QUANTUM MECHANICS AT THE PLANCK SCALE*

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ABSTRACT

Usual quantum mechanics requires a fixed, background, spacetime geometry and its associated causal structure. A generalization of the usual theory may therefore be needed at the Planck scale for quantum theories of gravity in which spacetime geometry is a quantum variable. The elements of generalized quantum theory are briefly reviewed and illustrated by generalizations of usual quantum theory that incorporate spacetime alternatives, gauge degrees of freedom, and histories that move forward and backward in time. A generalized quantum framework for cosmological spacetime geometry is sketched. This theory is in fully four-dimensional form and free from the need for a fixed causal structure. Usual quantum mechanics is recovered as an approximation to this more general framework that is appropriate in those situations where spacetime geometry behaves classically.

1. Introduction

Generalizations of usual quantum theory are required for physics at the Planck Scale. That is because the usual framework depends strongly on an assumed fixed, background, spacetime geometry. States are defined on spacelike surfaces in this geometry. States evolve unitarily in between such surfaces in the absence of measurement and by state vector reduction on them when a measurement occurs. The inner product between states is defined by integrals over fields on a spacelike surface. These are just some of the ways that a fixed spacetime geometry is central to the usual formulation of quantum mechanics. However, at the Planck scale, spacetime geometry is not fixed, but a quantum dynamical variable, — fluctuating and without definite value. Given two nearby points on a spacetime manifold it is not possible to say whether they are spacelike separated or not. Rather, the amplitudes for prediction are sums over different metrics on the manifold. Points separated by a spacelike interval in one metric may be timelike separated in another that con-

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tributes just as significantly to the sum. For this reason, usual quantum mechanics needs to be generalized to accommodate quantum spacetime. There are many approaches to this generalization whose difficulties have been lucidly reviewed in\textsuperscript{2,3,4,5}. This lecture describes another approach — the effort to provide a generalization of usual quantum mechanics that is in fully spacetime form and does not require a fixed spacetime geometry but which yields the usual formulation approximately in those physical situations where spacetime geometry is approximately fixed.

The rules for the calculation of $S$-matrix elements in field theory may be summarized without reference to states on spacelike surfaces except in asymptotic regions. It is thus possible to carry out investigations of the dynamics of quantum gravity as expressed by $S$-matrix elements without addressing the issues of generalization raised above. However, one of the principal applications of quantum gravity is to the very early universe where the physics of the Planck scale becomes important. In quantum cosmology we cannot evade the challenge of generalizing quantum theory. We live in the middle of this particular experiment.

The application of quantum mechanics to cosmology also requires another kind of generalization of the usual formulation. Usual quantum mechanics predicts the outcomes of “measurements” carried out on a system by another system outside it. But in cosmology there is no system outside. Cosmology requires a quantum mechanics of closed systems that is a generalization of the usual theory. Recent years have seen the emergence of such a generalization built on the work of Everett, Zeh, Zurek, Griffiths, Omnès, Gell-Mann, and many others.\textsuperscript{a} The most general predictions of this formulation of quantum mechanics are the probabilities of the individual members of sets of alternative histories of the closed system. Consistency of probability sum rules is the criterion determining the sets of histories which may be assigned probabilities rather than any notion of measurement. The absence of quantum mechanical interference between histories, or decoherence, is the sufficient condition for this consistency. The initial condition of the closed system and Hamiltonian determine which sets of histories decohere rather than the action of any external observer.

There is a relation between these two required generalizations. By abstracting the principles of the quantum mechanics of closed systems one arrives at a more general framework for quantum prediction — generalized quantum mechanics. Within generalized quantum mechanics one can construct fully spacetime quantum theories which do not require a fixed background spacetime. In this lecture we shall review

\footnote{For an introduction from the author’s perspective see Ref. [6].}
some simple examples of generalized quantum theories and sketch a generalized quantum mechanics of spacetime geometry. Our discussion is necessarily brief; for a more extensive one see Ref. [1].

II. Generalized Quantum Theory

Not every set of histories that may be described can be consistently assigned probabilities in quantum theory. In the two slit experiment it would be inconsistent to assign probabilities to the alternatives that the electron went through slit \( A \) or slit \( B \) on its way to detection at a point \( y \) on the screen in the absence of a measurement of which happened. The probability to arrive at \( y \) would not be the sum of the probability to go through \( A \) and arrive at \( y \), and the probability to go through \( B \) and arrive at \( y \), because of quantum interference. In quantum mechanics probabilities are squares of amplitudes, and

\[
|\psi_A(y) + \psi_B(y)|^2 \neq |\psi_A(y)|^2 + |\psi_B(y)|^2.
\]

A criterion is therefore needed to specify which sets of histories may be consistently assigned probabilities. In the familiar quantum mechanics that criterion is measurement — probabilities may be consistently assigned to histories of measured alternatives and not in general otherwise. In the two slit experiment if we measure which slit the electron passes through, then interference is destroyed, the probability sum rules are obeyed, and probabilities are predicted for the two alternative histories of the electron.

