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The global dynamics for a stochastic SIS epidemic model with isolation

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HIGHLIGHTS

• A stochastic SIS epidemic model with isolation and multiple noises perturbation is proposed.
• The criteria on the extinction and persistence in the mean with probability one are obtained.
• The sufficient conditions for the existence of unique stationary distribution are established.

ABSTRACT

In this paper, we investigate the dynamical behavior for a stochastic SIS epidemic model with isolation which is as an important strategy for the elimination of infectious diseases. It is assumed that the stochastic effects manifest themselves mainly as fluctuation in the transmission coefficient, the death rate and the proportional coefficient of the isolation of infective. It is shown that the extinction and persistence in the mean of the model are determined by a threshold value $R_0$. That is, if $R_0 < 1$, then disease dies out with probability one, and if $R_0 > 1$, then the disease is stochastic persistent in the mean with probability one. Furthermore, the existence of a unique stationary distribution is discussed, and the sufficient conditions are established by using the Lyapunov function method. Finally, some numerical examples are carried out to confirm the analytical results.

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1. Introduction

As is well-known, in the theory of epidemiology the quarantine/isolation is an important strategy for the control and elimination of infectious diseases. Such as, in order to control SARS, the Chinese government is the first to use isolation. The various types of classical epidemic models with quarantine/isolation have been investigated in many articles. See, for example [1–15] and the references cited therein.

Particularly, in [1], Herbert et al. studied the following SIS epidemic model with isolation

\[
\begin{align*}
S'(t) &= A - \beta IS - \mu S + \gamma I + \xi Q, \\
I'(t) &= \beta IS - (\mu + \gamma + \delta + \alpha)I, \\
Q'(t) &= \delta I - (\mu + \xi + \alpha)Q.
\end{align*}
\]

(1.1)

where $S(t)$ denotes the number of individuals who are susceptible to an infection, $I(t)$ denotes the number of individuals who are infectious but not isolated, $Q(t)$ is the number of individuals who are isolated. $A$ is the recruitment rate of $S(t)$, $\beta$...
is the transmission rate coefficient between compartment \( S(t) \) and \( I(t) \), \( \mu \) is natural death rate of \( S(t), I(t) \) and \( Q(t) \), \( \alpha \) is the disease-related death rate of \( I(t) \), \( \delta \) is the proportional coefficient of isolated for the infection, \( \gamma \) and \( \xi \) are the rates where individuals recover and return to \( S(t) \) from \( I(t) \) and \( Q(t) \), respectively. All parameters are usually assumed to be nonnegative.

In addition, we see that the quarantine/isolation strategies also are introduced and investigated in many practical epidemic model, such as the emerging infectious disease, two-strain avian influenza, childhood diseases, the Middle East respiratory syndrome, Ebola epidemics, Dengue epidemic, H1N1 flu epidemic, Hepatitis B and C, Tuberculosis, etc. See, for example [16–28] and the references cited therein.

As a matter of fact, epidemic systems are inevitably subjected to environmental white noise. Therefore, the studies for the stochastic epidemic models have more practical significance. In recent years, the stochastic epidemic models with the quarantine and isolation have been investigated in articles [29–32]. Particularly, in [29] Zhang et al. investigated the dynamics of the deterministic and stochastic SIQS epidemic model with an isolation and nonlinear incidence. The sufficient conditions on the extinction almost surely of the disease and the existence of stationary distribution of the model are established. Zhanget al. in [30] discussed the threshold of a stochastic SIQ epidemic model. The criteria on the extinction conditionson the extinction almost surely of the disease and the existence of stationary distribution of the model are established. Zhanget al. in [30] discussed the threshold of a stochastic SIQ epidemic model. The criteria on the extinction conditionson the extinction almost surely of the disease and the existence of stationary distribution of the model are established.

Motivated by the works [1,2,4,5,29–32], in this paper as an extension of model (1.1) we firstly assume that the disease-related death rates of isolation and no-isolation are different, respectively, denote by \( \alpha_2 \) and \( \alpha_3 \). Then, we further define \( \mu_1 = \mu, \mu_2 = \mu + \alpha_2 \) and \( \mu_3 = \mu + \alpha_3 \) for the convenience. It is clear that \( \mu_1 \leq \min\{\mu_2, \mu_3\} \). Next, we introduce randomness into model (1.1), by replacing the parameters \( \beta, \mu_i \) \( (i = 1, 2, 3) \) and \( \delta \) with \( \beta \to \beta + \sigma_1 W_1(t), \mu_2 \to \mu_2 + \sigma_2 W_2(t), \mu_3 \to \mu_3 + \sigma_3 W_3(t), \delta \to \delta + \sigma_2 W_4(t) \) and \( \mu_1 \to \mu_1 + \sigma_3 W_5(t) \), where \( W_i(t) \) \( (i = 1, 2, 3, 4, 5) \) are independent standard Brownian motion defined on some probability space \((\Omega, \mathcal{F}, P)\) and parameter \( \sigma_i \) \( i > 0 \) represents the intensity of \( W_i(t) \). Thus, we establish the following stochastic SIS epidemic model with multi-parameters white noises perturbations and the isolation of infection.

\[
\begin{align*}
\text{d}S &= [A - \beta IS - \mu_1 S + \gamma I + \xi Q] \text{d}t - \sigma_1 S \text{d}W_1(t) + \sigma_3 S \text{d}W_3(t), \\
\text{d}I &= [\beta IS - (\mu_2 + \gamma + \delta)I] \text{d}t + \sigma_1 I \text{d}W_1(t) + \sigma_2 I \text{d}W_2(t) - \sigma_4 I \text{d}W_4(t), \\
\text{d}Q &= [\xi I - (\mu_3 + \xi)Q] \text{d}t + \sigma_3 Q \text{d}W_3(t) + \sigma_4 Q \text{d}W_4(t).
\end{align*}
\]

(1.2)

Our purpose in this paper is to study the stochastic extinction and persistence, and the stationary distribution of model (1.2). We will establish a series of sufficient conditions to assure the extinction and persistence in the mean of the model with probability one, and the existence of unique stationary distribution for model (1.2) by using the theory of stochastic processes, the Ito's formula and the Liapunov function method.

This paper is organized as follows. In Section 2, we introduce the preliminaries and some useful lemmas. In Section 3, the criteria on the extinction and persistence in the mean with probability one for model (1.2) are stated and proved. In Section 4, the criteria on the existence of a unique stationary distribution for model (1.2) are stated and proved. In Section 5, the numerical examples are carried out to illustrate the main theoretical results.

2. Preliminaries

We denote \( \mathbb{R}^3_+ = \{(x_1, x_2, x_3) : x_i > 0, i = 1, 2, 3\} \). For an integrable function \( f(t) \) defined on \([0, \infty)\), denote \( \langle f(t) \rangle = \frac{1}{t} \int_0^t f(s) \text{d}s \).

As the preliminaries, we give the following lemmas.

Lemma 2.1. For deterministic model (1.1), let \( R_0 = \frac{\beta A}{\mu(\delta + \gamma + \mu + \alpha)} \). We have following conclusions.

1. If \( R_0 < 1 \), then model (1.1) has only a disease-free equilibrium \( E_0(\frac{A}{\mu}, 0, 0) \), which is globally asymptotically stable.
2. If \( R_0 > 1 \), then model (1.1) also has an endemic equilibrium \( E^*(S^*, I^*, Q^*) \), which is globally asymptotically stable, where

\[
S^* = \frac{A}{\mu R_0}, \quad I^* = \frac{A(1 - \frac{1}{R_0})}{(\mu + \alpha)(1 + \frac{\delta}{\mu + \xi + \alpha})}, \quad Q^* = \frac{\delta I^*}{\mu + \xi + \alpha}.
\]

The proof of Lemma 2.1 can be found in [1]. We hence omit it here.

Lemma 2.2. For any given initial value \((S(0), I(0), Q(0))\in\mathbb{R}^3_+\), model (1.2) has a unique global positive solution \((S(t), I(t), Q(t))\). That is, solution \((S(t), I(t), Q(t))\) is defined for all \( t \geq 0 \) and remains in \( \mathbb{R}^3_+ \) with probability one.

Lemma 2.2 can be proved by using the similar method given in [29].
Lemma 2.3. Let $(S(t), I(t), Q(t))$ be the solution of model (1.2) with initial value $(S(0), I(0), Q(0)) \in \mathbb{R}^3_+$, then

\[
\limsup_{t \to \infty} (S(t) + I(t) + Q(t)) < \infty \text{ a.s.}
\]

(2.1)

Moreover,

\[
\limsup_{t \to \infty} (S(t) + I(t) + Q(t)) \leq \frac{A}{\mu_1} \text{ a.s.}
\]

(2.2)

Proof. By model (1.2), we have

\[
\begin{align*}
d(S + I + Q) &= [A - \mu_1(S + I + Q) - \alpha_2 I - \alpha_3 Q]dt \\
&\quad + \sigma_1 dW_2(t) + \sigma_2 Q dW_3(t) + \sigma_5 dW_5(t),
\end{align*}
\]

where $\alpha_2 = \mu_2 - \mu_1 \geq 0$ and $\alpha_3 = \mu_3 - \mu_1 \geq 0$. Solving this equation, we further obtain that

\[
\begin{align*}
S(t) + I(t) + Q(t) &= \frac{A}{\mu_1} + \left[ (S(0) + I(0) + Q(0)) - \frac{A}{\mu_1} \right] e^{-\mu_1 t} - \alpha_2 \int_0^t e^{-\mu_1 (t-s)} I(s) ds \\
&\quad - \alpha_3 \int_0^t e^{-\mu_1 (t-s)} Q(s) ds + \sigma_2 \int_0^t e^{-\mu_1 (t-s)} I(s) dW_2(s) \\
&\quad + \sigma_3 \int_0^t e^{-\mu_1 (t-s)} Q(s) dW_3(s) + \sigma_5 \int_0^t e^{-\mu_1 (t-s)} S(s) dW_5(s) \\
&\leq \frac{A}{\mu_1} + \left[ (S(0) + I(0) + Q(0)) - \frac{A}{\mu_1} \right] e^{-\mu_1 t} + M(t) \text{ a.s.,}
\end{align*}
\]

(2.4)

where

\[
\begin{align*}
M(t) &= \sigma_2 \int_0^t e^{-\mu_1 (t-s)} I(s) dW_2(s) + \sigma_3 \int_0^t e^{-\mu_1 (t-s)} Q(s) dW_3(s) \\
&\quad + \sigma_5 \int_0^t e^{-\mu_1 (t-s)} S(s) dW_5(s).
\end{align*}
\]

Clearly, $M(t)$ is a continuous local martingale with $M(0) = 0$. Define

\[
X(t) = X(0) + A(t) - U(t) + M(t),
\]

where $X(0) = S(0) + I(0) + Q(0)$, $A(t) = \frac{A}{\mu_1} (1 - e^{-\mu_1 t})$ and $U(t) = (S(0) + I(0) + Q(0)) (1 - e^{-\mu_1 t})$. By (2.4) we have

\[
S(t) + I(t) + Q(t) \leq X(t) \text{ a.s. for all } t \geq 0.
\]

It is clear that $A(t)$ and $U(t)$ are continuous adapted increasing processes on $t \geq 0$ with $A(0) = U(0) = 0$. By Theorem 3.9 in [44], we obtain that $\lim_{t \to \infty} X(t) < \infty$ a.s. Thus, conclusion (2.1) is true.

