Abstract

In this work, we show a Bi-Ronchi test (BRT) proposal using point cloud for sensing the whole surface and wavefront of spherical mirrors as a fast computational test with efficiency comparable with the Ronchi test. We also show an application of the Irradiance Transport equation based on the differential form of the circular Zernike polynomials, to obtain the Phase Transport equation, with capabilities to study the wavefront propagation. To this aim, we experimentally analyze the propagation of $W$ with the BRT, to validate the experimental propagation performed with the Irradiance Transport Equation, giving place to a Phase Transport equation. For this purpose, we use square rulings to observe bi-Ronchigrams and their corresponding Transversal Aberrations as well as their $W$. We validated our results by comparing the BRT with the Ronchi Test for several measurements of a given wavefront $W$ with Ronchi rulings in the same position. Three different bRR were placed in several locations before and beyond the mirror curvature Radius in order to characterize the experimental propagation of $W$ and test the reflection relation of $W$ before and beyond the mirror Curvature Radius. Finally, we use the Phase Transport equation to analyze the propagation of an efficient $W$ in terms of the differential circular Zernike Polynomials in order to obtain a Riemann-integrable function to numerically find a solution for validating the experimental propagation, the symmetry of the wavefronts, the existence of a symmetry matrix, and the BRT.

1. Introduction

The underlying physical and mathematical analysis in propagation phenomena has a very solid scientific background, comprising resourceful experimental techniques, and giving place to secondary mathematical problems challenging the current state of the art giving place to new theories. For example, in quantum theory, the development of the theory related to the wave-particle duality, the matter states, and their propagation in media i.e. the radiation [1]. This theory arose beyond the classical limits describing the more atypical phenomena in undulatory physics. In this framework, the wavefront has been in the spotlight of diverse research works in mathematics, physics, electronics, and optics. The wavefront is intimately related to the Eikonal equation (Taylor [2], Adam [3], Christodoulou et al [4], and Stavroudis [5]) since it is considered as the envelope of the ray set produced by a given source. In the particular case of this work, we consider a mirror (reflecting the rays of a luminous source) to carry out the observations and the corresponding analysis. It should be noticed that the wavefront gradient is not necessarily parallel to the vectors defined from the rays ($R_v$, Ray Vectors) of the source. The wavefront is in general an envelope of an arbitrary type of electromagnetic radiation in the whole...
spectrum (visible or non-visible), in an arbitrary set of vectors or a vector space-field [6–8] generated by a given source.

In this work, we analyze the wavefronts of spherical mirrors using the Bi-Ronchi Test (BRT) along with the point cloud method [9] comparing ideal (simulated) and experimental Bi-Ronchigrams. We use bi-Ronchi Rulings (bRR) or square ruling similar to a chessboard or Checker grating [10, 11]. For the sake of validating the results obtained by applying the BRT, we analyzed the wavefront propagation (W(ρ, θ), referring to the Wave Aberration Function) [8, 12] along the propagation axis, placing the bRR at different positions before and beyond the point C (located at the curvature radius CR1) obtaining in each case a wavefront Wρ1 (ρ, θ). In order to validate the experimental wavefront propagation Wρ (ρ, θ), we use the ITE (derived by Teague [13]) to obtain a Phase Transport equation involving second-order derivatives of the circular Zernike polynomials [8, 14, 15]. These steps allow us to estimate several W beyond C (Curvature Radius), proving in addition the existence of a reflection matrix relating the W before and beyond C. Finally, in order to validate the wavefront propagation we used 3 bRR obtaining 6 Ws for each bRR, 3 before C and 3 beyond C. Our obtained RMS values along with the comparison with the RT (Ronchi Test) show the efficiency of the BRT. We illustrate the application of the BRT in the analysis of the wavefront propagation as well as its symmetry beyond C. Finally, we introduce the analytical proposal of the Phase Transport equation to numerically obtain the wavefront propagation from a Wefficient (efficient wavefront). We show our results and discussions, highlighting our numerical propagation results in the video included as supplementary material WPropagated.mov, where we can observe a Foucault phenomenon) [8].

2. Problem and background

2.1. Irradiance Transport equation (ITE)

The ITE was deduced in 1983 by Teague [13] assuming an electromagnetic perturbation satisfying the wave equation and the Fresnel diffractive principles. Teague [13] built a perturbation u(z) that satisfies the parabolic approximation of the wave equation

$$\frac{\partial}{\partial z} + \frac{\nabla^2_z}{2k} + k u_z(z, r) = 0,$$

with \( r = (x, y) \) a vector, \( i \) denoting the imaginary unity, \( k = 2\pi/\lambda \), and \( \nabla^2_z \) is the Laplacian transversal operator. We assume the irradiance term \( I(z, r) \) satisfying \( I(z, r) = u_0(z)u^*_z(r) \) (with \( u_0^* \) the conjugate complex of \( u_0 \)), and a phase term \( \phi_0(z) \), satisfying \( u_z(z, r) = \sqrt{I}e^{i\phi_0(z)} \). Hence, from equation (1) we obtain the ITE by Teague [13]. We highlight some cases regarding the term \( \partial \phi_0(z) / \partial z \) if the term \( \phi_0(z) \) satisfies \( \partial \phi_0(z) / \partial z = 0 \), and substituting \( u_0(z) \) in equation (1) we obtain a complex expression with real part

$$k^2 + \frac{\nabla^2_z I}{4I} - \frac{(\nabla^2_z \phi_0)^2}{2} - \frac{(\nabla^2_z I)^2}{8I^2} = -k \frac{\partial \phi_0}{\partial z},$$

