VERY WELL-COVERED GRAPHS WITH LOG-CONCAVE INDEPENDENCE POLYNOMIALS

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ABSTRACT. If $s_k$ equals the number of stable sets of cardinality $k$ in the graph $G$, then $I(G; x) = \sum_{k=0}^{\alpha(G)} s_k x^k$ is the independence polynomial of $G$ (Gutman and Harary, 1983). Alavi, Malde, Schwenk and Erdős (1987) conjectured that $I(G; x)$ is unimodal whenever $G$ is a forest, while Brown, Dilcher and Nowakowski (2000) conjectured that $I(G; x)$ is unimodal for any well-covered graph $G$. Michael and Traves (2003) showed that the assertion is false for well-covered graphs with $\alpha(G) \geq 4$, while for very well-covered graphs the conjecture is still open.

In this paper we give support to both conjectures by demonstrating that if $\alpha(G) \leq 3$, or $G \in \{K_1,n, P_n : n \geq 1\}$, then $I(G^*; x)$ is log-concave, and, hence, unimodal (where $G^*$ is the very well-covered graph obtained from $G$ by appending a single pendant edge to each vertex).

1. INTRODUCTION

Throughout this paper $G = (V,E)$ is a finite, undirected, loopless and without multiple edges graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The set $N(v) = \{u : u \in V, uv \in E\}$ is the neighborhood of $v \in V$, and $N[v] = N(v) \cup \{v\}$. As usual, a tree is an acyclic connected graph, while a spider is a tree having at most one vertex of degree $\geq 3$. $K_n, P_n, K_{n_1,n_2, \ldots, n_p}$ denote, respectively, the complete graph on $n \geq 1$ vertices, the chordless path on $n \geq 1$ vertices, and the complete $p$-partite graph on $n_1+n_2+ \ldots + n_p$ vertices, $n_1, n_2, \ldots, n_p \geq 1$. A graph is called claw-free if it has no induced subgraph isomorphic to $K_{1,3}$. The disjoint union of the graphs $G_1, G_2$ is the graph $G = G_1 \uplus G_2$ having $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. If $G_1, G_2$ are disjoint graphs, then their Zykov sum, $(\Box)$, is the graph $G_1 \boxplus G_2$ with $V(G_1 \boxplus G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \boxplus G_2) = E(G_1) \cup E(G_2) \cup \{v_1v_2 : v_1 \in V(G_1), v_2 \in V(G_2)\}$. In particular, $\square G$ and $\odot G$ denote the disjoint union and Zykov sum, respectively, of $n > 1$ copies of the graph $G$.

A stable set in $G$ is a set of pairwise non-adjacent vertices. The stability number $\alpha(G)$ of $G$ is the maximum size of a stable set in $G$. A graph $G$ is called well-covered if all its maximal stable sets are of the same cardinality, $\bbox{\alpha}$. If, in addition, $G$ has no isolated vertices and its order equals $2\alpha(G)$, then $G$ is very well-covered, $\bbox{V}$. By $G^*$ we mean the graph obtained from $G$ by appending a single pendant edge to each vertex of $G$. Let us remark that $G^*$ is well-covered (see, for instance, $\bbox{[3]}$), and $\alpha(G^*) = n$. In fact, $G^*$ is very well-covered.

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Let $s_k$ be the number of stable sets in $G$ of cardinality $k \in \{0, 1, \ldots, \alpha(G)\}$. The polynomial $I(G; x) = \sum_{k=0}^{\alpha(G)} s_k x^k = 1 + s_1 x + s_2 x^2 + \ldots + s_{\alpha} x^\alpha$, $\alpha = \alpha(G)$ is called the independence polynomial of $G$ (Gutman and Harary, [9]). In [9] was also proved the following equalities.

