The Complete Cohomology of the $W_3$ String

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ABSTRACT

We present a simple procedure for constructing the complete cohomology of the BRST operator of the two-scalar and multi-scalar $W_3$ strings. The method consists of obtaining two level–15 physical operators in the two-scalar $W_3$ string that are invertible, and that can normal order with all other physical operators. They can be used to map all physical operators into non-trivial physical operators whose momenta lie in a fundamental unit cell. By carrying out an exhaustive analysis of physical operators in this cell, the entire cohomology problem is solved.

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1. Introduction

Determining the physical spectrum of the $W_3$ string is equivalent to finding the cohomology of its BRST operator. We shall be concerned with the BRST operator of the critical $W_3$ string in this paper. The form of this operator in terms of the abstract $W_3$ algebra was first given in [1]. In the $W_3$ string, one realises the $W_3$ algebra in terms of a set of $n \geq 2$ free scalar fields [2,3], and the BRST operator takes the form [4]:

\[ Q_B = Q_0 + Q_1, \]
\[ Q_0 = \oint dz c (T^{\text{eff}} + T_{\varphi} + T_{\gamma,\beta} + \frac{1}{2} T_{c,b}), \]
\[ Q_1 = \oint dz \gamma (\partial \varphi)^3 + 3 \alpha \partial^2 \varphi \partial \varphi + \frac{19}{8} \partial^3 \varphi + \frac{9}{2} \partial \varphi \beta \partial \gamma + \frac{3}{2} \alpha \partial \beta \partial \gamma), \]

where the energy-momentum tensors are given by

\[ T_{\varphi} \equiv -\frac{1}{2} (\partial \varphi)^2 - \alpha \partial^2 \varphi, \]
\[ T_{\gamma,\beta} \equiv -3 \beta \partial \gamma - 2 \partial \beta \gamma, \]
\[ T_{c,b} \equiv -2 b \partial c - \partial b c, \]
\[ T^{\text{eff}} \equiv -\frac{1}{2} \partial X^\mu \partial X^\nu \eta_{\mu \nu} - i a_\mu \partial^2 X^\mu. \]

The background charge $\alpha$ for the scalar $\varphi$ is given by $\alpha^2 = \frac{49}{8}$, and the background-charge vector $a_\mu$ for $X^\mu$ is chosen so that $d - 12 a_\mu a^\mu = \frac{51}{2}$. The BRST operator is graded, with $Q_0^2 = Q_1^2 = \{Q_0, Q_1\} = 0$. Note that the BRST operator we are using here is the simplified one obtained in [4] by performing a non-linear field redefinition involving $\varphi$ and the ghost fields.

In addition to the general $n > 2$ multi-scalar string we shall be interested especially in the two-scalar string. For this case we take the energy-momentum tensor $T^{\text{eff}}$ to be given by

\[ T^{\text{eff}} = -\frac{1}{2} (\partial X)^2 - a \partial^2 X, \]

where $a^2 = \frac{49}{27}$.

Many physical states were found for both the multi-scalar and two-scalar $W_3$ strings in [4–8], by directly solving the physical-state conditions

\[ Q_B |\chi\rangle = 0, \quad |\chi\rangle \neq Q_B |\psi\rangle \]

at particular level numbers and ghost numbers. The most extensive list of such states can be found in [4]. From the examples that have been constructed some suggestive patterns and structures have emerged, especially in the case of the multi-scalar $W_3$ string, but such an approach is necessarily not exhaustive. A different approach was taken in [9], where
the cohomology of the non-critical $W_3$ BRST operator was studied by standard homological techniques. Results for the critical two-scalar $W_3$ string were obtained as a special case [9]. These results give the complete cohomology for states whose $\varphi$ and $X$ momenta $(p_1, p_2)$ lie within the “Seiberg bounds;” i.e. within a certain wedge ($SU(3)$ Weyl chamber) in the $(p_1, p_2)$ plane.*

In this paper we shall present a very simple and direct method that gives the complete cohomology of the two-scalar critical $W_3$ BRST operator for the entire $(p_1, p_2)$ plane. The method involves constructing certain invertible physical operators which have the property that they normal order with any physical operator to give another BRST non-trivial physical operator. Since these invertible operators carry momentum, they, together with their inverses, can be used to map any physical operator into one whose momentum lies within a fundamental rectangular unit cell in the $(p_1, p_2)$ plane. By carrying out an exhaustive analysis of all the physical states whose momenta lie within this fundamental cell, it follows that the complete cohomology problem for the BRST operator is then solved.

We shall begin the discussion with the two-scalar $W_3$ string in the next section. In section 3 we shall show how these results can be used to determine the complete cohomology of the multi-scalar $W_3$ string. The same method can be applied to the study of the cohomology of the one-scalar Virasoro string, which we shall do in section 4. We end with discussion and conclusions in Section 5.

2. The Cohomology of the Two-scalar $W_3$ String

2.1 Derivation of the cohomology

It was shown in [11] that the higher level physical states for the two-scalar Virasoro string could be built by taking powers of two ground ring generators $x$ and $y$. These operators have ghost number zero, which is understandable when one considers that the physical operators have a restricted range of ghost numbers $0 \leq G \leq 3$, regardless of their level number. It is natural to expect that the physical states of the $W_3$ string should similarly be constructable in terms of some analogues of the ground-ring generators. However here the range of ghost numbers of the physical operators is of the form $-r \leq G \leq r + 8$, where $r$ increases indefinitely with increasing level number. Thus the natural analogues of

* It should be emphasised that neither in this case nor indeed for the ordinary Virasoro string is the “Seiberg bound” in any sense a restriction on the physical acceptability of states in the free theory. Rather, it is a bound on the values of momentum that can occur in the exponential operators that approximate the true wavefunctions of the interacting Toda or Liouville theory in the region where the potential dies away. Since there exists a many-to-one map (6–1 and 2–1 respectively) from the free theory into the interacting theory, a condition analogous to the Seiberg bound arises as a statement that it suffices to consider free-field operators with momenta within one Weyl chamber in order to map into the entire Hilbert space of operators in the interacting theory [10]. However, if one considers the free theory in its own right then all physical operators are distinct, and there is no physical equivalence between operators with momenta related by Weyl reflections.
the ground-ring generators should have an appropriate negative ghost number. They should also have the property that they have well-defined normal-ordered products with all physical operators in the theory.

The candidates that fulfil these criteria are two physical operators with ghost number \( G = -2 \), at level \( \ell = 15 \). As we shall show, these operators, which we call \( x \) and \( y \), have the remarkable property that they have inverses \( x^{-1} \) and \( y^{-1} \), in the sense that there are two physical operators \( x^{-1} \) and \( y^{-1} \) whose normal-ordered products with \( x \) and \( y \) respectively yield the identity. This contrasts with the situation for the two-scalar Virasoro string, where the ground ring operators have no inverses. In this section we shall begin by constructing the \( x \) and \( y \) operators, and their inverses. Then we shall prove that normal ordering \( x, y, x^{-1} \) or \( y^{-1} \) with any physical operator will yield another BRST non-trivial physical operator. From this, we shall be able to show that all physical operators in the two-scalar \( W_3 \) string can be written in the form \( x^m y^n t_i \) and \( x^m y^n u_i \), where \( m \) and \( n \) are any integers; \( 1 \leq i \leq 6 \); and \( t_i \) and \( u_i \) denote sets of six “basic” physical operators at ghost numbers 3 and 2 respectively.

