Addendum: Triton and hypertriton binding energies calculated from $SU_6$ quark-model baryon-baryon interactions

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Previously we calculated the binding energies of the triton and hypertriton, using an $SU_6$ quark-model interaction derived from a resonating-group method of two baryon clusters. In contrast to the previous calculations employing the energy-dependent interaction kernel, we present new results using a renormalized interaction, which is now energy independent and reserves all the two-baryon data. The new binding energies are slightly smaller than the previous values. In particular the triton binding energy turns out to be 8.14 MeV with a charge-dependence correction of the two-nucleon force, 190 keV, being included. This indicates that about 350 keV is left for the energy which is to be accounted for by three-body forces.

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The QCD-inspired spin-flavor $SU_6$ quark model (QM) for the baryon-baryon interaction, developed by the Kyoto-Niigata group, has achieved accurate descriptions of available $NN$ and $YN$ experimental data. In particular, the most recent model fss2 gives in the $NN$ sector accuracy comparable to modern realistic meson-exchange potentials. Since the QM description of the short-range part is quite different from that of meson-exchange potentials, it is interesting to apply these interactions to calculate properties of three-baryon systems, namely the triton ($^3\text{H}$) and hypertriton ($^3\Lambda\text{H}$). For this purpose, we developed in Ref. a three-cluster equation which employs energy-dependent two-cluster quark-exchange kernels of the resonating-group method (RGM). Solving this equation, we obtained the following results for fss2: the triton binding energy, $B_t = 8.519$ MeV, and the $\Lambda$ separation energy, $B_\Lambda = 289$ keV for $^3\Lambda\text{H}$. We call this treatment $\varepsilon K$ prescription in the following.

Recently, an important progress is made to apply the energy-independent RGM kernel to the $3\alpha$ system through a standard procedure of eliminating the energy dependence of the RGM kernel. This renormalized kernel naturally gives different results in the application of the QM baryon-baryon interactions to many-body systems. We will report these new results in this paper.

In this formulation, the two-cluster RGM kernel $V_{\text{RGM}}$ is expressed as an energy-independent renormalized RGM kernel

$$V_{\text{RGM}} = V_D + G + W, \quad (1)$$

where $V_D$ is the direct potential, and $G$ is the sum of the exchange kinetic-energy and interaction kernels. The kernel $W$ is the term which appears through the elimination of the energy-dependence, and it is given by

$$W = \frac{1}{\sqrt{1 - K}} h \frac{1}{\sqrt{1 - K}} \Lambda - h. \quad (2)$$

Here $K$ is the exchange normalization kernel, $h$ denotes $h_0 + V_D + G$ with $h_0$ being the relative kinetic-energy operator, and $\Lambda = 1 - |u\rangle\langle u|$ is a two-cluster Pauli projection operator, where $|u\rangle$ is a Pauli forbidden state satisfying $K|u\rangle = |u\rangle$. An advantage of this procedure is that the two-cluster RGM equation takes the form of the usual Schrödinger equation in the allowed model space, and the relative wave function is properly normalized. This Schrödinger-type equation for the relative wave function gives the same asymptotic behavior as the original RGM equation, thus yielding the same phase shifts and physical observables for the two-cluster system. The difference between the previous energy-dependent RGM kernel, $V_{\text{RGM}}(\varepsilon) = V_D + G + \varepsilon K$, and $V_{\text{RGM}}$ in Eq. is essentially a replacement of $\Lambda(\varepsilon K)\Lambda$ with $W$. Here $\varepsilon$ is the two-cluster relative energy measured from its threshold, and it was determined in a self-consistent procedure in the previous $\varepsilon K$ treatment.

In the usual notation, $\alpha$, $\beta$, and $\gamma$, for three independent pairs of two-cluster subsystems, the three-cluster...
equation to be solved reads

\[ P \left[ E - H_0 - V_{\alpha}^{\text{RGM}} - V_{\beta}^{\text{RGM}} - V_{\gamma}^{\text{RGM}} \right] \Psi = 0 , \]

where \( E \) is the three-body energy, \( H_0 \) is the free three-body kinetic-energy operator, and \( V_{\alpha}^{\text{RGM}} \) stands for the RGM kernel in Eq. (1) for the \( \alpha \)-pair, etc. The three-body operator \( P \) projects on the Pauli-allowed space with a proper symmetry of clusters, and it is constructed from the orthogonality constraint that each pair of two-cluster subsystems is free from any Pauli forbidden states \([10, 11, 12]\). This definition of the three-cluster Pauli-allowed space may not be exactly equivalent to the standard definition given by the three-cluster normalization kernel. We however employed this orthogonality condition in the \( \varepsilon K \) prescription. See \([2, 3]\) for detail. We use the same definition of \( P \) in this paper as well.

