RELATIVELY MAXIMUM VOLUME RIGIDITY IN ALEXANDROV
GEOMETRY

NAN LI AND XIAOCHUN RONG

ABSTRACT. Given a compact metric space $Z$ with Hausdorff dimension $n$, let $X$ be a metric space such that there is a distance non-increasing onto map $f : Z \to X$. Then the Hausdorff $n$-volume $\text{vol}(X) \leq \text{vol}(Z)$. The relatively maximum volume conjecture says that if $X$ and $Z$ are both Alexandrov spaces and $\text{vol}(X) = \text{vol}(Z)$, then $X$ is isometric to a gluing space produced from $Z$ along its boundary $\partial Z$ and $f$ is length preserving. We will partially verify this conjecture, and give a further classification for compact Alexandrov $n$-spaces with relatively maximum volume in terms of a fixed radius and space of directions. We will also give an elementary proof for a pointed version of Bishop-Gromov relative volume comparison with rigidity in Alexandrov geometry.

INTRODUCTION

Let $Z$ be a compact metric space with Hausdorff dimension $\alpha$. Consider all compact metric spaces $X$ with Hausdorff dimension $\alpha$ such that there is a distance non-increasing onto map $f : Z \to X$. We let “vol” denote the Hausdorff measure (or volume) in the top dimension. Then $\text{vol}(X) \leq \text{vol}(Z)$. A natural question is to determine $X$ (in terms of $Z$) when $\text{vol}(X) = \text{vol}(Z)$. We will refer this as a relatively maximum volume rigidity problem.

A possible answer to the relatively maximum volume rigidity problem is closely related to the regularity of underlying geometric and topological structures. For instance, if $Z$ and $X$ are closed Riemannian $n$-manifolds, then $f$ is an isometry (see Corollary 0.2). On the other hand, taking any measure-zero subset $S$ in $Z$ (a Riemannian manifold) and identifying $S$ with a point $p \in S$, then the projection map, $Z \to X = Z/(S \sim p)$, is a distance non-increasing onto map, and it is hopeless to have some rigidity on $Y$ in terms of $X$.

In this paper, we will study the relatively maximum volume rigidity problem in Alexandrov geometry, partly because an Alexandrov space $X$ has a “right” geometric structure for this problem (see Conjecture 0.1 below). For instance, for $p \in X$, the gradient-exponential map, $g \exp_p : T_pX \to X$, becomes a distance non-increasing map, when $T_pX$ is equipped with the $\kappa$-cone metric via the cosine law on the space form $S^2_\kappa$ (cf. [BGP]). When taking $Z$ to be a closed $r$-ball at the vertex (for $\kappa > 0$, $r \leq \frac{\pi}{2\sqrt{\kappa}}$ or $r = \frac{\pi}{\sqrt{\kappa}}$), the relatively volume rigidity problem (see Theorem B) indeed extends the (absolutely) Maximum Radius-Volume Rigidity Theorem proved by Grove-Petersen ([GP], Theorem [1.3]).

The recent study of Alexandrov spaces was initiated by Burago-Gromov-Perel’man in the paper [BGP] and has gotten a lot attention lately. An Alexandrov space with curvature $\text{curv} \geq \kappa$ is a length metric space such that each point has a neighborhood in which Toponogov triangle

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comparison holds with respect to the space form of constant curvature $\kappa$. In the rest of the paper, we will freely use basic notions on an Alexandrov space from [BGP] and [Pet2] (e.g., the space of directions, the gradient-exponential maps, $(n, \delta)$-strained points, etc). Let $\text{Alex}^n(\kappa)$ denote the collection of compact Alexandrov $n$-spaces with $\text{curv} \geq \kappa$.

Note that the boundary gluing will automatically yield a distance non-increasing onto (projection) map, which also preserves the volume (see Example 2.14, 2.15). We propose the following relatively maximum volume rigidity conjecture for Alexandrov spaces.

**Conjecture 0.1.** Let $Z, X \in \text{Alex}^n(\kappa)$, and let $f : Z \to X$ be a distance non-increasing onto map. If $\text{vol}(Z) = \text{vol}(X)$, then $X$ is isometric to a gluing space produced from $Z$ along its boundary $\partial Z$ and $f$ is length preserving. In particular, $Z$ is isometric to $X$ if $\partial Z = \emptyset$ or if $f$ is injective.

Our goal in this paper is to partially verify Conjecture 0.1, and give a classification for the boundary gluing maps in a special case (see Theorem A, Corollary 0.2 and Theorem B).

We now begin to state the main results in this paper. Throughout this paper, $\tau(\delta)$ denotes a function in $\delta$ such that $\tau(\delta) \to 0$ as $\delta \to 0$. Our first result verifies conjecture 0.1 for the case that $f$ preserves non-$(n, \delta)$-strained points up to an error $\tau(\delta)$. For $X \in \text{Alex}^n(\kappa)$ and $\delta > 0$, let $X^\delta \subseteq X$ denote the set of all $(n, \delta)$-strained points. Then a small ball centered at an $(n, \delta)$-strained point is almost isometric to an open subset in $\mathbb{R}^n ([BGP])$.

**Theorem A.** Let $Z, X$ be Alexandrov $n$-spaces (not necessarily complete) with curvature $\text{curv} \geq \kappa$ and $\text{vol}(Z) = \text{vol}(X)$. Suppose that $f : Z \to X$ is a distance non-increasing onto map such that for any $\delta > 0$, $f^{-1}(X^\delta) \subseteq Z^{\tau(\delta)}$. Then $f$ is an isometry.

A point $z$ in $Z$ is called regular, if the space of directions $\Sigma_z$ is isometric to a unit sphere. Clearly, the space $Z$ with all points regular is a topological manifold but $Z$ may not be isometric to any Riemannian manifold (e.g., the doubling of two flat disks). Theorem A includes the following case:

**Corollary 0.2.** Let $Z, X \in \text{Alex}^n(\kappa)$ with $\text{vol}(Z) = \text{vol}(X)$ and all points in $Z$ are regular (e.g. $Z$ is a Riemannian manifold). If $f : Z \to X$ is a distance non-increasing onto map, then $f$ is an isometry.

In Alexandrov geometry, perhaps the most natural distance non-increasing onto map is the gradient-exponential map $g \exp_p : C_\kappa(\Sigma_p) \to X$, $p \in X \in \text{Alex}^n(\kappa)$, where $C_\kappa(\Sigma_p)$ denotes the tangent cone $T_pX$ equipped with a $\kappa$-cone metric via the cosine law in $S_\kappa^2 ([BGP])$. Since $g \exp_p$ is distance non-increasing and preserves any $r$-ball, one immediately gets the pointed version of Bishop type volume comparison:

$$\text{vol}(B_R(p)) \leq \text{vol}(C_\kappa^R(\Sigma_p)),$$

where $C_\kappa^R(\Sigma_p)$ denotes the open $R$-ball in $C_\kappa(\Sigma_p)$ at the vertex $\tilde{o}$. We will show that when the equality holds, $g \exp_p$ will satisfy the conditions in Theorem A (Lemma 2.4, Lemma 2.5) and thus open ball $C_\kappa^R(\Sigma_p)$ is isometric to $B_R(p)$ with respect to intrinsic metrics (see Theorem 2.1).

As an important case for Conjecture 0.1, one leads to classify Alexandrov spaces with relatively maximum volume: given any $\kappa, R > 0$ and $\Sigma \in \text{Alex}^{n-1}(1)$, let $\mathcal{A}_\kappa^R(\Sigma)$ denote the collection of
Alexandrov $n$-spaces $X \geq p$ satisfying
\[
\text{curv} \geq \kappa, \quad X = \bar{B}_R(p), \quad \Sigma_p = \Sigma.
\]
Then $\text{vol}(X) \leq \text{vol} \left( C^R \kappa(\Sigma) \right) = v(\Sigma, \kappa, R)$. When $\text{vol}(X) = v(\Sigma, \kappa, R)$, we say that $X$ has the relatively maximum volume.

**Theorem B** (Relatively maximum volume rigidity). Let $X \in A^R_\kappa(\Sigma)$ such that $\text{vol}(X) = v(\Sigma, \kappa, R)$. Then $X$ is isometric to $C^R_\kappa(\Sigma)/x \sim \phi(x)$ and $R \leq \frac{x}{2\sqrt{\kappa}}$ or $R = \frac{x}{\sqrt{\kappa}}$ for $\kappa > 0$, where $\phi : \Sigma \times \{R\} \to \Sigma \times \{R\}$ is an isometric involution (which can be trivial). Conversely, given any isometric involution $\phi$ on $\Sigma$, $C^R_\kappa(\Sigma)/x \sim \phi(x) \in A^R_\kappa(\Sigma)$ and has the relatively maximum volume.

Theorem B verifies Conjecture 0.1 for the case $f = g \exp_p : Z = C^R_\kappa(\Sigma_p) \to X$, together with a further classification for the boundary identification. Note that Theorem B implies that if $k > 0$ and $\frac{x}{2\sqrt{\kappa}} < R < \frac{x}{\sqrt{\kappa}}$, then max\{vol\}(X), $X \in A^R_\kappa(\Sigma) \} < v(\Sigma, \kappa, R)$. For the case that $X$ is a limit of Riemannian manifolds, a classification was given in [GP]. A general classification is more complicated, and we wish to discuss it elsewhere.

As mentioned earlier, Theorem B extends the radius-volume rigidity theorem in [GP], which are stated below.

**Theorem 0.3 ([GP]).** Let $M_i \xrightarrow{d_{GH}} X$ be a Gromov-Hausdorff convergent sequence of Riemannian $n$-manifold such that
\[
\text{sec}_{M_i} \geq \kappa, \quad \text{rad}(M_i) = R, \quad \text{vol}(M_i) = \text{vol}(C^R_\kappa(S^n_{i-1})),
\]
where $\text{rad}(M_i) = \min \{ r, \bar{B}_r(p) = M_i, \quad p \in M_i \}$. Then $X$ is isometric to $C^R_\kappa(S^n_{i-1})/x \sim \phi(x)$ and $R \leq \frac{x}{2\sqrt{\kappa}}$ or $R = \frac{x}{\sqrt{\kappa}}$ for $\kappa > 0$, where $\phi : \partial C^R_\kappa(S^n_{i-1}) \to \partial C^R_\kappa(S^n_{i-1})$ is either the antipodal map, or a reflection by a totally geodesic hypersurface. Moreover, $M_i$ is homeomorphic to an $n$-sphere or a real projective $n$-space.

Note that $\text{vol}(X) = \text{vol}(C^R_\kappa(S^n_{i-1}))$. Choosing $p_i \in M_i$ such that $M_i \to \bar{B}_R(p_i)$, then $p_i \to p \in X$ and $\Sigma_p = S^n_{i-1}$. By now Theorem B implies the rigidity part of Theorem 0.3 (a generalization of the homeomorphic rigidity in Theorem 0.3 will be given in Theorem C). Theorem B also implies the following extension of Theorem 0.3 by S. Shiegold.

**Theorem 0.4 ([Sh]).** Let $X \in A^R_\kappa(S^n_{i-1})$ with $\text{vol}(X) = v(S^n_{i-1}, \kappa, r)$. Then $X = C^r_\kappa(S^n_{i-1})/x \sim \phi(x)$, $x \in S^n_{i-1} \times \{ r \}$, where $\phi$ is the reflection on a $\ell$-dimensional totally geodesic subsphere, $1 \leq \ell \leq n$. ($\phi$ is trivial for $\ell = n$.)

A further problem concerning Theorem B is to determine the homeomorphic type of $X$. We have solved this problem for $X$ being a topological manifold (see Theorem 0.3).

**Theorem C.** Given $\Sigma \in \text{Alex}^{n-1}(1)$, $\kappa$ and $R > 0$, there is a constant $\epsilon = \epsilon(\Sigma, \kappa, R) > 0$ such that if $X \in A^R_\kappa(\Sigma)$ with $\text{vol}(X) > v(\Sigma, \kappa, R) - \epsilon$ and $X$ is a closed topological manifold, then $X$ is homeomorphic to $S^n_1$ or a real projective space $\mathbb{R}P^n$.

Note that $\Sigma$ in Theorem C is not necessarily a topological manifold; for instance, $X = C_1(C_1(N))$, the twice spherical suspensions over a Poincaré sphere $N$, satisfies Theorem C but $\Sigma = C_1(N)$ is not a topological manifold. However, $X$ is homeomorphic to a 5-sphere (cf. [Ka1]).
In the proof of Theorem B, we establish a pointed version of Bishop volume comparison with rigidity (Theorem 2.1). In general, we will prove the following pointed version of Bishop-Gromov relative volume comparison with rigidity.

For $p \in X \in \text{Alex}^n(\kappa)$, let $A^*_R(p)$ denote the annulus $\{x \in X : r < |px| < R\}$, $0 \leq r < R$, and let $A^*_R(\Sigma_p)$ denote the corresponding annulus in $C_\kappa(\Sigma_p)$.

**Theorem D** (Pointed Bishop-Gromov relative volume comparison). Let $X \in \text{Alex}^n(\kappa)$. Then for any $p \in X$ and $R_3 > R_2 > R_1 \geq 0$, 
\[
\frac{\text{vol}(A^*_{R_3}(p))}{\text{vol}(A^*_{R_2}(p))} \geq \frac{\text{vol}(A^*_{R_3}(\Sigma_p))}{\text{vol}(A^*_{R_2}(\Sigma_p))}, \quad \text{or equivalently}, \quad \frac{\text{vol}(A^*_{R_3}(p))}{\text{vol}(A^*_{R_2}(p))} \geq \frac{\text{vol}(A^*_{R_3}(\Sigma_p))}{\text{vol}(A^*_{R_2}(\Sigma_p))}.
\]

In particular, 
\[
\frac{\text{vol}(B_{R_3}(p))}{\text{vol}(B_{R_2}(p))} \geq \frac{\text{vol}(C^*_R(\Sigma_p))}{\text{vol}(C^*_R(\Sigma_p))}.
\]

If any of the above inequalities becomes equal, then the open ball $B_{R_3}(p)$ is isometric to $C^*_R(\Sigma_p)$ with respect to intrinsic metrics.

**Remark 0.5.** The Riemannian version of Bishop-Gromov relative comparison for Alexandrov spaces (i.e., the model space is $S^n_\kappa$) was stated in [BGP] (cf. [BBI]). A notable difference between Theorem D and the Riemannian version is in the rigidity part: the later is the absolute maximum volume rigidity and its model space is unique, while the former may be viewed as the relatively maximum volume rigidity (relatively to $\Sigma_p$), whose model spaces are of infinitely many possibilities. Moreover, the proof of Theorem D is considerably difficult; for instance, a dimension-inductive argument (which works in the Riemannian version) does not work.

