DILUTE BOSE-EINSTEIN CONDENSATE IN A TRAP: 
CHARACTERISTIC LENGTHS AND CRITICAL VELOCITIES

ALEXANDER L. FETTER
Departments of Physics and Applied Physics, Stanford University
Stanford, CA 94305-4060, USA
E-mail: fetter@leland.stanford.edu

The Bogoliubov approximation and the Gross-Pitaevskii equation characterize the effect of repulsive interactions on a dilute ideal Bose-Einstein gas in a spherical harmonic trap. For large $N$, the interactions expand the condensate relative to an ideal Bose gas; both the speed of sound and critical angular velocity $\Omega_c$ for creation of a quantized vortex depend crucially on the interparticle repulsion through the coherence length $\xi$.

1 Ideal Bose gas

An ideal Bose gas provides a valuable introduction to the physics of a real dilute Bose gas, for it emphasizes the fundamental role of the coherent condensate that contains a macroscopic number of particles in a single quantum state. The standard example treats $N$ noninteracting bosons in a volume $V$ with periodic boundary conditions and uniform density $n = N/V$. In the classical limit, the thermal de Broglie wavelength $\lambda_T \equiv (2\pi\hbar^2/k_B T)^{1/2}$ is much shorter than the interparticle spacing $l \sim n^{-1/3}$. As the temperature falls, however, the thermal de Broglie wavelength grows, and the system eventually becomes degenerate when $\lambda_T \sim l$. Equivalently, Bose-Einstein condensation in a uniform ideal gas occurs at $T_c \sim \hbar^2 n^{2/3}/k_B m$.

The situation is somewhat different for an ideal Bose gas in a harmonic trap (taken as isotropic for simplicity) with $V_{\text{trap}}(r) = \frac{1}{2}m\omega_0^2 r^2$; the corresponding oscillator length $d_0 = (\hbar/m\omega_0)^{1/2}$ characterizes the size of the (Gaussian) ground state. In the classical limit ($k_B T \gg \hbar \omega_0$), the density follows the Boltzmann distribution $n_{cl}(r) \propto \exp[-V_{\text{trap}}(r)/k_B T]$, which can be rewritten as $\exp(-r^2/2R_T^2)$, with $R_T \equiv d_0(k_B T/h\omega_0)^{1/2}$ the classical thermal radius of the trapped gas (note that $R_T \gg d_0$).

To estimate the temperature $T_c$ for the onset of Bose-Einstein condensation in a trap, use the same expression with $n \sim N/R_T^3$, so that $k_B T_c \sim \hbar^2 N^{2/3}/mR_T^5 \sim \hbar^2 \omega_0^2 N^{2/3}/k_B T_c$; equivalently, $k_B T_c \sim N^{1/3}h\omega_0 \gg h\omega_0$. Since $d_0 \ll R_T$, the condensate forms a narrow spike of width $d_0$ superimposed on the smooth background of width $R_T$, as is seen clearly in the original experiment with a few thousand $^{87}$Rb atoms. More recent experiments have condensed millions of atoms, with typical parameters $N \approx 5 \times 10^6$, $\omega_0/2\pi \approx 120$ Hz, and $d_0 \approx 1.9 \mu$m for sodium atoms.

2 Effect of Repulsive Interactions

The interparticle spacing is typically large compared to the range of the atomic interactions, so that the interparticle potential can be taken as $U(r) \approx U_0 \delta(r)$, with $U_0 = 4\pi a \hbar^2/2m$ and $a$ the $s$-wave scattering length ($a \approx 2.75$ nm for sodium atoms). The Bogoliubov approximation assumes that nearly all particles are in the
condensate; at zero temperature, the system forms a dilute Bose gas with \( na^3 \ll 1 \).

