HADAMARD WEIGHTED GEOMETRIC MEAN INEQUALITIES FOR THE SPECTRAL AND ESSENTIAL SPECTRAL RADIUS OF POSITIVE OPERATORS ON BANACH FUNCTION AND SEQUENCE SPACES

KATARINA BOGDANOVIĆ, ALJOŠA PEPERKO

Abstract. We prove new inequalities for the spectral radius, essential spectral radius, operator norm, measure of noncompactness and numerical radius of Hadamard weighted geometric means of positive kernel operators on Banach function and sequence spaces. Several inequalities appear to be new even in the finite dimensional case.

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1. Introduction and preliminaries

Let μ be a σ-finite positive measure on a σ-algebra M of subsets of a non-void set X. Let M(X, μ) be the vector space of all equivalence classes of (almost everywhere equal) complex measurable functions on X. A Banach space L ⊆ M(X, μ) is called a Banach function space if f ∈ L, g ∈ M(X, μ), and |g| ≤ |f| imply that g ∈ L and ∥g∥ ≤ ∥f∥. We assume that X is the carrier of L, that is, there is no subset Y of X of strictly positive measure with the property that f = 0 a.e. on Y for all f ∈ L (see [47]).

Let R denote the set {1, . . . , N} for some N ∈ N or the set N of all natural numbers. Let S(R) be the vector lattice of all complex sequences (x_n)_{n∈R}. A Banach space L ⊆ S(R) is called a Banach sequence space if x ∈ S(R), y ∈ L and |x| ≤ |y| imply that x ∈ L and ∥x∥_L ≤ ∥y∥_L. Observe that a Banach sequence space is a Banach function space over
a measure space \((R, \mu)\), where \(\mu\) denotes the counting measure on \(R\). Denote by \(\mathcal{L}\) the collection of all Banach sequence spaces \(L\) satisfying the property that \(\epsilon_n = \chi_{\{n\}} \in L\) and \(\|\epsilon_n\|_L = 1\) for all \(n \in R\). For \(L \in \mathcal{L}\) the set \(R\) is the carrier of \(L\).

Standard examples of Banach function spaces are Euclidean spaces, the space \(c_0\) of all null convergent sequences (equipped with the usual norms and the counting measure), the well-known spaces \(L^p(X, \mu)\) \((1 \leq p \leq \infty)\) and other less known examples such as Orlicz, Lorentz, Marcinkiewicz and more general rearrangement-invariant spaces (see e.g. [6], [10] and the references cited there), which are important e.g. in interpolation theory. Recall that the cartesian product \(L = E \times F\) of Banach function spaces is again a Banach function space, equipped with the norm \(||(f, g)||_L = \max\{\|f\|_E, \|g\|_F\}\).

If \(\{f_n\}_{n \in \mathbb{N}} \subset M(X, \mu)\) is a decreasing sequence and \(f = \inf\{f_n \in M(X, \mu) : n \in \mathbb{N}\}\), then we write \(f_n \downarrow f\). A Banach function space \(L\) has an order continuous norm, if \(0 \leq f_n \downarrow 0\) implies \(\|f_n\|_L \to 0\) as \(n \to \infty\). It is well known that spaces \(L^p, 1 \leq p < \infty\), have order continuous norms. Moreover, the norm of any reflexive Banach function space is order continuous. In particular, we will be interested in Banach function spaces \(L\) such that \(L\) and its Banach dual space \(L^*\) have order continuous norms. Examples of such spaces are \(L^p(X, \mu)\), \(1 < p < \infty\), while the space \(L = c_0 \in \mathcal{L}\) is an example of a non-reflexive Banach sequence (function) space, such that \(L\) and \(L^* = l^1 \in \mathcal{L}\) have order continuous norms.

By an operator on a Banach function space \(L\) we always mean a linear operator on \(L\). An operator \(K\) on \(L\) is said to be positive if it maps nonnegative functions to nonnegative ones, i.e., \(KL_+ \subset L_+\), where \(L_+ = \{f \in L : f \geq 0 \ a.e.\}\). Given operators \(K\) and \(H\) on \(L\), we write \(K \geq H\) if the operator \(K - H\) is positive. Recall that a positive operator \(K\) is always bounded, i.e., its operator norm

\[
\|K\| = \sup\{\|Kf\|_L : f \in L, \|f\|_L \leq 1\} = \sup\{\|Kf\|_L : f \in L_+, \|f\|_L \leq 1\}
\]

is finite (the second equality in \([11]\) follows from \(|Kf| \leq K|f|\) for \(f \in L\)). Also, its spectral radius \(r(K)\) is always contained in the spectrum.

In the special case \(L = L^2(X, \mu)\) we can define the numerical radius \(w(K)\) of a bounded operator \(K\) on \(L^2(X, \mu)\) by

\[
w(K) = \sup\{|\langle Kf, f \rangle| : f \in L^2(X, \mu), \|f\|_2 = 1\}.
\]

If, in addition, \(K\) is positive, then it is easy to prove that

\[
w(K) = \sup\{\langle Kf, f \rangle : f \in L^2(X, \mu)_+, \|f\|_2 = 1\}.
\]

From this it follows easily that \(w(K) \leq w(H)\) for all positive operators \(K\) and \(H\) on \(L^2(X, \mu)\) with \(K \leq H\).
An operator $K$ on a Banach function space $L$ is called a kernel operator if there exists a $\mu \times \mu$-measurable function $k(x, y)$ on $X \times X$ such that, for all $f \in L$ and for almost all $x \in X$,
\[
\int_X |k(x, y)f(y)| \, d\mu(y) < \infty \quad \text{and} \quad (Kf)(x) = \int_X k(x, y)f(y) \, d\mu(y).
\]
One can check that a kernel operator $K$ is positive iff its kernel $k$ is non-negative almost everywhere.

Let $L$ be a Banach function space such that $L$ and $L^*$ have order continuous norms and let $K$ and $H$ be positive kernel operators on $L$. By $\gamma(K)$ we denote the Hausdorff measure of non-compactness of $K$, i.e.,
\[
\gamma(K) = \inf \{ \delta > 0 : \text{there is a finite } M \subset L \text{ such that } K(D_L) \subset M + \delta D_L \},
\]
where $D_L = \{ f \in L : \|f\|_L \leq 1 \}$. Then $\gamma(K) \leq \|K\|$, $\gamma(K + H) \leq \gamma(K) + \gamma(H)$, $\gamma(KH) \leq \gamma(K)\gamma(H)$ and $\gamma(\alpha K) = \alpha \gamma(K)$ for $\alpha \geq 0$. Also $0 \leq K \leq H$ implies $\gamma(K) \leq \gamma(H)$ (see e.g. [30, Corollary 4.3.7 and Corollary 3.7.3]). Let $r_{ess}(K)$ denote the essential spectral radius of $K$, i.e., the spectral radius of the Calkin image of $K$ in the Calkin algebra. Then
\[
r_{ess}(K) = \lim_{j \to \infty} \gamma(K^j)^{1/j} = \inf_{j \in \mathbb{N}} \gamma(K^j)^{1/j}
\]
and $r_{ess}(K) \leq \gamma(K)$. Note that (2) is valid for any bounded operator $K$ on a given complex Banach space $L$ (see e.g. [30, Theorem 4.3.13]).

It is well-known that kernel operators play a very important, often even central, role in a variety of applications from differential and integro-differential equations, problems from physics (in particular from thermodynamics), engineering, statistical and economic models, etc (see e.g. [24], [4], [29], [11] and the references cited there). For the theory of Banach function spaces and more general Banach lattices we refer the reader to the books [47], [6], [1], [2].

Let $K$ and $H$ be positive kernel operators on $L$ with kernels $k$ and $h$ respectively, and $\alpha \geq 0$. The Hadamard (or Schur) product $K \circ H$ of $K$ and $H$ is the kernel operator with kernel equal to $k(x, y)h(x, y)$ at point $(x, y) \in X \times X$ which can be defined (in general) only on some order ideal of $L$. Similarly, the Hadamard (or Schur) power $K^{(\alpha)}$ of $K$ is the kernel operator with kernel equal to $(k(x, y))^\alpha$ at point $(x, y) \in X \times X$ which can be defined only on some order ideal of $L$.

