Spontaneous Symmetry Breaking in Tensor Theories

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Abstract

In this work we study spontaneous symmetry breaking patterns in tensor models. We focus on the patterns which lead to effective matrix theories transforming in the adjoint of $U(N)$. We find the explicit form of the Goldstone bosons which are organized as matrix multiplets in the effective theory. The choice of these symmetry breaking patterns is motivated by the fact that, in some contexts, matrix theories are dual to gravity theories. Based on this, we aim to build a bridge between tensor theories, quantum gravity and holography.

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1 Introduction

Nowadays, there is a wave of excitement in the tensor field community [1–20] triggered by the remarkable solvability of certain models in the large $N$ [1–11]. Especially since they were discovered to be related to holography [21], in the context of SYK duality [22–30]. These models involve very particular “melonic” interactions, which permit to exactly solve the Schwinger-Dyson equation. It is clear that those models do not exhaust the vast spectrum of tensor theories. Restricting ourselves to those solvable models, many physically relevant cases could be left out.

A more general scenario will presumably involve interaction terms that would take the theory out of its large $N$ solvability. In order to tackle generic tensor theories other techniques will be necessary. It is likely that the addition of interaction terms could provide theories with spontaneous symmetry breaking (SSB).

The purpose of this paper is to kinematically explore the vacuum in tensor theories. For this, we study SSB scenarios, especially when they lead to effective (multi-)matrix theories transforming in the adjoint of $U(N)$, which are the Goldstone bosons. The choice of such SSB patterns is mainly motivated by the fact that matrix theories are related to quantum gravity and holography.

Instead of aiming to fully solve the tensor theory, in this work we shall initiate the programme of studying the effective (matrix) theory arising from SSB. Assuming that at low energy the Goldstone bosons associated with SSB are the only massless degrees of freedom, the dynamics of the tensor theory could be handled in terms of matrix theories. Actually, constructing phenomenological actions for Goldstone bosons has been very successful, for instance the chiral symmetry breaking in QCD. Conversely, it could be the case that computations in a given matrix theory are hard but the uplifted tensor theory is easier to handle.

Matrix fields have been proven to furnish a dual description [31–34] of a quantum gravity theory. In this work, we see how matrix fields come up as effective degrees of freedom of a tensor theory. It is reasonable to think that the tensor theory could actually be describing the same dual theory, although at a higher energy regime. Schematically,
The main result of the paper is (4.42) with (3.10). It encompasses the identification of the matrix fields as multiplets of Goldstone bosons which come from SSB, together with the choice of the parameter which induces and specifies the SSB pattern. Explicitly,

\[
(B^{\alpha,k'})_{ij}(x) = \frac{1}{2} \left( \Phi_{i_1...i_d}^{j_1...j_d}(x) - v_{i_1...i_d}^{j_1...j_d} \Phi_{i_1...i_d}^{j_1...j_d}(x) \right),
\]

with

\[
u_{i_1...i_d}^{j_1...j_d} = v^\sigma \delta_{j_1...j_d}^{i_1...i_d} + v^{\sigma'} \delta_{j_1...j_d}^{i_1...i_d},
\]

where \(B^{\alpha,k'}(x)\) are \(N \times N\)-matrix fields transforming in the adjoint of \(U(N)\). There will be several matrix fields labeled by \((\alpha, k')\), the meaning of which will be explained later. The tensor parameter which induces SSB is \(v_{i_1...i_d}^{j_1...j_d}\), and \((\sigma, \sigma')\) are the only two permutations which are sufficient to specify the pattern. The tensor field is denoted by \(\Phi_{i_1...i_d}^{j_1...j_d}(x)\).

As usual in color tensor theories [1–5], there is a different \(U(N)\) symmetry group associated to every index. The full symmetry group will be defined as \(G_{dd}(N) = \prod_{k=1}^d [U_k(N) \times U_k(N)]\). In this work we will not consider any definite form of the interaction potential. Instead, we will assume that there exists theories that present SSB patterns into diagonal subgroups of \(G_{dd}(N)\).

In the classical example of chiral symmetry breaking, the scalar field is \(\Phi_j^{ij}(x)\), the pattern is \(U(N)_L \times U(N)_R\) into the diagonal subgroup \(U(N)\), and the \(v\) tensor is \(v^i_j = v^j_i\) [35]. Unlike chiral symmetry breaking, the SSB patterns for tensor models present a much richer structure. Because of the tensor nature of the field \(\Phi_{i_1...i_d}^{j_1...j_d}(x)\), the generalization of the chiral symmetry breaking will correspond to the different diagonal subgroups that can be made by suitably splitting \(G_{dd}(N)\). Accordingly, each SSB pattern is associated with a partition of \(d\).

The SSB patterns into the diagonal subgroups are induced by the invariant tensor \(v\). If we wish to break into diagonal subgroups, the tensor \(v\) should be a linear combination of a product of \(d\) Kronecker deltas. There are \(d!\) different delta monomials (driven by all possible permutations of \(d\) indices) and a linear combination \(v\) would, in principle, involve all of them. This would be the natural generalization of the choice \(v^i_j = v^j_i\) for chiral symmetry breaking. Remarkably enough, the suitable choice of only two monomials, as in equation (3.10), can induce whatever SSB pattern of the type we are interested in, and it holds for tensor theories of any rank.

The above statement is rigorously proven in section 3.2. For that, we first note that a single delta monomial in the \(\epsilon\)-term halves \(G_{dd}(N)\) into diagonal subgroups. The linear combination of two (or more) different monomials reduces the remaining symmetry group even more, to the intersection among the subgroups associated with each monomial. These observations, inspired us to build diagrams that represent the intersection of the groups that result from the combination of two delta monomials. Conversely, from a given diagram one can read off the two constituent monomials, and
its SSB pattern. It turns out that any SSB pattern can be realized in the diagrams. Therefore, any SSB can be induced by only two monomials (3.10).

In order to explore the SSB patterns and, more importantly, to identify the Goldstone bosons as in (4.42), we introduce the $\epsilon$-term technique in the path integral for tensor theories. As in the classical examples [36, 37], the role of the $\epsilon$-term in tensor theories is to restrict the symmetry of the vacuum. Besides, it allows us to derive the Ward-Takahashi (WT) identities, which relate different Green functions of the theory with SSB. The WT identities reveal the explicit form of the Goldstone bosons in terms of the 2d-rank tensor field $\Phi(x)$.

There is an increasing interest in clarifying the role of tensor theories in the context of holography [21, 38], see also [10] and references therein. We hope that the study of SSB in tensor models helps in this direction. Moreover, in more general terms, SSB could be also implemented in theories which are known to be solvable in the large $N$. The theories do no need to be perturbatively stable, they just require the existence of stationary points of the potential, see, for instance, section 19.3 of [39] and p. 246 of [40]. There is no need, either, for the symmetry group to be unitary. Actually, since SYK involves real fields, $O(N)$ theories are becoming popular. As a remark, there is a resemblance between the zero temperature limit of certain $O(N)$ theories [41] and the material presented in this paper.

The paper is organized as follows. As a warm-up, in section 2 we introduce the $\epsilon$-technique by reviewing the complex scalar case and its kinematical SSB [36]. We also fix the notation and conventions we will use throughout the paper. In section 3 we explore the SSB patterns which lead to diagonal subgroups of $G_{2d}(N)$; we introduce the “intersection-diagrams” which correspond to the different SSB patterns; and in section 3.2, we use the diagrammatic correspondence to prove the statement that leads to equation (3.10). In section 4, we develop the general formalism of SSB in tensor theories. We implement the $\epsilon$-term technique in these theories as a generalization of the method applied for the scalar field, reviewed in section 2. As far as we know, this has not been reported in the literature yet. Using this formalism (in particular, the WT identities), we identify the Goldstone bosons associated to a particular SSB pattern and we present our main result in equation (4.42). The classical example of chiral symmetry breaking is treated in our formalism in section 5.1. In order to illustrate the general treatment in a less trivial case, we present the 4-rank tensor example in section 5.2. After the conclusions, in appendix A, we show an example of how this formalism applies to other symmetry groups and different SSB patterns. In particular, by minimizing the potential, we find the SSB patterns and derive the Goldstone bosons in the model with tetrahedral interaction [42].
2 General setup

As a warm-up we will first review SSB in the case of a complex scalar field [36]. We shall show the $\epsilon$-term technique in the path integral to induce SSB. It allows us to identify the Goldstone boson by means of the WT identities. The technique has a straightforward extension to higher rank tensor theories.

2.1 Complex scalar field and its SSB

For the study of the SSB in the complex scalar field theory we will require an action invariant under the phase transformation

$$\Phi(x) \rightarrow e^{i\alpha} \Phi(x), \quad (2.1)$$

being $\alpha$ an arbitrary real constant, i.e.,

$$S[\Phi(x)] = \int d^4x L[e^{i\alpha} \Phi(x)] = \int d^4x L[\Phi(x)], \quad (2.2)$$

with $L$ the Lagrangian. In order to derive the WT identities for the SSB we define the generating functional

$$Z[J, J] = \frac{1}{N} \int D\Phi D\overline{\Phi} \exp\left[iS[\Phi(x)] + i \int d^4x (\overline{\Phi}(x) \Phi(x) + J(x) \overline{\Phi}(x))\right], \quad (2.3)$$

where the functional integration is restricted to a vanishing field at infinity. We suppose that the measure is also invariant under (2.1), and $N = Z[0, 0]$. In contrast, we could define the generating functional with different boundaries conditions, i.e., with a constant field at infinity. This change in the usual boundary conditions leads to the SSB. In order to implement the boundary conditions in the path integral we can define the new generating functional $Z_{\epsilon}[\overline{J}, J]$,

$$Z_{\epsilon}[\overline{J}, J] = \frac{1}{N} \int D\Phi D\overline{\Phi} \exp\left[iS[\Phi(x)] + i \int d^4x (\overline{\Phi}(x) \Phi(x) + J(x) \overline{\Phi}(x)) - \epsilon \int d^4x |\Phi(x) - v|^2\right], \quad (2.4)$$

where the limit $\epsilon \rightarrow 0$ is understood at the end of a given calculation and $N = Z_{\epsilon}[0, 0]$.

To elucidate the role of the $\epsilon$-term let us specialize (2.4) on the Lagrangian $L = \frac{1}{2} \partial_\mu \Phi \partial^\mu \overline{\Phi} - V[\Phi, \overline{\Phi}]$. The generating functional in this case can be computed as

$$Z_{\epsilon}[\overline{J}, J] = \exp\left[-i \int d^4x V\left[-i \frac{\delta}{\delta J(x)} - i \frac{\delta}{\delta \overline{J}(x)}\right]\right] Z_{0,\epsilon}[\overline{J}, J], \quad (2.5)$$

where $Z_{0,\epsilon}[\overline{J}, J]$ is (2.4) with the free Lagrangian.

Now we can solve the integral using the condition of stationary phase for the free action. Proceeding in the standard fashion we define $\Phi(x) = \Phi_{cl}(x) + \chi(x)$, where

$$\Box \Phi_{cl}(x) = i\epsilon (\Phi_{cl}(x) - v) + J(x), \quad (2.6)$$
we find
\[ \Phi_d(x) = v + \int d^d y (-\mathcal{L})^{-1}_{xy} J(y). \] (2.7)

Now (2.4) takes the form
\[ Z_{0,\epsilon}[^{\mathcal{J}, J}] = \exp \left[ i \int d^d x (v^{\mathcal{J}}(x) + \overline{\Phi}(x)) \right] \exp \left[ \int d^d x \int d^d y J(x)(\Box - i\epsilon)^{-1}_{xy} J(y) \right]. \] (2.8)

Notice that \( \langle \Phi(x) \rangle = v \) and \( \langle \overline{\Phi}(x) \rangle = \overline{v} \) for the free theory, while for the interacting one the only claim we can make so far is \( \langle \Phi(x) \rangle \neq 0 \) and \( \langle \overline{\Phi}(x) \rangle \neq 0 \). The subindex \( \epsilon \) in the expectation values indicates that they are taken with respect to \( Z_{\epsilon}[^{\mathcal{J}, J}] \).

For obtaining the WT identities we make use of the invariance of the generating function (2.4), under the transformation (2.1). After implementing (2.1) in the generating functional, we compute
\[ \frac{\partial}{\partial \alpha} Z_{\epsilon}[^{\mathcal{J}, J}] = 0, \] (2.9)

and we obtain
\[ \langle \Phi(x) \rangle = \langle \overline{\Phi}(x) \rangle = -\epsilon \int d^d y \langle v^{\mathcal{J}}(x), \overline{\Phi}(y) - v^{\mathcal{J}}(y) \rangle. \] (2.10)

We will identify the Goldstone bosons by doing certain linear combinations of derivatives of the form (2.9). This way the expressions are easily generalizable to the non abelian and higher rank tensors cases. Defining the combinations
\[ [\overline{v} \frac{\partial}{\partial \alpha} Z_{\epsilon}[^{\mathcal{J}, J}] \pm v \frac{\partial}{\partial \alpha} Z_{\epsilon}[^{\mathcal{J}, J}]] = 0, \] (2.11)

we get
\[ \langle \overline{\Phi}(x) + v^{\mathcal{J}}(x) \rangle = -\epsilon \int d^d y \langle \overline{\Phi}(x) - v^{\mathcal{J}}(x), \overline{\Phi}(y) - v^{\mathcal{J}}(y) \rangle, \] (2.12)

and
\[ \langle \overline{\Phi}(x) - v^{\mathcal{J}}(x) \rangle = -\epsilon \int d^d y \langle \overline{\Phi}(x) + v^{\mathcal{J}}(x), \overline{\Phi}(y) - v^{\mathcal{J}}(y) \rangle. \] (2.13)

Defining
\[ \varphi(x) = \overline{\Phi}(x) + v^{\mathcal{J}}(x), \] (2.14)
\[ B(x) = \overline{\Phi}(x) - v^{\mathcal{J}}(x), \] (2.15)

we can rewrite (2.12) and (2.13) in a more suggestive form
\[ \langle \varphi(x) \rangle = -\epsilon \int d^d y \langle B(x), B(y) \rangle, \] (2.16)
and
\[
\langle B(x) \rangle_\epsilon = -\epsilon \int d^d y \langle \varphi(x), B(y) \rangle_\epsilon. \tag{2.17}
\]

The full propagator, considering the interactions, in (2.16) is given by
\[
\langle B(x), B(y) \rangle_\epsilon = i \int \frac{d^4 p}{(2\pi)^4} \frac{Z_B}{p^2 - m_B^2 + i\epsilon a_B} e^{ip(x-y) + \text{(regular contributions)}}, \tag{2.18}
\]
where we have assumed at first that the \( B(x) \) field could be massive, and \( Z_B \) and \( a_B \) are renormalization constants. Now, we plug the propagator (2.18) in (2.16). After integration, (2.16) reduces to
\[
\langle \varphi(x) \rangle_\epsilon = \frac{i\epsilon Z_B}{-m_B^2 + i\epsilon a_B}. \tag{2.19}
\]

Now, since we assume that the one point function does not vanish, we immediately conclude that \( m_B = 0 \). The field \( B(x) \) is the Goldstone boson associated to the broken symmetry (2.1). This way we have illustrated how, when SSB occurs, the Goldstone boson is identified using the \( \epsilon \)-term technique.