2.1 **Quasiparticles and the Coherence Length**

A uniform Bose gas at \( T = 0 \text{ K} \) has only two characteristic single-particle energies: the kinetic energy 
\( \epsilon_k = \hbar^2 k^2 / 2m \) and the “Hartree” energy
\[
V_H(r) = \int d^3r' U(r - r') n(r') = U_0 n = 4\pi \hbar^2 n / m. \tag{1}
\]

Bogoliubov showed that a quasiparticle with wave vector \( k \) has an energy
\[
E_k = \sqrt{2V_H \epsilon_k + \epsilon_k^2} \approx \begin{cases} 
\hbar k, & \text{for } k \to 0 \text{ (phonons)}, \\
\hbar^2 k^2 / 2m, & \text{for } k \to \infty \text{ (free particles)},
\end{cases} \tag{2}
\]

where \( s^2 = U_0 n / m = 4\pi \hbar^2 n / m^2 \) is the squared speed of sound. The quasiparticle is a linear combination of a particle and a hole. For \( k \to 0 \), it is an equal particle-hole admixture, but it becomes a pure particle as \( k \to \infty \), with the cross-over occurring at \( k \approx \xi^{-1} \), where \( \xi = (8\pi n)^{-1/2} \) defines the “coherence” length. Note that the speed of sound can be rewritten as \( s = \kappa / 2\pi \sqrt{2\xi} \), where \( \kappa = \hbar / m \) is the quantum of circulation. When \( na^3 \ll 1 \), it is easy to verify that \( \xi \gg l \gg a \).

2.2 **Radius of Interacting Condensate in a Trap**

The presence of the trap introduces a third characteristic energy, and the Gross-Pitaevskii (GP) equation for the nonuniform condensate wave function \( \Psi(r) \) has the form
\[
(T + V_{\text{trap}} + V_H) \Psi = \mu \Psi, \tag{3}
\]
where \( T = -\hbar^2 \nabla^2 / 2m \) is the kinetic energy operator, \( V_{\text{trap}}(r) = \hbar^2 \omega_0 (r/d_0)^2 / 2m \) is the trap potential energy, \( V_H(r) = U_0 n(r) = 4\pi \hbar^2 a(r) / m \) is the Hartree potential energy of one particle with the remaining particles, and \( \mu \) is the chemical potential.

For very low density, the coherence length \( \xi \) exceeds the trap size \( d_0 \), and the system acts like an ideal Bose gas. As \( N \) increases, however, \( \xi \) shrinks; when \( \xi \) becomes comparable with \( d_0 \), the repulsive interactions begin to predominate, and the self-consistent radius \( R_0 \) of the condensate expands beyond \( d_0 \).

To estimate the actual radius, note that the kinetic energy per particle is of order \( T \sim \hbar^2 / m R_0^2 \sim \hbar \omega_0 / R^2 \), where \( R \equiv R_0 / d_0 \) is the dimensionless radius of the condensate. Similarly, the order of magnitude of the remaining single-particle energies are \( V_{\text{trap}} \sim \hbar \omega_0 / R^2 \), and
\[
V_H \sim U_0 n \sim \frac{\hbar^2 a}{m} n \sim \frac{\hbar^2 a}{m} \frac{N}{R_0^2} \sim \frac{\hbar \omega_0}{R^2} \frac{N a}{d_0}, \tag{4}
\]

