SCALAR GLUEBALL: ANALYSIS OF
THE \(IJ^{PC}=00^{++}\)−WAVE

A.V.Anisovich, V.V.Anisovich, and A.V.Sarantsev
St.Petersburg Nuclear Physics Institute
Gatchina, St.Petersburg 188350, Russia

Abstract

Basing on the previously performed \(K\)−matrix analysis of experimental data, we investigate, in the framework of the propagator matrix (\(D\)-matrix) technique, the 1100-1900 MeV mass region, where overlapping resonances \(f_0(1300), f_0(1500), f_0(1530 \pm 90 \pm 250),\) and \(f_0(1780)\) are located. Necessary elements of the \(D\)-matrix technique are developed. The \(D\)-matrix analysis confirms previous \(K\)-matrix results: in the region 900-1900 MeV five scalar/isoscalar states are located. Four of them are members of the two \(q\bar{q}\)-nonets, while one state is an extra for the \(q\bar{q}\) systematics, being a good candidate for the lightest scalar glueball. The \(D\)-matrix analysis shows that this extra state, a candidate for the lightest scalar glueball, is dispersed, due to a mixing with \(q\bar{q}\)-states, over three resonances: \(f_0(1300), f_0(1500),\) and \(f_0(1530 \pm 90 \pm 250).\) The broad resonance \(f_0(1530 \pm 90 \pm 250)\) is a descendant of the lightest glueball carrying about 50% of the gluonium component, the rest of the gluonium is shared between \(f_0(1300)\) and \(f_0(1500).\)
1 Introduction

At present time the mesons in the mass region 1000-2000 MeV are under intensive experimental studies. The problem these investigations face is the overlapping of resonances with identical quantum numbers: the widths of resonances in this region are comparatively large, of the same order as mass differences of neighbouring resonances \[1\]. This leads to an inevitable mixing of neighbouring resonances if they have common decay channels. Consideration of this mixing is needed for a restoration of the initial-meson masses as well as for the determination of the quark content of the mesons. Definition of mixing is in particular important in the search for exotic mesons, such as glueballs and hybrids. Glueball (or hybrid) may be located in the vicinity of \(q\bar{q}\) - mesons with identical quantum numbers that raises their mixing and leads to the dispersing of the glueball (or hybrid) component over several mesons.

This paper is devoted to the calculation of scalar/isoscalar resonance mixing, \(IJ^{PC} = 00^{++}\): experimental data prove that in this wave there is a state which is superflous for \(q\bar{q}\)- systematics \[2\], being possibly the lightest scalar glueball. The idea that in the mass region 1500-1700 MeV the lightest scalar glueball exists has been inspired by Lattice Gluodynamics calculations \[3, 4, 5\]. However, the quark degrees of freedom are not taken into account in these calculations properly, therefore they should be regarded as qualitative guide only: the \(q\bar{q}\)-components can easily shift the glueball mass by 100-300 MeV.

Glueball/\(q\bar{q}\)-meson mixing was considered in refs. \[6, 7\], though without an implication of resonance decays into the mixing machinery. The K-matrix analysis demonstrates \[2, 8\] that the resonance decay processes may cause the mass shift by the same value 100-300 MeV. The K-matrix amplitude, being unitary in the physical region and because of that treating the overlapping resonances correctly, does not reproduce left-hand side singularities of the amplitude which are related to the interaction of constituents. This means a necessity to complement the K-matrix consideration by an analysis which restores the analyticity of the amplitude.

Here we perform an analysis of the \(00^{++}\)-wave in terms of the propagator matrix (or D-matrix). Corresponding technique, based on the dispersion relation \(N/D\)-method, reconstructs the amplitude with correct analytic properties. The detailed presentation of propagator matrix method is made in Chapter 2: we consider the mixing of overlapping resonances, the mass shifts caused by mixing and decay processes, and the decomposition of the final (i.e. physically observed) state in a series of initial states. Before only the \(00^{++}\)-wave has been treated, however a generalization for other waves can be easily done, e.g. in the framework of the method given in ref. \[2, 14\].

Our D-matrix analysis of \(00^{++}\)-wave is made in terms of \(q\bar{q}\)-states, while K-matrix analysis of refs. \[2, 8\] used the hadron language. When going from the hadronic K-matrix to the quark-antiquark propator matrix, the quark-hadron duality problem should be considered. It can be illustrated using standard quark potential model. In this model the \(q\bar{q}\)-levels are defined by the potential which increases infinitely at large \(r\), \(V(r) \sim \alpha r\) (see Fig. 1). The infinitely increasing potential of confinement leads to the infinite set of stable \(q\bar{q}\) levels: of course, this is a simplified picture of what is seen in the experiment. Only the lowest \(q\bar{q}\) levels are stable with respect to hadronic decays. The higher \(q\bar{q}\) states, \(00^{++}\) included, decay into hadronic channels: an excited \((q\bar{q})_a\)-state produces new
q\bar{q} pair, then the quarks \((q\bar{q})_a + q\bar{q}\) recombine into mesons which leave the confinement trap, forming a continuous meson spectrum at large distances. Conventional picture of the decay is shown in Fig. 1b where the confinement interaction is depicted as certain barrier: the interaction at \(r < R_{confinement}\) creates the \(q\bar{q}\)-spectrum, while at \(r > R_{confinement}\) a continuous spectrum of mesons takes place. In the experiment one can observe a continuous meson spectrum, so to extract information on \(q\bar{q}\) spectra is a problem of

(i) transition from hadron language to that of quarks and gluons;
(ii) elimination of the influence of meson spectrum on \(q\bar{q}\)-levels at \(r > R_{confinement}\).

Sections 3 and 4 are devoted to the consideration of this problem. In Section 3 the necessary information is given concerning composite systems. In Section 4 the rules of quark combinatorics are summarized that allows to restore the quark content of a meson using the decays \(q\bar{q} - \text{meson} \rightarrow \text{two mesons}\). The same decays may serve as a signature of a glueball candidate. Section 5 is devoted to the calculation technique for the transition diagrams which are responsible for the mixing of levels created at \(r < R_{confinement}\). In Section 6 the discussed technique is applied to the analysis of \(0^0++\) wave. Using the quark language for the transition diagrams, we investigate 1100-1900 MeV mass region where four \(0^0++\) states are located. Previously [11], the mass region 1100-1700 MeV was studied with resonances \(f_0(1300)\), \(f_0(1500)\), and \(f_0\left(1530 \pm \frac{90}{250}\right)\) taken into consideration. The present investigation confirms the result of ref. [11]: the pure (gluodynamic) glueball is dispersed over three above-mentioned resonances, and the broad state \(f_0\left(1530 \pm \frac{90}{250}\right)\) is a descendant of the glueball. The resonance \(f_0(1780)\) has a small glueball admixture.

2 D-Matrix Technique

Here the D-matrix technique is presented in detail. First, we consider the propagator of non-stable particle (Section 2.1), then the propagator matrix for the two mixing resonances is constructed, followed by the generalization for an arbitrary number of resonances (Sections 2.2 and 2.3). Examples of a complete resonance overlapping are considered in Section 2.4: in this case one of the resonances created as a result of a mixing accumulates the widths of all other initial resonances.

2.1 Propagator of Non-stable Particle

Propagator of non-stable particle (resonance propagator) is given by the sum of diagrams shown in Fig. 2a, 2b, 2c, and so on. Scalar resonance propagator is equal to

\[
D(s) = (m^2 - s - B(s))^{-1}.
\]

Here \(s = p^2\), where \(p\) is the resonance four-momentum, and \(m\) is the input-state mass; \(d(s) = (m^2 - s)^{-1}\) is the propagator of an input state related to the diagram of Fig. 2a, and \(B(s)\) is the loop diagram which describes the transition of input state into two particles. For the decay of scalar resonance into two scalar or pseudoscalar particles, the
loop diagram $B(s)$ may be written in a form of the dispersion integral as follows:

$$B(s) = \frac{1}{\pi} \int_{(\mu_1+\mu_2)^2}^{\infty} ds' \frac{g^2(s')\rho(s')}{s' - s - i0}.$$  \hspace{1cm} (2)

$\mu_1$ and $\mu_2$ are meson masses in the loop diagram, $g(s)$ is a vertex of the transition resonance $\rightarrow$ two mesons, and $\rho(s)$ is invariant phase space of the two-meson state:

$$\rho(s) = \frac{1}{16\pi s} \sqrt{(s - (\mu_1 + \mu_2)^2)[s - (\mu_1 - \mu_2)^2]}.$$  \hspace{1cm} (3)

Complex resonance mass squared is determined by the propagator pole, $m^2 - s - B(s) = 0$. Below standard notation for it is used: $s = M^2 \equiv M_R^2 - i\Gamma M_R$. Near $s = M_R^2$, the real part of the loop diagram, $B(s)$, can be expanded in a series over $(s - M_R^2)$:

$$D^{-1}(s) = (1 + \text{Re}B'(M_R^2))(M_R^2 - s) - ig^2(s)\rho(s),$$  \hspace{1cm} (4)

$$M_R^2 = m^2 - \text{Re}B(M_R^2).$$

Here we take into account that

$$\text{Im}B(s) = g^2(s)\rho(s), \quad \text{Re}B(s) = P \int_{(\mu_1+\mu_2)^2}^{\infty} \frac{ds'}{\pi} \frac{g^2(s')\rho(s')}{s' - s}.$$  \hspace{1cm} (5)

The vertex $g(s)$ has only left singularities of the scattering amplitude, thus being nonsingular function in the resonance region. Smooth behaviour of $g(s)$ justifies standard Breit-Wigner approximation for $D(s)$, with resonance width being equal to: $M_R \Gamma = g^2(M_R^2)\rho(M_R^2)/(1 + \text{Re}B'(M_R^2))$.

For the resonance decay into several channels, one should make the following replacement in eq. (4):

$$B(s) \rightarrow \sum_n B^{(n)}(s),$$  \hspace{1cm} (6)

where index $n$ refers to different channels.

The loop diagram $B^{(n)}(s)$ is determined by eq. (4), with a specification of vertices and masses: $g(s) \rightarrow g_n(s)$ and $\mu_i \rightarrow \mu_{ni}$. The scattering amplitude in the channel $n$ is determined as

$$A_n(s) = g^{(n)}(s)D(s)g^{(n)}(s).$$  \hspace{1cm} (7)

The quantity $(g^{(n)}(M_R^2))^2\rho_n(M_R^2)/M_R = \Gamma_n$ defines partial width of the resonance.

### 2.2 Mixing of Two Resonances

In the two-resonance case, the propagator of state 1 is determined by the diagrams of figs. 2a, 2b, 2c type (transitions state 1 $\rightarrow$ state 1) and fig. 2d type (transitions state 1 $\rightarrow$ state 2). The sum of these diagrams gives:

$$D_{11}(s) = \left(m_1^2 - s - B_{11}(s) - \frac{B_{12}(s)B_{21}(s)}{m_2^2 - s - B_{22}(s)}\right)^{-1}.$$  \hspace{1cm} (8)
Here $m_1$ and $m_2$ are masses of input states 1 and 2, while loop diagram $B_{ab}(s)$ is determined by eq. (2), with the substitution $g^2(s) \rightarrow g_a(s)g_b(s)$. Let the propagator matrix be introduced:

$$
\hat{D} = \begin{pmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{pmatrix},
$$

(9)

where the non-diagonal term $D_{ab}$ is the transition propagator state $b \rightarrow state a$. The D-matrix is equal to:

$$
\hat{D} = \frac{1}{(M_1^2 - s)(M_2^2 - s) - B_{12}B_{21}} \begin{pmatrix}
M_2^2 - s, & B_{12} \\
B_{21}, & M_1^2 - s
\end{pmatrix}.
$$

(10)

The following notation is used here:

$$
M_a^2 = m_a^2 - B_{aa}(s) \quad a = 1, 2.
$$

(11)

Zeros of the denominator of propagator matrix provide the complex masses of resonances created as a result of the initial-state mixing:

$$
\Pi(s) = (M_1^2 - s)(M_2^2 - s) - B_{12}B_{21} = 0.
$$

(12)

Denote the complex masses of mixed states as $M_A, M_B$.

a) Model with constant $B_{ab}$

As the first step, we consider a simple model: let us assume that loop diagrams $B_{ab}(s)$ depend weakly on $s$ in the region $s \sim M_A^2$ and $s \sim M_B^2$. Let $M_a^2$ and $B_{12}$ be constants, then we have:

$$
M_{A,B}^2 = \frac{1}{2}(M_1^2 + M_2^2) \pm \sqrt{\frac{1}{4}(M_1^2 - M_2^2)^2 + B_{12}B_{21}}.
$$

(13)

Small widths of the resonances 1 and 2 affect small imaginary part of $B_{12}$. In this case eq. (13) gives the standard quantum mechanics result for splitting levels: a repulsion of mixed levels.

