Non-perturbative effective model for the Higgs sector of the Standard Model

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A non-perturbative effective model is derived for the Higgs sector of the standard model, described by a simple scalar theory. The renormalized couplings are determined by the derivatives of the Gaussian Effective Potential that are known to be the sum of infinite bubble graphs contributing to the vertex functions. A good agreement has been found with strong coupling lattice simulations when a comparison can be made.

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I. INTRODUCTION

The nature and eventual existence of the Higgs Boson is one of the major problems in particle physics. While direct searches already provide a lower bound above 100 GeV [1] for the Higgs mass, electroweak precision measurements still suggest a light Higgs, unless new physics should be found at the TeV scale by the LHC: in that case a heavy Higgs would require the study of non-perturbative effects on the effective theory which describes the Higgs sector.

On the other hand a light Higgs is surely consistent with a perturbative treatment of the Standard Model (SM), but does not rule out a large self-coupling of the Higgs sector, since a light Higgs mass has also been predicted[2–5] in the strong coupling regime by non-perturbative methods. Thus, even for a light Higgs, some important non-perturbative effects may come into play.

Moreover, even if the self coupling were weak, a strong Yukawa coupling is expected for the top quark, and non-perturbative effects cannot be ruled out in the resulting Higgs-top coupled theory. It has been recently shown[6] that the weak couplings of the full $SU(2) \times U(1)$ gauge theory play a very negligible role in the Higgs sector, and that the simple self-interacting $\lambda \phi^4$ scalar theory[2, 3] yields the same predictions of the full non-Abelian gauge theory. Thus we believe that the simple scalar theory, and its scalar-fermion extension[6] still represent a valid starting point in the study of the non-perturbative features of the Higgs sector.

The scalar theory has been extensively studied in the past, but few truly non-perturbative treatments have been reported, and most of them have failed in the attempt to give a fully consistent interacting theory independent of any regularization scheme. More recently the triviality of the theory has been generally accepted and nowadays the SM is regarded as an effective low energy model: it does not need to be weakly coupled nor renormalizable, but it is supposed to be only valid up to some energy scale. An energy cut-off may be used as a regulator of the diverging integrals, and must be left finite in order to avoid to front an useless non-interacting trivial theory. In this framework there has been a renewed interest in the non-perturbative behaviour of scalar theories with a large but finite energy cut-off[4, 5]. Lattice simulations are the most reliable non-perturbative approaches: even if they fail to provide any analytical description themselves, they can be regarded as the best benchmark for testing approximate analytical tools. Unfortunately there are not many of such tools, and in the case of a single scalar field $1/N$ expansions are ruled out.

The Gaussian Effective Potential (GEP)[8–17] is a simple variational tool which has been often used for describing the spontaneous symmetry breaking in the framework of the scalar theory for particle physics and condensed matter[18–20]. Moreover, as it has already been pointed out[21, 22], the GEP contains information about the renormalized one-particle-irreducible (1PI) n-point functions: the derivatives of the GEP are a variational estimate of the 1PI functions at zero external momenta. Thus a direct expansion of the GEP around the broken-symmetry vacuum can be regarded as an effective low-energy model with the derivatives that act as renormalized couplings.

In this paper we study the emerging effective model and discuss the meaning of the couplings in terms of infinite sums of special classes of graphs. In fact the couplings are known[21, 26] to be given by the infinite sum of all the bubble graphs which can be drawn as perturbative corrections for the variationally optimized Gaussian Lagrangian. At variance with previous work the derivation of the effective model and its renormalization is carried out at a finite cut-off exactly, without any further approximation that would require or assume a very large cut-off which should eventually be sent to infinite. Thus the calculation is in the spirit of recent lattice simulations[6, 7], and can be compared to such numerical calculations while providing an analytical non-perturbative low-energy effective model. When a direct comparison can be made, we find a very good agreement with lattice data[27] gaining confidence on
the reliability of the variational method.

It is worth pointing out that, while the method is not new by itself, no previous attempt had been made to compare the predictions of the finite cut-off effective model with the available lattice data. For instance we derive a very simple analytical expression for the transition point that fits the lattice data very well up to very large couplings. We think that the method could be extended and used for an analytical description of the Higgs-top model which has been recently addressed by lattice simulation.

Another important issue is the existence of an intermediate mass which plays the role of a variational parameter and appears as an intermediate energy scale: quite smaller than the cut-off and still rather larger than the physical masses. Since all internal lines in bubble graphs are evaluated with such intermediate mass, then the resulting n-point functions have a very weak dependence on the external momenta at low energy, and they can be regarded as constant renormalized couplings. In other words the physical low-energy amplitudes can be derived at tree-level by an effective lagrangian whose couplings already contain the sum of infinite bubble graphs. Moreover, the non-perturbative nature of the method does not require the self-coupling to be small.

The paper is organized as follows: in section II the effective model is defined for a self-interacting scalar field; in section III the couplings are recovered as the sum of bubble expansions around the Gaussian ansatz; in section IV the problem of field renormalization is addressed and the nature of the critical point is studied; in section V the phenomenological content of the effective model is discussed; some final remarks and directions for future work are reported in section VI.

II. THE VARIATIONAL EFFECTIVE MODEL

Neglecting couplings to other fields\[3\], the Higgs sector of the Standard Model can be described by a simple scalar theory: in the Euclidean formalism the Lagrangian reads

\[
\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + \frac{1}{2} m_B^2 \phi^2 + \frac{1}{4!} \lambda_B \phi^4. \tag{1}
\]

Denoting by \( \phi \) the expectation value of the scalar field \( \langle \phi \rangle \), and by \( h \) the Higgs field \( h = \phi - \langle \phi \rangle \), the Lagrangian can be split as

\[
\mathcal{L} = \mathcal{L}_{\text{GEP}} + \mathcal{L}_{\text{int}} \tag{2}
\]

where

\[
\mathcal{L}_{\text{GEP}} = \frac{1}{2} \partial^\mu h \partial_\mu h + \frac{1}{2} \Omega^2 h^2 \tag{3}
\]

and \( \mathcal{L}_{\text{int}} = \mathcal{L} - \mathcal{L}_{\text{GEP}} \). The GEP can be recovered\[3\] as the first order effective potential for the free theory described by \( \mathcal{L}_{\text{GEP}} \) in presence of the interaction \( \mathcal{L}_{\text{int}} \). The mass \( \Omega \) of the free theory is then determined by requiring that for any value of the average \( \phi \) the effective potential is at a minimum. A trivial calculation of the first order effective potential yields

\[
V_{\text{GEP}}(\varphi) = \frac{1}{2} m_B^2 \varphi^2 + \frac{1}{2} m_B^2 I_0(\Omega) + \frac{\lambda_B}{4!} \varphi^4 + \frac{\lambda_B}{4} \varphi^2 I_0(\Omega) + \frac{\lambda_B}{8} \left[ I_0(\Omega) \right]^2 - \frac{1}{2} \Omega^2 I_0(\Omega) + I_1(\Omega) \tag{4}
\]

where the Euclidean integrals \( I_0, I_1 \) are defined as

\[
I_0(X) = \int_{\Lambda} \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + X^2} \tag{5}
\]

\[
I_1(X) = \frac{1}{2} \int_{\Lambda} \frac{d^4 k}{(2\pi)^4} \log(k^2 + X^2). \tag{6}
\]

