ON REAL STRUCTURES OF RIGID SURFACES

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Abstract. We construct several rigid (i.e., unique in their deformation class) surfaces which have particular behavior with respect to real structures: in one example the surface has no any real structure, in the other one it has a unique real structure and this structure is not maximal with respect to the Smith-Thom inequality. So, it answers in negative to the following problems: existence of real surfaces in each complex deformation class and existence of maximal surfaces in each complex deformation class containing real surfaces. Besides, we prove that there is no real surfaces among the surfaces of general type with $p_g = q = 0$ and $K^2 = 9$.

The surfaces constructed provide new counter-examples to the “Dif=Def” problem.

§0. Introduction.

One of the principal settings in real algebraic geometry is to fix a deformation class of complex varieties and to study, inside this class, the varieties which can be equipped with a real structure (and then investigate their topological, as well as other invariant under real deformations, properties). Those, which are maximal with respect to the Smith-Thom bound, are of a special interest (since they have spectacular topological properties, see, for example, the survey [DK]; for surfaces this bound is reproduced below in Section 5). Thus, two natural questions arise: does any complex deformation class of compact complex varieties contain a real variety; and does any complex deformation class containing real varieties contain a maximal one? Up to our knowledge, in dimension $\geq 2$ the both questions remained open till now. We show that the response to the both questions is in negative. In

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our examples the varieties are surfaces which are rigid, where the latter means that
the moduli space of complex structures on the underlying smooth manifold is 0-
dimensional. Moreover, in our examples they are strongly rigid, i.e., the quotient of
the moduli space by the canonical complex conjugation (which replaces a complex
structure of the surface by the complex conjugated one, and thus holomorphic func-
tions by anti-holomorphic ones; the orientation of the underlying smooth 4-manifold
is preserved) is merely a point. It is worth noticing that in the first of our examples
the moduli space consists of two conjugated points, in the second one it reduces to
one real point, see the remarks in Section 4. Besides, the two conjugated surfaces
in the first example give one more counterexample to ”Dif=Def” problem (earliest
counterexamples were constructed by Manetti in [Ma]). In fact, in all our examples
the surfaces are of general type and with $c_1^2 = 3c_2$ (they are the so-called Miyaoka-
Yau surfaces; the fact that they are strongly rigid and, moreover, unique in their
homotopy type up to holomorphic and anti-holomorphic diffeomorphisms is well
known, see, for example, [BPV]). Following F. Hirzebruch [H] we construct such
rigid surfaces as (finite abelian) Galois coverings of the (blown-up) projective plane
branched along arrangements of lines. We start from giving in Sections 1 and 2
their explicit construction via the orbit spaces of the Ferma covering. In Section 3
we study the group of automorphisms and anti-automorphisms of the constructed
surfaces. In Section 4 three main examples are treated. In Section 5 we consider
fake projective planes (that is, the surfaces of general type with $c_2 = 3$ and $c_1^2 = 9$).
We prove that they have no anti-holomorphic diffeomorphisms and, in particular,
can not be equipped with a real structure. This section contains also several remarks
on other related topics.

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§1. Galois coverings of the plane branched over an arrangement of lines.

By a Galois covering of a smooth algebraic variety $Y$ we mean a finite morphism $f : X \to Y$ of a normal algebraic variety $X$ to $Y$ such that the function fields imbedding $\mathbb{C}(Y) \subset \mathbb{C}(X)$ induced by $f$ is a Galois extension. As is well known, a finite morphism $f : X \to Y$ is a Galois covering with Galois group $G$ if and only if $G$ coincides with the group of covering transformations and the latter acts transitively on every fiber of $f$. Besides, a finite branched covering is Galois if and only if the unramified part of the covering (i.e., the restriction to the complements of the ramification and branch loci) is Galois. In addition, a branched covering is determined up to isomorphism by its unramified part and, moreover, a covering morphism from the unramified part of one branched covering to the unramified part of another one induces a covering morphism between these branched coverings if the extension of the morphism of underlying varieties to the branch loci is given. Let us recall also that an unramified covering is Galois with Galois group $G$ if and only if it is a covering associated with an epimorphism of the fundamental group of the underlying variety to $G$, and, in particular, the Galois coverings with abelian Galois group $G$ are in one-to-one correspondence with epimorphisms to $G$ of the first homology group with integral coefficients. All these results are well known and their most nontrivial part can be deduced, for example, on the Grauert-Remmert existence theorem [G-R] (a detailed exposition of the basic results on branched coverings is found, f.e., in [N]).

In what follows we have deal only with coverings of the complex projective plane $\mathbb{P}^2$ ramified over an arrangement of lines $L = L_1 \cup \cdots \cup L_n$. Similarly to general abelian Galois coverings, a Galois covering $g : Y \to \mathbb{P}^2$ of $\mathbb{P}^2$ with abelian Galois group $G$ branched along $L$ is determined uniquely by an epimorphism $\varphi : H_1(\mathbb{P}^2 \setminus L, \mathbb{Z}) \to G$, and it exists for any such an epimorphism. Since $H_1(\mathbb{P}^2 \setminus L, \mathbb{Z}) \simeq$
\( \mathbb{Z}^{n-1} \), there exists, in particular, a covering \( g_u(m) : Y_u(m) \to \mathbb{P}^2 \) corresponding to the natural epimorphism \( \varphi : H_1(\mathbb{P}^2 \setminus L, \mathbb{Z}) \to H_1(\mathbb{P}^2 \setminus L, \mathbb{Z}/m\mathbb{Z}) = H_1(\mathbb{P}^2 \setminus L, \mathbb{Z}) \otimes (\mathbb{Z}/m\mathbb{Z}) \). We call it Fermat covering. The following statement is also an immediate consequence of the general results on branched coverings recalled in the beginning of this Section.

**Proposition 1.0.** If \( g : Y \to \mathbb{P}^2 \) is a Galois covering with Galois group \( G \simeq (\mathbb{Z}/m\mathbb{Z})^k \) branched along \( L \), then \( k \leq n-1 \) and for any epimorphism \( H_1(\mathbb{P}^2 \setminus L, \mathbb{Z}) \to G \) there exists a unique Galois covering \( f : Y_u(m) \to Y \) inducing this epimorphism and such that \( g_u(m) = g \circ f \).

In what follows we have deal with Galois coverings whose Galois group is \( G \simeq (\mathbb{Z}/m\mathbb{Z})^k \), and we construct them in a way described in the above proposition.

The simple loops \( \lambda_i, 1 \leq i \leq n \), around the lines \( L_i \) generate \( H_1(\mathbb{P}^2 \setminus L, \mathbb{Z}) \simeq \mathbb{Z}^{n-1} \). They are subject to the relation

\[
\lambda_1 + \cdots + \lambda_n = 0,
\]

and without loss of generality we can assume that the universal covering \( g_u(m) : Y_u(m) \to \mathbb{P}^2 \) is determined by the epimorphism \( \varphi : H_1(\mathbb{P}^2 \setminus L, \mathbb{Z}) \to (\mathbb{Z}/m\mathbb{Z})^{n-1} \) sending \( \lambda_n \) to \( (m-1, \ldots, m-1) \) and \( \lambda_i \) with \( 1 \leq i \leq n-1 \) to \( (0, \ldots, 0, 1, 0, \ldots, 0) \) with 1 in the \( i \)-th place. We choose an additional line \( L_\infty \subset \mathbb{P}^2 \) in general position with respect to \( L \) and introduce affine coordinates \( (x_1, x_2) \) in \( \mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty \). Let \( l_i(x_1, x_2) = 0 \) be a linear equation of \( L_i \cap \mathbb{C}^2 \). Put \( z_i = (l_i|_n^{m-1})^{1/m}, 1 \leq i \leq n-1 \).

Then the function field \( K_u(m) = \mathbb{C}(Y_u(m)) = \mathbb{C}(x_1, x_2, z_1, \ldots, z_{n-1}) \) of \( Y_u(m) \) is the abelian extension of the function field \( k = \mathbb{C}(x_1, x_2) \) of \( \mathbb{P}^2 \) of degree \( m^{n-1} \) with Galois group

\[
G = \{ \gamma = (\gamma_1, \ldots, \gamma_n) \in (\mathbb{Z}/m\mathbb{Z})^n \mid \sum \gamma_i \equiv 0 \pmod{m} \} \simeq (\mathbb{Z}/m\mathbb{Z})^{n-1}.
\]

(In other words, the pull-back of \( \mathbb{P}^2 \setminus L_\infty \) in \( Y_u(m) \) is naturally isomorphic to the normalization of the affine subvariety of \( \mathbb{C}^{n+1} \) given in coordinates \( x_1, x_2, z_1, \ldots, z_{n-1} \).)
by equations \( z_1^m = l_1 l_n^{m-1}, \ldots, z_n^m = l_{n-1} l_n^{m-1} \).

