Abstract. Several aspects of fusion rings and fusion rule algebras, and of their manifestations in two-dimensional (conformal) field theory, are described: diagonalization and the connection with modular invariance; the presentation in terms of quotients of polynomial rings; fusion graphs; various strategies that allow for a partial classification; and the role of the fusion rules in the conformal bootstrap programme.
1 Fusion rule algebras

In this paper I describe various aspects of fusion rules, or more precisely, of fusion rings and fusion rule algebras. Most of the contents is not entirely new, but rather a collection of known results, supplemented by examples. However, the presentation and emphasis is different from available expositions of the subject. For instance, I describe in detail the issue of diagonalisation of a fusion rule algebra, which is related to, but more fundamental than the Verlinde formula; to make this distinction explicit, I introduce the notion of a modular fusion rule algebra, which must not be confused with the issue of modular invariance in conformal field theory. I also clarify further the properties of fusion rings that are needed to represent them as local rings, which necessitates to distinguish carefully between the fusion rules as a ring over the integers and as an algebra over the complex numbers, respectively.

Fusion rule algebras are certain associative algebras over the complex numbers which arise in various areas of physics and mathematics, where they describe the possible couplings among three objects out of some given class. As examples let me mention:

- The decomposition of tensor products of finite-dimensional representations of reductive Lie algebras, of finite groups, and of associative (bi-) algebras, into irreducible representations.
- The composition of superselection sectors in the C*-algebraic approach to relativistic quantum field theory [1, 2, 3, 4].
- The multiplication of (equivalence classes of) polynomials in certain quotients of polynomial rings.
- Truncated tensor products of unitary representations of quantum groups with deformation parameter a root of unity [5, 6, 7, 8].
- The coupling of primary fields of $W$-algebras in two-dimensional conformal field theory [9, 10, 11].

(For a few further realizations see section 7 below.)

If the axioms of a fusion rule algebra are slightly relaxed, one can also describe:

- The multiplication of (classes of) polynomials in any quotient of a polynomial ring, e.g. the ring of chiral primary fields in $N = 2$ superconformal field theory [12, 13].
Operator products in topological field theory \[14, 15\].

In the present paper the main interest is in the realization of fusion rules in conformal field theory, but to motivate the concept of fusion rings and fusion rule algebras it seems to me most convenient to start with the first example in the above list, i.e. with the decomposition of tensor products of finite-dimensional representations, respectively modules, of a reductive Lie algebra. Thus let \( g \) denote a simple Lie algebra (the generalization to arbitrary reductive Lie algebras will be immediate). Any finite-dimensional module of \( g \) and any tensor product of such modules is fully reducible, i.e. the direct sum of irreducible modules. The finite-dimensional irreducible modules are highest weight modules labelled by a dominant integral highest weight \( \Lambda \) of \( g \); I denote them by \( L^\Lambda \), and their Kronecker tensor product and its decomposition into irreducible modules by

\[
L^\Lambda \times L^{\Lambda'} = \bigoplus_{\Lambda''} N_{\Lambda \Lambda'}^{\Lambda''} L^{\Lambda''}.
\]

(1.1)

(Here and below I use, for \( V \) a vector space and \( n \in \mathbb{Z}_{\geq 0} \), the short-hand notation \( nV \) in place of \( \bigoplus_{a=1}^{n} V^{(a)} \) with \( V^{(a)} \cong V \).)

Let me recall a few well-known properties of such tensor products:

(a) The addition \( \oplus \) and product \( \times \) are commutative, associative, and distributive.

(b) By definition, the numbers \( N_{\Lambda \Lambda'}^{\Lambda''} \) are non-negative integers.

(c) The number of highest weight modules \( L^\Lambda \) is infinite; but for fixed \( \Lambda \) and \( \Lambda' \), \( N_{\Lambda \Lambda'}^{\Lambda''} \) is nonzero only for a finite number of highest weights \( \Lambda'' \).

(d) The highest weight module with highest weight \( \Lambda = 0 \) (the trivial one-dimensional module) acts as the identity, i.e. \( L^\Lambda \times L_0 = L^\Lambda \), or in other words

\[
N_{\Lambda 0}^{\Lambda'} = \delta_{\Lambda,\Lambda'},
\]

(1.2)

for any dominant integral highest weight \( \Lambda \).

(e) To any module \( L^\Lambda \) there exists a unique conjugate module \( L^{\Lambda^+} \) which is again a finite-dimensional highest weight module (namely \( L^{\Lambda^+} = L^{\Lambda^+} \), with \( \Lambda^+ \) being minus the lowest weight of \( L^\Lambda \)), such that \( (L^{\Lambda^+})^+ = L_\Lambda \), that \( L_0 \) appears in \( L^\Lambda \times L^{\Lambda^+} \) precisely once, i.e.

\[
N_{\Lambda \Lambda^+}^0 = \delta_{\Lambda',\Lambda^+},
\]

(1.3)

and that the tensor product of conjugate modules is conjugate to the tensor product of the modules themselves, in the sense that

\[
N_{\Lambda^+ \Lambda'^+}^{\Lambda''+} = N_{\Lambda \Lambda'}^{\Lambda''}.
\]

(1.4)
The trivial module \( L_0 \) is self-conjugate.

It is sometimes convenient to describe the collection of all \( L_\Lambda \) of \( \mathfrak{g} \) and their direct sums from a category theoretic point of view. This collection constitutes a category with the objects being the finite-dimensional \( \mathfrak{g} \)-modules and the morphisms (arrows) being the intertwiners between them, i.e. the maps \( T \) that make the diagram
\[
\begin{array}{c}
L_\Lambda \\
\downarrow \pi_\Lambda(x) \\
L_\Lambda
\end{array}
\begin{array}{c}
\downarrow T \\
L_\Lambda
\end{array}
\]
\[
\begin{array}{c}
L_{\Lambda'} \\
\downarrow \pi_{\Lambda'}(x) \\
L_{\Lambda'}
\end{array}
\]
(with \( \pi_\Lambda \) the \( \mathfrak{g} \)-representation corresponding to the module \( L_\Lambda \)) commutative for all \( x \in \mathfrak{g} \). The Kronecker tensor product endows this category with a tensor product functor which satisfies obvious associativity and commutativity constraints and which allows for the notion of an identity object and dual (conjugate) objects; the commutativity isomorphisms are involutive, and hence the category has the structure of a rigid tensor (or monoidal) category.

The intertwiners between \( L_\Lambda \times L_{\Lambda'} \) and its irreducible submodules describe the Clebsch-Gordan decomposition of the product; that is, for \( v_\Lambda^\Lambda \in L_\Lambda \), etc., the tensor product states obey
\[
v_\Lambda^\Lambda \otimes v_{\Lambda'}^{\Lambda'} = \sum_{\Lambda''} C_{\Lambda\Lambda'}^{\Lambda'';a} v_{\Lambda'' + \Lambda'}^a.
\]

In this setting, the tensor product coefficients \( N_{\Lambda\Lambda'}^{\Lambda'';a} \) describe the basis independent contents of the Clebsch-Gordan decomposition, namely the dimensionality of the intertwiner spaces; thus the degeneracy index \( a \) of the Clebsch-Gordan coefficient \( C_{\Lambda\Lambda'}^{\Lambda'';a} \) runs over the values 1, 2, \ldots, \( N_{\Lambda\Lambda'}^{\Lambda''} \).

In the present context I am interested in an interpretation of the Kronecker product which puts a slightly different emphasis than the category theoretic point of view. Namely, consider the direct sum on the right hand side of (1.1) as a formal sum of the objects \( L_\Lambda \), and the tensor product as a formal product “\( \ast \)” on these objects; then the set of modules \( L_\Lambda \) spans a ring over the integers, with various additional properties corresponding to the respective properties of the tensor product decomposition. The ring structure obtained this way is an interesting object in its own right, irrespective of the particular realization in terms of (1.1). Consequently one introduces an abstract ring by formalizing (some of) these properties:
Definition: A fusion ring is a ring over the integers $\mathbb{Z}$, such that the following axioms are fulfilled:

(F1) Commutativity.

(F2) Associativity.

(F3) Positivity: Existence of a basis with non-negative structure constants.

(F4) Conjugation: Existence of an element of the basis required by (F3) such that the evaluation of the product with respect to this basis element furnishes an involutive automorphism.

As a set, such a ring $\mathcal{A}$ is isomorphic to the lattice $\mathbb{Z}^{\dim \mathcal{A}}$. To make the properties (F1) to (F4) more transparent, consider a basis $\{\phi_i \mid i \in I\}$, with $I$ some index set of cardinality $|I| := \dim \mathcal{A}$. Denote the structure constants in this basis by $N_{ij}^k$, i.e. write

$$\phi_i \ast \phi_j = \sum_{k \in I} N_{ij}^k \phi_k.$$  \hspace{1cm} (1.7)

Then commutativity means

$$(F1)_\mathcal{A} \quad N_{ij}^k = N_{ji}^k, \hspace{1cm} (1.8)$$

while associativity is expressed by

$$(F2)_\mathcal{A} \quad \sum_{k \in I} N_{ij}^k N_{kl}^m = \sum_{k \in I} N_{jl}^k N_{ik}^m. \hspace{1cm} (1.9)$$

The positivity axiom (F3) states that there exists a basis $\{\phi_i\}$ such that

$$(F3)_\mathcal{A} \quad N_{ij}^k \in \mathbb{Z}_{\geq 0}. \hspace{1cm} (1.10)$$

Finally, the axiom (F4) requires that for this choice of basis there exists an index $i_0 \in I$ such that, first, the conjugation map $\phi_i \mapsto \sum_{j \in I} C_{ij} \phi_j$ with

$$C_{jk} := N_{jk}^{i_0} \hspace{1cm} (1.11)$$

is an involution, i.e. that the matrix $C$ with entries (1.11) satisfies $C^2 = \mathbb{1}$; owing to (1.10) this implies that the map is a permutation of order two and hence can be written as

$$\phi_i \mapsto \phi_i^+ \equiv \phi_{i^+} := \sum_{j \in I} C_{ij} \phi_j \hspace{1cm} (1.12)$$
for some $i^+ \in I$, with $(i^+)^+ = i$, or in other words, that

\[(F4)^{(1)}_N \quad N_{jk}^{i_0} \equiv C_{jk} = \delta_{j,k^+}; \quad (1.13)\]

and second, (F4) requires that the conjugation map is an automorphism, i.e. satisfies $\phi_{i^+} \star \phi_{j^+} = \sum_i N_{ij}^{k^+} \phi_{k^+}$, i.e.

\[(F4)^{(2)}_N \quad N_{i^+ j^+ k^+} = N_{ij}^k. \quad (1.14)\]

The generator $\phi_{i^+}$ is called the element conjugate to $\phi_i$. The matrix $C$ is called the conjugation matrix; it can be used as a ‘metric’ which lowers or raises indices. In particular one can define structure constants with lower indices only,

\[N_{ijl} := \sum_{k \in I} N_{ij}^k C_{kl} = N_{ij}^{l+}. \quad (1.15)\]

Then $N_{ijk} \in \mathbb{Z}_{\geq 0}$, and it follows from associativity and commutativity that the $N_{ijk}$ are completely symmetric in the indices $i$, $j$, $k$; e.g. $N_{ijk} = \sum_{l \in I} N_{ik}^l N_{jl}^{i_0} = \sum_{l \in I} N_{ij}^l N_{lk}^{i_0} = N_{ijk}$. As a consequence of this total symmetry and of the automorphism property of $C$, one has

\[N_{i^0 j^+ k^+} = N_{i^0 j^+ k^0} = N_{j^+ k^0} = N_{j^+ k^0} = C_{j^+ k^0} = \delta_{j,k}. \quad (1.16)\]

This means that the generator $\phi_{i^0}$ is a (right, and analogously a left) unit, so that one writes $\phi_{i^0} = 1$; in basis independent notation, one has $1 \star \phi = \phi = \phi \star 1$ for all elements $\phi$ of the ring.

Thus $A$ is a unital ring, with the unit belonging to the preferred basis which has non-negative structure constants. Moreover, according to (1.16) one has $C_{i^0 j^+} = N_{i^0 j^+}^i = \delta_{i,j^0}$, i.e., as should be expected, the unit is self-conjugate, or in other words, the conjugation is a unital automorphism.

In the following, I will refer to any basis $\mathcal{B}_{\text{can}} \equiv \{ \phi_i \mid i \in I \}$ that is singled out by the above properties as a canonical basis of the fusion ring; in terms of the usual euclidean scalar product on $\mathbb{Z}^{|I|} \subset \mathbb{R}^{|I|}$, the elements of this basis are points of length one. Also, the symbol $\mathcal{N}_{ij}^k$ will always refer to the structure constants in a canonical basis $\mathcal{B}_{\text{can}}$; these structure constants are called fusion rule coefficients. Similarly, unless stated otherwise, the symbol $\phi_i$ will refer to an element of $\mathcal{B}_{\text{can}}$. Finally, it is conventional to use the symbol ‘0’ in place of $i_0$; thus e.g. the fact that $1$ is a unit reads in terms of the fusion rule coefficients

\[\mathcal{N}_{1 0}^j = \delta_{ij}. \quad (1.17)\]
In the context of finite-dimensional $g$-representations, it is natural to think of only integral linear combinations of the generators $\phi_i$. In contrast, in the application to conformal field theory, it often turns out to be more convenient to allow for arbitrary complex linear combinations. Correspondingly one views the fusion rules as an algebra over $\mathbb{C}$:

**Definition:** A fusion rule algebra $\mathcal{A}$ is an algebra over the complex numbers $\mathbb{C}$, such that the axioms (F1) to (F4) are fulfilled, with (F3) refined to

\[(F3)^{'}\] Existence of a basis with non-negative integral structure constants.

If the index set $I$ is finite, fusion rule algebras acquire a few particularly interesting properties. Therefore one defines:

**Definition:** A rational fusion rule algebra $\mathcal{A}$ is a fusion rule algebra for which the following axiom holds:

\[(R)\] Finite-dimensionality.

Note that the fusion rule algebras furnished by the tensor products (1.1) are not rational (by property (c), they are however quasi-rational in the sense that for fixed $i$ and $j$: $\sum_{k \in I} N_{ij}^k$ is still finite). In contrast, in the sequel the main interest will be in rational fusion rule algebras, and the property (R) will be assumed unless stated otherwise. Sometimes it is possible to construct from a non-rational fusion rule algebra an associated rational one by a certain process of ‘truncation’; examples of this phenomenon arise in the representation theory of quantum groups with deformation parameter a root of unity [5,6,7,8], and in the context of WZW theories that will be the subject of section 5. In the category theoretical setting, such a truncation implies that the commutativity isomorphisms can no longer be involutive, so that the category supplied by the fusion rules is a rigid quasitensor (or braided monoidal) category. As a consequence, generically the tensor category is not tannakian, i.e. there does not exist a compatible tensor functor from this category to the category of finite-dimensional vector spaces; this is in contrast to, for example, the non-rational fusion rule algebras that are supplied by $g$-tensor products, where such a functor is provided by the forgetful functor.

It is clear that two fusion rule algebras must be considered as equivalent if they have the same dimension and if there exists a bijection $\sigma$ of the
index sets $I^{(1)}$ and $I^{(2)}$ which is compatible with (F1) to (F4), i.e. is such that for $\phi_{\sigma(i)}^{(1)} = \phi_i^{(2)}$ one has $\phi_0^{(1)} = \phi_0^{(2)}$, and $\phi_{\sigma(i+)}^{(1)} = \phi_{(\sigma(i))}^{(1)}$, as well as $(N^{(1)})_{\sigma(i)\sigma(j)}^{(1)} = (N^{(2)})_{ij}^{(1)}$. This equivalence relation will be called isomorphism, and the map $\sigma$ a unital $^*$-isomorphism of the fusion rule algebras, or shortly a fusion rule isomorphism. Similarly, an endomorphism of a fusion rule algebra with analogous properties is called a fusion rule automorphism.

It should be pointed out that the axioms of a fusion ring (or a fusion rule algebra) do not require the canonical basis to be uniquely determined. For instance, consider the two-dimensional fusion ring with basis $\{1, \phi\}$ and products $1 \star 1 = 1$, $1 \star \phi = \phi = \phi \star 1$, and

$$\phi \star \phi = 1 + \phi.$$  \hspace{1cm} (1.18)

Clearly, this basis is canonical. But owing to

$$(1 - \phi) \star (1 - \phi) = 1 - 2 \phi + \phi \star \phi = 1 + (1 - \phi),$$  \hspace{1cm} (1.19)

the mapping $1 \mapsto 1$, $\phi \mapsto 1 - \phi$ defines a fusion rule automorphism, and hence in particular $\{1, 1 - \phi\}$ is a canonical basis as well. Still, the freedom that remains when choosing a canonical basis seems rather insignificant, so that in spite of this non-uniqueness I will stick to the qualification ‘canonical.’

To conclude this introduction into the subject, let me mention that there is a nice pictorial representation of fusion rule coefficients and their properties. Describe the fusion rule coefficients by labelled trivalent graphs with ordered oriented lines, according to

$$N_{ij}^{ik} = \begin{array}{c}
i \\
0 \\
j \\
\end{array} \begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\end{array} k.$$

Then the conjugation matrix corresponds to

$$C_{ij} = \begin{array}{c}
i \\
0 \\
j \\
\end{array} \begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\end{array} j \cong \begin{array}{c}
i \\
j \\
\end{array} \begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\end{array} j.$$

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so that the fusion rule coefficients with three lower indices are represented by graphs with three incoming lines,

\[ N_{ijk} = \begin{array}{c}
i \\ \downarrow \\
\end{array} k \begin{array}{c}
j \\ \downarrow \\
\end{array}, \quad (1.22) \]

while associativity together with commutativity result in the duality

\[ \sum_{m \in I} j \begin{array}{c}
i \\ \downarrow \\
\end{array} m \begin{array}{c}
j \ \downarrow \\
\end{array} k \begin{array}{c}
i \\ \downarrow \\
\end{array} l = \sum_{n \in I} j \begin{array}{c}
i \\ \downarrow \\
\end{array} n \begin{array}{c}
j \ \downarrow \\
\end{array} k \begin{array}{c}
i \\ \downarrow \\
\end{array} l = \sum_{p \in I} j \begin{array}{c}
i \\ \downarrow \\
\end{array} p \begin{array}{c}
j \ \downarrow \\
\end{array} k \begin{array}{c}
i \\ \downarrow \\
\end{array} l. \quad (1.23) \]

The remainder of this paper is organized as follows. In section 2 the diagonalization of the fusion rules is described, and the notion of modular fusion rule algebras is introduced. Section 3 provides a short introduction to those aspects of two-dimensional conformal field theory that are connected with the concept of fusion rules. In section 4 the relevance of the modular group to fusion rule algebras and to conformal field theory is explained. Section 5 provides a brief description of a particular class of conformal field theories, namely WZW theories, some of which are used as examples in the succeeding sections. In section 6 fusion rings are analyzed in terms of polynomial rings. In particular the possibility that the ideal that has to be divided out from a free polynomial ring in order to obtain the fusion ring derives from a potential is investigated in some detail. To provide examples for some of the concepts introduced in sections 2 to 6, various aspects of the fusion rules of the \( sl_n \) WZW theory at arbitrary level are discussed in section 7. In section 8 I explain the concept of fusion graphs and describe the graphs obtained in various special cases. Section 9 contains an overview of what is known about the classification of (modular) fusion rule algebras; in particular the list of polynomial fusion rule algebras with generator of fusion dimension \( \leq 2 \) is presented. I conclude in section 10 with further remarks on the connection between the fusion rules and the operator product algebra.
In short, there are a few sections dealing mainly with conformal field theory aspects (sections 3, 5, 10), while in the other sections fusion rules are mainly considered as abstract rings or algebras. There are of course various interrelations among the different aspects, but some sections may be read without the need to know the contents of all of the preceding sections. The logical interdependence is essentially as depicted in the following diagram:

\[ \begin{array}{c}
1 \rightarrow 2 \rightarrow 8 \rightarrow 9 \\
\downarrow \quad \uparrow \\
3 \rightarrow 4 \rightarrow 5 \\
\downarrow \downarrow \downarrow \\
10 \quad 6 \rightarrow 7 \\
\end{array} \]

2 Diagonalization

The structure constants of a commutative associative algebra play the role of the representation matrices of the algebra in its regular (or adjoint) representation. Accordingly, for a fusion rule algebra with structure constants \( N_{ij}^k \), the fusion matrices \( N_i, i \in I \), defined as the matrices with entries

\[ (N_i)^{jk} = N_{ij}^k, \quad (2.1) \]

furnish a representation

\[ \pi_{\text{reg}} : \phi_i \mapsto \pi_{\text{reg}}(\phi_i) := N_i \quad (2.2) \]

of the algebra. From the fact that the conjugation \( C \) is an automorphism, one learns that \( N_{ik} = (N_i)^t \), while associativity and commutativity of the fusion rules imply that the fusion matrices commute among another. In particular, \([N_i, (N_i)^t] = [N_i, N_{i+}] = 0\), i.e. the fusion matrices are normal.