All the known physical states of the two-scalar \( W_3 \) string have the property that their momenta \((p_1, p_2)\) have the form

\[
(p_1, p_2) = (\frac{1}{7}k_1 \alpha, \frac{1}{7}k_2 \alpha),
\]

where \( k_1 \) and \( k_2 \) are integers, and \( \alpha \) and \( a \) are the background charges for the \( \varphi \) and \( X \) directions, given by \( \alpha^2 = \frac{49}{8} \) and \( a^2 = \frac{49}{32} \). It can be argued in general that the momenta of all physical states must be of this form \[12\].

Thus it is convenient to characterise the momentum of a physical state by the pair of integers \((k_1, k_2)\). From (1.1)–(1.7) it is straightforward to show that \( k_1 \) and \( k_2 \) are related to the level number \( \ell \) by the mass-shell condition \[4\]

\[
3(k_1 + 7)^2 + (k_2 + 7)^2 = 4(12\ell + 1).
\]

In particular, this places severe restrictions on the momenta, and level numbers, that physical states can have.

The \( x \) and \( y \) operators that we require turn out to arise as physical states at level \( \ell = 15 \), with momenta \((k_1, k_2) = (8, 0) \) and \((4, 12)\) respectively. They have ghost number \( G = -2 \).

Finding their explicit form is straightforward, although somewhat tedious; one simply writes down the most general possible structure at the given level number and ghost number, and then solves the physical-state conditions (1.9). We made extensive use of the Mathematica package OPEdefs \[13\] in order to do this. In fact a convenient simplification can be achieved by introducing the associated screening currents \( S_x \) and \( S_y \), defined by

\[
S_x(z) \equiv \oint dw b(w) x(z)
\]

\[
(2.3)
\]

\[*
\]

This quantisation of the momenta as integer multiples of \( \frac{1}{7} \alpha \) and \( \frac{1}{7} \alpha \) is also seen in the results in \[9\].
(and similarly for $S_y$), in terms of which the physical-state condition reduces to the equation \( \{Q_B, S_x\} = \partial x \) (and similarly for \( y \)). The result for the screening current associated with the \( x \) operator is very simple:

\[
S_x = \partial^2 \beta \partial \beta \beta e^{\frac{8}{7} \alpha \varphi}.
\]  

(2.4)

The \( x \) operator itself has 16 terms, and takes the form \( x = (c \partial^2 \beta \partial \beta \beta + f(\varphi, \beta, \gamma)) e^{\frac{8}{7} \alpha \varphi} \); we need not give it explicitly here. The screening current associated with the \( y \) operator is much more complicated, and we shall not give it explicitly. Unlike \( S_x \), the screening current \( S_y \) involves the \( b \) antighost, and the structure of \( y \) is correspondingly much more complicated than that of \( x \) given above. \( S_y \) has the general form

\[
S_y = \left( \partial^2 \beta \partial \beta \beta + \cdots \right) e^{\frac{4}{7} \alpha \varphi} + \frac{12}{7} \alpha \varphi + \frac{12}{7} a X,
\]  

(2.5)

where we have suppressed about fifty further terms. The \( y \) operator itself has about 1000 terms.

The physical operators that we shall call \( x^{-1} \) and \( y^{-1} \) are much simpler. They are level 1 operators with ghost number +2, and momenta \((k_1, k_2) = (-8, 0)\) and \((-4, -12)\) respectively:

\[
x^{-1} = (c \gamma + \frac{15}{14} \alpha \partial \gamma \gamma) e^{-\frac{8}{7} \alpha \varphi},
\]  

(2.6)

\[
y^{-1} = c \gamma e^{-\frac{4}{7} \alpha \varphi} - \frac{12}{7} a X.
\]  

(2.7)

We have explicitly checked that the normal-ordered products \((x^{-1} x)\) and \((y^{-1} y)\) give non-vanishing constant multiples of the identity operator, thus justifying the names \( x^{-1} \) and \( y^{-1} \). (The precise values of the non-vanishing constants are inessential to the subsequent argument, and so we shall assume renormalisations such that \((x^{-1} x) = 1\) and \((y^{-1} y) = 1\).)

We now prove the following

**Lemma 2.1** If \( A \) and \( B \) are any two physical operators that have a well-defined normal-ordered product \((A B) \equiv \oint \frac{dz}{(z-w)} A(z) B(w)\), the commutator \([A, B] \equiv (A B) - (-1)^{ab} (B A)\) is BRST trivial.

To prove this, we note from [14] that

\[
[A, B] = (-1)^{ab} \sum_{r=1}^{\infty} \frac{(-1)^r}{r!} \partial^r [B A]_r,
\]  

(2.8)

where \([B A]_r\) denotes the coefficient of the \( r \)-th pole in the OPE of \( B \) with \( A \). (The sum in (2.8) terminates, since the degree of the highest pole in the OPE of \( B \) with \( A \) is bounded.) Thus \([A, B] = \partial \eta\) for some \( \eta \). Since \( A \) and \( B \) are physical, it follows that \((A B)\) and
consequently \([A, B]\) are annihilated by the BRST operator \(Q_B\). Therefore \(Q_B \partial \eta = 0\), implying \(\partial Q_B \eta = 0\) and hence \(Q_B \eta = 0\). Since \(\partial \eta = L_{-1} \eta = \{Q_B, b_{-1}\} \eta\), we see that

\[ [A, B] = \partial \eta = Q_B \xi, \tag{2.9} \]

where \(\xi = b_{-1} \eta\).

The main results of this paper then depend crucially on the following

**Theorem 2.2** If \(V\) is any BRST non-trivial physical operator, then the normal-ordered products \((x V)\), \((y V)\), \((x^{-1} V)\) and \((y^{-1} V)\) are all BRST non-trivial physical operators.

We shall present the proof for \((x V)\); the proofs for the other cases are identical. That \((x V)\) is annihilated by \(Q_B\) is obvious; what has to be proved is that it is BRST non-trivial. We begin by noting that for any operators \(A, B\) and \(C\),

\[ (A (B C)) = (-1)^{ab}(B (A C)) + ([A, B] C). \tag{2.10} \]

Thus we have

\[ (x^{-1} (x V)) \approx ((x^{-1} x) V) = V, \tag{2.11} \]

where \(\approx\) denotes equality up to BRST-trivial terms. This follows because the second term on the right-hand side of (2.10), and the extra terms that we incur by manipulating (2.10), with \(A = V\), \(B = x^{-1}\) and \(C = x\), into the form (2.11), all involve commutators of physical operators, and these commutators are BRST trivial by virtue of Lemma 2.1. Using \((A Q_B \xi) = Q_B (A \xi)\) for any physical operator \(A\), the result then follows, since \((x^{-1} (x V))\) can only yield the BRST non-trivial physical operator \(V\) if \((x V)\) is itself BRST non-trivial. In other words \((x V)\) must be non-trivial because we can act with the inverse of \(x\) to get back \(V\) itself. Clearly the proof works equally well for any invertible physical operator acting on \(V\), and thus in particular for \(y, x^{-1}\) and \(y^{-1}\) too.

