Topology Optimization with Frictional Self-Contact

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Abstract

Contact-aware topology optimization faces challenges in robustness, accuracy, and applicability to internal structural surfaces under self-contact. This work builds on the recently proposed barrier-based Incremental Potential Contact (IPC) model and presents a new self-contact-aware topology optimization framework. A combination of SIMP, adjoint sensitivity analysis, and the IPC frictional-contact model is investigated. Numerical examples for optimizing varying objective functions under contact are presented. The resulting algorithm proposed solves topology optimization for large deformation and complex frictionally contacting scenarios with accuracy and robustness.

Keywords: Topology optimization, Frictional contact, Self contact, Adjoint sensitivity analysis

1 Introduction

Topology optimization (TO) seeks to optimize material structural designs given user-specified inputs such as external loads and boundary conditions. It has been widely studied and developed for solving mechanical design problems across engineering fields (Sigmund and Maute, 2013; Li et al, 2021a). Due to the often impractical assumption of small deformation, existing studies largely ignore contacting mechanisms. When large deformation is considered, however, the importance of dealing with contact in structural topology optimization becomes immediately apparent. In addition to preventing non-physical behaviors, i.e., interpenetrations, resolution of contact leads to different optimal structural designs that correctly take contact into account. Despite the need to accurately model contact to predict real-world behaviors, there has been little progress in optimizing topology with contact, especially when self-contact is required during deformation. The main challenges lie in (1) the lack of an accurate and robust model of contact that can be included in TO and (2) the appropriate resolution of complex contact behaviors, especially considering self-contact with friction (Bluhm et al, 2021).
Fig. 1: Schematic graphical illustrations of three contact methods: mortar (left), air-mesh (mid), and IPC (right). Note that the contact boundary for the traditional mortar method must be manually specified and is typically a subset of the object boundary, while for IPC the contact boundary is the exact full boundary of the discrete mesh. The dashed line in the IPC graph is $d$–away from the boundary of the solid structure, within which contact is treated.

The literature on contact-aware topology optimization focuses primarily on two types of methods: mortar methods and fictitious domain methods. Fig. 1 shows a schematic illustration of both, as well as the IPC contact method we apply. Traditional mortar methods are generally limited to modeling the contact between a moving body and a fixed obstacle, often requiring a pre-specification (labeling) of contact surfaces. For each structural piece $i$, mortar methods pre-divide its boundary $\partial \Omega_i$ into $\Gamma_i \subseteq \partial \Omega_i$ where a potential contact, and its complement $\Gamma^c_i \subseteq \partial \Omega_i$ where contact is not considered and so cannot be resolved. Gap constraint functions are created, and a constrained optimization problem is solved. Satisfying the gap constraint ensures that nodes on the moving surface must not penetrate the element faces of the opposing obstacle surface (Hallquist et al., 1985), thus mimicking the contact between $\Gamma_i$'s. As contact interfaces require pre-specification, applications of mortar methods primarily focus on modeling either the interaction between elastic and fixed bodies as in (Kristiansen et al., 2020) or simple external contact such as (Fernandez et al., 2020; Niu et al., 2019; Mankame and Ananthasuresh, 2004). In addition, it is challenging for mortar methods to handle complex problems with nonlinear deformations and frictional effects. For example, Luo et al. (2016) model contact with nonlinear springs for large deformations of hyperelastic bodies but cannot extend to frictional-contact cases, while Han et al. (2022) performs a node-to-node frictional analysis, but are limited to linear elasticity.

Fictitious domain methods take a different path treating void regions between colliding bodies as a soft material with small stiffness. When two potentially colliding surfaces approach each other, the (filled) void region is compressed and so exerts large repulsion forces. The fictitious domain method was first introduced by Pagano and Alart (2008) to resolve self-contact. Many variations have followed. Wriggers et al. (2013) introduced the third medium contact method to handle external contact between bodies, which was then applied to self-contact (Bluhm et al., 2021; Müller et al., 2015). Despite the ability to resolve self-contact, these methods generally require manual hand-tuning of air-mesh parameters to avoid locking and parasitic transfer of non-physical forces. These methods also face significant challenges in modeling friction. Furthermore, the air mesh can potentially introduce large errors under significant distortions such as extreme shearing. Additional void regularization techniques are usually needed to alleviate such problems (Kruse et al., 2018; Wriggers et al., 2013).

Recently, Li et al. (2020a) propose a primal barrier-based Incremental Potential Contact (IPC) model for capturing the frictional contact of finite-strain elastic solids. Their method applies a smooth distance-based potential energy combined with a barrier-aware line search to avoid intersecting trajectories between surface primitives. Using a $C^2$ localized barrier, IPC automatically responds with contact forces between geometric pairs closer than a user-specified distance threshold and includes a corresponding variational friction model. As a result, IPC circumvents the
difficulties covered above while providing guaranteed resolution of all contacting geometries. Note that while IPC was originally developed for elastodynamics, we have modified it here to solve for static force equilibrium in topology optimization. We assume hyperelasticity and present both mathematical and algorithm details for incorporating the IPC formulation into an existing topology optimization framework.

To summarize, we propose a new contact-aware topology optimization framework that can handle complex frictional contact scenarios, including external contacts and self-contacts under large deformation. A narrow-band process is adopted to define contact boundaries that evolve along the optimization procedure. An artificial timestep method is developed to properly model frictions within the static simulation scenario. Additional contributions include a strain limiting mechanism for tackling numerical difficulties introduced by low-density elements. We present numerical experiments to demonstrate the efficacy of the presented method.

