On Upper Bounds on the Church-Rosser Theorem

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The Church-Rosser theorem in the type-free \(\lambda\)-calculus is well investigated both for \(\beta\)-equality and \(\beta\)-reduction. We provide a new proof of the theorem for \(\beta\)-equality with no use of parallel reductions, but simply with Takahashi’s translation (Gross-Knuth strategy). Based on this, upper bounds for reduction sequences on the theorem are obtained as the fourth level of the Grzegorczyk hierarchy.

1 Introduction

1.1 Background

The Church-Rosser theorem [3] is one of the most fundamental properties on rewriting systems, which guarantees uniqueness of computation and consistency of a formal system. For instance, for proof trees and formulae of logic the unique normal forms of the corresponding terms and types in a Pure Type System (PTS) can be chosen as their denotations [21] via the Curry-Howard isomorphism.

The Church-Rosser theorem for \(\beta\)-reduction states that if \(M \rightarrow N_1\) and \(M \rightarrow N_2\) then we have \(N_1 \rightarrow P\) and \(N_2 \rightarrow P\) for some \(P\). Here, we write \(\rightarrow\) for the reflexive and transitive closure of one-step reduction \(\Rightarrow\). Two proof techniques of the theorem are well known; tracing the residuals of redexes along a sequence of reductions [3, 1, 8], and working with parallel reduction [4, 1, 8, 19] known as the method of Tait and Martin-Löf. Moreover, a simpler proof of the theorem is established only with Takahashi’s translation [19] (the Gross-Knuth reduction strategy [1]), but with no use of parallel reduction [12, 5].

On the other hand, the Church-Rosser theorem for \(\beta\)-equality states that if \(M =_\beta N\) then there exists \(P\) such that \(M \rightarrow P\) and \(N \rightarrow P\). Here, we write \(M =_\beta N\) iff \(M\) is obtained from \(N\) by a finite series of reductions (\(\Rightarrow\)) and reversed reductions (\(\Leftarrow\)). As the Church-Rosser theorem for \(\beta\)-reduction has been well studied, to the best of our knowledge the Church-Rosser theorem for \(\beta\)-equality is always secondary proved as a corollary from the theorem for \(\beta\)-reduction [3, 4, 1, 8].

One of our motivations is to analyze quantitative properties in general of reduction systems. For instance, measures for developments are investigated by Hindley [7] and de Vrijer [13]. Statman [16] proved that deciding the \(\beta\eta\)-equality of typable \(\lambda\)-terms is not elementary recursive. Schwichtenberg [14] analysed the complexity of normalization in the simply typed lambda-calculus, and showed that the number of reduction steps necessary to reach the normal form is bounded by a function at the fourth level of the Grzegorczyk hierarchy \(\varepsilon^4\) [6], i.e., a non-elementary recursive function. Later Beckmann [2] determined the exact bounds for the reduction length of a term in the simply typed \(\lambda\)-calculus. Xi [22] showed bounds for the number of reduction steps on the standardization theorem, and its application to normalization. In addition, Ketema and Simonsen [9] extensively studied valley sizes of confluence and the Church-Rosser property in term rewriting and \(\lambda\)-calculus as a function of given term sizes and reduction lengths. However, there are no known bounds for the Church-Rosser theorem for \(\beta\)-equality.

In this study, we are also interested in quantitative analysis of the witness of the Church-Rosser theorem: how to find common contractums with the least size and with the least number of reduction
steps. For the theorem for $\beta$-equality ($M =_\beta N$ implies $M \rightarrow^{l_3} P$ and $N \rightarrow^{l_4} P$ for some $P$), we study functions that set bounds on the least size of a common contractum $P$, and the least number of reduction steps $l_3$ and $l_4$ required to arrive at a common contractum, involving the term sizes of $M$ and $N$, and the length of $=_\beta$. For the theorem for $\beta$-reduction ($M \rightarrow^{l_1} N_1$ and $M \rightarrow^{l_2} N_2$ implies $N_1 \rightarrow^{l_3} P$ and $N_2 \rightarrow^{l_4} P$ for some $P$), we study functions that set bounds on the least size of a common contractum $P$, and the least number of reduction steps $l_3$ and $l_4$ required to arrive at a common contractum, involving the term size of $M$ and the lengths of $l_1$ and $l_2$.

1.2 New results of this paper

In this paper, first we investigate directly the Church-Rosser theorem for $\beta$-equality constructively from the viewpoint of Takahashi translation [19]. Although the two statements are equivalent to each other, the theorem for $\beta$-reduction is a special case of that for $\beta$-equality. Our investigation shows that a common contractum of $M$ and $N$ such that $M =_\beta N$ is determined by (i) $M$ and the number of occurrences of reduction ($\rightarrow$) appeared in $=_\beta$, and also by (ii) $N$ and that of reversed reduction ($\leftarrow$). The main lemma plays a key role and reveals a new invariant involved in the equality $=_{\beta}$, independently of an exponential combination of reduction and reversed reduction. Next, in terms of iteration of translations, this characterization of the Church-Rosser theorem makes it possible to analyse how large common contractums are and how many reduction-steps are required to obtain them. From this, we obtain an upper bound function for the theorem in the fourth level of the Grzegorczyk hierarchy. In addition, the theorem for $\beta$-reduction is handled as a special case of the theorem for $\beta$-equality, where the key notion is contracting new redexes under development.

1.3 Outline of paper

This paper is organized as follows. Section 1 is devoted to background, related work, and new results of this paper. Section 2 gives preliminaries including basic definitions and notions. Following the main lemma, Section 3 provides a new proof of the Church-Rosser theorem for $\beta$-equality. Based on this, reduction length and term size for the theorem are analyzed in Section 4, and then we compare with related results. Section 5 concludes with remarks, related work, and further work.

2 Preliminaries

The set of $\lambda$-terms denoted by $\Lambda$ is defined with a countable set of variables as follows.

**Definition 1 ($\lambda$-terms)**

$$M, N, P, Q \in \Lambda ::= x \mid (\lambda x. M) \mid (MN)$$

We write $M \equiv N$ for the syntactical identity under renaming of bound variables. We suppose that every bound variable is distinct from free variables. The set of free variables in $M$ is denoted by $\text{FV}(M)$.

If $M$ is a subterm of $N$ then we write $M \sqsubseteq N$ for this. In particular, we write $M \sqsubseteq N$ if $M$ is a proper subterm of $N$. If $P \sqsubseteq M$ and $Q \sqsubseteq M$, and moreover there exist no terms $N$ such that $N \sqsubseteq P$ and $N \sqsubseteq Q$, then we write $P \parallel Q$ for this, i.e., $P$ and $Q$ have non-overlapping parts of $M$.

**Definition 2 ($\beta$-reduction)** One step $\beta$-reduction $\rightarrow$ is defined as follows, where $M[x := N]$ denotes a result of substituting $N$ for every free occurrence of $x$ in $M$.

1. $(\lambda x. M)N \rightarrow M[x := N]$
2. If \( M \rightarrow N \) then \( PM \rightarrow PN \), \( MP \rightarrow MP \), and \( \lambda x.M \rightarrow \lambda x.N \).

A term of the form of \( (\lambda x.P)Q \subseteq M \) is called a redex of \( M \). A redex is denoted by \( R \) or \( S \), and we write \( R : M \rightarrow N \) if \( N \) is obtained from \( M \) by contracting the redex \( R \subseteq M \). We write \( \rightarrow \) for the reflexive and transitive closure of \( \rightarrow \). If \( R_1 : M_0 \rightarrow M_1, \ldots, R_n : M_{n-1} \rightarrow M_n \) \((n \geq 0)\), then for this we write \( R_0 \ldots R_n : M_0 \rightarrow^n M_n \), and the reduction sequence is denoted by the list \([M_0, M_1, \ldots, M_n]\). For operating on a list, we suppose fundamental list functions, \( \text{append} \), \( \text{reverse} \), and \( \text{tail} (\text{cdr}) \).