**Proposition 1.** If $v \in V(G)$, then $I(G; x) = I(G - v; x) + x I(G - N[v]; x)$, and

$$I(G_1 \cup G_2; x) = I(G_1; x) \cdot I(G_2; x), \quad I(G_1 \uplus G_2; x) = I(G_1; x) + I(G_2; x) - 1.$$

A finite sequence of real numbers $(a_0, a_1, a_2, \ldots, a_n)$ is said to be unimodal if there is some $k$, called the mode of the sequence, such that $a_0 \leq \ldots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \ldots \geq a_n$, and log-concave if $a_i^2 \geq a_{i-1} \cdot a_{i+1}$ for $1 \leq i \leq n - 1$. It is known that any log-concave sequence of positive numbers is also unimodal. A polynomial is called unimodal (log-concave) if the sequence of its coefficients is unimodal (log-concave, respectively). For instance, $I(K_n \uplus (\cup 3 K_7); x) = 1 + (n + 21)x + 147x^2 + 343x^3$, $n \geq 1$, is (a) log-concave, if $147^2 - (n + 21) \cdot 343 \geq 0$, i.e., for $1 \leq n \leq 42$ (e.g., $I(K_{42} \uplus (\cup 3 K_7); x) = 1 + 63x + 147x^2 + 343x^3$), (b) unimodal, but non-log-concave, whenever $147^2 - (n + 21) \cdot 343 < 0$ and $n \leq 126$, that is, $43 \leq n \leq 126$ (for instance, $I(K_{123} \uplus (\cup 3 K_7); x) = 1 + 64x + 147x^2 + 343x^3$), (c) non-unimodal for $n \geq 127$ (e.g., $I(K_{127} \uplus (\cup 3 K_7); x) = 1 + 148x + 147x^2 + 343x^3$).

The graph $H = (\cup 3 K_{10}) \uplus K_{3, 3, \ldots, 3}$ is connected and well-covered, but not very well-covered, and its independence polynomial is unimodal, but not log-concave: $I(H; x) = 1 + 390x + 660x^2 + 1120x^3$. The product of two polynomials, one log-concave and the other unimodal, is not always log-concave, for instance, if $G = K_{40} \uplus (\cup 3 K_7)$, $H = K_{110} \uplus (\cup 3 K_7)$, then

$$I(G; x) \cdot I(H; x) = (1 + 61x + 147x^2 + 343x^3) \cdot (1 + 131x + 147x^2 + 343x^3)$$

$$= 1 + 192x + 8285x^2 + 28910x^3 + 87465x^4 + 100842x^5 + 117649x^6.$$

However, the following result, due to Keilson and Gerber, states that:

**Theorem 1.** [3] If $P(x)$ is log-concave and $Q(x)$ is unimodal, then $P(x) \cdot Q(x)$ is unimodal, while the product of two log-concave polynomials is log-concave.

Alavi et al. [11] showed that for any permutation $\sigma$ of $\{1, 2, \ldots, \alpha\}$ there is a graph $G$ with $\alpha(G) = \alpha$ such that $s_{\sigma(1)} < s_{\sigma(2)} < \ldots < s_{\sigma(\alpha)}$. Nevertheless, in [11] it is stated the following (still open) conjecture: $I(F; x)$ of any forest $F$ is unimodal.

In [2] it was conjectured that $I(G; x)$ is unimodal for each well-covered graph $G$. Michael and Traves [17] proved that this assertion is true for $\alpha(G) \leq 3$, but it is false for $4 \leq \alpha(G) \leq 7$. In [15] we showed that for any $\alpha \geq 8$, there exists a connected well-covered graph $G$ with $\alpha(G) = \alpha$, whose $I(G; x)$ is not unimodal. However, the conjecture of Brown et al. is still open for very well-covered graphs. In [14] an infinite family of very well-covered graphs with unimodal independence polynomials is described. We also showed that $I(G^*; x)$ is unimodal for any $G^*$ whose skeleton $G$ has $\alpha(G) \leq 4$ (see [14]).

Michael and Traves [17] formulated (and verified for well-covered graphs with stability numbers $\leq 7$) the following so-called "roller-coaster" conjecture: for any permutation $\pi$ of the set $\{\lfloor \alpha/2 \rfloor, \lceil \alpha/2 \rceil + 1, \ldots, \alpha\}$, there exists a well-covered graph $G$, with $\alpha(G) = \alpha$, whose sequence $(s_0, s_1, \ldots, s_{\alpha})$ satisfies the inequalities $s_{\pi(\lfloor \alpha/2 \rfloor)} < s_{\pi(\lceil \alpha/2 \rceil + 1)} < \ldots < s_{\pi(\alpha)}$. Recently, Matchett [16] showed that this conjecture is true for well-covered graphs with stability numbers $\leq 11$.
Recall also the following statement, due to Hamidoune.

