GENERALIZATIONS OF LERCH’S FORMULA BY
BARNES’ MULTIPLE ZETA FUNCTIONS

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ABSTRACT. The classical Lerch’s formula states the following normalized product:

\[ \prod_{n=0}^{\infty} (x + n) = \frac{\sqrt{2\pi}}{\Gamma(x)}, \quad \text{Re}(x) > 0, \]

where \( \Gamma(x) \) is the Euler gamma function.

In this note, by using Barnes’ multiple zeta function and its alternating form, we obtain two kinds of generalizations of Lerch’s formula, which imply the product

\[ \prod_{n=1}^{\infty} n = \sqrt{2\pi} \]

(in the sense of zeta regularization) and the product

\[ \frac{2 \cdot 2 \cdot 4 \cdot 6 \cdot 6}{1 \cdot 3 \cdot 5 \cdot 7 \cdots} = \frac{\pi}{2} \]

(Wallis’ formula in 1656), respectively.

1. INTRODUCTION

In the book “Number theory, 3, Iwasawa theory and modular forms” [9], the authors stated the following infinite product from the zeta regularization:

\[ \infty! := \prod_{n=1}^{\infty} n = \sqrt{2\pi}. \]

(See [9, Corollary 9.13]).

They explained it from the following interesting way. Given a finite sequence \( \mathbf{a} = (a_1, a_2, \ldots, a_N) \), define

\[ \zeta_\mathbf{a}(s) = \sum_{n=1}^{N} a_n^{-s}. \]

Then

\[ \zeta_\mathbf{a}'(0) = -\sum_{n=1}^{N} \log(a_n) = -\log \left( \prod_{n=1}^{N} a_n \right). \]
So
\[ \exp(-\zeta'_a(0)) = \prod_{n=1}^{N} a_n. \]
Thus for an infinite sequence \( a = (a_1, a_2, \ldots, a_N, \ldots) \), if the corresponding zeta function
\[ \zeta_a(s) = \sum_{n=1}^{\infty} a_n^{-s} \]
can be analytically continued to a neighborhood around \( s = 0 \), then we may define their normalized product or zeta regularization by
\[ (1.2) \quad \prod_{n=1}^{\infty} a_n = \exp(-\zeta'_a(0)). \]
Now let \( a = (1, 2, 3, \ldots) \), then the corresponding zeta function is just the Riemann zeta function
\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \]
and in 1859, Riemann \([12]\) proved that
\[ (1.3) \quad \zeta'(0) = -\frac{1}{2} \log(2\pi). \]
By \([12]\), we understand that
\[ (1.4) \quad \prod_{n=1}^{\infty} n = \exp(-\zeta'(0)) = \sqrt{2\pi}. \]
In 1894, Lerch generalized \((1.1)\) as the following product:
\[ (1.5) \quad \prod_{n=0}^{\infty} (x + n) = \frac{\sqrt{2\pi}}{\Gamma(x)}, \quad \text{Re}(x) > 0, \]
where \( \Gamma(x) \) is the Euler gamma function. (See \([9\), Theorem 9.12]). In fact, letting \( x = 1 \) in \((1.5)\), we get \((1.1)\).
Denote by
\[ Z_0^- = \{0, -1, -2, \ldots\}. \]
In 2004, the above Lerch’s formula was generalized by Kurokawa and Wakayama \([8]\) in the following way:
\[ (1.6) \quad \prod_{n=0}^{\infty} ((n+x)^m - y^m) = \frac{(\sqrt{2\pi})^m}{\prod_{\zeta=m=1}^{\infty} \Gamma(x - \zeta y)} \]
and in 2006, applying Stark’s summation formula \([13]\), Mizuno \([10]\) got a further extension:
\[ (1.7) \quad \prod_{m=0}^{\infty} \left( \prod_{j=1}^{n} (m + z_j) \right) = \frac{(\sqrt{2\pi})^n}{\prod_{j=1}^{n} \Gamma(z_j)} = \prod_{j=1}^{n} \left( \prod_{m=0}^{\infty} (m + z_j) \right) \]
for \( z_j \in \mathbb{C} \setminus Z_0^- \).
In this note, we obtain two kinds of generalizations of Lerch’s formula by using Barnes’ multiple zeta function and its alternating form, respectively.
Suppose that $\omega_1, \ldots, \omega_N$ are positive real numbers and $x$ is a complex number with positive real part. In 1904, Barnes [1] studied the multiple Hurwitz zeta function

$$\zeta(s, x; \omega_1, \ldots, \omega_N) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_N=0}^{\infty} \frac{1}{(x + \omega_1m_1 + \cdots + \omega_Nm_N)^s}, \quad \text{Re}(s) > N,$$

and the corresponding gamma function $\Gamma_N(x)$.

