Envy-free division in the presence of a dragon

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Abstract. Following a novel approach, where the emphasis is on configuration spaces and equivariant topology, we prove several results addressing the envy-free division problem in the presence of an unpredictable (secretive, non-cooperative) player, called the dragon. There are two basic scenarios. (1) There are \( r - 1 \) players and a dragon. Once the “cake” is divided into \( r \) parts, the dragon makes his choice and grabs one of the pieces. After that, the players should be able to share the remaining pieces in an envy-free fashion. (2) There are \( r + 1 \) players who divide the cake into \( r \) pieces. A ferocious dragon comes and swallows one of the players. The players need to cut the cake in advance in such a way that no matter who is the unlucky player swallowed by the dragon, the remaining players can share the tiles in an envy-free manner. We emphasize that in both settings, some players may prefer to choose degenerate pieces of the cake. Moreover, the players construct in advance both a cut of the cake and a decision tree, allowing them to minimize the uncertainty of what pieces can be given to each of the players.

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1. Introduction

Given some resource identified with the unit segment \( I = [0, 1] \) and a set of agents (players), one of the goals of welfare economics is to divide the resource among the agents in an envy-free manner. Envy-freeness is the principle where every player feels that their share is at least as good as the share of any other agent, and thus, no player feels envy.

In the literature, \( I \) is commonly referred to as the cake. A cut of the cake is a sequence of numbers \( x = (x_1, \ldots, x_{r-1}) \) where

\[
0 \leq x_1 \leq x_2 \leq \cdots \leq x_{r-1} \leq 1,
\]
so the set of all cuts is naturally identified as the standard \((r-1)\)-dimensional simplex \(\Delta^{r-1}\).

The pieces of the cake arising from the cut \((1)\) are the closed intervals (tiles) \(I_i = I_i(x) := [x_{i-1}, x_i] \quad (i = 1, \ldots, r)\), where \(x_0 = 0\) and \(x_r = 1\).

The non-degeneracy of tiles is a salient feature of some classical results, such as Gale’s equilibrium theorem [5]. However, as noticed by several authors in more recent publications [2,3,9,10,12], this condition may be too restrictive in some situations.

After a cut \(x\) is chosen, each of the players expresses her individual preferences, over the tiles arising from this particular cut, by pointing to one (or more) intervals which they like more than the rest.

This idea is formalized in the following definition, where the individual preferences are interpreted as subsets of the simplex \(\Delta^{r-1}\).

**Definition 1.1.** ([10, Definition 2.2]) The preferences of \(r\) players are a matrix of subsets \((A^j_i)_{i,j=1}^r\) of the standard simplex \(\Delta^{r-1}\). The subsets are interpreted as preferences in the usual sense as follows:

\[
x \in A^j_i \iff \text{in the cut } x \text{ the player } j \text{ prefers the tile } i.
\] (2)

The nature of the preferences may be unknown or hidden (black box preferences). There are, however, some conditions which are natural (intrinsic) or even unavoidable in the sense that they must be satisfied by all preferences.

The necessary assumption are the following:

\((P_{cl})\) The preferences (the sets \(A^j_i\)) are closed. That is, if a sequence of cuts \((x^{(n)})_{n \in \mathbb{N}}\) converges \(x^{(n)} \to x\) and if, in this sequence, a player always prefers the tile \(I_i(x^{(n)})\), then the player also prefers the limit tile \(I_i(x)\).

\((P_{cov})\) The preferences form a covering: \(\bigcup_{i=1}^r A^j_i = \Delta^{r-1} \quad \forall j = 1, \ldots, r\). This means that whatever a cut is, each player is expected to prefer at least one of the offered pieces.

If the players may prefer to choose degenerate tiles, the following condition is also quite natural.

\((P_{pe})\) (Partition equivalence) If two cuts produce one and the same collection of non-degenerate tiles, the preferences in these two cuts should be essentially the same (one can recovered from the other and vice versa).

Expressed more explicitly this condition says the following.

\(1\) A player prefers a non-degenerate tile for the first cut iff she prefers the same tile for the second cut.

\(2\) A player prefers a degenerate tile for the first cut iff she prefers any of the degenerate tiles for the second cut, and vice versa.

This condition implies, in particular, that the players do not distinguish the degenerate tiles.

Here is the summary of central results of the paper, presented in an informal and non-technical manner.
1. (Theorem 4.2) Assume that \( r \) is a prime power and that the preferences of \( r \) players are closed (\( P_{\text{cl}} \)) and covering (\( P_{\text{cov}} \)). Degenerate tiles may be preferred and the partition equivalence condition (\( P_{\text{pe}} \)) holds. One of the \( r \) players is a dragon whose preferences are secret. It is known that, once the cake is cut into \( r \) tiles, the dragon will be the one who chooses the first. Then, it is always possible to cut the cake in advance into \( r \) tiles, such that, whatever tile is taken by the dragon, the rest of the tiles may be allocated to the players, so that they do not feel envy, neither to the dragon, nor to each other. (It is self-understood that the dragon, being the first to choose, feels no envy as well.)