**Theorem 2.** \cite{[13]} The independence polynomial of a claw-free graph is log-concave.

As a consequence, we deduce that for any \( \alpha \geq 1 \), there exists a tree \( T \), with \( \alpha(T) = \alpha \) and whose \( I(T; x) \) is log-concave, e.g., the chordless path \( P_{2\alpha} \).

In this paper we show that the independence polynomial of \( G^* \) is log-concave, whenever: \( \alpha(G) \leq 3 \), or \( G^* \) is a well-covered spider (i.e., \( G = K_{1,n}, n \geq 1 \)), or \( G^* \) is a centipede (that is, \( G = P_n, n \geq 1 \)).

2. Results

**Lemma 1.** If \( G \) is a graph of order \( n \geq 1 \) and \( \alpha(G) = \alpha \), then \( \alpha \cdot s_\alpha \leq n \cdot s_{\alpha-1} \).

**Proof.** Let \( H = (\mathcal{A}, \mathcal{B}, \mathcal{W}) \) be the bipartite graph defined as follows: \( X \in \mathcal{A} \iff X \) is a stable set in \( G \) of size \( \alpha - 1 \), then \( Y \in \mathcal{B} \iff Y \) is a stable set in \( G \) of size \( \alpha(G) \), and \( XY \in \mathcal{W} \iff X \subseteq Y \) in \( G \). Since any \( Y \in \mathcal{B} \) has exactly \( \alpha(G) \) subsets of size \( \alpha - 1 \), it follows that \( |\mathcal{W}| = \alpha \cdot s_\alpha \). On the other hand, if \( X \in \mathcal{A} \), then \( |\{X \cup \{y\} : X \cup \{y\} \in \mathcal{B}\}| \leq n - |X| = n - \alpha + 1 \). Hence, any \( X \in \mathcal{A} \) has at most \( n - \alpha + 1 \) neighbors. Consequently, \( |\mathcal{W}| = \alpha \cdot s_\alpha \leq (n - \alpha + 1) \cdot s_{\alpha-1} \), and this leads to \( \alpha \cdot s_\alpha \leq n \cdot s_{\alpha-1} \). \( \square \)

In \cite{[13]} it was established the following result:

**Theorem 3.** \cite{[13]} If \( G \) is a graph of order \( n \geq 1 \) and \( I(G; x) = \sum_{k=0}^{\alpha(G)} s_kx^k \), then

\[
I(G^*; x) = \sum_{k=0}^{\alpha(G^*)} t_kx^k, \quad t_k = \sum_{j=0}^{k} s_{j} \cdot \binom{n-j}{n-k}, \quad 0 \leq k \leq \alpha(G^*) = n.
\]

In \cite{[14]} it was shown that \( I(G^*; x) \) is unimodal for any graph \( G \) with \( \alpha(G) \leq 4 \). Now we partially strengthen this assertion to the following result.

**Theorem 4.** If \( G \) is a graph with \( \alpha(G) \leq 3 \), then \( I(G^*; x) \) is log-concave.

**Proof.** Suppose that \( \alpha(G) = 3 \). Then \( n = |V(G)| \geq 3 \) and \( I(G; x) = 1 + nx + s_2x^2 + s_3x^3 \). According to Theorem \cite{[13]} for \( 2 \leq k \leq n - 1 \), we obtain: \( t_k = \binom{n}{k} + n\binom{n-1}{k-1} + s_2\binom{n-2}{k-2} + s_3\binom{n-3}{k-3} \). Therefore,

\[
t_k^2 - t_{k-1}t_{k+1} = A_0 + n^2A_1 + s_2^2A_2 + s_3^2A_3 + nA_{01} + s_2A_{02} + s_3A_{03} + ns_2A_{12} + ns_3A_{13} + s_2s_3A_{23},
\]