However, a criterion based on measurements or observers cannot be fundamental in a quantum theory that seeks to explain the early universe where neither existed. A more general criterion for closed systems assigns probabilities to just those sets of histories for which there is vanishing interference between its individual members as a consequence of the system’s initial quantum state\(^7,8,9\). Such sets of histories are said to decohere. Decoherent sets of histories are what may be used for prediction and retrodiction in quantum cosmology for they may be assigned probabilities.

The usual Hamiltonian formulation of quantum mechanics in a fixed background spacetime was the context for the treatment of the decoherent sets of histories of a closed system in Refs \([7,8,9]\). However, it is possible to abstract from these discussions principles for a wider class of quantum mechanical theories called generalized quantum theories. Within that framework, generalized quantum theories of spacetime geometry may be constructed which do not assume a fixed background spacetime.

\(^b\) This lecture may be thought of as an abridgement of the author’s lectures in Ref. [1].
The principles of generalized quantum mechanics were introduced in Ref. [10] and developed more fully in Refs [11] and [1]. The principles have been axiomatized in a rigorous mathematical setting by Isham\textsuperscript{12}. Three elements are needed to specify a generalized quantum theory:

(1) The sets of \textit{fine-grained histories}. These are the most refined possible description of a closed system.

(2) The allowed \textit{coarse grainings}. A coarse graining of a set of histories is generally a partition of that set into mutually exclusive classes \( \{c_\alpha\}, \alpha = 1, 2, \ldots \) called \textit{coarse-grained histories}. Each coarse-grained history is a set of fine-grained histories and the set of classes constitutes a set of coarse-grained histories with each history labeled by the discrete index \( \alpha \).

(3) A \textit{decoherence functional} defined for each allowed set of coarse-grained histories which incorporates a theory of the initial condition and dynamics of the closed system and measures the quantum mechanical interference between pairs of histories in the set. A decoherence functional \( D(\alpha', \alpha) \) must satisfy the following properties.

(i) \textit{Hermiticity}:

\[
D(\alpha', \alpha) = D^*(\alpha, \alpha') .
\]  
\[
(2.2a)
\]

(ii) \textit{Positivity}:

\[
D(\alpha, \alpha) \geq 0 .
\]  
\[
(2.2b)
\]

(iii) \textit{Normalization}:

\[
\sum_{\alpha' \in \alpha} D(\alpha', \alpha) = 1 .
\]  
\[
(2.2c)
\]

(iv) \textit{The Principle of Superposition}:

If \( \{\bar{c}_\alpha\} \) is a coarse graining of a set of histories \( \{c_\alpha\} \), that is, a further partition into classes \( \{\bar{c}_\alpha\} \), then

\[
D(\bar{\alpha}', \bar{\alpha}) = \sum_{\alpha' \in \bar{\alpha}'} \sum_{\alpha \in \bar{\alpha}} D(\alpha', \alpha) .
\]  
\[
(2.2d)
\]

Once these three elements are specified the process of prediction proceeds as follows: A set of histories is said to (medium) decohere if all the “off-diagonal” elements of \( D(\alpha', \alpha) \) are sufficiently small. The diagonal elements are the probabilities \( p(\alpha) \) of the individual histories in a decoherent set. These two definitions are summarized in the one relation

\[
D(\alpha', \alpha) \approx \delta_{\alpha' \alpha} p(\alpha) .
\]  
\[
(2.3)
\]
As a consequence of (twothree) and the four properties of (twotwo), the numbers $p(\alpha)$ lie between zero and one, sum to one, and satisfy the most general form of the probability sum rules
\[ p(\bar{\alpha}) = \sum_{\alpha \in \bar{\alpha}} p(\alpha) \tag{2.4} \]
for any coarse graining $\{\bar{c}_\alpha\}$ of the set $\{c_\alpha\}$. The $p(\alpha)$ are therefore probabilities. They are the predictions of generalized quantum mechanics for the possible coarse-grained histories of the closed system that arise from the theory of its initial condition and dynamics incorporated in the construction of $D$.

In the following we shall illustrate this general framework with examples designed to realize it in concrete form but also designed to show how to cast quantum theories into fully spacetime form. For these illustrative purposes we shall confine ourselves to sum-over-histories formulations which posit a unique set of fine-grained histories. Relations to operator formulations are discussed in Ref. [1].

3. Non-Relativistic Quantum Mechanics

The non-relativistic quantum mechanics of a particle moving in one dimension is the simplest example of generalized quantum theory when its three elements are identified as follows:

1. **Fine-Grained Histories.** The fine-grained histories are particle paths $x(t)$ which are single-valued functions of $t$ on a fixed time interval, say, $[0, T]$.