Setting $\omega_j = 1$, for $j = 1, 2, 3, \ldots, N$ in (1.8), we get the following special case:

$$\zeta(s, x) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_N=0}^{\infty} \frac{1}{(x + m_1 + \cdots + m_N)^s}, \quad \text{Re}(s) > N, x \in \mathbb{C} \setminus \mathbb{Z}_0^-.$$

As shown by Choi and Srivastava in [4, p. 503–504] or [17, p. 149, Theorem 2.6], for $\text{Re}(x) > 0$, the function $\zeta(s, x)$ can be analytic continued for all $s \in \mathbb{C}$ except for simple poles at $s = k$ $(1 \leq k \leq N; N, k \in \mathbb{N})$.

Thus the multiple (or, simply, $N$-ple) gamma function $\Gamma_N(x)$ can now be defined by

$$\Gamma_N(x) = \exp \left( \frac{\partial}{\partial s} \zeta_N(s, x) \bigg|_{s=0} \right), \quad \text{Re}(x) > 0.$$

(See [7], [15] and [17]). The following recurrence formula of $\Gamma_N(x)$ is well-known:

$$\Gamma_N(x + 1) = \frac{\Gamma_N(x)}{\Gamma_{N-1}(x)}$$

(e.g. see [2, Proposition 1.2 (1)]) and several series expansions and asymptotic formulas of $\Gamma_N(x)$ and $\log \Gamma_N(x)$ have been considered by Choi and Srivastava in [17, p. 375-382].

By using the multiple zeta function $\zeta(s, x)$ and the corresponding gamma function $\Gamma_N(x)$, we first prove the following generalization of the above Lerch’s formula (1.5).

**Theorem 1.1** (The first generalized Lerch formula). For $N \in \mathbb{N}$, in the sense of zeta regularization, we have

$$\prod_{n=0}^{\infty} (x + n)^{-\left(\frac{n+N-1}{N-1}\right)} = \Gamma_N(x), \quad \text{Re}(x) > 0.$$

The following relation between $\Gamma_1(x)$ and the Euler gamma function $\Gamma(x)$ will be proved in Section [2].

**Proposition 1.2** ([18, Lemma 2.1]).

$$\Gamma_1(x) = \frac{\Gamma(x)}{\sqrt{2\pi}}.$$
From the above proposition, we see that if letting $N = 1$ in (1.12), then we recover Lerch’s formula (1.5).

Now we go to the alternating case. For $\Re(s) > 0$, the multiple Barnes-Euler zeta function $\zeta_{E,N}(s, x; \omega_1, \ldots, \omega_N)$ is defined as a deformation of the Barnes’ multiple zeta function as follows

\[
(1.14) \quad \zeta_{E,N}(s, x; \omega_1, \ldots, \omega_N) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_N=0}^{\infty} \frac{(-1)^{m_1 + \cdots + m_N}}{(x + \omega_1 m_1 + \cdots + \omega_N m_N)^s}.
\]

Its analytic properties both in the complex plane $\mathbb{C}$ and in the $p$-adic complex plane $\mathbb{C}_p$ have been systematically studied by the authors in [6].

Letting $\omega_j = 1$, for $j = 1, 2, 3, \ldots, N$ in (1.14), we get the following special case:

\[
(1.15) \quad \zeta_{E,N}(s, x) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_N=0}^{\infty} \frac{(x + m_1 + \cdots + m_N)^s}{s}, \quad \Re(s) > 0, x \notin \mathbb{C} \setminus \mathbb{Z}_0^-.
\]

As shown by Choi and Srivastava in [4, Section 3], for $\Re(x) > 0$, the function $\zeta_{E,N}(s, x)$ can be continued analytically as an entire function of $s \in \mathbb{C}$. Thus the corresponding multiple (or, simply, $N$-ple) gamma function $\Gamma_N^*(x)$ can now be defined by

\[
(1.16) \quad \Gamma_N^*(x) = \exp \left( \frac{\partial}{\partial s} \zeta_{E,N}(s, x) \bigg|_{s=0} \right), \quad \Re(x) > 0.
\]