For a multi-index \( a = (\alpha_1, \ldots, \alpha_{n-1}) \), \( 0 \leq \alpha_i \leq m - 1 \), we put

\[
  z^a = \prod_{i=1}^{n-1} z_i^{\alpha_i}.
\]

The action of \( \gamma = (\gamma_1, \ldots, \gamma_n) \in G \) on \( K_{u(m)} \) is given by

\[
  \gamma(z^a) = \mu^{(\gamma, a)} z^a,
\]

where

\[
  (\gamma, a) = \sum_{j=1}^{n-1} \gamma_j \alpha_j
\]

and \( \mu = e^{2\pi i/m} \) is the \( m \)-th root of the unity. Thus,

\[
  K_{u(m)} = \bigoplus_{0 \leq \alpha_i \leq m-1} \mathbb{C}(x_1, x_2) z^a
\]

is a decomposition of the vector space \( K_{u(m)} \) over \( \mathbb{C}(x_1, x_2) \) into a finite direct sum of degree 1 representations of \( G \).

Let \( \varphi : H_1(\mathbb{P}^2 \setminus L, \mathbb{Z}) \to (\mathbb{Z}/m\mathbb{Z})^k \) be an epimorphism given by \( \varphi(\lambda_i) = (\alpha_{i,1}, \ldots, \alpha_{i,k}) \), where \( \alpha_{1,j} + \cdots + \alpha_{n,j} \equiv 0 \mod m \) for every \( j = 1, \ldots, k \), and let \( g : Y \to \mathbb{P}^2 \) be the corresponding Galois covering. Then, by Proposition 1.0, there exists a unique Galois covering \( f : Y_{u(m)} \to Y \). It determines the inclusion \( f^* : \mathbb{C}(Y) \to K_{u(m)} \) of the function field \( \mathbb{C}(Y) \) of \( Y \) into the function field \( K_{u(m)} = \mathbb{C}(Y_{u(m)}) \). Clearly, \( \mathbb{C}(Y) \) is the subfield \( K_\varphi = \mathbb{C}(x_1, x_2, w_1, \ldots, w_k) \) of \( K_{u(m)} \), where \( w_j = z_1^{\alpha_{1,j}} \cdots z_{n-1}^{\alpha_{n-1,j}} \), and

\[
  \text{Gal}(K_{u(m)}/K_\varphi) = \{ (\gamma_1, \ldots, \gamma_n) \in G \mid \sum_{i=1}^{n-1} \alpha_{i,j} \gamma_i \equiv 0 \mod m, 1 \leq j \leq k \}.
\]

By construction, \( Y \) is a normal surface with isolated singularities. The singular points of \( Y \) can appear only over the \( r \)-fold points of \( L \) with \( r \geq 2 \), i.e., over points lying on \( r \) lines \( L_{i_1}, \ldots, L_{i_r} \) of the arrangement.

In what follows we call \( r \) elements of \((\mathbb{Z}/m\mathbb{Z})^k\) linear independent over \( \mathbb{Z}/m\mathbb{Z} \) if they generate in \((\mathbb{Z}/m\mathbb{Z})^k\) a subgroup isomorphic to \((\mathbb{Z}/m\mathbb{Z})^r\) (and thus admitting \((\mathbb{Z}/m\mathbb{Z})^{k-r}\) as its complement).
Lemma 1.1. Let $p$ be a 2-fold point of $L$ and $\varphi(\lambda_{i_1})$ and $\varphi(\lambda_{i_2})$ are linear independent over $\mathbb{Z}/m\mathbb{Z}$ in $(\mathbb{Z}/m\mathbb{Z})^k$. Then the surface $Y$ is non-singular at each point of $f^{-1}(p)$.

Proof. Let $p = L_{i_1} \cap L_{i_2}$. Choose a small round neighborhood $U$ of $p$ in $\mathbb{P}^2$ and local analytic coordinates $y_1, y_2$ in $U$ such that $y_j = 0$ is an equation of $L_{i_j}$. Then, $H_1(U \setminus (L_{i_1} \cup L_{i_2}), \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}$. At any point $q \in g^{-1}(p)$ the germ $V \to U$ of the covering $Y \to \mathbb{P}^2$ is a $G'$-covering, where $G'$ is the image of $H_1(U \setminus (L_{i_1} \cup L_{i_2}), \mathbb{Z})$ under the composition $\varphi \circ i_*$ of $\varphi$ with the inclusion homomorphism $i_* : H_1(U \setminus (L_{i_1} \cup L_{i_2}), \mathbb{Z}) \to H_1(\mathbb{P}^2 \setminus L, \mathbb{Z})$. Moreover, this $G'$-covering is determined by $\varphi \circ i_*$. Identifying $\varphi(\lambda_{i_1})$, $\varphi(\lambda_{i_2})$ with the standard generators of $(\mathbb{Z}/m\mathbb{Z})^2$ we get an isomorphism between $V \to U$ and the covering determined by equations $z_1^m = y_1$, $z_2^m = y_2$. Thus, $V$ is nonsingular. □

In our further examples, to resolve the singularities of $Y$ over the $r$-fold points of $L$ with $r \geq 3$, we blow up all these points. Let $\sigma : \mathbb{P}^2 \to \mathbb{P}^2$ be this blow up, $L'_i$ the strict transform of $L_i$, $E_p$ the line blown up over a $r$-fold point $p$, and $\varepsilon_p \in H_1(\mathbb{P}^2 \setminus \sigma^{-1}(L), \mathbb{Z}) = H_1(\mathbb{P}^2 \setminus L, \mathbb{Z})$ a simple loop around $E_p$.

The identification $H_1(\mathbb{P}^2 \setminus \sigma^{-1}(L), \mathbb{Z}) = H_1(\mathbb{P}^2 \setminus L, \mathbb{Z})$ composed with $\varphi$ provides an epimorphism $\varphi : H_1(\mathbb{P}^2 \setminus \sigma^{-1}(L), \mathbb{Z}) \to (\mathbb{Z}/m\mathbb{Z})^k$. Let consider the associated Galois covering $f : X \to \mathbb{P}^2$.

The proof of the following statements is straightforward (to establish the relation given by the first statement it is sufficient to consider a generic line pencil; the second statement follows from Lemma 1.1).

Lemma 1.2. Let $p = L_{i_1} \cap \cdots \cap L_{i_r}$ be an $r$-fold point of $L$. Then $\varepsilon_p = \lambda_{i_1} + \cdots + \lambda_{i_r}$. □

Lemma 1.3. If for each $r$-fold point $p = L_{i_1} \cap \cdots \cap L_{i_r}$ of $L$ with $r \geq 3$ the pairs $\varphi(\varepsilon_p)$ and $\varphi(\lambda_{i_j})$, $1 \leq j \leq r$, are linear independent over $\mathbb{Z}/m\mathbb{Z}$ in $(\mathbb{Z}/m\mathbb{Z})^k$, then $X$ is nonsingular. □
As a consequence, the constructed surface $X$ is a resolution of singularities of $Y$. Indeed, the covering $f$ is included in the commutative diagram

$$
\begin{array}{c}
X \xrightarrow{\pi} Y \\
\downarrow f \downarrow g \\
\widetilde{\mathbb{P}^2} \xrightarrow{\sigma} \mathbb{P}^2.
\end{array}
$$

with a regular map $\pi$ (clearly, it is continuous, and thus its regularity follows, for example, from regularity on $X \setminus f^{-1}(\sigma^{-1}(L))$).

§2. $(\mathbb{Z}/5\mathbb{Z})^2$-Galois coverings branched over the arrangement of lines dual to the inflection points of a smooth cubic.

We use the notation of §1.

Let $L = L_1 \cup \cdots \cup L_9$ be an arrangement of nine lines in $\mathbb{P}^2$ dual to the nine inflection points of a smooth cubic $C$ in the dual plane. Let $t_r$ ($r \geq 2$) be the number of $r$-fold points of $L$, i.e., the number of points lying on exactly $r$ lines of the arrangement. As is well-known (and easy to check using the group law on the cubic), in this arrangement $t_3 = 12$, $t_r = 0$ if $r \neq 3$, and exactly four singular points of $L$ lie on each $L_i$, $1 \leq i \leq 9$. (Note that the arrangement of lines dual to the inflection points of a smooth cubic is rigid, i.e., any such arrangement can be transformed to another by a linear transformation of the projective plane.)

If $C$ is a cubic given by $x_1^3 + x_2^3 + x_3^3 = 0$, then the lines $L_1, \ldots, L_9$ are given by equations

$$
\begin{align*}
L_1 &= \{x_1 - x_3 = 0\}, \quad L_2 = \{x_1 - \mu^2 x_3 = 0\}, \quad L_3 = \{x_1 + \mu x_3 = 0\}, \\
L_4 &= \{x_2 - \mu^2 x_3 = 0\}, \quad L_5 = \{x_2 - x_3 = 0\}, \quad L_6 = \{x_2 + \mu x_3 = 0\}, \\
L_7 &= \{x_1 + \mu x_2 = 0\}, \quad L_8 = \{x_1 - \mu^2 x_2 = 0\}, \quad L_9 = \{x_1 - x_2 = 0\},
\end{align*}
$$

where $\mu = e^{\pi i/3}$.

