A NEW EQUIVALENCE BETWEEN SINGULARITY CATEGORIES OF COMMUTATIVE ALGEBRAS

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Abstract. We construct a triangle equivalence between the singularity categories of two isolated cyclic quotient singularities of Krull dimensions two and three, respectively. This is the first example of a singular equivalence involving connected commutative algebras of odd and even Krull dimension. In combination with Orlov’s localization result, this gives further singular equivalences between certain quasi-projective varieties of dimensions two and three, respectively.

1. Introduction

Singularity categories were introduced by Buchweitz providing a general framework for Tate cohomology [7]. More recently, they have been studied in birational geometry [28], in relation with string theory & homological mirror symmetry [25] and knot theory [21].

There are many known triangle equivalences between singularity categories $\mathcal{D}_{sg}(A) := \mathcal{D}^b(A)/\mathcal{K}^b(\text{proj-}A)$ of (noncommutative) connected Noetherian $\mathbb{C}$-algebras $A$. For example, for every ADE-surface singularity $R$ (except for $E_8$), we construct uncountably many algebras $A_i$, such that $\text{mod } A_i \ncong \text{mod } A_j$ for $i \neq j$ and $\mathcal{D}_{sg}(A_i) \cong \mathcal{D}_{sg}(R)$ for all $i$, see [14].

This is in stark contrast to the commutative case: until recently, Knörrer’s equivalences $\mathcal{D}_{sg}(S/(f)) \cong \mathcal{D}_{sg}(S[[x,y]]/(f-xy))$, for non-zero $f \in S := \mathbb{C}[z_0, \ldots, z_d]$, (1.1)

from 1987 [22] were the only [24] known source of singular equivalences between singular connected commutative Noetherian $\mathbb{C}$-algebras, which are not analytically isomorphic.

Using the relative singularity category techniques developed in our joint works [16, 17], one can construct the following additional family of singular equivalences

$$\mathcal{D}_{sg}\left(\mathbb{C}[y_1, y_2]^\frac{1}{n}(1,1)\right) \cong \mathcal{D}_{sg}\left(\mathbb{C}[z_1, \ldots, z_{n-1}]\left(\frac{z_1^{1/n}}{z_1, \ldots, z_{n-1}}\right)^2\right),$$

(1.2)

where for a primitive $n$th root of unity $\epsilon \in \mathbb{C}$, we use the notation:

$$\frac{1}{n}(a_1, \ldots, a_m) = \langle \text{diag} (\epsilon^{a_1}, \ldots, \epsilon^{a_m}) \rangle \subset \text{GL}(m, \mathbb{C}).$$

(1.3)

We learned about (1.2) from [29]. The equivalences (1.2) can also be deduced from [19] in combination with [26]. In our joint work with Joseph Karmazyn [15], we give another geometric proof of (1.2) and construct noncommutative finite dimensional algebras $K_{n,a}$ that generalize (1.2) to all cyclic quotient surface singularities $\mathbb{C}[y_1, y_2]^\frac{1}{n}(1,a)$.

We state the main result of this paper. To the best of our knowledge, this is the first example of a singular equivalence between rings of odd and even Krull dimension.

Theorem 1.1. There is a triangle equivalence between singularity categories

$$\mathcal{D}_{sg}\left(\mathbb{C}[x_1, x_2, x_3]^\frac{1}{2}(1,1,1)\right) \cong \mathcal{D}_{sg}\left(\mathbb{C}[y_1, y_2]^\frac{1}{4}(1,1)\right).$$

(1.4)
Moreover, these singularity categories are equivalent to the following singularity category:

\[ D_{sg} \left( \frac{\mathbb{C}[z_1, z_2, z_3]}{(z_1, z_2, z_3)^2} \right). \] (1.5)

Remark 1.2. (1) The category (1.5) is idempotent complete, Hom-infinite, not Krull–Schmidt and can be explicitly described using the Leavitt algebra of type (1, 2) [9, 23].

