Representations of q-Minkowski space algebra

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To Ludwig Dmitrievich Faddeev
on his sixtieth anniversary

Abstract

The properties of the quantum Minkowski space algebra are discussed. Its irreducible representations with highest weight vectors are constructed and relations to other quantum algebras: $su_q(2)$, $q$-oscillator, $q$-sphere are pointed out.

1 Introduction

The intensive development of quantum integrable systems in two dimensions in the last fifteen years was strongly influenced by the discovery of the quantum inverse scattering method (QISM) [1]. In this method the important concepts and names were introduced which are rather well known these days: the $R$-matrix, Yang-Baxter equation (YBE), quantum determinant, different forms of Bethe Ansatz (algebraic, analytic, functional) etc. (for reviews see [2]). One of the mathematical structures extracted from the QISM was the notion of quantum group introduced in [3, 4]. Although the development of the QISM is going continuously with impressive progress in the evaluation of correlation functions we see much higher activity in the quantum group field. It is certainly related to the fact that many properties of quantum groups (quasitriangular Hopf algebras) are very similar to those of Lie groups and Lie algebras. However, it seems that another reason is the formulation of the quantum group theory in terms of the $R$-matrix formalism [5].

1 On leave of absence from St.Petersburg Branch of the Steklov Mathematical Institute of the Russian Academy of Sciences.
The quantum group theory is used to describe more elaborated symmetries of physical models. It also gives rise to many explicit examples of non-commutative geometry (see e.g. [4]). For theoretical physicists it is very interesting to use the amazing possibility of constructing a non-commutative space-time according to the well-defined group-theoretical scheme where the Minkowski space \( M \) appears as a factor space of the Poincaré group \( P \) over the Lorentz group \( L \), \( M \sim P/L \). One of the difficulties for physicists who start working in this field is that the corresponding manifolds are absent in quantum groups (QG). Instead of them one has to use non-commutative analogs of algebra of functions on group \( \mathcal{F}(G) \).

One of the definitions of quantum (or \( q \)-deformed) Lorentz group \( L_q \) and corresponding \( q \)-Minkowski space algebra \( M_q \) was given in [7]. According to this definition \( M_q \) is an associative algebra (with \(*\)-operation) generated by four elements: \( \alpha, \beta, \gamma, \delta \) which satisfy the relations \((\lambda = q - q^{-1})\)

\[
\begin{align*}
\alpha \gamma &= q^2 \gamma \alpha, \\
\alpha \beta &= q^{-2} \beta \alpha, \\
\alpha \delta &= \delta \alpha, \\
[\alpha, \beta] &= \frac{\lambda}{q-1} (\alpha \delta - \alpha^2), \\
[\gamma, \delta] &= \frac{\lambda}{q-1} (\gamma \alpha).
\end{align*}
\]

The transformation of \( M_q \) under (co)action of the \( q \)-Lorentz group \( L_q \) will be defined in the next Section as well as \( L_q \) itself.

The aim of this paper is to construct irreducible representations of \( M_q \) with a highest weight vector (vacuum) and to discuss some algebraic problems related to non-commutative differential geometry (NCDG) of \( M_q \).

2 Reflection equation definition of \( q \)-Minkowski space algebras

The starting point of the papers [4] is the relation of the Lorentz group \( L \) to \( SL(2, C) \) and the spinorial construction of the Minkowski space coordinates \( x^\mu \). It is also possible to use for the definition of a \( q \)-Minkowski space the well-known \( 2 \times 2 \) matrix relation expressing the \( 2 \rightarrow 1 \) homomorphism between \( SL(2, C) \) and the Lorentz group,

\[
x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu, \quad \sigma_\mu x^\mu \rightarrow \sigma_\mu x'^\mu = A \sigma_\mu x^\mu A^\dagger,
\]

\( A, A^\dagger \in SL(2, C) \)

in the framework of the \( R \)-matrix formalism [5] and reflection equations. The covariance properties of \( \sigma_\mu x^\mu \) expressed by (2) are translated in this way to the deformed case. In this manner, all relations which define the quantum De Rham complex of \( M_q \) (coordinates (1), \( q \)-derivatives, \( q \)-1-forms) proposed in [8] can be written in compact unified form [3].

