A GENERALIZATION OF THE DENSITY ZERO IDEAL

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Abstract. Let $\mathcal{F} = (F_n)$ be a sequence of nonempty finite subsets of $\omega$ such that $\lim_n |F_n| = \infty$ and define the ideal

$$I(\mathcal{F}) := \{ A \subseteq \omega : |A \cap F_n|/|F_n| \to 0 \text{ as } n \to \infty \}.$$ 

The case $F_n = \{1, \ldots, n\}$ corresponds to the classical case of density zero ideal. We show that $I(\mathcal{F})$ is an analytic P-ideal. As a consequence, we show that the set of real bounded sequences which are $I(\mathcal{F})$-convergent to 0 is not complemented in $\ell_\infty$. We also present a second proof that $I(\mathcal{F})$ is an analytic P-ideal by using a classical result of Solecky.

1. Introduction

Let $\mathcal{I}$ be an ideal on the nonnegative integers $\omega$, that is, a collection of subsets of $\omega$ closed under subsets and finite unions. It is also assume, unless otherwise stated, that $\mathcal{I}$ is proper (i.e., $\omega \notin \mathcal{I}$) and admissible (i.e., $\mathcal{I}$ contains that ideal Fin of finite sets). $\mathcal{I}$ is said to be a P-ideal if it is $\sigma$-directed modulo finite sets. Moreover, $\mathcal{I}$ is said to be a density ideal if there exists a sequence $(\mu_n)$ of finitely additive measures $\mathcal{P}(\omega) \to \mathbb{R}$ supported on disjoint finite sets such that $\mathcal{I} = \{ A \subseteq \omega : \lim_n \mu_n(A) = 0 \}$, cf. [2]. Lastly, we endow $\mathcal{P}(\omega)$ with the Cantor-space-topology, hence we may speak about analytic ideals, $F_\sigma$-ideals, etc.

At this point, let $\mathcal{F} = (F_n)$ be a sequence of nonempty finite subsets of $\omega$ such that $\lim_n |F_n| = \infty$ and define the ideal

$$I(\mathcal{F}) := \{ A \subseteq \omega : |A \cap F_n|/|F_n| \to 0 \text{ as } n \to \infty \}.$$ 

(1.1)

This extends the classical density zero ideal $\mathcal{Z}$, which corresponds to the sequence $(F_n)$ defined by $F_n = \{1, \ldots, n\}$ for all $n \in \omega$. Similar ideals were considered in the literature, see e.g. [3, 4].

It is easy to see that the function

$$d^*_\mathcal{P} : \mathcal{P}(\omega) \to \mathbb{R} : A \mapsto \limsup_{n \to \infty} |A \cap F_n|/|F_n|.$$ 

is a monotone subadditive function, cf. also [6, Example 4] and the notion of abstract upper density given in [1]. It is not difficult to show that there exists a sequence $\mathcal{F}$ such that $I(\mathcal{F}) \neq \mathcal{Z}$: let $F_n := [n!, n! + n] \cap \omega$ for all $n$ and $A := \bigcup_n F_n$. Then $A \in \mathcal{Z} \setminus I(\mathcal{F})$. Our main result follows.

2. Main Results

Theorem 2.1. $I(\mathcal{F})$ is a density ideal.

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Proof. It follows by (1.1) that the ideal $\mathcal{I}(\mathcal{F})$ corresponds to
\[
\{ A \subseteq \omega : \lim_{n \to \infty} \mu_n(A) = 0 \},
\]
where, for each $n \in \omega$, $\mu_n : \mathcal{P}(\omega) \to \mathbb{R}$ is the finitely additive probability measure defined by
\[
\forall A \subseteq \omega, \quad \mu_n(A) = \frac{|A \cap F_n|}{|F_n|}.
\]
This concludes the proof. \qed

It is worth noticing that every density ideal is an analytic P-ideal, cf. [2]. It is known that every density ideal is also meager. Hence Theorem 2.1 implies, thanks to [5, Corollary 1.3], the following consequence:

**Corollary 2.2.** The set of bounded real sequences which are $\mathcal{I}(\mathcal{F})$-convergent to 0 is not complemented in $\ell_\infty$.

