Existence of torsion-free $G_2$-structures on resolutions of $G_2$-orbifolds using weighted Hölder norms

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Abstract

An alternative proof of the existence of torsion-free $G_2$-structures on resolutions of $G_2$-orbifolds considered in [JK17] is given. The proof uses weighted Hölder norms which are adapted to the geometry of the manifold. This leads to better control of the torsion-free $G_2$-structure and a simplification over the original proof.

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1 Introduction

In [Ber55], Berger presented a list of groups which can possibly occur as the holonomy groups of Riemannian manifolds. However, constructing manifolds which realise these holonomy
groups remained a wide open problem for decades. A milestone in this direction was the formu-
lation and proof of the Calabi conjecture in [Cal54, Cal57] and [Yau77, Yau78] respectively.
Among other things, the proof of this conjecture gives a powerful characterisation of man-
ifolds admitting a metric with holonomy SU(n), giving rise to a wealth of examples of such
manifolds. For the exceptional holonomy group G2, such a general characterisation remains
out of reach, and even the construction of examples persists to be a challenging task.

The first compact examples of Riemannian manifolds with holonomy equal to G2 were con-
structed in [Joy96b] by resolving an orbifold of the form T7/Γ, where Γ is a finite group of
isometries of T7. In [K.7], this construction was extended to resolutions of orbifolds of the
form M/Γ, where M is a manifold with holonomy contained in G2, but not necessarily flat.
In both cases, the orbifold singularities are resolved by glueing in families of Eguchi-Hanson
spaces. Near L = fix(Γ), these glued in bits are locally modelled on R3 × X, where X denotes
Eguchi-Hanson space, and carry a closed G2-structure with small torsion. In the case of T7/Γ,
the fixed point set L is the disjoint union of several flat T3, and the G2-structure on the glued
in part is exactly the product G2-structure of T3 × X, which is torsion-free.

Now, denote the resolution of M/Γ by N. On N, one can define a 1-parameter family of
G2-structures ϕI N by glueing together the G2-structure from the orbifold, and a G2-structure
around L that is constructed using the Hyperkahler triple on Eguchi-Hanson space. The para-
meter t ∈ (0, 1) controls the size of the glued in Eguchi-Hanson spaces. ϕI N is not torsion-free,
but its torsion tends to zero as t → 0. [Joy96b] Theorem A states: if the torsion of ϕI N satisfies
certain smallness estimates, then there exists a torsion-free G2-structure ϕ̃N I on N satisfying
\[ ∥ϕ̃N I − ϕI N∥L∞ ≤ ct^{1/2} \] (see theorem 3.8 for the statement). This theorem is very general and
applies to all manifolds that carry a G2-structure with small torsion, not only to resolutions of
M/Γ.

Now that existence of ϕ̃N I has been established we ask: is this estimate for the difference
between ϕ̃N I and ϕI N optimal, or are the two forms actually closer? This question is answered by
corollary 4.34

**Corollary.** Let ϕ I be the G2-structure on the resolution N of T7/Γ defined in eq. (1.6). Then, for
e ∈ (0, 1/2) and t small enough (depending on e) there exists η I ∈ Ω2(N) such that ϕ I := ϕ I + dη I
is a torsion-free G2-structure on N satisfying
\[ ∥ϕ I − ϕ I∥L∞ ≤ t^{1−e}. \]

Here, norms are defined using the metric induced by ϕ I.

Here, the first estimate is stated using a weighted Hölder norm, which amounts to saying that
away from the glueing region, the difference ϕ I − ϕ I is bounded by t4. We still get a mildly
improved control over the rest of the manifold: everywhere on N we have an upper bound of
\[ t^{1−e} \] rather than the original \[ ct^{1/2} \].

The case of M/Γ is different from the case of T7/Γ, in that the naively glued ϕI N does not
satisfy the necessary smallness estimates of the original existence theorem. In [K.7], the
authors overcome this problem by ingeniously constructing a correction ϕN I,corr of ϕI N which
satisfies the smallness estimates.

In this setting, we prove theorem 5.41, which is an improvement of the original [Joy96b] The-
orem A in the situation of M/Γ. This new theorem requires weaker estimates, which gives a
new, simpler proof for the fact that the resolution of $M/\Gamma$ carries a torsion-free $G_2$-structure, avoiding the construction of $\varphi_t^{N,\text{corr}}$, this is corollary \ref{cor:small_torsion}. On the other hand, if the new theorem is applied to $\varphi_t^{N,\text{corr}}$, it gives a better estimate for the difference $\varphi_t^{N,\text{corr}} - \varphi_t^{N,\text{corr}}$, see corollary \ref{cor:small_torsion}. Here is the statement of theorem \ref{thm:main}.

**Theorem.** Let $\beta \in (-4, -2)$ and let $(N_t, \varphi_t^N)$ be the resolution of $M/\langle i \rangle$ from definition \ref{def:resolution} endowed with the $G_2$-structure from eq. \ref{eq:G2_structure}. There exists $c_1, c_2 \in \mathbb{R}$ such that the following is true: If $\varphi$ is a closed $G_2$-structure on $N_t$ and $\vartheta \in \Omega^0(N_t)$ such that $d^* \vartheta = d^* \varphi$ and

$$||d^* \vartheta||_{C^{0,\alpha}_{\beta-2,2}} \leq c_1 t^\kappa,$$

$$||\vartheta||_{C^{0,\alpha}_{0,1}} \leq c_2$$

for $\kappa > 1 - \beta + \alpha$, then for $t$ small enough there exists $\eta \in \Omega^2(N_t)$ such that $\tilde{\varphi} := \varphi + d \eta$ is a torsion-free $G_2$-structure on $N_t$ satisfying

$$||\tilde{\varphi} - \varphi||_{C^{1,\alpha/2}_{\beta-1,1}} \leq t^\kappa.$$

Here, norms are defined using the metric induced by $\varphi_t^N$.

In section \ref{sec:analytic_results} we collect analytic results about Eguchi-Hanson space that will be needed in the later chapters. Section \ref{sec:necessary_background} contains an overview over the necessary background material of $G_2$-geometry.

Over the course of section \ref{sec:construction} we will review the construction of a $G_2$-structure with small torsion on the resolution of $T^7/\Gamma$, and then prove corollary \ref{cor:small_torsion} to give an estimate for the difference $\varphi_t - \varphi_t^N$. The analysis hinges on an estimate for the inverse of the Laplacian given in proposition \ref{prop:inverse_Laplacian}.

Lastly, in section \ref{sec:resolution} we bring the results from the previous section to the setting of $M/\langle i \rangle$. We explain the construction of $\varphi_t^N$ in detail, which is less involved compared to the construction of $\varphi_t^{N,\text{corr}}$ that was necessary in \cite{JK17}. All the analytic results are proven in much the same way as for $T^7/\Gamma$.

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## 2 Analysis on the Eguchi-Hanson Space

The singularities of the $G_2$-orbifolds that were studied in \cite{Joy96, Ki7} are locally modelled on $\mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\}$. In order to resolve these singularities, we study the resolution of the point singularity of $\mathbb{C}^2/\{\pm 1\}$ in this section.

Consider the blowup of $\mathbb{C}^2/\{\pm 1\}$, which is again a complex surface. It admits a Hyperkähler structure that is asymptotically locally Euclidean (ALE), see \cite[Section 7.2]{Joy96} and \cite{D99}.
for surveys listing these and more properties. In this section, we will define ALE Hyperkähler manifolds, write down explicit formulae for the metric and Kähler forms on Eguchi-Hanson space (cf. proposition 2.2), show that they satisfy the ALE Hyperkähler property (cf. proposition 2.6), identify the harmonic forms on Eguchi-Hanson space (cf. lemma 2.13), and prove a technical lemma that will be used in the later sections.

We begin with the definition of Hyperkähler manifolds.

Definition 2.1. Define the quaternions \( \mathbb{H} \) to be the associative, nonabelian real algebra

\[
\mathbb{H} = \{ x_0 + x_1 i + x_2 j + x_3 k : x_j \in \mathbb{R} \} = \mathbb{R}^4,
\]

endowed with the unique multiplication satisfying

\[
ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \quad i^2 = j^2 = k^2 = -1.
\]

Let \( \mathbb{H}^m \) be coordinates \((q^1, \ldots, q^m)\), with \( q^I = x_0^I + x_1^I i + x_2^I j + x_3^I k \in \mathbb{H} \) and \( x^I_\alpha \in \mathbb{R} \). Define a metric and 2-forms on \( \mathbb{H}^m \) by

\[
g = \sum_{I=1}^{m} \sum_{\alpha=0}^{3} (d x^I_\alpha)^2, \quad \omega_1 = \sum_{I=1}^{m} d x^I_0 \wedge d x^I_1 + d x^I_2 \wedge d x^I_3,
\]

\[
\omega_2 = \sum_{I=1}^{m} d x^I_1 \wedge d x^I_2 - d x^I_0 \wedge d x^I_3, \quad \omega_3 = \sum_{I=1}^{m} d x^I_0 \wedge d x^I_3 + d x^I_1 \wedge d x^I_2.
\]

Define complex structures \( I, J, K \) on \( \mathbb{H}^m \) to be left multiplication with \( i, j, k \) respectively. The subgroup of \( \text{GL}(4m, \mathbb{R}) \) preserving \( g, \omega_1, \omega_2, \omega_3 \) is \( \text{Sp}(m) \). It also preserves \( I, J, K \).

A \( 4m \)-dimensional Riemannian manifold \((M, g)\) is called Hyperkähler if \( \text{Hol}(g) \subset \text{Sp}(m) \).

Thus, on a Hyperkähler manifold we have the compatible data of a metric and three compatible complex structures and symplectic forms. Conversely, a metric together with three compatible parallel symplectic structures defines a Hyperkähler structure on a manifold.

We will now define the Eguchi-Hanson space and the Eguchi-Hanson metrics, which are a 1-dimensional family of Hyperkähler metrics, controlled by a parameter \( k \in \mathbb{R}_{\geq 0} \). For \( k > 0 \) we get a metric on a smooth 4-manifold (this is point one of the following proposition), and for \( k = 0 \) we get the standard metric on \( \mathbb{H}/\{\pm 1\} \) (this is point two of the following proposition).

Proposition 2.2. Let \( r \) be a coordinate on the \( \mathbb{R}_{\geq 0} \)-factor of \( \mathbb{R}_{\geq 0} \times \text{SO}(3) \). Let

\[
\eta^1 = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \eta^2 = 2 \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \eta^3 = 2 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(3)
\]

and denote the corresponding left-invariant 1-forms on \( \text{SO}(3) \) by the same symbols. For \( k \geq 0 \), let \( f : \mathbb{R}_{>0} \times \text{SO}(3) \rightarrow \mathbb{R}_{>0} \) be defined by \( f_k(r) = (k + r^2)^{1/4} \) and set

\[
dt = f_k^{-1}(r) \, dr, \quad e^1(r) = r f_k^{-1}(r) \eta^1, \quad e^2(r) = f_k(r) \eta^2, \quad e^3(r) = f_k(r) \eta^3.
\]

Define \( \omega_1^{(k)}, \omega_2^{(k)}, \omega_3^{(k)} \in \Omega^2(\mathbb{R}_{>0} \times \text{SO}(3)) \) to be

\[
\omega_1^{(k)} = dt \wedge e^1 + e^2 \wedge e^3, \quad \omega_2^{(k)} = dt \wedge e^2 + e^3 \wedge e^1, \quad \omega_3^{(k)} = dt \wedge e^3 + e^1 \wedge e^2,
\]

and denote by \( g^{(k)} \) the metric on \( \mathbb{R}_{>0} \times \text{SO}(3) \) that makes \( \{dt, e^1, e^2, e^3\} \) an orthonormal basis.
1. If \( k > 0 \), let \( \exp(\eta^1) = \SO(2) \subset \SO(3) \). Denote by \( V \approx \mathbb{R}^2 \) the standard representation of \( \SO(2) \). Define \( \Psi : \SO(3) \times \mathbb{R}_{>0} \rightarrow \SO(3) \times V \) as \( \Psi(g, r) = (g, (r, 0)) \). Denote \( X = \SO(3) \times_{\SO(2)} V \).

Then \( \Psi \) induces a map \( \tilde{\Psi} : \SO(3) \times \mathbb{R}_{>0} \rightarrow X \), and the forms \( \tilde{\Psi}_* \omega_i^{(k)} \) can be extended to smooth 2-forms on all of \( X \). Furthermore, \( \tilde{\Psi}_*(g(k)) \) can also be extended to a metric on all of \( X \), and \( (X, \tilde{\Psi}_*(g(k))) \) is a Hyperkähler manifold.

2. If \( k = 0 \): parametrise the quaternions as \( x_0 + x_1 i + x_2 j + x_3 k \) with \( x_0, x_1, x_2, x_3 \in \mathbb{R} \), embed \( S^3 \subset \mathbb{H} \), and fix the identification \( \phi : S^3/\{\pm 1\} \rightarrow \SO(3) \) that maps \( x \) onto the map \( y \mapsto xyx^{-1} \) for \( x \in S^3/\{\pm 1\} \subset \mathbb{H}/\{\pm 1\} \). Denote \( \Phi : \SO(3) \times \mathbb{R}_{>0} \rightarrow \mathbb{H}/\{\pm 1\} \)

\[ (x, t) \mapsto t \cdot \phi^{-1}(x). \]

Then \( \Phi^* \omega_i = \omega_i^{(0)} \) for \( i \in \{1, 2, 3\} \) and \( \Phi^* g = g_{(0)} \), where \( g, \omega_1, \omega_2, \omega_3 \in \Omega^2(\mathbb{H}) \) are defined as in definition 2.7.

By slight abuse of notation, we will denote the extensions of \( \omega_i^{(k)} \) for \( i \in \{1, 2, 3\} \) and \( g_{(k)} \) to \( X \) in the case \( k > 0 \) by the same symbol, suppressing the pushforward under \( \tilde{\Psi} \).

**Proof.** For \( k > 0 \): the fact that \( \omega_i^{(k)}, \omega_2^{(k)}, \omega_3^{(k)}, g_{(k)} \) can be extended to all of \( X \) was proven in [LM17 Section 2.4]. One checks using a direct computation that \( \omega_i^{(k)} \) for \( i \in \{1, 2, 3\} \) is closed and [Hit87 Lemma 6.8] implies that \( \omega_i^{(k)} \) is also parallel for \( i \in \{1, 2, 3\} \). Both the symplectic forms and the metric are defined using the same orthonormal basis, which proves that they are compatible. The case \( k = 0 \) is a direct calculation.

The previous proposition has established that \( X \) is a Hyperkähler manifold. Furthermore, it has the property that its Hyperkähler structure approximates the flat Hyperkähler structure on \( \mathbb{R}^4 \) for big values of \( r \). The following definition makes this notion precise, and proposition 2.6 proves that the Hyperkähler structure on \( X \) does indeed have this property.

**Definition 2.4** (Definition 7.2.1 in [Joyoo]). Let \( G \) be a finite subgroup of \( \Sp(1) \), and let \( (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3, \tilde{g}) \) be the Euclidean Hyperkähler structure on \( \mathbb{H} \), and \( \sigma : \mathbb{H}/G \rightarrow (0, \infty) \) the radius function on \( \mathbb{H}/G \). We say that a Hyperkähler 4-manifold \( (Y, \omega_1, \omega_2, \omega_3, g) \) is asymptotically locally Euclidean (ALE) asymptotic to \( \mathbb{H}/G \), if there exists a compact subset \( S \subset X \) and a map \( \pi : X \setminus S \rightarrow \mathbb{H}/G \) that is a diffeomorphism between \( X \setminus S \) and \( \{x \in \mathbb{H}/G : \sigma(x) > R\} \) for some \( R > 0 \), such that

\[
\tilde{\nabla}(\pi_*(g) - \tilde{g}) = O(\sigma^{-4-k}) \quad \text{and} \quad \tilde{\nabla}(\pi_*(\omega_i) - \tilde{\omega}_i) = O(\sigma^{-4-k})
\]

as \( \sigma \rightarrow \infty \), for \( i \in \{1, 2, 3\} \) and \( k \geq 0 \), where \( \tilde{\nabla} \) is the Levi-Civita connection of \( \tilde{g} \).

**Proposition 2.6.**

1. \( S^2 = \SO(3) \times_{\SO(2)} \{0\} \subset X \) has diameter \( \frac{\pi}{8} k^{1/4} \) and Riemannian volume \( \pi k^{1/2} \).
2. There exist $\tau_1^{(k)} \in \Omega^1(X)$ such that $\omega_1^{(k)} - \omega_1^{(0)} = d\tau_1^{(k)}$ and for any $l \in \mathbb{Z}$ there exists $c = c(l) \in \mathbb{R}$ such that
\[
\left|\nabla^{(k)}_1\right|_{g_1^{(k)}} \leq c \cdot k^{(k/4 + r/2)^{-3-l}} \text{ for } k \leq 1, r > 1, \quad (2.7)
\]
where $\nabla$ denotes the Levi-Civita connection of $g_1^{(0)}$. Furthermore, $\omega_2^{(k)} - \omega_2^{(0)} = 0$, and $\omega_3^{(k)} - \omega_3^{(0)} = 0$. In particular, $(X, \omega_1^{(k)}, \omega_2^{(k)}, \omega_3^{(k)}, g(k))$ is ALE asymptotic to $\mathbb{H}/\{\pm 1\}$.

