Physical Relaxation Terms for Compressible Two-Phase Systems

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Dedicated to the memory of Andro Mikelić

Abstract

In this note, we propose the first mathematical derivation of a macroscopic Baer-Nunziato type system for compressible two-phase flows allowing two pressure state laws depending on the different phases. By doing so, we extend the results obtained by the first author and M. Hillairet [Annales ENS (2019)] to cover this important physical situation. A relaxation term in the mass fraction equation is obtained without closure assumptions contrarily to theoretical-physics literature dedicated to mixture theory, see for instance “Thermo-Fluid dynamics of Two Fluid Flows” by M. Ishii. The relaxation parameter is linked to the viscosities of the different fluids (which may be small for applications) and the relaxed quantity is linked to the laws chosen at interfaces of the two-fluid system at a mesoscale. As this paper is intended for a large audience, we start with two formal arguments leading to the effective system. This provides formal procedures which could be useful for people working in environmental studies or industrial applications to understand how mixture models may be derived. Then we propose two mathematical proofs: One with a continuous approach (Hoff solutions for compressible NS equations with pressure depending on two transported quantities and the associated two-scale limit) and the other with a semi-discrete approach (ODE deduced from the discretization and its continuous limit). Finally, owing to the later approach, we describe some numerical experiments by comparing mesoscopic discretization and macroscopic discretization. This shows that theoretical proofs may be helpful in this topic to design appropriate numerical schemes.

Keywords: Homogenization, Compressible Navier-Stokes, Multi-fluid systems, Defect measures, Hoff solutions, Two-Scale limit, Numerical schemes.

Résumé

Dans cette note, nous étendons les résultats obtenus par le premier auteur et M. Hillairet [Annales ENS (2019)] pour couvrir des situations physiques importantes: Plus précisément, nous proposons la première justification mathématique d’un système de type Baer-Nunziato pour écoulements biphasés compressibles permettant deux lois d’état de pression dépendant des différentes phases et un terme de relaxation qui s’obtient sans hypothèse de fermeture contrairement aux ouvrages physiques dédiés à la théorie des mélanges comme le livre «Thermo-Fluid dynamic of Two Fluid Flows» par M. Ishii. La grandeur du paramètre de relaxation est liée aux viscosités des différents fluides (qui peuvent être petites) et la quantité relaxée est liée aux lois choisies aux interfaces du système bi-fluide à l’échelle mésoscopique. Comme cet article est destiné à un large public, nous commençons par deux arguments formels menant au système limite: Cela fournit des procédures formelles qui pourraient être utiles pour d’autres disciplines, par exemple, dans des études environnementales ou des applications industrielles. Nous proposons ensuite deux preuves mathématiques: l’une avec une approche continue (solutions à la Hoff d’équations de Navier-Stokes compressible avec pression dépendant de deux quantités transportées et sa limite à deux échelles) et l’autre avec une approche semi-discrète (Système d’ODE obtenu de la discrétisation et sa limite continue). Enfin, grâce à ces approches, nous présentons des illustrations numériques comparant la discrétisation mésoscopique et la discrétisation macroscopique. Ceci montre que les approches théoriques peuvent permettre sur ce sujet de définir des schémas numériques appropriés.

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Justification d’un terme de relaxation physique pour systèmes bi-phases.

Version française abrégée

Dans cette note, on propose de présenter l’obtention rigoureuse, en dimension un d’espace, d’un modèle de mélange à deux phases à une seule vitesse avec deux lois de pression différentes suivant la phase. Ce travail fait suite à un travail du premier auteur avec M. Hillairet (voir [5]) sur la justification de modèles de type Baer-Nunziato avec une pression commune aux deux phases. Il demande, pour une généralisation à deux pressions, de nouveaux résultats d’existence de solutions à la Hoff sur Navier-Stokes compressible avec pression dépendant de deux quantités satisfaisant chacune une équation de transport. Il est ensuite nécessaire d’introduire un paramètre d’homogénéisation et de justifier une asymptotique à deux échelles vers un modèle cinétique sous-jacent: L’obtention du modèle limite bi-phases provient alors de la caractérisation des mesures de défaux sous hypothèse initiale.

Plus précisément, on considère deux fluides compressibles (dont on connaît la loi d’état pour chacun) régis par les équations de Navier-Stokes avec chacun leur viscosité supposée constante. En considérant que le mélange de ces fluides est la limite de situations où les fluides sont séparés par des interfaces (approche multi-fluides à l’échelle mesoscopique) mais à une échelle ε de plus en plus fine, on obtient un système vérifié par la limite ε → 0, pour lequel on a une formule pour calculer la pression du mélange, ainsi qu’une équation pour la fraction volumique de chaque constituant avec un terme de relaxation totalement justifié faisant apparaître les différentes lois de pressions. Nous nous focalisons dans cette note sur le cas à deux composants mais le résultat se généralise au cas à plusieurs composants sans complication.

Pour commencer la note, nous présentons en section 2 les modèles mathématiques qu’ils soient méso-scopique ou macroscopique ainsi que les objectifs de preuves pour lier le modèle macroscopique bi-phases au système mesoscopique bi-fluides avec les données initiales. Nous présentons ensuite en section 3 deux approches formelles différentes (continue ou discrète) permettant d’obtenir le modèle limite. Nous commençons par une approche continue au travers d’une analyse à deux échelles de type WKB. Nous continuons ensuite par une approche discrète qui serait plus proche de ce que nous pouvons rencontrer dans le cadre d’applications. Ces résultats précisent les calculs formels que l’on peut trouver par exemple dans [20], [19], [1] en évitant l’hypothèse de fermeture formelle pour le terme de relaxation. Cette partie peut avoir un réel intérêt pour le lecteur intéressé par une compréhension formelle de l’obtention du modèle de mélange à partir du modèle mésoscopique avec interfaces.

Nous proposons ensuite dans la section 4 deux approches théoriques de justification mathématiques totalement complémentaires: I) Une approche d’homogénéisation sur le modèle mesoscopique sous sa forme continue où un théorème de type existence de solutions à la Hoff (pour NS compressible avec pression dépendant de deux paramètres transportés) et contrôle uniforme en le petit paramètre d’homogénéisation est nécessaire pour le passage à la limite. II) Une approche d’homogénéisation sous un angle plutôt discret qui consiste à considérer le système semi-discrétisé associé, d’en déduire un théorème d’existence locale pour le système d’ODE, de montrer un contrôle uniforme de versions discrètes des contrôles à la Hoff puis de passer à la limite vers le modèle continu justifiant ainsi une analyse formelle par un des auteurs dans [17]. Cette partie peut intéresser le lecteur motivé par une justification rigoureuse au travers d’une approche continue ou d’une approche discrétisée.

Nous terminons ensuite par la section 5 où nous montrons comment, grâce aux résultats théoriques, nous pouvons définir de bons schémas numériques et nous illustrons le tout par des résultats de simulation. Cette note est écrite pour présenter aux lecteurs, qu’ils soient mathématiciens, physiciens ou ingénieurs un nouveau cadre permettant d’aborder formellement, mathématiquement et numériquement l’obtention de certains modèles multiphasiques à partir de modèles multi-fluides mesoscopiques. Ce travail doit être vu comme une partie de la monographie complète [4] sur les systèmes multiphasiques avec deux pressions que les auteurs avec M. Hillairet se proposent de rédiger. Cette monographie rappellera les travaux précédents théoriques (de [8], [5], [6], [3], [12], [21], [22] par exemple) ainsi que les travaux de modélisation et aspects numériques importants ([20], [10], [19], [1], [9] et références contenues par exemple). Elle donnera également le détail des preuves et discutera quelques extensions possibles.
1 Introduction.

In this note, we propose to present the rigorous justification, in one space dimension, of a single velocity two-phase mixing model with two different pressure laws depending on each phase. This work follows a work by the first author with M. Hillairet (see [6]) on the justification of Baer-Nunziato type models with a pressure common to both phases. This asks, for a generalization to two possible different pressure state laws, new results of existence of solutions à la Hoff on compressible Navier-Stokes with pressure depending on two densities each satisfying a transport equation. It is then necessary to introduce a homogenization parameter and to justify a two scales asymptotic towards an underlying kinetic model: The obtaining of the two-phase limit model then comes from the characterization of the defect measures under initial hypothesis. More precisely, we consider two compressible fluids (of which we know the pressure state law for each) governed by the Navier–Stokes equations, each with their constant viscosity that we can write as a single compressible system with a pressure dependent on the averaged-density and a color function that is used to distinguish the fluid phase, both of which are advected by the flow. By considering that the mixture of these fluids is the limit of situations where the fluids are separated by interfaces (multi-fluid approach at the mesoscopic scale) but at an $\varepsilon$ scale more and more fine, we obtain a system verified by the limit $\varepsilon \to 0$, for which we have a formula to calculate the pressure of the mixture, as well as an equation for the volume fraction of each component with a completely justified relaxation term showing the different laws of pressures. We focus in this note on the two-component case but the result generalizes to the multi-component case.