Set

\[
\begin{align*}
M_2(t) &= \int_0^t I(s) dW_2(s), \quad M_2^*(t) = \int_0^t e^{-\mu_1 (t-s)} I(s) dW_2(s), \\
M_3(t) &= \int_0^t Q(s) dW_3(s), \quad M_3^*(t) = \int_0^t e^{-\mu_1 (t-s)} Q(s) dW_3(s), \\
M_5(t) &= \int_0^t S(s) dW_5(s), \quad M_5^*(t) = \int_0^t e^{-\mu_1 (t-s)} S(s) dW_5(s).
\end{align*}
\]

Since the quadratic variations

\[
\begin{align*}
\langle M_2(t), M_2(t) \rangle &= \int_0^t I^2(s) ds \leq (\sup_{t \geq 0} I^2(t)) t, \\
\langle M_2^*(t), M_2^*(t) \rangle &= \int_0^t e^{-2\mu_1 (t-s)} I^2(s) ds \leq (\sup_{t \geq 0} I^2(t)) t,
\end{align*}
\]

by the large number theorem for martingales (See [44,45]), we have

\[
\lim_{t \to \infty} \frac{1}{t} M_2(t) = 0, \quad \lim_{t \to \infty} \frac{1}{t} M_2^*(t) = 0 \text{ a.s.}
\]

(2.5)

Similarly, we also have

\[
\lim_{t \to \infty} \frac{1}{t} M_3(t) = 0, \quad \lim_{t \to \infty} \frac{1}{t} M_3^*(t) = 0, \quad \lim_{t \to \infty} \frac{1}{t} M_5(t) = 0, \quad \lim_{t \to \infty} \frac{1}{t} M_5^*(t) = 0 \text{ a.s.}
\]

(2.6)
Since
\[
(M(t)) = \frac{\sigma_2}{\mu_1} \int_0^t \int_0^t e^{-\mu_1(s-u)}I(u)dW_2(u)ds + \frac{\sigma_3}{\mu_1} \int_0^t \int_0^t e^{-\mu_1(s-u)}Q(u)dW_3(u)ds
\]
+ \frac{\sigma_5}{\mu_1} \int_0^t \int_0^t e^{-\mu_1(s-u)}S(u)dW_5(u)ds
\]
\[= \frac{\sigma_2}{\mu_1} \left( \int_0^t I(u)dW_2(u) - \int_0^t e^{-\mu_1(t-u)}I(u)dW_2(u) \right)
+ \frac{\sigma_3}{\mu_1} \left( \int_0^t Q(u)dW_3(u) - \int_0^t e^{-\mu_1(t-u)}Q(u)dW_3(u) \right)
+ \frac{\sigma_5}{\mu_1} \left( \int_0^t S(u)dW_5(u) - \int_0^t e^{-\mu_1(t-u)}S(u)dW_5(u) \right).
\]
by (2.5) and (2.6), we obtain \( \lim_{t \to \infty} (M(t)) = 0. \) Since
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \left[ (S(0) + I(0) + Q(0)) - \frac{A}{\mu_1} \right] e^{-\mu_1 s} ds
= \lim_{t \to \infty} \frac{1}{\mu_1 t} \left[ (S(0) + I(0) + Q(0)) - \frac{A}{\mu_1} \left( 1 - e^{-\mu_1 t} \right) \right] = 0.
\]
form (2.4), it follows that conclusion (2.2) is true. This completes the proof. \( \square \)

**Lemma 2.4.** Let \((S(t), I(t), R(t))\) be the solution of model (1.2) with initial value \((S(0), I(0), Q(0)) \in \mathbb{R}_+^3\) and \(N(t) = S(t) + I(t) + Q(t).\) Then
\[\langle S(t) \rangle = \frac{A}{\mu_1} - \left[ \frac{\mu_2}{\mu_1} + \frac{\delta \mu_3}{\mu_1(\mu_3 + \xi)} \right] \langle I(t) \rangle + K(t) \tag{2.7}\]
and
\[\langle N^2(t) \rangle = \frac{A}{\mu_1} \langle N(t) \rangle - \frac{\alpha_2}{\mu_1} \langle I(t)N(t) \rangle - \frac{\alpha_3}{\mu_1} \langle Q(t)N(t) \rangle + \frac{\alpha_4}{2\mu_1} \langle S^2(t) \rangle + \frac{\alpha_5}{2\mu_1} \langle I^2(t) \rangle + \frac{\alpha_6}{2\mu_1} \langle Q^2(t) \rangle + C(t). \tag{2.8}\]
where
\[C(t) = \frac{\sigma_2}{\mu_1} \frac{1}{t} \int_0^t N(s)I(s)dW_2(s) + \frac{\sigma_3}{\mu_1} \frac{1}{t} \int_0^t N(s)Q(s)dW_3(s)
+ \frac{\sigma_5}{\mu_1} \frac{1}{t} \int_0^t N(s)S(s)dW_5(s) + \frac{1}{2\mu_1 t} \langle N^2(0) - N^2(t) \rangle \tag{2.9}\]
and
\[K(t) = \frac{\alpha_3}{\mu_1(\mu_3 + \xi)t} \int_0^t Q(s)dW_3(s) + \frac{\mu_3}{\mu_1(\mu_3 + \xi)t} \langle Q(t) - Q(0) \rangle
+ \frac{\alpha_4}{\mu_1} \frac{1}{t} \int_0^t I(s)dW_2(s) - \frac{\alpha_4}{\mu_1} \frac{1}{t} \int_0^t I(s)dW_2(s)
+ \frac{\alpha_5}{\mu_1} \frac{1}{t} \int_0^t S(s)dW_5(s)
- \frac{1}{\mu_1 t} [S(t) + I(t) + Q(t) - \langle S(0) + I(0) + Q(0) \rangle]. \tag{2.10}\]

**Proof.** Using Ito’s formula, by (2.3) we have
\[dN^2(t) =LN^2(t)dt + 2N(t)[\sigma_2 I(t)dW_2(t) + \sigma_3 Q(t)dW_3(t) + \sigma_5 S(t)dW_5(t)], \tag{2.11}\]
where
\[LN^2(t) = 2AN(t) - 2\mu_1 N^2(t) - 2\alpha_2 I(t)N(t) - 2\alpha_3 Q(t)N(t)
+ \alpha_2 I^2(t) + \alpha_3 Q^2(t) + \alpha_5 S^2(t). \]
Integrating (2.11) from 0 to \( t \), we further obtain
\[
\begin{align*}
N^2(t) - N^2(0) &= 2A \int_0^t N(s)ds - 2\mu_1 \int_0^t N^2(s)ds - 2\alpha_2 \int_0^t I(s)N(s)ds \\
&\quad - 2\alpha_3 \int_0^t N(s)Q(s)ds + \alpha_2^2 \int_0^t I^2(s)ds + \alpha_3^2 \int_0^t Q^2(s)ds \\
&\quad + \alpha_3^2 \int_0^t S^2(s)ds + 2\alpha_2 \int_0^t N(s)I(s)W_2(s) \\
&\quad + 2\alpha_3 \int_0^t N(s)Q(s)W_3(s) + 2\alpha_5 \int_0^t N(s)S(s)W_5(s).
\end{align*}
\tag{2.12}
\]

Then, dividing \( t \) on both sides (2.12), it follows that
\[
\langle N^2(t) \rangle = \frac{A}{\mu_1} \langle N(t) \rangle - \frac{\alpha_2}{\mu_1} \langle I(t)N(t) \rangle - \frac{\alpha_3}{\mu_1} \langle Q(t)N(t) \rangle \\
+ \frac{\alpha_3^2}{2\mu_1} \langle S^2(t) \rangle + \frac{\alpha_2^2}{2\mu_1} \langle I^2(t) \rangle + \frac{\alpha_3^2}{2\mu_1} \langle Q^2(t) \rangle + C(t),
\]
where \( C(t) \) is given in (2.9). Thus, we finally obtain (2.8).

Taking the integration for the third equation of model (1.2) yields
\[
\begin{align*}
Q(t) - Q(0) &= \delta \int_0^t I(s)ds - (\mu_3 + \xi) \int_0^t Q(s)ds \\
&\quad + \sigma_3 \int_0^t Q(s)W_3(s) + \sigma_4 \int_0^t I(s)W_4(s).
\end{align*}
\tag{2.13}
\]

Dividing \( t \) on both sides of Eq. (2.13), we have
\[
\langle Q(t) \rangle = \frac{\delta}{\mu_3 + \xi} \langle I(t) \rangle + \frac{\sigma_3}{(\mu_3 + \xi)t} \int_0^t Q(s)W_3(s) \\
+ \frac{\sigma_4}{(\mu_3 + \xi)t} \int_0^t I(s)W_4(s) - \frac{1}{(\mu_3 + \xi)t} \langle Q(t) - Q(0) \rangle.
\tag{2.14}
\]

Integrating (2.3) from 0 to \( t \), and then dividing \( t \) on both sides, we have
\[
\frac{1}{t} (S(t) + I(t) + Q(t) - (S(0) + I(0) + Q(0)) \\
= A - \mu_1 \langle S(t) \rangle - \mu_2 \langle I(t) \rangle - \mu_3 \langle Q(t) \rangle \\
+ \sigma_2 \frac{1}{t} \int_0^t I(s)W_2(s) + \sigma_3 \frac{1}{t} \int_0^t Q(s)W_3(s) + \sigma_5 \frac{1}{t} \int_0^t S(s)W_5(s).
\]

Consequently,
\[
\langle S(t) \rangle = \frac{A}{\mu_1} - \frac{\mu_2}{\mu_1} \langle I(t) \rangle - \frac{\mu_3}{\mu_1} \langle Q(t) \rangle - \frac{\sigma_2}{\mu_1 t} \int_0^t I(s)W_2(s) \\
+ \frac{\sigma_3}{\mu_1 t} \int_0^t Q(s)W_3(s) + \frac{\sigma_5}{\mu_1 t} \int_0^t S(s)W_5(s) \\
- \frac{1}{\mu_1 t} \langle S(t) + I(t) + Q(t) - (S(0) + I(0) + Q(0)) \rangle.
\tag{2.15}
\]

By substituting (2.14) into (2.15), we obtain
\[
\langle S(t) \rangle = \frac{A}{\mu_1} - \left[ \frac{\mu_2}{\mu_1} + \frac{\delta \mu_3}{\mu_1 (\mu_3 + \xi)} \right] \langle I(t) \rangle + K(t),
\]
where \( K(t) \) is given in (2.10). Thus, we finally obtain (2.7). This completes the proof. \( \square \)

**Lemma 2.5.** Assume that functions \( Y \in C(R_+ \times \Omega, R_+) \) and \( Z \in C(R_+ \times \Omega, R_+) \) satisfies \( \lim_{t \to \infty} \frac{Z(t)}{t} = 0 \) a.s. If there are two constants \( v_0 > 0 \) and \( v > 0 \) such that
\[
\ln Y(t) = v_0 t - v \int_0^t Y(s)ds + Z(t) \text{ a.s.}
\]
for all $t \geq 0$, then
\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t Y(s) ds = \frac{v_0}{v} \quad \text{a.s.}
\]

Lemma 2.5 can be found in Liu et al. [46].