Let $K_1, \ldots, K_n$ be positive kernel operators on a Banach function space $L$, and $\alpha_1, \ldots, \alpha_n$ positive numbers such that $\sum_{j=1}^n \alpha_j = 1$. Then the Hadamard weighted geometric mean $K = K_1^{(\alpha_1)} \circ K_2^{(\alpha_2)} \circ \cdots \circ K_n^{(\alpha_n)}$ of the operators $K_1, \ldots, K_n$ is a positive kernel operator.
defined on the whole space $L$, since $K \leq \alpha_1 K_1 + \alpha_2 K_2 + \ldots + \alpha_n K_n$ by the inequality between the weighted arithmetic and geometric means.

A matrix $K = [k_{ij}]_{i,j \in R}$ is called nonnegative if $k_{ij} \geq 0$ for all $i, j \in R$. For notational convenience, we sometimes write $k(i, j)$ instead of $k_{ij}$. We say that a nonnegative matrix $K$ defines an operator on $L$ if $Kx \in L$ for all $x \in L$, where $(Kx)_i = \sum_{j \in R} k_{ij}x_j$. Then $Kx \in L_+$ for all $x \in L_+$ and so $K$ defines a positive kernel operator on $L$.

Let us recall the following result which was proved in [15, Theorem 2.2] and [33, Theorem 5.1 and Example 3.7] (see also e.g. [36, Theorem 2.1], [37], [34], [35], [17]).

**Theorem 1.1.** Let $K, K_1, \ldots, K_n$ and $\{K_{ij}\}_{i=1,j=1}^{l,n}$ be positive kernel operators on a Banach function space $L$. Assume $\alpha_1, \alpha_2, \ldots, \alpha_n$ are positive numbers such that $s_n = \sum_{j=1}^{n} \alpha_j = 1$ and define the positive kernel operator $H$ on $L$ by

$$
H := (K_1^{(\alpha_1)} \circ \cdots \circ K_n^{(\alpha_n)}) \cdots (K_{1l}^{(\alpha_1)} \circ \cdots \circ K_{ln}^{(\alpha_n)}).
$$

(i) Then the following inequalities hold for $\rho \in \{\| \cdot \|, r\}$:

$$
\rho(K_1^{(\alpha_1)} \circ K_2^{(\alpha_2)} \circ \cdots \circ K_n^{(\alpha_n)}) \leq \rho(K_1)^{\alpha_1} \rho(K_2)^{\alpha_2} \cdots \rho(K_n)^{\alpha_n},
$$

$$
H \leq (K_1 \cdots K_n)^{(\alpha_1)} \circ \cdots \circ (K_{1n} \cdots K_{ln})^{(\alpha_n)},
$$

$$
\rho(H) \leq \rho \left( (K_1 \cdots K_n)^{(\alpha_1)} \circ \cdots \circ (K_{1n} \cdots K_{ln})^{(\alpha_n)} \right)
$$

If, in addition, $L$ and $L^*$ have order continuous norms, then (3) and (5) hold also for $\rho \in \{\gamma, r_{ess}\}$.

If, in addition, $L = L^2(X, \mu)$ then (3) and (5) hold also for $\rho = w$.

(ii) If $L \in \mathcal{L}, t \geq 1$ and $s_n \geq 1$, then $K^{(t)}$, $K_1^{(\alpha_1)} \circ K_2^{(\alpha_2)} \circ \cdots \circ K_n^{(\alpha_n)}$ and $H$ define operators on $L$ and the inequalities

$$
k(i, j) \leq \| K \| \quad \text{for all} \quad i, j \in R,
$$

$$
K_1^{(t)} \cdots K_n^{(t)} \leq (K_1 \cdots K_n)^{(t)},
$$

$$
\rho(K_1^{(t)} \cdots K_n^{(t)}) \leq \rho(K_1 \cdots K_n)^{t},
$$

(3) and (5) hold for $\rho \in \{\| \cdot \|, r\}$.

In the finite-dimensional case Inequality (3) for the spectral radius goes back to Kingman [28] implicitly, and it was later considered by several authors ([21], [5], [19], [25],
Hadamard and ordinary products of operators (see, e.g., [42], [43], [35], [7], [17], [36], [38], [33], [35], [36], [38]) and have been applied to obtain several inequalities involving the results on the joint and generalized spectral radius and their essential versions (see e.g., continuous semigroups of operators. Let us also point out that Inequalities (5) are actually of Banach ordered spaces by Kato [26], who used extensively the theory of strongly continuous semigroups of operators. In [16] the generalization to the setting of infinite dimensional matrices was established and very recently in [40] a version for the essential spectral radius was proved. It should be mentioned that a very general extension of Cohen’s theorem was proved in the setting of Banach ordered spaces by Kato [26], who used extensively the theory of strongly continuous semigroups of operators. Let us also point out that Inequalities (5) are actually results on the joint and generalized spectral radius and their essential versions (see e.g., [33], [35], [36], [38]) and have been applied to obtain several inequalities involving the Hadamard and ordinary products of operators (see, e.g., [42], [43], [35], [7], [17], [36], [37], [38], [45], [31], [32], [41], [40]).

Let $K_1 = [k_1(i,j)]_{i,j \in R}, \ldots, K_n = [k_n(i,j)]_{i,j \in R}$ be nonnegative matrices and let $\alpha_1, \ldots, \alpha_n$ be nonnegative numbers such that $\sum_{i=1}^{n} \alpha_i = 1$. The nonnegative matrix $C(K_1, \ldots, K_n, \alpha_1, \ldots, \alpha_n) = [c(i,j)]_{i,j \in R}$ is defined by

\[ c(i,j) = \begin{cases} k_1^{\alpha_1}(i,j) \cdots k_n^{\alpha_n}(i,j) & \text{if } i \neq j \\ \alpha_1 k_1(i,i) + \cdots + \alpha_n k_n(i,i) & \text{if } i = j \end{cases} \]

In other words, the diagonal part of $C(K_1, \ldots, K_n, \alpha_1, \ldots, \alpha_n)$ is equal to the diagonal part of $\alpha_1 K_1 + \cdots + \alpha_n K_n$, while its nondiagonal part equals the nondiagonal part of $K_1^{(\alpha_1)} \circ K_2^{(\alpha_2)} \circ \cdots \circ K_n^{(\alpha_n)}$.

By the inequality between weighted geometric and weighted arithmetic means, we have

\[ K_1^{(\alpha_1)} \circ K_2^{(\alpha_2)} \circ \cdots \circ K_n^{(\alpha_n)} \leq C(K_1, \ldots, K_n, \alpha_1, \ldots, \alpha_n) \leq \alpha_1 K_1 + \cdots + \alpha_n K_n. \]

From the right-hand inequality it follows that the matrix $C(K_1, \ldots, K_n, \alpha_1, \ldots, \alpha_n)$ defines an operator on $L$ provided the matrices $K_1, \ldots, K_n$ define operators on $L \in \mathcal{L}$.

The following generalization of Cohen’s theorem was obtained in [16, Theorem 2.1] and in [40, Theorem 2.2].

**Theorem 1.2.** Given $L$ in $\mathcal{L}$, let $K_1, \ldots, K_n$ be nonnegative matrices that define operators on $L$ and $\alpha_1, \ldots, \alpha_n$ nonnegative numbers such that $\sum_{i=1}^{n} \alpha_i = 1$. Then for $\rho = r$ we have

\[ \rho(C(K_1, \ldots, K_n, \alpha_1, \ldots, \alpha_n)) \leq \alpha_1 \rho(K_1) + \cdots + \alpha_n \rho(K_n). \]

In particular, if $K_1, \ldots, K_n$ have the same non-diagonal part, then

\[ \rho(\alpha_1 K_1 + \cdots + \alpha_n K_n) \leq \alpha_1 \rho(K_1) + \cdots + \alpha_n \rho(K_n). \]
In other words, if \( D_1, \ldots, D_n \) are diagonal matrices and \( K \) a matrix such that \( K + D_1, \ldots, K + D_n \) are nonnegative matrices that define operators on \( L \), then we have

\[
\rho (\alpha_1 (K + D_1) + \cdots + \alpha_n (K + D_n)) \leq \alpha_1 \rho (K + D_1) + \cdots + \alpha_n \rho (K + D_n).
\]

If, in addition, \( L \) and \( L^* \) have order continuous norms then under the above conditions inequalities (10), (11) and (12) hold also for \( \rho = \rho_{ess} \).

Recall also the following well known inequality (see e.g. [33], [15], [45]) for nonnegative measurable functions and for \( \alpha \) and \( \beta \) nonnegative numbers such that \( \alpha + \beta \geq 1 \):

\[
f_1^\alpha g_1^\beta + \cdots + f_m^\alpha g_m^\beta \leq (f_1 + \cdots + f_m)^\alpha (g_1 + \cdots + g_m)^\beta.
\]

The rest of the article is organized as follows. In Section 2 we first state Theorem 2.1 which we will need in our proofs and follows directly from results of [40]. Then we prove new results on geometric symmetrizations of positive kernel operators and their weighted versions, which generalize several results from [14], [10], [15], [33] and [45] and we prove new versions of these results. We conclude the article with Section 3, where we establish some new additional results on Hadamard weighted geometric means of operators. In particular, in Section 3 we extend the main results of [48] and some results of [37].

