The Lexicographic Method for the Threshold Cover Problem

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Abstract. The lexicographic method is a technique that was introduced by Hell and Huang [Journal of Graph Theory, 20(3):361–374, 1995] as a way to simplify the problems of recognizing and obtaining representations of comparability graphs, proper circular-arc graphs and proper interval graphs. This method gives rise to conceptually simple recognition algorithms and leads to much simpler proofs for some characterization theorems for these classes. Threshold graphs are a class of graphs that have many equivalent definitions and have applications in integer programming and set packing problems. A graph is said to have a threshold cover of size $k$ if its edges can be covered using $k$ threshold graphs. Chvátal and Hammer conjectured in 1977 that given a graph $G$, a suitably constructed auxiliary graph $G'$ has chromatic number equal to the minimum size of a threshold cover of $G$. Although Cozzens and Leibowitz showed that this conjecture is false in the general case, Raschle and Simon [Proceedings of the Twenty-seventh Annual ACM Symposium on Theory of Computing, STOC ’95, pages 650–661, 1995] proved that $G$ has a threshold cover of size 2 if and only if $G'$ is bipartite. We show how the lexicographic method can be used to obtain a completely new and much simpler proof for this result. This method also gives rise to a simple new LexBFS-based algorithm for recognizing graphs having a threshold cover of size 2. Although this algorithm is not the fastest known, it is a certifying algorithm that matches the time complexity of the fastest known certifying algorithm for this problem. The algorithm can also be easily adapted to give a certifying recognition algorithm for bipartite graphs that can be covered by two chain subgraphs.

Keywords: Lexicographic method · Threshold cover · Chain graph cover.

1 Introduction

We consider only simple, undirected and finite graphs. A graph $G$ is said to be a threshold graph if it does not contain a pair of edges $ab, cd$ such that $ad, bc \notin E(G)$; or equivalently, $G$ is $(2K_2, P_4, C_4)$-free [1]. A graph $G = (V, E)$ is said to be covered by the graphs $H_1, H_2, \ldots, H_k$ if $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_k)$. A graph $G$ is said to have a threshold cover of size $k$ if it can be covered by $k$ threshold graphs. The threshold dimension of a graph $G$ is defined to be the smallest integer $k$ such that $G$ has a threshold cover of size $k$. Mahadev and Peled [12] give a comprehensive survey of threshold graphs and their applications.

Chvátal and Hammer [1] showed that the fact that a graph has a threshold cover of size $k$ is equivalent to the following: there exist $k$ linear inequalities on $|V(G)|$ variables such that the characteristic vector of a set $S \subseteq V(G)$ satisfies all the inequalities if and only if $S$ is an independent set of $G$ (see [13] for details). They further defined the auxiliary graph $G'$ (defined in Section 2) corresponding to a graph $G$ and showed that any threshold cover of $G$ must have size at least $\chi(G')$. This gave rise to the question of whether there exist any graph $G$ that does not have a threshold cover of size $\chi(G')$. Cozzens and Leibowitz [4] showed the existence of such graphs. In particular, they showed that for every $k \geq 4$, there exists a graph $G$ such that $\chi(G') = k$ but $G$ has no threshold cover of size $k$. The question of whether such graphs exist for $k = 2$ seems to have been intensely studied but remained open for a decade (see [11]). Ibaraki and Peled [7] showed that if $G$ is a split graph or if $G'$ contains at most two non-trivial components, then $\chi(G') = 2$ if and only if $G$ has a threshold cover of size 2. They further conjectured that every graph $G$ satisfying $\chi(G') \leq 2$ has a threshold cover of size 2. If the conjecture were to be true, it would have the important consequence that graphs having a threshold cover of size 2 can be recognized in polynomial time. In contrast, Yamakakis [10] showed that it is NP-complete to recognize graphs having a threshold cover of size 3. Cozzens and Halsey [3] studied some properties of graphs having a threshold cover of size 2 and showed that it can be decided in polynomial time whether the complement of a bipartite graph has a threshold cover of size 2. Finally, in 1995, Raschle and Simon [13] proved the conjecture of Ibaraki and Peled by extending the methods in [7]: they...
showed that every graph $G$ whose auxiliary graph $G'$ is bipartite has a threshold cover of size 2. This proof is very technical and involves the use of a number of complicated reductions and previously known results. In particular, they construct a set of edges that have a “threshold completion” by finding a 2-colouring of $G'$ that is so-called “AC$6_2$-free”, where $l \geq 2$ (a colouring of $G'$ is AC$6_2$-free, if for each colour class $S$, there is no cyclical sequence $v_1, v_2, \ldots, v_{2l}, v_1$ of vertices in $G'$ such that $v_1v_{i+1} \in S$ if and only if $i$ is odd). It is then shown that this reduces to finding a 2-colouring of $G'$ which is AC$6_2$-free. This further reduces to finding a so-called “$AP_6$-free” 2-colouring of $G'$ which further reduces to finding a so-called “double $AP_6$-free 2-colouring” of $G'$. The most intricate part is the proof of correctness of an algorithm that computes this particular kind of 2-colouring of $G'$.

