Abstract. Consider the stationary Boltzmann equation in 2D convex domains with diffusive boundary condition. In this paper, we establish the hydrodynamic limits while the boundary layers are present, and derive the steady Navier-Stokes-Fourier system with non-slip boundary conditions. Our contribution focuses on novel weighted $W^{1,\infty}$ estimates for the Milne problem with geometric correction. Also, we develop stronger remainder estimates based on an $L^{2m} - L^\infty$ framework.

Keywords: Boundary layer; geometric correction; $W^{1,\infty}$ estimates; $L^{2m} - L^\infty$ framework.

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1. Introduction

1.1. Problem Presentation. We consider the stationary Boltzmann equation in a two-dimensional smooth convex domain \( \Omega \ni \vec{x} = (x_1, x_2) \) with velocity \( \vec{v} = (v_1, v_2) \in \mathbb{R}^2 \). The density function \( \tilde{g}^\epsilon(\vec{x}, \vec{v}) \) satisfies

\[
\begin{cases}
\epsilon \vec{v} \cdot \nabla_x \tilde{g}^\epsilon = Q[\tilde{g}^\epsilon, \tilde{g}^\epsilon] & \text{in } \Omega \times \mathbb{R}^2, \\
\tilde{g}^\epsilon(\vec{x}_0, \vec{v}) = P^\epsilon[\tilde{g}^\epsilon](\vec{x}_0, \vec{v}) & \text{for } \vec{x}_0 \in \partial \Omega \text{ and } \vec{v} \cdot \vec{v}(\vec{x}_0) < 0,
\end{cases}
\] (1.1)

where \( \vec{v}(\vec{x}_0) \) is the unit outward normal vector at \( \vec{x}_0 \), the Knudsen number \( \epsilon \) satisfies \( 0 < \epsilon << 1 \), the diffusive boundary

\[
P^\epsilon[\tilde{g}^\epsilon](\vec{x}_0, \vec{v}) = \mu^\epsilon(\vec{x}_0, \vec{v}) \int_{\vec{u} \cdot \vec{v}(\vec{x}_0) > 0} \tilde{g}^\epsilon(\vec{x}_0, \vec{u}) |\vec{u} \cdot \vec{v}(\vec{x}_0)| \, d\vec{u}.
\] (1.2)

The boundary Maxwellian

\[
\mu^\epsilon(\vec{x}_0, \vec{v}) = \frac{\rho^\epsilon(\vec{x}_0)}{\theta^\epsilon(\vec{x}_0) \sqrt{2\pi}} \exp \left( -\frac{|\vec{v} - \vec{u}^\epsilon(\vec{x}_0)|^2}{2\theta^\epsilon(\vec{x}_0)} \right),
\] (1.3)

is a perturbation of the standard Maxwellian

\[
\mu_0(\vec{v}) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{|\vec{v}|^2}{2} \right).
\] (1.4)

It is normalized to satisfy

\[
\int_{\vec{v} \cdot \vec{v}(\vec{x}_0) > 0} \mu^\epsilon(\vec{x}_0, \vec{v}) |\vec{v} \cdot \vec{v}(\vec{x}_0)| \, d\vec{v} = \int_{\vec{v} \cdot \vec{v}(\vec{x}_0) > 0} \mu_0(\vec{v}) |\vec{v} \cdot \vec{v}(\vec{x}_0)| \, d\vec{v} = 1.
\] (1.5)

For simplicity, we just denote \( \mu = \mu^\epsilon \). We further assume that \( \rho^\epsilon, \vec{u}^\epsilon, \theta^\epsilon \) can be expanded into a power series with respect to \( \epsilon \),

\[
\rho^\epsilon(\vec{x}_0) = 1 + \sum_{k=1}^{\infty} \epsilon^k \rho_{\epsilon, k}(\vec{x}_0),
\] (1.6)

\[
\vec{u}^\epsilon(\vec{x}_0) = 0 + \sum_{k=1}^{\infty} \epsilon^k \vec{u}_{\epsilon, k}(\vec{x}_0),
\] (1.7)

\[
\theta^\epsilon(\vec{x}_0) = 1 + \sum_{k=1}^{\infty} \epsilon^k \theta_{\epsilon, k}(\vec{x}_0).
\] (1.8)

Hence, we may also expand the boundary Maxwellian \( \mu^\epsilon(\vec{x}_0, \vec{v}) \) into power series with respect to \( \epsilon \),

\[
\mu^\epsilon(\vec{x}_0, \vec{v}) = \mu(\vec{v}) + \mu^{\frac{1}{2}}(\vec{v}) \left( \sum_{k=1}^{\infty} \epsilon^k \mu_{k, \epsilon}(\vec{x}_0, \vec{v}) \right).
\] (1.9)

In particular, we have

\[
\mu_1(\vec{x}_0, \vec{v}) = \mu^{\frac{1}{2}}(\vec{v}) \left( \rho_{\epsilon, 1}(\vec{x}_0) + \vec{u}_{\epsilon, 1}(\vec{x}_0) \cdot \vec{v} + \theta_{\epsilon, 1}(\vec{x}_0) \frac{|\vec{v}|^2}{2} \right).
\] (1.10)

It is easy to check that

\[
\left| \langle \vec{v} \rangle^\theta \exp(\sigma^2) \frac{\mu^\epsilon - \mu}{\mu^{\frac{1}{2}}} \right| \leq C_0(\vartheta, \vartheta) \epsilon,
\] (1.11)

for any \( 0 \leq \vartheta < \frac{1}{4} \) and integer \( \vartheta \geq 3 \). We assume that \( C_0 > 0 \) is sufficiently small. Based on the expansion, we naturally have

\[
\int_{\vec{v} \cdot \vec{v}(\vec{x}_0) > 0} \frac{\mu_k(\vec{x}_0, \vec{v})}{\mu^{\frac{1}{2}}(\vec{v})} |\vec{v} \cdot \vec{v}(\vec{x}_0)| \, d\vec{v} = 0 \text{ for } k \geq 1.
\] (1.12)
Here we have the nonlinear collision term
\[ Q[F, G] = \int_{\mathbb{R}^2} \int_{\mathbb{S}^1} q(\vec{\omega}, |\vec{u} - \vec{v}|) \left( F(\vec{u}_s) G(\vec{v}_s) - F(\vec{u}) G(\vec{v}) \right) d\vec{\omega} d\vec{u}, \]  
with
\[ \vec{u}_s = \vec{u} + \vec{\omega} \left( (\vec{v} - \vec{u}) \cdot \vec{\omega} \right), \quad \vec{v}_s = \vec{v} - \vec{\omega} \left( (\vec{v} - \vec{u}) \cdot \vec{\omega} \right), \]
and the hard-sphere collision kernel
\[ q(\vec{\omega}, |\vec{u} - \vec{v}|) = q_0 \vec{\omega} \cdot (\vec{v} - \vec{u}), \]
for a positive constant \( q_0 \). We intend to study the behavior of \( \mathcal{F}^\epsilon \) as \( \epsilon \to 0 \).

1.2. Linearization. The solution \( \mathcal{F}^\epsilon \) can be expressed as a perturbation of the standard Maxwellian
\[ \mathcal{F}^\epsilon(x, \vec{v}) = M_0 \mu(\vec{v}) + \mu^\frac{1}{2}(\vec{v}) f^\epsilon(x, \vec{v}), \]
for some constant \( M_0 > 0 \) with the normalization condition
\[ \int_{\Omega} \int_{\mathbb{R}^2} f^\epsilon(x, \vec{v}) \mu^\frac{1}{2}(\vec{v}) d\vec{v} d\vec{x} = 0. \]
Then \( f^\epsilon \) satisfies the equation
\[
\begin{cases}
\epsilon \vec{v} \cdot \nabla_x f^\epsilon + \mathcal{L}[f^\epsilon] = \Gamma[f^\epsilon, f^\epsilon], \\
f^\epsilon(x_0, \vec{v}) = \mathcal{P}[f^\epsilon](x_0, \vec{v}) \text{ for } x_0 \in \partial \Omega \text{ and } \vec{v} \cdot \vec{v}(x_0) < 0,
\end{cases}
\]
where
\[ \mathcal{L}[f^\epsilon] = -2 \mu^{-\frac{1}{2}} Q \left[ \mu, \mu^\frac{1}{2} f^\epsilon \right], \]
\[ \Gamma[f^\epsilon, f^\epsilon] = \mu^{-\frac{1}{2}} Q \left[ \mu^\frac{1}{2} f^\epsilon, \mu^\frac{1}{2} f^\epsilon \right], \]
and
\[ \mathcal{P}[f^\epsilon](x_0, \vec{v}) = \mu^\epsilon(x_0, \vec{v}) \int_{|\vec{u} \cdot \vec{v}(x_0)| > 0} \mu^\frac{1}{2}(\vec{u}) f^\epsilon(x_0, \vec{u}) |\vec{u} \cdot \vec{v}(x_0)| d\vec{u} + \mu^{-\frac{1}{2}}(\vec{v}) \left( \mu^\epsilon(x_0, \vec{v}) - \mu(\vec{v}) \right). \]
Hence, in order to study \( \mathcal{F}^\epsilon \), it suffices to consider \( f^\epsilon \).

1.3. Background and Methods.

1.3.1. Asymptotic Analysis. Hydrodynamic limits are central to connecting the kinetic theory and fluid mechanics. Since early 20th century, this type of problems have been extensively studied in many different settings: stationary or evolutionary, linear or nonlinear, strong solution or weak solution, etc.

The early result dates back to 1912 by Hilbert himself, using the so-called Hilbert’s expansion, i.e. an expansion of the distribution function \( \mathcal{F}^\epsilon \) as a power series of the Knudsen number \( \epsilon \). Since then, a lot of works on Boltzmann equation in \( \mathbb{R}^n \) or \( \mathbb{T}^n \) have been presented, including [13, 20, 2, 3, 4, 5], for either smooth solutions or renormalized solutions.

The general theory of initial-boundary-value problems was first developed in 1963 by Grad [14], and then extended by Darrozes [8], Sone and Aoki [21, 22, 23, 24, 25], for both the evolutionary and stationary equations. In the classical books [21] and [25], Sone provided a comprehensive summary of previous results and gave a complete analysis of such approaches.

For stationary Boltzmann equation where the state of gas is close to a uniform state at rest, the expansion of the perturbation \( f^\epsilon \) consists of two parts: the interior solution \( F \), which is based on a hierarchy of linearized Boltzmann equations and satisfies a steady Navier-Stokes-Fourier system, and the boundary layer \( \mathcal{F} \), which is based on a half-space kinetic equation and decays rapidly when it is away from the boundary.

The justification of hydrodynamic limits usually involves two steps:
(1) Expanding $F = \sum_{k=1}^{\infty} \epsilon^k F_k$ and $\mathcal{F} = \sum_{k=1}^{\infty} \epsilon^k \mathcal{F}_k$ as power series of $\epsilon$ and proving the coefficients $F_k$ and $\mathcal{F}_k$ are well-defined. Traditionally, the estimates of interior solutions $F_k$ are relatively straightforward. On the other hand, boundary layers $\mathcal{F}_k$ satisfy one-dimensional half-space problems which lose some key structures of the original equations. The well-posedness of boundary layer equations are sometimes extremely difficult and it is possible that they are actually ill-posed (e.g. certain type of Prandtl layers).

(2) Proving that $R = f^\epsilon - \epsilon F_1 - \epsilon \mathcal{F}_1 = o(\epsilon)$ as $\epsilon \to 0$. Ideally, this should be done just by expanding to the leading-order level $F_1$ and $\mathcal{F}_1$. However, in singular perturbation problems, the estimates of the remainder $R$ usually involves negative powers of $\epsilon$, which requires expansion to higher order terms $F_N$ and $\mathcal{F}_N$ for $N \geq 2$ such that we have sufficient power of $\epsilon$. In other words, we define

$$R = f^\epsilon - \sum_{k=1}^{N} \epsilon^k F_k - \sum_{k=1}^{N} \epsilon^k \mathcal{F}_k$$

for $N \geq 2$ instead of $R = f^\epsilon - \epsilon F_1 - \epsilon \mathcal{F}_1$ to get better estimate of $R$.

Above formulation is for the convergence in the $L^\infty$ sense. If instead we consider $L^p$ convergence for $1 \leq p < \infty$, then the boundary layer $\mathcal{F}_1$ is of order $\epsilon^{1/2}$ due to rescaling, which is negligible compared with $F_1$. [10] justifies the $L^p$ convergence under the same formulation as ours without taking boundary layer expansion into consideration. On the other hand, the effect of boundary layers constitutes the major upshot of our paper.

1.3.2. Classical Approach. The classical construction of boundary layers requires the analysis of the flat Milne problem. In detail, let $\eta$ denote the rescaled normal variable with respect to the boundary, $\theta$ the tangential variable, and $\vec{v} = (v_\eta, v_\theta)$ the normal and tangential velocity. The boundary layer $F_1$ satisfies

$$v_\eta \frac{\partial F_1}{\partial \eta} + \mathcal{L}[F_1] = 0,$$

where $\mathcal{L}$ is the linearized Boltzmann operator.

Although a rigorous proof of such expansions has not been presented, it is widely believed that the idea of this approach is natural and the difficulties are purely technical. Besides the fact that this idea is an intuitive application of the Hilbert’s expansion, it is strongly supported by [1] which justifies the well-posedness and decay of the above flat Milne problem.

This idea is easily adapted to other kinetic models. As a linear prototype of Boltzmann equation, the case of neutron transport equation was carefully investigated. In particular, the hydrodynamic limit was proved in the remarkable paper [6] by Bensoussan, Lions and Papanicolaou. This is widely regarded as the foundation of rigorous analysis of boundary layers in kinetic equations.

Unfortunately, in [29], we demonstrated that both the proof and results of this formulation in [6] are invalid due to a lack of regularity in estimating $\frac{\partial \mathcal{F}_1}{\partial \theta}$. Similarly, counterexamples were proposed in [27] that this idea is also invalid in the Boltzmann equation. Basically, this pulls the whole study back to the starting point and any later results based on this type of boundary layers should be reexamined.

In detail, in order to show the hydrodynamic limits, we need $\frac{\partial \mathcal{F}_1}{\partial \theta} \in L^\infty$ since it is part of the remainder. However, though $\mathcal{F}_1 \in L^\infty$ as is shown in [11], we do not necessarily have $\frac{\partial \mathcal{F}_1}{\partial \eta} \in L^\infty$. Furthermore, the singularity $\frac{\partial \mathcal{F}_1}{\partial \eta} \notin L^\infty$ will be transferred to $\frac{\partial \mathcal{F}_1}{\partial \theta} \notin L^\infty$. This singularity was rigorously shown in [29] through a careful construction of the boundary data, i.e. the chain of estimates

$$R = o(\epsilon) \iff \mathcal{F}_2 \in L^\infty \iff \frac{\partial \mathcal{F}_1}{\partial \theta} \in L^\infty \iff \frac{\partial \mathcal{F}_1}{\partial \eta} \in L^\infty,$$

is broken since the rightmost estimate is wrong.

1.3.3. Geometric Correction. While the classical method breaks down, a new approach with geometric correction to the boundary layer construction has been developed to ensure regularity in the cases of disk and annulus in [29], [31], [28] and [27]. The new boundary layer $\mathcal{F}_1$ satisfies the $\epsilon$-Milne problem with geometric
correction,

\[ v_\eta \frac{\partial \mathcal{F}_1}{\partial \eta} - \epsilon \frac{R_\kappa - \epsilon \eta}{R_\kappa - \epsilon \eta} \left( v_\phi \frac{\partial \mathcal{F}_1}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{F}_1}{\partial v_\phi} \right) + \mathcal{L}[\mathcal{F}_1] = 0, \tag{1.24} \]

where \( R_\kappa \) is the curvature on the boundary curve. We proved that the solution recovers the well-posedness and exponential decay as in flat Milne problem, and the regularity in \( \theta \) is indeed improved, i.e. \( \frac{\partial \mathcal{F}_1}{\partial \theta} \in L^\infty \).

However, this new method fails to treat more general domains. Roughly speaking, we have two contradictory goals to achieve:

1. To prove hydrodynamic limits, the remainder estimates require higher-order regularity estimate of the boundary layer.
2. The geometric correction \( \frac{\epsilon}{R_\kappa - \epsilon \eta} \left( v_\phi \frac{\partial \mathcal{F}_1}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{F}_1}{\partial v_\phi} \right) \) in the boundary layer equation is related to the curvature of the boundary curve, which prevents higher-order regularity estimates.

In other words, the improvement of regularity is still not enough to close the proof. To be more specific, the discussion of domains is as follows:

- In the absence of the geometric correction \( \frac{\epsilon}{R_\kappa - \epsilon \eta} \left( v_\phi \frac{\partial \mathcal{F}_1}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{F}_1}{\partial v_\phi} \right) \), which is the flat Milne problem as in [24] and [25], the key tangential derivative \( \frac{\partial \mathcal{F}_1}{\partial \theta} \) is not bounded. Therefore, the expansion breaks down.
- In the domain of disk or annulus, when \( R_\kappa \) is constant, as in [27], \( \frac{\partial \mathcal{F}_1}{\partial \theta} \) is bounded, since the tangential derivative \( \frac{\partial}{\partial \theta} \) commutes with the equation, and thus we do not need the estimate of the singular term \( \frac{\partial \mathcal{F}_1}{\partial \eta} \).
- For general smooth convex domains, when \( R_\kappa \) is a function of \( \theta \), \( \frac{\partial \mathcal{F}_1}{\partial \theta} \) relates to the normal derivative \( \frac{\partial \mathcal{F}_1}{\partial \eta} \), which has been shown possibly unbounded in [27]. Therefore, we get stuck again at the regularity estimates.

1.3.4. Diffusive Boundary. In this paper, we will push the above argument from both sides (remainder estimates and regularity estimates) and prove the hydrodynamic limits for the nonlinear Boltzmann equation in 2D smooth convex domains. Our contribution consists of the following:

- **Remainder Estimates:**
  We first prove an almost optimal remainder estimates and reduce the regularity requirement of the expansion. In the remainder equation for \( R(\vec{x}, \vec{v}) = f - F - \mathcal{F} \),

\[ \epsilon \vec{v} \cdot \nabla_x R + \mathcal{L}[R] = S, \tag{1.25} \]

the estimate in [27] is

\[ \|R\|_{L^\infty} \lesssim \frac{1}{\epsilon^3} \|S\|_{L^2} + \text{higher order terms}. \tag{1.26} \]

We intend to show that \( \|R\|_{L^\infty} = o(\epsilon) \) as \( \epsilon \to 0 \). Since \( S \) contains the term related to \( \frac{\partial \mathcal{F}}{\partial \theta} \), the coefficients \( \epsilon^{-3} \) is too singularity. The key observation here is that due to the rescaling in the normal direction, the smaller \( p \geq 1 \) is, the better estimate \( \left\| \frac{\partial \mathcal{F}}{\partial \theta} \right\|_{L^p} \) will be. Therefore, we successfully prove a stronger estimate for \( m \geq 2 \),

\[ \|R\|_{L^\infty} \lesssim \frac{1}{\epsilon^{2+m-1}} \|S\|_{L^{2m+2m-1}} + \text{higher order terms}. \tag{1.27} \]

This is achieved by an innovative \( L^{2m} - L^\infty \) framework. The main idea is to introduce special test functions in the weak formulation to treat kernel and non-kernel parts of \( \mathcal{L} \) separately, and further to bootstrap to improve the \( L^\infty \) estimate by a modified double Duhamel’s principle. The proof relies
on a delicate analysis using interpolation and Young’s inequality.

- **Regularity of Boundary Layer:**
  Consider the boundary layer expansion
  \[ \mathcal{F}(\eta, \theta, \vec{v}) \sim \epsilon \mathcal{F}_1(\eta, \theta, \vec{v}) + \epsilon^2 \mathcal{F}_2(\eta, \theta, \vec{v}). \] (1.28)

  The diffusive boundary condition leads to an important simplification that \( \mathcal{F}_1 = 0 \). Thus the next-order boundary layer \( \mathcal{F}_2 \) must formally satisfy
  \[ v_\eta \frac{\partial \mathcal{F}_2}{\partial \eta} - \epsilon R_{\kappa} - \epsilon \eta \left( v_\phi^2 \frac{\partial \mathcal{F}_2}{\partial v_\phi} - v_\theta v_\phi \frac{\partial \mathcal{F}_2}{\partial v_\phi} \right) + L[\mathcal{F}_2] = 0. \] (1.29)

  The remainder estimate requires the estimate of \( \frac{\partial \mathcal{F}_2}{\partial \theta} \), whose boundedness had remained open.

  The key observation here is that the estimate of \( \frac{\partial \mathcal{F}_2}{\partial \theta} \) relies on \( v_\eta \frac{\partial \mathcal{F}_2}{\partial \eta} \), not \( \frac{\partial \mathcal{F}_2}{\partial \eta} \) itself. This extra \( v_\eta \) saves us and avoids singularity. Still, we cannot naively take \( \eta \) derivatives on both sides of (1.29) since the geometric correction forbids further estimates. Our main idea is to track the solution \( \mathcal{F}_2 \) along the characteristic curves in the presence of the non-local operator \( L \) and consider the intertwined \( \frac{\partial \mathcal{F}_2}{\partial \eta}, \frac{\partial \mathcal{F}_2}{\partial v_\eta}, \) and \( \frac{\partial \mathcal{F}_2}{\partial v_\phi} \) simultaneously.

  Our proof is intricate and relies on the weighted \( L^\infty \) estimates for the normal derivative, which is inspired by [18] and [19]. The convexity and invariant kinetic distance
  \[ \zeta(\eta, \theta, v_\eta, v_\phi) = \left( (v_\eta^2 + v_\phi^2) - \left( \frac{R_\kappa(\theta) - \epsilon \eta}{R_\kappa(\theta)} \right)^2 v_\phi^2 \right)^{\frac{1}{2}}, \] (1.30)

  plays the crucial role.

- **Expansion and Nonlinearity:**
  The matching procedure between the interior solution and boundary layer is actually a very tricky step, which is often ignored in the related literature. [24] and [25] offers a details justification of this procedure, but it has not taken the new boundary layer construction into consideration.

  Here, we provide a clear description on how to delicately determine the macroscopic variables and how to handle the nonlinearity in the remainder estimates. In particular, we enforce the well-known Boussinesq relation at the leading order through the conservation of mass and an intricate designing of the boundary layer expansion. Also, we use a special \( L^{2m} - L^\infty \) method to absorb the nonlinear terms contribution.

1.4. Main Theorem.

**Theorem 1.1.** For given \( M_0 > 0 \) and \( \mu_0 > 0 \) satisfying (1.9) and (1.11) with \( 0 < \epsilon << 1 \), there exists a unique positive solution \( \mathcal{F} = M_0 \mu + \mu^2 f^\epsilon \) to the stationary Boltzmann equation (1.1), and \( f^\epsilon \) fulfils that for integer \( \vartheta \geq 3 \) and \( 0 \leq \varrho < \frac{1}{4} \),

\[ \left\| \langle \vec{v} \rangle^\varrho e^{\varrho \langle \vec{v} \rangle^2} \left( f^\epsilon - \epsilon F \right) \right\|_{L^\infty} \leq C(\delta) \epsilon^{2-\delta}, \] (1.31)

for any \( 0 < \delta << 1 \), where

\[ F = \mu^2 \left( \rho + \vec{u} \cdot \vec{v} + \theta \| \vec{v} \|^2 - 2 \right), \] (1.32)
satisfies the steady Navier-Stokes-Fourier system

\[
\begin{cases}
\nabla_x (\rho + \theta) = 0, \\
\bar{u} \cdot \nabla_x \bar{u} - \gamma_1 \Delta_x \bar{u} + \nabla_x P_2 = 0, \\
\nabla_x \cdot \bar{u} = 0, \\
\bar{u} \cdot \nabla_x \theta - \gamma_2 \Delta_x \theta = 0,
\end{cases}
\]

(1.33)

where \(\gamma_1 > 0\) and \(\gamma_2 > 0\) are some constants, \(M(\bar{x}_0)\) is a constant such that the Boussinesq relation

\[
\rho + \theta \text{ is a constant such that the Boussinesq relation},
\]

(1.34)

and the normalization condition

\[
\int_{\Omega} \int_{\mathbb{R}^2} F(\bar{x}, \bar{v}) \mu^1(\bar{v}) d\bar{v} d\bar{x} = 0,
\]

(1.35)

hold.

**Remark 1.2.** The case \(\rho_{b,1}(\bar{x}_0) = 0\), \(\bar{u}_{b,1}(\bar{x}_0) = 0\) and \(\theta_{b,1}(\bar{x}_0) \neq 0\) is called the non-isothermal model, which represents a system that only has heat transfer through the boundary but has no work between the environment and the system. Based on above theorem, the hydrodynamic limit is a steady Navier-Stokes-Fourier system with non-slip boundary condition. This provides a rigorous derivation of this important fluid model.

Throughout this paper, \(C > 0\) denotes a constant that only depends on the parameter \(\Omega\), but does not depend on the data. It is referred as universal and can change from one inequality to another. When we write \(C(z)\), it means a certain positive constant depending on the quantity \(z\). We write \(a \lesssim b\) to denote \(a \leq Cb\).

This paper is organized as follows: in Section 2, we list some preliminary results on the linearized Boltzmann operator and the weak formulation; in Section 3, we present the asymptotic analysis of the equation (1.18); in Section 4, we establish the \(L^\infty\) well-posedness of the linearized Boltzmann equation; in Section 5, we prove the well-posedness and decay of the \(\epsilon\)-Milne problem with geometric correction; in Section 6, we study the weighted regularity of the \(\epsilon\)-Milne problem with geometric correction; finally, in Section 7, we prove the main theorem.

**Remark 1.3.** The general structure of this paper is very similar to that of [18] and [27]. In particular, Section 4 and 5 seem to be an adaption of the corresponding theorems there. However, our results hold for nonlinear Boltzmann equation in a finite domain and they are highly non-trivial, so it is better to start from scratch.
2. Preliminaries

2.1. Linearized Boltzmann Operator. [11] Chapter 3 provides the simplified linearized Boltzmann operator $\mathcal{L}$ as

$$\mathcal{L}[f] = -2\mu^{-\frac{1}{2}}Q[\mu, \mu^\gamma f] = \nu(\vec{v})f - K[f], \quad (2.1)$$

where

$$\nu(\vec{v}) = \int_{\mathbb{R}^2} \int_{S^1} q(\vec{\omega}, |\vec{u} - \vec{v}|)\mu(\vec{u})d\vec{\omega}d\vec{u}, \quad (2.2)$$

$$K[f](\vec{v}) = K_2[f](\vec{v}) - K_1[f](\vec{v}) = \int_{\mathbb{R}^2} k(\vec{u}, \vec{v})f(\vec{u})d\vec{u}, \quad (2.3)$$

for some kernel $k(\vec{u}, \vec{v})$.

Let $\langle \cdot, \cdot \rangle$ be the standard $L^2$ inner product in $\Omega \times \mathbb{R}^2$. We define the $L^p$ and $L^\infty$ norms in $\Omega \times \mathbb{R}^2$ as usual:

$$\|f\|_{L^p} = \left( \int_\Omega \int_{\mathbb{R}^2} |f(\vec{x}, \vec{v})|^p \, d\vec{v}d\vec{x} \right)^{\frac{1}{p}}, \quad (2.5)$$

$$\|f\|_{L^\infty} = \sup_{(\vec{x}, \vec{v}) \in \Omega \times \mathbb{R}^2} |f(\vec{x}, \vec{v})|. \quad (2.6)$$

Define the weighted $L^2$ norm as follows:

$$\|f\|_{L^2_w} = \left\| \nu^{\frac{1}{2}} f \right\|_{L^2}. \quad (2.7)$$

Define the weighted $L^\infty$ norm as follows:

$$\|f\|_{L^\infty_{\nu, e}} = \sup_{(\vec{x}, \vec{v}) \in \Omega \times \mathbb{R}^2} \left( \langle \vec{v} \rangle^\theta e^{\rho|\vec{v}|^2} |f(\vec{x}, \vec{v})| \right). \quad (2.8)$$

Define $d\gamma = |\vec{v} \cdot \vec{v}| \, d\vec{\omega}d\vec{v}$ on the boundary $\partial\Omega \times \mathbb{R}^2$ for $\vec{\omega}$ as the curve measure. Define the $L^p$ and $L^\infty$ norms on the boundary as follows:

$$|f|_{L^p} = \left( \int_\gamma |f(\vec{x}, \vec{v})|^p \, d\gamma \right)^{\frac{1}{p}}, \quad (2.9)$$

$$|f|_{L^p_{\pm}} = \left( \int_{\gamma_{\pm}} |f(\vec{x}, \vec{v})|^p \, d\gamma \right)^{\frac{1}{p}}, \quad (2.10)$$

$$|f|_{L^\infty} = \sup_{(\vec{x}, \vec{v}) \in \gamma} |f(\vec{x}, \vec{v})|, \quad (2.11)$$

$$|f|_{L^\infty_{\pm}} = \sup_{(\vec{x}, \vec{v}) \in \gamma_{\pm}} |f(\vec{x}, \vec{v})|. \quad (2.12)$$

Denote the Japanese bracket as

$$\langle \vec{v} \rangle = \left( 1 + |\vec{v}|^2 \right)^{\frac{1}{2}}. \quad (2.13)$$

Define the kernel operator $\mathbb{P}$ as

$$\mathbb{P}[f] = \mu^\gamma(\vec{v}) \left( a_f(\vec{x}) + \vec{v} \cdot \vec{b}_f(\vec{x}) + \frac{|\vec{v}|^2 - 2}{2} c_f(\vec{x}) \right), \quad (2.14)$$

where $\mathbb{P}$ is in the null space of $\mathcal{L}$, and the non-kernel operator $\mathbb{I} - \mathbb{P}$ as

$$(\mathbb{I} - \mathbb{P})[f] = f - \mathbb{P}[f]. \quad (2.15)$$
with

$$\int_{\mathbb{R}^2} (I - P)[f] \begin{pmatrix} 1 \\ \frac{1}{|\vartheta|^2} \end{pmatrix} \, d\vartheta = 0$$

\[(2.16)\]

**Lemma 2.1.** For the operator \( L = \nu I - K \), we have the estimates

$$\left\| \frac{\partial \nu}{\partial \vartheta} \right\|_{L^\infty} \leq C,$$

\[(2.17)\]



$$\nu_0(1 + |\vartheta|) \leq \nu(\vartheta) \leq \nu_1(1 + |\vartheta|),$$

\[(2.18)\]

$$\langle f, L[f] \rangle = \left\langle (I - P)[f], L \left[ (I - P)[f] \right] \right\rangle \geq C \left\| \nu^{\frac{1}{2}}(I - P)[f] \right\|^2_{L^2},$$

\[(2.19)\]

$$\left\| L \left[ (I - P)[f] \right] \right\|^2_{L^2} \geq C \left\| \nu^{\frac{1}{2}}(I - P)[f] \right\|^2_{L^2},$$

\[(2.20)\]

$$\left\| [f] \right\|_{L^2} \leq \left\| \nu[\bar{f}] \right\|_{L^2} \leq C \left\| [f] \right\|_{L^2}.$$ 

\[(2.21)\]

for \( \nu_0, \nu_1 \) and \( C \) positive constants.

**Proof.** See [11] Chapter 3].

**Lemma 2.2.** Let \( w_\xi(\vartheta) = w_{\xi,\beta,\varrho}(\vartheta) = \left( 1 + \xi^2 |\vartheta|^2 \right)^{-\frac{1}{2}} e^{\vartheta|\vartheta|^2} \), for \( \xi, \beta > 0 \) and \( 0 \leq \varrho \leq \frac{1}{4} \). Then there exists \( 0 \leq C_1(\varrho) < 1 \) and \( C_2(\varrho) > 0 \) such that for \( 0 \leq \delta \leq C_1(\varrho) \),

$$\int_{\mathbb{R}^2} e^{\delta|\vartheta|^2/2} k(\hat{u}, \vartheta) \frac{w_\xi(\vartheta)}{w_\xi(\hat{\vartheta})} d\vartheta \leq \frac{C_2(\varrho)}{1 + |\vartheta|},$$

\[(2.22)\]

$$\int_{\mathbb{R}^2} e^{\delta|\vartheta|^2/2} \frac{1}{|\hat{u}|} k(\hat{u}, \vartheta) \frac{w_\xi(\vartheta)}{w_\xi(\hat{\vartheta})} d\vartheta \leq C_2(\varrho),$$

\[(2.23)\]

$$\int_{\mathbb{R}^2} e^{\delta|\vartheta|^2/2} \nabla_v k(\hat{u}, \vartheta) \frac{w_\xi(\vartheta)}{w_\xi(\hat{\vartheta})} d\vartheta \leq C_2(\varrho).$$

\[(2.24)\]

For \( m \in \mathbb{N} \), we have

$$\| K[f] \|_{L^{2m}} \leq C \| f \|_{L^{2m}}.$$ 

\[(2.25)\]

**Proof.** See [16] Lemma 3].

**Lemma 2.3.** We have

$$\| K[f] \|_{L^\infty_{\varrho}} \leq C \left\| \frac{f}{\nu} \right\|_{L^\infty_{\varrho}},$$

\[(2.26)\]

$$\| \nabla_v K[f] \|_{L^\infty_{\varrho}} \leq C \| f \|_{L^\infty_{\varrho}}.$$ 

\[(2.27)\]

**Proof.** Consider the fact that for \( \vartheta = \beta \), we have

$$C_1 \langle \vartheta \rangle^\varrho e^{\vartheta|\vartheta|^2} \leq w_\xi \leq C_2 \langle \vartheta \rangle^\varrho e^{\vartheta|\vartheta|^2},$$

\[(2.28)\]

for some constant \( C_1, C_2 > 0 \). Then this is a natural corollary of Lemma 2.2. See [16] and [17].

**Lemma 2.4.** The nonlinear term \( \Gamma \) satisfies for \( \beta > 0 \) and \( 0 \leq \varrho \leq \frac{1}{4} \),

$$\| \nu^{-1} \Gamma[f, f] \|_{L^\infty_{\varrho}} \leq C \| f \|_{L^\infty_{\varrho}} \| f \|_{L^\infty_{\varrho}}^2,$$

\[(2.29)\]

$$\int_{\Omega \times \mathbb{R}^2} \Gamma[f, g] h \leq C \| h \|_{L^2_\varrho} \left( \| f \|_{L^2_\varrho} \| g \|_{L^\infty_{\varrho}} + \| g \|_{L^2_\varrho} \| f \|_{L^\infty_{\varrho}} \right),$$

\[(2.30)\]

$$\int_{\Omega \times \mathbb{R}^2} \Gamma[f, g] h \leq C \| h \|_{L^2_\varrho} \left( \| g \|_{L^2_\varrho} \| f \|_{L^\infty_{\varrho}} + \| f \|_{L^\infty_{\varrho}} \| g \|_{L^2_\varrho} \right).$$

\[(2.31)\]

**Proof.** See [15] Lemma 2.3 and [11] Chapter 3].

\[\square\]
2.2. **Formulation and Estimates.** Based on the flow direction, we can divide the boundary \( \gamma = \{(\vec{x}_0, \vec{v}) : \vec{x}_0 \in \partial \Omega, \vec{v} \in \mathbb{R}^2 \} \) into the in-flow boundary \( \gamma_- \), the out-flow boundary \( \gamma_+ \), and the grazing set \( \gamma_0 \) as

\[
\gamma_- = \{(\vec{x}_0, \vec{v}) : \vec{x}_0 \in \partial \Omega, \vec{v} \cdot \vec{v}(\vec{x}_0) < 0 \}, \tag{2.32}
\]
\[
\gamma_+ = \{(\vec{x}_0, \vec{v}) : \vec{x}_0 \in \partial \Omega, \vec{v} \cdot \vec{v}(\vec{x}_0) > 0 \}, \tag{2.33}
\]
\[
\gamma_0 = \{(\vec{x}_0, \vec{v}) : \vec{x}_0 \in \partial \Omega, \vec{v} \cdot \vec{v}(\vec{x}_0) = 0 \}. \tag{2.34}
\]

It is easy to see \( \gamma = \gamma_+ \cup \gamma_- \cup \gamma_0 \). Also, the boundary condition is only given on \( \gamma_- \).

**Lemma 2.5.** Define the near-grazing set of \( \gamma_+ \) or \( \gamma_- \) as

\[
\gamma_{\pm}^{\delta} = \left\{(\vec{x}, \vec{v}) \in \gamma_{\pm} : |\vec{v}(\vec{x}) \cdot \vec{v}| \leq \delta \text{ or } |\vec{v}| \geq \frac{1}{\delta} \text{ or } |\vec{v}| \leq \delta \right\}. \tag{2.35}
\]

Then

\[
\left| f \mathbf{1}_{\gamma_{\pm} \setminus \gamma_{\pm}^{\delta}} \right|_{L^1} \leq C(\delta) \left( \|f\|_{L^1} + \|\vec{v} \cdot \nabla_x f\|_{L^1} \right). \tag{2.36}
\]

**Proof.** See [9, Lemma 2.1]. \( \square \)

**Lemma 2.6.** (Green’s Identity) Assume \( f(\vec{x}, \vec{v}), g(\vec{x}, \vec{v}) \in L^2(\Omega \times \mathbb{R}^2) \) and \( \vec{v} \cdot \nabla_x f, \vec{v} \cdot \nabla_x g \in L^2(\Omega \times \mathbb{R}^2) \) with \( f, g \in L^2(\gamma) \). Then

\[
\int_{\Omega \times \mathbb{R}^2} \left( (\vec{v} \cdot \nabla_x f)g + (\vec{v} \cdot \nabla_x g)f \right)d\vec{x}d\vec{v} = \int_{\gamma_+} fgd\gamma - \int_{\gamma_-} fgd\gamma. \tag{2.37}
\]

**Proof.** See [9, Lemma 2.2]. \( \square \)
3. Asymptotic Analysis

In this section, we will construct the asymptotic expansion of the equation

\[
\begin{align*}
\begin{cases}
\epsilon \vec{v} \cdot \nabla_x f^\epsilon + \mathcal{L}[f^\epsilon] &= \Gamma[f^\epsilon, f^\epsilon], \\
f^\epsilon(\vec{x}_0, \vec{v}) &= \mathcal{P}^\epsilon[f^\epsilon](\vec{x}_0, \vec{v}) \text{ for } \vec{x}_0 \in \partial \Omega \text{ and } \vec{v} \cdot \vec{n}(\vec{x}_0) < 0,
\end{cases}
\end{align*}
\]  

(3.1)

with the normalization condition

\[
\int_\Omega \int_{\mathbb{R}^d} f^\epsilon(\vec{x}, \vec{v}) \mu^\epsilon(\vec{v}) d\vec{v} d\vec{x} = 0.
\]  

(3.2)

3.1. Interior Expansion. We define the interior expansion

\[
F(\vec{x}, \vec{v}) \sim \sum_{k=1}^{3} \epsilon^k F_k(\vec{x}, \vec{v}).
\]  

(3.3)

Plugging it into the equation (3.1) and comparing the order of \( \epsilon \), we obtain

\[
\begin{align*}
\mathcal{L}[F_1] &= 0, \\
\mathcal{L}[F_2] &= -\vec{v} \cdot \nabla_x F_1 + \Gamma[F_1, F_1], \\
\mathcal{L}[F_3] &= -\vec{v} \cdot \nabla_x F_2 + 2\epsilon^2[F_1, F_2].
\end{align*}
\]  

(3.4)  

(3.5)  

(3.6)

The following analysis is standard and well-known. We mainly refer to the method in [24, 25]. The solvability of

\[
\mathcal{L}[F_k] = S
\]  

(3.7)

requires that

\[
\int_{\mathbb{R}^2} S(\vec{v}) \psi(\vec{v}) d\vec{v} = 0
\]  

(3.8)

for any \( \psi \) satisfying \( \mathcal{L}[\psi] = 0 \). Then each \( F_k \) consists of three parts:

\[
F_k(\vec{x}, \vec{v}) = A_k(\vec{x}, \vec{v}) + B_k(\vec{x}, \vec{v}) + C_k(\vec{x}, \vec{v}),
\]  

(3.9)

where

\[
A_k(\vec{x}, \vec{v}) = \mu^k(\vec{v}) \left( A_{k,0}(\vec{x}) + A_{k,1}(\vec{x}) v_1 + A_{k,2}(\vec{x}) v_2 + A_{k,3}(\vec{x}) \left( \frac{|\vec{v}|^2 - 2}{2} \right) \right),
\]  

(3.10)

is the macroscopic part,

\[
B_k(\vec{x}, \vec{v}) = \mu^k(\vec{v}) \left( B_{k,0}(\vec{x}) + B_{k,1}(\vec{x}) v_1 + B_{k,2}(\vec{x}) v_2 + B_{k,3}(\vec{x}) \left( \frac{|\vec{v}|^2 - 2}{2} \right) \right),
\]  

(3.11)

is the connection part, with \( B_k \) depending on \( A_s \) for \( 1 \leq s \leq k - 1 \) as

\[
B_{k,0} = 0,
\]  

(3.12)

\[
B_{k,1} = \sum_{i=1}^{k-1} A_{i,0} A_{k-i,1},
\]  

(3.13)

\[
B_{k,2} = \sum_{i=1}^{k-1} A_{i,0} A_{k-i,2},
\]  

(3.14)

\[
B_{k,3} = \sum_{i=1}^{k-1} \left( A_{i,0} A_{k-i,3} + A_{i,1} A_{k-i,1} + A_{i,2} A_{k-i,2} + \sum_{j=1}^{k-1-i} A_{i,0} (A_{j,1} A_{k-i-j,1} + A_{j,2} A_{k-i-j,2}) \right),
\]  

(3.15)

and \( C_k(\vec{x}, \vec{v}) \) is the orthogonal part satisfying

\[
\int_{\mathbb{R}^2} \mu^k(\vec{v}) C_k(\vec{x}, \vec{v}) \begin{pmatrix} 1 \\ \vec{v} \\ |\vec{v}|^2 \end{pmatrix} d\vec{v} = 0,
\]  

(3.16)
with
\[ \mathcal{L}[C_k] = -\vec{v} \cdot \nabla_x F_{k-1} + \sum_{i=1}^{k-1} \Gamma[F_i, F_{k-i}], \]
which can be uniquely determined. Hence, we only need to determine \( A_k \). Traditionally, we write
\[ A_k = \mu^2 \left( \rho_k + \bar{u}_k \cdot \vec{v} + \theta_k \left( \frac{\mid \vec{v} \mid^2 - 2}{2} \right) \right), \]
where the coefficients \( \rho_k, u_k \) and \( \theta_k \) represent density, velocity and temperature in the macroscopic scale. Then the analysis in [24, 25] shows that \( A_k \) satisfies the equations as follows:

1\textsuperscript{st}-order expansion:
\[
\begin{align*}
P_1 - (\rho_1 + \theta_1) &= 0, \\
\nabla_x P_1 &= 0, \\
\nabla_x \bar{u}_1 &= 0,
\end{align*}
\]

2\textsuperscript{nd}-order expansion:
\[
\begin{align*}
P_2 - (\rho_2 + \theta_2 + \rho_1 \theta_1) &= 0, \\
\bar{u}_1 \cdot \nabla_x \bar{u}_1 - \gamma_1 \Delta_x \bar{u}_1 + \nabla_x P_2 &= 0, \\
\bar{u}_1 \cdot \nabla_x \theta_1 - \gamma_2 \Delta_x \theta_1 &= 0, \\
\nabla_x \bar{u}_2 + \bar{u}_1 \cdot \nabla_x \rho_1 &= 0.
\end{align*}
\]

Here \( P_1 \) and \( P_2 \) represent the pressure, \( \gamma_1 \) and \( \gamma_2 \) are constants.

### 3.2. Boundary Layer Expansion with Geometric Correction.

We will use the Cartesian coordinate system for the interior solution, and a local coordinate system in a neighborhood of the boundary for the boundary layer.

Assume the Cartesian coordinate is \( \vec{x} = (x_1, x_2) \). Using polar coordinates system \( (r, \theta) \in [0, \infty) \times [-\pi, \pi] \) and choosing pole in \( \Omega \), we assume \( \vec{x}_0 \in \partial \Omega \) is
\[
\begin{align*}
x_{1,0} &= r(\theta) \cos \theta, \\
x_{2,0} &= r(\theta) \sin \theta,
\end{align*}
\]
where \( r(\theta) > 0 \) is a given function describing the boundary curve. Our local coordinate system is a modification of the polar coordinate system.

In the domain near the boundary, for each \( \theta \), we have the outward unit normal vector
\[
\vec{v} = \left( \frac{r(\theta) \cos \theta + r'(\theta) \sin \theta}{\sqrt{r(\theta)^2 + r'(\theta)^2}}, \frac{r(\theta) \sin \theta - r'(\theta) \cos \theta}{\sqrt{r(\theta)^2 + r'(\theta)^2}} \right),
\]
where \( r'(\theta) = \frac{dr}{d\theta} \). We can determine each point \( \vec{x} \in \tilde{\Omega} \) as \( \vec{x} = \vec{x}_0 - N \vec{v} \) where \( N \) is the normal distance to the boundary point \( \vec{x}_0 \). In detail, this means
\[
\begin{align*}
x_1 &= r(\theta) \cos \theta - N \frac{r(\theta) \cos \theta + r'(\theta) \sin \theta}{\sqrt{r(\theta)^2 + r'(\theta)^2}}, \\
x_2 &= r(\theta) \sin \theta - N \frac{r(\theta) \sin \theta - r'(\theta) \cos \theta}{\sqrt{r(\theta)^2 + r'(\theta)^2}}.
\end{align*}
\]
It is easy to see that \( N = 0 \) denotes the boundary \( \partial \Omega \) and \( N > 0 \) denotes the interior of \( \Omega \). \((N, \theta)\) is the desired local coordinate system.