2. Results on weighted geometric symmetrizations

Given a nonnegative matrix \( K \) that defines an operator on \( L \) in \( \mathcal{L} \), let us denote \( \|K\|_\infty = \sup_{i,j \in \mathbb{R}} k(i,j) \). Then we have \( \|K\|_\infty \leq \|K\| \) by (3).

In the following result we state versions of (5). The result follows from Inequalities (4), and (5) combined with [40, Theorem 2.6] applied to the matrix \( H \) from (14).

Theorem 2.1. Given \( L \) in \( \mathcal{L} \), let \( \{K_{ij}\}_{i=1,j=1}^{l,n} \) be nonnegative matrices that define operators on \( L \) and \( \alpha_1, \alpha_2, \ldots, \alpha_n \) positive numbers such that \( s_n = \sum_{i=1}^n \alpha_i \geq 1 \). Let

\[
\begin{align*}
H : & \quad = \left( K_{11}^{(\alpha_1)} \circ \cdots \circ K_{1n}^{(\alpha_n)} \right) \cdots \left( K_{l1}^{(\alpha_1)} \circ \cdots \circ K_{ln}^{(\alpha_n)} \right), \\
H_i : & \quad = K_{1i} \cdots K_{li}, \quad M = \max_{i=1,\ldots,n} \|H_i\|_\infty, \\
\beta & \quad = M^{s_n-1} \quad \text{and} \quad \beta_i = \frac{\alpha_i}{s_n} \quad \text{for all} \quad i = 1, \ldots, n.
\end{align*}
\]
Then inequalities
\[
H \leq H_1^{(\alpha_1)} \circ \cdots \circ H_n^{(\alpha_n)} \leq \beta H_1^{(\beta_1)} \circ \cdots \circ H_n^{(\beta_n)}
\]
\[
\rho(H) \leq \beta C(H_1, \ldots, H_n, \beta_1, \ldots, \beta_n) \leq \beta(\beta_1 H_1 + \cdots + \beta_n H_n),
\]
\[
\rho(H) \leq \beta \rho(H_1)^{\beta_1} \cdots \rho(H_n)^{\beta_n} \leq \beta(\beta_1 \rho(H_1) + \cdots + \beta_n \rho(H_n))
\]
\[
(15)
\]
\[
\rho(H) \leq \rho(H_1^{(\alpha_1)} \circ \cdots \circ H_n^{(\alpha_n)}) \leq \beta \rho(H_1^{(\beta_1)} \circ \cdots \circ H_n^{(\beta_n)})
\]
\[
(16)
\]
hold for all \( \rho \in \{r, \| \cdot \| \} \) and inequalities
\[
d(H) \leq d(H_1^{(\alpha_1)} \circ \cdots \circ H_n^{(\alpha_n)}) \leq \beta d(H_1^{(\beta_1)} \circ \cdots \circ H_n^{(\beta_n)})
\]
\[
(17)
\]
hold for \( d = \| \cdot \| \).

If, in addition, \( L \) and \( L^* \) have order continuous norms then inequalities (15) and (16) hold also for all \( \rho \in \{r_{ess}, \gamma \} \) and inequalities (17) hold also for \( d = \gamma \).

If, in addition, \( L = L^2(R) \), then inequalities (15) and (17) (and (16)) hold also for \( \rho = w \) and for \( d = w \).

If, in addition, the matrices \( H_1, \ldots, H_n \) are \( m \times m \) matrices and the diagonal part of \( H_1^{(\alpha_1)} \circ \cdots \circ H_n^{(\alpha_n)} \) is equal to zero, then
\[
r(H) \leq r(H_1^{(\alpha_1)} \circ \cdots \circ H_n^{(\alpha_n)}) \leq (m-1)\delta,
\]
where \( \delta = \max\{M^*, 1\} \).

Let us recall the notion of geometric symmetrization of positive kernel operators on \( L^2(X, \mu) \). Let \( K \) be a positive kernel operator on \( L^2(X, \mu) \) with kernel \( k \). The geometric symmetrization \( S(K) \) of \( K \) is the positive selfadjoint kernel operator on \( L^2(X, \mu) \) with kernel equal to \( \sqrt{k(x,y)k(y,x)} \) at point \((x,y) \in X \times X \). Note that \( S(K) = K^{(1/2)} \circ (K^*)^{(1/2)} \), since the kernel of the adjoint operator \( K^* \) is equal to \( k(y,x) \) at point \((x,y) \in X \times X \).

Next we extend several results from [13, 40, 15, 33] and [45] to the setting of weighted geometric “symmetrizations” \( S_\alpha(\cdot) \) of positive kernel operators and prove new related results to those from the above references. Let \( K \) be a positive kernel operator on \( L^2(X, \mu) \) and \( \alpha \in [0,1] \). Denote \( S_\alpha(K) = K^{(\alpha)} \circ (K^*)^{(1-\alpha)} \), which is a kernel operator on \( L^2(X, \mu) \) with a kernel \( k^\alpha(x,y)k^{1-\alpha}(y,x) \). Observe that \((S_\alpha(K))^* = S_\alpha(K^*) = S_{1-\alpha}(K) \).
The following result generalizes and refines [40, Propositions 3.1 and 3.2].

**Proposition 2.2.** Let $K, K_1, \ldots, K_n$ be positive kernel operators on $L^2(X, \mu)$ and $\alpha \in [0, 1]$. Then we have

$$
\rho(S_\alpha(K_1) \cdots S_\alpha(K_n)) 
\leq \rho\left( (K_1 \cdots K_n)^{(\alpha)} \circ ((K_n \cdots K_1)^{1-\alpha}) \right) 
\leq \rho(K_1 \cdots K_n)\rho(K_n \cdots K_1)^{1-\alpha},
$$

(19)

$$
\rho(S_\alpha(K_1) + \cdots + S_\alpha(K_m)) \leq \rho(S_\alpha(K_1 + \cdots + K_m)) \leq \rho(K_1 + \cdots + K_m)
$$

(20)

for all $\rho \in \{r, r_{\text{ess}}, \gamma, \| \cdot \|, w\}$. In particular, for all $\rho \in \{r, r_{\text{ess}}, \gamma, \| \cdot \|, w\}$ we have

$$
\rho(S_\alpha(K)) \leq \rho(K).
$$

(21)

We also have

$$
\rho(S_\alpha(K_1)S_\alpha(K_2)) \leq \rho\left( (K_1^2)^{(\alpha)} \circ ((K_2^2)^{(1-\alpha)}) \right) \leq \rho(K_1K_2).
$$

(22)

for $\rho \in \{r, r_{\text{ess}}\}$.

**Proof.** By [5] we have

$$
\rho(S_\alpha(K_1) \cdots S_\alpha(K_n)) = \rho\left( (K_1^{(\alpha)} \circ (K_1^*)^{(1-\alpha)}) \cdots (K_n^{(\alpha)} \circ (K_n^*)^{(1-\alpha)}) \right)
\leq \rho\left( (K_1 \cdots K_n)^{(\alpha)} \circ ((K_n \cdots K_1)^{1-\alpha}) \right)
\leq \rho(K_1 \cdots K_n)^{\alpha}\rho((K_n \cdots K_1)^{1-\alpha} = \rho(K_1 \cdots K_n)^{\alpha}\rho(K_n \cdots K_1)^{1-\alpha}.
$$

This proves (19). Inequality (21) is a special cases of (19) while (22) follows from (19) and $\rho(K_1K_2) = \rho(K_2K_1)$ for $\rho \in \{r, r_{\text{ess}}\}$.

Inequalities (20) follow from (13) and (21).

If $K$ is a nonnegative matrix that defines an operator on $l^2(R)$ and if $\alpha$ and $\beta$ are nonnegative numbers such that $\alpha + \beta \geq 1$, then a nonnegative matrix

$$
S_{\alpha, \beta}(K) = K^{(\alpha)} \circ (K^*)^{(\beta)}
$$

also defines an operator on $l^2(R)$ by Theorem 1.1(ii). The following result is proved in a similar way as Proposition 2.2.