In order to \( q \)-deform the transformation (2) it is natural to consider instead of \( \sigma_\mu x^\mu \) just a \( 2 \times 2 \) matrix \( K \), the entries of which are the generators of the \( q \)-Minkowski space algebra \( M_q \) in question. Following [7] we introduce two isomorphic but mutually
non-commuting copies of the quantum group $SL_q(2, C)$. The commutation relations among generators of these quantum groups $(a, b, c, d; \bar{a}, \bar{b}, \bar{c}, \bar{d})$ in matrix form \cite{5, 7} look like this:

\begin{align*}
R_{12}M_1M_2 &= M_2M_1R_{12}, \\
R_{12}\tilde{M}_1\tilde{M}_2 &= \tilde{M}_2\tilde{M}_1R_{12}, \\
R_{12}M_1\tilde{M}_2 &= \tilde{M}_2M_1R_{12},
\end{align*}

where

\begin{equation}
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ det_qM = ad - qbc = 1, \ q \in R
\end{equation}

and $\tilde{M}$ is used for an isomorphic copy of $SL_q(2, C)$, so $det_q\tilde{M} = \tilde{a}\tilde{d} - \tilde{q}\tilde{b}\tilde{c} = 1$. We use the standard notations for the QISM as well as for the YBE and the quantum group theory (cf. \cite{5, 6}) e.g. $M_1 = M \otimes I, M_2 = I \otimes M$ and $R_{12}$ are $4 \times 4$ matrices in $C^2 \otimes C^2$, the $R$-matrix is well-known for the $SL_q(2, C)$ \cite{5}. This set of generators $(a, b, ...)$ define the quantum Lorentz group $L_q$. The $*$-operation or reality condition for $L_q$ is $M^\dagger = \tilde{M}^{-1}$ \cite{7}.

The transformation of the generators $(\alpha, \beta, \gamma, \delta)$ of $M_q$ is written as in the classical case (2)

\begin{equation}
\phi; K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longrightarrow K' = MK\tilde{M}^{-1}
\end{equation}

where it is assumed that the entries of $K$ commute with those of $M$ and $\tilde{M}$. This map $\phi$ is a coaction of the Hopf algebra $L_q$ on the algebra $M_q$ and as such it must be a homomorphism. To see that the defining relations (1) of $M_q$ are preserved by map (7) we can write them in the form of appropriate reflection equation (RE) (cf. \cite{10, 11} and refs. therein)

\begin{equation}
R_{12}K_1R_{21}K_2 = K_2R_{12}K_1R_{21}
\end{equation}

where

\begin{equation}
R_{21} = \mathcal{P}R_{12}\mathcal{P}
\end{equation}

and $\mathcal{P}$ is the permutation operator in $C^2 \otimes C^2$. Now it is just matrix algebra exercise to check that (8) is invariant under (7) taking into account the defining relations (3)-(5) of $L_q$ (it is so for any $GL_q(n)$ or quantum group, defined by the $R$-matrix relations).

Using the reality condition $\tilde{M}^{-1} = M^\dagger$ of $L_q$, the coaction (7) may be written as (2) $K' = MK\tilde{M}^\dagger$. This means that the reality ($*$-operation) of the $q$-Minkowski space $M_q$ may be expressed as in the classical case (2) by the hermiticity of $K$ i.e. $K^\dagger = K$. 

3
This requirement is consistent with (7) and the RE (2), because the latter one goes into itself after hermitian conjugation due to the $R$-matrix properties $R_{12}^\dagger = R_{21}$.

