Einstein Type Systems on Complete Manifolds

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Abstract

In the present paper, we study the coupled Einstein Constraint Equations (ECE) on complete manifolds through the conformal method, focusing on non-compact manifolds with flexible asymptotics. In particular, we do not impose any specific model for infinity. First, we prove an existence criteria on compact manifolds with boundary which applies to more general systems and can be seen as a natural extension of known existence theory for the coupled ECE. Building on this, we prove an $L^p$ existence based on existence of appropriate barrier functions for a family of physically well-motivated coupled systems on complete manifolds. We prove existence results for these systems by building barrier functions in the bounded geometry case. We conclude by translating our result to the $H^s$ formulation, making contact with classic works. To this end, we prove several intermediary $L^2$ regularity results for the coupled systems in dimensions $n \leq 12$ which fills certain gaps in current initial data analysis.

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1 Introduction

In this paper we shall analyse existence results for relativistic initial data via the well-known conformal method [25, 74, 62] focusing on constructions on complete non-compact manifolds without

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\^For some updated presentations of this method, see, for instance [8, Chapter 2], [13, Chapter VII] and [57].
special asymptotic conditions. This last method is associated to the Einstein constraint equations (ECE) of general relativity (GR), which stand as necessary conditions for general relativistic initial data sets to admit a well-posed evolution problem.\footnote{For thorough updated reviews on this topic see\cite{13,66}.} Let us briefly recall the setting of this last problem. To begin with, recall that within GR a space-time is defined as a Lorentzian manifold $(V^{n+1},\bar{g})$ satisfying the Einstein field equations\footnote{Although the case $n = 3$ is the one which corresponds to the physical space-time, we shall present our results, whenever possible, for any $n \geq 3$ since these cases are motivated by potential higher-dimensional modifications of GR.}

$$\text{Ric}_g - \frac{R_g}{2} g + \Lambda g = T(\bar{g}, \bar{\psi}),$$

\begin{equation}
(1)
\end{equation}

where $\text{Ric}_g$ and $R_g$ stand for the Ricci tensor and scalar curvature respectively; $\Lambda$ stands for the cosmological constant, and $T$ stands for the energy-momentum tensor field associated to some physical model, which will typically depend both on the space-time metric $\bar{g}$ and some collection of physical fields, here collectively denoted by $\bar{\psi}$. Demanding physically reasonable causality conditions imposes topological restrictions on $V^{n+1}$. In particular, global hyperbolicity, which excludes possibilities such as backwards in time travels, imposes that $V^{n+1} \cong M^n \times \mathbb{R}$ [11, 12, 26], and we shall always assume $V$ to be globally hyperbolic.

In the above context, an initial data set for GR is given by the manifold $M^n$ together with the necessary initial data for $\bar{g}$ on $M^n$ at $t = 0$. Such initial data must clearly involve the induced Riemannian metric on $M^n$ by $\bar{g}$ at $t = 0$, which we shall denote by $g$, as well as its initial time derivative. Basic extrinsic geometry of $M^n \hookrightarrow (V^n, \bar{g})$ shows that this initial time derivative is essentially given in terms of the extrinsic curvature $K$ of $M^n$ as an embedded hypersurface, which we define according to the convention

$$K(X, Y) = \langle \Pi(X, Y), n \rangle_g$$

for all $X, Y \in \Gamma(TM)$, where above $\Pi$ stands for the second fundamental form of $M^n \hookrightarrow (V^n, \bar{g})$ and $n$ for the future pointing unit normal to $M$. Then, the Gauss-Codazzi equations for hypersurfaces show us that the initial data $(M^n, g, K)$ for (1) are constrained, since they must satisfy the famous Einstein constraint equations:

$$R_g - |K|_g^2 + (\text{tr}_g K)^2 - 2\Lambda = 2\epsilon,$$

$$\text{div}_g K - d\text{tr}_g K = J,$$

\begin{equation}
(2)
\end{equation}

where above $\epsilon \equiv T(n, n)$ denotes the induced energy density by the matter fields, while $J$, defined via $J(X) \equiv -T(n, X)$ for all $X \in \Gamma(TM)$, denotes the momentum density associated to these fields. It is a remarkable fact that, for most cases of interest, the above equations are not only necessary, but also sufficient conditions for $(M^n, g, K)$ to admit (short-time) evolution into a space-time satisfying (1). These facts go back to the work of Y. Choquet-Bruhat in [24, 14], and recent self-contained and updated reviews of this topic can be found in [12, 26]. This, clearly, has raised plenty of attention into the analysis of (2) aimed to produce suitable initial data sets of the Einstein equations (1). Furthermore, the system (2) turns out to be quite subtle from an analytic standpoint and, being related to scalar curvature prescription problems, it is intertwined with classic problems in geometric analysis.

The method best understood to deal with (2) is the conformal-method, which transforms (2) into a determined system of elliptic PDEs. The idea is to split $(g, K)$ according to the choices

$$g = \phi^{-\frac{1}{2}} \gamma, \quad K = \phi^{-2} (\mathcal{L}_\gamma \gamma X + U) + \frac{\tau}{n} g,$$

\begin{equation}
(3)
\end{equation}

where $(M^n, \gamma)$ is a fixed Riemannian manifold, and therefore we fix the conformal class of $g$, and we have introduced the conformal Lie derivative, given by

$$\mathcal{L}_\gamma \gamma X \equiv \mathcal{L}_X \gamma - \frac{2}{n} \text{div}_\gamma X \gamma, \quad \text{for all } X \in \Gamma(TM)$$
whose kernel consists on the conformal Killing fields (CKFs) of $\gamma$. Also, in (3), $U$ stands for a traceless and transverse (TT) $(0,2)$ symmetric tensor field, where by transverse we mean that $\text{div}_g U = 0$, and $\tau = \tau_g K$ denotes the trace part of $K$ given by the mean curvature of the initial data set $(M, g, K)$. Appealing to the conformal splitting (3), by direct computation one finds that (2) is transformed into an elliptic system on $(\phi, X)$, given by

$$a_n \Delta_\gamma \phi - R_\gamma \phi + |\tilde{K}|^2 \phi - \frac{2n-2}{n} + \left(1 - \frac{n-2}{n} \right) \phi = 0,$$

$$\Delta_{\gamma, \text{conf}} X - \left(\frac{n-1}{n} \frac{d\tau}{dr} + J\right) \phi \frac{4n}{n-2} = 0,$$

where we have introduced the conformal Killing Laplacian (CKL) operator, $\Delta_{\gamma, \text{conf}} : \Gamma(TM) \mapsto \Gamma(T^*M)$, which is an elliptic operator defined by

$$\Delta_{\gamma, \text{conf}} X = \text{div}_\gamma (L_{\gamma, \text{conf}} X),$$

and we have denoted by $a_n = \frac{4(n-1)}{n-2}$ while $\tilde{K} \doteq L_{\gamma, \text{conf}} X + U$. In (4), fixing a given physical model determining the form of the sources $\epsilon$ and $J$, the equations form an elliptic system posed for $(\phi, X)$ with geometric data $I \doteq (\gamma, \tau, U)$, where $\gamma$ is a fixed Riemannian metric, $\tau$ a fixed function standing for the mean curvature of the initial data set and $U$ is a fixed $\gamma$-TT tensor. The scalar equation in (3) is referred to as the Lichnerowicz equation.

Let us notice that if the conformal data satisfies a constant mean curvature (CMC) condition and also $\epsilon, J = 0$, then the equations in (4) decouple. The momentum constraint then reduces to a selection of a CKF, and all of the analysis is centred on the corresponding Lichnerowicz equation. In this context the author of [14] provided in a classic paper a complete classification of the smooth CMC solutions on closed manifolds. Since then several refinements have been obtained. In particular, the decoupled constraint system has been been analysed on non-compact manifolds, for instance on asymptotically Euclidean manifolds (AE) [17, 18, 54], asymptotically cylindrical (AC) manifolds [19, 20], asymptotically hyperbolic (AH) manifolds [5, 3] and, recently, the decoupled Lichnerowicz equation has been analysed on general complete manifolds in [1, 2] where both existence and uniqueness results were obtained. Furthermore, low regularity results have been established for instance in [18, 54, 55, 17, 15, 16] and through these papers some non-vacuum situations have been incorporated. In particular, let us notice that whenever we impose a CMC condition, if the momentum density scales appropriately under conformal transformations, then the above equations decouple. This is explicitly the case when $J = \phi \frac{4n}{n-2} \hat{J}$, where $\hat{J}$ is a datum constructed from the conformal data $I$. This case is typically referred to as York-scaled momentum sources and is a common assumption to decouple the equations.

The conformal method has proven to be extremely powerful in the analysis of the ECE under decoupling conditions. Nevertheless, whenever the system is coupled the situation changes drastically and only recently have significant advances been made. In this direction, let us highlight that some near CMC results are known to hold through implicit function arguments, for instance from [15, 15]. Nevertheless, it was only in [10] that the first far-from-CMC results were made available, which addressed the coupled system (2) on closed manifolds via the equations (4) under the assumption of York-scaled sources and excluding the vacuum case. These remarkable results appeal to a clever fixed point argument, which was modified in [55] to account for the vacuum case. These two pioneering papers triggered several advances in the analysis of the coupled Gauss-Codazzi system (2), such as those of [23, 27, 28, 38, 39, 60, 62, 64, 72], where some important non-compact manifolds (namely, AE and AH manifolds) were analysed and also some model sources (in particular scalar fields) were incorporated. Furthermore, some non-uniqueness issues have been made clear [64, 56, 58, 59, 46, 73], which is a feature which does not occur in the CMC case and has further motivated some variations of the conformal method such as [53].

In the above context, the main objective of this paper is to analyse existence results for the coupled Gauss-Codazzi system (4) on general complete manifolds, without a specific asymptotic structure. This follows the spirit of the work done by G. Albanese and M. Rigoli in [1, 2], but now in the perspective of the developments commented above for the coupled system. As usual, this procedure will consist on two parts: first we prove a general existence criteria which relies on the
existence of appropriate barrier functions (sub and supersolutions) and then we provide explicit constructions for these barriers. Since our analysis will be sensitive to the specific non linearities present in our problem, we will focus on a fairly general and physically well-motivated situation, which is that of energy momentum sources which contain contributions from a perfect fluid and an electromagnetic field. In such a case, our system takes the following form:

\[ a_n \Delta \phi - R_{\gamma} \phi + \left( \tilde{K}(X) \right)_{\gamma}^2 \phi \frac{n-1}{n} \tau^2 \phi \frac{\omega_1}{\omega_2} + 2\epsilon_1 \phi \frac{\omega_1}{\omega_2} + 2\epsilon_2 \phi^{-3} + 2\epsilon_3 \phi \frac{\omega_1}{\omega_2} = 0, \]

\[ \Delta_{\gamma, \text{conf}} X - \frac{n-1}{n} d\tau \phi \frac{\omega_1}{\omega_2} - \omega_1 \phi \frac{\omega_1}{\omega_2} + \omega_2 = 0, \]

Above, the functions \( \epsilon_1, \epsilon_2 \) and \( \epsilon_3 \) represent the energy contributions of the physical sources involved, while \( \omega_1 \) and \( \omega_2 \) stand for the corresponding momentum densities.

Along the lines of the discussion presented in previous paragraphs, our main motivation (in contrast to the existence results quoted above for non-compact manifolds with specific asymptotics) is to prove existence results which can be applied to initial data sets which are as flexible as possible. In particular, the aim is to allow for initial data sets which (although controlled) need not decay at infinity. We shall show that, up to a reasonable extent, our goal is attainable.

In order to address the problem of existence of solutions to (5), we will first derive the following general existence criteria.

**Theorem A.** Let \((M^n, \gamma)\) be a Riemannian manifold of dimension \(n \geq 3\) with \(\gamma \in W^{2,p}_{\text{loc}}\), \(p > n\), a complete metric. Consider the system (5) with coefficients satisfying

\[ R_{\gamma}, \epsilon_2, \epsilon_3, |U|^2, \tau^2 \in L^p_{\text{loc}}(M) \quad \text{and} \quad \omega_1, \omega_2, d\tau \in L^2(M, dV_\gamma) \cap L^p_{\text{loc}}(M), \]

and assume such system admits a pair of compatible global barrier functions \(\phi_- \leq \phi \leq \phi_+ \in W^{2,p}_{\text{loc}}\), with \(0 < \phi_- \leq \phi \leq \phi_+ \leq m < \infty\), and that the first eigenvalue of the conformal Killing Laplacian satisfies

\[ \lambda_{1, \gamma, \text{conf}} > 0. \]

Then (5) admits a \(W^{2,p}_{\text{loc}}\) solution \((\phi, X)\).

In the above theorem, we have appealed to a few important concepts. The first one is that of global barriers, which were first introduced in the analysis of the ECE in [10]. In order to understand their importance, notice that we shall construct the solutions of the theorem by iteration of solutions to linear problems. Along such iteration procedure, at the \(k\)-th step, the coefficient \(\tilde{K}(X)_{\gamma}^2\) is constructed from a field \(X_{k-1}\) which is defined as a solution to the momentum constraint where \(\phi\) has been fixed as \(\phi_{k-1}\). Therefore, we see that this coefficient is changing at each step of the procedure and since conventional barriers depend on the coefficients of the equations, they would not provide us with uniform controls needed for the proof. The global barriers \(\phi_{\pm}\) are introduced to account for this, by demanding them to work for any \(\tilde{K}(X_0)_{\gamma}^2\) constructed from an \(X_0\) which arises as a solution of the momentum constraint with \(0 < \phi_- \leq \phi \leq \phi_+\) (see Definition 3.1).

Also, we have introduced the first eigenvalue of the CKL, which is defined by

\[ \lambda_{1, \gamma, \text{conf}} \leq \inf_{u \in C^\infty_0(M, M)} \frac{\int_M |L_{\gamma, \text{conf}} u|^2 dV_\gamma}{\int_M |u|^2 dV_\gamma}, \]

and its positivity allows us to guarantee the existence of solutions to linear equations which have the form of the momentum constraint in (5) on a complete manifold \(M\), and furthermore provides us with good estimates for such solutions on compacts (see Lemma 3.1). This spectral condition is key for the above existence theorem, since, in particular, it guarantees the invertibility of the CKL with Dirichlet boundary conditions on compact domains.

---

1. See Example 2.1. Equation (5) corresponds to the case \(q = 0\), and thus one avoids the electromagnetic constraint in the physically motivated equations (18).

2. Notice that if we have in mind some rigid decaying conditions for the initial data, then using specific models for infinity (i.e. AE, AH, AC) may be the most natural way to proceed.
Let us briefly comment on the flexibility of the functional hypotheses in the above theorem. First, notice that the $L^p_{\text{loc}}$-conditions imposed in (6) are mild, demanding integrability of the energy sources, the mean curvature and the TT-part of the initial data set only on compacts. On the other hand, the momentum sources as well as $d\tau$ are imposed with a global restriction, which forces a control at infinity. These global controls for the momentum sources arise from a necessity to have global controls on solutions for the momentum constraint throughout the iteration process, and we anticipate that a similar control is necessary in our strategy for constructing global supersolutions. It is worth noticing that the conditions in (6) clearly allow for a wide range of mean curvature behaviours, allowing for far-from-CMC initial data. In particular, the condition $d\tau \in L^2(M, dV_\gamma)$ can be interpreted as a near CMC-condition at infinity. Given the flexibility of the asymptotic structure of $M$, this can be seen as a substantial improvement on current existence theory for the conformally formulated ECE.

In order to construct global barrier function for (5), we will need to impose some further conditions on the global geometry and on the coefficients of the systems. In particular, in the following theorem establishing our main result, we will assume that $(M, \gamma)$ has bounded geometry (see Appendix A for detailed definitions).

**Theorem B.** Let $(M, \gamma)$ be a smooth Riemannian manifold of bounded geometry, let $n \geq 3$ be its dimension and $p > n$. We make the following assumptions:

\[ R_{\gamma}, \epsilon_1, \epsilon_2, \epsilon_3, |U|^2, \tau^2 \in L^p_{\text{loc}}(M) \] \[ \lambda_{1,\text{conf}} > 0, \] \[ a \doteq c_n R_{\gamma} + b_n \tau^2 \in L^\infty(M), \ a \geq a_0 > 0. \] \[ (8) \]

Assume further that:

\[ \begin{cases} 
\epsilon_2 + \epsilon_3 > 0 & \text{if } n \leq 6 \\
\epsilon_2 > 0 & \text{if } n > 6. 
\end{cases} \] \[ (9) \]

Then, there exists $C(n, M, \gamma, \lambda_{1,\text{conf}})$ such that if

\[ |R_{\gamma}| + \max \left( \|d\tau\|_{L^2(M)}, \|d\tau\|_{L^p(M)} \right) + \max \left( \|\omega_1\|_{L^2(M)}, \|\omega_1\|_{L^p(M)} \right) \]
\[ + \max \left( \|\omega_2\|_{L^2(M)}, \|\omega_2\|_{L^p(M)} \right) + |U| + \epsilon_1 + \epsilon_2 + \epsilon_3 \leq C\tau, \] \[ (10) \]

then (5) admits a $W^{2,p}_{\text{loc}}$ solution $(\phi, X)$.

Let us further highlight that manifolds of bounded geometry are manifolds with positive injectivity radius (and are therefore complete) and whose curvature tensor has bounded covariant derivatives to all orders. On such manifolds many local constructions can be globalised by means of a uniformly locally finite covering, and this provides several elliptic properties which resemble those of compact manifolds or manifolds with special asymptotics, where we can naturally appeal to weighted spaces which respect to such asymptotics. Thus, from the analytic standpoint, the analysis of the coupled system (5) on general manifolds of bounded geometry can be seen as a natural developing step in the current existence theory associated to the ECE. This also falls along the lines of recent work done on related problems in the context of bounded geometry, such as the Yamabe problem [32], and, more generally, we refer the reader to [50, 31, 68, 69, 29] and references therein for a review of analytic developments in this context. On the other hand, imposing these kind of controls on the curvature tensor and injectivity radius on the initial data seems to be reasonable physical hypotheses, which is further supported by the existence of analytic tools in this context which would allow one to evolve such initial data sets (see, for instance, [13, Theorem 4.14, Appendix III]).

Let us furthermore notice that (10) can be understood as a mean curvature restriction, given $\gamma$ and the energy-momentum sources. Notice that since $\tau \in W^{1,p}_{\text{loc}}$ with $p > n$, then $\tau \in C^0_{\text{loc}}$, and since $d\tau \in L^2(M, dV_\gamma) \cap L^p(M, dV_\gamma)$, then, in an appropriate $L^p$-sense, $d\tau \to 0$ at infinity. Therefore, assuming $M$ has only one end to simplify the discussion, if we regard $\tau_0$ as the asymptotic value of $\tau$, we suggest decomposing $\tau = \tau_0 + \tilde{\tau}$, with $\tilde{\tau} \in W^{1,p}(M, dV_\gamma)$ being the freely prescribed conformal datum, and leaving $\tau_0$ to be chosen so as to satisfy (10). In such a case, (10) fixes the a minimum possible value for the asymptotic value $\tau_0$ of $\tau$, given the conformal data $(\gamma, \tilde{\tau}, U, \epsilon_1, \epsilon_2, \epsilon_3, \omega_1, \omega_2)$. 


Let us highlight that this slight modification in the conformal decomposition allows for far-from-CMC solutions to (5) with large conformal data \((\gamma, \tilde{\tau}, U, \epsilon_1, \epsilon_2, \epsilon_3, \omega_1, \omega_2)\). To the best of our knowledge, this is a novelty, since even on closed manifolds far-from-CMC initial data typically force a trade-off with smallness conditions on the remaining coefficients of the system (see, for instance, [40, 53, 27]).

Concerning the above comments of the flexibility of (10) and the possibility of large initial data sets, let us highlight that the above discussion resonates with our objectives as described prior to Theorem A. That is, the goal was to obtain initial data sets without the need of strong decaying conditions. Notice that in case one wanted initial data sets with \(\tau \to 0\) at infinity (as one typically does on AE initial data sets), then (10) would become very restrictive, even demanding \(d\tau = 0\) and therefore decoupling the system. Nevertheless, for such situations, it would be much better to actually start with an AE model for infinity and obtain far-from-CMC constructions along the lines of papers such as [23, 38, 6]. Therefore, we regard Theorem B (as well as Theorem C below) as complementing results to these last ones, where the objectives of each initial data construction are quite different from the start.

As in the compact or AE case, the barrier method in the completely coupled situation solves the problem when the critical nonlinearity satisfies \(2\epsilon_1 - \frac{n-1}{n}r^2 \leq 0\) (here as a consequence of (10)) and represents a pivotal first step in a more general understanding of the ECE on manifolds without prescribing the asymptotic behavior. To consider the positive case required, in the aforementioned settings, mixing the barrier function approaches with new ideas (see for instance [21, 18, 65, 72]). It would be interesting to apply these methods to extend our existence result as a next step in the study of coupled systems on complete manifolds, now that the viability of the iteration scheme is ensured.

As it stands, Theorem A offers prospects on equations close to the ECE. Indeed the method employed is both robust (it considers a coupled system on a complete manifold in a general manner, and improves on pre-existing results in the way described above) and flexible (see for instance the \(L^2\) results or remark 3.3 for an exploration of how it can adapted to other systems). This points to the possibility to transpose these iteration schemes to constraint systems related to (5), such as those of fourth order gravity where the fourth order phenomena suggest stepping out of the AE framework (see [9] for an exploration of these phenomena and [61] for the associated constraint equations).

Due to the above comments, and with the authors’ objectives in mind, condition (9) could be regarded as the most restrictive condition from a mathematical standpoint, since it is avoids the special case of vacuum. This motivates the following result, which accounts for vacuum initial data, where the construction of the subsolution is modified by appealing to general results on Yamabe-type equations by [47].

**Theorem C.** Let \((M, \gamma)\) be a smooth Riemannian manifold of bounded geometry, let \(n \geq 3\) be its dimension, \(p > n\) and assume (5) holds. Let \(r = d_{\gamma}(p, \cdot) : M \to \mathbb{R}\) denote the distance function to a given point \(p \in M\). Letting \(H, A, B\) be real numbers, \(A, B > 0\), we make the following additional assumptions:

\[
\begin{align*}
\text{Ricc}_\gamma &\geq -(n-1)H^2(1 + r^2), \\
R_\gamma &\geq -A, \\
|\tau| &\geq B > 0 \text{ outside a compact set}, \\
\lambda_1^{\Delta_\gamma - c_n R_\gamma} &> 0, \\
\lambda_1^{\Delta_\gamma - c_n R_\gamma} &< 0, \\
\end{align*}
\]

where \(B_0 = \{x \in M : \tau(x) = 0\}\). Then, there exists \(C(n, M, \gamma, \lambda_{1, \text{conf}})\) such that if (10) holds then \(M\) admits a \(W^{2,p}_{\text{loc}}\) solution \((\phi, X)\).

The above theorem allows for vacuum initial data sets and complements our previous result, although the spectral condition \(\lambda_1^{\Delta_\gamma - c_n R_\gamma}\) may impose certain topological obstructions. Following [47], Chapter 6, given a non-empty open set \(\Omega \subset M\), let us recall the definition

\[
\lambda_1^{\Delta_\gamma - c_n R_\gamma}(\Omega) = \left\{ \inf_{\varphi \in W^{2,p}_{\text{loc}}(\Omega)} \int_\Omega (|\nabla \varphi|^2 + c_n R_\gamma \varphi^2) \, dV_\gamma : \int_\Omega \varphi^2 dV_\gamma = 1 \right\}.
\]
Finally, let us notice that within the proof of Theorem A we will need to produce a sequence of solutions along an exhaustion of $M$, where basically we will need to solve our PDE system on a sequence a compact manifolds with boundary, under appropriate boundary conditions. This type of problem has been analysed in the past, notably in [41, 39], and we will take this opportunity to enlarge this analysis by considering more general elliptic systems, with possibly mixed boundary conditions. These systems arise by coupling further equations to the model system [45] and are highly motivated by the constraint systems of initial data for the Einstein equations coupled with further fundamental fields, the easiest example being that of an electromagnetic field (see [6] and also [8, Chapter 2] for further details). Such systems have been analysed on AE manifolds by the first two authors in [6], where they are referred to as Einstein-type elliptic systems and the main existence criteria presented there can be naturally extended to the setting of compact manifolds with boundary, which complements the existence results previously mentioned, allowing to account, for instance, for the presence of charged fluids.

