Quadratic variation of càdlàg semimartingales as a.s. limit of the normalized truncated variations

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Abstract

For a real càdlàg path $x$ we define sequence of semi-explicit quantities, which do not depend on any partitions and such that whenever $x$ is a path of a càdlàg semimartingale then these quantities tend a.s. to the continuous part of the quadratic variation of the semimartingale. Next, we derive several consequences of this result and propose a new approach to define Föllmer’s pathwise integral.

Keywords: quadratic variation, truncated variation, càdlàg semimartingales, Föllmer’s pathwise integral

1. Introduction

In recent years there is significant interest in the pathwise approach to stochastic calculus. One of the most important quantities in stochastic calculus is arguably the quadratic variation of a semimartingale. It is usually defined as the limit of sums of squares of the increments of a semimartingale along sequence of deterministic partitions, as the meshes of the partitions tends to 0, and the convergence holds in probability. Unfortunately, when we allow random partitions, it may happen that this convergence and the limit (if it exists) depend on the partitions chosen (see for example [3, Theorem 7.1]). As a result one may obtain different values of Föllmer’s pathwise

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integral \( [4] \) with respect to specific path, along different sequences of partitions.

Fortunately, when the partitions are obtained from stopping times such that the oscillations of a path on the consecutive (half-open on the right) intervals of these partitions tend a.s. (almost surely) to 0, then there is no ambiguity and the sums of squares of the increments along these partitions tend a.s. (or a.s. along some subsequence of these partitions) to the quadratic variation (see for example \([3, Proposition 2.4 and Proposition 2.3]\)). One of such partition schemes dates back at least to Bichteler, see \([1, Theorem 7.14]\), \([5]\) and as a result one obtains a sequence of pathwise sums of squares of the increments which tend a.s. to the quadratic variation. Some modification of this scheme, so called Lebesgue partitions, was proposed by Vovk in \([16]\), to prove that the quadratic variation of typical, model-free càdlàg price paths with mildly restricted jumps (along the Lebesgue partitions) exist. Later, the same scheme was used in \([14]\) to prove that the quadratic variation of typical, model-free, non-negative càdlàg price paths exists.

Let \( D \) denote the family of càdlàg functions \( x : [0, +\infty) \to \mathbb{R} \). In this article, for any \( x \in D \) we will define another sequence of semi-explicit quantities, which \textit{do not depend on any partitions} and such that whenever \( X_t, t \geq 0 \), is a real càdlàg semimartingale on a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and that usual conditions hold (see \([15, Chapt. I, Sect. 1]\)), then for \( x = X(\omega), \omega \in \Omega \), these quantities tend \( \mathbb{P}\)-a.s. to the continuous part of the quadratic variation of \( X \). This result is a generalisation of \([12, Theorem 1]\) to the case of càdlàg semimartingales, however, the proof will be completely different from the proof of \([12, Theorem 1]\). We will mainly use results of \([8]\) and properties of so called double Skorohod map. Next, we will derive several consequences of this result and propose a new approach to define Föllmer’s pathwise integral.

In the sequel, refering to a càdlàg semimartingale \( X_t, t \geq 0 \), we will always assume that \( X \) is a semimartingale on a filtered probability space such that usual conditions hold.

2. Main result

To state our main result we need several definitions.

Let \(-\infty < a < b < +\infty\) and \( x : [a, b] \to \mathbb{R} \) be a real-valued path. The
truncated variation of \( x \) with the truncation parameter \( c \geq 0 \) is defined as

\[
TV^c(x, [a, b]) := \sup_n \sup_{a \leq t_0 < \ldots < t_n \leq b} \sum_{i=1}^n \max \{|x(t_i) - x(t_{i-1})| - c, 0\}.
\]

Thus \( TV^c(x, [a, b]) \) is obtained by taking supremum of sums of truncated increments \( \max \{|x(t_i) - x(t_{i-1})| - c, 0\} \) over all possible partitions \( \pi = \{a \leq t_0 < \ldots < t_n \leq b\} \) of \([a, b]\). It is possible to prove that \( TV^c(x, [a, b]) \) is finite for any \( c > 0 \) iff \( x \) is regulated, i.e. it has finite left limits \( x(t^-) \) for \( t \in (a, b] \) and finite right limits \( x(t^+) \) for \( t \in [a, b) \).