The centrality of the following two elements of $\mathcal{M}_q$

$$c_1 \equiv q^{-1}\alpha + q\delta,$$

$$c_2 = \alpha\delta - q^2\gamma\beta,$$

easily follows from the invariance property of the $q$-trace

$$tr_q K = tr_D K = q^{-1}\alpha + q\delta, \quad D = diag(q^{-1}, q),$$

with respect to the quantum group coaction

$$tr_q K = tr_q \{MKM^{-1}\}.$$

By the way, this is true for any $R$-matrix in (3) with $D = tr_p P_{12} \left( (R_{12}^t)^{-1} \right) \left( R_{12}^t \right)^{-1}$ [5, 11].

The algebra defining by the RE (8) appeared also as a lattice current algebra and as a braid group in [12, 13]. Among other properties of $\mathcal{M}_q$ let us mention an isomorphism of $\mathcal{M}_q$ to $\mathcal{M}_{1/q}$ and a characteristic equation for $K$ [10, 14]

$$qK^2 - c_1 K + q^{-1}c_2 I = 0.$$  \hspace{1cm} (13)

It was shown in [3] that the algebra $\mathcal{M}_q$ can be extended further according to the non-commutative differential geometry interpretation by adding algebras of $q$-forms $\Lambda_q$ and $q$-derivatives $\mathcal{D}_q$. Each of them are generated by four generators as well, which may be put also in the form of $2 \times 2$ matrices: $dK$ for generators of $\Lambda_q$ and $Y$ for $\mathcal{D}_q$. The additional to (1) set of $4 \times 16$ commutation relations among all these 12 generators [8] can be written in the form of reflection equations [8]

$$R_{12} K_1 R_{21} dK_2 = dK_2 R_{12} K_1 R_{12}^{-1},$$

$$R_{12} dK_1 R_{21} dK_2 = -dK_2 R_{12} dK_1 R_{12}^{-1},$$

$$R_{12} Y_1 R_{12}^{-1} Y_2 = Y_2 R_{21}^{-1} Y_1 R_{21},$$

$$R_{12} dK_1 R_{21} Y_2 = Y_2 R_{21}^{-1} dK_1 R_{21},$$

$$R_{12} K_1 R_{12}^{-1} Y_2 = Y_2 R_{12} K_1 R_{21} - q^2 R_{12} P_{12}.$$  \hspace{1cm} (18)

The quantum Lorentz group acts on this extended algebra (the quantum De Rham complex of $\mathcal{M}_q$) and relations (1), (8), (14)-(18) are invariant with respect to the transformations

$$K \longrightarrow MK\tilde{M}^{-1}, \quad dK \longrightarrow MdK\tilde{M}^{-1}, \quad Y \longrightarrow \tilde{Y}YM^{-1}. \hspace{1cm} (19)$$
The corresponding exterior derivative operator can be expressed as the $q$-trace of matrices of the $q$-1-forms and $q$-derivatives

$$d = tr_q\{(dK)Y\}.$$  \hspace{1cm} (20)

It is not easy to comment on several natural questions: is it possible to extend the constructed algebra further on, e.g. adding $q$-Grassmannian derivatives of $\Lambda_q$ to have supersymmetric complex? Are there some other covariant extensions of $M_q$, e.g. with the exterior derivative operator (20) which is nilpotent but without the Leibniz rule as in the case of the quantum $GL_q(n)$ space? By this reason we shall discuss in the next section the irreducible representations of $M_q$.

3 Representations of $M_q$

Due to the fact that one central element (10) is linear in generators one can change the basis of generators to $\alpha, \beta, \gamma,$ and $q\tau = c_1 = q^{-1}\alpha + q\delta$. Then we will have three non-trivial commutation relations ($\hat{\lambda} = \lambda/q = (1 - q^{-2})$)

$$\alpha\gamma = q^2\gamma\alpha, \quad \alpha\beta = q^{-2}\beta\alpha,$$

$$\beta\gamma = q^2\gamma\beta + \hat{\lambda}(l - \alpha^2)$$ \hspace{1cm} (21)

and centrality of $\tau$ and the quantum Minkowski length ($l \equiv c_2$ of (11))

$$l = \alpha\tau - \alpha^2/q^2 - q^2\gamma\beta = q^2(\alpha\tau - \alpha^2 - \beta\gamma)$$ \hspace{1cm} (22)

(second equality follows from (1)).

