LOGARITHMIC COEFFICIENTS AND A COEFFICIENT CONJECTURE FOR UNIVALENT FUNCTIONS

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Abstract. Let \( U(\lambda) \) denote the family of analytic functions \( f(z), f(0) = 0 = f'(0) - 1 \), in the unit disk \( D \), which satisfy the condition \( \left| (z/f(z))^2 f'(z) - 1 \right| < \lambda \) for some \( 0 < \lambda \leq 1 \). The logarithmic coefficients \( \gamma_n \) of \( f \) are defined by the formula \( \log(f(z)/z) = 2 \sum_{n=1}^{\infty} \gamma_n z^n \). In a recent paper, the present authors proposed a conjecture that if \( f \in U(\lambda) \) for some \( 0 < \lambda \leq 1 \), then
\[
|a_n| \leq \sum_{k=0}^{n-1} \lambda^k \quad \text{for } n \geq 2
\]
and provided a new proof for the case \( n = 2 \). One of the aims of this article is to present a proof of this conjecture for \( n = 3, 4 \) and an elegant proof of the inequality for \( n = 2 \), with equality for \( f(z) = z/(1 + z)(1 + \lambda z) \). In addition, the authors prove the following sharp inequality for \( f \in U(\lambda) \):
\[
\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{1}{4} \left( \frac{n^2}{6} + 2 \text{Li}_2(\lambda) + \text{Li}_2(\lambda^2) \right),
\]
where \( \text{Li}_2 \) denotes the dilogarithm function. Furthermore, the authors prove two such new inequalities satisfied by the corresponding logarithmic coefficients of some other subfamilies of \( S \).

1. Introduction

Let \( A \) be the class of functions \( f \) analytic in the unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) with the normalization \( f(0) = 0 = f'(0) - 1 \). Let \( S \) denote the class of functions \( f \) from \( A \) that are univalent in \( \mathbb{D} \). Then the logarithmic coefficients \( \gamma_n \) of \( f \in S \) are defined by the formula
\[
\frac{1}{2} \log \left( \frac{f(z)}{z} \right) = \sum_{n=1}^{\infty} \gamma_n z^n, \quad z \in \mathbb{D}.
\]
These coefficients play an important role for various estimates in the theory of univalent functions. When we require a distinction, we use the notation \( \gamma_n(f) \) instead of \( \gamma_n \). For example, the Koebe function \( k(z) = z(1 - e^{\theta} z)^{-2} \) for each \( \theta \) has logarithmic coefficients \( \gamma_n(k) = e^{\theta n}/n, n \geq 1 \). If \( f \in S \) and \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \), then by (1) it follows that \( 2 \gamma_1 = a_2 \) and hence, by the Bieberbach inequality, \( |\gamma_1| \leq 1 \). Let \( S^* \) denote the class of functions \( f \in S \) such that \( f(\mathbb{D}) \) is starlike with respect to the origin. Functions \( f \in S^* \) are characterized by the condition \( \text{Re} \left( z f'(z)/f(z) \right) > 0 \) in \( \mathbb{D} \). The inequality \( |\gamma_n| \leq 1/n \) holds for starlike functions \( f \in S \), but is false for the full class \( S \), even in order of magnitude. See [3, Theorem 8.4 on page 242]. In [2],

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Girela pointed out that this bound is actually false for the class of close-to-convex functions in $\mathbb{D}$ which is defined as follows: A function $f \in \mathcal{A}$ is called close-to-convex, denoted by $f \in \mathcal{K}$, if there exists a real $\alpha$ and a $g \in \mathcal{S}^*$ such that

$$\Re \left( e^{i\alpha} \frac{zf'(z)}{g(z)} \right) > 0, \quad z \in \mathbb{D}. $$

For $0 \leq \beta < 1$, a function $f \in \mathcal{S}$ is said to belong to the class of starlike functions of order $\beta$, denoted by $f \in \mathcal{S}^*(\beta)$, if $\Re \left( \frac{zf'(z)}{f(z)} \right) > \beta$ for $z \in \mathbb{D}$. Note that $\mathcal{S}(0) =: \mathcal{S}$ and $\mathcal{C}^*(\beta) = \mathcal{C}^*$. The class of all convex functions of order $\beta$, denoted by $\mathcal{C}(\beta)$, is then defined by $\mathcal{C}(\beta) = \{ f \in \mathcal{S} : zf''(z)/f'(z) \}$. The class $\mathcal{C}(0) =: \mathcal{C}$ is usually referred to as the class of convex functions in $\mathbb{D}$. With the class $\mathcal{S}$ being of the first priority, its subclasses such as $\mathcal{S}^*$, $\mathcal{K}$, and $\mathcal{C}$, respectively, have been extensively studied in the literature and they appear in different contexts. We refer to [4, 7, 10, 12] for a general reference related to the present study. In [5, Theorem 4], it was shown that the logarithmic coefficients $\gamma_n$ of every function $f \in \mathcal{S}$ satisfy

