Noncommutative phase space with rotational symmetry and hydrogen atom

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Abstract

We construct algebra with noncommutativity of coordinates and noncommutativity of momenta which is rotationally invariant and equivalent to noncommutative algebra of canonical type. Influence of noncommutativity on the energy levels of hydrogen atom is studied in rotationally invariant noncommutative phase space. We find corrections to the levels up to the second order in the parameters of noncommutativity and estimate the upper bounds of these parameters.

Key words: noncommutative phase space, rotational symmetry, hydrogen atom

1 Introduction

Studying noncommutativity has obtained recently a great interest because of development of String Theory and Quantum Gravity (see, for instance,\textsuperscript{1}[1, 2]). Idea of noncommutativity is quite old. The idea has been proposed by Heisenberg and formalized by Snyder in his article published in 1947\textsuperscript{3}.

In recent years physical systems have been intensively studied in the framework of noncommutative classical and quantum mechanics. Among them, for instance, harmonic oscillator [4, 5, 6], Landau problem [7, 8, 9, 10, 11], gravitational quantum well [12, 13], classical systems with various potentials [14, 15, 16, 17, 18, 19], many-particle systems [20, 21, 22, 23, 24, 25, 26] and many others.

In general noncommutative phase space of canonical type can be realized with the help of the following commutation relations for coordinates and momenta

\begin{align}
[X_i, X_j] &= i\hbar\theta_{ij}, \\
[X_i, P_j] &= i\hbar(\delta_{ij} + \gamma_{ij}), \\
[P_i, P_j] &= i\hbar\eta_{ij},
\end{align}

where \(\theta_{ij}, \eta_{ij}, \gamma_{ij}\) are elements of constant matrixes, \(\theta_{ij}, \eta_{ij}\) are called parameters of coordinate and momentum noncommutativity, respectively.

The coordinates \(X_i\) and momenta \(P_i\) which satisfy (1), (3) can be represented as

\begin{align}
X_i &= x_i - \frac{1}{2} \sum_j \theta_{ij}p_j, \\
P_i &= p_i + \frac{1}{2} \sum_j \eta_{ij}x_j,
\end{align}

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with $x_i, p_i$ being coordinates and momenta satisfying the ordinary commutation relations
\[ [x_i, x_j] = 0, \quad [x_i, p_j] = i\hbar \delta_{ij}, \quad [p_i, p_j] = 0. \]
Taking into account (4), (5), one has
\[ [X_i, P_j] = i\hbar \delta_{ij} + i\hbar \sum_k \frac{\theta_{ik} \eta_{jk} 4}{4}. \]
Therefore, parameters $\gamma_{ij}$ are considered to be defined as
\[ \gamma_{ij} = \sum_k \frac{\theta_{ik} \eta_{jk}}{4}. \]

It is important to note that noncommutativity of canonical type (1)-(3) causes rotational symmetry breaking [28, 29]. New classes of algebras with noncommutativity of coordinates were proposed to recover the rotational symmetry in noncommutative space. For example, in [30] the rotational invariance was preserved by foliating the space with concentric fuzzy spheres. In [31] the authors constructed rotationally symmetric noncommutative space as a sequence of fuzzy spheres. In the space the exact solution of the hydrogen atom problem was found. In [32] the curved noncommutative space was introduced to maintain the rotational symmetry and the hydrogen atom spectrum was studied. In paper [33] in order to preserve the rotation invariance the authors suggested promotion of the parameter of noncommutativity to an operator in Hilbert space and introduced the canonical conjugate momentum of this operator.

In our previous paper [34] we considered the idea of generalization of parameter of noncommutativity to a tensor and proposed rotationally invariant algebra with noncommutativity of coordinates. In the present paper we propose the way to preserve rotational symmetry in a space with noncommutativity of coordinates and noncommutativity of momenta. We construct noncommutative algebra which is rotationally invariant and equivalent to noncommutative algebra of canonical type (1)-(3). In addition we study energy levels of hydrogen atom in rotationally invariant noncommutative phase space.

