On the description of identifiable quartics

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ABSTRACT
In this paper, we study the identifiability of specific forms (symmetric tensors), with the target of extending recent methods for the case of 3 variables to more general cases. In particular, we focus on forms of degree 4 in 5 variables. By means of tools coming from classical algebraic geometry, such as Hilbert function, liaison procedure and Serre’s construction, we give a complete geometric description and criteria of identifiability for ranks \( \geq 9 \), filling the gap between rank \( \leq 8 \), covered by Kruskal’s criterion, and 15, the rank of a general quartic in 5 variables. For the case \( r = 12 \), we construct an effective algorithm that guarantees that a given decomposition is unique.

1. Introduction

We consider the study of symmetric tensors \( T \) over \( \mathbb{C} \), which we will identify with homogeneous polynomials (forms), with respect to their Waring (i.e. symmetric) rank and identifiability. Our point of view is the following. Assume that a specific form \( T \) is given and assume we know an expression of \( T \) as a sum of \( r \) powers of linear forms,

\[
T = \lambda_1 T_1 + \cdots + \lambda_r T_r.
\]

Then:

(Q1) is the length \( r \) minimal? In other words, is \( r \) the rank of \( T \)?

If the answer to the previous question is positive, then another question arises:

(Q2) is (1) the unique expression of minimal length for \( T \)?

Here unique means, of course, up to a permutation of the summands. When uniqueness holds, we will say that \( T \) is identifiable.

The most celebrated criterion that answers the questions is due to Kruskal (see [1]). It works provided that an inequality involving some invariants of the expression (Kruskal’s
ranks) is satisfied. Kruskal’s criterion, which indeed works for general multilinear tensors, is continuously employed in applications of tensor analysis to statistics, signal theory, chemistry, quantum information theory, artificial intelligence, etc. We mention, for one, the paper [2]. It is known that Kruskal’s criterion is sharp. Nothing better can be done if one uses solely Kruskal’s ranks of expression (1) (see [3]). Since Kruskal’s inequality cannot hold outside a precise range for the length \( r \), applications of Kruskal’s criterion are bounded to some, typically quite small, values of \( r \). The range in which Kruskal’s criterion possibly applies is much smaller, in fact, than the range in which identifiability holds for general forms (see [4]).

Actually, there are methods that determine the identifiability of \( T \) in a wider range for \( r \), provided that one computes some higher invariants of (1). For instance, the flattening procedure can determine the identifiability of \( T \) even when Kruskal’s inequality fails (see [5,6]). Yet, it seems hard to cover with flattening methods the whole range in which identifiability can hold.

In a series of papers [4,7–12], it is shown that a geometric approach can determine the identifiability of tensors in the whole range in which the property can hold. The new method starts by considering a finite set \( A = \{P_1, \ldots, P_r\} \) of points in a projective space \( \mathbb{P}^n \), for forms in \( n + 1 \) variables), naturally associated to (1). The analysis of the geometry of \( A \) can exclude the existence of a second expression for \( T \) of length \( \leq r \), for \( r \) ranging in a wide set of values.

In the new methods, properties of the finite set \( A \subset \mathbb{P}^n \) needed to examine the identifiability of \( T \) are rather deep and require advanced tools, both from algebra and geometry. They are based on an accurate analysis of the resolution of the homogeneous ideal, the Hilbert function and the Hilbert–Burch matrix of \( A \), together with liaison techniques. For ternary forms, when set \( A \) lives in \( \mathbb{P}^2 \), these properties describe the situation quite completely. When the number of variables grows, and \( A \) lives in higher dimensional projective spaces, even deeper theoretical results are necessary to understand the identifiability of specific forms. Not all the geometric tools necessary for the analysis are presently available. This turns out to be a stimulating challenge for algebraic geometry that could produce substantial advances both in the theory and for applications.

We produce a pattern that introduces some deeper geometric tools (minimal resolution conjecture, vector bundles on surfaces) and determines a criterion for the identifiability of some forms \( T \) in more than three variables. These advanced tools are not new, even if rather recent. What is new is their combination in a procedure that excludes the existence of sets of projective points forming an alternative decomposition of \( T \). We focus in particular on the case of quartics in 5 variables, even if a similar theory can handle more general forms. The identifiability of quartics in 2, 3, 4 variables turns out to be widely understood (for 4 variables, see, e.g. [9]). The case of quartics in 5 variables was not completely covered in the previous literature.

For quartics in 5 variables, we will study in detail the uniqueness of expression (1) when \( r \) is equal to 9, 10, 11, 12, 13. Notice that the range \( r < 9 \) is covered by the original Kruskal criterion. For higher values of \( r \), notice that \( r = 14 \) is a special case of the Alexander–Hirschowitz theorem [13], while 15 is the generic value for the rank. It follows that for \( r \geq 14 \) the standard map from the abstract secant variety to the secant variety of the 4-Veronese variety of \( \mathbb{P}^4 \) (as in Remark 2.1) has positive dimensional fibres, and this implies (see, e.g. [14]) that an expression of length \( r \geq 14 \) cannot be unique. Indeed, if one such expression exists, then infinitely many expressions of the same length must exist. So,
identifiability is excluded for \( r \geq 14 \). Thus, we have a method which tests the identifiability of \( T \) for all possible values of \( r \).

Case \( r \leq 11 \) turns out to be different from cases \( r = 12, 13 \). For \( r \leq 11 \), when \( A \) is sufficiently general (in a precise sense, that can be tested by computer algebra algorithms), then identifiability holds, and no forms spanned by the powers \( T_1, \ldots, T_r \) of (1), except trivially those spanned by proper subsets of the \( T_i \)'s, can have alternative expressions of length \( \leq r \).

Conversely for \( r = 12 \), even if the \( T_i \)'s are general, yet their linear span \( L \) contains forms with alternative expressions of length 13, but it also contains forms of rank 12 (not generated by proper subsets of \( \{ T_1, \ldots, T_{13} \} \)). Even in this case, we are able to produce a procedure that excludes the existence of alternative expressions of length \( \leq 13 \) for a specific \( T \).

In conclusion, we show that advanced geometric tools give a method to test the uniqueness (and minimality) of an expression (1) for quartics in 5 variables. The criterion is effective, in the sense of [4]: it will give a (positive) answer for forms which lie outside an algebraic (Zariski closed) subset in the space of forms \( S^4(\mathbb{C}^5) \).

The case \( r = 13 \) is even more involved. Not only the span \( L \) of general \( T_i \)'s contains forms with alternative expressions of length 13, but it also contains forms of rank 12 (not generated by proper subsets of \( \{ T_1, \ldots, T_{13} \} \)). Even in this case, we are able to produce a procedure that excludes the existence of alternative expressions of length \( \leq 13 \) for a specific \( T \).

The structure of the paper is the following. Section 2 is devoted to the main notation, concepts and results, used throughout the paper, coming both from tensors setting (such as Kruskal’s criterion and its generalizations) and from classical algebraic geometry (Hilbert function, Cayley–Bacharach property, Liaison procedure and Serre’s construction). By means of the above-mentioned fundamental tools, in Section 3, we give a geometric procedure to test the identifiability for quartics in five variables, complete in a dense subset, which covers also the cases of ranks 11, 12, 13 that, at the best of our knowledge, were not covered in the mathematical literature on this subject.

### 2. Preliminaries

#### 2.1. Notation

For \( d, n \in \mathbb{N} \), let \( \mathbb{C}^{n+1} \) be the space of linear forms in \( x_0, \ldots, x_n \), thus \( S^d\mathbb{C}^{n+1} \) is the space of forms of degree \( d \) in \( n+1 \) variables over \( \mathbb{C} \). Every \( T \in S^d\mathbb{C}^{n+1} \) defines an element of \( \mathbb{P}(S^d\mathbb{C}^{n+1}) \cong \mathbb{P}^N \) \((N = \binom{n+d}{d} - 1)\), which we still denote by \( T \). Moreover, \( \nu_d : \mathbb{P}^n \to \mathbb{P}^N \) is the Veronese embedding of \( \mathbb{P}^n \) of degree \( d \), i.e.

\[
\nu_d([a_0x_0 + \ldots + a_nx_n]) = [(a_0x_0 + \ldots + a_nx_n)^d].
\]

For any finite set \( A = \{ P_1, \ldots, P_r \} \subset \mathbb{P}^n \), \( \langle \nu_d(A) \rangle \) is the linear space in \( \mathbb{P}^N \) spanned by the points \( \nu_d(P_1), \ldots, \nu_d(P_r) \). The cardinality of \( A \) is usually denoted by \( \ell(A) \) and \( I_A, \mathcal{I}_A \) are,
respectively, the ideal of $A$ in the polynomial ring $R = \mathbb{C}[x_0, \ldots, x_n]$ and the ideal sheaf of $A$ on $\mathbb{P}^n$. We will denote with subscripts the homogeneous pieces of $R$ and its ideals.

**Definition 2.1:** Given a finite set $A \subset \mathbb{P}^n$ and a form $T \in S^d \mathbb{C}^{n+1}$, we say that

- $A$ computes $T$, or that $A$ is a decomposition of $T$, if $T \in \langle v_d(A) \rangle$;
- $A$ is non-redundant if $A$ computes $T$ and there are no proper subsets $A'$ of $A$ such that $A'$ computes $T$;
- $A$ is minimal if $A$ computes $T$ and there are no sets $B$, with $\ell(B) < \ell(A)$, such that $B$ computes $T$;
- $\ell(A)$ is the (Waring) rank of $T$ if $A$ is minimal;
- $T$ is identifiable if $A$ is the unique set such that $\ell(A)$ equals the rank of $T$.

### 2.2. Kruskal’s criterion for forms and its extensions

In the mathematical literature, one of the most famous criteria for detecting the identifiability of a tensor is due to Kruskal. It is based on the concept of $d$th Kruskal’s rank, $k_d$, of a finite set, for whose definition and main properties we refer to §2.2 of [12]. Here we recall one extension of this criterion, adapted to the case of forms.

**Theorem 2.2 (Reshaped Kruskal’s Criterion, see [4]):** Assume $d \geq 3$ and let $A \subset \mathbb{P}^n$ be a non-redundant decomposition of $T \in \mathbb{P}^1(S^d \mathbb{C}^{n+1})$. Fix a partition $d = d_1 + d_2 + d_3$ with $d_1 \geq d_2 \geq d_3 \geq 1$ and denote by $k_{d_i}(A)$ the $d_i$th Kruskal rank of $A$. If

$$\ell(A) \leq \frac{k_{d_1}(A) + k_{d_2}(A) + k_{d_3}(A) - 2}{2}$$

then $T$ has rank $\ell(A)$ and it is identifiable.

In the case of ternary forms, Theorem 2.2 has been recently extended in [11,12]. For the case of quartics in $n + 1$ variables, we refer to the following extension (see Section 6 of [9]):

**Theorem 2.3:** Let $T \in \mathbb{P}(S^4 \mathbb{C}^{n+1})$ and let $A = \{P_1, \ldots, P_{2n+1}\} \subset \mathbb{P}^n$ be a non-redundant decomposition of $T$. Write $X = v_4(\mathbb{P}^n)$ for the image of the Veronese embedding. If

(a) $\dim(v_4(P_1), \ldots, v_4(P_{2n+1})) = 2n + 1$,
(b) $k_1(A) = n + 1$,
(c) the linear span of the union of tangent spaces $\bigcup T_{X, v_4(P_i)}$ has the (expected) dimension $2n^2 + 3n + 1$,

then $T$ is identifiable of rank $2n + 1$.

**Remark 2.1:** Condition (c) in the previous statement is linked to the Terracini Lemma which describes the tangent space to secant varieties. For specific decompositions $A$ of $T$, the condition can be verified by a linear algebra algorithm introduced in [15].