To start the note, we present in section 2 the mathematical models whether they are mesoscopic or macroscopic as well as the proof objectives to link the two-phase macroscopic model (8) to the bi-fluid mesoscopic system (5) – (7) with the initial data (4). Then we present in section 3 two different formal approaches (continuous or discrete) allowing to obtain the limit model. We start with a continuous approach through a two-scale analysis of WKB type. We then continue with a discrete approach which would be closer to what we may encounter in applications. These results render mathematically rigorous the formal computations that can be found for example in [20], [19], [1] by avoiding the formal closure hypothesis for the relaxation term. This part may have a real interest for the reader interested in a formal understanding of obtaining the mixture model from the mesoscopic model with interfaces.

We then propose in the section 4 two completely complementary theoretical approaches to mathematical justification: I) A homogenization approach on the mesoscopic model under its continuous form where a theorem of type existence of solutions à la Hoff (for compressible Navier-Stokes with pressure depending on two transported parameters) and uniform control in the small parameter is necessary for the passage to the limit. II) A homogenization approach from a rather numerical-analysis-angle which consists in considering the associated semi-discrete system, hence to deduce a local existence theory for the ODE system, to show a uniform control of discrete versions of the energy estimates à la Hoff then to pass to the limit towards the continuous model thus justifying a formal analysis by one of the authors in [17]. This part is oriented towards the reader interested in a rigorous justification through a continuous approach or a discrete approach.

Owing to the theoretical results, in the last Section, we show how to define appropriate numerical schemes and we present the results of some numerical experiments which better illustrate our results. This note is addressed to a wide audience and aims at presenting a new framework allowing to approach formally, mathematically and numerically the derivation of some multiphase models with different pressure laws from mesoscopic descriptions.

This work should be seen as an introduction to the complete monograph [4] that the authors propose to write in collaboration with M. Hillairet. This monograph will recall previous important theoretical results (for instance [3], [21], [22], [12], [8], [1], [6], [4], [9] for example) as well as results dealing with mathematical modelling and important numerical aspects ([20], [10], [19], [1], [9] and references therein for example). It will also give full details of the proofs and discuss some possible extensions.

2 Mathematical models and objectives.

2.1 The mesoscale system

In this part, we detail the process towards the mathematical justification of the Baer-Nunziato model for multiphase mixture with a physical relaxation term.
Our main assumption is to consider a two-fluid mixture (multiphase fluid) as the limit of a sequence of situations where the two fluids are separated by sharps interfaces (this configuration is sometimes referred as a multifluid material). At a physical level this translates as follows: we assume that it is possible to zoom-in in the mixture at a very fine scale, which is referred to as being the mesoscopic scale. We assume that at this scale the two constituents are separated which allows us to model their behaviour with the well-known Navier-Stokes system plus an equation describing the evolution of the interface. In order to recover a macroscopic effective model, we have to zoom-out which mathematically is translated as a propagation of oscillations problem and its averaging.

Then, our second assumption is that the multifluid is governed by a Navier-Stokes type system where the viscosity and the pressure depend on the fluid, dependence that can be modelled by introducing a binary color parameter, which takes its values depending on the phase. More precisely, we consider a mixture of two compressible fluids, which will be referred in the following as $+$ and $-$ occupying a domain $\Omega$. In the following, physical quantities that characterize the phase $+/−$ will be explicitly denoted by a $+/−$ lower script.

Let us denote by $\Omega_+(t)$ the volume occupied by fluid $+$ and by $\Omega_-(t)$ the volume occupied by fluid $−$ at time $t$. We assume that initially we have $\Omega_+(0) \cap \Omega_-(0) = \emptyset$ and $\Omega_+(0) \cup \Omega_-(0) = \Omega$. Lest us denote by $c(t, \cdot)$ the characteristic function of $\Omega_+(0)$ (and note that thus $c$ is also the volume fraction or the mass fraction of fluid $+$). Let us denote $\rho_+$ and $\rho_−$ the densities of the two phases. The densities are initially defined in $\Omega_+(t)$ respectively in $\Omega_-(t)$. We extend $\rho_+$ and $\rho_−$ by $0$ in $\Omega$ and slightly abusing the notation we will still call them $\rho_{\pm}$. We see that owing to the separation hypothesis and the definition of $c$ we have that

$$\rho = c \rho_+ + (1 - c) \rho_-$$  \hspace{1cm} (1)

and

$$\rho_+ = c \rho, \rho_- = (1 - c) \rho.$$  \hspace{1cm} (2)

a.e. in $\Omega$.

We assume that fluid $+$ has a constant viscosity $\mu_+ > 0$ and fluid $-$ has a constant viscosity $\mu_- > 0$, and denote by $\mu(c)$ the viscosity of the multifluid:

$$\mu(c) = c \mu_+ + (1 - c) \mu_-.$$  \hspace{1cm} (3)

We assuming that both fluids are barotropic fluids with pressure laws $p_+(\rho)$ and $p_−(\rho)$, and denote by $p(c, \rho)$ the pressure law of the multifluid:

$$p(c, \rho) = c p_+(\rho) + (1 - c) p_−(\rho).$$  \hspace{1cm} (4)

We denote by $u(t, x)$ the velocity at time $t$ and position $x$. The above notations and definitions lead to model the multifluid Cauchy problem with the following mesoscopic system:

$$\begin{cases}
\partial_t c + u \partial_x c = 0 \text{ with } c(1 - c) = 0, \\
\partial_t \rho + \partial_x (\rho u) = 0, \\
\partial_t (\rho u) + \partial_x (\rho u^2) = -\partial_x (\mu(c) \partial_x u) + \partial_x p(c, \rho), \\
\mu(c) = c \mu_+ + (1 - c) \mu_-, \\
p(c, \rho) = c p_+(\rho) + (1 - c) p_−(\rho).
\end{cases}$$  \hspace{1cm} (5)

Let take a moment to resume what has been done up to the present. We made a number of hypothesis that allowed us to propose a system of equations governing the evolution of a mixture at a mesoscopic scale. Although very interesting at a mathematical level this system is very unlikely to be of use in practical applications. Thus, the necessity to un-zoom back at macroscopic scale comes naturally. Using similar systems as \cite{3}, performing a formal procedure of averaging and assuming some closure assumptions, several calculations may be encountered in physics-papers such as \cite{11, 15} or modelling or numerical papers such as \cite{20, 11}. We describe in the following lines how to obtain mathematically equations at a macroscopic scale. The key observation from \cite{7, 8} is that when un-zooming, an observer will witness rapid oscillations between the zones occupied by the two phases. Thus, a multi-phase fluid can be represented by the limit of solutions that widely oscillate in space. More precisely, following the formalism introduced in \cite{6, 7}, we obtain a macroscopic effective system for multiphase fluids by introducing a parameter $\varepsilon$ describing the oscillation scale. The idea
is to consider a sequence of solutions of \( \text{(3)} \) generated by a sequence of initial data widely oscillating in space and to analyse its limiting behaviour. A typical example of such initial data is given by

\[
c_0^\varepsilon(x) = c_0(x/\varepsilon), \quad \rho_0^\varepsilon(x) = \rho_0(x/\varepsilon)\rho_{+,0}(x) + (1 - c_0(x/\varepsilon))\rho_{-,0}(x)
\]  

(4)

where \( c_0 : \Omega \to \{0,1\} \) is a fixed profile and \( \rho_{0,+}, \rho_{0,-} \) are bounded initial data and where the initial velocity field \( u_0^\varepsilon(x) = u_0(x) \in H^1(\Omega) \). Thus, one considers \( (c^\varepsilon, \rho^\varepsilon, u^\varepsilon)_{\varepsilon>0} \) a sequence of solutions (in the sense of Hoff) of the bifluid system mentioned previously i.e.

\[
\begin{aligned}
\partial_t c^\varepsilon + u^\varepsilon \partial_x c^\varepsilon &= 0, \\
\partial_t \rho^\varepsilon + \partial_x (\rho^\varepsilon u^\varepsilon) &= 0, \\
\partial_t (\rho^\varepsilon u^\varepsilon) + \partial_x (\rho^\varepsilon u^\varepsilon^2) - \partial_x (\mu(c^\varepsilon)\partial_x u^\varepsilon) + \partial_x p(c^\varepsilon, \rho^\varepsilon) &= 0, \\
\mu(c^\varepsilon) &= (c^\varepsilon \mu_+ + (1 - c^\varepsilon)\mu_-), \\
p(c^\varepsilon, \rho^\varepsilon) &= c^\varepsilon p_+(\rho^\varepsilon) + (1 - c^\varepsilon)p_-(\rho^\varepsilon),
\end{aligned}
\]

(5)

with

\[
\begin{aligned}
c^\varepsilon|_{t=0} &= c_0^\varepsilon \text{ such that } c_0^\varepsilon(1 - c_0^\varepsilon) = 0, \\
\rho^\varepsilon|_{t=0} &= \rho_0^\varepsilon, \\
u^\varepsilon|_{t=0} &= u_0^\varepsilon,
\end{aligned}
\]