We have assumed in the above that the operators \(x, y, x^{-1}\) and \(y^{-1}\) have well-defined normal-ordered products with any physical operator in the theory. To see that this is the case, we must show that the product of the exponential operators, when normal ordered, gives an integer power of \((z - w)\). When \(x\), with momentum \((8, 0)\), is normal ordered with a physical operator with momentum given by \((k_1, k_2)\), this power will be \(-\left(\frac{8}{7} \alpha\right) \left(\frac{1}{7} k_1 \alpha\right) = -k_1\), which is clearly an integer. When \(y\), with momentum given by \((4, 12)\), is normal ordered instead, the power is \(-\left(\frac{4}{7} \alpha\right) \left(\frac{1}{7} k_1 \alpha\right) - \left(\frac{12}{7} a\right) \left(\frac{1}{7} k_2 a\right) = -\frac{1}{2}(k_1 + k_2)\). This is also always an integer, since any physical state must have a momentum \((k_1, k_2)\) that satisfies (2.2) for an integer level number \(\ell\), and one can immediately see from (2.2) that \(k_1\) and \(k_2\) are either both odd, or both even.

With the above theorem, we are now able to construct the entire cohomology of the BRST operator. Since \(x\) has momentum \((8, 0)\), and \(y\) has momentum \((4, 12)\), it is clear that
by acting with appropriate powers of $x$ and $y$ on an arbitrary physical state $V'(k_1', k_2')$ with momentum $(k_1', k_2')$, we can map it to a physical state $V(k_1, k_2)$ whose momentum $(k_1, k_2)$ lies in a fundamental rectangular unit cell with a width of 8 in the $k_1$ direction, and 12 in the $k_2$ direction. For reasons that will become clear presently, we choose our fundamental unit cell to be the one defined by

$$-8 \leq k_1 \leq -1, \quad -11 \leq k_2 \leq 0 . \quad (2.12)$$

By carrying out an exhaustive case-by-case analysis of the cohomology of all states with momenta satisfying (2.12), we will therefore have solved the complete cohomology problem for the two-scalar $W_3$ string, since all other physical state in the theory can then be obtained by acting with all possible integer powers of $x$ and $y$ on the fundamental physical states.

Not all of the 96 lattice points in the fundamental cell (2.12) can correspond to physical states, since $k_1$ and $k_2$ must satisfy the mass-shell condition (2.2) for integer level numbers $\ell$. By enumerating the possibilities we find that there are just 12 solutions to (2.2) that satisfy (2.12):

$$s_1 = (-6, -6)_0, \quad s_2 = (-6, -8)_0, \quad s_3 = (-8, -6)_0 ,$$

$$s_4 = (-8, -8)_0, \quad s_5 = (-7, -5)_0, \quad s_6 = (-7, -9)_0 ,$$

$$s_7 = (-8, 0)_1, \quad s_8 = (-6, 0)_1, \quad s_9 = (-4, -2)_1 ,$$

$$s_{10} = (-3, -5)_1, \quad s_{11} = (-3, -9)_1, \quad s_{12} = (-2, -2)_2 , \quad (2.13)$$

where the subscript on $(k_1, k_2)_{\ell}$ denotes the level number. Since the level numbers involved here are small, namely $\ell = 0, 1$ and 2, it is very easy to give a complete case-by-case analysis to find all of the physical states specified by (2.13). In fact the results are all already known; the six $\ell = 0$ states are tachyons, with ghost number $G = 3 [5,4]$:

$$t_1 = c \partial \gamma \gamma e^{\frac{6}{7} \alpha \varphi - \frac{6}{7} aX}, \quad t_2 = c \partial \gamma \gamma e^{\frac{6}{7} \alpha \varphi - \frac{8}{7} aX}, \quad t_3 = c \partial \gamma \gamma e^{\frac{8}{7} \alpha \varphi - \frac{6}{7} aX},$$

$$t_4 = c \partial \gamma \gamma e^{\frac{8}{7} \alpha \varphi - \frac{8}{7} aX}, \quad t_5 = c \partial \gamma \gamma e^{-\alpha \varphi - \frac{5}{7} aX}, \quad t_6 = c \partial \gamma \gamma e^{-\alpha \varphi - \frac{9}{7} aX}, \quad (2.14)$$

and the remaining six states, with $\ell = 1$ and $\ell = 2$, have ghost number $G = 2 [7,8,4]$:

$$u_1 = (c \gamma + \frac{15}{14} \alpha \partial \gamma \gamma) e^{-\frac{8}{7} \alpha \varphi}, \quad u_2 = (c \gamma - \frac{3}{7} \alpha \partial \gamma \gamma) e^{\frac{6}{7} \alpha \varphi},$$

$$u_3 = c \gamma e^{\frac{4}{7} \alpha \varphi - \frac{2}{7} aX}, \quad u_4 = c \gamma e^{-\frac{3}{7} \alpha \varphi - \frac{5}{7} aX},$$

$$u_5 = c \gamma e^{\frac{3}{7} \alpha \varphi - \frac{9}{7} aX}, \quad u_6 = c (\partial \varphi \gamma - \frac{3}{7} \alpha \partial \gamma \gamma) e^{-\frac{2}{7} \alpha \varphi - \frac{2}{7} aX} . \quad (2.15)$$

These operators are the prime operators for each of the momenta listed in (2.13). In other words, each of these is the lowest-ghost-number member of a quartet of physical operators, of which the other three members are $(a_\varphi V)$ and $(a_X V)$ (boosted by ghost number one), and $(a_\varphi a_X V)$ (boosted by ghost number two), for each prime operator $V [7,8,4]$. The ghost
boosters are defined by \( a_\varphi = [Q_B, \varphi] \), \( a_X = [Q_B, X] \). Since the processes of boosting and of normal ordering with \( x^m y^n \) commute (up to BRST trivial terms), we may always concentrate on the prime physical operators, it being understood that each such operator is accompanied by its three boosted partners. Note that \( x, y, x^{-1} \) and \( y^{-1} \) themselves are all prime physical operators, and that \( u_1 \) is the same operator as \( x^{-1} \).

