Magnetostatic Analogy of the Zero Energy State of Jackiw-Rebbi

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Abstract

We prepared a similarity between the Poisson equation in non-homogeneous magnetic material media and Dirac’s one-dimensional equation for the zero-energy state, which establishes a connection with the Jackiw-Rebbi model in one dimension for this same state.

Keywords — Transformation, analogy, Jackiw-Rebbi model, magnetostatic theory, quantum mechanics.

Introduction: The number of investigations that address how to achieve new forms of transition between classical mechanics and quantum mechanics has increased in the last decade, with mathematical analogies found between electromagnetic theory and quantum mechanics. Particular cases have developed within the classical framework that, under certain conditions, acquire a similarity and relationship with different quantum phenomena. These studies provide a manner to explore, at a macroscopic level, many quantum phenomena that are currently inaccessible in macroscopic quantum systems. At this point, the use of transformations that lead to a relationship between both theories is a very attractive proposal, as long as the transformations are made within a specific framework, established by the conditions that must be satisfied in the equation where the transformation is applied, such as e.g. the boundary conditions, among others.

The increase in the number of studies that address the problem of simulating relativistic quantum mechanics, using different physical platforms such as optical structures [1], metamaterials [2], and ion traps [3]. In the study of the classical-quantum analogies investigated so far, it appears that the most fruitful are given by the analogy between optics and quantum phenomena due to the duality between matter and optical. The study of quantum optical analogies is based on the formal similarity between the paraxial wave equation in dielectric media and the Schrödinger equation for a single particle [4]. Among the wide variety of quantum optical analogies, we can mention the Bloch oscillations, the Zener tunneling, the localization dynamics, the Anderson localization, the quantum Zeno effect, the Rabi flopping and the coherent population trapping. Consequently, this progress has led to investigate how relativistic systems can be imitated by optical waves or waves referring to potentials.

The Dirac equation is one of the fundamental equations in theoretical physics and fully explains quantum mechanics for elementary spin-1/2 particles [5] in the context of the Special Theory of Relativity, it also plays a key role in the explanation of many exotic physical phenomena such as the properties of graphene [6], topological insulators [7] and superconductors [8]. This equation was formulated by the physicist Dirac to exclude the negative values of energy and probability implicit in the Klein-Gordon equation [9], although, to his surprise, Dirac eventually discovered the existence of particles associated with negative energy called subsequently antiparticles. More recently, optical systems governed by the Dirac equation have been investigated experimentally such as the Klein Tunneling [10], Zitterbewegung [11] and...

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the Jackiw-Rebbi model [12]. To conclude, and eventually with what has been said so far, it should be clarified that the purpose of this contribution is to demonstrate that Magnetostatics can provide a laboratory tool where the physical phenomena described by the Dirac equation can be explored; in particular, we will show that the Poisson equation in non-homogeneous one-dimensional media can be mapped to the zero-energy eigenstate of the stationary one-dimensional Dirac equation with a scalar Lorentz potential (see section 1). By adopting magnetic permeability, we propose a magnetostatic experiment that simulates a historically important relativistic model known as the Jackiw-Rebbi model [12]; since its derivation, other models subsequent to Jackiw-Rebbi are the Ramajaran-Bell model [13], the massive Jackiw-Rebbi model [14], the coupled fermion-kink model [15], and the Jackiw-Rebbi model in distinct kinklike backgrounds [16]. The organization of this article is distributed as follows: in section 1, we present a revision of the Jackiw-Rebbi model and the one-dimensional Dirac equation. Section 2 introduces a transformation for the magnetic potential, in the case of a non-isotropic, non-homogeneous, linear magnetizable medium, in the one-dimensional Poisson equation that leads to a stationary Dirac equation. Section 3 exhibits a solution to the Dirac-type equation found in section 2 for some specific problems. Finally, the final sections contain the analysis and conclusions of our results.

1 Jackiw-Rebbi Model (zero-point energy)

The Jackiw-Rebbi model [12] describes a one-dimensional Dirac field coupled to a static background soliton field, it is known as one of the first theoretical descriptions of a topological insulator in which the zero-energy mode can be understood as the state of the edge or limit in a certain region. In particular, the Jackiw-Rebbi model has been studied by Su, Shrieffer and Heeger in the continuum polyacetylene limit [17]. The one-dimensional stationary equation of Dirac in the presence of an external field \( \varphi(x) \) with units \( \hbar = c = 1 \) is given in the following expression:

\[
\hat{H}_D \Psi(x) = [\sigma_y \hat{p} + \sigma_x \varphi(x)] \Psi(x) = \varepsilon \Psi(x) \quad (1)
\]

Where,

\[
\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Psi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}
\]

