Goldstone bosons in presence of charge density

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Abstract

We investigate spontaneous symmetry breaking in Lorentz-noninvariant theories. Our general discussion includes relativistic systems at finite density as well as intrinsically nonrelativistic systems. The main result of the paper is a direct proof that nonzero density of a non-Abelian charge in the ground state implies the existence of a Goldstone boson with nonlinear (typically, quadratic) dispersion law. We show that the Goldstone boson dispersion relation may in general be extracted from the current transition amplitude and demonstrate on examples from recent literature, how the calculation of the dispersion relation is utilized by this method. After then, we use the general results to analyze the nonrelativistic degenerate Fermi gas of four fermion species. Due to its internal SU(4) symmetry, this system provides an analog to relativistic two-color quantum chromodynamics with two quark flavors. In the end, we extend our results to pseudo-Goldstone bosons of an explicitly broken symmetry.

PACS numbers: 11.30.Qc

Keywords: Spontaneous symmetry breaking, Goldstone boson counting, Type-II Goldstone bosons, Two-color QCD
I. INTRODUCTION

The basic result concerning the physics of spontaneous breaking of global internal continuous symmetries in Lorentz-noninvariant systems was achieved by Nielsen and Chadha thirty years ago [1]. They showed that, under certain technical assumptions (the most important one being that of a sufficiently fast decrease of correlations, which ensures that no long-range forces are present), the energy of the Goldstone boson (GB) of the spontaneously broken symmetry is proportional to some power of momentum in the low-momentum limit, and classified the GBs as type-I and type-II according to whether this power is odd or even, respectively. Their general counting rule states that the number of type-I GBs plus twice the number of type-II GBs is greater than or equal to the number of broken generators.

Two classic examples of systems where the type-II GBs occur are the ferromagnet and the so-called A phase of superfluid $^3$He. The issue of GBs in Lorentz-noninvariant theories has regained considerable interest in the past decade due to the discovery of several other systems exhibiting the existence of type-II GBs. These fall into two wide classes: Relativistic systems at finite density and intrinsically nonrelativistic systems. The first class includes various color-superconducting phases of dense quark matter—the color-flavor-locked (CFL) phase with a kaon condensate [2, 3] and the two-flavor color superconductor [4]—as well as the Bose–Einstein condensation of relativistic Bose gases [5, 6] and the relativistic nuclear ferromagnet [7]. The second class covers the before mentioned ferromagnet and superfluid helium, and the rapidly developing field of the Bose–Einstein condensation of dilute atomic gases with several internal degrees of freedom [8].

Besides the investigation of particular systems, a few new general results also appeared. Leutwyler [9] showed within the low-energy effective field theory framework that the nonlinearity of the GB dispersion is tightly connected to the fact that some of the conserved charges develop nonzero density in the ground state. As a matter of fact, the nonzero charge density induces breaking of time-reversal invariance, which in turn leads to the appearance of a term in the effective Lagrangian with a single time derivative. This term is then responsible for the modification of the GB dispersion. In addition to Leutwyler’s results, Schaefer et al. [3] proved that nonzero ground-state density of a commutator of two broken charges is a necessary condition for an abnormal number of GBs.

In our previous work [10] we showed that, at least within a particular class of systems
described by the relativistic linear sigma model at finite density, and at tree level, the kotlinx to the theorem of Schaefer et al. is also true: Nonzero density of a commutator of two broken charges implies the existence of a single GB which has quadratic dispersion relation, i.e., is type-II. Moreover, there is a one-to-one correspondence between the pairs of broken generators whose commutator has nonzero density, and the type-II GBs. In Ref. [11] we analyzed the particular model of Miransky and Shovkovy [2] and Schaefer et al. [3] at one-loop level and asserted that the one-loop corrections do not alter the qualitative conclusions achieved at the tree level.

The purpose of this paper is to extend the previous results in a model-independent way. We are going to present a general argument, based on a Ward identity for the broken symmetry, that nonzero ground-state density of a non-Abelian charge leads to the existence of a GB with nonlinear (typically, quadratic) dispersion law. The plan of the paper is following. In the next section we give a detailed derivation of the main result. In the subsequent sections we then apply it to several systems discussed in recent literature, and show how it facilitates the calculation of the GB dispersion relations. Next we analyze in detail Cooper pairing in a nonrelativistic gas of four degenerate fermion species. Finally, we show how the proposed method may be generalized to include explicit symmetry breaking.

II. GENERAL ARGUMENT

In Ref. [10] we provided a general argument that nonzero density of a non-Abelian charge leads to modified GB counting. In particular, we invoked the standard proof of the Goldstone theorem [12] to show that

\[ i f_{abc} \langle 0 | j_c^0(0) | 0 \rangle = \langle 0 | [j_a^0(x), Q_b] | 0 \rangle = 2i \text{Im} \sum_n \langle 0 | j_a^0(0) | n \rangle \langle n | j_b^0(0) | 0 \rangle, \]

(1)

see Eq. (1) in Ref. [10] and the discussion below it. Here \( f_{abc} \) is the set of structure constants of the symmetry group and the index \( n \) runs over different zero-momentum states in the spectrum. The vacuum expectation value \( \langle 0 | j_a^0 | 0 \rangle \) represents the order parameter for symmetry breaking, while both currents \( j_a^0 \) and \( j_b^0 \) serve as interpolating fields for the GBs. \( Q_b \) is the charge operator associated with the current \( j_b^\mu \). Note that whereas in Lorentz-invariant systems there is a one-to-one correspondence between the GBs and the broken currents, at nonzero charge density the situation changes. As Eq. (1) clearly exhibits, a
single GB now couples to the two broken currents whose commutator has nonzero vacuum expectation value. Before we expand this simple observation into a more detailed argument, we discuss what the involved current transition amplitudes tell us about the GB dispersion relation.

A. GB dispersion from transition amplitude

The matrix element which couples the GB state \( |\pi(k)\rangle \) to the broken current operator \( j^\mu_a(0) \), may be generally parameterized as

\[
\langle 0 | j^\mu_a | \pi(k) \rangle = ik^\mu F_a(|k|) + i\delta^{\mu0} G_a(|k|). \tag{2}
\]

By the second term, \( G_a \), we explicitly account for the possibility of Lorentz violation. However, we still assume that the rotational invariance remains intact. The current conservation then implies the following equation for the GB dispersion (see also Ref. [13]), \( \omega(k) \),

\[(\omega^2 - k^2) F_a + \omega G_a = 0. \tag{3}\]

Let us now explicitly mention a few special cases, distinguished by the limiting behavior of the functions \( F_a \) and \( G_a \) at low momentum. First, when \( G_a = 0 \) for all values of \( k \), we recover the standard Lorentz-covariant parameterization of the transition amplitude (2), and the standard Lorentz-invariant dispersion relation, \( \omega = |k| \). Second, suppose that \( G_a/F_a \) is small in the limit \( |k| \to 0 \), of order \( O(|k|) \). Then the dispersion relation comes out linear, only the phase velocity is no more equal to one due to the Lorentz-violating term \( G_a \). Third and most importantly for the rest of the paper, when \( G_a \) has a finite limit, \( \lim_{|k| \to 0} G_a \neq 0 \), the dispersion relation takes the low-momentum form

\[
\omega(k) = k^2 \frac{F_a}{G_a}. \tag{4}
\]

Eq. (4) implies that the GB dispersion relation is nonlinear. However, in order to determine the corresponding power of momentum and thus classify the GB according to the Nielsen–Chadha scheme, we would have to further know the limiting behavior of \( F_a \).

One should keep in mind that so far, we have provided just a parameterization of the GB dispersion relation in terms of the current transition amplitude, which only assumed rotational invariance and current conservation. In order to decide which of the possibilities outlined above is actually realized, we need more information on the dynamics of the system, i.e., on the functions \( F_a, G_a \).
B. Transition amplitude from Ward identity

We now get back to Eq. (1). In order to turn it into a quantitative prediction of the GB dispersion relation, we replace the commutator with the time ordering, that is, consider the correlator of two currents, \( \langle 0 | T \{ j^\mu_a(x) j^\nu_b(y) \} | 0 \rangle \). Taking a divergence and using the current conservation, we arrive at a simple Ward identity,

\[
\partial_\mu \langle 0 | T \{ j^\mu_a(x) j^\nu_b(y) \} | 0 \rangle = \delta^4(x - y) i f_{abc} \langle 0 | j^c_v(x) | 0 \rangle. \tag{5}
\]

Of course, with the assumed rotational invariance the right hand side of Eq. (5) may only be nonzero for \( \nu = 0 \) so that it exactly encompasses our order parameter—the charge density.

