A MULTISTAGE STOCHASTIC PROGRAMMING FRAMEWORK FOR CARDINALITY CONSTRAINED PORTFOLIO OPTIMIZATION

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Abstract. This paper presents a multistage stochastic programming model to deal with multi-period, cardinality constrained portfolio optimization. The presented model aims to minimize investor’s expected regret, while ensuring achievement of a minimum expected return. To generate scenarios of market index returns, a random walk model based on the empirical distribution of market-representative index returns is proposed. Then, a single index model is used to estimate stock returns based on market index returns. Afterward, historical returns of a number of stocks, selected from Frankfurt Stock Exchange (FSE), are used to implement the presented scenario generation method, and solve the stochastic programming model. In addition, the impact of cardinality constraints, transaction costs, minimum expected return and predetermined investor’s target wealth are investigated. Results show that the inclusion of cardinality constraints and transaction costs significantly influences the investors risk-return tradeoffs. This is also the case for investors target wealth.

1. Introduction. Following the pioneering work of Markowitz [24], modern portfolio theory has been introduced as one of the main finance areas. During the last six decades, a large number of researchers have dealt with this area, and extended the Markowitz’s model form different viewpoints, to provide appropriate guidelines for institutional and individual investors. While the investment horizon is often more than one period, early studies in this area focus on single period portfolio selection models. However, the capital allocation to investment assets should be regarded as a dynamic decision making problem, whose planning horizon is often more than one time period.

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Mulvey et al. [25] emphasize the important role of multi-period portfolio selection models, regarding the presence of transaction costs, temporal dependence of asset returns and the role of liabilities in investment. These issues make multi-period portfolio selection models more consistent with real world applications than single period ones. In this paper, a multistage stochastic programming model is proposed to deal with multi-period portfolio optimization under real world assumptions. In the proposed model, investor’s risk is measured by expected regret risk metric, and proportional transaction costs, minimum transaction lots, cardinality constraints, and restrictions on the proportion of each asset in the portfolio are considered. To the best of our knowledge, previous studies do not formulate a multistage stochastic portfolio optimization model with the above-mentioned features. In addition, we extend the scenario generation method, proposed by Şakar and Köksalan [27], as unlike their study, we do not assume the returns of market index to be normally distributed. More precisely, our presented approach generates scenarios of stock returns based on empirical distributions of market index returns. This assumption is imperative, since experience shows that the distributions of stock returns are rarely normal, and follow fat-tailed distributions. In this regard, new random walk models based on Johnson transformations are proposed, and combined with a single index model to generate scenarios of stock returns. From now on in this section, a literature review about different aspects of the problem under consideration is proposed.

Stochastic programming is a prominent approach that handles uncertainties in multi-period portfolio optimization problems. To provide investors with appropriate investment decisions, it needs a set of scenarios that capture the uncertain behavior of financial and economic factors in forward periods [27].

Dantzig and Infanger [8] use multistage stochastic linear programs to deal with the multi-period portfolio optimization problem, with factor models as well as Markovian type processes to generate scenarios of asset returns. Ji et al. [21] propose a multistage stochastic linear goal programming formulation to cope with the multi-period portfolio optimization problem, with a linear moment matching model as well as Vector Auto-Regression (VAR) to generate scenarios of asset returns. Chen [4] develops a stochastic programming model to handle multi-period consumption and investment problems with short sales and proportional transaction costs. The study uses the MGARCH model to provide an appropriate description of the time-varying behavior of stock returns.

Numerous studies use stochastic programming models to deal with portfolio optimization problem with different types of assets. Barro and Canestrelli [1] formulate and solve multistage stochastic programming models to replicate an index as close as possible. They numerically test the proposed model by dynamically replicating the MSCI Euro index. Ferstl and Weissensteiner [12] formulate a multistage stochastic linear program to deal with cash management problem, using a riskless asset (cash), several default- and option-free bonds, and an equity investment. Their study considers interest rates and equity returns as uncertain parameters, and uses no-arbitrage interest rate models and moment matching for scenario generation. Consiglio and Staino [7] propose a multistage stochastic programming model to select portfolios of bonds, minimizing the cost of the issuance of government bonds. Rocha and Kuhn [26] propose a multistage stochastic mean-variance optimization model for the management of a portfolio of electricity derivative contracts, whose role is to hedge the financial risk imposed by deregulation of electricity markets.
Some of previous studies use stochastic programming to hedge against market and currency risk in international portfolios. Topaloglou et al. [30,31], and Yin and Han [33,34] incorporated derivatives in stochastic programming models to jointly allocate capital to international markets, select assets within each market, and determine appropriate hedging levels. Their studies shows that the stochastic programming framework provides a flexible and effective decision support tool for international portfolio management. Davari-Ardakani et al. [11] present a multistage stochastic programming model to explore the benefits of inclusion of option contracts to domestic portfolios of underlying assets. They assessed different option strategies to overall risk management on Greek letters.