It is a standard consequence of Terracini’s construction that if $T$ has infinitely many decompositions, then condition (c) cannot hold (see, e.g. Lemma 6.5 of [15] or Proposition 6 of [16]).
2.3. Hilbert function, first difference, $h$-vector of finite sets

Other important tools for our analysis, which come from classical Algebraic Geometry, are the Hilbert function, its first difference and, consequently, the $h$-vector of a finite set. For completeness, we briefly recall their definitions.

**Definition 2.4:** Let $Y$ be a set of homogeneous coordinates for a finite set $Z \subset \mathbb{P}^n$.

- The **Hilbert function** of $Z$ is the map
  \[
  h_Z : Z \rightarrow \mathbb{N}
  \]
  such that
  \[
  h_Z(j) = \begin{cases} 
  0 & \text{for } j < 0, \\
  \text{rank}(ev_Y(j)) & \text{for } j \geq 0,
  \end{cases}
  \]
  where $ev_Y(j) : S^j \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{\ell(Z)}$ is the evaluation map of degree $j$ on $Y = \{Y_1, \ldots, Y_{\ell(Z)}\}$, i.e. the linear map given by $ev_Y(j)(F) = (F(Y_1), \ldots, F(Y_{\ell(Z)}))$.

- The **first difference** of the Hilbert function $Dh_Z$ is given by
  \[
  Dh_Z(j) = h_Z(j) - h_Z(j - 1), \quad j \in \mathbb{Z}.
  \]

- The **$h$-vector** of $Z$ is the vector consisting of all the non-zero values of $Dh_Z$.

We refer to §2.3 of [12] for a list of many useful properties.

We just point out that when $h_Z(d) = \ell(Z)$ we say that $Z$ is separated in degree $d$. Notice that $Z$ is separated in degree $d$ when the evaluation map surjects. This implies that $Z$ is also separated in any degree $\geq d$. We stress, for reference, the following standard fact.

**Proposition 2.5:** Assume that $Z$ is separated in degree $d-1$. Then for a general linear form $\Lambda \in R$ the restriction map $(I_Z)_d \rightarrow (R/\Lambda)_d$ surjects.

**Proof:** We can identify $R/\Lambda$ with the polynomial ring in $n$ variables. The restriction map $R_d \rightarrow (R/\Lambda)_d$ surjects. The conclusion follows from the snake lemma applied to the diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & (I_Z)_{d-1} & \rightarrow & R_{d-1} & \rightarrow & \mathbb{C}^{\ell(Z)} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \| & & \downarrow \\
0 & \rightarrow & (I_Z)_d & \rightarrow & R_d & \rightarrow & \mathbb{C}^{\ell(Z)} & \rightarrow & 0 \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
& & & & (R/\Lambda)_d & & & & \rightarrow 0
\end{array}
\]

where the leftmost vertical maps are multiplication by $\Lambda$. □

The application of the Hilbert function to the identifiability of forms is based on the following well-known proposition (see, e.g. Lemma 1 of [7]).

**Proposition 2.6:** Let $T \in S^d \mathbb{C}^{n+1}$ and let $A, B \subset \mathbb{P}^n$ be non-redundant decomposition of $T$. Then $Dh_{A \cup B}(d + 1) > 0$. 
2.4. The CB property for finite sets

The Cayley–Bacharach property is fundamental to detect all possible $h$-vectors of $A \cup B$, where $A$ and $B$ are disjoint non-redundant decompositions of a form $T$.

**Definition 2.7**: A finite set $Z \subset \mathbb{P}^n$ satisfies the Cayley–Bacharach property in degree $i$, for simplicity $CB(i)$, if $H^0(I_{Z \setminus \{P\}}(i)) = H^0(I_Z(i))$ for all $P \in Z$.

The application of the $CB$ property to our analysis is based on the following results. For the proofs, we refer, respectively, to [9,11].

**Theorem 2.8**: Let $Z \subset \mathbb{P}^n$ be a finite set satisfying $CB(i)$ and let $j \in \{0, \ldots, i\}$. Then

$$Dh_Z(0) + Dh_Z(1) + \cdots + Dh_Z(j) \leq Dh_Z(i + 1 - j) + \cdots + Dh_Z(i + 1).$$

**Theorem 2.9**: Let $T \in S^d \mathbb{C}^{n+1}$ and let $A, B \subset \mathbb{P}^n$ be non-redundant decompositions of $T$ such that $A \cap B = \emptyset$. Then $A \cup B$ satisfies $CB(i)$, for any $i \in \{0, \ldots, d\}$.

2.5. Liaison and mapping cone

Several constructions will be based on the notion of liaison, or linkage, of finite sets, that we recall here briefly. We point to [17–19] for details and proofs.

**Definition 2.10**: We say that two finite sets $A, B \subset \mathbb{P}^n$ are linked when there exists a complete intersection $Z$ such that $I_B = I_Z : I_A$. When $A, B$ are disjoint, this simply means that $A \cup B = Z$. When $A, B$ are linked by $Z$, then we also say that $B$ is the residue of $A$ with respect to $Z$.

For a finite set $Z$, a resolution of the ideal $I_Z$ is an exact sequence of free modules and degree 0 maps:

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to I_Z \to 0.$$

When $Z$ is a complete intersection, a resolution is given by the Koszul complex (see [20, section 17]), so that $F_n$ has rank 1.

If $A \subset Z$, then the resolutions of the ideals of $A, Z$ determine a commutative diagram of resolutions:

$$
\begin{array}{ccccccc}
0 & \to & F_n & \to & F_{n-1} & \to & \cdots & \to & F_1 & \to & I_Z & \to & 0 \\
& & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow & & \\
0 & \to & G_n & \to & G_{n-1} & \to & \cdots & \to & G_1 & \to & I_A & \to & 0
\end{array}
$$

where the rightmost vertical arrow is the natural inclusion.

If furthermore $Z$ is a complete intersection of type $d_1, \ldots, d_n$, then a (non-necessarily minimal) resolution of the ideal of the residue $B$ is given by the mapping cone of the previous diagram

$$0 \to G_1^\vee \to F_1^\vee \oplus G_2^\vee \to \cdots \to F_{n-1}^\vee \oplus G_n^\vee$$

twisted by $-d_1 - \cdots - d_n$. 

As a consequence, the $h$-vectors $Dh_A$, $Dh_B$, and $Dh_Z$ of sets $A$, $B$ linked by a complete intersection $Z$ as above are related by the following formula

$$Dh_B(i) + Dh_A(d_1 + \cdots + d_n - n - i) = Dh_Z(i). \quad (3)$$

The previous construction generalizes to the case in which $Z$ is arithmetically Gorenstein.

**Definition 2.11:** We say that a finite set $Z$ is *arithmetically Gorenstein* if a minimal resolution of $I_Z$ is auto-dual, i.e. for all $i$ the dual of the map $F_i \to F_{i-1}$ is, up to twist, the map $F_{n-i+1} \to F_{n-i}$ (here we take $F_0 = I_Z$).

In particular, we have that $F_n$ has rank 1 and, for all $i$, $F_i$ is dual to $F_{n-i}$. This implies that the $h$-vector $Dh_Z$ of $Z$ is symmetric, i.e.

$$Dh_Z(i) = Dh_Z(s - i) \quad \forall i,$$

where $s$ is the maximum such that $Dh_Z(s) > 0$.

All complete intersection sets are arithmetically Gorenstein (but the converse is false).

If $Z = A \cup B$ is arithmetically Gorenstein, then the mapping cone of the diagram obtained by the resolutions of $I_Z, I_A$ provides, as in the case in which $Z$ is a complete intersection, a resolution for $I_B$.

**Proposition 2.12 (see [21]):** If the $h$-vector $(h_0, h_1, \ldots, h_s)$ of $Z$ is symmetric and $Z$ satisfies $CB(s)$, then $Z$ is arithmetically Gorenstein.

### 2.6. Rank 2 bundles and Serre construction

When a set $Z$ of points lies in a smooth surface $S \subset \mathbb{P}^n$, the Serre construction provides a link between the Cayley–Bacharach property and the existence of rank 2 vector bundles on $S$ associated with $Z$. We list in this section just the aspects of the connection that will be necessary for our analysis.

We recall that if $\mathcal{E}$ is a vector bundle of rank 2 on a smooth surface $S$, with Chern classes $c_1(\mathcal{E}) \in \text{Pic}(S)$ and $c_2(\mathcal{E}) \in \mathbb{Z}$, and $D$ is a divisor on $S$, then the Chern classes of the twist $\mathcal{E}(D)$ are given by

$$c_1(\mathcal{E}(D)) = c_1(\mathcal{E}) + 2D \quad c_2(\mathcal{E}(D)) = c_2(\mathcal{E}) + D \cdot c_1(\mathcal{E}) + D^2. \quad (4)$$

**Proposition 2.13 (Serre construction):** Let $S$ be a smooth surface in $\mathbb{P}^n$, with canonical divisor $K$ and hyperplane divisor $H$. Let $Z \subset S$ be a finite set which satisfies property $CB(d)$. Then there is a rank 2 vector bundle $\mathcal{E}$ on $S$, with Chern classes $c_1(\mathcal{E}) = dH + K$ and $c_2(\mathcal{E}) = \ell(Z)$ such that $Z$ is the zero-locus of a global section of $\mathcal{E}$. The ideal sheaf $\mathcal{I}_{Z,S}$ of $Z$ in $S$ fits in an exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{E} \to \mathcal{I}_{Z,S}(dH - K) \to 0,$$

in which the first map represents the global section that vanishes on $S$.

Conversely, if a finite set $Z$ is the zero-locus of a global section of a rank 2 bundle $\mathcal{E}$ as above, with first Chern class $dD + K$, then $Z$ satisfies the Cayley–Bacharach property $CB(d)$.
We will apply the previous proposition when $S$ in $\mathbb{P}^4$ is a complete intersection of two quadrics, so that $K = -H$. In this case, the exact sequence reads

$$0 \to \mathcal{O}_S \to \mathcal{E} \to \mathcal{I}_{Z,S}((d+1)H) \to 0.$$ \hspace{1cm} (5)

For the proof, we refer to [22].

The existence of a section of a vector bundle $\mathcal{E}$ whose zero-locus is finite is regulated by the following proposition (see Remark 1.0.1 of [23]).

**Proposition 2.14**: Assume that $\mathcal{E}$ has non-trivial global sections. If all global sections of $\mathcal{E}$ vanish in infinitely many points, then there exists an effective divisor $D$ on $S$ such that all sections of $\mathcal{E}$ are given by the product of a global section of $\mathcal{E}(D)$ times a global section of $\mathcal{O}(D)$.

### 3. Forms of degree 4 in five variables

We turn now to the case $n = 4$, $d = 4$ and let $T \in S^4\mathbb{C}^5$. Thus $T$ can be seen as a polynomial of degree 4 in five variables, which is associated with a hypersurface of degree 4 in $\mathbb{P}^4$.

**Remark 3.1**: For a general $T \in S^4\mathbb{C}^4$, according to the Alexander–Hirschowitz Theorem [13], the rank is 15, while the case of rank 14 is defective, in the sense of [14]. In both cases, the dimension of the secant variety is smaller than the dimension of the corresponding abstract secant variety, so that the identifiability cannot hold (this is a well-known fact, see, e.g. [24, Proposition 2.2] and its proof).

On the other hand, if a specific $T$ admits a non-redundant decomposition $A$ of cardinality $r \leq 8$, then Theorem 2.2 can be applied to establish the identifiability and rank of $T$, while if $r = 9$, then one can refer to the criterion developed in [9] and recalled in Theorem 2.3.

Therefore we assume, from now on, that $10 \leq r \leq 13$.

Let $A = \{P_1, \ldots, P_r\} \subset \mathbb{P}^4$ be a finite set that computes $T$. We often assume that $A$ satisfies the following conditions:

$$\begin{align*}
(i) & \quad A \text{ is non-redundant;} \\
(ii) & \quad k_1(A) = 5; \\
(iii) & \quad k_2(A) = r.
\end{align*}$$

It is a standard fact that when $r \leq 21$, for $A$ in a Zariski open subset of $(\mathbb{P}^4)^r$, then $A$ satisfies the previous conditions.