(6)

where the initial data are supposed to oscillate wildly in space. The property \( c_0^\varepsilon(1 - c_0^\varepsilon) = 0 \) is important because it implies that \( c^\varepsilon(1 - c^\varepsilon) = 0 \) at any time and that thus guarantees that it is legal to compute the pressure as

\[
p(c^\varepsilon, \rho^\varepsilon) = c^\varepsilon p_+(\rho^\varepsilon) + (1 - c^\varepsilon)p_-(\rho^\varepsilon),
\]

(7)

the phases are “pure” at any point of \( \Omega^\varepsilon \). The macroscopic bi-phase model is then derived letting \( \varepsilon \) tend to 0 and computing the limit system. We prove that, if the Cauchy data \( \text{(4)} \) converges weakly to \((\alpha_0, \rho_0, u_0)\) (note that the property \( \alpha_0(1 - \alpha_0) \) is of course lost), then, up to subsequence \( (c^\varepsilon, \rho^\varepsilon, u^\varepsilon) \) converges to \((\alpha, \rho, u)\) such that

\[
\begin{aligned}
\partial_t \alpha + u \partial_x \alpha &= \frac{\alpha(1 - \alpha)}{\alpha \mu_- + (1 - \alpha)\mu_+} (F_+ - F_-), \\
\partial_t \rho + \partial_x (\rho u) &= 0, \\
\partial_t (\rho u) + \partial_x (\rho u^2) - \partial_x (\mu_\text{eff} \partial_x u) + \partial_x p_\text{eff} &= 0, \\
\rho &= \alpha \rho_+ + (1 - \alpha)\rho_- , \\
\mu_\text{eff} &= \frac{\alpha \mu_- + (1 - \alpha)\mu_+}{\mu_+ + \mu_-}, \\
p_\text{eff} &= \frac{\alpha p_+(\rho_+)\mu_+ + (1 - \alpha)p_-(\rho_-)\mu_-}{\alpha \mu_- + (1 - \alpha)\mu_+}, \\
F_\pm &= -\mu_\pm \partial_x u + p_\pm (\rho_\pm).
\end{aligned}
\]

(8)

Observe that the resulting homogenized system contains only physical quantities such as the two viscosities, the two pressure laws and the volume fraction. We did not suppose a closure assumption with a relaxing parameter as usually in the physical literature.

In the sequel, we present two approaches in order to derive solutions of the bifluid system and to obtain at the limit the same macroscopic equations namely

- A so called continuous approach which follows the ideas introduced by D. Bresch and M. Hillairet in \([6,7,8]\). This consist in using the techniques introduced by D. Hoff in order to construct solutions of the bifluid system. The main novelty here is the presence of the new unknow \( c \) allowing to take into account different pressure laws. The bifluid system is a compressible Navier-Stokes system with a viscosity given in terms of \( c \) and a pressure law depending on \( c \) and \( \rho \).
- A semi-discretized approach which will render rigorous the result by one of the authors (see \([17]\)). This approach is interesting because it allows to numerically compute solutions for the macroscopic mixture model \([8]\) with a numerical scheme designed on the mesoscopic system \([5]\).

In Section \([3]\) we show that we may obtain the system \([3]\) from the system \([5]\) first by a formal two-scale analysis at a continuous level and then by a scale analysis at a discrete level. In Section \([4]\) we present the mathematical results justifying the derivation from the bifluid description to the biphase system: continuous approach and then semi-discrete approach. In the last section \([5]\) we illustrate the results stated in the paper.
More precisely, we design two numerical schemes: one to approximate the mesoscopic system, that is to say System (5) with a Cauchy datum (6), and one to approximate the macroscopic system (8) with any Cauchy datum.

3 Formal asymptotics

3.1 How to derive the relaxed equation (8) without closure assumptions?

3.1.1 Continuous level – Two scale analysis

In the homogenization process, two-scale asymptotic is a natural tool (see for instance [2], [21] and [22] and references cited therein). Let us recall formally, how it may be used in the compressible setting using some ellipticity properties of the effective flux and using the renormalized approach for the transport equation. Such formal computation has been performed in [8] but for the reader’s convenience we rewrite here such calculation. In the bifluid setting, to get the asymptotic system we formally assume the following ansatz

\[ c(t,x) = c(t, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}) \]

\[ \rho(t,x) = \rho^0(t, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}) + (1 - c(t, \frac{t}{\varepsilon}, \frac{x}{\varepsilon})) \rho^\varepsilon(t,x) \]

\[ u(t,x) = u_0(t, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}) + \varepsilon u_1(t, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(t, \frac{t}{\varepsilon}, \frac{x}{\varepsilon}) + O(\varepsilon^3) \]

assuming

\[ \rho^\varepsilon(t,x) = \rho^0(t,x) + O(\varepsilon), \quad c(t,\tau, x, y) \in \{0, 1\} \ a.e. \]

Plugging these informations in the bifluid system, we first get

\[ \partial_\tau c + u_0 \partial_y c = 0 \quad (9) \]

and

\[ \partial_t c + u_0 \partial_x c + u_1 \partial_y c = 0 \quad (10) \]

The first equation provides the behavior of \( c \) on a cell. This equation is compatible with the assumption that \( c \) is an indicator function. Averaging with respect to the fast variable \( y \) the second equation, we get the following equation

\[ \partial_\tau \alpha + \frac{u_0}{u_0} \partial_x = -u_1 \partial_y c \quad (11) \]

with \( \alpha = \overline{c} \) where we denote temporarily with the bar average with respect to \( y \) on a cell. Let us now quickly recall the different steps. Plugging now the ansatz in the momentum equation we get

\[ \partial_y((c \mu_+ + (1 - c) \mu_-) \partial_y u_0) = 0. \]

Multiplying this equation and integrate with respect to the \((t, \tau, x, y)\) and using that

\[ c \mu_+ + (1 - c) \mu_- \geq \min(\mu_+, \mu_-) > 0 \]

we get that

\[ \partial_y u_0 = 0 \]

and therefore coming back to the equation on \( c \) and averaging with respect to \( y \) that \( \alpha \) does not depend on \( \tau \). Looking now the main part of the momentum equation, we get

\[ \rho^0(\partial_x u_0 + u_0 \partial_y u_0) + \partial_y p^0 = \partial_y((c \mu_+ + (1 - c) \mu_-) \partial_y u_1) \]

\[ + \frac{1}{\partial_x}(c \mu_+ + (1 - c) \mu_-) \partial_y u_0) + \partial_y((c \mu_+ + (1 - c) \mu_-) \partial_x u_0). \]

Multiplying by \( \partial_x u_0 \) and integrating with respect to the \((t, \tau, x, y)\), we get that \( \partial_\tau u_0 = 0 \). Thus we get

\[ \partial_y((c \mu_+ + (1 - c) \mu_-) \partial_y u_1) - \partial_y p^0 = -\partial_y((c \mu_+ + (1 - c) \mu_-) \partial_x u_0). \]
Denoting \( \mu = c \mu + (1-c) \mu^- \), this gives
\[
\mu \partial_y u_1 - p^0 = \mu \partial_y u_1 - p^0 + (\overline{\rho} - \mu) \partial_x u^0.
\]
Note that using the expression of \( \mu \) and the equation of \( \alpha \), we get
\[
\mu \partial_y u_1 = (p^0 - \overline{\rho}) - (\mu - \overline{\rho}) \partial_x u_0 + (\mu_+ - \mu_-) (\partial_t \alpha + u_0 \partial_x \alpha)
\]
which may be rewritten as
\[
c \partial_y u_1 = \frac{c}{\mu} ((p^0 - \overline{\rho}) - (\mu - \overline{\rho}) \partial_x u_0 + (\mu_+ - \mu_-) (\partial_t \alpha + u_0 \partial_x \alpha))
\]
and therefore after calculation
\[
\frac{c \partial_y u^1}{\partial_t} = \frac{\alpha (1-\alpha)}{\mu_+} ((p_+ (\rho_+^0) - p_- (\rho_-^0)) - \partial_x u_0 (\mu_+ - \mu_-) + \alpha (1 - \frac{\mu_-}{\mu_+}) (\partial_t + u_0 \partial_x) \alpha
\]
using that
\[
\overline{u} \partial_y c = - c \partial_y u^1
\]
and inserting in (11), we get after simple calculations
\[
\partial_t \alpha + u_0 \partial_x \alpha = \frac{\alpha (1-\alpha)}{\alpha \mu_+ + (1-\alpha) \mu_+} ((p_+ (\rho_+^0) - p_- (\rho_-^0)) - \partial_x u_0 (\mu_+ - \mu_-))
\]

### 3.1.2 Discrete level – scale analysis

In this section we propose a formal procedure to derive the mixture model and we explain with very simple arguments why the volume fraction should satisfy the equation
\[
\partial_t \alpha + u \partial_x \alpha = \frac{\alpha (1-\alpha)}{\mu} (p_+ - p_-).
\]
We denote by \( D_t \alpha \) the Lagrangian time derivative of \( \alpha \):
\[
D_t \alpha = \partial_t \alpha + u \partial_x \alpha
\]
(this derivative will also be denoted \( \alpha'_{\tau} \) in the rest of the paper). Consider a situation where the fluids are separated (say, at a small scale \( \varepsilon \)), and a point \( x(t) \in \mathbb{T} \) at an interface between fluid \(+\) on its right and fluid \(-\) on its left, for any time \( t \). Denote by \( x_+(t) \) the center of the zone of pure fluid \(+\) on the right of \( x(t) \), by \( x_-(t) \) the center of the zone of pure fluid \(-\) on the left of \( x(t) \), and
\[
\varepsilon_+(t) = x_+(t) - x(t), \varepsilon_-(t) = x(t) - x_-(t)
\]
which are supposed to be small.