**Proposition 2.3.** Let $K, K_1, \ldots, K_n$ be nonnegative matrices that define operators on $l^2(R)$ and let $\alpha$ and $\beta$ be nonnegative numbers such that $\alpha + \beta \geq 1$. Then we have

$$
\rho(S_{\alpha, \beta}(K_1) \cdots S_{\alpha, \beta}(K_n)) 
\leq \rho\left( (K_1 \cdots K_n)^{(\alpha)} \circ ((K_n \cdots K_1)^{(\beta)}) \right) \leq \rho(K_1 \cdots K_n)^{\alpha}\rho(K_n \cdots K_1)^{\beta},
$$

(23)

$$
\rho(S_{\alpha, \beta}(K)) \leq \rho(K)^{\alpha+\beta}.
$$

(24)

$$
\rho(S_{\alpha, \beta}(K_1) + \cdots + S_{\alpha, \beta}(K_m)) \leq \rho(S_{\alpha, \beta}(K_1 + \cdots + K_m)) \leq \rho(K_1 + \cdots + K_m)^{\alpha+\beta}
$$

(25)
for all \( \rho \in \{r, \| \cdot \| \} \). We also have

\[
\rho (S_{\alpha,\beta}(K_1)S_{\alpha,\beta}(K_2)) \leq \rho(K_1K_2)^{\alpha+\beta}
\]

for \( \rho = r \). Moreover, we have

\[
\rho(S_{\alpha,\beta}(K_1) \cdots S_{\alpha,\beta}(K_n)) \leq \rho \left( (K_1 \cdots K_n)^{(\alpha)} \circ ((K_n \cdots K_1)^{(\beta)}) \right)
\]

\[
\leq \delta \rho \left( (K_1 \cdots K_n)^{(\alpha)} \circ ((K_n \cdots K_1)^{(\beta)}) \right) \leq \delta \cdot \rho(K_1 \cdots K_n)^{\alpha+\beta-1} \rho(K_n \cdots K_1)^{\beta},
\]

where \( \delta = \max\{\|K_1 \cdots K_n\|_{\infty},\|K_n \cdots K_1\|_{\infty}\}^{\alpha+\beta-1}, \) and

\[
\rho(S_{\alpha,\beta}(K)) \leq \|K\|_{\infty}^{\alpha+\beta-1} \rho \left( S_{\alpha,\beta}^{(\alpha)}(K) \right) \leq \|K\|_{\infty}^{\alpha+\beta-1} \rho(K),
\]

\[
\rho(S_{\alpha,\beta}(K_1) + \cdots + S_{\alpha,\beta}(K_m)) \leq \rho(S_{\alpha,\beta}(K_1 + \cdots + K_m))
\]

\[
\leq \|K_1 + \cdots + K_m\|_{\infty}^{\alpha+\beta-1} \rho \left( S_{\alpha,\beta}^{(\alpha)}(K_1 + \cdots + K_m) \right) \leq \|K_1 + \cdots + K_m\|_{\infty}^{\alpha+\beta-1} \rho(K_1 + \cdots + K_m)
\]

for all \( \rho \in \{r, r_{ess}; \gamma, \| \cdot \|, w\} \). We also have

\[
\rho(S_{\alpha,\beta}(K_1)S_{\alpha,\beta}(K_2)) \leq \rho \left( (K_1K_2)^{(\alpha)} \circ ((K_2K_1)^{(\beta)}) \right)
\]

\[
\leq \max\{\|K_1K_2\|_{\infty},\|K_2K_1\|_{\infty}\}^{\alpha+\beta-1} \rho \left( (K_1K_2)^{(\alpha)} \circ ((K_2K_1)^{(\beta)}) \right)
\]

\[
\leq \max\{\|K_1K_2\|_{\infty},\|K_2K_1\|_{\infty}\}^{\alpha+\beta-1} \rho(K_1K_2)
\]

for all \( \rho \in \{r, r_{ess}\} \).

**Proof.** Inequalities (23) are proved in similar way as Inequalities (19) by applying Theorem 1.1(ii). Inequalities (27) follow from (15). Inequalities (24) and (28) are special cases of (23) and (27), respectively. Inequalities (25) and (29) follow from (13), (24) and (28), while Inequalities (26) and (30) follow from (23) and (27). \( \square \)

The following two results generalize [14, Lemma 2.1], [14, Theorem 2.2] and [40, Theorem 3.5] by employing a similar but more general method of proof.

**Lemma 2.4.** (i) If \( K \) is a positive kernel operator on \( L^2(X, \mu) \) and \( \alpha \in [0, 1] \), then

\[
S_\alpha(K^2) \geq S_\alpha(K)^2.
\]

(ii) If \( K \) is a nonnegative matrix that defines an operator on \( l^2(R) \) and if \( \alpha \) and \( \beta \) are nonnegative numbers such that \( \alpha + \beta \geq 1 \), then

\[
S_{\alpha,\beta}(K^2) \geq S_{\alpha,\beta}(K)^2.
\]
Proof. The kernel of \( S_\alpha(K^2) \) at \((x, y) \in X \times X\) equals
\[
\left( \int_X k(x, z)k(z, y)d\mu(z) \right)^\alpha \left( \int_X k(y, z)k(z, x)d\mu(z) \right)^{1-\alpha}.
\]
By Hoelder’s inequality this is larger or equal to
\[
\int_X (k(x, z)k(z, y))^\alpha (k(y, z)k(z, x))^{1-\alpha} d\mu(z)
\]
and this equals the kernel of \( S_\alpha(K) \) at \((x, y)\), which proves (31).

Inequality (32) is proved in a similar way by [33, Proposition 4.1]. □

Theorem 2.5. (i) Let \( K \) be a positive kernel operator on \( L^2(X, \mu) \), \( \alpha \in [0, 1] \) and let \( \rho_n = \rho(S_\alpha(K^{2^n}))^{2^{-n}} \) for \( n \in \mathbb{N} \cup \{0\} \) and \( \rho \in \{r, r_{ess}\} \). Then for each \( n \)
\[
\rho(S_\alpha(K)) = \rho_0 \leq \rho_1 \leq \cdots \leq \rho_n \leq \rho(K).
\]

(ii) Let \( K \) be a nonnegative matrix that defines an operator on \( l^2(R) \) and \( \alpha \) and \( \beta \) nonnegative numbers such that \( \alpha + \beta \geq 1 \). If \( r_n = \rho(S_{\alpha,\beta}(K^{2^n}))^{2^{-n}} \) for \( n \in \mathbb{N} \cup \{0\} \) and \( \rho \in \{r, r_{ess}\} \), then
\[
\rho(S_{\alpha,\beta}(K)) = r_0 \leq r_1 \leq \cdots \leq r_n \leq \min\{\rho(K)^{\alpha+\beta}, \|K^{2^n}\|^{\frac{\alpha+\beta-1}{\alpha^2}} \rho(K)\} \text{ for } \rho = r,
\]
\[
\rho(S_{\alpha,\beta}(K)) = r_0 \leq r_1 \leq \cdots \leq r_n \leq \|K^{2^n}\|^{\frac{\alpha+\beta-1}{\alpha^2}} \rho(K) \text{ for } \rho = r_{ess}
\]
and
\[
r_n \leq \|K^{2^n}\|^{\frac{\alpha+\beta-1}{\alpha^2}} \rho\left(S_{\alpha,\beta}(K^{2^n})\right)^{2^{-n}} \leq \|K^{2^n}\|^{\frac{\alpha+\beta-1}{\alpha^2}} \rho(K).
\]

Proof. To prove (i) we first observe that by (31) (or by (21)) we have
\[
(33) \quad \rho(S_\alpha(K^2)) \geq \rho(S_\alpha(K^2)^2) = \rho(S_\alpha(K))^2.
\]
By (21) \( \rho(S_\alpha(K^{2^n})) \leq \rho(K^{2^n}) = \rho(K)^{2^n} \) and so \( \rho_n \leq \rho(K) \). Since \( \rho_{n-1} \leq \rho_n \) for all \( n \in \mathbb{N} \) by (33) the proof of (i) is completed.

In a similar way (ii) is proved by applying (21), (28) and (30). □

The following result generalizes and extends [43, Theorem 2.2 and Theorem 3.2 (3)].