and all \( A_i \geq 0, 0 \leq i \leq 3 \), where

\[
A_0 = \binom{n}{k}^2 - \binom{n}{k-1}\binom{n}{k+1}, \quad A_1 = \binom{n-1}{k-1}^2 - \binom{n-1}{k-2} \binom{n-1}{k}, \quad A_2 = \binom{n-2}{k-2}^2 - \binom{n-2}{k-3} \binom{n-2}{k-1}, \quad A_3 = \binom{n-3}{k-3}^2 - \binom{n-3}{k-4} \binom{n-3}{k-2},
\]
because the sequence \( \{ t_{n}^{2} \} \) is log-concave. Based on notation \( b = \binom{n}{k}^{2} \), we get

\[
A_{01} = \frac{2k (n + 1) b}{n (n - k + 1) (k + 1)}, \quad A_{02} = \frac{2kb (k - 2)n + 2k - 1}{(n - k + 1) (k + 1) (n - 1) n},
\]

\[
A_{03} = \frac{2kb (k - 1) ((k - 5)n + 4k - 2)}{(n - k + 1) (n - 2) (n - 1) n}, \quad A_{12} = \frac{2kb (k - 1)}{n (n - k + 1)},
\]

\[
A_{13} = \frac{2kb (k - 1) ((k - 3)n + k)}{(n - 2) (n - 1) n^{2} (n - k + 1)}, \quad A_{23} = \frac{2k^{2}b (k - 1) (k - 2)}{(n - 2) (n - 1) n^{2} (n - k + 1)},
\]

and all \( A_{ij} \geq 0 \) for \( k \geq 5 \). Hence, we must check that \( t_{k}^{2} - t_{k-1}t_{k+1} \geq 0 \) for \( k \in \{1, 2, 3, 4\} \). By Theorem 3 we obtain:

\[
t_{0} = 1, \quad t_{1} = 2n, \quad t_{2} = 3n(n - 1)/2 + s_{2}, \quad t_{3} = \frac{2n(n - 1)(n - 2)}{3} + (n - 2)s_{2} + s_{3},
\]

\[
t_{4} = \frac{5}{24} (n - 3)(n - 1)(n - 2) + \frac{1}{2} s_{2} (n - 2)(n - 3) + s_{3} n - 3 s_{3},
\]

\[
t_{5} = (n - 4)(n - 3) \left[ \frac{1}{20} n(n - 1)(n - 2) + \frac{1}{6} s_{2}(n - 2) + \frac{1}{2} s_{3} \right].
\]

Consequently, it follows \( t_{k}^{2} - t_{0}t_{1} = (n^{2} + 2(2n^{2} - s_{2}))/2 > 0 \). We also deduce

\[
t_{2}^{2} - t_{1}t_{3} = \frac{1}{12} (11n + 5)(n - 1)n^{2} + s_{2}^{2} + ns_{2} + n(ns_{2} - 2s_{3}) \geq 0,
\]

since \( 3s_{3} \leq ns_{2} \) is true according to Lemma 4.

Now, simple calculations lead us to

\[
144(t_{2}^{2} - t_{2}t_{4}) = (19n + 7)n^{2} - (n - 2)^{2} + (54n + 30)n(n - 1)(n - 2) s_{2}
\]

\[-24(n - 11) n(n - 1) s_{3} + 72n(n - 3) s_{2}^{2} + 144(s_{2}^{3} + (n - 1) s_{2} s_{3} + s_{3}^{2}).
\]

Let us notice that \( n(n - 1)((54n + 30)(n - 2) s_{2} - 24(n - 11) s_{3}) \geq 0 \), because Lemma 4 implies the inequality \( 54ns_{2} \geq 24s_{3} \). Hence, we infer that \( t_{3}^{2} - t_{2}t_{4} \geq 0 \), whenever \( n \geq 3 \).

