DEMOCRACY OF SHEARLET BASES WITH APPLICATIONS TO APPROXIMATION AND INTERPOLATION

DANIEL VERA

Abstract. Shearlets are based on wavelets with composite dilation. Thus, they inherit many important features from wavelets as the affine and basis-like structure. Shearlets provide (near) optimal approximation for the class of so-called cartoon-like images. Moreover, there are distribution spaces associated to them and there exist embeddings between these and classical (dyadic isotropic) inhomogeneous spaces.

We prove that the shear anisotropic inhomogeneous Besov and Triebel-Lizorkin sequence spaces verify the $p$-Temlyakov property (also known as $p$-space property and related to the concept of democratic bases) for certain parameters. Then, we prove embeddings (or characterizations) between approximation spaces and discrete weighted Lorentz spaces (in the framework of shearlet systems) and prove (that these embeddings are equivalent to) Jackson and Bernstein type inequalities. This allows us to find (real) interpolation between these highly anisotropic spaces.

1. Introduction.

Wavelets are reproducing systems that have been successfully applied in harmonic analysis, numerical and analytical solution of certain partial differential equations, signal processing and statistical estimation. Not only can they be used to characterize some classical spaces but there are also implementable fast algorithms. Approximation theory benefits from the unconditional bases provided by wavelet theory since “it is enough to threshold the properly normalized wavelet coefficients” to achieve good $N$-term nonlinear approximation (see [7]). It is also well known that the approximation order is closely related to the smoothness of the function. A generalization of the nonlinear approximation theory, called restricted nonlinear approximation (RNLA), was carried out by Cohen, DeVore and Hochmut in [4] in the setting of wavelet bases in Hardy and Besov spaces where they control the measure of the index set of the approximation elements instead of the number of terms in the approximation. This measure (generally other than the counting measure) is closely related to a weighting of the coefficients. One of the novelties in [4] is that the approximation spaces are not necessarily contained in the space in which the error is measured. Some extensions of the restricted nonlinear approximation were done in [18] and [17] for general bases in quasi-Banach distribution spaces and quasi-Banach lattice sequence spaces, respectively. We develop our results based in [17] since its framework is more general than [18] (in fact, [17] allows to recover results in [4], in contrast to [18]). Working
with sequence spaces will not be a limitation since, as in the case of wavelets, we have that shear anisotropic inhomogeneous Besov and Triebel-Lizorkin distribution spaces are a retract of their sequence spaces counterparts. Therefore, we can transfer the results established here to the function setting of [22] and [21]. Approximation theory is related to real interpolation (through Peetre’s $K$-functional) in a way that some identifications between approximation spaces, interpolations spaces, discrete Lorentz spaces and Besov spaces can be proved for certain parameters. These identifications turn out to be equivalent to what is known as $p$-Temlyakov property and to Jackson and Bernstein type inequalities (see [4], [18] and [17]). For the definition of some of these spaces and other concepts related to restricted nonlinear approximation in sequence spaces (RNLASS) see Section 2.1 and [17].

Although wavelets provide better approximation properties than Fourier techniques, they lack of high directional sensitivity in dimensions $d \geq 2$ since the number of wavelets (from a multi resolution analysis) remain constant across scales: $2^d - 1$. Some directional systems have been created to overcome this limitation. Two of them are the curvelets of Candès and Donoho ([2] and [3]) and the shearlets of Guo, Kutyniok and Labate ([13]). The number of directions in the shearlet or curvelet systems doubles at each (other) scale yielding sparser approximation. Consider the class $\mathcal{E}$ of the so-called cartoon-like images made up of bounded $C^2$-functions in $[0,1]^d$ except in $C^2$-discontinuities. It has been proved that, for $f \in \mathcal{E}(\mathbb{R}^2)$,

$$\|f - f_N^D\|_2^2 \approx N^{-2}(\log N)^3, \quad \text{as } N \to \infty,$$

where $f_N^D$ stands for the curvelet or shearlet approximation with $N$ terms. This is an optimal approximation except for the logarithmic factor. In contrast, wavelet approximation gives an error decay of only $O(N^{-1})$. When $d = 3$, the rate of error approximation is $O(N^{-1}(\log N)^2)$ for both directional systems, whilst for wavelets it decreases only as $O(N^{-1/2})$. For the previous statements see [10] and [14].

Both directional systems form tight frames. However, the shearlet systems (unlike the curvelet systems which are based on polar coordinates) are based on the composite dilations wavelets of Guo, Labate, Lim, Weiss and Wilson in [16] which take full advantage of the theory of affine systems. This fact allows a natural transition from the continuous to the discrete setting and an implementable framework.

A natural question is whether these directional systems can generate/characterize other function/distribution spaces as in the case of wavelet systems and, if so, what the relation with classical function/distribution spaces is. The first question has been answered affirmatively several times and, moreover, some embeddings between these new spaces and classical ones have been found, answering the second question. The theory of decomposition spaces was applied by Borup and Nielsen in [1] to develop what can be called the curvelet decomposition spaces. Short after Dahlke, Kutyniok, Steidl and Teschke defined in [6] the shearlet coorbit spaces through the theory of coorbit spaces. More recently, Labate, Mantovani and Negi applied again the general theory of decomposition spaces to describe and study the shearlet smoothness spaces in [19]. A more classical approach is done by the author in [22], following the ideas of Frazier and Jawerth in [11] to develop the shear anisotropic inhomogeneous Besov spaces. All of the above spaces are related to classical Besov spaces. This can easily be verified from their definition in $\ell^q(\ell^p)$ (quasi-)norms in sequence spaces ($\ell^q(\ell^p)$ quasi-norms in [22]) and from their embedding results. As another family of spaces
generated/characterized by the shearlet system (this time with $L^p(\ell^q)$ quasi-norms) is the shear anisotropic inhomogeneous Triebel-Lizorkin spaces developed by the author in [21], following again the ideas of Frazier and Jawerth in [12]. In order to develop a Triebel-Lizorkin type spaces with shearlets via decomposition spaces it is necessary to use the tools developed in [21], namely the Fefferman-Stein-Peetre maximal function with shear anisotropic dilations and related inequality (see Lemma 4.2.3 in [21]). We point out next some differences between these spaces. The shearlet coorbit spaces theory (with continuous parameters and non-uniform directional information) allows the study of homogeneous spaces with Banach frames (after discretizing the representation). The rest of spaces aforementioned are naturally inhomogeneous by construction but generate quasi-Banach spaces. In applications the amount of information is limited by the sampling operation and memory/transmission restrictions and so the inhomogeneous setting is well-adapted to computational procedures.

Most of the research on the approximation properties with shearlets has been focused on showing (near) optimality of sparse approximation to the class $\mathcal{E}$ of cartoon-like images in 2D and 3D with the error measured in the $L^2$-norm. However, the $L^2$-norm does not give necessarily the best visually faithful approximation. For this and other discussions (v.gr. statistical estimation) on the reasons to measure the error on norms different than $L^2$ see Section 10 of [7] and references therein. Here, we study approximation properties of shearlet systems when the target function belongs to a certain (shear anisotropic) smoothness space, more general than the class $\mathcal{E}$, and the error is measured on a different (shear anisotropic) smoothness space. For example, the class $\mathcal{E}(\mathbb{R}^2)$ is a small subset of the class of functions of bounded variation $BV(\mathbb{R}^2)$, which in turn lies between $B^{1,1}_1(\mathbb{R}^2)$ and $B^{1-\varepsilon,1}_1(\mathbb{R}^2)$ (see [5] for a discussion on the advantages of working with $B^{1,1}_1(\mathbb{R}^2)$ instead of $BV(\mathbb{R}^2)$).

The outline is as follows. In Section 2 we introduce the definitions, previous results and notation of i) the theory of RNLASS and ii) the shearlet systems and related spaces we will be working with. In Section 3 we prove democracy ($p$-space or $p$-Temlyakov property) of shearlet bases on some shear anisotropic inhomogeneous sequence spaces. Applications to approximation theory and interpolation are presented in Section 4. Finally, a straight extension of our results to other spaces generated by shearlets or curvelets is given in Section 5 as well as a brief discussion regarding the space $BV(\mathbb{R}^2)$.

