THE NONDEGENERATE GENERALIZED KÄHLER CALABI-YAU PROBLEM

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Abstract. We formulate a Calabi-Yau type conjecture in generalized Kähler geometry, focusing on the case of nondegenerate Poisson structure. After defining natural Hamiltonian deformation spaces for generalized Kähler structures generalizing the notion of Kähler class, we conjecture unique solvability of Gualtieri’s Calabi-Yau equation within this class. We establish the uniqueness, and moreover show that all such solutions are actually hyper-Kähler metrics. We furthermore establish a GIT framework for this problem, interpreting solutions of this equation as zeros of a moment map associated to a Hamiltonian action and finding a Kempf-Ness functional. Lastly we indicate the naturality of generalized Kähler-Ricci flow in this setting, showing that it evolves within the given Hamiltonian deformation class, and that the Kempf-Ness functional is monotone, so that the only possible fixed points for the flow are hyper-Kähler metrics. On a hyper-Kähler background, we establish global existence and weak convergence of the flow.

1. Introduction

Let $(M^{2m}, g, J)$ be a compact Kähler manifold, with $\Theta \in \wedge^m, \partial(M, J)$ a holomorphic volume form. Yau’s theorem (\cite{53}) asserts that in any Kähler class there exists a unique Calabi-Yau (Ricci-flat) metric. This result provides a wide class of examples of Ricci flat metrics, which play a central role in geometry and mathematical physics. Since Yau’s original proof a number of new analytic techniques have been brought to bear on the problem. For instance the Aubin-Yau $J$-functional (see e.g. \cite{4}) yields a variational characterization of Calabi-Yau metrics, which can be used to yield a variational proof of the Calabi-Yau theorem \cite{8}. Also, as shown by Cao \cite{14}, in this setting the Kähler-Ricci flow with arbitrary initial data exists for all time and converges to a Calabi-Yau metric.

In this paper we generalize aspects of this picture to generalized Kähler geometry.

Generalized Kähler structures first appeared through investigations into supersymmetric sigma models \cite{21}, and were rediscovered in a purely mathematical context in the work of Gualtieri \cite{28} and Hitchin \cite{34}, and have recently attracted interest in both the physics and mathematical communities as natural generalizations of Kähler structures. We will focus here entirely on the biHermitian description of generalized Kähler geometry (cf. \cite{3, 21}). Thus, a generalized Kähler manifold is a smooth manifold $M$ with a triple $(g, I, J)$ consisting of two integrable almost-complex structures, $I$ and $J$, together with a Riemannian metric $g$ which is Hermitian with respect to both, such that the Kähler forms $\omega_I$ and $\omega_J$ satisfy

$$d_I^* \omega_I = H = -d_J^* \omega_J, \quad dH = 0,$$

where the first equation defines $H$, and $d_I^* = \sqrt{-1} (\bar{\partial}_I - \partial_I)$.

A key feature of generalized Kähler geometry, observed by Hitchin in \cite{35} (cf. \cite{3, 43} for the 4-dimensional case) is that there are naturally associated Poisson structures. In particular, the tensor

$$\sigma = [I, J]g^{-1}$$

is a real Poisson structure, which is the real part of a holomorphic Poisson structure with respect to both complex structures $I$ and $J$. In this paper, we focus entirely on the case when this holomorphic Poisson structure is nondegenerate, in which case we will refer to the generalized Kähler structure.
itself as “nondegenerate.” These structures are also referred to as “type (0,0)” in the language of
generalized complex structures. In this case we define the corresponding symplectic form
\[ \Omega = \sigma^{-1}, \]
which is the common real part of holomorphic symplectic forms with respect to \( I \) and \( J \).

The simplest example of a nondegenerate generalized Kähler structure comes from hyper-Kähler
gometry. In particular, if \((M^{4n}, g, I, J, K)\) is hyper-Kähler then \((M^{4n}, g, I, J)\) is a nondegenerate
generalized Kähler structure with \( \Omega = -\frac{1}{2} \omega_K \). Later, Joyce ([3, 30, 35]) showed that one
then nondegenerate generalized Kähler structures. We rederive this construction in the purely biHermi-
tian context in \( \S 2.4 \) and moreover show that the proof adapts to show a more general statement,
namely that for an arbitrary nondegenerate generalized Kähler structure, \( \Omega \)-Hamiltonian isotopies
act locally to produce new nondegenerate generalized Kähler structures (Proposition 2.16).

Given this variational space for nondegenerate generalized Kähler structures, it is natural to
seek canonical representatives of this class. A Calabi-Yau equation in this setting was defined by
Gualtieri ([28] Definition 6.40). We define this equation only referencing biHermitian geometry in
\( \S 3.2 \) noting here that, in analogy with the classical Calabi-Yau equation, it asks for constancy of
a certain “Ricci potential,” which is defined as the ratio of the top exterior powers of the closed
spinors defining the two relevant generalized complex structures. Our first main result is that,
within our given deformation class, solutions to the equation are unique, and moreover more rigid
than expected: they are hyper-Kähler.

**Theorem 1.1.** (cf. Theorem 3.13) Let \((M^{4n}, g, I, J)\) be a nondegenerate generalized Kähler manifold. Any two solutions \((g_i, I, J_i)\), \(i = 1, 2\) of the generalized Kähler Calabi-Yau equation in the
\( \Omega \)-Hamiltonian deformation class agree, and moreover define a hyper-Kähler structure.

Taking inspiration from the Calabi-Yau theorem, this rigidity, as well as the further geometric
and analytic results described below, we conjecture unique solvability of the generalized Kähler
Calabi-Yau equation in the given \( \Omega \)-Hamiltonian deformation class.

**Conjecture 1.2.** Let \((M^{4n}, g, I, J)\) be a nondegenerate generalized Kähler manifold. There exists a
nondegenerate generalized Kähler structure \((g', I', J')\) in the \( \Omega \)-Hamiltonian deformation class solving
the generalized Kähler Calabi-Yau equation. Moreover, this resulting generalized Kähler structure is
unique, and hyper-Kähler.

Of course the uniqueness and rigidity statements have been established in Theorem 1.1. Observe
that this conjecture has the consequence that every nondegenerate generalized Kähler structure is
an \( \Omega \)-Hamiltonian deformation of a hyper-Kähler structure, i.e. given by the Joyce construction.

To give this conjecture more context, we next show that it fits into a formal GIT picture. Building
on the observation that there is a natural action of \( \Omega \)-Hamiltonian diffeomorphisms on generalized
Kähler structures, and taking inspiration from prior constructions in Kähler geometry, we define
a closed 1-form on the \( \Omega \)-Hamiltonian deformation class (seen as a Frechét space) which vanishes
if and only if the underlying GK structure is Calabi-Yau. After taking topological considerations
into account, one can construct a primitive \( F \) for this 1-form, which is a natural analogue of the
Aubin-Yau \( J \)-functional [4] in Kähler geometry. We go on to define a formal symplectic structure
and almost complex structure on each \( \Omega \)-Hamiltonian deformation class. We show (cf. Proposition
4.11) that the natural action of \( \Omega \)-Hamiltonians on the space of generalized Kähler structures is
itself Hamiltonian with respect to the symplectic structure we define, and compute the moment
map. Moreover, we show that the functional \( F \) serves as a Kempf-Ness functional, meaning that
the critical points of \( F \) are the zeroes of the momentum map. We note here that a different symplectic
action and moment map in the context of generalized Kähler geometry was recently discovered by
Boulanger [11] and Goto [27].
Next we provide a concrete analytic approach to Conjecture 1.2 through the use of the generalized Kähler-Ricci flow (GKRF), a natural notion of Ricci flow adapted to the context of generalized Kähler geometry introduced by the second author and Tian [51]. First we show that, starting with nondegenerate initial data, solutions to the flow preserve the $\Omega$-Hamiltonian deformation class. We also show convexity of the $F$-functional arising from the formal GIT picture (cf. Proposition 5.9 for a precise statement of part (3)).

**Theorem 1.3.** Let $(M^{4n}, g, I, J)$ be a nondegenerate generalized Kähler manifold, and let $(g_t, J_t)$ denote the solution to GKRF with this initial condition in the $I$-fixed gauge. Then

1. The structure $J_t$ evolves by the one-parameter family of $\Omega$-Hamiltonian isotopies determined by the time-dependent Ricci potential.
2. The only fixed points of the flow are solutions to the generalized Kähler Calabi-Yau equation. In particular they are hyper-Kähler.
3. The functional $F$ is convex along the flow.

Building on these natural formal properties, in line with Conjecture 1.2 it is natural to expect that in this setting the solution to GKRF exists for all time and converges to a hyper-Kähler metric (cf. Conjecture 5.11). With this in mind we establish some a priori estimates which are relevant to establishing the long time existence, and moreover lead to a definitive convergence statement assuming certain further a priori estimates. In particular, we begin by establishing the key fact that the Ricci potential evolves by the pure heat equation, in line with the corresponding behavior of Kähler-Ricci flow. Based on this we are able to obtain strong a priori estimates on the gradient of the Ricci potential. Combining this with previous regularity results for GKRF, we can establish a conditional resolution of Conjecture 1.2.

**Theorem 1.4.** (cf. Theorem 5.11) Let $(M^{4n}, g, I, J)$ be a nondegenerate generalized Kähler manifold. Let $(g_t, I, J_t)$ denote the solution to GKRF with this initial condition in the $I$-fixed gauge. Suppose there exists a constant $\Lambda > 0$ such that for all times $t$ in the maximal interval of existence, the solution satisfies

$$\Lambda^{-1} g_0 \leq g_t \leq \Lambda g_0, \quad |\partial \alpha|^2 \leq \Lambda,$$

where $\partial \alpha$ is an associated torsion potential. Then the solution exists for all time and converges to a hyper-Kähler metric. In particular, Conjecture 1.2 holds true.

In prior work of the second author [45] the global existence and weak convergence of the flow when $n = 1$ was established. The proof exploits the classification of complex surfaces, in particular using the existence of a background Kähler metric to obtain some necessary a priori estimates. Our final result extends this to arbitrary $n \geq 1$. In particular, we establish the global existence and weak convergence of GKRF under the assumption that a hyper-Kähler metric exists. Observe that even with the assumption that a hyper-Kähler metric exists on our given complex manifold, Conjecture 1.2 does not immediately follow, as it asks for deformability of an arbitrary GK structure to a hyper-Kähler one, and it is not a priori known if this space is connected. Thus, up to the weakness of the convergence, the flow verifies Conjecture 1.2 for hyper-Kähler backgrounds, in particular yielding the connectivity of the nonlinear space of generalized Kähler structures on these manifolds, a nonobvious fact.

**Theorem 1.5.** Let $(M^{4n}, I)$ be a compact hyper-Kähler manifold. Suppose $(g, I, J)$ is a nondegenerate generalized Kähler structure on $M$. The solution to generalized Kähler-Ricci flow in the $I$-fixed gauge with initial condition $(g, I, J)$ exists on $[0, \infty)$, and satisfies

$$||\Phi - \lambda||_{H^2}^2 \leq Ct^{-1}.$$

Moreover, there exists a sequence of times $\{t_j\} \to \infty$ such that $(\omega_t)_{t_j}$ converges in the sense of currents to a closed positive $(1, 1)$ current.
Here is an outline of the rest of this paper. In §2 we provide relevant background on nondegenerate generalized Kähler structures, precisely describing the deformations under consideration. In §3 we review the relevant Calabi-Yau type equation, give a precise setup for Conjecture 1.2 and prove Theorem 1.1. Next in §4 we establish a GIT framework for Conjecture 1.2. We begin our analysis of GKRF in §5 recalling background and proving Theorems 1.3 and 1.4. Finally, in §6 we develop further a priori estimates for GKRF in this setting and prove Theorem 1.5.

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2. Nondegenerate Generalized Kähler structures

In this section we establish some fundamental properties of nondegenerate generalized Kähler structures which we will use throughout the paper. In §2.1 we recall the basic definitions, and show that they lead to the existence of a holomorphic symplectic structure on the underlying complex manifolds. Next in §2.2 we recall some fundamental aspects of holomorphic symplectic manifolds. We also observe that the pluriclosed metrics associated to a nondegenerate generalized Kähler structure are symplectically tamed, and in §2.3 we establish some technical identities associated to such structures central to the results to follow. In §2.4 we show that Hamiltonian diffeomorphisms associated to the real part of the holomorphic symplectic structures can be used to produce nontrivial deformations of these generalized Kähler structures.

2.1. Background. In this subsection we recall the biHermitian formulation of generalized Kähler geometry, and the basic properties of the associated Poisson structures. We also here record definitions of connections and their curvature relevant to the rest of the paper, but not §2.1 directly.

Definition 2.1. Given a smooth manifold \( M \), we say that \((g, I, J)\) is a generalized Kähler structure (GK structure) if \( I \) and \( J \) are integrable complex structures, \( g \) is compatible with both \( I \) and \( J \), and furthermore

\[
d^c I \omega_I = H = -d^c J \omega_J, \quad dH = 0.
\]

Associated to this structure we define

\[
\sigma = g^{-1}[I, J] \in \wedge^2(TM).
\]

We will call a given GK structure nondegenerate if \( \sigma \) defines a nondegenerate pairing on each fibre of \( T^*M \), and our focus will be on nondegenerate structures throughout this paper.

In general, the tensor \( \sigma \) is the real part of a holomorphic \((2,0)\)-bivector with respect to both complex structures \( I \) and \( J \) [35] (cf. [3], [43] for the 4-dimensional case). Observe that when \( \sigma \) is nondegenerate, we can define

\[
\Omega = \sigma^{-1} = [I, J]^{-1} g.
\]

From the properties of \( \sigma \) it follows easily that \( \Omega \) is the real part of a holomorphic symplectic \((2,0)\)-form with respect to either complex structure \( I \) and \( J \), and moreover the real parts agree. We recall some fundamental aspects of complex manifolds admitting holomorphic symplectic \((2,0)\)-forms in §2.2. As we explain in Lemma 2.14 below, the data of distinct holomorphic symplectic structures with matching real parts determines a unique nondegenerate generalized Kähler structure.

Associated to generalized Kähler structures, and more generally Hermitian structures, are several relevant connections. We record these definitions here for convenience.
Definition 2.2. Let \((M^{2n}, g, I)\) be a complex manifold with a Hermitian metric \(g\). The Bismut connection is defined by
\[
\langle \nabla^B_X Y, Z \rangle = \langle \nabla_X Y, Z \rangle - \frac{i}{2} d^c \omega(X, Y, Z),
\]
where \(\nabla\) stands for the Riemannian connection of \(g\). Observe that in the context of generalized Kähler geometry we have two Bismut connections defined via
\[
\langle \nabla^B_X Y, Z \rangle = \langle \nabla_X Y, Z \rangle - \frac{i}{2} d^c \omega(X, Y, Z) = \langle \nabla_X Y, Z \rangle - \frac{1}{4} H(X, Y, Z),
\]
\[
\langle \nabla^B_J Y, Z \rangle = \langle \nabla_X Y, Z \rangle - \frac{i}{2} d^c \omega_J(X, Y, Z) = \langle \nabla_X Y, Z \rangle + \frac{1}{2} H(X, Y, Z).
\]

Definition 2.3. Let \((M^{2n}, g, I)\) be a compact complex manifold with a Hermitian metric \(g\). The Chern connection is defined by
\[
\langle \nabla^C_X Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \frac{i}{2} d^c \omega(X, IY, IZ).
\]

These are both Hermitian connections, which necessarily have torsion in the non-Kähler case. The curvatures of these connections arise in our analysis, and most important are the associated representatives of the first Chern class.

Definition 2.4. Let \((M^{2n}, g, I)\) be a compact complex manifold with pluriclosed metric \(g\). The Bismut Ricci curvature and Chern-Ricci curvature are defined by
\[
\rho_B(X, Y, e_i, Ie_i) = \frac{1}{2} R^B(X, Y, e_i, Ie_i),
\]
\[
\rho_C(X, Y, e_i, Ie_i) = \frac{1}{2} R^C(X, Y, e_i, Ie_i),
\]
where \(R^B\) and \(R^C\) are the curvature tensors associated to the Bismut and Chern connections respectively, and \(\{e_i\}\) is an orthonormal basis for \((TM, g)\).

2.2. Holomorphic symplectic manifolds. Recall that a holomorphic symplectic structure on a complex manifold \((M, I)\) is defined by a holomorphic \((2, 0)\)-form \(\Omega\) which is non-degenerate in the sense that at each point the complex linear map \(\Omega : T^{1,0}M \to \Lambda^{1,0}(T^*M)\) is non-degenerate. Elementary linear algebra shows that the non-degeneracy condition is equivalent to \(\text{Re}(\Omega)\) being non-degenerate (and therefore symplectic) or, equivalently, \(\Omega^n \wedge \overline{\Omega}^n \neq 0\), where \(4n\) is the real dimension of \((M, I)\). In particular, on any holomorphic-symplectic manifold \((M, I, \Omega)\), the form \(\Omega^n\) defines a trivialization of its canonical bundle \(\Lambda^{2n,0}(M, I)\).

The theory of compact, Kähler holomorphic-symplectic manifolds has been extensively developed by A. Beauville and F. Bogomolov, using the Calabi-Yau theorem [52]. We refer to [40] for an overview and recall below the following well-known Beauville-Bogomolov-Yau decomposition theorem (see e.g. [30], Prop. 6.2.2).

Theorem 2.5. [6, 10, 52] Let \((M, I)\) be a compact complex manifold which admits a Kähler metric and a holomorphic-symplectic form \(\Omega\). Then, in any Kähler class of \((M, I)\) there exists a unique Ricci-flat Kähler metric \(g\) with respect to which \(\Omega\) is parallel. Furthermore, up to a finite cover, \((M, I, g)\) is the product of irreducible simply-connected hyper-Kähler manifolds with a flat (hyper-Kähler) even dimensional complex torus.

However, there exist examples of holomorphic-symplectic structures on non-Kähler manifolds, for instance the Kodaira-Thurston surface or the higher dimensional examples of Guan [31, 32, 33]. Thus the complex structures underlying a nondegenerate generalized Kähler structure are not a priori Kähler, even though the only known examples arise by deformation away from hyper-Kähler structures (cf [22]), where the underlying complex structures remain Kähler. What our results suggest, and what would follow from our main conjecture, is that these examples are the only way to construct nondegenerate generalized Kähler structures, in the sense that they are all deformable to hyper-Kähler structures.
2.3. Symplectic type generalized Kähler structures. In this subsection we slightly generalize the discussion of nondegenerate generalized Kähler structures to those of symplectic type, and isolate some necessary properties and identities useful to what follows. To begin we recall the definition of a taming almost complex structure on a symplectic manifold.

**Definition 2.6.** An almost complex structure $I$ on a symplectic manifold $(M,F)$ is *tamed* by the symplectic form $F$ if $F(X,IX) > 0$ for any non-zero tangent vector $X$. It is easily seen that this is equivalent to the statement that

$$(2.4) \quad -IF = g + b,$$

where $g$ is a positive-definite $I$-invariant Riemannian metric and $b$ is a 2-form of type $(2,0)+(0,2)$.

