WELL-POSEDNESS OF LOW REGULARITY SOLUTIONS TO
THE SECOND ORDER STRICTLY HYPERBOLIC EQUATIONS
WITH NON-LIPSCHITZIAN COEFFICIENTS

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Abstract. In this paper, we establish the local well-posedness of low reg-
ularity solutions to the general second order strictly hyperbolic equation of
divergence form

\[ \partial_t (a_0 \partial_t u) + \sum_{j=1}^{n} [\partial_t (a_j \partial_j u) + \partial_j (a_j \partial_t u)] - \sum_{j,k=1}^{n} \partial_j (a_{jk} \partial_k u) + b_0 \partial_t u + \partial_t (c_0 u) + \sum_{j=1}^{n} [b_j \partial_j u + \partial_j (c_j u)] + du = f \]

in domain \( \Omega = (0, T_0) \times \mathbb{R}^n \), where the coefficients \( a_0, a_j, a_{jk} \in L^\infty(\Omega) \cap LL(\bar{\Omega}) \) (\( 1 \leq j, k \leq n \)), \( b_0, c_0, b_j, c_j \in L^\infty(\Omega) \cap C^\alpha(\bar{\Omega}) \) (\( 1 \leq j \leq n \)) for \( \alpha \in (\frac{1}{2}, 1) \), \( d \in L^\infty(\Omega) \), \( (u(0, x), Xu(0, x), X^2 u(0, x)) \in (H^{1-\theta+\beta \log}, H^{-\theta+\beta \log}) \) with \( \theta > 1 - \alpha, \beta \in \mathbb{R} \), and \( Xu = a_0 \partial_t u + \sum_{j=1}^{n} a_j \partial_j u \). Compared with previous references, except a little

more general initial data in the space \( (H^{1-\theta+\beta \log}, H^{-\theta+\beta \log}) \) (only \( \beta = 0 \)

is considered as before), we improve both the lifespan of \( u \) up to the precise

number \( T^* \) and the range of \( \theta \) to the left endpoint \( 1 - \alpha \) under some suitable

conditions.

1. Introduction. In this paper, we are concerned with the Cauchy problem for
the general second order strictly hyperbolic equations with divergence form:

\[
\begin{cases}
L u = L_2 u + L_1 u + du = f & \text{in } \Omega, \\
u|_{t=0} = u_0(x), \quad Xu|_{t=0} = u_1(x),
\end{cases}
\]

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where \( x \in \mathbb{R}^n, n \in \mathbb{N}, \Omega = (0, T_0) \times \mathbb{R}^n \), and

\[
L_2 = \partial_t (a_0 \partial_t \cdot) + \sum_{j=1}^n [\partial_j (a_j \partial_j \cdot) + \partial_j (a_j \partial_t \cdot)] - \sum_{j,k=1}^n \partial_j (a_{jk} \partial_k \cdot),
\]

(2)

\[
L_1 = b_0 \partial_t \cdot + \partial_t (c_0 \cdot) + \sum_{j=1}^n [b_j \partial_j \cdot + \partial_j (c_j \cdot)],
\]

(3)

\[
X = a_0 \partial_t \cdot + \sum_{j=1}^n a_j \partial_j \cdot.
\]

(4)

The coefficients in (1)-(4) satisfy the following assumptions:

- There are two constants \( \delta_0 > 0 \) and \( \delta_1 > 0 \) such that for all \( (t,x) \in \bar{\Omega} \) and \( \xi \in \mathbb{R}^n \),

\[
a_0 \geq \delta_0, \quad \sum_{j,k=1}^n (a_{jk} + \frac{a_j a_k}{a_0}) \xi_j \xi_k \geq \delta_1 |\xi|^2.
\]

(5)

- \( a_j = a_{kj}, \quad a_0, a_j, a_{jk} \in L^\infty(\Omega) \cap LL(\bar{\Omega}), \) where \( 1 \leq j, k \leq n. \)

(6)

- \( b_0, c_0, b_j, c_j \in L^\infty(\Omega) \cap C^\alpha (\bar{\Omega}) \) for some \( \alpha \in (\frac{1}{2}, 1) \), where \( 1 \leq j \leq n. \)

(7)

- \( d \in L^\infty(\Omega). \)

(8)

Here the Log-Lipschitz space in domain \( \bar{\Omega} (LL \text{ in brief}) \) is defined as

\[
a \in LL \iff |a(z_1) - a(z_2)| \leq C|z_1 - z_2| (1 + |\log|z_1 - z_2||), \text{ where } z_1, z_2 \in \Omega, \ z_1 \neq z_2,
\]

(9)

and the corresponding semi-norm \( |a|_{LL} \) stands for the best constant \( C > 0 \) in (9).

