The behavior of perturbations is studied in cosmological models which consist of two different homogeneous regions connected in a spherical shell boundary. The junction conditions for the metric perturbations and the displacements of the shell boundary are analyzed and the surface densities of the perturbed energy and momentum in the shell are derived, using Mukohyama’s gauge-invariant formalism and the Israel discontinuity condition. In both homogeneous regions, the perturbations of scalar, vector and tensor types are expanded using the 3-dimensional harmonic functions, but the mode coupling among them is caused in the shell by the inhomogeneity. By treating the perturbations with odd and even parities separately, it is found, however, that we can have consistent displacements and surface densities for given metric perturbations.

II. INTRODUCTION

In the real Universe, it is known at present that there are many superclusters and void-like objects with nonlinear overdense and underdense regions which are distributed everywhere on various scales. Such objects may have complicated structures in general, but their evolution has often been studied in comparatively large-scale cases, using a simplified assumption of spherical symmetry.

As general-relativistic examples of spherical treatments we have the analysis due to the Tolman-Bondi-Lemaître inhomogeneous dust model [1, 2], a single-shell model consisting of two different homogeneous regions connected with a spherical shell [3, 4], a self-similar model being described using the self-similar solutions [5, 6], a model consisting of two homogeneous regions connected with the self-similar intermediate region [7, 8, 9], and so on.

In order to study the dynamical behavior of the large-scale super clusters and void-like objects and their influence on the CMB anisotropy, moreover, it is necessary to consider the gravitational instability of these models. The general gauge-invariant formalism for linear perturbations in spherical symmetric inhomogeneous models was derived by Gerlach and Sengupta [10, 11, 12, 13] in four-dimensional spacetimes. Recently the junction of perturbations in two homogeneous regions was treated by Mukohyama [14, 15] and Kodama et al. [16] in higher dimensional spacetimes on the basis of the Israel discontinuity condition [17]. In this paper, we study the behavior of linear perturbations in the four-dimensional cosmological models with a single shell, using Mukohyama’s formalism. Here only one of his doubly gauge-invariant variables are used, and in both homogeneous regions the perturbations are classified into scalar, vector and tensor types in three dimensional space and expanded in terms of three dimensional harmonic functions [18, 19] in order to clarify the physical image of perturbations, though he classified them in a space of the shell surface. Among them the mode coupling is caused by the inhomogeneity in the shell. By treating the perturbations with odd and even parities separately, it is found however that the consistent expressions of the shell displacement and the energy-momentum tensor in the shell can be derived for given metric perturbations in both regions.

In §II, we describe briefly the single-shell model as the background model, in which the velocity of the shell is determined through the junction condition. In §III, we show the basic equations for gauge-invariant perturbations and the junction condition based on Mukohyama’s formalism [14, 15]. In §IV, we derive the intrinsic perturbations of metric and extrinsic curvature in the shell, and show the displacements of the shell and the intrinsic energy-momentum tensor in the shell which are derived from the junction conditions, by treating the perturbations with odd-parity and even-parity separately and imposing a localization condition on the wave-number dependence of the shell displacement. §V is given to the concluding remarks. In Appendix A, the harmonic functions in a homogeneous background model are shown, which are necessary for the analysis of the description of junction conditions. In Appendices B ad C, auxiliary quantities $J_{\mu\nu}$ and $\Theta_{\mu\nu}$ for the perturbed gravitational field and the intrinsic energy-momentum tensor in the shell, respectively, are shown.
II. BACKGROUND MODELS AND THE JUNCTION CONDITIONS

The background universe is assumed to consist of two spatially homogeneous regions connected by a spherical shell. The four-dimensional line-elements in the inner region $V^-$ and the outer region $V^+$ are described as

$$ds^2 = g_{\mu\nu}^\pm(dx^\mu)^2 \pm (dx^\nu)^2 = [a_\pm (\eta_\pm)]^2\left[-(d\eta_\pm)^2 + dl_\pm^2\right]$$

$$d\ell_\pm^2 = \gamma_\pm^i(dx^i)^2 = (d\chi_\pm)^2 + [\sigma_\pm(\chi_\pm)]^2d\Omega^2,$$

where the suffices $\mu, \nu$ and $i, j$ run from 0 to 3 and from 1 to 3, respectively, $x_\pm^1 = \chi_\pm$, $\sigma_\pm(\chi_\pm) = \sin \chi_\pm, \chi_\pm$ and $\sinh \chi_\pm$ for the spatial curvature $K_\pm = 1, 0, -1$, respectively, and $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$. The shell is a time-like hypersurface $\Sigma$ given as

$$x_\pm^\mu = Z_\pm^\mu(y^M),$$

where $y^M (M = 0, 2, 3)$ denote the intrinsic coordinates in the shell. In the following we specify the coordinates in the form

$$x_\pm^A = Z_\pm^A(y^0), \quad x_\pm^a = y^a,$$

where the suffices $A$ and $a$ take the values $(0, 1)$ and $(2, 3)$, respectively. When we define the tangential triads $e_{(M)}^\mu_\pm = (M = 0, 2, 3)$ by $e_{(M)}^\mu = \partial x^\mu / \partial y^M$, the intrinsic metric is given by

$$q^\pm_{MN} = e_{(M)}^\mu e_{(N)}^\nu g^\pm_{\mu\nu}.$$  

If we specify $y^0$ as the proper time $\tau$ in the shell, the components of the triads can be expressed as

$$e_{(0)}^\mu_\pm = \left(\partial x^\mu / \partial y^0, \partial x^1 / \partial y^0, 0, 0\right) = F_\pm(1, v_\pm, 0, 0),$$

$$e_{(2)}^\mu_\pm = (0, 0, 1, 0) \quad \text{and} \quad e_{(3)}^\mu_\pm = (0, 0, 0, 1), \quad \text{where} \quad F_\pm \equiv \gamma_\pm / a_\pm, \quad v_\pm \equiv dx_\pm^1 / dx_\pm^0 (= dx_\pm^1 / d\eta_\pm) \quad \text{and} \quad \gamma_\pm \equiv 1 / \sqrt{1 - (v_\pm)^2}. \quad \text{Here we take the units of the light velocity} \ c = 1. \quad \text{The unit normal vector} \ n_\pm^\mu \ \text{of} \ \Sigma \ \text{is defined by the conditions} \ n_\mu_\pm e_{(M)}^\mu_\pm = 0 (M = 0, 2, 3) \quad \text{and} \quad n_\mu_\pm n_\pm^\mu = 1, \ \text{and the components can be expressed as}$$

$$n_\pm^\mu = (-v_\pm, 1, 0, 0) \gamma_\pm / a_\pm.$$

The line-element in the shell in terms of the intrinsic coordinates is

$$ds^2 = q_{MN}dy^Mdy^N = -d\tau^2 + R^2 d\Omega^2,$$

where $y^0 = \tau, y^2 = \theta, y^3 = \varphi$, and $R \equiv a_+\sigma(\chi_+) = a_-\sigma(\chi_-)$.

Since the intrinsic geometry is regular, the induced metric should be continuous and we have

$$q_{MN+} = q_{MN-} \equiv q_{MN}.$$  

The extrinsic curvature of $\Sigma$ is defined by

$$K_{MN} = \frac{1}{2}e_{(M)}^\nu e_{(N)}^\rho \mathcal{L}_n g_{\mu\nu} = e_{(M)}^\mu e_{(N)}^\nu \left[n_{\mu\nu} + n_{\nu\mu}\right],$$

where $\mathcal{L}$ represents the Lie derivative, $n$ denotes the unit normal of $\Sigma$, and a suffix ; $\mu$ denotes the four-dimensional covariant derivative with respect to $x^\mu$. For the energy-momentum tensor $T^{\mu\nu}$, the corresponding surface component $S_{\mu\nu}$ in the shell is defined using the Gaussian normal coordinates as

$$S_{\mu\nu} = \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} T_{\mu\nu}d\zeta,$$

where $\zeta$ is the radial coordinate and $\zeta = 0$ is for the shell surface. The expression $S_{MN}$ in the intrinsic coordinates is given by

$$S_{MN} = e_{(M)}^\mu e_{(N)}^\nu S_{\mu\nu},$$
where \( e^{(M)} \) is the triad in the gaussian normal coordinates.

From the junction conditions derived by Israel, we obtain the relations between the jump of the extrinsic curvature and the surface energy-momentum tensor \( S_{MN} \) as

\[
[K_{MN}]^\pm = -\kappa^2 (S_{MN} - \frac{1}{2} q_{MN}),
\]

(13)

Together with the additional relations

\[
-S_{MN}^N = [T^N_M]^\pm
\]

(14)

and

\[
\frac{1}{2} (K_{N+}^M + K_{N-}^M) S_M^N = [T^N_n]^\pm,
\]

(15)

where \([\Phi]^\pm = \Phi^+ - \Phi^-\), \( S \equiv q_{MN} S_{MN} \), \( q_{MN} \) is the inverse of the induced metric \( q_{MN} \), \( \kappa^2 \equiv 8\pi G \), and

\[
T_{MN}^n = n^\nu e^{(M)}_\nu (T^n_\nu)^\pm \quad T^n_{n\pm} = n^\nu n^\nu_\pm (T^n_\nu)^\pm,
\]

(16)

\[
S_{N}^N = q^{NL} S_{LM}.
\]

(17)

In the case of dust matter, we have

\[
T_{\mu\nu} = \rho_\pm u_\mu u_\nu, \quad S_{MN} = \tilde{\sigma}_M v_N^\pm,
\]

(18)

where \( \tilde{\sigma} \) is the surface density of dust matter. For the metric (1), we have

\[
K_{0\pm}^0 = (\gamma^2 \hat{v}_\pm + (\dot{a}_\pm / a_\pm) v_\pm) \gamma / a_\pm,
\]

(19)

and

\[
K_{2\pm}^2 = K_3^3 \equiv |\sigma / \sigma + (\dot{a}_\pm / a_\pm) v_\pm| \gamma / a_\pm,
\]

(20)

where \( K_{N\pm}^M = q^{NL} K_{LN\pm} \), and a dot and a prime denote \( \partial / \partial \eta_\pm \) and \( \partial / \partial \chi_\pm \), respectively.