### 2.2 Conventions and notation

The theories we are considering in this paper are built on bosonic 2\( d \)-rank tensors, \( \Phi_{k_1...k_d}(x) \), transforming under the symmetry group
\[
G_{dd}(N) = \prod_{k=1}^{d}[U_k(N) \times U_k(N)], \tag{2.20}
\]
where \( U_k(N) \) and \( U_k(N) \) are different groups, acting on the upper and lower indices, respectively. A general element of \( G_{dd}(N) \) is
\[
(g_1, \ldots, g_d, g_1^*, \ldots, g_d^*). \tag{2.21}
\]

A field with one index downstairs will transform under \( U(N) \). For convenience we define the conjugate field with the index upstairs, which will transform under the same \( U(N) \). The transformation law will be
\[
\Phi'_l = g_l^\dagger \Phi_l \to \Phi g, \quad g \in U(N), \tag{2.22}
\]
\[
\Phi'^\dagger_l = g_l^\dagger \Phi_l \to g^\dagger \Phi, \quad g \in U(N). \tag{2.23}
\]

Note that, with this convention, the field conjugate \( \Phi \) will transform with the transpose conjugate\(^1\).

\(^1\)The transpose conjugate of \( g^\dagger_l \) is \( (g^\dagger_l)^t \). It is worth checking up the consistency of (2.22) and (2.23),
\[
\Phi g \to (\Phi^\dagger)^t_l = (g_l^\dagger)^t \Phi = (g_l^\dagger)^t \Phi^t \to g^\dagger \Phi. \]
This way, the contraction $\Phi \cdot \overline{\Phi} = \Phi_i \overline{\Phi}^i$ is invariant under $U(N)$. Note that if we had associated the field and the field conjugate both with the index downstairs the transformation law for the conjugate field would have been $\overline{g}$ instead of $g^\dagger$.

If the index of the field is upstairs, then the field and conjugate field transform as

$$
\Phi^i = g_l^i \Phi^l \rightarrow g^\dagger \Phi, \\
\overline{\Phi}^i = g_i^l \overline{\Phi}^l \rightarrow \overline{\Phi} g, \quad g \in U(N),
$$

so, $\overline{\Phi} \Phi^i$ is also an invariant. With this notation

- Upstairs indices always transform with the adjoint matrix.
- Contractions are always between downstairs and upstairs indices. The matrix multiplication is shown in the transformation laws above.

For a field with two indices we will have

$$
\Phi^i_j = (g^1_l)^i_j (g^2_l)^{jk} \Phi^k_j, \\
\overline{\Phi}^i_j = (g^1_l)^i_j (g^2_l)^{jk} \overline{\Phi}^k_j, \quad g_1, g_2 \in U_1(N), U_2(N).
$$

Notice that $g_1$ and $g_2$ are two different elements, which belong to two different groups. We could have called them $g_1$ and $g_2$, for instance, but we find the notation “1” and “2” convenient for the cases where more indices are involved. With this transformation law the contraction $\Phi \cdot \overline{\Phi} = \Phi_j^i \overline{\Phi}^j_i$ is invariant under $U_1(N) \times U_2(N)$.

The fields that we will consider are, in general, higher rank tensors $\Phi_{j_1 \ldots j_d}^{\phantom{j_1 \ldots j_d}i_1 \ldots i_d}(x)$, which transform under the group $G_{d\bar{d}}(N)$, defined in (2.20). They have $d$ indices upstairs and $d$ indices downstairs, each index transforming with a different $U(N)$ group. For tensors with $2d$ indices we have the transformation law

$$
\Phi_{j_1 \ldots j_d}^{i_1 \ldots i_d}(x)' = (g^1_l)^{i_1}_{j_1} \ldots (g^d_l)^{i_d}_{j_d} (g^1_{\bar{d}})^{i_1}_{\bar{d} l_1} \ldots (g^d_{\bar{d}})^{i_d}_{\bar{d} l_d} \Phi_{k_1 \ldots k_d}^{l_1 \ldots l_d}(x), \quad (2.26)
$$

$$
\overline{\Phi}_{i_1 \ldots i_d}^{j_1 \ldots j_d}(x)' = (g^1_l)^{i_1}_{j_1} \ldots (g^d_l)^{i_d}_{j_d} (g^1_{\bar{d}})^{i_1}_{\bar{d} l_1} \ldots (g^d_{\bar{d}})^{i_d}_{\bar{d} l_d} \overline{\Phi}_{k_1 \ldots k_d}^{l_1 \ldots l_d}(x), \quad (2.27)
$$

with $(g^1_1, \ldots, g^1_d, g^2_1, \ldots, g^2_d)$ an element of $G_{d\bar{d}}(N)$. Again, the contraction $\Phi \cdot \overline{\Phi} = \Phi_{j_1 \ldots j_d}^{i_1 \ldots i_d} \overline{\Phi}_{i_1 \ldots i_d}^{j_1 \ldots j_d}$ is invariant under the action of the group.

Now, suppose that the initial tensor theory is globally invariant under $G_{d\bar{d}}(N)$. This means that the Lagrangian, build on tensors $\Phi$ and $\overline{\Phi}$ which transform as (2.26) and (2.27), is invariant under $G_{d\bar{d}}(N)$. That is

$$
\mathcal{L}[\Phi(x)] = \mathcal{L}[\Phi'(x)]. \quad (2.28)
$$

Apart from the invariance under the gauge group $G_{d\bar{d}}(N)$, the details of the Lagrangian are irrelevant in the present discussion. We will assume that whatever the
Lagrangian is, it allows SSB. We will apply the same methodology as the scalar case, treated in section 2.

The path integral of the tensor theory is written as

$$Z_\epsilon[J, \mathcal{J}] = \frac{1}{\mathcal{N}} \int D\Phi D\bar{\Phi} \exp \left( i \int d^4x \left\{ \mathcal{L}[\Phi(x)] + \mathcal{J}(x) \cdot \Phi(x) + J(x) \cdot \bar{\Phi}(x) + i\epsilon |\Phi(x) - v|^2 \right\} \right),$$  \tag{2.29}

with

$$\mathcal{N} = Z_{[0,0]},$$  \tag{2.30}

and

$$|\Phi(x) - v|^2 = \Phi(x) \cdot \bar{\Phi}(x) - \Phi(x) \cdot \bar{v} - \bar{v} \cdot \Phi(x) + v \cdot \bar{v},$$  \tag{2.31}

where the dot indicates full contraction of indices,

$$X \cdot \bar{Y} = X^{i_1 \ldots i_d} Y_{i_1 \ldots i_d}.$$  \tag{2.32}

The interpretation of (2.29) is standard, with $J^{i_1 \ldots i_d}(x)$ and $\mathcal{J}^{j_1 \ldots j_d}(x)$ the external source fields which are used to extract the Green functions from (2.29) via derivation. For $\epsilon = 0$ the theory enjoys the full symmetry $G_{dd}(N)$. For $\epsilon \neq 0$, the term $|\Phi(x) - v|^2$ is chosen so as to break the $G_{dd}(N)$-symmetry, to induce SSB. After the calculation we take the limit $\epsilon \to 0$. The tensor $v^{i_1 \ldots i_d}$ is not spacetime-dependent. It may be thought as the boundary value of $\Phi(x)$ at infinity. When SSB occurs, the different degenerate vacua that the theory presents correspond to different choices of $v$. The effect of the $\epsilon$-term is to pick a particular vacuum of the theory. As said above, we are assuming that the Lagrangian presents SSB, which in this discussion means that the field configurations allow a non-zero $v$ value at infinity.

In this paper, we will be interested in $G_{dd}(N)$-invariant theories that spontaneously break into different patterns. The different choices of $v$ and their relation with the several patterns of symmetry breaking will be clarified in the next section.

3 Symmetry breaking patterns

Different choices of $v$ in the $\epsilon$-term lead to different symmetry breaking patterns. For instance, if $v = 0$ there is not symmetry breaking whereas for a generic $v$ the whole group $G_{dd}(N)$ gets spontaneously broken and there is no remaining continuous symmetry\(^2\). We will explore the choices of $v$ that induce SSB into diagonal subgroups

\(^2\)In more general SSB with generic $v$, some discrete symmetries might remain.
of $G_{dd}(N)$. The general idea is that $v$ must be an invariant of the subgroup we wish to break into. In this paper we are studying the patterns

$$G_{dd}(N) = \prod_{k=1}^{d}[U_k(N) \times U_k(N)] \longrightarrow G_\omega(N) = \prod_{\alpha=1}^{\omega} \text{Diag}[H_\alpha],$$  \hspace{1cm} (3.1)$$

where each $H_\alpha$ is a tensor product of $2n_\alpha$ different unitary groups ($n_\alpha$ barred and $n_\alpha$ unbarred) such that

$$H_\alpha \cap H_\beta = \emptyset, \quad \alpha \neq \beta, \quad \sum_{\alpha=1}^{\omega} (2n_\alpha) = 2d. \hspace{1cm} (3.2)$$

Be aware that each SSB pattern is associated with a partition of $d$ elements with $\omega$ parts, $\sum_{\alpha=1}^{\omega} n_\alpha = d$. Notice, as well, that $\text{Diag}[H_\alpha]$ is a unitary group and it will often be denoted $U_\alpha(N)$ along this paper. The SSB into the full diagonal group $G_1(N) = U(N)$ corresponds to $\omega = 1$, whereas for $\omega = d$ the remaining symmetry group is $G_d(N) = \prod_{\alpha=1}^{d} U_\alpha(N)$. The number of Goldstone boson that will result after a symmetry breaking $G \rightarrow H$ is counted by subtracting the number of generators of the initial and remaining symmetry groups, $d_G - d_H$. For the SSB pattern (3.1), we have

$$d_{G_{dd}(N)} - d_{G_\omega(N)} = 2dN^2 - \omega N^2 = (2d - \omega)N^2. \hspace{1cm} (3.3)$$

For the choices (3.1) the invariants under the corresponding subgroup can be constructed using Kronecker deltas. Thus, in general,

$$v_{j_1\ldots j_d} = \sum_{\sigma \in S_d} v_\sigma \delta_{j_1\ldots j_d}^{i_1\ldots i_d}, \hspace{1cm} (3.4)$$

will induce SSB into any $G_\omega(N)$ for suitable $v_\sigma$ parameters, where

$$\delta_{j_1\ldots j_d}^{i_1\ldots i_d} = \delta_{j_1}^{i_1} \cdots \delta_{j_d}^{i_d}. \hspace{1cm} (3.5)$$

To make the former statement clearer we will examine three examples; the case $d = 1$, for which $v_j \propto \delta_j^i$, and the case $d = 2$, for which $v_{j_1j_2} \propto \delta_{j_1j_2}^{i_1i_2}$ or $v_{j_1j_2} \propto \delta_{j_2j_1}^{i_1i_2}$. In the path integral, the $\epsilon$-term is of the form $\epsilon v \Phi$. We wish to know which subgroup of $G_{dd}(N)$ the $\epsilon$-term is invariant for a given $v$.

For $d = 1$, the tensor $\Phi$ transforms under $G_{11}(N) = U_1(N) \times U_1(N)$, The $\epsilon$-term is $\epsilon v_j^i \Phi_i^j \propto \delta_j^i \Phi_i^j$. It is clear in this simple case that the $\epsilon$-term, hence the path integral, is only invariant under the diagonal $U(N)$ subgroup of $G_{11}(N)$, i.e.,

$$\delta_j^i \Phi_i^j = \delta_j^i \delta_{l}^{i_1} \delta_{l}^{j_1} \Phi_i^{l}. \hspace{1cm} (3.6)$$

This tells us that the choice $v_j^i \propto \delta_j^i$ induce SSB into the diagonal subgroup of $G_{11}(N)$. We would like to stress that the role of $\delta$ is to link two indices, up and downstairs, which results in the identification of $U_1(N)$ and $U_1(N)$. 


In the case of $d = 2$, the choice $v_{j_1 j_2}^{i_1 i_2} \propto \delta_{j_1 j_2}^{i_1 i_2}$ will make the $\epsilon$-term invariant under the group

$$G_2(N) = \text{Diag}[U_1(N) \times U_1(N)] \times \text{Diag}[U_2(N) \times U_2(N)].$$

(3.7)

Using the notation (2.21), a general element of $G_2(N)$ is $(g, h, g, h)$. Whereas for $v_{j_1 j_2}^{i_1 i_2} \propto \delta_{j_2 j_1}^{i_1 i_2}$, the $\epsilon$-term will be invariant under

$$G_2'(N) = \text{Diag}[U_1(N) \times U_1(N)] \times \text{Diag}[U_2(N) \times U_1(N)],$$

(3.8)

where a general element of $G_2'(N)$ can be written as $(g, h, h, g)$.

The next step is to consider the linear combination $v_{j_1 j_2}^{i_1 i_2} = v_1 \delta_{j_1 j_2}^{i_1 i_2} + v_2 \delta_{j_2 j_1}^{i_1 i_2}$. The $\epsilon$-term will be invariant under the intersection $G_2(N) \cap G_2'(N)$, whose general element is $(g, g, g, g)$, that is, an element of $G_1(N)$.