2. **Allowed Coarse Grainings.** The fine-grained paths may be partitioned by their behavior with respect to regions of $x$ at definite moments of time. For instance, consider a division of the real line into intervals $\{\Delta_{\alpha_1}\}$ at time $t_1$ and $\{\Delta_{\alpha_2}\}$ at time $t_2$ where $\alpha_1$ and $\alpha_2$ run over a discrete set of labels. One coarse-grained history consists of all the paths which pass through, say, $\Delta_{\alpha_1}$ and $\Delta_{\alpha_2}$. This is the coarse-grained history in which the particle is localized in region $\Delta_{\alpha_1}$ at time $t_1$ and $\Delta_{\alpha_2}$ at time $t_2$. An exhaustive set of coarse-grained histories in this example is specified by considering all possible sequences of two regions in the sets $\{\Delta_{\alpha_1}\}$ and $\{\Delta_{\alpha_2}\}$.

3. **Decoherence Functional.** Given a partition of the set of fine-grained paths into classes $\{c_\alpha\}, \alpha = 1, 2, \cdots$ we may define a class operator for each class (coarse-grained history) by a sum over paths in the class, viz.: 
\[ \langle x'' \vert C_\alpha \vert x' \rangle = \int_{c_\alpha} \delta x \exp(i S[x(\tau)]) . \tag{3.1} \]

The functional $S[x(\tau)]$ is the classical action which defines quantum dynamics in path integrals. The sum is over paths which start at $x'$ at $t = 0$, proceed to $x''$
at \( t = T \) and lie in the class \( c_{\alpha} \). A sum of the form (threeone) over the class of all paths is just an expression for the propagator \( \langle x' | \exp(-iHT) | x'' \rangle \). Thus,

\[
\sum_{\alpha} C_{\alpha} = e^{-iHT} .
\]

With the class operators in hand, the decoherence functional for non-relativistic quantum mechanics is

\[
D(\alpha', \alpha) = Tr \left[ C_{\alpha'} \rho C_{\alpha}^\dagger \right]
\]

where \( \rho \) is the density matrix representing the initial condition of the closed system. It is not difficult to see from (threeone), (threetwo), and the properties of density matrices, that the four requirements decoherence functionals (twotwo) are satisfied.

The class operators are simply expressed for the coarse grainings by exhaustive sets of ranges of positions at definite movements of time that we have taken to define non-relativistic quantum mechanics above. If \( \{ P^{k}_{\alpha_k}(t_k) \} \) are a set of Heisenberg-picture projections onto ranges \( \{ \Delta^{k}_{\alpha_k} \} \) at time \( t_k \), then the class operator for the paths that pass through intervals \( \Delta^{1}_{\alpha_1}, \cdots, \Delta^{n}_{\alpha_n} \) at \( t_1 \cdots, t_n \) is

\[
C_{\alpha_n \cdots \alpha_1} = e^{-iHT} P^{n}_{\alpha_n}(t_n) \cdots P^{1}_{\alpha_1}(t_1) .
\]

In (threefour) one can see the two forms of evolution in usual quantum mechanics. Applied to a state, (threefour) evolves it unitarily in between times \( t_1, \cdots, t_n \) (Heisenberg-picture evolution of the \( P \)'s) and by state vector reduction (action of the \( P \)'s) at the times \( t_1, \cdots, t_n \).

This formulation of non-relativistic quantum mechanics is not in fully spacetime form. Quantum dynamics has been expressed in spacetime form through the use of Feynman’s path integrals over histories. But the alternatives are restricted to sequences of sets of alternative ranges of position at definite moments of time. However, this usual framework of non-relativistic theory is easily generalized to alternatives that are in spacetime form. Simply allow arbitrary partitions as coarse grainings and replace (2) by

\( (2') \) Allowed Coarse Grainings: Arbitrary partitions of the fine-grained paths.

This allows new types of spacetime alternatives which are not at definite moments of time. For example, given a region \( R \) with extent in both space and time, one could partition the paths into the class \( c_0 \) of paths that never cross \( R \) and the class \( c_1 \) that cross \( R \) sometimes. A way to see that this is a genuine generalization is to note that the resulting class operators are generally not unitary and neither are

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they projections or products of projections as in (threefour). Thus, this generalized quantum mechanics of a non-relativistic particle cannot be reformulated in terms of states on surfaces of constant time which evolve unitarily or by reduction. Even without states on spacelike surfaces the theory is predictive as described above and this generalization is now in fully spacetime form with respect to both dynamics and alternatives.

4. Abelian Gauge Fields

The three elements of a generalized quantum mechanics of the free electromagnetic field are:

(1) Fine-Grained Histories: Configurations of the potential $A_\mu(x)$ on a spacetime region between two constant time surfaces, say, those at $t = 0$ and $t = T$.

(2) Allowed Coarse-Grainings: Partitions of the fine-grained histories of the potential into gauge-invariant classes $\{c_\alpha\}$.