By substituting these equations (19) and (20) into the above Eqs. (13) - (15), we can obtain equations for \( v_\pm \) and \( \tilde{\sigma} \). By solving these equations, Sakai et al. [3] derived their time evolution and showed that \( |v_\pm|^2 \approx 10^{-5} \ll 1 \). Moreover, the following constraint equation for \( \tilde{\sigma} \) is obtained:

\[
[\gamma (\sigma' + vHR)]^\pm = -\frac{1}{2} \kappa^2 \bar{R} \tilde{\sigma},
\]

(21)

where \( \bar{R} \equiv a_+ \sigma_+ = a_- \sigma_- \).

In the following the suffice \( \pm \) is omitted except for the case when it is necessary. Here let us show the relations necessary for the later calculations:

\[
q_{00} = -1, \quad q_{22} = q_{33} / \sin^2 \theta = (a^2 \sigma)^2,
\]

(22)

\[
q^{00} = -1, \quad q^{22} = q^{33} \sin^2 \theta = (a^2 \sigma)^{-2},
\]

(23)

\[
K_{00} = -(\gamma / a)[\gamma^2 \dot{v} + (\dot{a} / a) v], \quad K_{22} = K_{33} / \sin^2 \theta = a \gamma \sigma^2 \gamma' / \sigma + (\dot{a} / a) v,
\]

(24)

where \( K \equiv q^{MN} K_{MN} = (\gamma / a)[2 \sigma' / \sigma + (\dot{a} / a) v + \gamma^2 \dot{v}] \) and \( K^{00} = K_{00}, K^{22} = K^{33} \sin^2 \theta = K_{22} / (a \sigma)^4 \).

\[
K_{00} - K_{00} = 2 (\gamma / a)[\sigma' / \sigma + (\dot{a} / a) v], \quad K_{22} - K_{22} = [K_{33} - K_{q33}] / \sin^2 \theta = -a \gamma \sigma^2 [\sigma' / \sigma + 2 (\dot{a} / a) v + \gamma^2 \dot{v}].
\]

(25)
III. PERTURBATIONS AND THE JUNCTION CONDITIONS

The perturbations of $g_{\mu\nu}$ and $Z^\mu$ around the background ($\bar{g}_{\mu\nu}$ and $\bar{Z}^\mu$) are considered:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \quad Z^\mu = \bar{Z}^\mu + \delta Z^\mu.$$  (26)

The perturbed tangent vectors are expressed as

$$e^\mu_{(M)} = \bar{e}^\mu_{(M)} - \mathcal{L}_Z \bar{e}^\mu_{(M)}.$$  (27)

For the perturbations of the induced metric:

$$q_{MN} = \bar{q}_{MN} + \delta q_{MN},$$  (28)

Mukohyama derived the expression

$$\delta q_{MN} = e^\mu_{(M)} e^\nu_{(N)} (\delta g_{\mu\nu} + \mathcal{L}_Z \bar{g}_{\mu\nu}).$$  (29)

The perturbed normal is expressed as

$$\delta n^\mu = \frac{1}{2} \bar{n}^\nu \bar{n}^\lambda \delta g_{\nu\lambda} + e^\mu_{(M)} q^{MN} \bar{n}_\nu \mathcal{L}_Z e^\nu_{(N)}$$  (30)

and

$$\delta K_{MN} = \frac{1}{2} e^\mu_{(M)} e^\nu_{(N)} [\mathcal{L}_Z \bar{g}_{\mu\nu} + \mathcal{L}_Z \bar{q}_{\mu\nu} - 2 n^\lambda \delta \Gamma_{\lambda\mu\nu}],$$  (31)

where

$$\delta \Gamma_{\lambda\mu\nu} = \frac{1}{2} (\delta g_{\lambda\mu\nu} - \delta g_{\lambda\nu\mu} - \delta g_{\mu\lambda\nu}).$$  (32)

This perturbed extrinsic curvature are reduced to

$$\delta K_{MN} = \frac{1}{2} I K_{MN} - \frac{1}{2} n^\lambda e^\nu_{(M)} e^\nu_{(N)} J_{\lambda\mu\nu},$$  (33)

where

$$I \equiv n^\nu n^\nu (\delta g_{\mu\nu} + 2 \delta Z_{\mu\nu})$$  (34)

and

$$J_{\lambda\mu\nu} = 2 \delta \Gamma_{\lambda\mu\nu} + \delta Z_{\lambda\mu\nu} + \delta Z_{\lambda\nu\mu} + (R_{\alpha\mu\lambda\nu} + R_{\alpha\nu\lambda\mu}) \delta Z^\alpha.$$  (35)

Under the gauge transformation

$$x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu,$$  (36)

$\delta g_{\mu\nu}$ and $\delta Z^\mu$ transform as

$$\delta g_{\mu\nu} \rightarrow \delta g_{\mu\nu} - \xi_{\mu\nu} - \xi_{\nu\mu}, \quad \delta Z^\mu \rightarrow \delta Z^\mu + \xi^\mu,$$  (37)

and it is found that $\delta q_{MN}$ and $\delta K_{MN}$ are invariant under this gauge transformation. The junction conditions for the perturbations of metric and extrinsic curvature which was shown by Mukohyama[15] are expressed as

$$\delta q_{MN+} = \delta q_{MN-}$$  (38)

and

$$[\delta \tilde{K}_{MN}]^+ = -\kappa^2 (\delta \tilde{S}_{MN} - \frac{1}{2} \delta \tilde{S} q_{MN}),$$  (39)
where
\[ \delta K_{MN} ≡ \delta K_{MN} - \frac{1}{2}(K^L_M\delta q_{LN} + K^L_N\delta q_{LM}), \]
and
\[ \delta S_{MN} ≡ \delta S_{MN} - \frac{1}{2}(S^L_M\delta q_{LN} + S^L_N\delta q_{LM}) \]
with \( S = q^{MN}S_{MN} \) and \( \tilde{S} = q^{MN}\delta S_{MN} \). The equation (39) is consistent with a relation
\[ [\delta \tilde{K}_{MN}]^± = -\kappa^2\{\delta S_{MN} - \frac{1}{2}(\delta q_{MN} + S\delta q_{MN})\}, \]
so that
\[ -\kappa^2\delta S_{MN} = [\delta K_{MN} - \delta K_{QMN} - K\delta q_{MN}]^±, \]
where \( \delta K = \delta q^{MN}K_{MN} + q^{MN}\delta K_{MN} = q^{MN}\delta K_{MN} - K^M^N\delta q_{MN} \).

When the metric perturbations in the two regions \( V^+ \) and \( V^- \) are given, we can determine \( \delta q_{MN}, \delta K_{MN} \) and \( \delta S_{MN} \) using Eqs. (29), (31) and (43), and find the conditions which are imposed on the shell displacements, using Eq. (38).

IV. PERTURBED QUANTITIES IN THE BOUNDARY SHELL

In this section we express first the perturbations of the induced metric and the extrinsic curvature, using the gauge-invariant variables representing metric perturbations and the shell displacement, and derive the perturbations of the energy-momentum tensors in the shell. The metric perturbations and the harmonic functions in the homogeneous regions are shown in Appendix A.

A. Metric perturbations

The metric perturbations in Eq. (29) are rewritten in terms of \( x^\mu \) as
\[ \delta g_{MN}dy^Mdy^N = (\delta g_{\mu\nu} + \delta Z_{\mu,\nu} + \delta Z_{\nu,\mu})dx^\mu dx^\nu, \]
where \( \delta Z_p \) represents the displacement of the boundary shell. \( \delta Z_0 \) and \( \delta Z_i \) are here treated as spatially 3-dimensional quantities in accord with the metric perturbations. In principle they are the quantities given in the shell. In Mukohyama’s formalism[15] the displacements of the shell were regarded as functions of only intrinsic variables \( \tau, \theta \) and \( \varphi \) which are spatially 2-dimensional. In order to recover the local nature of \( \delta Z_0 \) and \( \delta Z_i \) in our case, we impose later a localization condition, under which their behavior is constrained so that their values may be nonzero only in the neighborhood of the shell. The coefficients in the right-hand side of this equation are expressed as follows using the gauge-invariant variables.

1. Scalar perturbations

\[ \delta g_{00} + 2\delta Z_{0,0} = \int k^2dk[-2\alpha^2\Phi_A + 2\alpha(\phi_0/a)]Q, \]
\[ \delta g_{0i} + \delta Z_{c,0} + \delta Z_{0,i} = \int k^2dk[a^2(\phi_L/a^2)\gamma_i - \phi_0]Q_\gamma, \]
\[ \delta g_{ij} + \delta Z_{c,j} + \delta Z_{j,i} = \int k^2dk[(2\alpha^2\Phi_H - 2\alpha\phi_0)\gamma_{ij}Q - k\phi_L(Q_{ij} + Q_{ji})], \]
where the suffix \( |i \) denotes the covariant derivative in the 3-dimensional space with \( dt^2 = \gamma_{ij}dx^idx^j \). \( \delta Z_0 \) and \( \delta Z_i \) are expanded in terms of harmonic functions as \( \delta Z_0 = \int k^2dk\gamma_0Q \) and \( \delta Z_i = \int k^2dk\gamma_iQ \), and \( k \) is the wave-vector, whose length is \( k = |k| \). Here, \( \Phi_A, \Phi_H, \phi_0 \) and \( \phi_L \) are the gauge-invariant variables defined by
\[ \Phi_A ≡ \frac{A}{k} + \frac{\dot{A}}{k}B - \frac{1}{k^2}\left(\ddot{H}_T + \frac{\dot{a}}{a}\dot{H}_T\right), \]
\[
\Phi_H \equiv H_L + \frac{1}{3} H_T + \frac{1}{k} \frac{B}{a} - \frac{1}{k^2} \frac{H_T}{a}, \\
\phi_0 \equiv z_0 + (a^2/k)(B - H_T/k), \\
\phi_L \equiv -z_L/k + a^2 H_T/k^2, 
\]

where various expressions for the metric perturbations and harmonic functions are shown in Appendix A. Here we use normalized harmonic functions.