Together with the truncated variation we define two companion quantities - upward and downward truncated variations, which are defined in the following way:

\[
UTV^c(x, [a, b]) := \sup_n \sup_{a \leq t_0 < \ldots < t_n \leq b} \sum_{i=1}^n \max \{x(t_i) - x(t_{i-1}) - c, 0\}.
\]

and

\[
DTV^c(x, [a, b]) := \sup_n \sup_{a \leq t_0 < \ldots < t_n \leq b} \sum_{i=1}^n \max \{x(t_{i-1}) - x(t_i) - c, 0\}.
\]

**Remark 1.** The definitions of the (upward-, downward-) truncated variation may seem “pulled out of a hat”, however, these three quantities have very natural interpretation: they are (attainable) lower bounds for the total (resp. positive, negative) variation of any path approximating \( x \) with the accuracy \( c/2 \), see [11, displays (2.1)-(2.3)].

Now we are ready to state.

**Theorem 1.** Let \( X_t, t \geq 0 \), be a real càdlàg semimartingale on a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\) such that usual conditions hold. For each \( c > 0 \) and \( t \geq 0 \) let us define the following càdlàg processes

\[
T^c_t := c \cdot TV^c(X_t, [0, t]), \quad U^c_t := c \cdot UTV^c(X_t, [0, t]) \quad \text{and} \quad D^c_t := c \cdot DTV^c(X_t, [0, t]),
\]

then

\[
(T^c, U^c, D^c) \Rightarrow \left( [X]^{\text{cont}}, \frac{1}{2} [X]^{\text{cont}}, \frac{1}{2} [X]^{\text{cont}} \right) \quad \text{as} \ c \to 0+,
\]

where “\( \Rightarrow \)” denotes \( \mathbb{P} \)-a.s. convergence in the uniform convergence topology on compact subsets of positive half-line \([0, +\infty)\) and \([X]^{\text{cont}}\) denotes the continuous part of the quadratic variation of \( X \).
Proof. (I) Proof of the convergence of \( T^c \).

For \( t \geq 0 \) let \( \Delta X_t = X_t - X_{t-} \), where \( X_{0-} := 0 \) and for \( t > 0 \), \( X_{t-} = \lim_{s \to t-} X_s \), be the jump at the moment \( t \). For each \( c > 0 \) let \( X^c \) be the process constructed (for given process \( X \)) as in [8, Section 2, display (2.1)]. This construction is related to the double Skorohod map \( \Gamma^c \) on \([-c/c, c/c]\), see [8], [2], via \( X^c = X - \Gamma^c (X - X_0) \). However, it is not important to us but only the existence of \( X^c \), which has the following properties.

1. \( X^c \) has locally finite total variation;
2. \( X^c \) has càdlàg paths;
3. for every \( t \geq 0 \), \( |X_t - X^c_t| \leq c \);
4. for every \( t > 0 \), \( |\Delta X^c_t| \leq |\Delta X_t| \);
5. the process \( X^c \) is adapted to the filtration \( \mathcal{F} \);
6. \( X^c_0 = X_0 \).

Moreover, by [8, Lemma 5.1] for any \( t \geq 0 \) we have

\[
TV^{2c}(X, [0, t]) \leq TV(X^c, [0, t]) \leq TV^{2c}(X, [0, t]) + 2c, \tag{1}
\]

where \( TV(X^c, [0, t]) := TV^0(X^c, [0, t]) \) denotes the total variation of \( X^c \) on \([0, t]\).

Recall the classical Jordan decomposition and notice that \( UTV(X^c, [0, t]) := UTV^0(X^c, [0, t]) \) and \( DTV(X^c, [0, t]) := DTV^0(X^c, [0, t]) \) are nothing else but positive and negative parts of the total variation of \( X^c \). By [8, Lemma 5.2] we have that \( dUTV(X^c, [0, t]) \) and \( dDTV(X^c, [0, t]) \) are mutually singular measures carried by \( \{ t > 0 : X_t - X^c_t = c \} \) and \( \{ t > 0 : X_t - X^c_t = -c \} \), and on these sets we have

\[
dUTV(X^c, [0, t]) = dX^c \quad \text{and} \quad dDTV(X^c, [0, t]) = -dX^c.
\]

From all this it follows that

\[
c \cdot TV(X^c, [0, t]) = c \int_0^t dTV(X^c, [0, s])
\]

\[
= c \int_0^t dUTV(X^c, [0, s]) + c \int_0^t dDTV(X^c, [0, s])
\]

\[
= \int_0^t (X(X^c) - dUTV(X^c, [0, s]) - \int_0^t (X(X^c) - dDTV(X^c, [0, s])
\]

\[
= \int_0^t (X(X^c) \{ dUTV(X^c, [0, s]) - dDTV(X^c, [0, s]) \})
\]

\[
= \int_0^t (X(X^c) dX^c \quad \tag{2}
\]
Representation (2) together with the estimates (1) will be the main ingredients of the proof.

