Empirical processes with a bounded $\psi_1$ diameter

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Abstract

We study the empirical process $\sup_{f \in F} \{|N^{-1} \sum_{i=1}^{N} f^2(X_i) - E f^2|\}$, where $F$ is a class of mean-zero functions on a probability space $(\Omega, \mu)$ and $(X_i)_{i=1}^{N}$ are selected independently according to $\mu$.

We present a sharp bound on this supremum that depends on the $\psi_1$ diameter of the class $F$ (rather than on the $\psi_2$ one) and on the complexity parameter $\gamma_2(F, \psi_2)$. In addition, we present optimal bounds on the random diameters $\sup_{f \in F} \max_{|I|=m} (\sum_{i \in I} f^2(X_i))^{1/2}$ using the same parameters. As applications, we extend several well known results in Asymptotic Geometric Analysis to any isotropic, log-concave ensemble on $\mathbb{R}^n$.

1 Introduction

In this article we study the empirical process

$$\sup_{f \in F} \left| \frac{1}{N} \sum_{i=1}^{N} f^2(X_i) - E f^2 \right|,$$

where $F$ is a class of functions on the probability space $(\Omega, \mu)$ and $(X_i)_{i=1}^{N}$ are independent, distributed according to $\mu$. Properties of this process play an important part in Asymptotic Geometric Analysis and in Nonparametric Statistics, though even without considering the possible applications, (1.1) is a natural object. Indeed, a fundamental problem in Empirical Processes Theory is to understand the way the empirical (random) structure of a class would...
of functions, obtained by random sampling, captures the original structure determined by the underlying measure \(\mu\). More accurately, one wishes to relate, with high probability, \( N^{-1} \sum_{i=1}^{N} \ell(f(X_i)) \) to \( \mathbb{E} \ell(f) \), uniformly in \( f \in F \), for a reasonable real valued function \( \ell \). The two most natural functions that are considered in this context are \( \ell(t) = t \), which leads to the Uniform Law of Large Numbers, and \( \ell(t) = t^2 \), which is connected to properties of the Uniform Central Limit Theorem and gives information on the way the empirical \( \ell_2 \) structure of \( F \) is connected to the \( L_2(\mu) \) one (see [13, 37] for an extensive study of these topics).

Despite its importance, bounds on (1.1) are not satisfactory. Standard empirical processes methods allow one to bound (1.1) only in rather trivial cases, in which either the class \( F \) is bounded in \( L_\infty \), or if it has a well behaved envelope function (recall that an envelope function is \( W(\omega) = \sup_{f \in F} |f(\omega)| \)). In those cases it is possible to use contraction methods and control (1.1) using the linear process \( \sup_{f \in F} |N^{-1} \sum_{i=1}^{N} f(X_i) - \mathbb{E}f| \), which is a far simpler object than (1.1). However, if the function class is not uniformly bounded, or even if it is, but with a very weak uniform bound, contraction based methods lead to trivial estimates on (1.1).

An alternative approach is to control (1.1) using random parameters of \( F \) that depend on the geometry of typical coordinate projections

\[ P_\sigma F = \{(f(X_1), ..., f(X_N)) : f \in F\}, \]

for an independent sample \( \sigma = (X_1, ..., X_N) \). The downside of this approach is that the structure of \( P_\sigma F \) itself is often difficult to handle, let alone that of \( P_\sigma F^2 \) for \( F^2 = \{f^2 : f \in F\} \). Moreover, the standard way of relating the geometry of \( P_\sigma F^2 \) to that of \( P_\sigma F \) also involves contraction methods, resulting in the same type of problems that have been mentioned above.

To illustrate the difficulty, consider the following, seemingly simple problem. Let \( \Omega = \mathbb{R}^n \) and assume that \( \mu \) is a natural measure on \( \mathbb{R}^n \), say the canonical gaussian measure, the uniform measure on \( \{-1, 1\}^n \), or more generally, an isotopic log-concave measure (see the definitions in Section 2). Let \( F \) be the class of linear functionals on \( \mathbb{R}^n \) of Euclidean norm one, that is, \( F = \{\langle x, \cdot \rangle : x \in S^{n-1}\} \).

Note that \( F \) may consist of unbounded functions on \((\Omega, \mu)\), or, at best, of functions with an \( L_\infty \) bound that grows polynomially with the dimension \( n \). It is straightforward to show that contraction based methods lead to a very loose estimate on (1.1) in such a case. To make things worse, if one considers a typical sample \( (X_i)_{i=1}^{N} \), the structure of the ellipsoid \( P_\sigma F \) is hard to handle (certainly if all the information that one has on \( \mu \) is that
it is an isotropic, log-concave measure). And, finally, an attempt to bound \((1.1)\) using the structure of the class \(F^2 = \{f^2 : f \in F\}\) directly, without linearizing, will fail because \(P_\sigma F^2\) is a rather complicated object.

It would be highly desirable to bound \((1.1)\) using a deterministic parameter of \(F\), that is, a metric invariant of \(F\) that depends on \(\mu\) and not on \((X_i)_{i=1}^N\), since in many applications (the example mentioned above for one), \(F\) has a simple structure relative to a natural metric. Thus, our aim here is to obtain bounds on \((1.1)\) that depend on the deterministic structure of \(F\) as a class of functions on \((\Omega, \mu)\). All we will assume is that \(F\) consists of functions that have well behaved tails, but may be unbounded, and the class may be without a good envelope function.

It turns out that if one wishes to bound \(\mathbb{E} \sup_{f \in F} \left\{ N^{-1} \sum_{i=1}^N f(X_i) - \mathbb{E} f \right\}\) using a deterministic metric structure of \(F\), one has to consider metrics that are stronger than the \(L_p(\mu)\) ones (see Lemma 3.6 and Remark 3.7 for an exact formulation of this observation). More reasonable metrics for such a goal are the Orlicz norms \(\psi_\alpha\) for \(1 \leq \alpha \leq 2\). These norms are defined via the Young function \(\exp((x^\alpha) - 1)\) for \(\alpha \geq 1\) by

\[
\|f\|_{\psi_\alpha} = \inf \{ c > 0 : \mathbb{E} \exp(|f/c|^\alpha) \leq 2 \}.
\]

It is possible to bound the “linear” process using a natural complexity parameter of \(F\) that originated in the theory of Gaussian Processes. This complexity parameter is defined for any metric space \((T, d)\) and is denoted by \(\gamma_2(T, d)\) (see the book [36] and Section 2 for its definition and some of its properties). Indeed, it is standard to show that

\[\mathbb{E} \sup_{f \in F} \left| \frac{1}{N} \sum_{i=1}^N f(X_i) - \mathbb{E} f \right| \leq c \gamma_2(F, \psi_2) \frac{\sqrt{N}}{\sqrt{N}},\]

where \(c\) is an absolute constant, and that a similar bound holds with high probability (see Lemma 2.5). Moreover, as we will explain in Section 3.1, it is impossible to obtain such a bound using a weaker \(\psi_\alpha\) metric.

Unfortunately, if one is interested, as we are, in bounds on the empirical process indexed by \(F^2\) using complexity parameters of \(F\) itself, a contraction type argument only yields that

\[
\mathbb{E} \sup_{f \in F} \left| \frac{1}{N} \sum_{i=1}^N f^2(X_i) - \mathbb{E} f^2 \right| \leq c \left( \sup_{f \in F} \|f\|_\infty \right) \frac{\gamma_2(F, \psi_2)}{\sqrt{N}}, \tag{1.2}
\]

which is unsatisfactory when dealing with a class of unbounded or weakly bounded functions that only have nice tails. For such classes, \((1.2)\) is meaningless.
An improvement to this contraction based estimate appeared in [22] and later in [24], where it was shown that if $F$ is a symmetric subset of the $L_2(\mu)$ unit sphere (i.e. $\|f\|_{L_2(\mu)} = 1$ and if $f \in F$ then $-f \in F$), one has

$$\mathbb{E} \sup_{f \in F} \left| \frac{1}{N} \sum_{i=1}^{N} f^2(X_i) - \mathbb{E} f^2 \right| \leq c \max \left\{ \left( \sup_{f \in F} \|f\|_{\psi_2} \right)^2 \frac{\gamma_2(F, \psi_2)}{\sqrt{N}}, \frac{\gamma_2^2(F, \psi_2)}{N} \right\}. \tag{1.3}$$

Thus, the diameter of $F$ in $L_\infty$ may be replaced by its diameter in $\psi_2$.

There are many applications that follow from (1.3). For example, it was used in [24] to solve the approximate and exact reconstruction problems (studied, e.g., in [10, 11, 12]) in a rather general situation that includes any isotropic, subgaussian ensemble. However, even (1.3) still leaves something to be desired, since it too is meaningless for a large class of natural measures. Indeed, consider the volume measure on an isotropic convex body in $\mathbb{R}^n$, or more generally, an isotropic, log-concave measure on $\mathbb{R}^n$. Again, if we set $F = \{ \langle x, \cdot \rangle : x \in S^{n-1} \}$ – the class of linear functionals of Euclidean norm one, it may have a very bad diameter with respect to the $\psi_2(\mu)$ norm (as bad as $\sqrt{n}$), whereas, thanks to Borell’s inequality ([8], see also [27]), its $\psi_1(\mu)$ diameter is at most an absolute constant, independent of the dimension. Thus, it seems natural to ask whether one may replace $d_{\psi_2} = \sup_{f \in F} \|f\|_{\psi_2}$ in (1.3), with $d_{\psi_1} = \sup_{f \in F} \|f\|_{\psi_1}$. The main result of this article is a positive answer to this question.

**Theorem A.** There exists an absolute constant $c$ for which the following holds. If $F$ is a symmetric class of mean-zero functions on $(\Omega, \mu)$ then

$$\mathbb{E} \sup_{f \in F} \left| \frac{1}{N} \sum_{i=1}^{N} f^2(X_i) - \mathbb{E} f^2 \right| \leq c \max \left\{ d_{\psi_1} \frac{\gamma_2(F, \psi_2)}{\sqrt{N}}, \frac{\gamma_2^2(F, \psi_2)}{N} \right\}, \tag{1.4}$$

and a similar bound holds with high probability.

A key ingredient in the proof of Theorem A and our second main result, deals with the structure of random coordinate projections of a given class of functions that have nice tail properties. We will be interested in the growth of the Euclidean norm of monotone rearrangements of vectors in $P_\sigma F$: for every $1 \leq m \leq N$ consider

$$D_m = \sup_{v \in P_\sigma F} \left( \sum_{i=1}^{m} (v^2)_i \right)^{1/2} = \sup_{f \in F} \max_{|I| = m} \left( \sum_{i \in I} f^2(X_i) \right)^{1/2}.$$
where \((v^*_i)_{i=1}^N\) is a non-increasing rearrangement of \((|v_i|)_{i=1}^N\). We will present high probability, sharp bounds on the empirical diameters \(D_m\) and use them in the proof of Theorem A.

Let us consider a simple example that indicates which bound on \(D_m\) one can hope for. Let \(\mu\) be the canonical gaussian measure on \(\mathbb{R}^n\). Hence, if \((g_i)_{i=1}^n\) are independent, standard normal random variables and \(G = (g_1, ..., g_n)\), then for every Borel set \(A \subset \mathbb{R}^n\), \(\mu(A) = \Pr(G \in A)\). Let \(K \subset \mathbb{R}^n\), consider \(F = \{\langle x, \cdot \rangle : x \in K\}\), the class of linear functionals indexed by \(K\), and put \((X_i)_{i=1}^N\) to be independent, distributed according to \(\mu\). Thus, \((X_i)_{i=1}^N\) are independent copies of \(G\), and the coordinate projection of \(F\) is given by \(P_{\sigma} F = \{\langle x, x_i \rangle_{i=1}^N : x \in K\}\). Observe that there exists an absolute constant \(c\) such that for every \(1 \leq m \leq N\),

\[
\mathbb{E} \sup_{v \in \mathbb{R}_+^F} \left( \sum_{i=1}^m (v^*_i)^2 \right)^{1/2} \geq c \left( \mathbb{E} \sup_{x \in K} \sum_{i=1}^n g_i x_i + \sup_{x \in K} \|x\|_{\ell_2} \sqrt{m \log(eN/m)} \right),
\]

where \(\ell_2\) is the Euclidean norm on \(\mathbb{R}^n\). Indeed, this lower bound is evident because the first term is just the case \(m = N = 1\), while the second term is an estimate for a single point \(x \in K\) which has a maximal Euclidean norm.

The simple reasoning that leads to (1.5) gives the impression that the estimate is far from sharp. However, it turns out that there is an upper bound that holds in considerably more general situations, and that matches the lower bound in the gaussian case. The complexity parameter is, again, the \(\gamma_2\) functional with respect to the \(\psi_2\) norm, while the term that represents the behavior of the “worst” in the class is \(d_\alpha = \sup_{f \in F} \|f\|_{\psi_\alpha}\) for \(1 \leq \alpha \leq 2\).

**Theorem B.** For every \(1 \leq \alpha \leq 2\) there is a constant \(c_\alpha\) that depends only on \(\alpha\), and absolute constants \(c_1\) and \(c_2\) for which the following holds. Let \(F\) be a class of mean-zero functions. Then, for every \(u \geq c_1\), with probability at least \(1 - \exp(-c_2 u \log N)\), for every \(f \in F\) and every \(1 \leq m \leq N\),

\[
\max_{|I|=m} \left( \sum_{i \in I} f^2(X_i) \right)^{1/2} \leq c_\alpha u \left( \gamma_2(F, \psi_2) + d_{\psi_\alpha} m^{1/2} \log^{1/\alpha}(eN/m) \right).
\]

To put this result in the right perspective let us return to the gaussian example. If \(\mu\) is the canonical gaussian measure on \(\mathbb{R}^n\) then the \(\psi_2\) norm endowed on \(\mathbb{R}^n\) is equivalent to the Euclidean one. In particular, for every \(m \leq N\), \(\sup_{x \in K} \|x\|_{\ell_2} \sqrt{m \log(eN/m)}\) and \(\sup_{f \in F} \|f\|_{\psi_2} \sqrt{m \log(eN/m)}\) are
equivalent. Moreover, by the Majorizing Measures Theorem (see [36] and section 2) and since the Euclidean and the $\psi_2$ metrics are equivalent, so are $\mathbb{E} \sup_{x \in K} \sum_{i=1}^{n} g_i x_i$ and $\gamma_2(F, \psi_2)$. Hence, the bound in Theorem B is sharp (up to the absolute constants and the exact probabilistic estimate) for the class of linear functionals indexed by a subset of $\mathbb{R}^n$ and with respect to the Gaussian measure.

Theorem B reveals useful information on the way vectors in $P_\sigma F$ look like for a typical sample $(X_i)_{i=1}^N$. If $N$ is relatively small, namely, when $d_{\psi_\alpha} \sqrt{N} \ll \gamma_2(F, \psi_2)$, all the information one has is that the Euclidean norm of any $P_\sigma f$ is at most of the order of $\gamma_2(F, \psi_2)$. Then, for larger values of $N$ the situation changes. For every $f \in F$ and 

$$
\lambda = c'_\alpha d_{\psi_\alpha} \log^{1/\alpha}(eN d_{\psi_\alpha}^2 / \gamma_2^2(F, \psi_2)),
$$

the block $I = I(f) = \{i : |f(X_i)| \geq \lambda\}$ has small cardinality:

$$
\sup_{f \in F} |I(f)| \leq c_\alpha \gamma_2^2(F, \psi_2) / \lambda^2.
$$

Outside this block, a monotone rearrangement of any $P_\sigma f$ is dominated coordinate-wise by a rearrangement of the vector $(c_\alpha d_{\psi_\alpha} \log^{1/\alpha}(eN/i))$.

Note that the behavior of the “small” coordinates of each $P_\sigma f$ is natural for a single $\psi_\alpha$ random variable. Indeed, it is straightforward to verify that if $v$ is a $\psi_\alpha$ random variable and $(v_i)_{i=1}^N$ is a vector of independent copies of $v$, then with high probability, for every $i$, $v_i^* \leq c \|v\|_{\psi_\alpha} \log^{1/\alpha}(eN/i)$. Thus, our results show that for a random sample $\sigma$, the “small coordinates” of any $P_\sigma f$ are dominated by the typical behavior of a sample of the function in the class with the maximal $\psi_\alpha$ norm. From that point of view, each vector in $P_\sigma F$ can be decomposed into a “regular” part, which behaves as if $F$ has an envelope function whose $\psi_\alpha$ norm is $d_{\psi_\alpha}$, and a “peaky” part, which is supported on the block $I(f)$ and is bounded in $\ell_2^N$. The blocks $I(f)$ take care of the possibility that vectors in $P_\sigma F$ have a few “a-typical” large coordinates that are due to the complexity of the whole class.

To formulate a weak version of the decomposition result (the full one is presented in Theorem 4.1) we need two preliminary definitions. First, for sets $A, B \subset \mathbb{R}^n$, $A + B = \{a + b : a \in A, b \in B\}$ is the Minkowski sum of $A$ and $B$. Second, we denote by $B_p^N$ the unit ball of $\ell_p^N = (\mathbb{R}^N, \| \cdot \|_p)$ and by $B_{\psi_\alpha}^N$ the unit ball of the $\psi_\alpha$ norm on $\mathbb{R}^N$, when viewed as the space of functions on the probability space $\Omega = \{1, ..., N\}$ endowed with the uniform probability measure.

**Theorem C.** There exist absolute constants $c_1, ..., c_6$ for which the following holds. Let $F$ be a class of mean-zero functions. For $1 \leq \alpha \leq 2$ and for every
\( N \), set
\[
\lambda = c_1 \max \left\{ d_{\psi_2} \log^{1/\alpha} (c_2 N d_{\psi_2}^2 / \gamma_2^2 (F, \psi_2)), 1 \right\}.
\]
Then, for every \( t \geq c_3 \), with probability at least \( 1 - 2 \exp(-c_4 t \log N) \),
\[
P_\sigma F \subset c_5 t \left( \gamma_2 (F, \psi_2) B_2^N + (\lambda B_\infty^N \cap c_6 d_{\psi_2} B_\psi^N) \right).
\]

Theorem C extends and improves one of the main results from [25]. As we will explain in Section 4, it also extends an empirical processes version of a theorem due to Rudelson on selector processes from [33].

Let us turn to the applications of the three theorems described above that will be presented here. We will focus on properties of the random operator \( \Gamma = \sum_{i=1}^N \langle X_i, \cdot \rangle e_i \), considered as an operator between an arbitrary \( n \)-dimensional normed space \((\mathbb{R}^n, \| \cdot \|)\) and \( \ell_2^N \), where \((X_i)_{i=1}^N\) are independent, distributed according to an isotropic, log-concave measure \( \mu \) on \( \mathbb{R}^n \).

