Fast Factoring of Integers

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Abstract

An algorithm is given to factor an integer with \( N \) digits in \( \ln^m N \) steps, with \( m \) approximately 4 or 5. Textbook quadratic sieve methods are exponentially slower. An improvement with the aid of an a particular function would provide a further exponential speedup.
Factorization of large integers is important to many areas of pure mathematics and has practical applications in applied math including cryptography. This subject has been under intense study for many years [1]; improvements in the methodology are especially desired for computational reasons.

Given an integer \( N \) composed of approximately \( \ln_{10} N \) digits, standard textbook quadratic sieve methods generate the factorization of the number into primes in roughly

\[
e^{a\sqrt{\ln N \ln \ln N}}\]

(1)

moves. The steps require manipulations of large integers, of the size \( N \), with bit complexity of approximate \( \ln N \). The number \( a \) is approximately 2, depending on the variant used [1].

The presentation in this work generates a computational method to obtain the prime factorization in

\[
\ln^m N
\]

(2)

moves with integers of the same size. The factor \( m \) is specified by the convergence of the solution to a set of polynomial equations in \( \ln N \) variables, which numerically is approximately \( m = 3 \), after the root selection is chosen from small numbers to large (see, e.g. [2]).

Given a function \( C_N \) that counts the number of prime factors of a number, i.e.

\[
N = \prod_{j=1}^{r} p_{\sigma(j)}^{k_j} \quad \quad C_N = \sum_{i=1}^{r} k_j ,
\]

(3)

the factorization of the number \( N \) could be performed in approximately \( C_N^m \) steps. The bound on the number of prime factors of an integer \( N \) is set by \( \ln_2 N \), the product of the smallest prime number 2. The number of primes smaller than a number \( N \) is approximately \( N / \ln N \), and the \( C_N \) is roughly \( \ln \ln N \). Hence, given the function \( C_N \) a further exponential improvement is generically given. However, this function would drastically simplify the factorization of large numbers possessing only a few prime factors. The upper bound of \( C_N \sim \ln N \) describes the case discussed in the previous paragraph.
Consider a number with exactly $C_N$ factors. This number projects in base $x$ onto the form,

$$N = \sum_{i=0}^{C_N} a_i x^i .$$

(4)

The polynomial form in (4) admits a product form,

$$N = \prod_{i=1}^{C_N} (c_j x - b_j) ,$$

(5)

with $c_j x - b_j$ integral. The number scales into the form,

$$N = \gamma \prod_{i=1}^{C_N} (\alpha_j x - 1) ,$$

(6)

in which there are $C_N$ numbers $\alpha_j$ and a number $\gamma$. The same integer has the prime factorization

$$N = \prod_{i=1}^{C_N} p_{\sigma(i)} ,$$

(7)

with the set $\sigma(j)$ containing possible redundancy, for example, $p_{\sigma(1)} = p_{\sigma(2)} = 2$. Given an integer base $x$ the solution to the numbers $b_j$ generate the prime factors, as long as the value $C_N$ is correct.

Two examples are given. First, $15 = 3^2 + 2(3) = x(x + 2)$, which solves for the prime factors 3 and 5. Second, $10 = 2^3 + 2 = x(x^2 + 1)$, which solves for the prime factors 2 and 5. In the second example, even though there are two factors, the polynomial is a cubic with a vanishing zeroth order term; the origin of the cubic is that there is a complex root.

The polynomial base form of the number, i.e. $N = \sum a_i x^i$, will not in all cases factor into the form (5) with real coefficients $c_j$ and $b_j$. However, because the coefficients are real, the roots will enter in complex conjugate pairs. The product of these complex conjugates form a positive number. In order to test all possible cases, including the presence of a factor being represented as a product of two complex roots, all numbers from $C_N$ to $2C_N$ should be examined. Potential complex roots come in
pairs, and the maximum number of factors could take on the form of \( C_N \) products of two complex numbers. The cost of the additional complexity is of order unity.

The expansion of (5) generates a set of algebraic equations relating the integer coefficients \( b_j \) to those in \( a_j \). The form is,

\[
\gamma \alpha_1 \alpha_2 \ldots \alpha_{C_N} = a_{C_N} \\
\ldots \\
\gamma = a_0 ,
\]

in which the combinations

\[
c_jx - b_j
\]

must converge to integers or into pairs with the product being an integer, and for the maximal factorization to prime numbers \( p \) (of which there are an approximate \( N/\ln N \) of them for the number \( N \)). The determination of the numbers \( \alpha_j \) must be rational as \( (c_j/b_jx - 1) = n/b_j \), with \( n \) integral. The other case of interest is when \( c_j/b_j \) is complex and the relevant condition is \( |(c_j/b_jx - 1)|^2 = n^2/b_j^2 \); this is not satisfied in general by complex rational numbers. However, one may square the number \( N \) and then all terms in the product must be rational.