Throughout the paper we assume that the symmetry is not anomalous so that, in particular, the naive Ward identity (5) holds without any corrections due to, e.g., Schwinger terms. This assumption is partially justified by the models studied in the following sections, where our results prove equivalent to those obtained with different methods. In general, however, the validity of Eq. (5) should be checked case by case.

Next we use the Källén–Lehmann representation for the current–current correlator in the momentum space, plugging in the parameterization (2). We find that the condition (3), as expected, ensures cancelation of the one-particle poles in the correlator. Taking the limit \( |k| \to 0 \), it follows that in order to get a finite nonzero limit as the right hand side of the Ward identity (5) requires, the functions \( G_a, G_b \) have to have a nonzero limit as well. The resida of the poles then yield the density rule

\[
G_a G^*_b - G^*_a G_b = i f_{abc} \langle 0 | j^c_v(x) | 0 \rangle, \quad \text{or,}
\]

\[
\text{Im} G_a G^*_b = \frac{1}{2} f_{abc} \langle 0 | j^c_v(x) | 0 \rangle, \tag{6}
\]

where the \( G_s \) now stand for the zero-momentum limits of the respective functions defined by Eq. (2). Eq. (6) together with Eq. (4) constitute the main result of the paper. They say that nonzero density of a non-Abelian charge in the ground state requires the existence of one GB with nonlinear dispersion relation. In general (in fact, in all cases the author is aware of), this dispersion is quadratic unless \( F_a \) vanishes at \( k = 0 \). We will speculate on this possibility in the conclusions.

A few other comments are in order here. First, the fact that the function \( G_a \) has nonzero zero-momentum limit essentially means that the broken charge operator creates the GB even at zero momentum with a finite probability. This is in contrast to the Lorentz-invariant
case. Second, in dense relativistic systems the function $G_a$ must be proportional to the only Lorentz-violating parameter in the theory, the chemical potential. An example will be given in the next section. Last, in view of Eqs. (4) and (6), it is tempting to conclude that the energy of the GB is inversely proportional to the charge density. This indeed happens in the ferromagnet [4], but need not be true generally. Again in the next section, we shall see that both $F_a$ and $G_a$ may be proportional to the charge density so that it drops in the ratio.

III. LINEAR SIGMA MODEL

The first class of systems to which we shall now apply our general results will be that described by the relativistic linear sigma model at finite density (or chemical potential). This was already studied in Ref. [10] where we found that the chemical potential induces mixing terms in the Lagrangian and the excitation spectrum can be determined only upon an appropriate diagonalization of the matrix propagator. In this section we shall see that this complication may be circumvented by the use of Eq. (3), at least in the case of type-II GBs. We start with a detailed analysis of a simple example in order to illustrate the validity of the formulas (4) and (6).

A. Model with SU(2) x U(1) symmetry

We recall the model that was used to describe kaon condensation in the CFL phase of dense quark matter [2, 3]. It is defined by the Lagrangian,

$$ L = D_\mu \phi^\dagger D^\mu \phi - M^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2, $$

where the scalar field $\phi$ transforms as a complex doublet of the global SU(2) symmetry. In order to account for finite density, the chemical potential $\mu$ associated with the global U(1) invariance (particle number) is introduced via the covariant derivative, $D_0 \phi = (\partial_0 - i\mu)\phi$.

Once the value of the chemical potential exceeds the mass parameter $M$, the scalar field condenses. The ground state may be chosen so that only the lower component of $\phi$ develops expectation value. At tree level, $\langle \phi_2 \rangle = v/\sqrt{2}$, where $v = \sqrt{(\mu^2 - M^2)/\lambda}$. This breaks the original SU(2) x U(1) symmetry to its U(1)$_Q$ subgroup, generated by the matrix $Q = \frac{1}{2}(\mathbb{1} + \tau_3)$. The spectrum contains two (as opposed to three broken generators) GBs.
The type-II GB is annihilated by $\phi_1$ and its exact (tree-level) dispersion relation is given by

$$\omega = \sqrt{k^2 + \mu^2} - \mu.$$  \hfill (8)

The type-I GB is annihilated by a (energy-dependent) linear combination of $\phi_2$ and $\phi_2^\dagger$, and its dispersion relation corresponds to the “$−$” case of

$$\omega_2^2(k) = k^2 + 3\mu^2 - M^2 \pm \sqrt{4\mu^2k^2 + (3\mu^2 - M^2)^2}.$$  \hfill (9)

After shifting the field $\phi_2$ by its vacuum expectation value, the Noether currents $j_\mu$ associated with the SU(2) symmetry of the model pick up terms linear in $\phi$. From these, we may readily express the current transition amplitudes related to the type-II GB state $|G(k)\rangle$ as

$$\langle 0|j_\mu|G(k)\rangle = -i\langle 0|j_\mu^0|G(k)\rangle = \frac{v}{\sqrt{2}}(k^\mu + 2\mu\delta^{\mu0})\langle 0|\phi_1(0)|G(k)\rangle.$$  \hfill (10)

Note that this, according to Eq. (4), immediately leads to the dispersion relation $\omega = k^2/2\mu$, which is indeed the low-momentum limit of the exact result (8). Moreover, evaluating the amplitude $\langle 0|\phi_1(0)|G(k)\rangle$ explicitly gives

$$G_1G_2^* = -2i\mu^2v^2|\langle 0|\phi_1(0)|G(k)\rangle|^2 = -i\mu v^2$$

at $|k| = 0$. This is in accord with the density rule (6) for the ground-state isospin density is simply given by the constant term in $j^0_3$, $\langle 0|j^0_3|0\rangle = -\mu v^2$, and the relevant structure constant of the SU(2) group in the basis of Pauli matrices is $f_{123} = 2$.

Analogously, the coupling of the type-I GB state $|\pi(k)\rangle$ to the current $j_3^\mu$ is found to be

$$\langle 0|j_3^\mu|\pi(k)\rangle = \frac{v}{\sqrt{2}} \left[(k^\mu - 2\mu\delta^{\mu0})\langle 0|\phi_2^\dagger(0)|\pi(k)\rangle - (k^\mu + 2\mu\delta^{\mu0})\langle 0|\phi_2(0)|\pi(k)\rangle\right].$$

The basic complication as opposed to the type-II GB is that now the current amplitude is not proportional to a single scalar field amplitude as in Eq. (10), but contains two entangled amplitudes. Therefore, these do not simply drop in the ratio as before and have to be evaluated explicitly. Using the expressions for the scalar field amplitudes derived in Ref. [11], the final result for the transition amplitudes at $|k| \to 0$ becomes

$$\langle 0|j_3^0|\pi(k)\rangle = -\frac{v}{\sqrt{2}} \frac{1}{|k|} \left(\frac{\mu^2 - M^2}{3\mu^2 - M^2}\right)^{-1/4},$$

$$\langle 0|j_3|\pi(k)\rangle = -\frac{v}{\sqrt{2}} \frac{k}{|k|} \left(\frac{\mu^2 - M^2}{3\mu^2 - M^2}\right)^{1/4}.$$
A simple ratio of the coefficients of the spatial and temporal current transition amplitudes gives the dispersion relation

$$\omega = \sqrt{\frac{\mu^2 - M^2}{3\mu^2 - M^2}} |k|,$$

which is the correct low-momentum limit of the full dispersion (9).

**B. General case**

We checked the formulas (3) and (6) on the simple example of the linear sigma model with an SU(2) × U(1) symmetry. We saw that their application is particularly simple for the type-II GBs, which carry unbroken charge. On the other hand, the application to type-I GBs is obscured by mixing of scalar field transition amplitudes.

With this experience we are ready to investigate the general linear sigma model with arbitrary symmetry. In Ref. [10], we dealt with the Lagrangian

$$\mathcal{L} = D_\mu \phi^\dagger D^\mu \phi - V(\phi),$$

where $V(\phi)$ is the most general static potential containing terms up to fourth order in $\phi$ that is invariant under the prescribed symmetry. The covariant derivative, $D_\mu \phi = (\partial_\mu - i A_\mu) \phi$, includes the chemical potential(s) assigned to one or more mutually commuting generators of the symmetry group.

Once the chemical potential is large enough, the scalar field develops nonzero expectation value, $\phi_0$, and must be reparameterized as

$$\phi(x) = e^{i \Pi(x)} [\phi_0 + H(x)],$$

(11)

The field $\Pi(x)$ is a linear combination of the broken generators and includes the Goldstone degrees of freedom. The “radial” degrees of freedom are described by $H(x)$.