Here the symbol \( \int_{\Lambda} \) means that the integrals are regularized by insertion of a cut-off \( \Lambda \) so that \( k < \Lambda \): according to the well known triviality of the \( \lambda \phi^4 \) theory the Higgs sector is regarded as an effective model with a high energy scale \( \Lambda \) which plays the role of a further free parameter\[3\]. The GEP is given by the effective potential \( V_{\text{GEP}} \) in Eq.\[4\] provided that the mass parameter \( \Omega \) is regarded as an implicit function of \( \varphi \), defined by the minimum condition (gap equation) \( \partial V_{\text{GEP}} / \partial \Omega = 0 \) which reads\[17\]

\[
\Omega^2 = m_B^2 + \frac{\lambda_B}{2} \varphi^2 + \frac{\lambda_B}{2} I_0(\Omega). \tag{7}
\]
The phenomenological broken-symmetry minimum of the GEP occurs at \( \varphi = \varphi_0 \) where the partial derivative of the GEP vanishes. By insertion of the gap equation Eq. (7) the derivative of \( V_{GEP} \) reads

\[
\frac{\partial V_{GEP}}{\partial \varphi^2} = \frac{1}{2} \left( \Omega^2 - \frac{1}{3} \lambda_B \varphi^2 \right)
\]

and it vanishes at

\[
\varphi_0^2 = \left[ \frac{3\Omega^2}{\lambda_B} \right] \varphi = \varphi_0.
\]

The known phenomenology of the Standard Model requires that \( \varphi_0 = v = 247 \text{ GeV} \), and then Eq. (9) gives the mass parameter \( \Omega_0 = \Omega|_{\varphi=\varphi_0} \) at the minimum as a function of the bare self-coupling \( \lambda_B \). At the minimum point \( \varphi = \varphi_0 \) the gap equation Eq. (7) can be satisfied by a proper choice of the free parameter \( m^2_H \), and the theory has only one free parameter, i.e. the bare self-coupling \( \lambda_B \) (besides the cut-off \( \Lambda \)). We must stress that \( \Omega \) is a simple variational parameter and there is no reason to believe that its value has any physical relevance. In fact the mass \( \Omega \) is the mass of a free particle described by the unperturbed Lagrangian \( L_{GEP} \), while the true Higgs mass comes out from the 1PI 2-point function \( \Gamma_2 \) which is the infinite sum of 2-point graphs that arise from the interaction \( L_{int} \). Of course a comparison of Eq. (9) with the tree level perturbative Higgs mass \( M_h^2 = \lambda_B v^2 / 3 \) tells us that at the broken symmetry vacuum \( \varphi = \varphi_0 = v \), and for a small bare coupling \( \lambda_B \ll 1 \), the true mass is \( M_h \approx \Omega_0 \) as we expected since the residual interaction in \( L_{int} \) becomes very small.

In general, the exact quantum effective action \( \Gamma[\varphi] \) can be written as an expansion in powers of \((\varphi - \varphi_0)\) around the vacuum expectation value \( \varphi_0 \)

\[
\Gamma[\varphi] = \sum_n \frac{1}{n!} \prod_{i=1}^n \left[ \int \frac{dp_i}{(2\pi)^d} (\varphi(p_i) - \varphi_0(p_i)) \right] \Gamma_n(p_1, \ldots, p_n)
\]

where the functional derivatives \( \Gamma_n \) are the exact 1PI n-point functions. For the theory described by the Lagrangian \( \mathcal{L} = L_{GEP} + L_{int} \), the n-point functions are the sum of infinite n-point graphs where the vertices are read from the interaction \( L_{int} \) and the free propagator \( G_0(p) = (p^2 + \Omega^2)^{-1} \) arises from the optimized free-particle Lagrangian \( L_{GEP} \) with \( \Omega \) set at the vacuum value \( \Omega_0 \) according to Eq. (9). Any physical amplitude may be evaluated as a sum of connected tree graphs with vertices provided by the n-point functions \( \Gamma_n \) which play the role of renormalized couplings.

Now suppose, as it turns out to be the case, that the variational parameter \( \Omega_0 \) is quite large compared to the physical Higgs mass \( M_h \). Then the n-point functions have a very weak dependence on the external momenta as far as these are small or at least \( p_i \approx M_h \). An approximate low energy effective model can be recovered by taking these couplings at their zero momentum value \( \Gamma_n(0) = \Gamma_n(0, \ldots, 0) \) and going back to the direct space where an effective Lagrangian can be written as

\[
\mathcal{L}_{eff} = \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi + \frac{1}{2} \left[ \Omega_0^2 - \Gamma_2(0) \right] \varphi^2 + \frac{1}{4} \Gamma_4(0) \varphi^4 + \cdots
\]

having denoted by \( \Gamma_2 \) the sum of first and higher order contributions to the 1PI 2-point function, i.e. all the terms except the zeroth-order one which according to Eq. (3) is equal to \(-\Omega_0^2\). The effective Lagrangian in Eq. (11) describes a scalar Higgs field with a renormalized mass \( M_R^2 = -\Gamma_2(0) = \Omega_0^2 - \Gamma_2(0) \) and renormalized couplings \( g_R = -\Gamma_4(0), \lambda_R = -\Gamma_4(0) \), and so on. The advantage of the effective Lagrangian in Eq. (11) is that any low energy amplitude may be evaluated at tree level, neglecting all loops that have been already summed up to all orders in the renormalized couplings. In other words the mass \( M_R \) can be regarded as the true mass \( M_h \) of the Higgs boson, and the couplings \( g_R, \lambda_R \) are related to the phenomenological scattering amplitudes of the particle.

