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu} \right) \frac{\partial}{\partial X} \right\} w(X, \mu, \nu, t), \quad (3.8)$$

where $V'$ is the derivative of the potential with respect to the argument. Equation (3.8) may be checked to be equivalent to Boltzmann equation for a classical distribution of probability $f(q,p,t)$,

$$\frac{\partial f}{\partial t} + \frac{p}{m} \frac{\partial f}{\partial q} - \frac{\partial V}{\partial q} \frac{\partial f}{\partial p} = 0, \quad (3.9)$$

after performing the change of variables

$$w(X, \mu, \nu, t) = \frac{1}{2\pi} \int f(q,p,t) e^{i k (X - \mu q - \nu p)} dk \, dq \, dp; \quad (3.10)$$

Hence, the classical and quantum evolution equations only differ by terms of higher order in $\hbar$. Moreover, for potentials quadratic in $\dot{q}$, higher order terms cancel out and the quantum evolution equation coincides with the classical one. This leads to the remarkable result that there is no difference between the evolution of the distributions of probability
for quantum and classical observables, when the system is described by a Hamiltonian quadratic in positions and momenta. The generalization to \( N \) particles and eventually to field theory is straightforward and may be found in ref. [20].

The scheme just outlined is selfconsistent and doesn’t require at any step external structures such as the wave function or the density matrix. In this sense the probability description of quantum mechanics in terms of tomograms, once completely formulated, has to furnish a perfectly equivalent scheme to conventional ones [10]. Thus, it appears quite natural to investigate the symmetry properties of the MDF of identical particles with respect to permutations. To this regard we discuss for simplicity the case of two particles, while a generalization will be exhibited in the next section.

From eq. (3.3) the MDF of two identical particles is represented by:

\[
w(\xi_1, \mu_1, \nu_1; \xi_2, \mu_2, \nu_2) = k_2 \int D_2(z; z') e^{-i[\alpha_1 + \alpha_2]} \rho(z_1, z_2; z_1', z_2'; t) \quad (3.11)
\]

where we have defined

1) \( \alpha_i = \frac{z_i - z'_i}{\nu_i|\hbar|} \left[ \xi_i - \mu_i \frac{z_i + z'_i}{2} \right] \),
2) \( D_n(z; z'; x; \ldots; y) = (\Pi_{i=1}^n dz_i)(\Pi_{i=1}^n dz'_i)(\Pi_{i=1}^n dx_i) \ldots (\Pi_{i=1}^n dy_i) \),
3) \( k_n \equiv \Pi_{i=1}^n \frac{1}{2\pi \hbar |\nu_i|} \),

and the time-dependence has been omitted. We note that because

\[
\rho(z_1, z_2; z'_1, z'_2) = \rho_1(z_1, z'_1)\rho_2(z_2, z'_2) \quad (3.12)
\]

the tomogram of a system of two particles factorizes as:

\[
w(\xi_1, \mu_1, \nu_1; \xi_2, \mu_2, \nu_2) = w_1(\xi_1, \mu_1, \nu_1)w_2(\xi_2, \mu_2, \nu_2) \quad (3.13)
\]

with

\[
w_i(\xi_i, \mu_i, \nu_i) = \frac{1}{2\pi \hbar |\nu_i|} \int dz_i z'_i e^{-i\alpha_i} \rho_i(z_i, z'_i). \quad (3.14)
\]

Therefore it seems natural to identify the MDF relative to symmetrized and antisymmetrized states as

\[
w_\pm(\xi, \mu, \nu)_2 := k_2 \int D_2(z; z') e^{-i[\alpha_1 + \alpha_2]} \rho_\pm(z|z')_2 \quad (3.15)
\]

with \( (\xi, \mu, \nu)_n \equiv (\xi_1, \mu_1, \nu_1; \xi_2, \mu_2, \nu_2; \ldots \xi_n, \mu_n, \nu_n) \) (in the same way we may define \( w_1 \) and \( w_2 \) as the tomograms associated to \( \rho_1 \) and \( \rho_2 \).

From (2.18), which represents \( \rho_\pm \) in terms of the integral kernel, we can write (3.13) in the form

\[
w_\pm(\xi, \mu, \nu)_2 = k_2 \int D_2(z, y; z', y') K_\pm(y, y'|z, z')_2 e^{-i[\alpha_1 + \alpha_2]} \rho(y|y')_2, \quad (3.16)
\]
then, using the inverse formula (3.4) for the case of two particles

\[ \rho(y | y')_2 = \frac{1}{(2\pi)^2} \int D_2(m, x) e^{i[\beta_1 + \beta_2] w(x, m, y - y')_2} \]  

with \( \beta_i = x_i - m_i \frac{y_i + y'_i}{2} \) we obtain

\[ w_\pm(\xi, \mu, \nu)_2 = \frac{k_2}{(2\pi)^2} \int D_2(y, y', z, z', x, m, n) \delta_{n_1,y_1-y'_1} \delta_{n_2,y_2-y'_2} e^{-i[\alpha_1 + \alpha_2] e^{i[\beta_1 + \beta_2]} K_\pm(y, y'; z, z')_2 w(x, m, n)_2} \]