Since a fusion rule algebra is abelian, all its irreducible representations are one-dimensional. Moreover, because of their normality the \( N_i \) are diagonalizable by a unitary matrix, in such a way that the orthogonal eigenvectors \( v_j \) and hence the diagonalizing matrix do not depend on the index \( i \) of \( N_i \).

In other words, the regular representation of the fusion rule algebra is fully reducible, and is hence the direct sum of \(|I|\) inequivalent one-dimensional representations. I will denote the diagonalizing matrix by \( V \), but (in order to allow for a convenient normalization of the eigenvectors) do not require

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that $V$ be unitary. Thus $(V^{-1} \mathcal{N}_i V)_{jk} = \nu_j^{(i)} \delta_{jk}$, and the eigenvalues $\nu_l^{(i)}$ of the fusion matrices furnish one-dimensional representations of the algebra, i.e.

$$
\sum_{k \in I} \mathcal{N}_{ij}^k \nu_l^{(k)} = \nu^{(i)}_l \nu^{(j)}_l.
$$

(2.3)

Note that, being solutions to the characteristic equation $\det(\mathcal{N}_j - \nu^{(j)}_l \mathbf{1}) = 0$, the eigenvalues $\nu^{(j)}_l$ are algebraic integers.

Comparing (2.3) with the eigenvector equation $\sum_{k \in I} (\mathcal{N}_i)_j k (v_l)_k = \nu^{(i)}_l (v_l)_j$, one learns that the eigenvalues coincide with the (appropriately normalized) eigenvectors,

$$
V_{ji} \equiv (v_i)_j = \nu^{(j)}_l.
$$

(2.4)

In other words, there exists a set of $|I|$ vectors $v_i$ such that

$$
\pi_j : \phi_i \mapsto \pi_j(\phi_i) := (v_j)_i
$$

(2.5)

for $j \in I$ are one-dimensional representations of the fusion rule algebra, satisfying

$$
\sum_{k \in I} \mathcal{N}_{ij}^k (v_l)_k = (v_l)_i (v_l)_j.
$$

(2.6)

This may also be written as

$$
\mathcal{N}_{ij}^k = \sum_{l \in I} V_{il} V_{jl} (V^{-1})_{lk}.
$$

(2.7)

A fusion rule subalgebra $\tilde{\mathcal{A}}$ of a fusion rule algebra $\mathcal{A}$ is a subalgebra $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ which is itself a fusion rule algebra. The unit $\tilde{\phi}_0$ of $\tilde{\mathcal{A}}$ coincides with the unit $\phi_0$ of $\mathcal{A}$ ($\tilde{\phi}_0$ must belong to a canonical basis of $\tilde{\mathcal{A}}$, and extending this basis to a basis of $\mathcal{A}$ shows that $\tilde{\phi}_0$ is a unit for all of $\mathcal{A}$, which is however unique). Thus for $\phi \notin \tilde{\mathcal{A}}$, also $\tilde{\phi}_0 \ast \phi \notin \tilde{\mathcal{A}}$, implying that a fusion rule algebra does not possess any non-trivial (unital) ideals. Hence fusion rule algebras are simple. In particular, in the rational case, i.e. for $|I| < \infty$, the algebra is a finite-dimensional simple associative algebra. As a consequence [16,17], the one-dimensional representations are actually exhausted by the representations (2.3); in other words, the index set $I$ not only labels the generators, but also the inequivalent irreducible representations of the algebra.

Because of the orthogonality of the eigenvectors $v_i$, the right inverse of the matrix $V$ has entries

$$
(V^{-1})_{ij} = \eta_i^2 V_{ji}^*,
$$

(2.8)
where
\[ \eta_i := (\sum_{j \in I} |V_{ji}|^2)^{-1/2}. \] (2.9)

Thus (2.7) can be rewritten as
\[ \mathcal{N}_{ij}^k = \sum_{l \in I} V_{il} V_{jl} V_{kl}^* (\sum_{m \in I} |V_{ml}|^2)^{-1/2}. \] (2.10)

Since the right inverse must coincide with the left inverse, one not only has the orthogonality relation
\[ \sum_{l \in I} V_{il} V_{il}^* = \eta_i^{-2} \delta_{ij}, \] (2.11)
but also
\[ \sum_{l \in I} \eta_l^2 V_{il} V_{jl}^* = \delta_{ij}. \] (2.12)

Since the matrix elements of a fusion matrix $\mathcal{N}_i$ are non-negative, it follows from standard results in the theory of matrices (see e.g. [18, 19, 20]) that their largest eigenvalue is a positive real number (and is non-degenerate if the matrix $\mathcal{N}_i$ is indecomposable), and that there exists a unique normalized eigenvector to this eigenvalue with only positive entries, called the Perron-Frobenius eigenvector of $\mathcal{N}_i$. I will call the corresponding eigenvalue the fusion dimension and denote it by $D_i$; in other contexts, this Perron-Frobenius eigenvalue is known as statistical dimension of a superselection sector (in algebraic field theory [4]) or of an object of a quantum category [21], as quantum dimension (in conformal field theory [10]), and in the theory of quantum groups [22]), or as the square root of the index of an inclusion of von Neumann algebras [23,24]. As noticed after (2.3), the fusion dimension is an algebraic integer. But actually not any positive algebraic integer qualifies as a fusion dimension; for instance (see e.g. [25, 26, 27]), it follows from an old result by Kronecker [28] that
\[ D_i \in \left\{ 2 \cos \left( \frac{\pi}{n} \right) \mid n \in \mathbb{Z}_{\geq 3} \right\} \cup [2, \infty). \] (2.13)

Because of $\mathcal{N}_{i^+} = (\mathcal{N}_i)^*$, the matrices $\mathcal{N}_i$ and $\mathcal{N}_{i^+}$ have complex conjugate eigenvalues,
\[ ((v_i)_i)^* = (v_{i^+})_{i^+}. \] (2.14)
In particular, $D_i = D_{i^+}$. Since $V$ diagonalizes all of the matrices $N_i$, they have a unique common Perron-Frobenius eigenvector. Also, one may order the eigenvalues such that this is the vector $v_j$ with $j = 0$, so that

$$D_i = (v_0)_i = V_{i0}.$$  

(2.15)

On the other hand, the fusion matrix $N_0$ for the identity $\phi_0$ is just the unit matrix, so that

$$V_{0i} = 1$$

(2.16)

for all $i \in I$.

The diagonalizability implies among other things that apart from the basis $B_{\text{can}}$ of fields $\phi_i$ there exists another distinguished basis, whose elements are given by

$$e_i := \eta^2 \sum_{j \in I} V^*_{ji} \phi_j.$$  

(2.17)

Because of the relation (2.8), the inverse basis transformation reads $\phi_i = \sum_{j \in I} V_{ij} e_j$. By applying first (2.17), (1.7), and the inverse basis transformation, and then (2.17), to the fusion product $e_i \ast e_j$, one finds that

$$e_i \ast e_j = \delta_{ij} e_j;$$

(2.18)

thus the elements $e_i$ are primitive idempotents (the existence of a basis of primitive idempotents is guaranteed because fusion rule algebras are simple unital associative algebras). In other words, in the basis $\{e_i\}$ the fusion rules are diagonal, i.e. the structure constants read $\tilde{N}_{ij}^{k} = \delta_{ij} \delta_{ik}$, and the components of any element of $\mathcal{A}$ in this basis are just its eigenvalues in the regular representation. Further, as follows with the help of (2.16), the $e_i$ provide a partition of the unit element,

$$\phi_0 = \sum_{j \in I} e_j.$$  

(2.19)

For the subsequent discussion it will be convenient to pass to a normalization of eigenvectors different from the one used so far. Thus consider the matrix

$$Y_{ij} := \eta D_j V_{ij},$$

(2.20)

with $\eta = Y_{00}$ some positive real number to be fixed in (2.26) below. Note that $Y_{i0} = \eta D_i$ obeys $Y_{i0} = Y_{0i} = Y^*_i$. Let me make in the sequel the non-trivial assumption that this extends to a symmetry of the whole matrix $Y$; this may be reformulated as the following axiom.
(M1) Symmetry of $Y$:

$$D_i (v_i)_j = D_j (v_j)_i.$$  \hfill (2.21)

Let me point out that without this symmetry requirement, the labelling of the eigenvectors $v_i$ and that of the matrices $N_i$ are logically completely independent. In contrast, by imposing (M1) one correlates these two types of labelling. In other words, one defines a bijection (compatible with the choice (2.13)) between the representations $\pi_i$ and the represented objects $\phi_i$. (Incidentally, this also implies a partial fixing of the freedom that remains in the choice of the canonical basis.)

(M1) is also equivalent to the requirement that the numbers $\eta_i$ defined in (2.9) must be proportional to the fusion dimensions $D_i$. Also, along with (2.14) it follows immediately that $Y_{ij}^* = Y_{i+} = Y_{i+ j}$, i.e.

$$YC = Y^* = CY.$$  \hfill (2.22)

Moreover, since $V$ is invertible and diagonalizes the fusion rules, $Y$ is invertible, too, and $\sum_{k \in I} N_{ij}^k Y_{kl} = Y_{il} Y_{jl}/Y_{0l}$, as well as

$$\sum_{j,k \in I} (Y^{-1})_{mj} N_{ij}^k Y_{kl} = (Y_{im} Y_{0m}) \delta_{ml}.$$  \hfill (2.23)

The latter result may be transformed into an expression for the fusion rule coefficients analogous to (2.7),

$$N_{ij}^k = \sum_{l \in I} Y_{il} Y_{jl} (Y^{-1})_{kl}.$$  \hfill (2.24)

The inverse of $Y$ is related to the inverse of $V$, $(Y^{-1})_{ij} = (V^{-1})_{ij}/\eta D_i$, which by (2.3) implies that it is proportional to $Y^*$. Thus without loss of generality one may impose that $Y$ be unitary, which amounts to requiring

$$\eta_j = \eta D_j$$  \hfill (2.25)

for all $j \in I$, thereby fixing in particular the constant $\eta$ as

$$Y_{00} \equiv \eta = \eta_0 = \left( \sum_{j \in I} D_j^2 \right)^{-1/2}.$$  \hfill (2.26)

Due to $Y_{i0} = Y_{00} D_i$ and $D_i \geq 1$, one thus has

$$Y_{i0} \geq Y_{00} > 0.$$  \hfill (2.27)

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Also, together with (2.22) it follows that

\[ Y^2 = C, \]  

(2.28)

and that (2.24) may be rewritten more symmetrically as a formula for the fusion rule coefficients with three lower indices,

\[ N_{ijk} = \sum_{l \in I} \frac{Y_{il} Y_{jl} Y_{kl}}{Y_{0l}}. \]  

(2.29)

Let me stress that the constraint (2.21) is very selective; for a generic fusion rule algebra, \( Y \) cannot be chosen symmetric. As a simple counterexample, consider the three-dimensional fusion rule algebra with fusion matrices

\[ N_0 = \mathbb{1}, \quad N_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \]  

(2.30)

These are diagonalized by the unitary matrix

\[ V = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & \sqrt{2} & \sqrt{3} \\ 1 & \sqrt{2} & -\sqrt{3} \\ 2 & -\sqrt{2} & 0 \end{pmatrix} \]  

(2.31)

(namely, \( V^{-1} N_1 V = \text{diag}(1, 1, -1) \), \( V^{-1} N_2 V = \text{diag}(2, -1, 0) \)), but \( V \) is not symmetric. (More general constructions of fusion rule algebras with non-symmetric unitary diagonalization matrix can be found in [29].)

Rather than delving into the general case of non-symmetric unitary diagonalization matrix, I will turn my attention to a class of fusion rule algebras that is even more restricted. Namely, let me require that not only \( Y \) is symmetric (i.e. property (M1) holds), but in addition that there exists a diagonal \( |I| \times |I| \) matrix \( T \) with the following properties:

(M2) \( T \)-matrix: \( T = \text{diag}(t_i) \) satisfies

\[ T^* = T^{-1}, \quad CT = T, \]  

(2.32)

and

\[ YTY = T^*YT^*. \]  

(2.33)
Note that the 00-component of the latter relation reads

$$\sum_{j \in I} t_j D_j^2 = (t_0^*)^2 / Y_{00}. \quad (2.34)$$

For reasons to be clarified in section 4 below, I will refer to the properties (M1) and (M2) as *modularity constraints* and to (rational) fusion rule algebras satisfying them as *modular fusion rule algebras*. For modular fusion rule algebras, one finds that $Y$ can be expressed as

$$Y_{ij} = t_0^* t_i t_j Y_{00} \sum_{k \in I} N_{ik}^j t_k^* D_k, \quad (2.35)$$

as may be checked by first applying $(2.24)$, and then twice $(2.33)$, to the right hand side. Furthermore,

$$t_j D_j = t_0^* \sum_{k \in I} t_k^* D_k Y_{jk}, \quad (2.36)$$

as can be seen by combining the right hand side with $(2.35)$, then using the representation property of the fusion dimensions, and then applying $(2.34)$. Also [30], for a modular fusion rule algebra the fusion rule eigenvalues are not just algebraic integers (subject to the constraint $(2.13)$), but they belong to the algebraic integers of a cyclotomic extension of $\mathbb{Q}$.

As it turns out, the diagonalization matrix of a modular fusion rule algebra is essentially unique. Of course, $Y$ is not at all determined uniquely by the requirement that it diagonalizes the fusion rules. But requiring the matrix to be also symmetric and unitary, and to obey $Y^2 = C$ and $Y_{00} \geq Y_{00} > 0$, determines it up to permutations in the following sense [31]: if $Y^{(1)}$ and $Y^{(2)}$ are two matrices sharing all these properties, then there exists a fusion rule automorphism $\sigma$ such that $Y^{(1)}_{ij} = Y^{(2)}_{\sigma(i),j} = Y^{(2)}_{i,\sigma(j)}$. Imposing also the property $(2.33)$, it follows [32] that this automorphism must be of order three and commute with $T$, i.e. $t_{\sigma(j)} = t_j$. Thus one may recover the matrix $Y$ from the fusion rules up to at most a minor ambiguity, and in fact an efficient algorithm (employing standard routines for the numerical diagonalization of matrices) for doing so is available [33, 32].

In particular, the eigenvectors are essentially unique up to scalar multiplication. This implies that any modular fusion rule algebra $\mathcal{A}$ contains an element $x \in \mathcal{A}$ such that all eigenvalues of $\pi_{\text{reg}}(x)$ are distinct (if this were not so, then the freedom in defining the eigenvectors corresponding to degenerate eigenvalues would be a non-trivial unitary matrix).
3 Conformal field theory

Let me now describe how fusion rule algebras emerge in conformal field theory. To start, it is appropriate to recall a few basic facts of conformal field theory, or more precisely, of the bootstrap approach to two-dimensional conformal field theory. The two main ingredients are conformal invariance and the bootstrap idea. Conformal invariance \([34, 35, 36, 37]\) of a two-dimensional field theory implies \([38, 9]\) that the collection of all (properly interpreted) ‘fields’ of the theory carries a representation of the Virasoro algebra \(V\), which is the complex Lie algebra with basis \(\{C\} \cup \{L_m \mid m \in \mathbb{Z}\}\) and Lie brackets

\[
[L_m, L_n] = (m - n) L_{m+n} + \frac{1}{12} (m^3 - m) \delta_{m+n,0} C, \quad [L_m, C] = 0. \tag{3.1}
\]

More precisely, the fields \(\varphi(z, \bar{z})\) can be organized into a direct sum of irreducible highest weight modules of \(V\) which have a common eigenvalue of the central generator \(C\). This eigenvalue \(c\) is called the conformal central charge of the theory. Fields corresponding to highest weight vectors are referred to as primary fields and generically denoted by \(\phi\), while those corresponding to non-highest weight vectors are called descendant fields. Thus primary fields obey

\[
\mathcal{U}(V_+) \phi = 0, \quad \mathcal{U}(V_0) \phi = C \phi, \tag{3.2}
\]

while descendants are of the form

\[
\varphi \in \mathcal{U}(V_-) \phi, \tag{3.3}
\]

and the complete collection of descendants is obtained from the Verma module defined by these equations by taking the unique irreducible quotient. Here \(V_\pm\) denote the Lie subalgebras spanned by the generators \(L_n\) with \(n > 0\) and \(n < 0\), respectively, and \(V_0\) the subalgebra spanned by \(L_0\) and \(C\), while \(\mathcal{U}(L)\) stands for the universal enveloping algebra of a Lie algebra \(L\).

The second ingredient, the bootstrap hypothesis, amounts to the requirement that upon forming radially ordered products the fields of the theory constitute a closed associative operator algebra \([3, 40, 41, 42]\),

\[
\varphi(z, \bar{z}) \varphi'(w, \bar{w}) = \sum_{\varphi''} C_{\varphi \varphi'} \varphi''(z, w, \bar{z}, \bar{w}) \varphi''(w, \bar{w}). \tag{3.4}
\]

To be precise, this must be valid inside all correlation functions, which are defined (via the correspondence \([4]\) between fields \(\varphi\) and state vectors in
the physical Hilbert space of the theory) as the vacuum expectation values
\[ \langle \ldots \varphi(z, \bar{z}) \varphi'(w, \bar{w}) \ldots \rangle \]
of radially ordered products of fields. The radius of convergence of the expansion (3.4) is then determined by the remaining fields that are present in the correlator.