2. Definitions and notation

2.1. Restricted nonlinear approximation in sequence spaces. Denote by $S$ the space of all sequences $s = \{s_I\}_{I \in \mathcal{D}}$ of complex numbers indexed by a countable set $\mathcal{D}$ and by $\mathcal{E} = \{e_I\}_{I \in \mathcal{D}}$ its canonical basis. Let $f \subset S$ be a quasi-Banach sequence lattice (also called solid quasi-Banach space). Given a positive measure $\nu$ on $\mathcal{D}$ we define the restricted approximation spaces $A_{\xi,\mu}^f(\nu)$, $0 < \xi < \infty$, $0 < \mu < \infty$, as the set of all sequences $s \in S$ such that

$$
\left\| s \right\|_{A_{\xi,\mu}^f(\nu)} := \left( \int_0^\infty \left[ t^\xi \sigma_{\nu}(t,s) \right]^\mu \frac{dt}{t} \right)^{1/\mu} < \infty,
$$

(usual modifications if $\mu = \infty$), where

$$
\sigma_{\nu}(t,s) = \sigma_{\nu}(t,s)_f := \inf_{t \in \mathbb{E}_{t,\nu}} \| s - t \|_f,
$$

2. Definitions and notation
is the error of approximation (or $\mathcal{f}$-risk function) and $\Sigma_{t,\nu} := \{t = \sum_{i \in \mathcal{I}} t_i e_i : \nu(\Gamma) \leq t, \Gamma \subset \mathcal{D}\}$ where $\nu(\Gamma) = \sum_{i \in \mathcal{I}} \nu(I)$. When $\nu$ is the counting measure we are in the classical $N$-term nonlinear approximation setting. In this paper the space $\mathcal{f}$ will be any of the spaces $b^{p,q}(AB)$ or $l^{p,q}(AB)$ (see (2.5) and (2.6)) and thus will have associated an “integration” parameter $p$.

The **discrete Lorentz spaces** $\ell^{p,\mu}(\nu)$, $0 < p < \infty$, $0 < \mu \leq \infty$, are defined as the set of all sequences $s = \{s_I\}_{I \in \mathcal{D}} \in S$ such that

$$\|s\|_{\ell^{p,\mu}(\nu)} := \left( \int_0^\infty [t^{1/p} s^*_\nu(t)]^\mu \frac{dt}{t} \right)^{1/p},$$

(usual modifications if $\mu = \infty$), where $s^*_\nu(t) = \inf\{\lambda > 0 : \nu(\{I \in \mathcal{D} : |s_I| > \lambda\}) \leq t\}$ is the non-increasing rearrangement of $s$ with respect to the measure $\nu$. When $\mu = p$ we have $\ell^{p,p}(\nu) = \ell^p(\nu)$ and thus

$$\|s\|_{\ell^p(\nu)} = \left( \sum_{I \in \mathcal{D}} |s_I|^p \nu(I) \right)^{1/p}.$$

We use a weight sequence $u = \{u_I\}_{I \in \mathcal{D}}$, $u_I > 0$, to control the weight of each $s_I$ as follows. The spaces $\ell^{p,\mu}(u, \nu)$ are given by $\|s\|_{\ell^{p,\mu}(u, \nu)} := \|\{u_Is_I\}_{I \in \mathcal{D}}\|_{\ell^{p,\mu}(\nu)} < \infty$. See [17] for details and a more general framework.

From Theorems 2.6.1 and 2.7.2 in [17] we have the next equivalences.

**Theorem 2.1.** Let $(\mathcal{f}, \nu)$ be a standard scheme and let $u = \{u_I\}_{I \in \mathcal{D}}$ be a weight sequence. Fix $\xi > 0$ and $\mu \in (0, \infty)$. Then, for any $0 < p < \infty$ and $r$ such that $\frac{1}{r} = \frac{1}{p} + \xi$, the following are equivalent:

J1) The next upper $p$-Temlyakov property holds: There exists $C > 0$ such that for all $\Gamma \subset \mathcal{D}$ with $\nu(\Gamma) < \infty$,

$$\left\| \sum_{I \in \Gamma} \frac{e_I}{u_I} \right\|_f \leq C(\nu(\Gamma))^{1/p}.$$

J2) $\ell^{r,\mu}(u, \nu) \hookrightarrow A_1^\xi(\mathcal{f}, \nu)$.

J3) The space $\ell^{r,\mu}(u, \nu)$ satisfies the next Jackson’s inequality of order $\xi$: there exists $C > 0$ such that

$$\sigma_{\nu}(t, s) \leq Ct^{-\xi} \|s\|_{\ell^{r,\mu}(u, \nu)}, \text{ for all } s \in \ell^{r,\mu}(u, \nu).$$

With the same setting for $(\mathcal{f}, \tilde{\nu})$, the following are equivalent:

B1) The next lower $p$-Temlyakov property holds: There exists $C > 0$ such that for all $\Gamma \subset \mathcal{D}$ with $\tilde{\nu}(\Gamma) < \infty$,

$$\left\| \frac{1}{C(\tilde{\nu}(\Gamma))^{1/p}} \right\|_f \leq \left\| \sum_{I \in \Gamma} \frac{e_I}{u_I} \right\|_f.$$

B2) $A_1^\xi(\mathcal{f}, \tilde{\nu}) \hookrightarrow \ell^{r,\mu}(u, \tilde{\nu})$.

B3) The space $\ell^{r,\mu}(u, \tilde{\nu})$ satisfies the next Bernstein’s inequality of order $\xi$: there exists $C > 0$ such that

$$\|s\|_{\ell^{r,\mu}(u, \tilde{\nu})} \leq Ct^\xi \|s\|_f \text{ for all } s \in \Sigma_{t, \tilde{\nu}} \cap \mathcal{f}.$$
Obviously, when \( \nu = \tilde{\nu} \) we will say that the upper and lower democracy (p-Temlyakov property) are the same and have characterizations and identifications.

It is well known that \( N \)-term approximation and real interpolation are interconnected. If the Jackson and Bernstein’s inequalities hold for \( \nu = \) counting measure, \( N \)-term approximation spaces are characterized in terms of interpolation spaces (see e.g. Theorem 3.1 in [9] or Section 9, Chapter 7 in [8]). As stated in [4], the same scheme is true for the RNLA.

Below we state two more results we need in this paper. The proofs are straightforward modifications of those given in the references cited in the previous paragraph.

**Theorem 2.2.** Let \( (f, \nu) \) be a standard scheme. Suppose that the quasi-Banach lattice \( g \subset S \) satisfies the Jackson and Bernstein’s inequalities for some \( r > 0 \). Then, for \( 0 < \xi < r \) and \( 0 < \mu \leq \infty \) we have

\[
A_{\mu}^{(r)}(f, \nu) = (f, g)_{\xi/r, \mu}.
\]

It is not difficult to show that the spaces \( A_{\mu}^{(r)}(f, \nu), 0 < r < \infty, 0 < q \leq \infty, \) satisfy the Jackson and Bernstein’s inequalities of order \( r \), so that by Theorem 2.2,

\[
A_{\mu}^{(r)}(f, \nu) = (f, A_{\mu}^{(r)}(f, \nu))_{\xi/r, \mu},
\]

for \( 0 < \xi < r \) and \( 0 < \mu \leq \infty \). From here, and using the reiteration theorem for real interpolation (e.g. Ch. 6, Sec. 7 of [8]) we obtain the following result.

**Corollary 2.3.** Let \( 0 < \alpha_0, \alpha_1 < \infty, 0 < q, q_0, q_1 \leq \infty \) and \( 0 < \theta < 1 \). Then,

\[
(A_{\alpha_0}^{(q_0)}(f, \nu), A_{\alpha_1}^{(q_1)}(f, \nu))_{\theta,q} = A_{\alpha}^{(q)}(f, \nu), \quad \alpha = (1 - \theta)\alpha_0 + \theta\alpha_1
\]

for a standard scheme \((f, \nu)\).

For a comprehensive treatment of these and other results in RNLA see [17].

