The tail process revisited

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Abstract

The tail measure of a regularly varying stationary time series has been recently introduced. It is used in this contribution to reconsider certain properties of the tail process and establish new ones. A new formulation of the time change formula is used to establish identities, some of which were indirectly known and some of which are new.

1 Introduction

Let \( \{X_j, j \in \mathbb{Z}\} \) be a stationary regularly varying time series with values in \( \mathbb{R}^d \). This means that for all \( s \leq t \in \mathbb{Z} \) there exists a non-zero Radon measure \( \nu_{s,t} \) on \( (\mathbb{R}^d)^{[s,t]} \setminus \{0\} \) such that, as \( u \to \infty \),

\[
\frac{\mathbb{P}(u^{-1}(X_s, \ldots, X_t) \in \cdot)}{\mathbb{P}(|X_0| > u)} \to \nu_{s,t},
\]

where \( \to \) denotes vague convergence. Let \( |\cdot| \) denote an arbitrary norm on \( \mathbb{R}^d \). According to (Basrak and Segers, 2009, Theorem 2.1), this is equivalent to the existence of a sequence \( \{Y_j, j \in \mathbb{Z}\} \) with \( \mathbb{P}(|Y_0| > y) = y^{-\alpha} \) for \( y \geq 1 \) and such that for all \( s \leq t \in \mathbb{Z} \), as \( u \to \infty \),

\[
\mathcal{L}(u^{-1}X_s, \ldots, u^{-1}X_t | |X_0| > u) \to \mathcal{L}(Y_s, \ldots, Y_t),
\]

where \( \to \) denotes weak convergence. The sequence \( \{Y_j, j \in \mathbb{Z}\} \) is called the tail process of \( \{X_j, j \in \mathbb{Z}\} \). Furthermore, by (Basrak and Segers, 2009, Theorem 3.1) the process \( \{\Theta_j, j \in \mathbb{Z}\} \) defined by \( \Theta_j = Y_j / |Y_0| \) is independent of \( |Y_0| \) and is called the spectral tail process.

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In the unpublished manuscript Samorodnitsky and Owada (2012), the tail measure of a regularly varying time series was defined. It is the unique Borel measure $\nu$ on $\mathbb{R}^d \mathbb{Z}$ with respect to the product topology such that $\nu(\{0\}) = 0$ and for all $s \leq t \in \mathbb{Z}$

$$\nu \circ p_{s,t}^{-1} = \nu_{s,t}$$
on $\mathbb{R}^d[s,t] \setminus \{0\}$ where $p_{s,t}$ is the canonical projection of $\mathbb{R}^d \mathbb{Z}$ unto $\mathbb{R}^d[s,t]$. It follows easily that the tail measure has the following properties.

(i) $\nu$ is $\sigma$-finite;

(ii) $\nu$ is shift invariant;

(iii) $\nu$ is homogeneous with index $-\alpha$, i.e. $\nu(c \cdot) = c^{-\alpha} \nu$ for all $c > 0$;

(iv) For every non negative measurable functional on $\mathbb{R}^d \mathbb{Z}$,

$$\mathbb{E}[H(Y)] = \int_{(\mathbb{R}^d \mathbb{Z})} H(y) 1\{|y_0| > 1\} \nu(dy).$$  \hspace{1cm} (1.2)  \hspace{1cm} \{eq:tail-tail\}

The shift invariance of $\nu$ is a consequence of stationarity. An alternate construction of the tail measure (denoted $\mu^\infty$) was established in the more general framework of regular variation on metric spaces in Segers et al. (2017). Beyond its theoretical importance, the tail measure is an extremely efficient tool to prove new results and give a much shorter proof to known result.

The first such application will be in Section 2 where we establish an alternative proof of the time change formula (see (2.4)) which was first proved in Basrak and Segers (2009) by using stationarity of the original time series and expressing the tail process as a limit. Here, as in Samorodnitsky and Owada (2012), we will prove it using only the shift invariance and homogeneity of the tail measure $\nu$. Moreover, we will provide an equivalent formulation of the formula (see Lemma 2.2) which turns out to be also very useful.

In Section 3, we will restrict our attention to the case where the tail process tends to zero at infinity. This property holds for most usual heavy tailed time series. It holds for linear processes and for most Markov models of interest in time series such as GARCH-type processes and solutions to stochastic recurrence equations. Our first main result (Theorem 3.1) will be that under this assumption, the tail measure can be recovered from the spectral tail process conditioned to first achieve its maximum at time zero.

In Section 3.3 we will introduce the sequence $Q$ whose distribution is that of the tail process standardized by its maximum, conditionally on the event that $|Y_0|$ is the first exceedence of the tail process over 1. This sequence was introduced by Basrak and Segers (2009) where it appears in the limiting theory for the point process of exceedences and partial sums of the original time series and in the limits of the so-called cluster functionals (which will be introduced in Section 4). We will show that it can also be used to recover the tail measure
and is equivalent (in some sense to be made precise in Proposition 3.6) to the spectral tail process conditioned to have its first maximum at time zero.

Cluster functionals have also been investigated in Mikosch and Wintenberger (2016) where, as a consequence of using different techniques, expressions for their limits were obtained in terms of the spectral tail process. These two different types of expressions for the limit of the same quantities must therefore be equal but no direct proof of their equality had been given. Moreover, the sets of functionals for which limits have been obtained by one or the other method were not equal. We will directly prove that these expressions are the same. As a particular example, we will prove in Lemma 3.11 that $E[(\sum_{j \in \mathbb{Z}} |Q_j|)^\alpha] < \infty$ if and only if $E[(\sum_{j=0}^{\infty} |\Theta_j|)^{\alpha-1}] < \infty$. This equivalence is of importance, since those were the conditions under which limiting results were obtained in the literature, but it was not known if these conditions were equivalent.

Previously, we will have analyzed in Section 3.2 the so-called candidate extremal index $\vartheta = P(\sup_{j \geq 1} |Y_j| \leq 1)$, introduced in Basrak and Segers (2009), who proved that it is positive under a condition on the original time series referred to as the anticlustering condition (see (4.3)). Assuming only that $\lim_{|j| \to \infty} |Y_j| = 0$, we will prove that $\vartheta > 0$. This is useful since the anticlustering condition, which is a standard assumption in the literature, is often much harder to check than the convergence of the tail process to zero.

We will conclude Section 3 by extending and providing a very simple proof, based on the tail measure, of identities for quantities generalizing those introduced as cluster indices in Mikosch and Wintenberger (2014).

As already mentioned, the previous results are important in the context of limiting theory for heavy tailed time series and are used to characterize the limits of cluster functionals. Such convergence results were previously obtained by various methods and often by ad-hoc conditions for each functional at hand. In Section 4, following Basrak et al. (2016), we will consider clusters which are vectors of observations $(X_1, \ldots, X_{r_n})$ of non decreasing length $r_n$ as element of the space $\tilde{\ell}_0$ of shift equivalent sequences (see Section 4 for a precise definition).

In Basrak et al. (2016), it is proved that the suitably normalized distribution of the scaled clusters converge in the sense of $\mathcal{M}_0$ convergence of Hult and Lindskog (2006) under the anticlustering condition. We will prove in Lemma 4.1 that if the tail process tends to zero at infinity, then the cluster convergence mentioned above always holds for some sequence $\{r_n\}$. This result also has consequences for the convergence of the point process of clusters introduced in Basrak et al. (2016) which generalizes the point process of exceedences and is a key tool in the study of certain statistics and for establishing the (functional) convergence of the partial sum process to a stable process when $\alpha \in (0, 2)$.

We conclude the paper in Section 5 by recalling certain relations between spectral tail processes and max-stable processes. In particular, when already given a non negative process $\Theta$ satisfying the time change formula and $\lim_{|j| \to \infty} \Theta_j = 0$, we obtain an alternative construction, based on the tail measure and Theorem 3.1, of a max-stable process whose
This formula can be extended to functions $H = (x_i)_{i \in \mathbb{Z}} \in (\mathbb{R}^d)^{\mathbb{Z}}$, we write $x_{s,t} = (x_s, \ldots, x_t)$, $x^*_{s,t} = \max_{s \leq i \leq t} |x_i|$, $x^* = \max_{i \in \mathbb{Z}} |x_i|$ and $|x|_p = (\sum_{j \in \mathbb{Z}} |x_j|^p)^{1/p}$, $p > 0$.

Whenever convenient, we identity a vector $x_{s,t}$ with $-\infty \leq s \leq t \leq +\infty$ to an element of $(\mathbb{R}^d)^{\mathbb{Z}}$ by completing it with zeros to the left if $s > -\infty$ or to the right if $t < \infty$.

We consider the following subspaces of $(\mathbb{R}^d)^{\mathbb{Z}}$: $\ell_0 = \{x \in (\mathbb{R}^d)^{\mathbb{Z}} : \lim_{|j| \to \infty} |x_j| = 0\}$ and for $p > 0$, $\ell_p = \{x \in (\mathbb{R}^d)^{\mathbb{Z}} : |x|_p < \infty\}$.

We denote by $B$ the backshift operator, i.e. $(Bx)_j = x_{j-1}$ and by $B^k$ its $k$-th iterate for $k \in \mathbb{Z}$.

A function $H : (\mathbb{R}^d)^{\mathbb{Z}} \to \mathbb{R}$ is said to be homogeneous with degree $\alpha \in \mathbb{R}$ or simply $\alpha$-homogeneous if $H(tx) = t^\alpha H(x)$ for all $x \in (\mathbb{R}^d)^{\mathbb{Z}}$ and $t > 0$, and it said to be shift invariant if $H(Bx) = H(x)$ for all $x \in (\mathbb{R}^d)^{\mathbb{Z}}$. A subset $A$ of $(\mathbb{R}^d)^{\mathbb{Z}}$ is said to be homogeneous if $x \in A$ implies $tx \in A$ for all $t > 0$ and it is said to be shift invariant if $x \in A$ if and only if $Bx \in A$.

2 The time change formula

Since $\nu$ is homogeneous, it can be decomposed into “radial and angular” parts. Define $E^*_d = \{y \in (\mathbb{R}^d)^{\mathbb{Z}} : |y_0| > 0\}$ and $S_d = \{y \in (\mathbb{R}^d)^{\mathbb{Z}} : |y_0| = 1\}$. Let $\psi$ be the map defined by

$$\psi : (0, \infty) \times S_d \to E^*_d$$

$$(r, \theta) \mapsto r\theta .$$

Since $\nu(\{y \in (\mathbb{R}^d)^{\mathbb{Z}} : |y_0| > 1\} = 1$, the measure $\sigma$ defined by $\sigma = \nu(\{y : |y_0| > 1, y/|y_0| \in \cdot\})$ is a probability measure on $S_d$ and by the homogeneity of $\nu$ it follows that (cf. (Segers et al., 2017, Proposition 3.1, Property (4))

$$\nu \circ \psi(dr, d\theta) = \alpha r^{-\alpha-1} dr \sigma(d\theta) .$$

Equivalently, if $H$ is a measurable $\nu$-integrable or nonnegative function on $E^*_d$,

$$\int_{E^*_d} H(y) \nu(dy) = \int_0^\infty \int_{S_d} H(r\theta) \sigma(d\theta) \alpha r^{-\alpha-1} dr .$$

This formula can be extended to functions $H$ on $(\mathbb{R}^d)^{\mathbb{Z}}$ by adding the indicator $1\{y_0 \neq 0\}$, i.e.

$$\int_{(\mathbb{R}^d)^{\mathbb{Z}}} H(y) 1\{y_0 \neq 0\} \nu(dy) = \int_0^\infty \int_{S_d} H(r\theta) \sigma(d\theta) \alpha r^{-\alpha-1} dr . \tag{2.1}$$
The indicator $\mathbb{1}\{y_0 \neq 0\}$ in the left hand side of (2.1) cannot be dispensed with since it is possible that $\nu(\{y_0 = 0\}) = \infty$. If we denote by $\Theta$ a random element on $(\mathbb{R}^d)^\mathbb{Z}$ with distribution $\sigma$, we obtain that $\Theta$ is the spectral tail process of the time series $\{X_j\}$, i.e.

$$\mathbb{E}[H(Y)] = \int_1^\infty \mathbb{E}[H(r\Theta)]\alpha r^{-\alpha - 1}dr .$$

(2.2) \{eq:polar\}

**Example 2.1.** Let $\{X_j, j \in \mathbb{Z}\}$ be a sequence of i.i.d. nonnegative regularly varying random variables with tail index $\alpha > 0$. Then the tail process and the spectral tail process are trivial: $Y_j = \Theta_j = 0$ for all $j \neq 0$. Consider the function $H(y) = \mathbb{1}\{y_1 > 1\}$. Then

$$\int_0^\infty \mathbb{E}[H(r\Theta)]\alpha r^{-\alpha - 1}dr = \int_0^\infty \mathbb{E}[\mathbb{1}\{r\Theta_1 > 1\}]\alpha r^{-\alpha - 1}dr = 0 .$$

However, because of shift invariance,

$$\int_{(\mathbb{R}^d)^\mathbb{Z}} H(y)\nu(dy) = \int_{(\mathbb{R}^d)^\mathbb{Z}} \mathbb{1}\{y_0 > 1\}\nu(dy) = 1 .$$

This illustrates the necessity of the indicator in the left hand side of (2.1).

