\section*{Abstract}
The definition of a holomorphic function over a general measurable space $S$ endowed with a Markov process is defined by Zeghib and Barre. In this article we consider holomorphic functions over graphs whose ranges are a given finite field or a cyclic group. Also we consider a relation between $C$-holomorphic functions over regular trees and the field of $p$-adic numbers.

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\section{Introduction}
Let $R$ be a commutative ring. The definition of an $R$-holomorphic function over measurable space $S$ endowed by a Markov process is defined in [1]. In the sense of [1] a function $\phi : S \to R$ is holomorphic if both $\phi$ and $\phi^2$ are harmonic with respect to the natural notion of Laplacian over $S$. In the case of Riemannian manifolds this notion of $C$-holomorphic functions coincides with the concept of harmonic morphism from a manifold to $\mathbb{R}^2$. Harmonic morphism from a geometric graph to $\mathbb{C}$ are constant maps. Nevertheless there are nonconstant $C$-holomorphic function over graphs and therefore the definition of $C$-holomorphic functions is different from the notion of holomorphic function from a graph to $\mathbb{C}$. A harmonic function on a graph or a Riemannian manifold can be constructed by going to infinity (martin boundary) and then reproducing the object using a Poisson integral. Versus this global way of producing a holomorphic function, the construction of an $R$-holomorphic function will be done locally. In fact, there is a "special dynamics" which allows one to construct the holomorphic functions. We show this phenomenon with an example on 3-regular graphs. The main part of this article is devoted to $\mathbb{F}_p$ or $\mathbb{Z}_p$-valued holomorphic functions. In the case of $\mathbb{F}_p$ we provide a formula for the asymptotic behavior of the cardinality of the restriction of holomorphic functions to a neighborhood of a point in some regular trees. At the end of the article we determine a relation between $C$-valued holomorphic functions and the field of $p$-adic numbers. In fact we prove that a subgroup of automorphisms of this field is the same as special set of $C$-valued holomorphic functions on a $(p + 1)$-regular tree.

\section{Harmonic and holomorphic functions}
A graph is an ordered pair $X = (V, E)$ of disjoint sets $V$ and $E$ with a map $o : E \to V^2$, where $V^2 = \{(a, b) | a \in V \text{ and } b \in V\}$. Two elements $e \in V$ and $e' \in V$ are adjacent if $o^{-1}\{e, e'\}$ is not
an empty set. For an element \( e \in V \) let \( N(e) = \{ e' \in V \mid e' \sim e \} \), where \( e \sim e' \) means \( e \) is adjacent to \( e' \). Let \( C(X, F) \) be the set of all \( F \)-valued functions on the set \( V \). The elements of \( C(X, F) \) will be called the functions over the graph \( X \).

**Definition 1.** Let \( f \in C(X, F) \), the Laplacian \( \Delta f \) is the function:

\[
\Delta f(x) = \sum_{x \sim y} f(y) - f(x) \tag{1}
\]

**Definition 2.** A function \( g : X \to F \) is called harmonic if \( \Delta(g) = 0 \). The set of harmonic functions is a vector subspace of the set of all \( F \)-valued functions on \( X \).

The only \( \mathbb{R} \)-valued(\( \mathbb{C} \)-valued) harmonic functions on a finite graph are constant functions. Despite of this fact on \( \mathbb{R} \)-holomorphic functions, generally the set of harmonic \( F \)-valued functions are not constant on a finite graph.

**Example 1.** Let \( X \) be a cyclic graph with 6 vertices and let \( F = \mathbb{Z}_3 \). The set of \( \mathbb{Z}_3 \)-valued harmonic functions on \( X \) has nonconstant functions.

**Definition 3.** Let \( \phi : X \to F \), where \( X \) is a Riemannian manifold or a graph. We say that \( \phi \) is holomorphic if both \( \phi \) and its square \( \phi^2 \) are harmonic.

**Example 2.** Let \( X \) be a cyclic graph, then the set of \( \mathbb{F}_q \)-holomorphic functions on \( X \) (\( q \) is an odd number) is the same as the set of constant functions.

### 3 Holomorphic functions on finite fields

For the sake of simplicity we neglect finite fields of characteristic 2.