Setting $X_0^c := 0$ we calculate

$$c \cdot TV(X^c, [0, t]) = \int_0^t (X - X^c) \, dX^c = \int_0^t (X_0 - X^c + \Delta (X - X^c)) \, dX^c$$

$$= \int_0^t X_0 \, dX^c - \int_0^t X^c \, dX^c + \sum_{0 < s \leq t} \Delta (X_s - X^c_s) \Delta X^c.$$  

(3)

Let us now fix $T > 0$ and for $n = 1, 2, \ldots$, let us define $c(n) = 1/(2n)$. By [8, Theorem 3.2] we have

$$\lim_{n \to +\infty} \sup_{0 \leq t \leq T} \left| \int_0^t X_0 \, dX^{c(n)} + \int_0^t X^c \, dX - [X]^c_t \right| = 0 \quad \text{a.s.,}$$

(4)

where $\int_0^t X_0 \, dX^{c(n)}$ denotes the Lebesgue-Stieltjes integral (recall that $X^{c(n)}$ has finite total variation) and $\int_0^t X^c \, dX$ denotes the (semimartingale) stochastic integral. We may rewrite (4) in the following way:

$$\int_0^t X_0 \, dX^{c(n)} \Rightarrow \int_0^t X^c \, dX + [X]^c_t = \frac{1}{2} (X_t^2 - X_0^2 - [X]^c_t) + [X]^c_t$$

for $t \in [0, T]$, where $[X]^c_t = [X]^c_t + \sum_{0 < s \leq t} (\Delta X_s)^2$ denotes the quadratic variation of $X$. Next, by the integration by parts formula for the Lebesgue-Stieltjes integral for any $c > 0$ and $t \in [0, T]$ we calculate

$$\int_0^t X^c \, dX^c = \frac{1}{2} \left( (X_t^c)^2 - (X_0^c)^2 - \sum_{0 < s \leq t} (\Delta X_s^c)^2 \right).$$

Also, by properties 3. and 4. satisfied by $X^c$ and by the dominated convergence we have

$$\sup_{0 \leq t \leq T} \sum_{0 < s \leq t} \left| (\Delta X_s)^2 - (\Delta X_s^c)^2 \right| \leq \sum_{0 < s \leq T} \left| (\Delta X_s)^2 - (\Delta X_s^c)^2 \right| \to 0$$

and

$$\sup_{0 \leq t \leq T} \sum_{0 < s \leq t} |\Delta (X_s - X_s^c) \Delta X_s^c| \leq \sum_{0 < s \leq T} \min \{2c |\Delta X_s|, 2 |\Delta X_s|^2 \} \to 0$$
as \( c \to 0^+ \), where “\( \to \)” denotes \( \mathbb{P}\)-a.s. convergence. From (3) and last four relations we get

\[
c(n) \cdot TV(X^{c(n)}, [0, t]) = \int_0^t X_\cdot \text{d}X^{c(n)} - \int_0^t X_{-}^{c(n)} \text{d}X^{c(n)} + \sum_{0<s \leq t} \Delta(X_s - X_s^{c(n)}) \Delta X^{c(n)}
\]

\[
\Rightarrow \frac{1}{2} [X]_{t}^{\text{cont}}.
\]

Hence, from (1) and (5) we get

\[
2c(n) \cdot TV^{2c(n)}(X, [0, t]) = \frac{1}{n} \cdot TV^{1/n}(X, [0, t]) \Rightarrow [X]_{t}^{\text{cont}}.
\]

Finally, the convergence \( c_n \cdot TV^{c_n}(X, [0, t]) \Rightarrow [X]_{t}^{\text{cont}} \) for any sequence \( c_n \to 0^+ \) follows from the estimates

\[
\frac{\lfloor 1/c_n \rfloor}{1/c_n + 1} \cdot TV^{1/\lfloor 1/c_n \rfloor}(X, [0, t])
\]

\[
\leq c_n \cdot TV^{c_n}(X, [0, t]) \leq \frac{\lfloor 1/c_n \rfloor}{1/c_n - 1} \cdot TV^{1/\lfloor 1/c_n \rfloor}(X, [0, t])
\]

valid for \( c_n < 1 \), which stem directly from inequalities

\[
\frac{1}{1/c_n + 1} \leq c_n \leq \frac{1}{1/c_n - 1} \text{ and } \frac{1}{1/c_n} \leq c_n \leq \frac{1}{1/c_n}
\]

(valid for \( c_n < 1 \)), and the fact that the function \((0, +\infty) \ni c \mapsto TV^c(X, [0, t])\) is non-increasing.