We now obtain and prove a new version of the time change formula of Basrak and Segers (2009).

**Lemma 2.2.** Let $H$ be a non negative measurable functional on $(\mathbb{R}^d)^\mathbb{Z}$. Then, for all $k \in \mathbb{Z}$ and $t > 0$,

$$\mathbb{E}[H(B^kY)\mathbb{1}\{|Y_{-k}| > t\}] = t^{-\alpha}\mathbb{E}[H(tY)\mathbb{1}\{|Y_k| > 1/t\}] .$$

(2.3) \{eq:time-shift-Y\}

**Proof.** Applying (1.2), the homogeneity and shift invariance of $\nu$ yields

$$\mathbb{E}[H(B^kY)\mathbb{1}\{|Y_{-k}| > t\}] = \int_{(\mathbb{R}^d)^\mathbb{Z}} H(B^k y)\mathbb{1}\{|y_0| > 1\}\mathbb{1}\{|y_{-k}| > t\} \nu(dy)
\quad = \int_{(\mathbb{R}^d)^\mathbb{Z}} H(y)\mathbb{1}\{|y_k| > 1\}\mathbb{1}\{|y_0| > t\} \nu(dy)
\quad = t^{-\alpha} \int_{(\mathbb{R}^d)^\mathbb{Z}} H(tx)\mathbb{1}\{|x_k| > 1/t\}\mathbb{1}\{|x_0| > 1\} \nu(dx)
\quad = t^{-\alpha}\mathbb{E}[H(tY)\mathbb{1}\{|Y_k| > 1/t\}] .$$

By an application of the polar decomposition (2.1), it is easily seen that (2.3) is equivalent to the time change formula of Basrak and Segers (2009):

$$\mathbb{E}[H(B^k\Theta)\mathbb{1}\{|\Theta_{-k}| \neq 0\}] = \mathbb{E}[H(|\Theta_k|^{-1}\Theta)|\Theta_k|^\alpha] .$$

(2.4) \{eq:time-shift-Theta\}

where the quantity inside the expectation on the right hand side is understood to be 0 when $|\Theta_k| = 0$. A proof of this equivalence is in the appendix.
Remark 2.3. Note that (2.4) was proved in (Basrak and Segers, 2009, Theorem 3.1) by using the definition of the tail process as a limit and therefore restricting it to continuous functions. The present proof is without such restriction and arguably more straightforward.

Remark 2.4. If $H$ is homogeneous with degree 0, then (2.4) yields for all $k \in \mathbb{Z}$,

$$\mathbb{E}[H(B^k\Theta)1\{|\Theta_k| \neq 0\}] = \mathbb{E}[H(\Theta)|\Theta_k|^\alpha].$$

(2.5) \{eq:time-shift-homo-0\}

Conversely, by considering the function $x \mapsto H(|x_k|^{-1}x)$ it is easily seen that (2.5) is actually equivalent to (2.4). If $H$ is homogeneous with degree $\alpha$, then (2.4) yields for all $k \in \mathbb{Z}$,

$$\mathbb{E}[H(B^k\Theta)1\{|\Theta_k| \neq 0\}] = \mathbb{E}[H(\Theta)|\Theta_k|^\alpha].$$

(2.6) \{eq:time-shift-homo-alpha\}

If moreover $\sum_{i \in \mathbb{Z}} \mathbb{P}(|\Theta_i| = 0) = 0$, then we obtain, for all $k \in \mathbb{Z}$,

$$\mathbb{E}[H(B^k\Theta)] = \mathbb{E}[H(\Theta)].$$

(2.7) \{eq:time-shift-Theta-positive-homo-alpha\}

This property deceptively looks like stationarity, but it is only valid for functionals $H$ which are homogeneous with degree $\alpha$ and if $\mathbb{P}(|\Theta_k| = 0) = 0$ for all $k \in \mathbb{Z}$.

The shift invariance and homogeneity of $\nu$ allow to relate the null shift-invariant homogeneous sets for $\nu$ and for the distribution of $Y$.

Lemma 2.5. Let $A$ be a shift invariant, homogeneous measurable set in $(\mathbb{R}^d)\mathbb{Z}$. Then $\nu(A) \in \{0, \infty\}$ and the following statements are equivalent: (i) $\nu(A) = 0$; (ii) $\mathbb{P}(Y \in A) = 0$; (iii) $\mathbb{P}(\Theta \in A) = 0$.

Proof. Since $\nu(\{0\}) = 0$, we have, by the shift invariance of $\nu$ and $A$,

$$\nu(A \cap \{|y_0| > 0\}) \leq \nu(A) \leq \sum_{j \in \mathbb{Z}} \nu(A \cap \{|y_j| > 0\}) = \sum_{j \in \mathbb{Z}} \nu(A \cap \{|y_j| > 0\}).$$

(2.8) \{eq:nu-decompose\}

Applying the homogeneity and shift invariance of $\nu$ and $A$, the monotone convergence theorem and the definition of $Y$, we obtain

$$\nu(A \cap \{|y_0| > 0\}) = \lim_{\epsilon \to 0} \nu(A \cap \{|y_0| > \epsilon\})$$

$$= \lim_{\epsilon \to 0} \epsilon^{-\alpha} \nu(A \cap \{|y_0| > 1\}) = \lim_{\epsilon \to 0} \epsilon^{-\alpha} \mathbb{P}(Y \in A).$$

This proves that $\nu(A \cap \{|y_0| > 0\}) = 0$ if and only if $\mathbb{P}(Y \in A) = 0$ and that $\nu(A \cap \{|y_0| > 0\}) = \infty$ if $\nu(A \cap \{|y_0| > 0\}) > 0$. It now follows from (2.8) that $\nu(A) \in \{0, \infty\}$ and that the statements (i) and (ii) are equivalent. To finish the proof it just remains to notice that, since $A$ is homogeneous, $\mathbb{P}(\Theta \in A) = \mathbb{P}(Y \in A).$
3 Properties of the tail process when $\lim_{j \to \infty} |Y_j| = 0$

In this section, we restrict our attention to tail processes which satisfy the following condition.

$$\mathbb{P} \left( \lim_{|k| \to \infty} |Y_k| = 0 \right) = 1 . \tag{3.1} \{eq:Y_to_zero\}$$

This condition is satisfied by most time series models of interest. It will be further discussed in Section 4; here we simply admit it as our working assumption. By Lemma 2.5, the property (3.1) means that the tail measure $\nu$ is supported on the shift invariant and homogeneous set $\ell_0 = \{ \lim_{|j| \to \infty} |y_j| = 0 \}$.

3.1 Recovering the tail measure

An important consequence of (3.1) is that a.s. $Y^* < \infty$ and there is a first time index at which the maximum for the sequence $Y$ is achieved. To formalize this remark, we introduce the infargmax functional $I$, defined on $(\mathbb{R}^d)^\mathbb{Z}$ by

$$I(y) = \begin{cases} j \in \mathbb{Z} & \text{if } y_{-\infty,j-1} < |y_j| \text{ and } y_{j+1,\infty} \leq |y_j| , \\ -\infty & \text{if } y^* = y_{-\infty,j} \text{ for all } j \in \mathbb{Z} \\ +\infty & \text{if } y^* > y_{-\infty,j} \text{ for all } j \in \mathbb{Z} . \end{cases}$$

For instance, the infargmax of a constant sequence is $-\infty$. The infargmax is achieved in $\mathbb{Z}$ if there exists a first time when the maximum is achieved. The event $I(y) \in \mathbb{Z}$ can be expressed as

$$\sum_{j \in \mathbb{Z}} 1\{I(y) = j\} = 1 .$$

By Lemma 2.5, we have

$$\nu(\{I(y) \notin \mathbb{Z}\}) = 0 \iff \mathbb{P}(I(\Theta) \in \mathbb{Z}) = 1 . \{eq:noinfargmax\}$$

Theorem 3.1. Assume that $\mathbb{P}(I(\Theta) \in \mathbb{Z}) = 1$. Then $\mathbb{P}(I(\Theta) = 0) > 0$ and for all nonnegative measurable functions $H$,

$$\nu(H) = \sum_{j \in \mathbb{Z}} \int_0^\infty \mathbb{E}[H(rB_j\Theta)] 1\{I(\Theta) = 0\} \alpha r^{-\alpha-1} dr . \tag{3.2} \{eq:tailmeasure1\}$$

If moreover $\sum_{j \in \mathbb{Z}} \nu(\{|y_j| = 0\}) = 0$, or equivalently $\sum_{j \in \mathbb{Z}} \mathbb{P}(|\Theta_j| = 0) = 0$, then

$$\nu(H) = \int_0^\infty \mathbb{E}[H(r\Theta)] \alpha r^{-\alpha-1} dr . \tag{3.3} \{eq:tailmeasure2\}$$
Proof. Let \( H \) be a non-negative measurable function on \((\mathbb{R}^d)^Z\). Since \( \nu(\{I(y) \notin Z\}) = 0 \) by assumption, the shift invariance of \( \nu \) yields

\[
\nu(H) = \sum_{j \in Z} \int_{(\mathbb{R}^d)^Z} H(y) \mathbb{1}\{I(y) = j\} \nu(dy) = \sum_{j \in Z} \int_{(\mathbb{R}^d)^Z} H(B^j y) \mathbb{1}\{I(y) = 0\} \nu(dy).
\]

Since \( I(y) = 0 \) implies that \(|y_0| > 0\), applying the polar decomposition (2.1) to the function \( y \mapsto \sum_{j \in Z} H(B^j y) \mathbb{1}\{I(y) = 0\} \nu(dy) \) yields (3.2).

In the case \( \mathbb{P}(|\Theta_i| = 0) = 0 \) for all \( i \in Z \), we can apply the time change formula (2.7) to the function \( L(y) = \int_0^\infty H(r y) \alpha r^{-\alpha - 1} dr \) which is homogeneous with degree \( \alpha \) and we obtain

\[
\nu(H) = \sum_{j \in Z} \mathbb{E}[L(B^j \Theta) \mathbb{1}\{I(\Theta) = 0\}] = \sum_{j \in Z} \mathbb{E}[L(\Theta) \mathbb{1}\{I(B^{-j} \Theta) = 0\}]
\]

\[
= \sum_{j \in Z} \mathbb{E}[L(\Theta) \mathbb{1}\{I(\Theta) = j\}] = \mathbb{E}[L(\Theta)].
\]

This proves (3.3). Taking \( H(y) = \mathbb{1}\{|y_0| > 1\} \), (3.2) yields

\[
1 = \nu(\{|y_0| > 1\}) = \sum_{j \in Z} \int_0^\infty \mathbb{P}(r |\Theta_j| > 1, I(\Theta) = 0) \alpha r^{-\alpha - 1} dr
\]

\[
= \sum_{j \in Z} \mathbb{E}[|\Theta_j|^\alpha \mathbb{1}\{I(\Theta) = 0\}].
\]

This proves that \( \mathbb{P}(I(\Theta) = 0) > 0 \).

Using the representation of the tail measure, we can further refine Lemma 2.5.

Corollary 3.2. Assume that \( \mathbb{P}(I(\Theta) \in Z) = 1 \). If \( A \) is shift invariant and homogeneous, then the following statements are equivalent

1. \( \nu(A) = 0; \)
2. \( \mathbb{P}(\Theta \in A) = 0; \)
3. \( \mathbb{P}(\Theta \in A | I(\Theta) = 0) = 0. \)

Proof. The equivalence between (i) and (ii) was already stated in Lemma 2.5. The equivalence between (i) and (iii) follows from (3.2).

The equivalence (ii) and (iii) is useful in practice since as we will see right below it may be easier to prove that an event has a probability zero conditionally on the first maximum being achieved at time zero than unconditionally.

