ABSTRACT

Our study probes the impact of an exponentially decaying/growing time-dependent pressure gradient on unsteady Dean flow in a curved concentric cylinder. A two-step method of solution has been employed in the treatment of the governing momentum equation. Accordingly, the exact solution of the time-dependent partial differential equation is derived in terms of the Laplace variable. The Laplace domain solution is then transformed to the time domain using a numerical inversing scheme known as Riemann-sum approximation. The effect of the various dimensionless parameters involved in the problem on the Dean velocity, skin drags and Dean vortex are illustrated graphically. It was established that maximum Dean velocity is due to an exponentially growing time-dependent pressure gradient. However, the instability of the Dean vortex is rendered less effective by reducing time and applying an exponentially decaying time-dependent pressure gradient.

1. Introduction

Theoretical and experimental studies of unsteady flow with oscillating circumferential pressure gradient (Dean flow) has gained considerable attention over the past decades ascribed to its latest applications in biomedical engineering, biofluid mechanics, hemodynamics and mechanical engineering.

Early studies can be traced back to the work of Dean (1928), Goldstein (1938), Uchida (1956) and Bhatnagar (1975). Analytical study of viscous incompressible fluid flow due to an oscillating pressure gradient in a straight circular pipe with rigid walls was reported by Tsangaris (1984). Gupta, Poulikakos, and Kurtcuoglu (2008) presented analytical solution responsible for pulsatile viscous flow driven by a harmonically oscillating pressure gradient in a straight elliptic annulus with application to the motion of the cerebrospinal fluid in the human spinal cavity.

In an attempt to understand the effect of an imposed circumferential pressure gradient on oscillatory flow, several authors such as Dryden, Murnaghan, and Bateman (1956), Richardson and Tyler (1929), Hamza (2017), Gupta and Gupta (1996), Fan and Chao (1965) and Haslam and Zamir (1998) have studied extensively oscillatory flow in various geometries under different physical phenomenon.

Tsangaris, Kondaxakis, and Vlachakis (2006) scrutinized laminar fully developed flow in the region across two coaxial cylinders with an imposed oscillating circumferential pressure gradient (Finite gap oscillating Dean flow). In another related work, Tsangaris and Vlachakis (2007) examined laminar fully developed flow between two coaxial cylinders. In their work, they presented analytical solution of the equations of motion of a Newtonian fluid by taking into consideration the effect of oscillating pressure gradient on the flow formation. The exact solution for unsteady rotating flow of a generalized Maxwell fluid in an infinite straight circular cylinder with oscillating pressure gradient was reported by Zheng, Li, Zhang, and Gao (2011). They concluded that as time increases, the velocity increases as it attains its maximum, before decreasing and the oscillating pressure gradient leads to the fluctuations of the velocity.

It can be said that the theory of an exponential pressure gradient flow through channels as well as annulus helps in better understanding many technological and industrial problems. This phenomenon generates, in general, pressure gradient flow which is not constant but pulsates in some way about a non–zero pressure gradient.

In view of that, Yen and Chang (1961) examined the effect of time-dependent pressure gradient on magnetohydrodynamic flow in a channel in which three cases of time-dependent pressure gradient were considered, namely periodic, step and pulse pressure gradient. A numerical investigation was
carried out by Azad and Andallah (2017) in order to evaluate the unsteady flow of one-dimensional Navier-Stokes Equation (NSE) with time-dependent pressure gradient. Other related articles can be seen in references (Manos, Marinakis, & Tsangaris, 2006; McGinty, McKee, & McDermott, 2009; Mendiburu, Carrocci, & Carvalho, 2009; Mishra & Roy, 1968; Womersley, 1955).

In respect to the foregoing investigation, several experimental and analytical studies has been carried out to better understand this phenomenon. Jha and Yusuf (2018) conducted a semi-analytical investigation on transient pressure driven flow in a composite annulus due to the sudden application of azimuthal pressure gradient. The solutions of the governing momentum equations are derived using the Laplace transform technique and Riemann-sum approximation (RSA) as a tool for numerical inversion. Using a similar method of solution, Jha and Yahaya (2019a) performed an investigation on fully developed laminar transient flow formation in the region between two horizontally stationary impermeable concentric cylinders. The flow is driven by the applied circumferential pressure gradient in the annular gap. They found out that as time passes, the velocity is enhanced as it progressively attains steady state. Jha and Yahaya (2019b) further extended the work to the case when the walls of the cylinders are porous. They reported that in addition to the results obtained from their previous work (see Jha & Yahaya, 2019a) the velocity profile decreases with increase in suction/injection parameter. Yusuf and Gambo (2019), Yusuf and Gambo (2020) also adopted the use of Riemann-sum approximation for numerical inversion. Lately, reports on the action of an oscillating and exponential time-dependent pressure gradient on Dean formation was established by Jha and Gambo (2020a, 2020b) and Jha & Gambo (2020b).