**Definition 3 (\( \beta \)-equality)** A term \( M \) is \( \beta \)-equal to \( N \) with reduction sequence \( ls \), denoted by \( M =_\beta N \) with \( ls \) is defined as follows:

1. If \( M \rightarrow N \) with reduction sequence \( ls \), then \( M =_\beta N \) with \( ls \).
2. If \( M =_\beta P \) with \( ls_1 \) and \( P =_\beta N \) with \( ls_2 \), then \( M =_\beta N \) with \( \text{append}(ls_1, \text{tail}(ls_2)) \).

Note that \( M =_\beta N \) with reduction sequence \( ls \) iff there exist terms \( M_0, \ldots, M_n(n \geq 0) \) in this order such that \( ls = [M_0, \ldots, M_n] \), \( M_0 \equiv M, M_n \equiv N \), and either \( M_i \rightarrow M_{i+1} \) or \( M_{i+1} \rightarrow M_i \) for each \( 0 \leq i \leq n - 1 \). In this case, we say that the length of \( =_\beta \) is \( n \), denoted by \( =_\beta^n \). The arrow in \( M_i \rightarrow M_{i+1} \) is called a right arrow, and the arrow in \( M_{i+1} \rightarrow M_i \) is called a left arrow, denoted also by \( M_i \leftarrow M_{i+1} \).

**Definition 4 (Term size)** Define a function \( \| \| : \Lambda \rightarrow \mathbb{N} \) as follows.

1. \( \| x \| = 1 \)
2. \( \| \lambda x.M \| = 1 + |M| \)
3. \( \| MN \| = 1 + |M| + |N| \)

**Definition 5 (Takahashi’s \( * \) and iteration)** The notion of Takahashi translation \( M^* \)[19], that is, the Gross-Knuth reduction strategy [11] is defined as follows.

1. \( x^* = x \)
2. \( ((\lambda x.M)N)^* = M^*[x := N^*] \)
3. \( (MN)^* = M^*N^* \)
4. \( (\lambda x.M)^* = \lambda x.M^* \)

The 3rd case above is available provided that \( M \) is not in the form of a \( \lambda \)-abstraction. We write an iteration of the translation [20] as follows.

1. \( M^{0*} = M \)
2. \( M^{n*} = (M^{(n-1)*})^* \)

We write \( \sharp(x \in M) \) for the number of free occurrences of the variable \( x \) in \( M \).

**Lemma 1** \( \| M[x := N] \| = |M| + \sharp(x \in M) \times (|N| - 1) \).

**Proof.** By straightforward induction on \( M \). \( \square \)

**Definition 6 (Redex \( (M) \))** The set of all redex occurrences in a term \( M \) is denoted by \( \text{Redex}(M) \). The cardinality of the set \( \text{Redex}(M) \) is denoted by \( \sharp \text{Redex}(M) \).

**Lemma 2 (\( \sharp \text{Redex}(M) \))** We have \( \sharp \text{Redex}(M) \leq \frac{1}{2}|M| - 1 \) for \( |M| \geq 4 \).

**Proof.** Note that \( \sharp \text{Redex}(M) = 0 \) for \( |M| < 4 \). By straightforward induction on \( M \) for \( |M| \geq 4 \). \( \square \)
Lemma 3 (Substitution) If \( M_1 \rightarrow^{l_1} N_1 \) and \( M_2 \rightarrow^{l_2} N_2 \), then \( M_1[x := M_2] \rightarrow^{l} N_1[x := N_2] \) where \( l = l_1 + \#(x \in M_1) \times l_2 \).

Proof. By induction on the derivation of \( M_1 \rightarrow^{l_1} N_1 \). The case of \( l_1 = 0 \) requires induction on \( M_1 \equiv N_1 \).

We also need induction on the derivation of \( M_1 \rightarrow N_1 \), and we show here one of the interesting cases.

1. Case of \((\lambda y. M) N \rightarrow^{l} M[y := N]\):

\[
(\lambda y. M[x := M_2])(N[x := M_2]) \rightarrow^{m_1} (\lambda y. M[x := N_2])(N[x := M_2]) \text{ by IH1}
\]

\[
\rightarrow^{m_2} (\lambda y. M[x := N_2])(N[x := N_2]) \text{ by IH2}
\]

\[
\rightarrow^{1} (M[x := N_2])[y := (N[x := N_2])]
\]

Here, IH1 is \( \lambda y. M[x := M_2] \rightarrow^{m_1} \lambda y. M[x := N_2] \) with \( m_1 = \#(x \in M) \times l_2 \). IH2 is \( N[x := M_2] \rightarrow^{m_2} N[x := N_2] \) with \( m_2 = \#(x \in N) \times l_2 \). Therefore,

\[
l = m_1 + m_2 + 1
\]

\[
= 1 + \#(x \in M) \times l_2 + \#(x \in N) \times l_2
\]

\[
= 1 + \#(x \in ((\lambda y. M) N)) \times l_2. \quad \square
\]

Proposition 1 (Term size after \( n \)-step reduction) If \( M \rightarrow^{n} N \) \((n \geq 1)\) then

\[
|N| < 8 \left( \frac{|M|}{8} \right)^{2^n}.
\]

Proof. By induction on \( n \).

1. Case of \( n = 1 \), where \( M \rightarrow M_1 \):

The following inequality can be proved by induction on the derivation of \( M \rightarrow M_1 \):

\[
|M_1| \leq \frac{|M|^2}{2^3} - 1
\]

2. Case of \( n = k + 1 \), where \( M \rightarrow M_1 \rightarrow^{k} M_{k+1} \):

\[
|M_{k+1}| < 8 \left( \frac{|M_1|}{8} \right)^{2^k} \quad \text{from the induction hypothesis}
\]

\[
< 8 \left( \frac{|M_1|}{8} \right)^{2^k} \quad \text{from} \ |M_1| < \frac{1}{8} |M|^2
\]

\[
= 8 \left( \frac{|M_1|}{8} \right)^{2^{(k+1)}} \quad \square
\]

Lemma 4 (Size of \( M^* \)) We have \( |M^*| \leq 2^{|M|-1} \).

Proof. By straightforward induction on \( M \).

Definition 7 (Residuals \([3,8]\)) Let \( \mathcal{R} \subseteq \text{Redex}(M) \). Let \( R \in \mathcal{R} \), and \( R : M \rightarrow N \). Then the set of residuals of \( \mathcal{R} \) in \( N \) with respect to \( R \), denoted by \( \text{Res}(\mathcal{R}/R : M \rightarrow N) \) is defined by the smallest set satisfying the following conditions:
1. Case of \( S \in \mathcal{R} \) and \( S \parallel R \):
   Then we have \( S \in \text{Res}(\mathcal{R} / R : M \rightarrow N) \).

2. Case of \( S \in \mathcal{R} \) and \( S \equiv R \):
   Then we have \( S \notin \text{Res}(\mathcal{R} / R : M \rightarrow N) \).

3. Case of \( S \in \mathcal{R} \) and \( S \equiv (\lambda x.M_1)N_1 \) and \( R \sqsubset M_1 \) for some \( M_1, N_1 \sqsubset M \):
   Then we have \( S' \in \text{Res}(\mathcal{R} / R : M \rightarrow N) \) such that \( R : S \rightarrow S' \) for \( S' \sqsubset N \).

4. Case of \( S \in \mathcal{R} \) and \( S \equiv (\lambda x.M_1)N_1 \) and \( R \sqsubset N_1 \) for some \( M_1, N_1 \sqsubset M \):
   Then we have \( S' \in \text{Res}(\mathcal{R} / R : M \rightarrow N) \) such that \( R : S \rightarrow S' \) for \( S' \sqsubset N \).