(3) Decoherence Functional: Class operators on the physical Hilbert space of transverse degrees of freedom of the vector potential are defined by

$$\langle \vec{A}^{T''} | C_\alpha | \vec{A}^{T'} \rangle = \int_{c_\alpha} \delta A \Delta_\Phi[A] \delta[\Phi(A)] \exp(iS[A]) .$$

(4.1)

The functional $S[A]$ is the action for the free electromagnetic field. $\Phi(A)$ is a function such that $\Phi(A) = 0$ is a gauge fixing condition, and $\Delta_\Phi[A]$ is the associated Faddeev-Popov determinant. The sum is over all potentials $A_\mu(x)$ which match $\vec{A}^{T'}(x)$ on the initial surface $t = 0$ and $\vec{A}^{T''}(x)$ on the final surface $t = T$. In particular the values of the time component $A^t(x)$ and the longitudinal component of $\vec{A}(x)$ on these surfaces are summed over. The decoherence functional is

$$D(\alpha', \alpha) = Tr \left[ C_{\alpha'} \rho C_\alpha^\dagger \right]$$

(4.2)

where all operators and operations are defined in the Hilbert space of the transverse components of the vector potential.

When restricted to partitions by values of $\vec{A}^T(x)$ on spacelike surfaces, the generalized quantum mechanics specified above coincides with the familiar Hamiltonian quantum mechanics of the electromagnetic field. But more general alternatives are possible. One could consider partitions by the values of any gauge invariant functional, for instance the average of a magnetic field component over a spacetime region $R$

$$F[A] = \frac{1}{\Delta V} \int_R d^4x B_z(x) .$$

(4.3)
More specifically let \( \{ \Delta_\alpha \} \) be a set of mutually exclusive intervals making up the real line. The coarse-grained history \( c_\alpha \) consists of all those potentials \( A_\mu(x) \) for which \( F[A] \in \Delta_\alpha \). Such field averages are familiar from Bohr and Rosenfeld’s discussion of the measurability of the electromagnetic field.

Beyond functionals of the type (fourthree) which depend only on the transverse degrees of freedom one can also consider partitions by values of averages of quantities like \( \nabla \cdot E \) which are gauge invariant but involve non-transverse degrees of freedom. In this way the question of whether the constraint \( \nabla \cdot E = 0 \) is satisfied becomes a question of the probability of its value rather than a matter of the absence of the relevant degrees of freedom.

5. The Relativistic World Line

Classical dynamics in general relativity may be thought of as the evolution of the spatial geometry of a family of spacelike surfaces that foliate spacetime. This dynamics may be exhibited by decomposing the metric in 3 + 1 form with respect to these surfaces:

\[
ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^i + N^i dt) \; .
\]

Here, \( t \) is the label of a spacelike surface and \( x^i \) are three coördinates in it.

Two types of invariance of the action of general relativity may be distinguished when it is expressed in terms of the variables defined by (fiveone). First there are transformations of the coördinates \( x^i \) in the spacelike surfaces

\[
x^i \rightarrow \bar{x}^i(x^j) \; ,
\]

which for infinitesimal transformations \( x^i \rightarrow x^i + \xi^i(x^j) \) leads to

\[
h_{ij}(x) \rightarrow h_{ij}(x) + D_i(\xi_j)(x) \; .
\]

where \( D_i \) is the derivative in the surface. The transformation rule (fivethree) is not unlike that of gauge transformations in electromagnetism. It can be treated in a way analogous to that of Section 4 in the construction of a generalized quantum framework for general relativity.

A class of invariances of a different character are the reparameterizations of the time

\[
t \rightarrow \bar{t}(t) \; ,
\]

and these have their own special implications for the construction of a generalized quantum mechanics for gravitation. A simple model theory which exhibits time
reparameterization invariance classically is the relativistic world line. Its action may be taken to be

\[ S[x^\mu, N] = \frac{m}{2} \int_0^1 d\lambda N(\lambda) \left[ \left( \frac{\dot{x}^\mu(\lambda)}{N(\lambda)} \right)^2 - 1 \right]. \quad (5.5) \]

Here, \( x^\mu(\lambda) \) are the coordinates of the world line as functions of a parameter \( \lambda \) along it, a dot denotes a derivative with respect to \( \lambda \), and \( N(\lambda) \) is a multiplier enforcing the constraint \( (p^\mu)^2 = -m^2 \). This action is invariant under reparameterizations \( \lambda \to \bar{\lambda}(\lambda) \) that leave the endpoints fixed, with an appropriate transformation law for \( N \).

It is not possible to construct a quantum theory of a single relativistic particle interacting with an external field within the framework of usual Hamiltonian quantum mechanics. That is because the external field will produce pairs so that the only consistent theory is a many particle theory. However, it is possible to construct a generalized quantum theory of a single relativistic world line even interacting with an external field. This is not a theory of realistic relativistic particles such as the proton and electron for that is provided by field theory. However, in many ways this is a useful model for the quantum cosmology of a universe with a single fixed topology. We now sketch the elements of a generalized quantum theory of a single relativistic world line confining ourselves for simplicity to the non-interacting case.