We define \( \alpha(t) \) by
\[
\alpha(t) = \varepsilon_+(t) / (\varepsilon_+(t) + \varepsilon_-(t)).
\]
Indeed this quantity represents the local (at point \( x(t) \)) volume fraction of fluid \(+\). Obviously one has
\[
D_t \varepsilon_+(t) = u(t, x_+(t)) - u(t, x(t))
\]
and
\[
D_t (\varepsilon_+ + \varepsilon_-)(t) = u(t, x_+(t)) - u(t, x_-(t)).
\]
This allows to write
\[
D_t \alpha(t) = \frac{(\varepsilon_+ + \varepsilon_-) D_t \varepsilon_+ - \varepsilon_+ D_t (\varepsilon_+ + \varepsilon_-)}{(\varepsilon_+ + \varepsilon_-)^2} = \frac{\varepsilon_- (u(t, x_+(t)) - u(t, x(t))) - \varepsilon_+ (u(t, x(t)) - u(t, x_-(t)))}{(\varepsilon_+ + \varepsilon_-)^2}
\]
The regularity of the solution is expected to be the following: at any time \( t \), the pressure and the space derivative of the velocity should be continuous in space in each pure region (namely, in \((x_\alpha - \varepsilon_\alpha, x_\alpha + \varepsilon_\alpha)\)) and in \((x_\alpha - \varepsilon_\alpha, x_\alpha + \varepsilon_\alpha)\), but not at the point \( x(t) \). At this point, what is expected is that the effective flux \( p - \mu \partial_x u \) is continuous (and this continuity in space stands for the law of reciprocal forces of Newton). In the case where the two viscosity coefficients are equal, the formal computation is straightforward. Thus we propose to begin by assuming this equality, and to obtain the general law for \( \alpha \) in a second stage.

— Case where \( \mu_+ = \mu_- = \mu \)

The continuity of the effective flux together with the regularity on pure zones expresses as

\[
p_-(t) - \mu \frac{u(t, x(t)) - u(t, x_\alpha(-t))}{\varepsilon_-} = p_+(t) - \mu \frac{u(t, x_\alpha(+t)) - u(t, x(t))}{\varepsilon_+} + r(\varepsilon_- + \varepsilon_+),
\]

where \( p_\pm(t) \) denotes \( p_\pm(\rho(t, x_\pm(t))) \) and \( r \) is a function such that \( r(x) \to 0 \) as \( x \to 0^+ \). This rewrites

\[
p_+(t) - p_-(t) = \mu \frac{\varepsilon_-(u(t, x_\alpha(+t)) - u(t, x(t))) - \varepsilon_+(u(t, x(t)) - u(t, x_\alpha(-t)))}{\varepsilon_+\varepsilon_-} + r(\varepsilon_- + \varepsilon_+),
\]

and, thanks to (12) and letting \( \varepsilon \) go to 0,

\[
p_+(t) - p_-(t) = \mu \frac{(\varepsilon_+ + \varepsilon_-)^2}{\varepsilon_+\varepsilon_-} D_t \alpha_+ = \frac{\mu}{\alpha_+(1 - \alpha_+)} D_t \alpha_+,
\]

which is exactly what is stated in this paper.

— Case where \( \mu_+ \neq \mu_- \)

In this general case, it is convenient to define the approximate space derivatives of the velocity \( d_- (t) \) and \( d_+(t) \)

\[
d_- (t) = \frac{u(t, x(t)) - u(t, x_\alpha(-t))}{\varepsilon_-(t)}, \quad d_+(t) = \frac{u(t, x_\alpha(+t)) - u(t, x(t))}{\varepsilon_+(t)}.
\]

Equipped with this, we can rewrite (12) as

\[
D_t \alpha(t) = \frac{(\varepsilon_+ + \varepsilon_-)^2}{\varepsilon_+\varepsilon_-} (d_+(t) - d_-(t)).
\]

We would like to express the limit, as \( \varepsilon_- + \varepsilon_+ \) tends to 0, of the right-hand side term as a function of the limit quantities. Remark that \( u \) is intended to converge strongly but \( \partial_x u \) only weakly, thus \( d_+(t) \) and \( d_-(t) \) are not approximations of \( \partial_x u(t, x(t)) \); however \( \frac{\varepsilon_-}{\varepsilon_- + \varepsilon_+} d_- + \frac{\varepsilon_+}{\varepsilon_- + \varepsilon_+} d_+ \) is intended to converge toward \( \partial_x u \). The limit of the right-hand side should be expressed as a function of the limit unknowns \( \alpha_+, \alpha_-, p_+, p_-, \partial_x u \). We already know that \( \frac{\varepsilon_-\varepsilon_+}{(\varepsilon_- + \varepsilon_+)^2} \) converges to \( \alpha_+ - \alpha_- \). It remains to treat the term \( d_+ - d_- \). As \( \mu_+ d_+ - \mu_- d_- \) is intended to converge to \( p_+ - p_- \), it is quite natural to try to write

\[
d_+ - d_- = a(\mu_+ d_+ - \mu_- d_-) + (1 - a \mu_+) d_+ - (1 - a \mu_-) d_- \tag{13}
\]

with \( a \in \mathbb{R} \) such that there exists \( b \in \mathbb{R} \) satisfying

\[
1 - a \mu_+ = b \alpha \quad \text{and} \quad 1 - a \mu_- = -b(1 - \alpha),
\]

in which case one would have

\[
d_+ - d_- \to_{\varepsilon_- + \varepsilon_+ \to 0} a(p_+ - p_-) + b \partial_x u.
\]

The linear system in \( a \) and \( b \) has a unique solution, \( a = \frac{1}{(1 - \alpha) \mu_+ + \alpha \mu_-} \) and \( b = \frac{\mu_- - \mu_+}{(1 - \alpha) \mu_+ + \alpha \mu_-} \), which finally gives

\[
D_t \alpha_+ = \frac{\alpha(1 - \alpha)}{(1 - \alpha) \mu_+ + \alpha \mu_-} (p_+ - p_- - (\mu_+ - \mu_-) \partial_x u),
\]

which is exactly the first equation in (8).
3.2 How to derive the effective viscosity and the effective pressure?

It is interesting to note that the effective viscosity is computed in the same way as one computes the effective diffusion when considering homogenisation for 1d elliptic equations. The main remark is to use for the compressible the ellipticity of the effective flux $F = p(\rho) - \mu \partial_x u$. Looking at the order 0 terms in the momentum equation and averaging with respect to $y$ we get using that $u^0$ does not depend on $y$ that

$$\rho^0(\partial_t u^0 + u^0 \partial_x u^0) - \partial_x (\rho^0 \partial_x u^0) - \partial_x (\mu^0 \partial_x u^1) + \nabla p(\rho) = 0.$$ 

It remains to use the expression of $\mu^0 \partial_x u^1$ found previously to get the homogeneized momentum equation related to $\alpha_\pm$ and $u^0$.

4 Mathematical results

4.1 Continuous approach – Hoff solutions and its two-scale limit

First of all, we can adapt the techniques introduced by D. Hoff in order to obtain the well posessedness of system (3). More precisely, let us assume the following initial data conditions

$$\begin{cases}
  c_0 \in L^\infty(\mathbb{T}) &\text{such that } 0 \leq m = \inf_0 c_0 \leq c_0(x) \leq M = \sup_0 c_0 & \text{a.e. on } (0,1) \\
  \rho_0 \in L^\infty(\mathbb{T}) &\text{such that } 0 < \inf \rho_0 \leq \rho_0(x) & \text{a.e. on } (0,1) \\
  G(\rho_0, c_0) = \rho_0 \int_T^1 p(s, c_0)/s^2 < +\infty \\
  u_0 \in H^1(\mathbb{T})
\end{cases}$$

and let us assume that $p$ and $\mu$ satisfies for all $(\rho, c) \in [0, +\infty] \times [0,1]$ the following:

$$\begin{cases}
  p(\rho, c) \geq 0 & \text{and } \mu(c) \geq \mu_{\min} > 0 \\
  \exists \rho_0 > 0 \text{ such that } p(\rho, c) \leq C_0(\rho + G(\rho, c)) & \text{where } G(\rho, c) = \rho \int_T^1 p(\rho, c)/s^2 ds \\
  \rho \partial_1 p(\rho, c) \in L^\infty_{loc}([0, +\infty] \times [0,1]).
\end{cases}$$