5. Case of \( S \in \mathcal{R} \) and \( R \equiv (\lambda x.M_1)N_1 \) and \( S \sqsubset M_1 \) for some \( M_1, N_1 \sqsubset M \):
   Then we have \( S[x := N_1] \in \text{Res}(\mathcal{R} / R : M \rightarrow N) \) such that \( S[x := N_1] \sqsubset M_1[x := N_1] \) where \( R : (\lambda x.M_1)N_1 \rightarrow M_1[x := N_1] \).

6. Case of \( S \in \mathcal{R} \) and \( R \equiv (\lambda x.M_1)N_1 \) and \( S \sqsubset N_1 \) for some \( M_1, N_1 \sqsubset M \):
   Then we have \( S \in \text{Res}(\mathcal{R} / R : M \rightarrow N) \) for every occurrence \( S \) such that \( S \sqsubset M_1[x := N_1] \) where \( R : (\lambda x.M_1)N_1 \rightarrow M_1[x := N_1] \).

Definition 8 (Complete development [11]) Let \( \mathcal{R} \subseteq \text{Redex}(M) \). A reduction path \( R_0R_1 \ldots : M \equiv M_0 \rightarrow M_1 \rightarrow \cdots \) is a development of \( (M, \mathcal{R}) \) if and only if each redex \( R_i \sqsubset M_i \) is in the set \( \mathcal{R}_i \) (i.e., \( R \) properly contains no other \( S \in \mathcal{R} \)).

Let \( \mathcal{R} = \{R_0, \ldots, R_{n-1}\} \). Let \( \mathcal{R}_0 = \mathcal{R} \) and \( \mathcal{R}_i = \text{Res}(\mathcal{R}_{i-1} / R_i : M_{i-1} \rightarrow M_i) \). If \( \mathcal{R}_k = \emptyset \) for some \( k \), then the development is called complete.

Definition 9 (Minimal complete development [8]) Let \( \mathcal{R} \subseteq \text{Redex}(M) \). A redex occurrence \( R \in \mathcal{R} \) is called minimal if there is no \( S \in \mathcal{R} \) such that \( S \sqsupset R \) (i.e., \( R \) properly contains no other \( S \in \mathcal{R} \)).

Let \( \mathcal{R} = \{R_0, \ldots, R_{n-1}\} \). Let \( \mathcal{R}_0 = \mathcal{R} \) and \( \mathcal{R}_i = \text{Res}(\mathcal{R}_{i-1} / R_i : M_{i-1} \rightarrow M_i) \). A reduction path \( M \Rightarrow^0 N \) is a minimal complete development of \( \mathcal{R} \) if and only if we contract any minimal \( R_i \in \mathcal{R}_i \) at each reduction step. This development is also called an inside-out development that yields shortest complete developments [10][13].

We write \( M \Rightarrow N \) if \( N \) is obtained from \( M \) by a minimal complete development of a subset \( \{R_1, \ldots, R_n\} \) of \( \text{Redex}(M) \). In this case, we write \( R_1 \ldots R_n : M \Rightarrow^0 N \).

Note that we can repeat this development at most \( n \)-times with respect to \( \mathcal{R} = \{R_0, \ldots, R_{n-1}\} \) until no residuals of \( \mathcal{R} \) are left, since we never have the fifth or sixth case in Definition 7 and then we have \( R \notin \text{Res}(\mathcal{R} / R) \).

Definition 10 (Reduction of new redexes) Let \( R : M \Rightarrow N \). If there exists a redex occurrence \( S \in \text{Redex}(N) \) but \( S \notin \text{Res}(\text{Redex}(M) / R : M \rightarrow N) \), then we say that the reduction \( R : M \rightarrow N \) creates a new redex \( S \sqsubset N \), and \( N \) contains a created redex after contracting \( R \).

Let \( \sigma \) be a reduction path \( R_0R_1 \ldots : M \equiv M_0 \rightarrow M_1 \rightarrow \cdots \). We define the set of new redex occurrences denoted by \( \text{NewRed}(M_{i+1}) \) (i.e., \( i \geq 0 \)) as follows:

\[
\text{NewRed}(M_{i+1}) = \{ R \in \text{Redex}(M_{i+1}) \mid R \notin \text{Res}(\text{Redex}(M_i) / R_i) \}.
\]

A redex occurrence \( R_j \sqsubset M_j \) (i.e., \( 1 \leq j \)) in \( \sigma \) is called new if \( R_j \in \text{NewRed}(M_i) \) for some \( i \leq j \). The reduction path \( \sigma \) contains \( k \) reductions of new redexes if \( \sigma \) contracts \( k \) of the new redexes.
3 New proof of the Church-Rosser theorem for $\beta$-equality

Proposition 2 (Complete development) We have $M \rightarrow^l M^*$ where $l \leq \frac{1}{2}|M| - 1$ for $|M| \geq 4$.

Proof. By induction on the structure of $M$. Otherwise by the minimal complete development \([8]\) with respect to Redex$(M)$, where $l \leq 2\text{Redex}(M) \leq \frac{1}{2}|M| - 1$ for $|M| \leq 4$ by Lemma\([2]\) \qed

Definition 11 (Iteration of exponentials $2_n^m$, $F(m,n)$) Let $m$ and $n$ be natural numbers.

1. (1) $2_0^m = m$; (2) $2_n^{m+1} = 2_n^m$.
2. (1) $F(m,0) = m$; (2) $F(m,n+1) = 2^{F(m,n)-1}$. 

Proposition 3 (Length to $M^*$) If $M \rightarrow M^* \rightarrow \cdots \rightarrow M^{n*}$, then the reduction length $l$ with $M \rightarrow^l M^{n*}$ is bounded by $\text{Len}(|M|, n)$, such that 

$$\text{Len}(|M|, n) = \begin{cases} 0, & \text{for } n = 0 \\ \frac{1}{2} \sum_{k=0}^{n-1} F(|M|, k) - n, & \text{for } n \geq 1 \end{cases}$$

and then we have $\text{Len}(|M|, n) < 2_2^n$ for $n \geq 1$.

Proof. From Lemma\([4]\) we have $|M^*| \leq 2^n - 1$, and hence $|M^{k*}| \leq F(|M|, k) < 2_2^n$ for $k \geq 1$. Let $M \rightarrow^{l_1} M^* \rightarrow^{l_2} \cdots \rightarrow^{l_n} M^{n*}$. Then from Proposition\([2]\) each $l_k$ is bounded by $F(|M|, k-1)$:

$$l_k \leq \frac{1}{2}|M^{k-1*}| - 1 \leq \frac{1}{2} F(|M|, k-1) - 1$$

Therefore, $l$ is bounded by $\text{Len}(|M|, n)$ that is smaller than $2_2^n$ for $n \geq 1$.

$$l \leq \sum_{k=1}^{n} l_k \leq \frac{1}{2} \sum_{k=0}^{n-1} F(|M|, k) - n = \text{Len}(|M|, n) < \frac{1}{2} \sum_{k=0}^{n-1} 2_2^n - n < 2_2^n - n \quad \square$$

Lemma 5 ((Weak) Cofinal property) If $M \rightarrow N$ then $N \rightarrow^l M^*$ where $l \leq \frac{1}{2}|N| - 1$ for $|N| \geq 4$.

Proof. By induction on the derivation of $M \rightarrow N$. \qed

Lemma 6 $M^*[x := N^*] \rightarrow^l (M[x := N])^*$ with $l \leq |M^*| - 1$.

Proof. By induction on the structure of $M$. We show one case $M$ of $M_1M_2$.

1. Case $M_1 \equiv \lambda y.M_3$ for some $M_3$:

   $$(\lambda y.M_3)M_2)^*[x := N^*] = M_3[x := N^*][y := (M_2[x := N])^*]$$

   $\rightarrow^{m_1} M_3[x := N^*][y := (M_2[x := N])^*]$ by IH1

   $\rightarrow^{m_2} (M_3[x := N])^*[y := (M_2[x := N])^*]$ by IH2

Here, IH1 is $M_3^*[x := N^*] \rightarrow^{m_1} (M_2[x := N])^*$ with $n_1 \leq |M_2^*| - 1$, and then we have $m_1 = \#(y \in (M_3^*[x := N^*])) \times n_1$ from Lemma\([3]\).

