The R.I. Pimenov Unified Gravitation and Electromagnetism Field Theory as Semi-Riemannian Geometry*

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Abstract—More than forty years ago R.I. Pimenov introduced a new geometry—semi-Riemannian one—as a set of geometrical objects consistent with a fibering pr: Mn → Mm. He suggested the heuristic principle according to which the physically different quantities (meter, second, Coulomb, etc.) are geometrically modelled as space coordinates that are not superposed by automorphisms. As there is only one type of coordinates in Riemannian geometry and only three types of coordinates in pseudo-Riemannian one, a multiple-fibered semi-Riemannian geometry is the most appropriate one for the treatment of more than three different physical quantities as unified geometrical field theory. Semi-Euclidean geometry \( \mathbb{R}^4 \) with 1-dimensional fiber \( x^5 \) and 4-dimensional Minkowski space—time as a base is naturally interpreted as classical electrodynamics. Semi-Riemannian geometry \( \mathbb{R}^5 \) with the general relativity pseudo-Riemannian space—time \( \mathbb{V}^5 \), and 1-dimensional fiber \( x^5 \), responsible for the electromagnetism, provides the unified field theory of gravitation and electromagnetism. Unlike Kaluza–Klein theories, where the fifth coordinate appears in nondegenerate Riemannian or pseudo-Riemannian geometry, the theory based on semi-Riemannian geometry is free from defects of the former. In particular, scalar field does not arise.

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1. INTRODUCTION

The old problem of geometrical unification of gravity and electromagnetism goes back to the Kaluza–Klein theory [1, 2], where electromagnetism is described by curvature in an extra spacelike dimension, i.e., the fifth coordinate is introduced in pseudo-Riemannian geometry. However, the physical dimension of electromagnetism is different from the dimension of space, furthermore the fifth dimension is not observed as part of our everyday lives. To overcome last difficulty, the fifth dimension is supposed to be cyclic with very small radius. In addition there is also restriction on the set of transformations under which the theory is invariant. Indeed, rotation in a plane spanning the fifth coordinate \( x^5 \) and some space coordinate \( x \)

\[
x' = x \cos \varphi + x^5 \sin \varphi, \quad x^5 = x^5 \cos \varphi - x \sin \varphi,
\]

immediately leads to the dependence of space—time coordinates on the fifth dimension what is obviously unsatisfactory. Moreover, an additional scalar field arises in Kaluza–Klein type theories [3, 4], which makes Coulomb potential everywhere regular.

Nevertheless, the problem of unification of gravitation with electromagnetism in the framework of one geometry is still actual. This may help to incorporate some other interactions (weak interaction is the first candidate) in one geometrical picture. To overcome the disadvantages of Kaluza–Klein theory the fibered semi-Riemannian geometry with degenerate metrics need to be used.

The purpose of this paper is to explain the basic notion of semi-Riemannian geometry and to demonstrate on the example of Pimenov theory of gravitation and electromagnetism how different physical quantities may be unified with the help of this geometry.

More than forty years ago R.I. Pimenov suggested [5–7] a unified theory of gravitation and electromagnetism where both fundamental interactions are naturally incorporated into the 5-dimensional semi-Riemannian geometry with degenerate metric. This theory is free from disadvantages of Kaluza–Klein-type theories. Mathematical definition of multiferbed semi-Riemannian geometry of arbitrary dimension was given by Pimenov [8–10].

5-dimensional semi-Riemannian geometry can be formulated in a traditional way with the help of a real geometrical objects such as coordinates, components of metric tensor, etc. as well as can be obtained from
Riemannian geometry with the help of nilpotent fifth coordinate \[7\].

Recently \[11\] gravitation and electromagnetism have been unified in the 5-dimensional space with degenerate metric and nilpotent fifth coordinate. But the scalar field is presented in this theory and Coulomb potential is regular at small distance.

2. GEOMETRICAL MODELLING OF A PHYSICAL QUANTITIES

Geometrical modelling of physical quantities is understood as unification of different physical quantities (meter, second, Coulomb, etc.) within the framework of one geometry. Let us start with the simplest case of 2-dimensional space.

For a bundle of lines through a point in a plane one of the three possibilities is realized \[12\] (see Fig. 1):

I. Any line of the bundle is postulated.

II. There is one nonpostulated (isolated) line in the bundle.

III. There are two or more nonpostulated lines in the bundle.

The property of line to be postulated is conserved by the plane automorphisms.