In the above context, for the specific case of the conformally formulated initial data problem for a charged fluid, we will take this opportunity to present the corresponding results in the $L^2$-regularity setting, since $H^s$-initial data sets play a distinguished role within the evolution problem associated to initial data, as can be seen in [13, 66, 71, 13]. Notice that the appeal to $W^{2,p}$-solutions is particularly well-suited since $L^\infty$ estimates for the conformal factor can be obtained through the existence of barrier functions, and this translates into uniform bounds for $W^{2,p}$-solutions through elliptic estimates. Nevertheless, this can become trickier if one wants to start such a procedure directly in $H^s$, since, on the one hand in $H^2$ we lose some necessary regularity properties and, on the other hand, in the case of more regular $H^s$-spaces the $L^\infty$ a priori estimates on $\phi$ are not good enough. One way to circumvent this issue, is to start with sufficiently regular $H^s$-data and then appeal to Sobolev embeddings to guarantee such data lie within appropriate $W^{2,p}$-spaces ($p > n$) and then apply the $L^p$-existence theorems. Nevertheless, one would like to recover solutions in the same $H^s$ spaces as the data. Even when the system is decoupled, in low regularity, this is non-trivial as can be seen in [54], where some subtle lemmas are needed (see Lemma 2.5 and Proposition 6.2 in [54]). In the case of the coupled system, recovering the original regularity can become an even more subtle task due to the presence of non-linear terms coupling the equations (see Remark 2.3), which, for instance, do not allow us to appeal to the same type of arguments as in [54]. Therefore, in Section 2 besides providing certain intermediary results for Theorem A our main result (presented as Theorem 2.7) will be an existence criteria for $H^s$-initial data, based on the existence of appropriate global barriers, for the coupled constraint system associated to a charged fluid on a compact manifold with boundary. Let us highlight that, in this setting, in order to control the non-linearities in the bootstrap argument, a dimensional restriction appears in the form of $\dim(M) \leq 12$. We shall also present versions of Theorem B and C in $L^2$-regularity.

Taking all of the above into account, this paper will be organised as follows. First, in Section 2 we will analyse Einstein-type elliptic systems on compact manifolds with boundary, culminating with the above mentioned regularity results. In Section 3 these results will be used to prove Theorem A and some additional necessary technical results will be established. Then, in Section 4 we will proceed to construct the appropriate barrier functions for the application of Theorem A and prove Theorem B and C. Finally, in Appendix A we present several useful definitions and results associated to manifolds of bounded geometry and linear elliptic operators on them.

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2 Einstein-type systems on a compact manifold with boundary

In this section, we extend an existence result of [6] to compact manifolds with mixed boundary conditions. As mentioned in the introduction, we will do so for a wide class of equations to obtain the broadest result and make the closest contact with [6]. We will however only apply it with a decoupled system to find solutions of the constraint equations in the complete case (see section 3).
We also prove a $L^2$ existence result in a coupled case with Hamiltonian, momentum and electromagnetic constraints in low dimension. This result is transferable to the already studied case of AE manifolds, and in this very paper we will adapt it to the complete case (see also section 3).

2.1 Einstein-type systems

Let us consider $(M, \gamma)$ a compact Riemannian manifold of dimension $n$ with boundary $\partial M \cong \Sigma_1 \sqcup \Sigma_2$ and $\gamma \in W^{2,p}(M)$, $p > n$. On this manifold, let us consider an equation in $M$ of unknown $\Psi = (\phi, Y) \in \Gamma(E)$, where $E$ is a space of shape $(M \times \mathbb{R}) \oplus \bigoplus_{j=1}^{k} T_{\gamma_j}^* M$. This equation has the following form:

\[
\begin{align*}
\Delta \phi &= \sum_{j} a_j(Y)\phi^j \text{ in } M \\
L^i(Y^i) &= \sum_{j} a_j(Y)\phi^j \forall i \in 1 \ldots r \text{ in } M \\
-\partial_b \phi &= \sum_{k} b_k(Y)\phi^K \text{ on } \Sigma_1 \\
B^i(Y^i) &= \sum_{l} b_{l,i}(Y)\phi^K \forall i \in 1 \ldots r \text{ on } \Sigma_1 \\
\phi &= u \text{ on } \Sigma_2 \\
Y^i &= v^i \text{ on } \Sigma_2.
\end{align*}
\]

where

- $(L^i, B^i, Tr_{\Sigma_2})$ are linear elliptic operators with mixed Neumann-Dirichlet conditions, acting as maps $W^{2,p}(M) \to L^p(M) \times W^{1-\frac{1}{p},p}(\Sigma_1) \times W^{2-\frac{1}{p},p}(\Sigma_2)$. We assume they are invertible for $p > n$,
- $(a_j^\alpha)_{\alpha=0 \ldots r} : W^{2,p}(M) \to L^p(M)$ and $(b_{k,i})_{i=0 \ldots r} : W^{2,p}(M) \to W^{1-\frac{1}{p},p}(\Sigma_1)$ are maps which can depend on $Y$ and $\nabla Y$,
- $u, v^i \in W^{2-\frac{1}{p},p}(\Sigma_2)$,

with $p > n$. We further make the following hypotheses on the operators:

- **Boundedness:** for all $I$ and $K$, there exists $\rho > 0$, $f_I \in L^p(M)$ and $g_K \in W^{1-\frac{1}{p},p}(\Sigma_1)$ such that
  \[
  \forall Y^i \in B(0, \rho) \subset W^{2,p}(M), \quad \|a_j^\alpha(Y)\| \leq f_I, \quad \|b_k^\alpha(Y)\| \leq g_K \quad (14)
  \]
- **Continuity:** if $Y_k \to Y$ in $C^1(M)$,
  \[
  \forall \alpha = 0 \ldots r, \quad a_j^\alpha(Y_k) \to a_j^\alpha(Y) \text{ in } L^p(M), \quad b_{k,i}^\alpha(Y_k) \to b_{k,i}^\alpha(Y) \text{ in } W^{1-\frac{1}{p},p}(\Sigma_1) \quad (15)
  \]
- **Elliptic estimates:** in addition to the invertibility of the operators $(L^i, B^i, Tr_{\Sigma_2})$, we assume they satisfy the following estimates:
  \[
  \|Y\|_{W^{2,p}(M)} \leq C^i \left( \|L^i Y\|_{L^p(M)} + \|B^i Y\|_{W^{1-\frac{1}{p},p}(\Sigma_1)} + \|Tr_{\Sigma_2}(Y)\|_{W^{2-\frac{1}{p},p}(\Sigma_2)} \right) \quad (16)
  \]

**Definition 2.1.** We will call a system of the form (13) satisfying (14), (15) and (16) on a compact Riemannian manifold $(M, \gamma)$ a conformal Einstein-type system.

These kind of systems were introduced by the first two authors (see [4]), and their definition was shaped to encompass and mimic the Lichnerowicz equation with electromagnetic and momentum constraints.
Example 2.1. Let us consider the Einstein-Maxwell constraint equations associated to the space-time field equations with sources modelling a charged fluid (see our succinct summary in introduction, or \[8\], Chapter 2\) for a detailed accounting).

\[
\begin{align*}
R_g + \tau^2 - |K|^2 &= 2\epsilon, \\
\text{div}_g^2 K - d\tau &= J, \\
\text{div}_g E &= \tilde{q} \equiv q\tilde{N}u^0|_{t=0}, \\
dF &= 0,
\end{align*}
\]

(17)

where, with the usual notations, \(E\) is the 1-form \(E \equiv \tilde{F}(\cdot, n)|_{t=0}\) associated to the electric-field of the charged fluid; \(n\) represents the future pointing unit normal vector to \(M \cong M \times \{0\}\); \(\tilde{N}\) represents the lapse function associated with the orthogonal splitting of the Lorentzian metric, \(u\) is the velocity field associated to the charged fluid and \(F \in \Omega^2(M)\) is the induced 2-form that arises from restricting the electromagnetic 2-form \(\tilde{F}\), which described the space-time electromagnetic field, to tangent vectors to \(M\). Furthermore, recall that \(R_g\) stands for the scalar curvature of the Riemannian metric \(g\); \(K\) is a second rank symmetric tensor field, which, after solving the constraint equations, represents the extrinsic curvature of the embedded hypersurface \(\Sigma\). The first equation is referred to as the Lichnerowicz equation or \([8,\text{Chapter 2}]\) for a detailed accounting). Later, on \(X\), we will not reproduce here classical considerations, broached already in \([37,\text{ Section 3.2}, \text{ or } 49\], merely introduce \(H\) the mean curvature of \(\Sigma\), \(\nu\) its outward-pointing normal in \(M\), \(v > 0\) a function and \(\theta\) an expansion factor.

On the Dirichlet boundary \(\Sigma_2\), we simply impose trace conditions. They are somewhat artificial,

\[^{a}\text{this is an a priori condition}\]
but will make sense in the iteration scheme for complete manifolds.

\[
\begin{aligned}
&\Delta_{\gamma}\phi - c_n R_{\gamma}\phi + c_n \left| K(X) \right|^2_{\gamma} \phi^{-\frac{n+2}{n-2}} + c_n \left( 2\epsilon_1 - \frac{n-1}{n} \epsilon_2 \right) \phi^{-\frac{n}{n-2}} + 2c_n \left| \tilde{E} \right|^2 \phi^{-3} \\
&+ \frac{1}{2} c_n \left| \tilde{E} \right|^2 \phi^{-\frac{n}{n-2}} = 0 \text{ in } M \\
&\Delta_{\gamma,\text{conf}} X - \frac{n-1}{n} \nabla \tau \phi \frac{\tau}{n-2} - \omega_1 \phi^{2\frac{n-1}{n}} + \tilde{E}_{ik} \left( \nabla^k f + \nu^k \right) = 0 \text{ in } M \\
&\Delta_{\gamma} f = \bar{q} \phi^{\frac{2}{n-2}} \text{ in } M \\
&\partial_{\gamma} \phi = -a_n H \phi + (a_n \tau + d_n \theta_-) \phi \frac{\tau}{n-2} + \left( \frac{1}{2} \left| \theta_- \right| - r_n \tau \right) \nu \frac{\nu}{n-2} + U(\nu, \nu) \nu \text{ on } \Sigma_1 \\
&\partial_{\gamma} f = E_{\nu} \text{ on } \Sigma_1 \\
&\mathcal{L}_{\gamma,\text{conf}} X(\nu, \nu) = - \left( \frac{1}{2} \left| \theta_- \right| - c_n \tau \right) \nu \frac{\nu}{n-2} + U(\nu, \nu) \nu \text{ on } \Sigma_1 \\
&\phi = u > 0 \text{ on } \Sigma_2 \\
&X = v \text{ on } \Sigma_2 \\
&f = w \text{ on } \Sigma_2.
\end{aligned}
\]

(19)

The system (19) has the form of (15) and once we declare the free parameters in appropriate functional spaces, and take hypotheses ensuring the invertibility of $\Delta_{\gamma,\text{conf}}$ (see remark 2.1), we can show that it is of conformal Einstein type (the proof for the elliptic estimates can be consulted in D. Maxwell’s [8], see proposition 4). We refer the reader to [9] or the proof of proposition 2.2 for the details. System (13) can be seen as the special $q = 0$ case.

**Remark 2.1.** Thanks to the elliptic estimates, one simply needs to assume the kernel of the Conformal Killing Laplacian $\Delta_{\gamma,\text{conf}}$ is $\{0\}$ to obtain the invertibility. The classic proof, as presented in theorems 3.3.1 and 3.3.3 of [8], indeed ensures that since $\Delta_{\gamma,\text{conf}}$ is self-adjoint, the adjoint of the operator with boundary values also has a trivial kernel (which also requires the elliptic estimates), which yields that the CKL with boundary values is invertible.

The only difference in our case is that, in the AE case presented in [8], the asymptotic behavior is enough to ensure that the kernel is trivial (theorem 3.3.2), while in our compact case we will need to assume that there are no conformal Killing field to obtain the invertibility of the conformal Killing Laplacian. Such an hypothesis is generic, and can thus be assumed without losing too much generality (see (14)).

Later on, we will replace it with a stronger spectral hypothesis (see remark 3.2).

### 2.2 $W^{2,p}$ existence

We will use an iteration scheme very similar to the one in [6], relying on strong global barrier functions.

**Definition 2.2.** A conformal Einstein-type system on a compact Riemannian manifold $(M, \gamma)$ admits strong global barrier functions if there exist $(M_{Y^i})_{i=1, \ldots, r}$ such that, if we denote $B_{M_{Y^i}}$ the ball of radii $M_{Y^i}$, in $W^{2,p}(M)$ and $B_{M} = \times_i B_{M_{Y^i}}$, there exists a global subsolution $\phi_-$ and supersolution $\phi_+$ on $B_M$, meaning:

\[
\forall Y^i \in B_{M_{Y^i}}, \begin{cases}
\Delta_{\gamma} \phi_- \geq \sum I a_i^0(Y) \phi_-^I \text{ in } M \\
-\partial_{\nu} \phi_- \geq \sum K b_i^0(Y) \phi_+^K \text{ on } \Sigma_1 \\
\phi_- \leq u \text{ on } \Sigma_2,
\end{cases}
\]

(20)
and

\[ \forall Y_i \in B_{M_i}, \begin{cases} \Delta \phi_i \leq \sum_I a_i^0(Y) \phi_i^I \text{ in } M \\ -\partial \phi_i \leq \sum_K b_i^0(Y) \phi_i^K \text{ on } \Sigma_i \\ \phi_i \geq u \text{ on } \Sigma, \end{cases} \quad (21) \]

and two real numbers \( l \) and \( m \) such that \( 0 < l \leq \phi_- \leq \phi_+ \leq m \).

To simplify, we will denote \( \mathcal{P} \) the left-hand part of \((13)\) and \( F(\phi, Y) \equiv (h_Y(\phi), h^i_Y(\phi), g_Y(\phi), g^i_Y(\phi), u, v^i) \) its non-linear part. In the same manner, given \( a \in L^p(M) \) and \( b \in W^{1,\frac{1}{2}}(M) \), we will denote \( \mathcal{P}_{a,b} \) and \( \mathcal{F}_{a,b} \) the shifted operators:

\[ \begin{align*} h^b_Y(\phi) &= h_Y(\phi) - a\phi = \sum_I a^0_I(Y) \phi^I - a\phi \\ g^b_Y(\phi) &= g_Y(\phi) - b\phi = \sum_K b^0_K(Y) \phi^K - b\phi, \end{align*} \]

\[ (22) \]

\[ \mathcal{P}_{a,b} : \begin{cases} W^{2,p}(M) \to L^p(M) \times W^{1,\frac{1}{2}}(\Sigma_1) \times W^{2,\frac{1}{2}}(\Sigma_2) \\ \Psi = (\phi, Y) \mapsto (\Delta \phi - a\phi, L^i(Y^i), -\partial \phi - b\phi, B^i(Y^i), \mathcal{Tr}_{\Sigma_1}(\phi), \mathcal{Tr}_{\Sigma_2}(Y^i)), \end{cases} \]

\[ (23) \]

and

\[ \mathcal{F}_{a,b} : \begin{cases} W^{2,p}(M) \to L^p(M) \times W^{1,\frac{1}{2}}(\Sigma_1) \times W^{2,\frac{1}{2}}(\Sigma_2) \\ \Psi = (\phi, Y) \mapsto (h^a_Y(\phi), h^i_Y(\phi), g^b_Y(\phi), g^i_Y(\phi), u, v^i). \end{cases} \]

\[ (24) \]

By invertibility assumption and properties of the Laplace-Beltrami operator on compact manifolds with mixed Dirichlet-Neumann boundary conditions (see appendix B of \([37]\), emphasis on lemma B.8 with either \( \Sigma_2 \neq \emptyset \), or either \( a > 0 \) or \( b > 0 \)), \( \mathcal{P} \) and \( \mathcal{P}_{a,b} \) are invertible. In addition, given \( (16) \) and classical elliptic estimates on compact manifolds with boundaries (see lemma B.8 and (55) of \([37]\)) \( \mathcal{P}^{-1} \) and \( \mathcal{P}_{a,b}^{-1} \) are bounded.

Given a pair of strong global barrier functions on \( M \), in order to apply maximum principles, we will choose \( a \) and \( b \) such that \( h^a_Y \) and \( g^b_Y \) are non-increasing in \( \phi \). Given any \( Y \in \times_i B_{M_i} \), one has, thanks to \((14)\):

\[ |\partial_{\phi} h_Y(\phi)| \leq \sum_I a^0_I(Y) I \phi^{I-1} \leq \sum_I |I| |f_I| \sup_{l \leq \phi \leq m} \phi^{l-1}, \]

\[ |\partial_{\phi} g_Y(\phi)| \leq \sum_K b^0_K(Y) K \phi^{K-1} \leq \sum_K |K| g_K \sup_{l \leq \phi \leq m} \phi^{K-1}. \]

Thus, if we set

\[ \begin{align*} h^a_Y(\phi) &= h_Y(\phi) - a\phi \\ g^b_Y(\phi) &= g_Y(\phi) - b\phi, \end{align*} \]

\[ (25) \]

\[ (26) \]

one has \( \partial_{\phi} h^a_Y(\phi), \partial_{\phi} g^b_Y(\phi) < 0 \) on \([l, m] \) \( (l \) and \( m \) being the lower and upper bounds in definitions \((22)\), meaning that, as intended, both \( h^a_Y \) and \( g^b_Y \) are non-increasing functions on \([l, m] \) for all \( Y \in B_{M_i} \), which will lead to converging iterative procedures.

**Theorem 2.1.** Let \((M, \gamma)\) be a compact manifold with boundary of dimension \( n \geq 3 \), let \( p > n \) and consider a conformal Einstein-type system \((13)\) on \( M \). Assume that there exists a pair of strong global barrier functions \( 0 < l \leq \phi_- \leq \phi_+ \leq m \) and that for \( a \) and \( b \) as in \((27)\) the solution map:

\[ \mathcal{F}_{a,b} : \begin{cases} W^{2,p}(M) \to W^{2,p}(M) \\ \psi = (\phi, Y) \mapsto \mathcal{F}_{a,b} = \mathcal{P}^{-1}_{a,b} \circ \mathcal{F}_{a,b} \end{cases} \]

\[ (27) \]
is invariant on $B_{M^r}$ for $\phi_- \leq \phi \leq \phi_+$. Then the system admits a solution $\psi = (\phi, Y) \in W^{2,p}(M)$ with $\phi > 0$.

Proof. Iteration:

Let $Y_0 \in B_{M^r}$ and let us set $\Psi_0 = (\phi_0, Y_0)$ and $\phi_0 = \phi_-$. We define $\Psi_k$ as the solution to:

$$P_{a,b}(\Psi_{k+1}) = F_{a,b}(\Psi_k).$$  \hfill (28)

Since $P_{a,b}$ is invertible, $\Psi_k$ is well defined for all $k \in \mathbb{N}$.

We can show by induction that for all $k \in \mathbb{N}$, $l \leq \phi_- \leq \phi_k \leq \phi_+ \leq m$, and $Y_k \in B_{M^r}$.

- It is of course true for $k = 0$ given our first choice of $(\phi_0, Y_0)$.
- Assuming it is true for $k \in \mathbb{N}$, we write:

$$\begin{cases}
\langle \Delta_{\gamma} - a \rangle (\phi_{k+1} - \phi_-) \leq h^\nu_{Y_k}(\phi_k) - h^\nu_{Y_k}(\phi_-) \leq 0, \text{ in } M \\
- (\partial_{\gamma} + b)(\phi_{k+1} - \phi_-) \leq g^\nu_{Y_k}(\phi_k) - g^\nu_{Y_k}(\phi_-) \leq 0 \text{ on } \Sigma_1 \\
\phi_{k+1} - \phi_- \geq 0 \text{ on } \Sigma_2,
\end{cases}$$

since $\phi_-$ is a global subsolution, $Y_k \in B_{M^r}$, $\phi_k \geq \phi_-$ by hypothesis, and $h^\nu_{Y_k}$, $g^\nu_{Y_k}$ are non-increasing functions by construction. Using the maximum principle one thus has $\phi_{k+1} \geq \phi_-$. Using the same reasoning to compare $\phi_{k+1}$ to the supersolution $\phi_+$ we conclude that $\phi_{k+1} \leq \phi_+$.

- Since $l \leq \phi_- \leq \phi_{k+1} \leq \phi_+ \leq m$, the invariant hypothesis of $F_{a,b}$ on $B_{M^r}$ ensures that $Y_{k+1} \in B_{M^r}$. This thus stands for all $k \in \mathbb{N}$.

Our sequence $\Psi_k$ is thus well defined and uniformly bounded in $L^\infty \times W^{2,p}$. We will however need $W^{2,p}$ bounds on $\phi$ to ensure proper convergence. Using Calderón-Zygmund estimates (see lemma B.8 of [37]) and the definition of $(\Psi_k)$ we find:

$$\|\phi_{k+1}\|_{W^{2,p}(M)} \leq C \|F_{a,b}\|_{L^p(M) \times W^{1,\frac{1}{r}}(\Sigma_1) \times W^{2,\frac{1}{r}}(\Sigma_2)}$$

$$\leq C \left( \sum_k \|a_k^j(Y_k)\|_{L^p(M)} \|\phi_k^j\|_{L^\infty(M)} + \|a\|_{L^p(M)} \|\phi_k\|_{L^\infty(M)} \right) + \sum_k \|b^0_K(Y_k)\phi_k^K\|_{W^{1,\frac{1}{r}}(\Sigma_1)} + \|b\phi_k\|_{W^{1,\frac{1}{r}}(\Sigma_1)} + \|u\|_{W^{2,\frac{1}{r}}(\Sigma_2)} \right),$$

where $C$ depends on $M, \gamma, \Sigma_1, \Sigma_2$. Consider, for any $K$, $\bar{b}^0_K(Y_k)$ an extension of $b^0_K(Y_k)$ to the whole of $\partial M$ defined as

$$\bar{b}^0_K(Y_k)(x) = \begin{cases} b^0_K(Y_k)(x) \text{ if } x \in \Sigma_1 \\ 0 \text{ if } x \in \Sigma_2. \end{cases}$$

Consider then $\bar{b}^0_K(Y_k) \in W^{1,p}(M)$ an extension of $b^0_K(Y_k)$ such that

$$\left\|\bar{b}^0_K(Y_k)\right\|_{W^{1,p}(M)} \leq C(M, \Sigma_1, \Sigma_2) \|b^0_K(Y_k)\|_{W^{1,\frac{1}{r}}(\partial M)} \leq C(M, \Sigma_1, \Sigma_2) \|b^0_K(Y_k)\|_{W^{1,\frac{1}{r}}(\Sigma_1)} \leq C(M, \Sigma_1, \Sigma_2) \|b^0_K(Y_k)\|_{W^{1,\frac{1}{r}}(\Sigma_1)}. \hfill (30)$$

Thus $\bar{b}^0_K(Y_k)\phi^K_k \in W^{1,p}(M)$ is an extension of $b^0_K(Y_k)\phi^K_k \in W^{1,\frac{1}{r}}(\Sigma_1)$, and thus, thanks to a trace theorem

$$\|b^0_K(Y_k)\phi^K_k\|_{W^{1,\frac{1}{r}}(\Sigma_1)} \leq C \|\bar{b}^0_K(Y_k)\phi^K_k\|_{W^{1,p}(M)} \leq C \|b^0_K(Y_k)\|_{W^{1,p}(M)} \|\phi^K_k\|_{W^{1,p}(M)} \leq C \|b^0_K(Y_k)\|_{W^{1,\frac{1}{r}}(\Sigma_1)} \|\phi^K_k\|_{W^{1,p}(M)}, \hfill (31)$$

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depending on
Similarly, given the definition of $b$ one finds

$$\| \phi_k \|_{L^\infty(M)} \leq C.$$  \hfill (33)

Combining (31), (32) and (33) we find:

$$\| b^0(Y_k) \phi_k^0 \|_{W^{1-p/2}(\Sigma_1)} \leq C(M, \Sigma_1, \Sigma_2, m, K)(1 + \| \nabla \phi_k \|_{L^p(M)}).$$  \hfill (34)

Similarly, given the definition of $b$ one finds

$$\| b^0 \|_{W^{1-p/2}(\Sigma_1)} \leq C(M, \Sigma_1, \Sigma_2, b, K)(1 + \| \nabla \phi_k \|_{L^p(M)}).$$  \hfill (35)

Then, applying (14), (33) and (35) to (29) we deduce that for all $\varepsilon$

$$\| \phi_k+1 \|_{W^{2,p}(M)} \leq C(M, \Sigma_1, \Sigma_2, (f_1)_t, (g_K)_K, u) + C(M, \Sigma_1, \Sigma_2, (g_K)_K) \| \nabla \phi_k \|_{L^p(M)}$$

$$\leq C(M, \Sigma_1, \Sigma_2, (f_1)_t, (g_K)_K) \left( C^\varepsilon \| \phi_k \|_{L^p(M)} + \varepsilon \| \phi_k \|_{W^{2,p}(M)} \right)$$

$$\leq C(M, \Sigma_1, \Sigma_2, (f_1)_t, (g_K)_K, u) + C(M, \Sigma_1, \Sigma_2, (g_K)_K) \varepsilon \| \phi_k \|_{W^{2,p}(M)},$$

where we have used a Gagliardo-Nirenberg interpolation inequality.

