ON CERTAIN FOURIER EXPANSIONS FOR THE RIEMANN
ZETA FUNCTION

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ABSTRACT. We build on a recent paper on Fourier expansions for the Riemann
zeta function. We establish Fourier expansions for certain $L$-functions, and
offer series representations involving the Whittaker function $W_{\gamma,\mu}(z)$ for the
coefficients. Fourier expansions for the reciprocal of the Riemann zeta function
are also stated. A new expansion for the Riemann xi function is presented in
the third section by constructing an integral formula using Mellin transforms
for its Fourier coefficients.

Keywords: Riemann zeta function; Riemann Hypothesis; Fourier series

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1. INTRODUCTION AND MAIN RESULTS

The measure

$$
\mu(B) := \frac{1}{2\pi} \int_B \frac{dy}{\frac{x}{2} + y^2},
$$

for each $B$ in the Borel set $\mathcal{B}$, has been applied in the work of [7] as well as Coffey
[3], providing interesting applications in analytic number theory. For the measure
space $(\mathbb{R}, \mathcal{B}, \mu)$,

$$(1.1) \quad \|g\|_2^2 := \int_\mathbb{R} |g(t)|^2 d\mu,$$

is the $L^2(\mu)$ norm of $f(x)$. Here (1.1) is finite, and $f(x)$ is measurable [10, pg.326,
Definition 11.34]. In a recent paper by Elaissaoui and Guennoun [7], an interesting
Fourier expansion was presented which states that, if $f(x) \in L^2(\mu)$, then

$$
(1.2) \quad f(x) = \sum_{n \in \mathbb{Z}} a_n e^{-2in \tan^{-1}(2x)},
$$

where

$$
(1.3) \quad a_n = \frac{1}{2\pi} \int_\mathbb{R} f(y) e^{2in \tan^{-1}(2y)} \frac{dy}{\frac{x}{2} + y^2}.
$$
By selecting \( x = \frac{1}{2} \tan(\phi) \), we return to the classical Fourier expansion, since \( f(\frac{1}{2} \tan(\phi)) \) is periodic in \( \pi \). The main method applied in their paper to compute the constants \( a_n \) is the Cauchy residue theorem. However, it is possible (as noted therein) to directly work with the integral

\[
a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\frac{1}{2} \tan(\phi)) e^{inx} d\phi.
\]

Many remarkable results were extracted from the Fourier expansion (1.2)–(1.3), including criteria for the Lindelöf Hypothesis [7, Theorem 4.6].

Let \( \rho \) denote the nontrivial zeros of \( \zeta(s) \) in the critical region \((0,1)\), and \( \Re(\rho) = \alpha, \Im(\rho) = \beta \). The goal of this paper is to offer some more applications of (1.2)–(1.3), including a criteria for the Riemann hypothesis. Recall that the Riemann Hypothesis is the statement that \( \alpha \notin (\frac{1}{2}, 1) \).

**Theorem 1.1.** For \( \sigma > 1, x \in \mathbb{R} \),

\[
\frac{1}{\zeta(\sigma + ix)} = \frac{1}{\zeta(\sigma + \frac{1}{2})} + \sum_{n \geq 1} \tilde{a}_n e^{-2in \tan^{-1}(2x)},
\]

where

\[
\tilde{a}_n = \frac{1}{n!} \sum_{n > k \geq 0} \binom{n}{k} \frac{(-1)^n(n-1)!}{(k-1)!} \lim_{s \to 0} \frac{\partial^k}{\partial s^k} \frac{1}{\zeta(\sigma + \frac{1}{2} + s)}.
\]

Moreover, if the zeros of \( \zeta(s) \) are simple, we have

\[
\frac{1}{\zeta(\sigma - ix)} = \sum_{n \in \mathbb{Z}} \hat{a}_n e^{-2in \tan^{-1}(2x)},
\]

where for \( n \geq 1 \),

\[
\hat{a}_n = \frac{1}{n!} \sum_{n \geq k \geq 0} \binom{n}{k} \frac{(-1)^n(n-1)!}{(k-1)!} \lim_{s \to 0} \frac{\partial^k}{\partial s^k} \frac{1}{\zeta(\sigma - \frac{1}{2} + s)} - S(n, \sigma),
\]