where the dimensionless parameter \( \eta \equiv N a / d_0 \) characterizes the strength of the interactions. For \( \eta \ll 1 \), the system is effectively ideal, but for \( \eta \gg 1 \), the interactions become crucial (nevertheless, the system remains dilute with \( a \ll l \) for all practical trapped condensates). If \( \langle \cdots \rangle \equiv \int dV \Psi^* \cdots \Psi \) denotes a condensate ground-state expectation value, the total energy \( E = \langle T + V_{\text{trap}} + \frac{1}{2} V_H \rangle \) is of order \( \sim N \hbar \omega_0 (R^{-2} + R^2 + \eta R^{-3}) \), and \( R \) is determined by minimizing \( E \).
For small $\eta$, the minimum energy occurs at $R \approx 1$, and the condensate radius is just $d_0$, with $\mu \approx \frac{3}{2} \hbar \omega_0$ (the ground-state energy of the isotropic oscillator). For large $\eta$, in contrast, the kinetic energy is negligible, and the minimum total energy occurs for $R^2 \sim \eta$. In this limit, a detailed calculation\footnote{The resulting “Thomas-Fermi” (TF) approximation neglects the kinetic energy entirely, and the GP Eq. (3) then determines the condensate density by the condition $V_{\text{trap}}(r) + V_H(r) = \mu$, showing that the resulting density profile is parabolic. The corresponding central density $n(0) = R_0^3/8\pi a d_0^3 = (15/8\pi) N/R_0^3$ serves to define both the speed of sound $s^2 = 4\pi a \hbar^2 n(0)/m^2$ and the coherence length $\xi^2 = 1/8\pi a n(0)$ for a trapped condensate; as a corollary, the relation $\xi R_0 = d_0^2$ implies that $d_0$ is the geometric mean of $\xi$ and $R_0$. The previous expression $s = \kappa/2\pi \sqrt{2} \xi$ then shows that the speed of sound in a large trapped condensate increases linearly with $R_0$, reflecting the increased density. For the MIT experiments\footnote{In the Bogoliubov approximation, most of the particles remain in the condensate, which requires $\left[n(0)a^3\right]^{1/2} \ll 1$. For a trapped condensate at $T = 0$ K, this condition of small total noncondensate number holds for $R_0 a/d_0^2 \ll 1$. When this dimensionless ratio becomes comparable with 1, however, the repulsive interactions are so strong that the total noncondensate number becomes of order $N$ (this situation occurs in liquid helium, even at $T = 0$ K). Comparison with the TF expression $\eta = N a/d_0 \sim R^3$ shows that the Bogoliubov approximation fails for a trapped condensate when $N \sim (d_0/a)^6$, which is far larger than any current experimental value (for the MIT trap with sodium atoms, this limit merely requires that $N \ll 10^{15}$).} these relations yield $n(0) \approx 4 \times 10^{20}$ m$^{-3}$, $s \approx 1.04$ cm/s, and $\xi \approx 0.187$ $\mu$m.

In the Bogoliubov approximation, most of the particles remain in the condensate, which requires $\left[n(0)a^3\right]^{1/2} \ll 1$. For a trapped condensate at $T = 0$ K, this condition of small total noncondensate number holds for $R_0 a/d_0^2 \ll 1$. When this dimensionless ratio becomes comparable with 1, however, the repulsive interactions are so strong that the total noncondensate number becomes of order $N$ (this situation occurs in liquid helium, even at $T = 0$ K). Comparison with the TF expression $\eta = N a/d_0 \sim R^3$ shows that the Bogoliubov approximation fails for a trapped condensate when $N \sim (d_0/a)^6$, which is far larger than any current experimental value (for the MIT trap with sodium atoms, this limit merely requires that $N \ll 10^{15}$).

3 Critical Linear and Angular Velocities

Landau's original explanation of superfluidity involved the critical velocity $v_c$; it is the speed at which a moving macroscopic impurity can create quasiparticles and thus lose energy. A simple analysis shows that $v_c$ is the minimum value of the ratio $\omega_k/k$, where $\omega_k$ is the dispersion relation for the quasiparticles. From this perspective, a uniform ideal Bose gas has zero critical velocity, for the dispersion relation is simply $\hbar k^2/2m$. In contrast, Eq. (3) shows that a uniform dilute interacting Bose gas indeed has a nonzero critical velocity, with $v_c = s \propto a^{1/2}$, arising from the presence of the repulsive interactions.

3.1 Critical Angular Velocities in a Trapped Condensate

For a dilute trapped condensate, it is natural to take the speed of sound $s$ as the critical velocity. It is impractical to shoot an impurity through the condensate, but the equatorial speed of a rotating condensate can serve to define a corresponding critical angular velocity $\Omega_c = s/R_0 = \hbar md_0^2\sqrt{2} = \omega_0/\sqrt{2}$. Thus a large vortex-free condensate rotating at an angular velocity $\Omega \leq \Omega_c$ should remain superfluid (this expression also indicates that the frequencies of low-lying compressional modes in
a trapped condensate are independent of the radius and of order $\omega_0$.

