Research article

A certain subclass of bi-univalent functions associated with Bell numbers and \( q \)-Srivastava Attiya operator

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Abstract: In the present study, we introduced general a subclass of bi-univalent functions by using the Bell numbers and \( q \)-Srivastava Attiya operator. Also, we investigate coefficient estimates and famous Fekete-Szegö inequality for functions belonging to this interesting class.

Keywords: bi-univalent function; \( q \)-Srivastava Attiya operator; Bell numbers; coefficient estimates

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1. Introduction and preliminaries

Let \( \mathcal{A} \) be the class of all analytic functions of the form

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k
\]  

in the open unit disk \( D = \{z \in \mathbb{C} : |z| < 1\} \) normalized by the conditions \( f(0) = 0 \) and \( f'(0) = 1 \). The well-known Koebe one-quarter theorem \cite{8} ensures that the image of \( D \) under every univalent function \( f \in \mathcal{A} \) contains a disk of radius 1/4. Thus, every univalent function \( f \) has an inverse \( f^{-1} \) satisfying \( f^{-1}(f(z)) = z \) and

\[
f^{-1}(f(w)) = w, \quad (|w| < r_0(f), \ r_0(f) \geq 1/4)
\]

where

\[
f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - \cdots.
\]  

A function \( f \in \mathcal{A} \) is said to be bi-univalent in \( D \) if both \( f \) and \( g \) to \( D \) are univalent in \( D \), where \( g \) is the analytic continuation of \( f^{-1} \) to the unit disk \( D \). Let \( \Sigma \) denote the class of bi-univalent functions...
Lemma 1.1. For a brief history of functions in the class \( \Sigma \), see [3, 4, 16, 19]. Later, Srivastava et al.’s [24, 26–28] gave very important contributions to this theory. Recently, for coefficient estimates of the functions in some particular subclasses of bi-univalent functions, one may see [6, 7, 10, 15, 20, 25, 29, 30].

For analytic functions \( f \) and \( g \) in \( \mathbb{D} \), \( f \) is said to be subordinate to \( g \) if there exists an analytic function \( w \) such that \( w(0) = 0 \), \( |w(z)| < 1 \) and \( f(z) = g(w(z)) \). This subordination is denote by \( f(z) < g(z) \).

In particular, when \( g \) is univalent in \( \mathbb{D} \),

\[
f(z) < g(z) \iff f(0) = g(0) \text{ and } f(D) \subset g(D) \quad (z \in \mathbb{D}).
\]

The \( q \)-difference operator, which was introduced by Jackson [13], is define by

\[
\partial_q f(z) = \frac{f(qz) - f(z)}{(q - 1)z}, \quad (z \neq 0)
\]

for \( q \in (0, 1) \). It is clear that \( \lim_{q \to 1^+} \partial_q f(z) = f'(z) \) and \( \partial_q f(0) = f'(0) \), where \( f' \) is the ordinary derivative of the function. For more properties of \( \partial_q \) see [9, 11, 12].

Thus, for function \( f \in \mathcal{A} \) we have

\[
\partial_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1},
\]

where \([k]_q\) is given by

\[
[k]_q = \frac{1 - q^k}{1 - q}, \quad [0]_q = 0
\]

and the \( q \)-factorial is define by

\[
[k]_q! = \begin{cases} 
\prod_{n=1}^{k} [n]_q, & k \in \mathbb{N} \\
1, & k = 0
\end{cases}
\]

As \( q \to 1^− \), then we get \([k]_q \to k\). Thus, if we choose the function \( g(z) = z^k \), while \( q \to 1 \), then we have

\[
\partial_q g(z) = \partial_q z^k = [k]_q z^{k-1} = g'(z),
\]

where \( g' \) is the ordinary derivative.

In order to derive our main results, we have to recall here the following lemmas.

**Lemma 1.1.** [8] If \( p \in \mathcal{P} \) then \( |p_k| \leq 2 \) for each \( k \), where \( \mathcal{P} \) is the family of all functions \( p \) analytic in \( \mathbb{D} \) for which \( \Re p(z) > 0 \),

\[
p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots
\]

for \( z \in \mathbb{D} \).