(2) By [27, Ex 2.15 & 2.17], there are isomorphisms involving Grothendieck groups

\[ K_0 \left( D_{sg} \left( \mathbb{C}[x_1, x_2, x_3]^{\mathbb{I}_{1,1,1}} \right) \right) \cong \mathbb{Z}/4\mathbb{Z} \cong K_0 \left( D_{sg} \left( \mathbb{C}[y_1, y_2]^{\mathbb{I}_{1,1,1}} \right) \right), \] (1.6)

which provide evidence for (1.4) and will be used in our proof below.

(3) Let \( X \) and \( Y \) be complex quasi-projective varieties of Krull dimensions 2 and 3, respectively, with only isolated singularities \( \text{Sing}(X) = \{ s_1, \ldots, s_n \} \) and \( \text{Sing}(Y) = \{ t_1, \ldots, t_n \} \).

Assume that the \( m \)-adic completions of the local rings in the singular points satisfy:

\[ \hat{O}_{X,s_i} \cong \mathbb{C}[y_1, y_2]^{\mathbb{I}_{1,1,1}} \] and \( \hat{O}_{Y,t_i} \cong \mathbb{C}[x_1, x_2, x_3]^{\mathbb{I}_{1,1,1}}. \) (1.7)

There are triangle equivalences, where \((-)^\omega\) denotes the idempotent completion, cf. [26].

\[ D_{sg}(X)^\omega \cong \bigoplus_{i=1}^n D_{sg} \left( \mathbb{C}[y_1, y_2]^{\mathbb{I}_{1,1,1}} \right) \cong \bigoplus_{i=1}^n D_{sg} \left( \mathbb{C}[x_1, x_2, x_3]^{\mathbb{I}_{1,1,1}} \right) \cong D_{sg}(Y)^\omega. \] (1.8)

The equivalence in the middle is (1.4) and the equivalences on the left and on the right follow from [26], since the categories in Theorem 1.1 are idempotent complete, cf. part (1) of this remark. We can rephrase (1.8) as follows: the singularity categories \( D_{sg}(X) \) and \( D_{sg}(Y) \) are triangle equivalent up to taking direct summands.

1.1. Strategy of proof. By (1.2), it suffices to show that there is a singular equivalence between \( R_{1,1,1} := \mathbb{C}[x_1, x_2, x_3]^{\mathbb{I}_{1,1,1}} \) and \( K_{4,1} := \mathbb{C}[z_1, z_2, z_3]/(z_1, z_2, z_3)^2 \). Building on work of Auslander & Reiten [3] and Keller & Vossieck [20], we show that \( D_{sg}(R_{1,1,1}) \) is triangle equivalent to the Heller stabilization [11] of a left triangulated category, whose underlying \( \mathbb{C} \)-linear category is semisimple abelian with a unique simple object – this object corresponds to the maximal Cohen–Macaulay \( R_{1,1,1} \)-module \( M = \Omega^1(\omega_{R_{1,1,1}}) \). By Proposition 2.7, the structure of triangulated categories \( T \) arising in this way, is completely determined by the order of their Grothendieck groups \( K_0(T) \) – indeed, if \( |K_0(T)| = n \) then \( T \cong D_{sg}(K_{n,1}) \), where \( K_{n,1} = \mathbb{C}[z_1, \ldots, z_{n-1}]/(z_1, \ldots, z_{n-1})^2 \). This result is inspired by Chen [8]. Combining this with (1.6), shows Theorem 1.1.

We will prove the theorem in Section 3 using the preparations in Section 2.

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2. Preparation: stabilization of left triangulated categories

We fix a field $k$. All categories and functors below are $k$-linear. For a right Noetherian $k$-algebra $R$, we write $\text{mod} R$ for the category of finitely generated right $R$-modules.

Our main reference for Heller’s stabilization [11] of looped and left triangulated categories is [8], which builds on works of Keller & Vossieck [20] and Beligiannis [4].

**Lemma 2.1.** Let $\mathcal{H}$ be a semisimple abelian category, with an endofunctor $\Omega: \mathcal{H} \to \mathcal{H}$.

Then the only left triangulated structure on $(\mathcal{H}, \Omega)$ is the trivial structure – i.e. all left triangles are isomorphic to direct sums of trivial left triangles.