Since \( dx^0 = F dy^0, dx^1 = F dy^1, dx^2 = dy^2 \) and \( dx^3 = dy^3 \) with \( F \equiv e^{\delta_{(0)}} = \gamma/a \), we obtain

\[
\begin{align*}
\delta q_{00} &= \int k^2 dk \left[ (\zeta_0 Q + 2v \zeta_L Q_{,1} + \zeta_{LL} \gamma_{11} Q + 2k^2 \phi_L Q_{,11}) \right], \\
\delta q_{0a} &= \int k^2 dk \left[ \zeta_{LQ,a} + 2vk^2 \phi_L Q_{,1a} \right], \\
\delta q_{ab} &= \int k^2 dk \left[ \zeta_{LL} \gamma_{ab} Q + 2k^2 \phi_L Q_{ab} \right], 
\end{align*}
\]

where \( \zeta_0 \equiv -2a^2 \Phi_A + 2a(\phi_0/a), \zeta_L \equiv a^2(\phi_L/a^2) + \phi_0 \) and \( \zeta_{LL} \equiv 2a^2 \Phi_H - 2(\dot{a}/a)\phi_0 - \frac{2}{3} k^2 \phi_L \). For \( I \) defined in Eq.(34), we obtain

\[
(a/\gamma)^2 I = \int k^2 dk \left\{ \left[ 2a^2 \Phi_H - \frac{2}{a} \phi_0 \right] Q + 2\phi_L Q_{,11} + 2v[a^2(\phi_L/a^2) + \phi_0]Q_{,1} + v^2[-2a^2 \Phi_A + a(\phi_0/a)]Q \right\}. 
\]

Equations (47) are rewritten using spherical harmonics as follows:

\[
\begin{align*}
\delta q_{00} &= \int k^2 dk F^2 \left\{ [\zeta_0 \Pi + 2v \zeta_L \Pi' + v^2 \left( \zeta_{LL} \Pi + 2\phi_L \left( \Pi'' + \frac{k^2}{3} \Pi \right) \right)] \right\} Y_{lm}, \\
\delta q_{0a} &= \int k^2 dk F(\zeta_L \Pi + 2v \zeta_{LQ,a}(\Pi' - \frac{\sigma'}{\sigma}) Y_{lm,a}, \\
\delta q_{ab} &= \int k^2 dk \left[ \phi_L \Pi Y_{ab} + \left( \zeta_{LL} + 2\phi_L \left[ \Pi' \frac{\sigma'}{\sigma} + \left( \frac{k^2}{3} - \frac{l(l+1)}{2\sigma^2} \right) \right] \right] \right\} Y_{lm} \gamma_{ab}. 
\end{align*}
\]

where \( Q = \Pi_l(k,\chi) Y_{lm}(\theta,\varphi) \), the suffix \( l \) in \( \Pi_l \) is omitted, \( Y_{lm}(\theta,\varphi) \) is the spherical harmonics and \( \tilde{Y}_{ab} \) is a 2-dimensional traceless tensor defined by

\[
\tilde{Y}_{ab} \equiv Y_{lm} \gamma_{ab} + \frac{l(l+1)}{2\sigma^2} \gamma_{ab} Y_{lm}. 
\]

Here \( a \) and \( b \) take the value 2 or 3 and \( \| \) denotes the covariant derivative with respect to the space with \( dY^2 = d\theta^2 + \sin^2 \theta d\varphi^2 \).

In these equations, dots mean \( \partial/\partial \eta_+ \) and \( \partial/\partial \eta_- \) in \( V^+ \) and \( V^- \), respectively, primes mean \( \partial/\partial \chi_+ \) and \( \partial/\partial \chi_- \) in \( V^+ \) and \( V^- \), respectively, \( \Pi_+ \) and \( \Pi_- \) are equal to \( \Pi(\chi_+) \) and \( \Pi(\chi_-) \), and a suffix \( k \) is omitted here and in the following. Here, if we change the time variables \( \eta_+ \) and \( \eta_- \) to the common variable \( \tau(=y^0) \) in the shell by \( \partial/\partial \eta_\pm = (a_\pm/\gamma_\pm) \partial/\partial \tau \), then we have

\[
\begin{align*}
\zeta_0 &= -2a^2 \Phi_A + \frac{2a^2(\phi_0/a)}{\gamma}, \\
\zeta_L &= a^2(\phi_L/a^2) + \phi_0, \\
\zeta_{LL} &= 2a^2 \Phi_H - 2a_\tau \phi_0 - \frac{2}{3} k^2 \phi_L. 
\end{align*}
\]

2. *Vector perturbations*

The coefficients of Eq.(44) are

\[
\begin{align*}
\delta g_{00} + 2 \delta Z_{0,0} &= 0, \\
\delta g_{0i} + \delta Z_{i0} + \delta Z_{0i} &= \int k^2 dk [-a^2 \Psi + a^2 (\phi_T/a^2)] V_i, \\
\delta g_{ij} + \delta Z_{i,j} + \delta Z_{j,i} &= \int k^2 dk [-2k \phi_T] V_{ij}, 
\end{align*}
\]
where \( \delta Z_0 = 0 \) and \( \delta Z_i = \int k^2 dk z_T V_i \), and \( \Psi \) and \( \phi_T \) are the gauge-invariant variables defined as follows:

\[
\Psi = B - \frac{1}{k} \dot{H}_T^{(1)}, \quad \phi_T = z_T - \frac{1}{k} a^2 \dot{H}_T^{(1)}.
\]  

(53)

The induced metric perturbations are

\[
\delta q_{00} = -2F^2v \int k^2 dk \{[-a^2 \Psi + a^2(\phi_T/a^2)]V_1 + vk \phi_T V_1 \},
\]

\[
\delta q_{0a} = F \int k^2 dk \{[-a^2 \Psi + a^2(\phi_T/a^2)]V_a - 2vk \phi_T V_{1a} \},
\]

\[
\delta q_{ab} = -2 \int k^2 dk (k \phi_T) V_{ab},
\]

(54)

where \( V_{1a} = -\frac{1}{2k}[V_{1,a} + R^2(V_a/R^2), 1] \). For \( I \) we obtain

\[
(a/\gamma)^2 I = -2 \int k^2 dk \{v[a^2 \Psi + a^2(\phi_T/a^2)]V_1 + k \phi_T V_{11} \}.
\]

(55)

Equations (54) are rewritten using spherical harmonics in the cases of odd and even parities as follows:

(2-1) The odd-parity case

Since \( V_1 = 0 \) (cf. Appendix A), we have

\[
\delta q_{00} = 0,
\]

\[
\delta q_{0a} = F \int k^2 dk \{[-a^2 \Psi + (a^3/\gamma)(\phi_T/a^2), r]V_a + v \phi_T [V_{a,1} - 2(\sigma'/\sigma)V_a] \},
\]

\[
\delta q_{ab} = \int k^2 dk \phi_T [V_{a||b} + V_{b||a}],
\]

(56)

where \( (V_2, V_3) = \sigma(\chi)\Pi(-Y_{l,m,3}/\sin \theta, Y_{l,m,2} \sin \theta) \). These 3-dimensional vector perturbations are also 2-dimensional vector perturbations.

(2-2) The even-parity case

Harmonic functions are

\[
V_1 = l(l + 1)(\Pi/\sigma)Y_{lm},
\]

\[
(V_2, V_3) = (\Pi \sigma) (Y_{l,m,2}, Y_{l,m,3}),
\]

(57)

and the intrinsic metric perturbations are

\[
\delta q_{00} = -2F^2v \int k^2 dk \left[-a^2 \Psi + (a^3/\gamma)(\phi_T/a^2), r - v \phi_T \left(\frac{IV'}{\Pi} - \frac{\sigma'}{\sigma}\right)\right] V_1,
\]

\[
\delta q_{0a} = F \int k^2 dk \left\{[-a^2 \Psi + (a^3/\gamma)(\phi_T/a^2), r]V_a + v \phi_T [V_{a,1} - 2(\sigma'/\sigma)V_a + V_{1a}] \right\},
\]

\[
\delta q_{ab} = \int k^2 dk \phi_T [V_{a||b} + V_{b||a} + 2\gamma_{ab} \frac{\sigma'}{\sigma} V_1].
\]

(58)

These expressions are reduced to

\[
\delta q_{00} = -2F^2v \int k^2 dk \left[-a^2 \Psi + (a^3/\gamma)(\phi_T/a^2), r - v \phi_T \left(\frac{IV'}{\Pi} - \frac{\sigma'}{\sigma}\right)\right] l(l + 1)(\Pi/\sigma)Y_{lm},
\]

\[
\delta q_{0a} = F \int k^2 dk \{[-a^2 \Psi + (a^3/\gamma)(\phi_T/a^2), r](\Pi \sigma)' + v \phi_T [(\Pi \sigma)' - 2(\sigma'/\sigma)(\Pi \sigma)' + l(l + 1)(\Pi/\sigma)]\} Y_{lm,a},
\]

\[
\delta q_{ab} = \int k^2 dk \phi_T \{2(\Pi \sigma)' \dot{Y}_{ab} + \frac{l(l + 1)}{\sigma^2} [2\Pi \sigma' - (\Pi \sigma)']\} \gamma_{ab} Y_{lm}.
\]

(59)

In this form the dependence on spherical harmonics is found to be quite the same as that of scalar perturbations.
3. Tensor perturbations

The non-zero coefficient of Eq.(44) is

$$\delta g_{ij} + \delta Z_{i;j} + \delta Z_{j;i} = -2a^2 \int k^2 dk H_T^{(2)} G_{ij},$$

(60)

and the induced metric components are

$$\delta q_{00} = -2a^2 F v^2 \int k^2 dk H_T^{(2)} G_{11},$$
$$\delta q_{0a} = -2a^2 F v \int k^2 dk H_T^{(2)} G_{1a},$$
$$\delta q_{ab} = -2a^2 \int k^2 dk H_T^{(2)} G_{ab}.$$  

(61)

For $I$ we obtain

$$(a/\gamma)^2 I = -2a^2 \int k^2 dk H_T^{(2)} G_{11}.$$  

(62)

Equations (61) are rewritten using spherical harmonics in the cases of odd and even parities as follows:

(3-1) The odd-parity case

If we define a 2-dimensional axial vector $W_a$ by

$$W_a \equiv \left(-Y_{lm,3} / \sin \theta, Y_{lm,2} \sin \theta\right)$$

(63)

for $a = 2, 3$, we obtain

$$\delta q_{00} = 0,$$
$$\delta q_{0a} = F v \int k^2 dk H_T^{(2)} (l - 1)(l + 2) \Pi W_a,$$
$$\delta q_{ab} = \int k^2 dk H_T^{(2)} \left(\Pi \sigma^2 \right)^\prime (W_{a\parallel b} + W_{b\parallel a}).$$

(64)

When we compare $W_a$ with $V_a$ in the odd-parity case of vector perturbations, we have the relation $V_a = (\Pi \sigma) W_a$, so that with respect to the two-dimensional angular dependence the tensor perturbations in the odd-parity case have the same form as vector perturbations in the odd-parity case.