IH2 is $M_3^*[x := N^*] \rightarrow^{m_2} (M_3[x := N])^*$ with $m_2 \leq |M_3^*| - 1$. Hence, 

$$l = m_1 + m_2$$

$$\leq \#(y \in (M_3^*[x := N^*])) \times (|M_2^*| - 1 + |M_3^*| - 1)$$

$$= \#(y \in M_3^*) \times (|M_2^*| - 1 + |M_3^*| - 1) \quad \text{since } y \not\in \text{FV}(N^*)$$

$$= |M_3^*[y := M_3^*]| - 1.$$
2. Case $M_1 \not\equiv \lambda y.M_3$:
   (a) Case $(M_1[x := N]) \equiv (\lambda z.P)$ for some $P$:

   
   $$(M_1'[x := N])(M_2'[x := N]) \rightarrow^m (M_1[x := N])^*(M_2[x := N])^* \text{ by IH}$$

   $$= (\lambda z.P^*)(M_2[x := N])^*$$

   $$\rightarrow^1 P^*[z := (M_2[x := N])^*]$$

   $$= ((M_1M_2)[x := N])^*$$

   Now, IH are $M_1'[x := N] \rightarrow^{n_1} (M_1[x := N])^*$ with $n_1 \leq |M_1^*| - 1$, and $M_2'[x := N] \rightarrow^{n_2} (M_2[x := N])^*$ with $n_2 \leq |M_2^*| - 1$. Hence,

   $$l = m + 1$$

   $$\leq |M_1^*| - 1 + |M_2^*| - 1 + 1$$

   $$< |M_1^*M_2^*| - 1.$$

   (b) Case $(M_1[x := N]) \not\equiv (\lambda z.P)$:

   This case is handled similarly to the above case, and then

   $$l \leq m$$

   $$= |M_1^*| - 1 + |M_2^*| - 1$$

   $$< |M_1^*M_2^*| - 1. \quad \Box$$

Proposition 4 (Monotonicity) If $M \rightarrow N$ then $M^* \rightarrow^l N^*$ with $l \leq |M^*| - 1$.

Proof. By induction on the derivation of $M \rightarrow N$. We show some of the interesting cases.

1. Case of $(\lambda x.M)N \rightarrow M[x := N]$:

   $$(\lambda x.M)N)^* = M^*[x := N^*]$$

   $$\rightarrow^m (M[x := N])^*$$

   From Lemma 6, we have $m \leq |M^*[x := N^*]| - 1 = |((\lambda x.M)N)^*| - 1$.

2. Case of $PM \rightarrow PN$ from $M \rightarrow N$:

   (a) Case of $P \equiv \lambda x.P_1$ for some $P_1$:

   $$(\lambda x.P_1)M)^* = P_1'[x := M^*]$$

   $$\rightarrow^m P_1'[x := N^*] \text{ by IH}$$

   $$= ((\lambda x.P_1)N)^*$$

   Here, IH is $M^* \rightarrow^n N^*$ with $n \leq |M^*| - 1$, and $m = \#(x \in P_1^*) \times n$ from Lemma 5. Hence,

   $$l = m$$

   $$\leq \#(x \in P_1^*) \times (|M^*| - 1)$$

   $$\leq |P_1^*| + \#(x \in P_1^*) \times (|M^*| - 1) - 1$$

   $$= |P_1'[x := M^*]| - 1.$$
Lemma 7 (Main lemma) Let $M \equiv^k_\beta N$ with length $k = l + r$, where $r$ is the number of occurrences of right arrow $\rightarrow$ in $=^k_\beta$, and $l$ is that of left arrow $\leftarrow$ in $=^k_\beta$. Then we have both $M^{\ast r} \leftarrow N$ and $M \rightarrow N^{\ast s}$.

Proof. By induction on the length of $\equiv^k_\beta$.

(1) Case of $k = 1$ is handled by Lemma 5.

(2-1) Case of $(k + 1)$, where $M \equiv^k_\beta M_k \rightarrow M_{k+1}$:

From the induction hypothesis, we have $M_k \rightarrow M^{\ast r}$ and $M \rightarrow M_k^{\ast r}$ where $l + r = k$.

From $M_k \rightarrow M_{k+1}$, Lemma 5 gives $M_{k+1} \rightarrow M_k^r$, and then $M_k^r \rightarrow M^{(r+1)\ast}$ from the induction hypothesis $M_k \rightarrow M^{\ast r}$ and Proposition 4. Hence, we have $M_{k+1} \rightarrow M^{(r+1)\ast}$. On the other hand, we have $M_k^{\ast r} \rightarrow M_{k+1}^{\ast r}$ from $M_k \rightarrow M_{k+1}$ and the repeated application of Proposition 4. Then the induction hypothesis $M \rightarrow M_{k+1}^{\ast r}$ derives $M \rightarrow M_{k+1}^{\ast r}$, where $l + (r + 1) = k + 1$.

(2-2) Case of $(k + 1)$, where $M \equiv^k_\beta M_k \leftarrow M_{k+1}$:

From the induction hypothesis, we have $M_k \rightarrow M^{\ast r}$ and $M \rightarrow M_k^{\ast r}$ where $l + r = k$, and hence $M_{k+1} \rightarrow M^{\ast r}$. From $M_{k+1} \rightarrow M_k$ and Lemma 5, we have $M_k \rightarrow M_{k+1}^{\ast r}$, and then $M_k^{\ast r} \rightarrow M_{k+1}^{(l+1)\ast}$.

Hence, $M \rightarrow M_{k+1}^{(l+1)\ast}$ from the induction hypothesis $M \rightarrow M_k^{\ast r}$, where $(l + 1) + r = k + 1$.

Given $M_0 =^k_\beta M_k$ with reduction sequence $[M_0, \ldots, M_k]$, then for natural numbers $i$ and $j$ with $0 \leq i \leq j \leq k$, we write $\equiv^i_\beta [i, j]$ for the number of occurrences of right arrow $\rightarrow$ which appears in $M_i =^j_\beta M_j$, and $\equiv^j_\beta [i, j]$ for that of left arrow $\leftarrow$ in $M_i =^j_\beta M_j$. In particular, we have $\equiv^i_\beta [0, k] + \equiv^j_\beta [0, k] = k$.

Corollary 1 (Main lemma refined) Let $M_0 =^k_\beta M_k$ with reduction sequence $[M_0, M_1, \ldots, M_k]$. Let $r = \equiv^l_\beta [0, k]$ and $l = \equiv^m_\beta [0, k]$. Then we have $M_0 \rightarrow M_0^{\ast r_\ast}$ and $M_k^{\ast r_\ast} \leftarrow M_k$, where $m_1 = \equiv^m_\beta [0, r] \leq \min \{l, r\}$.

Proof. From the main lemma, we have two reduction paths such that $M_0 \rightarrow M_k^{\ast r_\ast}$ and $M_0 \leftarrow M_k^{\ast r_\ast}$, where the paths have a crossed point that is the term $M_r^{\ast r_\ast}$ for some $n \leq k$ as follows:

Let $m_i$ be $\equiv^i_\beta [0, r]$, then $\equiv^i_\beta [r, k] = (l - m_i)$ and $\equiv^i_\beta [r, k] = m_i$. Hence, from the main lemma, we have $M_0 \rightarrow M_0^{\ast m_1}$ and $M_k^{\ast m_1} \leftarrow M_k$ where $m_i \leq \min \{l, r\}$. Moreover, we have $M_r \rightarrow M_k^{(l-m_i)\ast}$ by the main lemma again, and then $M_r^{\ast m_1} \rightarrow M_k^{(l-m_i)+m_i\ast}$ from the repeated application of Proposition 4. Therefore, we indeed have $M_0 \rightarrow M_r^{\ast m_1} \rightarrow M_k^{\ast m_1 \ast}$ and $M_k^{\ast m_1 \ast} \leftarrow M_k$ as well.