The D-matrix can be written as a sum of pole terms, as follows:

$$
\hat{D} = \begin{pmatrix}
\cos^2 \theta + \frac{1}{M_A^2 - s} & -\cos \theta \sin \theta + \frac{\sin \theta \cos \theta}{M_B^2 - s} \\
-\cos \theta \sin \theta + \frac{\sin \theta \cos \theta}{M_A^2 - s} & \sin^2 \theta + \frac{\cos^2 \theta}{M_B^2 - s}
\end{pmatrix},
$$

(14)

where

$$
\cos^2 \theta = \frac{1}{2} + \frac{1}{2} \frac{1}{\sqrt{\frac{1}{4}(M_1^2 - M_2^2)^2 + B_{12}B_{21}}}
$$

(15)

The states $|A>$ and $|B>$ are superpositions of initial states $|1>$ and $|2>$:

$$
|A> = \cos \theta |1> - \sin \theta |2>,
$$

(16)

$$
|B> = \sin \theta |1> + \cos \theta |2>.
$$
b) General case. In general case, when the s-dependence of loop diagrams is not negligible and $|\text{Im}B_{ij}(s)|$ is not small, the states $|A>$ and $|B>$ can be presented as a superposition of initial states as well. For this aim, let us consider the propagator matrix near $s = M_A^2$:

$$\hat{D} = \frac{1}{\Pi(s)} \begin{vmatrix} M_2^2(s) - s & B_{12}(s) \\ B_{21}(s) & M_1^2(s) - s \end{vmatrix} \simeq \frac{-1}{\Pi(M_A^2)(M_A^2 - s)} \begin{vmatrix} M_2^2(M_A^2) - M_A^2 & B_{12}(M_A^2) \\ B_{21}(M_A^2) & M_1^2(M_A^2) - M_A^2 \end{vmatrix}.$$  \hspace{1cm} (17)

Only singular (pole) terms are taken into account here. In the right-hand side of eq. (17), the matrix determinant is equal to zero due to eq. (12). Indeed:

$$[M_2^2(M_A^2) - M_A^2][M_1^2(M_A^2) - M_A^2] - B_{12}(M_A^2)B_{21}(M_A^2) = 0 .$$  \hspace{1cm} (18)

Eq. (12), $\Pi(M_A^2) = 0$, allows us to introduce the complex mixing angle:

$$|A> = \cos \theta_A|1> - \sin \theta_A|2> .$$  \hspace{1cm} (19)

Then, the right-hand side of eq. (17) can be re-written, using $\theta_A$, as follows:

$$\left[\hat{D}\right]_{s \sim M_A^2} = \frac{N_A}{M_A^2 - s} \begin{vmatrix} \cos^2 \theta_A & -\cos \theta_A \sin \theta_A \\ -\sin \theta_A \cos \theta_A & \sin^2 \theta_A \end{vmatrix},$$  \hspace{1cm} (20)

where

$$N_A = \frac{1}{\Pi(M_A^2)} [2M_A^2 - M_1^2 - M_2^2],$$  \hspace{1cm} (21)

$$\cos^2 \theta_A = \frac{M_A^2 - M_2^2}{2M_A^2 - M_1^2 - M_2^2}, \quad \sin^2 \theta_A = \frac{M_A^2 - M_1^2}{2M_A^2 - M_1^2 - M_2^2}.$$ 

Remind that the functions $M_2^2(s)$, $M_1^2(s)$ and $B_{12}(s)$ in eq. (21) are fixed at $s = M_A^2$. In the case under consideration, when the angle $\theta_A$ is a complex magnitude, the quantities $\cos^2 \theta_A$ and $\sin^2 \theta_A$ do not relate to probabilities $|1>$ and $|2>$ in $|A>$. The factors $\sqrt{N_A} \cos \theta_A$ and $-\sqrt{N_A} \sin \theta_A$ are amplitudes for the transitions $|A> \rightarrow |1>$ and $|A> \rightarrow |2>$. This means that corresponding probabilities are equal to $|\cos \theta_A|^2$ and $|\sin \theta_A|^2$.

To decompose the state $|B>$ over initial states $|1>$ and $|2>$, one should likewise use the propagator matrix near $s = M_B^2$. Representing $|B>$ as

$$|B> = \sin \theta_B|1> + \cos \theta_B|2>,$$  \hspace{1cm} (22)

we have the following equation for $\hat{D}$ near $s = M_B^2$:

$$\left[\hat{D}\right]_{s \sim M_B^2} = \frac{N_B}{M_B^2 - s} \begin{vmatrix} \sin^2 \theta_B & \cos \theta_B \sin \theta_B \\ \sin \theta_B \cos \theta_B & \cos^2 \theta_B \end{vmatrix},$$  \hspace{1cm} (23)

where

$$N_B = \frac{1}{\Pi(M_B^2)} [2M_B^2 - M_1^2 - M_2^2].$$  \hspace{1cm} (24)
\[
\cos^2 \theta_B = \frac{M_B^2 - M_1^2}{2M_B^2 - M_1^2 - M_2^2}, \quad \sin^2 \theta_B = \frac{M_B^2 - M_2^2}{2M_B^2 - M_1^2 - M_2^2}.
\]

In eq. (24) the functions \(M_1^2(s), M_2^2(s)\) and \(B_{12}(s)\) are fixed at \(s = M_B^2\).

When \(B_{ab}(s)\) depends weakly on \(s\) and this \(s\)-dependence may be neglected, the angles \(\theta_A\) and \(\theta_B\) coincide. But in general case the angles are different. In this point the propagator matrix formulae differ essentially from those of standard quantum mechanics.

Another characteristic feature of the discussed formulae is related to the mass shift of mixed levels: in standard quantum mechanics, we have a repulsion of levels associated with a conservation of the mean value, \((E_1 + E_2)/2\) (see also eq.(13)). In the general case eq.(12) can yield both the repulsion of mixed levels and their attraction.

The scattering amplitude in the one-channel case is determined as

\[
A(s) = g_a(s)D_{ab}(s)g_b(s) .
\]

For a multichannel case, one should introduce

\[
B_{ab}(s) = \sum_n B_{ab}^{(n)}(s) ,
\]

where \(B_{ab}^{(n)}\) is a loop diagram for the channel \(n\) with the phase space \(\rho_n\) and vertices \(g_a^{(n)}, g_b^{(n)}\). The partial scattering amplitude in the channel \(n\) is equal to

\[
A_n(s) = g_a^{(n)}(s)D_{ab}(s)g_b^{(n)}(s) .
\]

### 2.3 Propagator Matrix for an Arbitrary Number of Resonances

Matrix elements \(D_{ab}\) responsible for the transition of the initial state \(b\) into the state \(a\), being a sum of diagrams of fig. 2 type, satisfy a set of linear equations:

\[
D_{ab} = D_{ac}B_{cb}(s)(m_b^2 - s)^{-1} + \delta_{ab}(m_b^2 - s)^{-1} ,
\]

where \(B_{ab}(s)\) is the loop diagram, and \(\delta_{ab}\) is Kroneker unit matrix. Let us introduce the diagonal propagator matrix \(\hat{d}\) for initial (non-mixed) states:

\[
\hat{d} = \begin{bmatrix}
(m_1^2 - s)^{-1} & 0 & 0 & \cdots \\
0 & (m_2^2 - s)^{-1} & 0 & \cdots \\
0 & 0 & (m_3^2 - s)^{-1} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix} .
\]

Then eq.(28) can be re-written in the matrix form,

\[
\hat{D} = \hat{D}\hat{B}\hat{d} + \hat{d} ,
\]

therefore one has:

\[
\hat{D} = \frac{I}{\hat{d}^{-1} - \hat{B}} .
\]
The matrix $\hat{d}^{-1}$ is diagonal, so the inverted propagator matrix $\hat{D}^{-1} = (\hat{d}^{-1} - \hat{B})$ is written in the following form:

$$
\hat{D}^{-1} = \begin{vmatrix}
M_A^2 - s & -B_{12}(s) & -B_{13}(s) & \cdots \\
-B_{21}(s) & M_B^2 - s & -B_{23}(s) & \cdots \\
-B_{31}(s) & -B_{32}(s) & M_C^2 - s & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{vmatrix},
$$

(32)

where $M_a^2$ is determined by eq. (31). Inverting the matrix, one has for $D_{ab}(s)$:

$$
D_{ab}(s) = \frac{(-1)^{a+b} \Pi_{ba}^{(N-1)}(s)}{\Pi^{(N)}(s)}.
$$

(33)

Here $\Pi^{(N)}(s)$ is the matrix determinant of $\hat{D}^{-1}$, and $\Pi_{ba}^{(N-1)}(s)$ is the matrix supplement to the element $[\hat{D}^{-1}]_{ba}$, i.e. the determinant of the matrix $\hat{D}^{-1}$ with the excluded $b$th row and $a$th column.

Let us write down $\Pi^{(N)}(s)$ as an example of the three-resonance case:

$$
\Pi^{(3)}(s) = (M_A^2 - s)(M_B^2 - s)(M_C^2 - s) - (M_A^2 - s)B_{21}B_{32} - (M_B^2 - s)B_{31}B_{13} -
$$

$$
- (M_C^2 - s)B_{12}B_{21} - B_{12}B_{23}B_{31} - B_{13}B_{32}B_{21}.
$$

(34)

Zeros of $\Pi^{(N)}(s)$ give poles of the propagator matrix, which correspond to physical resonances created after the mixing of initial states. Denote the complex masses of resonances (or zeros of $\Pi^{(N)}(s)$) as

$$
s = M_A^2, \quad M_B^2, \quad M_C^2, \ldots
$$

(35)

In the vicinity of the point $s = M_A^2$, the pole terms provide the leading contribution. Neglecting the next-to-leading terms in eq. (30), one has a system of homogenous equations for $D_{ab}(s)$:

$$
D_{ac}(s) (\hat{d}^{-1} - \hat{B})_{cb} = 0
$$

(36)

The solution of this system, found with an accuracy up to common normalization factor, has a factorized form:

$$
[\hat{D}^{(N)}]_{s \sim M_A^2} = \frac{N_A}{M_A^2 - s} \begin{vmatrix}
\alpha_1^2, & \alpha_1\alpha_2, & \alpha_1\alpha_3, & \cdots \\
\alpha_2\alpha_1, & \alpha_2^2, & \alpha_2\alpha_3, & \cdots \\
\alpha_3\alpha_1, & \alpha_3\alpha_2, & \alpha_3^2, & \cdots \\
\cdots & \cdots & \cdots & \ddots
\end{vmatrix},
$$

(37)

$N_A$ is a normalization factor which is chosen to satisfy:

$$
\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \ldots + \alpha_N^2 = 1.
$$

(38)

The complex coupling $\alpha_a$ is a normalized transition amplitude

$$
\alpha_a(\text{resonance } A \rightarrow \text{state } a),
$$

(39)
so the probability to find the state $a$ in the resonance $A$ is equal to:

$$W_a = |\alpha_a|^2 .$$  

(40)

Analogous expansion of the propagator matrix can be done in the vicinity of other poles:

$$D_{ab}^{(N)}(s \sim M_B^2) = N_B \frac{\beta_a \beta_b}{M_B^2 - s} , \quad D_{ab}^{(N)}(s \sim M_C^2) = N_C \frac{\gamma_a \gamma_b}{M_C^2 - s} .$$  

(41)

The couplings satisfy the normalization conditions similar to eq.(38):

$$\beta_1^2 + \beta_2^2 + \ldots + \beta_N^2 = 1 , \quad \gamma_1^2 + \gamma_2^2 + \ldots + \gamma_N^2 = 1 .$$  

(42)

But in general case, the completeness condition is not fulfilled for the invered expansion:

$$\alpha_a^2 + \beta_a^2 + \gamma_a^2 + \ldots \neq 1 .$$  

(43)

For the two-resonance case, this means that $\cos^2 \Theta_A + \sin^2 \Theta_B \neq 1$. Remind, however, that the right-hand side of eq.(43) is equal to unity in the model with constant $B_{ab}$: see eqs. (13)-(16) which correspond to the standard quantum mechanics formulae for mixed levels.