$$ \sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{\pi^2}{6}, $$

and the equality is attained for the Koebe function. The proof uses ideas from the work of Baernstein [3] on integral means. However, this result is easy to prove (see Theorem 1) in the case of functions in the class $\mathcal{U} := \mathcal{U}(1)$ which is defined as follows:

$$ \mathcal{U}(\lambda) = \left\{ f \in \mathcal{A} : \left| \left( \frac{z}{f(z)} \right)^2 \frac{f'(z)}{f(z)} - 1 \right| < \lambda, \quad z \in \mathbb{D} \right\}, $$

where $\lambda \in (0, 1]$. It is known that [11] every $f \in \mathcal{U}$ is univalent in $\mathbb{D}$ and hence, $\mathcal{U}(\lambda) \subset \mathcal{U} \subset \mathcal{S}$ for $\lambda \in (0, 1]$. The present authors have established many interesting properties of the family $\mathcal{U}(\lambda)$. See [10] and the references therein. For example, if $f \in \mathcal{U}(\lambda)$ for some $0 < \lambda \leq 1$ and $a_2 = f''(0)/2$, then we have the subordination relations

$$ \frac{f(z)}{z} < \frac{1}{1 + (1 + \lambda)z + \lambda z^2} = \frac{1}{(1 + z)(1 + \lambda z)}, \quad z \in \mathbb{D}, $$

and

$$ \frac{z}{f(z)} + a_2 z < 1 + 2\lambda z + \lambda z^2, \quad z \in \mathbb{D}. $$

Here $<$ denotes the usual subordination [4, 7, 12]. In addition, the following conjecture was proposed in [10].

**Conjecture 1.** Suppose that $f \in \mathcal{U}(\lambda)$ for some $0 < \lambda \leq 1$. Then $|a_n| \leq \sum_{k=0}^{n-1} \lambda^k$ for $n \geq 2$.

In Theorem 4 we present a direct proof of an inequality analogous to (2) for functions in $\mathcal{U}(\lambda)$ and in Corollary 1 we obtain the inequality (2) as a special case for $\mathcal{U}$. At the end of Section 2, we also consider estimates of the type (2) for some interesting subclasses of univalent functions. However, Conjecture 4 remains open for $n \geq 5$. On the other hand, the proof for the case $n = 2$ of this conjecture is due to [17] and an alternate proof was obtained recently by the present authors in [10].
Theorem 1]. In this paper, we show that Conjecture 1 is true for $n = 3, 4$. and our proof includes an elegant proof of the case $n = 2$. The main results and their proofs are presented in Sections 2 and 3.

2. LOGARITHMIC COEFFICIENTS OF FUNCTIONS IN $U(\lambda)$

**Theorem 1.** For $0 < \lambda \leq 1$, the logarithmic coefficients of $f \in U(\lambda)$ satisfy the inequality

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{1}{4} \left( \frac{\pi^2}{6} + 2\text{Li}_2(\lambda) + \text{Li}_2(\lambda^2) \right),$$

where $\text{Li}_2$ denotes the dilogarithm function given by

$$\text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = z \int_{0}^{1} \frac{\log(1/t)}{1 - tz} dt.$$

The inequality (4) is sharp. Further, there exists a function $f \in U$ such that $|\gamma_n| > (1 + \lambda^n)/(2n)$ for some $n$.

**Proof.** Let $f \in U(\lambda)$. Then, by (3), we have

$$\frac{z}{f(z)} < (1 - z)(1 - \lambda z)$$

which clearly gives

$$\sum_{n=1}^{\infty} \gamma_n z^n = \log \sqrt{\frac{f(z)}{z}} < -\log(1 - z) - \log(1 - \lambda z) = \sum_{n=1}^{\infty} \frac{1}{2n} (1 + \lambda^n) z^n.$$

Again, by Rogosinski’s theorem (see [4, 6.2]), we obtain

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \sum_{n=1}^{\infty} \frac{1}{4n^2} (1 + \lambda^n)^2 = \frac{1}{4} \left( \sum_{n=1}^{\infty} \frac{1}{n^2} + 2 \sum_{n=1}^{\infty} \frac{\lambda^n}{n^2} + \sum_{n=1}^{\infty} \frac{\lambda^{2n}}{n^2} \right),$$

and the desired inequality (4) follows. For the function $g_\lambda(z) = \frac{z}{(1 - z)(1 - \lambda z)}$, we find that $\gamma_n(g_\lambda) = (1 + \lambda^n)/(2n)$ for $n \geq 1$ and therefore, we have the equality in (4). Note that $g_1(z)$ is the Koebe function $z/(1 - z)^2$.