It is well known that many results in Asymptotic Geometric Analysis have been obtained using certain specific random selection methods, most often, according to the canonical gaussian measure on \( \mathbb{R}^n \), or with respect to the Haar measure on an appropriate Grassman manifold. These selection methods, combined with analogs of Theorem A and Theorem B for those models of randomness, lead to geometric information on the structure of convex bodies, most notably, to Dvoretzky type theorems and to low-\( M^* \) estimates (see, e.g. [27, 31]).

We will show that sometimes it is possible to use more general sampling methods and still obtain similar geometric results. In particular, we will show that parts of the classical, gaussian based theory, (e.g. “standard shrinking” and low-\( M^* \) estimates) may be extended to log-concave ensembles. In fact, the gaussian parameter associated with a convex body \( K \), \( \mathbb{E} \sup_{x \in K} \sum_{i=1}^n g_i x_i \), which is used as a complexity parameter in the classical gaussian based theory, is replaced in our results by \( \gamma_2 (K, \psi_2) \). And, although the two complexity parameters are seemingly different, it can be shown that they coincide if one resorts to the original sampling methods.

Because of their general nature, Theorems A, B and C have many other applications in very different directions, and these will not be presented here. For example (out of many), our results can be used to extend the analysis from the known cases to other ensembles of the reconstruction problem, approximate and exact (see, for example, [10, 11, 12, 24], of the statistical persistence problem [19, 6] and of various embedding problems. Some of the
applications are straightforward but others are more difficult, since obtaining sharp estimates on the complexity parameter $\gamma_2(F,\psi_2)$ can be nontrivial. To keep this article at a reasonable length and to maintain its focus on the structural, empirical processes oriented results, we chose to defer the presentation of most of the applications to a later work.

The article is organized as follows. In Section 2 we will present preliminary results and several definitions we will need. Then, in Section 3 we will prove Theorem B and Section 4 will be devoted to the proof of Theorem C. Theorem A will be proved in Section 5, and in Section 6 we will present some applications of the three theorems.

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2 Preliminaries

Let us begin with notational conventions. Throughout, all absolute constants are positive numbers, denoted by $c, c_0, c_1,...$ etc. Their value may change from line to line. We use $\kappa_0, \kappa_1,...$ for constants whose value will remain unchanged. By $A \sim B$ we mean that there are absolute constants $c$ and $C$ such that $cB \leq A \leq CB$, and by $A \lesssim B$ that $A \leq CB$. For $1 \leq p \leq \infty$, $\ell_p^n$ is $\mathbb{R}^n$ endowed with the $\ell_p$ norm, which we denote by $\| \cdot \|_p$, and $B^n_p$ is its unit ball. With a minor abuse of notation we denote by $\| \cdot \|$ the cardinality of a set and the absolute value.

We say that $K \subset \mathbb{R}^n$ is a convex body if it is a compact, convex and symmetric set (that is, if $x \in K$ then $-x \in K$) with a nonempty interior. If $K$ is a convex body we denote by $\| \cdot \|_K$ the norm on $\mathbb{R}^n$ whose unit ball is $K$ and set $K^o = \{y : \langle x, y \rangle \leq 1 \ \forall y \in K\}$ to be its polar body.

Given a probability measure $\mu$ and a sample $(X_i)_{i=1}^N$, we will sometimes write $P_N f = N^{-1} \sum_{i=1}^N f(X_i)$ and $P f = \mathbb{E} f$. Hence, the supremum of the empirical process indexed by $F$ is $\sup_{f \in F} |P_N f - P f|$. Given $f \in F$ and $\sigma \subset \{1, ..., N\}$ we set $P_\sigma f = (f(X_i))_{i \in \sigma}$.
A significant part of our discussion will use basic properties of sums of independent random variables that have nice tails. The proofs of the claims presented here may be found, for example, in [23], [18] or [37].

Recall that a random variable has a bounded $\psi_\alpha$ norm for $1 \leq \alpha \leq 2$ if there is some constant $C$ for which $\mathbb{E}\exp(|f|^\alpha/C^\alpha) \leq 2$, and in which case one sets

$$
\|f\|_{\psi_\alpha} = \inf \{ C : \mathbb{E}\exp(|f|^\alpha/C^\alpha) \leq 2 \}.
$$

One can show that there is an absolute constant $c$ such that if $f \in L_{\psi_\alpha}$, then for every $t \geq 1$, $\Pr(|f| \geq t) \leq 2 \exp(-ct^{\alpha}/\|f\|_{\psi_\alpha}^\alpha)$. Conversely, there is an absolute constant $c_1$ such that if $f$ displays a tail behavior dominated by $\exp(-t^\alpha/K^\alpha)$ for some $1 \leq \alpha \leq 2$, then $f \in L_{\psi_\alpha}$ and $\|f\|_{\psi_\alpha} \leq c_1 K$. We say that $X$ is a subgaussian random variable if $\|X\|_{\psi_2} < \infty$.

**Lemma 2.1** There exists an absolute constant $c$ for which the following holds. Let $X$ be a mean-zero, subgaussian random variable and let $X_1, \ldots, X_k$ be independent copies of $X$. Then, for any fixed $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$, $\|\sum_{i=1}^k a_i X_i\|_{\psi_2} \leq c\|X\|_{\psi_2}\|a\|_2$. Thus, for every $t > 0$,

$$
\Pr\left(\left|\sum_{i=1}^k a_i X_i\right| \geq ct\|X\|_{\psi_2}\|a\|_2\right) \leq 2 \exp(-t^2/2).
$$

In particular, if $(\varepsilon_i)_{i=1}^N$ are independent, symmetric $\{-1, 1\}$-valued random variables, then for every $(a_i)_{i=1}^N$,

$$
\Pr\left(\left|\sum_{i=1}^N a_i \varepsilon_i\right| \geq ct\|a\|_2\right) \leq 2 \exp(-t^2/2).
$$

For sums of independent $\psi_1$ random variables the situation is more delicate, and one should expect two types of behaviors: an early subgaussian decay followed by a subexponential one, as Bernstein’s inequality shows.

**Lemma 2.2** There exists an absolute constant $c$ for which the following holds. Let $X_1, \ldots, X_N$ be independent copies of a mean-zero random variable. Then, for any $t > 0$,

$$
\Pr\left(\left|\frac{1}{N} \sum_{i=1}^N X_i\right| > t\right) \leq 2 \exp\left(-c N \min\left(\frac{t}{\|X\|_{\psi_1}}, \frac{t^2}{\|X\|_{\psi_1}^2}\right)\right).
$$

This estimate may be extended to other values of $\alpha$. The next lemma is a standard outcome of Corollaries 2.9 and 2.10 from [35] (see [2] for the proof).
Lemma 2.3 Let $1 \leq \alpha \leq 2$ and let $(X_i)_{i=1}^N$ be independent, mean-zero random variables such that $\|X_i\|_{\psi_\alpha} \leq A$ for every $1 \leq i \leq N$. Then, for every $(a_i)_{i=1}^N \in \mathbb{R}^N$ and any $t > 0$,

$$Pr \left( \left| \sum_{i=1}^N a_i X_i \right| \geq tA \right) \leq 2 \exp \left( -c \min \left\{ \frac{t^2}{\|a\|^2_2}, \frac{t^\alpha}{\|a\|^\alpha_{\alpha^*}} \right\} \right),$$

where $1/\alpha + 1/\alpha^* = 1$ and $c$ is an absolute constant.

Next, let us turn to the main complexity parameter we will use - Talagrand’s $\gamma_2$ functional.

Definition 2.4 [36] For a metric space $(T,d)$, an admissible sequence of $T$ is a collection of subsets of $T$, $\{T_s : s \geq 0\}$, such that for every $s \geq 1$, $|T_s| \leq 2^s$ and $|T_0| = 1$. For $\beta \geq 1$, define the $\gamma_\beta$ functional by

$$\gamma_\beta(T,d) = \inf \sup_{t \in T} \sum_{s=0}^\infty 2^{s/\beta} d(t,T_s),$$

where the infimum is taken with respect to all admissible sequences of $T$. For an admissible sequence $(T_s)_{s \geq 0}$ we denote by $\pi_s t$ a nearest point to $t$ in $T_s$ with respect to the metric $d$.

When considered for a set $T \subset L_2$, $\gamma_2$ has close connections with properties of the canonical gaussian process indexed by $T$, and we refer the reader to [13, 36] for detailed expositions on these connections. One can show that under mild measurability assumptions, if $\{G_t : t \in T\}$ is a centered gaussian process indexed by a set $T$ then

$$c_1 \gamma_2(T,d) \leq \mathbb{E} \sup_{t \in T} G_t \leq c_2 \gamma_2(T,d),$$

where $c_1$ and $c_2$ are absolute constants and for every $s,t \in T$, $d^2(s,t) = \mathbb{E}|G_s - G_t|^2$. The upper bound is due to Fernique [14] and the lower bound is Talagrand’s Majorizing Measures Theorem [33]. Note that if $T \subset \mathbb{R}^n$, $(g_i)_{i=1}^n$ are standard, independent gaussians and $G_t = \sum_{i=1}^n g_i t_i$ then $d(s,t) = \|s - t\|_2$, and therefore

$$c_1 \gamma_2(T,\|\cdot\|_2) \leq \mathbb{E} \sup_{t \in T} \sum_{i=1}^n g_i t_i \leq c_2 \gamma_2(T,\|\cdot\|_2). \quad (2.1)$$
Note that a closely related complexity parameter that is used to describe geometric properties of a convex body \( K \) is

\[
M^*(K) = \int_{S^{n-1}} \|x\|_{K^*} d\sigma(x),
\]

where \( \sigma \) is the Haar measure on the sphere \( S^{n-1} \). This parameter is gaussian in nature and it is straightforward to verify that

\[
\sqrt{n}M^*(K) \sim E \sup_{x \in K} \sum_{i=1}^k g_i x_i.
\]

It is well known that chaining methods lead to simple bounds on empirical processes. Indeed, the following result is a combination of a chaining argument with Lemma 2.1 or with Lemma 2.2.

**Theorem 2.5** There exists an absolute constant \( c \) for which the following holds. If \( F \) is a class of functions on \((\Omega, \mu)\) then for every integer \( N \),

\[
E \sup_{f \in F} |P_N f - Pf| \leq c \left( \frac{\gamma_2(F, \psi_1)}{\sqrt{N}} + \frac{\gamma_1(F, \psi_1)}{N} \right),
\]

and

\[
E \sup_{f \in F} |P_N f - Pf| \leq c \frac{\gamma_2(F, \psi_2)}{\sqrt{N}}.
\]

Similar bounds hold with high probability.

Results of this flavor may be found in Chapters 1 and 2.7 of [36].

Finally, in Section 6 we will be interested in isotropic, log-concave measures on \( \mathbb{R}^n \).

**Definition 2.6** A symmetric probability measure \( \mu \) on \( \mathbb{R}^n \) is called isotropic if for every \( y \in \mathbb{R}^n \),

\[
\int |\langle x, y \rangle|^2 d\mu(x) = \|y\|^2_2.
\]

We say that a measure \( \mu \) on \( \mathbb{R}^n \) is \( L \)-subgaussian if for every \( x \in \mathbb{R}^n \),

\[
\|\langle x, \cdot \rangle\|_{L^2(\mu)} \leq L \|\langle x, \cdot \rangle\|_{L^2(\mu)}.
\]

The measure \( \mu \) is log-concave if for every \( 0 < \lambda < 1 \) and every nonempty Borel measurable sets \( A, B \subset \mathbb{R}^n \),

\[
\mu(\lambda A + (1 - \lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}.
\]

The canonical gaussian measure on \( \mathbb{R}^n \) is clearly isotropic and subgaussian, with \( L \) being an absolute constant. Lemma 2.1 implies that the same holds for the uniform measure on \( \{-1, 1\}^n \).
A typical example of a log-concave measure on $\mathbb{R}^n$ is the volume measure of a convex body in $\mathbb{R}^n$, a fact that follows from the Brunn-Minkowski inequality (see, e.g. [31]). Moreover, Borell’s inequality [8, 27] implies that there is an absolute constant $c$ such that if $\mu$ is an isotropic, log-concave measure on $\mathbb{R}^n$, then for every $x \in \mathbb{R}^n$, $\|\langle x, \cdot \rangle\|_{\psi_1} \leq c\|\langle x, \cdot \rangle\|_{L_2} = c\|x\|_2$. There are isotropic bodies with a subgaussian volume measure – for example, isotropic positions of $B_n^p$ for $p \geq 2$ [5]. However, the general situation is completely different, and there are many examples of volume measures of isotropic convex bodies in $\mathbb{R}^n$ for which linear functionals are far from exhibiting a bounded $\psi_2$ behavior. In fact, $\|\langle x, \cdot \rangle\|_{\psi_2}$ may be as large as $\sqrt{n}\|x\|_2$ (for example, $x = e_1$ and the volume measure on an isotropic position of $B_1^n$). We refer the reader to [16] for a survey on properties of the volume measure of isotropic convex bodies and, more generally, of isotropic log-concave measures on $\mathbb{R}^n$.

3 Bounding the diameter

This section is devoted to the proof of Theorem B. Although we will present a complete proof only for $\alpha = 1$, we will indicate the very minor modifications that are needed to prove it for any $1 \leq \alpha \leq 2$.

The first step in the proof of Theorem B is to construct a good cover of the Euclidean unit ball $B_2^N$, an idea which was used for a very similar goal in the proof of the main result in [1].

**Definition 3.1** If $A, B \subset \mathbb{R}^n$, we denote by $N(A, B)$ the smallest number of points $x_i \in A$ such that $A \subset \bigcup_i (x_i + B)$.

If $\| \|$ is a norm on $\mathbb{R}^n$ and $B = \{x : \|x\| \leq \varepsilon\}$, then the set $\{x_i\}$ is called an $\varepsilon$-cover of $A$ with respect to the norm $\| \|$.

Clearly, if $B$ is an $\varepsilon$ ball of some norm, then $N(A, B)$ is the smallest cardinality of a set $\{x_i\}$ such that for every $a \in A$, $\min_i \|a - x_i\| \leq \varepsilon$.

Fix an integer $N$ and define the following sets: for $1 \leq \ell \leq N/2$ put

$$A_\ell = \left\{z \in B_2^N : \supp(z) \leq \ell, \|z\|_\infty \leq 1/\sqrt{\ell}\right\}.$$  

Let $\varepsilon_\ell = \ell/N$, set $N_\ell \subset A_\ell$ to be an $\varepsilon_\ell$-cover of $A_\ell$ with respect to the $\ell_2^N$ norm and let $P_I : \mathbb{R}^N \to \mathbb{R}^N$ be the orthogonal projection onto the space spanned by the coordinates $(e_i)_{i \in I}$, that is, $P_I x = \sum_{i \in I} \langle e_i, x \rangle e_i$. A standard volumetric estimate shows that for every convex body $K \subset \mathbb{R}^N$, 

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\[ N(K, \varepsilon K) \leq (2/\varepsilon)^n. \] Therefore,
\[
|N_\ell| \leq \sum_{|I| = \ell} N(P_1 B_2^N, \varepsilon I B_2^N) \leq \left( \begin{array}{c} N \\ \ell \end{array} \right) \left( \frac{2}{\varepsilon \ell} \right) \leq \exp(c_0 \ell \log(\varepsilon N/\ell)) \quad (3.1)
\]
for a suitable absolute constant \( c_0 \).

Fix an integer \( m \leq N \) and assume that \( m = 2^{r_0} \) for some integer \( r_0 \). Define the sets \( B_m \) as follows:
\[
B_m = \left\{ z \in B_2^N : |\text{supp}(z)| \leq m, \text{supp}(z) = \bigcup_{r=0}^{r_0-1} I_r, P_I z \in N|I_r| \right\} \quad (3.2)
\]
where \((I_r)_{r=0}^{r_0-1}\) are disjoint sets of coordinates, \(|I_0| = 2\) and for \( r \geq 1\), \(|I_r| = 2^r\) (and thus the cardinality of their union is \( m \)).

It is evident that \( B_m \) consists of vectors in \( B_2^N \) that can be written as a sum over disjoint sets of coordinates \( I_r \) of cardinality \( 2^r \), and the projection onto each one of the “blocks” \( I_r \) belongs to the net \( N|I_r| \), and thus to \( A|I_r| \).

It is standard to verify that for every \( m = 2^{r_0} \),
\[
|B_m| \leq |N_2| \cdot \prod_{r=1}^{r_0-1} |N_{2^r}| \leq \prod_{r=0}^{r_0-1} \exp(c_0 2^r \log(eN/2^r)) \leq \exp(c_1 m \log(eN/m)).
\]

Let \( D_m = \sup_{f \in F} \sup_{|I| = m} (\sum_{i \in I} f^2(X_i))^{1/2} \). The next lemma shows that in order to bound \( D_m \) it is enough to consider the linearized process indexed by \( F \times B_m \) and defined by \((f, v) \rightarrow \sum_{i=1}^N f(X_i)v_i \). Although Lemma 3.2 is a purely deterministic result, it is formulated in the “random” context in which it will be used.

**Lemma 3.2** There exists an absolute constant \( C \) such that for every \( m \leq N/2 \) satisfying \( m = 2^{r_0} \) for some integer \( r_0 \), and for every \( X_1, ..., X_N \),
\[
D_m \leq C \sup_{f \in F} \sup_{v \in B_m} \sum_{i=1}^N v_i f(X_i).
\]

**Proof.** Let \( m = 2^{r_0} \) for some integer \( r_0 \) and assume that \( m \leq N/2 \). If \( v \in B_2^N \) for which \(|\text{supp}(v)| \leq m\), let \((v_i^*)_{i=1}^N\) be a monotone non-increasing rearrangement of \((|v_i|)_{i=1}^N\) and put \( v_{\sigma(j)} = v_j^* \), where \( \sigma \) is the suitable permutation of \( \{1, ..., N\} \). Consider the sets \((I_r)_{r=0}^{r_0-1}\) defined as follows: \( I_0 = \{\sigma(1), \sigma(2)\} \) are the largest two coordinate of \((|v_i|)_{i=1}^N\), \( I_1 = \{\sigma(3), \sigma(4)\} \) are
the two following that, and so on – \( I_r = \{ \sigma(2^r + 1), \sigma(2^{r+1}) \} \) for \( r \geq 1 \). Thus, \(|I_r| = 2^r \) for \( r \geq 1 \) and \(|I_0| = 2 \). Since \( v_i^* \leq 1/\sqrt{t} \) then for every \( j \),

\[
\|P_{I_j}v\|_\infty \leq 1/(|I_0|^{1/2}) = 2^{-j/2}
\]

and thus \( P_{I_j}v \in A_{2^j} \). Let \( \tilde{v} \in B_m \) be such that for every \( 1 \leq r \leq r_0 - 1 \), \( P_{I_{rj}}v, \tilde{v} \in N\|I\| = N_{2^r} \), \( \|P_{I_{rj}}v - P_{I_{rj}}\tilde{v}\|_2 \leq 2^r/N \) and \( \|P_{I_0}v - P_{I_0}\tilde{v}\|_2 \leq 2/N \).