**Theorem 1.** Let \( \phi \) be a \( \mathbb{F}_q \)-valued holomorphic function over \( X \) then we have the following equations at each vertex.

\[
\sum_i a_i^2 = \sum_i a_i = 0 \tag{2}
\]

when \( a_i = \begin{cases} \phi(e_i) - \phi(e_0) & \text{if } e_i \sim e_0 \\ 0 & \text{otherwise} \end{cases} \)

**Proof.** By the definition of a holomorphic function we have

\[
\Delta(\phi)(e_0) = \sum_{e_i \sim e_0} \phi(e_i) - \phi(e_0) = \sum_i a_i = 0
\]

and

\[
\Delta(\phi^2)(e_0) = \sum_{e_i \sim e_0} (\phi^2(e_i) - \phi^2(e_0)) = \sum_i (\phi(e_i)^2 + \phi(e_0)^2) - 2\phi(e_0) \sum_i \phi(e_i) = \\
\sum_i (\phi(e_i) - \phi(e_0))^2 = \sum_i a_i^2 = 0.
\]

**Theorem 2.** There are nontrivial \( \mathbb{F}_q \)-valued holomorphic functions on a tree with no vertex of degree less then 4.
Proof. By the extension of the Chevalley’s theorem we know that a set of equations consisting of a quadratic form and a linear one on more than 4 variables has nontrivial solutions on $\mathbb{F}_p$. So if we let $\phi(e_0) = c$, then we can find values for $\phi$ on $N(e_0)$ which are not equal with $c$. Now we extend this to the points which are joint to $e_0$ by a path of length 2. If $\phi(e_1) = d$ then we have the following set of equations at the vertex $e_1$.

$$
\begin{cases}
(c - d)^2 + \sum_{i=1}^{\deg(e_1)}(y_i - d)^2 = 0 \\
(c - d) + \sum_{i=1}^{\deg(e_1)}(y_i - d) = 0
\end{cases}
$$

Let $y_i - d = x_i$. Now, we obtain $\sum_{i=1}^{\deg(e_1)-2}x_i^2 = b$ for some $b \in \mathbb{F}_p$. We know that the equation

$$x_1^2 + x_2^2 = b$$

has at least $p-1$ solutions \[^4\]. Considering $\deg(e_1) - 2 \geq 2$, we are able to extend $\phi$ to a holomorphic function. \qed

**Theorem 3.** Let $\text{Hol}_{e_0,s}(\mathbb{F}_q)$ denotes the restriction of the set of $\mathbb{F}_p$-valued holomorphic functions $\phi$ such that $\phi(e_0) = s$, to $N(e_0)$. Then $\text{card}(\text{Hol}_{e_0,s}(\mathbb{F}_q)) = q^{\deg(e_0)-2} + (q - 1)q^{\deg(e_0)-3}\eta$ when $\eta \in \{1, -1\}$.

**Proof.** From the equation \[^2\] it is enough to find the number of solutions of the equation

$$\sum_{1 < i < j, i = 2}^{\deg(e_0)} (a_j - a_i)^2 = 0$$

The number of solution of this equation is equal with the number of solutions of the nondegenerated diagonal quadratic equation in $\deg(e_i) - 1$ variables. Now the theorem is a result of Lebesgue theorem on the counting of number of solutions of a quadratic equation. \qed

Now we provide some facts on the cardinality of the restriction of holomorphic functions to a finite neighborhood of a vertex.

**Theorem 4.** Let $G$ be a tree and $p$ divides the degree of each vertex when $p$ is the characteristic of the field $\mathbb{F}_q$. Let $\text{Hol}_{e_0,e_1,s,u}(\mathbb{F}_q)$ denotes the set of $\mathbb{F}_q$ valued holomorphic functions $\phi$ such that $\phi(e_0) = s, \phi(e_1) = u$, restricted to the set $N(e_1)$. Then $\text{card}(\text{Hol}_{e_0,e_1,s,u}(\mathbb{F}_p)) = q^{p-3}$.

**Proof.** For $e_j \sim e_1 \ , j = 1, ..., p-1$, let $x_j = \phi(e_j) - u$, also let $t = u - s$. Then, we have the following set of equations.

$$
\begin{cases}
\sum_{j=1}^{p-1} x_j^2 = -t^2 \\
\sum_{j=1}^{p-1} x_j = -t
\end{cases}
$$

Now we are in the set up of \[^4\] (page 341). Where $b = \sum_j b_j^2a_j^{-1} = p - 1$, and $c = b_0^2 - a_0b = (-t)^2 - (p - 1)(-t^2) = pt^2 = 0$. The number of variables is equal with $p - 1$ which is an even number so the cardinality of solutions is $q^{p-3}$. \qed

**Corollary 1.** In a $p$-regular graph the cardinality of the set of functions obtained by the restriction of $\mathbb{F}_q$-valued holomorphic functions to an $r$-neighborhood of $e_0$ is equal with

$$[q^{p-2} + (q - 1)q^{\frac{p-2}{2}}\eta]q^{(p-3)(r-1)}.$$
Proof. The proof is an straightforward application of theorems \textcolor[rgb]{0.75,0.75,0.75}{3} and \textcolor[rgb]{0.75,0.75,0.75}{4}. □

Corollary 2. In a 3-regular tree (Tr$_3$), there are only finitely many $\mathbb{F}_{3^n}$-valued holomorphic functions up to permutation.