**Lemma 1.2.** [17] If the function \( p \in \mathcal{P} \) is given by the series 1.8, then

\[
2p_2 = p_1^2 + x(4 - p_1^2),
\]

\[
4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z
\]

for some \( x, z \) with \( |x| \leq 1 \) and \( |z| \leq 1 \).
For a fixed non-negative integer \( n \), the Bell numbers \( B_n \) count the possible disjoint partitions of a set with \( n \) elements into non-empty subsets or, equivalently, the number of equivalence relations on it. The numbers \( B_n \) are named the Bell numbers after Eric Temple Bell (1883 – 1960) (see [1,2]) who called them the “exponential numbers”. The Bell numbers \( B_n \) \((n \geq 0)\) are generated by the function \( e^{e^z - 1} \) as follows: \( e^{e^z - 1} = \sum_{n=0}^{\infty} B_n \left( \frac{z^n}{n!} \right) (z \in \mathbb{R}) \). The Bell numbers \( B_n \) satisfy the following recurrence relation involving binomial coefficients: \( B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k \). Clearly, we have \( B_0 = B_1 = 1 \), \( B_2 = 2 \), \( B_3 = 5 \), \( B_4 = 15 \), \( B_5 = 52 \) and \( B_6 = 203 \). We now consider the function \( \varphi (z) := e^{e^z - 1} \) with its domain of definition as the open unit disk \( \mathbb{D} \). Recently Srivastava and co-authors studied geometric properties and coefficients bounds for starlike functions related to the Bell numbers (see [5,14]).

On the other hand, Shah and Noor [21] introduced the \( q \)-analogue of the Hurwitz Lerch zeta function by the following series:

\[
\phi_q (s,b;z) = \sum_{k=0}^{\infty} \frac{z^k}{[k+b]_q^s},
\]

where \( b \in \mathbb{C} \setminus \mathbb{Z}_0^-, s \in \mathbb{C} \) when \(|z| < 1\), and \( \text{Re} (s) > 1 \) when \(|z| = 1\). The a normalized form of 1.9 as follows:

\[
\psi_q (s,b;z) = [1+b]_q^s \left[ \phi_q (s,b;z) - [b]_q^{-s} \right] = z + \sum_{k=2}^{\infty} \left( \frac{[1+b]_q}{[k+b]_q} \right)^s z^k.
\]

From 1.10 and 1.1, Shah and Noor [21] defined the \( q \)-Srivastava Attiya operator \( J_{q,b}^s f (z) : \mathcal{A} \rightarrow \mathcal{A} \) by

\[
J_{q,b}^s f (z) = \psi_q (s,b;z) * f (z) = z + \sum_{k=2}^{\infty} \left( \frac{[1+b]_q}{[k+b]_q} \right)^s a_k z^k
\]

where * denotes convolution (or the Hadamard product).

We note that:

(i) If \( q \rightarrow 1^- \), then the function \( \phi_q (s,b;z) \) reduces to the Hurwitz-Lerch zeta function and the operator \( J_{q,b}^s \) coincides with the Srivastava-Attiya operator (see [22,23]).

(ii) \( J_{q,0}^1 f (z) = \int_0^z f(t) t^{-1} d_q t \) \( (q-\text{Alexander operator}) \).

(iii) \( J_{q,b}^1 f (z) = \frac{[1+b]_q}{[z]} \int_0^z f(t) t^{-1} d_q t \) \( (q-\text{Bernardi operator [18]}) \).

(iv) \( J_{q,1}^1 f (z) = \frac{[z]_q}{[z]} \int_0^z f(t) t^{-1} d_q t \) \( (q-\text{Libera operator [18]}) \).

In present paper, we defined a general subclass \( \Sigma H_{q,b}^{s} (\tau, \lambda, \mu) \) of bi-univalent functions related to the Bell numbers by using \( q \)-Srivastava Attiya operator. Using the principles of subordination, the estimates for the coefficients \( |a_2|, |a_3| \) and \( |a_3 - \delta a_2^2| \) of the functions of the form 1.1 in the class \( \Sigma H_{q,b}^{s} (\tau, \lambda, \mu) \) have been obtained. For some particular choices of \( \tau, \lambda, \mu \) and \( s \) the bounds determined.
2. Coefficient estimates

Let $\Omega$ be the class of analytic functions of the form

$$w(z) = w_1z + w_2z^2 + w_3z^3 + \ldots$$

in the unit disk $\mathbb{D}$ satisfying the condition $|w(z)| < 1$. There is an important relation between the classes $\Omega$ and $P$ as follows:

$$w \in \Omega \Leftrightarrow \frac{1 + w(z)}{1 - w(z)} \in P \text{ or } p \in P \Leftrightarrow \frac{p(z) - 1}{p(z) + 1} \in \Omega. \quad (2.1)$$

