COEFFICIENTS MULTIPLIERS OF WEIGHTED SPACES OF HARMONIC FUNCTIONS

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Abstract. Let \( h_\infty^g \) be the space of harmonic functions in the unit ball that are bounded by some increasing radial function \( g(r) \) with \( \lim_{r \to 1} g(r) = +\infty \); these spaces are called growth spaces. We describe functions in growth spaces by the Cesàro means of their expansions in harmonic polynomials and apply this characterization to study coefficient multipliers between growth spaces. Further, we introduce spaces of harmonic functions of regular growth and show that oscillation operators considered in [21] can be realized as multipliers mapping growth spaces to corresponding spaces of regular growth.

1. Introduction

1.1. Weighted spaces. Let \( g \) be a positive increasing continuous function such that \( \lim_{r \to +\infty} g(r) = +\infty \) and \( g(1) = 1 \), we say that \( g \) is a weight function. We study weighted (or growth) spaces of harmonic functions defined by

\[ h_\infty^g = \{ u : B \to \mathbb{R}, \Delta u = 0, |u(x)| \leq K g(1/(1 - |x|)) \text{ for some } K > 0 \}, \]

where \( B \) is the unit ball in \( \mathbb{R}^{N+1} \). For \( 1 \leq p < +\infty \) we also define

\[ h_p^g = \{ u : B \to \mathbb{R}, \Delta u = 0, \left( \int_S |u(rx)|^p ds(x) \right)^{1/p} \leq K g(1/(1 - r)) \}. \]

We assume always that the weight function \( g \) satisfies the following doubling condition

\[ g(2x) \leq D g(x). \]

Our main examples are \( g(x) = x^\alpha \) and \( g(x) = (1 + \log x)^\alpha \) for \( \alpha > 0 \).

In the present work we study operators between various growth spaces, first we characterize functions in the weighted spaces in terms of the Cesàro and de la Vallée-Poussin sums and of their Fourier series. We show that \( u \in h_p^g, 1 \leq p \leq \infty \), if and only if \( \| \sigma_d^p u \|_p \leq C g(n) \), where \( d > (N - 1)/2 \) and \( C \) does not depend on \( n \). The results on Cesàro sums in dimension two can be found in [3](see also references therein) and for more general spaces in [23, 24]. The de la Vallée-Poussin sums for growth spaces in dimension two were used in [19] to describe isomorphism classes of growth spaces. Our motivation comes from the study of the oscillation integral for harmonic functions in growth spaces, see [21, 14, 15], we also work in higher dimensions and use the standard results on Cesàro sums of spherical harmonics, [27, 5, 10], and some properties of the doubling weights. The characterization is

1991 Mathematics Subject Classification. 42B15, 46E15, 47B38.

Key words and phrases. Weighted spaces of harmonic functions, Cesàro means, multipliers, spherical harmonics, doubling weight.

The second author is supported by Project 213638 of the Research Council of Norway.
applied to describe the coefficient multipliers on the weighted spaces generalizing some results of [26]. Further we define new spaces, so called spaces of regular growth and show that coefficient multipliers that correspond to averaging radial operator map growth spaces to these new spaces of harmonic functions.

The weighted spaces of harmonic functions in the unit disk have been studied by A. L. Shields and D. L. Williams in [26], G. Bennett, D. A. Stegenga and R. M. Timoney in [3], M. Pavlović in [23, 24], W. Lusky in [19, 20], and others. In higher dimensions, the corresponding spaces of harmonic functions on the half spaces of $\mathbb{R}^{N+1}$ were recently characterized through wavelet expansions of the boundary values (represented by distributions on the boundary plane) in [15]. The spaces of distributions in question are generalizations of the Besov spaces $B^{-s,\infty}$, the latter appear for the particular choice of the weight $g(x) = x^s$. In this work we give a different description of the corresponding spaces of distributions on the unit sphere.

Literature on similar spaces of analytic functions, weighted Hardy spaces, is extensive. We mention just some works with technique and ideas that are close to our presentation, [25, 2, 28, 4, 22, 9, 16, 12]. In particular, functions in weighted Hardy spaces $H^p$ with doubling weights have been characterized by the partial sums of their Fourier series for $1 < p < \infty$ in a recent article by E. Doubtsov, [12], see also [23, 24]. This result is based on the boundedness of Fourier block projections that does not hold for $p = 1, \infty$ or $N > 1$.

1.2. Coefficient multipliers. Let $A$ and $B$ be sequence spaces consisting of sequences of the form $\{a_j\}_{j \in J}$ where $a_j \in \mathbb{R}$ and $J$ is some index set. A sequence $\lambda = \{\lambda_j\}_{j \in J}$ is called a multiplier from $A$ to $B$ if $\lambda a = \{\lambda_j a_j\} \in B$ for all $a \in A$. We consider real valued sequences and multipliers and denote the set of multipliers from $A$ to $B$ by $(A, B)$.

In this work the sequences are Fourier coefficients of some functions (or distributions) on the unit sphere in $\mathbb{R}^{N+1}$. Then it is convenient to choose $J$ to be the index set for a basis of spherical harmonics. For the two-dimensional case the index set can be identified with $\mathbb{Z}$. Let $\mathcal{H}^N_k$ be the space of spherical harmonics of degree $k$ on the $N$-dimensional sphere and let $\{Y_{kl}\}_{l=0}^{L_k}$ be an orthonormal basis for $\mathcal{H}^N_k$ and $L_k = \dim \mathcal{H}^N_k$. We denote by the same letters $Y_{kl}$ the corresponding homogeneous harmonic polynomials. We identify a harmonic function

$$u(x) = \sum_{j=(k,l) \in J} a_j Y_{kl}(x)$$

with the sequence $\{a_j\}$ the sequence of its coefficients in spherical harmonics expansion. Note that for multipliers we allow distinct factors for spherical harmonics of the same order, one may restrict the notion of multipliers only to the sequences $\lambda_j$ such that $\lambda_j = \lambda_{j'}$ when $j = (k, l)$ and $j' = (k, l')$, see for example [10] Chapter 2 and references therein. We also identify the sequence $\lambda$ with a formal series

$$\lambda(x, y) = \sum_{j=(k,l) \in J} \lambda_j Y_{kl}(x)Y_{kl}(y);$$

when the series converges we get a function on $B \times B$ harmonic in each variable. This series corresponds to the multiplier operator and does not depend on the choice of basis.
1.3. Multipliers of spaces of analytic and harmonic functions. A number of natural operators on holomorphic and harmonic functions can be considered as Fourier coefficient multipliers. Multipliers between various spaces of holomorphic functions have been studied by many authors; see for example [28], [22] and [9] for multipliers of Hardy and Bergman spaces and [4] for multipliers of more general spaces. J. M. Anderson and A. L. Shields have described multipliers between the Bloch space and $\ell^p$ in [2]. The case of spaces of harmonic functions is different, for the classical unweighted space it corresponds to the difference between multipliers of $L^p$ and $H^p$. Multipliers in weighted spaces $h_g^\infty$ of harmonic functions were studied in dimension two. G. Bennett, D. A. Stegenga and R. M. Timoney determined the multipliers from $h_g^\infty$ to $\ell^p$ in [3] in the case where $g$ grows fast, for example when $g(x) = x^\alpha$ for $\alpha > 0$, and this result was generalized to other weights with doubling in [13]. It was proved by A. L. Shields and D. L. Williams in [26] that for the two-dimensional case and under some regularity assumptions on $g$ a sequence $\lambda$ is a multiplier from $h_g^\infty$ to itself if and only if

$$\int_0^{2\pi} \left| \sum_{j=-\infty}^{\infty} g(|j|) \lambda_j r^{|j|} e^{ij\theta} \right| d\theta \leq C g \left( \frac{1}{1-r} \right).$$