In real world financial markets, there exist important features such as transaction costs, minimum transaction lots, cardinality constraints, and restrictions on the proportion of each asset in the portfolio to lie between lower and upper bounds [23]. Regarding these features, many studies investigate and formulate different portfolio selection problems, and solve them by developing numerous heuristic and metaheuristic algorithms. We only indicate some of these studies for brevity reasons. Gupta et al. [16] develop a credibilistic multi-objective expected value portfolio selection model with fuzzy parameters and cardinality constraints. They consider short-term return, long-term return, risk and liquidity as key financial criteria. Liu and Zhang [17] formulate a multi-period mean-semivariance fuzzy portfolio selection model, considering transaction costs, diversification degree, cardinality constraint and minimum transaction lots. The proposed model maximizes the terminal wealth and minimizes the cumulative risk over the whole investment horizon. Chen [3] presents a new mixed-integer possibilistic mean-semiabsolute deviation portfolio selection model with real-world constraints, namely transaction costs, cardinality and quantity constraints. Zhang [35] presents an interval mean-average absolute deviation model for multi-period portfolio selection by taking into account risk control, transaction costs, borrowing, threshold and cardinality constraints. The study generates an optimal investment policy to help investors achieve an optimal return, and have a good risk control.

Scenario generation is a non-detachable part of studies dealing with stochastic optimization models for portfolio optimization, since it provides a realistic view of market returns in the future, with which an investor can make informed investment decisions [11]. Høyland and Wallace [18] develop a nonlinear programming model to generate outcomes of uncertain variables, such that the distance between some specified properties of the generated outcomes, e.g. moments, and their target is minimized. Afterward, numerous studies used this method, individually or in combinations with other methods. For instance, one can refer to Ji et al. [21], Fleten et al. [13], Høyland et al. [19], Gpiniar et al. [15], Geyer et al. [14] and Chen and Xu [5]. In most cases, distributions of asset returns are leptokurtic. In other words, the probability distributions of asset returns exhibit fatter tails than those of normal distributions. However, a number of studies, e.g. Ji et al. [21], assume that asset returns are normally distributed. Saka and Köksalan [27] use scenarios of a market index as inputs of a single index model to generate scenarios of stock returns. In this regard, they assume that returns of market index are normally distributed, and use a random walk model to generate scenarios of index returns. As mentioned above, the normality assumption fails to hold in most financial markets. Davari-Ardakani et al. [9,10,21], without imposing any assumption on marginal distributions of asset returns, take advantage of Johnson transformation and Cholesky decomposition to
generate scenarios, such that marginal distributions of historical returns of assets are preserved.

As mentioned above, in this paper, a multistage stochastic programming model is proposed to deal with multi-period portfolio optimization under real world assumptions, namely presence of proportional transaction costs, minimum transaction lots, cardinality constraints, and restrictions on the proportion of all risky and risk-free assets. In addition, a random walk model, proposed based on Johnson transformation, is combined with a single index model to develop a new method that, unlike most previous studies as Şakar and Kóksalan [27], generates scenarios of stock returns, independently from the distributions of returns. To the best of our knowledge, the proposed multistage stochastic portfolio optimization model and the scenario generation method have not been addressed in previous studies.

The remainder of this paper is organized as follows. Section 2 presents a multistage stochastic programming model for multi-period portfolio optimization, considering features of real world financial markets. In Section 3, a new scenario generation method based on a random walk model and a single index model is proposed. In Section 4, a practical application of the proposed framework, the presented stochastic programming model as well as the developed scenario generation method, on Frankfurt Stock Exchange (FSE) is provided, and computational results are discussed. Finally, Section 5 concludes the paper.

2. The proposed multistage stochastic programming formulation. To incorporate features of real world financial markets in the multi-period portfolio selection problem, we use a multistage stochastic programming framework, utilizing a particular modelling of random parameters in the context of linear programming.

2.1. Assumptions. An investor with an initial wealth is considered, who sets a lower bound on his/her expected return, aiming to minimize his/her expected regret, during a predetermined planning horizon. This helps the investor control the bankruptcy risk during intermediate periods [21]. In addition, proportional transaction costs, minimum transaction lots, cardinality constraints, and restrictions on the proportion of each asset in the portfolio are considered. Moreover, short selling is not allowed. The uncertainty of asset prices is represented in terms of a scenario tree, as shown in Fig. 1.

2.2. Notations. Scenario tree notation

\[ N \] the set of nodes of the scenario tree
\[ n \in N \] a typical node of the scenario tree
\[ N_T \in N \] the set of leaf nodes of the scenario tree, at the last period T
\[ N_t \in N \] the set of nodes of the scenario tree, at the periods \( t = 0, 1, 2, \ldots, T \)
\[ p(n) \] the unique predecessor node of node \( n \in N \setminus \{0\} \)
\[ P_n \] the probability of the state associated with node \( n \in N \)
\[ \zeta (n) \] the time period associated with node \( n \in N \)

Sets

\[ I \] the set of stocks
\[ i \in I \] index of an stock

Deterministic input data
\[ W_0 \] investor’s initial wealth
Figure 1. The schematic representation of a scenario tree for $T$ periods

- $\nu$: proportional transaction cost for sales or purchases of stock $i \in I$
- $k$: maximum number of assets in the portfolio
- $u_i$: maximum proportion of investment in stock $i$
- $l_i$: minimum proportion of investment in stock $i$
- $r$: risk free interest rate
- $\zeta$: the target expected return of the portfolio
- $M$: a large number

