RANDOM WALKS ON COUNTABLE GROUPS

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ABSTRACT. We begin by giving a short and essentially self-contained proof of the equivalence between the vanishing of the drift of a finitely generated symmetric measured group with finite first moment and the absence of bounded harmonic functions; a result due to Kaimanovich-Vershik and Karlsson-Ledrappier.

Given a measured group \((G, \mu)\), we introduce the new notion of weak \((G, \mu)\)-mixing and show that the Poisson boundary is weakly \((G, \mu)\)-mixing. In particular, this gives a new proof of the fact that the “double” Poisson boundary is weakly mixing in the non-singular sense, which was first observed by Kaimanovich.

Finally, we show that (non-singular) weak mixing for ergodic \((G, \mu)\)-spaces is equivalent to the absence of a probability measure preserving factor with discrete spectrum.

1. VANISHING OF THE DRIFT OF RANDOM WALKS ON GROUPS

Let \(G\) be a countable group and suppose that \(\mu\) is a probability measure on \(G\) whose support generates \(G\) as a semi-group. For a positive integer \(n\), we define the \(n\)'th convolution of \(\mu\) by

\[
\mu^{\ast n}(s) = \sum_{s_1, \ldots, s_n \in G} \mu(s_1) \cdots \mu(s_n),
\]

where the sum is taken over all \(n\)-tuples \((s_1, \ldots, s_n)\) such that \(s_1 \cdots s_n = s\). We note that \(\mu^{\ast n}\) is again a probability measure on \(G\) and for every element \(s \in G\), there exists an integer \(n\) such that \(\mu^{\ast n}(s)\) is positive. We say that \(\mu\) is symmetric if the adjoint probability measure which is defined by \(\tilde{\mu}(s) := \mu(s^{-1})\) coincides with \(\mu\).

We shall refer to \((G, \mu)\) as a measured group. Given a right-invariant semi-metric \(d\) on \(G\), such that

\[
\sum_s d(s, e) \mu(s) < \infty,
\]

we define the drift \(\ell_d(\mu)\) of the triple \((G, \mu, d)\) by

\[
\ell_d(\mu) = \lim_{n \to \infty} \frac{1}{n} \sum_s d(s, e) \mu^{\ast n}(s),
\]

The limit exists by sub-additivity and is finite by (1.1).

Definition 1.1 (Harmonic function). Let \((G, \mu)\) be a measured group. A \(\mu\)-integrable real-valued function \(u\) on \(G\) is called right \(\mu\)-harmonic if

\[
\sum_s u(gs) \mu(s) = u(g), \quad \text{for all } g \in G,
\]

and we say that \(u\) is left \(\mu\)-harmonic if its adjoint function \(\tilde{u}(s) := u(s^{-1})\) is right \(\mu\)-harmonic. We denote by \(\mathcal{H}^{\infty}(G, \mu)\) the Banach space of all bounded right \(\mu\)-harmonic functions, equipped with the supremum norm. We say that \((G, \mu)\) is Liouville if \(\mathcal{H}^{\infty}(G, \mu)\) consists of only constant functions.

The main aim of this section is to provide a short proof of the following theorem.

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Theorem 1.1 (Zero drift and the Liouville property). Let $G$ be a finitely generated group and let $d$ be a right-invariant word-metric on $G$. Suppose that $\mu$ is a symmetric probability measure on $G$ such that \((1.1)\) holds. Then $(G, \mu)$ is Liouville if and only if $\ell_\phi(\mu) = 0$.

Remark 1.1. This result has a long history. The "only if"-direction is essentially contained in the main result of Kaimanovich-Vershik in [3], while the "if"-direction is much more recent and first established in the generality above by Karlsson-Ledrappier in [3].

1.1. Liouville implies zero drift.

Definition 1.2 (Quasi-harmonic function). Let $(G, \mu)$ be a measured group. A $\mu$-integrable real-valued function $\phi$ on $G$ is called right quasi-$\mu$-harmonic with distortion $\ell_\phi$ if

$$\sum_s \phi(gs) \mu(s) = \phi(g) + \ell_\phi, \quad \text{for all } g \in G.$$ 

Clearly, if $\phi$ is a right quasi-$\mu$-harmonic function with zero distortion, then $\phi$ is right $\mu$-harmonic.