In this article we give another proof of this result for weighted spaces of harmonic functions in the unit ball of $\mathbb{R}^{N+1}$, where $N \geq 1$. We also show that for doubling weights the regularity conditions are not essential, one can always replace the weight $g$ by a regular one without changing the corresponding space $h_g^\infty$. Then we discuss how the space of multipliers depends on the weight $g$ and give a number of examples. We also give a description of multipliers between some pairs of weighted spaces and show that weighted averaging along radii, studied in [21, 14] for the case $N = 1$ can be considered as a multiplier with $\lambda = g(k)^{-1}$. One of our main results, Theorem 3, gives a new description of the image of $h_g^p$ under the action of this multiplier.

2. Characterization of functions in weighted spaces

2.1. Cesàro means. We consider the standard difference operators on the space of sequences. Let $b = \{b_k\}_{k=0}^\infty$ be a sequence, then $\Delta b$ is a new sequence defined by

$$(\Delta b)_k = b_k - b_{k+1}.$$ 

For $l = 1, 2, \ldots$ we further define $\Delta^l b = \Delta^l \Delta^{l-1}(\Delta b)$, where $\Delta^1 = \Delta$ and $\Delta^0 = I$ is the identity operator. Then

$$\Delta^l b_k = (\Delta^l b)_k = \sum_{j=0}^{l} (-1)^j \binom{l}{j} b_{k+j}.$$ 

For $m \in \mathbb{R}$ the Cesàro $(C, m)$ means of the sequence $b = \{b_k\}$ are defined by

$$s^m_n(b) = \frac{1}{A^m_n} \sum_{k=0}^{n} A^m_{n-k} b_k,$$

where $A^m_k = \binom{k+m}{k} = (k+m)(k+m-1)\ldots(m+1)/(k!)$. We have also $A^m_k \leq C_m k^m$ for $k \in \mathbb{N}$. 
For \( m = 1, 2, \ldots \) the summation by parts formula is given by

\[
\sum_{k=0}^{\infty} a_k b_k = \sum_{k=0}^{\infty} \Delta^{m+1} a_k \sum_{j=0}^{k} A_{k-j}^m b_j = \sum_{k=0}^{\infty} (\Delta^{m+1} a_k) A_k^m s_k^m (b),
\]

if we assume that \( \lim_{k \to \infty} (\Delta^l a_k) A_k^l s_k^l (b) = 0 \) for \( l = 0, 1, \ldots, m \), see for example Appendix A.4 in [10].

For each \( k = 0, 1, \ldots \) the zonal harmonic of order \( k \) is defined by

\[
Z_k(x, y) = \sum_{l=0}^{\dim \mathcal{H}_0^N} Y_{kl}(x) Y_{kl}(y),
\]

and this function does not depend on the choice of basis. The value of \( Z_k(x, y) \) depends on \( \langle x, y \rangle \) only, and we write \( h_k(\langle x, y \rangle) = Z_k(x, y) \). Similarly to the usual Cesàro means in dimension two, we consider higher order Cesàro means of the expansions of harmonic polynomials in higher dimensions. An interesting observation due to Kogbetliantz [18] is that the Cesàro \((C, m)\) means of the zonal harmonics are positive for \( m \geq N \), see also [10] Theorem 2.4.3. We define \( W_k^m(x, y) \) to be the Cesàro \((C, m)\) means of \( Z_k(x, y) \), i.e.

\[
W_k^m(x, y) = \frac{1}{A_k^m} \sum_{j=0}^{k} A_{k-j}^m Z_j(x, y).
\]

For \( m \geq N \) we have \( W_k^m(x, y) \geq 0 \) and this yields

\[
\|W_k^m(\cdot, y)\|_1 = \int_S |W_k^m(x, y)| ds(x) = \int_S W_k^m(x, y) ds(x) = \int_S Z_0(x, y) ds(x) = 1.
\]

for each \( k \). Here and in what follows, the norms \( \| \cdot \|_p \) are with respect to the normalized surface measure on the unit sphere of \( \mathbb{R}^{N+1} \). Moreover, \( \sup_y \|W_k^m(\cdot, y)\|_1 \leq C \) when \( m > (N - 1)/2 \), see [10] Theorem 2.4.4 and references therein, in particular [27].

For the rest of the text we fix a positive integer \( d \) such that \( d > (N - 1)/2 \) (when \( N = 1 \) one may choose \( d = 1 \)).

We will also need some auxiliary harmonic polynomials. Let \( a_m = (1 - \frac{1}{m})^{-m} \), \( m \geq 0 \), then \( a_m \) are uniformly bounded. Consider the function \( q_m(t) = a_m^t \) on \( 0 \leq t \leq 1 \) and define a continuation \( q_m \) of this function for \( t > 1 \) such that \( q_m(t) = 0 \) for \( t \geq 2 \) and the derivatives of \( q_m \) up to order \( d + 1 \) are bounded by some uniform constant \( M = M(d) \) that does not depend on \( m \). Let also \( q_{m,n,k} = q_m(\frac{k}{n}) \). Now define

\[
Q_m,n(x, y) = \sum_{k=0}^{2n} q_{m,n,k} Z_k(x, y).
\]

**Lemma 1.** There exists a constant \( C \) such that the functions \( Q_{m,n} \) defined by (4) satisfy \( \|Q_m,n(\cdot, y)\|_1 = \|Q_m,n(\cdot, y)\|_1 \leq C \) for all \( m \) and \( n \).