Direct computation in [15] reveals that
\[
\begin{align*}
\frac{\partial \theta}{\partial x_1} &= \frac{MP}{P^3 + QN}, & \frac{\partial N}{\partial x_1} &= -\frac{N}{P}, \\
\frac{\partial \theta}{\partial x_2} &= \frac{NP}{P^3 + QN}, & \frac{\partial N}{\partial x_2} &= \frac{M}{P},
\end{align*}
\]
Therefore, noting the fact that for $C^2$ convex domains, the curvature
\[ \kappa(\theta) = \frac{r^2 + 2r'^2 - r''}{(r^2 + r'^2)^{3/2}} > 0, \]  
and the radius of curvature
\[ R_\kappa(\theta) = \frac{1}{\kappa(\theta)} = \frac{(r^2 + r'^2)^{3/2}}{r^2 + 2r'^2 - r''} > 0, \]
we define substitutions as follows:

Substitution 1: Coordinate Substitution
Let $(x_1, x_2) \to (\eta, \theta)$ with $0 \leq \eta < R_{\min}$ for $R_{\min} = \min_\theta R_\kappa$ as
\[
\begin{align*}
x_1 &= r(\theta) \cos \theta - \eta \frac{r(\theta) \cos \theta + r'(\theta) \sin \theta}{\sqrt{r(\theta)^2 + r'(\theta)^2}}, \\
x_2 &= r(\theta) \sin \theta - \eta \frac{r(\theta) \sin \theta - r'(\theta) \cos \theta}{\sqrt{r(\theta)^2 + r'(\theta)^2}},
\end{align*}
\]  
and then the equation (3.31) is transformed into
\[
\begin{align*}
\epsilon \left(v_1 \frac{-r \cos \theta - r' \sin \theta}{(r^2 + r'^2)^{3/2}} + v_2 \frac{-r \sin \theta + r' \cos \theta}{(r^2 + r'^2)^{3/2}}\right) \frac{\partial f^e}{\partial \eta} \\
+ \epsilon \left(v_1 \frac{-r \sin \theta + r' \cos \theta}{(r^2 + r'^2)^{3/2}} + v_2 \frac{r \cos \theta + r' \sin \theta}{(r^2 + r'^2)^{3/2}}\right) \frac{1}{1 - \kappa(\theta)} \frac{\partial f^e}{\partial \theta} + \mathcal{L}[f^e] = \Gamma[f^e, f^e],
\end{align*}
\]  
where
\[ f^e(0, \theta, \vec{v}) = \mathcal{P}^e[f^e](0, \theta, \vec{v}) \quad \text{for} \quad \vec{v} \cdot \vec{v} < 0, \]
and
\[ \mathcal{P}^e[f^e](0, \theta, \vec{v}) = \mu^e(\theta, \vec{v}) \mu^{-\frac{1}{2}}(\vec{v}) \int_{\vec{u} \cdot \vec{v}(\theta) > 0} \mu^e(\vec{u}) f^e(0, \theta, \vec{u}) |\vec{u} \cdot \vec{v}(\theta)| \, d\vec{u} + \mu^{-\frac{1}{2}}(\vec{v}) \left( \mu^e(\theta, \vec{v}) - \mu(\vec{v}) \right). \]  
Substitution 2: Velocity Substitution.
Define the orthogonal velocity substitution $\vec{v} = (v_1, v_2) \to \vec{v} = (v_\eta, v_\phi)$ as
\[
\begin{align*}
v_1 &= v_1 \frac{-r \cos \theta - r' \sin \theta}{(r^2 + r'^2)^{3/2}} + v_2 \frac{-r \sin \theta + r' \cos \theta}{(r^2 + r'^2)^{3/2}} = v_\eta, \\
v_1 &= v_1 \frac{-r \sin \theta + r' \cos \theta}{(r^2 + r'^2)^{3/2}} + v_2 \frac{r \cos \theta + r' \sin \theta}{(r^2 + r'^2)^{3/2}} = v_\phi.
\end{align*}
\]  
Then we have
\[ \frac{\partial}{\partial \theta} \to \frac{\partial}{\partial \eta} - \kappa(r^2 + r'^2)^{3/2} v_\phi \frac{\partial}{\partial v_\eta} + \kappa(r^2 + r'^2)^{3/2} v_\eta \frac{\partial}{\partial v_\phi}. \]  
The transport operator is
\[ \vec{v} \cdot \nabla_x = v_\eta \frac{\partial}{\partial \eta} - \frac{v_\phi}{R_\kappa - \eta \frac{(r^2 + r'^2)^{3/2}}{\partial \theta} - \frac{v_\phi}{R_\kappa - \eta \frac{\partial}{\partial v_\eta}} + \frac{v_\eta v_\phi}{R_\kappa - \eta \frac{\partial}{\partial v_\phi}}. \]
Hence, the equation (3.1) is transformed into
\[
\begin{aligned}
&\left\{ \begin{array}{c}
e v_0 \frac{\partial f^\epsilon}{\partial \eta} - \frac{v_0}{R_\kappa} - \frac{R_\kappa}{\eta (r^2 + r^2)^{\frac{1}{2}}} \frac{\partial f^\epsilon}{\partial \theta} - \frac{v_0^2}{R_\kappa} \frac{\partial f^\epsilon}{\partial \eta} + \frac{v_\eta v_\phi}{R_\kappa - \eta^2} \frac{\partial f^\epsilon}{\partial \phi} + L[f^\epsilon] = \Gamma[f^\epsilon, f^\prime],
\end{array} \right.
\end{aligned}
\]
\[
f^\epsilon(0, \theta, \vec{v}) = \mathcal{P}^\epsilon[f^\epsilon](0, \theta, \vec{v}) \text{ for } v_\eta > 0,
\]
where
\[
\mathcal{P}^\epsilon[f^\epsilon](0, \theta, \vec{v}) = \mu_0^\epsilon(\theta, \vec{v}) \mu_0 - \frac{1}{2} \left( \mu_0 \left( \frac{\partial}{\partial \eta} \right) \varphi \right) \int_{\vec{u}_\eta < 0} \mu_0^\epsilon(\vec{u}_\eta) f^\epsilon(0, \theta, \vec{u}_\eta) |\vec{u}_\eta| d\vec{u} + \mu_0^\epsilon(\vec{v}) \left( \mu_0^\epsilon(\theta, \vec{v}) - \mu(\vec{v}) \right).
\]

Substitution 3: Scaling Substitution.
We define the rescaled variable \( \eta = \frac{\eta}{\epsilon} \), which implies \( \frac{\partial}{\partial \eta} = \frac{1}{\epsilon} \frac{\partial}{\partial \eta} \). Then, under the substitution \( \eta \rightarrow \eta \), the equation (3.1) is transformed into
\[
\begin{aligned}
&\left\{ \begin{array}{c}
v_\eta \frac{\partial f^\epsilon}{\partial \eta} - \frac{v_0}{R_\kappa} - \frac{R_\kappa}{\eta (r^2 + r^2)^{\frac{1}{2}}} \frac{\partial f^\epsilon}{\partial \theta} - \frac{v_0^2}{R_\kappa} \frac{\partial f^\epsilon}{\partial \eta} + \frac{v_\eta v_\phi}{R_\kappa - \eta^2} \frac{\partial f^\epsilon}{\partial \phi} + L[f^\epsilon] = \Gamma[f^\epsilon, f^\prime],
\end{array} \right.
\end{aligned}
\]
\[
f^\epsilon(0, \theta, \vec{v}) = \mathcal{P}^\epsilon[f^\epsilon](0, \theta, \vec{v}) \text{ for } v_\eta > 0,
\]
where
\[
\mathcal{P}^\epsilon[f^\epsilon](0, \theta, \vec{v}) = \mu_0^\epsilon(\theta, \vec{v}) \mu_0 - \frac{1}{2} \left( \mu_0 \left( \frac{\partial}{\partial \eta} \right) \varphi \right) \int_{\vec{u}_\eta < 0} \mu_0^\epsilon(\vec{u}_\eta) f^\epsilon(0, \theta, \vec{u}_\eta) |\vec{u}_\eta| d\vec{u} + \mu_0^\epsilon(\vec{v}) \left( \mu_0^\epsilon(\theta, \vec{v}) - \mu(\vec{v}) \right).
\]

We define the boundary layer expansion as follows:
\[
\mathcal{F}(\eta, \theta, \vec{v}) \sim \sum_{k=1}^{2} \epsilon^k \mathcal{F}_k(\eta, \theta, \vec{v}),
\]
where \( \mathcal{F}_k \) can be defined by comparing the order of \( \epsilon \) via plugging (3.48) into the equation (3.46). Thus, in a neighborhood of the boundary, we have
\[
v_\eta \frac{\partial \mathcal{F}_1}{\partial \eta} - \frac{v_0}{R_\kappa} - \frac{R_\kappa}{\eta (r^2 + r^2)^{\frac{1}{2}}} \frac{\partial \mathcal{F}_1}{\partial \theta} + L[\mathcal{F}_1] = 0,
\]
(3.49)
\[
v_\eta \frac{\partial \mathcal{F}_2}{\partial \eta} - \frac{v_0}{R_\kappa} - \frac{R_\kappa}{\eta (r^2 + r^2)^{\frac{1}{2}}} \frac{\partial \mathcal{F}_2}{\partial \theta} + L[\mathcal{F}_2] = 2 \Gamma[\mathcal{F}_1, \mathcal{F}_1] + \Gamma[\mathcal{F}_1, \mathcal{F}_1] + \frac{v_0}{R_\kappa} - \frac{R_\kappa}{\eta (r^2 + r^2)^{\frac{1}{2}}} \frac{\partial \mathcal{F}_1}{\partial \theta}.
\]
(3.50)

3.3. Expansion of Boundary Conditions. The bridge between the interior solution and boundary layer is the boundary condition
\[
f^\epsilon(\vec{x}_0, \vec{v}) = \mathcal{P}^\epsilon[f^\epsilon](\vec{x}_0, \vec{v}),
\]
(3.51)
where
\[
\mathcal{P}^\epsilon[f^\epsilon](\vec{x}_0, \vec{v}) = \mu_0^\epsilon(\vec{x}_0, \vec{v}) \mu_0 - \frac{1}{2} \left( \mu_0 \left( \frac{\partial}{\partial \eta} \right) \varphi \right) \int_{\vec{u} \cdot \vec{v}(\vec{x}_0) > 0} \mu_0^\epsilon(\vec{u}) f^\epsilon(\vec{x}_0, \vec{u}, |\vec{u} \cdot \vec{v}(\vec{x}_0)|) d\vec{u} + \mu_0^\epsilon(\vec{v}) \left( \mu_0^\epsilon(\vec{x}_0, \vec{v}) - \mu(\vec{v}) \right).
\]
(3.52)

Plugging the combined expansion
\[
f^\epsilon \sim \sum_{k=1}^{3} \epsilon^k F_k + \sum_{k=1}^{2} \epsilon^k \mathcal{F}_k,
\]
(3.53)
A direct computation reveals that

\[ F_1 + \mathcal{F}_1 = \mu^{\frac{1}{2}}(\bar{v}) \int_{\bar{u} \cdot \bar{v}(\bar{x}) > 0} \mu^{\frac{1}{2}}(\bar{u})(F_1 + \mathcal{F}_1) |\bar{u} \cdot \bar{v}(\bar{x})| \, d\bar{u} + \mu_1(\bar{x}_0, \bar{v}), \]  

(3.54)

\[ F_2 + \mathcal{F}_2 = \mu^{\frac{1}{2}}(\bar{v}) \int_{\bar{u} \cdot \bar{v}(\bar{x}) > 0} \mu^{\frac{1}{2}}(\bar{u})(F_2 + \mathcal{F}_2) |\bar{u} \cdot \bar{v}(\bar{x})| \, d\bar{u} \]

+ \mu_1(\bar{x}_0, \bar{v}) \int_{\bar{u} \cdot \bar{v}(\bar{x}) > 0} \mu^{\frac{1}{2}}(\bar{u})(F_1 + \mathcal{F}_1) |\bar{u} \cdot \bar{v}(\bar{x})| \, d\bar{u} + \mu_2(\bar{x}_0, \bar{v}).

(3.55)



In particular, we do not further expand the boundary layer, so we directly require

\[ F_3 = \mu^{\frac{1}{2}}(\bar{v}) \int_{\bar{u} \cdot \bar{v}(\bar{x}) > 0} \mu^{\frac{1}{2}}(\bar{u}) F_3 |\bar{u} \cdot \bar{v}(\bar{x})| \, d\bar{u} \]

+ \mu_2(\bar{x}_0, \bar{v}) \int_{\bar{u} \cdot \bar{v}(\bar{x}) > 0} \mu^{\frac{1}{2}}(\bar{u})(F_1 + \mathcal{F}_1) |\bar{u} \cdot \bar{v}(\bar{x})| \, d\bar{u}

+ \mu_1(\bar{x}_0, \bar{v}) \int_{\bar{u} \cdot \bar{v}(\bar{x}) > 0} \mu^{\frac{1}{2}}(\bar{u})(F_2 + \mathcal{F}_2) |\bar{u} \cdot \bar{v}(\bar{x})| \, d\bar{u} + \mu_3(\bar{x}_0, \bar{v}).

(3.56)

Define

\[ \mathcal{P}[f](\bar{x}_0, \bar{v}) = \mu^{\frac{1}{2}}(\bar{v}) \int_{\bar{u} \cdot \bar{v}(\bar{x}) > 0} \mu^{\frac{1}{2}}(\bar{u}) f(\bar{x}_0, \bar{u}) |\bar{u} \cdot \bar{v}(\bar{x})| \, d\bar{u}. \]

(3.57)

Then we have

\[ F_1 + \mathcal{F}_1 = \mathcal{P}[F_1 + \mathcal{F}_1] + \mu_1(\bar{x}_0, \bar{v}), \]

(3.58)

\[ F_2 + \mathcal{F}_2 = \mathcal{P}[F_2 + \mathcal{F}_2] + \mu_1(\bar{x}_0, \bar{v}) \int_{\bar{u} \cdot \bar{v}(\bar{x}) > 0} \mu^{\frac{1}{2}}(\bar{u})(F_1 + \mathcal{F}_1) |\bar{u} \cdot \bar{v}(\bar{x})| \, d\bar{u} + \mu_2(\bar{x}_0, \bar{v}), \]

(3.59)

and

\[ F_3 = \mathcal{P}[F_3] + \mu_2(\bar{x}_0, \bar{v}) \int_{\bar{u} \cdot \bar{v}(\bar{x}) > 0} \mu^{\frac{1}{2}}(\bar{u})(F_1 + \mathcal{F}_1) |\bar{u} \cdot \bar{v}(\bar{x})| \, d\bar{u} \]

+ \mu_1(\bar{x}_0, \bar{v}) \int_{\bar{u} \cdot \bar{v}(\bar{x}) > 0} \mu^{\frac{1}{2}}(\bar{u})(F_2 + \mathcal{F}_2) |\bar{u} \cdot \bar{v}(\bar{x})| \, d\bar{u} + \mu_3(\bar{x}_0, \bar{v}).

(3.60)

This is the boundary conditions \( F_k \) and \( \mathcal{F}_k \) need to satisfy.

3.4. **Matching Procedure.** Define the length of boundary layer \( L = \epsilon^{-s} \) for \( 0 < s < \frac{1}{2} \). Also, denote \( \mathcal{A}[v_\eta, v_\phi] = (-v_\eta, v_\phi) \). We divide the construction of the asymptotic expansion into several steps for each \( k \geq 1 \):

Step 1: Construction of \( F_1 \) and \( \mathcal{F}_1 \).

A direct computation reveals that \( F_1 = A_1 + B_1 + C_1 \), where \( B_1 = C_1 = 0 \). Based on our expansion,

\[ \mu_1 = \mu^{\frac{1}{2}} \left( \rho_{b,1} + \bar{u}_{b,1} \cdot \bar{v} + \theta_{b,1} \frac{|\bar{v}|^2 - 2}{2} \right). \]

(3.61)

Define

\[ F_1 = \mu^{\frac{1}{2}} \left( \rho_1 + \bar{u}_1 \cdot \bar{v} + \theta_1 \frac{|\bar{v}|^2 - 2}{2} \right), \]

(3.62)
satisfying the Navier-Stokes-Fourier system as
\[
\begin{aligned}
\nabla_x (\rho_1 + \theta_1) &= 0, \\
\dot{u}_1 \cdot \nabla_x \dot{u}_1 - \gamma_1 \Delta_x \dot{u}_1 + \nabla_x P_2 &= 0, \\
\nabla_x \cdot \dot{u}_1 &= 0, \\
\dot{u}_1 \cdot \nabla_x \theta_1 - \gamma_2 \Delta_x \theta_1 &= 0,
\end{aligned}
\]
(3.63)
where \( M_1(\bar{x}_0) \) is a constant such that the Boussinesq relation
\[ \rho_1 + \theta_1 = \text{constant}, \]
(3.64)
is satisfied. Note that this constant is determined by the normalization condition.

\[
\int_\Omega \int_{\mathbb{R}^2} F_1(\bar{x}, \bar{v}) \mu^\frac{1}{2}(\bar{v}) d\bar{v} d\bar{x} = 0,
\]
(3.65)
and we are able to add \( M_1(\bar{x}_0) \) freely since \( \mu^\frac{1}{2} = \mathcal{P}[\mu^\frac{1}{2}] \). Then based on the compatibility condition of \( \mu_1 \) as
\[
\int_{\bar{u} \neq (\bar{x}_0)} \mu^\frac{1}{2}(\bar{u}) \mu_1(\bar{x}_0, \bar{u}) |\bar{u} \cdot \bar{v}(\bar{x}_0)| |d\bar{u}| = 0,
\]
(3.66)
we naturally obtain \( \mathcal{P}[F_1] = M_1 \mu^\frac{1}{2} \), which means
\[ F_1 = \mathcal{P}[F_1] + \mu_1 \text{ on } \partial\Omega. \]
(3.67)
Therefore, it is not necessary to introduce the boundary layer at this order and we simply take \( \mathcal{F}_1 = 0 \).

**Step 2: Construction of \( F_2 \) and \( \mathcal{F}_2 \).**

Define \( F_2 = A_2 + B_2 + C_2 \), where \( B_2 \) and \( C_2 \) can be uniquely determined following previous analysis, and
\[ A_2 = \mu^\frac{1}{2} \left( \rho_2 + \bar{u}_2 \cdot \bar{v} + \theta_2 |\bar{v}|^2 - \frac{2}{2} \right), \]
(3.68)
satisfying a more complicated fluid-type equation as in [24, 25]. On the other hand, \( \mathcal{F}_2 \) satisfies the \( \epsilon \)-Milne problem with geometric correction
\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{\partial \mathcal{F}_2}{\partial \eta} - \frac{\epsilon}{R_\kappa} \left( \frac{\partial \mathcal{F}_2}{\partial v_\phi} - v_\phi \frac{\partial \mathcal{F}_2}{\partial v_\phi} \right) + \mathcal{L}[\mathcal{F}_2] = 0 \text{ for } (\eta, \theta, \bar{v}) \in (0, L] \times [-\pi, \pi] \times \mathbb{R}^2, \\
\mathcal{F}_2(0, \theta, \bar{v}) = h(\theta, \bar{v}) - \tilde{h}(\theta, \bar{v}) \text{ for } v_\eta > 0, \\
\mathcal{F}_2(L, \theta, \bar{v}) = \mathcal{F}_2(L, \theta, \mathcal{F}[\bar{v}]),
\end{array} \right.
\]
(3.69)
with the in-flow boundary data
\[
h(\theta, \bar{v}) = \mu_1(\bar{x}_0, \bar{v}) \int_{\bar{u} \neq (\bar{x}_0)} \mu^\frac{1}{2}(\bar{u})(F_1 + \mathcal{F}_1) |\bar{u} \cdot \bar{v}(\bar{x}_0)| d\bar{u} + \mu_2(\bar{x}_0, \bar{v}) - \left( (B_2 + C_2) - \mathcal{P}[B_2 + C_2] \right).
\]
(3.70)
Based on Theorem 5.7.9 there exists
\[
\tilde{h}(\theta, \bar{v}) = \mu^\frac{1}{2} \left( \tilde{D}_0(\theta) + \tilde{D}_1(\theta)v_\eta + \tilde{D}_2(\theta)v_\phi + \tilde{D}_3(\theta) \left| \bar{v} \right|^2 - \frac{2}{2} \right),
\]
(3.71)
such that the \( \epsilon \)-Milne problem with geometric correction is well-posed and the solution decays exponentially fast. In particular, \( \tilde{D}_1 = 0 \). Then we further require that \( A_2 \) satisfies the boundary condition
\[ A_2(\bar{x}_0, \bar{v}) = \tilde{h}(\theta, \bar{v}) + M_2(\bar{x}_0) \mu^\frac{1}{2}(\bar{v}). \]
(3.72)
Here, the constant $M_2(\bar{x}_0)$ is chosen to enforce the Boussinesq relation
\[ P_2 - (\rho_2 + \theta_2 + \rho_1\theta_1) = 0. \]  
(3.73)

Similar to the construction of $F_1$, we can choose the constant to satisfy the normalization condition
\[ \int_{\Omega} \int_{\mathbb{R}^2} (F_2 + \mathcal{F}_2)(\bar{x}, \bar{v}) \mu^\perp(\bar{v}) d\bar{v} d\bar{x} = 0. \]  
(3.74)

Also, the construction implies that at boundary, we have
\[ A_2 + \mathcal{F}_2 = M_2 \mu^\perp + h \]
\[ = M_2 \mu^\perp + \mu_1(\bar{x}_0, \bar{v}) \int_{\mathbb{R}^2} \mu^\perp(\bar{u})(F_1 + \mathcal{F}_1) |\bar{u} \cdot \bar{v}(\bar{x}_0)| d\bar{u} + \mu_2(\bar{x}_0, \bar{v}) - (B_2 + C_2) - \mathcal{P}[B_2 + C_2]. \]  
(3.75)

Comparing this with the desired boundary expansion
\[ F_2 + \mathcal{F}_2 = \mathcal{P}[F_2 + \mathcal{F}_2] + \mathcal{B}_2, \]
we only need to verify that
\[ \mathcal{P}[A_2 + \mathcal{F}_2] = M_2 \mu^\perp. \]  
(3.77)

We may directly verify the zero mass-flux condition of $\mathcal{F}_2$ as
\[ \int_{\mathbb{R}^2} \mu^\perp(\bar{u}) \mathcal{F}_2(\bar{x}, \bar{u})(\bar{u} \cdot \bar{v}) d\bar{u} = 0, \]  
(3.78)

and the compatibility condition
\[ \int_{\mathbb{R}^2} \mu^\perp(\bar{u}) \mu_1(\bar{x}_0, \bar{u}) |\bar{u} \cdot \bar{v}(\bar{x}_0)| d\bar{u} = \int_{\mathbb{R}^2} \mu^\perp(\bar{u}) \mu_2(\bar{x}_0, \bar{u}) |\bar{u} \cdot \bar{v}(\bar{x}_0)| d\bar{u} = 0. \]  
(3.79)

Then we naturally derive
\[ \mathcal{P}[A_2 + \mathcal{F}_2] \]
\[ = \mu^\perp \int_{\mathbb{R}^2} \mu^\perp(\bar{u}) A_2(\bar{x}, \bar{u})(\bar{u} \cdot \bar{v}) d\bar{u} + \mu^\perp \int_{\mathbb{R}^2} \mu^\perp(\bar{u}) \mathcal{F}_2(\bar{x}, \bar{u})(\bar{u} \cdot \bar{v}) d\bar{u} \]
\[ = M_2 \mu^\perp + \mu^\perp \int_{\mathbb{R}^2} \mu^\perp(\bar{u}) \tilde{h}(\bar{x}, \bar{u})(\bar{u} \cdot \bar{v}) d\bar{u} + \mu^\perp \int_{\mathbb{R}^2} \mu^\perp(\bar{u}) \mathcal{F}_2(\bar{x}, \bar{u})(\bar{u} \cdot \bar{v}) d\bar{u} \]
\[ = M_2 \mu^\perp + \mu^\perp \int_{\mathbb{R}^2} \mu^\perp(\bar{u}) \tilde{h}(\bar{x}, \bar{u})(\bar{u} \cdot \bar{v}) d\bar{u} - \mu^\perp \int_{\mathbb{R}^2} \mu^\perp(\bar{u}) \mathcal{F}_2(\bar{x}, \bar{u})(\bar{u} \cdot \bar{v}) d\bar{u} \]
\[ = M_2 \mu^\perp + \mu^\perp \int_{\mathbb{R}^2} \mu^\perp(\bar{u}) \tilde{h}(\bar{x}, \bar{u})(\bar{u} \cdot \bar{v}) d\bar{u} - \mu^\perp \int_{\mathbb{R}^2} \mu^\perp(\bar{u}) \tilde{h}(\bar{x}, \bar{u})(\bar{u} \cdot \bar{v}) d\bar{u} \]
\[ = M_2 \mu^\perp + 0 - 0 \]
\[ = M_2 \mu^\perp. \]

$F_3$ can be defined in a similar fashion which satisfies an even more complicated fluid-type system.
4. Remainder Estimates

We consider the linearized stationary Boltzmann equation

\[
\begin{cases}
\epsilon \vec{v} \cdot \nabla_x f + \mathcal{L}[f] = S(\vec{x}, \vec{v}) \text{ in } \Omega, \\
f(\vec{x}_0, \vec{v}) = \mathcal{P}[f](\vec{x}_0, \vec{v}) + h(\vec{x}_0, \vec{v}) \text{ for } \vec{x}_0 \in \partial \Omega \text{ and } \vec{v} \cdot \vec{v} < 0,
\end{cases}
\]  

where

\[ \mathcal{P}[f](\vec{x}_0, \vec{v}) = \mu^+(\vec{v}) \int_{\vec{u}, \vec{v}(\vec{x}_0) > 0} \mu^+(\vec{u}) f(\vec{x}_0, \vec{u}) |\vec{u} \cdot \vec{v}(\vec{x}_0)| \, d\vec{u}, \]

provided the compatibility condition

\[ \int_{\Omega \times \mathbb{R}^2} S(\vec{x}, \vec{v}) \mu^+(\vec{v}) d\vec{x} d\vec{v} = 0, \quad \int_{\gamma_-} h(\vec{x}, \vec{v}) \mu^+(\vec{v}) d\gamma = 0. \]

It is easy to see if \( f \) is a solution to (4.1), then \( f + C \mu^+ \) is also a solution for arbitrary \( C \in \mathbb{R} \). Hence, we require that the solution should satisfy the normalization condition

\[ \int_{\Omega \times \mathbb{R}^2} f(\vec{x}, \vec{v}) \mu^+(\vec{v}) d\vec{x} d\vec{v} = 0. \]

Our analysis is based on the ideas in [9, 16]. Since the well-posedness of (4.1) is standard, we will focus on the a priori estimates here.

4.1. \( L^2 \) Estimates.

**Lemma 4.1.** The solution \( f(\vec{x}, \vec{v}) \) to the equation (4.1) satisfies the estimate

\[ \epsilon \| \mathcal{P}[f] \|_{L^2} \leq C \left( \epsilon \| (1 - \mathcal{P})[f] \|_{L^2}^2 + \| (1 - \mathcal{P})[f] \|_{L^2} + \| S \|_{L^2} + \epsilon \| h \|_{L^2} \right). \]

**Proof.** Applying Green’s identity in Lemma 2.6 to the equation (4.1). Then for any \( \psi \in L^2(\Omega \times \mathbb{R}^2) \) satisfying \( \vec{v} \cdot \nabla_x \psi \in L^2(\Omega \times \mathbb{R}^2) \) and \( \psi \in L^2(\gamma_-) \), we have

\[ \epsilon \int_{\gamma_+} f \psi d\gamma - \epsilon \int_{\gamma_-} f \psi d\gamma - \epsilon \int_{\Omega \times \mathbb{R}^2} (\vec{v} \cdot \nabla_x \psi) f = - \epsilon \int_{\Omega \times \mathbb{R}^2} \psi \mathcal{L}[f] + \int_{\Omega \times \mathbb{R}^2} S \psi. \]

Considering that \( f = \mathcal{P}[f] + (1 - \mathcal{P})[f] \) and \( \mathcal{L}[\mathcal{P}[f]] = 0 \), we may simplify

\[ \epsilon \int_{\gamma_+} f \psi d\gamma - \epsilon \int_{\gamma_-} f \psi d\gamma - \epsilon \int_{\Omega \times \mathbb{R}^2} (\vec{v} \cdot \nabla_x \psi) f = - \epsilon \int_{\Omega \times \mathbb{R}^2} \psi \mathcal{L}[f] + \int_{\Omega \times \mathbb{R}^2} S \psi. \]

Since

\[ \mathcal{P}[f] = \mu^+ \left( a + \vec{v} \cdot \vec{b} + \frac{|\vec{v}|^2}{2} \epsilon \right), \]

our goal is to choose a particular test function \( \psi \) to estimate \( a, \vec{b} \), and \( c \).

Step 1: Estimates of \( c \).

We choose the test function

\[ \psi = \psi_c = \mu^+ (\vec{v}) \left( |\vec{v}|^2 - \beta_c \right) \left( \vec{v} \cdot \nabla_x \phi_c(\vec{x}) \right), \]

where

\[ \begin{cases}
-\Delta_x \phi_c = c(\vec{x}) \text{ in } \Omega, \\
\phi_c = 0 \text{ on } \partial \Omega,
\end{cases} \]

and \( \beta_c \) is a real number to be determined later. Based on the standard elliptic estimates, we have

\[ \| \phi_c \|_{H^2} \leq C \| c \|_{L^2}. \]

With the choice of (4.9), the right-hand side (RHS) of (4.7) is bounded by

\[ \text{RHS} \leq C \| c \|_{L^2} \left( \| (1 - \mathcal{P})[f] \|_{L^2} + \| S \|_{L^2} \right). \]
We have
\[ \vec{v} \cdot \nabla_x \psi_c = \mu^\frac{1}{2}(\vec{v}) \sum_{i,j=1}^2 (|\vec{v}|^2 - \beta_c) v_i v_j \partial_{ij} \phi_c, \] (4.13)
so the left-hand side (LHS) of (4.13) takes the form
\[ \text{LHS} = \epsilon \int_{\partial \Omega \times \mathbb{R}^2} f \mu^\frac{1}{2}(\vec{v}) \left(|\vec{v}|^2 - \beta_c\right) \left(\sum_{i=1}^2 v_i \partial_i \phi_c\right) (\vec{v} \cdot \vec{v}) \] (4.14)
\[ \quad - \epsilon \int_{\Omega \times \mathbb{R}^2} f \mu^\frac{1}{2}(\vec{v}) \left(|\vec{v}|^2 - \beta_c\right) \left(\sum_{i,j=1}^2 v_i v_j \partial_{ij} \phi_c\right). \]

We decompose
\[ f = \mathcal{P}[f] + 1_{\gamma_+}(1 - \mathcal{P})[f] + 1_{\gamma_0} h \text{ on } \gamma, \] (4.15)
\[ f = \mu^\frac{1}{2}\left(a + \vec{v} \cdot \vec{b} + \frac{|\vec{v}|^2 - 2c}{2} \right) + (\mathbb{I} - \mathcal{P})[f] \text{ in } \Omega \times \mathbb{R}^2. \] (4.16)

We will choose \( \beta_c \) such that
\[ \int_{\mathbb{R}^2} \mu^\frac{1}{2}(\vec{v}) \left(|\vec{v}|^2 - \beta_c\right) v_i^2 d\vec{v} = 0 \text{ for } i = 1, 2. \] (4.17)
It is easy to check that this \( \beta_c \) can always be achieved. Now substitute (4.13) and (4.16) into (4.14). Then based on this choice of \( \beta_c \) and the oddness in \( \vec{v} \), there is no \( \mathcal{P}[f] \) contribution in the first term, and no \( a \) contribution in the second term of (4.14). Since \( \vec{b} \) contribution and the off-diagonal \( c \) contribution in the second term of (4.14) also vanish due to the oddness in \( \vec{v} \), we can simplify (4.14) into
\[ \text{LHS} = \epsilon \int_{\partial \Omega \times \mathbb{R}^2} 1_{\gamma_+} (1 - \mathcal{P})[f] \mu^\frac{1}{2}(\vec{v}) \left(|\vec{v}|^2 - \beta_c\right) \left(\sum_{i=1}^2 v_i \partial_i \phi_c\right) (\vec{v} \cdot \vec{v}) \] (4.18)
\[ \quad + \epsilon \int_{\partial \Omega \times \mathbb{R}^2} 1_{\gamma_0} h \mu^\frac{1}{2}(\vec{v}) \left(|\vec{v}|^2 - \beta_c\right) \left(\sum_{i=1}^2 v_i \partial_i \phi_c\right) (\vec{v} \cdot \vec{v}) \]
\[ \quad - \epsilon \sum_{i=1}^2 \int_{\mathbb{R}^2} \mu(\vec{v}) |v_i|^2 \left(|\vec{v}|^2 - \beta_c\right) \frac{|\vec{v}|^2 - 2}{2} d\vec{v} \int_{\Omega} c(\vec{x}) \partial_i \phi_c(\vec{x}) d\vec{x} \]
\[ \quad - \epsilon \int_{\Omega \times \mathbb{R}^2} (\mathbb{I} - \mathcal{P})[f] \mu^\frac{1}{2}(\vec{v}) \left(|\vec{v}|^2 - \beta_c\right) \left(\sum_{i,j=1}^2 v_i v_j \partial_{ij} \phi_c\right). \]

Since
\[ \int_{\mathbb{R}^2} \mu(\vec{v}) |v_i|^2 \left(|\vec{v}|^2 - \beta_c\right) \frac{|\vec{v}|^2 - 2}{2} d\vec{v} = C, \] (4.19)
we have
\[ \epsilon \left| \int_{\Omega} \Delta_x \phi_c(\vec{x}) c(\vec{x}) d\vec{x} \right| \leq C \|c\|_{L^2} \left(\epsilon \|(1 - \mathcal{P})[f]\|_{L^2} + \|(\mathbb{I} - \mathcal{P})[f]\|_{L^2} + \|S\|_{L^2} + \epsilon |h|_{L^2} \right), \] (4.20)
where we have used the elliptic estimates and the trace estimate: \|\nabla_x \phi_c\|_{L^2} \leq C \|\phi_c\|_{H^2} \leq C \|c\|_{L^2}. Since \(-\Delta_x \phi_c = c,\) we know
\[ \epsilon \|c\|_{L^2} \leq C \|c\|_{L^2} \left(\epsilon \|(1 - \mathcal{P})[f]\|_{L^2} + \|(\mathbb{I} - \mathcal{P})[f]\|_{L^2} + \|S\|_{L^2} + \epsilon |h|_{L^2} \right), \] (4.21)
which further implies
\[ \epsilon \|c\|_{L^2} \leq C \left(\epsilon \|(1 - \mathcal{P})[f]\|_{L^2} + \|(\mathbb{I} - \mathcal{P})[f]\|_{L^2} + \|S\|_{L^2} + \epsilon |h|_{L^2} \right). \] (4.22)

Step 2: Estimates of \( \vec{b}. \)
We further divide this step into several sub-steps:

Step 2.1: Estimates of \( \partial_{ij} \Delta_{x}^{-1} b_{j} \) for \( i, j = 1, 2 \).

We choose the test function

\[
\psi = \psi_{b}^{i,j} = \mu \hat{\xi}(\bar{v}) \left( v_{i}^{2} - \beta_{b} \right) \partial_{j} \phi_{b}^{i},
\]

where

\[
\begin{align*}
-\Delta_{x} \phi_{b}^{i} &= b_{j}(\bar{v}) \text{ in } \Omega, \\
\phi_{b}^{i} &= 0 \text{ on } \partial \Omega,
\end{align*}
\]

and \( \beta_{b} \) is a real number to be determined later. Based on the standard elliptic estimates, we have

\[
\| \phi_{b}^{i} \|_{H^{2}} \leq C \| \bar{b} \|_{L^{2}},
\]

With the choice of (4.23), the right-hand side (RHS) of (4.7) is bounded by

\[
\text{RHS} \leq C \| \bar{b} \|_{L^{2}} \left( \| (I - P) f \|_{L^{2}} + \| S \|_{L^{2}} \right).
\]

Hence, the left-hand side (LHS) of (4.7) takes the form

\[
\text{LHS} = \epsilon \int_{\Omega \times \mathbb{R}^{2}} f \mu \hat{\xi}(\bar{v}) \left( v_{i}^{2} - \beta_{b} \right) \partial_{j} \phi_{b}^{i} (\bar{v} \cdot \bar{v})
\]

\[
- \epsilon \int_{\Omega \times \mathbb{R}^{2}} f \mu \hat{\xi}(\bar{v}) \left( v_{i}^{2} - \beta_{b} \right) \left( \sum_{l=1}^{2} v_{l} \partial_{j} \phi_{b}^{l} \right).
\]

Now substitute (4.15) and (4.16) into (4.27). Then based on the oddness in \( \bar{v} \), there is no \( P[f] \) contribution in the first term, and no \( a \) and \( c \) contribution in the second term of (4.27). We can simplify (4.27) into

\[
\text{LHS} = \epsilon \int_{\Omega \times \mathbb{R}^{2}} 1_{\gamma_{+}} (1 - P)[f] \mu \hat{\xi}(\bar{v}) \left( v_{i}^{2} - \beta_{b} \right) \partial_{j} \phi_{b}^{i} (\bar{v} \cdot \bar{v})
\]

\[
+ \epsilon \int_{\Omega \times \mathbb{R}^{2}} 1_{\gamma_{-}} \mu \hat{\xi}(\bar{v}) \left( v_{i}^{2} - \beta_{b} \right) \partial_{j} \phi_{b}^{i} (\bar{v} \cdot \bar{v})
\]

\[
- \epsilon \sum_{l=1}^{2} \int_{\Omega \times \mathbb{R}^{2}} \mu(\bar{v}) v_{l}^{2} \left( v_{i}^{2} - \beta_{b} \right) \partial_{j} \phi_{b}^{l} b_{l}
\]

\[
- \epsilon \sum_{l=1}^{2} \int_{\Omega \times \mathbb{R}^{2}} (I - P)[f] \mu \hat{\xi}(\bar{v}) \left( v_{i}^{2} - \beta_{b} \right) v_{l} \partial_{j} \phi_{b}^{l}.
\]

We will choose \( \beta_{b} \) such that

\[
\int_{\mathbb{R}^{2}} \mu(\bar{v}) \left( |v_{i}|^{2} - \beta_{b} \right) d\bar{v} = 0 \text{ for } i = 1, 2.
\]

It is easy to check that this \( \beta_{b} \) can always be achieved. For such \( \beta_{b} \) and any \( i \neq l \), we can directly compute

\[
\int_{\mathbb{R}^{2}} \mu(\bar{v}) \left( |v_{i}|^{2} - \beta_{b} \right) v_{l}^{2} d\bar{v} = 0,
\]

\[
\int_{\mathbb{R}^{2}} \mu(\bar{v}) \left( |v_{i}|^{2} - \beta_{b} \right) v_{l}^{2} d\bar{v} = C \neq 0.
\]

Then we deduce

\[
- \epsilon \sum_{l=1}^{2} \int_{\Omega \times \mathbb{R}^{2}} \mu(\bar{v}) v_{l}^{2} \left( v_{i}^{2} - \beta_{b} \right) \partial_{j} \phi_{b}^{l} b_{l}
\]

\[
= - \epsilon \int_{\Omega \times \mathbb{R}^{2}} \mu(\bar{v}) v_{l}^{2} \left( v_{i}^{2} - \beta_{b} \right) \partial_{j} \phi_{b}^{l} b_{l} - \epsilon \sum_{l \neq 1} \int_{\Omega \times \mathbb{R}^{2}} \mu(\bar{v}) v_{l}^{2} \left( v_{i}^{2} - \beta_{b} \right) \partial_{j} \phi_{b}^{l} b_{l}
\]

\[
= C \int_{\Omega} (\partial_{ij} \phi_{b}^{k}) b_{j} = C \int_{\Omega} \left( \partial_{ij} \Delta_{x}^{-1} b_{j} \right) b_{i}.
\]
Hence, similar to (4.22), we may estimate
\[ \epsilon \left| \int_{\Omega} (\partial_{ij} \Delta_{x}^{-1} b_{j}) b_{i} \right| \leq C \left\| \frac{\partial}{\partial x} \right\| \varepsilon \left( |(1 - \mathcal{P})[f]|_{L^{2}} + ||(I - \mathcal{P})[f]|_{L^{2}} + ||S||_{L^{2}} + \epsilon |h|_{L^{2}} \right). \] (4.32)

Step 2.2: Estimates of \( (\partial_{ij} \Delta_{x}^{-1} b_{j}) b_{i} \) for \( i \neq j \).

We choose the test function
\[ \psi = \mu^{\frac{1}{2}}(\tilde{v}) |\tilde{v}|^{2} v_{i} v_{j} \partial_{j} \phi_{b}^{i} \text{ for } i \neq j. \] (4.33)

The right-hand side (RHS) of (4.7) is still bounded by
\[ \text{RHS} \leq C \left\| \frac{\partial}{\partial x} \right\| \varepsilon \left( ||(I - \mathcal{P})[f]|_{L^{2}} + ||S||_{L^{2}} \right). \] (4.34)

Also, the left-hand side (LHS) of (4.7) takes the form
\[ \text{LHS} = \epsilon \int_{\Omega} \int_{\mathbb{R}^{2}} f \mu^{\frac{1}{2}}(\tilde{v}) |\tilde{v}|^{2} v_{i} v_{j} \partial_{j} \phi_{b}^{i} (\tilde{v} \cdot \tilde{v}) \] (4.35)

Now substitute (4.15) and (4.10) into (4.35). Then based on the oddness in \( \tilde{v} \), there is no \( \mathcal{P}[f] \) contribution in the first term, and no \( a \) and \( c \) contribution in the second term of (4.35). We can simplify (4.35) into
\[ \text{LHS} = \epsilon \int_{\Omega} \int_{\mathbb{R}^{2}} 1_{\gamma_{+}} (1 - \mathcal{P})[f] \mu^{\frac{1}{2}}(\tilde{v}) |\tilde{v}|^{2} v_{i} v_{j} \partial_{j} \phi_{b}^{i} (\tilde{v} \cdot \tilde{v}) \] (4.36)

Then we deduce
\[ - \epsilon \int_{\Omega} \mu(\tilde{v}) |\tilde{v}|^{2} v_{l}^{2} v_{j}^{2} \left( \partial_{ij} \phi_{b}^{l} b_{l} + \partial_{j} \phi_{b}^{l} b_{l} \right) = C \left( \int_{\Omega} (\partial_{ij} \Delta_{x}^{-1} b_{j}) b_{i} \right). \] (4.37)

Hence, we may estimate that for \( i \neq j \),
\[ \epsilon \left| \int_{\Omega} (\partial_{ij} \Delta_{x}^{-1} b_{j}) b_{i} \right| \leq C \left\| \frac{\partial}{\partial x} \right\| \varepsilon \left( |(1 - \mathcal{P})[f]|_{L^{2}} + ||(I - \mathcal{P})[f]|_{L^{2}} + ||S||_{L^{2}} + \epsilon |h|_{L^{2}} \right) \] (4.38)

Moreover, by (4.32), for \( i = j = 1, 2 \),
\[ \epsilon \left| \int_{\Omega} (\partial_{ij} \Delta_{x}^{-1} b_{j}) b_{j} \right| \leq C \left\| \frac{\partial}{\partial x} \right\| \varepsilon \left( |(1 - \mathcal{P})[f]|_{L^{2}} + ||(I - \mathcal{P})[f]|_{L^{2}} + ||S||_{L^{2}} + \epsilon |h|_{L^{2}} \right). \] (4.39)

Step 2.3: Synthesis.