We say that a real-valued function $\phi$ on $G$ is right Lipschitz if

$$\sup_s |\phi(gs) - \phi(s)| < \infty, \quad \text{for all } g \in G.$$ 

We note that every homomorphism from $G$ into $\mathbb{R}$ is a right Lipschitz and right quasi-$\mu$-harmonic function for every probability measure $\mu$ on $G$.

Let $(G, \mu)$ be a measured group. If $\phi$ is right Lipschitz and right quasi-$\mu$-harmonic with distortion $\ell_\phi$, then for every $g$ in $G$, the function

$$\phi_g(s) := \phi(gs) - \phi(s), \quad s \in G,$$

is a bounded right $\mu$-harmonic function on $G$. In particular, if $(G, \mu)$ is Liouville, then $\phi_g$ is constant for every $g$ and thus

$$\phi(gs) - \phi(s) = \phi(g) - \phi(e), \quad \text{for all } g, s \in G.$$ 

We have thus established the following lemma.

Lemma 1.1. If $(G, \mu)$ is a Liouville measured group, then every right Lipschitz and right quasi-$\mu$-harmonic function $\phi$ with $\phi(e) = 0$ and distortion $\ell_\phi$ is a homomorphism and

$$\ell_\phi = \sum_s \phi(s) \mu(s).$$

In particular, if $\mu$ is symmetric, then $\ell_\phi = 0$.

The "if"-direction in Theorem 1.1 now follows immediately from the following proposition.

Proposition 1.1. Let $(G, \mu)$ be a measured group and suppose that $d$ is a right-invariant metric on $G$ such that \((1.1)\) holds. Then there exists a right Lipschitz and right quasi-$\mu$-harmonic function $\phi_d$ on $G$ with $\phi_d(e) = 0$ and distortion equal to $\ell_d(\mu)$.

Proof. By the triangle inequality, for every $g$, the sequence

$$c_k(g) = \sum_t (d(g, t) - d(t, e)) \tilde{\mu}^{*k}(t), \quad k \geq 1,$$

is bounded and one readily verifies that

$$\sum_s c_k(gs) \mu(s) = \sum_{s,t} (d(gs, t) - d(s, e)) \tilde{\mu}^{*k}(t) \mu(s)$$

$$= \sum_{s,t} (d(g, ts^{-1}) - d(ts^{-1}, e)) \tilde{\mu}^{*k}(t) \mu(s)$$

$$+ \sum_{s,t} (d(ts^{-1}, e) - d(s, e)) \tilde{\mu}^{*k}(t) \mu(s),$$
which translates to the equations
\[ \sum_s c_k(gs)\mu(s) = c_{k+1}(g) + \sum_s c_k(s)\mu(s), \quad (1.2) \]
for all \( g \in G \) and \( k \geq 1 \). Furthermore, we have
\[ |c_k(gs) - c_k(s)| \leq \sum_t |d(gs, t) - d(s, t)| \hat{\mu}^{*k}(t) \leq d(g, e) \quad (1.3) \]
for all \( s, g \) in \( G \), which shows that \( c_k \) is right Lipschitz for every \( k \). Also, we have \( c_k(e) = 0 \) for all \( k \).

By a simple diagonal argument, there exists a sequence \((n_j)\) such that the limits
\[ \phi_d(g) := \lim_{j} \frac{1}{n_j} \sum_{k=0}^{n_j-1} c_k(g) \]
exist for all \( g \). By (1.3), the function \( \phi_d \) is right Lipschitz on \( G \) and by (1.2), we have
\[ \sum_s \phi_d(gs)\mu(s) = \phi_d(g) + \sum_s \phi_d(s)\mu(s). \]
Furthermore, we have \( \phi_d(e) = 0 \), and
\[ \sum_s \phi_d(s)\mu(s) = \lim_{j} \frac{1}{n_j} \sum_{k=0}^{n_j-1} \left( \sum_s d(s, e) \hat{\mu}^{*(k+1)}(s) - \sum_s d(s, e) \hat{\mu}^{*k}(s) \right) \]
\[ = \lim_{j} \frac{1}{n_j} \sum_s d(s, e) \hat{\mu}^{*n_j}(s) \]
\[ = \lim_{j} \frac{1}{n_j} \sum_s d(s, e) \hat{\mu}^{*n_j}(s) = \ell_d(\mu), \]
where the second to last equality holds since \( d(s, e) = d(s^{-1}, e) \) for all \( s \) in \( G \).