**Proof.** By (3) we get

\[
Q_m,n(x, y) = \sum_{k=0}^{\infty} q_{m,n,k} Z_k(x, y) = \sum_{k=0}^{\infty} \Delta^{d+1} q_{m,n,k} A_k^d W_k^d(x, y).
\]

Since the functions \( q_m \) defined above have bounded derivatives up to order \( d + 1 \) and vanish for \( t \geq 2 \), we have \( |\Delta^{d+1} q_{m,n,k}| \leq C \frac{1}{n^{d+2}} \), and \( \Delta^{d+1} q_{m,n,k} = 0 \) for \( k > 2n \).
Then

\[ \|Q_{m,n}(\cdot, y)\|_1 \leq \sum_{k=0}^{\infty} |\Delta^{d+1}\alpha_{m,n,k}|A^d_k\|W^d(\cdot, y)\|_1 \leq C \sum_{k=0}^{2n} \frac{A^d_k}{n^{d+1}} \leq C \sum_{k=0}^{2n} \frac{1}{n \cdot d!} \leq C. \]

\[ \square \]

We will be interested in two particular cases of these function,

(5) \[ Q_n(x, y) = Q_{n,n}(x, y) \quad \text{and} \quad R_n(x, y) = Q_{0,n}(x, y). \]

For a harmonic function \( u \) in the unit ball we also define the operators

(6) \[ R_n u(rx) = \int_S R_n(x, y)u(ry)ds(y), \quad |x| = 1, \quad 0 \leq r < 1. \]

The definition depends on the smooth extension of the constant function \( q_0 \) from \([0, 1]\) to \([0, 2]\). In dimension two we may choose this extension such that the resulting operators are the de la Vallée-Poussin sums of \( u \). These operators in connection with growth spaces were used by W. Lusky in [19].

2.2. Characterization. First we prove that the doubling condition implies a useful estimate. We note also that a milder restriction on the growth of the weight function than doubling would work here.

**Proposition 1.** Let \( g \) fulfill (1), then for any \( m \geq 0 \) we have

\[ \sum_{k=0}^{\infty} r^k A^m_k g(k) \leq C_{m,D} g \left( \frac{1}{1-r} \right) (1-r)^{-m-1}. \]

For \( m = 1 \) this inequality can be found in [26].

**Proof.** Let \( r \) be given and choose \( N \) such that \( 1 - \frac{1}{N} < r \leq 1 - \frac{1}{N+1} \). Then

\[ \sum_{k=0}^{N} r^k A^m_k g(k) \leq g(N)A^m_N \sum_{k=0}^{N} r^k \leq C_m g(N)N^m \frac{1}{1-r} \leq C_m g \left( \frac{1}{1-r} \right) (1-r)^{-m-1} \]

and the rest \( \sum_{k=N+1}^{\infty} r^k A^m_k g(k) \) can be written as

\[ \sum_{j=0}^{\infty} r^{2^j N} \sum_{i=1}^{2^j N} r^i A^m_{2^j N+i} g(2^j N + i) \leq C_m \frac{g(N)}{1-r} \sum_{j=0}^{\infty} r^{2^j N} g(2^j N)(2^j N+1)^m \]

\[ \leq C_m g(N) \frac{1}{1-r} \sum_{j=0}^{\infty} e^{-2^j D_j+1 + 2(j+1)m} \leq C_{m,D} g \left( \frac{1}{1-r} \right) (1-r)^{-m-1}, \]

the required inequality follows. \( \square \)

Let \( u \) be a harmonic function in the unit ball, \( u(x) = \sum_{j=0}^{\infty} \sum_{l=1}^{L_j} a_{jl} Y_{jl}(x) \). The Cesàro \((C, m)\) means of \( u \) are denoted by

\[ \sigma^m_n u(x) = \frac{1}{A^m_n} \sum_{j=0}^{n} A^m_{n-j} \sum_{l=1}^{L_j} a_{jl} Y_{jl}(x). \]
For a polynomial of the form $P_n(x, y) = \sum_{j=0}^{n} \sum_{l=1}^{L_j} c_{jl} Y_{jl}(x) Y_{jl}(y)$, we write

$$u \ast P_n(x) = \int_{S} P_n(x', y) u(ry) ds(y) = \sum_{j=0}^{\infty} \sum_{l=1}^{L_j} a_{jl} c_{jl} Y_{jl}(x),$$

where $x = rx', x' \in S^N$. In particular, $\sigma_n^m u = u \ast W_n^m$.

In [15] we showed that harmonic functions in the upper half-space of $\mathbb{R}^{N+1}$ that are bounded by a majorant can be characterized by the size of their convolutions with functions having compactly supported Fourier transforms. For functions on the ball this corresponds to estimates of the convolutions with polynomials. The statement below ((a) equivalent to (c)) also generalizes the result for weighted spaces of harmonic functions on the unit disk proved in [3], to weighted spaces in several dimensions.

**Theorem 1.** Let $u(x) = \sum_{j=0}^{\infty} \sum_{l=1}^{L_j} a_{jl} Y_{jl}(x)$ be a harmonic function on the unit ball of $\mathbb{R}^{N+1}$, $1 \leq p \leq \infty$, and let $q$ be a weight function satisfying the doubling condition and $d > (N - 1)/2$. Then the following are equivalent:

(a) $u \in h^p_g$,

(b) There exists a constant $C$ such that $\|u \ast P_n\|_p \leq C \sup_x \|P_n(x, \cdot)\|_1 |g(n)|$ for any polynomial of the form $P_n(x, y) = \sum_{j=0}^{n} \sum_{l=1}^{L_j} c_{jl} Y_{jl}(x) Y_{jl}(y)$,

(c) $\|\sigma^m u\|_p \leq C g(n)$,

(d) $\|R_n u\|_p \leq C g(n)$.

**Proof.** First, we show that (a) implies (b). Suppose that $P_n$ is a polynomial as in (b), and let $Q_n = Q_n,S$ be the function in Lemma [1]. Let also $t = 1 - 1/n$, then $q_{n,j} = t^{-j}$ for $j \leq n$. We consider a new polynomial

$$\tilde{P}_n(x, y) = \sum_{k=0}^{n} \sum_{m=1}^{L_k} c_{km} t^{-k} Y_{km}(x) Y_{km}(y).$$

Then, applying Lemma [1] we obtain

$$\sup_x \|\tilde{P}_n(x, \cdot)\|_1 = \sup_x \left| \int_S \int_{S} \sum_{j=0}^{n} \sum_{l=1}^{L_j} c_{jl} Y_{jl}(x) Y_{jl}(z) \sum_{k=0}^{\infty} q_{n,k} Z_k(z, y) ds(z) \right| ds(y)$$

$$= \sup_x \left| \int_S \int_{S} P_n(x, z) Q_n(z, y) ds(z) \right| ds(y) \leq C \sup_x \|P_n(x, \cdot)\|_1.$$

Now let $u \in h^p_g$, then using $S_t = \sum_{k=0}^{\infty} t^k Z_k$, we get

$$\|u \ast P_n\|_p = \|u \ast S_t \ast \tilde{P}_n\|_p \leq$$

$$\leq C \left( g \left( \frac{1}{1-t} \right) \right)^p \sup_x \|\tilde{P}_n(x, \cdot)\|_T^p \leq C g(n)^p \sup_x \|P_n(x, \cdot)\|_T^p.$$

Hence (b) follows.