Define the functions $p$ and $s$ in $P$ given by

$$p(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots$$

and

$$s(z) = \frac{1 + v(z)}{1 - v(z)} = 1 + s_1z + s_2z^2 + s_3z^3 + \cdots.$$ 

It follows that

$$u(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{p_1z}{2} + 1 \left( p_2 - \frac{p_1^2}{2} \right) z^2 + \cdots \quad (2.2)$$

and

$$v(z) = \frac{s(z) - 1}{s(z) + 1} = \frac{s_1z}{2} + 1 \left( s_2 - \frac{s_1^2}{2} \right) z^2 + \cdots. \quad (2.3)$$

Definition 2.1. A function $f \in \Sigma$ is said to be in the class $\Sigma H^s_{q,b}(\tau, \lambda, \mu)$ if the following conditions hold true for all $z, w \in \mathbb{D}$:

$$1 + \frac{1}{\tau} \left( (1 - \lambda) \left[ \frac{J^s_{q,b,f}(z)}{z} \right] + \lambda \varphi_q \left( J^s_{q,b,f}(z) \right) \left( \frac{J^s_{q,b,f}(z)}{z} \right)^{\mu-1} - 1 \right) < \varphi(z)$$

and

$$1 + \frac{1}{\tau} \left( (1 - \lambda) \left[ \frac{J^s_{q,b,g}(w)}{w} \right] + \lambda \varphi_q \left( J^s_{q,b,g}(w) \right) \left( \frac{J^s_{q,b,g}(w)}{w} \right)^{\mu-1} - 1 \right) < \varphi(w)$$

where $\varphi(z) = e^{\varphi z}$, $g(w) = f^{-1}(w)$, $\tau \in \mathbb{C} \setminus \{0\}$, $\mu > 0$, $0 < q < 1$ and $\lambda \geq 0$.

Remark 2.1. We note that, for suitable choices parameters, the class $\Sigma H^s_{q,b}(\tau, \lambda, \mu)$ reduces to the following classes.

1) Let $\lambda = 1$ in $\Sigma H^s_{q,b}(\tau, \lambda, \mu)$. Then a function $f \in \Sigma$ is said to be in the class $\Sigma H^s_{q,b}(\tau, \mu)$ if the following subordinations hold for all $z, w \in \mathbb{D}$:

$$1 + \frac{1}{\tau} \left[ \varphi_q \left( J^s_{q,b,f}(z) \right) \left( \frac{J^s_{q,b,f}(z)}{z} \right)^{\mu-1} - 1 \right] < \varphi(z)$$
and

\[ 1 + \frac{1}{\tau} \left[ \partial_q \left( J_{q,b}^s g (w) \right) \left( J_{q,b}^s g (w) \right)^{\mu - 1} - 1 \right] < \varphi (w) \]

2) Let \( \lambda = 1 \) and \( \tau = 1 \) in \( \Sigma H_{q,b}^s (\tau, \lambda, \mu) \). Then a function \( f \in \Sigma \) is said to be in the class \( \Sigma H_{q,b}^s (\mu) \) if the following subordinations hold for all \( z, w \in \mathbb{D} \):

\[ \partial_q \left( J_{q,b}^s f (z) \right) \left( J_{q,b}^s f (z) \right)^{\mu - 1} < \varphi (z) \]

and

\[ \partial_q \left( J_{q,b}^s f (w) \right) \left( J_{q,b}^s f (w) \right)^{\mu - 1} < \varphi (w) \]

3) Let \( \mu = 1 \) in \( \Sigma H_{q,b}^s (\tau, \lambda, \mu) \). Then a function \( f \in \Sigma \) is said to be in the class \( \Sigma H_{q,b}^s (\tau, \lambda) \) if the following subordinations hold for all \( z, w \in \mathbb{D} \):

\[ 1 + \frac{1}{\tau} \left[ (1 - \lambda) \frac{J_{q,b}^s f (z)}{z} + \lambda \partial_q \left( J_{q,b}^s f (z) \right) - 1 \right] < \varphi (z) \]

and

\[ 1 + \frac{1}{\tau} \left[ (1 - \lambda) \frac{J_{q,b}^s g (w)}{w} + \lambda \partial_q \left( J_{q,b}^s g (w) \right) - 1 \right] < \varphi (w) \]