Therefore, \( \|v - \tilde{v}\|_2 \leq 2/N + \sum_{r=1}^{r_0-1} 2^r/N \leq m/N \), and thus, if we set \( U_m = \{ v \in B_2^N : |\text{supp}(v)| \leq m \} \) then

\[
D_m = \sup_{f \in F, |I| = m} \left( \sum_{i \in I} f^2(X_i) \right)^{1/2} = \sup_{f \in F, |I| = m} \sup_v \sum_{i \in I} v_if(X_i)
\]

\[
\leq \sup_{f \in F, |I| = m} \sup_v \sum_{i \in I} (v_i - \tilde{v}_i)f(X_i) + \sup_{f \in F, |I| = m} \sum_{i \in I} \tilde{v}_if(X_i)
\]

\[
\leq (m/N)D_m + \sup_{f \in F, v \in U_m} \sum_{i \in I} \tilde{v}_if(X_i)
\]

\[
\leq (m/N)D_m + \sup_{f \in F, v \in B_m} \sum_{i = 1}^N v_if(X_i).
\]

Since \( m \leq N/2 \) it is evident that \( D_m \leq 2\sup_{f \in F} \sup_{v \in B_m} \sum_{i=1}^N v_if(X_i) \), as claimed.

**Remark 3.3** Observe that for every \( 1 \leq j \leq r_0 - 1 \) and every \( v \in B_{2^{j+1}}, P_{\bigcup_{r \leq j} I_r}, v \in B_{2j} \), a fact which will be used in the dimension reduction procedure that is needed in the proof of Theorem B.

We will need two simple observations about sums of centered random variables, both of which follow from Lemma 2.1 and Lemma 2.3. First, if \( \mathbb{E}f = 0 \) then for every \( t > 0 \) and any \( I \subset \{1, ..., N\} \),

\[
Pr \left( \left| \sum_{i \in I} v_if(X_i) \right| \geq t\|P_{I}v\|_2\|f\|_\psi \right) \leq 2\exp(-c_0t^2).
\]

Second, if \( \mathbb{E}f = 0 \) then for every \( t > 0 \) and any \( I \subset \{1, ..., N\} \),

\[
Pr \left( \left| \frac{1}{|I|} \sum_{i \in I} v_if(X_i) \right| \geq t\|P_Iv\|_\infty\|f\|_\psi \right) \leq 2\exp(-c_0|I|\min(t^2, t)), \tag{3.3}
\]

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where in both cases $c_0$ is an absolute constant.

Before proving Theorem B we need a few more definitions. Let $E_\ell$ be the collection of all subsets of $\{1, \ldots, N\}$ of cardinality $\ell$. Note that there is an absolute constant $\kappa_0$ such that for every integer $1 \leq \ell \leq N$, $\exp(\kappa_0 \ell \log(eN/\ell)) \geq \max\{|E_\ell|, |B_\ell|\}$, and define $s_\ell$ to be the first integer which satisfies that $2^{2s} \geq \exp(\kappa_0 \ell \log(eN/\ell))$.

The chaining argument we will use for $f \to \sup_{v \in B_m} \sum_{i=1}^N v_i f(X_i)$ consists of three parts. First, when $s \geq s_m$, the number of vectors in $B_m$ is much smaller than the number of possible “links” in all the chains, and thus no special treatment is needed. In the middle part, when $s_2 \leq s < s_m$, there will be a simultaneous reduction in the level $s$ and in the dimension which will be achieved by passing from the set $B_m$ to the sets $B_m/2^r$ for the correct value of $r$. Finally, when $s \leq s_2$ no further chaining will be required because the cardinality of the indexing sets is small enough.

Let us reformulate Theorem B.

**Theorem 3.4** For every $1 \leq \alpha \leq 2$ there are constant $c_\alpha$ and $C_\alpha$ that depend only on $\alpha$, and there exist absolute constants $c_1 \geq 1$ and $c_2$ for which the following holds. Let $F$ be a class of mean-zero functions and let $(F_s)_{s \geq 0}$ be an admissible sequence of $F$. Then, for every $t \geq c_1$ and every integer $N$, with probability at least $1 - 2 \exp(-c_2 t \log N)$, for every $m \leq N$ and every $f \in F$,

$$\sup_{v \in B_m} \sum_{i=1}^N v_i f(X_i) \leq c_\alpha t \left( \sum_{s=0}^{\infty} 2^{s/2} \|\pi_s f - \pi_{s-1} f\|_{\psi_2} + d_{\psi_\alpha} \sqrt{m \log^{1/\alpha}(eN/m)} \right),$$

where $d_{\psi_\alpha} = \sup_{f \in F} \|f\|_{\psi_\alpha}$.

In particular, with that probability, for every $m \leq N$,

$$D_m \leq C_\alpha t \left( \gamma_2(F, \psi_2) + d_{\psi_\alpha} \sqrt{m \log^{1/\alpha}(eN/m)} \right).$$

As we said, we will present the proof of Theorem 3.4 only for $\alpha = 1$. The proof for $1 < \alpha \leq 2$ is identical, with the exception that (3.3) is replaced by an appropriate deviation estimate for $\psi_\alpha$ random variables, as stated in Lemma 2.3.

**Proof.** Let $\{F_s : s \geq 0\}$ be an admissible sequence of $F$ and without loss of generality, assume that $m = 2^{r_0}$ for some integer $r_0$.

To begin the first part of the chaining argument, for every fixed $f$ set $(\Delta_s f)_i = (\pi_s f - \pi_{s-1} f)(X_i)$. Then, for every $f \in F$ and $v \in B_m$,

$$\sum_{i=1}^N v_i f(X_i) = \sum_{s > s_m} \sum_{i=1}^N v_i (\Delta_s f)_i + \sum_{i=1}^N v_i (\pi_{s_m} f)(X_i).$$
Since the cardinality of the set \( \Delta_s = \{ \pi_s f - \pi_{s-1} f : f \in F \} \) is at most \( 2^{2s+1} \) and since \( |B_m| \leq \exp(\kappa_0 m \log(eN/m)) \), then by the definition of \( s_m \) and a \( \psi_2 \) estimate, if \( t \) is larger than an absolute constant, one has

\[
Pr \left( \exists f \in F, \ v \in B_m : \left| \sum_{s > s_m} \sum_{i=1}^{N} v_i(\Delta_s f)_i \right| \geq t \|v\|_2 \sum_{s > s_m} 2^{s/2} \|\Delta_s f\|_{\psi_2} \right) \\
\leq |B_m| \cdot 2 \sum_{s > s_m} |\Delta_s| \exp(-c_1 2^s t^2) \leq 2 \exp(-c_2 2^{s_m} t^2).
\]

Now, let us turn to the “middle part”, in which the structure of vectors that belong to \( B_m \) is used. First, consider the integers \( s_m, s_{m/2}, \) etc. From the definition of \( s_\ell \) it follows that there is an absolute constant \( c_3 \) such that for every \( 1 \leq \ell \leq N, \ s_\ell \) satisfies that

\[
2^{s_\ell} \geq c_3 \ell \log(eN/\ell), \quad 2^{s_{\ell-1}} < c_3 \ell \log(eN/\ell).
\]

In particular, \( 2^{s_\ell-2} \leq c_3 (\ell/2) \log(eN/\ell) < c_3 (\ell/2) \log(eN/(\ell/2)) \), implying that \( s_\ell - 1 \leq s_{\ell/2} \leq s_\ell \). A similar argument shows that \( s_{\ell/4} < s_\ell \) if \( \ell \leq N/2 \), and thus, either \( s_{\ell/2} = s_\ell - 1 \) or, if \( s_{\ell/2} = s_\ell \), then \( s_{\ell/4} = s_\ell - 1 \). In any case, if one considers the sequence \( s_\ell, s_{\ell/2}, \ldots, s_{\ell/2^r} \), it decreases in steps of at most one and remains constant on blocks of cardinality at most two.

Fix any \( v \in B_m \), and one may assume that \( |\text{supp}(v)| = m \). Let \( \ell_r = m/2^r \) and put \( I_{\ell_1} \subset \text{supp}(v) \) to be a set of \( m/2 \) coordinates such that \( \|P_{I_{\ell_1}} v\|_{\infty} \leq 1/(m/2)^{1/2} \) (such a set of coordinates exists by the definition of \( B_m \)). Denote by \( J_1 \) the complement of \( I_{\ell_1} \) in \( \text{supp}(v) \), and observe that \( P_{J_1} v \in B_{m/2} \) (where, of course, \( I_{\ell_1} \) and \( J_1 \) depend on \( v \)). Hence,

\[
\sum_{i=1}^{N} v_i(\pi_{s_m} f)(X_i) = \sum_{i \in I_{\ell_1}} v_i(\pi_{s_m} f)(X_i) + \sum_{i \in J_1} v_i(\pi_{s_m} f)(X_i),
\]

\[
\sum_{i \in I_{\ell_1}} v_i(\pi_{s_m} f)(X_i) = \sum_{i \in I_{\ell_1}} v_i(\pi_{s_m} f - \pi_{s_m/2} f)(X_i) + \sum_{i \in I_{\ell_1}} v_i(\pi_{s_m/2} f)(X_i),
\]

and

\[
\sum_{i \in J_1} v_i(\pi_{s_m} f)(X_i) = \sum_{i \in J_1} v_i(\pi_{s_m} f - \pi_{s_m/2} f)(X_i) + \sum_{i \in J_1} v_i(\pi_{s_m/2} f)(X_i). \tag{3.4}
\]

We will estimate the first part of (3.4) using a \( \psi_2 \) argument and the second one using the \( \psi_1 \) information. Indeed, there are at most \( 2^{s_{m+1}} \).
elements of the form $\pi_{s_m} f - \pi_{s_{m/2}} f$, and at most $|B_m|$ vectors $v$. Since $|B_m| \leq \exp(\kappa_0 m \log(eN/m))$ then from the definition of $s_m$ it follows that with probability at least $1 - 2 \exp(-c_1 t^2 2^{s_m})$, for every $f \in F$ and $v \in B_m$

$$| \sum_{i \in I_{t_1}} v_i(\pi_{s_m} f - \pi_{s_{m/2}} f)(X_i) | \leq t 2^{s_m/2} \| P_{I_{t_1}} v \|_2 \| \pi_{s_m} f - \pi_{s_{m/2}} f \|_2. \quad (3.6)$$

To handle the second term, recall that every $v \in B_m$, $\| P_{I_{t_1}} v \|_\infty \leq 1/(m/2)^{1/2}$ and $|I_{t_1}| = m/2$. Hence, for every $f \in F$, $v \in B_m$ and $u > 0$,

$$Pr \left( \left| \sum_{i \in I_{t_1}} v_i(\pi_{s_{m/2}} f)(X_i) \right| \geq u \| \pi_{s_{m/2}} f \|_2 \| I_{t_1} \|_1 \right) \leq 2 \exp(-c_5 |I_{t_1}| \min(u^2, u)).$$

In particular, if one takes $u = t 2^{s_m}/|I_{t_1}|$ (which is of the order of $\log(eN/m)$), then by our estimates on the cardinality of $B_m$ and the definition of $s_{m/2}$, it follows that with probability at least $1 - 2 \exp(-c_6 t 2^{s_m})$, for every $f \in F$ and every $v \in B_m$,

$$\left| \sum_{i \in I_{t_1}} v_i(\pi_{s_{m/2}} f)(X_i) \right| \leq t d \psi_1 \sqrt{m} \log(eN/m).$$

Turning to (3.5), the first term can be bounded exactly as in (3.6), while in the second term of (3.5), the required dimension reduction is achieved: all the vectors $P_{J_1} v$ belong to $B_{m/2}$ and the indexing class is $F_{s_{m/2}}$.

The same argument can be repeated, by breaking each $J_1$ into $I_{t_2}$ and its complement in $J_1$ (which we denote by $J_2$), just as in (3.4) and (3.5). At the $r$-th step one begins with vectors $P_{J_{r-1}} v$ that belong to $B_{m/2^{r-1}}$, and an indexing set $F_{s_{m/2^{r-1}}}$. It follows that with probability at least $1 - 4 \exp(-c_9 t^2 2^{s_{m/2^{r-1}}}) - 2 \exp(-c_6 t 2^{s_{m/2^{r-1}}})$, for every $f \in F$ and $v \in B_m$,

$$\left| \sum_{i=1}^N (P_{I_{t_{r-1}}}) v_i(\pi_{s_{m/2^{r-1}} f})(X_i) \right| \leq t 2^{s_{m/2^{r-1}}/2} \| \pi_{s_{m/2^{r-1}}} f - \pi_{s_{m/2^r}} f \|_2 \psi_2 (\| P_{I_{t_r}} v \|_2 + \| P_{I_r} v \|_2)$$

$$+ t d \psi_1 2^{s_{m/2^{r-1}}} \| P_{I_{t_{r-1}}} v \|_\infty$$

$$+ \sum_{i=1}^N (P_{I_r} v)_i(\pi_{s_{m/2^r}} f)(X_i).$$
Since $2^{s_m/2^r} \sim (m/2^r) \log(eN/(m/2^r))$ and $\|P_{I_r}v\|_\infty \leq 1/(m/2^r)^{1/2}$, then $2^{s_m/2^r-1}\|P_{I_r}v\|_\infty \lesssim (m/2^r-1)^{1/2} \log(eN/(m/2^r-1))$. Moreover, $\|P_{I_r}v\|_2 + \|P_{J_r}v\|_2 \leq 2\|P_{J_{r-1}}v\|_2 \leq 2$, and thus the first two terms are bounded by

$$c_7t2^{s_m/2^r-1/2}\|\pi_{s_m/2^r-1}f - \pi_{s_m/2^r}f\|_{\psi_2}$$
$$+ c_7td_{\psi_1}(m/2^r-1)^{1/2} \log(eN/(m/2^r-1)),$$

Hence, if we continue in this fashion until $s_2 = s_{m/2^{r_0}-1}$, it follows that for $t \geq c_8$, with probability at least

$$1 - 4\sum_{r=1}^{r_0} \left(\exp(-c_9t2^{s_{m/2^r}-1}) + \exp(-c_9t2^{s_{m/2^r}-1})\right),$$

(3.7)

for every $f \in F$ and every $v \in B_m$,

$$\left| \sum_{i=1}^{N} u_i(\pi_{s_m}f)(X_i) \right|$$
$$\leq c_9t \left( \sum_{r=1}^{r_0} \left( 2^{s_m/2^r/2} \|\pi_{s_m/2^r-1}f - \pi_{s_m/2^r}f\|_{\psi_2} \right) + d_{\psi_1}\sqrt{m} \log(eN/m) \right)$$
$$+ \left| \sum_{i=1}^{N} (P_{J_0-1}v)_i(\pi_{s_2}f)(X_i) \right|.$$

Observe that the elements of the sequence $(s_{m/2^r})_{r=1}^{r_0}$ belong to the interval $[s_1,s_m]$. Also, this sequence decreases in steps of at most one, and each integer is repeated at most twice. Hence,

$$\sum_{r=1}^{r_0} 2^{s_m/2^r/2} \|\pi_{s_m/2^r-1}f - \pi_{s_m/2^r}f\|_{\psi_2} \leq 2 \sum_{s=s_2}^{s_m} 2^{s/2} \|\Delta_s(f)\|_{\psi_2},$$

and the probabilistic estimate in (3.7) is at least $1 - 2\exp(-c_{10}2^{s_2}t) \geq 1 - 2\exp(-c_{11}t \log N)$, because $t \geq c_8$.

Finally, for the last step, consider the sets supported on at most two coordinate, and thus $\log |F_{s_2}|, \log |B_2| \lesssim \log N$. Therefore, with probability at least $1 - 2\exp(-c_{12}t \log N)$, for every $f \in F$ and $v \in B_m$,

$$\left| \sum_{i=1}^{N} (P_{J_{r_0-1}}v)_i(\pi_{s_2}f)(X_i) \right| \leq c_{13}td_{\psi_1} \log N.$$
Summing the three parts, it follows that for every $t \geq C_0$ and for every $m \leq N$, with probability at least $1 - C_1 \exp(-C_2 t \log N)$, for every $f \in F$ and every $v \in B_m$, 

$$\left| \sum_{i=1}^{N} v_i f(X_i) \right| \leq C_3 t \left( \sum_{s=1}^{\infty} 2^{s/2} \| \Delta_s(f) \|_{\psi_2} + d_{\psi_1} \sqrt{m \log(eN/m)} \right).$$

Since there are at most $N$ possible values of $m$, the same holds for all $m \leq N$ uniformly, as claimed.

Theorem B can be extended to other $\ell_p$ norms. Indeed, for $1 \leq p < 2$ and any $I \subset \{1, ..., N\}$, $\|x\|_{\ell_p} \leq \|x\|_{\ell_2}^{1/p-1/2} \|x\|_2$. Hence, with probability at least $1 - 2 \exp(-c_1 t \log N)$, for every $f \in F$ and $I \subset \{1, ..., N\}$,

$$(\sum_{i \in I} |f|^p(X_i))^{1/p} \leq c_\alpha t \left( \gamma_2(F, \psi_2) |I|^{1/p-1/2} + d_{\psi_1} |I|^{1/p} \log^{1/\alpha}(eN/|I|) \right).$$

For $p > 2$ let $m_0$ be the smallest integer for which

$$\gamma_2(F, \psi_2) \leq d_{\psi_1} \sqrt{m \log^{1/\alpha}(eN/m)}.$$ 

Then, by Theorem B, for every $|I| < m_0$,

$$(\sum_{i \in I} |f|^p(X_i))^{1/p} \leq (\sum_{i \in I} |f|^2(X_i))^{1/2} \leq 2c_\alpha t \gamma_2(F, \psi_2).$$

For larger values of $|I|$, if we denote $(u_i)_{i=1}^{N} = (f(X_i))_{i=1}^{N}$ then for $j \geq m_0$,

$$u_j^* \leq \left( \frac{1}{j} \sum_{i=1}^{j} (u_i^2)^* \right)^{1/2} \leq 2c_\alpha td_{\psi_1} \log^{1/\alpha}(eN/j).$$

Hence, by the triangle inequality, for $|I| \geq m_0$,

$$(\sum_{i \in I} |f|^p(X_i))^{1/p} \leq c_{\alpha,p} t \left( \gamma_2(F, \psi_2) + d_{\psi_1} |I|^{1/p} \log^{1/\alpha}(eN/|I|) \right).$$

(3.8)

(3.9)

Let us mention that the estimate for $p = 1$ (the weakest of all the estimates for $1 \leq p \leq 2$) was proved in [25] using a simpler chaining argument.
3.1 Optimality

We begin this section by recalling the observation made in the introduction, that Theorem B is sharp when $F$ is a class of linear functionals on $\mathbb{R}^n$ and $\mu$ is the canonical gaussian measure on $\mathbb{R}^n$:

Lemma 3.5 There exists an absolute constant $c$ for which the following holds. Let $K \subset \mathbb{R}^n$, set $F = \{\langle x, \cdot \rangle : x \in K\}$ and put $\mu$ to be the canonical gaussian measure on $\mathbb{R}^n$. Then, for every integer $N$ and any $1 \leq m \leq N$,

$$
E \sup_{f \in F} \max_{|I| = m} \left( \sum_{i \in I} f^2(x_i) \right)^{1/2} \geq c \left( \gamma_2(F, \psi_2) + d_{\psi_2} \sqrt{m \log(eN/m)} \right).
$$

Although Lemma 3.5 indicates that Theorem B cannot be improved, one might argue that it is a somewhat degenerate case, because of the equivalence between the $\psi_2$ norm and the $L_2$ one. The next lemma shows that in general, one cannot replace the $\psi_2$ norm in the $\gamma_2$ term with any other $\psi_\alpha$ norm for $\alpha < 2$.