3. For $k, k' > 0$ there exists a diffeomorphism $\phi_{k,k'} : X \to X$ s.t. $\phi_{k,k}^*(g(k)) = \lambda^2 g(k')$ for $\lambda^4 = \frac{k}{k'}$, which restricts to the identity on $S^2 \subset X$.

Proof.

1. Remember $\eta^2 \in \mathfrak{so}(3)$ from proposition [2.2]. Then $\gamma(s) = [\exp_{\text{Id}}(s\eta^2), 0]$ is a geodesic in $S^2 \subset X$ with $\gamma(0) = \gamma(\pi)$ of length $\pi k^{1/4}$. $S^2$ carries the spherical metric and has therefore Riemannian volume $\pi k^{1/2}$.

2. Set $\tau_1^{(k)} = (f_0^2 - f_0^2)\eta^1$. For $l = 0$ and $k = 1$, this satisfies the inequality with $c = 4$. $\left|\nabla^{(l)}_1\right|_{g_1^{(k)}} (k(k/4 + r/2)^{-3-l})^{-1}$ decreases as $k$ decreases, which shows the claim for $l = 0$. The proof for the case $l > 0$ is analogous. That shows that $\omega_1^{(k)}$ is asymptotic to $\omega_1^{(0)}$ for $i \in \{1, 2, 3\}$. As the three Kähler forms determine the metric, we also have that $g(k)$ is asymptotic to $g(0)$. By the second point of proposition [2.2], $(\omega_0^{(0)}, \omega_1^{(0)}, \omega_2^{(0)}, g(0))$ is the Euclidean Hyperkähler structure on $\mathbb{H}/\{\pm 1\}$, so $(X, \omega_0^{(k)}, \omega_1^{(k)}, \omega_2^{(k)}, g(k))$ is ALE asymptotic to $\mathbb{H}/\{\pm 1\}$.

3. Existence of $\phi$ and $\lambda$ is clear on abstract grounds, as there exists a classification of asymptotically locally Euclidean Hyperkähler metrics (this argument is used in [Joy, p. 154]). Explicitly, $\lambda^4 = \frac{k}{k'}$ and
\[
\phi : \text{SO}(3) \times_{\text{SO}(2)} V \to \text{SO}(3) \times_{\text{SO}(2)} V
\]
\[
[u, (r, 0)] \mapsto \left[u, (\lambda^2 r, 0)\right]. \quad (2.8)
\]
One sees from the definition of $\phi$ that $\phi$ restricted to $S^2$ is the identity.

The following lemma states that $r$ is essentially the squared distance from the exceptional orbit in $X$:

**Lemma 2.9.** For $S^2 = \text{SO}(3) \times_{\text{SO}(2)} \{0\} \subset X$ we have $\frac{r}{d(S^2, [\text{Id}, (r, 0)])} \to 1$ as $r \to \infty$.

**Proof.** If $k = 0$, then $\gamma(s) = [\text{Id}, (s^2, 0)]$ is a unit speed geodesic with respect to $g(0)$. Thus, $d(S^2, [\text{Id}, (r, 0)]) = d([\text{Id}, (0, 0)], [\text{Id}, (r, 0)]) = r^{1/2}$. This proves the claim for $k = 0$. For general $k$ we have that $f(r)/\sqrt{f} \to 1$ as $r \to \infty$, which gives the claim.

As alluded to in the introduction of this section, there is also a simple complex geometry description of $X$. Namely, $X$ as a complex surface is the blowup of $\mathbb{C}^2/\{\pm 1\}$ in the origin, which is made precise in the following lemma.
Lemma 2.10. For any $k > 0$, let $J_1^{(k)}$ be the complex structure on $X$ defined by $g_{(k)}(J_1^{(k)}, \cdot) = \omega_1^{(k)}$. Then the complex surface $(X, J_1^{(k)})$ is biholomorphic to the blowup of $C^2/\{\pm 1\}$ in the origin.

Proof. Denote by $SO(3) \times \mathbb{R}_{\geq 0}/(SO(3) \times \{0\})$ the space $SO(3) \times \mathbb{R}_{\geq 0}$ with all points $SO(3) \times \{0\}$ identified with each other. The map

$$\rho : X = SO(3) \times_{SO(2)} V \rightarrow SO(3) \times \mathbb{R}_{\geq 0}/(SO(3) \times \{0\})$$

\[u, (r, 0)) \mapsto (u, r) \quad (2.11)\]

away from $S^2 \subset X$ is smooth and biholomorphic with respect to the complex structure $J_1^{(k)}$ on $X$ and $J_1^{(0)}$ on $SO(3) \times \mathbb{R}_{\geq 0}$ for any $k \geq 0$. Here, $J_1^{(0)}$ is the complex structure on $SO(3) \times \mathbb{R}_{\geq 0}/(SO(3) \times \{0\})$ defined by $g_{(0)}(J_1^{(0)}, \cdot) = \omega_1^{(0)}$. Thus, $\rho$ is a blowup map. It follows from the second point of proposition 2.6 that $(SO(3) \times \mathbb{R}_{\geq 0}/(SO(3) \times \{0\}), J_0)$ is biholomorphic to $C^2/\{\pm 1\}$, which proves the claim. \(\square\)

Note that the previous lemma implies that $(X, J_1^{(k)})$ is the same complex surface for any $k > 0$. Furthermore, the map $\rho$ from eq. (2.11) does not depend on $k$, and composing it with the identification $SO(3) \times \mathbb{R}_{\geq 0}/(SO(3) \times \{0\}) \simeq C^2/\{\pm 1\}$ of complex surfaces defines a map $X \rightarrow C^2/\{\pm 1\}$ independent of $k$. Later on, we will refer to this map as the blowup map from $X$ to $C^2/\{\pm 1\}$ and will denote it by $\rho$.

In the following definition, as in the rest of the article, we will be interested in the Eguchi-Hanson metric $g_{(t)}$ for some $t \in (0, 1)$. $t$ (without exponent) will play a crucial role in the glueing constructions later on, and will be called the glueing parameter.

Definition 2.12. For $t \in (0, \infty)$ define the weight functions

$$w_t : X \rightarrow \mathbb{R}_{\geq 0}$$

\[x \mapsto t + d_{g_{(t)}}(x, S^2), \quad (x, y) \mapsto \min\{w_t(x), w_t(y)\}.\]

Let $U \subset X$. For $\alpha \in (0, 1)$, $\beta \in \mathbb{R}$, $k \in \mathbb{Z}_{\geq 0}$, and $f \in \Omega^k(X)$ define the weighted Hölder norm of $f$ via

$$[f]_{C_{\beta}^{0, \alpha}}(U) := \sup_{x, y \in U, d_{g_{(t)}}(x, y) \leq w_t(x, y)} w_t(x, y)^{\alpha-\beta} \frac{|f(x) - f(y)|_{g_{(t)}}}{d_{g_{(t)}}(x, y)^{\beta}},$$

$$\|f\|_{L_{\beta}^{\alpha, \alpha}}(U) := \left\|w_t^{-\beta} f\right\|_{L^{\alpha}(U)},$$

$$\|f\|_{L_{\beta}^{\alpha, k, \alpha}} := \sum_{j=0}^{k} \left\|\nabla^j f\right\|_{L_{\beta}^{\alpha, j}(U)} + \left\|\nabla^k f\right\|_{C_{\beta}^{0, \alpha}}(U) + \left\|\nabla^k f\right\|_{C_{\beta}^{0, \alpha}}(U)$$

$f(x) - f(y)$ in the first line denotes the difference between $f(x)$ and the parallel transport of $f(y)$ to the fibre $\Omega^k(X)|_x$ along one of the shortest geodesics connecting $x$ and $y$. When $U$ is not specified, take $U = X$.

Throughout the article we will set $\beta$ to be a negative number. Informally, an element in the $C_{\beta}^{0, \alpha}$ Hölder space decays like $d_{g_{(t)}}(\cdot, S^2)^{\beta}$, as $d_{g_{(t)}}(\cdot, S^2) \rightarrow \infty$.

Lemma 2.13 (Harmonic Forms on Eguchi-Hanson Space).
1. We have $H^2_{\text{sing}}(X) = H^2_{\text{deRham}} (X) = \mathbb{R}$. For $k > 0$ define $v_k \in \Omega^2(X)$ to be

$$v_k := f_k(r)^{-6} r \, d r \wedge \eta^1 - f_k(r)^{-2} \eta^2 \wedge \eta^3$$

and endow $X$ with the metric $g_{(k)}$. Then $v_k \in L^2(\Lambda^2(X))$, $\Delta_{g_{(k)}} v_k = 0$, $[v_k]$ generates $H^2_{\text{deRham}}(X)$, and $v_k$ is the unique element in $[v_k]$ satisfying $\Delta_{g_{(k)}} v_k = 0$. Moreover, for $t = k^{1/4}$, we have $v_k \in C^{2,\alpha}_{-4\pi}(\Lambda^2(X))$. Away from the exceptional orbit $S^2 = \rho^{-1}(0)$, we have that

$$v_k = d \lambda_k, \text{ where } \lambda_k = -f_k(r)^2 \eta^1.$$

2. The $L^2$-kernels of $\Delta_{g_{(k)}}$ acting on forms of different degrees are as follows:

$$\text{Ker}(\Delta_{g_{(k)}} : L^2(\Lambda^2(X)) \to L^2(\Lambda^2(X))) = \langle v_k \rangle,$$

$$\text{Ker}(\Delta_{g_{(k)}} : L^2(\Lambda^p(X)) \to L^2(\Lambda^p(X))) = 0 \text{ for } p \neq 2.$$

Proof.

1. $X = T^*S^2$ as smooth manifolds, therefore $H^2_{\text{sing}}(X) = \mathbb{R}$. On smooth manifolds $H^2_{\text{sing}} (X) = H^2_{\text{deRham}} (X)$ by de Rham’s Theorem.

One checks with a direct computation that $v_k$ from eq. (2.14) is closed and anti-self-dual, and therefore co-closed. $v_k = d \lambda_k$ follows from a direct computation as well.

For $k = 0$, eq. (2.14) still defines an element $v_0 \in \Omega^2(C^2/\{\pm1\} \setminus \{0\})$. One checks through direct calculation that this is in the corresponding Hölder space on $C^2/\{\pm1\} \setminus \{0\}$. Using the fact that $X$ is asymptotically locally Euclidean (cf. proposition in [Loc]), one gets the Hölder estimate on $X$. To see that $v_k \in L^2(\Lambda^2(X))$, note that

$$\int_X |v_k|^2 \, d \text{vol}_{g_{(k)}} \leq c \int_X (1 + r^{1/2})^{-8} \, d \text{vol}_{g_{(k)}} \leq c \int_{s \in [0, \infty)} (1 + s)^{-8} s^3 \, ds < \infty.$$

Here, $c$ is a positive constant, and $r : X \to \mathbb{R}_{\geq 0}$ is the function from proposition which is approximately the square root of the distance to the minimal 2-sphere in $X$ measured in $g_{(k)}$, according to lemma [Loc].

By Poincaré duality, we have $H^2_{\text{cs}}(X) = H^2_{\text{sing}}(X) = \mathbb{R}$, where $H^2_{\text{cs}}(X)$ denotes the de Rham cohomology with compact support. [Lott] Theorem 6.5.2 gives that the map

$$\mathcal{H}^2(X) := \{ \xi \in L^2(\Lambda^2 T^*X) : d^{*} \xi = d^{*} \xi = 0 \} \to \text{Im} \,(H^2_{\text{cs}}(X) \hookrightarrow H^2_{\text{deRham}}(X))$$

$$\xi \mapsto [\xi]$$

is an isomorphism. This is also [Loc] Example (0.15)] and the proof is given in [Loc] Theorem (7.9). Thus $[v_k]$ generates $H^2_{\text{deRham}}(X)$ and $v_k \in [v_k]$ is the unique element in $[v_k]$ satisfying $d v_k = 0$, $d^{*} v_k = 0$.

It remains to check that $v_k$ is also the unique element in $[v_k]$ satisfying $\Delta_{g_{(k)}} v = 0$, $\Delta_{g_{(k)}} v_k = 0$ and $(d + d^{*}) v = 0$ are equivalent by the same integration by parts argument.
as in the compact case, namely for $M > 0$:

\[
\int_{\{r \leq M\}} \langle (d^* + d^*) v_k, v_k \rangle \, d\text{vol}_{\gamma(k)} = \int_{\{r \leq M\}} \langle (d^*) v_k, v_k \rangle \, d\text{vol}_{\gamma(k)} + \int_{\{r \leq M\}} \langle (d^*) v_k, v_k \rangle \, d\text{vol}_{\gamma(k)}
\]

\[
= \int_{\{r \leq M\}} (d^* v_k, d^* v_k) \, d\text{vol}_{\gamma(k)} + \int_{\{r \leq M\}} (d^* v_k \wedge *v_k) + \int_{\{r \leq M\}} (d(v_k \wedge *d v_k))
\]

\[
+ \int_\partial_{\{r \leq M\}} (d^* v_k \wedge *v_k + v_k \wedge *d v_k),
\]

where we used $d(d^* v_k \wedge *v_k) = d^* v_k \wedge *v_k - d^* v_k \wedge *d v_k$ in the second step, and Stokes' Theorem in the last step. The last term tends to 0 as $M \to \infty$, because of the decay of elements in $C^{2,\alpha}(X)$. So, $\Delta_{\gamma(k)} v_k = 0$ implies that $d^* v_k = 0$, $d v_k = 0$, and the converse implication is trivial.

2. The first line is a restatement of the previous point. The other lines are [Loc87] Example (0.15) with proof in [Loc87] Theorem (7.9).

\[\Box\]

Remark 2.15. Note that $v$ from the lemma cannot have compact support by the unique continuation property for elliptic equations. We only have that $[v]$ contains a form of compact support.

Before ending this section, we will provide a technical lemma that will be used in the proof of proposition 4.17. There, we will switch from measuring in $g_{(t^1)}$ to measuring in $\gamma(1)$, which brings in a factor of $t$. But the weight function of the Hölder norms on $X$ was chosen to compensate this factor of $t$. The following definitions and lemmata make this statement precise.

Definition 2.16. For $\beta \in \mathbb{R}$ and $t > 0$, let $\phi_{t^1,1} : X \to X$ be the map from proposition 2.6 satisfying $\phi_{t^1,1}^* g_{(t^1)} = t^2 g_{(1)}$, and define

\[
\sigma_{\beta,t} : \Omega^2(X) \to \Omega^2(X)
\]

\[
\alpha \mapsto t^{-\beta-2} \phi_{t^1,1}^* \alpha.
\]

Lemma 2.17. For $\beta \in \mathbb{R}$ and $t > 0$, and any $a \in \Omega^2(X)$ we have:

\[
\|\sigma_{\beta,t} a\|_{C^{k,a}(X)} = \|a\|_{C^{k,a}(X)},
\]

and

\[
\Delta_{\gamma_{(t^1)}^a} a - \sigma_{-2,1}^{-1} \Delta_{\gamma(1)} \sigma_{\beta,t} a = 0.
\]

Here $\Delta_g$ denotes the Laplacian on $X$ with respect to the metric $g$.
Proof. For easier notation write $\phi = \phi_{t^2}$. Using this, we have
\[
\begin{align*}
\|\sigma_{\beta,t}a\|_{L^p_\infty(X)} &= \left\| (1 + d_{g(1)}(\cdot, S^2))^{-\beta} t^{-\beta - 2} \phi^* a \right\|_{L^\infty(X) g(1)} \\
&= \left\| (t + t \cdot d_{g(1)}(\cdot, S^2))^{-\beta} \phi^* a \right\|_{L^\infty(X) t^2 g(1)} \\
&= \left\| (t + d_{g(t^2)}(\cdot, S^2))^{-\beta} a \right\|_{L^\infty(X) g(t^2)} \\
&= \|a\|_{L^p_\infty(X)}.
\end{align*}
\]
In the second step, the $t^{-2}$ factor disappears because we changed from measuring in the norm $g(1)$ the norm $t^2 g(1)$. We also used
\[
t d_{g(1)}(\phi^{-1}(\cdot), S^2) = d_{t^2 g(1)}(\phi^{-1}(\cdot), S^2) = d_{g(t^2)}(\cdot, S^2)
\]
in the third step, which uses the fact that $\phi(S^2) = S^2$ from proposition 2.6. The estimates for the derivatives of $a$ are derived in the same way.

