BOCHNER-SCHOENBERG-EBERLEIN-TYPE INEQUALITY OF THE DIRECT SUM, IDEALS AND QUOTIENT OF FRÉCHET ALGEBRAS

M. AMIRI AND A. REJALI

Abstract. Let \( A \) and \( B \) be two commutative semisimple Fréchet algebras. We first give a characterization of the multiplier algebra of the direct sum of \( A \) and \( B \). We then prove that \( A \oplus B \) is a BSE-algebra if and only if \( A \) and \( B \) are BSE-algebras. Furthermore, for a closed ideal \( I \) of \( A \), we study multipliers of ideals and quotient algebras of \( A \) and show that \( I \) and \( A/I \) are BSE-algebras, under certain conditions.

1. Introduction and Preliminaries

BSE stands for the theorem Bochner-Schoenberg-Eberlein. This theorem proved for locally compact Abelian groups [3] and was generalized for commutative Banach algebras, by Takahasi and Hatori [9, 11]. Moreover, it developed by other authors such as Inoue and Takahasi [5] and later by Kaniuth and Ülger [7]. We studied the BSE property for the commutative Fréchet algebras; see [2].

Let \( A \) be a commutative Banach algebra without order and the space \( \Delta(A) \) denotes the set of all nonzero multiplicative linear functionals on \( A \) with respect to the Gelfand-topology. A bounded continuous function \( \sigma \) on \( \Delta(A) \) is called a BSE-function if there exists a positive real number \( \beta \) such that for every finite number of complex-numbers \( c_1, \cdots, c_n \) and the same number of \( \varphi_1, \cdots, \varphi_n \) in \( \Delta(A) \) the inequality

\[
\left| \sum_{i=1}^{n} c_i \sigma(\varphi_i) \right| \leq \beta \left\| \sum_{i=1}^{n} c_i \varphi_i \right\|_{A^*}
\]

holds. Following [12], the BSE norm of \( \sigma \), denoted by \( \left\| \sigma \right\|_{BSE} \), is defined to be the infimum of all such \( \beta \). The set of all BSE-functions is denoted by \( C_{BSE}(\Delta(A)) \) where it is a commutative semisimple Banach algebra, under \( \left\| \cdot \right\|_{BSE} \).

Let us recall from [8, 9] that a Fréchet space is a completely metrizable locally convex space where its topology is generated by a translation invariant metric. Furthermore, following [4], a complete topological algebra \( A \) is a Fréchet algebra if its topology is produced by a countable family of increasing submultiplicative seminorms \( (p_k)_{k \in \mathbb{N}} \). The class of Fréchet algebras is an important class of locally convex algebras has been widely studied by many authors. Note that every Fréchet (algebra) space is not necessarily a Banach (algebra) space. Some differences between Banach and Fréchet (algebras) spaces introduced in the survey paper [1].

Let \( (\mathcal{A}, p_k)_{k \in \mathbb{N}} \) be a Fréchet algebra. Consider \( \mathcal{A}^* \) the topological dual of \( \mathcal{A} \). The strong topology on \( \mathcal{A}^* \) is generated by seminorms \( (P_M) \) where \( M \) is a

