Scaling Limits for the 2D Metal-Insulator Transition at $B = 0$ in Si-MOSFETs

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We have reexamined data on the possible two dimensional metal-insulator transition at $B = 0$ in Si-MOSFETs using a nonlinear regression method to extract all scaling parameters in a single fit. By keeping track of the magnitude of errors in the data we can use the normalized mean square deviation $\chi^2$ of the fit as a quantitative measure of how well the data is compatible with scaling. We have used this method to study electric field scaling in three different samples. We find rather good agreement of the data with scaling in individual fits, i.e. $\chi^2$s of about 1, but also rather large variations in the fits depending on how cut-offs are introduced in the data. In particular, we report how fitted parameters vary when we cut away data that are either far from the critical point or at low excitation power, where temperature effects presumably dominate. In this way we find the critical $E$-field exponent $\beta$ to vary from about 3.2 to 3.9 with considerably smaller statistical error estimates in each fit.

I. INTRODUCTION

Several years ago the observation of a conductivity phase transition in the two-dimensional electron system (2DES) in Si metal-oxide-semiconductor field-effect-transistors (MOSFETs) was reported. This observation was unexpected and controversial because it was contrary to the widely accepted scaling theory for a noninteracting 2DES. Further experimental studies on MOSFETs, SiGe heterostructures, and GaAs heterostructures indicate that a conductivity transition is possible as a function of carrier density in a 2DES at $B = 0$. The observed conductivity transitions at $B = 0$ bear a marked similarity to transitions between states with different Hall resistances in the quantum Hall effect (QHE). This unexpected behavior and similarity to QHE transitions has also sparked considerable theoretical interest. Furthermore, these parallels make it advisable to consider developments in the QHE when studying the $B = 0$ conductivity transition in 2DES. In particular, a recent publication by Shahar et al. reports that the behavior of the resistivity as a function of temperature and electron density, $\rho(T, n_e)$, is inconsistent with scaling at temperatures above 0.5 K. Thus, they question the use of scaling analysis for the study of the QHE in general. We feel that a serious consideration of the appropriateness of the scaling analysis for the $B = 0$ conductivity transition in 2DES is warranted. We have data available that can address this question using a set of data manipulation routines in Mathematica™ especially developed for this purpose. Below we briefly describe the data used in this study and the data manipulation routines that allow us to consistently propagate errors and eventually place confidence limits on the applicability of this scaling analysis to the case of the $B = 0$ conductivity transition in 2DES. We feel that these methods have applicability beyond the particular system to the QHE and other quantum phase transitions.

Our aim has been to develop an objective method that gives a quantitative answer to the question of how well the data actually scales. We achieve this by analyzing our ability to fit the data to a general scaling model that assumes as little as possible about the functional form of the scaling function. We use the variance-weighted mean-square deviation from this fit, $\chi^2$, as a quantitative measure of the 'goodness' of the fit. Such a properly weighted and normalized $\chi^2$ close to one indicates good agreement of the data with the model. An objective estimate of $\chi^2$ requires a quantitative knowledge of experimental uncertainties. Therefore, great care has been taken to estimate the magnitude of these sources of uncertainty and to propagate them consistently to the final fit.

Using this method to investigate $E$-field scaling of the lowest temperature data available, we find rather good agreement with scaling in the best sample with a $\chi^2$ varying from 0.4 to 1.9 depending on our selection of the data. We find that $\chi^2$ gets smaller as more data at low power is discarded. Excluding more data at lower powers also tends to increase the consistency among the different samples.

II. DESCRIPTION OF THE DATA INCLUDING SOURCES OF ERROR

The data used in this study was amassed at the University of Oklahoma while studying the temperature $T$, electron density $n_e$, and excitation dependence of the resistivity $\rho$ of a number of high-mobility ($\mu_{\text{max}} \geq 25'000 \, \text{cm}^2/\text{Vs}$) metal-oxide-semiconductor field-effect transistors (MOSFETs). These devices are described in
This data includes large sequences of currents $I$ with their corresponding probe voltages $V$ recorded as a function of gate voltage $V_g \propto n_s$ and $T$ using a four-terminal technique. The resulting $I$–$V$ curves at low temperatures are highly nonlinear with distinctly different behaviors above and below a critical density $n_c$ or resistivity $\rho_c$. In this article we consider the nonlinear $I$–$V$ curves taken at the lowest temperature ($\sim 0.2$ K) and present a scaling analysis of the resistivity as a function of the applied electric field, $E$. This $E$ is taken to be the measured $V$ divided by the distance between the respective probes.