Then we have the following

**Theorem 4.1** Consider two functions $p \in C^1([0, \infty) \times [m, M])$ and $\mu \in C^1([0, M])$ verifying Hypothesis (14). Let $(c_0, \rho_0, u_0)$ satisfy (14), then there exists a unique weak solution $(c, \rho, u)$ of system (3) with initial data $(c_0, \rho_0, u_0)$ with

$$c, \rho \in C([0, \infty); L^q(\mathbb{T}^1)) \text{ for all } q < +\infty,$$

$$u \in L^\infty([0, \infty); H^1(\mathbb{T}^1)), \quad \partial_1 u \in L^2([0, \infty] \times \mathbb{T}^1).$$

Moreover, for any $T > 0$, there exists a constant $C(T)$ which depends only on the norms of the initial data and $T$ such that the following uniform bounds hold true:

$$\int_0^1 \frac{\rho u^2}{2} + \int_0^1 G(\rho, c) + \int_0^t \int_0^1 \mu(c) (\partial_x u)^2 \leq \int_0^1 \frac{\rho_0 u_0^2}{2} + \int_0^1 G(\rho_0, c_0),$$

$$\inf_{x \in [0,1]} c_0(x) \leq c(t, x) \leq \sup_{x \in [0,1]} c_0(x),$$

$$C(T)^{-1} \leq \rho(t, x) \leq C(T),$$

$$\frac{1}{2} \int_0^1 \mu(c) (\partial_x u)^2 + \int_0^t \int_0^1 \rho u^2 \leq C(T),$$

$$\frac{1}{2} \int_0^1 \sigma(t) \rho u^2 + \frac{1}{2} \int_0^t \int_0^1 \sigma(t) \mu(c) (\partial_x u)^2 \leq C(T),$$

$$\|u\|_{L^\infty([0,1] \times \mathbb{T}^1)} + \|u\|_{H^1([0,1] \times \mathbb{T}^1)} \leq C(T),$$

$$\sigma^\ast(t) \| (\partial_t u(t), \partial_x u(t)) \|_{L^\infty(\mathbb{T})} \leq C(T).$$

for all $t \in [0, T]$ where $\sigma(t) = \min \{t, 1\}$. 

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Remark 1 It is interesting to note that the previous theorem includes general pressure laws \( p(\rho, c) \) and viscosity \( \mu(c) \). It includes for instance the pressure law \( p(\rho, c) = \rho^\gamma \) that depends on the density \( \rho \) and the fraction \( c \) of each chemical/phase component. As described in [18], the function \( \gamma(c) \) depends on the constant heat capacity ratios of each component of the multifluid, the pressure \( p(\rho, c) \) effectively traces the thermodynamic "signature" of mixing chemicals/ phases in solution.

Let us now consider a sequence of initial data \( (\rho_0^\varepsilon, c_0^\varepsilon, u_0^\varepsilon)_{\varepsilon > 0} \) that satisfies:

\[
\begin{align*}
\rho_0^\varepsilon &\in L^\infty (T^1) \text{ with } 0 < \inf_{x \in T^1} \rho_0^\varepsilon (x) \leq \rho_0^\varepsilon (x) \leq \sup_{x \in T^1} \rho_0^\varepsilon (x) \leq M < +\infty, \\
c_0^\varepsilon &\in [0, 1], \\
\int_0^1 (c_0^\varepsilon p_+(\rho_0^\varepsilon) + (1 - c_0^\varepsilon) p_-(\rho_0^\varepsilon)) \leq M, \\
u_0^\varepsilon &\in H^1 (T^1) \text{ such that } \|u_0^\varepsilon\|_{H^1} \leq M
\end{align*}
\]  
with \( M > 0 \) independent of \( \varepsilon \). We note that these assumptions are satisfied in particular for initial configurations as depicted in (4). The bounds (22) allow us to conclude that there exists \( (\rho_0, c_0, u_0) \in L^\infty (T^1) \times L^\infty (T^1) \times H^1 (T^1) \) such that

\[
\rho_0^\varepsilon \to \rho_0 \text{ in } L^\infty (T^1) - w*, \quad c_0^\varepsilon \to c_0 \text{ in } L^\infty (T^1) - w*, \quad u_0^\varepsilon \to u_0 \text{ in } H^1 (T^1).
\]

Furthermore, given \( \varepsilon > 0 \) the initial data \( (\rho_0^\varepsilon, c_0^\varepsilon, u_0^\varepsilon)_{\varepsilon > 0} \) enters the scope of Theorem 4.1. So, we can associate to this initial data a solution \( (\rho^\varepsilon, c^\varepsilon, u^\varepsilon)_{\varepsilon > 0} \) to (22). Moreover, this sequence satisfies the following uniform bounds on any interval \([0, T]\) independent of \( \varepsilon \):

\[
\begin{align*}
\int_0^1 \rho^\varepsilon (u^\varepsilon)^2 + \int_0^1 G(\rho^\varepsilon, c^\varepsilon) + \mu \int_0^1 (\partial_x u^\varepsilon)^2 &\leq C, \\
(\partial_x u^\varepsilon)^2 + \int_0^1 (\partial_x u^\varepsilon)^2 + \int_0^1 |\partial_x(u\partial_x u^\varepsilon - p^\varepsilon)|^2 &\leq C(T), \\
\frac{1}{2} \int_0^1 \sigma(t) \rho^\varepsilon (\dot{u}^\varepsilon)^2 + \frac{1}{2} \int_0^1 \sigma(t) (\partial_x u^\varepsilon)^2 &\leq C(T), \\
\|u^\varepsilon\|_{L^\infty (0, T) \times T^1} + \|u^\varepsilon\|_{H^1 (0, T) \times T^1} &\leq C(T), \\
\sigma^\frac{1}{2} (t) \|\partial_x u^\varepsilon(t), \partial_x u^\varepsilon(t)\|_{L^\infty} &\leq C(T)
\end{align*}
\]

with \( \mu = \min(\mu_+, \mu_-) \). Using the uniform bounds of (23) - (28) we conclude that

\[
\begin{cases}
\rho^\varepsilon \to \rho, \quad p(\rho^\varepsilon, c^\varepsilon) \to \Pi \text{ in } L^\infty (\mathbb{R}; L^\infty (T^1)), \\
u^\varepsilon \to u \text{ in } L^\infty (\mathbb{R}; H^1 (T^1)), \\
Z^n := \mu \partial_x u^\varepsilon - p(\rho^\varepsilon, c^\varepsilon) \to Z \in L^2 (\mathbb{R}; H^1 (T^1)).
\end{cases}
\]

As explained previously, the density \( \rho^\varepsilon \) and the parameter \( c^\varepsilon \) are expected to oscillate widely in space. For this reason, it is hopeless to obtain stronger convergence on these sequences (and it would not be a good news for our plan either) than in a weak \( L^p \)-setting. It turns out that we may characterize weak*-limits of the form if we suppose that they are given at time \( 0 \). In the following lines we make rigourous the previous assertion. On the other hand, we need to recover some properties of the sequence \( p(\rho^\varepsilon, c^\varepsilon) \) to compute a limit system satisfied by \( (\rho, u, \Pi) \). To this end, we associate to the sequence \( (\rho^\varepsilon, c^\varepsilon) \) a sequence of measures on the space \( T^1_x \times \mathbb{R}_\varepsilon \times \mathbb{R}_\eta \) (here \( \mathbb{R}_\varepsilon \) must be understood as the range of the \( \rho^\varepsilon \) while \( \mathbb{R}_\eta \) is the range of the \( c^\varepsilon \)). Namely, given \( n \geq 0 \) and \( t \geq 0 \), we consider the measure on \( T^1_x \times \mathbb{R}_\varepsilon \times \mathbb{R}_\eta \) as defined by

\[
(\Theta^\varepsilon (t), b) := \int_{T^1_x} b(x, \rho^\varepsilon (t, x), c^\varepsilon (t, x)) \, dx, \quad \forall b \in C_c (T^1_x \times \mathbb{R}_\varepsilon \times \mathbb{R}_\eta)
\]
Proposition 4.1 For fixed $\varepsilon > 0$ fixed there holds

$$\Theta^\varepsilon \in C_w([0, \infty); \mathcal{M}_+(T^1_x \times \mathbb{R}_\xi \times \mathbb{R}_\eta))$$

with

$$\text{Supp}(\Theta^\varepsilon(t)) \subset T^1_x \times [C(t)^{-1}, C(t)] \times [0, 1] \quad (\Theta^\varepsilon, 1) = 1, \; \forall t \geq 0,$$

where $C(t)$ is given by [24].