Note, that in the framework of noncommutative quantum mechanics the hydrogen atom has been studied in [21, 22, 27, 28, 35, 36, 37, 38, 39, 40, 41]. In paper [28] the authors examined hydrogen atom in noncommutative phase space and found corrections to the energy levels of hydrogen atom up to the first order in the parameter of coordinate noncommutativity. In this paper corrections to the Lamb shift within the noncommutative quantum electrodynamic theory were also obtained. In [21] the hydrogen atom was studied as a two-particle system in a space with noncommutativity of coordinates. The authors studied the case when particles of opposite charges feel opposite noncommutativity. In [36] the quadratic Stark effect was examined. New result for shifts in the spectrum of hydrogen atom in noncommutative space was presented in [37]. In [38] the hydrogen atom energy levels were calculated in the framework of the noncommutative Klein-Gordon equation. The Dirac equation with a Coulomb field was considered in a space with noncommutativity of coordinates in [39, 40]. In the paper [22] the authors found corrections to the energy levels of hydrogen atom in noncommutative phase space up to the first order in the parameter of noncommutativity. In [41] the author studied spectrum of Hydrogen atom, Lamb shift and Stark effect in a space with noncommutativity of coordinates and
noncommutativity of momenta. Full phase-space noncommutativity in the Dirac equation was considered in [27]. In this article the influence of momentum noncommutativity on the spectrum of hydrogen atom was studied. Hydrogen atom was also considered in the case of space-time noncommutativity in [42, 43, 44].

In contrast to the previous papers in this paper we study hydrogen atom in noncommutative phase space with preserved rotational symmetry.

The article is organized as follows. In Section 2 we consider rotationally invariant algebra corresponding to noncommutative phase space. In Section 3 influence of noncommutativity on the energy levels of hydrogen atom is studied in rotationally invariant noncommutative phase space. Conclusions are presented in Section 4.

2 Rotationally invariant noncommutative phase space

In order to preserve rotational symmetry in noncommutative phase space let us consider generalization of parameters of noncommutativity $\theta_{ij}, \eta_{ij}$ to tensors. We propose to construct these tensors with the help of additional coordinates $a_i, b_i$ and conjugate momenta $p_a^i, p_b^i$ of them which correspond to rotationally symmetric systems. In our previous papers we propose to define tensor of coordinate noncommutativity as follows

$$\theta_{ij} = \frac{l_0}{\hbar} \epsilon_{ijk} a_k, \quad (8)$$

with $l_0$ being a constant with the dimension of length [34, 47]. For reason of simplicity and reason of dimension we propose to write tensor of momentum noncommutativity as follows

$$\eta_{ij} = \frac{p_0}{\hbar} \epsilon_{ijk} p_k, \quad (9)$$

here $p_0$ being a constant with the dimension of momentum. Here $a_i$ and $p_i^b$ are additional coordinates and additional momenta governed by rotationally symmetric systems. We suppose that these systems are harmonic oscillators with parameters $m_{osc}$ and $\omega$

$$H_{osc}^a = \frac{(p^a)^2}{2m_{osc}} + \frac{m_{osc} \omega^2 a^2}{2}, \quad (10)$$

$$H_{osc}^b = \frac{(p^b)^2}{2m_{osc}} + \frac{m_{osc} \omega^2 b^2}{2}. \quad (11)$$

We put

$$\sqrt{\frac{\hbar}{m_{osc} \omega}} = l_P, \quad (12)$$

where $l_P$ is the Planck length. We also consider the frequency $\omega$ to be very large. This leads to great distance between energy levels of harmonic oscillators $H_{osc}^a, H_{osc}^b$. So, these oscillators put into the ground states remain in them.
From (7), (8) and (9) we have

$$\gamma_{ij} = \frac{l_0 p_0}{4\hbar^2} (a \cdot p^b) \delta_{ij} - a_j p_i^b. \tag{13}$$

So, we propose the following noncommutative algebra

$$[X_i, X_j] = i\varepsilon_{ijk} l_0 a_k, \tag{14}$$
$$[X_i, P_j] = i\hbar \left( \delta_{ij} + \frac{l_0 p_0}{4\hbar^2} (a \cdot p^b) \delta_{ij} - \frac{l_0 p_0}{4\hbar^2} a_j p_i^b \right), \tag{15}$$
$$[P_i, P_j] = \varepsilon_{ijk} p_0 p_k^b. \tag{16}$$

Additional coordinates $a_i, b_i$ and momenta $p_i^a, p_i^b$ satisfy the ordinary commutation relations

$$[a_i, a_j] = [b_i, b_j] = [a_i, b_j] = 0, \tag{17}$$
$$[a_i, p_i^a] = [b_i, p_i^b] = i\hbar \delta_{ij}, \tag{18}$$
$$[p_i^a, p_j^b] = [p_i^b, p_j^a] = 0, \tag{19}$$
$$[a_i, p_i^b] = [b_i, p_i^a] = 0. \tag{20}$$