The low-lying hydrodynamic modes of a large spherical condensate have the dispersion relation $\omega_{nl} = \omega_0 [l + n(2n + 2l + 3)]^{1/2}$, where $n$ is the radial quantum number and $l$ is the orbital-angular-momentum quantum number. For fixed $l$, the radial wavenumber $k$ is $\approx n/R_0$, and a corresponding more precise Landau critical angular velocity is $\sqrt{2}\omega_0$ (this value occurs as $k \rightarrow \infty$).

It is helpful to review the effect of rotation $\Omega = \Omega \hat{z}$ on a sample of liquid helium. If the fluid is normal, the microscopically rough walls bring it into solid-body rotation $\mathbf{v}_{\text{sb}} = \Omega \times \mathbf{r}$, where $\mathbf{r}$ is the distance from the axis of rotation; this flow has uniform vorticity, with $\nabla \times \mathbf{v}_{\text{sb}} = 2\Omega$. In the rotating frame, the walls are stationary, and the relevant zero-temperature thermodynamic function is the “free energy” $F = E - \mathbf{\Omega} \cdot \mathbf{L}$, where $E$ is the total ground-state energy and $\mathbf{L}$ is the total ground-state angular momentum. For a circular cylinder of radius $R_0$, the free energy of a state with one vortex on the symmetry axis becomes lower than that with no vortex at a lower critical angular velocity $\Omega_{c1} = (\kappa/2\pi R_0^2) \ln(R_0/\xi)$, where $\xi$ represents the vortex core radius ($\xi$ is a few atomic diameters for superfluid $^4$He).

Assuming that a similar expression holds for the creation of a vortex in a large trapped condensate, the lower critical angular velocity is smaller than the trap frequency $\omega_0 = \kappa/2\pi d_0^2$ by a factor of order $(\xi/R_0) \ln(R_0/\xi) \ll 1$. In addition to the compressional modes with frequencies of order $\omega_0$, the presence of a vortex line introduces new dynamical degrees of freedom associated with “vortex waves.” At long wavelengths ($k\xi \ll 1$), the classical vortex-wave dispersion relation $\omega_k \approx (k\kappa^2/4\pi) \ln(1/k\xi)$ immediately suggests low-lying normal modes with $k \sim R_0^{-1}$ and frequencies of order $\Omega_{c1}$, which might serve to signal their presence.

### 3.2 Analogy with Type-II Superconductors

These results for $\Omega_c$ and $\Omega_{c1}$ are very similar to the thermodynamic critical field $H_{c} = \Phi_0/2\pi\sqrt{2}\lambda$ and lower critical field $H_{c1} \approx (\Phi_0/2\pi\lambda^2) \ln(\lambda/\xi)$, where $\Phi_0 = \hbar/2e$ is the flux quantum, $\lambda$ is the penetration length, and $\xi$ is vortex core radius (also the superconducting coherence length). A uniform bulk superconductor becomes unstable with respect to normal metal at the field $H_c$, and the formation of quantized flux lines (vortices) becomes favorable at the field $H_{c1}$. In a type-II superconductor (one with $\lambda > \xi/\sqrt{2}$), the thermodynamic instability at $H_c$ is preempted by vortex formation at $H_{c1} < H_c$.

The electromagnetic currents in a charged superfluid cut off the logarithmic intervortex interaction potential, screening it exponentially beyond the penetration length $\lambda$. This quantity diverges as the charge on each particle tends to zero, and the corresponding “screening” length in a neutral superfluid becomes either the radius of the container or the intervortex separation, whichever is smaller.