In terms of $v_\sigma$ the simplest cases to study are breaking patterns into $G_d(N)$. There are many different $v$’s which make that job, one for each monomial $\delta_{j_1^{(1)} \ldots j_{d^{(1)}}}^{i_1^{(1)} \ldots i_{d^{(1)}}}$. For $N \geq d$ there is one monomial per permutation $\sigma \in S_d$. So and overall of $d!$ monomials and, therefore, $d!$ different patterns. If $N < d$ then not all of them are linearly independent and the number of SSB patterns into $G_d(N)$ are counted by the formula

$$\text{Number of linearly independent monomials} = \sum_{R \vdash d, l(R) \leq N} d_R^n,$$

(3.9)

where $d_R$ is the dimension of the irrep of $S_d$. $R \vdash d$ tells that the Young diagram $R$ has $n$ boxes and $l(R)$ is the number of rows of $R$. Formula (3.9) counts the number of linearly independent monomials for any $N$. However, the interest is usually focused on large $N$ theories. In these cases, the condition $l(R) \leq N$ is fulfilled and the sum (3.9) is always $d!$.

If we wish to induce SSB into any other subgroup of the type $G_\omega(N)$ we need to consider a sum of monomials as indicated in (3.4). We shall see that there is a minimal choice of $v$ for any symmetry breaking pattern. Specifically, and this is one of the main results of the paper, we are showing in the next sections that $v$ built on just two summands in (3.4),

$$v_{j_1 \ldots j_d}^{i_1 \ldots i_d} = v_{\sigma} \delta_{j_1^{(1)} \ldots j_{d^{(1)}}}^{i_1^{(1)} \ldots i_{d^{(1)}}} + v_{\sigma'} \delta_{j_1^{(1)} \ldots j_{d^{(1)}}}^{i_1^{(1)} \ldots i_{d^{(1)}}},$$

(3.10)

appropriately chosen, induces any pattern of SSB in (3.1). For this purpose we will present a visual diagrammatic correspondence between the choices of $v$ and the SSB patterns.

---

Note that the irreducible representations (irreps) of $S_d$ are labeled by Young diagrams with $d$ boxes.
3.1 Diagrams

As commented above, the role of delta monomials is to identify up and downstairs groups. This suggests the following diagrammatical correspondence

\[
\delta_{\sigma(1)\ldots \sigma(d)} \rightarrow g_1 \cdots g_d | \cdots | .
\]

(3.11)

For \( d = 2 \), the two deltas that yield to the different symmetry breaking patterns shown in (3.7) and (3.8) can be mapped into the diagrams

\[
\delta_{j_1j_2} \rightarrow g_1 g_2 | \cdots | g_1 g_2, \quad \delta_{j_2j_1} \rightarrow g_1 g_2 | \cdots | .
\]

(3.12)

As said before, the linear combination of deltas comes along with the intersection of the groups that each delta induces. Precisely, it is when considering intersections where the diagrams will show their usefulness.

Diagrams representing intersections, which will be called “intersection-diagrams,” are constructed by concatenation of two diagrams of the type (3.11), and then joining the equal elements upstairs and their “bar” counterparts downstairs. This is exemplified in Fig.1, which represents the intersection of two monomials for \( d = 2 \).

\[
v_1 \delta_{j_1j_2} + v_2 \delta_{j_2j_1} \rightarrow \begin{array}{c}
\begin{array}{c}
| \\
g_1 \\
g_2 \\
g_2 \\
g_1 \\
\end{array}
\end{array}
\]

Figure 1. Intersection-diagram for \( d = 2 \) with \( \sigma = (1)(2) \) and \( \sigma' = (12) \).

The diagram of Fig.1 displays just one cycle, which is in correspondence with the only \( U(N) \) group that results from \( G_2(N) \cap G'_2(N) \). This is not a coincidence. It turns out that the cycle structure of the intersection-diagrams\(^4\) corresponds with the different symmetry breaking patterns. Therefore, the number of cycles of the

\(^4\)We will refer as the cycle structure the set of all loops, together with their length, that fully connect the diagram. Length is the number of elements the loop involves. Cycle structures are in one-to-one correspondence with partitions of \( d \).
diagram will be \( \omega \) in the remaining group \( G_\omega(N) \). This statement will be expanded in the subsequent paragraphs.

As a consistency check of the diagrammatic correspondence, let us see how it works when \( v \) is just one monomial, say, \( v_{i_j,j_2} = v_1 \delta_{j_1,j_2} \). This choice breaks into the group \( G_2(N) \) in the equation (3.7), which corresponds to the first diagram of (3.12). Trivially, \( v \) may be written as the sum \( \frac{1}{2} (v_1 \delta_{j_1,j_2} + v_1 \delta_{i_1,i_2}) \). On the one hand, we have two (same) monomials. In this case the sum is interpreted as a SSB term to the group \( G_2(N) \cap G_2(N) \). On the other hand, using the rules of the intersection discussed above, we get the diagram in Fig.2, which is a diagram with two cycles.

\[ \delta_{j_1,j_2} + \delta_{i_1,i_2} \quad \rightarrow \quad \begin{array}{cccc}
  g_1 & g_2 & g_1 & g_2 \\
  | & | & | & . \\
  g_1 & g_2 & g_1 & g_2 \\
\end{array} \]

**Figure 2.** Intersection-diagram of \( G_2(N) \cap G_2(N) \).

This tells us that the remaining symmetry group is \( G_2(N) \), in perfect agreement with \( G_2(N) \cap G_2(N) = G_2(N) \).

Let us discuss the case \( d = 3 \). As we show in Fig.3, for different choices of \( \sigma \) and \( \sigma' \) the diagram will display different cycle structures, which are not shown in Fig.3 because \( \sigma \) and \( \sigma' \) have not been specified yet. We will explore some explicit examples in the following.

\[ v_1 \delta_{j_{\sigma(1)},j_{\sigma(2)},j_{\sigma(3)}} + v_2 \delta_{j_{\sigma'(1)},j_{\sigma'(2)},j_{\sigma'(3)}} \quad \rightarrow \quad \begin{array}{cccccc}
  g_1 & g_2 & g_3 & g_1 & g_2 & g_3 \\
  | & | & | & | & | \\
  g_{\sigma(1)} & g_{\sigma(2)} & g_{\sigma(3)} & g_{\sigma'(1)} & g_{\sigma'(2)} & g_{\sigma'(3)} \\
\end{array} \]

**Figure 3.** Intersection-diagram for \( d = 3 \). Different choices of \( \sigma \) and \( \sigma' \) will lead to different cycle structures.
In Fig.4 we show the diagram of \( d = 3 \) where \( \sigma \) is the identity and \( \sigma' \) is a transposition. The identity induces the SSB into

\[
G_3(N) = \text{Diag}[U_1(N) \times U_1(N)] \times \text{Diag}[U_2(N) \times U_2(N)] \times \text{Diag}[U_3(N) \times U_3(N)]. \tag{3.13}
\]

Whereas \( \sigma' = (12)(3) \) induces the SSB into

\[
G'_3(N) = \text{Diag}[U_1(N) \times U_2(N)] \times \text{Diag}[U_2(N) \times U_1(N)] \times \text{Diag}[U_3(N) \times U_3(N)]. \tag{3.14}
\]

Now,

\[
G_3(N) \cap G'_3(N) = \text{Diag}[U_1(N) \times U_1(N) \times U_2(N) \times U_2(N)] \times \text{Diag}[U_3(N) \times U_3(N)] = U(N) = G_2(N). \tag{3.15}
\]

A detailed description of the symmetry breaking pattern is written in the second line of equation (3.15). Note that this information is encoded in the diagram in Fig.4 and can be easily read off.

![Diagram](https://example.com/diagram.png)

**Figure 4.** Diagrammatic representation of two deltas, with \( \sigma = (1)(2)(3) \) and \( \sigma' = (12)(3) \).

As a last example, we will regard the \( d = 3 \) with \( \sigma = (1)(23) \) and \( \sigma' = (12)(3) \). The monomial with the permutation \( \sigma \) induces SSB into

\[
G_3(N) = \text{Diag}[U_1(N) \times U_1(N)] \times \text{Diag}[U_2(N) \times U_3(N)] \times \text{Diag}[U_3(N) \times U_2(N)]. \tag{3.16}
\]

Whereas the monomial with \( \sigma' = (12)(3) \) leads to

\[
G'_3(N) = \text{Diag}[U_1(N) \times U_3(N)] \times \text{Diag}[U_2(N) \times U_1(N)] \times \text{Diag}[U_3(N) \times U_3(N)]. \tag{3.17}
\]

Now,

\[
G_3(N) \cap G'_3(N) = \text{Diag}[U_1(N) \times U_1(N) \times U_2(N) \times U_2(N) \times U_3(N) \times U_3(N)] = U(N) = G_1(N). \tag{3.18}
\]
Figure 5. Intersection-diagram corresponding to $\sigma = (1)(23)$ and $\sigma' = (12)(3)$. It displays only one cycle, which indicates that it breaks into the diagonal group.

For this case the SSB into the diagonal group is in perfect agreement with the diagram in Fig.5 which shows only one cycle.

With these examples we have shown in detail how a particular SSB pattern is associated with an intersection-diagram. Moreover, for $G_\omega(N)$,

$$\omega = \text{number of cycles of the intersection-diagram.} \quad (3.19)$$

3.2 Parameter space of SSB

Now, using the intersection-diagrams we can prove the statement leading to (3.10). That is, there exists a minimal choice of $v$ for any symmetry breaking pattern, namely, two summands. Be aware that because two deltas fix uniquely the intersection-diagram, in order to complete the proof it is sufficient to see that for an arbitrary SSB pattern there is always (at least) one intersection-diagram associated to it.

Note that any SSB can be associated to a partition of $d$, given by the $n_\alpha$ values. However, the choice of a partition does not specify completely the SSB pattern.

In order to associate an intersection-diagram to (3.1) we should proceed as follows. First, we draw a plain (with no numbers) diagram

\[
g \ldots g | g \ldots g \\
| \ldots | \ldots | \\
g \ldots g | g \ldots g \\
\hline
2d \text{ slots}
\]

(3.20)

Second, we draw in the diagram the cycle structure associated to a given SSB pattern, which can be read off from the set $\{H_\alpha\}$. For each of the $\omega H_\alpha$ groups we join $n_\alpha$ slots from the LHS of (3.20) with $n_\alpha$ slots of the RHS of (3.20) in a single loop. This turns (3.20) into a plain cycle-structured diagram with closed loops. We would like to emphasize that such procedure always fully connect the plain diagram. This
happens because \( n_1 + n_2 + \cdots + n_\omega = d \) is a partition of \( d \), and the groups \( H_\alpha \) do not intersect each other, as stated in (3.2).

The next step is to complete the diagram by writing the subscript of each element according to the given SSB pattern. For this purpose we focus on the cycle associated to \( H_\alpha \) and write, as subscripts, the labels of the different unitary groups \( H_\alpha \) contains. For instance, if \( n_\alpha = 2 \), we have the generic group

\[
H_\alpha = U_a \times U_b \times U_c \times U_d. \tag{3.21}
\]

Then the labelling process on the cycle of the diagram corresponding to \( H_\alpha \) can be chosen as

\[
g g g g \quad \rightarrow \quad g_a g_c g_d g_b
\]

A complete intersection-diagram is obtained by applying the same labelling prescription to each \( H_\alpha \). Now, the two delta monomials (3.10) can be read off from the full diagram. This concludes the proof.

As a remark, notice that the process of labelling is not unique. In general, there will be several pairs of deltas that would induce the same SSB pattern. This fact does not affect the conclusion of our argument since the aim is to show that any SSB pattern can be induced only with two parameters, as in (3.10).

As a second remark, from the two permutations \( \sigma \) and \( \sigma' \) in (3.10), there is a straight way of reading the partition associated to a SSB pattern, which is a valuable information of the SSB pattern. As seen in this section, the partition of the SSB pattern is in correspondence with the cycle structure of the intersection-diagram.

Now, it is interesting to read the intersection-diagrams as a composition of permutations. To make this point clearer let us picture the composition of two permutations that we used in the examples in Fig.4 and Fig.5

\[
\sigma'^{-1} \cdot \sigma = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}^{\cdot} = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 3 & 2 \end{pmatrix}, \quad \sigma'^{-1} \cdot \sigma = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}^{\cdot} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \tag{3.22}
\]

respectively. Note that the compositions pictured in (3.22) can be traced in the diagrams. The composition is realized as we follow a path starting on the upper right side of the diagram, move downstairs, then towards the left side and ending
on the upper left side of the diagram. From this observation we conclude that the 
cycle structure of the diagram coincides with the cycle structure of the permutation 
\( \sigma' - 1 \cdot \sigma \). Recall that the cycle structure of a permutation is indeed a partition \([43]\).

4 Matrix fields as Goldstone bosons

The next step in our discussion is to identify the Goldstone bosons associated to 
a certain SSB pattern which, as discussed in the previous section, is induced by a 
non-vanishing \( v \). To this aim we will derive the so-called WT identities which are 
identities among Green functions that arise from the path integral \((2.29)\) with the 
\( \epsilon \)-term above defined. This will be a generalization of the method exposed in section 
2.

4.1 Identities from the symmetry of the path integral

First, notice that path integral \((2.29)\) is invariant under the change of variables 
\( \Phi(x) \rightarrow \Phi'(x) \) and \( \overline{\Phi}(x) \rightarrow \overline{\Phi}'(x) \) as in \((2.26)\) and \((2.27)\), respectively. Let us write 
the elements of the special unitary group in exponential form as

\[
g_k = e^{i\theta_k^a T_a}, \quad g_{\bar{k}} = e^{i\theta_{\bar{k}}^a T_a}, \quad (4.1)
\]

where \( T_a \) are the generators of the algebra \( \mathfrak{u}(N) \). The labels \( k \) and \( \bar{k} \) have been added 
to indicate the slot in the tensor indices the element \( g_k \) is acting on. The algebra 
\( \mathfrak{u}(N) \) is generated by \( t_a \) which denote the generators of \( \mathfrak{su}(N) \) plus the identity, i.e., 
\( T_a = (\frac{1}{\sqrt{2N}} \mathds{1}, t_a) \). We will also choose the generators normalized as \( \text{Tr}(T_a T_b) = \frac{1}{2} \delta_{ab} \).