where

\[
S(n, \sigma) = \sum_{\beta: \zeta(\rho)=0} \left( \frac{\sigma - i\beta}{1 - \sigma + i\beta} \right)^n \frac{1}{\zeta'(\rho)(1 - \sigma + i\beta)(\sigma - i\beta)}
\]

\[
+ \sum_{k \geq 1} \left( \frac{1}{2} + \sigma + 2k \right)^n \frac{1}{\zeta'(-2k)(\frac{1}{2} - \sigma - 2k)(\frac{1}{2} + \sigma + 2k)},
\]

and \( \hat{a}_n = -S(n, \sigma) \) for \( n < 0 \), \( \hat{a}_0 = 1/\zeta(\sigma + \frac{1}{2}) \).
Corollary 1.1.1. For \( \sigma > 1 \),
\[
\frac{1}{2\pi} \int_{\mathbb{R}} \frac{d\mu}{|\zeta(\sigma + iy)|^2} = \frac{1}{\zeta^2(\sigma + \frac{1}{2})} + \sum_{k \geq 1} |\tilde{a}_k|^2,
\]
where the \( \tilde{a}_n \) are as defined in the previous theorem. Furthermore, even assuming
the Riemann Hypothesis, this integral diverges for \( \frac{1}{2} < \sigma < 1 \).

Next we consider a Fourier expansion with coefficients expressed as a series involving
the Whittaker function \( W_{\gamma,\mu}(z) \), which is a solution to the differential equation [8,
pg.1024, eq.(9.220)]
\[
d^2 W dz^2 + \left( -\frac{1}{4} + \frac{\gamma}{2} + \frac{1 - 4\mu^2}{4z^2} \right) W = 0.
\]
This function also has the representation [8, pg.1024, eq.(9.220)]
\[
W_{\gamma,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2} - \mu - \gamma)} M_{\gamma,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2} + \mu - \gamma)} M_{\gamma,-\mu}(z).
\]
Here the other Whittaker function \( M_{\gamma,\mu}(z) \) is given by
\[
M_{\gamma,\mu}(z) = z^{\mu + \frac{1}{2}} e^{-z/2} \mathbf{1}_F(\mu - \gamma + 1/2; 2\mu + 1; z),
\]
where \( \mathbf{1}_F(a;b;z) \) is the well-known confluent hypergeometric function.

Theorem 1.2. Let \( v \) be a complex number which may not be an even integer. Then
for \( 1 > \sigma > \frac{1}{2} \), we have the expansion
\[
\zeta(\sigma + ix) \cos^v(\tan^{-1}(2x)) = \frac{1}{2} \zeta(\sigma + \frac{1}{2}) + \sum_{n \in \mathbb{Z}} \tilde{a}_n e^{-2\pi n \tan^{-1}(2x)},
\]
where \( \tilde{a}_n = \frac{(2\sigma^2 - 4\sigma + 2)}{2(2\sigma - 2)^2 \left( \frac{3}{2} - \frac{\sigma}{2} \right)^n} \) for \( n < 0 \), and for \( n \geq 1 \),
\[
\tilde{a}_n = \frac{2\Gamma(v + 1)}{\Gamma(\frac{v}{2} + n + 1)\Gamma(\frac{v}{2} - n + 1)} + \frac{\pi}{2^{v/2+1}} \sum_{k > 1} k^{-\sigma} \left( \frac{\log(k)}{2} \right)^{v/2} \frac{W_{n,-\frac{v}{2}+1}(\log(k))}{\Gamma(1 + \frac{v}{2} + n)}.
\]

2. Proof of Main Theorems

In our proof of Corollary 1.1.1, we will require a well-known result [13, pg.331,
Theorem 11.45] on functions in \( L^2(\mu) \).