From the relation (5), we cannot conclude that $|\gamma_n(f)| \leq |\gamma_n(g_\lambda)| = \frac{1 + \lambda^n}{2n}$ for $f \in U(\lambda)$.

Indeed for the function $f_\lambda$ defined by

$$f_\lambda(z) = \frac{z}{(1 - z)(1 - \lambda z)(1 + (\lambda/(1 + \lambda))z)},$$

we find that

$$\frac{z}{f_\lambda(z)} = 1 + \frac{\lambda - (1 + \lambda)^2}{1 + \lambda} z + \frac{\lambda^2}{1 + \lambda} z^3.$$
Figure 1. The image of \( f_\lambda(z) = \frac{z}{(1-z)(1-\lambda z)(1+\lambda/(1+\lambda))z} \) under \( \mathbb{D} \) for certain values of \( \lambda \)

and

\[
\left( \frac{z}{f_\lambda(z)} \right)^2 f_\lambda'(z) - 1 = -\frac{2\lambda^2}{1+\lambda} z^3 = -\left( 1 - \frac{(1+2\lambda)(1-\lambda)}{1+\lambda} z^3 \right)
\]

which clearly shows that \( f_\lambda \in \mathcal{U}(\lambda) \). The images of \( f_\lambda(z) \) under \( \mathbb{D} \) for certain values of \( \lambda \) are shown in Figures 1(a)-(d). Moreover, for this function, we have

\[
\log \left( \frac{f_\lambda(z)}{z} \right) = -\log(1-z) - \log(1-\lambda z) - \log \left( 1 + \frac{\lambda}{1+\lambda} z \right)
\]

\[
= 2 \sum_{n=1}^{\infty} \gamma_n(f_\lambda) z^n,
\]

