CORRECTION TO "TOROIDAL AND KLEIN BOTTLE BOUNDARY SLOPES"

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Abstract. Let $M$ be a compact, connected, orientable, irreducible 3-manifold and $T_0$ an incompressible torus boundary component of $M$ such that the pair $(M, T_0)$ is not cabled. In the paper "Toroidal and Klein bottle boundary slopes" [5] by the author it was established that for any $\mathcal{K}$-incompressible tori $F_1, F_2$ in $(M, T_0)$ which intersect in graphs $G_{F_i} = F_i \cap F_j \subset F_i$, $\{i, j\} = \{1, 2\}$, the maximal number of mutually parallel, consecutive, negative edges that may appear in $G_{F_i}$ is $n_j + 1$, where $n_j = |\partial F_j|$. In this paper we show that the correct such bound is $n_j + 2$, give a partial classification of the pairs $(M, T_0)$ where the bound $n_j + 2$ is reached, and show that if $\Delta(\partial F_1, \partial F_2) \geq 6$ then the bound $n_j + 2$ cannot be reached; this latter fact allows for the short proof of the classification of the pairs $(M, T_0)$ with $M$ a hyperbolic 3-manifold and $\Delta(\partial F_1, \partial F_2) \geq 6$ to work without change as outlined in [5].

1. Introduction

Let $M$ be a compact, connected, orientable, irreducible 3-manifold and $T_0$ an incompressible torus boundary component of $M$ such that the pair $(M, T_0)$ is not cabled and irreducible (that is, $M$ is irreducible and $T_0$ is incompressible in $M$). A punctured torus $(F, \partial F) \subset (M, T_0)$ is said to be generated by a (an essential) Klein bottle if there is a (an essential, resp.) punctured Klein bottle $(P, \partial P) \subset (M, T_0)$ such that $F$ is isotopic in $M$ to the frontier of a regular neighborhood of $P$ in $M$. We also say that $F$ is $\mathcal{K}$-incompressible if $F$ is either incompressible or generated by an essential Klein bottle.

The main purpose of the paper [5] was to establish an upper bound for the maximal number of mutually parallel, consecutive, negative edges that may appear in either graph of intersection $G_{F_1}, G_{F_2}$ between $\mathcal{K}$-incompressible punctured tori $F_1, F_2$ in $(M, T_0)$. In [5, Proposition 3.4] it is proved that for $\{i, j\} = \{1, 2\}$ and $n_j = |\partial F_j|$, if $G_{F_i}$ contains such a collection of $n_j + 2$ negative edges then $M$ is homeomorphic to the trefoil knot exterior or to one of the manifolds $P \times S^1/[m], \ m \geq 1$ constructed in [5, §3.4], none of which is a hyperbolic manifold; consequently, if $M$ is not one of the manifolds listed in [5, Proposition 3.4], the upper bound for such a collection of negative edges was found to be $n_j + 1$.

In this paper we show that the list of options for the homeomorphism class of $M$ given in [5, Proposition 3.4] is incomplete, so that if $M$ is not one of the manifolds listed in [5, Proposition 3.4] then the correct bound for such families of negative edges in the graph $G_{F_i}$ is $n_j + 2$, and that if the upper bound $n_j + 2$ is reached then $(M, T_0)$ belongs to a certain family of examples each of which contains a separating essential twice punctured torus with boundary slope at distance 3 from that of $F_j$. 

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We will use the same notation set up in [5] except that polarized will be replaced by positive (see Section 2.1); in particular, the tori \( F_1, F_2 \) will now be denoted by \( S, T \), with \( s = |\partial S| \) and \( t = |\partial T| \). A graph \( G \) in a punctured surface \( F \) is a 1-submanifold properly embedded in \( F \) with vertices the components of \( \partial F \). The graph \( G \) is essential if no edge is parallel into \( \partial F \) and each circle component is essential in \( F \). The reduced graph \( \overline{G} \subset F \) of \( G \) is the graph obtained from \( G \) by amalgamating each maximal collection of mutually parallel edges \( e_1, \ldots, e_k \) of \( G \) into a single arc \( \overline{e} \subset F \); we then say that the size of \( \overline{e} \) is \( |\overline{e}| = k \). The symbol \((+g, b; \alpha_1/\beta_1, \ldots, \alpha_k/\beta_k)\) will be used to denote a Seifert fibered manifold over an orientable surface of genus \( g \geq 0 \) with \( b \geq 0 \) boundary components and \( k \geq 0 \) singular fibers of orders \( \beta_1, \ldots, \beta_k \).

The main technical result of this paper is the following.

**Proposition 1.1.** For each \( t \geq 4 \) there is an irreducible pair \((M_t, T_0)\) with \( \partial M_t = T_0 \) which contains properly embedded essential punctured tori \((S, \partial S)\) and \((T, \partial T) \subset (M, T_0)\) satisfying the following properties:

1. \( S \) is a separating twice punctured torus and \( T \) is a positive torus with \( |\partial T| = t \),
2. \( \Delta(\partial S, \partial T) = 3 \),
3. \( S, T \) intersect transversely and minimally in the essential graphs \( G_T = S \cap T \subset T \) shown in Figs. 16 and 17, where the reduced graph \( \overline{G}_S \) consists of 3 negative edges of sizes \( t + 2, t, t - 2 \),
4. \( M_t(\partial T) \) is a torus bundle over the circle with fiber \( \hat{T} \), and \( M_t \) is not homeomorphic to any of the manifolds \( P \times S^1/[m] \) constructed in [5] Proposition 3.4.

It is proved in [6] (preprint, in progress) that each manifold \( M_t \) in Proposition 1.1 is hyperbolic with \( M_t(\partial S) = (+0, 1 : \alpha_1/2, \alpha_2/(t+2)) \cup S(+0, 1 : \alpha_1/2, \alpha_2/(t-2)) \), so \( M_{t_1} \not\approx M_{t_2} \) for \( t_1 \neq t_2 \) and \( S \) is generated by a punctured Klein bottle iff \( t = 4 \).

The corrected version of [5, Proposition 3.4] can now be stated as follows.

**Proposition 1.2.** ( Correction to [5] Proposition 3.4 ) Let \((M, T_0)\) be an irreducible pair which is not cabled, \((T, \partial T) \subset (M, T_0)\) a \( K \)-incompressible torus with \( t = |\partial T| \geq 1 \), and \( R \subset M \) a surface which intersects \( T \) in essential graphs \( G_R, G_T \), such that \( G_R \) has at least \( t + 2 \) mutually parallel, consecutive negative edges. Then \( T \) is a positive torus and one of the following holds:

1. the conclusion of [5] Proposition 3.4 holds, so \( M \) is homeomorphic to the trefoil knot exterior \((t = 1)\) or to one of the manifolds \( P \times S^1/[m] \ (t \geq 2) \),
2. \( t \geq 4 \) and \((M, T_0)\) is homeomorphic to one of the pairs \((M_t, T_0)\) of Proposition 1.1, and \( |\overline{e}| \leq t + 2 \) holds for any edge of the reduced graph \( G_R \),
3. \( t = 2 \) with \( M = (+0, 1; -1/4, -1/4) \) and \( M(\partial T) = (+0, 0; 1/2, -1/4, -1/4) \), or \( t = 3 \) with \( M = (+0, 1; -1/3, -1/6) \) and \( M(\partial T) = (+0, 0; 1/2, -1/3, -1/6) \), where in each case the essential annulus \( A \subset (M, T_0) \) satisfies \( \Delta(\partial T, \partial A) = 2 \).