It is clear that (b) implies (c) and (d); we take convolutions with $W^d_n(x, y)$ and $R_n(x, y)$ (see [5] and Lemma [1]) and we are done since the $L^1$-norms of $W^d_n(x, \cdot)$ and $R_n(x, \cdot)$ are bounded uniformly on $n$ and $x \in S$.
To finish the proof we show that (c) implies (a) and (d) implies (c). First assume (c). By (3),

$$u(rx) = \sum_{j=0}^{\infty} \sum_{l=0}^{L_j} a_{jl} r^j Y_{jl}(x) = (1-r)^{d+1} \sum_{k=0}^{\infty} r^k A_{k}^d \sigma_k^d u(x).$$

Then (a) and Proposition 1 imply $$\|u(rx)\|_p \leq (1-r)^{d+1} \sum_{k=0}^{\infty} r^k A_{k}^d \sigma_k^d \leq Cg(\frac{1}{1-r}),$$
and then $$u \in h_p^g.$$

Finally, assume that (d) holds. Then $$u \ast P_n = \mathcal{R}_n u \ast P_n$$ and (b) follows since $$\|u \ast P_n\|_p \leq \|\mathcal{R}_n u\|_p \sup_x \|P_n(x, \cdot)\|_{L^1}.$$

\( \square \)

For the case $$N = 1$$ and $$1 < p < +\infty$$ (a) is equivalent to (c) with $$d = 0$$, see [12].
This does not hold for $$p = +\infty$$, see [3], the choice of optimal $$d$$ for particular $$p$$ is a delicate question, see [10] Chapter 2.5 for a related discussion.

3. Regular multipliers

3.1. Some multipliers from $$h_p^g$$ to $$h_p^g$$. Let $$f$$ be a positive, increasing function on $$[1, \infty)$$ which has derivatives up to order $$d$$ and satisfies (1). We also assume that there exists a constant $$C > 0$$ such that

$$|f^{(m)}(t)| \leq C \frac{f'(t)}{tm-1}$$

for $$1 \leq m \leq d+1$$. This assumption concerns the regularity of $$f$$ and does not restrict the growth further. Our usual examples $$f(x) = x^\alpha$$ and $$f(x) = (1 + \log x)^\alpha$$ for $$\alpha > 0$$ fulfill (4). In the next subsection we also show that for any weight $$g$$ satisfying the doubling condition one can construct an equivalent weight $$f$$ such that $$f(n) \simeq g(n)$$ and $$f$$ satisfies (4). Let $$u(x) = \sum_{j=0}^{\infty} \sum_{l=1}^{L_j} a_{jl} Y_{jl}(x)$$ and define an operator $$H_f$$ by

$$H_f u(x) = \sum_{j=0}^{\infty} \sum_{l=1}^{L_j} f(j) a_{jl} Y_{jl}(x).$$

This operator can also be considered as an operator on sequences by the identification of functions with their coefficients. We also define an operator $$H_f^{-1}$$ by

$$H_f^{-1} u(x) = \sum_{j=0}^{\infty} \sum_{l=1}^{L_j} \frac{a_{jl}}{f(j)} Y_{jl}(x).$$

We will now prove the following:

**Theorem 2.** Assume $$f$$ and $$g$$ satisfy (1) and $$f$$ satisfies (7), $$1 \leq p \leq \infty$$.\( \text{(a)} \) Let $$u \in h_p^g$$, then $$H_f u \in h_p^g$$.
\( \text{(b)} \) Assume there exists $$\varepsilon > 0$$ such that $$g(n)/f^{1+\varepsilon}(n) \nearrow \infty$$ when $$n \to \infty$$ and let $$u \in h_p^g$$, then $$H_f^{-1} u \in h_p^g$$.
\( \text{(c)} \) If $$g/f$$ is non-decreasing and $$u \in h_p^g$$, then $$H_f^{-1} u \in h_p^{(g/f) \log f}$$.

In [26] parts (a) and (b) were proved in the two-dimensional case. The proof given there is based on a duality construction which involves the measure whose moments coincide with $$f(n)$$. Our proof is different and employs Theorem 4.
Proof. (a) By Theorem 1 and 3,
\[ \|\sigma_n^d(Hf u)\|_p \leq C g(n) \left| \frac{1}{A_n^d} \sum_{j=0}^{n} A_{n-j}^d f(j) Z_j(x, \cdot) \right| \]
\[ \leq C g(n) \left| \frac{1}{A_n^d} \sum_{j=0}^{n} \Delta^{d+1}(A_{n-j}^d f(j)) \sum_{k=0}^{j} A_{j-k}^d Z_k(x, \cdot) \right| \]
\[ \leq C g(n) \sum_{j=0}^{n} \frac{\Delta^{d+1}(A_{n-j}^d f(j)) A_j^d}{A_n^d} \|W_j^d(x, \cdot)\|_1 \leq C g(n) \sum_{j=0}^{n} \frac{\Delta^{d+1}(A_{n-j}^d f(j)) A_j^d}{A_n^d}, \]
so if we show that
\[ \sum_{j=0}^{n} \frac{\Delta^{d+1}(A_{n-j}^d f(j)) A_j^d}{A_n^d} \leq C f(n), \]
we are done since Theorem 1 then implies that \( Hf u \in h^p_{\text{of}} \).

Define a function \( h_n \) by
\[ h_n(t) = M_{n,d}(t)f(t) = \left( 1 - \frac{t}{n+d} \right) \left( 1 - \frac{t}{n+d-1} \right) \ldots \left( 1 - \frac{t}{n+1} \right) f(t) \]
for \( 0 \leq t \leq n + d + 1 \). Then \( h_n(j) = A_{n-j}^d (A_n^d)^{-1} f(j) \). Further, it is easy to check that for \( t \leq n + d + 1 \) we have \( |M_{n,d}^{(m)}(t)| \leq C_{m,d} n^{-m} \) when \( m \leq d \) and \( M_{n,d}^{(d+1)}(t) = 0 \). Then, applying (7), we get
\[ |h_n^{(d+1)}(t)| \leq \sum_{j=0}^{d+1} \binom{d+1}{j} |M_{n,d}^{(j)}| |f^{(d+1-j)}(t)| \]
\[ \leq C d \sum_{j=1}^{d} \binom{d+1}{j} \frac{1}{n^j} \frac{f'(t)}{t^d j^d} \leq C d \frac{f'(t)}{t^d d} \]
for \( t \leq n + d + 1 \). We note that \( \Delta^{d+1}(A_{n-j}^d f(j))/A_n^d \) is equal to the divided differences \( h_n[j, j+1, \ldots, j+d+1] \) of the function \( h_n \) and express it on the Peano form,
\[ \Delta^{d+1}(A_{n-j}^d f(j))/A_n^d = h_n[j, j+1, \ldots, j+d+1] = \frac{1}{(d+1)!} \int_{j}^{j+d+1} h_n^{(d+1)}(t) B_d(t) dt, \]
for some B-spline \( B_d \). Then by (9),
\[ |\Delta^{d+1}(A_{n-j}^d f(j))/A_n^d| \leq \frac{1}{(d+1)!} \int_{j}^{j+d+1} |h_n^{(d+1)}(t)| B_d(t) |dt| \]
\[ \leq C \int_{j}^{j+d+1} \frac{f'(t)}{t^d d} |dt| \leq C \frac{f(j+d+1) - f(j)}{j^d}. \]
Hence
\[ \sum_{j=0}^{n} \frac{\Delta^{d+1}(A_{n-j}^d f(j)) A_j^d}{A_n^d} \leq C \sum_{j=0}^{n} j^d \left( \frac{f(j+d+1) - f(j)}{j^d} \right) \leq C f(n) \]
and (8) is proved.
(b) Applying Theorem 11 and 3, we obtain similarly to (a)

$$
\|s_n^d(H_f^{-1}u)\|_p = \left\| \frac{1}{A_n^d} \sum_{j=0}^n \Delta^{d+1} \left( \frac{A_{n-j}^d}{f(j)} \right) \sum_{k=0}^{L_k} \sum_{l=1}^{a_{j-l}} \sum_{i=1}^{A_i^d} A_j^d \right\|_p 
\leq \frac{1}{A_n^d} \sum_{j=0}^n \Delta^{d+1} \left( \frac{A_{n-j}^d}{f(j)} \right) A_j^d \|s_j^d\|_p \leq C \frac{1}{A_n^d} \sum_{j=0}^n \Delta^{d+1} \left( \frac{A_{n-j}^d}{f(j)} \right) A_j^d g(j).
$$