In the above, the variables \( z \) and \( \bar{z} \) are complex coordinates for the two-dimensional space-time on which the theory is living; for many aspects of the theory, these variables need not be considered as complex conjugates of each other, but can be treated as independent. The identity primary field \( 1 \), which by definition satisfies \( 1 \varphi(z, \bar{z}) = \varphi(z, \bar{z}) \) for all fields \( \varphi \) of the theory, does not depend on these variables at all. The generating function
\[ T(z) := \sum_{m \in \mathbb{Z}} L_m z^{-m-2} \]
plays the role of the energy-momentum tensor of the theory and is a descendant of the identity primary field. If there are further fields \( W(z) \) which do not depend on the antiholomorphic coordinate \( \bar{z} \), then it is convenient (and, for some aspects, even mandatory) to consider instead of the Virasoro algebra a larger algebra \( \mathcal{W} \) containing \( \mathcal{V} \) as a Lie subalgebra. Similarly as the Virasoro generators \( L_n \) are the Laurent modes of \( T(z) \), the additional generators of \( \mathcal{W} \) are the Laurent modes of the fields \( W(z) \). Furthermore, treating the antiholomorphic coordinate \( \bar{z} \) in an analogous manner, one arrives at a direct sum \( \mathcal{W} \oplus \mathcal{W} \), with \( \mathcal{W} \) generated by the modes of purely antiholomorphic fields \( \mathcal{W}(\bar{z}) \). The algebra \( \mathcal{W} \oplus \mathcal{W} \) is called the symmetry algebra of the theory, and its summands the (holomorphic, respectively antiholomorphic) chiral algebras of the theory. If all fields which depend either only on \( z \) or only on \( \bar{z} \) are included into the symmetry algebra, then it is called maximally extended or, shortly, maximal.

The remarks made above about the role of the Virasoro algebra extend to analogous statements about the symmetry algebra. Accordingly, in the sequel I will label the primaries of a given conformal field theory by an index set \( I \), and denote by \( [\phi_i] \), \( i \in I \), the collection of fields which correspond to the irreducible highest weight module of \( \mathcal{W} \) whose highest weight is carried by the primary \( \phi_i \); I will call \( [\phi_i] \) the \( \mathcal{W} \)-family of \( \phi_i \). The algebra \( \mathcal{W} \) is endowed with a \( \mathbb{Z} \)-gradation supplied by the mode numbers \( n \) of the non-central generators \( W_n \), and \( L_0 \) acts as a derivation,

\[ [L_0, W_n] = -n W_n. \tag{3.5} \]

The requirement of conformal, respectively \( \mathcal{W} \)-, symmetry imposes severe constraints on operator products. In particular, operator products of
primaries can be written as
\[
\phi_i(z, \bar{z}) \phi_j(w, \bar{w}) = \sum_{k \in I} C_{ij}^k (z-w)^{-\Delta_i - \Delta_j + \Delta_k} (\bar{z}-\bar{w})^{-\bar{\Delta}_i - \bar{\Delta}_j + \bar{\Delta}_k} [\phi_k(w, \bar{w}) + \ldots].
\]
(3.6)

Here $\Delta_i, \bar{\Delta}_i$ are the conformal dimensions of $\phi_i$, i.e. eigenvalues of $L_0$ and $\bar{L}_0$, the ellipsis stands for terms involving descendant fields, and $C_{ij}^k$ are complex numbers, known as operator product coefficients. By applying elements of $U(W)$ to both sides of (3.6), one obtains the so-called $W$-Ward identities which relate the operator product coefficients involving different members of a given $W$-family. As a consequence, one can extract the basis independent content of the operator product algebra in a manner analogous as one gets the decomposition (1.1) of tensor products of representations of a simple Lie algebra from the Clebsch-Gordan series (1.6). This way one arrives at the fusion rules [10] of a conformal field theory; they are written as in the formula (1.7), where $N_{ij}^k$ describes now the number of distinct couplings among the $W$-families headed by the primary fields $\phi_i, \phi_j, \phi_k$ that occur in the operator product (3.6).

The notion of fusion rings and fusion rule algebras as introduced in section 1 has in fact been tailored to fit to the fusion rules arising, in the manner just described, in conformal field theory. Namely, the canonical basis required by the axiom (F3) is supplied by the primary fields $\phi_i$, the commutativity (F1) follows from the fact that the families $[\phi_i]$ and $[\phi_j]$ appear symmetrically in the definition of $N_{ij}^k$, and the associativity (F2) is a consequence of the associativity of the operator product algebra. Finally, the unit $\phi_0$ is provided by the identity primary field $1$, and the field conjugate to $\phi_i$ whose existence is required by (F4) is the primary field $\phi_i^+$ that satisfies
\[
\langle \phi_i(z, \bar{z}) \phi_i^+(w, \bar{w}) \rangle = (z-w)^{-2\Delta_i} (\bar{z}-\bar{w})^{-2\bar{\Delta}_i}.
\]
(3.7)

(this conjugate field exists and is unique, as a consequence of the fact that any one-point function $\langle \phi \rangle$ vanishes except for $\phi = 1$, while not all two-point functions $\langle \phi \varphi \rangle$ can vanish [9]). Furthermore, there are large classes of conformal field theories which have a fusion rule algebra satisfying also the axiom (R), which means that the number of primary fields is finite. This possibility arises as a result of the presence of null vectors in the Verma modules of $W$ (these lead to the decoupling of $W$-families which naively would be expected to contribute to the operator product algebra). It is commonly supposed that the collection of all two-dimensional conformal field theories can be endowed with a topology such that the rational theories constitute a dense subspace.
Let me mention that one has to be rather careful if one wants to read off the fusion rule coefficients from the operator product algebra. This is so because coefficients $N_{ij}^k > 1$ are allowed, as typically happens if the zero mode (or ‘horizontal’) subalgebra $W_0$ of $W$ is nonabelian. In this situation the operator product coefficients as introduced above are actually not complex numbers, but rather complex numbers multiplied by appropriate invariant tensors of $W_0$. For a given triple of $W$-families there may exist several independent such tensors $\eta$; accordingly, $C_{ij}^k$ gets replaced by $C_{ij}^k;\eta_{ij}^{k;(1)} + C_{ij}^k;\eta_{ij}^{k;(2)} + \ldots$ (incidentally, the tensors $\eta_{ij}^{k;(a)}$ are always such that, with a suitable labelling of the degeneracy index $a$, they vanish for the grades $0, 1, \ldots, a - 2$; in particular, there is a unique coupling among the primaries $\phi_i, \phi_j$ and $\phi_k$, and when raising the grade by one unit, at most one new coupling arises \cite{27}.

Note that the fusion rule coefficients as introduced above for a conformal field theory refer only to the holomorphic symmetry algebra $W$, and not to its antiholomorphic counterpart $\overline{W}$. This is consistent provided that the maximal symmetry algebra is chosen; namely, it can be shown \cite{43, 44} that in this situation there exists a (possibly trivial) permutation of the index set $I$ which furnishes an isomorphism between the holomorphic and the antiholomorphic fusion rule algebras.

Let me also mention another aspect of the operator product algebra. From (3.6) and (3.7) it follows that the operator product coefficients can be read off the three-point functions of primary fields as $C_{ij}^k = \lim_{z, \bar{z} \to \infty} z^{-2\Delta_k} \bar{z}^{-2\bar{\Delta}_k} \langle \phi_i(0, 0) \phi_j(1, 1) \phi_k(z, \bar{z}) \rangle$, and in fact the three-point functions are, up to their normalization $C_{ij}^k$, uniquely fixed by the $W$-Ward identities. However, except for the case of free field theories it is not possible to read off these normalizations as well. But what one can often do, is to compute all four-point functions $F(z, \bar{z}) \equiv F_{ijkl}(z, \bar{z}) = \langle \phi_i(z, \bar{z}) \phi_j(0, 0) \phi_k(1, 1) \phi_l(\infty, \infty) \rangle$, \quad (3.8)
albeit again only up to normalization, and factorize them into three-point functions. The normalizations can then be determined (up to the freedom given by the normalization of the primary fields themselves) with the help of the associativity of the operator product algebra.

\footnote{In practice, one rarely proceeds in this direction, since there there does not exist a simple algorithm for computing operator products without using the fusion rules as an input.}
An important ingredient in the calculation of correlators such as (3.8) is the concept of chiral blocks, which arises naturally from a few basic properties of conformal field theories. Namely, first, the vacuum vector is invariant under the subalgebra of $V$ that is spanned by $L_0$, $L_1$ and $L_{-1}$; this allows to put the position of three of the primaries in the correlator to preferred values, say 0, 1, $\infty$, as has already been done in the formula (3.8). Next, the closure of the operator product algebra implies that $F(z, \bar{z})$ can be written as a sum of products of three-point functions, and because of $W$-invariance the contributions from all fields in a fixed $W$-family $[\phi_m]$ sum up to a function of definite analytic behaviour. The fact that the symmetry algebra is the direct sum of two chiral halves allows to separate the $z$- and $\bar{z}$-dependence of these functions. Thus the correlator becomes a sum of products of purely holomorphic or antiholomorphic pieces, the chiral blocks, according to

$$F(z, \bar{z}) = \sum_{m=1}^{M} \sum_{\bar{m}=1}^{\bar{M}} a_m \bar{a}_{\bar{m}} f_m(z) \bar{f}_{\bar{m}}(\bar{z}).$$

(3.9)

The number $M \equiv M_{ijkl}$ of blocks is determined through the fusion rule coefficients as

$$M = \sum_{n \in I} N_{ij}^n N_{nkl}.$$  

(3.10)

The same expression is valid for the integer $\bar{M}$ (here it is implicit that the chiral algebra is maximally extended), and one has actually $a_{m\bar{m}} = a_m \delta_{\bar{m}, \sigma(m)}$ for some permutation $\sigma$. If the chiral blocks are properly normalized, the coefficients $a_m$ coincide with the product $C_{ij}^m C_{mkl}$ of operator product coefficients $[9, 45]$.

The associativity of the operator product algebra implies certain identities, known as duality relations, for the system of chiral blocks of a theory. These lead in particular to the so-called polynomial equations for the fusing and braiding matrices, which play an important role in the classification of conformal field theories $[11, 44]$. (For additional information, see section 10.)

To conclude this section, let me point out that the fusion product defined in the manner described above does not provide a description of the tensor products of highest weight modules of the algebra $W$. For the latter, the eigenvalues of central operators such as $C$ add up, whereas all the highest weight modules appearing in a given conformal field theory have the same eigenvalues. Nevertheless it is possible to think of the collection of $W$-modules of a conformal field theory as the objects of a rigid braided monoidal category, with the tensor product of the category identified with the fusion
rules; the commutativity and associativity constraints of the category then coincide with the (genus zero) polynomial equations that were introduced in \cite{11}. As all the (non-trivial) \( W \)-modules are infinite-dimensional, there is no reason to expect that one can find a compatible tensor functor from this category to the category of finite-dimensional vector spaces, and indeed (compare e.g. \cite{21, 46}) such a functor does not exist.

4 The connection with the modular group

According to the formulæ (2.22), (2.28), (2.32), and (2.33), for the special class of (quasi)rational fusion rule algebras that satisfy the modularity constraints (M1) and (M2), the matrix \( S := Y \) diagonalizing the fusion matrices, the conjugation matrix \( C \), and some diagonal matrix \( T \) obey

\[
S^2 = C = (ST)^3
\]

and

\[
S = S^t, \quad S^* = S^{-1}, \quad T^* = T^{-1}.
\]

These are the defining relations for a unitary matrix representation of the group \( SL(2, \mathbb{Z}) \) of \( 2 \times 2 \)-matrices with integral entries and determinant 1, or what is the same, for a projective unitary matrix representation of the quotient \( PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{± \mathbb{I} \} \) of \( SL(2, \mathbb{Z}) \) by its center \( \{± \mathbb{I} \} \), which is known as the modular group. This justifies the qualification ‘modular’ fusion rule algebra that I ascribed to these structures in section 2.

A rather different role of the modular group in conformal field theory emerges as follows. The information about the conformal dimensions of the members of a \( \mathcal{W} \)-family \( [\phi_i] \) can be encoded into a function \( \chi \) of a complex variable \( \tau \) (well-defined for \( \text{Im}(\tau) > 0 \)) according to

\[
\chi_i(\tau) \equiv \chi_{\phi_i}(\tau) := \text{tr}_{[\phi_i]} e^{2\pi i(\ell_0 - c/24)} ; \tag{4.3}
\]

\( \chi_i \) is called the (Virasoro-specialized) character of the family. Conjugate families possess identical characters,

\[
\chi_{i^+} = \chi_i. \tag{4.4}
\]

The derivation property (3.5) of \( \ell_0 \) induces a \( \mathbb{Z} \)-gradation on the \( \mathcal{W} \)-modules. More precisely, one defines the grade of primary fields to be zero and, recursively, grade(\( W_{-m}\phi \)) = grade(\( \phi \)) + \( m \). Thus the conformal dimension of
a field $\varphi \in [\phi_i]$ is just the sum of the conformal dimension $\Delta_i$ of $\phi_i$ and the grade of $\varphi$, so that

$$\chi_i(\tau) = e^{2\pi i r(\Delta_i - c/24)} \sum_{n \in \mathbb{Z} \geq 0} d_n e^{2\pi i rn},$$

with $d_n$ the number of descendants at grade $n$. In particular, it follows immediately that

$$\chi_i(\tau + 1) = e^{2\pi i (\Delta_i - c/24)} \chi_i(\tau).$$

(4.5)

Now $PSL(2, \mathbb{Z})$ can abstractly be defined as the group generated freely by two elements $S$ and $T$ modulo the relations $S^2 = (ST)^3 = id$. These generators can be realized as the transformations $S: \tau \mapsto -1/\tau$, $T: \tau \mapsto \tau + 1$ of some complex variable $\tau$ which may be restricted to the upper complex half plane. According to (4.6), the transformation $T$ acts on the characters $\chi$ as multiplication by a diagonal matrix $T$ with entries

$$T_{jk} = \delta_{jk} t_k \equiv \delta_{jk} \exp(2\pi i (\Delta_j - c/24)).$$

(4.7)

Note that together with the formulæ (2.26), (2.34) and (2.36) this implies the relation

$$c = \frac{4}{\pi} \arg \left( \sum_{j \in I} D_j^2 e^{2\pi i \Delta_j} \right)$$

(4.8)

between the conformal dimensions, the fusion dimensions and the conformal central charge, which determines $c$ modulo 8. There are arguments, to be described in some detail further on, that the transformation $S$ acts on the collection of characters as multiplication by a matrix, too. Denoting this matrix by $S$, one arrives at the relations (1.1) and (4.2) above, with the entries of $T$ specified by the expression (4.7). That one obtains a representation of $SL(2, \mathbb{Z})$, and hence generically a projective representation of $PSL(2, \mathbb{Z})$, is not in conflict with the identities $S^2 = id = (ST)^3$, because due to the equality (4.4) the characters specify the $W$-families only up to conjugation. Moreover, further degeneracies are absent at least if $W$ is maximal. 

In more physical terms, both $S^2$ and $(ST)^3$ correspond to a combined ‘space’ and ‘time’ reflection (since, while they leave the homology cycles of the torus invariant, they invert their orientation) and hence should be equivalent to a charge conjugation.

Having arrived at the same structure in two different settings, it is natural to speculate that, as already anticipated in the notations, the matrix
$S = Y$ of equation (2.20) coincides with the modular matrix $S$ that implements the map $S$ on the characters. This is indeed true, so that for instance the relation (2.24) translates to the Verlinde formula \[ 10 \]

\[ N_{ij}^k = \sum_{l \in I} \frac{S_{il} S_{jl} (S^{-1})_{kl}}{S_{0l}}. \]  

(4.9)

In particular, the fusion rule algebra of any rational conformal field theory is a modular fusion rule algebra.

The formula (4.9) has been proven \[ 3 \] by performing certain formal manipulations of two-point correlators on the torus $T_\tau$ that is described in terms of $\tau$ as a parallelogram, with opposite edges identified, whose corners are at 0, 1, $\tau$ and $\tau + 1$. These manipulations lead \[ 10 \] to a relation like (4.9) with certain integers $N_{ij}^k$. Further, it can be shown \[ 44, 43 \] that (as a consequence of the pentagon identity applied to three-point correlators on the torus) these integers are indeed the fusion rule coefficients of the theory. An alternative proof of the formula is based on the connection between three-dimensional topological field theory, ribbon Hopf algebras \[ 47 \], and two-dimensional conformal field theory (in the topological setting, the matrix elements $S_{ij}$ are obtained as expectation values of Wilson lines) \[ 48, 49, 50, 51, 52, 53 \].

By the proof of the Verlinde formula (4.9), the identification of the diagonalization matrix $Y$ with the modular matrix $S$ follows from comparison with (2.24), provided that the (symmetric, unitary, etc.) diagonalization matrix is unique. As remarked at the end of section 2, this condition is almost always fulfilled; in fact, no modular fusion rule algebra is known for which the diagonalization matrix cannot be fixed completely.

Let me now come to the arguments in favor of the assertion that the transformation $S$ acts on the characters by matrix multiplication. One can think of the characters $\chi$ as the chiral blocks for the zero-point function $Z = \langle 1 \rangle_\tau$ on $T_\tau$, the so-called partition function of the theory. If the conformal field theory is regarded as the vacuum configuration of a relativistic string, then the partition function is closely related to the path integral for the vacuum-to-vacuum transition amplitude of the string theory. In order to be able to properly fix the gauge symmetries of this amplitude, the partition function must be modular invariant (see e.g. \[ 54, 55, 56 \]). \[ 4 \] By similar arguments \[ 57 \], modular invariance is a natural property of a two-dimensional

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3. It should be stressed that to arrive at this result, the characters must be those with respect to the maximally extended chiral algebra; otherwise, as mentioned in the previous section, the fusion rules would not be unambiguously defined.

4. To be precise, modular invariance is a sufficient condition. It is necessary only insofar
statistical mechanics system at the critical point of a second order phase transition. Together with the associativity of the operator product algebra, the requirement of modular invariance of $Z$ implies \cite{58} that the characters span a module over the modular group. Based on the experience with these important manifestations of conformal field theory, it has become common to require modular invariance more or less axiomatically \cite{59,60} (and apply this constraint e.g. \cite{51,52} to the classification of conformal field theories). More recently, a general argument for modular invariance in conformal field theory has been given in \cite{63}; it employs the C$^*$-algebraic approach to two-dimensional field theory, thereby relating in particular the partition function to a thermal state (at complex inverse temperature $\beta$ equal to $-2\pi i$ times the modular parameter $\tau$, as is implied by interpreting $\exp(2\pi i \tau L_0)$ as $\exp(-\beta H)$ \cite{64}) over the field algebra. The argument involves nontrivial results about the structure of the field algebra, which is taken to be a von Neumann algebra of type I$_1$ (such as the statement that any thermal state is a linear combination of the thermal states that are associated to the irreducible representations of the observable algebra, which are \cite{2,65,66} in one to one correspondence with the primary fields).

At this point it is worth recalling that most of the structural elements of two-dimensional conformal field theory are not particular to conformally invariant theories, but are rather generic properties of two-dimensional quantum field theory, as is especially transparent in the C$^*$-algebraic approach to local quantum physics \cite{1}. Somewhat surprisingly, even the relation with the modular group is already present in this much more general context. Namely, in the algebraic framework the fusion rules describe the composition of superselection sectors, which can be expressed in terms of the composition of certain endomorphisms of the local algebras of observables. It turns out \cite{67,68,51} that the matrix elements of the matrices $Y$ and $T$ introduced in section 2 can be entirely described \cite{1} in terms of these endomorphisms, and from general properties of the endomorphisms it follows \cite{6} that the modularity constraints (M1) and (M2), and hence the defining as no consistent treatment of the gauge degrees of freedom of a non-modular invariant theory has ever been conceived.