Before specializing to generalized Kähler structures, we record some facts relating taming complex structures and pluriclosed metrics.

**Lemma 2.7.** Let $(M,I)$ be a complex manifold, and suppose $I$ is tamed by a symplectic form $F$. Then the Hermitian structure $(g,I)$ defined by $(2.4)$ is pluriclosed, i.e. satisfies $dd^c_I\omega_I = 0$.

**Proof.** According to $(2.4)$, $F = \omega_I + Ib$ is closed, so that $d\omega_I = -db$, i.e. $d_I^c\omega_I = -Idb = db$ (cf. the end of the proof of Lemma 2.14). \quad \Box

The converse is also true, if we suppose that the complex manifold $(M,I)$ satisfies the $\partial_I\bar{\partial}_I$-Lemma.

**Lemma 2.8.** Suppose $(M,I)$ is a complex manifold on which the $\partial_I\bar{\partial}_I$-Lemma holds at degree $(1,2)$, meaning that the natural map from the Bott-Chern cohomology group $H^{1,2}_{BC}(M,I)$ to the Dolbeault cohomology group $H^1_{\bar{\partial}_I}(M,I)$ is an isomorphism. Then, any pluriclosed Hermitian metric $g$ on $(M,I)$ is obtained from a symplectic form $F$ which tames $I$, via $(2.4)$.

**Proof.** For any pluriclosed Hermitian metric $g$ on $(M,I)$, with Kähler form $\omega_I$, $\partial_I\omega_I$ is a $d$-closed $(1,2)$-form which defines a trivial class in the Dolbeault cohomology $H^1_{\bar{\partial}_I}(M,I)$ and a class in the Bott-Chern cohomology $H^{1,2}_{BC}(M,I)$. The $\partial_I\bar{\partial}_I$-Lemma implies that the class of $\partial_I\omega_I$ in $H^{1,2}_{BC}(M,I)$ must also be trivial, i.e. there exists a $(0,1)$-form $\xi$ such that $\partial_I\omega_I = \partial_I\bar{\partial}_I\xi$. Letting $F := \omega_I - \partial_I\xi - \partial_I\bar{\xi}$, we have $(2.4)$ with $Ib = -2\text{Re}(\partial_I\xi)$. Furthermore,

$$dF = \partial_I F + \bar{\partial}_I F = \partial_I\omega_I - \partial_I\bar{\partial}_I\xi + \partial_I\omega_I - \partial_I\bar{\partial}_I\xi = 0.$$ 

\quad \Box

**Definition 2.9.** A pluriclosed Hermitian structure $(g,I)$ on $M$, associated to a symplectic form taming $I$ via $(2.4)$, will be referred to as a *pluriclosed Hermitian metric of symplectic type*.

We next record an important identity for the Lee form associated to a pluriclosed Hermitian metric of symplectic type which will be central to various calculations to follow.

**Lemma 2.10.** Let $(M,g,I)$ be a pluriclosed Hermitian metric of symplectic type, $F$ a symplectic 2-form taming $I$, and $b$ the real $(2,0)+(0,2)$ form defined by $(2.4)$. Then the Lee form $\theta_I = I\delta\omega_I$ satisfies

$$(2.5) \quad \theta_I(X) = b(\theta_I^\sharp, X) - \langle b, (i_X db) \rangle_g + (\delta b)(X),$$

where $\delta$ is the $L^2$ adjoint of $d$. 
Proof. We shall use the following well-known expression for the covariant derivative of the Kähler form of a Hermitian structure \((g, I)\) (see e.g. [12, Ch. IX, Prop. 4.2] or [24, Prop. 1]):

\[
g(\nabla_X I)(Y, Z) = \frac{1}{2} \left( (d\omega_I)(X, Y, Z) - (d\omega_I)(X, IY, IZ) \right)
\]

(2.6)

\[
= \frac{1}{2} \left( (d\omega_I)(IX, IY, Z) + (d\omega_I)(IX, Y, IZ) \right)
\]

\[
= \frac{1}{2} \left( (d^c_I \omega_I)(X, Y, IZ) + (d^c_I \omega_I)(X, IY, Z) \right)
\]

where \(\nabla\) is the Levi-Civita connection of \(g\), \(\omega_I = gI\) is the fundamental form of \(I\), and we have used that \(I\) is integrable (so that \(d\omega_I\) and \(d^c_I \omega_I\) are of type \((1, 2) + (2, 1)\)) to go from the first line to the second and from the third to the fourth.

We denote by \(\Lambda\) the contraction with \(\omega_I\) acting on a \(p\)-form \(\psi\) by

\[
\Lambda(\psi) := \omega_I \lrcorner \psi = \frac{1}{2} \sum_{i=1}^{2m} \psi(e_i, Ie_i, \cdots, \cdot)
\]

(2.7)

where \(\{e_i\}\) is any \(I\)-adapted orthonormal frame and \(m\) is the complex dimension of \(M\). We then can express the Lee form as (see [23])

\[
d\omega_I^{m-1} = \theta_I \wedge \omega_I^{m-1},
\]

(2.8)

or, equivalently, by

\[
\theta_I = I(\delta \omega_I) = \Lambda(d\omega_I).
\]

(2.9)

Recall that under the hypothesis of Lemma 2.10 we have \(d^c_I \omega_I = db\), so we compute using (2.6):

\[
\theta_I(X) = \frac{1}{2} \sum_{i=1}^{2m} d\omega_I(e_i, Ie_i, X) = \frac{1}{2} \sum_{i=1}^{2m} db(e_i, Ie_i, IX)
\]

\[
= \frac{1}{2} \left( \sum_{i=1}^{2m} (\nabla_{IX} b)(e_i, Ie_i) + 2(\nabla_{e_i} b)(Ie_i, IX) \right)
\]

\[
= \langle \nabla_{IX} b, \omega_I \rangle + (\delta b)(X) + b(\theta_I^2, X) - \sum_{i,j=1}^{2m} b(e_i, e_j) \langle I(\nabla_{e_i} I)(X), e_j \rangle_g
\]

\[
= -\langle b, (\nabla_{IX} \omega_I) \rangle_g + (\delta b)(X) + b(\theta_I^2, X) - \sum_{i,j=1}^{2m} b(e_i, e_j) \langle (\nabla_{e_i} I)(e_j), IX \rangle_g
\]

\[
= -\langle b, (I_X d^c_I \omega_I) \rangle_g + (\delta b)(X) + b(\theta_I^2, X)
\]

\[
= -\langle b, (I_X db) \rangle_g + (\delta b)(X) + b(\theta_I^2, X)
\]

where for passing from the third to the fourth and from the fourth to the fifth lines we have used that \(b\) is of type \((2, 0) + (0, 2)\) (and hence is orthogonal to \(\omega_I\)) and the identity \((\nabla_{IX} I)(IY) = (\nabla_X I)(Y)\) (which holds for any Hermitian manifold by (2.6)).

We next shift attention to generalized Kähler structures with associated taming symplectic structures. We first give an equivalent formulation of a wide class of generalized Kähler structures encompassing the nondegenerate case.
Definition 2.11. Given a manifold $M$, a generalized Kähler structure of symplectic type on $M$ is a triple $(F, I, J)$ of a real symplectic 2-form $F$ and integrable complex structures $I$ and $J$, such that $I$ and $J$ are tamed by $F$ and

$$F(IX, Y) = -F(X, JY).$$

Equivalently,

$$FI = -g + b, \quad FJ = -g - b,$$

where $g$ is a positive-definite symmetric tensor and $b$ is a real 2-form of type $(2, 0) + (0, 2)$ (with respect to both $I$ and $J$). It follows from direct calculations (cf. Lemma 2.14 below) that $(g, I, J)$ is then a generalized Kähler structure with $d_I^* \omega_I = db = -d_J^* \omega_J$. By further direct calculations one can obtain that $I + J$ is invertible, and

$$F = -2g(I + J)^{-1}, \quad b = \frac{1}{2}F(I - J) = -g(I + J)^{-1}(I - J).$$

Note that any compact generalized Kähler 4-manifold is of symplectic type, provided that $I$ and $J$ induce the same orientation and $b_1(M)$ is even, see [3, Prop. 4] and [35, Prop. 4]. We next observe that a nondegenerate generalized Kähler structure is of symplectic type, and compute the relevant symplectic forms.

Lemma 2.12. Let $(M^{4n}, g, I, J)$ be a nondegenerate generalized Kähler structure. Then it is of symplectic type in two ways, with associated symplectic forms

$$F_\pm = -2g(I \pm J)^{-1}.$$

Proof. Letting

$$A := (I + J), \quad B := (I - J),$$

we can compute that

$$\sigma = -ABg^{-1},$$

$$\Omega_I = -gB^{-1}A^{-1} - \sqrt{-1}gIB^{-1}A^{-1},$$

$$\Omega_J = -gB^{-1}A^{-1} - \sqrt{-1}gJB^{-1}A^{-1}.$$

We introduce the $(0, 2)$-tensors

$$F_+ := -2gA^{-1}, \quad b := -gA^{-1}B,$$

or, equivalently,

$$-F_+A = 2g; \quad F_+B = 2b.$$

It follows by the very definitions of $F_+$ and $b$,

$$F_+I = \frac{1}{2}(F_+A + F_+B) = -g + b, \quad F_+J = \frac{1}{2}(F_+A - F_+B) = -g - b.$$

Using $AB = -BA$ and that $A$ and $B$ are skew with respect to $g$, we see that $b$ is a skew-symmetric tensor, i.e. a 2-form. Thus (2.17) implies that $F_+$ tames both $I$ and $J$, and that $b$ is of type $(2, 0) + (0, 2)$ with respect to either $I$ or $J$, i.e. (2.11) holds true. In order to conclude that $(g, I, J)$ is of symplectic type it is enough to show that $F_+$ is closed. Under the non-degeneracy assumption for $(g, I, J)$ we have

$$F_+ = -2gA^{-1} = -2gA^{-1} \sigma \sigma^{-1} = 2gA^{-1}(AB)g^{-1} \sigma^{-1} = 2B \sigma^{-1} = 2I \sigma^{-1} - 2J \sigma^{-1} = -2gIB^{-1}A^{-1} + 2gJB^{-1}A^{-1} = 2\left(\text{Im}(\Omega_I) - \text{Im}(\Omega_J)\right).$$
Then the
that
to construct large families of generalized Kähler structures generate generalized Kähler structures. The central observation is due to Joyce, who showed how
2.4. Variations of structure. In this subsection we exhibit a natural class of variations of nondegenerate generalized Kähler structures. The central observation is due to Joyce, who showed how to construct large families of generalized Kähler structures in the 4-dimensional case by deforming away from hyper-Kähler structures appropriately using Hamiltonian diffeomorphisms (cf. [1]). This construction was extended to arbitrary dimensions by Gualtieri ([10] Example 2.21). These constructions focused on deformation away from hyper-Kähler structure, and below we show that these ideas also yield deformations of arbitrary nondegenerate generalized Kähler structures. The first step is a higher dimensional extension of [10] Thm. 2].

Lemma 2.14. Suppose \((I, \Omega_I)\) and \((J, \Omega_J)\) are two holomorphic-symplectic structures on \(M^{4n}\), such that

1. \(\text{Re}(\Omega_I) = \text{Re}(\Omega_J)\),
2. The \((1, 1)\)-part with respect to \(I\) of the 2-form \(-\text{Im}(\Omega_I)\) is positive definite.

Then the \(I\)-Hermitian metric defined by \(g(X, X) = -2\text{Im}(\Omega_J)(X, IX)\) is also \(J\)-invariant, and \((g, I, J)\) defines a nondegenerate generalized Kähler structure with \(\Omega = \text{Re}(\Omega_I)\).
Proof. Because of condition (1), we can write the holomorphic symplectic forms as
\[ \Omega_I = \Omega + \sqrt{-1}I\Omega, \quad \Omega_J = \Omega + \sqrt{-1}J\Omega, \]
where \( \Omega \) is a real symplectic form on \( M \) of complex type \((2,0) + (0,2)\) with respect to both \( I \) and \( J \). Setting
\[ F := 2\left(\text{Im}(\Omega_I) - \text{Im}(\Omega_J)\right) = 2(\Omega + J\Omega) \]
we obtain another real symplectic form which, by condition (2), tames the complex structure \( I \). Let
\[ (2.20) \quad F(X,IY) = g(X,Y) + b(X,Y) \]
be the decomposition of \(-IF\) as the sum of a symmetric tensor \( g \) (which is a Riemannian metric compatible with \( I \) according to (2)) and a 2-form \( b \) (which is of complex type \((2,0) + (0,2)\) with respect to \( I \)). As
\[ F_I = 2(I\Omega - J\Omega) = 2(\Omega + JI\Omega) = JF, \]
or, equivalently,
\[ (2.21) \quad F(IX,Y) = -F(X,JIY), \]
we obtain that \( g \) is also \( J \)-invariant, and (by using (2.21))
\[ (2.22) \quad F(X,JY) = g(X,Y) - b(X,Y). \]
Let us denote by \( \omega_I = (F)^{1,1}_I \) and \( \omega_J = (F)^{1,1}_J \) the Kähler forms of \((g,I)\) and \((g,J)\), respectively. According to (2.20) and (2.22), we have
\[ F = \omega_I + Ib = \omega_J - Jb. \]
As \( F \) is closed,
\[ d\omega_I = -Idb, \quad d\omega_J = Jdb, \]
so that, setting \( I\eta = -\eta(I,I,I) \) for a 3-form \( \eta \in \Lambda^3(M) \),
\[ d^c_I \omega_I = Id\omega_I = -IdIb, \quad d^c_J \omega_J = Jd\omega_J = JdJb. \]
As \( b \) is of type \((2,0) + (0,2)\) with respect to \( I \) we can write
\[ b = b^{2,0} + b^{0,2}, \quad Ib = -ib^{2,0} + ib^{0,2}. \]
As \( dIb = -d\omega_I \) is of type \((2,1) + (1,2)\), we deduce
\[ d(Ib) = -i\partial\bar{\partial}b^{2,0} + i\partial\bar{\partial}b^{0,2}, \]
and therefore \( Id(Ib) = -\partial\bar{\partial}b^{2,0} - \partial\bar{\partial}b^{0,2} = -db \). Similarly, \( Jd(Jb) = -db \). We conclude, therefore, that \((g,I,J)\) is generalized Kähler with \( H = d^c_I \omega_I = db = -d^c_J \omega_J \). □

Lemma 2.14 yields a natural construction of generalized Kähler manifolds. In fact every nondegenerate generalized Kähler structure arises from this description. We state this in the next lemma, whose proof is contained in the proof of Lemma 2.12.

Lemma 2.15. Let \((M^{4n},g,I,J)\) be a nondegenerate generalized Kähler structure and \( \Omega_J \) the holomorphic symplectic structure associated to \( J \). Then the \((1,1)\) part of \(-2\text{Im}(\Omega_J)\) with respect to \( I \) is positive definite and equals \( \omega_I \), i.e. \((g,I,J)\) is given by the construction of Lemma 2.14.

With this description in place we can now exhibit a natural class of deformations of nondegenerate generalized Kähler structures.
Proposition 2.16. Let \((M^{4n}, g, I, J)\) be a nondegenerate generalized Kähler manifold. Let \(f_t\) be a smooth time dependent function on \(M\), and \(X_f\) be the \(\Omega\)-Hamiltonian vector field associated to \(f\), i.e.
\[
df = -X_f \cdot \Omega.
\]
Let \(\phi_t\) be the 1-parameter family of diffeomorphisms of \(M\) generated by \(X_f\). Then for all \(t\) such that the \((1, 1)\)-part with respect to \(I\) of \(-\text{Im}(\Omega_{\phi_t^*} J)\) is positive definite, the triple \((I, \phi_t^* J, \Omega)\) are the complex structures and symplectic structure associated to a unique nondegenerate generalized Kähler structure.

Proof. As in Lemma 2.14 associated to the given generalized Kähler structure is a pair of complex symplectic forms written as
\[
\Omega_I = \Omega + \sqrt{-1} I \Omega, \quad \Omega_J = \Omega + \sqrt{-1} J \Omega,
\]
where \(\Omega\) is a real symplectic form on \(M\) of complex type \((2, 0) + (0, 2)\) with respect to both \(I\) and \(J\). Now define a one-parameter family of 2-forms via \((\Omega_j)_t = \phi_t^* (\Omega_J)\) certainly by construction \((\phi^* J, \phi^* \Omega)\) remains a holomorphic symplectic structure. Since \(\phi\) is \(\Omega\)-Hamiltonian, it follows that
\[
(\Omega_I)_t = \phi^* (\Omega + \sqrt{-1} J \Omega) = \Omega + \sqrt{-1} (\phi^* J) \Omega,
\]
and so \(\text{Re}(\Omega_I)_t = \text{Re}(\Omega_I)\) for all \(t\). The result follows from Lemma 2.14. 

With this proposition in place we give a definition central to this work.

Definition 2.17. Let \((M^{4n}, g, I, J)\) be a nondegenerate generalized Kähler manifold. An \(\Omega\)-Hamiltonian diffeomorphism \(\phi\) is positive if
\[
- (\text{Im}(\Omega_{\phi^*} J))_I^{1,1} > 0.
\]
By Proposition 2.16 a positive \(\Omega\)-Hamiltonian diffeomorphism defines a unique generalized Kähler structure, which we denote \((g_\phi, I, J_\phi)\). It is important to note that while \(J_\phi = \phi_t^* J\), the metric \(g_\phi\) is defined implicitly from the triple \((I, \phi_t^* J, \Omega)\), and is not a pullback of the given \(g\).

With these constructions in place we can also recover the aforementioned construction of Joyce of nontrivial nondegenerate generalized Kähler structures by deformation away from hyper-Kähler structures.

Corollary 2.18. (cf. [30, 35]) Let \((M^{4n}, g, I, J, K)\) be a compact hyper-Kähler manifold. For any smooth function \(f\), denote by \(\phi_t\) the \(\omega_K\)-Hamiltonian flow determined by \(f\). Then, for \(|t|\) small enough, \(\phi_t\) is positive, and the associated structure \((g_t, I, J_t)\) is not Kähler, unless \(f \equiv \text{const}\).