The Noether current associated with a particular conserved charge $T_a$ reads

$$j^\mu_a = -2 \text{Im} \phi^\dagger T_a D^\mu \phi = -2 \text{Im} \phi^\dagger T_a \partial^\mu \phi + 2\phi^\dagger T_a A^\mu \phi.$$

Now we just have to insert the parameterization (11) and retain the terms linear in the field $\Pi$,

$$j^\mu_{a,\text{lin}} = -\phi_0^\dagger \{T_a, \partial^\mu \Pi\} \phi_0 + 2i \phi_0^\dagger A^\mu [T_a, \Pi] \phi_0.$$
According to Eq. (4), this is all we need to calculate the dispersions of the type-II GBs present in the system. The commutator and anticommutator will give factors that are purely geometrical, i.e., determined by the structure of the symmetry group and the symmetry breaking pattern. Without having to evaluate them explicitly, we simply observe that the constant ($G_a$) term is proportional to the chemical potential. This means that the dispersion relation always comes out as $\omega = c k^2/\mu$, where $c$ is some, yet unknown $\mu$-independent constant.

This constant can be determined in a different manner. Recall that (at least at tree level), once there is a (at least discrete) symmetry that prevents the potential $V(\phi)$ from picking a cubic term, the phase transitions between the normal and the Bose–Einstein condensed phases as well as those between two ordered phases are of second order. This means that the dispersion relations of the excitations must be continuous across the phase transitions. Hence, due to their particularly simple form, they can be matched to the dispersion in the normal phase. In the normal phase, a particle carrying charge $Q$ of the symmetry equipped with the chemical potential $\mu$, has dispersion $\omega = \sqrt{k^2 + M^2 - \mu Q}$, $M$ being its gap at $\mu = 0$. At $M = \mu Q$ the particle becomes a GB of the spontaneously broken symmetry and its low-momentum dispersion relation reads

$$\omega = \frac{k^2}{2\mu Q}. \quad (12)$$

Due to the argument sketched above, this dispersion relation then persists to the broken-symmetry phase for arbitrary values of the chemical potential.

IV. NAMBU–JONA-LASINIO MODEL

Now we turn our attention to models of the Nambu–Jona-Lasinio (NJL) type, i.e., purely fermionic models with local, nonderivative four-fermion interaction. There, symmetry is broken dynamically by a vacuum expectation value of a composite operator. At the same time, the GB of the spontaneously broken symmetry is a bound state of the elementary fermions, and it manifests itself as a pole in the correlation functions of the appropriate composite operators.

The standard way to calculate the dispersion relation of the GB is as follows (see e.g. Ref. [4]). One first performs the Hubbard–Stratonovich transformation, including the cor-
responding pairing channels. The purely scalar action is next expanded to second order in the auxiliary scalar field so as to get the GB propagator and find its pole or, equivalently, the zero of its inverse. However, the inverse propagator contains a constant term which prevents us from identifying the gapless pole directly. This term is removed with the help of the gap equation. Only after then may the inverse propagator be expanded to second power in momentum in order to establish the dispersion relation.

The approach proposed in this paper simplifies the calculation of the GB dispersion relation in three aspects. First, there is no need to use the gap equation in order to establish the existence of a gapless excitation. Indeed, since the only assumption, besides rotational invariance, is the current conservation, the broken symmetry is already built into the formalism. Second, the loop integrals are only expanded to first order in the momentum, again since one power of momentum is already brought about by the current conservation. This saves us one order in the Taylor expansion and thus reduces the analytic formulas considerably. (Of course, the final result must be the same whatever method we use. We just choose the less elaborate one.) Third, at least when calculating the dispersion relation of a type-II GB, we may take the zero-th component of the external momentum equal to zero from the very beginning. This means that, at finite temperature, we get a direct expression of the GB dispersion in terms of thermal loops, without having to analytically continue the result back to Minkowski spacetime afterwards.

A. Dispersion relation from fermion loops

In the models of the NJL type, the Noether currents are fermion bilinears. Their couplings to the Goldstone states are therefore given by the one-loop diagrams, symbolically,

$$\langle 0|j^\mu|\pi(k)\rangle =$$

The solid circles denote the full fermion propagators, which are in turn obtained by solving the gap equation, while the empty circle stands for the yet unknown vertex coupling the two fermions to the GB state. In case of a type-II GB the dispersion relation is then, according
to Eqs. (2) and (4), given by the ratio,

\[ \omega(k) = \frac{k \cdot j_{a} X}{j_{a}^{0}}, \tag{13} \]

where the symbols standing in front of the loops denote the operators inserted in place of the crosses.

Before we explain in detail how the Goldstone–fermion vertex may be extracted from another Ward identity, let us note that the explicit evaluation of this vertex may in fact often be circumvented. This will be demonstrated later on explicit examples. In general, the auxiliary scalar fields introduced by the Hubbard–Stratonovich transformation mix so that one has to look for poles of a matrix propagator. However, it seems to be a generic feature of type-II GBs that they carry charge of some unbroken symmetry. As a result, they may couple to just one of the auxiliary scalar fields and no mixing appears. (For instance, recall the model of Sec. IIIA where the type-II GB couples just to \( \phi_{1} \), while the type-I GB couples to both \( \phi_{2} \) and \( \phi_{2}^{\dagger} \).) In other words, we may devise an interpolating field \( \psi^{\dagger} X \psi \), \( X \) being some matrix in the flavor space, and calculate instead of the transition amplitude \( \langle 0|j^{\mu}|\pi(k)\rangle \) the correlation function of \( j^{\mu} \) and \( \psi^{\dagger} X \psi \). The point is that, by the Källén–Lehmann representation, this correlation function is proportional to the desired transition amplitude, and that the GB propagator as well as the unknown amplitude \( \langle \pi(k)|\psi^{\dagger} X \psi|0\rangle \) drop out in the ratio (13).

In practice this means that, instead of the unknown Goldstone–fermion vertex in the loops in Eq. (13), we may simply insert the operator \( \psi^{\dagger} X \psi \). In particular, in the nonrelativistic NJL model, the current \( j_{a}^{\mu} \) associated to the charge \( T_{a} \) reads

\[ j_{a}^{0} = \psi^{\dagger} T_{a} \psi, \quad j_{a} = \frac{1}{2i m}(\psi^{\dagger} T_{a} \nabla \psi - \nabla \psi^{\dagger} T_{a} \psi), \]

where \( m \) is the common mass of the degenerate fermion flavors. Inserting these operators into the fermion loops in Eq. (13), and making the Taylor expansion of the spatial loop to the first order in the external momentum, we arrive at the result

\[ \omega(k) = \frac{k^{2}}{2m} \left\{ 1 + \frac{4}{3} \int \frac{d^{3}q}{(2\pi)^{3}} \text{Tr} \left[ q^{2} \frac{\partial S}{\partial q}(q) X S(q) T_{a} \right] \right\}, \tag{14} \]
where the trace is performed over the flavor space and \( S \) denotes the full fermion propagator. At finite temperature \( 1/\beta \) the integral over \( q^0 \) in Eq. (14) becomes a sum over fermion Matsubara frequencies, \( \nu_n = (2n + 1)\pi/\beta \). The GB thus naturally remains gapless up to a possible thermal symmetry restoring phase transition, but otherwise its dispersion can depend on temperature.

### B. GB–fermion coupling from Ward identity

As already stressed in the discussion below Eq. (13), the formula (14) has a limited use. It applies only to the cases in which the GB couples to just one of the auxiliary scalar fields introduced by the Hubbard–Stratonovich transformation (no mixing). Unless this happens, the (unknown) couplings of the GB to the fermion bilinear operators do not drop in the ratio (13).