Taking $\varepsilon < \frac{1}{2C(M, \Sigma_1, \Sigma_2, (g_K)_K)}$, we deduce that

$$\| \phi_k+1 \|_{W^{2,p}(M)} \leq \frac{1}{2} \| \phi_k \|_{W^{2,p}(M)} + C,$$  \hfill (36)

which implies that for all $k \in \mathbb{N}$

$$\| \phi_k \|_{W^{2,p}(M)} \leq \frac{1}{2k} \| \phi_0 \|_{W^{2,p}(M)} + C.$$  \hfill (37)

In addition, since $Y_k \in \times_s B_M$, there exists $M_0 \in \mathbb{R}_+$ such that $\| \Psi_k \|_{W^{2,p}(M)} \leq M_0$. Consequently there exists $\Psi = (\phi, Y)$ such that

$$\Psi_k \rightarrow \Psi \text{ in } W^{2,p}(M)$$

$$\Psi_k \rightarrow \Psi \text{ in } C^1(M).$$  \hfill (38)

**Estimates**

Considering any $n, m \in \mathbb{N}$, since $P_{a,b}$ is linear, one has:

$$P_{a,b}(\Psi_m - \Psi_n) = F_{a,b}(\Psi_m - 1) - F_{a,b}(\Psi_{n-1}),$$

meaning that

$$\begin{cases}
(\Delta - a)(\phi_m - \phi_n) = \sum_I a^I_I(Y_{m-1})\phi^{I}_{m-1} - a^I_I(Y_{n-1})\phi^{I}_{n-1} - a(\phi_{m-1} - \phi_{n-1}) \text{ in } M \\
-(\partial_v + b)(\phi_m - \phi_n) = \sum_K b^K_K(Y_{m-1})\phi^{K}_{m-1} - b^K_K(Y_{n-1})\phi^{K}_{n-1} - b(\phi_{m-1} - \phi_{n-1}) \text{ on } \Sigma_1 \\
\phi_m - \phi_n = 0 \text{ on } \Sigma_2.
\end{cases}$$

\hfill 13
Using inequality (55) of [37], we find

\[
\|\phi_m - \phi_n\|_{W^2,p(M)} \leq C \left( \sum_I \| a_I^0(Y_{m-1}) \phi_{m-1}^I - a_I^0(Y_{n-1}) \phi_{n-1}^I \|_{L^p(M)} \right) + \| a \|_{L^p(M)} \| \phi_{m-1} - \phi_{n-1} \|_{L^\infty(M)} \\
+ \sum_K \| b_K^0(Y_{m-1}) \phi_{m-1}^K - b_K^0(Y_{n-1}) \phi_{n-1}^K \|_{W^{1,p}(M)} \right) + \| b \|_{W^{1,p}(M)} \| \phi_{m-1} - \phi_{n-1} \|_{W^{1,\infty}(M)} \right).
\]

Applying (39) and (15) to (40) ensures that \( \phi_m \) is Cauchy in \( W^{2,p}(M) \).

The same procedure on \( Y_k \) yields the same result (thanks to (10)). \( \Psi_k \) is thus Cauchy, and the convergence \( \Psi_k \to \Psi \) is strong in \( W^{2,p}(M) \). The limit \( \Psi \) is then a solution of the equation. \( \square \)

It must be pointed out that the equation in \( Y \) solved by \( F_{a,b} \) does not depend on \( a \) and \( b \). Since the invariance assumed in theorem 2.1 relies entirely on \( Y \) and its equation, it can be reformulated as:

**Theorem 2.2.** Let \( (M, \gamma) \) be a compact manifold with boundary of dimension \( n \geq 3 \), let \( p > n \) and consider a conformal Einstein-type system [13] on \( M \). Assume that there exists a pair of strong global barrier functions \( 0 < l \leq \phi_- \leq \phi_+ \leq m \) and that the solution map:

\[
F: \begin{cases} 
W^{2,p}(M) \to W^{2,p}(M) \\
\psi = (\phi, Y) \mapsto \tilde{F} = \mathcal{P}^{-1} \circ F(\psi)
\end{cases}
\]

is invariant on \( B_{M_Y} \) for \( \phi_- \leq \phi \leq \phi_+ \). Then the system admits a solution \( \psi = (\phi, Y) \in W^{2,p}(M) \) with \( \phi > 0 \).

**Proof.** One only has to show that the invariance of \( F \) is enough to ensure the invariance of \( F_{a,b} \) (\( a, b \) as defined in (29)). Assume \( \psi = (\phi, Y) \) is in \( (\mathbb{R} \times M) \times B_{M_Y} \) and let \( (\phi, \tilde{Y}) = \mathcal{F}_{a,b}(\psi) \), with \( (\phi, \tilde{Y}) = \tilde{F}(\psi) \). In both cases, since only the Lichnerowicz equation is impacted \( L(Y) = \sum_I a_I^0 \phi^I L(Y) \), \( B(\tilde{Y}) = B(\tilde{Y}) \) on \( \Sigma_1, \tilde{Y} = v = \tilde{Y} \) on \( \Sigma_2 \). By invertibility of the \( L \) operator: \( Y = \tilde{Y} \), and since \( F \) is invariant on \( B_{M_Y} \), \( \tilde{Y} \in B_{M_Y} \). In the end \( \mathcal{F}_{a,b}(\psi) \in (\mathbb{R} \times M) \times B_{M_Y} \), which proves the invariance. The theorem follows as a consequence of theorem 2.1. \( \square \)

**Remark 2.2.** In this section we did not use the global barrier functions alluded to in the introduction but, following [4], an apparently stronger notion: strong global barrier functions. We refer the reader to remark 3.3 below for an exploration of the difference between these notions and when coincide.

### 2.3 \( H^s \) data

So far in this section, as well as in [4], we have worked with \( W^{2,p} \) spaces. This is due to our reliance on the \( L^\infty \) estimates provided by the barrier functions in order to deal with the nonlinear terms: such bounds allow one to properly control all the \( \phi^I \). It would however be desirable, and in accordance with previous works (see [13], [71] or the appendix of [13]), to provide initial data for the Einstein equations in the Hilbert spaces \( H^s \). When considering coupled Einstein systems in \( H^s \),
Thus, (42) yields the following assumptions (since they do not depend on the invertible on $M$, there exists $\tau, \phi, c_n \in \mathbb{R}$ such that $\phi = R_c \phi + c_n \left( \hat{K}(X) \phi - \frac{n-1}{n+1} + c_n \right)$.

The non-linear terms will be precisely the main obstacle to finding a solution (we refer the reader to D. Maxwell’s [20] and [52] for a look at the decoupled case, and how the non-linearities are dealt with, using the algebraic properties of $H^s$ for $s$ big enough). Thus, to obtain an existence result, the precise nature of the non-linearities will be of pivotal importance, which will force us to step out of the general case of the Einstein-type systems. In fact, we will consider the physically inspired case displayed in example [21] with Hamiltonian, momentum and electromagnetic constraints, but without black hole boundary conditions, that is with $\Sigma_1 = \emptyset$. The system then becomes:

$$\begin{cases}
\Delta \gamma \phi - c_n R_c \phi + c_n \left( \hat{K}(X) \phi - \frac{n-1}{n+1} + c_n \right) \phi = 0 in M \\
\Delta \gamma \phi = -n-1 \nabla \tau \phi - \frac{n}{n+1} - c_n \left( \nabla \phi^2 + \phi \nabla \phi \right) = 0 in M \\
\Delta \gamma f = \tilde{q} \phi \frac{\phi^{2n}}{n} in M \\
0 < l \leq \phi < m in M \\
\phi = u > 0 on \Sigma \\
X = v on \Sigma \\
f = w on \Sigma.
\end{cases}$$

No longer considering a generic Einstein-type system is a restriction, but one with little practical consequences. Indeed, in theorem [24] the existence of a solution is entirely dependent on the construction of strong global barrier functions. Without a global method, any construction must necessarily be artifical and dependent on the exact shape of the equations (see section 3.2.3 of [52] for examples of construction in the AE configuration, and see section [1] for the case of complete manifolds). The restrictions to concrete systems of the type [11] is thus already hidden behind the barrier functions.

**Functional hypotheses**

Let us consider $s > \frac{3}{2} - 1$ and assume

$$\gamma \in H^s(M), \tau, U, V, F \in H^{s-1}(M), R_\gamma, \tilde{q}, \omega_1, \epsilon_1 \in H^{s-2}(M), u, v, w \in H^{s-\frac{3}{2}}(\Sigma).$$

**Proposition 2.3.** Under [12] and the hypotheses of theorem [23] for [11], assuming that $\Delta_{\gamma, \text{conf}}$ is invertible on $M$, there exists $p > n$ such that [11] is a $L^p$ conformal-type Einstein system as in definition [21].

**Proof.** The differential operators involved in [11] (respectively $(\Delta_\gamma, Tr|_{\Sigma})$ and $(\Delta_{\gamma, \text{conf}}, Tr|_{\Sigma})$) form linear invertible elliptic operators with Dirichlet boundary condition (since the Neumann boundary is supposed empty to simplify the considerations) $W^{2,p}(M) \to L^p(M) \times W^{2-\frac{1}{p},p}(\Sigma)$, and thus satisfy the first hypothesis of definition [21].

Moreover, we know that $H^\sigma \hookrightarrow L^p$ with $p = \frac{2n}{n-2\sigma} > n$ if and only if $n > \frac{3}{2} - 1$.

Thus, [12] yields the following $L^p$ controls on the data:

$$\gamma \in W^{2,p}(M), \tau, U, V, F \in W^{1,p}(M), R_\gamma, \tilde{q}, \omega_1 \in L^p, u, v, w \in W^{2-\frac{1}{p},p}(\Sigma).$$

The control on the boundary values $u, v, w$ is exactly the one required in the third hypothesis of definition [21]. In addition, since $p > n$ and $M$ is compact, $H^{1,p}(M) \hookrightarrow L^\infty(M) \subset L^p(M)$, and thus $c_n R_\gamma, c_n \left( \epsilon_1 - \frac{1}{2} \nabla \phi \right)$, $\nabla \tau \phi - \frac{n}{n+1} - c_n \left( \nabla \phi^2 + \phi \nabla \phi \right)$ and $\tilde{q} \in L^p(M)$. The corresponding $a_\phi$ and $a_{\tilde{q}}$ satisfy thus the second hypothesis of definition [2.1] as well as the boundedness and continuity assumptions (since they do not depend on $Y = (f, X)$).

Consequently, in order to show that [11] is a system of type [13], one only needs to check that the quadratic terms are in $L^p$ given a $W^{2,p}$ bound on $(f, X)$. Since such a bound implies, by Sobolev embeddings, an $L^\infty$ estimate on $\nabla f$ and $\nabla X$, one has:
concludes the proof. 

The continuity assumption comes immediately from the quadratic dependence on $\nabla Y$, which concludes the proof. 

$H^s$ solutions

Starting with $H^s$ hypotheses thus allows one to frame the Einstein-type system in $L^p$ terms, and thus, modulo the existence of strong global barrier functions, to find a solution in $W^{2,p}$. The question at the heart of this section is then: can we recover the $H^s$ regularity?

Let us first recall the context in which $H^s$ has algebra properties (see [54][lemma 2.3]):

**Lemma 2.1** (Multiplication lemma). Let $r_1$, $r_2$ and $\sigma$ satisfy $\sigma \leq \min\{r_1,r_2\}$, $r_1 + r_2 \geq 0$ and $r_1 + r_2 > \frac{n}{2} + \sigma$. Then one has:

$$H^{r_1}(M) \times H^{r_2}(M) \rightarrow H^\sigma(M)$$

$$(h_1,h_2) \mapsto h_1h_2.$$ 

We will apply it in the following specific cases:

**Corollary 2.1.** If $t > \frac{n}{2}$, $a, b \in H^t$, then $ab \in H^t$. 

**Proof.** We merely apply lemma 2.1 with $r_1 = r_2 = t$ and $\sigma = t < t + t - \frac{n}{2}$. 

**Corollary 2.2.** If $a \in H^t$, $b \in H^l$ with $l > \frac{n}{2}$, then $ab \in H^{\min(t,l)}$. 

**Proof.** We apply lemma 2.1 with $r_1 = t$ and $r_2 = l$, $\sigma = \min(t,l) < t + l - \frac{n}{2}$. 

We will need a lemma to splice the low regularities from the claims and the $H^s$ control from the hypotheses:

**Corollary 2.3.** If $f \in H^t$, $g \in H^{s-2}$, $t \geq 1$ and $s > \frac{n}{2} + 1$, then $fg \in H^{\min(t-1,s-2)}$. 

**Proof.** We apply lemma 2.1 with $r_1 = t$ and $r_2 = s - 2$, $\sigma = \min(t - 1, s - 2) < t + s - 2 - \frac{n}{2}$. Notice that if $\sigma = t - 1$, then the conditions of lemma 2.1 translate into $s > \frac{n}{2} + 1$, which is satisfied by hypothesis, and if $\sigma = s - 2$, then $t \geq s - 1 > \frac{n}{2} + 1$, and the conditions of lemma 2.1 translate precisely into $t > \frac{n}{2} + 1$. 

Let us now prove some further multiplication properties:

**Proposition 2.4** (General $L^p$-product). Consider $M^n$ to be a smooth compact manifold (possibly with boundary) with $n \geq 3$. Let $g \in L^p(M)$ and $h \in L^q(M)$ with $p > n$ and $q = \frac{n}{n - 2l}$, $1 \leq l < \frac{n}{2}$, then $hg \in L^2(M)$. 

**Proof.** We need to show that $g^2h^2 \in L^1(M)$. Since $g^2 \in L^\frac{n}{2}$, we know that this claim will follow as long as $h^2 \in (L^\frac{n}{2})' \cong L^{\frac{n}{2-n}}$. Also, we know that $h^2 \in L^{\frac{n}{2-n}}$ and thus, since $M$ is compact, $L^{\frac{n}{2-n}} \hookrightarrow L^{\frac{n}{2-n}}$ is guaranteed provided that 

$$\frac{n}{n - 2l} \geq \frac{p}{p - 2} \iff \frac{n(p - 2)}{p(p - 2)} \geq \frac{n - 2l}{p - 2l} \iff -n \geq -pl \iff p \geq \frac{n}{l}.$$ 

The last of the above conditions is guaranteed to hold since $l \geq 1$ and $p > n$. 

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Proposition 2.5. Consider $M^n$ to be a smooth compact manifold (possibly with boundary) with $n \geq 3$. Let $f \in H^m$ for some integer $m \geq 2$, then for any multi-indexes $\beta$ and $\beta_i$, $i = 1, \cdots, l \leq k$ such that $\sum_i |\beta_i| = |\beta|$, $|\beta_i| \geq 1$ and $|\beta| = k \leq m$, if $m > \frac{n}{2}$ then it holds that

$$\partial^{\beta_i} f \cdots \partial^{\beta_i} f \in L^2(M).$$

**(Proof.** Fix numbers $q_i \geq \frac{k}{|\beta_i|}$, so that $\sum_i \frac{1}{q_i} = 1$. Furthermore, if we assume that $1 + \frac{n}{2} > m > \frac{n}{2}$, then $2(m - |\beta_i|) < n$ and

$$2q_i \leq \frac{2n}{n - 2(m - |\beta_i|)} \iff k(n - 2m + 2|\beta_i|) \leq n|\beta_i| \iff k(n - 2m) \leq n - 2k,$$

Since $\frac{k}{|\beta_i|} \geq 1$ and $(n - 2m) < 0$, then $\frac{k}{|\beta_i|}(n - 2m) \leq n - 2m$. In turn, $n - 2m \leq n - 2k$ if $m \geq k$, which is satisfied. This implies that $\frac{k}{|\beta_i|}(n - 2m) \leq n - 2k$ and therefore $2q_i \leq \frac{2n}{n - 2(m - |\beta_i|)}$. Thus, under this restricting condition, we can appeal to the Sobolev embedding $H^{m - |\beta_i|} \hookrightarrow L^{q_i}$, with $r_i = 2q_i$. In particular,

$$\sum_{i=1}^l \frac{1}{r_i} = \frac{1}{2}.$$

From this, we conclude that $\partial^{\beta_i} f \in L^{q_i}$ and we can apply Hölder’s generalised inequality combined with the Sobolev continuous embedding to get

$$\|\partial^{\beta_i} f \cdots \partial^{\beta_i} f\|_{L^2} \leq \prod_{i=1}^l \|\partial^{\beta_i} f\|_{L^{q_i}} \leq \prod_{i=1}^l C_i \|\partial^{\beta_i} f\|_{H^{m - |\beta_i|}},$$

where the right-hand side is finite by hypothesis.

\[\square\]

Proposition 2.6. Consider $M^n$ to be a smooth compact manifold (possibly with boundary) with $n \geq 3$. Let $g \in H^{k_1} \cap L^p$ and $h \in H^{k_2}$ with $k_1, k_2 \geq 1$ and $p > n$, then $hg \in L^2(M)$.

**(Proof.** First, notice that the multiplication property gives the result if $n < 2(k_1 + k_2)$, thus at least for $n = 3$. Now, assume that $n \geq 2(k_1 + k_2)$ so that $k_2 < \frac{n}{2}$ and therefore $H^{k_2} \hookrightarrow L^{\frac{2n}{n - 2}}$. We can now apply Proposition 2.4 to guarantee that $hg \in L^{\frac{2n}{n - 2}} \otimes L^p \subset L^2$, which proves the claim for $n \geq 2(k_1 + k_2)$ and the result follows.

\[\square\]

To conclude, we recall the following well-known composition lemma. The proof goes along the same line as proposition 2.3 and is written in details [1, lemma A.2.1].

**Lemma 2.2.** Let $U$ be a bounded domain in $\mathbb{R}^n$ having the cone property. Let $F : I \to \mathbb{R}$ be a function of class $C^m$ on some open interval $I \subset \mathbb{R}$ and $f \in W^{m,p}(U)$ with $m > \frac{n}{p}$ and $1 \leq p < \infty$ satisfying $f(U) \subset I$. Then $F \circ f \in W^{m,p}(U)$.

Following are two claims that will be useful to prove the main result of this section:

**Claim 1:** Consider $M^n$ to be a smooth compact manifold (possibly with boundary) with $n \geq 3$. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a smooth function on a bounded open interval $I$, and let $\phi \in W^{2,p} \cap H^m$, with $p > n$ and $m \geq 2$ satisfy $\phi(M) \subset I$. Then:

1. If $m = 2, 3, 4, 5$, then $f \circ \phi \in H^m$ for all $n \geq 3$;
2. If $m = 6, 7$, then $f \circ \phi \in H^m$ for all $n \leq 12$.

**(Proof.** First, we know from lemma 2.2 that $f \circ \phi \in W^{2,p}$ and that $\partial^2 (f \circ \phi) = f' \partial^2 \phi + f'' \partial \phi \otimes \partial \phi \in L^p \hookrightarrow L^2$, which proves the case $m = 2$. Also, if $\phi \in H^3$, we have

$$\partial^3 (f \circ \phi) \approx f' \partial^3 \phi + f'' \partial \phi \otimes \partial^2 \phi + f''' \partial \phi \otimes \partial \phi \otimes \partial \phi,$$

which is explicitly seen to be in $L^2$, since $\partial \phi \in L^\infty$ due to $p > n$. Now, to consider the cases $m = 4, 5$, notice that if $\phi \in H^4$, we can apply to product rule to the above expression to find the
expression for $\partial^4(f \circ \phi)$. Also, notice that terms which do not involve at least one second derivative on $\phi$ are immediately in $L^\infty$ due to $p > n$, so we just need to analyse terms which involve some second order factor. Thus, let $m = 4$, these terms involve $\partial^4 \phi, \partial^3 \phi \circ \partial \phi, \partial^2 \phi \circ \partial^2 \phi$ and $\partial^2 \phi \circ \partial^2 \phi$ multiplied by $L^\infty$ factors. Using $p > n$, the only one which we need to consider separately is $\partial^2 \phi \circ \partial^2 \phi$. In this case, we have the following algorithm for $m \geq 4$:

1. If $n < 2m$, then $\partial^{m-2} \phi \circ \partial^2 \phi \in L^2$ by Proposition 2.3.
2. If $n \geq 2m$, then $0 < m - 2 < \frac{n}{2}$ and $\partial^{m-2} \phi \in H^2 \hookrightarrow L^{\frac{2n}{m-2}}$. Thus, $\partial^{m-2} \phi \circ \partial^2 \phi \in L^2$ by Proposition 2.4.

Our claim for $m = 4$ follows applying this algorithm.

Let us now move to $m = 5$. Again in this case, after differentiation, we just need to evaluate those terms in $\partial^5(f \circ \phi)$ involving at least one factor with second order derivatives. These are given by terms with the structure: $\partial^5 \phi, \partial^4 \phi \circ \partial \phi, \partial^3 \phi \circ \partial^2 \phi, \partial^2 \phi \circ \partial^3 \phi, \partial^3 \phi \circ \partial^3 \phi$. All the non-trivial term is last one, where we implement the same algorithm as above this time with $m = 5$.

The case $m = 6$ is borderline, since now we need to evaluate the terms $\partial^6 \phi, \partial^5 \phi \circ \partial \phi, \partial^4 \phi \circ \partial^2 \phi, \partial^3 \phi \circ \partial^3 \phi$ as well as $\partial^5 \phi \circ \partial^3 \phi, \partial^2 \phi \circ \partial^4 \phi$ and $\partial^3 \phi \circ \partial^3 \phi$.

Let us now move to $m = 4$. In this case, the second step in our algorithm does not work any longer, since it appealed to expression for $\phi$ on $\partial^2 \phi \circ \partial^2 \phi$. Thus, $\partial^{m-2} \phi \circ \partial^2 \phi \in L^2$ by Proposition 2.4.

The condition $p > n$ deals with the first kind of products and the case of $\partial \phi, \partial^2 \phi$ follows applying this algorithm. $\partial^2 \phi$ is dealt with as follows: since $\partial^2 \phi$ is in $L^2$, we apply Proposition 2.6 to obtain $\partial^2 \phi \in H^2$.

Now we need only worry about the terms $\partial \phi, \partial^2 \phi$ since $\partial \phi, \partial^2 \phi$ obeys the hypotheses of Proposition 2.6, and is therefore in $L^2$. Finally, we deal with $\partial^2 \phi \circ \partial^2 \phi$. Notice that the multiplication property already guarantees this term to be in $L^2$ if $n < 12$, but we shall see it also holds for $n = 12$.

Let $m = 7$. Then $abc \in H^{m-1}$ for all $n \geq 3$.

Proof. As in Claim 1, if $m = 2$, then $\partial (abc) = \partial abc + a \partial bc + ab \partial c$. Since $a, b, c, \partial a \in L^\infty$, this clearly lies in $L^2$ under our hypotheses. Let us now consider the case $m = 3$, and, to simplify, we will denote $L^\infty$ factors by $\ast$, so that, for instance $\partial^3 (abc) = \partial^3 a \ast + \partial^2 b \ast + \partial^2 c \ast + \partial \partial bc \ast + \partial bc \ast + \partial \partial c \ast$, and we intend to show this lies in $L^2$. In fact, we need only worry about the product term concerning first derivatives of $\partial bc$. In this case, due to Proposition 2.6, we find $\partial \partial bc \in (H^1 \cap L^p) \cap H^1 \subset L^2$.

Let us now move to $m = 4$. In this case we have $\partial^4 (abc) = \partial^4 a \ast + \partial^3 b \ast + \partial^3 c \ast + \partial^2 a \partial b \ast + \partial^2 a \partial c \ast + \partial^2 b \partial c \ast + \partial^2 a \partial b \ast + \partial^2 c \ast + \partial \partial b \ast + \partial \partial c \ast$, and we need only worry about the terms $\partial^2 \partial bc, \partial^2 b \partial c, \partial \partial bc$, noticing the terms obtained by replacing $b$ for $c$ follow by the same arguments since $b$ and $c$ obey the same hypotheses. But now $\partial^2 a \partial b \partial c \in H^2 \cap (H^3 \cap L^p)$ and $\partial \partial bc \in (H^3 \cap L^p) \cap H^3$ all obey the hypotheses of Proposition 2.6, and are therefore in $L^2$ for any $n \geq 3$.