## 2.2. Shearlets.

Let \( \mathbf{d} \) be a coordinate in the plane of frequencies \( \hat{\mathbb{R}}^d \). Let \( j \geq 0 \) and \( k \in \mathbb{Z}^d \) be scale and position, respectively, and \( \ell \) be the shear parameter such that \( \ell = (\ell_1, \ldots, \ell_{d-1}) \) with \(-2^j \leq \ell_i \leq 2^j, i = 1, \ldots, d-1\). Let \( A_{(q)} \) and \( B_{(q)} \) be the anisotropic and shear matrices, respectively (their value are not important for our results but only the knowledge that \(|\det A| = 2^{-(d+1)}\) and \(|\det B| = 1\)). Shearlets are constructed from functions \( \hat{\Psi}, \hat{\psi}_{(q)} \in C^\infty(\hat{\mathbb{R}}^d) \) of compact support such that

\[
\left| \hat{\Psi}(\xi) \right|^2 \chi_R(\xi) + \sum_{q=1}^d \sum_{j \geq 0} \sum_{||\ell|| \leq 2^j} \left| \hat{\psi}_{(q)}(\xi A^{-j}_{(q)} B^{-\ell}_{(q)}) \right|^2 \chi_{D^{(q)}}(\xi) = 1,
\]

(2.1)

for all \( \xi \in \hat{\mathbb{R}}^d \) and where \( \chi_R \) is the characteristic function of a square that covers the low frequencies and \( D^{(q)} \) are some truncated symmetric cones as in Figure 1. From (2.1), it can be shown that the shearlet system

\[
\{ \psi_{j,\ell,k}^{(q)}(x) = \left| \det A_{(q)} \right|^{1/2} \psi_{(q)}(B^{-\ell}_{(q)} A_{(q)} x - k) : q = 1, \ldots, d, j \geq 0, ||\ell|| \leq 2^j, k \in \mathbb{Z}^d \},
\]

is a Parseval frame for \( L^2(\mathbb{R}^d) \) (see [16], Section 5.2.1). This means that

\[
\sum_{q=1}^d \sum_{j \geq 0} \sum_{||\ell|| \leq 2^j} \sum_{k \in \mathbb{Z}^d} \left| \langle f, \psi_{j,\ell,k}^{(q)} \rangle \right|^2 = \|f\|_{L^2(\mathbb{R}^d)}^2,
\]

for all \( f \in L^2(\mathbb{R}^d) \).
for all \( f \in L^2(\mathbb{R}^d) \). From (2.1) and Littlewood-Paley theory it can be shown (\cite{22} and \cite{21}) that the shearlet coefficients characterize other spaces. We present these spaces below.

Observe that the characteristic functions \( \chi_{\mathbb{R}}, \chi_{\mathbb{D}}(d) \) in (2.1) destroy the localization in the space domain. With a slight variation on the above, Guo and Labate constructed in \cite{15} smooth Parseval frames of shearlets, which means that a) one can ignore the characteristic functions \( \chi_{\mathbb{R}}, \chi_{\mathbb{D}}(d) \) in (2.1) and b) one can prove that the composition of the analysis (see (2.4)) and synthesis operators is the identity on \( S' \) (see \cite{21}).

Since \( \mathcal{D}^{(b)} \) are orthogonal rotations of \( \mathcal{D}^{(1)} \) we will often drop the sub- or super-index and develop our results only for one direction.

For \( Q_0 = [0, 1)^d \), write

\[
Q_{j,\ell,k}^{(b)} = A_{(0)}^{-j} B_{(0)}^{-\ell} (Q_0 + k),
\]

with \( d = 1, \ldots, d, j \geq 0, ||\ell|| \leq 2^j \) and \( k \in \mathbb{Z}^d \). Therefore, \( \int \chi_{Q_{j,\ell,k}^{(b)}} = |Q_{j,\ell,k}^{(b)}| = 2^{-(d+1)j} = |\det A_{(0)}|^{-j} \). Let \( \mathcal{Q}_{AB} := \{ Q_{j,\ell,k}^{(b)} : d = 1, \ldots, d, j \geq 0, ||\ell|| \leq 2^j, k \in \mathbb{Z}^d \} \) and \( \mathcal{Q}_{(b)} := \{ Q_{j,\ell,k}^{(b)} : k \in \mathbb{Z}^d \} \). Thus, for fixed \( d, j, \ell \), \( \mathcal{Q}_{(b)} \) is a partition of \( \mathbb{R}^d \) as can be seen in Figure 2. Hence, for every \( j \geq 0 \) there exist \( 2^{(j+1)(d-1)} + 1 \) partitions since \( ||\ell|| \leq 2^j \). To shorten notation and clear exposition, we will identify the multi indices \( (j, \ell, k) \) and \( (i, m, n) \) with \( P \) and \( Q \), respectively. This way we write \( \psi_P = \psi_{j,\ell,k} \) or \( \psi_Q = \psi_{i,m,n} \), regardless the direction in question. We also write \( \tilde{\chi}_Q(x) = |Q|^{-1/2} \chi_Q(x) \).

We formally define the shearlet **analysis operator** as

\[
S_{\Psi,\psi}f = \{ \{ f, \Psi(\cdot - k) \} \}_{k \in \mathbb{Z}^d}, \{ \langle f, \psi_Q \rangle \}_{Q \in \mathcal{Q}_{AB}} \}. \tag{2.4}
\]
Figure 2. Sketch of the covering of the plane $\mathbb{R}^2$ with parallelograms in $Q^{1,2}$. Those parallelograms with solid lines cover the parallelogram $Q_{j,\ell,k} = Q_{0,1,0}$.

So the shearlet coefficients are $s = \{s_Q\}_{Q \in Q_{AB}} = S_\psi \mathcal{H} \hat{f}$.

The next two definitions are the sequence spaces associated to the shear anisotropic inhomogeneous Besov and Triebel-Lizorkin distribution spaces as defined in [22] and [21], respectively.

For $s \in \mathbb{R}$, $0 < p, q \leq \infty$, the shear anisotropic inhomogeneous Besov sequence space $b_{s,p}^{\gamma,q}(AB)$ is defined as the collection of all complex-valued sequences $s = \{s_Q\}_{Q \in Q_{AB}}$ such that

$$
\|s\|_{b_{s,p}^{\gamma,q}(AB)} := \left( \sum_{k \in \mathbb{Z}^d} |s_k|^p \right)^{1/p} + \left( \sum_{\beta=1}^d \sum_{j \geq 0} \sum_{||\ell|| \leq 2^j} \left( \sum_{Q \in Q_{AB}^{(s)}} ||Q||^{-s + \frac{1}{p} - \frac{1}{2}} |s_Q|^p \right) \right)^{q/p} < \infty.
$$

(2.5)

For $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. The shear anisotropic inhomogeneous Triebel-Lizorkin sequence space $f_{s,p}^{\gamma,q}(AB)$ is defined as the collection of all complex-valued sequences $s = \{s_Q\}_{Q \in Q_{AB}}$ such that

$$
\|s\|_{f_{s,p}^{\gamma,q}(AB)} = \left( \sum_{k \in \mathbb{Z}^d} |s_k|^p \right)^{1/p} + \left( \sum_{Q \in Q_{AB}} \left( ||Q||^{-s} |s_Q| |\tilde{\chi}_Q| \right)^q \right)^{1/q} \| \right|_{L^p} < \infty.
$$

(2.6)

It is not hard to prove that, for $0 < p < \infty$, $f_{s,p}^{\gamma,q}(AB) = b_{s,p}^{\gamma,q}(AB)$, either by straight calculations from the definitions above or by the embedding Theorem 4.1 iii) in [22] with $p = q$.

Next result is a particular case which identifies the shear anisotropic Triebel-Lizorkin or Besov sequence spaces just defined with discrete Lorentz sequence spaces indexed by $Q_{AB}$ for certain parameters.

Lemma 2.4. For $\beta, s \in \mathbb{R}$ and $0 < \tau < \infty$, let $\gamma = s + \frac{1-\beta}{\tau}$ and $u = \{|Q|^{-s-\frac{1}{2}}\}_{Q \in Q_{AB}}$. Then,

$$
\ell^{\gamma,\tau}(u, \nu_{\beta}, Q_{AB}) = b_{\tau}^{\gamma,\tau}(AB) = f_{\tau}^{\gamma,\tau}(AB),
$$

where $b_{\tau}^{\gamma,\tau}(AB)$ and $f_{\tau}^{\gamma,\tau}(AB)$ are the shear anisotropic inhomogeneous Besov and Triebel-Lizorkin sequence spaces, respectively.
with equal quasi-norms.