2 Problem Statement

Given a design domain $\Omega_0$, a standard density-based topology optimization problem seeks an optimal material distribution $\rho : \Omega_0 \rightarrow [0, 1]$ within $\Omega_0$ such that, under force equilibrium and a volume constraint, an applied objective function $F$ is minimized. Many approaches only consider external forces $f_{\text{ext}}$ and internal elastic forces $f_{\text{int}}$. Here we additionally include (normal-direction) contact forces and (tangential-direction) frictional forces. Formally, the problem is

$$\min_{\rho} F(\rho, u(\rho)) \quad \text{s.t.}$$

$$\begin{align*}
    f_{\text{int}}(u) + f_{\text{ext}} + f_{\text{cont}}(u) + f_{\text{fric}}(u) &= 0 \\
    u|_{\Gamma} &= u_0 \\
    V(\rho) &\leq \hat{V},
\end{align*}$$

where $u$ is the displacement field, $\rho$ is the unknown scalar field describing the material allocation in $\Omega_0$, and $F(\rho, u(\rho))$ is the design objective function of interest. Here $f_{\text{cont}}$ and $f_{\text{fric}}$ are contact and friction forces applied on one or more pieces of structures in $\Omega_0$ due to their relative contact. A portion $\Gamma$ of the structure boundary $\partial \Omega_0$ will have prescribed Dirichlet boundary condition $u_0$, while $V$ is the material’s volume, $\int_{\Omega_0} \rho dX$, that is constrained to be less than a user-specified upper bound criterion, $\hat{V}$.

For actual manufacture, the material density $\rho$ should be close to either zero or one. Consequently, a binarization is conducted to the entries in $\rho$ for re-evaluating the objective function.

Problem (1) applies to many widely studied topology optimization formulations. For instance, if $F$ is chosen to be the total elastic energy and $f_{\text{cont}} = f_{\text{fric}} = 0$, then we retrieve the classical structural compliance minimization problem (Andreassen et al, 2011; Ferrari and Sigmund, 2020; Sigmund, 1997).

3 Background

3.1 SIMP

The Solid Isotropic Material with Penalization Method (SIMP) (Andreassen et al, 2011; Sigmund, 2001; Ferrari and Sigmund, 2020) is widely applied in topology optimization. The local cell density $\rho_c \in [0, 1]$ represents the material distribution on an Eulerian grid. SIMP assumes that Young’s modulus of each grid cell $c$ is proportional to a polynomial of its cell density, i.e., $E_c = E_0 \rho_c^p$, where $E_0$ is the base Young’s modulus of the solid material. Typically $p$ is chosen to be 3 to reduce intermediate density values. Further enhancement of sharp material boundaries is achieved by the addition of a relaxed Heaviside projection (Ferrari and Sigmund, 2020). In our work, we follow these conventions and set $p = 3$; see Sec. 4.1.

It is well known that SIMP supports and can, in practice, generate unrealistic optimal solutions with checkerboard artifacts. A range of smoothing filters have been proposed to alleviate this issue (Andreassen et al, 2011; Sigmund, 2007; Sigmund and Maute, 2012); we also apply a density filter and a sensitivity filter in our work, see Sec. 4.5.

3.2 IPC

Contact Potential

The Incremental Potential Contact (IPC) model (Li et al, 2020a) is a potential-based contact model that guarantees nonpenetration for all configurations. For each surface contact pair, the potential
A point and \( e = (y, z) \) is an edge with endpoints \( y \) and \( z \). A point \( p \) non-incident to edge \( e \) implies that both \( (p, y) \) and \( (p, z) \) do not form boundary edges. The contact distance is then the minimal Euclidean distance between \( p \) and \( e \), i.e.,

\[
d = \min_{\beta} \|p - (y + \beta(z - y))\| \quad \text{s.t. } 0 \leq \beta \leq 1. \tag{4}
\]

This constrained optimization problem has two explicit solutions, classified as a point-point (PP) and a point-line (PL) distance. If \( \beta = 0 \) or 1, then the distance is a PP-distance, and

\[
d = \|p - y\| \text{ or } \|p - z\|; \tag{5}\]

otherwise, the distance is a PL-distance, and

\[
d = \frac{\|(y - p) \times (z - p)\|}{\|y - z\|}. \tag{6}\]

These two types of distances are illustrated in Fig. 2.

**Friction**

In the IPC framework, friction forces are defined per contact pair. For each contact pair \( k \), a consistently oriented sliding basis \( T_k(x) \in \mathbb{R}^{dm \times (d - 1)} \) is constructed, where \( d = 2 \) or 3 is the problem dimension and \( m \) is the number of nodes in the system. The corresponding local frictional force \( f_{\text{fric}}^k \) is then defined in terms of \( \tilde{\mathbf{u}}_k \), the local relative sliding displacement orthogonal to the distance gradient, and its corresponding discrete velocity \( \tilde{\mathbf{v}}_k = \tilde{\mathbf{u}}_k / \Delta t \). As suggested in (Moreau, 2011; Goyal et al, 1991a,b), \( f_{\text{fric}}^k \) can be defined by maximizing dissipation rate subject to the Coulomb constraint:

\[
f_{\text{fric}}^k(x) = T_k(x) \arg \min_{\beta \in \mathbb{R}^{d-1}} \beta^T \tilde{\mathbf{v}}_k \quad \text{s.t. } \|\beta\| \leq \mu \lambda_k, \tag{7}\]

where \( \lambda_k \) is the magnitude of contact force, and \( \mu \) is the friction coefficient. This can be equivalently written as

\[
f_{\text{fric}}^k(x) = -\mu \lambda_k T_k(x) f(\|\tilde{\mathbf{u}}_k\|) \left( \frac{\tilde{\mathbf{u}}_k}{\|\tilde{\mathbf{u}}_k\|} \right), \tag{8}\]

where the last term on the right-hand side can be any unit vector if the denominator vanishes. The nonsmooth friction magnitude function \( f \) is 1 if

**Distances**

An elastic body is discretized in this work by an axis-aligned regular grid. In 2D, the boundary of a structure consists of vertices and axis-aligned line segments. Each possible contact pair is then a non-incident point-edge pair \( \{p, e\} \), where \( p \) is
that low-density elements can be systematically removed and reintroduced. It has

\[ H(\rho; \beta) = \frac{\tanh(\frac{1}{2}\beta) + \tanh(\beta(\rho - \frac{1}{2}))}{2 \tanh(\frac{1}{2}\beta)} \]  

(13)

to further help with convergence.