Unfortunately we do not have the exact quantum effective action \( \Gamma[\varphi] \), but we can extract a variational estimate of the couplings \( \Gamma_n(0) \) from the GEP. In fact for constant background fields \( \varphi, \varphi_0 \) the effective action becomes the opposite of the effective potential, and if we set \( \varphi(p) = (2\pi)^d \delta^d(p) \varphi \) and \( \varphi_0(p) = (2\pi)^d \delta^d(p) \varphi_0 \) in Eq. (10) then the exact effective potential reads

\[
V_{eff}(\varphi) = -\sum_n \frac{1}{n!} \Gamma_n(0) (\varphi - \varphi_0)^n
\]

which is an expansion of the exact effective potential around \( \varphi = \varphi_0 \). Thus the renormalized couplings can be calculated by the simple derivatives of the effective potential as

\[
\Gamma_n(0) = -\left[ \frac{d^n V_{eff}}{d\varphi^n} \right]_{\varphi=\varphi_0}.
\]
On the other hand the GEP in Eq. (4) is a variational approximation to the exact effective potential $V_{\text{eff}}$ and we can evaluate a set of approximate couplings as

$$\Gamma_n(0) \approx - \left[ \frac{d^n V_{\text{GEP}}}{d\phi^n} \right]_{\phi = \phi_0}. \tag{14}$$

Insertion in Eq. (11) provides a simple way to perform non-perturbative low-energy calculations by the simple evaluation of connected tree graphs. In spite of the approximations, in the strong-coupling regime of the Higgs sector the predictions of this effective model are expected to be more reliable than perturbative calculations. In fact the approximate renormalized couplings in Eq. (14) are known to be the sum to all orders of bubble graphs for the vertex functions [23, 24, 25]. Moreover the derivatives of the GEP in Eq. (14) were shown to be a genuine variational approximation for the $n$-point functions [21, 23, 24] and were evaluated by several authors [21, 25, 26] in the past. Unfortunately such derivations strongly depend on the special regularization scheme and on a series of approximations which only make sense for a very large energy cut-off that should be eventually sent to infinite. In more recent years, the general consensus on the triviality of the scalar theory and the failure of any attempt to build a meaningful model with an infinite cut-off, has changed our view of the standard model, and the more modest aim of an effective model has been generally accepted as a reasonable compromise [6]. In this framework it would be desirable to discuss the variational $n$-point functions as defined in Eq. (14) exactly, without any further approximation, at a given cut-off $\Lambda$ which is supposed to be large but not too large, and plays the role of a physical parameter that points to the energy scale where new physics should become relevant. On the other hand, the resulting effective model could be easily compared to lattice calculations where a natural cut-off is supplied by the lattice spacing. In the next section we derive the explicit expressions for the variational $n$-point functions from the GEP, through Eq. (14) and by direct sum of the equivalent bubble expansions.

III. BUBBLE EXPANSIONS

The best way to calculate the approximate couplings $\Gamma_n(0)$ is by derivatives of the GEP [22]. However it is instructive to see that the variational approximation Eq. (14) is equivalent to the sum of all the tree bubble graphs that can be drawn for the vertex functions. By tree bubble graph we mean any 1PI graph which only contains chains of bubble insertions that do not make any loop: in other words the external lines of a graph become disconnected whenever two lines belonging to the same loop are cut. This set of graphs is a sub-class of the two particle point reducible graphs [29]. For instance in Fig.1 the 7-loop graph (a) is a tree bubble graph while the 8-loop graph (b) is not since its bubble chain makes a loop. Both of them are two particle point reducible.

![Fig. 1: Some two particle point reducible graphs. The 7-loop graph (a) is a tree bubble graph as it becomes disconnected whenever two lines belonging to the same loop are cut (i.e. whenever a bubble is cut); the 8-loop graph (b) is not since its bubble chain makes a loop. Both of them are two particle point reducible.](image)

In order to show the equivalence, let us write the explicit interaction $L_{\text{int}}$ that according to Eq. (2) and neglecting constant terms becomes

$$L_{\text{int}}(h) = \left( m_B^2 + \frac{\lambda_3 h^2}{3!} \right) \varphi_0 h + \frac{1}{2} \left( m_B^2 - \Omega_3^2 + \frac{\lambda_3 h^2}{2} \right) h^2 + \frac{1}{3!} \varphi_0 \lambda_B h^3 + \frac{\lambda_B}{4!} h^4. \tag{15}$$
The tree bubble graphs that contribute to the 1PI 2-point function $\Gamma'_2$ are shown in Fig. 2 where a straight line represents the free propagator $G_0 = (p^2 + \Omega_0^2)^{-1}$ as derived from the definition Eq. (3) of $L_{GEP}$ at the vacuum $\varphi = \varphi_0$. The first order contribution $\Gamma^{(1)}_2$ arises from the first two graphs of Fig. 2

$$\Gamma^{(1)}_2 = \Omega_0^2 - m^2_B - \frac{\lambda_B}{2} \varphi_0^2 - \frac{\lambda_B}{2} I_0(\Omega_0)$$

and by insertion of the gap equation Eq. (7) we get $\Gamma^{(1)}_2 = 0$ which is a well known property of the GEP. Thus $\Gamma'_2$ is given by the sum of all the higher order tree bubble graphs shown in Fig. 2.

$$\Gamma'_2 = \bigoplus \left[ G_0 = (p^2 + \Omega_0^2)^{-1} \right] + \bigoplus \left[ -2! \lambda_B \varphi_0; \right] + \bigoplus \left[ -\lambda_B / 4! \right].$$

Fig. 2: Tree bubble graphs contributing to the 1PI 2-point function. Explicit expressions for the free propagator and for the vertices are reported on the top.

Let us denote by $L_n$ the n-order 1-loop graph without vertex factors, without external lines and with external momenta set to zero, as shown in Fig. 3.

$$L_{n+1} = \int \frac{d^4 p}{\Lambda (2\pi)^4} \frac{1}{(p^2 + \Omega_0^2)^{n+1}} = \frac{1}{n!} \left| \frac{d^n I_0}{d(\Omega^2)^n} \right|_{\Omega = \Omega_0}$$

and let us denote by $B$ the bubble chain geometric expansion reported as an hatched bubble in Fig. 4

$$B = \sum_{n=0}^{\infty} \left[ -\frac{\lambda_B}{2} L_2 \right]^n = \frac{1}{1 + \frac{\lambda_B}{2} L_2}.$$ (18)

Fig. 3: Graphs for the integrals $L_2$, $L_3$ and $L_4$. They are n-order 1-loop graphs without vertices, without external lines and with external momenta set to zero.