In the practical applications of the QM baryon-baryon interactions to the Faddeev formalism, it is convenient to calculate \( W \) in Eq. (2) in the form of

\[ W = \mathcal{K}h + h\mathcal{K} + \mathcal{K}h\mathcal{K} \]

with

\[ \mathcal{K} = \Lambda \left( \frac{1}{1 - \Lambda} - 1 \right) \Lambda . \]

The kernel \( \mathcal{K} \) is calculated in the momentum representation, by using properties of exchange normalization kernels. There exists no Pauli forbidden state in the \( NN \) interaction (\( \Lambda = 1 \)), while one (0s) harmonic-oscillator Pauli forbidden state appears in the \( \Lambda N - \Sigma N \) system.

Table I lists three-nucleon bound state properties predicted by fss2 and FSS. The \( np \) interaction is employed in the isospin basis. The momentum discretization points for solving the Faddeev equations are the same as in Ref. [4]. The finite size corrections of the nucleons are made through \([13]\)

\[ \langle r^2 \rangle_{3\text{H}} = \left[ R_C^{(3\text{H})} \right]^2 + (0.8750)^2 + 2(-0.1161) , \]

\[ \langle r^2 \rangle_{3\text{He}} = \left[ R_C^{(3\text{He})} \right]^2 + (0.8750)^2 + \frac{1}{2}(-0.1161) , \]

where \( R_C^{(3\text{H})} \) stands for the squared charge radius for the point nucleons. In order to calculate \( R_C^{(3\text{H})} \) from the Faddeev components, we have improved the previous method using the power series expansion of the charge form factors. We have used the second-order numerical differentiation of the momentum variables in the fifth-order spline interpolation formula, based on the calculational scheme given in Ref. [14]. This approach yields a stable value for the rms radius within four digits, while in the previous method even the third digit fluctuates. In the present calculation, the Coulomb force and the relativistic correction terms \([17]\) of the charge current operator are entirely neglected.

### TABLE I: Three-nucleon bound state properties predicted by fss2 and FSS, using the energy-independent renormalized RGM kernels. The column \( n \) implies the number of three-nucleon channels, including the two-nucleon systems up to the total angular-momentum \( J \), \( E(3\text{H}) \) the ground state energy, and \( \sqrt{\langle r^2 \rangle_{3\text{H}}} \) is the charge rms radius for \( ^3\text{H} \) \( ^3\text{He} \) with the proton and neutron size corrections in Eq. (5). The value \( \Delta E \) is the energy change from the \( \varepsilon K \) prescription to the present approach.

| model | \( n \) | \( E(3\text{H}) \) (MeV) | \( \Delta E \) (keV) | \( \sqrt{\langle r^2 \rangle_{3\text{H}}} \) (fm) | \( \sqrt{\langle r^2 \rangle_{3\text{He}}} \) (fm) |
|-------|------|------------------|------------------|------------------|------------------|
| fss2  | 2 (S) | -7.952 -145      | 1.80             | 1.95             |
|       | 5 (S, D) | -8.261 -72    | 1.76             | 1.92             |
|       | 10 (J \( \leq 1 \)) | -7.962 55     | 1.77             | 1.95             |
| FSS   | 18 (J \( \leq 2 \)) | -8.228 211   | 1.75             | 1.93             |
|       | 26 (J \( \leq 3 \)) | -8.313       | 1.75             | 1.92             |
|       | 34 (J \( \leq 4 \)) | -8.322 192   | 1.75             | 1.92             |
|       | 42 (J \( \leq 5 \)) | -8.326       | 1.75             | 1.92             |
|       | 50 (J \( \leq 6 \)) | -8.326 193   | 1.75             | 1.92             |
| exp't | -8.482 | 1.755(86)       | 1.959(30)        | 1.9642(11) |

\( ^a \)Ref. [13]
\( ^b \)Ref. [14]

The final fss2 prediction for the triton binding energy is \(-8.326 \text{ MeV} \), which is 193 keV high, compared with the previous value \(-8.519 \text{ MeV} \). Since the experimental value is \( E_{\text{exp}}(3\text{H}) = -8.482 \text{ MeV} \), the calculated value is higher than the experiment by 156 keV. In fact, we have to take into account the effect of the charge dependence of the two-nucleon force, which is estimated to result in the energy loss by about 190 keV \([18]\). Therefore our calculation concludes that 346 keV, namely, about 350 keV is still missing. In order to compare with the results by the \( \varepsilon K \) prescription, we show the energy loss from the previous results in the column \( \Delta E \) in Table I. In both fss2 and FSS cases, we note that the 5-channel energy is already close to the converged value in the present approach, whereas the convergence was rather slow in the \( \varepsilon K \) calculation. We will see that this is also the case in the hypertriton calculation.