We find three sources of randomness that we assume are independent:

- The uncertainty in the measured probe voltage characterized by a root mean square (RMS) uncertainty $\sigma_V$.
- The uncertainty in the measured excitation current, $\sigma_I$, which we find is negligible.
- The uncertainty in $n_s$ due to the applied $V_g$ similarly characterized by $\sigma_n$.

For these samples at this $T$ we also have a source of systematic error that we associate with the well-known difficulty in producing ohmic contacts to a 2DES. The non-ohmic contacts caused the measured $I$–$V$ curves to be asymmetric with respect to $I = 0$ and to show a sharp anomaly centered at $I = 0$. By carefully accounting for all the voltage drops in the measurement circuit, we were able to determine the contribution due to the contacts. We found that a percentage of this contact potential (proportional to the resistivity of the sample) added to the measured $V$ essentially eliminated the observable effects of the contacts, see Fig. 1. Furthermore, we found that these contact potentials did not effect our analysis because they occurred at very low excitation currents where the physics of the system is still dominated by the finite $T$. The results presented here are independent of these contact anomalies and our treatment of them.

In order to measure the effective $\sigma_V$ for our data, we produced a modified data set that was equally spaced in excitation $I$ by using a spline interpolation scheme. The Fourier transform of the data was then examined. The transformed data is consistent with a low frequency signal superimposed on uniform white noise of constant amplitude. This white noise amplitude varies from about $2 \mu V$ in the most metallic curves up to $90 \mu V$ in the most insulating ones. We used this amplitude as a measure of $\sigma_V$ for each particular $V_g$ in the following analysis.

The uncertainty in the electron density, $\sigma_n$, is more difficult to estimate. To calibrate $n_s$ to $V_g$, we swept the gate voltage at a constant magnetic field to obtain $\rho_{xx}(V_g)$. Minima in $\rho_{xx}$ that correspond to known integral filling factors, and therefore known $n_s$, were then determined. The linear fit to the resulting $(n_s, V_g)$ pairs that gives $V_g(n_s)$ with a statistical error of $\sigma_n = 1 \times 10^3 \text{ cm}^{-2}$. This corresponds to a resolution of 0.8 millivolt in $V_g$. This determination is not obviously a measure of the density fluctuations for constant $V_g$. Furthermore, an effective gradient in the gate voltage will be imposed on the 2DES by the source-drain excitation. We made every attempt to make this potential drop symmetric with respect to the $V$ probes, but the $n_s$ gradient could not be fully eliminated. Thus, we estimate the density uncertainty, $\sigma_n$, by taking $\sigma_{\Sigma} \sim 1 \text{ mV}$. This estimate is compatible with our observations of the precision and reproducibility of producing $n_s$ with $V_g$ in our samples as evidenced by a repeatability of a given $\rho_{xx}(V_g)$ where $\rho_{xx}(n_s)$ is particularly steep.

Qualitatively, as has been reported previously, two different nonlinear $V(I)$ behaviors can be seen. At low densities the resistivity has a maximum at zero electric field and decreases for stronger fields. This is similar to a normal insulator where a larger electric field can excite more carriers and hence increase the conductivity. At high electron densities the opposite behavior is seen; the resistivity has a minimum at $E = 0$ and increases for stronger fields. Between these behaviors, a linear $V(I)$ is observed with a constant $\rho$, see Fig. 2. This is similar to the nonlinear region of a superconductor. However,
A quantum phase transition occurs at zero temperature whereas all conductivity measurements are taken at both finite bath temperature and nonzero excitation voltage to drive a current through the sample. Theoretical arguments suggest that an underlying quantum phase transition should give rise to scaling at finite temperatures and excitation $E$-fields, i.e. that the resistivity of the sample should have a specific temperature and $E$-field dependence:

$$\rho(n_s, T, E) = \rho(x, y) = f \left( \frac{\delta}{E^{1/\beta}} \right),$$

where $\delta = \frac{n_s - n_c}{n_c}$ is the relative deviation of the electron density from a critical density $n_c$, and $\nu, z$ and $\beta$ are critical exponents associated with the phase transition. Assuming the simplest $E$-field heating mechanism, $\beta$ would be related to the temperature exponents as $\beta = \nu(z+1)$. Here $x = \delta/E^{1/\beta}$ and $y = \delta/T^{1/\nu z}$ are called the scaled variables.