**Remark 1.2.** If \( G \) is a finitely generated group and \( \mu \) is a symmetric probability measure on \( G \) which is supported on a finite generating set of \( G \), such that \((G, \mu)\) is Liouville, then it is always possible to construct a non-constant right Lipschitz and left \( \mu \)-harmonic function on \( G \). Details of this construction can be found in the appendix of the paper [6].

### 1.2. Zero drift implies Liouville.

**Definition 1.3.** Let \((G, \mu)\) be a measured group and suppose that \( G \) acts bi-measurably on a standard Borel probability measure space \((X, \nu)\) such that the set of null sets of \( \nu \) is preserved by the \( G \)-action. If
\[ \sum_s \left( \int_X f(s \cdot x) \, d\nu(x) \right) \mu(s) = \int_X f(x) \, d\nu(x), \quad \text{for all } f \in L^\infty(X, \nu), \]
then we say that \( \nu \) is \( \mu \)-stationary and we refer to \((X, \nu)\) as a \((G, \mu)\)-space. We say that \((X, \nu)\) is ergodic if the only essentially \( G \)-invariant elements in \( L^\infty(X, \nu) \) are essentially constant.

Suppose that \((X, \nu)\) is a (not necessarily ergodic) \((G, \mu)\)-space. Since the \( G \)-action is assumed to preserve the set of null sets of \( \nu \), there exists by the Radon-Nikodym’s Theorem, for every \( s \in G \), a uniquely determined element \( \sigma_s(s, \cdot) \in L^1(X, \nu) \) such that
\[ \int_X f(s \cdot x) \, d\nu(x) = \int_X f(x) \sigma_s(s, x) \, d\nu(x), \quad \text{for all } f \in L^\infty(X, \nu). \]
One readily verifies that \( \sigma_s \) satisfies the equations
\[ \sigma_s(st, x) = \sigma_s(s, x) \sigma(t, s^{-1} \cdot x), \quad \text{for all } s, t \in G \text{ and } \nu\text{-a.e. } x \in X. \]
Furthermore, since
\[ \sum_s \left( \int_X f(s \cdot x) d\nu(x) \right) \mu^\ast n(s) = \int_X f(x) \left( \sum_s \sigma_v(s, x) \mu^\ast n(s) \right) d\nu(x), = \int_X f(x) d\nu(x), \]
for all \( f \in L^\infty(X, \nu) \) and for all \( n \), we can conclude that
\[ \sum_s \sigma_v(s, x) \mu^\ast n(s) = 1, \quad \text{for } \nu\text{-a.e. } x \in X, \quad (1.4) \]
and for all \( n \). By assumption, the support of \( \mu \) generates \( G \) as a semi-group, and thus there exists for every \( g \in G \), an integer \( n \) such that \( \mu^\ast n(g) \) is positive. In particular, by (1.4), we have
\[ \sigma_v(g, x) \mu^\ast n(g) \leq \sum_s \sigma_v(s, x) \mu^\ast n(s) = 1, \]
for \( \nu \)-almost every \( x \) in \( X \) and thus \( \sigma_v(g, \cdot) \) is essentially bounded. Also, again by (1.4), we see that \( \|\sigma_v(g, \cdot)\|_\infty \geq 1 \) for every \( g \in G \).

We conclude that the map \( r_v : G \to L^\infty(X, \nu) \) defined by
\[ r_v(s) = -\log \sigma_v(s, \cdot), \quad s \in G, \]
satisfies (by Jensen’s inequality)
\[ \int_X r_v(s) d\nu \geq 0, \quad \text{for all } s \in G \quad (1.5) \]
and is a cocycle in the following sense.

**Definition 1.4.** Let \((G, \mu)\) be a measured group and suppose that \((X, \nu)\) is a \((G, \mu)\)-space. A function \( c : G \to L^\infty(X, \nu) \) is called a cocycle if
\[ c(st) = c(s) + \pi(s)c(t), \quad \text{for all } s, t \in G, \]
where \( \pi \) denotes the left regular representation of \( G \) on \( L^\infty(X, \nu) \), that is to say, \( \pi(s)f = f(s^{-1}) \) for \( f \in L^\infty(X, \nu) \) and \( s \in G \). Clearly it follows that \( c(e) = 0 \).

