Weighted fractional generalized cumulative past entropy and its properties

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Abstract
In this paper, we introduce weighted fractional generalized cumulative past entropy of a non-negative absolutely continuous random variable. Various properties of the proposed weighted fractional measure are studied, including some bounds and stochastic orders. A connection between the proposed measure and the left-sided Riemann-Liouville fractional integral is established. Further, the proposed measure is studied for the proportional reversed hazard rate model. Next, a nonparametric estimator of the weighted fractional generalized cumulative past entropy is proposed based on empirical distribution function. Various examples with a real-life data set are considered for illustrative purpose. A validation of the proposed measure is provided using the logistic map and some applications are also discussed. Weighted fractional generalized cumulative paired entropy is proposed and some of its properties are explored. Finally, large-sample properties of the proposed empirical estimator are studied.

Keywords Weighted generalized cumulative past entropy · Fractional calculus · Reversed hazard rate model · Empirical cumulative distribution function · Logistic map

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1 Introduction
Entropy plays an important role in several areas of statistical mechanics and information theory. In statistical mechanics, the most widely applied form of entropy was proposed by Boltzmann and Gibbs, and in information theory, it was the one introduced by Shannon. Due to the growing applicability of entropy measures in different problems, various generalizations were proposed and their information theoretic properties were studied; see, for
instance, Rényi (1961) and Tsallis (1988). We recall that most of the generalized entropies were developed based on the concept of deformed logarithm. But, two generalized concept of entropies-fractal (see Wang (2003)) and fractional (Ubriaco (2009)) entropies-were proposed based on natural logarithm. To be specific, let \( P = (p_1, \ldots, p_n) \) be the probability mass function of a discrete random variable \( X \). Then, the Boltzmann-Gibbs-Shannon entropy of \( X \) can be defined through an equation involving the ordinary derivative as

\[
H_{BGS}(X) = - \sum_{i=1}^{n} p_i \ln p_i = - \frac{d}{du} \left. \sum_{i=1}^{n} p_i^u \right|_{u=1}.
\] (1.1)

Ubriaco (2009) proposed a new entropy measure, known as the fractional entropy, after replacing the ordinary derivative by the Weyl fractional derivative (see Ferrari (2018)) in (1.1). It is given by

\[
H_\alpha(X) = \sum_{i=1}^{n} p_i (-\ln p_i)^\alpha, \quad 0 \leq \alpha \leq 1.
\] (1.2)

The fractional entropy in (1.2) is positive, concave and non-additive. Further, one can recover the Shannon entropy (see Shannon (1948)) from (1.2) when \( \alpha = 1 \). From (1.2), we observe that the measure of information is mainly a function of probabilities of occurrence of various events. However, we often face with many situations (see Guiaşu (1971)) in different fields, wherein the probabilities and qualitative characteristics of events need to be taken into account for better uncertainty analysis. As a result, the concept of weighted entropy was introduced by Guiaşu (1971) in the form

\[
H^w(X) = - \sum_{i=1}^{n} w_i p_i \ln p_i,
\] (1.3)

where \( w_i \) is a non-negative number (known as weight) directly proportional to the importance of the \( i \)th elementary event. Note that the weights \( w_i \)'s can be equal. Following the same line as in (1.3), the weighted fractional entropy can be defined as

\[
H^w_\alpha(X) = \sum_{i=1}^{n} w_i p_i (-\ln p_i)^\alpha, \quad 0 \leq \alpha \leq 1.
\] (1.4)

Note that for \( w_i = 1, \ i = 1, \ldots, n \), (1.4) reduces to the fractional entropy given by (1.2). Further, (1.4) equals to the weighted entropy in (1.3) when \( \alpha = 1 \).

Recently, motivated by the notion of cumulative residual entropy due to Rao et al. (2004) given by

\[
CRE(X) = - \int_0^\infty \tilde{K}(x) \ln \tilde{K}(x) dx,
\]

where \( \tilde{K} = 1 - K \), and the fractional entropy in (1.2), Xiong et al. (2019) introduced a new information measure, known as the fractional cumulative residual entropy. The concept of multi-scale fractional cumulative residual entropy was presented by Dong and Zhang (2020). Moreover, inspired by the cumulative entropy (see Di Crescenzo and Longobardi (2009)) and (1.2), Di Crescenzo et al. (2021) proposed fractional generalized cumulative
entropy of a random variable $X$ with support $(0, s)$ as

$$CPE_\gamma(X) = \frac{1}{\Gamma(\gamma + 1)} \int_0^s K(x)[-\ln K(x)]^\gamma dx, \quad \gamma > 0, \quad (1.5)$$

where $K$ is the cumulative distribution function (CDF) of $X$. The fractional generalized cumulative entropy is a generalization of cumulative entropy and generalized cumulative entropy proposed by Di Crescenzo and Longobardi (2009) and Kayal (2016), respectively. It is known that cumulative entropy and generalized cumulative entropy are both independent of the location. This property appears as a drawback when quantifying information of an electronics device or neuron in different intervals having equal widths. Thus, to cope with these situations, various authors proposed length-biased (weighted) information measures. The weighted measures are also called shift-dependent measures by some researchers. For example, one may refer to Di Crescenzo and Longobardi (2007), Misagh et al. (2011), Misagh (2016), Das (2017), Kayal and Moharana (2017a, b), Misagh et al. (2017), Nourbakhsh and Yari (2017), Kayal (2018) and Kayal and Moharana (2019) for some weighted versions of various information measures. The existing weighted information measures and the fractional generalized cumulative entropy in (1.5) provide us a motivation for considering the weighted fractional generalized cumulative past entropy (WFGCPE), which is discussed in detail in the subsequent sections of this paper. The following definitions will be useful for establishing some ordering results for WFGCPE.

**Definition 1.1** Let $X_1$ and $X_2$ be two non-negative absolutely continuous random variables with CDFs $K_1$ and $K_2$, respectively. Then, $X_1$ is said to be smaller than $X_2$ in the sense of

(i) usual stochastic order, denoted by $X_1 \leq_{sf} X_2$, if $K_2(x) \leq K_1(x)$, for all $x \in \mathbb{R}$;

(ii) hazard rate order, denoted by $X_1 \leq_{hr} X_2$, if $\tilde{K}_2(x)/\tilde{K}_1(x)$ is nondecreasing in $x > 0$, where $\tilde{K}_1 = 1 - K_1$ and $\tilde{K}_2 = 1 - K_2$;

(iii) decreasing convex order, denoted by $X_1 \leq_{dcx} X_2$, if and only if $E(\tau(X_1)) \leq E(\tau(X_2))$ holds for all non-increasing convex real-valued functions $\tau$ for which the involved expectations exist.

The rest of the paper is organized as follows. In Section 2, we introduce WFGCPE and then study its various properties, including some ordering results and bounds. Further, a connection of the proposed measure to the fractional calculus is pointed out. The proportional reversed hazard model is considered and the WFGCPE is studied for it. Section 3 deals with the estimation of the introduced measure. An empirical WFGCPE estimator is proposed based on the empirical distribution function. Further, large-sample properties of the proposed estimator are studied. Validation of the WFGCPE is explored based on the data generated from a logistic map in Section 4. Some applications are also provided. Section 5 explores the concept of weighted fractional generalized cumulative paired entropy, and some of its properties are studied. Finally, Section 6 concludes with some brief discussions.