Now, consider the case $m = 5$, so that $\partial^5 (abc) = \partial^5 a \ast + \partial^4 b \ast + \partial^4 c \ast + \partial^3 a \partial b \ast + \partial^3 a \partial c \ast + \partial^3 b \partial c \ast + \partial^3 a \partial b \ast + \partial^3 b \partial c \ast + \partial^3 a \partial b \ast + \partial^3 b \partial c \ast + \partial^3 a \partial b \ast + \partial^3 b \partial c \ast + \partial^3 a \partial b \ast + \partial^3 b \partial c \ast$. An application of Proposition 2.6 deals with all the non-trivial terms except for $\partial^2 \partial^2 \partial \partial c$ and $\partial^2 a \partial b \partial c$. Concerning the former of these terms, we have $\partial^2 b \partial^2 c \in H^3 \cap L^p$ and we have seen at the end of the proof of Claim 1 that $H^3 \cap H^3 \subset L^2$ if $n \leq 12$. Now, regarding $\partial^2 a \partial b \partial c$, from the multiplication lemma we know that $H^4 \times H^4 \hookrightarrow H^3$ as long as $n < 14$, and therefore $\partial \partial bc \in H^1$ in all these cases. Then, since $\partial^2 a \in H^3 \cap L^p$, we apply Proposition 2.6 to obtain $\partial^2 a \partial b \partial c \in (H^3 \cap L^p) \cap H^1 \subset L^2$ for all $n < 14$. 

Claim 2: Consider $M^n$ to be a smooth compact manifold (possibly with boundary) with $n \geq 3$. Let $a \in W^{2,p} \cap H^m, b, c \in W^{1,p} \cap H^m$ with $p > n$ and $m \geq 2$.

1. If $m = 2, 3, 4$, then $abc \in H^{m-1}$ for all $n \geq 3$.
2. If $m = 5, 6, 7$, then $abc \in H^{m-1}$ for all $n \leq 12$.
In the case $m = 6$, we have $\partial_4(ab) = \partial_4 a + \partial_4 b + \partial_4 c + \partial_4 d + \partial_4 e + \partial_4 f + \partial_4 g + \partial_4 h + \partial_4 i + \partial_4 j + \partial_4 k$.

The last of these terms lies in $L^2$ for all $n < 16$ from the multiplicity property. Now, the case of $\partial_4 a b d e c f$ follows since $H^3 \otimes H^5 \hookrightarrow H^1$ as long as $n < 14$, implying that $\partial_4 a b d e f \in H^1$, and then $\partial_4 a b d e f \in H^1 \otimes (H^1 \cap L^p) \subset L^2$ follows from Proposition 2.6. A similar argument applies to show that $\partial_4 a b d e f \in H^1 \otimes H^5 \hookrightarrow H^1$ for $n < 14$ and thus $\partial_4 a b d e f \in H^1 \otimes (H^1 \cap L^p) \subset L^2$ for all $n < 14$.

Finally, $\partial_4 a b d e f \in H^3 \otimes H^4 \hookrightarrow H^1$ for $n < 16$ from the multiplicity property. Also, $\partial_4 a b d e f \in H^3 \otimes H^5 \hookrightarrow H^1$ for all $n < 16$ from the multiplicity property and hence $\partial_4 a b d e f \in H^3 \otimes (H^1 \cap L^p) \subset L^2$ from Proposition 2.6. Furthermore, $\partial_4 a b d e f \in H^3 \otimes H^5 \hookrightarrow H^1$ for all $n < 16$ from the multiplicity lemma. Next, consider $\partial_4 a b d e f \in H^4 \otimes H^5$, we have $\partial_4 a b d e f \in H^1 \otimes H^5 \hookrightarrow H^1$ for all $n < 16$. Finally, we let $\partial_4 a b d e f \in H^4 \otimes H^5 \hookrightarrow H^1$ if $n < 18$ from the multiplicity property, while in the case of $\partial_4 a b d e f$, we have $\partial_4 a b d e f \in H^5 \otimes H^5 \hookrightarrow H^1$ for all $n < 18$, which implies that $\partial_4 a b d e f \in (H^5 \cap L^p) \otimes H^1 \subset L^2$ from Proposition 2.6.

With the above tools, we can prove the $H^s$-regularity theorem associated to system (41):

**Theorem 2.7.** Let $(M, \gamma)$ be a compact manifold with boundary of dimension $n \geq 3$, let $s > \frac{n}{2} + 1$ and consider (41) satisfying (12). Assume that $\Delta_{H_{\gamma_{cont}}}$ is invertible on $M$, and that there exists a pair of strong global barrier functions $0 < \phi_- \leq \phi_+ \leq m$ and that the solution map:

$$F : \begin{cases} W^{2,p}(M) \to W^{2,p}(M) \\ \psi = (\phi, X, f) \mapsto F = \mathcal{P}^{-1} \circ F(\psi) \end{cases}$$

is invariant on $B_{\lambda_M(X, \phi)}$ for $\phi_- < \phi \leq \phi_+$. If, in addition $n \leq 12$, then the system admits a solution $\psi = (\phi, X, f) \in H^s(M)$ with $\phi > 0$.

**Proof.** Theorem 2.4 ensures there exists a solution $(\phi, X, f)$ in $W^{2,p}$ for a given $p > n$. Let us study for which $s$ one can show it is in $H^s(M)$.

A first consequence of corollaries 2.1 and 2.2 is that $H^s$ is an algebra for $l > \frac{n}{2}$, and that multiplying by $H^s$ conserves the regularity. Since $\gamma \in H^s$, multiplying by the metric or its first derivatives then does not impact the $H^s$ regularity. When considering tensors, we will not need to pay attention to their covariant and contravariant nature and we can take their local derivatives in charts or their covariant derivative without changing the final regularity.

**Step 1: Bootstrap with a solution in $H^s$, $t > \frac{n}{2} + 1$:** If $(\phi, X, f) \in H^t$ with $s > t > \frac{n}{2} + 1$, then $(\phi, X, f) \in H^s$.

As mentioned, corollary 2.1 implies that for $q > \frac{n}{2}$, $H^q$ is an algebra. Thus if we manage to obtain a solution $(\phi, X, f) \in H^t$ with $s > t > \frac{n}{2} + 1$, then:

$$\forall I \in \mathbb{R}, [\phi] \in H^t$$

$$|\nabla X|^2 \in H^{t-1}$$

$$|\nabla f|^2 \in H^{t-1}.$$ (45)

The first line clearly stands true for $I \in \mathbb{N}$ while we refer to the composition lemma 2.8 for $I \in \mathbb{R}$. Thus $|\nabla X|^2 \phi^{- \frac{2n}{n-2}}, |U|^2 \phi^{- \frac{2n}{n-2}}, (\nabla X, U)\phi^{- \frac{2n}{n-2}} \in H^{t-1}$, by applying corollary 2.2. Thus $|\tilde{K}(X)|^2 \phi^{- \frac{2n}{n-2}} \in H^{t-1}$. Similarly $|\nabla f|^2 \phi^{-3}, |\nabla f|^2 \phi^{-3}, (\nabla f, V)\phi^{-3} \in H^{t-1}$ and thus $|\tilde{E}|^2 \phi^{-3} \in H^{t-1}$. In addition for any $F \in H^{s-2}$, applying corollary 2.2 we deduce that $F \phi^t \in H^t_{\min(s-2, t-1)}$. 19
This ensures that: \( R, \phi, \partial r \phi, \partial w \) are in \( L^{\min(s-2, t-1)} \). In addition since \( V \in H^{s-1}(M) \), applying once more corollary \( 2.3 \) we conclude that \( \tilde{F}^{\gamma} \phi \) is in \( H^{\min(s-2, t-1)} \). In the same manner, since \( \tilde{F} \) and \( \tau \) are in \( H^{s-1} \) we has

\[
c_n R, \phi + c_n \left( 2c_1 - \frac{n-1}{n} c_2 \right) \phi + \frac{1}{2} c_n \left| \tilde{F} \right|^2 \phi \in H^{\min(s-2, t-1)}(M),
\]

To conclude: the right hand side of the system \( 11 \) is then in \( H^{\min(s-2, t-1)} \). Given \( 12 \) and the regularity assumed on \( u, v, w \), elliptic estimates give us that the solution is in \( H^{\min(s,t+1)}(M) \). This means that as soon as we manage to propel the solution in Step 2: \( \gamma \), \( \xi \), \( \eta \), regularity assumed on \( t > \frac{n}{2} + 1 \) to conclude that \( H \) case, all the terms on the right-hand side of the Lichnerowicz equation are then of the shape \( \tilde{K} X \). Similar, looking at the right-hand side of the momentum constraint ensures that \( \Delta \gamma \), conf \( \phi \) is in \( H^{\min(k+1, s-2)} \). Elliptic estimates then yield \( X \in H^{\min(k+1, s)} \).

Then, either \( s \leq k + 1 \), which implies that \( \Delta \gamma \), conf \( \phi \) is in \( H^{s-1} \), or \( k + 1 \leq s \). In the first case, all the terms on the right-hand side of the Lichnerowicz equation are then of the shape \( H^{s-2} \) which ensures, thanks to corollary \( 2.3 \) that \( \Delta \gamma \), conf \( \phi \) is in \( H^{\min(k+1, s-2)} \). In the second case:

\[ \Delta \gamma \), conf \( \phi \) is in \( H^{k} \). One can thus apply the claim 2 with \( a = \phi \), \( b = c = |\Delta \gamma \), conf \( X| \) to conclude that \( |\Delta \gamma \), conf \( X|^2 \phi \) is in \( H^{k-1} \). Similarly one can conclude that \( |\nabla f|^2 \phi \) is in \( H^{k-1} \).

Since \( U \) and \( V \) are more regular than \( \nabla X \) and \( \nabla f \) respectively we can once more apply the claim 2 to conclude that \( |K(X)|^2 \phi \) is in \( H^{k-1} \). Since all the other terms in the right-hand side of the Lichnerowicz equation can be controlled using corollary \( 2.3 \) we can conclude that \( \Delta \phi \) is in \( H^{\min(k+1, s-2)} \).

In all cases \( \Delta \phi \) is in \( H^{\min(k+1, s-2)} \) which, thanks to elliptic estimates, yields \( \phi \) is in \( H^{\min(k+1, s)} \). In conclusion, claims 1 and 2 ensure that if \( (\phi, X, f) \in H^{k} \) implies that \( (\phi, X, f) \in H^{\min(k+1, s)} \). There are then three possibilities:

1. \( \min(k + 1, s) = s \), which is the desired result,
2. \( \min(k + 1, s) = k + 1 \) but \( k > \frac{n}{2} \), in which case step 1 ensures the desired result,
3. \( \min(k + 1, s) = k + 1 \) and \( k \leq \frac{n}{2} \), in which case \( (\phi, X, f) \in H^{k+1} \) and we can go through the loop once more.

We can implement the above algorithm, starting with \( (\phi, X, f) \in H^{2} \) which is a consequence of \( (\phi, X, f) \in W^{2,p} \). This algorithm terminates only if \( 1 \) or \( 2 \) are satisfied, which is guaranteed to happen in a finite number of steps since claims 1 and 2 ensure we can go to \( k = 7 \), where \( 7 > \frac{n}{2} \) since \( n \leq 12 \).

\[ \square \]

\textbf{Remark 2.3:} It is interesting to compare this section with the work of D. Maxwell on the uncoupled equations \( 11 \) or \( 12 \) in order to understand why we could not do an iterative scheme of the manner of theorem \( 2.1 \). Indeed the nonlinear nature of the Lichnerowicz equation prevents one from simply using the algebra nature of \( H^{s} \): the iterative inequality \( 13 \) would become nonlinear, and thus non converging. D. Maxwell solved this issue with an ingenious and sharp lemma allowing to control the terms \( \phi \) thanks to the \( L^{\infty} \) norm of \( \phi \) (uniformly controlled by the barrier functions), times its \( H^{s} \) norm (see lemma 2.5 in \( 11 \)), keeping the linear nature of \( 12 \) in the uncoupled case. However, this falls short when dealing with the coupled terms \( |\nabla X|^2 \phi \), where no uniform estimate of \( |\nabla X|^2 \) is at hand in our framework.
Remark 2.4. Here the limitation $n \leq 12$ comes explcitly as the most restrictive constraint on the dimension in the proofs of claims 1 and 2. Without the starting regularity in $W^{2,p}$, this limitation would have been far stronger. A way to push this limitation is to make sure that the data allow for a bootstrap in $W^{k,p}$ regularity which can then be moved to the $H^s$ one. It is also interesting to notice how the triangular structure of the equations allowed us to push to dimension 12 by first improving the regularity on $(X,f)$ before moving back to $\phi$.

3 Einstein-type systems on a complete manifold

In this section, we first prove the existence of $L^p$ solutions to the constraint equations on a complete manifold, without the electromagnetic constraint (model case with $q = 0$). We briefly explain how this method can deal with the electromagnetic constraint. We then, as in the previous section, extend the existence to $H^s$ solutions.

3.1 $W^{2,p}$ existence: global estimates

We wish to show existence results for an Einstein-type system on a complete manifold $(M, \gamma)$ of dimension $n$. Since, once more, the exact shape of the equation (and the form of the coupled non-linearities) will be relevant, we will work with the simplified (albeit coupled) system highlighted in introduction. Let us thus recall system (4) on $(M, \gamma)$:

$$\begin{align*}
\Delta_\gamma \phi - c_n R_\gamma \phi + c_n \tilde{K}(X)^2 \frac{n-1}{n} n^{-2} \phi^{\frac{n-2}{n}} + 2 c_n \epsilon_1 \phi^{\frac{n-2}{n}} + 2 c_n \epsilon_2 \phi^{-3} + 2 c_n \epsilon_3 \frac{1}{\phi} = 0,
\Delta_{\gamma, conf} X - \frac{n-1}{n} \nabla \tau \phi \frac{2n}{n-2} - \omega_1 \phi^{\frac{n+2}{n-2}} + \omega_2 = 0,
\end{align*}$$

where given $p > n$, 
$$\gamma \in W^{2,p}_0(M), \ R_\gamma, \epsilon_2, \epsilon_3, |U|^2, \tau^2 \in L^p_{loc}(M) \ \text{and} \ \omega_1, \omega_2, \nabla \tau \in L^2(M) \cap L^p_{loc}(M).$$ (46)

Equation (46) corresponds to the case $q = 0$, and thus with no electromagnetic constraint in the physically motivated constraint equations (18).

Definition 3.1. We say that the system (46) on a complete manifold $(M, \gamma)$ admits global barrier functions on $M$ if there exists $m \in \mathbb{R}$ and $0 < \phi_- \leq \phi_+ \leq m < +\infty$ such that:

$$\forall \phi_- \leq \varphi \leq \phi_+, \ \forall Y \in L^2(M) \ s.t. \ \Delta_\gamma \phi Y - \frac{n-1}{n} \nabla \tau \phi \frac{2n}{n-2} - \omega_1 \phi^{\frac{n+2}{n-2}} + \omega_2 = 0$$

$$\Delta_\gamma \phi_- \geq c_n \left( R_\gamma \phi_- - \left| \tilde{K}(Y) \right|_2 \phi_-^{\frac{n-2}{n}} + c_n \frac{n-1}{n} \nabla \phi_-^{\frac{n+2}{n-2}} - 2 c_n \epsilon_1 \phi_-^{\frac{n-2}{n}} - 2 c_n \epsilon_2 \phi_-^{-3} - 2 c_n \epsilon_3 \phi_-^{\frac{n}{n-2}} \right) \text{ in } M$$

$$\Delta_\gamma \phi_+ \leq c_n \left( R_\gamma \phi_+ - \left| \tilde{K}(Y) \right|_2 \phi_+^{\frac{n-2}{n}} + c_n \frac{n-1}{n} \nabla \phi_+^{\frac{n+2}{n-2}} - 2 c_n \epsilon_1 \phi_+^{\frac{n-2}{n}} - 2 c_n \epsilon_2 \phi_+^{-3} - 2 c_n \epsilon_3 \phi_+^{\frac{n}{n-2}} \right) \text{ in } M$$

We can ensure, through a diagonal argument that will have many uses throughout these studies the existence of solutions to the momentum equations. This argument will rely on a spectral estimate for the conformal Killing Laplacian. To that end let us define:

Definition 3.2. Let $(M, \gamma)$ be a complete manifold of dimension $n$. The first eigenvalue of the conformal Killing Laplacian is defined as:

$$\lambda_{1, \gamma, \text{conf}} \doteq \inf_{u \in C^\infty_{p}((M, TM)) \setminus \{0\}} \frac{\int_M |\mathcal{L}_{\gamma, \text{conf}} u|^2 \ dV_\gamma}{\int_M |u|^2 \ dV_\gamma}.$$

Remark 3.1. In the following, we will rely on a control $\lambda_{1, \gamma, \text{conf}} > 0$ to deduce uniform $L^2$ controls. In addition, such a control ensures the invertibility of the conformal Killing Laplacian on bounded open sets $\Omega \subset M$ with Dirichlet boundary condition. Indeed, if $X$ belongs to the kernel of such an operator then since $\int_{\Omega} |\mathcal{L}_{\gamma, \text{conf}} X|^2 \ dV_\gamma = - \int_{\Omega} X \cdot \Delta_{\gamma, \text{conf}} X \ dV_\gamma = 0$, and since $X \in W^{1,2}_0$ it can be approached by compactly supported smooth functions. Then either $X = 0$, or $\lambda_{1, \gamma, \text{conf}} = 0$. 

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Lemma 3.1. Assume that on a complete manifold \((M, \gamma)\) of dimension \(n \geq 3\) the system \([\text{40}]\) admits two global barrier functions, and that the first eigenvalue of the conformal Killing Laplacian satisfies
\[
\lambda_{1, \gamma, \text{conf}} > 0.
\]

Then, for any \(\varphi \in L^\infty\) such that \(\phi_- \leq \varphi \leq \phi_+ \leq m\) there exists an \(L^2(M) \cap W^{2,p}_0(M)\) solution \(X_\varphi\) to the equation
\[
\Delta_{\gamma, \text{conf}} X_\varphi \equiv \frac{n-1}{n} \nabla \varphi \frac{2n}{n-2} - \omega_1 \varphi^{\frac{n+4}{n-4}} + \omega_2 = 0.
\]

In addition, for any compact with smooth boundary \(K \subset K' \subset M\) one has the following control on \(X_\varphi\):
\[
\|X_\varphi\|_{W^{2,p}(K')} \leq \frac{C(n, \gamma, K)}{\lambda_{1, \gamma, \text{conf}}} \left( \left\| \nabla \varphi \right\|_{L^2(M)} + \left\| \nabla \varphi \right\|_{L^p(K')} \right) m^{\frac{2n}{n-2}}
\]
\[
+ \left( \left\| \omega_1 \right\|_{L^2(M)} + \left\| \omega_1 \right\|_{L^p(K')} \right) m^{\frac{n+4}{n-4}} + \left( \left\| \omega_2 \right\|_{L^2(M)} + \left\| \omega_2 \right\|_{L^p(K')} \right).
\]

Proof. Step 1: Existence of a compact exhaustion
Let us fix \(\phi_- \leq \varphi \leq \phi_+ \leq m\) and \((\Omega_k)\) a compact exhaustion of \(M\) with smooth boundaries, and compact subdivisions \((U_k)\) and \((V_k)\) such that:
\[
\Omega_{k-1} \subset U_k \subset V_k \subset \Omega_k.
\]

Such a compact exhaustion follows from the completeness of the manifold and a classical regularization procedure to ensure the smoothness of the boundary (see remark 3.2. in \([\text{2}]\)).

Step 2: Uniform controls
Let us now consider \(X_k\) a solution of the Poisson-type equation with Dirichlet boundary condition:
\[
\begin{aligned}
\Delta_{\gamma, \text{conf}} X_k - \frac{n-1}{n} \nabla \varphi \frac{2n}{n-2} - \omega_1 \varphi^{\frac{n+4}{n-4}} + \omega_2 &= 0 \text{ in } \Omega_k, \\
X_k &= 0 \text{ on } \partial \Omega_k.
\end{aligned}
\]

The existence of the solution to this system directly follows from the invertibility of the operator \((\Delta_{\gamma, \text{conf}}, 0)\) on the compact \(\Omega_k\).

Let us now fix \(s < k\). Thanks to interior estimates for elliptic systems, one can establish that:
\[
\|X_k\|_{W^{2,2}(U_s)} \leq C(U_s, V_s, \gamma) \left( \left\| \frac{n-1}{n} \nabla \varphi \frac{2n}{n-2} + \omega_1 \varphi^{\frac{n+4}{n-4}} - \omega_2 \right\|_{L^2(V_s)} + \|X_k\|_{L^2(V_s)} \right).
\]

In addition, given the boundary condition \(X_k = 0\) on \(\partial \Omega_k\), we can estimate:
\[
\|L_{\gamma, \text{conf}} X_k\|_{L^2(\Omega_k)}^2 = - \int_{\Omega_k} X_k \left( \frac{n-1}{n} \nabla \varphi \frac{2n}{n-2} + \omega_1 \varphi^{\frac{n+4}{n-4}} - \omega_2 \right) dV, \\
= \left| \int_{\Omega_k} X_k \left( \frac{n-1}{n} \nabla \varphi \frac{2n}{n-2} + \omega_1 \varphi^{\frac{n+4}{n-4}} - \omega_2 \right) dV \right| \\
\leq \|X_k\|_{L^2(\Omega_k)} \left\| \frac{n-1}{n} \nabla \varphi \frac{2n}{n-2} + \omega_1 \varphi^{\frac{n+4}{n-4}} - \omega_2 \right\|_{L^2(\Omega_k)}.
\]

Further, since \(X_k \in W^{1,2}_0(\Omega_k)\) we can approach \(X_k\) in the \(W^{1,2}\) topology by compactly supported smooth vector fields and deduce:
\[
\lambda_{1, \gamma, \text{conf}} \leq \frac{\|L_{\gamma, \text{conf}} X_k\|_{L^2(\Omega_k)}^2}{\|X_k\|_{L^2(\Omega_k)}^2}.
\]