To analyze the irreducible representations in Hilbert space with positive metric the simple consequences of (21) are useful ($[n; q] = (q^n - 1)/(q - 1)$)

$$\beta\gamma^n = (q^2\gamma)^n\beta + \hat{\lambda}[n; q^2]\gamma^{n-1}(l - q^2(n-1)\alpha^2),$$ \hspace{1cm} (23)

$$\gamma^\beta n = (q^{-2}\gamma)^n\gamma + \hat{\lambda}/q^2[n; q^{-2}]\beta^{n-1}(l - q^{-2(n-1)}\alpha^2).$$ \hspace{1cm} (24)

The mentioned isomorphism of $M_q$ and $M_{1/q}$ is obvious for (21):

$$\alpha, l \longrightarrow \alpha', l', \beta \longrightarrow \gamma'/q^2, \gamma \longrightarrow \beta'/q^2, \tau \longrightarrow \tau'/q^2, \delta \longrightarrow (\delta'/q^2 + \hat{\lambda}\alpha').$$

We now have an associative algebra with three generators (one has to remember the relation (22) of the central element $l$ to $\alpha, \beta, \gamma$ and $\tau$). Due to the above equivalence $M_q \sim M_{q^{-1}}$ we can suppose that $q > 1$. The irreducible representations are parametrized by different values of $l$ and $\tau$.\footnote{This possibility for the quantum group non-commutative geometry was analyzed by L.D. Faddeev.}
0. $\alpha = 0$, then other three generators $\beta, \gamma, \delta$ commute among themselves. Hence, $\delta$ is real and arbitrary while $\beta = \gamma$ is arbitrary complex number, $\tau = \delta, l = -q^2|\gamma|^2$. This irrep is not faithful. It gives a one-dimensional representation of the REA (1), (8).

1. $l - \alpha^2 = 0$, $\delta = \alpha$, $\beta = \gamma = 0$, $\tau = \alpha(1 + q^{-2})$. This is also a one-dimensional representation, which is not faithful and corresponds to the stationary point of the coaction (7) of the quantum 'subgroup' $SU_q(2)$ of the $q$-Lorentz group $\mathcal{L}_q$.

2. $l > \alpha_0^2 > 0$, where $\alpha_0$ is the vacuum eigenvalue of $\alpha$ and $\beta|0\rangle = 0$. Then from (23) for unnormalized eigenvectors $|n\rangle = \gamma^n|0\rangle$ of $\alpha$ one gets

$$\langle n|n\rangle = (\lambda)^n[n; q^2]\Pi^n_{k=1}(l - q^{2(k-1)}\alpha_0^2).$$

(25)

For $\alpha_0 \neq 0$ and $q > 1$ this norm will be negative if the integer $n$ is sufficiently big. Because we are looking for irreps in a Hilbert space with positive metric there must be some $N = D - 1$ such that $\|\|N + 1\|\| \sim (l - q^{2N}\alpha_0^2) = 0$, or

$$\gamma|N\rangle = \gamma|D - 1\rangle = 0,$$

(26)

where $D$ is the dimension of the irrep and

$$l = q^{2D}\alpha_0^2/q^2, \quad \tau = (q^{2D} + 1)\alpha_0/q^2.$$ 

(27)

3. $0 < l < \alpha_0^2$, hence $(l - \alpha_0^2) < 0$ and from (23) one concludes that $\beta$ can not be annihilation operator. So we have to use (24) supposing that $\gamma|0\rangle = 0$. Then for $|n\rangle = \beta^n|0\rangle$ one gets

$$\langle n|n\rangle = (\lambda/q^2)^n[n; q^{-2}]\Pi^n_{k=1}(q^{-2(k-1)}\alpha_0^2 - l)$$

(28)

where from the same conclusion on finite dimensionality of this irrep follows like in the previous case

$$l = q^{-2(D-1)}\alpha_0^2, \quad \tau = (q^{-2D} + 1)\alpha_0.$$ 

(29)

4. $l \leq 0$, $\alpha \neq 0$, hence $(l - \alpha^2) < 0$ and one has to use (24) with $\gamma$ as the annihilation operator $\gamma|0\rangle = 0$. Then one has for $|n\rangle = \beta^n|0\rangle$ just (28)

$$\langle n|n\rangle = (\lambda/q^2)^n[n; q^{-2}]\Pi^n_{k=1}(q^{-2(k-1)}\alpha_0^2 - l)$$

which is now positive for any integer $n$. This irrep is infinite dimensional.