Moreover, each of the \( r \) pieces can be assigned in advance at most two names of the players, so that, whatever is the piece taken by the dragon, each of the players will be given a piece with her name on it. Equivalently, each human player is given in advance information about two tiles, one of which would be given to them under any circumstances.

2. (Theorem 5.2) Let \( r \) be a power of a prime. Assume that \( r+1 \) players have closed (\( P_{\text{cl}} \)), covering (\( P_{\text{cov}} \)) preferences, which satisfy the partition equivalence condition (\( P_{\text{pe}} \)), and the players are may prefer to choose degenerate tiles. It is known that once the cake is cut into \( r \) tiles, the dragon will appear and swallow one of the players. Then, it is always possible to cut the cake in advance into \( r \) tiles, such that regardless of which player is taken by the dragon, the rest of the tiles may be allocated to the players in an envy-free way.

As in the previous theorem, each of the \( r \) pieces can be assigned in advance at most two names of the \( r+1 \) players, so that, whoever is the player swallowed by the dragon, each of the surviving players will be given a piece with her name on it. Alternatively, each human player is informed in advance about at most two tiles, which include the tile that will be given to them if they are spared by the dragon.

Under the assumption that the degenerate tiles, if offered, are never chosen by the players, the condition (\( P_{\text{pe}} \)) is not needed and both Theorems 5.2 and 6.2 hold true with no assumption on \( r \) whatsoever (see Theorem 2.1 for illustration).

1.1. Brief historical overview and predecessors of our results

All our theorems derive inspiration from the results, both new and classical, about the envy-free-cake-cutting.

1. The original envy-free cake-cutting theorem was independently discovered by Stromquist [13] and Woodall [16] in 1980. It addresses the case of hungry players, where the degenerate tiles are never chosen by the players, and claims that (under the conditions (\( P_{\text{cl}} \)) and (\( P_{\text{cov}} \))) an envy-free division is always possible for any number of players \( r \).

2. D. Gale [5] is the author of the proof scheme where the result of Stromquist and Woodall is derived with the aid of the Birkhoff–von Neumann theorem about the polytope of bistochastic matrices.

Much more recent are the results where the existence of envy-free divisions is guaranteed under much less restrictive conditions that the players are
not necessarily hungry (which may motivate the players to choose degenerate tiles). However, in these results, the number of players is necessarily a prime power, \( r = p^k \).

(3) As shown by Avvakumov and Karasev [2, Theorem 3.3], see also [3] and [10], if \( r \) is a prime power, then an envy-free division (where all non-degenerate tiles are allocated) always exists if the preferences satisfy the conditions \( (P_{cl}), (P_{cov}) \) and \( (P_{pe}) \).

(4) This result was established earlier by Meunier and Zerbib [9, Theorem 1] in the cases when \( r \) is a prime or \( r = 4 \), see also Segal-Halevi [12] where the result was conjectured and proved in the case \( r = 3 \).

(5) As shown in [2], if \( r \) is not a prime power, the theorem of Avvakumov and Karasev is no longer true.

D.R. Woodall proved (already in [16]) a refined version of the envy-free-division-theorem (for hungry players) where a \emph{secretive player} hides her preferences.

(6) Woodall’s result was rediscovered by Asada et al. [1], with a much simpler proof, under the name \emph{Strong colorful KKM theorem}.

(7) Meunier and Su, applying the method of multilabeled Sperner lemmas, proved [8, Corollary 1.1] a result where one of the players (unknown in advance) is “voted off the group”.

Both (6) and (7) are proven for arbitrary \( r \) with the assumption that the players are hungry (so they never choose degenerate tiles).

1.2. Dragon marriage lemma

The following proposition [11, Proposition 5.4] is repeatedly used throughout the paper. It is a version of \emph{Hall’s marriage theorem} in the presence of a dragon: there are \( n-1 \) grooms and \( n \) brides, and one of the brides is taken by a dragon. For the reader’s convenience, we supply a short proof.

**Proposition 1.2.** (Dragon marriage lemma) \( J_1, \ldots, J_{n-1} \subseteq [n] \) be not necessarily distinct sets. Then, the following three conditions are equivalent:

1. For each \( k \)-element subset \( \{i_1, \ldots, i_k\} \subseteq [n-1] \), \( |J_{i_1} \cup \cdots \cup J_{i_k}| \geq k + 1 \).
2. For each \( j \in [n] \), there is a system of distinct representatives in the family \( J_1, \ldots, J_{n-1} \) that avoids \( j \).
3. There is a system of 2-element representatives \( \{a_i, b_i\} \subseteq J_i \), such that \( (a_1, b_1), \ldots, (a_{n-1}, b_{n-1}) \) are edges of a spanning tree in \( K_n \).