2010 Mathematics Subject Classification. 46J05, 46J20.
Key words and phrases. BSE-algebra, commutative Fréchet algebra, multiplier algebra.
bounded set in $A$; see [9] for more details. Following [2], a bounded complex-valued continuous function $\sigma$ defined on $\Delta(A)$ is called a BSE-function, if there exist a bounded set $M$ in $A$ and a positive real number $\beta_M$ such that for every finite number of complex-numbers $c_1, \cdots, c_n$ and the same number of $\varphi_1, \cdots, \varphi_n$ in $\Delta(A)$ the inequality
\[
\left| \sum_{i=1}^{n} c_i \sigma(\varphi_i) \right| \leq \beta_M P_M \left( \sum_{i=1}^{n} c_i \varphi_i \right)
\]
holds. Moreover by [2, Theorem 3.3], $C_{\text{BSE}}(\Delta(A))$ is a commutative semisimple Fréchet algebra and
\[
C_{\text{BSE}}(\Delta(A)) = A^{**}|_{\Delta(A)} \cap C_b(\Delta(A)).
\]
Let now $(A, p_T)$ be a commutative Fréchet algebra. A linear operator $T$ on $A$ is called a multiplier if it satisfies $a T(b) = T(ab)$, for all $a, b \in A$. The set $M(A)$ of all multipliers of $A$ with the strong operator topology, is a commutative unital complete locally convex algebra and not necessarily Fréchet algebra; see [2, Proposition 4.1]. Analogous to the Banach case, for each $T \in M(A)$, there exists a unique continuous function $\tilde{T}$ on $\Delta(A)$ such that $\varphi(T(a)) = \tilde{T}(\varphi) \varphi(a)$ for all $a \in A$ and $\varphi \in \Delta(A)$. The algebra $A$ is called BSE algebra if $M(A) = C_{\text{BSE}}(\Delta(A))$, where $M(A) = \{ \tilde{T} : T \in M(A) \}$. A bounded net $(e_a)_{a \in A}$ in $A$ is called a $\Delta$-weak approximate identity for $A$ if it satisfies $\varphi(e_a) \to a_1$ or equivalently $\varphi(e_a a) \to a \varphi(a)$ for every $a \in A$ and $\varphi \in \Delta(A)$. Moreover, $A$ has a bounded $\Delta$-weak approximate identity if and only if $M(A) \subseteq C_{\text{BSE}}(\Delta(A))$. In addition,
\[
\mathcal{M}(A) = \{ \Phi : \Delta(A) \to \mathbb{C} : \Phi \text{ is continuous and } \Phi \cdot \hat{A} \subseteq \hat{A} \}.
\]
This is another definition of the multiplier algebra of Fréchet algebras. If $A$ is a commutative semisimple Fréchet algebra, then $\mathcal{M}(A) = M(A) = M(A)$. In the present paper, we show that $\mathcal{M}(A \oplus B) = M(A) \times M(B)$ and
\[
C_{\text{BSE}}(\Delta(A \oplus B)) = C_{\text{BSE}}(\Delta(A)) \times C_{\text{BSE}}(\Delta(B)).
\]
Additionally, we prove that $A \oplus B$ is a BSE-algebra if and only if $A$ and $B$ are BSE. We also offer conditions under which closed ideals and quotient algebras of BSE algebras are BSE-algebras.

2. THE MULTIPLIER ALGEBRA FOR THE DIRECT SUM OF FRÉCHET ALGEBRAS

The direct sum of Banach algebras studied in [6]. In the following, we generalized it for Fréchet algebras. Let $(A, r_\ell)$ and $(B, s_\ell)$ be two commutative Fréchet algebras. Then, $A \oplus B = A \times B$ defined by
\begin{enumerate}[(i)]
\item $(a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1, a_2 b_2)$, for $a_1, a_2 \in A$, $b_1, b_2 \in B$;
\item $P_\ell(a, b) = r_\ell(a) + s_\ell(b)$, for $a \in A$, $b \in B$ and $\ell \in \mathbb{N}$;
\item $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$.
\end{enumerate}

By a standard argument, the following is straightforward.

**Lemma 2.1.** Let $(A, r_\ell)$ and $(B, s_\ell)$ be two Fréchet algebras. Then,
\begin{enumerate}[(i)]
\item $(A \oplus B, P_\ell)$ is a Fréchet algebra.
\item $(A \oplus B)^* = A^* \oplus B^*$ as homeomorphism.
\item $\Delta(A \oplus B) = (\Delta(A) \times \{0\}) \cup (\{0\} \times \Delta(B))$.
\item $A \oplus B$ is semisimple if and only if both $A$ and $B$ are semisimple.
\end{enumerate}
Lemma 2.2. Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative Fréchet algebras. Then,
$$C_{\text{BSE}}(\Delta(\mathcal{A} \oplus \mathcal{B})) = C_{\text{BSE}}(\Delta(\mathcal{A})) \times C_{\text{BSE}}(\Delta(\mathcal{B})).$$

Proof. Let $\sigma \in C_{\text{BSE}}(\Delta(\mathcal{A} \oplus \mathcal{B}))$. Then, by [2, Proposition 3.5(iii)] we have
$$\sigma \in C_b(\Delta(\mathcal{A} \oplus \mathcal{B})) \cap (\mathcal{A}^{**} \oplus \mathcal{B}^{**})|_{\Delta(\mathcal{A} \oplus \mathcal{B})}.$$