The discussion of the cohomology of the prime physical operators divides into the odd-ghost-number case and the even-ghost-number case. For odd ghost numbers, the prime physical operators are given by the normal-ordered products

\[
x^m y^n t_i ,
\]

with \( G = 3 - 2m - 2n \), where \( m \) and \( n \) are arbitrary integers (positive or negative) and \( 1 \leq i \leq 6 \). Using (2.2) we find that their momenta and level numbers are given by

\[
\begin{align*}
(k_1, k_2) & \quad \text{level number } \ell \\
x^m y^n t_1 : & \quad (8m + 4n - 6, 12n - 6), \quad 4m^2 + 4n^2 + 4mn + m + n \\
x^m y^n t_2 : & \quad (8m + 4n - 6, 12n - 8), \quad 4m^2 + 4n^2 + 4mn + m \\
x^m y^n t_3 : & \quad (8m + 4n - 8, 12n - 6), \quad 4m^2 + 4n^2 + 4mn - m \\
x^m y^n t_4 : & \quad (8m + 4n - 8, 12n - 8), \quad 4m^2 + 4n^2 + 4mn - m - n \\
x^m y^n t_5 : & \quad (8m + 4n - 7, 12n - 5), \quad 4m^2 + 4n^2 + 4mn + n \\
x^m y^n t_6 : & \quad (8m + 4n - 7, 12n - 9), \quad 4m^2 + 4n^2 + 4mn - n .
\end{align*}
\]

For even ghost numbers, the prime operators are given by

\[
x^m y^n u_i ,
\]

with \( G = 2 - 2m - 2n \), where \( m \) and \( n \) are arbitrary integers and \( 1 \leq i \leq 6 \). Again using (2.2) we find their momenta and level numbers are given by

\[
\begin{align*}
(k_1, k_2) & \quad \text{level number } \ell \\
x^m y^n u_1 : & \quad (8m + 4n - 8, 12n), \quad 4m^2 + 4n^2 + 4mn - m + 3n + 1 \\
x^m y^n u_2 : & \quad (8m + 4n - 6, 12n), \quad 4m^2 + 4n^2 + 4mn + m + 4n + 1 \\
x^m y^n u_3 : & \quad (8m + 4n - 4, 12n - 2), \quad 4m^2 + 4n^2 + 4mn + 3m + 4n + 1 \\
x^m y^n u_4 : & \quad (8m + 4n - 3, 12n - 5), \quad 4m^2 + 4n^2 + 4mn + 4m + 3n + 1 \\
x^m y^n u_5 : & \quad (8m + 4n - 3, 12n - 9), \quad 4m^2 + 4n^2 + 4mn + 4m + n + 1 \\
x^m y^n u_6 : & \quad (8m + 4n - 2, 12n - 2), \quad 4m^2 + 4n^2 + 4mn + 5m + 5n + 2 .
\end{align*}
\]

This completes the derivation of the entire cohomology of the two-scalar \( W_3 \) BRST operator.
2.2 Further observations

It is interesting to note that the above construction automatically gives the entire spectrum, which means that half of the quartets generated by this procedure are conjugate to the other half. The momenta and ghost numbers of the associated prime operators \( V \) and \( V' \) for a conjugate pair of quartets are related by \( k_1 + k'_1 = -14 \), \( k_2 + k'_2 = -14 \), \( G + G' = 6 \). This last equation follows because the operator conjugate to the prime operator \( V \) is actually the twice-boosted operator \( (a_{\varphi} a_x V') \) in the conjugate quartet, so that \( (a_{\varphi} a_x V') \) and \( V \) together have the correct total ghost number 8.

Another interesting observation is that although one might think that the mass-shell condition (2.2) is only a necessary condition for the occurrence of physical operators, in fact there are physical operators corresponding to every pair of integers \((k_1, k_2)\) that satisfy (2.2) for each integer \(\ell\). To see this, note that if \((k_1, k_2)\) satisfies (2.2) for some integer \(\ell\), then \((k_1 + 8p, k_2 + 24q)\) also satisfies (2.2) for some other integer \(\ell'\), where \(p\) and \(q\) are arbitrary integers. Thus if we can show that all solutions in one \(8 \times 24\) cell can be reached by acting with powers of \(x\) and \(y\) on the twelve basic states (2.14) and (2.15), then we have shown that all solutions can be reached, since we then can cover the whole \((k_1, k_2)\) plane by acting on the states in the \(8 \times 24\) cell with \(x^{p-q} y^{2q}\). There are 24 solutions to (2.2) in this \(8 \times 24\) cell, and it is easy to show that they can indeed all be obtained from the twelve fundamental states, thus completing the demonstration.

The mass-shell condition (2.2) is invariant under the action of the Weyl group of \(SU(3)\). This can easily be seen by defining the shifted momentum \((\hat{k}_1, \hat{k}_2)\) \(\equiv (k_1 + 7, k_2 + 7)\), in terms of which (2.2) becomes

\[
3\hat{k}_1^2 + \hat{k}_2^2 = 4(12\ell + 1) .
\] (2.20)

By writing the simple roots of \(SU(3)\) in the basis appropriate to our conventions, one can show that the Weyl-group transformations corresponding to the two simple roots are given by

\[
S_1 : (\hat{k}_1, \hat{k}_2) \rightarrow (\hat{k}_1, -\hat{k}_2) ,
S_2 : (\hat{k}_1, \hat{k}_2) \rightarrow \left( \frac{1}{2}(\hat{k}_2 - \hat{k}_1), \frac{1}{2}(3\hat{k}_1 + \hat{k}_2) \right) .
\] (2.21)

All six elements of the Weyl group can be generated from these. Clearly, these transformations leave (2.20) invariant.

Since, as we have seen, there are physical states associated with all solutions of (2.20), it follows that the action of the Weyl group on (2.20) maps physical states at any given level number into other physical states at the same level. For example, one easily sees that under (2.21), the six \(\ell = 0\) tachyons (2.14) map into each other. At higher levels, however, the set of six Weyl-related physical states will be at different ghost numbers. An immediate consequence of this Weyl-group symmetry is that the number \(N\) of prime physical operators at each level \(\ell\) is necessarily a multiple of 6. It is not obvious how to determine
$N$ as a function of $\ell$ without simply enumerating the solutions to (2.2). Some examples are 
\{ $\ell$, $N$ \} = \{ 0, 6 \}, \{ 1, 12 \}, \{ 2, 6 \}, \{ 4, 18 \}, \{ 7, 0 \}, \{ 11, 24 \}, \{ 53, 36 \}, \{ 144, 48 \}, \{ 690, 54 \}.

It will always be the case that exactly one member of each “Weyl sextet” satisfies the “Seiberg condition” \[ (2.2) \] which singles out one of the six Weyl chambers. As discussed in the introduction, however, the condition (2.22) should not be interpreted as a restriction on the validity of physical states in the free theory. The subset of the cohomology for which the physical states satisfy (2.22) was constructed in [9].