3. Persistence and extinction

Define
\[
R_0^c = \frac{1}{\mu_2 + \delta + \gamma} \left( \frac{A\beta}{\mu_1} - \frac{1}{2} \sigma_2^2 - \frac{1}{2} \sigma_4^2 - \frac{A^2 \sigma_1^2}{2\mu_1^2} \right).
\]

**Theorem 3.1.** Assume $\sigma_5 = 0$ in model (1.2). Let $(S(t), I(t), Q(t))$ be the solution of system (1.2) with initial value $(S(0), I(0), Q(0)) \in \mathbb{R}_+^3$. If $R_0^c > 1$, then $\liminf_{t \to \infty} (S(t)) > 0$, $\liminf_{t \to \infty} (I(t)) > 0$ and $\liminf_{t \to \infty} (Q(t)) > 0$ a.s. That is, model (1.2) is stochastic persistent in the mean with probability one.

**Proof.** Applying Ito’s formula, we have
\[
d\ln I(t) = [\beta S - (\mu_2 + \delta + \gamma) - \frac{1}{2} \sigma_2^2 - \frac{1}{2} \sigma_4^2 - \frac{1}{2} \sigma_1^2 \langle S^2(t) \rangle + \frac{1}{2} \sigma_1^2 \langle S^2(t) \rangle]
dt + \sigma_1 S dW_1(t) + \sigma_2 dW_2(t) - \sigma_4 dW_4(t).
\]

Integrating (3.1) from 0 to $t$ and then dividing $t$ on both sides, we have
\[
\frac{\ln I(t) - \ln I(0)}{t} = \beta \langle S(t) \rangle - (\mu_2 + \delta + \gamma) - \frac{1}{2} \sigma_2^2 - \frac{1}{2} \sigma_4^2 - \frac{1}{2} \sigma_1^2 \langle S^2(t) \rangle + \sigma_1^2 \langle S^2(t) \rangle + \frac{1}{t} \sigma_1 \int_0^t S(s) dW_1(s) + \frac{1}{t} \sigma_2 \int_0^t S(s) dW_2(s) - \sigma_4 \int_0^t W_4(s)
\]

From (2.7), we have
\[
\frac{\ln I(t) - \ln I(0)}{t} = \frac{A\beta}{\mu_1} - \beta \left[ \frac{\mu_2}{\mu_1} + \frac{\delta \mu_3}{\mu_1 (\mu_3 + \xi)} \right] \langle I(t) \rangle + \beta K(t)
\]
\[
- (\mu_2 + \delta + \gamma) - \frac{1}{2} (\sigma_2^2 + \sigma_4^2) + \frac{1}{2} \sigma_1 \int_0^t S(s) dW_1(s) + \sigma_2 \int_0^t S(s) dW_2(s) - \frac{1}{t} \sigma_4 \int_0^t W_4(s) - \frac{1}{2} \sigma_1^2 \langle S^2(t) \rangle.
\]

From (2.8) in Lemma 2.4, when $\sigma_5 = 0$ we have
\[
\langle N^2(t) \rangle \leq \frac{A}{\mu_1} \langle N(t) \rangle + \frac{\sigma_2^2}{2\mu_1} \langle I^2(t) \rangle + \frac{\sigma_4^2}{2\mu_1} \langle Q^2(t) \rangle + C(t).
\]

From Lemma 2.3, for solution $(S(t), I(t), Q(t))$ of model (1.2), without loss of generality, there is a constant $L > 0$ such that $I(t) \leq L$ and $Q(t) \leq L$ a.s. for all $t \geq 0$. Thus, we further obtain from (3.4),
\[
\langle N^2(t) \rangle \leq \frac{A}{\mu_1} \langle N(t) \rangle + \frac{L \sigma_2^2}{2\mu_1} \langle I(t) \rangle + \frac{L \sigma_4^2}{2\mu_1} \langle Q(t) \rangle + C(t).
\]

On the other hand, from (2.2) in Lemma 2.3 we have that for any enough small $\varepsilon > 0$ there is a $T > 0$ such that
\[
\langle N(t) \rangle < \frac{A}{\mu_1} + \varepsilon \quad \text{a.s.}
\]

for all $t \geq T$.

By substituting (2.14) and (3.6) into (3.5), we obtain for all $t \geq T$
\[
\langle N^2(t) \rangle \leq \frac{A}{\mu_1} \langle N(t) \rangle + \frac{L \sigma_2^2}{2\mu_1} \langle I(t) \rangle + \frac{L \sigma_4^2}{2\mu_1} \langle Q(t) \rangle + C(t) + H(t) + C(t),
\]

where
\[
H(t) = \frac{L \sigma_2^2 \beta}{2\mu_1} \left[ \frac{\sigma_3}{\mu_3 + \xi} t \right] \int_0^t Q(s) dW_2(s) + \frac{\sigma_4^2}{\mu_3 + \xi} \int_0^t I(s) dW_4(s)
\]
\[
- \frac{1}{\mu_3 + \xi} \left[ \frac{\sigma_3}{\mu_3 + \xi} t \right] \int_0^t Q(s) - Q(0).
\]
Because of $\langle S^2(t) \rangle \leq \langle N^2(t) \rangle$, substituting (3.7) into (3.3) we further have for all $t \geq T$

\[
\lim_{t \to \infty} \frac{\ln I(t) - \ln I(0)}{t} \geq \frac{A\beta}{\mu_1} - \beta \left( \frac{\mu_2}{\mu_1} + \frac{\delta \mu_3}{\mu_1(\mu_2 + \xi)} \right) (I(t)) + \beta K(t) - (\mu_1 + \sigma_1^2 + \delta + \gamma) - \frac{1}{2}(\sigma_2^2 + \sigma_4^2) + \sigma_1 \int_0^t S(s) dW_1(s) - \frac{1}{t} \int_0^t S(s) dW_1(s) - \sigma_4 W_2(t) - \frac{1}{2} \sigma_1^2 \left( \frac{A}{\mu_1} + \epsilon \right) - \frac{1}{2} \sigma_1^2 (H(t) + C(t)).
\]

Consequently, for all $t \geq T$

\[
\frac{1}{t} \left[ \beta \left( \frac{\mu_2}{\mu_1} + \frac{\delta \mu_3}{\mu_1(\mu_2 + \xi)} \right) + \frac{\sigma_1^2}{2} \left( \frac{La_2^2}{2\mu_1} + \frac{La_3^2 \xi}{2\mu_1(\mu_3 + \xi)} \right) \right] (I(t)) 
\geq \frac{A\beta}{\mu_1} + \beta K(t) - (\mu_2 + \delta + \gamma) - \frac{1}{2}(\sigma_2^2 + \sigma_4^2) + \sigma_1 \int_0^t S(s) dW_1(s) + \sigma_1 W_2(t) - \frac{1}{2} \sigma_1^2 A \left( \frac{A}{\mu_1} + \epsilon \right) - \frac{1}{2} \sigma_1^2 (H(t) + C(t))
- \frac{1}{t} \left( \ln I(t) - \ln I(0) \right).
\]

By the large number theorem for martingales, Lemmas 2.3 and 2.4, we have from (2.9), (2.10) and (3.8)

\[
\lim_{t \to \infty} C(t) = 0, \quad \lim_{t \to \infty} K(t) = 0, \quad \lim_{t \to \infty} H(t) = 0, \quad \lim_{t \to \infty} \frac{1}{t} (\ln I(t) - \ln I(0)) = 0
\]

and

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t S(t) dW_1(s) = 0, \quad \lim_{t \to \infty} \frac{1}{t} W_2(t) = 0, \quad \lim_{t \to \infty} \frac{1}{t} W_4(t) = 0.
\]

Therefore, from (3.9) and the arbitrariness of $\epsilon$ we finally obtain

\[
\liminf_{t \to \infty} (I(t)) \geq \frac{1}{B} \left( \frac{A}{\mu_1} - (\mu_2 + \delta + \gamma) - \frac{1}{2}(\sigma_2^2 + \sigma_4^2) - \frac{1}{2} \sigma_1^2 (\frac{A}{\mu_1})^2 \right)
\]

\[
\geq \frac{1}{B} \left( \frac{A}{\mu_1} - (\mu_2 + \delta + \gamma) - \frac{1}{2}(\sigma_2^2 + \sigma_4^2) - \frac{1}{2} \sigma_1^2 (\frac{A}{\mu_1})^2 \right),
\]

\[
\geq \frac{1}{B} \left( \frac{A}{\mu_1} - (\mu_2 + \delta + \gamma) - \frac{1}{2}(\sigma_2^2 + \sigma_4^2) - \frac{1}{2} \sigma_1^2 (\frac{A}{\mu_1})^2 \right),
\]

where

\[
B = \beta \left( \frac{\mu_2}{\mu_1} + \frac{\delta \mu_3}{\mu_1(\mu_2 + \xi)} \right) + \frac{\sigma_1^2}{2} \left( \frac{La_2^2}{2\mu_1} + \frac{La_3^2 \xi}{2\mu_1(\mu_3 + \xi)} \right).
\]

From the first equation of model (1.2), we easily obtain

\[
\frac{S(t) - S(0)}{t} = \frac{1}{t} \int_0^t [A - \beta S(t) - \mu_1 S + \mu_2 + \sigma_1^2 (\frac{A}{\mu_1})^2] ds - \sigma_1 W_2(t) - \frac{1}{t} \int_0^t S(s) dW_1(s)
\geq A - (\beta L + \mu_1) (S(t)) - \sigma_1 M_1(t) a.s.,
\]

where $M_1(t) = \int_0^t S(t) dW_1(s)$. Since the quadratic variation

\[
\langle M_1(t), M_1(t) \rangle = \int_0^t S^2(s) dW_1(s) \leq L^4 t,
\]

by the large number theorem for martingales we have $\lim_{t \to \infty} \frac{1}{t} M_1(t) = 0$. Therefore, by Lemma 2.3 and (3.12) we further have

\[
\lim_{t \to \infty} S(t) \geq \frac{A}{\beta L + \mu_1} > 0 a.s.
\]

From the third equation of model (1.2), we directly have

\[
\frac{Q(t) - Q(0)}{t} = \delta (I(t)) - (\mu_3 + \xi)(Q(t)) + \frac{\sigma_3}{t} \int_0^t Q(s) ds + \frac{\sigma_4}{t} \int_0^t I(s) ds a.s.
\]

Hence, we further have

\[
\lim_{t \to \infty} Q(t) = \frac{\delta}{\mu_3 + \xi} \lim_{t \to \infty} (I(t)) \geq \frac{\delta (\mu_2 + \delta + \gamma)}{B (\mu_2 + \xi)} (R^S_0 - 1) > 0 a.s.
\]
This shows that model (1.2) is persistent in the mean with probability one. This completes the proof. □

**Remark 3.1.** It is unfortunate that in Theorem 3.1 $\sigma_5 = 0$ is assumed. From the proof of Theorem 3.1 we see that this assumption only is used to deal with the term ($S^2(t)$) in (3.3). Therefore, an interesting open problem is to establish a similar result like Theorem 3.1 for model (1.2) in $\sigma_5 > 0$.