Therefore, there exist $\sigma_1 \in \mathcal{A}^{**}$ and $\sigma_2 \in \mathcal{B}^{**}$ where $\sigma_1|_{\Delta(\mathcal{A})} \in \mathcal{A}^{**}|_{\Delta(\mathcal{A})}$, $\sigma_2|_{\Delta(\mathcal{B})} \in \mathcal{B}^{**}|_{\Delta(\mathcal{B})}$ and $\sigma = (\sigma_1, \sigma_2)$. In addition, there exist a bounded set $M$ in $\mathcal{A} \oplus \mathcal{B}$ and a positive real number $\beta_M$ such that for every finite number of $c_1, \cdots, c_n \in \mathbb{C}$ and $(\varphi_1, \psi_1, \cdots, \varphi_n, \psi_n) \in \Delta(\mathcal{A} \oplus \mathcal{B})$ the inequality
$$\left| \sum_{i=1}^{n} c_i \sigma(\varphi_i, \psi_i) \right| \leq \beta_M P_M \left( \sum_{i=1}^{n} c_i \varphi_i, \psi_i \right)$$
holds. In particular, for each $(\varphi_1, 0, \cdots, \varphi_n, 0) \in \Delta(\mathcal{A} \oplus \mathcal{B})$ and $c_1, \cdots, c_n \in \mathbb{C}$, there exist bounded sets $N_1 \subseteq \mathcal{A}$ and $N_2 \subseteq \mathcal{B}$ such that
$$\left| \sum_{i=1}^{n} c_i \sigma_1(\varphi_i) \right| = \left| \sum_{i=1}^{n} c_i \sigma(\varphi_i, 0) \right| \leq \beta_M P_M \left( \sum_{i=1}^{n} c_i(\varphi_i, 0) \right) = \beta_M \sup \left\{ \left| \sum_{i=1}^{n} c_i(\varphi_i, 0) (a, b) \right| : (a, b) \in M \right\} = \beta_M \sup \left\{ \left| \sum_{i=1}^{n} c_i \varphi_i(a) \right| : a \in N_1 \right\} = \beta_M P_{N_1} \left( \sum_{i=1}^{n} c_i \varphi_i \right),$$
and similarly
$$\left| \sum_{i=1}^{n} c_i \sigma_2(\psi_i) \right| = \left| \sum_{i=1}^{n} c_i \sigma(0, \psi_i) \right| \leq \beta_M P_{N_2} \left( \sum_{i=1}^{n} c_i \psi_i \right).$$

We recall from [9] that $N_1 = \pi_1(M)$ and $N_2 = \pi_2(M)$ where $\pi_1$ and $\pi_1$ are projection mappings on $\mathcal{A} \oplus \mathcal{B}$. Moreover, $M \subseteq N_1 \times N_2$. By above arguments, $\sigma_1 \in C_{\text{BSE}}(\Delta(\mathcal{A}))$ and $\sigma_2 \in C_{\text{BSE}}(\Delta(\mathcal{B}))$, where $\sigma_1(\varphi) = \sigma(\varphi, 0)$ and $\sigma_2(\psi) = \sigma(0, \psi)$ for each $\varphi \in \Delta(\mathcal{A})$ and $\psi \in \Delta(\mathcal{B})$. This implies that
$$C_{\text{BSE}}(\Delta(\mathcal{A} \oplus \mathcal{B})) \subseteq C_{\text{BSE}}(\Delta(\mathcal{A})) \times C_{\text{BSE}}(\Delta(\mathcal{B})).$$

For the reverse conclusion, let $\sigma_1 \in C_{\text{BSE}}(\Delta(\mathcal{A}))$ and $\sigma_2 \in C_{\text{BSE}}(\Delta(\mathcal{B}))$. If $\xi \in \Delta(\mathcal{A} \oplus \mathcal{B})$, then $\xi = (\varphi, 0)$ or $\xi = (0, \psi)$ for some $\varphi \in \Delta(\mathcal{A})$ or $\psi \in \Delta(\mathcal{B})$. We define
$$\sigma(\xi) := \begin{cases} \sigma_1(\varphi) & \text{if } \xi = (\varphi, 0), \\ \sigma_2(\psi) & \text{if } \xi = (0, \psi). \end{cases}$$

By assumption, there exist a bounded set $N_1$ of $\mathcal{A}$ and a positive real number $\beta_{N_1}$ such that for every finite number of complex-numbers $c_1, \cdots, c_n$ and the same number of $\varphi_1, \cdots, \varphi_n$ in $\Delta(\mathcal{A})$ the inequality
$$\left| \sum_{i=1}^{n} c_i \sigma_1(\varphi_i) \right| \leq \beta_{N_1} P_{N_1} \left( \sum_{i=1}^{n} c_i \varphi_i \right)$$
holds. Also, there exist a bounded set $N_2$ of $\mathcal{B}$ and a positive real number $\beta_{N_2}$ such that for every finite number of complex-numbers $c_1, \ldots, c_n$ and the same number of $\psi_1, \ldots, \psi_n$ in $\Delta(\mathcal{B})$ the inequality
\[
|\sum_{i=1}^{n} c_i\sigma_2(\psi_i)| \leq \beta_{N_2}P_N\left(\sum_{i=1}^{n} c_i\psi_i\right)
\]
holds. Therefore,
\[
|\sum_{i=1}^{n} c_i\sigma(\xi)| \leq \beta_M P_M\left(\sum_{i=1}^{n} c_i(\xi)\right),
\]
where $M = N_1 \times N_2$ and $\beta_M = \max\{\beta_{N_1}, \beta_{N_2}\}$. Consequently,
\[
\sigma \in C_{\text{BSE}}(\Delta(\mathcal{A} \oplus \mathcal{B})),
\]
which completes the proof. \qed