The intersection of three distinct lines $L_i, L_j, L_k$ is nonempty if and only if $(i,j,k) \in T$, where

$$
T = \{(1,2,3), (4,5,6), (7,8,9), (1,4,7), (2,5,8), (3,6,9), \\
(1,5,9), (3,5,7), (1,6,8), (3,4,8), (2,4,9), (2,6,7)\}.
$$
Denote by \( p_{i,j,k}, (i, j, k) \in T \), the point of intersection of \( L_i, L_j, L_k \).

Consider a Galois covering \( g : Y \to \mathbb{P}^2 \) with Galois group \( G \simeq (\mathbb{Z}/5\mathbb{Z})^2 \) branched along \( L \) and determined by an epimorphism \( \varphi : H_1(\mathbb{P}^2 \setminus L, \mathbb{Z}) \to G \).

Denote by \( \sigma : \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2 \) the blow up with the centers at all the 3-fold points \( p_{i,j,k}, (i, j, k) \in T \), by \( E_{i,j,k} \) the exceptional divisor over \( p_{i,j,k} \), and by \( L_i' \) the strict transform of \( L_i \). Let \( \varepsilon_{i,j,k} \in H_1(\widetilde{\mathbb{P}}^2 \setminus \sigma^{-1}(L), \mathbb{Z}) \simeq H_1(\mathbb{P}^2 \setminus L, \mathbb{Z}) \) correspond to a simple loop around \( E_{i,j,k} \).

The epimorphism \( \varphi : H_1(\mathbb{P}^2 \setminus \sigma^{-1}(L), \mathbb{Z}) \to (\mathbb{Z}/5\mathbb{Z})^2 \) determines a Galois covering \( f : X \to \widetilde{\mathbb{P}}^2 \). Pose \( C_i = f^{-1}(L_i') \) and \( D_{i,j,k} = f^{-1}(E_{i,j,k}) \).

In what follows we assume that the epimorphism \( \varphi : H_1(\mathbb{P}^2 \setminus \sigma^{-1}(L), \mathbb{Z}) \to (\mathbb{Z}/5\mathbb{Z})^2 \) satisfies the following condition

\((S)\) \( \varphi(\varepsilon_{i_1,i_2,i_3}) \) and \( \varphi(\lambda_{i_j}), j = 1, 2, 3, \) are linear independent over \( \mathbb{Z}/5\mathbb{Z} \) for each triple \((i_1, i_2, i_3) \in T\).

From this assumption it follows that \( f \) is ramified with ramification index 5 in each \( C_i \) and each \( D_{i,j,k} \). Further, according to Lemma 1.3, \( X \) is non-singular under this assumption.

**Lemma 2.1.** Under above assumptions

(i) \( C_i^2 = -3 \) for each \( i = 1, \ldots, 9 \);

(ii) \( D_{i_1,i_2,i_3}^2 = -1 \) for each \((i_1, i_2, i_3) \in T \);

(iii) \( K_X^2 = 333 \), where \( K_X \) is the canonical class of \( X \);

(iv) the geometric genera of \( C_i, 1 \leq i \leq 9 \), and \( D_{i_1,i_2,i_3}, (i_1,i_2,i_3) \in T \), are \( g(C_i) = 4 \) and \( g(D_{i_1,i_2,i_3}) = 2 \).

**Proof.** (i) There are 4 triple points of \( L \) on each \( L_i \). Thus, \( (L_i', L_i') = -3 \). On the other hand

\[
\deg f \cdot (L_i', L_i') = (f^*(L_i'), f^*(L_i')) = (5C_i, 5C_i) = 25C_i^2.
\]

Therefore, \( C_i^2 = -3 \).
Proof of (ii) is similar to (i).

(iii) The 3-canonical class of $\tilde{\mathbb{P}}^2$ is $3K_{\tilde{\mathbb{P}}^2} = -\sum L_i$.

By the pull-back formula

$$K_X = f^*(K_{\tilde{\mathbb{P}}^2}) + 4(\sum C_i + \sum D_{i_1,i_2,i_3}),$$

and, hence,

$$(2.1) 3K_X = 7\sum C_i + 12\sum D_{i_1,i_2,i_3}.$$  

Thus, we have

$$9 \cdot K_X^2 = 49\sum C_i^2 + 144\sum D_{i_1,i_2,i_3}^2 + 168\sum (C_i, D_{i_1,i_2,i_3}) = 49 \cdot (-3) \cdot 9 + 144 \cdot (-1) \cdot 12 + 168 \cdot 4 \cdot 9.$$  

Therefore, $K_X^2 = 333$.

By (2.1),

$$(2.2) (C_i, K_X) = 9 \quad \text{and} \quad (D_{i_1,i_2,i_3}, K_X) = 3,$$  

and (iv) follows from the adjunction formula. $\square$

**Lemma 2.2.** $X$ is a surface of general type with ample canonical class.

**Proof.** According to the Moisheson-Nakai criterion it is sufficient to show that $(K_X, C) > 0$ for any algebraic curve $C \subset X$. It follows from (2.1) and (2.2) that $(K_X, C) \geq 0$ for any curve $C$. Assume that there is an irreducible curve $C$ such that $(K_X, C) = 0$. Then the intersection of $C$ and the effective divisor $3K_X = 7\sum C_i + 12\sum D_{i_1,i_2,i_3}$ is empty. Therefore the curve $\sigma(f(C))$ doesn’t meet any line $L_i, i = 1, \ldots, 9$. But it is impossible. $\square$
Lemma 2.3. The Euler characteristic \( e(X) \) of \( X \) is equal to 111, and, in particular, it satisfies the relation \( K_X^2 = 3e(X) \).

Proof. From \( e(\mathbb{P}^2) = 15 \) and \( e(L_i) = e(E_{i_1,i_2,i_3}) = 2 \) we deduce, by additivity of the Euler characteristic, that

\[
e(X) = 25e(\mathbb{P}^2 \setminus (\cup C_i \cup D_{i_1,i_2,i_3})) + 5\sum e(C_i \setminus \cup D_{i_1,i_2,i_3}) + \\
+ 5\sum e(D_{i_1,i_2,i_3} \setminus \cup C_i) + \sum (C_i, D_{i_1,i_2,i_3}) = \\
= 25(15 - 9 \cdot 2 - 12 \cdot 2 + 9 \cdot 4) + 5 \cdot 9(2 - 4) + 5 \cdot 12(2 - 3) + 9 \cdot 4 = 111.
\]

The relation \( K_X^2 = 3e(X) \) now follows from Lemma 2.1 (iii). □

Corollary 2.1. \( X \) is a strongly rigid surface (i.e., a surface whose moduli space reduces to \( X \) and \( \bar{X} \) or merely to \( X \), where \( \bar{X} \) stands for the complex conjugated surface). □

§3. Automorphisms of the coverings.

Let \( f : X \to \mathbb{P}^2 \) be a \((\mathbb{Z}/5\mathbb{Z})^2\)-Galois covering considered in §2. Denote by \( K_l \) the group of holomorphic and anti-holomorphic diffeomorphisms \( X \to X \). Clearly, if \( K_l \) contains at least one anti-holomorphic element, the holomorphic elements form in \( K_l \) a subgroup \( \text{Aut} \) of index 2. In other words, there is a short exact sequence \( 1 \to \text{Aut} \to K_l \to H \to 1 \), where \( H = \mathbb{Z}/2 \) or 0. We denote by \( k_l : K_l \to H \) the homomorphism of this sequence. Recall that, by definition, a real structure is an anti-holomorphic involution and note that \( H \) can be nontrivial even for varieties without real structure.

The group \( K_l \) acts most naturally on \( X \times \bar{X}, X \sqcup \bar{X} \) (\( \bar{X} \) is the surface complex conjugated to \( X \)), and the associated to them groups like \( \text{Div}, \text{Pic}, \) and \( H^* \), as well as on \( \mathbb{C}(X \times \bar{X}) \) and \( \mathbb{C}(X \sqcup \bar{X}) \) (where the latter is not a field, since \( X \sqcup \bar{X} \) is not reducible). There are different ways to extract from these actions an action of \( K_l \) extending the action of \( \text{Aut} \) on \( \mathbb{C}(X), \text{Div}(X), \text{Pic}(X), \) and \( H^*(X) \). We choose the
one which better fits to the needs of the present investigation. In addition, it is the one traditionally used in algebraic geometry.