**Proof.** By an axiom of left triangulated categories $L$ (cf. [5]), every morphism $B \xrightarrow{f} C$ in $L$ gives rise to a left triangle

$$\Omega(C) \to A \to B \xrightarrow{f} C. \quad (2.1)$$

One can check that every left triangle in $L$ is isomorphic to a left triangle of the form (2.1), by adapting the proof for triangulated categories (cf. e.g. [18, Prop. 10.1.15]) to the setup of left triangulated categories. Since $\mathcal{H}$ is semisimple, $f$ is isomorphic to

$$f': B' \oplus D \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & \text{id}_D \end{pmatrix}} C' \oplus D. \quad (2.2)$$

Completing $f'$ to a left triangle yields a direct sum of trivial left triangles, which is isomorphic to (2.1). \qed

**Corollary 2.2.** Let $\mathcal{H}$ be a full subcategory of a left triangulated category $(\mathcal{L}, \Omega)$. Assume that $\mathcal{H}$ is semisimple abelian and $\Omega(\mathcal{H}) \subseteq \mathcal{H}$.

Then $(\mathcal{H}, \Omega)$ is a left triangulated subcategory of $(\mathcal{L}, \Omega)$.

**Proof.** Following the proof of Lemma 2.1, shows that taking all left triangles in $(\mathcal{L}, \Omega)$, which lie in $\mathcal{H}$ induces the trivial left triangulated structure on $(\mathcal{H}, \Omega)$. By Lemma 2.1 this is the unique left triangulated structure on $(\mathcal{H}, \Omega)$. \qed

A pair of a category $L$ with an endofunctor $\Omega$ is called a looped category. A functor $F$ between looped categories $(L_1, \Omega_1), (L_2, \Omega_2)$, such that there is a natural isomorphism $F\Omega_1 \cong \Omega_2 F$, is called a looped functor. For any looped category $(L, \Omega)$, Heller constructs its stabilization $(S(L, \Omega), \Sigma)$, a looped category with an automorphism $\Sigma$ and a looped functor $S: L \to S(L, \Omega)$. The pair $(S(L, \Omega), \Sigma), S$ enjoys a universal property [11, Proposition 1.1]. Using this property, one can show that a looped functor $F$ between looped categories induces a functor $S(F)$ between their stabilizations. The functor $S(F)$ is called the stabilization of $F$. Finally, if $L$ is left triangulated, then $S(L, \Omega)$ is a triangulated category with shift functor $\Sigma^{-1}$ and triangles in $S(L, \Omega)$ are induced by left triangles in $L$, cf. [4, Section 3].

**Corollary 2.3.** We keep the assumptions in Corollary 2.2. In addition, we assume that for any $X \in L$, there exists $n(X) \in \mathbb{N}$, such that $\Omega^{n(X)} \in \mathcal{H}$. Then the stabilization of the inclusion $(\mathcal{H}, \Omega) \subseteq (\mathcal{L}, \Omega)$ yields a triangle equivalence $S(\mathcal{H}, \Omega) \cong S(\mathcal{L}, \Omega)$.

**Proof.** This follows from Corollary 2.2 in combination with [8, Cor 2.3. & 2.7]. \qed
Let $R$ be a right Noetherian ring and let $\Omega$ be the syzygy functor on the stable module category $\mod R := \mod R / \proj R$. Then the pair $(\mod R, \Omega)$ has a left triangulated structure [5, Theorem 3.1], where the left triangles are isomorphic to sequences $\Omega X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X$ arising from commutative diagrams in $\mod R$, where $P$ is projective and the rows are exact:

$$
\begin{array}{ccccccc}
0 & \rightarrow & \Omega X & \rightarrow & P & \rightarrow & X & \rightarrow & 0 \\
\downarrow & & \downarrow u & & \downarrow v & & \downarrow \text{id}_X & \\
0 & \rightarrow & Y & \rightarrow & Z & \rightarrow & X & \rightarrow & 0.
\end{array}
$$

The next result generalizes Buchweitz’s equivalence [7] from Gorenstein rings to Noetherian rings, see [20] & [4, Cor. 3.9(1)]. It is a key ingredient in our proof of Theorem 1.1.