**Proof.** This is a straight consequence of the respective definitions and the conditions on the parameters. So,

\[
\|s\|_{\ell^r,\tau(u,\nu,\Omega_{AB})}^r = \left\| \{ |s_Q| |Q|^{-\frac{r}{2}} \} Q \in \Omega_{AB} \right\|_{\ell^r,\tau(\nu,\Omega_{AB})}^r = \sum_{Q \in \Omega_{AB}} (\{ |s_Q| |Q|^{-\frac{r}{2}} \})^\tau |Q|^\beta = \sum_{Q \in \Omega_{AB}} (\{ |s_Q| |Q|^{-\gamma+\frac{1}{2} - \frac{d}{2}} \})^\tau
\]

\[= \|s\|_{b^{r,\tau}(AB)}^r. \]

\[\Box\]

### 3. Democracy of \(b^{s,p}(AB)\) and \(f^{s,q}(AB)\).

Our aim now is to prove that the spaces \(b^{s,p}(AB)\) and \(f^{s,q}(AB)\) verify points J1) and B1) of Theorem 2.1 with \(u\) related to a second space. For \(f^{s,q}(AB)\) we will not have the same upper and lower democracy. Hence, only embeddings can be proved. However, for \(f^{s,p}(AB) = b^{s,p}(AB)\) the upper and lower democracy will be the same and allow us to prove full characterizations.

To prove democracy for \(f^{s,q}(AB)\) we need a previous result.

**Lemma 3.1.** Let \(S^\gamma_T(x) := \sum_{P \in \Gamma} |P|^{\gamma} \chi_P(x)\), \(P^x\) be the largest “cube” that contains \(x\) for some scale and \(P_x\) be a smallest “cube” that contains \(x\) for some scale.

1. If \(\gamma > \frac{d-1}{d+1}\) and there exists \(P^x\) then,
   \[|P^x|^{\gamma} \chi_{P^x}(x) \leq S^\gamma_T(x) \leq C_\gamma |P^x|^{\gamma - \frac{d-1}{d+1}} \chi_{P^x}(x).\]

2. If \(\gamma < \frac{d-1}{d+1}\) and there exists \(P_x\) then,
   \[|P_x|^{\gamma} \chi_{P_x}(x) \leq S^\gamma_T(x) \leq C_\gamma |P_x|^{\gamma - \frac{d-1}{d+1}} \chi_{P_x}(x).\]

**Remark 3.2.** Observe that we refer to the largest “cube” \(P^x\) and to a smallest “cube” \(P_x\). For any scale \(j \geq 0\) there are \(2^{j+1}(d-1) + 1\) decompositions of the plane \(\mathbb{R}^d\) since \(|\{\ell\}| \leq 2^j\). Therefore, as the scale increases the number of decompositions increases and a “child” cube that contain \(x\) will not be unique.

**Proof of Lemma 3.1.** We start by proving a). It is clear that \(|P^x|^{\gamma} \chi_{P^x}(x) \leq S^\gamma_T(x)\), since the sum in \(S^\gamma_T(x)\) contains \(|P^x| \chi_{P^x}(x)\), at least. For the right-hand side of the inequality we enlarge the sum defining \(S^\gamma_T(x)\) to include all \(P\)’s in the same and finer scales that contain \(x\). Defining \(T^j := \Gamma \cap Q^j\) and since \(Q^j\) is a partition of \(\mathbb{R}^d\), we obtain

\[
S^\gamma_T(x) = \sum_{j \geq J} \sum_{||\ell|| \leq 2^j} \sum_{P \in T^j} |P|^{\gamma} \chi_P(x) \leq \sum_{j \geq J} \sum_{||\ell|| \leq 2^j} \sum_{P \in Q^j} |P|^{\gamma} \chi_P(x) \leq C_d \sum_{j \geq J} 2^{-j((d+1)\gamma-\gamma)} \chi_{P^x}(x)
\]

\[= C_d,\gamma |P^x|^{\gamma - \frac{(d-1)}{d+1}} \chi_{P^x}(x),\]

\[\Box\]
since $\gamma > \frac{d-1}{d+1}$ and $|P_x| = 2^{-J(d+1)}$.

The left-hand side of b) is clear since $\Gamma \ni P_x$, at least. For the right-hand side of b) we enlarge the sum $S_1^\gamma(x)$ to include all “cubes” in the same and coarser scales. With the same definition of $\Gamma^j,\ell$ and since $Q_j^\gamma,\ell$ is a partition of $\mathbb{R}^d$, for fixed $j, \ell$, we obtain

$$S_1^\gamma(x) = \sum_{j=0}^J \sum_{\|\ell\| \leq 2^j} \sum_{P \in \Gamma^j,\ell} |P|^\gamma \chi_{P}(x) \leq \sum_{j=0}^J \sum_{\|\ell\| \leq 2^j} \sum_{P \in \mathcal{Q}^j,\ell} |P|^\gamma \chi_{P}(x)$$

$$= \sum_{j=0}^J |P_j|^\gamma (2^{(d+1)(d-1)} + 1) \chi_{P}(x) \leq C_d \sum_{j=0}^J 2^{-j(d+1)\gamma -(d-1)} \chi_{P}(x)$$

$$\leq C_d 2^{-J(d+1)\gamma -(d-1)} \sum_{j=0}^{\infty} 2^{+j(d+1)\gamma -(d-1)} \chi_{P}(x) = C_{d,\gamma} |P_x|^{\gamma - \frac{d+1}{d-1}} \chi_{P}(x),$$

because $\gamma < \frac{d-1}{d+1}$.

The first main result is:

**Theorem 3.3.** Let $s_1, s_2 \in \mathbb{R}$, $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \leq \infty$ with $p_1 \neq q_1$ and $u = \{\|e_P\|_{t_2,q_2(AB)}\}_{P \in \mathcal{Q}_{AB}} = \{|P|^{-s_2 + \frac{1}{q_2}}\}_{P \in \mathcal{Q}_{AB}}$. If $\alpha = p_1(s_2 - \frac{1}{p_2} - 1 + \frac{1}{p_1})$, there exist $C, C' > 0$ depending only on $s_1, s_2, p_2$ and $q_1$ such that

$$C'(\nu_{\alpha + \frac{d+1}{d-1}}(\Gamma))^{1/p_1} \leq \left[ \sum_{P \in \Gamma} \frac{|e_P|}{\|e_P\|_{t_2,q_2(AB)}} \right]^{p_1/(p_1 + 1/2)} \leq C'(\nu_{\alpha - \frac{p_1(d-1)}{q_1(d+1)}}(\Gamma))^{1/p_1} \quad (3.1)$$

for all $\Gamma \subset \mathcal{Q}_{AB}$ such that $\nu_{\alpha - \frac{p_1(d-1)}{q_1(d+1)}}(\Gamma) < \infty$. Conversely, if (3.1) holds then,

$$p_1(s_2 - \frac{1}{p_2} - s_1 + \frac{1}{p_1}) - p_1(d-1)q_1(d+1) \leq \alpha \leq p_1(s_2 - \frac{1}{p_2} - s_1 + \frac{1}{p_1}) + d - 1 - d + 1.$$

**Proof.** Write $f_1 := f_{p_1,q_1}(AB)$ and $f_2 := f_{p_2,q_2}(AB)$ to simplify the notation in this proof. By definition we have that

$$\left[ \sum_{j=0}^J \sum_{\|\ell\| \leq 2^j} \sum_{P \in \Gamma^j,\ell} |P|^q_{\mathcal{Q}^j,\ell} \chi_{P}(x) \right]^{p_1/q_1}$$

$$= \int_{\mathbb{R}^d} \left[ \sum_{j=0}^J \sum_{\|\ell\| \leq 2^j} \sum_{P \in \Gamma^j,\ell} (|P|^{s_2-s_1 - \frac{1}{p_2}} \chi_{P}(x))^q \right]^{p_1/q_1} dx.$$