Using the hypothesis \(\lambda_{1, \gamma, \text{conf}} > 0\), we rephrase the previous inequality as:
\[
\|X_k\|_{L^2(\Omega_k)}^2 \leq \frac{1}{\lambda_{1, \gamma, \text{conf}}} \|L_{\gamma, \text{conf}} X_k\|_{L^2(\Omega_k)}^2.
\]
Injecting (51) into (52) then yields:
\[ \|X_k\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda_{1,\gamma,\text{conf}}} \|X_k\|_{L^2(\Omega)} \left( \frac{n-1}{n} \nabla \tau \varphi^{n/2} + \omega_1 \varphi^{3/2} - \omega_2 \right)_{L^2(\Omega)}, \]
which ensures the uniform estimate on \( V_n \):
\[ \|X_k\|_{L^2(\Omega)} \leq \frac{1}{\lambda_{1,\gamma,\text{conf}}} \left( \frac{n-1}{n} \|\nabla \tau\|_{L^2(M)} m^{n/2} + \|\omega_1\|_{L^2(M)} m^{3/2} + \|\omega_2\|_{L^2(M)} \right). \]  
Inserting (54) into (50) then yields:
\[ \|X_k\|_{W^{2,2}(U)} \leq \frac{C(n, U, V, \gamma)}{\lambda_{1,\gamma,\text{conf}}} \left( \|\nabla \tau\|_{L^2(M)} m^{n/2} + \|\omega_1\|_{L^2(M)} m^{3/2} + \|\omega_2\|_{L^2(M)} \right). \]  
Let us denote \( (p_j) \in \mathbb{N} \) the sequence defined by induction as: \( p_0 = 2, \ p_{j+1} = \frac{np_j}{n-2p_j} \), and \( j_{\text{max}} \) is the first integer for which \( p_{j_{\text{max}}} \geq \frac{n}{2} \). If \( j_{\text{max}} > 0 \), consider \( \Omega_{s-1} \subset \subset U_{j_{\text{max}}} \subset \subset \cdots \subset \subset U_s = U_0 \), and show by induction that for all \( i < j_{\text{max}} \), one has
\[ \|X_\varphi\|_{W^{2,p_i}(U)} \leq \frac{C(n, \gamma, U_1, \ldots, U_s, V)}{\lambda_{1,\gamma,\text{conf}}} \left( \|\nabla \tau\|_{L^2(M)} + \|\nabla \tau\|_{L^P(U_i)} \right) m^{n/2} + \left( \|\omega_1\|_{L^2(M)} + \|\omega_1\|_{L^P(U_i)} \right) m^{n/2} + \left( \|\omega_2\|_{L^2(M)} + \|\omega_2\|_{L^P(U_i)} \right). \]  
Inequality (54) shows that (55) stands for \( i = 0 \). If we assume that it stands for \( i < j_{\text{max}} \), then applying Sobolev estimates on the compact set \( U_i \) yield:
\[ \|X_k\|_{L^{P_{i+1}}(U_i)} \leq \frac{C(U_i, n)}{\lambda_{1,\gamma,\text{conf}}} \|X_\varphi\|_{W^{2,p_{i+1}}(U_i)} \]
\[ \leq \frac{C(n, \gamma, U_1, \ldots, U_s, V)}{\lambda_{1,\gamma,\text{conf}}} \left( \|\nabla \tau\|_{L^2(M)} + \|\nabla \tau\|_{L^P(U_i)} \right) m^{n/2} + \left( \|\omega_1\|_{L^2(M)} + \|\omega_1\|_{L^P(U_i)} \right) m^{n/2} + \left( \|\omega_2\|_{L^2(M)} + \|\omega_2\|_{L^P(U_i)} \right). \]  
while interior estimates yield:
\[ \|X_\varphi\|_{W^{2,p_{i+1}}(U_{i+1})} \leq \frac{C(n, \gamma, U_1, \ldots, U_{i+1})}{\lambda_{1,\gamma,\text{conf}}} \left[ \|\nabla \tau\|_{L^P(U_{i+1})} m^{n/2} + \|\omega_1\|_{L^P(U_{i+1})} m^{n/2} + \|\omega_2\|_{L^P(U_{i+1})} \right] \]
\[ \leq \frac{C(n, \gamma, U_1, \ldots, U_{i+1})}{\lambda_{1,\gamma,\text{conf}}} \left[ \|\nabla \tau\|_{L^P(U_i)} m^{n/2} + \|\omega_1\|_{L^P(U_i)} m^{n/2} + \|\omega_2\|_{L^P(U_i)} \right]. \]  
Injecting (56) into (57) then yields (58) for \( i + 1 \), which proves the result for \( p_{j_{\text{max}}} \) when \( j_{\text{max}} > 0 \), and since the estimate derives naturally from (54) when \( j_{\text{max}} = 0 \), it stands true in all cases. This time Sobolev embeddings ensure that \( X \in L^P(U_{j_{\text{max}}}) \), and interior estimates again show that \( X_k \) satisfies:
\[ \|X_k\|_{W^{2,p}(\Omega)} \leq \frac{C(n, \gamma, \Omega_{s-1}, U_1, \ldots, U_{s})}{\lambda_{1,\gamma,\text{conf}}} \left[ \|\nabla \tau\|_{L^2(M)} + \|\nabla \tau\|_{L^P(\Omega)} \right] m^{n/2} + \left( \|\omega_1\|_{L^2(M)} + \|\omega_1\|_{L^P(\Omega)} \right) m^{n/2} + \left( \|\omega_2\|_{L^2(M)} + \|\omega_2\|_{L^P(\Omega)} \right). \]  
Since we chose the \( U_j \), we can fix them once and for all for any \( \Omega_{s-1} \) and deduce:
\[ \|X_k\|_{W^{2,p}(\Omega_{s-1})} \leq \frac{C(n, \gamma, \Omega_{s-1})}{\lambda_{1,\gamma,\text{conf}}} \left[ \|\nabla \tau\|_{L^2(M)} + \|\nabla \tau\|_{L^P(\Omega_{s-1})} \right] m^{n/2} + \left( \|\omega_1\|_{L^2(M)} + \|\omega_1\|_{L^P(\Omega_{s-1})} \right) m^{n/2} + \left( \|\omega_2\|_{L^2(M)} + \|\omega_2\|_{L^P(\Omega_{s-1})} \right). \]
Notably, estimate \( (59) \) does not depend on \( k \).

**Step 3:** Diagonal extraction

Thanks to steps 1 and 2, we can produce a sequence of solutions \((X_k)\) on the compact exhaustion \((\Omega_k)\) of \( M \), uniformly bounded on interior compacts in \( W^{2,p} \).

We can then extract from \((\psi_k)\) a weak \( W^{2,p} \)-converging subsequence \((X_{n_k}(\Omega_1))\) on \( \Omega_1 \) toward a solution \( X_1 \) of \((18)\) on \( \Omega_1 \). From \((X_{n_k}(\Omega_1))\) we then extract a subsequence \((X_{n_k}(\Omega_2))\) which converges toward \( X_2 \) solving \((18)\) on \( \Omega_2 \), such that \( X_2|_{\Omega_1} = X_1 \). Similarly, we build \((X_{n_k}(\Omega_3))\) converging toward \( X_3 \), solving \((18)\) on \( \Omega_3 \), and extending the previous solutions. Thus the diagonal sequence \((\bar{X}_k) = (X_{n_k}(\Omega_k))\) satisfies

\[
\bar{X}_k \to \bar{X}_i \in W^{2,p}(\Omega_i), \quad \forall t \in \mathbb{N}.
\]

By Sobolev embeddings the convergence is strong in \( L^p_{\text{loc}} \), and given a compact \( K \), for \( k \) and \( k' \) large enough, \( \Delta_{\gamma,\text{conf}}(\bar{X}_{k'} - \bar{X}_k) = 0 \) on \( K \). Interior estimates then ensure that, given \( K \subset K' \):

\[
\|\bar{X}_{k'} - \bar{X}_k\|_{W^{2,p}(K)} \leq C(n, \gamma, K)\|\bar{X}_{k'} - \bar{X}_k\|_{L^p(K')},
\]

Since \( \bar{X}_k \) is \( L^p \) Cauchy, it is Cauchy in \( W^{2,p}_{\text{loc}} \).

Taking \( X_\varphi = \bar{X}_t \) on \( \Omega_t \), and taking \((59)\) to the limit then ensures the existence of a solution, and the uniform estimates on compact subsets. In addition, thanks to \((59)\) one can check that for any fixed \( s \)

\[
\|\bar{X}_k\|_{L^p(\Omega_t)} \leq \frac{1}{\lambda_{1,\gamma,\text{conf}}} \left( \frac{n-1}{n} \|\nabla \tau\|_{L^2(M)} \frac{2n}{m} \frac{m+1}{2n+2} + \|\omega_1\|_{L^2(M)} \frac{n+1}{m+1} + \|\omega_2\|_{L^2(M)} \right).
\]

Taking this inequality to the limit in \( k \) and then the supremum in \( s \) ensures the limit is in \( L^2(M) \) with the proper estimate.

**Theorem 3.1.** Assume that on a complete manifold \((M, \gamma)\) of dimension \( n \geq 3 \) the system \((18)\) satisfies \((47)\) and admits two global barrier functions \( 0 < \phi_- \leq \phi_+ \leq m < \infty \), and that the first eigenvalue of the conformal Killing Laplacian satisfies

\[
\lambda_{1,\gamma,\text{conf}} > 0.
\]

Then \((18)\) admits a \( W^{2,p}_{\text{loc}} \) solution.

**Proof.** Let us consider a compact exhaustion \((\Omega_k)\) of \( M \) with smooth boundary. Let \( \phi_0 = \frac{\phi_+ + \phi_-}{2} \) and \( X_0 \) the solution of \( \Delta_{\gamma,\text{conf}} X_0 - \frac{2n}{m} \nabla \tau \phi_0 \frac{2n}{m} - \omega_1 \phi_0 \frac{2n+2}{2n} + \omega_2 = 0 \) obtained thanks to lemma \[3.1\]. We now define \((\phi_k, X_k)\) by induction:

\[
\begin{align*}
\Delta_{\gamma,\text{conf}} X_{k+1} &= -\frac{n-1}{n} \nabla \tau \phi_k^n - \omega_1 \phi_k^n + \omega_2 = 0 \quad \text{in } M \\
\Delta_\gamma \phi_{k+1} &= -c_n R_\gamma \phi_{k+1} + c_n \left| K(X_{k+1}) \right|_\gamma \phi_{k+1}^\frac{2n-2}{m} - c_n \frac{n-1}{n} \phi_{k+1}^\frac{n+2}{2} \\
+ 2c_n \epsilon_1 \phi_{k+1}^\frac{2n-2}{m} + 2c_n \epsilon_2 \phi_{k+1}^\frac{2n-2}{m} + 2c_n \epsilon_3 \phi_{k+1}^\frac{2n-2}{m} = 0 \quad \text{in } \Omega_{k+1} \\
\phi_{k+1} &= \frac{\phi_+ + \phi_-}{2} \quad \text{on } \partial \Omega_{k+1},
\end{align*}
\]

where

\[
\varphi_k = \begin{cases} 
\phi_k & \text{on } \Omega_k \\
\frac{\phi_+ + \phi_-}{2} & \text{on } M \setminus \Omega_k.
\end{cases}
\]

Notice that in lemma \[3.1\], although \( \varphi \) plays the role of \( \phi \), we do not require the same regularity for it: the \( L^\infty \) estimates provided by the barrier functions are enough.

**Step 1:** \((\phi_k, X_k)\) is well defined and \( \phi_- \leq \phi_k \leq \phi_+ \leq m \).

Let us proceed by iteration.
• $\phi_- \leq \phi_0 \leq \phi_+ \leq m$ is well defined, and so is $X_0$ thanks to lemma 3.1

• Assuming that $(\phi_k, X_k)$ are well defined with $\phi_- \leq \phi_k \leq \phi_+ \leq m$, one has that $\phi_- \leq \varphi_k \leq \phi_+$, and thus thanks to lemma 3.1 $X_{k+1}$ is well defined. Since $\phi_-$ and $\phi_+$ are global barriers, they satisfy:

\[
\Delta \gamma \phi_- \geq c_n \left( R_{\gamma} \phi_- - |\tilde{K}(X_{k+1})| \right)^2 |\nabla \phi_- - \frac{n-1}{n} \tau^2 \phi_-^{n/2} - 2c_n \epsilon_1 \phi_-^{n/2} - 2\epsilon_2 \phi_-^{2/3} - 2\epsilon_3 \phi_-^{n/6}\right)
\]

\[
\Delta \gamma \phi_+ \leq c_n \left( R_{\gamma} \phi_+ - |\tilde{K}(X_{k+1})| \right)^2 |\nabla \phi_+ - \frac{n-1}{n} \tau^2 \phi_+^{n/2} - 2c_n \epsilon_1 \phi_+^{n/2} - 2\epsilon_2 \phi_+^{2/3} - 2\epsilon_3 \phi_+^{n/6}\right)
\]

in $M$, and thus on $\Omega_{k+1}$. They are thus barrier for the Lichnerowicz equation with Dirichlet boundary conditions (in fact strong global barriers since the equation is entirely decoupled) and thanks to theorem 2.2 it admits a solution $\phi_- \leq \phi_{k+1} \leq \phi_+$.

**Step 2:** Uniform estimates

Thanks to lemma 3.1, $(X_k)$ satisfies the uniform estimates on compacts $K \subset \subset K'$ of $M$:

\[
\|X_k\|_{W^{2,p}(K)} \leq C(n, \gamma, K') \lambda_{1, \text{conf}} \left( \left( \|\nabla \gamma\|_{L^2(M)} + \|\nabla \tau\|_{L^p(K')} \right) m^{n/2} \right) + \left( \|\gamma_1\|_{L^2(M)} + \|\gamma_0\|_{L^p(K')} \right) m^{n/2} \left( \|\gamma_2\|_{L^2(M)} + \|\gamma_0\|_{L^p(K')} \right).
\]

(60)

Considering $K \subset \subset K' \subset \Omega_k$ and applying uniform interior estimates with the Lichnerowicz equation yields:

\[
\|\phi_k\|_{W^{2,p}(K)} \leq \|R_{\gamma} \phi_+ \|_{L^p(K')} \|\phi_+\|_{L^\infty(M)} + \|\tilde{K}_{k+1}\|_{L^p(K')} \|\phi_-^{n/2}\|_{L^\infty(K')} + \|\tau^2 L^{p}(\phi_+^{n/2})\|_{L^p(K')} + \|\epsilon_1\|_{L^p(K')} \|\phi_-^{n/2}\|_{L^\infty(K')} + \|\epsilon_2\|_{L^p(K')} \|\phi_-^{n/2}\|_{L^\infty(K')} + \|\epsilon_3\|_{L^p(K')} \|\phi_+\|_{L^p(K')}.
\]

(61)

Notice that, $|\tilde{K}_{k+1}|^2 \leq 2 \left( L^{p}(\gamma_{\text{conf}} X_{k+1}^{2} + |U|^2) \right) \leq |DX_k|^2 + |U|^2$. Thus,

\[
\|\tilde{K}_{k+1}\|_{L^p(K')} \leq \|X_k\|_{W^{2,p}(K')} + \|U\|^2_{L^p(K')},
\]

using the Sobolev embeddings on the compact $K'$. Now thanks to (60) and (61) we deduce:

\[
\|\phi_k\|_{W^{2,p}(K')} \leq \|\phi_+\|_{L^\infty(M)} + \|\phi_-^{n/2}\|_{L^\infty(K')} + \|\phi_+\|_{L^\infty(M)} + \|\phi_+^{n/2}\|_{L^\infty(M)} + \|\phi_+\|_{L^\infty(K')},
\]

where the implicit constants depend on $K'$ (that is on the local $L^p$ norms of all the other terms), $\gamma, \lambda_1^{\text{conf}}, n, \|\frac{1}{\phi_-}\|_{L_\infty(K')}, \|\phi_+\|_{L_\infty(M)}, \|\nabla \gamma\|_{L^2(M)}, \|\nabla \tau\|_{L^p(K')}, \|\omega\|_{L^p(K')}, \|\epsilon_1\|_{L^p(K')}, \|U\|^2_{L^p(K')}$. We can then conclude:

\[
\|\phi_k\|_{W^{2,p}(K)} \leq C \left( K', \gamma, \lambda_1^{\text{conf}}, n, \|\frac{1}{\phi_-}\|_{L_\infty(K')}, \|\phi_+\|_{L_\infty(M)}, \|\nabla \gamma\|_{L^2(M)}, \|\nabla \tau\|_{L^p(K')}, \|\omega\|_{L^p(K')}, \|\epsilon_1\|_{L^p(K')}, \|U\|^2_{L^p(K')}. \right)
\]

(63)

**Step 3:** Diagonal extraction

We can then, thanks to steps 1 and 2, produce a sequence of solutions $\psi_k = (\phi_k, X_k)|_{\Omega_k}$ to:

\[
\begin{cases}
\Delta \gamma_{\text{conf}} X_{k+1} - \frac{n-1}{n} \nabla \tau \phi_k^{n/2} - \omega_1 \phi_k^{2n/3} + \omega_2 = 0 \text{ in } \Omega_k \\
\Delta \gamma \phi_{k+1} - c_n R_{\gamma} \phi_{k+1} + c_n \left| \tilde{K}(X_{k+1}) \right| \phi_{k+1}^{n/2} - c_n \frac{n-1}{n} \tau^2 \phi_{k+1}^{n/2} - 2c_n \epsilon_1 \phi_{k+1}^{n/2} - 2\epsilon_2 \phi_{k+1}^{2/3} - 2\epsilon_3 \phi_{k+1}^{n/6} = 0 \text{ in } \Omega_k.
\end{cases}
\]

(64)

We actually apply (60) to $K'$ and a slightly larger compact without loss of generality
on a compact exhaustion \((\Omega_k)\) of \(M\), uniformly bounded on the interior compacts.

We can thus extract a weakly \(W^{2,p}\)-converging subsequence \((\psi_{n_k}(\Omega_1))\) on \(\Omega_1\) toward a \(\bar{\psi}_1 \in W_{loc}^{2,p}\). The Sobolev embeddings then ensure that, since \(p > n\) the convergence is strong in \(C^1\) on compacts, and thus each term in the system except for the principal part is guaranteed to converge in \(L^p\) on compacts (see example 2.1 for the Einstein-type structure of system (40)). The equation (41) thus goes to the limit and yields a weak solution of the system (40) on \(\Omega_1\). In addition, as has already been done in the proof of lemma 3.1 interior estimates ensure that \((\psi_{n_k}(\Omega_1))\) is Cauchy in \(W^{2,p}\) on interior compacts of \(\Omega_1\), and thus that the convergence \((\psi_{n_k}(\Omega_1)) \to \bar{\psi}_1\) is strong in \(W^{2,p}\).

From \((\psi_{n_k}(\Omega_1))\) we then extract a subsequence \((\psi_{n_k}(\Omega_2))\) which converges toward \(\bar{\psi}_2\) on \(\Omega_2\), such that \(\bar{\psi}_{2|\Omega_1} = \bar{\psi}_1\). Similarly we build \((\psi_{n_k}(\Omega_m))\) converging toward \(\psi_m\), solving (40) on \(\Omega_m\), and extending the previous solutions. Thus the diagonal sequence \(\left(\bar{\psi}_k\right)\) satisfies:

\[
\bar{\psi}_k \to \bar{\psi}_m \in W^{2,p}(\Omega_t) \quad \forall t \in \mathbb{N}.
\]

The sequence \(\left(\bar{\psi}_k\right)\) is thus Cauchy on \(W^{2,p}_{loc}\) and thus yields a limit \(\psi\) in \(W^{2,p}_{loc}(M)\). Since it is an extension of each \(\bar{\psi}_i\), it is a solution of (40) on \(M\).

**Remark 3.2.** Classical elliptic regularity ensure that, in theorem 3.1, more regularity in the data yields more regularity for the solution. In particular, smooth data will yield smooth solutions to the Einstein constraint equations.

**Remark 3.3.** One might notice that here we switched from strong global barrier functions (as used by the first authors in [3] and utilized in section 2) to global barrier functions (see [39, 40, 41]). The difference lies in the space of admissible \(X\) for which the barriers are sub and super solutions of the Lichnerowicz equation (for all \(X\) in a \(W^{2,p}\) ball in the first case, for all \(X\) solving an equation in the second one) and the conditions required to solve the equation (a fixed domain hypotheses in the strong global framing, an a priori estimate on the equation for the global one). While on compacts or AE-manifolds, these two notions seem to be interchangeable (see [3]: the fixed domain hypothesis comes precisely from the type of a priori estimates required from the barrier functions), on the complete domain, with the scheme that we used, strong global barrier functions a priori demand expansion theorems in Sobolev spaces \(W^{2,p}\) with uniform constants to go from \(X \in W^{2,p}(\Omega_k)\) to \(\hat{X} \in W^{2,p}(M)\). On the other hand, in our proof using the global barrier functions we only need to expand the right-hand side of the equation in \(L^p\).

**Remark 3.4.** The matter of how flexible our iterative scheme is, underscores several interesting questions. For Einstein-type systems (13), where the \(Y\) equations are decoupled once \(\phi\) is considered a data of the equation, one can proceed as in lemma 3.1 and theorem 3.1 under the appropriate spectral and integrability assumptions.

If we no longer assume that the \(Y\)-system is decoupled but triangular, as in system (13):

\[
\begin{align*}
\Delta_{\gamma} \phi - c_n R_{\gamma} \phi + c_n \left(\nabla X |^2 + |U|^2\right) &\phi - \frac{n-2}{n} \nabla X \phi + c_n \left(2 \alpha_1 - \frac{n-1}{n} \tau^2\right) \phi - \frac{n+2}{n+2} \phi^3 + 2c_n \left(\nabla f |^2 + |V|^2\right) \phi^{-3} \\
+ \frac{1}{c_n} \left|\bar{F}\right|^2 &\phi^{-\frac{n+2}{n-2}} = 0 \text{ in } M \\
\Delta_{\gamma} \text{conf} X - \frac{n-1}{n} \nabla \tau \phi^{-\frac{n-1}{n}} - \omega_1 \phi^{-\frac{n+2}{n-2}} + \bar{F}_{ik} \left(\nabla^k f + V^k\right) = 0 &\text{ in } M \\
\Delta_{\gamma} f = \tilde{q} \phi^{-\frac{n+2}{n-2}} &\text{ in } M,
\end{align*}
\]

one can assume that \(\tilde{q} \in L^2(M)\), \(\lambda_{\gamma} \tilde{q} > 0\), and first find solutions \(f_{\phi}\), and then the corresponding \(X_{\phi}\). The latter however requires that \(\nabla \tau, \omega_1, \bar{F}_{ik} V^k\), and \(\bar{F}_{ik} \nabla^k f_{\phi}\) lie in \(L^2(M)\). This requires a uniform \(L^2\) control on \(\nabla f_{\phi}\), and thus a uniform \(W^{1,2}\) a priori estimate on \(f_{\phi}\). Working as we do below for [58] and [61] we can obtain such an estimate in the general case, and thus extend our existence result to such "triangular" systems. To remain concise we will not prove this result. It must however be pointed out that the spectral hypothesis here is topologically constraining and for instance excludes closed manifolds.
3.2 $H^s$ solutions

We will once more work on turning the $L^p$ initial data sets into $H^s$ solutions, this time on a complete manifold, starting with a set of initial conditions. Let us thus consider $s > \frac{n}{p} + 1$ and assume:

\[
\gamma \in H^s_{\text{loc}}(M), \tau, U \in H^{s-1}_{\text{loc}}(M), R_\gamma, \epsilon_1, \epsilon_2, \epsilon_3 \in H^{s-2}_{\text{loc}}(M).
\] (65)

As was evidenced in section 2.3 these assumptions imply the local $L^p$ controls for the terms in the equation. However, to work on a complete manifold, one needs the uniform control on $\|\nabla \tau\|_{L^p(M)}$ and $\|\omega\|_{L^p(M)}$ in order to obtain the starting uniform estimate on $\|X_k\|_{L^2(\Omega)}$ (see (63)). We will thus assume:

\[
\omega_1, \omega_2, \nabla \tau \in L^2(M) \cap H^{s-2}_{\text{loc}}(M).
\] (66)

**Theorem 3.2.** On a complete manifold $(M, \gamma)$ of dimension $n \geq 3$ we consider the system (46) satisfying (65) and (66) on $M$. Assume that there are two global barrier functions $0 < \phi_- \leq \phi_+ \leq m < \infty$, and that the first eigenvalue of the conformal Killing Laplacian satisfies

\[
\lambda_{1, \gamma, \text{conf}} > 0.
\]

Then, if $n \leq 12$ (46) admits a $H^s$ solution such that:

- $0 < \phi_- \leq \phi \leq \phi_+ \leq m < \infty$ on $M$.
- $\Psi \in H^s_{\text{loc}}(M)$.

**Proof.** As mentioned, thanks to Sobolev embeddings (65) ensures that the local $L^p$ controls of (65) are satisfied, while (66) contains the uniform Lebesgue controls required to apply theorem (64). We thus have a $W^{2,p}_{\text{loc}}$ solution to system (10). We can then proceed as in the proof of theorem 2.7 of which we briefly recall the steps:

1. Propositions 2.4, 2.5 and 2.6 as well as lemma 2.1 and corollaries 2.1, 2.2 and 2.3 stand on compact subsets of $M$.
2. Using the $W^{2,p}_{\text{loc}}$ estimates on the solution, we can control $\nabla \Psi \in W^{1,\infty}_{\text{loc}}$.
3. Then, using interior estimates one can reproduce the proof of theorem 2.7 in compact subsets of decreasing sizes of $M$.
4. Since the algorithm proving the theorem 2.7 only involves a finite number of steps, this concludes the proof.

\qed

4 Building global barrier functions

4.1 Barrier functions on Bounded Geometry

In this subsection we assume that $(M, \gamma)$ is a smooth complete Riemannian manifold of bounded geometry (see the appendix for quick review of manifolds of bounded geometry). We assume that $a = c_n R_\gamma + b_n \tau^2 \in L^p_{\text{loc}}(M) \cap L^\infty(M)$, $\tau \in W^{1,p}_{\text{loc}}$, with $p > n$, and $a \geq a_0 > 0$, where this last condition is in fact a restriction on the admissible mean curvatures, given $R_\gamma$.