Recent developments relating to the study can be found in Wazwaz (2007), Biazar, Badpeima, and Azimi (2009), Majeed, Zeeshan, and Ellahi (2016), Zeeshan, Shehzad, Abbas, and Ellahi (2019), Majeed et al. (2019), Marin, Ellahi, Vlase, and Bhatti (2020), Maitama and Zhao (2020), Yusuf, Gambo, and Olaife (2020), Jha and Yahaya (2020), Jafek et al. (2020), and Nikdoost and Rezaei (2020).

Despite the series of survey carried out, to the best of the authors’ knowledge no work has been done in order to semi-analytically study the role of an exponentially decaying/growing time-dependent pressure gradient on Dean formation. Motivated by its captivating applications, we set out to examine the effects of an exponentially decreasing/increasing time-dependent pressure gradient on Dean formation to obtain an empirical result for the Dean velocity, skin frictions and vortices of the fluid in a curvilinear cylinder. It is anticipated that our results can be beneficial in accurate design and fabrication of real time propulsion systems in the fields of bio-fluids mechanics and engineering.

2. Mathematical formulation

The unsteady fully developed flow of a viscous incompressible fluid in a curvilinear annulus formed by two infinite coaxial cylinders is considered in this study. The two cylinders are assumed to be fixed and the fluid is Newtonian. The axis of the cylinder is taken in the Z’ direction. The radii of the inner and outer cylinder are a and b respectively as depicted in Figure 1. Initially, at time \( t = 0 \), the fluid is assumed to be at rest. At \( t > 0 \), the flow is triggered by the applied exponential time-dependent pressure gradient \( \frac{e^{-\lambda t}}{r} \) in the annular region \( b - a \) (\( b > a \)). Two types of dependence on time are considered; one, exponentially decaying and the other exponentially growing.

The Navier-Stokes Equations (NSE) for unsteady flow of an incompressible fluid with constant viscosity \( \mu \), are used as a system with the continuity equation, both written in polar coordinates. By assuming fully developed flow \( (v_r = 0) \), the continuity equation is satisfied when developed flow condition across the Z’-axis of the annulus are valid for the function of the radial coordinate and time only, \( v = v(r', t') \). Using the corresponding Navier-Stokes Equation (NSE), the following partial differential equation can be written as:

\[
\begin{align*}
\frac{\partial (r' v_r)}{\partial r'} + \frac{\partial v}{\partial t} &= 0 \\
\rho \left( \frac{\partial v_r}{\partial t'} + v_r \frac{\partial v_r}{\partial r'} + v \frac{\partial v_r}{\partial \theta' r} - \frac{v^2}{r} \right) &= -\frac{\partial P}{\partial r'} \\
+ \mu \left( \frac{\partial^2 v_r}{\partial r'^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta'^2} \frac{v_r}{r'} - 2 \frac{\partial v}{\partial r'} \right) &= e^{-\lambda t'} \frac{\partial P}{\partial \theta'} \\
+ \mu \left( \frac{\partial^2 v}{\partial r'^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta'^2} \frac{v}{r'} + 2 \frac{\partial v}{\partial r'} \right) &= e^{-\lambda t'} \frac{\partial P}{\partial \theta'} \\
\end{align*}
\]

As earlier stated, the cylinders are of infinite length and the flow is fully developed \( (v_r = 0) \), since our interest is in the flow due to an exponentially decaying/growing time-dependent pressure gradient, following Tsangaris and Vlachakis (2007) and Jha and Gambo (2020a, 2020b) the momentum and continuity equations in equations (1)-(3) can be fashioned out into the following forms:

\[
\rho \frac{v^2}{r'} = \frac{\partial P}{\partial r'}
\]
The Laplace parameter, the expressions for equations (4) and (5) can be written in the dimensionless form as:

\[
\frac{\partial V}{\partial t} - \frac{e^{-\delta_0 t}}{\rho V} \frac{\partial P}{\partial \varphi} + \mu \left( \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial V}{\partial \rho} - \frac{V}{\rho^2} \right) = 0
\]  

with relevant boundary conditions

\[
\begin{align*}
V & = 0 \text{ at } R = 1 \\
\frac{\partial V}{\partial \varphi} & = 0 \text{ at } R = \lambda
\end{align*}
\]  

Following the approach of Tsangaris and Vlachakis (2007), Jha and Yahaya (2019a), Jha and Yahaya (2019b), Jha and Gambo (2020a), and Jha and Gambo (2020b), the non-homogeneous linear differential equation in equation (10) can be reduced using the given transformation below:

\[
\frac{\partial^2 \bar{V}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \bar{V}}{\partial \rho} - \frac{1}{R} \left( 1 + \frac{s R^2}{\lambda^2} \right) \bar{V} = \frac{1}{R s (s + \delta)}
\]  

where \( \bar{V}(R, s) \) is the homogeneous solution of equation (10).