Lemma 2.1. Suppose that \( f(x) = \sum_{k \in \mathbb{Z}} a_k \kappa_k \), where \( \{\kappa_n\} \) is a complete orthonormal set and \( f(x) \in L^2(\mu) \), then
\[
\int_X |f(x)|^2 d\mu = \sum_{k \in \mathbb{Z}} |a_k|^2.
\]
Proof of Theorem 1.1. First we rewrite the integral for $\sigma > 1$ as

\[(2.1) \quad \bar{a}_n = \frac{1}{2\pi i} \int_{\mathbb{R}} e^{2in \tan^{-1}(2y)} \frac{dy}{\zeta(\sigma + iy)(\frac{1}{2} + y^2)} dy = \frac{1}{2\pi i} \int_{(\frac{1}{2})} \left( \frac{s}{1 - s} \right)^n \frac{ds}{\zeta(\sigma - \frac{1}{2} + s)(1 - s)}.
\]

We replace $s$ by $1 - s$ and apply the residue theorem by moving the line of integration to the left. By the Leibniz rule, we compute the residue at the pole $s = 0$ of order $n + 1, n \geq 0$, as

\[(2.2) \quad \frac{1}{n!} \lim_{s \to 0} \frac{d^n}{ds^n} s^{n+1} \left( \left( \frac{1 - s}{s} \right)^n \frac{1}{\zeta(\sigma + \frac{1}{2} - s)(1 - s)} \right) = \frac{1}{n!} \lim_{s \to 0} \frac{d^n}{ds^n} \left( \frac{1 - s}{s} \right)^{n-1} \frac{1}{\zeta(\sigma + \frac{1}{2} - s)}
\]

\[= \frac{1}{n!} \sum_{n \geq k \geq 0} \binom{n}{k} (-1)^n (n-1)! \lim_{s \to 0} \frac{\partial^k}{\partial s^k} \frac{1}{\zeta(\sigma + \frac{1}{2} - s)}.
\]

The residue at $s = 0$ if $n = 0$ is $-1/\zeta(1 - \sigma)$. There are no additional poles when $n < 0$. Since the sum in (2.2) is zero for $k = n$ it reduces to the one stated in the theorem.

Next we consider the second statement. The integrand in

\[(2.3) \quad \frac{1}{2\pi i} \int_{(\frac{1}{2})} \left( \frac{1 - s}{s} \right)^n \frac{ds}{\zeta(\sigma - \frac{1}{2} + s)(1 - s)}
\]

has simple poles at $s = 1 - \sigma + i\beta$, where $\Im(\rho) = \beta$. The integrand in (2.3) also has simple poles at $s = \frac{1}{2} - \sigma - 2k$, and a pole of order $n + 1, n > 0$, at $s = 0$. We compute,

\[\frac{1}{n!} \lim_{s \to 0} \frac{d^n}{ds^n} s^{n+1} \left( \left( \frac{1 - s}{s} \right)^n \frac{1}{\zeta(\sigma + \frac{1}{2} - s)(1 - s)} \right) = \frac{1}{n!} \lim_{s \to 0} \frac{d^n}{ds^n} \left( \frac{1 - s}{s} \right)^{n-1} \frac{1}{\zeta(\sigma + \frac{1}{2} - s)}
\]

\[= \frac{1}{n!} \sum_{n \geq k \geq 0} \binom{n}{k} (-1)^n (n-1)! \lim_{s \to 0} \frac{\partial^k}{\partial s^k} \frac{1}{\zeta(\sigma + \frac{1}{2} - s)}.
\]

The residue at the pole $s = 1 - \sigma + i\beta$, is

\[\sum_{\beta:\zeta(\rho) = 0} \left( \frac{\sigma - i\beta}{1 - \sigma + i\beta} \right)^n \frac{1}{\zeta'(\rho)(1 - \sigma + i\beta)(\sigma - i\beta)}.
\]

and at the pole $s = \frac{1}{2} - \sigma - 2k$ is

\[\sum_{k \geq 1} \left( \frac{\frac{1}{2} + \sigma + 2k}{\frac{1}{2} - \sigma - 2k} \right)^n \frac{1}{\zeta'(-2k)(\frac{1}{2} - \sigma - 2k)(\frac{1}{2} + \sigma + 2k)}.
\]
The residue at the pole \( n = 0, s = 0 \), is \(-1/\zeta(\sigma - \frac{1}{2})\).