With the above notation the approximate 2-point coupling $\Gamma'_2$ is shown in Fig. 4 as a cross-hatched bubble with vertex and symmetry factors added at the external points, while the simple cross-hatched bubble represents the chain bubble sum without external vertex factors. Accordingly we can write

$$\Gamma'_2(0) = (3 \times 3 \times 2) \left( -\frac{1}{3!} \lambda_B \varphi_0 \right)^2 L_2 B = \frac{3\Omega_0^2}{2} \lambda_B L_2 B$$

where we have made use of Eq. (9). The renormalized mass $M^2_R = -\Gamma_2$ then follows

$$M^2_R = \Omega_0^2 - \Gamma'_2(0) = \Omega_0^2 \left[ 1 - \frac{\lambda_B L_2}{1 + \frac{\lambda_B}{2} L_2} \right].$$

(20)

On the other hand, the derivative of Eq. (8) yields

$$\frac{d^2 V_{GEP}}{d\varphi^2} = (\Omega^2 - \frac{1}{3} \lambda_B \varphi^2) + 2 \varphi^2 \left( \frac{d\Omega^2}{d\varphi^2} - \frac{\lambda_B}{3} \right)$$

and, at $\varphi = \varphi_0$, by insertion of Eq. (9) we get

$$\left[ \frac{d^2 V_{GEP}}{d\varphi^2} \right]_{\varphi = \varphi_0} = \frac{6\Omega_0^2}{\lambda_B} \left[ \frac{d\Omega^2}{d\varphi^2} \right]_{\varphi = \varphi_0} - \frac{\lambda_B}{3}.$$
\[ B = \quad \begin{array}{ccc} \circ \quad & \circ \quad & \circ \quad + \quad \end{array} = 1 + \quad \begin{array}{ccc} \circ \quad & \circ \quad & \circ \quad + \quad \end{array} + \quad \begin{array}{ccc} \circ \quad & \circ \quad & \circ \quad + \quad \end{array} + \ldots \]

\[ \begin{array}{ccc} \circ \quad & \circ \quad & \circ \quad = \quad \begin{array}{ccc} \circ \quad & \circ \quad & \circ \quad = L_2 \cdot B \end{array} \end{array} \]

\[ \Gamma' = \quad \begin{array}{ccc} \circ \quad & \circ \quad & \circ \quad = (3 \times 3 \times 2)(-\frac{1}{3!} \lambda_B \phi_0)^2 L_2 \cdot B. \end{array} \]

Fig. 4: The bubble chain geometric expansion denoted by \( B \) in the main text (hatched bubble). The cross-hatched bubble represents the sum of all the bubble chain graphs without external vertices. The coupling \( \Gamma'_L(0) \) is obtained by adding the external vertices and the correct symmetry factors to a cross-hatched bubble.

The derivative of \( \Omega^2 \) as a function of \( \varphi^2 \) can be easily obtained by the gap equation Eq. (7) whose derivative reads

\[ \frac{d\Omega^2}{d\varphi^2} = \lambda_B \frac{1}{2} + \lambda_B \frac{1}{2} \left( \frac{dI_0}{d\Omega^2} \right) \frac{d\Omega^2}{d\varphi^2}. \]  

(23)

Then the derivative of \( \Omega^2 \) is

\[ \frac{d\Omega^2}{d\varphi^2} = \lambda_B \frac{1}{2} \left[ 1 - \lambda_B \frac{1}{2} \left( \frac{dI_0}{d\Omega^2} \right) \right] \]  

(24)

which at the vacuum \( \varphi = \varphi_0 \) becomes

\[ \left( \frac{d\Omega^2}{d\varphi^2} \right)_{\varphi = \varphi_0} = \lambda_B \frac{1}{2} \left[ 1 + \frac{1}{2} \frac{dI_0}{d\Omega^2} \right]. \]  

(25)

Insertion of the derivative in Eq. (22) shows that, according to Eq. (14) which defines the couplings as derivatives of the GEP, Eq. (22) yields a renormalized mass \( M_R \) which is exactly the same as that obtained by the sum of tree bubble graphs in Eq. (20).

The equivalence can be extended to higher orders: the derivatives of the GEP are easily evaluated by Eq. (21) and give

\[ g_R = \left[ \frac{d^3V_{GEP}}{d\varphi^4} \right]_{\varphi = \varphi_0} = \varphi_0 \lambda_B \frac{M^2}{\Omega_0} + 4\varphi_0^2 \left[ \frac{d^2\Omega^2}{d(\varphi^2)^2} \right]_{\varphi = \varphi_0} \]  

(26)

\[ \lambda_R = \left[ \frac{d^4V_{GEP}}{d\varphi^4} \right]_{\varphi = \varphi_0} = \lambda_B^2 \frac{M^2}{\Omega_0^2} + 4! \frac{\varphi_0^2}{\Omega_0^2} \left[ \frac{d^3\Omega^2}{d(\varphi^2)^3} \right]_{\varphi = \varphi_0} + 8 \frac{\varphi_0^4}{\Omega_0^4} \left[ \frac{d^2\Omega^2}{d(\varphi^2)^2} \right]_{\varphi = \varphi_0} \]  

(27)

where the derivatives of \( \Omega^2 \) follow from Eq. (24) and can be written in compact form by insertion of Eq. (17) and Eq. (18)

\[ \left[ \frac{d^3\Omega^2}{d(\varphi^2)^3} \right]_{\varphi = \varphi_0} = \frac{\lambda^4_B}{4} B^3 L_3 \]  

(28)

\[ \left[ \frac{d^4\Omega^2}{d(\varphi^2)^2} \right]_{\varphi = \varphi_0} = \frac{3}{8} \lambda^3_B B^3 L_3^3 - \frac{3}{8} \lambda^4_B B^4 L_4. \]  

(29)
With the above notation the renormalized couplings read

\[ g_R = \varphi_0 \lambda_B \left[ \frac{M_R^2}{\Omega_0^2} + (\varphi_0 \lambda_B)^2 B^3 L_3 \right] \]  \hfill (30)

\[ \lambda_R = \lambda_B \left[ \frac{M_R^2}{\Omega_0^2} + 6\varphi_0^2 \lambda_B^2 B^3 L_3 - 3\varphi_0^4 \lambda_B^4 B^4 L_4 + 3\varphi_0^4 \lambda_B^4 B^5 L_5 \right] . \]  \hfill (31)

\[ \Gamma_3 = -g_R = \left( 3! \right) + 3 \left( 3 \times 4! \right) + \left( 3! \times 3! \times 3! \right) \]

Fig. 5: Tree bubble graphs contributing to the 1PI 3-point vertex function with their symmetry factors.

\[ \Gamma_4 = -\lambda_R = \left( 4! \right) + 3 \left\{ \left( 4 \times 3 \times 4! \right) + \right\} \left( 4! \times 3! \times 4! \right) + \left( 3! \times 3! \times 3! \times 3! \right) + \left( 4! \times 3! \times 3! \times 3! \right) \]

Fig. 6: Tree bubble graphs contributing to the 1PI 4-point vertex function with their symmetry factors.

It is not difficult to show that these couplings can be recovered by the sum of all the tree bubble graphs that can be drawn for the vertex functions. The sum of the corresponding geometric expansions are reported in Fig.5 and Fig.6. The tree bubble graphs that contribute to the 1PI 3-point vertex function are reported in Fig.5 with their symmetry factors. The sum of the first two terms is \( \varphi_0 \lambda_B - 3\Gamma_2' / \varphi_0 = \varphi_0 \lambda_B M_R^2 / \Omega_0^2 \), while the third graph yields \( \lambda_B^3 \varphi_0^3 B^3 L_3 \), in agreement with Eq. (30). With the same notation the tree bubble graphs contributing to the 1PI 4-point vertex function are reported in Fig.6: as for the 3-point function the sum of the first two terms is \( \lambda_B M_R^2 / \Omega_0^2 \) while the other graphs can be easily shown to reproduce the right hand side of Eq. (31) term by term.