where

\[
\gamma_n(f_\lambda) = \frac{1}{2} \left( \frac{1+\lambda^n}{n} + (-1)^n \frac{\lambda^n}{n(1+\lambda)^n} \right).
\]
This contradicts the above inequality at least for even integer values of \( n \geq 2 \). Moreover, with these \( \gamma_n(f_\lambda) \) for \( n \geq 1 \), we obtain
\[
\sum_{n=1}^{\infty} |\gamma_n(f_\lambda)|^2 = \frac{1}{4} \sum_{n=1}^{\infty} \left\{ \frac{(1+\lambda^2)^2}{n^2} + 2\frac{(-1)^n}{n^2} \left[ \left( \frac{\lambda}{1+\lambda} \right)^n + \frac{1}{n^2} \left( \frac{\lambda}{1+\lambda} \right)^{2n} \right] \right\}
\]
and by a computation, it follows easily that
\[
\sum_{n=1}^{\infty} |\gamma_n(f_\lambda)|^2 = \frac{1}{4} \left( \frac{\pi^2}{6} + 2\text{Li}_2(\lambda) + \text{Li}_2(\lambda^2) \right)
\]
\[
+ \frac{1}{2} \left[ \text{Li}_2 \left( \frac{-\lambda^2}{1+\lambda} \right) + \text{Li}_2 \left( \frac{-\lambda}{1+\lambda} \right) \right] + \frac{1}{4} \text{Li}_2 \left( \frac{\lambda^2}{(1+\lambda)^2} \right)
\]
\[
= \frac{1}{4} \left( \frac{\pi^2}{6} + 2\text{Li}_2(\lambda) + \text{Li}_2(\lambda^2) \right) + \frac{1}{4} A(\lambda)
\]
\[
< \frac{1}{4} \left( \frac{\pi^2}{6} + 2\text{Li}_2(\lambda) + \text{Li}_2(\lambda^2) \right) \quad \text{for } 0 < \lambda \leq 1,
\]
and we complete the proof, provided \( A(\lambda) > 0 \) for \( 0 < \lambda \leq 1 \). Now, we claim that
\[
A(\lambda) := 2 \left[ \text{Li}_2 \left( \frac{-\lambda^2}{1+\lambda} \right) + \text{Li}_2 \left( \frac{-\lambda}{1+\lambda} \right) \right] + \text{Li}_2 \left( \frac{\lambda^2}{(1+\lambda)^2} \right) < 0.
\]
Because \( \text{Li}_2(z^2) = 2(\text{Li}_2(z) + \text{Li}_2(-z)) \), the last claim is equivalent to
\[
\frac{A(\lambda)}{2} = 2 \text{Li}_2 \left( \frac{-\lambda}{1+\lambda} \right) + \left[ \text{Li}_2 \left( \frac{\lambda}{1+\lambda} \right) + \text{Li}_2 \left( \frac{-\lambda^2}{1+\lambda} \right) \right] < 0
\]
for \( 0 < \lambda \leq 1 \). According to the integral representation of \( \text{Li}_2(z) \) given in the statement of Theorem 1, we can write
\[
A(\lambda) = -2\lambda \int_0^1 B(\lambda, t) \log(1/t) \, dt,
\]
where
\[
B(\lambda, t) = \frac{2}{1+\lambda + t\lambda} - \frac{1}{1+\lambda - t\lambda} + \frac{\lambda}{1+\lambda + t\lambda^2}
\]
\[
= \frac{(1+\lambda) - 3t\lambda}{(1+\lambda)^2 - t^2\lambda^2} + \frac{\lambda}{1+\lambda + t\lambda^2}
\]
\[
= \frac{N(\lambda, t)}{(1+\lambda)^2 - t^2\lambda^2}[1+\lambda + t\lambda^2]
\]
with
\[
N(\lambda, t) = (1+\lambda)^3 - (3-\lambda)(1+\lambda)\lambda t - 4\lambda^3 t^2.
\]
Clearly, \( B(1, t) > 0 \) for \( t \in [0, 1) \) and it follows that, \( A(1) < 0 \). On the other hand, since \( N(\lambda, t) \) is a decreasing function of \( t \) for \( t \in [0, 1] \), we obtain that
\[
N(\lambda, t) \geq N(\lambda, 1) = (1+\lambda)^3 - (3-\lambda)(1+\lambda) - 4\lambda^3 = 1 - \lambda^3 + \lambda^2(1-\lambda) > 0
\]
for $0 < \lambda < 1$. Consequently, $B(\lambda, t) > 0$ for all $t \in [0, 1]$ and for $0 < \lambda < 1$. This observation shows that $A(\lambda) < 0$ for $0 < \lambda \leq 1$. This proves the claim and thus, the proof is complete. □

Corollary 1. The logarithmic coefficients of $f \in \mathcal{U}$ satisfy the inequality

$$
\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
$$

We have equality in the last inequality for the Koebe function $k(z) = z(1 - e^{i\theta}z)^{-2}$. Further there exists a function $f \in \mathcal{U}$ such that $|\gamma_n| > 1/n$ for some $n$.

Remark 1. From the analytic characterization of starlike functions, it is easy to see that for $f \in \mathcal{S}^*$,

$$
\frac{zf'(z)}{f(z)} - 1 = z \left( \log \left( \frac{f(z)}{z} \right) \right)' = 2 \sum_{n=1}^{\infty} n\gamma_n z^n < \frac{2z}{1 - z},
$$

and thus, by Rogosinski’s result, we obtain that $|\gamma_n| \leq 1/n$ for $n \geq 1$. In fact for starlike functions of order $\alpha$, $\alpha \in [0, 1)$, the corresponding logarithmic coefficients satisfy the inequality $|\gamma_n| \leq (1 - \alpha)/n$ for $n \geq 1$. Moreover, one can quickly obtain that

$$
\sum_{n=1}^{\infty} |\gamma_n|^2 \leq (1 - \alpha)^2 \frac{\pi^2}{6}
$$

if $f \in \mathcal{S}^*(\alpha)$, $\alpha \in [0, 1)$ (See also the proof of Theorem 2 and Remark 3). As remarked in the proof of Theorem 4 from the relation (7), we cannot conclude the same fact, namely, $|\gamma_n| \leq 1/n$ for $n \geq 1$, for the class $\mathcal{U}$ although the Koebe function $k(z) = z/(1 - z)^2$ belongs to $\mathcal{U} \cap \mathcal{S}^*$. For example, if we set $\lambda = 1$ in (6), then we have

$$
\frac{z}{f_1(z)} = (1 - z)^2\left(1 + \frac{z}{2}\right) = 1 - \frac{3}{2}z + \frac{z^3}{2},
$$