We also noted that this no-mixing property is most likely to be generic for type-II GBs, which carry charge of some unbroken symmetry. Nevertheless, in any case, the GB–fermion coupling can be evaluated directly by using yet another Ward identity [14]. Let us introduce the connected vertex function,

\[
G_{\mu a}^{\text{c}}(x, y, z) = \langle 0 | T \{ j_{\mu}^{a}(x) \psi(y) \psi^\dagger(z) \} | 0 \rangle.
\]

Taking the divergence and stripping off the full fermion propagators corresponding to the external legs, we find the Ward identity for the proper vertex,

\[
k_{\mu} \Gamma_{\mu a}^{\text{c}}(p + k, p) = T_{a} S^{-1}(p + k) - S^{-1}(p) T_{a}.
\]

This vertex exhibits a pole due to the propagation of the intermediate GB state. Near the GB mass shell, it may be approximated by a sum of the bare vertex and the pole contribution [15]. Writing the inverse nonrelativistic fermion propagator as \( S^{-1}(\omega, p) = \omega - \frac{p^2}{2m} + \mu - \Sigma(\omega, p) \), where \( \Sigma \) is the self-energy, the bare part of the propagator exactly cancels against the bare vertex in the Ward identity (15). The remaining term expresses the residuum at the GB pole in the vertex function—which is proportional to the desired GB–fermion coupling—in terms of the fermion self-energy, so that

\[
\Gamma_{\mu a}^{\text{c}} = \frac{1}{N} \left[ \Sigma(p) T_{a} - T_{a} \Sigma(p + k) \right].
\]

The normalization constant \( N \) is proportional to the coupling of the GB to the broken current itself, but this is not important since this factor only appears in the ratio (13).
The GB–fermion coupling is apparently determined by the symmetry-breaking part of the fermion self-energy. Once this is calculated, e.g., in the mean-field approximation, it may be plugged into the loops in Eq. (13) to determine the GB dispersion.

In conclusion of our general discussion let us remark that all the results presented so far in Sec. IV extend straightforwardly to cases including Cooper pairing of fermions. Once the GB is annihilated by a difermion operator instead of a fermion–antifermion one, we just have to flip the direction of some of the arrows in the fermion loops. The $S(q)$ in Eq. (14) then denotes the matrix propagator in both the flavor and the Nambu $(\psi, \psi^\dagger)$ space.

In addition, Eqs. (15) and (16) modify accordingly: The matrix $T_0$ must be replaced with the block matrix in the Nambu space, \[ \begin{pmatrix} T_0 & 0 \\ 0 & -T_0^T \end{pmatrix}. \] In the end, the GB–fermion coupling vertex comes out as

\[ \begin{array}{c} \includegraphics[width=0.2\textwidth]{diagram.png} \\ \end{array} = -\frac{1}{N} \left[ \Sigma_{\psi\psi}(p)T_a^T + T_a\Sigma_{\psi\psi}(p+k) \right], \tag{17} \]

where $\Sigma_{\psi\psi}$ denotes the anomalous (pairing) part of the fermion self-energy.

C. Three-flavor degenerate Fermi gas

The nonrelativistic Fermi gas consisting of three degenerate fermion species has been investigated theoretically in great detail [8, 16, 17]. The reason is that such a system has already been successfully realized in experiments using atoms of $^6$Li and $^{40}$K. At the same time it provides an interesting analogy with quantum chromodynamics (QCD). A recent theoretical analysis suggests that under certain conditions, bound clusters of three atoms may form, being singlets with respect to the global SU(3) symmetry, very much like baryons in QCD [18]. Such a three-flavor atomic Fermi gas thus creates a natural playground for testing properties of quark matter.

In the following, we are going to illustrate the utility of our general formulas and rederive the results of Ref. [8] concerning the type-II GBs in the three-flavor system. To that end, recall that with a proper choice of the interaction, the global SU(3) symmetry is broken by a difermion condensate, \( \langle \psi_i \psi_j \rangle \). Since this is antisymmetric under the exchange of $i, j$, it transforms in the $3$ representation of SU(3). This leads to the symmetry breaking pattern $SU(3) \times U(1) \rightarrow SU(2) \times U(1)$.

Once we choose the condensate to point in the third direction, the generator $\lambda_8$ acquires
FIG. 1: Fermion loop for the calculation of the type-II GB dispersion in the three-flavor Fermi gas. The numbers adjacent to the fermion propagators denote the respective flavors. The upper line represents the normal propagator of the (ungapped) third fermion flavor, while the lower line represents the pairing of the first two flavors.

nonzero vacuum expectation value. Based on the general discussion in Sec. [II] we anticipate that the four broken generators, $\lambda_4, \lambda_5, \lambda_6, \lambda_7$, couple to a doublet of type-II GBs with quadratic dispersion relations. Due to the unbroken symmetry, these two GBs have the same dispersion. For the explicit calculation, we choose the GB in the sector $(\lambda_4, \lambda_5)$.

To determine its dispersion relation, we need to evaluate the loop in Fig. 1 at nonzero external momentum $\mathbf{k}$. [In order to make contact with literature [8], we do not perform the Taylor expansion in the external momentum as in Eq. (14).] Note that since the type-II GB couples to two broken currents, $j_{\mu}^4$ and $j_{\mu}^5$, we are free to choose any of them to calculate its dispersion. Here, we take up $j_{\mu}^4$. The two propagators that appear in the loop are given by the finite-temperature expressions (see Ref. [8])

$$G_{33}(i\nu_n, \mathbf{k}) = \frac{1}{(i\nu_n - \xi_{\mathbf{k}})}$$

$$F_{12}(i\nu_n, \mathbf{k}) = -i\Delta/[((i\nu_n)^2 - E_{\mathbf{k}}^2)]$$

where $\Delta$ is the gap parameter, $\xi_{\mathbf{k}} = k^2/2m - \mu$ is the bare fermion energy measured from the Fermi surface, and $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}$. The interpolating field for the GB is chosen as $\psi^T \lambda_7 \psi$. This may be understood as follows. By our choice, the ground state condensate carries the third (anti)color of the representation $\overline{3}$. The current $j_{\mu}^4$ whose transition amplitude we calculate, then excites a difermion state with the first (anti)color. The GB thus carries the quantum numbers of a pair of fermions with the second and the third color, respectively. The same conclusion follows immediately from Eq. (17): The fermion self-energy is proportional to $\lambda_2$ and $\{\lambda_2, \lambda_4\} = -\lambda_7$.

The point of using Eq. (13) to calculate the GB dispersion relation is that all common algebraic factors including the trace over the flavor space cancel in the ratio. After stripping off such trivial factors, we are thus left with the loop integral

$$J(\mathbf{k}) = \frac{1}{\beta} \sum_n \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \left\{ \frac{1}{-\frac{k+2q}{2m}} \right\} \frac{1}{(i\nu_n)^2 - E_{\mathbf{q}-\mathbf{k}}^2} (i\nu_n - \xi_{\mathbf{q}}).$$

(18)
where the two options in the curly brackets stand for the insertion of the temporal and spatial components of the current, respectively. Note that the Matsubara frequency entering the loop graphs has been set to zero because, as already remarked above, in case of a type-II GB the transition amplitude of $j^0$ is nonzero even at vanishing energy.

After performing the Matsubara sum, Eq. (18) acquires the form

$$J(k) = \int \frac{d^3 q}{(2\pi)^3} \left\{ \frac{1}{2m} \left[ \frac{1}{E_{q-k} + E_{q} - k} \right] \right\},$$

where $f(x) = 1/(e^{\beta x} + 1)$ is the Fermi–Dirac distribution. This result is completely equivalent to Eq. (A9) in Ref. [8], though not identical, of course, for we calculate a different loop graph here. The GB dispersion may be directly inferred from here by means of Eq. (13).

We omit the explicit expression since it is rather cumbersome and not particularly revealing.

V. FOUR-FLAVOR DEGENERATE FERMI GAS

In this section we are going to discuss the nonrelativistic degenerate Fermi gas with four fermion flavors which, to our best knowledge, has not been analyzed in literature so far. Besides being the next-to-simplest nontrivial case of a nonrelativistic fermionic system exhibiting type-II GBs, it also provides an interesting analogy with the two-color QCD with two quark flavors [19, 20, 21].

We start by assuming the existence of four degenerate fermion species with a common chemical potential $\mu$, which interact by a contact four-fermion interaction invariant under the global SU(4) × U(1) symmetry. For the time being, all we need to know is that this interaction destabilizes the Fermi sea by the formation of Cooper pairs. The detailed form of the interaction will be specified later.

A. Symmetry breaking patterns

First we analyze the symmetry properties of the ordered ground state. Analogously to the three-flavor case, the original SU(4) × U(1) symmetry is spontaneously broken by a difermion condensate $\langle \psi_i \psi_j \rangle$, which transforms as an antisymmetric rank-2 tensor of SU(4).