We start by assuming that $\alpha = p_1(s_2 - \frac{1}{p_2} - s_1 + \frac{1}{p_1}) > 1 + \frac{p_1(d-1)}{q_1(d+1)} \iff \gamma = q_1(s_2 - s_1 - \frac{1}{p_2}) > \frac{d-1}{d+1}$. In this case, since $\nu_{\alpha - \frac{p_1(d-1)}{q_1(d+1)}}(\Gamma) < \infty$, the largest $P_x \in \Gamma$ exists for all $x \in \bigcup_{P \in \Gamma} P$. Applying part a) of Lemma 3.1 we get

$$\left[ \sum_{j=0}^J \sum_{\|\ell\| \leq 2^j} \sum_{P \in \Gamma^j,\ell} |P|^{q_1(s_2-s_1 - \frac{1}{p_2})} \chi_{P}(x) \right]^{p_1/q_1}$$

$$\leq C_\gamma |P|^\gamma \frac{d+1}{d-1} \chi_{P^x}(x)^{p_1/q_1}$$

$$= C_{\gamma,p_1} |P|^\gamma \frac{d+1}{d-1} \chi_{P^x}(x).$$
We also have from part a) of Lemma 3.1 that
\[ Q \subseteq \text{Temlyakov property} \]
for all \( x \in \bigcup_{P \in \Gamma} P \). From this we deduce the upper-Temlyakov property as
\[
\left\| \sum_{P \in \Gamma} \frac{e_P}{\|e_P\|_{f_1}} \right\|_{f_1} \leq C_\gamma \left( \int_{\mathbb{R}^2} \left| P \right|^{p_1(s_2-s_1-\frac{1}{p_2})+\frac{p_1(d-1)}{q_1(d+1)}} \chi_P(x) \right)^{1/p_1}
\]
\[
= C_\gamma \left( \sum_{P \in \Gamma} \left| P \right|^{p_1(s_2-s_1-\frac{1}{p_2})+1+\frac{p_1(d-1)}{q_1(d+1)}} \right)^{1/p_1}
\]
\[
= C_\gamma (\nu_{\alpha-\frac{p_1(d-1)}{q_1(d+1)}}(\Gamma))^{1/p_1}.
\]
We also have from part a) of Lemma 3.1 that
\[
\left[ \sum_{j \geq 0} \sum_{\|\ell\| \leq 2^j} \sum_{P \in \Gamma, \ell} \right] P^j_{1(s_2-s_1-\frac{1}{p_2})} \chi_P(x)^{p_1/q_1}
\]
\[
\geq \left[ \int_{\mathbb{R}^2} \left| P \right|^{p_1(s_2-s_1-\frac{1}{p_2})} \chi_P(x) \right]^{p_1/q_1}
\]
\[
= \left| P \right|^{p_1(s_2-s_1-\frac{1}{p_2})} \chi_P(x)
\]
\[
\geq \frac{1}{C_\gamma} \sum_{P \in \Gamma} \left| P \right|^{p_1(s_2-s_1-\frac{1}{p_2})+\frac{p_1(d-1)}{q_1(d+1)}} \chi_P(x).
\]
From this we obtain the lower-Temlyakov property as
\[
\left\| \sum_{P \in \Gamma} \frac{e_P}{\|e_P\|_{f_1}} \right\|_{f_1} \geq \left( \frac{1}{C_\gamma} \int_{\mathbb{R}^2} \sum_{P \in \Gamma} \left| P \right|^{p_1(s_2-s_1-\frac{1}{p_2})+\frac{p_1(d-1)}{q_1(d+1)}} \chi_P(x) \right)^{1/p_1}
\]
\[
= \frac{1}{C_\gamma p_1} \left( \sum_{P \in \Gamma} \left| P \right|^{p_1(s_2-s_1-\frac{1}{p_2})+1+\frac{p_1(d-1)}{q_1(d+1)}} \right)^{1/p_1}
\]
\[
= \frac{1}{C_\gamma p_1} \left( \sum_{P \in \Gamma} \left| P \right|^{p_1(s_2-s_1-\frac{1}{p_2})+\frac{p_1(d-1)}{q_1(d+1)}} \right)^{1/p_1}.
\]
Consider now the case \( \alpha = p_1(s_2-s_1-\frac{1}{p_2})+1 < 1 + \frac{p_1(d-1)}{q_1(d+1)} \Leftrightarrow \gamma = q_1(s_2-s_1-\frac{1}{p_2}) < \frac{d}{d+1} \). We can show that the set \( E_\alpha \) of all \( x \in \bigcup_{P \in \Gamma} P \) for which \( P_x \) does not exist has measure zero. To see this, we remind that \( Q^{j,\ell} = \{ P \in Q_{AB} : |P| = |Q_{j,\ell,k}| = |Q_j| = 2^{-j(d+1)}, j \geq 0 \} \). Then, for all \( i \geq 0 \), \( E_\alpha \subset \bigcup_{j \geq i} \bigcup_{\|\ell\| \leq 2^j} \bigcup_{P \in \Gamma, \ell} P \). Therefore,
\[
|E_\alpha| \leq \sum_{j \geq i} \sum_{\|\ell\| \leq 2^j} \sum_{P} |P|
\]
\[
= \sum_{j \geq i} \sum_{\|\ell\| \leq 2^j} \sum_{P \in \Gamma, \ell} |P|^{\alpha-\frac{p_1(d-1)}{q_1(d+1)}} |P|^{1-\alpha-\frac{p_1(d-1)}{q_1(d+1)}}.
\]
For the lower-Temlyakov property we consider two cases:

From part b), to obtain \( \alpha < \frac{1}{2} \), let \( i \to \infty \) we deduce \( |E_\alpha| = 0 \). We now apply Lemma 3.1, part b), to obtain

\[
\sum_{j \geq i} \sum_{|\ell| \leq 2^j} \sum_{P \in \Gamma_j} \chi_P(x) \left| P \right|^{p_1(s_2-s_1) - \frac{1}{p_2}} \leq C \gamma \sum_{P \in \Gamma} |P|^{p_1(s_2-s_1) - \frac{1}{p_2}} \chi_P(x),
\]

for all \( x \in \cup_{P \in \Gamma} P \setminus E_\alpha \). From this we deduce the upper-Temlyakov property as

\[
\left\| \sum_{P \in \Gamma} \frac{e_P}{\|e_P\|_{L^2}} \right\|_{L^1} \leq C \gamma \left( \sum_{P \in \Gamma} |P|^{p_1(s_2-s_1) - \frac{1}{p_2} + 1} \right)^{1/p_1} = C \gamma \left( \sum_{P \in \Gamma} |P|^{\alpha} \right)^{1/p_1} = C \gamma \left( \nu_\alpha \right)^{1/p_1}.
\]

For the lower-Temlyakov property we consider two cases: \( \gamma \leq 0 \) and \( 0 < \gamma < \frac{d-1}{d+1} \). From \( \gamma = q_1(s_2-s_1 - \frac{1}{p_2}) > 0 \), it follows that \( s_2-s_1 - \frac{1}{p_2} > 0 \Rightarrow p_1(s_2-s_1 - \frac{1}{p_2}) + \frac{d-1}{d+1} > \frac{d-1}{d+1} \). Hence, part a) of Lemma 3.1 yields

\[
\left\| \sum_{P \in \Gamma} \frac{e_P}{\|e_P\|_{L^2}} \right\|_{L^1} \geq \left( \frac{1}{C \gamma} \int_{\mathbb{R}^d} \sum_{P \in \Gamma} |P|^{p_1(s_2-s_1 - \frac{1}{p_2}) + \frac{d-1}{d+1}} \chi_P(x) \right)^{1/p_1} \geq \frac{1}{C \gamma} \left( \sum_{P \in \Gamma} |P|^{p_1(s_2-s_1 - \frac{1}{p_2}) + \frac{d-1}{d+1}} \chi_P(x) \right)^{1/p_1}.
\]

Similarly for \( \gamma = q_1(s_2-s_1 - \frac{1}{p_2}) \leq 0 \), \( s_2-s_1 - \frac{1}{p_2} \leq 0 \Rightarrow p_1(s_2-s_1 - \frac{1}{p_2}) + \frac{d-1}{d+1} \leq \frac{d-1}{d+1} \). Hence, part b) of Lemma 3.1 yields the same lower bound. In both cases we obtain

\[
\left\| \sum_{P \in \Gamma} \frac{e_P}{\|e_P\|_{L^2}} \right\|_{L^1} \geq \frac{1}{C \gamma} \left( \sum_{P \in \Gamma} |P|^{p_1(s_2-s_1 - \frac{1}{p_2}) + \frac{d-1}{d+1}} \chi_P(x) \right)^{1/p_1}
\]

\[
= \frac{1}{C \gamma} \left( \sum_{P \in \Gamma} |P|^{\alpha + \frac{d-1}{d+1}} \right)^{1/p_1} = \frac{1}{C \gamma} \left( \nu_{\alpha + \frac{d-1}{d+1}} \right)^{1/p_1}.
\]
For \( \alpha = p_1(s_2 - \frac{1}{p_2} - s_1 + \frac{1}{p_1}) = 1 + \frac{p_1(d-1)}{q_1(d+1)} \), \( \gamma = q_1(s_2 - s_1 - \frac{1}{p_2}) = \frac{d-1}{d+1} \), the set 
\[ E_{\alpha - \frac{p_1(d-1)}{q_1(d+1)}}(\alpha \in P(x)) \] of all \( x \in \bigcup_{P \in \Gamma} Q \) for which \( P_x \) does not exist has also measure zero. Indeed, since \( \alpha - \frac{p_1(d-1)}{q_1(d+1)} = 1 \),
\[
\left| E_{\alpha - \frac{p_1(d-1)}{q_1(d+1)}}(\alpha \in P(x)) \right| \leq \sum_{j \geq 1} \sum_{\|\ell\| \leq 2j} \sum_{P \in \Gamma_{j,\ell}} |P| = \sum_{j \geq 1} \nu_1(\Gamma^j),
\]
and the last sum tends to zero as \( i \to \infty \), since they are the tails of the convergent sum \( \sum_{j \geq 0} \nu_1(\Gamma^j) \leq \nu_{\alpha - \frac{p_1(d-1)}{q_1(d+1)}}(\Gamma) < \infty \), by hypothesis. This case follows the previous one and the sufficient condition on \( \alpha \) for \eqref{eq:main} to hold is proved.