\footnote{In this context one should note that, as is proven in \cite{65}, under rather general assumptions the observable algebra of a conformal field theory is isomorphic to the hyperfinite von Neumann factor of type III$_1$.}

\footnote{In particular, $Y$ is the so-called monodromy matrix. For its definition one needs the concepts \cite{1} of the ‘statistics operator’ and the ‘left inverse’ associated to an endomorphism.}

\footnote{Here a regularity property has to be assumed, which in the conformal case corresponds
properties (4.1) and (4.2) of the modular group, are satisfied. Moreover, for a conformal theory, $T$ indeed implements the modular transformation $T$, by definition of the characters. What is less clear is whether one can prove in a purely $C^*$-algebraic setting that the matrix $Y$ implements the modular transformation $S$, and whether $Y$ and $T$ possess any geometric interpretation (similar to, say, the modular operators of Tomita-Takesaki theory [8] for non-conformal theories as well.

According to the restriction (2.13) on fusion dimensions, the smallest possible fusion dimension is 1. Generators $\phi_i$ with fusion dimension equal to one are therefore particularly interesting; they are called simple currents. It follows from the fact that the quantities $\ell_i^{(n)} := S_{in}/S_{0n}$ furnish one-dimensional representations of the fusion rule algebra, i.e. obey the sum rules $\ell_i^{(n)} \ell_j^{(n)} = \sum_{k \in I} N_{ij}^k \ell_k^{(n)}$, that the fusion product of a simple current $\phi_i$ with any primary field $\phi_j$ contains only a single primary field. Accordingly one may write $\phi_i \star \phi_j = \phi_{i \star j}$. It can be shown [70,71] that the $S$-matrix elements corresponding to fields that are related by the fusion with a simple current are equal up to a root of unity, $S_{jk} = \exp[2\pi i q_{ij}/N_i] S_{j,i\star k}$, with $q_{ij}$ an integer determined by the conformal dimensions of $\phi_i$, $\phi_j$, and $\phi_{i\star j}$, and $N_i$ the order of the simple current, i.e. the smallest positive integer such that a multiple fusion product of $N$ factors of $\phi_i$ produces the unit $\phi_0$. As a consequence, simple currents play an important role in the construction of modular invariant partition functions, as well as for the so-called field identifications which arise in the coset construction of conformal field theories [70,71,72].

5 WZW theories

An important class of rational conformal field theories are the so-called (unitary) Wess-Zumino-Witten (WZW) theories [45,73]. All known rational conformal field theories are closely related to appropriate WZW theories, namely via the coset construction [74] or via Drinfeld-Sokolov Hamiltonian reduction [75,76]. For WZW theories the symmetry algebra $\mathcal{W}$ (and $\overline{\mathcal{W}}$ as well) is the semidirect sum of the Virasoro algebra with an untwisted affine Lie algebra $\hat{\mathfrak{g}}$. The latter is the Lie algebra with generators $\{K\} \cup \{J_m^a | m \in \mathbb{Z}, a = 1,2,..., \text{dim}(\mathfrak{g})\}$ and brackets

$$
[J_m^a, J_n^b] = f^{ab}_{\ c} J_m^c + \kappa^{ab} \delta_{m+n,0} K, \quad [J_m^a, K] = 0; \quad (5.1)
$$

to the assumption that the symmetry algebra is maximal.
κ<sub>ab</sub> and f<sub>abc</sub> denote the Killing form and structure constants of a semisimple Lie algebra g ⊂ ɡ, namely the subalgebra generated by the zero modes J<sub>0</sub><sup>a</sup>, i.e. the horizontal subalgebra of ɡ.

In addition, for a WZW theory the Virasoro generators L<sub>n</sub> are quadratic expressions in the ˆoJ<sub>a</sub>-generators J<sub>a</sub><sup>m</sup>. Also, all highest weight modules of ˆoJ<sub>a</sub> which appear in a given theory possess the same eigenvalues k and c of the central generators K and C, and these numbers are related by

c = k \dim(g) / (k + h),

with h the dual Coxeter number of oJ<sub>a</sub>. k is called the level of ˆoJ<sub>a</sub>; for unitarity it must be a positive integer (I normalize the inner product on the weight space of oJ<sub>a</sub> such that the highest root θ of g satisfies (θ, θ) = 2).

The spectrum \{φ<sub>Λ</sub><sup>0</sup></sup> of primary fields can be labelled by highest weights Λ of finite-dimensional oJ<sub>a</sub>-modules whose inner product with the highest root is not larger than the level, i.e.

\[ I = I_k = I(g, k) = \{ Λ ∈ (Z_{≥0})^{\text{rank}(g)} \mid (Λ, θ) ≤ k \}, \]  

(5.2)

and the conjugation of primary fields corresponds to the conjugation of highest oJ<sub>a</sub>-weights, (φ<sub>Λ</sub>)<sup>+</sup> = φ<sub>Λ</sub>+. For WZW theories the modular matrix T takes the form

\[ T<sub>ΛΛ</sub>' = δ<sub>ΛΛ'</sub> \exp \left[ πi (Λ, Λ + 2ρ) - k \dim(g) / 12 \right] / (k + h) \],

(5.3)

with ρ the Weyl vector of g. Also, as a consequence of the fact that the characters χ<sub>Λ</sub> are just specializations of the ordinary affine characters of ˆoJ<sub>a</sub>-modules, the matrix S is given by the following Kac-Peterson formula [77]:

\[ S<sub>ΛΛ'</sub> = \text{const} \prod_{α > 0} \sin \left( \frac{π(Λ + ρ, α)}{k + h} \right) \frac{\sum_{w ∈ W} \sigma(w) \exp \left[ - \frac{2πi}{k + h} (Λ + ρ, w(Λ' + ρ)) \right]}{\sum_{w ∈ W} \sigma(w) \exp \left[ - \frac{2πi}{k + h} (Λ + ρ, w(ρ)) \right]} ; \]

(5.4)

here the product is over the positive roots of g, the sums are over the Weyl group W of g, and σ(w) stands for the sign of the Weyl group element w.

In the notation appropriate to WZW theories, the fusion rules are written as φ<sub>Λ</sub> * φ<sub>Λ'</sub> = Σ<sub>Λ'' ∈ I_k</sub> N<sub>ΛΛ'</sub><sup>Λ''</sup> φ<sub>Λ''</sub>, and the Verlinde formula reads

\[ N<sub>ΛΛ'</sub><sup>Λ''</sup> = Σ_{μ ∈ I_k} \frac{S<sub>Λμ</sub>S<sub>Λ'μ</sub>S<sub>Λ''μ</sub>}{S<sub>0μ</sub>} . \]

(5.5)

Unfortunately, due to the summation over the Weyl group the calculation of the Kac-Peterson S-matrix is cumbersome for 'large' algebras, so that
the Verlinde formula is in itself of limited use. But fortunately there also exist other possibilities of computing the WZW fusion rules; all of them are in close relation with the representation theory of semisimple Lie algebras, and in particular require the knowledge of the tensor product multiplicities of \( \mathfrak{g} \), which (deviating, for further convenience, slightly from the notation that was used in (1.1)) will be denoted by \( N_{\Lambda \Lambda'}^{\Lambda''} \).

One of these algorithms is the so-called depth rule \([73, 78]\) which seems however also quite involved already for modestly large algebras, and which for general level so far has only been applied to \( \mathfrak{g} = A_1 \) \([73]\) and to \( \mathfrak{g} = A_2 \) \([79, 80]\). Another one expresses \([81, 82]\) the WZW fusion rule coefficients \( N_{\Lambda \Lambda'}^{\Lambda''} \equiv N_{\Lambda \Lambda'}^{\Lambda''}(k) \) as a weighted sum over the tensor product multiplicities \( N_{\mu \Lambda \Lambda'}^{\Lambda''} \).

\[
N_{\Lambda \Lambda'}^{\Lambda''} = \sum_{\hat{w} \in \hat{W}} \sigma(\hat{w}) N_{\Lambda \Lambda'}^{\hat{w}(\Lambda'')}.
\] (5.6)

Here \( \hat{W} \) denotes the horizontal projection of the Weyl group of \( \mathfrak{g} \), i.e. \( \hat{w} \in \hat{W} \) corresponds to a pair \( (w, \beta) \) with \( w \in W \) and \( \beta \) an element of the coroot lattice of \( \mathfrak{g} \), acting as \( \hat{w}(\Lambda) = w(\Lambda + \rho) - \rho + (k + h)\beta \). While \( \hat{W} \) is an infinite group, for any triple \( (\Lambda, \Lambda', \Lambda'') \in (I_k)^3 \) the sum in (5.6) contains only a finite number of non-vanishing terms, and there exists a finite algorithm to evaluate the formula. As a consequence, the formula is easily implemented on a computer. If \( \mathfrak{g} \) is a classical Lie algebra, it can also be translated into a Young-diagrammatic prescription, as has been done in \([79]\) for \( \mathfrak{g} = A_r \) and \( \mathfrak{g} = C_r \).

The result (5.6) is a rather straightforward consequence of the Kac-Peterson formula for \( S \); nevertheless it is instructive to mention a few details \([82, 18]\) of its proof. Namely, on one hand the numbers \( \ell_{\Lambda}^{(\mu)} := S_{\Lambda \mu}/S_0^{\mu} \) obey the representation property

\[
\ell_{\Lambda}^{(\mu)} \ell_{\Lambda'}^{(\mu)} = \sum_{\Lambda'' \in I_k} N_{\Lambda \Lambda'}^{\Lambda''} \ell_{\Lambda''}^{(\mu)}.
\] (5.7)

On the other hand, the Kac-Peterson formula implies \( \ell_{\Lambda}^{(\mu)} = \bar{\chi}_{\Lambda}(2\pi i(\mu + \rho)/(k + h)) \), with \( \bar{\chi}_{\Lambda} \) the character of the \( \mathfrak{g} \)-module \( L_{\Lambda} \) (not to be confused with the specialized affine character \( \chi_{\Lambda} \)), and hence the character sum rule for tensor products of \( \mathfrak{g} \)-modules implies

\[
\ell_{\Lambda}^{(\mu)} \ell_{\Lambda'}^{(\mu)} = \sum_{\Lambda'' \in I_\infty} \bar{N}_{\Lambda \Lambda'}^{\Lambda''} \ell_{\Lambda''}^{(\mu)}.
\] (5.8)
The two sum rules are compatible with each other because \( \ell^{(\mu)}_{\lambda} = \sigma(w) \ell^{(\mu)}_{\hat{w}(\lambda)} \) for any \( \hat{w} \in \hat{W} \), and because for given \( \Lambda \) there is at most one \( \hat{w} \in \hat{W} \) such that \( (\hat{w}(\Lambda), \theta) \leq k \). Finally, there are no further relations among the \( \ell^{(\mu)}_{\lambda} \), i.e. the set \( \{ \ell^{(\mu)}_{\lambda} \mid \Lambda \in I_k \} \) is linearly independent (otherwise the columns of \( S \) were dependent, in contradiction to \( S^4 = \mathbb{I} \)). Putting these results together, the formula (5.6) follows.

The prescription (5.6) is particularly simple if \( (\Lambda, \theta) = 1 \). In this case it follows that \( (\Lambda'', \theta) \leq k + 1 \) for all \( \Lambda', \Lambda'' \in I_k \) with \( \overline{N}_{\Lambda \Lambda'}^{\Lambda''} \neq 0 \), and

\[
N_{\Lambda \Lambda'}^{\Lambda''} = \begin{cases} 
\overline{N}_{\Lambda \Lambda'}^{\Lambda''} & \text{for } (\Lambda'', \theta) \leq k, \\
0 & \text{for } (\Lambda'', \theta) = k + 1.
\end{cases}
\] (5.9)

In other words, the fusion rules of fields satisfying the constraint \( (\Lambda, \theta) = 1 \) are obtained from the corresponding tensor products of \( \mathfrak{g} \)-representations by merely removing the couplings to fields with \( (\Lambda'', \theta) = k + 1 \).

From the results of the algorithms for WZW fusion rules, one can conversely deduce the matrix \( S \), in the manner explained at the end of the section 2. Actually [33], as input for the diagonalization procedure one usually needs only a single (non-trivial) fusion matrix, and hence only a single matrix of tensor product coefficients. Since the computation of the latter is the major obstacle that limits the use of the depth rule or of (5.6), the most efficient way to proceed is to use first the depth rule or (5.6) for the computation of a single fusion matrix, then diagonalize, thereby obtaining the modular matrix \( S \), and then calculate the remaining fusion matrices via the Verlinde formula.

As can be seen by investigating the depth rule [78], to each of the \( \overline{N}_{\Lambda \Lambda'}^{\Lambda''} \) distinct couplings \( (\Lambda, \Lambda', \Lambda''; p) \) that arise at some level of a WZW theory, one can associate a ‘threshold level’ \( k_p = k_{\Lambda, \Lambda', \Lambda''; p} \) such that the coupling is present for all levels \( k \geq k_p \), but absent for all \( k < k_p \). In particular, \( p \) takes the values \( p = 1, 2, ..., \overline{N}_{\Lambda \Lambda'}^{\Lambda''} \), the \( N_{\Lambda \Lambda'}^{\Lambda''} \) are majorized by the tensor product coefficients,

\[
0 \leq N_{\Lambda \Lambda'}^{\Lambda''}(k) \leq \overline{N}_{\Lambda \Lambda'}^{\Lambda''}
\] (5.10)

(this it is not manifest in the formula (5.6)), and

\[
N_{\Lambda \Lambda'}^{\Lambda''}(k) = 0 \quad \text{for } k < \min\{k_p \mid p = 1, ..., \overline{N}_{\Lambda \Lambda'}^{\Lambda''}\},
\]

\[
N_{\Lambda \Lambda'}^{\Lambda''}(k) = \overline{N}_{\Lambda \Lambda'}^{\Lambda''} \quad \text{for } k \geq \max\{k_p \mid p = 1, ..., \overline{N}_{\Lambda \Lambda'}^{\Lambda''}\}.
\] (5.11)

As an illustration, I list in the following table three specific fusion rule coefficients of the \( E_8 \) WZW theory at levels \( \geq 4 \) (for smaller levels, the
relevant field \( \phi_{\Lambda(1)+\Lambda(7)} \) does not belong to the spectrum):

\[
\begin{array}{c|ccc}
 k & 4 & 5 & \geq 6 \\
\hline
\mathcal{N}_{\Lambda(1)+\Lambda(7),\Lambda(1)+\Lambda(7),\Lambda(1)} & 1 & 2 & 2 \\
\mathcal{N}_{\Lambda(1)+\Lambda(7),\Lambda(1)+\Lambda(7),\Lambda(7)} & 2 & 3 & 3 \\
\mathcal{N}_{\Lambda(1)+\Lambda(7),\Lambda(1)+\Lambda(7),\Lambda(1)+\Lambda(7)} & 2 & 8 & 9 \\
\end{array}
\] (5.12)

Another property of WZW fusion rules that can easily be verified is that

\[
\mathcal{N}_{\omega(\Lambda),\omega'(\Lambda')} = \mathcal{N}_{\Lambda \Lambda'}
\] (5.13)

where \( \omega, \omega' \) is any pair of automorphisms of the extended Dynkin diagram of \( \mathfrak{g} \) that are not automorphisms of the unextended Dynkin diagram.

Let me finally mention that if the level is taken to be a non-real complex number or a real number smaller than \(-h\) (which means in particular that the theory is non-unitary), then it is possible to define a tensor product of \( \hat{\omega} \)-modules that does not change the level and hence is in this respect similar to the conformal field theory fusion rules. Further, there does exist a compatible tensor functor from the monoidal category with this tensor product to the category of finite-dimensional \( \mathfrak{g} \)-modules, and hence also such a functor to the category of finite-dimensional vector spaces.

6 Polynomial rings and fusion potentials

6.1 Fusion rings and polynomial rings

Any finite-dimensional commutative associative ring, and hence in particular any rational fusion ring \( \mathcal{A} \), can be presented as the quotient of a free polynomial ring by some ideal. Namely, consider the elements \( \phi_i \) of a canonical basis of \( \mathcal{A} \) as formal variables; then one has

\[
\mathcal{A} \cong \mathbb{Z}[\phi]/\mathcal{J},
\] (6.1)

with \( \mathbb{Z}[\phi] \equiv \mathbb{Z}[\phi_1, \phi_2, ..., \phi_{|I|}] \) the ring of polynomials (with integral coefficients) in these variables, and \( \mathcal{J} \) the subring generated by the fusion relations, i.e. by the polynomials

\[
P_{ij}(\phi) := \phi_i \phi_j - (\phi_i \star \phi_j)(\phi) = \phi_i \phi_j - \sum_{k \in I} \mathcal{N}_{ij}^k \phi_k.
\] (6.2)

\footnote{The \( \mathfrak{g} \)-modules must obey the usual restriction that their subspaces at any fixed grade are finite-dimensional.}
Due to commutativity and associativity, $\mathcal{J}$ is a two-sided ideal of $\mathbb{Z}[\phi]$, and as a consequence dividing it out as in (6.1) is a well-defined procedure so that indeed (6.1) can be used as a definition of the fusion ring.

Consider now the $\phi_i$ as complex variables rather than as formal indeterminates. Then owing to the representation property (2.3) of fusion rule eigenvalues, one has

$$P_{ij}(\phi) = 0 \iff \phi_l = \nu_l^{(i)} = Y_{kl}/Y_{k0} \text{ for all } l \in I \text{ and some } k \in I.$$  (6.3)

As a consequence, $\mathcal{J}$ may also be characterized as being generated by $|I|$ polynomials $P_k(\phi)$ with the property that $P_k(\phi) = 0$ iff $\phi_i = Y_{ki}/Y_{k0}$ for all $i \in I$. These polynomials can be chosen as $P_k(\phi) = \prod_{j \in I} (\phi_j - \nu_k^{(j)})$, but because not all of the $|I|^2$ numbers $\nu_l^{(i)}$ are distinct, one may also restrict the product to the terms with certain appropriate eigenvalues $\nu_k^{(j)}$. In particular, for $k = 0$ one can replace $P_0(\phi) = \prod_{j \in I} (\phi_j - \mathcal{D}_j)$ by $P_0(\phi) = \phi_0 - 1$, thereby effectively setting $\phi_0$ equal to one and hence eliminating it as an independent variable. Often the quotienting by $\mathcal{J}$ effectively eliminates some further generators $\phi_i$ as well; for example, as it turns out all primary fields of the fusion ring of the $A_1$ WZW theory are generated by the single element $\phi_{\Lambda(1)}$. It is then natural to consider $A$ as obtained by quotienting the free polynomial ring in the independent variables $\phi_{ji} =: x_i$, $i = 1, 2, \ldots, n$. Thus the fusion ring is written as

$$A \cong \mathbb{Z}[x]/\mathcal{J},$$  (6.4)

where now $\mathcal{J}$ is the ideal generated by some polynomial constraints $P_k(x) = 0$, $k = 1, 2, \ldots, n$, in the variables $x_i$, and these polynomials $P_k(x)$, vanish simultaneously iff for some $l \in \{1, 2, \ldots, n\}$ one has $x_j = \mu_l^{(j)}$ for all $j = 1, 2, \ldots, n$, where $\mu_l^{(j)}$ are the eigenvalues of $\pi_{\text{reg}}(x_j)$.

### 6.2 Local rings

Particularly interesting is certainly the situation where the polynomials $P_k(x)$ are integrable, which means that they are derived from some potential $V(x)$ in the sense that

$$P_k(x) = \frac{\partial}{\partial x_k} V(x)$$  (6.5)

for all $k$. In other words,

$$A \cong \mathbb{Z}[x]/dV(x),$$  (6.6)
i.e. \( A \) is the local ring of the potential \( V \).