**Remark 0.6.** By Lemma 2.1 in [LR], we see that $\frac{\text{vol}(C^*_R(\Sigma_p))}{\text{vol}(C^*_R(\Sigma_p))} = \frac{\text{vol}(C^*_R(S^n_\kappa))}{\text{vol}(C^*_R(S^n_\kappa))}$ and thus the monotonicity part of Theorem D coincides with that in the Riemannian version. We point out that our proof of the volume ratio monotonicity in Theorem D is different from one suggested by [BGP]; we take an elementary (calculus) approach via finding a (unconventional) partition suitable for triangle comparison arguments while a proof in [BBI] relies on a co-area formula for Alexandrov spaces.

We now give some indication on our approach to Theorem A and Theorem B. In the proof of Theorem A, we shall show that $f$ is a homeomorphism and $f$ preserves the length of curves. Based on basic properties of an Alexandrov space (not necessarily complete), any curve $c$ in $X$ can be approximated by piecewise geodesics $c_i$ in $X^{\delta_i} (\delta_i \to 0)$ such that lengths $L(c_i) \to L(c)$. Thus, it suffices to show that when restricting to $f^{-1}(X^{\delta})$ and $X^{\delta}$ respectively, $f$ is injective and $f^{-1}$ preserves the length of any geodesic up to an error $\tau(\delta) \to 0$ as $\delta \to 0$ respectively. We derive this with a volume formula for a tube-like $\epsilon$-balls in $X^{\delta}$ which can be treated as a replacement of the volume formula of a thin tube around a curve. The proof of the volume formula is a based on the fact that a small ball at an $(n, \delta)$-strained point can be almost isometrically embedded into $\mathbb{R}^n$ (see [BGP]).

Our approach to Theorem B consists of two steps: first, establishing the open ball rigidity: the gradient-exponential map $g \exp_p : C^*_R(\Sigma_p) \to B_R(p) \subset X$ is an isometry with respect to the intrinsic distance. We achieve this by showing that $g \exp_p$ satisfies the condition in Theorem
A (see Lemma 2.3 and Lemma 2.5). Consequently, \( X = \bar{C}^R_\kappa(\Sigma_p)/\sim \), where \( \sim \) is a relation on \( \Sigma_p \times \{R\} \): \( \bar{x} \sim \bar{y} \) if and only if \( g \exp_p(\bar{x}) = g \exp_p(\bar{y}) \). Observe that if \( \bar{x} \neq \bar{y} \in \Sigma_p \times \{R\} \) with \( \bar{x} \sim \bar{y} \), then the \( g \exp_p \)-images of the two geodesics \([\bar{a}_\bar{x}]\) and \([\bar{a}_\bar{y}]\) together form a local geodesic at \( g \exp_p \bar{x} = g \exp_p \bar{y} \). Because a geodesic does not bifurcate, any equivalent class contains at most two points and thus we obtain an involution \( \phi : \Sigma_p \times \{R\} \rightarrow \Sigma_p \times \{R\} \) such that \( X = \bar{C}^R_\kappa(\Sigma)/\bar{x} \sim \phi(\bar{x}), \bar{x} \in \Sigma_p \times \{R\} \). The main difficulty is to show that \( \phi \) is an isometry. Our main technical lemma is to show that \( \phi \) is almost 1-bi-Lipschitz up to a uniform error:

\[
\frac{|\phi(\bar{x})\phi(\bar{y})|}{|\bar{x}\bar{y}|} - 1 \leq 20|\bar{x}\bar{y}| \quad \text{for } |\bar{x}\bar{y}| \text{ small} \quad \text{(see Lemma 2.12)}.
\]

This implies that \( \phi \) is continuous and preserves the length of a path, and thus \( \phi \) is distance non-increasing. Consequently, \( \phi \) is an isometry since \( \phi \) is an involution. Note that without the curvature lower bound, this in general does not imply that the metric on \( X = \bar{C}^R_\kappa(\Sigma)/\bar{x} \sim \phi(\bar{x}) \) coincides with the induced metric. For example, \( X = \bar{C}_0^1(S^1)/\bar{x} \sim \bar{x} = \bar{B}_1(\mathbb{R}^2) \) is equipped with the length metric coincides with the Euclidean metric when restricted to the interior, and \( L(\gamma) \) is a half of the Euclidean arc length for any \( \gamma \subset \partial X \). Our proof relies on the curvature lower bound as well as the cone metric.

Let \( L_p(X) = \exp_p^1(\Sigma \times \{R\}) \), which locally divides a tubular neighborhood of \( L_p(X) \) into two components \( U_1, U_2 \). The main difficulty in proving the above inequality is that a geodesic in \( X \) connecting 2 points \( a, b \in L_p(X) \) may intersect with \( L_p(X) \) at many points other than \( a, b \) (called crossing points). We show that if a geodesic is not contained in \( L_p(X) \), then the crossing points are discrete (Corollary 2.9). Thus we can reduce the proof to the case that \( c_1 = [ab] \subset U_1 \) has no crossing point. It’s sufficient to construct a non-crossing piece-wise intrinsic geodesic \( c_2 \subset U_2 \) connecting \( a, b \), and show that \( \text{length}(c_2) \) is close to \( \text{length}(c_1) = |ab| \) up to a second order error (Lemma 2.12).

We remark that the present proof, in an essential way, relies on the \( \kappa \)-cone metric structure; and we believe that establishing a similar inequality in general will be the main obstacle in Conjecture 0.1.

The rest of the paper is organized as follows:

In Section 1, we will prove Theorem A.

In Section 2, we will prove Theorem B.

In Section 3, we will prove Theorem C.

In Section 4, we will prove Theorem D.

1. \((n, \delta)\)-strained isometry

Let \( f : Z \rightarrow X \) be as in Theorem A. We will establish that \( f \) is an isometry through the following properties:

(i) If a distance non-increasing onto map \( f \) preserves the volume of the total spaces, then \( f \) and \( f^{-1} \) preserve volumes of any subsets (see Lemma 1.1).

(ii) Based on a local bi-Lipschitz embedding property (see Lemma 1.2), we show that for \( \delta \) suitably small, \( f \) is injective on \( f^{-1}(X^\delta) \subseteq Z^{\tau(\delta)} \). In particular, for any curve \( c \subset X^\delta, f^{-1}(c) \subseteq Z^{\tau(\delta)} \) is a curve (see Lemma 1.3).

(iii) Our main technical lemma is a volume formula for a ‘tube’ of \( \epsilon \)-balls (which can be treated as a replacement for an \( \epsilon \)-tube around a curve, see Lemma 1.4). Together with (i) and (ii), this formula implies that \( f^{-1} \) preserves the length of any geodesic in \( X^\delta \) up to an error \( \tau(\delta) \). Because
for any small $\delta$ ($\delta < \frac{1}{8n}$), $X^\delta$ is dense in $X$ (see Lemma 1.6), we are able to show that $f$ is also distance non-decreasing and thus $f$ is an isometry.

**Lemma 1.1.** Let $f : Z \to X$ be a distance non-increasing onto map of two metric spaces of equal Hausdorff dimension. If $\text{vol}(X) = \text{vol}(Z)$, then for any subset $A \subseteq Z$ and $B \subseteq X$,

$$\text{vol}(A) = \text{vol}(f(A)), \quad \text{vol}(B) = \text{vol}(f^{-1}(B)).$$

**Proof.** We argue by contradiction; assuming that $\text{vol}(A) > \text{vol}(f(A))$. Then

$$\text{vol}(Z) = \text{vol}(A) + \text{vol}(Z - A) > \text{vol}(f(A)) + \text{vol}(f(Z - A)) \geq \text{vol}(f(Z)) = \text{vol}(X),$$

a contradiction. Similarly, one can check that $\text{vol}(f^{-1}(B)) = \text{vol}(B)$.

Let $X^\delta(p)$ denote the union of points with an $(n, \delta)$-strainer $\{(a_i, b_i)\}$ of radius $\rho > 0$, where $\rho = \min_{1 \leq i \leq n} \{|pa_i|, |pb_i|\} > 0$.

**Lemma 1.2** ([BGP] Theorem 9.4). Let $X \in \text{Alex}^n(\kappa)$. If $p \in X^\delta(p)$, then the map $\psi : X \to \mathbb{R}^n$ defined by $\psi(x) = [(a_1 x_1), \ldots, (a_n x_n)]$ maps a small neighborhood $U$ of $p$ $\tau(\delta, \delta_1)$-almost isometrically onto a domain in $\mathbb{R}^n$, i.e. $||\psi(x)\psi(y)|| - |xy| < \tau(\delta, \delta_1)|xy|$ for any $x, y \in U$, where $\delta_1 = \rho^{-1} \cdot \text{diam}(U)$. In particular, $\psi$ is an $\tau(\delta)$-almost isometric embedding when restricting to $B_{\delta\rho}(p)$.

A consequence of Lemma 1.2 is that

$$1 - \tau(\delta) \leq \frac{\text{vol}(B_{\epsilon}(p))}{\text{vol}(B_{\epsilon}(\mathbb{R}^n))} \leq 1 + \tau(\delta),$$

for any $p \in X^\delta(p)$ and $\epsilon \leq \delta \rho$.

**Lemma 1.3.** Let the assumptions be as in Theorem A. Then $f : f^{-1}(X^\delta) \to X^\delta$ is injective. Consequently, if $\gamma \subset X^\delta$ is a continuous curve, then $f^{-1}(\gamma)$ is also a continuous curve.

**Proof.** We argue by contradiction; assuming $z_1 \neq z_2 \in f^{-1}(X^\delta)$ such that $f(z_1) = f(z_2) = x$. We may assume that $z_1$ and $z_2$ have $\tau(\delta)$-strainer of radius $\rho > 0$. Choose $4\epsilon < |z_1 z_2|$ and $\epsilon < \delta \rho$. By Lemma 1.1 and the above consequence of Lemma 1.2, we get

$$1 = \frac{\text{vol}(f^{-1}(B_{\epsilon}(x)))}{\text{vol}(B_{\epsilon}(x))} \geq \frac{\text{vol}(B_{\epsilon}(z_1)) + \text{vol}(B_{\epsilon}(z_2))}{\text{vol}(B_{\epsilon}(x))} \geq 2(1 - \tau(\delta)),$$

a contradiction.

We now develop a formula which estimates the volume of an $\epsilon$-ball tube with a higher order error. Let $x_1, x_2, \ldots, x_{N+1}$ be $N + 1$ points in $X^\delta(\rho)$. We first give an estimate of the volume of the $\epsilon$-ball tube $\bigcup_{i=1}^{N+1} B_{\epsilon}(x_i)$ in terms of $\sum_{i=1}^{N} |x_i x_{i+1}|$ and $\epsilon, \delta$ with errors.
Lemma 1.4 (volume of an $\epsilon$-ball tube). Let $X \in \text{Alex}^\theta(\kappa)$ and $x_i \in X^\delta(\rho)$, $i = 1, 2, \cdots, N + 1$ satisfy that $0 < |x_i x_{i+1}| < 2\epsilon \ll \delta \rho$ and $B_{\epsilon}(x_i) \cap B_{\epsilon}(x_j) \cap B_{\epsilon}(x_k) = \emptyset$ for $i \neq j \neq k$. Then the volume of the $\epsilon$-ball tube $\bigcup_{i=1}^{N+1} B_{\epsilon}(x_i)$ (see Figure 1) satisfies:

\[
(1 + \tau(\delta)) \cdot \text{vol} \left( \bigcup_{i=1}^{N+1} B_{\epsilon}(x_i) \right)
\]

\[
= \text{vol} (B_{\epsilon}(\mathbb{R}^n)) + 2\epsilon \cdot \text{vol} \left( B_{\epsilon}(\mathbb{R}^{n-1}) \right) \sum_{i=1}^{N} \int_{\theta_i}^{\pi} \sin^n(t) dt,
\]

where $\theta_i \in [0, \frac{\pi}{2}]$ such that $\cos \theta_i = \frac{|x_i x_{i+1}|}{2\epsilon}$. If in addition, $|x_i x_{i+1}| \leq \epsilon^2$ for all $1 \leq i \leq N$, then

\[
(1 + \tau(\delta)) \cdot \text{vol} \left( \bigcup_{i=1}^{N+1} B_{\epsilon}(x_i) \right)
\]

\[
= \text{vol} (B_{\epsilon}(\mathbb{R}^n)) + \text{vol} \left( B_{\epsilon}(\mathbb{R}^{n-1}) \right) \sum_{i=1}^{N} |x_i x_{i+1}| + O(\epsilon^{n+1}) \sum_{i=1}^{N} |x_i x_{i+1}|,
\]

Figure 1

Because $B_{\epsilon}(x_{i-1}) \cup B_{\epsilon}(x_i) \cup B_{\epsilon}(x_{i+1}) \subset B_{\delta \rho}(x_i)$, which is $\tau(\delta)$-almost isometrically embedded into $\mathbb{R}^n$, one can divide $\bigcup_{i=1}^{N+1} B_{\epsilon}(x_i)$ into small pieces $\Gamma_{\epsilon}^\pm(x_i)$, whose volumes are $(1 + \tau(\delta))$-proportional to the volumes of the following “trapezoidal balls” $\Gamma_{\epsilon}^{h_\pm}(\mathbb{R}^n)$ in $\mathbb{R}^n$. This allows us to reduce the calculation to the Euclidean space.

We define the trapezoidal ball $\Gamma_{\epsilon}^h(\mathbb{R}^n)$ in $\mathbb{R}^n_+ = \{(x_1, x_2, \cdots, x_n) : x_n \geq 0\}$ as the following. Let $u \in \mathbb{R}^n_+$ be a point with $|ou| = h \leq r$. Then the hyper plane $H$ passing through $u$ and perpendicular to $\overrightarrow{ou}$ divides the half ball $B_{\epsilon}(\mathbb{R}^n) \cap \mathbb{R}^n_+$ into two subsets. Let $\Gamma_{\epsilon}^h(\mathbb{R}^n)$ be the subset which contains the origin (see Figure 3). It’s easy to see that $\text{vol} \left( \Gamma_{\epsilon}^h(\mathbb{R}^n) \right)$ depends only on $h$ and $r$, but not the direction $\overrightarrow{ou}$ as long as $H \cap B_{\epsilon}(\mathbb{R}^n) \subset \mathbb{R}^n_+$. 
Lemma 1.5. Let $\Gamma_h^\theta(\mathbb{R}^n)$ be a trapezoidal ball defined as the above. Then

$$\text{vol} \left( \Gamma_h^\theta(\mathbb{R}^n) \right) = r \cdot \text{vol} \left( B_r(\mathbb{R}^{n-1}) \right) \int_0^{\pi/2} \sin^n(t) \, dt,$$

where $\theta \in [0, \frac{\pi}{2}]$ such that $r \cos \theta = h$.