(3-2) The even-parity case

$$\delta q_{00} = (F v)^2 \int k^2 dk H_T^{(2)} L(\Pi / \sigma^2) Y_{lm},$$
$$\delta q_{0a} = F v \int k^2 dk H_T^{(2)} (l - 1)(l + 2) (\Pi \sigma)^\prime Y_{lm,a},$$
$$\delta q_{ab} = \int k^2 dk H_T^{(2)} \left[2G^m_{ab} Y_{ab} - L \Pi_{\gamma_{ab}} Y_{lm}\right],$$

(65)

where the definitions of $L$ and $G^m_{ab}$ are given in Appendix A. In this form the dependence on spherical harmonics is found to be quite the same as that of scalar perturbations, as well as that of vector perturbations in the even-parity case.

B. Perturbations of the extrinsic curvature and the energy-momentum tensor in the shell

Now let us calculate the perturbed extrinsic curvature $\delta K_{MN}$ using Eq.(33) and derive the perturbed energy-momentum tensor $\delta S_{MN}$ from Eq.(43). The former expression is rewritten as

$$\delta K_{MN} = \frac{1}{2} I K_{MN} - \frac{1}{2a} \int k^2 dk F_{MN}.$$  

(66)
where $F_{MN}$ is defined by $F_{MN} = n^\lambda e^\mu_{(M} e^\nu_{N)} J_{\lambda\mu\nu}$. They have the following components:

$$
\begin{align*}
F_{00} &= (\gamma/a)^2 [J_{100} + v(J_{000} + 2J_{101}) + v^2(J_{111} + 2J_{001}) + v^3 J_{011}], \\
F_{22} &= J_{122} + v(J_{022}), \\
F_{33} &= J_{133} + v(J_{033}), \\
F_{23} &= J_{123} + vJ_{023}, \\
F_{02} &= (\gamma/a)[J_{102} + v(J_{002} + J_{112}) + v^2 J_{012}], \\
F_{03} &= (\gamma/a)[J_{103} + v(J_{003} + J_{113}) + v^2 J_{013}],
\end{align*}
$$

(67)

where $F_{20} = F_{02}$, $F_{30} = F_{03}$, and $F_{32} = F_{23}$. The expressions of $J_{\lambda\mu\nu}$ in terms of gauge-invariant variables of metric perturbations and displacements of the boundary shell are shown in Appendix B.

The energy-momentum tensor in the shell is expressed as

$$
\tilde{\Theta}_{MN} = [\tilde{\Theta}_{MN}]^\pm = \frac{1}{2}(K_{MN} - K q_{MN}) - \frac{1}{2} a \int k^2 dk \tilde{F}_{MN} - K \delta q_{MN},
$$

(68)

where

$$
\tilde{F}_{MN} = F_{MN} - q^{KL} F_{KL} q_{MN}
$$

(69)

and their components are

$$
\begin{align*}
\tilde{F}_{00} &= \frac{1}{(a\sigma)^2} [J_{122} + vJ_{022} + (J_{133} + vJ_{033})/\sin^2 \theta], \\
\tilde{F}_{22} &= (\sigma)(\gamma) [J_{102} + v(J_{002} + 2J_{112}) + v^2(J_{111} + 2J_{001}) + v^3 J_{011}] - (J_{133} + vJ_{033})/\sin^2 \theta, \\
\tilde{F}_{33} &= (\sigma)(\gamma) \sin^2 \theta [J_{100} + v(J_{000} + 2J_{101}) + v^2(J_{111} + 2J_{001}) + v^3 J_{011}] \\
&\quad - (J_{122} + vJ_{022}) \sin^2 \theta.
\end{align*}
$$

(70)

The components of $\Theta_{MN}$ in terms of the gauge-invariant variables are obtained through lengthy calculations and are shown in Appendix C in the expanded form with respect to the shell velocity $v$.

### C. Junction of perturbations

From now let us discuss the junction of metric perturbations given by $[\delta q_{MN}]^\pm = 0$ in Eq.(38), in order to analyze the displacements of the shell. If the displacements are obtained for given metric perturbations in both regions, we can determine the perturbed energy-momentum tensor in the surface $(\delta S_{MN})$ using the above equations (68).

The following analyses in the shell are performed separately in the odd-parity and even-parity cases of metric perturbations, and all perturbations in each case are treated together in spite of the difference in their 3-dimensional behaviors. This is because the 2-dimensional angular dependence of metric perturbations in each case is the same and indistinguishable, as was shown in the previous subsection.

#### 1. Odd-parity perturbations

Here we consider vector and tensor perturbations together, and from the condition $[(\delta q_{MN})_{(vector)} + (\delta q_{MN})_{(tensor)}]^\pm = 0$, we obtain the following two independent equations:

$$
\begin{align*}
\left\lfloor k^2 dk \left\{-\gamma a \Psi + a^2 (\phi_T/a^2) \sigma \right\} (\Pi \sigma) + \frac{\gamma}{a} v [\phi_T (\Pi' \sigma - \Pi \sigma') + (l - 1)(l + 2) H^{(2)}_{\Pi} \Pi] \right\} \right\}^\pm &= 0, \\
\left\lfloor k^2 dk \left\{ \phi_T \Pi + H^{(2)}_{\Pi} (\sigma^2 \Pi') \right\} \right\}^\pm &= 0,
\end{align*}
$$

(71)

where the $x^0 (= \eta)$ derivative was changed to the $\tau$ derivative by $\partial/\partial \eta = (a/\gamma) \partial/\partial \tau$.

The metric perturbations $\Psi$ and $H^{(2)}_{\Pi}$ are given as functions of $\tau$ and $k$ in both regions. In order to recover the original local nature of $\phi_T$, on the other hand, we impose the following localization condition on it using a function $d(\chi)$:

$$
\int \phi_{T+}^\pm \Pi_{\pm} \sigma_{\pm} k^2 dk = \tilde{\phi}_{T+}(\tau) d_{\pm}(\chi),
$$

(72)
respectively, where \( \tilde{\phi}_{T \pm}(\tau) \) are functions of only \( \tau \) and the functions \( d_{\pm}(\chi) \) corresponding to the shell boundary (\( \chi = \chi_b \)) and the small width \( \epsilon \) are defined as
\[
d_{+}(\chi) = 1 \text{ for } \chi_b \leq \chi \leq \chi_b + \epsilon \text{ and } 0 \text{ for } 0 \leq \chi < \chi_b \text{ or } \chi > \chi_b + \epsilon,
\]
and
\[
d_{-}(\chi) = 1 \text{ for } \chi_b - \epsilon \leq \chi \leq \chi_b \text{ and } 0 \text{ for } 0 \leq \chi < \chi_b - \epsilon \text{ or } \chi > \chi_b.
\]
The above condition means that the total displacement (\( \int \phi_{T \pm}(\Pi \pm k^2dk) \)) has the nonzero value only in the neighborhood of the shell. Here we assume that \( \epsilon \) is small enough compared with \( \chi_b \). Then we find that \( \phi_{T \pm} \) are inversely expressed as
\[
\phi_{T \pm} = \tilde{\phi}_{T \pm}(\tau) \frac{1}{k^2} \int_{k^2}^{\infty} \Pi_{\pm}(\bar{\chi}) \sigma_{\pm}(\bar{\chi}) d_{\pm}(\bar{\chi}) d\bar{\chi} \approx \tilde{\phi}_{T \pm}(\tau) k^{-2} \Pi_{\pm}(k, \chi_b)
\]
in terms of \( \tilde{\phi}_{T \pm}(\tau) \), using the orthonormal relations ([21, 22]) of \( \Pi \) which are expressed as
\[
\int_{0}^{\infty} d\chi \Pi_{\pm}(k, \chi) \Pi_{\pm}(\bar{\chi}, \chi)\sigma^2(\chi) = \delta(k - \bar{k}),
\]
\[
\int_{0}^{\infty} dk \Pi_{\pm}(k, \chi) \Pi_{\pm}(\bar{\chi}, \chi)\sigma(\chi)\sigma(\bar{\chi}) = \delta(\chi - \bar{\chi})
\]
for \( \Pi_{\pm} \) with common \( l \). Here it should be noticed that the orthonormal relations require the integration interval \( [0, \infty) \) of \( \chi \), but it is formal and independent of the physical interval of \( \chi \) in the two homogeneous regions. Eq.(75) shows the \( k \) dependence of \( \phi_{T \pm} \). Under the above condition, the two unknown variables \( \tilde{\phi}_{T +} \) and \( \tilde{\phi}_{T -} \) are determined by solving the above two equations, as follows.