Proposition 2.6. Let \( K \) be a positive kernel operators on \( L^2(X, \mu) \) and \( \alpha \in [0, 1] \). Then for all \( \rho \in \{r, r_{ess}, \gamma, \|\cdot\|, w\} \) and \( n \in \mathbb{N} \) we have
\[
(34) \quad \rho(S(K)) \leq \rho(S_\alpha(K)) \leq \rho(K) \quad \text{and}
\]
\[
(35) \quad \rho(S(K^n))^{\frac{1}{n}} \leq \rho(S_\alpha(K^n))^{\frac{1}{n}} \leq \rho(K).
\]

Proof. Since \( S(K) = S(S_\alpha(K)) \) Inequalities (34) follow from (21). Inequalities (35) follow from (34). □
The following result generalizes and extends [45] Theorems 2.3 and 3.3. It is proved in similar way as [45] Theorem 2.3 by applying (21). To avoid too much repetition of ideas we omit the details of the proof.

**Theorem 2.7.** Let \( K \) be a positive kernel operators on \( L^2(X, \mu) \).

For \( \rho \in \{ r, r_{ess}, \gamma, \| \cdot \|, w \} \) and \( \alpha \in [0, 1] \) define \( f_\rho(\alpha) = \rho(S_\alpha(K)) \). Then \( f_\rho \) is decreasing in \([0, 0.5]\) and increasing in \([0.5, 1]\).

### 3. Additional results on weighted geometric means

The following refinement of inequality (3) was proved in [38, Corollary 3.10].

**Theorem 3.1.** Let \( K_1, \ldots, K_n \) be positive kernel operators on a Banach function space \( L \). If \( \alpha_1, \ldots, \alpha_n \) are positive numbers such that \( \sum_{i=1}^{n} \alpha_i = 1 \) and if \( m \in \mathbb{N} \) then

\[
\rho(K_1^{(\alpha_1)} \circ \cdots \circ K_n^{(\alpha_n)}) \leq \rho((K_1^{m})^{(\alpha_1)} \circ \cdots \circ (K_n^{m})^{(\alpha_n)})^{\frac{1}{m}} \leq \rho(K_1^{\alpha_1} \cdots K_n^{\alpha_n})
\]

for \( \rho = r \).

If, in addition, \( L \) and \( L^* \) have order continuous norms then Inequalities (36) hold also for \( \rho = r_{ess} \).

By iterating (36) we obtain its refinement.

**Corollary 3.2.** Let \( K_1, \ldots, K_n \) be positive kernel operators on a Banach function space \( L \). If \( \alpha_1, \ldots, \alpha_n \) are positive numbers such that \( \sum_{i=1}^{n} \alpha_i = 1 \) and if \( m, l \in \mathbb{N} \) then

\[
\rho(K_1^{(\alpha_1)} \circ \cdots \circ K_n^{(\alpha_n)}) \leq \rho((K_1^{m})^{(\alpha_1)} \circ \cdots \circ (K_n^{m})^{(\alpha_n)})^{\frac{1}{m}} \leq \rho(K_1^{\alpha_1} \cdots K_n^{\alpha_n})
\]

for \( \rho = r \).

If, in addition, \( L \) and \( L^* \) have order continuous norms then Inequalities (37) hold also for \( \rho = r_{ess} \).

The following result follows from (37) and Theorem 2.1.

**Corollary 3.3.** Given \( L \) in \( \mathcal{L} \), let \( K_1, \ldots, K_n \) be nonnegative matrices that define operators on \( L \) and \( \alpha_1, \ldots, \alpha_n \) nonnegative numbers such that \( s_n = \sum_{i=1}^{n} \alpha_i \geq 1 \). Let

\[
M = \max_{i=1, \ldots, n} \| K_i \|_{\infty}, \beta = M^{s_n - 1}
\]

and \( \beta_i = \frac{\alpha_i}{s_n} \) for \( i = 1, \ldots, n \).

Then

\[
\rho(K_1^{(\alpha_1)} \circ \cdots \circ K_n^{(\alpha_n)}) \leq \beta \rho(K_1^{(\beta_1)} \circ \cdots \circ K_n^{(\beta_n)}) \leq \beta \rho((K_1^{m})^{(\beta_1)} \circ \cdots \circ (K_n^{m})^{(\beta_n)})^{\frac{1}{m}} \leq \beta \rho(K_1^{\beta_1} \cdots K_n^{\beta_n})
\]

(38)
for all \( m, l \in \mathbb{N} \) and \( \rho = r \).

If, in addition, \( L \) and \( L^* \) have order continuous norms then Inequalities (38) hold also for \( \rho = r_{\text{ess}} \).

Next we prove (with a standard method from e.g. [15] and [33]) that in the case of sequence spaces \( L \in \mathcal{L} \) inequalities (37) for the spectral radius hold also under the condition \( \sum_{i=1}^{n} \alpha_i \geq 1 \). In this case we also prove additional refinements of (36).

**Theorem 3.4.** Given \( L \in \mathcal{L} \), let \( K_1, \ldots, K_n \) be nonnegative matrices that define operators on \( L \). If \( \alpha_1, \ldots, \alpha_n \) are nonnegative numbers such that \( s_n = \sum_{i=1}^{n} \alpha_i \geq 1 \) and if \( m, l \in \mathbb{N} \) and \( \beta_i = \frac{\alpha_i}{s_n} \) for all \( i = 1, \ldots, n \), then we have

\[
\begin{align*}
\left(r(K_1^{(\alpha_1)} \circ \cdots \circ K_n^{(\alpha_n)})\right)^m &\leq \left(r(K_1^{(m)})^{(\alpha_1)} \circ \cdots \circ (K_n^{(m)})^{(\alpha_n)}\right)^{\frac{1}{m}} \\
&\leq \left(r(K_1^{(m)})^{(\beta_1)} \circ \cdots \circ (K_n^{(m)})^{(\beta_n)}\right)^{\frac{m}{\beta_n}} \leq \left((K_1^{m})^{(\beta_1)} \circ \cdots \circ (K_n^{m})^{(\beta_n)}\right)^{\frac{m}{\beta_n}}
\end{align*}
\]

and

\[
\begin{align*}
\left(r(K_1^{(\alpha_1)} \circ \cdots \circ K_n^{(\alpha_n)})\right)^m &\leq \left(r(K_1^{(m)})^{(\alpha_1)} \circ \cdots \circ (K_n^{(m)})^{(\alpha_n)}\right)^{\frac{1}{m}} \\
&\leq \left(r(K_1^{(m)})^{(\beta_1)} \circ \cdots \circ (K_n^{(m)})^{(\beta_n)}\right)^{\frac{m}{\beta_n}} \leq \left((K_1^{m})^{(\beta_1)} \circ \cdots \circ (K_n^{m})^{(\beta_n)}\right)^{\frac{m}{\beta_n}}
\end{align*}
\]

**Proof.** First we prove that (37) holds also under our assumptions. By (4) we have

\[
\left(\bigcirc_{1}^{K_1^{(\alpha_1)} \circ \cdots \circ K_n^{(\alpha_n)}}\right)^m = \left(\bigcirc_{1}^{(K_1^{(m)})^{(\alpha_1)} \circ \cdots \circ (K_n^{(m)})^{(\alpha_n)}}\right) \leq \left(\bigcirc_{1}^{K_1^{m} \circ \cdots \circ K_n^{m}}\right)^{(\alpha_1)} \circ \cdots \circ (K_n^{m})^{(\alpha_n)}
\]

(41)

It follows from (41) and (3) that

\[
\begin{align*}
\left(r(K_1^{(\alpha_1)} \circ \cdots \circ K_n^{(\alpha_n)})\right)^m &= r\left(\left(\bigcirc_{1}^{K_1^{(m)} \circ \cdots \circ K_n^{(m)}}\right)^{(\alpha_1)} \circ \cdots \circ (K_n^{m})^{(\alpha_n)}\right) \\
&\leq r\left(\left(K_1^{m} \circ \cdots \circ (K_n^{m})^{(\beta_1)} \circ \cdots \circ (K_n^{m})^{(\beta_n)}\right)^{(\alpha_1)} \circ \cdots \circ (K_n^{m})^{(\alpha_n)}\right)
\end{align*}
\]

which proves (37) in this case. By iterating as before one obtains (37) under our assumptions.