To extend the action of Aut($X$) on $\mathbb{C}(X)$ to that of Kl($X$), we associate with an anti-holomorphic diffeomorphism $h$ the $\mathbb{C}$-anti-linear map $h^1 : \mathbb{C}(X) \to \mathbb{C}(X)$ defined by $h^1(f)(x) = \overline{f(h(x))}$, $f \in \mathbb{C}(X)$, $x \in X(\mathbb{C})$. The action $h^1$ on holomorphic differential forms is defined in a way that

$$h^1(df) = dh^1(f).$$

An anti-holomorphic diffeomorphism $h$ defines an action on Div($X$): if $C \in \text{Div}(X)$ is given by local equations $(U_\alpha, f_\alpha)$, then $h(C)$ is given by $(h^{-1}(U_\alpha), \overline{f_\alpha \circ h})$. We have

$$(3.1) \quad h^{-1}(\text{div } f) = \text{div } h^1(f), \; f \in \mathbb{C}(X).$$

According to (3.1), $h : \text{Div}(X) \to \text{Div}(X)$ induces an action $h^1 : \text{Pic}(X) \to \text{Pic}(X)$. Clearly, the canonical class $K_X \in \text{Pic}(X)$ is invariant under $h^1$ for any $h \in \text{Kl}$; here, and further, we put $h^1 = h^* \text{ for } h \in \text{Aut} X$. The intersection number is also preserved by any $h \in \text{Kl}$ (it is may be worth noting that the action on the Neron-Severi subgroup of $H^*(X)$ associated with $h^1 : \text{Pic}(X) \to \text{Pic}(X)$ is not the usual $h^* \text{ but } -h^*$, if $h \in \text{Kl} \setminus \text{Aut}$).

We say that $h \in \text{Kl}(X)$ is lifted from $\tilde{\mathbb{P}}^2$ if there exists $\tilde{h} \in \text{Kl} \tilde{\mathbb{P}}^2$ such that the following diagram is commutative

$$\begin{array}{ccc}
X & \xrightarrow{h} & X \\
\downarrow f & & \downarrow f \\
\tilde{\mathbb{P}}^2 & \xrightarrow{\tilde{h}} & \tilde{\mathbb{P}}^2.
\end{array}$$

**Proposition 3.1.** Every $h \in \text{Kl}(X)$ is lifted from $\tilde{\mathbb{P}}^2$. In particular, if $X$ has a real structure then for a proper chosen real structure of $\tilde{\mathbb{P}}^2$ the covering $f$ is defined over $\mathbb{R}$. 
Lemma 3.1. Let \( h \in \text{Kl}(X) \). Then \( h \) leaves fixed the sets \( \bigcup C_i \) and \( \bigcup D_{i_1,i_2,i_3} \).

Proof. Assume that \( h(C_{i_0}) \not\subset \bigcup C_i \) for some \( i_0 \). Then

\[
(h(C_{i_0}), \sum C_i) = a, a \geq 0.
\]

It follows from the difference of genera \( g(C_{i_0}) \neq g(D_{i_1,i_2,i_3}) \) that \( h(C_{i_0}) \neq D_{i_1,i_2,i_3} \).

Therefore,

\[
(h(C_{i_0}), \sum D_{i_1,i_2,i_3}) = b, b \geq 0.
\]

Since \( h^*(K_X) = K_X \), then by Lemma 2.1 and the adjunction formula,

\[
(h(C_{i_0}), K_X) = (C_{i_0}, K_X) = 9.
\]

Thus, in accordance with (2.1) and (2.2), we should have

\[
7a + 12b = 27
\]

for some non-negative integers \( a \) and \( b \), which is impossible.

The proof that \( h(D_{i_1,i_2,i_3}) \subset \bigcup (D_{i_1,i_2,i_3}) \) for every \( (i_1,i_2,i_3) \in T \) is similar. □

Proof of Proposition 3.1. The second statement is a straightforward consequence of the first one. To prove the latter it is sufficient to show that \( h \) acts on the fibers of \( f \), i.e., that for almost any \( p \in \mathbb{P}^2 \) one can find \( q \in \mathbb{P}^2 \) such that \( h(f^{-1}(p)) = f^{-1}(q) \).

Let us fix a point \( p_{i_0,j_0,k_0} \in \mathbb{P}^2 \). Since \( C_{i_0} \) and \( C_{j_0} \) meet \( D_{i_0,j_0,k_0} \), then \( h(C_{i_0}) \) and \( h(C_{j_0}) \) meet \( h(D_{i_0,j_0,k_0}) \). The curve \( C_{i_0} \) (respectively, \( C_{j_0} \)) intersects 3 other curves \( D_{i_r,j_r,k_r}, r = 1, 2, 3 \) (respectively, \( D'_{i_r,j_r,k_r}, r = 1, 2, 3 \)) distinct from \( D_{i_0,j_0,k_0} \). Thus, \( h(C_{i_0}) \) (respectively, \( h(C_{j_0}) \)) intersects each of \( h(D_{i_r,j_r,k_r}) \), \( r = 1, 2, 3 \) (respectively, \( h(D'_{i_r,j_r,k_r}) \), \( r = 1, 2, 3 \)).

By Lemma 3.1, \( h(C_{i_0}) = C_i \) and \( h(C_{j_0}) = C_j \) for some \( i \) and \( j \). We have

\[
\text{div } f^*(l_{i_0}^{-1}l_{j_0}^{-1}) = 5(C_{i_0} + \sum_{r=1}^{3} D_{i_r,j_r,k_r}) - 5(C_{j_0} + \sum_{r=1}^{3} D'_{i_r,j_r,k_r})
\]
and
\[
\text{div } f^*(l_i l_j^{-1}) = 5(h(C_{i_0}) + \sum_{r=1}^{3} h(D_{i_r,j_r,k_r})) - 5(h(C_{j_0}) + \sum_{r=1}^{3} h(D'_{i_r,j_r,k_r})).
\]

Therefore, there is a constant \( k_{i_0,j_0} \) such that
\[
(3.2) \quad h^i(f^*(l_i l_j^{-1})) = k_{i_0,j_0} f^*(l_{i_0} l_j^{-1}).
\]

Let us choose another point \( p'_{i_0,j_0,k_0} \in \mathbb{P}^2 \), \( p'_{i'_0,j'_0,k'_0} \in L'_{i_0} \cap L'_{j_0} \) and consider the curves \( C_{i'_0}, C_{j'_0} \), and their images \( h(C_{i'_0}) = C_{i'}, h(C_{j'_0}) = C_{j'} \). Arguing as above, we conclude that there exists a constant \( k_{i'_0,j'_0} \) such that
\[
(3.3) \quad h^i(f^*(l_{i'} l_{j'}^{-1})) = k_{i'_0,j'_0} f^*(l_{i'_0} l_{j'_0}^{-1}).
\]

Since every \( p \in \mathbb{P}^2 \setminus \cup D_{i_1,i_2,i_3} \) can be given as the intersection of fibers of two linear rational functions \( l_{i_0} l_j^{-1} \) and \( l'_{i_0} l_{j_0}^{-1} \), it follows from (3.2) and (3.3) that for any \( p \in \mathbb{P}^2 \setminus \cup D_{i_1,i_2,i_3} \) we have \( h(f^{-1}(p)) = f^{-1}(q) \) for some \( q \in \mathbb{P}^2 \). \( \square \)

§4. Three examples.

Example I. A non real rigid surface.

Let \( L = L_1 \cup \cdots \cup L_9 \) be an arrangement of nine lines in \( \mathbb{P}^2 \) dual to the nine inflection points of a smooth cubic \( C \) in the dual plane (see §2), and let \( f : X_1 \to \mathbb{P}^2 \) be the Galois covering associated with the epimorphism \( \varphi_1 : H_1(\mathbb{P}^2 \setminus L, \mathbb{Z}) \to (\mathbb{Z}/5\mathbb{Z})^2 \) given by
\[
\varphi_1(\lambda_1) = (1,1), \quad \varphi_1(\lambda_2) = (1,0), \quad \varphi_1(\lambda_3) = (1,1),
\]
\[
\varphi_1(\lambda_4) = (3,3), \quad \varphi_1(\lambda_5) = (3,0), \quad \varphi_1(\lambda_6) = (0,1),
\]
\[
\varphi_1(\lambda_7) = (0,1), \quad \varphi_1(\lambda_8) = (0,2), \quad \varphi_1(\lambda_9) = (1,1),
\]
see §1 (note that \( \sum \varphi_1(\lambda_i) = 0 \mod 5 \)).

Proposition 4.1. The surface \( X_1 \) is smooth and strongly rigid. The group \( K_l(X_1) \) coincides with the covering transformations group \( G = \mathbb{Z}/5 \times \mathbb{Z}/5 \). In particular,
there does not exist neither a real structure nor even an anti-holomorphic diffeomorphism on $X_1$.

Proof. The surface $X_1$ is smooth due to Lemma 1.3. According to Lemmas 2.1, 2.3 we have $K^2_{X_1} = 333$ and $e(X_1) = 111$, and the rigidity statement follows from Corollary 2.1.