Scenario dependent parameters
- $\theta^n$: investors target wealth at node $n \in N \setminus \{0\}$
- $\mu^n$: investors level of risk aversion at node $n \in N \setminus \{0\}$
- $u_b^n$: maximum investment allowed in the risk free asset at node $n \in N$
- $\Psi^n_i$: market price of asset $i \in I$ at node $n \in N$ multiplied by the size of transaction lots

Decision variables
- $x^n_i$: amount of stock $i \in I$ in terms of number of transaction lots purchased at node $n \in N$
- $y^n_i$: amount of stock $i \in I$ in terms of number of transaction lots sold at node $n$
\[ h_i^n \quad \text{amount of stock } i \in I \text{ in terms of number of transaction lots held at node } n \in N \]

\[ m^n \quad \text{amount of amount of investment in the risk free asset at node } n \in N \]

\[ \omega^n \quad \text{binary variable which will be 1 if stock } i \in I \text{ is held in the portfolio at node } n \in N \text{ and 0 otherwise} \]

**Auxiliary variables**

\[ C^n \quad \text{auxiliary variable to make the objective function linear at node } n \in N \{0\} \]

\[ V^n \quad \text{investor’s wealth at node } n \in N \]

### 2.3. Objective function

The aim of this paper is to minimize the investors expected regret, while ensuring achievement of a minimum expected return during the planning horizon. Such an approach helps investor(s) control the bankruptcy risk in intermediate periods.

\[
\min \left( \sum_{t} \sum_{n \in N \setminus \{0\}} \mu_n p_n \max(\theta^n - V^n, 0) (1 + r)^t \right)
\]

The objective function considers the minimization of the present value of the expected downside deviation of the investor’s wealth from a target level \( \theta^n \) (expected regret) during the planning horizon. To linearize the objective function, the term \( \max(\theta^n - V^n, 0) \) is replaced with the auxiliary variable \( C^n \) and two additional constraints namely \( C^n \geq \theta^n - V^n \) and \( C^n \geq 0 \) are appended to the set of constraints.

### 2.4. Constraints

At the root node, the investors initial wealth \( W_0 \) is invested in stocks and risk-free asset, regarding Eq. 2. In addition to the stock prices, the investor should pay proportional transaction cost, when he/she purchases stocks.

\[
\sum_{i \in I} x_0^i \Psi_i^0 (1 + \nu) + m^0 = W_0
\]

Eq. 3 ensures the balance for quantities of all stocks during the planning horizon. The holding amount of any stock on each node is calculated as the holding amount of that stock on the predecessor node plus the net amounts of purchasing (purchasing amount minus selling amount) that stock.

\[
h_i^{p(n)} + x_i^n - y_i^n = h_i^n \quad i \in I, n \in N \setminus \{N_T \cup 0\}
\]

In the cash balance Eq. 4, the principal and profits of the risk free investment in the previous time period and the fund provided by selling stocks are used to invest in the risk free asset and stocks. For both of selling and purchasing stocks, the investor should pay the proportional transaction cost. The interest accrued on the former investment in the risk-free asset is also considered.

\[
\sum_{i \in I} y_i^n \Psi_i^n (1 - \nu) + m^{p(n)} (1 + r) = \sum_{i \in I} x_i^n \Psi_i^n (1 + \nu) + m^n \quad n \in N \setminus \{0\}
\]

Eq. 5 and Eq. 6 guarantee a minimum level of expected return of the portfolio, regarding investor’s terminal wealth.

\[
\sum_{n \in N_T} p_n V^n / W_0 \geq \zeta
\]
\[ V^n = \sum_{i \in I} h^n \Psi^n + m^n \quad n \in N \] (6)

Eq. 7 and Eq. 8 ensure that the total number of assets held at each node are limited to a predefined number.

\[ \frac{h^n_i}{M} \leq \omega^n_i \leq h^n_i \quad i \in I, n \in N \] (7)

\[ \sum_{i \in I} \omega^n_i \leq k \quad n \in N \] (8)

Eq. 9 restricts the proportion of each asset in the portfolio to lie between lower and upper bounds.

\[ l_i V^n - M(1 - \omega^n_i) \leq h^n_i \Psi^n_i \leq u_i V^n + M(1 - \omega^n_i) \quad i \in T, n \in N \] (9)

Eq. 10 and Eq. 11 define nonnegative and binary variables.

\[ x^n_i \geq 0, y^n_i \geq 0, h^n_i \geq 0, m^n \geq 0 \quad i \in I, n \in N \] (10)

\[ \omega^n_i = \begin{cases} 1 & i \in I, n \in N \\ 0 & \end{cases} \] (11)

3. Scenario generation. The accurate prediction of financial asset prices is important for investors to make appropriate investment decisions. This issue is closely related to the efficient market hypothesis [27]. The efficiency of stock markets has been widely studied by many researchers.

Some studies investigate efficiency of German stock market as one of the largest markets in the world. Cheung and Lai [6] use a modified R/S test and a fractional differencing test to study market efficiency of major European stock markets in Austria, Belgium, Denmark, France, Germany, Italy, Netherlands, Norway, Spain, Sweden, Switzerland and the United Kingdom. Their study finds no evidence of long memory in these markets. Jacobsen [20] uses the modified S/R statistic, and shows that none of the return series of indexes of five major European stock markets in Netherlands, Germany, the United Kingdom, Italy and France exhibit long memory. The efficiency of German market is confirmed by other studies, as [22,32]. In agreement with the above discussion, we assume that German stock market is efficient, and hence, its market returns follow a random walk model. In this regard, we must assume that asset prices change randomly in consecutive periods. Applying the random walk model, returns of a market representative index are generated, in terms of a scenario tree. Then, Sharpe’s single index model is used to derive stock returns from market index returns.