Suppose now that \eqref{eq:main} holds. Fix \( j \geq 0, |[\ell]| \leq 2^j \), and \( N \in \mathbb{N} \). Let \( \gamma = q_1(s_2 - s_1 - \frac{1}{p_2}) \). Consider the set \( \Gamma_{N}^j = \{Q_{j,\ell,k}(x) : k = (k_1, k_2, \ldots, k_d), 0 \leq k_1, k_2, \ldots, k_d < N \} \) of \( N^d \) disjoint anisotropic parallelograms of area \( |Q_j| = 2^{-j(d+1)} \). On one hand we have that
\[
\left\| \sum_{P \in \Gamma_{j,\ell}^N} \frac{e_P}{\|e_P\|_{f_2}} \right\|_{f_1} = \left( \int_{\mathbb{R}^d} \left[ S_{\Gamma_{j,\ell}^N}^{\gamma}(x) \right]^{p_1/q_1} dx \right)^{1/p_1}.
\]
Since
\[
S_{\Gamma_{j,\ell}^N}^{\gamma}(x) = (2^{-j(d+1)})^{\gamma} \sum_{P_j \in \Gamma_{j,\ell}^N} \chi_{P_j}(x) = (2^{-j(d+1)})^{\gamma} \chi_{A_{-j}B^{-j}(NP_j)}(x),
\]
then,
\[
\left\| \sum_{P \in \Gamma_{j,\ell}^N} \frac{e_P}{\|e_P\|_{f_2}} \right\|_{f_1} = (2^{-j(d+1)})^{\gamma/q_1} \left[ N^d (2^{-j(d+1)}) \right]^{1/p_1}.
\]
On the other hand, we have from the right hand side of \eqref{eq:main} that
\[
\nu_{\alpha - \frac{p_1(d-1)}{q_1(d+1)}}(\Gamma_{j,\ell}^N) = \sum_{P \in \Gamma_{j,\ell}^N} |P|^{\alpha - \frac{p_1(d-1)}{q_1(d+1)}} = N^d 2^{-j(d+1)}(\alpha - \frac{p_1(d-1)}{q_1(d+1)}).
\]
Then, by hypothesis
\[(2^{-j(d+1)})^{\frac{q_1}{p_1}} (2^{-j(d+1)})^{\frac{1}{p_1}} \leq (2^{-j(d+1)})^{\frac{1}{p_1}} (\alpha - \frac{p_1(d-1)}{q_1(d+1)}).
\]
From this we deduce \( \frac{q_1}{p_1} + \frac{1}{p_1} \geq \frac{\alpha}{p_1} - \frac{(d-1)}{q_1(d+1)} \), which implies
\[
\frac{p_1}{p_2}(s_2 - \frac{1}{p_2} - s_1 + \frac{1}{p_1}) + \frac{p_1}{q_1}(d-1) \geq \alpha.
\]
Similarly, from the left hand side of \eqref{eq:main} we obtain \( \frac{1}{p_1}(\alpha + \frac{d-1}{d+1}) \geq \frac{q_1}{p_1} + \frac{1}{p_1} \), which implies
\[
\alpha \geq p_1(s_2 - \frac{1}{p_2} - s_1 + \frac{1}{p_1}) - \frac{d-1}{d+1},
\]
and the proof is complete.

The second main result is:
Theorem 3.4. Let $s_1, s_2 \in \mathbb{R}$, $0 < p_1, p_2 < \infty$, $0 < q_2 \leq \infty$ and

$$u = \left\{ \| e_P \|_{b^{s_2,q_2}_{s_2}(AB)} \right\}_{P \in \mathcal{Q}_{AB}} = \left\{ |P|^{-s_2 + \frac{1}{p_2} - \frac{1}{q_2}} \right\}_{P \in \mathcal{Q}_{AB}}.$$

Then,

$$\left\| \sum_{P \in \Gamma} \frac{e_P}{\| e_P \|_{b^{s_2,q_2}_{s_2}(AB)}} \right\|_{b^{s_1,p_1}_{s_1}(AB)} = (\nu_\alpha(\Gamma))^{1/p_1}, \quad (3.2)$$

for all $\Gamma \subset \mathcal{Q}_{AB}$ such that $\nu_\alpha(\Gamma) < \infty$ if and only if $\alpha = p_1[s_2 - \frac{1}{p_2} - s_1 + \frac{1}{p_1}]$.

Proof. Consider again the simplified notation. It is a straight consequence of the definitions

$$\left\| \sum_{P \in \Gamma} \frac{e_P}{\| e_P \|_{b^{s_2,q_2}_{s_2}(AB)}} \right\|_{b^{s_1,p_1}_{s_1}(AB)} = \left( \sum_{P \in \Gamma} |P|^{-p_1(s_2 - s_1 + \frac{1}{p_1} - \frac{1}{q_2})} \right)^{1/p_1} = \left( \sum_{P \in \Gamma} |P|^\alpha \right)^{1/p_1} = (\nu_\alpha(\Gamma))^{1/p_1}.$$

This result also applies to shear anisotropic inhomogeneous Triebel-Lizorkin spaces since $f_{p_2}^{s_1,p_1}(AB) = b^{s_1,p_1}_{s_1}(AB)$.

4. APPROXIMATION AND INTERPOLATION

Once we have found the lower and upper democracy bounds of the spaces $f_{p}^{s,q}(AB)$ and $b^{s,p}_{p}(AB)$ in (3.1) and (3.2), we can apply the results in Section 2.1.

Proofs for Theorems 4.1, 4.2 are straight applications of Theorem 2.1 to the right and left-hand sides of Theorem 3.3, respectively. Similarly, Theorem 4.3 is consequence of a straight application of Theorem 2.1 to Theorem 3.3 Part ii) in Theorems 4.1, 4.2, 4.3 and 4.4 is the special case when Lorenz spaces can be identified with spaces $b^{s,p}_{p}(AB) = f_{p}^{s,q}(AB)$ via Lemma 2.4.

Theorem 4.1. Let $s_1, s_2 \in \mathbb{R}$, $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \leq \infty$, $q_1 \neq p_1$ and $u = \left\{ \| e_P \|_{f^{s_2,q_2}_{s_2}(AB)} \right\}_{P \in \mathcal{Q}_{AB}}$. Fix $\alpha = p_1(s_2 - \frac{1}{p_2} - s_1 + \frac{1}{p_1})$, $\xi \in (0, \infty)$ and $\mu \in (0, \infty]$.

i) Let $r$ be such that $1/r = \xi/1 + 1/p_1$. The following are equivalent:

a) There exists $C > 0$ such that for all $\Gamma \subset \mathcal{Q}_{AB}$ with $\nu_\alpha(\Gamma) \leq \infty$,

$$\left\| \sum_{P \in \Gamma} \frac{e_P}{\| e_P \|_{f^{s_2,q_2}_{s_2}(AB)}} \right\|_{f^{s_1,q_1}_{-1}(AB)} \leq C \left( \nu_\alpha(\Gamma) \right)^{1/p_1}.$$

b) $\ell^{r,\mu}(u, \nu_{\alpha_{-1}(d-1)}, \mathcal{Q}_{AB}) \hookrightarrow A_\mu^{\xi}(f^{s_1,q_1}_{-1}(AB), \nu_{\alpha_{-1}(d-1)}, \mathcal{Q}_{AB})$.