Postulate I leads to the Euclidean geometry in the plane. Any two lines of the bundle are superposed by rotations around the point. This means that only one physical quantity can be modelled by Euclidean geometry. The same is true for \(n\)-dimensional Euclidean space.

Postulate III leads to the pseudo-Euclidean geometry in the plane, where there are three types of lines: with positive, negative, and zero length. These types of lines are not transformed to each other by rotations around the point (Lorentz transformations). Therefore only three different physical quantities can be incorporated in pseudo-Euclidean geometry. For kinematical interpretation these are time-like, space-like, and light-like lines. The same is true for \(n\)-dimensional pseudo-Riemannian space.

Postulate II leads to the fibered semi-Euclidean (or Galilean) geometry in the plane which is defined by the projection \(pr\) with 1-dimensional base \(\{t\}\) and 1-dimensional fiber \(\{x\}\). This plane is interpreted as classical \((1 + 1)\) kinematics with absolute time \(t\) and absolute space \(x\). Rotations (or Galilei boosts) superpose any two time-like lines but do not superpose time-like lines with space-like line. Two different physical quantities (space and time) are modelled by two types of lines on the semi-Euclidean plane.

Semi-Riemannian geometry is a fibered geometry with degenerate metric. If the base of semi-Riemannian space has Euclidean geometry, then the number of line types remains the same under adding of extra dimensions to the base. If it has pseudo-Riemannian geometry, then only one additional type of line with nonzero length appears. This exhausts all possibilities for incorporating new physical quantities.

A different situation takes place when adding extra dimensions to the fiber. Riemannian, pseudo-Riemannian or semi-Riemannian geometry can be realized in the fiber. For example, if the fiber \(\{x, y\}\) has semi-Riemannian geometry then there is projection \(pr_2\) with the base \(\{x\}\) and the new fiber \(\{y\}\) (see Fig. 2). So the third type of line appears in this geometry and it gives an opportunity for modelling some third physical quantity \(y \neq t \neq x\).

Similar algorithm for construction of multifibered spaces can be applied without limit. In other words, multifibered geometry with the subsequently enclosed projections can be successfully used for incorporating of any number of physical quantities in one geometry.

The above examples illustrates the heuristic principle \[7\]: quantities with different physical dimensions cannot be geometrically superposed by automorphisms.
3. SEMI-RIEMANNIAN GEOMETRY $^3V_5^4$ AS SPACE–TIME ELECTROMAGNETISM

3.1. Definition of Semi-Riemannian Space $^3V_5^4$

The semi-Riemannian space $^3V_5^4$ is described by the projection $pr: ^3V_5^4 \to V_4$ with 4-dimensional base $^3V_5^4$ and 1-dimensional fiber $\{x^5\}$. Pseudo-Riemannian geometry with the nondegenerate metrics $g_{\mu\nu}$, $\mu, \nu = 1, 2, 3, 4$ of signature $(+---)$ is set in the base. It follows from consistency with fibering that components $g_{\mu\nu}$ depend on base coordinates $x^1, \ldots, x^4$. Metrics in the fiber is defined by the component $g_{55} \neq 0$ depending on all coordinates $x^1, \ldots, x^5$. This component can be regarded as a scale for fifth coordinate. Remaining components $g_{5\mu}(x^1, \ldots, x^5) = g_{5\mu}(x^1, \ldots, x^5)$ define the 4-distribution

$$\omega = \{g_{5\mu}dx^\mu = 0\}, \quad (1)$$

which is transversal to the fiber. As a result, the metric flagtensor of the semi-Riemannian space $^3V_5^4$ has the form $g_{\alpha\beta} = (g_{\mu\nu}, g_{5\mu}, g_{55})$.