In addition,

\[
||a||_{LL} =: ||a||_{L^\infty} + ||a||_{LL}.
\]

For the requirements later on, we now introduce the following function spaces (see [7] or [16]).

**Definition 1.1.** Let \( S' \) be the space of temperate distributions in \( \mathbb{R}^n \) and \( \Lambda = Log\sqrt{2 - \Delta} \) be the pseudo-differential operator with the symbol \( \log \sqrt{2 + |\xi|^2} \). Then

1. the space \( H^{\sigma + \delta\log} \) consists of all functions \( u \in S' \) with

\[
||u||_{H^{\sigma + \delta\log}} =: ||(1 + |D|)\sigma A^\delta u||_{L^2(\mathbb{R}^n)} < \infty.
\]

(10)

2. the space \( C_{\sigma + \delta\log, \lambda}(T) \) consists of all functions \( u : [0, T] \to S' \) such that for any \( t \in [0, T], u \in C([0, t]; H^{\sigma - \lambda t + \delta\log}), \) which is equipped with the norm

\[
||u||_{C_{\sigma + \delta\log, \lambda}(T)} =: \sup_{t \in [0, T]} ||u(t)||_{H^{\sigma - \lambda t + \delta\log}}.
\]

(11)

3. the space \( \mathcal{H}_{\sigma + \delta\log, \lambda}(T) \) consists of all functions \( u : (0, T) \to S' \) with

\[
||u||_{\mathcal{H}_{\sigma + \delta\log, \lambda}(T)} =: ||(1 + |D|)\sigma - \lambda t A^\delta u(t)||_{L^2(0, T; L^2(\mathbb{R}^n))} < \infty.
\]

(12)

4. the space \( \mathcal{C}_{\sigma + \delta\log, \lambda}(T) \) consists of all functions \( u : (0, T) \to S' \) with

\[
||u||_{\mathcal{C}_{\sigma + \delta\log, \lambda}(T)} =: ||(1 + |D|)\sigma - \lambda t A^\delta u(t)||_{L^1(0, T; L^2(\mathbb{R}^n))} < \infty.
\]

(13)

So far there have been extensive results on the well-posedness of solutions for the second order strictly hyperbolic equations with non-regular coefficients (in the case
of regular coefficients, one can see [11] or [15] for the well-posedness of solutions). For instances, with respect to the wave equation in divergence form:

$$Lu = \partial_t^2 u - \sum_{i,j=1}^{n} \partial_j(a_{ij}(t, x)\partial_j u),$$  \hspace{1cm} (14)

the authors in [12] prove that the Cauchy problem of (14) is well-posed in $H^1(\Omega) \times L^2(\Omega)$ if the coefficients $a_{ij}$ are Lipschitzian in the variable $t$ and measurable in the variable $x \in \mathbb{R}^n$, and is well-posed in $H^{s+1}(\Omega) \times H^s(\Omega)$ for any $s \in \mathbb{R}$ when $a_{ij}$ are Lipschitzian in the variable $t$ and $C^\infty_0$ in $x$. Especially, for the latter case, the authors in [12] obtain the energy estimate of solution $u$ without loss of derivatives.

If the coefficients $a_{ij}(t, x) = a_{ij}(t)$ in (14) depend only on the time variable $t$ and the Lipschitzian continuity of $a_{ij}(t)$ is not fulfilled, for example, in the case of $a_{ij}(t) \in C^\alpha$ with $\alpha < 1$, it is proved in [4] that the Cauchy problem of (14) is ill-posed in $C^\infty$, furthermore, the well-posedness of (14) in Sobolev spaces may not be obtained. Precisely speaking, for $x \in \mathbb{R}^1$, it is shown in [6] that for any function $\phi$ with $\lim_{r \to 0^+} \phi(r) = +\infty$, there exists a function $a(t)$ with

$$|a(t) - a(s)| \leq C|t - s|\log|t - s|\phi(|t - s|)$$  \hspace{1cm} (15)

such that the Cauchy problem of $Lu = \partial_t^2 u - \partial_x(a(t)\partial_x u)$ is ill-posed in $C^\infty$ class. Here we point out that the functions in $C^\alpha(\alpha < 1)$ clearly satisfy condition (15) (at this time, the ill-posedness of solution $u$ can be also referred to [5] and [10]).

On the other hand, another kind of non-Lipschitzian regularity, that is, Log-Lipschitzian regularity (9) which is smoother than $C^\alpha$ with $\alpha < 1$, has been concerned for the coefficients of (14). Indeed, in [4], when the coefficients $a_{ij}(t, x) = a_{ij}(t) \in LL$ in (14), the authors obtain the $C^\infty$ well-posedness of weak solution $u$ to (14) with loss of derivatives. We remark that if the Lipschitzian continuity hypothesis of $a_{ij}(t)$ is not fulfilled, then the loss of regularity for the solution $u$ of (14) cannot be avoided as indicated in [3].