Once these measures are constructed, the rigorous justification of system (8) consists in the derivation of the kinetic limit and the characterization of the measures family, it reduces to

Theorem 4.2 Up to the extraction of a subsequence, we have $\Theta^\varepsilon \rightharpoonup \Theta$ in $C_w([0, \infty); \mathcal{M}_+(T^1_x \times \mathbb{R}_\xi \times \mathbb{R}_\eta))$ where $\Theta$ satisfies

$$\frac{\partial}{\partial t} \Theta + \partial_x (u \Theta) - \partial_t \left( \left( \frac{\xi Z^\infty}{\mu(\eta)} + \frac{\xi p(\xi, \eta)}{\mu(\eta)} \right) \Theta \right) = \left( \frac{Z^\infty}{\mu(\eta)} + \frac{p(\xi, \eta)}{\mu(\eta)} \right) \Theta = 0$$

with $(u, \Pi, Z^\infty)$ as defined in [20]. Moreover, if there exists $(\alpha_0, \rho_{+, 0}, \rho_{-, 0}) \in L^\infty(T^1)$ with $\alpha_0 \in [0, 1]$ a.e. and such that

$$\langle \Theta(0), b \rangle = \int_{T^1_x} (\alpha_0(x)b(x, \rho_{+, 0}(x), 0) + (1 - \alpha_0(x))b(x, \rho_{-, 0}(x), 1))dx,$$

$$\forall b \in C(T^1_x \times \mathbb{R}_\xi \times \mathbb{R}_\eta)$$

then there exists $(\alpha, \rho_{+}, \rho_{-}) \in [L^\infty([0, \infty); L^\infty(T))] \cap C([0, \infty); L^1(T))$ such that, for any $t \geq 0$, $\alpha(t) \in [0, 1]$ a.e. and

$$\langle \Theta(t), b \rangle = \int_{T^1_x} (\alpha(t, x)b(x, \rho_{+, 0}(t, x), 0) + (1 - \alpha(t, x))b(x, \rho_{-, 0}(t, x), 1))dx,$$

$$\forall b \in C(T^1_x \times \mathbb{R}_\xi \times \mathbb{R}_\eta),$$

Furthermore, $(\alpha, \rho_{+}, \rho_{-})$ together with $u$ verifies the biphase Baer-Nunziato type system (8).

Remark. It is interesting to understand that Hoff’s solution allows the density to be only bounded and the velocity to be sufficiently regular to characterize its evolution in time.

4.2 The semi-discrete approach – ODEs and its continuous limit

This part corresponds in some sense to a mathematical justification of the formal description indicated previously. We consider the following system of ODEs:

$$\begin{cases}
\dot{x}_{j+\frac{1}{2}} = u_{j+\frac{1}{2}}, \\
\dot{c}_j = 0, \\
\frac{d}{dt}(\rho_j \Delta x_j) = 0, \\
\rho_{j+\frac{1}{2}} \Delta x_{j+\frac{1}{2}} = u_{j+\frac{1}{2}} + p_{j+1} - p_j = \left\{ \mu(c_{j+1}) \frac{u_{j+\frac{1}{2}} - u_{j+\frac{1}{2}}}{\Delta x_{j+1}} - \mu(c_j) \frac{u_{j+\frac{1}{2}} - u_{j+\frac{1}{2}}}{\Delta x_j} \right\}, \\
\Delta x_{j+\frac{1}{2}} = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}, \\
\Delta x_{j+\frac{1}{2}} = \frac{\Delta x_j + \Delta x_{j+1}}{2}, \\
\rho_{j+\frac{1}{2}} = \frac{\rho_j \Delta x_j + \rho_{j+1} \Delta x_{j+1}}{\Delta x_j + \Delta x_{j+1}},
\end{cases}$$

for all $j \in 0, J - 1$ and

$$\begin{cases}
c_0 = c_J, \\
\rho_0 = \rho_J, \\
u_{\frac{1}{2}} = u_{J+\frac{1}{2}}.
\end{cases}$$
From the previous system of equations we also deduce that

\[
\begin{align*}
\Delta x_j &= u_j + \frac{1}{2} - u_j - \frac{1}{2}, \\
\frac{d}{dt} \left( \rho_j \Delta x_j + \frac{1}{2} \right) &= 0.
\end{align*}
\] (38)

System (38) is to be completed with initial data \((x_j^0 + \frac{1}{2})_{j=-1}^{J-1}, (c_j^0)_{j=0}^{J-1}, (\rho_j^0)_{j=0}^{J-1}, (u_j^0 + \frac{1}{2})_{j=0}^{J-1}\) such that:

\[
\begin{align*}
0 &< \rho_0^0 = \min_{j \in [0, J-1]} \rho_j^0 \leq \rho_0 = \max_{j \in [0, J-1]} \rho_j^0 < \infty, \\
0 &< \rho_0^0 = \min_{j \in [0, J-1]} \rho_j^0 \leq \rho_0 = \max_{j \in [0, J-1]} \rho_j^0 < \infty, \\
\left\| \left( u_j + \frac{1}{2} \right)_{j \in [0, J-1]} \right\|_{H}^2 &= \sum_{j=0}^{J-1} \left| u_j^0 \right|^2 \Delta x_j^0 + \sum_{j=0}^{J-1} \left| \frac{u_j^0 - u_{j-1}^0}{\Delta x_j^0} \right|^2 \Delta x_j^0 < \infty.
\end{align*}
\] (H)

System (38) is a system of ODEs which, owing to the fact that\((x_j^0 + \frac{1}{2})_{j=0}^{J-1} \in \mathbb{R}^d\) and \(u_j^0 + \frac{1}{2})_{j=0}^{J-1} \in \mathbb{R}^d\) and \(F : D \to \mathbb{R}^d\) where

\[
D = \left\{ (X, U) \in \mathbb{R}^d \times \mathbb{R}^d : x_\frac{1}{2} < x_\frac{1}{2} < \cdots < x_{\frac{J-1}{2}} \right\}
\]

which is an open set of \(\mathbb{R}^d \times \mathbb{R}^d\). Owing to the fact that \(F\) is \(C^\infty\) on \(D\) we obtain via the Cauchy-Lipschitz-Peano theorem that for any initial data \((X_0, U_0) \in D\), there exists a unique maximal solution for (39)

\[
(X, U) : [0, T_{\text{max}}) \to D
\]

with \(T_{\text{max}} = \infty\) or if \(T_{\text{max}} < \infty\) then

\[
\lim_{t \to T_{\text{max}}} (X(t), U(t)) \in \partial D
\] (40)

As expected, system (38) shares a lot of properties with its continuous version, in particular we can find a number of apriori estimates related to the mass and energy conservation which enable us to prove that (40) does not occur. We are thus able to show that the local solutions can be extended to global ones. The following energy functionals play a key role: the total mass Consider the mass

\[
M(t) = \sum_{j=0}^{J-1} \rho_j(t) \Delta x_j(t)
\] (41)

and basic energy functionals

\[
E(t) = \frac{1}{2} \sum_{j=0}^{J-1} \rho_j(t) \left| u_j + \frac{1}{2} \right| \Delta x_j + \frac{1}{2} \sum_{j=0}^{J-1} H(\rho_j(t), c_j(t)) \Delta x_j(t)
\]

\[
+ \int_0^{t} \sum_{j=0}^{J-1} \mu(c_j(\tau)) \left| \frac{u_j + \frac{1}{2}(\tau) - u_j - \frac{1}{2}(\tau)}{\Delta x_j(\tau)} \right|^2 \Delta x_j(\tau).
\] (42)

We formalize in the following theorem our first result.
Theorem 4.3 The system \( \text{(36)} \) along with initial data verifying hypothesis \( \text{(1)} \) admits an unique solution and, moreover, the following estimates hold true uniformly in \( J \):

\[
\begin{align*}
\frac{1}{C_{\text{ini}}^1(t)} & \leq \rho_j(t) \leq C_{\text{ini}}^1(t), \text{ for all } j \in \overline{0,J-1}, \\
\frac{1}{C_{\text{ini}}^2(t)} & \Delta x_j^0 \leq \Delta x_j(t) \leq \Delta x_j^0 C_{\text{ini}}^2(t), \text{ for all } j \in \overline{0,J-1}, \\
\frac{\rho_j(t)}{M(t)} & \leq c_j(t) \leq \frac{\rho_j(t)}{E(t)}, \text{ for all } t \in [0,E(t)].
\end{align*}
\]

where \( C_{\text{ini}}^1(\cdot) \), \( C_{\text{ini}}^2(\cdot) \) and \( C_{\text{ini}}^3(\cdot) \) are strictly positive increasing continuous functions that depend only on the initial data.

For any \( J \in \mathbb{N}^* \), having constructed the functions \( \left( c_j, \rho_j, u_j \right)_{j \in \overline{0,J-1}} \) as above we consider

\[
\dot{c}_j, \dot{\rho}_j, \dot{u}_j : [0, \infty) \times \mathbb{R} \to \mathbb{R}
\]

defined by

\[
\begin{align*}
\dot{c}_j(t,x) &= c_j(t) \text{ if } x \in [x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}), \quad (43) \\
\dot{\rho}_j(t,x) &= \rho_j(t) \text{ if } x \in [x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}), \quad (44) \\
\dot{u}_j(t,x) &= \frac{x-x_{j-\frac{1}{2}}}{\Delta x_j} u_{j-\frac{1}{2}}(t) + \frac{x_{j+\frac{1}{2}}-x}{\Delta x_j} u_{j+\frac{1}{2}}(t) \text{ if } x \in [x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}), \quad (45) \\
\dot{Z}_j(t,x) &= \frac{x-x_{j-\frac{1}{2}}}{\Delta x_j} \sigma_{j+1} + \frac{x_{j+1}-x}{\Delta x_j} \sigma_j \text{ if } x \in [x_j,x_{j+1}). \quad (46)
\end{align*}
\]

where

\[
Z_j = \mu \left( c_j \right) \frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{\Delta x_j} - p(c_j,\rho_j).
\]

First, we observe the following remarkable property which will be strongly use to get the limit

Proposition 4.2 The functions \( (\dot{c}_j, \dot{\rho}_j, \dot{u}_j) \) verify the following transport equations

\[
\begin{align*}
\partial_t \dot{c}_j + u_j \partial_x \dot{c}_j &= 0, \\
\partial_t \dot{\rho}_j + \partial_x \left( \dot{\rho}_j \dot{u}_j \right) &= 0,
\end{align*}
\]

with initial data

\[
\begin{align*}
\dot{\rho}_j|_{t=0} &= \rho_j^0, \\
\dot{c}_j|_{t=0} &= c_j^0,
\end{align*}
\]

in the sense of distributions.