Proof. Let $s = r \cos t \in [0, h]$ be the parameter for the height with the corresponding angle $t \in [\theta, \frac{\pi}{2}]$. Then

$$\text{vol} \left( \Gamma_h^\theta(\mathbb{R}^n) \right) = \int_0^h \text{vol} \left( B_{rs}(\mathbb{R}^{n-1}) \right) \, ds = \int_0^{\pi/2} \text{vol} \left( B_{rs}(\mathbb{R}^{n-1}) \right) r \sin(t) \, dt$$

$$= r \cdot \text{vol} \left( B_r(\mathbb{R}^{n-1}) \right) \int_0^{\pi/2} \sin^n(t) \, dt.$$

Proof of the volume formula, Lemma 1.4. Because $B_\epsilon(x_i) \cap B_\epsilon(x_{i+1}) \neq \emptyset$ and $B_\epsilon(x_i) \cap B_\epsilon(x_j) \cap B_\epsilon(x_k) = \emptyset$ for any $i \neq j \neq k$, we can decompose $\bigcup_{i=1}^{N+1} B_\epsilon(x_i)$ as the following (see Figure 0.2): let

$$A^+(x_i) = \{ q \in B_\epsilon(x_i) : |q x_i| \leq |q x_{i+1}| \},$$

$$A^-(x_i) = \{ q \in B_\epsilon(x_i) : |q x_i| \leq |q x_{i-1}| \}.$$ 

For $i = 2, 3, \cdots, N$, let

$$H^+(x_i) = A^+(x_i) \cap A^-(x_{i+1}) = \{ q \in B_\epsilon(x_i) \cap B_\epsilon(x_{i+1}) : |q x_i| = |q x_{i+1}| \},$$

$$H^-(x_i) = A^-(x_i) \cap A^+(x_{i-1}) = \{ q \in B_\epsilon(x_i) \cap B_\epsilon(x_{i-1}) : |q x_i| = |q x_{i-1}| \};$$

and

$$\Gamma^+(x_i) = \{ q \in A^+(x_i) \cap A^-(x_i) : d(q, H^+(x_i)) \leq d(q, H^-(x_i)) \},$$

$$\Gamma^-(x_i) = \{ q \in A^+(x_i) \cap A^-(x_i) : d(q, H^+(x_i)) \geq d(q, H^-(x_i)) \}.$$ 

By the construction,

$$\bigcup_{i=1}^{N+1} B_\epsilon(x_i) = A^-(x_1) \cup \left( \bigcup_{i=2}^N \Gamma^\pm(x_i) \right) \cup A^+(x_{N+1}).$$
Note that $H^+(x_i), i = 2, \cdots, N$ consist of all the possible intersections of any two of $A^-(x_1)$, $\Gamma^+(x_i), i = 2, \cdots, N$ and $A^+(x_{N+1})$ and $\text{vol}(H^+(x_i)) = 0$, we have
\[
\text{vol}\left(\bigcup_{i=1}^{N+1} B_\epsilon(x_i)\right) = \text{vol}(A^-(x_1)) + \text{vol}(A^+(x_{N+1})) + \sum_{i=2}^{N} \text{vol}(\Gamma^+(x_i)) + \sum_{i=2}^{N} \text{vol}(\Gamma^-(x_i)).
\]
(1.3)

Because $B_\epsilon(x_{i-1}) \cup B_\epsilon(x_i) \cup B_\epsilon(x_{i+1}) \subset B_\delta(x_i)$ which is homeomorphically and $\tau(\delta)$-almost isometrically embedded into $\mathbb{R}^n$, we have that
\[
(1 + \tau(\delta)) \cdot \text{vol}(\Gamma^-(x_i)) = \text{vol}\left(\Gamma^+_{\epsilon^i}(\mathbb{R}^n)\right),
\]
\[
(1 + \tau(\delta)) \cdot \text{vol}(A^+(x_1)) = \frac{1}{2} \text{vol}(B_\epsilon(\mathbb{R}^n)) + \text{vol}\left(\Gamma^+_{\epsilon^i}(\mathbb{R}^n)\right),
\]
\[
(1 + \tau(\delta)) \cdot \text{vol}(A^-(x_{N+1})) = \frac{1}{2} \text{vol}(B_\epsilon(\mathbb{R}^n)) + \text{vol}\left(\Gamma^+_{\epsilon^i}(\mathbb{R}^n)\right),
\]
where $h^+_i = \frac{1}{2}|x_i;x_{i+1}|$, $h^-_i = \frac{1}{2}|x_i;x_{i-1}|$. Note that it’s our convention that the same symbol $\tau(\delta)$ may represent different functions of $\delta$, as long as $\tau(\delta) \to 0$ as $\delta \to 0$. Together with (1.3) and the fact that $h^+_i = h^-_i$, we get
\[
(1 + \tau(\delta)) \cdot \text{vol}\left(\bigcup_{i=1}^{N+1} B_\epsilon(x_i)\right) = \text{vol}(B_\epsilon(\mathbb{R}^n)) + 2 \sum_{i=1}^{N} \text{vol}\left(\Gamma^+_{\epsilon^i}(\mathbb{R}^n)\right)
\]
(1.4)

Let $\theta_i \in [0, \frac{\pi}{2}]$ such that $\sin \theta_i = h^+_i / \epsilon = \frac{|x_i;x_{i+1}|}{2\epsilon}$. By Lemma 1.3, we have
\[
\text{vol}\left(\Gamma^+_{\epsilon^i}(\mathbb{R}^n)\right) = \epsilon \cdot \text{vol}(B_\epsilon(\mathbb{R}^{n-1})) \int_{\theta_i}^{\pi/2} \sin^n(t) \, dt.
\]

Plugging this into (1.4), we get (1.1).

To get (1.2), we need to write $\int_0^{\pi/2} \sin^n(t) \, dt$ in terms of $|x_i;x_{i+1}|$. Let $g(s) = \int_0^{s} \sin^n(t) \, dt$, where $\theta \in [0, \frac{\pi}{2}]$ with $\cos \theta = \frac{s}{2\epsilon}$. Noting that $\theta = \pi/2$ if and only if $s = 0$, we have $g(0) = 0$. Further more,
\[
g'(s) = -\sin^n \theta \cdot \frac{d\theta}{ds} = -\sin^n \theta \cdot \frac{1}{-2\epsilon \sin \theta} = \frac{\sin^{n-1} \theta}{2\epsilon};
\]
\[
g''(s) = \frac{1}{2\epsilon}(n-1)\sin^{n-2} \theta \cos \theta \cdot \frac{1}{-2\epsilon \sin \theta} = \frac{n-1}{-4\epsilon^2} \sin^{n-3} \theta \cos \theta;
\]
and thus $g'(0) = \frac{1}{2\epsilon}$, $g''(0) = 0$ and $g'''(0) = \frac{n}{2\epsilon^3}$. The Taylor expansion of $g$ at $s = 0$ is
\[
g(s) = \int_0^{s} \sin^n(t) \, dt = 0 + \frac{s}{2\epsilon} + \frac{1}{\epsilon^3} \cdot O(s^3).
\]

Let $s = |x_i;x_{i+1}| \leq \epsilon^2$, we get
\[
\int_{\theta_i}^{\pi/2} \sin^n(t) \, dt = \frac{1}{2\epsilon} |x_i;x_{i+1}| + O(\epsilon |x_i;x_{i+1}|).
\]

Plugging this into (1.1), we get (1.2). \qed
In the rest of this section we assume that \( f : Z \to X \) is a distance non-increasing onto map such that \( f^{-1}(X^\delta) \subset Z^{\tau(\delta)} \). By Lemma 1.3, \( f \) is homeomorphic on \( f^{-1}(X^\delta) \).

**Lemma 1.6.** Let the assumptions be as in Theorem A. Let \( x, y \in X^\delta \). For \( \delta > 0 \) sufficiently small, there exists a small constant \( c = c(\rho, \delta) > 0 \) such that if \( |xy| \leq c \), then \( |f^{-1}(x)f^{-1}(y)| \leq 2|x|y| \).

**Proof.** Assume that \( |xy| = \epsilon \ll \delta \rho \) and \( |f^{-1}(x)f^{-1}(y)| > 2\epsilon \), consider the metric balls \( B_\epsilon(x) \) and \( B_\epsilon(y) \). By Lemma 1.4

\[
(1 + \tau(\delta)) \cdot \vol (B_\epsilon(x) \cup B_\epsilon(y)) = \vol (B_\epsilon(\mathbb{R}^n)) + 2\epsilon \cdot \vol (B_\epsilon(\mathbb{R}^{n-1})) \int_{\pi/3}^{\pi/2} \sin^n(t) \, dt + O(\epsilon^{n+1}).
\]

Since \( B_\epsilon(f^{-1}(x)) \cap B_\epsilon(f^{-1}(y)) = \emptyset \), we have

\[
(1 + \tau(\delta)) \cdot \vol (B_\epsilon(f^{-1}(x)) \cup B_\epsilon(f^{-1}(y))) = 2\vol (B_\epsilon(\mathbb{R}^n)).
\]

Because \( f \) is distance non-increasing, \( B_\epsilon(f^{-1}(x)) \cup B_\epsilon(f^{-1}(y)) \subset f^{-1}(B_\epsilon(x) \cup B_\epsilon(y)) \). Together with that \( f^{-1} \) is volume preserving, we get

\[
1 = \frac{\vol (f^{-1}(B_\epsilon(x) \cup B_\epsilon(y)))}{\vol (B_\epsilon(x) \cup B_\epsilon(y))} \geq \frac{(1 - \tau(\delta)) \cdot 2\vol (B_\epsilon(\mathbb{R}^n))}{\vol (B_\epsilon(\mathbb{R}^n)) + 2\epsilon \cdot \vol (B_\epsilon(\mathbb{R}^{n-1})) \int_{\pi/3}^{\pi/2} \sin^n(t) \, dt + O(\epsilon^{n+1})}
\]

\[
= \frac{(1 - \tau(\delta)) \cdot 2 \int_{0}^{\pi/2} \sin^n(t) \, dt}{\int_{0}^{\pi/2} \sin^n(t) \, dt + \int_{\pi/3}^{\pi/2} \sin^n(t) \, dt + O(\epsilon)}.
\]

(see Lemma 1.5 \( \theta = 0 \))

This leads to a contradiction for sufficiently small \( \epsilon \) and \( \delta \). \( \square \)

In the proof of Theorem A, we will need the following result.

**Lemma 1.7** ([BGP] 10.6.1). Let \( X \in \text{Alex}^n(\kappa) \). For a fixed sufficiently small \( \delta > 0 \), the union of interior points which do not admit any \((n, \delta)\)-strainer has Hausdorff dimension \( \leq n - 2 \). In particular, \( X^\delta \) is dense.

**Proof of Theorem A.** Since \( f \) is distance non-increasing, it suffices to show that \( f \) is distance non-decreasing, i.e. for any \( \tilde{a}, \tilde{b} \in Z, |ab| \geq |\tilde{a}\tilde{b}| \), where \( a = f(\tilde{a}) \) and \( b = f(\tilde{b}) \).

For any small \( \epsilon_1 \), by Lemma 1.7 there are \( \tilde{a}_{\epsilon_1}, \tilde{b}_{\epsilon_1} \in Z^{\tau(\delta)}, a_{\epsilon_1} = f(\tilde{a}_{\epsilon_1}), b_{\epsilon_1} = f(\tilde{b}_{\epsilon_1}) \in X^\delta \), such that \( |aa_{\epsilon_1}| \leq |\tilde{a}\tilde{a}_{\epsilon_1}| < \epsilon_1, |bb_{\epsilon_1}| \leq |\tilde{b}\tilde{b}_{\epsilon_1}| < \epsilon_1 \).

Case 1. Assume that there exists a minimal geodesic \( [a_{\epsilon_1}b_{\epsilon_1}] \subset X \). Then \( [a_{\epsilon_1}b_{\epsilon_1}] \subset X^\delta \), because the spaces of directions are isometric along the interior of a geodesic ([Petrunin 98]). By Lemma 1.3 (which will be frequently used without mentioning), \( f^{-1}([a_{\epsilon_1}b_{\epsilon_1}]) \) is also a continuous curve. Because \( [a_{\epsilon_1}b_{\epsilon_1}] \) is compact, we may let \( \rho > 0 \) such that \( [a_{\epsilon_1}b_{\epsilon_1}] \subset X^{2\delta}(\rho) \) and \( f^{-1}([a_{\epsilon_1}b_{\epsilon_1}]) \subset Z^{\tau(\delta)}(\rho) \). Let \( \{x_i\}_{i=1}^{N+1} \) be an \( \epsilon \)-partition of \( [a_{\epsilon_1}b_{\epsilon_1}] \), where \( x_1 = a_{\epsilon_1}, x_{N+1} = b_{\epsilon_1} \). For \( \epsilon \ll \delta \rho \),
Because $[a_1, b_1]$ is a geodesic, Lemma 1.4 can be applied on the partition $\{x_i\}_{i=1}^{N+1}$. Thus we get

$$(1 + \tau(\delta)) \cdot \text{vol} \left( \bigcup_{i=1}^{N+1} B_\epsilon(x_i) \right)$$

$$= \text{vol} \left( B_\epsilon(\mathbb{R}^n) \right) + \text{vol} \left( B_\epsilon(\mathbb{R}^{n-1}) \right) \sum_{i=1}^{N} |x_i x_{i+1}| + O(\epsilon^{n+1}) \sum_{i=1}^{N} |x_i x_{i+1}|$$

$$= \text{vol} \left( B_\epsilon(\mathbb{R}^{n-1}) \right) \cdot |a_1, b_1| + O(\epsilon^n).$$

Let $z_i = f^{-1}(x_i)$. By Lemma 1.6, $|z_i z_{i+1}| \leq 2|x_i x_{i+1}| = 2\epsilon$. Together with that $f$ is distance non-increasing, one can easily check that $\bigcup_{i=1}^{N+1} B_\epsilon(z_i)$ satisfies the condition of Lemma 1.4. Then we have

$$(1 + \tau(\delta)) \cdot \text{vol} \left( \bigcup_{i=1}^{N+1} B_\epsilon(z_i) \right) = \text{vol} \left( B_\epsilon(\mathbb{R}^{n-1}) \right) \sum_{i=1}^{N} |z_i z_{i+1}| + O(\epsilon^n).$$

Because $f$ is distance non-increasing and volume preserving,

$$1 = \frac{\text{vol} \left( f^{-1}(\bigcup_{i=1}^{N+1} B_\epsilon(x_i)) \right)}{\text{vol} \left( \bigcup_{i=1}^{N+1} B_\epsilon(x_i) \right)} = \frac{\text{vol} \left( \bigcup_{i=1}^{N+1} B_\epsilon(z_i) \right)}{\text{vol} \left( \bigcup_{i=1}^{N+1} B_\epsilon(x_i) \right)}$$

$$= (1 - \tau(\delta)) \cdot \frac{\text{vol} \left( B_\epsilon(\mathbb{R}^{n-1}) \right) \sum_{i=1}^{N} |z_i z_{i+1}| + O(\epsilon^n)}{\text{vol} \left( B_\epsilon(\mathbb{R}^{n-1}) \right) \cdot |a_1, b_1| + O(\epsilon^n)},$$

$$= (1 - \tau(\delta)) \cdot \frac{\sum_{i=1}^{N} |z_i z_{i+1}| + O(\epsilon)}{|a_1 b_1| + O(\epsilon)}$$

$$\geq (1 - \tau(\delta)) \cdot \frac{|a_1 b_1| + O(\epsilon)}{|a_1 b_1| + O(\epsilon)}.$$