Given any fusion ring \( A \) one may attempt to construct a presentation as a local ring as follows. Consider for the moment \( A \) as an algebra over the field \( \mathbb{C} \) (actually, an appropriate finite algebraic extension of \( \mathbb{Q} \) already does the job), and take an element \( \tilde{x} \in A \) for which all eigenvalues of \( \pi_{\text{reg}}(\tilde{x}) \) are distinct; as pointed out at the end of section 2, such an element does exist. Now instead of the canonical basis \( \{ \phi_i \} \) of \( A \), consider a particular basis \( B \) that contains \( \tilde{x} \), namely \( B = \{ \phi_0, \tilde{x}, \phi_2, \phi_3, \ldots, \phi_{|I|} \} \); here it is assumed that the coefficient \( a_1 \) in the decomposition

\[
\tilde{x} = \sum_{j=0}^{|I|-1} a_j \phi_j
\]  

(6.7)

of \( \tilde{x} \) with respect to the canonical basis does not vanish, which can always be accomplished by appropriate labelling of the generators \( \phi_i \). Next eliminate \( \phi_0 \) and possibly further elements of \( B \) (but not \( \tilde{x} \), even if this were possible) in the manner described above. Writing from now on \( \tilde{x}_1 \) for \( \tilde{x} \), one arrives at a presentation of the form (6.4) of \( A \) in terms of the variables \( \{ \tilde{x}_1, x_2, x_3, \ldots, x_n \} \) (the tilde on the first variable is kept in order to emphasize that, generically, in contrast to the other variables it is not an element of the canonical basis).

Now denote the eigenvalues of \( \pi_{\text{reg}}(\tilde{x}_1) \) by \( \tilde{\mu}_k^{(1)} \), and, as before, those of \( \pi_{\text{reg}}(x_i) \) by \( \mu_i^{(i)} \) for \( i = 2, 3, \ldots, n \). Then make the ansatz [86]

\[
V(\tilde{x}_1, x_2, \ldots, x_n) = -\sum_{i=0}^{|I|-1} \mu_i^{(2)} \int \frac{\tilde{x}_1}{|I|} \prod_{j=0}^{|I|-1} \left( t - \tilde{\mu}_j^{(1)} \right) + x_2 \prod_{i=0}^{|I|-1} \left( \tilde{x}_1 - \tilde{\mu}_i^{(1)} \right) + \frac{1}{2} \sum_{i=0}^{|I|-1} \sum_{j=3}^n (x_j - \mu_j^{(j)})^2 \prod_{l=0}^{|I|-1} \left( \tilde{x}_1 - \tilde{\mu}_l^{(1)} \right).
\]  

(6.8)
follows that there exist polynomials which show that the coefficients can indeed be taken as rational.

The partial derivatives of (6.8) read

\[
\frac{\partial}{\partial x_1} V(x) = \sum_{i=0}^{\left| I \right|-1} \left( \prod_{j=0}^{\left| I \right|-1} (\tilde{x}_1 - \tilde{\mu}_i^{(1)}) \right) \cdot \left[ (x_2 - \mu^{(2)}_i) + \frac{1}{2} \sum_{j=3}^{n} (x_j - \mu^{(j)}_i)^2 \sum_{l=0}^{\left| I \right|-1} (\tilde{x}_1 - \tilde{\mu}_i^{(1)})^{-1} \right],
\]

\[
\frac{\partial}{\partial x_2} V(x) = \prod_{i=0}^{\left| I \right|-1} (\tilde{x}_1 - \tilde{\mu}_i^{(1)}),
\]

\[
\frac{\partial}{\partial x_j} V(x) = \sum_{i=0}^{\left| I \right|-1} (x_j - \mu^{(j)}_i) \prod_{l=0, l \neq k}^{\left| I \right|-1} (\tilde{x}_1 - \tilde{\mu}_i^{(1)}) \quad \text{for } j \geq 3.
\]

Thus requiring \( \partial V(x)/\partial x_2 = 0 \) enforces \( \tilde{x}_1 = \tilde{\mu}_k^{(1)} \) for some \( k \in I \); as all \( \tilde{\mu}_i^{(1)} \) are distinct, one has \( \partial V(\tilde{x}_1 = \tilde{\mu}_k^{(1)}; x_2, \ldots, x_n)/\partial x_j = (x_j - \mu^{(j)}_k) \prod_{l=0, l \neq k}^{\left| I \right|-1} (\tilde{x}_1 - \tilde{\mu}_l^{(1)}) \) for \( j \geq 3 \), so that \( \partial V(x)/\partial x_j = 0 \) enforces \( x_j = \mu^{(j)}_k \); finally, \( \partial V(\tilde{x}_1 = \tilde{\mu}_k^{(1)}; x_2, \ldots, x_n = \mu^{(n)}_k)/\partial \tilde{x}_1 = (x_2 - \mu^{(2)}_k) \prod_{l=0, l \neq k}^{\left| I \right|-1} (\tilde{x}_1 - \tilde{\mu}_l^{(1)}) \), so that \( \partial V(x)/\partial \tilde{x}_1 = 0 \) enforces \( x_2 = \mu^{(2)}_k \).

In short, one finds that \( \partial V(x)/\partial x_i = 0 \) iff \( x_j = \mu^{(k)}_j \) for all \( j \in I \) and some \( k \in I \), i.e. the standard property of the constraints \( \mathcal{P}_k(x) \) described above. Moreover, clearly the \( \left| I \right| \) polynomials \( 1, \tilde{x}_1, (\tilde{x}_1)^2, \ldots, (\tilde{x}_1)^{\left| I \right|-1} \) are linearly independent over \( \mathbb{C} \) (whereas \( (\tilde{x}_1)^{\left| I \right|} \) can be expressed through these owing to the explicit form of the constraint \( \partial V(x)/\partial x_2 = 0 \)). Since \( \left| I \right| \) is the dimension of the fusion ring, these polynomials provide a basis, and hence a reconstruction of the ring from the potential is possible. In particular it follows that there exist polynomials \( Q_j(\tilde{x}_1) \) and \( R_{ij}(\tilde{x}_1, x_2, \ldots, x_n) \) for \( j = 2, 3, \ldots, n \) and \( i = 1, 2, \ldots, n \) such that

\[
x_j = Q_j(\tilde{x}_1) + R_{1j}(\tilde{x}_1) \frac{\partial}{\partial \tilde{x}_1} V(x) + \sum_{i=2}^{n} R_{ij}(x) \frac{\partial}{\partial x_j} V(x), \tag{6.10}
\]

i.e. such that \( x_j = Q_j(\tilde{x}_1) \mod dV. \) As a consequence, as an algebra over \( \mathbb{C} \), \( \mathcal{A} \) can indeed be presented as the local algebra \( \mathbb{C}[\tilde{x}_1, x_2, \ldots, x_n]/dV. \) Whether as a ring \( \mathcal{A} \) can be written analogously as \( \mathbb{Z}[\tilde{x}_1, x_2, \ldots, x_n]/dV \) is, however, difficult to decide, as one would have to show that all coefficients in the polynomial \( V \) as defined in (5.8) are rational numbers \( \mathbb{R} \) (and hence, with

\[9\] There are however arguments (see [86], and O. Aharony, private communication) which show that the coefficients can indeed be taken as rational.
appropriate over-all normalization of \( V \), integers); if they are not, then \( \mathcal{A} \) is not a well-defined local ring over \( \mathbb{Z} \).

Moreover, by assumption there exist polynomials \( p_i \) such that 
\[
\phi_i = p_i(\tilde{x}_1, x_2, \ldots, x_n) \quad \text{for} \quad i = 2, 3, \ldots, |I|,
\]
and hence (6.7) and (6.10) imply
\[
\phi_1 = a_1^{-1} (\tilde{x}_1 - \sum_{i=0}^{[I]-1} a_i \phi_i)
\]
\[
= a_1^{-1} (\tilde{x}_1 - a_0 - \sum_{i=2}^{[I]} a_i p_i(\tilde{x}_1, Q_2(\tilde{x}_1), \ldots, Q_n(\tilde{x}_1))) \mod dV \quad (6.11)
\]
\[
= Q_1(\tilde{x}_1) \mod dV
\]
for the generator \( \phi_1 \) of the canonical basis. Note that the coefficients \( a_i \) introduced in the decomposition (6.7) are generically not rational. Thus even if \( \mathcal{A} \) could be presented as a local ring in the variables \( \tilde{x}_1, x_2, \ldots, x_n \), it would in general still not be a local ring in the canonical variables \( x_1 \equiv \phi_1, x_2, \ldots, x_n \).

In short, for any fusion algebra of a rational conformal field theory there exists a presentation of the form
\[
\mathcal{A} \cong \mathbb{C}[x]/dV(x) \quad (6.12)
\]
analogous to (6.6). It should, however, be realized that the potential \( V \) appearing here is typically far from being unique. Namely, first, for a generic fusion rule algebra the space of elements \( \tilde{x} \) for which all eigenvalues of \( \pi_{\text{reg}}(\tilde{x}) \) are distinct is of dimension larger than one. Second, for a fixed choice of \( \tilde{x} \) with this property, typically several coefficients \( a_j \) in the expansion (6.7) are non-vanishing, so that instead of trading \( \tilde{x} \) for \( \phi_1 \), one could trade it for other elements of the canonical basis as well. With respect to both of these ambiguities, different choices will usually lead to different presentations of the fusion ring. As a consequence, \( V \) must merely be considered as a condensed description of the fusion ring, and usually should not be expected to possess any independent meaning (such as, for instance, as the Landau–Ginzburg potential of some lagrangian field theory). But perhaps the situation is special if the presentation as a local algebra is in terms of canonical generators of the fusion ring (i.e. if \( \tilde{x}_1 = x_1 \) in the computations above). The latter situation is realized for the \( A_r \) \[4\] and the \( C_r \) \[5, 6\] WZW theories (see section 7 for some details on the \( A_r \) fusion rings), and – as has been advocated \[89\], based on the relation \[49\] between the canonically quantized three-dimensional Chern–Simons gauge theory and WZW models – persists for all other WZW theories as well. Also \[86\], for the \( c < 1 \)
unitary minimal models, there exists a presentation of the fusion ring as a local ring in two canonical generators (namely, \(x_1 = \phi_{(1,2)}\) and \(x_2 = \phi_{(2,1)}\) in the notation of [9]).

To conclude this subsection, let me point out that, although for a local ring the complete information on the ring structure is encoded in the single function \(V(x)\), this information is generically not sufficient to reconstruct the fusion ring in its canonical basis \(\{\phi_i\}\).

### 6.3 One-variable fusion potentials

A special class of local rings are those derived from the free polynomial ring in a single variable \(x\), for which the integrability condition is trivial. The present subsection is devoted to this particular situation. Thus assume that there exists an element \(\phi\) of a canonical basis together with \(|I|\) polynomials \(p_j(\phi)\) that are linearly independent over \(\mathbb{C}\), such that

\[
\phi_i = p_i(\phi)
\]

for all \(i \in I\). In particular, \(p_0(\phi) = 1\), and, numbering the generators such that \(\phi_1 = \phi\), the first polynomial is \(p_1(\phi) = \phi\). By evaluation of (6.13) in the regular representation, one has

\[
N_i = p_i(N_1),
\]

which implies that the fusion rule eigenvalues obey

\[
\nu_j^{(i)} = p_i(\nu_j^{(1)}).
\]

Moreover, because of the vanishing condition (6.3) the ideal \(\mathcal{J}\) in (6.4) is generated by

\[
\mathcal{P}(\phi) := \prod_j (\phi - \nu_j^{(1)}).
\]

Without changing the ring structure, the product on the right hand side may be restricted to those \(j \in I\) that correspond to distinct eigenvalues \(\nu_j^{(1)}\) of \(N_1\). But if the number of these were smaller than \(|I|\), then the assumption that in \(\mathbb{Z}[x]/\mathcal{J}\) there exist \(|I|\) independent polynomials \(p_i\) could not be fulfilled. One concludes [11] that all eigenvalues of \(N_1\) are distinct, i.e. non-degenerate (as a consequence, the information contained in \(N_1\) is sufficient to fix uniquely the eigenvectors, and hence the diagonalization matrix \(V\) via (2.4)).
In the above, it was assumed that the coefficients of the polynomials \( p_i \) take values in \( \mathbb{Z} \). Note, however, that it follows already from (6.14) (along with the fact that \( N_{ij}^k \in \mathbb{Z} \)) that the coefficients of the \( p_i \) are rational. One can show that the fact that the eigenvalues of \( N_1 \) are non-degenerate is not only necessary, but also sufficient for (6.13) to hold for independent polynomials with rational coefficients. Namely [91], define for any \( j \in I \) the polynomial \( p_j \) as the unique polynomial of order at most \(|I|−1\) that obeys (6.13) (and hence (6.14)). As a function of one complex variable \( \phi \), \( p_i \) then satisfies (6.13) at the points \( \phi = \nu_j^{(1)} \). If all the \(|I|\) eigenvalues \( \nu_j^{(1)} \) are distinct, this implies that (6.13) is true for any value of \( \phi \).

There exist conformal field theories for which all of the fusion matrices possess degenerate eigenvalues, e.g. the unitary minimal models (except the Ising model). But even for such theories it is still possible that the relation (6.13) is fulfilled, albeit with coefficients in an algebraic extension of \( \mathbb{Q} \) by some of the fusion eigenvalues (this happens for instance for all the minimal conformal models [91]). As a consequence, while not being independent over \( \mathbb{C} \), the polynomials \( p_i \) are still independent over \( \mathbb{Q} \). This makes such polynomial presentations again interesting, as independence over \( \mathbb{Q} \) is all that is needed for the presentation to be faithful, and hence for a reconstruction of the fusion rules from the potential \( V \) and from the polynomials \( p_i \).

To see how such polynomial presentations arise, fix some \( x \in \mathcal{A} \), and denote by \( \mu_j, j = 0, 1, ..., N \) (\( N \leq |I|−1 \)) the distinct eigenvalues of \( \pi_{\text{reg}}(x) \). Since \( \pi_{\text{reg}}(x) \) is linear combination of the matrices \( N_j \), its eigenvectors \( v_i \) coincide with those appearing in (2.3). Having interpreted the generators of \( \mathcal{A} \) as complex variables, the eigenvectors \( v_i \) may be formally considered as elements of \( \mathcal{A} \). Doing so, one finds that

\[
v_i = \sum_{j \in I} Y_{ij}^* \phi_j, \tag{6.17}
\]

since by use of the fusion rules this correctly reproduces the eigenvalue equation in the form \( \phi_l * v_j = (Y_{jl}/Y_{00}) v_j \). According to (6.17), the eigenvectors \( v_i \) correspond up to normalization to the minimal idempotents (2.17) of \( \mathcal{A} \). In particular,

\[
v_i * v_j = \delta_{ij} v_i/Y_{00}. \tag{6.18}
\]

Assume now that there exists a polynomial presentation of \( \mathcal{A} \), as an algebra over \( \mathbb{C} \), in terms of the element \( x \). Then [86] the constraint to be imposed on the free algebra \( \mathbb{C}[x] \) must be the minimal polynomial of \( \pi_{\text{reg}}(x) \),
\[ V'(x) = \prod_{j=0}^{N} (x - \mu_j). \]  

(6.19)

By assumption, in particular each of the elements (6.17) must be representable as a polynomial in \( x \). Ordering the eigenvectors in such a manner that \( v_i \) has eigenvalue \( \mu_i \) for \( i = 0, 1, \ldots, N \), the eigenvalue equations of these elements read \( (\pi_{\text{reg}}(x) - \mu_i \mathbb{1}) \cdot v_i(\pi_{\text{reg}}(x)) = 0 \), which by comparison with (6.19) implies

\[ v_i(x) \propto \prod_{\substack{l=0 \atop l \neq i}}^{N} \frac{x - \mu_l}{\mu_i - \mu_l}. \]  

(6.20)

Inserting this result into (6.18), it follows by specializing to \( i = j \) and \( x = \mu_i \) that the constant of proportionality must be either zero or equal to \( 1/Y_{i0} \). Moreover, using (6.18) with \( i \neq j \) along with \( \sum_{j \in I} Y_{ij} v_j(x) = \phi_0(x) = 1 \), one then deduces that in fact

\[ v_i(x) = (Y_{i0})^{-1} \prod_{\substack{l=0 \atop l \neq i}}^{N} \frac{x - \mu_l}{\mu_i - \mu_l} \quad \text{for} \quad i = 0, 1, \ldots, N, \]  

(6.21)

as well as

\[ v_i(x) \equiv 0 \quad \text{for} \quad i = N + 1, N + 2, \ldots, |I| - 1. \]  

(6.22)

Thus the polynomial representation of the generators reads

\[ \phi_i(x) = \sum_{j \in I} Y_{ij} v_j(x) = \sum_{j=0}^{N} Y_{ij} \prod_{\substack{l=0 \atop l \neq j}}^{N} \frac{x - \mu_l}{\mu_j - \mu_l}. \]  

(6.23)

It remains to be checked whether these polynomials are independent over \( \mathbb{C} \) or at least over \( \mathbb{Q} \), i.e. whether there exists a linear combination

\[ \tilde{\phi}(x) = \sum_{i \in I} a_i \phi_i(x) = \sum_{i,j \in I} a_i Y_{ij} v_j(x) \]  

(6.24)

that vanishes identically. Clearly, for arbitrary coefficients \( a_i \),

\[ \tilde{\phi} = \sum_{i=0}^{|I|-1} a_i \sum_{j=N+1}^{|I|-1} Y_{ij} v_j =: \sum_{j=N+1}^{|I|-1} \tilde{a}_j v_j \]  

(6.25)
satisfies this constraint, and from (6.19) and (6.21) it follows that all solutions are of this form. Thus, in agreement with the result above, independence over \( \mathbb{C} \) requires that \( N = |I| - 1 \). Concerning independence over \( \mathbb{Q} \), one must investigate whether there exist complex numbers \( \tilde{a}_j, j = N + 1, N + 2, \ldots, |I| - 1 \), for which \( a_i = \sum_{j=N+1}^{N+|I|-1} Y^*_{ij} \tilde{a}_j \) is rational for any \( i \in I \). So far no general answer to this question is available. For any given fusion ring, one may of course use the explicit form of the matrix \( Y \) to analyze the problem. A few theories for which in this manner one can show the absence of rational solutions for the \( a_i \), and hence independence of the polynomials over \( \mathbb{Q} \), are described in [86].

### 6.4 Quasihomogeneous polynomials

Another special class of local rings are those for which the potential \( V \) is a quasihomogeneous polynomial, i.e. satisfies

\[
V(\lambda^{m_1} x_1, \lambda^{m_2} x_2, \ldots, \lambda^{m_n} x_n) = \lambda^M V(x_1, x_2, \ldots, x_n)
\]

(6.26)

with some integers \( M \) and \( m_1, m_2, \ldots, m_n \) for any \( \lambda \in \mathbb{C} \setminus \{0\} \). The degree of quasihomogeneity provides a quantum number \( Q \) that is additively conserved under the ring product. As a consequence, the local ring of \( V \) is either infinite-dimensional, or else is nilpotent (in fact the ring is finite-dimensional only if \( V \) has an isolated singularity at \( x = 0 \)). But the existence of nilpotent elements is not allowed in a fusion rule algebra. Namely, from the properties of the unit \( \phi_0 \) and the conjugation one knows that \( \phi_i \ast \phi_j \ast \phi_i = \phi_0 + \ldots \), where the ellipsis stands for further generators, and hence \( \phi_i \ast (\phi_j \ast \phi_i) = \phi_i \ast (\phi_j \ast \phi_j) = \phi_i + \ldots \), implying in particular that the fusion product of any two generators cannot vanish, \( \phi_i \ast \phi_j \neq 0 \). To obtain the same result for arbitrary elements \( x \in \mathcal{A} \), it is most convenient to write them in the basis of minimal idempotents (2.17),

\[
x = \sum_{i \in I} a_i e_i,
\]

which shows that \( x^n = \sum_{i \in I} (a_i)^n e_i \) for any \( n \in \mathbb{Z}_{\geq 0} \) so that \( x^n \neq 0 \) unless \( x = 0 \).