Proof. Proposition 2.16 already yields that the deformation generates generalized Kähler structures, so we need to show that \((g_t, I)\) is not Kähler for \(|t|\) small enough, when the generating function \(f\) is not constant. To this end, consider the angle function defined by
\[
(2.23) \quad p_t = -\frac{1}{4n} \text{tr}(I J_t) = \frac{1}{4n} \langle I, J_t \rangle_{g_t}.
\]
Clearly, if \((g_t, I)\) were Kähler, so would be \((g_t, J_t)\), and hence \(p_t\) would be a constant function on \(M\) as \(I\) and \(J_t\) are \(g_t\)-parallel. One notes that with the given hyper-Kähler backgroud, \(X_f = \frac{1}{2} [I, J] g^{-1}(df)\). Hence differentiating \((2.23)\), we obtain
\[
\left. \frac{dp_t}{df} \right|_{t=0} = \frac{1}{4n} \text{tr} \left( I \circ (\mathcal{L}_{X_f} J) \right) = -\frac{1}{4n} \text{tr} \left( I \circ [\nabla X_f, J] \right) = \frac{1}{8n} \text{tr} \left( [I, J] \circ [I, J] \circ \nabla g^{-1}(df) \right) = -\frac{1}{2n} \Delta g f,
\]
where $\nabla$ is the Levi-Civita connection of the hyper-Kähler metric $g$, and $\Delta_g = -d\delta^g - \delta^g d$ will denote the ‘analytic’ Laplace operator in accordance with the notation of [31]. This maximum principle shows that if $f$ is not a constant function, $p_t$ cannot stay constant for $t$ sufficiently small. \hfill \Box

3. A Calabi-Yau conjecture for nondegenerate generalized Kähler manifolds

3.1. Generalized Calabi-Yau structures. In this subsection we recall some motivating ideas from the theory of generalized complex geometry which lead to the definition of the generalized Calabi-Yau equation. Our exposition is brief, and we refer to [28, 29, 34] for the general theory of generalized complex structures. The notion of an (even) generalized Calabi-Yau structure was introduced by Hitchin [34] as a special example of a generalized complex structure, i.e. an almost complex structure $\mathbb{I}$ defined on the vector bundle $TM \oplus T^*M$, which is orthogonal with respect to the natural non-degenerate inner product

\begin{equation}
\langle X + \xi, X + \xi \rangle := -\xi(X), \ X \in TM, \ \xi \in T^*M,
\end{equation}

and is integrable in the sense that the $i$-eigenspace $E \subset (TM \oplus T^*M) \otimes \mathbb{C}$ of $\mathbb{I}$ is closed under the Courant bracket on $TM \oplus T^*M$.

The case of (even) generalized Calabi-Yau structures studied in [34] corresponds to the special situation when the $i$-eigenspace of the generalized complex structure $\mathbb{I}$ coincides with the annihilator $E_\varphi$ of an even-degree closed form $\varphi \in \Gamma(\Lambda^{ev}(M) \otimes \mathbb{C})$, under the natural pointwise action of $TM \oplus T^*M$ on the exterior algebra $\Lambda^\bullet(M)$ of $T^*M$,

\[ (X, \xi) \cdot \varphi := i_X \varphi + \xi \wedge \varphi. \]

As $(X, \xi)^2 \cdot \varphi = -\langle X + \xi, X + \xi \rangle \varphi$, the annihilator $E_\varphi$ of a non-zero even form $\varphi$ is automatically isotropic with respect to the $\mathbb{C}$-linear extension of the product $\langle \cdot, \cdot \rangle$ on $(TM \oplus T^*M) \otimes \mathbb{C}$. It is shown in [34] that $d\varphi = 0$ implies that $E_\varphi$ is closed under the Courant bracket. Furthermore, the condition $E_\varphi \cap E_{\overline{\varphi}} = 0$ can be equivalently expressed in terms of the natural $\Lambda^{2m}(M)$-valued inner-product on $\Lambda^{ev}(M) \otimes \mathbb{C}$, defined by

\[ \langle \varphi, \psi \rangle = \sum_r (-1)^r \varphi_{2r} \wedge \psi_{2m-2r}, \]

where $2m$ denotes the real dimension of $M$, and we write $\varphi = \sum_r \varphi_{2r}$ for the degree decomposition of a form in $\Lambda^{ev}(M)$. It turns out that for any $\varphi, \psi \in \Lambda^{ev}(M)$, $E_\varphi \cap E_{\overline{\varphi}} \neq 0$ iff $\langle \varphi, \psi \rangle = 0$. Thus, fixing an orientation of $M$, $\varphi$ must satisfy (at every point)

\begin{equation}
\langle \varphi, \varphi \rangle = 0, \quad \langle \varphi, \overline{\varphi} \rangle > 0.
\end{equation}

Finally, the fact that $E_\varphi$ has maximal dimension ($= \dim_{\mathbb{R}}(M) = 2m$) leads to a complicated non-linear pointwise algebraic condition on $\varphi$, which is referred in the literature to as $\varphi$ being a pure spinor. To summarize,

**Definition 3.1.** An (even) generalized Calabi-Yau structure on $M$ is defined by a closed pure spinor $\varphi \in \Gamma(\Lambda^{ev}(M))$ satisfying (3.2) at each point of $M$.

The main observation in [34], and the reason for the terminology, is the following prototypical example, showing that the complex and Kähler structures underlying a Calabi-Yau manifold can both be interpreted as even generalized Calabi-Yau structures.

**Example 3.2.** If $\Theta$ is a holomorphic trivialization of the canonical bundle $K(M, I) = \Lambda^{m,0}(M, I)$ of a complex manifold $(M^{2m}, I)$, then $\Theta$ viewed as a closed complex $(m, 0)$ form on $M$, defines a closed pure spinor giving rise to the generalized complex structure

\[ \mathbb{I}_\Theta = \begin{pmatrix} -I & 0 \\ 0 & -I \end{pmatrix}. \]
Similarly, if $\omega$ is a real symplectic 2-form on $M$, then
\[
\varphi = \exp \sqrt{-1} \omega = 1 + \sqrt{-1} \omega - \frac{1}{2} \omega^2 + \cdots
\]
is a closed pure spinor, and the corresponding generalized complex structure $I_\varphi$ takes the form (see [28, Ex. 4.6])
\[
I_\varphi = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.
\]
In particular, any Calabi-Yau manifold, i.e., a compact complex manifold $(M, I)$ which admits both a Kähler metric $\omega$ and a trivialization $\Theta$ of its canonical line bundle, is naturally endowed with two generalized Calabi-Yau structures, given by the closed pure spinors $\varphi = \exp \sqrt{-1} \omega$ and $\varphi_2 = \Theta$.

To make the link with the non-degenerate generalized Kähler structures studied in this paper, we recall that Gualtieri [28, 30] has shown that the data $(g, I, J, b)$ of a Riemannian metric $g$, two $g$-orthogonal integrable almost-complex structures $(I, J)$, and a 2-form $b$ on $M$, satisfying the relation
\[
d^c \omega_I = db = -d^c_J \omega_J
\]
determine and are determined by a pair of commuting generalized complex structures $I_{\pm}$ on $TM \oplus T^*M$, such that $I_+ \circ I_-$ is positive-definite with respect to (3.1). Specifically, given $(g, I, J, b)$ as above, the generalized complex structures $I_{\pm}$ are
\[
I_{\pm} = \frac{1}{2} \epsilon^b \begin{pmatrix} (I \mp J) & -(\omega_I^{-1} \pm \omega_J^{-1}) \\ \omega_I \pm \omega_J & (I \mp J) \end{pmatrix} e^{-b},
\]
where $\epsilon^b := \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ is the exponential of $\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$ (and we note that our convention for the action of $I, J$ on $T^*M$ defers by a sign with the one used in [28]). By Lemma 2.12 for any non-degenerate GK structure $(g, I, J)$ in the sense of the present paper (see Definition 2.1), we can take $b = g(I - J)(I + J)^{-1}$. A straightforward computation shows that with this choice of $b$, (3.3) becomes
\[
I_- = e^{-4\Omega} \begin{pmatrix} 0 & -(F_-)^{-1} \\ F_- & 0 \end{pmatrix} e^{4\Omega}, \quad I_+ = \begin{pmatrix} 0 & -(F_+)^{-1} \\ F_+ & 0 \end{pmatrix},
\]
where $F_{\pm} = -2g(I \pm J)^{-1}$ are the symplectic forms taming $(I, \pm J)$ via Lemma 2.12 and $\Omega = g(I + J)^{-1}(I - J)^{-1}$ is the common real part of the holomorphic-symplectic forms $\Omega_I$ and $\Omega_J$. In particular, $I_{\pm}$ are generalized Calabi-Yau structures corresponding to the closed pure spinors $\varphi_- = e^{4\Omega + \sqrt{-1} F_-}$ and $\varphi_+ = e^{4\Omega - \sqrt{-1} F_+}$. The above formula extends to the non-degenerate generalized Kähler case the observations made in [28, Examples 6.30 & 6.31] in the hyper-Kähler and the nondegenerate 4-dimensional cases.

### 3.2. Generalized Calabi-Yau equation

In this subsection we recall the definition of the generalized Kähler Calabi-Yau equation, and describe it in terms of biHermitian data in our setting. To motivate the definition, recall that a classical Calabi-Yau Kähler metric on $(M^{2m}, I)$, with holomorphic volume form $\Theta$, is defined by an $I$-compatible symplectic 2-form $\omega$ such that $\Theta \wedge \overline{\Theta} = \lambda \omega^{2m}$, for some constant $\lambda$. We note that this condition is equivalently expressed in terms of the corresponding pure spinors $\varphi_- = \exp \sqrt{-1} \omega$, $\varphi_+ = \Theta$ as
\[
\langle \varphi_+, \varphi_+ \rangle = \lambda \langle \varphi_-, \varphi_- \rangle.
\]

**Definition 3.3.** Let $(M^{2m}, I_+, I_-)$ be a generalized Kähler manifold where $I_{\pm}$ are even generalized Calabi-Yau structures defined by pure spinors $\varphi_{\pm}$. Define the Ricci potential $\Phi$ via
\[
\Phi = \log \frac{\langle \varphi_+, \varphi_+ \rangle}{\langle \varphi_-, \varphi_- \rangle}.
\]
We say that the structure is generalized Kähler Calabi-Yau if
\[ \Phi \equiv \lambda \]
for some \( \lambda \in \mathbb{R} \).

The terminology “Ricci potential” certainly matches with the situation in the Kähler case, where we see that \( -\sqrt{-1}\partial \bar{\partial} \Phi = \rho \), the usual Ricci form. Moreover, in that setting the equation \( \Phi \equiv \lambda \) certainly defines a Ricci-flat, Calabi-Yau metric. We will justify this terminology in our setting in Proposition 3.9. We note that the geometry of pairs of even generalized Calabi-Yau structures satisfying (3.5) on K3 complex surface has been studied by D. Huybrechts in [38]. A detailed study of the local description of solutions to this equation appeared in [36]. In the next lemma we reduce the generalized Calabi-Yau equation to one involving purely biHermitian data in the nondegenerate case.

**Lemma 3.4.** Let \((M^{4n}, g, I, J)\) be a nondegenerate generalized Kähler manifold. Then
\[ \Phi = \log \frac{F^{2n}_{+}}{F^{2n}_{-}}. \]

**Proof.** Returning to the notation of the previous subsection, it follows from (3.4) that (see [28, Example 4.10]) both \( I_{\pm} \) are generalized Calabi-Yau structures corresponding to the closed pure spinors \( \varphi_{-} = e^{4\Phi + \sqrt{-1}F_{-}} \) and \( \varphi_{+} = e^{\sqrt{-1}F_{+}} \). Furthermore, as shown for instance in [34, pp. 286-287], we have
\[ \langle e^{4\Omega + \sqrt{-1}F_{-}} e^{-i\Phi} \rangle = e^{\sqrt{-1}F_{+}} e^{-\sqrt{-1}F_{-}} = \frac{2^{2n}}{(2n)!} (F_{-})^{2n}, \quad \langle e^{\sqrt{-1}F_{+}} e^{-\sqrt{-1}F_{-}} \rangle = \frac{2^{2n}}{(2n)!} (F_{+})^{2n}, \]
finishing the lemma. \( \square \)

### 3.3. A generalized Calabi conjecture.

**Proposition 2.16** shows that there exists a natural local action of the group \( \text{Ham}(M, \Omega) \) of Hamiltonian isotopies with respect to \( \Omega \) on the space of nondegenerate generalized Kähler structures, also fixing \( I \). Before stating our conjecture, we record a basic lemma indicating an integral invariant associated to this action, which determines the possible value of \( \lambda \) in a generalized Kähler Calabi-Yau structure.

**Lemma 3.5.** Let \((M^{4n}, g, I, J)\) be a nondegenerate generalized Kähler structure. The quantity
\[ \lambda := \log \left\{ \left( \int_{M} F^{2n}_{+} \right) / \left( \int_{M} F^{2n}_{-} \right) \right\} \]
is an invariant of the \( \Omega \)-Hamiltonian deformation class.

**Proof.** First, it is clear that the \( \Omega \)-Hamiltonian action preserves \( I \) and \( \Omega \), and so preserves \( \Omega_{I} \) and the deRham cohomology class of \( \Omega_{J} \). Since
\[ F_{\pm} = 2(\text{Im}(\pm \Omega_{I}) - \text{Im}(\Omega_{J})), \]
the lemma follows. \( \square \)

We now state our main conjecture:

**Conjecture 3.6.** Let \((M^{4n}, g, I, J)\) be a compact nondegenerate generalized Kähler manifold.

1. **Existence:** There exists a positive diffeomorphism \( \phi \in \text{Ham}(M, \Omega) \) such that
\[ \Phi(g_{\phi}, I, J_{\phi}) \equiv \lambda, \]
where \( \lambda \) is the constant introduced in (3.6).

2. **Uniqueness:** The induced generalized Kähler structure \((g_{\phi}, I, J_{\phi})\) is unique.

3. **Rigidity:** The induced generalized Kähler structure \((g_{\phi}, I, J_{\phi})\) is hyper-Kähler.
In particular the conjecture claims there is a generalized Kähler Calabi-Yau structure in every \( \Omega \)-Hamiltonian deformation class, which is unique, and which moreover is highly rigid: it is hyper-Kähler. Thus, Conjecture 3.6 would imply that any non-degenerate generalized Kähler structure on a compact hyper-Kähler manifold \( M^{4n} \) must arise from the construction of Proposition 2.18. We verify the last two parts of the conjecture below.

**Remark 3.7.** Recently, R. Goto \[27\] introduced a notion of a *generalized scalar curvature* of a generalized Kähler manifold of symplectic type. In the nondegenerate case, it is given by (see \[27\] p. 4 and (3.1))

\[
G_{\text{scal}}_g = 4\left( \frac{d(\Omega F^{-1}_g(\Delta \Phi)) \wedge \Omega^{2n-1}_g}{F_{2n}^g} \right) = -\frac{1}{2n} \left( \Delta_g \Phi + d\Phi \left( (\nabla e_i (I + J)^{-1})(I + J)(e_i) \right) \right).
\]

It follows by the maximum principle that on a compact nondegenerate generalized Kähler manifold \( (M^{4n},g,I,J) \), the generalized scalar curvature \( G_{\text{scal}}_g \) is constant if and only if \( \Phi \) is constant, thus giving a yet another motivation for Conjecture 3.6.

### 3.4. Uniqueness and rigidity

In this subsection we establish both the uniqueness and rigidity claims of Conjecture 3.6. To begin we establish a key curvature identity relating the Bismut-Ricci curvature of a nondegenerate generalized Kähler structure to the Ricci potential \( \Phi \). We recall that on complex surfaces, or more generally, when \( I,J \) are compatible with an almost-quaternion structure on \( M^{4n} \) (see \[1\]), \( I \) and \( J \) verify the identity

\[
IJ + JI = -2p \text{Id},
\]

where the smooth function \( p := -\frac{1}{4n}(trIJ) \) is the so-called the *angle function* of \((g,I,J)\). It follows from (3.7) that \((I \pm J)^2 = -2(1 \pm p)\text{Id} \), so we obtain \( \Phi = \log \left( \frac{1-p}{1+p} \right) \). This shows in particular that \( \Phi \) is constant when \( I,J \) belong to the same hyper-Kähler structure on \( M^{4n} \). More generally we will show that for a nondegenerate generalized Kähler structure, \( \Phi \) determines all of the relevant curvature and torsion quantities. We begin with a lemma on the differential of the Ricci potential.

**Lemma 3.8.** For any non-degenerate generalized Kähler structure \((g,I,J)\), one has

\[
[I,J](d\Phi) = -d\Phi \circ [I,J] = 2(\theta_I - \theta_J).
\]

**Proof.** By Lemmas 2.10 and 2.13 (and exchanging the roles of \( I \) and \( J \)), we have

\[
\theta_I = (I - J)(I + J)^{-1} \theta_I + \frac{1}{2} d \log \det(I + J) + \delta^g(I - J)(I + J)^{-1}
\]

\[
\theta_J = -(I - J)(I + J)^{-1} \theta_J + \frac{1}{2} d \log \det(I + J) - \delta^g(I - J)(I + J)^{-1}.
\]

Summing the above two equalities and observing the identities

\[
(I + J)^{-1} = + [I,J]^{-1}(I - J) = -(I - J)[I,J]^{-1},
\]

\[
(I - J)^{-1} = - [I,J]^{-1}(I + J) = +(I + J)[I,J]^{-1},
\]

we obtain

\[
\theta_I + \theta_J = -(I - J)^2[I,J]^{-1}(\theta_I - \theta_J) + d \log \det(I + J).
\]

Rewriting (3.8) with respect to \((g,I,-J)\) we also have

\[
\theta_I + \theta_J = (I + J)^2[I,J]^{-1}(\theta_I - \theta_J) + d \log \det(I - J).
\]

Then, (3.9)-(3.8) gives

\[
2d\Phi = \left( (I - J)^2 + (I + J)^2 \right)[I,J]^{-1}(\theta_I - \theta_J) = 4[I,J]^{-1}(\theta_I - \theta_J),
\]

as required. \( \square \)
Proposition 3.9. Let \((M^{4n}, g, I, J)\) be a non-degenerate generalized Kähler manifold. Then the Bismut-Ricci forms associated to \((g, I)\) and \((g, J)\) are respectively given by

\[(\rho_B)_I = -\frac{1}{2}dJd\Phi, \quad (\rho_B)_J = -\frac{1}{2}dId\Phi.\]

In particular, on a compact non-degenerate generalized-Kähler manifold \(M^{4n}, g, I, J\), \((\rho_B)_I = 0 \iff (\rho_B)_J = 0 \iff \Phi = \lambda.\)

Proof. Using the formulas (2.2) and (2.3) one can check that the induced unitary connections \(\nabla^B\) and \(\nabla^C\) on \((K^{-1}(M, I), g)\) satisfy (see also [21, Rem. 5])

\[\nabla^B_X = \nabla^C_X - i(I\theta_I)(X).\]  

It follows from (3.11) that the corresponding Ricci curvatures are related by

\[(\rho_B)_I = (\rho_C)_I + dI\theta_I.\]

Using that \(\Omega^m_I\) is a holomorphic section of \(K(M, I)\), the Chern-Ricci form is given by

\[(\rho_C)_I = \frac{1}{2}dI\left(\frac{\Omega^m_I \partial \Omega^m_I}{\omega^m_I}\right) = \frac{1}{2}dI\left(\frac{\Omega^m_I}{\omega^m_I}\right) = \frac{1}{4}dI \log \det[I, J]^{-1}\]

\[= -\frac{1}{4}dI \left(\log \det(I + J) + \log \det(I - J)\right)\]

\[= \frac{1}{2}dI \left(- (\theta_I + \theta_J) + (IJ + JI)[I, J]^{-1}(\theta_I - \theta_J)\right),\]

where for the last equality we have used (3.9) and (3.10). Substituting back in (3.12) and using Lemma 3.8 we obtain

\[(\rho_B)_I = \frac{1}{2}dI \left(\theta_I - \theta_J + (IJ + JI)[I, J]^{-1}(\theta_I - \theta_J)\right)\]

\[= \frac{1}{2}dI \left((IJ - JI)[I, J]^{-1}(\theta_I - \theta_J) + (IJ + JI)[I, J]^{-1}(\theta_I - \theta_J)\right)\]

\[= dI \left(IJ[I, J]^{-1}(\theta_I - \theta_J)\right)\]

\[= -\frac{1}{2}dJd\Phi,\]

as required. □

Proposition 3.10. Let \((M^{4n}, g, I, J)\) be a compact nondegenerate generalized Kähler manifold. Then, \((\rho_B)_I^{1,1} = 0 \iff (\rho_B)_J^{1,1} = 0\) and \((g, I)\) are Kähler Ricci-flat metrics belonging to the same hyper-Kähler structure on \(M^{4n}\). In particular, any nondegenerate generalized Kähler Calabi-Yau structure on \(M^{4n}\) is hyper-Kähler in the usual sense.