Further, we have

\[
2880(t_{3}^{2} - t_{2}t_{4}) = (29n^{8} - 252n^{7} - 108n^{5} + 818n^{6} + 12n^{3} - 1200n^{5} + 701n^{4}) +
\]

\[+ (672n + 2680n^{4} - 2520n^{5} + 64n^{2} + 136n^{6} - 1032n^{5}) s_{2} +
\]

\[+ (-3840n^{3} + 8400n^{2} - 4896n + 96n^{5} + 240n^{4}) s_{3} +
\]

\[+ (240n^{4} - 1920n^{3} + 5520n^{2} - 6720n + 2880) s_{2}^{2} +
\]

\[+ (10560n - 5760n^{2} + 960n^{3} - 5760) s_{3} s_{2} + (8640 + 1440n^{2} - 7200n) s_{3}^{2}.
\]

\[= (29n + 9)n^{2}(n - 1)^{2} (n - 2)^{2} (n - 3) +
\]

\[+ (136n + 56)n(n - 1)(n - 2)^{2}(n - 3) s_{2} +
\]

\[+ (96n + 816)n(n - 1)(n - 2)(n - 3) s_{3} + 240(n - 1)(n - 2)^{2}(n - 3) s_{2}^{2} +
\]

\[+ 960(n - 1)(n - 2)(n - 3) s_{2} s_{3} + 1440(n - 2)(n - 3) s_{3}^{2} \geq 0.
\]

Consequently, \( t_{k}^{2} - t_{k-1}t_{k+1} \geq 0 \), for \( 1 \leq k \leq n - 1 \), i.e., \( I(G^{*}; x) \) is log-concave.

The log-concavity for the cases \( \alpha(G) \in \{1, 2\} \) can be validated in a similar way, by observing that either \( s_{2} = s_{3} = 0 \) or only \( s_{3} = 0 \).

Since \( \alpha(K_{1,n}) = n, \alpha(P_{n}) = \lceil n/2 \rceil \), Theorem 3 is not useful in proving that \( I(K_{1,n}^{*}; x), I(P_{n}^{*}; x) \) are log-concave, as soon as \( n \) is sufficiently large. In 111, 122 we proved that \( I(K_{1,n}^{*}; x), I(W_{n}; x) \) are unimodal. Here we are strengthening these results.
The well-covered spider $S_n$, $n \geq 2$, has $n$ vertices of degree 2, one vertex of degree $n+1$, and $n+1$ vertices of degree 1 (see Figure 1). In fact, it is easy to see that $S_n = K^*_1,n$, $n \geq 2$.

![Figure 1. Well-covered spiders: $K_1, K_2, P_4, S_6$, and the centipede $W_n$.](image)

**Proposition 2.** [12] The independence polynomial of any well-covered spider is unimodal, moreover, $I(S_n; x) = (1 + x) \cdot \sum_{k=0}^{n} \left[ \binom{n}{k} \cdot 2^k + \binom{n}{k} \cdot 2^k \right] \cdot x^k$, $n \geq 2$, and its mode is unique and equals $1 + (n - 1) \text{mod} 3 + 2\left(\lceil n/3 \rceil - 1\right)$.

In [2] it was shown that $I(G; x)$ of any graph $G$ with $\alpha(G) = 2$ has real roots, and, hence, it is log-concave, according to Newton’s theorem (stating that if a polynomial with positive coefficients has only real roots, then its coefficients form a log-concave sequence). However, Newton’s theorem is not useful in solving the conjecture of Alavi et al., even for the particular case of very well-covered trees, since, for instance, $I(S_3; x) = 1 + 8x + 21x^2 + 23x^3 + 9x^4$ has non-real roots.

**Theorem 5.** The independence polynomial of any well-covered spider is log-concave.

**Proof.** Since $I(G; x)$ is log-concave for any graph $G$ with $\alpha(G) \leq 2$, we consider only well-covered spiders $S_n$ with $n \geq 2$. According to Proposition 2

\[
I(S_n; x) = (1 + x) \cdot \sum_{k=0}^{n} \left[ \binom{n}{k} \cdot 2^k + \binom{n}{k} \cdot 2^k \right] \cdot x^k = (1 + x) \cdot P(x).
\]

It is sufficient to prove that $P(x)$ is log-concave, because, further, Theorem 4 implies that $I(S_n; x)$ is log-concave, as well. Let us denote $c_k = \binom{n}{k} \cdot 2^k + \binom{n}{k} \cdot 2^k$, $0 \leq k \leq n$.