The following result has been proved in the related context of max-stable processes by Dombry and Kabluchko (2016) and recently by Janßen (2017). We provide an alternate straightforward proof. See also. Janßen (2017).
Corollary 3.3. The following statements are equivalent.

(i) \( P(I(\Theta) \in Z) = 1; \)

(ii) \( P(\lim_{|j|\to\infty} |\Theta_j| = 0) = 1; \)

(iii) \( P\left( \sum_{j \in Z} |\Theta_j|^{\alpha} < \infty \right) = 1. \)

Proof. The implications (iii) \( \implies \) (ii) and (ii) \( \implies \) (i) are obvious. We only need to prove the implication (i) \( \implies \) (iii). By Theorem 3.1, (i) implies the identity (3.4) which implies that

\[
P\left( \sum_{j \in Z} |\Theta_j|^{\alpha} < \infty \mid I(\Theta) = 0 \right) = 1. \]

By Corollary 3.2, this proves (iii). \( \square \)

Corollary 3.3 yields the following property which has been used in the literature, but to the best of our knowledge, never proved. If \( \lim_{|k|\to\infty} |Y_k| = 0 \) and \( \alpha \leq 1 \), then

\[
P\left( \sum_{j \in Z} |\Theta_j| < \infty \right) = 1. \tag{3.5} \]

3.2 The candidate extremal index

Following Basrak and Segers (2009), we define

\[
\vartheta = P\left( \sup_{j \geq 1} |Y_j| \leq 1 \right). \tag{3.6} \]

In terms of the tail measure, we have

\[
\vartheta = \nu(\{y_{1,\infty}^* \leq 1, |y_0| > 1\}). \tag{3.7} \]

The candidate extremal index turns out to be the true extremal index of many time series models. The relation between the candidate and true extremal index will be further developed in Sections 4 and 5. Decomposing the event \( \{\sup_{j \geq 1} |Y_j| > 1\} \) according to the first time the tail process is greater than 1 and applying the time change formula (2.3) (with \( t = 1 \)), we obtain

\[
P\left( \sup_{j \geq 1} |Y_j| > 1 \right) = \sum_{k \geq 1} P\left( \max_{1 \leq j \leq k-1} |Y_j| \leq 1, |Y_k| > 1 \right)
= \sum_{k \geq 1} P\left( \max_{-k+1 \leq j \leq -1} |Y_j| \leq 1, |Y_{-k}| > 1 \right)
= P\left( \sup_{j \leq -1} |Y_j| > 1 \right). \]
Thus the candidate extremal index can be expressed as the probability that the first exceedence over 1 happens at time 0:

$$\vartheta = \mathbb{P}\left(\sup_{j \leq -1} |Y_j| \leq 1\right).$$  \hspace{1cm} (3.8) \hspace{1cm} \{eq:altcandidate-BS09\}

This identity was obtained Guivarc’h and Le Page (2016) for solutions to stochastic recurrence equations. For general time series, Basrak and Segers (2009) proved that (3.8) holds under the so-called anticlustering condition (see (4.3) below) by using the original time series. Furthermore, it is also proved in the same manner in Basrak and Segers (2009) that the anticlustering condition implies that $\vartheta > 0$. Since the candidate extremal index is defined in terms of the tail process only, it is natural to give a proof using only the tail process under the assumption that the tail process tends to zero.

\textbf{Lemma 3.4.} If $\mathbb{P}(\lim_{|k| \to \infty} |Y_k| = 0) = 1$, then $\vartheta > 0$.

\textit{Proof.} Since $\mathbb{P}(|Y_0| > 1) = 1$ and by assumption there is always a last time when the tail process is bigger than 1, applying the time change formula (2.3), we have

$$1 = \mathbb{P}(Y_{0,\infty}^* > 1) = \sum_{j \geq 0} \mathbb{P}(Y_{j+1,\infty}^* \leq 1, |Y_j| > 1)$$

$$= \sum_{j \geq 0} \mathbb{P}(Y_{1,\infty}^* \leq 1, |Y_{-j}| > 1) \leq \sum_{j \geq 0} \mathbb{P}(Y_{1,\infty}^* \leq 1) = \infty \times \vartheta.$$  \hspace{1cm} \Box

This implies that $\vartheta > 0$.

\subsection{The sequence $Q$}

From now on, we assume that $\mathbb{P}(\lim_{|j| \to \infty} |Y_j| = 0) = 1$, which ensures that $\vartheta > 0$.

\textbf{Definition 3.5 (Basrak and Segers (2009))}. The sequence $Q = \{Q_j, j \in \mathbb{Z}\}$ is a random sequence whose distribution is that of $(Y^*)^{-1}Y$ (or $(\Theta^*)^{-1}\Theta$) conditionally on $Y_{-\infty,-1}^* \leq 1$.

The sequence $Q$ appears in limits of so-called cluster functionals. This will be further developed in Section 4. Here we study it formally. We first show that this sequence is closely related to the sequence $\Theta$ conditioned to have its first maximum at 0.

\textbf{Proposition 3.6}. Assume that $\mathbb{P}(\lim_{|k| \to \infty} |Y_k| = 0) = 1$. Let $H$ be a shift invariant, non negative measurable function on $(\mathbb{R}^d)^\mathbb{Z}$. Then

$$\vartheta \mathbb{E}[H(Q)] = \mathbb{E}[H(\Theta)\mathbb{1}\{I(\Theta) = 0\}].$$  \hspace{1cm} (3.9) \hspace{1cm} \{eq:identity-Q-infargmax\}
Proof. Let $K$ be a non negative measurable shift invariant functional. Applying the time change formula (2.3) yields

\[
\mathbb{E}[K(Y)1\{Y_{-\infty,-1}^* \leq 1\}] = \sum_{j \in \mathbb{Z}} \mathbb{E}[K(Y)1\{Y_{-\infty,-1}^* \leq 1\}1\{I(Y) = j\}]
\]

\[
= \sum_{j \in \mathbb{Z}} \mathbb{E}[K(Y)1\{Y_{-\infty,-1}^* \leq 1\}1\{I(Y) = j\}1\{|Y_j| > 1\}]
\]

\[
= \sum_{j \in \mathbb{Z}} \mathbb{E}[K(Y)1\{Y_{-\infty,-j-1}^* \leq 1\}1\{I(B^jY) = j\}1\{|Y_{-j}| > 1\}]
\]

\[
= \mathbb{E}[K(Y)1\{I(Y) = 0\}] \sum_{j \in \mathbb{Z}} 1\{Y_{-\infty,-j-1}^* \leq 1\}1\{|Y_{-j}| > 1\}
\]

\[
= \mathbb{E}[K(Y)1\{I(Y) = 0\}] . \tag{3.10} \]

Applying this identity to the function $K(y) = H(y/y^*)$ yields (3.9).

Let $H$ be a shift invariant non negative measurable functional on $(\mathbb{R}^d)^Z$. Applying the identity (3.10) we obtain for $t \geq 1$,

\[
\mathbb{E}[H(Y/Y^*)1\{Y^* > t\} \mid Y_{-\infty,-1}^* \leq 1] = \vartheta^{-1} \mathbb{E}[H(\Theta)1\{|Y_0| > t\}1\{I(\Theta) = 0\}]
\]

\[
= \vartheta^{-1} t^{-\alpha} \mathbb{E}[H(\Theta)1\{I(\Theta) = 0\}]
\]

\[
= t^{-\alpha} \mathbb{E}[H(Q)] . \tag{3.11} \]

The previous result implies that conditionally on $Y_{-\infty,-1}^* \leq 1$, $Y^*$ has the same Pareto distribution as $|Y_0|$ (unconditionally), but it does not imply that $Y^*$ and $(Y^*)^{-1}Y$ are independent, since (3.11) holds only for shift invariant functionals. However, the latter statement is true if one considers $(Y^*)^{-1}Y$ as a random element in the space $\tilde{\ell}_0$ of shift-equivalent sequences. This will be further developed in Section 4. These results were originally proved in Basrak and Tafro (2016) by different means under the antichlustering condition (4.3). The present proof, which assumes only $\mathbb{P}(\lim_{|y_j| \to \infty} |Y_{j}| = 0) = 1$, is simpler and moreover shows that these results are direct consequences of the fact that $|Y_0|$ is Pareto distributed and independent of the spectral tail process.

Another important consequence of Proposition 3.6 is that the tail measure can also be recovered from the sequence $Q$. For a non negative measurable $H$, by Theorem 3.1 we have

\[
\nu(H) = \vartheta \sum_{j \in \mathbb{Z}} \int_0^\infty \mathbb{E}[H(rB^jQ)] \alpha \nu^{-\alpha} dr . \tag{3.12} \]

Applying (3.12) to the function $H : x \mapsto 1\{|x_0| > 1\}$ yields

\[
1 = \nu(H) = \vartheta \sum_{j \in \mathbb{Z}} \mathbb{E}[|Q_j|^{\alpha}] . \tag{3.13} \]
The inequality \( \vartheta \sum_{j \in \mathbb{Z}} E[|Q_j|^\alpha] \leq 1 \) was obtained in (Davis and Hsing, 1995, Theorem 2.6) by an application of Fatou’s lemma. It was also stated there that equality holds under an additional uniform integrability assumption. We have thus proved that (3.13) holds without any additional assumption and this seems to be new.

We now prove more identities between quantities expressed in terms of the spectral tail process or in terms of the sequence \( Q \). The equality of some of these quantities was already indirectly known, since they appeared as limits of the same quantities, but obtained through different methods. For some of them, the case \( \alpha = 1 \) had not yet been treated. We first prove an identity which will be the main path from \( Q \) to \( \Theta \).

**Lemma 3.7.** Assume that \( \mathbb{P}(\lim_{|j| \to \infty} |Y_j| = 0) = 1 \). Let \( H \) be a non negative, shift invariant, \( \alpha \)-homogeneous measurable function on \( (\mathbb{R}^d)^\mathbb{Z} \). Then,

\[
\vartheta E[H(Q)] = E[H(\Theta)1\{I(\Theta) = 0\}] = E\left[\frac{H(\Theta)}{|\Theta|^\alpha}\right].
\]

(3.14)

**Proof.** The first equality in (3.14) is a repeat of (3.9). The function \( y \mapsto |y|^{-\alpha} H(y) \) is shift invariant and homogeneous with degree 0, thus applying the time change formula (2.5), we obtain

\[
E[H(\Theta)1\{I(\Theta) = 0\}] = \sum_{j \in \mathbb{Z}} E\left[ |\Theta_j|^{-\alpha} H(\Theta)1\{I(\Theta) = 0\} \right]
\]

\[
= \sum_{j \in \mathbb{Z}} E\left[ \frac{H(\Theta)}{|\Theta|^\alpha} 1\{I(B^j\Theta) = 0\} \right]
\]

\[
= \sum_{j \in \mathbb{Z}} E\left[ \frac{H(\Theta)}{|\Theta|^\alpha} 1\{I(\Theta) = -j\} \right] = E\left[\frac{H(\Theta)}{|\Theta|^\alpha}\right].
\]

We have used in the middle lines that \( I(B^j\Theta) = 0 \) implies \( |\Theta_{-j}| \neq 0 \) and is equivalent to \( I(\Theta) = -j \) and to conclude we used that \( \mathbb{P}(I(\Theta) \in \mathbb{Z}) = 1 \) as a consequence of the assumption \( \mathbb{P}(\lim_{|j| \to \infty} |Y_j| = 0) = 1 \).

As a consequence of Lemma 3.7, for every measurable shift invariant \( \alpha \)-homogeneous function \( H \) on \( (\mathbb{R}^d)^\mathbb{Z} \), the quantities

\[
E[|H(Q)|], \ E[|H(\Theta)|1\{I(\Theta) = 0\}], \ E\left[\frac{|H(\Theta)|}{|\Theta|^\alpha}\right],
\]

are simultaneously finite or infinite and in the former case, the identity (3.14) holds.