Consider, now, any $c \in \text{Kl}(X_1)$. By Proposition 3.1, $c$ is lifted from $\widetilde{\mathbb{P}}^2$, i.e., there is $\tilde{c} \in \text{Kl}(\widetilde{\mathbb{P}}^2)$ such that $f \circ c = \tilde{c} \circ f$.

As in §1, consider affine coordinates $x_1, x_2$ in $\mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty$ and the linear equations $l_i(x_1, x_2) = 0$ of $L_i \cap \mathbb{C}^2$. The function field $\mathbb{C}(X_1)$ of $X_1$ is identified with the sub-field

$$K_{\varphi_1} = \mathbb{C}(x_1, x_2, w_1, w_2)$$

of $K_{u(5)}$, where $w_1 = l_1 l_2 l_3 l_4 l_5 l_9$ and $w_2 = l_1 l_3 l_4 l_6 l_7 l_8 l_9$, so that

$$(4.1)$$

$$K_{\varphi_1} = \bigoplus_{a \in \text{pr} A_1} \mathbb{C}(x_1, x_2) z^a$$

is a subspace of the vector space

$$K_{u(m)} = \bigoplus_{a \in \text{pr} A} \mathbb{C}(x_1, x_2) z^a$$

over $\mathbb{C}(x_1, x_2)$, where

$$A = \{ \alpha = (\alpha_1, \ldots, \alpha_9) \in \mathbb{Z}^9 \mid 0 \leq \alpha_i \leq 4 \text{ and } \sum \alpha_i = 0 \mod 5 \},$$

pr : $A \mapsto \overline{A} \simeq (\mathbb{Z}/5\mathbb{Z})^8$ is the projection given by pr$(\alpha) = (\alpha_1, \ldots, \alpha_8)$ for $\alpha = (\alpha_1, \ldots, \alpha_9)$, and $A_1 \subset A$ consists from $0 = (0, 0, 0, 0, 0, 0, 0, 0, 0)$ and

$$(1, 1, 1, 3, 3, 0, 0, 0, 1), \ (2, 2, 2, 1, 1, 0, 0, 0, 2), \ (3, 3, 3, 4, 4, 0, 0, 0, 3), \ (4, 4, 4, 2, 2, 0, 0, 0, 4),$$
$$(1, 0, 1, 3, 0, 1, 1, 2, 1), \ (2, 0, 2, 1, 0, 2, 2, 4, 2), \ (3, 0, 3, 4, 0, 3, 3, 1, 3), \ (4, 0, 4, 2, 0, 4, 4, 3, 4),$$
$$(2, 1, 2, 1, 3, 1, 1, 2, 2), \ (4, 2, 4, 2, 1, 2, 2, 4, 4), \ (1, 3, 1, 3, 4, 3, 3, 1, 1), \ (3, 4, 3, 4, 2, 4, 4, 3, 3),$$
$$(3, 1, 3, 4, 3, 2, 2, 4, 3), \ (1, 2, 1, 3, 1, 4, 4, 3, 1), \ (4, 3, 4, 2, 4, 1, 1, 2, 4), \ (2, 4, 2, 1, 2, 3, 3, 1, 2),$$
$$(4, 1, 4, 2, 3, 3, 3, 1, 4), \ (3, 2, 3, 4, 1, 1, 1, 2, 3), \ (2, 3, 2, 1, 4, 4, 4, 3, 2), \ (1, 4, 1, 3, 2, 2, 2, 4, 1),$$
$$(0, 1, 0, 0, 3, 4, 4, 3, 0), \ (0, 2, 0, 0, 1, 3, 3, 1, 0), \ (0, 3, 0, 0, 4, 2, 2, 4, 0), \ (0, 4, 0, 0, 2, 1, 1, 2, 0).$$
The diffeomorphism $c$ induces an action $c^i$ on $\mathbb{C}(X_1)$ such that the restriction of $c^i$ to the subfield $\mathbb{C}(\overline{P}^2) = \mathbb{C}(\mathbb{P}^2)$ coincides with $\tilde{c}^i$ (see Section 3). By Lemma 3.1, the sets $\cup C_i$ and $\cup D_{i_1,i_2,i_3}$ are invariant under the action of $c$. Therefore the set $\cup L_4$ is invariant under the action of $\tilde{c}$. Thus, $c^i$ acts on the set of the one-dimensional subspaces $\mathbb{C}(x_1, x_2)z^a$, $a \in \text{pr} A_1$, of $K_{\varphi_1}$, and, thus, induces an action on $A_1$. We denote the latter action by $c^i$ also. For $a \in A_1$ denote by $r_i(a), i \in \mathbb{Z}/5\mathbb{Z}$, the number of coordinates of $a$ equal $i$.

**Lemma 4.1.** The functions $r_i$ are invariant under the action of $c^i$, i.e., $r_i(\alpha) = r_i(\beta)$ for $\beta = c^i(\alpha)$.

**Proof.** For each $j$, $1 \leq j \leq 9$ the coordinate $\alpha_j$ of $\alpha = (\alpha_1, \ldots, \alpha_9) \in A_1$ is congruent modulo 5 to the order of zero along $C_j$ of any of the functions in $C(x_1, x_2)z^a$, $a = \text{pr} \alpha$. It remains to note that due to Lemma 3.1 $c$ interchanges the curves $C_j$. \[ \square \]

By Lemma 4.1, the action of $c^i$ on $A_1$ is determined by a permutation $\pi$ of $1, \ldots, 9$.

Consider $\alpha = (1, 1, 1, 3, 3, 0, 0, 0, 1)$ and $\beta = (1, 0, 1, 3, 0, 1, 1, 2, 1)$. It is easy to see that $\alpha$ is the unique element in $A_1$ with $r_0 = 3$, $r_1 = 4$, $r_2 = 0$, $r_3 = 2$, $r_4 = 0$. Respectively, $\beta$ is the unique element in $A_1$ with $r_0 = 2$, $r_1 = 5$, $r_2 = 1$, $r_3 = 1$, $r_4 = 0$. Thus, by Lemma 4.1, $c^i(\alpha) = \alpha$ and $c^i(\beta) = \beta$. Since $r_2(\beta) = 1$ and $r_3(\beta) = 1$, we have $\tilde{c}(L_4) = L_4$ and $\tilde{c}(L_8) = L_8$. Further, $r_3(\alpha) = 2$ implies $\tilde{c}(L_5) = L_5$ and $r_0(\beta) = 2$ implies $\tilde{c}(L_2) = L_2$.

The above invariance properties of $L_2$, $L_4$, $L_5$, $L_8$ mean that these lines are invariant under the action of $\tilde{c}$. Hence, the points $p_{2,4,9} = L_2 \cap L_4$, $p_{2,5,8} = L_5 \cap L_8$, $p_{4,5,6} = L_4 \cap L_5$ and $p_{3,4,8} = L_4 \cap L_8$ are fixed points of $\tilde{c}$.

Since $r_0(\alpha) = 3$, there remain two possibilities: either $\tilde{c}(L_6) = L_7$ and $\tilde{c}(L_7) = L_6$, or $\tilde{c}(L_6) = L_6$ and $\tilde{c}(L_7) = L_7$. 
If \( \tilde{c}(L_6) = L_7 \) and \( \tilde{c}(L_7) = L_6 \), then their intersection point \( p_{2,6,7} \) is a fixed point. This is impossible. Indeed, in this case \( L_6 \) passes through two different fixed points \( p_{2,6,7} \) and \( p_{4,5,6} \), so should satisfy \( \tilde{c}(L_6) = L_6 \).

If \( L_6 \) and \( L_7 \) are invariant lines, then all lines \( L_i \) with \( 1 \leq i \leq 9 \), should be invariant. In fact, since \( L_5 \) and \( L_7 \) are invariant lines, their intersection point \( p_{3,5,7} \) is a fixed point. Then, \( L_3 \) is an invariant line, since \( L_3 \) passes through two fixed points \( p_{3,4,8} \) and \( p_{3,5,7} \). Therefore, the intersection points \( p_{3,6,9} \) of \( L_3 \) and \( L_6 \), \( p_{1,2,3} \) of \( L_2 \) and \( L_3 \), \( p_{1,4,7} \) of \( L_4 \) and \( L_7 \), and \( p_{7,8,9} \) of \( L_7 \) and \( L_8 \) are also fixed points. It implies, that \( L_1 \) and \( L_9 \), which go, respectively, through \( p_{1,2,3} \), \( p_{1,4,7} \) and \( p_{3,6,9} \), \( p_{7,8,9} \) are invariant under the action of \( \tilde{c} \). 0 As we have proved, the nine inflection points of \( C \), which is a smooth cubic, are fixed under the action induced on \( \mathbb{P}^2 \) by \( \tilde{c} \). Therefore, if \( \tilde{c} \in \text{Aut}(\mathbb{P}^2) \), then \( \tilde{c} = \text{Id} \) and, hence, \( c \) is a covering transformation. If \( \tilde{c} \notin \text{Aut}(\mathbb{P}^2) \), then \( \tilde{c}^2 \in \text{Aut}(\mathbb{P}^2) \) is the identity, and so \( \tilde{c} \) induces a real structure on \( \mathbb{P}^2 \) such that all inflection points of a smooth cubic \( C \) are real with respect to this structure, but it is impossible. \( \square \)

**Corollary 4.1.** The moduli space of complex structures on the underlying smooth 4-manifold consists of two distinct points \( X_1 \) and \( \bar{X}_1 \). In particular, \( X_1 \) and \( \bar{X}_1 \) give a counterexample to "Dif=Def" problem\(^1\).