We find that the expectation value, \( \varepsilon_{NN} = \langle P\Psi | h_{\alpha} + V_{RGM}^{\alpha} | P\Psi \rangle / \langle P\Psi | P\Psi \rangle \), for the triton is not very different from the previous value of the \( \varepsilon K \) prescription. For example, in the full 50 channel calculations with fss2, the
The previous result is $\varepsilon_{NN} = 4.492$ MeV, which is compared with the present result $\varepsilon_{NN} = 4.301$ MeV. The charge rms radii hardly change from the previous values.

For a realistic calculation of the $^3$H binding energy, it is important to use an $NN$ interaction which reproduces both the proper $D$-state probability $P_D$ of the deuteron and the effective range parameters of the $^1S_0$ scattering.

We show in Fig. 1 the updated plot of the fss2 and FSS values in the $B_t = -E(^3\text{H})$ vs. $P_D$ diagram. We find that fss2 gives a larger binding energy than the modern realistic meson-exchange potentials like Bonn-C and AV18, while the result of FSS is not very far from that of Bonn-C. It is interesting to note that our QM points are apparently off the line on which the data points of the modern meson-exchange potentials fall. The five-channel calculation of the model QCM-A by Takeuchi et al. gives almost the same result as Bonn-C.

The results of the hypertriton Faddeev calculations are listed in Table III. The $\Lambda$ separation energy of the hypertriton is $B_\Lambda = 262$ keV for fss2, which is by 27 keV less than the $\varepsilon K$ value, 289 keV. The corresponding FSS values are 790 keV in the present approach vs. 878 keV in the $\varepsilon K$ prescription. The difference is 88 keV. So far all the Faddeev calculations, using the energy-independent renormalized RGM kernels, yield less binding than the $\varepsilon K$ prescription, as long as the full model space with enough angular-momenta is taken into account. Compared with the experimental value, $B_\Lambda^{\text{exp}} = 130\pm50$ keV, the fss2 value is overbound by at least 82 keV. We conclude that the $\Lambda N$ interaction of fss2 is probably slightly too attractive. The model FSS has a problem that the attraction of the $^1S_0$ state is too strong, compared with that of the $^3S_1$ state.

From Table III we again find that the 15-channel calculation with $S$- and $D$-states only is a good approximation to the full calculation. We find that $B_\Lambda = 226$ keV for fss2 and 763 keV for FSS in the 15-channel calculation, and the energy gain to the full calculations is 36 keV and 27 keV, respectively. The $NN$ ($\varepsilon_{NN}$) and $\Lambda N$ ($\varepsilon_{\Lambda N}$) expectation values, and the admixture of the $\Sigma NN$ component ($P_\Sigma$) are also not much different from the previous values in the $\varepsilon K$ prescription. The converged values of $P_\Sigma$ are 0.83% for fss2 and 1.43% for FSS, which were previously 0.80% for fss2 and 1.36% for FSS, respectively. The decomposition of the $\varepsilon_{NN}$ value into the kinetic-energy and potential-energy contributions is 19.034 – 20.723 = −1.689 MeV for fss2, which was previously 19.376 – 21.032 = −1.657 MeV. As to the overbinding in the model fss2, we have discussed in Ref. [1] that a slight increase of the $\kappa$ meson mass will improve the fit to the experimental value, without changing good reproduction of the low-energy $\Lambda N$ cross section data. If we modify the $\kappa$-meson mass from the original value, $m_\kappa = 936$ MeV, to 995 MeV, we obtain $B_\Lambda = 134$ keV with $P_\Sigma = 0.56\%$, which is very close to the NSC89 prediction $B_\Lambda = 143$ keV with $P_\Sigma = 0.5\%$. The effective range parameters of this modified fss2 interaction are $a_s = -2.18$ fm, $r_s = 3.03$ fm, and $a_t = -1.78$ fm, $r_t = 2.88$ fm. The phase-shift difference is only 2.2° at $\theta_{lab} = 200$ MeV/c.