So, $\gamma_{ij}, \theta_{ij}, \eta_{ij}$, given by (13), (8), (9) commute with each other. Also, $a_i$ and $p_i^b$ commute with $X_i$ and $P_i$. Therefore,

$$[\theta_{ij}, X_k] = [\theta_{ij}, P_k] = [\eta_{ij}, X_k] = [\eta_{ij}, P_k] = [\gamma_{ij}, X_k] = [\gamma_{ij}, P_k] = 0 \tag{21}$$

So, $X_i, P_i, \theta_{ij}, \eta_{ij}$ and $\gamma_{ij}$ satisfy the same commutation relations as in the case of the canonical version of noncommutative phase space. In this sense algebra (14)-(16) is equivalent to algebra (11)-(13). Moreover algebra (14)-(16) is rotationally invariant. Commutation relations remains the same after rotation

$$[X'_i, X'_j] = i\varepsilon_{ijk} l_0 a'_k, \tag{22}$$
$$[X'_i, P'_j] = i\hbar \left( \delta_{ij} + \frac{l_0 p_0}{4\hbar^2} (a' \cdot p^b) \delta_{ij} - \frac{l_0 p_0}{4\hbar^2} a_j p_i^b \right), \tag{23}$$
$$[P'_i, P'_j] = \varepsilon_{ijk} p_0 p_k^b. \tag{24}$$

Here $X'_i = U(\varphi) X_i U^+(\varphi), a'_i = U(\varphi) a_i U^+(\varphi), p_i^b = U(\varphi) p_i^b U^+(\varphi)$. The rotation operator is the following

$$U(\varphi) = e^{i\varphi (\mathbf{L})} \tag{25}$$

with

$$\mathbf{L} = [\mathbf{r} \times \mathbf{p}] + [\mathbf{a} \times \mathbf{p}^a] + [\mathbf{b} \times \mathbf{p}^b] \tag{26}$$

being the total angular momentum. Here $\mathbf{r} = (x_1, x_2, x_3)$

Making straightforward calculations we can conclude that operator $\mathbf{L}$ commutes with scalar products

$$[L_i, (\mathbf{a} \cdot \mathbf{p})] = [\tilde{L}_i, (\mathbf{b} \cdot \mathbf{p})] = [\tilde{L}_i, (\mathbf{a} \cdot \mathbf{b})] = [\tilde{L}_i, (\mathbf{r} \cdot \mathbf{a})] = [\tilde{L}_i, (\mathbf{r} \cdot \mathbf{b})] =$$
$$= [\tilde{L}_i, (\mathbf{a} \cdot \mathbf{L})] = [\tilde{L}_i, (\mathbf{b} \cdot \mathbf{L})] = [\tilde{L}_i, (\mathbf{p}^a \cdot \mathbf{L})] = [\tilde{L}_i, (\mathbf{p}^b \cdot \mathbf{L})] = 0. \tag{27}$$
here $L = [r \times p]$. Also, $\tilde{L}$ commutes with $i^2, p^2, a^2, b^2, (p^a)^2$, and $(p^b)^2$. Therefore, the total momentum $\tilde{L}$ commutes with the operator of distance $R = \sqrt{\sum_i X^2_i}$ which taking into account (1), (5), (8), (9) can be written in the following form

$$R = \sqrt{\sum_i X^2_i} = \sqrt{r^2 + \frac{l_0^2}{4\hbar^2} a^2 p^2 - \frac{l_0^2}{4\hbar^2} (a \cdot p)^2 - \frac{l_0^2}{\hbar} (a \cdot L)}.$$  

(28)

So, the distance remains the same after rotation $R' = U(\varphi)RU^+(\varphi) = R$. Also, taking into account (27), we have

$$[\tilde{L}_i, P] = 0$$  

(29)

where

$$P = \sqrt{\sum_i P^2_i} = \sqrt{p^2 + \frac{p_0^2}{4\hbar^2} r^2 (p^b)^2 - \frac{p_0^2}{4\hbar^2} (r \cdot p^b)^2 + \frac{p_0}{\hbar} (p^b \cdot L)}.$$  

(30)

So, the absolute value of momentum remains the same after rotation $P' = U(\varphi)PU^+(\varphi) = P$.