Lemma 3.6 There exists an absolute constant $c_1$ for which the following holds. For every integer $N$, $1 \leq \alpha < 2$ and a number $R$, there is a probability space $(\Omega, \mu)$ and a class $F$ consisting of mean-zero functions on $(\Omega, \mu)$, such that if $(X_i)_{i=1}^N$ are independent, distributed according to $\mu$, then with probability at least $c_1$,

$$
\sup_{f \in F} \left| \sum_{i=1}^N f(X_i) \right| \geq R \gamma_2(F, \psi_\alpha) \sqrt{N},
$$

and in particular, $\sup_{f \in F} \left( \sum_{i=1}^N f^2(X_i) \right)^{1/2} \geq R \gamma_2(F, \psi_\alpha)$.

Remark 3.7 As we indicated in the introduction, Lemma 3.6 shows that in general, $E \sup_{f \in F} |N^{-1} \sum_{i=1}^N f(X_i) - \mathbb{E} f|$ cannot be controlled using a weaker deterministic parameter than $\gamma_2(F, \psi_2)/\sqrt{N}$.

For the proof of Lemma 3.6 we need the following formulation of the Paley-Zygmund inequality [18].

Lemma 3.8 Let $Z$ be a random variable. Then, for every $q > p \geq 1$ and $0 < \lambda < 1$,

$$
Pr \left( |Z| \geq \lambda \|Z\|_{L_p} \right) \geq \left( (1 - \lambda^p) \left( \|Z\|_{L_p}/\|Z\|_{L_q} \right)^p \right)^{q/(q-p)}.
$$
Proof of Lemma 3.6. Fix $1 \leq \alpha < 2$ and an integer $n$. Let $Y$ be a symmetric random variable with density $c_\alpha \exp(-|t|^\alpha)$ and set $X = (Y_1, \ldots, Y_n) \in \mathbb{R}^n$, a vector of independent copies of $Y$. Consider the probability space $(\mathbb{R}^n, \mu)$, with $\mu$ defined by $\mu(A) = \Pr(X \in A)$, let $(e_i)_{i=1}^n$ be the standard basis of $\mathbb{R}^n$, set $K = \{e_i/\sqrt{\log(i+1)} : 1 \leq i \leq n\}$ and put

$$F = \{\langle e_i, \cdot \rangle/\sqrt{\log(i+1)} : 1 \leq i \leq n\}.$$  

One can show (see, for example, Proposition 7 in [5]) that if $(x_i)_{i=1}^n$ is nonnegative and non-increasing, then for every $p \geq 1$,

$$\left\| \sum_{i=1}^n x_i Y_i \right\|_{L_p} \sim p^{1/\alpha} \left\| (x_i)_{i \leq p} \right\|_{\alpha^*} + \sqrt{p} \left\| (x_i)_{i > p} \right\|_2,$$  

(3.10)

where $\| \cdot \|_{\alpha^*}$ is the $\ell_{\alpha^*}$ norm for $\alpha^*$ satisfying $1/\alpha + 1/\alpha^* = 1$. Since $\alpha \leq 2$ then $\alpha^* \geq 2$ and thus $\left\| \sum_{i=1}^n x_i Y_i \right\|_{L_p} \leq c_1 p^{1/\alpha} \|x\|_2$. In particular, for every $x \in \mathbb{R}^n$, $\langle x, \cdot \rangle_{\psi_\alpha} \leq c_2 \|x\|_2$. Moreover, $\|\langle x, \cdot \rangle_{\psi_\alpha}\|_{\psi_\alpha} \geq c_3 \|x\|_2$, implying that the $\ell_2^n$ and the $\psi_\alpha(\mu)$ norms are equivalent on $\mathbb{R}^n$.

It is also straightforward to show that there is an absolute constant $c_4$ such that if $(g_i)_{i=1}^\infty$ are independent, standard gaussian variables then for every $m$,

$$\mathbb{E} \max_{1 \leq i \leq m} \frac{g_i}{\sqrt{\log(i+1)}} \leq c_4.$$  

Therefore, by the Majorizing Measures Theorem

$$\gamma_2(F, \psi_\alpha) \leq c_2 \gamma_2(K, \ell_2^n) \leq c_5 \mathbb{E} \max_{1 \leq i \leq n} \frac{g_i}{\sqrt{\log(i+1)}} \leq c_6.$$  

On the other hand, fix $N$ to be named later and consider $q > p \geq N$. Observe that by (3.10), for these values of $q, p$ and $N$, if $(Y_i)_{i=1}^N$ are independent copies of $Y$ then

$$\left\| \sum_{i=1}^N Y_i \right\|_{L_p} \sim p^{1/\alpha} N^{1-1/\alpha} \quad \text{and} \quad \left\| \sum_{i=1}^N Y_i \right\|_{L_q} \sim q^{1/\alpha} N^{1-1/\alpha}.$$  

Let $X_1, \ldots, X_N$ be independent copies of the random vector $X$, set $Y_{i,j}$ to be the $j$-th coordinate of $X_i$ and put $Z_j = \sum_{i=1}^N Y_{i,j}$. Applying the Paley-Zygmund inequality, it follows that there are $\beta > 1$ and $c_7$, both depend on $\alpha$, such that if $p = c_7 \log n$ and $q = \beta p$, then for every $j$,

$$\Pr \left( |Z_j| \geq c_8 (\log^{1/\alpha} n) N^{1-1/\alpha}/2 \right) = \Pr \left( |Z_j| \geq \|Z_j\|_{L_p}/2 \right) \geq 1/n.$$  

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Hence, by the independence of \( (Z_j)_{j=1}^n \),

\[
Pr \left( \exists 1 \leq j \leq n, \ |Z_j| \geq c_8 N^{1-1/\alpha} \log^{1/\alpha} n \right) \geq c_9.
\]

In particular, with that probability,

\[
\sup_{f \in F} \left| \sum_{i=1}^N f(X_i) \right| = \max_{1 \leq j \leq n} \left| \sum_{i=1}^N \left( \frac{e_j}{\sqrt{\log(j+1)}}, X_i \right) \right| = \max_{1 \leq j \leq n} \frac{1}{\sqrt{\log(j+1)}} \left| \sum_{i=1}^N Y_{i,j} \right|
\]

\[
\geq c_{10} N^{1-1/\alpha} \log^{1/\alpha} n \geq \sqrt{N} \left( \frac{\log n}{N} \right)^{1/\alpha - 1/2}.
\]

All that remains now is to find the connection between \( N \) and \( n \), where we already assumed that \( p = c_7 \log n \geq N \). Clearly, if \( N \ll \log n \) then \( (\log n/N)^{1/\alpha - 1/2} \) can be made to be arbitrarily large by increasing \( n \), as claimed.

4 Decomposing \( F \)

Here, we will present a decomposition of \( F \) into the sum of two sets, representing its peaky and regular parts. We will show that for every \( N \), one can truncate functions in \( F \) at the level

\[
\lambda \sim d_{\psi_\alpha} \log^{1/\alpha}(c_{\psi_\alpha} N^{1/2}/\gamma_2(F, \psi_2)).
\]

The resulting unbounded or peaky part of each \( f \in F \) has coordinate projections with a well behaved \( \ell_2^N \) norm and short support. On the other hand, the regular part of \( f \) is bounded in \( L_\infty \) by \( \lambda \), and, moreover, its typical coordinate projection is contained in \( c_{\psi_\alpha} B_{\psi_\alpha}^N \). Thus, the regular part of \( F \) behaves as if \( F \) has an envelope function \( \tilde{W}(x) = \sup_{f \in F} |f(x)| \) with

\[
\|W\|_{\psi_\alpha} \leq d_{\alpha}.
\]

This decomposition gives a hint of why it is reasonable to hope that the supremum of the empirical process \( \sup_{f \in F} |P_N f^2 - Pf^2| \) is well behaved. Although the peaky part of \( F \) exhibits no concentration, its \( \ell_2^N \) diameter is small, and thus there is no need for cancelation to control it. Since the regular part of \( F \) behaves as if it has a reasonable envelope function, powers concentrate around their mean uniformly.

To formulate the decomposition theorem (which implies Theorem C) we will use the following observations. Recall that if \( x \in \mathbb{R}^N \) then for \( 1 \leq \alpha \leq 2 \),
\[ \|x\|_{\psi_N} = \inf\{C : N^{-1} \sum_{i=1}^{N} \exp((|x_i|/C)^\alpha) \leq 2 \}. \]

It follows that for every \( x \in \mathbb{R}^N \),
\[ x^*_i \leq c \|x\|_{\psi_N} \log^{1/\alpha}(eN/i), \]
and, in fact, this behavior of a monotone rearrangement characterizes the \( \psi_N^\alpha \) norm. It is also standard to verify that if \( X \) is \( \psi_N^\alpha \) random variable on \( (\Omega, \mu) \) and \( (X_i)_{i=1}^{N} \) are independent copies of \( X \), then for \( t \geq c_0 \), with probability at least \( 1 - 2 \exp(-t^\alpha \log N) \), for every \( i \),
\[ X^*_i \leq c_1 t \|X\|_{\psi_N} \log^{1/\alpha}(eN/i). \]

Hence, a typical coordinate projection of an independent sample of a single function \( f \in L_{\psi_N^\alpha} \) satisfies that with high probability, \( \|f(X_i)\|_{\psi_N^\alpha} \leq c_2 \|f\|_{\psi_N} \). In what follows, given \( v = (f(X_i))_{i=1}^{N} \), we will sometimes denote the random norms \( \|f\|_{\psi_N^\alpha} \) and \( \|f\|_{\psi_N} \) by \( \|v\|_p \) and \( \|v\|_{\psi_N} \) respectively.

**Theorem 4.1** There exist absolute constants \( c_0, ..., c_7 \) for which the following holds. For any \( 1 \leq \alpha \leq 2 \) and an integer \( N \) set
\[ \lambda = c_0 d_{\psi_N} \max\left\{ \log^{1/\alpha} \left( c_0 d_{\psi_N}^2 N/\gamma_2^2(F, \psi_2) \right), 1 \right\}. \]

For any \( t \geq c_1 \) there are sets \( F_1 \) and \( F_2 \) that depend on \( N, \lambda \) and \( t \) such that \( F \subset F_1 + F_2 \), and with probability at least \( 1 - 2 \exp(-c_2 t \log N) \),
1. \( \sup_{f \in F_1} \|f\|_{L_2}^2 \leq c_3 \gamma_2(F, \psi_2) \), \( \sup_{f \in F_1} \|\text{supp}(P_{\alpha} f)\| \leq c_3 \gamma_2^2(F, \psi_2) / \lambda^2 \), \( \text{and} \sup_{f \in F_1} \|f\|_{L_2}^2 \leq c_3 \gamma_2^2(F, \psi_2) / N \).
2. \( \sup_{f \in F_2} \|f\|_{L_\infty} \leq \lambda t \) and \( \sup_{f \in F_2} \|f\|_{\psi_N} \leq c_4 t d_{\psi_\alpha} \).
3. For every \( u \geq c_5 \), with probability at least \( 1 - 2 \exp(-c_6 u^2) \), one has \( \sup_{f \in F_2} \|P_N f^2 - P f^2\| \leq c_7 u t \lambda \gamma_2(F, \psi_2) / \sqrt{N} \).

The proof of Theorem 4.1 requires all the information we have about the structure of the set \( P_{\alpha} F = \{(f(X_i))_{i=1}^{N} : f \in F \} \subset \mathbb{R}^N \). Our starting point is the next observation.

**Lemma 4.2** There exist absolute constants \( c_1, c_2 \) and \( c_3 \) for which the following holds. Let \( v \in \mathbb{R}^N \) for which there are \( A, B \) and \( 1 \leq \alpha \leq 2 \) such that for every \( I \subset \{1, ..., N\} \),
\[ \left( \sum_{i \in I} v_i^2 \right)^{1/2} \leq A + B \sqrt{|I| \log^{1/\alpha}(eN/|I|)}. \]
If \( \beta \geq c_1 B \max\{\log^{1/\alpha}(c_2 N B^2/A^2), 1\} \) and \( E_\beta = \{i : |v_i| \geq \beta\} \), then

\[
|E_\beta| \leq \max \left\{ \frac{4A^2}{\beta^2}, eN \exp(-2(\beta/2B)^\alpha) \right\} \quad \text{and} \quad \left( \sum_{i \in E_\beta} v_i^2 \right)^{1/2} \leq c_3 A.
\]

**Proof.** Clearly, for every integer \( n \), \( \|x\|_{l_1^n} \leq \sqrt{n} \|x\|_{l_2^n} \). Hence, for every \( I \subset \{1, \ldots, N\} \), \( \sum_{i \in I} |v_i| \leq A \sqrt{|I|} + B |I| \log^{1/\alpha}(eN/|I|) \). Let \( E_\beta = \{i : |v_i| \geq \beta\} \) and note that

\[
\beta |E_\beta| \leq \sum_{i \in E_\beta} |v_i| \leq A |E_\beta|^{1/2} + B |E_\beta| \log^{1/\alpha}(eN/|E_\beta|).
\]

If \( B |E_\beta| \log^{1/\alpha}(eN/|E_\beta|) \leq \beta |E_\beta|/2 \) then \( |E_\beta| \leq A^2/\beta^2 \). Otherwise, if the reverse inequality holds, then \( |E_\beta| \leq eN \exp(-2(\beta/2B)^\alpha) \). Thus,

\[
|\{i : |v_i| \geq \beta\}| \leq \max \left\{ \frac{4A^2}{\beta^2}, eN \exp(-2(\beta/2B)^\alpha) \right\}.
\] (4.1)

To complete the proof, let \( \beta \geq c_1 B \max\{\log^{1/\alpha}(c_2 N B^2/A^2), 1\} \). Therefore, \( |E_\beta| \leq A^2/\beta^2 \), and thus, for our choice of \( \beta \),

\[
\left( \sum_{i \in E_\beta} v_i^2 \right)^{1/2} \leq A + B \frac{A}{\beta} \log^{1/\alpha}(eN \beta^2/A^2) \leq c_3 A.
\]

**Proof of Theorem 4.1.** Fix \( t \geq c_0 \) and recall that by Theorem B, with probability at least \( 1 - 2 \exp(-c_0 t \log n) \), for every \( v \in P_c F \) the assumptions of Lemma 4.2 hold with \( A \sim t \gamma_2(F, \psi_2) \) and \( B \sim t d_{\psi_0} \). Just as in Lemma 4.2 set

\[
\beta \sim B \max\{\log^{1/\alpha}(c_2 N B^2/A^2), 1\} \equiv \lambda t.
\]

Let \( \phi(f) = \text{sgn}(f) \min\{|f|, \beta\} \) and \( \psi(f) = f - \phi(f) \), put \( F_1 = \{\psi(f) : f \in F\} \), \( F_2 = \{\phi(f) : f \in F\} \) and observe that \( F \subset F_1 + F_2 \).

Let us consider \( F_1 \), which is the unbounded part of \( F \). Note that for every \( f \in F \), if we set \( u_i = (\psi(f))(X_i) \), then \( \{i : |u_i| \geq \beta\} \subset \{i : |f(X_i)| \geq \beta\} = E_\beta \), and on that set \( |u_i| = |f(X_i)| - \beta \). Hence, by Lemma 4.2,

\[
\left( \sum_{i \in E_\beta} u_i^2 \right)^{1/2} \leq c_3 t \gamma_2(F, \psi_2).
\]
Also, since \(\|f\|_{\psi^2} \leq d_{\psi^2}\) then by integrating the tail, one may verify that
\[
\mathbb{E}|\psi(f)|^2 \leq c_4 \beta^2 \exp(-c_5 (\beta/d_{\psi^2})^\alpha) \leq c_6 \gamma_2^2(F, \psi^2)/N,
\]
proving the first part of the claim.

Turning to the second part, note that if \(f \in F\) and \(w = P_\sigma(\phi(f))\) then
\[
\|w\|_\infty \leq \beta.
\]
Let \(m = A^2/\beta^2\) and observe that
\[
\gamma_2(F, \psi^2) \sim d_{\psi^2} \sqrt{m \max\{\log 1/\alpha(\epsilon N/m), 1\}}.
\]

First, assume that \(m \leq N\). Therefore, since \(\beta \leq c_7 \tau d_{\psi^2} \log 1/\alpha(\epsilon N/m)\) then for every \(j \leq m\), \(w^*_j \leq \beta \leq c_7 \tau d_{\psi^2} \log 1/\alpha(\epsilon N/j)\).

Moreover, if \(m \leq N\) then by Theorem B,
\[
\sum_{i \leq j} (w^2)_i^{\star} \leq t \left( \gamma_2(F, \psi^2) + d_{\psi^2} \sqrt{J \log 1/\alpha(\epsilon N/j)} \right) \leq td_{\psi^2} \sqrt{J \log 1/\alpha(\epsilon N/j)}.
\]

Therefore,
\[
w^*_j \leq \left( \frac{1}{j} \sum_{i=1}^j (w^2)_i^{\star} \right)^{1/2} \leq c_8 \tau d_{\psi^2} \log 1/\alpha(\epsilon N/j),
\]
and thus, \(\sup_{f \in F_2} \|f\|_{\psi^N} \leq td_{\psi^2}\), as claimed.

On the other hand, if \(m \geq N\) then \(\lambda \sim d_{\psi^2}\). Hence, \(\sup_{f \in F_2} \|f\|_{L_\infty} \leq \beta \leq td_{\psi^2}\), implying that \(\sup_{f \in F_2} \|f\|_{\psi^N} \leq td_{\psi^2}\).