By a classical result of Solecky, an (not necessarily proper or admissible) ideal $\mathcal{I}$ is an analytic P-ideal if and only if
\[
\mathcal{I} = \{ A \subseteq \omega : \lim_{n \to \infty} \varphi(A \setminus [0, n]) = 0 \} = \text{Exh}(\varphi)
\]
for some lower semicontinuous submeasure $\varphi : \mathcal{P}(\omega) \to [0, \infty]$, cf. [2]. In our case, the associated lower semicontinuous submeasure associated with $\mathcal{I}(\mathcal{F})$ is
\[
\forall A \subseteq \omega, \quad \varphi(A) := \sup_{n \in \omega} |A \cap F_n|/|F_n|
\]
which we will prove in our next theorem. Next, by using this theorem we will give a second proof that $\mathcal{I}(\mathcal{F})$ is an analytic P-ideal.

**Theorem 2.3.** Let $\mathcal{F} = (F_n)$ be a sequence of nonempty finite subsets of $\omega$ such that $\lim_n |F_n| = \infty$. Then $\varphi : \mathcal{P}(\omega) \to [0, \infty]$, defined by
\[
\forall A \subseteq \omega, \quad \varphi(A) := \sup_{n \in \omega} |A \cap F_n|/|F_n|
\]
is a lower semicontinuous submeasure.

**Proof.** \begin{align*}
\varphi(A) &:= \sup_{n \in \omega} |A \cap F_n|/|F_n|, \quad A \subset \omega \\
\text{It is easy to check that } \varphi \text{ is a submeasure on } \omega. \text{ We will show that } \varphi \text{ is a lower semicontinuous submeasure on } \omega. \text{ Let } A \subset \omega. \text{ We will show that}\\n\varphi(A) &= \lim_{n \to \infty} \varphi(A \cap [0, n]) \\
\Rightarrow A \cap [0, n] &\subset A \\
\Rightarrow \varphi(A \cap [0, n]) &\leq \varphi(A \cap [0, n + 1]) \leq \varphi(A) \\
\Rightarrow \lim_{n \to \infty} \varphi(A \cap [0, n]) &= \sup_{n \in \omega} \varphi(A \cap [0, n]) \leq \varphi(A). \\
\text{Now we will show that } \varphi(A) &\leq \lim_{n \to \infty} \varphi(A \cap [0, n]). \\
\text{Let us choose } n_0 &\in \omega \text{ such that } F_n \subset [0, n_0]. \text{ Then we have} \\
A \cap F_n &\subset A \cap F_n \cap [0, n_0] \\
\Rightarrow \frac{|A \cap F_n|}{|F_n|} &\leq \frac{|A \cap F_n \cap [0, n_0]|}{|F_n|} \leq \sup_{n \in \omega} \frac{|A \cap [0, n_0] \cap F_n|}{|F_n|} \\
\end{align*}
⇒ \left| \frac{A \cap F_n}{|F_n|} \right| \leq \varphi(A \cap [0, n]) \leq \lim_{n \to \infty} \varphi(A \cap [0, n])

⇒ \varphi(A) \leq \lim_{n \to \infty} \varphi(A \cap [0, n])

So we have \( \varphi(A) = \lim_{n \to \infty} \varphi(A \cap [0, n]) \). This shows that \( \varphi \) is a lower semicontinuous submeasure on \( \omega \).

\[ \square \]

In our next theorem we show that \( I(\mathcal{F}) \), actually equal to the Exhaustive ideal generated by the lower semicontinuous submeasure \( \varphi \) defined in Theorem 2.3.

**Theorem 2.4.** \( I(\mathcal{F}) = \text{Exh}(\varphi) \) where \( \varphi : \mathcal{P}(\omega) \to [0, \infty] \) is defined in Theorem 2.3. Hence \( I(\mathcal{F}) \) is an analytic \( \mathcal{P} \)-ideal.

**Proof.** First of all suppose that \( A \subset \omega \) and \( A \in \text{Exh}(\varphi) \). So \( \lim_{n \to \infty} \varphi(A \setminus [0, n]) = 0 \). Let \( \varepsilon > 0 \). Then there exists \( m_0 \in \omega \) such that \( \varphi(A \setminus [0, m]) < \frac{\varepsilon}{2} \forall n \geq m_0 \).

⇒ \( \varphi(A \setminus [0, m_0]) < \frac{\varepsilon}{2} \)

⇒ \( \sup_{m \in \omega} \frac{|A \cap F_n \setminus [0, m_0]|}{|F_n|} < \frac{\varepsilon}{2} \).