We now proceed to apply the invariance of the path integral under the field 
change. With the parametrization \((4.1)\), it is expressed as

\[
\frac{\partial Z(J(x), J(x))}{\partial \theta_k^a} = \frac{\partial Z(J(x), J(x))}{\partial \theta_{\bar{k}}^a} = 0, \quad (4.2)
\]

where \( k \) and \( \bar{k} \) refer to the \( 2d \) transformations

\[
\Phi_{j_1, \ldots, j_d}^{i_1, \ldots, i_d}(x) \rightarrow (g_k)^{i_k}_{j_k} \Phi_{j_1, \ldots, j_d}^{i_1, \ldots, i_d}(x), \quad k = 1, \ldots, d, \\
\Phi_{j_1, \ldots, j_d}^{i_1, \ldots, i_d}(x) \rightarrow (g_{\bar{k}})^{i_{\bar{k}}}_{j_{\bar{k}}} \Phi_{j_1, \ldots, j_d}^{i_1, \ldots, i_d}(x), \quad \bar{k} = 1, \ldots, d, \quad (4.3)
\]

which lead, for \( \theta_k^a \), to the collection of identities

\[
\int d^4x (\overline{\Phi}(x) T_a^{(k)} \Phi(x) - \overline{\Phi}(x) T_a^{(k)} \Phi(x) - i\epsilon \Phi(x) T_a^{(k)} \overline{\Phi}(x) - i\epsilon \Phi(x) T_a^{(k)} \overline{\Phi}(x)) J_{\epsilon} = 0. \quad (4.4)
\]

and, derivating with respect to \( \theta_{\bar{k}}^a \) leads to

\[
\int d^4x (\Phi(x) T_a^{(k)} \overline{\Phi}(x) - \Phi(x) T_a^{(k)} \overline{\Phi}(x) - i\epsilon \Phi(x) T_a^{(k)} \overline{\Phi}(x) - i\epsilon \Phi(x) T_a^{(k)} \overline{\Phi}(x)) J_{\epsilon} = 0. \quad (4.5)
\]
We have used the notation
\[ \langle F(\Phi) \rangle_{J, \epsilon} = \frac{1}{N} \int D\Phi D\overline{\Phi} F(\Phi) \exp \left( i \int d^4 x \left\{ \mathcal{L}(\Phi(x)) + J(x) \cdot \Phi(x) + \overline{\Phi}(x) + i \epsilon |\Phi(x) - v|^2 \right\} \right) \],
(4.6)
and the shorthand notation for \( * = k, \bar{k}, \)
\[ XT^{(s)}_a Y = X^{j_1 \ldots j_d}_{i_1 \ldots i_d} (T_a)^i_{j_1 \ldots j_d} Y^{i_1 \ldots i_d}_{j_1 \ldots j_d}. \]
(4.7)
For later convenience, let us define
\[ B_a^k(x) = v^T (k) a \Phi(x) - \Phi(x) v (k) a, \]
\[ B_a^{\bar{k}}(x) = \Phi(x) v (\bar{k}) a - v^T (\bar{k}) a \Phi(x), \quad k, \bar{k} = 1, \ldots, d. \]
(4.8)
As we will see later on, the fields \( B_a^k(x) \) and \( B_a^{\bar{k}}(x) \) are indeed the Goldstone bosons, intimately related to the matrix modes. The Goldstone boson definition (4.8) is a general result, which holds for any Lie group of symmetry.

When we consider SSB into \( G_\omega(N) \) as in (3.1) there will be \( \omega \) constraints. These constraints appear as we perform the transformation of the field according to the SSB pattern \( \prod_{\alpha=1}^k \text{Diag}[H_\alpha] \), and derive with respect to the parameters. Since the diagonal group \( \text{Diag}[H_\alpha] \) "identifies" the unitary groups within it, there will be only one parameter for each diagonal group. So, for a SSB pattern we have a collection of parameters \( \{ \theta^a_\alpha | \alpha = 1, \ldots, \omega \} \). The \( \omega \) constraints are obtained from
\[ \frac{\partial Z(J(x), \overline{J}(x))}{\partial \theta^a_\alpha} = 0, \quad \alpha = 1, \ldots, \omega. \]
(4.9)
Now, let us apply the transformations of the diagonal group defined as
\[ \Phi^{i_1 \ldots i_d}_{j_1 \ldots j_d}(x) \rightarrow g_{m_1}^{i_1} \ldots g_{m_d}^{i_d} (g^d)^{j_1}_{i_1} \ldots (g^d)^{j_d}_{i_d} \Phi^{m_1 \ldots m_d}(x), \]
(4.10)
for elements \( g \) of the diagonal action of \( U(N) \) on \( \Phi \). The transformation (4.10) and the subsequent derivations (4.2), but in this case with \( \theta^a_\alpha = \cdots = \theta^a_\alpha = \theta^a_\alpha = \cdots = \theta^a_\alpha = \theta_a, \)
\[ \frac{\partial Z(J(x), \overline{J}(x))}{\partial \theta^a_\alpha} = 0, \]
(4.11)
lead to the “diagonal” constraint
\[ \sum_{k=1}^d \int d^4 x \langle J(x) T_a^{(k)} \Phi(x) - \overline{\Phi}(x) T_a^{(k)} J(x) \rangle_{J, \epsilon} - \]
\[ \sum_{k=1}^d \int d^4 x \langle \Phi(x) T_a^{(k)} \overline{J}(x) - J(x) T_a^{(k)} \overline{\Phi}(x) \rangle_{J, \epsilon} = 0, \]
(4.12)
where we have used the property $T_a^\dagger = T_a$. Be aware that no terms depending on $\epsilon$ do appear in the equation (4.12) in contrast to equations (4.4) and (4.5). This happens because we are assuming that for certain shapes of $v$, as discussed in section 3, the term $\epsilon|\Phi(x) - \nu|^2$ is invariant under the diagonal transformation. The non-appearance of the $\epsilon$-term is common for other SSB patterns, since it only reflects the fact that the $\epsilon$-term is invariant under the remaining group.

Using (4.4) and (4.5) we can rewrite the constraint (4.12) as

$$
\sum_{k=1}^{d} \int d^4x \langle (\nabla T_a^{(k)})\Phi(x) - \Phi(x)T_a^{(k)}v \rangle_{J,\epsilon} - \sum_{k=1}^{d} \int d^4x \langle (\Phi(x)T_a^{(k)}v - vT_a^{(k)}\Phi(x)) \rangle_{J,\epsilon} = 0,
$$

which must be satisfied for all $\Phi$ and $\bar{\Phi}$ and $\forall a$. Using the definition of the Goldstone bosons (4.8), the constraint (4.13) reads

$$
\langle \int d^4x \left( \sum_{k=1}^{d} B_a^k(x) - \sum_{k=1}^{d} B_a^k(x) \right) \rangle_{J,\epsilon} = 0.
$$

Notice that (4.13) imposes strong restrictions on $v$. The invariant tensors $v$ considered in section 3 for SSB into the diagonal group fulfill equation (4.14), which can be taken as a consistency check. In section 4.4 we show, for a generic SSB (i.e., a generic $v$), how constraints of the type (4.14) are fulfilled.

### 4.2 The Ward-Takahashi identities

Now, the WT identities, which relate Green functions, are obtained by functional differentiating (4.4) and (4.5) repeatedly with respect to the sources $J(y)$ and $\bar{J}(y)$ and then setting $J(y) = \bar{J}(y) = 0$. For example, if we operate with $\frac{\delta}{\delta J_{m_1...m_d}(y)} |_{J(y)=\bar{J}(y)=0}$ on (4.4) and (4.5), respectively, we obtain

$$
(T_a)^j_{m_k} \langle \Phi_{m_1...m_d}^{n_1...n_d}(y) \rangle_{\epsilon} = -\epsilon \int d^4x \langle \Phi_{m_1...m_d}^{n_1...n_d}(y), (\nabla T_a^{(k)})\Phi(x) - \Phi(x)T_a^{(k)}v \rangle_{\epsilon},
$$

$$
(T_a)^j_{m_k} \langle \Phi_{m_1...m_d}^{n_1...n_d}(y) \rangle_{\epsilon} = -\epsilon \int d^4x \langle \Phi_{m_1...m_d}^{n_1...n_d}(y), (\Phi(x)T_a^{(k)}v - vT_a^{(k)}\Phi(x)) \rangle_{\epsilon}.
$$

Here, we have written $\langle F(\Phi) \rangle_{\epsilon} = \langle F(\Phi) \rangle_{J=0,\epsilon}$.

Now, following the steps of the previous example of the scalar field, we will take linear combinations of differential operators as in (2.11),

$$
\nabla T_b^{(k)} \left[ \frac{\delta}{\delta J(y)} \right]_{J(y)=\bar{J}(y)=0} \pm \frac{\delta}{\delta J(y)} \left[ \frac{\delta}{\delta J(y)} \right]_{J(y)=\bar{J}(y)=0} T_b^{(k)}v,
$$

$$
\frac{\delta}{\delta J(y)} \left[ \frac{\delta}{\delta J(y)} \right]_{J(y)=\bar{J}(y)=0} T_b^{(k)}v \pm \frac{\delta}{\delta J(y)} \left[ \frac{\delta}{\delta J(y)} \right]_{J(y)=\bar{J}(y)=0} T_b^{(k)}v.
$$
onto (4.4) and (4.5). The subtraction in (4.16) applied on (4.4) results in
\[
\langle (T_a^{(k)})_i^j (T_b^{(k)})^i_j \Phi^{j_1 \ldots j_d}(y) 
\prod_{j_1 \ldots j_d} \rangle 
\]
\[
+ (T_b^{(k)})_i^j (T_a^{(k)})^i_j \Phi^{j_1 \ldots j_d}(y) 
\prod_{j_1 \ldots j_d} \rangle e = - \epsilon \int d^4x \left< B^k_b(y), B^k_a(x) \right> \epsilon.
\]

(4.18)

We now apply the product rule of generators of $U(N)$
\[
T_a T_b = \frac{1}{2} f_{abc} T_c + \frac{1}{2} d_{abc} T_c, \quad a, b, c = 0, 1, \ldots, N^2 - 1,
\]

(4.19)

where $f_{abc}$ are the structure constants, and $d_{abc}$ is a totally symmetric tensor.

\[
\frac{1}{2} i f_{abc} \langle \phi^k_c(y) \rangle_\epsilon + \frac{1}{2} d_{abc} \langle \phi^k_c(y) \rangle_\epsilon = - \epsilon \int d^4x \left< \phi^k_c(y), B^k_b(x) \right> \epsilon,
\]

(4.20)

where we have defined
\[
\varphi^k_a(y) = \overline{v} T_a^{(k)} \Phi(x) + \Phi(x) T_a^{(k)} v,
\]
\[
\varphi^l_{\bar{k}}(y) = \Phi(x) T_a^{(k)} v + v T_a^{(k)} \overline{\Phi}(x), \quad k, \bar{k} = 1, \ldots, d.
\]

(4.22)

Identical equations to (4.20) and (4.21) hold for $B^k_a$ and $\varphi^k_a$ defined in (4.8) and (4.22) after taking the addition and subtraction combinations in (4.17).

4.3 The Goldstone modes

We have named the fields $B^k_a(y)$ and $B^k_{\bar{a}}(y)$, defined in (4.8), Goldstone bosons. We should check, however, that they are massless. We are going to argue that this is the case using similar arguments as in section 2.

In momentum space, the propagators that appear on the RHS of (4.20) are written as
\[
\langle B^k_a(x) B^k_a(y) \rangle_\epsilon = i(2\pi)^{-4} \int d^4p \frac{Z_{B^k_a}}{p^2 - m^2_{B^k_a} + i\epsilon} e^{-ip(x-y)} + \text{(regular contributions)},
\]

(4.23)

where $Z_{B^k_a}$ and $a_{B^k_a}$ are renormalization constants. Now, because of the term $e^{-ip(x-y)}$ in (4.23), the integral over $x$ appearing in (4.20) picks a pole at $p = 0$. As in the scalar case, the LHS of (4.20) is non-vanishing\footnote{In analogy with the scalar case, we will take the fields $\varphi^k_a(y)$ to have non-vanishing expectation values. No further assumptions need to be taken with respect to the expectation value of the Goldstone bosons.}. So, the limit $\epsilon \to 0$ of the RHS of
\[(4.20)\) has to be non-zero. It implies that \(B^k_a(x)\) is massless. That is,
\[
\lim_{\epsilon \to 0} \int d^4x \left( B^k_a(x) B^k_a(y) \right)_\epsilon = -\delta_{ab} \frac{Z_{B^k}}{aB^k} \quad m_{B^k} = 0, \quad (4.24)
\]
which is nothing but the Goldstone theorem: when the symmetry gets spontaneously broken, to each broken generator corresponds a massless modes.

**4.4 Linear \((in)dependence of the Goldstone modes**

For a generic SSB pattern \(\prod_{\alpha = 1}^\omega \text{Diag}[H_{\alpha}]\), we will have \(\omega\) constraints given by the \(\omega\) equations \((4.9)\). In terms of the Goldstone bosons, those equations are
\[
\left\langle \int d^4x \left( \sum_{k'=1}^{n_\alpha} B^\alpha_{a,k'}(x) - \sum_{k'=1}^{n_\alpha} B^\alpha_{a,k'}(x) \right) \right\rangle_{J,\epsilon} = 0, \quad \alpha = 1, \ldots, \omega, \quad (4.25)
\]
where
\[
B^\alpha_{a,k'}(x) \equiv B^k_a(x), \quad B^\alpha_{a,k'}(x) \equiv B^k_a(x), \quad (4.26)
\]
is a reorganization of the labels, related to the partition \(\sum_{\alpha = 1}^\omega n_\alpha = d\). Notice that equation \((4.14)\), for the full diagonal group, is a particular case of \((4.25)\) when \(\omega = 1\).

We are going to see that \((4.25)\) are automatically satisfied provided
\[
\sum_{k'=1}^{n_\alpha} B^\alpha_{a,k'}(x) - \sum_{k'=1}^{n_\alpha} B^\alpha_{a,k'}(x) = 0, \quad \alpha = 1, \ldots, \omega, \quad (4.27)
\]
which, as we are going to show, is a consequence of the definition of the Goldstone bosons \((4.8)\).

Each of the equations of \((4.27)\) is associated to a diagonal group \(\text{Diag}[H_{\alpha}]\). So the discussion will focus on a particular \(H_\alpha\) (or equivalently, a single cycle in the intersection-diagram of length \(n_\alpha\)).