The previous identities involve the sequence \( Q \) and the sequence \( \Theta \). In practice, the quantities in terms of the spectral tail process are usually easier to compute explicitly or for use in simulation since they do not involve a conditioning contrary to those with the sequence \( Q \). Moreover, it is often relatively easy to compute the forward tail process \( \{\Theta_j, j \geq 0\} \) but
Lemma 3.8. Assume that $\mathbb{P}(\lim_{|j| \to \infty} |Y_j| = 0) = 1$. Let $H$ be a shift invariant, $\alpha$-homogenous measurable function defined on a subset $\mathcal{O}$ of $\ell_\alpha$ such that $\mathbb{P}(\Theta_{n,\infty} \in \mathcal{O}) = 1$ for all $n \in \mathbb{Z}$. Assume moreover

\begin{itemize}
  \item[(i)] $\mathbb{E}[|H(\Theta_{0,\infty}) - H(\Theta_{1,\infty})|] < \infty$;
  \item[(ii)] $\mathbb{P}(\lim_{n \to \infty} H(\Theta_{n,\infty}) = 0) = 1$;
  \item[(iii)] $\mathbb{P}(\lim_{n \to \infty} H(\Theta_{-n,\infty}) = H(\Theta)) = 1$.
\end{itemize}

Then $\mathbb{E}[|H(Q)|] < \infty$ and

$$\psi \mathbb{E}[H(Q)] = \mathbb{E}[H(\Theta_{0,\infty}) - H(\Theta_{1,\infty})].$$

Proof. Since the function $x \mapsto |x|_\alpha^{-\alpha} \{H(x_{0,\infty}) - H(x_{1,\infty})\}$ is $\alpha$-homogeneous and equal to 0 whenever $|x_0| = 0$, by the time change formula (2.5), we have

$$\sum_{j \in \mathbb{Z}} \mathbb{E}\left[\frac{H(\Theta_{-j,\infty}) - H(\Theta_{j+1,\infty})}{|\Theta|_\alpha^{\alpha}}\right] = \sum_{j \in \mathbb{Z}} \mathbb{E}\left[|\Theta_j|^{-\alpha} \frac{H(\Theta_{0,\infty}) - H(\Theta_{1,\infty})}{|\Theta|_\alpha^{\alpha}}\right] = \mathbb{E}[|H(\Theta_{0,\infty}) - H(\Theta_{1,\infty})|] < \infty.$$

Consequently, $\mathbb{P}(\sum_{j \in \mathbb{Z}} |H(\Theta_{-j,\infty}) - H(\Theta_{j+1,\infty})| < \infty) = 1$ and

$$\sum_{j \in \mathbb{Z}} \mathbb{E}\left[\frac{H(\Theta_{-j,\infty}) - H(\Theta_{j+1,\infty})}{|\Theta|_\alpha^{\alpha}}\right] = \mathbb{E}\left[\sum_{j \in \mathbb{Z}} \frac{H(\Theta_{-j,\infty}) - H(\Theta_{j+1,\infty})}{|\Theta|_\alpha^{\alpha}}\right] = \mathbb{E}[H(\Theta_{0,\infty}) - H(\Theta_{1,\infty})].$$

On the other hand, assumptions (ii) and (iii) and the dominated convergence theorem ensure that

$$\sum_{j \in \mathbb{Z}} \frac{H(\Theta_{-j,\infty}) - H(\Theta_{j+1,\infty})}{|\Theta|_\alpha^{\alpha}} = \lim_{n \to \infty} \sum_{-n < j \leq n} \frac{H(\Theta_{-j,\infty}) - H(\Theta_{j+1,\infty})}{|\Theta|_\alpha^{\alpha}} = \lim_{n \to \infty} \frac{H(\Theta_{-n,\infty}) - H(\Theta_{n,\infty})}{|\Theta|_\alpha^{\alpha}} = \frac{H(\Theta)}{|\Theta|_\alpha^{\alpha}}.$$

Hence, $\mathbb{E}[|\Theta|^{-\alpha} |H(\Theta)|| < \infty$ and $\mathbb{E}[|\Theta|^{-\alpha} H(\Theta)] = \mathbb{E}[H(\Theta_{0,\infty}) - H(\Theta_{1,\infty})]$. Lemma 3.7 finally yields that $\mathbb{E}[|H(Q)|] < \infty$ and that (3.15) holds.
Example 3.9. As a first illustration of the previous results, we provide other expressions for the candidate extremal index \( \vartheta \). These expressions might be used for statistical inference on the extremal index when it is known to be equal to the candidate extremal index. If \( \mathbb{P}(\lim_{k \to \infty} Y_k = 0) = 1 \), then

\[
\vartheta = \mathbb{P}(I(Y) = 0) = \mathbb{P}(I(\Theta) = 0) = \mathbb{E} \left[ \frac{(Y^*)^\alpha}{\sum_{j \in \mathbb{Z}} |Y_j|^\alpha} \right] = \mathbb{E} \left[ \frac{(\Theta^*)^\alpha}{\sum_{j \in \mathbb{Z}} |\Theta_j|^\alpha} \right]
\]

(3.16) \( \{eq:candidate\} \)

(3.17) \( \{eq:EI-debicky\} \)

(3.18) \( \{eq:vartheta-diff\} \)

(3.19) \( \{eq:candidate-hashorva\} \)

The identity (3.16) is obtained by applying 3.6 to the function \( H \equiv 1 \); (3.17) is obtained by applying Lemma 3.7 to the function \( H \) defined by \( H(x) = x^* \) and (3.18) is obtained by applying Lemma 3.8 to the same function. We only need to prove (3.19). The assumption implies that \( \mathbb{P}(\sum_{i \in \mathbb{Z}} 1\{ |Y_i| > 1 \} < \infty) = 1 \). Applying the time change formula (2.3), we obtain

\[
\vartheta = \mathbb{P}(Y^*_{-\infty,-1} \leq 1) = \mathbb{E} \left[ \frac{\sum_{j \in \mathbb{Z}} 1\{ |Y_j| > 1 \}}{\sum_{i \in \mathbb{Z}} 1\{ |Y_i| > 1 \}} \right] \mathbb{1}\{ Y^*_{-\infty,-1} \leq 1 \}
\]

\[
= \sum_{j \in \mathbb{Z}} \mathbb{E} \left[ \frac{1 \{ Y^*_{-\infty,-1} \leq 1 \} 1 \{ |Y_j| > 1 \}}{\sum_{i \in \mathbb{Z}} 1\{ |Y_i| > 1 \}} \right]
\]

\[
= \sum_{j \in \mathbb{Z}} \mathbb{E} \left[ \frac{1 \{ Y^*_{-\infty,-1} \leq 1 \} 1 \{ |Y_j| > 1 \}}{\sum_{i \in \mathbb{Z}} 1\{ |Y_i| > 1 \}} \right]
\]

\[
= \mathbb{E} \left[ \frac{\sum_{j \in \mathbb{Z}} 1\{ Y^*_{-\infty,-1} \leq 1 \} 1\{ |Y_j| > 1 \}}{\sum_{i \in \mathbb{Z}} 1\{ |Y_i| > 1 \}} \right] = \mathbb{E} \left[ \frac{1}{\sum_{i \in \mathbb{Z}} 1\{ |Y_i| > 1 \}} \right].
\]

The expressions (3.16) and (3.17) were obtained by Dębicki and Hashorva (2017) for max-stable processes, the expression (3.18) is due to Basrak and Segers, 2009, Remark 4.7) (3.19) is due to Enkelejd Hashorva (personal communication).

Example 3.10. In the case \( d = 1 \), for \( x \in \mathbb{R} \) define \( x^{(\alpha)} = \max(x, 0)^\alpha - \max(-x, 0)^\alpha = x|x|^{\alpha-1} \). Applying Lemma 3.8 to the function \( H(y) = \sum_{j \in \mathbb{Z}} y_j^{(\alpha)} \) which trivially satisfies its assumptions yields

\[
\vartheta \mathbb{E} \left[ \sum_{j \in \mathbb{Z}} Q_j^{(\alpha)} \right] = \mathbb{E}[\Theta_0^{(\alpha)}] = \mathbb{E}[\Theta_0].
\]

(3.20) \( \{eq:DH95-3.13\} \)
This identity was indirectly obtained in the proof of (Davis and Hsing, 1995, Theorem 3.2) in the case \( \alpha \in [1, 2] \) by identification of two terms. Recall that \( \mathbb{E}[\Theta_0] \) is the skewness of the tail of the marginal distribution of the original time series \( \{X_j, j \in \mathbb{Z}\} \).

In (Basrak et al., 2016, Theorem 4.5), the condition \( \mathbb{E}[\|Q_1\|^\alpha] < \infty \) is used in order to establish functional convergence of the partial sum process of a weakly dependent stationary regularly varying time-series. For \( \alpha \leq 1 \), the concavity of the function \( x \mapsto x^\alpha \) and (3.13) implies that \( \mathbb{E}[\|Q_1\|^\alpha] < \infty \). For \( \alpha > 1 \), the latter integrability condition does not always hold and we now obtain a sufficient condition in terms of the forward spectral tail process using Lemma 3.8.

**Lemma 3.11.** Assume that \( \mathbb{P}(\lim_{|t| \to \infty} |Y_t| = 0) = 1 \). Then

\[
\vartheta \mathbb{E} \left[ \left( \sum_{j \in \mathbb{Z}} |Q_j| \right)^{\alpha} \right] = \mathbb{E} \left[ \left( \sum_{j \in \mathbb{Z}} |\Theta_j| \right)^{\alpha-1} \right]. \tag{3.21} \]  

These quantities are always finite if \( \alpha \leq 1 \) and are simultaneously finite or infinite if \( \alpha > 1 \). Moreover, the following conditions are equivalent:

\[
\mathbb{E} \left[ \left( \sum_{j \in \mathbb{Z}} |Q_j| \right)^{\alpha} \right] < \infty, \tag{3.22} \]

\[
\mathbb{E} \left[ \left( \sum_{j=0}^{\infty} |\Theta_j| \right)^{\alpha-1} \right] < \infty. \tag{3.23} \]

**Proof.** Applying Proposition 3.6 to the shift invariant \( \alpha \)-homogeneous function \( y \mapsto |y|_1^{\alpha} \) and the time change formula (2.6) to the \( \alpha \)-homogeneous function \( y \mapsto |y_0| |y|_1^{\alpha-1} 1\{I(y) = j\} \) we obtain

\[
\vartheta \mathbb{E} \left[ \|Q_1\|^\alpha \right] = \mathbb{E} \left[ |\Theta_1|^{\alpha} 1\{I(\Theta) = 0\} \right] \\
= \sum_{j \in \mathbb{Z}} \mathbb{E} \left[ |\Theta_{-j}| |\Theta|_1^{\alpha-1} 1\{I(\Theta) = j\} \right] \\
= \sum_{j \in \mathbb{Z}} \mathbb{E} \left[ |\Theta_0| |\Theta|_1^{\alpha-1} 1\{I(\Theta) = j\} \right] \\
= \mathbb{E} \left[ |\Theta|_1^{\alpha-1} \sum_{j \in \mathbb{Z}} 1\{I(\Theta) = j\} \right] = \mathbb{E}[|\Theta|_1^{\alpha-1}] .
\]

Hence, (3.21) holds and in particular (3.22) implies (3.23). Moreover, since \( |\Theta|_1 \geq 1 \), (3.22) always holds when \( \alpha \leq 1 \) as already noted in the discussion preceding this lemma. Conversely, if (3.23) holds and \( \alpha > 1 \), applying Lemma 3.8 to the function \( H(x) = |x|_1^{\alpha} \) yields (3.22). Just note that the condition (i) is implied by the bound (3.26) below and the fact that \( |\Theta_{0,\infty}|_1 = 1 + |\Theta_{1,\infty}|_1 \). \( \square \)
Corollary 3.12. Assume that \( \mathbb{P}(\lim_{|t| \to \infty} |Y_t| = 0) = 1 \) and that (3.22) or (3.23) holds. Let \( H \) be a non negative, shift invariant, \( \alpha \)-homogenous measurable function on \((\mathbb{R}^d)^2\) such that

\[
|H(x) - H(y)| \leq C |x - y|_1 ,
\]

for some constant \( C > 0 \) and all \( x, y \in (\mathbb{R}^d)^2 \). Then \( \mathbb{E}[|H^\alpha(\Theta_{0,\infty}) - H^\alpha(\Theta_{1,\infty})|] < \infty \) and

\[
\vartheta \mathbb{E}[H^\alpha(Q)] = \mathbb{E}[H^\alpha(\Theta_{0,\infty}) - H^\alpha(\Theta_{1,\infty})] .
\]

Proof. Apply Lemma 3.8 to the function \( H^\alpha \) which satisfies its assumptions in view of (3.24) and the following bounds: for all \( a, b \geq 0 \),

\[
|a^\alpha - b^\alpha| \leq \begin{cases} 
|a - b|^\alpha & \text{if } \alpha \leq 1 , \\
\alpha|a - b|(a \vee b)^{\alpha - 1} & \text{if } \alpha > 1 .
\end{cases}
\]

\[
\text{(3.26)}
\]

Example 3.13. In the case \( d = 1 \), considering the functionals \( y \mapsto (\sum_{j \in \mathbb{Z}} y_j)_+ \) and \( y \mapsto (\sup_{k \in \mathbb{Z}} \sum_{j \leq k} y_j)_+ \), under condition (3.22), we obtain

\[
\vartheta \mathbb{E} \left[ \left( \sum_{j \in \mathbb{Z}} Q_j \right)_+^\alpha \right] = \mathbb{E} \left[ \left( \sum_{j=0}^\infty \Theta_j \right)_+^\alpha - \left( \sum_{j=1}^\infty \Theta_j \right)_+^\alpha \right] ,
\]

\[
\text{(3.27)}
\]

\[
\vartheta \mathbb{E} \left[ \left( \sup_{k \in \mathbb{Z}} \sum_{j \leq k} Q_j \right)_+^\alpha \right] = \mathbb{E} \left[ \left( \sup_{k \geq 0} \sum_{j=0}^k \Theta_j \right)_+^\alpha - \left( \sup_{k \geq 1} \sum_{j=1}^k \Theta_j \right)_+^\alpha \right] .
\]

\[
\text{(3.28)}
\]

The quantity in the left hand side of (3.27) appeared in Davis and Hsing (1995) in relation to the skewness of the limiting stable law of the partial sums when \( 0 < \alpha < 2 \). The right hand side appears in Mikosch and Wintenberger (2014) under the name cluster index and was also related to the limiting stable distribution of the partial sums in Mikosch and Wintenberger (2016) but for \( \alpha \neq 1 \). The quantity in the right hand side of (3.28) appears in Mikosch and Wintenberger (2016) in relation to ruin probabilities.