In Theorem 3.1 we only obtain the persistence in the mean of model (1.2). However, as a consequence of Theorem 3.1 we have the following result on the permanence in the mean for the disease in model (1.2).

**Corollary 3.1.** Assume $\sigma_5 = 0$ in model (1.2). Let $(S(t), I(t), Q(t))$ be the solution of model (1.2) with initial value $(S(0), I(0), Q(0)) \in \mathbb{R}_+^3$. If $R_0^S > 1$, and $\sigma_1 = 0$ or $\sigma_2 = 0$ and $\sigma_3 = 0$, then the disease $I(t)$ is permanent in the mean with probability one.

In fact, when $\sigma_1 = 0$ or $\sigma_2 = 0$ and $\sigma_3 = 0$, from (3.11) we have $B = \beta\left(\frac{\mu_2}{\mu_1} + \frac{\delta\mu_3}{\mu_1(\mu_3 + \xi)}\right)$, which is independent for $L$. Therefore, by Theorem 3.1, we obtain from (3.10) that

$$\liminf_{t \to \infty} \langle I(t) \rangle \geq \frac{\mu_2 + \delta + \gamma}{B} (R_0^S - 1) \text{ a.s.}$$

which implies that the disease $I(t)$ is permanent in the mean with probability one.

**Remark 3.2.** From the above Corollary 3.1, we can propose an important open problem. That is, when $R_0^S > 1$, $\sigma_1 > 0$ and $\sigma_2 > 0$ or $\sigma_3 > 0$, whether we can establish the permanence in the mean of the disease $I$ for model (1.2). An example will be given in Section 5 to show that this can hold.

**Theorem 3.2.** Assume $\sigma_1 = 0$ in model (1.2). Let $(S(t), I(t), Q(t))$ be the solution of system (1.2) with initial value $(S(0), I(0), Q(0)) \in \mathbb{R}_+^3$. If $R_0^S > 1$, then we have

$$\lim_{t \to \infty} \langle I(t) \rangle = \frac{\beta A}{\mu_1} - \frac{(\mu_2 + \delta + \gamma)}{\mu_1} \frac{1}{2}(\sigma_2^2 + \sigma_3^2)$$

with initial value $S(0)$.

**Proof.** Applying Ito’s formula, directly computing, we have

$$d\left(\ln I(t) + \frac{\beta}{\mu_1} N(t)\right) = \left(\beta S (\mu_2 + \delta + \gamma) - \frac{1}{2}(\sigma_2^2 + \sigma_3^2)\right)dt$$

$$+ \frac{\beta}{\mu_1} \sigma_2 dW_2(t) - \frac{\beta}{\mu_1} \sigma_4 dW_4(t) + \frac{\beta}{\mu_1} \left(A - \mu_1 S - \mu_2 I - \mu_3 Q\right)dt$$

$$+ \frac{\beta}{\mu_1} \sigma_3 dW_3(t) + \frac{\beta}{\mu_1} \sigma_5 dW_5(t) + \frac{\beta}{\mu_1} \sigma_3 dW_3(t) + \frac{\beta}{\mu_1} \sigma_5 dW_5(t)$$

$$+= \left(\beta S (\mu_2 + \delta + \gamma) - \frac{1}{2}(\sigma_2^2 + \sigma_3^2)\right)dt$$

$$- \frac{\beta}{\mu_1} \sigma_3 dW_3(t) + \frac{\beta}{\mu_1} \sigma_5 dW_5(t) + \frac{\beta}{\mu_1} \sigma_3 dW_3(t) + \frac{\beta}{\mu_1} \sigma_5 dW_5(t).$$

Integrating (3.13) and then dividing $t$ yields

$$\frac{1}{t} \left(\ln I(t) + \frac{\beta}{\mu_1} N(t)\right) - \frac{1}{t} \left(\ln I(0) + \frac{\beta}{\mu_1} N(0)\right)$$

$$= \frac{\beta A}{\mu_1} - (\mu_2 + \delta + \gamma) - \frac{1}{2}(\sigma_2^2 + \sigma_3^2) - \frac{\beta}{\mu_1} \mu_1 (I(t))$$

$$- \frac{\beta}{\mu_1} \sigma_3 dW_3(t) - \mu_3 \int_0^t I(s) dW_3(s)$$

$$+ \frac{\beta}{\mu_1} \sigma_5 dW_5(t) + \mu_5 \int_0^t S(s) dW_5(s).$$

From (2.14), we further have

$$\frac{1}{t} \ln I(t) = \frac{\beta A}{\mu_1} - (\mu_2 + \delta + \gamma) - \frac{1}{2}(\sigma_2^2 + \sigma_3^2)$$

$$- \left[\frac{\beta}{\mu_1} \mu_1 (\mu_3 + \xi)\right] (I(t)) + B(t).$$
that for any solution \( S \).

 Remark 3.3. This completes the proof.

 Furthermore, we also have that

\[
\lim_{t \to \infty} \langle S(t) \rangle = \frac{A}{\mu_1} \left[ \frac{\mu_2}{\mu_1} + \frac{\delta}{\mu_3 + \xi} \right] \mathbb{I}^* \quad \text{a.s.}
\]

This completes the proof. \( \square \)

**Remark 3.3.** Particularly, when \( \sigma_i = 0 \), \( i = 1, 2, 3, 4, 5 \) and \( \mu_2 = \mu_3 = \mu_1 + \alpha \), then the stochastic model (1.2) degenerates into the deterministic model (1.1). We also have \( R_0^S = R_0 = \frac{\beta A}{\mu_1(\delta + \gamma + \mu_1 + \alpha)} \). From Theorem 3.2, when \( R_0 > 1 \) we can obtain that for any solution \( S(t), I(t), Q(t) \) of model (1.1) with initial value \( (S(0), I(0), Q(0)) \) in \( \mathbb{R}^3_+ \).

\[
\lim_{t \to \infty} \langle S(t) \rangle = \frac{A}{\mu_1} R_0, \quad \lim_{t \to \infty} \langle I(t) \rangle = \frac{A(1 - \frac{1}{R_0})}{(\mu_1 + \alpha)(1 + \frac{\delta}{\mu_1 + \alpha + \xi})}, \quad \lim_{t \to \infty} \langle Q(t) \rangle = \frac{\delta}{\mu_1 + \alpha + \xi} \mathbb{I}^*.
\]

Therefore, Theorem 3.2 can be regarded as an extension of the conclusion (2) of Lemma 2.1 for the deterministic model (1.1) into the corresponding stochastic model (1.2).

**Remark 3.4.** It is a pity that in Theorem 3.2 \( \sigma_1 = 0 \) is assumed. Therefore, an interesting open problem is to establish a similar result for model (1.2) in \( \sigma_1 > 0 \).

**Theorem 3.3.** Let \( (S(t), I(t), Q(t)) \) be the solution of model (1.2) with initial value \( (S(0), I(0), Q(0)) \) in \( \mathbb{R}^3_+ \). Suppose that one of the following two conditions holds:

\( \big( A \big) \quad \frac{\beta^2}{2\sigma_1^2} - \left( \mu_2 + \delta + \gamma + \frac{1}{2}(\sigma_2^2 + \sigma_4^2) \right) < 0, \quad \big( B \big) \quad \beta \geq \frac{A}{\mu_1} \sigma_1^2, \quad R_0^S < 1. \)

Then the disease \( I(t) \) almost surely exponentially dies out. That is

\[
\lim_{t \to \infty} \sup \frac{\ln(I(t))}{t} \leq \frac{\beta^2}{2\sigma_1^2} - \left( \mu_2 + \delta + \gamma + \frac{1}{2}(\sigma_2^2 + \sigma_4^2) \right) < 0 \quad \text{a.s. if} \ (A) \text{ holds} \quad (3.14)
\]

and

\[
\lim_{t \to \infty} \sup \frac{\ln(I(t))}{t} \leq (\mu_2 + \delta + \gamma)(R_0^S - 1) < 0 \quad \text{a.s. if} \ (B) \text{ holds}. \quad (3.15)
\]

Furthermore, we also have that \( \lim_{t \to \infty} \langle S(t) \rangle = \frac{A}{\mu_1} \quad \text{a.s. and} \quad \lim_{t \to \infty} \frac{\ln(Q(t))}{t} \leq -c \quad \text{a.s. for some constant} \ c > 0. \) That is, \( S(t) \) in the mean almost surely converges to \( \frac{A}{\mu_1} \) and \( Q(t) \) almost surely exponentially converges to zero.