Of course, the estimates announced in Theorem 4.3 can be used in order to estimate various norms of the functions \( (\dot{c}_j, \dot{\rho}_j, \dot{u}_j, \dot{Z}_j) \) as in the Hoff solution proof.

More precisely, one has that

Theorem 4.4 Consider discrete initial data verifying the hypothesis \( \text{(1)} \) along with the globally defined solution of the system of ODEs \( \text{(30)–(37)} \). Furthermore, consider the functions \( (\dot{c}_j, \dot{\rho}_j, \dot{u}_j, \dot{Z}_j) \) given by
Then

$$\begin{align*}
\min_{j \in \mathbb{N}} c_j^0 \leq \hat{c}_j(t, x) \leq \max_{j \in \mathbb{N}} c_j^0,
\frac{1}{C_{\text{ini}}^{1}(t)} \leq \hat{\rho}_J(t, x) \leq C_{\text{ini}}^{1}(t),
\int_0^1 \int_0^1 H(\hat{\rho}_J(t, x), \hat{c}_J(t, x)) \, dx + \int_0^t \int_0^1 \mu(\hat{c}_J(\tau, x)) \left| \partial_x \hat{u}_J(\tau, x) \right|^2 \, dx \, d\tau \leq 2E_0, \\
\|\partial_x \hat{u}_J\|_{L^2_x} + \|\partial_t \hat{u}_J\|_{L^2_t} + \min \{1, t\} \|\partial_t \hat{u}_J\|_{L^2_t} \leq C(t), \\
\int_0^t \int_0^1 |\partial_x \hat{\sigma}_J(\tau, x)|^2 \, dx \, d\tau + \min \{1, t\} \int_0^1 |\partial_x \hat{\phi}_J(\tau, x)|^2 \, dx + \int_0^t (\sup_{\tau \in [0,1]} |\partial_x \hat{\sigma}_J(\tau, x)|)^4 \, d\tau \leq C(t) \\
\int_0^t \int_0^1 \sigma(\tau) |\partial_x \hat{\sigma}_J(\tau, x)|^2 \, dx \, d\tau \leq C(t).
\end{align*}$$

and basic energy functionals

For all $J \in \mathbb{N}^*$ and all $t \geq 0$, we consider the measure on $T_x^1 \times \mathbb{R}_x \times \mathbb{R}_\eta$ defined by

$$\langle \Theta^J(t), b \rangle \overset{\text{def.}}{=} \int_{T_x^1} b\left(x, \hat{x}^J(t, x), \hat{c}^J(t, x)\right) \, dx, \quad \forall b \in C_c(T_x^1 \times \mathbb{R}_x \times \mathbb{R}_\eta).$$

We may establish immediately similar results as in the previous section namely:

**Proposition 4.3** For fixed $J \in \mathbb{N}$ we have that

$$\Theta^J \in C_w([0, \infty); \mathcal{M}_+(T_x^1 \times \mathbb{R}_x \times \mathbb{R}_\eta))$$

with

$$\text{Supp}(\Theta^J(t)) \subset T_x^1 \times \left[C_{\text{ini}}^{1}(t)^{-1}, C_{\text{ini}}^{1}(t)^{-1}\right] \times [0, 1], \quad \langle \Theta^J, 1 \rangle = 1.$$  \hspace{1cm} (51)

where $C_{\text{ini}}^{1}(t)$ is given by (48).

We can also pass to the limit in the kinetic system to get the following result.

**Theorem 4.5** Up to the extraction of a subsequence, we have $\Theta^J \rightharpoonup \Theta$ in $C_w([0, \infty); \mathcal{M}_+(T_x^1 \times \mathbb{R}_x \times \mathbb{R}_\eta))$ where $\Theta$ satisfies

$$\partial_t \Theta + \partial_x (u \Theta) - \partial_x \left( \left( \frac{\xi Z \infty}{\mu(\eta)} + \frac{\xi p(\xi, \eta)}{\mu(\eta)} \right) \Theta \right) - \left( \frac{Z \infty}{\mu(\eta)} + \frac{p(\xi, \eta)}{\mu(\eta)} \right) \Theta = 0$$

with $(u, \Pi, Z)$ as defined in (36). Moreover, if there exists $(\alpha_0, \rho_{+0}, \rho_{-0}) \in L^\infty(T^1)$ such that $\alpha_0 \in [0, 1]$ a.e. and

$$\langle \Theta(0), b \rangle = \int_{T_x^1} (\alpha_0(x)b(x, \rho_{+0}(x), 0) + (1 - \alpha_0(x))b(x, \rho_{-0}(x), 1)) \, dx$$

$$\forall b \in C(T_x^1 \times \mathbb{R}_x \times \mathbb{R}_\eta),$$  \hspace{1cm} (53)

then there exists $(\alpha, \rho_+, \rho_-) \in [L^\infty([0, \infty); L^\infty(T^1)) \cap C([0, \infty); L^1(T^1))]^4$ such that, for any $t \geq 0$ we have $\alpha(t, \cdot) \in [0, 1]$ a.e. and

$$\langle \Theta(t), b \rangle = \int_{T_x^1} (\alpha(t, x)b(x, \rho_+(t, x), 0) + (1 - \alpha(t, x))b(x, \rho_-(t, x), 1)) \, dx$$

$$\forall b \in C(T_x^1 \times \mathbb{R}_x \times \mathbb{R}_\eta).$$  \hspace{1cm} (54)

Furthermore, $(\alpha, \rho_+, \rho_-)$ together with $u$ verifies the biphasic Baer-Nunziato system (36).
5 Numerical illustrations

In this last section we will illustrate the results stated in the paper. We design two numerical schemes: one to approximate the mesoscopic system, that is to say System (5) with a Cauchy datum (6), and one to approximate the macroscopic system (8) with any Cauchy datum.

5.1 Mesoscopic discretization

The numerical scheme we design here consists in a "brute force" discretization of System (5) where \( c_0(1 - c_0) = 0 \).

As the fluids have to remain pure (not mixed) in every cell, because, for modelling reasons, we want to use only the pure pressure laws (the mixture pressure law being unknown at this stage), the length of each pure zone has to be larger than a cell (and, more precisely, has to be large as an integer number of cells). Here, in the numerical tests, we choose to consider a numerical initial condition such that the fluid changes from one cell to the other (but of course this is not a restriction). The problem to achieve the aim here comes from the so-called numerical diffusion: the discretization of \( \partial_t c + u \partial_x c = 0 \) with a stable scheme usually brings a certain amount of diffusion, the effect of which being not to preserve the important feature \( c(t, \cdot)(1 - c(t, \cdot)) = 0 \) a.e.. In order to pass over this phenomenon, we consider a Lagrangian, or pseudo-Lagrangian\(^1\) scheme in which the cells follow the fluid in its transport, namely in which the edges of every cell moves at the fluid velocity. In this Lagrangian frame, the equation for the mass fraction is \( D_t c = 0 \) (recall that \( D_t = \partial_t + u \partial_x \)).

The spirit of the proposed scheme is the one of staggered schemes: it can be seen as a modification of the schemes in [16] and [13], this modification being that the present scheme is more explicit (precisely, the nonlinearity are time-discretized in a backward Euler way) and that it is a pseudo-Lagrange scheme. Staggered schemes are schemes in which different unknowns are associated to different points or cells in the mesh (for example, the density and the velocity, here). At last, this scheme is a time discretization of the semi-discrete scheme (36) that was proposed to determine the limit macroscopic system.

The discretization is the following. Let \( J \in \mathbb{N} \setminus \{0\} \) be the number of cells in \([0, 1)\). Let \( (x^0_j, \rho^0_j, c^0_j, u^0_j) \) be the collection of cell interface positions at time 0. One assumes \( 0 \leq x^0_{j-1/2} < x^0_{j+1/2} < 1 \) for any \( j = 1, \ldots, J - 1 \).

In order to take into account the fact that the problem under consideration is posed on \( \mathbb{T} \) in a simple manner, i.e. without taking care of the cells and quantities on the boundary, we extend all the data over \( \mathbb{R} \) and \( \mathbb{Z} \) by periodicity.

