Computing waveforms for spinning compact binaries in quasi-eccentric orbits

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Several scenarios have been proposed in which the orbits of binary black holes enter the band of a gravitational wave detector with significant eccentricity. To avoid missing these signals or biasing parameter estimation it is important that we consider waveform models that account for eccentricity. The ingredients needed to compute post-Newtonian (PN) waveforms produced by spinning black holes inspiralling on quasi-eccentric orbits have been available for almost two decades at 2 PN order, and this work has recently been extended to 2.5 PN order. However, the computational cost of directly implementing these waveforms is high, requiring many steps per orbit to evolve the system of coupled differential equations. Here we employ the standard techniques of a separation of timescales and a generalized Keplerian parameterization of the orbits to produce efficient waveforms describing spinning black hole binaries with arbitrary masses and spins on quasi-eccentric orbits to 1.5 PN order. We separate the fast orbital timescale from the slow spin-orbit precession timescale by solving for the orbital motion in a non-inertial frame of reference that follows the orbital precession. We outline a scheme for extending our approach to higher post-Newtonian order.

I. INTRODUCTION

Standard scenarios for the formation of black hole binaries predict that the orbits will have circularized by the time the system reaches the late inspiral phase probed by ground or space based gravitational wave detectors. There are however, alternative scenarios that produce systems with significant residual eccentricity for stellar mass systems; intermediate mass systems; and supermassive systems. Neglecting the effects of eccentricity on the waveforms will hurt detection of eccentric binaries with arbitrary masses and spins on quasi-eccentric orbits to 1.5 PN order. To avoid missing these signals or biasing parameter estimation it is important that we consider waveform models that account for eccentricity.

In most scenarios the individual black holes that make up the binary will be spinning, so what is needed are waveform templates that include the effects of spin and eccentricity. The equations of motion describing such systems, along with expressions for the instantaneous waveforms and energy and angular momentum fluxes were computed to 2nd post-Newtonian (PN) order (order $(v/c)^4$ in relative velocity of the system) by Kidder, Wiseman and Will in the early 1990’s, and more recently the calculation was carried to 2.5 PN order. By evolving these expressions for the coupled ordinary differential equations describing the orbital motion, spin precession and energy and angular momentum decay, it is possible to generate waveforms that can be used for data analysis and parameter estimation. The drawback of this direct approach is the high computational cost associated with accurately capturing the rapid orbital motion along with the smaller and slower effects of periastron precession, spin precession and orbital decay. A numerical implementation also loses control of the post-Newtonian expansion, introducing higher order effects that can lead to (potentially spurious) chaotic behavior.

Our goal here is to develop an efficient approach for producing waveforms for spinning eccentric binaries with arbitrary spins and eccentricities for use in gravitational wave astronomy. Given the long history of the problem, it is surprising that such waveforms do not already exist in the literature. Waveforms are available in the case of non-spinning eccentric binaries where analytic expressions for the orbital motion and phase evolution are available at 3 PN and 3.5 PN order, respectively. These can be combined with 2 PN accurate expression for the waveform amplitudes to produce ready to use gravitational waveforms. Efficient waveforms are also known for circular binaries with spin out to 1.5 PN order for precessing systems, and out to 2 PN order for non-precessing systems. For the more general case of eccentric binaries with spin, analytic expressions for the orbital motion and phase evolution have been computed to 1.5 PN order for equal mass systems and systems with one spinning body. These can be combined with the 1.5 PN order expression for the waveform amplitudes to produce ready to use waveforms for this sub-class of systems (however we caution that the waveform amplitudes quoted in Ref. contain several errors. We discuss and correct both in what follows).

Here we provide, for the first time, efficient waveforms at 1.5 PN order that describe eccentric binaries with arbitrary (but moderate) mass ratio and general spins. As is the case with the efficient waveform models described above, our derivation relies on the separation of timescales available in the problem that was first described by Damour. Our work extends the 1 PN accurate Post-Keplerian parameterization of the orbits introduced by Damour & Deruelle to include the leading order, 1.5 PN spin-orbit effects, for both the radial and angular motion (the radial component had previously been computed at this order by Wex). The key new element in our derivation, which allows us to analytically compute both the radial and angular components of the post-Keplerian description of the orbital motion, is that we re-cast the equations of motion in a non-inertial frame that follows the spin-orbit induced precession of the orbit. This allows us to maintain the separation between the fast orbital motion (which can be solve analytically), and the much slower spin-orbit
precession (which must be solve numerically for general masses and spins). We go on to compute the 1.5 PN order spin effects on the secular decay of the eccentricity and semi-major axis, building on the 1 PN accurate treatment of Junker and Schäfer \[33, 37\], and revisiting the earlier calculation of Gergely, Perjes and Vasuth \[39\], who used a different spin-supplementary condition. Our expressions for the orbital decay match - after converting from the eccentricity and semi-major axis to energy and angular momentum - with the gauge invariant expressions for the orbital decay found earlier by Rieth and Schäfer \[38\]. We also revisit the calculation of the waveform amplitudes \[34\] and correct several errors in the published expressions. Combining all these elements we arrive at ready to use, efficient, 1.5 PN order accurate expressions for the orbital decay found earlier by Rieth and Schäfer \[38\], and revisiting the treatment of Junker and Schäfer \[38\], and revisiting the earlier calculation of Gergely, Perjes and Vasuth \[39\], who used a different spin-supplementary condition. Our expressions for the orbital decay match - after converting from the eccentricity and semi-major axis to energy and angular momentum - with the gauge invariant expressions for the orbital decay found earlier by Rieth and Schäfer \[38\]. We also revisit the calculation of the waveform amplitudes \[34\] and correct several errors in the published expressions. Combining all these elements we arrive at ready to use, efficient, 1.5 PN order accurate waveforms for general, spinning, eccentric binaries. We conclude with some thoughts about continuing the calculation to higher post-Newtonian order.