Example 1 We demonstrate a simple example of $M_0 =^k_\beta M_k$ with length 4, and list 24 patterns of the reduction graph consisting of the sequence $[M_0, M_1, M_2, M_3, M_4]$. The sixteen patterns can be classified into 5 groups, in which $M_0$ and $M_4$ have a pair of the same common reducts $\langle M_0^{\ast s}, M_4^{\ast s} \rangle$ where $r + l = 4$:

1. Common reducts $\langle M_0^{\ast s}, M_4^{\ast s} \rangle$ and a crossed point $M_0^{\ast m_1} \equiv M_4^{\ast m_1}$:

(1) $M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow M_4$. 

2. Common reducts \(\langle M_0^3, M_3^4 \rangle\) and crossed points \(M_3^{m_0^*}\) of two kinds:
   (1) \(M_0 \leftarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow M_4\);
   (2) \(M_0 \rightarrow M_1 \leftarrow M_2 \rightarrow M_3 \rightarrow M_4\) with \(M_3^{m_1^*} \equiv M_3^*\);
   (3) \(M_0 \rightarrow M_1 \rightarrow M_2 \leftarrow M_3 \rightarrow M_4\);
   (4) \(M_0 \rightarrow M_1 \rightarrow M_2 \leftarrow M_3 \leftarrow M_4\) with \(M_3^{m_2^*} \equiv M_3^0\).

3. \(\langle M_0^2, M_3^2 \rangle\) and crossed points \(M_2^{m_2^*}\) of three kinds:
   (1) \(M_0 \leftarrow M_1 \rightarrow M_2 \leftarrow M_3 \rightarrow M_4\);
   (2) \(M_0 \leftarrow M_1 \leftarrow M_2 \rightarrow M_3 \rightarrow M_4\) with \(M_2^{m_2^*} \equiv M_2^*\);
   (3) \(M_0 \leftarrow M_1 \rightarrow M_2 \rightarrow M_3 \leftarrow M_4\);
   (4) \(M_0 \rightarrow M_1 \leftarrow M_2 \rightarrow M_3 \leftarrow M_4\) with \(M_2^{m_2^*} \equiv M_2^0\);
   (5) \(M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow M_4\);
   (6) \(M_0 \rightarrow M_1 \rightarrow M_2 \leftarrow M_3 \leftarrow M_4\) with \(M_2^{m_2^*} \equiv M_2^0\).

4. \(\langle M_0^1, M_4^1 \rangle\) and crossed points \(M_1^{m_1^*}\) of two kinds:
   (1) \(M_0 \leftarrow M_1 \rightarrow M_2 \leftarrow M_3 \rightarrow M_4\);
   (2) \(M_0 \leftarrow M_1 \leftarrow M_2 \rightarrow M_3 \rightarrow M_4\) with \(M_1^{m_1^*} \equiv M_1^*\);
   (3) \(M_0 \leftarrow M_1 \rightarrow M_2 \rightarrow M_3 \leftarrow M_4\);
   (4) \(M_0 \rightarrow M_1 \leftarrow M_2 \rightarrow M_3 \leftarrow M_4\) with \(M_1^{m_1^*} \equiv M_1^0\).

5. \(\langle M_0^0, M_4^0 \rangle\) and a crossed point \(M_0^{m_0^*} \equiv M_0^0\):
   (1) \(M_0 \leftarrow M_1 \rightarrow M_2 \rightarrow M_3 \leftarrow M_4\).

Observe that a crossed point \(M_0^{m_i^*}\) in Corollary gives a “good” common contractum such that the number \(m_i\), i.e., iteration of the translation \(\ast\) is minimum, see also the trivial cases above; Case 1, Case 2 (4), Case 3 (6), Case 4 (4), and Case 5. Consider two reduction paths: (i) a reduction path from \(M_0^{m_i^*}\) to \(M_i^0\), and (ii) a reduction path from \(M_i^{m_i^*}\) to \(M_k^0\), see the picture in the proof of Corollary. In general, the reduction paths (i) and (ii) form the boundary line between common contractums and non-common ones. Let \(B\) be a term in the boundary (i) or (ii). Then any term \(M\) such that \(B \rightarrow M\) is a common contractum of \(M_0\) and \(M_k\). In this sense, the term \(M_0^{m_i^*}\) where \(0 \leq m_i \leq \min\{l, r\}\) can be considered as an optimum common reduct of \(M_0\) and \(M_k\) in terms of Takahashi translation. Moreover, the refined lemma gives a divide and conquer method such that \(M_0 = \beta_1 M_k\) is divided into \(M_0 = \beta_2 M_r\) and \(M_r = \beta_3 M_k\), where the base case is a valley such that \(M_0 \rightarrow M_r \leftarrow M_k\) with minimal \(M_r\) and \(m_i = 0\), as shown by the trivial cases above.

The results of Lemma and Corollary can be unified as follows. The main theorem shows that every term in the reduction sequence \(ls\) of \(M_0 = \beta_1 M_k\) generates a common contractum: For every term \(M\) in \(ls\), there exists a natural number \(n \leq \max\{l, r\}\) such that \(M^{m^*_n}\) is a common contractum of \(M_0\) and \(M_k\). Moreover, there exist a term \(N\) in \(ls\) and a natural number \(m \leq \min\{l, r\}\) such that \(N^{m^*_n}\) is a common contractum of all the terms in \(ls\).

**Theorem 1 (Main theorem for \(\beta\)-equality)** Let \(M_0 = \beta M_k\) with reduction sequence \([M_0, \ldots, M_k]\). Let \(l = \#l[0, k]\) and \(r = \#r[0, k]\). Then there exist the following common reducts:

1. We have \(M_0 \rightarrow M_r^{[r-i, k]^*}\) and \(M_r^{[r-i, k]^*} \leftarrow M_k\) for each \(i = 0, \ldots, r\). We also have \(M_0 \rightarrow M_r^{[r, r+j]^*}\) and \(M_r^{[r, r+j]^*} \leftarrow M_k\) for each \(j = 0, \ldots, l\).

2. For every term \(M\) in the reduction sequence, we have \(M \rightarrow M_r^{m_i^*}\) where \(m_i = \#l[0, r]\).

*Proof.* Both 1 and 2 are proved similarly from Lemma and monotonicity. We show the case 2 here. Let \(M_i\) be a term in the reduction sequence of \(M_0 = \beta M_k\) where \(0 \leq i \leq r\). Take \(a = \#r[0, i]\), then \(M_a^{[0, i]}\) is a crossed point of \(M_0 \rightarrow M_r^{[0, i]^*}\) and \(M_i \rightarrow M_0^{[r, i]^*}\). From \(M_i \rightarrow M_r^{[r, i]^*}\) and monotonicity, we have \(M_i^{[0, l]} \rightarrow M_r^{m_i^*}\) where \(m_i = \#l[0, i] + \#l[i, r]\). Hence, we have \(M_i \rightarrow M_a^{[0, a]} \rightarrow M_r^{[0, a]} \rightarrow M_r^{m_i^*}\). The case of \(r \leq i \leq k\) is also verified similarly. Note that the case of \(i = r\) and \(j = l\) implies the main lemma, since \(\#r[0, r] = r\) and \(\#l[0, r + l] = \#l[0, k] = l\). Note also that the case of \(i = 0 = j\) implies the refinement, since \(\#l[0, r] = m_l = \#r[0, k]\).
Corollary 2 (Church-Rosser theorem for $\beta$-reduction) Let $P_n \leftarrow \cdots \leftarrow P_1 \leftarrow M \rightarrow Q_1 \rightarrow \cdots \rightarrow Q_m (1 \leq n \leq m)$. Then we have $P_n \rightarrow Q_m^*$ and $Q_m \rightarrow Q_m^*$. We also have $P_n \rightarrow Q_m^{(n-m)}$ and $Q_m \rightarrow Q_m^{(n-m)}$.