3.1. The random walk model. An exclusive characteristic of financial asset returns is that their probability distributions often exhibit fatter tails than those of normal distributions [9,11]. However, many studies, e.g. Ji et al. [21], assume that marginal distributions of asset returns are normally distributed. As another instance, Şakar and Kôksalan [27] use a random walk model to generate returns of a market index, based on the assumption that returns are normally distributed. Considering marginal distribution of market index returns, we adopt a random walk model to extend the scenario generation method, proposed by Şakar and
Köksalan [27]. We take advantage of Johnson transformation to generate scenarios of stock returns, based on the distribution of market index returns. There are three types of Johnson transformation, namely bounded system ($S_B$), log-normal system ($S_L$) and unbounded system ($S_U$). Eq.12-14 show these three types of Johnson transformation, where, $z$ represents the transformed value, $\gamma$ and $\eta$ denote shape parameters and $\varepsilon$ and $\lambda$ denote location and scale parameters.

\begin{align*}
    z &= \gamma + \eta \ln \left( \frac{x - \varepsilon}{\lambda + \varepsilon - x} \right) \tag{12} \\
    z &= \gamma + \eta \ln \left( \frac{x - \varepsilon}{\lambda} \right) \tag{13} \\
    z &= \gamma + \eta \text{arcsinh} \left( \frac{x - \varepsilon}{\lambda} \right) \tag{14}
\end{align*}

To generate returns of the market representative index, the historical non-normal returns of the index are transformed to the standard normal distribution. This needs to select the most appropriate Johnson transformation system. Then, a random walk model based on standard normal distribution in combination with the inverse of the selected Johnson transformation are used to generate scenarios of market index returns, such that the marginal distribution of market index returns is preserved. Eq.15-17 show the random walk models, proposed based on Johnson transformations, denoted by Eq. 12-14 respectively.

\begin{align*}
    r_t &= \varepsilon + \left( \lambda + \varepsilon \right) e^\frac{zt - \gamma}{\eta} \tag{15} \\
    r_t &= \varepsilon + \lambda e^\frac{zt - \gamma}{\eta} \tag{16} \\
    r_t &= \varepsilon + 0.5\lambda(e^\frac{zt - \gamma}{\eta} - e^{-\frac{zt - \gamma}{\eta}}) \tag{17}
\end{align*}

where, $r_t$ and $z_t$ are the market index return and a standard normal term at time $t$, respectively.

3.2. The single index model. Sharpe’s single index model investigates the relationship between each pair of securities via comparing each security to a macroeconomic factor. In case of a stock market, the macroeconomic factor may be a market representative index. Using regression analysis, one can relate returns of stocks to returns of the market index. Mathematically, the single index model [27] is expressed as follows:

\begin{equation}
    r_{it} = \alpha_i + \beta_i r_{Mt} + \varepsilon_{it} \tag{18}
\end{equation}

where, $r_{it}$ and $r_{Mt}$ are the return of stock $i$ and the return of the market index at time $t$, respectively. In addition, $\beta_i$ is the responsiveness of the stock $i$ to the market return, and $\alpha_i$ is the return of stock $i$ independent of the return of the market index. Moreover, $\varepsilon_{it}$ are residual returns with mean zero and the finite standard deviation $\sigma$. Residual returns are assumed to be independent identically distributed. We use the single index model to generate returns of stocks from returns of a market representative index.
3.3. The proposed approach. Following the above discussion about random walk and single index models, steps of the proposed scenario generation method are as follows:

Step 1: Transform historical stock prices to stock returns. Eq. 19 transforms a price series to a return series:

\[ r_{it} = \frac{P_{it}}{P_{i,t-1}} - 1 \]  

where, \( r_{it} \) is the return of asset \( i \) in period \( t \) and \( P_{it} \) is the price of asset \( i \) in period \( t \). This should also be performed for market index returns.

Step 2: Apply Johnson transformations, mentioned in Eq. 12-14, to historical returns of the market representative index, and select the best model to transform their marginal distribution to the standard normal distribution. Johnson transformation helps us generate scenarios of market index returns, without identification of marginal distribution of return series. Hence, it eliminates errors associated with fitting a marginal distribution on market index returns. To choose the best Johnson transformation, normality tests are utilized.

Step 3: Use the appropriate random walk model to generate scenarios of market index returns. In this regard, for each node of the scenario tree, generate a random standard normal number \( z_t \). Then, based on the Johnson transformation selected in step 2, use the appropriate random walk model, Eq. 15-17, to generate a return datum for the market index. This process should be repeated until a scenario tree with an arbitrary number of nodes is provided.

Step 4: For each stock, regress the stock return on market index return, and estimate alpha and beta values of the single index model, Eq. 18.

Step 5: Use single index models, obtained in step 4, to convert the scenario tree of market index returns to a scenario tree of stock returns.

Step 6: Transform the scenario tree of stock returns to a scenario tree of stock prices.