Conditions (i), (ii), (iii) imply that the Hilbert function of $A$ and its first difference verify

$$\begin{align*}
j & \quad 0 \quad 1 \quad 2 \quad 3 \quad \ldots \\
h_A(j) & \quad 1 \quad 5 \quad r \quad r \quad \ldots \\
Dh_A(j) & \quad 1 \quad 4 \quad r-5 \quad 0 \quad \ldots
\end{align*}$$ \hspace{1cm} (6)

We notice that conditions (i), (ii), (iii) can be easily controlled by a linear algebra algorithm (see the code available online at https://arxiv.org/src/2106.06730v1/anc/ancillary.txt that can be implemented in Macaulay2 [25]).
When $A$ is sufficiently general, then conditions (i), (ii), (iii) hold. Namely, a general set $A'$ of $r'$ points in $\mathbb{P}^4$, $5 \leq r' \leq 15$ satisfies $h_{A'}(1) = 5$, $h_{A'}(2) = r$. It is clear that if $A$ is general, then any subset of $A$ is general.

The following result is the basis for our analysis. It is a consequence of Theorem 1.2 of [8]. We provide here a different proof.

**Proposition 3.1:** Let $A$, $B$ be non-redundant sets that compute $T$. Assume that $A$ satisfies conditions (i), (ii), (iii), and assume that $A$, $B$ are disjoint. Then $B$ has a length equal to $A$.

Let $Q$ be the base locus of the linear system of quadric hypersurfaces passing through $A$. Then $Q$ contains $B$ too.

**Proof:** Set $\ell(A) = r$ and $Z = A \cup B \subset \mathbb{P}^4$. Notice that, being $A \subset Z$, from (6) we get that $Dh_Z(1) = 4$ and $Dh_Z(2) \geq r - 5$. Moreover, Proposition 2.6 implies $Dh_Z(5) \geq 1$.

Since $A$ and $B$ are non-redundant for $T$, then by Proposition 2.9, $Z$ has the Cayley–Bacharach property $CB(4)$. Therefore, from Theorem 2.8, we have

$$Dh_Z(3) + Dh_Z(4) + Dh_Z(5) \geq Dh_Z(0) + Dh_Z(1) + Dh_Z(2) \geq r.$$ 

Moreover, from the chain of inequalities

$$2r \geq \ell(Z) = \sum_{j \in Z} Dh_Z(j) \geq \sum_{j=0}^{5} Dh_Z(j) \geq r + Dh_Z(3) + Dh_Z(4) + Dh_Z(5),$$

we deduce that

$$Dh_Z(3) + Dh_Z(4) + Dh_Z(5) \leq r.$$ 

It turns out that

$$Dh_Z(3) + Dh_Z(4) + Dh_Z(5) = r.$$ 

Necessarily it has to be that $Dh_Z(2) = Dh_A(2) = r - 5$, $\ell(Z) = 2r$ and $\ell(B) = r$. In particular, $h_Z(2) = h_A(2)$ and $(I_Z)_2 = (I_A)_2$.

Thus, if we can compute that $Q$ is finite, of length $< 2r$, we obtain the non-existence of $B$. This can be easily achieved by a computer-aided procedure, as described in Remark 3.2.

### 3.1. Non-empty intersection

First, we consider the case $A \cap B \neq \emptyset$, following the argument used several times in [11].

Assume that $A = \{P_1, \ldots, P_r\}$ and let $B$ be another non-redundant decomposition of $T$ with $s = \ell(B) \leq r$ and define $Z = A \cup B$.

Assume that intersection $A \cap B$ is not empty.

Then we can reorder the points of $A$ so that $B = \{P_1, \ldots, P_j, P_{j+1}', \ldots, P_s'\}$ with $j \geq 1$ and $P_i \notin A$ for $i = j + 1, \ldots, s$. For any choice of representatives (i.e. coordinates) $T_1, \ldots, T_r$, and $T_{j+1}', \ldots, T_s'$ for the projective points $v_4(P_1), \ldots, v_4(P_r)$, and $v_4(P_{j+1}'), \ldots, v_4(P_s')$, respectively, there are non-zero scalars $a_i$, $b_i$ such that

$$T = a_1 T_1 + \cdots + a_r T_r,$$

$$T = b_1 T_1 + \cdots + b_j T_j + b_{j+1} T_{j+1} + \cdots + b_s T_s'.$$
Define
\[ T_0 = (a_1 - b_1)T_1 + \cdots + (a_j - b_j)T_j + a_{j+1}T_{j+1} + \cdots + a_rT_r \]
\[ = b_{j+1}T_{j+1}' + \cdots + b_sT_s'. \]

Now \( T_0 \) has the two decompositions \( A \) and \( B' = \{P'_{j+1}, \ldots, P'_s\} \), which are disjoint. If \( B' \) is redundant, then after rearranging the points, we may assume \( T_0 = c_{j+1}T_{j+1}' + \cdots + c_tT_t' \) for some \( t < s \), so that
\[ T = b_1T_1 + \cdots + b_jT_j + T_0 = b_1T_1 + \cdots + b_jT_j + c_{j+1}T_{j+1}' + \cdots + c_tT_t', \]
against the fact that \( B' \) is non-redundant. Thus \( B' \) must be non-redundant.

If \( A \) is non-redundant, define \( A' = A \).

If \( A \) is redundant, since the points \( v_4(P_1), \ldots, v_4(P_r) \) are linearly independent, then some coefficient \( (a_i - b_i) \) is 0. In this case, we may assume \( (a_i - b_i) = 0 \) if and only if \( i = 1, \ldots, q \leq j \), so we get a non-redundant decomposition \( A' = \{P_{q+1}, \ldots, P_r\} \) of \( T_0 \).

In conclusion, we find that \( T_0 \) has two different non-redundant decompositions \( A', B' \), with \( A' \subset A \) and \( \ell(B') \leq \ell(A') \), \( \ell(B') < r \). Notice that since \( A' \subset A \), then conditions (i), (ii), (iii) hold for \( A' \).

If \( \ell(A') \leq 8 \), then \( T_0 \) cannot exist, by Theorem 2.2. Thus \( \ell(A') \geq 9 \).

If \( \ell(B') < \ell(A') \) then \( T_0 \) cannot exist, by Proposition 3.1.

In conclusion, we have the following

**Proposition 3.2:** If \( T \) has two non-disjoint decompositions \( A, B \) of length \( 13 \geq \ell(A) \geq \ell(B) \) and \( A \) satisfies conditions (i), (ii), (iii) above, then \( \ell(B) = \ell(A) \geq 9 \).

Moreover, there are disjoint subsets \( A' \subset A \) and \( B' \subset B \), of the same length \( r' \geq 9 \), which are non-redundant decompositions of a form \( T_0 \) such that \( T = T_0 + \sum_{i=0}^{r-r'} a_i v_4(P_i) \), where \( \{P_1, \ldots, P_{r-r'}\} = A \cap B \).

It follows that in many arguments, after replacing \( T \) with \( T_0 \), we will assume that \( A, B \) are disjoint.

Mixing Proposition 3.1 and Proposition 3.2, we get a first result which determines the rank of \( T \).

**Theorem 3.3:** Let \( T \in S^4 \mathbb{C}^5 \) be a form with a decomposition \( A \) of length \( r \leq 13 \), which satisfies conditions (i), (ii), (iii) above. Then \( T \) cannot have a decomposition of length smaller than \( r \). In other words, \( r \) is the (Waring) rank of \( T \).

Next, we turn the attention to the identifiability of \( T \).

### 3.2. Case \( r = 9, 10, 11 \)

With the previous notation, we assume that there exists a second, non-redundant decomposition \( B \) of \( T \) of length \( r \).

**Remark 3.2:** Let \( Q' \) be the base locus of the linear system of quadric hypersurfaces passing through a finite set \( A' \). Let \( A' \subset \mathbb{P}^4 \) be general with \( \ell(A') = 9 \). Then it is a consequence of
the Minimal Resolution Conjecture (which holds in $\mathbb{P}^4$, see [26]) that the homogeneous ideal $IA'$ is generated by quadrics. Indeed, for a general set $A$ of 9 points, $IA$ has a free resolution of the form

$$0 \rightarrow R(-6)^{14} \rightarrow R(-5)^{12} \rightarrow R(-4)^9 \oplus R(-3)^4 \rightarrow R(-2)^{16} \rightarrow IA' \rightarrow 0. \quad (7)$$

In this case, $Q'$ coincides with $A'$ (see the file available online at https://arxiv.org/src/2106.06730v1/anc/ancillary.txt).

Similarly, for a general set $A'$ of 10 points, the homogeneous ideal $IA'$ is generated by quadrics and it has a free resolution of the form

$$0 \rightarrow R(-6)^5 \rightarrow R(-5)^{16} \rightarrow R(-4)^{15} \rightarrow R(-2)^5 \rightarrow IA' \rightarrow 0. \quad (8)$$

Thus also in this case $Q'$ coincides with $A'$ (see the file available online at https://arxiv.org/src/2106.06730v1/anc/ancillary.txt).

For a general set $A'$ of 11 points, the ideal $IA'$ has a free resolution of the form

$$0 \rightarrow R(-6)^6 \rightarrow R(-5)^{20} \rightarrow R(-4)^{21} \rightarrow R(-3)^4 \oplus R(-2)^4 \rightarrow IA' \rightarrow 0. \quad (9)$$

In this case, $Q'$ does not coincide with $A'$. On the other hand, for a general choice of $A'$ one computes (see the file available online at https://arxiv.org/src/2106.06730v1/anc/ancillary.txt) that four general quadrics in $IA'$ intersect properly, so that $Q'$ is a finite set of length 16.

As a consequence, we can use the second part of proposition 3.1 to prove the identifiability of $T$. We add the following condition for $A$:

(iv) the base locus $Q$ of the system of quadrics containing $A$ is finite.

Notice that a general set of $r \leq 11$ points will satisfy condition (iv). Moreover, if $A$ satisfies condition (iv), then trivially all subsets of $A$ also satisfy the condition.

**Proposition 3.4:** Let $T$ be a form of degree 4 in 5 variables. Assume

$$T = a_1 T_1 + \cdots + a_r T_r, \quad (10)$$

where $T_i$'s are powers of linear forms, corresponding to points $v_4(P_1), \ldots, v_4(P_r)$. Assume $r \leq 11$ and that the set $A = \{P_1, \ldots, P_r\}$ satisfies conditions (i), (ii), (iii), (iv).

Then $T$ is identifiable, of rank $r$, i.e. (10) provides the unique decomposition of length $r$ of $T$ (up to multiplication by a scalar and rearranging).

**Proof:** The assumptions say that $A$ is a non-redundant decomposition of $T$.

By condition (iv), four general quadrics containing $A$ meet in a finite set, so the base locus $Q$ of the system of quadrics containing $A$ is a finite set of length at most 16.

Consider a second decomposition $B$ of $T$ of length $\leq r$. Since $T$ has rank $r$ by Theorem 3.3, then in fact $\ell(B) = r$.

If $A, B$ are disjoint, then $A \cup B$ is contained in the intersection $Q$ of the quadrics containing $A$, by Proposition 3.1, so that $\ell(B) \leq 16 - \ell(A)$ and we get a contradiction.
If \( A, B \) are not disjoints, by Proposition 3.2, there are disjoint subsets \( A' \subset A \) and \( B' \subset B \), of the same length \( r' \geq 9 \), which are both decompositions of a form \( T_0 \). Since \( A' \) also satisfies conditions (i), (ii), (iii), (iv), then from the previous argument we get that \( B' \) is contained in the intersection \( Q' \) of quadrics containing \( A' \), which is a set of length at most \( 16 - \ell(A') \leq 7 \), a numerical contradiction. ■

Notice that the identifiability of \( T \) follows once we test that \( A \) is sufficiently general to satisfy conditions (i), (ii), (iii), (iv). The file at https://arxiv.org/src/2106.06730v1/anc/ancillary.txt contains an algorithm that checks if a specific \( A \) of length \( r \leq 11 \) satisfies the four conditions.