Thus a local algebra corresponding to a quasihomogeneous potential does not satisfy the axioms of a fusion rule algebra. But the structure is still close enough to that of a fusion rule algebra; the axioms (F1), (F2), and (F3) hold, and it is also possible to define a conjugation \( \tilde{C} \), but \( \tilde{C} \) cannot be unital.

\[\textit{10} \text{ That this is of course only a basis of } \mathcal{A} \text{ as an algebra over } \mathbb{C} \text{ does not affect the argument.}\]

\[\textit{11} \text{ In contrast, fusion rule algebras generically do possess zero divisors. For example, } \ast = 0 \text{ if } \phi_i \text{ is a simple current of order 2.}\]
particularly interesting nilpotent element of a finite-dimensional local ring
of a quasihomogeneous potential $V$ is

$$
\phi_c := \det_{i,j}(\partial_i \partial_j V).
$$

(6.27)

$\phi_c$ is the unique element with maximal charge $Q$. Namely, if $Q$ is normalized
such that $V$ has unit charge, i.e. $Q_i = m_i/M$, then the charge of $\phi_c$ is
$1 - 2 \sum_i Q_i$; on the other hand, the polynomial $P(t) := \text{tr}_A(t^{MQ})$, known
as the Poincaré polynomial of $\mathcal{A}$, can be shown \[12\] to read $P(t) = 
\prod_i (1 - t^{M-M_i})(1 - t^{m_i})^{-1}$, so that from the limit of large $t$ one can read off
that $1 - 2 \sum_i Q_i$ is the largest allowed charge and appears with multiplicity one.

A class of conformal field theories in which such structures arise are
the $N = 2$ superconformal field theories \[12\]. In this context, the charge quantum number $Q$ is associated with the $u_1$-subalgebra of the $N = 2$
superconformal algebra; as a consequence of the structure of the $N = 2$
algebra, $Q$ is bounded by the conformal dimension as $Q \leq 2\Delta$ (assuming
that the theory is unitary). The elements of the nilpotent ring are then the
so-called chiral primary fields, i.e. those primaries for which this bound on
the charge is saturated. The ring multiplication corresponds to the operator
product in the limit of vanishing distance, which is well-defined owing to
the conservation of the $u_1$-charge under operator products. (Moreover, it
is possible to remove all non-chiral primary fields and all descendants by
an appropriate ‘twisting’ procedure; the dependence of the fields on their
position in two-dimensional space-time then becomes irrelevant, and the
theory obtained this way is topological, possessing only a finite number
of degrees of freedom \[93, 49\].) Also, the charge $1 - 2 \sum_i Q_i$ of $\phi_c$ equals
one third of the conformal central charge. Furthermore, the conjugation $\tilde{C}$
amounts to forming the fusion product with the simple current $\phi_c$, followed
by conjugation in the conformal field theory sense (an equivalent description
of this operation is the following: first apply the so-called spectral flow
operator to pass from the Neveu-Schwarz sector to the Ramond sector of
the theory, then perform conjugation in the conformal field theory sense,
and afterwards flow back to the Neveu-Schwarz sector). This implies in
particular that

$$
\tilde{C}_{i0} = \delta_{i,c}.
$$

(6.28)

Due to the similarities between the two types of rings, it seems natural to
expect that there exist fusion rings which are obtained from the local ring of
a quasihomogeneous potential by a certain perturbation. It is also plausible
to require that the deformation must not change the number of critical points (counting multiplicities) of the potential (compare \textsuperscript{[94]}). Indeed it has been shown for various special cases \textsuperscript{[95,96,97,98]} that the deformation of the quasihomogeneous polynomial $V$ of the ring of chiral primary fields of an $N = 2$ theory by the polynomial corresponding to a single chiral primary field can result in a fusion ring. Note that if the deformation is by any integral linear combination of chiral primaries, then \textsuperscript{[99]} the fusion ring will contain a simple current, namely (the image under deformation of) the element $\phi_c$.

7 Example: the $\text{sl}_n$ fusion ring

As an example for conformal field theories for which some of the issues of the previous sections can be discussed in a rather explicit manner, consider the WZW theories based on one of the simple Lie algebras $\mathfrak{g} = \text{sl}_n$, at some level $k \in \mathbb{Z}_{>0}$. Denote by $I_k := I(\text{sl}_n, k)$ the index set \textsuperscript{(5.2)}, by $c_\Lambda$ the conjugacy class of $\Lambda \in I_k$, by $\Lambda_{(j)}$, $j = 1, 2, \ldots, n - 1$, the fundamental weights of $\text{sl}_n$, and by $\Lambda_{(0)}$ the zero weight. Then the fusion rules for any fixed choice of $n$ and of the level $k$ can be characterized uniquely (up to isomorphism) as follows:

- The generators $\phi_\Lambda$ are indexed by $I_k$, and
- the fusion rules of $\phi_{\Lambda_{(j)}}$ read\footnote{Because of $(\Lambda_{(j)}, \theta) = 1$ for $j = 1, 2, \ldots, n - 1$, this is a special case of \textsuperscript{(5.9)}.}

$$
\phi_\Lambda \ast \phi_{\Lambda_{(j)}} = \bigoplus_{\Lambda' \in M_j^\Lambda} \phi_{\Lambda'}
$$

(7.1)

for $j = 0, 1, \ldots, n - 1$, where

$$
M_j^\Lambda := \{ \Lambda' \in I_k \mid c_{\Lambda'} = c_\Lambda + j; \ 0 \leq (\Lambda')^i - \Lambda^i \leq 1 \ \forall \ i = 1, 2, \ldots, n - 1 \}.
$$

(7.2)

That the $(\text{sl}_n)_k$-fusion rules indeed have these properties can easily be deduced from the formula \textsuperscript{(5.6)}; alternatively, one may derive them with Young-diagrammatic methods. That the properties are enough to specify the fusion rules uniquely can be shown by proving \textsuperscript{[100,101]} that the abstract fusion ring possessing these properties is unique. Incidentally, the commutativity of the ring is not needed for the proof, but rather can be

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deduced from the other properties. (One may also verify these properties without the help of (5.6), and hence the statement provides an independent proof of (5.6) for the case $\mathfrak{g} = \mathfrak{sl}_n$.)

Being uniquely determined, these fusion rules coincide with ring structures that appear in other areas and possess the same basic properties. Examples of such structures are provided by the truncated tensor products of the quantum groups $U_q(\mathfrak{sl}_n)$ with deformation parameter $q = \exp(2\pi i/(k + N))$ [5, 6, 7, 8], by the Littlewood-Richardson coefficients for the so-called induction product of Hecke algebras at these roots of unity [101, 102], and by the spaces of edge variables in fusion-RSOS models [103].

The $(\mathfrak{sl}_n)_k$ fusion rules possess the following further properties:

- As a ring the fusion rules are generated by $\phi_{\Lambda(j)}$ with $j = 1, 2, \ldots, n - 1$. Thus all generators $\phi_{\Lambda}$ can be expressed as polynomials over $\mathbb{Z}$ in the $n - 1$ variables $x_j \equiv \phi_{\Lambda(j)}$. The explicit formula is similar [13] to the so-called Giambelli formula [104] for $\mathfrak{sl}_n$ tensor products; e.g. for $n = 2$ the polynomials are the Chebyshev polynomials of the second kind. By combining the polynomials $\phi_{\Lambda} = \phi_{\Lambda}(x_l)$ with (7.1), one can then obtain the full set of fusion rules explicitly.

- In agreement with (5.10), the structure constants are majorized by the tensor product coefficients of $\mathfrak{sl}_n$.

- The fusion ring is isomorphic to the quotient of the tensor product ring of $\mathfrak{sl}_n$ by the ideal that is generated by $\{\phi_{\Lambda} \mid \Lambda \in I_{k+1} \setminus I_k\}$ [100].

- The fusion ring is also isomorphic to the quotient $\mathbb{Z}[x_1, x_2, \ldots, x_{n-1}] / J$ of the ring of polynomials in $n - 1$ complex variables $x_j$ by the vanishing relations that are obtained when expressing the fusion rules of the $\phi_{\Lambda(j)}$ entirely in the variables $x_l$. The two-sided ideal $J$ is generated by $\{\phi_{(k+l)\Lambda(j)} \mid l = 1, 2, \ldots, n - 1\}$ [13].

- The relations generating $J$ can be integrated, $\phi_{(k+l)\Lambda(j)}(x) = \partial V(x) / \partial x_l$, with potential $V(x) = (x_1)^{n+k} + \ldots$. The full expression of $V$ in terms of the variables $x_l$ is rather lengthy (for $n = 2$, where there is a single variable $x = x_1$, $V(x)$ is a Chebyshev polynomial of the first kind). But when defining auxiliary variables $q_i, i = 1, \ldots, n - 1$ by

$$x_i = \sum_{1 \leq j_1 < j_2 < \ldots < j_i \leq n} q_{j_1} q_{j_2} \cdots q_{j_i}, \quad (7.3)$$

$$\sum_{1 \leq j_1 < j_2 < \ldots < j_i \leq n} q_{j_1} q_{j_2} \cdots q_{j_i}, \quad (7.3)$$
along with $q_n := (\prod_{i=1}^{n-1} q_i)^{-1}$ (so that the Jacobian $\partial x/\partial q$ of the transformation from $x$ to $q$ is the Vandermonde determinant $\prod_{i<j}(q_i - q_j)$), the potential acquires the simple form

$$V(x) = V_{n,k}(x) := \frac{1}{n + k} \sum_{i=1}^{n} q_i^{n+k}. \quad (7.4)$$

The expression in terms of the $x_i$ can be reduced, by means of a $n + 1$-term recurrence relation, to the result for low values of the level. For instance, for $n = 3$ there is the recurrence relation

$$V_{3,k}(x_1, x_2) = (k + 3)^{-1}[(k + 2) x_1 V_{3,k-1} - (k + 1) x_2 V_{3,k-2} + k V_{3,k-3}], \quad (7.5)$$

by which one can compute $V_{3,k}$ from $V_{3,0} = x_1^3/3 - x_1 x_2 + 1$, $V_{3,1} = x_1^4/4 - x_1^2 x_2 + x_1 + x_2^2/2$, and $V_{3,2} = x_1^5/5 - x_1^3 x_2 + x_1 x_2^2 + x_1^2 - x_2$.

The quasihomogeneous part of the potential $V(x)$ corresponds to the ring of chiral primary fields of the $\mathcal{N} = 2$ superconformal coset theory $(\mathfrak{sl}_n+1)/\mathfrak{sl}_n \oplus u_1$, and also to the cohomology ring of the Grassmannian manifold $U(n+k)/U(n) \times U(k)$ \cite{13,99,87}.

For $n = 3$, all eigenvalues of $\mathcal{N}_{\Lambda(1)}$ are distinct, so that the fusion ring is polynomial with generator $x = x_1 \equiv \phi_{\Lambda(1)}$ \cite{11}. The corresponding polynomial is of order $4k - 1$; e.g. one has $V(x) = \frac{1}{3}x^7 - x^4 - x$ at $k = 2$, and $V(x) = \frac{1}{17}x^{11} - \frac{9}{5}x^8 + \frac{2}{5}x^5 - 4x^2$ at $k = 3$.

The $(\mathfrak{sl}_n)_k$ fusion ring is particularly simple at level one. In this case the index set is $I_1 \cong \mathbb{Z}_n$, and the fusion matrices read

$$(\mathcal{N}_1)_{jk} = \delta^{(n)}_{i+j,k}, \quad (7.6)$$

where $\delta^{(n)}_{ij}$ is 1 if $i = j \mod n$, and zero else. Thus the fusion rule algebra is isomorphic to $\mathbb{C}\mathbb{Z}_n$. For any fusion algebra with this property, the modular matrix $S$ has $n$th roots of unity as its entries,

$$S_{jk} = n^{-1/2} \exp(2\pi i m jk/n), \quad (7.7)$$

where $m$ is some integer coprime with $n$, whose value depends on the precise identification of the fields with the elements of $\mathbb{Z}_n$. Without loss of generality one can put $m = 1$; for $\mathfrak{sl}_n$ at level one, this corresponds to the natural identification $j \in \mathbb{Z}_n \equiv \phi_{\Lambda(1)}$. 

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8 Fusion graphs

To any fusion matrix $\mathcal{N}_i$ one may associate a labelled bicolored graph $\Gamma_i^{\circ}$ and a labelled directed graph $\Gamma_i$ as follows (see e.g. [24]). To get $\Gamma_i^{\circ}$, associate to any $j \in I$ a ‘white’ vertex $w_j$ and a ‘black’ vertex $b_j$, and connect, for all $j, k \in I$, $w_j$ with $b_k$ by $(\mathcal{N}_i)_{jk}$ edges. Similarly, for $i \neq i^+$, $\Gamma_i$ is obtained by associating to any $j \in I$ a single vertex $v_j$ and connecting, for all $j, k \in I$, $v_j$ with $v_k$ by $(\mathcal{N}_i)_{jk}$ directed edges; on the other hand, if $i = i^+$ so that $\mathcal{N}_i$ is symmetric, then for any edge from $j$ to $k$ this prescription would yield precisely one edge from $k$ to $j$; therefore in this case one simply connects $v_j$ and $v_k$ by $(\mathcal{N}_i)_{jk}$ undirected edges. In short, $\Gamma_i$ is the graph whose incidence (or connectivity, or adjacency) matrix is $\mathcal{N}_i$. Also note that one can obtain $\Gamma_i$ from $\Gamma_i^{\circ}$ by identifying $w_i$ with $b_i$, and by supplementing the edges between the vertices with a direction ‘from white to black’.

These graphs serve as a compact description of the fusion matrix. Moreover, a lot of structural information on the fusion rule algebra can be read directly off the graphs. For instance, $\mathcal{N}_i$ is indecomposable iff $\Gamma_i^{\circ}$ is connected. Other examples are [105] certain properties of the Perron-Frobenius eigenvector, as well as the following [23]: For $D_i > 1$ a node in $\Gamma_i$ corresponds to a primary field that is a simple current iff it is reached by precisely one edge if $\phi_i$ is self-conjugate, respectively by precisely two edges (one incoming and one outgoing) if $\phi_i$ is non-selfconjugate.

If such a graph is isomorphic (as an unlabelled un-colored graph) to the Dynkin diagram of a simply laced simple or affine Lie algebra, the graph is conventionally denoted by the name of this algebra. For the graph corresponding to a simple Lie algebra $\mathfrak{g}$, the associated fusion rule matrix equals $2 \mathbb{I} - A$, with $A$ the Cartan matrix of $\mathfrak{g}$, and its eigenvalues are $2 \cos(\pi m_i/h)$, with $h$ the dual Coxeter number and $m_i$, $i = 1, 2, \ldots$, rank ($\mathfrak{g}$), the exponents (i.e., the orders of independent Casimir operators minus one) of $\mathfrak{g}$. Similar abbreviations as for the Dynkin diagrams can be used for other graphs, such
as
\[ \tilde{A}_r = \begin{array}{c}
\cdots \\
\end{array} \]
\[ \tilde{A}_{r-1}^{(1)} = \begin{array}{c}
\cdots \\
\end{array} \]
\[ \tilde{D}_{r-1}^{(1)} = \begin{array}{c}
\cdots \\
\end{array} \]
(8.1)
(each of these has by definition \( r \) nodes; thus, roughly, \( \tilde{A}_r = 'A_{2r}/\mathbb{Z}_2' \), \( \tilde{A}_{r-1}^{(1)} = 'A_{2r+1}/\mathbb{Z}_2' \), and \( \tilde{D}_{r-1}^{(1)} = 'D_{2r+1}/\mathbb{Z}_2' \), where \( \mathbb{Z}_2 \) refers to an appropriate automorphism of the relevant diagram). If a graph is not connected, then the notation \( \Gamma = \bigoplus_i \Gamma_i \), is used, with \( \Gamma_i \) the connected components.

Various examples of fusion graphs are given in the following list:

1. The simplest possibility for a pair of fusion graphs is given by
\[ \Gamma^\phi = \bigoplus |I| A_2, \quad \Gamma = \bigoplus |I| \tilde{A}_1. \]
(8.2)
It describes the fusion rules of the unit \( \phi_0 \); conversely, \( \phi_0 \) is uniquely characterized by having (8.2) as its fusion graphs.

2. Numbering the canonical elements that span the fusion rule algebra \( \mathbb{C} \mathbb{Z}_|I| \) according to \( \mathbb{Z}_|I| \), one has \( \Gamma_j^\phi \) as in (8.2) and \( \Gamma_j = \bigoplus |I|/p A_{p-1}^{(1)} \) for \( j = 0, 1, ..., |I| \), where \( p \geq 2 \) is defined by setting \( |I|/j = p/q \) with \( p, q \) coprime. For \( j = 0 \), this degenerates to (8.2), while for \( j \) coprime to \( |I| \), the graph is connected, \( \Gamma_j = A_{|I|-1}^{(1)} \).

3. One has \( \Gamma_j^\phi = \bigoplus |I| A_2 \) iff \( \phi_j \) is a simple current. Also, if \( \phi_j \) is a simple current, then \( \Gamma_j = (\tilde{A}_1)^{m} \oplus (A_2)^{n} \oplus \bigoplus_j A_{l_j}^{(1)} \) for some integers satisfying \( l_j \geq 2 \) and \( m + 2n + \sum_j (l_j + 1) = |I| \).

4. Consider the fusion matrix
\[ N_1 = \begin{pmatrix} 0 & 1 \\
1 & 1 \end{pmatrix} ; \]
(8.3)
this defines (along with, of course, \( N_0 = \mathbb{1} \)) the so-called the Lee-Yang fusion rules (these already appeared in (1.18) above). The associated
graphs are $\Gamma_1^\oplus = A_4$ and $\Gamma_1 = \bar{A}_2$.

Similarly, for

$$N_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (8.4)$$

known as the Ising fusion rules, the fusion graphs are $\Gamma_1^\oplus = A_3 \oplus A_3$, $\Gamma_1 = A_3$, and $\Gamma_1^\oplus = A_2 \oplus A_2 \oplus A_2$, $\Gamma_2 = \bar{A}_1 \oplus A_2$, respectively.

5. For $(\mathfrak{sl}_n)_k$, $\Gamma_{\Lambda(1)}$ is a graph obtained by ‘filling up’ an $(n+1)$-gon of edge length $k+1$ in an appropriate manner; the corners correspond to the simple currents. For $n = 2$, one simply has $\Gamma_{\Lambda(1)} = A_{k+1}$.