It remains to estimate the supremum of the empirical process indexed by \(|\phi(f)|^2\). Since \(\phi(x) = \text{sgn}(x) \min\{x, \beta\}\) is 1-Lipschitz, then for every \(f_1, f_2 \in F\), \(|\phi(f_1)|^2 - |\phi(f_2)|^2| \leq 2 \beta |f_1 - f_2|\) pointwise. In particular, \(|\phi(f_1)|^2 - |\phi(f_2)|^2|_{\psi^2} \leq 2 \beta \|f_1 - f_2\|_{\psi^2}\). Therefore, by a standard chaining argument, for every \(u \geq c_9\), with probability at least \(1 - 2 \exp(-c_{10} u^2)\), \(\sup_{h \in F_2} |P_N h^2 - P h^2| \leq c_{11} u \beta \gamma_2(F, \psi^2)/\sqrt{N}\), as claimed.

Theorem 4.1 should be compared with a result due to Rudelson (see [33]). Although Rudelson’s result was formulated for selector processes, an analogous result holds for empirical processes and with essentially the same proof as the original one.
Theorem 4.3 For every $0 < \delta < 1$ there is a constant $c(\delta)$ for which the following holds. If $F$ is a class of mean-zero functions then there are sets $F_1$ and $F_2$ such that $F \subset F_1 + F_2$, and with $\mu^N$-probability at least $1 - \delta$,

$$\sup_{f \in F_1} \|f\|_{\ell^2} \leq c(\delta)\sqrt{Nd_{L_2}}, \quad \sup_{f \in F_2} \|f\|_{\ell^1} \leq c(\delta)\mathbb{E}\sup_{f \in F} \sum_{i=1}^{N} \varepsilon_i f(X_i),$$

In particular, with probability at least $1 - \delta$,

$$P_\sigma F \subset c(\delta) \left( R_N\sqrt{NB_1^N} + d_{L_2}\sqrt{NB_2^N} \right),$$

where

$$R_N = \frac{1}{\sqrt{N}} \mathbb{E}\sup_{f \in F} \sum_{i=1}^{N} \varepsilon_i f(X_i) \quad \text{and} \quad d_{L_2} = \sup_{f \in F} (\mathbb{E}f^2)^{1/2}.$$

Let us compare Theorem 4.1 with Theorem 4.3. First, observe that the first two parts of Theorem 4.1 imply that

$$P_\sigma F \subset c(\delta) \left( \gamma_2(F, \psi_2)B_2^N + \lambda B_{\infty}^N \cap cB_{\psi_2}^N \right),$$

and that for large values of $N$, that is, when $\lambda \geq 1$, it is evident that $\gamma_2(F, \psi_2) \leq \sqrt{Nd_{\psi_2}}$. In particular, since $B_{\psi_2}^N \subset c_1\sqrt{NB_2^N}$, Theorem 4.1 implies that for large $N$,

$$P_\sigma F \subset c_2(\delta)d_{\psi_2} \sqrt{NB_2^N}.$$

Hence, if we are in a situation where the $\psi_2$ and the $L_2$ metrics are equivalent, then Theorem 4.1 is stronger than Theorem 4.3 since the $\sqrt{NB_1^N}$ component is not needed.

In fact, the gap between the two results can be considerable. As an example, let $F = \{\langle x, \cdot \rangle : x \in S^{n-1}\}$ and set $\mu$ to be the canonical gaussian measure on $\mathbb{R}^n$. Then, by Theorem 4.1 with high probability

$$P_\sigma F \subset c(\delta) \left( \sqrt{n}B_2^N + \lambda B_{\infty}^N \cap cB_{\psi_2}^N \right).$$

On the other hand, since

$$\frac{1}{\sqrt{N}} \mathbb{E}\sup_{f \in F} \sum_{i=1}^{N} \varepsilon_i f(X_i) \sim \frac{1}{\sqrt{N}} \mathbb{E}(\sum_{i=1}^{N} \|X_i\|_{\ell^2}^2)^{1/2} \sim \sqrt{n},$$
then Theorem 4.3 only yields that

\[ P_\sigma F \subset c(\delta) \left( \sqrt{nN}B_1^N + \sqrt{N}B_2^N \right), \]

which is a much weaker estimate.

The reason for the gap between the results is that Theorem 4.1 is tailored for situations in which one has additional information on the tails of functions in the class, and in return gets more structural information on the peaky part of coordinate projections. On the other hand, the assumptions of Theorem 4.3 can only give little information on the peaky part of \( F \), which is captured by the \( \ell_1^N \) component of that decomposition. Indeed, at best, for a fixed, reasonable class of functions \( F \), one may expect that

\[ \frac{1}{\sqrt{N}} \mathbb{E} \sup_{f \in F} \left| \sum_{i=1}^N \varepsilon_i f(X_i) \right| \leq c(F). \]

Thus Theorem 4.3 only yields

\[ P_\sigma F \subset c(\delta) \left( c(F)\sqrt{N}B_1^N + d_{L_2}\sqrt{N}B_2^N \right), \]

but with no further information on the way the coordinates are distributed in the \( \ell_1^N \) component of the decomposition. In the geometric applications we are interested in, the constant \( c(F) \) grows with the “dimension” of the class (in the example presented above, \( c(F) \sim \sqrt{n} \)), making Theorem 4.3 too weak for the analysis of such problems.

We end this section with a formulation of a simple application of the proof of Theorem 4.1.

**Corollary 4.4** There exist absolute constants \( c_0, c_1, c_2 \) and \( c_3 \) for which the following holds. Let \( F \) be a class of mean-zero functions and for every \( N \) set \( \lambda = c_0 d_{\psi_1} \max \{ \log(c_0 d_{\psi_1} N^{1/2}/\gamma_2(F, \psi_2), 1) \} \). Then, for every \( t \geq c_1 \), with probability at least \( 1 - 2 \exp(-c_2 \min \{ t \log N, t^2 \}) \),

\[ \sup_{f \in F} \left| \frac{1}{N} \sum_{i=1}^N f^2(X_i) - \mathbb{E}f^2 \right| \leq c_3 t^2 \max \left\{ \lambda \frac{\gamma_2(F, \psi_2)}{\sqrt{N}}, \frac{\gamma^2_2(F, \psi_2)}{N} \right\}. \]

It is important to note that using the \( L_\infty \) bound to obtain a concentration result for \( F_2 \) (as one does in Corollary 4.4) leads to a logarithmic looseness. Indeed, to obtain the correct estimate on the expectation of \( \sup_{f \in F} |P_N f^2 - Pf^2| \) one has to truncate functions at a level \( \sim d_{\psi_1} \). This is
impossible even if one considers a single Gaussian random variable. It is true that for small values of $N$—when $d_{\psi_1} N^{1/2} \ll \gamma_2(F, \psi_2)$, the level of truncation is the required one, but the resulting estimate on $\sup_{f \in F} |P_N f^2 - Pf^2|$ is trivial. Indeed, for those values of $N$ there is no real concentration and the bound reflects an estimate on the empirical diameter $\sup_{f \in F} (P_N f^2)^{1/2}$. On the other hand, when $d_{\psi_1} N^{1/2} \sim \gamma_2(F, \psi_2)$ and beyond, one starts seeing true concentration, but then the best possible level of truncation for those values of $N$ is off by a logarithmic factor from the required one. Thus, even with a sharp decomposition theorem at our disposal, a contraction based estimate on the empirical process indexed by $F$ leads to a superfluous log $N$ factor. Despite that, this type of a decomposition argument is strong enough for many applications (see, for example, [9, 15, 25], and most notably, in [1]), because in those cases the all the required information is when $d_{\psi_1} \sqrt{N}$ is proportional to the complexity parameter of the class, rather than for larger values of $N$.

If one wishes to obtain the correct estimate on $\sup_{f \in F} |P_N f^2 - Pf^2|$ for larger values of $N$, more accurate information on the “bounded part” of $F$ is needed. This is not surprising because decomposition theorems like Theorem 4.1 are based solely on deviation estimates and on bounds on the $\ell_2$ norms of monotone rearrangements of $(f(X_i))_{i=1}^N$. On the other hand, the correct rates require some sort of “local” concentration bounds, and those are at the heart of the proof of Theorem A.

## 5 From a bounded diameter to concentration

Here we will remove the superfluous logarithmic factor and prove Theorem A, by showing if $F$ is a symmetric class of mean-zero functions, then with high probability and in expectation

$$\sup_{f \in F} \left| \frac{1}{N} \sum_{i=1}^N f^2(X_i) - \mathbb{E} f^2 \right| \leq c \max \left\{ d_{\psi_1} \gamma_2(F, \psi_2) \sqrt{\frac{N}{N}}, \frac{\gamma_2^2(F, \psi_2)}{N} \right\}. \quad (5.1)$$

In particular, in the non-trivial range where there is actual concentration, the dominating term is $d_{\psi_1} \gamma_2(F, \psi_2)/\sqrt{N}$, which is a contraction type estimate with the maximal norm in $\psi_1$ taking the role of the maximal norm in $L_\infty$.

The source of difficulty in the proof of Theorem A is that the desired concentration does not follow from the individual concentration of each $N^{-1} \sum_{i=1}^N f^2(X_i)$ around its mean. Rather, it is a combination of two com-
ponents. First, a tail estimate on the diameter of the “ends” of chains
\[
\left( \sup_{f \in F} \frac{1}{N} \sum_{i=1}^{N} (f - \pi_{\tau_N} f)^2 (X_i) \right)^{1/2},
\]
whose role in the chaining process is to capture the “peaky behavior” of \( F \) that prevents concentration. The second component is an analysis of the Bernoulli process \( \sup_{f \in F} \sum_{i=1}^{N} \varepsilon_i (\pi_{\tau_N} f)^2 (X_i) \), conditioned on \( (X_i)_{i=1}^{N} \). It captures the part of \( F \) in which there is concentration. Moreover, the analysis of both parts has to be carried out without resorting to a “global” contraction argument, because the \( L_{\infty} \) or \( \psi_2 \) diameters of the relevant sets may be too large.

As a starting point of the proof of Theorem A, consider an almost optimal admissible sequence of \( F \) with respect to the \( \psi_2 \) metric. Let \( \tau_N \) be the integer \( s \) satisfying that \( N/2 \leq 2^s < N \). One can show \[24\] that with high probability
\[
\sup_{f \in F} \frac{1}{N} \sum_{i=1}^{N} (f - \pi_{\tau_N} f)^2 (X_i) \lesssim \frac{\gamma_2^2(F, \psi_2)}{N},
\]
which is of the desired order of magnitude. This estimate is based on Bernstein’s inequality, which implies that \( (N^{-1} \sum_{i=1}^{N} h^2(X_i))^{1/2} \) behaves like a sum of i.i.d. \( \psi_2 \) random variables for “large” deviations.

Next, one has to study
\[
\sup_{f \in F} \left| \frac{1}{N} \sum_{i=1}^{N} (\pi_{\tau_N} f)^2 (X_i) - \mathbb{E}(\pi_{\tau_N} f)^2 \right|,
\]
which, by a symmetrization argument behaves like
\[
\sup_{f \in F} \left| \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i \left((\pi_{\tau_N} f)^2 (X_i) - (\pi_0 f)^2 (X_i)\right) \right|.
\]
To analyze this Bernoulli process one uses a chaining argument with the same, non-random admissible sequence, and thus one has to study the increments
\[
Pr_{\varepsilon} \left( \left| \sum_{i=1}^{N} \varepsilon_i \left((\pi_s f)^2 - (\pi_{s-1} f)^2\right) (X_i) \right| > t \right), \quad (5.2)
\]
conditioned on \( (X_i)_{i=1}^{N} \). At every level \( s \leq \tau_N \) one has to control the \( 2^{2s+1} \) vectors in \( \mathbb{R}^N \) of the form \( (y_i)_{i=1}^{N} = ((\pi_s f - \pi_{s-1} f)(\pi_s f + \pi_{s-1} f)(X_i))_{i=1}^{N} \).
Proof. Recall that for every $\pi_s f \in F$ and thanks to Theorem B, one has very accurate information on the coordinate structure of $((\pi_s f + \pi_{s-1} f)(X_i))_{i=1}^N$. However, a similar result is required for the differences $((\pi_s f - \pi_{s-1} f)(X_i))_{i=1}^N$ which takes into account the $\psi_2$ distance between $\pi_s f$ and $\pi_{s-1} f$. The desired estimate is proved in Lemma 5.1 below.

Finally, to bound (5.2), observe that for every $\ell = 1$ since this is the only case we will actually use. The proof for $1 < \alpha$ takes into account the $\pi$ observation is used at the level $s$ of the chaining process for $t \sim 2^s$ and for different values of $\ell$ that depend both on $s$ and on the structure of each $(y_i)_{i=1}^N = ((\pi_s f - \pi_{s-1} f)(\pi_s f + \pi_{s-1} f)(X_i))_{i=1}^N$. The crucial point in determining $\ell$ is the number of coordinates on which $((\pi_s f - \pi_{s-1} f)(X_i))_{i=1}^N$ does not “behaves regularly” in the sense of Theorem 4.1.

We begin the proof with a “local” version of Theorem 3.4 – for a finite class $H$, in which for every $h \in H$ the bound on $(\sum_{i \in I} h^2(X_i))^{1/2}$ is given using $\|h\|_{\psi_2} \sqrt{\log |H|}$ and $\|h\|_{\psi_1}$ rather than using the global parameters $\gamma_2(H,\psi_2)$ and $d_{\psi_1}$ that are used in Theorem B.

Lemma 5.1. There exists absolute constants $c_1, c_2, c_3$ and $c_4$ for which the following holds. Let $H$ be a class of mean-zero functions and set $k = \log |H|$. Then, for every $u \geq c_1$, with probability at least $1 - 2\exp(-c_2 \max\{k, \log N\} u)$, for every $I \subset \{1, \ldots, N\}$ and every $h \in H$,

$$\left(\sum_{i \in I} h^2(X_i)\right)^{1/2} \leq c_4 u \left(\|h\|_{\psi_2} \sqrt{k} + \|h\|_{\psi_1} \sqrt{|I| \log(eN/|I|)}\right).$$

An analogous result holds for any $\psi_\alpha$ norm for $1 < \alpha \leq 2$, with $\log^{1/\alpha}(eN/|I|)$ taking the place of $\log(eN/|I|)$.

We will prove the lemma for $\alpha = 1$ since this is the only case we will actually use. The proof for $1 < \alpha \leq 2$ follows the same lines and is omitted.

The proof of Lemma 5.1 is very similar in nature to the proof of Theorem 3.4 and will use its notation. Again, we will denote by $E_m$ the collection of subsets of $\{1, \ldots, N\}$ of cardinality $m$.

Proof. Recall that for every $h \in H$,

$$\sup_{|I|=m} \left(\sum_{i \in I} h^2(X_i)\right)^{1/2} \leq C \sup_{v \in B_m} \sum_{i=1}^N v_i h(X_i)$$
where $B_m$ was defined in (5.2). First, assume that $|H| \geq N$ and that $m = 2^c$ satisfies that $\kappa_0 m \log(eN/m) \geq \max\{\log|E_m|, \log |B_m|\}$. We can assume without loss of generality that $\log|E_m| \geq \log |H|$. Indeed, if $\log|E_m| \lesssim \log |H|$ then the required estimate follows easily from a $\psi_2$ estimate and the union bound, since $\log(|H| \cdot |B_m|) \leq c_0 \log |H|$.

Recall that for every $v \in B_m$, $|\operatorname{supp}(v)| \leq m$ and that there is a set $I_{\ell_1}$ of cardinality $m/2$ such that $\|P_{I_{\ell_1}}v\|_\infty \leq 1/(m/2)^{1/2}$. Let $J_1$ be the complement of $I_{\ell_1}$ in $|\operatorname{supp}(v)|$, and so on for $\ell_r = m/2^r$, $r \leq r_1$, where $r_1$ will be named later. Observe that for every $v \in B_m$, $P_{J_r}v \in B_{\ell_r}$. Since $\max_{i \in I_{\ell_1}} \|v_i h(X_i)\|_{\psi_1} \leq \|h\|_{\psi_1}/\sqrt{r}$, then by Bernstein’s inequality, for every $u_1$ larger than an absolute constant,

$$
Pr \left( \exists h \in H, v \in B_m, r \leq r_1 \left| \sum_{i \in I_{\ell_r}} v_i h(X_i) \right| \geq u_1 \|h\|_{\psi_1} \sqrt{r} \log(eN/\ell_r) \right) \\
\leq 2|H| \sum_{r=1}^{r_1} |B_{\ell_r}| \exp(-c_1 u_1 \ell_r \log(eN/\ell_r)) \\
\leq 2|H| \exp(-c_2 u_1 \ell_{r_1} \log(eN/\ell_{r_1})) = (*) \tag{5.3}
$$

Since $\ell_r = m/2^r$, we set $r_1$ to be the largest integer for which

$$(m/2^{r_1}) \log(eN/(m/2^{r_1})) \gtrsim \log |H|$$

and since $\log|E_m| \gtrsim \log |H|$ such an integer exists. Thus, for $u_1 \geq c_3$ it is evident that $(*) \leq 2 \exp(-c_4 u_1 \log |H|)$.

Next, for every $v \in B_m$ consider the projection $P_{J_{r_1}}v$. Since $\|P_{J_{r_1}}v\|_2 \leq 1$ then $\|\sum_{i \in J_{r_1}} v_i h(X_i)\|_{\psi_2} \leq c_5 \|h\|_{\psi_2}$. Therefore,

$$
Pr \left( \exists v \in B_m, h \in H : \left| \sum_{i \in J_{r_1}} v_i h(X_i) \right| \geq u_2 \sqrt{\log |H| \|h\|_{\psi_2}} \right) \\
\leq 2|H| \cdot |B_{m/2^{r_1}}| \exp(-c_6 u_2^2 \log |H|) \leq \exp(-c_7 u_2^2 \log |H|),
$$

provided that $u_2 \geq c_8$.

Therefore, if $u$ is sufficiently large, then with probability at least $1 -$
\[ 2 \exp(-c_g u \log |H|), \] for every \( h \in H \)

\[
\sup_{v \in B_m} \sum_{i=1}^{N} v_i h(X_i) \leq c_{10} u \left( \|h\|_{\psi_2} \sqrt{\log H} + \|h\|_{\psi_1} \sum_{r=1}^{r_1} \ell_r \log(eN/\ell_r) \right) \\
\leq c_{11} u \left( \|h\|_{\psi_2} \sqrt{\log H} + \|h\|_{\psi_1} \sqrt{m} \log(eN/m) \right).
\]

Since \( 1 \leq m \leq N \) and \( |H| \geq N \), the claim holds for any such \( m \).