Fig. 2 provides a schematic representation of the proposed scenario tree generation method.

4. Computational results. In this section, the proposed scenario tree generation method is implemented to generate scenarios of prices for stocks selected from Frankfurt Stock Exchange (FSE). Then, the scenario tree is utilized to solve the proposed scenario based portfolio optimization model, and present computational results.

4.1. Generating scenarios of stock prices. We use monthly prices of 13 stocks, of different sectors, from Feb 1, 2008 to December 1, 2014. In addition, we use CDAX, a composite index of stocks traded in the Frankfurt Stock Exchange (FSE). We obtain the adjusted closing price of stocks as well as CDAX from finance.yahoo.com, and use MATLAB 7.9, Minitab 16 and GAMS 22.2 to implement the scenario tree generation method, and solve the proposed model.

To generate scenarios of asset returns, Eq. 19 is used to transform stock prices to stock returns. Similarly, returns of the stock market index, CDAX, are calculated. Table 1 shows descriptive statistics of historical CDAX returns.
Table 1. Descriptive statistics of historical CDAX returns

| Mean       | Standard Deviation | Median | Minimum | Maximum | Skewness | Kurtosis |
|------------|--------------------|--------|---------|---------|----------|----------|
| 0.0060     | 0.0569             | 0.0103 | -0.1795 | 0.1745  | -0.5381  | 1.7814   |

Figure 2. A schematic representation of the proposed scenario tree generation method
We perform an analysis about the probability distribution of historical CDAX returns. Distribution fitting results confirm our former discussion about the fat-tailed distributions of returns. In fact, goodness of fit test results reject the assumption that historical CDAX returns follow a normal distribution with 95% confidence (p-value = 0.015). Hence, Johnson Transformation is used to transform marginal distributions of historical index returns to the standard normal distribution. In this case, a high p-value, 0.931, is obtained. Thus, we cannot reject the assumption that transformed index returns follow a normal distribution. The Johnson transformation and its estimated parameters are as follows:

$$z_t = 0.407408 + 1.39206\text{arcsinh}\left(\frac{r_t - 0.0291861}{0.0600348}\right) \quad (20)$$

where $r_t$ and $z_t$ are original and transformed market index returns at time $t$, respectively.

After determining the appropriate Johnson transformation and its estimated parameters, scenario generation for index returns can be performed. We consider a multistage scenario tree with 3 stages and 10 branches from each node. Thus, the scenario tree has $(1 + 10 + 10^2 + 10^3 = 1111)$ nodes. Since we assume that returns follow a random walk model, for each node we independently sample from a standard normal distribution. Then, Eq. (21) is used to transform all standard normal samples to scenarios of CDAX returns.

$$r_t = 0.0291861 + 0.0300174\left(e^{\frac{z_t - 0.407408}{1.39206}} - e^{-\frac{z_t - 0.407408}{1.39206}}\right) \quad (21)$$

Afterward, we carry out regression analysis to estimate parameters of the single index model for all stocks. Table 2 shows $\alpha_i$ and $\beta_i$ values of the single index model for all stocks. Then, for all stocks, estimated single index models are used to transform index returns to stock returns. Now, scenarios of stocks returns are simply transformed to scenarios of stock prices.

4.2. Model implementation. After implementing the scenario tree generation method, the scenario tree is utilized to implement the multistage portfolio optimization model. An investor with $1000000$ initial wealth is considered, aiming to minimize his/her expected regret regarding a target wealth, subject to achievement of a minimum expected return.

We implement the multistage portfolio optimization model with different combinations of target wealth values, minimum expected returns, and proportional transaction costs. Thus, the model should be implemented several times. Table 3 shows investor’s expected regret, obtained by solving the model, considering different levels of target wealth, minimum expected return and proportional transaction costs. As shown in table 3, minimum expected return, target wealth level and proportional transaction costs have direct relationships with investor’s risk. Of course,

### Table 2. $\alpha_i$ and $\beta_i$ values of the single index model for all stocks

| Stock | B&G | LRS | LTEC | MZA | NEC1 | N2X | OTP |
|-------|-----|-----|------|-----|------|-----|-----|
| Intercept | 0.015231 | 0.0008692 | -0.0028 | 0.039533 | -3.1E-05 | 0.001772 | -0.01099 |
| Slope | 0.756845 | 1.211379 | 0.889253 | 1.837928 | 0.644086 | 0.971493 | 1.961487 |

| Stock | SIE | TA8 | BMW | XCY | O4B | ZYT |
|-------|-----|-----|-----|-----|-----|-----|
| Intercept | -0.00063 | 0.006095 | 0.0098 | 0.024254 | 0.001565 | 0.003712 |
| Slope | 1.091311 | 0.292933 | 1.186136 | 0.592039 | 0.564903 | 1.498048 |
for some combinations, the problem has no feasible solution, let alone an optimal one.

Similar analysis can be performed for the multistage portfolio optimization model with and without cardinality constraints. Table 4 focuses on the role of cardinality constraints on investor’s expected regret, considering different levels of target wealth and minimum expected return. Since cardinality constraints remarkably shrink the feasible region, they increase the level of investor’s risk. Again, for some combinations of target wealth and minimum expected return, feasible region is an empty set.