If \( c : G \to L^\infty(X, \nu) \) is a cocycle, then one readily verifies that
\[ \sum_s \left( \int_X c(s) d\nu(s) \right) \hat{\mu}^\ast n(s) = n \sum_s \left( \int_X c(s) d\nu(s) \right) \hat{\mu}(s). \quad (1.6) \]
In particular, since the support of \( \mu \) (and thus \( \hat{\mu} \)) is assumed to generate \( G \) as semi-group, we note that, by (1.5) and Jensen’s inequality, the probability measure \( \nu \) is \( G \)-invariant if and only if
\[ \sum_s \left( \int_X r_v(s) d\nu \right) d\hat{\nu}(s) = 0. \]

We summarize the discussion above as follows.

**Lemma 1.2.** For every \((G, \mu)\)-space \((X, \nu)\), the function
\[ r_v(s) = -\log \sigma_v(s, \cdot), \quad s \in G, \]
is a cocycle and \( \nu \) is \( G \)-invariant if and only if
\[ \sum_s \left( \int_X r_v(s) d\nu \right) d\hat{\nu}(s) = 0. \]

Let \( c : G \to L^\infty(X, \nu) \) be a cocycle. One readily verifies that the function \( \rho_c : G \to [0, \infty) \) defined by
\[ \rho_c(g) = \|c(g)\|_\infty, \quad g \in G, \]
satisfies \( \rho_c(e) = 0 \) and is sub-additive, i.e. the function
\[ d_c(s, t) = \rho_c(st^{-1}), \quad s, t \in G, \quad (1.7) \]
is a right invariant semi-metric on $G$. Furthermore, it is symmetric, i.e. $\rho_c(s^{-1}) = \rho_c(s)$ for all $s$ in $G$.

The following lemma relates the drift of the triple $(G, \mu, d_c)$ for a given cocycle $c$ and a certain integral involving $c$.

**Lemma 1.3.** For every $(G, \mu)$-space $(X, \nu)$ and every cocycle $c : G \to L^\infty(X, \nu)$, we have

$$\left| \sum_s \left( \int_X c(s) d\nu \right) d\hat{\mu}(s) \right| \leq \ell_d(\mu).$$

**Proof.** By (1.5) we have

$$\sum_s \left( \int_X c(s) d\nu \right) d\hat{\mu}(s) = n \cdot \sum_s \left( \int_X c(s) d\nu \right) \hat{\mu}(s)$$

for all $n$, and thus

$$\left| \sum_s \left( \int_X c(s) d\nu \right) d\hat{\mu}(s) \right| \leq \frac{1}{n} \sum_s \rho_c(s) \hat{\mu}^n(s) = \frac{1}{n} \sum_s \rho_c(s) \mu^n(s)$$

for all $n$. Upon letting $n$ tend to infinity we conclude that

$$\left| \sum_s \left( \int_X c(s) d\nu \right) d\hat{\mu}(s) \right| \leq \ell_d(\mu),$$

which finishes the proof. $\square$

**Proposition 1.2** (Furstenberg, [1]). For every (not necessarily symmetric) measured group $(G, \mu)$, there exists an ergodic $(G, \mu)$-space $(B, m)$, which we shall refer to as the Poisson boundary of $(G, \mu)$, such that the linear map $P_m : L^\infty(X, \nu) \to \mathcal{H}^\infty(G, \mu)$ defined by

$$P_m \phi(s) = \int_X \phi(s \cdot b) dm(b), \quad \phi \in L^\infty(B, m),$$

is an isometric isomorphism. In particular, $(G, \mu)$ is Liouville if and only if $m$ is $G$-invariant.

If $G$ is finitely generated, and $S \subseteq G$ is a given symmetric generating set, then the right invariant word metric $d$ on $G$ associated to $G$ has the property that for every right invariant semi-metric $d'$ on $G$, there exists a constant $C$, which only depends on the set $S$ such that

$$d'(s, e) \leq C \cdot d(s, e), \quad \text{for all } s \in G.$$ 

In particular, we have $\ell_{d'}(\mu) \leq C \cdot \ell_d(\mu)$ for every probability measure $\mu$ on $G$.

Hence, if $\mu$ is a probability measure on $G$ such that the support of $\mu$ generates $G$ and $\ell_d(\mu) = 0$, then $\ell_{d_m}(\mu) = 0$, where $d_m$ denotes the right-invariant semi-metric on $G$ associated to the Radon-Nikodym cocycle $r_m$ on the Poisson boundary $(B, m)$ of the measured group $(G, \mu)$, and thus

$$\sum_s \left( \int_X r_m(g) dm \right) \hat{\mu}(s) = 0$$

by Lemma 1.3. We conclude that $m$ is $G$-invariant by Lemma 1.2 and thus $(G, \mu)$ is Liouville.

