Identities on the $k$-ary Lyndon words related to a family of zeta functions
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Abstract

The main aim of this paper is to investigate and introduce relations between the numbers of $k$-ary Lyndon words and unified zeta-type functions which was defined by Ozden et al.\[15, p. 2785, Definition 3\]. Finally, we give some identities on generating functions for the numbers of $k$-ary Lyndon words and some special numbers and polynomials such as the Apostol-Bernoulli numbers and polynomials, Frobenius-Euler numbers, Euler numbers and Bernoulli numbers.

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1 Introduction

Throughout this paper, we consider the number of $k$-ary Lyndon words of length $n$, $L_k(n)$ as follows [4]:

$$L_k(n) = \frac{1}{n} \sum_{d|n} \mu(n/d) k^d,$$

where the arithmetic function $\mu$ is the Möbius function defined as follows [2]:

$$\mu(1) = 1; \quad \mu(n) = (-1)^k \quad \text{if } a_1 = a_2 = \ldots = a_k = 1,$$

that is, if $n$ is the product of $k$ distinct primes,

$$\mu(n) = 0 \quad \text{if } n \text{ is the product of non distinct primes}.$$

In [12], Lyndon words are studied as standard lexicographic sequences. According to [3, p. 36], a $k$-ary necklace is an equivalence class of $k$-ary strings under rotation. As a representative of such an equivalence class which is taken the smallest in the lexicographical order. A period $n$ necklace representative with $n$ digits is called a Lyndon word.
In addition to counting \( k \)-ary Lyndon words of length \( n \), Equation (1) is well known as Witt’s formula which is used to count the number of monic irreducible polynomials of degree \( n \) over Galois field (cf. [3]) and As we mentioned previously in [10], it is know that there are really interesting connections between this formula and dimension formula for the homogeneous subspaces of the free Lie algebra (cf. [3], [8], [18]) and the rank of the free abelian quotient (cf. [29], [12]). Furthermore, it is also called necklace polynomial (cf. [13]). For further information about \( L_k(n) \) and table including numerical values of \( L_k(n) \), the reader can consult [10], its references and also the references cited in each of these earlier works.

In [10], the authors gave the following explicit formula for the numbers of \( L_k(p) \) for \( p \) is a prime number, \( m \in \mathbb{N} \):

\[
L_k(p^m) = \frac{k^{p^{m-1}} \left(k^{p^{m-1}(p-1)} - 1\right)}{p^m}.
\]

In [10], the authors defined ordinary generating functions for the numbers of \( k \)-ary Lyndon words of prime length \( p \), \( L_k(p) \) for prime \( p \), as follows:

Let \( p \) is a prime number and \( m = 1 \) in the special case of Equation (2)

\[
f_L(t, p) = \sum_{k=1}^{\infty} L_k(p) t^k = \frac{1}{p} \sum_{k=2}^{\infty} (k^p - k) t^k,
\]

(\text{cf. [10]}). However, we modify (3) as follows:

\[
f_{L_y}(t, p) = \sum_{k=0}^{\infty} L_k(p) t^k.
\]

2 Relation between \( f_{L_y}(t, p) \) and a family of zeta functions

In this section, our aim is to give some identities on the generating functions for the numbers of \( k \)-ary Lyndon words of length prime related a family of zeta-type function, the Apostol-Bernoulli numbers and polynomials, Frobenius-Euler numbers and Euler numbers.

The Apostol-Bernoulli numbers, \( B_k(z) \) are defined by means of the following generating functions:

\[
\frac{t}{ze^t - 1} = \sum_{k=0}^{\infty} \frac{B_k(z)}{k!} t^k,
\]

(\text{cf. [7]}). The Apostol Benoulli polynomials, \( B_k(x, z) \) are also defined by means of the following generating functions:

\[
\frac{te^{zt}}{ze^t - 1} = \sum_{k=0}^{\infty} \frac{B_k(x, z)}{k!} t^k,
\]
where \(|t| < 2\pi\) when \(z = 1\) and \(|t| < |\log z|\) when \(z \neq 1\) and \(z \in \mathbb{C}\). From this generating function, one can easily see that

\[
B_m (x, z) = \sum_{j=0}^{m} \binom{m}{j} x^{m-j} B_j (z),
\]

and

\[
B_m (0, z) = B_m (z).
\]