2.4 Complete Overlapping of Resonances: Effect of Accumulation of Resonance Widths

We consider here the examples which describe ideal situation with a mixing of completely overlapping resonances. These examples demonstrate in intact form the effect of width accumulation by one of resonances.

)Complete overlapping of two resonances.

Example 1:

For the simplicity sake, consider the the weakly-s-dependent transition loop diagram, $B_{ab}$; this case allows us to use eq. (13). We suppose

$$M_1^2 = M_R^2 - i M_R \Gamma_1 , \quad M_2^2 = M_R^2 - i M_R \Gamma_2 ,$$  

(44)

that means

$$\text{Re} B_{12}(M_R^2) = P \int_0^\infty \frac{ds' g_1(s') g_2(s') \rho(s')}{s' - M_R^2} = 0 .$$  

(45)

Eq. (13) can be valid even for positive $g_1$ and $g_2$ if the integral over the region $s' < M_R^2$ compensates the integral over $s' > M_R^2$.

In this case

$$B_{12}(M_R^2) = i g_1(M_R^2) g_2(M_R^2) \rho(M_R^2) = i M_R \sqrt{\Gamma_1 \Gamma_2} .$$  

(46)

Substituting (44)-(46) into eq. (13), we have

$$M_{A,B}^2 = \frac{1}{2}(M_1^2 + M_2^2) \pm \sqrt{\frac{1}{4}(-i M_R \Gamma_1 + i M_R \Gamma_2)^2 + (i M_R \sqrt{\Gamma_1 \Gamma_2})^2} =$$
\[
\begin{align*}
M^2_R - iM_R(\Gamma_1 + \Gamma_2) \\
M^2_R
\end{align*}
\]

It is seen that, as a result of mixing, one state accumulates the widths of both initial states, \(\Gamma_A = \Gamma_1 + \Gamma_2\), while another one turns into a stable particle, \(\Gamma_B = 0\).

Example 2:

Consider one more example when \(Re M^2_1\) and \(Re M^2_2\) are different but \(M_1\Gamma_1 = M_2\Gamma_2\), namely:
\[
M^2_1 = M^2_{R1} - iM\Gamma, \quad M^2_2 = M^2_{R2} - iM\Gamma.
\]

Then eq. (13) reads:
\[
M^2_{A,B} = \frac{1}{2}(M^2_{R1} + M^2_{R2}) - iM\Gamma \pm \sqrt{\frac{1}{4}(M^2_{R1} - M^2_{R2})^2 - M^2\Gamma^2}.
\]

This equation allows one to see the dynamics of poles with an increase of \(\Gamma\). At \(2M\Gamma \ll |M^2_{R1} - M^2_{R2}|\), that corresponds to a suppressed mixing, one has two poles located near the positions given by eq. (18). With increasing \(\Gamma\), the poles move to each other along the real axis. At \(2M\Gamma = |M^2_{R1} - M^2_{R2}|\), the pole positions coincide with each other:
\[
M^2_{A,B} = \frac{1}{2}(M^2_{R1} + M^2_{R2}) - iM\Gamma
\]

With further increase of \(\Gamma\), the poles move along the imaginary axis: as a result, there are two poles, one above another. At \(|M^2_{R1} - M^2_{R2}| \ll 2M\Gamma\), one state is almost stable while the width of another resonance is close to \(2\Gamma\).

b) Complete overlapping of three resonances

Consider the equation
\[
\Pi^{(3)}(s) = 0
\]
in the same approximation as in Example 1. Remind, \(\Pi^{(3)}\) is determined by eq. (34). Correspondingly, we put
\[
M^2_a = M^2_R - s - iM_R\Gamma_a = x - i\gamma_a,
\]
\[
\text{Re}B_{ab}(M^2_R) = 0, \quad (a \neq b).
\]

Here \(x = M^2_R - s\) and \(M_R\Gamma_a = \gamma_a\). Then, with \(B_{ab}B_{ba} = i^2\gamma_a\gamma_b\) and \(B_{12}B_{23}B_{31} = i^3\gamma_1\gamma_2\gamma_3\), eq. (34) reads:
\[
x^3 + x^2(i\gamma_1 + i\gamma_2 + i\gamma_3) = 0.
\]

So, the propagator poles are at
\[
M^2_A = M^2_R - iM_R(\Gamma_1 + \Gamma_2 + \Gamma_3),
\]
\[
M^2_B = M^2_C = M^2_R.
\]

Resonance \(A\) accumulates the widths of all initial resonances, while the states \(B\) and \(C\) turn into being stable and degenerate. Remark, the degeneracy goes off, when a weak \(s\)—dependence of loop diagrams, \(B_{ab}\), is taken into account.
3 Composite systems

Up to now, the resonance is treated as a mixture of input states, and the loop diagrams $B_{ab}$ play crucial role in this mixing. In addition, they take another function, for they determine the compositeness of a particle.

3.1 Entirely-composite particle

Remind, following refs.[9] and [12], the main features of the N/D-method in a description of composite particles, like deuteron: this gives us an example of entirely-composite system. Entirely-composite particle appears as a result of the interaction of constituents and reveals itself as a pole of the scattering amplitude of constituents. Corresponding scattering amplitude for multichannel case is

$$\frac{g_n^2(s)}{1 - B(s)},$$

where $B(s)$ is a sum of loop diagrams given by eq. (5). This amplitude differs from that considered in Section 2 by the replacement $(m^2 - s) \to 1$: the absence of the input-particle propagator in the scattering amplitude is a signature of an entirely composite-state. The mass of the composite state, $M$, is determined by the condition:

$$1 - B(M^2) = 0.$$  

(56)

Suppose that the mass $M$ is located below all the thresholds, so we deal with a stable composite particle. In the vicinity of $s = M^2$, the scattering amplitude can be represented as a pole diagram of fig. 3d:

$$\frac{g_n^2(s)}{1 - B(s)} \approx \frac{g^2(s)}{B'(M^2)(M^2 - s)} = \frac{G_n^2(s)}{M^2 - s}.$$  

(57)

The vertex $G_n(s)$ describes the transition composite particle $\to n$-channel constituents:

$$G_n(s) = \frac{g_n(s)}{\sqrt{B'(M^2)}}.$$  

(58)

The wave function of an entirely-composite particle is the N-dimensional Fock column as follows:

$$\Psi = \begin{vmatrix} \psi_1(s) \\ \psi_2(s) \\ \vdots \\ \psi_N(s) \end{vmatrix},$$

(59)

where the partial wave function is determined as

$$\psi_n(s) = \frac{G_n(s)}{s - M^2}.$$  

(60)
Definition of the partial wave function proceeds from the calculation of the composite-particle electromagnetic form factor, given by the sum of triangle diagrams (fig. 3e), different channels taken into account in the intermediate state (for detail, see refs. [9, 12]):

\[
F(q^2) = \sum_n \int_4^{\infty} \frac{ds \, ds'}{\pi^2} \frac{G_n(s) \Delta_n(s, s', q^2) G_n(s')}{(s - M^2)(s' - M^2)},
\]

(61)

\[
\Delta_n(s, s', q^2) = \frac{(q^2 - s - s')q^2}{16\lambda^{3/2}} \Theta(-ss'q^2 - \mu_n^2\lambda),
\]

\[
\lambda = s^2 + s'^2 + q^4 - 2ss' - 2sq^2 - 2s'q^2.
\]

Here, to avoid cumbersome expressions, we put the masses of constituents in the intermediate state equal to each other, \(\mu_1 = \mu_2 \equiv \mu_n\). The function \(\Delta_n(s, s', q^2)\) in the limit \(q^2 \to 0\) satisfies the equation:

\[
\Delta_n(s, s', q^2 \to 0) = \pi \rho_n(s) \delta(s - s'),
\]

(62)

This leads to the following relation:

\[
F(0) = \sum_n \int_4^{\infty} \frac{ds}{\pi} \rho_n(s) \frac{G_n(s)}{s - M^2} \frac{G_n(s)}{s - M^2}.
\]

(63)

Substituting (58) into right-hand side of eq.(63), one sees that \(F(0)\) is equal to unity:

\[
F(0) = \sum_n \int_4^{\infty} \frac{ds}{\pi} \rho_n(s) \left[ \frac{G_n(s)}{s - M^2} \right]^2 = 1.
\]

(64)

In terms of the relative momenta of constituents, eq.(64) reads:

\[
1 = \sum_n \int \frac{d^3k_n}{(2\pi)^3} \psi_n^2(s),
\]

(65)

where \(k_n^2 = s/4 - \mu_n^2\). The quantity

\[
W_n = \int \frac{d^3k_n}{(2\pi)^3} \psi_n^2(s),
\]

(66)

is the probability for the composite particle to be in the state \(n\), so

\[
1 = \sum_n W_n.
\]

(67)

This means that the composite particle is entirely built of the constituents.
3.2 Non-Entirely Composite Particle

Now return to the case when the amplitude is determined by a set of diagrams of fig. 2 but with a pole of the amplitude located below all the thresholds. This means that we deal with a stable particle. In this case the Fock column which describes the wave function of the composite particle contains additional component, $\psi_0$, which corresponds to the point-like input state with bare mass $m$:

$$\Psi = \begin{pmatrix} \psi_0 \\ \psi_1(s) \\ \psi_2(s) \\ \vdots \\ \psi_N(s) \end{pmatrix}. \quad (68)$$

Consideration of $\psi_n(s)$ is made in a way similar to previous case. The scattering amplitude for multichannel case is equal to

$$\frac{g_n^2(s)}{m^2 - s - B(s)}. \quad (69)$$

Remind that $B(s)$ is given by eq. (6). The mass of the composite state, $M$, is determined by the following condition:

$$m^2 - M^2 - B(M^2) = 0. \quad (70)$$

Let us stress once again that $M$ is supposed to be below all the thresholds, so this is a stable composite particle. In the vicinity of $s = M^2$, the scattering amplitude can be represented as

$$\frac{g_n^2(s)}{m^2 - s - B(s)} \sim \frac{g^2(s)}{(1 + B'(M^2))(M^2 - s)} = \frac{G_n^2(s)}{M^2 - s}. \quad (71)$$

The vertex $G_n(s)$, responsible for the transition composite particle $\rightarrow$ $n$-channel constituents is equal to:

$$G_n(s) = \frac{g_n(s)}{\sqrt{1 + B'(M^2)}}. \quad (72)$$

Partial wave function in the channel $n$ reads:

$$\psi_n(s) = \frac{G_n(s)}{s - M^2}. \quad (73)$$

As before, the definition of the partial wave function proceeds from the calculation of electromagnetic form factor:

$$F(q^2) = W_0 + \sum_{n=1}^{\infty} \int_{4\mu_n^2}^{\infty} ds \, ds' \, \frac{\Delta_n(s, s', q^2)G_n(s')}{\pi^2} \frac{G_n(s)}{(s - M^2)(s' - M^2)}, \quad (74)$$

$\Delta_n(s, s', q^2)$ is determined by eq. (61). Here, as compared to eq.(61), an additional constant term should be included into the form factor, $W_0$, which corresponds to the
interaction of photon with a point-like component (input particle with mass $m$): $W_0$ is the probability for a particle to be in the input state. This leads to the following normalization condition:

$$F(0) = 1 = W_0 + \sum_{n=1}^{\infty} \int_{4\mu_n^2}^{\infty} \frac{ds}{\pi} \frac{G_n(s)}{s-M^2} \rho_n(s) \frac{G_n(s)}{s-M^2} =$$

$$= W_0 + \frac{B'(M^2)}{1 + B'(M^2)}.$$

So, one has:

$$W_0 = \frac{1}{1 + B'(M^2)}.$$  \hspace{1cm} (76)

The probability for the considered particle to be in the state $n$ is equal to

$$W_n = \frac{B_n'(M^2)}{1 + B'(M^2)} = \int \frac{d^3k_n}{(2\pi)^3} \psi_n^2(s).$$  \hspace{1cm} (77)

The magnitude $\sum_{n=1}^{N} W_n$ is the probability for a particle to be in the composite mode.