1. **Fine-grained histories**: The fine-grained histories are paths \( (x^\mu(\lambda), N(\lambda)) \) in the extended configuration space of paths in spacetime \( x^\mu(\lambda) \) and multiplier \( N(\lambda) \). The paths in spacetime are not single valued in the time of any Lorentz frame, but move both forward and backward in the time of any Lorentz frame.

2. **Allowed Coarse Grainings**: Any partition of the fine-grained histories into reparametrization invariant classes \( c_\alpha \) is an allowed coarse graining. For example, given a fixed spacetime region \( R \), one could partition the paths \( x^\mu(\lambda) \) into those which never cross \( R \) and the class which cross \( R \) at least once. One could partition the fine-grained histories between two such regions by the value of the reparameterization invariant “proper time”

\[ \int N(\lambda) d\lambda \quad (5.6) \]

between the last passage of the first region and the first passage of the second. More generally, one could partition the fine-grained histories by the values of any reparameterization invariant functional \( F[x^\mu(\lambda), N(\lambda)] \).
It is not possible to partition this class of paths by the position that the world line intersects a spacelike surface. The world line may cross a given surface, not just at one position, but at an arbitrarily large number of them.

(3) Decoherence Functional: For each class of fine-grained histories \( c_\alpha \) in an allowed coarse-graining, the matrix elements of a class operator may be defined by

\[
\langle x'' | C_\alpha | x' \rangle = \int_{c_\alpha} \delta x \, \delta N \, \Delta_\Phi [x, N] \, \delta [\Phi (x, N)] \, \exp(iS[x, N]) .
\] (5.7)

Here, the sum is over all fine-grained histories whose paths start at the spacetime point \( x''^{\mu} \), end at the point \( x'^{\mu} \), and lie in the class \( c_\alpha \) The functional \( S \) is the action (fivefive). The condition \( \Phi = 0 \) fixes a parameterization and \( \Delta_\Phi \) is the associated Faddeev-Popov determinant. The measure for the integration is that induced by the Liouville “\( dpdq \)” measure on phase space.

Initial and final conditions are represented by wave functions \( \psi (x) \) and \( \phi (x) \) on initial and final spacelike surfaces in spacetime, labeled \( \sigma' \) and \( \sigma'' \) respectively. We define class operators in linear spaces of such wave functions \( \{ \psi_j(x) \} \) and \( \{ \phi_i(x) \} \) by “attaching” wave functions to (fiveseven) using a bilinear (but not necessarily positive) inner product \( \circ \), viz.:

\[
\langle \phi_i | C_\alpha | \psi_j \rangle = \phi_i^*(x'') \circ \langle x'' | C_\alpha | x' \rangle \circ \psi_j(x') .
\] (5.8)

The Klein-Gordon inner product is an appropriate choice for \( \circ \) for the relativistic world line. Specifically, in the case that \( \sigma' \) and \( \sigma'' \) are surfaces of constant time in some Lorentz frame, we define

\[
\langle \phi_i | C_\alpha | \psi_j \rangle = -\int_{\sigma''} d^3 x'' \int_{\sigma'} d^3 x' \phi_i^*(x'') \frac{\partial}{\partial t''} \langle x'' | C_\alpha | x' \rangle \frac{\partial}{\partial t'} \psi_j(x') .
\] (5.9)

If the wave functions defined off the surfaces \( \sigma' \) and \( \sigma'' \) are consistent with the operator form of the constraint \( (p^\mu)^2 = -m^2 \), i.e. satisfy the Klein-Gordon equation:

\[
(-\nabla^2 + m^2) \phi_i(x) = 0 ,
\] (5.10)

then the construction (fivenine) is independent of deformations of \( \sigma' \) and \( \sigma'' \) provided they do not intersect.

For an initial condition represented by a single wave function \( \psi (x) \), and a final condition of equal probability for a set of wave functions \( \{ \phi_i(x) \} \), the decoherence functional is

\[
D(\alpha', \alpha) = N \sum_i \langle \phi_i | C_{\alpha'} | \psi \rangle \langle \psi | C_{\alpha} | \phi_i \rangle
\] (5.11a)
where $\mathcal{N}$ is a normalizing factor

$$\mathcal{N}^{-1} = \sum_{i} |\langle \phi_i | C_u | \psi \rangle|^2,$$  \hspace{1cm} (5.11b)

with $C_u$ being the class operator for all fine-grained histories. It is not difficult to verify that this decoherence functional satisfies the four general requirements (twotwo).

With the decoherence functional (fiveeleven), sets of alternative histories which decohere may be identified and the probabilities of the individual alternative histories computed. This generalized quantum mechanical framework cannot be reformulated in terms of states on spacelike surfaces, their unitary evolution and reduction. The inclusion of paths that move both forward and backward in time implies it is not meaningful to consider alternative values of position at moments of time so that identities like (threefour) can no longer hold. Thus, by moving beyond the strictures of usual Hamiltonian quantum mechanics, to a more general quantum framework, we have exhibited a quantum theory of a single relativistic world line in fully spacetime form.