Cell densities \( p_c \) are the design variables for material distribution. We follow Li et al (2021b) and solve for force equilibrium via the Material Point Method. Quadrature points within each computational cell share the same density value to avoid subcell QR-pattern artifacts (Li et al., 2021b). That is, \( p_q = p_c \) for all quadrature points \( q \) that belong to the same cell \( c \).

For a hyperelastic object, the total elastic energy induced by deformation is

\[ \epsilon_{\text{elasticity}}(\rho, u) = \int_{\Omega_0} \Psi(F) dX, \]  

(14)

where \( \Psi \) is the applied energy density function determined by the constitutive model, and \( F \) is the deformation gradient,

\[ F = \frac{\partial x}{\partial X} = I + \frac{\partial u}{\partial X}. \]  

(15)

with \( x(X) \) the world space mapping of a material point \( X \in \Omega_0 \) and \( u(X) = x(X) - X \) the displacement field. Following Li et al (2021b), we approximate the elastic energy by

\[ \epsilon_{\text{elasticity}}(\rho, u) \approx E_0 \sum_q \tilde{\rho}_q^3 \Psi(F_q)V_q, \]  

(16)

where \( \tilde{\rho}_q = H(\rho_q; \beta) \). Without loss of generality, we apply the compressible neo-Hookean energy density in this work:

\[ \Psi(F) = \frac{\mu}{2} \text{tr}(F^TF - d) - \mu \log(J) + \frac{\lambda}{2} \log^2(J), \]  

(17)

where \( J = \det(F) \), \( d = 2 \) or \( 3 \) is the dimension of the problem, and \( \mu \) and \( \lambda \) are the lamé parameters. Note that our numerical procedure guarantees a positive \( J \) throughout; see Sec. 4.4.

4.2 Incorporation of Contact

Contact Boundary Detection

The contact boundary is evolved based on the re-allocated density field during topology optimization iterations, after which it is fed into the contact solver. Bruns and Tortorelli (2003) introduced the narrow-band process to topology optimization so that low-density elements can be systematically removed and reintroduced. It has
then been widely used, for example, in (Liu et al, 2018; Zhang et al, 2021; Zhou et al, 2016), for filtering out low-density cells and thus avoiding singular stiffness matrices.

Our method adds a boundary detection procedure to the narrow-band process. Let $G$ denote the graph consisting of all cells $c$ defined by the actual adjacency between cells. Given a set of seed cells $S \subseteq G$ (which, by default, is chosen to be where boundary conditions are specified), a depth-first search (DFS) is performed to find all largest connected components $L_i \subseteq G$ such that

$$L_i \cap L_j = \emptyset \text{ if } i \neq j,$$

where for each $i$,

$$\rho_c > \eta, \forall c \in L_i,$$

and

$$\exists c \in S \text{ such that } c \in L_i.$$

Here $\eta$ is a thresholding parameter for density. Cells with density lower than $\eta$ and cells that are not connected to any component $L_i$ are removed and treated as zero-density void region. Meanwhile, the boundaries for all components $L_i$ are identified based on mesh connectivity to form the surface boundary for the contact potential. Further, each component $L_i$ is guaranteed to have boundary conditions, thus ensuring a nonsingular stiffness matrix. See Fig. 3 for an illustration of the full procedure.

**Contact Potential**

The IPC model is then integrated into the system by utilizing the detected codimension-1 boundary geometries. As covered in section 3.2, distances are calculated differently for point-point (PP) and point-line (PL) cases. Therefore, we divide the set $C$ of all the non-incident point-edge pairs into two groups $C_{PP}$ and $C_{PL}$ containing only PP pairs and PL pairs, respectively. The total contact potential (3) is then fully separated as

$$e_{\text{contact}}(u) = \sum_{i \in C_{PP}} h\hat{d}_i b(d_i, \hat{d}) + \sum_{j \in C_{PL}} h\hat{d}_j b(d_j, \hat{d})$$

and can be separately evaluated. Classifying the two cases at the energy level ensures the non-ambiguous evaluations of their gradient and Hessian during static solves (See Sec. 4.4).

**Frictional Contact via Artificial Timesteps**

The IPC friction model is parameterized by sliding velocities. While there is no velocity at a static solution, final resting equilibria are supported by friction forces given by the model’s sticking conditions. Inspired by (Fang et al, 2021), we propose to find equilibria under friction via artificial time-stepping. We apply a sequence of $N$ quasi-static solves over an artificial time period $t \in [0, T]$. In the following, we use $T = 1$ for all examples. Specifically, we solve a series of nodal
Fig. 4: An experiment showing the convergence of quasi-static solves under friction. Left: A square of width 20 cm is aligned against a fixed wall, their distance being 0.1\hat{d}. A displacement of \( u = 20 \) cm is prescribed on the top boundary of the square. \( T = 1, \mu = 0.2 \) and \( \epsilon_v = 10^{-6} \). Right: A sequence of \( N \) quasi-static solves are performed for different values of \( N \). Convergence can be observed as \( N \) becomes larger. For a balance between accuracy and efficiency, we choose \( N = 10 \).

4.3 Strain Limiting Relaxation

When large internal contact is present, cells with tiny densities near the contact interface tend to experience extreme deformation. This may cause numerical difficulties for the static solver. To alleviate this issue, we add a strain-limiting energy to those fragile cells to moderate distortion (Bridson et al, 2002; Goldenthal et al, 2007). Here we follow Li et al (2020b) and define a \( C^2 \) scalar function

\[
\psi(\sigma; \hat{s}, \bar{s}) = \begin{cases} 
-\left(\frac{\hat{s}-\sigma}{\hat{s}-\bar{s}}\right)^2 \log\left(\frac{\hat{s}-\sigma}{\hat{s}-\bar{s}}\right), & \sigma \geq \hat{s} \\
0, & \sigma < \hat{s}
\end{cases}
\]

so that \( \psi(\sigma) > 0 \) when \( \sigma > \hat{s} \) and \( \psi(\sigma) \to \infty \) when \( \sigma \to \bar{s} \); see Fig. 5. Given parameters \( \hat{s}, \bar{s}, \hat{p}, \) and \( \bar{p} \), the strain-limiting energy density function \( \Psi^{SL} \) is thus defined as

\[
\Psi^{SL}(F) = \Psi^{SL}(\sigma_i) = \sum_{i=1}^{d} \psi(\sigma_i; \hat{s}, \bar{s}) + \psi(-\sigma_i; -\hat{p}, -\bar{p}),
\]

where \( \sigma_i \)'s are the principal stretches defined by the singular values of \( F \). Intuitively, the strain is limited in a way that \( \sigma_i \) is not allowed to go beyond \( \bar{s} \) or fall below \( \hat{p} \). These bounds are guaranteed by the numerical procedure we apply, described in Sec. 4.4.
We present in Fig. 7 an experiment to demonstrate that the addition of strain-limiting as relaxation does not significantly alter the deformation of non-relaxed cells.