Throughout the paper, the random variables are considered as non-negative random variables. The terms increasing and decreasing are used in the wider sense. The differentiation and integration are assumed to exist whenever they are used. The notation $\mathbb{N}$ denotes the set of natural numbers. Further, throughout the paper, the conventions that $0 = 0$, $\ln 0 = 0$, $\ln \infty$ are adopted. The prime $'$ denotes the first-order ordinary derivative.
2 Weighted fractional generalized cumulative past entropy

In this section, we propose WFGCPE and then study its various properties. Consider a non-negative absolutely continuous random variable \( X \) with support \((0, s)\) and CDF \( K \) and PDF \( k \). Here, \( s \) can be \( \infty \) as well. Then, the WFGCPE of \( X \) with a general non-negative weight function \( \psi(x) \) \((\geq 0)\) is defined as

\[
CPE^\psi_{\gamma}(X) = \frac{1}{\Gamma(\gamma + 1)} \int_0^s \psi(x)K(x)[-\ln K(x)]^{\gamma} dx, \quad \gamma > 0, \quad (2.1)
\]

provided the involved integral is finite, where \( \Gamma \) denotes the gamma function. From (2.1), one can easily notice that the information measure \( CPE^\psi_{\gamma}(X) \) is always non-negative, and is equal to zero when \( X \) is degenerate. Note that the WFGCPE is non-additive. We recall that an information measure \( H \) is additive if

\[
H(A + B) = H(A) + H(B), \quad (2.2)
\]

for any two probabilistically independent systems \( A \) and \( B \). If (2.2) is not satisfied, then the information measure is said to be non-additive. Several information measures have been proposed in the literature since the introduction of the Shannon entropy. Among those, Shannon’s entropy and Renyi’s entropy (see Rényi (1961)) are additive while most other generalizations (see, for example, Tsallis 1988) are non-additive. For \( \gamma \in \mathbb{N} \) and \( s \to +\infty \), \( CPE^\psi_{\gamma}(X) \) reduces to the weighted generalized cumulative entropy proposed by Tahmasebi et al. (2020). Further, when \( \psi(x) = 1 \), we get the fractional generalized cumulative entropy due to Di Crescenzo et al. (2021). Now, let \( \Psi'(x) = \frac{d}{dx} \Psi(x) = \psi(x) \). Then, when \( \gamma \to 0^+ \) and \( 0 < s < +\infty \), we get from (2.1)

\[
CPE^\psi_{\gamma}(X) = \int_0^s \psi(x)dx - \int_0^s \psi(x)\bar{K}(x)dx
= \Psi(s) - \Psi(0) - \int_0^s \psi(x)\left( \int_x^s k(y)dy \right) dx
= \Psi(s) - \Psi(0) - \int_0^s \int_0^y \psi(x)k(y)dxdy
= \Psi(s) - E[\Psi(X)]. \quad (2.3)
\]

Thus,

\[
CPE^\psi_{\gamma}(X) = \begin{cases} 
\Psi(s) - E[\Psi(X)], & \text{if } \gamma \to 0^+ \text{ and } 0 < s < +\infty \\
+\infty, & \text{if } \gamma \to 0^+ \text{ and } s = +\infty.
\end{cases} \quad (2.4)
\]

Moreover, in the special case when \( \psi(x) = x \), we have

\[
CPE^\psi_{\gamma}(X) = \begin{cases} 
\frac{1}{2} \left[ s^2 - E(X^2) \right], & \text{if } \gamma \to 0^+ \text{ and } 0 < s < +\infty \\
CPE^{\psi_{\gamma}(x)=x}_{\frac{n}{n}}(X), & \text{if } \gamma = n \in \mathbb{N} \text{ and } s = +\infty \\
CPE^{\psi_{\gamma}(x)=x}^{(x)}(X), & \text{if } \gamma = 1 \text{ and } s = +\infty \\
+\infty, & \text{if } \gamma \to 0^+ \text{ and } s = +\infty,
\end{cases} \quad (2.5)
\]

where \( CPE^{\psi_{\gamma}(x)=x}_{\gamma}(X) \) and \( CPE^{\psi_{\gamma}(x)=x}(X) \) are the shift-dependent generalized cumulative past entropy of order \( n \) (see Eq. (1.4) of Kayal and Moharana (2019)) and weighted cumulative past entropy (see Eq. (10) of Misagh (2016)), respectively.
Next, we consider an example to show that even though the fractional generalized cumulative past entropy of two distributions are the same, but their WFGCPEs are not the same.

**Example 2.1** Consider two random variables $X_1$ and $X_2$ with respective CDFs $K_1(x) = x - a$, $0 < a < x < a + 1$ and $K_2(x) = x - b$, $b < x < b + 1 < +\infty$, with $b \neq a$. Then, the fractional generalized cumulative past entropy of $X_1$ and $X_2$ can be obtained, respectively, as

$$CPE_\gamma(X_1) = \frac{1}{2^{\gamma + 1}} = CPE_\gamma(X_2), \quad \gamma > 0;$$

thus, the fractional generalized cumulative past entropy of $X_1$ and $X_2$ are the same. Indeed, it is expected since the fractional generalized cumulative past entropy is shift-independent (see Proposition 2.2 of Di Crescenzo et al. (2021)). Let us now take $\psi(x) = x$. Then,

$$CPE_\gamma^{\psi(x)=x}(X_1) = \frac{1}{3^{\gamma + 1}} + \frac{a}{2^{\gamma + 1}} \quad \text{and} \quad CPE_\gamma^{\psi(x)=x}(X_2) = \frac{1}{3^{\gamma + 1}} + \frac{b}{2^{\gamma + 1}},$$

which show that the WFGCPEs of $X_1$ and $X_2$ are not the same. Further, let $\psi(x) = x^2$. In this case, we have

$$CPE_\gamma^{\psi(x)=x^2}(X_1) = \frac{1}{4^{\gamma + 1}} + \frac{2a}{3^{\gamma + 1}} + \frac{a^2}{2^{\gamma + 1}} \quad \text{and} \quad CPE_\gamma^{\psi(x)=x^2}(X_2) = \frac{1}{4^{\gamma + 1}} + \frac{2b}{3^{\gamma + 1}} + \frac{b^2}{2^{\gamma + 1}},$$

which also reveal that the WFGCPEs of $X_1$ and $X_2$ are different from each other.

From Example 2.1, we observe that while ignoring qualitative characteristics of a given data set, the fractional generalized cumulative past entropy of two distributions are the same, as treated from the quantitative point of view. However, when we do not ignore it, they are not the same. In Table 1, we provide closed-form expressions of the WFGCPE of various distributions for the two choices of $\psi(x)$. Let $K$ and $\tilde{K} = 1 - K$ be the distribution and survival functions of a symmetrically distributed random variable with bounded support $(0, s)$. Di Crescenzo et al. (2021) then showed that for this symmetric random variable, the fractional generalized cumulative residual entropy and the fractional generalized cumulative entropy are the same. However, this property does not hold for weighted versions of the fractional generalized cumulative residual entropy and fractional generalized cumulative entropy. Indeed,

$$CPE_\gamma^{\psi}(X) = \frac{1}{\Gamma(\gamma + 1)} \int_0^s \psi(s - x)\tilde{K}(x)[-\ln \tilde{K}(x)]' dx, \quad \gamma > 0. \quad (2.6)$$

Particularly, for a symmetric random variable $X$ with $\psi(x) = x$, we have

$$CPE_\gamma^{\psi}(X) = \frac{s}{\Gamma(\gamma + 1)} \int_0^s \tilde{K}(x)[-\ln \tilde{K}(x)]' dx - \frac{1}{\Gamma(\gamma + 1)} \int_0^s x\tilde{K}(x)[-\ln \tilde{K}(x)]' dx$$

$$= sCRE_\gamma(X) - CCRE_\gamma^{\psi(x)=x}(X), \quad \text{say}, \quad (2.7)$$

where $CRE_\gamma(X)$ and $CCRE_\gamma^{\psi(x)=x}(X)$ are, respectively, known as fractional generalized cumulative residual entropy and weighted fractional generalized cumulative residual entropy. Di Crescenzo et al. (2021) showed that the fractional generalized cumulative entropy of a non-negative random variable is indeed shift-independent.