Proof of Corollary 1.1.1. This result readily follows from application of Theorem 1.1 to Lemma 2.1 with \( X = \mathbb{R} \). In the first part of the theorem, note from [14, pg.191, Theorem 8.7], if \( \sigma > 1 \),

\[
\left| \frac{1}{\zeta(s)} \right| \leq \frac{\zeta(\sigma)}{\zeta(2\sigma)}.
\]

Hence

\[
\frac{1}{|\zeta(s)|^2 (t^2 + \frac{1}{4})} = O\left(\frac{1}{|t|^2}\right),
\]

as \( t \to \infty \), and \( 1/\zeta(\sigma + it) \in L^2(\mu) \), for \( \sigma > 1 \). The convergence of the series \( \sum_k |\bar{a}_k|^2 \) follows immediately from [10, pg.580, Lemma 12.6]. In the second part of the theorem, note from [14, pg.377] or [14, pg.372]

\[
\frac{1}{\zeta(s)} = O\left(\frac{|s|}{\sigma - \frac{1}{2}}\right).
\]

Hence

\[
\frac{1}{|\zeta(s)|^2 (t^2 + \frac{1}{4})} = O\left(\frac{|s|^2}{|t|^2}\right) = O(1),
\]

as \( t \to \infty \), and \( 1/\zeta(\sigma + it) \notin L^2(\mu) \), for \( \frac{1}{2} < \sigma < 1 \).

Proof of Theorem 1.2. It is clear that

\[
\cos^v(2 \tan^{-1}(2y)) = \left(\frac{1 - 4y^2}{1 + 4y^2}\right)^v = O(1).
\]
Comparing with [7, Theorem 1.2] we see our function belongs to \( L^2(\mu) \). We compute that
\[
\tilde{a}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\frac{1}{2} \tan(\frac{\phi}{2})) e^{i n \phi} d\phi
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \zeta(\sigma + \frac{i}{2} \tan(\frac{\phi}{2})) \cos^v(\frac{\phi}{2}) e^{i n \phi} d\phi
\]
\[
= \frac{1}{2\pi} \left( \int_0^{\pi} \zeta(\sigma + \frac{i}{2} \tan(\frac{\phi}{2})) \cos^v(\frac{\phi}{2}) e^{i n \phi} d\phi + \int_{-\pi}^{0} \zeta(\sigma + \frac{i}{2} \tan(\frac{\phi}{2})) \cos^v(\frac{\phi}{2}) e^{i n \phi} d\phi \right)
\]
\[
= \frac{1}{2\pi} \left( \int_0^{\pi} \zeta(\sigma + \frac{i}{2} \tan(\frac{\phi}{2})) \cos^v(\frac{\phi}{2}) e^{i n \phi} d\phi + \int_{0}^{\pi} \zeta(-\sigma + \frac{i}{2} \tan(\frac{\phi}{2})) \cos^v(\frac{\phi}{2}) e^{-i n \phi} d\phi \right)
\]
\[
= \frac{1}{\pi} \int_0^{\pi} \cos^v(\frac{\phi}{2}) \sum_{k \geq 1} k^{-\sigma} \cos \left( \frac{1}{2} \tan(\frac{\phi}{2}) \log(k) - n\phi \right) d\phi
\]
\[
= \frac{1}{\pi} \int_0^{\pi} \cos^v(\phi) \cos(n\phi) d\phi + \frac{1}{\pi} \int_0^{\pi} \cos^v(\frac{\phi}{2}) \sum_{k \geq 1} k^{-\sigma} \cos \left( \frac{1}{2} \tan(\frac{\phi}{2}) \log(k) - n\phi \right) d\phi
\]
\[
= \frac{2}{\pi} \int_0^{\pi/2} \cos^v(\phi) \cos(n2\phi) d\phi + \frac{2}{\Gamma(1 + \frac{v+\gamma}{2})}
\]
\[
\int_0^{\pi/2} \cos^v(y) \cos(a \tan(y) - \gamma y) dy = \frac{\pi a^{v/2}}{2^{v/2+1}} \frac{W_{\gamma/2,-\nu-1}(2a)}{\Gamma(1 + \frac{v+\gamma}{2})}.
\]

Hence, if we put \( b = 2n \) and replace \( v \) by \( v + 1 \) in (2.4), and select \( a = \frac{1}{2} \log(k) \) and \( \gamma = 2n \) in (2.5), we find
\[
\tilde{a}_n = \frac{2\Gamma(v+1)}{\Gamma(\frac{v}{2} + n + 1) \Gamma(\frac{v}{2} - n + 1)} + \frac{\pi}{2^{v/2+1}} \sum_{k \geq 1} k^{-\sigma} \left( \frac{\log(k)}{2} \right)^{v/2} \frac{W_{\nu-\frac{v+\gamma}{2}}(\log(k))}{\Gamma(1 + \frac{v+\gamma}{2})}.
\]

Hence \( v \) cannot be a negative even integer.