In addition, a type-II superconductor ultimately becomes normal at the upper critical field $H_{c2} = \Phi_0/2\pi\xi^2$, roughly when the vortex cores overlap. Unfortunately, the corresponding $\Omega_{c2} = \kappa/2\pi\xi^2$ is unattainably large in superfluid $^4$He, and mechanical instability for $\Omega \geq \omega_0$ may also render it unobservable in a rotating dilute trapped condensate (in this case, a trapped condensate in equilibrium could contain only relatively few vortices).
3.3 Non-Axisymmetric Rotating Traps

The magnetic field that confines a dilute atomic Bose condensate acts simply as a potential $V_{\text{trap}}(r)$; this situation differs greatly from the microscopically rough walls of a container for superfluid helium. Thus, a “rotating trap” is meaningful only to the extent that it is nonaxisymmetric, for the rotating time-dependent potential pushes the condensate, setting it into motion.

In contrast to the case of an axisymmetric potential such as a circular cylinder, the free energy $E - \Omega L$ of an irrotational vortex-free state in a nonaxisymmetric trap decreases with increasing $\Omega$ (like $-\frac{1}{2}\Omega^2$, where $I$ is an effective moment of inertia). In the limit of large distortion, $I$ can approach that of solid-body rotation. The negative free energy of the vortex-free state typically delays the onset of vortex formation, for the vorticity localized in the vortex cores becomes less necessary to “mimic” the uniform vorticity of $\mathbf{v}_{\text{db}}$. This effect is readily verified in simple cases. For a uniform superfluid in a long rotating elliptic cylinder with semiaxes $a$ and $b$, the preceding expression $\Omega_{c1} = (\kappa/2\pi b^2) \ln(b/\xi)$ applies if $b = a$, but the corresponding lower critical angular velocity $\Omega_{c1} \approx (\kappa/4\pi b^2) \ln(b/\xi)$ for $b \ll a$ can become significantly larger.

The GP Eq. (3) provides a basis for analyzing a rotating trap, where the confining potential $V_{\text{trap}}(r)$ is stationary in the rotating frame ($r$ now denotes the coordinate in the rotating frame). The free energy is given by $E - \Omega L_z$, where $E$ is the ground-state energy (assuming $T = 0$ K for simplicity). The condensate wave function $\Psi = |\Psi|e^{iS}$ can be expressed as a magnitude $|\Psi| = n^{1/2}$ and a phase $S$ that determines the superfluid velocity $\mathbf{v} = (\hbar/m)\nabla S$. The TF limit of a large condensate neglects the spatial variation of $n$, and the free energy becomes

$$F \approx \int d^3r \left( \frac{\hbar}{2m} \mathbf{v}^2 + nV_{\text{trap}} + \frac{\hbar}{4m} n^2 - mn\Omega \hat{z} \cdot \mathbf{r} \times \mathbf{v} \right).$$  \hspace{1cm} (5)

In a singly connected container, $\mathbf{v}$ can be written as a sum of two contributions: $\mathbf{v}_0 \propto \Omega$ arising from the moving walls and $\mathbf{v}_k$ arising from the vortices with circulation $\kappa$. Correspondingly, $F$ separates into a (vortex-free) contribution $F_0$ associated solely with $\mathbf{v}_0$ and $F_\kappa = \int d^3r \left( mn\mathbf{v}_0 \cdot \mathbf{v}_k + \frac{\hbar}{2m} mnv_k^2 - mn\Omega \hat{z} \cdot \mathbf{r} \times \mathbf{v}_k \right)$ that also depends on the angular velocity $\Omega$, both explicitly and through $\mathbf{v}_0$. The critical angular velocity $\Omega_{c1}$ for vortex creation occurs when $F_\kappa$ first vanishes.