Similarly to the proof of (a) we define a function $p_n(t) = \frac{M_{n,j}(t)}{f(t)}$ and estimate the corresponding coefficients $\frac{1}{A_n^d} \Delta^{d+1} \left( \frac{A_{n-j}^d}{f(j)} \right)$. It can be shown, by using (7), that

$$
\left| \left( \frac{1}{f(t)} \right) (m) \right| \leq C \frac{f'(t)}{t^{m-1} f(t)^2},
$$

where the constant also depends on the constant in (7). Then applying the estimates for $|M_{n,j}(t)|$ given in the proof of (a), we get for $t \leq n + d + 1$

$$
|p_n^{d+1}(t)| \leq C_d \sum_{j=1}^d (d + 1) \left( \frac{f'(t)}{t^{d-j} f(t)^2} \right) \leq C_d \frac{f'(t)}{t^d f(t)^2}.
$$

And, using the formula for divided differences once again, we get

$$
\Delta^{d+1} \left( \frac{A_{n-j}^d}{f(j)} \right) \frac{1}{A_n^d} \leq \frac{1}{(d+1)!} \int_j^{j+d+1} \frac{|p_n^{(d+1)}(t)||B_d(t)|dt}{f'(t)} \leq C_d \frac{1}{j^d} \left( \frac{1}{f(j)} - \frac{1}{f(j + d + 1)} \right).
$$

Let $n_0 = 1$ and for some $A > 1$ define $n_k$ by induction as

$$
n_{k+1} = \min \{ l \in \mathbb{N} : f(l) \geq Af(n_k) \},
$$

then $Af(n_k) \leq f(n_{k+1}) \leq ADf(n_k)$, where $D$ is the constant from (11). Let $N$ be such that $n_{N-1} < n \leq n_N$, then applying the inequality above and monotonicity of $g/f^{1+\varepsilon}$, we get

$$
\sum_{j=0}^n \Delta^{d+1} \left( \frac{A_{n-j}^d}{f(j)} \right) A_n^d g(j) \leq C_d \sum_{j=0}^n g(j) \left( \frac{1}{f(j)} - \frac{1}{f(j + d + 1)} \right) \leq C_d \frac{g(n)}{f(n)^{1+\varepsilon}} \sum_{j=0}^n \frac{f(j + d + 1) - f(j)}{f(j)^{1-\varepsilon}} \leq C_d \frac{g(n)}{f(n)^{1+\varepsilon}} \sum_{k=0}^{N-1} \frac{f(j + d + 1) - f(j)}{f(n_k)^{1-\varepsilon}} \leq C_d \frac{g(n)}{f(n)^{1+\varepsilon}} \sum_{k=0}^{N-1} f(n_{k+1})^\varepsilon \leq C_d \frac{g(n)}{f(n)},
$$

and we are done.
(c) The proof proceeds exactly like in (b), but we replace (11) by the following:

\[
\sum_{j=0}^{n} \Delta_{k+1} \left( \frac{A_{n-j}}{f(j)} \right) \frac{A_{n-j}^d g(j)}{\tilde{A}_n^d} \leq C_d \frac{g(n)}{f(n)} \sum_{j=0}^{n} j^d f(j) \left( \frac{1}{f(j)} - \frac{1}{f(j + d + 1)} \right) \\
\leq C_d \frac{g(n)}{f(n)} \sum_{j=0}^{n} \frac{f(j + d + 1) - f(j)}{f(j)} \leq C_d \frac{g(n)}{f(n)} \sum_{k=0}^{N-1} \sum_{j=n_k+1}^{n_{k+1}} \frac{f(j + d + 1) - f(j)}{f(n_k)} \\
\leq C_d D A \frac{g(n)}{f(n)} N \leq C_d D A \frac{g(n) \log f(n)}{f(n) \log A}.
\]

\[\square\]

3.2. Sharpness of the result. We give some examples showing that part (b) of Theorem 2 does not hold without the assumption \(g(n)/f^{1+\varepsilon}(n) \nearrow \infty\) and that the weight in part (c) is sharp. We consider the case \(p = \infty\).

A harmonic function on the disk \(D\) is called a Hadamard gap series if

\[(12)\quad u(z) = \Re \sum_{k=1}^{\infty} a_{n_k} z^{n_k}, \quad a_{n_k} \in C,\]

where \(n_{k+1} > \lambda n_k\) for some \(\lambda > 1\). In [14] Hadamard gap series were characterized in the following way:

**Theorem A.** Let \(\{n_k\}_{k=1}^{\infty}\) be a sequence of positive integers such that \(n_{k+1} \geq \lambda n_k\) for each \(k\), where \(\lambda > 1\). Assume \(g\) satisfies (1) and let \(u\) be given by (12) where the series converges in the unit disk. Then \(u \in h^g\) if and only if there exists \(C\) such that

\[\sum_{n_k \leq N} |a_{n_k}| \leq C g(N)\]

for any \(N \in \mathbb{N}\).

We use this to construct some examples.

Let \(u(z) = \sum_{j=0}^{\infty} 2^j z^{2^j}\), then by Theorem A, \(u \in h^g\) for \(g(x) = x\). Let \(f(x) = x/\log x\) and consider \(H_f^{-1} u(z) = \Re \sum_{j=0}^{\infty} \frac{2^j}{f(2^j)} z^{2^j}\). Then

\[\sum_{j=0}^{J} \frac{2^j}{f(2^j)} = \log 2 \sum_{j=0}^{J} j \approx J^2 \approx (\log 2^j)^2\]

hence by Theorem A, \(H_f^{-1} u \in h^g\) for \(\tilde{g} = (\log x)^2\), and \(H_f^{-1} u \notin h^g\) for any \(\tilde{g}\) which is \(o((\log x)^2)\). This shows that \(H_f^{-1} u\) grows faster than \(g/f = \log x\), and the growth assumption in (b) is necessary.