The interchange of the series and integral is justified by absolute convergence for \( \sigma > \frac{1}{2} \). To see this, note that [8, pg.1026, eq.(9.227), eq.(9.229)]
\[
W_{\gamma \mu}(z) \sim e^{-z/2} z^\gamma,
\]
as $|z| \to \infty$, and

$$W_{\gamma, \mu}(z) \sim \left(\frac{4z}{\gamma}\right)^{1/4} e^{-\gamma + \gamma \log(\gamma)} \sin(2\sqrt{\gamma z} - \gamma \pi - \frac{\pi}{4}),$$

as $|\gamma| \to \infty$. Using (2.6) as coefficients for $n < 0$ is inadmissible, due to the resulting sum over $n$ being divergent. On the other hand, it can be seen that

$$\tilde{a}_n = \frac{1}{2\pi i} \int_{\frac{1}{2}} e^{2i\pi \tan^{-1}(2y)} \frac{\zeta(\sigma + iy) \cos(\tan^{-1}(2y))dy}{(\frac{1}{4} + y^2)}$$

$$= \frac{1}{2\pi i} \int_{\frac{1}{2}} \frac{\zeta(\sigma - \frac{1}{2} + s)2(2s^2 - 2s + 1)}{(2s(1-s))^2} \left(\frac{s}{1-s}\right)^n ds$$

$$= \frac{1}{2\pi i} \int_{\frac{1}{2}} \frac{\zeta(\sigma + \frac{1}{2} - s)2(2s^2 - 2s + 1)}{(2s(1-s))^2} \left(\frac{1-s}{s}\right)^n ds.$$  

We will only use the residues at the pole $s = 0$ when $n < 0$ and $s = \sigma - \frac{1}{2}$, and outline the details to obtain an alternative expression for the $\tilde{a}_n$ for $n \geq 0$. The integrand has a simple pole at $s = \sigma - \frac{1}{2}$, a pole of order $n + 2$ at $s = 0$, and when $n < 0$ there is a simple pole when $n = -1$, at $s = 0$. The residue at the pole $s = 0$ for $n \geq 0$ is computed as

$$\frac{1}{n!} \lim_{s \to 0} \frac{d^{n+1}}{ds^{n+1}} s^{n+2} \left(\frac{\zeta(\sigma + \frac{1}{2} - s)2(2s^2 - 2s + 1)}{(2s(1-s))^2} \left(\frac{1-s}{s}\right)^n\right)$$

$$= \frac{1}{n!2} \lim_{s \to 0} \frac{d^{n+1}}{ds^{n+1}} \left(\frac{\zeta(\sigma + \frac{1}{2} - s)2(2s^2 - 2s + 1)(1-s)^{n-2}}{(2s(1-s))^2} \right).$$

And because the resulting sum is a bit cumbersome, we omit this form in our stated theorem. The residue at the simple pole when $n = -1$, at $s = 0$ is $\frac{1}{2}\zeta(\sigma + \frac{1}{2})$. Collecting our observations tells us that if $n < 0$,

$$\tilde{a}_n = \frac{(2\sigma^2 - 4\sigma + \frac{3}{2})}{2(\sigma - \frac{1}{2})^2(\frac{3}{2} - \sigma)^2} \left(\frac{\sigma - \frac{1}{2}}{\sigma - \frac{1}{2}}\right)^n.$$

\[\square\]

3. Riemann xi function

The Riemann xi function is given by $\xi(s) := \frac{1}{2}s(s - 1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$, and $\Xi(y) = \xi\left(\frac{1}{2} + iy\right)$. In many recent works [4, 5], Riemann xi function integrals have been shown to have interesting evaluations. (See also [11] for an interesting expansion for the Riemann xi function.) The classical application is in the proof of Hardy’s theorem that there are infinitely many non-trivial zeros on the line $\Re(s) = \frac{1}{2}$. 