The number \( \gamma \) must be an integer or the square of \( \gamma \) must be an integer, according to the presence of complex terms (roots) which square to an integer. If the solution does not satisfy these criteria, then there is not a valid factorization \( N \) into integers. Given rational solutions \( \alpha_j = c_j/b_j \) and the \( \gamma = \prod b_j \), the straightforward multiplication of \( \gamma \) into the \( C_N \) factors generates the factorization into \( N_1N_2\ldots N_{C_N} \), via eliminating the denominators in the individual terms of the rational numbers. The complex root case allows the numbers to be determined as \( N_j = N_j,+N_j,- \).

Solving these equations generates the prime factorization of the integer \( N \) into the set of primes \( p_{\sigma(j)} \). Numerically, solving a set of equations in \( n \) variables typically has convergence of \( n^3 \) if the initial starting values are chosen correctly.

In the case of \( C_N \) not known, but bounded by \( \ln N \), all cases of interest from the test cases of \( \tilde{C}_N = 1 \) to \( \tilde{C}_N = \ln N \) may be examined, at the cost of duplicating the process by the bound \( \ln N \). Typical true values of \( C_N \) are expected for generic numbers to be smaller than the bound, e.g. \( \ln \ln N \). The cases from \( C_N \) to \( 2C_N \) must also be examined in order to take into account the pairs of complex conjugates.

Computationally exploring all of the cases from \( \tilde{C}_N = 1 \) to \( \tilde{C}_N = 2\ln N \) (e.g. \( \sim C_N \)) for integer bases \( x \) finds all product forms of the integer \( N \) into products
\[ N = N_1 N_2 \quad N = N_1 N_2 N_3 \]  
\[ N = N_1 N_2 \ldots N_{CN} = \prod_{j=1}^{CN} p_{\sigma(j)} . \]

Solving for \( b_j \) and \( c_j \) (e.g. \( \alpha_j = b_j/c_j \)) in terms of \( a_j \) generates either integral values or non-integral values, or pairs or complex conjugates. In the case of integral values for all the \( b_j \) parameters, the base \( x \) is resubstituted into the factors \( c_j x - b_j \) of the total product,

\[ N = \tilde{C}_N \prod (c_j x - b_j) = \gamma \prod (\alpha_j x - 1) , \]

(12)

to find the values of the individual \( N_n \). The integral solutions generate the various factorizations. In the case of complex conjugate pairs, the integrality is tested by multiplying the individual terms,

\[ (c_j x - b_j)(c_j^* x - b_j^*) \sim (\alpha_j x - 1)(\alpha_j^* x - 1) , \]

(13)

and determining if it is an integral (the latter has to be rational). The maximum \( \tilde{C}_N \) that results in integral values of \( c_j x - b_j \) gives the prime factorization. Non-integral solutions to \( c_j x - b_j \) do not generate integer factorization of the number \( N \) into \( \tilde{C}_N \) numbers.

In the computation for the roots, the parameters \( \alpha_j \) and \( \gamma \) are determined. The integrality of the \( c_j x - b_j \) translates into \( \alpha_j \) being a rational number (when not a complex root allowing the complex \( c_j x - b_j \) square to be an integer). The rationality allows the denominators of all the \( \alpha_j \) parameters to be extracted and used to multiply the prefactor \( \gamma \). The case of the complex roots may be examined also by first taking the complex square and examining for integrality; the denominators must also be taken out of the products.

Computationally testing all cases from \( \tilde{C}_N = 1 \) to \( \ln N \) finds all product forms of the potentially large integer \( N \). The factorization process entails three steps: 1) projecting the number \( N \) into base \( x \), 2) solving the system of algebraic equations for integral \( c_j x - b_j \), 3) substituting the base \( x \) back into the \( c_j x - b_j \) for factor determination.
Computationally the first step requires specifying the base $x$ and projecting the number onto it. The base $x$ is specified by the two equations,

$$\ln x > \frac{\ln N}{C_N + 1} \quad \ln x \leq \frac{\ln N}{C_N}. \quad (14)$$

Following the determination of $x$, the coefficients $a_j$ are determined via starting with $a_{\tilde{C}_N} \leq x$, $j = \tilde{C}_N$, $N_{\tilde{C}_N} = N$ and following the procedure,

$$N_{j-1} = N_j - a_j x^j \quad (15)$$

with

$$\gamma_{j-1} = \frac{N_{j-1}}{x^j}. \quad (16)$$

Take $a_{j-1} = \lfloor \gamma_{j-1} \rfloor$ with a rounding down of $\gamma_{j-1}$; if $a_{j-1} \geq x$ then the procedure stops and the remaining $a_i$ are set to $a_i = 0$ with $i < j - 1$. Otherwise, the subtraction process continues. The procedure costs at most $3\tilde{C}_N$ operations with numbers of at most size $N$ (bit size $\ln N$). Due to the bound on $\tilde{C}_N$, this process is at most of the size $3 \ln N$ operations, one of which is division.