c) The space $\ell^{r,\mu}(u, \nu_{\alpha_{-1}(d-1)}, \mathcal{Q}_{AB})$ satisfies the next Jackson’s inequality of order $\xi$: there exists $C > 0$ such that

$$\sigma_\nu_{\alpha_{-1}(d-1)}(t, s)_{f^{s_1,q_1}_{-1}(AB)} \leq C t^{-\xi} \left\| s \right\|_{\ell^{r,\mu}(u, \nu_{\alpha_{-1}(d-1)}, \mathcal{Q}_{AB})}.$$

ii) If, additionally, $\mu = r$ and $\gamma = s_1 + \frac{d-1}{q_1(d+1)} + \xi \left[ 1 - (\alpha - \frac{d-1}{q_1(d+1)}) \right]$, the following are equivalent:
a) There exists $C > 0$ such that for all $\Gamma \in \mathcal{Q}_{AB}$ with $\nu_{\frac{q_1(d-1)}{q_1(d+1)}}(\Gamma) < \infty$,

$$
\left\| \sum_{P \in \Gamma} \frac{e_P}{u_P} \right\|_{\ell^{s_1,q_1}_{p_1}(AB)} \leq C \left( \nu_{\frac{q_1(d-1)}{q_1(d+1)}}(\Gamma) \right)^{1/p_1}.
$$

b) $b_{r,r}^{\mu}(AB) = \ell^r(u, \nu_{\frac{q_1(d-1)}{q_1(d+1)}}, \mathcal{Q}_{AB}) \hookrightarrow \mathcal{A}_\mu^{\xi}(f^{s_1,q_1}_{p_1}(AB), \nu_{\frac{q_1(d-1)}{q_1(d+1)}}).$

c) The space $b_{r,r}^{\mu}(AB) = \ell^r(u, \nu_{\frac{q_1(d-1)}{q_1(d+1)}}, \mathcal{Q}_{AB})$ satisfies the next Jackson’s inequality of order $\xi$: there exists $C > 0$ such that

$$
\sigma_{\nu_{\frac{q_1(d-1)}{q_1(d+1)}}}(t, s)_{r,1}^{s_1,q_1}(AB) \leq Ct^{-\xi} \|s\|_{\ell^{s_1,q_1}_{p_1}(AB)}.
$$

**Theorem 4.2.** Let $s_1, s_2 \in \mathbb{R}$, $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \leq \infty$, $q_1 \neq p_1$ and $u = \{\|e_Q\|_{p_2^{q_2}}\}_{Q \in \mathcal{Q}_{AB}}$. Fix $\alpha = p_1(s_2 - \frac{1}{p_2} - s_1 + \frac{1}{p_1})$, $\xi \in (0, \infty)$ and $\mu \in (0, \infty]$.

i) Let $r$ be such that $1/r = \xi + 1/p_1$. The following are equivalent:

a) There exists $C > 0$ such that for all $\Gamma \in \mathcal{Q}_{AB}$ with $\nu_{\alpha + \frac{q_1}{d+1}}(\Gamma) < \infty$,

$$
C \left( \nu_{\alpha + \frac{q_1}{d+1}}(\Gamma) \right)^{1/p_1} \leq \left\| \sum_{P \in \Gamma} \frac{e_P}{u_P} \right\|_{\ell^{s_1,q_1}_{p_1}(AB)}.
$$

b) $\mathcal{A}_\mu^{\xi}(f^{s_1,q_1}_{p_1}(AB), \nu_{\alpha + \frac{q_1}{d+1}}) \hookrightarrow \ell^{r,\mu}(u, \mathcal{A}_\mu^{\xi}(f^{s_1,q_1}_{p_1}(AB), \nu_{\alpha + \frac{q_1}{d+1}})).$

c) The space $\ell^{r,\mu}(u, \mathcal{A}_\mu^{\xi}(f^{s_1,q_1}_{p_1}(AB), \nu_{\alpha + \frac{q_1}{d+1}}))$ satisfies the next Bernstein’s inequality of order $\xi$: there exists $C > 0$ such that

$$
\|s\|_{\ell^{r,\mu}(u, \mathcal{A}_\mu^{\xi}(f^{s_1,q_1}_{p_1}(AB), \nu_{\alpha + \frac{q_1}{d+1}}))} \leq C\xi \|s\|_{\ell^{s_1,q_1}_{p_1}(AB)},
$$

for all $s \in \Sigma_{t,\nu_{\alpha + \frac{q_1}{d+1}}} \cap f^{s_1,q_1}_{p_1}(AB)$.

ii) If, additionally, $\mu = r$ and $\gamma = s_1 - \frac{d-1}{p_1(d+1)} + \xi(1 - (\alpha + \frac{d-1}{d+1}))$, the following are equivalent:

a) There exists $C > 0$ such that for all $\Gamma \in \mathcal{Q}_{AB}$ with $\nu_{\alpha + \frac{q_1}{d+1}}(\Gamma) < \infty$,

$$
C \left( \nu_{\alpha + \frac{q_1}{d+1}}(\Gamma) \right)^{1/p_1} \leq \left\| \sum_{P \in \Gamma} \frac{e_P}{u_P} \right\|_{\ell^{s_1,q_1}_{p_1}(AB)}.
$$

b) $\mathcal{A}_\mu^{\xi}(f^{s_1,q_1}_{p_1}(AB), \nu_{\alpha + \frac{q_1}{d+1}}) \hookrightarrow \ell^{r}(u, \mathcal{A}_\mu^{\xi}(f^{s_1,q_1}_{p_1}(AB), \nu_{\alpha + \frac{q_1}{d+1}})) = b_{r,r}^{\mu}(AB).$

c) The space $b_{r,r}^{\mu}(AB) = \ell^{r}(u, \mathcal{A}_\mu^{\xi}(f^{s_1,q_1}_{p_1}(AB), \nu_{\alpha + \frac{q_1}{d+1}}))$ satisfies Bernstein’s inequality of order $\xi$, that is, there exists $C > 0$ such that

$$
\|s\|_{b_{r,r}^{\mu}(AB)} \leq C\xi \|s\|_{\ell^{s_1,q_1}_{p_1}(AB)},
$$

for all $s \in \Sigma_{t,\nu_{\alpha + \frac{q_1}{d+1}}} \cap f^{s_1,q_1}_{p_1}(AB)$.

In the case $q_1 = p_1$, i.e. $b_{p_1}^{s_1,p_1}(AB) = f^{s_1,p_1}_{p_1}(AB)$, applying Theorem 2.1 to Theorem 3.4 yields the next characterizations.

**Theorem 4.3.** Let $s_1, s_2 \in \mathbb{R}$, $0 < p_1, p_2 \leq \infty$ and $u = \{\|e_Q\|_{b_{p_2}^{q_2}}\}_{Q \in \mathcal{Q}_{AB}}$. Fix $\alpha = p_1(s_2 - \frac{1}{p_2} - s_1 + \frac{1}{p_1})$, $\xi \in (0, \infty)$ and $\mu \in (0, \infty)$.
i) Let \( r \) be such that \( 1/r = \xi + 1/p_1 \). The following are equivalent:
   a) The next \( p_1 \)-Temlyakov property holds:
   \[
   \left\| \sum_{Q \in \Gamma} e_Q \right\|_{b_{p_1}^{s_1:p_1}(AB)} = (\nu_\alpha(\Gamma))^{1/p_1}.
   \]
   b) The next characterization of approximation spaces holds:
   \[
   \ell^{r,p}(u, \nu_\alpha, Q_{AB}) = A^\xi_{\mu}(b_{p_1}^{s_1:p_1}(AB), \nu_\alpha).
   \]
   c) The next Jackson’s inequality of order \( \xi \) holds: there exists \( C > 0 \) such that
   \[
   \sigma_{\nu_\alpha}(t, s)_{b_{p_1}^{s_1:p_1}(AB)} \leq C t^{-\xi} \left\| s \right\|_{\ell^{r,p}(u, \nu_\alpha, Q_{AB})}.
   \]
   The next Bernstein’s inequality of order \( \xi \) holds: there exists \( C' > 0 \) such that
   \[
   \left\| s \right\|_{\ell^{r,p}(u, \nu_\alpha, Q_{AB})} \leq C' t^\xi \left\| s \right\|_{b_{p_1}^{s_1:p_1}(AB)},
   \]
   for all \( s \in \Sigma_{t,\nu_\alpha} \cap b_{p_1}^{s_1:p_1}(AB) \).