**Proof.** The statements (1) and (2) are clearly equivalent in light of \emph{Hall’s marriage theorem}. Moreover (3) \( \Rightarrow \) (1) is obvious, so the interesting part of proposition is the implication (1) \( \Rightarrow \) (3).

Let \( G \subseteq [n-1] \times [n] \) be a bipartite graph where

\[(i, j) \in G \iff j \in J_i.\]

We prove the implication (1) \( \Rightarrow \) (3) by induction on the size (number of edges) of the graph \( G \).

Let us suppose that \( G \) is an inclusion minimal graph which still possesses the property (1), meaning that the \emph{Dragon marriage condition} (1) is no longer satisfied if any of the edges is removed from \( G \).
Let us call a proper subset \( I \subset [n-1] \) (\( \emptyset \neq I \neq [n-1] \)) tight if \( |G[I]| = |I| + 1 \), where \( G[I] := \{ j \in [n] \mid \exists i \in I \,(i,j) \in E \} \).

Note that the existence of tight subsets \( I \subset [n-1] \) is a consequence of minimality of \( G \). Indeed, for any edge \((i,j) \in E(G)\), the graph \( G_1 := G \setminus \{(i,j)\} \) does not satisfy (1); therefore, \( |G_1[I]| \leq |I| \) and \( |G[I]| = |I| + 1 \) for a subset \( I \subset [n-1] \) which contains \( i \). (The possibility that \( I = [n-1] \) can be easily ruled out by choosing an edge \((i,j)\), such that \((i',j) \in G \) for some \( i' \neq i \).)

Let \( I \) be a tight subset of maximal cardinality. Recall that \( |I| < n-1 \) and note that the restriction of \( G \) on \( I \times G[I] \) satisfies the condition (1) of Proposition 1.2. By the inductive hypothesis, there is a bipartite graph \( T_1 \subset I \times G[I] \) which defines \( [n-1] \) a spanning tree on \( G[I] \) with edges labelled by \( I \).

Let \( J := [n-1] \setminus I \) and \( V := [n] \setminus G[I] \). Note that \( G[I] \cap G[J] \neq \emptyset \), as a consequence of the fact that \( |G[I]| + |G[J]| \geq |I| + |J| + 2 = n + 1 \). Choose \( y \in G[I] \cap G[J] \) and define \( G' \) as the intersection \( G' := G \cap (J \times (V \cup \{y\})) \).

Let us show that the bipartite graph \( G' \) also satisfies the Dragon marriage condition (1). Suppose that \( J_1 \subset J \) is a proper subset of \( J \). Then, \( J_1 \cup I \) is a proper subset of \([n-1] \) and (since \( I \) is a maximal tight subset) \( |G[J_1 \cup I]| \geq |J_1| + |I| + 2 \). It immediately follows that \( |G'[J_1]| \geq |G[J_1] \setminus G[I]| \geq |J_1| + 1 \).

It remains to show that \( G'[J] = V \cup \{y\} \); however, this is obvious, since, by the previous argument, \( V \subset G'[J_1] \) for each subset \( J_1 \subset J \) of cardinality \( |J| - 1 \).

By the inductive hypothesis, there is a spanning tree \( T_2 \subset G' \) with \( V \cup \{y\} \) as the set of vertices and edges labelled by \( J \). The trees \( T_1 \) and \( T_2 \) have only the vertex \( y \) in common, so their union \( T = T_1 \cup T_2 \) is also a tree, with the vertex set \([n] \) and edges labelled by \([n-1] \). This completes the proof of the proposition.

Remark 1.3. Meunier and Su consider in [8] a general problem of envy-free division when the number of (hungry) players and the number of pieces of the cake differ by one, two, three, etc. (see [8, Theorem 2.4] for illustration). For this reason, it would be interesting to know how to modify the condition (3) in a version of Hall’s marriage theorem with \( k \) dragons, \( n-k \) grooms, and \( n \) brides, where \( k \) brides are stolen by the dragons.

2. Strong colorful KKM theorem revisited

The Strong colorful KKM theorem [1, Theorem 3.6] (see also [16, Theorem 4] and our Section 2.1) offers a solution to a problem of envy-free division in the presence of a dragon (the first scenario) in the classical case of KKM (Knaster–Kuratowski–Mazurkiewicz) preferences.

The following theorem refines that result by putting emphasis on a decision tree, which provides much more precise information how the pieces of the cake are distributed.
Theorem 2.1. Let $\Delta^n = \{ t \in \mathbb{R}^{n+1}_+ \mid \sum_{i=1}^{n+1} t_i = 1 \}$ be an $n$-dimensional simplex with facets $\Delta^n_i = \{ t \in \Delta^n \mid t_i = 0 \}$. Let $(A^j_i)$ be a $((n+1) \times n)$-matrix of closed subsets of $\Delta^n$, called the matrix of preferences. Assume that for each $j \in [n]$, the family $\{A^j_i\}_{i \in \{n+1\}}$ satisfies the condition of the KKM theorem. More explicitly

1. The family $\{A^j_i\}_{i \in \{n+1\}}$ is a covering of the simplex $\Delta^n$ for each $j \in [n]$.
2. The intersection $A^j_i \cap \Delta^n_i$ is empty for all $i \in \{n+1\}$ and $j \in [n]$.