In order not to overload the paper with notation we will consider the case \(n_\alpha = 3\) for a given \(\alpha\). Without loss of generality the cycle of length 3 in \(v\) under consideration can be chosen
\[
v^1_{j_1 \ldots j_d} = v_1^{j_1 j_2 j_3 \ldots j_d} + v_2^{j_1 j_2 j_3 \ldots j_d}, \quad (4.28)
\]
where the indices 4, \ldots, \(d\) correspond to the other diagonal groups in \(\prod_{\alpha = 1}^\omega \text{Diag}[H_{\alpha}]\), and appear in the rest of the equations of \((4.27)\).

According to the definition \((4.8)\) and the choice \((4.28)\), the Goldstone bosons are
\[
\begin{align*}
B^1_a(x) &= v_1 (T_a)_{j_1 j_2 j_3 \ldots j_d} - v_2 (T_a)_{j_1 j_2 j_3 \ldots j_d} - \text{c.c.}, \\
B^2_a(x) &= v_1 (T_a)_{j_1 j_2 j_3 \ldots j_d} - v_2 (T_a)_{j_1 j_2 j_3 \ldots j_d} - \text{c.c.}, \\
B^3_a(x) &= v_1 (T_a)_{j_1 j_2 j_3 \ldots j_d} - v_2 (T_a)_{j_1 j_2 j_3 \ldots j_d} - \text{c.c.}, \\
B^1_a(x) &= v_1 (T_a)_{j_1 j_2 j_3 \ldots j_d} - v_2 (T_a)_{j_1 j_2 j_3 \ldots j_d} - \text{c.c.}, \\
B^2_a(x) &= v_1 (T_a)_{j_1 j_2 j_3 \ldots j_d} - v_2 (T_a)_{j_1 j_2 j_3 \ldots j_d} - \text{c.c.}, \\
B^3_a(x) &= v_1 (T_a)_{j_1 j_2 j_3 \ldots j_d} - v_2 (T_a)_{j_1 j_2 j_3 \ldots j_d} - \text{c.c.}.
\end{align*}
\quad (4.29)
\]
The constraint equation for the group \( \text{Diag}[H] \) is
\[
B^{\alpha,1}_a(x) + B^{\alpha,2}_a(x) + B^{\alpha,3}_a(x) - B^{\alpha,1}_a(x) - B^{\alpha,2}_a(x) - B^{\alpha,3}_a(x) = 0. \tag{4.30}
\]
This equation is automatically fulfilled from (4.29). The cancellations among the Goldstone bosons have been depicted in colors. The crucial point is that the cancellation (4.30) occurs only when all the Goldstone bosons are involved as we can see in (4.30). No partial cancellations like \( B^{\alpha,1}_a(x) - B^{\alpha,3}_a(x) = 0 \) or \( B^{\alpha,1}_a(x) + B^{\alpha,2}_a(x) - B^{\alpha,1}_a(x) - B^{\alpha,2}_a(x) = 0 \) happen, which implies that only one Goldstone boson in (4.29) is linearly dependent. This is a general feature for any of the \( \omega \) groups \( \text{Diag}[H] \): for each \( \text{Diag}[H] \) there are \( 2n_\alpha - 1 \) linearly independent Goldstone bosons \( B_{a}(x) \), where \( a = 0, \ldots, N^2 - 1 \).

As a remark, there is a relation between the cancellation pattern shown in colors in (4.29) and the cycle corresponding to \( H_a \) in the intersection-diagram. Note that drawing lines that join the same colors, the whole set of equations (4.29) form one and only one loop. The similarity between these pictures goes beyond this observation, and suggests a deep connection which will be studied elsewhere.

Let us count the total number of Goldstone bosons in a SSB pattern \( \prod_{\alpha=1}^\omega \text{Diag}[H_{\alpha}] \).

As said above, for each \( \text{Diag}[H_{\alpha}] \) there are \( (2n_\alpha - 1)N^2 \) Goldstone bosons. So for \( \prod_{\alpha=1}^\omega \text{Diag}[H_{\alpha}] \) there are
\[
\sum_{\alpha=1}^{\omega} (2n_\alpha - 1)N^2 = (2d - \omega)N^2; \tag{4.31}
\]
Goldstone bosons, where we have used \( \sum_{\alpha=1}^{\omega} n_\alpha = d \). Equation \( \text{6} \) (4.31) is in perfect agreement with the direct counting of the number of broken symmetries in equation (3.3).

4.5 Matrix organization of the Goldstone modes

So far, we have described the appearance of the Goldstone bosons \( B^{a,k'}_{\alpha}(x) \) and \( B^{a,k'}_{\alpha}(y) \) as a consequence of the SSB \( G_{d\bar{d}}(N) \to \prod_{\alpha=1}^{\omega} \text{Diag}[H_{\alpha}] \). The number of Goldstone modes match the number of broken continuous symmetries, which is \( (2d - \omega)N^2 \). So in the effective theory we have a collection of linearly independent fields
\[
B_{\alpha} = \{ B^{a,k'}_{\alpha}(x), B^{a,k'}_{\alpha}(x) \mid a = 0, \ldots, N^2 - 1; \; k' = 1, \ldots, n_{\alpha}, \; \bar{k}' = 1, \ldots, n_{\alpha} - 1 \}, \tag{4.32}
\]

\( \text{6} \)There is a remarkable matching between the counting in equation (4.31) for the complete breaking of the symmetry \( \omega = 0 \) and the number of light modes reported in [41] for \( O(N) \). In the case of \( O(N) \) with \( d \) even, the total number of indices of the tensor \( \phi_{i_1 \ldots i_d} \) is \( d \), instead of \( 2d \) in equation (4.31). Moreover, since the tensor field for the \( O(N) \) group is real, the number of modes is half the number of the unitary case. With these considerations, the counting of the Goldstone modes for the orthogonal group, when the SSB breaks completely the original symmetry, is \( \frac{1}{2}dN^2 \), which is the number of light modes counted in p.5 of [41], with the identification \( d = q - 1 \).
where $\alpha = 1, \ldots, \omega$, and we have chosen the last fields $B^\alpha_{\alpha^n}(x)$ to be linearly dependent on the others, according to the constraints (4.27).

The question now is how these modes organize into multiplets. To answer this question we will consider an action of the group Diag$[H_\alpha]$ on the space $B_\alpha$, treated as a vector space, and find the irreducible representations of $B_\alpha$. The Goldstone modes multiplets will correspond, one-to-one, to those irreps.

The action of $\prod_{\alpha=1}^\alpha$ Diag$[H_\alpha]$ on the tensor field induces the transformation on each $B_\alpha$

\[
\begin{align*}
B^{\alpha,k'}_a(x) &= \tau[g_a T^{(k)}_a g_\alpha] \Phi(x) - \overline{\Phi}(x)[g_a T^{(k)}_a g_\alpha]^\dagger v, \\
B^\alpha_{\alpha,k}(x) &= \Phi(x)[g_a T^{(k)}_a g_\alpha] \tau - v[g_a T^{(k)}_a g_\alpha] \overline{\Phi}(x), \quad g_\alpha \in \text{Diag}[H_\alpha],
\end{align*}
\]

where the labels $(\alpha, k')$ and $k$ are related according to the map (4.26). Notice that the action of $g_\alpha$ is trivial in all the slots except for the slot $k$, where the generators hit. So the action of Diag$[H_\alpha]$ on $B_\alpha$ reduces to the adjoint action

\[
\text{Ad}_{g_\alpha} T_a = g_a T_a g_\alpha = \sum_b C^b_a (g_\alpha) T_b, \quad g_\alpha \in \text{Diag}[H_\alpha], \quad T_a \in U(N),
\]

on each of the slots labeled by $k$ and $\bar{k}$.

So for each $(\alpha, k')$ and $(\alpha, \bar{k'})$, all the modes $B^\alpha_{\alpha, k'}(x)$ and $B^\alpha_{\alpha, \bar{k'}}(x)$ with $a = 0, \ldots, N^2 - 1$ get arranged into a multiplet. The transformation (4.33) suggests that $B^\alpha_{\alpha, k'}(x)$ and $B^\alpha_{\alpha, \bar{k'}}(x)$ are the components of an $N \times N$-matrix field. In fact, we can map each $(\alpha, k')$ and $(\alpha, \bar{k'})$ collection $\{B^\alpha_{\alpha, k'}(x), B^\alpha_{\alpha, \bar{k'}}(x) | a = 0, \ldots, N^2 - 1\}$ into the matrices

\[
(B^\alpha_{\alpha, k'})_j^i(x) = \sum_a B^\alpha_{\alpha, k'}(x)(T_a)_j^i, \quad k' = 1, \ldots, n_\alpha,
\]

and

\[
(B^\alpha_{\alpha, \bar{k'}})_j^i(x) = \sum_a B^\alpha_{\alpha, \bar{k'}}(x)(T_a)_j^i, \quad \bar{k'} = 1, \ldots, n_\alpha - 1.
\]

These fields transform in the adjoint of $U_\alpha(N) = \text{Diag}[H_\alpha]$ as

\[
(B^\alpha_{\alpha, k'})_j^i(x) = (g_\alpha)_m^i (B^{\alpha, k'})_m^i(x)(g_\alpha)_j^m = \sum_{a,b} C^b_a (g_\alpha) B^\alpha_{\alpha, k'}(x)(T_b)_j^i = \sum_b B^\alpha_{b, k'}(x)(T_b)_j^i,
\]

where

\[
B^{\alpha, k'}_b(x) = \sum_a C^b_a (g_\alpha) B^\alpha_{\alpha, k'}(x).
\]

The same transformation law holds for $(\alpha, \bar{k'})$ fields.
The transformation (4.37) is perfectly compatible with (4.33). Indeed,

\[
B'_\alpha,k(x) = v [g_\alpha T^{(k)}(x) - \Phi(x)g^{(k)}_\alpha] v
= \sum_a C^a_b(g_\alpha) (v T^{(k)}(x) - \Phi(x)T^{(k)}_a v)
= \sum_a C^a_b(g_\alpha) B^{\alpha,k'}_a(x),
\]

which is the transformation law (4.38). We have shown that the Goldstone bosons \(B^{\alpha,k'}_a(x)\) and \(B^{\alpha,\bar{k}'}_a(x)\), with \(a = 0, \ldots, N^2 - 1\), are actually the components of a matrix transforming in the adjoint of \(U_\alpha(N)\).

Interestingly enough, we can write down the Goldstone boson matrix fields in terms the tensor fields and \(v\) exclusively, without the use of the generators\(^7\). Using the definitions (4.35) and (4.8), we have

\[
\begin{align*}
(B^{\alpha,k'})^i_j(x) &= \sum_a B^{\alpha,k'}_a(x)(T_a)^i_j \sum_a \left(\Phi(x) T^{(k)}_a v - v T^{(k)}_a \Phi(x)\right)(T_a)^i_j,
(B^{\alpha,\bar{k}})^i_j(x) &= \sum_a B^{\alpha,\bar{k}}_a(x)(T_a)^i_j \sum_a \left(\Phi(x) T^{(k)}_a v - v T^{(k)}_a \Phi(x)\right)(T_a)^i_j.
\end{align*}
\]

Equation (4.40) is completely general, valid for any symmetry group and any SSB. It means that the collection of Goldstone bosons which result from SSB of tensor theories organizes into matrix field multiplets. In other words, any tensor theory with SSB always leads to matrix theories. See appendix A for an application of (4.40) to a non-unitary group and a different SSB pattern.

Using the properties of the generators of \(U(N)\)

\[
(T_a)^m_n (T_a)^k_l = \frac{1}{2} \delta^m_l \delta^k_n,
\]

we arrive at

\[
\begin{align*}
(B^{\alpha,k'})^i_j(x) &= \frac{1}{2} \left(\Phi^{j_1 \ldots j_d}_{i_1 \ldots i_d} \Phi^{j_1 \ldots j_d}_{i_1 \ldots i_d}(x) - \Phi^{j_1 \ldots j_d}_{i_1 \ldots i_d} \Phi^{j_1 \ldots j_d}_{i_1 \ldots i_d}(x)\right),
(B^{\alpha,\bar{k}})^i_j(x) &= \frac{1}{2} \left(\Phi^{j_1 \ldots j_d}_{i_1 \ldots i_d} \Phi^{j_1 \ldots j_d}_{i_1 \ldots i_d}(x) - \Phi^{j_1 \ldots j_d}_{i_1 \ldots i_d} \Phi^{j_1 \ldots j_d}_{i_1 \ldots i_d}(x)\right),
\end{align*}
\]

where it is understood that the indices \(i, j\) are located at the \(k, \bar{k}\) slots of the tensor. Recall that \(k, \bar{k}\) are given in terms of \((\alpha, k')\) and \((\alpha, \bar{k}')\) by the map (4.26). Equations

\(^7\)Although equation (4.40) seems to involve the generators of the group, they appear in the combination \(T_a T_a\). Using the product rule for the given group, the explicit form of the generators is not needed, see equation (4.42).
are one of the main results of the paper. We would like to stress that the Goldstone bosons (4.42) are defined for any SSB pattern induced by \( v \) in unitary groups, where \( v \) is not necessarily constrained to the shape considered in this paper, namely linear combinations of Kronecker deltas.

The fields \( \varphi^k_a(x) \) and \( \varphi^\bar{k}_a(x) \) defined in (4.22) organise into matrix multiplets, for the same reason as the Goldstone bosons. Although these fields will be massive in the effective theory, hence they will not be considered at low energies.

At low energies, the massless modes (which we assume to be only the Goldstone bosons [39]) are the most relevant. According to the above discussion:

The massless field content of the effective theory that results from SSB of a tensor theory built on \( \Phi^{i_1\ldots i_d}_{j_1\ldots j_d}(x) \), into \( \prod_{\alpha=1}^\omega \text{Diag}[H_{\alpha}] \), which is associated with the partition \( \sum_{\alpha=1}^\omega n_{\alpha} = d \), is an overall of \( 2d - \omega \) \( N \times N \)-matrix fields organized as \( B^{\alpha,k'}(x) \) and \( \overline{B}^{\alpha,k'}(x) \), with \( k' = 1, \ldots, n_{\alpha} \) and \( \overline{k}' = 1, \ldots, n_{\alpha} - 1 \), transforming in the adjoint of \( U_{\alpha}(N) = \text{Diag}[H_{\alpha}] \), and \( \alpha = 1, \ldots, \omega \).