The smaller bound of \( n_j + 1 \) allows for the short proof of the classification of hyperbolic manifolds \((M, T_0)\) with toroidal or Kleinian Dehn fillings at distance \( 6 \leq \Delta \leq 8 \) given in [5] §4. The following result states that in the range \( 6 \leq \Delta \leq 8 \) the bound \( n_j + 2 \) is never reached, which implies that the proofs in [5] §4 work as written.
Lemma 1.3. Suppose \((M, T_0)\) is a hyperbolic manifold and \(S, T \subset (M, T_0)\) are essential tori such that \(t = |\partial T| \geq 3\) and \(\Delta(\partial S, \partial T) \geq 6\); then \(|e| \leq t + 1\) holds in \(G_S\).

Our last lemma summarizes the bounds on the sizes of negative edges of reduced graphs like \(G_R\) obtained from the above results.

Lemma 1.4. Let \((M, T_0)\) be an irreducible pair which is not cabled, \((T, \partial T) \subset (M, T_0)\) a \(K\)-incompressible torus with \(t = |\partial T| \geq 1\), and \(Q \subset M\) a surface which intersects \(T\) in essential graphs \(G_Q = Q \cap T \subset Q, G_T = Q \cap T \subset T\). Then one of the following holds:

1. \(t \leq 3\) and \(M\) is one of the Seifert manifolds \((+0, 1; 1/2, 1/3)(t = 1), (+0, 1; -1/4, -1/4)(t = 2),\) or \((+0, 1; -1/3, -1/6)(t = 3),\)
2. \(t \geq 2\) and \(M\) is one of the manifolds \(P \times S^1/[m]\) constructed in \([5,\) Proposition 3.4],
3. \(t \geq 4\) and \(M\) is homeomorphic to one of the manifolds \(M_t\) of Proposition 1.1, in which case the bound \(|e| \leq t + 1\) holds for all negative edges of \(G_Q\),
4. the bound \(|e| \leq t + 1\) holds for all negative edges of \(G_Q\).

In Section 2 we review the notation and constructions given in \([5, \S 2, \S 3]\), with which we assume the reader is familiar. In Section 3 we construct the manifolds \(M_t\) for \(t \geq 4\) of Proposition 1.1 and establish the results needed to prove Proposition 1.2 and Lemmas 1.3 and 1.4.

2. Slidable and non-slidable bigons

2.1. Generalities. For any slope \(r \in T_0 \subset \partial M, M(r)\) denotes the Dehn filling \(M \cup_{T_0} S^1 \times D^2\), where \(r\) bounds a disk in the solid torus \(S^1 \times D^2\). We denote the core of \(S^1 \times D^2\) by \(K_r \subset M(r)\).

Let \(F\) be a surface properly embedded in \((M, T_0)\) with \(|\partial F| \geq 1\) and boundary slope \(r\). Then the surface \(\hat{F} \subset M(r)\) obtained by capping off the components of \(\partial F\) with a disjoint collection of disks in \(S^1 \times D^2\) is a closed surface, which we always assume to intersect \(K_r\) transversely and minimally in \(M(\partial F)\).

If \(F\) is orientable, we say that \(F\) is neutral if \(\hat{F} \cdot K_r = 0\) in \(M(\partial F)\), where \(\hat{F} \cdot K\) denotes homological intersection number, and that \(F\) is positive if \(|\hat{F} \cdot K_r| = |\hat{F} \cap K_r|\); the latter is equivalent to the term polarized used in \([5]\).

For surfaces \(F_1, F_2 \subset (M, T_0)\) with transverse intersection, for \(i = 1, 2\), \(G_{F_i} = F_1 \cap F_2 \subset F_i\) will denote the graph of intersection of \(F_1\) and \(F_2\) in \(F_i\) (with vertices the boundary components of \(F_i\)).

Following \([3]\), for \(i = 1, 2\) we orient the components of \(\partial F_i\) and coherently on \(T_0\) and say that an edge \(e\) of \(G_{F_i}\) is positive or negative depending on whether the orientations of the components of \(\partial F_i\) (possibly the same) around a small rectangular regular neighborhood of \(e\) in \(F_i\) appear as in Fig 3.

The following lemma summarizes some of the general properties of graphs of intersection of surfaces in \((M, T_0)\) that will be relevant in the sequel.

Lemma 2.1. Let \(F_1, F_2\) be properly embedded surfaces in \((M, T_0)\) with essential graphs of intersection \(G_{F_1} = F_1 \cap F_2 \subset F_1\) and \(G_{F_2} = F_1 \cap F_2 \subset F_2\).
positive edge

negative edge

Figure 1.

(a) Parity Rule: for \( \{i, j\} = \{1, 2\} \), an edge of \( F_1 \cap F_2 \) is positive in \( G_{F_i} \) (cf. [3]); moreover, if \( (M, T_0) \) is not cabled,

(b) no two edges of \( F_1 \cap F_2 \) are parallel in both \( G_{F_1} \) and \( G_{F_2} \) ([1, Lemma 2.1]),

(c) if \( F_i \) is a torus and \( G_{F_j} \) has a family of \( n_i + 1 \) mutually parallel, consecutive, negative edges then \( F_i \) is a positive torus (cf. [5, Lemma 3.2]). \( \square \)

2.2. Review of constructions in [5, §2.3]. Suppose \( (M, T_0), T, R \) satisfy the hypothesis in Proposition 1.2, so \( (M, T_0) \) is an irreducible pair with \( T_0 \) a torus boundary component of \( M \) which is not cabled, \( T \) a \( K \)-incompressible torus in \( (M, T_0) \), and \( R \) any surface properly embedded in \( (M, T_0) \) which intersects \( T \) transversely in essential graphs \( G_T = R \cap T \subset T \) and \( G_R = R \cap T \subset R \).

We assume there is a collection \( E = \{e_1, e_2, \ldots, e_{t+1}, e_{t+2}\} \) of mutually parallel, consecutive, negative edges in \( G_R \). By Lemma 2.1(c), the torus \( T \) is actually positive and hence incompressible in \( M \).