**Theorem 2.4.** Let $R$ be a right Noetherian ring. Then there is a triangle equivalence

$$
S(\mod R, \Omega) \cong D_{sg}(R).
$$

**Definition 2.5.** We call a $k$-linear category $A$ split simple, if there is a $k$-linear equivalence

$$
A \cong \mod k.
$$

**Remark 2.6.** If $k$ is algebraically closed and $A$ is Hom-finite and semisimple abelian with a unique simple object (up to isomorphism), then $A$ is split simple.

Following [15], we denote the algebra $k[z_1, \ldots, z_{n-1}]/(z_1, \ldots, z_{n-1})^2$ by $K_{n,1}$. One of our main examples of a split simple category is the subcategory of semisimple $K_{n,1}$-modules, which we denote by $\text{ssmod } K_{n,1}$. Its objects are isomorphic to finite direct sums of the simple $K_{n,1}$-module $k[z_1, \ldots, z_{n-1}]/(z_1, \ldots, z_{n-1})$. We will write $\text{ssmod } K_{n,1}$ for the image of $\text{ssmod } K_{n,1}$ under the additive quotient functor

$$
q: \mod K_{n,1} \rightarrow \mod K_{n,1} := \mod K_{n,1} / \proj K_{n,1}.
$$

Since the semisimple $K_{n,1}$-modules are not projective, $q$ induces a $k$-linear equivalence $\text{ssmod } K_{n,1} \cong \text{ssmod } K_{n,1}$. The syzygy $\Omega_{K_{n,1}}(M)$ of a finitely generated $K_{n,1}$-module $M$ is a submodule of the radical $\rad(K_{n,1}^{\oplus c}) \cong (\rad K_{n,1})^{\oplus c}$, which is semisimple. Thus,

$$
\Omega_{K_{n,1}}(M) \in \text{ssmod } K_{n,1}.
$$

It follows from Corollary 2.2 that

$$
(\text{ssmod } K_{n,1}, \Omega_{K_{n,1}}) \subseteq (\mod K_{n,1}, \Omega_{K_{n,1}})
$$

is a left triangulated subcategory.

We have the following classification of stabilizations of split simple categories.

**Proposition 2.7.** Let $(\mathcal{H}, \Omega)$ be a left triangulated category, where $\mathcal{H}$ is a split simple category with simple object $s$. Then the following statements hold:

1. $\Omega(s) \cong s^{\oplus (n-1)}$ in $\mathcal{H}$, for some integer $n \geq 1$.
2. There is a triangle equivalence $S(\mathcal{H}, \Omega) \cong D_{sg}(K_{n,1})$.
3. There is an isomorphism of groups $K_0(S(\mathcal{H}, \Omega)) \cong \mathbb{Z}/n\mathbb{Z}$.

**Proof.** Since $\mathcal{H}$ is split simple, all objects in $\mathcal{H}$ are isomorphic to $s^{\oplus m}$ for some $m \geq 0$. This shows (1). Let us show (2). By (2.6), the inclusion of left triangulated categories...
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(2.7) satisfies the assumptions of Corollary 2.3. This gives a triangle equivalence between the stabilizations, cf. [8, Proof of Thm 4.4].

\[ S(\text{ssmod} K_{n,1}, \Omega_{K_{n,1}}) \cong S(\text{mod} K_{n,1}, \Omega_{K_{n,1}}). \]  

(2.8)

By Theorem 2.4, the stabilization of \( (\text{mod} K_{n,1}, \Omega_{K_{n,1}}) \) is triangle equivalent to \( D_{sg}(K_{n,1}) \).

Using Lemma 2.1 and (1), the left triangulated category \( (\text{ssmod} K_{n,1}, \Omega_{K_{n,1}}) \) is equivalent to \( (H, \Omega) \). In particular, their stabilizations are triangle equivalent. Summing up, we have a chain of triangle equivalences proving (2):

\[ S(H, \Omega) \cong S(\text{ssmod} K_{n,1}, \Omega_{K_{n,1}}) \cong S(\text{mod} K_{n,1}, \Omega_{K_{n,1}}) \cong D_{sg}(K_{n,1}). \]  

(2.9)

Finally, (3) follows from (2) and \( K_0(D_{sg}(A)) \cong \mathbb{Z}/\dim(A)\mathbb{Z} \) for finite dimensional local algebras \( A \).