Firstly, we notice that $c_k^2 - c_{k-1} \cdot c_{k+1} = \left(\frac{(n-1)^2}{n-1}\right) \cdot \left(\frac{(n-1)}{n-1}\right) \cdot \left(\frac{(n-1)}{n-1}\right) + \left(\frac{(n-1)^2}{n-1}\right) \cdot \left(\frac{(n-1)}{n-1}\right) \cdot \left(\frac{(n-1)}{n-1}\right) + \left(\frac{(n-1)^2}{n-1}\right) \cdot \left(\frac{(n-1)}{n-1}\right) \cdot \left(\frac{(n-1)}{n-1}\right)

Clearly, $\binom{n-k-1}{k}^2 - \binom{n-k-1}{k} \cdot \binom{n-k-1}{k+1} \geq 0$, since the sequence of binomial coefficients is log-concave, and $n(2n+2)2^k - k^2 (n+3) \geq 0$, because $n \cdot 2^k \geq k^2$ holds for any $k \in \{2, \ldots, n-1\}$. Thus, $c_k^2 - c_{k-1} \cdot c_{k+1} \geq 0$, for any $k \in \{1, 2, \ldots, n-1\}$.

The edge-join of two disjoint graphs $G_1, G_2$, is the graph $G_1 \odot G_2$ obtained by adding an edge joining a vertex from $G_1$ to a vertex from $G_2$. If both vertices are of degree at least two, then $G_1 \odot G_2$ is an internal edge-join of $G_1, G_2$. By $\triangle_n$ we mean the graph $\odot nK_3 = (\odot(n-1)K_3) \odot K_3$, $n \geq 1$ (see Figure 2).
Theorem 6. [10] A tree \( T \) is well-covered if and only if \( T \) is a well-covered spider, or \( T \) is the internal edge-join of a number of well-covered spiders.

A centipede is a well-covered tree defined by \( W_n = P^*_n, n \geq 1 \) (see Figure 3). For example, \( W_1 = K_2, W_2 = P_4, W_3 = S_2 \).

Theorem 7. The independence polynomial of any centipede is log-concave.

Proof. We show, by induction on \( n \geq 1 \), that

\[
I(W_{2n};x) = (1 + x)^n \cdot I(\triangle_n; x), \quad I(W_{2n+1};x) = (1 + x)^n \cdot I(\triangle_n \circ K_2; x),
\]

(for another proof of these equalities, see [12]).

For \( n = 1 \), the assertion is true, because

\[
I(W_2; x) = 1 + 4x + 3x^2 = (1 + x)(1 + 3x) = (1 + x) \cdot I(\triangle_1; x),
\]

\[
I(W_3; x) = 1 + 6x + 10x^2 + 5x^3 = (1 + x) \cdot I(\triangle_1 \circ K_2; x).
\]

Assume that the formulae are true for \( k \leq 2n + 1 \). By Proposition 1 we get:

\[
I(W_{2n+2}; x) = I(W_{2n+2} - b_{2n+1}; x) + x \cdot I(W_{2n+2} - N[b_{2n+1}]; x)
\]

\[
= (1 + x) (1 + 2x) \cdot I(W_{2n}; x) + x(1 + x)^2 \cdot I(W_{2n-1}; x)
\]

\[
= (1 + x)^{n+1} \cdot \{ I(K_2; x) \cdot I(\triangle_n; x) + x \cdot I(\triangle_n \circ K_2; x) \}. 
\]

On the other hand, if \( v \) is the vertex of degree 3 in the last triangle of \( \triangle_{n+1} \) (see Figure 3), then \( I(\triangle_{n+1}; x) = I(K_2; x)I(\triangle_n; x) + xI(\triangle_{n-1} \circ K_2; x) \), according to Proposition 1. In other words, \( I(W_{2n+2}; x) = (1 + x)^{n+1} \cdot I(\triangle_{n+1}; x) \).