**Proof.** Since for any \( t > 0, (\frac{1}{t} \int_0^t S(s)ds)^2 \leq \frac{1}{t} \int_0^t S^2(s)ds \) from (3.2) we have

\[
\frac{\ln(I(t))}{t} \leq \frac{\ln(I(0))}{t} + \beta \langle S(t) \rangle - (\mu_2 + \delta + \gamma) - \frac{1}{2}(\sigma_2^2 + \sigma_4^2) - \frac{1}{2}(\langle S(t) \rangle)^2 + \sigma_1 \frac{1}{t} \int_0^t S(s)ds + \sigma_2 \frac{1}{t} W_2(t) - 4 \frac{1}{t} W_4(t).
\]

Proof. Since for any \( t > 0, (\frac{1}{t} \int_0^t S(s)ds)^2 \leq \frac{1}{t} \int_0^t S^2(s)ds \) from (3.2) we have
If condition (B) holds, then from (2.7) and (3.16) we have
\[
\frac{\ln I(t)}{t} \leq \beta \frac{A}{\mu_1} - (\mu_2 + \delta + \gamma) - \frac{1}{2}(\sigma^2_2 + \sigma^2_4) - \beta \left[ \frac{\mu_2}{\mu_1} + \frac{\delta \mu_3}{\mu_1 (\mu_3 + \xi)} \right] I(t) \\
+ \beta K(t) - \frac{1}{2} \sigma_1^2 \left[ \frac{A}{\mu_1} - \left( \frac{\mu_2}{\mu_1} + \frac{\delta \mu_3}{\mu_1 (\mu_3 + \xi)} \right) (I(t)) + K(t) \right]^2 \\
+ \sigma_1 \frac{1}{t} \int_0^t S(s) dW_1(s) + \sigma_2 \frac{1}{t} W_2(t) - \sigma_4 \frac{1}{t} W_4(t) + \frac{\ln I(0)}{t}.
\]

Therefore,
\[
\frac{\ln I(t)}{t} \leq \beta \frac{A}{\mu_1} - (\mu_2 + \delta + \gamma) - \frac{1}{2}(\sigma^2_2 + \sigma^2_4) - \frac{A^2 \sigma^2_2}{2\mu^2_1} \\
- \left[ \frac{\mu_2}{\mu_1} + \frac{\delta \mu_3}{\mu_1 (\mu_3 + \xi)} \right] \left( \beta - \frac{A}{\mu_1} \right) I(t)) + \sigma_1 \frac{1}{t} \int_0^t S(s) dW_1(s) \\
- \left[ \frac{\mu_2}{\mu_1} + \frac{\delta \mu_3}{\mu_1 (\mu_3 + \xi)} \right] \left( I(t) \right)^2 + \Phi(t) + \frac{\ln I(0)}{t} + \sigma_2 \frac{1}{t} W_2(t) - \sigma_4 \frac{1}{t} W_4(t),
\]
where
\[
\Phi(t) = \beta K(t) - \frac{\sigma^2_1}{2} K^2(t) + \sigma^2_2 \left[ \frac{\mu_2}{\mu_1} + \frac{\delta \mu_3}{\mu_1 (\mu_3 + \xi)} \right] (I(t)) K(t) - A^2 I(t).
\]

By the large number theorem for martingales, Lemmas 2.3 and 2.4, we have \( \lim_{t \to \infty} \Phi(t) = 0 \) a.s. Therefore, we finally obtain
\[
\limsup_{t \to \infty} \frac{\ln I(t)}{t} \leq (\mu_2 + \delta + \gamma) (R_0^2 - 1) < 0 \text{ a.s.}
\]

If condition (A) holds, then from (3.2) we have
\[
\frac{\ln I(t)}{t} \leq \frac{\ln I(0)}{t} + \beta (S(t)) - (\mu_2 + \delta + \gamma) - \frac{1}{2}(\sigma^2_2 + \sigma^2_4) \\
- \frac{1}{2} \sigma_1^2 (S(t))^2 + \sigma_1 \frac{1}{t} \int_0^t S(s) dW_1(s) + \sigma_2 \frac{1}{t} W_2(t) - \sigma_4 \frac{1}{t} W_4(t) \\
= \frac{\ln I(0)}{t} + \frac{\beta^2}{2\sigma^2_1} - (\mu_2 + \delta + \gamma) - \frac{1}{2}(\sigma^2_2 + \sigma^2_4) \\
- \frac{1}{2} \sigma_1^2 (S(t)) - \frac{\beta^2}{\sigma^2_1} + \sigma_1 \frac{1}{t} \int_0^t S(s) dW_1(s) + \sigma_2 \frac{1}{t} W_2(t) - \sigma_4 \frac{1}{t} W_4(t) \\
\leq \frac{\ln I(0)}{t} + \frac{\beta^2}{2\sigma^2_1} - (\mu_2 + \delta + \gamma) - \frac{1}{2}(\sigma^2_2 + \sigma^2_4) \\
+ \sigma_1 \frac{1}{t} \int_0^t S(s) dW_1(s) + \sigma_2 \frac{1}{t} W_2(t) - \sigma_4 \frac{1}{t} W_4(t).
\]

Thus, we also have
\[
\limsup_{t \to \infty} \frac{\ln I(t)}{t} \leq \frac{\beta^2}{2\sigma^2_1} - (\mu_2 + \delta + \gamma + \frac{1}{2}(\sigma^2_2 + \sigma^2_4)) < 0 \text{ a.s.}
\]

From (3.14) and (3.15), there is a constant \( m > 0 \) such that for almost all \( \omega \in \Omega \) there exists a \( T_0 = T_0(\omega) > 0 \), when \( t \geq T_0 \) one has \( I(t, \omega) \leq e^{-mt} \). Without loss of generality, we assume that \( I(t, \omega) \leq e^{-mt} \) for all \( t \geq 0 \). It follows that from the third equation of model (1.2)
\[
dQ \leq [\delta e^{-mt} - (\mu_3 + \xi)Q] dt + \sigma_3 Q dW_3(t) + \sigma_4 dW_4(t).
\]

Hence,
\[
Q(t) \leq H_1(t) + H_2(t) + H_3(t), \tag{3.17}
\]
where
\[
H_1(t) = e^{-[\mu_3 + \xi] t + \sigma_3 W_3(t)} Q_0, \\
H_2(t) = e^{-[\mu_3 + \xi] t + \sigma_3 W_3(t)} \int_0^t e^{[\mu_3 + \xi] s - \sigma_3 W_3(s)} d\xi e^{-ms} ds, \\
H_3(t) = e^{-[\mu_3 + \xi] t + \sigma_3 W_3(t)} \int_0^t e^{[\mu_3 + \xi] s - \sigma_3 W_3(t)} \sigma_4 I(s) dW_4(s).
\]
Remark 3.5. Consider $H_3(t)$, choose the constants $\eta_0 > 0$ and $\varepsilon_0 > 0$ such that

\[ \mu_3 + \xi - \eta_0 - \sigma_3 \varepsilon_0 > 0, \quad m - \eta - 2\sigma_3 \varepsilon_0 > 0. \]

Since $\lim_{t \to \infty} \frac{W_3(t)}{t} = 0$, without loss of generality, we assume $|W_3(t)| \leq \varepsilon_0 t$ for all $t \geq 0$. Let $H_3^\varepsilon(t) = e^{\varepsilon t} H_3(t)$, then we have

\[ \langle H_3^\varepsilon(t), H_3^\varepsilon(t) \rangle = \int_0^t \left( e^{2\varepsilon t} e^{-[(\mu_3 + \xi) t + \sigma_3 W_3(t)]} \sigma_4 I(s)^2 \right) ds \]

\[ \leq \int_0^t e^{2\varepsilon t} e^{-2[(\mu_3 + \xi) t + 2\sigma_3 t \varepsilon + \sigma_3^2 t \varepsilon^2] + 2\sigma_3^2 \varepsilon^2} e^{-2m \varepsilon t} ds \]

\[ = \sigma_4^2 \int_0^t e^{-2[(\mu_3 + \xi - m) \varepsilon t + \sigma_3^2 \varepsilon^2] + t} e^{-2[m - \eta - 2\sigma_3 \varepsilon \varepsilon_0] \varepsilon t} ds < \infty. \]

By the large number theorem for martingales, we have $\lim_{t \to \infty} \frac{H_3^\varepsilon(t)}{t} = 0$. For any small enough $\varepsilon > 0$, we can obtain

\[ \lim_{t \to \infty} e^{-\eta_0 - \varepsilon t} \frac{|H_3(t)|}{t} = \lim_{t \to \infty} \frac{t}{e^{\eta_0 + \varepsilon t}} = 0. \]

Hence, $\lim_{t \to \infty} \frac{\ln H_3(t)}{t} = -\eta_0$. It follows that $\lim_{t \to \infty} \frac{\ln H_3(t)}{t} \leq -\eta_0$. Therefore, from (3.17) we finally have

\[ \limsup_{t \to \infty} \frac{\ln Q(t)(s)}{t} \leq -\min\{\eta_0, m, \mu_3 + \xi\} < 0 \text{ a.s.} \]

From the first equation of model (1.2) we have

\[ S(t) - S(0) \]

\[ = A - \beta(SI) - \mu_1(S) + \gamma(I) + \xi(Q) \]

\[ - \sigma_1 \int_0^t S dW_1(s) + \sigma_2 \int_0^t S dW_2(s). \]

By Lemma 2.3, the large number theorem of martingales, $\lim_{t \to \infty} I(t) = 0$ a.s. and $\lim_{t \to \infty} Q(t) = 0$ a.s., we have $\lim_{t \to \infty} S(t) = 0$, $\lim_{t \to \infty} I(t) = 0$, $\lim_{t \to \infty} Q(t) = 0$, $\lim_{t \to \infty} \frac{1}{t} (S(t) - S(0)) = 0$, $\lim_{t \to \infty} \frac{1}{t} \int_0^t S dW_1(s) = 0$ a.s. and $\lim_{t \to \infty} \frac{1}{t} \int_0^t S dW_2(s) = 0$ a.s. Therefore, $\lim_{t \to \infty} S(t) = \frac{A}{\mu_1}$ a.s. This completes the proof. \( \square \)

Remark 3.5. It is easy to see that when the condition (A) holds, then we have $R_0 ^\beta < 1$. Therefore, we can propose the following open problem. That is, when $R_0 > 1$, $\frac{\beta}{2\sigma_1^2} - (\mu_2 + \delta + \gamma + \frac{1}{2} \sigma_2^2 + \frac{1}{2} \sigma_4^2) > 0$ and $\beta < \frac{\Lambda}{\mu_1} \sigma_4^2$, whether we also can obtain the extinction of the disease $I$ with probability one for model (1.2). An example is given in Section 5 to show that the result can hold.

Remark 3.6. Comparing with the conclusion (1) of Lemma 2.1, we easily see that Theorem 3.3 can be regarded as an extension of conclusion (1) of Lemma 2.1 for the deterministic model (1.1) into the corresponding stochastic model (1.2).

In Theorem 3.3, when $\sigma_1 = 0$, then condition (A) does not hold, and condition (B) degenerates into

\[ R_0 ^\beta = \frac{1}{\mu_2 + \delta + \gamma} \left( \frac{A \beta}{\mu_1} - \frac{1}{2} \left( \sigma_2^2 + \sigma_4^2 \right) \right) < 1. \]  \( (3.18) \)

Therefore, as a consequence of Theorem 3.3, we have the following corollary.