**Proposition 2.3.** Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative Fréchet algebras. Then,
\[
\mathcal{M}(\mathcal{A} \oplus \mathcal{B}) = \mathcal{M}(\mathcal{A}) \times \mathcal{M}(\mathcal{B}).
\]

**Proof.** Let $\Phi \in \mathcal{M}(\mathcal{A})$ and $\Psi \in \mathcal{M}(\mathcal{B})$. Since $\Phi \cdot \hat{\mathcal{A}} \subseteq \hat{\mathcal{A}}$ and $\Psi \cdot \hat{\mathcal{B}} \subseteq \hat{\mathcal{B}}$, for all $(a, b) \in \mathcal{A} \oplus \mathcal{B}$ there are elements $c \in \mathcal{A}$ and $d \in \mathcal{B}$ such that
\[
((\Phi, \Psi) \cdot (a, b))(\varphi, 0) = (\Phi, \Psi)(\varphi, 0)(a, b)(\varphi, 0) = \Phi(\varphi)\widehat{a}(\varphi) = \widehat{\sigma}(\varphi),
\]
where $(\Phi, \Psi)(\varphi, 0) = \Phi(\varphi)$ for each $(\varphi, 0) \in \Delta(\mathcal{A}) \times \{0\}$. Similarly,
\[
((\Phi, \Psi) \cdot (a, b))(0, \psi) = (\Phi, \Psi)(0, \psi)(a, b)(0, \psi) = \Psi(\psi)\widehat{b}(\psi) = \widehat{\alpha}(\psi),
\]
where $(\Phi, \Psi)(0, \psi) = \Psi(\psi)$ for each $(0, \psi) \in \{0\} \times \Delta(\mathcal{B})$. Hence,
\[
((\Phi, \Psi) \cdot (a, b))(\varphi, 0) = (\widehat{c}, d)(\varphi, 0),
\]
and
\[
((\Phi, \Psi) \cdot (a, b))(0, \psi) = (\widehat{c}, d)(0, \psi).
\]
Thus, $(\Phi, \Psi) \cdot (\hat{\mathcal{A}} \oplus \hat{\mathcal{B}}) \subseteq (\hat{\mathcal{A}} \oplus \hat{\mathcal{B}})$ and $(\Phi, \Psi) \in \mathcal{M}(\mathcal{A} \oplus \mathcal{B})$.

Now, suppose that $F \in \mathcal{M}(\mathcal{A} \oplus \mathcal{B})$. Define $\Phi(\varphi) = F(\varphi, 0)$ and $\Psi(\psi) = F(0, \psi)$ for all $\varphi \in \Delta(\mathcal{A})$ and $\psi \in \Delta(\mathcal{B})$. Therefore, $F = (\Phi, \Psi)$. It is enough to show that $\Phi \in \mathcal{M}(\mathcal{A})$ and $\Psi \in \mathcal{M}(\mathcal{B})$. For each $a \in \mathcal{A}$, there exists $(a', b') \in \mathcal{A} \oplus \mathcal{B}$ such that
\[
\Phi(\varphi)\widehat{a}(\varphi) = F(\varphi, 0)(\widehat{a}, 0)(\varphi, 0) = (a', b')(\varphi, 0) = \widehat{a'}(\varphi).
\]
Consequently, $\Phi \cdot \hat{\mathcal{A}} \subseteq \hat{\mathcal{A}}$ and $\Phi \in \mathcal{M}(\mathcal{A})$. Similarly, $\Psi \in \mathcal{M}(\mathcal{B})$. Hence,
\[
\mathcal{M}(\mathcal{A} \oplus \mathcal{B}) = \{(\Phi, \Psi) : \Phi \in \mathcal{M}(\mathcal{A}), \Psi \in \mathcal{M}(\mathcal{B})\},
\]
and completes the proof. \qed

**Corollary 2.4.** Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative semisimple Fréchet algebras. Then,
\[
\mathcal{M}(\mathcal{A} \oplus \mathcal{B}) = \mathcal{M}(\mathcal{A}) \oplus \mathcal{M}(\mathcal{B}).
\]
We now state the main result of this paper.