First we obtain from the latter equation of Eq.(71)
\[
\tilde{\phi}_{T -} = \tilde{\phi}_{T +} + \int k^2dk [H_{T}^{(2)}(\sigma^2)']^\pm.
\]
Next let us differentiate the latter equation with respect to \( \tau \). Then we obtain
\[
\left[ \int k^2dk \left\{ \phi_{T, \tau}(\Pi \sigma) + H_{T, \tau}^{(2)}(\Pi \sigma^2)' + [\phi_{T}(\Pi \sigma)'] + H_{T}^{(2)}(\Pi \sigma^2)''(\Pi \sigma^2)' - (l - 1)(l + 2)\Pi \right\} \right]^\pm = 0.
\]
Eliminating \( \phi_{T, \tau} \) from the former of Eq.(71) and Eq.(78), we obtain
\[
\left[ \int k^2dk \left\{ 2\phi_{T} \left[ \frac{\alpha v}{a} + \gamma(v/a)\sigma' \right] \Pi + \alpha a \Psi \sigma + H_{T}^{(2)}(\Pi \sigma^2)' + \gamma(v/a)H_{T}^{(2)}(\Pi \sigma^2)''(\Pi \sigma^2)' - (l - 1)(l + 2)\Pi \right\} \right]^\pm = 0.
\]
Substituting Eqs.(72) and (75) to this equation, moreover, we obtain
\[
\left[ 2\tilde{\phi}_{T} \left( \frac{\alpha v}{a} + \gamma(v/a)\sigma \right) \right]^\pm + \left[ \int k^2dk \left\{ \alpha a \Psi \sigma + H_{T}^{(2)}(\Pi \sigma^2)' + \gamma(v/a)H_{T}^{(2)}(\Pi \sigma^2)''(\Pi \sigma^2)' - (l - 1)(l + 2)\Pi \right\} \right]^\pm = 0.
\]
where
\[
L \equiv \int dk \int d\bar{\chi} d(\bar{\chi}) \Pi(\bar{\chi})\sigma(\bar{\chi}) \Pi(\chi)\sigma'(\chi).
\]
In Appendix D, \( L \) is rewritten and reduced to a compact form. Eliminating \( \tilde{\phi}_{T -} \) using Eq.(77), therefore, we obtain
\[
2\tilde{\phi}_{T +} \left[ \frac{\alpha v}{a} + \frac{\gamma v}{a} \right]^\pm = \left[ \frac{\alpha v}{a} + \frac{\gamma v}{a} \right] - \left[ \int k^2dk H_{T}^{(2)}(\Pi \sigma^2)'' \right]^\pm - \left[ \int k^2dk \left\{ \gamma a \Psi \Pi \sigma + H_{T}^{(2)}(\Pi \sigma^2)' + \gamma(v/a)H_{T}^{(2)}(\Pi \sigma^2)''(\Pi \sigma^2)' - (l - 1)(l + 2)\Pi \right\} \right]^\pm.
\]
Thus Eqs.(77) and (82) give us \( \tilde{\phi}_{T +} \) and \( \tilde{\phi}_{T -} \) in terms of given 3-dimensional metric perturbations. Here let us assume that \( v \) vanishes for simplicity. This condition holds in a good approximation, because \( v (\approx 10^{-5}) \) is very small in our background models, as was shown by Sakai et al. (cf. §2). Then \( \tilde{\phi}_{T} \) is reduced to
\[
\tilde{\phi}_{T +} = -\frac{1}{2} \left( \frac{\alpha v}{a} \right)^{-1} \left\{ \int k^2dk a \Psi \Pi \sigma \right\}^\pm + \left[ \int k^2dk H_{T}^{(2)}(\Pi \sigma^2)'' \right]^\pm - \left( \frac{\alpha v}{a} \right) - \left[ \int k^2dk H_{T}^{(2)}(\Pi \sigma^2)'' \right]^\pm.
\]
2. Even-parity perturbations

Here we consider scalar, vector and tensor perturbations together, and from the condition $[(\delta q_{MN})_{(\text{scalar})} + (\delta q_{MN})_{(\text{vector})} + (\delta q_{MN})_{(\text{tensor})}]^\pm = 0$, we obtain the following four independent equations:

$$
\left[ F^2 \int k^2 dk \left\{ \zeta_0 \Pi + 2v \zeta_L \Pi' + v^2 [\zeta_{LL} \Pi + 2 \phi_L (\Pi'' + \frac{1}{3} k^2 \Pi)] - 2v \frac{l(l+1)\Pi}{\sigma} (-a^2 \Psi + (a^3/\gamma)(\phi_T/a^2),_\sigma)
- v \phi_T \left( \frac{\Pi'}{\Pi} - \frac{\sigma'}{\sigma} \right) + v^2 H_T^{(2)} \right\} \right]^\pm = 0,
$$

$$
\left[ F \int k^2 dk \left\{ \zeta_{LL} \Pi + 2v \phi_L [\Pi' - \frac{\sigma'}{\sigma}] + \left[ -a^2 \Psi + (a^3/\gamma)(\phi_T/a^2),_\sigma \right] (\Pi \sigma)' + v \phi_T [(\Pi \sigma)'' - 2 \frac{\sigma'}{\sigma} (\Pi \sigma)','_T
+ l(l+1)(\Pi/\sigma)] + v H_T^{(2)} (l-1)(l+2)(\Pi \sigma)'/\sigma \right\} \right]^\pm = 0,
$$

$$
\left[ \int k^2 dk \left\{ (\phi_L + 2 \phi_T (\Pi \sigma)',_T + 2 \phi_L H_T^{(2)} ) \right\} \right]^\pm = 0,
$$

$$
\left[ \int k^2 dk \left\{ (\zeta_{LL} \Pi + 2 \phi_L \left[ \Pi',_T + \left( \frac{k^2}{3} - \frac{l(l+1)}{2 \sigma^2} \right) \Pi \right] - \phi_L \frac{l(l+1)}{2 \sigma^2} (\Pi \sigma)' - 2 \sigma',_T \Pi - \frac{L}{2} H_T^{(2)} \Pi \right) \right\} \right]^\pm = 0, \tag{84}
$$

where the first and second equations come from the $00$ and $0a$ components of the condition and the last two equations come from $ab$ component. In these equations the metric perturbations $\Phi_A, \Phi_T, \Psi$ and $H_T^{(2)}$ are given as functions of $\tau$ and $k$ at both regions.

Now in the same way as in the odd-parity case, we impose a localization condition for $\phi_0, \phi_L$ and $\phi_T$ as follows, using the function $d_{\pm}(\chi)$ (defined in Eqs.(73) and (74), so that the original local property of the displacement may be recovered:

$$
\int \phi_{0\pm} \Pi_{\pm} k^2 dk = \bar{\phi}_{0\pm}(\tau) d_{\pm}(\chi),
$$

$$
\int \phi_{L\pm} \Pi_{\pm} k^2 dk = \bar{\phi}_{L\pm}(\tau) d_{\pm}(\chi),
$$

$$
\int \phi_{T\pm} (\Pi_{\pm} \sigma_{\pm}) k^2 dk = \bar{\phi}_{T\pm}(\tau) d_{\pm}(\chi), \tag{85}
$$

where $\bar{\phi}_{0\pm}(\tau), \bar{\phi}_{L\pm}(\tau)$ and $\bar{\phi}_{T\pm}(\tau)$ are functions of only $\tau$. This condition means similarly that the total displacements have the nonzero values only in the neighborhood of the shell. Inversely $\phi_{0\pm}$ and $\phi_{L\pm}$ are expressed as

$$
\phi_{0\pm} = \bar{\phi}_{0\pm}(\tau) \frac{1}{k^2} \int_0^\infty \Pi_{\pm} \sigma_{\pm}^2 d_{\pm}(\chi) d\chi \simeq \bar{\phi}_{0\pm}(\tau) e^{-k^2 \Pi_{\pm}(k, \chi_b)} \sigma_{\pm}^2(k, \chi_b),
$$

$$
\phi_{L\pm} = \bar{\phi}_{L\pm}(\tau) \frac{1}{k^2} \int_0^\infty \Pi_{\pm} \sigma_{\pm}^2 d_{\pm}(\chi) d\chi \simeq \bar{\phi}_{L\pm}(\tau) e^{-k^2 \Pi_{\pm}(k, \chi_b)} \sigma_{\pm}^2(k, \chi_b). \tag{86}
$$

Under this condition we find that for the above four equations there are six unknown variables $\bar{\phi}_{0\pm}, \bar{\phi}_{L\pm}$ and $\bar{\phi}_{T\pm}$, representing the displacement of the shell. But they are redundant, because only two kinds of displacements are allowed in the even-parity case of the 2-dimensional space, as in Mukohyama’s treatment. Here $\delta Z_a$ from 4-dimensional scalar and vector perturbations give $\phi_L Y_{lm,a}$ and $\phi_T (\Pi \sigma)' Y_{lm,a}$, respectively, in the similar manner. Accordingly we impose an additional condition to relate $\phi_{L\pm}$ and $\phi_{T\pm}$ respectively in the form of

$$
\int \phi_{T\pm} (\Pi_{\pm} \sigma_{\pm}) k^2 dk = \int \phi_{L\pm} \Pi_{\pm} k^2 dk \quad \text{or} \quad \bar{\phi}_{T\pm}(\tau) = \bar{\phi}_{L\pm}(\tau). \tag{87}
$$

Then we can eliminate $\bar{\phi}_{T\pm}$ using Eq.(87) and determine the remaining four variables solving the above four equations as follows.

Here let us assume $v = 0$ for simplicity. Then the above equations lead to

$$
\left[ - \int dkk^2 \Phi_A \Pi + (\bar{\phi}_0/a),_\tau \right]^\pm = 0,
$$

where
As the result we found that the perturbations with $\phi_0, \phi_L$ and $\phi_T$ to Eq.(68), in which $\phi_0, \phi_L$ and $\phi_T$ can be replaced by $\phi_0, \phi_L$ and $\phi_T$, and we find that the results are independent of the value of $\epsilon$, as long as $\epsilon/\chi_b << 1$.

V. CONCLUDING REMARKS

We have studied the junction conditions imposed on the metric perturbations in the shell boundary in a four-dimensional model with two homogeneous regions $V^\pm$. As the result we found that the perturbations with different parities can be separated and treated independently, and that due to the inhomogeneity in the shell the mode coupling is caused among perturbations with various types and different wave-numbers in each parity. The displacements of the shell and the perturbed energy and momentum density for given metric perturbations in both regions can however be consistently derived.

Our results do not mean that arbitrary perturbations in both regions can always be adjusted, because it depends on the physical situation of the shell whether the obtained energy-momentum tensor in the shell is allowed or not. For instance, if we consider an expanding or collapsing star with the empty external region, any energy and momentum are not expected to be stored in the boundary and only gravitational radiation is released in the external region, because there are no matter perturbations.

In the present study, though we treated the displacement of the boundary shell as spatially 3-dimensional quantities, we could recover the original local property of the shell displacement (that it has the values only in the neighborhood of the shell) by imposing the localization condition and could solve the equations for the displacements.