The next step requires the solution of the algebraic equations for $b_j$ in terms of $a_j$. There are $\tilde{C}_N + 1$ equations in $\tilde{C}_N + 1$ variables. The initial roots are chosen from the lowest prime neighborhood around $p = c_j x - b_j$, to larger. This procedure is natural for the root determinations in the case of the exact $C_N$. Another case is in choosing cascades in the range $10^n$ to $10^{n+1}$ for $n \leq \ln_{10} N$. Convergence is not analyzed, but for well chosen starting values, the number of iterations is typically $N^3$; this is $\tilde{C}_N^3$ for the case of $\tilde{C}_N$ variables and equations. The bound is $\ln^3 N$. Roughly, if the number of operations per iteration is $\tilde{C}_N^2$ (i.e. evaluating a set of similar polynomials) and there are $\tilde{C}_N$ roots, then the steps would number as $\tilde{C}_N^6$.

If any roots converge to a non-integral value of $b_j$, then the integer $N$ does not factor into $\tilde{C}_N$ numbers. This shortens the number of iterations and steps. The process of determining the factorization of $N$ into the products of two to $C_N$ numbers requires $\ln^m N$ steps, with $m$ denoting an average value from the root selection process, the number of variables at each step, the root solving and iteration process including shortcuts such as information from lower $\tilde{C}_N$ examples, and an averaging of the shortening the algorithm during the process of lower unknown roots or non-integerness. Perhaps, the average results in $m \sim 4$ or 5, less than $\ln^6 N$. 
To compare with the textbook quadratic sieve method, take the logarithm of the steps for both this method and the former,

\[ a\sqrt{\ln N \ln \ln N} \quad m \ln \ln N \quad (17) \]

which is,

\[ a^2 \ln N \ln \ln N \quad m^2 \ln^2 \ln N \quad (18) \]

The gain is clearly an exponential. The terms compare as \( a^2 \ln N = m^2 \ln \ln N' \) and \( N' = \exp(\exp(a^2/m^2 N)) \). Consider \( N = 10^{1000} \): the numbers are an approximate \( \exp(\exp(a^212000)^{1/2}) \) vs. \( \exp(m6)^{1/2} \).

In addition to prime factorization, the product form of the integer into various products of factors is determined; this is an additional byproduct of the procedure and its computational cost. Furthermore, an explicit knowledge of \( C_N \), the number of prime factors of a number, would provide a further exponential speedup.

The procedure here may be adapted to find various forms of number decompositions. An example is to find the form of a number written as a sum of products of primes.
References

[1] R. Crandall and C. Pomerance, *Prime Numbers, A Computational Perspective*, Springer-Verlag Inc., (2001).

[2] *Encylopedic Dictionary of Mathematics*, Iwanamic Shoten Publishes, Tokyo, 3rd Ed., (1985), English Transl. MIT Press (1993).
Addendum to Fast Factoring of Integers

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Abstract

An algorithm is given that generates the prime factorization of a number \( N \) in potentially \( \ln^2 N \) moves, with the complexity limited by the base determination of a number in a small degree polynomial. This fact could be a consequence of the Riemann hypothesis being true. This work adds previous work on prime factorizations, involving an LU (or QR) factorization. The modification to the previous methodology involves a special number on the base reduction of the number \( N \) into base \( x \).
1 Introduction

Factorization of large integers is important to many areas of pure mathematics and has practical applications in applied math including cryptography. This subject has been under intense study for many years [1]; improvements in the methodology are especially desired for computational reasons.

Previous work by the author allowed an estimate of $\ln^m N$ with $m$ equal to 5 or 6 due to computational reasons [2]. However, a condition has appeared on one of the variables that ultimately forces the prime factors to be found in $\ln^2 N$ steps, which is explained here. The greater Riemann hypothesis implies that the factorization could be performed in $\ln^4 N$ moves.