To derive the second equation, we use that $\Delta_{g(1)} = t^2 \Delta_{t^2 g(1)} = t^2 \Delta_{\phi^* g(t^2)}$, and therefore
\[
\Delta_{g(t^2)} a - \sigma_{\beta^2, t^4} \Delta_{g(1)} \sigma_{\beta, t} a = \Delta_{g(t^2)} a - (\phi^{-1})^* \Delta_{\phi^* g(t^2)} \phi^* a \\
= \Delta_{g(t^2)} a - (\phi^{-1})^* \phi^* \Delta_{g(t^2)} a = 0.
\]
\[
□
\]
Up to now, all geometry took place in four dimensions. When resolving singularities of $G_2$-orbifolds, we will glue in pieces that locally look like $\mathbb{R}^3 \times X$, and we will rescale these pieces throughout our proofs. Therefore, we will need the analog of the lemma on $\mathbb{R}^3 \times X$, which takes up the last part of this section.

**Definition 2.18.** For $t \in (0, \infty)$ define the weight function
\[
w_t : \mathbb{R}^3 \times X \to \mathbb{R}_{>0} \\
x \mapsto t + d_{g(t^2)}(x, \mathbb{R}^3 \times S^2)
\]
and the weighted Hölder norms $\|\cdot\|_{k^a_{\beta,t}(U)}$ for $k \in \mathbb{Z}_{>0}$, $\alpha \in (0, 1)$, $\beta \in \mathbb{R}$, $U \subset \mathbb{R}^3 \times X$ as in definition 2.12.

**Remark 2.19.** When defining the weight function in definition 2.18 we could have chosen to define it using the distance to $\{0\} \times S^2 \subset \mathbb{R}^3 \times X$, rather than $\mathbb{R}^3 \times S^2$. The reason for our choice is the following: in proposition 4.17 we will prove an estimate for the Laplacian on a resolution of $T^3 / \Gamma$. This resolution has regions that look like $T^3 \times X$ and contain a natural $T^3 \times S^2$, but there is no canonical choice of a point $x \in T^3$ in the resolution that would give an embedded $\{0\} \times S^2$. In fact, we will use a blowup argument and zoom into the resolution locus and in the limit get forms on $\mathbb{R}^3 \times X$ that decay in the $X$-direction, but are constant in the $\mathbb{R}^3$-direction, i.e. they do not decay as one moves away from $\{0\} \times S^2$ in the $\mathbb{R}^3$-direction.
Definition 2.20. Let

\[ s_{\beta,t} : \Omega^2(\mathbb{R}^3 \times X) \to \Omega^2(\mathbb{R}^3 \times X) \]
\[ \alpha \mapsto t^{-\beta-2}(t^{-1} \phi_{t^1})^* \alpha. \]

The following lemma is proved similarly to lemma 2.17 and we omit the proof here:

Lemma 2.21. For \( \beta \in \mathbb{R} \) and \( t > 0 \), and any \( a \in \Omega^2(\mathbb{R}^3 \times X) \) we have:

\[ \| s_{\beta,t}a \|_{c_{\beta,t}(\mathbb{R}^3 \times X)} = \| a \|_{c_{\beta,t}(\mathbb{R}^3 \times X)} , \quad \text{and} \]
\[ \Delta_{g_{t^1} \oplus g_{t^2}} - s_{\beta^{-1},t} \Delta_{g_{t^1} \oplus g_{t^2}} s_{\beta,t} a = 0. \]

Here \( \Delta \) denotes the Laplacian on \( \mathbb{R}^3 \times X \) with respect to the metric \( g \).

3 G\(_2\)-structures

3.1 Torsion of \( G_2 \)-structures on 7-manifolds

We now introduce \( G_2 \)-structures and their torsion, following the treatment in [Joy].

Definition 3.1 (Definition 10.1.1 in [Joy]). Let \((x_1, \ldots, x_7)\) be coordinates on \( \mathbb{R}^7 \). Write \( d x_{i_1} \wedge \cdots \wedge d x_{i_l} \) for the exterior form \( d x_{i_1} \wedge d x_{i_2} \wedge \cdots \wedge d x_{i_l} \). Define \( \varphi_0 \in \Omega^3(\mathbb{R}^7) \) by

\[ \varphi_0 = d x_{123} + d x_{145} + d x_{167} + d x_{246} - d x_{257} - d x_{347} - d x_{356}. \]

The subgroup of \( \text{GL}(7, \mathbb{R}) \) preserving \( \varphi_0 \) is the exceptional Lie group \( G_2 \). It also fixes the Euclidean metric \( g_0 = d x_1^2 + \cdots + d x_7^2 \), the orientation on \( \mathbb{R}^7 \), and \( \ast \varphi_0 \in \Omega^3(\mathbb{R}^7) \).

Definition 3.2. The skew-symmetric bilinear map \( \times : \mathbb{R}^7 \rightarrow \mathbb{R}^7 \) defined by

\[ \varphi_0(u, v, w) = g_0(u \times v, w) \]

for \( u, v, w \in \mathbb{R}^7 \) is called the cross product induced by \( \varphi \).

Definition 3.3. Let \( M \) be an oriented 7-manifold. A principal subbundle \( Q \) of the subbundle with structure group \( G_2 \) is called an oriented \( G_2 \)-structure, if \( Q \) induces the given orientation on \( M \). Viewing \( Q \) as a set of linear maps from tangent spaces of \( M \) to \( \mathbb{R}^7 \), there exists a unique \( \varphi \in \Omega^3(M) \) such that \( Q \) identifies \( \varphi \) with \( \varphi_0 \in \Omega^3(\mathbb{R}^7) \) at every point.

Oriented \( G_2 \)-structures are in 1-1 correspondence with 3-forms on \( M \) for which there exists an oriented isomorphism mapping it to \( \varphi_0 \) at every point. We will therefore also refer to such 3-forms as \( G_2 \)-structures.

Let \( M \) be a manifold with \( G_2 \)-structure \( \varphi \). We call \( \nabla \varphi \) the torsion of a \( G_2 \)-structure \( \varphi \in \Omega^3(M) \). Here, \( \nabla \) denotes the Levi-Civita induced by \( \varphi \) in the following sense: we have \( G_2 \subset SO(7) \), so \( \varphi \) defines a Riemannian metric \( g \) on \( M \), which in turn defines a Levi-Civita connection. As a shorthand, we also use the following notation: write \( \Theta(\varphi) = \ast \varphi \), where \( \ast \) denotes the Hodge star defined by \( g \). Using this, the following theorem gives a characterisation of torsion-free \( G_2 \)-manifolds:
Theorem 3.4 (Propositions 10.1.3 and 10.1.5 in [Joyo0]). Let $M$ be an oriented manifold with $G_2$-structure $\varphi$ with induced metric $g$. The following are equivalent:

(i) $\text{Hol}(g) \subseteq G_2$,

(ii) $\nabla \varphi = 0$ on $M$, where $\nabla$ is the Levi-Civita connection of $g$, and

(iii) $d \varphi = 0$ and $d \Theta(\varphi) = 0$ on $M$.

If these hold then $g$ is Ricci-flat.

The goal of the later sections is to construct $G_2$-structures that induce metrics with holonomy equal to $G_2$. A torsion-free $G_2$-structure alone only guarantees holonomy contained in $G_2$, but in the compact setting a characterisation of manifolds with holonomy equal to $G_2$ is available:

Theorem 3.5 (Proposition 10.2.2 and Theorem 10.4.4 in [Joyo0]). Let $M$ be a compact oriented manifold with torsion-free $G_2$-structure $\varphi$ and induced metric $g$. Then $\text{Hol}(g) = G_2$ if and only if $\pi_1(M)$ is finite. In this case the moduli space of metrics with holonomy $G_2$ on $M$, up to diffeomorphisms isotopic to the identity, is a smooth manifold of dimensions $b^3(M)$.

Note that this theorem makes no statement about the existence of a torsion-free $G_2$-structure in the first place. Finding a characterisation of manifolds which admit a torsion-free $G_2$-structure and even the construction of examples remain challenging problems in the field.

Later on, we will investigate perturbations of $G_2$-structures and analyse how that changes their torsion. To this end, we will use the following estimates for the map $\Theta$ defined before:

Proposition 3.6 (Proposition 10.3.5 in [Joyo0] and eqn. (21) of part II in [Joy96b]). Let $\varphi$ be a $G_2$-structure on $M$ with $d \varphi = 0$. Then there exists a neighbourhood $U \subset \Omega^3(M)$ of $\varphi$, such that for all $\chi \in U$: $\varphi + \chi$ is a $G_2$-structure, and

$$\Theta(\varphi + \chi) = *\varphi - T(\chi) - F(\chi),$$

(3.7)

where “$*$” denotes the Hodge star with respect to the metric induced by $\varphi$, $T : \Omega^3(M) \to \Omega^4(M)$ is a linear map (depending on $\varphi$), and $F : U \to \Omega^4(M)$ is a smooth map (also depending on $\varphi$) satisfying $F(0) = 0$, and

$$|F(\chi)| \leq c |\chi|^2,$$

$$|d(F(\chi))| \leq c \left\{ |\chi|^2 |d^* \varphi| + |\nabla \chi| \cdot |\chi| \right\},$$

$$[d(F(\chi))]_\alpha \leq c \left\{ |\chi|_\alpha \cdot |\chi|_{L^\infty} \cdot |d^* \varphi|_{L^\infty} + ||\chi||_{L^\infty} \cdot |d^* \varphi|_\alpha + |\nabla \chi|_\alpha \cdot |\chi|_{L^\infty} + ||\nabla \chi||_{L^\infty} \cdot |\chi|_\alpha \right\},$$

as well as

$$|\nabla(F(\chi))| \leq c \left\{ |\chi|^2 |\nabla \varphi| + |\nabla \chi| \cdot |\chi| \right\},$$

$$[\nabla(F(\chi))]_{\alpha\beta\gamma} \leq c \left\{ |\chi|_\alpha \cdot |\chi|_{L^\infty} \cdot |\nabla \varphi|_{L^\infty} + ||\chi||_{L^\infty} \cdot |\nabla \varphi|_\gamma + |\nabla \chi|_\alpha \cdot |\chi|_{L^\infty} + ||\nabla \chi||_{L^\infty} \cdot |\chi|_\alpha \right\}.$$

Here, $|\cdot|$ denotes the norm induced by $\varphi$, $\nabla$ denotes the Levi-Civita connection of this metric, and $[\cdot]_{C^\infty}$ denotes the unweighted Hölder semi-norm induced by this metric.

Finally, the landmark result on the existence of torsion-free $G_2$-structures is the following theorem. It first appeared in [Joy96b] part I, Theorem A, and we present a rewritten version in analogy with [JK17] Theorem 2.7:
Theorem 3.8. Let \( \alpha, K_1, K_2, K_3 \) be any positive constants. Then there exist \( \epsilon \in (0, 1] \) and \( K_4 > 0 \), such that whenever \( 0 < t \leq \epsilon \), the following holds.

Let \( M \) be a compact oriented 7-manifold, with \( G_2 \)-structure \( \varphi \) with induced metric \( g \) satisfying \( d \varphi = 0 \). Suppose there is a closed 3-form \( \psi \) on \( M \) such that \( d^* \varphi = d^* \psi \) and

(i) \( ||\psi||_{C^0} \leq K_1 t^\alpha, ||\psi||_{L^2} \leq K_1 t^{7/2 + \alpha}, \) and \( ||\psi||_{L^4} \leq K_1 t^{-1/2 + \alpha} \).

(ii) The injectivity radius \( \text{inj} \) of \( g \) satisfies \( \text{inj} \geq K_2 t \).

(iii) The Riemann curvature tensor \( R_m \) of \( g \) satisfies \( ||R_m||_{C^0} \leq K_3 t^{-2} \).

Then there exists a smooth, torsion-free \( G_2 \)-structure \( \tilde{\varphi} \) on \( M \) such that \( ||\tilde{\varphi} - \varphi||_{C^0} \leq K_4 t^\alpha \) and \( [\tilde{\varphi}] = [\varphi] \) in \( H^3(M, \mathbb{R}) \). Here all norms are computed using the original metric \( g \).

The main purpose of this article is to reprove this theorem, which holds true on any 7-manifold, in the setting of \([\text{IK}17]\). This is achieved by theorem 3.41. It turns out that adapting the analysis to the special construction explained in \([\text{IK}]17\) allows one to weaken the estimates from part (i) of the previous theorem, gives an improved estimate for the difference \( \tilde{\varphi} - \varphi \), and furthermore allows to construct new \( G_2 \)-manifolds by stretching different parts of the manifold at different length scales. All of this will be made precise in the following sections.

3.2 \( G_2 \)-manifolds and Hyperkähler 4-manifolds

On \( \mathbb{H} \) with coordinates \((y_0, y_1, y_2, y_3)\) we have the three symplectic forms \( \omega_1, \omega_2, \omega_3 \) from definition 2.1 given as

\[
\omega_0 = d y_0 \wedge d y_1 + d y_2 \wedge d y_3, \quad \omega_1 = d y_0 \wedge d y_2 - d y_1 \wedge d y_3, \quad \omega_2 = d y_0 \wedge d y_3 + d y_1 \wedge d y_2.
\]

Identify \( \mathbb{R}^2 \) with coordinates \((x_1, \ldots, x_7)\) with \( \mathbb{R}^3 \oplus \mathbb{H} \) with coordinates \(( (x_1, x_2, x_3), (y_1, y_2, y_3, y_4) ) \).

Then we have for \( \varphi_0, *\varphi_0 \) from definition 3.1

\[
\varphi_0 = d x_{123} - \sum_{i=1}^{3} d x_i \wedge \omega_i, \quad *\varphi_0 = \text{vol}_{\mathbb{H}} - \sum_{(i,j,k) = (1,2,3) \text{ and cyclic permutation}} \omega_i \wedge d x_{jk}. \quad (3.9)
\]

This linear algebra statement easily extends to product manifolds in the following sense: if \( X \) is a Hyperkähler 4-manifold, and \( \mathbb{R}^3 \) is endowed with the Euclidean metric, then \( \mathbb{R}^3 \times X \) admits a \( G_2 \)-structure. The \( G_2 \)-structure is given by the same formula as in the flat case, namely eq. (3.9), after replacing \((\omega_1, \omega_2, \omega_3)\) by the triple of parallel symplectic forms defining the Hyperkähler structure on \( X \). This product \( G_2 \)-structure will be glued into \( G_2 \)-orbifolds in the following sections.

4 Torsion-Free \( G_2 \)-Structures on the Generalised Kummer Construction

In the two articles \([\text{Joy}06b]\), Joyce constructed the first examples of manifolds with holonomy equal to \( G_2 \). The idea is start with the flat 7-torus, which admits a flat \( G_2 \)-structure. A quotient of the torus by maps preserving the \( G_2 \)-structure still carries a flat \( G_2 \)-structure, but
has singularities. The maps are carefully chosen, so that the singularities are modelled on $T^3 \times \mathbb{C}^2 / \{ \pm 1 \}$. By the results of section 2, $T^3 \times \mathbb{C}^2 / \{ \pm 1 \}$ has a one-parameter family of resolutions $T^3 \times X \to T^3 \times \mathbb{C}^2 / \{ \pm 1 \}$, where $X$ denotes the Eguchi-Hanson space, and the parameter defines the size of a minimal sphere in $X$. We can define a smooth manifold by gluing these resolutions over the singularities in the quotient of the torus.

$T^3 \times X$ carries the product $G_2$-structure from section 3. This means we have two torsion-free $G_2$-structures on our glued manifold: one coming from flat $T^7$, and the product $G_2$-structure near the resolution of the singularities. We will interpolate between the two to get one globally defined $G_2$-structure. This will no longer be torsion-free, but it will have small enough torsion in the sense of theorem 3.8. This is the argument that was used in [Joy96b] to prove the existence of a torsion-free $G_2$-structure, and the construction of this $G_2$-structure with small torsion is the content of section 4.1.

Sections 4.2 to 4.4 give an alternative proof of the existence of a torsion-free $G_2$-structure on this glued manifold.

### 4.1 Resolutions of $T^7 / \Gamma$

We briefly review the generalised Kummer construction as explained in [Joy96b]. Let $(x_1, \ldots, x_7)$ be coordinates on $T^7 = \mathbb{R}^7 / \mathbb{Z}^7$, where $x_i \in \mathbb{R} / \mathbb{Z}$, endowed with the flat $G_2$-structure $\varphi_0$ from definition 3.1. Let $\alpha, \beta, \gamma : T^7 \to T^7$ defined by

\[
\alpha : (x_1, \ldots, x_7) \mapsto (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7),
\]

\[
\beta : (x_1, \ldots, x_7) \mapsto \left( -x_1, \frac{1}{2} - x_2, x_3, x_4, -x_5, -x_6, -x_7 \right),
\]

\[
\gamma : (x_1, \ldots, x_7) \mapsto \left( \frac{1}{2} - x_1, x_2, \frac{1}{2} - x_3, x_4, -x_5, x_6, -x_7 \right).
\]

Denote $\Gamma := \langle \alpha, \beta, \gamma \rangle$. The next lemmata collect some information about the orbifold $T^7 / \Gamma$:

**Lemma 4.1** (Section 2.1 in part I, [Joy96b]). $\alpha, \beta, \gamma$ preserve $\varphi_0$, we have $\alpha^2 = \beta^2 = \gamma^2 = 1$, and $\alpha, \beta, \gamma$ commute. We have that $\Gamma \cong \mathbb{Z}_2^3$.