To construct a more general example, define a sequence \(\{m_k\}\) for \(g\) as in (10). The function \(u(z) = \sum_{j=1}^{\infty} g(m_k) z^{m_k}\) is in \(h^g\) by Theorem A. If \(f = g/\log g\), we get by a similar calculation as above that \(H_f^{-1} u \in h^g\) for \(\tilde{g}(x) = (\log g(x))^2\), and \(H_f^{-1} u\) does not belong to any weighted space with weight which grows slower than \(\tilde{g}\).

If the growth of \(g\) and \(f\) is even more similar, for example \(f = g/\log \log g\), we get for the function \(u(z) = \sum_{j=1}^{\infty} g(m_k) z^{m_k}\) that \(H_f^{-1} u \in h^g\) for \(\tilde{g}(x) = \log g \log \log g\). This shows that (c) is sharp, since \((g/f) \log f = \log g \log \log g\). When we take \(f = g\)
To estimate \( f \) it is a coefficient multiplier of the form \( H \). Let \( m > 1 \) and the same function \( u \) we obtain \( H^{-1}u(z) = \sum_{k=1}^{\infty} z^{m_k} \). Then Theorem A implies \( H^{-1}u \in h_{\log}^{\infty} \), we discuss this case in the next subsection.

3.3. Integral operators as multipliers. Let \( q(t) \) be an increasing weight that satisfies the doubling condition with some constant \( D \) and let \( w(t) = q(1/(1-t)) \), for \( t < 1 \). We consider the following integral operator defined on harmonic functions in the unit ball (see [21, 14, 15])

\[
I_{q}u(x) = \int_{1/2}^{1} u(tx)d(-1/w(t)).
\]

Let \( u(x) = \sum_{k,l} a_{kl} Y_{kl}(x) \) be the decomposition of \( u \) into spherical harmonics, then

\[
I_{q}u(x) = \sum_{k,l} a_{kl} \left( \int_{1/2}^{1} t^{k}d(-1/w(t)) \right) Y_{kl}(x).
\]

It is a coefficient multiplier of the form \( H^{-1} \), where

\[
f(n) = \left( \int_{1/2}^{1} t^{n}d(-1/w(t)) \right)^{-1}.
\]

We have the following elementary estimate from above

\[
f(n) \leq \left( \int_{1-1/n}^{1} t^{n}d(-1/w(t)) \right)^{-1} \leq 4q(n).
\]

To estimate \( f(n) \) from below we choose integers \( m \) and \( M \) such that \( 2^{M} > 2D \) but \( m > \alpha n \), where \( \alpha \) depends on \( D \) only, and \( 2^{M} < m \leq 2^{M+1} \). Then we get

\[
\int_{1/2}^{1-1/m} t^{n}d(-1/w(t)) \leq \sum_{j=1}^{M} \int_{1-2^{-j-1}}^{1-2^{-j}} t^{n}d(-1/w(t)) \leq \sum_{j=1}^{M} (1-2^{-j-1})^{n} \frac{1}{q(2^{j})}.
\]

We want to estimate the last sum by a multiple of its final term. It is enough to check that it grows faster than a geometric progression. Using an elementary inequality \( (1-x)/(1-2x) \geq 1 + x \), when \( 0 < x < 1/2 \), and the standard estimate \( (1 + (2m)^{-1})^{2m} \geq 2 \), we obtain

\[
\frac{(1-2^{-j-1})^{n}}{q(2^{j})} \geq D^{-1}(1 + 2^{-j-1})^{n} \geq D^{-1}(1 + (2m)^{-1})^{n} \geq D^{-1} 2^{M} > 2,
\]

for \( j = 2, \ldots, M \). Then

\[
f(n)^{-1} \leq \int_{1/2}^{1-1/m} t^{n}d(-1/w(t)) + \frac{1}{q(m)} \leq 2(1-2^{-M-1})^{n} \frac{1}{q(2^{M})} + \frac{1}{q(m)} \leq \frac{C}{q(n)},
\]

where \( C \) depends on \( D \) only.

For \( \alpha > 1 \) we define also

\[
f(\alpha) = \left( \int_{1/2}^{1} t^{\alpha}d(-1/w(t)) \right)^{-1}.
\]

To check the regularity of the function \( f \) we need estimates of the integrals

\[
j_{k}(\alpha) = \int_{1/2}^{1} |\log t|^{k}t^{\alpha}d(-1/w(t)),
\]
for \( k = 1, \ldots, d \). A calculation similar to the one above implies \( j_k(\alpha) \leq C\alpha j_{k-1}(\alpha) \) when \( k = 1, \ldots, d \) and \( C = C(d, D) \). Further, \( f' = j_1 f^2 \) and taking the derivatives on both sides and using induction we see that \( f^{(k)}(\alpha) \leq C f'(\alpha) \alpha^{1-k} \) when \( k = 1, \ldots, d \).

We summarize the estimates of this subsection in the following statement.

**Proposition 2.** Let \( q \) be a positive continuous increasing function that satisfies the doubling condition, define \( w(t) = q((1 - t)^{-1}) \) and let \( f \) be given by \( (13) \). Then \( f \) satisfies \((7)\) and there exists \( A = A(q) \) that depends on \( q \) only such that \( A^{-1} q(n) \leq f(n) \leq A q(n) \).

We say that \( f \) is the regularization of \( q \). Further, note that \( I_q = H_f^{-1} \) and we can apply Theorem 2 to this integral operator. In \([21, 14, 15]\) the operator \( I_q \) was studied on \( h^\infty_\omega \), which corresponds to the case \( f = g \) in Theorem 2 (c). A more delicate result, a version of the law of the iterated logarithm, was obtained for this case. In the next section we define new spaces that are better suited for the study of \( I_q \) and multipliers described in part (c) of Theorem 2.

### 3.4. Regular growth spaces

Let \( \mathcal{N} = \{ n_k \} \) be an increasing sequence with \( n_{k+1} > 2n_k, n_0 = 0 \). We introduce spaces of harmonic functions of regular growth with respect to \( \mathcal{N} \) in the following way

\[
h^p_\mathcal{N} = \{ u : B \to \mathbb{R}, \Delta u = 0, \sup_{k} \sup_{n_k \leq m \leq n_{k+1}} \|(\mathcal{R}_m - \mathcal{R}_n)u\|_p < \infty \}.\]

This definition formally depends on the construction of \( \mathcal{R}_n \), which is based on some smooth extension \( q_0 \) of the characteristic function of \([0,1]\). The theorem below shows however that \( h^p_\mathcal{N} \) has another description and does not depend on the choice of \( q_0 \).

Let \( f \) be a doubling weight and let \( \mathcal{N} \) be a sequence associated with \( f \) by \((10)\). Then Theorem 1 (d) implies that \( u \in h^p_f \) if and only if

\[
\sup_{n_k \leq m \leq n_{k+1}} \|(\mathcal{R}_m - \mathcal{R}_n)u\|_p \leq C f(n_{k+1})
\]

and \( h^p_\mathcal{N} \) is a subspace of \( h^p_{\log f} \). It is easy to see that it is a proper subspace. We claim that \( h^p_\mathcal{N} \) is the correct target space for the multiplier \( H_f^{-1} \).