Corollary 3.2. Assume that $\sigma_1 = 0$ in model (1.2). Let $(S(t), I(t), Q(t))$ be the solution of model (1.2) with initial value $(S(0), I(0), Q(0)) \in \mathbb{R}_+^3$. If condition (3.18) holds, then $S(t)$ in the mean almost surely converges to $\frac{A}{\mu_1}$, $I(t)$ and $Q(t)$ almost surely exponentially converge to zero.

4. Stationary distribution

In this section, we study the existence of unique stationary distribution of model (1.2). Before giving the main results, we introduce the following lemma.

Let $x(t)$ be a regular temporally homogeneous Markov process in $\mathbb{R}^d$ described by the stochastic differential equation

\[ dx(t) = b(x) dt + \sum_{i=1}^k \sigma_i(x) dB_i(t), \]  \( (4.1) \)
where $b(x) = (b_1(x), b_2(x), \ldots, b_d(x))$, $\sigma_r(x) = (\sigma^1_r(x), \sigma^2_r(x), \ldots, \sigma^d_r(x))$ and $B_r(t) (r = 1, 2, \ldots, k)$ are independent standard Brownian motions defined on some probability space $(\Omega, F, \mathbb{P})$. The diffusion matrix for Eq. (4.1) is defined as follows

$$A(x) = (a_{ij}(x))_{d \times d}, \quad a_{ij}(x) = \sum_{r=1}^{k} \sigma^r_i(x) \sigma^r_j(x).$$

**Lemma 4.1.** (See [44,45]) Assume that there exists a bounded domain $U \subset \mathbb{R}^d$ with regular boundary, satisfying the following properties.

(i) In the domain $U$ and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero.

(ii) If $x \in \mathbb{R}^d \setminus U$, the mean time $\tau$ at which a path issuing from $x$ reaches the set $U$ is finite, and $\sup_{x \in K} E_x \tau < \infty$ for every compact subset $K \subset \mathbb{R}^d$.

Then, the Markov process $x(t)$ of Eq. (4.1) has a stationary distribution $\mu_1(\cdot)$ with density in $\mathbb{R}^d$ such that $\lim_{t \to \infty} P\{x(t) \in B\} = \mu_1(B)$ for any Borel set $B \subset \mathbb{R}^d$, and

$$P\left\{ \lim_{t \to \infty} \frac{1}{T} \int_0^T f(x(t)) dt = \int_{\mathbb{R}^d} f(x) \mu_1(dx) \right\} = 1,$$

where $f(x)$ is a function integrable with respect to the measure $\mu_1$.

**Remark 4.1.** To verify condition (i), it is sufficient to show that there is a positive number $Q$ such that $\sum_{i=1}^d a_{ij}(x) \xi_i^2 \geq Q |\xi|^2$ for all $x \in U$ and $\xi \in \mathbb{R}^d$ (See [47,48]). To validate condition (ii), it is sufficient to show that there is a nonnegative $C^2$-function $V(x)$ and a bounded domain $U \subset \mathbb{R}^d$ with regular boundary such that for some constant $k > 0$ one has $LV(x) < -k$ for all $x \in \mathbb{R}^d \setminus U$ (See [49]).

When in model (1.2) there is not any stochastic perturbation, that is $\sigma_i = 0 \ (i = 1, 2, 3, 4, 5)$, then model (1.2) degenerates into the following deterministic model

$$S'(t) = A - \beta S I - \mu_1 S + \gamma I + \alpha Q,$$
$$I'(t) = \beta S I - (\mu_2 + \gamma + \alpha) I,$$
$$Q'(t) = \delta I - (\mu_3 + \xi) Q.$$ \hfill (4.2)

Let $\hat{R}_0 = \frac{\hat{\beta} A}{\mu_1(\hat{\beta} + \gamma + \mu_2)}$. We can prove that when $\hat{R}_0 > 1$ then model (4.2) has a unique endemic equilibrium $(S^*, I^*, R^*)$, where

$$S^* = \frac{A}{\mu_1 \hat{R}_0}, \quad I^* = \frac{A(1 - \frac{1}{\hat{R}_0})}{\mu_2 + \frac{\beta \mu_1}{\mu_3 + \xi}}, \quad Q^* = \frac{\delta I^*}{\mu_3 + \xi}.$$

Define the constants

$$\eta_1 = (a_3 + 1)\mu_1 - a_2 I^* \sigma_1^2 - (a_3 + 1) \sigma_2^2,$$
$$\eta_2 = a_3 (\mu_2 + \delta) + \mu_2 - (a_3 + 1) \sigma_2^2 - (a_1 + a_3) \sigma_4^2,$$
$$\eta_3 = a_1 (\mu_3 + \xi) + \mu_3 - (a_1 + 1) \sigma_3^2,$$
$$F = \frac{a_2 I^*}{2} (\sigma_2^2 + \sigma_4^2) + C_1 S^{*2} + C_2 I^{*2} + C_3 Q^{*2},$$
$$C_1 = a_2 I^* \sigma_1^2 + \sigma_3^2, \quad C_2 = (a_3 + 1) \sigma_2^2 + (a_1 + a_3) \sigma_4^2, \quad C_3 = (a_1 + 1) \sigma_3^2,$$

where

$$a_1 = \frac{\alpha_2}{\delta}, \quad a_2 = \frac{(\mu_1 + \mu_2 + \delta)(\mu_1 + \mu_3) + \xi (\mu_1 + \mu_2)}{\beta \xi}, \quad a_3 = \frac{\mu_1 + \mu_3}{\xi}.$$ 

Now, on the existence and uniqueness of stationary distribution for model (1.2) we have the following result.

**Theorem 4.1.** Assume that $\hat{R}_0 = \frac{\beta A}{\mu_1(\hat{\beta} + \gamma + \mu_2)} > 1$. If the conditions

$$\eta_i > 0 \ (i = 1, 2, 3), \quad F < \min\{\eta_1 S^{*2}, \eta_2 I^{*2}, \eta_3 Q^{*2}\}$$ \hfill (4.3)

are satisfied, then model (1.2) has a unique stationary distribution and ergodic property.

**Proof.** Define the Lyapunov function as follows.

$$V(S, I, Q) = a_1 V_1(Q) + a_2 V_2(I) + a_3 V_3(S, I) + V_4(S, I, Q).$$
where
\[ V_1 = \frac{1}{2} (Q - Q^*)^2, \quad V_2 = I - I^* - I^* \ln \frac{I}{I^*}, \]
\[ V_3 = \frac{1}{2} (S + I + Q - S^* - I^* - Q^*)^2, \quad V_4 = \frac{1}{2} (S + I + Q - S^* - I^* - Q^*)^2. \]

By computing, we have
\[
LV_1 = (Q - Q^*) \delta I - (\mu_3 + \xi) Q + \frac{1}{2} \sigma_2^2 Q^2 + \frac{1}{2} \sigma_4^2 I^2 \\
\leq \delta (I - I^*) (Q - Q^*) - (\mu_3 + \xi - \sigma_2^2) (Q - Q^*)^2 + \sigma_4^2 (I^*)^2 \\
+ \sigma_4^2 (I - I^*)^2 + \sigma_4^2 (I^*)^2.
\]
\[
LV_2 = (1 - \frac{I^*}{I}) (\beta S I - (\mu_2 + \delta + \gamma) I) + \frac{1}{2} I^* (\sigma_2^2 I^2 + \sigma_4^2 S^2 I^2 + \sigma_4^2 I^2) \\
\leq \beta (I - I^*) (S - S^*) + \frac{1}{2} (\sigma_2^2 + \sigma_4^2) + I^* \sigma_1^2 (S - S^*)^2 + S^* \sigma_2^2 S^2
\]
\[
LV_3 = (S + I + S^* - I^*) (a - \mu_1 S + \xi Q - (\mu_2 + \delta) I) + \frac{1}{2} \sigma_2^2 I^2 + \frac{1}{2} \sigma_4^2 I^2 + \frac{1}{2} \sigma_5^2 S^2 \\
\leq -(\mu_1 - \sigma_2^2) (S - S^*) - (\mu_2 + \delta - \sigma_2^2 - \sigma_5^2) (I - I^*)^2 \\
+ \xi (Q - Q^*) (S - S^*) - (\mu_1 + \mu_2 + \delta) (I - I^*) (S - S^*) \\
+ \xi (Q - Q^*) (I - I^*) + (\sigma_2^2 + \sigma_5^2) I^2 + \sigma_5^2 S^2
\]
and
\[
LV_4 = (S + I + Q - S^* - I^* - Q^*) (a - \mu_1 S - \mu_2 I) \\
- \mu_3 Q + \frac{1}{2} \sigma_2^2 I^2 + \frac{1}{2} \sigma_5^2 Q^2 + \frac{1}{2} \sigma_5^2 S^2 \\
\leq -(\mu_1 - \sigma_2^2) (S - S^*) - (\mu_2 + \delta - \sigma_2^2 - \sigma_5^2) (I - I^*)^2 \\
- (\mu_1 + \mu_2 + \gamma) (S - S^*) - (\mu_1 + \mu_3 + \gamma) (Q - Q^*) \\
+ (\sigma_3^2 I^2 + \sigma_5^2 S^2) + \sigma_5^2 S^2 - (\mu_1 - \sigma_2^2) (S - S^*)^2 \\
+ (\mu_2 - \sigma_2^2) (I - I^*)^2 + (\mu_3 - \sigma_2^2) (Q - Q^*)^2 \\
+ \sigma_5^2 I^2 + \sigma_5^2 Q^2 + \sigma_5^2 S^2
\]

Therefore, we have
\[
LV(S, I, Q) = a_1 LV_1(Q) + a_2 LV_2(I) + a_3 LV_3(S, I) + LV_4(S, I, Q), \\
\leq -a_1 (\mu_3 + \xi - \sigma_2^2) (Q - Q^*)^2 + a_1 \sigma_2^2 (Q^*)^2 + a_1 \sigma_4^2 I^2 \\
+ a_1 \sigma_4 (I - I^*)^2 + \frac{I^*}{2} (\sigma_2^2 + \sigma_4^2) + a_2 (\sigma_2^2 + \sigma_4^2) (S - S^*)^2 + a_2 I^* \sigma_1^2 S^2 \\
- a_3 (\mu_1 - \sigma_2^2) (S - S^*) - a_3 (\mu_2 + \gamma - \sigma_2^2) (I - I^*)^2 \\
+ a_3 (\sigma_2^2 I^2 + \sigma_5^2 S^2) + a_3 \sigma_5^2 S^2 - (\mu_1 - \sigma_2^2) (S - S^*)^2 \\
+ (\mu_2 - \sigma_2^2) (I - I^*)^2 + (\mu_3 - \sigma_2^2) (Q - Q^*)^2 \\
+ \sigma_5^2 I^2 + \sigma_5^2 Q^2 + \sigma_5^2 S^2
\]

If (4.3) holds, then the expression
\[
\eta_1 (S - S^*)^2 + \eta_2 (I - I^*)^2 + \eta_3 (Q - Q^*)^2 = F
\]
lie in the positive zone of \( \mathbb{R}^3_+ \). Hence, there exists a constant \( C > 0 \) and a compact set \( K \subset \mathbb{R}^3_+ \) such that for any
\[
x = (S, I, Q) \in \mathbb{R}^3_+ \setminus K \\
\eta_1 (S - S^*)^2 + \eta_2 (I - I^*)^2 + \eta_3 (Q - Q^*)^2 \geq F + C.
\]

Thus, we finally have
\[
LV(x) \leq -C, \quad x \in \mathbb{R}^3_+ \setminus K.
\]

From Remark 4.1, this shows that condition (ii) in Lemma 4.1 holds.