For $n = 3$, one has to fill the triangle of length $k+1$ with triangles of length one, leading to $\Gamma_{\Lambda(1)} = A_{2}^{(1)}$ for $k = 1$, and to

for the general case. Here the labelling of most of the nodes has been suppressed; it can easily be restored. In particular one has

$$2\Lambda_{(2)}$$

$$\Lambda_{(2)}$$

$\Lambda_{(1)} + \Lambda_{(2)}$
For \( n = 4 \), the situation is already a lot more complicated; e.g. at level two, the graph \( \Gamma_{\Lambda(1)} \) is given by

\[
\begin{align*}
&0 \quad \Lambda_1 \quad 2\Lambda_1 \\
&\Lambda_1 \quad + \Lambda_1 \quad \Lambda_2 + \Lambda_3 \\
&2\Lambda_1 \quad \Lambda_1 + \Lambda_2 \quad 2\Lambda_2
\end{align*}
\]

(8.7)

6. The fusion graphs of the \((B_r)_2\) WZW theory are: \( \Gamma_{2\Lambda(3)} = A_1^{(1)} \oplus A_1^{(1)} \oplus (\tilde{A}_1)^{\oplus r} \); \( \Gamma_{\Lambda(i)} \), \( i = 1, 2, \ldots, r-1 \), and \( \Gamma_{2\Lambda(r)} \) are all given by \( A_1^{(1)} \oplus D_r^{(1)} \); finally, \( \Gamma_{\Lambda(r)} \) looks like

\[
\begin{align*}
&2\Lambda(r) \\
&\Lambda_1 \\
&0 \quad \Lambda(r) \quad 2\Lambda_1 \\
&\Lambda(r) \quad + \Lambda(r) \\
&\Lambda(r-1)
\end{align*}
\]

(8.8)

and \( \Gamma_{\Lambda(1) + \Lambda(r)} \) is given by [8.8] with the labelling of the left- and right-most nodes interchanged.

7. Finally note that, of course, the fact that \( \Gamma_i^\circ = \Gamma_j^\circ \) (as unlabelled graphs) does not imply that \( \Gamma_i = \Gamma_j \), too. In particular, the connectivity of \( \Gamma_i \) is not fixed by \( \Gamma_i^\circ \). For instance, for the \((A_2)_2\) WZW
theory, one has $\Gamma_{\Lambda(1)} = A_4 \oplus A_4 \oplus A_4 = \Gamma_{\Lambda(1) + \Lambda(2)}$, but $\Gamma_{\Lambda(1)}$ is given by the graph above, whereas $\Gamma_{\Lambda(1) + \Lambda(2)} = A_2 \oplus A_2 \oplus A_2$.

Clearly, as soon as some fusion rule coefficients become large, the fusion graphs defined according to the above prescription become more and more unhandy. A modest improvement is achieved by modifying the prescription in the following manner. For self-conjugate $\phi_i$, join the vertices $j$ and $k$ by a single edge iff $N_{ij}^k \neq 0$, and attach the number $N_{ij}^k$ as a label to any edge with $N_{ij}^k > 1$, and similarly in the non-selfconjugate case.

9 Classification

A complete classification of fusion rule algebras (up to isomorphism) is far beyond reach. This problem may well be as hard as, say, the problem of classifying all discrete groups. In the modular rational case, however, several strategies which provide insight into the classification do exist.

1. Enumeration

The most direct approach is certainly to translate, for a given index set $I$, the various constraints provided by (F1) to (F4) and by (R) into a parametrization of the most general solution. This approach has been followed in [90] (see also [106, 107]). The complexity of the system of equations grows rapidly with the dimensionality $|I|$ of the algebra. A solution which exists for any $|I|$ is given by $\mathbb{C}Z_{|I|}$, as has already been observed in (7.6) above. More generally, for any finite abelian group $G$, $\mathbb{C}G$ is a rational fusion algebra. An enumeration of all solutions is easy for $|I| = 2$ and $|I| = 3$, but already for $|I| = 4$, this enumeration scheme becomes rather non-trivial so that only some partial results [90] are known.

For $|I| = 2$, besides $\mathbb{C}Z_2$ the only possibility is given by the Lee-Yang fusion rules (8.3). For $|I| = 3$, there are two solutions besides $\mathbb{C}Z_3$; one of them describes the Ising fusion rules (8.4), and the other one is

$$N_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (9.1)$$

Both of these are polynomial fusion rule algebras: they satisfy $\phi_2 = P_2(\phi_1) = \phi_1^2 - 1$, and the fusion potential is given by $V(\phi_1) = \frac{1}{2}\phi_1^4 - \phi_1^2$ and $V(\phi_1) =$
\( \frac{1}{3} \phi^1 - \frac{1}{3} \phi^2 - \phi_1^2 + \phi_1 \), respectively. The fusion dimensions are \( \mathcal{D}_1 = \sqrt{2}, \mathcal{D}_2 = 1 \) in the Ising case, and \( \mathcal{D}_1 = 2 \cos(\pi/7), \mathcal{D}_2 = 1 + 2 \cos(2\pi/7) \) for (9.1).

Many of the fusion rule algebras mentioned so far possess a realization as the fusion rules of some WZW theory. For example, according to section 7, a WZW theory realizing \( \mathbb{C} \mathbb{Z}_{|I|} \) is \( (s\mathbb{l}_{|I|})_1 \equiv (A_{|I|-1})_1 \); the Lee-Yang fusion rules are realized by \( (G_2)_1 \) and \( (F_4)_1 \), and the Ising fusion rules by \( (B_r)_1 \) for any rank \( r \) and by \( (E_8)_2 \).

Another approach to enumeration is to simply list the fusion rule algebras that arise in certain classes of conformal field theories for which they are calculable by some algorithm, say in WZW theories. While this is usually not very illuminating, it can nevertheless be helpful in specific circumstances. For example, with this procedure one may encounter theories with identical or closely related fusion rules, which in turn suggests that other quantities for the corresponding conformal field theories (such as the modular matrix \( S \) or correlation functions) are closely related as well. Let me present a few examples. First, as just mentioned, the fusion rule algebras of the WZW theories \( (F_4)_1 \) and \( (G_2)_1 \) coincide, and the same is true for the \( (B_r)_1 \) and \( (E_8)_2 \) WZW theories. Similarly, \( (F_4)_2 \) has the same fusion rules as \( (G_2)_4 \), and \( (E_8)_2 \) the same as \( (F_4)_2 \). There also exists an infinite series of such correspondences, namely \[ (9.2) \]

\[ \mathcal{A}((C_r)_k) \cong \mathcal{A}((C_k)_r) ; \]

this is known as level-rank duality. Further, even more often one arrives at identical structures after ‘modding out simple currents’, i.e. taking only one (specific) representative of each simple current orbit (the physical interpretation of this modding procedure still has to be clarified); for instance, one has \[ (9.3) \]

\[ \mathcal{A}((A_r)_k)/\mathbb{Z}_{r+1} \cong \mathcal{A}((A_{k-1})_{r+1})/\mathbb{Z}_{k} . \]

The isomorphisms among fusion algebras of WZW theories translate to similar relations for conformal field theories obtained from them via the coset construction. In some cases they can be used to show that some a priori different coset theories are actually identical as conformal field theories \[ \text{[108]} \text{[109]} \text{[110]} \text{[111]} \]

\[ \mathcal{A}((A_r)_k)/\mathbb{Z}_{r+1} \cong \mathcal{A}((A_{k-1})_{r+1})/\mathbb{Z}_{k} . \]

\[ \text{[112]} \text{[113]} \]

\[ ^{13} \text{At least modulo the ambiguities that can arise (see section 10 below) in the calculation of the operator product algebra from the fusion rules.} \]
2. Enumeration of characters

It can be shown that the conformal weights and conformal central charge of a rational conformal field theory are rational numbers \[114\], implying that the modular matrix \( T \) obeys \( T^m = 1 \) for some integer \( m \). Accordingly, it is expected that for any rational conformal field theory the relevant representation of the modular group factorizes through a finite index normal subgroup \( \Sigma \) of \( \text{PSL}(2, \mathbb{Z}) \), i.e. the characters are invariant under \( \Sigma \) so that the action of modular transformations on the characters generates the finite group \( \text{PSL}(2, \mathbb{Z})/\Sigma \) \[115,116\]. Accordingly, one may enumerate the possible forms of the modular matrices \( S \) and \( T \) by first making a list of all finite groups that possess a representation of dimension \( M \) equal to the number of distinct characters of the theory (which due to (4.4) is smaller than \( |I| \) if some fields are non-selfconjugate), then for each such group find the representation matrix for \( S \), and then calculate the fusion rules from the Verlinde formula \[116\]. The first part of this programme is again involved already for modestly large \( m \) (the full list of relevant finite groups is known only for \( M = 2 \) and \( M = 3 \) \[117\]). Further, for all rational conformal field theories for which the characters are known explicitly, one has in fact \( \Sigma = \text{SL}(2, \mathbb{Z}_m) \), so that one may employ \[115,118\] the representation theory of \( \text{SL}(2, \mathbb{Z}_m) \) to describe the structure of the associated fusion rule algebras.

3. Subalgebras

Given some fusion rule algebra \( \mathcal{A} \), various further fusion rule algebras are provided by its fusion rule subalgebras, among which one may hope to find some that were not known previously. In view of the poor prospects of the enumeration strategies, this is not an extremely attractive approach. But at least there exists a simple finite algorithm that provides all fusion rule subalgebras of a rational fusion rule algebra \[26\]. Namely, to any subset \( J \subseteq I \) one can associate a subset \( J^\infty \subseteq I \) such that \( \{ \phi_i \mid i \in J^\infty \} \) spans a fusion rule subalgebra, and this exhausts the collection of fusion rule subalgebras of \( \mathcal{A} \). The set \( J^\infty \) is determined as follows \[26\]. For \( J \subseteq I \) define \( J^+ := \{ j \in I \mid j^+ \in J \} \), and for any pair of subsets \( J, J' \subseteq I \) define \( J \ast J' := \{ j'' \in I \mid \sum_{j \in J, j' \in J'} N_{jj'}^{j''} \neq 0 \} \); write \( J \ast J \) as \( J^2 \), and recursively \( J^m := J^{m-1} \ast J \). Then

\[
J^\infty = \bigcup_{m,n \in \mathbb{Z}_{\geq 0}} J^m \ast (J^+)^n. \tag{9.4}
\]

A special example of this construction is given by the set

\[
J_o = \{ j \in I \mid D_j = 1 \} \tag{9.5}
\]
of simple currents. By elementary properties of simple currents, one has $J^+_o = J_o$ and $J^2_o = J_o$, and hence $J^{\infty}_o = J_o$. The associated fusion rule subalgebra is simply $\mathbb{C}J_o$, with $J_o$ regarded as an abelian group (with the multiplication $i \star j$ induced by the fusion product, and the inverse $i^{-1} = i^+$ induced by conjugation). In particular, $J_o \cong \mathbb{Z}_{|J_o|}$ if $|J_o|$ is not divisible by a square number.

Another simple example of subalgebras arises in connection with the tensor product of fusion rule algebras. By definition, the tensor product (also called crossed product in [26]) $A = A_1 \times A_2$ of two fusion rule algebras $A_1$ and $A_2$ with index sets $I_1$ and $I_2$ is the algebra with canonical basis

$$\{ \Phi_{ij} \mid i \in I_1, j \in I_2 \} = \{(i \otimes j)\}, \quad (9.6)$$

and product

$$(i \otimes j) \star (i' \otimes j') = ((i \star_1 i') \otimes (j \star_2 j')). \quad (9.7)$$

Clearly, this describes again a fusion rule algebra, with unit $1 = (0 \otimes 0)$ and conjugation $(i \otimes j)^+ = (i^{+1} \otimes j^{+2})$, and $A_1$ and $A_2$ are fusion rule subalgebras of $A$.

One may identify the sets $I_1$ with $\{(i \otimes 0) \mid i \in I_1\}$ and $I_2$ with $\{(0 \otimes j) \mid j \in I_2\}$, thereby considering them as subsets of the index set $I$ of $A$; they are then characterized by

$$I^+_1 = I_1 = I_1 \star I_1, \quad I^+_2 = I_2 = I_2 \star I_2, \quad I_1 \cap I_2 = \{0\}, \quad (9.8)$$

and by the property that any $i \in I \setminus (I_1 \cup I_2)$ obeys

$$\{i\} = \{i_1\} \star \{i_2\} \quad (9.9)$$

for a uniquely determined pair $i_1 \in I_1, i_2 \in I_2$ of indices. Conversely, any fusion rule algebra whose index set $I$ possesses subsets $I_1$ and $I_2$ with these properties is a tensor product of smaller fusion rule algebras, and hence need not be considered separately in the classification programme.

4. Grading

A fusion rule algebra $A$ is said to be $\mathbb{Z}_n$-graded iff there exists a partition

$$I = \bigcup_{p \in \mathbb{Z}_n} K_p \quad (9.10)$$

of the index set $I$ that satisfies

$$K_0 \ni 0, \quad K_{-p} = K_p^+, \quad \text{and} \quad K_p \star K_q = K_{p+q} \quad (9.11)$$
for all \(p, q \in \mathbb{Z}_n\); the integer \(p \in \mathbb{Z}_n\) is called the grading of the elements \(\phi_i\) for which \(i \in K_p\). (Below it will usually be assumed that the maximal possible grading of \(\mathcal{A}\) has been chosen; in the trivial case \(n = 1\), i.e. \(I = K_0\), one calls \(\mathcal{A}\) ungraded.) To any \(j \in I\), one can associate a \(\mathbb{Z}_n\)-gradation of the fusion rule subalgebra of \(\mathcal{A}\) that corresponds to the index set \(\{j\}\); this gradation is uniquely determined by the property that \(K_0 = (\{j\} \ast \{j^\perp\})^\infty\) (which implies that \(n_j \in \{1, 2\}\) if \(j\) is self-conjugate).

Given a \(\mathbb{Z}_n\)-gradation, the set \(J_0\) of simple currents obeys

\[
J_0/(J_0 \cap K_0) \cong \mathbb{Z}_m
\]

for some \(m \in \mathbb{Z}_{>0}\) that is a common divisor of \(n\) and \(|J_0|\). It can be shown that the entire algebra \(\mathcal{A}\) is already determined by its restriction to \(\bigcup_{p \in \mathbb{Z}_m} K_p\), supplemented by the fusion rules of a specific simple current that belongs to \(K_m\).

The grading of a fusion rule algebra becomes relevant to the classification problem because of the following two constructions of new fusion rule algebras from known ones. To any fusion rule algebra \(\mathcal{A}\) one can associate for any \(m \in \mathbb{Z}_{>0}\) a fusion rule algebra \(\mathcal{A}^{[m]}\), and for any \(j_0 \in J_0(\mathcal{A})\) a fusion rule algebra \(\mathcal{A}^{\{j_0\}}\). A canonical basis of \(\mathcal{A}^{[m]}\) is

\[
\{(\phi_i, p) \mid i \in I, p \in \mathbb{Z}_m\},
\]

while the canonical basis of \(\mathcal{A}^{\{j_0\}}\) coincides with the canonical basis of \(\mathcal{A}\). The fusion product of \(\mathcal{A}^{[m]}\) is defined by

\[
(\phi_i, p) \star_{[m]} (\phi_j, q) = (\phi_i \star \phi_j, p + q + r_{ij}),
\]

and the fusion product of \(\mathcal{A}^{\{j_0\}}\) by

\[
\phi_i \star_{\{j_0\}} \phi_j = (\phi_{j_0})^{r_{ij}} \star \phi_i \star \phi_j
\]

(and similarly for the conjugation), where \(r_{ij}\) is an integer determined by the grading of \(\phi_i\) and \(\phi_j\). Of course, these procedures can be iterated; one has

\[
(\mathcal{A}^{\{j_0\}})^{\{k_0\}} \cong \mathcal{A}^{\{j_0 \ast k_0\}}, \quad (\mathcal{A}^{[m]})^{[n]} \cong \mathcal{A}^{[mn]}, \quad (\mathcal{A}^{\{j_0\}})^{[m]} \cong (\mathcal{A}^{[m]})^{\{j_0\}^m}.
\]

In addition to generating new fusion rule algebras from known ones, these constructions allow conversely to reduce the classification of all fusion rule algebras to the classification of those with certain specific properties. For
instance [26], it is sufficient to restrict to even-graded fusion rule algebras, because any odd-graded algebra $\tilde{\mathcal{A}}$ appears in the list of even-graded ones as $\tilde{\mathcal{A}}[2] = \mathcal{A}$ with $\tilde{\mathcal{A}}$ such that there exists a $j_0 \in J_0(\mathcal{A})$ with $j_0 \star j_0 = 0$ and $j_0 \notin K_0$.

5. Polynomial fusion rule algebras

Sometimes progress can be made by restricting to a subclass of fusion rule algebras that satisfy some further axiom. A class for which some rather general results are available is given by the polynomial fusion rule algebras satisfying the polynomiality condition (6.13) [90]. Note that a fusion rule algebra is polynomial iff \{j\} = I for some $j \in I$. Also, if $\mathcal{A}$ is polynomial with generator $\phi = \phi_1$, the fusion graph $\Gamma(N_1)$ is connected.

Restricting to self-conjugate generator $\phi_1$, all fusion rule eigenvalues $\nu_j^{(1)}$ are real and distinct. Therefore the polynomials $p_i$ in (6.13), considered as functions over the real numbers, are orthogonal with respect to some positive definite measure, and hence satisfy a three-term recurrence relation. For the fusion matrices this implies that they are tridiagonal, i.e. with appropriate labelling they obey $(N_i)_{jk} = 0$ for $|i - j| \geq 2$. It then follows that the full information about a self-conjugate polynomial fusion rule algebra is contained in the $2|I| - 3$ integers $N_{1ii}$ with $i \in \{1, \ldots, |I| - 1\}$, and $N_{1i+1}$ with $i \in \{2, \ldots, |I| - 1\}$. These are still further restricted by associativity and by the recurrence relation just mentioned. If all of them are zero or one, then a complete classification has been obtained [90], while for the general case there are only partial results.

6. Small fusion dimensions.

A complete classification is available for (not necessarily rational) polynomial fusion rule algebras whose generator $\phi = \phi_1$ has fusion dimension $D_1 \leq 2$. It is based on the important mathematical result [25] that the only connected finite bicolorable graphs whose incidence matrices have Perron-Frobenius eigenvalue smaller than two, are

$$A_r, \quad D_r, \quad E_r,$$

i.e. the Dynkin diagrams of the simply-laced simple Lie algebras, and that those for which the Perron-Frobenius eigenvalue equals two, are

$$A_r^{(1)}, \quad r \geq 2, \quad D_r^{(1)}, \quad E_r^{(1)},$$

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i.e. the Dynkin diagrams of the simply-laced non-simple affine Lie algebras except $A_1^{(1)}$.

Using this result, the following classification of polynomial fusion rule algebras $\mathcal{A}$ with generator $\phi_1$ has been obtained in [26]:

- If $\mathcal{A}$ is $\mathbb{Z}_2$-graded, and $\phi_1$ is self-conjugate with fusion dimension $D_1 < 2$, then $\Gamma(N_1)$ is one of the Dynkin diagrams
  \[ A_r, \ r \geq 2, \quad D_{2r}, \quad E_6, \quad E_8, \quad (9.19) \]
  and each of these possibilities corresponds to precisely one fusion rule algebra.
  For $\Gamma(N_1) = A_r$, the fusion rules are those of the $A_1$ WZW theory at level $r - 1$, with $\phi_1$ corresponding to the defining representation of $A_1$. In particular, the conjugation is trivial, $J_\circ \cong \mathbb{Z}_2$, and $D_1 = 2 \cos(\pi/(r + 1))$.
  For $\Gamma(N_1) = D_{2r}$, one has $D_1 = 2 \cos(\pi/(4r - 2))$ and $J_\circ \cong \mathbb{Z}_3$ for $r = 2$, $J_\circ \cong \{1\}$ else; the conjugation is trivial for odd $r$, while for even $r$ it interchanges the generators that correspond to the end points of the two short legs of the diagram $D_{2r}$.
  For $\Gamma(N_1) = E_6$, one has $D_1 = 2 \cos(\pi/12) = (\sqrt{3} + 1)/\sqrt{2}$ and $J_\circ \cong \mathbb{Z}_2$, and the conjugation is trivial. Finally, for $\Gamma(N_1) = E_8$, one has $D_1 = 2 \cos(\pi/30) = \sqrt{3}(\sqrt{5} + 1) + \sqrt{2}\sqrt{5} - \sqrt{5}$ and $J_\circ \cong \{1\}$, and the conjugation is again trivial.