We conclude this section by an example in the case \( \alpha = 1 \), related to the location parameter of the limiting 1-stable law of the partial sum process of a weakly dependent regularly varying time series with tail index 1. An implicit and very involved expression was given in (Davis and Hsing, 1995, Theorem 3.2). An explicit expression is given in (Basrak et al., 2016, Theorem 4.5 and Remark 4.8) under the condition (3.29) below. The following lemma shows that this additional integrability condition is very light and moreover allows to express the location parameter in terms of the (forward) spectral tail process.
Lemma 3.14. Assume that $\alpha = 1$ and $\mathbb{P}(\lim_{|j|\to\infty} |Y_j| = 0) = 1$. The following conditions are equivalent:

\[
\sum_{j \in \mathbb{Z}} \mathbb{E} \left[ |Q_j| \log \left( |Q_j|^{-1} |Q|_1 \right) \right] < \infty , \tag{3.29} \]

\[
\mathbb{E} \left[ \log \left( \sum_{j=0}^{\infty} |\Theta_j| \right) \right] < \infty . \tag{3.30} \]

If either condition holds, then

\[
\mathbb{P} \left( \sum_{j \in \mathbb{Z}} |\Theta_j||\log(|\Theta_j|)| < \infty \right) = 1 . \tag{3.31} \]

If moreover $d = 1$, then

\[
\vartheta \mathbb{E} \left[ S_Q \log(|S_Q|) \right] - \vartheta \sum_{j \in \mathbb{Z}} \mathbb{E} [Q_j \log(|Q_j|)] = \mathbb{E} [S_0 \log(S_0) - S_1 \log(S_1)] , \tag{3.32} \]

with $S_Q = \sum_{i \in \mathbb{Z}} Q_i$ and $S_i = \sum_{j=i}^{\infty} \Theta_i$, $i = 0, 1$ and all the expectations in (3.32) are well defined and finite.

Proof. Note that $|x_j|^{-1} |x|_1 \geq 1$ for all $x \in \ell_0$ and $j \in \mathbb{Z}$. Applying Proposition 3.6 to the non negative shift invariant functional

\[
x \mapsto \sum_{j \in \mathbb{Z}} |x_j| \log \left( |x_j|^{-1} |x|_1 \right)
\]

with the convention $|x_j| \log (|x_j|^{-1} |x|_1) = 0$ when $|x_j| = 0$, and the time change formula (2.4), we obtain

\[
\vartheta \sum_{j \in \mathbb{Z}} \mathbb{E} \left[ |Q_j| \log \left( |Q_j|^{-1} |Q|_1 \right) \right] = \mathbb{E} \left[ |\Theta_j| \log \left( |\Theta_j|^{-1} |\Theta|_1 \right) \mathbb{1} \{I(\Theta) = 0\} \right] \tag{3.33} \]

\[
= \sum_{j \in \mathbb{Z}} \mathbb{E} \left[ \log \left( |\Theta_i| \right) \mathbb{1} \{I(B^j \Theta) = 0\} \right]
\]

\[
= \mathbb{E} \left[ \log \left( |\Theta_i| \right) \right] . \tag{3.34} \]

This proves that all these terms are simultaneously finite or infinite and thus (3.29) implies (3.30).

Conversely, assume that (3.30) holds. By the time change formula, we have for $j \geq 1$,

\[
0 \leq \mathbb{E} \left[ \log(|\Theta_{-j,\infty}|_1) - \log(|\Theta_{-j+1,\infty}|_1) \right] = \mathbb{E} \left[ |\Theta_j| \left\{ \log(|\Theta_j|^{-1} |\Theta_{0,\infty}|_1) - \log(|\Theta_j|^{-1} |\Theta_{1,\infty}|_1) \right\} \right]
\]

\[
= \mathbb{E} \left[ |\Theta_j| \left\{ \log(|\Theta_{0,\infty}|_1) - \log(|\Theta_{1,\infty}|_1) \right\} \right] .
\]
The quantity inside the expectation in right hand side is understood to be 0 if $|\Theta_j| = 0$. Also, $|\Theta_j| > 0$ implies that $|\Theta_{1,\infty}|_1 > 0$. Note now that if $y \geq x > 0$, then $0 \leq \log(y) - \log(x) \leq (y - x)/x$. Since $|\Theta_{0,\infty}|_1 - |\Theta_{1,\infty}|_1 = 1$, this yields
\[
0 \leq \mathbb{E} \left[ |\Theta_j| \left\{ \log(|\Theta_{0,\infty}|_1) - \log(|\Theta_{1,\infty}|_1) \right\} \right] \leq \mathbb{E} \left[ |\Theta_{1,\infty}|_1^{-1} |\Theta_j| \right].
\]
Since by assumption $0 \leq \mathbb{E}[\log(|\Theta_{0,\infty}|_1)] < \infty$, summing over $j$ yields, for $n \geq 1$,
\[
0 \leq \mathbb{E} \left[ \log(|\Theta_{-n,\infty}|_1) \right] \leq \mathbb{E}[\log(|\Theta_{0,\infty}|_1)] + \sum_{j=1}^n \mathbb{E} \left[ \frac{|\Theta_j|}{|\Theta_{1,\infty}|_1} \right] \leq \mathbb{E}[\log(|\Theta_{0,\infty}|_1)] + 1 < \infty.
\]
By monotone convergence, this proves that (3.30) implies $\mathbb{E}[\log(|\Theta_1|)] < \infty$ which by (3.34) implies (3.29).

As a first consequence, if (3.29) holds, then the identity (3.34) and Corollary 3.2 imply that
\[
0 \leq \sum_{j \in \mathbb{Z}} |\Theta_j| \log(|\Theta_j|^{-1} |\Theta_1|) < \infty, \quad \text{a.s.}
\]
and this in turn implies (3.31).

To prove the last statement, define the function $S$ on $\ell_1$ by $S(x) = \sum_{j \in \mathbb{Z}} x_j$. Some easy calculus (cf. Appendix B) yields the following properties: for all $x, y \in \ell_1$, such that $|S(x) - S(y)| \leq 1$, then
\[
|S(x) \log(|S(x)|) - S(y) \log(|S(y)|)| \leq 2 + \log_+(|x_1| \lor |y_1|).
\]
\[
\text{eq:borne-diff-slogs}
\]
Define a shift invariant, 1-homogeneous function $H$ on the subset of $\ell_1$ of sequences such that $\sum_{j \in \mathbb{Z}} |x_j| \log(|x_j|) < \infty$ by
\[
H(x) = S(x) \log(|S(x)|) - \sum_{j \in \mathbb{Z}} x_j \log(|x_j|).
\]
By (3.31), we know that $\Theta$ is in this set and since $|\Theta_0| = 1$, we have
\[
H(\Theta_{0,\infty}) - H(\Theta_{1,\infty}) = S(\Theta_{0,\infty}) \log(|S(\Theta_{0,\infty})|) - S(\Theta_{1,\infty}) \log(|S(\Theta_{1,\infty})|)
\]
Thus, (3.30) and (3.35) yield
\[
\mathbb{E}[|H(\Theta_{0,\infty}) - H(\Theta_{1,\infty})|] \leq 2 + \mathbb{E}[\log(|\Theta_{0,\infty}|_1)] < \infty.
\]
Thus condition (i) of Lemma 3.8 holds. Conditions (ii) and (iii) trivially hold under (3.31). Thus we can apply Lemma 3.8 to obtain (3.35).
3.4 Cluster indices

Let $H$ be a measurable shift invariant functional, homogeneous with degree 1, and continuous on the injection of $(\mathbb{R}^d)^k$ into $\ell_0$ for every $k \geq 1$. Then, we obtain by (1.1), 1-homogeneity of $H$ and continuous mapping that

$$\lim_{x \to \infty} \frac{\mathbb{P}(H(X_{1,k}) > x)}{\mathbb{P}(\|X_0\| > x)} = \frac{\mathbb{P}(H(x^{-1}X_{1,k}) > 1)}{\mathbb{P}(\|X_0\| > x)} = \nu_1, k(\{x \in (\mathbb{R}^d)^k : H(x) > 1\}) = \nu(\{y \in (\mathbb{R}^d)^\mathbb{Z} : H(y_{1,k}) > 1\}).$$

Let the limit on the left hand side or the expression in the right hand side be denoted by $b_k(H)$. For $d = 1$ and $H(x) = \sum_{j \in \mathbb{Z}} x_j$, the quantity $\lim_{k \to \infty} k^{-1}b_k(H)$ was called a cluster index of the time series $\{X_j, j \in \mathbb{Z}\}$ by Mikosch and Wintenberger (2014). We extend the notion of cluster index to a large class of functionals for which the limit $\lim_{k \to \infty} k^{-1}b_k(H)$ exists.

**Lemma 3.15.** Assume that $\mathbb{P}(\lim_{|t| \to \infty} |Y_t| = 0) = 1$. Let $H$ be a shift invariant, 1-homogeneous functional, continuous on the injection of $(\mathbb{R}^d)^k$ into $\ell_0$ for every $k \geq 1$ and such that $|H(y)| \leq C |y|_1$ for a constant $C > 0$ and all $y \in (\mathbb{R}^d)^\mathbb{Z}$. Then

$$b_{k+1}(H) - b_k(H) = \mathbb{E}[H_+^\alpha(\Theta_{0,k}) - H_+^\alpha(\Theta_{1,k})].$$

Assume moreover that $|H(x) - H(y)| \leq C |x - y|_1$ for all $x, y \in (\mathbb{R}^d)^\mathbb{Z}$. If $\alpha > 1$ assume in addition that

$$\mathbb{E} \left[ \sum_{j=0}^{\infty} |\Theta_j|^{\alpha-1} \right] < \infty.$$  \(\text{(3.37)}\)

Then,

$$\lim_{k \to \infty} \frac{b_k(H)}{k} = \mathbb{E}[H_+^\alpha(\Theta_{0,\infty}) - H_+^\alpha(\Theta_{1,\infty})] = \vartheta\mathbb{E}[H_+^\alpha(Q)].$$  \(\text{(3.38)}\)

**Proof.** Without loss of generality, we assume that $H$ is non-negative. By definition of the tail measure and by stationarity, we have

$$b_k(H) = \nu(\{H(y_{1,k}) > 1\}), \quad b_{k+1}(H) = \nu(\{H(y_{0,k}) > 1\}).$$

Note that $1\{H(y_{0,k}) > 1\} = 1\{H(y_{1,k}) > 1\}$ if $y_0 = 0$. Therefore, we can apply (2.1) and obtain

$$b_{k+1}(H) - b_k(H) = \int_{(\mathbb{R}^d)^\mathbb{Z}} (1\{H(y_{0,k}) > 1\} - 1\{H(y_{1,k}) > 1\}) 1\{y_0 \neq 0\} \nu(dy)$$

$$= \int_0^\infty \mathbb{E} [(1\{rH(\Theta_{0,k}) > 1\} - 1\{rH(\Theta_{1,k}) > 1\})] \alpha r^{-\alpha - 1} dr$$

$$= \mathbb{E}[H_+^\alpha(\Theta_{0,k}) - H_+^\alpha(\Theta_{1,k})].$$
We must now prove that the last expression has a limit when \( k \) tends to infinity. By (3.5) (if \( \alpha \leq 1 \)) and (3.37) (if \( \alpha > 1 \)), \( \sum_{j=0}^{\infty} |\Theta_j| < \infty \) almost surely, so the assumption on \( H \) implies that \( H(\Theta_{0,k}) \) converges almost surely to \( H(\Theta_{0,\infty}) \) which is well defined. Moreover, since \( |\Theta_0| = 1 \), we obtain

\[
|H^\alpha(\Theta_{0,k}) - H^\alpha(\Theta_{1,k})| \leq (\alpha \lor 1)C^\alpha \left( \sum_{j=0}^{\infty} |\Theta_j| \right)^{(\alpha-1)_+}.
\]

Thus, under assumption (3.37), the limit (3.38) holds by dominated convergence.