The torus \( T \) has orientation vector \( \vec{N} \) shown in Fig. 3 and each vertex of \( T \) is given the orientation induced by \( \vec{N} \) (see Fig. 3). Recall that the vertices of \( G_T, G_R \) are the components of \( \partial T, \partial R \) respectively; we denote the vertices of \( T \) by \( v_i \)'s and those of \( R \) by \( w_k \)'s. Any two edges \( e, e' \) of \( G_T \) that are incident to the oriented vertex \( v \) split \( v \) into two open subintervals \( (e, e') \subset v \) and \( (e', e) \subset v \), where \( (e, e) \) is the open subinterval whose left and right endpoints, as defined by the orientation of \( v \), come from \( e \) and \( e' \), respectively.

The collection of edges \( E \) induces a permutation \( \sigma \) of the form \( x \mapsto x + \alpha \) with \( 1 \leq \alpha \leq t \), where \( \gcd(t, \alpha) = 1 \) by [5, Lemma 3.2]; the definition of \( \sigma \) requires that a common orientation be given to the edges of \( E \), and reversing the orientation of such edges replaces \( \sigma \) with its inverse, hence \( \alpha \) with \( t - \alpha \).

Cutting \( M \) along \( T \) produces an irreducible manifold \( M_T = \text{cl}(M \setminus N(T)) \) with copies \( T^1, T^2 \subset \partial M_T \) of \( T \) on its boundary and strings \( I'_{1,2}, I'_{2,3}, \ldots, I'_{t,1} \subset \partial M_T \) such that \( M_T/\psi = M \) for some orientation preserving homeomorphism \( \psi : T^1 \to T^2 \), where \( T^1, T^2 \) are oriented by normal vectors \( \vec{N}^1, \vec{N}^2 \) as shown in Fig. 4.

We assume that the edges of \( E \) and \( T^1 \cap R, T^2 \cap R \) are arranged in \( G_R \) as shown in Fig. 2. Each edge \( e_i \) of \( E \) and vertex \( v_i \) of \( T \) gives rise to two copies of itself \( e_i^1, v_i^1 \subset T^1 \) and \( e_i^2, v_i^1 \subset T^2 \) such that \( \psi(e_i) = e_i^2 \) and \( \psi(v_i^1) = v_i^2 \).
The collection $E$ also gives rise to two essential cycles in $T$, $\gamma_1 = e_1 \cup e_2 \cup \cdots \cup e_t$ and $\gamma_2 = e_2 \cup e_3 \cup \cdots \cup e_t \cup e_{t+1}$, such that $\Delta(\gamma_1, \gamma_2) = 1$ holds in $\hat{T}$. The edges $e_1^1 \cup e_2^1 \cup \cdots \cup e_t^1$ form the essential cycle $\gamma_1^1$ in $T^1$, while $e_2^2 \cup e_3^2 \cup \cdots \cup e_{t+1}^2$ form the essential cycle $\gamma_2^2$ in $T^2$, such that the bigon faces $F_1', F_2', \ldots, F_t'$ bounded by $e_1, e_2, \ldots, e_{t+1}$ in $R \cap M_T$ form an essential annulus $A$ in $M_T(\partial T)$ as shown in Fig. 4. For simplicity, we refer to the union of the bigons $F_1', F_2', \ldots, F_t'$ in $M_T$ also as the annulus $A$.

The next result describes the embedding of $A \cup F_{t+1}'$ in $M_T$ and the structures of $M_T$ and $M_T(\partial T)$; its proof follows immediately from the arguments of [5, Lemma 3.3].

Lemma 2.2. ([5, Lemma 3.3]) Up to homeomorphism, the bigons $F_1', \ldots, F_{t+1}'$ lie in $M_T$ as shown in Fig. 4. In particular, $M_T \approx T \times I$ is a genus $t + 1$ handlebody with $F_1', \ldots, F_{t+1}'$ a complete disk system, $\partial M = T_0$, and $M(\partial T)$ is a torus bundle over the circle with fiber $\hat{T}$. \hfill \Box

Finally, we construct auxiliary circles $\mu_i$ in $T$ having the same slope in $\hat{T}$ as the cycle $e_1 \cup e_{t+1}$, oriented and labeled as shown in Fig. 3. The counterparts $\mu_i^1 \subset T^1$ of the $\mu_i$'s are shown in Fig. 4, while the circles $\mu_i^2 \subset T^2$ are represented abstractly in Fig. 4 since the location of the edge $e_i^1 \subset T^2$ is not given yet.

2.3. Review of the argument of [5, Proposition 3.4]. At this point the argument used in the proof of [5, Lemma 3.6] states that, for $t \geq 2$,
... the faces $F'_1$ and $F'_{t+1}$ can be isotoped in $M_T$ to construct an annulus $A_1 \subset M_T$ with boundary the circles $\mu^1_1 \cup \mu^2_2$, which under their given orientations remain coherently oriented relative to $A_1$. Via the product structure $M_T = T \times I$, it is not hard to see that each pair of circles $\mu^1_k, \mu^2_{k+1}$ cobounds such an annulus $A_k \subset M_T$ for $1 \leq k \leq t$, with the oriented circles $\mu^1_k, \mu^2_{k+1}$ coherently oriented relative to $A_k$; these annuli $A_k$ can be taken to be mutually disjoint and $I$-fibered in $M_T = T \times I$. Since $\psi(\mu^1_1) = \mu^2_2$ (preserving orientations), the union $A_1 \cup A_2 \cdots \cup A_t$ yields a closed nonseparating torus $T''$ in $M$, on which the circles $\mu_1, \mu_2, \ldots, \mu_t$ appear consecutively in this order and coherently oriented.

The problem with the above argument is that it is assumed from the beginning that the boundary of the annulus $A_1$ must necessarily be $\mu^1_1 \cup \mu^2_2$, which, as we shall see next, is not the case and leads to the present correction to [5, Proposition 3.4].

The isotopy of $F'_1$ and $F'_{t+1}$ in $M_T$ mentioned in the above quote can be thought of as the result of a sliding process, where the corners of the face $F'_{t+1}$ are slid onto the face $F'_1$ so as to coincide with each other, at which point the isotoped face $F'_{t+1}$ becomes the annulus $A_1$ properly embedded in $M_T$ with boundary the circles $\mu^1_1 \cup \mu^2_2$.

We will say that the bigon $F'_{t+1}$ is slidable (relative to the annulus $A$) whenever the annulus $A_1$ produced by the above isotopy of $F'_{t+1}$ satisfies $\partial A_1 = \mu^1_1 \cup \mu^2_2$, and otherwise that $F'_{t+1}$ is non-slidable. Equivalently, $F'_{t+1}$ is slidable iff the cycles $\psi(e^1_1 \cup e^2_{t+1}) = e^1_2 \cup e^2_{t+1} \subset T^2$ and $e^2_2 \cup e^2_{t+2} \subset T^2$ have the same slope in $\hat{T}$, that is iff the cycles $e_1 \cup e_{t+1}$ and $e_2 \cup e_{t+2}$ have the same slope in $\hat{T}$.