It is clear that (ii), (iii), (iv) only deal with the geometry of \( A \). It follows that if \( A \) satisfies (ii), (iii), (iv), then the identifiability holds for all tensors in the span of \( v_4(A) \), for which \( A \) is non-redundant, i.e. for all \( T \) of the form \( T = a_1T_1 + \cdots + a_rT_r \), where all the coefficients \( a_i \)'s are non-zero.

### 3.3. Case \( r = 12 \)

When \( r = 12 \), we still know from Theorem 3.3 that when \( A \) satisfies (i), (ii), (iii) then \( T \) has rank 12.

The situation for the identifiability of \( T \) changes. Indeed, ideal \( I_A \) of a general set of 12 points in \( \mathbb{P}^4 \) has a free resolution of the form

\[
0 \to R(-6)^{\oplus 7} \to R(-5)^{\oplus 24} \to R(-4)^{\oplus 27} \to R(-3)^{\oplus 8} \oplus R(-2)^{\oplus 3} \to I_A \to 0,
\]

and the Hilbert function and its difference are given by

\[
\begin{array}{c|cccccc}
\hline
j & 0 & 1 & 2 & 3 & \ldots \\
\hline
h_A(j) & 1 & 5 & 12 & 12 & \ldots \\
Dh_A(j) & 1 & 4 & 7 & 0 & \ldots \\
\hline
\end{array}
\]

Thus there are only three independent quadrics in \( I_A \); therefore, the intersection \( Q \) contains (and in general it is equal to) a curve of degree 8. It follows that condition (iv) cannot hold, and we cannot use the argument above to exclude the existence of a second decomposition \( B \) of length 12.

We replace condition (iv) with

(iv') for all subsets \( A' \subset A \) of length at most 11, the base locus \( Q' \) of the system of quadrics containing \( A' \) is finite.

Notice that (iv') can be checked only considering subsets of length 11.

When we start with a decomposition \( A \) of length \( r = 12 \), testing the identifiability of the situation becomes much more complicated.

By [27], we know that a decomposition of a general form of degree 4 in 5 variables and rank 12 is unique. But in order to get a criterion which determines the uniqueness of the decomposition, it is not sufficient to look at the points \( P_i \)'s. We will see that for a general choice of the set \( A = \{P_1, \ldots, P_{12}\} \), there are points \( T \) in the span of \( v_4(P_1), \ldots, v_4(P_{12}) \) for which \( A \) is the unique decomposition, and points \( T' \) for which \( A \) is non-redundant, but not unique.
Example 3.5: Let $A = \{P_1, \ldots, P_{12}\}$ be a general set of 12 points in $\mathbb{P}^4$.

The exact sequence (11) provides a minimal resolution for the ideal of $A$, which is thus generated in degree 3. This means that if we take 3 independent quadrics and a general cubic $F$ in $I_A$, the intersection $Z$ is a finite set of 24 distinct points in $\mathbb{P}^4$. $Z$ is a complete intersection set of points. The residue $B$ is a set of 12 points, different from $A$. Indeed, since $Z$ is reduced, for a general choice of $A$ and the cubic $F$, the two sets $A, B$ are disjoint.

We claim that the spans of $v_4(A)$ and $v_4(B)$ meet in a point $T$ representing a form of degree 4 in 5 variables, with rank 12 for which $A, B$ are two different minimal (non-redundant) decompositions. In other words, $T$ is non-identifiable.

In order to prove the claim, thanks to Theorem 3.3 and Proposition 2.19 of [11], it is sufficient to prove $v_4(A \cup B) = v_4(Z)$ is not linearly independent, i.e. $h_Z(4) < 24$. This is well known: it follows immediately from the free resolution of a complete intersection of type 2, 2, 2, 3 (i.e. a complete intersection of 3 quadrics and one cubic) in $\mathbb{P}^4$, which is given by the Koszul complex:

$$0 \to R(-9) \to R(-7) \oplus R(-6) \to R(-5) \oplus R(-4) \oplus R(-3) \oplus R(-2) \to I_Z \to 0.$$

We want to prove that, at least when $A$ is general, the previous example provides the unique possibility for a form $T \in \langle v_4(A) \rangle$, for which $A$ is non-redundant, to be non-identifiable.

Remark 3.3: Assume that $A$ satisfies conditions (i), (ii), (iii), (iv'). If $T$ has a second decomposition $B$ of length 12, then by arguing as in Proposition 3.4 we can prove that $A \cap B$ is empty.

Indeed, if $A \cap B$ is non-empty, then by Proposition 3.2 there are disjoint subsets $A' \subset A$ and $B' \subset B$, of the same length $r' \leq 11$, which are both decompositions of a form $T_0$. Since $A$ satisfies (i), (ii), (iii), (iv'), then $A'$ also satisfies conditions (i), (ii), (iii), (iv), so that $B'$ is contained in intersection $Q'$ of quadrics containing $A'$, which is a set of length at most $16 - \ell(A') \leq 7$, a contradiction.

Proposition 3.6: If $A$ is general, then the intersection of the quadric hypersurfaces in $I_A$ is an irreducible (complete intersection) curve $C$ of degree 8 and arithmetic genus 5.

Proof: The irreducibility of $C$ follows by a standard application of Bertini’s Theorem, see [28, Lemma V.1.2].

For the rest of the section, we will always assume that decomposition $A$ satisfies conditions (i), (ii), (iii), (iv'), and moreover,

(v) the base locus of the system of quadrics through $A$ is an irreducible curve, and the homogeneous ideal $I_A$ has a resolution as (11).

Condition (v) can be checked with the aid of computer algebra packages (see the file at https://arxiv.org/src/2106.06730v1/anc/ancillary.txt for details).

Proposition 3.7: Assume that $A$ satisfies conditions (i), (ii), (iii), (iv'), (v). If $T$ has a second decomposition $B$ of length 12, then $A \cup B$ is a complete intersection of type 2, 2, 2, 3.

Proof: We know that $A \cup B = Z$ is a set of length 24 which satisfies the Cayley–Bacharach property $CB(4)$. Thus, the difference of the Hilbert function of $Z$ satisfies

$$Dh_Z(0) = 1, \quad Dh_Z(1) = 4, \quad Dh_Z(2) \geq 7$$
\[ Dh_Z(3) + Dh_Z(4) + Dh_Z(5) \geq Dh_Z(0) + Dh_Z(1) + Dh_Z(2) \geq 12, \]
\[ Dh_Z(4) + Dh_Z(5) \geq Dh_Z(0) + Dh_Z(1) = 5, \]
\[ \sum_{i=0}^{5} Dh_Z(i) = 24, \]
which imply that \( Dh_Z(2) = Dh_A(2) = 7 \) and also \( h_Z(3) \leq 19 \). Thus the ideal \( I_Z \) coincides with \( I_A \) up to degree 2, hence \( I_Z \) contains three independent quadrics, whose intersection is an irreducible curve \( C \). Moreover, \( I_Z \) contains a cubic \( F \) which is not spanned by the quadrics of \( I_Z \). Hence \( C \cap F \) is a set of 24 points, so that \( Z = F \cap C \).

The linkage determines a resolution of the ideal \( I_B \) of \( B \), via the mapping cone procedure (see Section 2.5). Indeed, from the resolution of \( A \), we get a diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & R(-9) & \oplus & R(-5)^{\oplus 3} & \oplus & R(-3) & \rightarrow & I_Z & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & R(-6)^{\oplus 7} & \rightarrow & R(-5)^{\oplus 24} & \rightarrow & R(-4)^{\oplus 27} & \oplus & I_A & \rightarrow 0 \\
& & & & & & & R(-2)^{\oplus 3} & & & \\
\end{array}
\]

where \( Z = C \cap F \). We get that a resolution of \( I_B \) is numerically identical to the resolution of \( I_A \).

In order to present a specific algorithm which tests whether a given form \( T \in \langle v_4(A) \rangle \) has two decompositions or not, we need to describe how \( T \) is determined by \( A, B \). The procedure, which follows from Proposition 3.9 of [12], is presented in the following example.

**Example 3.8:** Fix a general set \( A \) of 12 points in \( \mathbb{P}^4 \). \( A \) is contained in three independent quadrics, whose intersection is a general (smooth) canonical curve \( C \). Cubics containing \( A \) and not \( C \) determine a 7-dimensional projective space \( \mathbb{P}(\langle I_A \rangle_3/\langle I_C \rangle_3) \). From the coordinates of set \( A \), one computes the ideal \( I_A \). From the choice of a cubic hypersurface \( F \) through \( A \) and the mapping cone procedure, one computes the ideal \( I_B \) of the set of 12 points, residue of the intersection \( F \cap C \) with respect to \( A \). If everything is general enough, then \( B \cap A \) is empty and \( h_Z(4) = 23 \), where \( Z = A \cup B \). It follows that in the polynomial ring \( R = \mathbb{C}[x_0, \ldots, x_4] \) the subspaces \( \langle I_A \rangle_4 \) and \( \langle I_B \rangle_4 \) of \( R_4 \) are both 58-dimensional, but their sum \( U = \langle I_A \rangle_4 + \langle I_B \rangle_4 \) is a linear subspace of codimension 1 in \( R_4 \). Coefficients for the equation of \( U \) provide coefficients for the form \( T \) in the intersection of \( \langle v_4(A) \rangle \) and \( \langle v_4(B) \rangle \). We refer to the file available online at [https://arxiv.org/src/2106.06730v1/anc/ancillary.txt](https://arxiv.org/src/2106.06730v1/anc/ancillary.txt) for details.

Thus, from the coordinates of \( A \) and \( B \), one determines \( T \) explicitly, and also an expression of \( T \) as combination of rank 1 tensors representing the points of \( A \). Experiments prove the existence of \( T \in \langle v_4(A) \rangle \cap \langle v_4(B) \rangle \) for which no coefficients in the decomposition given by \( A \) vanish. We refer for that to the file at [https://arxiv.org/src/2106.06730v1/anc/ancillary.txt](https://arxiv.org/src/2106.06730v1/anc/ancillary.txt).

If \( A \) is general enough, then the span of tangent spaces to the Veronese variety \( v_4(\mathbb{P}^4) \) at the points of \( v_4(A) \) determines a linear subspace of the expected projective dimension 59 in
the projective space spanned by \( v_4(P^4) \). This is easy to check directly for the points \( A \) above (see the file at https://arxiv.org/src/2106.06730v1/anc/ancillary.txt). From Remark 2.1, it follows that there are no positive-dimensional families of decompositions for the form \( T \).

**Corollary 3.9:** Fix a set \( A \) of 12 points in \( P^4 \), which satisfies conditions (i), (ii), (iii), (iv'), (v). Then the span of \( v_4(A) \), which is a \( P^{11} \), contains an irreducible 7-dimensional family \( T \) of forms \( T \) for which \( A \) is a non-redundant decomposition, and such that they have a second decomposition \( B \) obtained as in Example 3.8.

**Proof:** The family \( B \) of sets \( B \) of 12 points linked to \( A \) in a complete intersection of type 2, 2, 2, 3 is irreducible and 7-dimensional. For a general \( B \) in the family, \( h_{A\cup B}(4) = 23 \), so that \( \langle v_4(A) \rangle \) and \( \langle v_4(B) \rangle \) meet in one point \( T \). Call \( T \) the family of points \( T \) that we obtain in this way.