(II) Proof of the convergence of the whole triplet \((T^c, U^c, D^c)\).

To prove the convergence of the whole triplet \((T^c, U^c, D^c)\) for any \( c > 0 \) let us define the auxilary process

\[
\tilde{X}^c_t := X_0 + UTV^c(X, [0, t]) - DTV^c(X, [0, t]).
\]

The process \( \tilde{X}^c \) uniformly approximates \( X \) with accuracy \( c \). This is the consequence of [11] Theorem 4 and the classical Jordan decomposition. Indeed, let us fix some \( \omega \in \Omega \). By [11] Theorem 4 for càdlàg \( x = X(\omega) \) there exists some piecewise monotone \( x^c : [0, +\infty) \to \mathbb{R} \) such that \( x^c \) approximates \( x \) with accuracy \( c/2 \) and

\[
UTV^c(x, [0, t]) = UTV(x^c, [0, t]), \quad DTV^c(x, [0, t]) = DTV(x^c, [0, t]).
\]
Thus, by the classical Jordan decomposition,
\[
x^c(t) = x^c(0) + UT V(x^c, [0, t]) - DT V(x^c, [0, t])
= x^c(0) + UT V^c(x, [0, t]) - DT V^c(x, [0, t]).
\]

Since \(x^c\) approximates \(x = X(\omega)\) with accuracy \(c/2\) we must have that \(|X_0(\omega) - x^c(0)| \leq c/2\). From this, the definition of \(\tilde{X}_t^c\) and the triangle inequality we get
\[
|\tilde{X}_t^c(\omega) - X_t(\omega)| \leq |\tilde{X}_t^c(\omega) - x^c(t)| + |x^c(t) - X_t(\omega)|
= |X_0(\omega) - x^c(0)| + |x^c(t) - X_t(\omega)|
\leq c/2 + c/2 = c.
\]

From [11, Theorem 4] and (7) we also have the relation
\[
TV^c(X(\omega), [0, t]) = TV(x^c, [0, t])
= UT V(x^c, [0, t]) + DT V(x^c, [0, t])
= UT V^c(X(\omega), [0, t]) + DT V^c(X(\omega), [0, t]).
\]

Finally, from (8) we get that \(\tilde{X}_t^c = X_t + R_t^c\), where \(|R_t^c| \leq c\) for \(t \geq 0\) and from (6) and (9) we have the following representation
\[
UT V^c(X, [0, t]) = \frac{1}{2} (TV^c(X, [0, t]) + X_t - X_0 + R_t^c),
\]
\[
DT V^c(X, [0, t]) = \frac{1}{2} (TV^c(X, [0, t]) - X_t + X_0 - R_t^c).
\]

From this representation we obtain the convergence of the whole triplet \((T^c, U^c, D^c)\).

3. Some consequences of Theorem 1

From Theorem 1 we obtain pathwise formulas which in the limit tend a.s. to the quadratic covariation of two semimartingales.

**Corollary 1.** Let \(X_t\ and Y_t, t \geq 0\, be\ two\ real càdlàg semimartingales, then
\[
c \cdot \{TV^c(X + Y, [0, t]) - TV^c(X - Y, [0, t])\} \Rightarrow 4[X, Y]^{\text{cont}}, \text{ as } c \to 0+,
\]
where \([X, Y]^{\text{cont}}\) denotes the continuous part of the quadratic covariation of \(X\) and \(Y\), i.e. \([X, Y]_t = [X, Y]^{\text{cont}}_t + \sum_{0 < s \leq t} \Delta X_s \Delta Y_s\ for \ t \geq 0.\)
Also, using recent result \cite[Theorem 1]{9} and Theorem \cite[11] we obtain another pathwise formula for the quadratic variation, where the numbers of interval crossings play a role. To state this result we need to introduce the numbers of times the graph of regulated $x : [a, b] \to \mathbb{R}$ “(down-, up-) crosses” (on $[a, b]$) the closed value interval $[y, y + c]$.