It is also worth mentioning that the following commutation relations are satisfied

$$[X_i, \tilde{L}_j] = i\hbar \varepsilon_{ijk} X_k,$$  

(31)

$$[P_i, \tilde{L}_j] = i\hbar \varepsilon_{ijk} P_k,$$  

(32)

$$[a_i, \tilde{L}_j] = i\hbar \varepsilon_{ijk} a_k,$$  

(33)

$$[p^a_i, \tilde{L}_j] = i\hbar \varepsilon_{ijk} p^a_k,$$  

(34)

$$[b_i, \tilde{L}_j] = i\hbar \varepsilon_{ijk} b_k,$$  

(35)

$$[p^b_i, \tilde{L}_j] = i\hbar \varepsilon_{ijk} p^b_k,$$  

(36)

which are the same as in the ordinary space ($\theta_{ij} = \eta_{ij} = 0$).

Explicit representation for noncommutative coordinates $X_i$ and noncommutative momenta $P_i$, which taking into account (11), (13), (18), (19), reads

$$X_i = x_i + \frac{l_0}{2\hbar} [a \times p]_i,$$  

(37)

$$P_i = p_i - \frac{p_0}{2\hbar} [r \times p^b]_i,$$  

(38)

(here coordinates $x_i$ and momenta $p_i$ satisfy the ordinary commutation relations and commute with $a_i, p^a_i$) guarantees that the Jacobi identity is satisfied. This can be easily checked for all possible triplets of operators.

Note, that from (37), (38) follows that

$$[X_i, p^a_j] = i\varepsilon_{ijk} \frac{l_0}{2} p_k,$$  

(39)

$$[P_i, b_j] = i\varepsilon_{ijk} \frac{l_0}{2} x_k,$$  

(40)

$$[X_i, a_j] = [X_i, b_j] = [X_i, p^b_j] = [P_i, a_j] = [P_i, p^a_j] = [P_i, p^b_j] = 0.$$  

(41)
At the end of this section we would like to discuss possible physical meaning of additional coordinates $a_i, b_i$ which we considered in order to construct tensors of noncommutativity. The coordinates can be treated as some internal coordinates of a particle. Quantum fluctuations of these coordinates lead effectively to a non-point-like particle, size of which is of the order of the Planck scale.

3 Hydrogen atom in noncommutative phase space with preserved rotational symmetry

Let us study influence of noncommutativity on the energy levels of hydrogen atom in rotationally invariant noncommutative phase space [14]-[16]. Note that tensors of noncommutativity [8], [9] corresponding to [14]-[16] are defined with the help of additional coordinates and momenta which are governed by the harmonic oscillators. Therefore in order to examine energy levels of hydrogen atom in noncommutative phase space we have to consider the following hamiltonian

$$H = H_h + H^{a}_{osc} + H^{b}_{osc},$$

(42)

here

$$H_h = \frac{P^2}{2M} - \frac{e^2}{R},$$

(43)

where $R = \sqrt{\sum_i X_i^2}$, and $X_i, P_i$ satisfy (14)-(16) and $H^{a}_{osc}, H^{b}_{osc}$ are given by (10), (11).

Using representation (37), (38) we can write

$$H_h = \frac{P^2}{2M} + \frac{(\eta \cdot L)}{2M} + \frac{[\eta \times r]^2}{8M} - \frac{e^2}{\sqrt{r^2 - (\theta \cdot L) + \frac{1}{4}[\theta \times p]^2}},$$

(44)

here the following notations

$$\theta = \frac{l_0}{\hbar} a,$$

(45)

$$\eta = \frac{p_0}{\hbar} b,$$

(46)

are used for convenience.

Let us find corrections to the energy levels of the hydrogen atom up to the second order in the parameters of noncommutativity. To do that, let us write expansion of the hamiltonian $H_h$ in the series over $\theta$. Note, that for $1/R$ one has the following expansion

$$\frac{1}{R} = \frac{1}{\sqrt{r^2 - (\theta \cdot L) + \frac{1}{4}[\theta \times p]^2}} =$$

$$= \frac{1}{r} + \frac{1}{2r^3} (\theta \cdot L) + \frac{3}{8r^5}(\theta \cdot L)^2 - \frac{1}{16} \left( \frac{1}{r^2}[\theta \times p]^2 \frac{1}{r} + \frac{1}{r}[\theta \times p]^2 \frac{1}{r^2} + \frac{\hbar^2}{r^7}[\theta \times r]^2 \right).$$

(47)