As we shall see in the next section, there are two combinatorially different embeddings of the edge $e^2_1$ in $T^2$ which correspond to the annulus $A_1$ being slidable or not; the generic embeddings $e^2_1 \subset T^2$ are shown in Fig. 11, the slidable case which produces the circles $\mu^2_i \subset T^2$ shown in Fig. 11, and Fig. 12 the non-slidable case.
Using this notation we summarize [5] Proposition 3.4 as follows:

**Lemma 2.3.** [5 Proposition 3.4]

1. If \( t = 1 \) then \( M \) is the exterior of the trefoil knot \((+0, 1/2, 1/3)\).
2. If \( t \geq 2 \) and \( T_{t+1} \) is slidable then \( M \) is homeomorphic to one of the manifolds \( P \times S^1/[m] \) constructed in [5 §3.4], in which case \( M \) is not Seifert fibered and contains a closed nonseparating torus.

We shall see below that in most cases, which include those with \( t \geq 4 \) and \( \alpha \neq \pm 1 \mod t \), the bigon \( F_{t+1} \) is slidable, and that the exceptions with \( t \geq 4 \) form the family \( M_t \) of Proposition 1.1.

### 3. Main results

In this section we assume that \((M, T_0), T, R \) satisfy the hypothesis of Proposition 1.2, also, the mutually parallel edges \( E = \{e_1, e_2, \ldots, e_{t+1}, e_{t+2}\} \) in \( G_R \) are labeled as in Fig. 2; moreover, the cycles \( \gamma^1_{\alpha} = e_1 \cup e_2 \subset T^1 \) and \( \gamma^2_{\alpha} = e_3 \cup e_4 \subset T^2 \) and the bigon \( F'_3 \) cobounded by \( e_3, e_4 \) may be assumed to lie in \( M_T \) as shown in Fig. 4. The case \( t = 1 \) is considered in Lemma 2.3(1).

#### 3.1. The cases \( t = 2, 3 \).

**Lemma 3.1.** If \( t = 2, 3 \) then \((M, T_0)\) satisfies the conclusion of [5 Proposition 3.4] or of Proposition 1.2(3).

**Proof.** For \( t = 2 \) we must have \( \alpha = 1 \); for \( t = 3 \) we may also assume that \( \alpha = 1 \) after reversing the orientation of the edges of \( E \) if necessary.

We begin with a detailed analysis of the case \( t = 2 \). Fig. 5(a) shows the edges \( e_1 \) and bigons \( F'_1 \) of \( E \) in \( G_R \). By Lemma 2.2, up to homeomorphism, the annulus \( A \) cobounded by the cycles \( \gamma^1 = e_1 \cup e_2 \subset T^1 \) and \( \gamma^2 = e_3 \cup e_4 \subset T^2 \) and the bigon \( F'_3 \) cobounded by \( e_3, e_4 \) may be assumed to lie in \( M_T \) as shown in Fig. 5(b) or (c); for simplicity, the upper labels in the edges will not be shown in the figures representing \( M_T \).

Notice that the embeddings of the edges \( e_2 \) and \( e_3 \) are determined in both \( T^1 \) and \( T^2 \) at this point, but that the embeddings of \( e_1 \) and \( e_4 \) are so far determined only in \( T^1 \) and \( T^2 \), respectively.

To determine the possible embeddings of \( e_1 \) and \( e_4 \) we observe that either in Fig. 5(b),(c) the endpoint of \( e_1 \) in \( v_1 \) lies in the interval \((e_3, e_4)\), and that the same statement holds for the endpoint of \( e_1 \) in \( v_2 \). Since \( \psi \) maps each \( v_1 \subset T^1 \) to \( v_2 \subset T^2 \) and each \( e_1 \subset T^1 \) to \( e_4 \subset T^2 \), the endpoints of \( e_1 \) in \( v_1 \) and \( v_2 \) must also lie in the corresponding intervals \((e_3, e_4)\). As the edge \( e_1 \) is already embedded in \( T^2 \) and its endpoint on \( v_2 \) lies in \((e_3, e_4)\), it follows that the endpoint of \( e_1 \) in \( v_2 \) must lie in one of the intervals \((e_3, e_4)\) or \((e_1, e_2)\).

The first option is represented in Fig. 5(b); the placement of \( e_1 \) in \( T^2 \) is then uniquely determined using the fact that no two edges of \( E \) are mutually parallel in \( T \). Since the cycles \( e_1 \cup e_3 \) and \( e_1 \cup e_4 \) have the same slope in \( T^2 \), the bigon \( F'_3 \) is slidable and so by Lemma 2.3 the
Figure 5.
The second option is represented in Fig. 5(c), and here we have that $\Delta(\partial T, \partial A) \neq 2$ so $F'_3$ is not slidable. The endpoints of $e'_3$ in both vertices $v'_1, v'_2$ are now located in the intervals $(e'_1, e'_2)$, so the endpoints of $e'_1$ in both vertices $v'_1, v'_2$ must also be located in the intervals $(e'_3, e'_4)$; the endpoints of $e'_1$ are indicated in Fig. 5(c) by open circles in $v'_1, v'_2$. The only possible embedding of the edge $e'_4$ in $T^1$ is shown in Fig. 6(a).

Now, by Lemma 2.2 the bigons $F'_1, F'_2, F'_3$ form a complete disk system of the handlebody $M_T$. Observe that the endpoints of $e'_4$ can be connected via arcs $c_1, c_2$ along the strings $I'_{1,2}, I'_{2,1}$ that are disjoint from the corners of the bigons $F'_1, F'_2, F'_3$. It follows that the circle $e'_1 \cup e'_2 \cup c_1 \cup c_2 \subset \partial M_T$ is disjoint from $F'_1, F'_2, F'_3$ and hence bounds a disk $F'_4$ in $M_T$ disjoint from $F'_1, F'_2, F'_3$. It is not hard to see that the quotient $A = (F'_1 \cup F'_2 \cup F'_3 \cup F'_4)/\psi$ is a surface in $(M, T_0)$ which intersects $T$ transversely and minimally with $\Delta(\partial T, \partial A) = 2$ and $G_{T,A} = T \cap A \subset T$ the essential graph of Fig. 6(b). Since $T$ is positive, by the parity rule $A$ must be a neutral annulus, hence each face of $G_{T,A}$ is necessarily a Scharlemann cycle of length 4, and since $M$ is irreducible it follows that $A$ must separate $M$ into two
solid tori whose meridian disks, that is the faces of $G_{T,A}$, each intersect $A$ coherently in 4 arcs. Therefore $M$ is a Seifert fibered manifold over the disk with two singular fibers of indices $4, 4$, so $M(\partial T)$ is a Seifert fibered torus bundle over the circle with horizontal bundle fiber $\hat{T}$. By the classification of such torus bundles (cf [2, §2.2]), it follows that $M(\partial T) = (+0, 0; 1/2, -1/4, -1/4)$ and hence that $M = (+0, 1; -1/4, -1/4)$.