**Theorem 2.5.** Let \((A, r_\ell)\) and \((B, s_\ell)\) be two commutative semisimple Fréchet algebras. Then, \(A \oplus B\) is a BSE-algebra if and only if \(A\) and \(B\) are BSE-algebras.

**Proof.** Let \(A\) and \(B\) be BSE-algebras. By applying [2, Theorem 4.5], \(A\) and \(B\) have bounded \(\Delta\)-weak approximate identities. Suppose that \((e_\alpha)_\alpha\) and \((f_\beta)_\beta\) are bounded \(\Delta\)-weak approximate identities of \(A\) and \(B\), respectively. Therefore, \(\{(e_\alpha, f_\beta)\}_{(\alpha, \beta)}\) is a bounded \(\Delta\)-weak approximate identity for \(A \oplus B\). Indeed, for all \(\xi \in \Delta(A \oplus B)\) there exists \(\varphi \in \Delta(A)\) or \(\psi \in \Delta(B)\), where \(\xi = (\varphi, 0)\) or \(\xi = (0, \psi)\). Thus,

\[
\lim_{(\alpha, \beta)} \xi(e_\alpha, f_\beta) = \lim_{(\alpha, \beta)} (\varphi, 0)(e_\alpha, f_\beta) = \lim_{\alpha} \varphi(e_\alpha) = 1,
\]
or

\[
\lim_{(\alpha, \beta)} \xi(e_\alpha, f_\beta) = \lim_{(\alpha, \beta)} (0, \psi)(e_\alpha, f_\beta) = \lim_{\beta} \psi(f_\beta) = 1.
\]

Moreover, for each \(\ell \in \mathbb{N}\) we have \(\sup_{\alpha} r_\ell(e_\alpha) < \infty\) and \(\sup_{\beta} s_\ell(f_\beta) < \infty\). Also, \((A \oplus B, P_\ell)\) is a Fréchet algebra, where \(P_\ell(a, b) = r_\ell(a) + s_\ell(b)\) for each \(\ell \in \mathbb{N}\). Therefore,

\[
\sup_{(\alpha, \beta)} P_\ell(e_\alpha, f_\beta) = \sup_{\alpha} r_\ell(e_\alpha) + \sup_{\beta} s_\ell(f_\beta) < \infty.
\]

Hence, for all \(\xi \in \Delta(A \oplus B)\) we have \(\lim_{(\alpha, \beta)} \xi(e_\alpha, f_\beta) = 1\) and \(\{(e_\alpha, f_\beta)\}_{(\alpha, \beta)}\) is a \(\Delta\)-weak approximate identity for \(A \oplus B\). Consequently,

\[
M(A \oplus B) \subseteq C_{\text{BSE}}(\Delta(A \oplus B)).
\]

For the reverse conclusion, let \(\sigma \in C_{\text{BSE}}(\Delta(A \oplus B))\). Hence, there exist \(\sigma_1 \in C_{\text{BSE}}(\Delta(A))\) and \(\sigma_2 \in C_{\text{BSE}}(\Delta(B))\) such that \(\sigma(\varphi, \psi) = \sigma_1(\varphi) + \sigma_2(\psi)\) for all \(\varphi \in \Delta(A)\) and \(\psi \in \Delta(B)\). Since \(A\) and \(B\) are BSE algebras, \(\sigma_1 \in M(A)\) and \(\sigma_2 \in M(B)\), by applying Proposition 2.3. Therefore, \(\sigma \in M(A) \times M(B)\). Thus, \(\sigma \in M(A \oplus B)\) and

\[
C_{\text{BSE}}(\Delta(A \oplus B)) \subseteq M(A \oplus B).
\]

Consequently, \(A \oplus B\) is a BSE-algebra.