Summarizing (4.38) and (4.39), we may sum up over \( j = 1, 2 \) to obtain that for any \( i = 1, 2 \),
\[ \epsilon \left| b_{i} b_{j} \right|_{L^{2}} \leq C \left\| \frac{\partial}{\partial x} \right\| \varepsilon \left( |(1 - \mathcal{P})[f]|_{L^{2}} + ||(I - \mathcal{P})[f]|_{L^{2}} + ||S||_{L^{2}} + \epsilon |h|_{L^{2}} \right). \] (4.40)

which further implies
\[ \epsilon \left\| \frac{\partial}{\partial x} \right\| \varepsilon \left( |(1 - \mathcal{P})[f]|_{L^{2}} + ||(I - \mathcal{P})[f]|_{L^{2}} + ||S||_{L^{2}} + \epsilon |h|_{L^{2}} \right). \] (4.41)
Step 3: Estimates of $a$.
We choose the test function
$$
\psi = \psi_a = \mu^{\frac{1}{2}}(\bar{v}) \left( |\bar{v}|^2 - \beta_a \right) \left( \bar{v} \cdot \nabla x \phi_a(x) \right),
$$
(4.42)
where
$$
\left\{ \begin{array}{ll}
-\Delta_x \phi_a &= a(x) \text{ in } \Omega, \\
\frac{\partial \phi_a}{\partial \nu} &= 0 \text{ on } \partial \Omega,
\end{array} \right.
$$
(4.43)
and $\beta_a$ is a real number to be determined later. Based on the standard elliptic estimates with the normalization condition
$$
\int_\Omega a(x)d\bar{x} = \int_{\Omega \times \mathbb{R}^2} f(\bar{x}, \bar{v})\mu^{\frac{1}{2}}(\bar{v})d\bar{v}d\bar{x} = 0,
$$
(4.44)
we have
$$
\|\phi_a\|_{H^2} \leq C \|a\|_{L^2}.
$$
(4.45)
With the choice of (4.42), the right-hand side (RHS) of (4.47) is bounded by
$$
\text{RHS} \leq C \|a\|_{L^2} \left( \| (\mathbb{I} - P) [f] \|_{L^2} + \| S \|_{L^2} \right).
$$
(4.46)
We have
$$
\bar{v} \cdot \nabla x \psi_a = \mu^{\frac{1}{2}}(\bar{v}) \sum_{i,j=1}^2 \left( |\bar{v}|^2 - \beta_a \right) v_i v_j \partial_{ij} \phi_a(x),
$$
(4.47)
so the left-hand side (LHS) of (4.47) takes the form
$$
\text{LHS} = \epsilon \int_{\partial \Omega \times \mathbb{R}^2} f \mu^{\frac{1}{2}}(\bar{v}) \left( |\bar{v}|^2 - \beta_a \right) \left( \sum_{i=1}^2 v_i \partial_i \phi_a \right) (\bar{v} \cdot \bar{v})
$$
$$
- \epsilon \int_{\Omega \times \mathbb{R}^2} f \mu^{\frac{1}{2}}(\bar{v}) \left( |\bar{v}|^2 - \beta_a \right) \left( \sum_{i,j=1}^2 v_i v_j \partial_{ij} \phi_a \right).
$$
(4.48)
We will choose $\beta_a$ such that
$$
\int_{\mathbb{R}^2} \mu^{\frac{1}{2}}(\bar{v}) \left( |\bar{v}|^2 - \beta_a \right) \frac{|\bar{v}|^2 - 2}{2} v_i^2 d\bar{v} = 0 \text{ for } i = 1, 2.
$$
(4.49)
It is easy to check that this $\beta_a$ can always be achieved. Now substitute (4.15) and (4.16) into (4.48). Then based on this choice of $\beta_a$ and the oddness in $\bar{v}$, there is no $\bar{b}$ and $c$ contribution in the second term of (4.48). Since the off-diagonal $a$ contribution in the second term of (4.48) also vanishes due to the oddness in $\bar{v}$, we
can simplify (4.48) into

\[
\text{LHS} = \epsilon \int_{\partial \Omega \times \mathbb{R}^2} P[f] \mu^\gamma(\bar{v}) \left( |\bar{v}|^2 - \beta_a \right) \left( \sum_{i=1}^{2} v_i \partial_i \phi_a \right) (\bar{v} \cdot \bar{v}) \\
+ \epsilon \int_{\partial \Omega \times \mathbb{R}^2} 1_{\gamma_T} (1 - P)[f] \mu^\gamma(\bar{v}) \left( |\bar{v}|^2 - \beta_a \right) \left( \sum_{i=1}^{2} v_i \partial_i \phi_a \right) (\bar{v} \cdot \bar{v}) \\
+ \epsilon \int_{\partial \Omega \times \mathbb{R}^2} 1_{\gamma_V} h \mu^\gamma (\bar{v}) \left( |\bar{v}|^2 - \beta_a \right) \left( \sum_{i=1}^{2} v_i \partial_i \phi_a \right) (\bar{v} \cdot \bar{v}) \\
- \sum_{i=1}^{2} \epsilon \int_{\mathbb{R}^2} \mu(\bar{v}) |v_i|^2 \left( |\bar{v}|^2 - \beta_a \right) d\bar{v} \int_\Omega a(\bar{x}) \partial_i \phi_a(\bar{x}) d\bar{x} \\
- \epsilon \int_{\Omega \times \mathbb{R}^2} (1 - P)[f] \mu^\gamma(\bar{v}) \left( |\bar{v}|^2 - \beta_a \right) \left( \sum_{i,j=1}^{2} v_i v_j \partial_{ij} \phi_a \right).
\]

(4.50)

We make an orthogonal decomposition on the boundary

\[
\bar{v} = (\bar{v} \cdot \nu) \nu + \bar{v}_\perp = v_n \bar{v} + \bar{v}_\perp,
\]

(4.51)

where \( \nu \perp \bar{v}_\perp \). Then the contribution of \( P[f] = z_\gamma(\bar{x}) \mu^\gamma(\bar{v}) \) for a suitable function \( z_\gamma(\bar{x}) \) is

\[
\epsilon \int_{\partial \Omega \times \mathbb{R}^2} P[f] \mu^\gamma(\bar{v}) \left( |\bar{v}|^2 - \beta_a \right) \left( \sum_{i=1}^{2} v_i \partial_i \phi_a \right) (\bar{v} \cdot \bar{v}) \\
= \epsilon \int_{\partial \Omega \times \mathbb{R}^2} \mu(\bar{v}) \left( |\bar{v}|^2 - \beta_a \right) v_n (\bar{v} \cdot \nabla_x \phi_a) z_\gamma(\bar{x}) \\
= \epsilon \int_{\partial \Omega \times \mathbb{R}^2} \mu(\bar{v}) \left( |\bar{v}|^2 - \beta_a \right) v_n \frac{\partial \phi_a}{\partial \gamma} z_\gamma(\bar{x}) + \epsilon \int_{\partial \Omega \times \mathbb{R}^2} \mu(\bar{v}) \left( |\bar{v}|^2 - \beta_a \right) v_n (\bar{v}_\perp \cdot \nabla_x \phi_a) z_\gamma(\bar{x}).
\]

(4.52)

Based on the definition of \( \phi_a \) and the oddness of \( v_n \bar{v}_\perp \), we know the contribution of \( P[f] \) in the first term of (4.48) vanishes. Since

\[
\int_{\mathbb{R}^2} \mu^\gamma(\bar{v}) |v_i|^2 \left( |\bar{v}|^2 - \beta_a \right) d\bar{v} = C,
\]

(4.53)

we have

\[
-\epsilon \int_\Omega \Delta_x \phi_a(\bar{x}) a(\bar{x}) d\bar{x} \leq C \| a \|_{L^2} \left( \epsilon \|(1 - P)[f]\|_{L^2_T} + \|(1 - P)[f]\|_{L^2} + \|S\|_{L^2} + \epsilon |h|_{L^2_T} \right).
\]

(4.54)

Since \( -\Delta_x \phi_a = a \), similar to (4.22), we know

\[
\epsilon \| a \|_{L^2} \leq C \left( \epsilon \|(1 - P)[f]\|_{L^2_T} + \|(1 - P)[f]\|_{L^2} + \|S\|_{L^2} + \epsilon |h|_{L^2_T} \right).
\]

(4.55)

Step 4: Synthesis.
Collecting (4.22), (4.41) and (4.55), we deduce

\[
\epsilon \| P[f] \|_{L^2} \leq C \left( \epsilon \|(1 - P)[f]\|_{L^2_T} + \|(1 - P)[f]\|_{L^2} + \|S\|_{L^2} + \epsilon |h|_{L^2_T} \right).
\]

(4.56)

This completes our proof. □

**Theorem 4.2.** The solution \( f(\bar{x}, \bar{v}) \) to the equation (4.7) satisfies the estimate

\[
\frac{1}{\epsilon^2} \|(1 - P)[f]\|_{L^2_T} + \|f\|_{L^2} \leq C \left( \frac{1}{\epsilon^2} \|P[S]\|_{L^2} + \frac{1}{\epsilon} \|(1 - P)[S]\|_{L^2} + \frac{1}{\epsilon} |h|_{L^2_T} \right).
\]

(4.57)
Proof. We divide it into several steps:

Step 1: Energy Estimate.
Multiplying $f$ on both sides of (4.63) and applying Green’s identity imply
\[
\frac{\epsilon}{2} |f|_{L^2_1}^2 + \langle L[f], f \rangle = \frac{\epsilon}{2} |P[f] + h|_{L^2_2}^2 + \int_{\Omega \times \mathbb{R}^2} f S. \tag{4.58}
\]
A direct computation shows that
\[
|f|_{L^2_1}^2 - |P[f]|_{L^2_2}^2 = |(1 - P)[f]|_{L^2_2}^2. \tag{4.59}
\]
Also, from the spectral gap of $L$, we know
\[
\langle L[f], f \rangle \geq C \| (1 - P)[f] \|_{L^2_2}. \tag{4.60}
\]
Applying Cauchy’s inequality implies that
\[
|P[f] + h|_{L^2_2}^2 = |P[f]|_{L^2_2}^2 + |h|_{L^2_2}^2 + 2 \int_{\gamma} P[f] h d\gamma \leq (1 + \eta) |P[f]|_{L^2_2}^2 + \left( 1 + \frac{1}{\eta} \right) |h|_{L^2_2}^2, \tag{4.61}
\]
for some $\eta > 0$. In total, we have
\[
\frac{\epsilon}{2} |(1 - P)[f]|_{L^2_2}^2 + \| (1 - P)[f] \|_{L^2_2}^2 \leq \eta^2 |P[f]|_{L^2_2}^2 + \left( 1 + \frac{1}{\eta} \right) |h|_{L^2_2}^2 + \int_{\Omega \times \mathbb{R}^2} f S. \tag{4.62}
\]

Step 2: Estimate of $|P[f]|_{L^2_2}$.
Multiplying $f$ on both sides of the equation (4.63), we have
\[
c\bar{v} \cdot \nabla_x (f^2) = -2fL[f] + 2fS. \tag{4.63}
\]
Taking absolute value and integrating (4.63) over $\Omega \times \mathbb{R}^2$, we deduce
\[
\| \bar{v} \cdot \nabla_x (f^2) \|_1 \leq \frac{1}{\epsilon} \left( \| (1 - P)[f] \|_{L^2_2} + \int_{\Omega \times \mathbb{R}^2} f S \right). \tag{4.64}
\]
On the other hand, by Lemma 2.6, for any $\gamma \setminus \gamma^\delta$ away from $\gamma_0$, we have
\[
|1_{\gamma \setminus \gamma^\delta} f|_{L^2_2}^2 \leq C(\delta) \left( \| f \|_{L^2_2}^2 + \| \bar{v} \cdot \nabla_x (f^2) \|_1 \right). \tag{4.65}
\]
Based on the definition, we can rewrite $Pf = z_\gamma(x) \mu^\xi$ for a suitable function $z_\gamma(x)$ and for $\delta$ small, we deduce
\[
|P[1_{\gamma \setminus \gamma^\delta} f]|_{L^2_2}^2 = \int_{\partial \Omega} |z_\gamma(x)|^2 \left( \int_{|\bar{v} \cdot \nabla_x (x)| \geq \delta, |\bar{n}\| \leq \frac{\delta}{2}} \mu(\bar{v}) |\bar{v} \cdot \nabla_x (x)| \, d\bar{v} \right) \, dx \tag{4.66}
\geq \frac{1}{2} \left( \int_{\partial \Omega} |z_\gamma(x)|^2 \, dx \right) \left( \int_{\mathbb{R}^2} \mu(\bar{v}) |\bar{v} \cdot \nabla_x (x)| \, d\bar{v} \right)
= \frac{1}{2} |P[f]|_{L^2_2}^2,
\]
where we utilize the fact that
\[
\int_{|\bar{v} \cdot \nabla_x (x)| \leq \delta} \mu(\bar{v}) |\bar{v} \cdot \nabla_x (x)| \, d\bar{v} \leq C\delta, \tag{4.67}
\]
\[
\int_{|\bar{n}| \leq \delta \text{ or } |\bar{n}| \geq \frac{\delta}{2}} \mu(\bar{v}) |\bar{v} \cdot \nabla_x (x)| \, d\bar{v} \leq C\delta. \tag{4.68}
\]
Therefore, from
\[
|P[1_{\gamma \setminus \gamma^\delta} f]|_{L^2_2} \leq C \|1_{\gamma \setminus \gamma^\delta} f\|_{L^2_2}, \tag{4.69}
\]
we conclude

\[
\frac{1}{2} |\mathcal{P}[f]_L^2 | \leq |\mathcal{P}[1_{\gamma \gamma^*} f]_L^2 | \leq C |1_{\gamma \gamma^*} f|_L^2 \\
\leq C(\delta) \left( \|f\|_L^2 + \|\gamma \cdot \nabla_x (f^2)\|_1 \right) \\
\leq C \left( \|f\|_L^2 + \frac{1}{\epsilon} \|(I - \mathcal{P})[f]\|_L^2 + \frac{1}{\epsilon} \int_{\Omega \times \mathbb{R}^2} f S \right).
\]

Hence, we know

\[
|\mathcal{P}[f]_L^2 | \leq C \left( \|f\|_L^2 + \frac{1}{\epsilon} \|(I - \mathcal{P})[f]\|_L^2 + \frac{1}{\epsilon} \int_{\Omega \times \mathbb{R}^2} f S \right),
\]

which can be further simplified as

\[
|\mathcal{P}[f]_L^2 | \leq C \left( \|\mathbb{P}[f]\|_L^2 + \frac{1}{\epsilon} \|(I - \mathcal{P})[f]\|_L^2 + \frac{1}{\epsilon} \int_{\Omega \times \mathbb{R}^2} f S \right).
\]

Step 3: Synthesis.
Plugging (4.72) into (4.62) with \( \epsilon \) sufficiently small to absorb \( \|(I - \mathcal{P})[f]\|_L^2 \) into the left-hand side, we obtain

\[
\epsilon |(1 - \mathcal{P})[f]_L^2 + \|(I - \mathcal{P})[f]\|_L^2 \leq C \left( \eta \epsilon \|\mathbb{P}[f]\|_L^2 + \left( 1 + \frac{1}{\eta} \right) |h|_L^2 + \int_{\Omega \times \mathbb{R}^2} f S \right).
\]

We square on both sides of (4.53) to obtain

\[
\epsilon^2 \|\mathbb{P}[f]\|_L^2 \leq C \left( \epsilon^2 |(1 - \mathcal{P})[f]_L^2 + \|(I - \mathcal{P})[f]\|_L^2 + |S|_L^2 + \epsilon^2 |h|_L^2 \right).
\]

Multiplying a small constant on both sides of (4.74) and adding to (4.73) to absorb \( \epsilon^2 |(1 - \mathcal{P})[f]_L^2 \) and \( \|(I - \mathcal{P})[f]\|_L^2 \) into the left-hand side, we obtain

\[
\epsilon |(1 - \mathcal{P})[f]_L^2 + \epsilon^2 \|\mathbb{P}[f]\|_L^2 \leq C \left( \eta \epsilon^2 \|\mathbb{P}[f]\|_L^2 + \left( 1 + \frac{1}{\eta} \right) |h|_L^2 + \int_{\Omega \times \mathbb{R}^2} f S + |S|_L^2 \right).
\]

Taking \( \eta \) sufficiently small, we can further absorb \( \eta \epsilon^2 \|\mathbb{P}[f]\|_L^2 \) into the left-hand side to get

\[
\epsilon |(1 - \mathcal{P})[f]_L^2 + \epsilon^2 \|\mathbb{P}[f]\|_L^2 \leq C \left( |h|_L^2 + \int_{\Omega \times \mathbb{R}^2} f S + |S|_L^2 \right).
\]

Since

\[
\int_{\Omega \times \mathbb{R}^2} f S \leq C \epsilon^2 \|f\|_L^2 + \frac{1}{4C \epsilon^2} |S|_L^2,
\]

for \( C \) sufficiently small, we have

\[
\epsilon |(1 - \mathcal{P})[f]_L^2 + \epsilon^2 \|\mathbb{P}[f]\|_L^2 \leq C \left( |h|_L^2 + \frac{1}{\epsilon^2} |S|_L^2 \right).
\]

Hence, we deduce

\[
\frac{1}{\epsilon^2} |(1 - \mathcal{P})[f]_L^2 + |f|_L^2 \leq C \left( \frac{1}{\epsilon^2} |S|_L^2 + \frac{1}{\epsilon} |h|_L^2 \right).
\]

We may further improve the result based on the fact that

\[
\int_{\Omega \times \mathbb{R}^2} f S = \int_{\Omega \times \mathbb{R}^2} \left( (I - \mathcal{P})[f] + \mathbb{P}[f] \right) \left( (I - \mathcal{P})[S] + \mathbb{P}[S] \right)
\]

\[
= \int_{\Omega \times \mathbb{R}^2} \left( (I - \mathcal{P})[f](I - \mathcal{P})[S] + \mathbb{P}[f]\mathbb{P}[S] \right)
\]

\[
\leq C \|(I - \mathcal{P})[f]\|_L^2 + \frac{1}{4C} \|(I - \mathcal{P})[S]\|_L^2 + C \epsilon^2 \|\mathbb{P}[f]\|_L^2 + \frac{1}{4C \epsilon^2} \|\mathbb{P}[S]\|_L^2.
\]

Then the result is obvious.
4.2. $L^\infty$ Estimates - First Round. We first define the tracking back through the characteristics and diffusive reflection.

**Definition 4.3.** (Stochastic Cycle) For a fixed point $(\bar{x}, \bar{v})$ with $(\bar{x}, \bar{v}) \notin \gamma_0$, let $(t_0, \bar{x}_0, \bar{v}_0) = (0, \bar{x}, \bar{v})$. For $\bar{v}_{k+1}$ such that $\bar{v}_{k+1} \cdot \tilde{\nu}(\bar{x}_{k+1}) > 0$, define the $(k + 1)$-component of the back-time cycle as

$$ (t_{k+1}, \bar{x}_{k+1}, \bar{v}_{k+1}) = (t_k + t_b(\bar{x}_k, \bar{v}_k), \bar{x}_b(\bar{x}_k, \bar{v}_k), \bar{v}_{k+1}), $$

where

$$ t_b(\bar{x}, \bar{v}) = \inf \{ t > 0 : \bar{x} - et\bar{v} \notin \Omega \}, $$

$$ x_b(\bar{x}, \bar{v}) = \bar{x} - et_b(\bar{x}, \bar{v})\bar{v} \notin \Omega. $$

Set

$$ X_{ct}(s; \bar{x}, \bar{v}) = \sum_k 1_{ \{ t_b \leq s < t_{k+1} \} } (\bar{x}_k - e(t_k - s)\bar{v}_k), $$

$$ V_{ct}(s; \bar{x}, \bar{v}) = \sum_k 1_{ \{ t_b \leq s < t_{k+1} \} } \bar{v}_k. $$

Define $V_k = \{ \bar{v} \in \mathbb{R}^2 : \bar{v} \cdot \tilde{\nu}(\bar{x}_k) > 0 \}$, and let the iterated integral for $k \geq 2$ be defined as

$$ \int_{\Pi_{k-1}^{k-1}} \prod_{j=1}^{k-1} d\sigma_j = \int_{V_1} \cdots \left( \int_{V_{k-1}} d\sigma_{k-1} \right) \cdots d\sigma_1 $$

where $d\sigma_j = \mu(\bar{v}) |\bar{v} \cdot \tilde{\nu}(\bar{x}_j)| d\bar{v}$ is a probability measure. We define a weight function scaled with parameter $\xi$,

$$ w_\xi(\bar{v}) = w_{\xi, \beta, \varrho}(\bar{v}) = \left( 1 + \xi^2 |\bar{v}|^2 \right)^{\beta} e^{\varrho|\bar{v}|^2}, $$

and

$$ \hat{w}_\xi(\bar{v}) = \frac{1}{\mu(\bar{v}) w_\xi(\bar{v})} = \frac{\sqrt{2\pi} e^{(\frac{1}{2} - \varrho)|\bar{v}|^2}}{\left( 1 + \xi^2 |\bar{v}|^2 \right)^{\frac{\beta}{2}}}. $$

**Lemma 4.4.** For $T > 0$ sufficiently large, there exists constants $C_1, C_2 > 0$ independent of $T_0$, such that for $k = C_1T_0^{-\frac{3}{2}}$, and $(\bar{x}, \bar{v}) \in \bar{\Omega} \times \mathbb{R}^2$,

$$ \int_{\Pi_{k-1}^{k-1}} \prod_{j=1}^{k-1} d\sigma_j \leq \left( \frac{1}{2} \right)^{C_2T_0^{-\frac{3}{2}}}. $$

We also have, for $\beta > 2$,

$$ \int_{\Pi_{k-1}^{k-1}} \langle \hat{v} \rangle^{\frac{1}{2}} \hat{w}_\xi(\hat{v}) \prod_{j=1}^{k-1} d\sigma_j \leq \frac{C(\beta, \varrho)}{\xi^3}, $$

$$ \int_{\Pi_{k-1}^{k-1}} \langle \hat{v} \rangle^{\frac{1}{2}} \hat{w}_\xi(\hat{v}) \prod_{j=1}^{k-1} d\sigma_j \leq \frac{C(\beta, \varrho)}{\xi^3} \left( \frac{1}{2} \right)^{C_2T_0^{-\frac{3}{2}}}. $$

**Proof.** See [9] Lemma 4.1].

**Theorem 4.5.** The solution $f(\bar{x}, \bar{v})$ to the equation (4.1) satisfies the estimate for $\vartheta \geq 3$ and $0 \leq \varrho < \frac{1}{4}$,

$$ \left\| \langle \hat{v} \rangle^{\vartheta} e^{\varrho|\hat{v}|^2} f \right\|_{L^\infty} \leq C \left( \frac{1}{\varepsilon^2} \| \varepsilon [S] \|_{L^2} + \frac{1}{\varepsilon^2} \| (1 - \varepsilon [S]) \|_{L^2} + \frac{1}{\varepsilon^2} \| h \|_{L^2} + \left\| \langle \hat{v} \rangle^{\vartheta} e^{\varrho|\hat{v}|^2} S \right\|_{L^\infty} + \left\| \langle \hat{v} \rangle^{\vartheta} e^{\varrho|\hat{v}|^2} h \right\|_{L^\infty} \right). $$
Proof. We divide the proof into several steps:

Step 1: Mild formulation.
Denote
\begin{align}
g(\vec{x}, \vec{v}) &= w_\xi(\vec{v}) f(\vec{x}, \vec{v}), \\
K_{w_\xi(\vec{v})}[g](\vec{x}, \vec{v}) &= w_\xi(\vec{v}) K \left[ \frac{g}{w_\xi} \right](\vec{x}, \vec{v}) = \int_{\mathbb{R}^2} k_{w_\xi(\vec{v})}(\vec{v}, \vec{u}) g(\vec{x}, \vec{u}) d\vec{u},
\end{align}
where
\begin{equation}
k_{w_\xi(\vec{v})}(\vec{v}, \vec{u}) = k(\vec{v}, \vec{u}) \frac{w_\xi(\vec{v})}{w_\xi(\vec{u})}.
\end{equation}

We can rewrite the solution of the equation (4.11) along the characteristics by Duhamel’s principle as
\begin{equation}
g(\vec{x}, \vec{v}) = w_\xi(\vec{v}) h(\vec{x} - ct \vec{v}, \vec{v}) e^{-\nu(\vec{v})t_1} + \int_{0}^{t_1} w_\xi(\vec{v}) S \left( \vec{x} - \epsilon(t_1 - s) \vec{v}, \vec{v} \right) e^{-\nu(\vec{v})(t_1 - s)} ds + \frac{e^{-\nu(\vec{v})t_1}}{w_\xi(\vec{v})} \int_{\mathbb{R}^3} g(\vec{x}_1, \vec{v}_1) \tilde{w}_\xi(\vec{v}_1) d\sigma_1,
\end{equation}
where the last term refers to \( P[f] \). We may further rewrite the equation (4.11) along the stochastic cycle by applying Duhamel’s principle \( k \) times as
\begin{equation}
g(\vec{x}, \vec{v}) = w_\xi(\vec{v}) h(\vec{x} - ct \vec{v}, \vec{v}) e^{-\nu(\vec{v})t_1} + \int_{0}^{t_1} w_\xi(\vec{v}) S \left( \vec{x} - \epsilon(t_1 - s) \vec{v}, \vec{v} \right) e^{-\nu(\vec{v})(t_1 - s)} ds + \frac{1}{w_\xi(\vec{v})} \sum_{j=1}^{k-1} \int_{\mathbb{R}^3} g(\vec{x}_j, \vec{v}_j) \tilde{w}_\xi(\vec{v}_j) \left( \prod_{j=1}^{k} e^{-\nu(\vec{v}_j)(t_{j+1} - t_j)} d\sigma_j \right),
\end{equation}
where
\begin{align}
G[\vec{x}, \vec{v}] &= h(\vec{x} - ct_1 \vec{v}, \vec{v}) w_\xi(\vec{v}) + \int_{t_1}^{t_{1+1}} \left( S \left( \vec{x}_1 - \epsilon(t_{1+1} - s) \vec{v}_1, \vec{v}_1 \right) w_\xi(\vec{v}_1) e^{\nu(\vec{v}_1)s} \right) ds \\
H[\vec{x}, \vec{v}] &= \int_{t_1}^{t_{1+1}} \left( K_{w_\xi(\vec{v}_1)}[g] \left( \vec{x}_1 - \epsilon(t_{1+1} - s) \vec{v}_1, \vec{v}_1 \right) e^{\nu(\vec{v}_1)s} \right) ds.
\end{align}

Step 2: Estimates of source terms and boundary terms.
We set \( k = CT_0^2 \) and take absolute value on both sides of (4.98). Then all the terms in (4.98) related to the source term \( S \) and boundary term \( h \) can be bounded as
\begin{equation}
|w_\xi h|_{L^\infty} + \left\| \frac{|w_\xi S|}{\nu} \right\|_{L^\infty} \leq C \left( \left\| w_\xi \right\|_{L^\infty} \right),
\end{equation}
due to Lemma 4.4 and
\begin{equation}
\frac{1}{w_\xi} \leq C(\beta, \varphi) \xi^\beta.
\end{equation}
The last term in (4.98) can be decomposed as follows:
\begin{align}
\text{Part 1} &= \frac{1}{w_\xi(\vec{v})} \int_{\mathbb{R}^3} \left\{ \left\{ t_{k+1} \geq \frac{T_0}{2} \right\} g(\vec{x}_k, \vec{v}_k) \tilde{w}_\xi(\vec{v}_k) \left( \prod_{j=1}^{k} e^{-\nu(\vec{v}_j)(t_{j+1} - t_j)} d\sigma_j \right) \\
&\quad + \frac{1}{w_\xi(\vec{v})} \int_{\mathbb{R}^3} \left\{ t_{k+1} \leq \frac{T_0}{2} \right\} g(\vec{x}_k, \vec{v}_k) \tilde{w}_\xi(\vec{v}_k) \left( \prod_{j=1}^{k} e^{-\nu(\vec{v}_j)(t_{j+1} - t_j)} d\sigma_j \right) \right\},
\end{align}
Based on Lemma 4.4 we have
\[
\frac{1}{\tilde{w}_k(\bar{v})} \left| \int_{\Pi_{j=1}^{k} \mathbf{v}_j} 1_{\{t_{k+1} \leq \frac{t_0}{2^n}\}} g(\bar{x}_k, \bar{v}_k) \tilde{w}_k(\bar{v}_k) \left( \prod_{j=1}^{k} e^{-\nu(\bar{v})(t_{j+1} - t_j)} d\mathbf{\sigma}_j \right) \right| \leq C \left( \frac{1}{2} \right)^{-\frac{1}{2}} C_2 T_0 \|g\|_{L^\infty}.
\]

Based on Lemma 4.4 and \( \nu_0(1 + |\bar{v}|) \leq \nu(\bar{v}) \leq \nu_1(1 + |\bar{v}|) \), we obtain
\[
\frac{1}{\tilde{w}_k(\bar{v})} \left| \int_{\Pi_{j=1}^{k} \mathbf{v}_j} 1_{\{t_{k+1} \geq \frac{t_0}{2^n}\}} g(\bar{x}_k, \bar{v}_k) \tilde{w}_k(\bar{v}_k) \left( \prod_{j=1}^{k} e^{-\nu(\bar{v})(t_{j+1} - t_j)} d\mathbf{\sigma}_j \right) \right| \leq e^{-\frac{\nu_0 t_0}{2^n}} \|g\|_{L^\infty}.
\]

For \( T_0 \) sufficiently large and \( \epsilon \) sufficiently small, we get
\[
\text{Part } 2 \leq \delta \|g\|_{L^\infty}.
\] for \( \delta \) arbitrarily small.

Step 3: Estimates of \( K_{w_k} \) terms.
So far, the only remaining terms are related to \( K_{w_k} \). Define the back-time stochastic cycle from \( (s, X_{cl}(s; \bar{x}, \bar{v}), \bar{v}) \) as \( (t_1', \bar{x}', \bar{v}') \). Then we can rewrite \( K_{w_k} \) along the stochastic cycle as
\[
K_{w_k}(\bar{v})\{g(\bar{x} - \epsilon(t_1 - s)\bar{v}, \bar{v}) = K_{w_k}(\bar{v})\{g(X_{cl}, \bar{v}) = \int_{\mathbb{R}^2} k_{w_k}(\bar{v}, \bar{v}') g(X_{cl}, \bar{v}') d\mathbf{v}'
\]
\[
\leq \left| \int_{\mathbb{R}^2} \int_0^{t_1'} k_{w_k}(\bar{v}, \bar{v}') K_{w_k}(\bar{v}') \{g\}(X_{cl} - \epsilon(t_1' - r)\bar{v}', \bar{v}') e^{-\nu(\bar{v}')(t_1' - r)} dr d\mathbf{v}' \right|
\]
\[
+ \left| \int_{\mathbb{R}^2} \frac{1}{\tilde{w}_k(\bar{v})} \sum_{l=1}^{k-1} \int_{\Pi_{j=1}^{l} \mathbf{v}_j} k_{w_k}(\bar{v}, \bar{v}') H [X_{cl}, \bar{v}'] \tilde{w}_k(\bar{v}_l) \left( \prod_{j=1}^{l} e^{-\nu(\bar{v})(t_{j+1} - t_j)} d\mathbf{\sigma}_j \right) d\mathbf{v}' \right|
\]
\[
+ \left| \int_{\mathbb{R}^2} k_{w_k}(\bar{v}, \bar{v}') A d\mathbf{v}' \right|
\]
\[
= I + II + III.
\]

\( III \) can be directly estimated as
\[
III \leq C \left( |w_k|_{L^\infty} + |w_k S|_{L^\infty} + \delta \|g\|_{L^\infty} \right).
\]

We may further rewrite \( I \) as
\[
I = \left| \int_{\mathbb{R}^2} \int_0^{t_1'} k_{w_k}(\bar{v}, \bar{v}') k_{w_k}(\bar{v}) (\bar{v}', \bar{v}'') g(X_{cl} - \epsilon(t_1' - r)\bar{v}', \bar{v}'') e^{-\nu(\bar{v}')(t_1' - r)} dr d\mathbf{v}' d\mathbf{v}'' \right|
\]

which will be estimated in four cases:
\[
I = I_1 + I_2 + I_3 + I_4.
\]

Case I: \( |\bar{v}| \geq N \).

Based on Lemma 2.22 we have
\[
\left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} k_{w_k}(\bar{v}, \bar{v}') k_{w_k}(\bar{v}) (\bar{v}', \bar{v}'') d\mathbf{v}' d\mathbf{v}'' \right| \leq \frac{C}{1 + |\bar{v}|} \leq \frac{C}{N}.
\]

Hence, we get
\[
I_1 \leq \frac{C}{N} \|g\|_{L^\infty}.
\]
Case II: \(|\vec{v}| \leq N, |\vec{v}'| \geq 2N, \) or \(|\vec{v}'| \leq 2N, |\vec{v}''| \geq 3N\).
Notice this implies either \(|\vec{v} - \vec{v}| \geq N\) or \(|\vec{v} - \vec{v}'| \geq N\). Hence, either of the following is valid correspondingly:

\[
|k_{w_N}(\vec{v}, \vec{v}')| \leq Ce^{-\delta N^2} \left| k_{w_N}(\vec{v}, \vec{v}') e^{\delta |\vec{v} - \vec{v}'|} \right|,  \quad (4.113)
\]
\[
|k_{w_N}(\vec{v}', \vec{v}''')| \leq Ce^{-\delta N^2} \left| k_{w_N}(\vec{v}', \vec{v}'') e^{\delta |\vec{v}' - \vec{v}'''|} \right|.  \quad (4.114)
\]

Then based on Lemma 2.2,

\[
\int_{\mathbb{R}^2} \left| k_{w_N}(\vec{v}, \vec{v}') e^{\delta |\vec{v} - \vec{v}'|} \right| d\vec{v}' < \infty,  \quad (4.115)
\]
\[
\int_{\mathbb{R}^2} \left| k_{w_N}(\vec{v}', \vec{v}'') e^{\delta |\vec{v}' - \vec{v}'''|} \right| d\vec{v}'' < \infty.  \quad (4.116)
\]

Hence, we have

\[
I_2 \leq Ce^{-\delta N^2} \|g\|_{L^\infty}.  \quad (4.117)
\]

Case III: \(t_1' - r \leq \delta\) and \(|\vec{v}| \leq N, |\vec{v}'| \leq 2N, |\vec{v}''| \leq 3N\).
In this case, since the integral with respect to \(r\) is restricted in a very short interval, there is a small contribution as

\[
I_3 \leq \left| \int_{\mathbb{R}^2} \int_{t_1' - \delta}^{t_1'} k_{w_N}(\vec{v}, \vec{v}') k_{w_N}(\vec{v}', \vec{v}'') e^{-N(\vec{v}')} (t_1' - r) dr d\vec{v}'' \right| \|g\|_{L^\infty} \leq C\delta \|g\|_{L^\infty}.  \quad (4.118)
\]

Case IV: \(t_1' - r \geq \delta\) and \(|\vec{v}| \leq N, |\vec{v}'| \leq 2N, |\vec{v}''| \leq 3N\).
Since \(k_{w_N}(\vec{v}, \vec{v}')\) has possible integrable singularity of \(\frac{1}{|\vec{v} - \vec{v}'|}\), we can introduce the truncated kernel \(k_N(\vec{v}, \vec{v}')\) which is smooth and has compact support such that

\[
\sup_{|p| \leq 3N} \int_{|\vec{v}'| \leq 3N} \left| k_N(p, \vec{v}') - k_{w_N}(p, \vec{v}') \right| d\vec{v}' \leq \frac{1}{N}.  \quad (4.119)
\]

Then we can split

\[
k_{w_N}(\vec{v}, \vec{v}') k_{w_N}(\vec{v}', \vec{v}'') = k_N(\vec{v}, \vec{v}') k_N(\vec{v}', \vec{v}'') + \left( k_{w_N}(\vec{v}, \vec{v}') - k_N(\vec{v}, \vec{v}') \right) k_{w_N}(\vec{v}', \vec{v}'') + \left( k_{w_N}(\vec{v}', \vec{v}'') - k_N(\vec{v}', \vec{v}'') \right) k_N(\vec{v}, \vec{v'}).  \quad (4.120)
\]

This means we further split \(I_4\) into

\[
I_4 = I_{4,1} + I_{4,2} + I_{4,3}.  \quad (4.121)
\]

Based on the estimate (4.119), we have

\[
I_{4,2} \leq \frac{C}{N} \|g\|_{L^\infty},  \quad (4.122)
\]
\[
I_{4,3} \leq \frac{C}{N} \|g\|_{L^\infty}.  \quad (4.123)
\]

Therefore, the only remaining term is \(I_{4,1}\). Note we always have \(X_{cl} - \epsilon(t_1' - r)\vec{v} \in \Omega\). Hence, we define the change of variable \(\vec{v}' \rightarrow \vec{y}\) as \(\vec{y} = (y_1, y_2) = X_{cl} - \epsilon(t_1' - r)\vec{v}'\) such that

\[
\left| \frac{dy}{d\vec{v}'} \right| = \left| \begin{array}{cc} \epsilon(t_1' - r) & 0 \\ 0 & \epsilon(t_1' - r) \end{array} \right| = \epsilon^2(t_1' - r)^2 \geq \epsilon^2 \delta^2.  \quad (4.124)
\]
Since $k_N$ is bounded and $|\bar{v}|, |\bar{v}''| \leq 3N$, we estimate
\[
I_{4.1} \leq C \left| \int_{|\sigma| \leq 2N} \int_{|\sigma'| \leq 3N} \int_0^{t_1} 1_{\{X_{cl} - \epsilon(t_1' - r) \in \Omega\}} g \left( X_{cl} - \epsilon(t_1' - r) \bar{v}', \bar{v}'' \right) e^{-\nu(\bar{v}')}(t_1' - r) \, dr \, d\sigma' \, d\sigma'' \right|.
\]
Using the notation $f$, we have
\[
I_{4.1} \leq C \left( \int_{|\sigma| \leq 2N} \int_{|\sigma'| \leq 3N} \int_0^{t_1} 1_{\{X_{cl} - \epsilon(t_1' - r) \in \Omega\}} f \left( X_{cl} - \epsilon(t_1' - r) \bar{v}', \bar{v}'' \right) e^{-\nu(\bar{v}')}(t_1' - r) \, dr \, d\sigma' \, d\sigma'' \right)^{\frac{1}{2}}.
\]
Summarize all above in Case IV, we obtain
\[
I_4 \leq \frac{C}{N} \|g\|_{L^\infty} + \frac{C}{\epsilon \delta} \|f\|_{L^2}.
\]
Therefore, we already prove
\[
I \leq \left( Ce^{-\delta N^2} + \frac{C}{N} + \delta \right) \|g\|_{L^\infty} + \frac{C}{\epsilon \delta} \|f\|_{L^2}.
\]
Choosing $\delta$ sufficiently small and then taking $N$ sufficiently large, we have
\[
I \leq C \delta \|g\|_{L^\infty} + \frac{C}{\epsilon \delta} \|f\|_{L^2}.
\]
A similar technique can justify
\[
II \leq C \delta \|g\|_{L^\infty} + \frac{C}{\epsilon \delta} \|f\|_{L^2}.
\]
All the terms related to $K_{w_\xi}$ can be estimated in a similar fashion.

Step 4: Synthesis.
Collecting all above, based on the mild formulation (4.138) we have shown
\[
\|g\|_{L^\infty} \leq C \delta \|g\|_{L^\infty} + \frac{C}{\epsilon \delta} \|f\|_{L^2} + C \left| \frac{w_\xi S}{\nu} \right|_{L^\infty} + C |w_\xi h|_{L^\infty}.
\]
Taking $\delta$ is sufficiently small, based on Theorem (4.12) we obtain
\[
\|g\|_{L^\infty} \leq C \left( \frac{1}{\epsilon} \|f\|_{L^2} + \|w_\xi S\|_{L^\infty} + |w_\xi h|_{L^\infty} \right).
\]
It is easy to see for $\bar{v} = \beta$, we have
\[
C_1 \langle \bar{v} \rangle^\theta e^{O|\bar{v}|^2} \leq w_\xi \leq C_2 \langle \bar{v} \rangle^\theta e^{O|\bar{v}|^2},
\]
for some constant $C_1, C_2 > 0$. Then we must have

$$
\left\| \langle \vec{v} \rangle^2 e^{\gamma |\vec{v}|^2} f \right\|_{L^\infty} \leq C \left( \frac{1}{e^2} \| \mathbb{P}[S] \|_{L^2} + \frac{1}{e^2} \| (\mathbb{I} - \mathbb{P})[S] \|_{L^2} + h \right) + \left\| \langle \vec{v} \rangle^2 e^{\gamma |\vec{v}|^2} S \right\|_{L^\infty} + \left\| \langle \vec{v} \rangle^2 e^{\gamma |\vec{v}|^2} h \right\|_{L^\infty}.
$$

(4.133)

4.3. $L^{2m}$ Estimates. Here we consider $m \geq \mathbb{N}$ and $m > 2$. Let $o(1)$ denote a sufficiently small constant. Since this is very similar to the $L^2$ estimates, we will focus on the key differences.

**Lemma 4.6.** The solution $f(x, \vec{v})$ to the equation (4.1) satisfies the estimate

$$
\epsilon \| \mathbb{P}[f] \|_{L^{2m}} \leq C \left( \epsilon \| (1 - \mathbb{P})[f] \|_{L^\infty}^m + \| (\mathbb{I} - \mathbb{P})[f] \|_{L^2} + m \| (\mathbb{I} - \mathbb{P})[f] \|_{L^{2m}} + \| S \|_{L^2} + \epsilon \| h \|_{L^\infty} \right).
$$

(4.144)

**Proof.** Applying Green’s identity in Lemma 2.6 to the solution of the equation (4.1). Then for any $\psi \in L^2(\Omega \times \mathbb{R}^2)$ satisfying $\vec{v} \cdot \nabla_x \psi \in L^2(\Omega \times \mathbb{R}^2)$ and $\psi \in L^2(\gamma)$, we have

$$
\epsilon \int_{\gamma_+} f \psi d\gamma - \epsilon \int_{\gamma_-} f \psi d\gamma - \epsilon \int_{\Omega \times \mathbb{R}^2} (\vec{v} \cdot \nabla_x \psi) f = - \int_{\Omega \times \mathbb{R}^2} \psi \mathcal{L} [(\mathbb{I} - \mathbb{P})[f]] + \int_{\Omega \times \mathbb{R}^2} S \psi.
$$

(4.135)

Since

$$
\mathbb{P}[f] = \mu^\frac{1}{2} \left( a + \frac{|\vec{v}^2 - 2}{2} \right),
$$

(4.136)

our goal is to choose a particular test function $\psi$ to estimate $a, \tilde{b}$ and $c$.

Step 1: Estimates of $c$.

We choose the test function

$$
\psi = \psi_c = \mu^\frac{1}{2} (\vec{v}) \left( |\vec{v}|^2 - \beta_c \right) \left( \vec{v} \cdot \nabla_x \phi_c(\vec{x}) \right),
$$

(4.137)

where

$$
\begin{cases}
-\Delta_x \phi_c &= c^{2m-1}(\vec{x}) \text{ in } \Omega, \\
\phi_c &= 0 \text{ on } \partial \Omega,
\end{cases}
$$

(4.138)

and $\beta_c$ is a real number to be determined later. Based on the standard elliptic estimates, we have

$$
\| \phi_c \|_{W^{2, \frac{2m}{2m-1}}} \leq C \| c^{2m-1} \|_{L^{\frac{2m}{2m-1}}} \leq C \| c \|_{L^{2m}}^{2m-1}.
$$

(4.139)

Also, we know

$$
\| \psi_c \|_{L^2} \leq C \| \phi_c \|_{H^1} \leq C \| \phi_c \|_{W^{2, \frac{2m}{2m-1}}} \leq C \| c \|_{L^{2m}}^{2m-1},
$$

(4.140)

$$
\| \psi_c \|_{L^{2m}} \leq C \| \phi_c \|_{W^{1, \frac{2m}{2m-1}}} \leq C \| c \|_{L^{2m}}^{2m-1}.
$$

(4.141)

With the choice of (4.137), the right-hand side (RHS) of (4.135) is bounded by

$$
\text{RHS} \leq C \| c \|_{L^{2m}}^{2m-1} \left( \| (\mathbb{I} - \mathbb{P})[f] \|_{L^2} + \| S \|_{L^2} \right).
$$

(4.142)

We will choose $\beta_c$ such that

$$
\int_{\mathbb{R}^2} \mu^\frac{1}{2} (\vec{v}) \left( |\vec{v}|^2 - \beta_c \right) \vec{v}_i^2 d\vec{v} = 0 \text{ for } i = 1, 2.
$$

(4.143)
The left-hand side (LHS) of (4.135) takes the form

\[
\text{LHS} = \epsilon \int_{\partial \Omega \times \mathbb{R}^2} 1_{\gamma_+}(1 - \mathcal{P})[f]\mu^2(\tilde{v}) \left( |\tilde{v}|^2 - \beta_c \right) \left( \sum_{i=1}^{2} v_i \partial_i \phi_c \right) (\tilde{v} \cdot \tilde{v}) + \epsilon \int_{\partial \Omega \times \mathbb{R}^2} h\mu^2(\tilde{v}) \left( |\tilde{v}|^2 - \beta_c \right) \left( \sum_{i=1}^{2} v_i \partial_i \phi_c \right) (\tilde{v} \cdot \tilde{v}) \\
- \epsilon \sum_{i=1}^{2} \int_{\mathbb{R}^2} \mu(\tilde{v}) |v_i|^2 \left( |\tilde{v}|^2 - \beta_c \right) \frac{|\tilde{v}|^2 - 2}{2} \int_{\Omega} c(x) \partial_i \phi_c(x) dx \\
- \epsilon \int_{\Omega \times \mathbb{R}^2} (1 - \mathbb{P})[f]\mu^2(\tilde{v}) \left( |\tilde{v}|^2 - \beta_c \right) \left( \sum_{i,j=1}^{2} v_i v_j \partial_{ij} \phi_c \right).
\]

(4.144)

Since

\[
\int_{\mathbb{R}^2} \mu(\tilde{v}) |v_i|^2 \left( |\tilde{v}|^2 - \beta_c \right) \frac{|\tilde{v}|^2 - 2}{2} \tilde{v} = \epsilon \int_{\mathbb{R}^2} \Delta_x \phi_c(x)c(x) dx 
\]

(4.145)

we have

\[
\epsilon \int_{\Omega} \Delta_x \phi_c(x)c(x) dx \leq C \left| c \right|_{L^{2m-1}_m}^2 \left( \epsilon \left| (1 - \mathcal{P})[f] \right|_{L^m} + \left| (\mathbb{I} - \mathcal{P})[f] \right|_{L^2} + \epsilon \left| (\mathbb{I} - \mathcal{P})[f] \right|_{L^{2m}} \right) + \left| S \right|_{L^2} + \epsilon \left| h \right|_{L^m},
\]

(4.146)

where we have used the elliptic estimates, Sobolev embedding theorem, and the trace estimate:

\[
\left| \nabla_x \phi_c \right|_{L^m_{m-1}} \leq C \left| \nabla_x \phi_c \right|_{W^{2m-1}_m} \leq C \left| \nabla_x \phi_c \right|_{W^{1, m-1}} \leq C \left| \phi_c \right|_{W^{2m-1}_m} \leq C \left| c \right|_{L^{2m-1}_m}.
\]

(4.147)

Since \(-\Delta_x \phi_c = c^{2m-1}\), we know

\[
\epsilon \left| c \right|_{L^{2m}_m} \leq C \left| c \right|_{L^{2m-1}_m}^2 \left( \epsilon \left| (1 - \mathcal{P})[f] \right|_{L^m} + \left| (\mathbb{I} - \mathcal{P})[f] \right|_{L^2} + \epsilon \left| (\mathbb{I} - \mathcal{P})[f] \right|_{L^{2m}} + \left| S \right|_{L^2} + \epsilon \left| h \right|_{L^m} \right),
\]

(4.148)

which further implies

\[
\epsilon \left| c \right|_{L^{2m}_m} \leq C \left( \epsilon \left| (1 - \mathcal{P})[f] \right|_{L^m} + \left| (\mathbb{I} - \mathcal{P})[f] \right|_{L^2} + \epsilon \left| (\mathbb{I} - \mathcal{P})[f] \right|_{L^{2m}} + \left| S \right|_{L^2} + \epsilon \left| h \right|_{L^m} \right).
\]

(4.149)

Step 2: Estimates of \( \tilde{b} \).

We further divide this step into several sub-steps:

Step 2.1: Estimates of \( \left( \partial_{ij} \Delta_x^{-1} b_{ij} \right) b_i \) for \( i, j = 1, 2 \).

We choose the test function

\[
\psi = \psi^{i,j} \mu^2(\tilde{v}) \left( v_i^2 - \beta_b \right) \partial_j \phi_b^i,
\]

(4.150)

where

\[
\begin{align*}
-\Delta_x \phi_b^i &= b_{2m-1}^2(x) \text{ in } \Omega, \\
\phi_b^i &= 0 \text{ on } \partial\Omega,
\end{align*}
\]

(4.151)

and \( \beta_b \) is a real number to be determined later. Based on the standard elliptic estimates, we have

\[
\left| \phi_b^{i,j} \right|_{W^{2m-1}_m} \leq C \left| b_{j,2m-1} \right|_{L^{2m-1}_m} \leq C \left| b_j \right|_{L^{2m-1}_m}.
\]

(4.152)

Also, we know

\[
\left| \psi^{i,j} \right|_{L^2} \leq C \left| \phi_b^{i,j} \right|_{H^1} \leq C \left| \phi_b^{i,j} \right|_{W^{2m-1}_m} \leq C \left| b_j \right|_{L^{2m}}^2,
\]

(4.153)

\[
\left| \psi^{i,j} \right|_{L^2} \leq C \left| \phi_b^{i,j} \right|_{W^{2m-1}_m} \leq C \left| b_j \right|_{L^{2m}}^2.
\]

(4.154)
With the choice of (4.150), the right-hand side (RHS) of (4.135) is bounded by

\[ \text{RHS} \leq C \left\| \tilde{b}\right\|_{L^{2m}}^{2m-1} \left( \| (I - P) [f] \|_{L^2} + \| S \|_{L^2} \right). \]  

(4.155)

We will choose \( \beta_b \) such that

\[ \int_{\mathbb{R}^2} \mu(\tilde{v}) \left( |v_i|^2 - \beta_b \right) \, d\tilde{v} = 0 \quad \text{for} \quad i = 1, 2. \]  

(4.156)

Hence, the left-hand side (LHS) of (4.135) takes the form

\[ \text{LHS} = \epsilon \int_{\partial \Omega \times \mathbb{R}^2} 1_{\gamma^{+}} (1 - P) [f] \mu^{\frac{1}{2}} (\tilde{v}) (v_i^2 - \beta_b) \partial_j \phi^i_b (\tilde{v} \cdot \tilde{v}) \]

\[ + \epsilon \int_{\partial \Omega \times \mathbb{R}^2} 1_{\gamma^{-}} h \mu^{\frac{1}{2}} (\tilde{v}) (v_i^2 - \beta_b) \partial_j \phi^i_b (\tilde{v} \cdot \tilde{v}) \]

\[ - \epsilon \sum_{l=1}^2 \int_{\Omega \times \mathbb{R}^2} \mu(\tilde{v}) v_i^2 (v_i^2 - \beta_b) \partial_{ij} \phi^j_b b_l \]

\[ - \epsilon \sum_{l=1}^2 \int_{\Omega \times \mathbb{R}^2} (I - P) [f] \mu^{\frac{1}{2}} (\tilde{v}) (v_i^2 - \beta_b) v_l \partial_{ij} \phi^j_b. \]

For such \( \beta_b \) and any \( i \neq l \), we can directly compute

\[ \int_{\mathbb{R}^2} \mu(\tilde{v}) \left( |v_i|^2 - \beta_b \right) v_i^2 \, d\tilde{v} = 0, \]

(4.157)

\[ \int_{\mathbb{R}^2} \mu(\tilde{v}) \left( |v_i|^2 - \beta_b \right) v_i^2 \, d\tilde{v} = C \neq 0. \]  

(4.158)

Then we deduce

\[ - \epsilon \sum_{l=1}^2 \int_{\Omega \times \mathbb{R}^2} \mu(\tilde{v}) v_i^2 (v_i^2 - \beta_b) \partial_{ij} \phi^j_b b_l \]  

(4.159)

\[ = - \epsilon \int_{\Omega \times \mathbb{R}^2} \mu(\tilde{v}) v_i^2 (v_i^2 - \beta_b) \partial_{ij} \phi^j_b b_l - \epsilon \sum_{l \neq i} \int_{\Omega \times \mathbb{R}^2} \mu(\tilde{v}) v_i^2 (v_i^2 - \beta_b) \partial_{ij} \phi^j_b b_l \]

\[ = C \int_{\Omega} \left( \partial_{ij} \Delta^{-1} b_j \right) b_i. \]

Hence, by (4.149), we may estimate

\[ \epsilon \left| \int_{\Omega} \left( \partial_{ij} \Delta^{-1} b_j \right) b_i \right| \leq C \left\| \tilde{b} \right\|_{L^{2m}}^{2m-1} \left( \epsilon \| (I - P) [f] \|_{L^m} + \| (I - P) [f] \|_{L^2} + \epsilon \| (I - P) [f] \|_{L^{2m}} \right) \]

\[ + \| S \|_{L^2} + \epsilon \| h \|_{L^m} \). \]  

(4.160)

Step 2.2: Estimates of \( (\partial_{jj} \Delta^{-1} b_j) b_l \) for \( i \neq j \).