The case $t = 3$ is handled in a similar way: Fig. 7(a) shows the labeling of the edges $E = \{e_1, \ldots, e_5\}$ and the bigon faces $F'_1, \ldots, F'_4$ they cobound in $G_R$, while Fig. 7(b) shows

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{Figure 7.}
\end{figure}
the embedding of the bigons $F'_1, \ldots, F'_6$ in $M_T$, up to homeomorphism. Again there are only two possible embeddings for the edge $e^2_1$ in $T^2$ depending on whether the endpoint of $e^2_1$ in $v^2_2$ lies in $(e^2_4, v^2_5)$ or $(e^2_5, v^2_3)$. In the first case the bigon $F'_4$ is again slidable; we now sketch the details for the case corresponding to the later option, where $F'_4$ is non-slidable.

Fig. 6(b) shows the embedding of $e^2_4 \subset T^2$ which makes the bigon $F'_4$ non-slidable and forces the embedding of $e^2_5 \subset T^1$. At this point all the edges in $E^2 = \{e^2_1, \ldots, e^2_6\}$ have been embedded in $T^2$. Fig. 6(c) shows the embedding of a 6th edge $e^2_6$ in $T^2$ with endpoints on the intervals $(e^2_4, e^2_5) \subset v^2_5$ and $(e^2_5, e^2_1) \subset v^2_3$ and which is disjoint from and not parallel in $T^2$ to any of the edges $e^2_1, e^2_2, \ldots, e^2_5$. These properties imply that $e^2_6 = \psi^{-1}(e^2_6)$ must be the edge in $T^1$ sketched in Fig. 6(c).

As before, it is possible to connect the endpoints of $e^2_5$ and $e^2_6$ via corners $c_1 \subset I'_{2,3}$ and $c_2 \subset I'_{3,1}$, and the endpoints of $e^1_3$ and $e^2_1$ via corners $c_3 \subset I'_{3,1}$ and $c_4 \subset I'_{1,2}$, so that the circles $e^1_3 \cup e^2_1 \cup e^2_2 \cup c_1 \cup c_2 \subset \partial M_T$ and $e^1_1 \cup e^2_2 \cup e^2_3 \cup c_3 \cup c_4 \subset \partial M_T$ bound disks $F'_1 \subset M_T$ and $F'_6 \subset M_T$, respectively, with the property that the enlarged collection of disks $F'_1, \ldots, F'_6$ is disjoint (see Fig. 6(c)).

It follows that $A = (F'_1 \cup F'_2 \cup \cdots \cup F'_6)/\psi$ is a neutral annulus in $(M, T_0)$ which intersects $T$ transversely and minimally with $G_{T,A}$ the essential graph of Fig. 6 and $\Delta(\partial T, \partial A) = 2$. Since $|\partial A| = 2$, each face of $G_{T,A}$ is necessarily a Scharlemann cycle and hence $A$ must separate $M$ into two solid tori with meridian disks the faces of $G_{T,A}$. As $G_{T,A}$ has two trigon faces on one side of $A$ and a 6-sided face in the other side of $A$, we must have that $M$ is a Seifert manifold over the disk with two singular fibers of indices 3, 6, which by the classification of such torus bundles (cf [2 §2.2]) implies that $M(\partial T) = (+0, 1/2, -1/3, -1/6)$ and hence that $M = (+0, 1; -1/3, -1/6)$. The lemma follows.

3.2. The generic case $t \geq 4$. Here we assume that $t \geq 4$; we first show that for most values of $\alpha$ the bigon $F'_{t+1}$ is slidable.

Lemma 3.2. If $\alpha \not\equiv \pm 1 \mod t$ then the bigon $F'_{t+1}$ is slidable.

Proof. If $t = 4$ then $\alpha \equiv \pm 1 \mod t$. For $t \geq 5$ the condition $\alpha \not\equiv \pm 1 \mod t$ is equivalent to saying that the pairs of strings $\{I'_{1,2}, I'_{1+\alpha,2+\alpha}\}$ and $\{I'_{2,3}, I'_{2+\alpha,3+\alpha}\}$ are disjoint. These four
strings are shown in Fig. 9 along with the embeddings of the faces $F'_1, \ldots, F'_{t+1}$ from Fig. 11. As $e_{t+2}^2$ has one endpoint on the interval $(e_{2-\alpha}, e_2^2) \subset v_2^2$ and the other on $(e_{2+\alpha}, e_2^3) \subset v_2^{2+\alpha}$, the endpoints of $e_{t+2}^1 = \psi^{-1}(e_{t+2}^2)$ must lie on $(e_{2-\alpha}, e_2^1) \subset v_2^1$ and $(e_{2+\alpha}, e_2^3) \subset v_2^{1+\alpha}$. Therefore the edge $e_{t+2}^1$ must be embedded in $T^1$ as shown in Fig. 9 which implies that the cycles $e_1 \cup e_{t+1}$ and $e_2 \cup e_{t+2}$ have the same slope in $\widehat{T}$ and hence that $F'_{t+1}$ is slidable.

From here on we assume that $\alpha \equiv \pm 1 \mod t$, and after reversing the orientation of the edges of $E$, if necessary, we will take $\alpha = 1$.

**Lemma 3.3.** If $\alpha = 1$ and $t \geq 4$ then, up to homeomorphism, the bigons and edges of $E$ are embedded in $M_T$ and $T$, respectively, as shown in Fig. 11(a),(b) if $F'_{t+1}$ is slidable, or Fig. 12(a),(b) if $F'_{t+1}$ is nonslidable. In the latter case, if $\overline{\tau} \subset \overline{G}_R$ is the negative edge that contains the edges of $E$ then $\overline{\tau} = E$, so $|\overline{\tau}| = t + 2$.

**Proof.** By the argument of [5, Lemma 3.6] (see in particular [5, Fig. 11]) with $\alpha = 1$ we may assume that, up to homeomorphism, the bigons of $E$ are embedded in $M_T$ as shown in Fig. 10 where $t \geq 4$. Notice that the embedding of the bigons of $E$ in $M_T$ only determines the embeddings of the edges $e_1, e_2, \ldots, e_{t+1}$ in $T^1$ and of $e_2, e_3, \ldots, e_{t+2}$ in $T^2$. To determine the possible embeddings of $e_1$ in $T^2$ and $e_{t+2}$ in $T^1$ we proceed as follows.

From Fig. 10 the endpoints of $e_1^1$ in $T^1$ are located in the intervals $(e_{t+1}^1, e_1^1) \subset v_1^1$ and $(e_{t+1}^3, e_1^3) \subset v_1^2$; since $\psi(T^1) = T^2$, it follows that the endpoints of $e_1^2$ in $T^2$ must be located in the intervals $(e_{t+1}^2, e_1^2) \subset v_2^1$ and $(e_{t+1}^4, e_1^4) \subset v_2^2$.