- $(E_7$ and $D_{2r+1}$ do not appear in the list $(9.19)$. It can be shown directly that such graphs do not provide a consistent fusion rule algebra; for instance, having $\Gamma(N) = E_7$ would imply that one of the generators would have fusion dimension $2 \cos(5\pi/18) \simeq 1.28$, which is not contained in the allowed range $(2.13)$ [119, 120].

- If $\mathcal{A}$ is $\mathbb{Z}_2$-graded, and $\phi_1$ is non-selfconjugate with fusion dimension $D_1 < 2$, then the fusion rule algebra is one of
  \[ (A_{2r+1})^{(j_0)}, \ r \geq 2, \quad (E_6)^{(j_0)}, \quad (9.20) \]
  where $A_{2r+1}$ and $E_6$ stand for the fusion rule algebras that in the manner just described are associated to the respective Dynkin diagrams, and where $j_0$ corresponds to the non-trivial simple current of these algebras.
If $\mathcal{A}$ is ungraded, and $D_1 < 2$, then $\mathcal{A}$ is the fusion rule subalgebra spanned by the 0-graded generators of the fusion rule algebra corresponding to the $A_{2r}$ Dynkin diagram for some $r \geq 2$. The associated fusion graph is

$$\bar{A}_r, \ r \geq 2,$$

(9.21)

the fusion dimension is $D_1 = 2 \cos(\pi/(2r + 1))$, and the conjugation is trivial.

If $\mathcal{A}$ is $\mathbb{Z}_2$-graded, and $\phi_1$ is self-conjugate with fusion dimension $D_1 = 2$, then $\Gamma(\mathcal{N}_1)$ is one of the Dynkin diagrams

$$A_\infty, \ D_\infty, \ D_r^{(1)}, \ E_r^{(1)};$$

for $D_r^{(1)}$, there exist precisely two inequivalent fusion rule algebras, while for the other possibilities the fusion rule algebra is determined uniquely. For each of these algebras, all fusion dimensions are integral.

The fusion ring corresponding to the diagram $A_\infty$ (i.e. the graph obtained from the Dynkin diagram $A_r$ in the limit $r \to \infty$) is the fusion ring of the $A_1$ WZW theory in the limit of infinite level, i.e. the representation ring of the $A_1$ Lie algebra. In particular, the fusion dimension is just equal to the ordinary dimension of the relevant $A_1$-representation.

For $\Gamma(\mathcal{N}_1) = D_\infty$ (i.e. the graph obtained from the Dynkin diagram $D_r$ by extending the long leg infinitely), there is one simple current besides 1, and all other generators have $D = 2$. For the two algebras with $\Gamma(\mathcal{N}_1) = D_r^{(1)}$, one has $J_0 \cong \mathbb{Z}_4$ and $J_0 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, respectively, and again all remaining generators have fusion dimension 2.

For $\Gamma(\mathcal{N}_1) = E_6^{(1)}$, the fusion dimensions are as displayed in the following picture:

![Fusion Ring Diagram](9.23)

The fusion rules of the generators with $D \in \{1, 2\}$ follow from the $\mathbb{Z}_3$-grading of the algebra that corresponds to the $\mathbb{Z}_3$-symmetry of the
diagram, while for the field $\phi_3$ with fusion dimension 3, $\phi_3 \star \phi_3$ contains each of the simple currents once and $\phi_3$ twice.

- If $\mathcal{A}$ is $\mathbb{Z}_2$-graded, and $\phi_1$ is non-selfconjugate with fusion dimension $\mathcal{D}_1 = 2$, then $\Gamma(\mathcal{N}_1)$ is one of

$$E_6^{(1)}, \quad (E_7^{(1)})^{(j_0)}, \quad (D_4^{(1)})^{(j_0)}, \quad (9.24)$$

and if $\mathcal{A}$ is ungraded, with $\phi_1$ non-selfconjugate and $\mathcal{D}_1 = 2$, then $\Gamma(\mathcal{N}_1)$ is one of

$$\bar{D}_r^{(1)}, \quad r \geq 3. \quad (9.25)$$

Each of these corresponds to a unique fusion rule algebra. The algebra with $\Gamma(\mathcal{N}_1) = \bar{D}_r$ is the fusion rule subalgebra spanned by the 0-graded generators of one of the two algebras corresponding to the graph $D_{2r-1}$.

- If $\mathcal{D}_1 \leq 2$ and $\mathcal{A}$ is $\mathbb{Z}_n$-graded with $n > 2$, then there is a rather long list of possibilities for $\Gamma(\mathcal{N}_1)$ [26, Theorem 3.4.11]. All of the corresponding fusion rule algebras can be obtained with the help of the constructions $\mathcal{A} \mapsto \mathcal{A}^{[m]}$ and $\mathcal{A} \mapsto \mathcal{A}^{(j_0)}$ that were described above from the algebras appearing in the previous classifications (9.19) to (9.25), supplemented by two other series of algebras. One of the latter series corresponds to the graphs

$$\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\end{array} \quad (9.26)$$

while the graphs for the other series look much more complicated.

Non-polynomial fusion rule algebras are not directly accessible to the methods of [26]. However, for WZW theories the list of all primary fields with $\mathcal{D} \leq 2$ is known [27, 121], so that one can identify the corresponding fusion
rule algebras by inspection. The fusion graphs for WZW primaries $\phi_\Lambda$ with fusion dimension $D = 2\cos(\pi/m)$ with $m \in \mathbb{Z}_{\geq 4}$ are as displayed in the following table:

| $g$   | $k$ | $\Lambda$ | $m$ | $\Gamma_\Lambda$ |
|-------|-----|-----------|-----|------------------|
| $A_{r-1}$ | 2   | $\Lambda_{(r)}$ | $r + 2$ | $A_{r+1}$ |
| $C_r$   | 1   | $\Lambda_{(1)}$ |       |                 |
| $A_1$   | $r$ |           |       |                 |
| $B_r$   | 1   | $\Lambda_{(r)}$ | 4   | $A_3$          |
| $E_8$   | 2   | $\Lambda_{(1)}$ |       |                 |
| $F_4$   | 1   | $\Lambda_{(4)}$ | 5   | $\bar{A}_2$    |
| $G_2$   | 1   | $\Lambda_{(2)}$ |       | $\bar{A}_4$    |
| $G_2$   | 2   | $\Lambda_{(1)}$ | 9   |                 |
| $F_4$   | 2   | $\Lambda_{(1)}$ | 11  | $\bar{A}_5$    |
| $E_8$   | 3   | $\Lambda_{(8)}$ |       |                 |
| $F_4$   | 2   | $\Lambda_{(7)}$ | 4   | $A_3 \oplus A_3$ |
| $E_7$   | 2   | $\Lambda_{(1)}$ | 5   | $\bar{A}_2 \oplus \bar{A}_2 \oplus \bar{A}_2$ |
| $E_6$   | 2   | $\Lambda_{(2)}$ | 7   | $E_9$          |

Note that if $\Gamma_\Lambda$ is connected, then the fusion rule algebra is polynomial, and hence is one of the algebras described in the previous list. For theories containing simple currents, only one representative $\phi_\Lambda$ of a simple current orbit is written in column 3 of the table (9.27); also, the graph appearing in
The fusion graphs of WZW primaries with $\mathcal{D} = 2$ are shown in the next table – with the exception of the fields $\phi_{\Lambda(j)}$, with $j = 2, 3, \ldots, [r/2]$, of the $(D_r)_2$ theory. For the latter, the fusion graphs are disconnected, with connected components of the type $A_r^{(1)}$, $D_r^{(1)}$, $\bar{A}_r^{(1)}$, and/or $\bar{D}_r^{(1)}$, but the systematics of the decomposition into these connected components is somewhat complicated.

| $\mathfrak{g}$ | $k$ | $\Lambda$ | $\Gamma_{\Lambda}$ |
|------------|-----|-----------|-------------------|
| $C_4$      | 1   | $\Lambda_{(2)}$ | $\bar{A}_1^{(1)} \oplus D_2^{(1)}$ |
| $A_1$      | 4   | $2\Lambda_{(1)}$ | $A_1^{(1)} \oplus D_2^{(1)}$ |
| $A_2$      | 3   | $\Lambda_{(1)}$, $\Lambda_{(2)}$ | $T_{10}$ |
| $A_3$      | 2   | $\Lambda_{(2)}$ | $A_3^{(1)} \oplus D_5^{(1)}$ |
| $B_r$      | $2$ | $\Lambda_{(j)}$, $j=1, 2, \ldots, r-1; 2\Lambda_{(r)}$ | $A_1^{(1)} \oplus D_{r+1}^{(1)}$ |
| $D_4$      | 2   | $\Lambda_{(3)}$, $\Lambda_{(4)}$ | $A_3^{(1)} \oplus D_6^{(1)}$ |
| $D_r$      | 2   | $\Lambda_{(1)}$ | $A_3^{(1)} \oplus D_{r+2}^{(1)}$ |

Here $T_{10}$ stands for a triangular array of the type encountered in (8.5) above.

7. Local algebras.

According to the results of section 6 any modular rational fusion rule algebra can be viewed as the local algebra of some polynomial $V$. Thus one may
try to classify fusion rules to some extent by classifying local algebras. In particular, one can start from any quasihomogeneous polynomial and deform it in such a way as to obtain a fusion algebra. However, the conditions for a deformation to provide a fusion rule algebra (that is, essentially, to allow for a conjugation that is unital) are rather non-trivial and to my knowledge no general algorithm for obtaining solutions is known so far. On the other hand, the classification of quasihomogeneous polynomials (with isolated singularities) can be achieved for any fixed number \( n \) of variables, and indeed has been completed [122, 123] for small \( n \).

Note that even if one succeeded in classifying some class of allowed fusion potentials, one still has to investigate the equivalences among them, since as seen in section 6 a given fusion algebra can possess many distinct presentations as a local algebra.

10 The conformal bootstrap

Given some physical realization of a conformal field theory, the operator product coefficients of the theory are quantities which can, in principle, be measured experimentally. For example, if the conformal field theory is used in the inner sector of a string compactification, then the the Yukawa couplings of the massless particles that are present in the low energy limit are (products of) operator product coefficients (compare e.g. [124, 125, 126, 127]). Similarly, if the theory describes a statistical system at a second order phase transition, the operator product coefficients determine the corrections to finite size scaling [57]. Furthermore, if the operator product coefficients as well as the symmetry algebra of a conformal field theory are known, the theory can be considered as completely solved. For all these reasons, it is of considerable interest to compute the operator product coefficients of a theory. It is the central idea of the conformal bootstrap programme to perform this calculation on the basis of a few general requirements, among which the prominent part is played by the associativity of the operator product algebra.

By their definition, the fusion rules of a conformal field theory carry an important amount of information about the structure of the operator product algebra. However, as mentioned in section 9, many different conformal field theories may possess the same fusion rule algebra. For example, the \((B_r)_1\) WZW theories for any rank \( r \), and also the \((E_8)_2\) theory, all realize the Ising fusion rules; and even the trivial one-dimensional fusion rule algebra
described by $1 \star 1 = 1$ is shared by a huge class of unitary modular invariant theories. But still, the fusion rules of a two-dimensional conformal field theory at least provide nontrivial constraints that allow for a partial solution of the bootstrap programme. To explain this, I will now describe how far one can get in determining the operator products by using the fusion rules as an input. As mentioned towards the end of section 3, given the four-point functions of all primary fields, one can determine the operator product coefficients and hence solve the theory completely. The four-point functions, in turn, are to a large extent fixed by the analytic properties of their chiral blocks, and (part of) these analytic properties can be deduced from the fusion rules.

What is relevant are actually not the chiral blocks themselves, but rather how they combine, according to \((3.9)\), to the full correlation functions. The information carried by the coefficients in \((3.9)\) is equivalent to the information contained in the so-called fusing and braiding matrices $F = F[^{jk}_{il}]$ and $B = B[^{jk}_{il}]$ which, according to

\[
F_{ikl,jp}(z) = \sum_m F_{pm[^{jk}_{il}]} F_{ijkl,m}(z^{-1}) \quad (10.1)
\]

and

\[
F_{ikl,jn}(z) = \sum_m B_{nm[^{jk}_{il}]} F_{ijkl,m}(1 - z), \quad (10.2)
\]

implement the duality transformations ‘fusing’ and ‘braiding’ on the space of chiral blocks. Note that the chiral blocks may be represented pictorially as in the figure \([1.23]\), where the external lines are to be interpreted as the primary fields in $F$, and the internal lines as the families that are exchanged in the $s$-, $t$-, and $u$-channel, respectively; in this picture, the fusing and braiding matrices relate the chiral blocks of the $s$-channel to those of the $u$- and $t$-channel, respectively.

There exist two different strategies for determining the matrices $F$ and $B$. One possibility \([128]\) is to solve a set of consistency conditions known as (genus zero) polynomial equations which can be deduced \([11]\) by applying duality transformations to five-point functions. The first of these relations, the pentagon equation, describes the compatibility of fusing and braiding; the second, the hexagon equation, corresponds to the Yang-Baxter equation for the representation of the braid group that is induced by $B$. In the second approach \([131]\), linear differential equations for the correlators are
constructed (often these also follow from the presence of null vectors in the Verma modules of \( \mathcal{W} \); an example are the Knizhnik-Zamolodchikov equations \([18]\) for WZW correlators). Their independent solutions are the chiral blocks, and \( B \) and \( F \) can be determined from the explicit form of the blocks.

The fusion rules come into this game as follows. In order to know the dimensionality of the duality matrices, respectively the order of the relevant differential equation, one must know the number of chiral blocks that contribute to the four-point correlators. According to the formula (3.10), this information can be read off the fusion rules.

To be able to assess the strength of these ideas, it will be necessary to mention a few details \([130, 131]\) of the second strategy. Thus consider the four-point function (3.8). From the structure of the operator product algebra (3.6), it follows that the chiral blocks in the \( s \)-channel behave as

\[
z^{-\Delta_i - \Delta_j + \Delta^{(ij)}_m}
\]

for \( z \to 0 \), where \( \Delta^{(ij)}_m = \Delta_m^{(ij)} + \mu_m^{(ij)} \) is the conformal dimension of that field \( \varphi_m \), of grade \( \mu_m^{(ij)} \), in the family \( [\phi_m] \equiv [\delta_m^{(ij)}] \) that is responsible for the leading contribution to the coupling between \( \phi_i, \phi_j \) and \( [\phi_m] \). Similarly, in the \( t \)- and \( u \)-channel the singularities are of the form \( (1 - z)^{-\Delta_i - \Delta_k + \Delta^{(ik)}_n} \) for \( z \to 1 \), and \( (z^{-1})^{\Delta_i - \Delta_l + \Delta^{(il)}_p} \) for \( z \to \infty \), respectively. The systems of chiral blocks in the three channels are not independent. Associativity of the operator product algebra implies

\[
F_{ijkl}(z, \bar{z}) = F_{ikjl}(1 - z, 1 - \bar{z}) = z^{-2\Delta_i} \bar{z}^{-2\Delta_i} F_{iklj}(z^{-1}, \bar{z}^{-1}),
\]

which in turn requires \([132]\) that the systems are linearly related through analytic continuation; this is the contents of the relations (10.1) and (10.2). Moreover, in each channel the system of chiral blocks must be algebraically independent, which implies that the Wronskian determinant of the system must not vanish identically. Putting this information together, it follows from elementary results of the theory of ordinary linear differential equations that the chiral blocks are the \( M \) independent solutions of an \( M \)th order differential equation in the variable \( z \), with only regular singular points, among which there are in particular \( z = 0, 1, \infty \). The general solution of this equation is described by a so-called Riemann scheme which specifies the positions of the singularities together with the exponents \( \alpha^{(i)}_m \) at \( z_i \); the latter are the roots of the the \( M \)th order algebraic equation that one obtains in lowest order in \( z - z_i \) by inserting the ansatz \( F(z) = (z - z_i)\alpha^{(i)}_m \sum_{p=0}^{\infty} a_p (z - z_i)^p \) \( (a_0 \neq 0) \) into the differential equation. The Riemann scheme does not, in general,

\[14\] The notation used here applies directly only to the case \( N_{ij}^m \leq 1 \); otherwise a multiplicity index distinguishing the \( N_{ij}^m \) possible couplings is needed.
determine the chiral blocks uniquely because the number of parameters on which the differential equation, and hence its solutions, depend is generically larger than the number of exponents (the additional parameters needed to specify the differential equation uniquely are called accessory parameters).

The ideas described above can for example be employed to gain insight into the classification of (quasi)rational conformal field theories. Namely, suppose that the singularity structure of the four-point functions is fully known, which means that the conformal dimensions of the primaries, including the integer part (and, in the case of fusion rule coefficients larger than 1, also the grades of the exchanged fields which are responsible for the leading singular behaviour of a chiral block) are given. Then it can be proved \cite{129} that the duality matrices are uniquely determined, provided that the differential equations satisfied by the four-point functions do not possess any apparent singularities. (By definition, an apparent singularity of a differential equation is a singular point of the equation at which any of its solutions is regular.) In particular, in the absence of apparent singularities the values of all accessory parameters can be fixed by imposing the polynomial equations. On the other hand, the presence of apparent singularities spoils this uniqueness property, because the local monodromy of the solutions around an apparent singularity is trivial so that the number and positions of apparent singularities appear as free parameters in the Riemann monodromy problem.

Note that only the positions and exponents of the real singularities of correlation functions are part of the basic data of a conformal field theory, while the number and positions of apparent singularities must be considered as arbitrary, up to mild restrictions which result \cite{131} from crossing symmetry. (The polynomial equations, including those arising at genus one which involve the modular transformation matrices $S$ and $T$, do not lead to any further constraints on the apparent singularities; this is so because duality is the ‘square root’ of monodromy \cite{1}; technically, it follows by application of a simple result from the theory of isomonodromic deformations of differential equations.) If the conformal dimensions are only known up to integers, one can consider the integer parts, and similarly also the grades $\mu_{m}^{(i)}$ mentioned above, as additional parameters of the classification programme, and perform the analysis for each allowed set of parameters separately. One may even start from scratch and regard the fusion rule algebra as the single input of the programme: given the fusion rules, it is possible to determine the conformal dimensions of all primary fields of the theory up to a few integer
constants \[133, 132, 131\]; for each allowed set of values of these parameters one can then proceed as before.

The various types of parameters introduced here, such as the conformal dimensions, the grades \(\mu_m\), or the positions of apparent singularities and the associated exponents, distinguish among different conformal field theories which all possess a prescribed fusion rule algebra. After imposing the polynomial equations one is essentially left with those parameters which do not affect the monodromy representation of the relevant differential equation. Two types of such parameters can be distinguished: first, the integer parts of the exponents (i.e., the integer parts of conformal dimensions, the grades \(\mu_m\), and the exponents at the apparent singularities), and second, the positions of the apparent singularities. While the former are discrete parameters, the latter are a priori continuous, and (apart from the mild restrictions mentioned above) the principles of rational conformal field theory do not seem to give any information on them. In particular, one can continuously deform the positions of apparent singularities in such a way that the duality matrices are left invariant. It is an open question (for more details, see \[131\]) whether this means that to any allowed value of these positions there corresponds a consistent conformal field theory, which would imply the existence of continuous families of rational conformal field theories.

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