### 4.4 Static Solve with Projected Newton’s Method

Displacement $u$ under static force equilibrium is solved for sensitivity analysis.

In our framework, as each force (elasticity, the normal contact force, and the tangential frictional force) is associated with a corresponding potential energy, the force equilibrium can be expressed as

$$- \frac{\partial e_{\text{total}}}{\partial u}(\rho, u) + f_{\text{ext}} = 0,$$

(29)

where $e_{\text{total}} = e_{\text{elasticity}} + e_{\text{contact}}$. To incorporate forces due to friction or strain-limiting, it suffices to add the corresponding $e_{\text{friction}}$ or $e_{\text{SL}}$ to $e_{\text{total}}$. Solving (29) is equivalent to minimizing

$$\min_{u} e_{\text{total}}(\rho, u) - u \cdot f_{\text{ext}}$$

subject to boundary conditions. Following Li et al (2021b), we use the projected Newton’s method to solve the minimization problem (30), where the Hessian matrix is projected to be symmetric positive definite (SPD) and a line search procedure is performed to guarantee global convergence (Nocedal and Wright, 1999). Note that for each energy term $e_i$ in $e_{\text{total}}$, a corresponding stepsize upper bound $\alpha_i$ is needed to ensure that the energy $e_i$ is well defined. For instance, in the contact energy $e_{\text{contact}}$, the additive continuous collision detection method (Li et al, 2020b) is used to bound the stepsize to prevent trajectory intersection. For neo-Hookean elasticity and strain-limiting relaxation, an upper bound is derived to prevent the $\log(\cdot)$ term from approaching $\infty$. Finally, the global stepsize upper bound, $\alpha$, is the tightest determined bound, i.e.,

$$\alpha = \min_i \{\alpha_i\}.$$  

(31)

### 4.5 Sensitivity Analysis

Applying the adjoint method, we compute the sensitivity analysis for a general objective function $G(\rho, u(\rho))$ with respect to $\rho_c$, the density of cell $c$,
Fig. 7: An experiment using strain-limiting relaxation. Left: A schematic illustration. Mid: Simulation without strain-limiting. Right: Simulation with strain-limiting. The deformations of cells without relaxation are visually identical in the two cases despite the expected minor quantitative discrepancy.

\[
\frac{dG}{d\rho_c} = \left[ \frac{dE_c}{d\rho_c} \right]^T \frac{dG}{dE_c}\quad (32)
\]

with

\[
\frac{dG}{dE_c} = \frac{\partial G}{\partial E_c} - \frac{\partial^2 e}{\partial E_c \partial u} \left[ \frac{\partial^2 e}{\partial u^2} \right]^{-1} \frac{dG}{du} \quad (33)
\]

and

\[
\frac{dE_c}{d\rho_c} = 3E_0H^2(\rho_c, \beta) \frac{\partial H(\rho_c, \beta)}{\partial \rho_c} \quad (34)
\]

where \( E_c \) is defined to be \( E_c = E_0H^3(\rho_c, \beta) \) and \( e = e_{\text{total}} \) as in (30). Below we state the sensitivity for three particular objective functions that will be considered in our experiments.

**Compliance** If compliance \( G = e_{\text{elasticity}} \) is chosen to be the objective function, then

\[
\frac{\partial G}{\partial E_c} = \sum_{q \text{ in cell } c} V_q \Psi(F_q) \quad (35)
\]

and \( -\frac{\partial G}{\partial u} \) is the elastic force.

**Reaction Force** The reaction force on node \( B \) in direction \( n \) is defined to be

\[
R_{B,n} = \frac{\partial e}{\partial u_B} \cdot n \in \mathbb{R} \quad (36)
\]

Note that the node \( B \) must be a Dirichlet node, as a non-Dirichlet node satisfies force equilibrium and hence has zero reaction force. The sensitivity analysis for \( R_{B,n} \) is therefore

\[
\frac{dR_{B,n}}{dE_c} = \frac{\partial^2 e}{\partial E_c \partial u_B} \cdot n - \frac{\partial^2 e}{\partial E_c \partial u} \left[ \frac{\partial^2 e}{\partial u^2} \right]^{-1} \frac{\partial^2 e}{\partial u \partial u_B} \cdot n \quad (37)
\]

Let \( D \) denote the set of all nodes where the Dirichlet boundary condition is applied. Forces applied on multiple nodes \( B \subset D \) can be counted together as

\[
R = \sum_{B \in B \subset D} R_{B,n}, \quad \frac{dR}{dE_c} = \sum_{B \in B \subset D} \frac{dR_{B,n}}{dE_c} \quad (38)
\]

**Volume Fraction** One important constraint in topology optimization is the volume constraint. The requirement is usually stated as that the total volume fraction does not exceed a threshold \( \hat{V} \), i.e.,

\[
g(\rho) = \frac{\sum_i E_c}{\sum_i 1} \leq \hat{V} \in (0, 1) \quad (39)
\]

It follows that

\[
\frac{dg}{dE_c} = \frac{E_c}{\sum_i 1} \quad (40)
\]

where \( i \) loops over all cells.