On the other hand since \( |F_n| \to \infty \) as \( n \to \infty \), so choose \( n_0 \in \omega \) with \( n_0 \geq m_0 \) such that

\[ |F_n| > \frac{2}{\varepsilon} |A \cap [0, m_0]| \forall n \geq n_0. \]

⇒ \( \frac{|A \cap [0, m_0]|}{|F_n|} < \frac{\varepsilon}{2} \forall n \geq n_0 \).

Here we take \( m_0 \in \omega \) sufficiently large such that \( A \cap [0, m_0] \neq \phi \). Now

\[ \frac{|A \cap F_n|}{|F_n|} \leq \frac{|A \cap [0, m_0]|}{|F_n|} + \frac{|(A \cap F_n) \setminus [0, m_0]|}{|F_n|} < \varepsilon \forall n \geq n_0. \]

⇒ \( \lim_{n \to \infty} \frac{|A \cap F_n|}{|F_n|} = 0. \)

⇒ \( A \in I(\mathcal{F}) \).

Now we will show that \( I(\mathcal{F}) \subset \text{Exh}(\varphi) \). It is easy to show that the sequence \( \left\{ \varphi(A \setminus [0, m]) \right\}_{m \in \omega} \) is a decreasing sequence of positive real numbers. Now suppose that \( A \notin \text{Exh}(\varphi) \). So there exists \( \delta > 0 \) such that

\[ \varphi(A \setminus [0, m]) \geq \delta \forall m \in \omega \]

⇒ \( \sup_{m \in \omega} \frac{|A \cap F_n \setminus [0, m]|}{|F_n|} \geq \delta \forall m \in \omega \)

If \( A \in I(\mathcal{F}) \) then for \( \frac{\delta}{2} > 0 \) there exists \( n_0 \in \omega \) such that

\[ \frac{|A \cap F_n|}{|F_n|} < \frac{\delta}{2} \forall n \geq n_0. \]
Now choose \( p \in \omega \) such that
\[
\bigcup_{i=1}^{n_0-1} F_i \subset [0, p].
\]

Now
\[
\frac{|A \cap F_n \setminus [0, p]|}{|F_n|} \leq \frac{|A \cap F_n|}{|F_n|} < \frac{\delta}{2} \quad \forall \ n \geq n_0.
\]

and
\[
\frac{|A \cap F_i \setminus [0, p]|}{|F_i|} = 0 < \frac{\delta}{2} \quad \forall \ i = 1, 2, \ldots, n_0 - 1.
\]

which means
\[
\sup_{n \in \omega} \frac{|A \cap F_n \setminus [0, p]|}{|F_n|} \leq \frac{\delta}{2} < \delta
\]

which contradicts the fact that
\[
\Rightarrow \sup_{n \in \omega} \frac{|A \cap F_n \setminus [0, m]|}{|F_n|} \geq \delta \quad \forall \ m \in \omega.
\]

So our assumption is wrong. That means \( A \notin I(\mathcal{F}) \). Which proves that \( I(\mathcal{F}) = \text{Exh}(\varphi) \). \( \square \)

Now from [7], we recall
\[
\text{Fin}(\varphi) = \{ A \subset \omega : \varphi(A) < \infty \}.
\]

The lower semicontinuous submeasure \( \varphi \) is called Exhaustive if \( \text{Exh}(\varphi) = \text{Fin}(\varphi) \). So immediately we have the following theorem.

**Theorem 2.5.** The lower semicontinuous submeasure \( \varphi : \mathcal{P}(\omega) \to [0, \infty] \) defined in Theorem 2.3 is not Exhaustive.

**Proof.** It is easy to show that \( \text{Fin}(\varphi) = 2^\omega \) for lower semicontinuous submeasure \( \varphi : \mathcal{P}(\omega) \to [0, \infty] \) defined in Theorem 2.3. Also, \( \text{Exh}(\varphi) = I(\mathcal{F}) \). But we know that \( \omega \notin I(\mathcal{F}) \). This implies that \( \omega \notin \text{Exh}(\varphi) \). So \( \text{Exh}(\varphi) \neq \text{Fin}(\varphi) \). This proves that the lower semicontinuous submeasure defined in Theorem 2.3 is not Exhaustive. \( \square \)

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