The fact that the effective theory is invariant under \( \prod_{\alpha=1}^\omega U_{\alpha}(N) \) tells us that the fields \( B^{\alpha,k'}(x) \) and \( \overline{B}^{\alpha,k'}(x) \) must appear in the action of the effective theory with all the indices properly contracted. So, the vertices of the theory must appear as multi trace polynomials, where the matrix fields appearing within a given trace must involve only fields transforming under the same group, that is, with the same \( \alpha \). In addition, the fact that the fields are complex tells us that the monomials that appear in the action must include \( B^{\alpha,k'}(x) \) and \( \overline{B}^{\alpha,k'}(x) \) in an equal number. So, the action of the effective theory will contain interaction monomials of the type

\[
\text{Tr}(B^{1,1}\overline{B}^{1,2})\text{Tr}(\overline{B}^{2,1}B^{3,2})\text{Tr}(\overline{B}^{2,2}B^{2,3}B^{2,4})\ldots,
\]

where, for simplicity, we are omitting the spacetime derivatives of the matrix fields.

This concludes our discussion about the multiplets in the effective field theory. Apart from these particular considerations for our case, the general features of effective field theories [44–46] apply as well.

5 Examples

In this section we are going to apply the general formalism of SSB in tensor theories developed above in two examples.

5.1 SSB for the complex 2-tensor field

In order to make the discussion of the previous sections more explicit, using the techniques presented above, we shall study the well-known example of chiral symmetry breaking for the complex scalar field, see, for instance, [47]. This involves SSB of a complex 2-tensor field \( \Phi^i_j(x) \) transforming in the fundamental and anti-fundamental representation \( G_{1\bar{1}}(N) = U_1(N) \times U_{\bar{1}}(N) \) into the diagonal group. We will use the
notation defined in (2.25).
The elements of the group may be written as \((g_1, g_1)\), with
\[
\begin{align*}
g_i &= \exp(i\theta^{(i)}_a T_a) \in U_i(N), \\
g_i^\dagger &= \exp(-i\theta^{(i)}_a T_a^\dagger) \in U_i(N), \quad i = 1, \bar{1}.
\end{align*}
\] (5.1)
The generators \(T_a\) of the unitary group are Hermitian, so \(T_a = T_a^\dagger\). However, we will not make the substitution at this stage in order to keep track of the conjugate terms. For the 2-tensor case we will require an action and a measure invariant under the transformations (2.25). The generating functional \(Z[J, \bar{J}]\) is defined in the same fashion as for the scalar case in section 2,
\[
Z[J, \bar{J}] = \frac{1}{N} \int D\Phi \exp[iS[\Phi(x)] + i \int d^4x (\bar{J}_j^a(T_a)T^i_k \Phi^i_k(x) + c.c.) \\
- \epsilon \int d^4x ((\Phi(x) - v_j^a(T_a)k \bar{\Phi}^i_k(x) - \bar{\Phi}^i_k(x))].
\] (5.2)
Here we are only interested in the breaking of the symmetry to the diagonal group, which occurs when \(v_j^a\) is proportional to \(\delta^a_j\), as discussed in section 3.
Following the methodology of previous sections, we will compute the WT identities. First, we transform the fields as in (2.25) with the elements of the groups written as (5.1), and derive (5.2) with respect to the parameters \(\theta^{(i)}_a\). The path integral is invariant under the field transformation, so
\[
\frac{\partial}{\partial \theta^{(i)}_a} Z[J, \bar{J}] = 0, \quad i = 1, \bar{1}.
\] (5.3)
For \(\theta^{(1)}_a\) we obtain
\[
\int d^4x \langle \bar{J}_j^a(T_a)T^i_k \Phi^i_k(x) - J_j^a(T_a)k \bar{\Phi}^i_k(x) \rangle \epsilon \\
- i\epsilon \int d^4x \langle \bar{\Phi}^i_k(T_a)T^i_k \Phi^i_k(x) - v_j^a(T_a)k \bar{\Phi}^i_k(x) \rangle \epsilon = 0.
\] (5.4)
and for \(\theta^{(1)}_a\) we obtain the identity
\[
\int d^4x \langle \bar{J}_j^a(T_a)T^i_k \Phi^i_k(x) - J_j^a(T_a)k \bar{\Phi}^i_k(x) \rangle \epsilon \\
- i\epsilon \int d^4x \langle \bar{\Phi}^i_k(T_a)T^i_k \Phi^i_k(x) - v_j^a(T_a)k \bar{\Phi}^i_k(x) \rangle \epsilon = 0.
\] (5.5)
Again, the identities are obtained by derivating with respect to the sources. So, applying \(\frac{\delta}{\delta \phi^n_m(x)}\) on (5.4) and on (5.5) we have
\[
0 = (T_a^\dagger)^n_m \langle \phi^i_m(y) \rangle + \epsilon \int d^4x \langle \bar{J}_j^a(T_a)T^i_k \phi^i_k(x) - v_j^a(T_a)k \bar{\phi}^i_k(x) \rangle \epsilon, \quad (5.6)
\]
\[
0 = (T_a^\dagger)^n_m \langle \phi^i_m(y) \rangle + \epsilon \int d^4x \langle \bar{\phi}^i_k(T_a)T^i_k \phi^i_k(x) - v_j^a(T_a)k \bar{\phi}^i_k(x) \rangle \epsilon. \quad (5.7)
\]
To obtain the additional constraint we derive the path integral (5.2) with respect to \( \theta_a = \theta_a^{(1)} = \theta_a^{(2)} \), which parametrizes the diagonal group. Notice that under the action of the diagonal group the \( \epsilon \)-term is invariant. The constraint reads

\[
\int d^4x \langle \overline{\Phi}_i^j (T_a^i) \rangle \Phi_k^j(x) - v_j^i (T_a^i) \overline{\Phi}_i^j(x) \rangle \epsilon
- \int d^4x \langle \overline{\Phi}_i^j (T_a^i) \rangle \Phi_k^j(x) - v_j^i (T_a^i) \overline{\Phi}_i^j(x) \rangle \epsilon = 0. \tag{5.8}
\]

This holds for any configuration of the tensor field, i.e.,

\[
v_j^i (T_a^i) - v_j^i (T_a^i) = 0, \tag{5.9}
\]

which is satisfied for \( v_j^i = v \delta_j^i \). In terms of the Goldstone bosons (4.8), the constraint (5.8) and/or (5.9) follows from the cancellation (4.27), which in this case is

\[
B_a^i(x) - B_a^i(x) = 0. \tag{5.10}
\]

This proves the consistency of the choice of \( v \) for the case of two indices tensor. This result, namely that \( v \) is proportional to the identity, is well-known in the context of chirality breaking. Nevertheless, the generalization of it that we perform in section 3 has not been reported in the literature, as far as we know.

Now, we will use (5.7) to find an expression for the Goldstone bosons in the case of SSB into the diagonal group. We take \( v_j^i = v \delta_j^i \) where, without loss of generality, \( v \) is real. We will also use the Hermiticity of the generators \( T_a = T_a^\dagger \). So, we have

\[
(T_a^m) \langle \Phi_m^n (y) \rangle \epsilon = -\epsilon \int d^4x \langle \Phi_m^n (y) \rangle \epsilon \left( (T_a^m) v_j^i (x) - (T_a^m) \overline{\Phi}_i^j(x) \right) \epsilon. \tag{5.11}
\]

Conjugating the above equation we get

\[
(T_a^m) \langle \overline{\Phi}_n^m (y) \rangle \epsilon = \epsilon \int d^4x \langle \overline{\Phi}_n^m (y) \rangle \epsilon \left( (T_a^m) v_j^i (x) - (T_a^m) \overline{\Phi}_i^j(x) \right) \epsilon. \tag{5.12}
\]

We multiply (5.11) by \( (T_b)^m_n \) and (5.12) by \( (T_b)^m_n \) to obtain the couple of equations

\[
(T_b)^m_n (T_a^m) \langle \Phi_i^n (y) \rangle \epsilon = -\epsilon \int d^4x \langle (T_b)^m_n \Phi_i^n (y) \rangle \epsilon \left( (T_a^m) v_j^i (x) - (T_a^m) \overline{\Phi}_i^j(x) \right) \epsilon, \tag{5.13}
\]

\[
(T_b)^m_n (T_a^m) \langle \overline{\Phi}_n^m (y) \rangle \epsilon = \epsilon \int d^4x \langle (T_b)^m_n \overline{\Phi}_n^m (y) \rangle \epsilon \left( (T_a^m) v_j^i (x) - (T_a^m) \overline{\Phi}_i^j(x) \right) \epsilon. \tag{5.14}
\]

Multiplying by \( v \) (5.13) and (5.14), and summing them we obtain

\[
v \left( (T_b)^m_n (T_a^m) \langle \Phi_i^n (y) \rangle \epsilon + (T_b)^m_n (T_a^m) \langle \overline{\Phi}_n^m (y) \rangle \epsilon \right)
= \epsilon \int d^4x \langle (T_b)^m_n \Phi_i^n (y) \rangle \epsilon \left( (T_a^m) v_j^i (x) - (T_a^m) \overline{\Phi}_i^j(x) \right) \epsilon. \tag{5.15}
\]

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We will use the multiplication rule for the generators \((4.19)\). Now we define the fields

\[
B_a(x) = v(T_a)^k \left( \Phi_k(x) - \Phi_k(x) \right),
\]

\[
\varphi_a(x) = v(T_a)^k \left( \Phi_k(x) + \Phi_k(x) \right).
\]

Note that the field \(\varphi_a(x)\) is real whereas \(B_a(x)\) is purely imaginary. With these definitions, \((5.15)\) can be written as

\[
\frac{1}{2} i f_{abc} \langle B_c(y) \rangle_e + \frac{1}{2} d_{abc} \langle \varphi_c(y) \rangle_e = -\epsilon \int d^4 x \langle B_a(y), B_b(x) \rangle_e.
\]

Now, equations \((5.13)\) and \((5.14)\) could also be subtracted. In that case, and using the same fields as before we obtain the WT identity

\[
\frac{1}{2} i f_{abc} \langle \varphi_c(y) \rangle_e + \frac{1}{2} d_{abc} \langle B_c(y) \rangle_e = -\epsilon \int d^4 x \langle \varphi_a(y), B_b(x) \rangle_e.
\]

In order to identify the effective degree of freedom it is convenient to rewrite \((5.16)\) and \((5.17)\) as matrices. To this end we define the new fields

\[
B^m_n(x) = B_a(x) (T_a)_m^n,
\]

\[
\varphi^m_n(x) = \varphi_a(x) (T_a)_m^n,
\]

and using the identity

\[
(T_a)_m^n (T_a)_n^k = \frac{1}{2} \delta^m_i \delta_n^k,
\]

we get

\[
B^m_n(x) = \frac{v}{2} \left( \Phi^m_n(x) - \Phi^m_n(x) \right),
\]

\[
\varphi^m_n(x) = \frac{v}{2} \left( \Phi^m_n(x) + \Phi^m_n(x) \right),
\]

where now \((5.23)\) and \((5.24)\) are anti-hermitian and hermitian tensors, respectively. As discussed in section 4.5, the matrix field \(B^m_n(x)\) is the Goldstone boson matrix field. Particularly, \((5.23)\) is in accordance with \((4.42)\).

In the next section we will study a more involved case where the tensor field has four indices.

### 5.2 SSB for the complex 4-tensor field

Following the steps of the previous section, we will show how the SSB of \(G_{22}(N) = U_1(N) \times U_2(N) \times U_1(N) \times U_2(N)\) into the diagonal subgroup is triggered by the \(\epsilon\)-term. We will also study the Goldstone bosons, which display a much richer structure than
the two-tensor field. Using the notation (2.26) and (2.27), the fields will transform as

\[ \Phi_{j_1j_2}^{i_1i_2} = (g_1^{i_1}_{i_1'})(g_2^{j_2}_{j_2'})\Phi_{j_1j_2}^{i_1i_2}, \]

\[ \overline{\Phi}_{j_1j_2}^{i_1i_2} = (g_1^{i_1}_{i_1'})(g_2^{j_2}_{j_2'})\overline{\Phi}_{j_1j_2}^{i_1i_2}, \] (5.25)

where it is manifest that \( \Phi_{j_1j_2}^{i_1i_2} \overline{\Phi}_{j_1j_2}^{i_1i_2} \) is invariant under \( G_{22}(N) \).