By using the above generating function, several of Apostol Bernoulli numbers and polynomials are given as follows, respectively (cf. [1]):

\[
\begin{align*}
B_0 (z) &= 0, \\
B_1 (z) &= \frac{1}{z-1}, \\
B_2 (z) &= -\frac{2z}{(z-1)^2}, \\
B_3 (z) &= \frac{3(z+1)}{(z-1)^3}, \\
B_4 (z) &= -\frac{4z(z^2 + 4z + 1)}{(z-1)^4}, \\
B_5 (z) &= \frac{5z(z^3 + 11z^2 + 11z + 1)}{(z-1)^5}, \\
B_6 (z) &= -\frac{6z(z^4 + 26z^3 + 66z^2 + 26z + 1)}{(z-1)^6},
\end{align*}
\]

and

\[
\begin{align*}
B_0 (z, x) &= 0, \\
B_1 (z, x) &= \frac{1}{z-1}, \\
B_2 (z, x) &= \frac{x}{z-1} - \frac{2z}{(z-1)^2}, \\
B_3 (z, x) &= \frac{9z(z-1)}{(z-1)^3}x^2 - \frac{6z}{(z-1)^2}x + \frac{3z(z-1)}{(z-1)^3},
\end{align*}
\]

(cf. [1], [2], [13], [15], [22], [26], [27], [28]).

In the above numerical computation of the Apostol-Bernoulli numbers, we observe that all of these numbers are rational functions of parameter \(z\). \(z = 1\) is a pole of these functions.

The following generating function of the unification of the Bernoulli, Euler and Genocchi polynomials, \(\gamma_{n,\beta} (x; k, a, b)\), which was recently defined by Ozden [14] for \(k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\) (\(\mathbb{N} := \{1, 2, 3, \ldots\}\); \(a, b \in \mathbb{R}^+\); \(\beta \in \mathbb{C}\):

\[
\frac{2^{1-k} e^{tx}}{\beta^k e^t - a^k} = \sum_{n=0}^{\infty} \gamma_{n,\beta} (x; k, a, b) \frac{t^n}{n!},
\]

(4)
where \(|t + b \log \left( \frac{a}{b} \right)| < 2\pi; \ x \in \mathbb{R}\) and note that
\[
\mathcal{Y}_{n, \beta} (k, a, b) = \mathcal{Y}_{n, \beta} (0; k, a, b) = \mathcal{Y}_{n, \beta} (1; k, a, b).
\]

The following equation of the unified zeta-type functions \(\zeta_\beta (s, x; k, a, b)\), which was recently defined by Ozden et al.\[15\] p. 2785, Definition 3:
\[
\zeta_\beta (s, x; k, a, b) = \left( -\frac{1}{2} \right)^{k-1} \sum_{n=0}^{\infty} \frac{\beta^{bn}}{a^{b(n+1)} (n + x)^s}.
\] (5)

where \(\beta, s \in \mathbb{C}\) with \(\Re (s) < 1\) and \(|\beta| < 1\) and observe that if \(x = 1\), then
\[
\zeta_\beta (s; k, a, b) = \zeta_\beta (1; k, a, b) = \left( -\frac{1}{2} \right)^{k-1} \sum_{n=1}^{\infty} \frac{\beta^{b(n-1)}}{a^{b(n+1)} n^s}.
\]

Ozden et al.\[15\] p. 2789, Theorem 7] also proved the following relation for \(n \in \mathbb{N}\) and \(k \in \mathbb{N}_0\):
\[
\zeta_\beta (1-n: x; a, b) = (-1)^{k} \frac{(n-1)!}{(n + k - 1)!} \mathcal{Y}_{n+k-1, \beta} (x; k, a, b).\] (6)

Remark 1 Setting \(s \to -m, \ \beta \to t, \ x = 0, \ k = a = b = 1\ in \ (5), we have
\[
\zeta_\beta (-m; 0, 1, 1) = \sum_{n=0}^{\infty} n^m
\] (7)

and also setting \(1-n \to -m, \ \beta \to t, \ x = 0, \ k = a = b = 1\ in \ (6), we get
\[
\zeta_t (-m; 0, 1) = \frac{\mathcal{Y}_{1+m, t} (0; 1, 1)}{1+m}
\] (8)

It is well-know that \(\mathcal{Y}_{n, \beta} (x; 1, 1, 1)\ reduce to the Apostol-Bernoulli polynomials, \(B_n (x, \beta)\) and Apostol-Bernoulli numbers \(B_n (\beta)\), respectively. Thus
\[
\mathcal{Y}_{n, \beta} (0; 1, 1, 1) = B_n (0, \beta) = B_n (\beta).
\] (9)