We choose the test function

\[ \psi = \mu^{\frac{1}{2}} (\tilde{v}) |\tilde{v}|^2 v_i v_j \partial_{ij} \phi^i_b \quad i \neq j. \]  

(4.161)

The right-hand side (RHS) of (4.135) is still bounded by

\[ \text{RHS} \leq C \left\| \tilde{b} \right\|_{L^{2m}}^{2m-1} \left( \| (I - P) [f] \|_{L^2} + \| S \|_{L^2} \right). \]  

(4.162)
Hence, the left-hand side (LHS) of (4.135) takes the form

\[
\text{LHS} = \epsilon \int_{\partial \Omega \times \mathbb{R}^2} 1_{\gamma_+} (1 - P) [f] \mu \hat{\nu} (\tilde{\nu}) |\tilde{\nu}|^2 v_i v_j \partial_j \phi_b^i (\hat{\nu} \cdot \tilde{\nu}) + \epsilon \int_{\partial \Omega \times \mathbb{R}^2} 1_{\gamma_-} h \mu \hat{\nu} (\tilde{\nu}) |\tilde{\nu}|^2 v_i v_j \partial_j \phi_b^i (\hat{\nu} \cdot \tilde{\nu}) - \epsilon \int_{\Omega \times \mathbb{R}^2} \mu(\tilde{\nu}) |\tilde{\nu}|^2 v_i^2 v_j^2 \left( \partial_j \phi_b^i b_j + \partial_j \phi_b^j b_i \right) - \epsilon \sum_{l=1}^{2} \int_{\Omega \times \mathbb{R}^2} (I - P)[f] \mu \hat{\nu} (\tilde{\nu}) |\tilde{\nu}|^2 v_i v_j \partial_j \phi_b^l.
\]

(4.163)

Then we deduce

\[
- \epsilon \int_{\Omega \times \mathbb{R}^2} \mu(\tilde{\nu}) |\tilde{\nu}|^2 v_i^2 v_j^2 \left( \partial_j \phi_b^i b_j + \partial_j \phi_b^j b_i \right) = C \left( \int_{\Omega} \left( \partial_j \Delta_{\nu}^{-1} b_i \right) b_j + \int_{\Omega} \left( \partial_j \Delta_{\nu}^{-1} b_i \right) b_i \right).
\]

(4.164)

Hence, we may estimate for \( i \neq j \),

\[
\epsilon \int_{\Omega} \left( \partial_j \Delta_{\nu}^{-1} b_i \right) b_i \leq C \| \hat{b} \|^2_{L^2_{m}} \left( \epsilon \| (1 - P)[f] \|_{L^2_{m}} + \| (I - P)[f] \|_{L^2_{2}} + \epsilon \| (I - P)[f] \|_{L^2_{m}} + \| S \|_{L^2} + \epsilon |h|_{L^m} \right) + C \epsilon \int_{\Omega} \left( \partial_j \Delta_{\nu}^{-1} b_i \right) b_i,
\]

(4.165)

which implies

\[
\epsilon \int_{\Omega} \left( \partial_j \Delta_{\nu}^{-1} b_i \right) b_i \leq C \| \hat{b} \|^2_{L^2_{m}} \left( \epsilon \| (1 - P)[f] \|_{L^2_{m}} + \| (I - P)[f] \|_{L^2_{2}} + \epsilon \| (I - P)[f] \|_{L^2_{m}} + \| S \|_{L^2} + \epsilon |h|_{L^m} \right).
\]

(4.166)

Moreover, by (4.100), for \( i = j = 1, 2 \),

\[
\epsilon \int_{\Omega} \left( \partial_j \Delta_{\nu}^{-1} b_j \right) b_j \leq C \| \hat{b} \|^2_{L^2_{m}} \left( \epsilon \| (1 - P)[f] \|_{L^2_{m}} + \| (I - P)[f] \|_{L^2_{2}} + \epsilon \| (I - P)[f] \|_{L^2_{m}} + \| S \|_{L^2} + \epsilon |h|_{L^m} \right).
\]

(4.167)

Step 2.3: Synthesis.

Summarizing (4.166) and (4.167), we may sum up over \( j = 1, 2 \) to obtain, for any \( i = 1, 2 \),

\[
\epsilon \| b_i \|^2_{L^2_{m}} \leq C \| \hat{b} \|^2_{L^2_{m}} \left( \epsilon \| (1 - P)[f] \|_{L^2_{m}} + \| (I - P)[f] \|_{L^2_{2}} + \epsilon \| (I - P)[f] \|_{L^2_{m}} + \| S \|_{L^2} + \epsilon |h|_{L^m} \right).
\]

(4.168)

which further implies

\[
\epsilon \| \hat{b} \|^2_{L^2_{m}} \leq C \| \hat{b} \|^2_{L^2_{m}} \left( \epsilon \| (1 - P)[f] \|_{L^2_{m}} + \| (I - P)[f] \|_{L^2_{2}} + \epsilon \| (I - P)[f] \|_{L^2_{m}} + \| S \|_{L^2} + \epsilon |h|_{L^m} \right).
\]

(4.169)

Then we have

\[
\epsilon \| \hat{b} \|_{L^2_{m}} \leq C \left( \epsilon \| (1 - P)[f] \|_{L^2_{m}} + \| (I - P)[f] \|_{L^2_{2}} + \epsilon \| (I - P)[f] \|_{L^2_{m}} + \| S \|_{L^2} + \epsilon |h|_{L^m} \right).\]

(4.170)

Step 3: Estimates of \( a \).

We choose the test function

\[
\psi = \psi_a = \mu \hat{\nu} \left( |\tilde{\nu}|^2 - \beta_a \right) \left( \hat{\nu} \cdot \nabla \phi_a (\tilde{\nu}) \right),
\]

(4.171)
where
\[
\begin{align*}
-\Delta_x \phi_a &= a^{2m-1}(\vec{x}) - \frac{1}{|\Omega|} \int_{\Omega} a^{2m-1}(\vec{x})d\vec{x} \quad \text{in } \Omega, \\
\frac{\partial \phi_a}{\partial \nu} &= 0 \quad \text{on } \partial\Omega,
\end{align*}
\]
(4.172)
and \(\beta_a\) is a real number to be determined later. Based on the standard elliptic estimates with
\[
\int_{\Omega} \left( a^{2m-1}(\vec{x}) - \frac{1}{|\Omega|} \int_{\Omega} a^{2m-1}(\vec{x})d\vec{x} \right) d\vec{x} = \int_{\Omega \times \mathbb{R}^2} f(\vec{x}, \vec{v})d\vec{x}d\vec{v} = 0,
\]
we have
\[
\|\phi_a\|_{W^{2,\frac{2m}{2m-1}}} \leq C \|a^{2m-1}\|_{L^{\frac{2m}{2m-1}}} \leq C \|a\|^{2m-1}. \tag{4.173}
\]
(4.174)
Also, we know
\[
\|\psi_a\|_{L^2} \leq C \|\phi_a\|_{H^1} \leq C \|\phi_a\|_{W^{2,\frac{2m}{2m-1}}} \leq C \|a\|^{2m-1}, \tag{4.175}
\]
(4.175)
\[
\|\psi_a\|_{L^{\frac{2m}{2m-1}}} \leq C \|\phi_a\|_{W^{2,\frac{2m}{2m-1}}} \leq C \|a\|^{2m-1}. \tag{4.176}
\]
(4.176)
With the choice of (4.171), the right-hand side (RHS) of (4.135) is bounded by
\[
\text{RHS} \leq C \|a\|^{2m-1} \left( \|f\|_{L^2} + \|S\|_{L^2} \right). \tag{4.177}
\]
(4.177)
We will choose \(\beta_a\) such that
\[
\int_{\mathbb{R}^2} \mu^{\frac{2}{p}}(\vec{v}) \left( |\vec{v}|^2 - \beta_a \right) \frac{|\vec{v}|^2 - 2}{2} v_i^2 d\vec{v} = 0 \quad \text{for } i = 1, 2. \tag{4.178}
\]
The left-hand side (LHS) of (4.135) takes the form
\[
\text{LHS} = \epsilon \int_{\partial \Omega \times \mathbb{R}^2} 1_{\gamma_a} (1 - \mathcal{P}) [f] \mu^{\frac{2}{p}}(\vec{v}) \left( |\vec{v}|^2 - \beta_a \right) \left( \sum_{i=1}^{2} v_i \partial_i \phi_a \right) (\vec{v} \cdot \vec{v}) \tag{4.179}
\]
(4.179)
\[
+ \epsilon \int_{\partial \Omega \times \mathbb{R}^2} 1_{\gamma_a} h \mu^{\frac{2}{p}}(\vec{v}) \left( |\vec{v}|^2 - \beta_a \right) \left( \sum_{i=1}^{2} v_i \partial_i \phi_a \right) (\vec{v} \cdot \vec{v})
\]
\[
- \sum_{i=1}^{2} \epsilon \int_{\mathbb{R}^2} \mu(\vec{v}) |v_i|^2 \left( |\vec{v}|^2 - \beta_a \right) d\vec{v} \int_{\Omega} a(\vec{x}) \partial_i \phi_a (\vec{x}) d\vec{x}
\]
\[
- \epsilon \int_{\Omega \times \mathbb{R}^2} (\mathbb{I} - \mathcal{P}) [f] \mu^{\frac{2}{p}}(\vec{v}) \left( |\vec{v}|^2 - \beta_a \right) \left( \sum_{i,j=1}^{2} v_i v_j \partial_{ij} \phi_a \right).
\]
Since
\[
\int_{\mathbb{R}^2} \mu^{\frac{2}{p}}(\vec{v}) |v_i|^2 \left( |\vec{v}|^2 - \beta_a \right) d\vec{v} = C, \tag{4.180}
\]
we have
\[
-\epsilon \int_{\Omega} \Delta_x \phi_a (\vec{x}) a(\vec{x}) d\vec{x} \leq C \|a\|^{2m-1} \left( \epsilon |(1 - \mathcal{P})[f]|_{L^m} + \|(\mathbb{I} - \mathcal{P})[f]\|_{L^2} + \epsilon \|S\|_{L^2} + \epsilon |h|_{L^m} \right). \tag{4.181}
\]
(4.181)
Since \(-\Delta_x \phi_a = a^{2m-1} - \frac{1}{|\Omega|} \int_{\Omega} a^{2m-1}\), by (4.41), we know
\[
\epsilon \|a\|^{2m-1} \leq C \|a\|^{2m-1} \left( \epsilon |(1 - \mathcal{P})[f]|_{L^m} + \|(\mathbb{I} - \mathcal{P})[f]\|_{L^2} + \epsilon \|S\|_{L^2} + \epsilon |h|_{L^m} \right). \tag{4.182}
\]
(4.182)
This implies
\[ \epsilon \|a\|_{L^2} \leq C\left( \epsilon \|(1 - \mathcal{P})f\|_{L^2} + \|\mathcal{P}[f]\|_{L^2} + \epsilon \|\mathcal{P}[f]\|_{L^2} + \epsilon \|S\|_{L^2} + \epsilon \|h\|_{L^2}\right). \]

Step 4: Synthesis.
Collecting (4.141), (4.170) and (4.183), we deduce
\[ \epsilon \|\mathcal{P}[f]\|_{L^2} \leq C\left( \epsilon \|(1 - \mathcal{P})f\|_{L^2} + \|\mathcal{P}[f]\|_{L^2} + \epsilon \|\mathcal{P}[f]\|_{L^2} + \epsilon \|S\|_{L^2} + \epsilon \|h\|_{L^2}\right). \] (4.183)
This completes our proof. \( \square \)

**Theorem 4.7.** The solution \( f(\bar{x}, \bar{v}) \) to the equation \( f, f \) satisfies the estimate
\[ \frac{1}{\epsilon^2} |(1 - \mathcal{P})[f]|_{L^2} + \frac{1}{\epsilon} \|\mathcal{P}[f]\|_{L^2} \leq C\left( o(1) \frac{1}{\epsilon^2} \|f\|_{L^\infty} + \frac{1}{\epsilon^2} \|\mathcal{P}[f]\|_{L^2} \right) + \frac{1}{\epsilon} \|S\|_{L^2} + \frac{1}{\epsilon} \|h\|_{L^2}. \] (4.184)

**Proof.** We divide it into several steps:

Step 1: Energy Estimate. 
Multiplying \( f \) on both sides of (4.141) and applying Green’s identity imply
\[ \frac{\epsilon}{2} |f|_{L^2}^2 + \langle L[f], f \rangle = \frac{\epsilon}{2} |\mathcal{P}[f]|_{L^2}^2 + \int_{\Omega \times \mathbb{R}^2} fS. \] (4.185)

Considering the fact that
\[ |f|_{L^2}^2 - |\mathcal{P}[f]|_{L^2}^2 = |(1 - \mathcal{P})[f]|_{L^2}^2, \] (4.186)
We deduce from the spectral gap of \( \mathcal{L} \) and Cauchy’s inequality that
\[ \frac{\epsilon}{2} |(1 - \mathcal{P})[f]|_{L^2}^2 + \|\mathcal{P}[f]\|_{L^2}^2 \leq \eta \epsilon^2 |\mathcal{P}[f]|_{L^2}^2 + \left( 1 + \frac{1}{\eta} \right) |h|_{L^2}^2 + \int_{\Omega \times \mathbb{R}^2} fS. \] (4.187)

Following the same argument as in \( L^2 \) estimate, we obtain
\[ |\mathcal{P}[f]|_{L^2}^2 \leq C\left( \epsilon |(1 - \mathcal{P})[f]|_{L^2}^2 + \frac{1}{\epsilon} \|\mathcal{P}[f]\|_{L^2}^2 + \frac{1}{\epsilon} \int_{\Omega \times \mathbb{R}^2} fS \right). \] (4.188)

Plugging (4.188) into (4.187) with \( \epsilon \) sufficiently small to absorb \( \|\mathcal{P}[f]\|_{L^2}^2 \) into the left-hand side, we obtain
\[ \epsilon |(1 - \mathcal{P})[f]|_{L^2}^2 + \|\mathcal{P}[f]\|_{L^2}^2 \leq C\left( \eta \epsilon^2 |\mathcal{P}[f]|_{L^2}^2 + \left( 1 + \frac{1}{\eta} \right) |h|_{L^2}^2 + \int_{\Omega \times \mathbb{R}^2} fS \right). \] (4.189)

We square on both sides of (4.134) to obtain
\[ \epsilon^2 \|\mathcal{P}[f]\|_{L^2}^2 \leq C\left( \epsilon^2 |(1 - \mathcal{P})[f]|_{L^2}^2 + \|\mathcal{P}[f]\|_{L^2}^2 + \epsilon^2 \|\mathcal{P}[f]\|_{L^2}^2 + \epsilon^2 \|S\|_{L^2}^2 + \epsilon^2 |h|_{L^2}^2 \right). \] (4.190)

Multiplying a small constant on both sides of (4.190) and adding to (4.189) with \( \eta > 0 \) sufficiently small to absorb \( \epsilon^2 \|\mathcal{P}[f]\|_{L^2}^2 \) and \( \|\mathcal{P}[f]\|_{L^2}^2 \) into the left-hand side, we obtain
\[ \epsilon |(1 - \mathcal{P})[f]|_{L^2}^2 + \|\mathcal{P}[f]\|_{L^2}^2 + \epsilon^2 \|\mathcal{P}[f]\|_{L^2}^2 \leq C\left( \epsilon^2 |(1 - \mathcal{P})[f]|_{L^2}^2 + \epsilon^2 \|\mathcal{P}[f]\|_{L^2}^2 + \epsilon^2 \|S\|_{L^2}^2 + \epsilon^2 |h|_{L^2}^2 \right). \] (4.191)
Step 2: Interpolation Argument.

By interpolation estimate and Young’s inequality, we have

\[ |(1 - \mathcal{P})[f]|_{L^m} \leq |(1 - \mathcal{P})[f]|_{L^2}^{\frac{2}{m}} |(1 - \mathcal{P})[f]|_{L^\infty}^{\frac{m-2}{m}} \]

(4.192)

\[ = \left( \frac{1}{\epsilon^{\frac{m}{m^2}}} \right) |(1 - \mathcal{P})[f]|_{L^2}^{\frac{2}{m}} \left( \epsilon^{\frac{m-2}{m^2}} |(1 - \mathcal{P})[f]|_{L^\infty}^{\frac{m-2}{m}} \right) \]

\[ \leq C \left( \frac{1}{\epsilon^{\frac{m}{m^2}}} \right) |(1 - \mathcal{P})[f]|_{L^2}^{\frac{2}{m}} + o(1) \left( \epsilon^{\frac{m-2}{m^2}} |(1 - \mathcal{P})[f]|_{L^\infty}^{\frac{m-2}{m}} \right) \]

\[ \leq C \left( \frac{1}{\epsilon^{\frac{m}{m^2}}} \right) |(1 - \mathcal{P})[f]|_{L^2} + o(1) \epsilon^{\frac{m}{m^2}} \| (1 - \mathcal{P})[f] \|_{L^\infty} \].

Similarly, we have

\[ \|(1 - \mathcal{P})[f]\|_{L^2} \leq \|(1 - \mathcal{P})[f]\|_{L^2}^{\frac{1}{m}} \|(1 - \mathcal{P})[f]\|_{L^\infty}^{\frac{m-1}{m}} \]

(4.193)

\[ = \left( \frac{1}{\epsilon^{\frac{m}{m^2}}} \right) \|(1 - \mathcal{P})[f]\|_{L^2}^{\frac{1}{m}} \left( \epsilon^{\frac{m-1}{m^2}} \|(1 - \mathcal{P})[f]\|_{L^\infty}^{\frac{m-1}{m}} \right) \]

\[ \leq C \left( \frac{1}{\epsilon^{\frac{m}{m^2}}} \right) \|(1 - \mathcal{P})[f]\|_{L^2}^{\frac{1}{m}} + o(1) \left( \epsilon^{\frac{m-1}{m^2}} \|(1 - \mathcal{P})[f]\|_{L^\infty}^{\frac{m-1}{m}} \right) \]

\[ \leq C \left( \frac{1}{\epsilon^{\frac{m}{m^2}}} \right) \|(1 - \mathcal{P})[f]\|_{L^2} + o(1) \epsilon^{\frac{1}{m^2}} \|(1 - \mathcal{P})[f]\|_{L^\infty}. \]

We need this extra \( \epsilon^{\frac{m}{m^2}} \) for the convenience of \( L^\infty \) estimate. Then we know for sufficiently small \( \epsilon \),

\[ \epsilon^2 |(1 - \mathcal{P})[f]|_{L^2}^{\frac{2}{m}} \leq C \epsilon^{2 - \frac{m-2}{m}} |(1 - \mathcal{P})[f]|_{L^2}^{\frac{2}{m}} + o(1) \epsilon^{2+ \frac{m}{m^2}} |f|_{L^\infty}^{\frac{2}{m}} \]

(4.194)

\[ \leq o(1) \epsilon |(1 - \mathcal{P})[f]|_{L^2}^{\frac{2}{m}} + o(1) \epsilon^{2+ \frac{m}{m^2}} |f|_{L^\infty}^{\frac{2}{m}}. \]

Similarly, we have

\[ \epsilon^{2} \|(1 - \mathcal{P})[f]\|_{L^2} \leq \epsilon^{2 - \frac{2m-2}{m}} \|(1 - \mathcal{P})[f]\|_{L^2}^{\frac{2}{m}} + o(1) \epsilon^{2+ \frac{m}{m^2}} \| u \|_{L^\infty}^{\frac{2}{m}} \]

(4.195)

\[ \leq o(1) \|(1 - \mathcal{P})[f]\|_{L^2}^{\frac{2}{m}} + o(1) \epsilon^{2+ \frac{m}{m^2}} \| f \|_{L^\infty}^{\frac{2}{m}}. \]

In (4.191), we can absorb \( \epsilon |(1 - \mathcal{P})[f]|_{L^2}^{\frac{2}{m}} \) and \( \|(1 - \mathcal{P})[f]\|_{L^2}^{\frac{2}{m}} \) into left-hand side to obtain

\[ \epsilon |(1 - \mathcal{P})[f]|_{L^2}^{\frac{2}{m}} + \|(1 - \mathcal{P})[f]\|_{L^2}^{\frac{2}{m}} + \epsilon^{2} \| f \|_{L^2}^{\frac{2}{m}} \]

(4.196)

\[ \leq C \left( o(1) \epsilon^{2+ \frac{m}{m^2}} \| f \|_{L^\infty}^{\frac{2}{m}} + \| S \|_{L^2}^{\frac{2}{m}} + \epsilon^{2} \| h \|_{L^m}^{\frac{2}{m}} + \| h \|_{L^2}^{\frac{2}{m}} + \int_{\Omega \times \mathbb{R}^2} f S \right). \]

We can decompose

\[ \int_{\Omega \times \mathbb{R}^2} f S = \int_{\Omega \times \mathbb{R}^2} P[S]P[f] + \int_{\Omega \times \mathbb{R}^2} (1 - \mathcal{P}) S(I - \mathcal{P})[f]. \]

(4.197)

Hölder’s inequality and Cauchy’s inequality imply

\[ \int_{\Omega \times \mathbb{R}^2} P[S]P[f] \leq \| P[S] \|_{L^{\frac{2m}{m-1}}} \| P[f] \|_{L^{2m}} \leq C \| P[S] \|_{L^{\frac{2m}{m-1}}}^{\frac{2m}{m-1}} + o(1) \| P[f] \|_{L^{2m}}^{\frac{2m}{m-1}}, \]

(4.198)

and

\[ \int_{\Omega \times \mathbb{R}^2} (1 - \mathcal{P}) S(I - \mathcal{P})[f] \leq C \| \nu^{-\frac{1}{2}} (I - \mathcal{P}) S \|_{L^{2}}^{2} + o(1) \| (I - \mathcal{P})[f] \|_{L^{2}}^{2}. \]

(4.199)
Hence, absorbing \( \epsilon^2 \|P[f]\|_{L^2(\Omega \times \mathbb{S}^1)}^2 \) and \( \|((I - P)[f])\|_{L^2} \) into left-hand side of (4.196), we get

\[
\epsilon |(1 - P)[f]|_{L^2}^2 + \|((I - P)[f])\|_{L^2}^2 + \epsilon^2 \|P[f]\|_{L^{2m}}^2 
\leq C \left( o(1) \epsilon^2 \frac{1}{\epsilon} \|f\|_{L^\infty}^2 + \frac{1}{\epsilon^2} \|P[S]\|_{L^{2m}}^2 + \|S\|_{L^2}^2 + \epsilon^2 |h|_{L^2}^2 + |h|_{L^2}^2 \right).
\]

Therefore, we have

\[
\frac{1}{\epsilon^2} |(1 - P)[f]|_{L^2}^2 + \frac{1}{\epsilon} \|((I - P)[f])\|_{L^2}^2 + \|P[f]\|_{L^{2m}}^2
\leq C \left( o(1) \epsilon^2 \frac{1}{\epsilon} \|f\|_{L^\infty}^2 + \frac{1}{\epsilon^2} \|P[S]\|_{L^{2m}}^2 + \|S\|_{L^2}^2 + \frac{1}{\epsilon} |h|_{L^2}^2 + \|h|_{L^2}^2 \right).
\]

\( \square \)

4.4. \( L^\infty \) Estimates - Second Round.

**Theorem 4.8.** The solution \( f(\vec{x}, \vec{v}) \) to the equation (4.7) satisfies the estimate for \( \vartheta \geq 3 \) and \( 0 \leq \vartheta < \frac{1}{4} \).

\[
\|\langle \vartheta \rangle \vartheta e^{\vartheta |\vec{v}|^2} f \|_{L^\infty} \leq C \left( \frac{1}{\epsilon^2} \|P[S]\|_{L^{2m}}^2 + \frac{1}{\epsilon^2} \|S\|_{L^2}^2 + \frac{\|\vartheta \|_{L^\infty}^2}{\nu} \right)
\]

\[
+ \frac{1}{\epsilon^2} |h|_{L^2}^2 + \frac{1}{\epsilon} \|h|_{L^2}^2 + \langle \vartheta \vartheta e^{\vartheta |\vec{v}|^2} h \|_{L^\infty} \right).
\]

**Proof.** Following the argument in the proof of Theorem 4.3 by double Duhamel’s principle along the characteristic, the key step is to decompose \( f = P[f] + (I - P)[f] \) and utilize \( L^{2m} \) estimates and \( L^2 \) estimates separately as

\[
I_{4,1} \leq C \left( \int_{|\vec{v}| \leq 2N} \int_{|\vec{v}'| \leq 3N} \int_{0}^{t_1} \int_{(X_c - \epsilon(t_1' - r)\vec{v}) \in \Omega} f(X_c - \epsilon(t_1' - r)\vec{v}, \vec{v}')e^{-\vartheta \vartheta(t_1' - r)} dr dv' \right) \quad (4.203)
\]

\[
\leq C \left( \int_{|\vec{v}| \leq 2N} \int_{|\vec{v}'| \leq 3N} \int_{0}^{t_1} \int_{(X_c - \epsilon(t_1' - r)\vec{v}) \in \Omega} (P[f])_{L^{2m}}^2 + \|S\|_{L^2}^2 + \frac{\langle \vartheta \vartheta e^{\vartheta |\vec{v}|^2} h \|_{L^\infty} \right).
\]

All the other terms can be estimated in the similar fashion. In summary, we have

\[
\|\langle \vartheta \rangle \vartheta e^{\vartheta |\vec{v}|^2} f \|_{L^\infty} \leq C \left( \frac{1}{\epsilon^2} \|P[f]\|_{L^{2m}}^2 + \frac{1}{\epsilon} \|((I - P)[f])\|_{L^2}^2 + \frac{\|\vartheta \|_{L^\infty}^2}{\nu} \right)
\]

\[
+ \langle \vartheta \vartheta e^{\vartheta |\vec{v}|^2} h \|_{L^\infty} \right). \quad (4.204)
\]
Then using $L^2$ and $L^\infty$ estimates in Theorem 4.8, we know

\begin{equation}
\left\langle (\bar{v})^\theta e^{\bar{v}|\bar{v}|^2} f \right\rangle_{L^\infty} \leq C \left( o(1) \left\| (\bar{v})^\theta e^{\bar{v}|\bar{v}|^2} f \right\|_{L^\infty} + \frac{1}{\epsilon^{2+\frac{1}{m}}} \left\| P[S] \right\|_{L^{2\frac{m}{m-1}}} + \frac{1}{\epsilon^{1+\frac{1}{m}}} \left\| S \right\|_{L^2} + \frac{\left\langle (\bar{v})^\theta e^{\bar{v}|\bar{v}|^2} S \right\rangle}{\nu} \right)
\end{equation}

\begin{align*}
+ \frac{1}{\epsilon^{1+\frac{1}{m}}} \left| h \right|_{L^2} + \frac{1}{\epsilon^{\frac{1}{m}}} \left| h \right|_{L^\infty} + \left| \left\langle (\bar{v})^\theta e^{\bar{v}|\bar{v}|^2} h \right\rangle_{L^\infty} \right|.
\end{align*}

Since $o(1)$ is small, we can absorb $\left\| (\bar{v})^\theta e^{\bar{v}|\bar{v}|^2} f \right\|_{L^\infty}$ into the left-hand side to obtain

\begin{align*}
\left\| (\bar{v})^\theta e^{\bar{v}|\bar{v}|^2} f \right\|_{L^\infty} \leq C \left( \frac{1}{\epsilon^{2+\frac{1}{m}}} \left\| P[S] \right\|_{L^{2\frac{m}{m-1}}} + \frac{1}{\epsilon^{1+\frac{1}{m}}} \left\| S \right\|_{L^2} + \frac{\left\langle (\bar{v})^\theta e^{\bar{v}|\bar{v}|^2} S \right\rangle}{\nu} \right)
\end{align*}

\begin{align*}
+ \frac{1}{\epsilon^{1+\frac{1}{m}}} \left| h \right|_{L^2} + \frac{1}{\epsilon^{\frac{1}{m}}} \left| h \right|_{L^\infty} + \left| \left\langle (\bar{v})^\theta e^{\bar{v}|\bar{v}|^2} h \right\rangle_{L^\infty} \right|.
\end{align*}

\hfill \square
5. Well-Posedness of $\epsilon$-Milne Problem with Geometric Correction

We consider the $\epsilon$-Milne problem with geometric correction for $g(\eta, \theta, \vec{v})$ in the domain $(\eta, \theta, \vec{v}) \in [0, L] \times [-\pi, \pi) \times \mathbb{R}^2$ as

\[
\begin{cases}
    v_\eta \frac{\partial g}{\partial \eta} - \frac{\epsilon}{R_\kappa - c\eta} \left( v_\phi^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right) + \mathcal{L}[g] = 0, \\
    g(0, \theta, \vec{v}) = h(\theta, \vec{v}) \quad \text{for } v_\eta > 0, \\
    g(L, \theta, \vec{v}) = g(L, \theta, \mathcal{A}[\vec{v}]),
\end{cases}
\]

where $\mathcal{A}[\vec{v}] = (-v_\eta, v_\phi)$ and $L = e^{-\frac{\eta}{2 \epsilon}}$. For simplicity, we temporarily ignore the dependence of $\theta$, i.e. consider the $\epsilon$-Milne problem with geometric correction for $g(\eta, \vec{v})$ in the domain $(\eta, \vec{v}) \in [0, L] \times \mathbb{R}^2$ as

\[
\begin{cases}
    v_\eta \frac{\partial g}{\partial \eta} - \frac{\epsilon}{R_\kappa - c\eta} \left( v_\phi^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right) + \mathcal{L}[g] = 0, \\
    g(0, \vec{v}) = h(\vec{v}) \quad \text{for } v_\eta > 0, \\
    g(L, \vec{v}) = g(L, \mathcal{A}[\vec{v}]).
\end{cases}
\]

Since the null space of the operator $\mathcal{L}$ is spanned by $\mathcal{N} = \mu \frac{1}{2} \left\{ 1, v_\eta, v_\phi, \frac{|\vec{v}|^2 - 2}{2} \right\} = \{ \psi_0, \psi_1, \psi_2, \psi_3 \}$, we can decompose the solution as

\[
g = w_g + g_g, \quad \text{(5.3)}
\]

where

\[
g_g = \mu \frac{1}{2} \left( q_{g,0} v_\eta + q_{g,1} v_\phi + q_{g,2} v_\eta^2 - 2 + q_{g,3} v_\phi^2 \right) = q_{g,0} \psi_0 + q_{g,1} \psi_1 + q_{g,2} \psi_2 + q_{g,3} \psi_3 \in \mathcal{N}, \quad \text{(5.4)}
\]

and

\[
w_g \in \mathcal{N}^\perp, \quad \text{(5.5)}
\]

where $\mathcal{N}^\perp$ is the orthogonal space of $\mathcal{N}$. When there is no confusion, we will simply write $g = w + q$. Our main goal is to find

\[
\tilde{h}(\vec{v}) = \sum_{i=0}^{3} \tilde{D}_i \psi_i \in \mathcal{N}, \quad \text{(5.6)}
\]

with $\tilde{D}_i = 0$ such that the $\epsilon$-Milne problem with geometric correction for $\mathcal{G}(\eta, \vec{v})$ in the domain $(\eta, \vec{v}) \in [0, L] \times \mathbb{R}^2$ as

\[
\begin{cases}
    v_\eta \frac{\partial \mathcal{G}}{\partial \eta} - \frac{\epsilon}{R_\kappa - c\eta} \left( v_\phi^2 \frac{\partial \mathcal{G}}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathcal{G}}{\partial v_\phi} \right) + \mathcal{L}[\mathcal{G}] = 0, \\
    \mathcal{G}(0, \vec{v}) = h(\vec{v}) - \tilde{h}(\vec{v}) \quad \text{for } v_\eta > 0, \\
    \mathcal{G}(L, \vec{v}) = \mathcal{G}(L, \mathcal{A}[\vec{v}]),
\end{cases}
\]

is well-posed, and $\mathcal{G}$ decays exponentially fast as $\eta$ becomes larger and larger. The estimates and decaying rate should be uniform in $\epsilon$.

Let $G(\eta) = -\frac{\epsilon}{R_\kappa - c\eta}$. We define a potential function $W(\eta)$ as $G(\eta) = -\frac{dW}{d\eta}$ with $W(0) = 0$. It is easy to check that

\[
W(\eta) = \ln \left( \frac{R_\kappa}{R_\kappa - c\eta} \right). \quad \text{(5.8)}
\]
In this section, we introduce some special notation to describe the norms in the space $(\eta, \vec{v}) \in [0, L] \times \mathbb{R}^2$. Define the $L^2$ norm as follows:

\[
\|f(\eta)\|_{L^2} = \left( \int_{\mathbb{R}^2} |f(\eta, \vec{v})|^2 \, d\vec{v} \right)^{1/2},
\]

\[
\|f\|_{L^2L^2} = \left( \int_0^L \int_{\mathbb{R}^2} |f(\eta, \vec{v})|^2 \, d\vec{v} \, d\eta \right)^{1/2}.
\]

Define the inner product in $\vec{v}$ space

\[
\langle f, g \rangle (\eta) = \int_{\mathbb{R}^2} f(\eta, \vec{v}) g(\eta, \vec{v}) \, d\vec{v}.
\]

Define the weighted $L^\infty$ norm as follows:

\[
\|f(\eta)\|_{L^\infty} = \sup_{\vec{v} \in \mathbb{R}^2} \left| \langle \vec{v} \rangle^\theta e^{\epsilon|v|^2} |f(\eta, \vec{v})| \right|,
\]

\[
\|f\|_{L^\infty L^\infty} = \sup_{(\eta, \vec{v}) \in [0, L] \times \mathbb{R}^2} \left| \langle \vec{v} \rangle^\theta e^{\epsilon|v|^2} |f(\eta, \vec{v})| \right|,
\]

Define the mixed $L^2$ and weighted $L^\infty$ norm as follows:

\[
\|f\|_{L^\infty L^2} = \sup_{\eta \in [0, L]} \left( \int_{\mathbb{R}^2} e^{2\epsilon|v|^2} f(\eta, \vec{v})^2 \, d\vec{v} \right)^{1/2},
\]

for $0 \leq \epsilon < 1$ and an integer $\theta \geq 3$. Since the boundary data $h(\vec{v})$ is only defined on $v_\eta > 0$, we naturally extend above definitions on this half-domain as follows:

\[
\|h\|_{L^2} = \left( \int_{v_\eta > 0} |h(\vec{v})|^2 \, d\vec{v} \right)^{1/2},
\]

\[
\|h\|_{L^\infty L^\infty} = \sup_{v_\eta > 0} \left| \langle \vec{v} \rangle^\theta e^{\epsilon|v|^2} |h(\vec{v})| \right|.
\]

We assume

\[
\|h\|_{L^\infty L^\infty} \leq C,
\]

for some $C > 0$ uniform in $\epsilon$.

Here, we mainly refer to the procedure in [7] and [27], where $\eta \in [0, \infty)$. Since our domain for $\eta$ is bounded, we have to start from scratch and prove each result in the new settings.

5.1. $L^2$ Estimates. We write $g = w + q$ with $q = q_0\psi_0 + q_1\psi_1 + q_2\psi_2 + q_3\psi_3$.

Lemma 5.1. There exists a unique solution of the equation (5.2) satisfying the estimates

\[
\left\| \nu^{\frac{1}{2}} w \right\|_{L^2L^2} \leq C,
\]

\[
\left\| q - q_L \right\|_{L^2L^2} \leq C,
\]

for some $q_L$ satisfying $|q_L| \leq C$, where $C$ is a constant independent of $\epsilon$. Also, the solution satisfies the orthogonal relation

\[
\langle v_\eta \psi_i, w \rangle (\eta) = 0, \quad \text{for } i = 0, 2, 3.
\]

Proof. The existence and uniqueness follow from a standard argument by adding penalty term $\lambda g$ on the left-hand side of the equation for $0 < \lambda << 1$ and estimate along the characteristics (see [27]). Hence, we concentrate on the a priori estimates. We divide the proof into several steps:

Step 1: Estimate of $w$. Multiplying $g$ on both sides of (5.2) and integrating over $\vec{v} \in \mathbb{R}^2$, we have

\[
\frac{1}{2} \frac{d}{d\eta} \langle v_\eta g, g \rangle + G(\eta) \left( v_\eta^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right) g = - \langle g, L[g] \rangle.
\]
An integration by parts implies
\[
\left\langle v^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi}, g \right\rangle = \frac{1}{2} \left\langle v^2 \frac{\partial g^2}{\partial v_\eta}, g \right\rangle - \frac{1}{2} \left\langle v_\eta v_\phi \frac{\partial g^2}{\partial v_\phi}, g \right\rangle = \frac{1}{2} \left\langle v_\eta g, g \right\rangle. \tag{5.22}
\]
Therefore, using Lemma 2.1, we obtain
\[
\frac{1}{2} \frac{d}{d\eta} \langle v_\eta g, g \rangle + \frac{1}{2} G(\eta) \langle v_\eta g, g \rangle = - \langle w, \mathcal{L}[w] \rangle. \tag{5.23}
\]
Define
\[
\alpha(\eta) = \frac{1}{2} \langle v_\eta g, g \rangle (\eta), \tag{5.24}
\]
which implies
\[
\frac{d\alpha}{d\eta} + G(\eta) \alpha = - \langle w, \mathcal{L}[w] \rangle. \tag{5.25}
\]
Then we have
\[
\alpha(\eta) = \alpha(L) \exp \left( \int_0^L G(y) dy \right) + \int_0^L \exp \left( - \int_0^y G(z) dz \right) \left( \langle w, \mathcal{L}[w] \rangle (y) \right) dy \tag{5.26}
\]
\[
\alpha(\eta) = \alpha(0) \exp \left( - \int_0^\eta G(y) dy \right) + \int_0^\eta \exp \left( \int_y^\eta G(z) dz \right) \left( - \langle w, \mathcal{L}[w] \rangle (y) \right) dy. \tag{5.27}
\]
Since \(\alpha(L) = 0\) due to the reflexive boundary condition and the coercivity \(\langle w, \mathcal{L}[w] \rangle \geq \|v^2 w(\eta)\|_{L^2}\), \(5.30\) implies that
\[
\alpha(\eta) \geq 0. \tag{5.28}
\]
Considering
\[
\alpha(0) = \frac{1}{2} \int_{v_\eta > 0} v_\eta g^2(0, \bar{\nu}) d\bar{\nu} + \frac{1}{2} \int_{v_\eta < 0} v_\eta g^2(0, \bar{\nu}) d\bar{\nu} \leq \frac{1}{2} \int_{v_\eta > 0} v_\eta g^2(0, \bar{\nu}) d\bar{\nu}
\]
\[
= \frac{1}{2} \int_{v_\eta > 0} \nu_\eta \nu^2(0, \bar{\nu}) d\bar{\nu} \leq C,
\]
and \(5.27\), we obtain
\[
\alpha(\eta) \leq C. \tag{5.30}
\]
Hence, \(5.30\) and \(6.109\) lead to
\[
\int_0^L \exp \left( - \int_0^y G(z) dz \right) \left( \langle w, \mathcal{L}[w] \rangle (y) \right) dy \leq C, \tag{5.31}
\]
which, by Lemma 2.1, further yields
\[
\int_0^L \left\|v^2 w(\eta)\right\|_{L^2}^2 d\eta \leq C \tag{5.32}
\]
Step 2: Estimate of \(q\).
Multiplying \(v_\eta \psi_j\) with \(j \neq 1\) on both sides of \(5.2\) and integrating over \(\bar{\nu} \in \mathbb{R}^2\), we obtain
\[
\frac{d}{d\eta} \left\langle v^2 \psi_j, g \right\rangle + G(\eta) \left\langle v_\eta \psi_j, v^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right\rangle = - \left\langle v_\eta \psi_j, \mathcal{L}[w] \right\rangle. \tag{5.33}
\]
Define \(\tilde{q} = q - q_1 \psi_1\) and
\[
\beta_j(\eta) = \left\langle v^2 \psi_j, \tilde{q} \right\rangle (\eta), \tag{5.34}
\]
\[
\beta(\eta) = \left( \beta_0(\eta), \beta_1(\eta), \beta_2(\eta), \beta_3(\eta) \right)^T \tag{5.35}
\]
\[
\tilde{\beta}(\eta) = \left( \beta_0(\eta), \beta_2(\eta), \beta_3(\eta) \right)^T. \tag{5.36}
\]
Due to symmetry, it is easy to check that $\beta_1 = 0$. For $j \neq 1$, using integration by parts, we have
\[
\frac{d}{d\eta} \langle v_\eta^2 \psi_j, g \rangle = G(\eta) \left( \frac{\partial}{\partial v_\eta} (v_\eta v_\eta^2 \psi_j) - \frac{\partial}{\partial v_\phi} (v_\eta^2 v_\psi \psi_j), g \right) - \langle v_\eta \psi_j, [L[w]] \rangle, \tag{5.37}
\]
which further implies
\[
\frac{d\beta_j}{d\eta} = G(\eta) \left( \frac{\partial}{\partial v_\eta} (v_\eta v_\eta^2 \psi_j) - \frac{\partial}{\partial v_\phi} (v_\eta^2 v_\psi \psi_j), q_1 + w \right) - \langle v_\eta \psi_j, [L[w]] \rangle - \frac{d}{d\eta} \langle v_\eta^2 \psi_j, w \rangle. \tag{5.38}
\]
Then we can write
\[
\langle \frac{\partial}{\partial v_\eta} (v_\eta v_\eta^2 \psi_j) - \frac{\partial}{\partial v_\phi} (v_\eta^2 v_\psi \psi_j), q_1 \rangle (\eta) = \sum_i B_{ji} q_{i}(\eta), \tag{5.39}
\]
for $i, j = 0, 2, 3$, where
\[
B_{ji} = \left( \frac{\partial}{\partial v_\eta} (v_\eta v_\eta^2 \psi_j) - \frac{\partial}{\partial v_\phi} (v_\eta^2 v_\psi \psi_j), \psi_i \right). \tag{5.40}
\]
Moreover,
\[
\beta_j(\eta) = \sum_k A_{jk} q_k(\eta), \tag{5.41}
\]
for $k, j = 0, 2, 3$, where
\[
A_{jk} = \langle v_\eta^2 \psi_j, \psi_k \rangle, \tag{5.42}
\]
constitutes a non-singular matrix $A$ such that we can express back
\[
q_j(\eta) = \sum_k A^{-1}_{jk} \beta_k(\eta). \tag{5.43}
\]
Hence, (5.38) can be rewritten as
\[
\frac{d\tilde{\beta}}{d\eta} = G(BA^{-1})\tilde{\beta} + D, \tag{5.44}
\]
where
\[
D_j = G(\eta) \left( \frac{\partial}{\partial v_\eta} (v_\eta v_\eta^2 \psi_j) - \frac{\partial}{\partial v_\phi} (v_\eta^2 v_\psi \psi_j), q_1 + w \right) - \langle v_\eta \psi_j, [L[w]] \rangle - \frac{d}{d\eta} \langle v_\eta^2 \psi_j, w \rangle. \tag{5.45}
\]
We can solve for $\tilde{\beta}$ as
\[
\tilde{\beta}(\eta) = \exp \left( -W(\eta)BA^{-1} \right) \tilde{\beta}(0) + \int_0^\eta \exp \left( (W(\eta) - W(y))BA^{-1} \right) D(y)dy. \tag{5.46}
\]
We may further simplify the second term on the right-hand side. Consider the last term in (5.46). Define
\[
\zeta_j(\eta) = \langle v_\eta^2 \psi_j, w \rangle(\eta). \tag{5.47}
\]
We may directly simplify by parts to obtain
\[
\int_0^\eta \exp \left( (W(\eta) - W(y))BA^{-1} \right) \frac{d\zeta}{dy} dy = \zeta(\eta) - \exp \left( -W(\eta)BA^{-1} \right) \zeta(0) - \int_0^\eta \exp \left( (W(\eta) - W(y))BA^{-1} \right) G(y)(BA^{-1})\zeta(y)dy. \tag{5.48}
\]
Hence, we may rewrite (5.46) as
\[
\tilde{\beta}(\eta) = \exp \left( -W(\eta)BA^{-1} \right) \theta - \zeta(\eta) + \int_0^\eta \exp \left( (W(\eta) - W(y))BA^{-1} \right) Z(y)dy, \tag{5.49}
\]
where
\[
\theta_j = \langle v_\eta^2 \psi_j, g \rangle(0), \quad j = 0, 2, 3, \tag{5.50}
\]
and
\[
Z = D + \frac{d\zeta}{d\eta} + G(BA^{-1})\zeta. \tag{5.51}
\]
Hence, using the boundedness of $W(\eta)$ and $BA^{-1}$, we can directly estimate (5.49) to get
\[
|\beta_j(\eta)| \leq C|\theta_j| + |\zeta_j(\eta)| + C \int_0^\eta |Z_j(y)| \, dy \quad \text{for } i = 0, 2, 3. \tag{5.52}
\]
By Cauchy’s inequality and Lemma 2.1, we obtain
\[
|\zeta_j(\eta)| \leq \left\| \nu^\frac{1}{2} w(\eta) \right\|_{L^2} \tag{5.53}
\]
\[
|Z_j(\eta)| \leq C \left( \left\| \nu^\frac{1}{2} w(\eta) \right\|_{L^2} + q_1(\eta) \right). \tag{5.54}
\]
Multiplying $\psi_0$ on both sides of (5.2) and integrating over $\vec{v} \in \mathbb{R}^2$, we have
\[
\frac{d}{d\eta} \langle \psi_0 v_\eta, g \rangle = G(\eta) \left( \frac{\partial}{\partial v_\eta} (\psi_0 v_\eta^2) - \frac{\partial}{\partial v_\phi} (\psi_0 v_\eta v_\phi), g \right) = -G(\eta) \langle \psi_0 v_\eta, g \rangle, \tag{5.55}
\]
which is actually
\[
\frac{d}{d\eta} q_1 = -G(\eta) q_1. \tag{5.56}
\]
Since $q_1(L) = 0$, we have for any $\eta \in [0, L],$
\[
q_1(\eta) = 0. \tag{5.57}
\]
Also,
\[
\theta_j = \langle \nu^\frac{1}{2} \psi_j, g \rangle (0) \leq C \langle |v_\eta g(0)|, g(0) \rangle^{1/2} \langle \nu^\frac{1}{2}, \psi_j^2 \rangle^{1/2} \leq C \langle |v_\eta g(0)|, g(0) \rangle^{1/2}, \tag{5.58}
\]
\[
\langle |v_\eta g(0)|, g(0) \rangle = \int_{v_\eta > 0} v_\eta h^2(\vec{v})d\vec{v} - \int_{v_\eta < 0} v_\eta g^2(0, \vec{v})d\vec{v}. \tag{5.59}
\]
Since
\[
\int_{v_\eta > 0} v_\eta h^2(\vec{v})d\vec{v} + \int_{v_\eta < 0} v_\eta g^2(0, \vec{v})d\vec{v} = 2\alpha(0) \geq 0, \tag{5.60}
\]
we have
\[
\theta_j = \langle \nu^\frac{1}{2} \psi_j, g \rangle (0) \leq 2C \int_{v_\eta > 0} v_\eta h^2(\vec{v})d\vec{v} \leq C. \tag{5.61}
\]
In conclusion, collecting (5.52), (5.53), (5.54), (5.57), and (5.61), we have
\[
|\beta_j(\eta)| \leq C \left( 1 + \left\| \nu^\frac{1}{2} w(\eta) \right\|_{L^2} + \int_0^\eta \left\| \nu^\frac{1}{2} w(y) \right\|_{L^2} \, dy \right) \quad \text{for } j = 0, 2, 3, \tag{5.62}
\]
which further implies
\[
|q_j(\eta)| \leq C \left( 1 + \left\| \nu^\frac{1}{2} w(\eta) \right\|_{L^2} + \int_0^\eta \left\| \nu^\frac{1}{2} w(y) \right\|_{L^2} \, dy \right) \quad \text{for } j = 0, 2, 3, \tag{5.63}
\]
and $q_1(\eta) = 0$. An application of Cauchy’s inequality leads to our desired result.

Step 3: Orthogonal Properties.
It is easy to check that $q_1 = 0$ and
\[
\langle v_\eta \psi_i, q \rangle = 0, \quad i = 0, 2, 3. \tag{5.64}
\]
In the equation (5.2), multiplying $\psi_i$ for $i = 0, 2, 3$ on both sides and integrating over $\vec{v} \in \mathbb{R}^2$, we have
\[
\frac{d}{d\eta} \langle \psi_i v_\eta, g \rangle = G(\eta) \left( \frac{\partial}{\partial v_\eta} (\psi_i v_\eta^2) - \frac{\partial}{\partial v_\phi} (\psi_0 v_\eta v_\phi), g \right) = -G(\psi_i v_\eta, g). \tag{5.65}
\]
Since $\langle \psi_i v_\eta, g \rangle (L) = 0$ due to reflexive boundary condition, we have
\[
\langle v_\eta \psi_i, g \rangle (\eta) = \langle v_\eta \psi_i, w \rangle (\eta) = 0. \tag{5.66}
\]
Step 4: Estimate of $q_L$. 
The estimates in Step 2 is not strong enough to bound \( q \), so we need a different setting to further bound \( w \) and \( q \). Considering \( q_1(\eta) = 0 \) for any \( \eta \in [0, L] \), we do not need to bother with it.