\section{Equations of Motion}

We begin by deriving a semi-analytic solution to the dissipationless equations of motion at 1.5 PN order. This we accomplish in two steps - first we find an analytic solution to the equations of motion in a non-inertial frame that precesses with the orbital plane. Next we calculate the time dependent rotation between the inertial and precessing frame using a fast-slow decomposition of the spin-orbit precession equations.

The equations of motion at 1 PN order were solved by Damour & Deruelle \[36\] using a generalized Keplerian parameterization of the orbits. In what follows we will focus on the 1.5 PN order corrections, which can be added to the earlier result to give the complete solution at this order. The 1.5 PN equations of motion are mostly readily solved in the gauge defined by the Pray-Newton-Wigner \[41, 42\] (PNW) spin-supplementary condition. Suppressing 1 PN terms, the relative separation of the binary system \( r \), the individual spins \( S_1, S_2 \) and the orbital angular momentum \( L \) evolve according to the equations \[33, 57\] (in units where \( G = c = 1 \))

\[ \dot{r} = \frac{p}{\mu} + \frac{S_{\text{eff}} \times r}{r^3} \]  

(1)

and

\[ \frac{dL_N}{dt} = \frac{1}{r^3} S_{\text{eff}} \times L_N, \]  

(2)

\[ \frac{dS_1}{dt} = \frac{\delta_1}{r^3} L_N \times S_1, \]  

(3)

\[ \frac{dS_2}{dt} = \frac{\delta_2}{r^3} L_N \times S_2. \]  

(4)

Here \( M = m_1 + m_2 \) is the total mass, \( \mu = m_1 m_2 / M \) is the reduced mass, \( L_N = \mu r \times v \) is the Newtonian contribution to the orbital angular momentum and

\[ S_{\text{eff}} = \delta_1 S_1 + \delta_2 S_2 \]  

(5)

with

\[ \delta_1 = 2 \left(1 + \frac{3m_2}{4m_1}\right), \]  

\[ \delta_2 = 2 \left(1 + \frac{3m_1}{4m_2}\right). \]  

(6)

There are five constants of the motion: the magnitude of the angular momentum \( L \), the individual spin magnitudes \( S_1, S_2 \), the quantity \( L \cdot S_{\text{eff}} \), and the energy

\[ E = \frac{p^2}{2\mu} - \frac{\mu M}{r} + \frac{L \cdot S_{\text{eff}}}{r^3}. \]  

(8)

The angular momentum has post-Newtonian and spin corrections:

\[ L = L_N + L_{PN} + L_{SO} + \ldots, \]  

which to 1.5 PN order are given by

\[ L_{PN} = L_N \left(\frac{1}{2} v^2 (1 - 3\eta) + \frac{M}{r} (1 + 3\eta)\right), \]  

(9)

\[ L_{SO} = \frac{\mu}{r^3} r \times (r \times S_{\text{eff}}), \]  

(10)

with \( \eta = \mu / M \). The momentum and velocity can be decomposed into radial and angular contributions:

\[ p = p_r n + p_\perp m, \]  

\[ v = v_r n + v_\perp m, \]  

(11)

where \( n = r / r, \) \( m = \hat{L}_N \times n, \) \( v_r = \dot{r} \) and \( p_\perp = L \times r \). Combining the radial component of (11), \( p_r = \mu \dot{r} \), with (8) yields

\[ v^2 = \frac{2E}{\mu} + \frac{2M}{r} - \frac{L^2}{\mu^2 r^2} - \frac{2L \cdot S_{\text{eff}}}{\mu r^3}. \]  

(12)

Squaring (11) and dropping higher order terms yields

\[ v^2 = \dot{r} \cdot \dot{r} = \frac{p^2}{\mu^2} + \frac{2L \cdot S_{\text{eff}}}{\mu r^3}. \]  

(13)

from which it follows that

\[ v_\perp^2 = \frac{L^2}{\mu^2 r^2} + \frac{2L \cdot S_{\text{eff}}}{\mu r^3}. \]  

(14)

Solving the above set of equations is complicated by the precession of the orbital plane, which causes \( v_\perp \) to appear as a mix of azimuthal and longitudinal motion. The equations are more readily solved by transforming to a non-inertial frame that follows the precession of the orbital plane. The precessing frame is defined by the condition

\[ \frac{dL_N}{dt \big|_{\text{pre}}} = 0 = \frac{dL_N}{dt} - \omega \times \hat{L}_N, \]  

(15)

which implies that

\[ \omega = \frac{S_{\text{eff}}}{r^3}. \]  