Proof. From the main lemma and the refinement where $Q_0 \equiv M$. □

Theorem 2 (Improved Church-Rosser theorem for $\beta$-reduction) Let $P_n \leftarrow \cdots \leftarrow P_1 \leftarrow M \rightarrow Q_1 \rightarrow \cdots \rightarrow Q_m (1 \leq n \leq m)$. If $P_n \leftarrow \cdots \leftarrow P_1 \leftarrow M$ contains $a$-times reductions of new redexes $(0 \leq a \leq n - 1)$, and $M \rightarrow Q_1 \rightarrow \cdots \rightarrow Q_m$ contains $b$-times reductions of new redexes $(0 \leq b \leq m - 1)$, then we have both $P_n \rightarrow Q_m^{(a+1)*}$ and $Q_m \rightarrow P_n^{(b+1)*}$.

Proof. We show the claim that if a reduction path $\sigma$ of $R_0R_1\ldots R_n : M \equiv M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_{n+1}$ contains $a$-times reductions of new redexes $(1 \leq a \leq n-1)$ then $M_{n+1} \rightarrow M^{(a+1)*}$, from which the theorem is derived by repeated application of Proposition [4].

We prove the claim by induction on $a$.

1. Case of $a = 0$:
   
   We have $R_0R_1\ldots R_{n-1}R_{n+1} \ldots R_m : M \equiv M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_{n+1}$, where none of $R_i$ $(0 \leq i \leq n)$ is a new redex. The reduction path is a development of $M$ with respect to a subset of Redex($M$). Then we have $M_j \rightarrow M^* (0 \leq j \leq n+1)$, since all developments of Redex(M) are finite [7] and end with some $N$ such that $N \rightarrow M^*$.

2. Case of $a = k + 1$:
   
   We have $R_0R_1\ldots R_{n-1}R_nR_{n+1} \ldots R_m : M \equiv M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_{n+1} \rightarrow \cdots \rightarrow M_{m+1} (m \geq 0)$, where $R_0R_1\ldots R_{n-1} : M \equiv M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_n$ contains $k$ reductions of new redexes $(0 \leq k \leq n - 1)$. Moreover, the redex $R_n$ is a new redex, and $R_{n+1} \ldots R_m : M_{n+1} \rightarrow \cdots \rightarrow M_{m+1}$ contains no new redexes. Then the reduction path $R_nR_{n+1}\ldots R_m : M_n \rightarrow M_{n+1} \rightarrow \cdots \rightarrow M_{m+1}$ is a development of $M_n$ with respect to a subset of Redex($M_m$), and hence $M_{m+1} \rightarrow M_n^*$. On the other hand, from the induction hypothesis applied to the reduction path $R_0R_1\ldots R_{n-1} : M \equiv M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_n$ with $k$ reductions of new redexes, we have $M_n \rightarrow M^{(k+1)*}$. Therefore, we have $M_{m+1} \rightarrow M^{(k+2)*}$ by repeated application of Proposition [4]. □

4 Quantitative analysis and comparison with related results

4.1 Measure functions

For quantitative analysis, we list important measure functions, TermSize, Mon, and Rev.

Definition 12 (TermSize) We define TermSize($M = \beta N$) by induction on the derivation.

1. If $M \leftarrow^r N$ then TermSize($M = \beta N$) = $8(|M|/N)^2$.  

2. If $M = \beta N$ is derived from $N = \beta M$, then define TermSize($M = \beta N$) by TermSize($N = \beta M$).

3. If $M = \beta N$ is derived from $M = \beta P$ and $P = \beta N$, then define TermSize($M = \beta N$) as follows: max{TermSize($M = \beta P$), TermSize($P = \beta N$)}.

Proposition 5 (TermSize) Let $M_0 = \beta M_k$ with reduction sequence $ls$. Then $|M| \leq$ TermSize($M_0 = \beta M_k$) for each term $M$ in $ls$, and TermSize($M_0 = \beta M_k$) $\leq |N|^2$ for some term $N$ in $ls$.

Proof. By induction on the derivation of $= \beta$ together with Definition [12] and Proposition [1]. □
Definition 13 (Monotonicity)

\[
\text{Mon}(|M|, m, n) = \begin{cases} 
2^{|M|^{2n}} & \text{for } n = 1 \\
2^{2^{|M|^{(m-1)\times (n-2)}}} & \text{for } n > 1
\end{cases}
\]

Proposition 6 (Monotonicity) If \( M \rightarrow^m N \), then \( M^{rs} \rightarrow^l N^{rs} \) with \( l \leq \text{Mon}(|M|, m, n) \).

Proof. By induction on \( n \).

1. Case of \( n = 1 \):

   If \( M \rightarrow^m M_m \), then \( M^* \rightarrow^l M^* \) with \( l \leq 2^{|M|^{2m}} \). Indeed, from Proposition 6 we have \( |M_m| < |M|^{2m} \). If \( M_0 \rightarrow M_1 \) then we have \( M_0^* \rightarrow l_1^* M_1^* \) with \( l_1 < 2^{|M_0|^2} \) from Proposition 4 and Lemma 4. Hence, from \( M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_m \), we have \( M_0^* \rightarrow l M_1^* \rightarrow l_2 \cdots \rightarrow l_m M_m^* \) where

\[
l = \sum_{i=1}^{m} l_i < \sum_{i=0}^{m-1} 2^{|M_i|} < \sum_{i=0}^{m-1} 2^{|M_0|^2} < 2^{|M_0|^{2m}}.
\]

2. Case of \( n \geq 1 \):

   From the induction hypothesis, we have \( M^{rs} \rightarrow^l N^{rs} \) with \( l < \text{Mon}(|M|, m, n) \). Therefore, we have \( M^{(n+1)*} \rightarrow^l N^{(n+1)*} \) with

\[
l' < 2^{|M|^2} < 2^{|M_{n+1}|^2 \text{Mon}(|M|, m, n)} , \quad \text{where } |M^{rs}| < 2^{|M|^m}.
\]

Lemma 8 (Cofinal property) If \( M \rightarrow^n N \) \((n \geq 1)\), then \( N \rightarrow^l M^{rs} \) with \( l < \text{Rev}(|M|, n) \) as follows:

\[
\text{Rev}(|M|, n) = \begin{cases} 
\frac{1}{2}|M|^2, & \text{for } n = 1 \\
\frac{1}{2}|M|^{2n} + 2^{|M|^{(n-1)\times \text{Rev}(|M|, n-1)}} & \text{for } n > 1
\end{cases}
\]

Proof. The case \( \text{Rev}(|M|, 1) \) is by Lemma 5. For \( n > 1 \), \( \text{Rev}(|M|, n) \) follows \( \text{Mon}(|M|, n, 1) \) from Proposition 6 and \(|N| < |M|^{2n}\) from Proposition 1.

4.2 Quantitative analysis of Church-Rosser for \( \beta \)-reduction

We show two bound functions \( f(l, |M|, r) = \langle m, n \rangle \) such that for the peak \( N_1 \leftarrow^l M \rightarrow^r N_2 \), the valley size of \( N_1 \leftarrow^a P \leftarrow^b N_2 \) for some \( P \) is bounded by \( a \leq m \) and \( b \leq n \). The first function \( \text{CR-red}(l, M, r) = \langle m, N_1^{rs}, n \rangle \) provides a common reduct \( N_1^{rs} \) following the proof of the main lemma with \( M \). The second one \( V \)-size \((l, M, r) = \langle m, M^{rs}, n \rangle \) gives a common reduct \( M^{rs} \) simply using \( \text{Rev} \) provided that \( l \leq r \).