Thus the tomographic probability distribution corresponding to a symmetric or antisymmetric state of two particles may be finally written as

\[ w_\pm(\xi, \mu, \nu)_2 = \int D_2(x, m, n) \tilde{K}_\pm(\xi, \mu, \nu|x, m, n)_2 w(x, m, n)_2 \]  

with the kernel \( \tilde{K}_\pm \) given by

\[ \tilde{K}_\pm(\xi, \mu, \nu|x, m, n)_2 = \frac{k_2}{(2\pi)^2} \int D_2(y, y'; z, z') \delta_{n_1,y_1-y'_1} \delta_{n_2,y_2-y'_2} \times \exp[-i(\alpha_1 + \alpha_2) + i(\beta_1 + \beta_2)] K_\pm(y, y'|z, z')_2. \]  

Also in this case it is straightforward but not particularly illuminating to derive the integral kernels \( \tilde{K}_1, \tilde{K}_2 \), associated to \( w_1 \) and \( w_2 \), which complete the realization of the permutation group on the space of tomograms.

### 4 Generalization to \( N \) particles

The generalization to \( N \) particles is now straightforward: we define the kernel \( K_\pm(y, y'; z, z')_n \) as the following:

\[ K_\pm(y, y'; z, z')_n := \frac{1}{(n!)^2} \left( 1 + \sum_p \epsilon_p P_{j_1...j_n} \right) \otimes \left( 1 + \sum_{p'} \epsilon_{p'} P_{j'_1...j'_n} \right) \times \delta_{y_1,x_1} ... \delta_{y_n,x_n} \delta_{y'_1,x'_1} \delta_{y'_n,x'_n} \]

where \( \epsilon_p \) and \( \epsilon_{p'} \) are equal to 1 in the symmetric case, while representing, in the antisymmetric case, the sign of the permutations \( P \) and \( P' \) respectively. Therefore we obtain

\[ \tilde{K}_\pm(\xi, \mu, \nu)_n = \frac{k_n}{(2\pi)^n} \int D_n(y, y', z, z') \left[ \prod_{j=1}^{n} \delta_{y_j,y'_j} e^{-i\alpha_j} e^{i\beta_j} \right] K_\pm(y, y'; z, z')_n \]
so that we find
\[ w(\xi, \mu, \nu)_n = \int D_n(t, m, n) \tilde{K}_n(\xi, \mu, \nu|t, m, n) w(x, m, n). \] (4.3)

We may introduce a sort of generalized Slater determinant \[22\] to obtain \( K_\pm \), and, by means of (4.2), \( \tilde{K}_\pm \). We define
\[ \Delta_\pm(y|x)_n = \frac{1}{n!} \begin{vmatrix} \delta_{y_1, x_1} & \delta_{y_1, x_2} & \cdots & \delta_{y_1, x_n} \\ \delta_{y_2, x_1} & \delta_{y_2, x_2} & \cdots & \delta_{y_2, x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{y_n, x_1} & \delta_{y_n, x_2} & \cdots & \delta_{y_n, x_n} \end{vmatrix} \] (4.4)
with the convention that for the symmetric case we take always the plus sign in computing the determinant. In this way we can write the kernel \( K_\pm(y, y'|z, z') \) in a more conventional manner as:
\[ K_\pm(y, y'|z, z')_n := \Delta_\pm(y|z)_n \Delta_\pm(y'|z')_n. \] (4.5)

Summarizing, we have succeeded in realizing the symmetric and antisymmetric tomo-
graphic probability distribution associated to a system of \( N \) identical particles, eq. (4.3), in terms of the integral kernel (4.2). This result fits into the selfconsistent scheme outlined at the beginning of this section in the sense of yielding a realization of the permutation group on the space of tomograms which doesn’t require additional structures to be de-
dined. In the forthcoming section we will see in concrete how it works on an example and we will check the invertibility retriving the well known expressions for the symmetric and antysimmetric density matrices associated to a system of oscillators.

5 Application

Let us consider as an example the case of two independent harmonic oscillators. For this system the time-evolution equation (3.3) has been solved in ref. [21] and the solutions have the following expression:
\[ w_{nm}(x, \mu, \nu)_2 = \frac{\exp(-y_1^2-y_2^2)}{\pi r_1 r_2 n! m! 2^{n+m}} H_n^2(y_1) H_m^2(y_2) \] (5.1)
where \( r_j = \exp(it(\mu_j + i\nu_j)), y_j = x_j/|r_j|, j = 1, 2 \) and \( H_n(y) \) and \( H_m(y) \) are the Hermite polynomials of order \( n \) and \( m \) respectively.