The case of a single number factorizable into only two prime numbers is a special case. The factorization may be performed it seems in $\ln^2 N$ moves, given some further information about the base expansion $N = \sum b_i x^i$ of a number $N$ with a particular remainder $b_0$. This is remarkable given the currently known bounds on factoring the two-channel keys.

The factorization of a number into $m$ prime factors proceeds first by writing the number in a base $x$ format [2],

$$N = \sum_{i=0}^{n} b_i x^i = \gamma \prod_{i=0}^{m} (\alpha_i x + 1) ,$$

with $n$ ranging from $m$ to $2m$, with the case of $n = m$ being the simplest.

The numbers $b_i$ are converted into the $\alpha_i$ to find the factors. In the case of the prime factorization the number $m$ labels the maximal number of prime factors, in the case that $\gamma$ itself is not prime; this is the generic case as discussed in the following. Solving the system of equations in (1.1) for the $\alpha_i$ in terms of the $b_i$ results in the factorization of the number $N$ in terms of the factors $(\alpha_i x + 1)$ and $\gamma$ [2].

The case in which all $\alpha_i$ are real numbers, and not complex, is considered. The case in which the the $\alpha_i$ are complex is similarly analyzed with small changes in the algorithm. The solutions to the complex numbers $\alpha_i$ come in pairs, with the $(\alpha_i x + 1)(\alpha_i^* x + 1)$ generating the prime factors [2].

The number $b_0 = \gamma$ is a special number. If it is not prime itself, then it must be a factor of $10^a$, that is, a power of our base ten number system. Consider the factors

$$\alpha_i x + 1 ,$$

(1.2)
with \( x \) the base number used in the factoring of (1.1) and with \( \alpha_i \) solved for. For \( m \) the maximum number, this number must be prime, which means that the number is a potential decimal

\[
\alpha_i x + 1 = N_i / D_i , \quad (1.3)
\]
such as 2/10 or 7/100. Because \( N_i \) is prime, it does not divide into \( D_i \), which means that \( D_i \) must be a multiple \( 10^{a_i} \). In general these numbers \( N_i / D_i \) are decimals, which means that the denominator is a power of the base 10. Collecting all of the denominators \( 10^{a_i} \) results in a net factor,

\[
\prod_{i=1}^{m} 10^{a_i} = 10^a , \quad (1.4)
\]
and this number must cancel the \( \gamma \) factor, resulting in either a prime number or an integer. If the latter, then the factor of \( 10^a \) must cancel the \( \gamma \) number resulting in unity.

Thus we arrive at the result: In the case that \( m \) is the maximal integer and not all \( \alpha_i \) are fractions, \( b_0 \) is a number \( 10^a \). This means that one must only test the base decompositions

\[
N = \sum_{i=0}^{m} b_i x^i , \quad (1.5)
\]
in which \( b_0 \) is of the form \( 10^a \) with \( a \) an integer; the cost is a 'smeared' function of \( \ln N \) because there is potentially a function for determining the integrality of \( a \) without testing all \( a \) from 0 to \( \ln(10, N) \) (and \( a = -\infty \)). The solution for the \( \alpha_i \) follows from an LU (or QR) decomposition, which is of the order \( \ln^3 N \) steps.

**Exceptional Case:**

In the case that all \( \alpha_i \) are integral, with a number \( N \) factoring into \( n \) prime numbers, then

\[
\alpha_i x + 1 = N_i , \quad (1.6)
\]
with all \( N_i \) prime numbers. This includes the case of two prime numbers \( N = pq \). In this case \( \gamma \) itself must be a prime number. However, this case is quite special as the factorization in (1.1) must result in all integral values \( \alpha_i \). The result are a very special set of numbers,

\[
\begin{align*}
\gamma &= b_0 , \\
\gamma(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \ldots) &= b_1 \\
\gamma(\alpha_1\alpha_2\alpha_3 + \text{perms}) &= b_2 \\
\ldots ,
\end{align*}
\]

in which all \( \alpha_i \) are such that \( \alpha_i x + 1 \) is prime, with \( \alpha_i \) integer, and also with \( \gamma \) prime. These numbers, and the factorization, should be interesting to construct. Of course, the \( 10^a \) test of \( b_0 \) should rule out these numbers generically.

**Cubic and Quartic Cases**

The other two cases required to complete the analysis of the two prime numbers composing a number \( N \) are the cubic and quartic examples, as analyzed in [2]. These cases span the forms,

\[
N = \gamma \prod (\alpha_i x + 1) ,
\]

with \( \alpha_i \) in complex pairs and the pairwise product generating the prime factors. In the case of a cubic there is a single pair, and in the case of the quartic there are two pairs. The \( \gamma \) is \( b_0 \), which is \( 10^i \) or a prime number.