Table 5 shows results of sensitivity analysis on transaction cost and number of assets, in terms of investors expected regret, setting target wealth as $1100000. As shown in this table, when maximum number of assets, allowed to be in the portfolio, increases, the investors expected regret decreases. This is due to the fact that this enlarges the feasible region of the model, and hence, reduces the investors risk. Fig.3 shows the risk C return relationship for portfolios with and without cardinality constraints (Target wealth = $1000000). Note that investor’s risk is expressed in terms of $1000000. As shown in this figure, when cardinality constraints are added to the portfolio optimization model, if the minimum expected return is set to a value more than 0.08, the model would be infeasible. This is the case in the absence of cardinality constraints for an expected return greater than 0.13. In addition, in all cases, given a minimum expected return, the cardinality constraint portfolio optimization model leads to a more risky situation. Fig.4 compares the risk C return relationship obtained by setting different levels of target wealth ($1000000 and $1050000) for portfolios with and without cardinality constraints. Note that investor’s risk is expressed in terms of $1000000. In both cases, given a minimum expected return, a higher level of target wealth leads to a higher degree of risk imposed to the investor. Fig.5 shows the impact of proportional transaction costs on investor’s risk. As shown in this figure, given a minimum expected return,
Table 3. Investor’s expected regret considering different target wealth, minimum expected return and proportional transaction costs

| Target wealth | 1000000 | 1050000 | 1100000 |
|---------------|---------|---------|---------|
| 0.95          | 0       | 0.01    | 0.02    | 0       | 0.01    | 0.02    | 0       | 0.01    | 0.02    |
| 0.99          | 0       | 0       | 0       | 63429.6 | 85112.7 | 103678.9 | 157646.9 | 196749.4 | 223658.1 |
| 1.01          | 0       | 1290.1  | 2987.9  | 63429.6 | 85112.7 | 103678.9 | 157646.9 | 196749.4 | 223658.1 |
| 1.03          | 36.1    | 3953.9  | 9018.7  | 63429.6 | 85112.7 | 103678.9 | 157646.9 | 196749.4 | 223658.1 |
| 1.04          | 400.0   | 5567.1  | 12654.5 | 63429.6 | 85112.7 | 103678.9 | 157646.9 | 196749.4 | 223658.1 |
| 1.05          | 1142.6  | 7705.6  | 17739.6 | 63429.6 | 85112.7 | 103678.9 | 157646.9 | 196749.4 | 223658.1 |
| 1.06          | 2350.1  | 11121.8 | 26179.7 | 63429.6 | 85134.9 | 109571.7 | 157646.9 | 196749.4 | 223838.3 |
| 1.07          | 3904.2  | 15907.5 | 37152.7 | 63429.6 | 87879.1 | 118868.3 | 157646.9 | 198064.3 | 226873.5 |
| 1.08          | 5954.9  | 22562.9 | 49694.6 | 63429.6 | 95349.1 | 129104.5 | 157646.9 | 202106.9 | 232405.6 |
| 1.09          | 8525.3  | 36300.4 | 66271.2 | 63694.1 | 106415.1 | 140267.7 | 157646.9 | 208788.3 | 240209 |
| 1.10          | 12086.2 | 54243.7 | 88133.5 | 64838.1 | 120626.0 | 157811.5 | 157675.5 | 217600.2 | 251621.1 |
| 1.11          | 17279.9 | 74827.6 | -       | 67119.8 | 138148.1 | -       | 158591.5 | 228534.1 | -       |
| 1.12          | 24358.6 | 98255.0 | -       | 74520.5 | 159434.9 | -       | 163337.6 | 242911.9 | -       |
| 1.13          | 52656.5 | -       | -       | 104774.3 | -       | -       | 185917.5 | -       | -       |
| 1.14          | -       | -       | -       | -       | -       | -       | -       | -       | -       |
| Target wealth |
|---------------|
| Cardinality Constraints | No Cardinality Constraints |
|-------------------------|
| 0.95 | 0 | 81742.31 | 192944.5 | 0 | 63429.62 | 157646.9 |
| 0.99 | 0 | 81742.31 | 192944.5 | 0 | 63429.62 | 157646.9 |
| 1 | 0 | 81742.31 | 192944.5 | 0 | 63429.62 | 157646.9 |
| 1.01 | 0 | 81742.31 | 192944.5 | 0 | 63429.62 | 157646.9 |
| 1.02 | 197.428 | 81742.31 | 192944.5 | 0 | 63429.62 | 157646.9 |
| 1.03 | 893.094 | 81742.31 | 192944.5 | 36.097 | 63429.62 | 157646.9 |
| 1.04 | 2574.389 | 81742.31 | 192944.5 | 399.947 | 63429.62 | 157646.9 |
| 1.05 | 5261.672 | 81742.31 | 192944.5 | 1142.637 | 63429.62 | 157646.9 |
| 1.06 | 8917.349 | 82803.35 | 192944.5 | 2350.142 | 63429.62 | 157646.9 |
| 1.07 | 18358.44 | 87336.35 | 193443 | 3904.241 | 63429.62 | 157646.9 |
| 1.08 | 35077.99 | 96174.55 | 198126.4 | 5954.918 | 63429.62 | 157646.9 |
| 1.09 | - | - | - | 8525.318 | 63694.05 | 157646.9 |
| 1.10 | - | - | - | 12086.15 | 64838.09 | 157675.5 |
| 1.11 | - | - | - | 17279.88 | 67119.82 | 158591.5 |
| 1.12 | - | - | - | 24358.63 | 74520.5 | 163337.6 |
| 1.13 | - | - | - | 52656.51 | 104774.3 | 185917.5 |
| 1.14 | - | - | - | - | - | - |