### 4.6 Density filter, Sensitivity Filter, and Evolving Boundary

Following Sigmund (2007), density filtering and sensitivity filtering are implemented in our work. Given filter radius \( r_{\text{min}} \), the density field \( \rho_c \) is modified at the beginning of each iteration:

\[
\hat{\rho}_c = \frac{1}{\sum_{i \in N_c} H_{ci}} \sum_{i \in N_c} H_{ci} \cdot \rho_i \quad (41)
\]
and the sensitivities \( \frac{\partial F}{\partial \rho} \) are modified before feeding into optimizer:

\[
\frac{\partial \hat{G}}{\partial \rho_c} = \frac{1}{\max(\gamma, \rho_c)} \sum_{i \in N_c} H_{ci} \cdot \rho_i \frac{\partial G}{\partial \rho_c}, \tag{42}
\]

where

\[
H_{ci} = \max(0, r_{\min} - \text{dist}(c, i)), \tag{43}
\]

and

\[
N_c = \{ i \in \text{all cells}, \text{dist}(c, i) < r_{\min} \}. \tag{44}
\]

\( \gamma = 0.001 \) is a small number to avoid division by zero. Intuitively, the filtering functions as a local averaging of densities/sensitivities. As suggested in (Andreassen et al, 2011; Sigmund and Maute, 2012), it can ensure the existence of a solution and avoid formations of checkerboard patterns. We follow convention and choose \( r_{\min} \in [1.5, 3] \) (Li et al, 2021b). Moreover, in our framework, together with the narrow-band procedure, the filtering allows structures to naturally evolve (both shrink and enlarge) their boundaries, as shown in Fig. 8.

### 4.7 Optimizing Structures with MMA

We adopt the popular optimizer method of moving asymptotes (MMA) (Svanberg, 1987) to optimize structures. MMA is designed for general structural optimization problems with inequality constraints and box constraints. The original problem is approximated by a series of convex optimizations. At each iteration, two updated asymptotes are set up to constrain the searching interval. MMA typically requires careful parameter tuning for performance. Here we follow the setup in (Li et al, 2021b) and adopt an open-source C++ version of MMA (Dumas, 2018).

### 4.8 Overall Pipeline

The pipeline of this work is summarized in Algorithm 1.

#### Algorithm 1 Overall pipeline

1: Given: \( \rho^0, \hat{V}, f_{\text{ext}} \)

2: for \( j = 0, 1, 2, 3, \ldots \) do

3: Apply narrow-band to detect structure boundaries, see Sec. 4.2

4: Solve for static equilibrium \( u \) via projected Newton, see Algo.4.4

5: Compute \( \frac{dG}{dp}(\rho^j) \), see Sec. 4.5

6: \( \rho^{j+1} \leftarrow \text{MMA}(\rho^j, \frac{dG}{dp}(\rho^j)) \), see Sec. 4.7

7: end for

### 5 Results

#### 5.1 Comparison with Existing Contact Algorithms in Topology Optimization

Here we cover the advantages of the presented method compared to alternatives in topology optimization that handle contact via mortar or fictitious domain methods.

As discussed earlier in Sec. 1, a fundamental drawback of mortar methods is the required pre-specification of contact surfaces prior to simulation and so optimization. This restricts its application only to simple examples with just external contact obstacles where contact surfaces can be easily predetermined and assumed to be unchanged. Thus, once specified, contact boundaries are then generally unable to evolve, disallowing any large changes to be accounted for. Further difficulties then also arise when we require modeling of multiple contacting domains and/or when the accurate and precise resolution of contact requires intersection-free geometries. It is thus challenging to apply mortar methods to handle intricate and often changing internal contacts along evolving boundaries (Strömberg, 2013; Kristiansen et al, 2020).

Fictitious domain methods, on the other hand, are prone to numerical artifacts. As a comparison, we simulate a C-shaped structure deformed by pulling its top right corner downwards. Figs. 9a and 9b demonstrate the converged simulation results for this set-up using respectively an air-mesh and IPC model. The air-mesh model requires per-example fine-tuning of stiffness parameters to ensure that the resulting gap between geometries that should be in contact is neither too large (and
Fig. 8: The density and sensitivity filtering and self-evolution of boundary. Sensitivities \( \frac{dG}{d\rho} \) are only defined for non-void cells, so they are treated as zero for void cells. Sensitivity filtering and density filtering allow densities to spread from existing cells (e.g. cell \( A \)) to their neighbours (e.g. cells in circle). The originally void cells are now occupied by the structure, and the contact boundary detected by narrow-band procedure will enlarge correspondingly.

so not really in contact yet) nor negative (to avoid self-penetration artifacts) (Wriggers et al, 2013; Weißenfels and Wriggers, 2015). Moreover, the inversion of the domain and/or significant shearing of the fictitious domain is generally unavoidable. Bluhm et al (2021) can partially alleviate difficulties for air-mesh models by wrapping the domain with further layers of air-mesh, but these challenges remain.

Convergence of the forward simulation and accurate satisfaction of non-intersecting geometries are guaranteed independent of the choice of \( \hat{d} \). In turn, the parameter \( \hat{d} \) gives direct control of how close materials can be prior to application of contact forces (see dashed lines in Fig. 1). This enables users to decide how accurately conforming contact geometries should be per application. As smaller \( \hat{d} \) increases accuracy at the cost of more computation, it provides a parameter for directly controlling accuracy versus efficiency. In a similar manner, the convergence parameter for the Newton solve itself then allows a choice to balance efficiency versus accuracy for the simulation solves.

Another disadvantage of the air-mesh is that the fictitious domain elements generate non-physical artificial forces (and thus inaccurate deformations) when deformations of the domain are large. For example, when two boundaries are pulled sufficiently far away from each other, we see this effect in even simple examples like Fig. 9c. Clearly, these regions without material (voids) should not apply forces on the domain in these contexts. On the other hand, the IPC model, e.g. as simulated in Fig. 9d, ensures that forces are solely applied between true material surfaces and so avoids these artifacts altogether. In addition, by extending the simulated region and so increasing its range of deformation, we observe that air-mesh models also incur additional computational challenges. In our experiments, we see that for a single static solve, the air-mesh method generally takes about five to ten times more Newton iterations when compared to the IPC method to converge to the same tolerance for the same example. Finally, when it comes to modeling frictional contact air-mesh models lack the appropriate resolution of the necessary terms to model tangential resistance under contact. Friction modeling thus remains a major challenge for fictitious domain methods. Here, in contrast, the IPC model includes a direct and natural friction model (Sec. 4.2 and Sec. 5.4) that ensures proper coupling between well-defined normal forces and tangential friction forces.

