1/2-BPS vortex strings in $\mathcal{N}=2$ supersymmetric $U(1)^N$ gauge theories

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Abstract

Strings in $\mathcal{N}=2$ supersymmetric $U(1)^N$ gauge theories with $N$ hypermultiplets are studied in the generic setting of an arbitrary Fayet-Iliopoulos triplet of parameters for each gauge group and an invertible charge matrix. Although the string tension is generically of a square-root form, it turns out that all existing BPS (Bogomol’nyi-Prasad-Sommerfield) solutions have a tension which is linear in the magnetic fluxes, which in turn are linearly related to the winding numbers. The main result is a series of theorems establishing three different kinds of solutions of the so-called constraint equations, which can be pictured as orthogonal directions to the magnetic flux in SU(2)$_R$ space. We further prove for all cases, that a seemingly vanishing Bogomol’nyi bound cannot have solutions. Finally, we write down the most general vortex equations in both master form and Taubes-like form.
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1 Introduction and summary

In string theory there exists two kinds of string, namely fundamental strings or F-strings and D1-branes or D-strings \([1–5]\). Bound states between those two kinds of string yield a very distinctive tension formula

\[
T \propto \sqrt{p^2 + q^2 / g_s^2},
\]

(1.1)

where \(p\) is the number of F-strings and \(q\) the number of D-strings and \(g_s\) is the string coupling constant. A simple gauge theory mimicking \((p, q)\)-string bound states was proposed by Saffin \([6]\), but the string tension is unfortunately linear in the two winding numbers:

\[
T \propto p + \mu q,
\]

(1.2)

with \(\mu > 0\) a positive constant. Jackson then found that the square-root tension formula actually exists naturally in \(\mathcal{N} = 2\) supersymmetric gauge theories \([7]\), where the strings are bound states of D-term strings and F-term strings and the tension is exactly like Eq. (1.1). This happy discovery came to an early halt, as Jackson showed in the same paper that no BPS\(^1\) solutions – with the square-root tension – seemed to exist.

The gauge theory that Jackson studied is a \(U(1)^N\) \(\mathcal{N} = 2\) supersymmetric gauge theory with \(N\) hypermultiplets. Not all the field content will be used in the construction of vortices, so therefore it is simply \(N\) \(U(1)\) gauge fields with \(N\) pairs of complex scalar fields. It holds true that if a flavor of scalar fields possesses winding (topological charge), then either of the two (complex)

\(^1\)BPS is an abbreviation for Bogomol’nyi-Prasad-Sommerfield who where the authors first discovering a reduction from second-order Euler-Lagrange equations for a magnetic monopole to first-order partial differential equations (PDEs), now known as BPS equations. BPS equations and solutions are now used in many areas of theoretical physics, from gauge theory to string theory.
scalar fields in the pair must vanish everywhere. This can readily be realized by analyzing the self-dual equations, which are equal for the two scalars in the pair, except for having opposite (electric) charges. We confirm this fact, but do not restrict to the case that all winding numbers must be nonzero, which relaxes this statement. In Jackson’s analysis, it was found that the magnetic fluxes should be proportional to the column-vectors of the inverse of the charge matrix. Unfortunately, this is impossible, as if true, then the determinant of the inverse of the charge matrix vanishes and hence the charge matrix does not exist! Another option was pointed out, however, which is to align all the vectors of Fayet-Iliopoulos (FI) parameters in SU(2)_R space (which roughly is the space containing the D-term in one direction and the F-term in further two directions). This option, however, does not possess the square-root tension, but the normal linear tension formula (1.2). We confirm this possibility as a solution type and call them solutions of type A. The final step in Jackson’s analysis concludes from analyzing the asymptotic behavior of the fields that the charge matrix must be diagonal in order to have BPS solutions. We do not agree with this conclusion and we are able to find BPS-saturated solutions with non-diagonal charge matrices.

We consider briefly the vacuum solutions (vacuum expectation values (VEVs)) of the more general theory with M hypermultiplets, but conclude that we either have vacuum moduli (for M > N) or unbroken gauge symmetry (for M < N), unless M = N^2. Hence this is the case we will consider in this paper. It also makes the charge matrix a square matrix, which when possessing a nonvanishing determinant, is invertible. We will make this a requirement throughout the paper. First, we will perform an SU(2)_R rotation of the fields to put the FI vectors on a standard form, so that as many parameters as possible vanish – without loss of generality. Then we will consider the most general supersymmetry projection for a string pointed in the spatial x^3 direction and this gives the BPS equations that we will study. We confirm these BPS equations by writing down the Bogomol’nyi completion, which in turn yields the Bogomol’nyi bound, which is saturated for BPS solutions. Curiously, it is not clear that the Bogomol’nyi bound is always nonvanishing. We will, however, prove for all types of solutions, that the vanishing Bogomol’nyi bound does not possess any BPS solutions. When working with the BPS equations, it turns out to be convenient to perform a change of basis to a diagonal basis, which we shall call the Ψ basis, as opposed to the canonical basis, which we call the Φ basis. In the Ψ basis, we confirm Jackson’s result that either of the two fields in a (hypermultiplet) pair must vanish when the winding number is nonzero, see lemma 1. The BPS equations are not one equation for the magnetic flux equating the square-root of the potential as usual in N = 1 gauge theories, but

2Actually, there is a vacuum modulus (a complex phase) for each gauge group in the case of M = N, which corresponds to the base point of the broken U(1) symmetry. This parameter has no physical relevance.
contain another two equations in the $\Psi$ basis, which we shall coin constraint equations. A main result is to split the solutions of the constraint equation into two different types, which we shall denote type A and type B, see proposition 1. In type A solutions, all the FI vectors are parallel (see theorem 1), but in type B at least one vector is not parallel with the others. A further constraint comes from the fact that there exists no holomorphic U(1) bundles of negative degree and hence if the VEV is positive, the “fundamental field” must carry the winding, but if it is negative, the “anti-fundamental fields” must carry the winding. This is summarized in theorem 2 for type A solutions. With this result in hand, we can now prove for type A solutions that no solutions exist in the case where the Bogomol’nyi bound vanishes, see theorem 3.

For type B solutions the FI vectors are not all parallel. This means that the product of some pair of fields does not vanish in the vacuum and hence by the above holomorphicity constraint (coming from the pair of self-dual equations), those pairs cannot possess winding – we shall dub them non-winding fields or inert fields. We find that there are two different solutions to the constraint equations in the type B case, see theorem 4 and one is given by allowing only a single field to wind. This will be the type B1 solution. The other option is to restrict the charge matrix to be of a block-diagonal form between the winding and the non-winding fields. Exactly this form will avoid the mixing of fields observed by Jackson. For the winding fields of type B2, everything is analogous to the solutions of type A, but we prove every step carefully. First, we find that the FI vectors coupled to the winding fields, must be parallel, see lemma 2. We further prove that the non-winding fields in type B2 satisfy both the constraint equations (lemma 3) and would-be vortex equations (theorem 5) by means of their vacuum solution. This is possible only because of the restricted form of the charge matrix. We find, without loss of generality, that we can rotate the basis and relabel the gauge groups to obtain parallel FI vectors with real vacuum solutions, see theorem 6 and corollary 5. At this point, the same argument as in the type A case can be used to relate the sign of the vacuum solution to the sign of the winding number, see theorem 7. In turn we can prove analogously that the vanishing Bogomol’nyi bound has no solutions, see theorem 8. For the solutions of type B1, we still have to set either of the fields in the hypermultiplet pair to zero by corollary 4, but otherwise have no further constraints. Because of the simplicity of having only a single winding field, we are able to prove also in this last case, that the vanishing Bogomol’nyi bound has no solutions, see theorem 9.

Finally, we are in a position to write down the governing equations, which we call the master equations and they are given for type A, type B2 and type B1 solutions in theorems 10, 11 and 12, respectively. Since this form of the equations may be unfamiliar to many readers, we provide a change of variables to a form more similar to the Taubes equation in Sec. 3.6. We give some
explicit examples in Sec. 3.7.

As for the string tension, it turns out always to be of the linear form (1.2) and not of the square-root form (1.1). However, we find nontrivial BPS solutions which can have arbitrary charge matrices, but alignment of the FI vectors is a necessity for all winding flavors (in case there is more than one winding flavor).

Physically, we may interpret the vanishing Bogomol’nyi bound as a signal that the theory touches the Coulomb branch and hence possesses unbroken gauge symmetry, which we do not allow.

Apart from the situation with $M > N$, where we would have vacuum moduli and semi-local types of string [8], an interesting possibility has been left out in this work. That is, the case of the charge matrix with a vanishing determinant. It completely invalidates all analysis done in this paper, so we cannot say anything about such cases on the basis of our analysis. However, it is not necessarily unphysical to consider systems with charge matrices that do not possess an inverse (i.e. with vanishing determinant). The simplest two cases would be if all flavors have the same charge under each gauge groups and the other would be that one flavor has the same charge under all gauge groups (but not necessarily the same as other flavors). We leave this case for future work.

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2 The model

We consider an $\mathcal{N} = 2$ supersymmetric U(1)$^N$ gauge theory in 3 + 1 dimensions with $M$ “flavors” (hypermultiplets) $\Phi_A = (\phi_A, \tilde{\phi}_A)^T$, $A = 1, 2, \ldots, M$, with the following bosonic sector

$$\mathcal{L} = -\sum_a \frac{1}{4e_a^2} F_{\mu \nu}^a F^{a \mu \nu} - \sum_A D_\mu \Phi_A^+ D^\mu \Phi_A - \sum_a \frac{1}{e_a^2} \partial_\mu \Sigma^a \partial^\mu \Sigma^a$$

$$- \sum_{a, \alpha} \frac{1}{2e_a^2} (Y_a^\alpha)^2 - \sum_A \left( \sum_a Q_{Aa} \Sigma^a \right)^2 \Phi_A^+ \Phi_A.$$  \hfill (2.1)

The neutral scalars $\Sigma^a$ belong to the $\mathcal{N} = 2$ vector multiplet together with the photons $A^a_\mu$ which have the field strength

$$F_{\mu \nu}^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu,$$  \hfill (2.2)

the gauge couplings are $e_a > 0$, and finally the index $a = 1, 2, \ldots, N$ corresponds to the gauge groups U(1)$_a$. The gauge covariant derivative for the complex scalar fields is

$$D_\mu \Phi_A = \partial_\mu \Phi_A + i \sum_a Q_{Aa} A^a_\mu \Phi_A,$$  \hfill (2.3)

where $Q_{Aa}$ is a charge matrix with gauge and flavor index. Finally,

$$Y_a^\alpha \equiv e_a^2 \left( \sum_A \Phi_A^+ \sigma^a \Phi_A Q_{Aa} - r_a^\alpha \right),$$  \hfill (2.4)

is an SU(2)$_R$ vector of the D-terms ($\alpha = 3$) and the F-terms ($\alpha = 1, 2$), where $\sigma^\alpha$ are the standard Pauli matrices and $r_a^\alpha$ are $N$ Fayet-Iliopoulos (FI) triplets of parameters. Often we will write SU(2)$_R$ vectors as a boldfaced vector (e.g. $r_a$) (suppressing the SU(2)$_R$ index $\alpha$) and use standard vector notations, like the dot product. The indices $\mu, \nu$ are spacetime indices and are summed over with the Einstein summation convention (that is, any couple of indices are automatically summed over $\mu = 0, 1, 2, 3$). The internal indices, i.e. the gauge group index $a = 1, 2, \ldots, N$, the flavor index $A = 1, 2, \ldots, M$ and the SU(2)$_R$ index $\alpha = 1, 2, 3$ are not summed over by the Einstein convention throughout the paper and if a sum is intended, we write the sum explicitly. The metric signature used here is the mostly positive one, convenient for solitons.

The real fields $\Sigma^a$ are source-free fields and are not used in the construction of vortices. The minimum of the energy is attained by the vacuum solution for $\Sigma^a$ which is $\Sigma^a = 0$. Hence in the remainder of the paper, we will set $\Sigma^a = 0$. This yields the reduced Lagrangian density

$$\mathcal{L} = -\sum_a \frac{1}{4e_a^2} F_{\mu \nu}^a F^{a \mu \nu} - \sum_A D_\mu \Phi_A^+ D^\mu \Phi_A - \sum a, \alpha e_a^2 \left( \sum_A \Phi_A^+ \sigma^a \Phi_A Q_{Aa} - r_a^\alpha \right)^2.$$  \hfill (2.5)
We assume that each U(1) gauge group is compact and thus all elements of the charge matrix $Q_Aa \in \mathbb{Z}$ are integers.

Each U(1)$_a$ gauge group corresponds a redundancy known as a gauge symmetry. The symmetry acts on the fields as the following transformation

$$
\Phi_A \rightarrow \Phi_A \exp \left( i \sum_a Q_Aa \alpha_a(x) \right), \quad \alpha_a(x) \in \mathbb{R},
$$

$$
A^a_\mu \rightarrow A^a_\mu - \partial_\mu \alpha_a(x),
$$

which leaves the Lagrangian (2.1) (and (2.5)) invariant.

### 2.1 The FI parameters

We will now perform an SU(2)$_R$ transformation on the scalar fields, $\Phi \rightarrow U\Phi$, with $U$ a constant unitary matrix. This leaves the kinetic term invariant. The potential transforms, however, and in particular we get

$$
\sum_A \Phi_A^\dagger \sigma^\alpha \Phi_A Q_Aa - r^\alpha_a \rightarrow \sum_A \Phi_A^\dagger U^\dagger \sigma^\alpha U \Phi_A Q_Aa - r^\alpha_a,
$$

and using that $U^\dagger \sigma^\alpha U$ takes value in the su(2)$_R$ algebra, we can write

$$
U^\dagger \sigma^\alpha U = \frac{1}{2} \sum_\beta \text{Tr}[U^\dagger \sigma^\alpha U \sigma^\beta] \sigma^\beta \equiv \sum_\beta R_{\alpha\beta} \sigma^\beta.
$$

Using that $R \in \text{SO}(3)$ and in particular $\sum_\alpha R_{\alpha\beta} R_{\alpha\gamma} = \delta_{\beta\gamma}$, we can write

$$
\sum_{a,\alpha} \frac{e^2_a}{2} \left( \sum_A \Phi_A^\dagger U^\dagger \sigma^\alpha U \Phi_A Q_Aa - r^\alpha_a \right)^2 = \sum_{a,\alpha} \frac{e^2_a}{2} \left( \sum_\beta R_{\alpha\beta} \left( \sum_A \Phi_A^\dagger \sigma^\beta \Phi_A Q_Aa - \sum_\gamma r^\gamma_a R_{\gamma\beta} \right) \right)^2
$$

$$
= \sum_{a,\alpha} \frac{e^2_a}{2} \left( \sum_A \Phi_A^\dagger \sigma^\alpha \Phi_A Q_Aa - \tilde{r}^\alpha_a \right)^2,
$$

with the rotated FI parameters

$$
\tilde{r}^\alpha_a \equiv \sum_\beta r^\beta_a R_{\beta\alpha}.
$$

Now we can simplify the FI parameters using the SU(2)$_R$ rotations, depending on the total number of gauge groups, $N$. We assume that for each gauge group, there is an FI vector with nonvanishing length

$$
r_a \cdot r_a > 0, \quad \forall a = 1, 2, \ldots, N,
$$

7
where \( r_a \) is a 3-vector in SU(2)\(_R\) space for each gauge group U(1)\(_a\).

The first vector, say \( r_1 \), can be rotated to
\[
\tilde{r}_1 = (\alpha, 0, 0), \quad \alpha > 0,
\]
and using the residual U(1) \( \subset \) SU(2)\(_R\) symmetry, we can rotate the next vector to
\[
\tilde{r}_2 = (\beta, \gamma, 0), \quad \beta \in \mathbb{R}, \quad \gamma \geq 0.
\]
These two transformations use up our freedom to rotate within SU(2)\(_R\) and the remaining FI parameters (vectors) point in arbitrary directions. We will henceforth drop the tildes on the FI parameters.

The above considerations show that if we only consider two gauge groups \((N = 2)\), then without loss of generality, the FI parameters can be chosen to be in the \((\sigma^1, \sigma^2)\)-plane. For \( N > 2 \), however, we need the full 3-dimensional SU(2)\(_R\) space for the FI parameters.

### 2.2 The vacuum and the charge matrix

The vacuum equations are
\[
\sum_A \langle \Phi_A \rangle^\dagger \sigma^\alpha \langle \Phi_A \rangle Q_{Aa} = r_\alpha^a, \quad \forall \alpha = 1, 2, 3, \quad \forall a = 1, 2, \ldots, N,
\]
where \( \langle \Phi_A \rangle \) denotes the vacuum expectation value (VEV) of the complex scalar field doublet of flavor \( A \). Let us first consider the number of variables versus the number of constraints in this equation. Each flavor of scalar field doublets has 4 real components yielding a total of \( 4M \) degrees of freedom in the vacuum equation. The number of constraints on the other hand are 3 for each gauge group, so a total of \( 3N \) constraints. For each gauge group, there is 1 vacuum modulus – a free parameter. Thus if \( N > M \) then generically the vacuum equations are over-determined in which case one should minimize the potential (as it cannot vanish unless enough of the FI parameters are equal to each other). This, however, will lift the vacuum energy and break supersymmetry, so we will not consider such case further in this paper. On the other hand, if \( N < M \) there will be further vacuum moduli. These are typical characteristics of supersymmetric theories.

We will require the gauge symmetry to be completely broken which corresponds to the mass term for the photons
\[
\frac{1}{2} \sum_{a,b} (\mathcal{M}^2)_{ab} A^a_\mu A^{\mu b}, \quad (\mathcal{M}^2)_{ab} \equiv 2 \sum_A e_a e_b \langle \Phi_A \rangle^\dagger \langle \Phi_A \rangle Q_{Aa} Q_{Ab},
\]
not having one or more zero eigenvalues:

$$\det(M^2) > 0,$$

(2.16)

where $e_a$ are positive definite gauge coupling constants.

We will now consider the solutions to the vacuum equations, as they will be useful later. Defining

$$\langle \Phi_A \rangle \equiv \left( \begin{array}{c} \rho_A e^{-i\theta_A} \\ \tilde{\rho}_A e^{i\tilde{\theta}_A} \end{array} \right),$$

(2.17)

we get for each gauge group, $a = 1, \ldots, N$, the vacuum equations

$$2 \sum_A \rho_A \tilde{\rho}_A \exp \left[ i(\theta_A + \tilde{\theta}_A) \right] Q_{Aa} = r_a^1 + i r_a^2 \equiv r_a,$$

(2.18)

$$\sum_A (\rho_A^2 - \tilde{\rho}_A^2) Q_{Aa} = r_a^3.$$

(2.19)

The mass-squared matrix for the photons in terms of the new vacuum variables reads

$$(M^2)_{ab} = 2 \sum_A e_a e_b (\rho_A^2 + \tilde{\rho}_A^2) Q_{Aa} Q_{Ab}.$$  

(2.20)

The computation of the determinant of this matrix depends on $N$ and $M$.