Since \( L^2(\mathbb{R}^2) \to N^\perp \) with null space \( N \) and image \( N^\perp \), we have \( \bar{L} : L^2/N \to N^\perp \) is bijective, where \( L^2/N = N^\perp \) is the quotient space. Then we can define its inverse, i.e. the pseudo-inverse of \( L \) as \( \bar{L}^{-1} : N^\perp \to N^\perp \) satisfying \( \bar{L}^{-1}[f] = f \) for any \( f \in N^\perp \).

We intend to multiply \( \bar{L}^{-1}[v_\eta \psi_i] \) for \( i = 2, 3 \) on both sides of (5.2) and integrating over \( \bar{v} \in \mathbb{R}^2 \). Notice that \( v_\eta \psi_2 \in N^\perp \), but \( v_\eta \psi_3 \notin N^\perp \). Actually, it is easy to verify \( v_\eta(\psi_3 - \psi_0) \in N^\perp \). To avoid introducing new notation, we still use \( \psi_3 \) to denote \( \psi_3 - \psi_0 \) in the following proof and it is easy to see that there is no confusion. Then we get

\[
\frac{d}{d\eta} \langle \bar{L}^{-1}[\psi_i v_\eta], v_\eta g \rangle + G(\eta) \langle \bar{L}^{-1}[\psi_i v_\eta], \left( v_\phi^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right) \rangle = -\langle \bar{L}^{-1}[\psi_i v_\eta], L[w] \rangle. \tag{5.67}
\]

Since \( L \) is self-adjoint, combining with the orthogonal properties, we have

\[
\langle \bar{L}^{-1}[\psi_i v_\eta], L[w] \rangle(\eta) = \langle \bar{L} \left[ \bar{L}^{-1}[\psi_i v_\eta] \right], w \rangle(\eta) = \langle \psi_i v_\eta, w \rangle(\eta) = 0. \tag{5.68}
\]

Therefore, we have

\[
\frac{d}{d\eta} \langle v_\eta \bar{L}^{-1}[\psi_i v_\eta], g \rangle + G(\eta) \langle \bar{L}^{-1}[\psi_i v_\eta], \left( v_\phi^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right) \rangle = 0. \tag{5.69}
\]

We may integrate by parts to obtain

\[
\frac{d}{d\eta} \langle v_\eta \bar{L}^{-1}[\psi_i v_\eta], g \rangle - G(\eta) \left\langle \left( v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} - v_\eta \right) \bar{L}^{-1}[\psi_i v_\eta], g \right\rangle = 0. \tag{5.70}
\]

Since \( v_\eta \psi_0 = \psi_1 \in N \) and \( \bar{L}^{-1}[\psi_i v_\eta] \in N^\perp \), we have

\[
\langle v_\eta \bar{L}^{-1}[\psi_i v_\eta], \psi_0 \rangle = 0. \tag{5.71}
\]

For \( i, k = 2, 3 \), put

\[
N_{i,k} = \langle v_\eta \bar{L}^{-1}[\psi_i v_\eta], \psi_k \rangle, \tag{5.72}
\]

\[
P_{i,k} = \left\langle \left( v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} - v_\eta \right) \bar{L}^{-1}[\psi_i v_\eta], \psi_k \right\rangle. \tag{5.73}
\]

Thus,

\[
\Omega_i = \langle v_\eta \bar{L}^{-1}[\psi_i v_\eta], q \rangle = \sum_{k=2}^{3} N_{i,k} q_k(\eta), \tag{5.74}
\]

and

\[
\left\langle \left( v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} - v_\eta \right) \bar{L}^{-1}[\psi_i v_\eta], q \right\rangle = \sum_{k=2}^{3} P_{i,k} q_k(\eta). \tag{5.75}
\]

Since matrix \( N \) is invertible (see [12]), from (5.70) and integration by parts, we have for \( i = 2, 3 \),

\[
\frac{d\Omega_i}{d\eta} = -\frac{d}{d\eta} \langle v_\eta \bar{L}^{-1}[\psi_i v_\eta], w \rangle \tag{5.76}
\]

\[
+ \sum_{k=2}^{3} G(\eta) (PN^{-1})_{ik} \Omega_k + G(\eta) \left\langle \left( v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} - v_\eta \right) \bar{L}^{-1}[\psi_i v_\eta], w \right\rangle.
\]

Denote

\[
\hat{\Omega} = \exp \left( W(\eta) PN^{-1} \right) \Omega. \tag{5.77}
\]
This is an ordinary differential equation for \( \Omega \). Let \( \hat{\psi} = (\psi_2, \psi_3)^T \), we can solve

\[
\hat{\Omega}(\eta) = \langle v_\eta L^{-1}[\hat{\psi}v_\eta], g \rangle (0) - \exp \left( W(\eta)PN^{-1} \right) \langle v_\eta L^{-1}[\hat{\psi}v_\eta], w \rangle (\eta) \\
+ \int_0^\eta \exp \left( W(y)PN^{-1} \right) G(y) \left( \langle v_\eta^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} - v_\eta \rangle L^{-1}[\hat{\psi}v_\eta], w \rangle (y) \right) dy \\
+ \sum_{k=2}^3 PN^{-1} \langle v_\eta L^{-1}[\hat{\psi}v_\eta], w \rangle (y) \right) dy.
\]

By a similar method as in Step 2 to bound \( \theta_i(0) \), we can show

\[
\langle v_\eta L^{-1}[\hat{\psi}v_\eta], g \rangle (0) = \langle \hat{\psi}v_\eta, L[v_\eta g] \rangle (0) \leq C. \tag{5.79}
\]

Since \( w \in L^2([0, L] \times \mathbb{R}^2) \), considering \( W(\eta) \) and \( PN^{-1} \) are bounded, and \( G(\eta) \in L^\infty \), we define

\[
\hat{\Omega}_L = \langle v_\eta L^{-1}[\hat{\psi}v_\eta], g \rangle (0) + \int_0^L \exp \left( W(y)PN^{-1} \right) G(y) \left( \langle v_\eta^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} - v_\eta \rangle L^{-1}[\hat{\psi}v_\eta], w \rangle (y) \right) dy \\
+ \sum_{k=2}^3 PN^{-1} \langle v_\eta L^{-1}[\hat{\psi}v_\eta], w \rangle (y) \right) dy.
\]

Let \( \hat{q}_L = (q_{2,L}, q_{3,L})^T \). Then we can define

\[
\hat{q}_L = N^{-1} \exp \left( -W(L)PN^{-1} \right) \hat{\Omega}_L. \tag{5.81}
\]

Finally, we consider \( q_{0,L} \). Multiplying \( \psi_1 \) on both sides of (5.2) and integrating over \( \mathfrak{v} \in \mathbb{R}^2 \), we obtain

\[
\frac{d}{d\eta} \langle v_\eta \psi_1, g \rangle = -G \langle \psi_1, v_\phi \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \rangle = G \langle g, (v_\phi^2 - \hat{v}_\eta^2) \mu^\frac{1}{2} \rangle. \tag{5.82}
\]

Then integrating over \([0, \eta]\), we obtain

\[
\langle v_\eta \psi_1, g \rangle (\eta) = \langle v_\eta \psi_1, g \rangle (0) + \int_0^\eta G(y) \langle g, (v_\phi^2 - \hat{v}_\eta^2) \mu^\frac{1}{2} \rangle (y) dy \\
= \langle v_\eta g, v_\eta \rangle (0) + \int_0^\eta G(y) \langle w, (v_\phi^2 - \hat{v}_\eta^2) \mu^\frac{1}{2} \rangle (y) dy.
\]

Since \( w \in L^2([0, L] \times \mathbb{R}^2) \) and we can also bound \( \langle v_\eta g, v_\eta \rangle (0) \), we have

\[
\langle v_\eta \psi_1, g \rangle (L) = \langle v_\eta g, v_\eta \rangle (0) + \int_0^\eta G(y) \langle w, (v_\phi^2 - \hat{v}_\eta^2) \mu^\frac{1}{2} \rangle (y) dy \leq C. \tag{5.84}
\]

Note that

\[
\langle v_\eta \psi_1, \psi_1 \rangle (\eta) = \langle v_\eta \psi_1, \psi_2 \rangle (\eta) = 0. \tag{5.85}
\]

Then we define

\[
q_{0,L} = \frac{\langle v_\eta \psi_1, g \rangle (L) - q_{3,L} \langle v_\eta \psi_1, \psi_3 \rangle}{\langle v_\eta \psi_1, \psi_0 \rangle}. \tag{5.86}
\]

Naturally, we define \( q_{1,L} = 0 \). Then to summarize all above, we have defined

\[
q_L = q_{0,L} \psi_0 + q_{1,L} \psi_1 + q_{2,L} \psi_2 + q_{3,L} \psi_3, \tag{5.87}
\]

which satisfies \( |q_{i,L}| \leq C \) for \( i = 0, 1, 2, 3 \).

Step 5: \( L^2 \) Decay of \( w \).

The orthogonal property and \( q_1 = 0 \) imply

\[
\langle v_\eta q, w \rangle (\eta) = \sum_{k=0}^3 \langle v_\eta \psi_k, w \rangle (\eta) = 0. \tag{5.88}
\]
Also, we may directly verify that
\[ \langle v \eta q, q \rangle (\eta) = 0. \] (5.89)

Therefore, we deduce that
\[ \langle v \eta g, g \rangle (\eta) = \langle v \eta w, w \rangle (\eta). \] (5.90)

Multiplying \( e^{2K_0 \eta} g \) on both sides of (5.2) and integrating over \( \bar{\eta} \in \mathbb{R}^2 \), we obtain
\[ \frac{1}{2} \frac{d}{d\eta} \langle v \eta g, e^{2K_0 \eta} g \rangle + G(\eta) \langle v_\eta^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi}, e^{2K_0 \eta} g \rangle = K_0 e^{2K_0 \eta} \langle v \eta w, w \rangle - e^{2K_0 \eta} \langle g, L[g] \rangle. \] (5.91)

An integration by parts implies
\[ \langle v_\eta^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi}, e^{2K_0 \eta} g \rangle = \frac{1}{2} \langle v_\eta^2, \frac{\partial (e^{2K_0 \eta} g^2)}{\partial \eta} \rangle - \frac{1}{2} \langle v_\eta v_\phi, \frac{\partial (e^{2K_0 \eta} g^2)}{\partial \phi} \rangle = \frac{1}{2} \langle v \eta g, e^{2K_0 \eta} g \rangle \] (5.92)

Therefore, using Lemma 2.1, we obtain
\[ \frac{1}{2} \frac{d}{d\eta} \langle v \eta g, e^{2K_0 \eta} g \rangle + \frac{1}{2} G(\eta) \langle v \eta g, e^{2K_0 \eta} g \rangle = K_0 e^{2K_0 \eta} \langle v \eta w, w \rangle - e^{2K_0 \eta} \langle g, L[g] \rangle. \] (5.93)

Hence, we can rewrite it as
\[ \frac{1}{2} \frac{d}{d\eta} \left( e^{2K_0 \eta + W(\eta)} \langle v \eta w, w \rangle \right) = e^{2K_0 \eta + W(\eta)} \left( K_0 \langle v \eta w, w \rangle - \langle w, L[w] \rangle \right) = 0. \] (5.94)

Since
\[ \langle L[w], w \rangle \geq \langle \nu w, w \rangle, \] (5.95)
and \( W(\eta) \) is bounded, for \( K_0 \) sufficiently small, we have
\[ \langle L[w], w \rangle - K_0 \langle v \eta w, w \rangle \geq C \left\| v \eta \right\|_{L^2}. \] (5.96)

Then by a similar argument as in Step 1, we can show that
\[ \int_0^L e^{2K_0 \eta} \left\| v \eta \right\|_{L^2} \ d\eta \leq C. \] (5.97)

Step 6: Estimate of \( q - q_L \).
We first consider \( \hat{q} = (q_2, q_3)^T \), which satisfies
\[ \hat{q}(\eta) = N^{-1} \exp \left( -W(\eta)PN^{-1} \right) \hat{\Omega}(\eta), \] (5.98)

Let
\[ \delta = \left\langle v \eta L^{-1}[\hat{\psi} v_\eta], g \right\rangle (0) \] (5.99)
\[ \Delta = G \left( \left\langle v_\eta^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} - v_\eta \right \rangle L^{-1}[\hat{\psi} v_\eta], w \right\rangle + \sum_{k=2}^3 PN^{-1} \left\langle v_\eta L^{-1}[\hat{\psi} v_\eta], w \right\rangle \] (5.100)

Then we have
\[ \hat{q}(\eta) = N^{-1} \exp \left( -W(\eta)PN^{-1} \right) \delta - \hat{\Omega}^{-1} \left\langle v \eta L^{-1}[\hat{\psi} v_\eta], w \right\rangle (\eta) \] (5.101)
\[ + \int_0^\eta \exp \left( (W(y) - W(\eta))PN^{-1} \right) \Delta(y)dy. \]

Also, we know
\[ \hat{q}_L = N^{-1} \exp \left( -W(L)PN^{-1} \right) \delta + \int_0^L \exp \left( (W(y) - W(\infty))PN^{-1} \right) \Delta(y)dy. \] (5.102)
Then we have

\[
\hat{q}(\eta) - \hat{q}_L = N^{-1} \left( \exp \left( - W(\eta) PN^{-1} \right) - \exp \left( - W(L) PN^{-1} \right) \right) \delta - N^{-1} \left\langle v_\eta L^{-1} [\hat{\psi}_\eta], w \right\rangle (\eta) \tag{5.103}
\]

\[
+ N^{-1} \left( \exp \left( - W(\eta) PN^{-1} \right) - \exp \left( - W(L) PN^{-1} \right) \right) \int_0^L \exp \left( W(y) PN^{-1} \right) \Delta(y) dy 
\]

\[
+ N^{-1} \int_\eta^L \exp \left( (W(y) - W(\eta)) PN^{-1} \right) \Delta(y) dy.
\]

Then we have

\[
\|\hat{q} - \hat{q}_L\|_{L^2 L^2} \leq \left\| N^{-1} \left( \exp \left( - W(\eta) PN^{-1} \right) - \exp \left( - W(L) PN^{-1} \right) \right) \delta \right\|_{L^2 L^2} + \left\| N^{-1} \left\langle v_\eta L^{-1} [\hat{\psi}_\eta], w \right\rangle \right\|_{L^2 L^2} 

\]

\[
+ \left\| N^{-1} \left( \exp \left( - W(\eta) PN^{-1} \right) - \exp \left( - W(L) PN^{-1} \right) \right) \int_0^L \exp \left( W(y) PN^{-1} \right) \Delta(y) dy \right\|_{L^2 L^2} 

\]

\[
+ \left\| N^{-1} \int_\eta^L \exp \left( (W(y) - W(\eta)) PN^{-1} \right) \Delta(y) dy \right\|_{L^2 L^2}.
\]

We need to estimate each term on the right-hand side of (5.104). We have

\[
\left\| N^{-1} \left( \exp \left( - W(\eta) PN^{-1} \right) - \exp \left( - W(L) PN^{-1} \right) \right) \delta \right\|_{L^2 L^2}^2 \leq C \delta \left\| e^{-W(\eta)} - e^{-W(L)} \right\|_{L^2 L^2} \leq C. \tag{5.105}
\]

Since \( w \in L^2([0, L] \times \mathbb{R}^2) \), we have

\[
\left\| N^{-1} \left\langle v_\eta L^{-1} [\hat{\psi}_\eta], w \right\rangle \right\|_{L^2 L^2} \leq C \|w\|_{L^2 L^2} \leq C. \tag{5.106}
\]

Similarly, we can show

\[
\left\| N^{-1} \left( \exp \left( - W(\eta) PN^{-1} \right) - \exp \left( - W(L) PN^{-1} \right) \right) \int_0^L \exp \left( W(y) PN^{-1} \right) \Delta(y) dy \right\|_{L^2 L^2} \leq C \left\| e^{-W(\eta)} - e^{-W(L)} \right\|_{L^2 L^2} \|\Delta\|_{L^2 L^2} \leq C. \tag{5.107}
\]

For the last term, we have to resort to the exponential decay of \( w \) in Step 5. We estimate

\[
\left\| N^{-1} \int_\eta^L \exp \left( (W(y) - W(\eta)) PN^{-1} \right) \Delta(y) dy \right\|_{L^2 L^2} \leq C \int_0^L \left( \int_\eta^L \Delta(y) dy \right)^2 d\eta \leq C \int_0^L \left( \int_\eta^\infty e^{-2K_0 y} dy \right) \left( \int_\eta^L w^2(y) e^{2K_0 y} dy \right) d\eta
\]

\[
\leq \int_0^L Ce^{-2K_0 \eta} d\eta \leq C.
\]

Collecting all above, we have

\[
\|\hat{q} - \hat{q}_L\|_{L^2 L^2} \leq C. \tag{5.109}
\]

Then we turn to \( q_0 \). We have

\[
q_0(\eta) = \frac{\langle v_\eta \psi_1, q_0 \rangle (\eta) - \langle v_\eta \psi_1, w \rangle (\eta) - q_1(\eta) \langle v_\eta \psi_1, \psi_1 \rangle}{\langle v_\eta \psi_1, \psi_1 \rangle}. \tag{5.110}
\]
where

$$\langle \psi_1 g, v_\eta \rangle (\eta) = \langle \psi_1 g, v_\eta \rangle (0) + \int_0^n G(y) \left( w, \left( v_\phi^2 - v_\eta^2 \right) \mu_\phi^2 \right) (y) dy.$$  \hfill (5.111)

Also, we have

$$q_{0, L} = \frac{\langle v_\eta \psi_1, g \rangle (L) - q_{3, L} \langle v_\eta \psi_1, \psi_3 \rangle}{\langle v_\eta \psi_1, v_\eta \rangle}.$$  \hfill (5.112)

Therefore, we have

$$q_0(\eta) - q_{0, L} = \frac{\int_0^L G(y) \left( w, \left( v_\phi^2 - v_\eta^2 \right) \mu_\phi^2 \right) (y) dy - \langle v_\eta \psi_1, w \rangle (\eta) - (q_3(\eta) - q_{3, L}) \langle v_\eta \psi_1, \psi_3 \rangle}{\langle v_\eta \psi_0, v_\eta \rangle}.$$  \hfill (5.113)

Then we can naturally estimate

$$\| q_0 - q_{0, L} \|_{L^2 L^2} \leq C \left( \left\| \int_0^L G(y) \left( w, \left( v_\phi^2 - v_\eta^2 \right) \mu_\phi^2 \right) (y) dy \right\|_{L^2 L^2} + \| w \|_{L^2 L^2} + \| q_3(\eta) - q_{3, L} \|_{L^2 L^2} \right).$$  \hfill (5.114)

$$\| q_3(\eta) - q_{3, L} \|_{L^2 L^2}$$ is bounded due to the estimate of $$\| \hat{q}(\eta) - \hat{q}_L \|_{L^2 L^2}$$. Then by Cauchy’s inequality, we obtain

$$\left\| \int_0^L G(y) \left( w, \left( v_\phi^2 - v_\eta^2 \right) \mu_\phi^2 \right) (y) dy \right\|_{L^2 L^2} \leq \int_0^L \left( \int_0^L G(y) \| w(y) \|_{L^2} dy \right)^2 d\eta \leq \| w \|_{L^2 L^2}^2 \int_0^L \int_0^L G^2(y) dy d\eta \leq C.$$  \hfill (5.115)

Therefore, we have shown

$$\| q_0 - q_{0, L} \|_{L^2 L^2} \leq C.$$  \hfill (5.116)

In summary, we prove that

$$\| q - q_L \|_{L^2 L^2} \leq C.$$  \hfill (5.117)

\textbf{Lemma 5.2.} There exists a unique solution $$g(\eta, \vec{v})$$ to the $$\epsilon$$-Milne problem with geometric correction (5.2) satisfying

$$\| g - g_L \|_{L^2 L^2} \leq C,$$  \hfill (5.118)

for some $$g_L \in \mathcal{N}$$ satisfying $$|g_L| \leq C$$.

\textbf{Proof.} Taking $$g_L = q_L$$, we can naturally obtain the desired result. \hfill \Box

Then we turn to the construction of $$\tilde{h}$$ and the well-posedness of the equation (5.7).

\textbf{Theorem 5.3.} There exists $$\tilde{h}$$ satisfying the condition (5.6) such that there exists a unique solution $$\mathcal{G}(\eta, \vec{v})$$ to the $$\epsilon$$-Milne problem (5.7) with geometric correction satisfying

$$\| \mathcal{G} \|_{L^2 L^2} \leq C.$$  \hfill (5.119)

\textbf{Proof.} We want to find $$\mathcal{G}$$ such that $$\mathcal{G}_L = 0$$. The key part is the construction of $$\tilde{h}$$. Our main idea is to find $$\tilde{h} \in \mathcal{N}$$ such that the equation

$$\begin{cases}
  v_\eta \frac{\partial \tilde{g}}{\partial \eta} + G(\eta) \left( v_\phi^2 \frac{\partial \tilde{g}}{\partial \eta} - v_\eta v_\phi \frac{\partial \tilde{g}}{\partial v_\phi} \right) + \mathcal{L}[\tilde{g}] = 0, \\
  \tilde{g}(0, \vec{v}) = \tilde{h}(\vec{v}) \quad \text{for} \quad v_\eta > 0, \\
  \tilde{g}(L, \vec{v}) = \tilde{g}(L, \mathcal{S}[\vec{v}])), \\
  q_{\tilde{g}, L} = q_{g, L},
\end{cases}$$  \hfill (5.120)

for \( \tilde{g}(\eta, \vec{v}) \) is well-posed, where

\[
\tilde{g}_L(\vec{v}) = g_L(\vec{v}) = q_0, L \psi_0 + q_1, L \psi_1 + q_2, L \psi_2 + q_3, L \psi_3,
\]

is given by the equation (5.2) for \( g \). Note that

\[
\tilde{h}(\vec{v}) = \tilde{D}_0 \psi_0 + \tilde{D}_1 \psi_1 + \tilde{D}_2 \psi_2 + \tilde{D}_3 \psi_3,
\]

with \( \tilde{D}_1 = 0 \). We consider the endomorphism \( T \) in \( \tilde{N} \{ \psi_0, \psi_2, \psi_3 \} \) defined as \( T : \tilde{h} \to T[\tilde{h}] = \tilde{g}_L \). Therefore, we only need to study the matrix of \( T \) at the basis \{ \psi_0, \psi_2, \psi_3 \}. It is easy to check when \( \tilde{h} = \psi_0 \) and \( \tilde{h} = \psi_3 \), \( T \) is an identity mapping, i.e.

\[
\begin{align*}
T[\psi_0] &= \psi_0 \\
T[\psi_3] &= \psi_3
\end{align*}
\]

The main obstacle is when \( \tilde{h} = \psi_2 \). In this case, define \( \tilde{g}' = \tilde{g} - \psi_2 \). Then \( \tilde{g}' \) satisfies the equation

\[
\begin{align*}
\begin{cases}
\eta \frac{\partial \tilde{g}'}{\partial \eta} + G(\eta) \left( \psi^2 \frac{\partial \tilde{g}'}{\partial \psi} - \psi \frac{\partial \tilde{g}'}{\partial \psi} + \psi \tilde{g}' \right) + L[\tilde{g}'] = G(\eta) \mu \tilde{z}^2 \psi v^\phi, \\
\tilde{g}'(0, \vec{v}) = 0 & \text{for } \psi_0 > 0, \\
\tilde{g}'(L, \vec{v}) = \tilde{g}'(L, \vec{h}[\vec{v}]).
\end{cases}
\end{align*}
\]

Although \( G(\eta) \mu \tilde{z}^2 \psi v^\phi \) does not decay exponentially, it is easy to check that the \( L^1 \) and \( L^2 \) norm of \( G \) can be sufficiently small as \( \epsilon \to 0 \) and \( G(\eta) \mu \tilde{z}^2 \psi v^\phi \in \tilde{N} \). Using a natural extension of Lemma 5.1, we know \( |\tilde{q}'_L| \) is also sufficiently small, where \( \tilde{q}' \in \tilde{N} \) is defined in the decomposition \( \tilde{g}' = \tilde{w}' + \tilde{q}' \). Note that we do not need exponential decay of source term in order to show the bound of \( \tilde{q}'_L \). This means that

\[
T[\psi_0, \psi_2, \psi_3] = \begin{bmatrix} \tilde{q}_0, L & 0 \\ 0 & 1 + \tilde{q}_2, L & 0 \\ 0 & \tilde{q}_3, L & 1 \end{bmatrix}
\]

For \( \epsilon \) sufficiently small, this matrix is invertible, which means \( T \) is bijective. Therefore, we can always find \( \tilde{h} \) such that \( \tilde{g}_L = g_L \), which is desired. Then by Lemma 5.2 and the superposition property, when define \( \bar{G} = g - \tilde{g} \), the theorem naturally follows.

5.2. \( L^\infty \) Estimates. Consider the constant problem for \( g(\eta, \vec{v}) \)

\[
\begin{align*}
\begin{cases}
\eta \frac{\partial g}{\partial \eta} + G(\eta) \left( \psi^2 \frac{\partial g}{\partial \psi} - \psi \frac{\partial g}{\partial \psi} + \psi g \right) + \nu g = Q(\eta, \vec{v}), \\
g(0, \vec{v}) = h(\vec{v}) & \text{for } \psi_0 > 0, \\
g(L, \vec{v}) = g(L, R[\vec{v}]),
\end{cases}
\end{align*}
\]

We define the characteristics starting from \( \left( \eta(0), \psi_0(0), \psi_0(0) \right) \) as \( \left( \eta(s), \psi(s), \psi(s) \right) \) satisfying

\[
\begin{align*}
\frac{d\eta}{ds} &= \psi, \\
\frac{d\psi_0}{ds} &= G(\eta) \psi_0^2, \\
\frac{d\psi}{ds} &= -G(\eta) \psi v^\phi,
\end{align*}
\]

which leads to

\[
\begin{align*}
\psi_0^2(s) + \psi_0^2(s) &= C_1, \\
\psi(s) e^{-W(\eta(s))} &= C_2,
\end{align*}
\]

where \( C_1 \) and \( C_2 \) are two constants depending on the starting point. Along the characteristics, \( \psi_0^2 + \psi_0^2 \) and \( \psi_0 e^{-W(\eta)} \) are conserved quantities and the equation (5.127) can be rewritten as

\[
\psi \frac{\partial g}{\partial \eta} + \nu g = Q.
\]
Define the energy

\[ E_1 = v_\eta^2 + v_\phi^2, \]

\[ E_2 = v_\phi e^{-W(\eta)}. \]

Let

\[ v'_\phi(\eta, \vec{v}; \eta') = v_\phi e^{W(\eta') - W(\eta)}. \]

For \( E_1 \geq v_\phi^2 \), define

\[ v'_\eta(\eta, \vec{v}; \eta') = \sqrt{E_1 - v_\phi^2(\eta, \vec{v}; \eta')}, \]

\[ \vec{v}'(\eta, \vec{v}; \eta') = \left( v'_\eta(\eta, \vec{v}; \eta'), v'_\phi(\eta, \vec{v}; \eta') \right), \]

\[ \mathcal{H}[\vec{v}'(\eta, \vec{v}; \eta')] = \left( -v'_\eta(\eta, \vec{v}; \eta'), v'_\phi(\eta, \vec{v}; \eta') \right). \]

Basically, this means \((\eta, v_\eta, v_\phi)\) and \((\eta', v'_\eta, v'_\phi)\), \((\eta', -v'_\eta, v'_\phi)\) are on the same characteristics. Also, this implies \( v'_\eta \geq 0 \). Moreover, define an implicit function \( \eta^+(\eta, \vec{v}) \) by the equation

\[ E_1(\eta, \vec{v}) = v_\phi^2(\eta, \vec{v}; \eta^+). \]

We know \((\eta^+, 0, v'_\phi)\) at the axis \( v_\eta = 0 \) is on the same characteristics as \((\eta, \vec{v})\). Finally put

\[ H_{\eta, \eta'} = \int_\eta^{\eta'} \frac{\nu}{v'_\eta(\eta, \vec{v}; y)} dy, \]

\[ \mathcal{H}[H_{\eta, \eta'}] = \int_\eta^{\eta'} \nu \frac{\mathcal{H}[\vec{v}'(\eta, \vec{v}; y)]}{v'_\eta(\eta, \vec{v}; y)} dy. \]

Actually, since \( \nu \) only depends on \(|\vec{v}|\), we must have \( H_{\eta, \eta'} = \mathcal{H}[H_{\eta, \eta'}] \). This distinction is purely for clarify and does not play a role in the estimates. We can rewrite the solution to the equation (5.127) along the characteristics as

\[ g(\eta, \vec{v}) = K[h(\vec{v})] + \mathcal{T}[Q(\eta, \vec{v})], \]

where

Case I:
For \( v_\eta > 0 \),

\[ K[h(\vec{v})] = h\left( \vec{v}'(\eta, \vec{v}; 0) \right) \exp(-H_{\eta, 0}), \]

\[ \mathcal{T}[Q(\eta, \vec{v})] = \int_0^{\eta} \frac{Q(\eta, \vec{v}'(\eta, \vec{v}; \eta'))}{v'_\eta(\eta, \vec{v}; 0)} \exp(-H_{\eta, \eta'}) d\eta'. \]

Case II:
For \( v_\eta < 0 \) and \( v_\eta^2 + v_\phi^2 \geq v_\phi^2(\eta, \vec{v}; L) \),

\[ K[h(\vec{v})] = h\left( \vec{v}'(\eta, \vec{v}; 0) \right) \exp(-H_{L, 0} - \mathcal{H}[H_{L, \eta}]), \]

\[ \mathcal{T}[Q(\eta, \vec{v})] = \left( \int_0^{L} \frac{Q(\eta', \vec{v}'(\eta, \vec{v}; \eta'))}{v'_\eta(\eta, \vec{v}; \eta')} \exp(-H_{L, \eta'} - \mathcal{H}[H_{L, \eta}]) d\eta' \right) + \int_{\eta}^{L} \frac{Q(\eta', \mathcal{H}[\vec{v}'(\eta, \vec{v}; \eta')])}{v'_\eta(\eta, \vec{v}; \eta')} \exp(\mathcal{H}[H_{\eta, \eta'}]) d\eta'. \]
Proof. Based on Lemma 2.1, we know there is a positive Lemma 5.4.

\[ \|e^{\beta \eta} \mathcal{K}[h] \|_{L^\infty_{\beta, \nu}} \leq C \|h\|_{L^\infty_{\beta, \nu}}. \] (5.148)

**Proof.** Based on Lemma 2.1, we know

\[ \frac{\nu(\mathbf{v})}{v_{\eta}'} \geq \nu_0, \quad \frac{\nu(\mathcal{R}[\mathbf{v}])}{v_{\eta}'} \geq \nu_0. \] (5.149)

It follows that

\[ \exp(-H_{\eta, 0}) \leq e^{-\beta \eta} \] (5.150)

\[ \exp(-H_{\eta + \nu, 0} - \mathcal{R}[\eta, \nu]) \leq e^{-\beta \eta} \] (5.151)

Then our results are obvious.

**Lemma 5.5.** For any \( \beta \geq 0 \), \( \nu \geq 0 \) and \( 0 \leq \beta \leq \frac{\nu_0}{2} \), there is a constant \( C \) such that

\[ \|\mathcal{T}[Q]\|_{L^\infty_{\beta, \nu}} \leq C \left\| \frac{Q}{\nu} \right\|_{L^\infty_{\beta, \nu}}. \] (5.152)

Moreover, we have

\[ \|e^{\beta \eta} \mathcal{T}[Q]\|_{L^\infty_{\beta, \nu}} \leq C \left\| e^{\beta \eta} Q \right\|_{L^\infty_{\beta, \nu}}. \] (5.153)

**Proof.** The first inequality is a special case of the second one, so we only need to prove the second inequality. For \( \nu_0 > 0 \) case, we have

\[ \beta(\eta - \eta') - H_{\eta, \nu'} \leq \beta(\eta - \eta') - \frac{\nu_0(\eta - \eta')}{2} - \frac{H_{\eta, \nu'}}{2} \leq -\frac{H_{\eta, \nu'}}{2}. \] (5.154)

It is natural that

\[ \int_0^\eta \frac{\nu(\mathbf{v}(\eta'))}{v_{\eta}'} \exp \left( \beta(\eta - \eta') - H_{\eta, \nu'} \right) d\eta' \leq \int_0^\infty \exp \left( -\frac{z}{2} \right) dz = 2, \] (5.155)

for \( z = H_{\eta, \nu'} \). Then we estimate

\[ \left| (\mathbf{v})^\theta e \mathbf{v}^2 e^{\beta \eta} \mathcal{T}[Q] \right| \leq e^{3\eta} \int_0^\eta (\mathbf{v})^\theta e \mathbf{v}^2 \left| \frac{Q(\eta', \mathbf{v}(\eta'))}{v_{\eta}'} \right| \exp(-H_{\eta, \nu'}) d\eta' \]

\[ \leq \left\| e^{3\eta} Q \right\|_{L^\infty_{\beta, \nu}} \int_0^\eta \frac{\nu(\mathbf{v}(\eta'))}{v_{\eta}'} \exp \left( \beta(\eta - \eta') - H_{\eta, \nu'} \right) d\eta' \]

\[ \leq C \left\| e^{3\eta} Q \right\|_{L^\infty_{\beta, \nu}}. \]
The $v_\eta < 0$ case can be proved in a similar fashion, so we omit it here. \hfill \Box

**Lemma 5.6.** For any $\delta > 0$, $\theta > 2$ and $q \geq 0$, there is a constant $C(\delta)$ such that

$$
\|T[Q]\|_{L^\infty L^2_x} \leq C(\delta) \|\nu^{-\delta} Q\|_{L^\infty L^2_x} + \delta \|Q\|_{L^\infty L^\infty_x}.
$$

(5.157)

**Proof.** We divide the proof into several cases:

Case I: For $v_\eta > 0$,

$$
T[Q(\eta, \vec{v})] = \int_0^\eta Q\left(\eta', \vec{v}(\eta, \vec{v}; \eta')\right) \frac{\exp(-H_{\eta, \eta'})}{v_\eta'(\eta, \vec{v}; \eta')} \, d\eta'.
$$

(5.158)

We need to estimate

$$
\int_{\mathbb{R}^2} e^{2\sigma |\vec{v}|^2} \left( \int_0^\eta Q\left(\eta', \vec{v}(\eta, \vec{v}; \eta')\right) \frac{\exp(-H_{\eta, \eta'})}{v_\eta'(\eta, \vec{v}; \eta')} \, d\eta' \right)^2 \, d\vec{v}.
$$

(5.159)

Assume $m > 0$ is sufficiently small, $M > 0$ is sufficiently large and $\sigma > 0$ is sufficiently small, which will be determined in the following. We can split the integral into four parts

$$
I = I_1 + I_2 + I_3 + I_4.
$$

(5.160)

In the following, we use $\chi_i$ for $i = 1, 2, 3, 4$ to represent the indicator function of each case.

Case I - Type I: $\chi_1$: $M \leq v_\eta'(\eta, \vec{v}; \eta')$ or $M \leq v_\phi'(\eta, \vec{v}; \eta')$.

By Lemma 2.1, we have

$$
|\vec{v}(\eta')| + 1 \leq C\nu\left(\vec{v}(\eta')\right).
$$

(5.161)

Then for $\theta > 2$, since $|\vec{v}|$ is conserved along the characteristics, we have

$$
I_1 \leq C \|Q\|^2_{L^\infty L^\infty_{x, \phi}} \int_{\mathbb{R}^2} \chi_1 \left( \int_0^\eta \frac{1}{|\vec{v}|^\theta} \frac{\exp(-H_{\eta, \eta'})}{v_\eta'(\eta, \vec{v}; \eta')} \, d\eta' \right)^2 \, d\vec{v}
$$

(5.162)

$$
\leq \frac{C}{M^\theta} \|Q\|^2_{L^\infty L^\infty_{x, \phi}} \int_{\mathbb{R}^2} \frac{1}{|\vec{v}|^\theta} \left( \int_0^\eta \frac{\exp(-H_{\eta, \eta'})}{v_\eta'(\eta, \vec{v}; \eta')} \, d\eta' \right)^2 \, d\vec{v}
$$

$$
\leq \frac{C}{M^\theta} \|Q\|^2_{L^\infty L^\infty_{x, \phi}} \int_{\mathbb{R}^2} \frac{1}{|\vec{v}|^\theta} \, d\vec{v}
$$

$$
\leq \frac{C}{M^\theta} \|Q\|^2_{L^\infty L^\infty_{x, \phi}},
$$

since for $\nu = H_{\eta, \eta'}$,

$$
\left| \int_0^\eta \frac{\exp(-H_{\eta, \eta'})}{v_\eta'(\eta, \vec{v}; \eta')} \, d\eta' \right| \leq \int_0^\eta \frac{\nu}{v_\eta'(\eta, \vec{v}; \eta')} \exp(-H_{\eta, \eta'}) \, d\eta' \leq \int_0^\infty e^{-y} \, dy = 1.
$$

(5.163)

Case I - Type II: $\chi_2$: $v_\eta \geq \sigma$, $m \leq v_\eta'(\eta, \vec{v}; \eta') \leq M$ and $v_\phi'(\eta, \vec{v}; \eta') \leq M$.

Since along the characteristics, $|\vec{v}|^2$ can be bounded by $2M^2$ and the integral domain for $\vec{v}$ is finite, by
Cauchy’s inequality, we have

\[ I_2 \leq C e^{4eM^2} \int_{\mathbb{R}^2} \left( \int_0^{\eta} \frac{Q^2}{\nu} (\eta', \vec{v}(\eta, \vec{v}; \eta')) \, d\eta' \right) \left( \int_0^{\eta} \nu \left( \vec{w}(\eta, \vec{v}; \eta') / v'^2(\eta, \vec{v}; \eta') \exp(-2H_{\eta, \eta'}) \right) \, d\eta' \right) d\vec{v} \]  

(5.164)

\[ \leq C e^{4eM^2} / m \int_{\mathbb{R}^2} \left( \int_0^{\eta} \frac{Q^2}{\nu} (\eta', \vec{w}(\eta, \vec{v}; \eta')) \, d\eta' \right) \left( \int_0^{\eta} \nu \left( \vec{w}(\eta, \vec{v}; \eta') / v'^2(\eta, \vec{v}; \eta') \exp(-2H_{\eta, \eta'}) \right) \, d\eta' \right) d\vec{v} \]

\[ \leq C e^{4eM^2} / \sigma \left( \int_{\mathbb{R}^2} \left( \int_0^{\eta} \frac{Q^2}{\nu} (\eta', \vec{w}(\eta, \vec{v}; \eta')) \, d\eta' \right) d\vec{v} \right) \leq C \|v\|^2_{L^2 L^2}, \]

where for \( y = H_{\eta, \eta'} \),

\[ \int_0^{\eta} \nu \left( \vec{w}(\eta, \vec{v}; \eta') / v'^2(\eta, \vec{v}; \eta') \exp(-2H_{\eta, \eta'}) \right) \, d\eta' \leq \int_0^{\infty} e^{-2y} \, dy = \frac{1}{2}. \]  

(5.165)

and the Jacobian

\[ \left| \frac{d\vec{v}}{d\vec{v}'} \right| = \left| \frac{R - \epsilon \theta'}{R - \epsilon \theta} \right| \leq C \frac{v'}{v} \leq \frac{C M}{\sigma}. \]  

(5.166)

Case I - Type III: \( \chi_3: v_\eta \geq \sigma, 0 \leq v'(\eta, \vec{v}; \eta') \leq m \) and \( v'(\eta, \vec{v}; \eta') \leq M \).

We can directly verify the fact that

\[ 0 \leq v_\eta \leq v'(\eta, \vec{v}; \eta'), \]

for \( \eta' \leq \eta \). Then we know the integral of \( v_\eta \) is always in a small domain. We have for \( y = H_{\eta, \eta'} \),

\[ I_3 \leq C e^{4eM^2} \|Q\|^2_{L^2 L^2} \int_{\mathbb{R}^2} \frac{\chi_3}{\vec{w}} \left( \int_0^{\eta} \exp(-H_{\eta, \eta'}) / v'^2(\eta, \vec{v}; \eta') \, d\eta' \right) d\vec{v} \]  

(5.168)

\[ \leq C e^{4eM^2} \|Q\|^2_{L^2 L^2} \int_{\mathbb{R}^2} \frac{\chi_3}{\vec{w}} \left( \int_0^{\infty} e^{-y} \, dy \right) d\vec{v} \]

\[ \leq C e^{4eM^2} \|Q\|^2_{L^2 L^2} \int_{\mathbb{R}^2} \frac{\chi_3}{\vec{w}} d\vec{v} \]

\[ \leq C e^{4eM^2} m \|Q\|^2_{L^2 L^2}. \]

Case I - Type IV: \( \chi_4: v_\eta \leq \sigma, v'(\eta, \vec{v}; \eta') \leq M \) and \( v'(\eta, \vec{v}; \eta') \leq M \).

Similar to Case I - Type III, we know the integral of \( v_\eta \) is always in a small domain. We have for \( y = H_{\eta, \eta'} \),

\[ I_3 \leq C e^{4eM^2} \|Q\|^2_{L^2 L^2} \int_{\mathbb{R}^2} \frac{\chi_4}{\vec{w}} \left( \int_0^{\eta} \exp(-H_{\eta, \eta'}) / v'^2(\eta, \vec{v}; \eta') \, d\eta' \right) d\vec{v} \]  

(5.169)

\[ \leq C e^{4eM^2} \|Q\|^2_{L^2 L^2} \int_{\mathbb{R}^2} \frac{\chi_4}{\vec{w}} \left( \int_0^{\infty} e^{-y} \, dy \right) d\vec{v} \]

\[ \leq C e^{4eM^2} \|Q\|^2_{L^2 L^2} \int_{\mathbb{R}^2} \frac{\chi_4}{\vec{w}} d\vec{v} \]

\[ \leq C e^{4eM^2} \sigma \|Q\|^2_{L^2 L^2}. \]
Collecting all three types, we have

\[
I \leq C \frac{Me^{4\phi M^2}}{m\sigma} \left\| \nabla \chi Q \right\|^2_{L^2} + C \left( \frac{1}{M^\sigma} + e^{4\phi M^2} (m + \sigma) \right) \left\| Q \right\|^2_{L^\infty_{\beta,\varphi}}. \tag{5.170}
\]

Taking \( M \) sufficiently large, \( m < < e^{-4\phi M^2} \) and \( \sigma < < e^{-4\phi M^2} \) sufficiently small, we obtain the desired result.

Case II:

For \( v_\eta < 0 \) and \( v_\eta^2 + \varphi_\eta^2 \geq \varphi_\eta^2 (\eta, \vec{v}; L) \),

\[
T \left[ Q(\eta, \vec{v}) \right] = \left( \int_0^L \frac{Q(\eta', \vec{v}(\eta, \vec{v}; \eta'))}{v'_\eta(\eta, \vec{v}; \eta')} \exp(-H_{L,\eta'} - \mathcal{H}[H_{L,\eta}]) d\eta' \right)
\]

\[
+ \int_\eta^L \frac{Q(\eta', \mathcal{H}[\vec{v}(\eta, \vec{v}; \eta')])}{v'_\eta(\eta, \vec{v}; \eta')} \exp(\mathcal{H}[H_{\eta,\eta'}]) d\eta'.
\]

We first estimate

\[
\int_{\mathbb{R}^2} e^{2|\vec{v}|^2} \left( \int_\eta^L \frac{Q(\eta', \mathcal{H}[\vec{v}(\eta, \vec{v}; \eta')])}{v'_\eta(\eta, \vec{v}; \eta')} \exp(\mathcal{H}[H_{\eta,\eta'}]) d\eta' \right)^2 d\vec{v}.
\]

We can split the integral into four parts:

\[
II = II_1 + II_2 + II_3 + II_4.
\]

Case II - Type I: \( \chi_1 \): \( M \leq v'_\eta(\eta, \vec{v}; \eta') \) or \( M \leq v'_\phi(\eta, \vec{v}; \eta') \). Similar to Case I - Type I, we have

\[
II_1 \leq C \left\| Q \right\|^2_{L^\infty_{\beta,\varphi}} \int_{\mathbb{R}^2} \chi_1 \left( \int_\eta^L \frac{1}{(\vec{v})^2} \exp(\mathcal{H}[H_{\eta,\eta'}]) d\eta' \right)^2 d\vec{v}
\]

\[
\leq \frac{C}{M^\sigma} \left\| Q \right\|^2_{L^\infty_{\beta,\varphi}} \int_{\mathbb{R}^2} \left( \frac{1}{(\vec{v})^2} \right)^2 \left( \int_\eta^L \exp(\mathcal{H}[H_{\eta,\eta'}]) d\eta' \right)^2 d\vec{v}
\]

\[
\leq \frac{C}{M^\sigma} \left\| Q \right\|^2_{L^\infty_{\beta,\varphi}} \int_{\mathbb{R}^2} \frac{1}{(\vec{v})^3} d\vec{v}
\]

\[
\leq \frac{C}{M^\sigma} \left\| Q \right\|^2_{L^\infty_{\beta,\varphi}},
\]

since for \( y = H_{\eta,\eta'} \),

\[
\left| \int_\eta^L \frac{\exp(\mathcal{H}[H_{\eta,\eta'}]) d\eta'}{v'_\eta(\eta, \vec{v}; \eta')} \right| \leq \left| \int_\eta^L \frac{\exp(\mathcal{H}[H_{\eta,\eta'}])}{v'_\eta(\eta, \vec{v}; \eta')} d\eta' \right| \leq \int_{-\infty}^0 e^y dy = 1.
\]

Case II - Type II: \( \chi_2 \): \( m \leq v'_\eta(\eta, \vec{v}; \eta') \leq M \) and \( v'_\phi(\eta, \vec{v}; \eta') \leq M \). We can directly verify the fact that

\[
0 \leq v'_\eta(\eta, \vec{v}; \eta') \leq |v_\eta|,
\]
for $\eta' \geq \eta$. Similar to Case I - Type II, by Cauchy’s inequality, we have

$$H_2 \leq C \epsilon^4 e^{M^2} \int_{\mathbb{R}^2} \left( \int_0^\eta \int_{\mathbb{R}^2} \frac{Q^2}{\nu} \left( \eta', \bar{v}((\eta, \bar{v}; \eta')) \right) d\eta' \right) \left( \int_0^L \int_{\mathbb{R}^2} \frac{\nu(\eta, \bar{v}; \eta')}{v'_\eta(\eta, \bar{v}; \eta')} \right) d\nu \left( \int_0^L \int_{\mathbb{R}^2} \frac{e \left( 2\mathcal{R}[H_{\eta, \eta'}] \right)}{v'_\eta(\eta, \bar{v}; \eta')} d\eta' \right) d\nu \text{d}\bar{v} \quad (5.177)$$

$$\leq C \epsilon^4 e^{M^2} \int_{\mathbb{R}^2} \left( \int_0^\eta \int_{\mathbb{R}^2} \frac{Q^2}{\nu} \left( \eta', \bar{v}(\eta, \bar{v}; \eta') \right) d\eta' \right) \left( \int_0^L \int_{\mathbb{R}^2} \frac{\nu(\eta, \bar{v}; \eta')}{v'_\eta(\eta, \bar{v}; \eta')} \right) d\nu \left( \int_0^L \int_{\mathbb{R}^2} \frac{e \left( 2\mathcal{R}[H_{\eta, \eta'}] \right)}{v'_\eta(\eta, \bar{v}; \eta')} d\eta' \right) d\nu \text{d}\bar{v}$$

$$\leq C \epsilon^4 e^{M^2} \left( \int_{\mathbb{R}^2} \int_0^\eta \int_{\mathbb{R}^2} \frac{Q^2}{\nu} \left( \eta', \bar{v}(\eta, \bar{v}; \eta') \right) d\eta' d\nu \text{d}\bar{v} \right)$$

$$\leq C \epsilon^4 e^{M^2} \left( \int_{\mathbb{R}^2} \int_0^\eta \int_{\mathbb{R}^2} \frac{Q^2}{\nu} \left( \eta', \bar{v}(\eta, \bar{v}; \eta') \right) d\eta' d\nu \text{d}\bar{v} \right)$$

$$\leq C \epsilon^4 e^{M^2} \left( \int_{\mathbb{R}^2} \int_0^\eta \int_{\mathbb{R}^2} \frac{Q^2}{\nu} \left( \eta', \bar{v}(\eta, \bar{v}; \eta') \right) d\eta' d\nu \text{d}\bar{v} \right)$$

where for $y = H_{\eta, \eta'}$,

$$\int_0^\eta \frac{\nu \left( \bar{v}(\eta, \bar{v}; \eta') \right)}{v'_\eta(\eta, \bar{v}; \eta')} \exp(-2H_{\eta, \eta'}) d\eta' d\nu \leq \int_0^\infty e^{-2y} dy = \frac{1}{2}, \quad (5.178)$$

and the Jacobian

$$\left| \frac{d\nu}{d\bar{v}} \right| = \left| \frac{R_0 - \epsilon \eta}{R_0 - \epsilon \eta'} \right| \leq C \left| \frac{v'_\eta}{v'_\eta} \right| \leq C. \quad (5.179)$$

Case II - Type III: $\chi_3$: $0 \leq v'_\eta(\eta, \bar{v}; \eta') \leq m$, $v'_\phi(\eta, \bar{v}; \eta') \leq M$ and $\eta' - \eta \geq \sigma$.