We will need to utilize Mellin transforms to prove our theorems. By Parseval’s formula [12, pg.83, eq.(3.1.11)], we have

\[
\int_0^\infty f(y)g(y)dy = \frac{1}{2\pi i} \int_{(r)} \mathcal{M}(f(y))(s)\mathcal{M}(g(y))(1-s)ds,
\]

provided that \( r \) is chosen so that the integrand is analytic, and

\[
\int_0^\infty y^{-s-1}f(y)dy =: \mathcal{M}(f(y))(s).
\]

From [12, pg.95, eq.(3.3.27)] with \( n \geq 0, x > 1, c > 0 \), we have

\[
\frac{1}{2\pi i} \int_{(c)} x^s s^{n+1}ds = \left( \log(x) \right)^n n!.
\]

Now it is known [6, pg.207–208] that for any \( \Re(s) = u \in \mathbb{R} \),

\[
\Theta(y) = \frac{1}{2\pi i} \int_{(u)} \xi(s)y^{-s}ds,
\]

where

\[
\Theta(y) := 2y^2 \sum_{n \geq 1} (2\pi^2 n^4 y^2 - 3\pi n^2)e^{-\pi n^2 y^2},
\]

for \( y > 0 \). Define the operator \( D_{n,y}(f(y)) \) := \( y \frac{\partial}{\partial y} \ldots y \frac{\partial}{\partial y} (f(y)) \).

**Theorem 3.1.** For real numbers \( x \in \mathbb{R} \),

\[
\Xi(x) = \frac{1}{(\frac{1}{4} + x^2)} \sum_{n \in \mathbb{Z}} \tilde{a}_n e^{-2in \tan^{-1}(2x)},
\]

where \( \tilde{a}_0 = 0 \), and for \( n \geq 1 \),

\[
\tilde{a}_n = \frac{(-1)^n}{(n-1)!} \int_0^1 \log^{n-1}(y) D_{n,y}(\Theta(y))dy,
\]

and

\[
\tilde{a}_{-n} = -\frac{(-1)^n}{(n-1)!} \sum_{n-1 \geq k \geq 0} \binom{n-1}{k} \frac{n!}{(k+1)!} \xi^{(k)}(0).
\]

**Proof.** Applying the operator \( D_{n,y} \) to (3.3)–(3.4), then applying the resulting Mellin transform with (3.2) to (3.1), we have for \( c < 1, n \geq 1 \),

\[
\frac{(-1)^n}{(n-1)!} \int_0^1 \log^{n-1}(y) D_{n,y}(\Theta(y))dy = \frac{1}{2\pi i} \int_{(c)} \left( \frac{s}{1-s} \right)^n \xi(s)ds.
\]
On the other hand,

\begin{equation}
\hat{a}_n = \frac{1}{2\pi} \int_{\mathbb{R}} e^{2in\tan^{-1}(2y)} \left( \frac{1+y^2}{1+y^2} \right) dy = \frac{1}{2\pi i} \int_{(\frac{1}{2})} \left( \frac{s}{1-s} \right)^n \xi(s) ds
\end{equation}

This gives the coefficients for \( n \geq 1 \). If we place \( n \) by \(-n\) in the integrand of (3.6), we see that there is a pole of order \( n, n > 0 \), at \( s = 0 \). These residues are computed in the same way as before, and so we leave the details to the reader. Hence, for \( n > 0 \), \(-2 < r' < 0\),

\begin{equation}
\hat{a}_{-n} = \frac{1}{2\pi i} \int_{(\frac{1}{2})} \left( \frac{1-s}{s} \right)^n \xi(s) ds
\end{equation}

In the third line we implemented the fact that the remaining residue from the poles of \( \Gamma(\frac{s}{2}) \) at negative even integers is zero due to the trivial zeros of \( \zeta(s) \). □

Now according to Coffey [1, pg.527], \( \xi^{(n)}(0) = (-1)^n \xi^{(n)}(1) \), which may be used to recast Theorem 3.1 in a slightly different form. The integral formulae obtained in [2, pg.1152, eq.(28)] (and another form in [9, pg.11106, eq.(12)]) bear some resemblance to the integral contained in (3.5). It would be interesting to obtain a relationship to the coefficients \( \hat{a}_n \). Next we give a series evaluation for a Riemann xi function integral.