Our result will be useful to the analyses of CMB anisotropy observed in the nonlinear structures such as overdense regions (such as superclusters) and underdense regions (such as voids) in which we may live.
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APPENDIX A: PERTURBATIONS IN A HOMOGENEOUS MODEL

We show here the expressions for metric perturbations and harmonic functions in a homogeneous model with the line-element in the text, paying attention to the 2-dimensional parity operation $P: \theta \to \pi - \theta, \varphi \to \pi + \varphi$, where $x^2 = \theta$ and $x^3 = \varphi$.[13, 20]

1. Scalar perturbations

Metric perturbations are expressed as

$$
\begin{align*}
    g_{00} &= -a^2[1 + 2A(\eta)Q], \\
    g_{0i} &= -a^2B(\eta)Q_i, \\
    g_{ij} &= a^2\{[1 + 2H_L(\eta)Q]\gamma_{ij} + 2H_T(\eta)Q_{ij}\}.
\end{align*}
$$

(A1)

Here $Q$ is a scalar harmonic function with the wave-number $k$ satisfying the equation

$$
(\Delta + k^2)Q = 0,
$$

(A2)

where the Laplacian $\Delta$ is defined by $\Delta \varphi \equiv \gamma_{ij}\varphi_i\varphi_j$ and $i$ denotes the three-dimensional covariant derivative in the space $dl^2 = \gamma_{ij}dx^idx^j$. $Q_i$ and $Q_{ij}$ are defined by

$$
\begin{align*}
    Q_i &= -k^{-1}Q_i, \\
    Q_{ij} &= k^{-2}Q_{ij} + \frac{1}{3}\gamma_{ij}Q.
\end{align*}
$$

(A3)

Harmonic function $Q$ is expanded using the usual spherical harmonics $Y_{lm}$ as

$$
Q = \Pi^r_n(\chi)Y_{lm}(\theta, \varphi),
$$

(A4)

where $k^2 = n^2 - K$ for the spatial curvature $K = 1, 0, -1$. Then from Eq.(A2) we obtain

$$
\Pi'' + 2(\sigma'/\sigma)\Pi' + \left[n^2 - l(l + 1)/\sigma^2\right]\Pi = 0,
$$

(A5)

where we omitted suffixes of $\Pi^r_n$ for simplicity, a prime denotes the partial derivative with respect to $x^1 = \chi$ and $\sigma(\chi) = \sin \chi, \chi, \sinh \chi$ for $K = 1, 0, -1$, respectively. The normalized expressions of $\Pi$ are found in Wilson’s paper[21]. For the above-mentioned parity operation, we have

$$
P(Y_{lm}) = (-1)^lY_{lm}.
$$

(A6)

This property is called the even parity.

2. Vector perturbations

Metric perturbations are expressed as

$$
\begin{align*}
    g_{00} &= -a^2, \\
    g_{0i} &= -a^2B^{(1)}(\eta)V_i, \\
    g_{ij} &= a^2[\gamma_{ij} + 2H_T^{(1)}(\eta)V_{ij}].
\end{align*}
$$

(A7)

Here $V_i$ is a vector harmonic function with the wave-number $k$ satisfying the equations

$$
(\Delta + k^2)V_i = 0, \quad V_{i|i} = 0,
$$

(A8)

and $V_{ij}$ is defined by

$$
V_{ij} \equiv -(2k)^{-1}[V_{ij} + V_{ji}].
$$

(A9)

In vector perturbations there are two cases with different parities.
a. The case of odd parity

The components of vector harmonics $V_i$ are expressed as

$$
V_1 = 0, \\
V_2 = -\sigma \Pi Y_{lm,3}/\sin \theta, \\
V_3 = \sigma \Pi Y_{lm,2} \sin \theta.
$$

(A10)

In this case we have $P(V_i) = (-1)^{l-1}V_i$ and the property is called the odd parity.

b. The case of even parity

The vector harmonics $V_i$ in another case is

$$
V_1 = \frac{l(l+1)}{\sigma} \Pi Y_{lm}, \\
(V_2, V_3) = (\Pi \sigma),_1(Y_{lm,2}, Y_{lm,3}).
$$

(A11)

In this harmonics, we have $P(V_i) = (-1)^lV_i$, so that the property is called the even parity.

3. Tensor perturbations

Tensor harmonics $G_{ij}$ satisfies

$$
G_{ij}^{|j} = -k^2 G_{ik}, \quad G_{i|j}^k = 0, \quad G_i^i = 0,
$$

and the components are expressed as follows using $\Pi_l^n$ and $Y_{lm}$. The tensor perturbations also have two cases with different parities.

a. The case of odd parity

$$
G_{11} = 0, \\
G_{22} = (\Pi \sigma)_1(-X_{lm}/\sin \theta), \\
G_{33} = (\Pi \sigma)_1 X_{lm} \sin \theta, \\
G_{23} = (\Pi \sigma)_1 Z_{lm} \sin \theta, \\
G_{12} = (l-1)(l+2)\Pi(-Y_{lm,3}/\sin \theta), \\
G_{13} = (l-1)(l+2)\Pi Y_{lm,2} \sin \theta,
$$

(A13)

where

$$
X_{lm} \equiv 2(Y_{lm,23} - \cot \theta Y_{lm,3}), \\
Z_{lm} \equiv Y_{lm,22} - \cot \theta Y_{lm,2} - Y_{lm,33}/\sin^2 \theta.
$$

(A14)

(A15)

If we define a two-dimensional vector $W_a$ with the components $W_2 = -Y_{lm,3}/\sin \theta$ and $W_3 = Y_{lm,2} \sin \theta$, it is found that $G_{ab}$ and $G_{1a}$ is reduced to

$$
G_{ab} = (\sigma^2 \Pi)_1(W_a^b + W_b^a), \\
G_{1a} = (l-1)(l+2)\Pi W_a.
$$

(A16)

For the parity operation, we have the odd-parity property $P(G_{ij}) = (-1)^{l-1}G_{ij}$. 

b. The case of even parity

The tensor harmonics in another case are

\[
G_{11} = \frac{L}{\sigma^2} \Pi Y_{lm},
\]

\[
(G_{22}, G_{33}) = -\frac{L}{2} \Pi Y_{lm}(1, \sin^2 \theta) + G_1^0 Z_{lm}(1, -\sin^2 \theta),
\]

\[
G_{23} = G_1^0 X_{lm},
\]

\[
G_{12} = (l-1)(l+2)\frac{1}{\sigma} \Pi \sigma Y_{lm,2},
\]

\[
G_{13} = (l-1)(l+2)\frac{1}{\sigma} \Pi \sigma Y_{lm,3},
\]

(A17)

where \(L \equiv l(l+1)(l-1)(l+2)\), and

\[
G_i^l \equiv \sigma \sigma' \Pi_{ij} + \frac{1}{2} (l^2 + l + 2) - (k^2 - 2)\sigma^2 \Pi.
\]

(A18)

In this case we have the even-parity property \(P(G_{ij}) = (-1)^i G_{ij}\).

APPENDIX B: \(J_{\mu \nu}\)

The resulting expressions for \(J_{\mu \nu}\) for three types of metric perturbations are shown here.

1. Scalar perturbations

\[
J_{000} = 2\{ -a^2 \Phi_A - \ddot{\phi}_0 - 3\frac{\dot{a}}{a} \dot{\phi}_0 + [3(\frac{\dot{a}}{a})^2 - \frac{\dot{a}}{a}] \dot{\phi}_0 \} Q,
\]

\[
J_{00} = 2\{a^2 \Phi_A + \ddot{\phi}_L - 3\frac{\dot{a}}{a} \dot{\phi}_L + [2(\frac{\dot{a}}{a})^2 - \frac{\dot{a}}{a}] \dot{\phi}_L \} Q_i, \quad (i = 1, 2, 3)
\]

\[
J_{101} = 2\{a^2 \Phi_A - \ddot{\phi}_0 + (\frac{\dot{a}}{a})^2 \dot{\phi}_0 \} Q + 2(\dot{\phi}_L - 4\phi_L) Q_{111},
\]

\[
J_{106} = 2(\dot{\phi}_L - 4\phi_L)[Q_{1b} - \ddot{\phi}_L Q_{1b}], \quad (b = 2, 3)
\]

\[
J_{111} = [2a^2 \Phi_A - 4\frac{\dot{a}}{a} \dot{\phi}_0 - 2\frac{\dot{a}}{a} (\dot{\phi}_L - 2\phi_L)] Q_1 + 2\phi_L Q_{1111},
\]

\[
J_{116} = [2a^2 \Phi_A - 4\frac{\dot{a}}{a} \dot{\phi}_0] Q_b + 2\phi_L[Q_{111} - 2\frac{\sigma'}{\sigma} Q_1 + 2(\frac{\sigma'}{\sigma})^2 Q], \quad (b = 2, 3)
\]

\[
J_{122} = -2(\Phi_A a^2 + \frac{\dot{a}}{a} \Phi_L) \sigma^2 Q_1 + 2\phi_L[Q_{112} - 2\frac{\sigma'}{\sigma} Q_{112} + \sigma' Q_{111} - (\sigma')^2 - 2(\frac{\dot{a}}{a})^2 \sigma^2 Q_1],
\]

\[
J_{133} = -2(\Phi_A a^2 + \frac{\dot{a}}{a} \Phi_L) \sigma^2 \sin^2 \theta Q_{11} + 2\phi_L[Q_{133} - 2(\frac{\sigma'}{\sigma})^2 Q_{33} + \sin^2 \theta \sigma' Q_{11}]
\]

- \sin^2 \theta[(\sigma')^2 - 2(\frac{\dot{a}}{a})^2 \sigma^2 Q_1 + \sin \theta \cos \theta (Q_{11} - \frac{\sigma'}{\sigma} Q_{22}),
\]

\[
J_{123} = -2\phi_L \sigma^2 \sin \theta Q_1 \frac{\sigma^2 \sin \theta}{[\sigma^2 \sin \theta]^3},
\]

\[
J_{011} = -2(a^2 \Phi_H)^2 + 4a \dot{a} \Phi_A - 2\frac{\dot{a}}{a} \dot{\phi}_0 Q + 2\phi_A[Q_{11} + \frac{\dot{a}}{a} + (\frac{\dot{a}}{a})^2 Q] - 4\frac{\dot{a}}{a} \phi_L Q_{111},
\]

\[
J_{022} = -2(a^2 \Phi_H)^2 + 4a \dot{a} \Phi_A - 2\frac{\dot{a}}{a} \dot{\phi}_0 \sigma^2 Q + 2\phi_A[Q_{22} + \frac{\dot{a}}{a} + (\frac{\dot{a}}{a})^2 \sigma^2 Q + \sigma' Q_1]
\]

- \frac{\dot{a}}{a} \phi_L[Q_{22} + \sigma' Q_{11}],
\]

\[
J_{033} = -2(a^2 \Phi_H)^2 + 4a \dot{a} \Phi_A - 2\frac{\dot{a}}{a} \dot{\phi}_0 \sigma^2 \sin^2 \theta Q + 2\phi_A[Q_{33} + \frac{\dot{a}}{a} + (\frac{\dot{a}}{a})^2 \sigma^2 \sin^2 \theta Q + \sigma' \sin^2 \theta Q_1 + \sin \theta \cos \theta Q_{22}]
\]