**Remark 4.1.** One can deduce from Proposition 4.1 and the Mostow strong rigidity (using the Smith inequality for transformations of prime order and group cohomology arguments) that \( X_1 \) has no any nonidentical diffeomorphism of order \( \neq 5 \).

**Remark 4.2.** The irregularity of \( X_1 \) is equal to zero. It follows, for example, from [Is].

**Example II.** A non maximal rigid surface.

Now, suppose that the cubic \( C \) is given by \( x_1^3 + x_2^3 + x_3^3 = 0 \) and that the lines

\(^1\)First counter examples to "Dif=Def" problem were given by Manetti in [Ma].
$L_1, \ldots, L_9$ are numbered as in Section 2. In particular, under this choice, $L_1, L_5,$ and $L_9$ are real. Let $f : X_2 \to \mathbb{P}^2$ be the Galois covering associated with the epimorphism $\varphi_2 : H_1(\mathbb{P}^2 \setminus L, \mathbb{Z}) \to (\mathbb{Z}/m\mathbb{Z})^2$ given by $\varphi_2(\lambda_i) = (a_{i,1}, a_{i,2})$, where

\[
\begin{align*}
\varphi_2(\lambda_1) &= (0, 1), \quad \varphi_2(\lambda_2) = (1, 0), \quad \varphi_2(\lambda_3) = (1, 0), \\
\varphi_2(\lambda_4) &= (0, 1), \quad \varphi_2(\lambda_5) = (1, 0), \quad \varphi_2(\lambda_6) = (0, 1), \\
\varphi_2(\lambda_7) &= (1, 2), \quad \varphi_2(\lambda_8) = (1, 2), \quad \varphi_2(\lambda_9) = (0, 3).
\end{align*}
\]

**Proposition 4.2.** The surface $X_2$ is smooth and strongly rigid. It can be equipped with a real structure. Such a structure is unique, up to conjugation by covering transformations, and not maximal (where the latter means that $\sum \dim H_i(X_2(\mathbb{R}); \mathbb{Z}/2\mathbb{Z}) < \sum \dim H_i(X_2(\mathbb{C}); \mathbb{Z}/2\mathbb{Z})$). The group $\text{Kl}(X_2)$ is a semi-direct product of the group $\mu_2 \simeq \mathbb{Z}/2$ of order 2 and the covering transformations group $G \simeq \mathbb{Z}/5 \times \mathbb{Z}/5$. This $\mathbb{Z}/2$-extension is defined by relations $s\gamma s^{-1} = \gamma^{-1}, \gamma \in G, s \in \mu_2, s \neq 1$.

**Proof.** As in the proof of Proposition 4.1, $X_2$ is smooth due to Lemma 1.3. According to Lemmas 2.1, 2.3 we have $K_{X_2}^2 = 333$ and $e(X_2) = 111$, and the rigidity statement follows from Corollary 2.1.

As above, we identify the function field $\mathbb{C}(X_2)$ of $X_2$ with subfield $K_{\varphi_2} = \mathbb{C}(x_1, x_2, w_1, w_2)$ of $K_{u(5)}$, where $w_1 = l_2l_3l_5l_7l_8$ and $w_2 = l_1l_4l_6l_7^2l_8^3l_9$. Then

\[
K_{\varphi_2} = \bigoplus_{a \in \text{pr}A_2} \mathbb{C}(x_1, x_2)z^a
\]

is a subspace of the vector space

\[
K_{u(m)} = \bigoplus_{a \in \text{pr}A} \mathbb{C}(x_1, x_2)z^a
\]

over $\mathbb{C}(x_1, x_2)$, where $A$ and $\text{pr}$ are the same as in the previous example and where $A_2$ consists from $(0, 0, 0, 0, 0, 0, 0, 0, 0)$ and

\[
\begin{align*}
(0, 1, 0, 1, 0, 1, 1, 1, 0), & \quad (0, 2, 2, 0, 2, 0, 2, 2, 0), & \quad (0, 3, 3, 0, 3, 0, 3, 3, 0), & \quad (0, 4, 4, 0, 4, 0, 4, 4, 0), \\
(1, 0, 0, 1, 0, 1, 2, 2, 3), & \quad (2, 0, 0, 2, 0, 2, 4, 4, 1), & \quad (3, 0, 0, 3, 0, 3, 1, 1, 4), & \quad (4, 0, 0, 4, 0, 4, 3, 3, 2), \\
(1, 1, 1, 1, 1, 3, 3, 3), & \quad (2, 2, 2, 2, 2, 1, 1, 1), & \quad (3, 3, 3, 3, 3, 4, 4, 4), & \quad (4, 4, 4, 4, 4, 2, 2, 2), \\
(1, 2, 2, 1, 2, 1, 4, 4, 3), & \quad (2, 4, 4, 2, 4, 2, 3, 3, 1), & \quad (3, 1, 1, 3, 1, 3, 2, 2, 4), & \quad (4, 3, 3, 4, 3, 4, 1, 1, 2), \\
(1, 3, 3, 1, 3, 1, 0, 0, 3), & \quad (2, 1, 1, 2, 1, 2, 0, 0, 1), & \quad (3, 4, 4, 3, 4, 3, 0, 0, 4), & \quad (4, 2, 2, 4, 2, 4, 0, 0, 2), \\
(1, 4, 4, 1, 4, 1, 1, 1, 3), & \quad (2, 3, 3, 2, 3, 2, 2, 2, 1), & \quad (3, 2, 2, 3, 2, 3, 3, 4), & \quad (4, 1, 1, 4, 1, 4, 4, 4, 2).
\end{align*}
\]
Pose $\alpha = (0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0)$ and $\beta = (1, 0, 0, 1, 0, 1, 2, 2, 3)$ and consider any $c \in \text{Kl}(X_2)$, $c \neq \text{Id}$. The considerations as in the proof of Proposition 4.1 show that the line $L_9$ and each the unions $L_7 \cup L_8$, $L_1 \cup L_4 \cup L_6$, and $L_2 \cup L_3 \cup L_5$ are invariant under the action of $\tilde{c}$.

It is impossible that $\tilde{c}(L_7) = L_7$ and $\tilde{c}(L_8) = L_8$. In fact, otherwise, $\tilde{c}(p_{1,4,7}) = p_{1,4,7}$, since the arrangement $L_1 \cup L_4 \cup L_6 \cup L_7 \cup L_8$ has only two 3-fold points $p_{1,4,7}$ and $p_{1,6,8}$. It would imply $\tilde{c}(L_6) = L_6$, which together with $\tilde{c}(L_9) = L_9$ implies that $L_1$, and hence $L_4$ and subsequently all the lines, are invariant under the action of $\tilde{c}$, which contradicts to $\tilde{c} \neq \text{Id}$.

So, the only possibility is $\tilde{c}(L_7) = L_8$ and $\tilde{c}(L_8) = L_7$. Since the pair of the 3-fold points $\{p_{2,5,8}, p_{3,5,7}\}$ of $L_2 \cup L_3 \cup L_5 \cup L_7 \cup L_8$ is invariant under $\tilde{c}$, the line $L_5$ is invariant while $L_2$ and $L_3$ are permuted. The same arguments show that $\tilde{c}(L_1) = L_1$, $\tilde{c}(L_4) = L_6$, and $\tilde{c}(L_6) = L_4$.

Such an action of $\tilde{c}$ on $L = \cup L_i$ is the one induced by the standard complex conjugation on $\mathbb{RP}^2$ (see Section 2 or use the unicity) and thus coincides with it. It lifts to a real structure $s$ on $X_2$; in fact, $X_2$ can be seen as the minimal desingularization of the projective closure of the real surface given by equations

$$w_1^5 = (x_1^2 + x_1 + 1)(x_2 - 1)(x_1^2 + x_1 x_2 + x_2^2),$$
$$w_2^5 = (x_1 - 1)(x_2^2 + x_2 + 1)(x_1^2 + x_1 x_2 + x_2^2)^2(x_1 - x_2)^3.$$

This real surface is not maximal, since its real part is homeomorphic to $\mathbb{RP}^2$ with four blown up points (it is easy to check that there are only four real points among the blown up points $p_{i,j,k}, \{i, j, k\} \in T$). Since each $c \in \text{Kl}(X_2)$ is determined, up to composition with covering transformations, by $\tilde{c}$, the group $\text{Kl}(X_2)$ is generated by $s$ and the covering transformations. The commutation relations $s \gamma = \gamma^{-1}s$ follow from the above equations. These relations imply that each $s \gamma$ is a real structure and that these real structures are all equivalent. □
Remark 4.3. The surfaces in the both examples considered have the same $K^2$ and $e$. Thus, they belong to the same Hilbert scheme and provide an example of a Hilbert scheme whose connected components have different properties with respect to the existence of real structures on the surfaces representing these components. Note also that contrary to the first example in the second one the moduli space reduces to one point, which is real (and, moreover, corresponds to a surface with a real structure).