The analysis of the symmetry breaking patterns is much simplified by noting the Lie algebra isomorphism SU(4) ≃ SO(6). The six-dimensional antisymmetric tensor representation of SU(4) is equivalent to the vector representation of SO(6). This correspondence is
worked out in detail in the Appendix. The order parameter is thus represented by a complex 6-vector, \( \phi \), or equivalently, by two real 6-vectors, \( \phi_R = \text{Re}\, \phi \) and \( \phi_I = \text{Im}\, \phi \). The situation is pretty much similar to that in spin-one color superconductors where the order parameter is a complex 3-vector of SO(3) \[22, 23, 24\].

There are two qualitatively different possibilities, distinguished by the relative orientation of \( \phi_R \) and \( \phi_I \). First, if \( \phi_R \) and \( \phi_I \) are parallel, there is obviously an unbroken SO(5) rotational invariance. The symmetry breaking pattern is SO(6) \( \times \) U(1) \( \rightarrow \) SO(5), or SU(4) \( \times \) U(1) \( \rightarrow \) Sp(4), with six broken generators, transforming as a 5-plet and a singlet of the unbroken SO(5), respectively. Note that in this case, the U(1) symmetry of the Lagrangian may be used to even eliminate entirely the imaginary part of \( \phi \), i.e., to set \( \phi_I = 0 \). Let us remark that it is this phase of the nonrelativistic four-flavor Fermi gas that is analogous to the two-flavor two-color QCD. There, the reality of \( \phi \) is enforced by \( C \) and \( P \) conservation. Also, there is no global U(1) symmetry in two-color QCD as the corresponding phase transformations are the axial ones, being broken by the axial anomaly \[20, 21\].

The more interesting, but also more complicated possibility is that with nonzero \( \phi_R \) and \( \phi_I \) pointing in different directions. Then, the SO(6) \( \times \) U(1) symmetry may be exploited to cast the order parameter in the form \( \phi = (\alpha, \beta + i\gamma, 0, 0, 0, 0)^T \), with real \( \alpha, \beta, \gamma \), and both \( \alpha \) and \( \gamma \) nonzero. In general the SO(6) \( \times \) U(1) symmetry is now broken down to SO(4), thus breaking 10 generators, forming two vectors of the unbroken SO(4) as well as two singlets. However, when \( \beta = 0 \) and \( |\alpha| = |\gamma| \), there is an additional unbroken U(1) symmetry generated by a combination of a rotation in the \( (\phi_R, \phi_I) \) plane and the original U(1) phase transformation. There is thus one less broken generator and one less GB.

### B. Fermion propagator and excitation dispersions

According to our general discussion above, we need to know the charge densities in the ground state in order to determine the spectrum of GBs of the broken symmetry. The charge densities, \( \langle \psi^\dagger T_a \psi \rangle \), are obtained from the trace of the fermion propagator with the charge matrix \( T_a \). At this place, we have to specify the Lagrangian of the system.

Anticipating the Cooper pairing of the fermions, we choose the interaction Lagrangian in
a form suitable for a mean-field calculation,

$$\mathcal{L} = \psi^\dagger \left( i\partial_t + \frac{\nabla^2}{2m} + \mu \right) \psi + G \sum_{i=1}^{6} |\psi^T \Sigma_i \psi|^2,$$

(20)

where $\psi$ is the four-component fermion field and $\Sigma_i$ are a set of six basic antisymmetric $4 \times 4$ matrices, transforming as a vector of SO(6), see Eqs. (A1) and (A2). Thus, the Lagrangian (20) is manifestly SO(6) $\simeq$ SU(4) invariant.

Upon introduction of an auxiliary scalar field $\phi_i$ by the Hubbard–Stratonovich transformation, the Lagrangian becomes

$$\mathcal{L} = \psi^\dagger \left( i\partial_t + \frac{\nabla^2}{2m} + \mu \right) \psi - \frac{1}{4G} \phi_i^\dagger \phi_i + \frac{1}{2} \psi^T \Sigma_i \psi \phi_i^\dagger + \text{H.c.}.$$

(21)

The vacuum expectation value of $\phi$ provides the order parameter for symmetry breaking. In the mean-field approximation, $\phi$ is simply replaced with its vacuum expectation value and the semi-bosonized Lagrangian (21) then yields the inverse fermion propagator in the Nambu space,

$$G^{-1} = \begin{pmatrix} (G_0^+)^{-1} & \Phi^- \\ \Phi^+ & (G_0^-)^{-1} \end{pmatrix},$$

where $G_0^\pm$ are the bare propagators and $\Phi^- = \phi_i \Sigma_i^\dagger$, $\Phi^+ = (\Phi^-)^\dagger$. Inverting this block $2 \times 2$ matrix, we obtain

$$G = \begin{pmatrix} G^+ & \Xi^- \\ \Xi^+ & G^- \end{pmatrix},$$

where

$$G^\pm = [(G_0^+)^{-1} - \Phi^+ G_0^- \Phi^+]^{-1}, \quad \Xi^\pm = -G_0^- \Phi^+ G^\pm.$$

(22)

At finite temperature, the bare thermal propagators are simply $G_0^+(i\nu_n, k) = 1/(i\nu_n - \xi_k)$ and $G_0^-(i\nu_n, k) = 1/(i\nu_n + \xi_k)$. As explained in the previous section, we can exploit the SO(6) symmetry of the Lagrangian (20) so that only two components of the 6-vector $\phi$ are actually nonzero. Let us denote them generally as $a, b$ ($a \neq b$), so that $\Phi^- = \phi_a \Sigma_a^\dagger + \phi_b \Sigma_b^\dagger$. In the previous section we simply took $a = 1$ and $b = 2$, but when later calculating the charge densities, it will be more convenient to set $a = 1$ and $b = 6$.

According to Eq. (22), the full mean-field fermion propagator $G^-$ reads

$$G^-(i\nu_n, k) = -(i\nu_n - \xi_k)(\nu_n^2 + \xi_k^2 + \Phi^+ \Phi^-)^{-1}.$$

(23)
Using the properties of the $\Sigma_i$ matrices we have

$$
\Phi^+ \Phi^- = |\phi_a|^2 + |\phi_b|^2 - 2i \text{Im}(\phi_a \phi_b^*) \Sigma_a \Sigma_b^\dagger = |\phi_a|^2 + |\phi_b|^2 - 2 \text{Im}(\phi_a \phi_b^*)[P_{ab}^{(+)} - P_{ab}^{(-)}].
$$

The projectors $P_{ab}^{(\pm)}$ are defined in Eq. (A5). With projectors in the denominator, the matrix in the expression (23) for $G^-$ can easily be inverted and yields the final result

$$
G^-(i\nu_n, k) = -(i\nu_n - \xi_k) \left[ \frac{\nu_n^2 + \xi_k^2 + |\phi_a|^2 + |\phi_b|^2 + 2 \text{Im}(\phi_a \phi_b^*) [P_{ab}^{(+)} - P_{ab}^{(-)}]}{(\nu_n^2 + \xi_k^2 + |\phi_a|^2 + |\phi_b|^2)^2 - 4|\text{Im}(\phi_a \phi_b^*)|^2} \right]
$$

(24)

From Eq. (24) we can see that there are two doublets of fermionic excitations over the ordered ground state, with generally nonzero gaps. Their dispersion relations are most conveniently written in the form

$$
\omega^{(\pm)}(k) = \sqrt{\xi_k^2 + |\phi_a \mp i\phi_b|^2}.
$$

(25)

Two special cases deserve particular attention. First, in the SO(5) symmetric phase $\phi_b$ is set equal to zero so that all four excitations are degenerate, forming a fundamental quadruplet of $\text{SO}(5) \simeq \text{Sp}(4)$. Second, in the $\text{SO}(4) \times \text{U}(1)$ symmetric phase, $|\phi_a| = |\phi_b|$ with their complex phases differing by $\pi/2$. Eq. (25) then implies that two of the fermions are actually gapless. Altogether we have two pairs of degenerate fermions, which transform as doublets under the respective $\text{SU}(2)$ factors of $\text{SO}(4) \simeq \text{SU}(2) \times \text{SU}(2)$.