Golomb (1966) proposed an information generating (IG) function for a PDF $k$ as
Table 1 Closed-form expressions of the WFGCPE of different distributions. For the case of Fréchet distribution, we assume that $\gamma > 3/c$

| Model                  | Cumulative distribution function $\psi(x)$ = $x$ | $\psi(x)$ = $x^2$ |
|------------------------|-------------------------------------------------|--------------------|
| Power distribution     | $K(x) = \left(\frac{x^c}{b^c}\right)$, $0 < x < b$, $c > 0$, $\frac{b^2}{c(1 + \frac{2}{c})^{\gamma+1}}$ | $\frac{b^3}{c(1 + \frac{3}{c})^{\gamma+1}}$ |
| [3ex] Fréchet distribution | $K(x) = e^{-bx^{-c}}$, $x > 0$, $b, c > 0$ | $\frac{b^2 \Gamma(\gamma - \frac{2}{c})}{c \Gamma(\gamma + 1)}$ | $\frac{b^3 \Gamma(\gamma - \frac{3}{c})}{c \Gamma(\gamma + 1)}$ |

$$G_\beta(k) = \int_0^\infty k^\beta(x)dx, \ \beta > 0.$$ (2.8)

The derivatives of this IG function with respect to $\beta$ at $\beta = 1$ yield statistical information measures for a probability distribution. For example, the first-order derivative of $G_\beta(k)$ with respect to $\beta$ at $\beta = 1$ produces negative Shannon entropy measure. For detailed properties of the Shannon entropy, one may refer to Shannon (1948). Recently, Kharazmi and Balakrishnan (2021) considered the IG function and discussed some new properties that reveal its connections to some other well-known information measures. They have shown that the IG measure can be expressed based on different orders of fractional Shannon entropy. Kharazmi et al. (2021) studied IG function and relative IG function measures associated with maximum and minimum ranked set sampling schemes with unequal sizes. Along the similar lines, we define here a weighted cumulative past entropy generating function as

$$G_\beta(K) = \int_0^s \psi(x)K^\beta(x)dx, \ \beta > 0,$$ (2.9)

where $\psi(x)$ is a positive-valued weight function. Evidently,

$$\frac{d}{d\beta} G_\beta(K)|_{\beta=1} = \int_0^s \psi(x)K(x) \ln K(x)dx = -CPE_\gamma^\psi(X).$$ (2.10)

Indeed, higher-order derivatives of $G_\beta(K)$ yield correspondingly higher-order weighted cumulative past entropy measures.

In the following proposition, we establish that the WFGCPE is shift-dependent. This renders the proposed weighted fractional measure useful in many context-dependent situations as mentioned earlier.

**Proposition 2.1** Let $Y = aX + b$, where $a > 0$ and $b \geq 0$. Then,

$$CPE_\gamma^\psi(Y) = \frac{a}{\Gamma(\gamma + 1)} \int_0^s \psi(ax + b)K(x)[- \ln K(x)]^\gamma dx, \ \gamma > 0.$$ (2.11)

**Proof** The proof follows readily from $K_Y(x) = K(\frac{x-b}{a})$.

In particular, let us consider $\psi(x) = x$. Then, after some simplification, from (2.11), we get

$$CPE_\gamma^\psi(Y) = a^2CPE_\gamma^\psi(x) = x + abCPE_\gamma(X),$$ (2.12)
where
\[ CPE_\gamma^{(\psi)}(X) = \frac{1}{\Gamma(\gamma + 1)} \int_0^s x K(x)[-\ln K(x)]^\gamma dx \] (2.13)

and \( CPE_\gamma(X) \) is as given in (1.5). It is always of interest to express various information measures in terms of expectation of a function of the random variable of interest. Consider
\[ \mu(t) = \int_0^t K(x) \frac{K(t)}{K(x)} dx, \] (2.14)

which is known as the mean inactivity time of \( X \). Di Crescenzo and Longobardi (2009) then expressed cumulative entropy in terms of expectation of the mean inactivity time of \( X \). Recently, Di Crescenzo et al. (2021) showed that the fractional generalized cumulative entropy can be written as the expectation of a decreasing function of \( X \). We now obtain similar findings for the case of WFGCPE.

**Proposition 2.2** Let \( X \) be non-negative absolutely continuous random variable with distribution function \( K(\cdot) \) and density function \( k(\cdot) \) such that \( CPE_\gamma^{(\psi)}(X) < +\infty \). Then,
\[ CPE_\gamma^{(\psi)}(X) = E[\tau_\gamma^{(\psi)}(X)], \] (2.15)

where
\[ \tau_\gamma^{(\psi)}(u) = \frac{1}{\Gamma(\gamma + 1)} \int_u^s \psi(x)[-\ln K(x)]^\gamma dx, \quad \gamma > 0. \] (2.16)

**Proof** Noting \( K(x) = \int_0^x k(u)du \) and then applying Fubini’s theorem, we get from (2.1)
\[ CPE_\gamma^{(\psi)}(X) = \frac{1}{\Gamma(\gamma + 1)} \int_0^s \psi(x)[-\ln K(x)]^\gamma \left( \int_0^x k(u)du \right) dx \]
\[ = \frac{1}{\Gamma(\gamma + 1)} \int_0^s f(u) \left( \int_u^s \psi(x)[-\ln K(x)]^\gamma dx \right) du \]
\[ = E[\tau_\gamma^{(\psi)}(X)], \]

where \( \tau_\gamma^{(\psi)}(.) \) is as in (2.16). This completes the proof.

Note that (2.15) reduces to Eq. (20) of Tahmasebi et al. (2020) for \( \gamma = n \in \mathbb{N} \) and \( s = +\infty \). For the choice of \( \psi(x) = 1 \), Proposition 2.2 reduces to Proposition 2.1 of Di Crescenzo et al. (2021). As with normalized cumulative entropy, Di Crescenzo et al. (2021) proposed a normalized fractional generalized cumulative entropy of a random variable \( X \) with non-negative support. Here, we define a normalized WFGCPE. It is assumed that the weighted cumulative past entropy with a general non-negative weight function is non-zero and finite, and it is given by (see Suhov and Sekeh (2015))
\[ CPE_\gamma^{(\psi)}(X) = -\int_0^s \psi(x) K(x) \ln K(x) dx, \quad \psi(x) \geq 0. \] (2.17)

The normalized WFGCPE of \( X \) can then be defined as
\[ NCPE_\gamma^{(\psi)}(X) = \frac{CPE_\gamma^{(\psi)}(X)}{(CPE_\gamma^{(\psi)})^\gamma} = \frac{1}{\Gamma(\gamma + 1)} \frac{\int_0^s \psi(x) K(x)[-\ln K(x)]^\gamma dx}{\left( \int_0^s \psi(x) K(x)[-\ln K(x)] dx \right)^\gamma}, \quad \gamma > 0. \] (2.18)
Fig. 1 Graphs of the normalized WFGCPE with \( a \) \( \psi(x) = x \) and \( b \) \( \psi(x) = x^2 \) for the power distribution in Table 2.

Note that
\[
\lim_{\gamma \to 0^+} NCPE_\gamma^\psi (X) = \int_0^x \psi(x) K(x) dx \quad \text{and} \quad \lim_{\gamma \to 1} NCPE_\gamma^\psi (X) = 1.
\]

The closed-form expressions of the normalized WFGCPE of power and Fréchet distributions are presented in Table 2 for two choices of the weight function. The graphs of the normalized WFGCPE for power distribution with distribution function as given in Table 1 have been plotted in Fig. 1 for two different choices of \( \psi(x) \).