To conclude this section, we shall show that the calculation of the normal-ordered product of any of $x$, $y$, $x^{-1}$ or $y^{-1}$ with a prime physical operator $V$ can equivalently be performed by acting on a ghost-boosted version of $V$ with the corresponding screening currents $S_x$, $S_y$, etc. From a practical point of view, this can often simplify the calculation of higher-level operators considerably. Thus we prove the following

**Theorem 2.3** If $V$ is a prime physical operator, then $[S_x (RV)]_1 \approx (x V)$, and similarly for $y$, $x^{-1}$ or $y^{-1}$, where $S_x$ is the screening current for $x$, defined by (2.3), and $R$ is a certain linear combination of the $a_\phi$ and $a_x$ ghost boosters.

Note that here, as in Lemma 2.1, $[AB]_r$ denotes the coefficient of the $r$’th pole in the OPE of $A$ with $B$. In particular, $[AB]_0$ means the normal-ordered product of $A$ with $B$, which we often also write as $(AB)$. The expression $[S_x (a V)]_1$ means $\oint dz S_x(z) (RV)(w)$, which is the commutator of the screening charge $\oint dz S_x(z)$ with the boosted physical operator. As in Theorem 2.2, $\approx$ denotes equality up to BRST-trivial terms.

To prove the theorem, we use the following identity for any operators $A$, $B$ and $C$ [14]:

\[ [A [BC]_0]_q = (-1)^{ab} [B [AC]_q]_0 + \sum_{r=0}^{q-1} \left( \frac{q - 1}{r} \right) ([AB]_{q-r} C)_r , \tag{2.23} \]

where $q \geq 1$. Thus we have

\[ [S_x (RV)]_1 = [S_x RV]_0 = -[R [S_x V]_1]_0 + [[S_x R]_1 V]_0 . \tag{2.24} \]

Now from (2.3) we have $S_x = [b x]_1$, so $[S_x V]_1 = [[b x]_1 V]_1$ and $[S_x R]_1 = [[b x]_1 R]_1$. We may again use (2.23), with $q = 2$, to evaluate these expressions:

\[ [[b x]_1 V]_1 = [b [x V]_0]_2 - [x [b V]_2]_0 - [[b x]_2 V]_0 , \]
\[ [[b x]_1 R]_1 = [b [x R]_0]_2 - [x [b R]_2]_0 - [[b x]_2 R]_0 . \tag{2.25} \]

The first of these equations implies that $[[b x]_1 V]_1 \approx 0$. This is because $V$, $x$ and $(x V)$ are all prime physical operators, which means that the BRST-closed operators $[b V]_2$, $[b x]_2$ and
must be BRST trivial, there being, by the definition of a “prime operator,” no BRST
non-trivial physical operator at the lower ghost number. The second equation in (2.25) implies that 
\([\{b, R\}]_1 \approx x\) (ignoring an unimportant non-zero constant factor). This is because 
\(R = (\text{const.}) \partial c + \cdots\), so \([bR]_2 = \text{const.},\text{ and } [b[x R]_0]_2 \approx (\text{const.})x,\) since it is annihilated by \(Q_B\) and has the same ghost number as the prime operator \(x\). Substituting these results into (2.24), the theorem immediately follows.

In order to see that \([\{b, R\}]_1 \) above is a non-vanishing constant multiple of \(x\), which is essential to the proof, it is useful to look more closely at the ghost boosters \(a_\varphi \equiv [Q_B, \varphi]\) and \(a_X \equiv [Q_B, X]\). They are given by [4]

\[
\begin{align*}
    a_\varphi &= c \partial \varphi - \alpha \partial c - \frac{19}{8} \partial^2 \gamma - \frac{9}{2} \beta \partial \gamma \gamma - 3(\partial \varphi)^2 \gamma + 3\alpha \partial \varphi \partial \gamma, \\
    a_X &= c \partial X - a \partial c.
\end{align*}
\]

(2.26)

For calculational purposes, it is most convenient to choose the combination \(R\) of \(a_\varphi\) and \(a_X\) in Theorem 2.3 so that \([bR]_2 = 0\). It is easy to see that the required combination is

\[
R = (k_2 + 7)a_\varphi - (k_1 + 7)\alpha a_X,
\]

(2.27)

where the momentum of the physical operator \(V\) is given by \((k_1, k_2)\). It is then straightforward to show that \([\{b, R\}]_1 \approx \frac{1}{7}a\alpha((k_1 + 7)k_2 - (k_2 + 7)k_1)x,\) where \((k_1, k_2)\) is the momentum of \(x\) (or, mutatis mutandis, \(y\)). Elementary algebra using (2.2) then shows that the right-hand side of this expression is always non-vanishing.

Let us close with a sample calculation using the screening current \(S_x\) to build a higher-
level physical operator. We begin with the tachyon \(t_1\), which after boosting according to
(2.27), becomes \(c \partial^2 \gamma \partial \gamma \gamma e^{-\frac{6}{7}\alpha \varphi - \frac{6}{7}aX}\) (up to an irrelevant constant factor). It is convenient to bosonise the \((\beta, \gamma)\) ghosts, so we write \(\gamma = e^{i\rho}\) and \(\beta = e^{-i\rho}\). The boosted tachyon then becomes \(2c e^{3i\rho} e^{-\frac{6}{7}\alpha \varphi - \frac{6}{7}aX}\). The screening current (2.4) bosonises to

\[
S_x = 2e^{-3i\rho} e^{\frac{8}{7}a\varphi},
\]

(2.28)

and so \(\oint dz S_x(z)\) acting on the boosted tachyon immediately gives

\[
4 \oint \frac{dz}{(z-w)^3} e^{-3i\rho(z) + \frac{8}{7}a\varphi(z) + 3i\rho(w) - \frac{6}{7}a\varphi - \frac{6}{7}aX(w)}
\]

(2.29)

after normal ordering the exponentials. Thus re-expressing the result in terms of \(\beta\) and \(\gamma\) again, we obtain the \(\ell = 5\) prime physical operator

\[
\frac{16}{7} \alpha c \left(6\sqrt{2} \partial \beta \gamma - 3\sqrt{2} \beta \partial \gamma + 12 \partial \varphi \beta \gamma + 4\sqrt{2}(\partial \varphi)^2 + 2\partial^2 \varphi\right)e^{\frac{8}{7}a\varphi - \frac{6}{7}aX}.
\]

(2.30)
This agrees with the result in [4], where it was found by directly solving the physical-state conditions (1.9). It should perhaps be emphasised, however, that the constructions we are presenting in this paper are primarily of utility for deriving general results about the spectrum of physical states, rather than constructing specific explicit examples. In practice, it is easier to obtain specific examples by directly solving the conditions (1.6).