Similarly, again by Proposition 1 we obtain:

\[
I(W_{2n+3}; x) = I(W_{2n+3} - b_{2n+2}; x) + x \cdot I(W_{2n+3} - N[b_{2n+2}]; x)
\]

\[
= (1 + x) (1 + 2x) \cdot I(W_{2n+1}; x) + x(1 + x)^2 \cdot I(W_{2n}; x)
\]

\[
= (1 + x)^{n+1} \cdot \{ I(K_2; x) \cdot I(\triangle_n \circ K_2; x) + x(1 + x) \cdot I(\triangle_n; x) \}.
\]
On the other hand, if \( v \) is the vertex of degree 3 belonging to the last triangle of \( \Delta_{n+1} \circ K_2 \) (see Figure 3(b)) and adjacent to one of the vertices of \( K_2 \), we have

\[
I(\Delta_{n+1} \circ K_2; x) = I(\Delta_{n+1} \circ K_2 - v; x) + xI(\Delta_{n+1} \circ K_2 - N[v]; x)
\]

\[
= I(K_2; x) \cdot I(\Delta_n \circ K_2; x) + x(1 + x) \cdot I(\Delta_n; x)).
\]

In other words,

\[
I(W_{2n+3}; x) = (1 + x)^{n+1} \cdot I(\Delta_{n+1} \circ K_2; x).
\]

While Theorem 2 assures that \( I(\Delta_n; x), I(\Delta_n \circ K_2; x) \) are log-concave, finally Theorem 1 implies that \( I(W_n; x) \) is log-concave, as claimed. \( \square \)

**Corollary 1.** (i) If the graph \( H \) has as connected components well-covered spiders/centipedes and/or graphs with stability number \( \leq 2 \), and/or claw-free graphs, and/or graphs that may be represented as \( G^* \) whose \( G \) has \( \alpha(G) \leq 3 \), then its independence polynomial \( I(H; x) \) is log-concave.

(ii) If \( H_n \in \{S_n, W_n\} \), then the independence polynomial of \( \psi mH_n \) is log-concave, for any \( m \geq 2, n \geq 1 \).

**Proof.** (i) Let \( G_i, 1 \leq i \leq m \), be the connected components of \( G \). According to Theorems 2 and 4 and any \( I(G_i; x) \) is log-concave. Further, Theorem 1 implies that \( I(G; x) \) is also log-concave, as \( I(G; x) = I(G_1; x) \cdot \ldots \cdot I(G_m; x) \).

(ii) Since \( I(H_n; x) \) is log-concave, and \( I(\psi mH_n; x) = m \cdot I(H_n; x) - (m - 1) \), it follows that \( I(\psi mH_n; x) \) is log-concave, as well. \( \square \)

3. Conclusions

In this paper we showed that for any \( \alpha \), there is a very well-covered tree \( T \) with \( \alpha(T) = \alpha \), whose independence polynomial \( I(T; x) \) is log-concave. We conjecture that the independence polynomial of any (well-covered) forest is log-concave.

![Figure 4. Two (very) well-covered trees.](image)

In 1990, Hamidoune [7] conjectured that the independence polynomial of any claw-free graph has only real roots. Recently, Chudnovsky and Seymour [3] validated this conjecture. Consequently, \( I(P_n; x) \) has all the roots real. Moreover, the roots of \( I(W_n; x) \) are real (see the proof of Theorem 1).

For general (very well-covered) spiders/trees the structure of the roots of the independence polynomial is more complicated. For instance, the independence polynomial of the claw graph \( I(K_{1,3}; x) = 1 + 4x + 3x^2 + x^3 \) has non-real roots. Figure 4 provides us with some more examples:

\[
I(T_1; x) = (1 + x)^2(1 + 2x)(1 + 6x + 7x^2),
\]

\[
I(T_2; x) = (1 + x)(1 + 7x + 14x^2 + 9x^3),
\]

where only \( I(T_1; x) \) has all the roots real. It seems to be interesting to characterize (well-covered) trees whose independence polynomials have only real roots.
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