Then, one can choose for each $j \in [n]$ two distinct elements $u_j$ and $v_j$ in $[n]$, such that

(a) The collection $E = \{e_j\}_{j \in [n]}$ of two element sets $e_j = \{u_j, v_j\}$ is the edge-set of a tree $T = (V, E)$ (connected graph without cycles) on $V = [n+1]$ as the set of vertices.
(b) $\bigcap_{e \text{ incident to } e} A^e_n = \bigcap_{j \in [n]} (A^j_{u_j} \cap A^j_{v_j}) \neq \emptyset.$ \hspace{2cm} (3)

Proof. Let $(O^j_i)$ be a $((n+1) \times n)$-matrix of open sets in $\Delta^n$, such that $A^j_i \subseteq O^j_i \subseteq \Delta^n \setminus \Delta^n_i$ for each $i$ and each $j$. Let $(f^j_i)$ be a matrix of functional preferences (fuzzy set preferences) associated with the matrix of preferences $(A^j_i)$, subordinated to $(O^j_i)$. More explicitly $(f^j_i)$ is a collection of functions $f^j_i : \Delta^n \rightarrow [0, 1]$ satisfying the following conditions.

1. For each $j \in [n]$, the collection $\{f^j_i\}_{i=1}^{n+1}$ is a partition of unity
   $$f^j_1 + f^j_2 + \cdots + f^j_{n+1} = 1.$$ 
2. For each $i$ and $j$
   $$A^j_i \subseteq \{ x \mid f^j_i(x) > 0 \} \subseteq O^j_i \subseteq \Delta^n \setminus \Delta^n_i.$$ 

Let $h_i = \frac{1}{n}(f^1_i + f^2_i + \cdots + f^n_i)$. The restriction of the map
$$h = (h_1, h_2, \ldots, h_{n+1}) : \Delta^n \rightarrow \Delta^n$$
to the boundary $\partial \Delta^n$ is homotopic (by the linear homotopy) to the identity map. It follows that the degree of this map is equal to one and, as a consequence, $h(x) = (\frac{1}{n+1}, \ldots, \frac{1}{n+1})$ for some $x \in \Delta^n$.

Let $M$ be the matrix $M = (f^j_i(x))$ and $\Omega = (\omega^j_i) = (\text{Sign}(f^j_i(x)))$ the associated 0-1 matrix of signs ($\text{Sign}(\cdot) \in \{-1, 0, +1\}$).

Lemma. The matrix $\Omega$ interpreted as a bipartite graph (with $(n+1)$ brides and $n$ grooms) satisfies the “Dragon marriage condition” of Postnikov [11, Section 5] (see Proposition 1.2) saying that for each subset $S \subseteq [n]$ and the corresponding set $\Omega[S] = \{ i \in [n+1] \mid \exists j \in [n] f^j_i(x) > 0 \}$

$$|\Omega[S]| \geq |S| + 1.$$ 

Indeed
$$|S| = \sum_{i \in [n+1]} f^j_i \leq \sum_{i \in \Omega[S]} f^j_i = \frac{n}{n+1}|\Omega[S]| \Rightarrow |\Omega[S]| \geq |S| + 1.$$
By Proposition 1.2, there exists a system of 2-element representatives $e_j = \{u_j, v_j\} \subseteq \{i \mid f_i^j(x) > 0\}$, such that $E = \{e_j\}$ is the edge-set of a tree on $[n + 1]$.

The proof is completed by choosing smaller and smaller open neighborhoods $O_{i,j}$ of closed sets $A_{i,j}$ and by passing to the limit. □

2.1. Strong colorful KKM theorem

Theorem 2.1 implies the standard form of the *strong colorful KKM theorem* [1, Theorem 3.6] which claims that, given $n$ KKM-coverings $\{A_{i,j}\}_{j=1}^n$ of $\Delta^n$, there exists $x \in \Delta^n$ and $n + 1$ bijections $\pi_i : [n] \to [n + 1] \setminus \{i\}$, such that for all $i \in [n]

\[ x \in A_{\pi_i(1)}^1 \cap \cdots \cap A_{\pi_i(n)}^n. \tag{4} \]

Indeed, if a vertex $i \in [n + 1]$ is chosen to be the root of the tree $T = (V, E) = ([n+1],[n])$, then there is a canonical bijection $\pi_i : V \setminus \{i\} \to E$, such that $k$ is incident to $e_k := \pi_i(k)$ for each $k \in [n + 1] \setminus \{i\}$, and (4) is an immediate consequence of (3).

A salient feature of Theorem 2.1 is that, given a matrix of preferences $(A_{i,j}^j)$, one should be able to determine both $x$ and a tree $T$ (not just $x$), which automatically leads to a very small collection of $n + 1$ bijections $\pi_i$, arising from the tree $T$.