**Definition 1.** Given a function $x : [a, b] \to \mathbb{R}$, for $c \geq 0$ we put $\sigma_0^c = a$ and for $n = 0, 1, \ldots$

$$\tau_n^c = \inf \{ t > \sigma_n^c : t \leq b, x(t) > y + c \}, \quad \sigma_{n+1}^c = \inf \{ t > \tau_n^c : t \leq b, x(t) < y \}.$$ 

Next, we set

$$d_y^c (x, [a, b]) := \max \{ n : \sigma_n^c \leq b \}.$$ 

Similarly we define.

**Definition 2.** Given a function $x : [a, b] \to \mathbb{R}$, for $c \geq 0$ we put $\tilde{\sigma}_0^c = a$ and for $n = 0, 1, \ldots$

$$\tilde{\tau}_n^c = \inf \{ t > \tilde{\sigma}_n^c : t \leq b, x(t) < y \}, \quad \tilde{\sigma}_{n+1}^c = \inf \{ t > \tilde{\tau}_n^c : t \leq b, x(t) > y + c \}.$$ 

Next, we set

$$u_y^c (x, [a, b]) := \max \{ n : \tilde{\sigma}_n^c \leq b \}. \tag{10}$$

In all definitions we apply the convention that $\inf \emptyset = +\infty$.

The number $d_y^c (x, [a, b])$ can be viewed as the number of times the graph of $x$ “downcrosses” (on $[a, b]$) the closed value interval $[y, y + c]$, while the number $u_y^c (x, [a, b])$ can be viewed as the number of times the graph of $x$ “upcrosses” the value interval $[y, y + c]$.

At last, for $x$ and the interval $[a, b]$ as in two preceding definitions, we define the number of times the graph of $x$ crosses (on $[a, b]$) the value interval $[y, y + c]$ as

$$n_y^c (x, [a, b]) := d_y^c (x, [a, b]) + u_y^c (x, [a, b]).$$

\cite[Theorem 1]{9} states that

$$UTV^c(x, [a, b]) = \int_{\mathbb{R}} u_y^c (x, [a, b]) \, dy, \tag{11}$$

$$DTV^c(x, [a, b]) = \int_{\mathbb{R}} d_y^c (x, [a, b]) \, dy \tag{12}$$
and

\[ TV^c(x, [a, b]) = \int_{\mathbb{R}} n^y_c (x, [a, b]) \, dy. \tag{13} \]

Using (11)-(13) and Theorem 1 we get

**Corollary 2.**

\[ \int_{\mathbb{R}} c \cdot u^y_c (X, [0, \cdot]) \, dy \Rightarrow \frac{1}{2} [X]^{cont}, \]

\[ \int_{\mathbb{R}} c \cdot d^y_c (X, [0, \cdot]) \, dy \Rightarrow \frac{1}{2} [X]^{cont}, \]

\[ \int_{\mathbb{R}} c \cdot n^y_c (X, [0, \cdot]) \, dy \Rightarrow [X]^{cont} \]

as \( c \to 0^+ \).

Much stronger result of this type, namely that \( c \cdot n^y_c (X, [0, \cdot]) \) tends a.s. to the local time of \( X \) at all but countably many real \( y \)s, was proven in [6, Theorem 3.3], but only for semimartingales satisfying condition \( \sum_{0 < s \leq t} |\Delta X_s| < +\infty \) a.s.

Theorem 1 and the construction used in its proof imply also

**Corollary 3.** A real càdlàg semimartingale \( X \) may be uniformly approximated with accuracy \( c \) by finite variation and adapted (to the natural filtration of \( X \)) processes, whose total variation is of order \( O(c^{-1}) \) as \( c \to 0^+ \). Moreover, if \( X \) is a pure-jump semimartingale, then it may be uniformly approximated with accuracy \( c \) by finite variation, adapted processes whose total variation is of order \( o(c^{-1}) \) as \( c \to 0^+ \).

**Remark 2.** If \( X \) is a strictly \( \alpha \)-stable process, \( \alpha \in (1, 2) \), using scaling properties of \( X \) it may be proven that \( TV^c(X, [0, T]) \) is of order \( c^{1-\alpha} \).