+ \sigma' \sin^2 \theta Q_{11} + \sin \theta \cos \theta Q_{22}] - 4\frac{\dot{a}}{a} \phi_L[Q_{33} + \sigma' \sin^2 \theta Q_{11} + \sin \theta \cos \theta Q_{22}],
\]

\[
J_{023} = 2(\phi_0 - 2\frac{\dot{a}}{a} \phi_L)(Q_{23} - \cot \theta Q_{33}),
\]
\[ J_{01b} = 2(\phi_0 - \frac{\dot{a}}{a}\phi_L)(Q_{,tb} - \frac{\sigma'}{\sigma}Q_{,b}), \quad (b = 2, 3) \]
\[ J_{00i} = \left[ -2a^2\Phi_A + 2(\phi_0 - \frac{\dot{a}}{a}\phi_0) - \frac{\dot{a}}{a}(\phi_L - \frac{\dot{a}}{a}\phi_L) \right]Q_{,i}, \quad (i = 1, 2, 3). \] (B1)

2. Vector perturbations

\[ J_{000} = 0, \]
\[ J_{100} = 2[\ddot{\phi}T - \frac{\dot{a}}{a}\phi_T + 2(-\frac{\dot{a}}{a} + 2\frac{\ddot{a}}{a})\phi_T - a^2(\dot{\Psi} + \frac{\dot{a}}{a}\Psi)]V_1, \]
\[ J_{101} = 2[\dot{\phi}T - \frac{\dot{a}}{a}\phi_T]V_{1,11}, \]
\[ J_{10b} = a^2\Psi(V_{b,1} - V_{1,b}) + 2(\phi_T - \frac{\dot{a}}{a}\phi_T)(V_{1,b} - \frac{\sigma'}{\sigma}V_{b}), \quad (b = 2, 3) \]
\[ J_{111} = [2a\dot{a}\Psi - 2\frac{\dot{a}}{a}(\phi_T - \frac{\dot{a}}{a}\phi_T)]V_1 + 2\phi_TV_{1,11}, \]
\[ J_{122} = \frac{2}{a}\sigma'(a^2\Psi - \phi_T) + 2\phi_T [V_{1,22} + \sigma'V_{1,1} - 2\frac{\sigma'}{\sigma}V_{2,2} - (\sigma')'V_1 + 2(\frac{\dot{a}}{a})^2\sigma^2 V_1], \]
\[ J_{133} = \frac{2}{a}\sigma'^2\sin^2 \theta(a^2\Psi - \phi_T) + 2\phi_T[V_{1,33} + \sigma'\sin^2 \theta V_{1,1} - 2\frac{\sigma'}{\sigma}V_{3,3} \]
\[ - (\sigma')'\sin^2 \theta V_1 + 2(\frac{\dot{a}}{a})^2\sigma^2 \sin^2 \theta V_1 + \cos \theta \cos (V_{1,2} - 2\frac{\sigma'}{\sigma}V_2)], \]
\[ J_{123} = 2\phi_T[V_{1,23} - \frac{\sigma'}{\sigma}(V_{2,3} + V_{3,2}) + \cot \theta(-V_{1,3} + 2\frac{\sigma'}{\sigma}V_3)], \]
\[ J_{11b} = 2[\phi_T[V_{1,1a} - \frac{\sigma'}{\sigma}(V_{b,1} + V_{1,b}) + 2(\sigma')^2V_2], \quad (b = 2, 3) \]
\[ J_{00i} = [2a\dot{a}\Psi - 2\frac{\dot{a}}{a}(\phi_T - \frac{\dot{a}}{a}\phi_T)]V_i, \quad (i = 1, 2, 3) \]
\[ J_{011} = -2(a^2\Psi + 2\frac{\dot{a}}{a}\phi_T)V_{1,11}, \]
\[ J_{022} = -2(a^2\Psi + 2\frac{\dot{a}}{a}\phi_T)(V_{2,2} + \sigma'V_1), \]
\[ J_{033} = -2(a^2\Psi + 2\frac{\dot{a}}{a}\phi_T)(V_{3,3} + \sigma'\sin^2 \theta V_1), \]
\[ J_{023} = -(a^2\Psi + 2\frac{\dot{a}}{a}\phi_T)(V_{2,3} + V_{3,2} - 2\cot \theta V_3), \]
\[ J_{01b} = -(a^2\Psi + 2\frac{\dot{a}}{a}\phi_T)(V_{1,b} + V_{b,1} - 2\frac{\sigma'}{\sigma}V_b). \quad (b = 2, 3) \] (B2)

3. Tensor perturbations

\[ J_{000} = J_{001} = J_{002} = J_{003} = J_{100} = 0, \]
\[ J_{i0j} = -2(a^2\dot{H}_T^{(i)}G_{i,j}), \quad (i, j = 1, 2, 3) \]
\[ J_{111} = 2a^2\dot{H}_T^{(2)}G_{11,1}, \]
\[ J_{122} = 2a^2\dot{H}_T^{(2)}[2G_{12,2} - G_{22,1} + 2\sigma'G_{11}], \]
\[ J_{133} = 2a^2\dot{H}_T^{(2)}[2G_{13,3} - G_{33,1} + 2\sigma'\sin^2 \theta G_{11} + 2\sin \theta \cos \theta G_{12}], \]
\[ J_{123} = 2a^2\dot{H}_T^{(2)}[G_{12,3} + G_{31,2} - G_{23,1} - 2\cot \theta G_{13}], \]
\[ J_{11b} = 2a^2\dot{H}_T^{(2)}[G_{11,b} - 2\frac{\sigma'}{\sigma}G_{1b}], \quad (b = 2, 3) \] (B3)
APPENDIX C: THE INTRINSIC QUANTITY $\Theta_{MN}$ IN THE SHELL

The expressions of $\Theta_{MN}$ are shown for three types of metric perturbations, where $M$ and $N$ take the values 0, 2 and 3.

1. Scalar perturbations

\[
\Theta_{00} = \frac{2}{a^3} \left\{ \frac{\sigma'}{\sigma} a^2 (\Phi_H + 2\Phi_A)Q + a^2 \Phi_H Q_1 \right\} + \frac{2}{a^3} \frac{\sigma'}{\sigma} (-2\ddot{\phi}_0 + \dot{a} \frac{\dot{\phi}_0}{a})Q \\
+ \frac{2}{a^3} \dot{\phi}_L Q_{1,1} - \frac{1}{a^3} \dot{\phi}_L \left\{ \frac{1}{\sigma^2} ((Q_1 - 2\frac{\sigma'}{\sigma} Q)_{2,2} + (Q_1 - 2\frac{\sigma'}{\sigma} Q)_{3,3} \sin^2 \theta) \right\} \\
- 2\frac{\sigma''}{\sigma} + (\frac{\sigma'}{\sigma})^2 - 2(\frac{\dot{a}}{a})^2 |Q_{1,1} + \frac{1}{\sigma^2} \cot \theta (Q_1 - 2\frac{\sigma'}{\sigma} Q)_{2,1} \right\} \\
+ \frac{2}{a^3} \frac{\sigma'}{\sigma} \dot{\phi}_L + \frac{\dot{a}}{a} (3\Phi_H + \Phi_A)Q + \frac{2}{a^3} a^2 \Phi_A Q \\
- \frac{v}{a^3} \frac{\phi_0}{2} \left\{ \frac{2\dot{a}}{a} + 3(\frac{\dot{a}}{a})^2 - \frac{l(l+1)}{\sigma^2} \right\} |Q_{2,1} + \frac{4\sigma' Q_{1,1}}{\sigma} \right] - \frac{2}{a^3} (3\frac{\dot{a}}{a} + \ddot{v})(\phi_0 - \frac{\dot{a}}{a} \phi_0)Q \\
+ \frac{2}{a^3} \dot{\phi}_L Q + \frac{2}{a^3} \frac{\phi_0}{2} \dot{\phi}_L |Q_{1,1} + \frac{1}{\sigma^2} (Q_{2,2} + Q_{3,3} \sin^2 \theta) \\
+ \frac{4\sigma' Q_{1,1}}{\sigma} + \frac{1}{\sigma} \cot \theta Q_{2,1} - \frac{2\sigma'}{\sigma} \frac{\phi_0}{2} Q_{1,1} + 0(v^2),
\]

\[
\Theta_{22} = \Theta_s + \frac{1}{a} \dot{\phi}_L \{(Q_{1,33} - 2\frac{\sigma'}{\sigma} Q_{3,33}) / \sin^2 \theta + \cot \theta (Q_{1,12} - 2\frac{\sigma'}{\sigma} Q_{2}) - 4\frac{\sigma'}{\sigma} Q_{22}) + 0(v),
\]

\[
\Theta_{33} = \Theta_s \sin^2 \theta + \frac{1}{a} \dot{\phi}_L \{((Q_{1,22} - 2\frac{\sigma'}{\sigma} Q_{22}) \sin^2 \theta - 4\frac{\sigma'}{\sigma} (Q_{3,3} + \sin \theta \cos \theta Q_{2,1}) \} + 0(v),
\]

\[
\Theta_{06} = -\frac{1}{a^2} \frac{\phi_0}{2} \frac{\phi_0}{2} |(Q_1 + \frac{\sigma'}{\sigma} Q)_{,b} - \frac{2}{a^4} \frac{\sigma'}{\sigma} \phi_0 Q_{,b} - \frac{2}{a^2} \frac{\sigma'}{\sigma} \phi_0 Q_{,b} - \frac{2}{a^2} \frac{\sigma'}{\sigma} \phi_0 Q_{,b} |Q_{,b} \\
- \frac{v}{a^2} \left\{ [2\dot{a}^2 (\Phi_H - \Phi_A) + \ddot{\phi}_0 + \frac{4}{a} (\phi_0 - \frac{\dot{a}}{a} \phi_0)] Q_{,b} + 2\phi_0 |Q_{,11} + 2\frac{\sigma'}{\sigma} Q_{,1} \\
- \frac{2}{\sigma^2} \frac{\sigma'}{\sigma} Q_{,b} \} + 0(v^2), \quad (b = 2, 3)
\]

\[
\Theta_{23} = \frac{1}{a} \dot{\phi}_L |(Q_{1,12} - 2\frac{\sigma'}{\sigma} Q_{2}) - \cot \theta (Q_{1,1} - 2\frac{\sigma'}{\sigma} Q_1), 3 - \phi_0 - \frac{2}{a^4} \phi_0 |Q_{,2} - \cot \theta Q_{,3} \right\} \right\} + 0(v^2),
\]

where

\[
\Theta_s / \sigma^2 \equiv -\frac{1}{a} a^2 (\Phi_A + \Phi_H) Q_{,1} - 5\frac{\sigma'}{\sigma} \frac{1}{a^2} (a^2 \Phi_H + \frac{\dot{a}}{a} \phi_0) Q - \frac{1}{2a} (\phi_0 - \frac{\dot{a}}{a} \phi_0) Q_{,1} \\
+ \frac{1}{a} \dot{\phi}_L \left\{ -\frac{\sigma'}{\sigma} Q_{,11} + \frac{\sigma''}{\sigma} (Q_{3,3})^2 Q_{,1} + \frac{\dot{a}}{a} - \frac{\dot{a}}{a^2} a^2 Q_{,1} \right\} + 0(v).
\]