Proof. This follows essentially from the arguments in [21] (see also [39], Theorem 4.1). For convenience of the reader, we reproduce them below. We will work only with the Hermitian structure \((g, I)\), and to simplify notation we drop the index \(I\). We thus need to show that for any compact pluriclosed Hermitian manifold \((M, g, I)\) with trivial canonical bundle satisfying

\[(\rho_B)^{1,1} = 0,\]

must be Kähler. We are going to establish this under the weaker assumption, namely supposing that the scalar curvature \(s_B := 2\langle \rho_B, \omega \rangle_g\) of \(\nabla^B\) is identically zero, i.e.

\[s_B = 0.\]

To this end, we are going to use the following relation between the scalar curvatures \(s_C = 2\langle \rho_C, \omega \rangle_g\) and \(s_B\) (which follows by taking a trace in (3.12) with respect to \(\omega\))

\[s_B = s_C + \langle dI\theta, \omega \rangle_g\]

\[= s_C - \delta \theta - |\theta|^2_g.\]
A key observation ([2], Eq. (2.13)) is that the pluriclosedness of \((g, I)\) implies the identity
\[
|d\omega|^2_g = \delta \theta + |\theta|^2_g,
\]
so that \(3.17\) reduces to
\[
s_C = s_B + |d\omega|^2_g = |d\omega|^2_g \geq 0
\]
under the assumption \(3.16\). Since, furthermore, the canonical bundle \(K(M, I)\) is trivial, it follows by ([22], Proposition 13 and Théorème de Classification) that \(s_C \equiv 0\), i.e. \(d\omega = 0\), meaning that \((g, I)\) is Kähler. As on a Kähler manifold the Bismut connection coincides with the Levi-Civita connection, \(\rho_B\) is therefore the usual Ricci form and \(3.15\) means that \((g, I)\) is a Calabi-Yau metric. Similarly \((g, J)\) is a Calabi-Yau Kähler metric, so that \((g, I, J)\) gives rise to a hyper-Kähler metric (cf. Theorem [2.5]).

**Proposition 3.11.** Suppose \(\phi_1, \phi_2 \in \text{Ham}(M, \Omega)\) satisfy the hypotheses of Conjecture 3.6. Then, the corresponding generalized Kähler structures \((g_{\phi_1}, I, J_{\phi_1})\) and \((g_{\phi_2}, I, J_{\phi_2})\) coincide, i.e. \(g_{\phi_1} = g_{\phi_2}, J_{\phi_1} = J_{\phi_2}\).

**Proof.** By assumption, each \(\phi_i\) belongs to the connected component of the identity of the group of diffeomorphisms on \(M\). It follows that each \(\phi_i\) acts trivially on deRham cohomology, so that the symplectic 2-forms \(F_{\phi_1} := -2\left(\text{Im}(\Omega_{J_{\phi_1}}) - \text{Im}(\Omega_I)\right)\) and \(F_{\phi_2} := -2\left(\text{Im}(\Omega_{J_{\phi_2}}) - \text{Im}(\Omega_I)\right)\) taming \(I\) belong to the same deRham class (see Lemma [2.12]). Furthermore, as \((g_{\phi_1}, I)\) and \((g_{\phi_2}, I)\) are both Kähler, the \((1, 1)\)-parts of \(F_{\phi_1}\) and \(F_{\phi_2}\) with respect to \(I\) are the Kähler forms \(\omega_{\phi_1}\) and \(\omega_{\phi_2}\) of \((g_{\phi_1}, I)\) and \((g_{\phi_2}, I)\), respectively, which thus belong to the same Aeppli cohomology class. Using the Kähler condition again, it then follows that \(\omega_{\phi_1}\) and \(\omega_{\phi_2}\) define Calabi-Yau metrics on \((M, I)\) in the same Kähler class. By the uniqueness of the Calabi-Yau metric in its Kähler class, we conclude \(\omega_{\phi_1} = \omega_{\phi_2}\) or, equivalently, \(g_{\phi_1} = g_{\phi_2}\).

Notice that for any GK structure \((g, I, J)\) of symplectic type with symplectic 2-form \(F\) taming \(I\) and \(J\) as in \([2,10]\), we have (see \([2,11]\)) \(F(I + J) = -2g\). It thus follows that in order to show \(J_{\phi_1} = J_{\phi_2}\) it is enough to establish that \(F_{\phi_1} = F_{\phi_2}\). As we have already proved that the \((1, 1)\)-parts with respect to \(I\) of \(F_{\phi_i}\) coincide and are closed, it follows that \((2, 0) + (0, 2)\)-part of \(F_{\phi_1} - F_{\phi_2}\) is closed and exact. It must therefore be zero, being also parallel with respect to \(g_{\phi_1} = g_{\phi_2}\) (this follows from the fact that \((g_{\phi_1}, I, J_{\phi_1})\) is hyper-Kähler and \([2,13]\)). We thus conclude \(F_{\phi_1} = F_{\phi_2}\) which in turn implies \(J_{\phi_1} = J_{\phi_2}\). \(\square\)

**Remark 3.12.** By the very definition of \(J_{\phi_i}\), the identity \(J_{\phi_1} = J_{\phi_2}\) in Proposition 3.11 is equivalent to \(\phi_1 \circ \phi_2^{-1} \in \text{Aut}(M, J)\). If \(\text{Aut}(M, J) = \{\text{id}\}\), we can conclude \(\phi_1 = \phi_2\). In general, for a holomorphic-symplectic Kähler manifold \((M, J)\) whose first Betti number is zero (see Theorem [2.5]) the group \(\text{Aut}(M, J)\) is discrete. This follows because, on the one hand, the connected component of the identity \(\text{Aut}_0(M, J)\) coincides with the group of reduced automorphisms \(\text{Aut}'(M, J)\), and, on the other hand, \(\text{Aut}'(M, J)\) is trivial on a Calabi-Yau manifold by Matsushima’s theorem (see e.g. [25]). Furthermore, \(\rho_1 \circ \rho_2^{-1}\) acts trivially on \(H^2(M, \mathbb{R})\) (being in \(\text{Diff}_0(M)\)). This implies that it must belong to the isometry group of any Calabi-Yau metric on \((M, J)\), showing that \(\phi_1 \circ \phi_2^{-1}\) is also of finite order. It is well-known (see e.g. [37, Ch. 15, Cor. 2.6]) that on a K3-surface these conditions imply \(\phi_1 \circ \phi_2^{-1} = \text{id}\) but we are not aware of a general argument for an arbitrary holomorphic-symplectic Kähler manifold.

We can now prove Theorem 1.1 which we restate for convenience.

**Theorem 3.13.** (cf. Theorem 1.1) Let \((M^{4n}, g, I, J)\) be a nondegenerate generalized Kähler manifold. Any two solutions \((g_{i}, I, J_{i}), i = 1, 2\) of the generalized Kähler Calabi-Yau equation in the \(\Omega\)-Hamiltonian deformation class agree, and moreover define a hyper-Kähler structure.

**Proof.** This is restating Propositions [3.10] and [3.11]. \(\square\)
4. A Formal GIT picture

In this section we establish a formal moment map interpretation of the generalized Kähler Calabi-Yau equation. A closely related problem was addressed in [19], giving a formal moment map interpretation of the problem of prescribing the volume form of symplectic structures within a given de Rham class. A similar formal framework concerning the problem of finding constant scalar curvature Kähler metrics within a given Kähler class is given in [20, 18], and has been extended to the generalized Kähler case in [11, 27] (inspired from the setting in [26]).

4.1. Setup. Let \((g, I, J)\) be a nondegenerate generalized Kähler metric, denote the holomorphic-
symplectic structures by \(\Omega_I\) and \(\Omega_J\), and denote by

\[
\Psi_1 := I\Omega = \text{Im}(\Omega_I), \quad \Psi_2 := J\Omega = \text{Im}(\Omega_J)
\]

the closed imaginary parts of \(\Omega_I\) and \(\Omega_J\). Furthermore, let \(\alpha = [\Omega_I]\) and \(\beta = [\Omega_J]\) be the corresponding deRham classes in \(H^2(M, \mathbb{C})\). Now set

\[
\mathcal{GK}_{\alpha,\beta} = \{\text{GK triples } (g', I', J') \mid [\Omega_{I'}] \in \alpha, \ [\Omega_{J'}] \in \beta \}.
\]

As the considerations in this section are purely formal, we endow \(\mathcal{GK}_{\alpha,\beta}\) with the \(C^\infty\) topology.

By using Lemmas 2.14 and 2.15 we can interpret paths in \(\mathcal{GK}_{\alpha,\beta}\) as smooth families of pairs of symplectic forms. In particular, consider a smooth family \(\Omega^t_j = (\Omega^t + \sqrt{-1}\Psi^t_j), j = 1, 2\) of closed complex-valued 2 forms \(\Omega^t_j\) on \(M^{4n}\), such that for all \(t\)

(a) \(\Omega^t\) and \(\Psi^t_j\) are real symplectic 2-forms and \([\Omega^t_1] = \alpha, \ [\Omega^t_2] = \beta\).

(b) the endomorphisms \(I_t := -(\Psi^t_1)^{-1}\Omega^t\) and \(J_t := -(\Psi^t_2)^{-1}\Omega^t\) define almost-complex structures;

(c) the \((1, 1)\)-part with respect to \(I_t\) of \(-\Psi^t_2\) is positive definite.

It is observed in [7] that \(I_t\) and \(J_t\) are automatically integrable (as an easy consequence of the closeness of \(\Omega^t_j\)), and thus, by Lemma 2.14 \((\Omega^t_j, j = 1, 2)\) give rise to a smooth curve in \(\mathcal{GK}_{\alpha,\beta}\). Conversely, by Lemma 2.15 any path in \(\mathcal{GK}(\alpha, \beta)\) has this form.

By Moser’s lemma with respect to the path of cohomologous symplectic forms \(\Omega^t\), we can pull-back \((\Omega^t_j, j = 1, 2)\) by an isotopy of diffeomorphisms and assume that \(\text{Re}(\Omega^t_j) = \Omega\) is a fixed symplectic form on \(M\). We shall thus be interested, without loss, in a restricted space, namely

\[
\mathcal{GK}^K_{\alpha,\beta}(\Omega) = \{ (g', I', J') \in \mathcal{GK}_{\alpha,\beta} \mid \Omega = \text{Re}(\Omega_I) = \text{Re}(\Omega_J) \}.
\]

Note that points of \(\mathcal{GK}^K_{\alpha,\beta}(\Omega)\) are equivalently parametrized by pairs of complex structures \((I, J)\) (which in turn determine \(\Psi_1\) and \(\Psi_2\) via (b)). Also, to simplify some of the discussion below, we make a further restriction, namely we set

\[
\mathcal{GK}^K_{\alpha,\beta}(\Omega) = \{ (g', I', J') \in \mathcal{GK}_{\alpha,\beta}(\Omega) \mid I', J' \text{ Kähler} \}.
\]

The next result shows the openness of the orbits of the natural \(\Omega\)-Hamiltonian action in this setting.

Lemma 4.1. Let \((M^{4n}, \Omega)\) be a hyper-Kähler manifold and suppose \((g, I, J)\) is a generalized Kähler metric in \(\mathcal{M}\) an open path connected subset of \(\mathcal{GK}^K_{\alpha,\beta}(\Omega)\). Let \(\mathcal{O}_I\) and \(\mathcal{O}_J\) denote the orbits of the 2-forms \(\text{Im}(\Omega_I)\) and \(\text{Im}(\Omega_J)\) under the (right) action of \(\text{Ham}(M, \Omega)\). Then, in the \(C^\infty\) topology, \(\mathcal{M}\) is an open subset of \(\mathcal{O}_I \times \mathcal{O}_J\). If furthermore \(\text{b}_1(M) = 0\), then \(\mathcal{M}\) is finitely covered by an open subset in \(\text{Ham}(M, \Omega) \times \text{Ham}(M, \Omega)\).

Proof. Any path in \(\mathcal{M}\) starting at \((g, I, J)\) is determined by a smooth family \((\Psi^t_j, j = 1, 2)\) of symplectic forms satisfying the conditions (a), (b) and (c) above with \(\Omega^t = \Omega\). Thus, the 2-forms \(\gamma^t_j := \frac{\partial}{\partial t}\Psi^t_j\) are exact (by (a)). Writing \(\gamma^t_j = da^t_j\) for some 1-forms \(a^t_j\), the fact that \((\Omega + \sqrt{-1}\Psi^t_j)^{2n+1} = 0\) imply that \(da^t_1\) (resp. \(da^t_2\) is of type \((1, 1)\) with respect to \(I_t\) (resp. \(J_t\)). By the \(\partial\bar{\partial}\)-Lemma for \((1, 1)\) forms (which holds for each of the complex structures \(I_t\) and \(J_t\) for any element in \(\mathcal{M}\)) we conclude that there are unique smooth functions \(f^t_j\),
normalized by \( \int_M f_j^t \Omega^{2n} = 0 \), such that \( \gamma_1^t = dI_t df_1^t \) and \( \gamma_2^t = dJ_t df_2^t \). Observe that Hodge theory with respect to some Kähler metric implies that \( f_j^t \) vary smoothly in \( t \). We now apply Moser’s lemma to each of the families \( \Psi_1^t \) and \( \Psi_2^t \). It shows that 
\[
\Psi_1^t = (\phi_1^t)^*(\text{Im}(\Omega_I)); \quad \Psi_2^t = (\phi_2^t)^*(\text{Im}(\Omega_J)),
\]
where \( \phi_1^t \) are the flows of the time-dependent vector fields \( X_1^t = -\Psi_1^t(I_t, X_1^t, \cdot) \) and \( X_2^t = -\Psi_2^t(J_t, X_2^t, \cdot) \).

We claim that \( X_1^t \) are Hamiltonian with respect to \( \Omega \). Indeed, using (b), we have 
\[
\iota_{X_1^t}(\Omega) = -\Psi_1^t(I_t, X_1^t, \cdot) = I_t \Psi_1^t(I_t, X_1^t) = df_1^t.
\]
and similarly for \( X_2^t \), finishing the first claim.

The second claim follows from Remark 4.1 above, using that \( b_1(M) = 0 \).

Remark 4.2. The setting above extends naturally to the general (not necessarily Kähler) case. On any holomorphic-symplectic manifold \((M^{4n}, I, \Omega_I = \Omega + i\Omega)\), the subgroup of \( \text{Ham}(M, \Omega) \) leaving \( \text{Im}(\Omega_I) = I\Omega \) invariant is a closed subgroup of \( \text{Aut}(M, I) \) with Lie algebra identified with the \( \Omega \)-Hamiltonian vector fields \( X = -\Omega^{-1}(df) \) with \( L_X(I\Omega) = 0 \), i.e. satisfying \( dI df = 0 \). It follows that the latter is trivial when \( M \) is compact, i.e. the stabilizer of a generalized Kähler structure \((g, I, J)\) under the local action of \( \text{Ham}(M, \Omega) \times \text{Ham}(M, \Omega) \) is a discrete subgroup. Thus, generalizing the setting of Lemma 4.1 above, we let \( \mathcal{M} \subset \mathcal{G}_{\alpha,\beta}(\Omega) \) be a path-connected component of an orbit for the local action of \( \text{Ham}(M, \Omega) \times \text{Ham}(M, \Omega) \) on \( \mathcal{G}_{\alpha,\beta}(\Omega) \). The proof of Lemma 4.1 shows that any tangent vector of \( \mathcal{M} \) at a point \((I, J)\) is identified with a pair of exact forms \((-dI df, dJ dg)\) for uniquely determined \( \Omega \)-normalized smooth functions \( f, g \). Thus, we have an identification
\[
(4.5) \quad T_{I,J} \mathcal{M} \cong C^\infty_0(M) \oplus C^\infty_0(M),
\]
which we use throughout this section.

4.2. The Aubin-Yau functional. In this subsection we establish a variational characterization of the generalized Kähler Calabi-Yau equation, in analogy with the Aubin-Yau functional for the classical Calabi-Yau equation. To begin we recall some fundamental aspects of Hamiltonian actions on symplectic manifolds. The group \( \mathcal{H} := \text{Ham}(M, \Omega) \) can be thought as as an infinite dimensional analog of a compact Lie group. Indeed, it is simple by a result of Banyaga [5]. Furthermore, its Lie algebra is identified with the space \( C^\infty_0(M) \) of smooth functions on \( M \) with zero mean with respect to \( \Omega^{2n} \), endowed with the Poisson bracket \( \{f, g\} = \langle \Omega^{-1}, df \wedge dg \rangle := \frac{1}{2} \text{tr} \left( \Omega^{-1} \circ (df \wedge dg) \right) \). Then, the \( L^2 \)-product
\[
(4.6) \quad g(f, g) = \frac{1}{(2\pi)^n} \int_M fg \Omega^{2n}
\]
defined for any \( f, g \in C^\infty_0(M) \) gives rise to an ad-invariant inner-product on \( \text{Lie}(\mathcal{H}) \), and thus to an Ad-invariant Riemannian metric on \( \mathcal{H} \), a property characterizing the finite dimensional compact simple Lie groups. It is known that (see e.g. [11]) the geodesics with respect to \( g \) are the flows of time-independent Hamiltonian vector fields on \((M, \Omega)\).