The following structure result is a direct consequence. This will not be used later.

**Corollary 2.8.** For \( i = 1, 2 \), let \( (H_i, \Omega_i) \) be left triangulated categories, where the \( H_i \) are split simple categories. Then the following statements are equivalent.

(1) There is a triangle equivalence between the stabilizations \( S(H_1, \Omega_1) \cong S(H_2, \Omega_2) \).

(2) There is an isomorphism of Grothendieck groups \( K_0(S(H_1, \Omega_1)) \cong K_0(S(H_2, \Omega_2)) \).

**Corollary 2.9.** Let \( R \) be a Noetherian \( k \)-algebra with syzygy functor \( \Omega_R \colon \text{mod} R \to \text{mod} R \). Assume there is a finitely generated \( R \)-module \( M \) such that the following conditions hold:

(s1) \( \text{End}_R(M) \cong k \).

(s2) \( \Omega_R(M) \cong M^{\oplus n-1} \) for an integer \( n \in \mathbb{Z}_{>0} \).

(s3) For every finitely generated \( R \)-module \( N \) there is an integer \( d \in \mathbb{Z}_{>0} \) such that \( \Omega_R^d(N) \cong M^{\oplus m} \).

Then there is an isomorphism of groups

\[ K_0(D_{sg}(R)) \cong \mathbb{Z}/n\mathbb{Z} \]  

(2.10)

and there is a triangle equivalence

\[ D_{sg}(R) \cong D_{sg}(k[z_1, \ldots, z_{n-1}]/(z_1, \ldots, z_{n-1})^3). \]  

(2.11)

**Proof.** Consider the additive subcategory \( \mathcal{H} := \text{add} M \subseteq \text{mod} R \) generated by \( M \). Condition (s1) implies that \( \mathcal{H} \) is split simple with simple object \( M \) and condition (s2) ensures that the assumptions of Corollary 2.2 are satisfied. Using (s3), we can apply Corollary 2.3 to get a triangle equivalence \( S(\mathcal{H}, \Omega_R) \cong S(\text{mod} R, \Omega_R) \). By Theorem 2.4, we have a triangle equivalence \( S(\text{mod} R, \Omega_R) \cong D_{sg}(R) \). Applying Proposition 2.7 to \( (\mathcal{H}, \Omega_R) \) completes the proof.

\[ \square \]

3. PROOF OF THE THEOREM

First, we recall the following special case of the singular equivalence (1.2)

\[ D_{sg}\left(\mathbb{C}\llbracket y_1, y_2\rrbracket, 3^{(1,1)}\right) \cong D_{sg}\left(\frac{\mathbb{C}[z_1, z_2, z_3]}{(z_1, z_2, z_3)^2}\right). \]  

(3.1)

In order to prove Theorem 1.1, it remains to prove the following equivalence

\[ D_{sg}(R_{1,1,1}) := D_{sg}\left(\mathbb{C}\llbracket x_1, x_2, x_3\rrbracket, 3^{(1,1,1)}\right) \cong D_{sg}\left(\frac{\mathbb{C}[z_1, z_2, z_3]}{(z_1, z_2, z_3)^2}\right). \]  

(3.2)
which follows from Corollary 2.9, using Proposition 3.2 and $K_0(D_{sg}(R_{1,1,1})) \cong \mathbb{Z}/4\mathbb{Z}$ (cf. [27, Ex 2.17]) to check that the assumptions of Corollary 2.9 are satisfied for $n = 4$.

The proof of Proposition 3.2 uses some facts about the stable category of maximal Cohen–Macaulay $R_{1,1,1}$-modules $\text{CM} R_{1,1,1} := \text{CM} R_{1,1,1}/\text{proj} R_{1,1,1}$. We have collected this information in the following proposition, cf. [3] & [30].