The equivalence explains the content of the variational effective model and makes it even more evident that the GEP provides a set of genuine non-perturbative approximate couplings that represent the sum to all orders of tree bubble graphs for the vertex functions.
IV. RENORMALIZED COUPLINGS AND CRITICAL POINT

Before we can compare the predictions of the model with the phenomenology, we must address the problem of field renormalization. As we already said, the variational mass $\Omega$ turns out to be quite larger than the physical Higgs mass $M_h$, thus reducing any momentum dependence of the n-point functions. As a consequence we expect a negligible field renormalization and a physical mass $M_R^2$ which is basically equal to $M_R^2 = -\Gamma_2(0)$. However, we would like to avoid any further approximation, and we prefer to evaluate both the physical mass $M_h$ and the field renormalization factor $Z_h$ from the 2-point function $\Gamma_2(p)$ according to the usual definitions:

$$\Gamma_2(p)\big|_{p^2=-M_h^2} = 0$$

$$Z_h^{-1} = -\frac{\partial \Gamma_2(p)}{\partial p^2}\big|_{p^2=-M_h^2}$$

Thus our finding that $Z_h \approx 1$ and $M_h \approx M_R$ can be regarded as a check that the momentum dependence can be neglected, and a confirmation that the 2-point function can be taken as $-\Gamma_2(p) = (p^2 + M_h^2)$ as it was already assumed without proof in previous work on the full gauge theory\cite{4}. The same conclusion had been reached by numerical simulations on the lattice\cite{6,7}. Accordingly, the physical renormalized couplings must be defined as

$$\lambda_h = -Z_h^2 \Gamma_4(0) = Z_h^2 \lambda_R.$$  

$$g_h = -Z_h^{3/2} \Gamma_3(0) = Z_h^{3/2} g_R.$$  

The full 2-point $\Gamma_2(p)$ function may be recovered by the sum of the bubble expansion in Fig.4 with the loop $L_2$ replaced by the function $L_2(p)$, defined by the same graph in Fig.3 with the external momentum restored

$$L_2(p) = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(p+k)^2 + \Omega_0^2} \frac{1}{[k^2 + \Omega^2]}$$  

Then according to Eq.(38) the 2-point function reads

$$-\Gamma_2(p) = (p^2 + \Omega_0^2) - \frac{3\Omega_0^2}{2} \left( 1 + \frac{\lambda_{BS} L_2(p)}{2} \right).$$

The same result can be found by the covariant Gaussian approximation\cite{24}, or by direct functional differentiation of the Gaussian effective action\cite{21}, and can be shown to be a genuine variational bound of the exact 2-point function\cite{21,23}. According to the definitions in Eqs.(32),(33) the physical Higgs mass $M_h$ is obtained as a solution of the equation

$$M_h^2 = \Omega_0^2 \left[ 1 - \frac{\lambda_{BS} L_2(p)}{1 + \frac{\lambda_{BS}}{2} L_2(p)} \right]_{p^2=-M_h^2}$$

while the field renormalization constant is given by

$$Z_h^{-1} = 1 - \frac{3}{2} \Omega_0^2 \lambda_{BS} \left[ \frac{\partial L_2(p)}{\partial p^2} \left( 1 + \frac{\lambda_{BS}}{2} L_2(p) \right)^{-2} \right]_{p^2=-M_h^2}.$$

The integral in Eq.(30) must be evaluated inside the four-dimensional hyper-sphere $k < \Lambda$, so that the usual Feynman formula cannot be used. However, by a tedious calculation the integral can be shown to yield the following exact result

$$L_2(p) = \frac{1}{32 \pi^2 p^2} \left[ \Lambda^2 + p^2 \log \frac{\Omega_0^2 + \Lambda^2}{\Omega_0^2} - I(p) \right]$$

where the integral $I(p)$

$$I(p) = \int_{\Omega_0^2}^{\Omega_0^2 + \Lambda^2} \frac{dx}{x} \sqrt{(x-p^2)^2 + 4p^2 \Omega_0^2}$$
is given by

$$I(p) = p^2 \left[ t_1 (t_2 - t_1) - \log \left| \frac{t_2}{t_1} \right| + \left( t_1 \frac{t_2 - 1}{t_2} \right) - \sqrt{1 + 4t_1^2} \log \left| \frac{(t_2 - t_-)(t_1 - t_+)}{(t_2 - t_+)(t_1 - t_-)} \right| \right]$$  \hspace{1cm} (42)

with the variables $t_i$, $t_\pm$ defined according to

$$t_\pm = \frac{1}{2\Omega_0} \left[ \pm \sqrt{p^2 + 4\Omega_0^2} - p \right]$$  \hspace{1cm} (43)

$$t_1 = \frac{\Omega_0}{p}$$  \hspace{1cm} (44)

$$t_2 = \frac{1}{2p\Omega_0} \left[ \Lambda^2 - p^2 + \Omega_0^2 + \sqrt{4p^2\Omega_0^2 + (\Omega_0^2 + \Lambda^2 - p^2)^2} \right].$$  \hspace{1cm} (45)

In spite of the appearance, the loop function $L_2(p)$ in Eq. (40) has a logarithmic divergence only: in fact for $M_h, \Omega_0 \ll \Lambda$ we can write, up to order $1/\Lambda^2$

$$L_2(p) \approx \frac{1}{16\pi^2} \left\{ 1 + 2 \log \left( \frac{\Lambda}{\Omega_0} \right) - \sqrt{1 + 4\Omega_0^2} \frac{p^2}{\Omega_0^2} \log \left[ 1 + \frac{p^2}{2\Omega_0^2} - \frac{p^2}{\Omega_0^2} \left( 1 + \frac{p^2}{4\Omega_0^2} \right) \right] \right\}$$  \hspace{1cm} (46)

yielding at the point $p^2 = -M_h^2$

$$L_2(p)|_{p^2=-M_h^2} \approx \frac{1}{16\pi^2} \left\{ 1 + 2 \log \left( \frac{\Lambda}{\Omega_0} \right) - \sqrt{4\Omega_0^2 - M_h^2} \frac{M_h}{\Omega_0} \arctan \left( \frac{M_h}{2\Omega_0^2 - M_h^2} \right) \right\}$$  \hspace{1cm} (47)

We notice that as far as $\Lambda$ is finite, large but not too large compared to $\Omega_0$, the constant terms must be retained as they can be of the same order as the logarithm term $\log(\Lambda/\Omega_0)$. From Eq. (47) we see that $L_2(iM_h)$ is analytical in the limit $M_h \to 0$ and its limit value $L_2(0)$ is

$$L_2(0) = \frac{1}{16\pi^2} \left[ 2 \log \left( \frac{\Lambda}{\Omega_0} \right) - 1 \right]$$  \hspace{1cm} (48)

which is exactly the same large $\Lambda$ behaviour which would come out from a direct calculation through the definitions of $L_n$ and $I_0$ by Eqs. (17), (5).