Table 4. Investor’s expected regret considering different target wealth and minimum expected return with and without cardinality.
Table 5. Investor’s expected regret considering different proportional transaction costs and number of assets

| Number of assets | 6         | 0.01     | 0.02     | 12         | 0.01     | 0.02     |
|------------------|----------|----------|----------|------------|----------|----------|
| Proportional transaction cost | 0.95     | 229314.2 | 275191.5 | 318875.4  | 192944.5 | 225372.1 | 258752.1 |
|                   | 0.99     | 229314.2 | 275191.5 | 318875.4  | 192944.5 | 225372.1 | 258752.1 |
|                   | 1.01     | 229314.2 | 275191.5 | 318875.4  | 192944.5 | 225372.1 | 258752.1 |
|                   | 1.03     | 229314.2 | 275191.5 | 318875.4  | 192944.5 | 225372.1 | 258752.1 |
|                   | 1.04     | 229314.2 | 275191.5 | 318875.4  | 192944.5 | 225372.1 | 258752.1 |
|                   | 1.05     | 229314.2 | 275191.5 | 318875.4  | 192944.5 | 225372.1 | 258752.1 |
|                   | 1.06     | 229314.2 | 277364.2 | 338961.2  | 192944.5 | 225372.1 | 258752.1 |
|                   | 1.07     | 229314.2 | 280367.1 | -         | 192944.5 | 230553.7 | 263452.1 |
|                   | 1.08     | 229314.3 | -        | -         | 192944.5 | 235638.9 | -        |
|                   | 1.09     | 231175.9 | -        | -         | 193443.0 | 239987.4 | -        |
|                   | 1.10     | -        | -        | -         | 198126.4 | -        | -        |
|                   | 1.11     | -        | -        | -         | -        | -        | -        |

Figure 4. Risk vs. expected return obtained by setting different levels of target wealth ($1000000 and $1050000) for portfolios with and without cardinality constraints.

Increasing the proportional transaction cost, would lead to higher investor’s risk. Note that investor’s risk is expressed in terms of $1000000. For example, changing the proportional transaction cost from %0.5 to %3.0, increases investor’s expected regret from nearly $5000 to $35000. The remarkable impact of transaction costs on investor’s risk shows the importance of making informed decisions subject to multiple time periods rather than a single time period ahead.

5. Conclusion. This paper proposed a multistage stochastic programming model to deal with the multi-period cardinality constrained portfolio optimization problem. The presented model aims to minimize investor’s expected regret, while ensuring a minimum level of expected return. The investor is allowed to invest in some risky assets and a riskless one. Taking any position on risky assets is subject to proportional transaction costs. To generate scenarios of risky asset returns, a random walk model, based on the empirical distribution of market index returns, as well as single index models were utilized. Then, historical returns of a number of stocks, selected from Frankfurt Exchange (FSE), were used to implement the scenario tree generation method, and solve the proposed stochastic programming model.

A detailed sensitivity analysis on different combinations of minimum expected returns, proportional transaction costs and target wealth levels was performed. Results showed that increasing each of the above-mentioned factors, puts the investor...
into a more risky situation. In addition, investor’s risk increased in the presence of cardinality constraints. As previously indicated, the direct impact of transaction costs on investor’s risk is one of the main reasons that multi-period portfolio optimization models are preferable to single period ones.

Appendix A. Johnson transformation [29]. The Johnson system utilizes three families of distributions to transform variables to standard normal distribution. The standard normal variables are generated by transformations of the following form

\[ z = \gamma \eta k_i(x; \lambda, \varepsilon) \]  

(A1)

where \( z \) is a standard normal variable and the \( k_i(x; \lambda, \varepsilon) \) are chosen to cover a wide range of possible shapes. Johnson suggested the following functions:

\[ k_1(x; \lambda, \varepsilon) = \arcsinh \left( \frac{x - \varepsilon}{\lambda} \right) \]  

(A2)

\[ k_2(x; \lambda, \varepsilon) = \ln \left( \frac{x - \varepsilon}{\lambda + \varepsilon - x} \right) \]  

(A3)

\[ k_3(x; \lambda, \varepsilon) = \ln \left( \frac{x - \varepsilon}{\lambda} \right) \]  

(A4)

These functions are referred to as the SU distribution, SB distribution and SL distribution, respectively.

Consider any of these transformations. For any fixed positive value of \( z \), points -3\( z \), -\( z \), +\( z \) and +3\( z \) determine three intervals with equal length. Any of these transformations yields four values of \( x \) which are no longer equally spaced. Let \( x-3z \), \( x-z \), \( xz \) and \( x+3z \) be the values corresponding to -3\( z \), -\( z \), +\( z \) and +3\( z \) under any transformation.