4) Let \( \mu = 1 \) and \( \tau = 1 \) in \( \Sigma H_{q,b}^s (\tau, \lambda, \mu) \). Then a function \( f \in \Sigma \) is said to be in the class \( \Sigma H_{q,b}^s (\lambda) \) if the following subordinations hold for all \( z, w \in \mathbb{D} \):

\[ (1 - \lambda) \frac{J_{q,b}^s f (z)}{z} + \lambda \partial_q \left( J_{q,b}^s f (z) \right) < \varphi (z) \]

and

\[ (1 - \lambda) \frac{J_{q,b}^s g (w)}{w} + \lambda \partial_q \left( J_{q,b}^s g (w) \right) < \varphi (w) \]

5) Let \( \mu = 1, \tau = 1 \) and \( \lambda = 0 \) in \( \Sigma H_{q,b}^s (\tau, \lambda, \mu) \). Then a function \( f \in \Sigma \) is said to be in the class \( \Sigma H_{q,b}^s \) if the following subordinations hold for all \( z, w \in \mathbb{D} \):

\[ \frac{J_{q,b}^s f (z)}{z} < \varphi (z) \]

and

\[ \frac{J_{q,b}^s g (w)}{w} < \varphi (w) \]

6) Let \( s = 0 \) in \( \Sigma H_{q,b}^s (\tau, \lambda, \mu) \). Then a function \( f \in \Sigma \) is said to be in the class \( \Sigma H (\tau, \lambda, \mu) \) if the following subordinations hold for all \( z, w \in \mathbb{D} \):

\[ 1 + \frac{1}{\tau} \left[ (1 - \lambda) \left( f (z) \right)^\mu + \lambda \partial_q \left( f (z) \right) \left( f (z) \right)^{\mu - 1} - 1 \right] < \varphi (z) \]

and

\[ 1 + \frac{1}{\tau} \left[ (1 - \lambda) \left( g (w) \right)^\mu + \lambda \partial_q \left( g (w) \right) \left( g (w) \right)^{\mu - 1} - 1 \right] < \varphi (w) \]
Theorem 2.1. Let \( f \) given by (1.1) be in the class \( \Sigma H_{q,b}^s (\tau, \lambda, \mu) \). Then

\[
|a_2| \leq \left| \left[ \frac{2 + b}{1 + b} \right] \right|^s \min \left\{ \frac{\tau}{\mu + \lambda q}, \sqrt{\frac{\tau}{\mu + \frac{2\tau}{\mu(1 + \mu) + 2\lambda q(\mu + q)}}} \right\} \tag{2.4}
\]

and

\[
|a_3| \leq \left| \left[ \frac{3 + b}{1 + b} \right] \right|^s \frac{\tau}{\mu + \lambda q(1 + q)} \min \left\{ 1, \frac{\tau(\mu + \lambda q(1 + q))}{(\mu + \lambda q)^2} \right\}. \tag{2.5}
\]

Proof. Let \( f \in \Sigma H_{q,b}^s (\tau, \lambda, \mu) \) and \( g = f^{-1} \). Then, there are analytic functions \( u, v \in \Omega \) satisfying

\[
1 + \frac{1}{\tau} \left[ 1 - \lambda \right] \left[ \frac{J_{q,b}^s f(z)}{z} \right]^\mu + \lambda \partial_\mu \left( J_{q,b}^s f(z) \right) \left( \frac{J_{q,b}^s f(z)}{z} \right)^{\mu-1} = \varphi(u(z)) \tag{2.6}
\]

and

\[
1 + \frac{1}{\tau} \left[ 1 - \lambda \right] \left[ \frac{J_{q,b}^s g(w)}{w} \right]^\mu + \lambda \partial_\mu \left( J_{q,b}^s g(w) \right) \left( \frac{J_{q,b}^s g(w)}{w} \right)^{\mu-1} = \varphi(v(z)). \tag{2.7}
\]

In other words, by using 2.1 in 2.6 and 2.7 we write

\[
1 + \frac{1}{\tau} \left[ 1 - \lambda \right] \left[ \frac{J_{q,b}^s f(z)}{z} \right]^\mu + \lambda \partial_\mu \left( J_{q,b}^s f(z) \right) \left( \frac{J_{q,b}^s f(z)}{z} \right)^{\mu-1} = \varphi \left( \frac{p(z) - 1}{p(z) + 1} \right) = e^{\frac{p(z)-1}{p(z)+1}} \tag{2.8}
\]

and

\[
1 + \frac{1}{\tau} \left[ 1 - \lambda \right] \left[ \frac{J_{q,b}^s g(w)}{w} \right]^\mu + \lambda \partial_\mu \left( J_{q,b}^s g(w) \right) \left( \frac{J_{q,b}^s g(w)}{w} \right)^{\mu-1} = \varphi \left( \frac{s(z) - 1}{s(z) + 1} \right) = e^{\frac{s(z)-1}{s(z)+1}}. \tag{2.9}
\]