where $f_1 \in \mathcal{U}$ and for this function, we obtain

$$
\sum_{n=1}^{\infty} |\gamma_n(f_1)|^2 = \sum_{n=1}^{\infty} \left( \frac{1}{n} + (-1)^n \frac{1}{n2^{n+1}} \right)^2
= \frac{\pi^2}{6} + \frac{1}{4} \text{Li}_2 \left( \frac{1}{4} \right) + \text{Li}_2 \left( \frac{-1}{2} \right)
= \frac{\pi^2}{6} + \frac{1}{2} \left[ \text{Li}_2 \left( \frac{1}{2} \right) + 3\text{Li}_2 \left( \frac{-1}{2} \right) \right],
$$

where we have used the fact that $\text{Li}_2(z^2) = 2(\text{Li}_2(z) + \text{Li}_2(-z))$. From the proof of Theorem 7, we conclude that

$$
\sum_{n=1}^{\infty} |\gamma_n(f_1)|^2 < \frac{\pi^2}{6},
$$
because
\[ \text{Li}_2 \left( \frac{1}{2} \right) + 3 \text{Li}_2 \left( -\frac{1}{2} \right) < 0. \]

As a direct approach, it is easy to see that
\[ \text{Li}_2(z) + 3 \text{Li}_2(-z) = \sum_{n=1}^{\infty} \frac{1}{n^2} (1 + 3(-1)^n)z^n = \sum_{k=1}^{\infty} \frac{z^{2k}}{k^2} - 2 \sum_{k=1}^{\infty} \frac{z^{2k-1}}{(2k-1)^2} \]
and thus, we obtain that
\[ \sum_{n=1}^{\infty} n^2 |\gamma_n(f_1)|^2 = \frac{\pi^2}{6} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{4^k} \left( \frac{1}{k^2} - \frac{1}{(k-1/2)^2} \right) = \frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{1}{4^k} \left( \frac{4k-1}{k^2(2k-1)^2} \right) \]
and thus,
\[ \sum_{n=1}^{\infty} |\gamma_n(f_1)|^2 < \frac{\pi^2}{6}. \]

On the other hand, it is a simple exercise to verify that \( f_1 \notin S^* \). The graph of this function is shown in Figure 1(d).

Let \( G(\alpha) \) denote the class of locally univalent normalized analytic functions \( f \) in the unit disk \(|z| < 1\) satisfying the condition
\[ \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) < 1 + \frac{\alpha}{2} \quad \text{for } |z| < 1, \]
and for some \( 0 < \alpha \leq 1 \). Set \( G(1) =: G \). It is known (see [13, Equation (16)]) that \( G \subset S^* \) and thus, functions in \( G(\alpha) \) are starlike. This class has been studied extensively in the recent past, see for instance [9] and the references therein. We now consider the estimate of the type \( (2) \) for the subclass \( G(\alpha) \).

**Theorem 2.** Let \( 0 < \alpha \leq 1 \) and \( G(\alpha) \) be defined as above. Then the logarithmic coefficients \( \gamma_n \) of \( f \in G(\alpha) \) satisfy the inequalities
\[ \sum_{n=1}^{\infty} n^2 |\gamma_n|^2 \leq \frac{\alpha}{4(\alpha + 2)} \]
and
\[ \sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{\alpha^2}{4} \text{Li}_2 \left( \frac{1}{(1+\alpha)^2} \right). \]
Also we have
\[ |\gamma_n| \leq \frac{\alpha}{2(\alpha + 1)n} \quad \text{for } n \geq 1. \]

**Proof.** If \( f \in G(\alpha) \), then we have (see eg. [8, Theorem 1] and [13])
\[ \frac{zf'(z)}{f(z)} - 1 \prec \frac{(1+\alpha)(1-z)}{1+\alpha-z} - 1 = -\alpha \left( \frac{z/(1+\alpha)}{1-(z/(1+\alpha))} \right), \quad z \in \mathbb{D}, \]
which, in terms of the logarithmic coefficients $\gamma_n$ of $f$ defined by (1), is equivalent to

\[(12) \quad \sum_{n=1}^{\infty} (-2n\gamma_n)z^n < \alpha \sum_{n=1}^{\infty} \frac{z^n}{(1+\alpha)^n}.\]

Again, by Rogosinski’s result, we obtain that

\[\sum_{n=1}^{\infty} 4n^2|\gamma_n|^2 \leq \alpha^2 \sum_{n=1}^{\infty} \frac{1}{(1+\alpha)^{2n}} = \frac{\alpha}{\alpha + 2}\]

which is (5).