The limit $M_h \to 0$ is very important as it turns out to be the critical point of the Higgs sector, since the vanishing of the mass may be associated to a change of sign for the second derivative of the effective potential at its stationary point which becomes a maximum. The exact equation for $M_h$, Eq. (38), can be easily seen to admit the solution $M_h = 0$ whenever the condition

$$\lambda_B = \frac{1}{L_2(0)}$$  \hspace{1cm} (49)

is fulfilled. Thus we observe that, for a large but finite cut-off, the bare self-coupling $\lambda_B$ does not need to be small at the critical point: assuming that $\Omega_0/\Lambda$ is small enough we can make use of Eq. (48) and write the critical condition Eq. (49) as

$$\Omega_0 = \frac{\Lambda}{\sqrt{e}} e^{-\frac{8\pi^2}{\lambda_B}}.$$  \hspace{1cm} (50)

We can check that for a coupling as large as $\lambda_B \approx 8\pi^2 \approx 80$ we still find $\Omega_0/\Lambda \approx 0.22$ which is small enough to be consistent with the use of the approximate Eq. (48) for $L_2(0)$.

From Eq. (50) we see that there is no way to keep $\Omega_0$ finite when the cut-off $\Lambda$ is sent to infinity unless the bare coupling $\lambda_B$ is taken to be infinitesimal. This is just what happens in the autonomous renormalization[17] where the bare self-coupling is taken to be $\lambda_B = 1/L_2(0)$ while $L_2(0)$ diverges logarithmically. That is exactly the required condition Eq. (49) for the vanishing of the Higgs mass. Thus it is not a great surprise that in autonomous renormalization the GEP predicts a vanishing Higgs mass[21].
From Eq. (50) we also see that the GEP predicts a weak first order transition because $\Omega_0$ does not vanish at the critical point. According to Eq. (49) at the minimum of the effective potential the vacuum expectation value of the scalar field is $\phi_0^2 = 3\Omega_0^2/\lambda_B$, and if $\Omega_0$ does not vanish at the transition, the expectation value $\phi_0$ jumps to zero at the critical point. That seems to be an unavoidable shortcoming of the GEP, since the transition is believed to be continuous. However by Eq. (50) we see that the jump $\Delta\phi$ of the expectation value at the transition is small

$$\Delta\phi = \frac{\sqrt{2}\Lambda}{\sqrt{e\lambda_B}} e^{-\frac{x_0^2}{2}}. \quad (51)$$

It reaches its maximum at $\lambda_B = 16\pi^2 \approx 158$ where $\Delta\phi/\Lambda \approx 0.05$ and it goes to zero in both the limits $\lambda_B \to 0$ and $\lambda_B \to \infty$. Thus we may neglect the jump in most cases, as far as we do not go too close to the transition point. That is not a major problem at all, as we know that the physical range of interest cannot be too close to the critical point where the Higgs mass eventually vanishes. We cannot use the GEP at criticality, but we expect that the order of the transition should not affect the behaviour of the resulting effective model provided that we do not reach the critical point. Actually we will see in the next section that the vacuum expectation value $\phi_0$ seems to be continuous up to the transition point for any reasonable plot resolution.

Whenever $\Lambda \gg \Omega_0$ we can use the approximate Eq. (16) for $L_2(p)$ in order to get the field renormalization constant $Z_h$. Insertion of Eq. (16) in Eq. (39) yields

$$Z_h^{-1} = 1 + \frac{\lambda_B}{96\pi^2} \frac{(2 + x^2)^2}{(4 - x^2)} \left[ 1 - \frac{32\pi^2 f(x)}{x^2} \right] \quad (52)$$

where $x = M_h/\Omega_0$ and

$$f(x) = L_2(iM_h) - L_2(0) = \frac{1}{16\pi^2} \left[ 2 - \frac{\sqrt{4 - x^2}}{x} \arctan \left( \frac{x\sqrt{4 - x^2}}{2 - x^2} \right) \right]. \quad (53)$$

We can simply check that $Z_h \approx 1$ in the small-coupling limit and close to the critical point. That is quite obvious in the small-coupling limit $\lambda_B \to 0$ where $M_h \approx \Omega_0$ and $x \to 1$. In this limit $Z_h^{-1} - 1 \approx 2 \cdot 10^{-3} \cdot \lambda_B$, which is vanishing small. In the opposite strong coupling limit we can explore the critical range where $M_h \to 0$ ($x \to 0$) and, according to Eq. (49), $\lambda_B L_2 \approx 1$. In this limit $96\pi^2 f(x) \approx x^2$ and $Z_h^{-1} - 1 \approx \lambda_B/(144\pi^2) \approx 0.7 \cdot 10^{-3} \cdot \lambda_B$. This is just one per cent for a coupling as large as $\lambda_B \approx 15$. In actual calculations we have never found values larger than few per cent in the broken symmetry phase, in agreement with previous numerical findings[6, 7].

Quite interesting, around criticality we find a hierarchy of energy scales with an Higgs mass $M_h$ quite smaller than the mass parameter $\Omega_0$ which is supposed to be much smaller than the energy cut-off $\Lambda$. The physical meaning of the intermediate scale $\Omega_0$ is related to the vacuum expectation value of the scalar field $v = 247$ GeV according to Eq. (39) which reads $\Omega_0 = v/\sqrt{\lambda_B}/3$. For a strong coupling $\lambda_B \approx 20$ we get $\Omega_0 \approx 640$ GeV while $M_h$ can be made as small as we like by a large cut-off.

V. RENORMALIZED COUPLINGS AND PHENOMENOLOGY

In order to make contact with Monte Carlo calculations[6, 7] we take all energies in units of the cut-off $\Lambda$. In these units the vacuum expectation value $\phi_0 = v/\Lambda$ can be seen as representing the inverse of the cut-off in units of $v = 247$ GeV. In fact at the critical point the vanishing of $\phi_0$ can be regarded as a restoration of symmetry at a fixed cut-off, or as the effect of an infinite cut-off on a fixed vacuum expectation value. Following Ref. [6] we take a constant value for the bare self-coupling $\lambda_B$ and change the bare mass $m_B^2$ up to the critical point which is reached when $M_h$ vanishes at $m_B^2 = m_c^2$. In the broken-symmetry phase we measure the distance from the critical point by the adimensional parameter $\tau = [1 - m_B^2/m_c^2]$. The physical Higgs mass $M_h$ and the vacuum expectation value $\phi_0$ are reported in Fig.7 up to the transition point for a moderate $\lambda_B = 10$. We observe that the first-order jump of $\phi_0$ is so small at the transition that it cannot be even seen in the plot at any reasonable scale.