and imaginary part given by

$$\nabla_z[I(\nabla^2_z \phi)] = I(\nabla^2_z \phi_0) + (\nabla^2_z \phi_0) \cdot (\nabla^2_z I) = -k \frac{\partial I}{\partial z},$$

with ‘*’ the dot product and equation (3) the so-called ITE, obtained by assuming \( \partial \phi_0(z) / \partial z = 0 \). We recall that complex or imaginary equations and their solutions are commonly associated with absorptions in the media, components, or elements [1, 16]. In the literature, the irradiance is written in terms of the analyzed phenomenon [8, 16, 17]. In this work, we assume a function \( I(x, y, z) = I_0e^{i\phi_0(x, y, z)} \) (with the aim of analyzing the wavefront propagation using equation (3) [18, 19] with \( I_0 = |I(x, y, z)| \). Recalling the series expansion \( e^z = \sum_{n=0}^{\infty} (z^n/n!) \) [1, 20, 21] and the paraxial conditions in the ITE (equation (3)) [8, 12, 13, 20], we write \( I(x, y, z) \) in terms of a first-order truncated series as follows \( I(x, y, z) \sim I_0[1 + i\phi_0(x, y, z)] \). Substituting this expression in the ITE (equation (3)), we obtain

$$-k \frac{\partial I_0}{\partial z} = -k \frac{\partial I_0}{\partial z} - ik \left[ \frac{\partial}{\partial z}(I_0 \phi_0) \right],$$

$$\nabla_z[I(\nabla_z \phi)] = \nabla_z[I_0(\nabla_z \phi_0)] + i \nabla_z[I_0 \phi_0] \nabla_z \phi_0,$$

from equations (4) and (5) we obtain a complex expression. Recalling the works by Gureyev et al [22], Nugent et al [23], Shomali et al [24] and Arriaga et al [25], we consider certain mild and smooth changes in \( I_0 \), applicable to our proposal, where we analyze spherical surfaces, with its Bi-Ronchigrams showing those changes in intensity rather than in the centroid (as we will address in the following). Under these conditions, we can assume \( \nabla_z I_0 \to 0 \), \( y \to 0 \), leading us to two expressions, considering the ITE and equations (4)–(5).

The first expression corresponds to the real part \(-k \frac{\partial I_0}{\partial z} = I_0 \nabla_z^2 \phi_0\), resembling the ITE (as a reduced version of
equation (3) in terms of the amplitude $I_0$. The second expression, the imaginary part, is our target

$$i I_0 \varphi_z \left( \nabla_T \varphi_z \right) + i I_0 \left( \nabla_T \varphi_z \right) \cdot \left( \nabla_T \varphi_z \right) = i I_0 \left[ \varphi_z \left( \nabla_T \varphi_z \right) \right] \nabla_T \varphi_z \left( \nabla_T \varphi_z \right) = -i \frac{\partial \varphi_z}{\partial z}.$$  

(6)

We refer to equation (6) as the Phase Transport equation (in the following sections we use equation (6) to propagate the wavefront). This equation is similar to the ITE (equation (3)) in terms of the propagation axis $z$ and the phase $\varphi_z$.

2.2. Zernike polynomials and the wavefront

The wavefront is a scalar surface $W$ satisfying the Eikonal equation $|\nabla T \Phi(x, y, z)|^2 = n^2(x, y, z) [1-3, 5, 26-29]$, with $\Phi(x, y, z)$ the phase for which $\phi = (2\pi/\lambda)W$ holds, with $n(x, y, z)$ the refraction index. However, the wavefront analysis is simplified by carrying it out in terms of the transversal plane $(x, y)$ and polar coordinates $(\rho, \theta)$. Thus, we can regard $W$ as a function analyzed in specific points along the propagation axis $z$, for example in the points $z_j$. Based on the theory by Teague [13], Rayces [12], and Wolf [17] we denote the wavefront in a given point $z_j$ by $W_{z_j} = W(\rho, \theta, z_j)$. Furthermore, Rayces’ equations provide an explicit relation between the Transversal Aberrations (TA) and $W_{z_j}$, which could be simplified as follows

$$\begin{aligned}
\frac{\partial W_{z_j}}{\partial x} & \sim \frac{TA_x}{r}, \\
\frac{\partial W_{z_j}}{\partial y} & \sim \frac{TA_y}{r},
\end{aligned}$$

(7)

as shown in Malacara [8] with $TA_x$ and $TA_y$ the projections of TA along the x and y axes, respectively, and $r$ the mirror Curvature Radius (CR1). It is not possible to obtain a straightforward relation between TA and $W$ from equation (7) (in the sense of considering it as a non-linear or non-proportional relation, as the expression TA in Malacara [8]), and recalling the relation between $W$ and the Eikonal equation. However, it is possible to establish a vector relation [2, 3, 5, 19, 29] as well as an integral relation by means of the calculus fundamental theorem and from the definition of function primitive (regarding vector spaces) [30-32]. These relations are given by

$$\| \nabla_T W \| = \frac{TA}{CR1}$$

(8)

$$\int_{\Omega} (\nabla_T W) \cdot dS = W = \int_{0}^{\frac{TA}{CR1}} ds,$$

(9)