### 2.1 Ordering results

In this subsection, we obtain ordering properties for the WFGCPE. It can be shown that the function \( \tau_\gamma^{\psi} (u) \) in (2.16) is decreasing and convex when \( \psi(x) \) is decreasing in \( x \). Thus, for decreasing \( \psi \), we have
\[
X_1 \leq_{dcx} X_2 \Rightarrow E[\tau_\gamma^{\psi} (X_1)] \leq E[\tau_\gamma^{\psi} (X_2)] \Rightarrow CPE_\gamma^{\psi} (X_1) \leq CPE_\gamma^{\psi} (X_2). \tag{2.19}
\]

Next, we will study whether the usual stochastic order implies the ordering of the WFGCPE. In doing so, we consider two random variables \( X_1 \) and \( X_2 \) with respective distribution functions \( K_1(x) = x^{c_1} \) and \( K_2(x) = x^{c_2} \), \( 0 < x < 1 \), \( c_1, c_2 > 0 \). For \( c_1 \leq c_2 \), clearly \( K_1(x) \geq K_2(x) \) implies \( X_1 \leq_{st} X_2 \). Now, we plot graphs of the difference of the WFGCPEs of \( X_1 \) and \( X_2 \) in Fig. 2, for some values of \( \gamma \), which reveal that in general, the ordering between the WFGCPEs may not hold.

| Model          | \( \psi(x) = x \)                                                                 | \( \psi(x) = x^2 \)                                                                 |
|----------------|----------------------------------------------------------------------------------|----------------------------------------------------------------------------------|
| Power distribution | \[
\frac{(c + 2)^{\gamma - 1}}{\Gamma(\gamma + 1)b^{2(\gamma - 1)}}
\]              | \[
\frac{(c + 3)^{\gamma - 1}}{\Gamma(\gamma + 1)b^{3(\gamma - 1)}}
\]              |
| Fréchet distribution | \[
\frac{c^{\gamma - 1}}{(\Gamma(\gamma + 1))^2} - \frac{2}{b^2(\gamma - 1)} \frac{\Gamma(\gamma - \frac{2}{c})}{(\Gamma(1 - \frac{2}{c}))^\gamma}
\] | \[
1 - \frac{c^{\gamma - 1}}{(\Gamma(\gamma + 1))^2} \frac{\Gamma(\gamma - \frac{2}{c})}{b^2(\gamma - 1) (\Gamma(1 - \frac{3}{c}))^\gamma}
\] |
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Fig. 2 The plots of $CPE_{\psi}^{\gamma}(X_1) - CPE_{\psi}^{\gamma}(X_2)$ for (a) $\gamma = 0.5$, (b) $\gamma = 0.75$, (c) $\gamma = 1.5$, and (d) $\gamma = 2.5$. Here, we have taken $\psi(x) = x$.

The next result compares WFGCPE measures when two random variables are ordered in the sense of the usual stochastic order. Here, prime denotes the ordinary derivative.

**Proposition 2.3** Consider two non-negative absolutely continuous random variables $X_1$ and $X_2$ with CDFs $K_1$ and $K_2$, respectively, such that $X_1 \leq_{st} X_2$. Further, assume that the means of $X_1$ and $X_2$ are finite, but unequal. Then, for $\tau_{\psi}^{\gamma}(x) < +\infty$ and $E[\tau_{\psi}^{\gamma}(X)] < \infty$, we have

$$CPE_{\psi}^{\gamma}(X_1) = E[\tau_{\psi}^{\gamma}(X_2)] + E[\tau_{\psi}^{\gamma}(V)][E(X_1) - E(X_2)],$$

where $V$ is a non-negative absolutely continuous random variable with density function

$$k_V(x) = \frac{\tilde{K}_2(x) - \tilde{K}_1(x)}{E(X_2) - E(X_1)}, \quad x > 0. \quad (2.21)$$

**Proof** We note that $CPE_{\psi}^{\gamma}(X_1) = E[\tau_{\psi}^{\gamma}(X_1)]$ (see Proposition 2.2). Now, the rest of the proof follows from Theorem 4.1 of Di Crescenzo (1999).

It can be easily seen that when $\psi(x) = 1$, Proposition 2.3 simply reduces to Proposition 3.4 of Di Crescenzo et al. (2021). Moreover, when $\gamma = n \in \mathbb{N}$ and $\psi(x) = 1$, Proposition 2.3 coincides with Proposition 3.8 of Di Crescenzo and Toomaj (2017). Here, $\tau_{\psi}^{\gamma}(v) \leq 0$, for $v > 0$. Thus, under the assumptions of Proposition 2.3, a lower bound of $CPE_{\psi}^{\gamma}(X_1)$ can be obtained as

$$CPE_{\psi}^{\gamma}(X_1) \geq E[\tau_{\psi}^{\gamma}(X_2)].$$

In the next subsection, we discuss various bounds of WFGCPE in (2.1).
2.2 Bounds

In this subsection, we present bounds of WFGCPE in (2.1).

**Proposition 2.4** For a non-negative random variable with support \((0, s)\) and \(\psi(x) = [\xi(x)]^\gamma, \xi(x) \geq 0\), we have

\[
CPE_\gamma^{\psi}(X) \begin{cases} 
\geq \frac{s^{1-\gamma}}{\Gamma(\gamma+1)} [CPE_\gamma^\xi(X)]^\gamma, \text{ if } \gamma \geq 1 \\
\leq \frac{s^{1-\gamma}}{\Gamma(\gamma+1)} [CPE_\gamma^\xi(X)]^\gamma, \text{ if } 0 < \gamma \leq 1,
\end{cases}
\]

(2.22)

where \(CPE_\gamma^\xi(X) = -\int_0^s \xi(x) K(x) \ln K(x) dx\) is known as the weighted cumulative past entropy with weight function \(\xi(x)\).

**Proof** Let \(\gamma \geq 1\). Then, for \(0 < x < s\), \(K(x) \geq [K(x)]^\gamma\). So, under the assumptions made, from (2.1), we obtain

\[
CPE_\gamma^{\psi}(X) \geq \frac{1}{\Gamma(\gamma+1)} \int_0^s \alpha_\gamma(\beta(x)) dx,
\]

(2.23)

where \(\beta(x) = -\xi(x) K(x) \ln K(x) \geq 0\) and \(\alpha_\gamma(t) = t^\gamma\). It can be shown that \(t^\gamma\) is convex in \(t \geq 0\), for \(\gamma \geq 1\). Then, from Jensen’s integral inequality, the rest of the proof follows. The case for \(0 < \gamma \leq 1\) can be proved in an analogous manner.

**Proposition 2.5** For a non-negative random variable with support \((0, s)\) and \(\gamma > 0\), we have

\[
CPE_\gamma^{\psi}(X) \begin{cases} 
\geq \psi(s) CPE_\gamma(X), \text{ if } \psi \text{ is decreasing} \\
\leq \psi(s) CPE_\gamma(X), \text{ if } \psi \text{ is increasing},
\end{cases}
\]

(2.24)

where \(CPE_\gamma(X) = \frac{1}{\Gamma(\gamma+1)} \int_0^s K(x) [-\ln K(x)]^\gamma dx\) is known as the fractional generalized cumulative past entropy.

**Proof** The proof is straightforward, and is therefore omitted.

**Proposition 2.6** Let \(X\) be an absolutely continuous random variable with support \((0, s)\) with mean \(E(X) = \mu < +\infty\). Then,

(i) \(CPE_\gamma^{\psi}(X) \geq \frac{1}{\Gamma(\gamma+1)} \int_0^s \psi(x) K(x) [1 - K(x)]^\gamma dx\);

(ii) \(CPE_\gamma^{\psi}(X) \geq D(\gamma)e^{H(X)}, \text{ where } D(\gamma) = e^{\int_0^1 \ln[\psi(K^{-1}(u))u(-\ln u)]^\gamma du} \text{ and } H(X) \text{ is the differential entropy of } X\);

(iii) \(CPE_\gamma^{\psi}(X) \geq \tau_\gamma^{\psi}(\mu), \text{ provided } \psi \text{ is decreasing}.

**Proof** The first part follows from the relation \(\ln u \leq u - 1\), for \(0 < u < 1\). To prove the second part, from the log-sum inequality, we have

\[
\int_0^s k(x) \ln \frac{k(x)}{\psi(x) K(x)[-\ln K(x)]^\gamma} dx \geq -\ln \frac{1}{\int_0^s \psi(x) K(x)[-\ln K(x)]^\gamma dx} = -CPE_\gamma^{\psi}(X).
\]

(2.25)

Now, the rest of the proof follows using the same arguments as in the proof of Theorem 2 of Xiong et al. (2019). Third part follows readily from Jensen’s inequality.
We end this subsection with the following result, which provides bounds of WFGCPE of \( X_2 \), where the CDF of \( X_2 \) is

\[
K_2(x) = [K_1(x)]^\eta, \quad x \in \mathbb{R}, \quad \eta > 0,
\]

(2.26)

where \( K_1 \) is the baseline distribution function (see Gupta et al. (1998), Di Crescenzo (2000), and Gupta and Gupta (2007)).