Under boundary conditions (11), the exact solution of equation (10) in the Laplace domain is obtained by substituting the homogeneous solution of equation (10) into equation (12) and is given below as:

\[
\bar{V}(R, s) = A_1 I_1 \left( \lambda \sqrt{s} \right) + A_2 K_1 \left( \lambda \sqrt{s} \right) + \frac{1}{R s (s + \delta)}
\]  

where \( A_1 = \frac{K_1 \left( \lambda \sqrt{s} \right) - \lambda^{-1} K_1 \left( \sqrt{s} \right)}{s (s + \delta) [I_1 \left( \lambda \sqrt{s} \right) - I_1 \left( \sqrt{s} \right) K_1 \left( \lambda \sqrt{s} \right)]} \)

and \( A_2 = \frac{\lambda^{-1} I_1 \left( \sqrt{s} \right) - I_1 \left( \lambda \sqrt{s} \right) K_1 \left( \sqrt{s} \right)}{s (s + \delta) [I_1 \left( \lambda \sqrt{s} \right) - I_1 \left( \sqrt{s} \right) K_1 \left( \lambda \sqrt{s} \right)]} \).

The skin frictions at the cylinder surfaces in the Laplace domain are obtained by differentiating equation (13) as follows:

\[
\tau(R, \lambda) = R \frac{d}{dR} \left( \frac{\bar{V}(R, s)}{R} \right) \bigg|_{R=1} = \sqrt{s} \left[ A_1 I_2 \left( \lambda \sqrt{s} \right) - A_2 K_2 \left( \lambda \sqrt{s} \right) \right] - \frac{2}{s (s + \delta)}
\]

The fluid vorticity otherwise known as Dean vortex \( \tau(R, s) \) which is produced in annular region due to...
the rotation of the fluid as a result of the curved nature of the geometry is computed by differentiating equation (13) and is given below as:

\[
\tau(R, s) = \frac{1}{R^2 \partial R} \left( R^2 s (R s) \right) = \sqrt{s} \left[ A_1 s^2 - A_2 s^3 \right] - \frac{2}{R^2 (s + \delta)}
\]  

(16)

2.3. Riemann-sum approximation (RSA)

It is paramount to note that the equations (13)-(16) are in the Laplace domain and are to be inverted in order to determine the Dean velocity, skin frictions and vorticity in the time domain. Due to the complex nature of these solutions, a numerical

Table 1. Comparison of the present results obtained using the Riemann-sum approximation approach (RSA) with Jha and Yahaya (2019a) for the transient state velocity.

| $t$ | $R$ | Present work RSA ($\delta = 0$) | Jha and Yahaya (2019a) Exact solution |
|-----|-----|---------------------------------|--------------------------------------|
| 0.2 | 1.2 | 0.0530                          | 0.0600                               |
|     | 1.4 | 0.0707                          | 0.0813                               |
|     | 1.6 | 0.0645                          | 0.0745                               |
|     | 1.8 | 0.0402                          | 0.0460                               |
| 0.4 | 1.2 | 0.0591                          | 0.0600                               |
|     | 1.4 | 0.0800                          | 0.0813                               |
|     | 1.6 | 0.0733                          | 0.0745                               |
|     | 1.8 | 0.0453                          | 0.0460                               |
| 3.0 | 1.2 | 0.0600                          | 0.0600                               |
|     | 1.4 | 0.0813                          | 0.0813                               |
|     | 1.6 | 0.0745                          | 0.0745                               |
|     | 1.8 | 0.0460                          | 0.0460                               |

Table 2. Comparison of the present results obtained using the Riemann-sum approximation approach (RSA) with Jha and Yahaya (2019a) for the transient state skin frictions.