The paper of Raschle and Simon also gives an $O(|E(G)|^2)$ algorithm that checks whether a graph $G$ has a threshold cover of size 2 and outputs two threshold graphs that cover $G$ in case it has. If the input graph $G$ does not have a threshold cover of size 2, the algorithm detects an odd cycle in the auxiliary graph $G''$. This odd cycle gives edges $e_1, e_2, \ldots, e_k$ in $G$, where $k$ is odd, such that the edges $e_i, e_{i+1}$, for $1 \leq i < k$, and the edges $e_k, e_1$, can never both belong to any threshold subgraph of $G$ (because their endpoints induce a $2K_2$, $P_4$ or $C_4$ in $G$). In this way, the algorithm provides an easily verifiable “certificate” for the fact that there does not exist two threshold graphs that cover $G$. If $G$ does have a threshold cover of size 2, then the two threshold graphs returned by the algorithm that cover $G$ form an easily verifiable certificate for that fact. Such algorithms are called certifying algorithms [5].

Since as noted above, an odd cycle in the auxiliary graph $G'$ corresponds to a structure present in $G$ that serves as an “obstruction” to it having a threshold cover of size 2, the result of Raschle and Simon can also be seen as a “forbidden structure characterization” of graphs having a threshold cover of size 2. That is, a graph $G$ has a threshold cover of size 2 if and only if the said obstruction is not present in $G$. In fact, this is the only known forbidden structure characterization for this family of graphs. Such characterizations exist for many different classes of graphs — for example, interval graphs [9] and circular-arc graphs [8].

In this paper, we propose a completely different and self-contained proof for the theorem of Raschle and Simon that a graph $G$ can be covered by two threshold graphs if and only if $G'$ is bipartite. Our proof is short and direct, and also gives rise to a simpler (although having the same asymptotic worst case running time of $O(|E(G)|^2)$) certifying recognition algorithm for graphs having a threshold cover of size 2.

Note that faster algorithms for determining if a graph has a threshold cover of size 2 are known. After the algorithm of Raschle and Simon [13], Sterbini and Raschle [15] used some observations of Ma [10] to construct an $O(|V(G)|^3)$ algorithm for the problem. But this algorithm is not a certifying algorithm in the sense that if the input graph $G$ does not have a threshold cover of size 2, it does not produce an obstruction in $G$ that prevents it from having a threshold cover of size 2. Note that there is an obvious way to make this algorithm a certifying algorithm: if the algorithm answers that the input graph $G$ does not have a threshold cover of size 2, run a secondary algorithm that constructs $G'$ and finds an odd cycle in it (this odd cycle can serve as a certificate). But a naive implementation of the secondary algorithm will have worst-case running time $\Theta(|E(G)|^2)$, and it is not clear if it can be implemented to run in time $o(|E(G)|^2)$.

In the current work, we show that a graph $G$ has a threshold cover of size 2 if and only if its auxiliary graph $G'$ is bipartite using a technique called the lexicographic method which was introduced by Hell and Huang [6]. Hell and Huang demonstrated how this method can lead to shorter proofs and simpler recognition algorithms for certain problems that can be viewed as orienting the edges of a graph satisfying certain conditions — for example, they showed how this method can lead to simpler characterization proofs and recognition algorithms for comparability graphs, proper interval graphs and proper circular-arc graphs. The method starts by taking an arbitrary ordering of the vertices of the graph. It then prescribes choosing the lexicographically smallest (with respect to the given vertex ordering) edge to orient and then orienting it in one way or the other, along with all the edges whose orientations are forced by it. Hell and Huang showed that the lexicographic approach makes it easy to ensure that the orientation so produced satisfies the necessary conditions, if such an orientation exists. We adapt this technique to the problem of generating two threshold graphs that cover a given graph, if two such graphs exist. This shows that the applicability of the lexicographic method may not be limited to only problems involving orientation of edges. However, it should be noted that in our proof, we start with a Lex-BFS ordering of the vertices of the graph instead of an arbitrary ordering. It is an ordering of the vertices that gives the order in which a Lex-BFS, or lexicographic breadth first search, a graph searching algorithm that was introduced by Rose, Tarjan and Lueker [14], may visit the vertices of the graph. A Lex-BFS ordering always gives an order in which a breadth-first search can visit the vertices...
of the graph, but has some additional properties. Lex-BFS can be implemented to run in time linear in the size of the input graph and Rose, Tarjan and Lueker originally used this algorithm to construct a linear-time algorithm for recognizing chordal graphs. Later, Lex-BFS based algorithms were discovered for the recognition of many different graph classes (see [2] for a survey).