Let $\epsilon \to 0$, we get

$$|a_1 b_1| \geq (1 - \tau(\delta))|a_1 b_1|.$$

Case 2. Assume that there is no minimal geodesic in $X^\delta$ from $a_1$ to $b_1$ (since $X$ may not be complete). Because spaces of directions along the interior of geodesic are isometric to each other ([Pet1]), we may assume a curve $c_1$ in $X^\delta$ from $a_1$ to $b_1$ such that $L(c_1) < |a_1 b_1| + \epsilon_1$. Since $c_1(t)$ is a compact subset in the open set $X^\delta$, we may assume $\eta > 0$ such that an $\eta$-tube of $c_1$ is also contained in $X^\delta$. Consequently, we may assume a piecewise geodesic $c$ in $X^\delta$ such that $L(c) \leq L(c_1) \leq |a_1 b_1| + \epsilon_1$. Applying Case 1 to each geodesic segment of $c$, we conclude that

$$|a_1 b_1| \geq L(c) - \epsilon_1 \geq (1 - \tau(\delta))|a_1 b_1| - \epsilon_1.$$

In either Case 1 or Case 2, we have

$$|ab| \geq |a_1 b_1| - 2\epsilon_1 \geq (1 - \tau(\delta))|a_1 b_1| - 3\epsilon_1$$

$$\geq (1 - \tau(\delta)) \cdot (|\tilde{a} \tilde{b}| - 2\epsilon_1) - 3\epsilon_1.$$

Let $\delta \to 0$, $\epsilon_1 \to 0$, we get $|ab| \geq |\tilde{a} \tilde{b}|$. \qed
Our proof of the classification part in Theorem B is divided into the following two theorems: open ball rigidity (Theorem 2.1) and isometric involution (Theorem 2.2). Recall that $\phi$ denotes the vertex of the cone $\bar{C}R(\Sigma_p)$ and thus $g \exp_p(\bar{o}) = p$.

**Theorem 2.1.** Under the assumptions of Theorem B, $g \exp_p : \bar{C}R(\Sigma) \to B_R(p)$ is an isometry with respect to the intrinsic metrics. In particular, $g \exp_p = \exp_p$.

By Theorem 2.1, $X = \bar{C}R(\Sigma_p)/x \sim x'$, where the equivalent relation $x \sim x'$ if and only if $\exp_p(x) = \exp_p(x')$ and $x, x' \in \Sigma_p \times \{R\}$.

**Theorem 2.2.** Let $X = \bar{C}R(\Sigma_p)/x \sim x' \in \text{Alex}^n(\kappa)$ defined as the above, then each equivalent class contains at most 2 points. Moreover, the induced involution $\phi : \Sigma_p \times \{R\} \to \Sigma_p \times \{R\}$, $\phi(x) = x'$ (where $x \sim x'$) is an isometry.

Recall that the induced gradient-exponential map $g \exp_p : \bar{C}R(\Sigma) \to \bar{B}_R(p) = X$ is distance non-increasing and onto. Indeed, the open ball rigidity is essentially a consequence of Theorem A and a general property of $\exp^{-1}_p : X \to T_p X$: $\exp^{-1}_p$ preserves $(n,\delta)$-strained points up to a constant depending on $\delta$ (see Lemma 2.4). In the proof, let’s recall the following property from [BGP]:

**Lemma 2.3** ([BGP] Lemma 7.5 and 11.2). Let $p \in X \in \text{Alex}^n(\kappa)$. Then for any $\delta > 0$, there is a small neighborhood $U_p$ of $p$ such that for any triangle $\triangle pq$ with $a, b \in U_p$ each angle of $\triangle pq \subset X$ differs from the comparison angle of $\triangle pq \subset S^2_\kappa$ by less than $\delta$.

**Lemma 2.4.** Let $q \in X^\delta$. Then for any $p \in X$, $\tau^p_\delta \in \Sigma_p^{\tau(\delta)}$. Consequently, $\exp^{-1}_p(q) \in \bar{C}R(\Sigma_p)^{\tau(\delta)}$.

**Proof.** Since $q \in X^\delta$, by Lemma 1.2 we may assume an $(n,2\delta)$-strainer $\{(a_i,b_i)\}$ for $q_1 \in [pq]$ and near $q$, such that $b_n = q$, $a_n \in [pq_1]$. Because the spaces of directions are isometric along the interior of a geodesic ([Petrunin 98]), there is $q' \in [pq] \cap U_p$ which has an $(n,\tau(\delta))$-strainer $\{(a'_i,b'_i)\}$. By the same reason as the above, we can assume that $a'_n \in [pq']$ and $b'_n \in [q'q]$.

In addition, we can assume that $|q'a'_i|, |q'b'_i|$ are short so that $a'_i, b'_i \in U_p$ and $|\triangle a'_ipq', \triangle b'_ipq' < 5\delta$. We claim that $\{(\tau^p_{a'_i}, \tau^p_{b'_i})\}_{i=1}^{n-1}$ forms an $(n-1,\tau(\delta))$-strainer at $\tau^p_\delta \in \Sigma_p$. It’s easy to see that $\triangle a'_ipq' = \tilde{\triangle} a'_iq'x_j + \tau(\delta)$. Thus

$$
\cos \tilde{\triangle} a'_iq'x_j = \frac{|q'a'_i|^2 + |x_jq'|^2 - |a'_ix_j|}{2|a'_ix_j||x_jq'|} + \tau(\delta) = \cos \tilde{\triangle} a'_iq'x_j + \tau(\delta),
$$

where $i,j = 1,2,\cdots, n-1, x_j = a'_j$ or $b'_j$. \qed

To conclude the open ball rigidity by applying Theorem A, we need to check that $g \exp^{-1}_p(X^\delta) \subseteq \bar{C}R(\Sigma_p)^{\tau(\delta)}$. We obtain this by showing $g \exp_p = \exp_p$ when $\text{vol}(X) = v(\Sigma_p,\kappa, R)$.

**Lemma 2.5.** If $\text{vol}(B_R(p)) = \text{vol}(\bar{C}R(\Sigma_p))$, then the gradient exponential map is actually an exponential map $\exp_p : \bar{C}R(\Sigma_p) \to B_R(p)$ which preserves the distance along the radio direction.
Proof. Clearly, the map $\exp_p^{-1} : B_R(p) \to \tilde{C}_R^\kappa(\Sigma_p)$ (If there is more than one image, we will pick one) is distance non-decreasing. Because

$$\vol(C_R^\kappa(\Sigma_p)) = \vol(X) \leq \vol(\exp_p^{-1}(X)) \leq \vol(C_R^\kappa(\Sigma_p)),$$

$\exp_p^{-1}(X)$ is dense in $C_R^\kappa(\Sigma_p)$. For any $z \in C_R^\kappa(\Sigma_p)$, there is a sequence $x_i \in X$, such that $\exp_p^{-1}(x_i) = z_i \to z$. Let $\exp_p : C_R^\kappa(\Sigma_p) \to X; \exp_p(z) = \lim_{i \to \infty} x_i$. Such $\exp_p$ is well defined, since if there is another sequence $\exp_p^{-1}(x_i') = z_i' \to z$, then

$$d(\lim_{i \to \infty} x_i, \lim_{i \to \infty} x_i') = \lim_{i \to \infty} d(x_i, x_i') \leq \lim_{i \to \infty} d(z_i, z_i') = 0.$$

It’s clear that $\exp_p$, defined as an extension of $\exp_p^{-1}$, is distance non-increasing. Moreover, it preserves the distance along the radio direction.

We now show that any geodesic from $p = \exp_p(\tilde{o})$ to $q = \exp_p(\tilde{q}) \in B_R(p)$ can be extended. Therefore $\exp_p$ is a bijection since geodesic does not bifurcate. Let $[\tilde{o} \tilde{q}]$ be the geodesic in $C_R^\kappa(\Sigma_p)$ such that $\exp_p([\tilde{o} \tilde{q}]) = [pq]$ and $\tilde{q}' \in C_R^\kappa(\Sigma_p)$ be the extended point of $[\tilde{o} \tilde{q}]$. Then

$$|pq| + |q\tilde{q}'| \leq |\tilde{o} \tilde{q}| + |\tilde{q} \tilde{q}'| = |\tilde{o} \tilde{q}'| = |pq|,$$

which forces $[pq] \cup [q\tilde{q}']$ being a geodesic. □

Proof of Theorem 2.1. For $X \in A^R_\kappa(\Sigma)$ with $\vol(X) = v(\Sigma, \kappa, R)$, by Lemma 2.4 and Lemma 2.5 we see that $\exp_p : C_R^\kappa(\Sigma) \to B_R(p)$ is a distance non-increasing onto map which satisfies the assumptions in Theorem A (note that $\exp_p : C_R^\kappa(\Sigma_p) \to B_R(p) = X$ may not satisfy the assumption of Theorem A).

In the proof of Theorem 2.2, our main technique lemma is Lemma 2.12. Let $\phi : \Sigma \times \{R\} \to \Sigma \times \{R\}$ be defined as in Theorem 2.2. We first observe that $\phi$ is an involution. Let $L_p(X) = \exp_p(\Sigma \times \{R\}) = \{x \in X : |px| = R\}$.

Lemma 2.6. Let $X = \tilde{C}_R^\kappa(\Sigma)/x \sim x' \in Alex^\kappa(\Sigma)$ be as in Theorem 2.2. For any $q \in L_p(X)$, if $\tilde{q}_1 \neq \tilde{q}_2$ with $\exp_p(\tilde{q}_1) = \exp_p(\tilde{q}_2) = q$, then the loop $\exp_p([\tilde{o} \tilde{q}_1]) \cup \exp_p([\tilde{o} \tilde{q}_2])$ forms a local geodesic at $q$. Consequently, $\exp_p^{-1}(q)$ contains at most 2 points.

Proof. It’s clear that $\exp_p([\tilde{o} \tilde{q}_i])$ are minimal geodesics, $i = 1, 2$. Let $x_1, x_2 \in X$ be a point on $\exp_p((\tilde{o} \tilde{q}_i))$ and $\tilde{x}_i = \exp_p^{-1}(x_i)$, $i = 1, 2$. We claim that if $x_1, x_2$ are both close to $q$ enough, the geodesic $[x_1 x_2]$ intersects with $L_p(X)$. If not, then $[x_1 x_2] \subset B_R(p)$. By the assumption, $|x_1 x_2| = |\tilde{x}_1 \tilde{x}_2| c_\kappa(\Sigma)$. Let $x_1, x_2 \to q$. We get that $|x_1 x_2| \to 0$ and $|\tilde{x}_1 \tilde{x}_2| c_\kappa(\Sigma) = |\tilde{q}_1 \tilde{q}_2| c_\kappa(\Sigma) > 0$, a contradiction.

Let $a \in [x_1 x_2] \cap L_p(X)$, it remains to show that $a = q$. For $i = 1, 2$,

$$|x_i a| \geq |pa| - |px_i| = |pq| - |px_i| = |x_i q|.$$

Thus

$$|x_1 q| + |x_2 q| \leq |x_1 a| + |x_2 a| = |x_1 x_2|,$$

which forces both of the above inequalities to be equalities, and thus $a = q$. □

As a corollary of Lemma 2.6, we conclude that for $X \in A^R_\kappa(\Sigma)$, $\kappa > 0$ and $\frac{\pi}{2\sqrt{\kappa}} < R < \frac{\pi}{\sqrt{\kappa}}$, $\vol(C_R^\kappa(\Sigma))$ is not the optimal upper bound for $\vol(X)$ (c.f. [GP]). Equivalently, we have
Corollary 2.7. Assume $X \in \mathcal{A}_R^K(\Sigma)$ with $\text{vol}(X) = \text{vol}(\bar{C}_R^K(\Sigma))$ and $\kappa > 0$, then $R \leq \frac{\pi}{2\sqrt{\kappa}}$ or $R = \frac{\pi}{2\sqrt{\kappa}}$. In the second case, $X = C_\kappa(\Sigma)$ which is the k-suspension of $\Sigma$.

Proof. Assume $\frac{\pi}{2\sqrt{\kappa}} < R < \frac{\pi}{\sqrt{\kappa}}$. Let $p \in X$ such that $\Sigma_p = \Sigma$. It’s clear that $\text{rad}_p(X) = R$. We claim that $L_p(X) = \{q\}$ has only one point. Then by Lemma 2.6, $\Sigma_p \times \{R\} = \text{exp}_p^{-1}(\{q\})$ contains at most 2 points, a contradiction. Let $a \neq b \in L_p(X)$, consider the triangle $\Delta pab$ and the compared triangle $\bar{\Delta}pab \in S^2_\kappa$. Take $c \in [ab]$ and the corresponding $\bar{c} \in [\bar{a}\bar{b}]$ with $|ac| = |\bar{ac}|$. By the triangle comparison, $|pc| \geq |\bar{p}\bar{c}| > R$, a contradiction.

Note that the case $R = \frac{\pi}{\sqrt{\kappa}}$ follows from Theorem 2.4. \qed

It remains to show that $\phi$ is an isometry. The following lemma plays an important role in the study of the angles in the gluing space $X$.

Lemma 2.8. Let $a, b \in C_\kappa(\Sigma)$. Then $\angle apb = \bar{\angle}apb$ and $\angle pab = \bar{\angle}pab$.

Proof. The proofs are essentially same for different $\kappa$. For simplicity, we only give a proof for $\kappa = 0$. Note that $\angle apb = \bar{\angle}apb$ by the definition of $C_\kappa(\Sigma)$.

To see $\angle pab = \bar{\angle}pab$, shortly extend the geodesic $[pa]$ to $a'$ and apply the cosine law to the triangles $\triangle aa'b$, $\triangle pa'b$ and $\triangle pab$. We get

\begin{align}
(2.1) \quad |a'b|^2 &= |aa'|^2 + |ab|^2 - 2|aa'||ab| \cos \bar{\angle}a'ab, \\
(2.2) \quad |a'b|^2 &= |pa'|^2 + |pb|^2 - 2|pa'||pb| \cos \angle apb \\
&\quad = (|pa| + |aa'|)^2 + |pb|^2 - 2(|pa| + |ad'|)|pb| \cos \angle apb, \\
(2.3) \quad |ab|^2 &= |pa|^2 + |pb|^2 - 2|pa||pb| \cos \angle pab.
\end{align}

Calculating $\text{2.1} + \text{2.3} - \text{2.2}$, we get

\begin{align}
0 &= |ab| \cos \bar{\angle}a'ab + |pa| - |pb| \cos \angle apb \\
&\geq |ab| \cos \bar{\angle}a'ab + |pa| - |pb| \cos \angle apb \\
&= -|ab| \cos \angle pab + |pa| - |pb| \cos \angle apb.
\end{align}

Since $\angle pab \geq \bar{\angle}pab$ and $\angle apb = \bar{\angle}apb$, the above inequality implies

\begin{align}
|pa| &\leq |ab| \cos \angle pab + |pb| \cos \angle apb \\
&\leq |ab| \cos \bar{\angle}pab + |pb| \cos \angle apb = |pa|,
\end{align}

which forces $\angle pab = \bar{\angle}pab$. \qed

Corollary 2.9. Let $x, y \in X$ be two points. If $[xy] \cap L_p(X) \neq \emptyset$, then either $[xy] \subset L_p(X)$ or $[xy] \cap L_p(X)$ is finite.