Rather small modification of the present results from the previous $\varepsilon K$ prescription is related to a simple structure of the quark-exchange normalization kernel $\Lambda K\Lambda$ in the Pauli allowed space. For the $NN$ interaction, $\Lambda = 1$ since there is no Pauli forbidden state. For the positive-parity states, the largest eigenvalue of $|K|$ is $1/9$ for the $0s$ harmonic oscillator state. Although almost Pauli forbidden states appear in the $P$-states, such partial waves give rather minor contributions to the binding energy of the triton. For the $\Lambda N$-$\Sigma N$ interaction, we have a Pauli forbidden state classified by the SU$_3$ label $(11)_s$. Once this component is properly eliminated, the eigenvalues of $\Lambda K\Lambda$ also become very small. These are the main reasons why the present treatment by the energy-independent renormalized RGM kernel gives the results rather similar to the previous energy-dependent $\varepsilon K$ prescription. On the contrary, the difference between $\Lambda(\varepsilon K)\Lambda$ and $W$ is rather large in the nuclear cluster systems, which leads to appreciable difference between these two prescriptions [7, 8].

In summary, we have recalculated triton and hypertriton binding energies in a new semi-microscopic three-cluster equation, using the energy-independent renormal-
TABLE II: Results of the $^3\Lambda$H Faddeev calculations by fss2 and FSS, using the energy-independent renormalized RGM kernels. The momentum discretization points are the same as in Ref. [6]. The calculated deuteron binding energy is $\varepsilon_d = 2.2247$ MeV for fss2 and 2.2561 MeV for FSS ($\varepsilon_d^{\exp} = 2.2246$ MeV). The column $n$ implies the number of three-baryon channels, including the two-baryon systems up to the total angular-momentum $J$, $E$ the $^3\Lambda$H energy measured from the $N + N + \Lambda$ threshold, and $B_\Lambda$ the $\Lambda$ separation energy. The experimental value is $B_\Lambda^{\exp} = 130 \pm 50$ keV. The column $P_2$ shows the squared amplitudes of the $\Sigma NN$ admixture in percent. The $\Delta B_\Lambda$ is the energy change from the previous $\varepsilon K$ prescription to the present energy-independent renormalized RGM kernels.

| model | $n$ | $E$ (MeV) | $B_\Lambda$ (keV) | $\Delta B_\Lambda$ (keV) | $P_2$ (%) |
|-------|-----|----------|------------------|-----------------|----------|
| 6 (S) | -2.392 | 167 | -30 | 0.566 |
| 15 (SD) | -2.451 | 226 | -28 | 0.775 |
| 30 (J ≤ 1) | -2.404 | 179 | -1 | 0.67 |
| fss2 | 54 (J ≤ 2) | -2.467 | 243 | 31 | 0.792 |
| 78 (J ≤ 3) | -2.483 | 259 | 27 | 0.824 |
| 102 (J ≤ 4) | -2.486 | 261 | 27 | 0.828 |
| 126 (J ≤ 5) | -2.487 | 262 | 27 | 0.830 |
| 150 (J ≤ 6) | -2.487 | 262 | 27 | 0.830 |
| 6 (S) | -2.978 | 722 | -68 | 1.251 |
| 15 (SD) | -3.019 | 763 | -53 | 1.421 |
| 30 (J ≤ 1) | -2.926 | 670 | 21 | 1.318 |
| FSS | 54 (J ≤ 2) | -3.030 | 774 | 92 | 1.412 |
| 78 (J ≤ 3) | -3.041 | 785 | 87 | 1.427 |
| 102 (J ≤ 4) | -3.045 | 789 | 88 | 1.430 |
| 126 (J ≤ 5) | -3.046 | 790 | 89 | 1.431 |
| 150 (J ≤ 6) | -3.046 | 790 | 88 | 1.431 |

The calculation of the model fss2 is slightly reduced. We still have a large ambiguity in $^1S$ and $^3S\Lambda N$ interactions, before further details, such as the charge symmetry breaking of $\Delta p$ and $\Lambda n$ interactions, come into play. The comparison of the fss2 value, 262 keV, with the experimental one, $B_\Lambda^{\exp} = 130 \pm 50$ keV, shows that the $^1S$ interaction of fss2 is still too attractive, which can be corrected by choosing a slightly heavier $\pi$-meson mass.

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