Let us comment on obvious relations of the $q$-Minkowski algebra to the well known $q$-algebras: $sl_q(2)$, $q$-oscillator algebra $A(q)$ and $q$-sphere.

For $l > 0$, considering $\alpha$ as positive element of $\mathcal{M}_q$ and defining generators

$$X_+ = \alpha^{-1/2}\beta\lambda/(lq^2)^{1/4}, \quad X_- = \alpha^{-1/2}\lambda/(lq^2)^{1/4}, \quad q^J = (\alpha/l)^{-1/2}$$

(30)

one gets from (21) the defining relations of the quantum algebra $sl_q(2)$

$$q^J X_\pm = q^\pm X_\pm q^J, \quad [X_+, X_-] = [2J]_q.$$ 

(31)
This correspondence easily follows from results of [3, 12] related \( L^+, L^- \) the 2 \( \times \) 2 matrices of quantum algebra \( sl_q(2) \) generators with a solution to the reflection equation

\[
K = L^-(L^+)^{-1}.
\]

For \( l = 0 \), one gets the \( q \)-oscillator algebra \( \mathcal{A}(q_0) \)

\[
[A, A^\dagger] = q_0^{-2N}, [N, A] = -A, [N, A^\dagger] = A^\dagger,
\]

(32)

where \( q_0 = q, A^\dagger \sim \alpha^{-1/2} \beta, A \sim \gamma \alpha^{-1/2}, \alpha = q^{-2N} \) for \( q > 1 \) and \( q_0 = 1/q, A \sim \alpha^{-1/2} \beta, A^\dagger \sim \gamma \alpha^{-1/2}, \alpha = q^{2N} = (1/q)^{-2N} \) for \( q < 1 \).

Hence one has always in (32) \( q_0 > 1 \) and among different inequivalent irreducible representations of \( \mathcal{A}(q_0) \) [14, 15] in this case only the standard oscillator one survives. This is also consistent with the fact that the central element

\[
z = A^\dagger A - [N; q_0^{-2}]
\]

(33)

of the \( q \)-oscillator algebra is zero for the given realization due to \( l = 0 \).

There is a map of \( L_q \) on the quantum group \( SU_q(2) \). If we denote by \( U \) the generator matrix of \( SU_q(2) \), which satisfies (3) and 'unitarity' \( U^\dagger = U^{-1} \), then the mentioned map is: \( M \rightarrow U, \bar{M} \rightarrow U \). Now both central elements of \( \mathcal{M}_q \) are invariant under the reduced coaction \( K \rightarrow UKU^\dagger \). Once \( \tau \) and \( l \) are fixed the relations (21), (22) coincide with defining relations of the quantum sphere algebra [16, 17].

The algebra \( \mathcal{M}_q \) is isomorphic to the \( q \)-derivative or \( q \)-momentum algebra \( \mathcal{D}_q \), hence some of the representations coincide with those found in [18] for the \( q \)-deformed Poincare algebra, which has the algebra \( \mathcal{D}_q \) as a subalgebra.