2 Preliminaries

Let \( G = (V, E) \) be any graph. Two edges \( ab, cd \) are said to form a pair of cross edges in \( G \) if \( ad, bc \notin E(G) \). If \( ab, cd \) form a pair of cross edges in \( G \), we say that the set \( \{a, b, c, d\} \) is a crossing set in \( G \) (such a set is called an \( AC_4 \) in [13]). It is easy to see that threshold graphs are exactly the graphs that contain no pairs of cross edges, or equivalently no crossing set.

For a graph \( G \), the auxiliary graph \( G' \) is defined to be the graph with \( V(G') = E(G) \) and \( E(G') = \{e_1e_2 : e_1, e_2 \text{ form a pair of cross edges in } G\} \). We shall refer to the vertex of \( G' \) corresponding to an edge \( ab \in E(G) \) alternatively as \( \{a, b\} \) or \( ab \), depending upon the context. The following lemma is just a special case of the observation of Chvátal and Hammer [1] that a graph \( G \) cannot have a threshold cover of size less than \( \chi(G') \).

**Lemma 1.** If a graph \( G = (V, E) \) has a threshold cover of size two then \( G' \) is bipartite.

**Proof.** Let \( G \) be covered by two threshold graphs \( H_1 \) and \( H_2 \). By the definition of \( G' \), if \( \{ab, cd\} \in E(G') \) then \( ad, bc \notin E(G) \). The fact that \( H_1 \) and \( H_2 \) are threshold subgraphs of \( G \) then implies that neither \( H_1 \) nor \( H_2 \) can contain both the edges \( ab \) and \( cd \). We therefore conclude that the sets \( E(H_1) \) and \( E(H_2) \) are both independent sets in \( G' \). Since \( G \) is covered by \( H_1 \) and \( H_2 \), we have that \( V(G') = E(H_1) \cup E(H_2) \). Thus, \( \{E(H_1), E(H_2) \setminus E(H_1)\} \) forms a bipartition of \( G' \) into two independent sets. This completes the proof. \( \square \)

Our goal is to provide a new proof for the following theorem of Raschle and Simon [13].

**Theorem 1.** A graph \( G \) can be covered by two threshold graphs if and only if \( G' \) is bipartite.

By Lemma 1 it is enough to prove that if \( G' \) is bipartite, then \( G \) can be covered by two threshold graphs. In order to prove this, we find a specific 2-coloring of the non-trivial components of \( G' \) using the lexicographic method of Hell and Huang [4].

Let \( < \) be an ordering of the vertices of \( G \). Given two \( k \)-element subsets \( S = \{s_1, s_2, \ldots, s_k\} \) and \( T = \{t_1, t_2, \ldots, t_k\} \) of \( V(G) \), where \( s_1 < s_2 < \cdots < s_k \) and \( t_1 < t_2 < \cdots < t_k \), \( S \) is said to be lexicographically smaller than \( T \), denoted by \( S < T \), if \( s_j < t_j \) for some \( j \in \{1, 2, \ldots, k\} \), and \( s_i = t_i \) for all \( i \leq j \leq k \). In the usual way, we let \( S \leq T \) denote the fact that either \( S < T \) or \( S = T \). For a set \( S \subseteq V(G) \), we abbreviate \( \min_S \) to just \( \min S \). Note that the relation \( < \) (“is lexicographically smaller than”) that we have defined on \( k \)-element subsets of \( V(G) \) is a total order. Therefore, given a collection of \( k \)-element subsets of \( V(G) \), the lexicographically smallest one among them is well-defined.

3 Proof of Theorem 1

Assume that \( G' \) is bipartite. Let \( < \) denote a Lex-BFS ordering of the vertices of \( G \). The following observation states a well-known property of Lex-BFS orderings [2].

**Observation 1** For \( a, b, c \in V(G) \), if \( a < b < c, \ ab \notin E(G) \) and \( ac \in E(G) \), then there exists \( x \in V(G) \) such that \( x < a < b < c, \ xb \in E(G) \) and \( xc \notin E(G) \).