### 3.3 Non-Stable Particles: Small and Large Distances in Transition Diagrams

Above we have discussed the stable composite particle. In the language of a potential, this corresponds to a level inside potential well: see fig. 4a. Now consider the case of non-stable particle (resonance): corresponding shape of the potential well and the level location is illustrated by fig. 4b. The level is mostly created by the potential at $r < r_0$ (where $r_0$ corresponds to the potential barrier maximum), while at $r \gg r_0$ the wave function describes outgoing particles, which are the resonance decay products. One may introduce the notions “inside the well” and ”outside the well” in the simplest way, using step-function $\Theta(r_0 - r)$, then the probability for constituents to be bound is determined by

$$w_n = \int d^3r \phi_n^2(r) \Theta(r_0 - r).$$  \hspace{1cm} (78)

Actually, we have introduced here the wave function

$$\phi_n^{\text{in}}(r) = \phi_n(r) \Theta(r_0 - r),$$  \hspace{1cm} (79)

which is equal to $\phi_n(r)$ at $r < r_0$ and to zero at $r > r_0$. Likewise, we can introduce the wave function for constituents at $r > r_0$:

$$\phi_n^{\text{out}}(r) = \phi_n(r) \Theta(r - r_0).$$  \hspace{1cm} (80)

In the momentum representation, the wave functions read:

$$\psi_n^{\text{in/out}}(s) = \int d^3r \phi_n^{\text{in/out}}(r) e^{iK \cdot r},$$  \hspace{1cm} (81)
where $\vec{k} = \frac{1}{2}(\vec{k}_1 - \vec{k}_2)$ and $\vec{r} = \vec{r}_1 - \vec{r}_2$.

Presentation of the composite-particle wave function as a sum of short- and long-range pieces is the basic point in the analysis of $0^{++}$ wave. The $q\bar{q}$-resonance being a composite state of quarks is determined by the diagrams of the type shown in fig. 5a: quarks interact by gluon exchanges. The quark interaction at $r < R_{\text{confinement}}$ results in the creation of bound states: in our analysis we approximate the small-$r$ amplitude as a set of pole terms (see fig. 5b):

$$\sum_a \frac{g_a^2}{m_a^2 - s}. \quad (82)$$

However, the sum of pole terms does not take into account the interaction outside the confinement barrier, at $r > R_{\text{confinement}}$. The quark loop with $r > R_{\text{confinement}}$ (fig. 5c) should be saturated by a set of hadron loop diagrams (figs. 5d, 5e, and so on), in consequence with quark-hadron duality. Therefore, the inclusion of hadron loop diagrams into consideration is a necessary step in accounting for the large-$r$ quark interaction.

In terms of the $k$-representation, the separation of small-$r$ and large-$r$ contributions occurs due to a choice of vertices $g_a(s)$: suppression of the large-$s$ region means suppression of the small-$r$ domain, and vice versa. Of course, the use of the step function $\Theta(r - r_0)$ for a separation of the large-$r$ contribution is a simplification, which leads to the violation of analytic properties of the transition loop diagrams, $B(s)$, in the physical region: actually, the small-$r$ region switch-off should be done in a softer way. To keep the analyticity of $B(s)$'s, one should care for analytic properties of vertices, $g_a(s)$.

4 The Couplings: Rules of $1/N$-Expansion and Quark Combinatoric Relations

When analysing experimental data, we approximate the amplitude at $r > R_{\text{confinement}}$ by the set of pole terms, see eq. (82): these poles are determined by the quark-gluon interaction. Quark-gluon content of the states related to poles reveals itself in coupling constants which govern the decay of these states into mesons, thus allowing to find out from the experiment the quark-gluon composition of input states.

In this Section, first, the order of coupling constant magnitude is evaluated within the rule of $1/N$ expansion [13] (Section 4.1). Then, the decay constants are calculated (Section 4.2) using quark combinatoric rules. These latter were used rather long ago for the calculation of secondaries in hadron-hadron collisions at high energies [14] and in the $J/\Psi$ decay [15]. Calculation of the decay glueball $\rightarrow$ two pseudoscalar mesons within the quark combinatorics was carried out in refs. [4, 14, 18]. In the same guideline, the decay couplings are calculated for the process $q\bar{q} - \text{meson} \rightarrow \text{two pseudoscalar mesons}$ in Section 4.3.
4.1 Estimation of the Decay Couplings in the Framework of 1/\(N\)-Expansion Rules

The rules of the 1/\(N\)-expansion \([13]\) (\(N = N_c = N_f\) where \(N_c\) and \(N_f\) are numbers of colours and light flavours) can be used as a guide in the soft quark-gluon physics. Let us estimate, in terms of 1/\(N\)-expansion, the order of magnitude of the decay coupling constants \(\text{Glueball} \to \text{two mesons}\) and \(\text{Glueball} \to q\bar{q}\) denoted below as \(g_{G \to mm}\) and \(g_{G \to q\bar{q}}\).

For this aim, consider the gluon loop diagram in the glueball propagator, fig. 6a, which is determined by the coupling \(\text{Glueball} \to \text{two gluons}\) denoted as \(g_{G \to gg}\). This loop diagram is of the order of unity if the glueball is a two-gluon composite system. From the other side, this loop diagram is proportional to \(g_{G \to gg}^2 N_c^2\) (\(N_c^2\) is a number of colour states in a loop at large \(N_c\)), therefore

\[
B(G \to gg \to G) \sim g_{G \to gg}^2 N_c^2 \sim 1, \quad (83)
\]

or \(g_{G \to gg} \sim 1/N_c\).

The coupling \(g_{G \to q\bar{q}}\) is determined by the diagram of fig. 6b. Similar estimation leads to

\[
g_{G \to q\bar{q}} \sim g_{G \to gg} g_{QCD} N_c \sim 1/N_c. \quad (84)
\]

Here \(g_{QCD}\) is gluon-quark coupling which is of the order of \(\sqrt{1/N_c}\), while the factor \(N_c\) is related to the number of colour states in the triangle diagram. Then the process shown in fig. 6c determines the coupling of the \(\text{Glueball} \to \text{two mesons}\) decay in the leading terms of the 1/\(N_c\)-expansion:

\[
g_{G \to mm}^f \sim g_{G \to q\bar{q}} g_{m \to q\bar{q}}^2 N_c \sim 1/N_c. \quad (85)
\]

Here we take into account that the coupling \(g_{m \to q\bar{q}}\) is of the order of \(1/\sqrt{N_c}\): it is because the quark loop diagram in the meson propagator \(B(m \to q\bar{q} \to m)\) (see fig.6d) is of the order of unity:

\[
B(m \to q\bar{q} \to m) \sim g_{m \to q\bar{q}}^2 N_c \sim 1. \quad (86)
\]

The diagram of fig. 6c provides the contribution which is of the next-to-leading order of magnitude:

\[
g_{G \to mm}^{NL} \sim g_{G \to gg} g_{QCD}^4 g_{m \to q\bar{q}}^2 N_c^2 \sim 1/N_c^2. \quad (87)
\]

Low-lying glueballs may decay into the following two-pseudoscalar channels:

\[
n = \pi\pi, K\bar{K}, \eta\eta, \eta\eta', \eta'\eta'. \quad (88)
\]

The ratios of couplings for the processes of fig. 6c can be calculated in the framework of the quark combinatoric rules. The important point here is the flavour symmetry violation in the production of the light quark pairs: \(u\bar{u}, d\bar{d}\) and \(s\bar{s}\). The production probabilities refer as

\[
uu : \dd : s\bar{s} = 1 : 1 : \lambda, \quad (89)
\]
with $\lambda = 0.45 \pm 0.05$ [17, 18]. The couplings for the glueball decay into two pseudoscalar mesons calculated within quark combinatorics are presented in Table 1 for both leading and next-to-leading terms, $g_{G\to mm}^L$ and $g_{G\to mm}^{NL}$. Unknown dynamics of the decay is hidden into parameters $G^L$ and $G^{NL}$. The decay coupling to the channel $n$ is equal to the sum of both contributions:

$$g_{G\to mm}(n) + g_{G\to mm}^{NL}(n)$$ (90)

The second term is suppressed, compared to the first one, by the factor $1/N_c$: an experience with calculation of quark diagrams gives us evidence that it leads to a suppression by a factor about $1/10$. The sums of coupling constants squared obey the sum rules:

$$\sum_n (g_{G\to mm}^L(n))^2 I(n) = \frac{1}{2} G^2_L (2 + \lambda)^2,$$

$$\sum_n (g_{G\to mm}^{NL}(n))^2 I(n) = \frac{1}{2} G^2_{NL} (2 + \lambda)^2,$$ (91)

where $I(n)$ is an identity factor for the particles produced (see Table 1). These sum rules reflect the fact that the decays under consideration are resulted from the cutting of the two-loop diagrams shown in figs. 6f and 6g, respectively: each quark loop contains the factor $(2 + \lambda)$, see eq. (89).

### 4.2 Quark Combinatoric Calculations of the $q\bar{q}$-Meson Decay Couplings

Quark combinatoric rules can be applied to the $(q\bar{q})_a$-meson decay if we know the quark content of the meson. Let the $(q\bar{q})_a$-meson be the following mixture of non-strange and strange quarks:

$$(q\bar{q})_a = n\bar{n} \cos \Phi \, + \, n\bar{n} \sin \Phi,$$ (92)

where $n\bar{n} = (u\bar{u} + d\bar{d})/\sqrt{2}$. Decay couplings of this meson to two pseudoscalar meson channels, eq. (88), are determined by the diagrams of fig. 7a and 7b types.

The process of fig. 7a provides the leading contribution, in terms of $1/N_c$-expansion:

$$g_{m(a)\to mm}^L \sim \frac{1}{\sqrt{N_c}}.$$ (93)

Similarly, one has for the process of fig. 7b:

$$g_{m(a)\to mm}^{NL} \sim \frac{1}{N_c \sqrt{N_c}}.$$ (94)

The couplings of the decays $m(a) \to \pi\pi$, $K\bar{K}$, $\eta\eta$, $\eta\eta'$, $\eta'\eta'$ are presented in Table 2 both for leading and next-to-leading terms: $g^L$ and $g^{NL}$ are parameters in which the unknown dynamics of the decay process is hidden. The decay coupling to channel $n$ is a sum of the leading and next-to-leading terms,

$$g_{m(a)\to mm}(n) + g_{m(a)\to mm}^{NL}(n).$$ (95)
These two terms are coupling constants in the most general form, thus giving all the varieties of decays. Examples with a special choice of coupling constants may be found in refs. [16,18], where it was suggested that $g^{NL}_{m(a)\to mm}$ is not small as compared to $g^L_{m(a)\to mm}$.