The next step in the representation theory is related to construction of a representation in the tensor product of two irreducible representations. It depends on existence of a bialgebra ( or a Hopf algebra ) structure for the algebra \( \mathcal{M}_q \) or a homomorphism from \( \mathcal{M}_q \) to \( \mathcal{M}_q \times \mathcal{M}_q \). Existence of such a map could be interpreted physically as the \( q \)-Lorentz group covariance for two- (or multi-) particle system. There are few propositions for a possible "coproduct" \( \Delta : \mathcal{M}_q \rightarrow \mathcal{M}_q \times \mathcal{M}_q \). These propositions use:

1. the relation of \( \mathcal{M}_q \) to the quantum algebra \( sl_q(2) \), extending it to isomorphism (modulo some additional requirements ) and introducing the bialgebra structure through the factorization [12] ( here and below the indices (1), (2) refer to the factors )

\[
K = L^+(L^-)^{-1} = L^+(1)K(2)(L^-(1))^{-1}.
\]

2. appropriate non-commutativity of the factors in the "tensor product" \( \mathcal{M}_q \times \mathcal{M}_q \) ("braiding" [13] ), then the matrix product of two matrices

\[
K = K(2)K(1)
\]

will satisfy RE (8) and its entries will generate the algebra isomorphic to \( \mathcal{M}_q \) [1, 3];
3. non-commutativity between generators of two factors in such a way, that the sum of two matrices satisfies (8) $K = K_{(1)} + K_{(2)}$.

4. additional matrix $O$ constructed from the $q$-Lorentz algebra generators acting on the first factor such that the matrix

$$K = K_{(1)} + O_{(1)} \times K_{(2)}$$

will satisfy the RE (8) [8], while the entries of $K_{(1)}$ and $K_{(2)}$ commute (a kind of "undressing" of the preceding case).

Though last two cases look physically reasonable, they together with 2. are not symmetric with respect to the permutation of factors and not all irreps of $K_{(1)}$ and $K_{(2)}$ are compatible.

It would be interesting to study which of the above constructed representations may be extended to representations of a bigger algebra including $D_q$ as well as $M_q$. This algebra is defined by 8 generators (entries of $K, Y$) and relations (1) or (8), (16), (18). Introducing explicitly the matrix elements of $Y$ (we use the notations of [8])

$$Y = \begin{pmatrix} \partial_D & \partial_A/q \\ q\partial_B & \partial_C \end{pmatrix}$$

one could find, that $\partial_B$ and $\partial_C$ together with $M_q$ generate a closed subalgebra. Most of the constructed irreps can be easily extended to this subalgebra. However, these extensions usually have singular $q$ dependence for $q \to 1$, e.g. for one-dimensional representation $\alpha = 0$ one has $\partial_C = 0$

$$\partial_A = q^4/\gamma(q^2 - 1), \; \partial_B = q^2/\beta(q^2 - 1), \; \partial_D = -q\delta/(q^2 - 1).$$

In the next Section a central element of this algebra ($M_q, D_q$) will be found.

4 Differential calculus and the $R$-matrix formalism

The importance of covariant and contravariant vectors (tensors), their contractions, invariant differential operators etc. is well-known in tensor calculus and, in particular, in the special relativity theory. Examples of such elements were given in Sec. 2: if $dK$ is covariant vector, then $Y$ is a contravariant one, while their contraction, given by the $q$-trace, results in the invariant operator i.e. the exterior derivative (20). In this Section other invariant operators and relations among them and the generators of $M_q, D_q$ and $\Lambda_q$ will be defined in the frame of the $R$-matrix formalism. Most of the final formulas in terms of components (generators) can be found in [8]. However, the $R$-matrix approach demonstrates where one can use a general $R$-matrix or where such properties as the two eigenvalue characteristic equation

$$(\hat{R} - q)(\hat{R} + 1/q) = 0, \; \hat{R} = qP_+ - 1/qP_-,$$
or $n = 2$, rank$P_\perp = 1$ are essential.