Coordinate transformations consistent with the fiber can be written as

$$\begin{cases} x^\mu = f^\mu(x^1, \ldots, x^4), \\ x^5 = f^5(x^1, \ldots, x^4, x^5), \end{cases} \quad (2)$$

$$(D^\mu_\beta) = \begin{pmatrix} (D^\mu_\nu) & 0 \\ (D^5_\nu) & D^5_\beta \end{pmatrix}, \quad (3)$$

$$\det(D^\mu_\beta) \neq 0, \quad D^5_\beta \neq 0,$$

where $D^\alpha_\beta = \partial_\beta x^\alpha$. The metric flagtensor $g_{\alpha\beta} = (g_{\mu\nu}, g_{5\mu}, g_{55})$ is transformed under (2), (3) by the formulas

$$g_{\mu'\nu'} = g_{\mu\nu}D^\mu_\mu D^\nu_\nu, \quad g_{55'} = g_{55}D^5_5 \cdot D^5_5', \quad (4)$$

Here and below, summation on repeating indices is to be understood. It is always possible to make $g_{55} = 1$ by the scale changing $D^5_5 = \frac{\partial x^5}{\partial x^5} = \frac{1}{\sqrt{g_{55}}}$; assuming this, we have ($D^5_5 = D^5_5' = 1$)

$$g_{55'} = g_{5\mu}D^5_\mu D^\mu_\mu + g_{55}D^5_5 D^5_5' = g_{5\mu}D^\mu_\mu + D^5_5'. \quad (5)$$

If one assumes that in the base, which is interpreted as space–time, nothing happens, i.e., $D^\mu_\mu = \delta^\mu_\mu$, then the transformations of metric flagtensor components $g_{5\mu}$ look like adding arbitrary function gradients

$$g_{5\mu} \mapsto g_{5\mu} + \partial_\mu x^5, \quad g_{55} \mapsto g_{55} + \partial_\mu x^5. \quad (6)$$

Invariants of semi-Riemannian geometry $^3V_5^4$ are pseudo-Riemannian length in the base

$$ds = \sqrt{g_{\mu\nu}dx^\mu dx^\nu}, \quad (7)$$

and Euclidean length in the fiber

$$ds(2) = \sqrt{g_{55}dx^5 dx^5} = |dx^5| \quad (with g_{55} = 1). \quad (8)$$

As far as the angle between general position vector $dx = (dx^\mu, dx^5)$ and fiber vector $\delta x = (0, \delta x^5)$ equals infinity, the supplemental angle $\psi$ is taken as the third invariant

$$\psi = g_{55} \frac{dx^\mu}{ds} + \frac{dx^5}{ds}, \quad (9)$$

$\psi$ is the angle between vector $dx$ and its projection on the base in 2-plane formed by $dx$ and $\delta x$.

Thus, semi-Riemannian geometry $^3V_5^4$ is defined by the set of objects described above with transformation properties (2)–(4).

In order to avoid terminological misunderstanding let us stress that the fibering $pr$ has nothing to do with the principal bundle, where some group acts on the fiber. In the last approach something similar to a tangent space $(dx^1, \ldots, dx^n)$ is built

Fig. 2. Types of lines in twice fibered space.
on the space \((x^1, \ldots, x^n)\) and then unified object \((x^1, \ldots, x^n, dx^1, \ldots, dx^n)\) is regarded as having base \((x^1, \ldots, x^n)\) and fiber \((dx^1, \ldots, dx^n)\). In semi-Riemannian geometry the fibering ‘takes place’ in the space \((x^1, \ldots, x^n)\) itself. The reader should not confuse the terminology and methods of these different notions of fibering.

3.2. Interpretation of Semi-Euclidean Geometry as Classical Electrodynamics

The fiber can be regarded as an “inner” space of a particle. The fiber geometry in semi-Riemannian space is completely defined by the metrics \(g_{55}(x^1, \ldots, x^5)\) and does not depend on the base metrics \(g_{\mu\nu}(x^1, \ldots, x^4)\) (which is not the case in non-degenerate geometry). The remaining components \(g_{\mu5}\) are functions of all coordinates \(x^1, \ldots, x^4\).

Let us impose on the space \(^3V^3\) an additional condition, namely, the fiber has Euclidean geometry which does not depend on base coordinates \(x^1, \ldots, x^4\). Then in a neighborhood (instead of just one point) there exists such coordinate system that all components of the metric are equal to zero and \(\partial_5 g_{55} = 0, \partial_5 g_{45} = 0, \partial_5 g_{54} = 0\), which implies \(g_{55} = \text{const} = 1, g_{\mu5} = g_{5\mu}(x^1, \ldots, x^4)\).