The elements of the group may be written as

\[ g_i = \exp(i\theta_a^{(i)} T_a) \in U_i(N), \]

\[ g_i^\dagger = \exp(-i\theta_a^{(i)} T_a^\dagger) \in U_i(N), \quad i = 1, 2, \bar{1}, \bar{2}. \] (5.26)

The path integral in this case reads

\[ Z_c[J, \overline{J}] = \frac{1}{N} \int D\Phi \exp\left[ iS[\Phi(x)] + i \int d^4x (\overline{J}^{i_1i_2}_{j_1j_2}(x)\Phi^{j_1j_2}_{i_1i_2}(x) + c.c) \right] \]

\[ -\epsilon \int d^4x \left( (\Phi(x) - v)^{i_1i_2}_{j_1j_2}(x) - (\overline{\Phi}(x) - \overline{v})^{i_1i_2}_{j_1j_2}(x) \right). \] (5.27)

We will again transform the fields with the parametrizations (5.26) and find the four basic identities through the derivatives

\[ \frac{\partial}{\partial \theta_a^{(i)}} Z_c[J, \overline{J}] = 0, \quad i = 1, 2, \bar{1}, \bar{2}. \] (5.28)

For \( \theta_a^{(1)} \) we obtain

\[ \int d^4x \langle \overline{J}^{i_1i_2}_{j_1j_2}(T_a^\dagger)_{i_1i_2} \Phi^{j_1j_2}_{i_1i_2}(x) - J^{i_1i_2}_{j_1j_2}(T_a)_{i_1i_2} \overline{\Phi}^{j_1j_2}_{i_1i_2}(x) \rangle \epsilon = 0, \] (5.29)

and for \( \theta_a^{(1)} \) we obtain the identity

\[ \int d^4x \langle \overline{J}^{i_1i_2}_{j_1j_2}(T_a^\dagger)_{i_1i_2} \Phi^{j_1j_2}_{i_1i_2}(x) - J^{i_1i_2}_{j_1j_2}(T_a)_{i_1i_2} \overline{\Phi}^{j_1j_2}_{i_1i_2}(x) \rangle \epsilon = 0, \] (5.30)

For \( \theta_a^{(2)} \), the generator hits on the second position as indicated by \( k = 2 \). We obtain

\[ \int d^4x \langle \overline{J}^{i_1i_2}_{j_1j_2}(T_a^\dagger)_{j_1j_2} \Phi^{j_1j_2}_{i_1i_2}(x) - J^{i_1i_2}_{j_1j_2}(T_a)_{j_1j_2} \overline{\Phi}^{j_1j_2}_{i_1i_2}(x) \rangle \epsilon = 0, \] (5.31)

29
and for $\theta_a^{(2)}$ we obtain the identity

$$\int d^4x \langle \overline{\Phi}^{ji_{12}}_{j_1j_2} (T_a)^{i_1i_2}_{j_1j_2} (x) - J^{ji_{12}}_{j_1j_2} (T_a)_{j_1j_2} \Phi^{ji_{12}}_{i_1i_2} (x) \rangle \epsilon = -i \epsilon \int d^4x \langle \overline{\Phi}^{ji_{12}}_{j_1j_2} (T_a)^{i_1i_2}_{j_1j_2} (x) - v^{ji_{12}}_{j_1j_2} (T_a)_{j_1j_2} \Phi^{ji_{12}}_{i_1i_2} (x) \rangle \epsilon = 0.$$  

(5.32)

Note that equation (5.31) is analogous to (5.29) except for the generator hitting on the second slot of the fields. The same happens to equations (5.32) and (5.30). For a tensor of 2d indices we will have d similar equations to (5.29) and other d equations similar to (5.30), where the generators hit on each of the d slots.

In order to obtain WT identities we derive with respect the sources. So, applying

$$\delta \frac{\delta}{\delta J_{mn}^j} \mid_{J=0}$$
on (5.29), (5.30), (5.31) and (5.32) we obtain

$$(T_a)^{i_1}_{m_1} \langle \Phi^{ji_{12}}_{m_1m_2} (y) \rangle = -\epsilon \int d^4x \langle \Phi^{ni_{12}}_{m_1m_2} (y), \overline{\Phi}^{ji_{12}}_{j_1j_2} (T_a)^{i_1i_2}_{j_1j_2} (x) - v^{ji_{12}}_{j_1j_2} (T_a)_{j_1j_2} \Phi^{ji_{12}}_{i_1i_2} (x) \rangle \epsilon,$n

(5.33)

As before, under the action of the diagonal group, the $\epsilon$-term is invariant. So, for the elements of the diagonal group parametrized by $\theta_a^{(1)} = \theta_a^{(1)} = \theta_a^{(2)} = \theta_a^{(2)} = \theta_a$, we take derivatives with respect to $\theta_a$, and obtain the extra constraint

$$\int d^4x \langle \overline{\Phi}^{ji_{12}}_{j_1j_2} (T_a)^{i_1i_2}_{j_1j_2} (x) - v^{ji_{12}}_{j_1j_2} (T_a)_{j_1j_2} \Phi^{ji_{12}}_{i_1i_2} (x) \rangle \epsilon$$

$$- \int d^4x \langle \overline{\Phi}^{ji_{12}}_{j_1j_2} (T_a)^{i_1i_2}_{j_1j_2} (x) - v^{ji_{12}}_{j_1j_2} (T_a)_{j_1j_2} \Phi^{ji_{12}}_{i_1i_2} (x) \rangle \epsilon$$

$$+ \int d^4x \langle \overline{\Phi}^{ji_{12}}_{j_1j_2} (T_a)^{i_1i_2}_{j_1j_2} (x) - v^{ji_{12}}_{j_1j_2} (T_a)_{j_1j_2} \Phi^{ji_{12}}_{i_1i_2} (x) \rangle \epsilon$$

$$- \int d^4x \langle \overline{\Phi}^{ji_{12}}_{j_1j_2} (T_a)^{i_1i_2}_{j_1j_2} (x) - v^{ji_{12}}_{j_1j_2} (T_a)_{j_1j_2} \Phi^{ji_{12}}_{i_1i_2} (x) \rangle \epsilon = 0.$$  

(5.34)

This constraint must hold for any configuration of the field $\Phi (y)$, what means that the relation

$$- v^{ji_{12}}_{j_1j_2} (T_a)^{i_1}_{i_1} + v^{ji_{12}}_{j_1j_2} (T_a)^{j_1}_{j_1} - v^{ji_{12}}_{j_1j_2} (T_a)^{i_2}_{i_2} + v^{ji_{12}}_{j_1j_2} (T_a)^{j_2}_{j_2} = 0,$$

(5.35)

must be satisfied for a suitable choice of $v$ which breaks the symmetry into the diagonal group. In terms of the Goldstone bosons (4.8), the constraint (5.34) and/or
(5.35) follows from the cancellation (4.27), which for this case reads

$$B^1_u(x) - B^1_u(x) + B^2_u(x) - B^2_u(x) = 0. \quad (5.36)$$

The only option for $v$ to fulfill the constraint (5.35) is

$$v^{i_1i_2}_{j_1j_2} = v_1\delta^{i_1i_2}_{j_1j_2} + v_2\delta^{i_1i_2}_{j_2j_1}. \quad (5.37)$$

With $v$ as in (5.37), the Goldstone bosons are

$$
\begin{align*}
B^1_u(x) &= \left[ v_1(\Phi^{j_1i_1}_{i_1i_1}(x) - \Phi^{j_1i_1}_{i_1i_1}(x)) + v_2(\Phi^{j_1i_1}_{i_1i_1}(x) - \Phi^{j_1i_1}_{i_1i_1}(x)) \right] (T_u)^j_{i_1}, \\
B^2_u(x) &= \left[ v_1(\Phi^{j_1i_1}_{i_1i_1}(x) - \Phi^{j_1i_1}_{i_1i_1}(x)) + v_2(\Phi^{j_1i_1}_{i_1i_1}(x) - \Phi^{j_1i_1}_{i_1i_1}(x)) \right] (T_u)^{j_2}_{i_2}, \\
B^2_u(x) &= \left[ v_1(\Phi^{j_1i_1}_{i_1i_1}(x) - \Phi^{j_1i_1}_{i_1i_1}(x)) + v_2(\Phi^{j_1i_1}_{i_1i_1}(x) - \Phi^{j_1i_1}_{i_1i_1}(x)) \right] (T_u)^{j_2}_{i_2}, \\
B^2_u(x) &= \left[ v_1(\Phi^{j_1i_1}_{i_1i_1}(x) - \Phi^{j_1i_1}_{i_1i_1}(x)) + v_2(\Phi^{j_1i_1}_{i_1i_1}(x) - \Phi^{j_1i_1}_{i_1i_1}(x)) \right] (T_u)^{j_2}_{i_2},
\end{align*}
$$

(5.38)

where as before we have depicted the terms which cancel in (5.36) in colors. Notice that all the Goldstone bosons are needed for the cancellation, and no partial cancellations occur. Equivalently, there is one loop when joining equal colors in the equations. So, there are three independent Goldstone boson matrix fields, matching the number of broken symmetries in the SSB pattern

$$U_1(N) \times U_2(N) \times U_1(N) \times U_2(N) \rightarrow U(N). \quad (5.39)$$

6 Conclusion and outlook

The $\epsilon$-term technique has long been proven a powerful tool to tackle SSB, here implemented for the first time in tensor models. This technique leads us to identify the Goldstone bosons as matrix fields, which is one of the central results of the paper. It also enables the discussion of the SSB patterns characterized by the tensor $v$. In this paper, we focus on SSB patterns leading to diagonal subgroups of $G_{dd}(N)$, for which the more general $v$ is a linear combination of Kronecker deltas, as in (3.4). In order to understand the intricate relation between the monomial constituents of $v$ and the SSB patterns, we develop a diagrammatic correspondence. The correspondence provides a visual and straightforward way of interpreting the SSB on the diagrams. Unexpectedly, from the diagram inspection, we conclude that any SSB pattern can be induced by only two (complex) parameters.

This work comprises a kinematic study of vacua in tensor models. We mainly focus on the cases where the effective theory is a matrix theory transforming in the adjoint of $U(N)$, although the treatment holds for other scenarios, see appendix A. We claim that any tensor theory with SSB always leads to matrix theories. In fact, equation (4.40) is a general result, which holds for any symmetry group and any

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8See Fig.1.
SSB pattern. There are other attempts in the literature to relate matrix and tensor theories [11, 20, 22, 48–52], but those are referred to specific models. In contrast, our approach is applicable to any tensor theory as long as the theory presents SSB, which requires that the potential has nonzero stationary points, as in (A.4).

One of the undoubtable succeeds of tensor theories is the simple large $N$ structure of some models, which makes them exactly solvable. However, this might not be the case for all physically relevant models. SSB provides valuable information of the system, particularly useful when the large $N$ solvability is lost. Additionally, as SSB generically leads to matrix models, this mechanism could conceptually clarify the relation between tensor theories and quantum gravity, holography, etc.

Let us now describe a possible scenario. Suppose we have a tensor theory with a nontrivial potential which presents several stationary points at which the SSB patterns lead to diagonal subgroups of $G_{dd}(N)$, hence to $U(N)$ matrix theories. This is shown in Fig.6, where we have marked the minima.

![Figure 6. Schematic landscape of matrix theories, with three minima.](image)

Given a theory with a potential as in Fig.6, by means of SSB we could arrive at any minimum point of the landscape, where different (multi-)matrix theories sit. Notice that the collection of matrix theories is encoded in an underlying single tensor theory. We believe that this remarkable fact deserves to be explored in depth. On this line, finding specific models which realized the aforementioned correspondence is our next goal. As a remark, if the tensor theory were not solvable in the large $N$, then it could be used to perform non-trivial calculations in the (non-solvable) matrix setups.

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A SSB for the model with tetrahedral vertex

In this appendix we show an example of symmetry breaking occurring in a well-studied tensor model. The pattern of symmetry breaking for this model is different from the patterns studied in this paper. However, it is worth mentioning it for its potential relevance in future works. It also supports our believe that SSB is a common feature in tensor models and deserves further attention.

The real $O_1(N) \times O_2(N) \times O_3(N)$ scalar model with tetrahedral vertex is a $\phi^4$ real massless scalar tensor model with potential

$$V(\phi) = \phi_{ijk} \phi_{ilm} \phi_{njm} \phi_{nlk},$$

(A.1)

where the tensor field transforms under $O_1(N) \times O_2(N) \times O_3(N)$ as

$$\phi'_{ijk} = (g_1)_{il} (g_2)_{jm} (g_3)_{kn} \phi_{lmn}, \quad g_i \in O_i(N).$$

(A.2)

To study SSB we proceed as usual by searching for stationary points of the Lagrangian which happens at the solutions

$$\left. \frac{\partial}{\partial \phi} V(\phi) \right|_{\phi=v} = 0.$$  

(A.3)

If there are non-trivial solutions of this equation, namely $v \neq 0$, then the theory allows for SSB around the configurations $\phi(x)$ which have a non-zero value at infinity\(^9\). It is interesting to see that although the usual massless $\phi^4$ theory do not present any SSB the tensor theory does. To see this consider first the case $N = 3$. It is not hard to check that a solution of the equation

$$-2v = v_{123} = v_{213} = v_{321} = v_{132} = v_{231} = v_{312},$$

$$-2v = v_{111} = v_{122} = v_{333},$$

$$v = v_{112} = v_{121} = v_{211} = v_{113} = v_{131} = v_{311} = v_{221} = v_{212} = v_{122}$$

$$= v_{223} = v_{322} = v_{331} = v_{313} = v_{332} = v_{323} = v_{233},$$

(A.4)

for any $v \in \mathbb{R}$ is a solution of the collection of equations

$$\frac{\partial}{\partial \phi_{ijk}} V(v) = v_{lnm} \phi_{ijm} \phi_{lkn} = 0, \quad i, j, k = 1, 2, 3.$$  

(A.5)

We would like to emphasize that the existence of a solution like (A.4) is a purely tensor effect, since the only solution of $V'(\phi) = 0$ for the scalar case is $\phi = 0$.