Conversely, suppose that \(A \oplus B\) is a BSE-algebra. Then,

\[
M(A \oplus B) = M(A \oplus B) = C_{\text{BSE}}(\Delta(A \oplus B)).
\]

Let \(\sigma_1 \in C_{\text{BSE}}(\Delta(A))\) and \(\sigma_2 \in C_{\text{BSE}}(\Delta(B))\). Then \((\sigma_1, 0)\) and \((0, \sigma_2)\) belong to \(C_{\text{BSE}}(\Delta(A \oplus B))\). Also, \(\sigma_1 \in M(A)\) and \(\sigma_2 \in M(B)\) and so

\[
C_{\text{BSE}}(\Delta(A)) \subseteq M(A) \quad \text{and} \quad C_{\text{BSE}}(\Delta(B)) \subseteq M(B).
\]

Conversely, suppose that \(\sigma_1 \in M(A)\) and \(\sigma_2 \in M(B)\). Thus, \((0, \sigma_2)\) and \((\sigma_1, 0)\) belong to \(M(A) \times M(B)\), where

\[
M(A) \times M(B) = M(A \oplus B)
\]

\[
= C_{\text{BSE}}(\Delta(A \oplus B))
\]

\[
= C_{\text{BSE}}(\Delta(A)) \times C_{\text{BSE}}(\Delta(B)).
\]

Thus, \(\sigma_1 \in C_{\text{BSE}}(\Delta(A))\) and \(\sigma_2 \in C_{\text{BSE}}(\Delta(B))\). Hence, \(M(A) \subseteq C_{\text{BSE}}(\Delta(A))\) and \(M(B) \subseteq C_{\text{BSE}}(\Delta(B))\). Consequently, \(A\) and \(B\) are BSE algebras. \(\square\)
3. IDEALS AND QUOTIENT ALGEBRAS OF BSE-FRÉCHET ALGEBRAS

Let \((A, p_\ell)\) be a commutative Fréchet algebra and \(I\) be a closed ideal of \(A\). Then, \((I, p_\ell|_I)\) and \((\frac{A}{I}, s_\ell)\) are also Fréchet algebras where

\[ s_\ell(x + I) := \inf \{ p_\ell(x + y) : y \in I \}. \]

In the sequel we will call \(I\) an essential ideal, when \(I\) equals the closed linear span of \(\{ax : a \in A, x \in I\}\). Moreover, we will give conditions under which \(I\) and \(\frac{A}{I}\) are BSE algebras.

Following [9, Corollary 26.25], if \(A\) is a Fréchet Schwartz space and \(I\) be a closed ideal of \(A\), then

\[ I^* \cong \frac{A^*}{I^0} \quad \text{and} \quad (\frac{A}{I})^* \cong I^0, \]

where \(I^0 = \{ f \in A^* : |f(x)| \leq 1, \text{ for all } x \in I \}\) and equivalently

\[ I^0 = \{ f \in A^* : f(x) = 0, \text{ for all } x \in I \}. \]

Analogous to the Banach case, if \(A\) is a BSE-algebra with discrete carrier space, then \(C_0(\Delta(A)) = \ell_\infty(\Delta(A))\). Hence, \(C_{\text{BSE}}(\Delta(A)) = A^{**}|_{\Delta(A)}\). Moreover, by similar arguments to the proof of [10, Theorem 3.1.18], if \(I\) is a closed ideal of \(A\), then \(\Delta(I)\) is discrete.

**Lemma 3.1.** If the Fréchet algebra \(A\) has a bounded \(\Delta\)-weak approximate identity and \(I\) is a closed essential ideal of \(A\), then \(\frac{A}{I}\) has a bounded \(\Delta\)-weak approximate identity.

**Proof.** Let \((e_\alpha)_\alpha\) be a bounded \(\Delta\)-weak approximate identity of \(A\). Then, for each \(\varphi \in \Delta(A)\), \(\varphi(e_\alpha) \to_\alpha 1\). For each \(\alpha\), we define \(f_\alpha := e_\alpha + I\) and show that the net \((f_\alpha)_\alpha\) is a bounded \(\Delta\)-weak approximate identity for \(\frac{A}{I}\). Suppose that \(\psi \in \Delta\left(\frac{A}{I}\right)\). Therefore, there exists \(\varphi \in \Delta(A)\) such that \(\varphi|_I = 0\) and \(\psi(a + I) := \varphi(a)\), for each \(a \in A\). Hence,

\[ \psi(f_\alpha) = \varphi(e_\alpha) \to_\alpha 1. \]

Thus, \((f_\alpha)_\alpha\) is a bounded \(\Delta\)-weak approximate identity for \(\frac{A}{I}\). \(\square\)

Let \((A, p_\ell)\) be a Fréchet algebra. For each \(\ell \in \mathbb{N}\), consider

\[ M_\ell = \{ a \in A : p_\ell(a) < 1 \}, \]

and

\[ M^\ell_\ell = \{ f \in A^* : |f(a)| \leq 1 \text{ for all } a \in M_\ell \}. \]

As mentioned in section 3.3, by applying [2, Theorem 3.3], \((C_{\text{BSE}}(\Delta(A)), r_\ell)\) is a commutative semisimple Fréchet algebra such that for each \(\ell \in \mathbb{N}\) and \(\sigma \in \Delta(A)\) we have

\[ r_\ell(\sigma) = \sup \{ |\sigma(f)| : f \in M^\ell_\ell \cap <\Delta(A)> \}. \]

Now, the following result is immediate.