Remark 3.16. As noted in Example 2.1, the fact that we integrate a function which vanishes when \( y_0 = 0 \) is essential. The identity (3.36) was obtained in (Mikosch and Wintenberger, 2014, Lemma 3.1) by means of a rather lengthy proof which made repeated use of the definition of the spectral tail process and the time change formula.

Example 3.17. We pursue Example 3.13. If \( \Pr(\lim_{|t| \to \infty} |Y_t| = 0) = 1 \) and (3.37) hold, then we can apply Lemma 3.15 to the functionals of Example 3.13 and obtain

\[
\lim_{k \to \infty} \lim_{x \to \infty} \Pr(X_1 + \cdots + X_k > x) = \mathbb{E} \left[ \left( \sum_{j=0}^{\infty} \Theta_j \right)^\alpha - \left( \sum_{j=1}^{\infty} \Theta_j \right)^\alpha \right], \quad (3.39)
\]

\[
\lim_{k \to \infty} \lim_{x \to \infty} \frac{\Pr(\sup_{1 \leq j \leq k}(X_1 + \cdots + X_j) > x)}{k \Pr(|X_0| > x)} = \mathbb{E} \left[ \left( \sup_{k \geq 0} \sum_{j=0}^{k} \Theta_j \right)^\alpha - \left( \sup_{k \geq 1} \sum_{j=1}^{k} \Theta_j \right)^\alpha \right]. \quad (3.40)
\]

The identity (3.39) was proved for geometrically ergodic Markov chains by (Mikosch and Wintenberger, 2014, Theorem 3.2).

4 Convergence of clusters

The quantities studied in Section 3 appear as limits of so-called cluster functionals. To be precise, a cluster is a vector \( X_{1,r_n} = (X_1, \ldots, X_{r_n}) \) where \( \{r_n\} \) is a non decreasing sequence of integers such that \( \lim_{n \to \infty} r_n = \infty \). The vector \( X_{1,r_n} \) can be embedded in \((\mathbb{R}^d)^Z\) and cluster functional may be simply defined as measurable function \( H \) on \((\mathbb{R}^d)^Z\). Limiting theory for regularly varying time series relies fundamentally on the convergence of functionals of renormalized clusters, that is the convergence of

\[
\frac{\mathbb{E}[H(c_n^{-1}X_{1,r_n})]}{r_n \Pr(|X_0| > c_n)}, \quad (4.1)
\]

where \( \{c_n\} \) an increasing sequence such that \( \lim_{n \to \infty} c_n = \infty \) and

\[
\lim_{n \to \infty} r_n \Pr(|X_0| > c_n) = 0. \quad (4.2)
\]
The convergence of the quantity in (4.1) has been established under various conditions on the functions \( H \), in particular some form of shift invariance, and more essentially under the following so-called anticlustering condition, originally introduced in (Davis and Hsing, 1995, Condition (2.8)): for all \( u > 0 \),

\[
\lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P}\left( \max_{m \leq |i| \leq r_n} |X_i| > c_n u \mid |X_0| > c_n u \right) = 0 .
\]  

(4.3)

It is proved in (Basrak and Segers, 2009, Proposition 4.1) that (4.3) implies (3.1), i.e. \( \mathbb{P}\left( \lim_{|k| \to \infty} |Y_k| = 0 \right) = 1 \). In full generality, condition (4.3) cannot bring more information on the tail process since it is proved in Dębicki and Hashorva (2017) that for max-stable stationary processes with Fréchet marginal, (4.3) and (3.1) are equivalent; see Section 5.

In order to give a rigorous meaning to the convergence of clusters, following Basrak et al. (2016), we consider clusters as element of the space \( \tilde{\ell}_0 \) of shift equivalent sequences. More precisely, we say that \( \tilde{x} \sim \tilde{y} \) if there exists \( j \in \mathbb{Z} \) such that \( B^j \tilde{x} = \tilde{y} \). The space \( \tilde{\ell}_0 \) is the space of equivalence classes. It is readily checked that the space \( \tilde{\ell}_0 \) endowed with the metric \( \tilde{d} \) defined by

\[
\tilde{d}(\tilde{x}, \tilde{y}) = \inf_{\tilde{x} \in \tilde{x}, \tilde{y} \in \tilde{y}} |\tilde{x} - \tilde{y}|_\infty
\]

is a complete separable metric space. See (Basrak et al., 2016, Lemma 2.1).

Define the measure \( \nu_{n,r_n} \) on \( \tilde{\ell}_0 \) by

\[
\nu_{n,r_n} = \frac{\mathbb{P}\left( c_n^{-1} X_{1,r_n} \in \cdot \right)}{r_n \mathbb{P}(|X_0| > c_n)} .
\]

The convergence of the quantity in (4.1) can now be related to the convergence of the measure \( \nu_{n,r_n} \) on the space \( \tilde{\ell}_0 \setminus \{0\} \) in the following sense.

Let \( M_0 \) be the set of boundedly finite Borel measures on \( \tilde{\ell}_0 \setminus \{0\} \), that is Borel measures \( \mu \) such that \( \mu(A) < \infty \) for all Borel sets \( A \subset \tilde{\ell}_0 \) which are bounded away from \( 0 \) i.e. for which there exists \( \epsilon > 0 \) such that \( \tilde{x} \in A \) implies that \( \tilde{x}^* > \epsilon \).

Following Hult and Lindskog (2006) or (Kallenberg, 2017, Chapter 4), we say that a sequence of measures \( \mu_n \in M_0 \) converge to \( \mu \) in \( M_0 \) if \( \lim_{n \to \infty} \mu_n(f) = \mu(f) \) for all bounded continuous functions \( f \) on \( \tilde{\ell}_0 \setminus \{0\} \) with support bounded away from zero. As shown in (Kallenberg, 2017, Lemma 4.1), the class of test functions can be restricted to Lipschitz continuous functions. Let \( \nu^* \) be the measure defined on \( \ell_0 \) by

\[
\nu^*(H) = \vartheta \int_0^\infty \mathbb{E}[H(rQ)] \alpha r^{-\alpha-1} \, dr ,
\]

for non negative measurable functions \( H \) defined on \( \ell_0 \). Since \( Q^* = 1 \), the measure \( \nu^* \) is boundedly finite on \( \ell_0 \). By Proposition 3.6, if \( H \) is shift invariant then \( \nu^*(H) \) has the alternative expression

\[
\nu^*(H) = \int_0^\infty \mathbb{E}[H(r \Theta) 1\{I(\Theta) = 0\}] \alpha r^{-\alpha-1} \, dr .
\]
By a slight abuse of notation, in the following we consider $\nu^*$ as a measure on $\tilde{\ell}_0$. It is proved in (Basrak et al., 2016, Lemma 3.3) that if $X$ is a stationary regularly varying time series with tail measure $\nu$ and which satisfies condition (4.3), then $\nu_{n,r_n} \to \nu^*$ in $\mathcal{M}_0$. This convergence implies that for all bounded measurable shift invariant functions $H$ on $\ell_0$ (which can be identified with functions on $\tilde{\ell}_0$) with support bounded away from zero and almost surely continuous w.r.t. $\nu^*$,

$$\lim_{n \to \infty} \mathbb{E}[H(c_n^{-1}X_{1,r_n})] = \lim_{n \to \infty} \nu_{n,r_n}(H) = \nu^*(H).$$

This approach unifies and extends similar results in Basrak and Segers (2009) and Mikosch and Wintenberger (2014, 2016). The extension of the above convergence to unbounded functions or functions whose support is not bounded away from 0 can be obtained by usual uniform integrability arguments.

The previous results were proved under the anticlustering condition (4.3). In particular, as already mentioned, this always implies that the tail process tends to zero at infinity. However, whereas for most time series models (such as linear models or solutions to stochastic recurrence equations), it is relatively easily checked that the tail process tends to zero, proving condition (4.3) is relatively hard and may require very stringent conditions.

We next show that the anticlustering condition (4.3) is actually not essential. Recall that $\mathbb{P} \left( \lim_{|k| \to \infty} |Y_k| = 0 \right) = 1$ is equivalent to tail measure $\nu$ being supported on $\ell_0$.

**Lemma 4.1.** Let $X$ be a regularly varying time series with tail measure $\nu$ supported on $\ell_0$. Then for every sequence $\{c_n\}$ such that $\lim_{n \to \infty} c_n = \infty$, there exists a non decreasing sequence of integers $\{r_n\}$ such that (4.2) holds and $\nu_{n,r_n} \to \nu^*$ in $\mathcal{M}_0$.

**Proof.** For each integer $m \geq 1$ and for every non negative shift invariant function $H$ on $\ell_0$ such that $H(\mathbf{x}) = 0$ if $\mathbf{x}^* \leq \epsilon$ and continuous with respect to the distribution of $Y$, we have, by regular variation,

$$\lim_{n \to \infty} \frac{\mathbb{E}[H(c_n^{-1}X_{1,m})]}{m \mathbb{P}(|X_0| > c_n)} = \frac{\epsilon^{-\alpha} \sum_{j=1}^{m} \mathbb{E}[H(\epsilon Y_{1-j,m-j}) 1\{|Y_{1-j,m-j}| \leq 1\}]}{m}.$$

The limit is independent of $\epsilon$ and therefore defines a boundedly finite measure $\nu_m^*$ on $\tilde{\ell}_0 \setminus \{0\}$. By (Kallenberg, 2017, Lemma 4.1), it suffices to prove that for all bounded Lipshitz continuous (with respect to the uniform norm) functions $H$ on $\tilde{\ell}_0 \setminus \{0\}$ with support bounded away from zero, we have

$$\lim_{m \to \infty} \nu_m^*(H) = \nu^*(H).$$

The class of test functions can be further restricted to functions which depend only on coordinate greater than some $\eta > 0$. Indeed, let $T_\eta$ be the operator on $\ell_0$ which puts all coordinates no greater than $\eta$ to zero:

$$T_\eta(\mathbf{x}) = (\mathbf{x}_j 1\{|\mathbf{x}_j| > \eta\})_{j \in \mathbb{Z}},$$

**lem:anticlustering-useless**
and identify \( T_\eta \) to an operator on \( \tilde{\ell}_0 \setminus \{0\} \) in an obvious way. Then if \( H \) is Lipshitz continuous, there exists a constant \( C \) (depending only on \( H \)) such that for all \( x \in \ell_0 \setminus \{0\} \),
\[
|H(x) - H \circ T_\eta(x)| \leq C\eta .
\]
Moreover, \( H \circ T_\eta \) is almost surely continuous with respect to the distribution of \( Y \) since \( |Y_0| \) has a continuous distribution and is independent of \( \Theta \) so \( \mathbb{P}(\exists j \in \mathbb{Z}, |Y_j| = \eta) = 0 \) for all \( \eta > 0 \). Consider now a function \( H \) with support bounded away from \( 0 \), which depends only on coordinates greater than \( \eta \) (that is such that \( H = H \circ T_\eta \)) and almost surely continuous with respect to the distribution of \( Y \). Then we can write
\[
\nu^*_m(H) = \frac{\epsilon^{-\alpha}}{m} \sum_{j=1}^{m} \mathbb{E}[H(\epsilon Y_{1-j,m-j})1\{Y^*_1 \leq 1\}] = \epsilon^{-\alpha} \int_0^1 g_m(t)dt
\]
with \( g_m(t) = \mathbb{E}[H(\epsilon Y_{1-[mt],m-[mt]})1\{Y^*_{1-[mt],-1} \leq 1\}] \) (where \([x]\) denotes the smallest integer larger than or equal to the real number \( x \)). Since \( H \) is shift invariant, depends only on coordinates greater than \( \eta \) and \( \mathbb{P}(\lim_{|j| \to \infty} |Y_j| = 0) = 1 \), for every \( t \in (0,1) \), it holds that
\[
H(\epsilon Y_{1-[mt],m-[mt]})1\{Y^*_{1-[mt],-1} \leq 1\} = H(\epsilon Y)1\{Y^* \leq 1\}
\]
for large enough \( m \). Also, \( \lim_{m \to \infty} H(\epsilon Y)1\{Y^*_{1-[mt],-1} \leq 1\} = H(\epsilon Y)1\{Y^*_\infty \leq 1\} \).