In the case of $N = M$, it is easy to show that the determinant of the mass-squared matrix reads

$$\det(M^2) = 2^N \left( \prod_a e_a^2 \right) (\det Q)^2 \prod_A (\rho_A^2 + \tilde{\rho}_A^2).$$

(2.21)

In order to Higgs the gauge symmetry completely, we must have $\det Q \neq 0$, all couplings positive $e_a > 0$ as well as each flavor must have a nonvanishing VEV in either the fundamental or the antifundamental fields (or in both).

If $N > M$ we have

$$\det(M^2) = 2^N \sum_{b_1, \ldots, b_N} e_{b_1} \cdots e_{b_N} \left[ e_1 e_{b_1} \sum_{A_1} Q_{A_11} Q_{A_1 b_1} R_{A_1}^2 \right] \cdots \left[ e_N e_{b_N} \sum_{A_N} Q_{A_N N} Q_{A_N b_N} R_{A_N}^2 \right],$$

(2.22)

where we have defined $R_A^2 \equiv \rho_A^2 + \tilde{\rho}_A^2$. In order for the determinant not to vanish, we must have a different flavor for each bracket, as the same flavor is eliminated by the epsilon tensor. For $N > M$, that is impossible and the determinant always vanishes. For $N = M$, we get the result of Eq. (2.21).
For $N < M$, the determinant can be nonvanishing. It will be instructive to consider first the case of $M = N + 1$. The result will be a sum of $M$ minor determinants

$$\det(M^2) = 2^N \left( \prod_a e_a^2 \right) \sum_{B=1}^M (\det Q^B)^2 \prod_{A \neq B} (\rho_A^2 + \tilde{\rho}_A^2),$$

where $\det Q^B$ is the minor determinant where the $B$th row is removed. The general case of $N < M$ thus reads

$$\det(M^2) = 2^N \left( \prod_a e_a^2 \right) \sum_{B_1=1}^M \sum_{B_2>B_1}^M \cdots \sum_{B_{M-N}>B_{M-N-1}}^M (\det Q^{B_1 \ldots B_{M-N}})^2 \prod_{A \neq B_1, B_2, \ldots, B_{M-N}} (\rho_A^2 + \tilde{\rho}_A^2),$$

where $\det Q^{B_1 \ldots B_{M-N}}$ is the minor determinant with the rows $B_1, \ldots, B_{M-N}$ removed. Hence, in the case of $N < M$ it is possible that the determinant of the mass-squared matrix of the photons vanishes due to several terms of either sign contributing to the determinant. Therefore we shall not consider this case further in this paper.

From this point on, we shall consider only the case $N = M$. It will be convenient in the remainder of the paper to define two new variables for solving the vacuum equations

$$z_A \equiv 2 \rho_A \tilde{\rho}_A e^{i(\vartheta_A + \tilde{\vartheta}_A)}, \quad y_A \equiv \rho_A^2 - \tilde{\rho}_A^2,$$

in terms of which the general vacuum solution reads

$$z_A = \sum_a (r_a^1 + i r_a^2) Q^{-1}_{aa}, \quad y_A = \sum_a r_a^3 Q^{-1}_{aa},$$

where it is understood that $Q^{-1}_{aa}$ is the inverse matrix of $Q_{aa}$. The following relations will come in handy in various computations

$$\langle \phi_A \tilde{\phi}_A \rangle = \frac{1}{2} z_A^*, \quad |\langle \phi_A \rangle|^2 - |\langle \tilde{\phi}_A \rangle|^2 = y_A, \quad |\langle \phi_A \rangle|^2 + |\langle \tilde{\phi}_A \rangle|^2 = \sqrt{y_A^2 + |z_A|^2}.$$

It will now be convenient to consider the vacua and mass matrices on a case-by-case basis.

### 2.2.1 $N = M = 1$

For a single gauge group with a single flavor of hypers, the FI vector can without loss of generality be taken to be that of Eq. (2.12). The solution to the vacuum equations (2.18), (2.19) is

$$\rho = \tilde{\rho} = \sqrt{\frac{\alpha}{|Q|}}, \quad \vartheta + \tilde{\vartheta} = \left( \frac{1 - \text{sign}(Q)}{2} \right) \pi,$$

where we have suppressed the flavor index and $\vartheta - \tilde{\vartheta}$ is a vacuum modulus. The gauge symmetry is spontaneously broken and the photon mass reads

$$M^2 = 4 e^2 |Q| \alpha > 0.$$
2.2.2 \( N = M = 2 \)

For the \( N = M = 2 \) case, we have two FI vectors and they can without loss of generality be taken to be those of Eqs. (2.12) and (2.13) and hence the vacuum equation (2.19) forces \( \rho_A = \tilde{\rho}_A \) for both flavors. Changing variables to

\[
\mathbf{z} \equiv \begin{pmatrix} 2\rho_1^2 e^{(\vartheta_1 + \tilde{\vartheta}_1)} \\ 2\rho_2^2 e^{(\vartheta_2 + \tilde{\vartheta}_2)} \end{pmatrix}^T,
\]

the vacuum solution reads

\[
\mathbf{z} = \left( \alpha, \beta + i\gamma \right) Q^{-1}.
\]

In case of \( \gamma = 0 \), which corresponds to the situation where the two FI vectors were parallel (proportional to each other) before the SU(2)_R rotation to the standard form (2.12), (2.13), the solution \( \mathbf{z} \) is real valued and hence

\[
\vartheta_A + \tilde{\vartheta}_A = w_A \pi, \quad w_A \in \mathbb{Z}, \quad \forall A = 1, 2,
\]

correspond to the signs of the solution (2.31).

The determinant of the mass-squared matrix can thus be written as

\[
\det(M^2) = 16\epsilon_1^2 \epsilon_2^2 (\det Q)^2 \rho_1^2 \rho_2^2,
\]

where we recall that \( \rho_A = \tilde{\rho}_A \). Using now that \(|z_A| = 2\rho_A^2\) and

\[
(\det Q)\mathbf{z} = \begin{pmatrix} \alpha Q_{22} - Q_{21}(\beta + i\gamma) \\ -\alpha Q_{12} + Q_{11}(\beta + i\gamma) \end{pmatrix},
\]

we have

\[
\det(M^2) = 4\epsilon_1^2 \epsilon_2^2 \sqrt{(\gamma^2 Q_{11}^2 + (\alpha Q_{12} - \beta Q_{11})^2)(\gamma^2 Q_{21}^2 + (\alpha Q_{22} - \beta Q_{21})^2)},
\]

and for \( \gamma = 0 \), this can clearly vanish even if \( \det Q \neq 0 \). Of course it is a special case of aligned FI vectors, which in turn align with the charge lattice.

Considering the exceptional case of \( \gamma = 0 \), \( \det Q \neq 0 \) for which the gauge symmetry is not completely broken, we see from Eq. (2.35) that for either flavor \( A \), we have

\[
\frac{Q_{A1}}{Q_{A2}} = \frac{\alpha}{\beta}.
\]
which in turn must be rational for the elements of $Q$ to be able to take such values. The vacuum solution in this case ($\gamma = 0, \det Q \neq 0$), reads

$$\rho_B = \sqrt{\sum_{A=1}^{2} \frac{\epsilon_{BA}(\alpha Q_{A2} - \beta Q_{A1})}{2 \det Q}}, \quad e^{i(\theta_A + \hat{\theta}_A)} = \text{sign} \left( \sum_{A=1}^{2} \frac{\epsilon_{BA}(\alpha Q_{A2} - \beta Q_{A1})}{2 \det Q} \right). \tag{2.37}$$

Now if the contrived alignment between the FI vector is in turn aligned with the charge lattice according to Eq. (2.36), the symmetry is not completely broken spontaneously

$$\rho_1 = 0, \quad \rho_2 = \sqrt{\frac{\beta}{Q_{22}}}, \quad e^{i(\theta_A + \hat{\theta}_A)} = \text{sign} \left( \frac{\beta}{Q_{22}} \right). \tag{2.38}$$

In summary we have the condition for broken gauge symmetry: $\det Q \neq 0$ which is also necessary for invertibility of $Q$ for the vacuum solution. If $\gamma = 0$, we have the additional conditions: $\frac{Q_{A1}}{Q_{A2}} \neq \frac{a}{\beta}, \forall A = 1, 2$.

### 2.2.3 $N = M = 3$

Although we can simplify the first 2 FI vectors, the third vector is generic even after the SU(2)$_R$ rotations. Hence, we generically have $r_1 = (\alpha > 0, 0, 0), r_2 = (\beta, \gamma \geq 0, 0)$ and $r_3 = (\delta, \kappa, \eta)$. Changing variables to

$$z \equiv \begin{pmatrix} 2\rho_1 \hat{\rho}_1 e^{i(\theta_1 + \hat{\theta}_1)} \\ 2\rho_2 \hat{\rho}_2 e^{i(\theta_2 + \hat{\theta}_2)} \\ 2\rho_3 \hat{\rho}_3 e^{i(\theta_3 + \hat{\theta}_3)} \end{pmatrix} \in \mathbb{C}^3, \quad y \equiv \begin{pmatrix} \rho_1^2 - \hat{\rho}_1^2 \\ \rho_2^2 - \hat{\rho}_2^2 \\ \rho_3^2 - \hat{\rho}_3^2 \end{pmatrix} \in \mathbb{R}^3, \tag{2.39}$$

the vacuum equations (2.18) and (2.19) are solved by

$$z = \begin{pmatrix} \alpha, \beta + i\gamma, \delta + i\kappa \end{pmatrix} Q^{-1}, \quad y = \begin{pmatrix} 0, 0, \eta \end{pmatrix} Q^{-1}. \tag{2.40}$$

The determinant of the mass-squared matrix of the photons for this case is again given by Eq. (2.21), which we now can write as

$$\det(M^2) = 8e_1^2 e_2^2 e_3^2 (\det Q)^2 \prod_{A=1}^{3} (\rho_A^2 + \hat{\rho}_A^2)$$

$$= 8e_1^2 e_2^2 e_3^2 (\det Q)^2 \prod_{A=1}^{3} \sqrt{|z_A|^2 + y_A^2}$$

$$= 8e_1^2 e_2^2 e_3^2 |\det Q|^{-1} \prod_{A=1}^{3} \sqrt{(|\gamma A_{11} - \beta \gamma A_{21} + \delta \gamma A_{32})^2 + (\gamma \gamma A_{21} - \kappa \gamma A_{32})^2 + \eta^2 \gamma A_{33}}. \tag{2.41}$$
where $\Upsilon_{Aa} \equiv \det Q^{Aa}$ is the minor determinant of $Q$ with the $A$th row and the $a$th column removed. Note that $\eta \Upsilon_{A3} \neq 0$, $\forall A$ and $\det Q \neq 0$ are sufficient conditions for ensuring that the gauge symmetry is completely broken (spontaneously).

If on the other hand, $\eta = 0$ or one of the three $\Upsilon_{A3} = 0$, it is possible to find accidental solutions with a vanishing eigenvalue of the mass-squared matrix even though $\det Q \neq 0$. Starting with $\eta = 0$, we have the equations giving rise to a vanishing determinant of the mass-squared matrix

$$\alpha \Upsilon_{A1} + \delta \Upsilon_{A3} = \beta \Upsilon_{A2}, \quad \gamma \Upsilon_{A2} = \kappa \Upsilon_{A3}, \quad (2.42)$$

and choosing the flavor giving rise to a vanishing eigenvalue to be $A = 3$, we can write the solution as

$$z = \frac{1}{\det Q} \begin{pmatrix} \alpha \Upsilon_{11} - (\beta + i\gamma) \Upsilon_{12} + (\delta + i\kappa) \Upsilon_{13} \\ -\alpha \Upsilon_{21} + (\beta + i\gamma) \Upsilon_{22} - (\delta + i\kappa) \Upsilon_{23} \\ 0 \end{pmatrix}^T, \quad y = 0. \quad (2.43)$$

The other exceptional case which could result in a partially unbroken gauge symmetry is $\Upsilon_{A3} = 0$, $\eta \neq 0$ and $\gamma = 0$. We will again choose the guilty flavor to be $A = 3$. The fact that $\gamma$ must vanish as well in this exceptional case is because the requirement $\det Q \neq 0$ does not allow us to have $\Upsilon_{32} = 0$ for $\Upsilon_{33} = 0$. We thus have

$$\alpha \Upsilon_{31} = \beta \Upsilon_{32}, \quad (2.44)$$

and the vacuum solution is

$$z = \frac{1}{\det Q} \begin{pmatrix} \alpha \Upsilon_{11} - \beta \Upsilon_{12} + (\delta + i\kappa) \Upsilon_{13} \\ -\alpha \Upsilon_{21} + \beta \Upsilon_{22} - (\delta + i\kappa) \Upsilon_{23} \\ 0 \end{pmatrix}^T, \quad y = \frac{1}{\det Q} \begin{pmatrix} \eta \Upsilon_{13} \\ -\eta \Upsilon_{23} \\ 0 \end{pmatrix}^T. \quad (2.45)$$

In summary we have the condition for broken gauge symmetry $\det Q \neq 0$, which is also a necessity for the invertibility of the charge matrix $Q$. If, however, $\eta = 0$ or $\gamma = 0$, a conspiracy amongst the FI parameters and the charges can occur so that the gauge symmetry is partially unbroken.

### 3 1/2-BPS strings

In this section, we will consider 1/2-BPS strings (states), which by the nature of supersymmetry algebra are always parallel strings. Thus, without loss of generality, we can take the string to
point in the $x^3$ direction and hence the nontrivial behavior is all contained in the $(x^1, x^2)$-plane. Next, we will consider the supersymmetry projections that will spell out the BPS equations.

3.1 Supersymmetry projections

A fermion field is described by a $2^\lfloor d/2 \rfloor$-dimensional spinor, which is a $2^\lfloor d/2 \rfloor$-dimensional representation of an SU(2) field. In particular, in $d + 1 = 6$ dimensions, a fermion field has 8 components, whereas in $d + 1 = 4$ dimensions (like nature) it has only 4 components. This makes it convenient to use $\mathcal{N} = 1$ supersymmetry in $d + 1 = 6$ dimensions, which naturally possesses 8 supercharges (the eight components of the fermionic spinor), as a shortcut to $\mathcal{N} = 2$ supersymmetry in $d + 1 = 4$ dimension (which has $(\mathcal{N} = 2) \times 4 = 8$ supercharges as well). An appropriate dimensional reduction gives the field content and field equations in $d + 1 = 4$ dimensions, see e.g. Ref. [9]. In this section, we will use the invariance of $\mathcal{N} = 1$ supersymmetry in $d + 1 = 6$ dimensions as a method to obtain the BPS equations for the $\mathcal{N} = 2$ theory (2.1) in $d + 1 = 4$ dimensions.

The supersymmetry transformation is realized by means of an SU(2) Majorana-Weyl spinor, $\epsilon_i$, with $i = 1, 2$ that satisfies

\[
\sum_j \Gamma^{012345}_{ij} \epsilon_j = \epsilon_i, \quad (3.1)
\]

\[
B \bar{\epsilon}^i = \sum_j \epsilon^{ij} \epsilon_j, \quad (3.2)
\]

where we have defined the 6-dimensional “$\gamma^5$” as

\[
\Gamma^{012345} = -\Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \Gamma^5, \quad (3.3)
\]

and $B$ can be defined as

\[
B = \Gamma^{012345} \Gamma^3 \Gamma^5. \quad (3.4)
\]

In general

\[
\Gamma^{K_1 \ldots K_n} = \frac{1}{n!} \Gamma^{[K_1 \ldots K_n]}, \quad (3.5)
\]

is the normalized totally antisymmetric product of $n \Gamma$ matrices and $K_1, \ldots, K_n$ are 6-dimensional spacetime indices. Because of the Clifford algebra

\[
\{ \Gamma^{K_1}, \Gamma^{K_2} \} = 2\eta^{K_1 K_2}, \quad (3.6)
\]
the anti-symmetrized product of $\Gamma$ matrices, for $K_1$ and $K_2$ different from each other ($K_1 \neq K_2$), is simply the product of the two Gamma matrices, and so on for higher $n$. For more details on the $\Gamma$ matrices and supersymmetry in higher dimensions, see Appendix B of Polchinski’s book [5]. The Majorana-Weyl spinor with eight supercharges is given by

$$\epsilon_1 = (p, 0, 0, q, 0, r, s, 0),$$
$$\epsilon_2 = (-\bar{q}, 0, 0, \bar{p}, 0, -\bar{s}, \bar{r}, 0),$$

(3.7)

where $p, q, r, s$ are complex Grassmann variables.

As mentioned above, we can without loss of generality choose the string to point in the $x^3$ direction, for which the appropriate combination of $\Gamma$ matrices is $\Gamma^{12} = \Gamma^1 \Gamma^2$. The most general 1/2-BPS supersymmetry projection can thus be written as

$$P^{m}_{ij} = \Gamma^{12} \otimes (i \mathbf{m} \cdot \sigma)_{ij},$$

(3.8)

with $\mathbf{m} = (m_1, m_2, m_3)$ a unit 3-vector ($\mathbf{m} \cdot \mathbf{m} = 1$) and $\sigma$ a 3-vector of the three Pauli SU(2) matrices.

The invariance

$$\sum_j P^{m}_{ij} \epsilon_j = \epsilon_i,$$

(3.9)

yields the following relations among the supercharges

$$p = im_\tau \bar{r} - im_3 s, \quad q = m_\tau \bar{s} + im_3 r, \quad m_{\pm} \equiv m_1 \pm im_2,$$

(3.10)

which confirms that there are 4 conserved supercharges and hence it is a 1/2-BPS projection, as promised. Notice that the unit vector $\mathbf{m}$ rotates the combinations of $r$ and $s$ in the 1/2-BPS supersymmetry relations between the supercharges.