Definition 14 (CR-red)

1. \( \text{CR-red}(l, M, 1) = \langle \frac{1}{2}|M|^2, N_1^{rs}, \frac{1}{2}|M|^2 + 2^{|M|^{2l}} \rangle \)

2. \( \text{CR-red}(l, M, r) = \)

   let \( \langle m, N_1^{(r-1)*}, n \rangle \) be \( \text{CR-red}(l, M, r-1) \) in \( \langle 2^{|M|^{2l}}, N_1^{rs}, \frac{1}{2}|M|^2 + 2^{|M|^{(r-1)*}} \rangle \) for \( r > 1 \)

Proposition 7 (CR-red) If \( N_1 \leftarrow^l M \rightarrow^r N_2 \), then we have \( \text{CR-red}(l, M, r) = \langle m, N_1^{rs}, n \rangle \) such that \( N_1 \leftarrow^a N_1^{rs} \leftarrow^b N_2 \) with \( a \leq m \) and \( b \leq n \).

Proof. By induction on \( r \).
1. Case $r = 1$:

We have $M^* \rightarrow^a N_2$ with $a \leq \frac{1}{2} |N_2|$. Then $N_1^* \rightarrow^b M^*$ with $b \leq \text{Mon}(|M|, l, 1) = 2|M|^2$. On the other hand, we have a common contractum $N_1^*$ such that $N_1 \rightarrow^e N_1^*$ with $e \leq \frac{1}{2} |N_1| \leq \frac{1}{2} |M|^2$.

2. Case of $r > 1$:

From the induction hypothesis, we have $(m, N_1^{(r-1)}, n) = \text{CR-red}(|l, M, r-1|)$ such that $M \rightarrow^{(r-1)} N_3 \rightarrow N_2$ and $N_1^{(r-1)} \rightarrow^b N_3$ with $b \leq n$ for some $N_3$. Then we have $N_3^* \rightarrow^c N_2$ with $c \leq \frac{1}{2} |N_3| \leq \frac{1}{2} |M|^2$, and hence $N_1^{*r} = \frac{d}{d}$ where

$$d \leq \text{Mon}(|N_3|, n, 1) \leq \text{Mon}(|M|^{2^{(r-1)}}, n, 1) = 2(|M|^{2^{(r-1)}})^2 = 2|M|^{2^{(r+\alpha-1)}}.$$

Therefore, we have a common reduct $N_1^{*r}$ such that $N_1 \rightarrow^e N_1^{*r}$ with $e \leq \text{Len}(|N_1|, r) \leq 2|M|^2$.

**Definition 15 (V-size)**

\[ V\text{-size}(l, M, r) = \langle \text{Rev}(|M|, l) + 2_l^{\lfloor |M| \rfloor}, M^{*r}, \text{Rev}(M, r) \rangle \text{ for } 1 \leq l \leq r. \]

**Proposition 8 (V-size)** If $N_1 \rightarrow^l M \rightarrow^r N_2$ with $l \leq r$, then we have $V\text{-size}(l, M, r) = \langle m, M^{*r}, n \rangle$ such that $N_1 \rightarrow^a M^{*r} \rightarrow^b N_2$ with $a \leq m$ and $b \leq n$.

**Proof.** Suppose that $l \leq r$. We have $N_1 \rightarrow^a M^{*r}$ with $a \leq \text{Rev}(|M|, l)$ and $M^{*r} \rightarrow^b N_2$ with $b \leq \text{Rev}(|M|, r)$, respectively. From $l \leq r$, we have $M^{*r} \rightarrow^c M^{*r}$ where

$$c \leq \text{Len}(|M^{*r}|, r-l) \leq 2^{\lfloor |M^{*r}| \rfloor} \leq 2^{\lfloor |M| \rfloor} = 2^{\lfloor |M| \rfloor}.$$

On the other hand, Ketema and Simonsen [9] showed that an upper bound on the size of confluence diagrams in $\lambda$-calculus is $\text{bl}(l, |M|, r)$ for $P \rightarrow^l M \rightarrow^r Q$. The valley size $a$ and $b$ of $P \rightarrow^a N \rightarrow^b Q$ for some $N$ is bounded by $\text{bl}(l, |M|, r)$ as follows:

$$\text{bl}(l, |M|, r) = \begin{cases} |M|^{2^{l+2}}, & \text{for } r = 1 \\ |M|^{2^{\lfloor l, |M|, r-1|, |bl(l, |M|, r-1)| + 1}} & \text{for } r > 1 \end{cases}$$

Their proof method is based on the use of the so-called Strip Lemma, and in this sense our first method CR-red is rather similar to theirs. However, for a large term $M$, $\text{bl}$ can give a shorter reduction length than that by CR-red from the shape of the functions. The reason can be expounded as follows: From given terms, we explicitly constructed a common reduct via $*$-translation, so that more redexes than a set of residuals can be reduced, compared with those of $\text{bl}$. To overcome this point, an improved version of Theorem 2 is introduced such that $*$-translation is applied only when new redexes are indeed reduced.

The basic idea of the second method V-size is essentially the same as the proof given in [11]. In summary, the functions $\text{bl}$ and CR-red including a common reduct are respectively defined by induction on the length of one side of the peak, and V-size is by induction on that of both sides of the peak. All the functions belong to the fourth level of the Grzegorczyk hierarchy.

### 4.3 Quantitative analysis of Church-Rosser for $\beta$-equality

Let $M_0 =^k_\beta M_k$ with length $k = l + r$ where $l = \sharp[0, k]$ and $r = \sharp r[0, k]$, and $M$ be $\text{TermSize}(M_0 =^k_\beta M_k)$. Then we show a bound function $\text{CR-eq}(M_0 =^k_\beta M_k) = \langle m, M_0^{*r}, n \rangle$ such that $M_0 \rightarrow^a M_0^{*r}$ and $M_0^{*r} \rightarrow^b M_k$ with $a \leq m$ and $b \leq n$. This analysis reveals the size of the valley described in Lemma 7.
**Definition 16** Given $M_0 =_\beta M_k$ with length $k = l + r$ where $l = \sharp l[0,k]$ and $r = \sharp r[0,k]$. Let $M$ be TermSize($M_0 =_\beta M_k$). A measure function CR-eq is defined by induction on the length of $=_\beta$, where $\cdot$ denotes an arbitrary term.

1. CR-eq($M_0 \leftarrow \cdot$) = $\langle 0, M_0^{0*}, 1 \rangle$;
2. CR-eq($M_0 \rightarrow \cdot$) = $\langle \frac{1}{2} |M_0|, M_0^{0*}, \frac{1}{2} |M_0|^2 \rangle$;
3. CR-eq($M_0 =_\beta \cdot \rightarrow \cdot$) = let $\langle a, M_0^{r*}, b \rangle$ be CR-eq($M_0 =_\beta \cdot$) in $\langle a + \frac{1}{2} |M_0|, M_0^{r* + 1}, \frac{1}{2} M + 2^{M_0^{r*}} \rangle$;

Note that in the definition of CR-eq, as shown by the use of $\cdot$, we use no information on $N$ such that $M_0 =_\beta N$, but only by the use of the length of $=_\beta$ and case analysis of $\rightarrow$ or $\leftarrow$. From Definition [12] and Proposition TermSize($M_0 =_\beta M_k$) is well-defined by induction on $=_\beta$. From the definition above, CR-eq is also a function in the fourth level of the Grzegorczyk hierarchy (non-elementary).

**Proposition 9 (Church-Rosser for $\beta$-equality)** If $M_0 =_\beta M_k$ with length $k = l + r$ where $l = \sharp l[0,k]$ and $r = \sharp r[0,k]$, then we have CR-eq($M_0 =_\beta M_k$) = $(m, M_0^{r*}, n)$ such that $M_0 \rightarrow^n M_0^{r*}$ and $M_0^{r*} =_\beta M_k$ with $a \leq m$ and $b \leq n$.