Proof. Let $x \notin L_p(X)$, we shall show that $[xy] \cap L_p(X)$ is finite. Let $a \in [xy] \cap L_p(X)$ be the accumulation point which is closest to $x$. Clearly $a \neq x$ since $x \notin L_p(X)$. Thus there is a geodesic segment $[ba]$ of $[xy]$ with that $[ba] - \{a\} \subset B_R(p)$. Since $|pb| < |pa| = R$, by Lemma 2.8

\begin{align}
\angle pab = \bar{\angle}pab < \frac{\pi}{2}.
\end{align}
On the other hand, because there are \( a_i \in [xy] \cap L_p(X) \) with \( a_i \to a \) as \( i \to \infty \) and \( |pa| = |pa_i| = R \), by the first variation formula, we get

\[
\angle pax = \frac{\pi}{2}.
\]

Therefore \( \pi = \angle pab + \angle pay < \pi \), a contradiction. \( \square \)

As another corollary, we prove Theorem 2.2 for the special case \( \kappa > 0 \) and \( R = \frac{\pi}{2\sqrt{\kappa}} \).

**Corollary 2.10.** Theorem 2.2 holds for the case \( \kappa > 0 \) and \( R = \frac{\pi}{2\sqrt{\kappa}} \).

**Proof.** Let \( x, y \in L_p(X) \), \( \tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2 \in \Sigma \times \{R\} \) with \( \exp_p(\tilde{x}_1) = \exp_p(\tilde{x}_2) = x \), \( \exp_p(\tilde{y}_1) = \exp_p(\tilde{y}_2) = y \). We will show that \( |\tilde{x}_1\tilde{y}_1|_{\kappa R(\Sigma)} = |\tilde{x}_2\tilde{y}_2|_{\kappa R(\Sigma)} \). Assume \( |\tilde{x}_1\tilde{y}_1|_{\kappa R(\Sigma)} > |\tilde{x}_2\tilde{y}_2|_{\kappa R(\Sigma)} \). Then there is a point \( a \notin L_p(X) \) (take \( \exp_p^{-1}(a) \) be close to \( x_1 \)) such that \( [ay] \cap L_p(X) \) contains a point \( b \neq y \). Because \( \exp_p \) is distance non-increasing and \( \Sigma \times \{\frac{\pi}{\sqrt{\kappa}}\} \) is totally geodesic, \( [by] \subset L_p(X) \), which contradicts to Corollary 2.9. \( \square \)

Let \( \text{Fix}(\phi) = \{ \tilde{x} \in \Sigma \times \{R\} : \phi(\tilde{x}) = \tilde{x} \} \) be the fixed points set. Let \( L_p^1(X) = \exp_p(\text{Fix}(\phi)) \) denote the image. Due to Lemma 2.6 let \( L_p^2(X) = L_p(X) - L_p^1(X) \) denote the points that are identified from exactly two points, i.e. for any \( x \in L_p^2(X) \), \( \exp^{-1}(x) = \{ \tilde{x}^+, \tilde{x}^- \} \) contains exactly two points.

In the rest proof of Theorem 2.2 by Corollary 2.9 2.10 and their proofs, we can always assume \( R < \frac{\pi}{2\sqrt{\kappa}} \) for \( \kappa > 0 \) and that for any \( x, y \in X \), \( [xy] \cap L_p(X) \) is finite if it is not empty. More over, the following corollary shows that \( [xy] \cap L_p(X) \subset L_p^2(X) \), where \( [xy] \) denotes the geodesic connecting \( x, y \) without the end points.

**Corollary 2.11.** Let the assumption be as in Theorem 2.2. Assume \( R < \frac{\pi}{2\sqrt{\kappa}} \) when \( \kappa > 0 \). For any \( x, y \in X \), if \( q \in [xy] \cap L_p(X) \), then \( q \in L_p^2(X) \).

**Proof.** Not losing generality, assume \( x, y \notin L_p(X) \) and \( [xy] \cap L_p(X) = \{q\} \). If \( q \in L_p^1(X) \), by Lemma 2.8 \( \angle xqp = \angle xqp < \frac{\pi}{2} \) and \( \angle yqp = \angle yqp < \frac{\pi}{2} \). Thus \( \angle xqp + \angle yqp < \pi \), which contradicts to the fact that \( [xy] \) is a geodesic. \( \square \)

Now we are ready to prove our main technique lemma. Let \( x \in L_p^2(X) \) and \( \{\tilde{x}^+, \tilde{x}^-\} = \exp_p^{-1}(x) \) denote the pre-image. Then there are exactly two geodesics \( \exp_p([\tilde{o} \tilde{x}^+]), \exp_p([\tilde{o} \tilde{x}^-]) \) connecting \( x \) to \( p \). To distinguish geodesics and angles, we use the following notation.

- Let \( [px^+] \) and \( [px^-] \) denote \( \exp_p([\tilde{o} \tilde{x}^+]) \) and \( \exp_p([\tilde{o} \tilde{x}^-]) \) respectively.

In addition, for \( y \in L_p^2(X) \) and \( \exp_p^{-1}(y) = \{\tilde{y}^+, \tilde{y}^+\} \),

- Let \( [x^+y^+] \) denote \( \exp_p([\tilde{x}^+\tilde{y}^+]) \);
- Let \( |x^+y^| \) denote the length of the geodesics \( [x^+y^+] \);
- Let \( \angle x^+py^+ \) denote the angle between \( [px^+] \) and \( [py^+] \) at \( p \);
- Let \( \angle px^+y^+ \) denote the angle between \( [px^+] \) and \( [x^+y^+] \) at \( x \).

**Lemma 2.12.** Let the assumption be as in Theorem 2.2. Assume \( R < \frac{\pi}{2\sqrt{\kappa}} \) when \( \kappa > 0 \). Then for any \( \tilde{x} \neq \tilde{y} \in \Sigma \times \{R\} \) with \( |\tilde{x}\tilde{y}| \) sufficiently small,

\[
\left| \frac{\phi(\tilde{x})\phi(\tilde{y})}{|\tilde{x}\tilde{y}|} - 1 \right| \leq 20|\tilde{x}\tilde{y}|.
\]
Proof. For simplicity, we give a proof for the case \( \kappa = 0 \). The other cases can be carried out similarly. Throughout the proof, we will frequently use Lemma 2.6, 2.8 and Corollary 2.11 without mentioning. We will also assume that for any \( a, b \in X \), \([ab] \cap L_p(X)\) is finite if it is not empty.

Clearly, \( \phi \) preserves the distance when \( x \) and \( y \) are both in \( L^1_\text{p}(X) \). Let \( x \in L^2_\text{p}(X) \), \( y \in L_\text{p}(X) \) (if \( y \in L^1_\text{p}(X) \), \( \tilde{y}^+ = \tilde{y}^- \) will denote the same point and the argument will still go through). Because \([xy] \cap L_p(X)\) is finite, not losing generality, assume \([xy] = [x^-y^-]\). Thus \( \angle x^-py^- \leq \angle x^+py^+ \). Let \( \beta_0 = \angle x^-py^- \). Since \( |x^-y^-| = 2R \sin \frac{\beta_0}{2} \) and \( |x^+y^+| = 2R \sin \frac{\angle x^+py^+}{2} \), it’s sufficient to show that

\[
10\beta_0^2 + \beta_0 \geq \angle x^+py^+.
\]

Take \( u_0 \in [px^+] \) with \( |u_0x^+| = \epsilon \). Let \([u_0y]\) be a geodesic. If \([u_0y]\) \( \cap L_p(X) \neq \emptyset \), let \( a_1(\neq y) \) and \( b_1 \) (\( b_1 \) can be \( y \)) be the first and second intersection points in \([u_0y]\) \( \cap L_p(X) \) along the direction \( \uparrow u_0 \) (see Figure 3). Assign \( \pm \) to \( \exp^{-1}_p(a_1), \exp^{-1}_p(b_1) \) such that \( \angle pa_1^-u_0 < \frac{\pi}{2} \). Let \( \alpha_1 = \angle x^+pa_1^+ \) and \( \beta_1 = \angle a_1^-pb_1^- \). In the case of \([u_0y]\) \( \cap L_p(X) = \emptyset \), we take \( b_1 = a_1 = y \) and \( \beta_1 = 0 \).

Because \([u_0a_1^+] \ast [a_1^-b_1^-] \ast [b_1^+y]\) is a minimal geodesic, by triangle inequality,

\[
|u_0x| + |xy| \geq |u_0a_1^+| + |a_1^-b_1^-| + |b_1y|.
\]

This implies

\[
\epsilon + 2R \sin \frac{\beta_0}{2} \geq |u_0a_1^+| + 2R \sin \frac{\beta_1}{2}.
\]  

Applying the cosine law (the form in Lemma 4.7 (5)) in \( \triangle pu_0a_1 \) with the angle \( \angle u_0pa_1^+ = \alpha_1 \), we get that

\[
|u_0a_1^+| = \sqrt{\epsilon^2 + 4R(R - \epsilon) \sin^2 \frac{\alpha_1}{2}} \geq 2(R - \epsilon) \sin \frac{\alpha_1}{2}.
\]

Thus

\[
\epsilon + 2R \sin \frac{\beta_0}{2} \geq 2(R - \epsilon) \sin \frac{\alpha_1}{2} + 2R \sin \frac{\beta_1}{2}.
\]
If \([ (u_0y) ] \cap L_p(X) = \emptyset\), we stop here. If \([ (u_0y) ] \cap L_p(X) \neq \emptyset\), we proceed with \(u_1 \in [pa^+_1]\) and \(|u_1a_1| = \epsilon\). Let \([u_1b_1]\) be a geodesic. Again, if \([ (u_1b_1) ] \cap L_p(X) \neq \emptyset\), let \(a_2(\neq y)\) and \(b_2\) (can be \(b_1\)) be the first and second intersection points in \([u_1b_1] \cap L_p(X)\) along the direction \(\nu^{b_1}_{u_1}\). Assign \(\pm\) to \(\exp_p^{-1}(a_2), \exp_p^{-1}(b_2)\) such that \(\angle pa^+_2 u_1 < \frac{\pi}{2}\). Let \(\alpha_2 = \angle a_1^+ pa^+_2\) and \(\beta_2 = \angle a_2^- pb_2^-\). If \([ (u_1b_1) ] \cap L_p(X) = \emptyset\), then \(a_2 = b_2 = b_1\), \(\beta_2 = 0\) and we stop the process. Proceed inductively until \([ (u_Nb_N) ] \cap L_p(X) = \emptyset\), which yields that \(a_{N+1} = b_{N+1} = b_N\) and \(\beta_{N+1} = 0\). We claim that \(N\) is finite, and moreover,

\[(2.7) \quad (N + 1)\epsilon < 5R \cdot \beta_0^2.\]

For each \(0 \leq i \leq N\), we have

\[(2.8) \quad \epsilon + 2R \sin \frac{\beta_i}{2} \geq |u_i a^+_{i+1}| + 2R \sin \frac{\beta_{i+1}}{2},\]

and

\[(2.9) \quad \epsilon + 2R \sin \frac{\beta_i}{2} \geq 2(R - \epsilon) \sin \frac{\alpha_{i+1}}{2} + 2R \sin \frac{\beta_{i+1}}{2},\]

where \(\alpha_i = \angle a^+_i pa^+_{i+1}\), \(\beta_i = \angle a^-_i pb^-_i\). Summing up \((2.9)\) for \(i = 0, 1, \cdots, N\) and applying \((2.7)\), we get

\[
5R \cdot \beta_0^2 + 2R \sin \frac{\beta_0}{2} \geq (N + 1)\epsilon + 2R \sin \frac{\beta_0}{2} \\
\geq 2(R - \epsilon) \sum_{i=1}^{N} \sin \frac{\alpha_i}{2} + 2(R - \epsilon) \sin \frac{\sum_{i=1}^{N} \alpha_i}{2} \\
\geq 2(R - \epsilon) \sin \frac{\angle x^+ pb_N}{2}.
\]

Since \(b_N \to b_1 \to y^+\) when taking \(\epsilon \to 0\), \((2.4)\) follows.