### 3.2 BPS equations

The most general SUSY projection (3.8) is all that is needed for generating the 1/2-BPS equations that we want to study. Indeed, we consider the supersymmetry transformations of the gaugino and the hyperino – which are fermion fields belonging to the gauge and the scalar sectors – by the Majorana-Weyl spinor $\epsilon$ in 6 dimensions, which read [9]

$$\delta_\epsilon \chi^a = \frac{1}{2} \Gamma^{K_1 K_2} F_{K_1 K_2}^a \epsilon + i Y_a \cdot \sigma \epsilon,$$

(3.11)

$$\delta_\epsilon \psi_A = \Gamma^K D_K \Phi_A^T \sigma^2 \epsilon,$$

(3.12)
where \( a = 1, 2, \ldots, N \), is the gauge group index for \( U(1)_a \), \( A = 1, 2, \ldots, M \), is the flavor index and \( Y_a \) is an SU(2)_{\mathcal{R}} triplet of D and F terms. We will now trivialize the fields in the extra-dimensional spatial directions \( K = 4, 5 \) as well as in the spatial direction \( K = 3 (\mu = 3) \) (due to translational invariance in this direction). Forcing the above two fermion transformations to vanish, we have

\[
\Gamma^{12}F_{12}^a \epsilon + iY_a \cdot \sigma \epsilon = 0, \tag{3.13}
\]

\[
\Gamma^1 (D_1 \Phi_A^T + \Gamma^{12} D_2 \Phi_A^T) \sigma^2 \epsilon = 0. \tag{3.14}
\]

Utilizing the SUSY projection (3.8), we get

\[
-i F_{12}^a (m \cdot \sigma) \epsilon + iY_a \cdot \sigma \epsilon = 0, \tag{3.15}
\]

\[
\Gamma^1 (D_1 \Phi_A^T - iD_2 \Phi_A^T \sigma^2 (m \cdot \sigma) \sigma^2) \sigma^2 \epsilon = 0. \tag{3.16}
\]

Using that \( \sigma^2 (m \cdot \sigma) \sigma^2 = -(m \cdot \sigma)^T \), we can readily transpose the second equation, obtaining the set of 1/2-BPS equations

\[
(m \cdot \sigma) F_{12}^a = e_a^2 \sigma \cdot \left( \sum_A Q_A \Phi_A^\dagger \sigma \Phi_A - r_a \right), \tag{3.17}
\]

\[
D_1 \Phi_A + i(m \cdot \sigma) D_2 \Phi_A = 0, \tag{3.18}
\]

where we have inserted the D- and F-terms \( Y_a \) of Eq. (2.4).

### 3.3 Bogomol’nyi bound

By knowing the BPS equations (3.17), (3.18) as well as the Lagrangian (2.5), it is fairly straightforward to write down the Bogomol’nyi bound for the theory

\[
\mathcal{E} = \sum_A |D_1 \Phi_A + im \cdot \sigma D_2 \Phi_A|^2 + \sum_{a,\alpha} \frac{1}{2e_a^2} \left[ m^\alpha F_{12}^a - e_a^2 \left( \sum_A \Phi_A^\dagger \sigma^\alpha \Phi_A Q_A - r_a^\alpha \right) \right]^2
\]

\[
- \sum_a (m \cdot r_a) F_{12}^a, \tag{3.19}
\]

where we have assumed that no field is dependent on \( x^3 \), that all \( A^3_a = 0 \) vanish and we have used the fact that

\[
[D_1, D_2] \Phi_A = i \sum_a Q_{Aa} F_{12}^a \Phi_A. \tag{3.20}
\]

Since the first two terms of Eq. (3.19) are positive semi-definite, the total energy in the \((x^1, x^2)\)-plane, which is the string tension, is bounded from below. Integrating over the \((x^1, x^2)\)-plane
thus gives the Bogomol’nyi bound
\[ T = \int_{\mathbb{R}^2} \mathcal{E} \, dx^1 dx^2 \geq 2\pi \sum_a \mathbf{m} \cdot \mathbf{r}_a k^a, \]  
where we have defined the magnetic fluxes
\[ k^a \equiv -\frac{1}{2\pi} \int_{\mathbb{R}^2} F_{12}^a \, dx^1 dx^2. \]  
We have seen explicitly in Sec. 3.1 that the choice of \( \mathbf{m} \) corresponds to the choice of SUSY projection. The most stringent bound appears when \( \mathbf{m} \) is parallel with \( \mathbf{r}_a k^a \) in SU(2)_R space, which makes it an obvious choice for the SUSY projection, which thus reads
\[ \mathbf{m} = \sum_a \frac{\mathbf{r}_a k^a}{\sqrt{\sum_{b,c} \mathbf{r}_b \cdot \mathbf{r}_c k^b k^c}}, \]  
and is normalized to unit length.

The Bogomol’nyi bound can thus be written as
\[ T \geq T_{\text{BPS}} \equiv 2\pi \sqrt{\sum_{a,b} \mathbf{r}_a \cdot \mathbf{r}_b k^a k^b} = 2\pi \left| \sum_a \mathbf{r}_a k^a \right|. \]  
We thus see that the minimal string tension depends not only on the FI vectors, but also on the magnetic fluxes and which gauge groups they are turned on in. Since the BPS tension (i.e. the tension given by the saturated Bogomol’nyi bound) is given by the length of the vector
\[ \sum_a \mathbf{r}_a k^a, \]  
the bound can only vanish if this vector is a null vector, viz. a vector of length 0. \( \mathbf{m} \) on the other hand is, by definition, a unit 3-vector and hence cannot be a null vector. In the case that the vector (3.25) vanishes, \( \mathbf{m} \) cannot be determined by Eq. (3.23) but can be chosen freely.

### 3.4 Vortex and constraint equations

In this section we will study the conditions for the existence of solutions to the 1/2-BPS equations, which are nontrivial due to the fact that Eq. (3.17) has three components for each gauge group. This generally gives us one vortex equation and two constraint equations.

First, analogously to rotating the FI vectors, we will use a more convenient basis for the vortex fields, which is not the \( \Phi \) basis, but is defined as
\[ \Phi_A = U \Psi_A = U \begin{pmatrix} \psi_A \\ \bar{\psi}_A^* \end{pmatrix}, \]  
(3.26)
with the SU(2)$_R$ transformation

$$U = \sqrt{\frac{1 + m_3}{2}} \begin{pmatrix} 1 & \frac{m_3 - 1}{m_+} \\ \frac{1 - m_3}{m_-} & 1 \end{pmatrix},$$ (3.27)

which is readily checked to have determinant one. We call this the Ψ basis. This matrix diagonalizes all m vectors into $\sigma^3$. Notice that if $m = (0, 0, 1)$ the diagonalization matrix $U$ above, is just the unit two-matrix.

The BPS equations (3.17) and (3.18) can thus be put in the form

$$F_{12}^a = e^2_a \left( \sum_A Q_{Aa} \Psi_A^\dagger \sigma^3 \Psi_A - m \cdot r_a \right),$$ (3.28)

$$D_1 \Psi_A + i \sigma^3 D_2 \Psi_A = 0,$$ (3.29)

which we shall coin the vortex equations and

$$\sum_A \Phi_A^\dagger \vec{\ell} \cdot \sigma \Phi_A Q_{Aa} = \ell \cdot r_a,$$ (3.30)

are the constraint equations, where we have defined the SU(2)$_R$ complex vector that is orthonormal to m, (i.e. $m \cdot \vec{\ell} = 0$):

$$\vec{\ell} \equiv \frac{\ell_1 - i \ell_2}{\sqrt{2}} = \frac{1}{\sqrt{2m_+}} \begin{pmatrix} m_1 m_3 + i m_2 \\ m_2 m_3 - i m_1 \\ -m_+ m_- \end{pmatrix}.$$ (3.31)

Changing to the Ψ basis, the constraint equation reads

$$\frac{1}{\sqrt{2}} \sum_A \Psi_A^\dagger \Psi_A Q_{Aa} = \sqrt{2} \sum_A \psi_A \tilde{\psi}_A Q_{Aa} = \ell \cdot r_a,$$ (3.32)

which is valid for all cases.$^4$ It will prove convenient to rewrite the above equation into an equation with a free flavor index $A$, instead of a free gauge index $a$:

$$\psi_A \tilde{\psi}_A = \frac{1}{\sqrt{2}} \sum_a \vec{\ell} \cdot r_a Q_{aA}^{-1} = \frac{1}{\sqrt{2}} \ell \cdot \Omega_A,$$ (3.33)

where in the last equality we have defined

$$\Omega_A \equiv \sum_a r_a Q_{aA}^{-1}.$$ (3.34)

$^3$Strictly speaking, we should take the limiting value of $U$ for $m = (0, 0, \pm 1)$, i.e. $\lim_{m_3 \to \pm 1} U$.

$^4$Like for $U$, $\vec{\ell}$ should be taken as the limiting value for $m_3 = \pm 1$, i.e. $\lim_{m_3 \to \pm 1} \vec{\ell}$, which for $m_3 \to 1$ yields $\vec{\ell} = (1, -i, 0)/\sqrt{2}$. However, the limit $m_3 \to -1$ is ambiguous and this introduces a spurious phase $\alpha$, i.e. $m = (\sqrt{1 - m_3^2} \cos \alpha, \sqrt{1 - m_3^2} \sin \alpha, m_3)$, which survives the limit of $m_3 \to -1$. Noticing that $m_3 \to -1$ corresponds simply to swapping $\psi_A$ with $\tilde{\psi}_A$, we can fix the ambiguous phase $\alpha$ so the limit reads $\lim_{m_3 \to -1} \vec{\ell} = (1, i, 0)/\sqrt{2}$.
In components, the vortex equations (3.28) and (3.29) read

\[ D_{\bar{z}}\psi_A = 0, \quad (3.35) \]
\[ D_{\bar{z}}\tilde{\psi}_A = 0, \quad (3.36) \]
\[ 2iF^a_{\bar{z}z} = e_a^2 \left( \sum_A |\psi_A|^2 Q_{Aa} - \sum_A |\tilde{\psi}_A|^2 Q_{Aa} - \mathbf{m} \cdot \mathbf{r}_a \right), \quad (3.37) \]

where we have switched to complex coordinates in the \((x^1, x^2)\)-plane, i.e. \(z = x^1 + ix^2\). Although the two first BPS equations look identical, the field \(\psi_A\) has charges \(Q_{Aa}\), whereas \(\tilde{\psi}_A\) is an antifundamental field and thus has the charges \(-Q_{Aa}\). This sign has severe consequences for the existence of solutions.

It will prove instructive to flesh out the self-dual equation (3.35) as

\[ D_{\bar{z}}\psi_A = \partial_{\bar{z}}\psi_A + i \sum_a Q_{Aa} A^a_{\bar{z}} \psi_A = 0, \quad \forall A. \quad (3.38) \]

We can solve for the gauge fields, obtaining

\[ A^a_{\bar{z}} = i \sum_A Q_{aA}^{-1} \partial_{\bar{z}} \log \psi_A. \quad (3.39) \]

We may split the field \(\psi_A\) into a holomorphic and a nonanalytic field

\[ \psi_A(z, \bar{z}) = \varsigma_A^{-1}(z, \bar{z}) h_A(z). \quad (3.40) \]

The gauge field does not depend on \(h_A(z)\) and thus reads

\[ A^a_{\bar{z}} = -i \sum_A Q_{aA}^{-1} \partial_{\bar{z}} \log \varsigma_A. \quad (3.41) \]

The field strength tensor can now be written as

\[ F^a_{12} = 2iF^a_{\bar{z}z} = -2 \sum_A Q_{aA}^{-1} \partial_{\bar{z}} \partial_z \log |\varsigma_A|^2. \quad (3.42) \]

Integrating over the \((x^1, x^2)\)-plane, we get

\[ k^a = -\frac{1}{2\pi} \int_{\mathbb{R}^2} F^a_{12} \, dx^1 dx^2 = \frac{1}{\pi} \sum_A Q_{aA}^{-1} \int_{\mathbb{R}^2} \partial_{\bar{z}} \partial_z \log |\varsigma_A|^2 \, dx^1 dx^2 = \frac{1}{2\pi i} \sum_A Q_{aA}^{-1} \oint_{\partial\mathbb{R}^2} \partial_z \log |\varsigma_A|^2 \, dz, \quad (3.43) \]

where we have used Green’s theorem.
In order to know the asymptotic behavior of $\varsigma_A$, we need to transform the vacuum solution from the $\Phi$ basis to the $\Psi$ basis

\[
|\langle \psi_A \rangle|^2 = \frac{1}{2} \left((1 + m_3)|\langle \phi_A \rangle|^2 + (1 - m_3)|\langle \tilde{\phi}_A \rangle|^2 + m_+|\langle \phi_A \tilde{\phi}_A \rangle + m_-|\langle \phi^*_A \tilde{\phi}_A \rangle\right),
\]

(3.44)

\[
|\langle \tilde{\psi}_A \rangle|^2 = \frac{1}{2} \left((1 - m_3)|\langle \phi_A \rangle|^2 + (1 + m_3)|\langle \tilde{\phi}_A \rangle|^2 - m_+|\langle \phi_A \tilde{\phi}_A \rangle - m_-|\langle \phi^*_A \tilde{\phi}_A \rangle\right).
\]

(3.45)

Using now the vacuum solution (2.26), we can write the vacuum in the $\Psi$ basis as

\[
|\langle \psi_A \rangle|^2 = \frac{1}{2} \left(\sqrt{y^2_A + |z_A|^2} + m_3 y_A + \frac{m_+}{2} z_A^* + \frac{m_-}{2} z_A \right),
\]

(3.46)

\[
|\langle \tilde{\psi}_A \rangle|^2 = \frac{1}{2} \left(\sqrt{y^2_A + |z_A|^2} - m_3 y_A - \frac{m_+}{2} z_A^* - \frac{m_-}{2} z_A \right).
\]

(3.47)

Utilizing the useful relations (2.27) and the definition (3.34), we arrive at

\[
|\langle \psi_A \rangle|^2 = \frac{1}{2} (|\Omega_A| + m \cdot \Omega_A), \quad |\langle \tilde{\psi}_A \rangle|^2 = \frac{1}{2} (|\Omega_A| - m \cdot \Omega_A),
\]

(3.48)

with $|\Omega_A| \equiv \sqrt{\Omega_A \cdot \Omega_A}$. Clearly, the sum

\[
|\langle \psi_A \rangle|^2 + |\langle \tilde{\psi}_A \rangle|^2 = |\Omega_A| \neq 0,
\]

(3.49)

is always nonvanishing whereas if $\Omega_A$ is parallel (anti-parallel) with $m$ then $|\langle \psi_A \rangle|^2 > 0 (|\langle \tilde{\psi}_A \rangle|^2 > 0)$ is nonvanishing. Therefore, if one of the two VEVs $|\langle \psi_A \rangle|^2$, $|\langle \tilde{\psi}_A \rangle|^2$ vanishes, the other must be nonvanishing.

We will now assume that $|\langle \psi_A \rangle|^2 > 0$; therefore, the boundary condition on $\lim_{|z| \to \infty} |\psi_A| \to |\langle \psi_A \rangle|$ leads to

\[
\lim_{|z| \to \infty} \left| \frac{\varsigma_A(z, \bar{z})}{h_A(z)} \right|^2 = \frac{1}{|\langle \psi_A \rangle|^2},
\]

(3.50)

Because there exists no holomorphic $U(1)$ bundle of negative degree, we must have

\[
h_A(z) = \sum_{p=0}^{n_A} h_{A,p} z^p,
\]

(3.51)

with $h_{A,p} \in \mathbb{C}$ being constants, therefore

\[
\lim_{|z| \to \infty} |\varsigma_A(z, \bar{z})|^2 = \frac{|z|^{2n_A}}{|\langle \psi_A \rangle|^2} = \frac{z^{n_A} \bar{z}^{n_A}}{|\langle \psi_A \rangle|^2},
\]

(3.52)

and in turn by Eq. (3.43), we obtain

\[
k^a = \frac{1}{2\pi i} \sum_A Q^{-1}_{aA} \oint \frac{n_A}{z} \, dz = \sum_A Q^{-1}_{aA} n_A,
\]

(3.53)
where \( n_A \) is the winding number of the flavor \( A \). Notice that \( k^a \) are not necessarily integers, but must be rational numbers. Solving for the winding numbers, we get

\[
n_A = \sum_a Q_{Aa} k^a.
\] (3.54)

Now we can repeat the calculation for \( \tilde{\psi}_A \) which is completely analogous, except that its charge is \( -Q_{aA} \) and we now assume that \(|\langle \tilde{\psi}_A \rangle|^2 > 0\), hence

\[
\tilde{n}_A = -\sum_a Q_{Aa} k^a.
\] (3.55)

Since it is impossible to have holomorphic U(1) bundles of negative degrees, neither \( n_A \) nor \( \tilde{n}_A \) can be negative. Unfortunately, as they are determined by the same expression, a non-negative \( \sum_a Q_{Aa} k^a \) will inevitably turn on a positive \( n_A \) and a negative \( \tilde{n}_A \) or vice versa. The only way out is the following lemma.

**Lemma 1.** For winding flavors of 1/2-BPS vortices in the theory (2.5), i.e. \( n_A \neq 0 \), either \( \langle \tilde{\psi}_A \rangle = 0 \) or \( \langle \psi_A \rangle = 0 \), since if nonvanishing they will possess winding and they cannot both be winding since that would make one of them winding with a negative holomorphic function and in turn be singular.

**Proof:** Asymptotically the behavior of the scalar fields is

\[
\lim_{|z| \to \infty} \psi_A = |\langle \psi_A \rangle| e^{in_A \theta},
\] (3.56)

\[
\lim_{|z| \to \infty} \tilde{\psi}_A = |\langle \tilde{\psi}_A \rangle| e^{-in_A \theta},
\] (3.57)

where we have used that \( \tilde{n}_A = -n_A \). Since neither field can have negative winding, \( \langle \tilde{\psi}_A \rangle \) must vanish if \( n_A > 0 \) and contrarily \( \langle \psi_A \rangle \) must vanish if \( n_A < 0 \). \( \square \)

**Corollary 1.** Since either \( \langle \tilde{\psi}_A \rangle = 0 \) or \( \langle \psi_A \rangle = 0 \) for all winding flavors \( (n_A \neq 0) \) by lemma 1, the vector \( m \) must correspondingly be parallel or anti-parallel with \( \Omega_A \).

**Proof:** By inspection of Eq. (3.48) the corollary immediately follows. \( \square \)

We will now contemplate the solutions to the constraint Eq. (3.33). Considering first asymptotic distances, we only need to know the scalar fields VEVs \( \langle \psi_A \rangle, \langle \tilde{\psi}_A \rangle \), the charge matrix \( Q \), the FI vectors \( r_a \) and the vector \( m \) which is given by Eq. (3.23) in terms of \( k^a \) and \( r_a \). Each winding flavor must have \( \langle \psi_A \rangle = 0 \) or \( \langle \tilde{\psi}_A \rangle = 0 \) and for that flavor, the right-hand side of Eq. (3.33) must accordingly vanish; this puts constraints of the allowed fluxes \( k^a \) for given FI vectors and charge.
matrix. Considering now coming in from infinity to non-asymptotic distances. The field that has a vanishing VEV, say \( \tilde{\psi}_{A} \), must remain strictly vanishing throughout the entire \((x^1, x^2)\)-plane or else there is no smooth way to satisfy the constraint Eq. (3.33). The other field – the winding field – vanishes only at the \( n_A \) vortex centers, which in turn have Lebesgue measure zero.

These considerations can be summarized in the following proposition:

**Proposition 1.** The 1/2-BPS vortex solutions in the theory (2.5) which solve the constraint Eq. (3.33) and is compatible with the result of Lemma 1 can be classified into the following categories, based on whether \( \ell \cdot r_a \) vanishes for all \( a \) or not:

- **Type A solutions:** Either \( \psi_A \equiv 0 \) or \( \tilde{\psi}_A \equiv 0 \) throughout \( \mathbb{R}^2 \) and hence \( \sum_a \ell \cdot r_a Q^{-1}_{aA} = 0 \), for every flavor \( A \). This allows all flavors to have nonvanishing winding numbers: \( n_A \neq 0, \forall A \).