**2. Weak $(G, \mu)$-mixing for the Poisson boundary**

We begin by giving the central definition of this section.

**Definition 2.1** (Weak $(G, \mu)$-mixing). Let $(G, \mu)$ be a measured group and suppose that $(X, \nu)$ is a $(G, \mu)$-space. We say that the $G$-action on $(X, \nu)$ is weakly $(G, \mu)$-mixing if for every ergodic $(G, \tilde{\mu})$-space $(Y, \eta)$, the diagonal $G$-action on $(X \times Y, \nu \otimes \eta)$ is ergodic.
If \((X, \nu)\) is a standard Borel probability measure space equipped with an action of \(G\) by bi-measurable transformations which preserve the set of null sets of \(\nu\), then we say that \((X, \nu)\) is a \textit{non-singular} \(G\)-space. We say that \((X, \nu)\) is ergodic if the only essentially \(G\)-invariant elements in \(L^\infty(X, \nu)\) are essentially constant, and we say that \((X, \nu)\) is \textit{weakly mixing} (in the non-singular sense) if for every probability measure preserving \(G\)-space \((Z, \xi)\), the diagonal \(G\)-action on the direct product
\[(X \times Z, \nu \otimes \xi),\]
which clearly is non-singular, is an ergodic \(G\)-space. We note that if \((G, \mu)\) is a measured group and the \(G\)-action on \((X, \nu)\) is weakly \((G, \mu)\)-mixing, then \((X, \nu)\) is weakly mixing in the non-singular sense as well. Indeed, every ergodic probability measure preserving \(G\)-space \((Z, \xi)\) is an ergodic \((G, \mu)\)-space.

**Lemma 2.1.** Let \((G, \mu)\) be a measured group and suppose that \((X, \nu)\) is a weakly \((G, \mu)\)-mixing space and \((Y, \eta)\) is a weakly \((G, \hat{\mu})\)-mixing space. Then the diagonal \(G\)-action on the direct product
\[(X \times Y, \nu \otimes \eta)\]
is weakly mixing in the non-singular sense.

The main result of this section can now be formulated as follows.

**Theorem 2.1.** For every measured group \((G, \mu)\), the \(G\)-action on the Poisson boundary \((B, \mu)\) is weakly \((G, \mu)\)-mixing. In particular, if \((\hat{B}, \hat{\mu})\) denotes the Poisson boundary of \((G, \hat{\mu})\), then the diagonal \(G\)-action on
\[(B \times \hat{B}, \mu \otimes \hat{\mu})\]
is weakly mixing in the non-singular sense.

**Remark 2.1.** The author has not been able to locate the first part of this theorem in the existing literature. However, the second part is a special case of the main result in [5].

We begin the proof by stating a simple consequence of Theorem 1.2.

**Corollary 2.1.** The linear span of the set
\[\mathcal{R} = \{\sigma_m(s, \cdot) : s \in G\} \subset L^1(B, \mu)\]
is norm-dense in \(L^1(B, \mu)\).

**Proof.** If the linear span is not dense, then, by Hahn-Banach’s Theorem, there exists at least one element \(\phi \in L^\infty(B, \mu) = L^1(B, \mu)^*\), which is not identically zero, such that
\[
\int_B \phi(b) \sigma_m(s, b) dm(b) = \int_B \phi(s \cdot b) dm(b) = (P_m \phi)(s) = 0,
\]
for all \(s \in G\). Since \(P_m : L^\infty(B, \mu) \to \mathcal{H}^\infty(G, \mu)\) is an isometric isomorphism by Theorem 1.2, we conclude that \(\phi\) is identically zero, which is a contradiction. \(\square\)