Next, we show that condition (i) holds in Lemma 4.1. The diffusion matrix associated to model (1.2) is
\[
A(x) = (a_i(x))_{3 \times 3} = \begin{pmatrix}
\sigma_1^2 S^2 I^2 + \sigma_2^2 S^2 & -\sigma_1^2 S^2 I^2 & 0 \\
-\sigma_1^2 S^2 I^2 & \sigma_1^2 S^2 I^2 + \sigma_2^2 I^2 + \sigma_4^2 I^2 & -\sigma_4^2 I^2 \\
0 & -\sigma_4^2 I^2 & \sigma_2^2 Q^2 + \sigma_4^2 I^2
\end{pmatrix},
\] (4.4)
where \( x = (S, I, Q) \). Choose \( M = \min_{(S,I,Q)\in U} \{ \sigma_2^2 I^2, \sigma_3^2 Q^2, \sigma_5^2 S^2 \} \). We have \( M > 0 \). For any \((S, I, Q) \in \overline{U}\) and \((\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3_+\), from (4.4) we have

\[
\sum_{i,j=1}^{3} a_{ij}(x)\zeta_j = \sigma_1^2 S^2 I^2 (\zeta_1 - \zeta_2)^2 + \sigma_2^2 I^2 \zeta_2^2 + \sigma_3^2 Q^2 \zeta_3^2 + \sigma_4^2 I^2 (\zeta_2 - \zeta_3)^2 + \sigma_5^2 S^2 \zeta_4^2 \geq \min_{(S,I,Q)\in U} \{ \sigma_2^2 S^2, \sigma_3^2 I^2, \sigma_5^2 Q^2 \} |\zeta_1 + \zeta_2 + \zeta_3| = M|\zeta|^2,
\]

where \( |\zeta| = (\zeta_1^2 + \zeta_2^2 + \zeta_3^2)^{\frac{1}{2}} \). From Remark 4.1 this shows that condition (i) in Lemma 4.1 is verified. Therefore, model (1.2) has a unique stationary distribution and the ergodic property. This completes the proof. □

**Remark 4.2.** It is clear that there exists a constant \( \sigma_0 > 0 \) such that when \( 0 \leq \sigma_i \leq \sigma_0 (i = 1, 2, 3, 4, 5) \) the condition (4.3) holds. This implies that as long as \( \tilde{R}_0 = \frac{\beta A}{\sigma_1^2 (\gamma + \mu_2)} > 1 \) then the conclusions of Theorem 4.1 hold when the stochastic perturbations in model (1.2) are small enough. However, the condition (4.3) are also very strong. We easily see that along with the increase of \( \sigma_i (i = 1, 2, 3, 4, 5) \) the condition (4.3) will not satisfy. Thus, Theorem 4.1 will not be applicable.

In the following, we consider a special case of model (1.2): \( \sigma_1 = \sigma_4 = 0 \). Here, model (1.2) degenerates into the following form

\[
\begin{align*}
\frac{dS}{t} &= [A - \beta IS - \mu_1 S + \gamma I + \xi Q]dt + \sigma_3 dW_5(t), \\
\frac{dI}{t} &= [\mu_2 I - (\mu_3 + \gamma + \delta) I]dt + \sigma_3 dW_3(t), \\
\frac{dQ}{t} &= [\beta I - (\mu_3 + \xi) Q]dt + \sigma_3 dW_2(t).
\end{align*}
\]

We will give a new conclusion on the existence of unique stationary distribution for model (4.5). Define the constant

\[
\tilde{R}_0 = \frac{\beta A}{(\mu_1 + \frac{1}{2} \sigma_5^2)(\mu_2 + \delta + \gamma + \frac{1}{2} \sigma_5^2)}.
\]

**Theorem 4.2.** Assume that \( \tilde{R}_0 > 1 \). Then model (4.5) has a unique stationary distribution and the ergodic property.

**Proof.** Let a \( C^2 \)-function \( H(S, I, Q) \) in the following form

\[
H(S, I, Q) = MV_1 + V_2 - \ln S - \ln Q,
\]

where

\[
V_1 = -c_1 \ln S - c_2 \ln I, \quad V_2 = \frac{1}{\theta + 1} (S + I + Q)^{\theta + 1}
\]

whit \( \theta \) is a constant satisfying \( 0 < \theta < \frac{2\mu_3}{\sigma_2^2 \sigma_3^2 \sigma_5^2} \), constant \( M > 0 \) will be determined later, and \( c_1 = \frac{2A}{2\mu_3 + \sigma_5^2} \).

\[
c_2 = \frac{2A}{2\mu_3 + \gamma + \delta + \mu_5}.
\]

It is easy to see that

\[
\lim_{k \to \infty} \inf_{(S,I,Q)\in \mathbb{R}_+^3 \setminus \mathbb{U}_k} H(S, I, Q) = \infty,
\]

where \( \mathbb{U}_k = \left( \frac{1}{k}, k \right) \times \left( \frac{1}{k}, k \right) \times \left( \frac{1}{k}, k \right) \) with integer \( k > 1 \). At the same time, \( H(S, I, Q) \) is a continuous function. Hence, \( H(S, I, Q) \) has a minimum value \( H(S_0, I_0, Q_0) \) in the interior of \( \mathbb{R}_+^3 \). Then, we define a nonnegative \( C^2 \)-function \( V \) in the following form

\[
V(S, I, Q) = H(S, I, Q) - H(S_0, I_0, Q_0)
\]

By the Itô's formula, for any solution \((S(t), I(t), Q(t))\) of model (1.2) we have

\[
\begin{align*}
L(-\ln S) &= -\frac{A}{S} + \beta I + \mu_1 - \frac{\gamma I}{S} - \frac{\xi Q}{S} + \frac{1}{2} \sigma_5^2, \\
L(-\ln Q) &= -\frac{\beta I}{Q} + \mu_1 + \xi + \alpha_3 + \frac{1}{2} \sigma_5^2, \\
LV_1 &= -\frac{c_1 A}{S} + c_1 \beta I + c_1 \mu_1 - c_1 \frac{\gamma I}{S} - c_1 \frac{\xi Q}{S} + c_1 \frac{1}{2} \sigma_5^2 \\
&\quad - c_2 \beta S + c_2 (\mu_1 + \gamma + \delta + \alpha_2) + c_2 \frac{1}{2} \sigma_5^2 \\
&\leq -2[(Ac_1 \beta I)^2 - A] + c_1 \beta I - c_1 \frac{\gamma I}{S} - \frac{\xi Q}{S} \\
&= -2[A(\tilde{R}_0)^2 - 1] + c_1 \beta I - c_1 \frac{\gamma I}{S} - \frac{\xi Q}{S}.
\end{align*}
\]
Therefore, the differential operator $L$ acting on the $V$ yields

$$LV \leq -2AM\eta + c_1 M \beta l - c_1 M \frac{\epsilon Q}{S} - c_1 M \frac{\epsilon Q}{S} - \mu^*(S^{\theta+1} + I^{\theta+1} + Q^{\theta+1}) + C - A \frac{\gamma l}{S} + \beta l + \mu_1 - \gamma l \frac{I}{S} - \frac{\epsilon Q}{S} - \frac{\delta I}{Q} + \mu_1 + \xi + \alpha_3 + \frac{1}{2} \sigma_3^2 + \frac{1}{2} \sigma_5^2 \leq -2AM\eta + C + 2\mu_1 + \xi + \alpha_3 + \frac{1}{2} \sigma_3^2 + \frac{1}{2} \sigma_5^2 - \mu^*(S^{\theta+1} + I^{\theta+1} + Q^{\theta+1})$$

$$- A \frac{\delta I}{Q} + (c_1 M \beta + \beta) l,$$

where $\eta = \left(\overline{R}_0\right)^{\frac{1}{2}} - 1$.

Now, we construct a compact subset $D$ such that the condition (ii) in Lemma 4.1 holds. Define the bounded closed set

$$D = \{(S, I, Q) : \epsilon_1 \leq S \leq \frac{1}{\epsilon_1}, \epsilon_2 \leq I \leq \frac{1}{\epsilon_2}, \epsilon_3 \leq Q \leq \frac{1}{\epsilon_3}\},$$

where $\epsilon_i (i = 1, 2, 3)$ are small enough positive constants, which will be determined later.

For convenience, we divide $\mathbb{R}_+^3 \backslash D$ into six domains.

$$D_1 = \{(S, I, Q) \in \mathbb{R}_+^3, 0 < S < \epsilon_1\}, \ D_2 = \{(S, I, Q) \in \mathbb{R}_+^3, 0 < I < \epsilon_2, S \geq \epsilon_1\},$$

$$D_3 = \{(S, I, Q) \in \mathbb{R}_+^3, 0 < Q < \epsilon_3, S \geq \epsilon_1, I \geq \epsilon_2\}, \ D_4 = \{(S, I, Q) \in \mathbb{R}_+^3, S \geq \epsilon_1\},$$

$$D_5 = \{(S, I, Q) \in \mathbb{R}_+^3, I \geq \epsilon_2\}, \ D_6 = \{(S, I, Q) \in \mathbb{R}_+^3, Q \geq \frac{1}{\epsilon_3}\}.$$

We will prove that $LV(S, I, Q) \leq -1$ on $\mathbb{R}_+^3 \backslash D$, which is equivalent to show it on the above six domains.