The interval $(e_{t+1}^2, e_1^2) \subset v_2^2$ is split into the two subintervals $(e_{t+1}^2, e_{t+2}^2)$ and $(e_{t+2}^2, e_1^2)$ by the endpoint of $e_{t+2}^1$ in $v_2^2$, which gives rise to two possible locations of the endpoint of $e_1^2$ in $v_2^2$. The endpoints of $e_1^2$ are denoted by open circles in $v_1^2, v_2^2 \subset T^2$ in Fig. 11. Since all faces of the graphs on $T^1, T^2$ of Fig. 10 are disks, the embedding of $e_1^2$ in $T^2$ is completely determined by the location of its endpoints; therefore there are exactly two possible embeddings of $e_1^2$ in $T^2$.
Case 1. The endpoint of $e_1^2$ in $v_2$ lies in $(e_{t+1}^2, e_{t+2}^2)$.

This case is represented in Fig. 11(a). Now the endpoints of $e_{t+2}^2$ in $T^2$ lie in the intervals $(e_1^2, e_2^2) \subset v_2$ and $(e_3^2, e_4^2) \subset v_3$, and so the endpoints of $e_{t+2}^2 = \psi^{-1}(e_{t+2}^2)$ in $T^1$ must lie in $(e_1^1, e_2^1) \subset v_1$ and $(e_3^1, e_4^1) \subset v_3$. Therefore $e_{t+2}^1$ must be the edge in $T^1$ shown in Fig. 11(a). The graph in $T$ produced by the edges of $E$ is shown in Fig. 11(b); as the cycles $e_1 \cup e_{t+1}$ and $e_2 \cup e_{t+2}$ have the same slope in $\hat{T}$, the bigon $F'_{t+1}$ is slidable.

Case 2. The endpoint of $e_1^2$ in $v_2$ lies in $(e_{t+1}^2, e_{t+2}^2)$.

The embedding of $e_1^2$ in $T^2$ is shown in Fig. 12(a). The endpoints of $e_{t+2}^2$ in $T^2$ lie in the intervals $(e_{t+1}^2, e_1^2) \subset v_2$ and $(e_3^2, e_4^2) \subset v_3$, so the endpoints of $e_{t+2}^1$ in $T^1$ lie in $(e_{t+1}^1, e_1^1) \subset v_1$ and $(e_3^1, e_4^1) \subset v_3$; thus $e_{t+2}^1$ must lie in $T^1$ as shown in Fig. 12(a). The graph in $T$ produced by the edges of $E$ is shown in Fig. 12(b); this time the cycles $e_1 \cup e_{t+1}$ and $e_2 \cup e_{t+2}$ have slopes in $\hat{T}$ at distance 1 and so the bigon $F'_{t+1}$ is nonslidable.

Suppose there is a negative edge $e_0 \in \bar{\tau} \setminus E$ in $G_R$ which cobounds a bigon $F_0$ with $e_1$, so that $F_0' \subset M_R$ is cobounded by the edges $e_0^1 \subset T^1$ and $e_1^2 \subset T^2$. Since $e_1^2$ has endpoints in $(e_{t+1}^2, e_1^2) \subset v_2$ and $(e_{t+2}^2, e_3^2) \subset v_3$, the corners of $F_0'$ must lie in the strings $I_{t+1}'$ and $I_{t+2}'$ and so the endpoints of $e_0^1$, represented by squares in Fig. 12, necessarily must lie in $(e_{t+1}^1, e_1^1) \subset v_1$ and $(e_{t+2}^1, e_3^1) \subset v_1$. This is impossible since such endpoints are separated by the edge $e_{t+2} \subset T$ (see Fig. 12(a),(b)), so no such edge $e_0$ exists in $\bar{\tau}$. A similar argument shows that $\bar{\tau} \setminus E$ does not contain any edge adjacent to $e_{t+2}$. Therefore $E = \bar{\tau}$ and so $|\tau| = t + 2$.

**Corollary 3.4.** If $t \geq 4$ and $M$ is not homeomorphic to any of the manifolds $P \times S^1/[m]$ then $|\tau| \leq t + 2$ holds for any negative edge $\tau$ of $\overline{G}_R$.

**Proof.** Let $\tau$ be any negative edge of $\overline{G}_R$ with $|\tau| \geq t + 2$, and let $E = \{e_1, \ldots, e_{t+2}\}$ be any collection of $t + 2$ consecutive edges in $\tau$. Since $M$ is not any of the manifolds $P \times S^1/[m]$,
by Lemmas 2.3 and 3.2 the permutation induced by $E$ must be of the form $x \to x \pm 1$, and hence $|\tau| = t + 2$ holds by Lemma 3.3.

We now prove Lemma 1.3:

**Proof of Lemma 1.3.** If $t = 3$ the bound $|\tau| \leq t + 1$ holds for any negative edge $\tau$ of $\overline{G}_S$ by Lemma 3.1, so we will assume that $t \geq 4$. Since none of the manifolds $P \times S^1/[m]$ is hyperbolic (cf \cite{5}, Proposition 3.4), by Corollary 3.4 the bound $|\tau| \leq t + 2$ holds for any negative edge $\tau$ of $\overline{G}_S$.

Suppose now there is an edge $\tau$ in $\overline{G}_S$ with $|\tau| = t + 2$, so that $T$ is a positive surface and hence, by the parity rule, in $G_S$ all edges are negative, so any cycle in $G_S$ is even sided.

By \cite{5} Lemma 2.2(b), the reduced graph $\overline{G}_S$ has a vertex $u$ of degree $n \leq 4$. Counting endpoints of edges of $G_S$ around $v$ yields the relations

$$6t \leq \Delta \cdot t \leq n \cdot (t + 2),$$

which along with the restriction $n \leq 4$ imply that $n = t = 4$, $\Delta = 6$, and that each of the 4 edges $\overline{\tau}_1, \overline{\tau}_2, \overline{\tau}_3, \overline{\tau}_4$ incident to $u$ in $\overline{G}_S$ has size $t + 2 = 6$, as shown in Fig. 13. We orient the edges of $\overline{\tau}_1, \overline{\tau}_2$ away from $u$ as indicated in Fig. 13.
By Lemma 3.2, reversing the orientation of the vertices of $G_S$ and relabeling the vertices of $G_T$, if necessary, we may assume that the permutation induced by the oriented edge $e_1 = \{e_1, \ldots, e_6\}$ around $u$ is of the form $x \mapsto x + 1$, while the permutation induced by the oriented edge $e_2 = \{a_1, \ldots, a_6\}$ is of the form $x \mapsto x \pm 1$.