In cases where either the $E$-field or the temperature has been judged negligible compared to the other, data has been examined for one-parameter scaling, i.e. whether or not it collapses to a function of only one scaled variable. At the core of all arguments for a phase transition lies the fact that one in many cases can find a critical density $n_c$ and exponent, $\nu z \beta$, so that data falls onto a single curve. This is usually done by trying out different critical densities and exponents. The best fit is then determined by eye as the parameter choice that gives the best data collapse. Because the question of a phase transition in our system still is controversial, we developed an objective method to determine the compatibility of the data with scaling.

Our goal is now to give a quantitative measure of the consistency of the data with a single curve when we plot the resistivities taken at the lowest temperature as a function of the scaled $E$-field, $x = \frac{(n_s - n_c)}{n_c} E^{-1/\beta}$.

We want to assume as little as possible about the form of the scaling function, $f(x)$, but previous investigations suggest that $\log[f(x)]$ is a reasonably smooth curve without too much curvature. We therefore investigate a model where $\log[f(x)]$ is a polynomial with only a few terms, $f(x) = \exp[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots]$. Because theory does not tell us the form of $f(x)$, any functional form for $f$ that is general enough is acceptable, provided it contains a small number of free parameters compared to the number of data points being fitted. If we truncate this polynomial at say third order, we obtain a scaling model for $\rho(n_s, E)$ that contains the six parameters $n_c, \beta, a_0, a_1, a_2$ and $a_3$:

$$\rho_{\text{model}}(n_s, E) = e^{a_0 + a_1 x + a_2 x^2 + a_3 x^3}$$

where

$$x = \frac{(n_s - n_c)}{n_c} E^{-1/\beta}. \quad (3)$$

We can now address the question of the consistency of the data with scaling by investigating our ability to fit the
measured data to this scaling model. We do this with a nonlinear regression routine in Mathematica\textsuperscript{TM} that fits all parameters at once and hence obtain values for $n_c$ and $\beta$ with statistical error estimates. At the same time we get an objective measurement of the ‘goodness of fit’ via the $\chi^2$ of the fit.

In order to have an unbiased fit and get $\chi^2$ in the correct units we need to keep track of the estimated uncertainty for each experimental point. In this case we need to consider not only the error in $\rho$ itself but also the uncertainty in $n_s$ and in $E$. We find that the uncertainty in density, $\sigma_n$, normally dominates the uncertainty of points in the $f(x)$-plot. To take this into account we propagate all errors to a single effective error in the resistivity, $\sigma_{\rho,\text{eff}}$:

$$\sigma_{\rho,\text{eff}}^2 = [\sigma_\rho]^2 + \left[\sigma_n \frac{\partial \rho_{\text{model}}}{\partial n_s}\right]^2 + \left[\sigma_V \frac{\partial \rho_{\text{model}}}{\partial V}\right]^2.$$  \hspace{1cm} (4)

We estimate the error in the resistivity $\sigma_\rho$ by propagating the errors from the probe voltage $\sigma_V$ either via the convolution \textsuperscript{11} or the ratio $V/I$. When calculating the total effective error $\sigma_{\rho,\text{eff}}$, we still assume $\sigma_\rho$ and $\sigma_V$ to be independent. We then weigh the points in the fit with a weight $1/\sigma_{\rho,\text{eff}}^2$. Because the weights depend on the fitted polynomial, we have to start with an unweighted fit, then propagate the errors and iterate the fit until a self-consistent solution is found.

**IV. RESULTS**

Using either the derivative $dV/dI$ or the ratio $V/I$ as the resistivity $\rho$, we have used the scaling model described in the previous section to investigate $E$-field scaling in three different samples. Here we used a fifth order polynomial \textsuperscript{12}. We also achieved reasonable results with lower order polynomials. Because data is taken at a finite temperature, ($\sim 0.2K$) we know that the resistivity is dominated by the temperature for small enough excitation $E$-fields; hence, we introduce a low power cut-off to look for scaling in the $E$-field only.