We shall now construct a partial 2-coloring of the vertices of \( G' \) using the colors \( \{1, 2\} \). Notice that choosing a color for any vertex in a component of \( G' \) fixes the colors of all the other vertices in that component. Recall that every vertex of \( G' \) is a two-element subset of \( V(G) \). For every non-trivial component \( C \) of \( G' \), perform the following operation: Choose the lexicographically smallest vertex in \( C \) (with respect to the ordering \( < \)) and assign the color 1 to it. This fixes the colors of all the other vertices in \( C \). Note that after this procedure, every vertex of \( G' \) that is in a non-trivial component has been colored either 1 or 2.
For \(i \in \{1, 2\}\), let \(F_i = \{e \in V(G') : e \text{ is colored } i\}\). Further, let \(F_0\) denote the set of all isolated vertices in \(G'\). Clearly, \(F_0\) is exactly the set of uncolored vertices of \(G'\) and we have \(V(G') = F_0 \cup F_1 \cup F_2\). Consider the subgraphs \(H_1 = (V, F_1 \cup F_0)\) and \(H_2 = (V, F_2 \cup F_0)\) of \(G\). We claim that \(H_1\) and \(H_2\) are two threshold graphs that cover \(G\). Clearly \(E(G) = E(H_1) \cup E(H_2)\); so it only remains to be proven that both \(H_1\) and \(H_2\) are threshold graphs. Note that for any edge \(ab \in E(G), ab \notin E(H_1) \Rightarrow ab \in F_2\) and \(ab \notin E(H_2) \Rightarrow ab \in F_1\).

**Observation 2** If \(ab, cd\) form a pair of cross edges in \(G\), then exactly one of the following is true:

1. \(ab \in F_1\) and \(cd \in F_2\), or
2. \(ab \in F_2\) and \(cd \in F_1\).

Therefore, \(ab\) and \(cd\) cannot be present together in either \(H_1\) or \(H_2\).

\[\square\]

For \(i \in \{1, 2\}\), let \(\mathcal{P}_i = \{(x, y, z, w) : xy, zw \in E(H_i), xw \notin E(G)\text{ and }yz \in E(G) \setminus E(H_i)\}\) and \(\mathcal{C}_i = \{(x, y, z, w) : xy, zw \in E(H_i), xw, yz \in E(G) \setminus E(H_i)\}\). By Observation 2, it can be seen that the crossing sets in \(H_i\) are exactly the elements of \(\mathcal{P}_i \cup \mathcal{C}_i\). Define \(\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2\) and \(\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2\). Notice that in order to show that both \(H_1\) and \(H_2\) are threshold graphs, we only need to prove that \(\mathcal{P} \cup \mathcal{C} = \emptyset\). We shall first show that \(\mathcal{P} = \emptyset\). Suppose not. Let \(\{a, b, c, d\}\) be the lexicographically smallest element in \(\mathcal{P}\).

**Lemma 2.** \(\{a, b, c, d\} \notin \mathcal{P}_1\).

\[\square\]

Proof. Suppose for the sake of contradiction that \(\{a, b, c, d\} \in \mathcal{P}_1\). By definition of \(\mathcal{P}_1\), we can assume without loss of generality that \(ab, cd \in E(H_1), ad \notin E(G)\) and \(bc \in E(G) \setminus E(H_1)\). Since \(E(G) = E(H_1) \cup E(H_2)\), we have that \(bc \in E(H_2) \setminus E(H_1)\), which implies that \(bc \in F_2\). By the definition of \(F_2\), we have that \(bc\) belongs to a non-trivial component \(C\) of \(G'\) and has been colored 2. Therefore \(\{b, c\}\) is not the lexicographically smallest vertex in \(C\). Let \(\{b, c\}\) be the lexicographically smallest vertex in \(C\) (\(k\) is defined below). Then we have \(\{b_k, c_k\} < \{b, c\}\) and by our construction the vertex \(\{b_k, c_k\}\) must have received color 1. Let \(bc = b_0c_0, b_1c_1, \ldots, b_{i-1}c_{i-1}, b_ic_k\) be a path in \(C\) between \(\{b, c\}\) and \(\{b_k, c_k\}\), where for \(0 \leq i < k\), \(b_{i+1}c_i \notin E(G)\). Note that \(k\) is odd, \(b_ic_i \in F_2\) for each even \(i\) and \(b_ic_i \in F_1\) for each odd \(i\), where \(0 \leq i \leq k\).

We claim that \(ab_i, c_id \in E(H_1)\) for each even \(i\) and \(ab_i, c_id \in E(H_2)\) for each odd \(i\), where \(0 \leq i \leq k\). We prove this by induction on \(i\). The case where \(i = 0\) is trivial as \(b_0 = b = c_0 = c\). So let us assume that \(i > 0\). Consider the case where \(i\) is odd. As \(i-1\) is even, by the induction hypothesis we have, \(ab_{i-1}, c_{i-1}d \in E(H_1)\). As \(ab_{i-1}, b_ic_i \in E(H_1)\) and \(b_{i-1}c_i \notin E(G)\), by Observation 2 we have that \(ab_i \in E(G)\). Now as \(ab_i, c_{i-1}d\) form a pair of cross edges in \(G\) and \(c_{i-1}d \in E(H_1)\) the same observation then implies that \(ab_i \in E(H_2)\). Similarly, as \(c_{i-1}d, b_ic_i \in E(H_1)\) and \(b_{i-1}c_i \notin E(G)\), we have \(c_id \in E(G)\). Again, as \(c_id, ab_{i-1}\) form a pair of cross edges in \(G\) and \(ab_{i-1} \in E(H_1)\) we have \(c_id \in E(H_2)\). The case where \(i\) is even is also similar and hence the claim.