Note, that the last term in (47) is caused by noncommutativity of operators $[\theta \times p]^2$ and $r^2$ under the square root. So, using (47) we can write

$$H = H_0 + V,$$

(48)
with $V$ being the perturbation caused by the noncommutativity of coordinates

$$\begin{align*}
V &= \frac{(\eta \cdot L)}{2M} + \frac{[\eta \times r]^2}{8M} - \\
&\quad -\frac{e^2}{2r^3}(\theta \cdot L) - \frac{3e^2}{8r^5}(\theta \cdot L)^2 + \frac{e^2}{16} \left( \frac{1}{r^2} [\theta \times p]^2 \frac{1}{r} + \frac{1}{r} [\theta \times p]^2 \frac{1}{r^2} + \frac{\hbar^2}{2r^3} [\theta \times r]^2 \right),
\end{align*}$$

(49)

and

$$H_0 = \frac{p^2}{2M} - \frac{\eta^2}{2} + H_{osc}^a + H_{osc}^b. \quad (50)$$

Let us find corrections to the energy levels of hydrogen atom caused by noncommutativity. It is worth mentioning that

$$\begin{align*}
\frac{p^2}{2M} - \frac{\eta^2}{2} H_{osc}^a &= \frac{p^2}{2M} - \frac{\eta^2}{2} H_{osc}^b = [H_{osc}^a, H_{osc}^b] = 0, \quad (51)
\end{align*}$$

operator $p^2/2M - \eta^2/2$ is the Hamiltonian of the hydrogen atom in the ordinary space ($\theta_{ij} = \eta_{ij} = 0$). So, we can write eigenvalues and eigenstates of $H_0$

$$E_n^{(0)} = -\frac{\omega^2}{2a_B n^2} + \hbar \omega (n_1^a + n_2^a + n_3^a + n_1^b + n_2^b + n_3^b + 3), \quad (52)$$

$$\psi_{n,l,m}^{(0)} = \psi_{n,l,m}^{a} \psi_{n_1^a,n_2^a,n_3^a}^{b} \psi_{n_1^b,n_2^b,n_3^b}^{b}, \quad (53)$$

where $\psi_{n,l,m}^{a}$, $\psi_{n_1^a,n_2^a,n_3^a}^{b}$, $\psi_{n_1^a,n_2^a,n_3^a}^{b}$ are well known eigenfunctions of the hydrogen atom and the three-dimensional harmonic oscillators $H_{osc}^a$, $H_{osc}^b$ in the ordinary space ($\theta_{ij} = \eta_{ij} = 0$), $a_B$ is the Bohr radius. Let us note ones again that we consider the case when harmonic oscillators $H_{osc}^a$, $H_{osc}^b$ are in the ground states. So, taking this into account and using perturbation theory we can write the corrections to the energy levels of hydrogen atom as follows

$$\Delta E_n^{(1)} = \langle \psi_{n,l,m}^{(0)}, \{0\} | V | \psi_{n,l,m}^{(0)}, \{0\} \rangle. \quad (54)$$

First, let us mention that

$$\langle \psi_{0,0,0}^{a} | \psi_{0,0,0}^{a} \rangle = 0, \quad (55)$$

$$\langle \psi_{0,0,0}^{b} | \psi_{0,0,0}^{b} \rangle = 0, \quad (56)$$

therefore, corrections to the energy levels caused by the terms of the first order in the parameters of noncommutativity vanish. Namely,

$$\begin{align*}
\langle \psi_{n,l,m}^{(0)}, \{0\} | \frac{(\eta \cdot L)}{2M} | \psi_{n,l,m}^{(0)}, \{0\} \rangle &= 0, \quad (57) \\
\langle \psi_{n,l,m}^{(0)}, \{0\} | \frac{e^2}{2b^3} (\theta \cdot L) | \psi_{n,l,m}^{(0)}, \{0\} \rangle &= 0. \quad (58)
\end{align*}$$

Let us calculate corrections caused by the term $[\eta \times r]^2/8M$

$$\Delta E_2^{(1)} = \langle \psi_{n,l,m}^{(0)}, \{0\} | \frac{[\eta \times r]^2}{8M} | \psi_{n,l,m}^{(0)}, \{0\} \rangle = \frac{a_B^2 n^2}{24M} \langle \eta^2 \rangle. \quad (59)$$
Here we take into account that

\[
\langle \psi_{0,0,0}^b | \eta_i \eta_j | \psi_{0,0,0}^b \rangle = \frac{m_{\text{osc}} \omega P_0^2}{2\hbar} \delta_{ij} = \frac{1}{3} \langle \eta^2 \rangle \delta_{ij},
\]

(60)

with

\[
\langle \eta^2 \rangle = \frac{p_0^2}{\hbar^2} \langle \psi_{0,0,0}^b | (p^b)^2 | \psi_{0,0,0}^b \rangle = \frac{3m_{\text{osc}} \omega P_0^2}{2\hbar} = \frac{3p_0^2}{2l_P^2}.
\]