For example, just from the knowledge of $x$ and $T$, each player will know in advance which two tiles are the only candidates to be allocated to them. Moreover, from the shape of the tree, they would be able to calculate the probability of getting each of the two selected tiles, etc.

3. Envy-free division via configuration spaces

Following the general framework of [10], in this section, we outline a new approach to envy-free division based on equivariant topology and configuration spaces (chessboard complexes).

As before there are $r$ players, but now they are not necessarily hungry and may prefer to choose degenerate pieces. Originally, the cake $I = [0,1]$ was divided by $(r-1)$ cut-points into $r$ pieces. Now, we consider cuts $x$ with a larger number of cut-points

\[ 0 \leq x_1 \leq x_2 \leq \cdots \leq x_{2r-2} \leq 1 \tag{5} \]

together with associated allocation functions $\alpha : [2r - 1] \to [r]$, which place the tiles $\{I_i\}_{i=1}^{2r-1}$ (created by the cut $x$) into $r$ “boxes” [10, Section 2]. The pair $(x, \alpha)$ is referred to as a *partition/allocation* of the cake $[0,1]$.

We allow only “admissible” allocation functions $\alpha : [2r - 1] \to [r]$, permitting at most one non-degenerate tile in each of the boxes. As a consequence, at least $r - 1$ tiles are degenerate and many of the cut-points (5) coincide.

Two admissible allocation functions $\alpha, \beta : [2r - 1] \to [r]$ and the corresponding partitions/allocations $(x, \alpha)$ and $(x, \beta)$ are equivalent if they differ
only by positions of the degenerate tiles. More explicitly
\[(x, \alpha) \sim (x, \beta) \iff (\forall i \in [2r - 1])(\alpha(i) \neq \beta(i) \Rightarrow I_i \text{ is degenerate})\,.

Following [10, Section 5], we define \(C\) as the configuration space of all equivalence classes \([(x, \alpha)]\) of pairs \((x, \alpha)\), where \(x\) is a cut and \(\alpha\) is an associated admissible allocation function.

Alternatively, the configuration space \(C\) can be described [10, Proposition 5.1] as the chessboard complex \(\Delta_{r,2r-1}\) of all non-attacking configurations of rooks in a \([r] \times [2r - 1]\) chessboard. (Two rooks are in a non-attacking position if they are not allowed to be in the same row or in the same column of the chessboard.)

In this interpretation, the rows of the \([r] \times [2r - 1]\)-chessboard correspond to the boxes, columns correspond to the tiles, and empty columns correspond to degenerate tiles. The condition “a non-attacking collection of rooks” is translated as “at most one non-degenerate tile per box is allowed”, etc.

More information about the history and applications of chessboard complexes in combinatorics and discrete geometry can be found in [4,7,15,17,18].

**Definition 3.1.** ([10, Definition 2.2]) The \(C\)-preferences of \(r\) players are a matrix of subsets \((B^j_i)_{i,j=1}^r\) of the configuration space \(C \cong \Delta_{r,2r-1}\). More explicitly
\[(x, \alpha) \in B^j_i \iff \text{in the cut } x \text{ and the allocation } \alpha \text{ the player } j \text{ prefers the content } \alpha^{-1}(i) \text{ of the box } i. \quad (6)\]

The symmetric group \(S_r\) acts on the configuration space \(C \cong \Delta_{r,2r-1}\) by renumbering the boxes
\[\sigma(x, \alpha) := (x, \sigma \circ \alpha)\]
or equivalently, by permuting the rows of the chessboard \([r] \times [2r - 1]\).

We say that \(C\)-preferences are **equivariant** if for each \(\sigma \in S_r\)
\[(x, \alpha) \in B^j_i \iff \sigma(x, \alpha) \in B^j_{\sigma(i)} \,. \quad (7)\]

The condition (7) is quite natural. It follows from (6) that \((x, \sigma \circ \alpha) \in B^j_{\sigma(i)}\) if and only if, in the cut \(x\) and the allocation \(\sigma \circ \alpha\), the player \(j\) prefers the content of the box \(\sigma(i)\). Since \((\sigma \circ \alpha)^{-1}(\sigma(i)) = \alpha^{-1}(i)\), the condition (7) expresses the idea that players make their decisions solely on the content of the boxes, not on their current labels.

We say that the preferences \((B^j_i)\) are **closed** if \(B^j_i\) are closed subsets of the configuration space \(C \cong \Delta_{r,2r-1}\). The preferences \((B^j_i)\) are **covering** if
\[\bigcup_{i=1}^r B^j_i = C\]
for each \(j \in [r]\), meaning that each player must choose one of the (possibly degenerate) tiles, displayed in the boxes by the corresponding allocation functions.
4. First scenario: the dragon takes a piece of the cake

The following result is a refinement of [10, Theorem 5.1].