Let us emphasize a very important feature of the meson decay couplings, $g_{m(a)\to mm}$: at $\Phi = \Phi_{Glueball}$ where

$$\tan \Phi_{Glueball} = \sqrt{\frac{\lambda}{2}}, \quad (96)$$

the ratios of meson couplings coincide with those for the glueball both for leading and next-to-leading terms. Namely,

$$[g^L_{m(a)\to mm}(n)]_{\Phi = \Phi_{Glueball}} \rightarrow g^L_{G\to mm}(n), \quad (97)$$

$$[g^{NL}_{m(a)\to mm}(n)]_{\Phi = \Phi_{Glueball}} \rightarrow g^{NL}_{G\to mm}(n).$$

It is due to the two-stage structure of the glueball decay: on the first stage, the $q\bar{q}$-pair is produced with the flavour content given by eq. (89) that corresponds to $\Phi$ given by eq. (96). The final stage is a decay of this $q\bar{q}$-state. An important consequence of the two-stage structure of the glueball decay reads: basing on the ratio of the decay couplings to hadron channels only it is impossible to find out either we deal with the glueball or $q\bar{q}$-state which has $n\bar{n}/s\bar{s}$ content close to that of eq. (89).

5 Resonance Mixing in Terms of the Quark Transition Diagrams

Quark–hadron duality means that one can analyse the 00++-wave using quark or hadron languages. For our analysis this means the choice of a language for the description of loop diagrams — the transitions may be described using either quark or hadron states. In this Section the presentation of loop diagrams, $B_{ab}(s)$, is made in terms of quark states. For the description of quark states, the light cone variables are often used, so, as the first step, in Section 5.1 the loop diagram is written in these variables. As was mentioned above, the quark model deals with wave functions in the $r$-representation; in Section 5.2, $B_{ab}(s)$ is written using not vertices but wave functions. Simple wave functions with one-parameter dependence are presented in Section 5.3.

5.1 Loop Diagram as a Function of Light Cone Variables

Consider the loop diagram of eq.(2) using light cone variables. For this aim we write the phase space factor $\rho(s)$ in general form:

$$\rho(s) \rightarrow d\Phi(P, k_1, k_2) = \frac{1}{2} \frac{(2\pi)^4}{(2\pi)^3} \delta^4(P - k_1 - k_2) \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3}$$

Rewriting $B(s)$ with light cone variables can be done imposing $P_z \rightarrow \infty$. Then, denoting

$$k_{iz}/P_z = x_i, \quad m^2 + k_\perp^2 = m_\perp^2,$$
one has for the right-hand side of eq. (98):

\[ \frac{1}{4(2\pi)^2} \int_0^1 \int_0^1 dx_1 dx_2 \delta(1-x_1-x_2) d^2k_1 \perp d^2k_2 \perp \delta(\vec{P}_\perp - \vec{k}_1 \perp - \vec{k}_2 \perp) \cdot \delta(s + P_- - \frac{m_{1\perp}^2}{x_1} - \frac{m_{2\perp}^2}{x_2}). \]  

(100)

Putting \( \vec{P}_\perp = 0 \), we rewrite \( B(s) \) in terms of the light cone variables:

\[ B(s) = \frac{1}{4(2\pi)^2} \int_0^1 \int_0^1 dx_1 dx_2 \delta(1-x_1-x_2) d^2k_1 \perp d^2k_2 \perp \delta(\vec{k}_1 \perp + \vec{k}_2 \perp) \frac{g^2(s')}{s' - s - i0}. \]  

(101)

Here \( s' = m_{1\perp}^2/x_1 + m_{2\perp}^2/x_2 \).

### 5.2 Loop Diagram in the \( r \)-Representation

To present the loop diagram as an integral in the coordinate space, let us rewrite it in the c.m. \( \vec{k} \)-representation, where \( \vec{k} = (\vec{k}_1 - \vec{k}_2)/2 \) and \( \vec{k}_1 + \vec{k}_2 = 0 \). Then, the equation for \( B_{ab}(s) \) reads:

\[ B_{ab}(s) = \int \frac{d^3k}{(2\pi)^3} \frac{k_{10} + k_{20}}{2k_{10}k_{20}} \frac{g_a(s')g_b(s')}{s' - s - i0}, \]  

(102)

\[ s' = (k_{10} + k_{20})^2, \quad k_{10} = \sqrt{m^2 - k^2}. \]

In terms of wave functions of the states \( a \) and \( b \), it is equal to:

\[ B_{ab}(s) = \int \frac{d^3k}{(2\pi)^3} \psi_a(s') \frac{(s' - m_a^2)(s' - m_b^2)}{s' - s - i0} \psi_b(s'). \]  

(103)

Here

\[ \psi_a(s) = \sqrt{\frac{k_{10} + k_{20}}{2k_{10}k_{20}}} \frac{g_a(s)}{s' - m_a^2}. \]  

(104)

In the coordinate representation one has for \( B_{ab}(s) \):

\[ B_{ab}(s) = \int d^3r \int d^3r' \phi_a(r) v^{(ab)}(\vec{r} - \vec{r}'; s) \phi_b(r'), \]  

(105)

\[ v^{(ab)}(\vec{r}; s) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{r}} \frac{(s' - m_a^2)(s' - m_b^2)}{s' - s - i0}. \]

The wave function in the coordinate representation is determined as usually:

\[ \phi_a(r) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{r}} \psi_a(s). \]  

(106)

Constituent quark model makes it possible to estimate quantitatively the \( ^3P_0q\bar{q} \) wave functions: making use of them, eqs. (103) and (105) allow us to calculate the transition loop diagrams.
5.3 Quark Transition Diagrams

As was stressed above, when the resonance mixing has been analysed, two alternative languages can be used: the hadron language or the quark one. An advantage of the quark language is that the results of the quark phenomenology may be used as a quantitative guide.

The quark loop diagram for the transition scalar state \(b\) \(\rightarrow\) scalar state \(a\) is equal to:

\[
B_{ab}(s) = \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{g_a(s') \rho(s') g_b(s')}{s' - s - i0} \text{Tr}[(\hat{k} + m)(\hat{P} - \hat{k} - m)] = \tag{107}
\]

\[
= \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{g_a(s') \rho(s') g_b(s')}{s' - s - i0} 2(s' - 4m^2),
\]

where \(m\) is quark mass. In the trace calculations, we put \(P^2 = s'\) and \(k^2 = m^2\); there is no momentum conservation in the dispersion representation of the loop diagram, but intermediate particles are on mass shell (for more detail, see refs. \[10, 19\]).

In Section 3, a receipt is given for the construction of composite-particle wave function. As a basis, one can use \(B'_{aa}(M_a^2)\) where \(M_a\) is the composite-particle mass; this quantity is equal, up to normalization factor \(\zeta\), to the wave function squared integrated over \(d^3k\). Applying this receipt to eq. (107), we get:

\[
B'_{aa}(M_a^2) = \int \frac{d^3k}{(2\pi)^3} \left[ \frac{4g_a(s)}{s^{1/4}(s - M_a^2)} \right]^2 (\vec{S}^+ \cdot \vec{k})(\vec{S} \cdot \vec{k}) \rightarrow \tag{108}
\]

\[
\rightarrow \zeta \int \frac{d^3k}{(2\pi)^3} \Psi^+_a(k)\Psi_a(k).
\]

Here \(\zeta = B'_{aa}(M_a^2)\) and

\[
\Psi_a(k) = \frac{4G_a(s)}{s^{1/4}(s - M_a^2)} (\vec{S} \cdot \vec{k}), \tag{109}
\]

\[
G_a(s) = \frac{g_a(s)}{\sqrt{B'_{aa}(M_a^2)}},
\]

In eq.(109) we introduce the spin-1 operator \(\vec{S}\). In standard representation it reads:

\[
S_0 = (0, 0, 1) \quad S_{\pm 1} = \left( \frac{1}{\sqrt{2}}, \pm \frac{i}{\sqrt{2}}, 0 \right) \tag{110}
\]

The wave function

\[
\Psi_a(k) = (\vec{S} \cdot \vec{k})\psi_a(k) \tag{111}
\]

can be rewritten in the \(r\)-representation as follows:

\[
\Phi_a(r) = (\vec{S} \cdot \vec{k})\phi_a(r) = \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\vec{r}}\Psi_a(k). \tag{112}
\]
Here
\[(\vec{S} \cdot \vec{r})\phi_a(r) = i(\vec{S} \cdot \frac{\partial}{\partial \vec{r}})\phi_a(r), \quad (113)\]
\[\phi_a(r) = \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k} \cdot \vec{r}} \psi_a(k). \quad (114)\]

Examples
Now we present simple examples of the \(N^3P_0q\bar{q}\)-wave functions (\(N = 1, 2\)), with a one parameter dependence. We suppose an exponential decrease of wave functions at large \(r\). Then, the wave function of the \(N = 1\) state in the \(r\)-representation reads:
\[\Psi_{10^{++}}(r) = \left(\frac{2\alpha}{3}\right)^{1/2}(\frac{\alpha}{\pi})^{3/4}(\vec{S} \cdot \vec{r})e^{-\frac{\alpha r^2}{2}}. \quad (115)\]
Parameter \(\alpha\) is related to the mean radius squared \(<r^2>\) as follows:
\[<r^2>_{1^3P_0} = \frac{5}{2\alpha}. \quad (116)\]
The same wave function in the \(k^2\)-representation reads:
\[\Psi_{10^{++}}(k^2) = \frac{4}{\sqrt{3}}(\frac{\pi}{\alpha})^{3/4} \frac{i}{\sqrt{\alpha}}(\vec{S} \cdot \vec{k})e^{-\frac{k^2}{2\alpha}}. \quad (117)\]
The wave function of the \(N = 2\) state in the \(r\)- and \(k\)-representations reads:
\[\Psi_{20^{++}}(r) = \left(\frac{5\alpha}{3}\right)^{1/2}(\frac{\alpha}{\pi})^{3/4}(\vec{S} \cdot \vec{r})(1 - \frac{2}{5\alpha r^2})e^{-\frac{\alpha r^2}{2}}, \quad (118)\]
\[\Psi_{20^{++}}(k^2) = \frac{2\sqrt{2}}{\sqrt{15}}(\frac{\pi}{\alpha})^{3/4} \frac{i}{\sqrt{\alpha}}(\vec{S} \cdot \vec{k})(-5 + 2\frac{k^2}{\alpha})e^{-\frac{k^2}{2\alpha}}. \quad (119)\]
The mean radius squared of the \(2^3P_0q\bar{q}\)-state is equal to
\[<r^2>_{2^1P_0} = \frac{9}{2\alpha}. \quad (120)\]
Mean radii squared, eqs. (115) and (118), can be used for fixing the wave functions of the \(1^3P_0\) and \(2^3P_0\) states.

6 \(D\)-Matrix Amplitude and K-Matrix Representation

In this Section we consider the connection between the D-matrix amplitude and the amplitude in the K-matrix representation. Usually the analysis of experimental data is performed in the framework of the K-matrix amplitude: such an analysis has been carried out in refs. [2, 8]. Performing a more complicated analysis, such as the D-matrix one, it is helpful to use the results of the K-matrix analysis at full scale.