Let

\[ m = x_{3z} - x_z, n = x_{-z} - x_{-3z}, p = x_z - x_{-z} \]  

(A5)
It can be proved that for any $S_U$ distribution, $\frac{mn}{p^2} > 1$, for any $S_B$ distribution, $\frac{mn}{p^2} < 1$ and for any $S_L$ distribution, $\frac{mn}{p^2} = 1$. This property can be used to discriminate among the three families.

To select the appropriate transformation, a value of $z$ is chosen. Then, from the tables of areas for the standard normal distribution, the percentages $\varphi_z$ corresponding to $\zeta = -3z, -z, z$ and $3z$ are determined. For each $\zeta$, the percentile $x^{(i)}$ corresponding to $\varphi_z$ is obtained using the relationship $\varphi_z = (i - 1/2)/N$, where $N$ is the number of data points, and $x_z$ is set equal to $x^{(i)}$. Since $i$ is not necessarily an integer, interpolation may be required. Afterward, the sample values of $m, n$ and $p$ are computed with Eq. (A5) and the appropriate transformation is selected. Since the probability that $\frac{mn}{p^2} = 1$ is zero, if one wishes to use $S_L$ distribution, it will be necessary to allow a tolerance interval around 1. After the selection process is completed, the next problem is to estimate parameters of the chosen distribution. There exist various parameter estimation techniques for the Johnson system. Here, a uniform approach of matching percentiles is introduced. The estimates are given in terms of the chosen value of $z$ and the formerly computed values of $m, n$ and $p$.

For each family, the formulas are obtained by starting with a given Johnson distribution and fixed positive $z$, and then solving explicitly for the parameters in terms of $z$ and the population values of $m, n$ and $p$. It should be mentioned that the parameter values are functions of $m, n$ and $p$ which, in turn, are functions of $x_{-3z}, x_{-z}, x_z$ and $x_{3z}$.

For the three families, the estimations are given by the following formulas:

(a) Johnson unbounded system ($S_U$ distribution)

\[
z = \gamma + \eta \text{arcsinh}
\left(\frac{x - \varepsilon}{\lambda}\right)
\]  

Estimates of the parameters in this case are as follows:

\[
\eta = \frac{2z}{\text{arccosh}\left(\frac{m}{p} + \frac{n}{p}\right)}, \quad (\eta > 0) \tag{A7}
\]

\[
\eta = \frac{2z}{\text{arccosh}\left(\frac{n}{p} - \frac{m}{p}\right)}, \quad (\eta > 0) \tag{A8}
\]

\[
\lambda = \left(\frac{2p\left(\frac{mn}{p^2} - 1\right)^{\frac{1}{2}}}{\left(\frac{m}{p} + \frac{n}{p} - 2\right)\left(\frac{m}{p} + \frac{n}{p} + 2\right)^{\frac{1}{2}}}\right), \quad (\lambda > 0) \tag{A9}
\]

\[
\varepsilon = \frac{x_z + x_{-z}}{2} + \frac{p\left(\frac{n}{p} - \frac{m}{p}\right)}{2\left(\frac{m}{p} + \frac{n}{p} - 2\right)} \tag{A10}
\]

(b) Johnson bounded system ($S_B$ distribution)

\[
z = \gamma + \eta \ln
\left(\frac{x - \varepsilon}{\lambda + \varepsilon - x}\right) \tag{A11}
\]
\[ \eta = \frac{Z}{\text{arccosh} \left( \frac{1}{2} \left( 1 + \frac{p}{m} \right) \left( 1 + \frac{p}{n} \right) \right) \frac{1}{2}} \]  \hspace{1cm} (A12)

\[ \gamma = \eta \text{arcsinh} \left( \frac{\left( \frac{p}{m} - \frac{p}{n} \right) \left[ 1 + \frac{p}{m} \right] \left( 1 + \frac{p}{n} \right) - 4}{\frac{1}{2} \left( \frac{p^2}{mn} - 1 \right)} \right)^{\frac{1}{2}} \]  \hspace{1cm} (A13)

\[ \lambda = \frac{p \left[ \left( 1 + \frac{p}{m} \right) \left( 1 + \frac{p}{n} \right) - 2 \right]^2 - 4}{\frac{p^2}{mn} - 1}, \quad (\lambda > 0) \]  \hspace{1cm} (A14)

\[ \varepsilon = \frac{x_z + x_{-z}}{2} - \frac{\lambda}{2} + \frac{p \left( \frac{p}{m} - \frac{p}{n} \right)}{\frac{p^2}{mn} - 1} \]  \hspace{1cm} (A15)

(c) Johnson log-normal system \((S_L\text{ distribution})\)

\[ z = \gamma + \eta \ln(x - \varepsilon) \]  \hspace{1cm} (A16)

Note that in case of log-normal system, we have \(\frac{p}{m} = \frac{p}{m}\). Estimates of the parameters in this case are as follows:

\[ \eta = \frac{2z}{\ln \left( \frac{m}{p} \right)} \]  \hspace{1cm} (A17)

\[ \lambda = \eta \ln \left( \frac{\frac{m}{p} - 1}{\left[ \frac{m}{p} \right]^{\frac{1}{2}}} \right) \]  \hspace{1cm} (A18)

\[ \varepsilon = \frac{x_z + x_{-z}}{2} - \frac{p \left[ \frac{m}{p} + 1 \right]}{\left[ \frac{m}{p} - 1 \right]} \]  \hspace{1cm} (A19)

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