- **Type B solutions:** All winding flavors have either \( \psi_A \equiv 0 \) or \( \tilde{\psi}_A \equiv 0 \) throughout \( \mathbb{R}^2 \) for \( A \in S_{WF} \) (i.e. the set of winding flavors (WF)) for which \( \sum_a \ell \cdot r_a Q^{-1}_{aA} = 0 \), and the remaining “inert” flavors are constant solutions obeying

\[
\psi_A \tilde{\psi}_A = \frac{1}{\sqrt{2}} \sum_a \ell \cdot r_a Q^{-1}_{aA}, \quad A \notin S_{WF},
\]

which must be nonvanishing for at least one flavor \( A \notin S_{WF} \) because we demand that

\[
\ell \cdot r_a \neq 0,
\]

for at least one gauge group \( a \). Although the product \( \psi_A \tilde{\psi}_A \) for a given inert flavor \( A \notin S_{WF} \) is constant, the individual fields \( \psi_A \) and \( \tilde{\psi}_A \) do not have to be constant, as they may be coupled to nontrivial magnetic fluxes.

### 3.4.1 Type A solutions

For the type A solutions, we must have

\[
\psi_A = \frac{(1 + m_3) \phi_A + m_\phi^* A}{\sqrt{2(1 + m_3)}}, \quad \text{or} \quad \tilde{\psi}_A = \frac{(1 + m_3) \tilde{\phi}_A - m_\phi^* A}{\sqrt{2(1 + m_3)}} = 0, \quad \forall A,
\]

which by corollary 1 implies that \( \Omega_A \propto \pm m \).

The constraint \( \sum_a \ell \cdot r_a Q^{-1}_{aA} = 0 \) can be reformulated geometrically as

\[
\Omega_A = \sum_a r_a Q^{-1}_{aA} \propto m \propto \sum_a r_a k_a.
\]

One could be mislead to think that this relates the magnetic fluxes \( k_a \) and the charges in the charge matrix \( Q_{Aa} \). However, the geometric constraint is really a constraint on \( r_a \) as stated in the following theorem.
Theorem 1. The constraint \((3.33)\) arising in the theory \((2.5)\), in the case of 1/2-BPS type A solutions (i.e. \(\psi_A = 0\) or \(\tilde{\psi}_A = 0\) for each \(A\)), implies that \(m\) must be proportional to \(r_1 = (\alpha, 0, 0)\), where the latter direction is a choice of fixing \(SU(2)_R\) rotational freedom. Hence, in this case \(m\) must be \((\pm 1, 0, 0)\).

Proof: Using the constraint equation in the formulation of Eq. \((3.61)\), we must have

\[
\Omega_A = \sum_a r_a Q^{-1}_{aA} \propto m, \tag{3.62}
\]

which has two solutions: Either all the vectors \(V_a\)

\[
Q^{-1}_{aA} = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{pmatrix}, \tag{3.63}
\]

must be parallel or all the vectors \(r_a\) must be parallel. If the vectors \(V_a\) are parallel it implies that \(\det Q^{-1} = 0\), which in turn implies that \(Q\) does not exist, and so we must discard this solution. The only remaining solution is that all \(r_a\) are parallel for all \(a\). Since we have – without loss of generality – rotated \(r_1\) to \(r_1 = (\alpha > 0, 0, 0, )\), we must have

\[
r_a \propto r_1, \quad \forall a. \tag{3.64}
\]

Finally, since \(m\) is a unit vector proportional to a linear combination vectors \(r_a\) which are all proportional to \(r_1\) we have \(m = (\pm 1, 0, 0)\) and in particular by Eq. \((3.23)\), we have

\[
m = \left( \text{sign} \left( \sum_a r_a^1 k^a \right), 0, 0 \right), \tag{3.65}
\]

except for the exceptional case where \(\sum_a r_a k^a = 0\) vanishes. In that case, the sign of the first component of \(m\) may be chosen freely. \(\square\)

This theorem thus immediately leads to the following corollary:

Corollary 2. The vacuum of 1/2-BPS type A vortices of the theory \((2.5)\) is given by the real solution

\[
z_A = \sum_a r_a^1 Q^{-1}_{aA}, \tag{3.66}
\]

and the sign of \(m_1 z_A\) determines whether the winding field is \(\psi_A\) (for \(m_1 z_A > 0\)) and hence \(n_A > 0\) or \(\tilde{\psi}_A\) (for \(m_1 z_A < 0\)) and hence \(\tilde{n}_A = -n_A > 0\). In particular, if \(n_A = \sum_a Q_A k^a\) is nonvanishing, it must have the same sign as that of \(m_1 z_A\).
Proof: For type A solutions, due to theorem 1, we have $r_a^2 = r_b^3 = 0$, for all $a$, which by Eq. (2.19) implies that $\rho_A = \tilde{\rho}_A$ and by Eq. (2.18) yields the real solution for $z_A$

$$z_A \equiv 2\rho_A^2 e^{\imath(\theta_A + \tilde{\theta}_A)} = \sum_a r_a^1 Q_a^{-1}.$$  

(3.67)

Since $\rho_A > 0$, the sign of $z_A$ is the sign of the real quantity $e^{\imath(\theta_A + \tilde{\theta}_A)} = \pm 1$, i.e. $\text{sign}(z_A) = e^{\imath(\theta_A + \tilde{\theta}_A)}$. By definition (2.17), the VEV of $\Phi$ is determined as

$$\langle \Phi \rangle = e^{\imath \tilde{\theta}_A} \left( \rho_A e^{-\imath(\theta_A + \tilde{\theta}_A)} \right).$$

(3.68)

For $\mathbf{m} = (\pm 1, 0, 0)$ in accord with theorem 1, we have by Eq. (3.60),

$$\langle \psi_A \rangle = e^{-\imath \theta_A} \left( 1 + m_1 e^{\imath(\theta_A + \tilde{\theta}_A)} \right) \rho_A,$$

(3.69)

$$\langle \tilde{\psi}_A \rangle = e^{-\imath \tilde{\theta}_A} \left( -1 + m_1 e^{\imath(\theta_A + \tilde{\theta}_A)} \right) \rho_A.$$

(3.70)

Hence, for positive (negative) sign$(m_1 z_A)$ we have $\langle \tilde{\psi}_A \rangle = 0$ ($\langle \psi_A \rangle = 0$) and the winding field is $\psi_A$ ($\tilde{\psi}_A$). Using now lemma 1 it follows that if $n_A$ is nonvanishing, it must have the same sign as $m_1 z_A$. □

By using corollary 2 we can relate the magnetic fluxes to the FI vectors by means of the charge matrix and a set of positive real numbers.

**Theorem 2.** 1/2-BPS solutions of type A in the theory (2.5) must have magnetic fluxes

$$k^a = m_1 \sum_{A,b} Q_{aA}^{-1} c_A^2 r_b^1 Q_{bA}^{-1}, \quad c_A^2 z_A \in \mathbb{Z},$$

(3.71)

where the $c_A$ are adjusted so all winding numbers $|n_A| = c_A^2 |z_A| \in \mathbb{Z}$ are integers.

Proof: Solutions of type A must have $n_A$ to be either vanishing or of the same sign as $m_1 z_A$ and thus we can write $n_A = c_A^2 m_1 z_A$, where $c_A$ are chosen such that $n_A \in \mathbb{Z}$. This yields the relation

$$n_A = c_A^2 m_1 z_A = c_A^2 m_1 \sum_a r_a^1 Q_{aA}^{-1} \in \mathbb{Z}.$$  

(3.72)

c $A$ may vanish for some $A$, but not for all flavors $A$, since that corresponds to the trivial solution (no winding numbers in any fields). On the other hand, we have

$$n_A = \sum_a Q_{Aa} k^a.$$  

(3.73)
Eliminating \( n_A \) from the above two equations and multiplying by the inverse of \( Q \), we arrive at Eq. (3.71).

With theorem 2 in hand, we can finally prove that the somewhat bizarre situation where the vector \( \sum_a r_a k^a \) vanishes, which in turn leads to a vanishing energy bound, cannot happen for type A solutions.

**Theorem 3.** 1/2-BPS solutions of type A in the theory (2.5) cannot have \( \sum_a r^1_a k^a = 0 \), which would imply a vanishing energy bound \( T \geq T_{\text{BPS}} = 2\pi |\sum_a r^1_a k^a| = 0 \).

**Proof:** The energy bound (3.24) simplifies due to theorem 1 to

\[
T \geq T_{\text{BPS}} = 2\pi \left| \sum_a r^1_a k^a \right|.
\]  

(3.74)

Now multiply Eq. (3.71) by \( r^1_a \) and sum over \( a \) to get

\[
\sum_a r^1_a k^a = m_1 \sum_{a,A,b} r^1_a Q^{-1}_{aA} c_A r^1_b Q^{-1}_{bA} \]

\[
= m_1 \sum_A \left( c_A \sum_a r^1_a Q^{-1}_{aA} \right)^2,
\]

(3.75)

where the parenthesis is positive (a square of a real number) semi-definite and \( m_1 \) is a sign \((\pm 1)\) by theorem 1. Thus no cancellation among the terms can happen and the expression can only vanish if all \( c_A = 0 \), which we cannot allow as that is the trivial solution (the non-winding vacuum).

\[ \square \]

### 3.4.2 Type B solutions

For the type B solutions, we split the flavors \( A \) into the set of winding flavors \( \tilde{A} \in S_{\text{WF}} \) and the rest \( \hat{A} \not\in S_{\text{WF}} \). One might think the winding flavors are subject to the same constraints as in the type A case; this, however, would only be true if the charge matrix is block diagonal in the winding and non-winding flavors, respectively, so it does not induce mixing. Therefore, this type of solutions is nontrivial and the constraints should be reconsidered carefully.

For the winding flavors, we shall impose

\[
n_{\tilde{A}} = \sum_a Q_{\tilde{A}a} k^a \neq 0, \quad \tilde{A} \in S_{\text{WF}},
\]

(3.76)

and in turn we have by lemma 1

\[
\psi_{\tilde{A}} \tilde{\psi}_{\tilde{A}} = 0, \quad \tilde{A} \in S_{\text{WF}}.
\]

(3.77)
For the non-winding flavors, $\hat{A} \notin S_{WF}$, on the other hand, we have

$$
2\psi_{\hat{A}}\bar{\psi}_{\hat{A}} = \sqrt{2} \sum_a \bar{\ell} \cdot r_a Q^{-1}_{a\hat{A}}
$$

$$
= \left( (1 + m_3)\phi_{\hat{A}}\bar{\phi}_{\hat{A}} - \frac{m_2}{1 + m_3} \phi_{\hat{A}}^{*}\bar{\phi}_{\hat{A}}^{*} - m_{-}(|\phi_{\hat{A}}|^{2} - |\bar{\phi}_{\hat{A}}|^{2}) \right)
$$

$$
= \sum_a \left( \frac{m_1 m_3 + i m_2}{m_{+}} r_{a}^{1} + \frac{m_2 m_3 - i m_1}{m_{+}} r_{a}^{2} - m_{-} r_{a}^{3} \right) Q^{-1}_{a\hat{A}}, \quad \hat{A} \notin S_{WF}, \tag{3.78}
$$

which may or may not vanish and the magnetic fluxes are determined by

$$
k^{a} = \sum_{\hat{A} \in S_{WF}} Q^{-1}_{a\hat{A}} n_{\hat{A}}, \tag{3.79}
$$

where all the $n_{\hat{A}}$ in the sum are nonvanishing by Eq. (3.76). By definition of the type B vortex solutions, we must have

$$
\bar{\ell} \cdot r_{a} \neq 0, \tag{3.80}
$$

for some gauge groups $a$, but not necessarily for all $a$. This will in turn induce a number of nonvanishing $\psi_{\hat{A}}\bar{\psi}_{\hat{A}}$. This leads us to the following lemma:

**Lemma 2.** 1/2-BPS solutions of type B in the theory (2.5) must have FI vectors $r_{a}$ and charge matrix $Q$ which satisfies

$$
\sum_a r_{a} Q^{-1}_{a\hat{A}} \propto m \propto \sum_{a} \sum_{\hat{B} \in S_{WF}} r_{a} Q^{-1}_{a\hat{B}} n_{\hat{B}}, \quad \forall \hat{A} \in S_{WF}, \tag{3.81}
$$

for nonvanishing windings $n_{\hat{B}} \neq 0$.

Proof: For type B solutions, we must have nonvanishing $\sum_a \bar{\ell} \cdot r_a Q^{-1}_{a\hat{A}} \neq 0$ for some flavors $A$. These flavors in turn cannot posses nonzero winding numbers, so those flavors must be $\hat{A} \notin S_{WF}$. On the contrary, for all winding flavors, we must have $\psi_{\hat{A}}\bar{\psi}_{\hat{A}} = 0$ by lemma 1 and in turn

$$
\sum_a \bar{\ell} \cdot r_a Q^{-1}_{a\hat{A}} = 0. \tag{3.82}
$$

Since $\bar{\ell}$ is a complex vector composed by the only two orthogonal directions to $m$, this means that

$$
\sum_a r_{a} Q^{-1}_{a\hat{A}} \propto m. \tag{3.83}
$$

As the vector $m$ is given by Eq. (3.23) and is proportional to $\sum_a r_a k^a$ and $k^a$ is given by Eq. (3.79), the lemma follows. \[\square\]
Theorem 4. For 1/2-BPS vortices of type B in the theory (2.5), the obvious solution to Eq. (3.81) of lemma 2 is that all FI vectors are proportional to each other, \( r_a \propto r_1 \), but we have to discard that solution because it is the solution of type A. There are two remaining ways to solve Eq. (3.81):

- **Type B1 solutions**: There is only 1 winding flavor and the remaining flavors are inert.

- **Type B2 solutions**: The charge matrix is block diagonal, such that the winding flavors are all charged under one block of the charge matrix, which in turn only turn on magnetic fluxes under gauge groups \( \tilde{a} \) for which all \( r_{\tilde{a}} \) are proportional to each other, but not to at least one of the FI vectors of the remaining gauge groups \( r_{\tilde{a}} \).

**Proof**: First let us prove that with all FI vectors parallel to each other, the solution must be of type A. Since \( m \propto \sum_a r_a k^a \) (see Eq. (3.23)), which are all proportional to each other, all \( r_a \) must be orthogonal to \( \ell \), since \( \ell \) is orthogonal to \( m \) by definition and by Eq. (3.31). If all \( r_a \) are orthogonal to \( \ell \), i.e. \( \ell \cdot r_a = 0 \) for all \( a \), then by proposition 1, the solution is of type A.

Let us now prove that a single winding flavor always satisfies lemma 2. With just a single winding flavor \( \tilde{A} \in S_{WF} \), \( Q^{-1}_{a\tilde{A}} \) is just a vector in gauge group space, \( v_a \equiv Q^{-1}_{a\tilde{A}} \) and clearly

\[
\sum_a r_a v_a \propto \sum_a r_a v_a n_{\tilde{A}},
\]  
(3.84)

with \( n_{\tilde{A}} \neq 0 \) being the winding number of the only winding flavor.

Finally, let us prove the last statement of theorem 4. All the winding flavors \( n_{\tilde{A}} \in S_{WF} \) are charged under a block in the charge matrix \( Q \), which is block diagonal with respect to the non-winding flavors \( \tilde{A} \notin S_{WF} \):

\[
Q_{Aa} = \begin{pmatrix}
\tilde{Q}_{A\tilde{a}} & 0 \\
0 & \tilde{Q}_{\tilde{A}\tilde{a}}
\end{pmatrix},
\]  
(3.85)

and therefore the inverse of the charge matrix is given by

\[
Q^{-1}_{a\tilde{A}} = \begin{pmatrix}
\tilde{Q}^{-1}_{a\tilde{A}} & 0 \\
0 & \tilde{Q}^{-1}_{\tilde{a}\tilde{A}}
\end{pmatrix}.
\]  
(3.86)

Due to the block-diagonal property of the inverse of the charge matrix, the winding numbers turn on the following magnetic fluxes:

\[
k^a = \sum_{\tilde{A} \in S_{WF}} Q^{-1}_{a\tilde{A}} n_{\tilde{A}} = \sum_{\tilde{A} \in S_{WF}} \tilde{Q}^{-1}_{a\tilde{A}} n_{\tilde{A}} = k^{\tilde{a}},
\]  
(3.87)

where we have used that only \( n_{\tilde{A}} \) are nonvanishing. The ramification of this for the vector \( m \) is

\[
m \propto \sum_a r_a k^a = \sum_{\tilde{a}} r_{\tilde{a}} k^{\tilde{a}}.
\]  
(3.88)
Now using that all \( r_a \) are proportional to each other, it follows that
\[
\sum_a r_a Q^{-1}_{aA} = \sum_a \tilde{r}_a \tilde{Q}^{-1}_{a\tilde{A}}, \tag{3.89}
\]
and
\[
\sum_a \sum_{B \in S_{WF}} r_a Q^{-1}_{aB} n_B = \sum_a \sum_{B \in S_{WF}} \tilde{r}_a \tilde{Q}^{-1}_{a\tilde{B}} n_{\tilde{B}}. \tag{3.90}
\]
Therefore, the two latter equations are proportional to each other as required by lemma 2 and the last statement of theorem 4 follows.

**Corollary 3.** A special case in which theorem 4 holds for type B2 solutions, is the case where the charge matrix \( Q_{Aa} \) is diagonal.

It is important that those \( \psi_A \tilde{\psi}_A \) that do not vanish do not wind and those that do wind do vanish. In particular, it may seem nontrivial that the nonvanishing \( \psi_A \tilde{\psi}_A \neq 0 \) satisfy Eq. (3.78) with their vacuum solution (2.26). Intuitively, this clearly works out, because we have just changed the basis of the constraint (3.30), which is indeed part of the vacuum equations.

**Lemma 3.** For 1/2-BPS vortex solutions of type B, the nonwinding flavors obey the constraint (3.78) by means of their vacuum solution (2.26).

**Proof:** Using the general vacuum solution (2.26) and the useful relations (2.27), we can write Eq. (3.78) as
\[
2 \langle \psi_A \tilde{\psi}_A \rangle = \left( \frac{1}{2} (1 + m_3) (r_a^1 - ir_a^2) - \frac{m^2_{+}}{2(1 + m_3)} (r_a^1 + ir_a^2) - m_- r_a^3 \right) Q^{-1}_{aA} \]
\[
= \left( \frac{m_1 m_3 + im_2}{m_+} r_a^1 + \frac{m_2 m_3 - im_1}{m_+} r_a^2 - m_- r_a^3 \right) Q^{-1}_{aA}, \tag{3.91}
\]
and is automatically satisfied by the vacuum solution for all FI vectors and all \( m \), by using that \( m_+ m_- + m_3^2 = 1 \).

**Corollary 4.** For 1/2-BPS vortex solutions of type B, the winding flavors satisfying Eq. (3.81) of lemma 2 will automatically have either \( \psi_A \equiv 0 \) or \( \tilde{\psi}_A \equiv 0 \).