(16)
The velocity in the precessing frame is given by
\[
\frac{dr}{dt}_{\text{pre}} = \frac{dr}{dt} - \omega \times r ,
\]
from which it follows that the orbital plane remains fixed in the precessing frame, \( \mathbf{L}_N \cdot \dot{\mathbf{r}}_{\text{pre}} = 0 \), and the radial motion is unchanged, \( \dot{r}_{\text{pre}} = \dot{r} \). Introducing the azimuthal coordinate \( \phi \) in the orbital plane we have \( \dot{r}_{\text{pre}}^2 = r^2 + r^2 \phi^2 \) and
\[
\dot{\phi}^2 = \frac{L^2}{\mu^2 r^4} .
\]

The equations for \( \dot{r} \) and \( \dot{\phi} \) can be solved by introducing the generalized Keplerian parameterization of the orbits [36]:
\[
\begin{align*}
\mathbf{r} &= r \cos \phi \hat{\mathbf{p}} + r \sin \phi \hat{\mathbf{l}} \\
n t &= u - e_t \sin u \\
r &= a(1 - e_r \cos u) \\
\phi &= 2(k + 1) \tan^{-1} \left[ \left( \frac{1 + e_\phi}{1 - e_\phi} \right)^{1/2} \tan \frac{u}{2} \right]
\end{align*}
\]
where \( u \) is the eccentric anomaly, \( n = 2\pi f \) is the mean motion with orbital frequency \( f \), \( k \) is the fractional periastron advance per orbit, \( a \) is the semi-major axis, and the regular Keplerian eccentricity has split into the triad of time \( e_t \), radial \( e_r \), and angular \( e_\phi \), eccentricities. Here we have used a coordinate system defined by the Newtonian angular momentum \( \mathbf{L}_N \) and the line of sight vector to the source \( \hat{\mathbf{N}} \):
\[
\begin{align*}
\mathbf{p} &= \mathbf{L}_N \times \hat{\mathbf{N}} / |\mathbf{L}_N \times \hat{\mathbf{N}}| , \\
\hat{\mathbf{l}} &= \mathbf{L}_N \times \mathbf{p} .
\end{align*}
\]
Following Ref. [36] and restoring the 1 PN contributions, we find
\[
\begin{align*}
a &= -\frac{\mu M}{2E} \left[ 1 + \frac{1}{2} \left( 7 - \eta \right) \frac{E}{\mu} - \frac{2\eta \mathbf{L} \cdot \mathbf{S}_{\text{eff}} E}{L^2} \frac{E}{\mu} \right] \frac{E}{\mu} \\
e_r^2 &= 1 + 2 \frac{EL^2}{\mu^3 M^2} + \frac{E}{\mu} \left[ 2(\eta - 6) + 5(\eta - 3) \frac{EL^2}{\mu^2 M^2} \right] + 8 \left[ 1 + \frac{EL^2}{\mu^3 M^2} \right] \frac{\eta \mathbf{L} \cdot \mathbf{S}_{\text{eff}}}{L^2} \frac{E}{\mu} \\
n &= \frac{1}{M} \left[ -\frac{2E}{\mu} \right]^{3/2} \left( 1 + \frac{1}{4} (15 - \eta) \frac{E}{\mu} \right) \\
e_t &= e_r \left( 1 + (8 - 3\eta) \frac{E}{\mu} - \frac{2\eta \mathbf{L} \cdot \mathbf{S}_{\text{eff}} E}{L^2} \frac{E}{\mu} \right) \\
k &= \frac{3\mu^2 M^2}{L^2} \left( 1 - \frac{\eta \mathbf{L} \cdot \mathbf{S}_{\text{eff}}}{L^2} \right) \\
e_\phi &= e_r \left( 1 - \frac{E}{\mu} \left( \eta - \frac{2\eta \mathbf{L} \cdot \mathbf{S}_{\text{eff}}}{L^2} \right) \right) .
\end{align*}
\]
For completeness we have included the 1.5 PN correction to the perihelion precession, \( k \), even though it is formally a 2.5 PN order term.

The next step is to solve the spin-orbit precession equations [2] to establish the time-dependent transformation between the inertial and precessing frames of reference. We begin by writing \( \mathbf{L}_N = \mathbf{L} + \delta \mathbf{L} \) where \( \mathbf{L} \) denotes the slowly changing, orbit averaged angular momentum, and \( \delta \mathbf{L} \) is a small periodic correction that varies on the orbital timescale. Adopting a similar decomposition for the two spins we find
\[
\delta \mathbf{L} = g(t) \left( \frac{M}{a} \right)^{3/2} \left( \frac{\mathbf{S}_{\text{eff}} \times \mathbf{L}}{M^2} \right) ,
\]
where
\[
g(t) = \frac{\phi - u}{(1 - e^2)^{3/2}} - \frac{\eta u}{(1 - e^2)} \left( \frac{1}{\sqrt{1 - e^2}} - \frac{1}{1 - e \cos u} \right) .
\]
In the above expression it is understood that we are using the Newtonian limit for \( u, \phi, e \), and \( \eta \). Note that the function \( g(t) \) is periodic with period \( T = 1/f \). The \( \delta \mathbf{L} \) term causes a periodic variation in the observed waveforms of the same order as the 1.5 PN amplitude corrections discussed below. The slowly varying orbit averaged expressions for the spins and angular momentum are found by numerically integrating the coupled set of differential equations
\[
\begin{align*}
\frac{d\mathbf{L}}{dt} &= \hat{\mathbf{S}}_{\text{eff}} \times \mathbf{L} , \\
\frac{d\mathbf{S}_1}{dt} &= -\frac{\delta_1}{a^3 (1 - e^2)^{3/2}} \mathbf{L} \times \mathbf{S}_1 , \\
\frac{d\mathbf{S}_2}{dt} &= -\frac{\delta_2}{a^3 (1 - e^2)^{3/2}} \mathbf{L} \times \mathbf{S}_2 .
\end{align*}
\]
III. DISSIPATION