A direct comparison with numerical simulations requires that the correct scale factor should be fixed between energies. In lattice calculations energies are usually taken in units of the inverse of the lattice spacing $a$ which also provides a natural cut-off. Thus we expect that the scale factor $c = \Lambda a$ should be of order unity, but we have no direct and unique way to determine it. For instance we could require that the unit cell of the inverse lattice has the same extension of our four-dimensional hyper-sphere $p < \Lambda$, and obtain $c = 2^{5/4}\sqrt{2} \approx 4.2$. Here we prefer to make a more empirical choice and take the scale factor $c$ as the ratio between the critical bare masses $m_c$, assuming that the
critical masses should be the same if the energy scales are correctly handled. For a moderate $\lambda_B \approx 10 - 100$ there are not too many data to compare with: some numerical results have been reported for the symmetric phase in Ref. [27], and a comparison of the critical bare masses for $\lambda_B = 10$ yields the empirical scale factor $c = 4.38$.

Once the scale factor has been fixed we may compare our results with the lattice data. The critical point $m^2_c$ is shown as a function of $\lambda_B$ in Fig. 8 where the lattice data of Ref. [27] have also been included for comparison. Our result fits the lattice data very well up to $\lambda_B \approx 150$. For larger couplings the GEP underestimates the strength of the critical $m^2_B$ as a natural consequence of the predicted first order transition. On the other hand we may extract from the GEP a yet simpler analytical approximate function that interpolates the data very well for a moderate bare coupling. In fact for a moderate coupling $\lambda_B < 100$ we already know that, according to Eq. (50), the ratio $\Omega_0/\Lambda$ is small enough at the critical point. Therefore we can neglect powers higher than $\Omega_0^2/\Lambda^2$ in the expansions of $L_2(0)$ and $I_0(\Omega_0)$, and inserting Eqs. (50), (9) in the gap equation Eq. (7) we gain the simple result

$$m^2_c = -\frac{\lambda_B}{32\pi^2} \left[ 1 - e^{-\left(1+\frac{4\lambda_B}{\pi^2}\right)} \right].$$

As shown in Fig. 8, the approximate Eq. (54) gives the correct critical point up to $\lambda_B \approx 200$, while for stronger couplings the strength of the critical $m^2_B$ is overestimated. We observe that Eq. (54) has an essential singularity at the point $\lambda_B = 0$ and thus it cannot be recovered by any perturbative expansion in powers of $\lambda_B$. Despite being very simple, the analytical result in Eq. (54) is a genuine non-perturbative prediction, and provides an example of the capabilities of the variational method.

With the same scale factor, the available lattice data [27] for the mass have been inserted in Fig. 7 for comparison, and again they are in good agreement with the present variational calculation. Both calculations are consistent with a square root behaviour $M_h \sim \sqrt{\tau}$. Furthermore, once the correct scale factor has been inserted, the numerical values seem to be almost the same, with our data slightly larger than the lattice predictions. That was not entirely expected as in the lattice simulation of Ref. [27] the renormalized mass was measured in the symmetric phase while our results are for the broken symmetry phase, and the critical behaviour is expected to agree in the two phases up to constant factors: integration constants that are usually different in the two phases. Some perturbative arguments have been reported in the past giving evidence for the equivalence of the integration constants in the two phases [30], while lattice data have shown [7] that, for a very strong coupling $\lambda_B = 600$, the constant factors are different, with the broken...
symmetry mass being almost twice the mass of the symmetric phase. Thus we argue that the slight difference in Fig. 7 can be taken as a measure of non-perturbative effects, still small for $\lambda_B = 10$. Actually we have checked that when $\lambda_B$ increases the difference also increases with the effect becoming quite large for $\lambda_B = 50$ already. Unfortunately, as discussed at the end of the previous section, we cannot allow $\lambda_B$ to become too large at the transition point since the transition is predicted to be first order by the GEP. That precludes a direct comparison with the lattice data of Ref. [7] in the broken symmetry phase.

In Fig. 9 the physical adimensional renormalized couplings $3M_h^2/v^2$, $g_h/v$ and $\lambda_h$ are reported. For a large $\tau$ (i.e. a small cut-off) they all tend to the same bare value $\lambda_B$. At the critical point $\tau \to 0$ they all seem to vanish yielding a trivial model. However it should be observed that the renormalized coupling $\lambda_h$ is not equal to $3M_h^2/v^2$ as it would be suggested by the standard tree-level relation. Such a condition can only be satisfied if $M_h \approx \Omega_0$ in Eq. (5), i.e. for a very small bare self-coupling or an unphysically small cut-off (large $\tau$). The same adimensional renormalized couplings are reported in Fig. 10 for the critical range $\tau < 1$. For comparison in the same Fig. 10 some data points from Ref. [27] are reported as an indirect measure of $3M_h^2/v^2$ on the lattice. According to scaling, these points can be assumed to be equal to the value of $\lambda_R$ in the symmetric phase. In fact for $\lambda_R$ we could not find any direct simulation data in the broken symmetry phase, nor any direct lattice measure of the renormalized couplings $\lambda_h$ and $g_h/v$.

For small values of the parameter $\tau$ (i.e. for a large cut-off) all the couplings decrease and show a logarithmic behaviour. We observe that $\lambda_h$ is always larger than its perturbative value $3M_h^2/v^2$, yielding a strongly coupled effective model even when the Higgs mass falls well below the electroweak energy scale $v$. Moreover the 3-point adimensional coupling $g_h/v$ and the 4-point vertex $\lambda_h$ turn out to be different at variance with the case of a weak coupling where they are the same. For a very light Higgs (close to the critical point), we may assume the autonomous condition Eq. (49), take $B = 2/3$ in Eqs. (30)-(31) and for a large cut-off write the ratio $g_h/(v\lambda_h)$ as

$$\frac{g_h}{v\lambda_h} \approx \frac{1}{4 + \frac{\lambda_B}{\pi^2}}$$  \hspace{1cm} (55)

We observe that the ratio between the couplings is $g_h/(v\lambda_h) \approx 0.25$ for any moderate bare coupling, and still smaller if $\lambda_B$ gets very large, to be compared with the weak coupling limit $g_h/(v\lambda_h) \approx 1$. The weakening of the 3-point coupling would give less chances of finding bound states in the Higgs sector, since it is the 3-point vertex that can give a binding contribution in a strongly self-coupled Higgs sector [3, 31].