Example III. A rigid surface with two non-equivalent real structures.

Here, we call two structures equivalent if they can be transformed one into another by an automorphism of the surface.

Let $L = L_1 \cup \cdots \cup L_6$ be a complete quadrilateral. Note that two complete quadrilaterals are projectively equivalent. In this arrangement $t_2 = 3$, $t_3 = 4$, and $t_r = 0$ for $r \geq 4$. After suitable numbering, we can assume that the set of 2-fold points consists of $\{L_1 \cap L_4, L_2 \cap L_5, L_3 \cap L_6\}$ and the set of 3-fold points of $\{L_1 \cap L_2 \cap L_6, L_2 \cap L_3 \cap L_4, L_1 \cap L_3 \cap L_6, L_4 \cap L_5 \cap L_6\}$.

Let $f : X_3 \to \widetilde{\mathbb{P}}^2$ be the Galois covering associated with the epimorphism $\varphi_3 : H_1(\mathbb{P}^2 \setminus L, \mathbb{Z}) \to (\mathbb{Z}/5\mathbb{Z})^2$ given by $\varphi_3(\lambda_i) = (a_{i,1}, a_{i,2})$, where

$$\varphi_3(\lambda_1) = (1, 0), \quad \varphi_3(\lambda_2) = (1, 0), \quad \varphi_3(\lambda_3) = (1, 2),$$
$$\varphi_3(\lambda_4) = (0, 1), \quad \varphi_3(\lambda_5) = (0, 1), \quad \varphi_3(\lambda_6) = (2, 1),$$

and $\widetilde{\mathbb{P}}^2$ is the blow up of $\mathbb{P}^2$ at the 3-fold points of $L$. As above, denote by $\sigma : \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$ the blow up with the centers at the 3-fold points, by $E_{i,j,k}$ the exceptional divisor over the 3-fold point $p_{i,j,k}$, and by $L_i'$ the strict transform of $L_i$. Pose $C_i = f^{-1}(L_i')$ and $D_{i,j,k} = f^{-1}(E_{i,j,k})$.

As in §1, consider affine coordinates $x_1, x_2$ in $\mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty$ and the linear equations $l_i(x_1, x_2) = 0$ of $L_i \cap \mathbb{C}^2$. Then, by Lemma 1.3, $X_3$ is isomorphic to the
minimal desingularization of the projective closure of the surface given by equations

\begin{align}
(4.2) \quad w_1^5 &= l_1 l_2 l_3 l_6^2, \\
(4.3) \quad w_2^5 &= l_4 l_5 l_6.
\end{align}

The computations as in the proof of Lemmas 2.1 and 2.2 show that $X_3$ is a surface of general type with $K_{X_3}^2 = 45$ and $e(X_3) = 15$. Therefore $X_3$ is a strongly rigid surface.

**Lemma 4.3.** Let $h \in \text{Kl}(X_3)$. Then $h$ leaves fixed the set $(\cup C_i) \cup (\cup D_{i_1, i_2, i_3})$.

*Proof* is similar to that of Lemma 3.1.

**Proposition 4.3.** Every $h \in \text{Kl}(X_3)$ is lifted from $\overline{\mathbb{P}^2}$. In particular, if $X_3$ has a real structure then for a proper chosen real structure of $\overline{\mathbb{P}^2}$ the covering $f$ is defined over $\mathbb{R}$.

*Proof* is similar to that of Proposition 3.1.

**Proposition 4.4.** The surface $X_3$ can be equipped with 2 non-equivalent real structures.

*Proof.** Consider two real structures of $\mathbb{P}^2$. For the first one, all the lines of $L$ are real, and for the second one, the lines $L_3, L_6$ are real and the lines $L_1, L_2$, respectively $L_4$ and $L_5$, are complex conjugated. Then these two real structures induce two real structures on $X_3$, since in the both cases the polynomials in (4.2) and (4.3) are defined over $\mathbb{R}$.

These two real structures of $X_3$ are non-equivalent. Indeed, by Lemma 4.3, each automorphism of $X_3$ leaves fixed the set $(\cup C_i) \cup (\cup D_{i_1, i_2, i_3})$ while, on one hand, all the curves $C_i$ and $D_{i_1, i_2, i_3}$ are real with respect to the first real structure, but, on the other hand, only $C_3$ and $C_6$ (among $C_1 \ldots, C_6$) are real curves with respect to the second real structure.
§5. Non reality of fake projective plane and remarks.

A. We call a surface of general type with \( p_g = q = 0 \) and \( K^2 = 9 \) a fake projective plane. The existence of fake projective planes was proved by D. Mumford [Mu].

**Theorem 5.1.** A fake projective plane has no anti-holomorphic diffeomorphisms.

*Proof.* Let \( X \) be a fake projective plane. Then (see [Mi], [Y]), the universal covering space of \( X \) is a ball.

First, let us show that there is no an anti-holomorphic involution on \( X \).

So, assume that \( X \) can be equipped with a real structure and denote by \( X_\mathbb{R} \) the real point set of \( X \). According to the Lefschetz trace formula applied to the involution defining the structure, \( e(X_\mathbb{R}) = 1 \). Thus, \( X_\mathbb{R} \) is in nonempty and contains at least one component diffeomorphic either to sphere or real projective plane. To lift the real structure to the universal covering pick a point \( p \) on such a component and identify the points of the universal covering with homotopy classes of the paths starting at \( p \). The real part of the covering covers (without ramification) the chosen real component of \( X \). On the other hand, since the universal covering space is a ball, its real part has no compact components.

From the above it follows now that if there exists an anti-holomorphic diffeomorphism \( h \), then its order can not be \( 2n \), where \( n \) is odd. In fact, if \( n \) is odd, then \( h^n \) is the anti-holomorphic involution. Thus, Theorem 5.1 follows from the following Lemma.

**Lemma 5.1.** The group \( \text{Aut} \, X \) have no elements of even order.

*Proof of Lemma.* Assume that there is \( h \in \text{Aut} \, X \) of order 2. One dimensional components \( C \) of the fixed point set of \( h \) are nonsingular. By Enoki-Hirzebruch [BHH] relative proportionality,

\[
e(C) = 2C^2.
\]
Thus $C = \emptyset$, since otherwise $C^2 > 0$ and $e(C) < 0$ (the latter inequalities can be deduced, for example, from $C = rK$ with positive $r \in \mathbb{Q}$).

Since $\dim H^i(X, \mathbb{C}) = 1$ for $i = 0, 2, 4$ and 0 for $i = 1, 3$, the topological Lefschetz trace formula shows that the number of fixed points of $h$ should be equal to 3 for any nontrivial holomorphic automorphism without one dimensional components in the fixed point set. Next, applying the holomorphic Lefschetz formula to such a $h$ (of order 2), we get

$$\sum_{i=1}^{3} \frac{1}{\det(Id - D_i)} = 1,$$

where $D_i, i = 1, 2, 3$, are the Jacobi matrices of $h$ at its fixed points. On the other hand, $\det(Id - D_i) = 4$ at each fixed point, and thus the Lefschetz formula turns into $\frac{3}{4} = 1$, i.e., we get a contradiction, which proves the Lemma and finishes the proof of Theorem 5.1. \(\square\)

**Corollary 5.1.** For any fake projective plane $X$ the moduli space of complex structures on the underlying smooth 4-manifold consists of two distinct points $X$ and $\bar{X}$.

**B.** Arguments used in the proof of Theorem 5.1 to exclude anti-holomorphic involutions can be replaced by the following general result.

**Theorem 5.2.** If $X$ is a compact complex Kähler surface of negative sectional curvature, then for any real structure the real part of $X$ has no real component diffeomorphic to sphere, real projective plane, torus or Klein bottle.

**Proof.** Let $p : B \to X$ be the universal covering. According to Cartan-Hadamard theorem, $B$ is diffeomorphic to $\mathbb{R}^4$. Each connected component $M$ of the pull back $p^{-1}(F)$ of a real component $F$ of $X$ is a real component of some real structure on $B$. Hence, by Smith theorem, $M$ has the homology of a point and, hence, it is diffeomorphic to $\mathbb{R}^2$. This excludes sphere and real projective plane as $F$ and gives the injectivity of $\pi_1(F) \to \pi_1(X)$. It remains to note that $\pi_1(X)$, as
the fundamental group of a compact manifold of negative curvature, contains no
subgroup isomorphic to \( \mathbb{Z} \oplus \mathbb{Z} \), see [P]. □

It may be interesting to compare this observation with Kollár conjecture (and
Viterbo theorem, see [Kh]) according to which an algebraic variety of dimension
\( \geq 3 \) is of general type as soon as one of its real components, with respect to some
real structure, is hyperbolic.