Let us fix an exhaustion of $M$ by precompact sets $\{\Omega_k\}$ with smooth boundaries, two constants $0 < c_- \leq c_+$, $\Lambda_+ \in L^\infty(M) \cap C^\infty(M)$ positive functions, $\Lambda_+ \geq \Lambda_-$, and analyse the sequence of problems:

\[
\begin{cases}
\Delta_\gamma \varphi_- - a \varphi_- = c_- a - \Lambda_- \quad \text{in} \quad \Omega_k, \\
\varphi_- = 0 \quad \text{on} \quad \partial \Omega_k,
\end{cases}
\] (67)

\[
\begin{cases}
\Delta_\gamma \varphi_+ - a \varphi_+ = c_+ a - \Lambda_+ \quad \text{in} \quad \Omega_k \\
\varphi_+ = 0 \quad \text{on} \quad \partial \Omega_k.
\end{cases}
\] (68)

From (67) and (68) we construct two sequences of solutions $(\varphi^-_k)$ and $(\varphi^+_k)$ and define $u_k = \varphi^-_k + c_-$ and $v_k = \varphi^+_k + c_+$, which are solutions to
\[
\begin{align*}
\{ \Delta_{\gamma} u_k - au_k = -\Lambda_\ + \text{ in } \Omega_k, \\
u_k = c_\ - \text{ on } \partial\Omega_k, \\
\}
\] (60)

\[
\{ \Delta_{\gamma} v_k - av_k = -\Lambda_\ - \text{ in } \Omega_k, \\
v_k = c_\ + \text{ on } \partial\Omega_k. \\
\}
\] (70)

**Lemma 4.1.** Under the above conditions there is a constant \( c > 0 \), depending on the fixed quantities \( c_\pm, \Lambda_\pm, a_0 \) such that
\[
0 < u_k \leq v_k \leq c < \infty \text{ for all } k \in \mathbb{N}.
\] (71)

**Proof.** Since \( a > 0 \) and \( 0 \leq c_\ - \leq c_\ + \), then \( h_k = u_k - v_k \) satisfies
\[
\begin{align*}
\{ \Delta_{\gamma} h_k - ah_k = \Lambda_\ - \Sigma \geq 0 \text{ in } \Omega_k, \\
h_k = c_- - c_\ + \leq 0 \text{ on } \partial\Omega_k.
\}
\]
we can apply the maximum and comparison principles to get:
\[
0 < u_k \leq v_k,
\] (72)
where the first inequality follows from the maximum principle since \( u_k \geq 0 \) and can only attain a non-positive interior minimum if it is constant.

Now, let \( c > 0 \) be a constant to be determined and consider the difference \( \bar{v}_k = v_k - c \) on \( \Omega_k \). Then,
\[
\begin{align*}
\{ \Delta_{\gamma} \bar{v}_k - a\bar{v}_k = -\Lambda_\ + ac \text{ in } \Omega_k, \\
\bar{v}_k = c_\ + c \text{ on } \partial\Omega_k.
\}
\]
Then, let us chose \( c \) satisfying
\[
c \geq \max \{ \sup M \frac{\Lambda_\ +}{a}, c_\ + \} < \infty,
\]
where the last inequality holds since \( a \geq a_0 > 0 \) and \( \Lambda_\ + \in C^\infty(M) \cap L^\infty(M) \) imply \( \sup M \frac{\Lambda_\ +}{a} < \infty \), and notice that this choice is independent of \( v_k \) and \( \Omega_k \). Therefore, for any \( k \in \mathbb{N} \), we find
\[
\begin{align*}
\{ \Delta_{\gamma} \bar{v}_k - a\bar{v}_k \geq 0 \text{ in } \Omega_k, \\
\bar{v}_k \leq 0 \text{ on } \partial\Omega_k.
\}
\] (73)
An application of the maximum principle then gives \( \bar{v}_k \leq 0 \), which is equivalent to \( v_k \leq \max \{ \sup M \frac{\Lambda_\ +}{a}, c_\ + \} \).

Appealing to the above lemma, let us now show that the sequences \( \{u_k\}_{k=1}^\infty \) and \( \{v_k\}_{k=1}^\infty \) admit subsequences that converge to solutions of an associated problem over all of \( M \).

**Lemma 4.2.** Consider the same setting as above. Then, the PDE problems
\[
\begin{align*}
\{ \Delta_{\gamma} u - au = -\Lambda_- \text{ on } M, \\
u > 0
\}
\]
(74)

\[
\begin{align*}
\{ \Delta_{\gamma} v - av = -\Lambda_+ \text{ on } M, \\
v > 0
\}
\]
(75)
admit solutions \( u, v \in W^{2,p}_{loc}(M) \) satisfying \( 0 < u \leq v \leq c \), where \( c(a_0, \Lambda_+, c_\ +) \) is the same constant appearing in Lemma 4.1.
Proof. Consider the sequence of solutions \( \{u_k\}_{k=1}^{\infty} \) and \( \{v_k\}_{k=1}^{\infty} \) associated to (69) and (70). Let us now fix some \( \Omega_k \subset U_{k'} \subset \Omega_{k'+1} \), consider \( \{u_k\}_{k>k'} \) and appeal to the interior elliptic estimates

\[
\|u_k\|_{H^2(\Omega_{k'})} \leq C(\Omega_{k'}, U_{k'}) \left( \| (\Delta_{\gamma} - a) u_k \|_{L^2(U_{k'})} + \| u_k \|_{L^2(U_{k'})} \right)
= C(\Omega_{k'}, U_{k'}) \left( \| \Lambda_- \|_{L^2(U_{k'})} + \| u_k \|_{L^2(U_{k'})} \right).
\]

Using lemma 4.1, we know that \( u_k \leq c \), with \( c \) independent of \( k \) for all \( k \). Then \( \|u_k\|_{L^2(U_{k'})} \leq c \text{Vol}(U_{k'})^{\frac{1}{2}} \), where \( k' \) is fixed. Therefore, we find

\[
\|u_k\|_{H^2(\Omega_{k'})} \leq C(\Omega_{k'}, U_{k'}) \text{ for all } k \geq k'.
\]

We can work similarly on \( \{v_k\}_{k=k'} \) and obtain:

\[
\|v_k\|_{H^2(\Omega_{k'})} \leq C(\Omega_{k'}, U_{k'}) \left( \| \Lambda_+ \|_{L^2(U_{k'})} + \| v_k \|_{L^2(U_{k'})} \right).
\]

Since \( \Lambda \in L^\infty \) and \( v_k \leq c \) for all \( k \) from Lemma 4.1 one again finds a uniform upper bound

\[
\|v_k\|_{H^2(\Omega_{k'})} \leq C(\Omega_{k'}, U_{k'}) \text{ for all } k \geq k'.
\]

Doing as in (54)-(59) we can bootstrap (76) and (78) into a uniform \( W^{2,p} \) estimate on the \( \Omega_{k'} \). Thus, the sequence is uniformly bounded in \( W^{2,p}(\Omega_{k'}) \) and working as in step 3 of the proof of theorem 3.1 we find \( u, v \in W^{2,p}_\text{loc}(M) \) solutions of \( \Delta_{\gamma} u - au = -\Lambda_- \) and \( \Delta_{\gamma} v - av = -\Lambda_+ \). Furthermore, since \( 0 \leq u_k \leq v_k \leq c \) for all \( k \), we find

\[
0 \leq u \leq v \leq c.
\]

We still need to exclude the possibility of \( u = 0 \). Assume there is some point \( p \in M \) such that \( u(p) = 0 \). Since \( u \geq 0 \), then such point is a minimum of \( u \). But, it also holds that

\[
\Delta_{\gamma} u(p) = au(p) - \Lambda_-(p) = -\Lambda_-(p) < 0,
\]

which contradicts the fact that \( p \) is a minimum of \( u \geq 0 \), and therefore \( u > 0 \).

The functions \( u \) and \( v \) constructed above are good starting points for barrier functions. Let us show that one can choose \( 0 < \alpha \leq \beta \), and \( 0 < \Lambda_- \leq \Lambda_+ \) such that \( \phi_- \equiv au \) and \( \phi_+ \equiv \beta (1 + v) \) are respectively uniform sub and supersolutions. Since \( \alpha \leq \beta \), they will still satisfy \( 0 < \phi_- \leq \phi_+ \).

**\( \phi_+ \)** is a global supersolution

With \( \phi_+ = \beta (1 + v) \), let \( \phi \) and \( Y \) be given such that

\[
\phi_- \leq \phi \leq \phi_+,
\]

\[
\Delta_{\gamma, \text{conf}} Y - \frac{n-1}{n} \nabla \tau \phi \frac{2\phi}{\sqrt{\phi}} - \omega_1 \phi^{\frac{2n+4}{n-2}} + \omega_2 = 0.
\]

Then

\[
\Delta_{\phi_+} = \beta \left( c_n R_+ + b_n \tau^2 \right) v - \beta \Lambda_+ = -\beta \left( c_n R_+ + b_n \tau^2 \right) \phi_+ - \beta \Lambda_+ + \beta \Lambda_+,
\]

and

\[
\mathcal{H}(\phi_+) \equiv \Delta_{\phi_+} \phi_+ - c_n R_+ \phi_+ - b_n \tau^2 \phi_+^{\frac{2(n+2)}{n-2}} + c_n |\tilde{K}|^2 \phi_+^{\frac{2n-4}{n-2}} + 2c_n \epsilon_1 \phi_+^{\frac{2n+4}{n-2}} + 2c_n \epsilon_2 \phi_+^{-3} + 2c_n \epsilon_3 \phi_+^{-\frac{2n+4}{n-2}} \leq -\beta \Lambda_+ + \left( \phi_+ - \phi_- \right) b_n \tau^2 + c_n |\tilde{K}|^2 \phi_+^{\frac{2n-4}{n-2}} + 2c_n \epsilon_1 \phi_+^{\frac{2n+4}{n-2}} + 2c_n \epsilon_2 \phi_+^{-3} + 2c_n \epsilon_3 \phi_+^{-\frac{2n+4}{n-2}} \leq \beta \left( -\Lambda_+ + (1 + c) \left( 1 - \frac{1}{\beta^{\frac{n-2}{n}}} \right) b_n \tau^2 + c_n |\tilde{K}|^2 \beta^{-\frac{2n-4}{n-2}} + 2c_n \epsilon_1 (1 + c) \beta^{\frac{n}{n-2}} \phi_+^{\frac{2n}{n-2}} + 2c_n \epsilon_2 \beta^{-n} + 2c_n \epsilon_3 \beta^{-\frac{n}{n-2}} (1 + c) \frac{n}{n-2} \right)
\]

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where \( c = \max \left( \sup_M \frac{\Delta \varphi}{n}, c_+ \right) \) is the same constant appearing in Lemma 4.1. Let us now recall that

\[
|\tilde{\gamma}|^2 \leq 2 \left( |\mathcal{L}_{\gamma,conf} Y|^2 + |\bar{U} |^2 \right).
\]  

(82)

To obtain that \( \phi_+ \) is a global supersolution, i.e. that \( \mathcal{H}(\phi_+) \leq 0 \), we need an a priori bound on solutions \( Y \) of the momentum constraint with \( \varphi \leq \phi_+ \).

**Proposition 4.1.** On \((M, \gamma)\) a complete smooth Riemannian manifold of bounded geometry and dimension \( n \geq 3 \) such that \( \lambda_{1,conf}(M) > 0 \) let \( p > n \) and assume that:

\[
\nabla \tau, \omega_1, \omega_2 \in L^2(M) \cap L^p(M).
\]  

(83)

Then for all \( \varphi \leq \phi_+ \leq c \), any \( L^2 \) solution of the momentum constraint

\[
\Delta_{\gamma,conf} Y = \frac{n-1}{n} \nabla \varphi - \omega_1 \varphi^{\frac{n+2}{n-2}} + \omega_2 = 0
\]  

satisfies:

\[
|\mathcal{L}_{\gamma,conf} Y|^2 \leq \frac{C(n, M, \gamma)}{\lambda_{1,conf}} \left[ \sum_{j=0}^{j_{\max}} \|\nabla \tau\|_{L^p(M)} + \|\nabla \tau\|^2_{L^p(M)} \right] (1 + c)^\frac{2n}{2 - n}
\]

\[
+ \left( \sum_{j=0}^{j_{\max}} \|\omega_1\|_{L^p(M)} + \|\omega_1\|_{L^p(M)} \right) (1 + c)^\frac{2n}{2 - n} + \left( \sum_{j=0}^{j_{\max}} \|\omega_2\|_{L^p(M)} + \|\omega_2\|_{L^p(M)} \right),
\]  

where \((p_j)_{j \in \mathbb{N}}\) is the sequence defined by induction as: \( p_0 = 2 \), \( p_{j+1} = \frac{n p_j}{n - 2p_j} \), and \( j_{\max} \) is the first integer for which \( p_{j_{\max}} \geq \frac{n}{2} \).

Thanks to Hölder estimates one can interpolate the \( L^q \) norm for \( 2 \leq q \leq p \) by the \( L^2 \) and \( L^p \) norms:

\[
\|V\|_{L^q} = \left( \int |V|^q \right)^{\frac{1}{q}} = \left( \int |V|^{\frac{2(p-q)}{p-2}} |V|^{\frac{2(q-n)}{p-2}} \right)^{\frac{1}{q}}
\]

\[
\leq \left( \left( \int |V|^2 \right)^{\frac{p-2}{p}} \left( \int \left( |V|^{\frac{2(p-q)}{n-2q} - \frac{2(p-q)}{2(q-n)} - \frac{2(q-n)}{p-2}} \right)^{\frac{p-2}{p}} \right)^{\frac{p}{p-2}} \right)^{\frac{1}{q}}
\]

\[
\leq \|V\|_{L^2}^\frac{p(q-n)}{2(p-q)} \|V\|_{L^p}^\frac{p(q-n)}{2(p-q)}
\]

\[
\leq \max (\|V\|_{L^2}, \|V\|_{L^p})^{\frac{2(p-q)}{n-2q} + \frac{2(p-q)}{2(q-n)}}
\]

\[
\leq \max (\|V\|_{L^2}, \|V\|_{L^p})^{\frac{2(p-q)}{n-2q} + \frac{2(p-q)}{2(q-n)}}
\]  

(85)

Estimate (81) is then finite. Below, we will reframe it as a function of the \( L^2 \) and \( L^p \) norms but we chose to present here what is the most immediate shape.

**Proof.** The proof will proceed in several steps. We will first prove a Güneysu - Pigola inspired lemma (see [34]) to ensure that we use the estimate of the first eigenvalue of the conformal Killing Laplacian to control the \( L^2 \) norm of \( Y \). We will then proceed with a bootstrap in bounded geometry to prove the estimate.

**Lemma 4.3.** Let \((M^n, \gamma)\) be a complete smooth Riemannian manifold. Then, there is a constant \( C = C(n) \) such that the following estimate holds for all \( X \in W^{1,2}_{loc}(M) \) and \( \Delta_{\gamma,conf} X \in L^2(M) \) such that \( \Delta_{\gamma,conf} X \in L^2(M) \):

\[
\|\mathcal{L}_{\gamma,conf} X\|_{L^2(M)}^2 \leq C\|\langle X, \Delta_{\gamma,conf} X \rangle\|_{L^1(M)},
\]  

(86)

30
Proof. Consider \( X \in W^{2,2}_\text{loc}(M) \cap L^2(M) \) and a function \( \varphi \in C^\infty_0(M), \varphi \geq 0 \), and notice that, in a distributional sense:

\[
\text{div}_Y (\varphi^2 \mathcal{L}_{\gamma,\text{conf}} X(Y, \cdot)) = 2\varphi \mathcal{L}_{\gamma,\text{conf}} X(Y, \nabla \varphi) + \varphi^2 \langle \Delta_{\gamma,\text{conf}} X(Y, \varphi) \rangle + \varphi^2 \langle \mathcal{L}_{\gamma,\text{conf}} X(Y, \nabla \varphi) \rangle, \\
= 2\varphi \mathcal{L}_{\gamma,\text{conf}} X(Y, \nabla \varphi) + \varphi^2 \langle \Delta_{\gamma,\text{conf}} X(Y, \varphi) \rangle + \frac{1}{2} \varphi^2 |\mathcal{L}_{\gamma,\text{conf}} X(Y, \nabla \varphi)|^2.
\]

Integrating the above equation we find

\[
\frac{1}{2} \int_M \varphi^2 |\mathcal{L}_{\gamma,\text{conf}} X(Y, \nabla \varphi)|^2 dV_Y = -\int_M 2\varphi \mathcal{L}_{\gamma,\text{conf}} X(Y, \nabla \varphi) dV_Y - \int_M \varphi^2 \langle \Delta_{\gamma,\text{conf}} X(Y, \nabla \varphi) \rangle dV_Y.
\]

Now, apply the point wise estimates almost everywhere \( \mathcal{L}_{\gamma,\text{conf}} X(Y, \nabla \varphi) \leq C(n) \langle \nabla_{\gamma,\text{conf}} X \rangle \langle \nabla \varphi \rangle \), and then, given \( \epsilon > 0 \), apply Young’s inequality along a sequence of first order cut-off functions \( \{\varphi_j\}_{j=1}^\infty \) (which exists since \( M \) is complete, see B. Gineysu’s \cite{[32]} theorem 2.2 or M. Shubin’s \cite{[37]} proposition 4.1), using monotone and dominated convergence, one finds the desired estimate.

Picking \( \epsilon \) sufficiently small, we can absorb the first term in the right-hand side into the left-hand side, so as to find a fixed constant \( C > 0 \) such that

\[
\int_M \varphi^2 |\mathcal{L}_{\gamma,\text{conf}} X(Y, \nabla \varphi)|^2 dV_Y \leq C \left( \int_M |\nabla \varphi|^2 dV_Y + \int_M \varphi^2 \langle \Delta_{\gamma,\text{conf}} X(Y, \nabla \varphi) \rangle dV_Y \right). 
\]

Now, using the above inequality along a sequence of first order cut-off functions \( \{\varphi_j\}_{j=1}^\infty \) (which exists since \( M \) is complete, see B. Gineysu’s \cite{[32]} theorem 2.2 or M. Shubin’s \cite{[37]} proposition 4.1), using monotone and dominated convergence, one finds the desired estimate. \( \square \)

From this lemma, we can deduce that our solution \( Y \) satisfies:

\[
\|\mathcal{L}_{\gamma,\text{conf}} Y\|_{L^2(M)} \leq C \|Y\|_{L^2(M)} \|\Delta_{\gamma,\text{conf}} Y\|_{L^2(M)} \\
\leq C \|Y\|_{L^2(M)} \left( \|\nabla Y\|_{L^2(M)}(1 + \varphi)^{\frac{n}{2}} + \|\nabla_\gamma Y\|_{L^2(M)}(1 + \varphi)^{\frac{n+1}{2}} + \|\nabla Y\|_{L^2(M)} \right).
\]

We can then proceed as in the proof of lemma \ref{lem:appx} and say:

\[
\|Y\|^2_{L^2(M)} \leq \frac{1}{\Lambda_{1,\gamma,\text{conf}}} \|\mathcal{L}_{\gamma,\text{conf}} Y\|^2_{L^2(M)},
\]

which with \ref{lem:appx} yields:

\[
\|Y\|_{L^2(M)} \leq \frac{C(n)}{\Lambda_{1,\gamma,\text{conf}}} \left( \|\nabla Y\|_{L^2(M)}(1 + \varphi)^{\frac{n}{2}} + \|\nabla_\gamma Y\|_{L^2(M)}(1 + \varphi)^{\frac{n+1}{2}} + \|\nabla Y\|_{L^2(M)} \right).
\]

We will now prove by induction that for all \( i \leq j_{\text{max}} \), one has

\[
\|Y\|_{W^{2,\nu_i}(M)} \leq \frac{C(n, \gamma, M)}{\Lambda_{1,\gamma,\text{conf}}} \left( \sum_{j=0}^i \|\nabla Y\|_{L^{p_j}(M)} \right) (1 + \varphi)^{\frac{2n}{2n+1}} + \left( \sum_{j=0}^i \|\nabla_\gamma Y\|_{L^{p_j}(M)} \right) (1 + \varphi)^{\frac{n+1}{2n+1}} + \left( \sum_{j=0}^i \|\nabla Y\|_{L^{p_j}(M)} \right) 
\]

\[
\leq \frac{C(n, \gamma, M)}{\Lambda_{1,\gamma,\text{conf}}} \left( \sum_{j=0}^i \|\nabla Y\|_{L^{p_j}(M)} \right) (1 + \varphi)^{\frac{2n}{2n+1}} + \left( \sum_{j=0}^i \|\nabla_\gamma Y\|_{L^{p_j}(M)} \right) (1 + \varphi)^{\frac{n+1}{2n+1}} + \left( \sum_{j=0}^i \|\nabla Y\|_{L^{p_j}(M)} \right),
\]

\[
(91)
\]
• when $i = 0$, $n_i = 2$, and injecting the $L^2$ estimate \(91\) into Shubin’s elliptic regularity estimates in bounded geometry (see lemma \[A.2]\) this ensures that:

$$
\|Y\|_{W^{2,2}(M)} \leq \frac{C(n, M, \gamma)}{\lambda_{1,\gamma,\text{conf}}} \left( \|\nabla \tau\|_{L^2(M)}(1 + c)^\frac{2n}{p_j} + \|\omega_1\|_{L^2(M)}(1 + c)^\frac{2N+1}{p_j} + \|\omega_2\|_{L^2(M)} \right).
$$

• Assuming the result for $i < j_{\text{max}}$, the Sobolev embeddings $W^{2,p_j} \subset L^{p_j+1}$ (true in bounded geometry) and the inductive hypothesis ensure that $Y \in L^{p_j+1}$ with:

$$
\|Y\|_{L^{p_j+1}} \leq \frac{C(n, M, \gamma)}{\lambda_{1,\gamma,\text{conf}}} \left[ \left( \sum_{j=0}^{j_{\text{max}}} \|\nabla \tau\|_{L^{p_j}(M)} \right)(1 + c)^\frac{2n}{p_j} + \left( \sum_{j=0}^{j_{\text{max}}} \|\omega_1\|_{L^{p_j}(M)} \right)(1 + c)^\frac{2N+1}{p_j} + \left( \sum_{j=0}^{j_{\text{max}}} \|\omega_2\|_{L^{p_j}(M)} \right) \right],
$$

Shubin’s elliptic regularity estimates in bounded geometry then yield:

$$
\|Y\|_{W^{p_j+1}(M)} \leq C(n, M, \gamma) \left( \|\Delta_{\gamma,\text{conf}} Y\|_{L^{p_j+1}(M)} + \|Y\|_{L^{p_j+1}(M)} \right).
$$

Injecting the estimate on $\Delta_{\gamma,\text{conf}} Y$ and \(92\) into the above yields the $p_{i+1}$ estimate. Estimate \(91\) then stands true for $j_{\text{max}}$. Then:

• If $p_{j_{\text{max}}} > \frac{2n}{\gamma}$, Sobolev embeddings $W^{2,p_j} \subset C^0$ ensure that:

$$
\|Y\|_{L^\infty(M)} \leq \frac{C(n, M, \gamma)}{\lambda_{1,\gamma,\text{conf}}} \left[ \left( \sum_{j=0}^{j_{\text{max}}} \|\nabla \tau\|_{L^{p_j}(M)} \right)(1 + c)^\frac{2n}{p_j} + \left( \sum_{j=0}^{j_{\text{max}}} \|\omega_1\|_{L^{p_j}(M)} \right)(1 + c)^\frac{2N+1}{p_j} + \left( \sum_{j=0}^{j_{\text{max}}} \|\omega_2\|_{L^{p_j}(M)} \right) \right].
$$

Since in addition $Y \in L^2(M)$, one can conclude that $Y \in L^p(M)$:

$$
\|Y\|_{L^p(M)} \leq \left( \int_M |Y|^{p-2}|Y|^2 dV_\gamma \right)^{\frac{1}{p}} \leq \|Y\|_{L^\infty(M)} \|Y\|_{L^2(M)} \leq \frac{C(n, M, \gamma, \gamma)}{\lambda_{1,\gamma,\text{conf}}} \left[ \left( \sum_{j=0}^{j_{\text{max}}} \|\nabla \tau\|_{L^{p_j}(M)} \right)(1 + c)^\frac{2n}{p_j} + \left( \sum_{j=0}^{j_{\text{max}}} \|\omega_1\|_{L^{p_j}(M)} \right)(1 + c)^\frac{2N+1}{p_j} + \left( \sum_{j=0}^{j_{\text{max}}} \|\omega_2\|_{L^{p_j}(M)} \right) \right]^{\frac{1}{p}} \times \left( \left( \sum_{j=0}^{j_{\text{max}}} \|\nabla \tau\|_{L^2(M)}(1 + c)^\frac{2n}{p_j} + \|\omega_1\|_{L^2(M)}(1 + c)^\frac{2N+1}{p_j} + \|\omega_2\|_{L^2(M)} \right)^{\frac{1}{2}} \right)^{\frac{1}{p}}
$$

$$
\leq \frac{C(n, M, \gamma)}{\lambda_{1,\gamma,\text{conf}}} \left[ \left( \sum_{j=0}^{j_{\text{max}}} \|\nabla \tau\|_{L^{p_j}(M)} \right)(1 + c)^\frac{2n}{p_j} + \left( \sum_{j=0}^{j_{\text{max}}} \|\omega_1\|_{L^{p_j}(M)} \right)(1 + c)^\frac{2N+1}{p_j} + \left( \sum_{j=0}^{j_{\text{max}}} \|\omega_2\|_{L^{p_j}(M)} \right) \right]^{\frac{1}{p}} \times \left( \left( \sum_{j=0}^{j_{\text{max}}} \|\nabla \tau\|_{L^2(M)}(1 + c)^\frac{2n}{p_j} + \|\omega_1\|_{L^2(M)}(1 + c)^\frac{2N+1}{p_j} + \|\omega_2\|_{L^2(M)} \right)^{\frac{1}{2}} \right)^{\frac{1}{p}}
$$

$$
\leq \frac{C(n, M, \gamma)}{\lambda_{1,\gamma,\text{conf}}} \left[ \left( \sum_{j=0}^{j_{\text{max}}} \|\nabla \tau\|_{L^{p_j}(M)} \right)(1 + c)^\frac{2n}{p_j} + \left( \sum_{j=0}^{j_{\text{max}}} \|\omega_1\|_{L^{p_j}(M)} \right)(1 + c)^\frac{2N+1}{p_j} + \left( \sum_{j=0}^{j_{\text{max}}} \|\omega_2\|_{L^{p_j}(M)} \right) \right]^{\frac{1}{p}} \times \left( \left( \sum_{j=0}^{j_{\text{max}}} \|\nabla \tau\|_{L^2(M)}(1 + c)^\frac{2n}{p_j} + \|\omega_1\|_{L^2(M)}(1 + c)^\frac{2N+1}{p_j} + \|\omega_2\|_{L^2(M)} \right)^{\frac{1}{2}} \right)^{\frac{1}{p}}
$$

$$
\leq \frac{C(n, M, \gamma)}{\lambda_{1,\gamma,\text{conf}}} \left[ \left( \sum_{j=0}^{j_{\text{max}}} \|\nabla \tau\|_{L^{p_j}(M)} \right)(1 + c)^\frac{2n}{p_j} + \left( \sum_{j=0}^{j_{\text{max}}} \|\omega_1\|_{L^{p_j}(M)} \right)(1 + c)^\frac{2N+1}{p_j} + \left( \sum_{j=0}^{j_{\text{max}}} \|\omega_2\|_{L^{p_j}(M)} \right) \right]^{\frac{1}{p}},
$$

(94)
where we used (90), (93) and the non optimal but simplifying estimate:

\[
\|\nabla \tau\|_{L^2(M)} (1 + c)^{\frac{4n}{n-4}} + \|\omega_1\|_{L^2(M)} (1 + c)^{\frac{2n}{n-4}} + \|\omega_2\|_{L^2(M)} \leq \left( \sum_{j=0}^{j_{\text{max}}} \|\nabla \tau\|_{L^{p_j}(M)} \right) (1 + c)^{\frac{4n}{n-4}} + \left( \sum_{j=0}^{j_{\text{max}}} \|\omega_1\|_{L^{p_j}(M)} \right) (1 + c)^{\frac{2n}{n-4}} + \left( \sum_{j=0}^{j_{\text{max}}} \|\omega_2\|_{L^{p_j}(M)} \right).
\]

Applying once more Shubin’s elliptic estimates yield $W^{2,p}$ estimate which we translate into the proper $L^\infty$ control on $\mathcal{L}_r, \text{conf} Y$ thanks to Sobolev embeddings, which conclude the proof.