**Proposition 3.1.** (a) There is a $\mathbb{C}$-linear equivalence $\text{CM} R_{1,1,1} \cong \text{add } \omega \oplus M$, where $\omega$ denotes the canonical $R_{1,1,1}$-module and $M := \Omega(\omega)$ is its first syzygy. Both $\omega$ and $M$ are indecomposable.

(b) $\text{End}_R(M) \cong \mathbb{C}$.

**Proof.** We first describe the Auslander–Reiten quiver of $\text{CM} R_{1,1,1}$ using [30, Prop. 16.10] and then explain how both statements (a) and (b) can be deduced from this:

$$
\begin{align*}
\text{(i) } & \quad \omega \xrightarrow{} M \xrightarrow{} R_{1,1,1} \\
\text{(ii) } & \quad \omega \xrightarrow{} M
\end{align*}
$$

The quiver (3.3) (i) is the Auslander–Reiten quiver in [30, (16.10.5)]\(^1\) translated to our notation: firstly, $R_{1,1,1} \cong \hat{R}(4(2, -1, -1, -1)) = \hat{R}$, by the comment between [30, Prop. 16.10] and its proof. Moreover, $\omega$ is its first syzygy. Both $\omega$ and $M$ are indecomposable objects in the stable category. This shows (a). There are no arrows starting in the vertex $M$ of the quiver (3.3) (ii). This implies that the only non-trivial stable endomorphisms of $M$ are of the form $\mathbb{C} \cdot \text{id}_M$ and completes the proof. \(\square\)

By the discussion above, the following proposition completes the proof of Theorem 1.1.

**Proposition 3.2.** Let $\omega \in \text{CM} R_{1,1,1}$ be the canonical module. Then $M := \Omega(\omega)$ satisfies the conditions (s1) – (s3) of Corollary 2.9 for some integer $n \in \mathbb{Z}_{>0}$.

**Proof.** Condition (s1) holds by Proposition 3.1 (b). To see (s2), we first note that the syzygy $\Omega(N)$ of a maximal Cohen–Macaulay module $N$ is again maximal Cohen–Macaulay, [30, Prop. 1.3]. In combination with Proposition 3.1 (a), this shows that

$$
\Omega(M) \cong M^{\oplus n-1} \oplus \omega^{\oplus k} \quad \text{for } n \in \mathbb{Z}_{>0} \text{ and } k \in \mathbb{Z}_{>0}. \tag{3.4}
$$

If $k > 0$, then $0 \neq \text{Hom}_{R_{1,1,1}}(\Omega(M), \omega) \cong \text{Ext}^1_{R_{1,1,1}}(M, \omega)$, contradicting the fact that $\omega$ is an injective object in $\text{CM} R_{1,1,1}$, cf. [30, Cor. 1.13]. This shows that $k = 0$ and therefore (3.4) implies (s2). We prove that condition (s3) holds by showing

$$
\Omega^4(X) \in \text{add } M, \tag{3.5}
$$

for all $X \in \text{mod } R_{1,1,1}$. Indeed, since $R_{1,1,1}$ is Cohen–Macaulay of Krull dimension 3, we see that $\Omega^4(X)$ is a maximal Cohen–Macaulay module, cf. [30, Prop. 1.4]. Thus Proposition 3.1 (a) shows that $\Omega^4(X) \cong M^{\oplus m} \oplus \omega^{\oplus l}$. Now $M = \Omega(\omega)$ and condition (s2) show (3.5). \(\square\)

\(^1\)There are two small typos in [30, (16.10.5)]: both $R$ and $S_{-1}$ should be replaced by $\hat{R}$ and $\hat{S}_{-1}$, respectively.
4. Appendix: Graded versus ungraded singular equivalences

For group graded algebras $A$, one can define graded singularity categories

$$D^g_s(A) = D^b(\text{grmod} A)/K^b(\text{grproj} A).$$

(4.1)

In this appendix, we describe graded commutative algebras $R_{d+1}$ and $R_{d+2}$ of Krull dimensions $d + 1$ and $d + 2$, respectively, which are known [12, Corollary 3.23 (d)] to have equivalent graded singularity categories. Then we show that their ‘ungraded’ singularity categories are not triangle equivalent, by proving the following more general statement:

$$D_{sg}(S/(f)) \not\cong D_{sg}(S[X_0]/(f - X_0^2)),$$  

(4.2)

such that $(S/(f))_p$ is non-singular for all prime ideals $p \neq (X_1, \ldots, X_{d+1})$. In other words, for isolated singularities, Knörrer’s equivalences [22] are not 1-periodic. This answers a question of E. Shinder.