**Proposition 2.7** Let \( X_1 \) and \( X_2 \) be two random variables with CDFs \( K_1 \) and \( K_2 \), respectively. Further, assume that the random variables satisfy the proportional reversed hazard model in (2.26). Then,

\[
CPE_\gamma(X_2) \leq \eta \gamma CPE_\gamma(X_1), \quad \text{for } \eta \geq 1
\]

\[
CPE_\gamma(X_2) \geq \eta \gamma CPE_\gamma(X_1), \quad \text{for } 0 < \eta \leq 1.
\]

**Proof** The proof is simple, and is therefore omitted.

### 2.3 Connection to fractional calculus

Fractional calculus and its applications have received considerable attention. One may refer to Miller and Ross (1993) and Gorenflo and Mainardi (2008) for recent developments on fractional calculus. Several known forms of fractional integrals have been proposed in the literature. Among these, the Riemann-Liouville fractional integral of order \( \gamma > 0 \) has been discussed extensively; see Dahmani et al. (2010), Romero et al. (2013) and Tunc (2013). Let \( \gamma > 0 \) and \( f \in L^1(a, b) \), \( a \geq 0 \). Then, the left-sided Riemann-Liouville fractional integral in the interval \([a, b]\) is defined as

\[
J^\gamma_a f(t) = \frac{1}{\Gamma(\gamma)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\gamma}} d\tau, \quad t \in [a, b],
\]

(2.27)

where \( f \) is a real-valued continuous function. It is worth mentioning that the notion of left-sided Riemann-Liouville fractional integral in (2.27) can be elongated with respect to a strictly increasing function \( h(.) \). In addition to this strictly increasing property, we further assume that the first-order derivative \( h'(.) \) is continuous in the interval \((a, b)\). Then, for \( \gamma > 0 \), the left-sided Riemann-Liouville fractional integral of \( f \) with respect to \( h \) is given by

\[
J^{\gamma}_{a^+;h} f(t) = \frac{1}{\Gamma(\gamma)} \int_a^t \frac{h'(\tau) f(\tau)}{(h(t)-h(\tau))^{1-\gamma}} d\tau, \quad t \in [a, b].
\]

(2.28)

One may refer to Samko et al. (1993) (Section 18.2) for the representation in (2.28). Now, we will show that WFGCPE can be expressed in terms of the limits of the integral in (2.28) after suitable choices of the functions \( f(x) \) and \( h(x) \), that is, the fractional nature of the proposed measure gets justified. Let us take

\[
h(x) = \ln K(x) \quad \text{and} \quad f(x) = \frac{\psi(x)(K(x))^2}{k(x)}.
\]

Then,

\[
\lim_{a \to 0, \ t \to s} J^{\gamma+1}_{a^+;h} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^s \psi(x)K(x)[-\ln K(x)]^\gamma dx
\]

\[
= CPE_\gamma(X).
\]

(2.29)
2.4 Proportional reversed hazards model

Let $X$ be a non-negative absolutely continuous random variable with distribution function $K$ and density function $k$. Here, $X$ may be viewed as the lifetime of a unit. If $\lambda(t) = \frac{d}{dt} \ln K(t)$ denotes the reversed hazard rate of $X$, then $\lambda(t)dt$ represents the conditional probability the unit stopped working in an infinitesimal interval of width $dt$ preceding $t$, given that the unit failed before time $t$. In other words, $\lambda(t)dt$ is the probability of failing in the interval $(t - dt, t)$ given that the unit is found to have failed by time $t$. Let $X_1$ and $X_2$ be two random variables with PDFs $k_1$ and $k_2$, CDFs $K_1$ and $K_2$, and reversed hazard rate functions $\lambda_1$ and $\lambda_2$, respectively. It is well known that $X_1$ and $X_2$ have proportional reversed hazard rate model if

$$\lambda_2(x) = \eta \lambda_1(x) = \eta \frac{k_1(x)}{K_1(x)},$$

(2.30)

where $\eta > 0$ is known as the proportionality constant. Note that (2.30) is equivalent to (2.26). Thus, the PDF of $X_2$ is

$$k_2(x) = \eta (K_1(x))^{\eta-1} k_1(x), \quad x > 0, \quad \eta > 0.$$  

(2.31)

Next, we evaluate WFGCPE of $X_2$. Making use of (2.31), from (2.1) and (2.26), we obtain

$$CPE_{\gamma}^\psi(X_2) = \frac{1}{\Gamma(\gamma + 1)} \int_0^s \psi(x)(K_1(x))^{\eta}[−\ln(K_1(x))^\eta]^\gamma dx$$

$$= -\frac{1}{\Gamma(\gamma + 1)} \int_0^s x\psi(x)[−\ln(K_1(x))^\eta]^\gamma \eta(K_1(x))^{\eta-1} k_1(x)dx$$

$$− \gamma \int_0^s x\psi(x)[−\ln(K_1(x))^\eta]^\gamma \eta(K_1(x))^{\eta-1} k_1(x)dx$$

$$+ \int_0^s \frac{x\psi'(x)}{\eta \lambda_1(x)}[−\ln(K_1(x))^\eta]^\gamma \eta(K_1(x))^{\eta-1} k_1(x)dx$$

(2.32)

$$= -\frac{1}{\Gamma(\gamma + 1)} \int_0^s x\psi(x)[−\ln K_2(x)]^\gamma k_2(x)dx$$

$$− \gamma \int_0^s x\psi(x)[−\ln K_2(x)]^\gamma k_2(x)dx$$

$$+ \int_0^s \frac{x\psi'(x)}{\eta \lambda_1(x)}[−\ln K_2(x)]^\gamma k_2(x)dx.$$

Now, let us denote

$$\mathcal{E}_2^\eta(\gamma) = \frac{1}{\Gamma(\gamma)} E \left[ X_2 \psi(X_2)[−\ln K_2(X_2)]^\gamma \right]$$

(2.33)

and

$$\tilde{\mathcal{E}}_2^\eta(\gamma) = \frac{1}{\Gamma(\gamma)} E \left[ \frac{X_2 \psi'(X_2)}{\lambda_1(X_2)}[−\ln K_2(X_2)]^\gamma \right].$$

(2.34)

Then, using (2.33) and (2.34) in (2.32), the following proposition follows.

Proposition 2.8 Let (2.26) be true. Then, WFGCPE of $X_2$ can be expressed as

$$CPE_{\gamma}^\psi(X_2) = \mathcal{E}_2^\eta(\gamma) − \mathcal{E}_2^\eta(\gamma + 1) − \eta^{-1} \tilde{\mathcal{E}}_2^\eta(\gamma + 1), \quad \gamma > 0,$$

(2.35)

provided the associated expectations are finite.
We note that when $\psi(x) = 1$, (2.35) reduces to Eq. (19) of Di Crescenzo et al. (2021). An illustration of the result in Proposition 2.8 is provided in the following example when $\psi(x) = x$.

**Example 2.2** Let $K_1(x) = x$, $0 < x < 1$, be the baseline distribution function. Then, we will find WFGCPE of a random variable $X_2$ with distribution function $K_2(x) = [K_1(x)]^c = x^c$, $0 < x < 1$, $c > 0$. Under this set up, from (2.33), for $\psi(x) = x$, we get

$$\mathcal{E}_2^n(\gamma) = \frac{c^\gamma}{(2 + c)^\gamma} = \tilde{\mathcal{E}}_2^n(\gamma).$$

(2.36)

Now, using (2.36) in (2.35), we find

$$CPE_{\gamma}^{\psi(x)=x}(X_2) = \frac{c^\gamma}{(2 + c)^\gamma} - \frac{c^{\gamma+1}}{(2 + c)^{\gamma+1}} - \frac{c^\gamma}{(2 + c)^{\gamma+1}}$$

$$= \frac{c^\gamma}{(2 + c)^{\gamma+1}},$$

which coincides with the case of the Power distribution in Table 1 for $b = 1$.