Let \((Y, \eta)\) be an ergodic \((G, \hat{\mu})\)-space and suppose that \(f \in L^\infty(B \times Y, \mu \otimes \eta)\) is an essentially \(G\)-invariant element. We may assume that
\[
\int_B \int_Y f(b, y) dm(b) d\eta(y) = 0,
\]
and we wish to prove that \(f\) vanishes almost everywhere, or equivalently: For every \(\phi \in L^1(B, \mu)\), we have
\[
\int_B f(b, y) \phi(b) dm(b) = 0, \quad \text{for \(\eta\)-a.e.} \ y \in Y.
\]
By Corollary 2.1 this is equivalent to proving that
\[ \int_B f(b,y) \sigma_m(s,b) dm(b) = \int_B f(s \cdot b,y) dm(b) = \int_B f(b,s^{-1} \cdot y) dm(b) = 0, \]
for every \( s \in G \) and \( \eta \)-a.e. \( y \in Y \). Since \((Y, \eta)\) is a non-singular \(G\)-space, it suffices to show that the element \( F \in L^\infty(Y, \eta) \) defined by
\[ F(y) = \int_B f(b,y) dm(b), \quad y \in Y, \]
is essentially constant. We note that
\[ \sum_s F(s \cdot y) \tilde{\mu}(s) = \sum_s F(s^{-1} \cdot y) \mu(s) = \sum_s \left( \int_B f(s \cdot b,y) dm(b) \right) \mu(s) = F(y), \]
for \( \eta \)-a.e. \( y \in Y \), since \( m \) is \( \mu \)-harmonic. Theorem 2.1 can now be deduced from the following lemma applied to the ergodic \((G, \tilde{\mu})\)-space \((Y, \eta)\).

**Lemma 2.2.** Let \((X, \nu)\) be a \((G, \mu)\)-space. If \( F \in L^\infty(X, \nu) \) satisfies
\[ \sum_s F(s \cdot x) \mu(s) = F(x), \quad \text{for } \nu\text{-a.e. } x \in X, \]
then \( F \) is \( G \)-invariant. In particular, if \((X, \nu)\) is ergodic, then \( F \) is essentially constant.

**Proof.** We may assume that \( F \) is real-valued. Since the support of \( \mu \) generates \( G \) as a semi-group, it suffices to show that
\[ \sum_s \left( \int_X |F(s \cdot x) - F(x)|^2 d \nu(x) \right) \mu^k(s) = 0 \]
for all \( k \). Upon expanding the square and using the \( \mu \)-harmonicity of \( \nu \), we note that
\[ \sum_s \left( \int_X (F(s \cdot x) - F(x))^2 d \nu(x) \right) \mu^k(s) = 2 \left( \int_X F^2 d \nu - \int_X F(x) \left( \sum_s F(s \cdot x) \mu^k(s) \right) d \nu(x) \right). \]
By our assumption on \( F \), we conclude that these expressions vanish for every \( k \).

3. **A Characterization of Non-singular Weak Mixing for \((G, \mu)\)-Spaces**

Let \( G \) be a countable group and suppose that \((X, \nu)\) is a \( G \)-space. Recall that another \( G \)-space \((Y, \eta)\) is a *factor* of \((X, \nu)\) if there exist co-null Borel sets \( X' \subset X \) and \( Y' \subset Y \) and a \( G \)-equivariant Borel measurable map \( p : X' \to Y' \). Clearly, if \((X, \nu)\) is ergodic (weakly mixing), then every factor of \((X, \nu)\) is ergodic (weakly mixing).

Standard examples of a non-weakly mixing *probability measure preserving* \( G \)-space are given by compact homogeneous spaces. Let \( K \) be a non-trivial compact group and let \( L \subset K \) be a closed (not necessarily normal) subgroup. Suppose that there exist a homomorphism \( \tau : G \to K \) with dense image. Then the \( G \)-action on \( K/L \) given by
\[ g \cdot kL = \tau(g)kL, \quad g \in G, kL \in K/L \]
preserves the Haar measure \( m \) on the quotient space \( K/L \) and is ergodic. We shall refer to \( G \)-spaces of the form \((K/L, m)\) as above as *isometric*.

**Definition 3.1** (Discrete spectrum). Let \( G \) be a countable group and suppose that \((\mathcal{H}, \pi)\) is a unitary representation of \( G \). We say that \((\mathcal{H}, \pi)\) has *discrete spectrum* if \((\mathcal{H}, \pi)\) decomposes as an orthogonal sum of *finite-dimensional* sub-representations.

The following result of Mackey [7] characterizes isometric \( G \)-spaces among the probability measure preserving \( G \)-spaces.
Proposition 3.1. Every ergodic probability measure preserving $G$-space $(Y, \eta)$ whose associated Koopman $G$-representation has discrete spectrum is isomorphic as a $G$-space to an isometric $G$-space.

The main result of this section can now be formulated as follows. Recall that a countable group is minimally almost periodic if every ergodic probability measure preserving $G$-space is weakly mixing. For instance, every finitely generated simple infinite group is minimally almost periodic.