**Lemma 4.2** (Lemma 2.1.1 in part I, [Joy96b]). The elements $\beta \gamma$, $\gamma \alpha$, $\alpha \beta$, and $\alpha \beta \gamma$ of $\Gamma$ have no fixed points on $T^7$. The fixed points of $\alpha$ in $T^7$ are 16 copies of $T^3$, and the group $\langle \beta, \gamma \rangle$ acts freely on the set of 16 3-tori fixed by $\alpha$. Similarly, the fixed points of $\beta$, $\gamma$ in $T^7$ are each 16 copies of $T^3$, and the groups $\langle \alpha, \gamma \rangle$ and $\langle \alpha, \beta \rangle$ acts freely on the sets of 16 3-tori fixed by $\beta$, $\gamma$ respectively.

**Lemma 4.3** (Lemma 2.1.2 in part I, [Joy96b]). The singular set $L$ of $T^7 / \Gamma$ is a disjoint union of 12 copies of $T^3$. There is an open subset $U$ of $T^7 / \Gamma$ containing $L$, such that each of the 12 connected components of $U$ is isometric to $T^3 \times \left( B^4_\zeta / \{ \pm 1 \} \right)$, where $B^4_\zeta$ is the open ball of radius $\zeta$ in $\mathbb{R}^4$ for some positive constant $\zeta$ ($\zeta = 1/9$ will do).

We now define a compact 7-manifold $M$, which can be thought of as a resolution of the orbifold $T^7 / \Gamma$, and a one-parameter family of closed $G_2$-structures $\varphi^t$ thereon. We can choose an
identification $U \simeq L \times \left( B^4_{\ell} / \{ \pm 1 \} \right)$ such that we can write on $U$

$$
\varphi_0 = \delta_1 \wedge \delta_2 \wedge \delta_3 - \sum_{i=1}^{3} \omega_i \wedge \delta_i, \quad \ast \varphi_0 = \frac{1}{2} \omega_1 \wedge \omega_1 - \sum_{(i,j,k) = (1,2,3)} \omega_i \wedge \delta_j \wedge \delta_k,
$$

where $\delta_1, \delta_2, \delta_3$ are constant orthonormal 1-forms on $L$, and $\omega_1, \omega_2, \omega_3$ are the Hyperkähler triple from definition 2.1, cf. section 3.2.

As before, denote by $X$ the Eguchi-Hanson space, and by $r$ the function $r : X \to \mathbb{R}_{\geq 0}$ from proposition 2.2. Remember that $r$ is approximately the square root of the distance to the exceptional fibre in Eguchi-Hanson space, measured in the Eguchi-Hanson metric (cf. lemma 2.9).

For $t \in (0, 1)$, let $\tilde{U}_t := L \times \{ x \in X : r(x)^{1/2} < \zeta \}$. Define

$$
M := \left( (T^7 / \Gamma) \setminus L \sqcup \tilde{U} \right) / \sim, \quad (4.4)
$$

where $x \sim y$ if $(\text{Id}, \rho)(x) = y$, with projection $\pi : M \to T^7 / \Gamma$ induced by $(\text{Id}, \rho)$. Here, $\rho : X \to \mathbb{C}^2 / \{ \pm 1 \}$ denotes the blowup map from lemma 2.10.

Now choose a non-decreasing function $\chi : [0, \zeta] \to [0, 1]$ such that $\chi(s) = 0$ for $s \leq \zeta/4$ and $\chi(s) = 1$ for $s \geq \zeta/2$, and set

$$
\tilde{\omega}_{i,t} := \omega_i^{(t)} - d \left( \chi(d_{T^7 / \Gamma}(\pi(\cdot), L)) \tau_i^{(t)} \right). \quad (4.5)
$$

The $\tau_i^{(t)}$ were defined in proposition 2.6 and are the difference between the flat Hyperkähler triple on $\mathbb{C}^2 / \{ \pm 1 \}$ and the Hyperkähler triple $(\omega_1^{(t)}, \omega_2^{(t)}, \omega_3^{(t)})$ on $X$. $d$ denotes the distance on $T^7 / \Gamma$ with respect to the metric induced by $\varphi_0$. Notice that $\tilde{\omega}_{i,t} = \omega_i$ where $|\rho| > \zeta/2$, and $\tilde{\omega}_{i,t} = \omega_i^{(t)}$ where $|\rho| < \zeta/4$. Now define a 3-form $\varphi_t \in \Omega^3(M)$ and a 4-form $\theta \in \Omega^4(M)$ by

$$
\varphi_t := \delta_1 \wedge \delta_2 \wedge \delta_3 - \sum_{i=1}^{3} \tilde{\omega}_{i,t} \wedge \delta_i, \quad (4.6)
$$

$$
\theta_t := \frac{1}{2} \tilde{\omega}_{1,t} \wedge \tilde{\omega}_{1,t} - \sum_{(i,j,k) = (1,2,3)} \tilde{\omega}_{i,t} \wedge \delta_j \wedge \delta_k. \quad (4.7)
$$

This definition mimics the product situation explained in section 3.2. For small $t$, $\varphi_t$ is a $G_2$-structure and therefore induces a metric $g_t$. Both $\varphi_t$ and $\theta_t$ are closed forms, so, if $\ast \varphi_t = \theta_t$, then $\varphi_t$ would be a torsion-free $G_2$-structure by theorem 4.4. However, in general this does not hold, and $\varphi_t$ is not a torsion-free $G_2$-structure. The following 3-form $\psi_t$ is meant to measure the torsion of $\varphi_t$:

$$
\ast \psi_t = \Theta(\varphi_t) - \theta_t. \quad (4.8)
$$

Its crucial properties are:

**Lemma 4.9.** Let $\psi_t \in \Omega^3(M)$ as in eq. (4.8). There exists a positive constant $c$ such that

$$
\|\psi_t\|_{C^{0,\alpha}} \leq c t^4,
$$

where the Hölder norm is defined with respect to the metric $g_t$ and its induced Levi-Civita connection.
Proof. \( d^* \psi^f = d^* \varphi^f \) follows from eq. (4.8) and the fact that \( \vartheta^f \) is closed. 

\( \nabla_X \) and \( * \) commute for every vector field \( X \) on \( M \) (cf. [1512]), therefore it suffices to estimate \( * \psi^f \) rather than \( \psi^f \). Write \( \varphi^{\varphi^f}_{\psi^{\psi^f}} := \delta_1 \wedge \delta_2 \wedge \delta_3 - \sum_{i=1}^3 \omega_i^{(t^i)} \wedge \delta_i \) for the product \( G_2 \)-structure on \( X \times L \) and denote the induced metric (which is the product metric) by \( g^{(t^i)}_{\psi^{\psi^f}} \). Recall the linear map \( T \) and the non-linear map \( F \) from proposition [3,6] satisfying \( \Theta(\varphi + \chi) = * \varphi - T(\chi) - F(\chi) \) for a \( G_2 \)-structure \( \varphi \) and a small deformation \( \chi \). Using this notation, we get:

\[
\Theta(\varphi^f) - \vartheta^f = \Theta \left( \varphi^{(t^i)}_{\psi^{\psi^f}} - \delta_1 \wedge d \left( \chi(d(\pi(-), L)) r_1^{(r^i)} \right) \right) \\
\quad - * g^{(t^i)}_{\psi^{\psi^f}} + \delta_2 \wedge \delta_3 \wedge d \left( \chi(d(\pi(-), L)) r_1^{(r^i)} \right) \\
\quad = -T \left( \delta_1 \wedge d \left( \chi(d(\pi(-), L)) r_1^{(r^i)} \right) \right) - F \left( \delta_1 \wedge d \left( \chi(d(\pi(-), L)) r_1^{(r^i)} \right) \right) \\
\quad \quad + \delta_2 \wedge \delta_3 \wedge d \left( \chi(d(\pi(-), L)) r_1^{(r^i)} \right)
\]

Here we used the equality \( \omega^{(k)}_1 - \omega_1 = d r_1^{(k)} \) from proposition [2,6] in the first step and the definition of \( T \) and \( F \) in the second step.

Note that \( \Theta(\varphi^f) - \vartheta^f \) is supported on \( \{ x \in M : \zeta/4 < d_{\Gamma''/\Gamma'}(\pi(x), L) < \zeta/2 \} \). Therefore, using the estimates for \( T \) and \( F \) from proposition [3,6] together with eq. [2,7] we get the claim. \( \square \)

### 4.2 The Laplacian on \( \mathbb{R}^3 \times X \)

In proposition [13,17] we will prove an estimate for the Laplacian on 2-forms on \( M \). There, we will use a blowup argument to essentially reduce the analysis on \( M \) to the analysis on \( T''/\Gamma' \) and \( \mathbb{R}^3 \times X \). In this section we will cite a general result for uniformly elliptic operators on product manifolds \( \mathbb{R}^n \times Y \) from [Walsh13], where \( Y \) is a Riemannian manifold, and use this to find that harmonic 2-forms on \( \mathbb{R}^3 \times X \) are wedge products of parallel forms on \( \mathbb{R}^3 \) and harmonic forms on \( X \).

**Lemma 4.10** (Lemma 2.76 in [Walsh13]). Let \( E \) be a vector bundle of bounded geometry over a Riemannian manifold \( Y \) of bounded geometry and with subexponential volume growth, and suppose that \( D : C^\infty(Y, E) \to C^\infty(Y, E) \) is a uniformly elliptic operator of second order whose coefficients and their first derivatives are uniformly bounded, that is non-negative, i.e., \( \langle Da, a \rangle \geq 0 \) for all \( a \in W^{2,2}(Y, E) \), and formally self-adjoint. If \( a \in C^\infty(\mathbb{R}^n \times Y, E) \) satisfies

\[
(\Delta_{\mathbb{R}^n} + D) a = 0
\]

and \( \|a\|_{L^\infty} \) is finite, then \( a \) is constant in the \( \mathbb{R}^n \)-direction, that is \( a(x, y) = a(y) \). Here, by slight abuse of notation, we denote the pullback of \( E \) to \( \mathbb{R}^n \times Y \) by \( E \) as well.

**Theorem 4.11** (Theorem 1.2.1 in [Lam86]). Let \( N \) be a smooth manifold, \( m \in \mathbb{Z}_{\geq 0} \cup \{ \infty \}, m \geq 1 \). Let \( G \subset C^m(N) \) and denote by \( \mathbb{R}[G] \) the algebra generated by \( G \).

Then \( \mathbb{R}[G] \) is dense in \( C^m(N) \) with respect to the topology of compact convergence up to the \( m \)-th derivative, if and only if the following conditions hold:

1. \( G \) is strongly separating, that is for all \( x, y \in N \) there exists \( g \in G \) such that \( g(x) \neq g(y) \) and for all \( z \in N \) there exists \( g \in G \) such that \( g(z) \neq 0 \).
2. For every \( x \in N \) and \( v \in T_xN, v \neq 0 \), there exists \( g \in G \) such that \( d_g(v) \neq 0 \).

**Corollary 4.12.** Let \( Y \) be a manifold of bounded geometry and with subexponential volume growth. If \( a \in \Omega^2(\mathbb{R}^3 \times Y) \) satisfies \( ||a||_{L^\infty} < \infty \) and

\[
\Delta_{g_{k+1} \circ g_1}(a) = 0,
\]

then \( a \) is a sum of terms of the form \( a_1 \wedge a_2 \), where \( a_1 \in \Omega^k(\mathbb{R}^3) \) is parallel, and \( a_2 \in \Omega^l(Y) \) satisfies \( \Delta_{g_1}(a_2) = 0 \).

**Proof.** Step 1: Check the statement for \( a \in \Omega^2(\mathbb{R}^3 \times Y) \) in separated variables.

Let \( a = a_1 \wedge a_2 \in \{ b_1 \wedge b_2 \in \Omega^2(\mathbb{R}^3 \times Y) : b_1 \in \Omega^k(\mathbb{R}^3), b_2 \in \Omega^l(Y) \text{ for } k, l \in \mathbb{Z}_{\geq 0} \} \) with \( \Delta_{g_{k+1} \circ g_1}(a) = 0 \). Then

\[
\Delta_{g_{k+1} \circ g_1}(a) = \Delta_{g_{k+1}}(a) + \Delta_{g_1}(a).
\]

Then, by lemma 4.10, \( a \) is constant in the \( \mathbb{R}^3 \)-direction, therefore \( a_1 \) is parallel and \( \Delta_{g_1}(a_2) = 0 \).

Step 2: Check the statement for arbitrary \( a \in \Omega^2(\mathbb{R}^3 \times Y) \).

By theorem 3.11 \( \mathbb{R} \{ b_1 \wedge b_2 \in \Omega^2(\mathbb{R}^3 \times Y) : b_1 \in \Omega^k(\mathbb{R}^3), b_2 \in \Omega^l(Y) \text{ for } k, l \in \mathbb{Z}_{\geq 0} \} \) is dense in \( \Omega^2(\mathbb{R}^3 \times Y) \) with respect to the topology of compact convergence of all derivatives (we only need dense in the topology of pointwise convergence). This shows the claim. \( \square \)

### 4.3 The Laplacian on \( M \)

We now move on to the heart of the argument: an operator bound for the inverse of the Laplacian on \( M \), cf. proposition 4.17. The Laplacian on 2-forms has a kernel of dimension \( b^2(M) \), so we can only expect such a bound for forms which are not in the kernel. Elliptic regularity would give an estimate for forms orthogonal to the kernel. This estimate would depend on the glueing parameter \( t \), but we want a uniform estimate, i.e. an estimate independent of \( t \). To this end, we will first replace the kernel of the Laplacian by an approximate kernel.

**Definition 4.13 (Approximate Kernel).** Let \( \{\alpha_1, \ldots, \alpha_k\} \) be an orthonormal basis of \( \text{Ker}(\Delta_{T^\infty/T}) : \Omega^2(T^\infty/T) \to \Omega^2(T^\infty/T) \). Let \( U_1 \cup \cdots \cup U_{12} \) be the connected components of the neighbourhould of the singular set \( U \subset T^\infty/T \) from lemma 4.3. For \( t \in (0, 1) \) and \( i \in \{1, \ldots, 12\} \) denote by \( \nu_{t,i} \in \Omega^2(M) \) the 2-form which is equal to \( \nu_i \) (cf. lemma 2.13) on \( \pi^{-1}(U_i) \) and zero everywhere else. Note that \( \nu_{t,i} \) is not continuous, but \( \chi(d(\pi(\cdot), L)) \cdot \nu_{t,i} \) is. Let

\[
\mathcal{K}_{ap}^{(t)} := \text{span}\{(1 - \chi(d(\pi(\cdot), L)))(\pi^*\alpha_i)\}_{i \in \{1, \ldots, k\}} \oplus \text{span}\{\chi(d(\pi(\cdot), L)) \cdot \nu_{t,i}\}_{i \in \{1, \ldots, 12\}}.
\]

**Definition 4.15.** For \( t \in (0, 1) \) define the weight functions

\[
w_t : M \to \mathbb{R}_{>0}, \quad x \mapsto t + d_{g_0}(x, \pi^{-1}(L)),
\]

\[
w : \mathbb{R}^3 \times \mathbb{R}^4 \to \mathbb{R}_{>0}, \quad (x, y) \mapsto ||y||,
\]

\[
w_t : \mathbb{R}^3 \times X \to \mathbb{R}_{>0}, \quad x \mapsto t + d_{g_{k+1} \circ g_1}(x, \mathbb{R}^3 \times S^2)
\]

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and for \( k \in \mathbb{Z}_{\geq 0}, \alpha \in (0, 1), \beta \in \mathbb{R} \) the weighted Hölder norms \(||\cdot||_c^{\beta,\alpha}_{p,t}
olimits\) on \( M \) and \(||\cdot||_c^{\beta,\alpha}_{\beta-2,t}
olimits\) on \( \mathbb{R}^3 \times \mathbb{R}^4 \) respectively as in definition 2.12.

The following proposition is the crucial part of the argument: an estimate for the inverse of the Laplacian on \( M \). Note that the proof is similar to the proof of [Wal3a, Proposition 8.7].

**Proposition 4.17.** Let \( \beta \in (-4, -2) \). Then there exists \( T > 0 \) and \( c > 0 \) such that for \( a \in C^2,\alpha \nolimits\( M \) with \( a \perp K_{ap}^{(t)} \) with respect to the \( L^2 \) inner product:

\[
|a|_{C^{\beta,\alpha}_{\beta-2,t}} \leq c \Delta a \quad (4.18)
\]

for all \( t \in (0, T) \).