**Theorem 3.2.** Let \(A\) be a BSE-Fréchet Schwartz algebra with discrete carrier space and \(I\) be an essential closed ideal of \(A\). Then,

(i) \(C_{\text{BSE}}(\Delta(I)) \subseteq \overline{M(I)}\).

(ii) \(M(\frac{A}{I}) = C_{\text{BSE}}(\Delta(\frac{A}{I}))\).
Proof. (i) Suppose that \( w \in C_{\text{BSE}}(\Delta(I)) \). Note that
\[
\Delta(I) = \{ \varphi | I : \varphi \in \Delta(A) \setminus I^\circ \}.
\]
We define \( \sigma \in \Delta(A) \) as follow
\[
\sigma(\varphi) := \begin{cases} 
  w(\varphi | I) & \varphi \in \Delta(A) \setminus I^\circ, \\
  0 & \varphi \in \Delta(A) \cap I^\circ.
\end{cases}
\]
We show that \( \sigma \in C_{\text{BSE}}(\Delta(A)) \). In fact, there exists a bounded set \( M \) in \( I \) and a positive real number \( \beta_M \) such that for every finite number of complex-numbers \( c_1, \ldots, c_n \) and the same number of \( \varphi_1, \ldots, \varphi_n \in \Delta(A) \), we have
\[
\left| \sum_{i=1}^{n} c_i \sigma(\varphi_i) \right| = \left| \sum_{\varphi_i \in \Delta(A) \setminus I^\circ} c_i \sigma(\varphi_i) \right|
\leq \beta_M P_M \left( \sum_{\varphi_i \in \Delta(A) \setminus I^\circ} c_i |\varphi_i| \right)
\leq \beta_M P_M \left( \sum_{i=1}^{n} c_i |\varphi_i| \right)
\leq \beta_M P_M (\sum_{i=1}^{n} c_i |\varphi_i|).
\]
Thus, \( \sigma \in C_{\text{BSE}}(\Delta(A)) \). Since \( A \) is BSE-algebra, take \( T \in M(A) \) such that \( \widehat{T} = \sigma \) and put \( S = T | I \). Then, \( S \in M(I) \). Also, for any \( x \in I \) and \( \varphi \in \Delta(A) \setminus I^\circ \), we have
\[
\widehat{S}(\varphi | I) \widehat{x}(\varphi | I) = (\widehat{S}x)(\varphi | I) = (\widehat{T}x)(\varphi) = \widehat{T}(\varphi) \widehat{x}(\varphi)
= \sigma(\varphi) \widehat{x}(\varphi | I) = w(\varphi | I) \widehat{x}(\varphi | I).
\]
Since \( \varphi | I \in \Delta(A) \), there exists \( x \in I \) such that \( \varphi | I(x) \neq 0 \). Then, \( \widehat{x}(\varphi | I) \neq 0 \).
Therefore, \( \widehat{S}(\varphi | I) = w(\varphi | I) \). Consequently, \( w = \widehat{S} \), where \( \widehat{S} \in M(I) \). Hence, \( C_{\text{BSE}}(\Delta(I)) \subseteq M(I) \).

(ii) Since \( A \) is BSE algebra, it has a bounded \( \Delta \)-weak approximate identity and by Lemma 3.1, \( \frac{A}{I} \) also has an approximate identity. Thus, \( \widehat{M}(\frac{A}{I}) \subseteq C_{\text{BSE}}(\Delta(\frac{A}{I})) \).