However, there exist pure-jump semimartingales for which \( TV^c(X, [0, T]) \) is of order greater than \( c^{-\beta} \) for any \( \beta < 1 \). An example of such a semimartingale (with a.s. infinite 2-variation norm) is given in [7, Proposition 3(a)] and the fact that for such \( X \), \( TV^c(X, [0, T]) \) is of order greater than \( c^{-\beta} \) for any \( \beta < 1 \) follows from [13, Proposition 2].
4. Föllmer’s pathwise integral

Inspired by equation (2) of Section 1, in this section we will define an integral with respect to a càdlàg path \( x : [0, +\infty) \to \mathbb{R} \) which may be uniformly approximated for any \( c > 0 \) with accuracy \( c \) by some path \( x^c : [0, +\infty) \to \mathbb{R} \) with finite total variation and such that the limit

\[
\langle x \rangle_t := 2 \lim_{c \to 0^+} \int_0^t (x - x^c) \, dx^c
\]  

exists for any \( t \geq 0 \). The integral in (14) is understood as the classical Lebesgue-Stieltjes integral. Note that the function \( \langle x \rangle(t) := \langle x \rangle_t \) is continuous.

Let \( \mathcal{X} = (x^c)_{c > 0} \) be the family of functions \( x^c \). Now, for a measurable function \( f : \mathbb{R} \to \mathbb{R} \) we define

\[
(\mathcal{X}) \int_0^t f(x_s) \, dx_s := \lim_{c \to 0^+} \int_0^t f(x^c_s) \, dx_s^c
\]

if this limit exists. We have the following result similar to [4, THÉORÈME]

**Theorem 2.** Let \( x : [0, +\infty) \to \mathbb{R} \) be a càdlàg path such that \( \sum_{0 < s \leq t} \left( \Delta x_s \right)^2 < +\infty \) for any \( t \geq 0 \). Assume that \( \mathcal{X} = (x^c)_{c > 0} \) is such that \( \|x - x^c\|_\infty := \sup_{s \geq 0} |x(s) - x^c(s)| \leq c \) and the limit \((14)\) exists. Moreover, assume that

\[
\sup_{c \to 0^+} \int_0^t |x - x^c| \, dx^c < +\infty
\]  

and there exists a constant \( K \) such that

\[
|\Delta x^c_t| \leq K |\Delta x_t|
\]

for any \( t \geq 0 \). Then, for any \( f : \mathbb{R} \to \mathbb{R} \) of class \( C^1 \) the integral \((\mathcal{X}) \int_0^t f(x_s) \, dx_s\) exists, moreover, we have the following formula

\[
F(x_t) - F(x_0) = (\mathcal{X}) \int_0^t f(x_s) \, dx_s - \frac{1}{2} \int_0^t f'(x_s) \, d\langle x \rangle_s
+ \sum_{0 < s \leq t} \left\{ F(x_s) - F(x^-_s) - f(x^-_s) \Delta x_s \right\},
\]

where \( F \) is an antiderivative of \( f \).
Proof. For any $c > 0$ we write $f(x_s) - f(x_c^s) = f'(\bar{x}_c^s)(x_s - x_c^s)$ for some $\bar{x}_c^s \in \left[ \min\{x_s, x_c^s\}, \max\{x_s, x_c^s\} \right]$. Thus we have

$$
\int_0^t f(x_{s-}) \, dx_c^s = \int_0^t f(x_s) - \Delta f(x_s) \, dx_c^s
$$

$$
= \int_0^t f'(\bar{x}_c^s)(x_s - x_c^s) \, dx_c^s + \int_0^t f(x_c^s) \, dx_c^s - \sum_{0 < s \leq t} \Delta f(x_s) \Delta x_c^s.
$$

By the continuity of $f'$, (14) and condition (15) we get

$$
\int_0^t f'(\bar{x}_c^s)(x_s - x_c^s) \, dx_c^s \rightarrow \frac{1}{2} \int_0^t f'(x_s) \, d\langle x \rangle_s = \frac{1}{2} \int_0^t f'(x_{s-}) \, d\langle x \rangle_s,
$$

where the last equality follows from the continuity of the function $\langle x \rangle(t) := \langle x \rangle_t$. Note also that due to (15), $\langle x \rangle$ has finite total variation. Next, from the properties of the Lebesgue-Stieltjes integral we obtain

$$
\int_0^t f(x_c^s) \, dx_c^s = F(x_c^t) - F(x_c^0) - \sum_{0 < s \leq t} \{ \Delta F(x_c^s) - f(x_c^s) \Delta x_c^s \}.
$$

(18)

Putting together relations (17) and (18) and then proceeding to the limit we obtain (using (16) and (15) to justify the dominated convergence) the assertion. \qed

Remark 3. Defining $(X)' \int_0^t f(x_{t-}) \, dx_t := \lim_{c \to 0^+} \int_0^t f(x_c^{t-}) \, dx_t$ under conditions of Theorem 2 we obtain an integral satisfying the “usual” Itô formula.

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