**Proof.** By induction on the length of $=_{(l+r)}$. The outline of the proof is the same as that of Lemma [7].

1. Base cases of $k = 1$:
   - CR-eq($M_0 \leftarrow \cdot$) = $\langle 0, M_0^{0*}, 1 \rangle$:
     We have $M_0 \equiv M_0^{0*} \leftarrow M_1$ for some $M_1$.
   - CR-eq($M_0 \rightarrow \cdot$) = $\langle \frac{1}{2} |M_0|, M_0^{r*}, \frac{1}{2} |M_0|^2 \rangle$:
     We have $M_0 \rightarrow M_1$ for some $M_1$, and then $M_0 \rightarrow^n M_0^{r*}$ with $a \leq \frac{1}{2} |M_0|$ and $M_0^{r*} =_\beta M_1$ with $b \leq \text{Rev}(|M_0|), 1 = \frac{1}{2} |M_0|^2$.

2. Step cases:
   - CR-eq($M_0 =_\beta \cdot \leftarrow \cdot$) = let $\langle a, M_0^{r*}, b \rangle$ be CR-eq($M_0 =_\beta \cdot$) in $\langle a, M_0^{r*}, b + 1 \rangle$:
     From the induction hypothesis, we have $M_0 \rightarrow^m M_0^{r*}$ with $m \leq a$ and $M_0^{r*} =_\beta M_2 \leftarrow M_3$ for some $M_2, M_3$ with $n \leq b$. Then we have the same common reduct $M_0^{r*}$ and $n + 1 \leq b + 1$ from $M_0^{r*} =_\beta M_3$.
   - CR-eq($M_0 =_\beta \cdot \rightarrow \cdot$) = let $\langle a, M_0^{r*}, b \rangle$ be CR-eq($M_0 =_\beta \cdot$) in $\langle a + \frac{1}{2} |M_0|, M_0^{r* + 1}, \frac{1}{2} |M_0| + 2^{M_0^{r*}} \rangle$:
     From the induction hypothesis, we have $M_0 \rightarrow^m M_0^{r*}$ with $m \leq a$ and $M_0^{r*} =_\beta M_2 \rightarrow M_3$ for some $M_2, M_3$ with $n \leq b$. We also have $M_2^{r+1} =_\beta M_3$ with $c \leq \frac{1}{2} |M_2| \leq \frac{1}{2} M$, and then $M_0^{(r+1)*} =_\beta M_2^{r+1}$ where
     $$d \leq \text{Mon}(|M_2|, b, 1) \leq \text{Mon}(M, b, 1) = 2^{M_0^{r*}}.$$  
     Hence, we have a common reduct $M_0^{(r+1)*}$ such that $M_0 \rightarrow^m M_0^{r*} \rightarrow^c M_0^{(r+1)*}$ where
     $$m + e \leq a + \frac{1}{2} |M_0^{r*}| \leq a + \frac{1}{2} |M_0|.$$ □

**Example 2** The Church numerals $c_n = \lambda f x. f^n(x)$ are defined as usual due to Rosser [17], where we write $F^0(M) = M$, and $F^{n+1}(M) = F(F^n(M))$. We define $N$ such that $N_1 = c_2$, and $N_{n+1} = N_n c_2$. We also define $M_1 = c_1 p(N_4 p q)$ and $M_2 = N_4 p c_1 p q$ with fresh variables $p$ and $q$ for $n \geq 4$. We might have
$M_1 =_\beta M_2$, but the length of $=_\beta$ is not trivial. From the fact that $N_n \rightarrow^a \lambda f.\lambda x.f^2(x)$ with $a \leq 2_n^1$, indeed we prove $M_1 =_\beta M_2$ as follows:

$$M_1 \rightarrow c_1 p((\lambda f.\lambda x.f^2(x))pq) \rightarrow^2 c_1 p(p^2(q)) \rightarrow^2 p(p^2(q)), \text{ and similarly } p^2_i(q) \leftarrow M_2.$$ 

Hence, the length of $=_\beta$ is at most $2 \times (4 + 2_n^1)$, and the size of the common reduct is $1 + 2 \times (2_{n+1}^1 + 1)$, although $|M_1| = |M_2| = 8n + 1$. The example suggests that there is plenty of room for improvement of the upper bound. Note that $M_1 \rightarrow p^{2^i+1}(q) \leftarrow M_2$ is regarded as a base case in the sense of Example 1.

5 Concluding remarks and further work

The main lemma revealed that a common contractum $P$ from $M_0$ and $M_k$ with $M_0 =_\beta M_k$ can be determined by (i) $M_0$ and the number of occurrences of $\rightarrow$ in $=_\beta$, and also by (ii) $M_k$ and that of $\leftarrow$. In general, we have $2^k$ patterns of reduction graph for $=_\beta$ as a combination of $\rightarrow$ and $\leftarrow$ with length $k$. This lemma means that $2^k$ patterns of graph can be grouped into $(k + 1)$ classes with $\binom{k}{i}$ patterns $(i = 0, \ldots, k)$, like Pascal’s triangle. As demonstrated by Example 1, we have common contractums $\langle M_0^{(k-1)i}, M_k^{i\beta} \rangle$ for each class $(i = 0, \ldots, k)$, contrary to an exponential size of the patterns of reduction graph. Moreover, Corollary [1] provides an optimum common contractum $M^{m_i\beta}$ for $M_0 =_\beta M_k$ in terms of Takahashi translation, which is one of important consequences of the main lemma.

The main lemma depends only on Proposition [4] and Lemma [5] which can be expounded geometrically as parallel and flipped properties respectively. Hence, if there exists an arbitrary reduction strategy $\ast$ that satisfies both properties, then the main lemma can be established. In fact, the main lemma holds even for $\beta\eta$-equality, because for $\beta\eta$-reduction, under an inside-out development we still have Lemma [5], Proposition [4] and Proposition [2] without bounds as observed already in [11]. This implies that under a general framework with such a strategy, it is possible to analyze quantitative properties of rewriting systems in the exactly same way, and indeed $\lambda$-calculus with $\beta\eta$-reduction and weakly orthogonal higher-order rewriting systems [17] are instances of these systems. Moreover, this general approach is available as well for compositional $Z$ [13] that is an extension of the so-called $Z$ property [5] (property of a reduction strategy that is cofinal and monotonic), which makes it possible to apply a divide and conquer method for proving confluence.

In order to analyze reduction length of the Church-Rosser theorem, we provided measure functions Len, TermSize, Mon, and Rev. In terms of the measure functions, bound functions are obtained for the theorem for $\beta$-reduction and $\beta$-equality, explicitly together with common contractums. A bound on the valley size for the theorem for $\beta$-equality is obtained by induction on the length of $=_\beta$. Compared with [9], the use of TermSize is important to set bounds to the size of terms, in particular, for the theorem for $\beta$-equality. Given $M =_\beta N$, then there exists some constant TermSize($M =_\beta N$), and under the constant bound functions can be provided by induction only on the length of $=_\beta$ with neither information on $M$ nor $N$, including the size of a common contractum.

In addition, based on Corollary [1], it is also possible to analyze the valley size of $M_0 =_\beta M_{l+r}$ in terms of $M_d^{m_i\beta}$: In the base case of $m_i = 0$, the valley size is bounded simply by $l$ and $r$, for instance, see Example 2; in the maximum case of $m_i = \min\{l, r\}$, the valley size is at most that of the theorem for $\beta$-reduction as observed in Example 1; and this analysis will be discussed elsewhere.

Towards a tight bound, our bound depends essentially on Proposition [2] and Lemma [4]. Proposition [2] provides an optimal reduction, since we adopted the so-called minimal complete development [8][10][15]. For the bound on the size of $M^\ast$, Lemma [4] can be proved, in general, under some function $f(x)$ such that $f(x) \times f(y) \leq f(x + y)$, which may lead to a non-elementary recursive function, as described by Len.
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