In specific examples one computes that the tangent spaces to \( v_4(P^4) \) at the points of \( v_4(A) \) span a space of dimension 59 (see the ancillary file at https://arxiv.org/src/2106.06730v1/anc/ancillary.txt). Thus, there are no positive dimensional families of decompositions \( B \) which map to the same general \( T \) in the family. \( \Box \)

**Remark 3.4:** The family described in Corollary 3.9 is parametrized by the choice of a cubic in the ideal of \( A \), modulo the span of the three quadrics. This corresponds geometrically to the choice of a point in a projective space \( P^7 \). By the corollary, the map \( \nu \) from \( P^7 \) to \( P^{11} \) which determines the family is birational. One can see that \( \nu \) is described by polynomials of degree 11.

Namely, following the procedure of Example 3.8, by the mapping cone, generators of the ideal of \( B \) depend linearly on the parameter of the choice of a cubic through \( A \). The space \( (I_A)_4 + (I_B)_4 \) is then the column space of a matrix \( N \) of size 70 \( \times \) 69. In the matrix (see the file available at https://arxiv.org/src/2106.06730v1/anc/ancillary.txt) the first 58 columns contain constant entries, while the last 11 columns are linear (in the 8 affine parameters). This implies that the coordinates of a general form \( T \in T \), which are given by the maximal minors of \( N \), are polynomials of degree 11 in the parameters.

In other words, it turns out that \( T \) is a (special) projection of the 11th Veronese image of \( P^7 \).

We describe an algorithm that tests if the decomposition \( A \) satisfies conditions (i), (ii), (iii), (iv'), (v), and moreover, if the given form \( T \) belongs to the family \( T \). The algorithm has been implemented in the file at https://arxiv.org/src/2106.06730v1/anc/ancillary.txt.

Since all the conditions controlled by the algorithm are Zariski open, then the algorithm is effective, in the sense of [4]: it will give a positive answer for all \( T \) lying in a Zariski open, dense subset of the space of all quartics in five variables.

Notice that steps 1–4 and 8 require only standard methods of linear algebra for matrix analysis, while steps 5, 6 need also computer algebra packages.

Notice also that, when the tests 1–6 are positive, then a negative answer to test 8 can be refined to provide generators for the ideal of a second decomposition \( B \) of length 12 for \( T \).

**Example 3.10 (An identifiable quartic):** Let

\[
T_1 = \sum_{i=1}^{12} L_i^4
\]
Algorithm 1

input: a Waring expression \( T = \lambda_1 L_1^4 + \cdots + \lambda_{12} L_{12}^4 \) of a quartic \( T \) in five variables, where each \( L_i \) is a linear form, represented by the vector \( u_i \) of coefficients, and the corresponding set \( A \subset \mathbb{P}^2 \).

1. test (i): check that \( \dim(v_4(u_1), \ldots, v_4(u_{12})) = 12 \);
2. test (ii): check that \( k_1(A) = 5 \);
3. test (iii): check that \( k_2(A) = 12 \);
4. Terracini test: check that the span of the tangent spaces of \( v_4(\mathbb{P}^2) \) at the points of \( v_4(A) \) has the expected dimension 59;
5. test (iv): check that, for all \( i \), the base locus of the system of quadrics through the points corresponding to \( L_j, j \neq i \), is finite, of length \( \leq 16 \);
6. test (v): check that the base locus of the system of quadrics through \( v_4(A) \) is an irreducible curve;
7. parametrization of \( T \): construct a parametrization of alternative decompositions \( B \) of forms in the span of \( v_4(A) \);
8. final test: test that for no choice of the parameters for \( B \) the form \( T \) is orthogonal to generators of \( (I_B + I_A)_4 \).

If any of the tests above answers negatively, then the algorithm terminates by returning that it cannot handle the expression \( T = \lambda_1 L_1^4 + \cdots + \lambda_{12} L_{12}^4 \). Otherwise the algorithm returns that the expression is minimal and unique.

\[
\begin{align*}
0 &= [2454x^4 - 14837x^3y - 9546x^2y^2 + 12272xy^3 + 5779y^4 + 9852x^3z + 11840x^2yz \\
&\quad + 4479xy^2z - 6699y^3z + 5245x^2z^2 + 7347xyz^2 + 979yz^2z^2 - 14274xzt^2 \\
&\quad - 10753yz^3 - 15547z^4 + 3625x^2t + 1511x^2yt - 7021xy^2t + 8756y^3t - 12116x^2zt \\
&\quad - 11133xyzt - 4526y^2zt - 8491xz^2t + 12057yz^2t - 9401z^3t - 10613x^2t^2 \\
&\quad - 6878xzt^2 + 8208y^2t^2 + 3405xzt^2 + 10766yzt^2 - 13732z^2t^2 + 14028xrt^3 \\
&\quad - 9572yrt^3 - 11158zrt^3 - 2774t^4 - 5103x^3w + 5136x^2yw + 10632xy^2w \\
&\quad - 15393y^3w - 4914x^2zw + 8047xyzw - 4020y^5zw - 1609x^2w + 14390yz^2w \\
&\quad - 5791z^3w + 8743x^2tw + 14600xtyw + 11388y^2tw + 6681xztw + 15846yztw \\
&\quad + 9266z^2tw + 3649xt^2w - 4887y^2tw + 12361zt^2w + 8699t^3w + 12211x^2w^2 \\
&\quad - 10563xyw^2 - 13952y^2w^2 + 2139xzw^2 - 12182yzw^2 - 7237z^2w^2 - 113tzw^2 \\
&\quad - 1224ytw^2 - 2612ztw^2 + 13999r^2w^2 - 6977xw^3 - 8368yw^3 + 1738zw^3 \\
&\quad - 14977tw^3 + 3637w^4]
\end{align*}
\]
be a quartic in five variables, where \( L_1, \ldots, L_{14} \) are linear forms represented by a random collection of 12 vectors generated in Macaulay2:

\[
A = [u_i]_{i=1}^{12} = \begin{bmatrix}
-1960 & 7185 & 2948 & 1986 & -7270 \\
8416 & -14232 & 8567 & 14988 & -12297 \\
4210 & -11055 & -6249 & 530 & 6066 \\
-6981 & 1313 & 6692 & 12883 & 4597 \\
8211 & -5857 & 6853 & -5758 & -1890 \\
8633 & 6895 & 14963 & 12883 & -405 \\
12697 & -10281 & 10647 & 1414 & 11296 \\
-15107 & 4696 & -6212 & 6064 & 8777 \\
-14194 & -13431 & -2768 & 6063 & -1066 \\
-687 & 7327 & 9904 & 11696 & 10323 \\
-262 & -14530 & 5673 & 10210 & 5157 \\
-5397 & 6232 & -7867 & -10827 & -653
\end{bmatrix}.
\]

The matrix obtained by applying to each of the \( u_i \)'s the Veronese map \( v_4 \) of degree 4 has full rank 12, so that test (i) has positive answer. Moreover, all the 792 maximal minors of \( A \) do not vanish, which implies that \( k_1(A) = 5 \). Similarly, by applying the Veronese map \( v_2 \) to the rows of \( A \), we get a matrix with all the 455 maximal minors different from 0, i.e. test (iii) is successful. Terracini test provides a positive answer since the 60 \( \times \) 70 matrix associated to the span of the tangent spaces of \( v_4(\mathbb{P}^2) \) at the points of \( v_4(A) \) is of maximal rank. By removing one vector at a time from the rows of \( A \), we get that for each one the 12 sets of 11 points the base locus of the system of quadrics through them is finite dimensional, that is, test \( (iv') \) is successful. Since the ideal of the base locus of the system of quadrics through the points represented by the \( u_i \)'s is irreducible, also test (v) answers positively. Final test translates into check that a linear system with 11 equations and 8 unknowns (which represent the choices of affine parameters for alternative decomposition \( B \) for \( T_1 \)) admits only the trivial solution: since for \( T_1 \) the matrix (MatEqns, in the notation of the file at \url{https://arxiv.org/src/2106.06730v1/anc/ancillary.txt}) associated to this system is of full rank 8, then final test provides a positive answer for \( T_1 \). Therefore, \( T_1 \) is a rank-12 identifiable quartic.

**Example 3.11 (A non-identifiable quartic):** Consider the 12 linear forms \( L_1, \ldots, L_{12} \) in \( x, y, z, t, w \), whose coefficients appear in the rows of the following matrix, generated randomly in Macaulay2:

\[
A = [u_i]_{i=1}^{12} = \begin{bmatrix}
39 & -33 & -11 & 5 & 24 \\
-30 & -28 & 44 & -19 & -32 \\
-15 & 19 & -50 & 43 & 48 \\
-35 & 5 & -9 & -31 & 4 \\
30 & 8 & 5 & -28 & -44 \\
-31 & -33 & 39 & 41 & 47 \\
-6 & -20 & 34 & -7 & 27 \\
-23 & -9 & 31 & -24 & 13 \\
-16 & -34 & -36 & 4 & 5 \\
9 & 49 & 42 & -36 & 34 \\
14 & 50 & 44 & -4 & -17 \\
12 & 48 & 6 & -32 & -26
\end{bmatrix}.
\]
Then, by reversing the procedure illustrated in the section, we constructed a quartic

\[ T = \sum_{i=1}^{12} \lambda_i I_i^4 \]

which has a second decomposition of length 12.

The computer guided construction generated coefficients \( \lambda_i \)'s which are considerably too long to be reported here. We list them explicitly, together with the coefficients of \( T \) in the standard monomial basis of degree 4, in the file available at https://arxiv.org/src/2106.06730v1/anc/ancillary1.txt.

We checked that all steps 1–6 in the algorithm provide positive answers for \( T \). On the other hand, the final test does not succeed, being the \( 11 \times 8 \) matrix MatEqsns, constructed as in the previous example, of rank 7. Indeed, \( T \) has a second decomposition given by linear forms described in the following matrix:

\[
B = [v_i^1]^{12} =
\begin{bmatrix}
7.35766 + 11.8909i & -6.51802 + 2.18767i & 11.0495 + 0.44639i & -3.19976 - 2.02225i & 1 \\
7.35766 - 11.8909i & -6.51802 - 2.18767i & 11.0495 - 0.44639i & -3.19976 + 2.02225i & 1 \\
-2.99409 + 5.9125i & 3.81836 + 6.94562i & -1.56734 + 9.41357i & 7.62593 - 1.87791i & 1 \\
-2.99409 - 5.9125i & 3.81836 - 6.94562i & -1.56734 - 9.41357i & 7.62593 + 1.87791i & 1 \\
0.301069 + 0.983283i & 0.710226 + 0.540009i & -1.10161 - 1.42291i & 0.489521 + 0.792445i & 1 \\
0.301069 - 0.983283i & 0.710226 - 0.540009i & -1.10161 + 1.42291i & 0.489521 - 0.792445i & 1 \\
-0.430153 & -1.88032 & 1.33533 & 1 \\
-6.49097 & 0.732771 & 0.858328 & 1 \\
-0.59154 + 0.0315252i & -0.891556 + 0.00946299i & 0.202355 - 0.0971274i & 0.511408 - 0.174549i & 1 \\
-0.59154 - 0.0315252i & -0.891556 - 0.00946299i & 0.202355 + 0.0971274i & 0.511408 + 0.174549i & 1 \\
-0.512054 & -2.11703 & 0.424299 & 1
\end{bmatrix}
\]

The coordinates of these points have been found by matching eigenvalues corresponding to the same eigenvector of certain companion matrices associated to the polynomial system given by a set of minimal generators of \( I_B \).

We notice that the form \( T \) constructed in Example 3.11 is nevertheless identifiable over the reals.

Indeed, we know from Remark 3.3 and Proposition 3.7 that for any decomposition \( B' \) of \( T \) alternative to \( A \), the union \( A \cup B' \) is a complete intersection of three quadrics and one cubic. The quadrics are uniquely determined by \( A \), while the cubic depends on parameters. Our algorithm in the ancillary file https://arxiv.org/src/2106.06730v1/anc/ancillary.txt for the final test (step 8 above) shows that there exists a unique choice of the parameters for the cubic that produce a set \( B' \) such that \( T \) is orthogonal to the generators of \( (I_{A'} + I_A)^4 \).