**Proof.** The Schauder estimate

\[
|a|_{C^{\beta,\alpha}_{\beta-2,t}} \leq c \left( |\Delta a|_{C^{\beta,\alpha}_{\beta-2,t}} + |a|_{L^\infty_{\beta,t}} \right) \quad (4.19)
\]

can be derived as in [Wal7, Proposition 8.15]. It then suffices to show that there exists \( c \) such that \(|a|_{L^\infty_{\beta,t}} \leq c |\Delta a|_{C^{\beta,\alpha}_{\beta-2,t}}\) for small \( t \). Assume \( T \) and \( c \) as in the proposition statement exist, then let \( t_i \to 0, a_i \in \Omega^2(M^{t_i}) \) with \( a_i \perp K_{ap}^{(t_i)} \), \( x_i \in M^{t_i} \) such that

\[
|a|_{C^{\beta,\alpha}_{\beta,t_i}} \leq c \left( |w_{\beta,t_i}(x_i)a_i(x_i)| = 1, \text{ and } |\Delta a_i|_{C^{\beta,\alpha}_{\beta-2,t_i}} \to 0. \quad (4.20)\right)
\]

Without loss of generality we can assume to be in one of three following cases, and we will arrive at a contradiction in each of them.

**Case 1:** \( x_i \) concentrates on one ALE space, i.e. \( t_i^{-1}d(x_i, \pi^{-1}(L)) \to c < \infty \) (see fig. 1).

![Blowup analysis near the associative is reduced to the analysis of the Laplacian on \( \mathbb{R}^3 \times X \).](image)

**Figure 1:** Blowup analysis near the associative is reduced to the analysis of the Laplacian on \( \mathbb{R}^3 \times X \).

By passing to a subsequence, we can assume that \( x_i \) concentrates near one fixed connected component of \( L \). Let \( U_j \) be the connected component of the tubular neighbourhood from lemma 4.3 containing an accumulation point of the sequence \( x_i \). As before, we can view \( U_j \) as a subset of \( L \times X \), so we can write \( x_i = (y_i, z_i) \in L \times X \). Denote by \( \tilde{U} \subset \mathbb{R}^3 \times X \) the preimage of \( U_j \) under \((\exp_{y_i}^L, \text{Id}) : \mathbb{R}^3 \times X \to T_{y_i}L \times X \to L \times X \).

Restrict \( a_i \) to \( U_j \) and let \( \tilde{a}_i := s_{\beta,t_i} \left( (\exp_{y_i}^L, \text{Id})^* a_i \right) \in \Omega^2((\text{Id}, \phi)^{-1}(\tilde{U})) \), then \( \tilde{a}_i \) satisfies

\[
|\tilde{a}_i|_{C^{\beta,\alpha}_{\beta-1}} \leq c \text{ and } d(\tilde{x}_i, \mathbb{R}^3 \times S^2_{\beta})^{-\beta} |s_{\beta,t_i} \tilde{a}_i(\tilde{x}_i)| \geq c,
\]

\[\text{for small } t_i \to 0.\]
where $\tilde{x}_i := (0, \phi^{-1}(z_i)) \in \mathbb{R}^3 \times X$. This follows from lemma 2.20. Here we used $s_{\beta, t_i}$ from definition 2.20, which was defined as a map $s_{\beta, t_i} : \Omega^2(\mathbb{R}^3 \times X) \to \Omega^2(\mathbb{R}^3 \times X)$, but the same expression also defines a map $\Omega^2(\tilde{U}) \to \Omega^2((\text{Id}, \phi)^{-1}(\tilde{U}))$. Now the weight function no longer has $t_i$ in it (cf. definition 4.15) and distances and tensors are measured using the metric $g_{R^3 \oplus g(1)}$.

Similarly, we can estimate the first summand of eq. (4.20) for any $\epsilon > 0$.

Taking $\epsilon := \lim_{i \to \infty} \epsilon_i \in \mathbb{R}^3 \times X$. Note that $\phi^{-1}(\tilde{U})$ depends implicitly on $t$, as $\phi$ depends implicitly on $t$, and the sets $\phi^{-1}(\tilde{U})$ exhaust $\mathbb{R}^3 \times X$ as $t \to 0$ by eq. (2.3).

Using the Arzela-Ascoli theorem and a diagonal argument, we can extract a limit $a^* \in \Omega^2(\mathbb{R}^3 \times X)$ of the sequence $s_{\beta, i} \tilde{a}_i$ satisfying:

$$||a^*||_{\beta, \epsilon} \leq c, \quad \Delta_{g_{R^3 \oplus g(1)}} a^* = 0, \quad d(x^*, \mathbb{R}^3 \times S^2) |a^*(x^*)| > c.$$  

We have a splitting $T^* \mathbb{R}^3 \times X = T^*\mathbb{R}^3 \oplus T^*X$, and therefore a splitting of $\Lambda^k(\tilde{U})$. Write $[a_i]_{(0,2)}$ for the $T^*X \otimes T^*X$ part of $a_i$. Then, by the assumption $a_i \perp \mathcal{K}_{\text{app}}(L_j)$ and by splitting $M$ into one part close to $\pi^{-1}(L_j)$ and one part far away from $\pi^{-1}(L_j)$:

$$0 = \left\langle [a_i]_{(0,2)}, \chi(d(\pi(\cdot), L)) \cdot v_{t_i,j} \right\rangle_{(x \in M_i, d(x, \pi^{-1}(L_j)) \leq \epsilon)} \cdot t_i^d \geq \left\langle [a_i]_{(0,2)}, \chi(d(\pi(\cdot), L)) \cdot v_{t_i,j} \right\rangle_{(x \in M_i, d(x, \pi^{-1}(L_j)) \leq \epsilon/4)} \cdot t_i^d \right.$$

$$- \left\langle [a_i]_{(0,2)}, \chi(d(\pi(\cdot), L)) \cdot v_{t_i,j} \right\rangle_{(x \in M_i, \epsilon/4 \leq d(x, \pi^{-1}(L_j)) \leq \epsilon/2)} \cdot t_i^d. \right.$$  

Write $\tilde{a}_i := (\text{Id}, \phi)^* a_i$, then we can estimate the second summand of eq. (4.24) as

$$\left| \left\langle [a_i]_{(0,2)}, v_{t_i,j} \right\rangle_{x \in M_i, \epsilon/4 \leq d(x, \pi^{-1}(L_j)) \leq \epsilon/2} \cdot t_i^d \right| = \left| \left\langle [\tilde{a}_i]_{(0,2)}, v_1 \right\rangle_{x \in L \times X, \epsilon/4 \leq d_{g_{R^3 \oplus g(1)}} \leq \epsilon/2} \right| \leq c \int_{\epsilon/4 \leq \epsilon/2} \int_{t_i^d} r^d \cdot r^2 \cdot d r \leq c t_i^d \to 0 \text{ as } i \to \infty,$$

where the $t_i^d$ in the first step disappeared because we changed from measuring in $g_L \oplus g(t_i^d)$ to measuring in $g_L \oplus g(1)$. This gives a factor of $t_i^d$ for the length of the tensor, and the change of the volume form affects the integral with another factor of $t_i^{-d}$.

Similarly, we can estimate the first summand of eq. (4.24) for any $0 < l < t_i^{-1} \epsilon/4$ as

$$\left| \left\langle [a_i]_{(0,2)}, v_{t_i,j} \right\rangle_{x \in M_i, d(x, \pi^{-1}(L_j)) \leq \epsilon/4} \cdot t_i^d \right| \geq \left| \left\langle [\tilde{a}_i]_{(0,2)}, v_1 \right\rangle_{x \in L \times X, d_{g_{R^3 \oplus g(1)}} \leq l} \right| - c l^d$$

for a constant $c$ independent of $l$ and $i$. As before, we can extract a $C_{\text{loc}}^{2a/2}$-limit of the sequence $[\tilde{a}_i]_{(0,2)}$. Denote it by $b^* \in \Omega^2(X \times L)$. Then, taking the limit $i \to \infty$ in eq. (4.24) gives:

$$0 \geq \left| \left\langle b^*, v_1 \right\rangle_{x \in L \times X, d_{g_{R^3 \oplus g(1)}} \leq l} \right| - c l^d$$

Taking $l \to \infty$ shows that $0 \geq |\langle b^*, v_1 \rangle|$, i.e. $b^* \perp v_1$. By corollary 4.17 (applied to the case $\mathbb{R}^3 \times X$), we have that $a^*$ is independent of the $\mathbb{R}^3$-direction. By the second point of lemma 2.13.
By abuse of notation, we denote this form by $b^*$ as well. $b^* \in \text{Ker} \Delta_X$ by eq. (4.22), but $b^* \perp v_1$. However, by the first point of lemma 2.13, we have that the $C^2_{\beta}$-kernel of $\Delta_X$ is spanned by $v_1$. Thus, $b^* = 0$, which is a contradiction to eq. (4.23).

Case 2: $x_i$ concentrates on the regular part, i.e. $d(x_i, \pi^{-1}(L)) \to c > 0$ (see fig. 2).

Figure 2: Blowup analysis away from the associative is reduced to the analysis of the Laplacian on $T^3 / \Gamma$.

Using the Arzela-Ascoli theorem and a diagonal argument, we can extract a limit $a^* \in \Omega^2(T^3 / \Gamma)$. Denote, furthermore, $\lim_{i \to \infty} x_i = x^*$. We have $|a^*| < c \cdot d(\cdot, L)^{\beta}$, so we have that $a^*$ is a well defined distribution on $M / \langle i \rangle$ because $\beta > -4$. $\Delta a^* = 0$, so $a^*$ is smooth by elliptic regularity.

Furthermore,

$$\langle a^*, (1 - \chi) \circ (d(\cdot, L)) \cdot a_i \rangle_{T^3 / \Gamma} = \lim_{i \to \infty} \langle a_i, (1 - \chi_i) \cdot \pi^* a_i \rangle_{M / \langle i \rangle} = 0.$$

By the unique continuation property for elliptic PDEs, the inner product

$$\langle \cdot, (1 - \chi) \circ (d(\cdot, L)) \cdot \cdot \rangle$$

is non-degenerate on harmonic forms. $a^*$ is a harmonic form that is orthogonal to all harmonic forms with respect to this inner product, therefore $a^* = 0$. But this contradicts $a^*(x^*) > c$.

Case 3: $x_i$ concentrates on the neck region, i.e. $t_i^{-1} d(x_i, \pi^{-1}(L)) \to \infty$, but $d(x_i, \pi^{-1}(L)) \to 0$ (see fig. 3).

Figure 3: Blowup analysis in the neck region is reduced to the analysis of the Laplacian on $\mathbb{R}^3 \times \mathbb{R}^4$.

Define $\tilde{a}_i \in \Omega^2(\mathbb{R}^3 \times X)$ and $\tilde{x}_i \in \mathbb{R}^3 \times X$ as in case 1. In this case, we have that $|\rho(\tilde{x}_i)| \to \infty$. In order to be able to obtain a limit of this sequence, let $R_i \to \infty$ be a sequence such that
$R_i/|\rho(\tilde{x}_i)| \to 0$. Cutting out the exceptional locus of the Eguchi-Hanson space, we can consider \( \{R_i \leq |\rho| \leq \zeta I_i\} \) as a subset of \( \mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\} \). On \( \mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\} \), we have the rescaling map \((\cdot |\rho(\tilde{x}_i))\).

We now define:

\[
\tilde{a}_i := (\cdot |\rho(\tilde{x}_i))^\ast (\tilde{a}_i |_{\{R_i \leq |\rho| \leq \zeta I_i\}}) \cdot |\rho(\tilde{x}_i)|^{2-\beta} \\
\in \Omega^2(\mathbb{R}^3 \times \{R_i/|\rho(\tilde{x}_i)| \leq |\rho| \leq R\zeta I_i/|\rho(\tilde{x}_i)|\}),
\]

(4.26)

\[
\tilde{x}_i := \tilde{x}_i/|\rho(\tilde{x}_i)|.
\]

This sequence satisfies

\[
\left\|\tilde{a}_i\right\|_{C^{2,\alpha}_\beta} \leq c \text{ and } \left|\tilde{a}_i(\tilde{x}_i)\right| > c.
\]

(4.27)

\(\tilde{a}_i\) and \(\tilde{x}_i\) are defined on (subsets of) \( \mathbb{R}^3 \times \mathbb{C}^2/\{\pm 1\} \). We use the same symbols to denote their pullbacks under the quotient map \( \mathbb{C}^2 \to \mathbb{C}^2/\{\pm 1\} \).

As before, we extract a \( C^{2,\alpha/2}_\text{loc} \)-limit \( a^* \in \Omega^2(\mathbb{R}^3 \times \mathbb{R}^4 \setminus \{0\}) \) satisfying

\[
\Delta_\mathbb{R}^2 a^* = 0, \text{ and } ||a^*||_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^4)} \leq c.
\]

\(a^*\) defines a distribution on all of \( \mathbb{R}^7 \), and is smooth by elliptic regularity on all of \( \mathbb{R}^7 \).

We also get an \( L^\infty\)-bound for \( a^*\) as follows: away from \( \mathbb{R}^3 \times \{0\} \), this is given by eq. (4.27). To see that \( a^*\) does not blow up in the \( \mathbb{R}^3\)-direction near \( \mathbb{R}^3 \times \{0\} \), consider any \( y \in \mathbb{R}^3 \times \{0\} \). Let \( 1 < p < -4/\beta \), then \( ||a^*||_{L^p(\mathbb{R}^3 \times \{y\})} \leq c \), independent of \( y \), by eq. (4.27). So, by elliptic regularity \( ||a^*||_{L^m(\mathbb{R}^3 \times \{y\})} \leq c \) for any \( m \in \mathbb{N} \), and by the Sobolev embedding we have \( ||a^*||_{L^\infty} \leq c \), where all of these estimates were independent of \( y \).

By corollary 4.17 (applied to \( \mathbb{R}^3 \times \mathbb{R}^4 \)), \( a^* \) is constant in the \( \mathbb{R}^3 \) direction. \( a^* \) is therefore the pullback of a harmonic, bounded function of \( \mathbb{R}^4 \), and must thus vanish, which is a contradiction to the second part of eq. (4.27).

The following proposition indicates that we made the right definition for \( \mathcal{K}^{(t)}_{ap} \): we got the desired estimate for the inverse of the Laplacian in proposition 4.17 but by restricting to the orthogonal complement of \( \mathcal{K}_{ap}^{(t)} \) we are not forgetting about any important 2-forms — the image of the Laplacian remains the same when restricted to this orthogonal complement!

**Proposition 4.28.** Let \( T > 0 \) be the constant from proposition 4.17 and consider the Laplacian \( \Delta_{M^t} : C^{2,\alpha}(\Lambda^2(M^t)) \to C^{0,\alpha}(\Lambda^2(M^t)) \) acting on 2-forms. For all \( t \in (0, T) \) we have

\[
\text{Im} \left( \Delta_{M^t} |_{(\mathcal{K}_{ap}^{(t)})^\perp} \right) = \text{Im} \Delta_{M^t}.
\]

(4.29)

**Proof.** **Step 1:** Show that the \( L^2\)-orthogonal projection \( q : \text{Ker} \Delta_{M^t} \to (\mathcal{K}_{ap}^{(t)})^\perp \) is an isomorphism.

Assume there exists \( 0 \neq a \in \Omega^2(M^t) \) with \( \Delta a = 0 \) such that \( q(a) = 0 \), i.e. \( a \perp (\mathcal{K}_{ap}^{(t)}) \). Then \( \Delta a \neq 0 \) by proposition 4.17 which is a contradiction. Now note \( \text{dim}(\text{Ker} \Delta_{M^t}) = b^0(L) + b^2(T^7/T) = 12 + k \), which is proved using the Künneth formula (see [JK17, Proposition 6.1]).
By construction, \( \dim(\mathcal{K}_{\text{ap}}^{(t)}) = 12 + k \), so \( q \) is a surjective linear map between vector spaces of the same dimension, and therefore injective.

**Step 2:** Check \( \operatorname{Im} \left( \Lambda \big|_{(\mathcal{K}_{\text{ap}}^{(t)})^\perp} \right) = \operatorname{Im} \Delta \).

It suffices to check that \( \operatorname{Im} \Delta \subset \operatorname{Im} \left( \Lambda \big|_{(\mathcal{K}_{\text{ap}}^{(t)})^\perp} \right) \). Let \( y \in \operatorname{Im} \Delta \), and \( \Delta x = y \). Denote the \( L^2 \)-orthogonal projection onto \( \mathcal{K}_{\text{ap}}^{(t)} \) by \( \text{proj}_{\mathcal{K}_{\text{ap}}^{(t)}} \). Let

\[
    z := q^{-1}(\text{proj}_{\mathcal{K}_{\text{ap}}^{(t)}}(-x)).
\]

Then \( \Delta(x + z) = y \), and \( \text{proj}_{\mathcal{K}_{\text{ap}}^{(t)}}(x + z) = 0 \) because of \( \text{proj}_{\mathcal{K}_{\text{ap}}^{(t)}} \circ q^{-1} = \text{Id} \), i.e. \( x + z \perp \mathcal{K}_{\text{ap}}^{(t)} \) which completes the proof. \( \square \)

### 4.4 The Existence Theorem

We will now prove the theorem which guarantees the existence of a torsion-free \( G_2 \)-structure when starting from a \( G_2 \)-structure with small torsion. The structure of the proof is the same as the structure of the proof of [Joyce Proposition 11.8.1].