### C. Charge densities and GBs

The ground state densities of the conserved charges are obtained from the propagator $G^-$ as

$$
\langle \psi^\dagger T_a \psi \rangle = -\frac{1}{\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \text{Tr}[T_a^T G^-(i\nu_n, k)].
$$

For sake of this section, it is convenient to fix the direction of the order parameter $\phi$ in the $\text{SO}(6)$ space. As already remarked above, we choose $a = 1$ and $b = 6$. With the explicit form of the $\Sigma_i$ matrices given in Eq. (A2), we find

$$
i\Sigma_1 \Sigma_6^\dagger = P_{16}^{(+)} - P_{16}^{(-)} = -\begin{pmatrix} \tau_3 & 0 \\ 0 & \tau_3 \end{pmatrix}.
$$
Since $i\Sigma_1 \Sigma_6^\dagger = M_{16}$ is one of the SO(6) generators $M_{ij}$ defined in Eq. (A6), the orthogonality relation (A8) implies that it can also be the only one which acquires nonzero density in the ground state,

$$\langle M_{16} \rangle = 8 \text{Im}(\phi_1 \phi_6^*) \frac{1}{\beta} \sum n \int \frac{d^3k}{(2\pi)^3} \frac{(i\nu_n - \xi_k)}{(\nu_n^2 + \xi_k^2 + |\phi_1|^2 + |\phi_6|^2)^2 - 4|\text{Im}(\phi_1 \phi_6^*)|^2}$$

$$= -\int \frac{d^3k}{(2\pi)^3} \xi_k \left[ \frac{1}{\omega^2(k)} - \frac{1}{\omega(k)} - 2f(\omega^2(k)) \right],$$

after the Matsubara summation.

The spectrum of GBs may be understood as follows. First, in the SO(5) symmetric phase, $\phi_6 = 0$ and there is no charge density in the ground state. According to our general results, there are then six type-I GBs—five corresponding to the coset SO(6)/SO(5), and one to the broken U(1).

From now on, we shall analyze the SO(4) symmetric phase of the model (20), i.e., assume that both $\phi_1$ and $\phi_6$ are nonzero and have different complex phases. The condensate $\phi_1$ breaks all SO(6) generators of the form $M_{1i}$, where $i = 2, \ldots, 6$. On the other hand, the condensate $\phi_6$ breaks all SO(6) generators of the form $M_{j6}$, where $j = 1, \ldots, 5$. This is altogether nine spontaneously broken generators of SO(6). One of them, $M_{16}$, is a singlet of the unbroken SO(4). The remaining eight ones group into two SO(4) vectors, $M_{1i}$ and $M_{i6}$, with $i = 2, \ldots, 5$. Note that, by Eq. (A7), $[M_{1i}, M_{i6}] = 2iM_{16}$ (no summation over $i$!). Since $M_{16}$ has nonzero density in the ground state, the two broken generators $M_{1i}$ and $M_{i6}$ couple to a type-II GB with a quadratic dispersion relation. These eight broken generators are thus associated to an SO(4) vector of type-II GBs.

Finally, for $\phi_6 = \pm i\phi_1$ there is a residual unbroken U(1) symmetry generated by $1 \pm M_{16}$. Note that the combinations $M_{1i} \pm iM_{i6}$ are charged under this unbroken symmetry, carrying opposite charges. It may also be illuminating to observe that, for instance, when $\phi_6 = -i\phi_1$, the matrix $\Phi^-$ has the form

$$\Phi^- = 2\phi_1 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. $$

In words, only the first and third fermion flavors participate in the pairing. The second and fourth do not, and represent the two gapless excitations predicted by Eq. (25).
Let us now calculate the dispersion relation of the type-II GBs in this SO(4) \times U(1) symmetric phase. (As will be shown in the next section, the relation \( \phi_6 = \pm i \phi_1 \) is one of the two possible solutions of the mean-field gap equation, the other one corresponding to the SO(5) symmetric phase.) As already noted above, there are four type-II GBs, that couple to the broken generators \( M_{1i} \) and \( M_{i6} \). Since these form a multiplet of the unbroken SO(4), they have all the same dispersions. We choose \( i = 3 \) for which the broken generators have a particularly simple form,

\[
M_{13} = - \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_1 \end{pmatrix}, \quad M_{36} = \begin{pmatrix} \tau_2 & 0 \\ 0 & \tau_2 \end{pmatrix}.
\]

Together with \( M_{16} \) these generate an SU(2) subalgebra of SU(4).

The generators \( M_{13} \) and \( M_{36} \) must both couple to the sought type-II GB. When acting on the ground state, they both excite the third component of \( \phi \), so the interpolating field for the GB will be \( \psi^T \Sigma_3 \psi \). In order to determine the GB dispersion, we evaluate its coupling to the broken current of \( M_{13} \), which is given by the loop graph in Fig. 2, analogous to that for the thee-flavor case in Fig. 1. After performing the Matsubara sum, and throwing away trivial numerical factors, we arrive at the loop integral, analogous to Eq. (19),

\[
J(k) = \int \frac{d^3 q}{(2\pi)^3} \left\{ \frac{1}{-k+2m} \right\} \frac{\xi_q}{2\omega_{q-k}^{(\pm)} \omega_q^{(+)}} \left[ 1 - f(\omega_{q-k}^{(\pm)}) - f(\omega_q^{(+)}) \right] \frac{\omega_{q-k}^{(+)}}{\omega_q^{(+)}} + \frac{f(\omega_{q-k}^{(\pm)}) - f(\omega_q^{(+)})}{\omega_q^{(+)}} \right],
\]

where the upper and lower signs were obtained with \( \phi_6 = \pm i \phi_1 \), respectively. Ultimately, both yield the same dispersion, which is found by the Taylor expansion to first order in momentum and plugging the result into Eq. (19).
D. Gap equation

The actual value of the order parameter $\phi$ is determined by the gap equation. To that end, we need to know the anomalous part of the matrix propagator which follows from Eq. (22),

$$\Xi^-(i\nu_n, k) = -\frac{1}{i\nu_n - \xi_k}(\phi_a\Sigma^I_a + \phi_b\Sigma^I_b)G^-(i\nu_n, k).$$

In the mean-field approximation, we require the self-consistency at the one-loop level, which leads to the gap equation

$$\Phi^- = 2G\frac{1}{\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \Sigma^I_i \text{Tr}[\Sigma_i \Xi^-(i\nu_n, k)].$$

After working out the algebraic trace and the Matsubara sum, this set of gap equations for $\phi_a$ and $\phi_b$ may be rewritten in a particularly symmetric form,

$$\phi_a + i\phi_b = 4G(\phi_a + i\phi_b)I^-(\phi),$$
$$\phi_a - i\phi_b = 4G(\phi_a - i\phi_b)I^+(\phi),$$

where

$$I^{(\pm)}(\phi) = \int \frac{d^3k}{(2\pi)^3} \frac{1 - 2f(\omega^{(\pm)}(k))}{\omega^{(\pm)}(k)}.$$

The gap equations (26) apparently admit two qualitatively different solutions. The first one corresponds to $\text{Im}(\phi_a\phi_b^*) = 0$, i.e., the SO(5) symmetric phase. The gap equation then simply reads

$$1 = 4GI^{(\pm)}(\phi).$$

Either sign can be used in the integrals since, in this case, the $\omega^{(\pm)}$ dispersions are degenerate.

The second solution has $\text{Im}(\phi_a\phi_b^*) \neq 0$. Eq. (26) then implies that $\phi_a = \pm i\phi_b$. For the two possibilities, the gap equations are

$$1 = 4GI^{(-)}(\phi), \quad \text{for } \phi_a = +i\phi_b,$$
$$1 = 4GI^{(+)}(\phi), \quad \text{for } \phi_a = -i\phi_b.$$  

Note that this solution of the gap equations (26) precisely corresponds to the SO(4) $\times$ U(1) phase discussed in Sec. V A. In either case, $\phi_a = \pm i\phi_b$, only the two gapped fermion flavors contribute to the gap equation (28).

It is interesting to observe that the mean-field analysis does not support the most general form of the order parameter $\phi$. Rather, either the real and imaginary parts of $\phi$ are parallel,
breaking the SO(6) × U(1) symmetry to SO(5), or they are perpendicular with the same magnitude, breaking the symmetry to SO(4) × U(1).

Of course, which of these two possibilities is the actual ground state may depend on the magnitude of the coupling constant and the chemical potential, and can only be decided by solving the gap equations (27) and (28) and comparing their free energies. This task is however, beyond the scope of the present paper where the model (20) only serves as an illustration of the general features of symmetry breaking in presence of charge density.

VI. PSEUDO-GOLDSTONE BOSONS

So far we have been dealing exclusively with exact symmetries for which, once spontaneously broken, the Goldstone theorem ensures the existence of exactly gapless excitations. Real world is, however, more complicated. Almost as a rule, global internal symmetries are usually only approximate, leading to approximately conserved currents and, when spontaneously broken, to the existence of excitations with a small gap, so called pseudo-Goldstone bosons (pGBs). (In the following, we shall keep using the term Goldstone boson unless the existence of a nonzero gap is important for our discussion.)