In Section 6.1 the transition from D-matrix amplitude to the K-matrix representation is considered for the case of a separate resonance decaying into a number channels. In Section 6.2 the case of two overlapping resonances is investigated. The results of the K-matrix analysis of the \(00^{++}\) wave obtained in ref. [2], are quoted in Section 6.3. These results are used in the next Section as a guideline for the construction of the D-matrix amplitude.
### 6.1 One-Resonance Amplitude

The multichannel amplitude for a separate resonance is given by eq. (7). The transition from this amplitude to the K-matrix representation is rather simple:

\[
\sqrt{\rho_n(s)} A_{nn'}(s) \sqrt{\rho_{n'}(s)} = \frac{g^{(n)}(s)g^{(n')}\rho_n(s)\rho_{n'}(s)}{m^2 - s - \sum_{n''} (R e B^{(n'')}(s) - i \rho_{n''}(s)g^{(n'')2}(s))} = \frac{K_{nn'}(s)}{1 - i \sum_{n''} K_{n''n''}(s)},
\]

where

\[
K_{nn'}(s) = \frac{g^{(n)}(s)g^{(n')}\rho_n(s)\rho_{n'}(s)}{m^2 - s - \sum_{n''} R e B^{(n'')}(s)}.
\]

Let us emphasize that the D-matrix amplitude gives factorized K-matrix elements that means, for example, \( K_{nn}K_{n'n'} = K_{nn'}K_{n'n} \). This specific property results in a simplified K-matrix presentation of the amplitude what is seen at eq. (119). The K-matrix pole is defined by the equation:

\[
m^2 - s - R e B(s) = 0, \quad B(s) = \sum_{n''} B^{(n'')}(s).
\]

Suppose that this equation is satisfied at some point \( s = \mu^2 \). Then in the vicinity of this K-matrix pole the K-matrix elements are of the form:

\[
K_{nn'}(s) = \sqrt{\rho_n(s)} \left[ \frac{g^{(n)}(\mu^2)g^{(n')}(\mu^2)}{(\mu^2 - s)(1 + R e B'(\mu^2))} + f_{nn'} \right] \sqrt{\rho_{n'}(s)}.
\]

This is exactly the form used in refs. [2, 8] for fitting experimental data. The pole position of the K-matrix elements is shifted, as compared to the propagator pole, in the magnitude \( R e B(\mu^2) \). Position of the K-matrix pole also differs significantly from the position of the amplitude pole, provided the value \( I m B(s) \) is not suppressed. This imaginary part should be saturated by meson-meson states, therefore their contribution can be considered as an additional ”dressing” of bound state by a cloud of real mesons. Because of that we call K-matrix poles, which do not include this dressing, as poles of ”bare” states [2, 8].

Another important consequence of eq. (122) is that the K-matrix coupling constants are connected with the D-matrix couplings as follows:

\[
g^{(n)}(K - matrix) = \frac{g^{(n)}(\mu^2)}{\sqrt{1 + R e B'(\mu^2)}}.
\]

Therefore, if the D-matrix couplings satisfy quark combinatoric relations, the same relations are valid for the K-matrix coupling constants: the difference between \( g^{(n)}(K - matrix) \) and \( g^{(n)}(\mu^2) \) lays only in the factor \( (1 + R e B'(s))^{1/2} \) which is common for all channels.

### 6.2 Two-Resonance/Two Channel Amplitude

Now we consider the case when two overlapping resonances decay into two meson-meson channels. Being rather simple for the investigation, this example contains all principal points of a representation of the D-matrix amplitude in the K-matrix form.
The D-matrix for the case under discussion is given by eq. (11) while the transition amplitude is written in eq. (27). For the scattering amplitude in the channel 1 one has:

\[ \rho_1(s)A_{11}(s) = \frac{g_1^{(1)}(M_2^2 - s)g_1^{(1)} + g_1^{(1)}B_{12}g_2^{(1)} + g_2^{(1)}B_{12}g_1^{(1)} + g_2^{(1)}(M_1^2 - s)g_2^{(1)}}{(M_1^2 - s)(M_2^2 - s) - B_{12}B_{21}} = \]

\[ = \frac{K_{11} + i(K_{12}K_{21} - K_{11}K_{22})}{1 - i(K_{11} + K_{22}) + (K_{12}K_{21} - K_{11}K_{22})}. \]

Remind that \( M_a^2 = m_a^2 - B_{aa}(s) \). The main question which should be answered here is: under what condition the quark combinatoric rules are valid for both D-matrix and K-matrix coupling constants? For this purpose, consider in more detail the matrix elements \( K_{11} \) and \( K_{22} \). The K-matrix form of eq. (124) gives us the following expressions for \( K_{11} \) and \( K_{22} \):

\[ K_{nn} = \frac{g_1^{(n)}(\mu_2^2 - s)g_1^{(n)} + g_2^{(n)}(\mu_1^2 - s)g_2^{(n)} + 2ReB_{12}g_1^{(n)}g_2^{(n)}}{(\mu_1^2(s) - s)(\mu_2^2(s) - s) - (ReB_{12})^2}, \]

where

\[ \mu_a^2(s) = m_a^2 - Re B_{aa}(s) = m_a^2 - \sum_n Re B_{aa}^{(n)}(s). \]

Below we show that the K-matrix couplings sustain the quark combinatoric rules if \( ReB_{12} \) is small. Indeed, in this case one has:

\[ K_{nn} \simeq \frac{(g_1^{(n)}(s))^2}{\mu_1^2(s) - s} + \frac{(g_2^{(n)}(s))^2}{\mu_2^2(s) - s} - \frac{(g_1^{(n)}(K-matrix))^2}{\mu_1^2(s) - s} + \frac{(g_2^{(n)}(K-matrix))^2}{\mu_2^2(s) - s} + f_{nn}, \]

An expansion of the right-hand side of eq. (127) near the K-matrix poles gives:

\[ K_{nn} \simeq \frac{(g_1^{(n)}(K-matrix))^2}{\mu_1^2(s) - s} + \frac{(g_2^{(n)}(K-matrix))^2}{\mu_2^2(s) - s} + f_{nn}, \]

where

\[ g_a^{(n)}(K-matrix) = \frac{g_a^{(n)}(\mu_a^2)}{\sqrt{1 + Re B_{aa}'(\mu_a^2)}}. \]

so the K-matrix couplings \( g_a^{(n)}(K-matrix) \) satisfy the same relations as \( g_a^{(n)}(\mu_a^2) \) because they differ by the factor \( \sqrt{1 + Re B_{aa}'(\mu_a^2)} \) which is common for all channels. The K-matrix elements in the form given by eq. (11) were used in refs. [2, 8] for fitting experimental data.

One sees that using quark combinatorics for the K-matrix couplings is justified if \( ReB_{12} \) is small or, more generally, if the real parts of the non-diagonal loop transition diagrams are small. In the case of \( q\bar{q} \)-states, a suppression of the non-diagonal terms \( ReB_{ab}(s) \) occurs because of orthogonality of \( (q\bar{q})_a \) and \( (q\bar{q})_b \) wave functions. For mesons — members of the same nonet — the suppression happens due to cancellation of the \( n\bar{n} \) and \( s\bar{s} \)-loop diagrams (this cancellation is precise, provided the flavour SU(3)-symmetry is not broken). For mesons which are members of different nonets, the suppression of \( ReB_{ab}(s) \) is the result of orthogonality of radial wave functions.
The situation with imaginary parts of non-diagonal loop diagrams may be quite different: the imaginary parts are determined by decays into channels with real mesons, and therefore they strongly depend on the location of thresholds. For example, two lowest K-matrix bare states, which are located below 1.2 GeV [2, 8], despite of their being members of the same nonet, strongly mix due to large imaginary part of the loop transition diagram. The reason is that for the low-mass region only imaginary part of the non-strange quark loop diagram is properly saturated by \( \pi \pi \) mesons, while the imaginary part of \( s \bar{s} \) loop diagram, which should be saturated by \( K \bar{K}, \eta \) and \( \eta' \) mesons, is small because of the threshold suppression of phase spaces, and it is not able to cancel the \( \pi \pi \) contribution.

Another situation with the D-matrix and K-matrix couplings may happen when \( q \bar{q} \) states mix with a particle of another origin, for instance, with a glueball. This case is a subject of our further investigation.

### 6.3 Results of K-Matrix Fit of \( 00^{++} \) Amplitude

Analysis of scalar/isoscalar states in the mass region below 1900 MeV was performed in the framework of the K-matrix approach in ref. [2] basing on the following data set: GAMS data for \( \pi^- p \to \pi^0 \pi^0 n \) [20], \( \eta \eta \) [21], \( \eta 
\eta' \) [22]; CERN-Münich data for \( \pi^- p \to \pi^+ \pi^- n \) [23]; Crystal Barrel data for \( p \bar{p} \to \pi^0 \pi^0 \pi^0, \pi^0 \pi^0 \eta, \pi^0 \eta \eta \) [24, 25]; BNL data for \( \pi \pi \to K \bar{K} \) [26]. The K-matrix elements were used in the form similar to eq. (127):

\[
K_{nn'} = \sum_{n} g_{\alpha}^{(n)}(K - \text{matrix})g_{\alpha}^{(n')}\frac{(K - \text{matrix})}{\mu_{\alpha} - s} + f_{nn'},
\]

with five channels taken into account \( n = \pi \pi, K \bar{K}, \eta \eta, \pi \pi \pi \pi \). Couplings \( g_{\alpha}^{(n)}(K - \text{matrix}) \) are supposed to obey quark combinatoric relations that allows to reconstruct the \( q \bar{q} \)/glueball content of bare states below 1900 MeV. The simultaneous fit of the two-meson spectra of refs. [24 - 26] gives two solutions which describe well the data sets:

**Solution I.** Two bare states \( f_0^{\text{bare}}(720 \pm 100) \) and \( f_0^{\text{bare}}(1260 \pm 30) \) are members of the \( 1^3P_0 \) \( q \bar{q} \)-nonet, with \( f_0^{\text{bare}}(720) \) being \( s \bar{s} \)-rich state, \( \phi(720) = -69^\circ \pm 12^\circ \) (the flavour wave function is defined as \( n\bar{n}\cos\phi + s\bar{s}\sin\phi \)). The bare states \( f_0^{\text{bare}}(1600 \pm 50) \) and \( f_0^{\text{bare}}(1810 \pm 30) \) are members of the \( 2^3P_0 \)-nonet; \( f_0^{\text{bare}}(1600) \) is dominantly \( n\bar{n} \)-state: \( \phi(1600) = -6^\circ \pm 15^\circ \). The state \( f_0^{\text{bare}}(1235 \pm 50) \) is superfluous for \( q \bar{q} \)-classification, being a candidate for the lightest glueball: its couplings to two-meson channels obey quark combinatoric relations for glueball.

Concluding, we have the following states in the solution I:

\[
\begin{align*}
\text{Type of state} & & \text{Flavour wave function} \\
f_0^{\text{bare}}(720) \to 1^3P_0 q \bar{q} & & 0.37n\bar{n} - 0.93s\bar{s} \\
f_0^{\text{bare}}(1260) \to 1^3P_0 q \bar{q} & & 0.93n\bar{n} + 0.37s\bar{s} \\
f_0^{\text{bare}}(1600) \to 2^3P_0 q \bar{q} & & 0.995n\bar{n} - 0.10s\bar{s} \\
f_0^{\text{bare}}(1810) \to 2^3P_0 q \bar{q} & & 0.10n\bar{n} + 0.995s\bar{s} \\
f_0^{\text{bare}}(1235) \to \text{Glueball} & & 0.91n\bar{n} + 0.42s\bar{s} .
\end{align*}
\]

For the glueball, "flavour wave function" refers the \( q \bar{q} \) intermediate state in the decay process (see fig. 6c).
Solution II. Basic nonet members are the same as in solution I. The members of the $2^3P_0$-nonet are the following: $f_0^{\text{bare}}(1235)$ and $f_0^{\text{bare}}(1810)$; both these states have significant $s\bar{s}$-component: $\phi(1235) = 42^\circ \pm 10^\circ$ and $\phi(1810) = -53^\circ \pm 10^\circ$. The state $f_0^{\text{bare}}(1560 \pm 30)$ is superfluous for the $q\bar{q}$-classification, being a good candidate for the lightest glueball.