Hurwitz-Lerch zeta function \(\Phi (z, s, a)\) is defined by (cf.\[28\ p. 121 et seq.], \[28\ p. 194 et seq.]):
\[
\Phi (z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n + a)^s},
\]
where \(a \in \mathbb{C} \setminus \mathbb{Z}_0^\ast; \ s \in \mathbb{C}\) when \(|z| < 1; \ \Re (s) > 1\ when \ |z| = 1. \ One can also easily see that a relation between \(\zeta_\beta (s, x; k, a, b)\) and \(\Phi (z, s, a)\) is given by
\[
\zeta_\beta (s; k, a, b) = \left( -\frac{1}{2} \right)^{k-1} \frac{a^b}{a^{b(n+1)} n^s} \Phi \left( \frac{\beta^{b}}{a^{b}}, s, x \right)
\]
Hurwitz–Lerch zeta function is related to not only Riemann zeta function and the Hurwitz zeta function (see, for details, [28, Chapter 2], see also [27, [22]):

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \Phi(1, s, 1),
\]

and

\[
\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \Phi(1, s, a).
\]

**Remark 2** Let \( n \geq 1 \). Then in [9], [23] and [24], one can see that

\[
E_{n}(z) = \frac{n}{z-1} H_{n-1} \left( \frac{1}{z} \right), \tag{10}
\]

where \( H_{n}(z) \) denotes the Frobenius-Euler numbers which are defined by means of the following generating function:

\[
\frac{1 - z}{e^{t} - z} = \sum_{n=0}^{\infty} H_{n}(z) \frac{t^{n}}{n!}.
\]

for \( z = -1 \), we have \( H_{n}(-1) = E_{n} \) which is defined by

\[
\frac{2}{e^{t} + 1} = \sum_{n=0}^{\infty} E_{n}(z) \frac{t^{n}}{n!},
\]

(cf. [7]-[28]; and the references cited therein).

Now, by combining Equation (7) with Equation (8) and Equation (9), we obtain the following explicit formula of generating functions for the numbers of \( k \)-ary Lyndon words in terms of the Apostol-Bernoulli numbers:

**Theorem 3** Let \( p \) is a prime number. Then

\[
f_{L_{k}}(t, p) = \frac{B_{2}(t)}{2p} - \frac{B_{p+1}(t)}{p(p+1)}
\]

where \( B_{p+1}(z) \) denotes the Apostol-Bernoulli numbers.

**Remark 4** If we substitute \( p = 2 \) into Theorem 3, we arrive at

\[
f_{L_{k}}(t, 3) = \frac{t^{2}}{(1-t)^{3}}
\]

which was given in [10, p. 3].
Remark 5 Observe that degree of polynomial in the numerator of \( f_{L_y}(t, 2) \) is lower than its the denominator. Also, when \( p = 3 \) into Theorem 3 we also arrive at

\[ f_{L_y}(t, 3) = \frac{2t^2}{(t-1)^4}. \]

By combining Theorem 3 with Equation (10), we give the following Remark:

Theorem 6 Let \( p \) is a prime number. Then

\[ f_{L_y}(t, p) = \frac{t \left( \mathcal{H}_1 \left( \frac{1}{t} \right) - \mathcal{H}_p \left( \frac{1}{t} \right) \right)}{p} \]  \hspace{1cm} (11)

where \( \mathcal{H}_p \left( \frac{1}{t} \right) \) denotes the Frobenius-Euler number.

Since \( \mathcal{H}_n (-1) = E_n \), then Equation (11) is reduced to the following Corollary:

Corollary 7

\[ f_{L_y}(-1, p) = \frac{E_p - E_1}{p} \]

where \( E_n \) denotes Euler numbers.

3 Further identities related to Bernoulli numbers

In this section, we give some identities on the numbers of \( k \)-ary Lyndon words related to the Apostol-Bernoulli numbers. Now, we recall definition the Bernoulli polynomials \( B_n (x) \) which are defined by means of the following generating function:

\[ \frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n (x) \frac{t^n}{n!}, \]

where \( |t| < 2\pi \) and also

\[ B_n = B_n (0) \]

which denotes the Bernoulli numbers (cf. [7]-[28]; and the references cited therein).

The sum of the powers of integers is related to the Bernoulli numbers and polynomials:

\[ \sum_{k=0}^{m} k^n = \frac{B_{n+1} (m+1) - B_{n+1}}{n+1} \]  \hspace{1cm} (12)

(cf. [5], [28], [29]; see also the references cited in each of these earlier works).

After applying mobius inversion formula to Equation (3), we have

\[ k^n = \sum_{d|n} dL_k (d). \]  \hspace{1cm} (13)
Hence, summing Equation (13) over all $0 \leq k \leq m$ and combining with Equation (12), we obtain the following Theorem:

**Theorem 8** Let $n \geq 1$. Then

$$\sum_{d|n} \frac{m}{k=0} L_k(d) = \frac{B_{n+1}(m+1) - B_{n+1}}{n+1},$$

where $B_{n+1}(m)$ denotes Bernoulli polynomials.

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