We consider in detail the case where $e_2$ induces the permutation $x \mapsto x - 1$; the case where this permutation is $x \mapsto x + 1$ follows by a similar argument. We may assume that the endpoints of the edges in $e_1, e_2$ are labeled as in Fig. 13. We may also assume that the edges of $e_1$ lie in $T$ as shown in Fig. 14; this figure is obtained from the graph of Fig. 12(b) with $t = 4$, except here $T$ is shown cut along the edges $e_1, e_5$ of $e_1$.

The edges $a_2, a_6$ of $e_2$ are parallel in $G_S$ and have endpoints on the vertices $v_3, v_4$ of $G_T$, so by Lemma 2.1(b) for some $i \in \{2, 6\}$ the edge $a_i$ is not parallel in $T$ to $e_3$; it is not hard to see that there is only one embedding of $a_i$ in $T$, as shown in Fig. 14.

Let $F$ be the bigon of $e_2$ cobounded by $a_{i-1}$ and $a_i$; from the point of view of $M_T$, the edge $a_{i-1}$ of $F$ lies in $T^1$ while the edge $a_i$ lies in $T^2$. Since the edge $a_2^2 \subset T^2$ has one endpoint in the interval $(e_2^2, e_6^2) \subset v_3^2$ and its other endpoint in the interval $(e_4^2, e_3^2) \subset v_2^2$, represented by open circles in Fig. 14 by following the corners of $F$ up along the strings $I_{2,3}'$ and $I_{3,4}'$ in Fig. 12(a) (with $t = 4$) we can see that $a_{i-1}^1 \subset T^1$ has one endpoint in the interval $(e_1^1, e_5^1) \subset v_2^1$ and its other endpoint in the interval $(e_3^1, e_2^1)$ of $v_3^1$. The possible locations of the endpoints of $a_{i-1}$ around $v_2$ and $v_3$ are indicated by open squares in Fig. 14; this situation is
impossible since the edges of $\overline{e_1} \cup \{a_i\} \subset T$ separate the endpoints of $a_{i-1}$. This contradiction shows that $|\overline{e}| \leq t + 1$ holds in $G_S$.

\[\square\]

3.3. **The manifolds** $(M_t, T_0)$, $t \geq 4$. For each $t \geq 4$ let $T$ be a $t$-punctured torus and $M_t$ the manifold $M_T/\psi$, where $M_T = T \times I$ is the solid handlebody with complete disk system $F'_1, F'_2, \ldots, F'_{t+1}$ of Fig. 12(a) considered in Case 2 of Lemma 3.3 and $\psi : T^1 = T \times 0 \to T^2 = T \times 1$ is the homeomorphism uniquely determined up to isotopy by the conditions $\psi(v_i^1) = v_i^2$ for $1 \leq i \leq t$ and $\psi(e_j^1) = e_j^2$ for $1 \leq j \leq t + 1$. The basic properties of $M_t$ are summarized in the next lemma.
Lemma 3.5. The manifold $M_t$ is orientable and $\partial M_t$ is a torus $T_0$. Moreover, the pair $(M_t,T_0)$ is irreducible and $T = T^1/\psi$ is a properly embedded essential punctured torus in $(M_t,T_0)$.

Proof. If $T^1 \subset \partial M_T$ and $T^2 \subset \partial M_T$ are oriented by the normal vectors $\vec{N}^1, \vec{N}^2$ in Fig. 14 then the conditions $\psi(v^1_i) = v^2_i$ for $1 \leq i \leq t$ and $\psi(e^1_j) = e^2_j$ for $1 \leq j \leq t + 1$ imply that the homeomorphism $\psi : T^1 \to T^2$ is orientation preserving, hence $M_t$ is orientable. Clearly, $\partial M_t = (I'_{1,2} \cup I'_{2,3} \cup \cdots \cup I'_{t,1})/\psi$ is a single torus $T_0$, and hence $T = T_1/\psi$ is a properly embedded torus in $(M_t,T_0)$.

If $T = T_1/\psi$ compresses in $M_t = M_T/\psi$ then, as $M_T$ is obtained by cutting $M_t$ along $T$, it follows that $T^1$ or $T^2$ compresses in $M_T$, which is not the case since $M_T = T \times I$. Therefore $T$ is incompressible in $M_t$.

Since $M_T$ is a handlebody, hence irreducible, it now follows that $M_t$ is also irreducible. So, if $T_0$ compresses in $M_t$ then $M_t$ must be a solid torus, contradicting the fact that $T$ is incompressible in $M_t$. Therefore $T_0$ is incompressible and $T$ is essential in $(M_t,T_0)$.  

We now show that the bigons in the complete disk system of $M_T$ give rise to a twice-punctured torus $S$ embedded in $(M_t,T_0)$.

Lemma 3.6. For each $t \geq 4$ there is a separating, essential, twice-punctured torus $S$ in $(M_t,T_0)$ such that $\Delta(\partial T, \partial S) = 3$ and $G_S = S \cap T \subset S$, $G_{T,S} = S \cap T \subset T$ are the graphs shown in Figs. 16 and 17.

Proof. Let $t \geq 4$, and consider the faces $F'_1, \ldots, F'_{t+1}$ and the edges $e^1_j$ for $1 \leq j \leq t + 2$, $i = 1,2$, embedded in the handlebody $M_T = T \times I$ and $T^i$, respectively, as shown in Fig. 12(a), with $\psi(e^1_j) = e^2_j$ for all $j$. We add the following objects to $M_T$ as indicated in Fig. 15:

- one bigon face parallel to $F'_i$ for $i = 2, t$,
- two bigon faces parallel to $F'_i$ for $3 \leq i \leq t - 1$,
- one more edge parallel to each of the edges $e^2_3, e^3_3$ and $e^1_1, e^1_{t+1}$.

Therefore, in $T$, the edges $e_1, e_{t+2}$ get no parallel edges, each of the edges $e_i$ for $i = 2, t + 1$ gets one parallel edge denoted by $e'_i$, and each of the edges $e_i$ for $3 \leq i \leq t$ gets two parallel edges denoted $e'_i, e''_i$. The new collections of edges $e_i, e'_i, e''_i$ produce graphs $G_i \subset T^i$ for $i = 1, 2$ isomorphic to the graph in Fig. 16 such that, after a slight isotopy of the edges $e'_i, e''_i$ if necessary, satisfy $\psi(G^1) = G^2$.

Now connect all the old and newly added edges of $T^1, T^2$ via mutually disjoint corners as shown in Fig. 15. Observe that the 6-cycle $C$ in $\partial M_T$ containing the edges $e^2_1, (e'_1)^1, (e'_3)^2, e^1_{t+2}, (e'_2)^2, (e''_1)^1$ is disjoint from the complete disk system $F'_1, \ldots, F'_{t+1}$ of $M_T$, hence $C$ bounds a 6-sided disk ‘face’ $F'_C$ in $M_T$ disjoint from all the other bigons in $M_T$.