**Theorem 4.1.** Let $r$ be a prime power. Let

$$(C^j_i)_{i,j \in [r]} \subseteq C \cong \Delta_{r,2r-1}$$

be a $r \times (r - 1)$-matrix of preferences which are closed, covering, and equivariant, in the sense of Sect. 3. Then, one can choose for each $j \in [r - 1]$ two distinct elements $u_j$ and $v_j$ in $[r]$, such that

(a) The collection $E = \{e_j\}_{j \in [r - 1]}$ of two element sets $e_j = \{u_j, v_j\}$ is the edge-set of a tree $T = (V, E)$ on $V = [r]$.

(b) $\bigcap_{v \text{ incident to } e} C^e_v = \bigcap_{j \in [r - 1]} (C^j_{u_j} \cap C^j_{v_j}) \neq \emptyset$. (8)

**Proof.** The proof relies on methods from equivariant topology, combined with the ideas used in the proof of Theorem 2.1.

Let $(O^j_i)$ be a $(r \times (r - 1))$-matrix of open sets in $\Delta_{r,2r-1}$, such that $C^j_i \subseteq O^j_i$ for all $i$ and $j$. By averaging, it can be assumed that $(O^j_i)$ is also equivariant.

Let $(f^j_i)$ be a matrix of functional preferences associated with the matrix of preferences $(C^j_i)$, subordinated to $(O^j_i)$, which is also equivariant in the sense that

$$f^j_{\sigma(i)}(\sigma(x, \alpha)) := f^j_{\sigma(i)}(x, \sigma \circ \alpha) = f^j_i(x, \alpha).$$

In particular, $(f^j_i)$ is a collection of functions $f^j_i : \Delta_{r,2r-1} \to [0, 1]$ satisfying the following conditions.

1. For each $j \in [r - 1]$, the collection $\{f^j_i\}_{i=1}^r$ is a partition of unity

$$f^1_i + f^2_i + \cdots + f^r_i = 1.$$  

2. For each $i$ and $j$

$$C^j_i \subseteq \{x \mid f^j_i(x) > 0\} \subseteq O^j_i.$$  

Let $h_i = \frac{1}{r-1}(f^1_i + f^2_i + \cdots + f^{r-1}_i)$. The map

$$h = (h_1, h_2, \ldots, h_r) : \Delta_{r,2r-1} \longrightarrow \Delta^{r-1}$$

is equivariant and, since $\Delta_{r,2r-1}$ is $(r - 2)$-connected, Volovikov’s theorem [14] guarantees that $h(x, \alpha) = (\frac{1}{r}, \ldots, \frac{1}{r})$ for some $(x, \alpha) \in \Delta_{r,2r-1}$.

Let $M$ be the matrix $M = (f^j_i(x))$ and $\Omega = (\omega^j_i) = (\text{Sign}(f^j_i(x)))$ the associated 0-1 matrix of signs.

As in the proof of Theorem 2.1, one obtains the inequality

$$|\Omega[S]| \geq |S| + 1$$

for each subset $S \subseteq [r - 1]$ and the corresponding set

$$\Omega[S] = \{i \in [r] \mid \exists j \in [r - 1], f^j_i(x) > 0\}.$$
As before by [11, Proposition 5.4], there exists a system of 2-element representatives \( e_j = \{u_j, v_j\} \subseteq \{i \mid f_i^j(x) > 0\} \), such that \( E = \{e_j\} \) is the edge-set of a tree on \([r]\).

Finally, the proof is completed (as in the case of Theorem 2.1) by choosing smaller and smaller open neighborhoods \( O_i^j \) of closed sets \( A_i^j \) and by passing to the limit. □

4.1. Classical preferences with a secretive player

Theorem 4.1 allows us to prove a relative of both Theorem 2.1 and [2, Theorem 4.1], where the players may choose degenerate pieces of the cake and there is a secretive or non-cooperative player (the dragon).

Theorem 4.2. Assume that \( r \) is a prime power. Assume we have the old-style closed, covering preferences \((A_i^j)_{i \in [r]}\), \( A_i^j \subseteq \Delta^{r-1} \) satisfying the \((P_{pe})\) condition.

Then, one can choose for each \( j \in [r-1] \) two distinct elements \( u_j \) and \( v_j \) in \([r]\), such that

1. The collection \( E = \{e_j\}_{j \in [r-1]} \) of two element sets \( e_j = \{u_j, v_j\} \) is the edge-set of a tree \( T = (V, E) \) on \( V = [r] \) as the set of vertices.