From 2.8 and 2.9, we have

\[
1 + \frac{(\mu + \lambda q)}{\tau} \left( \frac{1 + b}{2 + b} \right)^s a_2 \frac{z}{1 + b} + \frac{1}{\tau} \left( \frac{(\mu - 1) (\mu + 2 \lambda q)}{2} \right) \left( \frac{1 + b}{2 + b} \right)^{2s} a_2^2 + \left( \mu + \lambda q (1 + q) \right) \left( \frac{1 + b}{3 + b} \right)^s a_3 \frac{z^2}{2} + \cdots
\]

\[
= 1 + \frac{p_1}{2} + \frac{p_2}{2} \frac{z^2}{2} + \cdots
\]

and

\[
1 - \frac{(\mu + \lambda q)}{\tau} \left( \frac{1 + b}{2 + b} \right)^s a_2 w + \frac{1}{\tau} \left( \lambda q (2q + \mu + 1) + \frac{\mu (\mu + 3)}{2} \right) \left( \frac{1 + b}{2 + b} \right)^{2s} a_2^2 + \left( -\mu - \lambda q (1 + q) \right) \left( \frac{1 + b}{3 + b} \right)^s a_3 \frac{w^2}{2} + \cdots
\]
Comparing the coefficients on the both sides of above last equalities, we have the relations

\[ \frac{1}{r} (\mu + \lambda q) a_2 \left( \frac{[1 + b]_q}{[2 + b]_q} \right)^s = \frac{p_1}{2}, \]  
\[ (2.10) \]

\[ \frac{1}{r} \left( \frac{(\mu - 1)(\mu + 2\lambda q)}{2} \right) \left( \frac{[1 + b]_q}{[2 + b]_q} \right)^{2s} + (\mu + \lambda q) (1 + q) \left( \frac{[1 + b]_q}{[3 + b]_q} \right)^s a_3 = \frac{p_2}{2}, \]  
\[ (2.11) \]

\[ - \frac{1}{r} (\mu + \lambda q) a_2 \left( \frac{[1 + b]_q}{[2 + b]_q} \right)^s = \frac{s_1}{2} \]  
\[ (2.12) \]

and

\[ \frac{1}{r} \left( \frac{\lambda q(2q + \mu + 1) + \mu(\mu + 3)}{2} \right) \left( \frac{[1 + b]_q}{[2 + b]_q} \right)^{2s} a_2^2 - (\mu + \lambda q) (1 + q) \left( \frac{[1 + b]_q}{[3 + b]_q} \right)^s a_3 = \frac{s_2}{2}. \]  
\[ (2.13) \]

Therefore, from the Eqs 2.10 and 2.12, we find that

\[ p_1 = -s_1 \]  
\[ (2.14) \]

and

\[ \left[ \frac{1}{r} (\mu + \lambda q) \left( \frac{[1 + b]_q}{[2 + b]_q} \right)^s \right]^2 a_2^2 = \frac{1}{8} \left( p_1^2 + s_1^2 \right), \]  
\[ (2.15) \]

which upon applying Lemma 1.1, yields

\[ |a_2| \leq \left| \left( \frac{[2 + b]_q}{[1 + b]_q} \right)^s \right| \frac{|r|}{\mu + \lambda q}. \]

On the other hand, by using 2.11 and 2.13, we obtain

\[ \frac{1}{r} \left( \mu^2 + \mu + 2\lambda q\mu + 2\lambda q^2 \right) \left( \frac{[1 + b]_q}{[2 + b]_q} \right)^{2s} a_2^2 = \frac{p_2 + s_2}{2}, \]  
\[ (2.16) \]

which yields

\[ |a_2| \leq \left| \left( \frac{[2 + b]_q}{[1 + b]_q} \right)^s \right| \sqrt{\frac{2 |r|}{\mu^2 + \mu + 2\lambda q\mu + 2\lambda q^2}}. \]

We now, investigate the upper bound of \(|a_3|\). For this, by using 2.11 and 2.13, we have

\[ \frac{2}{r} (\mu + \lambda q(1 + q)) \left( \frac{[1 + b]_q}{[2 + b]_q} \right)^{2s} a_2^2 - \left( \frac{[1 + b]_q}{[3 + b]_q} \right)^s a_3 = \frac{s_2 - p_2}{2}. \]  
\[ (2.17) \]

