Strong Asymmetric Limit of the Quasi-Potential of the Boundary Driven Weakly Asymmetric Exclusion Process

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Abstract: We consider the weakly asymmetric exclusion process on a bounded interval with particles reservoirs at the endpoints. The hydrodynamic limit for the empirical density, obtained in the diffusive scaling, is given by the viscous Burgers equation with Dirichlet boundary conditions. In the case in which the bulk asymmetry is in the same direction as the drift due to the boundary reservoirs, we prove that the quasi-potential can be expressed in terms of the solution to a one-dimensional boundary value problem which has been introduced by Enaud and Derrida [16]. We consider the strong asymmetric limit of the quasi-potential and recover the functional derived by Derrida, Lebowitz, and Speer [15] for the asymmetric exclusion process.

1. Introduction

The study of steady states of non-equilibrium systems has motivated a lot of work over the last decades. It is now well established that the steady states of non-equilibrium systems exhibit in general long-range correlations and that the thermodynamic functionals, such as the free energy, are neither local nor additive.

The analysis of the large deviations asymptotics of stochastic lattice gases with particle reservoirs at the boundary has proven itself to be an important step in the physical description of nonequilibrium stationary states and a rich source of mathematical problems. We refer to [6,14] for two recent reviews on this topic.

We consider a boundary driven one-dimensional lattice gas whose dynamics can be informally described as follows. Fix an integer \( N \geq 1 \), an external force \( E \) in \( \mathbb{R} \) and boundary densities \( 0 < \rho_- < \rho_+ < 1 \). At any given time each site of the interval \( \{-N+1, \ldots, N-1\} \) is either empty or occupied by one particle. In the bulk, each particle attempts to jump to the right at rate \( 1 + E/2N \) and to the left at rate \( 1 - E/2N \). To respect the exclusion rule, the particle jumps only if the target site is empty, otherwise nothing happens. At the boundary sites \( \pm(N-1) \) particles are created and removed for
the local density to be $\rho_{\pm}$: at rate $\rho_{\pm}$ a particle is created at $\pm(N - 1)$ if the site is empty and at rate $1 - \rho_{\pm}$ the particle at $\pm(N - 1)$ is removed if the site is occupied.

The dynamics just described defines an irreducible Markov process on a finite state space which has a unique stationary state denoted by $\mu^N$. Let $\varphi_{\pm} := \log(\rho_{\pm} / (1 - \rho_{\pm}))$ be the chemical potential of the boundary reservoirs and set $E_0 := (\varphi_+ - \varphi_-)/2$. When $E = E_0$, the drift caused by the external field $E$ matches the drift due to the boundary reservoirs, and the process becomes reversible.

In the limit $N \uparrow \infty$, the typical density profile $\bar{\rho}_E$ under the stationary state $\mu^N$ can be described as follows. For each $E \leq E_0$ there exists a unique $J_E \leq 0$ such that

$$\frac{1}{2} \int_{\rho_-}^{\rho_+} dr \frac{1}{E \chi(r) - J_E} = 1,$$

where $\chi$ is the mobility of the system: $\chi(a) = a(1 - a)$. The profile $\bar{\rho}_E$ is then obtained by solving

$$\bar{\rho}_E - E \chi(\bar{\rho}_E) = -J_E$$

with the boundary condition $\bar{\rho}_E(-1) = \rho_-.$

In the same limit $N \uparrow \infty$, the probability of observing a density profile $\gamma$ different from $\bar{\rho}_E$ can be expressed as

$$\mu^N_E\{\gamma\} \sim \exp\{-N V_E(\gamma)\}. \quad (1.1)$$

The large deviations functional $V_E$, which also depends on $\rho_-, \rho_+$, is an extension of the notion of free energy in the context of non-equilibrium systems.

The free energy of a boundary driven lattice gas has first been derived for the symmetric simple exclusion process by Derrida, Lebowitz and Speer [15] based on the so-called matrix method, introduced by Derrida, which permits to express the stationary state $\mu^N_E$ as a product of matrices. Bertini et al. [4] derived the same result through a dynamical approach which we extend here to the weakly asymmetric case.

We consider only the situation $E < E_0$ for the bulk asymmetry to be in the same direction as the drift due to the boundary. The reversible case $E = E_0$ lacks interest because the stationary state is product and does not exhibit long range correlations. In contrast, the analysis of the quasi-potential $V_E$ for $E > E_0$, not treated here, appears a most interesting problem. For instance, a representation of $V_E$ as a supremum of trial functionals analogous to (2.14) below seems to be ruled out.

In the boundary driven weakly asymmetric exclusion process, for $E < E_0$, the quasi-potential takes the following form:

$$V_E(\gamma) := \int_{-1}^{1} du \left\{ \gamma \log \gamma + (1 - \gamma) \log(1 - \gamma) + (1 - \gamma) \varphi - \log(1 + e^\varphi) 
+ \frac{1}{E} \left[ \varphi' \log \varphi' - (\varphi' - E) \log(\varphi' - E) \right] - A_E \right\}, \quad (1.2)$$

where $A_E$ is the constant given by

$$A_E := \log(-J_E) + \frac{1}{2} \int_{\gamma_-}^{\gamma_+} dr \frac{1}{E \chi(r)} \log \left[ 1 - \frac{E \chi(r)}{J_E} \right];$$