First of all let us deduce commutation relations of the linear central elements of $\mathcal{M}_q$ and $\mathcal{D}_q$ (time variable and corresponding derivative $\partial_0$) with the generators of another algebra. This is easy, for it is enough to take the $q$-trace of (18) with respect to the first space for $x_0 \sim c_1 = tr_q K$ and the $q$-trace of slightly transformed (18) w.r.t. the second space for $\partial_0 = tr_q Y$. One gets

$$Y c_1 = c_1 Y + q^4 I - q^2 \lambda Y K, \quad (34)$$

$$\partial_0 K = K \partial_0 + I - q^{-2} \lambda KY, \quad (35)$$

where the invariance of the $q$-trace and the relations

$$R_{21}^{-1} = R_{12} - \lambda P_{12} = P_{12}(q^{-1} I - [2]_q P_{-(12)}),$$

$$tr_{q(1)}(R_{12} P_{12})^{\pm 1} = q^{\pm 2} I_2 \quad (36)$$

were used. To get (35) we multiply (18) by $R_{12}^{-1}$ from the left and by $R_{21}^{-1}$ from the right. The formulas (34), (35) demonstrate that there is no "naive" reduction $c_1 = 0$, $\partial_0 = 0$ to a three dimensional space algebra.

The algebras $\mathcal{M}_q$ and $\mathcal{D}_q$ are graded ones. It is natural to introduce a grading operator $N$ with relations

$$[N, K] = K, \quad [N, Y] = -Y. \quad (37)$$

On the basis of our experience with the $q$-oscillator algebra $\mathcal{A}(q)$ (see, e.g. [14]) we know that once the grading operator $N$ is independent of the generators $A, A^\dagger$ it is possible to find a non-trivial central element (33). The latter one is useful, in particular, for the irreducible representation description. The corresponding central element for $\mathcal{M}_q$ and $\mathcal{D}_q$ has to be related to the invariant operator (scaling or dilatation)

$$s = tr_q KY. \quad (38)$$

Its commutativity with $q$-1-forms: $sdK = dKs$ follows easily from the defining relations (17) and (14) multiplying the first one by $R_{21} K_2$ from the left, by $R_{21}^{-1}$ from the right, using (14) and taking $tr_{q(2)}$. The relation with coordinates $K$ is more complicated.

Multiplying (18) by $R_{21} K_2$ from the left by $R_{21}^{-1}$ from the right and taking $tr_{q(2)}$, where the index in the brackets like in (36) refers to the number of space, one gets

$$sK = Ks - q^2 \lambda KY K + q^4 K. \quad (39)$$

For the covariant combination $KYK$ using as an intermediate step

$$Y_1 K_1 = q^{-2} tr_{q(2)}(R_{(21)} K_2 R_{(21)}^{-1} Y_2 R_{(21)}^{-1}) + [2]_q I,$$

we get
\[ KYK = q^{-3}Ks + [2]_q K + q^{-3}lY^\varepsilon, \]  
(40)

where the notation for the covariant vector \( Y^\varepsilon \) is introduced \((\phi : Y^\varepsilon \rightarrow MY^\varepsilon \tilde{M}^{-1})\)

\[ Y^\varepsilon_1 = [2]_q tr_q(2)P_{12}P_{-21}Y_2R_{-21}^{-1} \]  
(41)

The resulting relation is

\[ sK = q^{-2}Ks + K - (1 - q^{-2})lY^\varepsilon, \]  
(42)

where \( l = c_2 \) is the \( q \)-Minkowski length \((11), (22)\).

To cancel 'unwanted' term with \( lY^\varepsilon \) in (42) let us calculate the commutation relation of the \( q \)-D'Alembertian operator \( \Box_q \), which is a central element for \( D_q \),

\[ \Box_q P_{-(12)} = -q^{-1}P_{-(12)}Y_1\tilde{R}_{12}^{-1}Y_1 = -q^{-1}(\partial_D\partial_C - q^{-2}\partial_A\partial_B) \]  
(43)

with coordinates. This combination is easily going through \( K \) once we construct such a product of matrices

\[ Y_3R_{32}^{-1}Y_2R_{13}R_{12}K_1R_{21} \]

and apply two times the relation (18) as well as the YBE itself for different reordering of the \( R \)-matrices, e.g. \( R_{32}^{-1}R_{13}R_{12} = R_{12}R_{13}R_{32}^{-1} \). Multiplying the final equality by \( P_{-(32)} \) from the left and by \( R_{21}^{-1}P_{23} \) from the right one gets

\[ \Box_q K = q^{-2}K\Box_q - Y^\varepsilon. \]  
(44)