Theorem 3.1. Let $(G, \mu)$ be a (countable) measured group and suppose that $(X, \nu)$ is an ergodic $(G, \mu)$-space which is not weakly mixing in the non-singular sense. Then $(X, \nu)$ admits a non-trivial probability measure preserving factor which is isometric. In particular, if $G$ is minimally almost periodic, then every ergodic $(G, \mu)$-space is weakly mixing in the non-singular sense.

We note that if $(\mathcal{H}, \pi)$ is a unitary representation of $G$ on a separable Hilbert space $\mathcal{H}$, then the unit ball $B_1(\mathcal{H})$ is sequentially compact in the weak topology, and $G$ acts on $B_1(\mathcal{H})$ by homeomorphisms by

$$g \cdot v = \pi(g)v, \quad g \in G, v \in B_1(\mathcal{H}).$$

By the definition of the weak topology and the Stone-Weierstrass Theorem, the complex conjugate invariant algebra generated by functions of the form $\langle \cdot, u \rangle$, where $u$ ranges over $\mathcal{H}$ is dense in $C(B_1(\mathcal{H}))$, and it is not hard to show that if $(\mathcal{H}, \pi)$ has discrete spectrum, then for every $G$-invariant probability measure $\eta$ on $B_1(\mathcal{H})$ (if such exist), the associated Koopman representation on $L^2(B_1(\mathcal{H}), \eta)$ has discrete spectrum.

We recall that every unitary $G$-representation $(\mathcal{H}, \pi)$ can be decomposed into an orthogonal sum of two sub-representations $\mathcal{H}_c$ and $\mathcal{H}_{wm}$, where $\mathcal{H}_c$ is the closure of the direct sum of all finite-dimensional sub-representations of $\pi$ and where $\mathcal{H}_{wm}$ completely lacks finite-dimensional sub-representations.

If $u \in \mathcal{H}$, then we denote by $u^*$ the element in $\mathcal{H}^*$ defined by $u^*(v) = \langle u, v \rangle$ for all $v \in \mathcal{H}$. We identify $\mathcal{H}$ and $\mathcal{H}^*$ via the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle u^*, v^* \rangle = \overline{\langle u, v \rangle}, \quad u, v \in \mathcal{H}.$$

The contragradient of $(\mathcal{H}, \pi)$ is the unitary $G$-representation $\pi^*$ on $\mathcal{H}^*$ defined by

$$(\pi^*(g)u^*)(v) = \overline{\langle u, \pi(g^{-1})v \rangle}, \quad u, v \in \mathcal{H},$$

where we have adopted the convention that the first variable is anti-linear over the complex numbers. Clearly, $(\mathcal{H}, \pi)$ and $(\mathcal{H}^*, \pi^*)$ are isomorphic as unitary $G$-representations.

We can write

$$|\langle u, v \rangle|^2 = \overline{\langle u, v \rangle} \cdot \langle u, v \rangle = \langle u^* \otimes u, v^* \otimes v \rangle,$$

so that if $\eta$ is a $G$-invariant probability measure on $B_1(\mathcal{H})$, then

$$\int_{B_1(\mathcal{H})} |\langle u, v \rangle|^2 d\eta(v) = \langle u^* \otimes u, z \rangle,$$

where the element

$$z = \int_{B_1(\mathcal{H})} v^* \otimes v d\eta(v) \in \mathcal{H}^* \otimes \mathcal{H},$$

is invariant under the tensor representation $\pi^* \otimes \pi$ on $\mathcal{H}^* \otimes \mathcal{H}$. In particular, the compact operator $K_z : \mathcal{H} \to \mathcal{H}^*$ defined by

$$\langle u, K_z v \rangle = \langle z, u^* \otimes v \rangle, \quad u, v \in \mathcal{H}.$$
intertwines the representations $\pi$ and $\pi^*$, and upon identifying $\mathcal{H}^*$ and $\mathcal{H}$, we see that $K_z$ is hermitian. We also note that if $\mathcal{H}$ (and thus $\mathcal{H}^*$) does not admit any finite-dimensional sub-representations, then
\[
\int_{B_1(\mathcal{H})} |\langle u, v \rangle|^2 \, d\eta(v) = \langle u^*, K_z u \rangle = 0, \quad \text{for all } u \in \mathcal{H},
\]
by the spectral theorem for compact hermitian operators, and thus $\eta$ must be concentrated at zero. In particular, since $\mathcal{H}_{wm}$ lacks finite-dimensional sub-representations and
\[
B_1(\mathcal{H}) = \{ (u, v) \in \mathcal{H}_c \oplus \mathcal{H}_{wm} : \|u\|^2 + \|v\|^2 \leq 1 \},
\]
we conclude that every $G$-invariant probability measure on $B_1(\mathcal{H})$ is concentrated on $B_1(\mathcal{H}_{c})$ and thus the $G$-space $(B_1(\mathcal{H}), \eta)$ has discrete spectrum, and is isometric by the result of Mackey stated above.