Since \( H \) is bounded, we obtain by dominated convergence that
\[
\lim_{m \to \infty} g_m(t) = \mathbb{E}[H(\epsilon Y)1\{Y^*_\infty \leq 1\}]
\]
for all \( t \in (0,1) \). The functions \( g_m \) are uniformly bounded thus by dominated convergence again, we obtain
\[
\lim_{m \to \infty} \nu^*_m(H) = \epsilon^{-\alpha} \lim_{m \to \infty} \int_0^1 g_m(t)dt = \epsilon^{-\alpha} \mathbb{E}[H(\epsilon Y)1\{Y^*_\infty \leq 1\}] = \nu^*(H) .
\]

This proves that \( \nu^*_m \) converges to \( \nu^* \) in \( \mathcal{M}_0 \). Since convergence in \( \mathcal{M}_0 \) is metrizable (cf. (Hult and Lindskog, 2006, Theorem 2.3)), there exists a sequence \( r_n \) such that \( \nu_{r_n} \to \nu^* \).

As a consequence, we obtain for all \( u > 0 \),
\[
\lim_{n \to \infty} \frac{\mathbb{P}(X^*_{1,r_n} > c_n u)}{r_n\mathbb{P}(|X_0| > c_n)} = \vartheta u^{-\alpha} . \tag{4.4}
\]

This in turn implies that \( \lim_{n \to \infty} r_n\mathbb{P}(|X_0| > c_n) = 0 \). Otherwise \( r_n\mathbb{P}(|X_0| > c_n) \to c \in (0, \infty) \) (possibly along a subsequence) which implies that \( \mathbb{P}(X^*_{1,r_n} > c_n u) \to c\vartheta u^{-\alpha} \) for all \( u > 0 \). This is impossible since the latter quantity is greater than 1 for small \( u \). \qed
The convergence (4.4) was proved under condition (4.3) by Basrak and Segers (2009). Here, we have bypassed the anticlustering condition (4.3). The sequence \( \{r_n\} \) is not explicitly known, but neither is it when condition (4.3) is simply assumed as often happens in the literature.

We next show that the convergence \( \nu_{n,r_n} \to \nu^* \) is equivalent to (4.4) and convergence in distribution of the normalized block \( (X^{*}_{1,r_n})^{-1}X_{1,r_n} \) conditionally on \( X^{*}_{1,r_n} > c_nu \) in \( \tilde{\ell}_0 \) to the sequence \( Q \). Note that, since the convergence takes place in the space \( \tilde{\ell}_0 \), by Proposition 3.6 the limit has the same distribution as \( \Theta \) conditionally on \( I(\Theta) = 0 \).

**Lemma 4.2.** Let \( X \) be a stationary regularly varying time series with tail measure \( \nu \) supported on \( \ell_0 \) and let \( \{c_n\} \) and \( \{r_n\} \) be sequences satisfying (4.2). Then \( \nu_{n,r_n} \to \nu^* \) in \( M_0 \) if and only if for every \( u > 0 \) (4.4) holds and

\[
\mathcal{L} \left( (X^{*}_{1,r_n})^{-1}X_{1,r_n} \mid X^{*}_{1,r_n} > c_nu \right) \xrightarrow{w} \mathcal{L}(Q) \tag{4.5}
\]

as \( n \to \infty \) in \( \tilde{\ell}_0 \).

**Proof.** Assume first that for every \( u > 0 \), (4.4) and (4.5) hold. It suffices to show that then for every \( u > 0 \)

\[
\mathcal{L} \left( (c_nu)^{-1}X_{1,r_n} \mid X^{*}_{1,r_n} > c_nu \right) \xrightarrow{w} \mathcal{L}(Y \cdot Q) \tag{4.6}
\]

in \( \tilde{\ell}_0 \), where \( Y \) is a Pareto distributed random variable independent of \( Q \), since the fact that (4.4) and (4.6) imply \( \nu_{n,r_n} \to \nu^* \) follows as in (Basrak et al., 2016, Lemma 3.2). Fix \( u > 0 \) and take an arbitrary \( v \geq 1 \) and a Borel subset \( B \) of \( \tilde{\ell}_0 \setminus \{0\} \) such that \( \mathbb{P}(Q \in \partial B) = 0 \). Then by (4.4), (4.5) and regular variation of \( |X_0| \), as \( n \to \infty \)

\[
\mathbb{P} \left( X^{*}_{1,r_n} > c_nuv, (X^{*}_{1,r_n})^{-1}X_{1,r_n} \mid X^{*}_{1,r_n} > c_nu \right) = \frac{\mathbb{P} \left( X^{*}_{1,r_n} > c_nuv \right)}{\mathbb{P} \left( X^{*}_{1,r_n} > c_nu \right)} \cdot \mathbb{P} \left( (X^{*}_{1,r_n})^{-1}X_{1,r_n} \mid X^{*}_{1,r_n} > c_nuv \right) \to v^{-\alpha} \cdot \mathbb{P}(Q \in B).
\]

This implies that for every \( u > 0 \)

\[
\mathcal{L} \left( (c_nu)^{-1}X_{1,r_n}, (X^{*}_{1,r_n})^{-1}X_{1,r_n} \mid X^{*}_{1,r_n} > c_nu \right) \xrightarrow{w} \mathcal{L}(Y, Q)
\]

in \( (1, \infty) \times \tilde{\ell}_0 \) and (4.6) now follows by an continuous mapping argument.

For the converse, assume that \( \nu_{n,r_n} \to \nu^* \) in \( M_0 \). As already noted in the proof of Lemma 4.1, this implies that (4.4) holds for every \( u > 0 \). Fix an \( u > 0 \) and take an arbitrary bounded continuous function \( H \) on \( \tilde{\ell}_0 \). Note that the function \( \tilde{y} \mapsto H((\tilde{y}^{*})^{-1}\tilde{y})\mathbb{1}\{\tilde{y}^{*} > u\} \) on \( \tilde{\ell}_0 \setminus \{0\} \) is bounded, has support bounded away from 0 and is almost surely continuous.
with respect to $\nu^*$ since $\nu^*(\{\hat{y} : \hat{y}^* = u\}) = 0$ by the definition of $\nu^*$ and the fact that $Q^* = 1$. Now by the convergence $\nu_{n,r_n} \to \nu^*$ in $\mathcal{M}_0$ and (4.4), as $n \to \infty$

$$
\mathbb{E} \left[ H((X_{1,r_n})^{-1}X_{1,r_n} | X_{1,r_n} > c_n u) \right] = \frac{r_n \mathbb{P}(|X_0| > c_n)}{\mathbb{P}(X_{1,r_n} > c_n u)} \mathbb{E} \left[ H((X_{1,r_n})^{-1}X_{1,r_n}) 1 \{X_{1,r_n} > c_n u\} \right]
$$

It must be noted that if $r_n \to 0$ and $c_n \to 0$ then $\mathbb{E} e^{-N_n^I(f)} \to 0$.

Assume now that $n \mathbb{P}(|X_0| > c_n) \sim 1$ and that $\{c_n\}$ and $\{r_n\}$ satisfy the assumption of Lemma 4.1. Define $k_n = [n/r_n]$, $X_{n,i} = c_n^{-1} \{X(X(i-1)r_n+1, \ldots, X_ir_n)\}$, $i = 1, \ldots, k_n$ and the point process of clusters

$$
N_n^I = \sum_{i=1}^{k_n} \delta_{X_{n,i}}.
$$

The point process $N_n^I$ is a generalization introduced in Basrak et al. (2016) of the point process of exceedences $N_n = \sum_{k=1}^{n} \delta_{X_k/c_n}$ and of its functional version $N_n = \sum_{k=1}^{n} \delta_{i/n, X_k/c_n}$ considered in Davis and Hsing (1995) and Basrak et al. (2012). The convergence of these point processes is a central tool in obtaining limit theorems for heavy tailed time series.

The convergence of $N_n^I$ to a Poisson point process on $[0,1] \times \tilde{\ell}_0 \setminus \{0\}$ with mean measure $\text{Leb} \times \nu^*$ is proved in (Basrak et al., 2016, Theorem 3.6) under the anticlustering condition (4.3) and the following mixing condition:

$$
\mathbb{E} \left[ e^{-N_n^I(f)} \right] - \prod_{i=1}^{k_n} \mathbb{E} \left[ e^{-f(i/k_n, X_{n,i})} \right] \to 0,
$$

where $f$ is a continuous non negative function on $[0,1] \times \tilde{\ell}_0 \setminus \{0\}$ with support bounded away from $[0,1] \times \{0\}$ and $X_{n,i}$ is identified to an element of $\ell_0$. The condition (4.7) has been shown in (Basrak et al., 2016, Lemma 6.5) to hold under $\beta$-mixing and it probably also holds under $\alpha$-mixing. However, many processes of interest are neither $\beta$- nor $\alpha$-mixing, for instance, linear processes without stringent assumptions on the distribution of the innovation or long memory linear processes.

It must be noted that if $\nu_{n,r_n} \to \nu^*$ in $\mathcal{M}_0$, then condition (4.7) is a necessary and sufficient condition for the convergence of $N_n^I$ to a Poisson point process $N^I$ with mean measure $\text{Leb} \times \nu^*$ on $[0,1] \times \tilde{\ell}_0 \setminus \{0\}$. Indeed, the convergence $\nu_{n,r_n} \to \nu^*$ implies that

$$
\lim_{n \to \infty} \prod_{i=1}^{k_n} \mathbb{E} \left[ e^{-f(i/k_n, X_{n,i})} \right] = \mathbb{E} \left[ e^{-N^I(f)} \right].
$$
for all functions \( f \) as before (cf. (Resnick, 1987, Proposition 3.21)) and this is also the limit of \( \mathbb{E} \left[ e^{-N_n''(f)} \right] \) if \( N_n'' \) converges weakly to \( N'' \). Thus, the two quantities in (4.7) have the same limit and their difference tends to zero.

Consider again linear processes. Since the convergence \( N_n'' \) to \( N'' \) is known to hold for any \( r_n \) such that \( r_n \rightarrow \infty \) and \( r_n/n \rightarrow 0 \), (by an argument of \( m \)-dependent approximation, cf. (Basrak et al., 2016, Proposition 3.8)), thus (4.7) holds (for the same sequence \( r_n \) we get from Lemma 4.1) even when the linear process is not mixing.

In view of these remarks, it is not surprising that conditions (4.3) and (4.7) are relatively hard to check since they are nearly necessary and sufficient conditions for the point process convergence. Unfortunately, no more easily checked sufficient conditions (other than \( \beta \)-mixing) are known.

## 5 Max-stable processes with a given tail measure

In this section, we recall some connections between max-stable processes and spectral tail processes. In particular, we provide an alternative construction based on the tail measure of (Janßen, 2017, Theorem 3.2) which states that given a non negative process \( \Theta \) satisfying the time change formula and \( \mathbb{P}(\Theta_0 = 1) = 1 \), there exists a stationary max-stable process with spectral tail process \( \Theta \). For brevity, we only consider non negative real valued max-stable processes. The extension to the \( d \)-dimensional case is straightforward. We only consider the case \( \lim_{|j| \rightarrow \infty} \Theta_j = 0 \) here. The general (non negative) case is considered in (Janßen, 2017, Theorem 4.2). Further generalizations in connection with the tail measure are considered in Dombry et al. (2017).