**Theorem 4.30.** Let \( \beta \in (-4, -2) \). There exist \( T' > 0 \) and \( c_1, c_2 > 0 \) such that the following is true: If \( t \in (0, T') \), \( \varphi \) is a closed \( G_2 \)-structure on \( M, \psi \in \Omega^2(M) \) such that \( d^* \psi = d^* \varphi \) and

\[
    \|d^* \psi\|_{C^{\beta-2,t}} \leq c_1 t^\kappa, \\
    \|\psi\|_{C^{\beta,t}} \leq c_2,
\]

for \( \kappa > 1 - \beta + \alpha \), then there exists \( \eta \in \Omega^2(M) \) such that \( \tilde{\varphi} := \varphi + d \eta \) is a torsion-free \( G_2 \)-structure on \( M \) satisfying

\[
    \|\tilde{\varphi} - \varphi\|_{C^{\beta-1,t}} \leq t^\kappa.
\]

**Proof.** Let \( T \) be the constant from proposition [4.17] let \( t \in (0, T) \) and assume \( \varphi \) and \( \psi \) are given as specified in the theorem. We start off by constructing a sequence \( \eta_j \subset \Omega^2(M) \) solving

\[
    \Delta \eta_j = d^* \psi + d^* (f_{j-1} \psi) + * d F(d \eta_{j-1}) \text{ and } f_j \psi = \frac{7}{3} \pi_1(d \eta_j), \\
    \eta_0 = 0, \\
    \|\eta_j\|_{C^{\beta,t}} \leq t^\kappa.
\]

Given \( \eta_j \) for \( j \geq 0 \), note that

\[
    \sigma = d^* (\psi + f_j \psi + * F(d \eta_j))
\]

is in \( \operatorname{Im} \Delta \), because \( \operatorname{Im} d^* \perp \text{Ker } d \supset \text{Ker } \Delta \).

By proposition [4.28] there exists \( \eta_{j+1} \perp \mathcal{K}_{\text{ap}}^{(t)} \) such that \( \Delta \eta_{j+1} = \sigma \) which also satisfies

\[
    \|\eta_{j+1}\|_{C^{\beta,t}} \leq c_2 \|\sigma\|_{C^{\beta-2,t}}
\]

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for a constant $c_2$ independent of $t$ and $j$. Then

$$||\sigma||_{C^0_{\beta-2,t}} \leq ||d^* \psi||_{C^0_{\beta-2,t}} + ||d^*(f_j \psi)||_{C^0_{\beta-2,t}} + ||d F(d \eta_j)||_{C^0_{\beta-2,t}},$$

where we have

$$||d^* \psi||_{C^0_{\beta-2,t}} \leq c_1 t^\kappa,$$

$$||d^*(f_j \psi)||_{C^0_{\beta-2,t}} \leq ||f_j||_{C^0_{\beta-1,t}} \cdot ||d^* \psi||_{C^0_{\beta-1,t}} t^{\beta-1} + ||\nabla f_j||_{C^0_{\beta-2,t}} \cdot ||\psi||_{C^0_{\beta-1,t}} \leq c_1^2 t^{2k+\beta-1} + c_1 c_2 t^{\kappa} \leq t^{\kappa}$$

where in the first step we used $d^*(f_j \psi) = f_j d^* \psi - \nabla(f) \cdot \psi$, and in the last step we used that $t$ is small enough to absorb the constant $c_1^2$ and $c_2$ was chosen such that $c_1 c_2 \leq \frac{1}{2}$. For small $t$, proposition 3.6 gives

$$||d F(d \eta_j)||_{L^\infty_{\beta-1,t}} \leq c \left( ||d \eta_j||_{L^m_{\beta-1,t}}^2 ||d^* \phi||_{L^m_{\beta-1,t}} t^{2\beta-2} + ||\nabla d \eta_j||_{L^m_{\beta-1,t}} ||d \eta_j||_{L^m_{\beta-1,t}} t^{\beta-1} \right) \leq c(t^{3k+2\beta-2} + t^{2k+\beta-1}) \leq c_1 t^\kappa$$

and

$$[d F(d \eta_j)]_{C^0_{\beta-2,t}} \leq c \left( ||d \eta_j||_{C^0_{\beta-1,t}} ||d \eta_j||_{L^m_{\beta-1,t}} ||d^* \phi||_{L^m_{\beta-1,t}} t^{2\beta-2-\alpha} + ||d \eta_j||_{L^m_{\beta-1,t}} [d^* \phi]_{C^0_{\beta-2-\alpha,t}} t^{2\beta-2-\alpha} + ||\nabla d \eta_j||_{C^0_{\beta-2-\alpha,t}} ||d \eta_j||_{L^m_{\beta-1,t}} t^{\beta-1-\alpha} + ||\nabla d \eta_j||_{L^m_{\beta-1,t}} [d \eta_j]_{C^0_{\beta-1-\alpha,t}} t^{\beta-1-\alpha} \right) \leq c(t^{3k+2\beta-2-\alpha} + t^{2k+\beta-1-\alpha}) \leq c_1 t^\kappa.$$
Proof. Let $\kappa = 4$, $\beta = -2 - \epsilon$, $\alpha = \epsilon$, then $\kappa > 1 - \beta + \alpha$, and theorem 4.30 gives the first estimate. The $L^\infty$-estimate follows from using $w_\ell^\beta \leq t^\beta$ for $\beta < 0$. □

Remark 4.32. In [Joy96b, Joy00] the estimate $||\bar{\varphi} - \varphi||_{L^\infty} \leq ct^{1/2}$ was shown. In this sense, corollary 4.34 is an improvement.

5 Torsion-Free $G_2$-Structures on Joyce-Karigiannis Manifolds

In [JK17], the authors constructed new examples of compact manifolds with holonomy $G_2$ by generalising Joyce’s original construction that was described in section 4.2. As in section 4.1 they first use a glueing procedure to construct a $G_2$-structure with small torsion. They then apply theorem 3.8 to perturb this $G_2$-structure into a torsion-free $G_2$-structure.

The main difference to Joyce’s original construction is the following: if one uses the cutoff procedure from the $T^7/\Gamma$ case in the new setting, one produces a $G_2$-structure that does not satisfy the necessary estimates to apply theorem 3.8. The $G_2$-structure obtained this way is denoted by $\varphi_t^N$ in section 5.3. The authors of [JK17] overcome this problem by constructing a $G_2$-structure with even smaller torsion, to which theorem 3.8 can be applied.

In the case of the generalised Kummer construction on $T^7/\Gamma$, we observed the following: when comparing theorem 3.8 (Joyce’s theorem for the existence of torsion-free $G_2$-structures) with theorem 4.30 (theorem for the existence of torsion-free $G_2$-structures using weighted H"older norms), we see that theorem 4.30 requires weaker estimates for the torsion of the $G_2$-structure than theorem 3.8. That did not matter in that case, because the torsion of the glued $G_2$-structure was small enough to satisfy both estimates. In this section we have a different situation: even though the torsion of $\varphi_t^N$ is too big to apply theorem 3.8, it will be small enough to apply the analogue of theorem 4.30, namely theorem 5.1.

5.1 Ingredients for the Construction

Let $M$ be a compact manifold endowed with a torsion-free $G_2$-structure $\varphi$. Write $g$ for the metric induced by $\varphi$. Let $\iota : M \to M$ be a $G_2$-involution, i.e. satisfying $\iota^2 = \text{Id}$, $\iota \neq \text{Id}$, $\iota^* \varphi = \varphi$. We then have:

Proposition 5.1 (Proposition 2.13 in [JK17]). Let $L = \text{fix}(\iota)$ and assume $L \neq \emptyset$. Then $L$ is a smooth, orientable 3-dimensional compact submanifold of $M$ which is totally geodesic, and, with respect to a canonical orientation, is associative.

Assumption 5.2. We assume that $L$ is nonempty, and we assume we are given a closed, coclosed, nowhere vanishing 1-form $\lambda$ on $L$.

Such a 1-form need not exist, and cases in which its existence can be guaranteed are discussed in [JK17, Section 7.1].
5.2 $G_2$-structures on the normal bundle $\nu$ of $L$

The metric defined by $\varphi$ defines a splitting

$$TM|_L \cong \nu \oplus TL,$$ \hspace{1cm} (5.3)

which is orthogonal with respect to $g$. Write $g_L$ for the metric on $L$ induced by $g$ and $g|_L = h_v \oplus g_L$. Write $\nabla^\nu$ for the restriction of the Levi-Civita connection of $g$ to $\nu \to L$. Fix $R > 0$ and let

$$U_R = \{(x, \alpha) \in \nu : |\alpha|_{h_v} < R\}.$$

Write $\pi : U_R \to L$ for the projection $(x, \alpha) \mapsto x$. Then for $R$ small enough, the map

$$\exp : \nu \to M$$

$$(x, \alpha) \mapsto \exp_x (\alpha)$$

is a diffeomorphism satisfying $\exp (x, -\alpha) = \iota \circ \exp (x, \alpha)$, because $\iota$ preserves $\varphi$, and in particular is an isometry.

Write $(\cdot t) : \nu \to \nu$ for the dilation map $(x, \alpha) \mapsto (x, t\alpha)$, and write $\delta$ for the vector field on $\nu$ which has $(\cdot t)$ as its flow. For $t \neq 0$, define $\exp_t = \exp \circ (\cdot t) : U_{|t|^{-1}R} \to M$. From the fact that $\exp_t (x, \alpha) = \exp_t (x, -\alpha)$ we see that the Taylor series of $\exp^*_t \varphi$ and $\exp^*_t (\ast \varphi)$ in $t$ at $t = 0$ contain no odd powers of $t$, thus there exists $\varphi^{2n} \in \Omega^2 (\nu)$, $\psi^{2n} \in \Omega^2 (\nu)$ such that:

$$\exp^*_t (\varphi) \sim \sum_{n=0}^{\infty} t^{2n} \varphi^{2n},$$ \hspace{1cm} (5.4)

$$\exp^*_t (\ast \varphi) \sim \sum_{n=0}^{\infty} t^{2n} \psi^{2n}.$$ \hspace{1cm} (5.5)

Here, '$\sim$' means that for $S \subset \nu$ compact, so that $S \subset U_{|t|^{-1}R}$ for sufficiently small $t$, we have

$$\sup_S \left| \exp^*_t (\varphi) - \sum_{n=0}^{k} t^{2n} \varphi^{2n} \right| = o(t^k) \quad \text{as } t \to 0 \text{ for all } k = 0, 1, \ldots,$$

and similarly for $\exp^*_t (\psi)$.

$\nabla^\nu$ defines a splitting

$$T\nu = V \oplus H, \quad \text{where } V \cong \pi^* (\nu) \text{ and } H \cong \pi^* (TL),$$ \hspace{1cm} (5.6)

where $V$ and $H$ are the vertical and horizontal subbundles of the connection. This induces a splitting

$$\Lambda^k T^* \nu = \bigoplus_{\substack{i+j=k, \ 0 \leq i \leq k, \ 0 \leq j \leq k \ \text{and} \ \iota+j=k \ \text{for sufficiently small } t, \text{ we have}}} \Lambda^i V^* \otimes \Lambda^j H^*,$$ \hspace{1cm} (5.7)

and we write $\varphi^{2n}_{i,j}$, $\psi^{2n}_{i,j}$ for the components of $\varphi^{2n}$, $\psi^{2n}$ in $\Lambda^i V^* \otimes \Lambda^j H^*$ with respect to this splitting.
For $\beta = \beta_{i,j}$ of type $(i, j)$ we have $d\beta = (d\beta)_{i+1,j} + (d\beta)_{i,j+1} + (d\beta)_{i-1,j+2}$, i.e., $d\beta$ only has components of three different types. That can be seen from writing $d\varphi$ as a wedge product of vertical 1-forms and closed forms on $L$ pulled back under $\pi$. From $d\varphi = 0, d\psi = 0$ we have that $d\varphi^{2n} = 0$ and $d\psi^{2n} = 0$ for all $n \in \mathbb{Z}_{>0}$. Thus, by Cartan’s formula, we have

$$\varphi^{2n} = \frac{1}{2n} d(\delta \varphi^{2n}), \quad \psi^{2n} = \frac{1}{2n} d(\delta \psi^{2n}) \quad \text{for } n = 1, 2, \ldots, \quad (5.8)$$

and thus there exist $\varphi_{i,j}^{2n}, \varphi_{i+1,j}^{2n}, \varphi_{i,j+1}^{2n}$, and $\psi_{i,j}^{2n}, \psi_{i+1,j}^{2n}, \psi_{i,j+1}^{2n}$ of type $(i, j)$ for $n > 0$ satisfying

$$\frac{1}{2n} d(\delta \varphi_{i,j}^{2n}) = \varphi_{i,j}^{2n} + \varphi_{i-1,j+1}^{2n} + \varphi_{i-2,j+2}^{2n},$$

$$\frac{1}{2n} d(\delta \psi_{i,j}^{2n}) = \psi_{i,j}^{2n} + \psi_{i-1,j+1}^{2n} + \psi_{i-2,j+2}^{2n}. \quad (5.9)$$

So by eq. (5.3) we have

$$\varphi_{i,j}^{2n} = \varphi_{i,j}^{2n} + \varphi_{i-1,j+1}^{2n} + \varphi_{i-2,j+2}^{2n}, \quad \psi_{i,j}^{2n} = \psi_{i,j}^{2n} + \psi_{i-1,j+1}^{2n} + \psi_{i-2,j+2}^{2n}. \quad (5.10)$$

Combining eqs. (5.9) and (5.6), we have that $T \nu \simeq \pi^*(TM|L)$. Denote by

$$\varphi^\nu \in \Omega^3(v), \quad \psi^\nu \in \Omega^3(v), \quad \text{and} \quad g^\nu \in S^2(v) \quad (5.11)$$

the structures obtained from $\varphi, \psi,$ and $g$ via this isomorphism.

The functions $\varphi_{i,j}^{2n} |_{g^\nu}, \psi_{i,j}^{2n} |_{g^\nu}$ are homogenous of degree $2n - i$. As the degree of homogeneity cannot be negative, we get that $\varphi_{i,j}^{2n} |_{g^\nu} = 0, \psi_{i,j}^{2n} |_{g^\nu} = 0$ if $i > 2n$. Using this, one finds as in [JK17] Eqns. (3.24), (3.25)]:

$$\exp^* \varphi \sim \varphi_{0,3} + t^2 \varphi_{2,1} \quad \left[ \varphi_{0,3} + t^2 \varphi_{2,1} \right] = O(1)$$

$$+ t^2 \varphi_{1,2}^2 + t^2 \varphi_{1,2}^4 + t^4 \varphi_{3,0}^4 \quad \left[ \varphi_{1,2}^2 + t^2 \varphi_{1,2}^4 + t^4 \varphi_{3,0}^4 \right] = O(t^2 \nu^2)$$

$$+ t^4 \varphi_{0,3}^4 + t^4 \varphi_{0,3}^2 + t^4 \varphi_{3,1}^4 + t^4 \varphi_{3,1}^2 + t^2 \varphi_{0,3}^4 + t^2 \varphi_{0,3}^2 + t^2 \varphi_{3,1}^4 + t^2 \varphi_{3,1}^2 + t \varphi_{1,2}^4 + \ldots, \quad (5.12)$$

$$\exp^* \psi \sim \psi_{0,4} + t^2 \psi_{2,0} + t^2 \psi_{2,0}^4 \quad \left[ \psi_{0,4} + t^2 \psi_{2,0} + t^2 \psi_{2,0}^4 \right] = O(1)$$

$$+ t^2 \psi_{1,3}^2 + t^2 \psi_{1,3}^4 + t^2 \psi_{1,3}^2 + t^2 \psi_{1,3}^4 \quad \left[ \psi_{1,3}^2 + t^2 \psi_{1,3}^4 + t^2 \psi_{1,3}^2 \right] = O(t^2 \nu^2)$$

$$+ t^4 \psi_{0,3}^4 + t^4 \psi_{0,3}^2 + t^4 \psi_{3,1}^4 + t^4 \psi_{3,1}^2 + t^4 \psi_{0,3}^4 + t^4 \psi_{0,3}^2 + t^4 \psi_{3,1}^4 + t^4 \psi_{3,1}^2 + t \psi_{1,2}^4 + \ldots. \quad (5.13)$$

Here we also compared homogeneity degrees to deduce that the leading order terms of the series expansions are equal to $\varphi^\nu$ and $\psi^\nu$ respectively. In this grid, columns sum up to closed forms. So, we see that in general neither $\varphi^\nu$ nor $\psi^\nu$ will be closed. They have the benefit of satisfying $\Theta(\varphi^\nu) = \psi^\nu$, though.