Using the matrices of Pauli [18] \( \sigma_y \) and \( \sigma_z \) to obtain a real two-component spinor \( \Psi(x) \). This Hamiltonian of Dirac (1) also has a chiral symmetry defined by the operator \( \sigma_z \) (Pauli’s matrix), which anticommutes with hamiltoniante of Dirac\( \{\hat{H}_D, \sigma_z\} = 0 \). The chiral symmetry implies that the eigenstates come in pairs with positive and negative energy \( \pm \varepsilon \), respectively. However, it is possible for an eigenstate to be its own partner for \( \varepsilon = 0 \), i.e., that it acquires positive and negative values to be its own pair, if this is the case, then the state is topologically protected. The eigenstate of resulting zero energy is protected by the topology of the scalar field,

whose existence is guaranteed by the index theorem, which is localized around the soliton (non-linear wave that propagates without deforming) [12]. The Jackiw-Rebbi model uses \( \varphi(x) = m \tanh(x\lambda) \) for the external scalar field, with \( m > 0 \) and \( \lambda > 0 \). For simplicity, we will consider an external scalar field given by:

\[
\varphi(x) = m \frac{x}{|x|} \quad (2)
\]

Forming a domain wall at \( x = 0 \) where \( \varphi(x = 0) = 0 \). The scalar field given by equation (2) is a simplification of the Jackiw-Rebbi model first proposed by Rajaraman-Bell [13]. The precise form of the external scalar potential is not important if it approaches asymptotically an opposite sign at \( x \to \pm \infty \). The own function may change corresponding to a particular form of the external scalar potential, but the existence of the zero-energy state is determined solely by the fact that the mass is positive on one side and negative on the other one. Therefore, the mass of the fermion makes the potential asymmetric so that its asymptotic values in two spatial infinities are different [14]. Therefore, the solution is very robust against
Figure 1: Shows the external scalar potential \( \varphi(x) \) which changes the sign at the interface \( x = 0 \).

The external scalar potential. The solution of the Dirac equation (1) at exactly zero energy for the scalar field (2) is obtained by solving the following equation:

\[
\begin{pmatrix}
0 & -\partial_x + \varphi(x) \\
\partial_x + \varphi(x) & 0
\end{pmatrix}
\begin{pmatrix}
\psi_1(x) \\
\psi_2(x)
\end{pmatrix} = 0
\]

Which gives

\[
\psi_i(x) = C_{1,2} \exp\{m |x|\}_{1,2} \quad i = 1, 2
\]

Where \( C_{1,2} \) is the normalization constant. If the function is normalized (3), it is evident that \( C_2 \) must tend to zero to avoid problems of divergence. Next, Figure 2 shows the wave function for the zero-energy state of the Jackiw-Rebbi model.

Figure 2: Jackiw-Rebbi zero-energy mode given by Equation (3) for the external scalar field depicted in figure 1, the zero-energy state is localized around the interface \( x = 0 \).

## 2 Magnetostatic Analog of the Jackiw-Rebbi Model

As an extension to the study of the analogies found by González [19] and Rokaj [20], this section will establish a relationship between the Poisson one-dimensional equation in non-homogeneous, non-isotropic magnetizable media and Dirac’s one-dimensional equation for zero-point energy. It is very likely that this relationship can also be obtained for the electrostatic case, but we are only interested in analyzing the magnetostatic case in this article.
Considering that now the magnetic permeability is a function that depends on the position, for the one-dimensional case the Poisson equation must be expressed as:

\[
\frac{d}{dx} \left( \mu(x) \frac{dA_z}{dx} \right) = -J_z(x) \quad (4)
\]

\[
\frac{d^2 A_z}{dx^2} - \frac{\mu'(x)}{\mu(x)} \frac{dA_z}{dx} = -\mu(x) J_z(x) \quad (5)
\]

Applying in the equation (5) the transformation (6), which establishes a distribution of magnetostatic fields different from those proposed by Rokaj in [20], in the following way:

\[
A_z(x) = A_0 \ln \left( \frac{\psi_1(x)}{\nu} \right) \quad (6)
\]

Where \( A_0 \) (constant magnetostatic potential) and \( \nu \) are constants that ensure the dimensionality, and \( \psi_1(x) \) an arbitrary function:

\[
\frac{dA_z}{dx} = A_0 \frac{\psi_1'(x)}{\psi_1(x)} \quad \frac{d^2 A_z}{dx^2} = A_0 \frac{\psi_1''(x)}{\psi_1(x)} - A_0 \left( \frac{\psi_1'(x)}{\psi_1(x)} \right)^2
\]

Assuming a magnetostatic field (figure ??) that behaves in the following way:

\[
B^2 = B \cdot B = -A_0^2 \left( \frac{\psi_1'(x)}{\psi_1(x)} \right)^2 \quad B_y(x) \approx \frac{dA_z}{dx} = -A_0 \frac{\psi_1'(x)}{\psi_1(x)} \quad (7)
\]