Let us just note that, in fact, the only generic type of an exact global symmetry is the particle number conservation. This is, however, only Abelian and hence does not lead to the nontrivial phenomena studied in this paper. In addition, gauge symmetries are always exact. Here, once the gauge is appropriately fixed and the remaining global symmetry spontaneously broken, there are no physical GB states as they are “eaten” by the gauge bosons, making them massive by the Higgs mechanism. It turns out that the number of massive gauge bosons is always equal to the number of broken generators even if the number of would-be GBs is smaller [25].

A. Counting of pseudo-Goldstone bosons

It should be stressed that for approximate symmetries, there is no rigorous theorem analogous to the Goldstone’s, which would guarantee the existence of “approximately gapless” excitations. On a heuristic level, however, the conclusions of the theorems of Goldstone and of Nielsen and Chadha may be carried over to this less perfect, yet more common and phys-
cal situation. Before a more detailed discussion of approximate symmetries, let us illustrate this on a simple example.

Both Goldstone and Nielsen–Chadha theorems make predictions about the low-momentum limit of the GB dispersion relation without actually specifying what the low momentum means. What is the type and number of pGBs associated with a spontaneously broken approximate symmetry, therefore depends on the scale at which the symmetry is broken explicitly. Consider the simple model studied in Sec. IIIA. The type-II GB has a massive partner which is annihilated by $\phi_1^\dagger$, i.e., its antiparticle, and their exact tree-level dispersions may be jointly expressed as

$$\omega = \sqrt{k^2 + \mu^2} \pm \mu,$$

see Eq. (8). At low momentum, we clearly see a gapless mode with a quadratic dispersion and a mode with the gap $2\mu$. On the other hand, at high momentum both modes have almost linear dispersion.

So, once the SU(2) $\times$ U(1) symmetry of the model (7) is explicitly broken, the resulting pGB spectrum depends on whether the scale of the explicit breaking is smaller or larger than the chemical potential $\mu$. For weak breaking, we still see an approximately gapless mode with a quadratic dispersion and a mode with a gap approximately $2\mu$. On the other hand, for strong explicit breaking, larger than $\mu$, we can no longer resolve the quadratic onset of the dispersion, and instead find two pGBs with linear dispersion relations. Note that in both cases the Nielsen–Chadha counting rule is satisfied even for pGBs.

**B. Ward identity and dispersion relations**

Let us now consider a symmetry which is explicitly broken by adding a small term to the Lagrangian of the theory. The associated current then becomes only approximately conserved and we find $\partial_\mu j_\mu^a = \Delta_a \mathcal{L}$, where $\Delta_a \mathcal{L}$ is the variation of the Lagrangian under the symmetry transformation. The basic assumption of an “approximate” spontaneously broken symmetry is that there is still a mode $|\pi(k)\rangle$ which couples to the current $j_\mu^a$. The parameterization (2), which is based just on rotational invariance, thus still holds. What does change is, of course, Eq. (3) which now becomes

$$(\omega^2 - k^2)F_a + \omega G_a = \langle 0|\Delta_a \mathcal{L}|\pi(k)\rangle.$$  

(29)
Up to this change, we proceed along the same line of argument as in Sec. II. The Ward identity (5) acquires a new term that accounts for the explicit symmetry breaking,

\[ \partial_\mu \langle 0| T \{ j_\mu^a (x) j_\nu^b (y) \}|0\rangle = \langle 0| T \{ \Delta_a \mathcal{L} (x) j_\nu^b (y) \}|0\rangle + \delta^4(x-y) i f_{abc} \langle 0| j_\nu^c (x)|0\rangle. \]

Since both the broken currents and \( \Delta_a \mathcal{L} \) couple to the GB state, both correlators can be expressed via the Källén–Lehmann representation. We find that cancelation of the one-particle poles exactly requires the energy and momentum to fulfill the dispersion relation (29). In the limit \( |k| \to 0 \), the Ward identity leads to the modified density rule,

\[ \text{Im} [G_a G^*_b + \omega(0) G_a F^*_b] = \frac{1}{2} f_{abc} \langle 0| j_0^c |0\rangle, \]

cf. Eq. (6). The explicit breaking of the symmetry enters here in terms of the (nonzero) energy gap, \( \omega(0) \).

Eq. (29) may be used to give a precise meaning to the otherwise vague term “approximately gapless mode”, as well as to distinguish type-I and type-II GBs in the case of an explicitly broken symmetry. Let us thus, again, assume that \( G_a(|k|) \) has a nonzero limit as \( |k| \to 0 \), and denote \( \langle 0| \Delta_a \mathcal{L} |\pi(k)\rangle = H_a(|k|) \), for sake of brevity.

According to Eq. (29), the energy gap \( \omega(0) \) of the pGB is given by the solution of the equation

\[ \omega^2(0) F_a + \omega(0) G_a = H_a, \]

which has exactly one positive solution provided all coefficients, \( F_a, G_a, H_a \), are positive,

\[ \omega(0) = \frac{1}{2 F_a} \left( -G_a + \sqrt{G_a^2 + 4 F_a H_a} \right). \]

The low-momentum behavior of the GB dispersion is apparently determined by the relative value of \( G_a^2 \) and \( F_a H_a \). If the explicit symmetry breaking is weak enough, i.e., if \( F_a H_a \ll G_a^2 \), the term quadratic in \( \omega \) may be neglected in comparison with that linear in \( \omega \), and the low-momentum dispersion relation of the pGB reads

\[ \omega(k) = \frac{H_a}{G_a} + k^2 \frac{F_a}{G_a}, \]  \hspace{1cm} (30)

as opposed to Eq. (4). We thus find a type-II pGB.

If, on the other hand, \( F_a H_a \gg G_a^2 \), the \( \omega^2 \) term dominates over the \( \omega \) term in Eq. (29), and the resulting low-momentum dispersion relation,

\[ \omega(k) = \sqrt{\frac{H_a}{F_a} + k^2}, \]  \hspace{1cm} (31)

is one of a type-I pGB with nonzero mass squared \( H_a/F_a \).
C. Nambu–Jona-Lasinio model

The formulas (29)–(31) are absolutely general but, unfortunately, not of much use unless we are somehow able to calculate the new amplitude, \( \langle 0 | \Delta_a \mathcal{L} | \pi(k) \rangle \). Nevertheless, there is a wide class of systems where this can be determined in the same way as the current amplitudes \( F_a \) and \( G_a \): They are the models of the NJL type where the symmetry-breaking operator is bilinear in the fermion fields.

First of all, note that this is all we need, for instance, to provide a realistic description of dense quark matter since both quark masses and the various chemical potentials can be taken into account by adding appropriate bilinear operators into the Lagrangian.

The point is that in such a case, the dispersion relation of the GB (30) [or (31) as well] is given by the ratio of fermion loops, just like in Eq. (13), which now becomes

\[
\omega(k) = \frac{\Delta_a \mathcal{L}}{\hat{j}_a^0} + k \cdot j_a.
\]

All that was said in Sec. IV A then holds without change also in the presence of explicit symmetry breaking.

VII. CONCLUSIONS

In this paper, we analyzed the general problem of spontaneous symmetry breaking in presence of nonzero charge density. Based on a Ward identity for the broken symmetry, we showed in Sec. II that nonzero density of a commutator of two broken currents implies the existence of a GB with nonlinear dispersion relation. Under fairly general circumstances, this dispersion is quadratic. The only assumption is the existence of a nonzero low-momentum limit of the transition amplitude of the spatial current, \( F_a \).

It is interesting to compare this result to that of Leutwyler [9]. In his approach, nonzero density of a non-Abelian charge leads to the presence of a term with a single time derivative in the effective Lagrangian. Together with the standard two-derivative kinetic term, this results in a quadratic dispersion of the GB. In fact, however, Leutwyler’s kinetic term is proportional to the same constant \( F_a \) that we use here just to parameterize the current.
transition amplitude. Thus, regarding the possibility of the existence of GBs with energy proportional to some higher power of momentum, our results are equivalent. Nevertheless, the comparison with Leutwyler’s approach gives us new insight into this open question: In order to have a GB with the power of momentum higher than two in the dispersion relation, the standard kinetic term in the low-energy effective Lagrangian would have to vanish. It is hard to imagine how such a theory would be defined even perturbatively, yet such a possibility is not theoretically ruled out.

In Sec. III we demonstrated the general results of Sec. II on the linear sigma model, which we studied in detail in our previous work [10, 11]. We were thus able to prove on a very general ground, that the tree-level dispersion of type-II GBs in such a class of models is of the form (12).