where $\Omega$ is a vector set in $\mathbb{R}^2$ containing the transversal plane [21, 30]. Moreover, $\Omega$ is defined over the scalar field constituted by TA. $S \in \Omega$, and $O$ a subset of scalars in $\mathbb{R}^2$ (for all $s \in O$). Each vector of the vector set $\Omega$ can be built in $O$ as a linear combination of scalars in $O$ and elements of the base $\Omega$ [30, 32]. It must be satisfied that the envelopes (wavefronts) of the rays in $O$ should be contained in $\Omega$. In order to perform a study of the Ronchi Test [33, 34] it is a common practice to consider only one of the relations in equation (7). However, in the particular case of this work, both relations in equation (7) are required, given that we apply the BRT using the bRR (square rulings or Checker gratings [10, 11, 35]) with periods both in $x$ and $y$. Hence, equations (8) and (9) are consistent with the phenomenon under analysis. On the other hand, the points $z_j$ correspond to the points the bRR are located at, where the wavefront is observed to define a transversal plane (perpendicular to the propagation axis $z$), which is in agreement with $\nabla_T$ and the Zernike polynomials [14, 15]. Thus, it is possible to re-write $W(W_A(\rho, \theta))$ in terms of the Zernike aberrations polynomials as follows

$$W_z(\rho, \theta) = \sum_{i=0}^{n} \sum_{j=-i}^{i} a_{ij} Z_{ij}(r, \theta),$$

(10)

with $a_{ij}$ constant coefficients, and $Z_{ij}$ the Zernike polynomials [14]. Due to the similarities between the BRT and the RT [8, 33–35] applied to our experimental setup, it is plausible to re-write $Z_{ij}(\rho, \theta)$ in terms of the normalization constant $N_j'$ associated with the circular polynomials $R_j'(\rho)$ regarding the azimuthal term $j$ for $\Theta_j(\theta)$ [8, 14, 36, 37]. Hence, we re-write the Zernike polynomials as $Z_{ij}(\rho, \theta) = N_j' R_j'(\rho) \Theta_j(\theta)$, with

$$\Theta_j(\theta) = \begin{cases} \cos(j \theta) & \text{if } j \geq 0, \\
-\sin(j \theta) & \text{if } j < 0, \end{cases}$$

(11)

$$R_j'(\rho) = \sum_{h=0}^{[j/2]} \frac{(-1)^h (i-h)!}{(h!)((i/2)-h)!} \rho^{i-2h},$$

(12)
where the symbol \([\cdot]\) denotes the integer part (for example \([7.5] = 7\)) \([21]\). From the previously described prescription, we are able to obtain the equations to estimate the numerical approximation solving the \(W_z\) propagation along \(z\) from a reliable, verified efficient \(W(W_{\text{eff}})\) using the Phase Transport equation (equation \((6)\)) in terms of the wavefront. Moreover, we can experimentally validate the propagation by observing several Bi-Ronchigrams giving place to different \(W_{\text{eff}}\).

### 3. Experimental setup

For this work purposes, we tested a flawed spherical mirror (figure 1(G)). The mirror was tested applying the BRT and the RT using the same experimental setup. A first surface aluminized procedure was performed in the 108.5 mm mirror. The mirror Curvature Radius (CR1) is 937.5 mm. The mirror alignment procedure is verified using a quasi-point source constituted by a He–Ne laser with \(\lambda = 632.3\) nm of \(3mW\), along with a microscope objective \(40\times\) with a numerical aperture of 0.65 and a pinhole with 5 \(\mu m\) of diameter. The ruling with a period in \(x\) or Ronchi ruling (RR, figure 1(L)) used to apply the RT has a period of 2 lines per mm (50.8 lines per inch). The 3 bRRs (figures 1(M) and (N)) were built using aluminum deposition in glass (keeping the same periods in \(x\) and \(y\)). The first ruling (bRR1) has a period of 3 lines per millimeter (76.2 lines per inch) in a whole sample size of 9.5 \(\times\) 9.5 mm\(^2\) (0.374 \(\times\) 0.374 inch\(^2\)), the second (bRR2), 2 lines per millimeter (50.8 lines per inch) in a whole sample size of 4 \(\times\) 4 mm\(^2\) (0.158 \(\times\) 0.158 inch\(^2\)), and, the third (bRR3), 4 lines per millimeter (101.6 lines per inch) in a whole sample size of 9.8 \(\times\) 9.8 mm\(^2\) (0.386 \(\times\) 0.386 inch\(^2\)). We used as illumination source in both tests, BRT and RT, a LED light Bead (voltage range \([3.2, 3.4]\) V, 3W power and luminous flux of \([260, 280]\)LM) automatically controlled by a KEITHLEY 2231A-30-3 195W programmable source (figure 1(T), indicating also the LED source). A SONY \(\alpha\) reflex camera 3000 with a resolution of 4592 \(\times\) 3056 pixels (figure 1(J)) was used to observe the Bi-Ronchigrams in figure 1(K).

In figure 1 we show the experimental setup used to apply the BRT. The mount or steel base (D) is placed on the alignment micrometric screws A and B (the setup contains 3 screws, however only two can be observed). Hence, A, B, D, E, and F constitute the mount required for the alignment procedure of the mirror G. The LED I (placed on its own mount) and the mount H for the bRR M, are symmetrically placed at a distance of 5 mm (0.197 inches) alongside the propagation axis \(z\). The used bRR have different periods as shown in M and N. We include the RR (Ronchi ruling) L, since the RT was used to test the mirror G and validate the BRT. The RR is also placed on the mount H allowing us to observe its corresponding Ronchigram. We stress the application of the RT to validate the wavefronts obtained by applying the BRT along with the same experimental setup as well as carrying out the same aligning procedure. Following this, we obtained 10 Bi-Ronchigrams for a given bRR in a
fixed position and 10 Ronchigrams from a given RR in the same position before C. Finally, the 3D model and experimental setup are separated by a red line. Pictures of the experimental setup in the laboratory are shown on the right side of the line indicating the model described elements, highlighting the KEITHLEY LED Source and the Quasi-Point source (with a laser, used for alignment purposes only).