Thus \( A \) and set \( B \) described above are the unique decompositions of length 12 of \( T \), and clearly \( A \) is the unique one defined over the reals.

Compare with [29] for similar examples of tensors which are identifiable over the reals, but not identifiable over \( \mathbb{C} \).

### 3.4. Case \( r = 13 \)

This turns out to be the most difficult case, because even when \( A \) is general there are some degenerate cases to consider. From now on, in this section, we assume that \( A \) satisfies conditions (\( i \)), (\( ii \)), (\( iii \)), (\( iv \)) above, so that its Hilbert function is described in table \( (6) \).
When $r = 13$, the ideal $I_A$ of a general set of $r$ points in $\mathbb{P}^4$ has free resolution of the form
\[
0 \to R(-6)^{\oplus 8} \to R(-5)^{\oplus 28} \to R(-4)^{\oplus 33} \to R(-3)^{\oplus 12} \oplus R(-2)^{\oplus 2} \to I_A \to 0.
\]
(13)

**Remark 3.5:** For a general choice of $A$, there are only two independent quadrics containing $A$. Moreover, the quadrics intersect in a smooth surface $S$ of degree 4.

The surface $S$ has a well-known structure. It corresponds to the blow-up of $\mathbb{P}^2$ at 5 points $Q_1, \ldots, Q_5$, embedded with the linear system of cubics through the points $Q_i$'s (a Del Pezzo surface).

The Picard group of $S$ is generated by the strict transform of a line $h$ and the five exceptional divisors $e_1, \ldots, e_5$. The hyperplane divisor is $H = 3h - \sum e_i$. The canonical class of $S$ is $K_S = -H$.

**Example 3.12:** For $r = 13$, even if the set $A$ is general, the span $L$ of $v_4(A)$ contains forms of rank 13 with another decomposition $B$ which intersects $A$.

Namely, write $A = \{P_1, \ldots, P_{13}\}$ and consider $A' = \{P_1, \ldots, P_{12}\} \subset A$. $A'$ is a general set of 12 points, and we saw in the previous section that the span of $v_4(A')$, which is a hyperplane in $L$, contains forms $T'$ of rank 12 with a second decomposition $B'$ of length 12. If $T$ is a general form in the line joining $T'$ with $v_4(P_{13})$, then necessarily $T$ has rank 13 and two minimal decompositions $A = A' \cup \{P_{13}\}$ and $B = B' \cup \{P_{13}\}$.

Thus, in order to prove that the decomposition $A$ is unique, it is necessary to compute, for $i = 1, \ldots, 13$, the tensor $T_i$ in the intersection of the line joining $v_4(P_i)$, $T$ with the linear span of the $v_4(P_j)$'s, $j \neq i$, and run the procedure illustrated in the previous section, to guarantee that $T_i$ has no other decompositions of length $\leq 12$, different from $A \setminus \{P_i\}$.

We want to prove that Example 3.12 is the unique case in which $T$ has a second decomposition $B$, different from $A$ and not disjoint from $A$.

**Proposition 3.13:** Let $A \subset \mathbb{P}^4$ be a set of 13 points satisfying conditions (i), (ii), (iii), (iv') above. Then the span of $v_4(A)$ contains no tensors of rank $\leq 11$, outside the 10 dimensional subspaces generated by the images subsets of $A$ of length 11.

**Proof:** Assume that a form $T$, which sits in the span of $v_4(A)$ and not in the span of subsets of length 11 in $v_4(A)$, has a second decomposition $B$ of length $\leq 11$. By proposition 3.2, we get $A \cap B = \emptyset$. Then $Z = A \cup B$ satisfies the Cayley–Bacharach property $CB(5)$. Thus, we have
\[
\sum_{i=0}^{5} Dh_Z(i) \leq 24,
\]
\[
Dh_Z(0) = 1, \quad Dh_Z(1) = 4, \quad Dh_Z(2) \geq 8,
\]
\[
Dh_Z(3) + Dh_Z(4) + Dh_Z(5) \geq Dh_Z(0) + Dh_Z(1) + Dh_Z(2) \geq 13,
\]
which all together provide a contradiction. ■
**Proposition 3.14:** Let $A \subset \mathbb{P}^4$ be a set of 13 points satisfying conditions (i), (ii), (iii), (iv') above, and let $T$ be a form in the span of $v_4(A)$, such that $A$ is non-redundant for $T$. Assume that $T$ has a second decomposition $B$ of length $s \leq 13$ such that $A \cap B \neq \emptyset$.

Then $s = 13$ and there exists a point $P \in A$ and a form $T'$ in the span of $v_4(A \setminus \{P\})$ such that $T'$ has rank 12 and 2 disjoint decompositions of length 12: $A \setminus \{P\}$ and $B'$, moreover $B = B' \cup \{P\}$.

In particular, $\ell(B) = 13$ and $A \cap B$ is a singleton.

**Proof:** The fact that $s = 13$ follows from Proposition 3.2. With the usual notation, write $B = \{P_1, \ldots, P_j, P_{j+1}', \ldots, P_{13}'\}$, $j \geq 1$ and choose representatives (i.e. coordinates) $T_1, \ldots, T_{13}, T_{j+1}', \ldots, T_s'$ for the projective points $v_4(P_1), \ldots, v_4(P_{13})$ and $v_4(P_{j+1}'), \ldots, v_4(P_s')$, respectively. Then:

$$T = a_1 T_1 + \cdots + a_{13} T_{13},$$

$$T = b_1 T_1 + \cdots + b_j T_j + b_{j+1} T_{j+1}' + \cdots + b_s T_s'.$$

The form

$$T_0 = (a_1 - b_1) T_1 + \cdots + (a_j - b_j) T_j + a_{j+1} T_{j+1} + \cdots + a_{13} T_{13}$$

$$= b_{j+1} T_{j+1}' + \cdots + b_s T_s'$$

has the two decompositions $A$ and $B' = \{P_{j+1}', \ldots, P_s'\}$, which are disjoint.

If $\ell(A') \leq 11$, we get a contradiction with Proposition 3.4. Thus $\ell(A') = \ell(B') = 12$, so that $j = 1$ and the claim is proved.

Now, let us turn to the case of disjoint decompositions $A, B$, with $A$ general of length 13 and $B$ of length $\leq 13$.

**Proposition 3.15:** Let $A \subset \mathbb{P}^4$ be a set of 13 points satisfying conditions (i), (ii), (iii), (iv'), and let $T$ be a form in the span of $v_4(A)$, such that $A$ is non-redundant for $T$, and assume that $T$ has a second decomposition $B$ of length $s \leq 13$ such that $A \cap B = \emptyset$.

Then $\ell(B) = 13$, the ideal of $Z = A \cup B$ coincides with the ideal of $A$ up to degree 2, and $h_Z(3) \leq 21$.

**Proof:** The set $Z$ satisfies the Cayley–Bacharach property $CB(5)$. Thus, we have

$$\sum_{i=0}^{5} Dh_Z(i) \leq 26,$$

$$Dh_Z(0) = 1, \quad Dh_Z(1) = 4, \quad Dh_Z(2) \geq 8,$$

$$Dh_Z(3) + Dh_Z(4) + Dh_Z(5) \geq Dh_Z(0) + Dh_Z(1) + Dh_Z(2) \geq 13.$$

The previous conditions give a contradiction if $\ell(B) < 13$. They imply $Dh_Z(0) + Dh_Z(1) + Dh_Z(2) = 13$ and since $Dh_Z(0) = 1$, $Dh_Z(1) = 4$, then necessarily $h_Z(2) = h_A(2) = 13$. From $CB(5)$, we also get $Dh_Z(5) + Dh_Z(4) \geq Dh_Z(0) + Dh_Z(1) = 5$. Thus $h_Z(3) \leq 21$. 

■
Putting together Proposition 3.14 and Proposition 3.15, we get

**Theorem 3.16:** Let $A \subset \mathbb{P}^4$ be a set of 13 points satisfying conditions (i), (ii), (iii), (iv'). Let $T$ be a form in the span of $v_4(A)$, such that $A$ is non-redundant for $T$. Then $T$ has rank 13.

Next, we want to find conditions for the identifiability of a quartic form in five variables, of rank 13. The following example shows how one can construct non-identifiable forms in the span of $v_4(A)$, even if $A$ is general.

**Example 3.17:** Let $A$ be a general set of 13 points in $\mathbb{P}^4$.

From the resolution of the ideal $I_A$ illustrated by sequence (13), we know that $A$ is contained in the complete intersection of two quadric hypersurfaces. By Bertini’s Theorem for irreducibility, the two quadrics intersect in a quartic surface $S$, whose section with a general hyperplane determined by the linear form $\Lambda$ is an elliptic normal curve $\Gamma$ of degree 4 in $\mathbb{P}^3$.

Fix a general set $W$ of 10 points in $\Gamma$. $W$ is the residue in $\Gamma$ of two points in a complete intersection of type $(2, 2, 3)$. Since 2 points determine a linear series $g^3_2$ on $\Gamma$, then $W$ is contained in a pencil of cubic surfaces of $\mathbb{P}^3$. The $h$-vector of $W$ in $\mathbb{P}^3$ is $(1, 3, 4, 2)$. Thus $W$ is separated by cubics. It follows that $(1, 3, 4, 2)$ is also the $h$-vector in $\mathbb{P}^4$ of $W$ (which is a set of points in the hyperplane $\Lambda$).

There are two cubic surfaces in $\mathbb{P}^3$ which intersect $\Gamma$ properly and contain $W$. The two cubics lift to two cubic hypersurfaces in $\mathbb{P}^4$ which contain $A$. Namely $A$ is separated by quadrics, thus the restriction map $(I_A)_{3} \rightarrow (R/\Lambda)_{3}$ surjects (see Proposition 2.5). It follows that $W' = A \cup W$ is contained in a complete intersection of type $(2, 2, 3, 3)$. The residue is a finite scheme $B$ of length $36-13-10 = 13$.

We want to prove that for a general choice of $A$, $\Lambda$, and $W$ the residue $B$ is a set of 13 points, which determines a second decomposition of a form in the span of $v_4(A)$.

To get the result, we claim first that $A \cup W$ is separated by cubics. Indeed, in $\mathbb{P}^3$ the ideal of $W$ is generated by two quadrics and two cubics, because $W$ is linked $(2, 2, 3)$ to a pair of points. Moreover, the four generators have no syzygies of degree 3. The two quadrics and the two cubics spanning the ideal of $W$ in $\mathbb{P}^3$ lift to two quadrics $Q_1, Q_2$ and two cubics $K_1, K_2$, all containing $A$, as explained above. Thus, any cubic $K$ containing $A \cup W$ is of the form

$$K = \Lambda Q + Q_1 \Lambda_1 + Q_2 \Lambda_2 + c_1 K_1 + c_2 K_2,$$

where $Q$ is a quadric, $\Lambda_1, \Lambda_2$ are linear forms and $c_1, c_2 \in \mathbb{C}$. Since $Q_1, Q_2, K_1, K_2$ contain $A$ and $\Lambda$ is generic, then $Q$ contains $A$. Then $Q$ belongs to the ideal of $A$, which is generated in degree 2 by $Q_1, Q_2$. Thus the ideal of $A \cup W$ is generated by $Q_1, Q_2, K_1, K_2$ up to degree 3. Since the four forms cannot have syzygies in degree 3, then one computes $h_{A \cup W}(3) = 23$, and the claim holds.

When $A$ and $W$ are general, by semi-continuity a general set $B$ linked to $A \cup W$ by a complete intersection of type $2, 2, 3, 3$ is smooth and disjoint from $A \cup W$; hence, it consists of 13 distinct points.