So, we have in the solution II the following set of states:

| Type of state | Flavour wave function |
|---------------|-----------------------|
| $f_0^{\text{bare}}(720)$ | $1^3P_0q\bar{q}$ $0.37n\bar{n} - 0.93s\bar{s}$ |
| $f_0^{\text{bare}}(1260)$ | $1^3P_0q\bar{q}$ $0.93n\bar{n} + 0.37s\bar{s}$ |
| $f_0^{\text{bare}}(1235)$ | $2^3P_0q\bar{q}$ $0.74n\bar{n} + 0.67s\bar{s}$ |
| $f_0^{\text{bare}}(1810)$ | $2^3P_0q\bar{q}$ $0.67n\bar{n} - 0.74s\bar{s}$ |
| $f_0^{\text{bare}}(1560)$ | Glueball $\rightarrow$ $0.91n\bar{n} + 0.42s\bar{s}$ |

For both solutions five scalar resonances with nearly coinciding positions of the amplitude poles were found:

| Resonance | Position of pole on the complex-$M$ plane, in MeV units |
|-----------|-------------------------------------------------------|
| $f_0(980)$ | $1015 \pm 15 - i(43 \pm 8)$ |
| $f_0(1300)$ | $1300 \pm 20 - i(120 \pm 20)$ |
| $f_0(1500)$ | $1499 \pm 8 - i(65 \pm 10)$ |
| $f_0(1750)$ | $1750 \pm 30 - i(125 \pm 70)$ |
| $f_0(1200/1600)$ | $1530^{+90}_{-250} - i(560 \pm 140)$ |

Comparison of the positions of bare states and those of resonances shows that the observed states are strong mixtures of the bare states and two/four-meson states into which these resonances decay.

Concluding this section, let us discuss the problem of applying quark combinatorics to the K-matrix decay couplings for the bare states in the mass region 1200-1600 MeV, $f_0^{\text{bare}}(1260)$, $f_0^{\text{bare}}(1235)$ and $f_0^{\text{bare}}(1600)$: one of them is supposed to be the glueball. This means that corresponding non-diagonal loop diagrams for the transitions Glueball $\rightarrow 1^3P_0q\bar{q}$ and Glueball $\rightarrow 2^3P_0q\bar{q}$ are not small, and the arguments of Section 6.2 do not work. Nevertheless, eqs. (131) and (132) show us that quark combinatoric rules can be applied to these resonances as well. The reason is that the discussed $q\bar{q}$-states have approximately the same $n\bar{n}/s\bar{s}$ content as the intermediate state in the glueball decay (the flavour wave function of this intermediate state is written in eqs. (131) and (132)). In this case, it is obvious that quark combinatoric relations are almost the same both for the K- and D-matrix amplitudes, see eq. (125).

7 D-Matrix Fit of the $00^{++}$ Amplitude

In this Section we consider the mixing of the glueball state with neighbouring $q\bar{q}$-states. We restrict our consideration by the region 1100-1900 MeV: the calculation of the lowest state $f_0(980)$ requires to take into account the mechanism which saturates the imaginary part of the quark-antiquark loop diagram by meson-meson states in the precise form. In
the large-mass region, above the main meson-meson thresholds, the detailed description of the saturation is not important.

7.1 Parameters and Results

For the calculation of the quark loop diagrams, we should fix the vertices $g_a(s)$. We parametrize the vertices for the transition state $a \rightarrow n\bar{n}$ in the simple form:

$$1^3P_0 \, q\bar{q} - \text{state :} \quad g_1(s) = \gamma_1 \sqrt{s} \frac{k_1^2 + \sigma_1}{k_1^2 + \sigma_1};$$

$$2^3P_0 \, q\bar{q} - \text{first state :} \quad g_2(s) = \gamma_2 \sqrt{s} \left[ \frac{k_2^2 + \sigma_2}{k_2^2 + \sigma_2} - d \frac{k_2^2 + \sigma_2}{k_2^2 + \sigma_2 + h} \right];$$

$$\text{Glueball :} \quad g_3(s) = \gamma_3 \sqrt{s} \frac{k_3^2 + \sigma_3}{k_3^2 + \sigma_3};$$

$$2^3P_0 \, q\bar{q} - \text{second state :} \quad g_4(s) = g_2(s).$$

Here $k^2 = \frac{s}{4} - m^2$ and $k_a^2 = \frac{m^2}{4} - m^2$ where $m$ is the constituent quark mass (fixed at 350 MeV for non-strange quark and at 500 MeV for strange one); $m_a$, $\gamma_a$ and $\sigma_a$ are parameters. Factor $d$ is due to orthogonality of the $1^3P_0q\bar{q}$ and $2^3P_0q\bar{q}$ states: we put $\text{Re} \, B_{12}(s_0) = 0$ at $\sqrt{s_0} = 1.5$ GeV. (In the case of the $s$-dependent B-functions the orthogonality requirement for loop transition diagrams cannot be fixed at all values of $s$).

The parameters $m_a$, $\sigma_a$, $h$ and $\gamma_a$ ($a = 1, 2, 3$) should be determined by masses and widths of the physical resonances. However, the masses $m_a$ are approximately fixed by the K-matrix fit of ref. [2], where masses of the K-matrix poles, $\mu_{a}^\text{bare}$, are defined: $\mu_a^2 \simeq m_a^2 - B_{aa}^2(\mu_a^2)$ (see also eqs. (131) and (132)). Let us stress that $m_3$ is the mass of pure gluonic glueball which is a subject of Lattice QCD calculation.

Parameters which are found in our fit of the $00^{++}$ amplitude in the mass region 1200-1900 MeV are given in Table 3 for the solutions I and II. Using these parameters, we calculate the couplings $\alpha_a$ which are introduced by eqs. (37) and (38):

$$|f_0(1300) > \rightarrow |f_0(1500) > \rightarrow |f_0(1530) > \rightarrow |f_0(1780) > \rightarrow .$$

These couplings determine relative weight of the initial state $a$ in the physical resonance $A$:

$$W_a(A) = |\alpha_a|^2$$

The probabilities $W_a$ are given in Table 3 together with masses of physical resonances, $M_A$, and masses of input states, $m_a$.

7.2 Glueball/$q\bar{q}$-State Mixing

In order to analyze the dynamics of the glueball/$q\bar{q}$ mixing, we use the following method: in final formulae the vertices are replaced in a way:

$$g_a(s) \rightarrow \xi g_a(s),$$

76
with a factor $\xi$ running in the interval $0 \leq \xi \leq 1$. The case $\xi = 0$ corresponds to switching off the mixing of input states. Input states are stable in this case, and corresponding poles of the amplitude are at $s_a = m_a^2$. Fig. 8 shows the pole position at $\xi = 0$ for solution I (fig. 8a) and solution II (fig. 8b). For the glueball state, $m_3$ is the mass of a pure glueball, without $q\bar{q}$ degrees of freedom. In solution I the pure-glueball mass is equal to

$$m_{\text{pure glueball}}(\text{Solution I}) = 1225 \text{ MeV},$$

that definitely disagrees with Lattice-Gluodynamics calculations for the lightest glueball. In solution II

$$m_{\text{pure glueball}}(\text{Solution II}) = 1633 \text{ MeV}.$$  

This value is in a good agreement with recent Lattice-Gluodynamics results: $1570\pm85(\text{stat})\pm100(\text{syst})$ MeV [3, 4] and $1707\pm64$ MeV [3]. With increasing $\xi$ the poles are shifted into lower part of the complex mass plane. Let us discuss in detail the solution II which is consistent with Lattice result.

At $\xi \simeq 0.1 - 0.5$ the glueball state of solution II is mixing mainly with $2^3P_0 \, q\bar{q}$-state, at $\xi \simeq 0.8 - 1.0$ the mixture with $1^3P_0 \, q\bar{q}$-state becomes significant. As a result, the descendant of the pure glueball state has the mass $M = 1450 - i450$ MeV. Its gluonic content is 47% (see Table 3). We should emphasize: the definition of $W_a$ suggests that $\sum_{A=1,2,3,4} W_{\text{glueball}}(A) \neq 1$ because of the $s$-dependent $B_{ab}$ in the propagator matrix.

Hypothesis that the lightest scalar glueball is strongly mixed with neighbouring $q\bar{q}$ states was discussed previously (see refs. [7], [27], and references therein). However, the attempts to reproduce a quantitative picture of the glueball/$q\bar{q}$-state mixing and the mass shifts caused by this mixing cannot be successful within standard quantum mechanics approach that misses two phenomena:

(i) Glueball/$q\bar{q}$-state mixing described by the propagator matrix can give both a repulsion of the mixed levels, as in the standard quantum mechanics, and an attraction of them. The latter effect may happen because the loop diagrams $B_{ab}$ are complex magnitudes, and the imaginary parts $\text{Im}B_{ab}$ are rather large in the region 1500 MeV.

(ii) Overlapping resonances yield a repulsion of the amplitude pole positions along imaginary-$s$ axis. In the case of full overlap of two resonances the width of one state tends to zero, while the width of the second state tends to the sum of the widths of initial states, $\Gamma_{\text{first}} \simeq 0$ and $\Gamma_{\text{second}} \simeq \Gamma_1 + \Gamma_2$. For three overlapping resonances the widths of two states tend to zero, while the width of the third state accumulates the widths of all initial resonances, $\Gamma_{\text{third}} \simeq \Gamma_1 + \Gamma_2 + \Gamma_3$.

Therefore, in the case of nearly overlapping resonances, what occurs in the region near 1500 MeV, it is inevitable to have one resonance with a large width. It is also natural that it is the glueball descendant with large width: the glueball mixes with the neighbouring $1^3P_0 \, q\bar{q}$ and $2^3P_0 \, q\bar{q}$ states, which are both $n\bar{n}$ rich, without suppression.

8 Conclusion

On the basis of N/D method we have developed the matrix propagator approach (D-matrix method) for the analysis of hadron spectra in the case of overlapping resonances. This technique preserves the unitarity of a scattering amplitude as well as its analyticity.
in the whole complex-s plane. Being rather similar to the K-matrix technique for the case of overlapping $q\bar{q}$ resonances the D-matrix technique can give different results when a particle of another nature (e.g. glueball or hybrid) interacts with a $q\bar{q}$ resonance. The case of the interaction of the scalar glueball with neighbouring $q\bar{q}$ states in the region 1100-1800 MeV is considered on the language of quarks and gluons. The content of the initial $q\bar{q}$ and glueball states in the physical resonances was calculated. It appeared that the lightest gluodynamic glueball is dispersed over neighbouring resonances mixing mainly to $1^3P_0$ $q\bar{q}$ and $2^3P_0$ $q\bar{q}$ states. With this mixing the glueball descendant transforms into broad resonance, $f_0(1530^{\pm 90}_{-250})$. This resonance contains (40-50)% of the glueball component. Another part of the glueball component is shared between comparatively narrow resonances, $f_0(1300)$ and $f_0(1500)$ which are descendants of $1^3P_0$ $q\bar{q}$ and $2^3P_0$ $q\bar{q}$ states.

The important development for understanding of meson spectra would be the direct application of the D-matrix approach (instead of the K-matrix one) to the analysis of the meson production amplitude. Such approach will give not only the correct position and mixing of the exotic states with the $q\bar{q}$-mesons but also clarify the situation at low energies where a usual K-matrix approach starts to violate analytical properties of the scattering amplitude.

9 Acknowledgement

We thank D.V.Bugg, F.E.Close, L.G.Dakhno, S.S.Gershtein, A.K.Likhoded, L.Montanet, and Yu.D.Prokoshkin for helpful discussions and remarks. We are indebted to H.Koch for kind invitation to participate at LEAP’96-Conference (August 27-31, 1996, Dinkelsbuehl, Germany) where preliminary results of the analysis were presented. This work was supported by Russian Fundamental Investigation Fund, grant 96-02-17934.
References

[1] L. Montanet et al. (Particle Data Group), Phys. Rev., D50 (1994) 1173.

[2] V.V. Anisovich, Yu.D. Prokoshkin, and A.V. Sarantsev, ”Nonet classification of scalar/isoscalar resonances below 1900 MeV: the existence of extra scalar state in the region 1200-1600 MeV”, hep-ph/9610414, Phys. Lett. B, in press.