Dissipational effects first enter the equations of motion at 2.5 PN order. Rather than directly integrating these equations, we adiabatically evolve the system by incrementing the energy and angular momentum according to the flux equations. Because the dissipation occurs on a much longer timescale than the orbital or precession motion, we begin by orbit averaging the instantaneous flux equations. Junker and Schäfer [38] carried out the calculation to 1 PN order, and we now extend their calculation to include the spin-orbit effects at 1.5 PN order. The instantaneous expressions for the 1.5 PN fluxes were computed by Kidder [17] using the covariant spin supplementary condition, and by Zeng and Will [45] using the PNW spin supplementary condition. We use the latter expressions to be consistent with the choice we made for the dissipationless equations of motion.

The orbit-averaged flux at 1.5 PN order has two contributions: one from averaging the 0 PN flux over a 1.5 PN order orbit, \langle F_{0,1.5} \rangle; the another from averaging the 1.5 PN flux over a 0 PN order orbit, \langle F_{1.5,0} \rangle. For the energy these are:

\[ \langle \dot{E}_{0,1.5} \rangle = \frac{M^2 \mu^2}{15a^5(1-e^2)^{3/2}} \left[ \mathbf{L} \cdot \mathbf{S}_{\text{eff}} (61 + 276e^2 + 471e^4) \right] \tag{36} \]

and

\[ \langle \dot{E}_{1.5,0} \rangle = \frac{M^2 \mu}{30a^7(1-e^2)^{1/2}} \left[ \mathbf{L} \cdot \mathbf{S} (784 + 5480e^2 + 3810e^4 + 195e^6) + \mathbf{L} \cdot \mathbf{Z} (432 + 2928e^2 + 1962e^4 + 96e^6) \right]. \tag{37} \]

Here we have introduced the spin combinations

\[ \mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 \]

\[ \mathbf{Z} = \frac{m_2}{m_1} \mathbf{S}_1 + \frac{m_1}{m_2} \mathbf{S}_2 , \tag{39} \]

which are related to \( \mathbf{S}_{\text{eff}} \) by

\[ \mathbf{S}_{\text{eff}} = 2 \mathbf{S} + \frac{3}{2} \mathbf{Z} \tag{40} \]

The contributions to the decay of the angular momentum are:

\[ \langle \dot{L}_{0,1.5} \rangle = \frac{2M^2 \mu^2 \mathbf{L} \cdot \mathbf{S}_{\text{eff}} (16 + 33e^2 + 26e^4)}{5a^5(1-e^2)^{7/2}} \tag{41} \]

and

\[ \langle \dot{L}_{1.5,0} \rangle = \frac{M^2 \mu^2}{15a^5(1-e^2)^{3/2}} \left\{ \mathbf{Z}_\parallel (144 + 592e^2 + 144e^4) + \mathbf{Z}_\perp (296 + 1032e^2 + 237e^4) - \frac{\mathbf{Z}_\parallel}{4}(480 + 2496e^2 + 671e^4) - \frac{\mathbf{Z}_\perp}{2}(332 + 1572e^2 + 435e^4) \right\} \tag{42} \]

where \( \mathbf{Z}_\parallel = \mathbf{L} \cdot \mathbf{S}, \mathbf{Z}_\perp = \mathbf{S} - \mathbf{Z}_\parallel, \) and similarly for \( \mathbf{Z}_0 \) and \( \mathbf{Z}_1 \). The terms parallel to the orbital angular momentum contribute to the decay of the orbit (since \( \dot{L} = \mathbf{L} \cdot \dot{\mathbf{L}} \)). The orthogonal terms introduce additional, higher order precessional effects. Rieth and Schäfer [40] found expressions for the spin-orbit corrections to the energy and angular momentum loss using a clever trick that allowed them to solve for the orbit averaged fluxes without having to solve the equations of motion. They quote their expressions in terms of \( E, L \) and \( r, \) but we can invert (25) to find expressions for \( E \) and \( L \) in terms of \( a \) and \( e_r \):

\[ E = -\frac{\mu}{2} \left( \frac{M}{a} \right) \left[ 1 - \frac{7 - \eta}{4} \left( \frac{M}{a} \right)^{3/2} \right] \tag{43} \]

\[ L = \mu M \sqrt{1 - e^2} \sqrt{\frac{a}{M}} \left[ 1 + \frac{(4 + 2 \eta - \eta e^2)}{2(1 - e^2)} \left( \frac{M}{a} \right)^{3/2} \right] \tag{44} \]

and after substituting these expressions into their equation (61) and converting the result into an energy flux by dividing through by \( -\mu \), we get perfect agreement with the sum of the terms in our equations (36) and (37). Similarly, after correcting their equation (70) by restoring a missing factor of \( 1/3^4 \) in the second term, we get perfect agreement with our expression for the rate of change of the angular momentum from equations (41) and (42). It is worth noting that the expressions for the orbit-averaged energy and angular momentum loss quoted by Rieth and Schäfer [40] are gauge invariant (i.e. they do not depend on the choice of SSC), while our expressions are not gauge invariant since they involve non-invariant quantities such as \( a \) and \( e_r \).