6. General Relativity

The question of whether there is a consistent manageable quantum theory of Einstein’s general relativity is still open. Whatever the answer, a generalized quantum theory of general relativity provides a model of the kind of conceptual issues that must be faced in any quantum theory of gravity. Further, such a generalized quantum theory, made finite by truncating its ultraviolet divergences, may be a useful tool in quantum cosmology for investigating the predictions of a theory of the initial condition for the very low energy phenomena of the universe like the galaxy-galaxy correlation function.

As mentioned above, the classical theory of general relativity exhibits symmetries analogous both to gauge symmetries and reparameterizations of time. Building on the treatments of gauge theories in Section 4 and the relativistic world line in Section 5 it is possible to sketch the elements of a generalized quantum theory for quantum spacetime. For simplicity we restrict attention to spatially closed cosmological four-geometries.

(1) Fine-grained histories. A class of metrics on four dimensional manifolds are the fine-grained histories of a generalized quantum theory of spacetime geometry. The framework is broad enough to allow for different manifolds and thus discuss topology change, but for simplicity in this discussion we restrict attention to the
case where the manifold is fixed and of form $\mathbb{R} \times M^3$ where $M^3$ is a closed three manifold that supports spatially closed cosmological geometries. What behavior is permitted the fine-grained histories on very small scales, and what singularities are allowed on large scales, are two issues related to the ultraviolet behavior of the theory that we shall not discuss. The geometries should at least be such that the action for general relativity is finite.

2. **Allowed Coarse Grainings.** The general idea is a partition of the fine-grained metrics into four-dimensional *diffeomorphism-invariant* classes $\{c_\alpha\}$. For example, we could partition all cosmological four metrics into the class that contain no three surface with a volume bigger than $V_0$ and the class that contain at least one such three surface. The probability of the second of these alternatives can be thought of as the probability that the universe expands to a volume bigger than $V_0$. The set of fine-grained histories could be partitioned into the class of geometries which is homogeneous and isotropic to some standard at large volumes and the class of geometries which is not homogeneous and isotropic to that standard. The probability of the first class is the probability that the universe becomes homogeneous and isotropic far from its singularities. The set of fine-grained histories could be partitioned into geometries which obey the Einstein equations to some accuracy at large volumes and the class which do not obey the Einstein equation. The probability of the first class is the probability that the universe becomes classical at large size. With further partitions of this class the probabilities of individual classical histories could be calculated.

In general when we ask for the probability of any property of the universe which can be expressed in terms of spacetime geometry and matter fields there is a corresponding partition of the fine-grained histories into the class which has this property and the class which does not have it. If it is not possible to tell which four-dimensional geometries have the property, and which do not, then the property is not well defined. Coarse grainings by partitions of the fine-grained histories into diffeomorphism-invariant classes is thus the most general notion in quantum theory of a set of alternatives expressible in terms of spacetime geometry.

(3) **Decoherence Functional.** The construction of a decoherence functional for general relativity parallels the construction of that for the relativistic particle. We sketch it here referring the reader to [1] for more details. Consider histories on a manifold $M = I \times M^3$ with two end boundaries $\partial M'$ and $\partial M''$. Any four-dimensional metric and matter field configuration on $M$ induces a three-metric $h'_{ij}(x)$ and a field configuration $\chi'(x)$ on $\partial M'$. Similarly three-metrics and fields are induced
on the other end of the history $\partial M''$. Class operator matrix elements for a class $c_\alpha$ may be defined by

$$\langle h''_{ij}, \chi'' \| C_\alpha \| h'_{ij}, \chi' \rangle = \int_{c_\alpha} \delta g \delta \phi \Delta_\Phi \left[ g \right] \delta \left[ \Phi(g) \right] \exp \left( iS[g, \phi] \right). \quad (6.1)$$

Here the sum is over four-dimensional metrics $g$ and field configurations that induce the assigned three-metrics and fields on the boundaries of $M$ and lie in the class $c_\alpha$. The functional $S$ is the action for metric coupled to matter, and $\Phi$ is a suitable gauge fixing condition.

Pure initial or final conditions are represented by wave functions on the super-space of three-metrics and spatial matter field configurations that satisfy operator versions of the four constraints of general relativity

$$H_\mu(x) \Psi \left[ h_{ij}(x), \chi(x) \right] = 0. \quad (6.2)$$

One of these is the Wheeler-DeWitt equation. The action of the class operators on such wave functions is specified by a linear (but not necessarily positive) inner product $\circ$. Thus,

$$\langle \Phi | C_\alpha | \Psi \rangle = \Phi^* \left[ h''_{ij}, \chi'' \right] \circ \langle h''_{ij}, \chi'' \| C_\alpha \| h'_{ij}, \chi' \rangle \circ \Psi \left[ h'_{ij}, \chi' \right]. \quad (6.3)$$

There are various candidates for this $\circ$ product. The most immediate is the analog of the Klein-Gordon inner product — the DeWitt inner product on hypersurfaces in superspace.