**Proof:** By lemma 2, \( \Omega_{\tilde{A}} \propto m \) for all winding flavors \( \tilde{A} \in S_{WF} \) and therefore by Eq. (3.48) either \( |\langle \psi_{\tilde{A}} \rangle|^2 = 0 \) or \( |\langle \tilde{\psi}_{\tilde{A}} \rangle|^2 = 0 \). Since the product \( \psi_{\tilde{A}} \tilde{\psi}_{\tilde{A}} = 0 \) must remain vanishing throughout \( \mathbb{R}^2 \) according to the constraint (3.33), it follows that either \( \psi_{\tilde{A}} \equiv 0 \) or \( \tilde{\psi}_{\tilde{A}} \equiv 0 \).

The constraint equation (3.33) is now satisfied for all flavors. It remains to be proven that the would-be vortex equations are satisfied by the inert flavors \( \hat{A} \). This leads us to the next theorem.
Theorem 5. For non-winding flavors $\hat{A}$ of 1/2-BPS vortex solutions of type B2 in the theory (2.5), the would-be vortex equations are satisfied by the vacuum solution and moreover both the fields $\psi_{\hat{A}}$ and $\tilde{\psi}_{\hat{A}}$ are individually constant throughout $\mathbb{R}^2$.

Proof: The field strength contracted with the charge matrix can be written as

$$\sum_a Q_{Aa} F_{12}^a = 2i \sum_a Q_{Aa} F_{zz}^a = 2\partial_x \partial_z \log |\psi_{\hat{A}}|^2 = -2\partial_x \partial_z \log |\tilde{\psi}_{\hat{A}}|^2. \quad (3.92)$$

For winding flavors, the expression with the nonvanishing field (i.e. either $\psi_{\hat{A}}$ or $\tilde{\psi}_{\hat{A}}$) should be used. However, for the non-winding flavors $\hat{A}$ there are two cases: If the left-hand side of Eq. (3.92) vanishes, both $\psi_{\hat{A}}$ and $\tilde{\psi}_{\hat{A}}$ must be constant and coincide with their respective VEVs. If the left-hand-side of Eq. (3.92) does not vanish, they are forced to be inversely proportional to each other.

Now the right-hand side of the vortex equation (3.37), multiplied by the charge matrix, reads

$$\sum_a Q_{Aa} e_a^2 \left[ \sum_B \left( |\psi_B|^2 - |\tilde{\psi}_B|^2 \right) Q_{Ba} - m \cdot r_a \right] = \sum_a Q_{\hat{A}a} \left[ \sum_{\hat{B} \in S_{WF}} \left( |\psi_{\hat{B}}|^2 - |\tilde{\psi}_{\hat{B}}|^2 \right) \hat{Q}_{\hat{B}a} - m \cdot r_a \right], \quad (3.93)$$

which we have split into winding flavors $\hat{B}$ and non-winding flavors $\hat{B}$ in the last line.

Using now the block-diagonal property of the charge matrix (3.85) for the type B2 case, we obtain for the non-winding flavors

$$\sum_a Q_{\hat{A}a} F_{12}^a = \sum_{\hat{a}} \hat{Q}_{\hat{A}\hat{a}} F_{12}^{\hat{a}}, \quad (3.94)$$

for the left-hand side of the vortex equation (3.37), and

$$\sum_{\hat{a}} \hat{Q}_{\hat{A}\hat{a}} e_{\hat{a}}^2 \left[ \sum_{\hat{B} \in S_{WF}} \left( |\psi_{\hat{B}}|^2 - |\tilde{\psi}_{\hat{B}}|^2 \right) \hat{Q}_{\hat{B}\hat{a}} - m \cdot r_{\hat{a}} \right] = \sum_{\hat{a}} \hat{Q}_{\hat{A}\hat{a}} \left[ \sum_{\hat{B} \notin S_{WF}} \left( |\psi_{\hat{B}}|^2 - |\tilde{\psi}_{\hat{B}}|^2 \right) \hat{Q}_{\hat{B}\hat{a}} - m \cdot r_{\hat{a}} \right]. \quad (3.95)$$

This latter equation contains only the non-winding fields and thus allows for the constant vacuum solution (2.26). By insertion, we can verify that

$$\sum_{\hat{a}} \hat{Q}_{\hat{A}\hat{a}} e_{\hat{a}}^2 \left[ \sum_{\hat{B} \notin S_{WF}} \left( |\psi_{\hat{B}}|^2 - |\tilde{\psi}_{\hat{B}}|^2 \right) \hat{Q}_{\hat{B}\hat{a}} - m \cdot r_{\hat{a}} \right]$$

$$= \sum_{\hat{a}} \hat{Q}_{\hat{A}\hat{a}} e_{\hat{a}}^2 \left[ \sum_{\hat{B} \notin S_{WF}} m \cdot \Omega_{\hat{B}} \hat{Q}_{\hat{B}\hat{a}} - m \cdot r_{\hat{a}} \right]$$

$$= \sum_{\hat{a}} \hat{Q}_{\hat{A}\hat{a}} e_{\hat{a}}^2 \left[ m \cdot r_{\hat{a}} - m \cdot r_{\hat{a}} \right] = 0, \quad (3.96)$$
where we have used Eq. (3.48) and that
\[
\sum_B \Omega_B \tilde{Q}_{\tilde{B}\tilde{a}} = \sum_{B,b} r_{\tilde{b}} \tilde{Q}_{bB}^{-1} \tilde{Q}_{\tilde{B}\tilde{a}} = r_{\tilde{a}}.
\] (3.97)
Since the vacuum solution (3.48) solves the right-hand side of the vortex equation (3.37) throughout \( \mathbb{R}^2 \), the left-hand side vanishes as well and this is consistent because the fluxes \( k^{\tilde{a}} \) are decoupled from the winding flavors. Therefore the constant solution is allowed and uniquely determined by the vacuum equations. □

It should be clear from the latter proof that for type B1 solutions, the would-be vortex equations do not decouple in general, and hence they will induce nontrivial behavior in \( \tilde{\psi}_A \) and \( \tilde{\psi}_A \) – even though they are non-winding flavors – but with \( \psi_A \tilde{\psi}_A = \langle \psi_A \tilde{\psi}_A \rangle \) everywhere constant and equal to their (product of) VEV(s).

Analogously to the situation of type A solutions, we must have a positive \( n_A > 0 \) if \( \langle \psi_A \rangle \neq 0 \) and positive \( \tilde{n}_A = -n_A > 0 \) if \( \langle \tilde{\psi}_A \rangle \neq 0 \). A slight complication arises because we can no longer ensure that the FI vectors \( r_{\tilde{a}} \) are proportional to \( r_1 \). This, however, can quickly be remedied by a subsequent SU(2) \( _R \) rotation and a suitable relabeling of the gauge groups.

**Theorem 6.** For 1/2-BPS vortices of type B2 in the theory (2.5), the winding flavors \( \tilde{A} \) turn on magnetic fluxes \( k^{\tilde{a}} \) only in a subset of gauge groups \( \tilde{a} \), whose FI vectors \( r_{\tilde{a}} \) are all parallel by theorem 4 and furthermore they can be chosen to be parallel with \( r_1 = (\alpha > 0, 0, 0) \) without loss of generality.

**Proof:** All the FI vectors \( \tilde{a} \) are parallel to each other by theorem 4 and can be rotated, without loss of generality to \( (\alpha' > 0, 0, 0) \), for some positive constant \( \alpha' \). Now we will relabel the gauge groups so that one of them is \( \tilde{a} = 1 \). Dropping the prime \( \alpha' \rightarrow \alpha \) completes the proof. □

**Corollary 5.** The vacuum of winding flavors \( \tilde{A} \) in 1/2-BPS type B2 vortices of the theory (2.5) is given by the real solution
\[
z_{\tilde{A}} = \sum_{\tilde{a}} r_{\tilde{a}}^{-1} \tilde{Q}_{\tilde{a}\tilde{A}}^{-1},
\]
(3.98)
and the sign \( m_1 z_{\tilde{A}} \) determines whether the winding field is \( \psi_{\tilde{A}} \) (for \( m_1 z_{\tilde{A}} > 0 \)) and hence \( n_{\tilde{A}} > 0 \) or \( \tilde{\psi}_{\tilde{A}} \) (for \( m_1 z_{\tilde{A}} < 0 \)) and hence \( \tilde{n}_{\tilde{A}} = -n_{\tilde{A}} > 0 \).

**Proof:** Using the block-diagonal property of the charge matrix (3.85) for the type B2 solutions and theorem 6, the proof is analogous to that of corollary 2. □
Theorem 7. 1/2-BPS solutions of type B2 in the theory (2.5) must have magnetic fluxes

\[ k^\tilde{a} = m_1 \sum_{\tilde{A} \in SWF, \tilde{b}} \tilde{Q}_{a\tilde{A}}^{-1} c^2_{\tilde{A}} r^1_{\tilde{b}} \tilde{Q}_{\tilde{b}\tilde{A}}^{-1}, \quad c^2_{\tilde{A}} z_{\tilde{A}} \in \mathbb{Z}, \]

where the \( c_{\tilde{A}} \) are adjusted so all winding numbers \( |n_{\tilde{A}}| = c^2_{\tilde{A}} |z_{\tilde{A}}| \) are integers.

Proof: Using the block-diagonal property of the charge matrix (3.85) for the type B2 solutions and corollary 5, the proof is analogous to that of theorem 2. □

With theorem 7 in hand, we can finally rule out the possibility of a solution with vanishing BPS bound.

Theorem 8. 1/2-BPS solutions of type B2 in the theory (2.5) cannot have \( \sum_a r^1_a k^a = 0 \), which would imply a vanishing energy bound \( T \geq T_{\text{BPS}} = 2\pi |\sum_a r^1_a k^a| = 0 \).

Proof: Using theorem 6 implying that \( r_\hat{a} = (r^1_\hat{a}, 0, 0) \), the block-diagonal property of the charge matrix (3.85) implies via theorem 7 that only \( k^\tilde{a} \) are nonvanishing (whereas \( k^\hat{a} = 0 \)). Finally, multiplying by \( r^1_{\hat{a}} \), the proof is completed analogously to that of theorem 3. □

For type B1 solutions, the single winding flavor turns on fluxes in potentially all gauge groups, making it a complicated type of solution. We can nevertheless prove that it is impossible to have a vanishing BPS bound also in this case.

Theorem 9. 1/2-BPS solutions of type B1 in the theory (2.5) cannot have \( \sum_a r_a k^a = 0 \), which would imply a vanishing energy bound \( T \geq T_{\text{BPS}} = 2\pi \sqrt{\sum_{a,b} r_a \cdot r_b k^a k^b} = 0 \).

Proof: By the definition of the type B1 solution according to theorem 4, only a single flavor possesses nonvanishing winding number \( n_{\tilde{A}} \neq 0 \) and therefore the magnetic fluxes are given by

\[ k^a = Q_{a\tilde{A}}^{-1} n_{\tilde{A}}, \]

where there is no sum over \( \tilde{A} \) since it is just a single flavor. We can thus write

\[ \sum_a r_a k^a = \sum_a r_a Q_{a\tilde{A}}^{-1} n_{\tilde{A}} = \Omega_{\tilde{A}} n_{\tilde{A}} \neq 0, \]

where we have used Eq. (3.34) and the fact that neither \( n_{\tilde{A}} \) can vanish nor can \( \Omega_{\tilde{A}} \). The latter follows from Eq. (3.49), which states that \( |\Omega_{\tilde{A}}| = 0 \) corresponds to \( |\langle \psi_{\tilde{A}} \rangle|^2 + |\langle \tilde{\psi}_{\tilde{A}} \rangle|^2 = 0 \), which in turn implies unbroken gauge symmetry which we cannot allow. □

In the type B1 solution, it remains whether there is any restriction on the sign of the (only nonvanishing) winding number \( n_{\tilde{A}} \) with regards to the vacuum solution. This leads us to the following lemma.
Lemma 4. The sign of the only nonvanishing winding number $n_{\tilde{A}} \neq 0$, in 1/2-BPS solutions of type B1 in the theory (2.5), is not restricted by the vacuum solution.

Proof: The unit vector $m$ is directed as

$$m \propto \sum_a r_a k^a = \sum_a r_a Q^{-1}_{a\tilde{A}} n_{\tilde{A}},$$

(3.102)

and the sign of $n_{\tilde{A}}$ thus determines whether $m$ is parallel ($n_{\tilde{A}} > 0$) or anti-parallel ($n_{\tilde{A}} < 0$) with $\Omega_{\tilde{A}}$ as

$$\Omega_{\tilde{A}} \equiv \sum_a r_a Q^{-1}_{a\tilde{A}}.$$  

(3.103)

Thus, if they are parallel ($n_{\tilde{A}} > 0$) then by Eq. (3.48), $\tilde{\psi}_{\tilde{A}} \equiv 0$ and vice versa if they are anti-parallel ($n_{\tilde{A}} < 0$) then $\psi_{\tilde{A}} \equiv 0$. Since, there is only a single winding field, no incompatibility can arise and the lemma follows.

3.5 Master equations

For supersymmetric (BPS) solitons, it is often the case that there is a selfdual equation of the form $D_2 \Phi = 0$ and another BPS equation of the form $F_{12} = \cdots$, which generically can be combined into a resulting gauge-invariant equation in terms of a non-holomorphic field, independent of the number of codimensions (i.e. whether the system contains domain walls, vortices, etc.) or of the gauge symmetry (Abelian or non-Abelian) or other specifics of the system, see Refs. [10–12]. In our case, that field is $\varsigma_A(z, \bar{z})$ and the master equation is the governing equation for that field.

We will now write down the master equations for the different types of solution in turn, starting with type A.

3.5.1 Type A solutions

By proposition 1, type A solutions solve the constraint equation (3.32) by having $\psi_A \tilde{\psi}_A = 0$ for all flavors throughout $\mathbb{R}^2$, which means either $\psi_A$ or $\tilde{\psi}_A$ can be a winding field and the other vanishes everywhere. By theorem 1, $r_a \propto r_1$ for all gauge groups $a$ and thus without loss of generality $r_a = (r_1^a, 0, 0)$ and in turn $m = (\pm 1, 0, 0)$. The sign $m_1$ is given by the sign of $\sum_a r_1^a k^a$, which cannot have vanishing length by theorem 3 and the magnetic fluxes take the form

$$k^a = m_1 \sum_{A,b} Q^{-1}_{AB} c_B^A r_1^b Q^{-1}_{b\tilde{A}}, \quad c_A^2 z_A \in \mathbb{Z},$$

(3.104)
where \( c_A \in \mathbb{R} \) are constants adjusted so the winding numbers \( |n_A| = c_A^2 |z_A| \) are all integers, by theorem 2. Therefore the BPS tension is always nonvanishing for all existing solutions of type A and the vortex equations can be written in the form of the following master equations.

**Theorem 10.** 1/2-BPS solutions of type A in the theory (2.5), with the above described properties, are governed by the master equations

\[
-2 \text{sign}(z_A) \partial_z \partial_{\bar{z}} \log |\varsigma_A(z, \bar{z})|^2 = \sum_{a,B} Q_{Aa} e_a^2 (Q^T)_{aB} z_B \left( \left| \frac{h_B(z)}{\varsigma_B(z, \bar{z})} \right|^2 - 1 \right),
\]

with \( z_A = \sum_a r_a^1 Q^{-1}_{aA} \) and \( h_B(z) \) a holomorphic polynomial of degree \( |n_B| \) of the form

\[
h_B(z) = \prod_{i=1}^{n_B} (z - Z_i^B),
\]

with \( Z_i^B \in \mathbb{C} \) being position moduli and if \( n_B = 0 \), \( h_B = 1 \) can be chosen without loss of generality. Finally, the boundary condition on \( \varsigma_A(z, \bar{z}) \) is given by \( \lim_{|z| \to \infty} |\varsigma_A(z, \bar{z})| = |z|^{2|n_A|} \).

**Proof:** We start by multiplying both sides of the vortex equation (3.37) by \( Q_{Aa} \) (and sum over the gauge index \( a \))

\[
\sum_a Q_{Aa} F^a_{12} = \sum_a Q_{Aa} e_a^2 \left( \sum_B Q^T_{aB} \left( |\psi_B|^2 - |\bar{\psi}_B|^2 \right) - m_1 r_a^1 \right),
\]

where we have used that only \( r_a^1 \) is nonvanishing (of the FI vector components). For each flavor we will switch variables into the moduli functions \( h_A(z) \) and auxiliary fields \( \varsigma_A(z, \bar{z}) \), according to which field is the winding field:

\[
\psi_A = \sqrt{|z_A|} \frac{h_A(z)}{\varsigma_A(z, \bar{z})}, \quad \text{or} \quad \bar{\psi}_A = \sqrt{|z_A|} \frac{h_A(z)}{\varsigma_A(z, \bar{z})}.
\]

That in turn is decided by the sign of \( m_1 z_A = m_1 \sum_a r_a^1 Q^{-1}_{aA} \), which when nonvanishing has the same sign as \( n_A \) according to corollary 2 and we have multiplied by the norm of the field’s VEV \( \sqrt{|z_A|} \). According to lemma 1 and proposition 1, this sign determines whether \( \psi_A \) or \( \bar{\psi}_A \) is the nonvanishing flavor: for \( n_A > 0 \) we have \( \bar{\psi}_A \equiv 0 \) and \( \psi_A \) is the winding fields and *vice versa* for negative \( n_A \). When \( n_A = 0 \), the nonvanishing field is still determined by the sign of \( m_1 z_A \) and its would-be vortex equation is the same master equation as that of a winding flavor. We therefore have

\[
-2 \text{sign}(m_1 z_A) \partial_z \partial_{\bar{z}} \log |\varsigma_A(z, \bar{z})|^2 = \sum_{a,B} Q_{Aa} e_a^2 (Q^T)_{aB} \text{sign}(m_1 z_B) |z_B| \left| \frac{h_B(z)}{\varsigma(z, \bar{z})} \right|^2 - \sum_a Q_{Aa} e_a^2 m_1 r_a^1.
\]
Now, since \(\text{sign}(AB) = \text{sign}(A)\text{sign}(B)\) and \(\text{sign}(m_1) = m_1 = \pm 1\), \(m_1\) drops out of the entire master equation, \(\text{sign}(z_B)z_B = z_B\), and only the sign of the vacuum solution \(z_A\) determines the sign of the kinetic term. The constant (last) term can be rewritten as follows

\[
- \sum_{a} Q_A e_a e_a^2 r_1^a = - \sum_{a,b,B} Q_A e_a e_a^2 Q_{Bb} r_1^a Q_{Bb}^{-1} = - \sum_{a,b,B} Q_A e_a^2 (Q^T)_{ab} z_B. \tag{3.110}
\]

We are thus left with Eq. (3.105). Finally, the degree of the polynomial \(h_A(z)\) for \(n_A > 0\) must be \(n_A\), however, since we have given the same name to the moduli function of the anti-fundamental field, the degree in the case that \(\tilde{\psi}_A\) is nonvanishing must have \(h_A(z)\) of degree \(\tilde{n}_A = -n_A\). In all cases, the degree of the polynomial is thus given by \(|n_A|\) and the theorem follows.