Proof. It is a result of Theorem \textcolor[rgb]{0.75,0.75,0.75}{4}. □

Theorem 5. Let $L$ be a finite graph with $n$ vertices and $|E|$ edges. Let $|E| > 3n$ ($\chi(L) > 2n$). Then there are nontrivial $\mathbb{F}_q$-valued holomorphic functions over $L$.

Proof. By the Theorem 2 we must solve a system of $2n$ equations on $E$ variables. $n$ equations have a degree equal with 2 and the other equations have degree equal with 1. Summation of equation’s degree are equal with $3n$. Therefore, by Theorem 6.8. of \textcolor[rgb]{0.75,0.75,0.75}{4} there exists nontrivial solutions for this set of equations. So, we have nontrivial holomorphic functions on these finite graphs. □

4 $\mathbb{Z}_p$-valued Holomorphic functions and dynamics

Theorem 6. There are nontrivial $\mathbb{Z}_p$ $p > 5$ valued holomorphic functions on a tree with no vertex of degree less than 6.

Proof. The first part of the proof is similar to the proof of theorem 2. For the last part of the discussion we must prove that the equation $\sum_{i=1}^{n} x_i^2 = b \pmod{p}$ for $n > 4$ has a nontrivial solution. Now we use the Lagrange’s four-square theorem [3] which asserts that every natural number is equal with the summation of at most 4 square numbers. Therefore every choice of fifth variable leads to a solution which is not trivial. □

Regular trees are important in the study of $p$-adic field $\mathbb{Q}_p$ and $p$-adic ring $\mathbb{Z}_p$ which is the inverse limit of the finite rings $\mathbb{Z}_{p^n}$. The Bruhat-Tits tree of a $p$-adic field is a regular $p + 1$- tree.

Let $s,t$ be two classes of neighboring elements of the Bruhat-Tits tree over $\mathbb{Q}_2$. This tree is isomorphic to Tr$_3$. Then we have the following Theorem.

Theorem 7. Let $\text{Aut}_{s,t}(\mathbb{Q}_2)$ be the set of automorphisms of the 2-adic field $\mathbb{Q}_2$ which preserve $s$ and $t$. Then there is a one-one correspondence between $\text{Aut}_{s,t}(\mathbb{Q}_2)$ and $\mathbb{C}$-valued holomorphic functions over the Bruhat-tits tree which have fixed values $\alpha$ and $\beta$, $\alpha \neq \beta$, on $s$ and $t$.

Proof. Let $A_1$ and $A_2$ be two connected subtrees of Tr$_3$ obtained by deleting the edge $st$. Let $A(s,t) = A_1 \cup \tilde{st}$, $A(t,s) = A_2 \cup st$. Let $H_{\alpha,\beta}$ be the space of holomorphic functions on $A(s,t)$ with prescribed values $\phi(s) = \alpha$ and $\phi(t) = \beta$. Let $\text{Aut}(A)$ be the automorphism group of the graph $A(s,t)$. $s$ and $t$ do not change under the action of the elements of $\text{Aut}(A)$. Also, for any element $w$ of $A(s,t)$, there are elements of the group which exchange two neighborhoods of $w$ which are not between $w$ and $s$. On the other hand, this exchanging was the only freedom in determining the holomorphic function (which is obtained by the induced dynamics to choose $j$ or $j^2$ [1]). Equivalently, right action of $\text{Aut}(A(s,t)) \times \text{Aut}(A(t,s))$ on $\mathbb{C}$-valued holomorphic functions over the Bruhat-tits tree with the fixed values $\alpha \neq \beta$ on $s$, $t$ is simply transitive. This completes the proof. □

Let we consider the special case of $G = Tr_3$. There is no $\mathbb{Z}_{3^n}$-holomorphic function on $G = Tr_3$. Now we consider the set of Holomorphic functions from $G$ to $\mathbb{Z}_9$. The equation $a + b + c = a^2 + b^2 + c^2 = 0$ has 3 different solutions (up to permutation) $(0,0,0), (3,3,3), (0,3,6)$ so we can
construct the set of holomorphic functions using a dynamics over $G$. In fact let we consider the dynamics obtained by the random choice of elements 0, 3 or 6 on edge initiated from each vertex. This dynamics is equivalent to all holomorphic functions on $T_{r_3}$ when we neglect the neighboring edges with constant values 0 or 3.

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