4. Obtaining the efficient $W$

With the aim of obtaining the wavefront by applying the BRT ($W_{BRT}$), we analyze the Bi-Ronchigram observed after placing a bRR (figure 1(K)) at the same point the RR is placed at (point located at a distance $d$ before C). The ideal Bi-Ronchigram is simulated considering a completely spherical mirror, and the results in the literature where square rulings are used (for example in Cordero et al [35, 38] and the diffraction analysis in [7, 17, 20]).

Subsequently, the centroids of the ideal Bi-Ronchigram maxima with coordinates $(\rho_j, \theta_j)$ (for each centroid $j$), are numerically identified. Each centroid could be regarded as an element of a given order $s$, similar to the diffraction orders of a bRR [8, 20] (bright zones or maxima), with order 0 in the higher intensity center, and increasing its value moving outwards, maintaining a value expression of $(2s + 1)^2$ under the order $s$ (with $s = 0$ for the first order corresponding to the Bi-Ronchigram center). Hence, the first order coordinates are $m_0 = (0, 0)$, and for the second order $(s = 1), (2s + 1)^2 = 9$ bright zones around the $(0, 0)$ maximum are found. We start the centroids identification with the counterclockwise order in polar coordinates $m_{s} = (\rho_{s}, \theta_{s})$ (with $j = 1, 2, ..., (2s + 1)^2 = 9$). This is the general rule used to determine the coordinates of the centroids $(m_{s} = (\rho_{s}, \theta_{s})$) with the maxima according to their order $(s)$ and expansion $(j = 1, 2, ..., (2s + 1)^2)$ in the ideal Bi-Ronchigram centroids. On the other hand, in order to determine the centroids of the experimental Bi-Ronchigram (with the distortions produced by the aberrations), we designed an algorithm to attenuate the Bi-Ronchigram and to highlight the centroids of the bright zones or maxima. The algorithm ensures the last order $S$ of the centroids in the Bi-Ronchigram edges are scarcely numerically distinguishable (it should be noticed that the algorithm identifies the value 1 in terms of pixels, whereas the human eye cannot resolve it).

The algorithm allows us to obtain the minimum centroids coordinates $M_{s} = (\rho_{s}, \theta_{s})$ as in the case of the ideal Bi-Ronchigram. We built the matrices $M$ and $m$ from the centroid coordinates in both Bi-Ronchigrams $M_{s}$ (experimental) and $m_{s}$ (ideal), respectively. Since determining $TA$ is our ansatz, we numerically estimate a point cloud surface [9, 39] from $ta = M - m$. In order to obtain a surface from such a point cloud, similar to a spline or interpolation [9], we apply the multi-linear least-square method [32, 39] to fit $ta$ to a polar surface. Such a fitting surface corresponds to the matrix $TA$, i.e. the Transversal Aberrations surface (TA) obtained by the BRT. We apply equation (9) to $TA$ to determine the wavefront. These are the last steps of the proposal to obtain $W$ applying the BRT observing a Bi-Ronchigram after placing a bRR in a given position. This procedure allows us to obtain a resolution of the order of $10 \times 10^{-6}$ mm in the wavefront measurement, due to the dependence of the point cloud on the number of points to determine the resolution [9, 39] (the larger the number of points the larger the resolution). On the other hand, this procedure depends also on the least-square fitting method in its multi-linear or 2D form [39]. Proceeding analogously to the RT procedure, we estimate 10 $W_{BRT}$. We subsequently consider the $W_{BRT}$ with the least-dispersion value $(0.55 \times 10^{-3}$ mm) in the comparison between the $W_{BRT}$ and $W_{RT}$ (obtained from the RT). The wavefronts should coincide. We refer to the least-dispersion wavefront as $W_{eff}$. Finally, we perform a multi-linear fit following equation (10) for $n = 60$ to $W_{BRT}$. We obtained $W_{fit-BRT}$ from the fitting, which we call the efficient wavefront $W_{eff}$.

We illustrate the previously stressed arguments in figure 2. In (a) we show the Ronchigram associated with $W_{RT}$. In (b) we show an ideal Bi-Ronchigram to illustrate the orders $s$ and their increase or decrease in intensity while moving away or toward the center, respectively, in addition to the increase in the maxima number, where the first order ($S = 0$) maximum coordinates are $m_0 = (0, 0)$ and the rest follow the pattern $m_{s} = (\rho_{s}, \theta_{s})$ (with $j = 1, 2, ..., (2s + 1)^2$). In (c) we show a zoom-in to the central region in (b), to illustrate the procedure performed to obtain the coordinates $m_{s} = (\rho_{s}, \theta_{s})$ from the centre. We also enumerate the maxima with respect to the order $S$, in this case for $s = 1$ and $j = 1$. In (d) we show the intensity changes of the experimental Bi-Ronchigram to numerically identify the centroids. In (e) we illustrate the coordinates capture in the experimental Bi-Ronchigram orders. Finally, in (f) we show the Bi-Ronchigram in (e) in translucent tone, with a given minimum intensity. We include the ideal Bi-Ronchigram maxima in red squares, and the experimental Bi-Ronchigram centroids in black circles (translucent), in order to show the difference between the centroids, with the purpose of visually understanding the numerical differences contained in the matrix $ta$.