With this additional condition the semi-Riemannian space of zero curvature (or semi-Euclidean space) can be defined. The base of semi-Euclidean space is Minkowskian space—time with coordinates \(x^1 = t, x^2 = x, x^3 = y, x^4 = z\) and metric tensor \(g_{\mu\nu} = \text{diag}(1, -1, -1, -1)\). The fiber is Euclidean line \(x^5\) and \(g_{55} = 1\). Besides, the metric lagrangian has nonzero components \(g_{5\mu}(t, x, y, z)\). Invariants \((7), (8)\) can be written in integral form

\[
\begin{align*}
S = &\int ds + b\int \psi ds = S_p + S_{\text{int}}, \\
&= -mc\int_A^B \sqrt{dt^2 - dx^2 - dy^2 - dz^2} \\
&= -mc^2\int_{t_1}^{t_2} \sqrt{1 - \frac{v^2}{c^2}} dt,
\end{align*}
\]

i.e., the action for a free particle [13, § 8]. Taking (9) into account we have \(S_{\text{int}} = \int A_\mu dx^\mu + \int dx^5\). The second term can be omitted, and for \(b = -e/c\), where \(e\) is the electron charge, we obtain

\[
S_{\text{int}} = \frac{e}{c} \int_A^B A_\mu dx^\mu,
\]

i.e. the action which describes the interaction of charged particle moving in a given electromagnetic field with this field [13, § 16].

The 4-dimensional vector-potential \(A_\mu\) of electromagnetic field is defined up to adding the gradient \(\partial_\mu f\) of an arbitrary function of space—time coordinates. It follows from (8) that the components \(g_{\mu5}\) of metric tensor have just similar properties and therefore can be identified with vector-potential \(A_\mu = g_{\mu5}\). Maxwellian field stress tensor is obtained in a standard way \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\).

The action must be invariant under the automorphism group. Let us form it from the invariants \((7), (8)\) of semi-Euclidean space \(^3R^4\)

\[
S = a \int_A^B ds + b\int \psi ds = S_p + S_{\text{int}},
\]

where \(a, b\) are dimension factors. The third invariant \(\int dx^5 = x^5(x^\mu)\) (8) is an arbitrary function of space—time coordinates, therefore it can be omitted. For \(a = -mc\), where \(m\) is the particle mass, \(c\) is the speed of light we have

\[
\begin{align*}
S_p = &\int ds \\
&= -mc\int_A^B \sqrt{dt^2 - dx^2 - dy^2 - dz^2} \\
&= -mc^2\int_{t_1}^{t_2} \sqrt{1 - \frac{v^2}{c^2}} dt,
\end{align*}
\]

i.e. the action which describes the interaction of charged particle moving in a given electromagnetic field with this field [13, § 16].

There is no field action among geometrical invariants of the semi-Euclidean space \(^3R^4\). However, there is the electromagnetic field stress tensor \(F_{\mu\nu}\). Therefore, introducing a new postulate: the field action must be an invariant function of electromagnetic field stress tensor, we obtain in a standard way [13, § 27]

\[
S_f = -\frac{1}{16\pi} \int F_{\mu\nu} F^{\mu\nu} d^4x.
\]
Sum of the terms (13)–(15)

\[ S = S_p + S_{int} + S_f \]  

(16)
gives the complete action for a particle in an electromagnetic field.

3.3. $3V^4$ as Space–Time–Electromagnetism

In semi-Riemannian space $3V^4$ for invariantly selected coordinate system, where $g_{55} = 1$, the metric components $g_{\mu 5}(x) = A_\mu(x), x = (x^1, x^2, x^3, x^4)$ are interpreted as 4-vector potential of electromagnetic field. The pseudo-Riemannian metrics of the general relativity space $3V^4$ is specified by the components $g_{\mu \nu}(x)$. The fiber geometry does not affect the geometry of the base $3V^4$.

To obtain gravitation field equations in the space $3V^4$ its scalar curvature $R$ is used instead of invariant (7). The variation of sum of the action for gravitation field [13, §93]

\[ S_g = -\frac{1}{16\pi G} \int R \sqrt{-g} \, d^4x, \]  

(17)

where $G$ is gravitation constant, $g = \det(g_{\mu \nu})$, and the action (15) for electromagnetic field gives in result Einstein equations

\[ R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} + \Lambda g_{\mu \nu} = -8\pi G T_{\mu \nu}, \]  