It remains to show \((2.7)\). A sum of \((2.8)\) for \(i = 0, 1, \cdots, N\) indicates that the upper bound of \(N\) relies on an estimate of \(|u_i a^+_{i+1}|\) in terms of \(\epsilon\) and \(\beta_{i+1}\). Note that \(a_{i+1} = [u_{i+1} \cap (pa^+_{i+1} \cap pa^-_{i+1})\) and \([pa^+_{i+1} \cap pa^-_{i+1}]\) is a local geodesic at \(a_{i+1}\), we have \(\angle pa^+_{i+1} u_i = \angle pa^+_{i+1} b_{i+1} = \frac{\pi}{2} - \beta_{i+1}/2\). Applying the cosine law in triangle \(\triangle pu_i a^+_{i+1}\), we get

\[
(R - \epsilon)^2 = R^2 + |u_i a^+_{i+1}|^2 - 2R |u_i a^+_{i+1}| \sin \frac{\beta_{i+1}}{2},
\]

i.e.

\[
|u_i a^+_{i+1}|^2 - 2R \sin \frac{\beta_i}{2} \cdot |u_i a^+_{i+1}| + R \epsilon - \epsilon^2 = 0.
\]

Solving for \(|u_i a^+_{i+1}|\) and taking in account that \(\epsilon > 0\) is small, we have

\[
|u_i a^+_{i+1}| \geq R \sin \frac{\beta_{i+1}}{2} - \sqrt{(R \sin \frac{\beta_{i+1}}{2})^2 - (R \epsilon - \epsilon^2)} > \frac{\epsilon}{4 \sin \frac{\beta_{i+1}}{2}}.
\]

Note that \(\beta_i\) is decreasing, which is implied by \((2.8)\) and \(|u_i a^+_{i+1}| > |u_i a^+_1| = \epsilon\). We get

\[(2.10) \quad |u_i a^+_{i+1}| > \frac{\epsilon}{4 \sin \frac{\beta_0}{2}}.
\]
Plugging (2.10) into (2.8), we get

\[(2.11) \quad \epsilon + 2R \sin \frac{\beta_i}{2} > \frac{\epsilon}{4 \sin \frac{\beta_0}{2}} + 2R \sin \frac{\beta_{i+1}}{2}.\]

Summing up (2.11) for \(i = 0, 1, \cdots, N\), we get

\[(N + 1) \epsilon + 2R \sin \beta_0 > (N + 1) \frac{\epsilon}{4 \sin \frac{\beta_0}{2}}.\]

Therefore

\[(N + 1) \epsilon < \frac{8R \sin^2 \frac{\beta_0}{2}}{1 - 4 \sin^2 \frac{\beta_0}{2}} < 5R \cdot \beta_0^2.\]

\[\square\]

**Proof of Theorem 2.2** (Assuming \(R < \frac{\pi}{2\sqrt{\kappa}}\) when \(\kappa > 0\)). By Lemma 2.12, \(\phi\) is a continuous involution and thus a homeomorphism. It reduces to show that \(\phi : \Sigma \times \{R\} \to \Sigma \times \{R\}\) preserves length of any curve \(c : [0, 1] \to \Sigma \times \{R\}\). Given \(\delta, \epsilon > 0\), we may assume a partition \(P : 0 = t_0 < t_1 < \cdots < t_N = 1\) with \(|c(t_i)c(t_{i+1})| \leq \delta\) such that the length of the curves satisfy

\[L(c) < \sum_{i=0}^{N-1} |c(t_i)c(t_{i+1})| + \frac{\epsilon}{2}, \quad L(\phi(c)) < \sum_{i=0}^{N-1} |\phi(c(t_i))\phi(c(t_{i+1}))| + \frac{\epsilon}{2}.\]

Then

\[|L(c) - L(\phi(c))| \leq \sum_{i=0}^{N-1} |c(t_i)c(t_{i+1})| - |\phi(c(t_i))\phi(c(t_{i+1}))| + \epsilon\]

\[\leq \sum_{i=0}^{N-1} 20|c(t_i)c(t_{i+1})|^2 + \epsilon\]

\[\leq 20\delta \cdot \sum_{i=0}^{N-1} |c(t_i)c(t_{i+1})| + \epsilon\]

\[\leq 20\delta \cdot L(c) + \epsilon.\]

Since \(\epsilon > 0, \delta > 0\) can be chosen arbitrarily small, we conclude the desired result. \(\square\)

**Completion of Proof of Theorem B.** By Theorems 2.1 and 2.2, we identify \(X\) with \(\bar{C}^R_\kappa(\Sigma_p)/x \sim \phi(x)\). We shall show that the metric on \(X\) coincides with the metric induced from the identification \(x \sim \phi(x)\). It’s equivalent to show that \(\exp_p : \bar{C}^R_\kappa(\Sigma_p) \to X\) preserves lengths of geodesics. Let \(\gamma \subset \bar{C}^R_\kappa(\Sigma_p)\) be a geodesic and \(\sigma = f(\gamma)\). Since \(L(\gamma) \geq L(\sigma)\), it remains to show that \(L(\sigma) \geq L(\gamma)\). Because either \(\gamma \subset \Sigma \times \{R\}\) or \(\gamma \cap (\Sigma \times \{R\})\) has at most 2 points, we only need to check for the case \(\gamma \subset \Sigma \times \{R\}\) i.e., \(\sigma \subset L_p(\Sigma)\). For any \(\epsilon > 0\), let \(\{x_i\}_{i=0}^{2N+1} \subset \sigma\) be an \(\epsilon\)-partition and

\[L(\sigma) = \lim_{\epsilon \to 0} \sum_{i=0}^{2N} |x_i x_{i+1}|.\]
Let $a_i \in \gamma$ so that $\exp_p(a_i) = x_i$. Choose $b_{2k} \in C^R_k(\Sigma)$, $k = 0, 1, \cdots, N$, with $|a_{2k} - b_{2k}| < \epsilon^4$. Let $b_{2k+1} = a_{2k+1}$ for $k = 0, 1, \cdots, N$ and $y_i = \exp_p(b_i)$ for $i = 0, 1, \cdots, 2N + 1$. Then $|y_i - x_i| < \epsilon^4$ and thus

$$L(\sigma) = \lim_{\epsilon \to 0} \sum_{i=0}^{2N} |y_i y_{i+1}|.$$  

We claim that $[y_i y_{i+1}] \cap L_p(X)$ is either $y_i$ or $y_{i+1}$. By Corollary \[24\] let $u, v \in [y_i y_{i+1}] \cap L_p(X)$ and there is no crossing point in between. Not losing generality, assume $y_i \notin L_p(X)$ and $|y_i u| < |y_i v|$. Let $[u-v] \subset [y_i y_{i+1}]$. Because the involution $\phi$ is an isometry (Theorem \[2.2\]), $L([u+v]) = L([u-v])$. Thus $[y_i u] \cup [u+v] \neq [y_i u] \cup [u-v]$ is also a geodesic, which yields a bifurcation of geodesics.

By the claimed property, we have that $|y_i y_{i+1}| = |b_i b_{i+1}|$. Since $\sum_{i=0}^{2N} |b_i b_{i+1}| \geq L(\gamma)$, we have

$$L(\sigma) = \lim_{\epsilon \to 0} \sum_{i=0}^{2N} |y_i y_{i+1}| = \lim_{\epsilon \to 0} \sum_{i=0}^{2N} |b_i b_{i+1}| \geq L(\gamma).$$

It remains to show that for $\Sigma \in \text{Alex}^{n-1}(1)$, if $\phi : \Sigma \times \{R\} \to \Sigma \times \{R\}$ is a symmetric involution, then $X = C^R_\kappa(\Sigma)/(x \sim \phi(x)) \in \text{Alex}^n(\kappa)$.

Case 1. Assume $\partial \Sigma = \varnothing$. Take two copies of $C^R_\kappa(\Sigma)$, marked as $C^R_\kappa(\Sigma)_1$ and $C^R_\kappa(\Sigma)_2$, whose vertices are $p_1$ and $p_2$ respectively. Gluing along their boundaries by $\phi$, we obtain a double space $\tilde X = C^R_\kappa(\Sigma)_1 \cup_\phi C^R_\kappa(\Sigma)_2$. By the gluing theorem ([Petrunin 97]), $\tilde X \in \text{Alex}^n(\kappa)$.

Now we extend the isometric $\mathbb{Z}_2$-action on $\Sigma$ to an isometric $\mathbb{Z}_2$-action on $\tilde X$ such that $X = \tilde X/\mathbb{Z}_2$, and thus $X \in \text{Alex}^n(\kappa)$. For any $u \in C^R_\kappa(\Sigma)_1$, extend the geodesic $[p_1 u] C^R_\kappa(\Sigma)_1$ to $u_1 \in (\Sigma \times \{R\})_1$. Let $\hat{\phi}(u)$ be the point on the geodesic $[p_2 \phi(u_1)] C^R_\kappa(\Sigma)_2$ such that $|p_2 \hat{\phi}(u)| = |p_1 u|$ (so $\hat{\phi} : C^R_\kappa(\Sigma)_1 \to C^R_\kappa(\Sigma)_2$). Switching the role of $C^R_\kappa(\Sigma)_1$ and $C^R_\kappa(\Sigma)_2$, we extend $\phi$ to an isometric involution $\hat{\phi} : C^R_\kappa(\Sigma)_2 \to C^R_\kappa(\Sigma)_1$. Clearly, $\hat{\phi} : \tilde X \to \tilde X$ is an isometric involution such that $X = \tilde X/\hat{\phi}$.

Case 2. Assume $\partial \Sigma \neq \varnothing$. Let $\hat{\Sigma} = \Sigma^+ \cup \Sigma^-$ denote the double of $\Sigma$. We first extend the isometric involution $\phi$ on $\Sigma$ to $\hat{\phi} : \hat{\Sigma} \to \hat{\Sigma}$ by $\hat{\phi}(x_+) = \phi(x)_+$, where $x_+ = x_- \in \Sigma$. We then define another isometric involution $\psi : \hat{\Sigma} \to \hat{\Sigma}$ by the reflation on $\partial \Sigma$, $\psi(x_+) = x_-$. Then $\hat{\psi}(\phi(x_+)) = \hat{\psi}(\phi(x)_+) = \phi(x)_+ = \hat{\phi}(x_+)$ and $\hat{\psi}(\phi(x_-)) = \hat{\psi}(\phi(x)_-) = \hat{\phi}(\psi(x_+))$. This implies that $\hat{\Sigma}$ admits an $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-action. Clearly, the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-action extends uniquely to an isometric $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-action on $C^\tau_\kappa(\Sigma)$. By Case 1, we extend only the $\phi$-action to $\hat{\Sigma}$ such that $C^\tau_\kappa(\Sigma)/x \sim \phi(x) \in \text{Alex}^n(\kappa)$. Then $X = [C^\tau_\kappa(\Sigma)/x \sim \phi(x)]/\hat{\phi} \in \text{Alex}^n(\kappa)$.

By Theorem B, the isometric classification of $X \in \text{Alex}^n(\Sigma)$ with relatively maximum volume reduces to the isometric classification of all $(n-1)$-dimensional Alexandrov spaces $\Sigma$ with curv $\geq 1$ and the equivariant isometric $\mathbb{Z}_2$-actions on $\Sigma$. For $n = 2$, one easily gets a complete list:

**Corollary 2.13.** Any 2-dimensional compact Alexandrov space with curv $\geq \kappa$ and relatively maximum volume is isometric to one of the following:

$$C^\tau_\kappa(S^1_\theta)/\phi_i \quad (i = 1, 2, 3), \quad C^\tau_\kappa([0, \theta])/\psi_i \quad (i = 1, 2),$$

where $S^1_\theta$ denotes a circle of length $\theta$ with $0 < \theta \leq \pi$, $\phi_i : S^1_\theta \to S^1_\theta$ (resp. $\psi_i : [0, \theta] \to [0, \theta]$) is trivial, reflection or ancipital respectively for $i = 1, 2$ and 3 (resp. $i = 1$ and 2).
Example 2.14 (One-to-one Self-Gluing). This is an example for self-gluing (c.f. [GP]). Let $Z = \mathbb{D}^2$ be a 2-dimensional flat unit disk. Then $\partial Z = S^1(1)$ is a unit circle. Let $\phi : \partial Z \to \partial Z$ be a one-to-one map and $X = \mathbb{D}^2/x \sim \phi(x)$ be the glued space via identification $z \sim \phi(z)$. By Theorem B, $X$ is an Alexandrov space if and only if $\phi$ is a reflection, antipodal map or identity, where $X$ is homeomorphic to $S^2$, $\mathbb{RP}^2$ and $\mathbb{D}^2$ respectively.

Example 2.15 (Three Points Glued in a Self-Gluing). Let $Z$ be a triangle. We identify points on each side via a reflection about the mid point, i.e., $[Ab]$ glued with $[Cb]$, $[Ac]$ glued with $[Bc]$, $[Ba]$ glued with $[Ca]$ and $A$, $B$ and $C$ are glued to one point. The glued space $X$ is a tetrahedron, which belongs to $\text{Alex}^2(0)$.

3. RELATIVELY ALMOST MAXIMUM VOLUME

In the proof of Theorem C, we need the following result.

Theorem 3.1 (Theorem 5.5 in [Br]). Let $M$ be a $G$-manifold, $G$ is a finite group. Assume that for a given prime $p$ and all $p$-subgroups $P \subseteq G$ satisfies that

$$H_i(M^P; \mathbb{Z}_p) = 0, \quad i \leq q \quad \text{(including } P = \{e\}).$$

Then $H_i(M/G; \mathbb{Z}_p) = 0$ for all $i \leq q$. Moreover, if this holds for all prime $p$ and $H_i(M; \mathbb{Z}) = 0$ for $i \leq q$, then $H_i(M/G; \mathbb{Z}) = 0$ for $i \leq q$.

Proof of Theorem C. We first show that if $X \in \mathcal{A}_r^\kappa(\Sigma)$ with $\text{vol}(X) = v(\Sigma, \kappa, r)$, then $X$ is homeomorphic to $S^n$ or $\mathbb{CP}^n$.

By Theorem B, $X$ is isometric to $\mathcal{C}^R_\kappa(\Sigma)/x \sim \phi(x)$, $\phi : \Sigma \to \Sigma$ is an isometric involution. To determine the homeomorphism type of $X$, we consider the double space $\hat{X} = \mathcal{C}^R_\kappa(\Sigma)^+ \cup_{\phi} \mathcal{C}^R_\kappa(\Sigma)^-$. As seen in the proof of Theorem B, $\hat{X} \in \text{Alex}^n(\kappa)$ and $\phi$ extends an isometric $\mathbb{Z}_2$-action on $\hat{X}$ such that $\hat{X}/\mathbb{Z}_2$.

We claim that $\hat{X}$ is a homeomorphism sphere. First, $\hat{X}$ is a topological manifold if every point $\hat{q} \in \partial \mathcal{C}^R_\kappa(\Sigma) \subset \hat{X}$ is a manifold point. According to [Wu], a point $x$ in an Alexandrov space is a manifold point if and only if $\Sigma_x$ is simply connected. Because $\Sigma_{\hat{q}}$ is a suspension of $\Sigma(\Sigma)$, $\hat{q}$ is a manifold point. By the Poincaré conjecture (in all dimensions), our claim reduces to that $\hat{X}$ is an integral homotopy sphere. Because $\hat{X}$ is a suspension, $\hat{X}$ is simply connected, and thus it suffices to show that $\hat{X}$ is a homology sphere. Because $\mathcal{C}^R_\kappa(\Sigma)$ is contractible, from Mayer-Vietoris exact sequence of $(\mathcal{C}^R_\kappa(\Sigma))^+, \mathcal{C}^R_\kappa(\Sigma)^-)$ it is easy to see that $\hat{X}$ is an integral homology sphere.

If the $\mathbb{Z}_2$-action is free, then $X = \hat{X}/\mathbb{Z}_2$ is homeomorphic to $\mathbb{RP}^n$. Otherwise, $X$ is a simply connected topological manifold (the induced map, $\pi_1(\hat{X}) \to \pi_1(\hat{X})$ is an onto map). Again, it
suffices to show that $X$ is an integral homology sphere. By Smith theorem, the $\mathbb{Z}_2$-fixed point set $\hat{X}^{\mathbb{Z}_2}$ is an $\mathbb{Z}_2$-homology sphere. By now we can apply Theorem 4.1 to conclude the claim.