Let us prove (40). Since \( s_n \geq 1 \) and \( \alpha_i = \beta_i \) then it follows by the first inequality in (37) in the case \( s_n \geq 1 \), (8), (3) and (37) that

\[
\begin{align*}
\left(r(K_1^{(\alpha_1)} \circ \cdots \circ K_n^{(\alpha_n)})\right)^m &\leq \left(r\left(K_1^{m} \circ \cdots \circ (K_n^{m})^{(\alpha_1)} \circ \cdots \circ (K_n^{m})^{(\alpha_n)}\right)^{(\beta_1)} \circ \cdots \circ (K_n^{m})^{(\beta_n)}\right)^{\frac{m}{\beta_n}} \\
&\leq \left(r\left(K_1^{m} \circ \cdots \circ (K_n^{m})^{(\beta_1)} \circ \cdots \circ (K_n^{m})^{(\beta_n)}\right)^{\frac{m}{\beta_n}} \leq \left(r\left(K_1^{m} \circ \cdots \circ (K_n^{m})^{(\beta_1)} \circ \cdots \circ (K_n^{m})^{(\beta_n)}\right)^{\frac{m}{\beta_n}}
\end{align*}
\]

(40)

\[
\begin{align*}
&= r\left(K_1^{(\alpha_1)} \circ \cdots \circ (K_n^{m})^{(\alpha_n)}\right)^{\frac{m}{\beta_n}} \leq r\left(K_1^{m} \circ \cdots \circ (K_n^{m})^{(\alpha_1)} \circ \cdots \circ (K_n^{m})^{(\alpha_n)}\right)^{\frac{m}{\beta_n}}
\end{align*}
\]
which proves (40). Now (39) follows from (37) in the case $s_n \geq 1$ and (40), which completes the proof.

We conclude the article by extending the main results of [48] and some results of [37]. The following result is a new variation of [37], Theorem 4.1] for even $m$. By $\sigma_m$ we denote the group of permutations of the set $\{1, \ldots, m\}$.

**Theorem 3.5.** Let $m$ be even, $\{\tau, \nu\} \subset \sigma_m$ and let $H_1, \ldots, H_m$ be positive kernel operators on $L^2(X, \mu)$. For $j = 1, \ldots, \frac{m}{2}$ denote $A_j = H^*_{\tau(2j-1)}H_{\tau(2j)}$ and $A_{\frac{m}{2}+j} = A_j^* = H^*_{\tau(2j)}H_{\tau(2j-1)}$. Let $P_i = A_{\nu(i)} \cdots A_{\nu(m)}A_{\nu(1)} \cdots A_{\nu(i-1)}$ for $i = 1, \ldots, m$.

(i) Then

$$\|H_{\frac{1}{m}}^1 \circ \cdots \circ H_{\frac{1}{m}}^m\| \leq r\left(A_{\frac{1}{m}}^1 \circ \cdots \circ A_{\frac{1}{m}}^m\right)^{\frac{1}{2}}$$

(ii) If $H_1, \ldots, H_m$ are nonnegative matrices that define operators on $l^2(R)$ and if $\alpha \geq \frac{1}{m}$, then

$$\|H^{(\alpha)}_1 \circ \cdots \circ H^{(\alpha)}_m\| \leq r\left(A_{\frac{1}{m}}^{(\alpha)} \circ \cdots \circ A_{\frac{1}{m}}^{(\alpha)}\right)^{\frac{1}{2}}$$

(iii) If $H_1, \ldots, H_m$ are nonnegative matrices that define operators on $l^2(R)$ and if $\alpha \geq \frac{1}{m}$, then

$$\|H^{(\alpha)}_1 \circ \cdots \circ H^{(\alpha)}_m\| \leq r\left(A_{\frac{1}{m}}^{(\alpha)} \circ \cdots \circ A_{\frac{1}{m}}^{(\alpha)}\right)^{\frac{1}{2}}$$

**Proof.** First we prove (42). By

$$\|H\| = r(H^*H)^{\frac{1}{2}} = r(HH^*)^{\frac{1}{2}},$$

and commutativity of Hadamard product we have

$$\|H_{\frac{1}{m}}^1 \circ \cdots \circ H_{\frac{1}{m}}^m\| = r\left((H_{\frac{1}{m}}^1 \circ \cdots \circ H_{\frac{1}{m}}^m)^* (H_{\frac{1}{m}}^1 \circ \cdots \circ H_{\frac{1}{m}}^m)\right)^{\frac{1}{2}} =$$

$$r\left[(H_{\tau(1)}^*(\frac{1}{m})) \circ \cdots \circ (H_{\tau(m-1)}^*(\frac{1}{m})) (H_{\tau(2)}^*\circ \cdots \circ (H_{\tau(m)}^*(\frac{1}{m}))\right)^{\frac{1}{2}} \leq$$

$$r\left((H_{\tau(1)}^*H_{\tau(2)}^*)^* \circ \cdots \circ (H_{\tau(m-1)}H_{\tau(m)}^*)^* \circ \cdots \circ (H_{\tau(2)}^*H_{\tau(1)}^*)^* \circ \cdots \circ (H_{\tau(m)}^*H_{\tau(m-1)}^*)^*\right)^{\frac{1}{2}} =$$

$$r\left(A_{\frac{1}{m}}^1 \circ \cdots \circ A_{\frac{1}{m}}^m\right)^{\frac{1}{2}} = r\left(A_{\nu(1)}^1 \circ \cdots \circ A_{\nu(m)}^m\right)^{\frac{1}{2}},$$

which proves the first inequality in (35). The second and the third inequality in (42) follow from [37], Inequalities (4.2)].

Inequalities (43) are proved in a similar manner by applying Theorem 1.1(ii).

By interchanging $H_i$ with $H_i^*$ for all $i$ in Theorem 3.5 we obtain the following result.
Corollary 3.6. Let \( m \) be even, \( \tau \in \sigma_m, \beta \in [0, 1] \) and let \( H_1, \ldots, H_m \) be positive kernel operators on \( L^2(X, \mu) \). Let \( A_j \) for \( j = 1, \ldots, m \) be as in Theorem 3.5 and denote \( B_j = H_{r(2j-1)} H_{r(2j)}^* \) and \( B_{2j} = B_j^* = H_{r(2j)} H_{r(2j-1)}^* \) for \( j = 1, \ldots, m/2 \).

(i) Then
\[
\|H_{1}^{(\frac{1}{m})} \circ \cdots \circ H_{m}^{(\frac{1}{m})}\| \leq r(B_{1}^{(\frac{1}{m})} \circ \cdots \circ B_{m}^{(\frac{1}{m})})^{\frac{1}{2}}
\]
and
\[
\|H_{1}^{(\frac{1}{m})} \circ \cdots \circ H_{m}^{(\frac{1}{m})}\| \leq r(A_{1}^{(\frac{1}{m})} \circ \cdots \circ A_{m}^{(\frac{1}{m})})^{\frac{1}{2}} r(B_{1}^{(\frac{1}{m})} \circ \cdots \circ B_{m}^{(\frac{1}{m})})^{\frac{1}{2} - \beta}.
\]

(ii) If \( H_1, \ldots, H_m \) are nonnegative matrices that define operators on \( l^2(R) \) and if \( \alpha \geq \frac{1}{m} \), then
\[
\|H_{1}^{(\alpha)} \circ \cdots \circ H_{m}^{(\alpha)}\| \leq r(B_{1}^{(\alpha)} \circ \cdots \circ B_{m}^{(\alpha)})^{\frac{1}{2}}
\]
and
\[
\|H_{1}^{(\alpha)} \circ \cdots \circ H_{m}^{(\alpha)}\| \leq r(A_{1}^{(\alpha)} \circ \cdots \circ A_{m}^{(\alpha)})^{\frac{1}{2}} r(B_{1}^{(\alpha)} \circ \cdots \circ B_{m}^{(\alpha)})^{\frac{1}{2} - \beta}.
\]

Remark 3.7. In the special case of the identity permutation \( \mu \) in Theorem 3.5 it holds
\[
r(A_{\mu(1)} \cdots A_{\mu(m)})^{\frac{1}{2}} = \|H_{\tau(1)}^* H_{\tau(2)} H_{\tau(3)} H_{\tau(4)} \cdots H_{\tau(m-1)}^* H_{\tau(m)}\|
\]
by (44).

The following two results extend, generalize and refine \[48\] Theorem 2.8] and give an extension and a different refinement of \[37\] Inequality (4.16)] in the case \( \alpha \geq \frac{2}{m} \).