- If $p_{j_{\text{max}}} = \frac{n}{2}$, let us consider the $j_{\text{max}}-1$ estimate, and use the Sobolev embedding $W^{2,p_{j_{\text{max}}}-1} \subset L^{p_{j_{\text{max}}}}$. Using estimate (85) (and dealing with the maximum by injecting the $L^2$ estimate as in (91)), one can obtain a $L^{p_{j_{\text{max}}}-\varepsilon}$ estimate for a small $\varepsilon > 0$. The induction proof ensures that we obtain a control in $W^{2,p_{j_{\text{max}}}}$ for $p_{j_{\text{max}}} = \min \left( p, \frac{n(p_{j_{\text{max}}}-\varepsilon)}{n-2(p_{j_{\text{max}}}-\varepsilon)} \right) > p_{j_{\text{max}}} > \frac{n}{2}$. We are then back in the first case, which concludes the proof. We do not change the notations on the right-hand side of the inequality for simplicity. Since all the $L^{p_j}$ and $L^p$ estimates are obtained thanks to (85), this bears no impact on the final result.

\[\square\]

**Remark 4.1.** The bounded geometry hypothesis was used as a way to ensure the availability of the two tools we used: Sobolev embeddings and elliptic estimates. Any other context with the same property may lead to similar constructions.

To lighten notations let us write

\[
M_{\mathcal{V}_r} = 2 \frac{C_n M \tau}{\lambda_1, \gamma, \text{conf}} \left( \sum_{j=0}^{j_{\text{max}}} \|\nabla \tau\|_{L^{p_j}(M)} + \|\nabla \tau\|_{L^p(M)} \right)
\]

\[
M_{\omega_1} = 2 \frac{C_n M \tau}{\lambda_1, \gamma, \text{conf}} \left( \sum_{j=0}^{j_{\text{max}}} \|\omega_1\|_{L^{p_j}(M)} + \|\omega_1\|_{L^p(M)} \right)
\]

\[
M_{\omega_2} = 2 \frac{C_n M \tau}{\lambda_1, \gamma, \text{conf}} \left( \sum_{j=0}^{j_{\text{max}}} \|\omega_2\|_{L^{p_j}(M)} + \|\omega_2\|_{L^p(M)} \right)
\]

(95)

Injecting (81) into (81) yields:

\[
\mathcal{H}(\phi_+) \leq \beta \left( -\Lambda_+ + (1 + c) \left( 1 - \beta \frac{n}{n-4} \right) b_n \tau^2 + c_n (M \nabla \tau + M_{\omega_1} + M_{\omega_2}) (1 + c)^{\frac{4n}{n-4}} \beta^{\frac{n}{n-4}} \right)
+ 2c_n |U|^2 \beta^{\frac{4n}{n-4}} + 2c_n \epsilon_1 (1 + c)^{\frac{4n+2}{n-4}} \beta^{\frac{n}{n-4}} + 2c_n \epsilon_2 \beta^{-4} + 2c_n \epsilon_3 \beta^{-\frac{n}{n-4}} (1 + c)^{\frac{n-6}{n-4}}
\]

At this point it is worth noticing that the maximum $c = \max \left( \sup \frac{\Lambda_+}{\alpha}, c_+ \right)$ depends on both parameters $\Lambda_+$ and $\alpha$ (and thus $\tau$). To deal with this dependance and make $c$ independant of $\tau$, we will chose $\Lambda_+ = \alpha$. To visually represent this we will denote $m = 1 + c \geq 1$. This then implies:

\[
\mathcal{H}(\phi_+) \leq \beta \left( -a + m \left( 1 - \beta \frac{n}{n-4} \right) b_n \tau^2 + c_n (M \nabla \tau + M_{\omega_1} + M_{\omega_2}) m^{\frac{4n}{n-4}} \beta^{\frac{n}{n-4}} \right)
+ 2c_n |U|^2 \beta^{\frac{4n}{n-4}} + 2c_n \epsilon_1 m^{\frac{n+2}{n-4}} \beta^{\frac{n}{n-4}} + 2c_n \epsilon_2 \beta^{-4} + 2c_n \epsilon_3 \beta^{-\frac{n}{n-4}} m^{\frac{n-6}{n-4}}
\]

Taking $\beta = 1$ in the above then yields:

\[
\mathcal{H}(\phi_+) \leq -a + c_n (M \nabla \tau + M_{\omega_1} + M_{\omega_2}) m^{\frac{4n}{n-4}} + 2c_n |U|^2 + 2c_n \epsilon_1 m^{\frac{n+2}{n-4}} + 2c_n \epsilon_2 + 2c_n \epsilon_3 m^{\frac{n-6}{n-4}}
\leq -b_n \tau^2 - c_n R_\tau + c_n (M \nabla \tau + M_{\omega_1} + M_{\omega_2}) m^{\frac{4n}{n-4}} + 2c_n |U|^2 + 2c_n \epsilon_1 m^{\frac{n+2}{n-4}} + 2c_n \epsilon_2 + 2c_n \epsilon_3 m^{\frac{n-6}{n-4}}.
\]
Then, there exists a constant $C(n, M, \gamma, \lambda_1, \gamma_{\text{conf}}, c_\pm)$ such that if
\begin{equation}
|R_\gamma| + \sum_{j=0}^{j_{\text{max}}} \|\nabla \tau\|_{L^p(M)} + \|\nabla \tau\|_{L^p(M)} + \sum_{j=0}^{j_{\text{max}}} \|\omega_1\|_{L^p(M)} + \|\omega_1\|_{L^p(M)} + \|\omega_2\|_{L^p(M)} + \|\omega_2\|_{L^p(M)} + |U| + \epsilon_1 + \epsilon_2 + \epsilon_3 \leq C\tau^2,
\end{equation}
(96)
$\phi_+$ is a global supersolution.

Using (33), one can simplify: $\|\nabla \tau\|_{L^p(M)} \leq \max (\|\nabla \tau\|_{L^2(M)}, \|\nabla \tau\|_{L^p(M)})$ and similarly $\|\omega_1\|_{L^p(M)} \leq \max (\|\omega_1\|_{L^2(M)}, \|\omega_1\|_{L^p(M)}).$ Thus, there exists a constant $C(n, M, \gamma, \lambda_1, \gamma_{\text{conf}}, c_\pm)$ such that if
\begin{equation}
|R_\gamma| + \max (\|\nabla \tau\|_{L^2(M)}, \|\nabla \tau\|_{L^p(M)}) + \max (\|\omega_1\|_{L^2(M)}, \|\omega_1\|_{L^p(M)}) + \max (\|\omega_2\|_{L^2(M)}, \|\omega_2\|_{L^p(M)}) + |U| + \epsilon_1 + \epsilon_2 + \epsilon_3 \leq C\tau^2,
\end{equation}
(97)
is satisfied, and thus $\phi_+$ is a global supersolution. This simplified hypothesis will feature in the final theorems.

To clarify condition (97), let us notice that the hypotheses on $\nabla \tau$ induce an Asymptotically CMC (ACMC) behavior for the mean curvature (in a $W^{1,p}$ sense). As a simplifying assumption, one can write $\tau = \tau_0 + \tilde{\tau}$ with $\tau_0$ a real constant and $\tilde{\tau}$ a function going toward 0 at infinity, then (96) is satisfied if
\begin{equation}
|R_\gamma| + \tilde{\tau}^2 + \max (\|\nabla \tau\|_{L^2(M)}, \|\nabla \tau\|_{L^p(M)}) + \max (\|\omega_1\|_{L^2(M)}, \|\omega_1\|_{L^p(M)}) + \max (\|\omega_2\|_{L^2(M)}, \|\omega_2\|_{L^p(M)}) + |U| + \epsilon_1 + \epsilon_2 + \epsilon_3 \leq C\tilde{\tau}_0^2.
\end{equation}
(98)
This can then be considered either as a smallness assumption of the parameters compared with the asymptotic behavior of the mean curvature, or, considering this limit as a parameter of the problem, given $R_\gamma, \omega, \epsilon_1, \epsilon_2, \epsilon_3$ and $\tilde{\tau}$ one can fix the asymptotic behavior of the mean curvature to ensure the existence of a global supersolution (and, provided one can introduce a global subsolution, of a solution to the constraint equations).

\subsection*{\phi_- is a uniform subsolution (non-vacuum and $n \leq 6$)}

Let us start by choosing $\alpha > 0$ small enough so that $0 < \phi_- \leq 1,$ so that $\phi_-^3, \phi_-^\frac{n-4}{2} \geq 1.$ Thus,
\begin{align*}
\mathcal{H}(\phi_-) &= \Delta_\gamma \phi_- - c_n R_\gamma \phi_- - b_n \tau^2 \phi_-^\frac{n+2}{2} + c_n \left|K(X)\right|^{\frac{2}{\gamma}} \phi_-^{\frac{n-2}{2}} + 2c_n \epsilon_1 \phi_-^\frac{n+2}{2} + 2c_n \epsilon_2 \phi_-^3 + 2c_n \epsilon_3 \phi_-^\frac{n-4}{2} \\
&\geq \Delta_\gamma \phi_- - c_n R_\gamma \phi_- - b_n \tau^2 \phi_-^\frac{n+2}{2} + 2c_n \epsilon_2 + 2c_n \epsilon_3 \\
&\geq \alpha b_n \tau^2 u - \alpha \Lambda_- - b_n \tau^2 \alpha \frac{\phi_-^n}{u} + 2c_n \epsilon_2 + 2c_n \epsilon_3 \\
&\geq \alpha b_n \tau^2 \left(1 - (\alpha u)^{\frac{n-2}{n}}\right) + 2c_n \epsilon_2 + 2c_n \epsilon_3 - \alpha \Lambda_-
\end{align*}
From our choice of $\alpha$ so that $\phi_- \leq 1,$ we find that $1 - (\alpha u)^{\frac{n-2}{n}} \geq 1 - \phi_-^\frac{n-2}{n} \geq 0.$ Finally, if $\epsilon_2 + \epsilon_3 > 0$ on $M,$ then there is a choice of $\Lambda_- > 0$ (or $\alpha < 1$) satisfying
\begin{equation}
2c_n \epsilon_2 + 2c_n \epsilon_3 \geq \alpha \Lambda_- > 0.
\end{equation}
(99)
After fixing such a choice we have that $\mathcal{H}(\phi_-) \geq 0,$ and therefore $\phi_-$ is a global subsolution.

\subsection*{\phi_- is a uniform subsolution (non-vacuum and $n > 6$)}

The reasoning is very similar. The difference lies in the majorization of the $\phi_-^\frac{n-4}{2} \geq 0$ terms. The condition thus falls entirely on $\epsilon_2$ and one can find an admissible $\Lambda_-$ and $\alpha$ if and only if
\begin{equation}
\epsilon_2 > 0.
\end{equation}
(100)
4.2 Global subsolutions with Yamabe-type equations

An issue with the construction of barrier functions in section 4.1 is that the conditions (99) and (100) do not allow vacuum conditions. To counterbalance this, one can notice that the supersolution does allow for $c_1, c_2, c_3 = 0$, and is bounded from below (in fact with our construction $\phi_0 \geq 1$). If we thus manage to build a subsolution $\phi_- \leq 1$ with another method, we would thus be able to obtain compatible barrier functions.

Reminders on Yamabe-type equations

We here choose to build a bounded subsolution by finding a solution to Yamabe-type equations. To that end, we first recall several theorems from P. Mastrolia, M. Rigoli and A. Setti’s [47] on a priori estimates and existence conditions for Yamabe-type equations.

First is [47][Theorem 4.4] which yields an a priori estimate to subsolutions of Yamabe-type equations:

**Theorem 4.2.** Let $(M,g)$ be a complete manifold with Ricci tensor satisfying

$$\text{Ric} \geq -(n-1)H^2(1+r^2)^{2},$$

(101)

where $H, \delta > 0$ are real constants and $r(x) = d(p,x)$ for a given $p \in M$. Let $a(x), b(x) \in C^0(M)$ satisfy

$$a(x) \leq Ar(x)^{\alpha}, \quad \alpha \geq \frac{\delta}{2} - 1,$$

(102)

$$b(x) \geq Br(x)^{\beta}, \quad \beta \leq 1 - \frac{\delta}{2} + \alpha,$$

(103)

when $r(x) \gg 1$ and $A$ and $B$ are two strictly positive constants. Then any nonnegative solution $u \in C^2(M)$ of

$$\Delta u + a(x)u - b(x)u^{\sigma} \geq 0, \quad \sigma > 1 \text{ on } M$$

(104)

satisfies

$$u(x) \leq Cr(x)^{-\frac{\delta-\sigma}{\sigma-1}}$$

(105)

for $r(x) \gg 1$ and $C$ a strictly positive constant.

Then comes an existence theorem of [47][Theorem 6.7]:

**Theorem 4.3.** Let $a(x), b(x) \in C^{0,\alpha}_{\text{loc}}(M)$ for some $0 < \alpha \leq 1$. Assume that $b(x) \geq 0$, $b(x) > 0$ outside a compact set, and that $\lambda_1(L)$ is nonnegative, where $\lambda_1(L) = \{x \in M : b(x) = 0\}$ and $L = \Delta + a(x)$. Assume

$$\lambda_1(M) < 0.$$ (106)

Then the equation

$$\Delta u + a(x)u - b(x)u^{\sigma} = 0$$

(107)

possesses a minimal and a maximal (possibly coinciding) positive solutions.

Our method of building convenient subsolutions requires an existence result and an upper bound in order to associate it with a compatible supersolution. For convenience we will thus assemble the previous theorems into a ready-for-application result.

**Theorem 4.4.** Let $a(x), b(x) \in C^{0,\alpha}_{\text{loc}}(M)$ for some $0 < \alpha \leq 1$. Assume that:

$$b(x) \geq 0, b(x) > 0 \text{ outside a compact set},$$

(108)

$$\lambda_1(L) > 0 \text{ where } L = \{x \in M : b(x) = 0\} \text{ and } L = \Delta + a(x),$$

(109)

$$a(x) \leq A, \quad b(x) \leq b(x)$$

(110)

$$\text{Ric} \geq -(n-1)H^2(1+r^2)$$

(111)

$$\lambda_1(M) < 0.$$ (112)

Then the equation (107) possesses a positive solution satisfying

$$0 < u(x) \leq C,$$

for a constant $C > 0$. 

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Proof. Hypotheses (108), (109), (112) ensure theorem 4.3 applies: there exist positive solutions to (107). On the other hand hypotheses (111) means that (111) stands with δ = 2, while (110) means that we can apply theorem 4.2 with α = β = 0 to obtain the uniform upper bound. □

Remark 4.2. A quick way to ensure hypothesis (109) is to impose b > 0 on the whole of M, which indeed implies B_0 = ∅ and thus by definition λ^2_c(B_0) = +∞ > 0 (we refer the reader to the discussion in [12] p.159 as well as the proofs of theorems 6.2 and 6.7 for more details).

We will then seek a global uniform subsolution of (46) from a bounded solution φ= of an equation

\[ \Delta u + a(x) u - b(x) u^\alpha = 0, \]

with a and b to be chosen according our needs. We will then build our subsolution φ_- = κφ= such that

\[ \mathcal{H}(φ_-) \geq \Delta φ_- - c_n R_γ φ_- + c_n |K(X)|^2 \phi_+ 3 \phi_- n - 3 - b_n \phi_- + 2c_n c_1 \phi_- n - 3 + 2c_n c_2 \phi_- n - 3 \geq 0. \]

First choice: a = -c_n R_γ, b = c_n r_n \tau^2

If φ_- is a solution of

\[ \Delta φ_- - c_n R_γ φ_- + c_n r_n \tau^2 \phi_- n = 0, \]

let φ_- = κφ=.

Then for any X

\[ \mathcal{H}(φ_-) \geq \Delta φ_- - c_n R_γ φ_- c_n r_n \tau^2 \phi_- n + \phi_- n + \phi_- n = 0 \]

for κ ≤ 1. Thus if there exists a bounded solution to (114), one can find a strictly positive subsolution to (46) compatible with the supersolution φ=, defined above.

Theorem 4.5. Let (M, γ) be a complete manifold of dimension n ≥ 3. We assume that R_γ ∈ C^{0,0}_loc (M) and τ ∈ C^{0,0}_loc (M) for some 0 < α ≤ 1. We further assume

\[ \text{Ricc} \geq -(n - 1)H^2(1 + r^2), \]

\[ R_γ \geq -A, \]

\[ |\tau| \geq B > 0 \text{ outside a compact set}, \]

\[ λ_1^\text{R_γ} = c_0(R_γ(B_0)) > 0 \]

\[ λ_2^\text{R_γ} < 0, \]

where B_0 = \{x ∈ M : τ(x) = 0\}. Then for any m > 0 the Lichnerowicz equation (10) admits a strictly positive subsolution φ_- satisfying φ_- ≤ m.

Proof. We apply theorem 4.3 with the preceding analysis. □

This result is enough for our goals, but to highlight the variety of constructions of barrier functions, we will offer another possible choice for a and b. We will not translate it into an existence result as it would add little to the one we already have.
The proof is merely an application of theorem 4.4 with the preceding analysis.

Second choice: \( R_\gamma \leq 0, \ a = 2c_n\epsilon_3, \ b = c_n r_n \tau^2 \)

If \( \varphi_- \) is a solution of

\[
\Delta \varphi_- + 2c_n \epsilon_3 \varphi_- - c_n r_n \tau^2 \varphi_-^{\frac{a+2}{a-2}} = 0,
\]

(118)

let \( \phi_- \equiv \kappa \varphi_- \). Then, if in addition \( R_\gamma \leq 0 \)

\[
H(\phi_-) \geq \Delta \phi_- + 2c_n \epsilon_3 \frac{\phi_-^{\frac{a+2}{a-2}}}{\phi_-} - c_n r_n \tau^2 \phi_-^{\frac{a+2}{a-2}}
\]

\[
\geq \kappa \Delta \phi_- + 2c_n \epsilon_3 \frac{\phi_-^{\frac{a+2}{a-2}}}{\phi_-} - c_n r_n \tau^2 \kappa^{\frac{a+2}{a-2}} \phi_-^{\frac{a+2}{a-2}}
\]

\[
\geq -2c_n \epsilon_3 \varphi_- + c_n r_n \tau^2 \kappa^{\frac{a+2}{a-2}} \phi_-^{\frac{a+2}{a-2}} + 2c_n \epsilon_3 \frac{\phi_-^{\frac{a+2}{a-2}}}{\phi_-} - c_n r_n \tau^2 \kappa^{\frac{a+2}{a-2}} \phi_-^{\frac{a+2}{a-2}}
\]

\[
\geq c_n r_n \tau^2 \phi_-^{\frac{a+2}{a-2}} \kappa \left( 1 - \kappa^{\frac{1}{a-2}} \right) + 2c_n \epsilon_3 \kappa \varphi_- - \left( (\kappa \varphi_-)^{\frac{1}{a-2}} - 1 \right) \geq 0
\]

for \( \kappa \leq \min \left( 1, \frac{1}{\sup \varphi_-} \right) \). Thus, if there exists a solution to (118) bounded from above, one can find a strictly positive subsolution to (10), as small as required.

Theorem 4.6. Let \((M, \gamma)\) be a complete manifold of dimension \( n \geq 3 \). We assume that \( \epsilon_3 \in C^0_{\text{loc}}(M) \) and \( \tau \in C^0_{\text{loc}}(M) \) for some \( 0 < \alpha \leq 1 \). We further assume

\[
\text{Ric} \geq -(n-1)H^2(1+\tau^2),
\]

(119)

\[
R_\gamma \leq 0,
\]

\[
\epsilon_3 \leq A
\]

(120)

\[
|\tau| \geq B > 0 \text{ outside a compact set},
\]

\[
\lambda_1^{\Delta + 2c_n \epsilon_3}(B_0) > 0
\]

(121)

where \( B_0 = \{ x \in M : \tau(x) = 0 \} \).

Then for any \( m > 0 \) the Lichnerowicz equation (10) admits a strictly positive subsolution \( \varphi_- \) satisfying \( \varphi_- \leq m \).

Proof. The proof is merely an application of theorem 4.4 with the preceding analysis.

Remark 4.3. As mentioned in remark 4.2, a quick way to satisfy both (116) and the first inequality of (117) (respectively (120) and the first inequality of (121)) is to impose \( \tau^2 > 0 \) on \( M \).

Remark 4.4. This result showcases the flexibility of this method of construction: while the first choice of a and b does not allow for a simultaneous cancellation of \( R \) and \( \tau \) (due to (117), (120) and (121) allow one to consider the case \( R = 0 \) and \( \tau = 0 \) on a compact \( K \subset M \) with \( \tau^2 > 0 \) on \( M \setminus K \). The constraint is then shifted onto the magnetic data \( \epsilon_3 \) with (121).

4.3 Existence results

We can now build global barrier functions and obtain existence results for equation (10) on a complete manifold in bounded geometry. Our first one results entirely from the construction in section 4.1.