Density matrices representing non-pure initial and final conditions may be constructed in linear spaces of wave functions $\{\Psi_k[h_{ij}, \chi]\}$ satisfying the constraints (sixtwo). For example

$$\langle h''_{ij}, \chi'' \| \rho^i \| h'_{ij}, \chi' \rangle = \sum_k \Psi_k \left[ h''_{ij}, \chi'' \right] p^i_k \Psi^*_k \left[ h'_{ij}, \chi' \right] \quad (6.4)$$

and similarly for $\rho^f$. The decoherence functional is then constructed in the obvious way on these linear spaces

$$D(\alpha', \alpha) = \mathcal{N} \text{Tr} \left[ \rho^f C_{\alpha'} \rho \dagger C_{\alpha} \right] \equiv \mathcal{N} \sum_{ij} p^i \langle \Phi_i | C_{\alpha'} | \Psi_j \rangle p^j \langle \Psi_j | C_{\alpha} | \Phi_i \rangle \quad (6.5)$$

where $\mathcal{N}$ is a normalizing factor ensuring $\sum_{\alpha' \alpha} D(\alpha', \alpha) = 1$.

The generalized quantum mechanics defined by the above three elements does not require a fixed background spacetime or its associated causal structure, at least formally. No special set of spacelike surfaces has been singled out in the specification
of sets of fine-grained histories, the sets of possible alternatives, or the amplitudes that define the decoherence functional. This theory of four-dimensional quantum spacetimes is in fully four-dimensional form.

One cannot expect that this theory of quantum spacetime can be reformulated in terms of states on spacelike surfaces, not least for the reasons given in the Introduction. More specifically, this follows from the nature of the fine-grained histories. Spacelike surfaces may be defined intrinsically in a cosmological four-geometry by their three-geometries including such properties as the total spatial volume. However, there will be fine-grained quantum geometries in which surfaces of a given three-geometry occur an arbitrary number of times. In this sense, the histories of quantum general relativity move both “forward” and “backward” in any time defined by three-geometry. The factorization of histories represented by relations like (threefour) cannot be expected to hold and states on spacelike surfaces in superspace cannot be defined.

Yet, the usual formalism of quantum mechanics with states on spacelike surfaces in spacetime must hold in some approximation when geometry is approximately fixed and classical. To see how this comes about consider a partition of the fine-grained histories into coarse-grained classes \( \{ c_\alpha \} \) representing different possible classical and non-classical evolutions defined to distance accuracies well above the Planck scale. Denote the possible classical evolutions by \( \{ c_\gamma \} \) and suppose these are further refined by partitions defining different behaviors of the matter fields in these geometries into classes \( \{ c_\gamma \beta \} \). Suppose further that the initial and final conditions are such that, when these classes are sufficiently coarse grained, the integral over metrics from (sixone) in (sixthree) may be done by the method of steepest descents with but a single classical metric \( g_\gamma \) contributing in each classical class and no metrics at all in the non-classical classes. The class operator matrix elements in this approximation vanish for the non-classical classes and are determined by functional integrals of the form

\[
\int_{c_\gamma \beta} \delta \phi \exp\left( iS[g_\gamma, \phi] \right)
\]

for those classes \( c_\gamma \beta \) that define classical behavior of geometry. Functional integrals like (sixsix) define a matter field theory in a fixed background spacetime \( g_\gamma \). These amplitudes can be equivalently constructed from states of the matter field on spacelike surfaces and their evolution. The classical metric \( g_\gamma \) fixes the meaning of “spacelike” and defines the notion(s) of time.

In this way the usual formulation of quantum mechanics is recovered as an approximation to the more general theory for those coarse-grainings and initial and
final conditions in which spacetime geometry is approximately fixed, behaves classically, and can define the necessary fixed causal structure.

7. Conclusions

The accompanying table shows where we have been. We have used the framework of generalized quantum theory to exhibit a series of generalizations of usual quantum theory. In each case we have specified the three elements: fine-grained histories, allowed coarse grainings, and decoherence functional. The successive generalizations have incorporated spacetime alternatives, gauge symmetries, and histories that move forward and backward in time — all expected features of a theory of quantum spacetime. We were able to build on these generalizations to formally sketch a quantum theory of geometry and matter fields that does not require a fixed background spacetime geometry or causal structure as familiar quantum theory does.