| $t$ | $\lambda$ | Skin friction at $R = 1$ Present work RSA ($\delta = 0$) | Jha and Yahaya (2019a) Exact solution | Skin friction at $R = \lambda$ Present work RSA ($\delta = 0$) | Jha and Yahaya (2019a) Exact solution |
|-----|-----------|----------------------------------------------------------|--------------------------------------|----------------------------------------------------------|--------------------------------------|
| 0.2 | 1.2       | 0.0968                                                   | 0.0967                               | 0.0857                                                   | 0.0856                               |
|     | 1.4       | 0.1871                                                   | 0.1870                               | 0.1496                                                   | 0.1495                               |
|     | 1.6       | 0.2706                                                   | 0.2713                               | 0.1982                                                   | 0.1987                               |
|     | 1.8       | 0.3390                                                   | 0.3502                               | 0.2292                                                   | 0.2376                               |
| 0.4 | 1.2       | 0.0968                                                   | 0.0967                               | 0.0858                                                   | 0.0856                               |
|     | 1.4       | 0.1871                                                   | 0.1870                               | 0.1496                                                   | 0.1495                               |
|     | 1.6       | 0.2714                                                   | 0.2713                               | 0.1988                                                   | 0.1987                               |
|     | 1.8       | 0.3499                                                   | 0.3502                               | 0.2373                                                   | 0.2376                               |
| 3.0 | 1.2       | 0.0967                                                   | 0.0967                               | 0.0856                                                   | 0.0856                               |
|     | 1.4       | 0.1870                                                   | 0.1870                               | 0.1495                                                   | 0.1495                               |
|     | 1.6       | 0.2713                                                   | 0.2713                               | 0.1987                                                   | 0.1987                               |
|     | 1.8       | 0.3502                                                   | 0.3502                               | 0.2376                                                   | 0.2376                               |

Figure 2. Velocity distribution for different values of time ($\delta = -2.0$).
procedure used in Jha and Yusuf (2018), Jha and Yahaya (2019a, 2019b), Yusuf and Gambo (2019, 2020), and Jha and Gambo (2020a, 2020b) known as the Riemann-sum approximation (RSA) which is notable for its accuracy is applied to invert Equations (13)-(16) to the time domain. Here, any function of the Laplace domain can be transformed to time domain as follows:

Figure 3. Velocity distribution for different values of time ($\delta = 2.0$).

Figure 4. Variation of skin friction at $R = 1$ for different values of time ($\delta = -2.0$).
Figure 5. Variation of skin friction at $R = 1$ for different values of time ($\delta = 2.0$).

Figure 6. Variation of skin friction at $R = \lambda$ for different values of time ($\delta = -2.0$).
\[ V(R, t) = \frac{e^{\text{st}}}{t} \left[ V(R, \epsilon) + \text{Re} \left( \sum_{n=1}^{M} V \left( R, \epsilon + \frac{\text{i} \pi n}{t} \right) (-1)^n \right) \right] \]
\[ 1 \leq R \leq \lambda \]  
(17)

\[ \tau(R, t) = \frac{e^{\text{st}}}{t} \left[ \tau(R, \epsilon) + \text{Re} \left( \sum_{n=1}^{M} \tau \left( R, \epsilon + \frac{\text{i} \pi n}{t} \right) (-1)^n \right) \right] \]
\[ 1 \leq R \leq \lambda \]  
(18)

\[ \eta(R, t) = \frac{e^{\text{st}}}{t} \left[ \eta(R, \epsilon) + \text{Re} \left( \sum_{n=1}^{M} \eta \left( R, \epsilon + \frac{\text{i} \pi n}{t} \right) (-1)^n \right) \right] \]
\[ 1 \leq R \leq \lambda \]  
(19)

Where \( \text{Re} \) is the real part of the summation, \( i = \sqrt{-1} \) the imaginary number, \( M \) is the number of terms involve in the summation and \( \epsilon \) is the real part of the Bromwich contour that is used in inverting Laplace transform. The Riemann-sum approximation (RSA) for the Laplace inversion involves a single summation for the numerical computation, of which its exactness is dependent on the value of \( \epsilon \) and the truncation error prescribed by \( M \). Following Tzou (1997), taking \( \epsilon t \) to be 4.7 ensures the stability of the computation.

In order to further establish the accuracy of the numerical inversion technique used in transforming the Laplace domain solution to time domain, Tables of comparison are presented between the result of the present analysis and previously established result given by Jha and Yahaya (2019a) when the coefficient of the time-dependent pressure gradient is taken to be zero \( (\delta = 0) \). A perfect match is observed when the numerical values obtained using Riemann-sum approximation at large time is compared with the exact solutions reported by Jha and Yahaya (2019a) (See Tables 1 and 2).