While in the framework of the linear sigma model we merely verified results that could be obtained in a different manner, in the models of the NJL type our general results are practically useful. In Sec. IV we suggested to calculate the GB dispersion relation by a direct evaluation of its coupling to the broken current. This procedure considerably reduces both the length and the complexity of the analytic computation as compared with the conventional approach using the Hubbard–Stratonovich transformation and the consecutive diagonalization of the quadratic part of the action.

In Sec. V we analyzed a particular example of a system exhibiting type-II GBs: The non-relativistic gas of four fermion flavors with an SU(4) × U(1) global symmetry. We showed that the mean-field gap equation admits two qualitatively different solutions, i.e., two different ordered phases. In the first one, all fermion flavors pair in a symmetric way, leaving unbroken an SO(5) subgroup of the original symmetry. The six broken generators give rise to six type-I GBs.

In the other phase, only two of the four fermion flavors participate in the pairing. The other two flavors remain gapless. In this case, the unbroken symmetry is SO(4) × U(1) and one of the broken charges—corresponding to the difference of the number of the paired and unpaired fermions—acquires nonzero density. Consequently, eight of the nine broken generators give rise to four type-II GBs with a quadratic dispersion. Which of these two phases is the actual ground state of the system, can only be decided upon the solution of the gap equation.

Finally, in Sec. VI we extended the results achieved in the preceding parts of the paper to
spontaneously broken approximate symmetries. We discovered that once the notions of the type and low-momentum dispersion relation of the GB are defined and treated carefully, the general results remain valid. In particular, in the NJL model this allows their application to the realistic description of dense quark matter.

Acknowledgments

The author is indebted to M. Buballa and J. Hošek for fruitful discussions and critical reading of the manuscript. He is also obliged to J. Noronha, D. Rischke, and A. Smetana for discussions. Furthermore, he gratefully acknowledges the hospitality offered him by ECT* Trento and by the Frankfurt Institute for Advanced Studies, where parts of the work were performed. The present work was in part supported by the Institutional Research Plan AV0Z10480505 and by the GA CR grant No. 202/06/0734.

APPENDIX A: ALGEBRA OF ANTISYMMETRIC $4 \times 4$ MATRICES

In Ref. [19] we showed that a general unitary antisymmetric $4 \times 4$ matrix $\Sigma$ with unit determinant may be cast in the form $\Sigma = n_i \Sigma_i, i = 1, \ldots , 6$, where $\Sigma_i$ is a set of six basis matrices, satisfying the constraint

$$\Sigma_i^\dagger \Sigma_j + \Sigma_j^\dagger \Sigma_i = 2 \delta_{ij} \mathbb{1},$$

and $n_i$ is an either real or pure imaginary unit vector. By this construction we provided an explicit mapping between the cosets SU(4)/Sp(4) and SO(6)/SO(5). We also found a particular explicit solution to Eq. (A1),

$$\Sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} \tau_2 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix} 0 & i \tau_1 \\ -i \tau_1 & 0 \end{pmatrix},$$

$$\Sigma_4 = \begin{pmatrix} i \tau_2 & 0 \\ 0 & -i \tau_2 \end{pmatrix}, \quad \Sigma_5 = \begin{pmatrix} 0 & i \tau_2 \\ i \tau_2 & 0 \end{pmatrix}, \quad \Sigma_6 = \begin{pmatrix} 0 & i \tau_3 \\ -i \tau_3 & 0 \end{pmatrix}. \quad (A2)$$

By a simple complex conjugation, using the antisymmetry of the $\Sigma_i$s, Eq. (A1) translates to

$$\Sigma_i \Sigma_j^\dagger + \Sigma_j \Sigma_i^\dagger = 2 \delta_{ij} \mathbb{1}. \quad (A3)$$
In the context of the present paper, \( n_i \) no is longer a unit vector, but rather an arbitrary complex six-dimensional vector. We start by showing that its real and imaginary parts realize independent real representations of the group \( SU(4) \cong SO(6) \). This acts on \( \Sigma \) as \( \Sigma \to U\Sigma U^T \). For an infinitesimal transformation, \( U = 1 + iX \) with a traceless, Hermitian \( X \). Using the orthogonality relation

\[
\text{Tr} \Sigma_i^\dagger \Sigma_j = 4\delta_{ij},
\]

we derive the change of the coordinate \( n_i \) induced by the transformation \( U \),

\[
\delta n_i = \frac{in_j}{4} \text{Tr}[\Sigma_i^\dagger (X \Sigma_j + \Sigma_j X^T)] = \frac{in_j}{2} \text{Tr}(X \Sigma_j \Sigma_i^\dagger).
\]

(We transposed the second term and used the antisymmetry of \( \Sigma_i \).) Now for traceless \( X \), the matrix \( i \text{Tr}(X \Sigma_j \Sigma_i^\dagger) \) is, with a little help from Eq. (A.3), readily seen to be real. As an immediate consequence, the real part of \( n_i \) transforms into the real part and analogously for the imaginary part, as was to be proven.

In fact, just the six matrices \( \Sigma_i \) generate the whole algebra of \( SU(4) \). To see this, note that \( i\Sigma_i \Sigma_j^\dagger, i \neq j \), are 15 independent matrices with the following properties: 1. They are Hermitian. 2. Their trace is zero. 3. Their square is equal to unit matrix.

Proof. Hermiticity follows from \( (i\Sigma_i \Sigma_j^\dagger)^\dagger = -i\Sigma_j \Sigma_i^\dagger = i\Sigma_i \Sigma_j^\dagger \), by Eq. (A.3). The same identity together with the unitarity of \( \Sigma_i \) implies \( (i\Sigma_i \Sigma_j^\dagger)^2 = -\Sigma_i \Sigma_j^\dagger \Sigma_i \Sigma_j^\dagger = \Sigma_i \Sigma_j^\dagger \Sigma_j \Sigma_i^\dagger = 1 \). The tracelessness is an immediate consequence of Eq. (A.4).

The fact that the matrices \( i\Sigma_i \Sigma_j^\dagger \) square to one tells us that their eigenvalues are \( \pm 1 \) and, moreover, both with the twofold degeneracy, in order to ensure zero trace. We may then write \( i\Sigma_i \Sigma_j^\dagger = P_{ij}^{(+)} - P_{ij}^{(-)} \), where the projectors \( P_{ij}^{(\pm)} \) are conveniently expressed as

\[
P_{ij}^{(+)} = (\Sigma_i - i\Sigma_j)(\Sigma_i - i\Sigma_j)^\dagger,
\]

\[
P_{ij}^{(-)} = (\Sigma_i + i\Sigma_j)(\Sigma_i + i\Sigma_j)^\dagger.
\]

The identity (A.3) implies that the product \( i\Sigma_i \Sigma_j^\dagger \) is, for \( i \neq j \), antisymmetric under the exchange of \( i \) and \( j \). To make this antisymmetry explicit, let us define a new set of matrices,

\[
M_{ij} = \frac{i}{2}(\Sigma_i \Sigma_j^\dagger - \Sigma_j \Sigma_i^\dagger) = i(\Sigma_i \Sigma_j^\dagger - \delta_{ij} 1).
\]

By means of Eqs. (A.1) and (A.3), these may be shown to satisfy the commutation relation

\[
[M_{ij}, M_{kl}] = 2i(\delta_{il} M_{jk} + \delta_{jk} M_{il} - \delta_{ik} M_{jl} - \delta_{jl} M_{ik}).
\]
The matrices $M_{ij}$ thus provide an explicit mapping between the Lie algebras of SU(4) and SO(6).

Finally, Eqs. (A1), (A3), and (A4) imply the identity, analogous to that for the Dirac $\gamma$ matrices,

$$\text{Tr}(\Sigma_i \Sigma_j \Sigma_k \Sigma_l) = 4(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

As a consequence we have the orthogonality relation for the SO(6) generators $M_{ij}$,

$$\text{Tr}(M_{ij} M_{kl}) = 4(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$  \hfill (A8)

Note also that none of the identities proven in this Appendix relies on the explicit representation of the matrices $\Sigma_i$, Eq. (A2). All the results thus hold independently of the particular realization of $\Sigma_i$. In a sense, the construction worked out here is similar to that of the Clifford algebra of the spinor representations of the orthogonal groups. However, in case of SO($2n$), this is realized by matrices of rank $2n \times 2n$, while here we need just $4 \times 4$ matrices for the group SO(6).

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