ii) If, additionally, \( \mu = r \) and \( \gamma = s_1 + \xi (1 - \alpha) \), the following are equivalent:
   a) The next \( p_1 \)-Temlyakov property holds:
   \[
   \left\| \sum_{Q \in \Gamma} e_Q \right\|_{b_{p_1}^{s_1:p_1}(AB)} = (\nu_\alpha(\Gamma))^{1/p_1}.
   \]
   b) The next characterization of approximation spaces holds:
   \[
   b_{\gamma,r}^r(AB) = \ell^r(u, \nu_\alpha, Q_{AB}) = A^\xi_{\mu}(b_{p_1}^{s_1:p_1}(AB), \nu_\alpha).
   \]
   c) The next Jackson’s inequality of order \( \xi \) holds: there exists \( C > 0 \) such that
   \[
   \sigma_{\nu_\alpha}(t, s)_{b_{p_1}^{s_1:p_1}(AB)} \leq C t^{-\xi} \left\| s \right\|_{b_{\gamma,r}^r(AB)}.
   \]
   The next Bernstein’s inequality of order \( \xi \) holds: there exists \( C' > 0 \) such that
   \[
   \left\| s \right\|_{b_{\gamma,r}^r(AB)} \leq C' t^\xi \left\| s \right\|_{b_{p_1}^{s_1:p_1}(AB)},
   \]
   for all \( s \in \Sigma_{t,\nu_\alpha} \cap b_{p_1}^{s_1:p_1}(AB) \).

We finish with a result on interpolation of shear anisotropic inhomogeneous spaces.

**Theorem 4.4.** Let \( s_1, s_2 \in \mathbb{R} \), \( 0 < p_1, p_2 < \infty \), \( \alpha = p_1(s_2 - \frac{1}{p_2} - s_1 + \frac{1}{p_1}) \) and \( u = \{ |Q|^{-s_2 + \frac{1}{p_2}} \}_{Q \in Q_{AB}} \). Fix \( \xi_i \in (0, \infty) \) and \( \mu_i \in (0, \infty) \) and let \( r_i \) such that \( 1/r_i = \xi_i + 1/p_1 \) for \( i = 0, 1 \). Then,

i) For all \( \theta \in (0, 1) \) and \( \mu \in (0, \infty) \),
\[
(A^\xi_{\mu_0}(b_{p_1}^{s_1:p_1}(AB), \nu_\alpha), A^\xi_{\mu_1}(b_{p_1}^{s_1:p_1}(AB), \nu_\alpha))_{\theta, \mu} = A^\xi_{\mu}(b_{p_1}^{s_1:p_1}(AB), \nu_\alpha),
\]
where \( \xi = (1 - \theta)\xi_0 + \theta\xi_1 \) and
\[
(\ell^{r_0,\mu_0}(u, \nu_\alpha, Q_{AB}), \ell^{r_1,\mu_1}(u, \nu_\alpha, Q_{AB}))_{\theta, \mu} = \ell^{r_\mu}(u, \nu_\alpha, Q_{AB}),
\]
where \( 1/r = (1 - \theta)/r_0 + \theta/r_1 = (1 - \theta)(\xi_0 + \frac{1}{p_1}) + \theta(\xi_1 + \frac{1}{p_1}) = \xi + \frac{1}{p_1} \).
one has\[ \begin{align*}
(\mathbf{b}^{r_0-r_1}(AB), \mathbf{b}^{r_1-r_2}(AB))_{\theta, r} = \mathbf{b}^{r_2-r_1}(AB),
\end{align*} \]
where \( \gamma = (1-\theta)\gamma_0 + \theta\gamma_1 = s_1 + (1-\theta)(\xi_0(1-\alpha)) + \theta(\xi_1(1-\alpha)) = s_1 + \xi(1-\alpha). \)

**Remark 4.5.** It is not hard to see that one can choose \( s_1, s_2, p_1, p_2 \) in Theorem 4.4 such that we have interpolation of shear anisotropic inhomogeneous Besov spaces in point ii) of Theorem 4.4 for any \( r_i \in (0, \infty), \gamma_i = s_1 + \xi_i(1-\alpha) \in \mathbb{R}, \theta \in (0, 1) \) and \( r \in (0, \infty), \) since \( r_i = r_i(\xi, p_i), \) \( u = (s_2, p_2) \) and \( \alpha = \alpha(s_1, p_1, s_2, p_2) \) can be chosen independently. Observe also that \( \gamma \) in point ii) of Theorem 4.4 coincides with that in point ii) of Theorem 4.3.

5. Comments

5.1. Extension to parabolic molecules. With Theorem 2 in [1], about equivalent admissible coverings, it was proved in [19] that the shearlet smoothness spaces \( S^2_{p,q} \) and the curvelet (first and second generation) decomposition spaces \( G^2_{p,q} \) [11] are equivalent with equivalent norms. Since the shear anisotropic inhomogeneous Besov sequence spaces \( b^p_{\alpha,q} \) are based on (almost) the same bounded admissible partition of unity (BAPU) of \( S^2_{p,q}, \) Theorems 4.3 and 4.4 are immediately extended to \( S^2_{p,p} \) and \( G^2_{p,p}, \) previous normalization of the partition and normalization in the smoothness parameters that we explain next.

The BAPU in \( S^2_{p,q} \) and \( G^2_{p,q} \) is done in dyadic cartesian or polar coronae \( |\xi| \sim 2^j \) and, therefore, \( ||\xi|| \lesssim 2^{(j/2)}, \) which means that the number of curvelets or shearlets is doubled at each other scale. But nothing avoids us to develop the highly anisotropic partition for cartesian or polar coronae concentrated in \( |\xi| \sim 2^{2j} \) (4-adic partition) in whose case the number of curvelets or shearlets is doubled at each scale. In fact, Candés and Donoho in [3] develop the second generation of curvelets with the dyadic partition since they “find this choice more consistent with standard literature”.

When, besides of the 4-adic partition, we choose a weight \( 2^{(d+1)\alpha} \) instead of \( 2^{i\beta} \) (as in [1] and in [19]) the spaces \( S^2_{p,q} \) and \( G^2_{p,q} \) coincide with the spaces \( b^p_{\alpha,q}(\mathbf{A}) \) whose natural “weights” are \( |Q|^{-\alpha} = 2^{(d+1)\alpha}, \) i.e. the volumes of the “cubes” in [2,3].

5.2. Shearlet spaces and \( BV(\mathbb{R}^2). \) As mentioned in the Introduction, the inclusions \( B^{1,1}_1(\mathbb{R}^2) \hookrightarrow BV(\mathbb{R}^2) \hookrightarrow B^{1-\varepsilon,1}_1(\mathbb{R}^2) \) were proved in [5]. From the results in [4] or [17] one has \( A^{1}_1(L^2) = B^{1,1}_1. \) Therefore, one conjectures that the approximation error measured in \( L^2 \) of a \( f \in BV \) decays (Jackson’s inequality) as \( N^{-\xi} \) with \( \xi \leq 1/2. \) Nevertheless, the sharp value \( \xi = 1/2 \) is not proved by the results in neither [4] nor [17] but by the more thorough arguments in [5]. One is tempted to think that shearlets might be equally as good as wavelets to approximate a function in \( BV(\mathbb{R}^2). \) However, the shearlet representation is more redundant than that of wavelets since it involves directionality. Let us show what we can get from our results. From Theorem 4.2 in [22] one has \( B^{1-\varepsilon,1}_1(\mathbb{R}^2) \hookrightarrow B^{1-\varepsilon,1}_1(\mathbf{A}) \) for \( \gamma < -\frac{1}{2}(1+2\varepsilon), \) and from Theorem 4.3 in [22] one has \( B^{2,2}_2(\mathbf{A}) \hookrightarrow L^2(\mathbb{R}^2) \) for \( s > 1. \) Hence, Theorem 4.3 shows that the inequality
\[
\sigma_{\nu_\gamma}(t, s)_{\mathbf{B}^{2,2}_2(\mathbf{A})} \lesssim t^{-\xi} ||s||_{\mathbf{B}^{1,1}_1(\mathbf{A})},
\]

with $\xi = \frac{1}{2}$ and $\gamma$ as above, can only occur if we do not impose at the same time $N$-term approximation ($\alpha = 0$) and a norm of the approximation error comparable to the $L^2$ norm ($s > 1$).

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Daniel Vera, Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain. Tel: +34 914974795, Fax: +34 914974889. 
E-mail address: daniel.vera@uam.es