5.2 Fixed-Interface Contact

We first test our method in an experiment designed to ensure internal contacts to the design domain will occur at pre-specified, fixed interface. Importantly, as our results will demonstrate, a resulting optimal design can still consider and include contact interfaces in other regions as well. For the experimental set up, please see Fig. 10. Here the design domain is of width \( L_1 = 1.8m \) and height \( H_1 = 1m \). An internal “always-void” region of width \( L_2 = 1m \) and height \( H_2 = 0.08m \) is applied in the center where we keep \( \rho = 0 \) fixed throughout. Downward forces of \( F_1 = 0.72N \)
are evenly loaded on the top surface, length $\frac{2}{3}L_1$, and corresponding upward forces of $F_2 = 1N$ are evenly loaded on the entire bottom face surface of the domain. A homogeneous Dirichlet boundary condition is enforced along both the two side walls. Simulation resolution is $180 \times 80$, with a mesh resolution of $h = 0.01m$. The IPC parameter $\hat{d}$ is $0.1h$, $\mu = 0$ and $E_0$ is set to 100. To facilitate the formation of contact supporting structures, two additional “always-solid” regions (marked in dark grey) are specified above and beneath the “always-void” region with a fixed density of $\rho = 1$. We also set another two thin layers of “always-solid” regions: one at the top and one at the bottom of the domain. Non-fixed portions of the design domain are then initialized with $\rho = 0.42$ with a volume constraint of 42%. Here, we begin with a goal objective of minimizing structure compliance with equilibrium given by the force balance of elastic and contact forces. Convergence can be observed after around 370 iterations, where both the density field (Fig. 11a) and the value of the objective function (Fig. 12a) approach invariance. A final binarization

$$B(\tilde{\rho}) = \begin{cases} 1 & \text{if } \tilde{\rho} \geq 0.5 \\ 0 & \text{if } \tilde{\rho} < 0.5 \end{cases}$$

is then applied to get the binary 0-1 solution shown in Fig. 11b. Compliance of the optimized solution decreases from a start of $5.234 \times 10^{-3}$ to $1.981 \times 10^{-3}$, while the binarized solution reduces a bit further to $1.976 \times 10^{-3}$ (the red horizontal line in Fig. 12a). The volume constraint $V(\rho)/\hat{V} \times 100\%$ is 100.09\% for the converged result.
and 100.01% for the binarized result. It can be observed in Fig. 12b that two surfaces designed for contact here do indeed touch closely along their entire interface. Here the final optimized deformed structure is primarily supported by two vertical beams connecting the top plate and the bottom plate and by contacts along the “fixed” contact interface. Peripheral structures then also connect the left and right (where the homogeneous Dirichlet condition is specified) to the bottom plate and to the vertical beams for additional support.

5.3 Two-Stage Min-Max Problem

Next, we apply our contact-aware topology optimization algorithm to design a structure that will handle switching from a loose configuration to close contact under varying magnitudes of a prescribed displacement. The problem setup is shown in Fig. 13. Here the design domain has length $L_1 = 1.0m$ and height $H_1 = 0.5m$, with an inset square of width $L_3 = \frac{2}{13}L_1$ and and supporting legs with bottom length $L_2 = \frac{6}{13}L_1$. The gap, initially set to be void, separates the inner square and the outer piece. The bottom layer of the inner square $\Gamma$ is set to be “always-solid”, and a displacement $u$ is prescribed for it. Here the design goal for this problem is to minimize the reaction force on $\Gamma$ for small displacement $u$ and, at the same time, to maximize reaction forces when $u$ is large. Specifically, the applied design objective is

$$\min_{\rho} (R_{\Gamma,n}(\rho, u = u_1n))^2 - (R_{\Gamma,n}(\rho, u = u_2n))^2,$$

where $n = (1, 0)$ is the unit vector pointing to the right with small and large displacements respectively $u_1 = \frac{1}{9}L_1$, and $u_2 = \frac{2}{9}L_1$. Here the grey region in Fig. 13 is initialized with $\rho = 0.32$, the applied volume constraint is 32%, with $E_0 = 100$, $\mu = 0$, and a discretization resolution of 180 × 90. Our problem setup thus motivates from a comparable design experiment in (Bluhm et al, 2021). Severe internal contact induced by the larger (but not smaller) displacement of the self-evolving boundary highlights the difficulty of this problem. Along with elasticity and the contact forces, we also apply the above-described strain-limiting relaxation to help reduce numerical difficulty from the large deformations. Convergence can be obtained after 850 iterations. We plot the density field after 290 iterations, 580 iterations, and 850 iterations, together with the binarized structure in Fig. 14 to show the optimization process.

Notice from Fig. 14d that the inner structure and the outer structure, as expected, are separated by a small gap. This reflects the trade-off that the optimizer takes when balancing reaction
forces under different values of displacement. The reaction force is minimized when a smaller displacement $u_1$ is prescribed. Ideally the two pieces should remain fairly isolated under prescribed displacement $u_1$ so that there will be no deformation and hence zero reaction force. On the other hand, the reaction force is maximized when a larger displacement $u_2$ is prescribed. Ideally the two pieces should be completely connected so that the prescribed displacement can yield the largest deformation (hence the largest reaction force). Balancing the two goals, the optimal design shown in Fig. 14d separates the inner piece and the outer piece by a gap just enough to achieve zero reaction force under displacement $u_1$, while allowing contact to take place for $u > u_1$. See the red curve in Fig. 15a. Also, note that compared with the initial design (reaction forces under the two levels of displacement are marked by black crosses), both the optimized design and the final binarized design have achieved significant gain in optimizing the objective.

