Modular Hamiltonian of a chiral fermion on the torus

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We consider a chiral fermion at non-zero temperature on a circle (i.e., on a torus in the Euclidean formalism) and compute the modular Hamiltonian corresponding to a subregion of the circle. We do this by a very simple procedure based on the method of images, which is presumably generalizable to other situations. Our result is non-local even for a single interval, and even for Neveu-Schwarz boundary conditions. To the best of our knowledge, there are no previous examples of a modular Hamiltonian with this behavior.

I. INTRODUCTION

In recent years, the study of entanglement and its measures has proven to be very useful in unveiling some of the deepest properties of Quantum Field Theory (QFT). Entanglement measures are based on the reduced density matrix, or equivalently on (minus) its logarithm, the modular Hamiltonian. Among other applications, the knowledge of modular Hamiltonians was essential for the proof of the averaged null energy condition \cite{1}, the derivation of quantum energy inequalities \cite{2,3} and the formulation of a well-defined version of the Bekenstein bound \cite{4}. Modular Hamiltonians also played a key role in applications to holography, the most notable case probably being the derivation of the linearized Einstein equations in the bulk from entanglement properties of the boundary Conformal Field Theory (CFT) \cite{5–8}.

There are only few cases where modular Hamiltonians have been computed. The result is universal and local for the vacuum of any QFT reduced to Rindler space \cite{9,10}, and from this result one can also derive CFT expressions for the vacuum reduced to a ball in the plane \cite{11}, for a thermal state reduced to an interval in the plane in 1+1 dimensions \cite{12} and for the vacuum reduced to an interval in the cylinder in 1+1 dimensions \cite{13}. In these cases the modular Hamiltonian turns out to be local, but non-local contributions are expected to appear in general. This was first shown explicitly with the calculation of the modular Hamiltonian of the vacuum state reduced to an arbitrary set of disjoint intervals in 1+1 dimensions for free chiral fermions on the plane \cite{14}, later for the cylinder \cite{15} and more recently for free chiral scalars on the plane \cite{16}. Another notable result is the modular Hamiltonian for the vacuum state of any QFT reduced to regions ending on a null plane \cite{17}.

In this paper we compute a new modular Hamiltonian, namely that corresponding to a chiral fermion on the circle at non-zero temperature (i.e., on the torus in Euclidean language). Our analysis is based on the method of images applied to the calculation of the Euclidean propagator, which enables us to map the problem to a similar problem on the plane. The method turns out to be very simple, and we expect it to have applications beyond the case of chiral fermions. Our result is non-local even for a single interval, even for Neveu-Schwarz (antiperiodic) boundary conditions.

II. MODULAR HAMILTONIAN FROM THE RESOLVENT

Consider a chiral fermion $\psi$ on a circle of length $L$. The Hamiltonian is

$$H = \pm i \int_{-L/2}^{L/2} dx \psi^\dagger \psi',$$

where the sign depends on the chirality. Suppose that the field is in a thermal state with inverse temperature $\beta$. The purpose of this paper is to compute the reduced density matrix $\rho_V$ corresponding to a subset $V$ of the circle or, equivalently, the modular Hamiltonian

$$H_V = -\log \rho_V.$$  

Since the global state is Gaussian, the reduced density matrix is also Gaussian and hence the modular Hamiltonian has the form

$$H_V = \int_V dx dy \psi^\dagger(x) K_V(x,y) \psi(y).$$  

As shown in \cite{13}, the kernel $K_V$ is related to the two-point function $G_V(x,y) = \langle \psi(x) \psi^\dagger(y) \rangle$ ($x, y \in V$) by

$$K_V = -\log \left( G^{-1}_V - 1 \right),$$  

where both $K_V$ and $G_V$ are viewed as operators acting on functions on $V$. This equation can be rewritten as

$$K_V = -\int_{1/2}^{\infty} d\xi \left[ R_V(\xi) + R_V(-\xi) \right],$$

where $R_V$ is the resolvent of $G_V$,

$$R_V(\xi) = \frac{1}{G_V + \xi - 1/2}. $$
as can be easily checked by explicitly performing the integral in (9). Thus, the problem of computing the modular Hamiltonian reduces to that of finding the resolvent of $G_V$.