2. \[
\bigcap_{v \text{ incident to } e} A_v^e = \bigcap_{j \in [r-1]} (A_{u_j}^j \cap A_{v_j}^j) \neq \emptyset.
\] (10)

Proof. Fundamental configuration space used in Theorem 4.1 is the chessboard complex \( \Delta_{r,2r-1} \), while the corresponding object in Theorem 2.1 is the simplex \( \Delta^{r-1} \). For the comparison of these spaces and transfer from one to another, we use the diagram (11), where

\[
I(r) := \{A \subset [0, 1] \mid \{0, 1\} \subseteq A \text{ and } |A| \leq r + 1\}
\]

is a configuration space of finite sets (symmetric product) with the topology induced by the Hausdorff metric, while \( \phi \) and \( \psi \) are obvious forgetful maps

\[
\begin{array}{ccc}
\Delta^{r-1} & \xrightarrow{\psi} & I(r) \\
\Delta_{r,2r-1} & \xrightarrow{\phi} & I(r)
\end{array}
\] (11)

Starting with the old-style (classical) preferences \((A_i^j)\), we construct new-style (equivariant) preferences \((C_i^j)\) defined on the configuration space \( C_2 = \Delta_{r,2r-1} \).

As in [10, Section 4], this construction can be presented in the form of an algorithm [where the diagram (11) and the maps \( \phi \) and \( \psi \) are implicitly used but not explicitly mentioned].

1. Given a cut \( x \) of the segment \( I \) with \( 2r-2 \) cut-points create an induced cut \( y \) of \( I \) with \( r-1 \) cut-points \( (y \in \Delta^{r-1}) \), preserving the same collection of non-degenerate intervals \( \{I_k'\}_{k=1}^s \). In other words, we eliminate \( r-1 \) superfluous (multiple) cuts. Note that this step can be performed in many different ways.
2. If the preferences \( \{A^j_i\} \) of a player dictate the choice of some non-degenerate tile \( i \), add the box \( \alpha(i) \) to the preferences of the player. Note that the property \((P_{\text{pe}})\) of \( (A^j_i) \) implies that no matter how the superfluous cuts are eliminated, the result will be one and the same.

3. If the preferences \((A^j_i)\) dictate a player to choose a degenerate interval (which occurred after \( r-1 \) cuts), observe that there necessarily exists an empty box, since, in this case, the number of non-degenerate tiles is at most \( r-1 \). Add all empty boxes to the preferences of the player.

As before, we emphasize that the condition \((P_{\text{pe}})\) is used in the proof that the preferences \( C^j_i \) are well defined.

It is not difficult to check (see also Example 4.4) that the preferences \((C^j_i)\) satisfy all conditions of Theorem 4.1. Then, the existence of an element \((x, \alpha)\) in the intersection (8) and the corresponding tree \( T \) are used, by a transfer in the opposite direction, for the proof that the intersection (10) is non-empty and the completion of the proof of Theorem 4.2.

\[\square\]

Remark 4.3. As emphasized above, the technical part of the proof of Theorem 4.2 consists of constructing equivariant (new-style) preferences \((C^j_i)\) from the given old-style preferences \((A^j_i)\), and proving that the construction is well defined (if \((P_{\text{pe}})\) is satisfied).

Example 4.4. There is a class of preferences, arising from continuous valuations (continuous, signed measures \( \nu \) are a special case), where this proof is particularly transparent. More explicitly, a valuation is a function \( v : [0, 1]^2 \to \mathbb{R} \) (the domain is the set of all intervals \([x, y] \subseteq [0, 1] \)) such that \( v(x, y) = v([x, y]) = 0 \) if \( x = y \) (this accounts for the \((P_{\text{pe}})\) condition).

By definition \( x \in A^j_i = A^j_i(v_j) \) iff \( v_j(I_i(x)) = \max\{v_j(I_k(x))\}_{k=1}^r \), where \( v_j \) is the valuation associated with the player \( j \). The corresponding equivariant preferences \((C^j_i)\) are, in agreement with Eq. (6) (Sect. 3), defined as follows.

Given a partition/allocation \((x, \alpha) \in \Delta_{2r-1,r}\), let \( N\!d(x) \) be the set of non-degenerate tiles of the partition \( x \). For each \( i \in [r] \) let \( I(i) \) be the unique non-degenerate tile in \( \alpha^{-1}(i) \), if it exists. Otherwise, if \( \alpha^{-1}(i) \cap N\!d(x) = \emptyset \), let \( I(i) \) be an arbitrary degenerate tile in \( \alpha^{-1}(i) \). Then, by definition

\[
(x, \alpha) \in C^j_i \iff v_j(I(i)) = \max\{v_j(I_k(x))\}_{k=1}^r.
\]

Corollary 4.5. Under the conditions of Theorem 4.2, there always exists a partition of \([0, 1] \) into at most \( r \) non-degenerate intervals, such that, even if one of the intervals is removed (taken by the dragon), each of the remaining non-degenerate intervals is given to a different player, the rest of the players are not given anything (they are given “empty pieces”), and this distribution is envy-free from the viewpoint of each of the players (the dragon included).
5. Second scenario: the dragon takes a player

Theorem 5.1. Let $r$ be a prime power. Let

$$(C^j_i)_{i \in [r]} \subseteq C \cong \Delta_{r,2r-1}$$

be a $r \times (r + 1)$-matrix of preferences which are closed, covering, and equi-

variant.