**Theorem 3.** Assume \( f \) satisfies \((1)\) and \((7)\), \( 1 \leq p \leq \infty \), and let \( \mathcal{N} \) be associated with \( f \). Then \( H_f \) maps \( h^p_\mathcal{N} \) into \( h^p_f \) and its inverse \( H_f^{-1} \) maps \( h^p_f \) into \( h^p_\mathcal{N} \).

**Proof.** We need the following statement, if \( f \) satisfies \((1)\) and \((7)\), and then for any \( n \) there exists polynomial \( T_{n,f}(x,y) = \sum_{j} a_j Z_j(x,y) \) such that \( a_j = f(j) \) when \( j \leq n \) and \( \|T_{n,f}(x,\cdot)\|_1 \leq C f(n) \). The proof is similar to one we gave when proving Theorem 2 (a) and (b). Let \( a \) be a function with \( d+1 \) bounded derivatives such that \( a = 1 \) on \([0,1]\) and \( a = 0 \) on \((2, +\infty)\). Define \( a_j = f(j) a(j/n) \), since \( \|(f(t)a(t/n))((d+1)!)\|_1 \leq C_k(f(t)n^{-d-1} + f'(t)t^{-d}) \) when \( t < 2n \), we have

\[
\Delta^{d+1} a_j \leq C_d(f(2n)n^{-d-1} + f(j) + f(j)-j^{-d}).
\]

Then \( \|T_{n,f}(x,\cdot)\|_1 \leq \sum_{j=1}^{2n} A_j (\Delta^{d+1} a_j) \leq C_d f(2n) \).

Suppose that \( u \in h^p_\mathcal{N} \) then for \( n_k \leq m \leq n_{k+1} \) we have \( (\mathcal{R}_m - \mathcal{R}_n_k)(H_f u) = T_{2m,f} \ast (\mathcal{R}_m - \mathcal{R}_n_k)u \) and

\[
\|(\mathcal{R}_m - \mathcal{R}_n_k)(H_f u)\|_p \leq C f(4m) \|(\mathcal{R}_m - \mathcal{R}_n_k)u\|_p \leq CD^2 f(n_{k+1}).
\]

Thus \( H_f u \in h^p_f \).
To prove the inverse we need another auxiliary function $S_{n,m,f} = \sum_j b_j Z_j(x,y)$ where $b_j = f^{-1}(j)$ when $n \leq j \leq m$. Let function $b$ with bounded derivatives up to order $d+1$ be such that $b = 1$ on $(0,1)$ and $b = 0$ on $(2, +\infty)$. Consider $b_j = (b(j/m) - b(2j/n)/f(j))$. We have $|b^{(d+1)}(t)| \leq C(n^{-d-1}f(n)^{-1} + f'(t)f(t)^{-2}t^{-d})$ when $n/2 \leq t \leq n$ and $|b^{(d+1)}(t)| \leq C(m^{-d-1}f(n)^{-1} + f'(t)f(t)^{-2}t^{-d})$ when $n \leq t \leq 2m$. Hence

$$\|S_{n,f}(x,\cdot)\|_1 \leq \sum_{j=n/2}^n A_j^d|\Delta^{d+1}b_j| + \sum_{j=n}^{2m} A_j^d|\Delta^{d+1}b_j| \leq C_d f(n)^{-1}.$$  

Now suppose that $u \in H_f^p$, we have $(\mathcal{R}_m - \mathcal{R}_{n_k})(H_f^{-1}u) = S_{n_k,2m,f} * (\mathcal{R}_m - \mathcal{R}_{n_k})u$ and 

$$\|\mathcal{R}_m - \mathcal{R}_{n_k}u\|_p \leq C f((nk)^{-1})\|\mathcal{R}_m - \mathcal{R}_{n_k}u\|_p \leq C.$$

Thus $H_f^{-1}u \in h^p_{\mathcal{X}}$.  

4. Fourier multipliers on weighted spaces

4.1. Characterization of multipliers. First, we give a description of all multipliers between two weighted spaces $h^\infty_g$ and $h^\infty_\tilde{g}$ when the weight $\tilde{g}$ does not grow much slower than $g$, for example the condition of the theorem holds for $\tilde{g} = g$. We remind that a multiplier is considered as a formal series and define the operator $H_f$ on $\lambda$ by

$$H_f \lambda(x,y) = \sum_{k,l} \lambda_{kl} f(k) Y_{kl}(x) Y_{kl}(y).$$

Then $H_f \lambda$ is also a formal series but its partial sums are well-defined functions on $S \times S$.

**Theorem 4.** Let $g$ and $\tilde{g}$ satisfy \([11]\). Assume that $\tilde{g}(n)/g(n)^\epsilon \nearrow \infty$ as $n \to \infty$ for some $\epsilon > 0$. Then $\lambda$ is a multiplier from $h^\infty_\tilde{g}$ to $h^\infty_g$ if and only if

$$\|\sigma_n^d(H_f \lambda)))(\cdot, y)\|_1 \leq C \tilde{g}(n),$$

where $f$ is the regularization of $g$ defined in Proposition \([2]\).

This generalizes \([2]\), which is Theorem 6 in \([26]\). We main job is already done; we will deduce this result from Theorem \([2]\) in a way similar to that in \([26]\).

**Proof.** Let $\lambda \in (h^\infty_{\tilde{g}}, h^\infty_g)$. By Theorem \([2]\) (a) $H_f$ is a bounded operator from $h^\infty$ to $h^\infty_{\tilde{g}} = h^\infty_g$, where $h^\infty$ is the space of bounded harmonic functions in the ball. Let $u = \sum_{k,l} a_{kl} Y_{kl}(x) \in h^\infty$ be an arbitrary function with $\|u\|_\infty \leq 1$. Then since $\lambda(H_f u) \in h^\infty_{\tilde{g}}$, Theorem \([1]\) implies,

$$\left| \int_S \sigma_n^d(H_f \lambda)(x,y) u(x) ds(x) \right| = \left| \sum_{k,l} a_{kl} f(k) \lambda_{kl} \frac{A_d}{A_n}^{n-k} Y_{kl}(y) \right| = |\sigma_n^d(\lambda(H_f u))(y)| \leq C \tilde{g}(n).$$

Hence $\|\sigma_n^d(H_f \lambda)(\cdot, y)\|_1 \leq C \tilde{g}(n)$.
Now suppose \( \| \sigma_n^d(H_f\lambda)(\cdot, y) \|_1 \leq C g(n) \) and let \( v \in h^{\infty}_f \). Then by Theorem \([1]\),

\[
|\sigma_n^d(\lambda(H_f v))(y)| = \left| \int_S v(x) \sigma_n^d(H_f\lambda)(x, y) ds(x) \right| \\
\leq C g(n) \| (\sigma_n^d(H_f\lambda))(\cdot, y) \|_1 \leq C_1 f(n) g(n).
\]

Thus, \( \lambda(H_f v) \in h^{\infty}_f \) but \( \lambda(v) = H_f^{-1}(\lambda(H_f v)) \) and by Theorem \([2]\) (b) \( \lambda(u) \in h^{\infty}_g \).