The following recurrence formula of $\Gamma_N^*(x)$ is implied by a formula of the authors [6, Lemma 2.1 (1)]:

\[
(1.17) \quad \Gamma_N^*(x + 1) = \frac{\Gamma_{N-1}^*(x)}{\Gamma_N^*(x)}.
\]

By using the alternating multiple zeta function $\zeta_{E,N}(s, x)$ and the corresponding gamma function $\Gamma_N^*(x)$, we prove another generalization of the above Lerch’s formula (1.5).

**Theorem 1.3** (The second generalized Lerch formula). For $N \in \mathbb{N}$, we have

\[
(1.18) \quad \prod_{n=0}^{\infty} (x + n)^{(-1)^{n+1}(n+N-1)} = \Gamma_N^*(x), \quad \Re(x) > 0.
\]

The following relation between $\Gamma_1^*(x)$ and the Euler gamma function $\Gamma(x)$ will be proved in Section 3.

**Proposition 1.4.**

\[
(1.19) \quad \Gamma_1^*(x) = \frac{\Gamma \left( \frac{x}{2} \right)}{\sqrt{2\Gamma \left( \frac{x+1}{2} \right)}}, \quad \Re(x) > 0.
\]
Remark 1.5. In [11, p. 742], during the proof for the claim that the Students t-distribution is a continuous probability density, Miller obtained the following integral

\[
\int_{-\infty}^{\infty} \left(1 + \frac{t^2}{x}\right)^{-\frac{x+1}{2}} dt = \frac{\sqrt{\pi x \Gamma\left(\frac{x}{2}\right)}}{\Gamma\left(\frac{x+1}{2}\right)}.
\]

From Proposition 1.4, the above equality in fact leads to the following integral representation of \(\Gamma^*_1(x)\):

\[
\Gamma^*_1(x) = \frac{\Gamma\left(\frac{x}{2}\right)}{\sqrt{2\Gamma\left(\frac{x+1}{2}\right)}} = \frac{1}{\sqrt{2\pi x}} \int_{-\infty}^{\infty} \left(1 + \frac{t^2}{x}\right)^{-\frac{x+1}{2}} dt.
\]

By setting \(N = 1\) in (1.18), from (1.19), we immediately get

Corollary 1.6 (Lerch type formula).

\[
\prod_{n=0}^{\infty} (x + n)^{(-1)^{n+1}} = \frac{1}{\sqrt{2\pi x}} \Gamma\left(\frac{1}{2}\right), \quad \text{Re}(x) > 0.
\]

Then letting \(x = 1\) in (1.22), and recalling that \(\Gamma(1) = 1\) and \(\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\), we immediately get

Corollary 1.7 (Wallis’ formula).

\[
\frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots} = \frac{\pi}{2}.
\]

Remark 1.8. As early as in 1656, the British mathematician John Wallis [19] showed the above remarkable formula in his book “Arithmetica Infinitorum”. After his work, there appears many methods to prove it, including the well-known ones based on the formula for integrals of powers of \(\sin x\) from the inductive method or based on the infinite product expansion of \(\sin x\). Recently, Miller [11] found a probabilistic proof by using the Students t-distribution, and Friedmann and Hagen [5] presented an quantum mechanical derivation based on the spectrum of the hydrogen in the physical three dimensions. In 1994, Sondow [16] derived (1.3) from Wallis’ formula by using Euler’s transformation of series.

2. Barnes’ multiple zeta function and the first generalized Lerch formula

The main aim of this section is to prove the first generalized Lerch’s formula (Theorem 1.1 above).