### III. THE METHOD OF IMAGES

Our strategy for computing the resolvent is based on the method of images applied to the calculation of the Euclidean propagator $G$, which is defined by

$$G(x,t; y,u) = \theta(t-u)\langle e^{H(t-u)}\psi(x)e^{-H(t-u)}\rangle$$

for $t-u \in (-\beta, \beta)$ and by analytic continuation for other values of $t$ and $u$, where $\theta$ is the step function. Depending on the spin structure chosen on the circle, the Euclidean propagator can be either periodic or antiperiodic in $x$ with period $L$, and it is antiperiodic in $t$ with period $\beta$. Due to these quasiperiodicity properties, we may view $G$ as a section of a line bundle over a torus of circumferences $L$ and $\beta$. Inserting (11) in (7) one sees that the Euclidean propagator satisfies

$$(\partial_x^2 + i\partial_t)G = \delta(x-y)\delta(t-u)$$

(8)

for $(x,t)-(y,u) \in (-L, L) \times (-\beta, \beta)$. Identifying $\mathbb{R}^2$ with $\mathbb{C}$ via the map $(x,t) \mapsto x + it$, this equation says precisely that $G(z,w)$ is analytic in $z$ for $z \neq w$ and has a simple pole at $z = w$ with residue $1/(2\pi i)$. In other words,

$$G(z,w) = \frac{1}{2\pi i} \frac{1}{z-w} + F(z,w),$$

(9)

where $F$ is analytic in $z$ for $z - w \in (-L, L) \times (-\beta, \beta)$. In the language of complex variables the quasiperiodicity conditions read

$$G(z + P_1, w) = (-1)^{\nu_1}G(z, w),$$

(10)

$$G(z + P_2, w) = (-1)^{\nu_2}G(z, w),$$

and $\nu \cdot \lambda = \nu_1 \lambda_1 + \nu_2 \lambda_2$. Indeed, the function (12) clearly has the form (9), and one can easily check that it satisfies the quasiperiodicity conditions (10).

Let now $x,y \in V$. It is clear from (7) that $G_V(x,y) = G(x,0^+; y,0)$ (note that it is important to take the limit $t \to 0$ from above, because if we take it from below we pick a delta function). With our identification of $\mathbb{R}^2$ with $\mathbb{C}$, we thus have

$$G_V(x,y) = G(x \mp i\epsilon, y),$$

(14)

so, by (12),

$$G_V(x,y) = \sum_{\lambda \in \Lambda} (-1)^{\nu \cdot \lambda}G_{0V\Lambda}(x + \lambda, y),$$

(15)

where $V_\Lambda = \bigcup_{\lambda \in \Lambda}(V + \lambda)$, see Fig. 1 and

$$G_{0V\Lambda}(u,v) = G_0(u \mp i\epsilon, v)$$

(16)

for $u, v \in V_\Lambda$. The main reason why the method of images is useful for us is that Eq. (15) also holds for the powers of the operators involved,

$$(G^n_V)(x,y) = \sum_{\lambda \in \Lambda} (-1)^{\nu \cdot \lambda}(G^n_{0V\Lambda})(x + \lambda, y)$$

(17)

for any $n \in \mathbb{N}$, where $(A^n)(u,v)$ denotes the kernel of the operator $A^n$ (not to be confused with $[A(u,v)]^n$). We can see this by induction. First, the above equation is satisfied for $n = 1$ (this is Eq. (15)). And second, if it holds for some $n \in \mathbb{N}$ we have

\[ (G^n_V)(x,y) = \sum_{\lambda \in \Lambda} (-1)^{\nu \cdot \lambda}(G^{n-1}_{0V\Lambda})(x + \lambda, y) \]

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\[ (G^n_V)(x,y) = \sum_{\lambda \in \Lambda} (-1)^{\nu \cdot \lambda}(G^{n-1}_{0V\Lambda})(x + \lambda, y) \]
\[(G^0_{\lambda})(x, y) = \int dV \, G_\lambda(x, z) (G^0_{\lambda})(z, y)\]

\[= \sum_{\lambda, \mu \in \Lambda} (-1)^{\mu} \int dV \, G_{\lambda \mu}(x + \lambda, z) (G^0_{\lambda \mu})(z + \mu, y)\]

\[= \sum_{\lambda, \mu \in \Lambda} (-1)^{\mu} \int dV \, G_{\lambda \mu}(x + \lambda, z + \mu) (G^0_{\lambda \mu})(z + \mu, y)\]

\[= \sum_{\lambda', \mu \in \Lambda} (-1)^{\mu} \int dV \, G_{\lambda' \mu}(x + \lambda', z') (G^0_{\lambda' \mu})(z', y)\]

\[= \sum_{\lambda' \in \Lambda} (-1)^{\lambda'} (G^0_{\lambda'})(x + \lambda', y).\] (18)