By the above claim, \(ab_k, c_kd \in E(H_2)\). Since \(b_kc_k \in F_1\), \(b_kc_k \notin E(H_2)\). Recalling that \(ad \notin E(G)\), we now have that \(\{a, b, c, d\} \notin \mathcal{P}_2\). Since \(\{b, c\} < \{b, c\}\), we have that \(\{a, b, c, d\} < \{a, b, c, d\}\), which is a contradiction.

\[\square\]

**Lemma 3.** \(\{a, b, c, d\} \notin \mathcal{P}_2\).

\[\square\]

Proof. Suppose for the sake of contradiction that \(\{a, b, c, d\} \in \mathcal{P}_2\). By definition of \(\mathcal{P}_2\), we can assume without loss of generality that \(ab, cd \in E(H_2), ad \notin E(G)\) and \(bc \notin E(H_2)\). Recall that \(bc \notin E(H_2) \Rightarrow bc \in F_1\). As \(bc \in F_1\), the vertex \(bc\) belongs to a non-trivial component of \(G'\). Then there exists a neighbor \(b'c'\) of \(bc\) in \(G'\) such that \(b'c' \notin E(G)\). By Observation 2, \(bc \in F_1\) implies \(b'c' \in F_2\). Further, \(ab', b'c' \in E(H_2)\) and \(b'c' \notin E(G)\) implies that \(ab' \in E(G)\). Now \(ab', cd\) form a pair of cross edges in \(G\). Since \(cd \in E(H_2)\), we now have by Observation 2 that \(cd \in F_2\) and \(ab' \in F_1\). This implies that \(cd\) is in a non-trivial component \(C_1\) of \(G'\). Similarly, as \(cd, b'c' \in E(H_2)\) and \(b'c' \notin E(G)\) we have that \(c'd \in E(G)\). Now \(c'd, ab\) form a pair of cross edges in \(G'\) with \(ab \notin E(G)\) and \(cd \notin E(H_2)\). This is a contradiction.
edges in \( G \). Since \( ab \in E(H_2) \), we have by Observation 2 that \( ab \in F_2 \) and \( cd \in F_1 \). This implies that \( ab \) is in a non-trivial component \( C_2 \) of \( G' \).

We now prove two claims using the fact that \( < \) is a Lex-BFS ordering of \( V(G) \).

Claim 1. \( d < a < c \) is not possible.

Suppose not. Note that \( da \notin E(G) \) and \( dc \notin E(G) \). Then by Observation 1 there exists \( x \in V(G) \) such that \( x < d < a < c, xa \in E(G) \) and \( xc \notin E(G) \). Now \( cd, xa \) form a pair of cross edges in \( G \). By Observation 2, \( cd \in F_2 \) implies that \( xa \in F_1 \). As \( bc \in F_1 \), \( xc \notin E(G) \) and \( ab \notin E(H_1) \) (recall that \( ab \in F_2 \)) we then have that \( \{ x, a, b, c \} \in P_1 \). Further, \( x < d \) implies that \( \{ x, a, b, c \} \) which is a contradiction.

The next claim is symmetric to the claim above, so the proof is omitted.

Claim 2. \( a < d < b \) is not possible.

As \( cd \in F_2 \), \( cd \) must have received color 2 in the partial 2-coloring of \( G' \). This means that \( cd \) is not the lexicographically smallest vertex in the component \( C_1 \). Let \( \{ c_k, d_k \} \) be the lexicographically smallest vertex in \( C_1 \). Then we have \( \{ c_k, d_k \} < \{ c, d \} \) and by our construction, the vertex \( \{ c_k, d_k \} \) must have received color 1. Let \( cd = c_0d_0, c_1d_1, \ldots, c_{k-1}d_{k-1}, c_kd_k \) be a path in \( C_1 \) between \( cd \) and \( d_kc_k \), where for \( 0 \leq i < k, c_i d_{i+1}, c_{i+1} d_i \notin E(G) \). Note that \( k \) is odd, \( c_i d_i \in F_2 \) for each even \( i \) and \( c_i d_i \in F_1 \) for each odd \( i \), where \( 0 \leq i \leq k \).

Claim 3. \( c_i b, c_i d_i \in F_1 \) for each even \( i \) and \( c_i b, c_i d_i \in F_2 \) for each odd \( i \), where \( 0 \leq i \leq k \).