As above (see Remark 4.2), \( \mathcal{M} \subset \mathcal{G}_{\alpha,\beta}(\Omega) \) will denote a path connected component of an orbit for the local action of \( \mathcal{G} = \mathcal{H} \times \mathcal{H} \) on \( \mathcal{G}_{\alpha,\beta}(\Omega) \) (the whole group \( \mathcal{G} \) acts only locally on \( \mathcal{G}_{\alpha,\beta}(\Omega) \) because of the open condition (c)) and notice that the diagonal action of \( \mathcal{H} \) on \( \mathcal{G} = \mathcal{H} \times \mathcal{H} \) descends to a well-defined global action of \( \mathcal{H} \) on \( \mathcal{M} \), by pulling back each generalized Kähler structure \((g, I, J)\) via the natural right action of the diffeomorphisms on \( M \). We also want to emphasize that our formal manifold \( \mathcal{M} \) is locally a subset of the infinite dimensional Lie group \( \mathcal{G} \), thus we have a non-abelian version of the familiar Calabi-Yau setting.

Using the identification (4.5), for any pair of normalized functions \((f, g) \in C^\infty_0(M) \oplus C^\infty_0(M) = \text{Lie}(\mathcal{G}) \), we have a canonically associated vector field on \( \mathcal{M} \), defined by
\[
(4.7) \quad (f, g)_{I,J} := (-dI df, -dJ dg).
\]
Notice that \((f, g)\) are the vector fields induced by the local (right) action of \(G\) on \(M\) and will play a key role in the computations below. We shall use that such vector fields span each tangent space of \(M\), and

\[
[(f_1, g_1), (f_2, g_2)]_M = (\{f_1, f_2\}, \{g_1, g_2\}),
\]

where \([\cdot, \cdot]_M\) stands for the Lie bracket of vector fields on \(M\), and \(\{\cdot, \cdot\}\) stands for the Poisson bracket with respect to \(\Omega\).

**Definition 4.3.** Given the setup above, we define the 1-form \(\sigma\) on \(M\), defined at a point \((I, J)\) and a tangent vector \((f, g) \in C^\infty_0(M) \oplus C^\infty_0(M) \cong T_{I,J}(M)\) by

\[
\sigma_{I,J}(f, g) := \frac{1}{(2n)!} \int_M (f - g) \left( (F_+)^{2n} - e^\lambda (F_-)^{2n} \right),
\]

where, we recall, \(F_\pm = 2(\pm \Psi_1 - \Psi_2) = -2g(I \pm J)^{-1}\) are the real symplectic forms taming both \(I\) and \(J\), and \(\lambda\) is the cohomological constant associated to \(M\) (cf. Lemma 3.5).

**Lemma 4.4.** Given the setup above, \(\sigma\) is invariant under the diagonal action of \(\text{Ham}(M, \Omega)\) on \(M\) and is closed.

**Proof.** The claim of invariance is obvious. To show \(\sigma\) is closed, by using (4.8) it is enough to show that for any fundamental vector fields \((f_1, g_1)\) and \((f_2, g_2)\) on \(M\) defined via (4.7), we have

\[
0 = d\sigma((f_1, g_1), (f_2, g_2)) = (f_1, g_1) \cdot \sigma(f_2, g_2) - (f_2, g_2) \cdot \sigma(f_1, g_1) - \sigma(\{f_1, f_2\}, \{g_1, g_2\}).
\]

The induced flow \(\phi^t\) by \((f, g)\) on \(M\) is defined at a point \((\Psi_1, \Psi_2)\) by

\[
\phi^t : (\Psi_1, \Psi_2) = ((\phi^t_1)^*(\Psi_1), (\phi^t_2)^*(\Psi_2)),
\]

where \(\phi^t_1\) is the flow of \(-\Omega^{-1}(df)\) and \(\phi^t_2\) is the flow of \(-\Omega^{-1}(dg)\). It follows that the induced symplectic forms \((F_\pm)_t\) satisfy along the flow

\[
\frac{\partial}{\partial t}(F_\pm)_t = \mp 2dI_t df + 2dJ_t dg.
\]
We thus calculate
\[
(f_1, g_1) \cdot \sigma(f_2, g_2) = \int_M (f_2 - g_2) \frac{\partial}{\partial t} \bigg|_{t=0} \left( \frac{(F_+)^{2n} - e^\lambda (F_-)^{2n}}{(2n)!} \right)
\]
\[
= 2 \int_M (f_2 - g_2)(-dIdf_1 + dJdg_1) \wedge (F_+)^{2n-1} - 2e^\lambda \int_M (f_2 - g_2)(dIdf_1 + dJdg_1) \wedge (F_-)^{2n-1} \frac{(2n-1)!}{(2n)!}
\]
\[
= 2 \int_M d(f_2 - g_2) \wedge (Idf_1 - Jdg_1) \wedge (F_+)^{2n-1} + 2e^\lambda \int_M d(f_2 - g_2) \wedge (Idf_1 + Jdg_1) \wedge (F_-)^{2n-1} \frac{(2n-1)!}{(2n)!}
\]
\[
= 2 \int_M \left\langle d(f_2 - g_2) \wedge (Idf_1 - Jdg_1), (F_+)^{-1} \right\rangle \left( \frac{(F_+)^{2n}}{(2n)!} \right)
\]
\[
+ 2e^\lambda \int_M \left\langle d(f_2 - g_2) \wedge (Idf_1 + Jdg_1), (F_-)^{-1} \right\rangle \left( \frac{(F_-)^{2n}}{(2n)!} \right)
\]
\[
= \int_M \left\langle (I + J)d(f_2 - g_2), (Idf_1 - Jdg_1) \right\rangle \left( \frac{(F_+)^{2n}}{(2n)!} \right)
\]
\[
+ e^\lambda \int_M \left\langle (I - J)(d(f_2 - g_2), (Idf_1 + Jdg_1) \right\rangle \left( \frac{(F_-)^{2n}}{(2n)!} \right)
\]
\[
= \frac{1}{2} \int_M \left\langle (I + J)(df_2 - dg_2), (I + J)(df_1 - dg_1) \right\rangle \left( \frac{(F_+)^{2n}}{(2n)!} \right)
\]
\[
+ \frac{1}{2} \int_M \left\langle (I - J)(df_2 - dg_2), (I - J)(df_1 - dg_1) \right\rangle \left( \frac{(F_-)^{2n}}{(2n)!} \right)
\]
\[
+ \frac{1}{2} \int_M \left\{ f_1 + g_1, f_2 - g_2 \right\} \left( \frac{(F_+)^{2n} - e^\lambda (F_-)^{2n}}{(2n)!} \right).
\]

The identity (1.11) then follows easily. \qed

We observe that \( \sigma \) restricts to zero on each orbit of the diagonal action of \( \mathcal{H} = \text{Ham}(M, \Omega) \) on \( \mathcal{M} \), since the tangent vectors to this orbit are generated by the fundamental vector fields \( (f, f) \). Moreover, it vanishes at \( (I, J) \in \mathcal{M} \) if and only if \( (F_+)^{2n} = e^\lambda (F_-)^{2n} \), i.e. \( (I, J) \) is hyper-Kähler \( \text{GK} \) structure in \( \mathcal{M} \), (cf. Theorem 1.1). We thus want to integrate \( \sigma \) on the space \( \mathcal{M}/\mathcal{H} \) and study the critical points of a primitive \( F \) there, which in turn parametrize the hyper-Kähler \( \text{GK} \) structures in \( \mathcal{M} \) modulo the diagonal (isometric) action of \( \text{Ham}(M, \Omega) \). However, as \( \mathcal{H} = \text{Ham}(M, \Omega) \) may in principle have a complicated topology (in particular, \( \pi_1(\mathcal{H}) \neq \{1\} \) in general) one needs to work with the universal cover \( \widetilde{\mathcal{M}} \) of \( \mathcal{M} \) (and of \( \mathcal{M}/\mathcal{H} \)).

**Proposition 4.5.** Given the setup above, there exists a functional
\[
F : \widetilde{\mathcal{M}} \to \mathbb{R}
\]
such that \( \delta F = \pi^* \sigma \), where \( \pi : \widetilde{\mathcal{M}} \to \mathcal{M} \) is the canonical projection. In particular, the critical points of \( F \) correspond to hyper-Kähler metrics.

**Proof.** This follows directly from the fact that \( \sigma \) is closed via formal path integration. \qed

This functional \( F \) is a natural analogue of the Aubin-Yau functional in the classical (abelian) setting. However, to avoid the use of the universal cover, we can also directly define path integrals of \( \sigma \). To define the relevant paths we turn to the more geometrically natural space \( \mathcal{M}/\mathcal{H} \), which can be thought of as an infinite dimensional “orbifold” because of Lemma 4.1. Recall that for any Lie group \( \mathcal{H} \) acting diagonally on \( \mathcal{G} = \mathcal{H} \times \mathcal{H} \), we have the smooth identification
\[
\mathcal{G}/\mathcal{H} \cong \mathcal{H}.
\]
given by the map \([\phi_1, \phi_2] \rightarrow \phi_1^{-1}\phi_2\) with inverse
\begin{equation}
\mathcal{H} \cong \frac{G}{\mathcal{H}}, \\
\phi \mapsto [\text{id}, \phi].
\end{equation}

In view of Lemma 4.11 we can identify \(\mathcal{M}/\mathcal{H}\) with the path-connected component in \(\mathcal{O}_J\) of \(\tilde{J}\)'s such that \((I, \tilde{J})\) satisfy the condition (c). This is precisely the setting of Section 5.3 above.

Recall from Lie theory that for any compact simple Lie group \(H\), the pair \((H \times H, H)\) with \(H\) acting diagonally defines a symmetric pair, i.e. \(H\) can be viewed as a Riemannian-symmetric space with respect to any \(\text{Ad}\)-invariant Riemannian metric (i.e. defined by negative multiple of the Killing form on \text{Lie}(H)). Furthermore, in terms of the isomorphism (4.11), the flows of the left-invariant vector fields of \(H\) are the geodesics of the Riemannian-symmetric space \(H \cong (H \times H)/H\). Similarly, in our infinite dimensional setting, the flows of the fundamental vector fields \((0, g)\) are the geodesics with respect to the \(L^2\) Riemannian metric \(g\) on \(\mathcal{H}\) (see (31)). Thus, the flows of the fundamental vector fields \((0, g)\) acting on \(\mathcal{O}_J\) (and locally on \(\mathcal{M}/\mathcal{H}\)) are the geodesics with respect to the \(L^2\) Riemannian metric \(g\) defined on \(\mathcal{O}_J\) (and hence on \(\mathcal{M}/\mathcal{H}\)) by (11.5) (by using the identification of the tangent space of \(\mathcal{O}_J\) with \(C_0^\infty(M)\)). With this background we define a corresponding functional.

**Definition 4.6.** Given the setup above, fix an \(\Omega\)-normalized time independent function \(g \in C^\infty(M)\), and define a path \((\Psi_1^n = \Psi_1, \Psi_2^n = \phi^n_1\Psi_2)\) in \(\mathcal{M}\), where \(\phi^n_1\) is the flow of \(-\Omega^{-1}(dg)\). We thus have
\[
\Psi^t_1 = 0, \quad \Psi^t_2 = -dJ_t dg.
\]

Along this path we define
\begin{equation}
\delta F_g(t) := \sigma_{J_t, I_t}(0, g) = -\frac{1}{2\pi^2} \int_M g \left( (F_+)_t^{2n} - e^\lambda (F_-)_t^{2n} \right).
\end{equation}

As the next proposition shows, this functional is monotone, corresponding formally to geodesic convexity of \(F\).

**Proposition 4.7.** The function \(\delta F_g(t)\) is monotone nondecreasing.

**Proof.** Using the main calculation of Lemma 4.3 noting that \(f_t \equiv 0\) and \(g_t \equiv g\) we obtain
\[
\frac{d}{dt} \delta F(t) = \frac{1}{2} \int_M \left( (I_t + J_t) dg, (I_t + J_t) dg \right) \frac{(F_+)_t^{2n}}{g_t (2n)!} \\
+ \frac{\lambda}{2} \int_M \left( (I_t - J_t) dg, (I_t - J_t) dg \right) \frac{(F_-)_t^{2n}}{g_t (2n)!} \\
= \frac{1}{2} \int_M |(I_t + J_t) dg|^{2n} \frac{(F_+)_t^{2n}}{(2n)!} + \frac{\lambda}{2} \int_M |(I_t - J_t) dg|^{2n} \frac{(F_-)_t^{2n}}{(2n)!}.
\]

as required. \(\square\)

We close by noting that Proposition 4.7 points towards the uniqueness of a hyper-Kähler metric in \(\mathcal{M}/\mathcal{H}\), shown in Proposition 3.11 and provides an alternative proof assuming geodesic convexity of \(\mathcal{M}/\mathcal{H} \subset \mathcal{O}_J\).

### 4.3 Symplectic form and moment map.

In this subsection we define a symplectic structure on \(\mathcal{M}\), and then compute the moment map of our action. As in the finite dimensional case, the product \(\mathcal{G} = \mathcal{H} \times \mathcal{H}\) admits a natural left-invariant almost-complex structure \(I\), defined on its Lie algebra \(\text{Lie}(\mathcal{G}) = C_0^\infty(M) \oplus C_0^\infty(M)\) by
\[
I(f, g) = (-g, f),
\]
for all \(f, g \in C_0^\infty(M)\). One can easily check that the left and right diagonal action of \(\mathcal{H}\) on \(\mathcal{G}\) preserves \(I\), however \(I\) need not be integrable. The almost-complex structure \(I\) on \(\mathcal{G}\) induces an almost-complex structure on \(\mathcal{M}\), which we still denote by \(I\).
Definition 4.8. Given the setup above, let
\[(4.13) \quad \mathbf{I}_{I,J}(-dIdf, -dJdg) := (dIdg, -dJdf).\]

With this in place, we can follow finite dimensional constructions to define a symplectic form on \( M \) as well. We give the definition, then show that it is indeed symplectic in Lemma 4.10 below.

Definition 4.9. Given the setup above, let
\[(4.14) \quad \Omega := d\mathbf{I}\sigma.\]

Lemma 4.10. Given the setup above, \( \Omega \) is closed and tames \( \mathbf{I} \), so is non-degenerate.

Proof. By definition \( \Omega \) is exact, so is closed. To show that \( \Omega \) tames \( \mathbf{I} \), we first note that by definition, on fundamental vector fields \((f_1, g_1)\) and \((f_2, g_2)\) we have (see (4.1) and (4.8))
\[(4.15) \quad \Omega_{I,J}(f_1, g_1), (f_2, g_2) = d(I\sigma)_{I,J}(f_1, g_1), (f_2, g_2) = (f_1, g_1) \cdot \sigma(g_2, -f_2) - (f_2, g_2) \cdot \sigma(g_1, -f_1) - \sigma(f_1, g_2) - \sigma(f_2, g_1).
\]

Using this and the main calculation of Lemma 4.3 we obtain
\[(4.16) \quad \Omega_{I,J}(f_1, g_1), (f_2, g_2) = \int_M (I + J)df_1, (I + J)dg_2 \cdot \frac{(F_+)^{2n}(F_-)^{2n}}{(2n)!} - \int_M (I + J)df_2, (I + J)dg_1 \cdot \frac{(F_+)^{2n}(F_-)^{2n}}{(2n)!} + e^\lambda \int_M (I - J)df_1, (I - J)dg_2 \cdot \frac{(F_+)^{2n}(F_-)^{2n}}{(2n)!} - e^\lambda \int_M (I - J)df_2, (I - J)dg_1 \cdot \frac{(F_+)^{2n}(F_-)^{2n}}{(2n)!} + \int_M (f_1, g_2) - (f_2, g_1) \cdot \frac{(F_+)^{2n} - e^\lambda(F_-)^{2n}}{(2n)!}.
\]

This implies that
\[
\Omega_{I,J}(f, g), (I(f, g)) = \int_M |(I + J)df|^2_g \cdot \frac{(F_+)^{2n}(F_-)^{2n}}{(2n)!} + \int_M |(I + J)dg|^2_g \cdot \frac{(F_+)^{2n}(F_-)^{2n}}{(2n)!} + e^\lambda \int_M |(I - J)df|^2_g \cdot \frac{(F_+)^{2n}(F_-)^{2n}}{(2n)!} + e^\lambda \int_M |(I - J)dg|^2_g \cdot \frac{(F_+)^{2n}(F_-)^{2n}}{(2n)!} \geq 0.
\]

This shows that \( \Omega \) tames \( \mathbf{I} \), so that it is non-degenerate, as required. \( \square \)

We close by observing that the diagonal action of \( \text{Ham}(M, \Omega) \) is Hamiltonian with respect to \( \Omega \), and then compute the moment map.

Proposition 4.11. The diagonal action of \( \text{Ham}(M, \Omega) \) on \( M \) is Hamiltonian with respect to the symplectic form \( \Omega \), with momentum map
\[(4.17) \quad \mu(I, J) = 2\left((F_+)^{2n} - e^\lambda(F_-)^{2n}\right),
\]
seen as an element of dual vector space of \( C^\infty_0(M) \cong \text{Lie}(\text{Ham}(M, \Omega)) \) via integration over \( M \).

Proof. Given the setup above, let
\[
\mu_f(I, J) := (I\sigma)_{I,J}(f, f) = \sigma(f, -f) = 2\int_M f \left((F_+)^{2n} - e^\lambda(F_-)^{2n}\right)/(2n)!.
\]
Next, we note that (4.16) implies that $\Omega$ is invariant under the diagonal action of $\mathcal{H}$ on $\mathcal{M}$. This also shows that for each fundamental vector field $(f, f)$ on $\mathcal{M}$ (manifestly induced by the diagonal action of $\mathcal{H}$), one has, using (4.16) and the main calculation of Lemma 4.4

$$
\Omega_{I,J}(f, f) = -d\left(I\sigma_{I,J}(f, f)\right) = (f_2, g_2) - \sigma(f, -f).
$$

Lastly, we observe using Lemma 4.3 and (4.17), that

$$(d\mu_I)(g, g) = (g, g) \cdot \mu_I = (g, g) \cdot \sigma(f, -f) = \mu_{\{f, g\}},$$

showing that $\mu$ is $\mathcal{H}$-equivariant. The proposition follows. \(\square\)

We finally notice that, although the almost complex structure $I$ is neither integrable nor compatible with $\Omega$, we can still think of the functional $F$ on $\mathcal{M}/\mathcal{H}$ (whenever it is defined) as a “Kempf-Ness functional”, in the sense that the pull-back of $F$ to $\mathcal{M}$ satisfies $(\text{Id} F)(f, f) = \sigma(f, -f) = \mu_I$ (see (4.12) and (4.17)), and, as it is strictly-convex along the geodesics of $(\mathcal{M}/\mathcal{H}, g)$ by Lemma 4.7 its only critical point on $(\mathcal{M}/\mathcal{H}, g)$ corresponds to the $\mathcal{H}$-orbit of the zero of $\mu$.