(18)

where $R_{\mu \nu} = R^\rho_{\mu \rho \nu}$ is Ricci tensor, curvature tensor $R^\pi_{\mu \rho \nu} = \partial_\rho \Gamma^\pi_{\mu \nu} - \partial_\nu \Gamma^\pi_{\mu \rho} + \Gamma^\pi_{\lambda \rho} \Gamma^\lambda_{\mu \nu} - \Gamma^\pi_{\lambda \nu} \Gamma^\lambda_{\mu \rho}$ is expressed by Christoffel symbols $\Gamma^\pi_{\mu \nu} = \frac{1}{2} g^{\pi \tau} (\partial_\nu g_{\mu \tau} + \partial_\tau g_{\nu \mu} - \partial_\mu g_{\nu \tau} - \partial_\tau g_{\nu \mu})$, $g^{\mu \nu} g_{\nu \tau} = \delta^\mu_\tau, R = R^\mu_\nu = g^{\mu \rho} R_{\rho \nu}$ is scalar curvature, $\Lambda$ is cosmological constant, $T_{\mu \nu}$ is energy–momentum tensor. Thus left-hand side of Einstein equations is determined by the metrics $g_{\mu \nu}(x)$ and their right-hand side is determined by the energy–momentum tensor of the system under consideration

\[ T_{\mu \nu} = g_{\mu \rho} g_{\nu \tau} T^{\rho \tau}, \]  

(19)

where

\[ T^{\rho \tau} = \frac{1}{4\pi} \left( - F^{\rho \mu} F^\tau_\mu + \frac{1}{4} g^{\rho \tau} F_{\mu \nu} F^{\mu \nu} \right) \]  

(20)
is the energy–momentum tensor of electromagnetic field without charges.

So, the space $3V^4$ gives rise to the standard equations for interactions of gravitational and electromagnetic fields demonstrating that they are incorporated in the framework of semi-Riemannian geometry.

3.4. Comparison with the Kaluza–Klein Theories

The unification of general relativity and electromagnetism within the framework of 5-dimensional geometry goes back to papers [1, 2]. In Kaluza–Klein type theories 4-dimensional pseudo-Riemannian space–time of general relativity is extended to 5-dimensional one by adding on extra dimension responsible for electromagnetism [3, 4, 14]. Since the fifth coordinate is introduced within the framework of pseudo-Riemannian geometry, it can be superposed with any space–time coordinate by an automorphism and therefore cannot model a new physical quantity. To avoid this difficulty the fifth dimension is supposed to be “curled up small”, so that it cannot be directly considered as ordinary space or time dimension. Moreover, additional scalar field $\chi$ appears as $g_{55} = 1 + \chi$ in Kaluza–Klein type models. This scalar field modifies electromagnetic potential in such a way that it is everywhere regular and is close to the Coulomb potential at all, except very small, lengths scales.

The use of semi-Riemannian geometry $3V^4$ allows to overcome the above-mentioned difficulties, namely:

1. The independence of $g_{\mu 5}$ of $x^5$ follows from the geometrical axioms. It is possible, however not necessary, to suppose the fifth coordinate to be cyclic with very small radius.

2. The independence of $g_{\mu 5}$ of $x^5$ follows from the additional assumption that the fiber has Euclidean geometry which does not depend on base coordinates.

3. No scalar field arise which modifies Coulomb potential at small distances.

4. space–time and the fiber geometries can be taken absolutely independent of each other.

5. Einstein equations in space–time impose no restrictions on extra coordinates.

6. Gradient invariance of electromagnetic vector-potential is automatically a consequence of transformations of metric flagtensor components.

7. The unified theory can be applied to the empty space–time of special relativity, where electromagnetic field is present. (This is not possible in pseudo-Riemannian case when $g_{55} = 1, \partial_5 g_{\mu \nu} = 0$ because of the formula $2R_{\mu 5 \rho 5} = 2\partial_\mu \partial_5 g_{\rho 5} - \partial_5 \partial_\rho g_{\mu 5} - \partial_\mu \partial_5 g_{\rho 5}$, which is not the case in semi–Riemannian geometry.) The action for charged particle interacting with a given electromagnetic field is the sum of invariants of semi–Euclidean geometry.