To show the reverse inclusion, let \( \sigma' \in C_{\text{BSE}}(\Delta(\frac{A}{I})) \). Since both \( \Delta(A) \) and \( \Delta(\frac{A}{I}) \) are discrete,
\[
C_{\text{BSE}}(\Delta(A)) = A^{**}|_{\Delta(A)}
\]
and
\[
C_{\text{BSE}}(\Delta(\frac{A}{I})) = (\frac{A}{I})^{**}|_{\Delta(\frac{A}{I})}.
\]
Thus, we can take \( F' \in (\frac{A}{I})^{**} \) with \( F'|_{\Delta(\frac{A}{I})} = \sigma' \). Set \( I^\circ = \{ f \in A^*: f| I = 0 \} \).
For each \( f \in I^\circ \), define \( f' \) by the relation \( f'(a') = f(a) \) \( (a' = a + I \in \frac{A}{I}) \). The map \( f \mapsto f' \) is an isometric isomorphism of \( I^\circ \) onto \( (\frac{A}{I})^{*} \). We define \( F' \) by the relation \( F'(f) = F'(f') \) \( (f \in I^\circ) \). Then, \( F' \in (I^\circ)^* \) and there exists \( F \in A^{**} \) such that \( r_F(F) = r_{F'}(F') \) and \( F|I^\circ = F' \) by using the general framework of the Hahn-Banach theorem [9, Proposition 22.12]. By the BSE property of \( A \), there exists \( T \in M(A) \) with \( \widehat{T} = F|_{\Delta(A)} \). Since \( I \) is essential and \( T(ab) = aTb \) for all
a, b ∈ A, we have T(I) ⊆ I. Then, T′ defined by $T′(a′) = (Ta)' (a ∈ A)$ belongs to $M(\hat{\mathcal{A}})$. In this case, $\hat{T}' = \sigma'$. Actually for any $a ∈ A$ and $\varphi ∈ \Delta(A) \cap I^0$, we have
\[
\hat{\alpha}(\varphi)\hat{T}'(\varphi)' = \hat{\alpha}'(\varphi)\hat{T}'(\varphi)' = \hat{(Ta)'}(\varphi)'
\]
\[
= (Ta)'(\varphi)' = (Ta)(\varphi) = \hat{\alpha}(\varphi)\hat{T}(\varphi),
\]
so that $\hat{T}' = \hat{T}(\varphi)$ and
\[
\sigma'(\varphi') = F'(\varphi') = F(\varphi) = \hat{T}(\varphi) = \hat{T}'(\varphi').
\]
Therefore, $\sigma' = \hat{T}$, since $\Delta(\hat{\mathcal{A}}) = \{\varphi' : \varphi ∈ \Delta(A) \cap I^0\}$. Thus, $\sigma' ∈ M(\hat{\mathcal{A}})$. Hence, $C_{\text{BSE}}(\Delta(\hat{\mathcal{A}})) ⊆ M(\hat{\mathcal{A}})$. Consequently, $\mathcal{A}$ is BSE.

Corollary 3.3. In above Theorem, if I has a bounded $\Delta$-weak approximate identity, then I is a BSE algebra.

References

[1] Z. Alimohammadi and A. Rejali, Fréchet algebras in abstract harmonic analysis, arXiv:1811.10987v1 [math.FA].
[2] M. Amiri and A. Rejali, The Bochner-Schoenberg-Eberlein Property for commutative Fréchet Algebras, submitted.
[3] S. Bochner, A theorem on Fourier-Stieltjes integrals, Bull. Amer. Math. Soc. 40(1934), 271-276.
[4] H. Goldmann, Uniform Fréchet algebras, North-Holland Mathematics Studies, 162. North-Holland (Amsterdam-New York, 1990).
[5] J. Inoue and S. E. Takahasi, On characterizations of the image of Gelfand transform of commutative Banach algebras. Math. Nachr. 280(2007), 105-126.
[6] Z. Kamali, M. Lashkarizadeh Bami, The multiplier algebra and bse property of the direct sum of banach algebras, Bull. Aust. Math. Soc. 88(2013), 250-258.
[7] E. Kaniuth and A. Ülger, The Bochner-Schoenberg-Eberlein property for commutative Banach algebras, especially Fourier-Stieltjes algebras, Trans. Amer. Math. Soc., 362(2010), 4331-4356.
[8] R. Larsen, An Introduction to the theory of multipliers, Springer-Verlag, New York-Heidelberg, (1971).
[9] R. Meise and D. Vogt, Introduction to functional analysis, Oxford Science Publications, (1997).
[10] C. E. Rickart, General theory of Banach algebras, Van Nostrand, Princeton, New Jersey.
[11] I. J. Schoenberg, A remark on the preceeding note by Bochner, Bull. Amer. Math. Soc. 40(1934), 277-278.
[12] S. E. Takahasi and O. Hatori, Commutative Banach algebras which satisfy a Bochner-Schoenberg-Eberlein-type theorem, Proc. Amer. Math. Soc. 110(1990), 149-158.

M. Amiri
Department of Pure Mathematics, University of Isfahan, Isfahan, Iran
mitra.amiri@sci.ui.ac.ir
mitra75amiri@gmail.com

A. Rejali
Department of Pure Mathematics, University of Isfahan, Isfahan, Iran
rejali@sci.ui.ac.ir