### 3.5.2 Type B2 solutions

By proposition 1, the type B solutions are characterized by having \(\tilde{e} \cdot r_a \neq 0\) for some gauge group indices \(a\) and therefore not all FI vectors are parallel. For convenience, we split the flavors into winding flavors \(\tilde{\tilde{A}} \in S_{WF}\) and non-winding flavors \(\hat{\tilde{A}} \notin S_{WF}\). For type B2 solutions according to theorem 4 and by lemma 2 all winding flavors \(\tilde{\tilde{A}}\) are via the charge matrix coupled only to parallel FI vectors \(\tilde{a}\) due to the block-diagonal property (3.85) of the charge matrix and by theorem 6, those FI vectors can all be taken to be proportional to \(r_1 = (\alpha > 0, 0, 0)\) without loss of generality. The block-diagonal property (3.85) of the charge matrix further implies that magnetic fluxes are only turned on in gauge groups with parallel FI vectors and therefore \(m = (\pm 1, 0, 0)\). Furthermore, the non-winding flavors satisfy – by means of their vacuum solution – the constraint equations according to lemma 3 and the would-be vortex equations according to theorem 5. For the winding flavors \(\tilde{\tilde{A}}\), either \(\psi_{\tilde{\tilde{A}}} \equiv 0\) or \(\tilde{\psi}_{\tilde{\tilde{A}}} \equiv 0\) by corollary 4 and lemma 2 and by corollary 5 if \(m_1 z_{\tilde{\tilde{A}}} > 0\) then \(\psi_{\tilde{\tilde{A}}} \equiv 0\) and \(\tilde{\psi}_{\tilde{\tilde{A}}}\) is the winding field with winding number \(n_{\tilde{\tilde{A}}} \geq 0\), whereas if \(m_1 z_{\tilde{\tilde{A}}} < 0\) then \(\psi_{\tilde{\tilde{A}}} \equiv 0\) and \(\tilde{\psi}_{\tilde{\tilde{A}}}\) is the winding field with winding number \(\tilde{n}_{\tilde{\tilde{A}}} = -n_{\tilde{\tilde{A}}} \geq 0\). By theorem 7 the magnetic fluxes are given by

\[
k_{\tilde{a}} = m_1 \sum_{\tilde{\tilde{A}} \in S_{WF}, \tilde{\tilde{a}}} \tilde{Q}_{\tilde{a}\tilde{A}} e_{\tilde{a}} e_\tilde{a}^2 Q_{\tilde{Bb}}^{-1} c_{\tilde{A}}^2 z_{\tilde{\tilde{a}}}, \quad c_{\tilde{A}}^2 z_{\tilde{\tilde{A}}} \in \mathbb{Z}, \tag{3.111}
\]

where the \(c_{\tilde{A}}\) are adjusted so all winding numbers \(|n_{\tilde{\tilde{A}}}| = c_{\tilde{A}}^2 |z_{\tilde{\tilde{A}}}|\) are integers and \(m_1\) is given by the sign of \(\sum_{a} r_1^a k_{\tilde{a}}\), which in turn cannot vanish by theorem 8. Therefore the BPS tension is always nonvanishing for all existing solutions of type B2 and the vortex equations can be written in the form of the following master equations.

**Theorem 11.** All winding flavors \(\tilde{\tilde{A}}\) in 1/2-BPS solutions of type B2 in the theory (2.5), with
the above described properties, are governed by the master equations

\[-2 \text{sign}(z_A) \partial_z \partial_{\bar{z}} \log |\xi_A(z, \bar{z})|^2 = \sum_{\tilde{a}, \tilde{B}} \tilde{Q}_{\tilde{a}} e^2_{\tilde{a}}(\tilde{Q}^T)_{\tilde{a}\tilde{B}} \left( \frac{h_{\tilde{B}}(z)}{|\xi_{\tilde{B}}(z, \bar{z})|^2} - 1 \right), \tag{3.112}\]

with \(z_A = \sum_{\tilde{a}} r_{\tilde{a}} \tilde{Q}^{-1}_{\tilde{a}A}\) and \(h_{\tilde{B}}(z)\) a holomorphic polynomial of degree \(|n_{\tilde{B}}|\) of the form

\[h_{\tilde{B}}(z) = \prod_{i=1}^{\frac{|n_{\tilde{B}}|}{2}} (z - Z^B_i), \tag{3.113}\]

with \(Z^B_i \in \mathbb{C}\) being position moduli and if \(n_{\tilde{B}} = 0\) (which we shall allow to cover all solution types), \(h_{\tilde{B}} = 1\) can be chosen without loss of generality. Finally, the boundary condition on \(\xi_A(z, \bar{z})\) is given by \(\lim_{|z| \to \infty} |\xi_A(z, \bar{z})| = |z|^{2|n_A|} \).

**Proof:** Using the block-diagonal property (3.85) of the charge matrix and the above mentioned properties of type B2 solutions, the proof is analogous to that of theorem 10. \(\square\)

### 3.5.3 Type B1 solutions

By proposition 1, the type B solutions have \(\tilde{e} \cdot r_a \neq 0\) for some gauge group indices \(a\) and therefore the FI vectors are not all parallel. The type B1 solution, by theorem 4 is defined by having only a single winding flavor \(\tilde{A}\) for which we must have \(n_{\tilde{A}} \neq 0\). The unit vector \(m \propto \sum_a r_a k_a\) and the latter cannot be vanishing by theorem 9. Either \(\psi_{\tilde{A}}\) is winding (for \(n_{\tilde{A}} > 0\)) and \(\psi_{\tilde{A}} \equiv 0\) or \(\psi_{\tilde{A}}\) is winding (for \(n_{\tilde{A}} = -n_{\tilde{A}} > 0\)) and \(\psi_{\tilde{A}} \equiv 0\). By lemma 4 there is no restriction on the sign of \(n_{\tilde{A}}\).

Since there is no restriction on the charge matrix (other than \(\det Q \neq 0\)), the master equations change to the following form.

**Theorem 12.** 1/2-BPS solutions of type B1 in the theory (2.5), with the above described properties, are governed by the master equations

\[-2 \Upsilon_A \partial_{\bar{z}} \partial_{\bar{z}} \log |\xi_A(z, \bar{z})|^2 = \sum_a Q_{Aa} e^2_a(Q^T)_{aA} \text{sign}(n_{\tilde{A}}) |\Omega_{\tilde{A}}| \left( \frac{h_{\tilde{A}}(z)}{|\xi_{\tilde{A}}(z, \bar{z})|^2} - 1 \right) \tag{3.114}\]

\[+ \frac{1}{2} \sum_{a, \tilde{B} \notin \text{SWF}} Q_{Aa} e^2_a(Q^T)_{a\tilde{B}} (|\Omega_{\tilde{B}}| + m \cdot \Omega_{\tilde{B}}) \left( \frac{1}{|\xi_{\tilde{B}}(z, \bar{z})|^2} - 1 \right), \]

\[- \frac{1}{2} \sum_{a, \tilde{B} \notin \text{SWF}} Q_{Aa} e^2_a(Q^T)_{a\tilde{B}} (|\Omega_{\tilde{B}}| - m \cdot \Omega_{\tilde{B}}) (|\xi_{\tilde{B}}(z, \bar{z})|^2 - 1), \]

with

\[\Upsilon_A = \begin{cases} \text{sign}(n_{\tilde{A}}), & A = \tilde{A}, \\ 1, & \text{otherwise}, \end{cases} \quad \Omega_{\tilde{A}} = \sum_a r_a Q_{aA}, \quad m = \frac{\sum_a r_a k_a}{\sqrt{\sum_{b,c} r_b \cdot r_c k_b k_c}}, \tag{3.115}\]
and \( h_{\bar{A}}(z) \) a holomorphic polynomial of degree \(|n_{\bar{A}}|\) of the form

\[
h_{\bar{A}}(z) = \prod_{i=1}^{|n_{\bar{A}}|} (z - Z_i),
\]

(3.116)

with \( Z_i \in \mathbb{C} \) being position moduli. Finally, the boundary condition for the (only) winding auxiliary field \( \varsigma_{\bar{A}}(z, \bar{z}) \) is \( \lim_{|z| \to \infty} \varsigma_{\bar{A}}(z, \bar{z}) = |z|^{2|n_{\bar{A}}|} \) and for the non-winding auxiliary fields \( \varsigma_{\bar{A}}(z, \bar{z}) \) it is \( \lim_{|z| \to \infty} \varsigma_{\bar{A}}(z, \bar{z}) = 1 \).

**Proof:** Using the vortex equation (3.37) multiplied by \( Q_{Aa} \) (and summed over \( a \)), we have Eq. (3.107). Now for the winding flavor \( \bar{A} \), we will switch variables into the moduli function \( h_{\bar{A}}(z) \) and the auxiliary field \( \varsigma_{\bar{A}}(z, \bar{z}) \), according to which fields is the winding field:

\[
\psi_{\bar{A}} = \sqrt{\frac{\Omega_{\bar{A}}}{\varsigma_{\bar{A}}(z, \bar{z})}} h_{\bar{A}}(z), \quad \text{or} \quad \bar{\psi}_{\bar{A}} = \sqrt{\frac{\Omega_{\bar{A}}}{\varsigma_{\bar{A}}(z, \bar{z})}} h_{\bar{A}}(z),
\]

(3.117)

where we have used that the vacuum of the nonvanishing (and hence winding) field is \( \sqrt{\Omega_{\bar{A}}} \) and which field is winding is determined by \( \text{sign}(n_{\bar{A}}) \). We thus have

\[
\sum_a Q_{Aa} F_{a12}^a = -2 \text{sign}(n_{\bar{A}}) \partial_{\bar{z}} \partial_z \log |\varsigma_{\bar{A}}(z, \bar{z})|^2.
\]

(3.118)

Solving the self-dual equations \( D_{\bar{z}} \psi_A = D_{\bar{z}} \bar{\psi}_A = 0 \), works by picking the nonvanishing field and determining \( A_{\bar{z}} \) from that field (\( \psi_A \) or \( \bar{\psi}_A \)) and it is consistent, because the other field being everywhere zero solves the other self-dual equation. For the non-winding flavors, the product \( \psi_A \bar{\psi}_A \) may not vanish, but is everywhere constant and equal to its VEV \( \langle \psi_A \bar{\psi}_A \rangle \), according to lemma 3. Therefore, in order to solve both self-dual equations, we set

\[
\psi_{\bar{A}} = \sqrt{\frac{\Omega_{\bar{A}} + m \cdot \Omega_{\bar{A}}}{\sqrt{2} \varsigma_{\bar{A}}(z, \bar{z})}}, \quad \bar{\psi}_{\bar{A}} = \sqrt{\frac{\Omega_{\bar{A}} - m \cdot \Omega_{\bar{A}}}{\sqrt{2} \varsigma_{\bar{A}}(z, \bar{z})}},
\]

(3.119)

which is consistent with

\[
\sum_a Q_{Aa} F_{a12}^a = 2 \partial_{\bar{z}} \partial_z \log |\psi_{\bar{A}}|^2 = -2 \partial_{\bar{z}} \partial_z \log |\bar{\psi}_{\bar{A}}|^2 = -2 \partial_{\bar{z}} \partial_z \log |\varsigma_{\bar{A}}(z, \bar{z})|^2,
\]

(3.120)

and the sign is fixed by the Ansatz (definition) (3.119). Inserting the fields of Eqs. (3.117) and (3.119) into the right-hand side of Eq. (3.107), we obtain the result (3.114) apart from the
constant term, which needs a bit more massage
\[-\sum_a Q_A e_a^2 m \cdot r_a = -\sum_{a,b} Q_A e_a^2 (Q^T)_{aB} Q_{bB}^{-1} m \cdot r_b\]
\[= -\sum_{a,B} Q_A e_a^2 (Q^T)_{aB} m \cdot \Omega_B\]
\[= -\sum_a Q_A e_a^2 (Q^T)_{a\tilde{A}} m \cdot \Omega_{\tilde{A}} - \sum_{a,B \notin WF} Q_A e_a^2 (Q^T)_{aB} m \cdot \Omega_B\]
\[= -\sum_a Q_A e_a^2 (Q^T)_{a\tilde{A}} \text{sign}(n_{\tilde{A}}) |\Omega_{\tilde{A}}| - \sum_{a,B \notin WF} Q_A e_a^2 (Q^T)_{aB} m \cdot \Omega_B. \tag{3.121}\]
where we have used the definition (3.34), split the sum over the flavor index \(B\) into the (single) winding index \(\tilde{A}\) and the non-winding flavors \(\hat{B}\), and finally used that \(\Omega_{\tilde{A}}\) must be parallel or anti-parallel with \(m\) (which is a unit vector) by corollary 4 and Eq. (3.48); the sign of whether the vectors are parallel or anti-parallel is that of \(n_{\tilde{A}}\) by definition. We thus arrive at the result in Eq. (3.114). The proof of the moduli function is analogous to that of theorem 10. \(\square\)

### 3.6 Vortex equations in standard form

It will prove convenient to rewrite the master equations of the last subsection to a more commonly used form, i.e. like the Taubes equation.

#### 3.6.1 Type A solutions

Starting with type A solutions, we make the change of variables \(u_A \equiv -2 \log |\frac{h_A(z)}{z_A(z,\bar{z})}|\), and obtain from theorem 10,
\[
\nabla^2 u_A = \sum_B A_{AB} (e^{u_B} - 1) + 4\pi \sum_{i=1}^{n_A} (z - Z_i^A), \tag{3.122}
\]
where we have defined the matrix
\[
A_{AB} \equiv 2 \sum_a Q_A e_a^2 Q_{Ba} \text{sign}(z_A) z_B, \tag{3.123}
\]
which is real and positive definite, but not symmetric. The boundary conditions on \(u_A\) are \(\lim_{|z| \to \infty} u_A = 0, \forall A\).

In the special case where the matrix \(A\) is symmetric, the existence and uniqueness (as well as sharp decay estimates) have been proven by Yang in Ref. [13]. The matrix can be made symmetric, by setting the same size for all the VEVs as
\[
z_B = \pm \alpha, \tag{3.124}
\]
with a sign that can be chosen arbitrarily for each flavor.

In the general case, where the matrix is only positive definite, but not symmetric, we can use the proof of existence in Yang’s book [14, Chapter 4.7]. In order to use this existence proof, we must have that

$$\sum_A A^{-1}_{BA} g_A > 0,$$  \hspace{1cm} (3.125)

for the equation

$$\nabla^2 u_A = \sum_B A_{AB} e^{u_B} - g_A.$$  \hspace{1cm} (3.126)

Since $A$ is invertible (it is not possessing any vanishing eigenvalues), we can readily calculate the condition (3.125) for our case:

$$\sum_{A,C} (A^{-1})_{BA} A_{AC} = \sum_C \delta_{BC} = 1 > 0, \hspace{1cm} \forall B,$$  \hspace{1cm} (3.127)

which indeed is positive. We do not know of any uniqueness results in this case.

### 3.6.2 Type B2 solutions

Clearly, the same equation is obtained for type B2 solutions, from theorem 11, for the winding flavors

$$\nabla^2 u_\tilde{A} = \sum_{\tilde{B}} A_{\tilde{A}\tilde{B}} (e^{u_{\tilde{B}}} - 1) + 4\pi |n_\tilde{A}| \sum_i (z - Z_\tilde{i}),$$  \hspace{1cm} (3.128)

and the matrix is now

$$A_{\tilde{A}\tilde{B}} \equiv 2 \sum_\tilde{a} \tilde{Q}_{\tilde{A}\tilde{a}} e^{2\tilde{Q}_{\tilde{B}\tilde{a}}} \text{sign}(z_\tilde{A}) z_\tilde{B},$$  \hspace{1cm} (3.129)

which is still real and positive definite, but not symmetric. Clearly the same results of existence and uniqueness applies to this case as to the case of type A solutions, see the previous subsubsection. The boundary conditions on $u_\tilde{A}$ are $\lim_{|z| \to \infty} u_\tilde{A} = 0, \forall \tilde{A}$.

### 3.6.3 Type B1 solutions

Finally, for the type B1 solutions a new type of vortex equation is found in theorem 12, and in standard form it reads

$$\nabla^2 u_A = A_A (e^{u_N} - 1) + \sum_{B=1}^{N-1} B_{AB} (e^{u_B} - 1) - \sum_{B=1}^{N-1} C_{AB} (e^{-u_B} - 1) + 4\pi \delta A_N \sum_i |n_N| \delta(z - Z_i),$$  \hspace{1cm} (3.130)
where we have selected the last flavor as the winding flavor and defined the vector

\[ \mathcal{A}_A \equiv 2 \sum_a Q_A e_a^2 Q_N a \text{sign}(n_N) |\Omega_N|, \]  

(3.131)
as well as the \(N\)-by-\((N - 1)\) matrices

\[ B_{AB} \equiv \sum_a \sum_{B \neq N} Q_A e_a^2 Q_B a \left( |\Omega_B| + m \cdot \Omega_B \right), \]  

(3.132)
\[ C_{AB} \equiv - \sum_a \sum_{B \neq N} Q_A e_a^2 Q_B a \left( |\Omega_B| - m \cdot \Omega_B \right). \]  

(3.133)
The boundary conditions on \(u_A\) are \(\lim_{|z| \to \infty} u_A = 0, \forall A\) as well as for \(A = N\). We do not know of any existence or uniqueness results for this case, which is to the best of our knowledge a new type of vortex equation.

### 3.7 Explicit examples

We will now flesh out explicit examples of 1/2-BPS solutions to the constraint equation (3.33) and display the corresponding master equations obtained by theorems 10-12 as well as vortex equations in the standard form, found in the previous subsection.

#### 3.7.1 \(N = M = 1\)

For one gauge group (\(N = 1\)) and one flavor (\(M = 1\)), the indices are just cluttering, so we will suppress them here. The vacuum solution is given by Eq. \((2.28)\), the FI vector is \(r = (\alpha > 0, 0, 0)\), and the solution is always of type A and either fully fundamental or fully anti-fundamental\(^5\), with string tension

\[ T_{\text{BPS}} = 2\pi |k|\alpha. \]  

(3.134)
The master equation is thus given by theorem 10 and reads

\[ -2\partial_z \partial_{\bar{z}} \log |\varsigma(z, \bar{z})|^2 = |Q| e^2 \alpha \left( \left| \frac{h(z)}{\varsigma(z, \bar{z})} \right|^2 - 1 \right), \]  

(3.135)
which holds for any combination of signs of \(Q\) and \(k\) (and in turn of \(n = Qk\)). It will prove instructive to see explicitly how the constraint equations are solved in this simple situation. The SU(2)\(_R\) vectors read

\[ m = (\text{sign}(k), 0, 0), \quad \ell = -\frac{1}{\sqrt{2}}(0, i, \text{sign}(k)), \]  

(3.136)
\(^5\)We do not allow \(n = 0, Q = 0\) and therefore neither \(k = 0\), since that is the (trivial) vacuum solution.
by theorem 1 so it is manifest that $\ell$ and $r$ are orthogonal; on the other hand it is obvious that
the constraint (3.61) is always satisfied. The constraint (3.60) is more nontrivial and becomes
\[
\text{sign}(n) = \begin{cases} 
-1, & \Rightarrow \quad \psi(z, \bar{z}) \equiv 0, \quad \tilde{\psi}(z, \bar{z}) = \zeta^{-1}(z, \bar{z}) h(z), \\
+1, & \Rightarrow \quad \tilde{\psi}(z, \bar{z}) \equiv 0, \quad \psi(z, \bar{z}) = \zeta^{-1}(z, \bar{z}) h(z). 
\end{cases}
\tag{3.137}
\]
By rescaling the lengths $z \rightarrow \sqrt{2|Q|a}e z$ and defining a field $u \equiv 2 \log |\frac{h(z)}{\zeta(z, \bar{z})}|$, we can write the
master equation in the form
\[
\nabla^2 u = e^u - 1 + 4\pi |n| \sum_{i=1}^{n} \delta(z - Z_i),
\tag{3.138}
\]
which is Taubes equation (Eq. (3.122)). The existence and uniqueness has been proved by Taubes in Ref. [15].