**Comments on Base Determination**

Empirical observation, with the use of a Matlab program, indicate that the factorization of a number \( N \) into the quadratic case

\[
N = ax^2 + bx + 10^i ,
\]

can be performed with an approximate \( \ln N \) solutions. That is, given the remainder term \( 10^i \), there appears to be at most \( \ln N \) solutions to the the base \( x \); this bound seems to be even lower in some examples.
Knowledge of the totient function $\phi(N)$, and its inverse, might be responsible for this delimiting factor (the $\sigma_k(N)$ which are generally determined in terms of $\phi(N)$). The totient has been bounded in general by

$$\phi(N) > \frac{N}{e^\gamma \ln \ln N + \frac{3}{\ln \ln N}}.$$  \hspace{1cm} (1.10)

For prime $N$, the totient hits a local maximum, $\phi(N) = N - 1$. The function oscillates strongly near these local maximum, being partly due to the presence of small prime factors such as 2 or 3 which make the factor $1 - 1/p_j$ small. However, for numbers $N$ which are between these prime $N$’s, or local maximum, the oscillations are much smaller (e.g. of strength $\ln N$) and the function is relatively stable. A reasonable initial value for a numerical routine is chosen for $\phi(N)$ and $b_i$; the parameters can be computed from the system of equations relating $\phi(N)$ and its moments $\sigma_k(N)$. Apparently, the case of a number $N$ factoring into two numbers $p$ and $q$ which are of the same size in digits is easier to factor in this approach than with two numbers of different sizes.

*Case of $N = \prod p_j^{k_j}$: Multiple Factors*

The case of multiple prime numbers in the expansion of a number, such as $N = \prod p_j^{k_j}$, is examined by a generalization of the two prime factor case. According as in [2], the polynomials grow in size in the base expansion,

$$N = \sum_{j} b_j x^j,$$  \hspace{1cm} (1.11)

with $m$ bounded from $C_N$ to $2C_N$. The $C_N$ is $\sum k_j$. Including the required divisor functions $\sigma_i(N)$ allows the systems of equations to be solved for. The divisors can be written in terms of the totient function $\phi(N)$, for classes of numbers with $C_N$ fixed.

Solving the systems of equations for $b_i, x$ and $\phi(N)$ generates the prime numbers $p_j$ and coefficients $k_j$, at fixed $C_N$. The number of possibilities increases, however, because there are from $C_N$ to $2C_N$ equations in the individual system sets. There are also additional branch cuts; the use of the branch cuts may generate as check on the actual unique solution to these parameters. The solution to the prime factors is again achieved in $O(1)$ operations, set by the number of prime factors.

The cases of $b_0 = 10^i$ or $b_0$ prime are handled individually by the direct solution of the systems of equations for $b_i, x$. 

4
Discussion

Previous work in [2] generated an algorithm for determining the prime factors of a number \( N \), using base expansions with the base determination the limiting factor on the complexity.

The factoring of numbers can be achieved in polynomial time, albeit at an order which is unexpected in the literature. The number of remainder terms in the base expansion, the \( 10^i \), is generically \( \ln N \) in number; the number \( 10^i \) could be replaced by a prime number, which is a special case. The further inverse determination of the base \( x \) following from each case, although the algorithm is apparently unknown, appears to number in at most \( \ln N \) cases; this results in the complexity of factoring a number to be in \( \ln^2 N \) computational steps. An algorithm for the base determination from the remainder term could in principle be performed in \( O(1) \) steps, even if a table is known (this table is similar to a table of pythagorean triples, upon further manipulation of the base determination with the roots \( \alpha_i \) being the rational numbers).

In the case of a number containing \( C_N \) prime factors, solving a system of coupled algebraic equations of degree \( C_N \) to \( 2C_N \) is required [2], with remainder term in the base expansion of the form \( 10^i \) or a prime number. These polynomials are of degree 2 to 4 in the case of numbers \( N = pq \), i.e. two prime factors.

These equations are Diophantine in the sense that the solutions to the parameters are integral. Solving these equations requires the techniques for analyzing low degree polynomials, potentially beyond the quintic. Numerical methods are straightforward to implement.

The analysis could also generate an indirect determination of the totient function \( \phi \), and the related divisor functions \( \sigma_k \), given an algorithm for the base determination.
References

[1] R. Crandall and C. Pomerance, *Prime Numbers, A Computational Perspective*, Springer-Verlag Inc., (2001).

[2] Gordon Chalmers, *Fast Factoring of Integers*, physics/0503159