For the two-dimensional case the kernel may be obtained in a closed form by integrating Eq. (3.20) over the \( z, z' \) and the \( y, y' \) variables. After some lengthy calculation we are able to write
\[ \tilde{K}_\pm(\xi, \mu, \nu)_2 = \pm \left[ f(\xi_1, \mu_1, \nu_1; \xi_2, \mu_2, \nu_2) + f(\xi_2, \mu_2, \nu_2; \xi_1, \mu_1, \nu_1) \right] \] (5.2)
• \( f(\xi_2, \mu_2, \nu_2; \xi_1, \mu_1, \nu_1) = \frac{k_2}{4} \delta \left( m_1 - \frac{\mu_1 n_1}{\nu_1} \right) \delta \left( m_2 - \frac{\mu_2 n_2}{\nu_2} \right) \exp \left\{ i \left( x_1 + x_2 - \frac{m_1}{\nu_1} \xi_1 - \frac{m_2}{\nu_2} \xi_2 \right) \right\} \);
• \( g(\xi_2, \mu_2, \nu_2; \xi_1, \mu_1, \nu_1) = \frac{k_2}{2} \frac{\nu_2 \nu_1}{\mu_2 \mu_1 - \mu_1 \nu_2} \exp \left\{ \frac{1}{2} v^T A v + B \cdot v + i(x_1 + x_2) \right\} \);
• \( v = \{m_1, n_1, m_2, n_2\} \);
• \( A = \begin{pmatrix}
\frac{i \nu_1 \mu_1 - \mu_2 \nu_1}{2(\mu_2 \nu_1 - \mu_1 \nu_2)} & \frac{i \mu_2 \nu_1 + \mu_1 \nu_2}{2 \mu_2 \nu_1 - \mu_1 \nu_2} & 0 & 0 \\
0 & 0 & \frac{i \mu_2 \nu_1 + \mu_1 \nu_2}{2 \mu_2 \nu_1 - \mu_1 \nu_2} & \frac{i \nu_1 \mu_1 - \mu_2 \nu_1}{2(\mu_2 \nu_1 - \mu_1 \nu_2)} \\
0 & 0 & \frac{i \nu_1 \mu_1 - \mu_2 \nu_1}{2(\mu_2 \nu_1 - \mu_1 \nu_2)} & \frac{i \mu_2 \nu_1 + \mu_1 \nu_2}{2 \mu_2 \nu_1 - \mu_1 \nu_2} \\
\end{pmatrix} \);
• \( B = \left\{ \frac{i \nu_2 \mu_2 - \mu_2 \nu_2}{\mu_2 \nu_2 - \mu_2 \nu_2}, \frac{i \mu_2 \xi_2 - \mu_2 \xi_1}{\mu_1 \nu_2 - \mu_2 \nu_2}, \frac{i \mu_2 \xi_1 - \mu_2 \xi_2}{\mu_2 \nu_2 - \mu_2 \nu_2}, \frac{i \nu_2 \mu_2 - \mu_2 \nu_2}{\mu_2 \nu_2 - \mu_2 \nu_2} \right\} \)

where \( v^T \) denotes the transposed vector and \( \cdot \) is the usual matrix product, rows by columns. This is all what we need to obtain the symmetrized and antisymmetrized MDF corresponding to all the states of the two oscillators, although Eq. (3.19) is in general not easy to integrate.

For the first non trivial case, i.e. the first excited state, \( w^0_{1+}(\xi, \mu, \nu)_2 \), we obtain, substituting Eq. (5.2) and Eq. (5.1) for \( w^0_{01} \), we obtain, substituting Eq. (5.2) and Eq. (5.1) for \( w^0_{01} \) into Eq. (3.19)

\[
\begin{align*}
\xi_1 \xi_2 & + 2 \xi_1 \xi_2 
\frac{\xi_1^2}{\mu_1^2 + \nu_1^2} & + \frac{\xi_2^2}{\mu_2^2 + \nu_2^2} \cdot 
\end{align*}
\]

As we can see, only the first two terms of the sum are probability distributions, whereas the last one is an interference term.

For our edification we can check, using the inverse formula (3.17), that we obtain the correct expression for the symmetrized and antisymmetrized density matrices associated to a system of two harmonic oscillators, that is

\[
\rho^0_1 (x, x')_2 = \frac{1}{4} \exp \left\{ -\frac{1}{2} (x_1^2 + x_2^2 + x_1'^2 + x_2'^2) \right\} [4x_2x'_2 + 4x_1x'_1 \pm (4x_1x'_2 + 4x_2x'_1)].
\]

6 Conclusions

In summary, we have studied the permutational symmetry of tomograms of quantum states for systems of identical particles of bosonic and fermionic nature. We showed that
the tomographic probability distributions of identical particles in the probability representation of quantum mechanics are associated to the realization of the action of $G \otimes G$, where $G$ is the permutation group and $\otimes$ denotes the direct product. It is demonstrated that the tomographic probability distribution, that is the MDF is a sum of terms, some of them directly recognizable as tomograms, some others which are interference terms, that is not associated to a probability distribution by themselves, but only in the given combination. The main results of our analysis are summarized by Eqs. (4.2), (4.3), where tomograms of identical particles are explicitly derived in terms of an integral kernel.

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