Fig. 8: The critical parameter $m_c^2 = m_2^2$ as a function of the bare coupling $\lambda_B$. The solid line is the result of the present variational method. The squares are the numerical lattice data of Ref. [27] scaled according to $\Lambda a = 4.38$. The dahed line is the outcome of the approximate analytical function reported in Eq. (54).
VI. DISCUSSION

The effective model that emerges variationally from the GEP could provide a reliable way to describe the experimental data, together with numerical lattice simulations. In fact we have shown that the non-perturbative predictions of the GEP can be as effective as numerical simulations, provided that the comparison is not pushed too close to the critical point where the GEP is known to fail. That precludes the study of extremely strong couplings $\lambda_B \approx 100 - 1000$ as according to Eq.(51) the jump of the vacuum expectation value $\varphi_0$ would not be negligible at the transition point. Unfortunately that has not allowed us to compare our predictions with some numerical data[7] on the broken-symmetry phase that have been reported for $\lambda_B = 600$.

On the other hand most of the numerical simulations that have been reported for moderate couplings $\lambda_B \approx 10 - 100$ happen to be too far from the relevant physical range of parameters as the cut-off is usually too small[27, 32]. That represents a common problem of numerical simulations since the finite size of the sample limits the accuracy when the correlation length $\xi$ reaches the size $L$ of the sample. Usually $L = na$ where $a$ is the lattice constant and $n$ is a small integer of order ten at most, therefore taking $M_h = 1/\xi$ and $\Lambda \approx 1/a$ we get a bound $\Lambda < nM_h$. For a light Higgs $M_h \approx 100$ GeV and for $n = 10$ we get a cut-off $\Lambda < 1$ TeV, too small for exploring the Higgs sector. Actually, by a variational argument, a threshold has been predicted at $\Lambda \approx 3.5$ TeV for the existence of a strongly interacting light Higgs[2].

However some simulation data have been reported for the symmetric phase and a moderate strong coupling[27], and have been found in perfect agreement with the predictions of the present method. Larger lattices should be studied for a comparison in the physical interesting range of parameters, while the real experiments seem to be an interesting and viable way to test the predictions of the present effective model.

It must be mentioned that before a full comparison can be made between the real phenomenology and the effective model, the couplings to fermions and gauge fields must be included in the simple scalar theory. Inclusion of gauge fields is not a major issue as we have already derived the GEP for the full $SU(2) \times U(1)$ non-Abelian gauge theory[4], and shown that the weak gauge couplings do not play any relevant role in the Higgs sector. The couplings with fermions are expected to be more important and, since they are proportional to masses, the Top quark should be considered at least. The present variational method can be extended to the Higgs-Top model that has been recently studied by the GEP[33] and by numerical lattice simulation[6]. While we do not expect any dramatic change of the
general phenomenology with respect to the present simple theory, the resulting effective model should give a more reliable description of the Higgs sector.

Fig. 10: The physical adimensional renormalized couplings $3M^2_h/v^2$ (lower solid curve), $g_h/v$ (central dashed curve) and $\lambda_h$ (upper dotted curve) as functions of the adimensional parameter $\tau = |1 - m^2_B/m^2_c|$, in the critical range of the broken symmetry phase, for a moderately strong bare coupling $\lambda_B = 10$. Note the logarithmic decreasing of the couplings for $\tau \to 0$. Data points are indirect lattice simulations for $3M^2_h/v^2$, and have been obtained by scaling from the symmetric phase data of Ref. [27].

[1] R. Barate et al. [LEP Working Group for Higgs boson searches], Phys. Lett. B 565, 61 (2003). arXiv:hep-ex/0306033
[2] F. Siringo, Phys. Rev. D 62, 116009 (2000).
[3] F. Siringo, Europhys. Lett. 59, 820 (2002).
[4] F. Siringo, L. Marotta, Phys. Rev. D 78, 016003 (2008).
[5] R. Ibañez-Meier, I. Stancu, P.M. Stevenson, Z. Phys. C 70, 307 (1996).
[6] Z. Fodor, K. Holland, J. Kuti, D. Nogradi, C. Schroeder, arXiv:0710.3151v1
[7] J. Kuti, Y. Shen, Phys. Rev. Lett. 60, 85 (1988).
[8] L.I. Schiff, Phys. Rev. 130, 458 (1963).
[9] G. Rosen, Phys. Rev. 172, 1632 (1968).
[10] T. Barnes and G. I. Ghandour, Phys. Rev. D 22, 924 (1980).
[11] J. Kuti (unpublished), as discussed in J. M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. D 10, 2428 (1974).
[12] J. Chang, Phys. Rev. D 12, 1071 (1975); Phys. Rep. 23, 301 (1975); Phys. Rev. D 13, 2778 (1976).
[13] M. Weinstein, S. Drell, and S. Yankielowicz, Phys. Rev. D 14, 487 (1976).
[14] K. Huang and D. R. Stump, Phys. Rev. D 14, 223 (1976).
[15] W. A. Bardeen and M. Moshe, Phys. Rev. D 28, 1372 (1983).
[16] M. Peskin, Ann. of Phys. 113, 122 (1978).
[17] P.M. Stevenson, Phys. Rev. D 32, 1389 (1985).
[18] M. Camarda, G.G.N. Angilella, R. Pucci, F. Siringo, Eur. Phys. J. B 33, 273 (2003).
[19] L. Marotta, M. Camarda, G.G.N. Angilella and F. Siringo, Phys. Rev. B 73, 104517 (2006).
[20] L. Marotta, F. Siringo, arXiv:0806.4560
[21] R. Ibanez-Meier, Phys. Lett. B 295, 89 (1992).
[22] R. Ibanez-Meier, L. Polley, U. Ritschel, Phys. Lett. B 279, 106 (1992).
[23] A. Kovner, B. Rosenstein, Phys. Rev. D 39, 2332 (1989).
[24] B. Rosenstein, A. Kovner, Phys. Rev. D 40, 504 (1989).
[25] M. Consoli, G. Passarino, Phys. Lett. B 165, 113 (1985).
[26] V. Branchina, M. Consoli, N.M. Stivala, Z. Phys. C 57, 251 (1993).
[27] I.T. Drummond, S. Duane, R.R. Horgan, Nucl. Phys. B 280, 25 (1987).
[28] S. Weinberg, The quantum theory of fields, vol. II, Cambridge University Press (1996).
[29] H. Verschelde, M. Coppens, Phys. Lett. B 287, 133 (1992).
[30] M. Lüscher, P. Weisz, Nucl. Phys. B 285, 65 (1988).
[31] G. Rupp, Phys. Lett. B 288, 99 (1992).
[32] B. Freedman, P. Smolensky, D. Weingarten, Phys. Lett. B 113, 481 (1982).
[33] F. Siringo and L. Marotta, Phys. Rev. D 74, 115001 (2006).