Now define

$$\tilde{\varphi}^\nu = \varphi^\nu + t^2 \varphi_{1,2}^2 + t^2 \varphi_{0,3}^2, \quad \tilde{\psi}^\nu = \psi^\nu + t^2 \psi_{1,3}^2 + t^4 \psi_{3,1}^4 + t^4 \psi_{2,2}^4 \quad (5.14)$$

which are two closed forms, which do not satisfy $\Theta(\tilde{\varphi}^\nu) = \tilde{\psi}^\nu$. From eqs. (5.8), (5.12) and (5.13) we get: 

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Proposition 5.15 (Section 3.5 in [IK17]). There exist \( \eta \in \Omega^2(U_R) \), \( \zeta \in \Omega^3(U_R) \) satisfying
\[
\frac{d}{dx} \eta = \exp^* \varphi - \tilde{\varphi} \big| \mid U_R, \\
\frac{d}{dx} \zeta = \exp^* (\ast \varphi) - \tilde{\varphi} \big| \mid U_R,
\]
and
\[
|\eta|_{g^v} = O(r^2), \\
|d\eta|_{g^v} = O(r^3), \\
|\xi|_{g^v} = O(r^2), \\
|d\xi|_{g^v} = O(r^3).
\]

5.3 G2-structures on the resolution P of \( v/\{\pm 1\} \)

The G2-structure \( \varphi \in \Omega^3(M) \) defines for all \( x \in M \) a cross product \( \times : T_xM \times T_xM \rightarrow T_xM \) as in definition \[3.2\]. We then have a complex structure \( I \in \text{End}(v) \) given by
\[
I(V) = \frac{\lambda}{|A|} \times V \quad \text{for} \quad V \in v_x, x \in L.
\]

Recall the metric \( h_v \) on \( v \) defined by \( g|_L = h_v \oplus g_L \), cf. section \[3.7\]. Then \( I \) and \( h_v \) together define a \( U(2) \)-reduction of the frame bundle of \( v \). Denote by \( X \) the Eguchi-Hanson space with Hyperkähler triple \( \omega_1^{(1)}, \omega_2^{(1)}, \omega_3^{(1)} \) from proposition \[2.6\]. Denote by \( \rho : X \rightarrow C^2/\{\pm 1\} \) the blowup map of the blowup with respect to the complex structure induced by \( \omega_1^{(1)} \) and let
\[
P = \text{Fr} \times_{U(2)} X.
\]

Denote by \( \sigma : P \rightarrow L \) the projection of this bundle. Analogously, we have
\[
v/\{\pm 1\} = \text{Fr} \times_{U(2)} C^2/\{\pm 1\}.
\]

Let \( L' \subset L \) be a nonempty, open set on which we can extend \( e_i := \frac{\lambda}{|A|} \in T^*(L') \) to an orthonormal basis \( (e_1, e_2, e_3) \). Then there exist \( \tilde{\omega}^J, \tilde{\omega}^J, \tilde{\omega}^K \in \Omega^2((v/\{\pm 1\})|_{L'}) \) such that \( \varphi^v \) from eq. \[5.11\] has the form
\[
\varphi^v = e_1 \wedge e_2 \wedge e_3 - \tilde{\omega}^J \wedge e_1 - \tilde{\omega}^J \wedge e_2 - \tilde{\omega}^K \wedge e_3.
\]

We define \( \tilde{\omega}^J, \tilde{\omega}^J, \tilde{\omega}^K \in \Omega^2(P|_{L'}) \) as follows: For \( x \in L' \), let \( f \in \text{Fr}_x \) such that \( f : (v/\{\pm 1\})_x \rightarrow C^2/\{\pm 1\} \) satisfies
\[
f^* (\omega_1^{(0)}, \omega_2^{(0)}, \omega_3^{(0)}) = (\tilde{\omega}^J|_{v_x}, \tilde{\omega}^J|_{v_x}, \tilde{\omega}^K|_{v_x}),
\]
where \( (\omega_1^{(0)}, \omega_2^{(0)}, \omega_3^{(0)}) \) denotes the Hyperkähler triple on \( C^2/\{\pm 1\} \) from proposition \[2.6\]. This choice of \( f \) defines isomorphisms of complex surfaces \( P_x \cong X \) and \( (v/\{\pm 1\})_x \cong C^2/\{\pm 1\} \). Let \( \tilde{\omega}^J, \tilde{\omega}^J, \tilde{\omega}^K \in \Omega^2(P_x) \) be the pullback of \( \omega_1^{(1)}, \omega_2^{(1)}, \omega_3^{(1)} \in \Omega^2(X) \) under this isomorphism. This is independent of the choice of \( f \), and therefore defines \( \omega_1^{(1)}, \omega_2^{(1)}, \omega_3^{(1)} \in \Omega^2(P|_{L'}) \). The following diagram sums up the situation:

\[
\begin{array}{ccc}
(P_x, \tilde{\omega}^J|_{P_x}, \tilde{\omega}^J|_{P_x}, \tilde{\omega}^K|_{P_x}) & \xrightarrow{=} & (X, \omega_1^{(1)}, \omega_2^{(1)}, \omega_3^{(1)}) \\
\downarrow^\rho & & \downarrow^\rho \\
(v_x/\{\pm 1\}, \tilde{\omega}^J|_{v_x/\{\pm 1\}}, \tilde{\omega}^J|_{v_x/\{\pm 1\}}, \tilde{\omega}^K|_{v_x/\{\pm 1\}}) & \xrightarrow{=} & (C^2/\{\pm 1\}, \omega_1^{(0)}, \omega_2^{(0)}, \omega_3^{(0)})
\end{array}
\]
Here, by abuse of notation we denoted the map $P_x \to \nu_x/\{\pm 1\}$ which makes the diagram commutative also by $\rho$. Horizontal arrows pull Hyperkähler triples back to one another. Hyperkähler triples connected by vertical arrows are asymptotic in the sense of proposition 2.6.

We are now ready to define $\phi^P_i \in \Omega^3(P|_{L'})$, $\psi^P_i \in \Omega^4(P|_{L'})$ via

$$
\phi^P_i := \phi_{0,3} + t^2 \phi_{2,1}
$$

$$
:= \sigma^* (e_1 \wedge e_2 \wedge e_3) - t^2 \left( \sigma^* (e_1) \wedge \partial I - \sigma^* (e_2) \wedge \partial I - \sigma^* (e_3) \wedge \partial K \right),
$$

$$
\psi^P_i := t^2 \psi_{2,2} + t^4 \psi_{4,0}
$$

$$
:= \frac{1}{2} \partial I \wedge \partial I - \sigma^* (e_2 \wedge e_3) \wedge \partial I - \sigma^* (e_1 \wedge e_1) \wedge \partial I - \sigma^* (e_1 \wedge e_2) \wedge \partial K.
$$

One checks that these expressions are independent of the choice of $(e_2, e_3)$, and therefore define forms $\phi^P_i \in \Omega^3(P), \psi^P_i \in \Omega^4(P)$, not just forms over $L' \subseteq L$. Let also $\phi^Q_i$ denote the metric induced by $\phi^P_i$. Here, $\phi_{0,3}$ and $\phi_{2,1}$ correspond to $\phi^Q_{0,3}$ and $\phi^Q_{2,1}$ from eq. (5.12) and $\phi_{2,2}$ and $\phi_{4,0}$ correspond to $\phi^Q_{2,2}$ and $\phi^Q_{4,0}$ from eq. (5.13) in the following sense:

$\phi^P_i$ and $\psi^P_i$ are not closed, but satisfy $\Theta(\phi^P_i) = \psi^P_i$. Write $\check{r} := r \circ \rho : P \to [0, \infty)$, where $r : \nu/\{\pm 1\} \to [0, \infty)$ is the radius function. Then $\phi_{0,3}, t^2 \phi_{2,1}, t^2 \psi_{2,2}, t^4 \psi_{4,0}$ are asymptotic to $\rho^*(\phi^Q_{0,3}), \rho^*(t^2 \phi^Q_{2,1}), \rho^*(t^2 \psi_{2,2}), \rho^*(t^4 \psi_{4,0})$ as $\check{r} \to \infty$ in $P$. This can be seen from proposition 2.6.

As in the previous section, we add terms to $\phi^P_i$ and $\psi^P_i$ to define closed forms on $P$, and we have the following control over how they are asymptotic to forms on $\nu/\{\pm 1\}$:

**Proposition 5.20 (Section 4.5 in [JK17]).** There exist $\xi_{1,2}, \xi_{0,3} \in \Omega^3(P), \tau_{1,1} \in \Omega^2(\{x \in P : \check{r}(x) > 1\})$, such that

$$
\bar{\phi}^P_i := \phi^P_i + t^2 \xi_{1,2} + t^2 \xi_{0,3}
$$

is closed and satisfies

$$
\bar{\phi}^P_i = \rho^* \tilde{\phi}^\nu_i + t^2 \, d \tau_{1,1}
$$

where $\check{r} > 1$. These forms satisfy the following estimates:

$$
\left| \nabla^k (t^2 \xi_{1,2}) \right|_{g^P} = \begin{cases} O(t^{1-k}), & \check{r} \leq 1, \\
O(t^{1-k} \check{r}^{-3-k}), & \check{r} > 1, \end{cases}
$$

$$
\left| \nabla^k (t^2 \xi_{0,3}) \right|_{g^P} = \begin{cases} O(t^{2-k}), & \check{r} \leq 1, \\
O(t^{2-k} \check{r}^{-2-k}), & \check{r} > 1, \end{cases}
$$

$$
\left| \nabla^k (t^2 \tau_{1,1}) \right|_{g^P} = O(t^{1-k} \check{r}^{-3-k}).
$$

**Proposition 5.23 (Section 4.5 in [JK17]).** There exist $\chi_{1,3}, \theta_{3,1}, \theta_{2,2} \in \Omega^4(P), \psi_{1,2} \in \Omega^3(\{x \in P : \check{r}(x) > 1\})$, such that

$$
\bar{\psi}^P_i := \psi^P_i + t^2 \chi_{1,3} + t^4 \theta_{3,1} + t^4 \theta_{2,2}
$$

is closed and satisfies

$$
\bar{\psi}^P_i = \rho^* \tilde{\psi}^\nu_i + t^2 \, d \psi_{1,2}
$$
where $\tilde{r} > 1$. These forms satisfy the following estimates:

$$
\begin{align*}
\nabla^k(t^2 \mathcal{X}_{1,3})_{\mathcal{L}_\rho} &= \begin{cases} 
O(t^{1-k}), & \tilde{r} \leq 1, \\
O(t^{1-k} \tilde{r}^{-3-k}), & \tilde{r} > 1,
\end{cases} & (5.26) \\
\nabla^k(t^4 \theta_{3,1})_{\mathcal{L}_\rho} &= \begin{cases} 
O(t^{1-k}), & \tilde{r} \leq 1, \\
0, & \tilde{r} > 1,
\end{cases} & (5.27) \\
\nabla^k(t^4 \theta_{2,2})_{\mathcal{L}_\rho} &= \begin{cases} 
O(t^{2-k}), & \tilde{r} \leq 1, \\
O(t^{2-k} \tilde{r}^{-2-k}), & \tilde{r} > 1,
\end{cases} & (5.28) \\
\nabla^k(t^2 v_{1,2})_{\mathcal{L}_\rho} &= O(t^{1-k} \tilde{r}^{-3-k}). & (5.29)
\end{align*}
$$

Figure 4: An overview over the three $G_2$-structures being glued together. $\varphi$ is a torsion-free $G_2$-structure on the orbifold $M/\langle \iota \rangle$. $\varphi^\nu$ is a $G_2$-structure which is "constant on the fibres" of $\nu$ (see eq. 5.24). $\varphi^\rho_1$ is a non-closed $G_2$-structure on $P$, $\varphi^\rho_1$ is a correction of $\varphi^\rho_1$ which is closed (see section 5.3).

### 5.4 $G_2$-structures on the Resolution $N_\iota$ of $M/\iota$

We are now ready to glue together $P$ and $M/\langle \iota \rangle$ to a manifold, and define a $G_2$-structure with small torsion on it. As lamented in the introduction of section 5, this $G_2$-structure is defined mimicking the glueing construction from the $T^2/\Gamma$ case, while the original construction in [IK17] is more involved.

**Definition 5.30.** Define

$$
N_\iota := \rho^{-1}(U_{\iota^{-1}R}/\{\pm 1\}) \bigsqcup (M \setminus L)/\langle \iota \rangle /\sim,
$$

where $x \sim \exp_\iota \circ \rho(x)$ for $x \in \rho^{-1}(U_{\iota^{-1}R}/\{\pm 1\})$. 


Definition 5.32. Let \( a : [0, \infty) \to \mathbb{R} \) be a smooth function with \( a(x) = 0 \) for \( x \in [0, 1] \), and \( a(x) = 1 \in [2, \infty) \), and let \( B \in [-1, 0] \). Define then

\[
\psi_t^N = \begin{cases} 
\widehat{\varphi}_t^p, & \text{if } \hat{r} \leq t^B, \\
\frac{\varphi_t^p - d((1 - a(t\hat{r}))\tau_{1,1}) + d(a(t\hat{r})\exp_\eta)}{\varphi_t^p + d\exp_\eta} = \varphi, & \text{if } t^B \leq \hat{r} \leq 2t^B, \\
\frac{\varphi_t^p}{\varphi_t^p + d\exp_\eta}, & \text{elsewhere},
\end{cases}
\]

and

\[
\rho_t^N = \begin{cases} 
\tilde{\psi}_t^p, & \text{if } \hat{r} \leq t^B, \\
\frac{\psi_t^p - d((1 - a(t\hat{r}))\alpha_{1,2}) + d(a(t\hat{r})(\exp_p)\zeta)}{\psi_t^p + d(\exp_p)\zeta = \ast\varphi}, & \text{if } t^B \leq \hat{r} \leq 2t^B, \\
\frac{\psi_t^p}{\psi_t^p + d(\exp_p)\zeta}, & \text{elsewhere}.
\end{cases}
\]

Here \( \eta \in \Omega^2(U_R), \zeta \in \Omega^3(U_R) \), where \( U_R \subset v \), are the forms defined in proposition 5.35. They interpolate between the forms \( \varphi \) and \( \ast\varphi \) on \( M \) and the forms \( \tilde{\varphi}^p \) and \( \tilde{\psi}^p \) on \( v \).

Remark 5.35. \( B \) will control how far away from the bolts of Eguchi-Hanson space we will interpolate between \( \tilde{\varphi}_t^p \) and \( \varphi \). To make the torsion of \( \psi_t^N \) for our particular choice of norm very small, we will set \( B = -\frac{1}{2} \) in section 5.5. When measuring the torsion in other norms, there would be other optimal values for \( B \), so we leave it undetermined for now.

The important estimates for these forms are stated in the following theorem. The proof is similar to [JK17][Proposition 6.2].

Theorem 5.36. In the situation above, we have

\[
\left| \Theta(\psi_t^N) - \psi_t^N \right|_{\theta_t^N} = \begin{cases} 
O(t), & \text{if } \hat{r} \leq t^B, \\
O(t^{-4B} + t^{1B}), & \text{if } t^B \leq \hat{r} \leq 2t^B, \\
0, & \text{elsewhere},
\end{cases}
\]

\[
\left| \nabla \left( \Theta(\psi_t^N) - \psi_t^N \right) \right|_{\theta_t^N} = \begin{cases} 
O(1), & \text{if } \hat{r} \leq t^B, \\
O(t^{-1-5B} + 1), & \text{if } t^B \leq \hat{r} \leq 2t^B, \\
0, & \text{elsewhere},
\end{cases}
\]

\[
\left| \nabla \left( \Theta(\psi_t^N) - \psi_t^N \right) \right|_{\alpha, U} = \begin{cases} 
O(t^{-a}), & \text{if } U = \{ x : \hat{r}(x) \leq t^B \}, \\
O(t^{-1-5B-a(1+B)} + t^{-a(1+B)}), & \text{if } U = \{ x : t^B \leq \hat{r}(x) \leq 2t^B \}, \\
0, & \text{if } U = \{ x : 2t^B \leq \hat{r}(x) \}.
\end{cases}
\]

Proof. Case 1: \( \hat{r} \leq t^B \)

In this case,

\[
\Theta\psi_t^N - \psi_t^N = \Theta\tilde{\varphi}_t^p - \tilde{\psi}_t^p
\]

\[
= \Theta \left( \varphi_t^p + t^2 \xi_{1,2} + t^2 \xi_{0,3} \right) - \psi_t^p - t^2 \chi_{1,3} - t^4 \theta_{3,1} - t^4 \theta_{2,2}
\]

\[
= \left( T_{\varphi_t^p} + F_{\varphi_t^p} \right) \left( t^2 \xi_{1,2} + t^2 \xi_{0,3} \right) - t^2 \chi_{1,3} - t^4 \theta_{3,1} - t^4 \theta_{2,2},
\]

where we used \( \Theta(\varphi_t^p) = \psi_t^p \) together with eq. (3.7) in the last step. We made the dependence of maps \( T \) and \( F \) on the \( G_2 \)-form explicit by adding it as an index. \( T_{\varphi_t^p} \) is linear and by the first
estimate in proposition\[3.6\] we find that
\[
\begin{align*}
\left| \Theta \phi_t^N - \psi_t^N \right|_{g_t^N} &\leq c \left( |t^2 \xi_{1,2}| + |t^2 \xi_{0,3}| + |t^4 \chi_{2,3}| + |t^4 \theta_{3,1}| + |t^4 \theta_{2,2}| \right) \\
&= \begin{cases} 
O(t), & \text{if } \tilde{r} \leq 1, \\
O(t \tilde{r}^{-3}), & \text{if } 1 \leq \tilde{r} \leq t^B,
\end{cases}
\end{align*}
\]
by propositions \[5.20\] and \[5.23\] where we also used the fact that \( |g_t^p - g_t^N|_{g_t^p} = O(1) \). In fact propositions \[5.20\] and \[5.23\] imply better estimates for \( |g_t^p - g_t^N|_{g_t^p} \), but \( O(1) \) is all we need here. Similarly, by the two last estimates of proposition \[3.6\] we get
\[
\begin{align*}
\left| \nabla \left( \Theta \phi_t^N - \psi_t^N \right) \right|_{g_t^N} &= \begin{cases} 
O(1), & \text{if } \tilde{r} \leq 1, \\
O(\tilde{r}^{-4}), & \text{if } 1 \leq \tilde{r} \leq t^B,
\end{cases} \\
\left| \nabla \left( \Theta \phi_t^N - \psi_t^N \right) \right|_{\alpha, U} &= \begin{cases} 
O(t^{-a}), & \text{if } U = \{ x : \tilde{r}(x) \leq 1 \}, \\
O(t^{-a} \tilde{r}^{-4-a}), & \text{if } U = \{ x : 1 \leq \tilde{r}(x) \leq t^B \}.
\end{cases}
\end{align*}
\]
Case 2: \( t^B \leq \tilde{r} \leq 2t^B \)