Now, assume that \( |H| < N \). Then, set \( r_1 = r_0 \), and by (5.3), with probability at least \( 1 - 2 \exp(-c_{12} u \log N) \), for every \( h \in H \) and \( v \in B_m \)

\[
\left| \sum_{i=1}^{N} v_i h(X_i) \right| \leq c_{13} u \|h\|_{\psi_1} \sqrt{m} \log(eN/m). 
\]
Again, summing the probabilities over every \( 1 \leq m \leq N \) the claim follows.

Recall that \( \tau_N \) is the integer \( s \) for which \( N/2 \leq 2^s < N \). The sets \( H \) we will be interested in are the sets of links at the level \( s \), namely, \( \Delta_s = \{ \Delta_s(f) = \pi_s f - \pi_{s-1} f : f \in F \} \), for \( s \leq \tau_N \), where \( (F_s)_{s \geq 0} \) is an almost optimal admissible sequence of \( F \) with respect to the \( \psi_2 \) norm.

Let us summarize the information we have on the set

\[
\left\{ ((\Delta_s(f))(X_i) : (\pi_s f + \pi_{s-1} f)(X_i) \right\}^N_{i=1} : f \in F \}.
\]

Consider the following events: let

\[
A_t = \left\{ (X_i)_{i=1}^N : \forall I \subset \{1, \ldots, N\}, \sup_{f \in F} \left( \sum_{i \in I} f^2(X_i) \right)^{1/2} \leq \kappa_1 t \left( \gamma_2(F, \psi_2) + d_{\psi_1} \sqrt{|I| \log(eN/|I|)} \right) \right\},
\]

and

\[
B^s_t = \left\{ (X_i)_{i=1}^N : \forall f \in F, \forall I \subset \{1, \ldots, N\}, \left( \sum_{i \in I} (\Delta_s f)^2(X_i) \right)^{1/2} \leq \kappa_1 t \left( 2^{s/2} \||\Delta_s f\||_{\psi_2} + \||\Delta_s f\||_{\psi_1} \sqrt{|I| \log(eN/|I|)} \right) \right\},
\]

where \( \kappa_1 \) is a suitable absolute constant.
By Theorem 3.4 for every \( t \geq c_1 \), \( Pr(A_t) \geq 1 - 2 \exp(-c_2 t \log N) \), while applying Lemma 5.1 it is evident that for every \( t \geq c_3 \) and every \( s \leq \tau_N \), \( Pr(B_t^s) \geq 1 - 2 \exp(-c_4 \max\{2^s, \log N\}t) \).

Let us consider the Bernoulli process \( \sum_{i=1}^{N} \varepsilon_i ((\pi_{\tau_N} f)^2(X_i) - f_0^2(X_i)) \) conditioned on the set \( \Omega_t = A_t \cap \left( \bigcap_{s \leq \tau_N} B_t^s \right) \). Observe that on \( \Omega_t \) we have enough information to identify the cardinality of each set of “large" coordinates of individual functions \( \Delta_s(f) \), and thus the point from which each vector \( ((\Delta_s(f))(X_i))_{i=1}^{N} \) behaves regularly. The parameter we will use to identify the point from which the regular behavior begins is

\[
\begin{align*}
m(\Delta_s(f)) &= \min \left\{ m : 2^{s/2}||\Delta_s(f)||_{\psi_2} \leq ||\Delta_s(f)||_{\psi_1} \sqrt{m \log(eN/m)} \right\}.
\end{align*}
\]

**Theorem 5.2** There exist absolute constants \( c_1, c_2, c_3 \) and \( c_4 \) for which the following holds. If \( f_0 \in F \) then for every \( t \geq c_1 \), \( u \geq c_2 \) and \( (X_i)_{i=1}^{N} \in \Omega_t \),

\[
\begin{align*}
Pr_{\varepsilon} \left( \exists f \in F, \left| \sum_{i=1}^{N} \varepsilon_i ((\pi_{\tau_N} f)^2(X_i) - f_0^2(X_i)) \right| \geq u \rho_t \right) &\leq 2 \exp(-c_3 u^2),
\end{align*}
\]

where

\[
\rho_t = c_4 t^2 \left( \sqrt{N \gamma_2(F, \psi_2)} + \gamma_2^2(F, \psi_2) \right).
\]

For the proof we will need the following definition.

**Definition 5.3** Let \( u = (u_i)_{i=1}^{N} \), \( I \subset \{1, \ldots, N\} \) and \( v = (v_i)_{i=1}^{|I|} \). We say that \( v \) dominates \( u \) on \( I \) if for every \( i \in I \), \( (P_I u)_i \leq (P_I v)_i \). In other words, if a monotone rearrangement of \( P_I u \) is smaller than that of \( v \) coordinate-wise on \( I \).

**Proof.** Fix \( f_0 \in F \) and for every \( f \in F_{\tau_N} \) write \( f^2 - f_0^2 = \sum_{s=1}^{\tau_N} (\pi_s f)^2 - (\pi_{s-1} f)^2 \). Let \( 1 \leq s \leq \tau_N \) and consider a link \( (\pi_s f)^2 - (\pi_{s-1} f)^2 \). Set \( h_- = \pi_s f - \pi_{s-1} f \), \( h_+ = \max\{\pi_s f, \pi_{s-1} f\} \) and for every \( (X_i)_{i=1}^{N} \) let \( v_- = (h_-(X_i))_{i=1}^{N} \) and \( v_+ = (h_+(X_i))_{i=1}^{N} \). Also, for every \( h_- \), recall that \( m(h_-) \) is the smallest integer such that \( 2^{s/2}\|h_-\|_{\psi_2} \leq \|h_-\|_{\psi_1} \sqrt{m \log(eN/m)} \) and if the smallest is \( m \geq N \), set \( m(h_-) = N \).

Since \( (X_i)_{i=1}^{N} \in B_t^s \) then for every \( I \subset \{1, \ldots, N\} \)

\[
\left( \sum_{i \in I} h_i^2(X_i) \right)^{1/2} \leq \kappa_1 t \left( 2^{s/2}\|h_-\|_{\psi_2} + \|h_-\|_{\psi_1} \sqrt{|I| \log(eN/|I|)} \right),
\]

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and let us consider two cases. The first is when \( m(h_-) \leq 2^s \) and the second is when the reverse inequality holds.

To handle the first case, when \( m(h_-) \leq 2^s \), observe that by the subgaussian inequality for Bernoulli sums, for every \( u > 0 \), with probability at least 
\[ 1 - 2 \exp(-c_1 u^{2^s}) \]

\[
\left| \sum_{i=1}^{N} \varepsilon_i((\pi_s f)^2 - (\pi_{s-1} f)^2) \right| \leq \sum_{i=1}^{2^s} (v_- v_+)^*_i + \sum_{i>2^s} |\varepsilon_i(v_- v_+)^*_i| \\
\leq \sum_{i=1}^{2^s} (v_- v_+)^*_i + u2^{s/2} \left( \sum_{i>2^s} (v_-^2 v_+^2)^*_i \right)^{1/2}
\]

where, as always, \((x_i^*_i)_{i \geq 1}\) denotes a non-increasing rearrangement of \(|x_i|_{i \geq 1}\).

Clearly, \[ \sum_{i=1}^{2^s} (v_- v_+)^*_i \leq \left( \sum_{i=1}^{2^s} (v_-^2)^*_i \right)^{1/2} \left( \sum_{i=1}^{2^s} (v_+^2)^*_i \right)^{1/2} \]. Since \( (v_+)_i \leq \max\{|(\pi_s f)(X_i)|,|(\pi_{s-1} f)(X_i)|\} \) and \((X_i)_{i=1}^N \in A_t\) then
\[
\left( \sum_{i=1}^{2^s} (v_+^2)^*_i \right)^{1/2} \leq 2 \sup_{f \in F} \sup_{|I| = 2^s} \left( \sum_{i \in I} f^2(X_i) \right)^{1/2} \\
\leq c_2 t \left( \gamma_2(F, \psi_2) + d_{\psi_1} 2^{s/2} \log(eN/2^s) \right).
\]

Also, because \( m(h_-) \leq 2^s \) and \((X_i)_{i=1}^N \in B_t^{s^*}\), it is evident that
\[
\left( \sum_{i=1}^{2^s} (v_-^2)^*_i \right)^{1/2} \leq t \kappa_1 \left( 2^{s/2} \|h_-\|_{\psi_2} + m(h_-) 2^{s/2} \log(eN/2^s) \right) \\
\leq c_3 t \|h_-\|_{\psi_1} 2^{s/2} \log(eN/2^s).
\]

Hence, recalling that
\[ 2^{s/2} \|h_-\|_{\psi_1} = 2^{s/2} \|\Delta_s(f)\|_{\psi_1} \leq 2^{s/2} \|\Delta_s(f)\|_{\psi_2} \leq \gamma_2(F, \psi_2), \]

one has
\[
\sum_{i=1}^{2^s} (v_- v_+)^*_i \\
\leq c_4 t^2 \left( \gamma_2(F, \psi_2) \|h_-\|_{\psi_1} 2^{s/2} \log(eN/2^s) + d_{\psi_1} \|h_-\|_{\psi_1} 2^s \log^2(eN/2^s) \right) \\
\leq 2c_4 t d_{\psi_1} \gamma_2(F, \psi_2) 2^{s/2} \log^2(eN/2^s).
\]

(5.6)
Next, let us consider the term

\[
\left( \sum_{i \geq 2^s} (v^2 v^2_i) \right)^{1/2} = \inf_{\{J: |J| = N - 2^s\}} \left( \sum_{i \in J} (v^2 v^2_i) \right)^{1/2}.
\]

Let \( J \) be the set of the \( N - 2^s \) smallest coordinates of \( v_- \). Observe that for every \( I \subset \{1, \ldots, N\} \), \( |I| \geq 2^s \) one has

\[
\left( \sum_{i \in I} (v_-^2)_i \right)^{1/2} \leq \kappa_1 t \left( 2^{s/2} \|h_-\|_{\psi_2} + \|h_-\|_{\psi_1} \sqrt{|I| \log(eN/|I|)} \right)
\]

\[
\leq 2\kappa_1 t \|h_-\|_{\psi_1} \sqrt{|I| \log(eN/|I|)},
\]

since \( m(h_-) \leq 2^s \).

Thus, for every \( i > 2^s \),

\[
(v_-)_i^* \leq 2\kappa_1 t \|h_-\|_{\psi_1} \log(eN/i),
\]

and in particular, \( v_- \) is dominated by \( (2\kappa_1 t \|h_-\|_{\psi_1} \log(eN/i))_{i > 2^s} \) on the set \( J \) and

\[
\|P_J v_-\|_{\infty} \leq 2\kappa_1 t \|h_-\|_{\psi_1} \log(eN/2^s).
\]

To obtain a similar control over the vector \( v_+ \), let \( m_0 \) be the smallest integer such that \( \gamma_2(F, \psi_2) \leq d_{\psi_1} \sqrt{m} \log(eN/m) \), and if the smallest one is larger than \( N \), set \( m_0 = N \). Just as we did for \( v_- \), if \( f \in F \) and \( (X_i)_{i=1}^N \in A_t \), and if we set \( (u_i)_{i=1}^N = (f(X_i))_{i=1}^N \), then for every \( i \geq m_0 \), \( u_i^* \leq 2\kappa_1 d_{\psi_1} \log(eN/i) \).

Therefore, if \( I_+ \) is the set of the \( m_0 \) largest coordinates of \((v_+^i)_{i=1}^N\), then \( (u_i)_{i=1}^N \) is dominated by \( (2\kappa_1 d_{\psi_1} \log(eN/i))_{i > m_0} \) on \( I_+^c \). Therefore,

\[
\left( \sum_{i \in I \cap I_+^c} (v_+^2 v_+^2_i) \right)^{1/2} \leq 8\kappa_1^2 t^2 d_{\psi_1} \|h_-\|_{\psi_1} \left( \sum_{i=1}^N \log^4(eN/i) \right)^{1/2}
\]

\[
\leq c_5 t^2 d_{\psi_1} \|h_-\|_{\psi_1} \sqrt{N},
\]

implying that
\[
\left( \sum_{i>2^s} (v_-^2 v_+^2)_i \right)^{1/2} \leq \left( \sum_{i \in J \cap I_+} (v_-^2 v_+^2)_i \right)^{1/2} + \left( \sum_{i \in J \cap I_+^c} (v_-^2 v_+^2)_i \right)^{1/2}
\]
\[
\leq \left( \sum_{i=1}^{m_0} (v_+^2)_i \right)^{1/2} \parallel P_J v_- \parallel _\infty + c_5 t^2 \parallel h_- \parallel _\psi_1 d \sqrt{N}
\]
\[
\leq c_6 t^2 \left( \gamma_2(F, \psi_2) \parallel h_- \parallel _{\psi_1} \log(e N/2^s) + d \parallel h_- \parallel _{\psi_1} \sqrt{N} \right).
\]

Thus, if \( m(h_-) \leq 2^s \) then

\[
\sum_{i=1}^{2^s} (v_- v_+^*)_i + u 2^{s/2} \left( \sum_{i \geq 2^s} (v_-^2 v_+^2)_i \right)^{1/2}
\]
\[
\leq c_7 t^{2^{s/2}} \left( d \gamma_2(F, \psi_2) \log^2(e N/2^s) \right)
\]
\[
+ u \left( \gamma_2(F, \psi_2) \parallel h_- \parallel _{\psi_1} \log(e N/2^s) + d \parallel h_- \parallel _{\psi_1} \sqrt{N} \right)
\]
\[
\leq c_8 u 2^s d \psi_1 \left( \gamma_2(F, \psi_2) 2^{s/2} \log^2(e N/2^s) + 2^{s/2} \parallel h_- \parallel _{\psi_1} \sqrt{N} \right),
\]

(5.7)

provided that \( u \geq 1 \).

Next, we turn to the case when \( m(h_-) > 2^s \). Let \( I_- \) be the set of the \( m(h_-) \) largest coordinates of \( v_- \) and again, \( I_+ \) is the set of the \( m_0 \) largest coordinates of \( v_+ \). Therefore, if \( I \subset \{1, ..., N\} \) and \( |I| \geq m(h_-) \) then

\[
\left( \sum_{i \in I} (v_-^2)_i \right)^{1/2} \leq 2 \kappa_1 t \parallel h_- \parallel _{\psi_1} \sqrt{|I|} \log(e N/|I|),
\]

and thus, for \( i \geq m(h_-) \),

\[
(v_-)_i^* \leq 2 \kappa_1 t \parallel h_- \parallel _{\psi_1} \log(e N/i).
\]

Also, from the definition of \( m(h_-) \) it is evident that

\[
\left( \sum_{i=1}^{m(h_-)} (v_-^2)_i \right)^{1/2} \leq 4 \kappa_1 t 2^{s/2} \parallel h_- \parallel _{\psi_2},
\]

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implying that
\[
\|P_I^c v_+\|_\infty \leq 4\kappa_1 t(2^s/m(h_-))^1/2\|h_-\|_{\psi_2} \leq c_9t\|h_-\|_{\psi_2},
\]
because \(2^s \leq m(h_-).\)

Set \(I = I_- \cup I_+ = (I_+ \setminus I_-) \cup I_-\). Note that
\[
\sum_{i \in I_-} |(v_- v_+)_i| \leq \left(\sum_{i \in I_-} (v_-^2)_i\right)^{1/2} \left(\sum_{i \in I_-} (v_+^2)_i\right)^{1/2}
\leq c_{10} t^{2^s/2} \|h_-\|_{\psi_2} \left(\gamma_2(F, \psi_2) + d_{\psi_1} \sqrt{N}\right),
\]
where we have used that
\[
\left(\sum_{i \in I_-} (v_-^2)_i\right)^{1/2} \leq 2 \sup_{f \in F} \left(\sum_{i=1}^N f^2(X_i)\right)^{1/2}
\leq t \left(\gamma_2(F, \psi_2) + d_{\psi_1} \sqrt{N}\right).
\]

Moreover, applying the bound on \(\|P_I^c v_-\|_\infty\),
\[
\left(\sum_{i \in I_+ \setminus I_-} (v_- v_+^2)_i\right)^{1/2} \leq \|P_I^c v_-\|_\infty \left(\sum_{i=1}^N (v_+^2)_i\right)^{1/2}
\leq c_{11} t^2 \|h_-\|_{\psi_2} \left(\gamma_2(F, \psi_2) + d_{\psi_1} \sqrt{N}\right).
\]

That leaves us with the coordinates that are outside \(I\), that is, outside both \(I_-\) and \(I_+\). Observe that \(v_-\) is dominated on \(I^c\) by \((2\kappa_1 t\|h_-\|_{\psi_1} \log(e N/i))\) and \(v_+\) is dominated on \(I^c\) by \((2\kappa_1 t d_{\psi_1} \log(e N/i))\). Hence,
\[
\left(\sum_{i \in I^c} (v_- v_+^2)_i\right)^{1/2} \leq 4\kappa_1^2 t^2 \|h_-\|_{\psi_1} d_{\psi_1} \left(\sum_{i=1}^N \log^4(e N/i)\right)^{1/2}
\leq c_{11} t^2 \|h_-\|_{\psi_1} d_{\psi_1} \sqrt{N}.
\]

Therefore, if \(m(h_-) \geq 2^s\), then with \((\varepsilon_i)_{i=1}^N\)-probability at least \(1 - 2 \exp(-c_{12} u^2 2^s)\),
\[
\left|\sum_{i=1}^N \varepsilon_i (v_- v_+)\right| \leq c_{13} u^2 t^{2^s/2} \left(\|h_-\|_{\psi_2} \left(\gamma_2(F, \psi_2) + d_{\psi_1} \sqrt{N}\right) + \|h_-\|_{\psi_1} d_{\psi_1} \sqrt{N}\right)
\leq c_{14} u^2 t^{2^s/2} \|h_-\|_{\psi_2} \left(\gamma_2(F, \psi_2) + d_{\psi_1} \sqrt{N}\right),
\]

(5.8)
provided that \( u \geq 1 \).