3.7.2 $N = M = 2$

We now turn to the case of two gauge groups with two flavors ($N = M = 2$) in which case there
are solutions of all types.

3.7.2.1 Type A solution. By theorem 1 the FI vectors are parallel and hence read $r_1 = (\alpha, 0, 0)$ and $r_2 = (\beta, 0, 0)$ with $\alpha > 0$ and $\beta \neq 0$ and in turn $m = (\pm 1, 0, 0)$ (by theorem 1). The
vacuum solution is given by
\[
z_A = \sum_{a} r_a Q^{-1}_a (\text{corollary 2})
\]
and the string tension reads
\[
T_{\text{BPS}} = 2\pi |\alpha k^1 + \beta k^2| > 0.
\tag{3.139}
\]
We will now select an explicit example for the charge matrix with nonvanishing determinant
\[
Q = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}, \quad q \in \mathbb{Z},
\tag{3.140}
\]
for which the vacuum solution is given by
\[
z_A = (\alpha, \beta - q\alpha),
\tag{3.141}
\]
and the master equations by theorem 10 read
\[
-2\partial_z \partial_{\bar{z}} \log |\varsigma_1(z, \bar{z})|^2 = (e_1^2 + q^2 e_2^2) \alpha \left( \left| \frac{h_1(z)}{\varsigma_1(z, \bar{z})} \right|^2 - 1 \right) + q e_2^2 (\beta - q\alpha) \left( \left| \frac{h_2(z)}{\varsigma_2(z, \bar{z})} \right|^2 - 1 \right).
\tag{3.142}
\]
\[
-2\partial_z \partial_{\bar{z}} \log |\varsigma_2(z, \bar{z})|^2 = \text{sign}(\beta - q\alpha) q e_2^2 \alpha \left( \left| \frac{h_1(z)}{\varsigma_1(z, \bar{z})} \right|^2 - 1 \right) + e_2^2 |\beta - q\alpha| \left( \left| \frac{h_2(z)}{\varsigma_2(z, \bar{z})} \right|^2 - 1 \right).
\tag{3.143}
\]
Note that \( \beta - q\alpha \neq 0 \) cannot vanish as that would make the theory touch the Coulomb branch on the second flavor, or in other words, the gauge symmetry would be unbroken.

By theorem 2, the magnetic fluxes are given by

\[
\begin{pmatrix} k^1 \\ k^2 \end{pmatrix} = \begin{pmatrix} n_1 - q n_2 \\ n_2 \end{pmatrix},
\]

(3.144)

and the signs of \((n_1, n_2)\) (when nonvanishing) must obey

\[
\text{sign}(n_1) = \text{sign}(\alpha k^1 + \beta k^2), \quad \text{sign}(n_2) = \text{sign}(\alpha k^1 + \beta k^2) \text{sign}(\beta - q\alpha),
\]

(3.145)

and finally by theorem 3, \(\alpha k^1 + \beta k^2 \neq 0\) cannot vanish.

We can now make a change of variables: \(u_A = 2 \log \left| h_A(z) \right| \), for which we can write the system of PDEs as Eq. (3.122):

\[
\nabla^2 u_A = \sum_B A_{AB} (e^{u_B} - 1) + 4\pi \sum_{i=1}^{[n_A]} \delta(z - Z_i^A),
\]

(3.146)

where we have defined the real matrix

\[
A_{AB} \equiv 2 \begin{pmatrix} (e_1^2 + q^2 e_2^2)\alpha & q e_2^2 (\beta - q\alpha) \\ \text{sign}(\beta - q\alpha) q e_2^2 \alpha & e_2^2 |\beta - q\alpha| \end{pmatrix}.
\]

(3.147)

Note that although the diagonal entries are all positive definite, the matrix \(A\) is not symmetric, but always positive definite. We can see this from the determinant

\[
\det A = 4 e_1 e_2 \beta - q\alpha) > 0,
\]

(3.148)

which can never switch sign or vanish as \(\beta - q\alpha \neq 0\) and \(\alpha > 0\). This is also (1/4 times) the determinant of the photon mass-squared matrix in Eq. (2.33) and the nonvanishing of that matrix is guaranteed by requiring completely broken gauge symmetry.

In the special case where the matrix \(A\) is symmetric, the existence and uniqueness (as well as sharp decay estimates) have been proved by Yang in Ref. [13]. Such a symmetric case corresponds (for this specific example) to:

\[
\alpha = |\beta - q\alpha|,
\]

(3.149)

which has several solutions

\[
\beta = 0, \quad q = \pm 1,
\]

(3.150)

\[
\beta = (q + 1)\alpha,
\]

(3.151)

etc. The origin of this criterion of symmetricity is that \(\sum_a Q_{Aa} e_a^2 Q_{Ba}\) of theorem 10 is always a real symmetric matrix, but with the addendum of the factor of \(z_B\), this no longer holds true.
3.7.2.2 Type B2 solution. By theorem 4, the B2 solution has a block diagonal charge matrix, which for \( N = M = 2 \) means diagonal

\[
Q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}, \quad p, q \in \mathbb{Z}\backslash\{0\},
\]

(3.152)

and the FI vectors cannot all be parallel and since there are only two of them \( (N = 2) \), we take the most generic case of FI vectors (after the SU(2)\(_R\) symmetry simplification): \( \mathbf{r}_1 = (\alpha, 0, 0) \) and \( \mathbf{r}_2 = (\beta, \gamma, 0) \) with \( \alpha > 0, \beta \in \mathbb{R} \) and \( \gamma \neq 0 \). By theorem 6, the winding flavor is coupled to \( \mathbf{r}_1 \), without loss of generality, which in our case means that the winding flavor is \( \tilde{A} = 1 \). By corollary 5, the vacuum of the winding flavor is

\[
z_1 = \frac{\alpha}{p},
\]

(3.153)

and the magnetic flux is given by

\[
k^1 = \frac{n_1}{p},
\]

(3.154)

so the sign \( m_1 = \text{sign}(\alpha k^1) \) and the sign determining which field is winding is \( \text{sign}(m_1 z_1) = \text{sign}(n_1) \), which is unconstrained (because it is only a single winding flavor in this case). By theorem 8, \( \alpha k^1 \neq 0 \) cannot vanish and that is certainly true as we must have \( n_1 \neq 0 \). The non-winding flavors obey the constraint equations by lemma 3 and the vortex equation by theorem 5. The non-winding of the second flavor, \( n_2 = 0 \), implies that \( k^2 = 0 \) and hence the string tension reads

\[
T = T_{\text{BPS}} = 2\pi \alpha |k^1|.
\]

(3.155)

Finally, by theorem 11, we have the master equation

\[
-2\partial_z \partial_{\bar{z}} \log |\varsigma_1(z, \bar{z})|^2 = |p| \epsilon^2 \alpha \left( \left| \frac{h_1(z)}{\varsigma_1(z, \bar{z})} \right|^2 - 1 \right).
\]

(3.156)

Comparing with the master equation (3.135), this case reduces to the standard Taubes equation and the details are identical to the case discussed in Sec. 3.7.1.

It will prove instructive to see explicitly how the constraint equations are solved in this simple case. The SU(2)\(_R\) vectors read

\[
\mathbf{m} = (\text{sign}(k^1), 0, 0), \quad \mathbf{\ell} = -\frac{1}{\sqrt{2}}(0, i, \text{sign}(k^1)), \quad \Omega_1 = \left( \frac{\alpha}{p}, 0, 0 \right), \quad \Omega_2 = \left( \frac{\beta}{q}, \gamma, 0 \right),
\]

(3.157)
so by the fact that the FI vectors are not parallel and by theorem 4 this solution type solves the constraint equation (3.33) nontrivially. As promised by corollary 4, the winding flavor must have a vanishing product $\psi_1 \tilde{\psi}_1 = 0$ as

$$\psi_1 \tilde{\psi}_1 = \frac{1}{\sqrt{2}} \overline{\ell} \cdot \Omega_1 = 0. \quad (3.158)$$

As a requirement of theorem 4, for the non-winding flavor, the inner product of $\overline{\ell}$ and $\Omega_A$ does not vanish

$$\psi_2 \tilde{\psi}_2 = \frac{1}{\sqrt{2}} \overline{\ell} \cdot \Omega_2 = -\frac{i\gamma}{2q}. \quad (3.159)$$

Calculating now the VEV $\langle \psi_2 \tilde{\psi}_2 \rangle$, we have

$$\langle \psi_2 \tilde{\psi}_2 \rangle = \frac{1}{2} \left[ \phi_2 \tilde{\phi}_2 - \phi_2^* \tilde{\phi}_2^* - \text{sign}(k^1) \left( |\phi_2|^2 - |\tilde{\phi}_2|^2 \right) \right] = -\frac{1}{2} \left[ i \Im(z_2) + \text{sign}(k^1) y_2 \right] = -\frac{i\gamma}{2q}, \quad (3.160)$$

where we have used the useful relations (2.27) and the vacuum solution for the non-winding flavor

$$z_2 = \frac{\beta + i\gamma}{q}, \quad y_2 = 0. \quad (3.161)$$

This solution remains a solution throughout $\mathbb{R}^2$, because the magnetic flux $k^2$ is not turned on and the vortex equation for the winding field decouples due to the diagonality of the charge matrix (3.152).

### 3.7.2.3 Type B1 solution.

By theorem 4, the B1 solution can have an arbitrary charge matrix with integer elements and nonvanishing determinant, but only a single winding flavor. We take the charge matrix to be

$$Q = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}, \quad q \in \mathbb{Z}, \quad (3.162)$$

and the winding flavor to be $\tilde{A} = 2$. Also by theorem 4 there is no restriction on the FI vectors, so we will take the most general case $r_1 = (\alpha, 0, 0)$ and $r_2 = (\beta, \gamma, 0)$, with $\alpha > 0$ and $|r_2| \neq 0$. The vacuum solution is given by

$$z_A = (\alpha, \beta - q\alpha), \quad (3.163)$$

and the magnetic fluxes by

$$k^a = (-qn_2, n_2). \quad (3.164)$$
The string tension reads

\[ T = T_{\text{BPS}} = 2\pi \sqrt{(\alpha k^1 + \beta k^2)^2 + (\gamma k^2)^2} = 2\pi \sqrt{(\beta - q\alpha)^2 + \gamma^2 |n_2|}, \] (3.165)

which is linear in the (only) winding number \( n_2 \), but has a square-root form of the FI parameters. By theorem 9, the vector

\[ \sum_a r_a k^a = (\beta - q\alpha, \gamma, 0)n_2, \] (3.166)

cannot have a vanishing length, i.e.

\[ \sqrt{(\beta - q\alpha)^2 + \gamma^2 |n_2|} \neq 0, \] (3.167)

and therefore the BPS bound cannot vanish as stated in the theorem. The \( \mathbf{m} \) vector is thus

\[ \mathbf{m} = \frac{(\beta - q\alpha, \gamma, 0)}{\sqrt{(\beta - q\alpha)^2 + \gamma^2}} \text{sign}(n_2). \] (3.168)

Finally, by theorem 12, the master equations read

\[-2\partial_z \partial_{\bar{z}} \log |\varsigma_1(z, \bar{z})|^2 = \text{sign}(\beta - q\alpha)q e_2^2 \sqrt{(\beta - q\alpha)^2 + \gamma^2} \left( \frac{h_2(z)}{\varsigma_2(z, \bar{z})} \right)^2 - 1 \]

\[+ \frac{e_2^2 + q^2 e_2^2}{2} \left[ \left( \alpha + \frac{\alpha|\beta - q\alpha|}{\sqrt{(\beta - q\alpha)^2 + \gamma^2}} \right) \left( \frac{1}{|\varsigma_1(z, \bar{z})|^2} - 1 \right) 

- \left( \alpha - \frac{\alpha|\beta - q\alpha|}{\sqrt{(\beta - q\alpha)^2 + \gamma^2}} \right) (|\varsigma_1(z, \bar{z})|^2 - 1) \right], \] (3.169)

\[-2\partial_z \partial_{\bar{z}} \log |\varsigma_2(z, \bar{z})|^2 = e_2^2 \sqrt{(\beta - q\alpha)^2 + \gamma^2} \left( \frac{h_2(z)}{\varsigma_2(z, \bar{z})} \right)^2 - 1 \]

\[+ \frac{qe_2^2}{2} \left[ \left( \text{sign}(\beta - q\alpha)\alpha + \frac{\alpha(\beta - q\alpha)}{\sqrt{(\beta - q\alpha)^2 + \gamma^2}} \right) \left( \frac{1}{|\varsigma_1(z, \bar{z})|^2} - 1 \right) 

- \left( \text{sign}(\beta - q\alpha)\alpha - \frac{\alpha(\beta - q\alpha)}{\sqrt{(\beta - q\alpha)^2 + \gamma^2}} \right) (|\varsigma_1(z, \bar{z})|^2 - 1) \right], \] (3.170)

where \( \beta - q\alpha \neq 0 \) cannot vanish as that would imply unbroken gauge symmetry.

We can now make a change of variables: \( u_1 = -2 \log |\varsigma_1(z, \bar{z})|, \) \( u_2 = 2 \log \left| \frac{h_2(z)}{\varsigma_2(z, \bar{z})} \right|, \) for which we can write the system of PDEs as Eq. (3.130):

\[ \nabla^2 u_A = \mathcal{A}_A (e^{u_2} - 1) + \sum_{B \neq 2} \mathcal{B}_{AB} (e^{u_B} - 1) - \sum_{B \neq 2} \mathcal{C}_{AB} (e^{-n_B} - 1) + 4\pi \delta^{A_2} \sum_{i=1}^{n_2} \delta(z - Z_i), \] (3.171)
where we have defined

\[ A_A \equiv 2 \left( \frac{\text{sign}(\beta - q\alpha)qe_2^2 \sqrt{(\beta - q\alpha)^2 + \gamma^2}}{e_2^2 \sqrt{(\beta - q\alpha)^2 + \gamma^2}} \right), \tag{3.172} \]

\[ B_{AB} \equiv \begin{pmatrix} e_1^2 + q^2 e_2^2 & \alpha + \frac{\alpha(\beta - q\alpha)}{\sqrt{(\beta - q\alpha)^2 + \gamma^2}} \\ qe_2^2 \text{sign}(\beta - q\alpha) & \alpha + \frac{\alpha(\beta - q\alpha)}{\sqrt{(\beta - q\alpha)^2 + \gamma^2}} \end{pmatrix}, \tag{3.173} \]

\[ C_{AB} \equiv \begin{pmatrix} e_1^2 + q^2 e_2^2 & \alpha - \frac{\alpha(\beta - q\alpha)}{\sqrt{(\beta - q\alpha)^2 + \gamma^2}} \\ qe_2^2 \text{sign}(\beta - q\alpha) & \alpha - \frac{\alpha(\beta - q\alpha)}{\sqrt{(\beta - q\alpha)^2 + \gamma^2}} \end{pmatrix}, \tag{3.174} \]

with \( A \) an \( N \)-vector and \( B \) and \( C \) are \( N \)-by-(\( N - 1 \)) matrices (and \( N = 2 \)). We do not know of any existence of uniqueness results for the equation (3.171).

### 3.7.3 \( N = M = 3 \)

We now turn to the case of three gauge groups with three flavors (\( N = M = 3 \)).

**3.7.3.1 Type A solution.** By theorem 1, the FI vectors are parallel and hence read \( r_1 = (\alpha, 0, 0) \), \( r_2 = (\beta, 0, 0) \) and \( r_3 = (\delta, 0, 0) \) with \( \alpha > 0 \), and \( \beta, \delta \neq 0 \) and in turn \( m = (\pm 1, 0, 0) \) (by theorem 1). The vacuum solution is given by \( z_A = \sum_a r_a^1 Q_a^{-1} \) (corollary 2) and the string tension reads

\[ T_{\text{BPS}} = 2\pi|\alpha^1 + \beta^2 + \delta^3| > 0. \tag{3.175} \]

We will now select an explicit example for the charge matrix with nonvanishing determinant

\[ Q = \begin{pmatrix} 1 & p & q \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}, \quad p, q, r \in \mathbb{Z}, \tag{3.176} \]

for which the vacuum solution is given by

\[ z_A = (\alpha, \beta - p\alpha, (pr - q)\alpha - r\beta + \delta), \tag{3.177} \]
and the master equations by theorem 10 read

\[ -2\partial_z \partial_{\bar{z}} \log |s_1(z, \bar{z})|^2 = (e_1^2 + p^2 e_2^2 + q^2 e_3^2)\alpha \left( \frac{|h_1(z)|}{|s_1(z, \bar{z})|} - 1 \right) \]

\[ + (pe_2^2 + qre_3^2)(\beta - p\alpha) \left( \frac{|h_2(z)|}{|s_2(z, \bar{z})|} - 1 \right) \]

\[ + qe_3^2((pr - q)\alpha - r\beta + \delta) \left( \frac{|h_3(z)|}{|s_3(z, \bar{z})|} - 1 \right), \]

(3.178)

\[ -2\partial_z \partial_{\bar{z}} \log |s_2(z, \bar{z})|^2 = \text{sign}(\beta - p\alpha)(pe_2^2 + qre_3^2)\alpha \left( \frac{|h_1(z)|}{|s_1(z, \bar{z})|} - 1 \right) \]

\[ + (e_2^2 + r^2 e_3^2)|\beta - p\alpha| \left( \frac{|h_2(z)|}{|s_2(z, \bar{z})|} - 1 \right) \]

\[ + \text{sign}(\beta - p\alpha)re_3^2((pr - q)\alpha - r\beta + \delta) \left( \frac{|h_3(z)|}{|s_3(z, \bar{z})|} - 1 \right), \]

(3.179)

\[ -2\partial_z \partial_{\bar{z}} \log |s_3(z, \bar{z})|^2 = \text{sign}((pr - q)\alpha - r\beta + \delta)qe_3^2\alpha \left( \frac{|h_1(z)|}{|s_1(z, \bar{z})|} - 1 \right) \]

\[ + \text{sign}((pr - q)\alpha - r\beta + \delta)re_3^2(\beta - p\alpha) \left( \frac{|h_2(z)|}{|s_2(z, \bar{z})|} - 1 \right) \]

\[ + e_3^2|(pr - q)\alpha - r\beta + \delta| \left( \frac{|h_3(z)|}{|s_3(z, \bar{z})|} - 1 \right). \]

(3.180)

Note that \( \beta - p\alpha \neq 0 \) and \( (pr - q)\alpha - r\beta + \delta \neq 0 \) cannot vanish as that would make the theory touch the Coulomb branch on the second and third flavors, respectively, or in other words, the gauge symmetry would be unbroken.