We know

$$H_{\eta, \eta'} \leq -\frac{\sigma}{m}. \quad (5.180)$$

Then after substitution $y = H_{\eta, \eta'}$, the integral is not from zero, but from $-\frac{\sigma}{m}$. In detail, we have

$$I_3 \leq C \epsilon^4 e^{M^2} \left\| \frac{\nu \left( \bar{v}(\eta, \bar{v}; \eta') \right)}{v'_\eta(\eta, \bar{v}; \eta')} \right\|_{L^2_{\bar{v}, \eta}} \left( \int_0^L \int_{\mathbb{R}^2} \frac{\nu(\eta, \bar{v}; \eta')}{v'_\eta(\eta, \bar{v}; \eta')} \right)^2 d\bar{v} \quad (5.181)$$

$$\leq C \epsilon^4 e^{M^2} \left\| \frac{\nu \left( \bar{v}(\eta, \bar{v}; \eta') \right)}{v'_\eta(\eta, \bar{v}; \eta')} \right\|_{L^2_{\bar{v}, \eta}} \left( \int_0^L \int_{\mathbb{R}^2} \frac{e \left( 2\mathcal{R}[H_{\eta, \eta'}] \right)}{v'_\eta(\eta, \bar{v}; \eta')} \left( \int_0^L \int_{\mathbb{R}^2} \frac{\nu(\eta, \bar{v}; \eta')}{v'_\eta(\eta, \bar{v}; \eta')} \right) d\eta' \right)^2 d\bar{v}$$

$$\leq C \epsilon^4 e^{M^2} \left\| \frac{\nu \left( \bar{v}(\eta, \bar{v}; \eta') \right)}{v'_\eta(\eta, \bar{v}; \eta')} \right\|_{L^2_{\bar{v}, \eta}} \left( \int_0^L \int_{\mathbb{R}^2} \frac{\nu(\eta, \bar{v}; \eta')}{v'_\eta(\eta, \bar{v}; \eta')} \right)^2 d\bar{v}$$

$$\leq C \epsilon^4 e^{M^2} \left\| Q \right\|_{L^2_{\bar{v}, \eta}} \left( \int_0^\infty \left\| e^y \right\|_{L^2_{\bar{v}, \eta}} \right)^2 d\bar{v}$$

Case II - Type IV: $\chi_4$: $0 \leq v'_\eta(\eta, \bar{v}; \eta') \leq m$, $v'_\phi(\eta, \bar{v}; \eta') \leq M$ and $\eta' - \eta \leq \sigma$. 

For $\eta' \leq \eta$ and $\eta - \eta \leq \sigma$, we have

\[ v_{\eta} = \sqrt{v_{\eta}^2(\eta, \vec{v}; \eta') + v_{\phi}^2(\eta, \vec{v}; \eta') - v_{\phi}^2} \]

\[ = \sqrt{v_{\eta}^2(\eta, \vec{v}; \eta') + v_{\phi}^2(\eta, \vec{v}; \eta') - v_{\phi}^2(\eta, \vec{v}; \eta')} - \frac{R_{\eta} - \epsilon \eta'}{R_{\eta} - \epsilon \eta} \]

\[ \leq \sqrt{v_{\eta}^2(\eta, \vec{v}; \eta') + 2R_{\eta}M^2(\eta' - \eta)} \]

\[ \leq C\sqrt{m^2 + \epsilon M^2} \leq C(m + M\sqrt{\epsilon \sigma}). \]

Therefore, the integral domain for $v_{\eta}$ is very small. We have the estimate for $v = H_{\eta, \eta'}$

\[ I_1 \leq Ce^{4eM^2} ||Q||^2 \int_{L^\infty L^\infty} \int_R^2 \frac{\chi_1}{(\vec{v})^\eta} \left( \int_0^L \exp(\mathcal{R}[H_{\eta, \eta'}])d\eta' \right)^2 d\vec{v} \]

\[ \leq Ce^{4eM^2} ||Q||^2 \int_{L^\infty L^\infty} \int_R^2 \frac{\chi_1}{(\vec{v})^\eta} \left( \int_{-\infty}^0 \exp(\mathcal{R}[H_{\eta, \eta'}])d\eta' \right)^2 d\vec{v} \]

\[ \leq Ce^{4eM^2} (m + \sqrt{\epsilon \sigma}) ||Q||^2 \int_{L^\infty L^\infty}. \]

Collecting all four types, we have

\[ II \leq Ce^{4eM^2} m \left[ ||v_{\eta}^{-2}||^2 + C \left( \frac{1}{M^\theta} + e^{4eM^2} m \right) \right] ||Q||^2 \int_{L^\infty L^\infty}. \]

Taking $M$ sufficiently large, $\sigma < e^{-8eM^2}$ sufficiently small and $m << \min\{\sigma, e^{-4eM^2}\}$ sufficiently small, we obtain the desired result.

Note that we have the decomposition

\[ \int_0^L Q\left( \frac{H_{\eta, \eta'} - \mathcal{R}[H_{\eta, \eta}]}{v_{\eta}^2(\eta, \vec{v}; \eta')} \right) \exp(-H_{\eta, \eta'} - \mathcal{R}[H_{\eta, \eta}])d\eta' \]

\[ = \int_0^L \left( \frac{Q\left( \frac{H_{\eta, \eta'} - \mathcal{R}[H_{\eta, \eta}]}{v_{\eta}^2(\eta, \vec{v}; \eta')} \right)}{v_{\eta}^2(\eta, \vec{v}; \eta')} \right) \exp(-H_{\eta, \eta'} - \mathcal{R}[H_{\eta, \eta}])d\eta' + \int_0^L \left( \frac{Q\left( \frac{H_{\eta, \eta'} - \mathcal{R}[H_{\eta, \eta}]}{v_{\eta}^2(\eta, \vec{v}; \eta')} \right)}{v_{\eta}^2(\eta, \vec{v}; \eta')} \right) \exp(-H_{\eta, \eta'} - \mathcal{R}[H_{\eta, \eta}])d\eta'. \]

Then this term can actually be bounded using the techniques in Case I and Case II.

Case III:
For $v_{\eta} < 0$ and $v_{\phi}^2 + v_{\phi}^2 \leq v_{\phi}^2(\eta, \vec{v}; L)$,

\[ T[Q(\eta, \vec{v})] = \left( \int_0^{+} Q\left( \frac{H_{\eta, \eta'} - \mathcal{R}[H_{\eta, \eta}]}{v_{\eta}^2(\eta, \vec{v}; \eta')} \right) \exp(-H_{\eta, \eta'} - \mathcal{R}[H_{\eta, \eta}])d\eta' \right) + \int_0^{+} Q\left( \frac{H_{\eta, \eta'} - \mathcal{R}[H_{\eta, \eta}]}{v_{\eta}^2(\eta, \vec{v}; \eta')} \right) \exp(-H_{\eta, \eta'} - \mathcal{R}[H_{\eta, \eta}])d\eta'. \]

This is a combination of Case I and Case II, so it naturally holds.

\[ \square \]

**Lemma 5.7.** The solution $g(\eta, \vec{v})$ to the $\epsilon$-Milne problem satisfies for $\varrho \geq 0$ and an integer $\vartheta \geq 3$,

\[ ||g - g_L||_{L^\infty L^\infty_{\epsilon}} \leq C + C ||g - g_L||_{L^2 L^2}. \]
Proof. Define \( u = g - g_L \). Then \( u \) satisfies the equation

\[
\begin{align*}
\begin{cases}
\nu \left( \frac{\partial u}{\partial \eta} + G(\eta) \left( v_0^2 \frac{\partial u}{\partial \eta} - v_\eta v_0 \frac{\partial u}{\partial \phi} \right) \right) + \nu u - K[u] = S = g_{2,L}G(\eta)\mu^2 v_\eta v_0, \\
 u(0,\vec{\nu}) = p(\vec{\nu}) = h(\vec{\nu}) - g_L(\vec{\nu}) \quad \text{for} \quad v_\eta > 0, \\
 u(L,\vec{\nu}) = u(L,\vec{\nu}[\vec{\nu}]).
\end{cases}
\end{align*}
\]

(5.188)

Since \( u = K[p] + T[K[u]] + T[S] \), based on Lemma 5.6, we have

\[
\|u\|_{L^\infty L^2_0} \leq \|K[p]\|_{L^\infty_0} + \|T[K[u]]\|_{L^\infty L^2_0} + \|T[S]\|_{L^\infty L^2_0} \leq \|p\|_{L^\infty_0} + C(\delta) \|\nu^{-\frac{1}{2}}K[u]\|_{L^2 L^2_0} + \delta \|\nabla K[u]\|_{L^\infty L^2_0} + \|\nu^{-1}S\|_{L^\infty L^2_0},
\]

where we can directly verify

\[
\|\nu^{-\frac{1}{2}}K[u]\|_{L^2 L^2_0} \leq \|u\|_{L^2 L^2_0}.
\]

(5.190)

In [11] Lemma 3.3.1, it is shown that

\[
\|K[u]\|_{L^\infty_0} \leq \|u\|_{L^\infty_{-1,0}},
\]

(5.191)

\[
\|K[u]\|_{L^\infty_0} \leq \|u\|_{L^\infty_0}.
\]

(5.192)

Since \( u = K[p] + T[K[u] + S] \), by Lemma 5.4 and Lemma 5.5 for \( \epsilon \) and \( \delta \) sufficiently small, we can estimate

\[
\|u\|_{L^\infty L^2_0} \leq C \left( \|T[K[u]]\|_{L^\infty L^2_0} + \|T[S]\|_{L^\infty L^2_0} + \|K[p]\|_{L^\infty_0} \right) \leq C \left( \|K[u]\|_{L^\infty L^2_0} + \|\nu^{-1}S\|_{L^\infty L^2_0} + \|p\|_{L^\infty_0} \right)
\]

\[
\leq \ldots \leq C \left( \|u\|_{L^\infty L^2_0} + \|\nu^{-1}S\|_{L^\infty L^2_0} + \|p\|_{L^\infty_0} \right)
\]

\[
\leq C(\delta) \|u\|_{L^2 L^2_0} + \delta \|K[u]\|_{L^\infty L^2_0} + C \left( \|\nu^{-1}S\|_{L^\infty L^2_0} + \|p\|_{L^\infty_0} \right).
\]

Therefore, absorbing \( \delta \|K[u]\|_{L^\infty L^2_0} \) into the right-hand side of the second inequality implies

\[
\|K[u]\|_{L^\infty L^2_0} \leq C \left( \|u\|_{L^2 L^2_0} + \|\nu^{-1}S\|_{L^\infty L^2_0} + \|p\|_{L^\infty_0} \right).
\]

(5.194)

Therefore, we have

\[
\|u\|_{L^\infty L^2_0} \leq \|K[p]\|_{L^\infty_0} + \|T[K[u]]\|_{L^\infty L^2_0} + \|T[S]\|_{L^\infty L^2_0} \leq C \left( \|p\|_{L^\infty_0} + \|\nu^{-1}K[u]\|_{L^\infty L^2_0} + \|\nu^{-1}S\|_{L^\infty L^2_0} \right)
\]

\[
\leq C \left( \|p\|_{L^\infty_0} + \|K[u]\|_{L^\infty L^2_0} + \|\nu^{-1}S\|_{L^\infty L^2_0} \right)
\]

\[
\leq C \left( \|u\|_{L^2 L^2_0} + \|\nu^{-1}S\|_{L^\infty L^2_0} + \|p\|_{L^\infty_0} \right).
\]

(5.195)
Then our result naturally follows.

Lemma 5.8. There exists a unique solution \( g(\eta, \vec{v}) \) to the \( \epsilon \)-Milne problem (5.3) satisfying for \( \varrho \geq 0 \) and an integer \( \vartheta \geq 3 \),

\[
\|g - g_L\|_{L^\infty_{\vartheta,\varrho}} \leq C.
\] (5.196)

Proof. Based on Lemma 5.2 and Lemma 5.8, this is obvious.

Theorem 5.9. There exists a unique solution \( G(\eta, \vec{v}) \) to the \( \epsilon \)-Milne problem (5.7) satisfying for \( \varrho \geq 0 \) and an integer \( \vartheta \geq 3 \),

\[
\|G\|_{L^\infty_{\vartheta,\varrho}} \leq C.
\] (5.197)

Proof. Based on Theorem 5.3 and Lemma 5.8, this is obvious.

5.3. Exponential Decay.

Theorem 5.10. For sufficiently small \( K_0 \), there exists a unique solution \( G(\eta, \vec{v}) \) to the \( \epsilon \)-Milne problem (5.7) satisfying for \( \varrho \geq 0 \) and an integer \( \vartheta \geq 3 \),

\[
\left\| e^{K_0\eta}G \right\|_{L^\infty_{\vartheta,\varrho}} \leq C.
\] (5.198)

Proof. Define \( U = e^{K_0\eta}G \). Then \( U \) satisfies the equation

\[
\begin{cases}
\nabla U = G(\eta) \left( v^{2}_\eta \frac{\partial U}{\partial v^{2}_\eta} - v^{3}_\eta v^{3}_\nu \frac{\partial U}{\partial v^{3}_\nu} \right) + C[U] = K_0v^{3}_\eta U, \\
U(0, \vec{v}) = h(\vec{v}) - \tilde{h}(\tilde{\vec{v}}) \quad \text{for} \quad v^{3}_\eta > 0, \\
U(L, \vec{v}) = U(L, \mathcal{H}[\vec{v}])
\end{cases}
\]

We divide the proof into several steps:

Step 1: \( L^2 \) Estimates.

In the proof of Lemma 5.1, we already show

\[
\int_0^L e^{2K_0\eta} \langle w_G, w_G \rangle (\eta) d\eta \leq C.
\] (5.200)

where we decompose \( \mathcal{G} = w_G + q_G \) with \( q_G \in \mathcal{N} \) and \( w_G \in \mathcal{N}^{\perp} \). Based on the construction of \( \mathcal{G} \), we naturally have \( q_G, L = 0 \). Then using the orthogonal relation, we have

\[
\int_0^L e^{2K_0\eta} \int_{\mathbb{R}^2} G^2(\eta, \vec{v}) d\vec{v} d\eta = \int_0^L e^{2K_0\eta} \langle q_G, q_G \rangle (\eta) d\eta + \int_0^L e^{2K_0\eta} \langle w_G, w_G \rangle (\eta) d\eta.
\] (5.201)

Similar to Step 6 in the proof of Lemma 5.1, using the exponential decay of \( w_G \), we have

\[
\int_0^L e^{2K_0\eta} \langle q_G, q_G \rangle (\eta) d\eta \leq C + C \int_0^L e^{2K_0\eta} \langle w_G, w_G \rangle (\eta) d\eta.
\] (5.202)

This shows that

\[
\|U\|_{L^2_{\vartheta,\varrho}} < C.
\] (5.203)

Step 2: \( L^\infty \) Estimates.

Since \( U = K[p] + T[K[U]] + T[K_0v^{3}_\eta U], \) similar to the proof of Lemma 5.7, we have

\[
\|U\|_{L^\infty_{\vartheta,\varrho}} \leq C \left( \|U\|_{L^2_{\vartheta,\varrho}} + \|v^{-1}K_0v^{3}_\eta U\|_{L^\infty_{\vartheta,\varrho}} + \|p\|_{L^\infty_{\vartheta,\varrho}} \right) + C \left( \|U\|_{L^2_{\vartheta,\varrho}} + K_0\|U\|_{L^\infty_{\vartheta,\varrho}} + \|p\|_{L^\infty_{\vartheta,\varrho}} \right).
\] (5.204)
When $K_0 > 0$ is sufficiently small, we may absorb $K_0 \|U\|_{L^\infty L^\infty_{\vartheta, \kappa}}$ into the left-hand side to obtain

$$\|U\|_{L^\infty L^\infty_{\vartheta, \kappa}} \leq C \left( \|U\|_{L^2 L^2} + \|P\|_{L^\infty_{\vartheta, \kappa}} \right).$$

Then we naturally obtain the result. \qed
6. Regularity of $\epsilon$-Milne Problem with Geometric Correction

In this section, we consider the regularity of the $\epsilon$-Milne problem with geometric correction for $G(\eta, \theta, \vec{v})$ in the domain $(\eta, \theta, \vec{v}) \in [0, L] \times [-\pi, \pi] \times \mathbb{R}^2$ as

$$\begin{aligned}
\begin{cases}
\frac{v_\eta}{\partial \eta} - \frac{\epsilon}{R_\kappa - \epsilon \eta} \left( v_\phi^2 \frac{\partial G}{\partial v_\phi} - v_\eta v_\phi \frac{\partial G}{\partial v_\phi} \right) + \mathcal{L}[G] = 0, \\
G(0, \theta, \vec{v}) = p(\theta, \vec{v}) & \text{for } v_\eta > 0, \\
G(L, \theta, \vec{v}) = G(L, \theta, \mathcal{A}[\vec{v}]),
\end{cases}
\end{aligned}$$

(6.1)

where $\mathcal{A}[\vec{v}] = (-v_\eta, v_\phi)$ and $L = \epsilon^{-\frac{1}{2}}$. For simplicity, we temporarily ignore the dependence of $\theta$, i.e. consider the $\epsilon$-Milne problem with geometric correction for $G(\eta, \vec{v})$ in the domain $(\eta, \vec{v}) \in [0, L] \times \mathbb{R}^2$ as

$$\begin{aligned}
\begin{cases}
\frac{v_\eta}{\partial \eta} - \frac{\epsilon}{R_\kappa - \epsilon \eta} \left( v_\phi^2 \frac{\partial G}{\partial v_\phi} - v_\eta v_\phi \frac{\partial G}{\partial v_\phi} \right) + \mathcal{L}[G] = 0, \\
G(0, \vec{v}) = p(\vec{v}) & \text{for } v_\eta > 0, \\
G(L, \vec{v}) = G(L, \mathcal{A}[\vec{v}]).
\end{cases}
\end{aligned}$$

(6.2)

Define a weight function

$$\zeta(\eta, v_\eta, v_\phi) = \left( (v_\eta^2 + v_\phi^2) - \left( \frac{R_\kappa - \epsilon \eta}{R_\kappa} \right)^2 v_\phi^2 \right)^{\frac{1}{2}}.$$  

(6.3)

It is easy to see that the closer a point $(\eta, v_\eta, v_\phi)$ is to the grazing set $(\eta, v_\eta, v_\phi) = (0, 0, v_\phi)$, the smaller $\zeta$ is. In particular, at the grazing set, $\zeta(0, 0, v_\phi) = 0$.

**Lemma 6.1.** We have

$$v_\eta \frac{\partial \zeta}{\partial \eta} - \frac{\epsilon}{R_\kappa - \epsilon \eta} \left( v_\phi^2 \frac{\partial \zeta}{\partial v_\phi} - v_\eta v_\phi \frac{\partial \zeta}{\partial v_\phi} \right) = 0.$$  

(6.4)

**Proof.** We may directly compute

$$\frac{\partial \zeta}{\partial \eta} = \frac{1}{\zeta} \left( \frac{R_\kappa - \epsilon \eta}{R_\kappa} \right) v_\phi^2, \quad \frac{\partial \zeta}{\partial v_\eta} = \frac{1}{\zeta} v_\eta, \quad \frac{\partial \zeta}{\partial v_\phi} = \frac{1}{\zeta} \left( v_\phi - \left( \frac{R_\kappa - \epsilon \eta}{R_\kappa} \right)^2 v_\phi \right).$$  

(6.5)

Then we know

$$v_\eta \frac{\partial \zeta}{\partial \eta} - \frac{\epsilon}{R_\kappa - \epsilon \eta} \left( v_\phi^2 \frac{\partial \zeta}{\partial v_\phi} - v_\eta v_\phi \frac{\partial \zeta}{\partial v_\phi} \right) = \frac{1}{\zeta} \left( \frac{R_\kappa - \epsilon \eta}{R_\kappa} v_\eta v_\phi^2 - \frac{\epsilon}{R_\kappa - \epsilon \eta} \left( v_\phi^2 v_\eta^2 - v_\eta v_\phi \frac{\partial \zeta}{\partial v_\phi} \left( \frac{R_\kappa - \epsilon \eta}{R_\kappa} \right)^2 v_\phi \right) \right) = 0.$$

\[\square\]

6.1. Mild Formulation. Consider the $\epsilon$-transport problem for $\mathcal{A} = \zeta \frac{\partial G}{\partial \eta}$ as

$$\begin{aligned}
\begin{cases}
v_\eta \frac{\partial \mathcal{A}}{\partial \eta} + G(\eta) \left( v_\phi^2 \frac{\partial \mathcal{A}}{\partial v_\phi} - v_\eta v_\phi \frac{\partial \mathcal{A}}{\partial v_\phi} \right) + v_\mathcal{A} = \mathcal{A} + S_\mathcal{A}, \\
\mathcal{A}(0, \vec{v}) = p_\mathcal{A}(\vec{v}) & \text{for } v_\eta > 0, \\
\mathcal{A}(L, \vec{v}) = \mathcal{A}(L, \mathcal{A}[\vec{v}]),
\end{cases}
\end{aligned}$$

(6.7)

where $p_\mathcal{A}$ and $S_\mathcal{A}$ will be specified later with

$$\mathcal{A}(\eta, \vec{v}) = \int_{\mathbb{R}^2} \frac{\zeta(\eta, \vec{u})}{\zeta(\eta, \vec{v})} k(\vec{u}, \vec{v}) \mathcal{A}(\eta, \vec{u}) d\vec{u}.$$  

(6.8)
Here we utilize Lemma 6.1. We need to derive the a priori estimate of $\mathcal{A}$. Define the energy as before

$$
E_1 = v_0^2 + v_0^2,
$$

(6.9)

$$
E_2 = v_0 e^{-W(\eta)}.
$$

(6.10)

We can easily check that the weight function $\zeta = \sqrt{E_1 - E_2}$. Along the characteristics, where $E_1$, $E_2$ and $\zeta$ are constants, the equation (6.7) can be simplified as follows:

$$
v_0 \frac{d\mathcal{A}}{d\eta} = \mathcal{A} + S_{\mathcal{A}}.
$$

(6.11)

Similar to the $\epsilon$-Milne problem with geometric correction, we can define the solution along the characteristics as follows:

$$
\mathcal{A}(\eta, \vec{v}) = \mathcal{K}[p_{\mathcal{A}}] + T[\mathcal{A} + S_{\mathcal{A}}],
$$

(6.12)

where

Region I:

For $v_0 > 0$,

$$
\mathcal{K}[p_{\mathcal{A}}] = p_{\mathcal{A}} \left( \vec{v}(\eta, \vec{v}; 0) \right) \exp(-H_{\eta,0}),
$$

(6.13)

$$
T[\mathcal{A} + S_{\mathcal{A}}] = \int_0^\eta \left( \mathcal{A} + S_{\mathcal{A}} \right) \left( \eta', \vec{v}(\eta, \vec{v}; \eta') \right) \frac{v_0'(\eta, \vec{v}; \eta')}{{v_0'}(\eta, \vec{v}; \eta')} \exp(-H_{\eta,\eta'}) d\eta'.
$$

(6.14)

Region II:

For $v_0 < 0$ and $v_0^2 + v_0^2 \geq v_0^2(\eta, \vec{v}; L)$,

$$
\mathcal{K}[p_{\mathcal{A}}] = p_{\mathcal{A}} \left( \vec{v}(\eta, \vec{v}; 0) \right) \exp(-H_{L,0} - \mathcal{R}[H_{L,\eta}]),
$$

(6.15)

$$
T[\mathcal{A} + S_{\mathcal{A}}] = \left( \int_0^L \left( \mathcal{A} + S_{\mathcal{A}} \right) \left( \eta', \vec{v}(\eta, \vec{v}; \eta') \right) \frac{v_0'(\eta, \vec{v}; \eta')}{{v_0'}(\eta, \vec{v}; \eta')} \exp(-H_{L,\eta'}) d\eta' \right) + \int_\eta^L \left( \mathcal{A} + S_{\mathcal{A}} \right) \left( \eta', \mathcal{R}[\vec{v}(\eta, \vec{v}; \eta')] \right) \frac{v_0'(\eta, \vec{v}; \eta')}{{v_0'}(\eta, \vec{v}; \eta')} \exp(\mathcal{R}[H_{\eta,\eta'}]) d\eta'.
$$

(6.16)

Region III:

For $v_0 < 0$ and $v_0^2 + v_0^2 \leq v_0^2(\eta, \vec{v}; L)$,

$$
\mathcal{K}[p_{\mathcal{A}}] = p_{\mathcal{A}} \left( \vec{v}(\eta, \vec{v}; 0) \right) \exp(-H_{L,0} - \mathcal{R}[H_{L,\eta}]),
$$

(6.17)

$$
T[\mathcal{A} + S_{\mathcal{A}}] = \left( \int_0^{\eta'} \left( \mathcal{A} + S_{\mathcal{A}} \right) \left( \eta', \vec{v}(\eta, \vec{v}; \eta') \right) \frac{v_0'(\eta, \vec{v}; \eta')}{{v_0'}(\eta, \vec{v}; \eta')} \exp(-H_{L,\eta'}) d\eta' \right) + \int_\eta^{\eta'} \left( \mathcal{A} + S_{\mathcal{A}} \right) \left( \eta', \mathcal{R}[\vec{v}(\eta, \vec{v}; \eta')] \right) \frac{v_0'(\eta, \vec{v}; \eta')}{{v_0'}(\eta, \vec{v}; \eta')} \exp(\mathcal{R}[H_{\eta,\eta'}]) d\eta'.
$$

(6.18)

Then we need to estimate $\mathcal{K}[p_{\mathcal{A}}]$ and $T[\mathcal{A} + S_{\mathcal{A}}]$ in each region. We assume $0 < \delta << 1$ and $0 < \delta_0 << 1$ are small quantities which will be determined later.

6.2. Region I: $v_0 > 0$. Based on Lemma 6.4 and Lemma 6.5 we can directly obtain

$$
\|\mathcal{K}[p_{\mathcal{A}}]\|_{L^\infty_{\eta,\vec{v}}} \leq \|p_{\mathcal{A}}\|_{L^\infty_{\eta,\vec{v}}},
$$

(6.19)

$$
\|T[S_{\mathcal{A}}]\|_{L^\infty_{\eta,\vec{v}}} \leq \left\| S_{\mathcal{A}} \right\|_{L^\infty_{\eta,\vec{v}}}. 
$$

(6.20)
Hence, we only need to estimate
\[ I = T[\omega'] = \int_0^\eta \omega' \left( \eta', \vec{\omega}(\eta, \vec{v}; \eta') \right) \exp(-H_{\eta, \eta'}) d\eta'. \] (6.21)

Since we always assume that \((\eta, \vec{v})\) and \((\eta', \vec{\omega})\) are on the same characteristics, in the following, we will simply write \(\vec{\omega}(\eta')\) or even \(\vec{\omega}\) instead of \(\vec{\omega}(\eta, \vec{v}; \eta')\) when there is no confusion. We can use this notation interchangeably when necessary.

We divide it into several steps:

Step 0: Preliminaries.
We have
\[ E_2(\eta', \nu') = \frac{R_\kappa - \epsilon \eta'}{R_\kappa} \nu'. \] (6.22)

Then we can directly obtain
\[ \zeta(\eta', \vec{\omega}) = \frac{1}{R_\kappa} \sqrt{R_\kappa^2 (v_{\eta}^2 + v_{\phi}^2) - \left( (R_\kappa - \epsilon \eta') \nu_{\phi} \right)^2} = \frac{1}{R_\kappa} \sqrt{R_\kappa^2 v_{\eta}^2 + \left( R_\kappa^2 - (R_\kappa - \epsilon \eta')^2 \right) v_{\phi}^2}, \] (6.23)
\[ \leq \frac{1}{R_\kappa} \sqrt{R_\kappa^2 v_{\eta}^2 + \frac{1}{R_\kappa} \sqrt{\left( R_\kappa^2 - (R_\kappa - \epsilon \eta')^2 \right) v_{\phi}^2}} \leq C \left( \nu'_{\eta} + \sqrt{\epsilon \eta' v_{\phi}} \right) \leq C \nu(\vec{\omega}), \]
and
\[ \zeta(\eta', \vec{\omega}) = \frac{1}{2} \left( \frac{1}{R_\kappa} \sqrt{R_\kappa^2 v_{\eta}^2 + \frac{1}{R_\kappa} \sqrt{\left( R_\kappa^2 - (R_\kappa - \epsilon \eta')^2 \right) v_{\phi}^2}} \right) \geq C \left( \nu'_{\eta} + \sqrt{\epsilon \eta' v_{\phi}} \right) \geq C \sqrt{\epsilon \eta' |\vec{v}|}. \] (6.24)

Also, we know for \(0 \leq \eta' \leq \eta\),
\[ v_{\eta} = \sqrt{v_{\eta}^2 + v_{\phi}^2} = \sqrt{v_{\eta}^2 + v_{\phi}^2 - v_{\phi}^2 \left( \frac{R_\kappa - \epsilon \eta}{R_\kappa - \epsilon \eta'} \right)^2} \] (6.25)
\[ = \sqrt{(R_\kappa - \epsilon \eta')^2 v_{\phi}^2 + (2R_\kappa - \epsilon \eta - \epsilon \eta')(\epsilon \eta - \epsilon \eta') v_{\phi}^2} \] (6.26)
\[ \frac{1}{2 \sqrt{v_{\eta}^2 + \epsilon (\eta - \eta') v_{\phi}^2}} \leq \frac{1}{v_{\eta}'} \leq \frac{1}{v_{\eta}}. \] (6.29)

Therefore,
\[ -\int_0^\eta \frac{1}{v_{\eta}'} (\eta, \vec{v}; y) dy \leq -\int_0^\eta \frac{1}{2 \sqrt{v_{\eta}^2 + \epsilon (\eta - \eta') v_{\phi}^2}} dy = \frac{1}{\epsilon v_{\phi}^2} \left( v_{\eta} - v_{\eta} + \sqrt{v_{\eta}^2 + \epsilon (\eta - \eta') v_{\phi}^2} \right) \] (6.30)
\[ = -\frac{\eta - \eta'}{v_{\eta} + \sqrt{v_{\eta}^2 + \epsilon (\eta - \eta') v_{\phi}^2}} \leq -\frac{\eta - \eta'}{2 \sqrt{v_{\eta}^2 + \epsilon (\eta - \eta') v_{\phi}^2}}. \]

Define a cut-off function \(\chi \in C^\infty[0, \infty)\) satisfying
\[ \chi(v_{\eta}) = \begin{cases} 1 & \text{for } |v_{\eta}| \leq \delta, \\ 0 & \text{for } |v_{\eta}| \geq 2\delta, \end{cases} \] (6.31)
In the following, we will divide the estimate of $I$ into several cases based on the value of $v_\eta$, $v_\eta'$, $\epsilon \eta'$ and $\epsilon (\eta - \eta')$. Let $\mathbf{1}$ denote the indicator function. Assume the dummy variable $\mathbf{u} = (u_\eta, u_\eta)$. We write

$$I = \int_0^\eta 1_{\{\nu \geq \delta_0\}} + \int_0^\eta 1_{\{0 \leq \nu \leq \delta_0\}} 1_{\{\chi(u_\eta) < 1\}} + \int_0^\eta 1_{\{0 \leq \nu \leq \delta_0\}} 1_{\{\chi(u_\eta) = 1\}} 1_{\{|\varepsilon \eta'| \geq v_\eta'\}} (6.32)$$

Step 1: Estimate of $I_1$ for $v_\eta \geq \delta_0$.

In this step, we will prove estimates based on the characteristics of $G$ itself instead of $\mathcal{A}$. Here, we rewrite the equation (6.2) along the characteristics as

$$v_\eta \frac{dG}{d\eta} + \nu G = K[G]. \quad (6.33)$$

In the following, we will repeatedly use simple facts (SF):

- Based on the well-posedness and decay theorem for $G$, we know $\|G\|_{L_{s,\infty}^\infty} \leq C$.
- Based on Lemma 2.3, we get $\|K[G]\|_{L_{s,\infty}^\infty} \leq \|G\|_{L_{s,\infty}^\infty} \leq C$ and $\|\nabla_v K[G]\|_{L_{s,\infty}^\infty} \leq \|G\|_{L_{s,\infty}^\infty} \leq C$.
- Since $E_1$ is conserved along the characteristics, we must have $\|\mathbf{u}\| = \|\mathbf{v}\|$.
- For $\eta' \leq \eta$, we must have $v_\eta' \geq v_\eta \geq \delta_0$.
- Using substitution $y = H_{\eta, \eta'}$, we know

$$\left| \int_0^\eta \frac{\nu \left(\mathbf{v}(\eta, \mathbf{v}; \eta')\right)}{v_\eta' \left(\eta, \mathbf{v}; \eta'\right)} \exp(-H_{\eta, \eta'}) d\eta \right| \leq \left| \int_0^\infty e^{-y} dy \right| = 1. \quad (6.34)$$

For $v_\eta \geq \delta_0$, we do not need the mild formulation for $\mathcal{A}$. Instead, we directly estimate

$$\left| \left\langle \mathbf{v} \right\rangle^{\theta} e^{|\mathbf{v}|^2} I_1 \right| \leq \left| \left\langle \mathbf{v} \right\rangle^{\theta} e^{|\mathbf{v}|^2} \frac{\partial G}{\partial \eta} \right|. \quad (6.35)$$

We rewrite the equation (6.2) along the characteristics as

$$G(\eta, \mathbf{v}) = \exp(-H_{\eta, 0}) \left( p \left( \mathbf{v}(\eta, \mathbf{v}; 0) \right) + \int_0^\eta K[G] \left( \eta', \mathbf{v}(\eta, \mathbf{v}; \eta') \right) \frac{v_\eta' (\eta, \mathbf{v}; \eta')}{v_\eta' (\eta, \mathbf{v}; \eta')} \exp(H_{\eta', 0}) d\eta' \right). \quad (6.36)$$

where $(\eta', \mathbf{v}')$ and $(\eta, \mathbf{v})$ are on the same characteristic with $v_\eta' \geq 0$, and

$$H_{\eta, s} = \int_s^t \frac{\nu \left(\mathbf{v}(\eta, \mathbf{v}; y)\right)}{v_\eta' (\eta, \mathbf{v}; y)} dy. \quad (6.37)$$

for any $s, t \geq 0$.

Taking $\eta$ derivative on both sides of (6.36), we have

$$\frac{\partial G}{\partial \eta} = X = X_1 + X_2 + X_3 + X_4 + X_5 + X_6. \quad (6.38)$$
where

\[ X_1 = - \exp( -H_{\eta, 0}) \frac{\partial H_{\eta, 0}}{\partial \eta} \left( p \left( \bar{\omega}^\prime (\eta, \bar{v}; 0) \right) + \int_0^\eta K[G] \left( \eta', \bar{v}(\eta, \bar{v}; \eta') \right) \frac{\partial v^\prime_\eta (\eta')}{\partial \eta'} \exp (H_{\eta', 0}) \, d\eta' \right), \quad (6.39) \]

\[ X_2 = \exp ( -H_{\eta, 0}) \frac{\partial p (\bar{v}(\eta, \bar{v}; 0))}{\partial \eta}, \quad (6.40) \]

\[ X_3 = \frac{K[G](\eta, \bar{v})}{v_\eta}, \quad (6.41) \]

\[ X_4 = - \exp ( -H_{\eta, 0}) \int_0^\eta \left( K[G] (\eta', \bar{v}(\eta, \bar{v}; \eta')) \exp (H_{\eta', 0}) \frac{1}{v_\eta^0 (\eta, \bar{v}; \eta')} \frac{\partial v^\prime_\eta (\eta, \bar{v}; \eta')}{\partial \eta} \, d\eta' \right), \quad (6.42) \]

\[ X_5 = \exp ( -H_{\eta, 0}) \int_0^\eta \frac{K[G] (\eta', \bar{v}(\eta, \bar{v}; \eta'))}{v_\eta^0 (\eta, \bar{v}; \eta')} \exp (H_{\eta', 0}) \frac{\partial H_{\eta', 0}}{\partial \eta} \, d\eta', \quad (6.43) \]

\[ X_6 = \exp ( -H_{\eta, 0}) \int_0^\eta \frac{1}{v_\eta^0 (\eta, \bar{v}; \eta')} \left( \nabla_\nu K[G] \left( \eta', \bar{v}(\eta, \bar{v}; \eta') \right) \frac{\partial \bar{v}(\eta, \bar{v}; \eta')}{\partial \eta} \right) \exp (H_{\eta', 0}) \, d\eta'. \]

We need to estimate each term. Note that

\[
\frac{\partial H_{\eta, s}}{\partial \eta} = \int_s^t \frac{\partial}{\partial \eta} \left( \frac{\nu (\bar{v}(\eta, \bar{v}; y))}{v_\eta^0 (\eta, \bar{v}; y)} \right) \, dy \quad (6.44)
\]

\[
= \int_s^t \frac{1}{v_\eta^0 (\eta, \bar{v}; y)} \frac{\partial \nu (\bar{v}(\eta, \bar{v}; y))}{\partial \bar{v}} \left( v_\eta^0 (\eta, \bar{v}; y) \frac{\partial v^\prime_\eta (\eta, \bar{v}; y)}{\partial \eta} + v_\phi^\prime (\eta, \bar{v}; y) \frac{\partial v^\prime_\phi (\eta, \bar{v}; y)}{\partial \eta} \right) \, dy
\]

\[- \int_s^t \frac{\nu (\bar{v}(\eta, \bar{v}; y))}{v_\eta^0 (\eta, \bar{v}; y)} \frac{\partial v^\prime_\eta (\eta, \bar{v}; y)}{\partial \eta} \, dy.
\]

Considering

\[
v_\phi^\prime (\eta, \bar{v}; y) = v_\phi e^{W(\eta') - W(\eta)} = v_\phi \frac{R_\kappa - \epsilon \eta}{R_\kappa - \epsilon \eta'}, \quad (6.45)
\]

\[
v_\eta^\prime (\eta, \bar{v}; y) = \sqrt{v_\eta^2 + v_\phi^2 - v_\eta^2} = \sqrt{v_\eta^2 + v_\phi^2 - v_\phi^2 \left( \frac{R_\kappa - \epsilon \eta}{R_\kappa - \epsilon \eta'} \right)^2}, \quad (6.46)
\]

we know

\[
\frac{\partial v^\prime_\eta (\eta, \bar{v}; y)}{\partial \eta} = \frac{\epsilon v_\phi}{R_\kappa - \epsilon \eta'}, \quad \frac{\partial v^\prime_\phi (\eta, \bar{v}; y)}{\partial \eta} = \frac{2 \epsilon v_\phi^2}{v_\eta^0 (\eta, \bar{v}; y)} \frac{R_\kappa - \epsilon \eta}{R_\kappa - \epsilon \eta'}. \quad (6.47)
\]

This implies

\[
\left| \frac{\partial v^\prime_\eta (\eta, \bar{v}; y)}{\partial \eta} \right| \leq \frac{C \epsilon |\bar{v}|}{v_\eta^0 (\eta, \bar{v}; y)} \leq \frac{C \epsilon |\bar{v}|}{\delta_0}, \quad \left| \frac{\partial v^\prime_\phi (\eta, \bar{v}; y)}{\partial \eta} \right| \leq C \epsilon |\bar{v}|. \quad (6.48)
\]
The method to estimate \( X_i \) is standard and we simply use the facts (SF) and direct computation, so we omit the details and only list the result
\[
\left| \langle \vec{v} \rangle^0 e^{\epsilon|\vec{v}|^2} X_3 \right| \leq \frac{C}{\delta_0} \| |G| \|_{L^\infty L^\infty_o} \quad \text{(6.49)}
\]
\[
\left| \langle \vec{v} \rangle^0 e^{\epsilon|\vec{v}|^2} X_4 \right| \leq \frac{C}{\delta_0} \left( \left\| \frac{\partial p}{\partial v_1} \right\|_{L^\infty_o} + \left\| \frac{\partial p}{\partial \phi} \right\|_{L^\infty_o} \right) \quad \text{(6.50)}
\]
\[
\left| \langle \vec{v} \rangle^0 e^{\epsilon|\vec{v}|^2} X_5 \right| \leq \frac{C}{\delta_0} \left( \left\| \frac{\partial p}{\partial v_1} \right\|_{L^\infty_o} + \left\| \frac{\partial p}{\partial \phi} \right\|_{L^\infty_o} + \| |G| \|_{L^\infty L^\infty_o} \right) \quad \text{(6.51)}
\]
\[
\left| \langle \vec{v} \rangle^0 e^{\epsilon|\vec{v}|^2} X_6 \right| \leq \frac{C}{\delta_0} \left( \left\| \frac{\partial p}{\partial v_1} \right\|_{L^\infty_o} + \left\| \frac{\partial p}{\partial \phi} \right\|_{L^\infty_o} + \| |G| \|_{L^\infty L^\infty_o} \right). \quad \text{(6.52)}
\]

In summary, we have
\[
\left| \langle \vec{v} \rangle^0 e^{\epsilon|\vec{v}|^2} I_1 \right| \leq \frac{C}{\delta_0} \left( \left\| \frac{\partial p}{\partial v_1} \right\|_{L^\infty_o} + \left\| \frac{\partial p}{\partial \phi} \right\|_{L^\infty_o} + \| |G| \|_{L^\infty L^\infty_o} \right). \quad \text{(6.55)}
\]

Step 2: Estimate of \( I_2 \) for \( 0 \leq v_\eta \leq \delta_0 \) and \( \chi(u_\eta) < 1 \).

We have
\[
I_2 = \int_0^\eta \left( \int_{\mathbb{R}^2} \frac{\zeta(\eta', \tilde{u}')}{\zeta(\eta, \tilde{u})} \left( 1 - \chi(u_\eta) \right) k(\tilde{u}, \tilde{u}') \alpha' (\eta', \tilde{u}) d\tilde{u} \right) \frac{1}{v_\eta} \exp(-H_{\eta, \eta'}) d\eta' \quad \text{(6.56)}
\]
\[
= \int_0^\eta \left( \int_{\mathbb{R}^2} \left( 1 - \chi(u_\eta) \right) k(\tilde{u}, \tilde{u}') \frac{G(\eta', \tilde{u})}{\eta'} d\tilde{u} \right) \frac{\zeta(\eta', \tilde{u}')}{v_\eta} \exp(-H_{\eta, \eta'}) d\eta'.
\]

Based on the \( \epsilon \)-Milne problem with geometric correction of \( G \) as
\[
u_\eta \frac{\partial G(\eta', \tilde{u})}{\partial \eta'} + G(\eta') \left( u_\phi^2 \frac{\partial G(\eta', \tilde{u})}{\partial u_\phi} - u_\eta u_\phi \frac{\partial G(\eta', \tilde{u})}{\partial u_\phi} \right) + \nu G(\eta', \tilde{u}) - K[G](\eta', \tilde{u}) = 0,
\]
we have
\[
\frac{\partial G(\eta', \tilde{u})}{\partial \eta'} = -\frac{1}{u_\eta} \left( G(\eta') \left( u_\phi^2 \frac{\partial G(\eta', \tilde{u})}{\partial u_\eta} - u_\eta u_\phi \frac{\partial G(\eta', \tilde{u})}{\partial u_\phi} \right) + \nu G(\eta', \tilde{u}) - K[G](\eta', \tilde{u}) \right). \quad \text{(6.58)}
\]

Hence, we have
\[
\tilde{A} := \int_{\mathbb{R}^2} \left( 1 - \chi(u_\eta) \right) k(\tilde{u}, \tilde{u}') \frac{G(\eta', \tilde{u})}{\eta'} \tilde{u} \quad \text{(6.59)}
\]
\[
= -\int_{\mathbb{R}^2} \left( 1 - \chi(u_\eta) \right) k(\tilde{u}, \tilde{u}') \frac{1}{u_\eta} \left( \nu G(\eta', \tilde{u}) - K[G](\eta', \tilde{u}) \right) \tilde{u}
\]
\[
= -\int_{\mathbb{R}^2} \left( 1 - \chi(u_\eta) \right) k(\tilde{u}, \tilde{u}') \frac{1}{u_\eta} G(\eta') \left( u_\phi^2 \frac{\partial G(\eta', \tilde{u})}{\partial u_\eta} - u_\eta u_\phi \frac{\partial G(\eta', \tilde{u})}{\partial u_\phi} \right) \tilde{u}
\]
\[
= \tilde{A}_1 + \tilde{A}_2.
\]

Using \( G \) estimates and \( |u_\eta| \geq \delta \), we may directly obtain
\[
\left| \langle \vec{v} \rangle^0 e^{\epsilon|\vec{v}|^2} \tilde{A}_1 \right| \quad \text{(6.60)}
\]
\[
\leq \left| \langle \vec{v} \rangle^0 e^{\epsilon|\vec{v}|^2} \int_{\mathbb{R}^2} \left( 1 - \chi(u_\eta) \right) k(\tilde{u}, \tilde{u}') \frac{1}{u_\eta} \left( \nu G(\eta', \tilde{u}) - K[G](\eta', \tilde{u}) \right) d\tilde{u} \right|
\]
\[
\leq \frac{C}{\delta} \| |G| \|_{L^\infty L^\infty_o}.
\]
On the other hand, an integration by parts yields

\[
\hat{A}_2 = \int_{\mathbb{R}^2} \frac{\partial}{\partial u_\phi} \left( \frac{u_\phi^2}{u_\eta} G(\eta') \left( 1 - \chi(u_\eta) \right) k(u_\eta, \bar{v}') \right) G(\eta', \bar{u}) d\bar{u} \\
- \int_{\mathbb{R}^2} \frac{\partial}{\partial u_\phi} \left( u_\phi G(\eta') \left( 1 - \chi(u_\eta) \right) k(u_\eta, \bar{v}') \right) G(\eta', \bar{u}) d\bar{u},
\]

which further implies

\[
\left| \langle \bar{v} \rangle \theta e^{e^{|\bar{v}|^2} \hat{A}_2} \right| \leq \frac{C}{\delta^2} \|G\|_{L^\infty \mathbb{R}_\theta^2}.
\]

As before we can use substitution to show that

\[
\left| \int_0^\eta \zeta(\eta', \bar{v}(\eta, \bar{v}; \eta')) \exp(-H_{t_\theta, \eta'}) d\eta' \right| \leq \left| \int_0^\eta \nu \left( \bar{v}(\eta, \bar{v}; \eta') \right) \exp(-H_{t_\theta, \eta'}) d\eta' \right| \leq 1,
\]

and \( |\bar{v}| \) is a constant along the characteristics. Then we have

\[
\left| \langle \bar{v} \rangle \theta e^{e^{|\bar{v}|^2} I_3} \right| \leq \frac{C}{\delta^2} \|G\|_{L^\infty \mathbb{R}_\theta^2}.
\]

Step 3: Estimate of \( I_3 \). For \( 0 \leq v_\eta \leq \delta_0 \), \( \chi(u_\eta) = 1 \) and \( \sqrt{\eta'} v'_\phi \geq v'_\eta \).