The adiabatic decay of the orbits is found by applying the chain rule to our expressions (25) for the semi-major axis and radial eccentricity:

\[ \langle \dot{e}_r \rangle = \left[ \frac{\partial e_r^2}{\partial E} \langle E \rangle + \frac{\partial e_r^2}{\partial L} \langle L \rangle \right] \left( 2e_r \right). \tag{45} \]

and similarly for \( a \). The eccentricity evolution provides a good cross check for our calculations as the collection of terms in square brackets have to cancel to order \( e_r^2 \) to avoid unphysical behavior in the circular limit.
Putting everything together and restoring the 0 PN and 1 PN terms we have

\[
\langle \dot{e}_r \rangle = -\frac{1}{15} \frac{\mu}{M^2} \left( \frac{M}{a} \right)^4 \frac{e_r}{(1 - e_r^2)^{5/2}} \left( 304 + 121 e_r^2 \right)
- \left( \frac{M}{a} \right)^{3/2} \left[ 8(16705 + 4676\eta) 
+ 12(9082 + 2807\eta) e_r^2 - (25211 - 3388\eta) e_r^4 \right]
+ \left( \frac{M}{a} \right)^{3/2} \left[ 7032 \hat{L} \cdot S 
+ 4408 \hat{L} \cdot Z \right]
+ (5592 \hat{L} \cdot S + 2886 \hat{L} \cdot Z)e_r^2
+ (1313 \hat{L} \cdot S + 875 \hat{L} \cdot Z)e_r^4 \right], \tag{46}
\]

and

\[
\langle \dot{a} \rangle = -\frac{1}{15} \left( \frac{M}{a} \right)^3 \frac{\eta}{(1 - e_r^2)^{7/2}} \left( 2(96 + 292e_r^2 + 37e_r^4) \right)
+ \left( \frac{M}{a} \right)^{3/2} \left[ 14(16008 + 4704\eta) 
+ (80124 + 21560\eta) e_r^2 + (17325 + 10458\eta) e_r^4 \right]
- \frac{1}{2} \left( 5501 - 1036\eta \right) e_r^6 \right]
+ \left( \frac{M}{a} \right)^{3/2} \left[ 2128 \hat{L} \cdot S + 1440 \hat{L} \cdot Z \right]
+ (7936 \hat{L} \cdot S + 4770 \hat{L} \cdot Z)e_r^2
+ (3510 \hat{L} \cdot S + 1737 \hat{L} \cdot Z)e_r^4
+ (363 \hat{L} \cdot S + 222 \hat{L} \cdot Z)e_r^6 \right]. \tag{47}
\]

as functions of \( a \) and \( e_r \):

\[
n = \frac{1}{M} \left( \frac{M}{a} \right)^{3/2} \left[ 1 - \left( \frac{M}{a} \right) \left( \frac{9 - \eta}{2} \right) \right]
+ \left( \frac{M}{a} \right)^{3/2} \left( \frac{3\hat{L} \cdot \text{Seff}}{2M^2\sqrt{1 - e_r^2}} \right) \tag{48}
\]

\[
e_t = \left( \frac{M}{a} \right)^{3/2} \left( \frac{3\hat{L} \cdot \text{Seff}}{2M^2\sqrt{1 - e_r^2}} \right) \tag{49}
\]

\[
k = \frac{3}{(1 - e_r^2)} \left( \frac{M}{a} \right) \tag{50}
\]

\[
e_{\phi} = \left. \left( 1 + \frac{\eta}{2} \left( \frac{M}{a} \right) \right) \left( \frac{M}{a} \right)^{3/2} \right( \frac{\hat{L} \cdot \text{Seff}}{M^2\sqrt{1 - e_r^2}} \right) \tag{51}
\]

IV. WAVEFORMS

With the orbital motion in hand, the final step is to derive expressions for the polarization states of the gravitational waves. These can be written as

\[
h_+ = \frac{1}{2}(p_i p_j - q_i q_j)h^{ij}_{\text{TT}} \tag{52}
\]

\[
h_\times = \frac{1}{2}(p_i q_j + q_i p_j)h^{ij}_{\text{TT}} \tag{52}
\]

where \( h^{ij}_{\text{TT}} \) are the transverse-traceless components of the metric perturbation and \( p \) and \( q \) are unit vectors orthogonal to the line of sight vector \( \text{N} \) defined by

\[
p = \hat{L}_N \times \hat{N} / |\hat{L}_N \times \hat{N}|, \tag{53}
\]

\[
q = \hat{N} \times p. \tag{53}
\]