(61)

and use the following result (see, for example, [35])

\[
\langle \psi_{n,l,m} | r^2 | \psi_{n,l,m} \rangle = a_B^2 n^2 \frac{2}{2}(5n^2 + 1 - 3l(l + 1)).
\]

(62)

Also, we have

\[
\langle \psi_{n,l,m,\{0\}}^{(0)} | \frac{3e^2}{8r^5} (\theta \cdot L)^2 + \frac{e^2}{16} \left( \frac{1}{r^2} (\theta \times p)^2 \frac{l + 1}{l} + \frac{1}{r} (\theta \times p)^2 \frac{1}{l^2} + \frac{h^2}{rL} (\theta \times r)^2 \right) | \psi_{n,l,m,\{0\}}^{(0)} \rangle = \frac{\hbar^2 e^2 \langle \theta^2 \rangle}{a_B^2 n^5} \left( \frac{1}{6l(l + 1)(2l + 1)} - \frac{6n^2 - 2l(l + 1)}{3l(l + 1)(2l + 1)(2l + 3)(2l - 1)} + \frac{5n^2 - 3l(l + 1) + 1}{2l + 2)(2l + 1)(2l + 3)(l - 1)(2l - 1)} \right) - \frac{5}{6} \frac{l(l + 1)(l + 2)(2l + 1)(2l + 3)(l - 1)(2l - 1)}{l(l + 1)(l + 2)(2l + 1)(2l + 3)(l - 1)(2l - 1)}.
\]

(63)

here

\[
\langle \theta^2 \rangle = \frac{\hbar^2}{a_B^2} \langle \psi_{0,0,0}^b | a^2 | \psi_{0,0,0}^b \rangle = \frac{3l_0^2}{2\hbar} \left( \frac{1}{m_{\text{osc}}} \right) = \frac{3l_0^2}{2h^2}.
\]

(64)

The details of calculations of integrals in (63) can be found in our paper [34].

So, taking into account (57), (58), (59) and (63) corrections to the energy levels of hydrogen atom in the first order of perturbation theory read

\[
\Delta E_{n,l}^{(1)} = \Delta E_{n,l}^{(\eta)} + \Delta E_{n,l}^{(\theta)},
\]

(65)

with

\[
\Delta E_{n,l}^{(\eta)} = a_B^2 n^2 \langle \eta^2 \rangle \frac{24M}{(5n^2 + 1 - 3l(l + 1))},
\]

(66)

being corrections caused by noncommutativity of momenta and

\[
\Delta E_{n,l}^{(\theta)} = \frac{-\hbar^2 e^2 \langle \theta^2 \rangle}{a_B^2 n^5} \left( \frac{1}{6l(l + 1)(2l + 1)} - \frac{6n^2 - 2l(l + 1)}{3l(l + 1)(2l + 1)(2l + 3)(2l - 1)} + \frac{5n^2 - 3l(l + 1) + 1}{2l + 2)(2l + 1)(2l + 3)(l - 1)(2l - 1)} \right) - \frac{5}{6} \frac{l(l + 1)(l + 2)(2l + 1)(2l + 3)(l - 1)(2l - 1)}{l(l + 1)(l + 2)(2l + 1)(2l + 3)(l - 1)(2l - 1)},
\]

(67)

being corrections caused by coordinates noncommutativity.
In order to find corrections to the energy levels of hydrogen atom up to the second order in the parameters of noncommutativity we need to consider also the second order of the perturbation theory. We have

$$\Delta E_{n,l,m,\{0\}}^{(2)} = \sum_{n',l',m',\{n^a\},\{n^b\}} \frac{|\langle \psi_{n',l',m',\{n^a\},\{n^b\}}^{(0)} | V | \psi_{n,l,m,\{0\},\{0\}}^{(0)} \rangle|^2}{E_n^{(0)} - E_{n'}^{(0)} - \hbar \omega(n_1^a + n_2^3 + n_3^3 + n_4^l + n_5^l + n_6^l)} ,$$  \hspace{1cm} (68)

here the set of numbers $n', l', m', \{n^a\}, \{n^b\}$ does not coincide with the set $n, l, m, \{0\}, \{0\}$. We also use notation $E_n^{(0)}$ for the unperturbed energy of the hydrogen atom

$$E_n^{(0)} = -\frac{e^2}{2a_B n^2} .$$  \hspace{1cm} (69)