**Corollary 3.1.1.** If the coefficients \( \hat{a}_n \) are as defined in Theorem 3.1., then

\[ \int_{\mathbb{R}} (\frac{1}{4} + y^2)^2 \Xi^2(y) dy = \sum_{n \in \mathbb{Z}} |\hat{a}_n|^2. \]

**Proof.** This is an application of Theorem 3.1 to Lemma 2.1 with \( X = \mathbb{R} \). □

4. On the partial Fourier series

Here we make note of some interesting consequences of our computations related to the partial sums of our Fourier series. First, we recall [10, pg.69] that
\[
\sum_{n=-N}^{N} a_ne^{inx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y)DN(y)dy,
\]
where
\[
DN(x) = \frac{\sin((N + \frac{1}{2})x)}{\sin(\frac{x}{2})}.
\]

Now making the change of variable \( y = 2\tan^{-1}(2y) \), we find (4.1) is equal to
\[
\frac{1}{2\pi} \int_{\mathbb{R}} f(x - 2\tan^{-1}(2y)) \frac{DN(2\tan^{-1}(2y))}{\frac{1}{4} + y^2} dy.
\]

Recall [10, pg.71] that \( K_N(x) \) is the Fejér kernel if
\[
K_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} D_n(x).
\]

**Theorem 4.1.** Let \( K_N(x) \) denote the Fejér kernel. Then, assuming the Riemann hypothesis, \[\lim_{N \to \infty} \frac{1}{2\pi} \int_{\mathbb{R}} \frac{K_N(x_0 - 2\tan^{-1}(2y))}{\zeta(\sigma + iy)(\frac{1}{4} + y^2)} dy = \frac{1}{\zeta(\sigma + \frac{i}{2}\tan(\frac{x_0}{2}))}, \]
for \( x_0 \in (-\pi, \pi), \frac{1}{2} < \sigma < 1. \)

**Proof.** Notice that \( 1/\zeta(\sigma + \frac{i}{2}\tan(\frac{x_0}{2})) \) is continuous for \( y \in (-\pi, \pi) \) if there are no singularities for \( \frac{1}{2} < \sigma < 1 \). Hence, we may apply [10, pg.29, Theorem 1.26] to find \( 1/\zeta(\sigma + \frac{i}{2}\tan(\frac{x_0}{2})) \) would then be Riemann integrable on \( (-\pi, \pi) \) if there are no singularities for \( \frac{1}{2} < \sigma < 1 \). It is also periodic in \( \pi \). Applying Fejér’s theorem [10, pg.73, Theorem 1.59] with \( f(y) = 1/\zeta(\sigma + \frac{i}{2}\tan(\frac{y}{2})) \) implies the result. \(\square\)

Note that if \( 1/\zeta(\sigma + \frac{i}{2}\tan(\frac{y}{2})) \) has even finitely many points of discontinuity for \( \frac{1}{2} < \sigma < 1 \), we would not be able to apply Fejér’s theorem. This is because the function is unbounded by Montgomery’s omega result [14, pg.209], and therefore not Riemann integrable by [10, pg.31, Proposition 1.29].

5. CONCLUDING REMARKS

The Fourier series for the Riemann zeta function contained herein, just like those in [7], are pointwise convergent. Seeing as how there exists a Fourier series for \( \zeta(\sigma + it) \) in the region \( \frac{1}{2} < \sigma < 1 \), that is pointwise convergent, it would be interesting if one existed that were absolutely convergent. Wiener’s result [15, pg.14, Lemma IIe] says the following:
Lemma 5.1. (Wiener [15]) Suppose \( f(x) \) has an absolutely convergent Fourier series and \( f(x) \neq 0 \) for all \( x \in \mathbb{R} \). Then its reciprocal \( 1/f(x) \) also has an absolutely convergent Fourier series.

Therefore, an application of the Riemann hypothesis would then imply the existence of an absolutely convergent Fourier series for \( 1/\zeta(\sigma + it) \), when \( \frac{1}{2} < \sigma < 1 \).

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