Since the $h$-vector of a complete intersection $(2, 2, 3, 3)$ is $(1, 4, 8, 10, 8, 4, 1)$, as one can easily compute, then the $h$-vector of $Z = A \cup B$, which is linked to $W$, is $(1, 4, 8, 8, 4, 1)$ (see
formula (3)). Thus $A \cup B$ does not impose independent conditions to quartics. The intersection of the spans of $v_4(A)$ and $v_4(B)$ is a point $T$. Experiments on numerical data prove that, for a general choice of $A$, $W$ and the linkage, then $A$, $B$ are two non-redundant decompositions of $T$ (see the file at https://arxiv.org/src/2106.06730v1/anc/ancillary.txt). Thus $T$ has rank 13 (by Theorem 3.16) and two different minimal decompositions.

In the previous example, notice that $I_A$ and $I_{A\cup W}$ coincide in degree 2, and $A \cup W$ is separated by cubics. Thus, the Hilbert function of $A \cup W$ and its difference are given by

$$
\begin{array}{c|cccccc}
  j & 0 & 1 & 2 & 3 & 4 & \ldots \\
  h_{A\cup W}(j) & 1 & 5 & 13 & 23 & 23 & \ldots \\
  Dh_{A\cup W}(j) & 1 & 4 & 8 & 10 & 0 & \ldots \\
\end{array}
$$

It follows by the mapping cone that $B$ has an $h$-vector $(1, 4, 8)$ identical to the $h$-vector of $A$.

We want to prove how we can detect tensors in the span of $v_4(A)$, for $A$ general of length $\ell(A) = 13$, with a second decomposition $B$ of length 13. After Proposition 3.15, we can limit ourselves to the case in which $A \cap B$ is empty.

Call $Z$ the union $Z = A \cup B$. $Z$ has cardinality 26 and, by Theorem 2.9, it satisfies $CB(4)$.

**Remark 3.6:** When $A$ is general, then $I_A$ has two generators of degree 2 and the two hypersurfaces intersect in a smooth quartic surface $S$, whose geometry is outlined in Remark 3.5. By Proposition 3.15, $S$ contains $Z$.

Since $Z$ satisfies $CB(4)$, the Serre construction (see Section 2.6) provides a rank 2 vector bundle $E$ with Chern classes $c_1 = 4H - K_S = 5H$ and $c_2 = 26$ on $S$, such that $Z$ is the zero-locus of a global section of $E$. The ideal sheaf $\mathcal{I}_{Z,S}$ of $Z$ in $S$ sits in the exact sequence

$$
0 \to \mathcal{O}_S \to E \to \mathcal{I}_{Z,S}(5H) \to 0.
$$

(14)

Since $Z$ is contained in 14 independent cubics, while there are only 10 independent cubics containing $S$, then $\mathcal{E}(-2H)$ has global sections. Indeed, $h^0(\mathcal{I}_{Z,S}(4H)) \geq 4$, so one has

$$
h^0(\mathcal{E}(-2H)) \geq 4.
$$

(15)

One computes that the Chern classes of $\mathcal{E}(-2H)$ are $c_1(\mathcal{E}(-2H)) = c_1(\mathcal{E}) - 4H = H$ and $c_2(\mathcal{E}(-2H)) = c_2(\mathcal{E}) - 2H \cdot c_1(\mathcal{E}) + 2H \cdot 2H = 2$.

We want to prove that $\mathcal{E}(-2H)$ has a section which vanishes in codimension 2. This is true, by Proposition 2.14, unless all the global sections of $\mathcal{E}(-2H)$ are obtained by taking one global section of a twist $h^0(\mathcal{E}(-2H - D)) > 0$, with $D$ non-trivial and effective, and multiplying it by elements of $\mathcal{O}_S(D)$. We get the result by exploiting the possibilities for $D$.

**Remark 3.7:** Let $D = ah + \sum_{i=1}^5 b_i e_i$ be a divisor in $S$. Then $h^0(\mathcal{O}_S(ah)) \geq h^0(\mathcal{O}_S(D))$.

Indeed clearly if $b_i \leq 0$ then $h^0(\mathcal{O}_S(D - b_i e_i)) \geq h^0(\mathcal{O}_S(D))$ because $-b_i e_i$ is effective. If $b_i > 0$ then $De_i < 0$, so that $e_i$ is a fixed component in $D$, hence $h^0(\mathcal{O}_S(D - e_i) = h^0(\mathcal{O}_S(D)))$ and one obtains the claim, arguing by induction.

**Proposition 3.18:** A general section of $\mathcal{E}(-2H)$ vanishes in a zero-dimensional scheme of length 2.
Proof: We need to prove that there are no effective divisors $D$ such that every section of $E(-2H)$ is a product of a fixed section $s$ of $E(-2H - D)$ by divisors in $O_S(D)$. By (15) the claim is obvious when $h^0(O_S(D)) < 4$.

Write $D = ah + \sum_{i=1}^{5} b_i e_i$ and order the $b_i$’s so that $b_1 \leq \cdots \leq b_5$.

Since $h^0(O_S(h)) = 3$, by the previous remark, we can exclude the cases in which $a \leq 1$.

Assume $a = 2$. Since $h^0(O_S(2h)) = 6$, we have $b_1, b_2 \geq -1$, $b_3 = b_4 = b_5 \geq 0$. From $h^0(E(-2H - D)) > 0$ and from (14) we get that $Z$ belongs to a divisor of type $3H - D = 7h - \sum_{i=1}^{5} (b_i + 3)e_i$. From the previous inequalities, we get $h^0(O_S(3H - D)) \leq h^0(O_S(7h - 2e_1 - 2e_2 - 3e_3 - 3e_4 - 3e_5)) = 36 - 3 - 3 - 6 - 6 - 6 = 12$. It follows that divisors of type $3H - D$ cannot contain a general set $A$ of 13 points of $S$, thus they cannot contain $Z$, a contradiction.

Assume $a = 3$. Since $h^0(O_S(3h)) = 10$, then $(b_1, \ldots, b_5)$ is greater or equal than the following 5-tuples:

1. $(-3, 0, 0, 0, 0)$,
2. $(-2, -2, 0, 0, 0)$,
3. $(-2, -1, -1, -1, 0)$,
4. $(-1, -1, -1, -1, -1)$.

In case (1), one computes that $h^0(O_S(3H - D)) \leq h^0(O_S(6h - 3e_2 - 3e_3 - 3e_4 - 3e_5)) = 28 - 6 - 6 - 6 - 6 = 4$. In case (2), one computes that $h^0(O_S(3H - D)) \leq h^0(O_S(6h - e_1 - e_2 - 3e_3 - 3e_4 - 3e_5)) = 28 - 1 - 6 - 6 - 6 = 8$. In case (3), one computes that $h^0(O_S(3H - D)) \leq h^0(O_S(6h - e_1 - 2e_2 - 2e_3 - 2e_4 - 3e_5)) = 28 - 1 - 3 - 3 - 3 - 3 = 13$. In any event $h^0(O_S(3H - D)) \leq 13$. Thus divisors of type $3H - D$ cannot contain a general set $A$ of 13 points of $S$, thus they cannot contain $Z$, a contradiction.

Assume $a = 4$. Since the general set $A$ of 13 points is contained in a divisor of type $3H - D = 5h - \sum_{i=1}^{5} (b_i + 3)e_i$, so that $h^0(O_S(3H - D)) \geq 14$, and since $h^0(O_S(5h)) = 21$, then $(b_1 + 3, \ldots, b_5 + 3)$ is greater or equal than the following 5-tuples:

1. $(-3, -1, 0, 0, 0)$,
2. $(-2, -2, -1, 0, 0)$,
3. $(-2, -1, -1, -1, -1)$.

In any case, one computes that $D$ cannot pass through 13 general points, a contradiction.

The cases in which $a \geq 5$ are similar but much easier.

It follows that a general section of $E(-2H)$ vanishes in a finite set $Z'$ of length

\[ \ell(Z') = c_2(E(-2H)) = c_2(E) - 2H \cdot c_1(E) + (2H)^2 = 26 - 10H^2 + 4H^2 \]

\[ = 26 - 24 = 2. \]

It follows that $Z$ and $Z'$ are zero-loci of sections of two different twists of $E$. The vanishing loci of two different twists of $E$ are connected in a double linkage. We explain the procedure in one example.
**Example 3.19:** Consider a general set $A$ of 13 points in $\mathbb{P}^4$ and take a set $Z'$ of 2 general points in the quartic surface $S$, complete intersection of type 1, 1, 1, 2, determined by $A$. The ideal of $Z'$ has the Koszul resolution:

$$0 \to R(-5) \to \bigoplus R(-3)^3 \to R(-2) \to I_{Z'} \to 0.$$

Fix a general hyperplane $\pi$ through $Z'$ and consider a general cubic $F$ which contains $Z'$ and $A$. Since $Z', A$ are general, for a general choice of $F$ the curve $F \cap S$ is irreducible. $F$ exists since $h_A(3) = 13$ so $(I_A)_3$ has dimension 7. Link $Z'$ to a set $Z''$ by the intersection of $S, \pi, F$. Since $A$ is general then $F$ is a general cubic through $Z'$ and the two quadrics which determine $S$ are general quadrics through $Z'$. Being $I_{Z'}$ generated in degree 2, we get that $Z''$ is a set of 10 points in the hyperplane $\pi$. A resolution of $I_{Z''}$ is determined by the mapping cone associated to the diagram

$$0 \to R(-8) \to \bigoplus R(-3)^2 \to R(-5)^2 \to R(-1)^2 \to I_{Z''} \to 0,$$

$$0 \to R(-5) \to \bigoplus R(-3)^3 \to R(-3)^3 \to R(-1)^3 \to I_{Z'} \to 0.$$  

where $W' = S \cap F \cap \pi$. We get

$$0 \to R(-7)^2 \to \bigoplus R(-3)^3 \to R(-5)^2 \to R(-3)^2 \to I_{Z''} \to 0.$$  

Notice from the resolution that $(I_{Z''})_3$ has dimension 27. Since the linear space of cubics containing $S$ is 10-dimensional, then there exists a space of dimension at least $27 - 10 - 13 = 4$ of cubics containing $A$ and not containing $S$. These cubics determine a linear system of divisors $3H$ on $S$ which contain $Z'' \cup A$. Since $F \cap S$ is irreducible, for a general choice of a cubic $F'$ through $Z'' \cup A$ the intersection $S \cap F \cap F'$ is a finite set $W''$ of length $4 \cdot 3 \cdot 3 = 36$. Linking $Z''$ with $S, F, F'$ we obtain thus a set $Z$ of length 26 containing $A$. Since $A, Z''$ are general and the ideal of $Z'$ is generated in degree 2, then by [18] we may assume that $Z$ is a set of 26 distinct points. The resolution of $Z$ comes from the mapping
cone of the diagram

\[
0 \rightarrow R(-9) \rightarrow R(-7)\oplus R(-6) \rightarrow R(-5)\oplus R(-4) \rightarrow R(-3)\oplus R(-2) \rightarrow I_Z \rightarrow 0.
\]

so that we obtain

\[
0 \rightarrow R(-10) \rightarrow R(-8)\oplus R(-7)\oplus R(-6) \rightarrow R(-5)\oplus R(-4) \rightarrow R(-3)\oplus R(-2) \rightarrow I_{W''} \rightarrow 0
\]

Thus the \( h \)-vector of \( Z \) is \( (1,4,8,8,4,1) \). Since the last module of the resolution is 1-dimensional, the set \( Z \) is arithmetically Gorenstein, by Proposition 2.12. In particular, the resolution is auto-dual. The residue of \( A \) in \( Z \) is a set \( B \) of 13 points.

Since \( h_Z(4) = 25 = 26 - 1 \), the spaces \( \langle v_4(A) \rangle \) and \( \langle v_4(B) \rangle \) meet in exactly one point \( T \), which represents a form with 2 different decompositions of length 13.