Now, since the sequence $A_n = \frac{1}{(1+\alpha)^n}$ is convex decreasing, we obtain from (12) and [15, Theorem VII, p.64] that

\[| -2n\gamma_n | \leq \alpha A_n = \frac{\alpha}{1 + \alpha},\]

which implies the desired inequality (10). As an alternate approach to prove this inequality, we may rewrite (11) as

\[\sum_{n=1}^{\infty} (2n\gamma_n)z^n = z \left( \log \left( \frac{f(z)}{z} \right) \right)' \prec \phi(z) = -\alpha \left( \frac{z/(1+\alpha)}{1 - (z/(1+\alpha))} \right)\]

and, since $\phi(z)$ is convex in $\mathbb{D}$ with $\phi'(0) = -\alpha/(1 + \alpha)$, it follows from Rogosinski’s result (see also [4, Theorem 6.4(i), p.195]) that $|2n\gamma_n| \leq \alpha/(1 + \alpha)$. Again, this proves the inequality (10).

Finally, we prove the inequality (9). From the formula (12) and the result of Rogosinski (see also [12, Theorem 2.2] and [4, Theorem 6.2]), it follows that for $k \in \mathbb{N}$ the inequalities

\[\sum_{n=1}^{k} n^2 |\gamma_n|^2 \leq \frac{\alpha^2}{4} \sum_{n=1}^{k} \frac{1}{(1+\alpha)^{2n}}\]

are valid. Clearly, this implies the inequality (8) as well. On the other hand, consider these inequalities for $k = 1, \ldots, N$, and multiply the $k$-th inequality by the factor $\frac{1}{k^2} - \frac{1}{(k+1)^2}$, if $k = 1, \ldots, N - 1$ and by $\frac{1}{N^2}$ for $k = N$. Then the summation of the multiplied inequalities yields

\[\sum_{k=1}^{N} |\gamma_k|^2 \leq \frac{\alpha^2}{4} \sum_{k=1}^{N} \frac{1}{k^2(1+\alpha)^{2k}} \leq \frac{\alpha^2}{4} \sum_{k=1}^{\infty} \frac{1}{k^2(1+\alpha)^{2k}} = \frac{\alpha^2}{4} \text{Li}_2 \left( \frac{1}{(1+\alpha)^2} \right) \text{ for } N = 1, 2, \ldots,\]

which proves the desired assertion (9) if we allow $N \to \infty$. \hfill \square
Corollary 2. The logarithmic coefficients $\gamma_n$ of $f \in \mathcal{G} := \mathcal{G}(1)$ satisfy the inequalities
\[ \sum_{n=1}^{\infty} n^2 |\gamma_n|^2 \leq \frac{1}{12} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{|\gamma_n|^2}{n^2} \leq \frac{1}{4} \text{Li}_2 \left( \frac{1}{4} \right). \]
The results are the best possible as the function $f_0(z) = z - \frac{1}{2}z^2$ shows. Also we have $|\gamma_n| \leq 1/(4n)$ for $n \geq 1$.

Remark 2. For the function $f_0(z) = z - \frac{1}{2}z^2$, we have that $\gamma_n(f_0) = -\frac{1}{n2^{n+1}}$ for $n = 1, 2, \ldots$ and thus, it is reasonable to expect that the inequality $|\gamma_n| \leq \frac{1}{n2^{n+1}}$ is valid for the logarithmic coefficients $\gamma_n$ of each $f \in \mathcal{G}$. But that is not the case as the function $f_n$ defined by $f_n(z) = (1 - z^n)^{\frac{1}{2}}$ shows. Indeed for this function we have
\[ 1 + \frac{zf''_n(z)}{f'_n(z)} = \frac{1 - 2z^n}{1 - z^n} \]
showing that $f_n \in \mathcal{G}$. Moreover,
\[ \log \frac{f_n(z)}{z} = -\frac{1}{n(n+1)}z^n + \cdots, \]
which implies that $|\gamma_n(f_n)| = \frac{1}{2n(n+1)}$ for $n = 1, 2, \ldots$, and observe that $\frac{1}{2n(n+1)} > \frac{1}{n2^{n+1}}$ for $n = 2, 3, \ldots$. Thus, we conjecture that the logarithmic coefficients $\gamma_n$ of each $f \in \mathcal{G}$ satisfy the inequality $|\gamma_n| \leq \frac{1}{2n(n+1)}$ for $n = 1, 2, \ldots$ Clearly, Corollary 2 shows that the conjecture is true for $n = 1$.