5. Nondegenerate Generalized Kähler-Ricci flow

In this section we show that in the setting of generalized Kähler structures with pure spinors, the generalized Kähler Ricci flow preserves the natural variation classes of objects defined above. In particular, we show that the flow reduces to a family of $\Omega$-Hamiltonian diffeomorphisms, which is generated by the Ricci potential. We then show that the Ricci potential itself evolves by the pure time-dependent heat equation along the flow, which leads to a number of delicate a priori estimates along the flow. Then we relate the flow to the GIT picture, in particular showing convexity of $F$, finishing the proof of Theorem 1.3. We end by showing a general result showing that uniform equivalence of the time dependent metrics suffices to show long time existence and convergence to hyper-Kähler for the flow.

5.1. Background on GKRF. In this subsection we review the construction of generalized Kähler Ricci flow (GKRF) from [51]. To begin we review the pluriclosed flow [49], defined by

$$
\frac{\partial}{\partial t}\omega = -2(\rho_B)^{1,1}.
$$

This equation can also be expressed using the curvature of the Chern connection. In [49] we showed that this flow preserves the pluriclosed condition and agrees with Kähler-Ricci flow when the initial data is Kähler. Moreover, the induced pairs of metrics and Bismut torsions $(g_t, H_t)$ satisfy [51], Proposition 6.3,

$$
\frac{\partial}{\partial t}g = -2\text{Re}^g + \frac{1}{2}\mathcal{H} - \mathcal{L}_{g^t}g,
$$

$$
\frac{\partial}{\partial t}H = \Delta_g H - \mathcal{L}_{g^t}H,
$$

where $\mathcal{H}_{X,Y} := \sum_{i,j=1}^{2n} H(X, e_i, e_j)H(Y, e_i, e_j)$. $\text{Re}^g$ is the Ricci tensor of $g$.

As explained in [51], with a generalized Kähler initial condition one can unify the two pluriclosed flow lines given by the distinct pluriclosed structures by removing the gauge terms to arrive at the generalized Kähler-Ricci flow system

$$
\frac{\partial}{\partial t}g = -2\text{Re}^g + \frac{1}{2}\mathcal{H}, \quad \frac{\partial}{\partial t}H = \Delta_g H,
$$

$$
\frac{\partial}{\partial t}I = \mathcal{L}_{g^t}I, \quad \frac{\partial}{\partial t}J = \mathcal{L}_{g^t}J.
$$

In obtaining estimates for the flow, we need to use two different points of view, each of which makes certain estimates possible. Some estimates will use the system (5.3) directly, which we will call a solution “in the RG flow gauge.” Other times it is easier to work with pluriclosed flow directly, so we pull back the flow to the fixed complex manifold $(M^{2n}, I)$. In other words by pulling back the entire system by the family of diffeomorphisms $(\phi_t^I)^{-1}$ we return to pluriclosed flow on $(M^{2n}, I)$,
which encodes everything about the GKRF except the other complex structure, which is given by a certain diffeomorphism pullback. We will refer to this point of view on GKRF as occurring “in the I-fixed gauge.” For concreteness, we record the evolution equations for the GKRF in the I-fixed gauge,

\[ \frac{\partial}{\partial t} \omega_I = -2(\rho_B)^{1,1}_I, \quad \frac{\partial}{\partial t} J = \mathcal{L}_{\phi^*_j - \phi^*_t}. \]

**Remark 5.1.** Two distinct Laplacians are relevant to the analysis to follow. First, we have the Riemannian Laplacian acting on functions

\[ \Delta_g f := \langle \nabla^2 f, g \rangle. \]

Also, we will use the Chern Laplacian associated to a Hermitian structure \((g, I)\), which takes the form

\[ \Delta^C_g f := \langle d \xi^*_I f, \omega_I \rangle = \Delta_g f - \langle df, \theta_I \rangle_g. \]

This formulas clarify an important point we will exploit in several places. In particular, the two Laplacians differ by the natural action of the Lee vector field on the function. This is also the form vanishes, we compute

\[ \frac{\partial}{\partial t} \sigma_t = g_t^{-1}[I, \mathcal{L}_{\phi^*_j - \phi^*_t} J] - 2g_t^{-1}(\rho_B)^{1,1}_I g_t^{-1}[I, J_t] \]

\[ = g_t^{-1}[I, \mathcal{L}_{\phi^*_j - \phi^*_t} J] - 2g_t^{-1}(\rho_B)^{1,1}_I \sigma_t \]

\[ = g_t^{-1}[I, \mathcal{L}_{\phi^*_j - \phi^*_t} J] - g_t^{-1}(\rho_B) I + (\rho_B) I \sigma_t \]

\[ = g_t^{-1}[I, \mathcal{L}_{\phi^*_j - \phi^*_t} J] - (\rho_B) I \sigma_t, \]

where for the last equality we have also used that \( \sigma_t \) anti-commutes with \( I \). To show that this in fact vanishes, we compute

\[ (\mathcal{L}_{\phi^*_j - \phi^*_t} J) \Omega = \mathcal{L}_{\phi^*_j - \phi^*_t} (J \Omega) - J \mathcal{L}_{\phi^*_j - \phi^*_t} \Omega \]

\[ = d \left( J \Omega (\phi^*_j - \phi^*_t) \right) - J d \left( \Omega (\phi^*_j - \phi^*_t) \right) \]

\[ = -\frac{1}{2} d J d \Phi - \frac{1}{2} J d d \Phi \]

\[ = -\frac{1}{2} d J d \Phi \]

\[ = (\rho_B) I, \]

where for the second equality we used that both \( \Omega \) and \( J \Omega \) are closed, for the third equality we used Lemma 3.8 whereas the fifth equality follows from Proposition 3.9. Plugging (5.6) into (5.5) yields (2).
Lastly, to establish (3), using that \( \frac{\partial}{\partial t} \Omega_t = 0 \) as just established, equations (5.4) and (5.6) then imply
\[
\frac{\partial}{\partial t} \Omega_J = \sqrt{-1}(\frac{\partial}{\partial t} J) \Omega = \sqrt{-1} \left( L_{\theta_J^t - \theta_I^t} J \right) \Omega \sqrt{-1}(\rho_B) t,
\]
as required. \( \square \)

With this lemma in place we can establish that the GKRF evolves by \( \Omega \)-Hamiltonian diffeomorphisms.

**Proposition 5.3.** Suppose \((g_t, I, J_t)\) is a smooth solution of the \(I\)-fixed gauge GKRF with nondegenerate initial data. Let \( \phi_t \) denote the flow of the time dependent, \( \Omega \)-Hamiltonian vector field \( X_t := -\frac{i}{2} \sigma d\Phi_t \). Then the induced family of generalized Kähler structures \((g_{\phi_t}, I, J_{\phi_t})\) obtained via Proposition 2.16 coincides with \((g_t, I, J_t)\).

**Proof.** By Lemma 3.8 \( X_t = (\theta_I^t - \theta_J^t)^\sharp \), where \( \theta_I^t \) and \( \theta_J^t \) are the Lee forms along the GKRF and \( ^\sharp \) denotes \( g_t^{-1} \). It thus follows that
\[
\frac{\partial}{\partial t}(J_{\phi_t} - J_t) = 0,
\]
showing that \( J_{\phi_t} = J_t \) as they equal \( J \) at \( t = 0 \). According to Lemma 5.2 and by a computation identical to (5.6), the corresponding symplectic 2-forms \( F_t = 2(\text{Im}(\Omega_I - \Omega_{J_t})) \) and \( F_{\phi_t} = 2(\text{Im}(\Omega_I - (\phi_t)^* \Omega_{J_t})) \) satisfy
\[
\frac{\partial}{\partial t} F_t = \frac{\partial}{\partial t} F_{\phi_t} = dJ_t d\Phi_t,
\]
so that \( F_t = F_{\phi_t} \). Taking (1,1)-part with respect to \( I \) gives \( g_t = g_{\phi_t} \), as required. \( \square \)

### 5.3. A priori estimates.

In this subsection we derive a priori estimates associated to the Ricci potential along solutions to GKRF with nondegenerate initial data. First we show that in these settings the associated Ricci potential satisfies the pure time dependent heat equation. This remarkably simple evolution equation can be exploited to obtain decay of the gradient of the Ricci potential.

**Proposition 5.4.** Let \((M^{4n}, g_t, I, J_t)\) be a solution to GKRF in the \(I\)-fixed gauge with nondegenerate initial data. Let \( \Phi_t = \log \frac{F_{2n}}{F_{-2n}} \) denote the associated family of Ricci potentials. Then
\[
\left( \frac{\partial}{\partial t} - \Delta_g^{C} \right) \Phi = 0.
\]

**Proof.** Let \((F_{\pm})_t = 2(\pm \text{Im}(\Omega_I) - \text{Im}(\Omega_{J_t})) = -2g_t(I \pm J_t)^{-1} \) denote the two symplectic forms given by the construction of Lemma 2.12. Lemma 5.2 yields
\[
\frac{\partial}{\partial t} F_{\pm} = dJ_t d\Phi_t.
\]
On the other hand, it follows by the arguments in Lemma 2.12 that
\[
(F_{\pm})^2 = 2^n (\det(I \mp J))^\frac{1}{2} \text{Re}(\Omega_I)^{2n},
\]
so that, using Lemma 5.2 part (1) again, we derive from (5.8)
\[
\frac{\partial}{\partial t}(F_{\pm})^{2n}/(2n!) = (dJ_t d\Phi_t) \wedge (F_{\pm})^{2n-1}/(2n - 1)!
\]
\[
= \frac{1}{2} \text{tr} \left( (dJ_t d\Phi_t) \circ (F_{\pm})^{-1} \right) \left( (F_{\pm})^{2n}/(2n!) \right)
\]
\[
= \frac{1}{2} \langle dJ_t d\Phi_t, \omega_I \pm \omega_J \rangle_g \left( (F_{\pm})^{2n}/(2n!) \right).
\]
Together with (5.9), this implies
\[
\frac{\partial}{\partial t} \Phi = \langle dJ_t d\Phi_t, \omega_I \rangle_g = \Delta_g \Phi - \langle d\Phi, \theta_I \rangle_g = \Delta_g \Phi - \langle d\Phi, \theta_I \rangle_g = \Delta_g \Phi,
\]
where the second equality follows easily from (2.6) and the third equality follows from Lemma 3.8 and the fact that \([I, J]\) is skew. \( \square \)
Corollary 5.5. Let \((M^{2n}, g_t, I_t, J_t)\) be a solution to GKRF in the nondegenerate case. Then
\[
\sup_{M \times [0, T]} |\Phi| \leq \sup_{M \times \{0\}} |\Phi|.
\]

Proof. This follows immediately from Proposition 5.4 and the maximum principle. \(\square\)

Next we exploit the simple evolution equation of (5.4) to obtain a gradient estimate for the Ricci potential. To begin we recall a basic fact about solutions to the heat equation with background flow along the \((g, H)\) equations of (5.3).

Lemma 5.6. (45) Lemma 4.3) Let \((M^n, g_t, H_t)\) be a solution to the \((g, H)\) evolution equations of (5.3), and let \(\phi_t\) be a solution to
\[
\frac{\partial}{\partial t} \phi = \Delta g_t \phi.
\]
Then
\[
\frac{\partial}{\partial t} |\nabla \phi|^2 = \Delta |\nabla \phi|^2 - 2 |\nabla^2 \phi|^2 - \frac{1}{2} \langle \mathcal{H}, \nabla \phi \otimes \nabla \phi \rangle.
\]

Proposition 5.7. Let \((M^{2n}, g_t, I_t, J_t)\) be a solution to generalized Kähler Ricci flow with nondegenerate initial data. Let \(\Phi_t\) be the associated family of Ricci potentials. Then
\[
\left( \frac{\partial}{\partial t} - \Delta g_t \right) |\nabla \Phi|^2 = -2 |\nabla^2 \Phi|^2 - \frac{1}{2} \langle \mathcal{H}, \nabla \Phi \otimes \nabla \Phi \rangle.
\]

Proof. This follows directly from Proposition 5.4 and Lemma 5.6. \(\square\)

Proposition 5.8. Let \((M^{2n}, g_t, I_t, J_t)\) be a solution to generalized Kähler Ricci flow with nondegenerate initial data. Let \(\Phi_t\) be the associated family of Ricci potentials. Then
\[
\sup_{M \times \{t\}} |\nabla \Phi|^2 \leq t^{-1} \left( \sup_{M \times \{0\}} |\Phi|^2 \right).
\]

Proof. Let
\[
W = t |\nabla \Phi|^2 + \Phi^2.
\]
Combining Proposition 5.4 with Proposition 5.7 we see
\[
\left( \frac{\partial}{\partial t} - \Delta g_t \right) W = -2t |\nabla^2 \Phi|^2 - \frac{t}{2} \langle \mathcal{H}, \nabla \Phi \otimes \nabla \Phi \rangle - |\nabla \Phi|^2.
\]
As the tensor \(\mathcal{H}\) is positive definite, we conclude that \(W\) is a subsolution to the heat equation, and so we conclude from the maximum principle that
\[
\sup_{M \times \{t\}} W \leq \sup_{M \times \{0\}} W = \sup_{M \times \{0\}} \Phi^2.
\]
The result follows upon rearranging this. \(\square\)

5.4. GIT framework and GKRF. In this subsection we observe here monotonicity of the Aubin-Yau differential \(\sigma\) under the generalized Kähler-Ricci flow, and record the proof of Theorem 1.3.

Proposition 5.9. Let \((M^{2n}, g_t, I_t, J_t)\) be a solution to generalized Kähler Ricci flow in the \(I\)-fixed gauge with nondegenerate initial data. Then
\[
\frac{d^2}{dt^2} F(I_t) = \frac{d}{dt} \sigma_{I_t J_t} (0, \Phi_t) = \int_M |\nabla \Phi_t|^2 \left( (F_+)^{2n} + e^\lambda (F_-)^{2n} \right) / (2n)!.}
\]
Proof. By Proposition 5.3, the GKRF is the Hamiltonian flow of $X_t = -\frac{1}{2}\sigma^{-1}d\Phi_t$. Using this and the definition of $F$, the first equation follows directly. For the second, following a calculation similar to Proposition 4.7 yields

$$\frac{d}{dt} \sigma_{I,J}(0, \Phi_t) = \frac{1}{2} \int_M \left| (I + J) d\Phi_t \right|^2 \frac{F_{2n}^2}{(2n)!} + \frac{\Lambda}{4} \int_M \left| (I - J) d\Phi_t \right|^2 \frac{F_{2n}^2}{(2n)!} - \int_M \frac{\partial_t \Phi_t}{\Phi_t} \left( F_{2n}^2 - \epsilon^\lambda F_{2n}^2 \right) / (2n)!.$$  

We use Lemma 5.2 to compute the last term. To this end, as $F_+ = -2(I + J)^{-1}g$, we have $(F_+)^n/(2n)! = 2^{2n}(\det(I + J))^{-\frac{1}{2}}dV_g$, and we compute using (3.8) and Lemma 3.8

$$\int_M \Delta_g \Phi (F_+)^{2n} = 2^{2n} \int_M \Delta_g \Phi (\det(I + J))^{-\frac{1}{2}}dV_g = \frac{1}{2} \int_M \langle d\Phi, d \log \det(I + J) \rangle_g (F_+)^{2n}.$$

(5.15)

A similar calculation yields

$$\int_M \Delta_g \Phi (F_+)^{2n} = \frac{1}{2} \int_M \langle d\Phi, \theta_I + \theta_J \rangle_g (F_+)^{2n} + \frac{1}{2} \int_M \langle d\Phi, (I - J)^{-1}d\Phi \rangle_g (F_+)^{2n}.$$

(5.16)

The claim follows by substituting (5.11) in (5.14), and using (5.15) and (5.16) together with the basic fact

$$g((I - J)X, (I - J)Y) + g((I + J)X, (I + J)Y) = 4g(X,Y).$$

□

Proof of Theorem 1.3. The claims follow directly from Propositions 5.3, 3.10, and 5.9. □

5.5. Conjectural picture. Given the overall picture we have now shown, we make a natural conjecture concerning the GKRF, adjoining Conjecture 3.6.

Conjecture 5.10. Let $(M^{4n}, g, I, J)$ be a nondegenerate generalized Kähler structure. Then the solution to generalized Kähler Ricci flow exists for all time and converges to a hyper-Kähler metric.

Reiterating the introduction, this conjecture is a natural analogue of Cao’s theorem [14] establishing global existence and convergence of Kähler-Ricci flow when $c_1 = 0$. In [45] the second author established the existence portion of this conjecture in the case $n = 1$, as well as a form of convergence to a kind of weak hyper-Kähler structure. We next show a result which indicates the main analytic hurdle left to overcome to establish this conjecture.

Theorem 5.11. Let $(M^{4n}, g, I, J)$ be a nondegenerate generalized Kähler structure. Let $(g_t, I_t, J_t)$ denote the solution to GKRF with this initial condition in the $I$-fixed gauge. Suppose there exists a constant $\Lambda > 0$ such that for all times $t$ in the maximal interval of existence, the solution satisfies

$$\Lambda^{-1}g_0 \leq g_t \leq \Lambda g_0, \quad |\partial \alpha|^2 \leq \Lambda,$$

where $\partial \alpha$ is an associated torsion potential. Then the solution exists for all time and converges to a hyper-Kähler metric $(g_\infty, I, J_\infty)$. Furthermore $J_\infty = \phi_\infty^*J_0$ for some $\phi_\infty \in \text{Ham}(M, \Omega)$. 

□
Proof. To show the long time existence we exploit regularity results from [16], assuming some technical familiarity with that paper. In particular, as our manifold \((M^{4n}, I)\) admits a holomorphic volume form \((\Omega^{2,0})^{2n}\), there exists some background Hermitian metric \(h\) such that \(\rho_C(h) = 0\), obtained by conformal modification of any given Hermitian metric. Thus we can set up a formal background for the pluriclosed flow in this setting with the choices \(\hat{\omega}_t = (\omega_t)_0\), the \(h\) described above, and \(\mu = 0\), and the existence of a maximal solution \(\alpha_t\) to the \((\hat{g}_t, h, \mu)\)-reduced pluriclosed flow. Using the a priori metric bound and torsion potential bound, we can apply (14) Theorem 1.7 to conclude uniform higher order regularity of the time varying metric. The global existence now follows from standard arguments.