3. Cohomology of the Multi-scalar $W_3$ String

The BRST operator for the multi-scalar realisation is related to that for the two-scalar realisation in a very simple way, namely by replacing the energy-momentum tensor (1.8) for $X$ by the one (1.7) for $X^\mu$. Owing to this fact, the cohomology of the multi-scalar $W_3$ string can be understood by examining the physical states of the two-scalar $W_3$ string. In particular, the multi-scalar states that are tachyonic in the effective spacetime described by $X^\mu$ can be obtained from the subset of the states of the two-scalar $W_3$ string that generalise to the multi-scalar case. The remaining multi-scalar states can then be obtained by replacing the tachyonic effective spacetime operator $e^{ip\cdot X}$ by arbitrary excited highest-weight operators of the same conformal dimension under $T_{\text{eff}}$. Therefore the key problem is to identify the subset of the states of the two-scalar $W_3$ string that generalise to the multi-scalar case.

Physical operators in the two-scalar $W_3$ string will generalise to multi-scalar operators if one of the following conditions holds:

1) If there is a pair of two-scalar prime physical operators with momenta $(k_1, k_2)$ and $(k_1, -14 - k_2)$, both at the same ghost number. This pair, which have the same conformal dimension $\Delta = -\frac{1}{48}(k_2 + 7)^2 + \frac{49}{48}$ under $T_{\text{eff}}$, generalise to a continuous momentum multi-scalar operator where $e^{ip\cdot X}$ has the same dimension $\Delta = \frac{1}{2}p^\mu(p_\mu + 2a_\mu)$.

2) If there is a two-scalar prime physical operator with momentum $(k_1, 0)$. This generalises to a discrete multi-scalar operator with $p_\mu = 0$ in the effective spacetime, and hence $\Delta = 0$.

3) If there is a two-scalar prime physical operator with momentum $(k_1, -14)$. This generalises to a discrete multi-scalar operator with $p_\mu = -2a_\mu$ in the effective spacetime, where $a_\mu$ is the constant background-charge vector appearing in (1.7). Again, this has conformal dimension $\Delta = 0$ as measured by $T_{\text{eff}}$.

By examining the entire cohomology of the two-scalar $W_3$ string, given by (2.16)–(2.19), one can easily see that the subset of states that fulfil these requirements can be described as follows:

$$x^m t_i, \quad x^m u_i, \quad x^m(xy^{-1}u_1), \quad x^m(xy^{-1}u_2), \quad x^m(y^{-1}u_3), \quad x^m(y^{-1}u_6), \quad (3.1)$$
where \( m \) is an arbitrary integer in each case. In fact
\[
(xy^{-1}u_1) = c\gamma e^{-\frac{4}{7}\alpha \varphi - \frac{12}{7}aX},
\]
\[
(xy^{-1}u_2) = c(\partial \varphi \gamma - \frac{3}{7}\alpha \partial \gamma)e^{-\frac{2}{7}\alpha \varphi - \frac{12}{7}aX}.
\] (3.2)

In other words, these two operators are nothing but the \( X^\mu \)-space momentum conjugates of \( u_3 \) and \( u_6 \) respectively. The remaining operators involving \( y^{-1} \) in (3.2), namely \((xy^{-1}u_3)\) and \((y^{-1}u_6)\), have momenta \((-8, -14)\) and \((-6, -14)\); these are the \( G = 4 \) prime physical operators for the quartets conjugate to those of \( u_2 \) and \( u_1 \) respectively.

There is an equivalent but more convenient way to describe the cohomology (3.1) of the multi-scalar \( W_3 \) string, by starting with a set of “basic” physical operators that are expressed directly in the multi-scalar formalism. Thus we take the following basis of prime physical operators. At \( G = 3 \), there are [5,6,4]:
\[
\tilde{t}_1 = c\partial \varphi \gamma e^{-\frac{6}{7}\alpha \varphi} V_1(X),
\]
\[
\tilde{t}_2 = c\partial \varphi \gamma e^{-\frac{8}{7}\alpha \varphi} V_1(X),
\]
\[
\tilde{t}_3 = c\partial \varphi \gamma e^{-\alpha \varphi} V_{15} \frac{1}{16}(X),
\] (3.3)
where \( V_\Delta(X) \) denotes an effective-spacetime physical operator, which is highest-weight under \( T_{\text{eff}} \) with conformal dimension \( \Delta \). At \( G = 2 \), there are further prime operators with continuous spacetime momentum [7,8,4]:
\[
\tilde{u}_1 = c\gamma e^{-\frac{3}{7}\alpha \varphi} V_{15} \frac{1}{16}(X),
\]
\[
\tilde{u}_2 = c\gamma e^{-\frac{4}{7}\alpha \varphi} V_1(X),
\]
\[
\tilde{u}_3 = c(\partial \varphi \gamma - \frac{3}{7}\alpha \partial \gamma)e^{-\frac{2}{7}\alpha \varphi} V_1 \frac{1}{2}(X).
\] (3.4)
Finally, there are discrete prime operators \( \tilde{d}_1 \) and \( \tilde{d}_2 \) at \( G = 2 \) [7], and \( \tilde{d}_3 \) and \( \tilde{d}_4 \) in the conjugate quartets, at \( G = 4 \):
\[
\tilde{d}_1 = (c\gamma + \frac{15}{14}\alpha \partial \gamma \gamma)e^{-\frac{8}{7}\alpha \varphi},
\]
\[
\tilde{d}_2 = (c\gamma - \frac{3}{7}\alpha \partial \gamma \gamma)e^{-\frac{6}{7}\alpha \varphi},
\]
\[
\tilde{d}_3 = c(\partial \varphi \partial^2 \gamma \partial \gamma \gamma - \frac{4}{21}\alpha \partial^3 \gamma \partial \gamma \gamma)e^{-\frac{6}{7}\alpha \varphi - 2ia.X},
\]
\[
\tilde{d}_4 = c(\partial \varphi \partial^2 \gamma \partial \gamma \gamma - \frac{1}{21}\alpha \partial^3 \gamma \partial \gamma \gamma)e^{-\frac{8}{7}\alpha \varphi - 2ia.X}.
\] (3.5)

From the basic prime operators (3.3)–(3.5), all the prime physical operators of the multi-scalar \( W_3 \) string are obtained, by normal ordering them with \( x^m \) for arbitrary integers \( m \). (Note that both \( x \) and \( x^{-1} \) themselves generalise to the multi-scalar case, since they have zero momentum in the \( X^\mu \) directions.) Thus the complete spectrum of prime physical operators in the multi-scalar \( W_3 \) string is as follows:
Operator $k_1$ $G$ $\Delta$ $\ell$