The WFGCPE of $X_2$ can be represented in terms of WFGCPE with different weight functions as follows:

$$CPE_{\gamma}^{\psi}(X_2) = -CPE_{\gamma}^{\psi_1}(X_2) - \eta CPE_{\gamma}^{\psi_2}(X_2) + \eta \gamma^{-1} CPE_{\gamma+1}^{\psi_2}(X_2),$$

(2.37)

where $\psi$ is increasing, $\psi_1(x) = x^\eta(x)$ and $\psi_2(x) = x^\eta(x)\lambda_1(x)$. Next, we show that a recurrence relation can be constructed for WFGCPE of $X_2$. It is shown that WFGCPE of $X_2$ of order $(\gamma + 1)$ can be expressed in terms of that of order $\gamma$. From (2.35) we find

$$CPE_{\gamma+1}^{\psi}(X_2) = \mathcal{E}_2^n(\gamma + 1) - \mathcal{E}_2^n(\gamma + 2) - \eta^{-1}\tilde{\mathcal{E}}_2^n(\gamma + 2)$$

$$= \mathcal{E}_2^n(\gamma) - \mathcal{E}_2^n(\gamma + 2) - \eta^{-1}[\tilde{\mathcal{E}}_2^n(\gamma + 1) + \tilde{\mathcal{E}}_2^n(\gamma + 2)]$$

(2.38)

Further, when $\psi(x) = 1$, (2.38) reduces to Eq. (22) of Di Crescenzo et al. (2021). We note that the recurrence relation in (2.38) can be generalized for any integer $n \geq 1$, as presented in the following proposition.

**Proposition 2.9** Let $n$ be a positive integer. Then, under the model in (2.26), for $\eta > 0$ and $\gamma > 0$, we have

$$CPE_{\gamma+n}^{\psi}(X_2) = \mathcal{E}_2^n(\gamma + n) - \mathcal{E}_2^n(\gamma + n + 1) + (-1)^{n-1}[\mathcal{E}_2^n(\gamma) - \mathcal{E}_2^n(\gamma + 1)]$$

$$+ \eta^{-1}[(-1)^n\tilde{\mathcal{E}}_2^n(\gamma + 1) - \tilde{\mathcal{E}}_2^n(\gamma + n + 1)] + (-1)^n CPE_{\gamma}^{\psi}(X_2).$$

(2.39)

**Proof** The proof follows using arguments similar to those in the proof of Proposition 2.4 of Di Crescenzo et al. (2021). It is therefore omitted.

We note that for the weight function $\psi(x) = 1$, Proposition 2.9 coincides with Proposition 2.4 of Di Crescenzo et al. (2021). In this case, both the terms $\tilde{\mathcal{E}}_2^n(\gamma + 1)$ and $\tilde{\mathcal{E}}_2^n(\gamma + n + 1)$ become zero.
3 Empirical WFGCPE

Let \( T = (T_1, \ldots, T_n) \) be a random sample of size \( n \) drawn from a population with CDF \( K \). The order statistics of the sample \( T \) are the ordered sample values, denoted by \( T_{1:n} \leq \ldots \leq T_{n:n} \). Denote the indicator function of the set \( A \) by \( I_A \), where

\[
I_A = \begin{cases} 
1, & \text{if } A \text{ is true} \\
0, & \text{otherwise.}
\end{cases}
\]

The empirical CDF on the basis of the random sample \( T \) is given by

\[
\hat{K}_n(x) = \frac{1}{n} \sum_{i=1}^{n} I_{\{T_i \leq x\}} = \begin{cases} 
0, & \text{if } x < T_{1:n} \\
\frac{l}{n}, & \text{if } T_{l:n} \leq x < T_{l+1:n} \\
1, & \text{if } x \geq T_{n:n},
\end{cases}
\]

(3.1)

where \( l = 1, \ldots, n - 1 \). Using (3.1), for \( \gamma > 0 \) and \( \psi(x) \geq 0 \), the WFGCPE given in (2.1) can be expressed as

\[
CPE_{\psi}^{\gamma} (\hat{K}_n) = \frac{1}{\Gamma(\gamma + 1)} \int_0^s \psi(x) \hat{K}_n(x) [-\ln \hat{K}_n(x)]^\gamma dx = \frac{1}{\Gamma(\gamma + 1)} \sum_{l=1}^{n-1} \int_{T_{l:n}}^{T_{l+1:n}} \psi(x) \hat{K}_n(x) [-\ln \hat{K}_n(x)]^\gamma dx
\]

(3.2)

where \( Z_l = \Psi(T_{l+1:n}) - \Psi(T_{l:n}) \) and \( \Psi(x) = \int_0^x \psi(x) dx \). Note that when \( \psi(x) = 1 \), we get the empirical fractional generalized cumulative entropy (see Di Crescenzo et al. (2021)) from (3.2). For \( \psi(x) = x \) and \( \gamma = 1 \), (3.2) coincides with the empirical weighted cumulative entropy proposed by Misagh et al. (2011). Further, let \( \gamma \) be a natural number. Then, for \( \psi(x) = x \), (3.2) reduces to the empirical shift-dependent generalized cumulative entropy due to Kayal and Moharana (2019). We thus observe that the proposed empirical estimate in (3.2) is a generalization of several empirical estimates discussed so far in the literature. In the following theorem, we show that the empirical WFGCPE converges to WFGCPE almost surely.

**Theorem 3.1** Let \( X \) be a non-negative absolutely continuous random variable with CDF \( K \). Then, for \( X \in L^p, \ p > 2 \), we have

\[
CPE_{\psi}^{\gamma} (\hat{K}_n) \to CPE_{\psi}^{\gamma} (X)
\]

almost surely.

**Proof** We have

\[
\frac{\Gamma(\gamma + 1)}{(-1)^\gamma} CPE_{\psi}^{\gamma} (\hat{K}_n) = \int_0^1 \psi(x) \hat{K}_n(x) [\ln \hat{K}_n(x)]^\gamma dx + \int_1^s \psi(x) \hat{K}_n(x) [\ln \hat{K}_n(x)]^\gamma dx
\]

(3.3)

\[
= I_1 + I_2, \ \text{say}.
\]

Now, using dominated convergence theorem and Glivenko-Cantelli theorem, the rest of the proof follows along the lines of Theorem 14 of Tahmasebi et al. (2020).
Let us now consider a data set, analyzed earlier by Abouammoh et al. (1994). It represents the ordered lifetimes (in days) of 43 blood cancer patients, observed from one of the Ministry of Health Hospitals in Saudi Arabia.

115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1165, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 15999, 1603, 1605, 1696, 1735, 1799, 1815, 1852, 1899, 1925, 1965.

Based on this data set, let us now compute the values of WFGCPE with weight functions \( \psi(x) = \sqrt{x} \), \( \psi(x) = x \) and \( \psi(x) = x^2 \) for various values of \( \gamma \), and the obtained values are presented in Table 3. Indeed, we can compute the values of WFGCPE with any positive valued weight functions. From Table 3, we observe that as the value of \( \gamma \) increases, the values of WFGCPE decrease for all the above choices of \( \psi(x) \). The graphs of the empirical WFGCPE based on the ordered lifetimes of 43 blood cancer patients are plotted in Fig. 3 for different choices of \( \gamma \) and \( \psi(x) \). We observe that the values of the empirical WFGCPE increase with respect to \( n \). Further, the values of empirical WFGCPE decrease with respect to \( \gamma \).

Next, we present some examples to illustrate the proposed empirical measure.

**Example 3.1** Let \( T = (T_1, \ldots, T_n) \) be a random sample drawn from a population with CDF \( K(x) = x^2, 0 < x < 1 \). Consider \( \psi(x) = x \). It can be shown that \( T_{i,l}^2, l = 1, \ldots, n - 1, \) follow uniform distribution in the interval (0, 1). Further, the sample spacings \( T_{i+1,n}^2 - T_{i,n}^2, l = 1, \ldots, n - 1, \) are independently distributed as beta with parameters 1 and \( n \); for details,

| \( \gamma \) | \( \psi(x) = \sqrt{x} \) | \( \psi(x) = x \) | \( \psi(x) = x^2 \) |
|---|---|---|---|
| 0.25 | 24004.3 | 881460 | 1.27542 \times 10^9 |
| 0.5 | 20065.8 | 707724 | 9.59358 \times 10^8 |
| 0.75 | 16858.4 | 570814 | 7.23578 \times 10^8 |
| 1.5 | 10279.3 | 309581 | 3.22149 \times 10^8 |
| 2.75 | 4489.63 | 114320 | 8.89639 \times 10^7 |
Consider a random sample

Example 3.2

Further, for fixed $\gamma$, that for fixed sample sizes, as the sample size increases, the mean and variance of the proposed estimator decrease. Further, for fixed $\gamma$, the mean and variance increase and decrease, respectively, as the sample size increases.