Here we get
\[
\begin{align*}
\Theta \phi_t^N - \psi_t^N &= \Theta \left[ \tilde{\phi}_t^p - d((1 - a(t\tilde{r}))t^2 \tau_{1,1}) + d(a(t\tilde{r}) \exp_\tau \eta) \right] \\
&= \Theta \tilde{\phi}_t^p - \tilde{\psi}_t^p \\
&\quad + (T_{\tilde{\phi}_t^p} + F_{\tilde{\phi}_t^p}) \left( - d((1 - a(t\tilde{r}))t^2 \tau_{1,1}) + d(a(t\tilde{r}) \exp_\tau \eta) \right) \\
&\quad + d((1 - a(t\tilde{r}))v_{1,2}) - d(a(t\tilde{r}) \exp_\tau \zeta) \\
&= \Theta \tilde{\phi}_t^p - \tilde{\psi}_t^p + (T_{\tilde{\phi}_t^p} + F_{\tilde{\phi}_t^p}) \left( - d((1 - a(t\tilde{r}))t^2 \tau_{1,1}) + d(a(t\tilde{r}) \exp_\tau \eta) \right) \\
&\quad + d((1 - a(t\tilde{r}))v_{1,2}) - d(a(t\tilde{r}) \exp_\tau \zeta). \tag{5.37}
\end{align*}
\]
We estimate the summands individually. First, we find by the analysis of case 1:
\[
\left| \Theta \tilde{\phi}_t^p - \tilde{\psi}_t^p \right|_{g_t^N} = O(t \tilde{r}^{-3}).
\]
For the second row we have
\[
\left| (T_{\tilde{\phi}_t^p} + F_{\tilde{\phi}_t^p}) \left( - d((1 - a(t\tilde{r}))t^2 \tau_{1,1}) + d(a(t\tilde{r}) \exp_\tau \eta) \right) \right|_{g_t^N} \\
\leq \left| d((1 - a(t\tilde{r}))t^2 \tau_{1,1}) \right|_{g_t^N} + \left| d(a(t\tilde{r}) \exp_\tau \eta) \right|_{g_t^N} \\
= O(\tilde{r}^{-4}) + O(t\tilde{r}),
\]
where we used proposition \[5.6\] in the first step and eqs. \[5.16\] and \[5.22\] in the second step. For the last part of eq. \(5.37\) we find:
\[
\left| d((1 - a(t\tilde{r}))v_{1,2}) - d(a(t\tilde{r}) \exp_\tau \zeta) \right| = O(t^{-4B}) + O(t\tilde{r}),
\]
where we used eqs. \[5.17\] and \[5.29\]. So, altogether
\[
\left| \Theta \phi_t^N - \psi_t^N \right| = O(t^{-4B}) + O(t^{1+B}).
\]

Adding a derivative adds a factor of $t^{-1} \tilde{r}^{-1}$ in all the estimates. So, applying derivatives to eq. (3.37) and using the last two estimates of proposition 3.6, we get

$$\left| \nabla (\Theta \phi_t^N - \psi_t^N) \right| = O(t^{-1} \tilde{r}^{-5}) + O(1) = O(t^{-1-5B} + 1),$$

$$\left[ \nabla (\Theta \phi_t^N - \psi_t^N) \right]_{\alpha, \{x \in I^B \leq \tilde{r}(x) \leq 2t^B\}} = O(t^{-1-\alpha} \tilde{r}^{-5-\alpha}) + O(t^{-\alpha} \tilde{r}^{-\alpha}).$$

$$= O(t^{-1-5B-\alpha(1+B)} + t^{-\alpha(1+B)}).$$

Case 3: $\tilde{r} \geq 2t^B$

Here $\phi_t^N = \phi$ and $\psi_t^N = * \phi$, which shows the claim. \hfill \Box

5.5 Torsion-Free $G_2$-Structures on the Resolution $N_t$

We have defined a $G_2$-structure $\phi_t^N$ with small torsion in the previous section, and gave pointwise estimates for its torsion in theorem 5.36. Following the spirit of section 5.4, our next moves will be to define weighted Hölder norms on $N_t$ (compare definition 4.15 with definition 5.38), estimate $\phi_t^N$ in this norm, and prove the existence of a small deformation of $\phi_t^N$ that is torsion-free (compare theorem 4.30 with theorem 5.41).

Definition 5.38. For $t \in (0, 1)$ define the weight function

$$w_t : N_t \rightarrow \mathbb{R}$$

$$x \mapsto \begin{cases} (t + t \cdot \tilde{r}(x)), & \text{if } x \in \rho^{-1}(U_{t-1 R} / \{ \pm 1 \}), \\ R & \text{otherwise,} \end{cases} \quad (5.39)$$

and for $k \in \mathbb{Z}_{\geq 0}$, $\alpha \in (0, 1)$, $\beta \in \mathbb{R}$ the weighted Hölder norms $||| \cdot |||_{C^{k, \alpha}_{\beta, t}}$ on $N_t$ as in definition 4.4.3.

Then we get the following corollary of theorem 5.36.

Corollary 5.40. In the situation above, let $B = -\frac{1}{8}$ and define $\gamma_t^N \in \Omega^3(N_t)$ via $\gamma_t^N = \Theta(\phi)_t^N - \psi_t^N$, then $\text{d}^* \gamma_t^N = \text{d}^* \gamma_t^N$ and there exists $c_1 \in \mathbb{R}$ such that for all $t \in (0, T)$, $\alpha \in (0, 1)$, $\beta \in \mathbb{R}$ we have

$$\left\| \gamma_t^N \right\|_{C^{0, \alpha}_{0, t}} \leq c_1,$$

$$\left\| \text{d}^* \gamma_t^N \right\|_{C^{0, \alpha}_{\beta-2, t}} \leq c_1 t^{7/8(-\beta+2) - 3/8}.$$

Proof. Note that $\psi_t^N$ is closed, and therefore $\text{d}^* \gamma_t^N = \text{d}^* \phi_t^N$.

The first estimate follows from theorem 5.36 by observing that

$$\left\| \gamma_t^N \right\|_{C^{0, \alpha}_{0, t}} \leq \left\| \gamma_t^N \right\|_{C^1} = O(1).$$

For the second estimate, note that

$$\left\| \text{d}^* \gamma_t^N \right\|_{C^{0, \alpha}_{\beta-2, t}} \leq \left\| \nabla \gamma_t^N \right\|_{C^{0, \alpha}_{\beta-2, t}} = \left\| \nabla \psi_t^N \right\|_{L^\infty} + \left\| \nabla \gamma_t^N \right\|_{C^{0, \alpha}_{\beta-2, t}}.$$
and theorem 5.36 gives

\[
\left\| \nabla g_t^N w_t^{-\beta+2} \right\|_{L^\infty} = \begin{cases} 
O(t^{-\beta+2}), & \text{if } \tilde{t} \leq 1, \\
O((t\tilde{t})^{-\beta+2}), & \text{if } 1 \leq \tilde{t} \leq t^B, \\
O((t^{-1-5B} + 1) \cdot t^{-\beta+2} \tilde{t}^{-\beta+2}), & \text{if } t^B \leq \tilde{t} \leq 2t^B, \\
0, & \text{elsewhere}
\end{cases}
\]

as well as

\[
\left[ \nabla g_t^N \right]_{C^{\alpha,\beta}_{t^{-2},t^{-1}}}(U) = \begin{cases} 
O(t^{-\beta+2}), & \text{if } \tilde{t} \leq 1, \\
O(t^{7/8}(-\beta+2)), & \text{if } 1 \leq \tilde{t} \leq t^{-1/8}, \\
O(t^{7/8}(\beta+2)-3/8), & \text{if } t^{-1/8} \leq \tilde{t} \leq 2t^{-1/8}, \\
0, & \text{elsewhere}
\end{cases}
\]

Now definition 4.13 and propositions 4.17 and 4.28 can be adapted in a straightforward way to give the following analogue of theorem 4.39.

**Theorem 5.41.** Let \( \beta \in (-4, -2) \) and let \( (N_t, \varphi_t^N) \) be the resolution of \( M/\langle \iota \rangle \) from definition 5.36, endowed with the \( G_2 \)-structure from eq. (5.33). There exists \( c_1, c_2 \in \mathbb{R} \) such that the following is true: If \( \varphi \) is a closed \( G_2 \)-structure on \( N_t \) and \( \delta \in \Omega^1(N_t) \) such that \( d^* \delta = d^* \varphi \) and

\[
\left\| d^* \delta \right\|_{C^{0,\alpha}_{t^{-2},t^{-1}}} \leq c_1 t^\kappa, \\
\left\| \delta \right\|_{C^{0,\alpha}_{t^{-1},t^{-1}}} \leq c_2
\]

for \( \kappa > 1 - \beta + \alpha \), then for \( t \) small enough there exists \( \eta \in \Omega^2(N_t) \) such that \( \tilde{\varphi} := \varphi + d \eta \) is a torsion-free \( G_2 \)-structure on \( N_t \) satisfying

\[
\left\| \tilde{\varphi} - \varphi \right\|_{C^{1,\alpha}_{t^{-1},t^{-1}}} \leq t^\kappa.
\]

Here, norms are defined using the metric induced by \( \varphi_t^N \).

And, as in section 4, we get the following corollary:

**Corollary 5.42.** Let \( \varphi_t^N \) be the \( G_2 \)-structure on the resolution \( N_t \) of \( M/\langle \iota \rangle \) defined in eq. (5.33). Then, for \( \epsilon \in (0, \frac{1}{10}) \) and \( t \) small enough (depending on \( \epsilon \)) there exists \( \eta \in \Omega^2(N_t) \) such that \( \tilde{\varphi}_t := \varphi_t^N + d \eta \) is a torsion-free \( G_2 \)-structure on \( N_t \) satisfying

\[
\left\| \tilde{\varphi}_t - \varphi_t^N \right\|_{L^\infty} \leq t^{1/8-\epsilon}.
\]

**Proof.** Let \( \beta = -2 - \epsilon, \kappa = 25/8, \alpha = \epsilon \). Then, by corollary 5.40 \( \psi_t^N \) defined in eq. (5.33) and \( \varphi_t^N := \varphi_t^N - \Theta(\psi_t^N) \) satisfy the assumptions of theorem 5.41 which implies the claim.
5.6 A Better Estimate Obtained by Correcting the Approximate Solution

The $G_2$-structure defined in eq. (5.33) has, roughly speaking, torsion of order $t^{\tilde{r}}$. (This has been made precise in theorem 5.36) In [K7] the authors corrected this $G_2$-structure and defined a $G_2$-structure that has, roughly speaking, torsion of order $t^{2}\tilde{r}$. To be precise, they get the following, where we changed one interpolation region from $t^{-4/5} \leq \tilde{r} \leq 2t^{-4/5}$ to $t^{-1} \leq \tilde{r} \leq 2t^{-1}$:

**Theorem 5.43** (Proposition 6.2 in [K7]). Let $N_i$ as in definition 5.30 and assume without loss of generality that $R \geq 2$. For $\gamma \in \mathbb{R}$ such that $0 < \gamma \ll 1$, the following is true: For small $t$, there exists a closed $G_2$-structure $\varphi_t^N \in \Omega^3(N_i)$ and $\tilde{\varphi}_t^N \in \Omega^3(N_i)$ such that $d^* \varphi_t^N = d^* \tilde{\varphi}_t^N$ and $\tilde{\varphi}_t^N$ satisfies the following estimates:

$$|\tilde{\varphi}_t^N| = \begin{cases} 
O(t^2), & \text{if } \tilde{r} \leq 1 \\
O(t^{2}\tilde{r}^{2}), & \text{if } 1 \leq \tilde{r} \leq t^{-1/9} \\
O(t^{16/9}), & \text{if } t^{-1/9} \leq \tilde{r} \leq 2t^{-1/9} \\
O(t^{2}\tilde{r}^{-2\gamma}), & \text{if } 2t^{-1/9} \leq \tilde{r} \leq t^{-1} \\
O(t^{4-\gamma}), & \text{if } t^{-1} \leq \tilde{r} \leq 2t^{-1} \\
0, & \text{elsewhere,}
\end{cases}$$

$$|\nabla \tilde{\varphi}_t^N| = \begin{cases} 
O(t), & \text{if } \tilde{r} \leq 1 \\
O(t\tilde{r}), & \text{if } 1 \leq \tilde{r} \leq t^{-1/9} \\
O(t^{3/9}), & \text{if } t^{-1/9} \leq \tilde{r} \leq 2t^{-1/9} \\
O(t^{3-3\gamma}), & \text{if } 2t^{-1/9} \leq \tilde{r} \leq t^{-1} \\
O(t^{4-\gamma}), & \text{if } t^{-1} \leq \tilde{r} \leq 2t^{-1} \\
0, & \text{elsewhere.}
\end{cases}$$

Analogously to corollary 5.40, we can derive the following corollary:

**Corollary 5.44.** In the situation above there exists $c_1 \in \mathbb{R}$ such that for all $t \in (0, T)$, $\alpha \in (0, 1)$, $\beta \in (-\infty, -2)$ we have

$$\|\tilde{\varphi}_t^N\|_{C^{\alpha,\beta} (\Omega)} \leq c_1, \quad \|d^* \tilde{\varphi}_t^N\|_{C^{\alpha,\beta} (\bar{\Omega})} \leq c_1 t^{4-\gamma}.$$

Combining this with theorem 5.41 we get the following corollary in the same way as we got corollary 5.44.

**Corollary 5.45.** Let $\varphi_t^N$ be the $G_2$-structure on the resolution $N_i$ of $M/(i)$ from theorem 5.33. Then, for $\epsilon > 0$ and $t$ small enough (depending on $\epsilon$) there exists $\eta \in \Omega^2(N_i)$ such that $\overline{\varphi}_t^N := \varphi_t^N + d\eta$ is a torsion-free $G_2$-structure on $N_i$ satisfying

$$\|\overline{\varphi}_t^N - \varphi_t^N\|_{L^\infty} \leq t^{1-\epsilon}.$$

**Remark 5.46.** This correction brings the difference between the torsion-free $G_2$-structure and the $G_2$-structure with small torsion down to $t^{1-\epsilon}$. This is the same as the estimate for the (uncorrected) $G_2$-structure on the resolution of $T^2/\Gamma$ from corollary 4.31.
6 Open questions

The following interesting questions remain, ordered from easy to answer to hard to answer, according to the author’s estimate:

1. In [Joy6a], compact manifolds with holonomy Spin(7) were constructed. In the simplest case, one constructs Spin(7)-structures with small torsion by glueing together the product Spin(7)-structure on $T^4 \times X$ to the flat Spin(7)-structure on $T^8$. This glueing construction is analog to the definition of the $G_2$-structure in eq. (4.6). In contrast to the $G_2$-situation, however, Joyce’s theorem about the existence of torsion-free Spin(7)-structures cannot immediately be applied, because the torsion of the glued structure is too big. He overcame this problem by constructing a correction of the glued structure by hand which has smaller torsion, to which the existence theorem can be applied. By considering $T^7 / \Gamma \times S^1$, one gets a corrected $G_2$-structure on the resolution of $T^7 / \Gamma$. Using this corrected structure, one would get even better control over the difference between glued structure and torsion-free structure than what is known from corollary 4.3.

2. Resolving different connected components of the singular set at different length-scales was envisioned in [JK7, Section 7.2] but has not been proven yet.

3. It was expected in [JK7, Section 8], that an analogue of the construction from the article can be carried out in the Spin(7)-setting. As explained in the first point: even in the simplest case of resolutions of $T^8 / \Gamma$, one has to construct a formidable correction of the glued solution, before being able to apply Joyce’s existence theorem (cf. [Joy00, Theorem 13.6.1]) and perturb it to a torsion-free Spin(7)-structure. It is likely that an analogue of theorem 4.4 can be applied without constructing such a correction, as was the case in the $G_2$-setting.
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