Fig. 15a plots the reaction force on $\Gamma$ as the displacement specified on $\Gamma$ ranges from zero to $u_2\hat{n}$ for the optimized design and the binarized design. The reaction forces corresponding to $u_1$ and $u_2$ are accentuated, while the reaction forces for the initial design at $u_1$ and $u_2$ are also marked. For the initial design, there is no contact at $u_1$, resulting in a zero reaction force. At $u_2$ the reaction force is merely $0.0120N$. The optimized design has significant gain at $u_2$ where the reaction force becomes $0.2016N$, with a little sacrifice at $u_1$ where there is now slight contact. After removing the peripheral low-density cells, the binarized result re-achieves the state of no contact at $u_1$. Reaction force at $u_2$ is subsequently reduced to $0.1842N$, though still much larger than that in the initial design. $V(\rho)/V$ is 100.21% for the optimized design and 100.92% for the binarized design.

5.4 Screwdriver with Friction

Lastly, we explore the effect of friction using the proposed algorithm. The problem setup is shown in Fig. 16. The design domain is the outer geometry, representing a screwdriver that is rotated to drive the screw inside. The inner rectangle, representing a screw, is fixed in position and remains an “always-solid” region throughout the optimization process. A gap of width $1.5\hat{d}$ is set between the
inner and the outer geometries. (Recall that two objects are treated as in contact when their gap is less than \( \hat{d} \).) In this experiment, we set the design domain with \( L_1 = 92 \text{cm} \), \( H_1 = 78 \text{cm} \), \( L_2 = 44 \text{cm} \), and \( H_2 = 10 \text{cm} \), and a simulation resolution of \( 92 \times 78 \). Here, the inner layer of the design domain is also kept solid throughout the optimization to ensure the design obtains sliding contact between the screw and the screwdriver. The design domain (light grey region in Fig. 16) is initialized with \( \rho = 0.32 \) with a volume constraint set to be 32%. The rotation is prescribed by displacements of \((-16 \text{cm}, 16 \text{cm})\) at the lower left corner and \((16 \text{cm}, -16 \text{cm})\) the upper right corner while \( E_0 \) is set to 100. Here our design goal objective is to maximize the compliance of the final structure.

For optimizing, we find that incrementally allocating the prescribed displacement \( u \) into ten static solves (so each progressing displacement by \( \frac{1}{10} u \)) is sufficient to obtain convergent displacement field with friction. See Sec. 4.2 for details.

We solve the design optimization problem for three increasing values of friction: \( \mu = 0, \mu = 0.2, \) and \( \mu = 0.4 \). 440 iterations were run, and convergence can be observed in all three scenarios. Results are summarized in Table 1, with change in compliance demonstrated in Fig. 17. Post-evaluation reveals that the compliance of a binarized structure (first row in Fig. 1) differs no more than 2% from the compliance of a corresponding optimized result (second row in Table 1). The volume constraints for the optimized results are reported to be 99.98%, 99.98%, and 100.77% for \( \mu = 0, \mu = 0.2, \) and \( \mu = 0.4 \). The volume constraints for the binarized results are reported to be 99.96%, 100.01%, and 100.73%, respectively. Both the objective values and the volume constraints confirm that convergence has been reached in each case.

Several observations can be made from these results. First, in the last row of Table 1 we see that structures corresponding to larger friction coefficients \( \mu \) have less relative displacement at the contact interface. This demonstrates that the friction model, and so downstream optimization, captures the effect of larger \( \mu \) applying larger impedance. Second, we see that larger friction values generate optimized designs with more and smaller poles, while with no friction, we see even fewer and larger poles. One possible explanation for this effect is that compared with the case of no friction, over-clustering of mass around where contact happens will yield larger contact forces and hence larger local frictional impedance. This, in turn, will limit deformation to smaller regions. On the other hand, a more uniform distribution of supporting structures will allow deformation to be spread out and can hence achieve a larger compliance. Finally, as we see in Table 1, a larger friction coefficient \( \mu \) generates an optimal design with larger compliance. This is because when the same boundary condition is applied on the lower left and upper right corners, larger impedance due to larger friction yields more distortion within the “screwdriver,” and hence larger compliance.
Table 1: Results for different values of friction coefficient $\mu$. The top row shows the optimized design. The mid row shows the binarized design. The bottom row shows the world space deformation of binarized design.

6 Discussion

We have proposed a new framework to handle frictional self-contact in topology optimization. To do so, the IPC model is incorporated into the SIMP algorithm. The presented method provides the first frictional contact-aware topology optimization framework with guaranteed non-interpenetration satisfaction covering both external- and self-contact without the need for pre-specification or labeling. As demonstrated, this framework now enables optimizations to explore evolving contact interfaces, and so new designs of structures can be generated that are able to take advantage of self-contact. Potential applications of this framework include soft robotic grippers, energy-absorbing cushions, and meta-materials with microstructures in contact.

While the narrow-band procedure was originally introduced to topology optimization to speed up convergence, here, the proposed method augments it with contact boundary detection. This procedure, nevertheless, is then non-smooth in $\rho$. When compared with a smooth boundary, the grid-aligned contact boundary generated by the narrow-band procedure may also lead to an unnatural concentration of contact in small regions. Thus a promising direction for future work is to form a differentiable and smooth contact surface using techniques such as (Remelli et al, 2020). Last but not least, our current framework uses MMA for design optimization, which usually requires parameter tuning for high performance. In our experiments, a default standard MMA setup is applied without further fine-tuning. Exploring the effects of alternate optimization methods as well as a thorough analysis of their parameters on the current method should be a useful direction for further improvements in practical performance.

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Declarations

**Conflict of interest.** The authors declare that there are not competing interests or conflict of interests.

**Ethical approval.** This project does not contain any studies with human participants or animals.

**Replication of results.** The presented methodology is implemented in C++. The program is compiled with the GNU C++ compiler and executed on the Ubuntu OS. The code and data are freely available to readers upon request.