To obtain $W_{fit-BRT}$ and $W_{fit-RT}$ we analyzed 10 wavefronts. For the rest of the wavefronts, we only analyzed a single wavefront per bRR. For example to validate the propagation we obtained a $W_{BRT}$ for each bRR in the position $z$. The choice of carrying out the fitting for $n = 60$ is related to the dependence of the RMS (error) between the interpolating polynomial (Zernike polynomials in equation (10)) and the fitted curve (wavefront
obtained by the BRT on the polynomial degree. With larger polynomials degrees leading to lower RMS values. However, high-degree polynomials require high-computing power.

5. W propagation

5.1. Experimental propagation of W
For each bRR (with \(a = 1, 2, 3\)) a Bi-Ronchigram is observed, from which we obtain \(W_{zj}\) (with \(j = 1, 2, 3, 4, 5, 6\)), considering three positions before \(C\) and three after \(C\) (see figures 1(C) and (H)). Each \(W_{zj}\) was obtained following the prescription in section 4, determining \(TA\), applying equation (9), and fitting equation (10) for \(n = 60\) to the obtained result. The latter procedure allows us to explicitly determine the fitting coefficients, which are manipulated according to equation (10) following the ANSI format and given the aberration and polynomial they are associated with. The bRRs were used as planes orthogonal to the propagation axis, where the \(R_v\) coming from the mirror are projected into. The different positions \(z_j\) are measured from the vertex along the propagation axis \(z\), following these schemes for each \(a\) (before and beyond \(C\) with \(CR1 = 937.5\) mm). For \(a = 1\) the positions are 920 mm, 920.5 mm, 969.5 mm and 1003.5 mm; for \(a = 2\) the positions are 917.5 mm, 921.5 mm, 972.0 mm and 981.5 mm; for \(a = 3\) the positions are 920 mm, 921.5 mm, 965.5 mm and 965.5 mm. The positions 916.5 mm and 958.5 are common for \(a = 1, 2, 3\). In figure 3 we show these positions along with a line joining the vertex and the point \(C\), showing the behavior of the \(R_v\) set coming from the mirror (source), similar to a cone. In (a) we show the bRR located before and beyond \(C\) as well as their different positions \(z_j\), where the Bi-Ronchigrams are observed (Figure 1(K)) with the camera (figure 1(J)). We observe the \(W_{zj}\) obtained by applying the BRT in (b), for each position of the bRR, where the wavefront propagation is also shown. A relation between the \(W_{z3}\) and \(W_{z4}\), as well as between \(W_{z2}\) and \(W_{z5}\), is observable. Such a relation is a consequence of the mirror geometry (figure 1(G)), which is spherical, and the Zernike polynomials intrinsic geometry \([8, 15, 20]\). Hence, we glimpse a relation between the wavefronts \(W_{z3}\) and \(W_{z4}\) or \(W_{z2}\) and \(W_{z5}\), as a transformation or symmetry-reflection matrix \([32]\).

5.2. Numerical propagation of W
Recalling the relation \(\phi = (2\pi/\lambda)W\), equation (6) and removing the common factor \(k^2\), we obtain the following expression

![Figure 2](image-url)
we will refer to equation (13) as the wavefront transport equation. We recall equations (11)–(12) in the analysis of equation (13) to build the constant term $A_i$ absorbing the fitting coefficients $a_{ij}$ as $A_i = a_{ij} N_j$. We consider the nabla operator $
abla_{\rho, \theta} = \frac{\partial}{\partial \rho} \hat{e}_\rho + \frac{1}{\rho} \frac{\partial}{\partial \theta} \hat{e}_\theta$ and the differential operators $\frac{\partial}{\partial \rho}$ and $\frac{\partial}{\partial \theta}$ applied to $\Theta_i$ and $R_i$
\[ \frac{\partial R_i}{\partial \rho} = \left[ \sum_{h=0}^{\infty} \left( \frac{(-1)^h}{(h)!} \left( \frac{1}{2} \right)^{i-j} \right) \right] \frac{i-2h}{\rho^{i-2h}}, \] applying $\nabla_{\rho, \theta}$ to equation (10)
\[ \nabla_{\rho, \theta} W = \sum_{i=0}^{n} \sum_{j=-i}^{i} A_i \left[ \Theta_i(\rho) \frac{\partial R_j}{\partial \rho} \hat{e}_\rho + R_i \frac{1}{\rho} \frac{\partial \Theta_i}{\partial \theta} \hat{e}_\theta \right], \]
recalling $(\nabla_{\rho, \theta} W)_2 = (\nabla_{\rho, \theta} W)_3$ and the orthogonality of $\hat{e}_\rho$ and $\hat{e}_\theta$, i.e. an orthonormal base (with $\hat{e}_\rho = (\cos(\theta), \sin(\theta))$ and $\hat{e}_\theta = (-\sin(\theta), \cos(\theta))$), we obtain
\[ (\nabla_{\rho, \theta} W)^2 = \sum_{i=0}^{n} \sum_{j=-i}^{i} A_i \left[ \left( \frac{\partial R_j}{\partial \rho} \right)^2 + \left( \frac{\partial \Theta_i}{\partial \theta} \right)^2 \right], \]
\[ (\nabla_{\rho, \theta} W) \cdot (\nabla_{\rho, \theta} W) = \sum_{i=0}^{n} \sum_{j=-i}^{i} A_i \left[ \Theta_i \frac{\partial R_j}{\partial \rho} + R_i \frac{1}{\rho} \frac{\partial \Theta_i}{\partial \theta} \right], \]
with
\[ \frac{\partial^2 \Theta_i}{\partial \theta^2} = \begin{cases} -j^2 \cos(\theta) & \text{if } ij \geq 0, \\ j^2 \sin(\theta) & \text{if } ij < 0, \end{cases} \]
\[ \frac{\partial^2 R_i}{\partial \rho^2} = \sum_{h=0}^{\infty} \left( \frac{(-1)^h}{(h)!} \left( \frac{1}{2} \right)^{i-j} \right) \frac{(i-2h)(i-2h-1)}{\rho^{i-2h-2}}. \]