Remark 3. Let $f \in \mathcal{C}(\alpha)$, where $0 \leq \alpha < 1$. Then we have [15]
\[ (13) \quad \frac{zf'(z)}{f(z)} - 1 < G_\alpha(z) - 1 = \sum_{n=1}^{\infty} \delta_n z^n, \]
where $\delta_n$ is real for each $n$,
\[ G_\alpha(z) = \begin{cases} 
\frac{(2\alpha - 1)z}{(1-z)[(1-z)^{1-2\alpha} - 1]} & \text{if } \alpha \neq 1/2, \\
\frac{-z}{(1-z)\log(1-z)} & \text{if } \alpha = 1/2,
\end{cases} \]
and
\[ \beta(\alpha) = G_\alpha(-1) = \inf_{|z|<1} G_\alpha(z) = \begin{cases} 
\frac{1 - 2\alpha}{2[1 - 2\alpha - 1]} & \text{if } 0 \leq \alpha \neq 1/2 < 1, \\
\frac{1}{2\log 2} & \text{if } \alpha = 1/2
\end{cases} \]
so that $f \in \mathcal{S}^*(\beta(\alpha))$. Also, we have [10]
\[ \frac{f(z)}{z} < K_\alpha(z) = \begin{cases} 
\frac{(1-z)^{2\alpha-1} - 1}{(1-2\alpha)z} & \text{if } 0 \leq \alpha \neq 1/2 < 1, \\
\frac{1 - \log(1-z)}{z} & \text{if } \alpha = 1/2,
\end{cases} \]
and $K_\alpha(z)/z$ is univalent and convex (not normalized in the usual sense) in $\mathbb{D}$. 

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Now, the subordination relation \[ (13) \], in terms of the logarithmic coefficients \( \gamma_n \) of \( f \) defined by \[ (1) \], is equivalent to
\[
2 \sum_{n=1}^{\infty} n \gamma_n z^n < G_{\alpha}(z) - 1 = \sum_{n=1}^{\infty} \delta_n z^n, \quad z \in \mathbb{D},
\]
and thus,
\[
(14) \quad \sum_{n=1}^{k} n^2 |\gamma_n|^2 \leq \frac{1}{4} \sum_{n=1}^{k} \delta_n^2 \quad \text{for each } k \in \mathbb{N}.
\]
Since \( f \) is starlike of order \( \beta \), it follows that
\[
\frac{z K_{\alpha}'(z)}{K_{\alpha}(z)} - 1 = G_{\alpha}(z) - 1 < 2(1 - \beta) \frac{z}{1 - z}
\]
and therefore, \( |\delta_n| \leq 2(1 - \beta) \) for each \( n \geq 1 \). Again, the relation \[ (14) \] by the previous approach gives
\[
\sum_{k=1}^{N} |\gamma_k|^2 \leq \frac{1}{4} \sum_{k=1}^{N} \delta_k^2 \leq (1 - \beta)^2 \sum_{k=1}^{N} \frac{1}{k^2}
\]
for \( N = 1, 2, \ldots, \) and hence, we have
\[
\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{1}{4} \sum_{n=1}^{\infty} \frac{\delta_n^2}{n^2} \leq (1 - \beta)^2 \sum_{n=1}^{\infty} \frac{1}{n^2} = (1 - \beta)^2 \pi^2/6
\]
and equality holds in the first inequality for \( K_{\alpha}(z) \). In particular, if \( f \) is convex then \( \beta(0) = 1/2 \) and hence, the last inequality reduces to
\[
\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{\pi^2}{24}
\]
which is sharp as the convex function \( z/(1 - z) \) shows.

3. Proof of Conjecture \[ \text{for } n = 2, 3, 4 \]

Theorem 3. Let \( f \in \mathcal{U}(\lambda) \) for \( 0 < \lambda \leq 1 \) and let \( f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \). Then
\[
|a_n| \leq \frac{1 - \lambda^n}{1 - \lambda} \quad \text{for } 0 < \lambda < 1 \text{ and } n = 2, 3, 4,
\]
and \( |a_n| \leq n \) for \( \lambda = 1 \) and \( n \geq 2 \). The results are the best possible.