Proof of Theorem 1.1. According to Barnes [11], as the classical Riemann zeta functions \(\zeta(s)\), the multiple zeta function \(\zeta_N(s, x)\) defined in (1.9) may also be represented by the Mellin transform as follows

\[
\zeta_N(s, x) = \frac{1}{\Gamma(s)} \int_0^\infty t^s e^{-xt} (1 - e^{-t})^{-N} \frac{dt}{t},
\]

for \(\text{Re}(s) > N\) and \(\text{Re}(x) > 0\).
This can be shown as follows. Start with the power series

\[(2.2) \quad (1 - e^{-t})^{-N} = \sum_{m_1, \ldots, m_N = 0}^{\infty} e^{-t(m_1 + \cdots + m_N)}, \quad t > 0.\]

Note that

\[(2.3) \quad w^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^s e^{-wt} \frac{dt}{t} = \frac{1}{\Gamma(s)} \int_1^\infty u^{-w} (\log u)^{s-1} \frac{du}{u} \]

(see \[15\] (2.7) and \[17\] (1))). Then substituting (2.2) into (2.1) we have

\[(2.4) \quad \zeta_N(s, x) = \frac{1}{\Gamma(s)} \int_0^\infty t^s e^{-xt} (1 - e^{-t})^{-N} \frac{dt}{t} \]

\[= \sum_{m_1, \ldots, m_N = 0}^{\infty} \frac{1}{\Gamma(s)} \int_0^\infty t^s e^{-(x+m_1+\cdots+m_N)t} \frac{dt}{t} \]

for \(\Re(s) > N\) and \(\Re(x) > 0\). (See \[15\] (3.14)).

By (2.1) we have

\[(2.5) \quad \zeta_N(s, x) = \frac{1}{\Gamma(s)} \int_0^\infty t^s e^{-xt} (1 - e^{-t})^{-N} \frac{dt}{t} \]

\[= \frac{1}{\Gamma(s)} \int_1^\infty \left(\frac{1}{1-u^{-1}}\right)^N u^{-x} (\log u)^{s-1} \frac{du}{u} \]

\[= \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} (-1)^n \binom{-N}{n} \int_1^\infty u^{-x-n} (\log u)^{s-1} \frac{du}{u} \]

(by using (2.3) with \(w = x + n\))

\[= \sum_{n=0}^{\infty} \binom{n+N-1}{N-1} (x+n)^{-s} \]

(see \[4\] p. 505, (1.17)). The above equation is established for \(\Re(x) > 0\) and for all \(s \in \mathbb{C}\) except

\[s = k \quad (1 \leq k \leq N; N, k \in \mathbb{N}).\]

In fact, (2.5) is another way of analytic continuation for \(\zeta_N(s, x)\) by Choi in \[3\]. Then from (1.10), in considering of the concept of normalized product or zeta regularization introduced at the beginning of this paper, we have

\[(2.6) \quad \Gamma_N(x) = \exp \left( \frac{\partial}{\partial s} \zeta_N(s, x) \bigg|_{s=0} \right) \]

\[= \exp \left( - \sum_{n=0}^{\infty} \binom{n+N-1}{N-1} \log(x+n) \right) \]

\[= \prod_{n=0}^{\infty} (x+n)^{-\binom{n+N-1}{N-1}},\]
which is the desired result.

**Proof of Proposition 1.2.** In the history, the gamma function $\Gamma(s)$ is defined by Euler in 1729 from the integral

$$
\Gamma(s) = \int_0^\infty e^{-t^s} \frac{dt}{t}.
$$

Note that this integral is well-defined if $\text{Re}(s) > 0$. Integrating by parts yields

$$
\Gamma(s + 1) = s\Gamma(s).
$$

This implies that if $n$ is a nonnegative integer then

$$
\Gamma(n + 1) = n!,
$$

thus $\Gamma(s)$ generalizes the factorial function.

Letting $N = 1$ in (1.9), we recover the Hurwitz zeta function introduced by Hurwitz in 1882:

$$
\zeta(s, x) = \sum_{n=0}^\infty \frac{1}{(n + x)^s}.
$$

By (1.10), we have

$$
\Gamma_1(x) = e^{\zeta'(0, x)}.
$$

The following equality comes from the difference functional equation of the Hurwitz zeta function $\zeta(s, x)$:

$$
\zeta'(0, x + 1) = \zeta'(0, x) + \log x.
$$

(See [18, p. 497]). Then substituting (2.9) into (2.8), we get

$$
\Gamma_1(x + 1) = x\Gamma_1(x).
$$

So by Bohr-Mollerup Theorem (see e.g., [14, p. 44], the uniqueness of gamma functions), we have

$$
\Gamma_1(x) = \Gamma(x)R
$$

for a constant $R$ and by (2.7) and (1.3)

$$
R = \frac{\Gamma_1(1)}{\Gamma(1)} = e^{\zeta'(0, 1)} = e^{\zeta'(0)} = e^{-\log \sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}.
$$

since $\zeta(s, 1) = \zeta(s)$. Therefore we have

$$
\Gamma_1(x) = \frac{\Gamma(x)}{\sqrt{2\pi}},
$$

which is what we want.