Considering the dependence of the circular polynomials $R_i(\rho)$ on $\rho$ in the transversal plane along with the $R\rho$ propagation cone (figure 3), we observe $W_{z_{C_i}}$ is either moving away from $C$ and moving toward the mirror vertex (figure 3(b)) or moving away from $C$ and also from the mirror vertex, causing $\rho$ to increase or decrease, respectively. Taking this into account, we integrate equation (13) with the magnitude changes in $\rho$ along the propagation axis, starting from the vertex. Hence, we consider the mirror radius ($D/2 = 54.25$ mm) in the point $C$ and the angle $\alpha$ between the outer edge of the cone and the propagation axis $z$. This angle is common to all planes where the wavefront is observed, with the $\rho$ magnitude increasing or decreasing in terms of the distance to $C$ from the vertex. Moreover, the $\rho$ magnitude change is in terms of $\delta$ (for example, $\rho - \delta$), for each $\delta z$-step considered from the vertex $C$. Hence, the integration procedure starts from the mirror vertex (figure 1(G) and figure 4). Thus, $\delta$ can be straightforwardly determined from $\delta z$. Therefore, using trigonometric arguments, the
6.1. RT and BRT

The RT is a widely-used test, powerful, and with high accuracy [8, 33, 34]. Considering this we use the RT to validate the BRT. Moreover, both the BRT and the RT can be applied to the same experimental setup to test surfaces replacing the RR by the bRR. Hence, the RT is the most suitable technique to be compared with the BRT due to their experimental similarities, and their shared underlying theories, allowing us to use the Rayces’ equations [12] in both cases. However, in the BRT, both Rayces’ expressions are used as seen in equation (9) to obtain TA as a surface. Thus, we analyzed 10 Ronchigrams (section 4) using APEX and 10 Bi-Ronchigrams using the RR and bRR at the same distance \( d \) before C. A sample of 10 wavefronts was obtained for each test. The mean
RMS for the $W_{RT}$ (wavefronts obtained from the RT) is $2 \times 10^{-8}$ mm and for the $W_{BRT}$ (wavefronts obtained from the BRT) is $6 \times 10^{-7}$ mm. In figure 5 we show the wavefronts $W_{RT}$ and $W_{BRT}$ in (a) and (b), respectively. These wavefronts have the largest dispersion values among the tests. However, the differences between the wavefronts are very small, as can be seen in (c), with a value of $6 \times 10^{-7}$ mm. Recalling the arguments stressed in section 6.2. Experimental propagation works properly when establishing equation (9) integrating the TA numerical values obtained by point cloud [8, 39]. Such integral is defined in the scalar region $O$, in a plane transversal to the propagation axis, contained in the pink region in figure 3. The BRT has the ability to reduce the measurement times of $W$ in the laboratory since the BRT does not require the 90-degree rotation and subsequent wavefront superposition (vertical and horizontal RRs) as in the case of some RT applications, to characterize a surface of $W$ integrating both measurements $W(x, \theta)$ and $W(\rho, \theta)$. The BRT estimates TA and the wavefront as surfaces in polar coordinates from a single measurement. Thus, our BRT proposal using point cloud allows us to determine the wavefront ($W(\rho, \theta)$) variations from a single measurement.

6.2. Experimental propagation

For the wavefronts obtained at this point, we use the BRT along with the setup for which the dispersion value was minimum. Hence, for each $W_z$, we observed and analyzed a Bi-Ronchigram at the position $z_j$ (section 5 section 5.1) following the prescription in section 4 to estimate TA and applying equation (9), obtained from the Rayces’ relations [12], for each bRR$_m$ ($a = 1, 2, 3$). The mean RMS values for each BRR are $5.8 \times 10^{-7}$ mm for $a = 1, 2.03 \times 10^{-7}$ mm for $a = 2$, and $0.74 \times 10^{-7}$ mm for $a = 3$. Moreover, in section 5 section 5.1, 6.2 are mentioned for each bRR. We emphasize that the analysis was performed in the same mirror in all cases, giving as a result the same wavefront for similar positions, regardless of the used bRR. Hence, in figure 6 we show 6 different positions for 3 bRR placed before C to validate the experimental propagation. In (a) we show the wavefront obtained after observing the Bi-Ronchigram placing the bRR$_1$ (as in the case of bRR$_2$ and bRR$_3$) at $z_j = 916.5$ mm, in (b) placing the bRR$_2$ at $z_j = 917.5$ mm, in (c) placing the bRR$_3$ at $z_j = 918.5$ mm, (d) placing the bRR$_4$ (as in the case of bRR$_5$) at $z_j = 920$ mm, in (e) placing the bRR$_5$ at $z_j = 920.5$ mm, and, in (f) placing the bRR$_6$ at $z_j = 921.5$ mm. The propagation is properly observed despite the absence of the Focault phenomenon [8], assuming the bRR is placed at a point C. The phenomenon was not experimentally observed, however, we observed it numerically, as can be seen in the video WPropagated.mov. The propagation before C is observed in the first 8 s and the Focault [8] in the second 8. The video contains 420 frames in 16 s. Each frame corresponds to [Figure 5. Wavefronts $W_{RT}$ and $W_{BRT}$ in (a) and (b), respectively. Such wavefronts have the highest dispersion values of the sample, for the RT using APEX and for the BRT when comparing each $W_{RT}$ with the $W_{RT}$, In (c) we show the difference between (a) and (b). We observe the similitude of (a) with $W_{RT}$ regarding the fitting coefficients for $n = 60$ (in the supplementary material mentioned as Coeff$W_{eff}$RT.mat), and the similitude of (b) with $W_{eff}$ with their corresponding fitting coefficients (fit for $n = 60$ in Coeff$W_{eff}$mat).]
a different wavefront observed in a simulated position \(z_0\), as seen in the numerical propagation (Section 6, section 5.2).