4.2. Solid spaces and weighted mix-norm spaces. In this subsection we collect some basic facts on solid spaces that will be needed to discuss various examples of multipliers. A sequence space \( A \) is called solid if \( a\lambda \in A \) whenever \( a \in A \) and \( \lambda \in \ell^\infty \).

If \( A \) is not solid, we can instead find solid spaces contained in it or containing it. The smallest solid space containing \( A \) is denoted \( S(A) \), and the largest solid space contained in \( A \) is \( s(A) \). The study of smallest and largest solid spaces in connection with multipliers problems was initiated in \([2]\), see also \([1, 7]\).

The space \( h^{\infty}_g \) is not solid, and this means that functions in \( h^{\infty}_g \) cannot be characterized in terms of the absolute values of their coefficients. For the two-dimensional case it is known that \( S(h^{\infty}_g) \) is the space of functions whose coefficients satisfy

\[
\left( \sum_{j=-n}^n |a_j|^2 \right)^{1/2} \leq C g(n),
\]

and \( s(h^{\infty}_g) \) consists of functions satisfying

\[
\sum_{j=-n}^n |a_j| \leq C g(n),
\]

see \([3]\). The most difficult part is based on a theorem of de Leeuw, Katznelson and Kahane, see \([11]\). Applying this result one easily gets a similar characterization in higher dimensions.

Let \( J = \{(k, l) : k \in \mathbb{Z}_+, 1 \leq l \leq L_k \} \) be the index set for the spherical harmonics expansion. Let further \( \{n_m\}_m \) be an increasing sequence of positive integers, we define \( J_0 = \{(k, l) \in J : 0 \leq k \leq n_0 \} \) and \( J_m = \{(k, l) \in J : n_{m-1} < k \leq n_m \} \) for \( m \geq 1 \). Let \( \lambda \) be a sequence and let

\[
\| \lambda \|_{p,q} = \left( \sum_{m=0}^{\infty} \left( \sum_{j \in J_m} |\lambda_j|^p \right)^{q/p} \right)^{1/q},
\]

where \( 1 \leq p, q < \infty \). If \( p \) or \( q \) are infinite, we replace the corresponding sum by a supremum. The space of sequences \( \{\lambda_j\}_{j \in J} \) for which \( \| \lambda \|_{p,q} \) is finite is called a mixed-norm space. If \( p = q \) this is just the usual \( \ell^p \) space. Multipliers between such spaces were determined by C. N. Kellogg in \([17]\) for the sequence \( n_k = 2^k \).

The mixed-norm can be generalized further by introducing a positive weight \( f \):

\[
\| \lambda \|_{p,q,f,N} = \left( \sum_{m=0}^{\infty} \left( \frac{\sum_{j \in J_m} |\lambda_j|^p}{f(n_m)} \right)^{q/p} \right)^{1/q}.
\]
Let $\ell^p_{f,N}$ be the set of sequences for which $\|\lambda\|_{p,q,f,N}$ is finite; we call such spaces weighted mixed-norm spaces. Different mixed-normed spaces appeared in [6].

Let $g$ satisfy the doubling condition and let $N = \{n_m\}_m$ be defined by (10) with $g$ in place of $f$. We consider the corresponding partition $N$ of the index set. Then (11) and (15) can be written as

$$S(h_g^\infty) = l^2_{g,N}, \quad s(h_g^\infty) = l^1_{g,N}.$$  

4.3. **Examples of multipliers on $h_g^\infty$.** The sequence $\lambda_j = 1$ is obviously a multiplier on $h_g^\infty$ (i.e., a multiplier from $h_g^\infty$ to $h_g^\infty$), and a sequence cannot be a multiplier on $h_g^\infty$ unless it is in $\ell^\infty$. But not all bounded sequences are multipliers since $h_g^\infty$ is not solid. We determine the largest solid subspace of the multipliers from $h_g^\infty$ to itself.

Multipliers between the standard (without weight) mixed normed spaces were described in [17]. Similar results hold for the weighted mixed-norm spaces, see [13], in particular,

$$(l^2_{g,N}, l^1_{g,N}) = l^2_N,$$

where the last space in unweighted. It is easy to see that $s((A,B)) = (S(A), s(B))$ thus we have the following

$$s((h_g^\infty, h_g^\infty)) = l^2_N.$$  

This space depends on the weight (through the partition $N$) and it always contains $l^2$. For the weights $g(x) = x^\alpha$ we can take $n_k = 2^k$ and we get the standard mixed-norm spaces $l^2_N$. For $g(x) = (1 + \log x)^\beta$ we get $n_k = 2^{2^k}$ and the space $l^2_N$ is smaller than $l^2_N$. For any $N$ the sequence $\lambda_j = 1/\sqrt{n_k}$ for $n_k < j \leq n_k$ is in $l^2_N$. This gives some examples of multipliers.

It is proved in [20] (in dimension two) that if $\tilde{g}/g \to \infty$ as $x \to \infty$, then $(h_g^\infty, h_g^\infty) \subset (h_g^\infty, h_g^\infty)$ and if in addition there is an integer $m > 1$ such that $g^m/\tilde{g} \to \infty$ as $x \to \infty$, then $(h_g^\infty, h_g^\infty) = (h_g^\infty, h_g^\infty)$. Theorem 2 shows that the same is true in higher dimensions. The inclusion is strict for pairs of weights of different growth, for example when $g(x) = x^\alpha$ and $\tilde{g}(x) = (1 + \log x)^\beta$. We see that if $l^2_{g,N} \not\subset l^2_{\tilde{g},N}$ then $(h_g^\infty, h_g^\infty) \not\subset (h_g^\infty, h_g^\infty)$ since they have distinct solid parts.

The multipliers we looked at so far are small ones. In [8] a vector space of analytic functions $X$ is said to have the small multiplier property when there exists $r > 0$ such that $\lambda = \{\lambda_j\} \in (X,X)$ for any $\lambda$ with $\lambda_j = O(j^{-r})$, $j \to \infty$. We see that all weighted spaces $h_g^\infty$ have the small multiplier property since $l^2 \subset (h_g^\infty, h_g^\infty)$ (for example we can take any $r > 1$ when $d = 1$). Further Theorem 2 (a) shows that every regular bounded sequence is a multiplier. Then any sequence $\{\lambda_{kl} = f(k) + d_{kl}\}$, where $f$ is bounded and satisfies (7) and $\{d_{kl}\} \in l^2_N$, is a multiplier from $h_g$ to $h_g^\infty$.

**Acknowledgements**

This work is an extended version of the last chapter of the Ph.D. Thesis of Kjersti Solberg Eikrem. We are grateful to Dragan Vukotić and Catherine Bénéteau who read the Thesis prior to the defense for their useful comments and suggestions. The work was carried out at Center of Advanced Study at the Norwegian Academy of Sciences and Letters in Oslo and the Department of Mathematical Sciences at NTNU, Trondheim, Norway, and it is a pleasure to thank the Center and Department for the support.
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