The frequency of the harmonic oscillator $\omega$ is considered to be very large. Taking into account that $\langle \psi_{n',l',m',\{n^a\}}^{(0)} | V | \psi_{n,l,m,\{0\},\{0\}}^{(0)} \rangle$ does not depend on $\omega$ because of (12). In the limit $\omega \to \infty$ we have

$$\lim_{\omega \to \infty} \Delta E_{n,l,m,\{0\}}^{(2)} = 0 .$$  \hspace{1cm} (70)

Therefore, up to the second order over the parameters of noncommutativity the corrections to the energy levels read

$$\Delta E_{n,l} = \Delta E_{n,l}^{(1)} .$$  \hspace{1cm} (71)

It is important to note that obtained result (71) is divergent in the case when $l = 0$ or $l = 1$. This means that to find finite result for corrections to $ns$ and $np$ levels we can not use expansion of hamiltonian into the series over the parameters of noncommutativity. In order to estimate the value of parameters of noncommutativity we are interested in the corrections to the $ns$ energy levels because they are measured with high precision. To find corrections to the $ns$ energy levels let us rewrite perturbation $V$ as follows

$$V = \frac{(\eta \cdot L)}{2M} + \frac{[\eta \times r]^2}{2M} - \frac{e^2}{R} + \frac{e^2}{r} = -\frac{e^2}{\sqrt{r^2 - (\theta \cdot L) + \frac{1}{4}[\theta \times p]^2}} + \frac{e^2}{r} ,$$  \hspace{1cm} (72)

Therefore the corrections to these levels read

$$\Delta E_{ns} = \left| \frac{\langle \psi_{n,0,0,\{0\},\{0\}}^{(0)} | (\eta \cdot L) 2M + [\eta \times r]^2 8M - e^2 r - \frac{e^2}{\sqrt{r^2 - (\theta \cdot L) + \frac{1}{4}[\theta \times p]^2}} | \psi_{n,0,0,\{0\},\{0\}}^{(0)} \rangle}{\sqrt{\sum_i \theta_i^2 |\psi_{0,0,0}^{(0)}|^2} \frac{2l_0 l_p}{\sqrt{\pi \hbar}} \sqrt{\frac{a_B^2 n^2}{24M}} (5n^2 + 1) + 1.72 \frac{\hbar (\theta) \pi e^2}{8a_B^2 n^3}} \right| .$$  \hspace{1cm} (73)

with

$$\langle \theta \rangle = \langle \psi_{0,0,0}^{(0)} | \sum_i \theta_i^2 |\psi_{0,0,0}^{(0)} \rangle = \frac{2l_0 l_p}{\sqrt{\pi \hbar}} \sqrt{\frac{a_B^2 n^2}{24M}} (5n^2 + 1) + 1.72 \frac{\hbar (\theta) \pi e^2}{8a_B^2 n^3} .$$  \hspace{1cm} (74)
Here we use (59) and results of calculation of corresponding integrals presented in our papers [46, 47].

Analyzing obtained results (71), (73), we can conclude that there is an important difference between influences of coordinates noncommutativity and momentum noncommutativity on the energy levels of hydrogen atom. For energy levels with large quantum numbers $n$ we have that corrections caused by noncommutativity of momenta $\Delta E_{n,l}^{(\theta)}$ (67) are proportional to $n^4$, in contrast corrections $\Delta E_{n,l}^{(\eta)}$ (66) are proportional to $1/n^3$. So, energy levels with large quantum numbers $n$ are more sensitive to the noncommutativity of momenta than noncommutativity of coordinates. Effect of coordinates noncommutativity better appears for energy levels with small quantum numbers $n$. Note also that corrections to the $ns$ energy levels (73) include terms proportional to $\langle \theta \rangle$ and terms proportional to $\langle \eta^2 \rangle$. In the case of $l > 1$ we found that corrections (71) include terms proportional to $\langle \theta^2 \rangle$ and $\langle \eta^2 \rangle$. So, we can conclude that $ns$ energy levels are more sensitive to the noncommutativity of coordinates (14).