Example 3.3

Let $T_i (i = 1, \ldots, n)$ be a random sample from a population with absolutely continuous CDF $K$ and PDF $k$. Let $\psi(x) = k(x)$. Then, $Z_l = K(T_{l+1:n}) - K(T_{l:n})$, $l = 1, \ldots, n - 1$, are independent random variables distributed as beta with parameters 1 and $n$. We refer to Pyke (1965) for details. Thus, as in (3.4) and (3.5), we have

\[
E[CPE_\psi(\hat{K}_n)] = \frac{1}{\Gamma(\gamma + 1)} \sum_{l=1}^{n-1} \frac{1}{2(1+n)} \left( \frac{1}{n} \right) \left( -\ln \frac{l}{n} \right)^\gamma \quad (3.6)
\]

and

\[
Var[CPE_\psi(\hat{K}_n)] = \frac{1}{\Gamma(\gamma + 1)^2} \sum_{l=1}^{n-1} \frac{n}{4(1+n)^2(2+n)} \left( \frac{1}{n} \right) \left( -\ln \frac{l}{n} \right)^{2\gamma}. \quad (3.7)
\]

We present the computed values of the means and variances of the empirical estimator of WFGCPE under the same set up as in Example 3.1 in Table 4. From Table 4, we observe that for fixed sample sizes, as $\gamma$ increases, the mean and variance of the proposed estimator decrease. Further, for fixed $\gamma$, the mean and variance increase and decrease, respectively, as the sample size increases.

**Table 4** Numerical values of $E[CPE_\psi(\hat{K}_n)]$ and $Var[CPE_\psi(\hat{K}_n)]$ for the distribution considered in Example 3.1

| $\gamma$ | $n$ | $E[CPE_\psi(\hat{K}_n)]$ | $Var[CPE_\psi(\hat{K}_n)]$ | $\gamma$ | $n$ | $E[CPE_\psi(\hat{K}_n)]$ | $Var[CPE_\psi(\hat{K}_n)]$ |
|----------|-----|-------------------------|-----------------------------|----------|-----|-------------------------|-----------------------------|
| 0.25     | 5   | 0.153878                | 0.004609                     | 0.5      | 5   | 0.135721                | 0.003395                     |
| 10       | 0.181591                | 0.003434                     | 10       | 0.156472                | 0.002416                     |
| 15       | 0.191238                | 0.002627                     | 15       | 0.163420                | 0.001822                     |
| 30       | 0.200941                | 0.001507                     | 30       | 0.170268                | 0.001034                     |
| 50       | 0.204774                | 0.000956                     | 50       | 0.172941                | 0.000653                     |
| 0.75     | 5   | 0.116302                | 0.002472                     | 1.5      | 5   | 0.066611                | 0.000968                     |
| 10       | 0.132732                | 0.001734                     | 10       | 0.077849                | 0.000712                     |
| 15       | 0.138160                | 0.001304                     | 15       | 0.081549                | 0.000538                     |
| 30       | 0.143500                | 0.000738                     | 30       | 0.085119                | 0.000306                     |
| 50       | 0.145593                | 0.000466                     | 50       | 0.086481                | 0.000194                     |

One may see Pyke (1965). Thus, from (3.2), for $\gamma > 0$, we get

\[
E[CPE_\psi(\hat{K}_n)] = \frac{1}{\Gamma(\gamma + 1)} \sum_{l=1}^{n-1} \frac{1}{2(1+n)} \left( \frac{1}{n} \right) \left( -\ln \frac{l}{n} \right)^\gamma \quad (3.4)
\]

and

\[
Var[CPE_\psi(\hat{K}_n)] = \frac{1}{\Gamma(\gamma + 1)^2} \sum_{l=1}^{n-1} \frac{n}{4(1+n)^2(2+n)} \left( \frac{1}{n} \right) \left( -\ln \frac{l}{n} \right)^{2\gamma}. \quad (3.5)
\]
\[
E[CPE_{\gamma}^{\psi}(\hat{K}_n)] = \frac{1}{\Gamma(\gamma + 1)} \sum_{l=1}^{n-1} \frac{1}{(1 + n)} \left( \frac{l}{n} \right) \left[ -\ln \frac{l}{n} \right]^{\gamma} \] (3.8)

and
\[
Var[CPE_{\gamma}^{\psi}(\hat{K}_n)] = \frac{1}{(\Gamma(\gamma + 1))^2} \sum_{l=1}^{n-1} \frac{n}{(1 + n)(2 + n)} \left( \frac{l}{n} \right)^2 \left[ -\ln \frac{l}{n} \right]^{2\gamma}. \] (3.9)

We now present central limit theorems for the empirical WFGCPE when the random samples are drawn from (i) a Weibull distribution with \( \psi(x) = x \) and (ii) a general CDF \( K(x) \) with \( \psi(x) = k(x) = \frac{d}{dx} K(x) \).

**Theorem 3.2** Consider a random sample \( T_i \) \( (i = 1, \ldots, n) \) from a population with PDF \( k(x) = 2\lambda x e^{-\lambda x^2}, \ x > 0, \ \lambda > 0 \). Then, for \( \gamma > 0 \) and \( \psi(x) = x \), we have
\[
\frac{CPE_{\gamma}^{\psi}(\hat{K}_n) - E(CPE_{\gamma}^{\psi}(\hat{K}_n))}{\sqrt{Var(CPE_{\gamma}^{\psi}(\hat{K}_n))}} \rightarrow N(0, 1)
\]
in distribution, as \( n \rightarrow \infty \).

**Proof** The proof is similar to that of Theorem 5.1 of Kayal and Moharana (2019). Hence, it is omitted.

**Theorem 3.3** Consider a random sample \( T_i \) \( (i = 1, \ldots, n) \) from a population with CDF \( K(x) \). Then, for \( \gamma > 0 \) and \( \psi(x) = k(x) \), we have
\[
\frac{CPE_{\gamma}^{\psi}(\hat{K}_n) - E(CPE_{\gamma}^{\psi}(\hat{K}_n))}{\sqrt{Var(CPE_{\gamma}^{\psi}(\hat{K}_n))}} \rightarrow N(0, 1)
\]
in distribution, as \( n \rightarrow \infty \).

**Proof** The proof is similar to that of Theorem 15 of Tahmasebi et al. (2020). Hence, it is omitted.

# 4 Validation of WFGCPE using logistic map and some applications

In this section, we validate the proposed weighted fractional generalized cumulative past entropy using the logistic map, which is one of the simplest examples of chaos. The logistic map consists of an iterative polynomial form
\[
x_{n+1} = cx_n(1 - x_n), \] (4.1)
where \( c \) is the bifurcation parameter. Usually, the values of \( x_n \) lie in the interval \([0, 1]\) and \( c \in [0, 4] \). Depending on the values of \( c \), the simulated data using logistic map provide different characteristics, such as periodic and chaotic. Here, the choices of \( c = 3.5, \ 3.6, \ 3.7, \ 4 \), with initial value \( x_0 = 0.1 \), have been considered for the generation of data. The bifurcation diagram is shown in Fig. 4a. For the validation of WFGCPE, based on the generated data, we plot the graphs of empirical WFGCPE for \( c = 3.5, \ 3.6, \ 3.7 \) and 4 in Fig. 4b, which reveal that the newly proposed measure responds well when the series is periodic and chaotic.
expected, the graph of the empirical WFGCPE when $c = 4$ lies above all other graphs, as it provides highest uncertainty.