Given the smooth global existence and uniform estimates for the metric now in place, we may choose any sequence of times \(\{t_i\} \to \infty\) and obtain a subsequential limit \(\{(\hat{g}_{t_i}, J_{t_i})\} \to (\hat{g}_\infty, J_\infty)\), with convergence in \(C^\infty\). Proposition 3.8 implies that this limiting metric must satisfy \(\Phi_\infty \equiv \lambda\). Proposition 3.9 then implies that the limiting metric is \(I\)-Bismut Ricci flat, hence by Proposition 3.10 \((\hat{g}_\infty, I)\) and \((\hat{g}_\infty, J_\infty)\) belong to the same hyper-Kähler structure. Given this subsequential convergence and moreover the existence of the hyper-Kähler structure, we apply (18) Theorem 1.2 to conclude that the entire flow converges exponentially fast to this same hyper-Kähler structure. With this exponential convergence in place, it follows that the time dependent vector field \(\nabla_\Phi \omega\) driving the family of diffeomorphisms \(\phi_t\) of the \(J\) evolution as well as the normalized Hamiltonian function \(\Phi_t := \Phi_t - \lambda\) converges exponentially fast to zero. Letting \(t := \tan(s), s \in [0, \frac{\pi}{2}]\), we obtain that the reparametrized isotopy \(\tilde{\phi}_t := \phi_{\tan(s)}(\cdot)\) is generated by the \(\Omega\) hamiltonian function \(H_s := \frac{1}{\cos^2 s} \tilde{\Phi}_{\tan(s)}\) which, because of the exponential rate of convergence of \(\tilde{\Phi}\), can be extended to a smooth function defined on \([0, \frac{\pi}{2}] \times M\) by letting \(H_{\frac{\pi}{2}} \equiv 0\). It thus follows that \(\phi_t\) itself converges to a limiting \(\Omega\) Hamiltonian diffeomorphism \(\phi_\infty\) such that \(\phi_\infty^* J_0 = J_\infty\). \(\square\)

6. Global existence and weak convergence on hyper-Kähler manifolds

In this section we prove Theorem 1.5. We break the proof into three phases. First we use the background Kähler structure to set up a simplified reduction of the pluriclosed flow system. Next we establish the weak convergence at infinity. Finally we establish the weak convergence at infinity.

6.1. Reduction of pluriclosed flow. In this section we reduce the pluriclosed flow to a certain system coupling an evolution for a \((1,0)\)-form with a scalar evolution. We exploit the hyper-Kähler background to simplify the background terms needed to define this reduced equation. To begin we exhibit a proposition allowing us to express an arbitrary pluriclosed metric as a Kähler metric plus a term coming from a potential \((1,0)\)-form.

**Proposition 6.1.** Let \(g\) be a pluriclosed Hermitian metric on a compact complex manifold \((M, I)\). If \((M, I)\) admits a Kähler metric, then the Aeppli class of \(\omega_I\) contains a Kähler metric \(\omega\), i.e.

\[
\omega_I = \omega + \partial \bar{\alpha} + \bar{\partial} \alpha
\]

for some \(\alpha \in \Lambda^{1,0}(M, I)\) and a positive definite closed \((1,1)\) form \(\omega\).

**Proof.** By Lemma 2.8 \(\omega_I\) is tamed by a symplectic 2-form \(\psi\). Let \(g'\) be any Kähler metric on \((M, I)\). By the Hodge theorem, the \(g'\)-harmonic part \(\psi_H\) of the closed 2-form \(\psi\) decomposes as the sum \((\psi_H)^{1,1} + (\psi_H)^{(2,0)} + (0,2)\) of harmonic 2-forms of type \((1,1)\) and \((2,0) + (0,2)\), respectively. Letting \(\psi' := \psi - (\psi_H)^{(2,0)} + (0,2)\) we obtain a new closed 2-form with \((\psi')^{1,1} = \psi^{1,1} = \omega_I\) and whose \(g'\)-harmonic part is of type \((1,1)\). We write \(\psi\) instead of \(\psi'\) and we thus have shown that \(\omega_I\) is the \((1,1)\) part of a closed 2-form \(\psi\) which determines a deRham class \([\psi] \in H^{1,1}(M, I)\). We want to prove that this class contains a Kähler form \(\omega\), since then writing \(\psi = \omega + d\alpha\) and taking \((1,1)\) part concludes the proof.
Lemma 6.2. Let \( \omega \) when the background metric is fixed and Kähler, if one has a family of functions description of (6.2) which is parabolic. In particular, as exhibited in ([46] Proposition 3.9) in the case equivalent solutions. In [46] the second author resolved this ambiguity by giving a further reduced Because of this "gauge-invariance," the equation (6.2) is not parabolic, and admits large families of

\[
(6.1) \quad \int_A (\alpha + t[\omega_{\text{Kah}}])^p > 0.
\]

In our situation, \( \alpha = [\psi] \) is represented by an \( I \)-taming symplectic 2-form, and therefore so is any class \( \alpha + t[\omega_{\text{Kah}}] \) for \( t > 0 \). Indeed, the \((1,1)\) part of \( \psi + t\omega_{\text{Kah}} \) is \( \omega_I + t\omega_{\text{Kah}} > 0 \). It thus follows that \( (\psi + t\omega_{\text{Kah}})^p \) defines a strictly positive measure on the regular part \( A_{\text{reg}} \) of \( A \), thus showing that (6.1) holds. Note that we have used Lelong’s theorem to define the integral \( \int_A (\psi + t\omega_{\text{Kah}})^p \), see e.g. ([15] Chapter 4), and more specifically to say that the integral of \( (\psi + t\omega_{\text{Kah}})^p \) over the regular part of \( A \) exists and is independent of the choice of representative of \( \alpha + t[\omega_{\text{Kah}}] \).

We now describe our reduction of the generalized Kähler-Ricci flow in the \( I \)-fixed gauge. Fix \( (M^{4n}, I) \) a Kähler, holomorphic symplectic manifold, and suppose \( (M^{4n}, g, I, J) \) is some nondegenerate generalized Kähler structure. By Proposition 6.1 we can choose \( \alpha \in \Lambda^{1,0}(M, I) \) such that

\[
\omega_I = \omega_{\text{HK}} + \overline{\partial\alpha} + \overline{\partial\alpha},
\]

where \( \omega_{\text{HK}} \) denotes the Kähler form of the hyper-Kähler metric on \( (M^{4n}, I) \) in the Aeppli class of \( \omega_I \). We will always normalize the initial data for our flow in this way without further comment.

Lemma 6.2. Let \( (M, I, g_I) \) be a solution to pluriclosed flow, and suppose \( \alpha_t \in \Lambda^{1,0}(M, I) \) satisfies

\[
\begin{align*}
\partial_t \alpha & = \overline{\partial}^* \omega_t - \frac{\sqrt{-1}}{2} \partial \log \det g_t \\
\alpha(0) & = \alpha_0,
\end{align*}
\]

then the one-parameter family of pluriclosed metrics \( \omega_\alpha = \omega_{\text{HK}} + \overline{\partial}\alpha + \overline{\partial}\alpha \) is the given solution to pluriclosed flow.

Proof. This is a simple modification of ([46] Lemma 3.2). \( \square \)

We note that the natural local decomposition of a pluriclosed metric as \( \omega = \omega_{\text{HK}} + \overline{\partial}\alpha + \overline{\partial}\alpha \) is not canonical, as one may observe that \( \alpha + \partial f \) describes the same Kähler form for \( f \in C^\infty(M, \mathbb{R}) \). Because of this "gauge-invariance," the equation (6.2) is not parabolic, and admits large families of equivalent solutions. In [46] the second author resolved this ambiguity by giving a further reduced description of (6.2) which is parabolic. In particular, as exhibited in ([46] Proposition 3.9) in the case when the background metric is fixed and Kähler, if one has a family of functions \( f_t \) and \((1,0)\)-forms \( \beta_t \) which satisfy

\[
\begin{align*}
\frac{\partial}{\partial t} \beta & = \Delta_{g_t} \beta - T_{g_t} \circ \overline{\partial} \beta \\
\frac{\partial}{\partial t} f & = \Delta_{g_t} f + \text{tr}_{g_t} g_{\text{HK}} + \log \frac{\det g_t}{\det g_{\text{HK}}} \\
\alpha_0 & = \beta_0 - \sqrt{-1} \partial f_0,
\end{align*}
\]

then \( \alpha_t := \beta_t - \sqrt{-1} \partial f_t \) is a solution to (6.2). The term \( T \circ \overline{\partial} \beta \) is defined by

\[
(T \circ \overline{\partial} \beta)_i = g^{ik} g^{jp} T_{ik} \nabla_j \overline{\partial} \beta_p.
\]

6.2. Evolution equations. In this subsection we record several evolution equations and a priori estimates directly associated to a solution to nondegenerate generalized Kähler-Ricci flow. We will in places refer to a solution to (6.3), assuming the setup of §6.1.

Lemma 6.3. Given a solution to (6.3) as above, one has

\[
(\frac{\partial}{\partial t} - \Delta_{g_t}) \frac{\partial f}{\partial t} = \langle \frac{\partial g}{\partial t}, \overline{\partial} \beta + \overline{\partial} \overline{\partial} \beta \rangle.
\]
Proof. This follows directly from the calculation of (45 Proposition 4.10), which we record here for convenience.

\[
\frac{\partial \beta}{\partial t} = \frac{\partial}{\partial t} \left[ n - \text{tr}_{g_t} (\overline{\partial \beta} + \partial \overline{\beta}) + \log \frac{\text{det} g_t}{\text{det} g_{1K}} \right]
\]

\[
= \left\langle \frac{\partial}{\partial t}, \overline{\partial \beta} + \partial \overline{\beta} \right\rangle - \text{tr}_{g_t} \left[ \frac{\partial}{\partial t} (\overline{\partial \beta} + \partial \overline{\beta}) \right] + \text{tr}_{g_t} \frac{\partial \gamma}{\partial t}
\]

\[
= \left\langle \frac{\partial}{\partial t}, \overline{\partial \beta} + \partial \overline{\beta} \right\rangle + \text{tr}_{g_t} \frac{\partial f}{\partial t}
\]

\[
= \Delta_{g_t} f_t + \left\langle \frac{\partial}{\partial t}, \overline{\partial \beta} + \partial \overline{\beta} \right\rangle,
\]
as required.

**Lemma 6.4.** Given a solution to (6.3) as above, one has

\[
\left( \frac{\partial}{\partial t} - \Delta_{g_t} C \right) |\beta|^2 = -|\nabla \beta|^2 - |\nabla \overline{\beta}|^2 - \langle Q, \beta \otimes \overline{\beta} \rangle + 2 \Re \langle \beta, T \circ \overline{\beta} \rangle,
\]

where

\[
Q_{ij} = g^{ik} g^{jp} T_{ik}; T_{jp}.
\]

**Proof.** This is a simple modification of (46 Proposition 4.4).

**Corollary 6.5.** Given a solution to (6.3) as above one has

\[
\frac{\partial}{\partial t} |\beta|^2 \leq \Delta |\beta|^2 - |\nabla \beta|^2.
\]

In particular, one has

\[
\sup_M |\beta|^2 \leq \sup_M |\beta_0|^2.
\]

**Proof.** A similar inequality and estimate was claimed in (46 Corollary 4.5) in the case \( n = 1 \). The central point is to obtain an inequality bounding the final inner product term in (6.5). In general we see using (6.4) that in fact

\[
2 \Re \langle \beta, T \circ \overline{\beta} \rangle = 2 \Re \left( g^{ik} \overline{\beta} (g^{jp} T_{ik}; T_{jp}) \right)
\]

\[
= 2 \Re \left( g^{ik} \overline{\beta} (g^{jp} T_{ik}; T_{jp}) \right)
\]

\[
= 2 \Re \langle \beta, T \circ \overline{\beta} \rangle.
\]

Also we note using (6.6) that

\[
\langle Q, \beta \otimes \overline{\beta} \rangle = \overline{\beta} g^{ij} Q_{ij} \beta \beta_i
\]

\[
= \overline{\beta} g^{ij} \left( g^{jp} T_{ij} \right) \beta \beta_i
\]

\[
= \overline{\beta} g^{ij} \left( g^{jp} T_{ij} \right) \beta \beta_i
\]

\[
= |\beta|^2 |T|^2.
\]

Using these calculations we see by the Cauchy-Schwarz inequality that

\[
\frac{\partial}{\partial t} |\beta|^2 = \Delta |\beta|^2 - |\nabla \beta|^2 - |\nabla \overline{\beta}|^2 - \langle Q, \beta \otimes \overline{\beta} \rangle + 2 \Re \langle \beta, T \circ \overline{\beta} \rangle
\]

\[
= \Delta |\beta|^2 - |\nabla \beta|^2 - |\nabla \overline{\beta}|^2 - |\beta|^2 |T|^2 + 2 \Re \langle \nabla \beta, \beta_T \rangle
\]

\[
\leq \Delta |\beta|^2 - |\nabla \beta|^2,
\]
as required. The estimate (6.8) now follows directly from the maximum principle.
Lemma 6.8. Given a solution to (6.2) as above, one has
\[(\frac{\partial}{\partial t} - \Delta^C_{g_t}) |\partial \alpha|^2 = - |\nabla \partial \alpha|^2 - |T_{g_t}|^2 - 2 \langle Q, \partial \alpha \otimes \overline{\partial} \alpha \rangle.\]
In particular,
\[(\frac{\partial}{\partial t} - \Delta^C_{g_t}) |\partial \alpha|^2 \leq \sup_M |\partial \alpha|^2_{g_t} \leq \sup_M |\partial \alpha|^2_{g_0}.\]
Proof. It follows from the proof of ([46] Proposition 4.9), with \(\hat{g} = g_{HK}\) a Kähler metric, and \(\mu = 0\), that
\[\frac{\partial}{\partial t} \partial \alpha = \Delta_{g_t} \partial \alpha - \text{tr}_{g_t} \nabla^{g_t} T_{\hat{g}} = \Delta_{g_t} \partial \alpha.\]
Equation (6.9) now follows from ([47] Lemma 4.7). The estimate (6.10) follows directly from the maximum principle.

Next we record a few basic evolution equations associated to solutions of pluriclosed flow.

Lemma 6.7. Let \((M^{2n}, I, g_t)\) be a solution to pluriclosed flow, and suppose \(h\) is another Hermitian metric on \((M, I)\). Then
\[\left(\frac{\partial}{\partial t} - \Delta^C_{g_t}\right) \text{tr}_h g = - |\Upsilon(g, h)|^2_{g^{-1}, h^{-1}, g} + \text{tr}_h Q - \det_h (\Omega^h)^{T_k}_{p} g_{k\bar{l}}.\]

Lemma 6.8. Let \((M^{2n}, I, g_t)\) be a solution to pluriclosed flow, and let \(h\) denote another Hermitian metric on \((M, I)\). Then
\[\left(\frac{\partial}{\partial t} - \Delta^C_{g_t}\right) \text{tr}_h g = - |\Upsilon(g, h)|^2_{g^{-1}, h^{-1}, g} + \text{tr}_h Q - \det_h (\Omega^h)^{T_k}_{p} g_{k\bar{l}},\]
where \(\Upsilon(g, h) = \nabla^C_g - \nabla^C_h\) is the difference of the Chern connections associated to \(g\) and \(h\).

Proof. To begin we establish a general evolution equation for pluriclosed flow which is implicit in ([50] Proposition 2.4). Using the formula for pluriclosed flow in complex coordinates (cf. [49] (1.3)) we obtain
\[\frac{\partial}{\partial t} \text{tr}_h g = \frac{\partial}{\partial t} h^{T_i} g_{T_i} = h^{T_i} \left[ g^{\overline{p} q} g_{\overline{p} q} - g^{\overline{p} q} g_{\overline{p} q} g_{r \sigma} g_{r \sigma} + Q_{\overline{p} q} \right].\]

On the other hand
\[\Delta \text{tr}_h g = g^{p \overline{q}} \left[ h^{T_i} g_{T_i} \right]_{p \overline{q}} = g^{p \overline{q}} \left[ - h^{T_i} h_{k \sigma} h^{T_i} g_{\overline{q} \sigma} + h^{T_i} g_{T_i} \right]_{p \overline{q}} = g^{p \overline{q}} \left[ h^{T_i} h_{r \sigma} g^{T_i} h_{k \sigma} h^{T_i} g_{\overline{q} \sigma} - h^{T_i} h_{k \sigma} g^{T_i} g_{\overline{q} \sigma} + h^{T_i} h_{r \sigma} h^{T_i} h_{k \sigma} h^{T_i} g_{\overline{q} \sigma} \right].\]

Combining the above calculations yields
\[\left(\frac{\partial}{\partial t} - \Delta\right) \text{tr}_h g = - h^{T_i} g^{p \overline{q}} g_{p \overline{q}} g_{r \sigma} g_{r \sigma} + \text{tr}_h Q - g^{p \overline{q}} \left[ h^{T_i} h_{r \sigma} h^{T_i} h_{k \sigma} h^{T_i} g_{\overline{q} \sigma} - h^{T_i} h_{k \sigma} g^{T_i} g_{\overline{q} \sigma} + h^{T_i} h_{r \sigma} h^{T_i} h_{k \sigma} h^{T_i} g_{\overline{q} \sigma} \right] = - |\Upsilon(g, h)|^2_{g^{-1}, h^{-1}, g} + \text{tr}_h Q - \det_h (\Omega^h)^{T_k}_{p} g_{k\bar{l}},\]
as required.

Lemma 6.9. Let \((M^{2n}, I, g_t)\) be a solution to pluriclosed flow, and let \(h\) denote another Hermitian metric on \((M, I)\). Then there exists a constant \(C\) depending on \(h\) such that
\[\left(\frac{\partial}{\partial t} - \Delta^C_{g_t}\right) \log \text{tr}_h g \leq |T|^2_g + C \text{tr}_h h.\]
Proof. A direct calculation using Lemma 6.8 implies that
\[
\left(\frac{\partial}{\partial t} - \Delta^C_g\right) \log \text{tr}_h g = \frac{1}{\text{tr}_h g} \left[-\left|\mathbf{Y}(g, h)\right|_{g^{-1}, h^{-1}, g}^2 + \text{tr}_h Q - \text{tr}_h h \text{tr}_h g + n \text{tr}_h g\right] + \frac{\left|\nabla \text{tr}_h g\right|^2}{\text{tr}_h g^2}.
\]
Since \( h \) is Kähler we may choose complex normal coordinates at any given point such that
\[
h_\mathbf{\sigma} = \delta_{ij}, \quad g_\mathbf{\sigma} = g_\mathbf{\sigma} \delta_{ij}, \quad \partial_{h_\mathbf{\sigma}} = 0.
\]
Then we may estimate using the Cauchy-Schwarz inequality
\[
\frac{|\nabla \text{tr}_h g|^2}{\text{tr}_h g^2} = \left(\sum_i g_{i\mathbf{\sigma}}\right)^{-1} g^{ij} \nabla_j \text{tr}_h g \nabla_i \text{tr}_h g
\]
\[
= \left(\sum_i g_{i\mathbf{\sigma}}\right)^{-1} \sum_j g^{ij} \left[\sum_k \nabla_j g_{k\mathbf{\sigma}} \sum_l \nabla_l g_{j\mathbf{\sigma}}\right]
\]
\[
= \left(\sum_i g_{i\mathbf{\sigma}}\right)^{-1} \sum_j \left[\sum_k \left(\left(g^{jj}\right)^{1/2} \mathbf{Y}_{jk}^k (g_{kk})^{1/2}\right) (g_{kk})^{1/2} \sum_l \left(\left(g^{jj}\right)^{1/2} \mathbf{Y}_{jl}^l (g_{ll})^{1/2}\right) g_{lj}\right]
\]
\[
\leq \left(\sum_i g_{i\mathbf{\sigma}}\right)^{-1} \left(\sum_{j,k} g^{jj} g_{kk}^k \sum_l \mathbf{Y}_{jk}^l \mathbf{Y}_{jl}^l\right)^{1/2} \left(\sum_k g_{kk}\right)^{1/2} \left(\sum_l g_{ll}\right)^{1/2}
\]
\[
\leq |\mathbf{Y}(g, h)|_{g^{-1}, h^{-1}, g}^2.
\]
Moreover, note that in these same coordinates it follows that
\[
(\text{tr}_h g)^{-1} \text{tr}_h Q = \left(\sum_i g_{i\mathbf{\sigma}}\right)^{-1} \sum_j Q_{j\mathbf{\sigma}} \leq \sum_i g_{i\mathbf{\sigma}}^{-1} Q_{i\mathbf{\sigma}} = |T|^2.
\]
The result follows. \(\square\)

6.3. Global existence. In this subsection we establish global existence of the generalized Kähler-Ricci flow on a hyper-Kähler background.