To prove Theorem 3.1 we now argue as follows. Suppose that $(X, \nu)$ is an ergodic $(G, \mu)$-space which is not weakly mixing in the non-singular sense. Then there exists a ergodic probability measure preserving $G$-space $(Y, \eta)$ and an essentially $G$-invariant element $f \in L^{\infty}(X \times Y, \nu \otimes \eta)$ which is not essentially constant. We may assume that the essential supremum of $f$ is bounded above by one, so that the image of the $G$-equivariant map $p_f : X \to L^2(Y, \eta)$ defined by (after choosing a representative of $f$)
\[
p_f(x) = f(x, \cdot) \in L^2(Y, \eta), \quad x \in X,
\]
is contained in $B_1(L^2(Y, \eta))$. The image measure $\zeta = (p_f)_*\nu$ is an ergodic $\mu$-harmonic probability measure on $B_1(L^2(Y, \eta))$, where $G$ acts via the Koopman representation on $L^2(Y, \eta)$, so that Theorem 3.1 will be a direct consequence of the following observation, and the discussion above.

**Proposition 3.2.** If $(\mathcal{H}, \pi)$ is a unitary representation of $G$ and $\nu$ is a $\mu$-harmonic probability measure on $B_1(\mathcal{H})$, then $\nu$ is $G$-invariant.

We note that it is enough to show that whenever $u_1, \ldots, u_k$ and $w_1, \ldots, w_l$ are elements in the Hilbert space $\mathcal{H}$, then
\[
\int_{B_1(\mathcal{H})} \phi(g \cdot v) \, d\nu(v) = \int_{B_1(\mathcal{H})} \phi(v) \, d\nu(v), \quad \text{for all } g \in G,
\]
where
\[
\phi(v) = \langle w_1, v \rangle \cdots \langle w_l, v \rangle \cdots \langle u_1, v \rangle \cdots \langle u_k, v \rangle, \quad v \in \mathcal{H}.
\]
However, since $\nu$ is assumed to be $\mu$-harmonic, and thus
\[
\sum_g \left( \int_{B_1(\mathcal{H})} \phi(g \cdot v) \, d\nu(v) \right) \mu(g) = \int_{B_1(\mathcal{H})} \phi(v) \, d\nu(v),
\]
for all $u_1, \ldots, u_k, w_1, \ldots, w_l \in \mathcal{H}$, we note that the element
\[
u = \int_{B_1(\mathcal{H})} (v^*)^{\otimes l} \otimes \nu^{\otimes k} \, d\nu(v) \in (\mathcal{H}^*)^{\otimes l} \otimes \mathcal{H}^{\otimes k}
\]
satisfies
\[
\int_{B_1(\mathcal{H})} \phi(g \cdot v) \, d\nu(v) = \langle w_1^* \otimes \ldots \otimes w_l^* \otimes u_1 \otimes \ldots \otimes u_k, \pi_{k,l}(g)u \rangle
\]
for all $g \in G$, where $\pi_{k,l} = (\pi^*)^{\otimes l} \otimes \pi^{\otimes k}$ and
\[
\sum_g \mu(g) \cdot \pi_{k,l}(g)u = u.
\]
We wish to show that $u$ is $\pi_{k,l}(g)$-invariant for every $g \in G$. If $u = 0$, then this is clear, and if $u$ is non-zero, then we may assume that its norm equals one, and then the last equation simply
means that a convex combination of unitary translates of $u$ equals $u$, which cannot happen unless all of these translates are trivial (due to the strict convexity of the unit ball in tensor product $(\mathcal{H}^\alpha)\otimes \mathcal{H}^\beta$). We conclude that $u$ is invariant, and thus $\nu$ is $G$-invariant.

Remark 3.1. An alternative proof of Proposition 3.2 (using the fact that the $G$-action on $B_1(\mathcal{H})$ is weakly almost periodic) is given in the paper [2] by Furstenberg and Glasner.

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