The obtained \(W_1\) during the analysis of the observed Bi-Ronchigrams after placing a bRR before and beyond \(C\), follow a relation given in terms of the reflection matrix \(T_R\) with respect to the vertical axis in the transversal plane for each symmetric position \(z_2\) regarding \(C\) (at the same distance before and beyond \(C\)) \([1, 21, 31, 32]\). The RMS values correspond to symmetrical \(W\) before and beyond \(C\), for each bRR. We report the dispersion values between the comparison of \(W_{z2-}\) and \(W_{z2+}\), with the obtained wavefront before \(C\) applying the reflection matrix \((T_R)_{a}\) applied to \(W_{z2-}\), i.e. \(T_RW_{z2-}\). Thus, analytically speaking, it should hold \(T_RW_{z2-} = W_{z2+}\). Therefore, we compare 6 positions to apply the \(T_R\) in 3 cases for the wavefronts \(T_RW_{z2-} = W_{z2+}\). The resulting RMS in these three cases are close to \(3.2 \times 10^{-3}\) mm for \(T_RW_{z2+}\), with \(t = 21\) mm, \(t = 17\) mm, and \(t = 19\) mm. However, the dispersion values between \(T_RW_{z2-}\) (numerically obtained from the reflection matrix) and \(W_{z2+}\) (experimentally obtained) for \(t = 21\) mm is \(3.27 \times 10^{-3}\) mm, for \(t = 19\) mm is \(7.502 \times 10^{-3}\) mm, and, for \(t = 17\) mm is \(3.27 \times 10^{-3}\) mm. In figure 7 we show the examples for \(t = 21\) mm in (a) for \(W_{z2-}\), and (b) for \(T_RW_{z2-}\). Moreover, (a) is also shown in figure 6(a). Similarly, in the \(t = 17\) case in (c) for \(W_{z2-}\), and in (d) for \(T_RW_{z2+}\). We also show (c) in figure 6(e), as seen in the video WPropagated.mov for symmetric times before and after 8 s, since our numerical propagation proposal for 8 s predicts \(W\) at the point \(C\).

6.3. Numerical propagation

The solution in equation (24) takes into account the integral in equation (23). It should be noticed that the integral property regarding the existence of an intermediate point \([21, 31, 39]\) is required for obtaining the \(W\) propagation in the cone beyond \(C\). For instance, if we consider the intervals \([C - t, C + t]\), the existence of a point \(C \in [C - t, C + t]\) is clear, with \(t\) satisfying \(t + \delta > t\) and \(t > \delta\). The \(\delta\) value can be used as a term to avoid the singularity in the cone in \(C\), integrating equation (23) as close to \(C\) as the continuity holds for the terms in the symbol \(\nabla W\) (section 5.2) \([31, 39]\). However, such a discontinuity does not affect the analysis in our proposal, since, for \(\delta > 0\), \(C_{r} \rightarrow CR_{1}\), \(\rho \rightarrow i\delta\) holds, and in the case of the propagation beyond \(C\), the indices in the sum change to \(\sum_{i=0}^{N}\) with \(N\), the integer contained in \(t/\varepsilon\) and \(t\) the maximum distance the wavefront propagation can be performed (observed), beyond \(C_{r}, C + t\). We bear in mind that the observed Bi-Ronchigram should experimentally match the Ronchigram since the Foucault phenomenon \([8]\) should be observed due to diffractive criteria \([8, 17, 20, 33, 36]\). However, given the nature of our analysis, such a