$x^m \tilde{t}_1$ $8m - 6$ $3 - 2m$ 1 $4m^2 + m$

$x^m \tilde{t}_2$ $8m - 8$ $3 - 2m$ 1 $4m^2 - m$

$x^m \tilde{t}_3$ $8m - 7$ $3 - 2m$ $\frac{15}{16}$ $4m^2$

$x^m \tilde{u}_1$ $8m - 3$ $2 - 2m$ $\frac{15}{16}$ $4m^2 + 4m + 1$

$x^m \tilde{u}_2$ $8m - 4$ $2 - 2m$ $\frac{1}{2}$ $4m^2 + 3m + 1$

$x^m \tilde{u}_3$ $8m - 2$ $2 - 2m$ $\frac{1}{2}$ $4m^2 + 5m + 2$

$x^m \tilde{d}_1$ $8m - 8$ $2 - 2m$ 0 $4m^2 - m + 1$

$x^m \tilde{d}_2$ $8m - 6$ $2 - 2m$ 0 $4m^2 + m + 1$

$x^m \tilde{d}_3$ $8m - 6$ $4 - 2m$ 0 $4m^2 + m + 1$

$x^m \tilde{d}_4$ $8m - 8$ $4 - 2m$ 0 $4m^2 - m + 1$

Table 1. Prime physical operators in the multi-scalar $W_3$ string

The level numbers $\ell$ in the table are for the case where the effective-spacetime highest-weight operators $V_\Delta(X)$ are purely tachyonic, $V_\Delta(X) = e^{ip \cdot X}$. The total level number of any physical operator would be given by the sum of $\ell$ and the level number of the excitation number of $V_\Delta(X)$. From Table 1, we see that physical operators in the $\Delta = 1$ sector occur at levels $\ell = 0, 3, 5, 14, 18, 33, 39, \ldots$; in the $\Delta = \frac{15}{16}$ sector at $\ell = 0, 1, 4, 9, 16, 25, 36 \ldots$; and in the $\Delta = \frac{1}{2}$ sector at $\ell = 1, 2, 8, 11, 23, 28, 46, \ldots$. The discrete physical operators in the $\Delta = 0$ sector occur at levels $\ell = 1, 4, 6, 15, 19, 34, 40, \ldots$.

Note that for all six sequences of physical operators $x^m \tilde{t}_i$ and $x^m \tilde{u}_i$, the set with $m < 0$ correspond to the conjugates of the set with $m \geq 0$. These all have continuous on-shell spacetime momentum $p_\mu$, subject only to the mass-shell condition $\Delta = \frac{1}{2} p^\mu (p_\mu + 2a_\mu)$, with intercepts $\Delta$ as given in the first six lines of Table 1 above. All of the discrete operators $x^m \tilde{d}_1$ and $x^m \tilde{d}_2$ have $p_\mu = 0$, and all of the discrete operators $x^m \tilde{d}_3$ and $x^m \tilde{d}_4$ have $p_\mu = -2a_\mu$. The complete cohomology of physical operators that we find here in the multi-scalar $W_3$ string accords with the pattern that was observed in many low-level examples in [8,4]. However, the discrete operators $x^m \tilde{d}_3$ and $x^m \tilde{d}_4$ with spacetime momentum $p_\mu = -2a_\mu$ were not found previously.

One can easily verify that the subset of the two-scalar cohomology (2.16)–(2.19) that satisfies any of the three criteria for generalisability given at the beginning of this section is the same as the multi-scalar cohomology given in the table above, where one restricts the effective-spacetime highest-weight operators $V_\Delta(X)$ to be purely tachyonic. Many, indeed the majority, of the two-scalar operators are “lost” when one passes to the multi-scalar $W_3$ string. The reason for this is that the $y$ operator of the two-scalar $W_3$ string does not
generalise to the multi-scalar case, and so it can no longer play a role in generating higher-level physical operators from the basic $G = 3$ and $G = 2$ physical operators. Thus whilst the entire cohomology of the two-scalar $W_3$ string is generated by acting with all possible integer powers of the $G = -2$ operators $x$ and $y$ on a basis of $G = 3$ and $G = 2$ operators, in the multi-scalar $W_3$ string the entire cohomology is generated by acting with all possible powers of just the $x$ operator on a basis of $G = 3$ and $G = 2$ continuous-momentum operators, and $G = 4$ and $G = 2$ discrete operators. A consequence of this is that at any given level number $\ell$, the range of ghost numbers at which prime physical operators occur is much sparser in the multi-scalar case than in the two-scalar case. In fact in the multi-scalar $W_3$ string, if continuous momentum prime operators occur at all at a given level, then they occur at just two ghost numbers; $G = g$ and $6 - g$ (corresponding to a quartet and its conjugate), where $g$ becomes increasingly negative as $\ell$ increases. If discrete prime operators occur at a given level, then they occur at four ghost numbers; $G = g$, $g + 2$, $4 - g$, and $6 - g$ (corresponding to two quartets* and their conjugates). In the two-scalar $W_3$ string, by contrast, prime operators can occur at many of the ghost numbers $G = g$, $g + 2$, $g + 4$, $\cdots$, $2 - g$, $4 - g$, $6 - g$. (It can sometimes also happen that even and odd ghost-number sequences occur at the same level.) For example, at level $\ell = 14$, the two-scalar $W_3$ string has 18 prime operators, at ghost numbers $G = -1, 1, 3, 5$ and 7; in the multi-scalar case, there are just two prime operators, with $\Delta = 1$ and $G = -1$ and $G = 7$. At level $\ell = 46$, there are 24 prime operators in the two-scalar $W_3$ string, at ghost numbers $G = -5, -4, -1, 2, 4, 7, 10$ and 11; in the multi-scalar case there are two prime operators, with $\Delta = \frac{1}{2}$ and $G = -4$ and 10. At level $\ell = 30$, there are 18 prime operators in the two-scalar $W_3$ string, at ghost numbers $G = -4, -3, -1, 0, 1, 5, 6, 7, 9$ and 10; in the multi-scalar case, there are no physical operators at this level.

4. The Cohomology of the One-scalar String

The methods that we have used to study the cohomology of the $W_3$ string can also be applied to the simpler problem of the one-scalar Virasoro string. The BRST operator in this case is simply given by

$$Q_B = \oint dz \, c \left( T + \frac{1}{2} T_{c,b} \right),$$

(4.1)

* Since in the multi-scalar $W_3$ string there are $(d + 1)$ ghost boosters $a_\varphi$ and $a_{x^\mu}$, the structure of boosted operators can be more complicated than simply quartets. As discussed in [4], one still gets quartets from prime operators of the form $c U(\varphi, \beta, \gamma) e^{ip \cdot X}$, but a discrete prime operator at $G = g$ with $p_\mu = 0$, which has the form $V = c U_1(\varphi, \beta, \gamma) + U_2(\varphi, \beta, \gamma)$, gives rise to a multiplet at ghost numbers $\{g, g + 1, g + 2, g + 3\}$, with multiplicities $\{1, d + 1, 2d - 1, d - 1\}$. It is convenient to view these as a standard quartet, generated by $a_\varphi$ and $a^\parallel$ acting on the prime operator $V$, and a set of $(d - 1)$ further quartets, spanning the ghost numbers $\{g + 1, g + 2, g + 3\}$, generated by acting with these same two boosters on the $(d - 1)$ operators $a^\perp_{\mu} V$ at $G = g + 1$. (a$^\parallel$ denotes $a_{x^\mu}$ projected parallel to the background-charge vector $a_\mu$, and $a^\perp_{\mu}$ denotes the remaining $a_{x^\mu}$ ghost boosters perpendicular to $a_\mu$.) These $(d - 1)$ quartets can alternatively be viewed as special cases of the $\Delta = 1$ continuous momentum physical operators that always occur at the next-lower level number, in which the spacetime tachyon $e^{ip \cdot X}$ is replaced by excited $\Delta = 1$ operators of the form $\xi_\mu \partial X^\mu$.  