In the third equality we have used the translational invariance of \(G_\lambda\), and in the fourth we have defined \(\lambda' = \lambda + \mu\) and \(z' = z + \mu\). Eq. (17) implies that the method of images works for any function of \(G_\lambda\) which can be expressed as a power series. In particular, it works for the resolvent,

\[R_\lambda(\xi; x, y) = \sum_{\lambda, \mu \in \Lambda} (-1)^{\mu} R_{\lambda}(\xi; x + \lambda, y).\] (19)

In the case of zero temperature, \(\beta \to \infty\), the terms with \(\lambda_2 \neq 0\) do not contribute to the sum (12), so the lattice \(\Lambda\) effectively reduces to \(\{m L, m \in \mathbb{Z}\}\) and, in consequence, the region \(V_{\lambda}\) reduces to an arrangement of segments in the real line. The resolvent \(R_{\lambda\lambda}\) is well-known in that case \cite{14}, so we can use it to obtain \(R_\lambda\) via the above equation. To the best of our knowledge, \(R_{\lambda\lambda}\) is not known for generic temperatures, where \(\Lambda\) is a collection of segments distributed all over the complex plane, but it can be easily computed as we will explain in the next section.

**IV. THE RESOLVENT FOR A GENERIC SET OF SEGMENTS IN THE PLANE**

Consider a curve \(\gamma\) in the complex plane with both endpoints at infinity, and a subset \(A \subset \gamma\), see Fig. 2. As shown in the figure, \(\gamma\) divides the plane into two regions: the one to the left of the curve (+ region) and the one to the right (− region); if \(\gamma\) is the real line these regions are the upper and lower half-planes respectively. The purpose of this section is to compute the resolvent of the operator \(G_\lambda\) with kernel

\[G_\lambda(u, v) = G_\lambda(u^\mp, v) = \pm \frac{1}{2\pi i} \frac{1}{u^\mp - v}\] (20)

for \(u, v \in A\), where \(F(u^\mp)\) denotes the limit of \(F\) as \(u\) is approached from the \(\mp\) region. This resolvent is known in the case where \(\gamma\) is the real line \cite{14}; as we will see, the computation for \(\gamma\) generic is remarkably simple.

We will first obtain an expression for the powers of \(G_\lambda\). Then we will insert that expression into the expansion of the resolvent in powers of \(G_\lambda\) and find that it is easy to perform the sum. For the square we have

\[G^2_\lambda(u, v) = \int_A \frac{dw}{u - w} G_\lambda(w, v) G_\lambda(w, v)\]

\[= \frac{1}{(2\pi i)^2} \int_A \frac{dw}{u - w} (w - v)\]

\[= \frac{1}{2\pi i} G_\lambda(u, v) [\omega_\lambda^+ (u) + \omega_\lambda^- (v)],\] (21)

where

\[\omega_\lambda^+(u) = \int_A \frac{dv}{u^+ - v}.\] (22)

Note that

\[\omega_\lambda^+ + \omega_\lambda^- = 2\pi i.\] (23)

Indeed, if \(A^\pm\) is a slight deformation of \(A\) which has the same endpoints but travels through the \(\pm\) region we have

\[\omega_\lambda^+(u) + \omega_\lambda^-(u) = \int_A \frac{dv}{u - v} - \int_A \frac{dv}{u^+ - v}\]

\[= \int_A^+ \frac{dv}{u - v} - \int_A^- \frac{dv}{u - v}\]

\[= \int \frac{dv}{u - v} = 2\pi i,\] (24)

where the contour in the last integral encircles \(A\), and hence \(u\). We can rewrite (21) as an operator equation,

\[G^2_\lambda = \frac{1}{2\pi i} (\omega_\lambda^+ G_\lambda + G_\lambda \omega_\lambda^-)\]. (25)