Combining \((5.7)\) and \((5.8)\), and since there are at most \(2^{s+1}\) links at the \(s\)-level, it is evident that for every \( t, u \geq c_{15} \) and every \( s \leq \tau_N \),

\[
Pr \left( \exists f \in F : \left| \sum_{i=1}^{N} \varepsilon_i \left( (\pi_s(f))^2(X_i) - (\pi_{s-1}(f))^2(X_i) \right) \right| \geq \rho(s, f) u \right) \\
\leq 2 \exp(-c_{16} u^2 2^s),
\]

where

\[
\rho(s, f) \sim ut^2 d_{\psi_1} \left( \gamma_2(F, \psi_2) 2^{s/2} \log^2(e N/2^s) + 2^{s/2} \|\Delta_s(f)\|_{\psi_1} \sqrt{N} \right) \\
+ ut^2 \left( 2^{s/2} \|\Delta_s(f)\|_{\psi_2} \right) \left( \gamma_2(F, \psi_2) + d_{\psi_1} \sqrt{N} \right).
\]

It remains to show that for every \( f \in F \),

\[
\sum_{\{s : 2^s \leq N\}} \rho(s, f) \lesssim ut^2 \left( \sqrt{N} d_{\psi_1} \gamma_2(F, \psi_2) + \gamma_2^2(F, \psi_2) \right),
\]

which is straightforward because for an almost optimal admissible sequence,

\[
\sum_{s \geq 1} 2^{s/2} \|\Delta_s(f)\|_{\psi_1} \leq 2 \gamma_2(F, \psi_2) \quad \text{and} \quad \sum_{\{s : s \leq \tau_N\}} 2^{s/2} \log^2(e N/2^s) \sim \sqrt{N}.
\]

We need an additional preliminary result which allows one to move freely between the empirical process and the Bernoulli one – the Giné-Zinn symmetrization Theorem [17]:

**Theorem 5.4** Let \( H \) be a class of functions and set \( \alpha^2 = \sup_{h \in H} \mathbb{E}(h - \mathbb{E}h)^2 \). For every integer \( N \) and any \( t \geq 2^{1/2} \alpha N^{1/2} \),

\[
Pr \left( \sup_{h \in H} \left| \sum_{i=1}^{N} (h(X_i) - \mathbb{E}h) \right| > t \right) \leq 4Pr \left( \sup_{h \in H} \left| \sum_{i=1}^{N} \varepsilon_i h(X_i) \right| > t/4 \right).
\]

Combining Theorem 5.4 and Theorem 5.2 we obtain the next result on the “beginning” of every chain.

**Theorem 5.5** There exist absolute constants \( c_1, c_2 \) and \( c_3 \) for which the following holds. Let \( F \) be a class of mean-zero functions and let \((F_s)_{s \geq 0}\) be
an almost optimal admissible sequence with respect to the $\psi_2$ norm. Then, for every $f_0 \in F$ and $x \geq c_1$, with probability at least $1 - 2 \exp(-c_2 x^{2/5})$

$$\sup_{f \in F} \left| \sum_{i=1}^{N} (\pi_{\tau N} f)^2(X_i) - f_0^2(X_i) - \mathbb{E}((\pi_{\tau N} f)^2 - f_0^2(X_i)) \right| \leq c_3 x \left( \sqrt{N} d_{\psi_1} \gamma_2(F, \psi_2) + \gamma_2^2(F, \psi_2) \right).$$

**Remark 5.6** The power of $x^{2/5}$ in the exponent is likely to be an artifact of the proof. We made no effort to optimize this power since it is not of major importance in the problems we wish to address, and because any exponential tail estimate would give us the integrability properties we need.

**Proof.** Fix $f_0 \in F$ and let $H = \{(\pi_{\tau N} f)^2 - f_0^2 : f \in F\}$. It is standard to verify that $a^2 = \sup_{h \in H} \mathbb{E}(h - \mathbb{E}h)^2 \leq c_0 d_{\psi_1}^4$. Since $F$ is symmetric then $\gamma_2(F, \psi_2) \geq d_{\psi_1}$ and $x \sqrt{N} d_{\psi_1} \gamma_2(F, \psi_2) \geq 2\sqrt{N} x$ provided that $x \geq 2c_0$.

If we set $\rho = \left( \sqrt{N} d_{\psi_1} \gamma_2(F, \psi_2) + \gamma_2^2(F, \psi_2) \right)$ then by Theorem 5.4 and the definition of $H$, for every $x \geq 2c_0$,

$$\Pr \left( \sup_{f \in F} \left| \sum_{i=1}^{N} (\pi_{\tau N} f)^2(X_i) - f_0^2(X_i) - \mathbb{E}((\pi_{\tau N} f)^2 - f_0^2(X_i)) \right| > x \rho \right) \leq 4 \Pr \left( \sup_{f \in F} \left| \sum_{i=1}^{N} \varepsilon_i((\pi_{\tau N} f)^2(X_i) - f_0^2(X_i)) \right| > x \rho/4 \right)$$

$$= 4 \mathbb{E}_X \mathbb{P}_{\tau N} \left( \sup_{f \in F} \left| \sum_{i=1}^{N} \varepsilon_i((\pi_{\tau N} f)^2(X_i) - f_0^2(X_i)) \right| > x \rho/4 \right),$$

by Fubini’s Theorem. Using the notation of (5.4) and (5.5), for $t > c_1$, let

$$\Omega_t = A_t \cap \left( \bigcap_{s \leq \tau N} B_t^s \right)$$

and observe that

$$\Pr(\Omega_t^c) \leq \Pr(A_t^c) + \sum_{s=1}^{\tau N} \Pr(B_t^s) \leq 2 \exp(-c_2 t \log N).$$

Thus, if we set $u_t = x/t^2$, then as long as $u_t \geq c_4$ (or in other words, for every $t$ such that $x \geq c_4 t^2$), Theorem 5.4 implies that

$$\mathbb{E}_X \mathbb{P}_{\tau N} \left( \sup_{f \in F} \left| \sum_{i=1}^{N} \varepsilon_i((\pi_{\tau N} f)^2(X_i) - f_0^2(X_i)) \right| \geq u_t^2 \rho \right) (1_{\Omega_t} + 1_{\Omega_t^c})$$

$$\leq 2 \left( \exp(-c_5 x^2/t^4) + \exp(-c_5 x t) \right) \leq 2 \exp(-c_6 x^{2/5}),$$

where the last inequality holds if we take $t = x^{2/5} \geq 1$. 

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The last component in the proof of Theorem A is an estimate on the “end” of each chain, that is, \( f - \pi_{\tau_N} f = \sum_{s>\tau_N} \Delta_s(f) \). Its proof is a combination of Bernstein’s inequality and a chaining argument (see Lemma 1.5 in [24]), and the key point is the observation that for every \( f, g \) and every \( u \geq 1 \), with probability at least \( 1 - 2 \exp(-cNu^2) \), \( (P_N(f-g))^2)^{1/2} \leq u \|f-g\|_{\psi_2} \). In particular one has

**Lemma 5.7** [24] There exist absolute constants \( c_1, c_2, c_3 \) and \( c_4 \) for which the following holds. Let \( (F_s)_{s \geq 0} \) be an almost optimal admissible sequence of \( F \) with respect to the \( \psi_2 \) norm. Then, for every \( u \geq c_1 \), with probability at least \( 1 - 2 \exp(-c_2Nu^2) \), for every \( f \in F \),

\[
\sup_{f \in F} (P_N(f - \pi_{\tau_N}(f))^2)^{1/2} \leq c_3 u \gamma_2(F, \psi_2) \sqrt{N},
\]

and

\[
\mathbb{E} \sup_{f \in F} (P_N(f - \pi_s(f))^2)^{1/2} \leq c_4 \gamma_2(F, \psi_2) \sqrt{N}.
\]

Finally, let us reformulate Theorem A.

**Theorem 5.8** There exist absolute constants \( c_1, c_2, c_3 \) and \( c_4 \) for which the following holds. If \( F \) is a symmetric class of mean-zero functions, then for every \( x \geq c_1 \), with probability at least \( 1 - 2 \exp(-c_2x^2/5) \),

\[
\sup_{f \in F} \left| \frac{1}{N} \sum_{i=1}^{N} f^2(X_i) - \mathbb{E} f^2 \right| \leq c_3 x \left( d_{\psi_1} \frac{\gamma_2(F, \psi_2)}{\sqrt{N}} + \frac{\gamma_2^2(F, \psi_2)}{N} \right).
\]

In particular,

\[
\mathbb{E} \sup_{f \in F} \left| \frac{1}{N} \sum_{i=1}^{N} f^2(X_i) - \mathbb{E} f^2 \right| \leq c_4 \left( d_{\psi_1} \frac{\gamma_2(F, \psi_2)}{\sqrt{N}} + \frac{\gamma_2^2(F, \psi_2)}{N} \right).
\]
Proof. Let \((F_s)_{s \geq 0}\) and \(\tau_N\) be as above. Then, for every \(f \in F\),
\[
\sum_{i=1}^{N} (f^2(X_i) - \mathbb{E}f^2) = \sum_{i=1}^{N} (f^2(X_i) - (\pi_{\tau_N} f)^2(X_i))
\]
\[
+ \sum_{i=1}^{N} ((\pi_{\tau_N} f)^2(X_i) - \mathbb{E}(\pi_{\tau_N} f)^2) + N \mathbb{E}((\pi_{\tau_N} f)^2 - f^2)
\]
\[
\leq 2 \left( \sum_{i=1}^{N} (f - \pi_{\tau_N} f)^2(X_i) \right)^{1/2} \sup_{f \in F} \left( \sum_{i=1}^{N} f^2(X_i) \right)^{1/2}
\]
\[
+ 2N \sup_{f \in F} (\mathbb{E}(f - \pi_{\tau_N} f)^2)^{1/2} \cdot \sup_{f \in F} (\mathbb{E}f^2)^{1/2}
\]
\[
+ \sup_{f \in F} \sum_{i=1}^{N} ((\pi_{\tau_N} f)^2(X_i) - \mathbb{E}(\pi_{\tau_N} f)^2).
\]

By Lemma 5.7 combined with Theorem 3.4, with probability at least 1 - \(2 \exp(-c_1 N t) - 2 \exp(-t^{1/2} \log N)\) the first and second terms are at most
\[
c_2 t \gamma_2(F, \psi_2) \left( \gamma_2(F, \psi_2) + d_{\psi_1} \sqrt{N} \right)
\]
for \(t \geq c_3\). The third term may be bounded using Theorem 5.5. Indeed, for every such \(t\),
\[
\sum_{i=1}^{N} ((\pi_{\tau_N} f)^2(X_i) - f_0^2(X_i)) - \mathbb{E}(\pi_{\tau_N} f)^2 - f_0^2)
\]
\[
\leq c_4 t \gamma_2(F, \psi_2) \left( \gamma_2(F, \psi_2) + d_{\psi_1} \sqrt{N} \right).
\]

with probability at least 1 - \(2 \exp(-c_5 t^{2/5})\).

Finally, a similar argument to the one used in the proof of Theorem 5.2 shows that for every such \(t\), with probability at least 1 - \(2 \exp(-c_5 t^{2/5})\),
\[
\sum_{i=1}^{N} (f_0^2(X_i) - \mathbb{E}f_0^2) \leq c_6 t (d_{\psi_2}^2 + d_{\psi_1} \sqrt{N})
\]
\[
\leq c_6 t (\gamma_2^2(F, \psi_2) + d_{\psi_1} \gamma_2(F, \psi_2) \sqrt{N}).
\]

Hence, with probability at least 1 - \(4(\exp(-c_1 t^{1/2} \log N) - \exp(-c_5 t^{2/5}))\),
\[
\sup_{f \in F} \left| \frac{1}{N} \sum_{i=1}^{N} f^2(X_i) - \mathbb{E}f^2 \right| \leq C t \left( \frac{\gamma_2(F, \psi_2)}{\sqrt{N}} + \frac{\gamma_2^2(F, \psi_2)}{N} \right),
\]

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as required.

The claim regarding the expectation follows from an integration argument and is omitted.

6 Applications

In this final section we will present several geometric applications of our three main results, though as pointed out in the introduction, there are numerous other applications in Empirical Processes Theory, Nonparametric Statistics and Asymptotic Geometric Analysis that will not be mentioned here.

It is well known that many results in Asymptotic Geometric Analysis are based on a random selection argument, for example, a random choice of a section or of a projection of a convex body in $\mathbb{R}^n$. Historically, the motivation was to understand the geometry of convex bodies and thus the models of random selection that had been studied were rather limited. Indeed, in classical results such as Dvoretzky’s Theorem, low-$M^*$ estimates and many others (see, e.g. [27, 31]), the selection was performed using a random point on a Grassman manifold $G_{n,k}$ relative to the Haar measure, or by applying a gaussian operator $\sum_{i=1}^{k}\langle G_i, \cdot \rangle e_i$ to the given body, with $(G_i)_{i=1}^{k}$ selected independently according to the canonical gaussian measure on $\mathbb{R}^n$.

In recent years, the distribution of volume in a convex body has become a central area of interest in Asymptotic Geometric Analysis. Hence, it is natural to ask whether the classical results in the area can be extended to other random selection methods, endowed by these volume measures, or, more generally, by isotropic, log-concave measures. It is, perhaps, surprising that extending the classical gaussian-based results even to natural subgaussian selection methods, for example, the uniform measure on $\{-1,1\}^n$, is not simple at all, and in some cases the extension is simply not true. Moreover, going beyond the subgaussian realm and proving such results for arbitrary isotropic, log-concave measures is even more difficult, mainly because the tail estimate that one has for linear functionals is rather weak. Indeed, in the isotropic, log-concave case the $\psi_1$ and the $\ell_2^n$ norms are equivalent, but $\|\langle x, \cdot \rangle\|_{\psi_2}$ might have a strong dependence on the dimension.

Here, we will study the way a random operator $\Gamma = \sum_{i=1}^{N}\langle X_i, \cdot \rangle e_i$ acts on a convex body, where $(X_i)_{i=1}^{N}$ are selected according to an isotropic, log-concave measure on $\mathbb{R}^n$. We will show that many parts of the gaussian theory remain true for such an operator, with the main difference being that
the classical parameter

$$\sqrt{n}M^*(K) = \sqrt{n} \int_{S^{n-1}} \|x\|_{K^*} d\sigma \sim E \sup_{x \in K} \sum_{i=1}^n g_i x_i$$

that is used to quantify the phenomena one sees for a gaussian operator is replaced by \(\gamma_2(K, \psi_2)\) (and recall that \((K, \psi_2)\) is the set of functions \(\{\langle x, \cdot \rangle : x \in K\}\) endowed with the \(\psi_2(\mu)\) norm). Another difference is that the probabilistic estimates we will obtain for a general random, isotropic, log-concave operator are much weaker than in the gaussian or subgaussian cases.

Assume that \(K \subset \mathbb{R}^n\) is symmetric. Then \(d_{\psi_\alpha} \sim \text{diam}(K, \psi_\alpha)\) and \(d_{\ell_2^n} \sim \text{diam}(K, \ell_2^n)\).

For \(\alpha = 1, 2\) and an isotropic measure \(\mu\), let \(Q_\alpha(\mu) = \sup_{\theta \in S^{n-1}} \|\langle \theta, \cdot \rangle\|_{\psi_\alpha}\) – the equivalence constant between the \(\psi_\alpha\) norm restricted to linear functionals on \(\mathbb{R}^n\) and the \(\ell_2^n\) norm. For example, if \(\mu\) is an isotropic, log-concave measure on \(\mathbb{R}^n\) then by Borell’s inequality, \(Q_1(\mu) \sim 1\). On the other hand, \(Q_2(\mu)\) can grow polynomially in \(n\).

### 6.1 The norm of random matrices

Let \(K \subset \mathbb{R}^n\) be a convex body and let \(\Gamma : \mathbb{R}^n \rightarrow \mathbb{R}^N\) be the random operator \(\sum_{i=1}^N \langle X_i, \cdot \rangle e_i\), where \((X_i)_{i=1}^N\) are independent, selected according to an isotropic, log-concave measure on \(\mathbb{R}^n\). Our goal is to estimate \(E \|\Gamma\|_{K \rightarrow \ell_2^N}\), and for the sake of brevity we will consider the case \(p \geq 2\), although the case \(1 \leq p < 2\) can be handled using similar means.

Let us begin with the relatively simple subgaussian case, when \(Q_2(\mu) \sim 1\).

**Theorem 6.1** There exists an absolute constant \(c\) for which the following holds. If \(p \geq 2\) and \(K \subset \mathbb{R}^n\) is a convex body, then for every integer \(N\),

$$E \|\Gamma\|_{K \rightarrow \ell_2^N} \leq c \left( \gamma_2(K, \psi_2) + Q_2(\mu) \text{diam}(K, \ell_2^n) \cdot N^{1/p} \right)$$

Since the proof of Theorem 6.1 is rather standard, we will only sketch it here.

**Proof.** Let \(p'\) be the conjugate index of \(p\). Consider the random process indexed by \(K \times B_{p'}^N\), defined by \(Z_{x,y} = \sum_{i=1}^N \langle X_i, x \rangle y_i\) and note that for every \((x, y)\) and \((x', y')\),

$$\|Z_{x,y} - Z_{x',y'}\|_{\psi_2} \leq d_{\psi_2} \|y - y'\|_2 + \text{diam}(B_{p'}^N, \ell_2^N) \|\langle X, x - x' \rangle\|_{\psi_2}.$$
Therefore, applying a chaining argument,
\[
E \sup_{x \in K, \ y \in B^{N}_{\ell^2}} Z_{x,y} \leq c_1 \left( d_{\psi_2} \gamma_2(B^{N}_{\ell^2}, \ell^2) + \text{diam}(B^{N}_{\ell^2}, \ell^2) \gamma_2(K, \psi_2) \right).
\]

To complete the proof, if \( G = (g_1, \ldots, g_N) \) is the standard gaussian vector in \( \mathbb{R}^N \) then by the Majorizing Measures Theorem,
\[
\gamma_2(B^{N}_{\ell^2}, \ell^2) \leq c_2 \mathbb{E} \sup_{y \in B^{N}_{\ell^2}} \sum_{i=1}^{k} g_i y_i = c_2 \mathbb{E} \| G \|_{\ell^p} \leq c_3 N^{1/p}.
\]

Also, since \( p \geq 2 \) then \( \text{diam}(B^{N}_{\ell^2}, \ell^2) = 1 \) and clearly \( d_{\psi_2} \lesssim Q_2(\mu) d_{\ell^2} \). Therefore,
\[
E \| \Gamma \|_{K \rightarrow \ell^N} \lesssim \gamma_2(K, \psi_2) + Q_2(\mu) d_{\ell^2} N^{1/p},
\]
as claimed.

\[ \blacksquare \]

It is simple to verify that Theorem 6.1 cannot be improved, up to the constants involved. Indeed, if \( \mu \) is the standard gaussian measure on \( \mathbb{R}^n \) then \( Q_2(\mu) \) is an absolute constant and \( \gamma_2(K, \psi_2) \sim \gamma_2(K, L_2) = \gamma_2(K, \ell^2_2) \). Let \( (G_i)_{i=1}^{N} \) be independent copies distributed according to \( \mu \) and since \( e_1 \in B^{N}_{\ell^2} \) then
\[
E \| \Gamma \|_{K \rightarrow \ell^N} = E \sup_{x \in K} \sup_{y \in B^{N}_{\ell^2}} \sum_{i=1}^{N} \langle G_i, x \rangle y_i \geq E \sup_{x \in K} \sum_{i=1}^{n} g_i x_i.
\]

Also, if \( \| x_0 \|_2 = d_{\ell^2} \) then
\[
E \| \Gamma \|_{K \rightarrow \ell^N} \geq E \| x_0 \|_{\ell^p} \| \Gamma \|_{K \rightarrow \ell^N} \geq c_2 \| x_0 \|_2 N^{1/p},
\]
showing that the estimate in Theorem 6.1 is sharp in this case.