2. Vector perturbations

\[
\Theta_{00} = -\frac{1}{a^3} \left\{ \frac{\dot{a}}{a} (a^2 \Psi - \dot{\phi}_T) V_1 + \phi_T (-2\frac{\sigma'}{\sigma} V_{1,1} + 4(\frac{\dot{a}}{a})^2 V_1 + \frac{1}{\sigma^2} (V_{1,22} - 2\frac{\sigma'}{\sigma} V_{2,2} \\
+ (V_{1,33} - 2\frac{\sigma'}{\sigma} V_{3,3}) / \sin^2 \theta + \cot \theta (V_{1,2} - 2\frac{\sigma'}{\sigma} V_2) + 2\frac{\sigma'}{\sigma} V_{1,1} - 2\frac{\sigma'}{\sigma^2} V_1) \right\} \\
+ \frac{v}{a^3} \left\{ \frac{\dot{a}}{a} \phi_T V_{1,1} - 2(a^2 \Psi + a^2 (\phi_T / a^2)) \right\} \frac{\sigma'}{\sigma} V_1 \\
+ (a^2 \Psi + 2\frac{\dot{a}}{a} \phi_T) \left\{ \frac{1}{\sigma^2} (V_{2,2} + V_{3,3} / \sin^2 \theta) + 2\frac{\sigma'}{\sigma} V_1 \\
+ \sigma' \left\{ -a^2 \Psi + a^2 (\phi_T / a^2) V_1 \right\} + 0(v^2),
\]
\[ \Theta_{22} = \Theta_v + \frac{1}{a} \phi_T [(V_{1,33} - 2 \sigma' V_{3,3})/\sin^2 \theta - \frac{1}{\sigma} \sigma' V_{2,2} + \cot \theta (V_{1,2} - 2 \sigma' V_2)] \]
\[ - \frac{2}{a} \phi_T (3 \dot{a}/a + v) V_{2,2} - \frac{v}{a} (a^2 \Psi + 2 \frac{\dot{a}}{a} \phi_T) V_{3,3}/\sin^2 \theta + 0(v^2), \]
\[ \Theta_{33} = \Theta_v \sin \theta + \frac{1}{a} \phi_T [(V_{1,22} - 2 \sigma' V_{2,2}) \sin^2 \theta - \frac{1}{\sigma} (V_{3,3} + \sin \theta \cos \theta V_2)] - \frac{2}{a} \phi_T (3 \dot{a}/a + v) (V_{3,3} + \sin \theta \cos \theta V_2) \]
\[ + \sin \theta \cos \theta V_2 - \frac{v}{a} (a^2 \Psi + 2 \frac{\dot{a}}{a} \phi_T) V_{2,2} \sin^2 \theta + 0(v^2), \]
\[ \Theta_{23} = - \frac{1}{a} \phi_T [V_{1,23} + \frac{\sigma}{\sigma} (V_{2,3} + V_{2,3})] - \cot \theta (V_{1,3} + 2 \sigma' V_3) \]
\[ + \frac{v}{2a} [a^2 \Psi + 2 (\frac{\dot{a}}{a} - \frac{2 \sigma'}{\sigma}) \phi_T] (V_{2,3} + V_{3,2} - 2 \cot \theta V_3), \]
\[ \Theta_{06} = - \frac{1}{a^2} \frac{1}{2} a^2 \Psi (V_{b,1} - V_{b,1}) + (\phi_T - 2 \frac{\dot{a}}{a} \phi_T) (V_{b,1} - \frac{\sigma'}{\sigma} V_b) \]
\[ + (2 \frac{\sigma'}{\sigma} + 3 \frac{\dot{a}}{a} v + \dot{v}) [-a^2 \Psi + a^2 (\phi_T/a^2)] V_b \]
\[ - \frac{v}{a^2} [(a \dot{\phi} - \frac{\dot{a}}{a} (\phi_T - 2 \frac{\dot{a}}{a} \phi_T)] V_b + \phi_T [V_{1,1b} - \frac{\sigma'}{\sigma} (V_{1,1b} + V_{1,1b}) + 2 (\frac{\sigma'}{\sigma})^2 V_b] \]
\[ + [-\frac{1}{a} v \phi_T (2 \frac{\sigma'}{\sigma} + 3 \frac{\dot{a}}{a} v + \dot{v})] (V_{1,1b} + V_{1,1b} - 2 \frac{\sigma'}{\sigma} V_b) + 0(v^2). \quad (b = 2, 3) \] (C3)

where

\[ \Theta_v = - \frac{\sigma^2 \sigma'}{a} \phi_T [V_{1,1} + \frac{\sigma'}{\sigma} V_1] - \frac{\sigma^2}{\sigma} \phi_T - \frac{\dot{a}}{a} \phi_T - 2 (\frac{\dot{a}}{a})^2 \phi_T \]
\[ + a^2 \psi a \phi_T - \frac{\ddot{a}}{a} \phi_T + \frac{\dot{a}}{a} \phi_T + (\frac{\ddot{a}}{a} - \frac{\ddot{a}}{a} \phi_T) V_{1,1} + \phi_T \frac{\sigma'}{\sigma} V_{1,1} \]
\[ + 2 (\frac{\dot{a}}{a})^2 V_1 - \frac{\sigma'}{\sigma} \sigma^{-2} V_1] - \frac{\sigma^2}{\sigma} (2 (v \phi_T - \frac{\dot{a}}{a} \phi_T + \ddot{\phi} \phi_T) V_{1,1} + \frac{\dot{a}}{a} V_1) \]
\[ a^2 (\phi_T/a^2)] V_1] - 2 \frac{\sigma^2}{\sigma} \phi_T (3 \frac{\dot{a}}{a} v + \dot{v}) \frac{\sigma'}{\sigma} V_{1,1} - \frac{\dot{a}}{a} a^2 \Psi + 2 (\frac{\dot{a}}{a} \phi_T) \sigma' V_1 + 0(v^2). \quad (C4) \]

3. Tensor perturbations

\[ \Theta_{00} = -a^{-1} H_T^{(2)} \right[ \sigma^{-2} (2 G_{12,2} - G_{22,1} + (2 G_{13,3} - G_{33,1})/\sin^2 \theta) + 6 \frac{\sigma'}{\sigma} G_{11} \]
\[ + 2 \cot \theta \sigma^{-2} G_{12} + 0(v), \]
\[ \Theta_{22} = a H_T^{(2)} [3 \sigma' G_{11} + (2 G_{13,3} - G_{33,1})/\sin^2 \theta + 2 \cot \theta G_{12}] + 0(v), \]
\[ \Theta_{33} = a H_T^{(2)} [3 \sigma' G_{11} + (2 G_{12,2} - G_{22,1}) \sin^2 \theta + 0(v), \]
\[ \Theta_{23} = a H_T^{(2)} [3 \sigma' G_{23} - (G_{12,3} + G_{13,2} - G_{23,1} - 2 \cot \theta G_{13})] + 0(v), \]
\[ \Theta_{06} = -\bar{H}_T^{(2)} G_{1b} + 0(v). \quad (b = 2, 3) \] (C5)

APPENDIX D: INTEGRALS L AND M

The integrals L and M in Eqs. (81) and (89) are rewritten here using the orthonormal relations (76). First we have

\[ L = |\sigma'(\chi)/\sigma(\chi)| \int d\tilde{\chi} d\tilde{\phi} \int dk \Pi(\tilde{\chi}) |\sigma'(\tilde{\chi})| \Pi(\chi) |\sigma'(\chi)| d\chi. \] (D1)

Since \( d(\chi) = 1 \) in the neighborhood of the shell, we obtain

\[ L_\pm = (\sigma'/\sigma)_\pm. \] (D2)
Next we notice that
\[
\frac{\Pi\sigma'}{\Pi\sigma} = \frac{\Pi}{\sigma} \left( 1 - \frac{\Pi\sigma'}{\Pi\sigma} \right)
\]
and the factor \(|\Pi\sigma'/\Pi\sigma|\) is \(\ll 1\), if assume that the perturbations shorter enough than the shell size contribute dominantly to the integral. Then \(\mathcal{M}\) reduces to
\[
\mathcal{M} = \frac{\partial}{\partial \chi} \int d\tilde{\chi} d(\tilde{\chi}) \frac{\Pi(\chi)\sigma(\chi)}{\sigma(\chi)} \left[ \left( \frac{\Pi(\chi)\sigma(\chi)}{\sigma(\chi)} \right) - \frac{\Pi(\chi)\sigma'(\chi)}{\sigma(\chi)} \right] \\
- l(l+1) \int d\tilde{\chi} d(\tilde{\chi}) \frac{\Pi(\chi)\sigma(\chi)}{\sigma(\chi)} \left( 1 - \frac{\Pi(\chi)\sigma'(\chi)}{\Pi\sigma} \right) \\
= \left\{ \sigma \left[ \frac{(\sigma')'}{\sigma^2} - \frac{(\sigma')'}{\sigma^2} \right] - \frac{l(l+1)}{\sigma^2} \left( 1 - \frac{\Pi\sigma'}{\Pi\sigma} \right) \right\} d(\chi) \\
= -\frac{1}{\sigma^2} \left[ (\sigma')^2 + l(l+1) \left( 1 - \frac{\Pi\sigma'}{\Pi\sigma} \right) \right] d(\chi),
\]
where \(\Pi\sigma'/\Pi\sigma\) represents the root-mean-square of \(\Pi\sigma'/\Pi\sigma\) in the \(k\)-space. Accordingly, we obtain
\[
\mathcal{M}_\pm \simeq -\frac{1}{\sigma^2} \left[ (\sigma')^2 + l(l+1) \right].
\]

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