Theorem 4.7. Let \((M, \gamma)\) be a smooth complete Riemannian manifold of bounded geometry, let \( n \geq 3 \) be its dimension and \( p > n \). We make the following assumptions:

\[
R_\gamma, \epsilon_1, \epsilon_2, \epsilon_3, |U|^2, \tau^2 \in L^p_{\text{loc}}(M) \text{ and } \omega, \nabla \tau \in L^2(M) \cap L^p(M).
\]

(122)

\[
\lambda_{1, \text{cont}} > 0,
\]

(123)

\[
a = c_n R_\gamma + b_n \tau^2 \in L^\infty(M), \ a \geq a_0 > 0.
\]

(124)
Assume further that:

\[
\begin{cases}
\epsilon_2 + \epsilon_3 > 0 & \text{if } n \leq 6 \\
\epsilon_2 > 0 & \text{if } n > 6.
\end{cases}
\]  

Then, there exists \( C(n, M, \gamma, \lambda_{1, \text{conf}}) \) such that if

\[
|R_\gamma| + \max \left( \|d\tau\|_{L^p(M)}, \|d\tau\|_{L^p(M)} \right) + \max \left( \|\omega_1\|_{L^p(M)}, \|\omega_1\|_{L^p(M)} \right) + \max \left( \|\omega_2\|_{L^p(M)}, \|\omega_2\|_{L^p(M)} \right) + |U| + \epsilon_1 + \epsilon_2 + \epsilon_3 \leq C\tau^2,
\]

then (46) admits a \( W^{2,p}_{\text{loc}} \) solution.

**Proof.** Hypotheses (122), (123) allow one to use theorem 3.1 and conclude that if there exists global barrier functions, system (46) admits a \( W^{2,p}_{\text{loc}} \) solution. Then, hypothesis (126) in bounded geometry corresponds to our construction of a global supersolution in section 4.1 with \( \Lambda^+ = a \) (and \( c_+ \) arbitrary), while (125) allows for the construction of a compatible subsolution with \( \Lambda^- = \frac{1}{2} (\epsilon_2 + \epsilon_3) \) or \( \frac{1}{2} \epsilon_3 \) (while (126) ensures that \( \Lambda^- \leq \Lambda_+ \) if \( C(n, M, \gamma, \lambda_{1, \text{conf}}) \geq 1 \)).

In the end, the global barrier functions ensure the existence of a solution to the constraint equations.

\[\square\]

We reiterate our comments from the end of section 4.1 (see (98)): this is an \( L^p \) existence result in non vacuum conditions, in an asymptotically CMC case under a relative smallness condition of the physical terms compared to the asymptotic value of the mean curvature.

We can deal with the vacuum case under different hypotheses:

**Theorem 4.8.** Let \((M, \gamma)\) be a smooth complete Riemannian manifold of bounded geometry, let \( n \geq 3 \) be its dimension and \( p > n \). We make the following assumptions:

\[
R_\gamma, \tau \in C^{0,\alpha}_{\text{loc}}(M) \cap L^p_{\text{loc}}(M), \epsilon_1, \epsilon_2, \epsilon_3, |U|^2, \tau^2 \in L^p_{\text{loc}}(M) \quad \text{and} \quad \omega, \nabla \tau \in L^2(M) \cap L^p(M). \tag{127}
\]

\[
\lambda_{1, \text{conf}} > 0, \tag{128}
\]

\[
a \doteq c_n R_\gamma + b_n \tau^2 \in L^\infty(M), \quad a \geq a_0 > 0. \tag{129}
\]

Assume further that:

\[
\begin{align*}
\text{Ricc} & \geq -(n - 1) H^2 (1 + r^2), \\
R_\gamma & \geq -A, \\
|\tau| & \geq B > 0 \quad \text{outside a compact set}, \\
\lambda_{1, \text{conf}}^{-1} - c_n R_\gamma & > 0, \\
\lambda_{1, \text{conf}}^{-1} - c_n R_\gamma & < 0,
\end{align*}
\]  

where \( B_0 = \{ x \in M : \tau(x) = 0 \} \).

Then, there exists \( C(n, M, \gamma, \lambda_{1, \text{conf}}) \) such that if

\[
|R_\gamma| + \max \left( \|d\tau\|_{L^p(M)}, \|d\tau\|_{L^p(M)} \right) + \max \left( \|\omega_1\|_{L^p(M)}, \|\omega_1\|_{L^p(M)} \right) + \max \left( \|\omega_2\|_{L^p(M)}, \|\omega_2\|_{L^p(M)} \right) + |U| + \epsilon_1 + \epsilon_2 + \epsilon_3 \leq C\tau^2,
\]

then (46) admits a \( W^{2,p}_{\text{loc}} \) solution.

**Proof.** Once more hypotheses (127), (128) allow one to use theorem 3.1 and conclude that if there exists global barrier functions, system (46) admits a \( W^{2,p}_{\text{loc}} \) solution. Then, hypothesis (131) in bounded geometry corresponds to our construction of a global supersolution in section 4.1 with \( \Lambda_+ = a \) (and \( c_+ \) arbitrary) bounded from below, while (130) allows for the construction of a compatible subsolution with the Yamabe-type equation method thanks to theorem 4.5.

In the end, the global barrier functions ensure once more the existence of a solution to the constraint equations.

\[\square\]
This theorem can thus give an existence for ACMC data with the relative smallness condition and spectral hypotheses. The constraint on \( \lambda_1 \) is void if we can set \( \tau^2 > 0 \).

Thanks to the work done in sections 2.3 and 5.2 we can give the \( H^s \) counterparts to these theorems, respectively.

**Theorem 4.9.** Let \((M, \gamma)\) be a smooth complete Riemannian manifold of bounded geometry, let \( n \geq 3 \) be its dimension and \( s > \frac{n}{2} + 1 \). We make the following assumptions:

\[
\gamma \in H^s_{\text{loc}}(M), U, \tau \in H^{s-1}_{\text{loc}}(M), R_\gamma, \epsilon_1, \epsilon_2, \epsilon_3 \in H^{s-2}_{\text{loc}}(M) \quad \text{and} \quad \omega, \nabla \tau \in L^2(M) \cap L^p(M) \cap H^s_{\text{loc}}(M). 
\]

\[
\lambda_{1, \text{conf}} > 0, \\
\gamma \triangleq c_n R_\gamma + b_n \tau^2 \in L^\infty(M), \quad a \geq a_0 > 0.
\]

Assume further that:

\[
\begin{cases}
\epsilon_2 + \epsilon_3 > 0 & \text{if } n \leq 6 \\
\epsilon_2 > 0 & \text{if } n > 6.
\end{cases}
\]

Then, there exists \( C(n, M, \gamma, \lambda_{1, \text{conf}}) \) such that if

\[
|R_\gamma| + \max \left( \|d\tau\|_{L^2(M)}, \|d\tau\|_{L^p(M)} \right) + \max \left( \|\omega_1\|_{L^2(M)}, \|\omega_1\|_{L^p(M)} \right) + \max \left( \|\omega_2\|_{L^2(M)}, \|\omega_2\|_{L^p(M)} \right) + |U| + \epsilon_1 + \epsilon_2 + \epsilon_3 \leq C \tau^2,
\]

then, if \( n \leq 12 \) admits a \( H^s_{\text{loc}} \) solution.

and

**Theorem 4.10.** Let \((M, \gamma)\) be a smooth complete Riemannian manifold of bounded geometry, let \( n \) be its dimension and \( s > \frac{n}{2} + 1 \). We make the following assumptions:

\[
\gamma \in H^s_{\text{loc}}(M), U, \tau \in H^{s-1}_{\text{loc}}(M), R_\gamma, \epsilon_1, \epsilon_2, \epsilon_3 \in H^{s-2}_{\text{loc}}(M) \quad \text{and} \quad \omega, \nabla \tau \in L^2(M) \cap L^p(M) \cap H^s_{\text{loc}}(M). 
\]

\[
\lambda_{1, \gamma, \text{conf}} > 0, \\
\gamma \triangleq c_n R_\gamma + b_n \tau^2 \in L^\infty(M), \quad a \geq a_0 > 0.
\]

Assume further that:

\[
\text{Ric} \geq -(n-1)H^2(1 + r^2), \\
R_\gamma \geq -A, \\
|\tau| > B > 0 \quad \text{outside a compact set}, \\
\lambda_{1, \gamma, \text{conf}} > 0, \\
\lambda_{1, \gamma, \text{conf}} < 0,
\]

where \( B_0 = \{ x \in M : \tau(x) = 0 \}. \)

Then, there exists \( C(n, M, \gamma, \lambda, \lambda_{1, \gamma, \text{conf}}) \) such that if

\[
|R_\gamma| + \max \left( \|d\tau\|_{L^2(M)}, \|d\tau\|_{L^p(M)} \right) + \max \left( \|\omega_1\|_{L^2(M)}, \|\omega_1\|_{L^p(M)} \right) + \max \left( \|\omega_2\|_{L^2(M)}, \|\omega_2\|_{L^p(M)} \right) + |U| + \epsilon_1 + \epsilon_2 + \epsilon_3 \leq C \tau^2,
\]

then if \( n \leq 12 \) admits a \( H^s_{\text{loc}} \) solution.

### A Manifolds of bounded geometry

In order to apply elliptic estimates to complete manifolds, we will need to work in the bounded geometry context. To suit these needs, we here recall notions of bounded geometry taken from Shubin’s [68] (see also [70, Exposé V]).

**Definition A.1.** We will say \((M, \gamma)\) is a (smooth) manifold of bounded geometry if:

1. \( r_{\text{inj}} > 0 \),
2. \(|\nabla^k \text{Riem}|_\gamma \leq C_k\) for all \(k \in \mathbb{N}\),
where \(r_{inj}\) stands for the injectivity radius of \((M, \gamma)\) and \(C_k\) for constants depending on \(k\).

Given a point \(x \in M\) and \(r \in (0, r_{inj})\) we have normal coordinate systems given by \(\exp_x : B_r(x) \to U_{x,r} \subset \mathbb{R}^n\) such coordinate systems will be called canonical. The second condition above guarantees that the transition matrices (together with their derivatives up to any order) between such coordinate systems are bounded.

Let \(E \to M\) be a vector bundle over \(M\). We say that \(E\) is a bundle of bounded geometry if trivializations of \(E\) over canonical coordinate systems \(U, U'\), with \(U \cap U' \neq \emptyset\), give rise to transition functions \(g^{U \to U'}\) such that \(\partial^\alpha_y g^{U \to U'}(y)\) are bounded by constants \(C_\alpha\) which do not depend on the pair \(U, U'\). Tensor bundles over manifolds of bounded geometry are bundles of bounded geometry.

In the above context we have the following useful result, which is extracted from [68, Lemma 1.3].

**Lemma A.1.** Let \((M, \gamma)\) be a manifold of bounded geometry and fix \(\varepsilon < \frac{1}{2}\) with \(r \in (0, r_{inj})\). Then, there exists a sequence of points \(\{x_i\}_{i=1}^\infty \subset M\) such that \(M = \bigcup_i B_r(x_i)\) and a partition of unity \(1 = \sum_{i=1}^\infty \varphi_i\) on \(M\) such that:

1) \(\varphi_i \geq 0\), \(\varphi_i \in C_\infty^0(M)\) with \(\text{supp}(\varphi_i) \subset B_{2r}(x_i)\);
2) \(\|\partial^\alpha_y \varphi_i(y)\| \leq C_\alpha\), for every multiindex \(\alpha\) in canonical coordinates uniformly with respect to \(i\) (i.e. with the constants \(C_\alpha\) which do not depend on \(i\)).

Using such a partition of unity, we define the Sobolev spaces \(W^{s,p}\), with \(s \in \mathbb{R}\) and \(1 \leq p \leq \infty\) as the closure of \(C_0^\infty\) with respect to the norm
\[
\|u\|_{W^{s,p}} = \sum_{i=1}^\infty \|\varphi_i u\|_{L^{s,p}(B_{2r}(x_i))},
\]
where the spaces \(W^{s,2}(M)\) have a Hilbert structure, and we will denote then by \(H^s(M)\). Above, following [68], we understand the local Sobolev norms in the sense of Bessel potentials. These spaces agree with standard Sobolev spaces when \(r_{inj}\) stands for the injectivity radius of \((M, \gamma)\) and \(C_k\) for constants depending on \(k\).

One could write a short proof of the previous proposition appealing to (142), the corresponding local embeddings, the finite multiplicity of our special covering provided by Lemma A.1 and embeddings for summable sequence spaces \(\ell^p\). For instance, regarding the first of the above inequalities, the classical embeddings apply on each \(B_{2r}(x_i)\), with a uniform constant in \(i\). Equality (142) combined with an embedding \(\ell^p \subset \ell^{\frac{n}{n \cdot p}}\) for the sequence \(\|\varphi_i u\|_{W^{s,p}(B_{2r}(x_i))}\) allows one to recover the desired inequality. Such a sketch highlights the pivotal role played by the non trivial partition offered by Lemma A.1.

Let us now consider \(E, F \to M\) be two vector bundles of bounded geometry; \(A : C^\infty(M; E) \to C^\infty(M; F)\) a differential operator with smooth coefficients. We will call it \(C^\infty\)-bounded if in any canonical coordinate system \(A\) is written in the form
\[
A = \sum_{|\alpha| \leq k} a_\alpha(y) \partial^\alpha_y,
\]
with \(a_\alpha \in \text{Hom}(E, F)\) satisfying uniform estimates of the form \(|\partial^\beta_y a_\alpha(y)| \leq C_\beta\) for any multiindex \(\beta\) and where \(C_\beta\) does not depend on the canonical coordinate system. We now have the following
regularity result for elliptic operators, extracted again from [68]. We shall present a self-contained proof for the benefit of the reader.

**Lemma A.2.** Let $A$ be a $C^\infty$-bounded linear uniformly elliptic operator like (143) acting between two vector bundles of bounded geometry. For any $s, t \in \mathbb{R}$, $t < s$, and $1 < p < \infty$, if $u \in W^{-\infty, p}(M; E)$ and $Au \in W^{s-k, p}(M; F)$, then $u \in W^{s,p}(M; E)$ and there exists a constant $C > 0$ such that

$$
\|u\|_{W^{s,p}(M)} \leq C \left( \|Au\|_{W^{s-k,p}(M)} + \|u\|_{W^{s,p}(M)} \right).
$$

**Proof.** Let us first notice that the local regularity statement given by

$$
u \in W^{-\infty, p}(M) \text{ and } Au \in W^{s-k, p}(M) \implies u \in W^{s,p}_{\text{loc}}$$

follows from the regularity claim in [42, Theorem 13.3.3]. Then, let us assume that the following local estimate holds:

$$
\|u\|_{W^{s,p}(B_\epsilon(x_i))} \leq C \left( \|Au\|_{W^{s-k,p}(B_\epsilon(x_i))} + \|u\|_{W^{s,p}(B_\epsilon(x_i))} \right) \forall u \in W^{s,p}(B_\epsilon(x_i)) \text{ and } t < s.
$$

(146)

If $u \in W^{s,p}(M)$ for some $\sigma \in \mathbb{R}$, then writing $u = \sum_{i=1}^\infty \varphi_i u$ with $\varphi_i$ a locally finite partition of unity as in (142), we find

$$
\|u\|_{W^{s,p}(M)} \lesssim \sum_{i} \left( \|Au\|_{W^{s-k,p}(B_4(0))} + \|\varphi_i u\|_{W^{s-1,p}(B_4(0))} \right).
$$

Then,

$$
\|Au_i\|_{W^{s-k,p}(B_4(x_i))} \lesssim \|\varphi_i Au\|_{W^{s-k,p}(B_4(x_i))} + \|[A, \varphi_i]u\|_{W^{s-k,p}(B_4(x_i))},
$$

$$
\lesssim \|\varphi_i Au\|_{W^{s-k,p}(B_4(x_i))} + \|u\|_{W^{s-1,p}(B_4(x_i))},
$$

$$
\lesssim \|\varphi_i Au\|_{W^{s-k,p}(B_4(x_i))} + \sum_{j \in I_i} \|\varphi_j u\|_{W^{s-1,p}(B_4(x_i))},
$$

$$
\lesssim \|\varphi_i Au\|_{W^{s-k,p}(B_4(x_i))} + \sum_{j \in I_i} \|\varphi_j u\|_{W^{s-1,p}(B_4(x_i))},
$$

where we have used that multiplication by smooth compactly supported functions is continuous from $W^s,p(U) \to W^{s,p}(U)$ for any $s \in \mathbb{R}$ and $1 < p < \infty$ due to [42, Theorem 10.1.15], and denoted by $I_i$ the finite subset indices $j$ such that $B_{2\epsilon}(x_j) \cap B_{2\epsilon}(x_i) \neq \emptyset$. The last line stands since $\varphi_j$ is compactly supported in $B_{2\epsilon}(x_j)$. This implies that

$$
\|Au_i\|_{W^{s-k,p}(B_4(x_i))} \lesssim \|\varphi_i Au\|_{W^{s-k,p}(B_4(x_i))} + \sum_{j \in I_i} \|\varphi_j u\|_{W^{s-1,p}(B_4(x_i))},
$$

and therefore

$$
\|u\|_{W^{s,p}(M)} \lesssim \sum_{i=1}^\infty \left( \|\varphi_i Au\|_{W^{s-k,p}(B_4(x_i))} + \|\varphi_i u\|_{W^{s-1,p}(B_4(x_i))} \right) + \sum_{i=1}^\infty \sum_{j \in I_i} \|\varphi_j u\|_{W^{s-1,p}(B_4(x_i))}.
$$

(147)

Then, denoting by $N$ the multiplicity of the locally finite covering we have

$$
\sum_{i=1}^\infty \sum_{j \in I_i} \|\varphi_j u\|_{W^{s-1,p}(B_4(x_i))} \leq (N + 1) \|u\|_{W^{s-1,p}(M)}.
$$

(148)

The above would imply

$$
\|u\|_{W^{s,p}(M)} \lesssim \|Au\|_{W^{s-k,p}(M)} + \|u\|_{W^{s-1,p}(M)}.
$$
Now given \((148)\), if \(Au \in W^{s-m+p}(M)\) for some \(\epsilon > 0\), then one can proceed inductively to get global \(W^{s-k+p}(M)\)-regularity as follows. From the regularity claim \((145)\) one has \(u \in W^{s-p}_{loc}\) and then the local estimate \((146)\) one gets
\[
\|u\|_{W^{s+p}(B_{1}(x_{1}))} \leq C \left( \|Au\|_{W^{s-k+p}(B_{2z}(x_{1}))} + \|u\|_{W^{s-p}(B_{2z}(x_{1}))} \right).
\]
Then, proceeding as above and using \((148)\), we find
\[
\|u\|_{W^{s+p}(M)}^{p} \leq \|Au\|_{W^{s-k+p}(M)}^{p} + \|u\|_{W^{s-p}(M)}^{p} \leq \|Au\|_{W^{s-k+p}(M)}^{p} + \|u\|_{W^{s-1+p}(M)}^{p}. \tag{149}
\]
Then, if \(Au \in W^{s-k+p}(M), s > \sigma\), repeating the argument as many times as necessary we find the global estimate
\[
\|u\|_{W^{s+p}(M)}^{p} \leq \|Au\|_{W^{s-k+p}(M)}^{p} + \|u\|_{W^{s-1+p}(M)}^{p} < \infty \tag{150}
\]
showing that the claim \((145)\) can be improved into
\[
u \in W^{-\infty,p}(M) \text{ and } Au \in W^{s-k,p}(M) \implies u \in W^{s,p}(M), \tag{151}
\]
with an improved global estimate
\[
\|u\|_{W^{s+p}(M)}^{p} \leq C \left( \|Au\|_{W^{s-k+p}(M)}^{p} + \|u\|_{W^{s-1+p}(M)}^{p} \right) \tag{152}
\]
valid for any \(t < s\).

In the above line of reasoning, we are still missing a self-contained proof of \((146)\). Let us then argue as follows. From \([12, \text{Theorem } 13.3.3]\) we know that in a small enough neighbourhood \(X\) of any point \(x_{0}\) there is a mapping \(E : \mathcal{E}'(\mathbb{R}^{n}) \to \mathcal{E}'(\mathbb{R}^{n})\) such that for \(s \in \mathbb{R}\):
\[
EAu = u \text{ in } X \text{ if } u \in \mathcal{E}'(X),
\]
\[
\|Ef\|_{W^{s-p}(\mathbb{R}^{n})} \leq C\|f\|_{W^{s-k,p}(\mathbb{R}^{n})} \text{ for all } f \in \mathcal{E}'(\mathbb{R}^{n}) \cap W^{s-k,p}(\mathbb{R}^{n}). \tag{153}
\]

Then, consider \(\varphi \in C_{c}^{\infty}(B_{r}(x_{0}))\) with \(r\) sufficiently small, \(\varphi u \in W^{s,p}(\mathbb{R}^{n}) \cap \mathcal{E}'(\mathbb{R}^{n})\) and apply the above estimates to get
\[
\|\varphi u\|_{W^{s,p}(B_{r}(x_{0}))} = \|EA(\varphi u)\|_{W^{s,p}(B_{r}(x_{0}))} \lesssim \|\varphi u\|_{W^{s-k,p}(B_{r}(x_{0}))},
\]
\[
\lesssim \|\varphi Au\|_{W^{s-k,p}(B_{r}(x_{0}))} + \|u\|_{W^{s-1,p}(B_{r}(x_{0}))},
\]
\[
\lesssim \|Au\|_{W^{s-k,p}(B_{r}(x_{0}))} + \|u\|_{W^{s-1,p}(B_{r}(x_{0}))},
\]
\[
\text{where the last line follows from } [12, \text{Theorem } 10.1.15]. \text{ Now, if we take } r = \alpha, 1 < \alpha < 2 \text{ and } \varphi \geq 0 \text{ as above satisfying } \varphi = 1 \text{ on } B_{1}(x_{0}), \text{ then}
\]
\[
\|u\|_{W^{s,p}(B_{1}(x_{0}))} \leq \|\varphi u\|_{W^{s,p}(B_{1}(x_{0}))} \lesssim \|Au\|_{W^{s-k,p}(B_{1}(x_{0}))} + \|u\|_{W^{s-1,p}(B_{1}(x_{0}))}. \tag{154}
\]

Given any arbitrary \(l \in \mathbb{N}\), one can use the above argument inductively by choosing a sequence of cut-off functions \(\{\varphi_{j}\}\) supported in \(B_{\alpha_{j}}(x_{0})\), with \(1 < \alpha_{1} < \alpha_{2} < \cdots < 2\), and satisfying \(\varphi_{j} \equiv 1\) on \(B_{\alpha_{j-1}}(x_{0})\) for \(j \geq 2\), which allow us to estimate
\[
\|u\|_{W^{s-1,p}(B_{\alpha_{j}}(x_{0}))} \lesssim \|Au\|_{W^{s-1-m,p}(B_{\alpha_{j+1}}(x_{0}))} + \|u\|_{W^{s-1-k+p}(B_{\alpha_{j+1}}(x_{0}))}, \tag{155}
\]
\[
\text{Then, given any } t \in \mathbb{R} \text{ satisfying } t < s - 1, \text{ picking in the above estimate } l > s - t - 1 \text{ and putting together } \text{(153)-(155)} \text{ along the (finite) inductive sequence, we find}
\]
\[
\|u\|_{W^{s,p}(B_{t}(x_{0}))} \lesssim \|Au\|_{W^{s-k,p}(B_{t+1}(x_{0}))} + \|u\|_{W^{t,p}(B_{t+1}(x_{0}))}. \tag{156}
\]
Now the extension of the above estimate to \(t < s\) is trivial, since \([12, \text{Theorem } 13.3.3]\) implies that if \(Au \in W^{s-k,p}\), then \(u \in W^{s}_{loc}\), and therefore \(u \in W^{s,p}(B_{t}(x_{0}))\) for all \(t < s\). \qed
Two operators are of particular interest in this work: $\Delta_\gamma$ and $\Delta_{\gamma,\text{conf}}$. We will spell out the regularity results for these two operators for convenience and self-containedness.

**Lemma A.3.** Let $(M, \gamma)$ be a smooth manifold of bounded geometry and let $A$ stand for either $\Delta_\gamma$ or $\Delta_{\gamma,\text{conf}}$. For any $s, t \in \mathbb{R}$, $t < s$, and $1 < p < \infty$, if $u \in W^{-\infty,p}(M; E)$ and $Au \in W^{s-2,p}(M; F)$, then $u \in W^{s,p}(M; E)$ and there exists a constant $C > 0$ such that

$$\|u\|_{W^{s,p}(M)} \leq C (\|Au\|_{W^{s-2,p}(M)} + \|u\|_{W^{t,p}(M)}). \quad (157)$$

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