Based on (6.23), this implies

\[
\zeta(\eta', \bar{v}) \leq C \sqrt{\eta'} v'_\phi.
\]

Also, we know that

\[
\zeta(\eta', \bar{u}) \geq C \sqrt{\eta'} |\bar{u}|.
\]

Then we may decompose

\[
\hat{A} := \int_{\mathbb{R}^2} \frac{\zeta(\eta', \bar{v})}{\zeta(\eta', \bar{u})} \chi(u_\eta) k(u_\eta, \bar{v}) \mathcal{A}(\eta', \bar{u}) d\bar{u}
\]

\[
\leq \int_{|\bar{u}| \geq \sqrt{\delta}} \frac{v'_\phi}{|\bar{u}|} \chi(u_\eta) k(u_\eta, \bar{v}) \mathcal{A}(\eta', \bar{u}) d\bar{u} + \int_{|\bar{u}| \leq \sqrt{\delta}} \frac{v'_\phi}{|\bar{u}|} \chi(u_\eta) k(u_\eta, \bar{v}) \mathcal{A}(\eta', \bar{u}) d\bar{u}
\]

\[
= v'_\phi \left( \hat{A}_1 + \hat{A}_2 \right).
\]

Using Lemma 1.2.2, we directly estimate

\[
\left| \langle \bar{v} \rangle \theta e^{e^{|\bar{v}|^2} \hat{A}_1} \right| \leq C \left| \langle \bar{v} \rangle \theta e^{e^{|\bar{v}|^2} \int_{|\bar{u}| \geq \sqrt{3}} \frac{1}{|\bar{u}|} \chi(u_\eta) k(u_\eta, \bar{v}) \mathcal{A}(\eta', \bar{u}) d\bar{u} \right|
\]

\[
\leq C \sqrt{\delta} \| \mathcal{A} \|_{L^\infty \mathbb{R}_\theta^2}.
\]

Also, based on Lemma 1.2.2, we obtain

\[
\left| \langle \bar{v} \rangle \theta e^{e^{|\bar{v}|^2} \hat{A}_2} \right| \leq C \left| \langle \bar{v} \rangle \theta e^{e^{|\bar{v}|^2} \int_{|\bar{u}| \leq \sqrt{\delta}} k(u_\eta, \bar{v}) \mathcal{A}(\eta', \bar{u}) d\bar{u} \right|
\]

\[
\leq C \delta \| \mathcal{A} \|_{L^\infty \mathbb{R}_\theta^2}.
\]

Hence, since \( |\bar{v}| \) is a constant along the characteristics, we have

\[
\left| \langle \bar{v} \rangle \theta e^{e^{|\bar{v}|^2} I_3} \right| \leq C \sqrt{\delta} \| \mathcal{A} \|_{L^\infty \mathbb{R}_\theta^2} \left( \int_0^\eta \frac{v'_\phi}{v'_\eta} \exp(-H_{t_\theta, \eta'}) d\eta' \right)
\]

\[
\leq C \sqrt{\delta} \| \mathcal{A} \|_{L^\infty \mathbb{R}_\theta^2} \left( \int_0^\eta \frac{\nu(\bar{v})}{v'_\eta} \exp(-H_{t_\theta, \eta'}) d\eta' \right) \leq C \sqrt{\delta} \| \mathcal{A} \|_{L^\infty \mathbb{R}_\theta^2}.
\]
Step 4: Estimate of $I_4$ for $0 \leq v_\eta \leq \delta_0$, $\chi(u_\eta) = 1$, $\sqrt{c_\eta}v_\phi \leq v_\eta'$ and $v_\eta^2 \leq \epsilon(\eta - \eta')v_\phi^2$.

Based on (6.23), this implies

$$\zeta(\eta', \vec{v}') \leq C v'_\eta.$$  \hfill (6.68)

Based on (6.30), we have

$$-H_{\eta, \eta'} = - \int_{\eta'}^{\eta} \nu(\vec{u}) \frac{d\eta}{v'_\eta(y)} \leq - \frac{\nu(\vec{u})(\eta - \eta')}{2v_\phi \epsilon(\eta - \eta')} \leq - \frac{C' \nu(\vec{u})}{v_\phi} \sqrt{\eta - \eta'}. \hfill (6.69)$$

Hence, since $|\vec{u}|$ is a constant along the characteristics, we know

$$\left| \langle \vec{v} \rangle^\theta e^{[c][r]^2} I_4 \right| \leq C \left| \langle \vec{v} \rangle^\theta e^{[c][r]^2} \right| \int_{\mathbb{R}^2} \frac{1}{|u|} \chi(u_\eta)k(\vec{u}, \vec{v}') \varphi'(\eta', \vec{v}')d\eta' \leq C \sqrt{\delta} || \varphi ||_{L^\infty L_{\vec{v}, \varphi}^\infty}.$$  \hfill (6.70)

Hence, we have

$$\left| \langle \vec{v} \rangle^\theta e^{[c][r]^2} I_4 \right| \leq C \sqrt{\delta} || \varphi ||_{L^\infty L_{\vec{v}, \varphi}^\infty} \left( \int_{0}^{\eta} \frac{1}{\sqrt{c_\eta}} \exp(-H_{\eta, \eta'})d\eta' \right).$$

Define $z = \frac{\eta'}{\epsilon}$, which implies $d\eta' = \epsilon dz$. Substituting this into above integral, we have

$$\left| \langle \vec{v} \rangle^\theta e^{[c][r]^2} I_4 \right| \leq C \sqrt{\delta} || \varphi ||_{L^\infty L_{\vec{v}, \varphi}^\infty} \left( \int_{0}^{\frac{\eta}{\epsilon}} \frac{1}{\sqrt{z}} \exp(-C' \nu(\vec{u}) \sqrt{\frac{\eta}{\epsilon} - z}) dz + \int_{\frac{\eta}{\epsilon}}^{\frac{\eta'}{\epsilon}} \frac{1}{\sqrt{z}} \exp\left(-C' \nu(\vec{u}) \sqrt{\frac{\eta}{\epsilon} - z}\right) dz \right).$$

We can estimate these two terms separately.

$$\int_{0}^{\frac{\eta}{\epsilon}} \frac{1}{\sqrt{z}} \exp\left(-C' \nu(\vec{u}) \sqrt{\frac{\eta}{\epsilon} - z}\right) dz \leq \int_{0}^{1} \frac{1}{\sqrt{z}} dz = 2. \hfill (6.73)$$

$$\int_{1}^{\frac{\eta}{\epsilon}} \frac{1}{\sqrt{z}} \exp\left(-C' \nu(\vec{u}) \sqrt{\frac{\eta}{\epsilon} - z}\right) dz \leq \int_{1}^{\frac{\eta}{\epsilon}} \frac{1}{\sqrt{z}} \exp\left(-C' \nu(\vec{u}) \sqrt{\frac{\eta}{\epsilon} - z}\right) dz \leq \frac{\epsilon^2 - \frac{\eta}{\epsilon}}{2} \int_{0}^{\infty} te^{-t \frac{c' \nu(\vec{u})}{v_\phi}} dt \leq C \left( \frac{v_\phi}{\nu(\vec{u})} \right)^2 \leq C.$$  \hfill (6.74)

Therefore, we know

$$\left| \langle \vec{v} \rangle^\theta e^{[c][r]^2} I_4 \right| \leq C \sqrt{\delta} || \varphi ||_{L^\infty L_{\vec{v}, \varphi}^\infty}. \hfill (6.75)$$
Step 5: Estimate of $I_5$ for $0 \leq v_\eta \leq \delta_0$, $\chi(u_\eta) = 1$, $\sqrt{c\eta'v_\phi} \leq v_\eta'$ and $v_\eta^2 \geq \epsilon(\eta - \eta')v_\phi^2$.

Based on (6.23), this implies

$$\zeta(\eta', \vec{v}') \leq CV_\eta'.$$

Based on (6.30), we have

$$-H_{\eta, \eta'} = -\int_0^{\eta'} \frac{\nu(\vec{v}')}{v_\eta'} dy \leq -\frac{CV(\vec{v}')(\eta - \eta')}{v_\eta}.$$

Hence, we know

$$\left| \langle \vec{v}' \rangle^\theta_0 e^{|\vec{v}'|^2} I_5 \right| \leq C \int_0^{\eta'} \left( \langle \vec{v}' \rangle^\theta e^{|\vec{v}'|^2} \right) \int_{\mathbb{R}^2} \frac{1}{\zeta(\eta', \vec{u})} \chi(u_\eta) k(\vec{u}, \vec{v}') \mathcal{A}(\eta', \vec{u}) d\vec{u} \frac{1}{v_\eta} \exp(-H_{\eta, \eta'}) dy.'$$

Using Hölder’s inequality, we obtain

$$\left| \langle \vec{v}' \rangle^\theta_0 e^{|\vec{v}'|^2} \right| \left( \int_{\mathbb{R}^2} \frac{1}{\zeta(\eta', \vec{u})} \chi(u_\eta) k(\vec{u}, \vec{v}') \mathcal{A}(\eta', \vec{u}) d\vec{u} \right) \left( \int_{\mathbb{R}^2} \frac{1}{\zeta(\eta', \vec{v})} \chi(u_\eta) k(\vec{v}, \vec{u}') \mathcal{A}(\eta', \vec{u}) d\vec{u} \right) \leq C \left( \int_{\mathbb{R}^2} \frac{1}{\zeta(\eta', \vec{u})} \chi(u_\eta) k(\vec{u}, \vec{v}') \mathcal{A}(\eta', \vec{u}) d\vec{u} \right)^\frac{s}{2} \left( \int_{\mathbb{R}^2} \frac{1}{\zeta(\eta', \vec{v})} \chi(u_\eta) k(\vec{v}, \vec{u}') \mathcal{A}(\eta', \vec{u}) d\vec{u} \right)^\frac{s}{2},$$

where $0 < s << 1$. Note the fact that in 2D, $k(\vec{u}, \vec{v}')$ does not contain the singularity of $|\vec{u} - \vec{v}'|^{-1}$. Using a similar argument as in Step 3, we may directly compute

$$\left( \left( \langle \vec{v}' \rangle^\theta e^{|\vec{v}'|^2} \right) ^\frac{1+s}{2} \right) \int_{\mathbb{R}^2} \frac{1}{\langle \vec{v}' \rangle^{\frac{s}{1+s}}} k(\vec{u}, \vec{v}') \mathcal{A}(\eta', \vec{u}) d\vec{u} \leq C \left\| \mathcal{A} \right\|_{L^\infty_{\vec{v}' \phi}}.$$

Then we estimate

$$\int_{\mathbb{R}^2} \frac{1}{\zeta(\eta', \vec{u})} \frac{1}{|u_\phi|^{\frac{s}{1+s}}} \chi^\phi(\eta_\phi, \vec{u}) d\vec{u} \leq \int_{\mathbb{R}} \left( \int_{\delta}^{\infty} \frac{1}{\zeta(\eta', \vec{u})} \chi(u_\eta) du_\eta \right) \frac{1}{|u_\phi|^{\frac{s}{1+s}}} du_\phi.$$

(6.80)
We may directly compute
\begin{equation}
\int_{-\delta}^{\delta} \frac{1}{\zeta(\eta', \bar{u})} \chi(u_\eta) d\eta \leq \int_{-\delta}^{\delta} \frac{1}{\sqrt{(R^2_\kappa(u_\eta)^2 + (R^2_\kappa - (R^2_\kappa - \epsilon \eta')^2)(u_\phi)^2}} \chi(u_\eta) d\eta 
\end{equation}
where
\begin{equation}
r = \frac{R^2_\kappa - (R^2_\kappa - \epsilon \eta')^2}{R^2_\kappa} \leq C \sqrt{\epsilon \eta'}.
\end{equation}
We may further compute
\begin{equation}
\int_{-\delta}^{\delta} \frac{1}{\sqrt{(u_\eta)^2 + \sqrt{2}(u_\phi)^2}} d\eta = 2 \int_{0}^{\delta} \frac{1}{\sqrt{\eta}} \chi(\eta) d\eta 
\end{equation}
where
\begin{equation}
\int_{-\delta}^{\delta} \frac{1}{\sqrt{(u_\eta)^2 + \sqrt{2}(u_\phi)^2}} d\eta = 2 \int_{0}^{\delta} \frac{1}{\sqrt{(u_\eta)^2 + \sqrt{2}(u_\phi)^2}} d\eta
\end{equation}
Then considering \( s + \frac{1}{2s} > 1 \), we know
\begin{equation}
\int_{\mathbb{R}^2} \frac{1}{\zeta(\eta', \bar{u})} \chi(\eta) k(\bar{u}, \bar{v}) \alpha(\eta', \bar{u}) d\bar{u} \leq C \int_{\mathbb{R}} \left( 1 + \ln(r) + \ln(u_\phi) \right) \frac{1}{|\phi|^s} d\phi \n\end{equation}
Hence, we can obtain
\begin{equation}
\left| \langle \bar{v} \rangle^{\epsilon |\bar{v}|^2} \int_{\mathbb{R}^2} \frac{1}{\zeta(\eta', \bar{u})} \chi(\eta) k(\bar{u}, \bar{v}) \alpha(\eta', \bar{u}) d\bar{u} \right| \leq C \left( \frac{\langle \bar{v} \rangle^{\epsilon |\bar{v}|^2}}{\epsilon^\eta} \right) \left\| \alpha \right\|_{L^\infty} \left( 1 + \ln(\epsilon) + \ln(\eta') \right) \exp \left( -\frac{C \nu(\bar{v})(\eta - \eta')}{\bar{v}} \right) d\eta'.
\end{equation}
Hence, we know
\begin{equation}
\left| \langle \bar{v} \rangle^{\epsilon |\bar{v}|^2} I_5 \right| \leq C \left( \frac{\langle \bar{v} \rangle^{\epsilon |\bar{v}|^2}}{\epsilon^\eta} \right) \left\| \alpha \right\|_{L^\infty} \left( 1 + \ln(\epsilon) + \ln(\eta') \right) \exp \left( -\frac{C \nu(\bar{v})(\eta - \eta')}{\bar{v}} \right) d\eta'.
\end{equation}
Hence, we first estimate
\begin{equation}
\left| \int_0^\eta \left( \frac{\langle \bar{v} \rangle^{\epsilon |\bar{v}|^2}}{\eta^{\frac{1}{1+2s}}} \right) \ln(\eta') \exp \left( -\frac{C \nu(\bar{v})(\eta - \eta')}{\bar{v}} \right) d\eta' \right|
\end{equation}
If \( 0 \leq \eta \leq 2 \), using Hölder’s inequality, we have
\begin{equation}
\left| \int_0^\eta \left( \frac{\langle \bar{v} \rangle^{\epsilon |\bar{v}|^2}}{\eta^{\frac{1}{1+2s}}} \right) \ln(\eta') \exp \left( -\frac{C \nu(\bar{v})(\eta - \eta')}{\bar{v}} \right) d\eta' \right|
\end{equation}
\begin{equation}
\leq \left( \int_0^\eta \frac{1}{\eta^{\frac{1}{2}} \eta^{\frac{1}{2}}} \ln(\eta') \frac{\langle \bar{v} \rangle}{\eta^{\frac{1}{2}}} \frac{\langle \bar{v} \rangle}{\eta^{\frac{1}{2}}} \right) d\eta' \leq \left( \frac{1}{\eta^{\frac{1}{2}}} \eta^{\frac{1}{2}} \right) \leq C \delta_0 \leq \sqrt{\delta_0}.
\end{equation}
If $\eta \geq 2$, it suffices to estimate
\begin{equation}
\left| \int_{\eta^2}^{\eta} (\langle \vec{v}^2 \rangle)^{\frac{3}{2\eta}} \ln(\eta^2) \exp \left( -\frac{C \nu(\vec{v}) (\eta - \eta^2)}{v_\eta} \right) d\eta \right| \leq \ln(L) \left| \int_{\eta^2}^{\eta} (\langle \vec{v}^2 \rangle)^{\frac{3}{2\eta}} \exp \left( -\frac{C \nu(\vec{v}) (\eta - \eta^2)}{v_\eta} \right) d\eta \right| \leq C |\ln(\eta)| v_\eta \leq C |\ln(\eta)| \delta_0.
\end{equation}
(6.89)

With a similar argument, we may justify
\begin{equation}
\left| \int_{0}^{\eta} \left( 1 + |\ln(\eta)| \right) (\langle \vec{v}^2 \rangle)^{\frac{3}{2\eta}} \exp \left( -\frac{C \nu(\vec{v}) (\eta - \eta^2)}{v_\eta} \right) \right| \leq C \sqrt{\delta_0 + |\ln(\eta)| \delta_0}.
\end{equation}
(6.90)

Hence, we have
\begin{equation}
\left| \langle \vec{v}^2 \rangle^\delta \nu |\vec{v}|^2 I_5 \right| \leq \frac{C}{\epsilon^2} \left( \sqrt{\delta_0 + |\ln(\eta)| \delta_0} \right) \|\mathcal{A}\|_{L^\infty_{|\vec{v}|,\epsilon}}.
\end{equation}
(6.91)

Step 6: Synthesis.
Collecting all the terms in previous steps, we have proved
\begin{equation}
\left| \langle \vec{v}^2 \rangle^\delta \nu |\vec{v}|^2 \right| \leq \frac{C}{\epsilon^2} \left( 1 + |\ln(\eta)| \right) \sqrt{\delta_0 \|\mathcal{A}\|_{L^\infty_{|\vec{v}|,\epsilon}}} + \frac{C}{\delta_0} \|\mathcal{A}\|_{L^\infty_{|\vec{v},D_{\vec{v}}}}} + \frac{C}{\delta_0} \left( \left\| \frac{\partial \rho}{\partial v_\eta} \right\|_{L^\infty_{|\vec{v},D_{\vec{v}}}}} + \left\| \frac{\partial \rho}{\partial v_\phi} \right\|_{L^\infty_{|\vec{v},D_{\vec{v}}}}} + \|\mathcal{G}\|_{L^\infty_{|\vec{v},D_{\vec{v}}}}} \right).
\end{equation}
(6.92)

6.3. Region II: $v_\eta < 0$ and $v_\eta^2 + v_\epsilon^2 \geq v_\phi^2 (\eta, \vec{v}; L)$. Based on Lemma 5.4 and Lemma 5.5, we can directly obtain
\begin{equation}
|\mathcal{K}[p_{\mathcal{A}}]| \leq \|p_{\mathcal{A}}\|_{L^\infty_{|\vec{v}|,\epsilon}},
\end{equation}
(6.93)
\begin{equation}
|\mathcal{T}[S_{\mathcal{A}}]| \leq \left\| \frac{S_{\mathcal{A}}}{\nu} \right\|_{L^\infty_{|\vec{v}|,\epsilon}}.
\end{equation}
(6.94)

Hence, we only need to estimate
\begin{equation}
II = \mathcal{T}[\mathcal{A}] = \int_{0}^{L} \mathcal{A} \left( \eta', \vec{v}(\eta, \vec{v}; \eta') \right) \exp \left( -H_{L,\eta'} - \mathcal{R}[H_{L,\eta}] \right) d\eta' + \int_{\eta}^{L} \mathcal{A} \left( \eta', \mathcal{R}[\vec{v}(\eta, \vec{v}; \eta')] \right) \exp \left( \mathcal{R}[H_{\eta,\eta'}] \right) d\eta'.
\end{equation}
(6.95)

In particular, we can decompose
\begin{equation}
\mathcal{T}[\mathcal{A}] = \int_{0}^{\eta} \mathcal{A} \left( \eta', \vec{v}(\eta, \vec{v}; \eta') \right) \exp \left( -H_{L,\eta'} - \mathcal{R}[H_{L,\eta}] \right) d\eta' + \int_{\eta}^{L} \mathcal{A} \left( \eta', \mathcal{R}[\vec{v}(\eta, \vec{v}; \eta')] \right) \exp \left( \mathcal{R}[H_{\eta,\eta'}] \right) d\eta'.
\end{equation}
(6.96)

The integral $\int_{0}^{\eta}$ part can be estimated as in Region I, so we only need to estimate the integral $\int_{\eta}^{L}$ part. Also, noting that fact that
\begin{equation}
\exp \left( -H_{L,\eta'} - \mathcal{R}[H_{L,\eta}] \right) \leq \exp \left( -\mathcal{R}[H_{\eta',\eta}] \right),
\end{equation}
(6.97)
we only need to estimate
\begin{equation}
\int_{\eta}^{L} \mathcal{A} \left( \eta', \vec{v}(\eta, \vec{v}; \eta') \right) \exp \left( -H_{\eta',\eta} \right) d\eta'.
\end{equation}
(6.98)
Here the proof is almost identical to that in Region I, so we only point out the key differences.

Step 0: Preliminaries.
We need to update one key result. For $0 \leq \eta \leq \eta'$,

$$v'_\eta = \sqrt{E_1 - v''_\phi} = \sqrt{E_1 - \left(\frac{R_\kappa - \epsilon_\eta}{R_\kappa - \epsilon_\eta'}\right)^2} v''_\phi \leq v_\eta. \tag{6.99}$$

Then we have

$$- \int_{\eta}^{\eta'} \frac{1}{v'_\eta(y)} dy \leq - \frac{\eta' - \eta}{v_\eta}. \tag{6.100}$$

In the following, we will divide the estimate of $II$ into several cases based on the value of $v_\eta$, $v'_\eta$ and $\epsilon\eta'$. We write

$$II = \int_{\eta}^{L} 1_{\{v_\eta \leq -\delta_0\}} + \int_{\eta}^{L} 1_{\{-\delta_0 \leq v_\eta \leq 0\}} 1_{\{\chi(u_\eta) < 1\}} \tag{6.101}$$

$$+ \int_{\eta}^{L} 1_{\{-\delta_0 \leq v_\eta \leq 0\}} 1_{\{\chi(u_\eta) = 1\}} 1_{\{\sqrt{\epsilon\eta'} v'_\phi \geq v'_\eta\}} + \int_{\eta}^{L} 1_{\{-\delta_0 \leq v_\eta \leq 0\}} 1_{\{\chi(u_\eta) = 1\}} 1_{\{\sqrt{\epsilon\eta'} v'_\phi \leq v'_\eta\}}$$

$$= II_1 + II_2 + II_3 + II_4.$$

Step 1: Estimate of $II_1$ for $v_\eta \leq -\delta_0$.
We first estimate $v'_\eta$. Along the characteristics, we know

$$e^{-W(\eta')} v'_\phi = e^{-W(\eta)} v'_\phi,$$  \tag{6.102}

which implies

$$|v'_\phi| = e^{W(\eta') - W(\eta)} |v'_\phi| \leq e^{W(L) - W(0)} |v'_\phi| \leq e^{W(L) - W(0)} \sqrt{E_1 - \delta_0^2}. \tag{6.103}$$

Then we can further deduce that

$$|v'_\phi| \leq \left(1 - \frac{\epsilon_\eta}{R_\kappa}\right)^{-1} \sqrt{E_1 - \delta_0^2}. \tag{6.104}$$

Then we have

$$v'_\eta \geq \sqrt{E_1 - \left(1 - \frac{\epsilon_\eta}{R_\kappa}\right)^{-2} (E_1 - \delta_0^2)} \geq \delta_0 - C\epsilon^+ > \frac{\delta_0}{2}, \tag{6.105}$$

when $\epsilon$ is sufficiently small. Then this implies that for $|E_1(\eta, \vec{v})| \geq \left|v'_\phi(\eta, \vec{v}; L)\right|$, for $\epsilon$ sufficiently small, we know $\min v'_\eta \geq \delta_0$ where $(\eta', \vec{v}')$ is on the same characteristics as $(\eta, \vec{v})$ with $v'_\eta \geq 0$.

Similar to the estimate of $I_1$, in this step, we will prove estimates based on the characteristics of $\mathcal{G}$ itself instead of $\mathcal{A}$. Here, we rewrite the equation (6.2) along the characteristics as

$$v_\eta \frac{d\mathcal{G}}{d\eta} + \nu \mathcal{G} = K[\mathcal{G}]. \tag{6.106}$$

Also, we will still use simple facts (SF):

- Based on the well-posedness and decay theorem for $\mathcal{G}$, we know $\|\mathcal{G}\|_{L^\infty_{\nu, \theta}} \leq C$.
- Based on Lemma 2.3 we get $\|K[\mathcal{G}]\|_{L^\infty_{\nu, \theta}} \leq \|\mathcal{G}\|_{L^\infty_{\nu, \theta}} \leq C$ and $\|
abla_\nu K[\mathcal{G}]\|_{L^\infty_{\nu, \theta}} \leq \|\mathcal{G}\|_{L^\infty_{\nu, \theta}} \leq C$.
- Since $E_1$ is conserved along the characteristics, we must have $|\vec{v}| = |\vec{w}|$.
- For $\eta' \leq \eta$, we must have $v'_\eta \geq |v'_\eta| \geq \delta_0$.
- Using substitution $y = H_{\eta, \eta'}$, we know

$$\left|\int_{\eta}^{L} \nu \left(\frac{\vec{v}(\eta, \vec{v}; \eta')}{v'_\eta(\eta, \vec{v}; \eta')}\right) \exp(H_{\eta, \eta'}) d\eta'\right| \leq \left|\int_{-\infty}^{0} e^y dy\right| = 1. \tag{6.107}$$
For \( v_\nu \leq -\delta_0 \), we do not need the mild formulation for \( \mathcal{A} \). Instead, we directly estimate

\[
\left| \langle \ddot{v} \rangle \text{e}^{\|\ddot{v}\|^2} L \right| \leq \left| \langle \ddot{v} \rangle \text{e}^{\|\ddot{v}\|^2} \frac{\partial G}{\partial \nu} \right|.
\]

We rewrite the equation along the characteristics as

\[
\mathcal{G} = \left( \frac{v}{v}\right) \exp(-H_{L,0} - \mathcal{R}[H_{L,\nu}])
\]

\[
+ \int_0^L \frac{K(G)}{v'}(\nu, y') \exp(-H_{L,\nu} - \mathcal{R}[H_{L,\nu}]) d\eta'
\]

\[
+ \int_\eta^L \frac{K(G)}{v'(\nu, y')} \exp(\mathcal{R}[H_{\nu,\nu}]) d\eta',
\]

where \( \nu'(\eta') = \nu'(\eta, y') \) satisfying \( (\eta', \nu') \) and \( (\eta, \nu) \) are on the same characteristic with \( \nu' \geq 0 \), and

\[
H_{\nu, s} = \int_s^t \nu(\nu, y) d\eta.
\]

for any \( s, t \geq 0 \).

Then taking \( \eta \) derivative on both sides of (6.109) yields

\[
\frac{\partial G}{\partial \eta} = Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7 + Y_8 + Y_9,
\]

where

\[
Y_1 = \frac{\partial p(\nu, y)}{\partial \eta} \exp(-H_{L,0} - \mathcal{R}[H_{L,\nu}]),
\]

\[
Y_2 = -p(\nu, y) \exp(-H_{L,0} - \mathcal{R}[H_{L,\nu}]) \left( \frac{\partial H_{L,0}}{\partial \eta} + \frac{\partial \mathcal{R}[H_{L,\nu}]}{\partial \eta} \right),
\]

\[
Y_3 = \int_0^L \frac{K(G)}{v'}(\nu, y') \frac{\partial v'}{\partial \eta} \exp(-H_{L,\nu} - \mathcal{R}[H_{L,\nu}]) d\eta',
\]

\[
Y_4 = -\int_0^L \frac{K(G)}{v'}(\nu, y') \exp(-H_{L,\nu} - \mathcal{R}[H_{L,\nu}]) \left( \frac{\partial H_{L,\nu}}{\partial \eta} + \frac{\partial \mathcal{R}[H_{L,\nu}]}{\partial \eta} \right) d\eta',
\]

\[
Y_5 = \int_0^L \frac{1}{v'}(\nu, y') \exp(-H_{L,\nu} - \mathcal{R}[H_{L,\nu}]) \left( \nabla \nu K(G) \left( \nu', y' \right) \right) d\eta',
\]

\[
Y_6 = \int_0^L \frac{K(G)}{v'}(\nu, y') \frac{\partial v'}{\partial \eta} \exp(\mathcal{R}[H_{\nu,\nu}]) d\eta',
\]

\[
Y_7 = \int_0^L \frac{1}{v'}(\nu, y') \mathcal{R}[H_{\nu,\nu}] \exp(\mathcal{R}[H_{\nu,\nu}]) d\eta',
\]

\[
Y_8 = \int_0^L \frac{1}{v'}(\nu, y') \exp(\mathcal{R}[H_{\nu,\nu}]) \left( \nabla \nu K(G) \left( \nu', y' \right) \right) d\eta',
\]

\[
Y_9 = -\frac{K[G]}{\nu}.
\]
We need to estimate each term. Since the techniques are very similar to the estimate of $I_1$ without introducing new tricks, we just list the results here:

\[
\left| \langle \mathbf{v} \rangle^{\theta} e^{\bar{\bar{v}}^2} Y_1 \right| \leq C \left( 1 + \frac{1}{\delta_0} \right) \left( \left\| \frac{\partial p}{\partial v_\eta} \right\|_{L^\infty_{\delta,\phi}} + \left\| \frac{\partial p}{\partial v_\phi} \right\|_{L^\infty_{\delta,\phi}} \right), \tag{6.121}
\]

\[
\left| \langle \mathbf{v} \rangle^{\theta} e^{\bar{\bar{v}}^2} Y_2 \right| \leq C \left( 1 + \frac{1}{\delta_0} \right) \| p \|_{L^\infty_{\delta,\phi}}, \tag{6.122}
\]

\[
\left| \langle \mathbf{v} \rangle^{\theta} e^{\bar{\bar{v}}^2} Y_3 \right| \leq C \left( 1 + \frac{1}{\delta_0} \right) \| \mathcal{G} \|_{L^\infty_{\delta,\phi}}, \tag{6.123}
\]

\[
\left| \langle \mathbf{v} \rangle^{\theta} e^{\bar{\bar{v}}^2} Y_4 \right| \leq C \left( 1 + \frac{1}{\delta_0} \right) \| \mathcal{G} \|_{L^\infty_{\delta,\phi}}, \tag{6.124}
\]

\[
\left| \langle \mathbf{v} \rangle^{\theta} e^{\bar{\bar{v}}^2} Y_5 \right| \leq C \left( 1 + \frac{1}{\delta_0} \right) \| \mathcal{G} \|_{L^\infty_{\delta,\phi}}, \tag{6.125}
\]

\[
\left| \langle \mathbf{v} \rangle^{\theta} e^{\bar{\bar{v}}^2} Y_6 \right| \leq C \left( 1 + \frac{1}{\delta_0} \right) \| \mathcal{G} \|_{L^\infty_{\delta,\phi}}, \tag{6.126}
\]

\[
\left| \langle \mathbf{v} \rangle^{\theta} e^{\bar{\bar{v}}^2} Y_7 \right| \leq C \left( 1 + \frac{1}{\delta_0} \right) \| \mathcal{G} \|_{L^\infty_{\delta,\phi}}, \tag{6.127}
\]

\[
\left| \langle \mathbf{v} \rangle^{\theta} e^{\bar{\bar{v}}^2} Y_8 \right| \leq C \left( 1 + \frac{1}{\delta_0} \right) \| \mathcal{G} \|_{L^\infty_{\delta,\phi}}, \tag{6.128}
\]

\[
\left| \langle \mathbf{v} \rangle^{\theta} e^{\bar{\bar{v}}^2} Y_9 \right| \leq C \left( 1 + \frac{1}{\delta_0} \right) \| \mathcal{G} \|_{L^\infty_{\delta,\phi}}. \tag{6.129}
\]

In summary, we have

\[
\left| \langle \mathbf{v} \rangle^{\theta} e^{\bar{\bar{v}}^2} II_1 \right| \leq \frac{C}{\delta_0} \left( \left\| \frac{\partial p}{\partial v_\eta} \right\|_{L^\infty_{\delta,\phi}} + \left\| \frac{\partial p}{\partial v_\phi} \right\|_{L^\infty_{\delta,\phi}} + \| \mathcal{G} \|_{L^\infty_{\delta,\phi}} \right). \tag{6.130}
\]

Step 2: Estimate of $II_2$ for $-\delta_0 \leq v_\eta \leq 0$ and $\chi(u_\eta) < 1$.

This is similar to the estimate of $I_2$ based on the integral

\[
\int_{\eta}^{L} \nu \left( \mathbf{v} \langle \eta, \mathbf{v}, \eta' \rangle \right) \exp(-H_{\eta', \eta}) d\eta' \leq 1. \tag{6.131}
\]

Then we have

\[
\left| \langle \mathbf{v} \rangle^{\theta} e^{\bar{\bar{v}}^2} II_2 \right| \leq \frac{C}{\delta_2} \| \mathcal{G} \|_{L^\infty_{\delta,\phi}}. \tag{6.132}
\]

Step 3: Estimate of $II_3$ for $-\delta_0 \leq v_\eta \leq 0$, $\chi(u_\eta) = 1$ and $\sqrt{\epsilon \eta} v_\phi \geq v_\eta'$.

This is identical to the estimate of $I_3$, we have

\[
\left| \langle \mathbf{v} \rangle^{\theta} e^{\bar{\bar{v}}^2} II_3 \right| \leq C \sqrt{\delta} \| \mathcal{G} \|_{L^\infty_{\delta,\phi}}. \tag{6.133}
\]

Step 4: Estimate of $II_4$ for $-\delta_0 \leq v_\eta \leq 0$, $\chi(u_\eta) = 1$ and $\sqrt{\epsilon \eta} v_\phi \leq v_\eta'$.

This step is different. We do not need to further decompose the cases. Based on (6.100), we have,

\[
-H_{\eta, \eta'} \leq - \frac{\nu(\mathbf{v}) (\eta' - \eta)}{v_\eta}. \tag{6.134}
\]

Then following the same argument in estimating $I_5$, we know

\[
\left| \langle \mathbf{v} \rangle^{\theta} e^{\bar{\bar{v}}^2} II_4 \right| \leq \frac{C}{\epsilon} \| \mathcal{G} \|_{L^\infty_{\delta,\phi}} \int_{\eta}^{L} \left( \frac{\langle \mathbf{v} \rangle^{\theta} e^{\bar{\bar{v}}^2}}{\eta^{\frac{\theta}{\alpha}} \eta'} \right) \left( 1 + |\ln(\epsilon)| + |\ln(\eta')| \right) \exp \left( - \frac{C \nu(\mathbf{v}) (\eta - \eta')}{v_\eta} \right) d\eta'.
\]
Hence, we first estimate
\[
\left| \int_\eta^L \frac{\langle \vec{\psi} \rangle^\frac{1}{\eta^+}}{\eta^+} |\ln(\eta')| \exp \left( -\frac{C \nu(\vec{\psi})(\eta - \eta')}{v_\eta} \right) d\eta' \right|. \tag{6.135}
\]
If \( \eta \geq 2 \), we have
\[
\left| \int_\eta^L \frac{\langle \vec{\psi} \rangle^\frac{1}{\eta^+}}{\eta^+} |\ln(\eta')| \exp \left( -\frac{C \nu(\vec{\psi})(\eta - \eta')}{v_\eta} \right) d\eta' \right| \leq \ln(L) \left| \int_\eta^L \frac{\langle \vec{\psi} \rangle^\frac{1}{\eta^+}}{\eta^+} \exp \left( -\frac{C \nu(\vec{\psi})(\eta - \eta')}{v_\eta} \right) d\eta' \right| \leq C |\ln(\epsilon)| v_\eta \leq C |\ln(\epsilon)| \delta_0. \tag{6.136}
\]
If \( 0 \leq \eta \leq 2 \), using Hölder’s inequality, it suffices to estimate
\[
\left| \int_0^2 \frac{\langle \vec{\psi} \rangle^\frac{1}{\eta^+}}{\eta^+} |\ln(\eta')| \exp \left( -\frac{C \nu(\vec{\psi})(\eta - \eta')}{v_\eta} \right) d\eta' \right| \leq \left( \int_0^2 \frac{1}{\eta^+} |\ln(\eta')|^{\frac{1+2}{\eta^+}} d\eta' \right)^\frac{\eta^+}{1+2} \left( \int_0^2 \langle \vec{\psi} \rangle \exp \left( -\frac{(1+2\epsilon)\nu(\vec{\psi})C(\eta - \eta')}{v_\eta} \right) d\eta' \right)^\frac{1}{1+2} \leq C \left( \frac{v_\eta \langle \vec{\psi} \rangle}{\nu(\vec{\psi})} \right)^{\frac{1}{\eta^+}} \leq C\delta_0^{\frac{1}{\eta^+}} \leq \sqrt{\delta_0}. \tag{6.137}
\]
With a similar argument, we may justify
\[
\left| \int_\eta^L \left( 1 + |\ln(\epsilon)| \right) \langle \vec{\psi} \rangle^\frac{1}{\eta^+} \exp \left( -\frac{C \nu(\vec{\psi})(\eta - \eta')}{v_\eta} \right) \right| \leq C \left( \sqrt{\delta_0} + |\ln(\epsilon)| \delta_0 \right). \tag{6.138}
\]
Hence, we have
\[
\left| \langle \vec{\psi} \rangle^\frac{\eta}{\nu} e^{\epsilon |\theta|^2} II_4 \right| \leq \frac{C}{c \epsilon} \left( 1 + |\ln(\epsilon)| \right) \sqrt{\delta_0} \| \mathcal{A} \|_{L^\infty L^\infty_{\bar{\sigma}, \rho}}. \tag{6.139}
\]
Hence, we have
\[
\left| \langle \vec{\psi} \rangle^\frac{\eta}{\nu} e^{\epsilon |\theta|^2} II_5 \right| \leq \frac{C}{c \epsilon} \left( \sqrt{\delta_0} + |\ln(\epsilon)| \delta_0 \right) \| \mathcal{A} \|_{L^\infty L^\infty_{\bar{\sigma}, \rho}}. \tag{6.140}
\]
Step 5: Synthesis.
Collecting all the terms in previous steps, we have proved
\[
\left| \langle \vec{\psi} \rangle^\frac{\eta}{\nu} e^{\epsilon |\theta|^2} II \right| \leq \frac{C}{c \epsilon} \left( 1 + |\ln(\epsilon)| \right) \sqrt{\delta_0} \| \mathcal{A} \|_{L^\infty L^\infty_{\bar{\sigma}, \rho}} + C \sqrt{\delta_0} \| \mathcal{A} \|_{L^\infty L^\infty_{\bar{\sigma}, \rho}} + C \frac{\eta}{\delta_0} \left( \left\| \frac{\partial p}{\partial v_\theta} \right\|_{L^\infty_{\bar{\sigma}, \rho}} + \left\| \frac{\partial p}{\partial v_\phi} \right\|_{L^\infty_{\bar{\sigma}, \rho}} + \| \mathcal{G} \|_{L^\infty L^\infty_{\bar{\sigma}, \rho}} \right). \tag{6.141}
\]
6.4. Region III: \( v_\eta < 0 \) and \( v_\rho^2 + v_\bar{\rho}^2 \leq v_\rho^2(\eta, \bar{\psi}; L) \). Based on Lemma 5.13 and Lemma 5.16, we still have
\[
| \mathcal{K}[p_{\mathcal{A}}] | \leq \| p_{\mathcal{A}} \|_{L^\infty_{\bar{\sigma}, \rho}}, \tag{6.142}
\]
\[
| \mathcal{T}[S_{\mathcal{A}}] | \leq \left\| S_{\mathcal{A}} \right\|_{L^\infty L^\infty_{\bar{\sigma}, \rho}}. \tag{6.143}
\]
Hence, we only need to estimate
\[
III = \mathcal{T}[\mathcal{A}] = \int_0^{\eta^-} \frac{\mathcal{A}(\eta, \bar{\psi}; \eta', \eta')}{v_\eta(\eta, \bar{\psi}; \eta')} \exp \left( -H_{\eta^-} \right) d\eta' \tag{6.144}
\]
\[
+ \int_{\eta^-}^{\eta} \frac{\mathcal{A}(\eta, \bar{\psi}; \eta, \eta')}{v_\eta(\eta, \bar{\psi}; \eta')} \exp \left( \mathcal{P}[H_{\eta^-} \eta'] \right) d\eta'. \tag{6.145}
\]
In particular, we can decompose

\[ T[\tilde{\omega}] = \int_0^\eta \frac{\tilde{\omega}(\eta', \tilde{\omega}(\eta, \tilde{\omega}'))}{v_\eta^{\prime}(\eta, \tilde{\omega}; \eta')} \exp(-H_{\eta^+, \eta'} - \mathcal{A}[H_{\eta^+, \eta}])\,d\eta' \]  

(6.145)

\[ + \int_\eta^{\eta^+} \frac{\tilde{\omega}(\eta', \tilde{\omega}(\eta, \tilde{\omega}'))}{v_\eta^{\prime}(\eta, \tilde{\omega}; \eta')} \exp(-H_{\eta^+, \eta'} - \mathcal{A}[H_{\eta^+, \eta}])\,d\eta' \]

\[ + \int_\eta^{\eta^+} \frac{\tilde{\omega}(\eta', \tilde{\omega}(\eta, \tilde{\omega}'))}{v_\eta^{\prime}(\eta, \tilde{\omega}; \eta')} \exp(\mathcal{A}[H_{\eta, \eta'}])\,d\eta'. \]

Then the integral \( \int_0^\eta (\cdots) \) is similar to the argument in Region I, and the integral \( \int_\eta^{\eta^+} (\cdots) \) is similar to the argument in Region II. The only difference is in Step 1 when estimating \( \int_\eta^{\eta^+} (\cdots) \) part for \( \eta \leq -\delta_0 \). Here, we introduce a special trick.

We first estimate \( v_\eta \) in term of \( v_{\phi} \). Along the characteristics, we know

\[ e^{-W(L)} v'_{\phi}(L) = e^{-W(\eta)} v_{\phi}, \]

(6.146)

which implies

\[ |v'_{\phi}(L)| = e^{W(L) - W(\eta)} |v_{\phi}| \leq e^{W(L) - W(0)} |v_{\phi}| = \left( 1 - \frac{\varepsilon^2}{R_K} \right)^{-1} |v_{\phi}|. \]

(6.147)

Then we can further deduce that

\[ v_\eta^2 + v_{\phi}^2 \leq \left( 1 - \frac{\varepsilon^2}{R_K} \right)^{-2} v_{\phi}^2 \leq \left( 1 - \frac{\varepsilon^2}{R_K} \right)^{-2} v_{\phi}^2. \]

(6.148)

Then we have

\[ |v_\eta| \leq \sqrt{\left( 1 - \frac{\varepsilon^2}{R_K} \right)^{-2} v_{\phi}^2 - v_{\phi}^2} \leq \varepsilon^2 |v_{\phi}| \leq \delta_0 |v_{\phi}|, \]

(6.149)

when \( \varepsilon \) is sufficiently small.

- Therefore, if \( |v_{\phi}| \leq 1 \), then Step 1 is not necessary at all since we already have \( |v_\eta| \leq \delta_0 \). We directly apply the argument in Estimation II to obtain

\[ \left| \langle \tilde{\omega} \rangle e^{i|\tilde{\omega}|^2} III \right| \leq \frac{C}{e^s} (1 + |\ln(\varepsilon)|) \sqrt{\delta_0} \| \omega \|_{L^\infty L^\infty_{\tilde{\omega}, \omega}} + C \sqrt{\delta} \| \omega \|_{L^\infty L^\infty_{\tilde{\omega}, \omega}} + \frac{C}{a^2} \| \mathcal{G} \|_{L^\infty L^\infty_{\tilde{\omega}, \omega}}. \]

(6.150)

- However, if \( v_{\phi} \geq 1 \), let \( (\eta, v_\eta, v_{\phi}) \) and \( (\tilde{\eta}, -\delta_0, \tilde{v}_{\phi}) \) be on the same characteristics. Then we have the mild formulation

\[ \mathcal{G}(\eta, \tilde{\omega}) = \mathcal{G}(\tilde{\eta}, -\delta_0, \tilde{v}_{\phi}) \exp(-H_{\tilde{\eta}, \eta}) + \int_\eta^{\tilde{\eta}} \frac{K[\mathcal{G}] (\eta', \tilde{\omega}(\eta, \tilde{\omega}; \eta'))}{v_\eta^{\prime}(\eta, \tilde{\omega}; \eta')} \exp(H_{\eta', \eta})\,d\eta'. \]

(6.151)

In other words, we try to use a mild formulation and avoid go through \( \eta^+ \) point. Then similar to the estimate of \( H_1 \), taking \( \eta \) derivative in the mild formulation, we obtain

\[ \left| \langle \tilde{\omega} \rangle e^{i|\tilde{\omega}|^2} \frac{\partial \mathcal{G}}{\partial \eta} \right| \leq C \left( 1 + \frac{1}{\delta_0} \right) \| \mathcal{G} \|_{L^\infty L^\infty_{\tilde{\omega}, \omega}} + C \left| \langle \tilde{\omega} \rangle e^{i|\tilde{\omega}|^2} \frac{\partial \mathcal{G}(\tilde{\eta}, -\delta_0, \tilde{v}_{\phi})}{\partial \eta} \right|. \]

Also, we may directly verify that

\[ \left| \langle \tilde{\omega} \rangle e^{i|\tilde{\omega}|^2} \frac{\partial \mathcal{G}(\tilde{\eta}, -\delta_0, \tilde{v}_{\phi})}{\partial \eta} \right| \leq \left| \langle \tilde{\omega} \rangle e^{i|\tilde{\omega}|^2} \frac{\partial \mathcal{G}(\tilde{\eta}, -\delta_0, \tilde{v}_{\phi})}{\partial \eta} \right| + \left| \langle \tilde{\omega} \rangle e^{i|\tilde{\omega}|^2} \frac{\partial \mathcal{G}(\tilde{\eta}, -\delta_0, \tilde{v}_{\phi})}{\partial \tilde{v}_{\phi}} \frac{\partial \tilde{v}_{\phi}}{\partial \eta} \right| \]

\[ \leq \left| \langle \tilde{\omega} \rangle e^{i|\tilde{\omega}|^2} \frac{\partial \mathcal{G}(\tilde{\eta}, -\delta_0, \tilde{v}_{\phi})}{\partial \eta} \right|, \]

(6.152)
This is much simpler than normal derivative, since \( \tilde{\eta} \) is achieved since now \( |\tilde{\eta}| \leq \delta_0 \).