Theorem 3.8. Let \( m \) be even, \( \alpha \geq \frac{2}{m} \), \( \tau \in \sigma_m \) and let \( H_1, \ldots, H_m \) be nonnegative matrices that define operators on \( l^2(R) \). Let \( A_j \) for \( j = 1, \ldots, m \) be as in Theorem 3.5 and denote \( S_i = A_i \cdots A_{\frac{m}{2}} A_1 \cdots A_{i-1} \) for \( i = 1, \ldots, m/2 \). Then
\[
\|H_{1}^{(\alpha)} \circ \cdots \circ H_{m}^{(\alpha)}\| \leq r(A_{1}^{(\alpha)} \circ \cdots \circ A_{m}^{(\alpha)})^{\frac{1}{2}} \leq r(A_{1}^{(\alpha)} \circ \cdots \circ A_{\frac{m}{2}}^{(\alpha)})
\]
\[
= r((H_{\tau(1)}^* H_{\tau(2)})^{(\alpha)} \circ \cdots \circ (H_{\tau(m-1)}^* H_{\tau(m)})^{(\alpha)})
\]
\[
\leq r(S_{1}^{(\alpha)} \circ S_{2}^{(\alpha)} \circ \cdots \circ S_{\frac{m}{2}}^{(\alpha)})^{\frac{2}{\alpha}} \leq r(H_{\tau(1)}^* H_{\tau(2)} H_{\tau(3)}^* H_{\tau(4)} \cdots H_{\tau(m-1)}^* H_{\tau(m)})^{\alpha}.
\]

Proof. By the first inequality in (43) and (3) in Theorem 3.5(ii) we have
\[
\|H_{1}^{(\alpha)} \circ \cdots \circ H_{m}^{(\alpha)}\| \leq r(A_{1}^{(\alpha)} \circ \cdots \circ A_{\frac{m}{2}}^{(\alpha)})^{\frac{1}{2}}
\]
\[
= r(A_{1}^{(\alpha)} \circ \cdots \circ A_{\frac{m}{2}}^{(\alpha)}) r(A_{\frac{m}{2}}^{(\alpha)} \circ \cdots \circ A_{m}^{(\alpha)})^{\frac{1}{2}} \leq r(A_{1}^{(\alpha)} \circ \cdots \circ A_{\frac{m}{2}}^{(\alpha)}) r((A_{\frac{m}{2}}^{(\alpha)} \circ \cdots \circ A_{m}^{(\alpha)})^*)^{\frac{1}{2}} = r(A_{1}^{(\alpha)} \circ \cdots \circ A_{\frac{m}{2}}^{(\alpha)})
\]
\[
= r((H_{\tau(1)}^* H_{\tau(2)})^{(\alpha)} \circ (H_{\tau(3)}^* H_{\tau(4)})^{(\alpha)} \cdots \circ (H_{\tau(m-1)}^* H_{\tau(m)})^{(\alpha)}).}
Since
\[ ((H^*_\tau(1)H_{\tau(2)})^{(a)} \circ (H^*_\tau(3)H_{\tau(4)})^{(a)} \circ \cdots \circ (H^*_\tau(m-1)H_{\tau(m)})^{(a)})^{\frac{1}{\alpha}} = \]
\[ ((H^*_\tau(1)H_{\tau(2)})^{(a)} \circ \cdots \circ (H^*_\tau(m-1)H_{\tau(m)})^{(a)})((H^*_\tau(3)H_{\tau(4)})^{(a)} \circ \cdots \circ (H^*_\tau(1)H_{\tau(2)})^{(a)}) \]
\[ \cdots ((H^*_\tau(m-1)H_{\tau(m)})^{(a)} \circ \cdots \circ (H^*_\tau(m-3)H_{\tau(m-2)})^{(a)}), \]
we obtain by [15] that
\[ r((H^*_\tau(1)H_{\tau(2)})^{(a)} \circ (H^*_\tau(3)H_{\tau(4)})^{(a)} \circ \cdots \circ (H^*_\tau(m-1)H_{\tau(m)})^{(a)}) \leq \]
\[ r(S_1^{(a)} \circ S_2^{(a)} \circ \cdots \circ S_m^{(a)})^{\frac{2}{m}} \leq (r(S_1)^{\alpha} \cdots r(S_m)^{\alpha})^{\frac{2}{m}} = r(H^*_\tau(1)H_{\tau(2)}H^*_\tau(3)H_{\tau(4)} \cdots H^*_\tau(m-1)H_{\tau(m)})^{\alpha}, \]
where the last equality follows from \( r(S_1) = \cdots = r(S_m). \)

\begin{corollary}
Let \( m \) be even, \( \alpha \geq \frac{2}{m}, \tau \in \sigma_m, \beta \in [0,1] \) and let \( H_1, \ldots, H_m \) be nonnegative matrices that define operators on \( l^2(R) \). Let \( A_j \) and \( B_j \) for \( j = 1, \ldots, m \) be as in Corollary 3.8 and denote \( S_i = A_i \cdots A_{\frac{m}{2}} A_1 \cdots A_{i-1} \) and \( T_i = B_i \cdots B_{\frac{m}{2}} B_1 \cdots B_{i-1} \) for \( i = 1, \ldots, \frac{m}{2} \). Then
\end{corollary}

\begin{equation}
\| H_1^{(a)} \circ \cdots \circ H_m^{(a)} \| \leq r(A_1^{(a)} \circ \cdots \circ A_m^{(a)})^{\frac{2}{\alpha}} r(B_1^{(a)} \circ \cdots \circ B_m^{(a)})^{\frac{1-\beta}{m}} \leq r((H^*_\tau(1)H_{\tau(2)})^{(a)} \circ (H^*_\tau(3)H_{\tau(4)})^{(a)} \circ \cdots \circ (H^*_\tau(m-1)H_{\tau(m)})^{(a)})^{\beta}.
\end{equation}

\( r((H^*_\tau(1)H_{\tau(2)})^{(a)} \circ (H^*_\tau(3)H_{\tau(4)})^{(a)} \circ \cdots \circ (H^*_\tau(m-1)H_{\tau(m)})^{(a)})^{1-\beta} \leq r(H^*_\tau(1)H_{\tau(2)}H^*_\tau(3)H_{\tau(4)} \cdots H^*_\tau(m-1)H_{\tau(m)})^{\alpha(1-\beta)}. \)

The following result extends [18, Theorem 2.13] and [37, Theorem 4.1].

\begin{theorem}
Let \( H_1, \ldots, H_m \) be positive kernel operators on \( L^2(X, \mu) \) and \( \{\tau, \nu\} \subset \sigma_m \). Denote \( Q_j = H^*_\tau(j)H^*_\nu(j) \cdots H^*_\tau(m)H^*_\nu(m)H^*_\tau(1)H^*_\nu(1) \cdots H^*_\tau(j-1)H^*_\nu(j-1) \) for \( j = 1, \ldots, m \).

(i) Then
\begin{equation}
\| H_1^{(\frac{1}{m})} \circ \cdots \circ H_m^{(\frac{1}{m})} \| \leq r((H^*_\tau(1)H^*_\nu(1))^{\frac{1}{m}} \circ \cdots \circ (H^*_\tau(m)H^*_\nu(m))^{\frac{1}{m}})^{\frac{1}{2}} \leq r(Q_1^{(\frac{1}{m})} \cdots \circ Q_m^{(\frac{1}{m})})^{\frac{1}{2m}} \leq r(H^*_\tau(1)H^*_\nu(1) \cdots H^*_\tau(m)H^*_\nu(m))^{\frac{1}{2m}}.
\end{equation}