Thanks to Theorem B it is possible to replace \( Q_2(\mu) \) in Theorem 6.1 by \( Q_1(\mu) \), which, in the log-concave case, is of the order of an absolute constant.

\[ \textbf{Theorem 6.2} \]

There exists an absolute constant \( c \) for which the following holds. Let \( K \) be a convex body in \( \mathbb{R}^n \). Then for every \( p \geq 2 \) and any integer \( N \), a random isotropic, log-concave operator \( \Gamma \) satisfies that
\[
E \| \Gamma \|_{K \rightarrow \ell^N} \leq c \left( \gamma_2(K, \psi_2) + \text{diam}(K, \ell^2_2) \cdot N^{1/p} \right).
\]

\[ \text{Proof.} \] Since \( \text{diam}(F, \psi_1) \sim \text{diam}(K, \ell^2_2) \), the claim follows immediately from Theorem B and its extensions to other \( \ell^p \) norms for \( m = N \) and \( F = \{ \langle x, \cdot \rangle : x \in K \} \). \[ \blacksquare \]
An interesting case in which Theorem 6.2 can be used is the “standard shrinking” phenomenon. Simply put, standard shrinking is the observation that for every $x \in \mathbb{R}^n$, and with high probability with respect to the uniform measure on the Grassman manifold $G_{n,k}$, the random orthogonal projection $P_E$ satisfies that $\|P_E x\|_2 \leq c \sqrt{k/n} \|x\|_2$. This property can be extended to a more general situation. Indeed, one can show that if $K \subset \mathbb{R}^n$ is a convex body, $k^* = \sqrt{nM^*(K)/d_{\ell_2^n}}$ and $k \geq c_1 k^*$, then with high probability in $G_{n,k}$, $\text{diam}(P_E K, \ell_2^n) \leq c_2 d_{\ell_2^n} \sqrt{k/n}$. Moreover, this result is sharp, since Milman’s version of Dvoretzky’s Theorem (see, for example, [27]) implies that if $k \leq c_3 k^*$, then with high probability $P_E K \supset c_4 M^*(K) B_2^n$, and the diameter cannot decrease further.

The shrinking of the diameter for $k \geq k^*$ extends to other random operators, but even in a relatively simple case, when $\Gamma$ is selected according to the uniform measure on $\{-1,1\}^n$, some nontrivial machinery is required [3], particularly if one wishes to recover the probabilistic estimate $\sim \exp(-ck)$. The methods developed in [24] (see Corollary 1.9 there) show that the same is true — and with the same probability estimate, as long as $Q_2(\mu) \sim 1$.

Theorem 6.2 implies that shrinking does happen for a random isotropic, log-concave operator — though with a weaker probabilistic estimate. Indeed, consider the operator $A = \Gamma/\sqrt{n}$, let $K \subset \mathbb{R}^n$ be a convex body and set $k' = \gamma_2(K, \psi_2)/d_{\ell_2^n}$. Then, with high probability,

$$\text{diam}(AK, \ell_2^n) \leq \frac{1}{\sqrt{n}} \left( \gamma_2(K, \psi_2) + \text{diam}(K, \ell_2^n) \sqrt{k} \right) \leq \sqrt{\frac{k}{n}} \text{diam}(K, \ell_2^n),$$

as long as $k \geq k'$. Since

$$c_1 \sqrt{nM^*(K)} \leq \gamma_2(K, \psi_2) \leq c_2 Q_2(\mu) \sqrt{nM^*(K)},$$

it follows that if $\mu$ happens to be subgaussian, i.e. if $Q_2(\mu) \sim 1$, then $k'$ and $k^*$ are equivalent.

### 6.2 Low–$M^*$ estimates

Given a convex body $K \subset \mathbb{R}^n$ and $k \leq n$, one would like to find a subspace $E \subset \mathbb{R}^n$ for which the Euclidean diameter of $K \cap E$ is as small as possible. We refer the reader to [26, 24] for a brief description of the progress made on this problem.

In [28, 29] it was shown that if $E$ is the kernel of a random orthogonal projection (or of a gaussian projection), and if

$$r^*_N = \inf \left\{ r > 0 : \sqrt{nM^*(K \cap r S^{n-1})}/\sqrt{N} \leq cr \right\},$$

(6.1)
then $\text{diam}(E \cap K) \leq r_N^*$, where $c$ is an absolute constant.

Since the original proof of this result is based on the structure of gaussian variables or that of the Haar measure on $G_{n,k}$, extending it to other natural random operators is not trivial. Equation (6.1) was extended to the subgaussian case in [24] using a subgaussian version of Theorem A. It was shown that if $\mu$ is isotropic and $\Gamma = \sum_{i=1}^{N} \langle X_i, \cdot \rangle e_i$ (with $X_1, \ldots, X_N$, independent, distributed according to $\mu$), then with high probability,

$$\text{diam}(K \cap \ker \Gamma) \leq \inf \left\{ r > 0 : Q_2(\mu) \gamma_2(K \cap r S^{n-1}, \psi_2)/\sqrt{N} \leq cr \right\}. \quad (6.2)$$

Therefore, if $\mu$ is isotropic and $Q_2(\mu) \sim 1$, (that is, if $\Gamma$ is an isotropic, subgaussian operator) then (6.1) is true. Applying Theorem A, the fact that for an isotropic, log-concave measure $Q_1(\mu) \sim 1$ and the proof from [24], one has

**Theorem 6.3** There exist absolute constants $c$ and $c_1$ for which the following holds. Let $\Gamma : \ell_2^n \to \ell_2^N$ be a random isotropic log-concave operator. Then for a convex body $K \subset \mathbb{R}^n$ one has

$$\mathbb{E} (\text{diam}(K \cap \ker \Gamma)) \leq c_1 \inf \left\{ r > 0 : \gamma_2(K \cap r S^{n-1}, \psi_2)/\sqrt{N} \leq cr \right\},$$

and a similar estimate holds with high probability.

Again, Theorem 6.3 extends the classical result to any isotropic log-concave case ensemble, with $\gamma_2(K, \psi_2)$ taking the place of $\sqrt{n}M^*(K)$ – though with a weaker probabilistic estimate.

### 6.3 The process indexed by $S^{n-1}$

This section is devoted to a problem that is far from being fully solved – the behavior of the process

$$\sup_{\theta \in S^{n-1}} \left| \frac{1}{N} \sum_{i=1}^{N} \langle X_i, \theta \rangle^2 - 1 \right|, \quad (6.3)$$

where $X_1, \ldots, X_N$ are selected independently according to an isotropic, log-concave measure on $\mathbb{R}^n$.

In [1] the authors solved the following facet of this problem: Given $\varepsilon > 0$ and $0 < \delta < 1$, how many random points $X_1, \ldots, X_N$ are needed to ensure that with probability $1 - \delta$,

$$\sup_{\theta \in S^{n-1}} \left| \frac{1}{N} \sum_{i=1}^{N} \langle \theta, X_i \rangle^2 - 1 \right| < \varepsilon?$$
An equivalent formulation of this question is to find the smallest $N$ that would still guarantee that a random, isotropic, log-concave operator $\Gamma$ embeds $\ell^2_2$ in $\ell^N_2$, $1 + \varepsilon$ isomorphically.

This problem has been studied extensively in recent years (e.g. [21, 9, 15, 32, 20, 30, 25, 4]), in which the estimate has been improved from the initial $N \geq c(\varepsilon, \delta) n^2$ in [21] to the best possible estimate of $N \geq c(\varepsilon, \delta)n$, proved in [1]. In fact, what was actually proved in [1] is the following:

**Theorem 6.4** There exist absolute constants $C$, $c$ and $c_1$ for which the following holds. Let $\mu$ be an isotropic, log-concave measure on $\mathbb{R}^n$ and let $(X_i)_{i=1}^N$ be independent, distributed according to $\mu$. Then, for every $t \geq 1$ and every $1 \leq N \leq \exp(\sqrt{n})$, with probability at least $1 - 2 \exp(-c t \sqrt{n})$, for every $I \subset \{1, \ldots, N\}$,

$$\sup_{\theta \in S^{n-1}} \left( \sum_{i \in I} \langle \theta, X_i \rangle^2 \right)^{1/2} \leq C \left( \sqrt{n} + \sqrt{|I| \log(eN/|I|)} \right). \quad (6.4)$$

Moreover, for every $c_1 n \leq N \leq \exp(\sqrt{n})$ and every $s, t \geq 2$,

$$\sup_{\theta \in S^{n-1}} \left| \frac{1}{N} \sum_{i=1}^N \langle \theta, X_i \rangle^2 - 1 \right| \leq C \left( ts \sqrt{\frac{n}{N} \log(eN/n)} + s \frac{n}{N} \right) \quad (6.5)$$

with probability at least $1 - 2 \exp(-c s \sqrt{n}) - 2 \exp(-c \min\{u, v\})$, where $u = t^2 s^2 n \log^2(eN/n)$ and $v = (t/s) \sqrt{nN} / \log(eN/n)$.

Although Theorem 6.4 beautifully resolves the case $N \sim n$, its proof has certain weaknesses from the point of view of empirical processes theory and the general understanding of the process (6.3). First of all, (6.5) is derived from (6.4) using a decomposition and contraction argument, just like our Theorem C is derived from Theorem B. Hence, there is an intrinsic logarithmic looseness in (6.5) – a superfluous factor of $\log N$ for $N \geq c(\beta)n^{1+\beta}$ for any $\beta > 0$.

Second, the proof of Theorem 6.4 relies on the Euclidean nature of the problem in a very strong way: that the given class is a class of linear functionals on $\mathbb{R}^n$, that the indexing set is the entire sphere and that the measure is isotropic, log-concave (in particular, that $Q_1(\mu) \sim 1$ and that the Euclidean norm of a random point concentrates around $\sqrt{n}$). Hence, the method of [1] cannot be extended beyond this limited setup, even to obtain an analogous result for a small subset of the sphere as an indexing class. Naturally, it is also impossible to obtain an “empirical processes” result like Theorem A in
this way. A consequence of this limitation is that the method of [1] cannot be used to prove the applications presented in the two previous sections (i.e., estimates on the norm $\|\Gamma\|_{K \to \ell^N_2}$, the shrinking phenomenon, and low-$M^*$ estimates) since those applications require accurate information on the way $\Gamma$ acts on arbitrary subsets of $\mathbb{R}^n$ rather than on the entire sphere.

Process (6.3) is very far from being understood when one goes beyond the case $N \sim n$. A reasonable conjecture is that for any $N \gg n$, with high probability/in expectation,

$$\sup_{\theta \in S^{n-1}} \left| \frac{1}{N} \sum_{i=1}^{N} \langle \theta, X_i \rangle^2 - 1 \right| \leq c \sqrt{\frac{n}{N}},$$

which is the situation for the gaussian ensemble.

Below, we will indicate some of the problems one faces when trying to verify this conjecture, with the main one being that very little is known on the metric structure endowed on $S^{n-1}$ by a log-concave measure.

Currently, the best estimate on (6.3) in the range $c(\beta)n^{1+\beta} \leq N \leq \exp(\sqrt{n})$ for any $\beta > 0$ is $c_1(\beta)\sqrt{n \log n}/N$. This is a corollary of Theorem A, and the suboptimal estimate from [25], that $\gamma_2(S^{n-1}, \psi_2) \lesssim \sqrt{n \log n}$ for $\mu$ that is supported in $c_2\sqrt{n}B^n_2$ (the so-called small diameter case). Note that the small diameter assumption can be made without loss of generality as long as $N \leq \exp(\sqrt{n})$ thanks to the result of Paouris [30] which states that for $N \leq \exp(\sqrt{n})$, $\mathbb{E} \max_{i \leq N} \|X_i\|_2 \lesssim \sqrt{n}$. Hence, for those values of $N$, one may assume that $\mu$ is supported in $c_2\sqrt{n}B^n_2$, implying that if $c(\beta)n^{1+\beta} \leq N \leq \exp(\sqrt{n})$ then Theorem A improves Theorem 6.4 and gives the best known estimate on (6.3).

We believe that under the small diameter assumption, the extra logarithmic term in $\gamma_2(S^{n-1}, \psi_2)$ could be removed. Indeed, if $\mu$ is supported on a ball of radius $\sim \sqrt{n}$, “most” directions $\theta \in S^{n-1}$ have a $\psi_2$ norm that is bounded by an absolute constant (see, for example, [16]). Unfortunately, even under a small diameter assumption, there is very little information on the geometry of the set of these “good” directions, except that it is a very large subset of the sphere.

The second step towards a complete solution, and most likely the more difficult one, is when $N \geq \exp(\sqrt{n})$. Here, one can no longer assume that $\mu$ is supported in a ball of radius $\sim \sqrt{n}$, and thus both Theorem 6.4 and the bound on $\gamma_2(S^{n-1}, \psi_2)$ from [25] fail. Moreover, when leaving the small diameter case, it is not known whether there is even a single direction $\theta$ for which $\|\theta\|_{\psi_2} \sim 1$. 

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6.3.1 The unconditional case

We end this note with an example of how the $\sqrt{\log n}$ factor may be removed in a special case, when $\mu$ is unconditional. This example illustrates the difficulties that one is likely to encounter in the general case, where there is little structure at our disposal.

The argument has two parts. First, we will show that one may consider a slightly different “small diameter” assumption, and second, that under this assumption, the metric entropy $\log N(S^{n-1}, \epsilon B_{\psi_2})$ is well behaved.

For the first part, note that by the Bobkov-Nazarov Theorem $[7, 16]$, if $N \sim n^\alpha$ and if we denote the $j$-th coordinate of a monotone rearrangement of the coordinates of the vector $X_i$ by $(X_i)^*_j$, then with high probability, for every $1 \leq i \leq N$ and $1 \leq j \leq n$, $(X_i)^*_j \leq c_\alpha \log(en/j)$. Hence, without loss of generality we may assume that $\mu$ is supported in $c_1(\alpha) B_{\psi_1^n}$. This gives more accurate information than the standard small diameter assumption, that $\mu$ is supported in $c_\sqrt{n} B_{\psi_2^n}$. In particular, we may assume that almost surely, for every $j \leq n$, $(X)^*_j \leq c_\alpha \log(en/j)$. Since $\mu$ is unconditional, then for every $\theta \in S^{n-1}$ the random variable $\langle X, \theta \rangle$ has the same distribution as $\sum_{j=1}^n \epsilon_j |\langle X, e_j \rangle| \theta_j$, where $(\epsilon_j)_{j=1}^n$ are i.i.d. Bernoulli random variables. Hence, for any $p \geq 1$,

\[
\left( E_X |\langle X, \theta \rangle|^p \right)^{1/p} \leq c_\sqrt{p} \left( E_X \left| \sum_{j=1}^n \epsilon_j |\langle X, e_j \rangle| \theta_j \right|^p \right)^{1/p} \leq c_\sqrt{p} \left( E_X \left| \sum_{i=1}^n (X^2)_i (\theta^2)_i \right|^p \right)^{1/p} \leq c_1(\alpha) \sqrt{p} \left( \sum_{j=1}^n (\theta^2)_j^* \log^2(en/j) \right)^{1/2}.
\]

In particular, for every $\theta \in S^{n-1}$,

\[
||\theta||_{\psi_2} \leq c(\alpha) \left( \sum_{j=1}^n (\theta^2)_j^* \log^2(en/j) \right)^{1/2}
\]

and $\text{diam}(S^{n-1}, \psi_2) \leq c(\alpha) \log n$.

Now, just as in $[25]$ one may show that for every $\epsilon \leq 2$, the covering numbers satisfy $N(S^{n-1}, \epsilon B_{\psi_2}) \leq (c_2/\epsilon)^n$. Thus, it remain to estimate the covering numbers for larger scales.
To that end, we will use a minor modification of the sets \( N_\ell \) and \( B_m \) that appeared in Section 3.

Let \( A_\ell = \left\{ z \in B_n^2 : |\text{supp}(z)| \leq \ell, \|z\|_\infty \leq 1/\sqrt{\ell} \right\} \), fix \( r \) such that \( 2r \leq n/10 \) and let \( \varepsilon_r = \log(en/2r) \). Set \( N_{2j} \subset A_{2j} \) to be an \( \varepsilon_r(2^j/n) \)-cover of \( A_{2j} \) with respect to the \( \ell_2^n \) norm and define

\[
B_r = \left\{ z \in B_n^2 : |\text{supp}(z)| \leq 2^r, \text{supp}(z) = \bigcup_{j=0}^{r-1} I_j, P_{I_j}z \in N_{2j} \right\},
\]

where \( I_j \) are disjoint sets of coordinates with \( |I_0| = 2 \) and \( |I_j| = 2^j \) for \( j \geq 1 \).

It is standard to verify that \( |B_r| \leq \exp(c_02^r \log(en/2^r)) \) and that for every \( \theta \in S^{n-1} \) there is some \( \tilde{\theta} \in B_r \) whose support is denoted by \( I \), such that

\[
\|\theta - \tilde{\theta}\|_{\psi_2} \leq c_1 \left( \sum_{j=0}^{r-1} \|P_{I_j}(\theta - \tilde{\theta})\|_2 \log(en/2^j) + \|P_{I_r}\theta\|_2 \log(en/2^r) \right)
\]

\[
\leq c_1 \left( \frac{\varepsilon_r}{n} \sum_{j=0}^{r-1} 2^j \log(en/2^j) + \log(en/2^r) \right) \leq c_2\varepsilon_r
\]

for \( c_1 \) and \( c_2 \) that depend on \( \alpha \).

Therefore, \( B_r \) is a \( c_2\varepsilon_r \)-cover of \( S^{n-1} \) with respect to the \( \psi_2 \) norm, implying that

\[
\log N(S^{n-1},c_2\varepsilon_r,\psi_2) \leq c_02^r \log(en/2^r).
\]

It is well known [36] that if \((T,d)\) is a metric space then

\[
\gamma_2(T,d) \lesssim \int_0^{\text{diam}(T,d)} \frac{1}{\sqrt{\log(N(T,\varepsilon,d))}} d\varepsilon,
\]

and thus a simple calculation of this entropy integral shows that

\[
\gamma_2(S^{n-1},\psi_2) \leq c_3(\alpha)\sqrt{n},
\]

proving our claim. \[\blacksquare\]
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