Proof. The case \( \lambda = 1 \) is well-known because \( \mathcal{U} = \mathcal{U}(1) \subset \mathcal{S} \) and hence, by the de Branges theorem, we have \( |a_n| \leq n \) for \( f \in \mathcal{U} \) and \( n \geq 2 \). Here is an alternate proof without using the de Branges theorem. From the subordination result \[ (3) \] with \( \lambda = 1 \), one has
\[
\frac{f(z)}{z} \prec \frac{1}{(1 - z)^2} = \sum_{n=1}^{\infty} n z^{n-1}
\]
and thus, by Rogosinski’s theorem [4, Theorem 6.4(ii), p. 195], it follows that \( |a_n| \leq n \) for \( n \geq 2 \).
So, we may consider \( f \in \mathcal{U}(\lambda) \) with \( 0 < \lambda < 1 \). The result for \( n = 2 \), namely, \(|a_2| \leq 1 + \lambda\) is proved in \([10, 17]\) and thus, it suffices to prove (15) for \( n = 3, 4 \) although our proof below is elegant and simple for the case \( n = 2 \) as well. To do this, we begin to recall from (3) that

\[
\frac{f(z)}{z} < \frac{1}{(1 - z)(1 - \lambda z)} = 1 + \sum_{n=1}^{\infty} \frac{1 - \lambda^{n+1}}{1 - \lambda} z^n
\]

and thus

\[
\frac{f(z)}{z} = \frac{1}{(1 - z \omega(z))(1 - \lambda z \omega(z))},
\]

where \( \omega \) is analytic in \( \mathbb{D} \) and \(|\omega(z)| \leq 1\) for \( z \in \mathbb{D} \). In terms of series formulation, we have

\[
\sum_{n=1}^{\infty} a_{n+1} z^n = \sum_{n=1}^{\infty} \frac{1 - \lambda^{n+1}}{1 - \lambda} \omega^n(z) z^n.
\]

We now set \( \omega(z) = c_1 + c_2 z + \cdots \) and rewrite the last relation as

(16) \[
\sum_{n=1}^{\infty} (1 - \lambda) a_{n+1} z^n = \sum_{n=1}^{\infty} (1 - \lambda^{n+1})(c_1 + c_2 z + \cdots) z^n.
\]

By comparing the coefficients of \( z^n \) for \( n = 1, 2, 3 \) on both sides of (16), we obtain

(17) \[
\begin{align*}
(1 - \lambda) a_2 &= (1 - \lambda^2) c_1 \\
(1 - \lambda) a_3 &= (1 - \lambda^2) c_2 + (1 - \lambda^3) c_1^2 \\
(1 - \lambda) a_4 &= (1 - \lambda^2) (c_3 + \mu c_1 c_2 + \nu c_1^3),
\end{align*}
\]

where

\[
\mu = 2 \frac{1 - \lambda^3}{1 - \lambda^2} \quad \text{and} \quad \nu = \frac{1 - \lambda^4}{1 - \lambda^2}.
\]

It is well-known that \(|c_1| \leq 1\) and \(|c_2| \leq 1 - |c_1|^2\). From the first relation in (17) and the fact that \(|c_1| \leq 1\), we obtain

\[
(1 - \lambda)|a_2| = (1 - \lambda^2)|c_1| \leq 1 - \lambda^2,
\]

which gives a new proof for the inequality \(|a_2| \leq 1 + \lambda\).

Next we present a proof of (15) for \( n = 3 \). Using the second relation in (17), \(|c_1| \leq 1\) and the inequality \(|c_2| \leq 1 - |c_1|^2\), we get

\[
(1 - \lambda)|a_3| \leq (1 - \lambda^2)|c_2| + (1 - \lambda^3)|c_1|^2 \leq (1 - \lambda^2)(1 - |c_1|^2) + (1 - \lambda^3)|c_1|^2 = 1 - \lambda^2 + (\lambda^2 - \lambda^3)|c_1|^2 \leq 1 - \lambda^3,
\]

which implies \(|a_3| \leq 1 + \lambda + \lambda^2\).

Finally, we present a proof of (15) for \( n = 4 \). To do this, we recall the sharp upper bounds for the functionals \(|c_3 + \mu c_1 c_2 + \nu c_1^3|\) when \( \mu \) and \( \nu \) are real. In \([14]\), Prokhorov and Szynal proved among other results that

\[
|c_3 + \mu c_1 c_2 + \nu c_1^3| \leq |\nu|
\]
if $2 \leq |\mu| \leq 4$ and $\nu \geq (1/12)(\mu^2 + 8)$. From the third relation in (17), this condition is fulfilled and thus, we find that

$$(1 - \lambda)|a_4| = (1 - \lambda^2)|c_3 + \mu c_1 c_2 + \nu c_3^2| \leq (1 - \lambda^2) \left(\frac{1 - \lambda^4}{1 - \lambda^2}\right) = 1 - \lambda^4$$

which proves the desired inequality $|a_4| \leq 1 + \lambda + \lambda^2 + \lambda^3$. □

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