C. Miyaoka-Yau surfaces can provide interesting examples related to the ”Rags-
dale bound”, i.e., examples of real surfaces \( X \) with \( \beta_1^R = \dim H_1(X_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) \) close
or above \( h^{1,1}(X) \). (First examples with \( \beta_1^R > h^{1,1}(X) \) were found in early 80th by
I. Itenberg [It].) Recall that a real surface \( X \) is called maximal (or \( M \)-surface), if
the Smith bound (see, f.e., the survey [DK])

\[
\sum \beta_i^R \leq \sum \beta_i^C = 2 + 4(h^{1,0} + \nu) + 2h^{2,0} + h^{1,1},
\]

where \( \nu \) is the rank of the 2-torsion in \( H_1(X; \mathbb{Z}) \) and \( \beta_i^C = \dim H_i(X; \mathbb{Z}/2\mathbb{Z}) \), turns
into equality.

By the Lefschetz formula, for any real surface

\[
\beta_0^R - \beta_1^R + \beta_2^R = 1 + \text{tr } \text{P}^{1,1},
\]

where \( \text{P}^{1,1} \) is the primitive part of \( H^{1,1} \) (which is, in fact, of codimension 1 in \( H^{1,1} \)).
Hence, for an \( M \)-surface

\[
\beta_1^R = 1 + 2(h^{1,0} + \nu) + h^{2,0} + p_{-1}^{1,1},
\]

where \( p_{-1}^{1,1} \) stands for the dimension of the anti-invariant part of the action of the
real structure in \( P^{1,1} \). On the other hand, for Miyaoka-Yau surfaces

\[
3(2 + 2h^{2,0} - h^{1,1}) = 2 - 4h^{1,0} + 2h^{2,0} + h^{1,1}
\]

and thus

\[
h^{1,1} = h^{2,0} + h^{1,0} + 1.
\]
Finally, for any maximal real Miyaoka-Yau surface

\[(5.5) \quad \beta_1^R = h^{1,1} + p_{-1}^{1,1} + h^{1,0} + 2\nu.\]

It implies that either for all maximal real Miyaoka-Yau surfaces with \(h^{1,0} = 0\) it holds \(p_{-1}^{1,1} = \nu = 0\) (which would be strange) or there are (maximal) real Miyaoka-Yau surfaces with \(h^{1,0} = 0\) and \(\beta_1^R > h^{1,1}\) (which is more plausible).

The next propositions show that if maximal real Miyaoka-Yau surfaces of general type exist their topology should be very restricted. Note also that Theorem 5.2 provides a below bound on \(|e(F) - 1|\) for the real components \(F\) of \(X\), while the more traditional results give the upper bounds of \(|e(X_\mathbb{R}) - 1|\) where \(X_\mathbb{R}\) is the whole real point set (see, f.e., the survey [DK]).

**Proposition 5.1.** There is no any maximal real Miyaoka-Yau surface with \(h^{2,0} \leq 3\).

*Proof.* Let \(X\) be a maximal real Miyaoka-Yau surface. Denote by \(k\) the number of connected components of \(X_\mathbb{R}\). By (5.2),

\[(5.6) \quad 2k - \beta_1^R = 1 + p_1^{1,1} - p_{-1}^{1,1}.\]

Substituting \(\beta_1^R\) from (5.5) into (5.6), and via (5.4), we have

\[(5.7) \quad 2k = h^{1,1} + h^{1,0} + 2\nu + p_1^{1,1} + 1.\]

By Theorem 5.2, \(\dim H_1(S;\mathbb{Z}/2\mathbb{Z}) \geq 3\) for any connected component \(S\) of \(X_\mathbb{R}\) (the theorem is applied, since the universal covering of \(X\) is a ball, see [Mi],[Y]). Therefore \(\beta_1^R \geq 3k\) and, by (5.2) and (5.7)

\[2p_{-1}^{1,1} \geq h^{1,1} + h^{1,0} + 2\nu + 3p_1^{1,1} + 3\]

or, equivalently,

\[(5.8) \quad h^{1,1} \geq h^{1,0} + 2\nu + 5p_1^{1,1} + 5.\]
and
\[ h^{2,0} \geq 2\nu + 5p_{+}^{1,1} + 4. \]
Therefore \( h^{2,0} \geq 4 \). □

**Proposition 5.2.** Let \( X \) be a maximal real Miyaoka-Yau surface. Then there are at least 3 connected components of \( X_\mathbb{R} \) diffeomorphic to a sphere with 3 points blown up.

**Proof.** Denote by \( k_3 \) the number of connected components of \( X_\mathbb{R} \) diffeomorphic to a sphere with 3 points blown up. Then, by Theorem 5.2, \( \dim H_1(S; \mathbb{Z}/2\mathbb{Z}) \geq 4 \) for all other connected components \( S \) of \( X_\mathbb{R} \). Therefore \( \beta_1^{\mathbb{R}} \geq 4k - k_3 \). It follows from (5.5) and (5.7) that in this case we should have the following inequality
\[
h^{1,1} + h^{1,0} + 2\nu + p_{-}^{1,1} \geq 2h^{1,1} + 2h^{1,0} + 4\nu + 2p_{+}^{1,1} + 2 - k_3
\]
which contradicts to \( h^{1,1} > p_{-}^{1,1} \) if \( k_3 < 3 \). □

**D.** The Miyaoka-Yau surfaces are quasi-simple in the following sense: two real structures of such a surface are conjugated by an automorphism, if and only if they are conjugated by a diffeomorphism.\(^2\) This follows from Mostow strong rigidity and the fact that the only isometry of a compact hyperbolic riemannian manifold acting identically on the fundamental group is the identity map. (Note, that two real structures are conjugated by an element of Aut as soon as they are conjugated by an element of Kl.)

**Added in proof.** Using the nonreal surface constructed in Section 4 or fake projective planes (see Section 5), one can obtain examples of varieties \( X \) of any dimension \( \geq 3 \) having the same property, i.e., examples such that \( X \) and \( \bar{X} \) belong to distinct connected components of the moduli space. It is sufficient to consider

\(^2\)In general, the real quasi-simplicity of a deformation class of varieties should mean that two real structures are real deformation equivalent, if and only if they are conjugated by a diffeomorphism.
products of these surfaces with tori. The statement on the components of the moduli space will then follow from the well known properties of the Albanese map and Siu’s rigidity theorem.

F. Catanese informed us that he also constructed examples of surfaces where the complex conjugation interchanges the components of their moduli space. His surfaces are covered by the bi-disc $D \times D \subset \mathbb{C}^2$.

References

[BPV] Barth W., Peters C., Van de Ven A., *Compact Complex Surfaces*, Springer-Verlag, 1984.
[BHH] Barthel G., Hirzebruch F., Höfer Th., *Geradenkonfigurationen und Algebraische Flächen*, Friedr. Vieweg & Sohn, 1987.
[DK] Degtyarev A., Kharlamov V., *Topological properties of real algebraic varieties*, Uspekhi Mat. Nauk 55 (2000), 129–212.
[G-R] Grauert H. and Remmert R., *Komplexe Räume*, Math. Ann. 136 (1958), 245-318.
[H] Hirzebruch F., *Arrangements of lines and algebraic surfaces*, Arithmetics and Geometry, vol. II, Prog. Math. 36, Birkhäuser, 1983, pp. 113-140.
[Is] Ishida M.-N., *The Irregularities of Hirzebruch’s Examples of Surfaces of General Type with $c_2^1 = 3c_2$*, Math. Ann. 262 (1983), 407–420.
[It] Itenberg I., *Contre-exemples à la conjecture de Ragsdale*, C.R.Acad.Sci. Paris 317 (1993), 277–282.
[Kh] Kharlamov V., *Variétés de Fano réelles*, Sém. Bourbaki exp. 872 (2000).
[Ma] Manetti M., *On the moduli space of diffeomorphic algebraic surfaces*, Invent. Math. 143 (1) (2001), 29–76.
[Mi] Miyaoka Y., *On algebraic surfaces with positive index*, Classification of algebraic and analytic manifolds, Prog. Math. 39, Birkhäuser, 1983, pp. 281-301.
[Mu] Mumford D., *An algebraic surface with $K$ ample, $K^2 = 9$, $p_g = q = 0*, Amer. J. Math. 101 (1979), 233–244.
[N] Namba M., *Branched coverings and algebraic functions*, Longman Scientific & Technical, 1987.
[P] Preissman A., *Quelques propriétés globales des espaces de Riemann*, Comment. Math. Helv. 15 (1943), 175–216.
[Y] Yau S-T., *Calaby’s conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. USA 74 (1977), 1798-1799.