Using the ratio $V/I$ for the resistivity $\rho$ we obtain a $\chi^2$ reasonably close to one. Figure \textsuperscript{3} shows the result from sample one, using a low power cut-off $P_c = 50$ pW. The critical exponent we find in this way is comparable to previous investigations, \textsuperscript{13} even though the exponent and $\chi^2$ do depend somewhat on the particular cut-offs used.

We have most data from sample one. It also shows the best behavior; $\chi^2$ is about one and the physically interesting fit parameters, in particular the exponent $\beta$, are relatively stable as the low-power cut-off is increased, see Table \textsuperscript{II}. The results in the other samples show stronger dependence on the cut-off. Only for quite large low power cut-offs ($\sim 500pW$) are the exponents from different samples compatible with each other, see Table \textsuperscript{II} for results from sample two. We note that at an excitation power of 1 nW temperature increases in the sample were observed via the temperature monitoring RuO$_x$ resistor. Thus, data at these high excitations may correspond to the hydrodynamic regime, discussed in this context earlier.

Sample one shows rather good consistency with the scaling model. Here all the fitted parameters are reasonably independent of the low power cut-off, $P_c$. The number of points in a fit varies from 864 for the smallest cut-off to 467 for the largest. Actual variations in the fitted parameters are consistent with the uncertainty estimates based on the statistical deviations from the model as the low power cut-off is changed. For example, the critical density $n_c$ is virtually unchanged for all cut-offs. This is also reflected in very small statistical uncertainties for $n_c$ in each fit. The exponent $\beta = \nu (z + 1)$ is consistent with $\beta = 3.8 \pm 0.05$, and the critical resistivity $\rho_{\text{crit}}$ is approximately $2.9h/e^2$. The mean square deviation per point, $\chi^2$, however, does vary substantially with the cut-off, decreasing as the cut-off is increased. This could be understood as an effect of the finite temperature the data is taken at; at low powers the temperature is not negligible and the model \textsuperscript{14} is too crude to give a good fit. It is also possible that there are two regimes.

### Table I. Results for sample one. Listed are the low-power cut-offs $P_c$ in pW, the critical density $n_c$ in units of $10^{11}$ cm$^{-2}$, the logarithm $a_0$ of the critical resistivity in log $h/e^2$ and the average $\chi^2$ for the fit. The errors are estimated from the statistical deviations of the data from the fitted model only.

| $P_c$ (pW) | $n_c$ | $\beta$ | $a_0$ | $\chi^2$ |
|-----------|-------|---------|-------|---------|
| 50        | 1.0142 $\pm$ 0.0006 | 3.83 $\pm$ 0.03 | 1.074 $\pm$ 0.009 | 1.86 |
| 100       | 1.0138 $\pm$ 0.0006 | 3.76 $\pm$ 0.03 | 1.083 $\pm$ 0.009 | 1.18 |
| 200       | 1.0134 $\pm$ 0.0006 | 3.75 $\pm$ 0.04 | 1.094 $\pm$ 0.009 | 0.70 |
| 400       | 1.0136 $\pm$ 0.0009 | 3.81 $\pm$ 0.05 | 1.093 $\pm$ 0.013 | 0.45 |
| 500       | 1.0143 $\pm$ 0.0010 | 3.89 $\pm$ 0.06 | 1.084 $\pm$ 0.014 | 0.38 |

### Figure 3. Scaled data points, $\rho(x) = V/I$, from sample one and fitted fifth order model polynomial. The fitted polynomial is only visible where it is not covered by data points. Here a low power cut-off of 50pW was used and the fit has $\chi^2 = 1.86$. 

![Figure 3: Scaled data points, $\rho(x) = V/I$, from sample one and fitted fifth order model polynomial. The fitted polynomial is only visible where it is not covered by data points. Here a low power cut-off of 50pW was used and the fit has $\chi^2 = 1.86$.](image-url)
of nonlinearity as mentioned above. The fact that \( \chi^2 \) is less than one for the best fits may indicate that we have overestimated the statistical uncertainties in the data.