Therefore for substituting 2.15 in 2.17, we have

\[ \left( \frac{[1 + b]_q}{[3 + b]_q} \right)^s a_3 = \frac{r^2 \left( p_1^2 + s_1^2 \right)}{8 (\mu + \lambda q)^2} + \frac{\tau (p_2 - s_2)}{4 (\mu + \lambda q(1 + q))} \]  
\[ (2.18) \]
or
\[ a_3 = \left( \frac{[3 + b]_q}{[1 + b]_q} \right)^s \frac{\tau}{4(\mu + \lambda q(1 + q))} \left( p_2 - s_2 \right) + \frac{\tau (\mu + \lambda q(1 + q))}{(\mu + \lambda q)^2} p_1^2. \]  

(2.19)

On the other hand, according to the Lemma 1.2 and 2.14, we write
\[ 2p_2 = p_1^2 + x \left( 4 - p_1^2 \right) \]  
\[ 2s_2 = s_1^2 + y \left( 4 - s_1^2 \right) \]  
\[ \implies p_2 - s_2 = \frac{4 - p_1^2}{2} (x - y) \]  

(2.20)

and so, from 2.19 and 2.20, we have
\[ a_3 = \left( \frac{[3 + b]_q}{[1 + b]_q} \right)^s \frac{\tau}{4(\mu + \lambda q(1 + q))} \left[ \frac{4 - p_1^2}{2} (x - y) + \frac{\tau (\mu + \lambda q(1 + q))}{(\mu + \lambda q)^2} p_1^2 \right]. \]  

(2.21)

If we apply triangle inequality to equation 2.21, we obtain
\[ |a_3| \leq \left| \left( \frac{[3 + b]_q}{[1 + b]_q} \right)^s \frac{\tau}{4(\mu + \lambda q(1 + q))} \left[ \frac{4 - p_1^2}{2} (|x| + |y|) + \frac{\tau (\mu + \lambda q(1 + q))}{(\mu + \lambda q)^2} p_1^2 \right] \right|. \]

Since the function \( p(e^{\theta z}) \) (\( \theta \in \mathbb{R} \)) is in the class \( \mathcal{P} \) for any \( p \in \mathcal{P} \), there is no loss of generality in assuming \( p_1 > 0 \). Write \( p_1 = p, p \in [0, 2] \). Thus, for \( |x| \leq 1 \) and \( |y| \leq 1 \) we obtain
\[ |a_3| \leq \left| \left( \frac{[3 + b]_q}{[1 + b]_q} \right)^s \frac{\tau}{4(\mu + \lambda q(1 + q))} \left[ 4 + \frac{\tau (\mu + \lambda q(1 + q))}{(\mu + \lambda q)^2} - 1 \right] p_1^2 \right|, \]

which upon applying Lemma 1.1, yields upper bound of \( |a_3| \). \( \square \)

**Theorem 2.2.** If \( f(z) \) given by (1.1) be in the class \( \Sigma H^{2}_{q,b} (\tau, \lambda, \mu) \) and \( \delta \in \mathbb{C} \), then
\[ |a_3 - \delta a_2^2| \leq |\tau| (|K + L| + |K - L|) \]

where
\[ K = \left( \frac{[3 + b]_q}{[1 + b]_q} \right)^s - \frac{\tau}{(\mu^2 + \mu + 2\lambda q \mu + 2\lambda q^2),} \]  

(2.22)

\[ L = \left( \frac{[3 + b]_q}{[1 + b]_q} \right)^s - \frac{1}{2(\mu + \lambda q + \lambda q^2).} \]

**Proof.** From the Eqs 2.16 and 2.18 we obtain
\[ a_2^2 = \frac{\left( [2 + b]_q \right)^{2s}}{([1 + b]_q)^2} \frac{\tau (p_2 + s_2)}{2(\mu^2 + \mu + 2\lambda q \mu + 2\lambda q^2)} \]  

(2.23)

and
\[ a_3 = \frac{\tau}{2} \left( \frac{[3 + b]_q}{[1 + b]_q} \right)^s \left( \frac{p_2 + s_2}{(\mu^2 + \mu + 2\lambda q \mu + 2\lambda q^2)} - \frac{s_2 - p_2}{2(\mu + \lambda q + \lambda q^2)} \right). \]  

(2.24)