\[\text{Wavefronts obtained for 6 different positions where the 3 bRR were placed: (a) the bRR}_{1}, (\text{as in the case of bRR}_{2}, \text{and bRR}_{3}) \quad \text{at} \quad z_{2} = 916.5\ \text{mm}, \quad (b) \text{the bRR}_{2}, \text{at} \quad z_{2} = 917.5\ \text{mm}, \quad (c) \text{the bRR}_{3}, \text{at} \quad z_{2} = 918.5\ \text{mm}, \quad (d) \text{the bRR}_{4}, \text{and bRR}_{5}, \text{at} \quad z_{2} = 920\ \text{mm}, \quad (e) \text{the bRR}_{6}, \text{at} \quad z_{2} = 920.5\ \text{mm}, \quad \text{and (f) placing the bRR}_{7}, \text{at} \quad z_{2} = 921.5\ \text{mm}. \quad \text{This is illustrated in the video WPropagated.mov in the first 8 s, corresponding to W before C.}\]
phenomenon is only observable in the video WPropagated.mov at \( t = 8 \) s, when considering the numerical contribution of \( \Theta_i(f) \) and the corresponding differential operators in the frame corresponding to the referred time. Hence, we should consider that equation (24) obtains the wavefronts estimating a propagation along \( z \) rather than estimating the Bi-Ronchigrams. On the other hand, our proposal numerically removes the discontinuity in the cone, allowing us to perform the integration procedure (equation (23)). If \( \hat{\text{d}}z \rightarrow \text{CR1} \) then \( (\rho - i\delta) \rightarrow 0 \). The latter predicts an exit pupil with a diffractive pattern remarkably close to the center (see video WPropagated.mov at time 8 s). At the beginning of the propagation \( (z=0, \text{simulating} W \text{in the mirror vertex, with} i=0 \text{and} \rho = D/2) \), we should observe a bright exit pupil. We do not show the latter behavior due to its similitude with the previous video. Hence, for individual cases, the \( W_0 \) before \( C \) for a propagation proposal have a mean dispersion of \( 3.01 \times 10^{-4} \) mm for the bRR1, \( 3.75 \times 10^{-4} \) mm for bRR2, and \( 9.25 \times 10^{-4} \) mm for bRR3, keeping an RMS of the order of \( 1 \times 10^{-8} \) mm. On the other hand, beyond \( C \), in general, the dispersion values are close to \( 1 \times 10^{-7} \) mm, being larger but acceptable given their RMS values close to \( 7.5 \times 10^{-6} \) mm. This leads to less precision, giving place to a prediction without prior knowledge of the efficient wavefront beyond \( C \), as can be seen in the video WPropagated.mov. The video validates the efficiency of our proposal, being the main result of the wavefront propagation from a given \( W_{\text{eff}} \), implying the previously stressed arguments. The experimental wavefronts shown in figures 6 and 7 are indeed predicted and obtained. We draw special attention to the video duration and wavefronts, which take into account the results shown in previous sections, for a step in the integral in equation (24) to estimate \( W_{\text{propagated}} \delta z = 0.1 \) mm. Hence, 10 000 frames would be required before and after \( C \) (to validate the reflection), increasing the time, resolution, and computing time. We focused on validating the experimental observations. Therefore, the video (WPropagated.mov) starts with the wavefront \( (W_{\delta z-(t=0)} \) shown in figures 6(a) and 7(a), and ends with the wavefront \( W_{\delta z+(t=21)} \), constituting a video of 420 frames and a duration of 16 s. The first 209 frames (first 8 s) correspond to the wavefronts \( W_{\delta z-(t)} \). The frame at \( t = 8 \) s corresponds to the wavefront obtained by placing a bRR at the point \( C \), giving place to the simulated Focault phenomenon [8]. The following 8 s, the frames correspond to the simulated wavefronts \( W_{\delta z+(t)} \), symmetric with respect to the \( W_{\delta z-(t)} \), with equation (24) validating the symmetry matrix \( T_0 \) for each \( t \).

Finally, in figure 8 we show experimental Bi-Ronchigrams for given bRR \( (a = 1, 2, 3) \) (obtaining the previous \( W \) using the BRT). Hence, we observe the Bi-Ronchigram placing the bRR1 in (a) at \( z_i = 920.5 \) mm and in (b) at \( z_i = 958.5 \) mm. For the bRR2 at \( z_i = 918.5 \) mm we observe the Bi-Ronchigram in (c) and at \( z_i = 958.5 \) mm we observe the Bi-Ronchigram in (d). For the bRR3 at \( z_i = 920 \) mm and at \( z_i = 968.5 \) mm we observe the Bi-Ronchigrams in (e) and (f), respectively.
7. Conclusions

The main objective of this work is given by the wavefront propagation using the Phase Transport equation, giving place to equation (13) from the wavefront $W_{\text{eq}}$ obtained by the BRT using point cloud (section 4). Such an objective was successfully fulfilled, as seen in the evidence shown in the video WPropagated.mov, allowing us to propagate the wavefront beyond CR1. The only shortcoming in this procedure is given by the mandatory necessity of prior knowledge of at least one wavefront in the given position ($W_0$). The obtained propagation from equation (24) requires large numerical processing, which in our case was properly carried out since the function is Riemann-integrable and the singularity was removed when reducing the radius ($\rho$) as long as the observation plane moves away from the surface and toward C. On the other hand, the differential form (up to second order) of the circular Zernike polynomials was properly obtained when considering the terms $N_i^r, R_i^z$, and $\Theta_j$, allowing us to obtain the numerical data in equation (13) in polar coordinates. On the other hand, the $W$ reflection with respect to C successfully validates the reflection matrix as mentioned in section 6 section 6.3. Moreover, the experimental propagation in section 6.3 validates the results of the reflection matrix (figure 7) and the numerical propagation. This is possible by obtaining a BRT using an efficient point cloud (as verified by its RMS and dispersion values resulting from the comparison with the RT).

We also applied the BRT using point cloud to test spherical mirrors, and to determine its wavefront from the Bi-Ronchigram observed after placing a bRR at a given distance from C. Subsequently, for several bRR placed at different positions along the propagation axis $z$, we analyzed the wavefront propagation along $z$. The obtained results suggest the existence of a symmetry matrix relating the wavefronts before and beyond C. Such results allowed us to build an equation to numerically estimate the wavefront propagation from a given $W_{\text{eq}}$. From the comparison between the BRT and RT, we found a mean and minimum dispersion of $1 \times 10^{-3}$ mm and $8 \times 10^{-5}$ mm, respectively, considering the RMS of the RT from the APEX software and the corresponding to the BRT of our proposal. At a first glance, the BRT is an excellent proposal. It should be noticed that the proposal for determining TA is numerically efficient and easy to implement since it is based on the maxima estimation of the ideal and experimental Bi-Ronchigram images. Such values are subtracted to obtain a point cloud constituted by fitting points corresponding to the TA in polar coordinates.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Disclosures

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