Proposition 6.10. Let \((M^{4n}, g, I, J)\) be a nondegenerate generalized Kähler manifold, and suppose \( I \) is a Kähler complex structure. Then the solution to generalized Kähler Ricci flow with initial condition \((g, I, J)\) exists on \([0, \infty)\).

Proof. We use a solution \((\beta_t, f_t)\) to (6.3) as above. Our main goal is to establish, for each finite time interval, uniform upper and lower bounds for the metric tensor and the torsion potential estimate. The proposition will then follow along the lines of the proof of Theorem 5.11. To that end we first apply Lemma 6.7 choosing \( h = g_{\text{HK}} \), so that \( \rho_C(h) = \rho_C(g_{\text{HK}}) = 0 \), to obtain
\[
\left(\frac{\partial}{\partial t} - \Delta^C_{g_t}\right) \log \frac{\det g_t}{\det g_{\text{HK}}} = |T|^2 \geq 0.
\]
The maximum principle then implies
\[
(6.11) \quad \inf_{M \times \{t\}} \log \frac{\det g_t}{\det g_{\text{HK}}} \geq \inf_{M \times \{0\}} \log \frac{\det g_t}{\det g_{\text{HK}}}.
\]
Also, we can set
\[
W_1 = \log \frac{\det g_t}{\det g_{\text{HK}}} + |\partial \alpha|^2,
\]
and then combining Lemma 6.6 with Lemma 6.7 we obtain
\[
\left(\frac{\partial}{\partial t} - \Delta^C_{g_t}\right) W_1 \leq 0.
\]
The maximum principle then implies
\begin{equation}
\sup_{M \times \{t\}} \log \frac{\det g}{\det g_{HK}} \leq \sup_{M \times \{t\}} W_1 \leq \sup_{M \times \{0\}} W_1 \leq C.
\end{equation}
Hence we have established uniform upper and lower bounds on the volume form. To finish the proof of uniform metric equivalence it suffices to show an upper bound for the metric.

To that end we let
\[ W_2 = \log \text{tr}_{g_{HK}} g + |\partial \alpha|^2 - Af, \]
where \( A \) is a constant to be determined. Combining Lemmas 6.6, 6.9 and recalling (6.3) we see
\[ (\frac{\partial}{\partial t} - \Delta g^C) W_2 \leq (C - A) \text{tr}_{g_{HK}} - A \log \frac{\det g}{\det g_{HK}}. \]
Choosing \( A \) sufficiently large with respect to \( C \) and recalling the previously established bound for the volume form yields
\[ (\frac{\partial}{\partial t} - \Delta g^C) W_2 \leq C. \]
Applying the maximum principle we see that
\[ \sup_{M \times \{t\}} \log \text{tr}_{g_{HK}} g - Af \leq \sup_{M \times \{t\}} W_2 \leq \sup_{M \times \{0\}} W_2 + Ct \leq C(1 + t). \]
Rearranging yields
\begin{equation}
\sup_{M \times \{t\}} \text{tr}_{g_{HK}} g \leq \sup_{M \times \{t\}} e^{C(1+t+f)}.
\end{equation}
Hence, to finish the proof it suffices to estimate \( f \). Since we are only concerned with finite time intervals, it suffices to estimate \( \frac{\partial f}{\partial t} \).

Thus set
\[ W_3 = \frac{\partial f}{\partial t} + |\beta|^2 - A_1 \log \frac{\det g}{\det g_{HK}} + A_2 |\nabla \Phi|^2, \]
where \( A_1 \) and \( A_2 \) are positive constants to be determined below. Combining Lemmas 6.3, 6.4 and 6.7 with Proposition 5.7 (n.b. the conversion from Riemannian Laplacian to Chern Laplacian when changing from B-field gauge to I-fixed gauge), we obtain
\[ (\frac{\partial}{\partial t} - \Delta g^C) W_3 = \left\langle \frac{\partial g}{\partial t}, \bar{\beta} + \partial \bar{\beta} \right\rangle + \left[ -|\nabla \beta|^2 - |\nabla \bar{\beta}|^2 - \left\langle Q, \beta \otimes \bar{\beta} \right\rangle + 2 \Re \left\langle \beta, T \circ \bar{\beta} \right\rangle \right] - A_1 |T|^2 + A_2 \left[ -2 |\nabla^2 \Phi|^2 - \frac{1}{2} \langle \mathcal{H}, \nabla \Phi \otimes \nabla \Phi \rangle \right]. \]
First observe that by the Cauchy-Schwarz inequality and the a priori estimate for \( \beta \) we have
\[ 2 \Re \left\langle \beta, T \circ \bar{\beta} \right\rangle \leq C |T| |\nabla \beta| \leq \frac{1}{2} |\nabla \beta|^2 + C |T|^2. \]
Thus choosing \( A_1 \) sufficiently large and applying the Cauchy-Schwarz inequality to \( \left\langle \frac{\partial g}{\partial t}, \bar{\beta} + \partial \bar{\beta} \right\rangle \), and dropping negative terms we obtain
\begin{equation}
(\frac{\partial}{\partial t} - \Delta g^C) W_3 \leq \left| \frac{\partial g}{\partial t} \right|^2 - \frac{A_1}{2} |T|^2 - 2A_2 |\nabla^2 \Phi|^2.
\end{equation}
Now note from Proposition 3.9 that \( \frac{\partial g}{\partial t} \) can be expressed as the (1, 1) projection of the J-Chern Hessian of the Ricci potential \( \Phi \). Combining this with (2.3), there is a uniform constant \( C \) such that
\begin{equation}
\left| \frac{\partial g}{\partial t} \right|^2 \leq C \left[ |\nabla^2 \Phi|^2 + |T|^2 |\nabla \Phi|^2 \right].
\end{equation}
Since $|\nabla \Phi|^2$ is uniformly bounded by Proposition 5.8, plugging (6.15) into (6.14) and choosing $A_1$ and $A_2$ sufficiently large with respect to the initial data, we have
\begin{equation}
\left( \frac{\partial}{\partial t} - \Delta_{g_t}^C \right) W_3 \leq 0.
\end{equation}
The a priori estimate for $W_3$ follows by the maximum principle. Since the volume form is bounded below uniformly, this implies an upper bound for $\frac{\partial f}{\partial t}$ as required. A directly analogous estimate can yield a lower bound for $\frac{\partial f}{\partial t}$, finishing the proof. \hfill \Box

6.4. Weak convergence. In this subsection we finish the proof of Theorem 1.5. First we establish the convergence of the Ricci potential in $H^2_1$. Then we derive specialized estimates to get the convergence to a closed current in the limit.

**Proposition 6.11.** Let $(M^{4n}, g, I, J)$ be a nondegenerate generalized Kähler manifold, and suppose that $I$ is a Kähler complex structure. If the solution to generalized Kähler Ricci flow with initial condition $(g, I, J)$ exists on $[0, \infty)$, then there exists a constant $C$ such that
$$||\Phi - \lambda||^2_{H^2_1} \leq C t^{-1}.$$  

**Proof.** By the Poincaré Lemma for some background Kähler metric $(\tilde{g}, I, \tilde{\omega})$, it suffices to obtain the estimate for $||d\Phi||^2_{L^2_3}$. We proceed to estimate, using properties of exterior algebra and Proposition 5.8,
\begin{align*}
||d\Phi||^2_{L^2_3(\tilde{g})} &= \int_M \sqrt{-1} \partial \bar{\partial} \Phi \wedge \tilde{\omega}^{(m-1)} \\
&\leq C \int_M |d\Phi|^2 |\omega_t \wedge \tilde{\omega}^{(m-1)}| \\
&\leq C t^{-1} \int_M |\omega_t \wedge \tilde{\omega}^{(m-1)}| \\
&= C t^{-1} \int_M (\omega_{HK} + \partial \overline{\alpha}_t + \overline{\partial} \alpha_t) \wedge \tilde{\omega}^{(m-1)} \\
&= C t^{-1},
\end{align*}
where the last line follows by Stokes Theorem and the fact that $\tilde{\omega}$ is a closed $(1,1)$-form, and we recall that $m = 2n$ is the complex dimension of $(M, I)$. \hfill \Box

**Proposition 6.12.** Let $(M^{4n}, g, I, J)$ be a nondegenerate generalized Kähler manifold, and suppose $I$ is a Kähler complex structure. Let $(g_t, I_t, J_t)$ be the solution to generalized Kähler Ricci flow in the $I$-fixed gauge. Choose $A > 0$ so that
$$1 \leq W := \left( A - \log \frac{\det g}{\det g_{HK}} \right) \leq C,$$
which exists by the estimates (6.11) and (6.12) of Proposition 6.10. For $p > 1$ sufficiently large, one has
$$\frac{d}{dt} \int_M W^p dV_g \leq -\int_M |T|^2 dV_g + Ct^{-1}.$$  

**Proof.** An elementary calculation yields
$$\left( \frac{\partial}{\partial t} - \Delta_C \right) f^p = p f^{p-1} \left( \frac{\partial}{\partial t} - \Delta_C \right) f - p(p-1) f^{p-2} |\nabla f|^2.$$
Note also that (6.13) implies the general integration identity
$$\int_M \Delta_C f dV_g = \int_M f \left( |\theta|^2 - \frac{1}{2} |T|^2 \right) dV_g.$$
Applying Proposition 5.8. The proposition follows.

\[ \frac{d}{dt} \int_M W^p dV_g = \int_M \left[ \left( \frac{d}{dt} W^p \right) + W^p (\text{tr}_{\omega_j} dJd\Phi) \right] dV_g \]

\[ = \int_M \left\{ \left[ \Delta C W^p + pW^{p-1} \left( - |T|^2 \right) - p(p-1)W^{p-2} |\nabla W|^2 \right] + W^p (\text{tr}_{\omega_j} dJd\Phi) \right\} dV_g \]

\[ \leq \int_M \left\{ \left( \frac{2p}{C} W^p |T|^2 - p(p-1)W^{p-2} |\nabla W|^2 + W^p \text{tr}_{\omega_j} dJd\Phi \right) \right\} dV_g \]

\[ \leq \int_M \left\{ \frac{3}{2p} W^p |T|^2 - p(p-1)W^{p-2} |\nabla W|^2 + W^p \text{tr}_{\omega_j} dJd\Phi \right\} dV_g \]

\[ = A_1 + A_2 + A_3. \]

Note that the second inequality follows by choosing \( p \) large with respect to the bounds on \( W \). It remains to estimate \( A_3 \). To that end we have

\[ A_3 = \int_M W^p dJd\Phi \wedge \omega_I^{n-1} \]

\[ = \int_M pW^{p-1} dW \wedge Jd\Phi \wedge \omega_I^{n-1} + W^p Jd\Phi \wedge d\omega_I \wedge \omega_I^{n-2} \]

\[ \leq \delta_1 \int_M pW^{p-2} |\nabla W|^2 dV_g + \delta_2 \int_M W^p |T|^2 dV_g + \delta_2 \int_M |\nabla \Phi|^2 dV_g \]

\[ \leq \frac{1}{2} A_1 + \frac{1}{2} A_2 + C \delta_1^{-1}, \]

where the last line follows by choosing \( \delta_1 \) and \( \delta_2 \) small with respect to universal constants, then applying Proposition 5.8. The proposition follows.

Lemma 6.13. Let \((M^{2m}, \omega, J)\) be a Hermitian manifold, with \( \omega' \) another Hermitian metric. Given \( \mu \in \wedge^{m,m-2}(M, I) \) one has

\[ ||\mu||_{L^2(g)}^2 \leq \sup_M |\mu|_{\omega'}^2 \left( \frac{\det g}{\det g'} \right)^{-1} \int_M \omega \wedge \omega \wedge (\omega')^{m-2}. \]

Proof. Fix a point \( p \in M \) and choose complex coordinates such that

\[ \omega'_{ij} = \delta_{ij}, \quad \omega'_{ij} = \lambda_i \delta_{ij} \]

Then we observe that

\[ |\mu|_{\omega'}^2 dV_g = \mu_{i_1 \ldots i_m j_1 \ldots j_{m-2}} g^1_{j_1 \ldots j_{m-2}} \ldots g^m_{i_1 \ldots i_m j_1 \ldots j_{m-2}} \left( \prod_{k=1}^m g_{k k} dV_{g'} \right) \]

\[ \leq |\mu|_{\omega'}^2 \sum_{1 \leq j_1 < \ldots < j_{m-2} \leq m} \lambda_{j_1}^{-1} \ldots \lambda_{j_{m-2}}^{-1} dV_{g'} \]

\[ = |\mu|_{\omega'}^2 \left( \frac{\det g}{\det g'} \right)^{-1} \sum_{1 \leq j_1 < \ldots < j_{m-2} \leq m} \lambda_{j_1}^{-1} \ldots \lambda_{j_{m-2}}^{-1} dV_{g'} \]

\[ = |\mu|_{\omega'}^2 \left( \frac{\det g}{\det g'} \right)^{-1} \sum_{1 \leq j_1 < j_2 \leq m} \lambda_{j_1} \lambda_{j_1} dV_{g'} \]

\[ = |\mu|_{\omega'}^2 \left( \frac{\det g}{\det g'} \right)^{-1} \omega \wedge \omega \wedge (\omega')^{m-2}. \]

Integrating yields the result. □
Proposition 6.14. Let \((M^{4n}, g, I, J)\) be a nondegenerate generalized Kähler manifold, and suppose \(I\) is a Kähler complex structure. Let \((g_t, I_t, J_t)\) be the solution to generalized Kähler Ricci flow in the \(I\)-fixed gauge. Suppose \(\{t_j\} \to \infty\) is a sequence such that
\[
\lim_{j \to \infty} (\omega_I)_{t_j} = \omega_I^\infty, \quad \lim_{j \to \infty} \int_M |T|_{g_{t_j}}^2 \, dV_{g_{t_j}} = 0,
\]
where \(\omega_I^\infty\) is a positive \((1, 1)\) current and the convergence is in the topology of currents. Then \(\omega_I^\infty\) is closed.

Proof. We will denote \((g_t, (\omega_I)_t)\) by \((g, \omega)\) in this proof for notational simplicity. We fix a form \(\mu \in \wedge^{m-1, m-2}(M, I)\) and compute
\[
\int_M \omega \wedge \overline{\partial} \mu = \int_M \overline{\partial} \omega \wedge \mu
= \int_M \overline{\partial} \beta \wedge \mu
= \int_M \beta \wedge \overline{\partial} \mu
\leq ||\beta||_{L^2(g)} ||\partial \mu||_{L^2(g)}.
\]
Note first the estimate using Lemma 6.13 and the fact that the deRham class of the corresponding symplectic form \(F_+ = (F_+)_t\) does not change along the flow
\[
||\partial \mu||_{L^2(g)}^2 \leq C \sup_M \left( \frac{\det g}{\det g_{HK}} \right)^{-1} \int_M \omega \wedge \omega_{HK}^{-2}
 \leq C \sup_M \left( \frac{\det g}{\det g_{HK}} \right)^{-1} \int_M F_+ \wedge F_+ \wedge \omega_{HK}^{-2} \leq C.
\]
Also we estimate using Corollary 6.5 and Lemma 6.6
\[
||\overline{\partial} \beta||_{L^2(g)}^2 = \int_M \overline{\partial} \beta \wedge \partial \beta \wedge \omega^{m-2}
= \int_M \overline{\partial} \beta \wedge \partial \beta \wedge \omega^{m-2} + (m-2) \int_M \beta \wedge \partial \beta \wedge \overline{\partial} \omega \wedge \omega^{m-3}
\leq \sup_M |\beta| \int_M |T| \, dV_g + C \sup_M |\beta| ||\overline{\partial} \beta|| \int_M |T| \, dV_g
\leq C \left( \int_M |T|^2 \, dV_g \right)^{1/2} \text{Vol}(g)^{1/2}
= o(j^{-1}).
\]
Combining these estimates it follows that
\[
\int_M \omega_I^\infty \wedge \overline{\partial} \mu = 0,
\]
as required. \(\square\)

Proposition 6.15. Let \((M^{4n}, g, I, J)\) be a nondegenerate generalized Kähler manifold, and suppose \(I\) is a Kähler complex structure. Let \((g_t, I_t, J_t)\) be the solution to generalized Kähler Ricci flow in the \(I\)-fixed gauge. There exists a sequence \(\{t_j\} \to \infty\) and a closed positive current \(\omega_I^\infty\) such that
\[
\lim_{j \to \infty} (\omega_I)_{t_j} = \omega_I^\infty.
\]
Proof. First, since the quantity $W$ of Proposition 6.12 is positive, it follows that there must exist some sequence $\{t_j\} \to \infty$ such that
\[
\lim_{j \to \infty} \int_M |T|_{g_{t_j}}^2 \, dV_{g_{t_j}} = 0.
\]
Indeed, if $\liminf_{t \to \infty} \int_M |T|^2 \, dV_g = \delta > 0$, then for sufficiently large $t > 0$ Proposition 6.12 yields
\[
\frac{d}{dt} \int_M W^p \, dV_g \leq -\frac{\delta}{2},
\]
which eventually yields a negative value for $\int_M W^p \, dV_g$, a contradiction. Furthermore, by Lemma 6.2 we have $(\omega_I)_t = \omega_{HK} + \partial\alpha_t + \bar{\partial}\alpha_t$, and therefore
\[
\int_M (\omega_I)_t \wedge \omega_{HK}^{n-1} = \int_M \omega_{HK}^{2n}.
\]
By Banach-Alaoglu Theorem (see [16], Chapter III Proposition 1.23), the sequence $(\omega_I)_{t_j}$ weakly subsequently converges to a positive current $\omega_I^\infty$. We have thus obtained a sequence of times satisfying the hypotheses of Proposition 6.14 and hence $\omega_I^\infty$ is closed. □

Proof of Theorem 7.5. The claims of the theorem follow from Propositions 6.10, 6.11 and 6.15. □

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