In this way we obtain a collection $\mathcal{F}$ of $3t - 2$ disjoint disk faces embedded in $M_T$: $2(3) + 3(t - 3) = 3t - 3$ bigons and one 6-sided disk face. The condition $\psi(G^1) = G^2$ guarantees that $S = \mathcal{F}/\psi \subset (M,T_0)$ is a properly embedded surface which intersects $T$ transversely in the graph $G_{T,S} = T \cap S \subset T$ of Fig. 16. Moreover, the collection of corners of faces in $\mathcal{F}$ whose
endpoints are capped with a small closed disk in Fig. [15] form one boundary component $\partial_1 S$ of $S$, while the remaining corners form a second boundary component $\partial_2 S$; thus $|\partial S| = 2$.

Therefore all faces of the graph $G_S = S \cap T \subset S$ are disks, and $G_S$ has 2 vertices, $3t$ edges, and $|F| = 3t - 2$ faces, so $\hat{S}$ is a surface with Euler number 0; since each vertex of $G_{T,S}$ has degree 6, it follows that $\Delta(\partial S, \partial T) = 3$.

Now, the faces of $G_{T,S}$ can be colored black or white as shown in Fig. [16] and this coloring induces a corresponding black and white coloring of the components of $M_T \setminus \cup F$ such that each face in $\mathcal{F}$ is adjacent to opposite colored components, which implies that $S$ separates $M_t$ and hence that $S$ is a 2-punctured separating torus. We denote by $S^B, S^W$ the closures of the components of $M_t \setminus S$.

The graph $G_S$ can be determined by following the boundary circle $\partial_1 S$ in Fig. [15] in the direction of increasing labels; starting at the endpoint of $e_1^1$ in $v_1 \subset \partial T$ we find that the collections of edges \{\begin{align*} e_1, \ldots, e_{t+2} \end{align*}\}, \{\begin{align*} e_2', \ldots, e_{t+1}' \end{align*}\}, and \{\begin{align*} e_3'', \ldots, e_t'' \end{align*}\} form distinct parallelism...
classes in $T$ and are read in this order as we traverse the circle $\partial_1 S$ in Fig. 15, and so $G_S$ must be the graph shown in Fig. 17.

For each $\ast \in \{B,W\}$ the manifold $S^\ast$ is irreducible with $\partial S^\ast$ a genus two surface. Notice that $G_{T,S}$ contains Scharlemann cycle disk faces in $S^\ast$ of distinct lengths; any such Scharlemann cycle disk face is nonseparating in $S^\ast$, whence $S^\ast$ is a genus two handlebody with complete disk system any pair of Scharlemann cycle disk faces of $G_{T,S}$ in $S^\ast$ of different sizes.

For $S^B$ we can take as complete disk system the bigon $x$ of $G_{T,S}$ containing the edge $e_2$ and the $t$-sided face $y$ containing $e_{t+2}$, so that $\pi_1(S^B) = \{x, y \mid \} \text{ and } \partial_1 S \subset \partial S^B$ is represented by the word in $\pi_1(S^B)$ obtained by reading the consecutive intersections of $\partial_1 S$ with the disks $x$ and $y$. Following $\partial_1 S$ around in Fig. 15 we can see that $\partial_1 S = (yx)^2y^{t-2}$, which is not a primitive word in $\pi_1(S^B) = \{x, y \mid \}$; therefore $S$ is incompressible in $S^B$ by [4, Lemma 5.2]. Similarly, in $S^W$ we can take as complete disk system the white bigon $X$ in $G_{T,S}$ with corners along $v_3$ and $v_4$ and the white $t+4$-sided face $Y$ containing $e_{t+2}$; we then compute that $\partial_1 S \subset \partial S^W$ is represented by the word $Y^{t+3}XYX$ in $\pi_1(S^W) = \{X, Y \mid \}$, which is not primitive, so $S$ is incompressible in $S^W$ too. Therefore $S$ is incompressible, hence essential, in $M_t$. \hfill \Box

We now complete the proofs of the remaining main results of this paper.

**Proof of Proposition 1.1.** Parts (1), (2) and (3) follow immediately from Lemmas 3.5 and 3.6. By Lemma 2.2, the manifold $M_t(\partial T)$ is a torus bundle over a circle with fiber $\tilde{T}$. Finally, by [5] Proposition 3.4(c), if $(P \times S^1/[m], T_0)$ contains two $\mathcal{K}$-incompressible tori $T, T'$
then $\Delta(T, T') \in \{0, 1, 2, 4\}$; since $M_t$ contains the essential torus $S$ with $\Delta(\partial T, \partial S) = 3$, it follows that $M_t$ is not homeomorphic to any of the manifolds $P \times S^1/[m]$, so part (4) holds.

Proof of Proposition 1.2: That $T$ is positive follows from Lemma 2.1(c). Suppose $(M, T_0)$ does not satisfy parts (1) and (3) of the proposition. By Lemmas 2.3(1) and 3.1 we then have that $t \geq 4$, and so by Lemmas 2.3(2), 3.2, 3.3 and the definition of $M_t$, we have that $(M, T_0)$ is homeomorphic to $(M_t, T_0)$. Since, by Proposition 1.1 $M_t$ is not a manifold of the form $P \times I/[m]$, the bound $|\tau| \leq t+2$ holds for all negative edges $\tau$ of $G_R$ by Corollary 3.4. Therefore part (2) holds.

Proof of Lemma 1.4: Assume parts (1), (2) and (4) of the lemma do not hold; then $t \geq 4$ and $(M, T_0) \approx (M_t, T_0)$ by Proposition 1.2(2), and there is a negative edge in $G_Q$ of length $|\tau| \geq t + 2$. By Corollary 3.4 we then have that $|\tau| = t + 2$, so the lemma holds.

References

[1] C. M. Gordon, Boundary slopes of punctured tori in 3-manifolds, Trans. Amer. Math. Soc. 350 (5) (1998) 1713–1790.
[2] A. Hatcher, Notes on basic 3-manifold topology, Available at http://www.math.cornell.edu/hatcher/3M/3Mdownloads.html (2000).
[3] S. Lee, S. Oh, M. Teragaito, Reducing Dehn fillings and small surfaces, Proc. London Math. Soc. 92 (1) (2006) 203–223.
[4] L. G. Valdez-Sánchez, Seifert Klein bottles for knots with common boundary slopes, in: Proceedings of the Casson Fest, Vol. 7 of Geom. Topol. Monogr., Geom. Topol. Publ., Coventry, 2004, pp. 27–68 (electronic).
[5] L. G. Valdez-Sánchez, Toroidal and Klein bottle boundary slopes, Topology Appl. 154 (3) (2007) 584–603.
[6] L. G. Valdez-Sánchez, Toroidal boundary slopes at distance 3: the positive case, preprint, in progress.

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