The cells itselfs are denoted by \( \omega^0_j = [x^0_{j-1/2}, x^0_{j+1/2}] \) for \( j \in \mathbb{Z} \). We denote by \( \Delta x^n_j = x^n_{j+1/2} - x^n_{j-1/2} \) their length. The minimum length of these cells is intended to be small (and to tend to 0 as \( J \) tends to \( \infty \) to reach convergence). We also will need the distance between two centers of consecutive cells: \( \Delta x^n_{j+1/2} = (\Delta x^n_j + \Delta x^n_{j+1})/2 \).

Each time step of the scheme, given a discrete datum \( (x^n_j, \rho^n_j, c^n_j, u^n_{j-1/2}) \) \( j \in \mathbb{Z} \), consists in defining

\(^1\) It can be called pseudo-Lagrangian because, although the solution is actually expressed in the classical Euler variable, the scheme strongly uses the Langrange formulation of the system.
appropriately $\Delta t^n > 0$ and constructing $\left( x_{j-1/2}^{n+1}, \rho_j^{n+1}, c_j^{n+1}, u_{j-1/2}^{n+1} \right)_{j \in \mathbb{Z}}$ by the formula

\[
\begin{align*}
\rho_{j+1/2}^{n+1} &= \frac{\Delta x_j^n \rho_j^n + \Delta x_{j+1}^n \rho_{j+1}^n}{\Delta x_j^n + \Delta x_{j+1}^n}, & j \in \mathbb{Z}, \\
\alpha_j^{n+1} &= \alpha_j^n, & j \in \mathbb{Z}, \\
\rho_{j+1/2}^n \Delta x_{j+1/2}^{n+1/2} u_{j+1/2}^{n+1} &= \rho_{j+1/2}^n \Delta x_{j+1/2}^{n+1/2} u_{j+1/2}^{n+1} - \Delta t^n \left( p(c_{j+1}^n, \rho_{j+1}^n) - p(c_j^n, \rho_j^n) \right) \\
&\quad + \Delta t^n \left( \mu(c_{j+1}^n, \rho_{j+1}^n) \frac{u_{j+1/2}^{n+1} - u_{j+1/2}^n}{\Delta x_{j+1}^n} - \mu(c_j^n, \rho_j^n) \frac{u_{j+1/2}^n - u_{j+1/2}^{n-1}}{\Delta x_j^n} \right), & j \in \mathbb{Z},
\end{align*}
\]

In the system above,
- The first equation defines a density associated to the nodes $x_{j+1/2}^n$, density that is used in the third equation,
- The second equation is a (non-diffusive) discretization of $D_t \rho = 0$,
- The third equation is the discretization of $\partial_t p\mu + \partial_\alpha (\rho u^2 + p) = \partial_x (\mu \partial_x u)$: indeed notice that thanks to the last equation of the system, this third equation rewrites

\[
\begin{align*}
\rho_{j+1/2}^n \Delta x_{j+1/2}^{n+1/2} u_{j+1/2}^{n+1} &= \rho_{j+1/2}^n \Delta x_{j+1/2}^{n+1/2} u_{j+1/2}^{n+1} - \Delta t^n \left( p(c_{j+1}^n, \rho_{j+1}^n) - p(c_j^n, \rho_j^n) \right) \\
&\quad + \Delta t^n \left( \mu(c_{j+1}^n, \rho_{j+1}^n) \frac{u_{j+1/2}^{n+1} - u_{j+1/2}^n}{\Delta x_{j+1}^n} - \mu(c_j^n, \rho_j^n) \frac{u_{j+1/2}^n - u_{j+1/2}^{n-1}}{\Delta x_j^n} \right), & j \in \mathbb{Z},
\end{align*}
\]

which is consistent with the partial differential equation,
- The fourth equation is the translation of the mesh,
- Fifth and sixth equations redefine quantities that are used in the scheme,
- The last equation expresses the conservation of mass in a material volume $\partial_t \rho + \partial_x (\rho u) = 0$.

It is possible to prove that if the time step $\Delta t^n$ is sufficiently small, $x_{j-1/2}^n < x_{j+1/2}^{n+1}$ for all $j$ implies $x_{j-1/2}^{n+1} < x_{j+1/2}^{n+1}$ for all $j$.

5.2 Macroscopic discretization

For the macroscopic homogenized system \(\text{(55)}\), we use the same type of scheme. The only difference is that the volume fraction of fluid $\alpha$ does not satisfy $\alpha_+ (1 - \alpha_+) = 0$ but

\[
D_t \alpha = \frac{\alpha (1 - \alpha)}{\alpha_+ - (1 - \alpha) \rho_+} \left( p_+ (\rho_+) - p_-(\rho_-) - (\mu_+ - \mu_-) \partial_x u \right).
\]

In the following we choose to discretize this equation in a forward Euler way (but a backward Euler scheme has also been tested and validated):

\[
\alpha_j^{n+1} = \alpha_j^n + \Delta t^n \left( \frac{\alpha_j^n (1 - \alpha_j^n)}{\alpha_j^+ - (1 - \alpha_j^+) \rho_+} \left( p_+ (\rho_+^{n+1}) - p_- (\rho_-^{n+1}) - (\mu_+ - \mu_-) \frac{u_{j+1/2}^{n+1} - u_{j-1/2}^{n+1}}{x_{j+1/2}^{n+1} - x_{j-1/2}^{n+1}} \right) \right).
\]

All the other variables are approximated in a very standard and natural way.
5.3 Experiments

We propose two test-cases with \( p_+ (x) = x \) and \( p_- (x) = x^2 \). They are associated with a Cauchy datum of Riemann type:

\[
\begin{array}{l}
\alpha_0 (x) = 1/2, \quad x \in \mathbb{T}_x, \\
\rho_+ (x) = \rho_- (x) = \begin{cases} 
1/8 & \text{if } x \in [0, 1/4) \cup [3/4, 1), \\
2 & \text{if } x \in [1/4, 3/4),
\end{cases} \\
u(x) = 0, \quad x \in \mathbb{T}_x,
\end{array}
\]

and we propose to compare the numerical solutions obtained at time \( t = 0.1 \) with 1000 cells
— with the homogenized scheme of Section 5.2,
— and with the mesoscopic scheme of Section 5.1 by setting

\[
(\alpha_0^0, \rho_0^0, \rho_0^0) = \begin{cases} 
(1, \rho_0^0, 0) & \text{if } j \text{ is even}, \\
(0, 0, \rho_0^0) & \text{if } j \text{ is odd},
\end{cases}
\]

and with a mesh with constant space step, which indeed corresponds in the weak limit to \( \alpha = 1/2 \).

Note that the pressure is largely oscillating in this initial condition for the mesoscopic system.

In the first test, we take \( \mu_+ = \mu_- = 0.1 \) while in the second one we choose \( \mu_+ = 0.1 \) and \( \mu_- = 0.02 \). Figures 1 to 5 allow to compare the density, velocity, pressure and volume fraction. We observe a very good agreement between the mesoscopic and the macroscopic results. Note that for the mesoscopic computation, we consider that there is only one density and one pressure, thus these quantities oscillate very fast (at the scale of the cell, which is the scale of the mixture). We observe, especially on the zoom of the density proposed by Figure 2, that these oscillations occur between two functions that are very close to \( \rho_+ \) and \( \rho_- \) computed by the macroscopic scheme. With the mesoscopic scheme, the volume fraction of fluid \( + \) should oscillate between 0 and 1. In order to evaluate a volume fraction of \( + \) in the limit mixture, what we here (Figure 5) call \( \alpha_j^n \) is computed by

\[
\alpha_j^n = \frac{c_j (x_{j+1/2}^n - x_{j-1/2}^n) + c_{j-1} (x_{j-1/2}^n - x_{j-3/2}^n) + c_{j+1} (x_{j+3/2}^n - x_{j+1/2}^n) / 2}{x_{j+3/2}^n - x_{j-3/2}^n}
\]

(recall that \( c_j \) is equal to 0 or 1 and does not depend on the time index).

The organization and the comments for the case with different viscosities, from Figure 6 to Figure 10 are the same.

5.3.1 Case with equal viscosities

![Figure 1 – Densities. On the left, the 3 densities of the mixture, on the right, the density of the unmixed fluid.](image-url)
Figure 2 – Densities. Zoom of the preceding figures.

Figure 3 – Velocities. On the left, the velocity of the mixture, on the right, the velocity of the unmixed fluid.
Figure 4 – Pressures. On the left, the 3 pressures in the mixture, on the right, the pressure in the unmixed fluid.

Figure 5 – Volume fractions. On the left, the volume fraction $\alpha_+$ in the mixture, on the right, the estimate of the volume fraction in the unmixed fluid.
5.3.2 Case with different viscosities

Figure 6 – Densities. On the left, the 3 densities of the mixture, on the right, the density of the unmixed fluid.

Figure 7 – Densities. Zoom of the preceding figures.
Figure 8 – Velocities. On the left, the velocity of the mixture, on the right, the velocity of the unmixed fluid.

Figure 9 – Pressures. On the left, the 3 pressures in the mixture, on the right, the pressure in the unmixed fluid.
Figure 10 – Volume fractions. On the left, the volume fraction $\alpha_+$ in the mixture, on the right, the estimate of the volume fraction in the unmixed fluid.

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