We prove this by induction on \( i \). The case \( i = 0 \) is trivial as \( c_0 = c \) and \( d_0 = d \). So let us assume that \( i > 0 \). Consider the case where \( i \) is odd. As \( i - 1 \) is even, we have by the induction hypothesis that \( c_{i-1} b, c_{i-1} d_{i-1} \in F_1 \). As \( c_i d_i \) is odd can be proved in the same way. Hence the claim.

Recall that \( ab \in F_2 \), \( ab \) is in a non-trivial component \( C_2 \) of \( G' \), and it has color 2 in the partial 2-coloring of \( G' \). Therefore, there exists a lexicographically smallest vertex \( \{ a_k, b_k \} \in C_2 \) which has been colored 1. Clearly, \( \{ a_k, b_k \} < \{ a, b \} \). Let \( ab = a_0b_0, a_1b_1, \ldots, a_{k-1}b_{k-1}, a_k b_k \), be a path in \( C_2 \) between \( \{ a_k, b_k \} \) and \( \{ a, b \} \), where for \( 0 \leq i < k, a_i b_{i+1}, a_{i+1} b_i \notin E(G) \). Note that \( k \) is odd, \( a_i b_i \in F_2 \) for each even \( i \) and \( a_i b_i \in F_1 \) for each odd \( i \), where \( 0 \leq i \leq k \).

The following claim is symmetric to Claim 3 and hence we omit the proof.

Claim 4. \( a_i b_i, c_i b_i \in F_1 \) for each even \( i \) and \( a_i b_i, c_i b_i \in F_2 \) for each odd \( i \), where \( 0 \leq i \leq k \).

Recall that \( c_k d_k \in F_1 \). By Claim 3 \( c_k \in F_1 \) and \( c_k b \in F_2 \), implying that \( c_k b \notin E(H_1) \). As \( c_{k-1} d_k \notin E(G) \) we then have \( \{ c_{k-1}, b, c_k, d_k \} \in P_1 \). Similarly, as \( a_k b_k \in F_1, a_k b_k \notin E(G) \), and by Claim 4, we have \( \{ a_k, b_k, c, b_{k-1} \} \in P_1 \). We get the final contradiction from the following claim.

Claim 5. Either \( \{ a_k, b_k, c, b_{k-1} \} < \{ a, b, c, d \} \) or \( \{ c_{k-1}, b, c_k, d_k \} < \{ a, b, c, d \} \), and we are done. So we shall assume that \( d < a \). By Claim 2 we now have that \( c < a \), implying that \( a > \max \{ c, d \} \). If \( \min \{ c_k, d_k \} < \min \{ c, d \} \), then we have \( \min \{ c_k, d_k \} < a, c, d \), which implies that \( c_{k-1}, b, c_k, d_k < \{ a, b, c, d \} \), proving the claim. So we shall assume that \( \min \{ c_k, d_k \} \geq \min \{ c, d \} \). Therefore, since \( \{ c_k, d_k \} < \{ c, d \} \), we have \( \min \{ c_k, d_k \} = \min \{ c, d \} \) and \( \max \{ c_k, d_k \} < \max \{ c, d \} \). Thus we have \( a > \max \{ c_k, d_k \} \), implying that \( \{ c_{k-1}, b, c_k, d_k \} < \{ a, b, c, d \} \). \( \square \)

From Lemma 2 and Lemma 3 it follows that \( P = \emptyset \).

Lemma 4. \( C = \emptyset \).

Proof. Suppose for the sake of contradiction that \( C \neq \emptyset \). Then there exists \( i \in \{ 1, 2 \} \) such that \( C_i \neq \emptyset \). Consider an element \( \{ a, b, c, d \} \in C_i \). We can assume without loss of generality that \( ab, cd \in E(H_i) \), \( ad, bc \in E(G) \setminus E(H_i) \). As \( ad \in E(G) \setminus E(H_i) \), it belongs to a non-trivial component of \( G' \). Therefore there exists a
neighbor $a'd'$ of $ad$ in $G'$ such that $ad', a'd' \notin E(G)$. Therefore by Observation we have that $a'd' \in E(H_i)$. As $ab, a'd' \in E(H_i)$, where $ad' \notin E(G)$, by the same observation we then have $a'b \in E(G)$. Now if $a'b \in E(H_i)$, then the fact that $cd \in E(H_i)$, $be \in E(G) \setminus E(H_i)$ and $a'd \notin E(G)$ implies that $\{a', b, c, d\} \in \mathcal{P}$, which is a contradiction to our earlier observation that $\mathcal{P} = \emptyset$. Therefore $a'b \in E(G) \setminus E(H_i)$. As $ab, a'd' \in E(H_i)$ and $ad' \notin E(G)$, it then follows that $\{a, b, a', d'\} \in \mathcal{P}$, which again contradicts the fact that $\mathcal{P} = \emptyset$. This completes the proof.