At the end of this section let us estimate the values of parameters of noncommutativity. In order to obtain the upper bounds of the parameters of noncommutativity we suppose that corrections to the hydrogen atom transition energies which are caused by noncommutativity do not exceed the accuracy of the transitions measurements. Therefore, to estimate upper bounds of the parameters of noncommutativity we use result for $1s - 2s$ transition frequency because it is measured with height precision. In paper [48] the authors presented $f_{1s-2s} = 2466061413187018(11)\text{Hz}$ with relative uncertainty of $4.5 \times 10^{-15}$. So, we consider the following inequality

$$\left| \frac{\Delta_1^{\theta} + \Delta_2^{\eta}}{E_2^{(0)} - E_1^{(0)}} \right| \leq 4.5 \times 10^{-15}, \quad (75)$$

here $\Delta_1^{\theta}$, $\Delta_2^{\eta}$ are corrections to the energy of $1s - 2s$ transition caused by coordinates noncommutativity and momentum noncommutativity, respectively, $E_n^{(0)}$ is given by (69). To estimate the order of values of parameters of coordinate and momentum noncommutativity, it is sufficiently to consider the following inequalities

$$\left| \frac{\Delta_1^{\theta}}{E_2^{(0)} - E_1^{(0)}} \right| \leq 2.25 \times 10^{-15}, \quad (76)$$

$$\left| \frac{\Delta_1^{\eta}}{E_2^{(0)} - E_1^{(0)}} \right| \leq 2.25 \times 10^{-15}. \quad (77)$$

Taking into account (78) we have

$$\Delta_1^{\theta} = -\frac{3\hbar \langle \theta \rangle \pi e^2}{16a_B^3}, \quad (78)$$

$$\Delta_1^{\eta} = \frac{13a_B^2 \langle \eta^2 \rangle}{4M}. \quad (79)$$

So, we obtain

$$\hbar \langle \theta \rangle \leq 10^{-36} \text{ m}^2, \quad (80)$$

$$\hbar \sqrt{\langle \eta^2 \rangle} \leq 10^{-61} \text{ kg}^2 \text{ m}^2 / \text{s}^2. \quad (81)$$
It is worth mentioning that our estimation for parameters of coordinate and momentum noncommutativity (80), (81) are in the agreement with the results obtained in the literature. For instance, they are in agreement with upper bounds obtained from the spectrum of gravitation quantum well [12], from the spectrum of hydrogen atom considered in space with noncommutativity of momenta without preserved rotational symmetry [27], and on the basis of the data on the Lamb shift [28].

4 Conclusion

In the paper we have proposed the way to preserve rotational symmetry in a space with noncommutativity of coordinates and noncommutativity of momenta (1)-(3). Rotationally invariant noncommutative algebra was constructed with the help of generalization of parameters of noncommutativity to tensors. The tensors have been defined with the help of additional coordinates and conjugate momenta of them. The additional coordinates have been considered to be governed by spherically-symmetric systems. For reason of simplicity, we have considered the systems to be harmonic oscillators. As a result, we have proposed algebra with noncommutativity of coordinates and noncommutativity of momenta (14)-(16) which is rotationally invariant, moreover it is equivalent to the noncommutative algebra of canonical type (11)-(13).

We have studied the hydrogen atom in rotationally invariant noncommutative phase space (14)-(16). We have found corrections to the energy levels of the atom up to the second order in the parameters of noncommutativity (71). On the basis of obtained results we conclude that effect of momentum noncommutativity better appears for energy levels of hydrogen atom with large principal quantum numbers. These levels are more sensitive to the noncommutativity of momenta than noncommutativity of coordinates. Effect of coordinates noncommutativity better appears for energy levels with small quantum numbers \( n \). In addition we have obtained that corrections to the \( ns \)-energy levels caused by noncommutativity (73) are proportional to \( \langle \theta \rangle \). In the contrast, corrections to energy levels with \( l > 1 \) (71) are proportional to \( \langle \theta^2 \rangle \). So, we have concluded that \( ns \) energy levels are more sensitive to the noncommutativity of coordinates.

We have estimated the upper bounds of the parameters of noncommutativity on the basis of assumption that corrections to the energies caused by noncommutativity do not exceed the accuracy of measurements. So, for this purpose we have considered hight precision measurement of \( 1s-2s \) transition of hydrogen atom. Comparing corrections to the energy of \( 1s-2s \) transition caused by noncommutativity with the accuracy of experimental results for \( 1s-2s \) transition frequency measurement we have estimated the upper bounds for parameters of coordinate and momentum noncommutativity (80), (81) in rotationally-invariant noncommutative phase space. The upper bounds are in the agreement with that presented in the literature. Among them, for example, are upper bounds obtained from the spectrum of gravitation quantum well [12], from the spectrum of hydrogen atom considered in a space with noncommutativity of momenta without preserved rotational symmetry [27], and on the basis of the data on the Lamb shift [28].
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