(ii) If \( H_1, \ldots, H_m \) are nonnegative matrices that define operators on \( l^2(R) \) and if \( \alpha \geq \frac{1}{m} \), then
\begin{equation}
\| H_1^{(a)} \circ \cdots \circ H_m^{(a)} \| \leq r((H^*_\tau(1)H^*_\nu(1))^{(a)} \circ \cdots \circ (H^*_\tau(m)H^*_\nu(m))^{(a)})^{\frac{1}{2}} \leq r(Q_1^{(a)} \circ \cdots \circ Q_m^{(a)})^{\frac{1}{2m}} \leq r(H^*_\tau(1)H^*_\nu(1) \cdots H^*_\tau(m)H^*_\nu(m))^{\frac{1}{2}}.
\end{equation}
\end{theorem}
Proof. First we prove (47). By (44) and (5) we have
\[
\|H_1^{(\frac{1}{m})} \circ \cdots \circ H_m^{(\frac{1}{m})}\| = r((H_1^{(\frac{1}{m})} \circ \cdots \circ H_m^{(\frac{1}{m})})^*) (H_1^{(\frac{1}{m})} \circ \cdots \circ H_m^{(\frac{1}{m})})^\frac{1}{2} =
\]
\[
r((H_1^{(\frac{1}{m})} \circ \cdots \circ H_m^{(\frac{1}{m})})^*) (H_1^{(\frac{1}{m})} \circ \cdots \circ H_m^{(\frac{1}{m})}) (H_1^{(\frac{1}{m})} \circ \cdots \circ H_m^{(\frac{1}{m})})^\frac{1}{2} \leq r((H_1^{(\frac{1}{m})} \circ \cdots \circ H_m^{(\frac{1}{m})})^*) (H_1^{(\frac{1}{m})} \circ \cdots \circ H_m^{(\frac{1}{m})})^\frac{1}{2}.
\]
Notice that
\[
((H_{\tau(1)}^* H_{\nu(1)})^{(\frac{1}{m})} \circ \cdots \circ (H_{\tau(m)}^* H_{\nu(m)})^{(\frac{1}{m})})^m = ((H_{\tau(1)}^* H_{\nu(1)})^{(\frac{1}{m})} \circ \cdots \circ (H_{\tau(m)}^* H_{\nu(m)})^{(\frac{1}{m})})^m
\]
\[
((H_{\tau(2)}^* H_{\nu(2)})^{(\frac{1}{m})} \circ \cdots \circ (H_{\tau(m)}^* H_{\nu(m)})^{(\frac{1}{m})}) \cdots ((H_{\tau(m-1)}^* H_{\nu(m-1)})^{(\frac{1}{m})} \circ \cdots \circ (H_{\tau(m-1)}^* H_{\nu(m-1)})^{(\frac{1}{m})})
\]
It follows by (5) that
\[
r((H_{\tau(1)}^* H_{\nu(1)})^{(\frac{1}{m})} \circ \cdots \circ (H_{\tau(m)}^* H_{\nu(m)})^{(\frac{1}{m})})^\frac{1}{2} \leq r(Q_1^{(\frac{1}{m})} \circ \cdots \circ Q_m^{(\frac{1}{m})})^\frac{1}{2m}
\]
\[
\leq (r(Q_1) \cdots r(Q_m))^\frac{1}{2m} = r(H_{\tau(1)}^* H_{\nu(1)} \cdots H_{\tau(m-1)}^* H_{\nu(m-1)})^\frac{1}{2m},
\]
where the last equality follows from \( r(Q_1) = \ldots = r(Q_m) \). This completes the proof of (47). The proof of (48) is similar by applying Theorem 1.1(ii). \( \square \)

The following corollary is a refinement of [37, Inequality (4.11)], which differs from refinements in [37, Inequalities (4.15) and (4.17)]. It also extends and generalizes [48, Corollary 2.15].

**Corollary 3.11.** Let \( m \) be odd and let \( H_1, \ldots, H_m \) be positive kernel operators on \( L^2(X, \mu) \).

(i) Then
\[
\|H_1^{(\frac{1}{m})} \circ \cdots \circ H_m^{(\frac{1}{m})}\| \leq r((H_1^* H_2)^{\frac{1}{m}} \circ \cdots \circ (H_{m-2}^* H_{m-1})^{\frac{1}{m}} \circ (H_m^* H_1)^{\frac{1}{m}} \circ (H_2^* H_3)^{\frac{1}{m}} \circ \cdots \circ (H_{m-1}^* H_m)^{\frac{1}{m}})^{\frac{1}{2}}
\]
\[
\leq r(H_1^* H_2 \cdots H_{m-2}^* H_{m-1}^* H_m^* H_1 H_2 H_3 \cdots H_{m-1}^* H_m)^{\frac{1}{2m}}.
\]

(ii) If \( H_1, \ldots, H_m \) are nonnegative matrices that define operators on \( l^2(R) \) and if \( \alpha \geq \frac{1}{m} \), then
\[
\|H_1^{(\alpha)} \circ \cdots \circ H_m^{(\alpha)}\| \leq r((H_1^* H_2)^{(\alpha)} \circ \cdots \circ (H_{m-2}^* H_{m-1})^{(\alpha)} \circ (H_m^* H_1)^{(\alpha)} \circ (H_2^* H_3)^{(\alpha)} \circ \cdots \circ (H_{m-1}^* H_m)^{(\alpha)})^{\frac{1}{2}}
\]
\[
\leq r(H_1^* H_2 \cdots H_{m-2}^* H_{m-1}^* H_m^* H_1 H_2 H_3 \cdots H_{m-1}^* H_m)^{\frac{1}{2m}}.
\]

**Proof.** The result follows by taking the permutations \( \tau(j) = 2j - 1 \) for \( 1 \leq j \leq \frac{m+1}{2} \); \( \tau(j) = 2(j - \frac{m+1}{2}) \) for \( \frac{m+1}{2} \leq j \leq m \) and \( \nu(j) = 2j \) for \( 1 \leq j \leq \frac{m-1}{2} \); \( \nu(j) = 2(j - \frac{m-1}{2}) - 1 \) for \( \frac{m+1}{2} \leq j \leq m \) in Theorem 3.10. \( \square \)
The following corollary gives new lower bounds for the operator norm of the Jordan triple product $ABA$, which differ from the one obtained in [37, Corollary 4.10]. The result follows from Corollary 3.11 and Theorem 3.10 by taking $H_1 = A$, $H_2 = B^*$ and $H_3 = A$.

**Corollary 3.12.** Let $A$ and $B$ be positive kernel operators on $L^2(X, \mu)$.

(i) Then
\[
\|A^{\frac{1}{3}} \circ (B^*)^{\frac{1}{3}} \circ A^{\frac{1}{3}}\| \\
\leq \frac{r^2}{2} \left( (A^*B^*)^{\frac{1}{3}} \circ (A^*A)^{\frac{1}{3}} \circ (BA)^{\frac{1}{3}} \right) \\
\leq \frac{r^2}{2} \left( (A^*B^*A^*ABA)^{\frac{1}{3}} \circ (A^*ABAA^*B^*)^{\frac{1}{3}} \circ (BAA^*B^*A^*)^{\frac{1}{3}} \right) \\
(51) \\
\leq \|ABA\|^{\frac{1}{3}}.
\]

(ii) If $A$ and $B$ are nonnegative matrices that define operators on $l^2(R)$ and if $\alpha \geq \frac{1}{3}$, then
\[
\|A^{(\alpha)} \circ (B^*)^{(\alpha)} \circ A^{(\alpha)}\| \\
\leq \frac{r^2}{2} \left( (A^*B^*)^{(\alpha)} \circ (A^*A)^{(\alpha)} \circ (BA)^{(\alpha)} \right) \\
\leq \frac{r^2}{2} \left( (A^*B^*A^*ABA)^{(\alpha)} \circ (A^*ABAA^*B^*)^{(\alpha)} \circ (BAA^*B^*A^*)^{(\alpha)} \right) \\
(52) \\
\leq \|ABA\|^{\alpha}.
\]

The following result generalizes [48, Inequality (2.12)].

**Lemma 3.13.** Let $\alpha \geq \frac{1}{2}$ and let $C$ be a nonnegative matrix that defines an operator on $l^2(R)$. Then
\[
r(C^{(\alpha)} \circ (C^*)^{(\alpha)}) \leq r(C^{(\alpha)} \circ C^{(\alpha)}) \leq r(C)^{2\alpha}.
\]

**Proof.** By (3) in Theorem 1.1(ii) applied twice we have
\[
r(C^{(\alpha)} \circ (C^*)^{(\alpha)}) = r((C^{(\alpha)} \circ C^{(\alpha)})^{\frac{1}{2}} \circ ((C^*)^{(\alpha)} \circ (C^*)^{(\alpha)})^{\frac{1}{2}}) \\
\leq r(C^{(\alpha)} \circ C^{(\alpha)})^\frac{1}{2} r((C^*)^{(\alpha)} \circ (C^*)^{(\alpha)})^\frac{1}{2} = r(C^{(\alpha)} \circ C^{(\alpha)}) \leq r(C)^{2\alpha},
\]
which completes the proof. \[\square\]

The following result generalizes [48, Theorem 2.17] and refines [37, Inequalities (4.9)]. It follows e.g. from Theorem 3.10 (or [37, Inequalities (4.9)]) and Lemma 3.13.

**Corollary 3.14.** Let $\alpha \geq \frac{1}{2}$ and let $A$ and $B$ be nonnegative matrices that define operators on $l^2(R)$. Then
\[\left\|A^{(\alpha)} \circ B^{(\alpha)}\right\| \leq r_1^{\frac{1}{2}} \left(\left(A^*B\right)^{(\alpha)} \circ \left(B^*A\right)^{(\alpha)}\right)\]

\[\leq r_1^{\frac{1}{2}} \left(\left(A^*B\right)^{(\alpha)} \circ \left(A^*B\right)^{(\alpha)}\right) \leq r_1^{\alpha} (A^*B),\]

(53)

**Remark 3.15.** Several results of Section 3 can be further refined by applying Theorems 3.1 and 3.4 in the proofs. We omit the details.

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