Now, using (23) and (25) it is a simple matter to check that the operator-valued function

\[G_\lambda(s) = (1 + s)^{-\omega_\lambda^- / (2\pi i)} G_\lambda (1 + s)^{-\omega_\lambda^- / (2\pi i)}\] (26)
satisfies $G'_{0A} = -G''_{0A}$. In turn, this implies for the $n$-th derivative $G^{(n)}_{0A} = (-1)^n n! G^{n+1}_{0A}(0)$, as can be easily shown by induction. Noting that $G_{0A}(0) = G_{0A}$, we thus obtain

$$G^{n+1}_{0A} = \frac{(-1)^n}{n!} G^{(n)}_{0A}(0).$$  

Inserting this expression into the expansion of the resolvent in powers of $G_{0A}$ (which is a geometric series) we recognize the Taylor series of $G_{0A}$,

$$R_{0A}(\xi) = \frac{1}{G_{0A} + \xi - 1/2}$$

$$= \frac{1}{\xi - 1/2} \left[ 1 - \frac{1}{\xi - 1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{\xi - 1/2} G^{n+1}_{0A} \right]$$

$$= \frac{1}{\xi - 1/2} \left[ 1 - \frac{G_{0A}(1/(\xi - 1/2))}{\xi - 1/2} \right]$$

$$= \frac{1}{\xi - 1/2} \left[ 1 - \frac{e^{-i k(\xi)\omega^+_{0A} - \xi/2}}{\xi + 1/2} \right],$$  

where in the last step we have used (23) and defined

$$k(\xi) = \frac{1}{2\pi} \log \frac{\xi - 1/2}{\xi + 1/2}.$$  

Eq. (28) gives the resolvent for a generic subset $A$ of a generic curve. Let us particularize it to the case where $A$ is a collection of horizontal segments, $A = \bigcup_{a_\alpha} \{(a_\alpha, b_\alpha) + i \eta_\alpha\}$ with $a_\alpha, b_\alpha, \eta_\alpha \in \mathbb{R}$. In this case the integral (22) is easily computed,

$$\omega^\pm_A(u) = \pm \sum_{\alpha} \log \frac{a_\alpha + i \eta_\alpha - u \pm i \epsilon}{b_\alpha + i \eta_\alpha - u \pm i \epsilon},$$  

and, using the relation $1/(x \mp i \epsilon) = 1/x \pm i \epsilon \delta(x)$ (there is a principal part implicit in the first term), the resolvent (28) takes the form

$$R_{0A}(\xi; u, v) = \frac{1}{\xi^2 - 1/4} \left\{ \xi \delta(u - v) \right.\right.$$ 

$$\left. + \frac{e^{-i k(\xi)\omega^+_{0A}(u) - \omega^+_{0A}(v)}}{2\pi i (u - v)} \right\}.$$  

This result agrees with that of [14] in the case where $A$ is contained in the real line.

V. MODULAR HAMILTONIAN ON THE TORUS

Our last step is to use the resolvent just computed to obtain the resolvent on the torus by the method of images, Eq. (19), and from it the modular Hamiltonian. For simplicity, we concentrate on the case where $V$ is a single interval, $V = (a, b)$, but the analysis that follows extends straightforwardly to the general case of multiple intervals. Setting $A = V_\Lambda$ in (30) yields

$$\omega^\pm_{V_\Lambda}(u) = \pm \sum_{\lambda \in \Lambda} \log \frac{a_u - u + \lambda \pm i \epsilon}{b_u - u + \lambda \pm i \epsilon}.$$  

This series is ambiguous because it is not absolutely convergent. However, its second derivative is unambiguous,

$$(\omega^\pm_{V_\Lambda})''(u) = \mp \sum_{\lambda \in \Lambda} \left[ \frac{1}{(a_u - u + \lambda \pm i \epsilon)^2} - \frac{1}{(b_u - u + \lambda \pm i \epsilon)^2} \right]$$

$$= \mp \left[ \varphi(a_u - u \pm i \epsilon) - \varphi(b_u - u \pm i \epsilon) \right],$$  

where $\varphi$ is the Weierstrass elliptic function (see [19] for a review). Since the latter is related to the Weierstrass sigma function

$$\sigma(z) = \prod_{\lambda \neq 0} \left( 1 + \frac{z}{\lambda} \right) e^{-\frac{z}{\lambda} + \frac{1}{2} \beta(\lambda)^2},$$  

by $\varphi = -(\log \sigma)'$, we conclude that

$$\omega^\pm_{V_\Lambda}(u) = \omega^\pm(u) + e^\pm u + d^\pm,$$  

where

$$\omega^\pm(u) = \pm \log \frac{\sigma(a_u - u \pm i \epsilon)}{\sigma(b_u - u \pm i \epsilon)}$$

and $e^\pm, d^\pm$ are undetermined constants. The Weierstrass sigma function is quasiperiodic,

$$\sigma(z + \lambda) = (-1)^{\lambda_1 + \lambda_2 + \lambda_3} e^{\lambda_1 \zeta(P/2)(z + \lambda)} \sigma(z),$$  