References

Andreassen E, Clausen A, Schevenels M, et al (2011) Efficient topology optimization in matlab using 88 lines of code. Structural and Multidisciplinary Optimization 43(1):1–16

Bluhm GL, Sigmund O, Poulios K (2021) Internal contact modeling for finite strain topology optimization. Computational Mechanics 67:1099–1114

Bridson R, Fedkiw R, Anderson J (2002) Robust treatment of collisions, contact and friction for cloth animation. In: Proceedings of the 29th annual conference on Computer graphics and interactive techniques, pp 594–603

Bruns TE, Tortorelli DA (2003) An element removal and reintroduction strategy for the topology optimization of structures and compliant mechanisms. International journal for numerical methods in engineering 57(10):1413–1430

Dumas J (2018) Mma and gemma. https://github.com/jdumas/mma

Fang Y, Li M, Jiang C, et al (2021) Guaranteed globally injective 3d deformation processing. ACM Trans Graph 40(4):75–1

Fernandez F, Puso MA, Solberg J, et al (2020) Topology optimization of multiple deformable bodies in contact with large deformations. Computer Methods in Applied Mechanics and Engineering 371:113,288

Ferrari F, Sigmund O (2020) A new generation 99 line matlab code for compliance topology optimization and its extension to 3d. Structural and Multidisciplinary Optimization 62(4):2211–2228

Goldenthal R, Harmon D, Fattal R, et al (2007) Efficient simulation of inextensible cloth. In: ACM SIGGRAPH 2007 papers. p 49–es

Goyal S, Ruina A, Papadopoulos J (1991a) Planar sliding with dry friction part 1. limit surface and moment function. Wear 143(2):307–330

Goyal S, Ruina A, Papadopoulos J (1991b) Planar sliding with dry friction part 2. dynamics of motion. Wear 143(2):331–352

Hallquist J, Goudreau G, Benson D (1985) Sliding interfaces with contact-impact in large-scale lagrangian computations. Computer methods in applied mechanics and engineering 51(1-3):107–137

Han Y, Xu B, Duan Z, et al (2022) Stress-based topology optimization of continuum structures for the elastic contact problems with friction. Structural and Multidisciplinary Optimization 65(2)

Kristiansen H, Poulios K, Aage N (2020) Topology optimization for compliance and contact pressure distribution in structural problems with friction. Computer Methods in Applied Mechanics and Engineering 364:112,915

Kruse R, Nguyen-Thanh N, Wriggers P, et al (2018) Isogeometric frictionless contact analysis with the third medium method. Computational Mechanics 62(5):1009–1021

Li M, Ferguson Z, Schneider T, et al (2020a) Incremental potential contact: intersection-and inversion-free, large-deformation dynamics. ACM Trans Graph 39(4):49

Li M, Kaufman DM, Jiang C (2020b) Codimensional incremental potential contact. arXiv preprint arXiv:201204457

Li X, McWilliams J, Li M, et al (2021a) Soft hybrid aerial vehicle via bistable mechanism.
Li X, Fang Y, Li M, et al (2022) Bfemp: Interpenetration-free mpm–fem coupling with barrier contact. Computer Methods in Applied Mechanics and Engineering 390:114,350

Li Y, Li X, Li M, et al (2021b) Lagrangian–eulerian multidensity topology optimization with the material point method. International Journal for Numerical Methods in Engineering 122(14):3400–3424

Liu H, Hu Y, Zhu B, et al (2018) Narrow-band topology optimization on a sparsely populated grid. ACM Transactions on Graphics (TOG) 37(6):1–14

Luo Y, Li M, Duan Z, et al (2016) Topology optimization of hyperelastic structures with frictionless contact supports. International Journal of Solids and Structures 81:373–382. https://doi.org/10.1016/j.ijsolstr.2015.12.018

Mankame ND, Ananthasuresh G (2004) Topology optimization for synthesis of contact-aided compliant mechanisms using regularized contact modeling. Computers & structures 82(15-16):1267–1290

Moreau JJ (2011) On unilateral constraints, friction and plasticity. In: New variational techniques in mathematical physics. Springer, p 171–322

Müller M, Chentanez N, Kim T, et al (2015) Air meshes for robust collision handling. ACM Transactions on Graphics (TOG) 34(4):1–9

Niu C, Zhang W, Gao T (2019) Topology optimization of continuum structures for the uniformity of contact pressures. Structural and Multidisciplinary Optimization 60(1):185–210

Noceval J, Wright SJ (1999) Numerical optimization. Springer

Pagano S, Alart P (2008) Self-contact and fictitious domain using a difference convex approach. International journal for numerical methods in engineering 75(1):29–42

Remelli E, Lukoianov A, Richter SR, et al (2020) Meshsdf: Differentiable iso-surface extraction. CoRR abs/2006.03997. https://arxiv.org/abs/2006.03997

Rozvany G (2000) The simp method in topology optimization-theoretical background, advantages and new applications. In: 8th Symposium on Multidisciplinary Analysis and Optimization, p 4738

Sigmund O (1997) On the design of compliant mechanisms using topology optimization. Journal of Structural Mechanics 25(4):493–524

Sigmund O (2001) A 99 line topology optimization code written in matlab. Structural and multidisciplinary optimization 21(2):120–127

Sigmund O (2007) Morphology-based black and white filters for topology optimization. Structural and Multidisciplinary Optimization 33(4):401–424

Sigmund O, Maute K (2012) Sensitivity filtering from a continuum mechanics perspective. Structural and Multidisciplinary Optimization 46(4):471–475

Sigmund O, Maute K (2013) Topology optimization approaches. Structural and Multidisciplinary Optimization 48(6):1031—-1055

Strömberg N (2013) The influence of sliding friction on optimal topologies. In: Recent advances in contact mechanics. Springer, p 327–336

Svanberg K (1987) The method of moving asymptotes—a new method for structural optimization. International journal for numerical methods in engineering 24(2):359–373

Weißfels C, Wriggers P (2015) A contact layer element for large deformations. Computational Mechanics 55(5):873–885

Wriggers P, Schröder J, Schwarz AA (2013) A finite element method for contact using a third medium. Computational Mechanics volume 52:837–847
Zhang X, Li Y, Wang Y, et al (2021) Narrow-band filter design of phononic crystals with periodic point defects via topology optimization. International Journal of Mechanical Sciences 212:106,829

Zhou Y, Zhang W, Zhu J, et al (2016) Feature-driven topology optimization method with signed distance function. Computer Methods in Applied Mechanics and Engineering 310:1–32