Expressions for \( h^{ij}_{\text{TT}} \) out to the requisite order have been computed by Kidder [17], using the covariant spin-supplementary condition. To our knowledge, a similar calculation has not been done using the PNW spin supplementary condition, but transforming Kidder’s expressions to the PNW frame reveals no change to 1.5 PN order, which is all we need for our current purposes (the first changes appear at 2.5 PN order). Majár and Vasúth [34] have provided formal expressions for the instantaneous polarization states using a corotating coordinate system. They first define an new “invariant” source frame using the line of sight vector \( \text{N} \) and the total angular momentum vector \( J = L + S \), which then relate to a corotating system attached to the orbital angular momentum \( L_N \) and orbital separation vector \( r \). When dissipation is included, \( J \) slowly evolves and it is necessary to introduce a new invariant coordinate system. A convenient choice might be a frame related to the detectors, such as the Barycenter or Geocenter frames, which are approximately invariant on the observational timescale of a space based or ground
based detector. Thus our description of the waveforms involves three reference frames: the detection frame; the source frame; and the orbital frame. The rotation between the source frame and orbital frame occurs on the orbital timescale, while the rotation between the source frame and the detection frame occurs on the more sedate dissipation timescale.

The source frame is described by the triad \( \{ \hat{i}, \hat{j}, \hat{k} \} \), where \( \hat{k} = \hat{J}, \hat{i} = \hat{j} \times \hat{J}, \hat{j} = \hat{J} \times \hat{N}/\sin \gamma \), and \( \cos \gamma = \hat{J} \cdot \hat{N} \). The comoving orbital frame is described by the triad \( \{ \hat{x}, \hat{y}, \hat{z} \} \), where \( \hat{x} = \hat{r}, \hat{z} = \hat{L} \times \hat{N} \) and \( \hat{y} = \hat{L} \times \hat{r} \). The corotating system is related to the source frame by three time dependent Euler angles \( \{ \Phi, i_S, \Psi \} \), where \( \Psi \) is the orbital phase in the source frame, \( \Phi \) is the precession angle of the orbital plane, and \( i_S \) is the precession cone angle. Applying these rotations to \( \hat{N} \), we find it has the following components in the corotating frame:

\[
\begin{align*}
N_x &= \cos \Psi \cos \Phi \sin \gamma - \sin \Psi (\sin \Phi \sin \gamma \cos i_S - \cos \gamma \sin i_S) \\
N_y &= -\sin \Psi \cos \Phi \sin \gamma - \cos \Psi (\sin \Phi \sin \gamma \cos i_S - \cos \gamma \sin i_S) \\
N_z &= \sin i_S \sin \Phi \sin \gamma + \cos \gamma \cos i_S.
\end{align*}
\]

(54)

The components of \( p \) and \( q \) in the corotating frame are then

\[
\begin{align*}
p_i &= -N_y/\sqrt{N_x^2 + N_y^2} \\
p_j &= N_x/\sqrt{N_x^2 + N_y^2} \\
p_k &= 0,
\end{align*}
\]

(55) and

\[
\begin{align*}
q_i &= -N_x N_z/\sqrt{N_x^2 + N_y^2} \\
q_j &= N_y N_z/\sqrt{N_x^2 + N_y^2} \\
q_k &= \sqrt{N_x^2 + N_y^2}.
\end{align*}
\]

(56)

The mapping between the detection frame and the source frame can be accomplished by evolving the components of the spins and angular momentum in the detection frame, then solving for the angles that appear in the corotating frame using the relations

\[
\begin{align*}
\cos i_S &= \hat{J} \cdot \hat{L}_N \\
\cos \gamma &= \hat{J} \cdot \hat{N} \\
\cos \Phi &= \frac{(\hat{N} \times \hat{J}) \cdot (\hat{J} \times \hat{L}_N)}{|\hat{N} \times \hat{J}| |\hat{J} \times \hat{L}_N|},
\end{align*}
\]

(57, 58, 59)

and

\[
\dot{\Psi} = \dot{\phi} - \cos i_S \dot{\Phi}.
\]

(60)