We now have that $\mathcal{P} \cup \mathcal{C} = \emptyset$, or in other words, there is no crossing set in either $H_1$ or $H_2$. Therefore $H_1$ and $H_2$ are two threshold graphs that cover $G$. We have thus shown that if $G'$ is bipartite then $G$ has a threshold cover of size two. As we already have Lemma this completes the proof of Theorem.

4 A Certifying Algorithm

Our proof of Theorem gives an algorithm which when given a graph $G$ as input, either constructs two threshold graphs that cover $G$, or produces an odd cycle in $G'$ as a certificate that $G$ cannot be covered by two threshold graphs.

Algorithm 2-Threshold-Cover

Input: A graph $G$.

Output: If $G$ has a threshold cover of size 2, two threshold graphs $H_1, H_2$ such that they cover $G$, otherwise the auxiliary graph $G'$ and an odd cycle in it.

1. Run the Lex-BFS algorithm on $G$ (starting from an arbitrarily chosen vertex) to produce a Lex-BFS ordering $\prec$ of $V(G)$.
2. Construct the auxiliary graph $G'$.
3. Initialize $V(H_1) = V(H_2) = V(G)$ and $E(H_1) = E(H_2) = \{e \in V(G') : e \text{ belongs to a trivial component of } G'\}$.
4. While there exist uncolored vertices in a non-trivial component $C$ of $G'$, do
   (i) Choose the lexicographically smallest vertex $uv$ in $C$ and assign the color 1 to it.
   (ii) Complete the 2-coloring of $C$ by doing a BFS starting from the vertex $uv$. If an odd cycle is detected, return $G'$ along with the cycle and exit. Otherwise update $E(H_1) = E(H_1) \cup \{e \in V(C) : e \text{ is colored 1 in } G'\}$, $E(H_2) = E(H_2) \cup \{e \in V(C) : e \text{ is colored 2 in } G'\}$.
5. Output $H_1$ and $H_2$.

Correctness of the algorithm follows from the proof of Theorem. The Lex-BFS on $G$ can be done in $O(|V(G)| + |E(G)|)$ time and the remaining steps in $O(|V(G')| + |E(G')|)$ time. As $G'$ contains at most $|E(G)|$ vertices and at most $|E(G)|^2$ edges, the running time of this algorithm is $O(|E(G)|^2)$.

5 The Chain Subgraph Cover Problem

A bipartite graph $G = (A, B, E)$ is called a chain graph if it does not contain a pair of edges whose endpoints induce a $2K_2$ in $G$. A collection of chain graphs $\{H_1, H_2, \ldots, H_k\}$ is said to be a $k$-chain subgraph cover of a bipartite graph $G$ if it is covered by $H_1, H_2, \ldots, H_k$. The problem of deciding whether a bipartite graph $G$ can be covered by $k$ chain graphs, i.e. whether $G$ has a $k$-chain subgraph cover, is known as the $k$-chain subgraph cover ($k$-CSC) problem. He showed that 3-CSC is NP-complete and pointed out that using the results of Ibaraki and Peled [7], the 2-CSC problem can be solved in polynomial time as it can be reduced to the problem of determining whether a split graph can be covered by two threshold graphs. Ma and Spinrad [11] note that a direct implementation of this approach to the 2-CSC problem only gives an $O(|V(G)|^4)$ algorithm and instead propose an $O(|V(G)|^2)$ algorithm for the problem. This algorithm works by reducing the 2-CSC problem to the problem of deciding whether a partial order has Dushnik-Miller dimension at most 2. Note that this algorithm does not produce a directly verifiable certificate, such as a forbidden structure in the graph, in case the input graph does not have a 2-chain subgraph cover. Our algorithm can be easily modified to make it an $O(|E(G)|^2)$ certifying algorithm for deciding if an input bipartite graph $G$ has a 2-chain subgraph cover as explained below. In fact, the only modification that is needed is to change the definition
of \(G'\) so that two edges of \(G\) are adjacent in \(G'\) if and only if they induce \(2K_2\) in \(G\). As shown below, we can start with an arbitrary ordering of vertices in this case, i.e. we do not need to run the Lex-BFS algorithm to produce a Lex-BFS ordering of the input graph as the first step.