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where

\[ T = -\frac{1}{2}(\partial \varphi)^2 - \alpha \partial^2 \varphi , \quad (4.2) \]
\[ T_{c,b} = -2b \partial c - \partial b c , \quad (4.3) \]

where the background charge parameter \( \alpha \) now satisfies \( \alpha^2 = \frac{25}{12} \). Physical operators have momenta \( p = \frac{1}{5}k \alpha \), where \( k \) is an integer, which implies that the mass-shell condition takes the form

\[ (k + 5)^2 = 24\ell + 1 , \quad (4.4) \]

where \( \ell \) is the level number. Thus \( k \) is always an even integer, and physical operators can occur at levels \( \ell = 0, 1, 2, 5, 7, 12, 15, 22, \ldots \).

We find that there is a prime physical operator at \( \ell = 5 \) with momentum given by \( k = 6 \). This means that it has a well-defined normal-ordered product with all physical operators. It has ghost number \( G = -1 \), and takes the form

\[ x = \left( -\frac{1}{2}\partial \varphi \partial b - \sqrt{3} \partial b c + \partial^2 \varphi b + \frac{1}{2\sqrt{3}} \partial^2 b \right) e^{\frac{6}{5}\alpha \varphi} . \quad (4.5) \]

The inverse of this operator occurs at \( \ell = 0 \) with \( k = -6 \) and \( G = 1 \):

\[ x^{-1} = c e^{-\frac{6}{5}\alpha \varphi} . \quad (4.6) \]

Similar arguments to those in section 2 imply that \( x^m \) acting on any physical operator gives another BRST-non-trivial physical operator. Thus the cohomology of the one-scalar string may be determined by finding all physical operators whose momenta lie within an interval \( n \leq k \leq n + 5 \) for any convenient integer \( n \). We shall choose \( n = -6 \). From (4.4), it follows that the only physical operators with \( -6 \leq k \leq -1 \) are the \( \ell = 0 \) tachyons, which have \( G = 1 \):

\[ t_1 = c e^{-\frac{4}{5}\alpha \varphi}, \quad t_2 = c e^{-\frac{6}{5}\alpha \varphi} . \quad (4.7) \]

Thus the complete cohomology of prime physical operators in the one-scalar string is given by

\[ x^m t_1, \quad x^m t_2 \quad , \quad (4.8) \]

where \( m \) is an arbitrary integer. The momenta, ghost numbers and level numbers are given by:

| Operator   | \( k \)   | \( G \)   | \( \ell \)      |
|------------|----------|----------|----------------|
| \( x^m t_1 \) | 6m - 4   | 1 - m    | \( \frac{1}{2}m(3m + 1) \) |
| \( x^m t_2 \) | 6m - 6   | 1 - m    | \( \frac{1}{2}m(3m - 1) \) |

Table 2.  Prime physical operators in the one-scalar string
Each of the prime operators given above is associated with a doublet of physical operators, where the second member is obtained from the prime operator by normal ordering with the ghost booster \( a_\varphi = c \partial \varphi - \alpha \partial c \). This gives the complete cohomology of the one-scalar string. One can verify that it agrees with the results in \([15,16]\) for the two-scalar Virasoro string when the central charge of the matter field \( X \) is chosen to be zero.

5. Discussion and Conclusions

In this paper, we have studied the complete spectrum of physical states in the two-scalar and multi-scalar \( W_3 \) strings and in the one-scalar Virasoro string. In all of these theories, there exist certain special physical operators that are invertible, and by normal ordering arbitrary powers of these operators with a set of basic operators, all physical operators can be constructed.

In earlier work on the spectrum of the \( W_3 \) string, it was found that there are two \( G = 0 \) prime operators at level \( \ell = 6 \) in the two-scalar \( W_3 \) string, which we shall call \( \tilde{x} \) and \( \tilde{y} \), with momenta given by \( (k_1, k_2) = (2, 0) \) and \( (1, 3) \) respectively \([7]\). The operator \( \tilde{x} \) generalises to the multi-scalar \( W_3 \) string, whilst \( \tilde{y} \) does not. In terms of the construction in this paper, they are given by \( \tilde{x} = xu_2 \) and \( \tilde{y} = y u_5 \) respectively. It was suggested in \([7]\) that one might use these operators in order to build higher-level physical states from a basis of low-level states. In fact, because the trend is for the ghost numbers of prime operators to cover a wider and wider range of values as the level number increases, one would need also to use the \( G = -1 \) screening currents built from \( \tilde{x} \) and \( \tilde{y} \) by inserting \( \oint dz b(z) \) operators. Some examples were presented in \([17]\) for the special case of the multi-scalar \( W_3 \) string. There are many disadvantages to trying to use the \( \tilde{x} \) and \( \tilde{y} \) operators for building up the entire cohomology of the theory. First of all, these operators do not have inverses, which means that it is difficult to be sure that the higher-level states that one builds are BRST non-trivial. Indeed in general, unless one adjusts the number of insertions of \( \oint dz b(z) \) carefully, the result will certainly be BRST trivial, since it will have a ghost number at which no non-trivial states exist. Only if the ghost numbers and momenta of the physical operators are already known from some other argument does one know how to make the appropriate number of \( \oint dz b(z) \) insertions. Even then, it is not obvious that the physical state that one arrives at will necessarily be BRST non-trivial. Another related difficulty with using the \( \tilde{x} \) and \( \tilde{y} \) operators is that they do not in general give integer-degree poles when normal ordered with other physical states, or, indeed, with themselves. This leads to tedious complications in performing the multiple contour integrals that arise when building higher-level states. It was observed in \([7]\) that the fourth powers of \( \tilde{x} \) and \( \tilde{y} \) do always give integer poles. The reason for this is that their momenta correspond to those of the \( x \) and \( y \) operators at \( \ell = 15 \). Indeed, all of the above-mentioned difficulties are avoided by using the \( \ell = 15 \) operators \( x \) and \( y \), as we have seen in this paper.
One of the outstanding problems for $W_3$ strings is to understand the “$W_3$ geometry” on the worldsheet, and its rôle in governing the structure of the physical spectrum and the interactions. In some sense the multi-scalar $W_3$ string is somewhat trivial, in that the physical states all factorise into products of effective-spacetime states with primary fields of the Ising model realised by the $(\varphi, \beta, \gamma)$ system [5,6,8,4]. The rôle of the $W_3$ symmetry is rather trivial in this case. However, it may be that the much richer spectrum of the two-scalar $W_3$ string is associated with a more non-trivial action of the symmetry. Possibly this can be related to the discussion of $W_3$ geometry given in [18].

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