| Fine-Grained Histories \( \{f\} \) | Non-Relativistic Quantum Mechanics | Gauge Field Theory | Single Relativistic World Line | General Relativity |
|--------------------------------------|-----------------------------------|--------------------|-------------------------------|-------------------|
| Paths \( x(t) \) that move forward in time. | Four-dimensional single-valued configurations of the potential \( \mathcal{A}^\mu(x) \). | Paths in spacetime \( x^\alpha(\lambda) \) that move both forward and backward in time and multiplier \( N(\lambda) \). | Four-dimensional manifolds, \( M \) with metrics \( g_{\mu\nu}(x) \), and matter fields \( \phi(x) \). |
| Coarse Grainings \( \{c_\alpha\} \) | Partitions of the paths into classes. | Partitions of the potential into gauge invariant classes. | Partitions of the paths into repurification invariant classes. | Partitions of manifolds, metrics, and fields into diffeomorphism invariant classes. |
| e.g., (i) By the position of crossing a surface of constant time, (ii) By whether a path crosses a given spacetime region or does not. | e.g., By the values of the field averaged over a given spacetime region, or any other gauge-invariant functional. | e.g., (i) By whether the path crosses a given spacetime region or does not, (ii) By values of the “proper time” \( \int N d\lambda \). | e.g., By whether a 4-geometry contains a spacelike surface with a given 3-geometry or does not. |

Decoherence Functional

\[
\langle \text{end}''|C_\alpha|\text{end}'' \rangle = \Sigma_{j\in\alpha} \exp(iS[f])
\]

\[
\langle \phi_i|C_\alpha|\psi_j \rangle = \phi_i \circ \langle \text{end}''|C_\alpha|\text{end}'' \rangle \circ \psi_j
\]

\[
D(\alpha', \alpha) = \mathcal{N} \Sigma_{ij} p''_i \langle \phi_i|C_\alpha'|\psi_j \rangle \langle \phi_i|C_\alpha|\psi_j \rangle^* p'_j
\]

\( o = L_2 \) inner product on functions of \( x \). \( o = L_2 \) inner product on “true degrees of freedom”. \( o = \text{Klein-Gordon inner product on a surface in spacetime.} \)

\( o = \text{DeWitt inner product on a surface in superspace.} \)
quantum theory. Rather it is an approximation to a more general theory sketched above that is free from the prerequisite of a fixed background spacetime and therefore applicable at the Planck scale.

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9. References

1. J.B. Hartle, *Spacetime Quantum Mechanics and the Quantum Mechanics of Spacetime* in Gravitation and Quantizations, Proceedings of the 1992 Les Houches Summer School, edited by B. Julia and J. Zinn-Justin, Les Houches Summer School Proceedings Vol. LVII, North Holland, Amsterdam (1995); UCSBTH-92-21, gr-qc/9304006.

2. K. Kuchař, in Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics, ed. by G. Kunstatter, D. Vincent, and J. Williams, World Scientific, Singapore, (1992).

3. C. Isham, in Recent Aspects of Quantum Fields, ed. by H. Mitter and H. Gausterer, Springer-Verlag, Berlin (1992).

4. C. Isham, in Integrable Systems, Quantum Groups, and Quantum Field Theories, ed by L.A. Ibort and M.A. Rodriguez, Kluwer Academic Publishers, London (1993).

5. W. Unruh, in Gravitation: A Banff Summer Institute, ed. by R. Mann and P. Wesson, World Scientific, Singapore (1991).

6. J.B. Hartle, in Directions in General Relativity, Volume 1: A Symposium and Collection of Essays in honor of Professor Charles W. Misner’s 60th Birthday, ed. by B.-L. Hu, M.P. Ryan, and C.V. Vishveshwara, Cambridge University Press, Cambridge (1993); and in Gravitation and Relativity 1989: Proceedings of the 12th International Conference on General Relativity and Gravitation ed. by N. Ashby, D.F. Bartlett, and W. Wyss, Cambridge University Press, Cambridge (1990), gr-qc/9210006

7. R. Griffiths, *J. Stat. Phys.* **36** (1984) 219.

8. R. Omnès, *J. Stat. Phys.* **53** (1988) 893, *ibid* **53** (1988) 933; *ibid* **53** (1988)
957; ibid 57 (1989) 357; Rev. Mod. Phys. 64 (1992) 339; Interpretation of Quantum Mechanics, (Princeton University Press, Princeton, 1994).

9. M. Gell-Mann and J.B. Hartle in Complexity, Entropy, and the Physics of Information, SFI Studies in the Sciences of Complexity, Vol. VIII, ed. by W. Zurek, Addison Wesley, Reading, MA or in Proceedings of the 3rd International Symposium on the Foundations of Quantum Mechanics in the Light of New Technology ed. by S. Kobayashi, H. Ezawa, Y. Murayama, and S. Nomura, Physical Society of Japan, Tokyo (1990).

10. J.B. Hartle, The Quantum Mechanics of Cosmology, in Quantum Cosmology and Baby Universes: Proceedings of the 1989 Jerusalem Winter School for Theoretical Physics, ed. by S. Coleman, J.B. Hartle, T. Piran, and S. Weinberg, World Scientific, Singapore (1991) pp. 65-157.

11. J.B. Hartle, Phys. Rev. D44 (1991) 3173.

12. C.J. Isham, J. Math. Phys. 35 (1994) 2157.