For completeness, and also to correct several errors in the original paper, we re-compute and quote the explicit expression for the polarization states out to 1.5 PN order. The 1.5 PN terms are separated into spin-orbit (SO) and other contributions.
\[ h_+^0 = \left( \frac{\dot{r}^2}{r} - \frac{M}{r} \right) \left( p_x^2 - q_x^2 \right) + 2v_\perp \dot{r} (p_x p_y - q_x q_y) + v_\perp^2 (p_y^2 - q_y^2), \]  
\[ h_+^{0.5} = \frac{\delta m}{M} \left( \frac{\dot{r}}{r} \left[ \frac{2M}{r} - \dot{r}^2 \right] N_x + v_\perp \left[ \frac{M}{2r} - \dot{r}^2 \right] N_y \right) \left( p_x^2 - q_x^2 \right) \] 
\[ + v_\perp \left[ \frac{3M}{r} - 2\dot{r}^2 \right] N_x - 2v_\perp \dot{r} N_y \right) (p_x p_y - q_x q_y) \] 
\[ - v_\perp^2 (\dot{r} N_x + v_\perp N_y) (p_y^2 - q_y^2), \]  
\[ h_+^1 = \frac{1}{6} \left[ (1 - 3\eta) \left[ \left( \frac{-21\dot{r}^2 M}{r} + 3M v^2 \right) + 6v_\perp \left( \frac{M}{r} \right)^2 \right] \right] N_x^2 \] 
\[ + 4v_\perp \left( \frac{-6M}{r} + 3\dot{r}^2 \right) N_x N_y + 2v_\perp^2 \left[ \frac{3\dot{r}^2 - \frac{M}{r}}{N^2_y} \right] \] 
\[ + \left[ \frac{(19 + 9\eta)\dot{r}^2 M}{r} + (3 - 9\eta)v^2 \right] \left( \frac{M}{r} \right)^2 \right] \right] \left( p_x^2 - q_x^2 \right) \] 
\[ + \frac{v_\perp}{6} \left( (1 - 3\eta) \left[ 6\dot{r} \left( \frac{-5M}{r} + 2\dot{r}^2 \right) N_x^2 + 8v_\perp \left( \frac{-4M}{r} + 3\dot{r}^2 \right) N_x N_y + 12v_\perp \dot{r} N_y \right] \right) \] 
\[ + \frac{v_\perp^2}{6} \left( (1 - 3\eta) \left[ 2 \left( \frac{-7M}{r} + 3\dot{r}^2 \right) N_x^2 + 12v_\perp \dot{r} N_x N_y + 6v_\perp^2 N_y \right] \right) \] 
\[ + \left[ \frac{-(4 - 6\eta)M}{r} + (3 - 9\eta)v^2 \right] \right] \left( p_y^2 - q_y^2 \right), \]  
\[ h_+^{SO} = -\frac{1}{r^2} \left[ (\nabla \cdot q_x) p_x + (\nabla \cdot p) q_x \right], \]
\[ h_1 = \frac{\delta m}{M} \left\{ (1 - 2\eta) \left[ \frac{3\dot{r}^2 M}{4r} - \frac{v^2 M}{r} - \frac{41M^2}{12r^2} - r^4 \right] N_x^3 \right. \\
+ v_\perp \left[ \frac{85\dot{r}^2 M}{8r} - \frac{9v^2 M}{8r} - \frac{7M^2}{2r^2} - 3r^4 \right] N_x^2 N_y \\
+ 3v_\perp^2 \left[ \frac{2M}{r^2} - r^2 \right] N_x N_y^2 + v_\perp^3 \left[ \frac{M}{4r} - r^2 \right] N_y^3 \left\} \right. \\
+ \dot{r} \left[ \frac{(10 + 7\eta)^2 M}{2r} + \frac{(2 + \eta)v^2 M}{2r} \right. \\
- \frac{(11 - 6\eta)v^2 M}{6r^2} - \frac{(7 - 2\eta)M^2}{2} - \frac{(1 - 5\eta)v^2 r^2}{2} \right] N_y \\
+ v_\perp \left[ \frac{- (25 + 26\eta)\dot{r}^2 M}{8r} + \frac{(7 - 2\eta)v^2 M}{8r} \right. \\
- \frac{(26 - 3\eta)M^2}{6r^2} - \frac{(1 - 5\eta)v^2 r^2}{2} \right] N_y \\
+ 3v_\perp^2 \left[ \frac{5M}{2r} - 2r^2 \right] N_x N_y^2 - 2v_\perp^3 \dot{r} N_y^3 \right\} \left( p_x^2 - q_x^2 \right) \\
- \dot{r} \left[ \frac{- (7 + 3\eta)M}{r} + \frac{(1 - 5\eta)v^2}{2} \right] N_x + v_\perp \left[ \frac{3 - 8\eta M}{4r} - \frac{(1 - 5\eta)v^2}{2} \right] N_y \left( p_y^2 - q_y^2 \right), \right. \]
\begin{align}
h_0^3 &= 2 \left( r^2 - \frac{M}{r} \right) p_x q_x + v_\perp \dot{r} (p_x q_y + q_x p_y) + v_\perp^2 p_y q_y, \\
h_{05}^3 &= \frac{\delta m}{M} \left[ \left( r \left( 4 \frac{M}{r} - 2 r^2 \right) \right) n_x + v_\perp \left( \frac{M}{r} - 2 r^2 \right) n_y \right] p_x q_x \\
&\quad + v_\perp \left( 3 \frac{M}{r} - 2 r^2 \right) n_x - 2 v_\perp \dot{r} n_y \right) (p_x q_y + q_x p_y) \\
&\quad - 2 v_\perp^2 \left( \dot{r} n_x \div v_\perp n_y \right) p_y q_y, \\
h_x^3 &= \frac{1}{3} \left( 1 - 3 \eta \right) \left[ \left( \begin{array}{c} 21 r^2 M \\
+ 3 M v^2 \\
+ 6 r^4 + \frac{7 M^2}{r^2} \end{array} \right) N_x^2 \\
+ 4 v_\perp \dot{r} \left( -6 \frac{M}{r} + 3 r^2 \right) n_x n_y + 2 v_\perp \left( 3 r^2 - \frac{M}{r} \right) N_y^2 \right] \\
&\quad + \left( 19 - 9 \eta \right) \dot{r}^2 M \left( \begin{array}{c} (3 - 9 \eta) v^2 r^2 \\
- (10 + 3 \eta) v^2 r^2 + \frac{29 M^2}{r^2} \end{array} \right) p_x q_x \\
&\quad + \frac{v_\perp}{6} \left( 1 - 3 \eta \right) \left( \begin{array}{c} 6 r \left( -5 \frac{M}{r} + 2 r^2 \right) N_x^2 \\
+ 8 v_\perp \left( -4 \frac{M}{r} + 3 r^2 \right) N_x n_y + 12 v_\perp \dot{r} N_y^2 \end{array} \right) \\
&\quad + 6 \dot{r} \left( 2 + 4 \eta \right) M \left( \begin{array}{c} (1 - 3 \eta) v^2 r^2 \end{array} \right) \left( p_x q_y + q_x p_y \right) \\
&\quad + \frac{v_\perp}{3} \left( 1 - 3 \eta \right) \left( 2 \left( -\frac{7 M}{r} + 3 r^2 \right) N_x^2 + 12 v_\perp \dot{r} n_x n_y + 6 v_\perp^2 N_y^2 \right) \\
&\quad - \left( 4 - 6 \eta \right) M \left( \begin{array}{c} -3 - 9 \eta \end{array} \right) v^2 r^2 \right) p_y q_y, \\
h_{SO}^3 &= -\frac{1}{r^2} \left( \left( \Delta \cdot q \right) q_x - \left( \Delta \cdot p \right) p_x \right) .
\end{align}
where \( \delta m = m_2 - m_1 \), and \( \Delta = M(S_2/m_2 - S_1/m_1) \).