\(^9\)It was notice in [42] that the potential (A.1) has a negative direction. Thus, solution (A.4) is not a minimum. However, as stated in [39, 40], the discussion of SSB also applies to stationary points of the potential.
The multiplication rule for the orthogonal group is

$$T \in \text{O}(N)$$

using (4.40) adapted to the orthogonal group we may write

$$a \in \text{O}(N)$$

The invariance of $v$ under $S_3$ is stated as

$$v_{ijk} = v_{\sigma(i)\sigma(j)\sigma(k)}, \quad \sigma \in S_3.$$ (A.7)

We argue that for configurations $\varphi(x)$ which take values (A.4) at infinity with $v \neq 0$, the theory present SSB and the remaining symmetry is precisely $\text{Diag}[S_3]$. The solution (A.4) can be extended to larger values of $N$ and apply (A.4) with two independent and nonzero values of $v$ on each subspace. In general, for arbitrary $N$, we could chose $k \text{ O(3)}$ subspaces, where the solution pattern (A.4) applies, and break the symmetry

$$O_1(N) \times O_2(N) \times O_3(N) \rightarrow O_1(N - 3k) \times O_2(N - 3k) \times O_3(N - 3k) \times \text{Diag}[S_3]_1 \times \cdots \times \text{Diag}[S_3]_k.$$ (A.8)

In order to complete the discussion let us calculate the Goldstone bosons. Without loss of generality we will take $N = 3$. Using the definition (4.8) adapted to $O(3)$ we may write

$$B^{(1)}_a(x) = v_{i_1i_2i_3}(T_a)_{i_1k} \varphi_{k_2i_3}(x)$$

$$B^{(2)}_a(x) = v_{i_1i_2i_3}(T_a)_{i_2k} \varphi_{i_1k_3}(x)$$

$$B^{(3)}_a(x) = v_{i_1i_2i_3}(T_a)_{i_3k} \varphi_{i_1i_2}(x),$$ (A.9)

where $T_a$ are the generators of the algebra of $O(3)$. They can be written as

$$(T_{mn})_{ik} = -i(\delta_{mk}\delta_{in} - \delta_{nk}\delta_{im}),$$ (A.10)

where the index $a = 1, 2, 3$ has been mapped to the pair $(mn) = (12), (13), (23)$. As the in the unitary case, the Goldstone bosons in (A.9) get arranged into matrices. Using (4.40) adapted to the orthogonal group we may write

$$(B^{(1)})_{ij}(x) = v_{i_1i_2i_3}[(T_a)_{i_1k}(T_a)_{ij}] \varphi_{k_2i_3}(x)$$

$$(B^{(2)})_{ij}(x) = v_{i_1i_2i_3}[(T_a)_{i_2k}(T_a)_{ij}] \varphi_{i_1k_3}(x)$$

$$(B^{(3)})_{ij}(x) = v_{i_1i_2i_3}[(T_a)_{i_3k}(T_a)_{ij}] \varphi_{i_1i_2}(x).$$ (A.11)

The multiplication rule for the orthogonal group is

$$(T_a)_{i_1k}(T_a)_{ij} = (T_{mn})_{i_1k}(T_{mn})_{ij} = -(\delta_{kj}\delta_{i_1i} - \delta_{ji}\delta_{ki}),$$ (A.12)
which implemented on (A.11) yields

\[
\begin{align*}
(B^{(1)})_{ij}(x) &= -(v_{i2i3}\varphi_{ji2i3}(x) - v_{j2i3}\varphi_{ij2i3}(x)) \\
(B^{(2)})_{ij}(x) &= -(v_{1i2i3}\varphi_{1i3j}(x) - v_{1i3j}\varphi_{1i3j}(x)) \\
(B^{(3)})_{ij}(x) &= -(v_{1i2i3}\varphi_{ii1i2}(x) - v_{1i1i2}\varphi_{ii1i2}(x)).
\end{align*}
\]

(A.13)

Since we are considering \(O(3)\) the continuous symmetry gets completely broken, leading to the three independent Goldstone boson matrices (A.13). Notice that the nine generators of the original symmetry group \(O(3) \times O(3) \times O(3)\) match the number of independent components of the three \(3 \times 3\)-antisymmetric matrices in (A.13).

Although there is no continuous symmetry left, it still remains a discrete \(S_3\) symmetry. Interestingly, we can track how the Goldstone bosons transform under this symmetry. Let us see how it goes for \(B^{(1)}\). First, realize that

\[
\begin{align*}
(B^{(1)})_{ij}(x) &= -(v_{i2i3}\varphi_{ji2i3}(x) - v_{j2i3}\varphi_{ij2i3}(x)) \\
&= -(v_{i\sigma(i_2)\sigma(i_3)}\varphi_{j\sigma(i_2)\sigma(i_3)}(x) - v_{j\sigma(i_2)\sigma(i_3)}\varphi_{i\sigma(i_2)\sigma(i_3)}(x)).
\end{align*}
\]

(A.14)

The second equality in (A.14) is just a rearrangement of the sum over \(i_2\) and \(i_3\) induced by the permutation \(\sigma\). According to (A.6) the transformation of \(B^{(1)}\) under the diagonal \(S_3\) is

\[
\begin{align*}
(B'^{(1)})_{ij} &= (B^{(1)})_{\sigma(i)\sigma(j)}(x) \\
&= -(v_{\sigma(i)\sigma(i_2)\sigma(i_3)}\varphi_{\sigma(j)\sigma(i_2)\sigma(i_3)}(x) - v_{\sigma(j)\sigma(i_2)\sigma(i_3)}\varphi_{\sigma(i)\sigma(i_2)\sigma(i_3)}(x)).
\end{align*}
\]

(A.15)

Using the invariance of \(v\) (A.7), the transformed Goldstone boson reads

\[
(B'^{(1)})_{ij} = -(v_{i2i3}\varphi_{j\sigma(i_2)\sigma(i_3)}(x) - v_{j2i3}\varphi_{i\sigma(i_2)\sigma(i_3)}(x)).
\]

(A.16)

This transformation is the analog to the adjoint action (4.37) for the unitary group.

The procedure described above can be straightforwardly extended to the general SSB (A.8). It would be interesting to study the effective theory related to those symmetry breaking patterns. We leave it for a future work.

References

[1] R. Gurau, “Colored Group Field Theory,” Commun. Math. Phys. 304, 69 (2011), [arXiv:0907.2582 [hep-th]].

[2] R. Gurau, “The 1/N expansion of colored tensor models,” Annales Henri Poincare 12, 829 (2011), doi:10.1007/s00023-011-0101-8 [arXiv:1011.2726 [gr-qc]].

[3] R. Gurau and V. Rivasseau, “The 1/N expansion of colored tensor models in arbitrary dimension,” EPL 95, no. 5, 50004 (2011), [arXiv:1101.4182 [gr-qc]].
[4] R. Gurau, “The complete 1/N expansion of colored tensor models in arbitrary dimension,” Annales Henri Poincare 13, 399 (2012), [arXiv:1102.5759 [gr-qc]].

[5] R. Gurau and J. P. Ryan, “Colored Tensor Models - a review,” SIGMA 8, 020 (2012), [arXiv:1109.4812 [hep-th]].

[6] I. R. Klebanov and G. Tarnopolsky, “On Large $N$ Limit of Symmetric Traceless Tensor Models,” JHEP 1710, 037 (2017), [arXiv:1706.00839 [hep-th]].

[7] S. Giombi, I. R. Klebanov and G. Tarnopolsky, “Bosonic tensor models at large $N$ and small $\epsilon$,” Phys. Rev. D 96, no. 10, 106014 (2017), [arXiv:1707.03866 [hep-th]].

[8] K. Bulycheva, I. R. Klebanov, A. Milekhin and G. Tarnopolsky, “Spectra of Operators in Large $N$ Tensor Models,” Phys. Rev. D 97, no. 2, 026016 (2018), [arXiv:1707.09347 [hep-th]].

[9] S. Giombi, I. R. Klebanov, F. Popov, S. Prakash and G. Tarnopolsky, “Prismatic Large $N$ Models for Bosonic Tensors,” arXiv:1808.04344 [hep-th].

[10] I. R. Klebanov, F. Popov and G. Tarnopolsky, “TASI Lectures on Large $N$ Tensor Models,” arXiv:1808.09434 [hep-th].

[11] F. Ferrari, V. Rivasseau and G. Valette, “A New Large N Expansion for General Matrix-Tensor Models,” arXiv:1709.07366 [hep-th].

[12] J. Ben Geloun and S. Ramgoolam, “Counting Tensor Model Observables and Branched Covers of the 2-Sphere,” arXiv:1307.6490 [hep-th].

[13] J. Ben Geloun and S. Ramgoolam, “Tensor Models, Kronecker coefficients and Permutation Centralizer Algebras,” JHEP 1711, 092 (2017), [arXiv:1708.03524 [hep-th]].

[14] H. Itoyama, A. Mironov and A. Morozov, “Rainbow tensor model with enhanced symmetry and extreme melonic dominance,” Phys. Lett. B 771, 180 (2017), [arXiv:1703.04983 [hep-th]].

[15] A. Mironov and A. Morozov, “Correlators in tensor models from character calculus,” Phys. Lett. B 774, 210 (2017), [arXiv:1706.03667 [hep-th]].

[16] H. Itoyama, A. Mironov and A. Morozov, “Cut and join operator ring in tensor models,” Nucl. Phys. B 932, 52 (2018), [arXiv:1710.10027 [hep-th]].

[17] H. Itoyama, A. Mironov and A. Morozov, “From Kronecker to tableau pseudo-characters in tensor models,” arXiv:1808.07783 [hep-th].

[18] P. Diaz and S. J. Rey, “Orthogonal Bases of Invariants in Tensor Models,” JHEP 1802, 089 (2018), [arXiv:1706.02667 [hep-th]].

[19] P. Diaz and S. J. Rey, “Invariant Operators, Orthogonal Bases and Correlators in General Tensor Models,” Nucl. Phys. B 932, 254 (2018), [arXiv:1801.10506 [hep-th]].

[20] P. Diaz, “Tensor and Matrix models: a one-night stand or a lifetime romance?,” JHEP 1806, 140 (2018), [arXiv:1803.04471 [hep-th]].

[21] E. Witten, “An SYK-Like Model Without Disorder,” arXiv:1610.09758 [hep-th].
[22] T. Azeyanagi, F. Ferrari and F. I. Schaposnik Massolo, “Phase Diagram of Planar Matrix Quantum Mechanics, Tensor, and Sachdev-Ye-Kitaev Models,” Phys. Rev. Lett. 120, no. 6, 061602 (2018), [arXiv:1707.03431 [hep-th]].

[23] J. Yoon, “Supersymmetric SYK Model: Bi-local Collective Superfield/Supermatrix Formulation,” JHEP 1710, 172 (2017), [arXiv:1706.05914 [hep-th]].

[24] J. Yoon, “SYK Models and SYK-like Tensor Models with Global Symmetry,” JHEP 1710, 183 (2017), [arXiv:1707.01740 [hep-th]].

[25] P. Narayan and J. Yoon, “Supersymmetric SYK Model with Global Symmetry,” JHEP 1808, 159 (2018), [arXiv:1804.09934 [hep-th]].

[26] T. Nosaka, D. Rosa and J. Yoon, “The Thouless time for mass-deformed SYK,” JHEP 1809, 041 (2018), [arXiv:1804.09934 [hep-th]].

[27] A. Jevicki, K. Suzuki and J. Yoon, “Bi-Local Holography in the SYK Model,” JHEP 1607, 007 (2016), [arXiv:1603.06246 [hep-th]].

[28] A. Jevicki and K. Suzuki, “Bi-Local Holography in the SYK Model: Perturbations,” JHEP 1611, 046 (2016), [arXiv:1608.07567 [hep-th]].

[29] S. R. Das, A. Jevicki and K. Suzuki, “Three Dimensional View of the SYK/AdS Duality,” JHEP 1709, 017 (2017), [arXiv:1704.07208 [hep-th]].

[30] S. R. Das, A. Ghosh, A. Jevicki and K. Suzuki, “Space-Time in the SYK Model,” JHEP 1807, 184 (2018), [arXiv:1712.02725 [hep-th]].

[31] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, “M theory as a matrix model: A Conjecture,” Phys. Rev. D 55, 5112 (1997), [hep-th/9610043].

[32] P. Di Francesco, P. H. Ginsparg and J. Zinn-Justin, “2-D Gravity and random matrices,” Phys. Rept. 254, 1 (1995), [hep-th/9306153].

[33] D. Berenstein, “A Toy model for the AdS / CFT correspondence,” JHEP 0407, 018 (2004), [hep-th/0403110].

[34] R. de Mello Koch, J. Smolic and M. Smolic, “Giant Gravitons - with Strings Attached (I),” JHEP 0706, 074 (2007), [hep-th/0701066].

[35] S. R. Coleman and E. Witten, “Chiral Symmetry Breakdown in Large N Chromodynamics,” Phys. Rev. Lett. 45, 100 (1980).

[36] H. Matsumoto, N. J. Papastamatiou and H. Umezawa, “The formulation of spontaneous breakdown in the path-integral method,” Nucl. Phys. B 68, 236 (1974).

[37] H. Matsumoto, H. Umezawa, G. Vitiello and J. K. Wyly, “Spontaneous breakdown of a nonAbelian symmetry,” Phys. Rev. D 9, 2806 (1974).

[38] R. Mello Koch, D. Gossman and L. Tribelhorn, “Gauge Invariants, Correlators and Holography in Bosonic and Fermionic Tensor Models,” JHEP 1709, 011 (2017), [arXiv:1707.01455 [hep-th]].

[39] S. Weinberg, “The quantum theory of fields. Vol. 2: Modern applications,” (Cambridge University Press, 1995).
[40] J. Polchinski, “An Introduction to the Bosonic String”, String Theory Vol. 1 (Cambridge University Press, Cambridge, 405 1998).

[41] S. Choudhury, A. Dey, I. Halder, L. Janagal, S. Minwalla and R. Poojary, “Notes on melonic $O(N)^{q-1}$ tensor models,” JHEP 1806, 094 (2018), [arXiv:1707.09352 [hep-th]].

[42] I. R. Klebanov and G. Tarnopolsky, “Uncolored random tensors, melon diagrams, and the Sachdev-Ye-Kitaev models,” Phys. Rev. D 95, no. 4, 046004 (2017), [arXiv:1611.08915 [hep-th]].

[43] W. Fulton and J. Harris, “Representation Theory”, (Springer-Verlag New York, 2004).

[44] S. Weinberg, “Effective Gauge Theories,” Phys. Lett. 91B, 51 (1980).

[45] E. D’Hoker and S. Weinberg, “General effective actions,” Phys. Rev. D 50, R6050 (1994), [hep-ph/9409402].

[46] S. Weinberg, “Effective field theories in the large N limit,” Phys. Rev. D 56, 2303 (1997), [hep-th/9706042].

[47] Y. Bai and B. A. Dobrescu, “Minimal $SU(3) \times SU(3)$ Symmetry Breaking Patterns,” Phys. Rev. D 97, no. 5, 055024 (2018), [arXiv:1710.01456 [hep-ph]].

[48] V. Bonzom and F. Combes, “Tensor models from the viewpoint of matrix models: the case of loop models on random surfaces,” Ann. Inst. H. Poincare Comb. Phys. Interact. 2, no. 2, 1 (2015), [arXiv:1304.4152 [hep-th]].

[49] F. Ferrari, “The Large D Limit of Planar Diagrams,” arXiv:1701.01171 [hep-th].

[50] C. Krishnan, S. Sanyal and P. N. Bala Subramanian, “Quantum Chaos and Holographic Tensor Models,” JHEP 1703, 056 (2017), [arXiv:1612.06330 [hep-th]].

[51] C. Krishnan, K. V. P. Kumar and S. Sanyal, “Random Matrices and Holographic Tensor Models,” JHEP 1706, 036 (2017), [arXiv:1703.08155 [hep-th]].

[52] H. Itoyama, A. Mironov and A. Morozov, “Ward identities and combinatorics of rainbow tensor models,” JHEP 1706, 115 (2017) [arXiv:1704.08648 [hep-th]].