where $\zeta = (\log \sigma)'$ and $\lambda \cdot \zeta(P/2) = \lambda_1 \zeta(P_1/2) + \lambda_2 \zeta(P_2/2)$. Therefore, $\omega^\pm$ is also quasiperiodic,

$$\omega^\pm(u + \lambda) = \omega^\pm(u) \pm 2i \lambda \cdot \zeta(P/2),$$  

where $l = b - a$. Substituting the resolvent (31) with $A = V_\Lambda$ into (19), and using (35), (36) and (38), we obtain for the resolvent on the torus

$$R_{\Lambda}(\xi; x, y) = \frac{1}{\xi^2 - 1/4} \left\{ \xi \delta(x - y) \right.\right.$$ 

$$\left. - \frac{e^{\mp i k(\xi)\Delta \omega(x, y)}}{2\pi i} \right\} \frac{F^\pm(\xi; x, y)}{F^\pm(\xi; x, y)},$$  

where

$$\Delta \omega(x, y) = \log \frac{\sigma(a - x) \sigma(b - y)}{\sigma(b - x) \sigma(a - y)},$$  

which does not need the regulator $\epsilon$ because the argument of the logarithm is real and positive, and

$$F^\pm(\xi; z, w) = \sum_{\lambda \in \Lambda} (-1)^{\nu \lambda} e^{\mp 2i k(\xi)\lambda \cdot \zeta(P/2)}$$

$$\times \frac{e^{-i k(\xi)\zeta(z + \lambda - w)}}{z + \lambda - w}. $$  

\[41\]
Note that the constant $d^\pm$ has dropped out because $R_{0}\forall\lambda$ only involves the difference $\omega_{\pm}(u) - \omega_{\pm}(v)$, see [31]. Note also that the summand above may diverge exponentially as $\lambda$ grows; preventing this divergence fixes $e^{\pm} = \mp 2\iota(\delta_{l}/2)/\delta_{l}$. Therefore, the ambiguity in $\omega_{\pm}$ disappears from the resolvent on the torus. Now, $F^{\pm}$ has the following properties: (i) it is analytic in $z$ for $z - w \in (-L, L) \times (-\beta, \beta)$ except for a simple pole at $z = w$ with residue $\pm 1$; and (ii) it is quasiperiodic,

$$F^{\pm}(\xi; z + P_{l}, w) = (-1)^{\nu_{l}} e^{\pm 2i k(\xi) l / 2} F^{\pm}(\xi; z, w).$$

(42)

By the argument we gave in Eq. [10], there is only one function with this property. This function is

$$F^{\pm}(\xi; z, w) = \frac{1}{\sigma(z - w)} \frac{\sigma_{\nu}(z - w \pm ik(\xi) l)}{\sigma_{\nu}(\pm ik(\xi) l)} \frac{2\iota(\delta_{l})}{\sigma(\pm ik(\xi) l)},$$

(43)

where

$$\sigma_{\nu}(z) = e^{-\nu_{l} \zeta(P_{l}/2) + \zeta(P_{l}/2) z} \sigma(z + \nu_{l} P_{l}/2 + P_{l}/2)$$

(44)

(recall that $\nu_{l} = 1$). Indeed, $\sigma$ is analytic, has a simple zero at the origin with $\sigma(0) = 1$ and does not vanish anywhere else in the region $(-L, L) \times (-\beta, \beta)$. Together with quasiperiodicity, this implies that the second ratio in [43] is analytic, from which property (i) follows. On the other hand, Eq. [37] and the relation $P_{l} \zeta(P_{l}/2) - P_{l} \zeta(P_{l}) = \iota k l$ imply

$$\sigma_{\nu}(z + P_{l}) = (-1)^{\nu_{l} + 1} e^{\zeta(P_{l}/2) (2z + P_{l})} \sigma_{\nu}(z),$$

(45)

from which property (ii) follows. Thus, Eqs. (39) and [43] give the resolvent on the torus. Inserting it into [5], noting that the terms with a delta function cancel and changing the variable of integration from $\xi$ to $k(\xi)$ yields the modular Hamiltonian,

$$K(x, y) = \frac{\mp i}{\sigma(x - y)} \int_{-\infty}^{\infty} dk f(k; x, y)$$

$$f(k; x, y) = e^{-ik \Delta \omega(x, y)} \frac{\sigma_{\nu}(x - y + ikl)}{\sigma_{\nu}(ikl)}.$$
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