We have checked that the waveforms we get by substituting in our solution for the orbital motion and spin-orbit precession are in perfect agreement with the Lang and Hughes waveforms \([40]\) in the circular limit.

V. FUTURE WORK

It would be desirable to extend our treatment to higher post-Newtonian order. The necessary building blocks are known to 2.5 PN order, and while there is no fundamental barrier to going to higher order, there are some new effects to contend with. We will briefly describe some of the issues that crop up at 2 PN order. At this order the effects to contend with. We will briefly describe some of the issues that crop up at 2 PN order. At this order the precession equations read

\[
\begin{align*}
\frac{dS_1}{dt} &= \frac{\delta_1}{r^3} L_N \times S_1 + \frac{3}{r^3} (S_1 \times S_2 + (n \cdot S_2)(n \times S_1)), \\
\frac{dS_2}{dt} &= \frac{\delta_2}{r^3} L_N \times S_2 + \frac{3}{r^3} (S_2 \times S_1 + (n \cdot S_1)(n \times S_2)), \\
\frac{dL_N}{dt} &= \frac{1}{r^3} S_{\text{eff}} \times L_N - \frac{3}{r^3} ((n \cdot S_2)(n \times S_1) \\
&\quad+ (n \cdot S_1)(n \times S_1)), \quad (71)
\end{align*}
\]

from which it follows that \( L \cdot S_{\text{eff}} \) and the \( L_N \) are no longer constant. The condition \( dL_N/dt|_{\text{pre}} = 0 \) demands that the precessing frame rotates with angular velocity

\[
\omega = \frac{S_{\text{eff}}}{r^3} \left( \frac{3n}{r^3} \left( (n \cdot S_2)(L_N \cdot S_1) + (n \cdot S_1)(L_N \cdot S_2) \right) \right). \quad (72)
\]

Note that the new terms in \( \omega \) have no affect on the mapping between the velocity in the inertial and precessing frames. As before, the equations for \( \dot{r} \) and \( \dot{\phi} \) can be written as polynomials in \( 1/r \), with new spin\(^2 \) terms from the 2PN order Hamiltonian. In contrast to what we found at 1.5 PN order, the spin-dependent coefficients in the polynomial expansion are no longer constant. The time dependence of the coefficients prevents us from finding an exact generalized Keplerian solution to the equations of motion in the precessing frame beyond 1.5 PN order. On the other hand, we know that the spin dependent coefficients are approximately constant on the orbital time scale since the time dependence enters at 2 PN order. Thus we can find a solution for the Keplerian parameters by treating the spin dependent terms as constants, which are then updated adiabatically via the precession equations, just as the total energy and angular momentum are updated adiabatically via the dissipation equations.

As first noted by Damour and Schäfer\([47]\), and later solved by Schäfer and Wex\([48]\), the generalized Keplerian parameterization has to be further generalized to handle 2 PN terms in the equations of motion for non-spinning bodies. In particular, they found it necessary to introduce terms involving the quantity

\[
v = 2 \tan^{-1} \left[ \left( \frac{1 + e_\phi}{1 - e_\phi} \right)^{1/2} \tan \frac{u}{2} \right], \quad (73)
\]

which is closely related to the orbital angle \( \phi \). Some of the spin-spin terms that appear at 2 PN order will require addition terms that depend on \( v \). In particular, there are terms in the equation of motion of the form \( (S_1 \cdot n)(S_2 \cdot n) \), which produce terms like \( (S_1 \cdot q)(S_2 \cdot p) \cos v \) in the generalization Keplerian parameterization. In the adiabatic approach, spin dependent coefficients such as \( S_1 \cdot q, S_2 \cdot p \) and \( S_1 \cdot S_2 \) are treated as constants when solving the equations of motion. Expressions for the 2PN spin-spin corrections to the radial motion have already been derived using the PNW spin supplementary condition\([43]\). However, given that the terms involving \( L \cdot S_{\text{eff}} \) are no longer constant when 2 PN effects are taken into account, there is little advantage to using the PNW spin supplementary condition beyond 1.5 PN order. Indeed, it may be wise to adopt the covariant spin supplementary condition since this is the gauge in which the higher order corrections to the instantaneous waveforms have already been derived.

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