**Remark 3.8:** In the previous example, \( A \) and \( B \) are linked by the arithmetically Gorenstein scheme \( Z \). Thus, by the Gorenstein liaison procedure (see Section 2.5), a resolution of the ideal of \( B \) comes out from the resolution of the ideals \( I_Z \) and \( I_A \), by taking the mapping cone of the diagram

\[
0 \rightarrow R(-9) \rightarrow R(-7)\oplus R(-6) \rightarrow R(-5)\oplus R(-4) \rightarrow R(-3)\oplus R(-2) \rightarrow I_Z \rightarrow 0.
\]

One realizes immediately that the Betti numbers of a resolution of \( B \) coincide with the Betti numbers of a resolution of \( A \).

The following proposition is an exercise of duality.

**Proposition 3.20:** Assume \( A \) is a general set of 13 points in \( \mathbb{P}^4 \) and let \( T \) be a form in the span of \( v_4(A) \) such that \( A \) is a non-redundant decomposition of \( T \). If \( T \) has a second decomposition \( B \) of length 13 with \( A \cap B = \emptyset \), then \( B \) is obtained from \( A \) by a double linkage as illustrated in Example 3.19, starting with a non-necessarily reduced scheme \( Z' \) of length 2 in \( S \). In particular, the set \( Z = A \cup B \) has \( h \)-vector \( (1,4,8,8,4,1) \).
\textbf{Proof:} We know that two general quadrics containing $A$ intersect in a smooth irreducible quartic surface $S$. Let $B$ be another minimal decomposition of $T$, disjoint from $A$, and write $Z = A \cup B$. By Proposition 3.15, $Z$ is contained in $S$. By Theorem 2.9, $Z$ satisfies CB(4), hence it is associated with a rank 2 vector bundle $E$ on $S$. By Proposition 3.18 $E(-2H)$ has a section vanishing in a zero-dimensional scheme $Z'$ of length 2. It is classically known that schemes of length 2 on a smooth surface $S$ form an irreducible family whose general element is a set of 2 distinct points. Any scheme of length 2 is separated by hyperplanes, thus it is aligned and complete intersection of type 1, 1, 1, 2. If $Z'$ is not smooth, then it consists of a length 2 scheme concentrated at a point $P \in S$ and in a tangent line $L$ to $S$ at $P$. Any tangent line $L$ uniquely determines $Z'$.

Let $s$ be a global section of $E$ which vanishes at $Z$ and $s'$ be a section of $E(-2H)$ which vanishes at $Z'$. By the exact sequence

$$0 \to O_S \to E(-2H) \to I_{Z'}(H) \to 0,$$

where $I_{Z'}$ is the ideal sheaf of $Z'$ on $S$, the section $s$ determines a section of $I_{Z'}(3H)$, i.e. a divisor $D_E$ of type $3H$ which vanishes where $s \land s'$ vanishes. Hence $D_E$ lifts to a cubic $F$ containing $Z' \cup Z$. Fix a general hyperplane $\pi$ containing $Z'$, which determines a section of $E(-2H)$. Call $Z''$ the residual scheme $Z'$ of $Z'$ with respect to the intersection $F \cap \pi$ on $S$. The mapping cone of the diagram

$$
\begin{array}{ccccccccc}
0 & \to & O_S(-4H) & \to & O_S(-3H) \oplus O_S(-H) & \to & I_{F \cap \pi \cap S} & \to & 0 \\
0 & \to & O_S(-H) & \to & E(-3H) & \to & I_{Z'} & \to & 0 \\
\end{array}
$$

determines the following exact sequence for the ideal sheaf $I_{Z''}$ of $Z''$ on $S$

$$0 \to E^\vee(-H) \to O_S(-3H) \oplus O_S(-3H) \oplus O_S(-H) \to I_{Z''} \to 0. \quad (16)$$

Notice that, in the diagram, the map $E^\vee(-H) \to O_S(-3H)$ is the dual of the map $O_S(-H) \to E(-H)$ defined by $s$. Furthermore, the central vertical map corresponds to the section $s$ of $E$ and a section $s_0$ of $E(-2H)$, where $s_0 \land s'$ defines the divisor $\pi \cap S$. The map $E^\vee(-H) \to O_S(-3H) \oplus O_S(-3H) \oplus O_S(-H)$ in the mapping cone is the dual of the map $O_S(-3H) \oplus O_S(-3H) \oplus O_S(-H)$ in the mapping cone is the dual of the map $O_S(-3H) \oplus O_S(-3H) \oplus O_S(-H) \to E$ defined by $s', s_0, s$. Thus the three divisors, of type $3H, 3H, H$ containing $Z''$, defined by the map $O_S(-3H) \oplus O_S(-3H) \oplus O_S(-H) \to I_{Z''}$, correspond to the zero-loci of the wedge products $s' \land s, s_0 \land s, s' \land s_0$, respectively. Let $F'$ be a cubic in $\mathbb{P}^4$ such that $F \cap S$ corresponds to the zero-locus of $s_0 \land s$. Clearly, $F'$ contains both $Z$ and $Z''$. Sequence (16) proves that $Z''$ is cut set theoretically by $F, F', \pi, S$, thus $F \cap F' \cap \pi$ is zero-dimensional. The mapping cone of the diagram

$$
\begin{array}{ccccccccc}
0 & \to & O_S(-6H) & \to & O_S(-3H) \oplus O_S(-3H) & \to & I_{F \cap F' \cap S} & \to & 0 \\
0 & \to & E^\vee(-H) & \to & O_S(-3H) \oplus O_S(-3H) \oplus O_S(-H) & \to & I_{Z''} & \to & 0 \\
\end{array}
$$

does not show the residue of $Z''$ with respect to $F \cap F' \cap S$ is exactly the zero-locus of the section defined by the dual of the map $E^\vee(-H) \to O_S(-H)$. Hence the residue is $Z$.

When $Z'$ is a set of two distinct points, we get back exactly the double linkage described in Example 3.19. Since every set $Z'$ of length 2 is a limit of a family of reduced sets $Z'$ and
every hyperplane containing $Z'$ is a limit of a family of hyperplanes containing $Z'$, then, even if $Z'$ is non-reduced, the scheme $Z$ is a limit of schemes obtained as in Example 3.19.

For any scheme $Z'$ of length 2 in $S$ we have $h^0(I_{Z'}(2H)) = 11$. Thus $h^0(\mathcal{E}(-H)) = 16$, so that $h^0(I_Z(4H)) = 16$. Hence $\dim(I_Z)_4 = h^0(I_Z(4H)) + \dim(I_{S})_4 = 16 + 29 = 45$. It follows that $h_Z(4) = 70 - 45 = 25 = \ell(Z) - 1$. Hence $\langle v_4(A) \rangle \cap \langle v_4(B) \rangle = \{ T \}$, i.e. $T$ is uniquely determined by $B$.

The claim follows.

Next, the results explain how one can parametrize sets $B$ such that the spans of $v_4(A), v_4(B)$ meet at a point.

**Proposition 3.21:** Fix a general set $A$ of 13 points in $\mathbb{P}^4$ and let $S$ be the (smooth) quartic surface intersection of the quadrics containing $A$. Let $C$ be a general hyperplane section of $S$. Then for a general choice of a set $Z''$ of 10 points on $C$ the $h$-vector of $A \cup Z''$ is $(1, 4, 8, 10)$. Thus, the residue $B$ of $A \cup Z''$ is a set such that the spans of $v_4(A), v_4(B)$ meet at a point $T$.

**Proof:** Adding to $A$ 10 points in some hyperplane section of $S$ obtained as in Example 3.19, we get a set of points whose residue with respect to a complete intersection of type 2, 2, 3, 3 is a set $B$ as in the previous proposition. Since the residue of such $B$ has $h$-vector $(1, 4, 8, 10)$, the same holds for the addition to $A$ of 10 general points in $C$. The second claim follows immediately by the mapping cone.

It remains to prove that for a general choice of a set $Z''$ of 10 points in a hyperplane section of $S$ the residue $B$ of $A \cup Z''$ in a complete intersection of $S$ and two cubics defines a quartic $T = \langle v_4(A) \rangle \cap \langle v_4(B) \rangle$ such that $A$ is a non-redundant decomposition for $T$.

In practice, this amounts to finding the expression of a general such $T$ in terms of coordinates of the points of $A$ and show that no coefficients of the expression vanish.

It is clear that the previous property is open; thus, it is sufficient to check it in one example.

**Example 3.22:** Fix a general set $A$ of 13 points in $\mathbb{P}^4$, which determines a general (smooth) surface $S$, the complete intersection of two quadrics. Fix a general set $Z''$ of 10 points on a general hyperplane section of $S$. From the coordinates of set $A$ one computes the ideal $I_A$. From the coordinates of the set $A \cup Z''$ and the mapping cone procedure, one computes the ideal $I_B$. If everything is general enough, then $B \cap A$ is empty and $h_Z(4) = 25$, where $Z = A \cup B$. It follows that in the polynomial ring $R = \mathbb{C}[x_0, \ldots, x_4]$ the subspaces $(I_A)_4$ and $(I_B)_4$ of $R_4$ are both 57-dimensional, but their sum $U = (I_A)_4 + (I_B)_4$ is a linear subspace of codimension 1 in $R_4$. Coefficients for the equation of $U$ provide coefficients for the form $T$ in the intersection of $\langle v_4(A) \rangle$ and $\langle v_4(B) \rangle$. We refer to [11] for details.

Thus, from the coordinates of $A$ and $Z''$, one determines $T$ explicitly, and also an expression of $T$ as a combination of rank 1 tensors representing the points of $A$. Experiments prove the existence of $T$ for which no coefficients vanish. We refer for that to the file available at https://arxiv.org/src/2106.06730v1/anc/ancillary.txt.

If $A$ is general enough, then the span of tangent spaces to the Veronese variety $v_4(\mathbb{P}^4)$ at the points of $v_4(A)$ determines a linear subspace of the expected projective dimension 64 in the projective space spanned by $v_4(\mathbb{P}^4)$. This is easy to check directly for the points $A$ above.
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(see the file at https://arxiv.org/src/2106.06730v1/anc/ancillary.txt). From Remark 2.1, it follows that there are no positive-dimensional families of decompositions for the form $T$.

**Corollary 3.23:** Fix a general set $A$ of 13 points in $\mathbb{P}^4$. Then the span of $v_4(A)$, which is a $\mathbb{P}^{12}$, contains an irreducible 10-dimensional family $T$ of points $T$ which have a second decomposition $B$ obtained as in Example 3.19.

**Proof:** The family $Z''$ of sets $Z''$ of 10 points contained in some hyperplane section of $S$ is irreducible and 14-dimensional and maps to the family of points $T \in \langle v_4(A) \rangle$ obtained as in Example 3.19. It is thus sufficient to prove that the general fibre is 4-dimensional.

Indeed, by the previous example, the family of points $T$ is a finite image of the family $B$ of sets $B$ linked to $A \cup Z''$, when $Z''$ moves. For a general choice of $B$ in the family $B$, the sets $Z''$ determining $B$ are obtained as the residue in a complete intersection of $S$ and two more cubics. Thus the fibre over $B$ of the map $Z'' \to B$ is determined by the choice of a pencil of cubics through $A \cup B$. Since the vector space of cubics through $A \cup B$ is 4-dimensional, the fibre corresponds to the choice of a line in a $\mathbb{P}^3$. The claim follows. ■

**Remark 3.9:** In principle, as in the case $r = 12$, also for an expression $A$ of length 13 one could try to construct an algorithm that checks if $A$ is minimal and unique.

In particular, the algorithm should test if $A$ satisfies $(i)$, $(ii)$, $(iii)$, if for any choice of two points $P_i, P_j \in A$ the set $A \setminus \{P_i, P_j\}$ satisfies $(iv)$, and if for any choice of the index $i = 1, \ldots, 13$ the algorithm for $r = 12$ gives the identifiability of $T - T_i$.

Then, in order to detect if $T$ belongs to the bad family $T$ or not, the algorithm should construct a parametrization of the bad family. This step is much more delicate than the corresponding step 7 for the case $r = 12$, because it requires to parametrize the choice of a set of 10 points in an elliptic curve.

We intend to devote a future paper to practical solutions of this last technical step.

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