In order to show an application of the proposed measure, here, we consider an example, in which we compare the standard deviation (SD), Gini’s mean difference and WFGCPE. Let $X$ and $Y$ be non-negative independent random variables with identical distributions with distribution function $F(.)$. Then, the Gini’s mean difference, denoted by $\Delta_1$, is defined as

$$\Delta_1 = E(|X - Y|) = 2 \int_0^\infty (2F(x) - 1)x f(x) dx.$$  \hspace{1cm} (4.2)

**Example 4.1** Let $X$ be a random variable having exponential distribution with distribution function $F(x) = 1 - e^{-\lambda x}$, $x > 0$, $\lambda > 0$. For $\psi(x) = x$, we then obtain

$$CPE_\gamma^\psi(X) = -\frac{1}{\lambda^2 \Gamma(\gamma + 1)} \int_0^1 \frac{u}{1 - u} \ln(1 - u)(-\ln u)^\gamma du,$$  \hspace{1cm} (4.3)

and $SD(X) = \frac{1}{\lambda} = \Delta_1$. The graphs of $SD(X)$ and $CPE_\gamma^\psi(X)$ for $\psi(x) = x$ and $x^2$ are plotted in Fig. 5, when $\gamma = 0.5, 1, 1.5$. It shows that the newly proposed measure indeed behaves like a variability measure.

We now consider a financial stock data and explore the behaviour of the empirical WFGCPE. Specifically, we take the closing price data of each day of National Stock Exchange

![Fig. 4](image1)  \hspace{1cm} ![Fig. 4](image2)

*Fig. 4* a Bifurcation diagram of the logistic map. b Plots of the empirical WFGCPE for $c = 3.5, 3.6, 3.7, 4$ (from bottom to top)

![Fig. 5](image3)  \hspace{1cm} ![Fig. 5](image4)

*Fig. 5* The graphs of $SD(X)$ and $CPE_\gamma^\psi(X)$ for exponential distribution when $\gamma = 0.5, 1, 1.5$ for a $\psi(x) = x$ and b $\psi(x) = x^2$

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Fig. 6 The graphs of the empirical WFGCPE when (a) $\psi(x) = x^2$, (b) $\psi(x) = x$ and (c) $\psi(x) = 1$

of India (NIFTY50) from 26th February, 2022 to 28th August, 2022 during phase-I and phase-II of the war between Russia and Ukraine. The data set is available in the link “https://finance.yahoo.com/?guccounter=1”. The graphs of the empirical WFGCPE are plotted in Fig. 6 for three choices of $\psi$, namely $\psi(x) = x^2$, $\psi(x) = x$ and $\psi(x) = 1$. The graphs have been provided in three different figures for better visualization. From them, it can be seen that all the graphs decrease with respect to $\gamma > 0$.

Next, we consider the daily closing price data of four different countries presented in Table 5 for the period 26th February, 2022 to 28th August, 2022, which are available in “https://finance.yahoo.com/?guccounter=1”. The graphs of the empirical WFGCPE for $\psi(x) = x^2$ have been depicted in Fig. 7, which shows that during phase-I and phase-II of the war between Russia and Ukraine, the stock price of JAPAN was more unstable if we compare it with those of USA, INDIA and FRANCE.

5 Weighted fractional generalized cumulative paired entropy

In this section, we study weighted fractional generalized cumulative (WFGC) paired entropy of a non-negative random variable $X$.

**Definition 5.1** Let $X$ be a nonnegative random variable with support $[0, s]$, CDF $K$, and survival function $\bar{K} = 1 - K$. Then, for $\gamma > 0$, the WFGC paired entropy is defined as

$$PE_\gamma^{\psi}(X) = \frac{1}{\Gamma(\gamma + 1)} \int_0^s \psi(x) K(x)[- \ln K(x)]^\gamma dx + \frac{1}{\Gamma(\gamma + 1)} \int_0^s \psi(x) \bar{K}(x)[- \ln \bar{K}(x)]^\gamma dx$$

(5.1)

| Table 5 | Stock indices of different countries |
|---------|--------------------------------------|
| Country | Stock index                          | Abbreviations |
| The USA | Nasdaq Stock Market                  | NASDAQ        |
| INDIA   | National Stock Exchange              | NIFTY50       |
| FRANCE  | CAC 40 Index                         | CAC           |
| JAPAN   | Tokyo Stock Price Index              | NIKKEI225     |


Fig. 7 The graphs of the empirical WFGCPE with $\psi(x) = x^2$ for four different countries

where $CPE^\psi_{\gamma}(X)$ is as in (2.1) and $CRE^\psi_{\gamma}(X) = \frac{1}{\Gamma(\gamma+1)} \int_0^x \psi(x) \bar{K}(x)[-\ln \bar{K}(x)]^\gamma dx$ is known as the weighted fractional generalized cumulative residual entropy.

We note that the WFGC paired entropy is useful for the description of information in dynamic reliability systems, wherein the uncertainty is related to past as well as future. We now present some properties of the WFGC paired entropy.

**Proposition 5.1** Let $X$ be a non-negative random variable and $Y = aX + b$, where $a > 0$ and $b \geq 0$ be an affine transformation. Then, for $\psi(x) = x$, we have

$$PE^\psi_{\gamma}(Y) = a^2[E^x_{\gamma}(X) + \bar{E}^x_{\gamma}(X)] + ab[CPE^\gamma_{\gamma}(X) + CRE^\gamma_{\gamma}(X)],$$

(5.2)

where $E^x_{\gamma}(X)$ and $\bar{E}^x_{\gamma}(X)$ are the weighted fractional generalized cumulative past and residual entropies, respectively, with weight function $\psi(x) = x$. Further, $CPE^\gamma_{\gamma}(X)$ and $CRE^\gamma_{\gamma}(X)$ are the fractional generalized cumulative past and residual entropies, respectively.

**Proof** The proof of the proposition follows upon using the facts that $F_Y(x) = F_X(\frac{x-b}{a})$ and $\bar{F}_Y(x) = \bar{F}_X(\frac{x-b}{a})$.

### 6 Concluding remarks and some discussions

In this paper, we have proposed a weighted fractional generalized cumulative past entropy of a non-negative random variable. A number of results for the proposed weighted fractional measure have been obtained when the weight is a general non-negative function. It
is observed that WFGCPE is shift-dependent and can be expressed as the expectation of a decreasing function of a random variable. Some ordering results and bounds are established. Based on these properties, it can be seen that the proposed measure is a variability measure. Further, a connection between the proposed weighted fractional measure and the fractional calculus is provided. The weighted fractional generalized cumulative past entropy measure is studied for the proportional reversed hazards model. A nonparametric estimator of the weighted fractional generalized cumulative past entropy is introduced based on the empirical cumulative distribution function. Some examples are considered for the computation of the mean and variance of the estimator. A validation of the proposed fractional measure has been explained based on the data generated from a logistic map. The empirical estimator of WFGCPE has been compared with the standard deviation and Gini’s mean difference for exponential distribution. The stability of the stock prices for four different countries during phase-I and phase-II war between Russia and Ukraine has been examined. The concept of WFGC paired entropy and its properties have also been studied. Finally, a large-sample property of the estimator has been established.

The proposed measure is not appropriate when uncertainty is associated with past. Suppose a system has started working at time \( t = 0 \). At a pre-specified inspection time, say \( t \in (0, s) \), the system is found to be down. Then, the random variable \( X(t) = [X|X \leq t] \), where \( t \in (0, s) \) is known as the past lifetime. The dynamic weighted fractional generalized cumulative past entropy of \( X(t) \) is defined in this case as

\[
CPE^\psi_{\gamma}(X; t) = \frac{1}{\Gamma(\gamma + 1)} \int_0^t \psi(x) \frac{K(x)}{K(t)} \left( -\ln \frac{K(x)}{K(t)} \right) \gamma \, dx, \quad \gamma > 0, \quad \psi(x) \geq 0. \quad (6.1)
\]

We can then establish similar properties for \( CPE^\psi_{\gamma}(X; t) \) as for the measure in (2.1).

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**Declarations**

**Conflict of interest** All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

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