We then prove Theorem C by contradiction; assuming a sequence $X_i \in A'_R(\Sigma)$ such that $\text{vol}(X_i) > \text{vol}(C^R(\Sigma)) - \epsilon_i (\epsilon_i = i^{-1})$, and none of $X_i$ is homeomorphic to $S^n$ or $\mathbb{R}P^n$. Without loss of generality, we may assume that $(X_i, p_i) \overset{d_{GH}}{\to} (X, p) \in \text{Alex}^n(\kappa)$, where $X_i = \tilde{B}_r(p_i)$. By Perelman’s stability theorem ([Ka2], [Pe]), $X_i$ is homeomorphic to $X$ for $i$ large. In particular, $X$ is a topological manifold. We claim that $X \in A'_R(\Sigma_p)$ satisfies that $\text{vol}(X) = v(\Sigma_p, \kappa, r)$. By the above, we then conclude that $X$ is homeomorphic to $S^n$ or $\mathbb{R}P^n$, and thus $X_i$ is homeomorphic to $X$ for $i$ large, a contradiction.

To see the claim, $\text{vol}(X) = \lim_{i \to \infty} \text{vol}(X_i) = \lim_{i \to \infty} (\text{vol}(C^R(\Sigma)) - \epsilon_i) = \text{vol}(C^R(\Sigma)).$

On the other hand, we shall construct a distance non-increasing map, $\phi : \Sigma \to \Sigma_p$. Consequently, $\text{vol}(\Sigma_p) \leq \text{vol}(\Sigma)$ and thus $\text{vol}(X) \leq \text{vol}(C^R(\Sigma_p)) \leq \text{vol}(C^R(\Sigma)) \leq \text{vol}(X)$.

Let $A = \{v_1\} \subset \Sigma$ be a countable dense subset, and let $f_i : (X_i, p_i) \to (X, p)$ be a sequence of $\epsilon_i$-Gromov-Hausdorff approximation, $\epsilon_i \to 0$. For $v_1$, the sequence $\{f_i(\exp_{p_i} v)\} \subset X$ contains a converging subsequence $f_{i_1}(\exp_{p_{i_1}} q(v)) \to x_1 \in X$. Then $[px_1] = w_1 \in \Sigma_p$ (which may not be unique). We define $\phi(v_1) = w_1$. For $v_2$ and $\{f_{i_1}\}$, repeating the above we obtain $w_2 \in \Sigma_p$ and define $\phi(v_2) = w_2$. Iterating this process, we define a map $\phi : A \to \Sigma_p, \phi(v_1) = w_1$. It is easy to check that $\phi$ is distance non-increasing and thus $\phi$ extends uniquely to distance non-increasing map from $\Sigma$ to $\Sigma_p$.

4. Pointed Bishop-Gromov relative volume comparison

Assuming the monotonicity in Theorem D, the rigidity part follows by Lemma 4.3 and Theorem 2.1. For $p \in X \in \text{Alex}^n(\kappa)$, let $A'_R(p)$ (or briefly $A'_R$) denote the annulus $\{x \in X : r < |px| \leq R\}$, $0 \leq r < R$, and let $A'_R(\Sigma_p)$ (or briefly $A'_R(\Sigma_p)$) denote the corresponding annulus in $C^\kappa(\Sigma_p)$. Let $B_r$ denote $A^0_r, \tilde{B}_r$ denote $\tilde{A}^0_r$. Let’s recall the following two lemmas from [LR].

**Lemma 4.1** ([LR] Lemma 2.1). Let $\Sigma \in \text{Alex}^{p-1}(1)$ and $0 < r \leq \frac{r}{\sqrt{\kappa}}$. Then
$$\text{vol}(C^\kappa(\Sigma)) = \text{vol}(\Sigma) \cdot \int_0^R \frac{sn_{n-1}(t)}{t} dt.$$  

**Lemma 4.2** ([LR] Theorem B). Let $U$ be an open subset in $X \in \text{Alex}^n(\kappa)$. Then there is a constant $c(n)$ depending only on $n$ such that
$$V_{\kappa}(U) = V_{\kappa}(U) = c(n) \cdot \text{Haus}_n(U) = c(n) \cdot \text{Haus}_n(U),$$

where $V_{\kappa}$ and $\text{Haus}_n$ represent the $n$-dimensional rough volume and Hausdorff measure respectively.

**Lemma 4.3.** If the monotonicity in Theorem B holds. then
$$\frac{\text{vol}(B_r)}{\text{vol}(B_s)} = \frac{\text{vol}(B_R)}{\text{vol}(B_R)}$$
for some $0 < r < R$ ($R \leq \frac{\pi}{\sqrt{\kappa}}$ for $\kappa > 0$) if and only if $\text{vol}(B_R) = \text{vol}(\tilde{B}_R)$.

**Proof.** Assume $\text{vol}(B_R) = \text{vol}(\tilde{B}_R)$. The desired equation follows by the monotonicity:

$$1 = \frac{\text{vol}(B_R)}{\text{vol}(\tilde{B}_R)} \leq \frac{\text{vol}(B_r)}{\text{vol}(\tilde{B}_r)} \leq \lim_{r \geq t \to 0} \frac{\text{vol}(B_t)}{\text{vol}(\tilde{B}_t)} = 1.$$ 

Assume $\frac{\text{vol}(B_r)}{\text{vol}(\tilde{B}_r)} = \frac{\text{vol}(B_R)}{\text{vol}(\tilde{B}_R)}$, for some $0 < r < R$. Then for any $t < r$,

$$\frac{\text{vol}(B_t)}{\text{vol}(A_R^t)} + \frac{\text{vol}(A_R^t)}{\text{vol}(A_R^t)} = \frac{\text{vol}(B_R)}{\text{vol}(A_R^t)} = \frac{\text{vol}(\tilde{B}_R)}{\text{vol}(A_R^t)} = \frac{\text{vol}(\tilde{B}_t)}{\text{vol}(A_R^t)} + \frac{\text{vol}(\tilde{A}_R^t)}{\text{vol}(A_R^t)}.$$ 

By the monotonicity, we have $\frac{\text{vol}(A_R^t)}{\text{vol}(A_R^t)} \geq \frac{\text{vol}(\tilde{A}_R^t)}{\text{vol}(A_R^t)}$. Also,

$$\frac{\text{vol}(B_t)}{\text{vol}(A_R^t)} = \frac{\text{vol}(B_t)}{\text{vol}(A_R^t)} \frac{\text{vol}(A_R^t)}{\text{vol}(A_R^t)} \geq \frac{\text{vol}(\tilde{B}_t)}{\text{vol}(A_R^t)} \frac{\text{vol}(\tilde{A}_R^t)}{\text{vol}(A_R^t)} = \frac{\text{vol}(\tilde{B}_t)}{\text{vol}(A_R^t)}.$$ 

Consequently $\frac{\text{vol}(B_t)}{\text{vol}(A_R^t)} = \frac{\text{vol}(\tilde{B}_t)}{\text{vol}(A_R^t)}$, or equivalently, $\frac{\text{vol}(B_t)}{\text{vol}(A_R^t)} = \frac{\text{vol}(A_R^t)}{\text{vol}(A_R^t)}$. Let $t \to 0$, we get $\text{vol}(A_R^t) = \text{vol}(\tilde{A}_R^t)$. Thus

$$1 \geq \frac{\text{vol}(B_R)}{\text{vol}(A_R^t)} = \frac{\text{vol}(A_R^t)}{\text{vol}(A_R^t)} = 1.$$ 

□

By now, it remains to show the monotonicity in Theorem D. We take an elementary approach by expressing the monotonicity as a form of “Riemann sum” (see (4.5)) and using the Toponogov triangle comparison to bound each term in terms of the desired form (see Corollary 4.6). To achieve this goal, we choose a special infinite partition (see (1.5) and (4.0)).

We start the proof of Theorem D by deriving an equivalent form of the monotonicity. For $0 \leq R_1 < R_2 < R_3 (\leq \frac{\pi}{\sqrt{\kappa}})$ when when $\kappa > 0$, and $p \in X$, by Lemma 3.1, the monotonicity has the following integral form

$$\frac{\text{vol}(A_R^{R_1})}{\text{vol}(A_R^{R_2})} \leq \frac{\int_{R_1}^{R_3} \text{sn}_\kappa^{n-1}(t) \, dt}{\int_{R_2}^{R_3} \text{sn}_\kappa^{n-1}(t) \, dt},$$

which is equivalent to

$$I_1 = \log \left[ \frac{\text{vol}(A_R^{R_1})}{\text{vol}(A_R^{R_2})} \right] \leq \log \left[ \frac{\int_{R_1}^{R_3} \text{sn}_\kappa^{n-1}(t) \, dt}{\int_{R_2}^{R_3} \text{sn}_\kappa^{n-1}(t) \, dt} \right] = I_2.$$ 

Fixing a small $\delta > 0$, let $m = \frac{[R_3-R_2]}{\delta} + 1$, $\Delta = \frac{R_3-R_2}{m} \approx \delta$, and $r_j = R_2 + j \cdot \Delta$, $0 \leq j \leq m$. Then

$$A_{R_2}^{R_1} = A_{R_2}^{R_1} \subset A_{r_1}^{R_1} \subset \cdots \subset A_{r_m}^{R_1} = A_{R_1}^{R_1}.$$
Using the Taylor expansion \( \log \frac{1}{x} = 1 - x + O((1 - x)^2) \), we may rewrite the left hand side of \((4.1)\) as:

\[
I_1 = \sum_{j=1}^{m} \log \frac{\text{vol}(A_{R_1}^{r_j})}{\text{vol}(A_{R_1}^{r_{j-1}})} = \sum_{j=1}^{m} \left[ \left( 1 - \frac{\text{vol}(A_{R_1}^{r_{j-1}})}{\text{vol}(A_{R_1}^{r_j})} \right) + O(\delta^2) \right]
= \sum_{j=1}^{m} \frac{\text{vol}(A_{R_1}^{r_{j-1}})}{\text{vol}(A_{R_1}^{r_j})} + O(\delta).
\]  

Let \( \phi(r) = \int_{R_1}^{r} \text{sn}^{n-1}(t) \, dt \). Then the right hand side of \((4.1)\) can be written as:

\[
I_2 = \log \frac{\phi(R_3)}{\phi(R_2)} = \int_{R_2}^{R_3} \frac{\phi'(t)}{\phi(t)} \, dt
= \sum_{j=1}^{m} \frac{\phi'(r_j)}{\phi(r_j)} \delta + \tau(\delta)
= \sum_{j=1}^{m} \delta \cdot \text{sn}^{n-1}(r_j) + \tau(\delta).
\]

Comparing \((4.1)\) to \((4.2)\) and \((4.3)\), it’s sufficient to show

\[
\frac{\text{vol}(A_{R_1}^{r_{j-1}})}{\text{vol}(A_{R_1}^{r_j})} \leq \frac{\delta \cdot \text{sn}^{n-1}(r_j)}{\int_{R_1}^{r_j} \text{sn}^{n-1}(t) \, dt}.
\]  

We further divide \( A_{R_1}^{r_j} \) into thinner annulus: given a monotonic sequence \( \{a_i\}_{i=1}^{\infty} \subset [0, 1] \) such that \( a_j \to 0 \). Then \( \{a_{i}r_{j}\}_{i=1}^{\infty} \) is an infinite partition for \([0, r_{j}]\), and \((4.4)\) is equivalent to

\[
\frac{\text{vol}(A_{R_1}^{a_i r_{j}})}{\text{vol}(A_{R_1}^{r_{j-1}})} = \sum_{i=1}^{\infty} \frac{\text{vol}(A_{R_1}^{a_{i+1} r_{j}})}{\text{vol}(A_{R_1}^{r_{j-1}})} \geq \frac{\int_{R_1}^{r_j} \text{sn}^{n-1}(t) \, dt}{\delta \cdot \text{sn}^{n-1}(r_j)}.
\]  

To show \((4.5)\), we need to estimate \( \frac{\text{vol}(A_{R_1}^{a_{i+1} r_{j}})}{\text{vol}(A_{R_1}^{r_{j-1}})} \) from below (see Corollary \(4.6\)). Assume \( \delta \) is so small that \( R - \delta > 0 \) and \( r - \lambda \delta > 0 \). Let \( x \in A_{R_1}^{r_{j-1}} \). We define a map, \( \phi : A_{R_1}^{r_{j-1}} \to \tilde{A}_{r - \lambda \delta}^{r_{j}}, \)

where \( f(x) \) is the point on a minimal geodesic \([px]\) (if not unique, we pick one of them) such that

\[
|pf(x)| = r - \lambda(R - |px|).
\]

Because a geodesic in \( X \) does not branch, \( \phi \) is well-defined and is injective.

In the proof of Theorem D, the following is a main technical lemma, which asserts that \( \phi \) behaves like a bi-Lipschitz function.

**Lemma 4.4.** Let \( \delta > 0 \) sufficiently small, \( \lambda = \frac{\text{sn} \, r}{\text{sn} \, R} \) and \( \phi : \tilde{A}_{R_1}^{r_{j-1}} \to \tilde{A}_{r - \lambda \delta}^{r_{j}} \) be defined as the above. Then

\[
\epsilon(\kappa, \delta) \cdot \lambda \leq \frac{\text{sn} \, \epsilon(\phi(x), \phi(y))}{\text{sn} \, \epsilon(\phi(x), \phi(y))} \leq \epsilon(\kappa, \delta)^{-1} \lambda,
\]
Let \( \epsilon \)

Corollary 4.6. Let \( U \) and \( V \) be two open subsets of \( X \in \text{Alex}^n(\kappa) \), and let \( \phi : V \to U \) be an injection. If \( \phi \) satisfies that \( \text{sn}_\kappa \frac{\|\phi(x)\phi(y)\|}{2} \geq c \cdot \text{sn}_\kappa \frac{|xy|}{2} \) for any \( x, y \in V \), then \( \text{vol}(U) \geq \epsilon^n \cdot \text{vol}(V) \), where \( c \) is a constant.

Proof. By Lemma 4.2, it suffices to prove for rough volume. Recall that the \( n \)-dimensional rough volume of a subset \( V \) is

\[
V_{\text{rn}}(V) = \lim_{\epsilon \to 0} \epsilon^n \cdot \beta_V(\epsilon),
\]

where \( \beta_V(\epsilon) \) denotes the number of points in an \( \epsilon \)-net \( \{x_i\} \) on \( V \).