Let \(G = (A, B, E)\) be a bipartite graph. We now redefine the meaning of the term “cross edges”. Two edges \(ab, cd \in E(G)\) are now said to be cross edges if and only if \(a, c \in A, b, d \in B\) and \(ad, bc \notin E(G)\). Note that the meaning of the auxiliary graph \(G'\) now changes, but our proof that \(\chi(G') \leq 2\) if and only if there exists two graphs \(H_1, H_2\), each containing no cross edges, such that \(E(G) = E(H_1) \cup E(H_2)\) still works almost verbatim—the only change to be made is that all occurrences of expressions of the form “\(xy \notin E(G)\)” (where \(x\) and \(y\) are some two vertices in \(V(G)\)) in the proof of Theorem 1 have to be be now read as “\(x\) and \(y\) belong to different parts of the bipartition \((A, B)\) and \(xy \notin E(G)\)”. As noted before, while adapting the proof, we could let the ordering \(<\) on \(V(G)\) be any arbitrary ordering. In that case, we cannot use Observation 1 and any argument that uses it. Note that Observation 1 is used only in the proof of Lemma 4 in particular, only in Claims 1 and 2, which in turn are used only in Claim 5. Remove Claims 1 and 2 and replace the proof of Claim 5 with the following proof.

**Claim 5.** Either \(\{a_k, b_k, c, b_{k-1}\} < \{a, b, c, d\}\) or \(\{c_{k-1}, b, c_k, d_k\} < \{a, b, c, d\}\).

As \(c_k = b, c \in F_1\) and \(c'b \notin E(G)\), we have \(c'd_k < E(G)\). From Claim 3 we have \(c'd_k < E(G)\). Then, \(c'd_k < E(G)\), where \(c'b \notin E(G)\) implies that \(c'd_k < E(G)\). As \(c_k d_k < E(G)\), we can conclude that \(c \neq c_k\) and \(d \neq d_k\). Since \(c, c_k \in A\) and \(d, d_k \in B\), we further have that \(c \neq d_k\) and \(d \neq c_k\). Therefore we get,

\[
c, d > \min\{c_k, d_k\} \quad \text{(as \(c_k d_k < c d\))}
\]

From Claim 4 we have \(a_k - 1 b' \in F_1\). Now \(a_k - 1 b', c \in F_1\) where \(c'b \notin E(G)\) implies that \(a_k - 1 b \in E(G)\). Since \(c_k b_{k-1}, c'b \notin E(G)\), we have \(a_k b_{k-1} \in E(G)\). As \(a_k b_{k-1}, a_k b_k \notin E(G)\) we can conclude that \(a \neq a_k\) and \(b \neq b_k\). Since \(a, a_k \in A\) and \(b, b_k \in B\), we further have that \(a \neq b_k\) and \(b \neq a_k\). Therefore we get,

\[
a, b > \min\{a_k, b_k\} \quad \text{(as \(a_k b_k < a b\))}
\]

If \(a \leq \min\{c_k, d_k\}\) and \(d \leq \min\{a_k, b_k\}\), we get by 1 and 2 that \(a \leq \min\{c_k, d_k\} < d \leq \min\{a_k, b_k\} < a\), which is a contradiction. Therefore, either \(a > \min\{c_k, d_k\}\) or \(d > \min\{a_k, b_k\}\). If \(a > \min\{c_k, d_k\}\), then by 1, we get \(\{c_k, c_k, d_k\} < \{a, b, c, d\}\), and we are done. Similarly, if \(d > \min\{a_k, b_k\}\), then by 2, we have \(\{a_k, b_k, c, b_{k-1}\} < \{a, b, c, d\}\), again we are done. This proves the claim.

Thus Algorithm 2-Threshold-Cover can be modified into a certifying recognition algorithm for deciding if a bipartite graph has a 2-chain subgraph cover by just changing the definition of \(G'\). Moreover, this algorithm can choose any arbitrary ordering of the vertices of the input graph to start with and hence does not require the implementation of the Lex-BFS algorithm. Note that we do not know the answer to the following question: Would Algorithm 2-Threshold-Cover correctly decide whether the input graph \(G\) has a threshold cover of size 2 even if it lets \(<\) be an arbitrary ordering of \(V(G)\)?

6 Conclusion

Chvátal and Hammer [1] showed that the problem of deciding whether an input graph has a threshold cover of size at most \(k\) is NP-complete, when \(k\) is part of the input. Yannakakis [16] observes that a bipartite graph \(G = (A, B, E)\) has a \(k\)-chain subgraph cover if and only if the split graph \(H\) obtained from \(G\) by making every pair of vertices in \(A\) adjacent to each other has a threshold cover of size \(k\). He notes that therefore, his proof of the NP-completeness of the 3-CSC problem implies that the problem of deciding if an input graph has a threshold cover of size at most 3 is also NP-complete.

We believe that our result demonstrates once again the power of the lexicographic method in yielding short and elegant proofs for certain kinds of problems that otherwise seem to need more complicated proofs. Further research could establish the applicability of the method to a wider range of problems.

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