We first describe the algebras $R_{d+1} \cong R$ and $R_{d+2} \cong R'$ appearing in [12, Corollary 3.23 (d)]. Note that our presentation (4.3) for $R_{d+2}$ is obtained from the ‘normal form’ [12, top of p.13] by renaming the variables as follows: $X_i \rightarrow X_{i-1}$. Let $P = \mathbb{C}[X_1, \ldots, X_{d+2}]$, then

$$R_{d+1} = P/(f) \quad \text{and} \quad R_{d+2} = P[X_0]/(f - X_0^2),$$

(4.3)

where

$$f = X_{d+2}^{p_{d+2}} - \cdots - X_1^{p_1} \quad \text{for} \ (p_1, \ldots, p_{d+2}) \in \mathbb{Z}_{\geq 2}$$

(4.4)

and the grading groups $\mathbb{Z}_{d+1}$ and $\mathbb{Z}_{d+2}$ are abelian of rank 1, possibly with torsion elements.

Assume there is a singular equivalence (4.5). We show that this yields a contradiction.

$$D_{sg}(R_{d+1}) \cong D_{sg}(R_{d+2}).$$

(4.5)

The algebras $R_{d+1}$ and $R_{d+2}$ have unique isolated singularities in $m_{d+1} = (X_1, \ldots, X_{d+2})$ and $m_{d+2} = (X_0, \ldots, X_{d+2})$, respectively, cf. [12, Prop. 2.36]. Therefore, passing to the idempotent completions of (4.5) yields singular equivalences between $m_i$-adic completions

$$D_{sg}(\hat{R}_{d+1}) \cong D_{sg}(\hat{R}_{d+2}),$$

(4.6)

cf. [13, Theorem 3.2(b)], which follows from [26]. The algebras $\hat{R}_{d+1}$ and $\hat{R}_{d+2}$ are special cases of the algebras appearing in (4.2). In particular, (4.6) contradicts (4.2), showing that there are no singular equivalences (4.6) or (4.5).

We now prove (4.2) by assuming that there is a singular equivalence

$$D_{sg}(S/(f)) \cong D_{sg}(S[X_0]/(f - X_0^2)),$$

(4.7)

as in (4.2) and show that this yields a contradiction. Let $R = S/(f)$ and $R_1 = S[X_0]/(f - X_0^2)$. By assumption, $R$ has isolated singularities and is Gorenstein of Krull dimension $d$. Therefore, $D_{sg}(R)$ is Hom-finite and has Serre functor $S = [d - 1]$, by [2] and [1], respectively. The equivalence (4.7) implies that $D_{sg}(R_1)$ is also Hom-finite and thus $R_1$ also has isolated singularities, see [2]. By applying [1] to the $(d+1)$-dimensional Gorenstein algebra $R_1$, we see that the Serre functor of $D_{sg}(R_1)$ is $S_1 = [d]$. Since Serre functors are unique up to isomorphism [6], we see that an equivalence (4.7) would imply $[d - 1] \cong S \cong S_1 \cong [d]$, which would show $[1] \cong \text{id}$ in $D_{sg}(R) \cong D_{sg}(R_1)$. In particular, $X[1] \cong X$ for every indecomposable object $X$ in $D_{sg}(R)$. Identifying singularity categories with homotopy categories of matrix factorizations (via [10, 7]), it follows from [22, Prop. 2.7. i)
& Lemma 2.5. ii)] that there is an indecomposable object $Y$ in $D_{sg}(R_1)$ such that $Y[1] \not\cong Y$. This contradicts $[1] \cong \text{id}$ in $D_{sg}(R_1)$ and shows that (4.7) is impossible.

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