We first recall some results about stationary max-stable processes with Fréchet marginals. Let \( \zeta \) be a max-stable process which admits the representation

\[
\zeta_j = \bigvee_{i=1}^{\infty} P_i Z_j^{(i)}, \quad j \in \mathbb{Z},
\]

where \( \{P_i, i \in \mathbb{N}\} \) are the points of a Poisson point process on \((0, \infty)\) with mean measure \( \alpha x^{-\alpha-1} dx \) and \( \{Z_j^{(i)}, j \in \mathbb{Z}\}, i \geq 1 \) are i.i.d. copies of a non negative process \( Z \) such that \( \mathbb{E}[Z_j^\alpha] = 1 \) for all \( j \in \mathbb{Z} \). The marginal distributions are standard \( \alpha \)-Fréchet and the condition for stationarity is that \( Z \) satisfies

\[
\mathbb{E} \left[ \bigvee_{i=s}^{t} \frac{Z_i^\alpha}{x_i^\alpha} \right] = \mathbb{E} \left[ \bigvee_{i=s}^{t} \frac{Z_{i-k}^\alpha}{x_i^\alpha} \right],
\]

for all \( k, s \leq t \in \mathbb{Z} \) and \( x_i > 0 \) for \( i = s, \ldots, t \). The marginal distribution of \( \zeta_0 \) is unit Fréchet, the process \( \zeta \) is regularly varying and it is proved in (Hashorva, 2016, Section 6.2) that the distribution of its spectral tail process \( \Theta \) is given, for all \( h \in \mathbb{Z} \) and bounded
measurable functions $F$ on $(\mathbb{R}^d)^Z$, by
\[\mathbb{E}[F(\Theta)] = \mathbb{E}[Z_h^+ F(B^h Z / Z_h) 1\{Z_h \neq 0\}] .\]  
{(5.2) \eq{Z-theta}}

It is also proved in (Hashorva, 2016, Section 6.2) that the distribution of $\zeta$ is characterized by its spectral tail process via the infargmax formula:
\[-\log \mathbb{P}(\zeta_j \leq y_j, j \in Z) = \sum_{h \in Z} \frac{1}{y_h} \mathbb{P}\left(\inf \arg \max_{j \in Z} \frac{\Theta_j}{y_j + h} = 0\right),\]  
{(5.3) \eq{infargmax-formula}}

where only finitely many of the positive numbers $y_j$ are finite.

Janßen (2017) proves that given a non negative sequence $\Theta$ which satisfies $\Theta_0 = 1$ and the time change formula, there exists a max-stable process $\zeta$ whose spectral tail process is $\Theta$. We provide a proof of this fact based on the tail measure when the tail process tends to zero.

Let $\Theta$ be a non negative sequence with satisfies the time change formula (2.4) and such that $\mathbb{P}(\Theta_0 = 1) = 1$ and $\lim_{|j| \to \infty} \Theta_j = 0$. Define $\vartheta = \mathbb{P}(I(\Theta) = 0)$ and the measure $\nu$ on $\mathbb{R}^Z$ by
\[\nu(H) = \sum_{j \in Z} \int_0^\infty \mathbb{E}[H(r B^j \Theta) 1\{I(\Theta) = 0\}] \alpha r^{-\alpha - 1} dr .\]  
{(5.4) \eq{def-nu}}

Let $\sum_{i \geq 1} \delta_{W(i)}$ be a Poisson point process on $[0, \infty)^Z$ with mean measure $\nu$. Define the max-stable process $\zeta$ by
\[\zeta_j = \bigvee_{i \geq 1} W(j)^{(i)} , \quad j \in Z .\]  
{(5.5) \eq{def-zeta}}

Let $Y_0$ be a Pareto random variable independent of $\Theta$ and define $Y = Y_0 \Theta$. Let $Q$ be as in Definition 3.5. The following result proves the existence of a max-stable process with a given spectral tail process and provides an M3 representation for it. For a review of the M3 representation of max-stable processes, see Dombry and Kabluchko (2016).

**Theorem 5.1.** The measure $\nu$ given by (5.4) is $\sigma$-finite, $\nu(\{0\}) = 0$, $\nu$ is homogeneous and shift invariant. The max-stable process $\zeta$ defined by (5.5) is stationary, has tail measure $\nu$, spectral tail process $\Theta$, extremal index $\vartheta > 0$ and it admits the M3 representation
\[\{\zeta_j, j \in Z\} \overset{d}{=} \{\bigvee_{i \geq 1} P_i Q_{j-T_i}^{(i)}, \quad j \in Z\} .\]  
{(5.6) \eq{M3-Q}}
where \(\sum_{i=1}^{\infty} \delta_{r_i}\) is a Poisson point process on \((0, \infty)\) with mean measure \(\alpha x^{-\alpha-1}dx\), \(Q^{(i)}\), \(i \geq 1\) are i.i.d. copies of the sequence \(Q\) and are independent of the previous point process and \(\sum_{i=1}^{\infty} \delta_{r_i}\) is a point process on \(\mathbb{Z}\) with mean measure \(\vartheta\) times the counting measure.

**Remark 5.2.** It seems that the link between the sequence \(Q\) and the M3 representation was not known.

**Proof.** The fact that \(\nu(\{0\}) = 0\), the homogeneity and shift-invariance of \(\nu\) are straightforward consequences of the definition. We prove that \(\nu\) is \(\sigma\)-finite. In view of homogeneity and shift-invariance, it suffices to prove that \(\nu(\{y_0 > 1\}) < \infty\). For a measurable \(A\), we have

\[
\nu(A \cap \{y_0 > 1\}) = \sum_{j \in \mathbb{Z}} \int_0^\infty \mathbb{P}(r B_j^\Theta \in A, r(B_j^\Theta)_0 > 1, I(\Theta) = 0) \alpha r^{-\alpha - 1}dr
\]

\[
= \sum_{j \in \mathbb{Z}} \mathbb{E} \left[ \int_0^\infty 1_{\{r B_j^\Theta \in A\}} 1_{\{r(B_j^\Theta)_0 > 1\}} 1_{\{I(B_j^\Theta) = j\}} \alpha r^{-\alpha - 1}dr \right].
\]

The function \(y \rightarrow \int_0^\infty 1_{\{ry \in A\}} 1_{\{ry_0 > 1\}} 1_{\{I(y) = j\}} \alpha r^{-\alpha - 1}dr\) is \(\alpha\)-homogeneous and is equal to zero if \(y_0 = 0\). Thus, applying the time change formula (2.6) yields

\[
\nu(A \cap \{y_0 > 1\}) = \sum_{j \in \mathbb{Z}} \mathbb{E} \left[ \int_0^\infty 1_{\{r \Theta \in A\}} 1_{\{r \Theta_0 > 1\}} 1_{\{I(\Theta) = j\}} \alpha r^{-\alpha - 1}dr \right]
\]

\[
= \mathbb{E} \left[ \int_0^\infty 1_{\{r \Theta \in A\}} 1_{\{r \Theta_0 > 1\}} \alpha r^{-\alpha - 1}dr \right]
\]

\[
= \mathbb{E} \left[ \int_1^\infty 1_{\{r \Theta \in A\}} \alpha r^{-\alpha - 1}dr \right] = \mathbb{P}(Y \in A).
\]

Taking \(A = \mathbb{R}^\mathbb{Z}\) yields \(\nu(\{y_0 > 1\}) = 1\).

By Theorem 3.1, to prove that \(\nu\) is the tail measure of \(\zeta\), it suffices to prove that \(Y\) is its tail process. By definition of \(\zeta\), we have, for \(y \in [0, \infty]^\mathbb{Z}\) with finitely many finite coordinates, as \(u \to \infty\),

\[
\mathbb{P}(\zeta \in u[0, y] \mid \zeta_0 > u) = \frac{e^{-u^{-\alpha}} \nu([0, y]^c)}{1 - e^{-u^{-\alpha}}}
\]

\[
\to \nu(\{y_0 > 1\} \cup [0, y]^c) - \nu([0, y]^c)
\]

\[
= \nu(\{y_0 > 1\} \cap [0, y]) = \mathbb{P}(Y \in [0, y]).
\]

This proves that \(Y\) is the tail process of \(\zeta\) and that \(\nu\) is the tail measure of \(\zeta\).

To prove (5.6), it suffices to note that for \(x \in [0, \infty]^\mathbb{Z}\) with only finitely many finite coordinates, denoting \(\xi\) the process in the right hand side of (5.6), we have

\[-\log \mathbb{P}(\xi_j \leq x_j, j \in \mathbb{Z}) = \vartheta \sum_{j \in \mathbb{Z}} \int_0^\infty \mathbb{P}\left(r \sum_{j \in \mathbb{Z}} \frac{Q_{j-i}}{x_j} \leq 1\right) \alpha r^{-\alpha - 1}dr = \nu(\{y, y_j \leq x_j, j \in \mathbb{Z}\}).\]
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**Appendix**

**A Proof of the equivalence between (2.3) and (2.4)**

Assume first that (2.3) holds. It suffices to prove (2.4) for a non negative measurable functional $H$, homogeneous with degree 0. Applying (2.2) and the monotone convergence theorem, we obtain

$$
\lim_{t \to 0} E[H(B^kY)1\{|Y_k| > t\}] = \lim_{t \to 0} E[H(B^k\Theta)1\{|\Theta_k| > t\}]
$$

$$
= E[H(B^k\Theta)1\{|\Theta_k| > 0\}].
$$

On the other hand, applying again (2.2) and the monotone convergence theorem yields

$$
\lim_{t \to 0} t^{-\alpha}E[H(tY)1\{|Y_k| > 1/t\}] = \lim_{t \to 0} t^{-\alpha}E\left[H(\Theta) \int_1^{\infty} 1\{r |\Theta_k| > 1/t\} \alpha r^{-\alpha-1}dr\right]
$$

$$
= \lim_{t \to 0} E[H(\Theta) |\Theta_k|^\alpha]
$$

$$
= E[H(\Theta) |\Theta_k|^\alpha].
$$

Since we started from quantities which are equal by (2.3), this proves (2.4) for a 0-homogeneous functional.
Conversely, assume that (2.4) holds and let $H$ be a non negative measurable functional and $t > 0$. Then by (2.1)

\[
\mathbb{E}[H(B^kY)\mathbb{1}\{|Y_k| > t\}]
= \int_1^\infty \mathbb{E}[H(rB^k\Theta)\mathbb{1}\{r|\Theta_k| > t\}]\alpha r^{-\alpha-1}dr
= \int_1^\infty \mathbb{E}[H(rB^k\Theta)\mathbb{1}\{r|B^k\Theta|_0 > t\}]\mathbb{1}\{|\Theta_k| \neq 0\}\alpha r^{-\alpha-1}dr
= \int_1^\infty \mathbb{E}[H(r|\Theta_k|^{-1}\Theta)\mathbb{1}\{r|\Theta_k|^{-1}|\Theta_0| > t\}]|\Theta_k|^\alpha\alpha r^{-\alpha-1}dr
= t^{-\alpha}\int_1^\infty H(tu\Theta)\mathbb{1}\{u|\Theta_k| > 1/t\}\alpha u^{-\alpha-1}du = t^{-\alpha}\mathbb{E}[H(tY)\mathbb{1}\{|Y_k| > 1/t\}],
\]

where the last line was obtained by the change of variable $u|\Theta_k|t = r$. Thus (2.3) holds.

\section*{B Proof of (3.35)}

Let $g$ be defined on $\mathbb{R}$ by $g(x) = x \log(|x|)$ with the convention $0 \log 0 = 0$. Then $|g(x)| \leq 1$ for $x \in [-1, 1]$ and if $|x| \vee |y| \geq 1$ and $|x - y| \leq 1$, which implies that $x$ and $y$ are of the same sign and $||x| - |y|| = |x - y| \leq 1$, we have

\[
|g(x) - g(y)| = \int_{|x|\wedge|y|}^{||x|\vee|y||} (1 + \log(s))ds \leq ||x| - |y||(|\log(|x| \vee |y|) + 1) \leq \log(|x| \vee |y|) + 1.
\]

For $x, y \in \ell_1$ such that $|S(x) - S(y)| \leq 1$, this yields

\[
|g(S(x)) - g(S(y))| \\
\leq 2 \cdot \mathbb{1}\{|S(x)| \vee |S(y)| \leq 1\} + (\log(|x|_1 \vee |y|_1) + 1)\mathbb{1}\{|S(x)| \vee |S(y)| \geq 1\} \\
\leq 2 + \log_+ (|x|_1 \vee |y|_1).
\]

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