Hence, we have

\[
\begin{align*}
\left| \langle \tilde{v} \rangle e^{i|\tilde{v}|^2} III \right| & \leq \frac{C}{\epsilon^3} \left( 1 + |\ln(\epsilon)| \right) \sqrt{\delta_0} \| \mathcal{A} \|_{L^\infty_{\tilde{\eta},\phi}} + C \sqrt{\delta} \| \mathcal{A} \|_{L^\infty_{\tilde{\eta},\phi}} \\
& + \frac{C}{\delta} \| \mathcal{G} \|_{L^\infty_{\tilde{\eta},\phi}} + \frac{C}{\delta_0} \| \mathcal{G} \|_{L^\infty_{\tilde{\eta},\phi}}.
\end{align*}
\]  

(6.153)

6.5. Estimates of Normal Derivative. Combining the analysis in these three regions, we have for \( 0 < s < 1 \),

\[
\| \mathcal{A} \|_{L^\infty_{\tilde{\eta},\phi}} \leq \frac{C}{\epsilon^3} \left( 1 + |\ln(\epsilon)| \right) \sqrt{\delta_0} \| \mathcal{A} \|_{L^\infty_{\tilde{\eta},\phi}} + C \sqrt{\delta} \| \mathcal{A} \|_{L^\infty_{\tilde{\eta},\phi}} + C \| \mathcal{G} \|_{L^\infty_{\tilde{\eta},\phi}}
\]  

(6.154)

Then we choose these constants to perform absorbing argument. First we choose \( 0 < \delta < 1 \) sufficiently small such that

\[
C \sqrt{\delta} \leq \frac{1}{4}.
\]

(6.155)

Then we take \( \delta_0 = \sqrt{\epsilon^3} |\ln(\epsilon)|^{-1} \) such that

\[
\frac{C}{\epsilon^3} \left( 1 + |\ln(\epsilon)| \right) \sqrt{\delta_0} \leq 2C \delta \leq \frac{1}{2}.
\]

(6.156)

for \( \epsilon \) sufficiently small. Note that this mild decay of \( \delta_0 \) with respect to \( \epsilon \) also justifies the assumption in Case II and Case III that

\[
\epsilon^* \leq \delta_0,
\]

(6.157)

for \( \epsilon \) sufficiently small. Here since \( \delta \) and \( C \) are independent of \( \epsilon \), there is no circulant argument. Hence, we can absorb all the term related to \( \| \mathcal{A} \|_{L^\infty_{\tilde{\eta},\phi}} \) on the right-hand side of (6.154) to the left-hand side to obtain the desired result.

**Lemma 6.2.** We have

\[
\| \mathcal{A} \|_{L^\infty_{\tilde{\eta},\phi}} \leq C \left( \| p_\mathcal{A} \|_{L^\infty_{\tilde{\eta},\phi}} \right.
\]

\[
+ \frac{S_\mathcal{A}}{\nu} \left. \right|_{L^\infty_{\tilde{\eta},\phi}} + C \| \mathcal{G} \|_{L^\infty_{\tilde{\eta},\phi}}.
\]

(6.158)

6.6. Estimates Velocity Derivative. Consider the general \( \epsilon \)-Milne problem with geometric correction for \( \mathcal{B} = \zeta \frac{\partial \mathcal{G}}{\partial \nu} \) as

\[
\begin{align*}
\frac{\partial \mathcal{B}}{\partial \nu} + G(\eta) \left( \nu^2 \frac{\partial \mathcal{B}}{\partial \nu} - \nu v_\phi \frac{\partial \mathcal{B}}{\partial v_\phi} \right) + \nu \mathcal{B} = \tilde{\mathcal{B}} + S_{\mathcal{B}},
\end{align*}
\]

(6.159)

\[
\mathcal{B}(0, \tilde{v}) = p_{\mathcal{B}}(\tilde{v}) \quad \text{for} \quad v_\eta > 0,
\]

\[
\mathcal{B}(L, \tilde{v}) = \mathcal{B}(L, \mathcal{B}[\tilde{v}]),
\]

where \( p_{\mathcal{B}} \) and \( S_{\mathcal{B}} \) will be specified later with

\[
\tilde{\mathcal{B}}(\eta, \tilde{v}) = \int_{\mathbb{R}^2} \zeta(\eta, \tilde{v}) \partial_{v_\phi} k(\tilde{u}, \tilde{v}) G(\eta, \tilde{u}) d\tilde{u}.
\]

(6.160)

This is much simpler than normal derivative, since \( \mathcal{B} \) does not contain \( \mathcal{B} \) directly. Then by a similar argument as before, we obtain the desired result.
Lemma 6.3. We have
\[
\| \mathcal{B} \|_{L^\infty L^\infty_{\tilde{\omega}, \tilde{\nu}}} \leq C \left( \| p \phi \|_{L^\infty_{\tilde{\omega}, \tilde{\nu}}} + \left\| \frac{S \phi}{\nu} \right\|_{L^\infty L^\infty_{\tilde{\omega}, \tilde{\nu}}} \right) 
+ C \left( \left\| \frac{\partial p}{\partial \nu} \right\|_{L^\infty_{\tilde{\omega}, \tilde{\nu}}} + \left\| \frac{\partial p}{\partial \nu} \right\|_{L^\infty L^\infty_{\tilde{\omega}, \tilde{\nu}}} + \| G \|_{L^\infty L^\infty_{\tilde{\omega}, \tilde{\nu}}} \right). 
\] (6.161)

In a similar fashion, consider the general $\epsilon$-Milne problem with geometric correction for $\hat{\mathcal{C}} = \hat{\zeta} \frac{\partial \mathcal{G}}{\partial \nu \phi}$ as
\[
\begin{cases}
\frac{\partial \mathcal{C}}{\partial \eta} + G(\eta) \left( \frac{v^2}{v^2} \frac{\partial \mathcal{C}}{\partial \eta} - v \frac{\partial \mathcal{C}}{\partial \nu \phi} \right) + \nu \mathcal{C} = \mathcal{G} + \mathcal{S}_\phi, \\
\mathcal{C}(0, \tilde{\nu}) = p \phi(\tilde{\nu}) \quad \text{for} \quad v_0 > 0, \\
\mathcal{C}(L, \tilde{\nu}) = \mathcal{C}(L, \mathcal{B}[\tilde{\nu}]),
\end{cases}
\] (6.162)

where $p \phi$ and $\mathcal{S}_\phi$ will be specified later with
\[
\hat{\mathcal{C}}(\eta, \tilde{\nu}) = \int_{\mathbb{R}^2} \zeta(\eta, \tilde{\nu}) \partial_{\nu \phi} k(\tilde{u}, \tilde{v}) \mathcal{G}(\eta, \tilde{u}) d\tilde{u}.
\] (6.163)

This is also much simpler than normal derivative, since $\hat{\mathcal{C}}$ does not contain $\mathcal{C}$ directly. Then by a similar argument as before, we obtain the desired result.

Lemma 6.4. We have
\[
\| \mathcal{C} \|_{L^\infty L^\infty_{\tilde{\omega}, \tilde{\nu}}} \leq C \left( \| p \phi \|_{L^\infty_{\tilde{\omega}, \tilde{\nu}}} + \left\| \frac{S \phi}{\nu} \right\|_{L^\infty L^\infty_{\tilde{\omega}, \tilde{\nu}}} \right) 
+ C \left( \left\| \frac{\partial p}{\partial \nu} \right\|_{L^\infty_{\tilde{\omega}, \tilde{\nu}}} + \left\| \frac{\partial p}{\partial \nu} \right\|_{L^\infty L^\infty_{\tilde{\omega}, \tilde{\nu}}} + \| G \|_{L^\infty L^\infty_{\tilde{\omega}, \tilde{\nu}}} \right). 
\] (6.164)

6.7. Estimates of Tangential Derivative. In this subsection, we combine above a priori estimates of normal and velocity derivatives.

Theorem 6.5. We have
\[
\left\| \frac{\zeta}{\partial \eta} \right\|_{L^\infty L^\infty_{\tilde{\omega}, \tilde{\nu}}} + \left\| \frac{\zeta}{\partial \nu \phi} \right\|_{L^\infty L^\infty_{\tilde{\omega}, \tilde{\nu}}} + \left\| \frac{\zeta}{\partial \nu \phi} \right\|_{L^\infty L^\infty_{\tilde{\omega}, \tilde{\nu}}} \leq C |\ln(\epsilon)| \epsilon^{-s},
\] (6.165)

for some $0 < s < 1$.

Proof. Collecting the estimates for $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ in Lemma 6.2, Lemma 6.3, and Lemma 6.4, we have
\[
\| \mathcal{A} \|_{L^\infty L^\infty_{\tilde{\omega}, \tilde{\nu}}} \leq C \left( \| p \phi \|_{L^\infty_{\tilde{\omega}, \tilde{\nu}}} + \left\| \frac{S \phi}{\nu} \right\|_{L^\infty L^\infty_{\tilde{\omega}, \tilde{\nu}}} \right) + C_0 |\ln(\epsilon)| \epsilon^{-s},
\] (6.166)

\[
\| \mathcal{B} \|_{L^\infty L^\infty_{\tilde{\omega}, \tilde{\nu}}} \leq C \left( \| p \phi \|_{L^\infty_{\tilde{\omega}, \tilde{\nu}}} + \left\| \frac{S \phi}{\nu} \right\|_{L^\infty L^\infty_{\tilde{\omega}, \tilde{\nu}}} \right) + C_0,
\] (6.167)

\[
\| \mathcal{C} \|_{L^\infty L^\infty_{\tilde{\omega}, \tilde{\nu}}} \leq C \left( \| p \phi \|_{L^\infty_{\tilde{\omega}, \tilde{\nu}}} + \left\| \frac{S \phi}{\nu} \right\|_{L^\infty L^\infty_{\tilde{\omega}, \tilde{\nu}}} \right) + C_0,
\] (6.168)

where
\[
C_0 = \| p \|_{L^\infty_{\tilde{\omega}, \tilde{\nu}}} + \left\| \frac{\partial p}{\partial \nu \phi} \right\|_{L^\infty L^\infty_{\tilde{\omega}, \tilde{\nu}}} + \left\| \frac{\partial p}{\partial \nu \phi} \right\|_{L^\infty L^\infty_{\tilde{\omega}, \tilde{\nu}}} + \| G \|_{L^\infty L^\infty_{\tilde{\omega}, \tilde{\nu}}}.
\] (6.169)
Taking derivatives on both sides of (6.2) and multiplying $\zeta$, we have

$$p_{st} = -\frac{\epsilon}{Rm} \left( \nu \frac{\partial p}{\partial \eta} - v_\eta v_\phi \frac{\partial p}{\partial \psi} \right) + \nu p - K[G](0, \vec{v}),$$  \hspace{1cm} (6.170)

$$p_\beta = v_\eta \frac{\partial \rho}{\partial \eta},$$  \hspace{1cm} (6.171)

$$p_\epsilon = v_\eta \frac{\partial \rho}{\partial \psi},$$  \hspace{1cm} (6.172)

$$S_{st} = \frac{\partial G}{\partial \eta} \left( v_\phi \mathcal{B} - v_\eta v_\phi \mathcal{E} \right),$$  \hspace{1cm} (6.173)

$$S_\beta = \mathcal{A} - G v_\phi \mathcal{E},$$  \hspace{1cm} (6.174)

$$S_\epsilon = G \left( 2v_\phi \mathcal{B} - v_\eta \mathcal{E} \right).$$  \hspace{1cm} (6.175)

We can directly verify that

$$\|p_{st}\|_{L^\infty_{\varphi,\eta}} + \|p_\beta\|_{L^\infty_{\varphi,\eta}} + \|p_\epsilon\|_{L^\infty_{\varphi,\eta}} \leq C_0.$$  \hspace{1cm} (6.176)

Since $|G(\eta)| + \left| \frac{\partial G}{\partial \eta} \right| \leq \epsilon$, from (6.175), we obtain

$$\|\mathcal{A}\|_{L^\infty_{\varphi,\eta}} \leq C_0 + C_0 \epsilon \left( \left\| \frac{v_\phi \mathcal{B}}{\nu} \right\|_{L^\infty_{\varphi,\eta}} + \left\| \frac{v_\eta \mathcal{E}}{\nu} \right\|_{L^\infty_{\varphi,\eta}} \right),$$  \hspace{1cm} (6.177)

which further implies

$$\|\mathcal{A}\|_{L^\infty_{\varphi,\eta}} \leq C_0 + C_0 \epsilon \|\mathcal{B}\|_{L^\infty_{\varphi,\eta}}.$$  \hspace{1cm} (6.178)

Plugging (6.178) into (6.174), we obtain

$$\|\mathcal{B}\|_{L^\infty_{\varphi,\eta}} \leq C_0 + C_0 \epsilon \left( \left\| \frac{\mathcal{A}}{\nu} \right\|_{L^\infty_{\varphi,\eta}} + \epsilon \left\| \frac{v_\phi \mathcal{E}}{\nu} \right\|_{L^\infty_{\varphi,\eta}} \right),$$  \hspace{1cm} (6.179)

which further implies

$$\|\mathcal{B}\|_{L^\infty_{\varphi,\eta}} \leq C_0 + C_0 \epsilon \left\| \frac{\mathcal{A}}{\nu} \right\|_{L^\infty_{\varphi,\eta}},$$  \hspace{1cm} (6.180)

$$\|\mathcal{E}\|_{L^\infty_{\varphi,\eta}} \leq C_0 + C_0 \epsilon \left\| \frac{\mathcal{A}}{\nu} \right\|_{L^\infty_{\varphi,\eta}}.$$  \hspace{1cm} (6.181)

Plugging (6.180) and (6.181) into (6.173), we get

$$\|\mathcal{A}\|_{L^\infty_{\varphi,\eta}} \leq C_0 \|\mathcal{A}\|_{L^\infty_{\varphi,\eta}} \epsilon^{-s} + C_0 \epsilon \left( \left\| \frac{v_\phi^2 \mathcal{B}}{\nu^2} \right\|_{L^\infty_{\varphi,\eta}} + \left\| \frac{v_\eta v_\phi \mathcal{E}}{\nu^2} \right\|_{L^\infty_{\varphi,\eta}} \right),$$  \hspace{1cm} (6.182)

$$\leq C_0 \|\mathcal{A}\|_{L^\infty_{\varphi,\eta}} \epsilon^{-s} + C_0 \epsilon \left( \left\| \frac{v_\phi^2 \mathcal{A}}{\nu^2} \right\|_{L^\infty_{\varphi,\eta}} + \epsilon \left\| \frac{v_\eta v_\phi \mathcal{A}}{\nu^2} \right\|_{L^\infty_{\varphi,\eta}} \right),$$  \hspace{1cm} (6.183)

$$\leq C_0 \|\mathcal{A}\|_{L^\infty_{\varphi,\eta}} \epsilon^{-s} + C_0 \epsilon \|\mathcal{A}\|_{L^\infty_{\varphi,\eta}}.$$  \hspace{1cm} (6.184)
which implies
\[ \| \mathcal{A} \|_{L^\infty L^\infty_{\theta, \phi}} \leq C_0 |\ln(\epsilon)| \epsilon^{-s}. \] (6.183)

Hence, we derive
\[ \mathcal{A} \leq C |\ln(\epsilon)| \epsilon^{-s}, \] (6.184)
\[ \mathcal{B} \leq C |\ln(\epsilon)| \epsilon^{-s}, \] (6.185)
\[ \mathcal{C} \leq C |\ln(\epsilon)| \epsilon^{-s}. \] (6.186)

Above theorems only provide a priori estimates. The rigorous proof relies on a penalty method and an iteration argument. This step is standard as in [18], so we omit it here.

**Theorem 6.6.** For \( K_0 > 0 \) sufficiently small, we have
\[ \left\| e^{K_0 \eta} \frac{\partial \mathcal{G}}{\partial \theta} (\eta, \theta, \phi) \right\|_{L^\infty L^\infty_{\theta, \phi}} \leq C |\ln(\epsilon)| \epsilon^{-s}, \] (6.187)
for some \( 0 < s << 1 \).

**Proof.** This proof is almost identical to Theorem 6.5. The only difference is that \( S, \mathcal{A} \) is added by \( K_0 \nu_0 \mathcal{A} \), \( \mathcal{B} \) added by \( K_0 \nu_0 \mathcal{B} \), and \( \mathcal{A} \) added by \( K_0 \nu_0 \mathcal{A} \). When \( K_0 \) is sufficiently small, we can also absorb them into the left-hand side. Hence, this is obvious.

Now we pull \( \theta \) dependence back and study the tangential derivative.

**Theorem 6.7.** We have
\[ \left\| e^{K_0 \eta} \frac{\partial \mathcal{G}}{\partial \theta} (\eta, \theta, \phi) \right\|_{L^\infty L^\infty_{\theta, \phi}} \leq C |\ln(\epsilon)| \epsilon^{-s}, \] (6.188)
for some \( 0 < s << 1 \).

**Proof.** Let \( \mathcal{W} = \frac{\partial \mathcal{G}}{\partial \theta} \). Taking \( \theta \) derivative on both sides of (6.2), we have that \( \mathcal{W} \) satisfies the equation
\[
\begin{align*}
\nu_{\theta} \frac{\partial \mathcal{W}}{\partial \eta} + G(\eta) \left( \nu_{\theta} \frac{\partial \mathcal{W}}{\partial \nu_{\theta}} - \nu_{\eta} \mathcal{W} \right) + \nu \mathcal{W} - K \mathcal{W} &= \frac{R_0^\prime}{R_0^\prime - \epsilon \eta} G(\eta) \left( \nu_{\theta} \frac{\partial \mathcal{G}}{\partial \nu_{\theta}} - \nu_{\eta} \mathcal{G} \right), \\
\mathcal{W}(0, \theta, \nu) &= \frac{\partial \mathcal{W}}{\partial \theta}(0, \theta, \nu) \quad \text{for} \quad \sin \phi > 0, \\
\mathcal{W}(L, \theta, \nu) &= \mathcal{W}(L, \theta, \mathcal{W}(|\mathcal{W}|)),
\end{align*}
\] (6.189)
where \( R^\prime_0 \) is the \( \theta \) derivative of \( R_0 \). For \( \eta \in [0, L] \), we have
\[ \frac{R^\prime_0}{R_0^\prime - \epsilon \eta} \leq C \max_{\theta} R^\prime_0 \leq C. \] (6.190)

Since \( \zeta(\eta, \nu) \geq \nu_{\eta} \), based on Theorem 6.6 and the equation (6.2), we know
\[ \left\| e^{K_0 \eta} \nu_{\eta} \frac{\partial \mathcal{W}}{\partial \eta} \right\|_{L^\infty L^\infty_{\theta, \phi}} \leq C |\ln(\epsilon)| \epsilon^{-s}, \] (6.191)
which further implies
\[ \left\| e^{K_0 \eta} G(\eta) \left( \nu_{\theta} \frac{\partial \mathcal{W}}{\partial \nu_{\theta}} - \nu_{\eta} \mathcal{W} \right) \right\|_{L^\infty L^\infty_{\theta, \phi}} \leq C |\ln(\epsilon)| \epsilon^{-s}, \] (6.192)
for some \( 0 < s << 1 \). Therefore, the source term in the equation (6.189) is in \( L^\infty \) and decays exponentially. By Theorem 6.10 we have that
\[ \left\| e^{K_0 \eta} \mathcal{W}(\eta, \theta, \phi) \right\|_{L^\infty L^\infty_{\theta, \phi}} \leq C |\ln(\epsilon)| \epsilon^{-s}, \] (6.193)
for some \( 0 < s << 1 \).
7. Hydrodynamic Limits

**Theorem 7.1.** For given $M_0 > 0$ and $\mu_0 > 0$ satisfying (1.7) and (1.11) with $0 < \epsilon << 1$, there exists a unique positive solution $\mathfrak{F} = M_0\mu + \mu^2 f^*$ to the stationary Boltzmann equation (1.1), and $f^*$ fulfils that for integer $\vartheta \geq 3$ and $0 < \vartheta < \frac{1}{4}$,

$$\left\| \langle \tilde{v} \rangle^\vartheta e^{i|\tilde{v}|^2} (f^* - \epsilon F) \right\|_{L^\infty} \leq C(\vartheta) \epsilon^{2-\delta},$$

(7.1)

for any $0 < \delta << 1$, where

$$F = \mu^\frac{1}{2} \left( \rho + \tilde{u} \cdot \tilde{v} + \vartheta \frac{|\tilde{v}|^2 - 2}{2} \right),$$

(7.2)

satisfies the steady Navier-Stokes-Fourier system

$$\begin{cases}
\nabla_x (\rho + \vartheta) = 0, \\
\tilde{u} \cdot \nabla_x \tilde{u} - \gamma_1 \Delta_x \tilde{u} + \nabla_x P_2 = 0, \\
\nabla_x \cdot \tilde{u} = 0, \\
\tilde{u} \cdot \nabla_x \vartheta - \gamma_2 \Delta_x \vartheta = 0, \\
\rho(x_0) = \rho_{b,1}(x_0) + M(x_0), \\
\tilde{u}(x_0) = \tilde{u}_{b,1}(x_0), \\
\vartheta(x_0) = \vartheta_{b,1}(x_0),
\end{cases}$$

(7.3)

where $\gamma_1 > 0$ and $\gamma_2 > 0$ are some constants, $M(x_0)$ is a constant such that the Boussinesq relation $\rho + \vartheta = \text{constant}$,

(7.4)

and the normalization condition

$$\int_{\Omega} \int_{\mathbb{R}^2} F(x, \tilde{v}) \mu^\frac{1}{2}(\tilde{v}) d\tilde{v} d\tilde{x} = 0,$$

(7.5)

hold.

**Proof.** The asymptotic analysis already reveals that the construction of the interior solution and boundary layer is valid. Here, we focus on the remainder estimates. We divide the proof into several steps:

Step 1: Remainder definitions.

Define the remainder as

$$R = \frac{1}{\epsilon^3} \left( f^* - \left( \epsilon F_1 + \epsilon^2 F_2 + \epsilon^3 F_3 \right) - \left( \epsilon \mathfrak{F}_1 + \epsilon^2 \mathfrak{F}_2 \right) \right) = \frac{1}{\epsilon^3} \left( f^* - Q - \mathcal{L} \right),$$

(7.6)

where

$$Q = \epsilon F_1 + \epsilon^2 F_2 + \epsilon^3 F_3,$$

(7.7)

$$\mathcal{L} = \epsilon \mathfrak{F}_1 + \epsilon^2 \mathfrak{F}_2.$$  

(7.8)

In other words, we have

$$f^* = Q + \mathcal{L} + \epsilon^3 R.$$  

(7.9)

We write $\mathcal{L}$ to denote the linearized Boltzmann operator as follows:

$$\mathcal{L}[f] = \epsilon \tilde{v} \cdot \nabla_x u + \mathcal{L}[f]$$

$$= v_\eta \frac{\partial f}{\partial \eta} - \frac{\epsilon}{R_\kappa - \epsilon \eta} \left( v_\eta^2 \frac{\partial f}{\partial v_\eta} - v_\eta v_\phi \frac{\partial f}{\partial v_\phi} \right) - \frac{\epsilon}{R_\kappa - \epsilon \eta} \frac{R_\kappa}{(r^2 + r_x^2)^{\frac{3}{2}}} v_\phi \frac{\partial f}{\partial \theta} + \mathcal{L}[f].$$

(7.10)

Step 2: Representation of $\mathcal{L}[R]$.

The equation (1.18) is actually

$$\mathcal{L}[f^*] = \Gamma[f^*, f^*],$$

(7.11)
which means

\[ \mathcal{L}[Q + D + e^3R] = \Gamma[Q + D + e^3R + Q + D + e^3R]. \]  

(7.12)

Note that the nonlinear term can be decomposed as

\[ \Gamma[Q + D + e^3R + Q + D + e^3R] = e^3 \Gamma[R, R] + 2e^3 \Gamma[Q, Q + D] + \Gamma[Q + D, Q + D]. \]  

(7.13)

The interior contribution can be represented as

\[ \mathcal{L}[Q] = e^4 \epsilon \cdot \nabla_x (eF_1 + e^2F_2 + e^3F_3) + \mathcal{L}[eF_1 + e^2F_2 + e^3F_3] \]  

(7.14)

\[ = e^4 \epsilon \cdot \nabla_x F_3 + e^2 \Gamma[F_1, F_1] + 2e^3 \Gamma[F_1, F_1]. \]

The nonlinear term will be handled by \( \Gamma[Q + D, Q + D] \). On the other hand, we consider the boundary layer contribution. Since \( \mathcal{F}_1 = 0 \), we may directly compute

\[ \mathcal{L}[D] = e^2 \left( v_\nu \left( \frac{1}{R_\nu} - \frac{\epsilon}{\epsilon \eta} \left( \epsilon \nu \phi \left( \frac{\partial \mathcal{F}_2}{\partial \phi} \right) - \frac{\epsilon}{\epsilon \eta} \left( R_\eta \left( \frac{\partial \mathcal{F}_2}{\partial \eta} \right) + \mathcal{L}[\mathcal{F}_2] \right) \right) \right) \]  

(7.15)

\[ = - \frac{e^4}{R_\nu - \epsilon \eta \left( r^2 + r^2 \right)} v_\nu \frac{\partial \mathcal{F}_2}{\partial \theta}. \]

Therefore, we have

\[ \mathcal{L}[R] = e^3 \Gamma[R, R] + 2e^3 \Gamma[R, Q + D] + S_1 + S_2, \]  

(7.16)

where

\[ S_1 = - e^3 \epsilon \cdot \nabla_x F_3 + \frac{1}{R_\nu - \epsilon \eta \left( r^2 + r^2 \right)} v_\nu \frac{\partial \mathcal{F}_2}{\partial \phi}, \]

(7.17)

\[ S_2 = 2e^3 \Gamma[F_1, F_2] + 2e^3 \Gamma[F_1, F_3] + e^3 \Gamma[F_2, F_2] + 2e^3 \Gamma[F_2, F_1] + e^3 \Gamma[F_3, F_3]. \]  

(7.18)

Step 3: Representation of \( R - \mathcal{P}[R] \).

Since

\[ f'(\bar{x}_0, \bar{v}) = \mu_0'(\bar{x}_0, \bar{v}) \mu^{-1}(\bar{v}) \int_{\bar{u} \cdot \nu(\bar{x}_0) > 0} \mu^{-1}(\bar{u}) f'(\bar{x}_0, \bar{u}) |\bar{u} \cdot \nu(\bar{x}_0)| d\bar{u} + \mu^{-1}(\bar{v}) \left( \mu_0'(\bar{x}_0, \bar{v}) - \mu(\bar{v}) \right), \]

where both sides are linear, we may directly write

\[ R(\bar{x}_0, \bar{v}) - \mathcal{P}[R](\bar{x}_0) = H[R](\bar{x}_0, \bar{v}) + h(\bar{x}_0, \bar{v}), \]  

(7.19)

where

\[ H[R](\bar{x}_0, \bar{v}) = \left( \mu_0'(\bar{x}_0, \bar{v}) - \mu(\bar{v}) \right) \mu^{-1}(\bar{v}) \int_{\bar{u} \cdot \nu(\bar{x}_0) > 0} \mu^{-1}(\bar{u}) R(\bar{x}_0, \bar{u}) |\bar{u} \cdot \nu(\bar{x}_0)| d\bar{u}, \]  

(7.20)

and

\[ h(\bar{x}_0, \bar{v}) \]

\[ = \left( \mu_0'(\bar{x}_0, \bar{v}) - \mu(\bar{v}) \right) \mu^{-1}(\bar{v}) \int_{\bar{u} \cdot \nu(\bar{x}_0) > 0} \mu^{-1}(\bar{u}) F_3(\bar{x}_0, \bar{u}) |\bar{u} \cdot \nu(\bar{x}_0)| d\bar{u} + \left( \mu_0'(\bar{x}_0, \bar{v}) - \mu(\bar{v}) - \epsilon \mu_1(\bar{x}_0, \bar{v}) \right) \mu^{-1}(\bar{v}) \int_{\bar{u} \cdot \nu(\bar{x}_0) > 0} \mu^{-1}(\bar{u}) \epsilon^{-1} F_2 + \epsilon^{-1} F_3(\bar{x}_0, \bar{u}) |\bar{u} \cdot \nu(\bar{x}_0)| d\bar{u} \]

\[ + \left( \mu_0'(\bar{x}_0, \bar{v}) - \mu(\bar{v}) - \epsilon \mu_1(\bar{x}_0, \bar{v}) - \epsilon^2 \mu_2(\bar{x}_0, \bar{v}) \right) \mu^{-1}(\bar{v}) \]

\[ \int_{\bar{u} \cdot \nu(\bar{x}_0) > 0} \mu^{-1}(\bar{u}) \epsilon^{-3}(Q + D)(\bar{x}_0, \bar{u}) |\bar{u} \cdot \nu(\bar{x}_0)| d\bar{u} \]

\[ + \epsilon^{-3} \mu^{-1}(\bar{v}) \left( \mu_0'(\bar{x}_0, \bar{v}) - \mu(\bar{v}) - \epsilon \mu_1(\bar{x}_0, \bar{v}) - \epsilon^2 \mu_2(\bar{x}_0, \bar{v}) - \epsilon^3 \mu_3(\bar{x}_0, \bar{v}) \right). \]
Step 4: $L^{2m}$ Estimates of $R$.
Using Theorem 4.7, we have the $L^{2m}$ estimate of $R$
\begin{align}
\frac{1}{\epsilon} \| (I - P) [R] \|_{L^{2}} + \frac{1}{\epsilon^2} |(I - P) [R]|_{L^{2}} + \| P [R] \|_{L^{2m}} \\
\leq C \left( o(1) \epsilon^{\frac{1}{2}} \| R \|_{L^{\infty}} + \frac{1}{\epsilon^2} \| \mathcal{L} [R] \|_{L^{\frac{2m-1}{2}}} + \frac{1}{\epsilon} \| \mathcal{L} [R] \|_{L^{2}} + |R - P [R]|_{L^{m}} + \frac{1}{\epsilon} |R - P [R]|_{L^{2}} \right) \\
\leq C \left( o(1) \epsilon^{\frac{1}{2}} \| R \|_{L^{\infty}} + \frac{1}{\epsilon^2} \| P [S] \|_{L^{2m}} \\
+ \frac{1}{\epsilon} \left( \| 3 \Gamma [R, R] \|_{L^{2}} + \| 2 \Gamma [R, Q + \mathcal{Q}] \|_{L^{2}} + \| S_{1} \|_{L^{2}} + \| S_{2} \|_{L^{2}} \right) \\
+ |H [R]|_{L^{m}} + |h|_{L^{m}} \\
+ \frac{1}{\epsilon} |H [R]|_{L^{2}} + \frac{1}{\epsilon} |h|_{L^{2}} \right).
\end{align}
(7.22)

Note that here we do not have other source terms in the $L^{2m-1}$ norm because for any $f, g \in L^{2}$,
$$P [\Gamma (f, g)] = 0.$$  
(7.23)

We need to estimate each term. It is easy to check
\begin{align}
\| \epsilon \bar{v} \cdot \nabla x F_{3} \|_{L^{2}} &\leq C \epsilon, \\
\| \epsilon \bar{v} \cdot \nabla x F_{3} \|_{L^{\frac{2m}{2m-1}}} &\leq C \epsilon,
\end{align}
(7.24)\hspace{1cm}(7.25)

and also by Theorem 0.7, using the rescaling and exponential decay, we have
\begin{align}
\left\| \frac{1}{R_{\kappa} - \epsilon \eta (r^{2} + r'^{2})^{\frac{1}{2}}} v_{\phi} \frac{\partial \mathcal{F}_{2}}{\partial \theta} \right\|_{L^{2}} \leq C \left( \int_{-\pi}^{\pi} \int_{0}^{R_{\kappa} - \eta} \left\| \frac{\partial \mathcal{F}_{2}}{\partial \theta} (\eta, \theta) \right\|_{L_{\infty, \phi}}^{2} d\eta d\theta \right)^{1/2} \\
\leq C \epsilon^{\frac{1}{2}} \left( \int_{-\pi}^{\pi} \int_{0}^{R_{\kappa} - \eta} \left\| \frac{\partial \mathcal{F}_{2}}{\partial \theta} (\eta, \theta) \right\|_{L_{\infty, \phi}}^{2} d\eta d\theta \right)^{1/2} \\
\leq C \epsilon^{\frac{1}{2}} \left( \int_{-\pi}^{\pi} \int_{0}^{R_{\kappa} - \eta} e^{-2K_{0} \eta [\ln (\epsilon)]^{2} e^{-2s} d\eta d\theta} \right)^{1/2} \\
\leq C \epsilon^{-\frac{1}{2} - s} [\ln (\epsilon)],
\end{align}
(7.26)

for some $0 < s << 1$. Similarly, we can prove that
\begin{align}
\left\| \frac{1}{R_{\kappa} - \epsilon \eta (r^{2} + r'^{2})^{\frac{1}{2}}} v_{\phi} \frac{\partial \mathcal{F}_{2}}{\partial \theta} \right\|_{L^{\frac{2m}{2m-1}}} \leq C \epsilon^{-\frac{1}{2m} - s} [\ln (\epsilon)].
\end{align}
(7.27)

In total, we have
\begin{align}
\| S_{1} \|_{L^{2}} \leq C \epsilon^{\frac{1}{2} - s} [\ln (\epsilon)], \\
\| P [S_{1}] \|_{L^{\frac{2m}{2m-1}}} \leq C \epsilon^{-\frac{1}{2m} - s} [\ln (\epsilon)].
\end{align}
(7.28)\hspace{1cm}(7.29)

On the other hand, by Lemma 2.4, we know
\begin{align}
\| 2 \Gamma [R, Q + \mathcal{Q}] \|_{L^{2}} \leq C \left( \| R \|_{L^{2}} \| Q + \mathcal{Q} \|_{L_{\infty, \phi}} + \| R \|_{L^{2}} \| Q + \mathcal{Q} \|_{L^{\infty}} \right).
\end{align}
(7.30)

Based on the smallness assumption (1.11) on the boundary Maxwellian, it is easy to check that
\begin{align}
\| R \|_{L^{2}} \| Q + \mathcal{Q} \|_{L_{\infty, \phi}} \leq o(1) \epsilon \| R \|_{L^{2}}.
\end{align}
(7.31)

Also, we may decompose
\begin{align}
\| R \|_{L^{2}} \| Q + \mathcal{Q} \|_{L^{\infty}} \leq \| (I - P) [R] \|_{L^{2}} \| Q + \mathcal{Q} \|_{L^{\infty}} + \| P [R] \|_{L^{2}} \| Q + \mathcal{Q} \|_{L^{\infty}} \\
\leq o(1) \epsilon \| (I - P) [R] \|_{L^{2}} + o(1) \epsilon \| P [R] \|_{L^{2}}.
\end{align}
(7.32)
Then we may derive that
\[
\|2\Gamma[R, Q + R]\|_{L^2} \leq o(1)\epsilon \left( \|P[R]\|_{L^2} + \|(I - P)[R]\|_{L^2} \right). \tag{7.33}
\]

Also, using the smallness assumption \([1.11]\) again, we can directly estimate
\[
|H[R]|_{L^m} \leq o(1)\epsilon \|R\|_{L^m} \leq o(1)\epsilon \|R\|_{L^\infty}, \tag{7.34}
\]
\[
|H[R]|_{L^2} \leq o(1)\epsilon \|P[R]\|_{L^2}. \tag{7.35}
\]

Using Lemma \([2.3]\) it is easy to check
\[
\|S_2\|_{L^2} \leq C\epsilon^{\frac{1}{2} - s} |\ln(\epsilon)|, \tag{7.36}
\]
\[
|h|_{L^m} \leq C\epsilon, \tag{7.37}
\]
\[
|h|_{L^2} \leq C\epsilon. \tag{7.38}
\]

Summarizing all of them, we have proved that
\[
\frac{1}{\epsilon} \|\Gamma[R]\|_{L^2} + \frac{1}{\epsilon^2} \|(1 - P)[R]\|_{L^2} + \|P[R]\|_{L^2} \tag{7.39}
\]
\[
\leq C \left( o(1)\epsilon^{\frac{1}{2} - s} \|R\|_{L^\infty} + \epsilon^{-1 - \frac{1}{2m} - s} |\ln(\epsilon)| \right.
\]
\[
+ \epsilon^2 \|\Gamma[R, R]\|_{L^2} + o(1) \|P[R]\|_{L^2} + o(1) \|(I - P)[R]\|_{L^2} + \epsilon^{\frac{1}{2} - s} |\ln(\epsilon)| \| \epsilon^{\frac{1}{2} - s} |\ln(\epsilon)|
\]
\[
+ o(1) \|R\|_{L^\infty} + \epsilon
\]
\[
+ o(1) \|P[R]\|_{L^2} + 1 \right). \]

Absorbing \(||(I - P)[R]\|_{L^2}||\) into the left-hand side, we obtain
\[
\frac{1}{\epsilon} \|\Gamma[R]\|_{L^2} + \frac{1}{\epsilon^2} \|(1 - P)[R]\|_{L^2} + \|P[R]\|_{L^2} \tag{7.40}
\]
\[
\leq C \left( o(1)\epsilon^{\frac{1}{2} - s} \|R\|_{L^\infty} + \epsilon^{-1 - \frac{1}{2m} - s} |\ln(\epsilon)| \right.
\]
\[
+ \epsilon^2 \|\Gamma[R, R]\|_{L^2} + o(1) \|P[R]\|_{L^2} + o(1) \|(I - P)[R]\|_{L^2} \right). \]

Here, we apply the estimate \([4.71]\) to bound
\[
|P[R]|_{L^2} \leq C \left( \|P[R]\|_{L^2} + \frac{1}{\epsilon^2} \||\Omega - P\|P[R]\|_{L^2} + \frac{1}{\epsilon^2} \left( \int_{\Omega \times R^2} |\Omega - P|^2 \right)^{\frac{1}{2}} \right) \tag{7.41}
\]
\[
\leq C \left( \|P[R]\|_{L^2} + \frac{1}{\epsilon^2} \||\Omega - P\|P[R]\|_{L^2} + \frac{1}{\epsilon^2} \left( \int_{\Omega \times R^2} |\Omega - P| |\Omega - P|^2 \right)^{\frac{1}{2}} \|P[R]\|_{L^2} \right). \]

Then using Lemma \([4.11]\) we obtain
\[
\|P[R]\|_{L^2} \leq C \left( \|(I - P)[R]\|_{L^2} + \frac{1}{\epsilon} \||\Omega - P\|P[R]\|_{L^2} + \frac{1}{\epsilon} \|P[R]\|_{L^2} + \|R - P[R]\|_{L^2} \right). \tag{7.42}
\]

Therefore, we have
\[
|P[R]|_{L^2} \leq C \left( \|P[R]\|_{L^2} + \frac{1}{\epsilon^2} \||\Omega - P\|P[R]\|_{L^2} + \frac{1}{\epsilon^2} \left( \int_{\Omega \times R^2} |\Omega - P|^2 \right)^{\frac{1}{2}} \right) \tag{7.43}
\]
\[
\leq C \left( \|(I - P)[R]\|_{L^2} + \frac{1}{\epsilon} \||\Omega - P\|P[R]\|_{L^2} \right.
\]
\[
+ \frac{1}{\epsilon} \left( \int_{\Omega \times R^2} |\Omega - P|^2 \right)^{\frac{1}{2}} \|P[R]\|_{L^2} + \frac{1}{\epsilon} \|P[R]\|_{L^2} + \|R - P[R]\|_{L^2} \left. \right) \right). \]

Note that the right-hand side of \([7.33]\) has been estimated in \([7.32]\) and \([7.30]\), so we obtain
\[
|P[R]|_{L^2} \leq C \left( o(1)\epsilon^{\frac{1}{2} - s} \|R\|_{L^\infty} + \epsilon^{-1 - \frac{1}{2m} - s} |\ln(\epsilon)| \right.
\]
\[
+ \epsilon^2 \|\Gamma[R, R]\|_{L^2} + o(1) \|P[R]\|_{L^2} + o(1) \|P[R]\|_{L^2} \right). \tag{7.44}
\]
Absorbing $|\mathcal{P}[R]|_{L^2_+}$ into the left-hand side, we obtain

$$
|\mathcal{P}[R]|_{L^2_+} \leq C \left( o(1) \epsilon \frac{1}{m^2} \| R \|_{L^\infty} + \epsilon^{-1 - \frac{1}{4m} - s} \| \ln(\epsilon) \| + \epsilon^2 \| \Gamma[R, R] \|_{L^2} + o(1) \| \mathcal{P}[R] \|_{L^2} \right). \tag{7.45}
$$

Plugging (7.45) into (7.40), we have

$$
\frac{1}{\epsilon} \| (\mathcal{I} - \mathcal{P})[R] \|_{L^2_+} + \frac{1}{\epsilon^2} \| (1 - \mathcal{P})[R] \|_{L^2_+} + \| \mathcal{P}[R] \|_{L^{2m}} \leq C \left( o(1) \epsilon \frac{1}{m^2} \| R \|_{L^\infty} + \epsilon^{-1 - \frac{1}{4m} - s} \| \ln(\epsilon) \| + \epsilon^2 \| \Gamma[R, R] \|_{L^2} + o(1) \| \mathcal{P}[R] \|_{L^2} \right). \tag{7.46}
$$

Note that the estimate of $\| \mathcal{P}[R] \|_{L^2}$ has been incorporated in above analysis for $|\mathcal{P}[R]|_{L^2_+}$, so we may further simplify

$$
\frac{1}{\epsilon} \| (\mathcal{I} - \mathcal{P})[R] \|_{L^2_+} + \frac{1}{\epsilon^2} \| (1 - \mathcal{P})[R] \|_{L^2_+} + \| \mathcal{P}[R] \|_{L^{2m}} \leq C \left( o(1) \epsilon \frac{1}{m^2} \| R \|_{L^\infty} + \epsilon^{-1 - \frac{1}{4m} - s} \| \ln(\epsilon) \| + \epsilon^2 \| \Gamma[R, R] \|_{L^2} \right). \tag{7.47}
$$

Step 5: $L^{\infty}$ Estimates of $R$

Based on Theorem 158, we have

$$
\left\| \langle \hat{v} \rangle^\theta e^{\hat{v}\hat{v}}^2 R \right\|_{L^\infty} + \left\| \langle \hat{v} \rangle^\theta e^{\hat{v}\hat{v}}^2 \right\|_{L^\infty} \leq C \left( \frac{1}{\epsilon} \| \mathcal{P}[R] \|_{L^2} + \frac{1}{\epsilon^2} \| (\mathcal{I} - \mathcal{P})[R] \|_{L^2} + \| \mathcal{P}[R] \|_{L^{2m}} \right) \tag{7.48}
$$

Hence, using (7.47), we know

$$
\left\| \langle \hat{v} \rangle^\theta e^{\hat{v}\hat{v}}^2 R \right\|_{L^\infty} + \left\| \langle \hat{v} \rangle^\theta e^{\hat{v}\hat{v}}^2 \right\|_{L^\infty} \leq C \left( o(1) \| R \|_{L^\infty} + \epsilon^{-1 - \frac{1}{4m} - s} \| \ln(\epsilon) \| + \epsilon^2 \| \Gamma[R, R] \|_{L^2} + \| \mathcal{P}[R] \|_{L^{2m}} \right) \tag{7.49}
$$

We can directly estimate

$$
\| \epsilon^2 \Gamma[R, R] \|_{L^2} \leq C \| \langle \hat{v} \rangle^\theta e^{\hat{v}\hat{v}}^2 \|_{L^\infty}^2 \tag{7.50}
$$

$$
\epsilon^3 \left\| \langle \hat{v} \rangle^\theta e^{\hat{v}\hat{v}}^2 \right\|_{L^\infty} \leq C \| \langle \hat{v} \rangle^\theta e^{\hat{v}\hat{v}}^2 \|_{L^\infty}^2 \tag{7.51}
$$

$$
\left\| \langle \hat{v} \rangle^\theta e^{\hat{v}\hat{v}}^2 \right\|_{L^\infty} \leq C \| \langle \hat{v} \rangle^\theta e^{\hat{v}\hat{v}}^2 \|_{L^\infty}^2 \tag{7.52}
$$
Also, we know

$$\left\| \langle \vec{v} \rangle^\theta e^{\varepsilon |\vec{v}|^2} S_1 \right\|_{L^\infty} \leq C, \quad \text{(7.53)}$$

$$\left\| \langle \vec{v} \rangle^\theta e^{\varepsilon |\vec{v}|^2} S_2 \right\|_{L^\infty} \leq C, \quad \text{(7.54)}$$

$$\left| \langle \vec{v} \rangle^\theta e^{\varepsilon |\vec{v}|^2} H[R] \right|_{L^\infty} \leq \varepsilon \left| \langle \vec{v} \rangle^\theta e^{\varepsilon |\vec{v}|^2} R \right|_{L^+}, \quad \text{(7.55)}$$

$$\left| \langle \vec{v} \rangle^\theta e^{\varepsilon |\vec{v}|^2} h \right|_{L^\infty} \leq \varepsilon. \quad \text{(7.56)}$$

Hence, in total, we have

$$\left\| \langle \vec{v} \rangle^\theta e^{\varepsilon |\vec{v}|^2} R \right\|_{L^\infty} + \left\| \langle \vec{v} \rangle^\theta e^{\varepsilon |\vec{v}|^2} R \right\|_{L^+} \leq C (o(1) \left\| \langle \vec{v} \rangle^\theta e^{\varepsilon |\vec{v}|^2} R \right\|_{L^\infty} + o(1) \left\| \langle \vec{v} \rangle^\theta e^{\varepsilon |\vec{v}|^2} R \right\|_{L^+} + \varepsilon^{1-\frac{1}{2m}} |\ln(\varepsilon)| + \varepsilon^2 \left\| \langle \vec{v} \rangle^\theta e^{\varepsilon |\vec{v}|^2} R \right\|_{L^\infty}^2).$$

Absorbing $o(1) \left\| \langle \vec{v} \rangle^\theta e^{\varepsilon |\vec{v}|^2} R \right\|_{L^\infty}$ and $o(1) \left\| \langle \vec{v} \rangle^\theta e^{\varepsilon |\vec{v}|^2} R \right\|_{L^+}$ into the left-hand side, we obtain

$$\left\| \langle \vec{v} \rangle^\theta e^{\varepsilon |\vec{v}|^2} R \right\|_{L^\infty} + \left\| \langle \vec{v} \rangle^\theta e^{\varepsilon |\vec{v}|^2} R \right\|_{L^+} \leq C \left( \varepsilon^{1-\frac{1}{2m}} |\ln(\varepsilon)| + \varepsilon^2 \left\| \langle \vec{v} \rangle^\theta e^{\varepsilon |\vec{v}|^2} R \right\|_{L^\infty}^2 \right), \quad \text{(7.58)}$$

which further implies

$$\left\| \langle \vec{v} \rangle^\theta e^{\varepsilon |\vec{v}|^2} R \right\|_{L^\infty} + \left\| \langle \vec{v} \rangle^\theta e^{\varepsilon |\vec{v}|^2} R \right\|_{L^+} \leq C \varepsilon^{1-\frac{1}{2m}} |\ln(\varepsilon)|, \quad \text{(7.59)}$$

for $\varepsilon$ sufficiently small. This means we have shown

$$\frac{1}{\varepsilon^3} \left\| \langle \vec{v} \rangle^\theta e^{\varepsilon |\vec{v}|^2} \left( f^\varepsilon - \left( \varepsilon F_1 + \varepsilon^2 F_2 + \varepsilon^3 F_3 \right) - \left( \varepsilon \mathcal{F}_1 + \varepsilon^2 \mathcal{F}_2 \right) \right) \right\|_{L^\infty} \leq C \varepsilon^{1-\frac{1}{2m}} |\ln(\varepsilon)|. \quad \text{(7.60)}$$

Therefore, we know

$$\left\| \langle \vec{v} \rangle^\theta e^{\varepsilon |\vec{v}|^2} \left( f^\varepsilon - \varepsilon F_1 - \varepsilon \mathcal{F}_1 \right) \right\|_{L^\infty} \leq C \varepsilon^{2-\frac{1}{2m}} |\ln(\varepsilon)|. \quad \text{(7.61)}$$

Since $\mathcal{F}_1 = 0$, then we naturally have for $F = F_1$.

$$\left\| \langle \vec{v} \rangle^\theta e^{\varepsilon |\vec{v}|^2} \left( f^\varepsilon - \varepsilon F \right) \right\|_{L^\infty} \leq C \varepsilon^{2-\frac{1}{2m}} |\ln(\varepsilon)|. \quad \text{(7.62)}$$

Here $0 < s << 1$, so we may further bound

$$\left\| \langle \vec{v} \rangle^\theta e^{\varepsilon |\vec{v}|^2} \left( f^\varepsilon - \varepsilon F \right) \right\|_{L^\infty} \leq C(\delta) \varepsilon^{2-\delta}, \quad \text{(7.63)}$$

for any $0 < \delta << 1$. Also, this justifies that the solution $f^\varepsilon$ to the equation (1.18) exists and is well-posed. The uniqueness and positivity follow from a standard argument as in [9]. \qed
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