Hence, the relation of the invariant product \( l\Box_q \) with \( K \) will have the same 'unwanted' term as in (42). The coefficients \( x \) and \( y \) in linear combination \( x + yl\Box_q \) are defined from the requirement that it commutes with \( K \) (and \( Y \) as the grading operator \( q^{-2}N \) (37): \( x = 1/(q^{-2} - 1) \), \( y = (q^{-2} - 1) \). As in the \( q \)-oscillator case the element

\[ z = q^{2N}([N; q^{-2}] - s - (q^{-2} - 1)l\Box_q), \]  
(45)

\[ [n; q] = (q^n - 1)/(q - 1), \]

is central in the algebras \( M_q, D_q \). It is central also in the algebra \( A_q \), due to the relations

\[ \Box_q(dK) = q^{2}(dK)\Box_q, \quad l(dK) = q^{-2}(dK)l. \]  
(46)

The latter one, for example, follows from (14) multiplying it by \( R_{32}K_3R_{23}R_{13} \) from the left, using the YBE, the relation (14) for \( dK_2, K_3 \) and finally multiplying by the \( q \)-antisymmetriser \( P_{-(31)} \) from the left.

Multiplying (18) by \( R_{(31)}R_{(32)}K_3R_{(21)}^{-1}P_{(13)} \) from the right and by \( P_{-(13)} \) from the left one gets a "dual" analog of (44)

\[ Yl = q^{-2}lY - q^{2}K^\varepsilon \]  
(47)
where $K_\varepsilon = [2]_q tr_q(\tilde{R}_{(12)} K_1 P_{-(12)})$ is the contravariant vector (the inverse transformation to (41) and proportional to the inverse of $K : K_\varepsilon = l K^{-1}$). This results to reduction of the $q$-derivatives on the functions $f(l)$ of the invariant length $l$ to a $q$-difference operator $D_q : f(l) \rightarrow ((1 - q^{-2})l)^{-1}(f(l) - f(q^{-2}l))$. In this manner one could analyze the kernel of $\Box_q$ in the algebra $\mathcal{M}_q$. In particular,

$$trCP_{(12)} K_\varepsilon(2) R_{(12)} K_1 \in Ker \Box_q, \quad \text{if} \quad trC = 0.$$

## 5 Concluding remarks

The above results demonstrate the rich structure of the $q$-deformed Minkowski space algebras and usefulness of the $R$-matrix formalism. However, although the $q$-deformed relativistic one-particle states as unitary irreducible representations of the quantum Poincaré algebra were defined, the physical features of the future complete theory were not discussed thoroughly. It is so happened that a $q$-deformed space-time have been formulated first in frame of the quantum theory without a classical counterpart. At the same time the exact quantum relations are often useful for some constructions in the classical theory, e.g. the quasiclassical limit of the main ingredients of the QISM gave rise to the classical $r$-matrix and the classical YBE. If we would introduce Planck’s constant just by multiplying the defining relations of the $q$-algebras by $\hbar$ and then take independent limits $q \to 1, \hbar \to 0$ the resulting relations would be nothing but the standard Poisson brackets $\{x_\mu, p_\nu\} = g_{\mu\nu}$ for commuting coordinates and momenta of the scalar relativistic particle. If on the other hand the Planck’s constant and the deformation parameter are directly related, e.g. $q = \exp(\gamma \hbar)$ then one has an additional dimensional parameter in the theory and the Poisson brackets in the quasiclassical limit are highly nontrivial, e.g. for coordinates

\[
\{K_1, K_2\} = \gamma([r_{12}, K_1 K_2 P_{12}] + [K_1 \tilde{r}_{12} K_1, P_{12}]).
\]

In this case even the Poincaré group would be dynamical because its parameters would have also nontrivial Poisson brackets (a Lie-Poisson group). The straightforward application of the Dirac theory of the constrained systems results in non-autonomous equations, though with conserved momentum. This gives rise to additional questions of interpretation if one would like to preserve the usual mathematical structure of a physical theory.

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