Figure 3: Magnetostatic configuration with a surface current \( \kappa = \mp \kappa \hat{z} \) [21], [22]. Between the plates, there is a linear, non-homogeneous, non-isotropic, magnetizable material.

\[
A_0 \frac{\psi_1''(x)}{\psi_1(x)} - \frac{B^2}{A_0} + \frac{\mu'(x)}{\mu(x)} B_y = -\mu(x) J_z(x) \quad (8)
\]

Relating equation (7) with equation (5) and entering the resulting term in (8):

\[
A_0 \frac{\psi_1''(x)}{\psi_1(x)} - \frac{B^2}{A_0} + \frac{\mu'(x)}{\mu(x)} B_y = -\mu(x) \left( \frac{\mu'(x)}{\mu(x)} B_y - \frac{1}{\mu(x)} \frac{dB_y}{dx} \right) \quad \frac{dA_z}{dx} = \frac{B^2}{A_0} = 0
\]
\[
\frac{d^2 \psi_1 (x)}{dx^2} + \frac{1}{A_0} dB_y \psi_1 (x) - \frac{B^2}{A_0^2} \psi_1 (x) = 0
\]  \hspace{1cm} (9)

A brief caveat is that equation (9) does not depend on magnetic permeability. Adding and subtracting \( B_y \psi_1 (x) / A_0 \) = 0 in it we have:

\[
\frac{d^2 \psi_1 (x)}{dx^2} + \frac{1}{A_0} dB_y \psi_1 (x) - \frac{B^2}{A_0^2} \psi_1 (x) + B_y \psi_1 (x) / A_0 - B_y \psi_1 (x) / A_0 = 0
\]

In the limit, where \( \epsilon \) will represent the zero-point energy:

\[
\lim_{\epsilon \to 0} \left\{ \frac{d}{dx} \left[ \psi_1^{(x)} (x) + B_y \psi_1 (x) / A_0 \right] - \frac{B^2}{A_0^2} \psi_1 (x) - B_y \psi_1 (x) / A_0 + \epsilon^2 \psi_1 (x) \right\} = 0 \hspace{1cm} (10)
\]

Where \( \epsilon \) is an auxiliary constant to be determined. If the substitution \( \psi_1^{(x)} (x) + B_y \psi_1 (x) / A_0 = \epsilon \psi_2 (x) \) is made in equation (10), we find:

\[
\epsilon \psi_2^{(x)} (x) - \frac{B^2}{A_0^2} \psi_1 (x) - B_y \psi_1 (x) / A_0 + \epsilon^2 \psi_1 (x) = 0 \hspace{1cm} (11)
\]

\[
\psi_2^{(x)} (x) + B_y \psi_2 (x) / A_0 = \epsilon \psi_1 (x) \hspace{1cm} (12)
\]

Being (11) and (12) two coupled differential equations that can be rewritten similarly (mathematically) to the one-dimensional stationary equation of Dirac (1), with units \( h = c = 1 \), such that:

\[
\hat{H}_D \Psi (x) = \left[ \sigma_y \sigma + \sigma_x \left( B_y (x) / A_0 \right) \right] \Psi (x) = \epsilon \Psi (x) \hspace{1cm} (13)
\]

In this way, a solution to equation (13) is proposed, which is related to the solution (3) due to the existence of a magnetostatic field \( B_y (x) \) that produces a behavior homologous to that established by the function of Jackiw-Rebbi in said solution (see Sect. 1). That is, it shows how the Jackiw- Rebbi zero energy state can be generated at the interface of two magnetic materials separated by an infinite plate with a surface current density. Analogous to this case, we found that the use of infinite charged plates to emulate physical systems has been widely used for a variety of applications, such as a simple parallel plate capacitor [23] or in the one-dimensional study of Coulomb gas [24].

Equation (13) can be simplified into two Schrödinger uncoupled equations \( \hat{H}_i \psi_i = 0 \) for \( i = 1, 2 \).

\[
\hat{H}_i \psi_i = \left( \frac{\partial^2}{\partial x^2} + U_i (x) \right) \psi_i = 0
\]

Where,

\[
U_{1,2} (x) = \left[ - \left( \frac{B_y (x)}{A_0} \right)^2 + \frac{\epsilon^2}{A_0^2} \right] + \frac{1}{A_0} dB_y
\]