The integral for $F$ provides a variational expression in the rotating frame; as an approximate trial function for the phase, take $S = -(\hbar/m)\Phi$, where $\Phi$ is the classical velocity potential for a uniform irrotational fluid (conventionally defined so that $\mathbf{v} = -\nabla \Phi$). This velocity field is automatically irrotational apart from the vortex cores and satisfies the condition that its normal derivative match the normal velocity of the rotating boundary. In addition, $\nabla \cdot \mathbf{v} = 0$, which is the appropriate TF limit of the continuity equation $\nabla \cdot (n\mathbf{v}) = 0$. The actual trapped-condensate density differs from that for $\Omega = 0$ only because of the Coriolis and centrifugal forces of order $m\Omega \times \mathbf{v}$ and $m\Omega \times (\Omega \times \mathbf{r})$, respectively. Each of these is small compared to the force of the harmonic trap so long as $\Omega \ll \omega_0$, in which case it is permissible to neglect the change in density (apart from the formation of the small vortex core of radius $\approx \xi$).

As a result, the free energy $F_0$ for an irrotational condensate in a rotating trap has the same form as that for a uniform incompressible irrotational fluid, but with
the particle density taken from the solution of the GP equation. Since the actual
density varies slowly in the TF limit, $F_{\Omega}$ differs from the classical expression only
through factors of order unity, representing various moments of the actual density.
Similarly, the additional free energy $F_\kappa$ arising from the presence of a vortex line
also differs from the classical expression for a uniform incompressible fluid only
by factors of order unity, because the TF density profile cuts off the logarithmic
divergence in the kinetic energy at a distance of order $\xi$, which can be identified with
the classical vortex radius. Consequently, the classical analysis for uniform fluid
in a rotating elliptical cylinder provides a qualitative guide to the corresponding
problem of vortex nucleation in a rotating elliptical trap.

This correspondence remains valid for a multiply connected container, when the
velocity field $v$ includes an additional contribution $v_\Gamma$ arising from the quantized
circulation $\Gamma = j\kappa$ around the various internal boundaries (here, $j$ is an integer). For
example, consider an incompressible fluid in an asymmetric annular region between
two nonconcentric cylinders (outer radius $R_2$ and inner radius $R_1 < R_2$), with the
centers displaced a distance $d \leq R_2 - R_1$. If the system rotates in equilibrium
at angular velocity $\Omega$ around the symmetry axis of the inner cylinder, the critical
angular velocity $\Omega_\kappa$ for the creation of vortex-free quantized circulation $\kappa$ around
the inner boundary can be evaluated for arbitrary $d$. In the symmetrical limit
($d \to 0$), the result $\Omega_\kappa = (\kappa/2\pi)(R_2^2 - R_1^2)^{-1}\ln(R_2/R_1)$ is similar to that for a
cylinder; in the small-gap limit, however, the resulting expression $\Omega_\kappa \approx \kappa/4\pi R_1 R_2$
has no logarithmic factor. These expressions can serve to estimate the corresponding
quantities for a trapped toroidal condensate (created, for example, by piercing the
condensate with an off-axis laser beam). The circulation-induced deformation of
the condensate might serve to detect the presence of a persistent current.

Acknowledgments

I am grateful to M. R. Andrews and D. Rokhsar for valuable discussions and insights.
This research is supported in part by the National Science Foundation, under Grant
No. DMR 94-21888.

References

1. M.H. Anderson et al., Science 269, 198 (1995).
2. M.-O. Mewes et al., Phys. Rev. Lett. 77, 416 (1996).
3. N.N. Bogoliubov, J. Phys. (USSR) 11, 23 (1947)
4. E.P. Gross, Nuovo Cimento 20, 454 (1961).
5. L.P. Pitaevskii, Sov. Phys.-JETP 13, 451 (1961).
6. G. Baym and C.J. Pethick, Phys. Rev. Lett. 76, 6 (1996).
7. S. Stringari, Phys. Rev. Lett. 77, 2360 (1996).
8. F. Dalfovo and S. Stringari, Phys. Rev. A 53, 2477 (1996).
9. See, for example, M. Tinkham, Introduction to Superconductivity (McGraw-
   Hill, NY, 1996), 2nd ed., Chaps. 4 and 5.
10. A.L. Fetter, J. Low Temp. Phys. 16, 533 (1974).
11. M.R. Andrews, private communication.