Therefore, by using the equalities 2.23 and 2.24 for \( \delta \in \mathbb{C} \), we have
\[ a_3 - \delta a_2^2 = \frac{\tau}{2} \left( \frac{[3 + b]_q}{[1 + b]_q} \right)^s \left( \frac{p_2 + s_2}{(\mu^2 + \mu + 2\lambda q \mu + 2\lambda q^2)} - \frac{s_2 - p_2}{2(\mu + \lambda q + \lambda q^2)} \right). \]
By substituting 2.16 in 2.26, we have

\[
-\delta \left( \frac{[1 + b]_q}{[3 + b]_q} \right)^{2s} \frac{\tau (p_2 - s_2)}{2 (\mu^2 + \mu + 2\lambda q \mu + 2\lambda q^2)}.
\]

After the necessary arrangements, we rewrite the above last equality as

\[
a_3 - \delta a_2^2 = \frac{\tau}{2} ((K + L) p_2 + (K - L) s_2)
\]

(2.25)

where \( K \) and \( L \) are given by 2.22. Taking the absolute value of 2.25, from Lemma 1.1 we obtain the desired inequality.

\[\Box\]

**Theorem 2.3.** If \( f(z) \) given by (1.1) be in the class \( \Sigma H_{q,b}^{s}(\tau, \lambda, \mu) \) and \( \delta \in \mathbb{C} \), then

\[
\left| \left( \frac{[1 + b]_q}{[3 + b]_q} \right)^s a_3 - \delta \left( \frac{[1 + b]_q}{[2 + b]_q} \right)^{2s} a_2^2 \right| \leq 2 |\tau| \left\{ \frac{1}{2 (\mu + \lambda q + \lambda q^2)} 0 \leq |\Psi(\delta)| \leq \frac{1}{2 (\mu + \lambda q + \lambda q^2)} \right\}
\]

where

\[
\Psi(\delta) = \frac{1 - \delta}{\mu^2 + \mu + 2\lambda q \mu + 2\lambda q^2}.
\]

**Proof.** From Eq 2.17, we write

\[
\left( \frac{[1 + b]_q}{[3 + b]_q} \right)^s a_3 - \delta \left( \frac{[1 + b]_q}{[2 + b]_q} \right)^{2s} a_2^2 = \frac{\tau (p_2 - s_2)}{4 (\mu + \lambda q + \lambda q^2)} + (1 - \delta) \left( \frac{[1 + b]_q}{[2 + b]_q} \right)^{2s} a_2^2.
\]

(2.26)

By substituting 2.16 in 2.26, we have

\[
\left( \frac{[1 + b]_q}{[3 + b]_q} \right)^s a_3 - \delta \left( \frac{[1 + b]_q}{[2 + b]_q} \right)^{2s} a_2^2 = \frac{\tau (p_2 - s_2)}{4 (\mu + \lambda q + \lambda q^2)} + (1 - \delta) \left( \frac{[1 + b]_q}{[2 + b]_q} \right)^{2s} a_2^2.
\]

\[
= \frac{\tau (p_2 - s_2)}{4 (\mu + \lambda q + \lambda q^2)} + (1 - \delta) \frac{\tau (s_2 + p_2)}{2 (\mu + \lambda q + \lambda q^2)}
\]

\[
= \frac{\tau}{2} \left( \Psi(\delta) + \frac{1}{2 (\mu + \lambda q + \lambda q^2)} \right) p_2 + \left( \Psi(\delta) - \frac{1}{2 (\mu + \lambda q + \lambda q^2)} \right) s_2
\]

where

\[
\Psi(\delta) = \frac{1 - \delta}{\mu^2 + \mu + 2\lambda q \mu + 2\lambda q^2}.
\]

Therefore, we conclude that

\[
\left| \left( \frac{[1 + b]_q}{[3 + b]_q} \right)^s a_3 - \delta \left( \frac{[1 + b]_q}{[2 + b]_q} \right)^{2s} a_2^2 \right| \leq 2 |\tau| \left\{ \frac{1}{2 (\mu + \lambda q + \lambda q^2)} 0 \leq |\Psi(\delta)| \leq \frac{1}{2 (\mu + \lambda q + \lambda q^2)} \right\},
\]

which evidently complete the proof of the theorem.