8. The heuristic principle demanding that different physical quantities should not be geometrically superposed by automorphisms is fulfilled.
4. SEMI-RIEMANNIAN SPACE $V^4_5$ WITH NILPOTENT FIFTH COORDINATE

In the previous sections semi-Riemannian geometry was constructed for real objects as the structure consistent with fiber. At the same time geometry with degenerate metrics can be realized with the help of nilpotent objects. In particular, fiber space can be obtained from Riemannian space by multiplying fiber coordinates by nilpotent unit $\iota$, $\iota^2 = 0$. Similar approach has been useful for investigating group contractions [15].

Let us obtain $V^4_5$ from Riemannian space $V_5$ by substituting $\iota x^5$ instead of $x^5$. We shall demand that the following heuristic rules be fulfilled: for a real $a$ the expression $a/\iota$ is defined only for $a = 0$, however $\iota/\iota = 1$; furthermore

$$ \sqrt{a^2 + \iota^2b^2} = \begin{cases} |a|, & \text{if } a \neq 0, \\ |\iota b|, & \text{if } a = 0. \end{cases}$$

Substitution $x^5 \rightarrow \iota x^5$ induces another one $g_{\iota 5} \rightarrow \iota g_{5 \iota}$, i.e., metric tensor has nilpotent components

$$ g_{\alpha\beta} = \begin{pmatrix} g_{\mu\nu} & \iota g_{5\iota} \\ \iota g_{\iota 5} & g_{55} \end{pmatrix}, \quad (21) $$

The nondegeneracy condition $\det(g_{\alpha\beta}) = \det(g_{\mu\nu})g_{55} \neq 0$ implies nondegeneracy in the base $\det(g_{\mu\nu}) \neq 0$ and in the fiber $g_{55} \neq 0$. Inverse tensor $g^{\alpha\beta}$ is easily obtained from equations $g^{\alpha\gamma}g_{\gamma\beta} = \delta^\alpha_\beta$ and is as follows

$$ g^{\alpha\beta} = \begin{pmatrix} g^{\mu\nu} & -\iota g_{5\iota}^{-1}g^{\mu\nu}g_{5\iota} \\ -\iota g_{5\iota}^{-1}g_{\iota 5}g^{\mu\nu} & g_{55}^{-1} \end{pmatrix}, \quad (22) $$

where $g^{\mu\nu}g_{\mu\nu} = \delta^\mu_\nu$. For $g_{\mu\nu} = \text{diag}(1, -1, -1, -1) = g^{\mu\nu}$, $g_{55} = 1$, we have $g^{15} = -g_{15}$, $g^{55} = g_{55}$, $k = 2, 3, 4$.

The entries $D^\mu_{5\iota} = \frac{\partial x^\mu}{\partial x^5}$ of transformation matrix are substituted for $D^\mu_{5\iota} = \frac{1}{\iota} \frac{\partial x^\mu}{\partial x^5}$, i.e., $\frac{\partial x^\mu}{\partial x^5} = 0$, and entries $D^\nu_{\mu\iota} = \frac{\partial x^\nu}{\partial x^\mu}$ are substituted for $\frac{\partial x^\nu}{\partial x^\mu} = \iota D^\nu_{\mu\iota}$. Therefore differentials of coordinate functions are transformed as

$$ \left( \frac{dx^{\mu'}}{\iota dx^5} \right) = \begin{pmatrix} \left( D^\mu_{5\iota} \right) & 0 \\ \iota D^\nu_{\mu\iota} & D^\nu_{5\iota} \end{pmatrix} \left( \frac{dx^{\mu}}{\iota dx^5} \right), \quad (23) $$

and coordinate transformations are consistent with fibering form (2), (3). From general formula $g_{\alpha\iota'\beta'} = g_{\alpha\beta}D^\alpha_{\alpha'}D^\beta_{\beta'}$ for metrics (21) with the help of matrix (23) we obtain the transformations (4) of metric tensor.

Invariants of semi-Riemannian space $V^4_5$ are obtained from those of Riemannian space $V_5$, namely

$$ ds = \sqrt{g_{\alpha\beta}dx^{\alpha}dx^\beta} \quad (24) $$

$$ = \sqrt{g_{\mu\nu}dx^{\mu}dx^{\nu} + i^2 (g_{55}dx^5dx^5 + 2g_{5\iota}dx^{\iota}dx^5)} $$

$$ = \left\{ \begin{array}{ll} \sqrt{g_{\mu\nu}dx^{\mu}dx^{\nu}} = ds, & \text{if } \exists \mu dx^\mu \neq 0 \text{— length in the base}, \\ |\iota dx^5| = |ds(2)|, & \text{if } \forall \mu dx^\mu = 0 \text{— length in the fiber}. \end{array} \right. $$