By theorem 2, the magnetic fluxes are given by

\[
\begin{pmatrix}
k^1 \\
k^2 \\
k^3
\end{pmatrix}
= \begin{pmatrix}
n_1 - pn_2 + (pr - q)n_3 \\
n_2 - rn_3 \\
n_3
\end{pmatrix},
\]

(3.181)

and the signs of \((n_1, n_2, n_3)\) when nonvanishing must obey

\[
\text{sign}(n_1) = \text{sign}(\alpha k^1 + \beta k^2 + \delta k^3),
\]

(3.182)

\[
\text{sign}(n_2) = \text{sign}(\alpha k^1 + \beta k^2 + \delta k^3) \text{sign}(\beta - p\alpha),
\]

(3.183)

\[
\text{sign}(n_3) = \text{sign}(\alpha k^1 + \beta k^2 + \delta k^3) \text{sign}((pr - q)\alpha - r\beta + \delta),
\]

(3.184)

and finally by theorem 3, \( \alpha k^1 + \beta k^2 + \delta k^3 \neq 0 \) cannot vanish.
We can now make a change of variables: $u_A = 2 \log \left| \frac{h_A(z)}{s_A(z, \beta)} \right|$, for which we can write the system of PDEs as Eq. (3.122):

$$
\nabla^2 u_A = \sum_B A_{AB} (e^{u_B} - 1) + 4\pi \sum_{i=1}^{\lfloor n_A \rfloor} \delta(z - z_i^A),
$$

(3.185)

where we have defined the real matrix

$$
A_{AB} = 2 \begin{pmatrix}
(e_1^2 + p^2e_2^2 + q^2e_3^2)\alpha & (pe_2^2 + qre_3^2)(\beta - p\alpha) & qe_3^2((pr - q)\alpha - r\beta + \delta) \\
S_2(pe_2^2 + qre_3^2)\alpha & (e_2^2 + r^2e_3^2)|\beta - p\alpha| & S_2re_3^2((pr - q)\alpha - r\beta + \delta) \\
S_3qe_3^2\alpha & S_3re_3^2(\beta - p\alpha) & e_3^2((pr - q)\alpha - r\beta + \delta)
\end{pmatrix},
$$

(3.186)

with

$$
S_2 \equiv \text{sign}(\beta - p\alpha), \quad S_3 \equiv \text{sign}((pr - q)\alpha - r\beta + \delta).
$$

(3.187)

Note that although the diagonal entries are all positive definite, the matrix $A$ is not symmetric, but always positive definite. We can see this from the determinant

$$
\det A = 8e_1^2e_2^2e_3^2\alpha |(\beta - p\alpha)((pr - q)\alpha - r\beta + \delta)| > 0,
$$

(3.188)

which can never switch sign or vanish as $\beta - p\alpha \neq 0$, $(pr - q)\alpha - r\beta + \delta \neq 0$ and $\alpha > 0$. This is also $(1/8$ times) the determinant of the mass-squared matrix (2.41) for the photons, which must be positive definite in order to ensure broken gauge symmetry.

Similarly to the $N = M = 2$ example, in the special case where the matrix $A$ is symmetric, the existence and uniqueness (as well as sharp decay estimates) have been proven by Yang in Ref. [13] (same system). Such a symmetric case corresponds (for this specific example) to:

$$
\alpha = |\beta - p\alpha|, \quad \beta \neq 0 \quad \text{and} \quad |r_3| \neq 0.
$$

(3.189)

### 3.7.3.2 Type B2 solution.

By theorem 4, the B2 solution has a block diagonal charge matrix, which in this example we shall choose in the form

$$
Q = \begin{pmatrix}
1 & q & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad q \in \mathbb{Z}\setminus\{0\},
$$

(3.190)

and the FI vectors cannot all be parallel by theorem 4, but by theorem 6, the first two FI vectors (due to the winding block of the above matrix being 2-by-2) must be parallel. We thus set the FI vectors to be: $r_1 = (\alpha, 0, 0)$, $r_2 = (\beta, 0, 0)$ and $r_3 = (\delta, \kappa, \eta)$ with $\alpha > 0$, $\beta \neq 0$ and $|r_3| \neq 0$. The winding flavors are $\tilde{A} = 1, 2$. By corollary 5, the vacuum of the winding flavors is

$$
z_{1,2} = (\alpha, \beta - q\alpha),
$$

(3.191)
and the magnetic flux is given by

\[ k^{1,2} = (n_1 - qn_2, n_2), \]  

(3.192)

so the sign

\[ m_1 = \text{sign}(\alpha k^1 + \beta k^2). \]  

(3.193)

By theorem 8, \( \alpha k^1 + \beta k^2 \neq 0 \) cannot vanish. The non-winding flavors obey the constraint equations by lemma 3 and the vortex equations by theorem 5. The non-winding of the third flavor, \( n_3 = 0 \), implies that \( k^3 = 0 \) and hence the string tension reads

\[ T = T_{\text{BPS}} = 2\pi |\alpha k^1 + \beta k^2|. \]  

(3.194)

Finally, by theorem 11, we have the master equations

\[ -2\bar{z}_{\bar{z}} \log |\varsigma_1(z, \bar{z})|^2 = (e_1^2 + q^2 e_2^2)\alpha \left( \left| \frac{h_1(z)}{\varsigma_1(z, \bar{z})} \right|^2 - 1 \right) + q e_2^2 (\beta - q\alpha) \left( \left| \frac{h_2(z)}{\varsigma_2(z, \bar{z})} \right|^2 - 1 \right), \]  

(3.195)

\[ -2\bar{z}_{\bar{z}} \log |\varsigma_2(z, \bar{z})|^2 = \text{sign}(\beta - q\alpha) q e_2^2 \left( \left| \frac{h_1(z)}{\varsigma_1(z, \bar{z})} \right|^2 - 1 \right) + e_2^2 |\beta - q\alpha| \left( \left| \frac{h_2(z)}{\varsigma_2(z, \bar{z})} \right|^2 - 1 \right). \]  

(3.196)

Comparing with the master equations (3.142) and (3.143), this case reduces to the equations found in Sec. 3.7.2.1.

3.7.3.3 Type B1 solution. By theorem 4, the B1 solution can have an arbitrary charge matrix with integer elements and nonvanishing determinant, but only a single winding flavor. We take the charge matrix to be

\[ Q = \begin{pmatrix} 1 & p & q \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}, \]  

(3.197)

and the winding flavor to be \( \tilde{A} = 3 \). Also by theorem 4, there is no restriction on the FI vectors, so we will take the most general case \( r_1 = (\alpha, 0, 0) \), \( r_2 = (\beta, \gamma, 0) \) and \( r_3 = (\delta, \kappa, \eta) \), with \( \alpha > 0 \), \( |r_2| \neq 0 \) and \( |r_3| \neq 0 \). The vacuum solution is given by

\[ z_A = \begin{pmatrix} \alpha \\ \beta - p\alpha + i\gamma \\ (pr - q)\alpha - r(\beta + i\gamma) + \delta + i\kappa \end{pmatrix}^T, \quad y_A = \begin{pmatrix} 0 \\ 0 \\ \eta \end{pmatrix}^T, \]  

(3.198)
and the magnetic fluxes by

\[ k^a = \begin{pmatrix} (pr - q)n_3 \\ -rn_3 \\ n_3 \end{pmatrix}. \quad (3.199) \]

The string tension reads

\[ T = T_{\text{BPS}} = 2\pi \sqrt{\frac{\alpha k^1 + \beta k^2 + \delta k^3}{2} + (\gamma k^2 + \kappa k^3)^2 + (\eta k^3)^2} \]
\[ = 2\pi \sqrt{(pr - q)\alpha - r\beta + \delta}^2 + (\kappa - r\gamma)^2 + \eta^2 |n_3|, \quad (3.200) \]

which is linear in the (only) winding number \( n_3 \), but has a square-root form of the FI parameters. By theorem 9, the vector

\[ \sum_a r_a k^a = \begin{pmatrix} (pr - q)\alpha - r\beta + \delta \\ \kappa - r\gamma \\ \eta \end{pmatrix} n_3 \quad (3.201) \]

cannot have a vanishing length, i.e.

\[ \sqrt{(pr - q)\alpha - r\beta + \delta}^2 + (\kappa - r\gamma)^2 + \eta^2 |n_3| \neq 0, \quad (3.202) \]

and therefore the BPS bound cannot vanish as stated in the theorem. The \( \mathbf{m} \) vector reads

\[ \mathbf{m} = \frac{1}{\sqrt{(pr - q)\alpha - r\beta + \delta}^2 + (\kappa - r\gamma)^2 + \eta^2} \begin{pmatrix} (pr - q)\alpha - r\beta + \delta \\ \kappa - r\gamma \\ \eta \end{pmatrix} \text{sign}(n_3). \quad (3.203) \]

Finally, by theorem 12, the master equations read

\[ -2\partial_z \partial_{\bar{z}} \log |\varsigma_1(z, \bar{z})|^2 = \text{sign}(n_3)q e_3^2 \sqrt{(pr - q)\alpha - r\beta + \delta}^2 + (\kappa - r\gamma)^2 + \eta^2 \left( \frac{h_3(z)}{\varsigma_3(z, \bar{z})} \right)^2 - 1 \]
\[ + \frac{e_1^2 + p^2 e_2^2 + q^2 e_3^2}{2} \left( \alpha + \frac{\text{sign}(n_3)\alpha((pr - q)\alpha - r\beta + \delta)}{\sqrt{(pr - q)\alpha - r\beta + \delta}^2 + (\kappa - r\gamma)^2 + \eta^2} \right) \left( \frac{1}{\varsigma_1(z, \bar{z})^2} - 1 \right) \]
\[ - \left( \alpha - \frac{\text{sign}(n_3)\alpha((pr - q)\alpha - r\beta + \delta)}{\sqrt{(pr - q)\alpha - r\beta + \delta}^2 + (\kappa - r\gamma)^2 + \eta^2} \right) (|\varsigma_1(z, \bar{z})|^2 - 1) \]
\[ + \frac{pe_2^2 + qre_3^2}{2} \left( \sqrt{\beta - p\alpha}^2 + \gamma^2 \right) \]
\[ + \frac{pe_2^2 + qre_3^2}{2} \left( \sqrt{\beta - p\alpha}^2 + \gamma^2 - \text{sign}(n_3) \frac{(\beta - p\alpha)((pr - q)\alpha - r\beta + \delta) + \gamma(\kappa - r\gamma)}{\sqrt{(pr - q)\alpha - r\beta + \delta}^2 + (\kappa - r\gamma)^2 + \eta^2} \right) \left( \frac{1}{\varsigma_2(z, \bar{z})^2} - 1 \right) \]
\[ - \left( \sqrt{\beta - p\alpha}^2 + \gamma^2 - \text{sign}(n_3) \frac{(\beta - p\alpha)((pr - q)\alpha - r\beta + \delta) + \gamma(\kappa - r\gamma)}{\sqrt{(pr - q)\alpha - r\beta + \delta}^2 + (\kappa - r\gamma)^2 + \eta^2} \right) (|\varsigma_2(z, \bar{z})|^2 - 1) \],

(3.204)
\[-2\partial_z \partial_{\bar{z}} \log |\varsigma_2(z, \bar{z})|^2 = \text{sign}(n_3)re_3^2 \sqrt{((pr-q)\alpha - r\beta + \delta)^2 + (\kappa - r\gamma)^2 + \eta^2} \left( \frac{h_3(z)}{|\varsigma_3(z, \bar{z})|} - 1 \right) + \frac{pe_2^2 + qe_2^2}{2} \left( \alpha + \frac{\text{sign}(n_3)\alpha((pr-q)\alpha - r\beta + \delta)}{\sqrt{((pr-q)\alpha - r\beta + \delta)^2 + (\kappa - r\gamma)^2 + \eta^2}} \right) \left( \frac{1}{|\varsigma_1(z, \bar{z})|^2} - 1 \right) - \left( \alpha - \frac{\text{sign}(n_3)\alpha((pr-q)\alpha - r\beta + \delta)}{\sqrt{((pr-q)\alpha - r\beta + \delta)^2 + (\kappa - r\gamma)^2 + \eta^2}} \right) (|\varsigma_1(z, \bar{z})|^2 - 1) \right] \]

\[
\frac{e_3^2 + r^2e_3^2}{2} \left[ \sqrt{(\beta - pa)^2 + \gamma^2} + \text{sign}(n_3)\sqrt{(\beta - pa)((pr-q)\alpha - r\beta + \delta) + (\kappa - r\gamma)^2)} \left( \frac{1}{|\varsigma_2(z, \bar{z})|^2} - 1 \right) - \left( \text{sign}(n_3)\sqrt{(\beta - pa)^2 + \gamma^2 - \text{sign}(n_3)\sqrt{(\beta - pa)((pr-q)\alpha - r\beta + \delta) + (\kappa - r\gamma)^2)}} \right) (|\varsigma_2(z, \bar{z})|^2 - 1) \right], \tag{3.205}
\]

\[-2\partial_z \partial_{\bar{z}} \log |\varsigma_3(z, \bar{z})|^2 = e_3^2 \sqrt{((pr-q)\alpha - r\beta + \delta)^2 + (\kappa - r\gamma)^2 + \eta^2} \left( \frac{h_3(z)}{|\varsigma_3(z, \bar{z})|} - 1 \right) + \frac{qe_2^2}{2} \left( \text{sign}(n_3)\alpha + \frac{\alpha((pr-q)\alpha - r\beta + \delta)}{\sqrt{((pr-q)\alpha - r\beta + \delta)^2 + (\kappa - r\gamma)^2 + \eta^2}} \right) \left( \frac{1}{|\varsigma_1(z, \bar{z})|^2} - 1 \right) - \left( \text{sign}(n_3)\alpha - \frac{\alpha((pr-q)\alpha - r\beta + \delta)}{\sqrt{((pr-q)\alpha - r\beta + \delta)^2 + (\kappa - r\gamma)^2 + \eta^2}} \right) (|\varsigma_1(z, \bar{z})|^2 - 1) \right] \]

\[
+ \frac{re_3^2}{2} \left( \text{sign}(n_3)\sqrt{(\beta - pa)^2 + \gamma^2 - \text{sign}(n_3)\sqrt{(\beta - pa)((pr-q)\alpha - r\beta + \delta) + (\kappa - r\gamma)^2)}} \right) (|\varsigma_2(z, \bar{z})|^2 - 1) \right], \tag{3.206}
\]

where \((\beta - pa)^2 + \gamma^2 \neq 0\) and \((pr-q)\alpha - r\beta + \delta)^2 + (\kappa - r\gamma)^2 \neq 0\) cannot vanish as either would imply unbroken gauge symmetry.

We can now make a change of variables: \(u_1 = -2 \log |\varsigma_1(z, \bar{z})|, u_2 = -2 \log |\varsigma_2(z, \bar{z})|, u_3 = 2 \log \frac{h_3(z)}{|\varsigma_3(z, \bar{z})|}\), for which we can write the system of PDEs as Eq. (3.130):

\[
\nabla^2 u_A = A_A(e^{u_A} - 1) + \sum_{B \neq 3} B_{AB}(e^{u_B} - 1) - \sum_{B \neq 3} C_{AB}(e^{-u_B} - 1) + 4\pi \delta^{A_3} \sum_{i=1}^{|n_3|} \delta(z - Z_i), \tag{3.207}
\]

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where we have defined

\[
\mathcal{A}_A \equiv 2 \begin{pmatrix}
\text{sign}(n_3) q e_3^2 \\
\text{sign}(n_3) r e_3^2 \\
e_3^2
\end{pmatrix} \sqrt{((pr-q)\alpha - r\beta + \delta)^2 + (\kappa - r\gamma)^2 + \eta^2},
\tag{3.208}
\]

\[
\mathcal{B}_{A1} \equiv \begin{pmatrix}
e_1^2 + p^2 e_2^2 + q^2 e_3^2 \\
pe_2^2 + qre_3^2 \\
\text{sign}(n_3) q e_3^2
\end{pmatrix} \left( \alpha + \frac{\text{sign}(n_3)\alpha((pr-q)\alpha - r\beta + \delta)}{\sqrt{((pr-q)\alpha - r\beta + \delta)^2 + (\kappa - r\gamma)^2 + \eta^2}} \right),
\tag{3.209}
\]

\[
\mathcal{B}_{A2} \equiv \begin{pmatrix}
pe_2^2 + qre_3^2 \\
e_2^2 + r^2 e_3^2 \\
\text{sign}(n_3) re_3^2
\end{pmatrix} \left( \sqrt{(\beta - p\alpha)^2 + \gamma^2 + \text{sign}(n_3)(\beta - p\alpha)((pr-q)\alpha - r\beta + \delta) + \gamma(\kappa - r\gamma)} \right) / \sqrt{((pr-q)\alpha - r\beta + \delta)^2 + (\kappa - r\gamma)^2 + \eta^2},
\tag{3.210}
\]

\[
\mathcal{C}_{A1} \equiv - \begin{pmatrix}
e_1^2 + p^2 e_2^2 + q^2 e_3^2 \\
pe_2^2 + qre_3^2 \\
\text{sign}(n_3) q e_3^2
\end{pmatrix} \left( \alpha - \frac{\text{sign}(n_3)\alpha((pr-q)\alpha - r\beta + \delta)}{\sqrt{((pr-q)\alpha - r\beta + \delta)^2 + (\kappa - r\gamma)^2 + \eta^2}} \right),
\tag{3.211}
\]

\[
\mathcal{C}_{A2} \equiv \begin{pmatrix}
pe_2^2 + qre_3^2 \\
e_2^2 + r^2 e_3^2 \\
\text{sign}(n_3) re_3^2
\end{pmatrix} \left( \sqrt{(\beta - p\alpha)^2 + \gamma^2 - \text{sign}(n_3)(\beta - p\alpha)((pr-q)\alpha - r\beta + \delta) + \gamma(\kappa - r\gamma)} \right) / \sqrt{((pr-q)\alpha - r\beta + \delta)^2 + (\kappa - r\gamma)^2 + \eta^2},
\tag{3.212}
\]

with \( \mathcal{A} \) an \( N \)-vector and \( \mathcal{B} \) and \( \mathcal{C} \) are \( N \times (N-1) \) matrices (and \( N = 3 \)). We do not know of any existence of uniqueness results for the equation (3.207).

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