In this way, it is observed that \( \hat{H}_{1,2} \) are supersymmetric Hamiltonians that can be factored as \( \hat{H}_1 = \hat{A} \hat{A} - \epsilon^2 \) and \( \hat{H}_2 = \hat{A} \hat{A}^\dagger - \epsilon^2 \), where \( \hat{A} = \left( \sigma_y \sigma + \frac{B_y (x)}{A_0} \right) \) and \( \hat{A}^\dagger = \left( - \sigma_y + \frac{B_y (x)}{A_0} \right) \) are analogous to ladder operators. Likewise, the eigenvalue of the zero-energy mode is constructed by setting \( \epsilon = 0 \) in equation (13), and by solving the first-order differential equations uncoupled for \( \psi_{1,2}^{(x)} (x) \) (two-component constant spinor):

\[
\psi_{1,2}^{(x)} (x) = \psi_{1,2}^{(x)} \exp \left\{ \int \left( \frac{B_y (x)}{A_0} \right) dx \right\}_{1,2}
\]

Assuming a magnetostatic configuration described in Figure 3,
\[ B_y (x) = \begin{cases} \frac{-\mu_2}{2} \hat{y}, & \text{for } x < 0 \\ \frac{\mu_1}{2} \hat{y}, & \text{for } x > 0 \end{cases} \] (15)

The magnetostatic field given by equation (15) is simple to obtain and can be found in reference [21], having as a basis the configuration of figure ??.. Curiously, this field has the same form of the external scalar field given by equation (2), which allows the existence of the own state of zero energy in the Jackiw-Rebbi model. The magnetostatic vector potential given by the field (15) is determined by solving (7) in the following way:

\[ A_z = -\int_0^x B_y (x) \, dx = \begin{cases} \frac{\mu_2}{2} \hat{y}, & \text{for } x < 0 \\ -\frac{\mu_1}{2} \hat{y}, & \text{for } x > 0 \end{cases} \] (16)

Normalizing the solution function (14) and introducing (16) into it,

\[ |\psi_i(x)| = |v_{1,2}|^2 \int_0^\infty \exp \left( \frac{2\mu_1 k^2}{A_0} x \right) dx = 1 \]

\[ |v_{1,2}|^2 \frac{A_0}{\mu_1 k^2} \left( e^\infty - e^0 \right) = 1 \]

For the above function to be normalizable, it must be fulfilled that \( \nu_2 = 0 \), resulting the value of \( \nu_1 \) to be:

\[ |\nu_1| = \sqrt{\frac{(\mu_1 + \mu_2) k}{A_0}} \] (17)

Replacing the normalization constant (17) in (14) and adding the magnetostatic potential (16), we obtain a proper function for the zero mode of the Dirac equation (13), given by:

\[ \Psi (x) = \sqrt{\frac{(\mu_1 + \mu_2) k}{A_0}} \left( \begin{array}{c} e^{\frac{4x}{A_0}} \\ 0 \end{array} \right) \] (18)

3 Results

To check whether the proper function (18) models its own function for the zero-energy state of the Jackiw-Rebbi model, initially, the own function (18) is simulated for different values of magnetic permeability, starting with values close to the characteristic ones in superparamagnetic materials, and later of the order of the ferromagnetic ones. When simulating the own solution function in Figure 4, some constant parameters \( A_0 = 3; k = 1 \), in Figure 20, \( A_0 = 2000; k = 1 \), are taken as constant parameters, where the magnetostatic potential must be \( A_0 \geq 3 \) and \( A_0 \geq 2000 \) so that the values acquired in each figure by the own solution function (18) agree with the normalization.

Effectively, in both figures (4 and 5) it is observed that the proper function (18) is located around the zero on the x axis, in the same way as the own wave function of the own state of zero energy of the Jackiw-Rebbi model in Figure 2. Now, undoubtedly, Figure 5 presents a greater similarity with Figure 2; however, it has higher intensities because the magnetostatic field is reinforced by the strong magnetization of the medium used between the plates.

Opposite case is Figure 4, where the material used does not have such a large magnetization, due to this, characteristic behavior curves are obtained in each medium employed. It is noteworthy to observe that the closer the relative magnetic permeability constants, i.e., \( \mu_1 \approx \mu_2 \) the greater the similarity with the Jackiw-Rebbi model in Figure 2.
Figure 4: The proper function (18) for a one-dimensional, non-homogeneous medium with constants $\mu_1 = 1$ and $\mu_2 = 2$ [25].

Figure 5: The proper function (18) for a one-dimensional, non-homogeneous medium with constants $\mu_1 = 983$ and $\mu_2 = 1043$ [26].
4 Conclusions

By using a non-homogeneous medium, it was shown that the one-dimensional Poisson equation can be used to simulate the Jackiw-Rebbi model in one dimension for the eigenstate of zero energy. As a characteristic of this finding, it is also concluded that the magnetostatic potential must be proportional to the sum of the relative magnetic permeability constants of the means used. Finally, based on the findings of this paper, a magnetostatic platform has been introduced to replicate the zero-energy state of the Jackiw-Rebbi model, thus allowing a laboratory test.

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