Supplementary angle between general position vector $dx = (dx^\mu, \iota dx^5)$ and fiber vector $d\iota x = (0, \iota dx^5)$ is deduced from the formula for sine of supplementary angle $\psi$ between vectors $dx$ and $d\iota x$ in Riemannian space

$$ \cos \left( \frac{\pi}{2} - \psi \right) = \sin \psi = \frac{g_{\alpha\beta}dx^{\alpha}d\iota x^{\beta}}{|dx||d\iota x|} \quad (25) $$

by substitutions $x^5 \rightarrow \iota x^5$, $g_{\iota 5} \rightarrow \iota g_{5\iota}$, $\psi \rightarrow \iota \psi$. As far as $|dx| = \sqrt{g_{\mu\nu}dx^{\mu}dx^{\nu}} = ds$, $|d\iota x| = \iota |dx^5|$, $\sin \psi = \iota \psi$ (functions of nilpotent arguments are defined by their Taylor expansion) and $g_{\alpha\beta}dx^{\alpha}d\iota x^{\beta} = i_g_{5\iota}d\iota x^5 = \iota |dx^5| = i_g_{5\iota}dx^5 + i_g_{5\iota}dx^{\iota}) = i^2 dx^5 \times (g_{5\iota}dx^{\iota} + dx^5)$, then from (25) we have

$$ \iota \psi = \frac{i^2 dx^5 (g_{5\iota}dx^{\iota} + dx^5)}{dsd\iota x^5} = \iota \left( \frac{g_{\iota 5}dx^{\iota}}{ds} + \frac{dx^5}{d\iota x^5} \right), \quad (26) $$

what coincides with (9) after cancelling $\iota$.

The fact that the base geometry does not depend on fiber geometry comes automatically. Indeed, derivative $\frac{1}{\iota} \frac{\partial g_{\mu\nu}}{\partial x^5}$ is defined only for $\frac{\partial g_{\mu\nu}}{\partial x^5} = 0$, i.e., $g_{\mu\nu}(x^1, \ldots, x^4)$ does not depend on $x^5$.

In a similar way pseudo-Riemannian geometry $^{3}V_4$ in the base can be obtained from Riemannian one $V_4$ by “analytic continuation” of locally orthogonal coordinates substituting $ix^k$ for $x^k$, $k = 2, 3, 4$.

Let us note that recently the interest was shown to the problem of unifying gravitation with electromagnetism in a space with nilpotent fifth coordinate [11]. However, semi-Riemannian geometry was not constructed there, therefore the scalar field is present in this theory and electromagnetic potential is everywhere regular.
5. CONCLUSIONS

R.I. Pimenov’s unified geometrical theory of gravitation and electromagnetism is based on the multifibered semi-Riemannian geometry developed by himself. This theory is formulated taking into account the heuristic principle according to which physically different quantities cannot be superposed by automorphisms. In other words, for unification of gravitation (or space–time) with some other fundamental interactions (electromagnetic or possible weak) it is necessary to use $D$-dimensional fiber space ($D > 4$) with degenerate metrics. The base of this space is 4-dimensional pseudo-Riemannian space–time of general relativity and extra dimensions, which are responsible for other interactions, belong to the fiber. The particular case of single-fibered semi-Euclidean space $^3R^4_5$ with 1-dimensional fiber $x^5$ and 4-dimensional Minkowski space–time as the base is naturally interpreted as classical electrodynamics. Both fundamental gravitational and electromagnetic interactions are incorporated in one semi-Riemannian geometry $^3V^4_5$ with general relativity space–time as the base of the fibering and one-dimensional fiber $x^5$ responsible for electromagnetism.

Unlike Kaluza–Klein-type theories, where fifth dimension appears in the context of nondegenerate Riemannian or pseudo-Riemannian geometry, Pimenov theory based on semi-Riemannian geometry does not have restrictions on admissible transformations of 5-dimensional space as well as additional scalar field which modifies Coulomb potential at small distances. The last property is not compatible with experimental data.

5-dimensional semi-Riemannian geometry is formulated in real geometrical notions (coordinates, metric tensor components, etc.) but can also be obtained from Riemannian geometry by using nilpotent fifth coordinate.

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