By the assumption, \( \{\phi(x_i)\} \) is a \( 2\text{sn}_\kappa \left( c \cdot \text{sn}_\kappa \frac{\epsilon}{2} \right) \)-net in \( U \). We get

\[
\beta_U \left( 2\text{sn}_\kappa^{-1} \left( c \cdot \text{sn}_\kappa \frac{\epsilon}{2} \right) \right) \geq \beta_V(\epsilon),
\]

or as the following form:

\[
\frac{\epsilon^n}{\left(2\text{sn}_\kappa^{-1} \left( c \cdot \text{sn}_\kappa \frac{\epsilon}{2} \right) \right)^n} \cdot \left(2\text{sn}_\kappa^{-1} \left( c \cdot \text{sn}_\kappa \frac{\epsilon}{2} \right) \right)^n \cdot \beta_U \left( 2\text{sn}_\kappa^{-1} \left( \text{sn}_\kappa \frac{\epsilon}{2} \right) \right) \geq \epsilon^n \beta_V(\epsilon).
\]

Let \( \epsilon \to 0 \), we get \( \frac{1}{c^n} V_{\text{rn}}(U) \geq V_{\text{rn}}(V) \). \( \Box \)

Corollary 4.6. Let \( p \in X \in \text{Alex}^n(\kappa) \), \( \delta > 0 \) small. Then

\[
\frac{\text{vol}(A^{r-\delta}_R)}{\text{vol}(A^{R-\delta}_R)} \geq (1 - \tau(\delta)) \cdot \left( \frac{\text{sn}_\kappa r}{\text{sn}_\kappa R} \right)^n.
\]

Proof. Consider the map \( \phi : A^{R-\delta}_R \to A^{r-\delta}_R \) and \( \tilde{\phi} : A^{R-\delta}_R \to A^{r-\delta}_R \) defined as the above. For any \( x, y \in A^{R-\delta}_R \), take two points \( \tilde{x}, \tilde{y} \in C_\kappa(2p) \) such that \( \|\tilde{x}\tilde{y}\| = |px|, \|\tilde{y}\tilde{y}\| = |py| \) and \( |\tilde{x}\tilde{y}| = |xy| \).

By condition B (see [BGP]), it’s easy to see that \( |f(x)f(y)| \geq |\tilde{\phi}(\tilde{x})\tilde{\phi}(\tilde{y})| \). Thus by Lemma 4.4 we have

\[
\text{sn}_\kappa \frac{|f(x)f(y)|}{2} \geq \text{sn}_\kappa \frac{|\tilde{\phi}(\tilde{x})\tilde{\phi}(\tilde{y})|}{2} \geq (1 - \tau(\delta)) \cdot \text{sn}_\kappa \frac{|\tilde{x}\tilde{y}|}{2} = (1 - \tau(\delta)) \cdot \text{sn}_\kappa \frac{|xy|}{2}.
\]

Then we get the desired estimate by Lemma 4.5. \( \Box \)

Proof of the monotonicity in Theorem D.

Continuing from the early discussion, the proof reduces to verify (4.5). We now take \( \delta > 0 \) sufficiently small, and choose the sequence \( \{a_i\}_{i=0}^{\infty} \) as:

\[
a_0 = 1, a_{i+1} = a_i - \frac{\text{sn}_\kappa (a_i r_j)}{r_j \cdot \text{sn}_\kappa r_j} \cdot \delta, \quad i = 0, 1, \ldots
\]

Then

\[
0 < a_{i+1} \leq \begin{cases} (1 - \frac{\delta}{r_j}) a_i, & \text{if } \kappa \geq 0, \\ (1 - \frac{\delta}{\text{sn}_\kappa r_j}) a_i, & \text{if } \kappa < 0, \end{cases}
\]
and thus \( a_i \to 0 \) and is monotonically decreasing. For each \( 0 \leq i < \infty \) and \( 0 \leq j \leq m \), consider the map, \( \phi : A_{r_j}^{(j)} - \delta \to A_{r_j}^{(j)} - \lambda_i \delta = A_{r_j}^{(j) + i} \), with \( \lambda_i = \frac{\sin(\delta r_j)}{\sin(\delta r_j)} \). By Corollary 4.6, we obtain that

\[
\frac{\text{vol}(A_{r_j}^{(j) + i})}{\text{vol}(A_{r_j}^{(j) - \delta})} \geq (1 - \tau(\delta)) \left( \frac{\sin(\delta r_j)}{\sin(\delta r_j)} \right)^{\nu}.
\]

Observe that for \( \delta \to 0 \), \( \{a_i\} \) will become more dense, and thus we can take \( N_\delta > 0 \) such that \( a_{N_\delta} r_j \geq R_1 \) and \( a_{N_\delta} r_j \to R_1 \) as \( \delta \to 0 \). Summing up for \( i = 0, 1, \cdots , N_\delta \), we get

\[
\frac{\text{vol}(A_{r_j}^{(j)})}{\text{vol}(A_{r_j}^{(j) - \delta})} \geq \sum_{i=0}^{N_\delta} (1 - \tau(\delta)) \left( \frac{\sin(\delta r_j)}{\sin(\delta r_j)} \right)^{\nu}
\]

\[
\geq (1 - \tau(\delta)) \cdot \frac{1}{\delta \cdot \sin(\delta r_j)} \sum_{i=0}^{N_\delta} \sin^{n-1}(\delta r_j) \cdot \frac{\delta \cdot \sin(\delta r_j)}{\sin(\delta r_j)}
\]

\[
= (1 - \tau(\delta)) \cdot \frac{1}{\delta \cdot \sin(\delta r_j)} \left( \int_{R_1}^{r_j} \sin^{n-1}(t) \, dt + \tau(\delta) \right)
\]

\[
= (1 - \tau(\delta)) \cdot \frac{\int_{R_1}^{r_j} \sin^{n-1}(t) \, dt}{\delta \cdot \sin(\delta r_j)}
\]

or the following equivalent form:

\[
\frac{\text{vol}(A_{r_j}^{(j)})}{\text{vol}(A_{r_j}^{(j) - \delta})} \leq (1 + \tau(\delta)) \cdot \frac{\delta \cdot \sin^{n-1}(r_j)}{\int_{R_1}^{r_j} \sin^{n-1}(t) \, dt}.
\]

Summing up for all \( j \) and together with [4.2] and [4.3], we get

\[
I_1 + O(\delta) \leq (1 + \tau(\delta))I_2 + \tau(\delta).
\]

Let \( \delta \to 0 \), we get the desired inequality. \( \square \)

The rest of this section is devoted to a proof of Lemma 4.4. The following are some properties used in the proof.

Lemma 4.7.

1. For \( \lambda \in [0, 1] \) and \( x \in [0, \pi] \), \( \sin \lambda x \geq \lambda \sin x \).
2. For \( \lambda \in [0, 1] \) and \( x \geq 0 \), \( \sinh \lambda x \leq \lambda \sinh x \).
3. For \( \lambda \geq 0 \) and \( x \geq 0 \), \( \frac{\sinh \lambda x}{\lambda \sinh x} \geq 1 - (\lambda x)^2/6 \).
4. For \( \lambda \geq 0 \) and \( x \geq 0 \), \( \frac{\sinh \lambda x}{\lambda \sinh x} \geq \frac{x}{\sinh x} \geq 1 - x \).
5. Let \( \triangle abp \) be a triangle in \( S_\kappa^2 \). The cosine law can be written as

\[
\sin^2 \frac{|ab|}{2} = \sin^2 \frac{|pa| - |pb|}{2} + \sin^2 \frac{\angle apb}{2} \sin |pa| \sin |pb|.
\]

Proof. (1) Let \( h(x) = \sin \lambda x - \lambda \sin x \), then

\[
h'(x) = \lambda \cos \lambda x - \lambda \cos x = \lambda (\cos \lambda x - \cos x) \geq 0
\]

since \( 0 \leq \lambda x \leq x \leq \pi \).
(2) Let \( h(x) = \sinh \lambda x - \lambda \sinh x \), then
\[
h'(x) = \lambda \cosh \lambda x - \lambda \cosh x = \lambda (\cosh \lambda x - \cosh x) \leq 0
\]
since \( 0 \leq \lambda x \leq x \).
(3) For \( x > 0 \), one can show that \( x \geq \sin x \geq x - x^3/6 \). Then
\[
\frac{\sin \lambda x}{\sin x} \geq \frac{\lambda x - (\lambda x)^3/6}{\lambda x} = 1 - (\lambda x)^2/6.
\]
(4) The first equality is easy to see through \( \sinh \lambda x \geq \lambda x \). Obviously, the second equality is true for \( x \geq 1 \). For \( 0 < x < 1 \),
\[
\sinh x = x + \frac{x^3}{6} + \cdots \leq x(1 + x + x^2 + \cdots) = \frac{x}{1 - x}.
\]
(5) Follows by trigonometric metric identities. □

**Proof of Lemma 4.4.** By scaling, we only need to check for \( \kappa = 1, -1 \) and \( \kappa = 0 \).

**Case 1.** \( \kappa = 1 \). Noting that
\[
\frac{|px'| - |py'|}{|px| - |py|} = \frac{\lambda (|px| - |py|)}{|px| - |py|} = \lambda,
\]
by Lemma 4.7(3) and \( 0 \leq ||px| - |py||| \leq \delta < \frac{1}{2} \sin R \), we have
\[
\sin \left( \frac{||px| - |py||}{2} \right) = \sin \left( \lambda \cdot \frac{||px| - |py||}{2} \right)
\geq \left( 1 - \frac{(\lambda \delta)^2}{6} \right) \lambda \cdot \sin \left( \frac{||px| - |py||}{2} \right)
\geq \left( 1 - \frac{\delta^2}{6 \sin^2 R} \right) \lambda \cdot \sin \left( \frac{||px| - |py||}{2} \right)
\geq \left( 1 - \frac{2\delta}{\sin R + \delta} \right) \lambda \cdot \sin \left( \frac{||px| - |py||}{2} \right)
= \tau_1 \lambda \cdot \sin \left( \frac{||px| - |py||}{2} \right).
\]
Thus
\[
(4.7) \quad \tau_1 \lambda \leq \frac{\sin \left( \frac{||px| - |py||}{2} \right)}{\sin \left( \frac{||px| - |py||}{2} \right)} \leq \frac{\lambda \cdot \frac{||px| - |py||}{2}}{\sin \left( \frac{||px| - |py||}{2} \right)} \leq \lambda \cdot \frac{\delta}{\sin \delta} \leq \tau_1^{-1} \lambda.
\]
For any \( x \in A^R_{R-\delta} \), by Lemma 4.7(1), we have
\[
\sin |px'| \geq \frac{|px'|}{r} \sin r \geq \frac{r - \lambda \delta}{r} \sin r = \frac{r - \sin \frac{r}{\sin R} \delta}{r} \sin r \geq \left( 1 - \frac{\delta}{\sin R} \right) \sin r,
\]
Together with \( \sin |px'| - \sin r = 2 \sin \frac{|px'| - r}{2} \cos \frac{|px'| + r}{2} \leq r - |px'| \leq \lambda \delta \), we get
\[
\left( 1 - \frac{\delta}{\sin R} \right) \sin r \leq \sin |px'| \leq \sin r + \lambda \delta = \left( 1 + \frac{\delta}{\sin R} \right) \sin r.
\]
Similarly,
\[
\sin |px| \geq \frac{|px|}{R} \sin R \geq \frac{R - \delta}{R} \sin R \geq \left( 1 - \frac{\delta}{\sin R} \right) \sin R
\]
and
\[
\sin |px| - \sin R = 2 \sin \left( \frac{|px|}{2} - \frac{R}{2} \right) \cos \left( \frac{|px|}{2} + \frac{R}{2} \right) \leq R - |px| \leq \delta,
\]
hence
\[
\left( 1 - \frac{\delta}{\sin R} \right) \sin R \leq \sin |px| \leq \sin R + \delta = \left( 1 + \frac{\delta}{\sin R} \right) \sin R.
\]
So
\[
\frac{c_1 \sin r}{\sin R} \leq \frac{\sin |px'|}{\sin |px|} \leq c_1^{-1} \frac{\sin r}{\sin R}.
\]

Let \( \theta = \angle xpy \). Since \( \frac{|xy|}{2} \leq \frac{\pi}{4} \), by the cosine law and inequalities \((4.7),(4.8)\),
\[
c_2 \lambda^2 \leq \frac{\sin^2 \frac{|x'y'|}{2}}{\sin^2 \frac{|xy|}{2}} \leq \frac{\sin^2 \frac{|px'| - |py'|}{2}}{\sin^2 \frac{|px| - |py|}{2}} + \sin^2 \frac{\theta}{2} \sin |px'| \sin |py'| \leq c_1^2 \lambda^2.
\]

Case 2, \( \kappa = -1 \). By Lemma \(4.7(2)\), \( \lambda \delta = \frac{\sinh r}{\sinh R} \cdot \frac{R}{\cosh R} < \frac{r}{\pi} \cdot R = r \). Together with Lemma \(4.7(4)\), we get
\[
\lambda \geq \frac{\sinh \left( \frac{|px'| - |py'|}{2} \right)}{\sinh \left( \frac{|px| - |py|}{2} \right)} \geq \frac{\sinh \left( \frac{\lambda \delta}{2} \right)}{\sinh \left( \frac{\lambda \delta}{2} \right)} \geq (1 - \delta) \lambda \geq c_{-1} \lambda,
\]
since \( \frac{\cosh R}{R} \geq \frac{1 + R^2/2}{2} > 1 \). If \( \delta < \frac{R}{\cosh R} < R \), then \( \frac{\lambda \delta}{2} < \frac{r}{\pi} \cdot \frac{\delta}{2} = \frac{\delta}{2} < 1 \). Hence we can apply Lemma \(4.7(2)\) with \( \lambda = \frac{\sinh r}{\sinh R} \leq \frac{r}{\pi} \), to get
\[
\frac{\sinh r - \sinh (r - \lambda \delta)}{\sinh r} \leq \frac{2 \sinh (\lambda \delta/2) \cdot \cosh R}{\sinh r} \leq \frac{\lambda \delta}{r} \cdot \cosh r \leq \delta \cdot \cosh R.
\]
thus
\[
\sinh (r - \lambda \delta) \geq \left( 1 - \frac{\cosh R}{R} \right) \sinh r.
\]
For \( x' \in \tilde{A}_R - \lambda \delta \), \( (1 - \frac{\cosh R}{R}) \sinh r \leq \sinh (r - \lambda \delta) \leq \sinh |px'| \leq \sinh r \). For \( x \in \tilde{A}_R - \lambda \delta \),
\[
\sinh r - \sinh(R - \delta) \leq \frac{2 \sinh(\delta/2) \cosh R}{\sinh R} \leq \frac{\delta \cdot \cosh R}{R},
\]
and \( (1 - \frac{\cosh R}{R}) \sinh R \leq \sinh (R - \lambda \delta) \leq \sinh |px| \leq \sinh R \). Then
\[
\frac{c_{-1} \sinh r}{\sinh R} \leq \frac{\sinh |px'|}{\sinh |px|} \leq c_{-1}^{-1} \frac{\sinh r}{\sinh R}.
\]
By inequalities \((4.9),(4.10)\) and the cosine law, we get
\[
c_{-1}^2 \lambda^2 \leq \frac{\sin^2 \frac{|x'y'|}{2}}{\sin^2 \frac{|xy|}{2}} = \frac{\sin^2 \frac{|px'| - |py'|}{2}}{\sin^2 \frac{|px| - |py|}{2}} + \sin^2 \frac{\theta}{2} \sinh |px'| \sinh |py'| \leq c_{-1}^2 \lambda^2.
\]

Case 3. \( \kappa = 0 \). This is straight forward. \( \Box \)
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E-mail address: nl2@nd.edu

E-mail address: rong@math.rutgers.edu