3. **Euler-Barnes multiple zeta function and the second generalized Lerch formula**

The main aim of this section is to prove the second generalized Lerch’s formula (Theorem 1.3 above).
Proof of Theorem 1.3. The proof goes a similar way as Theorem 1.1.

The multiple zeta function $\zeta_{E,N}(s, x)$ defined in (1.15) may also be represented by the Mellin transform as follows

\[(3.1) \quad \zeta_{E,N}(s, x) = \frac{1}{\Gamma(s)} \int_0^\infty t^s e^{-xt} (1 + e^{-t})^{-N} \frac{dt}{t},\]

for $\text{Re}(s) > 0$ and $\text{Re}(x) > 0$.

This can be shown as follows. Start with the power series

\[(3.2) \quad (1 + e^{-t})^{-N} = \sum_{m_1, \ldots, m_N=0}^\infty (-1)^{m_1 + \cdots + m_N} e^{-t(m_1 + \cdots + m_N)}, \quad t > 0.\]

Then substituting (3.2) into (3.1) we have

\[(3.3) \quad \zeta_{E,N}(s, x) = \sum_{m_1, \ldots, m_N=0}^\infty \frac{(-1)^{m_1 + \cdots + m_N}}{\Gamma(s)} \int_0^\infty t^s e^{-(x+m_1+\cdots+m_N)t} \frac{dt}{t} = \sum_{m_1, \ldots, m_N=0}^\infty \frac{(-1)^{m_1 + \cdots + m_N}}{(x+m_1 + \cdots + m_N)^s},\]

for $\text{Re}(s) > 0$ and $\text{Re}(x) > 0$.

By (3.1) we have

\[(3.4) \quad \zeta_{E,N}(s, x) = \frac{1}{\Gamma(s)} \int_0^\infty t^s e^{-xt} (1 + e^{-t})^{-N} \frac{dt}{t} = \frac{1}{\Gamma(s)} \int_1^\infty \frac{1}{1 + u^{-1}} u^{-\left(N \left(\sum \frac{-N}{n} \int_1^\infty u^{-x-n}(\log u)^{s-1} \frac{du}{u}\right)\right)} (by \ using \ (2.3) \ with \ w = x + n) = \sum_{n=0}^\infty (-1)^n \left(\frac{n + N - 1}{N - 1}\right) (x + n)^{-s}.\]

The above equation is established for $x > 0$ and for all $s \in \mathbb{C}$ by analytic continuation. So by (1.16) we have

\[(3.5) \quad \Gamma_N^*(x) = \exp \left( \frac{\partial}{\partial s} \zeta_{E,N}(s, x) \bigg|_{s=0} \right) = \exp \left( \sum_{n=0}^\infty (-1)^{n+1} \left(\frac{n + N - 1}{N - 1}\right) \log(x + n) \right) = \prod_{n=0}^\infty (x + n)^{(-1)^{n+1} \left(\frac{n + N - 1}{N - 1}\right)},\]

which is the desired result.
Proof of Proposition 1.4. Letting $N = 1$ in (1.15), we recover the alternating Hurwitz zeta function:

$$\zeta_E(s, x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + x)^s}.$$ 

The following derivative formula of $\zeta_E(s, x)$ is shown by Williams and Zhang in [20, Proposition 3]:

$$\zeta'_E(0, x) = \log \frac{\Gamma\left(\frac{x}{2}\right)}{\Gamma\left(\frac{x+1}{2}\right)} - \frac{1}{2} \log 2,$$

where $\Gamma(x)$ is the Euler gamma function. Hence by (1.16) we have

$$\Gamma_1^*(x) = \exp \left( \left. \frac{\partial}{\partial s} \zeta_E(s, x) \right|_{s=0} \right)$$

(3.6)

$$= \frac{\Gamma\left(\frac{x}{2}\right)}{\sqrt{2\Gamma\left(\frac{x+1}{2}\right)}},$$

which is what we want.

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