Then, one can choose for each $i \in [r]$ two distinct elements $u_i$ and $v_i$ in

$[r+1]$, such that

1. The collection $E = \{e_i\}_{i \in [r]}$ of two element sets $e_i = \{u_i, v_i\}$ is the
edge-set of a tree $T = (V, E)$ on vertices $V = [r+1]$.  
2.  

$$\bigcap_{v \text{ incident to } e} C_v^u = \bigcap_{i \in [r]} (C^u_i \cap C^v_i) \neq \emptyset.$$  

(12)

Proof. The proof uses similar ideas as the proof of Theorem 5.1.

We choose $(O^j_i)$, a $(r \times (r + 1))$-matrix of open sets in $\Delta_{r,2r-1}$, such that

$C^j_i \subseteq O^j_i$ for all $i$ and $j$. As before, by averaging, it can be assumed that $(O^j_i)$
is also equivariant

$$(x, \alpha) \in O^j_i \iff (x, \sigma \circ \alpha) \in O^j_{\sigma(i)}.$$  

Let $(f^j_i)$ be a matrix of functional preferences (associated with the ma-
trix of preferences $(C^j_i)$), which for each $j$ forms a partition of unity sub-
or-ordinated to the cover $(O^j_i)_{i=1}^r$. Moreover, $(f^j_i)$ is equivariant in the sense that

$$f^j_{\sigma(i)}(\sigma(x, \alpha)) = f^j_i(x, \alpha).$$  

(13)

More explicitly, $(f^j_i)$ is a collection of functions $f^j_i : \Delta_{r,2r-1} \to [0, 1]$ which, in addition to (13), satisfy the following conditions.

1. For each $j \in [r-1]$, the collection $\{f^j_i\}_{i=1}^r$ is a partition of unity

$$f^j_1 + f^j_2 + \cdots + f^j_r = 1.$$  

2. For each $i$ and $j$

$$C^j_i \subseteq \{x \mid f^j_i(x) > 0\} \subseteq O^j_i.$$  

Let $h_i = \frac{1}{r+1} (f^1_i + f^2_i + \cdots + f^{r+1}_i)$. The map

$$h = (h_1, h_2, \ldots, h_r) : \Delta_{r,2r-1} \longrightarrow \Delta^{r-1}$$
is equivariant and the Volovikov’s theorem again guarantees the existence of an element $(x, \alpha) \in \Delta_{r,2r-1}$, such that $h(x, \alpha) = (\frac{1}{r}, \ldots, \frac{1}{r})$.

Let $M$ be the matrix $M = (f^j_i(x))$ and $\Omega = (\omega^j_i) = (\text{Sign}(f^j_i(x))$ the
associated 0-1 matrix of signs. Note that this is a $r \times (r + 1)$ matrix, as op-
opposed to the $r \times (r - 1)$ matrix arising in the proof of Theorem 2.1.

(Informally speaking the brides and grooms in the corresponding bipar-
tite graph interchange the places.)

By mimicking the proof of the lemma, used in the proof of Theorem 2.1, one obtains the inequality

$$|\Omega[S]| \geq |S| + 1$$
for each subset $S \subseteq [r]$ and the corresponding set
\[ \Omega(S) = \{ j \in [r+1] \mid \exists i \in [r] f_i^j(x) > 0 \}. \]

As before by [11, Proposition 5.4], there exists a system of 2-element representatives $e_i = \{ u_i, v_i \} \subseteq \{ j \mid f_i^j(x) > 0 \}$, such that $E = \{ e_i \}$ is the edge-set of a tree on $[r+1]$.

Finally, the proof is completed (as in the case of Theorem 2.1) by choosing smaller and smaller open neighborhoods $O^j_i$ of closed sets $A^j_i$ and by passing to the limit. □

5.1. Classical preferences with $r+1$ players and a dragon

Theorem 5.1 implies the following theorem which extends and refines [8, Corollary 1.1], by allowing degenerate tiles as preferences, and by specifying in advance a decision tree describing what pieces can be given to each of the players.

**Theorem 5.2.** Assume that $r$ is a prime power. Assume we have the old-style closed, covering preferences $(A^j_i)_{i \in [r]}$, $A^j_i \subseteq \Delta^{r-1}$ satisfying the $(P_{pe})$ condition.

Then, one can choose for each $i \in [r]$ two distinct elements $u_i$ and $v_i$ in $[r+1]$, such that

1. The collection $E = \{ e_i \}_{i \in [r]}$ of two element sets $e_i = \{ u_i, v_i \}$ is the edge-set of a tree $T = (V, E)$ on $V = [r+1]$ as the set of vertices.
2. \[ \bigcap_{e \text{ incident to } e} A^u_e = \bigcap_{i \in [r]} (A^{u_i}_i \cap A^{v_i}_i) \neq \emptyset. \] (14)

**Proof.** The proof is similar to the proof of Theorem 4.2, with Theorem 4.1 playing the role of Theorem 4.1, so we omit the details. □

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