Sample two shows a larger dependence on the low power cut-off, and only at large cut-offs (\( \sim 400\) pW) is the exponent \( \beta \) compatible with the results in sample one. Clearly the exponent varies much more than the statistical error estimates in each of the fits with lower cut-offs. Again \( \chi^2 \) varies in the same way as in sample one. At the largest cut-offs, however, both \( \beta \) and \( \chi^2 \) are close to the values found in sample one. The critical resistivity is, however, clearly not the same as in sample one, \( e^{a_0} \approx 7.5 h/e^2 \).

Sample three shows a behavior similar to sample one (exponent \( \beta \approx 3.7 \) and \( e^{a_0} \approx 3.9 \)) but with much poorer statistics.

In order to gain a further understanding of the data, and our fitting procedure, we have also studied the effect of introducing a cut-off in the scaled variable \( x \). By keeping data where \( |x| \leq x_{\text{cut}} \) we effectively cut away points that are taken far from the critical point. Note that the criteria for a specific density to be “near” or “far” from the critical density is determined also by the excitation field; a small field corresponds to a more narrow range of densities. Since \( x \) depends also on the critical density and exponent given by the fit, this cut has to be made self-consistently. Ideally one would expect fitted parameters to be independent of the actual value of \( x_{\text{cut}} \), provided \( x_{\text{cut}} \) is not too large. Figure 4 shows how the fitted exponent \( \beta \) varies with \( x_{\text{cut}} \) in sample one. Each point in the figure corresponds to a complete fit using both the \( |x| \leq x_{\text{cut}} \) cut-off and a low power cut-off \( (P_c) \). The statistical uncertainty in each fit, shown as errorbars on the points, are clearly smaller than the actual variation in \( \beta \). This behavior may be an indication that the idea of multiple regimes is appropriate, but further investigations are clearly necessary to settle the question. The large uncertainties in the points with very small \( x_{\text{cut}} \) reflect the fact that there are no longer enough data left for a good fit.

We have also investigated the derivative \( dV/dI \) for scaling. In this case we find similar but more varied results with generally higher \( \chi^2 \). The \( \chi^2 \)'s determined for these fits also depend on the specific Super-Lancos filter used \( (\text{fit}) \). The derivative scaling also requires the introduction of cut-offs for high power data. This was done by discarding data outside three times the FWHM (Full Width Half Maximum) of the central structure. Results depend strongly on this FWHM-cut. The results, presented above, based on ratios show very little dependence on such high power cut-offs and we have used all data consistent with other cut-offs.

### V. CONCLUSIONS

We have developed and used a general scaling model to fit all parameters in \( E \)-field scaling at once. We find some results that are consistent with scaling, \( \chi^2 \) of about one, but also surprisingly large variations in the fitted parameters when different physically appropriate cut-offs are introduced.

It would be desirable to generalize and use this method for a full two-parameter scaling test, treating the \( E \)-field and temperature at equal footing. Any one-parameter scaling suffers from the fact that the other (unscaled) parameter is not zero. As we have discussed above, any \( E \)-field scaling will always suffer from the finite temperature of the surrounding bath. In the same way, temperature scaling is sensitive to the nonzero excitation field used to measure resistivities, and heating of the 2DES will always be a problem for low enough temperatures. Both these problems are reduced in a two-parameter scaling test where the scaling model \( (\text{fit}) \) is supposedly valid for any \( E \) and \( T \) such that \( x \) and \( y \) are close enough to the transition.

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| \( P_c \) | \( n_c \) | \( \beta \) | \( a_0 \) | \( \chi^2 \) |
|---|---|---|---|---|
| 50 | 0.9717 ± 0.0004 | 3.14 ± 0.02 | 2.164 ± 0.006 | 0.94 |
| 100 | 0.9751 ± 0.0004 | 3.25 ± 0.03 | 2.117 ± 0.006 | 0.63 |
| 200 | 0.9780 ± 0.0006 | 3.36 ± 0.04 | 2.078 ± 0.008 | 0.46 |
| 400 | 0.9818 ± 0.0010 | 3.53 ± 0.08 | 2.028 ± 0.013 | 0.37 |
| 500 | 0.9826 ± 0.0014 | 3.69 ± 0.13 | 2.018 ± 0.017 | 0.36 |

**TABLE II. Results for sample two.**
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