[3] G.S. Bali et al., Phys. Lett., B309 (1993) 378.

[4] J. Sexton, A. Vaccarino, and D. Weingarten, Phys. Rev. Lett., 75 (1995) 4563.

[5] F.E. Close and M.J. Teper ”On the lightest scalar glueball”, preprint RAL-96-040 (1996).

[6] M. Genovese, Phys. Rev., D46 (1992) 5204.

[7] C. Amsler and F.E. Close, Phys. Rev., D53 (1996) 295; C. Amsler and F.E. Close, Phys.Lett., B353 (1995) 385.

[8] V.V. Anisovich et al. Phys. Lett., B355 (1995) 363; V.V. Anisovich and A.V. Sarantsev, Phys. Lett., B382 (1996) 429.

[9] V.V. Anisovich, M.N. Kobrinsky, D.I. Melikhov, and A.V. Sarantsev, Nucl.Phys., A544 (1992) 747.

[10] A.V. Anisovich and A.V. Sarantsev, Sov. J. Nucl. Phys., 55 (1992) 1200.

[11] A.V. Anisovich, V.V. Anisovich, Yu.D. Prokoshkin, and A.V. Sarantsev, ”Observation of the lightest scalar glueball”, Z. Phys. A, in press.

[12] V.V. Anisovich, D.I. Melikhov, B.Ch. Metsch, and H.R. Petry, Nucl.Phys., A563 (1993) 549.

[13] G. t’Hoft, Nucl.Phys., B72 (1974) 461; G. Veneziano, Nucl.Phys., B117 (1976) 519.

[14] V.V. Anisovich and V.M. Shekhter, Nucl. Phys., B55 (1974) 455; J.D. Bjorken and G.E. Farrar, Phys. Rev., D9 (1974) 1449.

[15] M.A. Voloshin, Yu.P. Nikitin, and P.I. Porirov, Sov. J. Nucl. Phys., 36 (1982) 586.

[16] S.S. Gershstein, A.K. Likhoded, Yu.D. Prokoshkin, Z.Phys., C24 (1984) 305.

[17] V.V. Anisovich, M.N. Kobrinsky, J. Nyiri, Yu.M. Shabelski, ”Quark model and high energy collisions”, World Scientific, Singapore (1985).

[18] V.V. Anisovich, Phys.Lett., B364 (1995) 195.
[19] V.V. Anisovich, ”Elements of scattering theory”, in ”Hadron Spectroscopy and the Confinement Problem”, pp. 73-130, ed. by D.V. Bugg, NATO ASI Series, Series B: Physics v. 353, Plenum Press, New York and London (1996); A.V. Sarantsev, ”Dispersion diagram method and deuteron as two nucleon composite system”, Lecture Notes in Physics, ”Quantum Inversion Theory and Applications”, H.V. Geramb (Ed.) (1993) 465.

[20] D. Alde et al., Z. Phys. C66 (1995) 375; A.A. Kondashov et al., Proc. 27th Intern. Conf. on High Energy Physics, Glasgow (1994) 1407; Yu.D. Prokoshkin et al., Physics-Doklady 342 (1995) 473; A.A. Kondashov et al, Preprint IHEP 95-137, Protvino (1995).

[21] F. Binon et al., Nuovo Cim. A78 (1983) 313.

[22] F. Binon et al., Nuovo Cim. A80 (1984) 363.

[23] B. Hyams et al., Nucl. Phys. B64 (1973) 134.

[24] V.V. Anisovich et al., Phys. Lett. B323 (1994) 233.

[25] C. Amsler et al., Phys. Lett. B342 (1995) 433.

[26] S.J. Lindenbaum and R.S. Longacre, Phys. Lett. B274 (1992) 492; A. Etkin et al., Phys. Rev. D25 (1982) 1786.

[27] V.V. Anisovich, Physics-Uspekhi, 38 (1995) 1202.
Table 1

Coupling constants given by quark combinatorics for a glueball decaying into two pseudoscalar mesons in the leading and next-to-leading terms of the 1/N expansion. \( \Phi \) is the mixing angle for \( n\bar{n} \) and \( s\bar{s} \) states, and \( \Theta \) is the mixing angle for \( \eta - \eta' \) mesons:

\[
\eta = n\bar{n}\cos \Theta - s\bar{s}\sin \Theta \quad \text{and} \quad \eta' = n\bar{n}\sin \Theta + s\bar{s}\cos \Theta.
\]

| Channel | The glueball decay couplings in the leading terms of 1/N expansion (Fig. 1e) | The decay couplings in the next-to-leading terms of 1/N expansion (Fig. 1f) | Identity factor in phase space |
|---------|--------------------------------------------------------------------------------|--------------------------------------------------------------------------|-----------------------------|
| \( \pi^0\pi^0 \) | \( G^L \) | 0 | 1/2 |
| \( \pi^+\pi^- \) | \( G^L \) | 0 | 1 |
| \( K^+K^- \) | \( \lambda G^L \) | 0 | 1 |
| \( K^0\bar{K}^0 \) | \( \lambda G^L \) | 0 | 1 |
| \( \eta\eta \) | \( G^L (\cos^2 \Theta + \lambda \sin^2 \Theta) \) | \( G^{NL}(\cos \Theta - \sqrt{\frac{\lambda}{2}} \sin \Theta)^2 \) | 1/2 |
| \( \eta\eta' \) | \( G^L (1 - \lambda) \sin \Theta \cos \Theta \) | \( G^{NL}(\cos \Theta - \sqrt{\frac{\lambda}{2}} \sin \Theta) \times \) | 1 |
| \( \eta'\eta' \) | \( G^L (\sin^2 \Theta + \sqrt{\lambda} \cos^2 \Theta) \) | \( G^{NL}(\sin \Theta + \sqrt{\frac{\lambda}{2}} \cos \Theta)^2 \) | 1/2 |
Table 2
Coupling constants given by quark combinatorics for a \((q\bar{q})_a\)-meson decaying into two pseudoscalar mesons; here \((q\bar{q})_a = n\bar{n}\cos\Phi + s\bar{s}\sin\Phi\).

| Channel     | The \((q\bar{q})_a\)-meson decay couplings in the leading terms of \(1/N_c\) expansion (Fig. 7a) | The \((q\bar{q})_a\)-meson decay couplings in the next-to-leading terms of \(1/N_c\) expansion (Fig. 7b) |
|-------------|-------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------|
| \(\pi^0\pi^0\) | \(g^L \cos\Phi/\sqrt{2}\)                                                                      | 0                                                                                                 |
| \(\pi^+\pi^-\) | \(g^L \cos\Phi/\sqrt{2}\)                                                                      | 0                                                                                                 |
| \(K^+K^-\)   | \(g^L(\sqrt{2}\sin\Phi + \sqrt{\lambda}\cos\Phi)/\sqrt{8}\)                                  | 0                                                                                                 |
| \(K^0\bar{K}^0\) | \(g^L(\sqrt{2}\sin\Phi + \sqrt{\lambda}\cos\Phi)/\sqrt{8}\)                                  | 0                                                                                                 |
| \(\eta\eta\)  | \(g^L \left(\cos^2\Theta \cos\Phi/\sqrt{2} + \sqrt{\lambda} \sin\Phi \sin^2\Theta\right)\)   | \(2g^{NL}(\cos\Theta - \sqrt{\frac{\lambda}{2}}\sin\Theta)\)                                  |
|              |                                                                                                | \((\cos\Phi \cos\Theta - \sin\Phi \sin\Theta)\)                                                 |
| \(\eta\eta'\) | \(g^L \sin\Theta \cos\Theta \left(\cos\Phi/\sqrt{2} - \sqrt{\lambda} \sin\Phi\right)\)     | \(\frac{1}{2}g^{NL}(\cos\Theta - \sqrt{\frac{\lambda}{2}}\sin\Theta)\)                         |
|              |                                                                                                | \((\cos\Phi \sin\Theta + \sin\Phi \cos\Theta)\)                                                 |
|              |                                                                                                | \(+\sin\Theta + \sqrt{\frac{\lambda}{2}}\cos\Theta)\)                                             |
|              |                                                                                                | \((\cos\Phi \sin\Theta - \sin\Phi \cos\Theta)\)                                                 |
| \(\eta'\eta'\) | \(g^L \left(\sin^2\Theta \cos\Phi/\sqrt{2} + \sqrt{\lambda} \sin\Phi \cos^2\Theta\right)\) | \(2g^{NL}(\sin\Theta + \sqrt{\frac{\lambda}{2}}\cos\Theta)\)                                 |
|              |                                                                                                | \((\cos\Phi \cos\Theta + \sin\Phi \sin\Theta)\)                                                 |
Table 3

Masses of the initial states, coupling constants and $q\bar{q}$/glueball content of physical states.

| Initial state      | $1^3P_0$ | $2^3P_0$ | Glueball | $2^4P_0$ |
|--------------------|----------|----------|----------|----------|
|                    | $n\bar{n}$-rich | $n\bar{n}$-rich | $s\bar{s}$-rich | $s\bar{s}$-rich |
| $m_i$ (GeV)        | 1.457    | 1.536    | 1.230    | 1.750    |
| $\gamma_i$ (GeV$^{3/4}$) | 0.708    | 1.471    | 0.453    | 1.471    |
| $\sigma_i$ (GeV$^2$) | 0.075    | 0.225    | 0.375    | 0.225    |

| $W[f_0(1300)]$ | 32% | 12% | 55% | 1% |
| $W[f_0(1500)]$ | 25% | 70% | 3%  | 2% |
| $W[f_0(1530)]$ | 44% | 24% | 27% | 4% |
| $W[f_0(1780)]$ | 1%  | 1%  | –   | 98% |

$h = 0.25 \text{ GeV}^2$, $d = 1.01$

Solution II

| Initial state      | $1^3P_0$ | $2^3P_0$ | Glueball | $2^4P_0$ |
|--------------------|----------|----------|----------|----------|
|                    | $n\bar{n}$-rich | $n\bar{n}$-rich | $s\bar{s}$-rich | $s\bar{s}$-rich |
| $m_i$ (GeV)        | 1.107    | 1.566    | 1.633    | 1.702    |
| $\gamma_i$ (GeV$^{3/4}$) | 0.512    | 0.994    | 0.446    | 0.994    |
| $\sigma_i$ (GeV$^2$) | 0.175    | 0.275    | 0.375    | 0.275    |

| $W[f_0(1300)]$ | 35% | 26% | 38% | 0.4% |
| $W[f_0(1500)]$ | 1% | 64% | 35% | 0.4% |
| $W[f_0(1530)]$ | 12% | 41% | 47% | 0.3% |
| $W[f_0(1780)]$ | 0.1% | 0.2% | 0.2% | 99.5% |

$h = 0.625 \text{ GeV}^2$, $d = 1.16$
Fig. 1. c) Meson states in the infinitely increasing quark model potential b) Conventional picture for the transition \( q\bar{q} \rightarrow \) states → meson states.
Fig. 2. Diagrams responsible for the decay/mixing process.
Diagrams for entirely-composite particle. Fig. 3.
Fig. 4. Stable (a) and non-stable (b) levels in the potential model.
Fig. 5. The $q\bar{q}$-interaction diagrams at $r < R_{\text{confinement}}$ (a) presented as a set of pole terms (b). Saturation of the $q\bar{q}$-loop diagram at $r > R_{\text{confinement}}$ (c) by a set of meson loop diagrams (d,e).
Fig. 6. Diagrams responsible for glueball decay.
Fig. 7. Diagrams of $({q\bar{q}})_a$-meson decay.
Fig. 8. Complex-$\sqrt{s}$ plane ($M = Re\sqrt{s}$, $-\Gamma/2 = Im\sqrt{s}$): location of $00^{++}$ amplitude poles after replacing $g_a \rightarrow \xi g_a$. The case $\xi = 0$ gives the positions of masses of input $q\bar{q}$ states and gluodynamic glueball; $\xi = 1$ corresponds to the real case.