\[\Box\]
Corollary 2.1. Let $f$ given by (1.1) be in the class $\Sigma H^s_{q,b} (\tau, \mu)$. Then

$$|a_2| \leq \left| \frac{[2 + b]_q}{[1 + b]_q} \right|^s \min \left\{ \frac{|\tau|}{\mu + q}, \frac{2|\tau|}{\mu(1 + \mu) + 2q(\mu + q)} \right\},$$

$$|a_3| \leq \left| \frac{[3 + b]_q}{[1 + b]_q} \right|^s \frac{|\tau|}{\mu + q(1 + q)} \min \left\{ 1, \frac{|\tau|(\mu + q(1 + q))}{(\mu + q)^2} \right\},$$

$$|a_3 - \delta a_2^2| \leq |\tau|(|K_1 + L_1| + |K_1 - L_1|)$$

and

$$\left| \frac{[1 + b]_q}{[3 + b]_q} \right|^s a_3 - \delta \left| \frac{[1 + b]_q}{[2 + b]_q} \right| a_2^2 \leq 2|\tau| \left\{ \frac{1}{2(\mu + q + q^2)}, \quad 0 \leq |\Psi_1(\delta)| \leq \frac{1}{2(\mu + q + q^2)} \right\},$$

where

$$K_1 = \left( \frac{[3 + b]_q}{[1 + b]_q} \right)^s - \delta \left( \frac{[2 + b]_q}{[1 + b]_q} \right)^{2s} \frac{1}{\mu^2 + \mu + 2q(\mu + q)},$$

$$L_1 = \left( \frac{[3 + b]_q}{[1 + b]_q} \right)^s \frac{1}{2(\mu + q + q^2)},$$

$$\Psi_1(\delta) = \frac{1 - \delta}{\mu^2 + \mu + 2q(\mu + q)}. $$

Corollary 2.2. Let $f$ given by (1.1) be in the class $\Sigma H^s_{q,b} (\tau, \lambda)$. Then

$$|a_2| \leq \left| \frac{[2 + b]_q}{[1 + b]_q} \right|^s \min \left\{ \frac{|\tau|}{1 + \lambda q}, \frac{|\tau|}{1 + \lambda q(1 + q)} \right\},$$

$$|a_3| \leq \left| \frac{[3 + b]_q}{[1 + b]_q} \right|^s \frac{|\tau|}{1 + \lambda q(1 + q)} \min \left\{ 1, \frac{|\tau|(1 + \lambda q(1 + q))}{(1 + \lambda q)^2} \right\},$$

$$|a_3 - \delta a_2^2| \leq |\tau|(|K_2 + L_2| + |K_2 - L_2|)$$

and

$$\left| \frac{[1 + b]_q}{[3 + b]_q} \right|^s a_3 - \delta \left| \frac{[1 + b]_q}{[2 + b]_q} \right| a_2^2 \leq 2|\tau| \left\{ \frac{1}{2(1 + \lambda q + \lambda q^2)}, \quad 0 \leq |\Psi_2(\delta)| \leq \frac{1}{2(1 + \lambda q + \lambda q^2)} \right\},$$

where

$$K_2 = \left( \frac{[3 + b]_q}{[1 + b]_q} \right)^s - \delta \left( \frac{[2 + b]_q}{[1 + b]_q} \right)^{2s} \frac{1}{2(1 + \lambda q + 2\lambda q^2)},$$

$$L_2 = \left( \frac{[3 + b]_q}{[1 + b]_q} \right)^s \frac{1}{2(1 + \lambda q + \lambda q^2)},$$

$$\Psi_2(\delta) = \frac{1 - \delta}{2(1 + \lambda q + \lambda q^2)}.$$
Corollary 2.3. Let \( f \) given by (1.1) be in the class \( \Sigma H(\tau, \lambda, \mu) \). Then

\[
|a_2| \leq \min \left\{ \frac{|\tau|}{\mu + \lambda q}, \sqrt[2]{\frac{2|\tau|}{\mu(1+\mu) + 2\lambda q(\mu + q)}} \right\},
\]

\[
|a_3| \leq \frac{|\tau|}{\mu + \lambda q(1 + q)} \min \left\{ 1, \frac{|\tau| (\mu + \lambda q(1 + q))}{(\mu + \lambda q)^2} \right\}
\]

and

\[
|a_3 - \delta a_2^2| \leq 2|\tau| \left\{ \frac{1}{2(1+\lambda q+\lambda q^2)}, \quad 0 \leq |\Psi_3(\delta)| \leq \frac{1}{2(1+\lambda q+\lambda q^2)} \right\},
\]

where

\[
\Psi_3(\delta) = \frac{1 - \delta}{2(1 + \lambda q + \lambda q^2)}.
\]

3. Conclusions

In this paper, we defined a general subclass of bi-univalent functions related with \( q \)-Srivastava Attiya operator by using the Bell numbers and subordination. For the functions belonging to this class, we obtained non-sharp bounds for the initial coefficients and the Fekete-Szegö functional. Some interesting corollaries and applications of the results are also discussed.

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