Case 1. If $(S, I, Q) \in D_1$, we can obtain

$$LV \leq -A \frac{I}{S} + F_1 = -\frac{A}{\epsilon_1} + F_1,$$

where

$$F_1 = \sup_{(S, I, Q) \in \mathbb{R}_+^3} \left\{ C + 2\mu_1 + \xi + \alpha_3 + \frac{1}{2} \sigma_3^2 + \frac{1}{2} \sigma_5^2 - \frac{1}{2} \mu^*(S^{\theta+1} + I^{\theta+1} + Q^{\theta+1}) + (c_1 M \beta + \beta) l \right\}.$$

We choose a constant $\epsilon_1 > 0$ small enough such that $-\frac{A}{\epsilon_1} + F_1 \leq -1$, then it follows that

$$LV \leq -1 \quad \text{for all} \quad (S, I, Q) \in D_1. \quad (4.6)$$

Case 2. If $(S, I, Q) \in D_2$, we can obtain

$$LV \leq -2AM\eta + (c_1 M \beta + \beta) l + F_2 \leq -2AM\eta + (c_1 M \beta + \beta) \epsilon_2 + F_2,$$

where

$$F_2 = \sup_{(S, I, Q) \in \mathbb{R}_+^3} \left\{ C + 2\mu_1 + \xi + \alpha_3 + \frac{1}{2} \sigma_3^2 + \frac{1}{2} \sigma_5^2 - \mu^*(S^{\theta+1} + Q^{\theta+1}) \right\}.$$
Fig. 1. The numerical simulation of solution (S(t), I(t), Q(t)) with initial value (S(0), I(0), Q(0)) = (3, 2, 3) in Example 5.1. This shows that S(t) is permanent in the mean, I(t) and Q(t) are extinct with probability one.

Choose constants $M > 0$ large enough and $\varepsilon_2 > 0$ small enough such that

$$- 2AM\eta + (c_1 M\beta + \beta)\varepsilon_2 + F_2 \leq -1,$$

then it follows that

$$LV \leq -1 \text{ for all } (S, I, Q) \in D_2.$$  \hspace{1cm} (4.7)

Case 3. If $(S, I, Q) \in D_3$, we can obtain

$$LV \leq -\frac{\delta I}{Q} + F_1 \leq -\frac{\delta \varepsilon_2}{\varepsilon_3} + F_1.$$ 

Choose a constant $\varepsilon_3 > 0$ small enough such that $-\frac{\delta \varepsilon_2}{\varepsilon_3} + F_1 \leq -1$, then it follows that

$$LV \leq -1 \text{ for all } (S, I, Q) \in D_3.$$  \hspace{1cm} (4.8)

Case 4. If $(S, I, Q) \in D_4$, we can obtain

$$LV \leq -\frac{1}{2} \mu^*\varepsilon_1^{\phi + 1} + F_1 \leq -\frac{1}{2} \mu^*(\frac{1}{\varepsilon_1})^{\phi + 1} + F_1.$$ 

Choose a constant $\varepsilon_1 > 0$ small enough such that $-\frac{1}{2} \mu^*(\frac{1}{\varepsilon_1})^{\phi + 1} + F_1 \leq -1$, then we have

$$LV \leq -1 \text{ for all } (S, I, Q) \in D_4.$$  \hspace{1cm} (4.9)
**Fig. 2.** The numerical simulation of solution \((S(t), I(t), Q(t))\) with initial value \((S(0), I(0), Q(0)) = (3, 2, 3)\) in Example 5.2. This shows that \(S(t), I(t)\) and \(Q(t)\) are permanent in the mean.

Case 5. If \((S, I, Q) \in D_5\), we can obtain

\[
LV \leq -\frac{1}{2} \mu^* I^{\theta+1} + F_1 \leq -\frac{1}{2} \mu^* \left(\frac{1}{\varepsilon_2}\right)^{\theta+1} + F_1.
\]

Choose a constant \(\varepsilon_2 > 0\) small enough such that \(-\frac{1}{2} \mu^* \left(\frac{1}{\varepsilon_2}\right)^{\theta+1} + F_1 \leq -1\), then we have

\[
LV \leq -1 \quad \text{for all} \quad (S, I, Q) \in D_5. \tag{4.10}
\]

Case 6. If \((S, I, Q) \in D_6\), we can obtain

\[
LV \leq -\frac{1}{2} \mu^* Q^{\phi+1} + F_1 \leq -\frac{1}{2} \mu^* \left(\frac{1}{\varepsilon_3}\right)^{\phi+1} + F_1.
\]

Choose a constant \(\varepsilon_3 > 0\) small enough such that \(-\frac{1}{2} \mu^* \left(\frac{1}{\varepsilon_3}\right)^{\phi+1} + F_1 \leq -1\), then we get

\[
LV \leq -1 \quad \text{for all} \quad (S, I, Q) \in D_6. \tag{4.11}
\]

Finally, from (4.6)–(4.11) we obtain

\[
LV \leq -1 \quad \text{for all} \quad (S, I, Q) \in \mathbb{R}_+^3 \setminus D.
\]

Therefore, by Remark 4.1 the condition (ii) in Lemma 4.1 is satisfied.
Next, we show that condition (i) holds in Lemma 4.1. In fact, the diffusion matrix associated to model (1.2) is

\[ A(x) = (a_{ij})_{3 \times 3} = \begin{pmatrix} \sigma_2 I^2 & 0 & 0 \\ 0 & \sigma_3 Q^2 & 0 \\ 0 & 0 & \sigma_5 I^2 \end{pmatrix}, \]

where \( x = (S, I, Q). \) It is easily proved that by Remark 4.1 condition (i) in Lemma 4.1 hold. Thus, we finally obtain that model (1.2) has a unique stationary distribution and is ergodic. This completes the proof. \( \square \)

**Remark 4.3.** When \( \sigma_1 > 0 \) or \( \sigma_4 > 0 \) in model (1.2), then whether model (1.2) also is ergodic and has a unique stationary distribution still is an interesting open problem. However, the numerical example given in below Section 5 shows that model (1.2) when \( \sigma_1 > 0 \) or \( \sigma_4 > 0 \) may have not a stationary distribution.

### 5. Numerical examples

In this section, we further analyze the stochastic model (1.2) by means of the numerical examples.

**Example 5.1.** In model (1.2) we take the parameters \( A = 2.5, \beta = 0.08, \mu_1 = 0.1, \sigma_1 = 0.06, \sigma_2 = 0.7, \sigma_3 = 0.2, \sigma_4 = 0.6, \sigma_5 = 0.1, \gamma = 0.16, \xi = 0.1, \mu_2 = 0.2, \mu_3 = 0.2 \) and \( \delta = 0.1. \) We obtain by computing \( R_0 = 0.9783 < 1, \) \( \beta = 0.08 < \frac{A}{\mu_1} \sigma_1^2 = 0.09, \frac{\beta^2}{2 \sigma_1} - (\mu_2 + \gamma + \frac{1}{2} (\sigma_2^2 + \sigma_4^2)) = 0.0039 > 0. \) Therefore, Theorem 3.3 is not applicable. However, from the numerical simulations given in Fig. 1, we can see that the infective \( I(t) \) and isolation \( Q(t) \) in model (1.2) are extinct with probability one, and the susceptible \( S(t) \) in model (1.2) is permanent in the mean with probability one.
Example 5.2. In model (1.2), we take the parameters $A = 2$, $\beta = 0.1$, $\mu_1 = 0.1$, $\mu_2 = 0.1$, $\sigma_1 = 0.01$, $\sigma_2 = 0.12$, $\sigma_3 = 0.001$, $\sigma_4 = 0.14$, $\sigma_5 = 0.01$, $\gamma = 0.1$, $\xi = 0.05$, $\mu_3 = 0.11$, $\mu_3 = 0.22$ and $\delta = 0.11$. We obtain $R_0^S = 6.1344 > 1$. From the numerical simulations given in Fig. 2, we can see that the infective $I(t)$, isolation $Q(t)$ and susceptible $S(t)$ in model (1.2) are not only persistent in the mean with probability one, but also permanent in the mean with probability one.

Example 5.3. In model (1.2), we take the parameters $A = 10$, $\beta = 0.6$, $\mu_1 = 1$, $\xi = 0.001$, $\gamma = 0.1$, $\mu_2 = 1.26$, $\mu_3 = 1.01$, $\delta = 0.2$, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, $\sigma_3 = 0.15$, $\sigma_4 = 0.05$, $\sigma_5 = 0.12$. We obtain the threshold value $R_0^S = 3.5120 > 1$ and the endemic equilibrium of deterministic model (4.2) is $(S^*, I^*, Q^*) = (2.6, 6.4535, 1.2767)$. The conditions in Theorem 4.1 are checked as follows: $R_0 = 3.8462 > 1$, $\eta_1 = 1450 > 0$, $\eta_2 = 2850.4 > 0$, $\eta_3 = 2.2725 > 0$, and $F = 8287.2 > \min\{\eta_1S^*, \eta_2I^*, \eta_3Q^*\} = \min\{9801.8, 1187.1, 3.7039\}$. Hence, the condition (4.3) does not hold. This shows that Theorem 4.1 is not applicable. But, from the numerical simulations given in Fig. 3, we can see that the solution $(S(t), I(t), Q(t))$ of model (1.2) still has a unique stationary distribution.

Example 5.4. In model (1.2), we take the parameters $A = 0.9$, $\beta = 0.1$, $\mu_1 = 0.1$, $\mu_2 = 0.12$, $\alpha_3 = 0.11$, $\sigma_1 = 0.01$, $\sigma_2 = 0.12$, $\sigma_3 = 0.001$, $\sigma_4 = 0.14$, $\sigma_5 = 0.001$, $\sigma_6 = 0.7$, $\gamma = 0.1$, $\xi = 0.05$, and $\delta = 0.11$. We obtain $R_0^S = 2.6932 > 1$, $R_0 = 0.7736 < 1$. This shows that Theorem 4.2 is not applicable. But, from the numerical simulations given in Fig. 4, we can see that the solutions of model (1.2) $(S(t), I(t), Q(t))$ may not exist the stationary distribution.

6. Conclusion

In this paper, we have investigated the global dynamics for a stochastic SIS epidemic model with isolation of the infection. The stochastic effects are assumed as the fluctuations in the transmission coefficient, disease-related rate and the
proportional coefficient of isolated infection. The research given in this paper shows that the extinction and persistence in the mean of the model are determined by a threshold value $R_0^S$. Concretely, we have proved that if $R_0^S < 1$ then disease dies out with probability one (Theorem 3.3), if $R_0^S > 1$, then the model is stochastic persistent or permanent in the means with probability one (Theorems 3.1 and 3.2). Furthermore, we also established the sufficient conditions for the existence of a unique stationary distribution (Theorems 4.1 and 4.2) by constructing the new suitable Lyapunov function. Particularly, we also see that the researches given in this paper extend the results on the global stability of the disease-free and endemic equilibria for the corresponding deterministic model given in Lemma 2.1.

We see that, in order to deal with the isolation term for the stochastic SIS epidemic model, some novel interesting research techniques are proposed. They are presented in Lemma 2.4 and the proofs of Theorems 3.1–3.3 and 4.2. In addition, we also see that there are still many problems for the considered model. These problems have been shown in Remarks 3.1, 3.2, 3.4, 3.5 and 4.3, which are interesting and valuable to be further investigated in the future.

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