### 3. Results and discussion

A semi-analytical investigation on the influence of an exponentially decaying/growing time-dependent pressure gradient on Dean flow in a horizontal curved coaxial cylinders has been carried out. In order to have an insight into the physical problem, a MATLAB program is written to determine and generate line graphs and numerical values for the velocity, skin frictions and vorticity. The flow is governed by the annular gap \( (\lambda) \), time \( (t) \) and the coefficient of time in the exponentially decaying/growing time-dependent pressure gradient \( (\delta) \). Our present analysis has been performed over a reasonable range of values with the coefficient of time in the exponentially decaying/growing time-dependent pressure gradient taken over \(-2.0 \leq \delta \leq 2.0 \) and time \( 0.06 \leq t \leq 0.2 \). In the course of this study, \( \delta > 0 \) has been used to represent an exponentially decaying pressure gradient, whereas \( \delta < 0 \) has been used to simulate a growing pressure gradient in the annular gap. The effect of the pertinent parameters on the flow formation are depicted pictorially in Figures 2–9.
Figures 2 and 3 show the influence of time and an exponentially decaying/growing time-dependent pressure gradient on the fluid velocity. With a growing pressure gradient, it is observed that there is an increase in the fluid velocity as time passes as seen in Figure 2. In Figure 3, the velocity increases.

Figure 8. Dean vortex profile for different values of time ($\delta = -2.0$).

Figure 9. Dean vortex profile for different values of time ($\delta = 2.0$).
with time and a decaying pressure gradient. It is evident that steady state is attained faster when an exponentially decaying pressure gradient is applied. Furthermore, maximum velocity is borne out of an increasing time and an exponentially increasing pressure gradient. As anticipated, the role of an exponentially time-dependent pressure gradient is to enhance Dean velocity. This finding is key in fabricating devices conveying biofluids such as diaphragm pumps and dialysis machine which are propelled by pressure rather than convective current.

The effects of an exponentially decaying/growing time-dependent pressure gradient on skin friction at the outer surface of the inner cylinder at different values of time are presented in Figures 4 and 5. Result shows that as the exponential time-dependent pressure gradient increases/decreases, the drag on the inner wall is seen to increase. Although, the magnitude of the skin friction is seen to be higher when the time-dependent pressure gradient is growing exponentially.

On the other hand, Figures 6 and 7 exhibit the influence of an exponentially decaying/growing time-dependent pressure gradient on skin friction at the inner surface of the outer cylinder for different values of time. It is concluded that skin friction is enhanced with a growing exponential pressure gradient and slowly drops as the coefficient of time in the pressure gradient decrease. This is attributed to the fact that a decaying exponential pressure gradient retards the fluid flow and thus, diminishing the drag on the wall as time passes.

The resultant effects of Dean vortices due to fluid rotation in the annular gap when an exponential time-dependent pressure gradient is imposed are demonstrated in Figures 8 and 9. It is interesting to note that as time increases, the instability of the Dean vortex is more pronounced towards the wall of the inner cylinder when an increasing/decreasing exponential time-dependent pressure gradient is applied. However, a subtle behaviour is seen towards the wall of the outer cylinder. In addition, the instability of the Dean vortex is more pronounced with an exponentially growing time-dependent pressure gradient as seen in Figure 8.

4. Conclusion

A two-dimensional mathematical model has been analysed semi-analytically in order to study the effect of an exponentially decaying/growing time-dependent pressure gradient on Dean formation. A combination of Laplace transforms technique and a numerical approach known as Riemann-sum approximation (RSA) has been used to derive the transient solution of the problem under consideration. The effect of dimensionless parameters governing the flow has been illustrated graphically. We concluded that Dean velocity and skin friction can be maximized by inducing an exponentially growing time-dependent pressure gradient. In addition, as time passes, the drag on the walls can be amplified by increasing the coefficient of time in the time-dependent pressure gradient. Furthermore, the instability of the Dean vortices can be minimized by applying an exponentially decaying time-dependent pressure gradient with relatively a small amount of time.

Nomenclature

- $a$: Radius of the inner cylinder (m)
- $b$: Radius of the outer cylinder (m)
- $P$: Static pressure (Kg/m^3)
- $R$: Dimensionless radius
- $s$: Laplace parameter
- $t$: Dimensionless time (s)
- $U_0$: Reference velocity (m/s)
- $v_r$: Radial velocity (m/s)
- $v$: Circumferential velocity (m/s)
- $V$: Dimensionless velocity

Greek letters

- $\delta$: Coefficient of time-dependent pressure gradient
- $\lambda$: Radii ratio (b/a)
- $\rho$: Fluid density (Kg/m^3)
- $\eta$: Dean vortex
- $\tau$: Skin friction
- $\mu$: Dynamic viscosity of the fluid (Kg/ms)

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