Interesting results have also been obtained when the coefficients $a_{ij}(t, x)$ in (14) depend both on time variable $t$ and space variable $x$. The authors in [13] and [14] establish the well-posedness of (14) in Gevery spaces when the coefficients $a_{ij}(t, x)$ are $C^\alpha$ ($0 < \alpha < 1$) in $t$ and Gevery class in $x$. For $a_{ij}(t, x) \in LL$ in (14), the authors in [6] prove that there exists $T > 0$ such that for $0 \leq t < T$, the solution $u$ of (14) satisfies the following estimate:

$$||u(t, \cdot)||_{H^{1-\theta-\alpha t}} + ||\partial_t u(t, \cdot)||_{H^{-\theta-\alpha t}}$$

$$\leq C(||a_0||_{H^{1-\theta}} + ||a_1||_{H^{-\theta}} + \int_0^T ||Lu(s, \cdot)||_{H^{-\theta-\alpha t}}ds).$$  \hspace{1cm} (16)

This implies that the solution $u$ to (14) for $a_{ij}(t, x) \in LL$ will have the loss of regularities with the development of time $t$. With respect to the wave equations in non-divergence form

$$Lu = \partial_t^2 u - \sum_{i,j=1}^{n} a_{ij}(t, x)\partial_{ij}^2 u,$$

the authors in [6] also obtain $C^\infty$ well-posedness with a loss of regularity increasing in time, where $a_{ij}$ are $LL$ in $t$ and smooth in $x$. In [8], for $x \in \mathbb{R}^1$ and $a_{11}(t, x)$ is
the authors have obtained the energy estimate similar to \((16)\) for the weak solution \(u\) of \((14)\). Recently, the 1-D result in \([8]\) has been extended to the case of \(n \geq 1\) by the authors of \([9]\), where the related para-differential calculus with parameters is used to derive the positivity of some corresponding para-product operators.

With respect to the more general second order strictly hyperbolic equation in \((1)\), the authors in \([7]\) prove that for \((u_0, u_1) \in H^{1-\theta} \times H^{-\theta}\) with \(\theta \in (1-\alpha, \alpha)\) and \(\alpha > 1\), problem \((1)\) admits a unique solution

\[ u(t, \cdot) \in H^{1-\lambda t-\theta+\frac{1}{2}\log}, \quad \partial_t u(t, \cdot) \in H^{-\lambda t-\theta+\frac{1}{2}\log}, \]

where \(\lambda > 0\) is a constant depending on the \(LL\)-norms of the coefficients, the constants \(\delta_0\) and \(\delta_1\) of hyperbolicity and \(\alpha\) in \((5)-(8)\). Note that for \(b \in C^{\alpha}\) and \(v \in H^s\), \(bv \in H^s\) generally holds only for \(|s| < \alpha\), and note \(H^{-\lambda t-\theta+\frac{1}{2}\log} \subset H^{-\lambda t-\theta}\) for any \(t \in [0, T]\), then in order to let the first-order terms \(b_j \partial_j u\) in \((1.1)\) make sense in distribution, it is required to pose \(T^* > \frac{2\lambda t}{\alpha}\) for some other constant \(\lambda' > 0\), however, direct analysis easily shows that the constant \(\lambda'\) has to depend on \(\theta\). Indeed, if taking \(T* = \frac{\alpha-\theta}{\alpha}\) and \(\alpha - \theta < 2\), then \(\frac{\alpha-\theta}{\alpha} \leq \frac{\alpha-\theta}{\lambda'}\) implies that \(\lambda' \geq \frac{4\lambda t}{\alpha-\theta}\to \infty\) as \(\theta \to 1-\alpha\). In the present paper, we intend to improve the lifespan of solution \(u\) for \((1)\) up to the precise number \(T^* = \frac{\alpha-\theta}{\lambda}\) for any \(\theta \in (1-\alpha, \alpha)\) and study the regularity of \(u\) up to the left endpoint \(\theta = 1-\alpha\) under some suitable conditions.

We next state the main results in this paper.

**Theorem 1.2.** Let \(\theta \in (1-\alpha, \alpha)\) and \(\beta > \frac{1}{2}\). Then there exists a constant \(\lambda > 0\) which depends on the \(LL\)-norms of \(a_{ij}\) \((0 \leq i \leq n)\), \(a_{ijk} \) \((1 \leq j, k \leq n)\) in \((2)\), the positive numbers \(\delta_0, \delta_1\) in \((5)\), and \(\alpha, \beta, \lambda\), such that for

\[ T^* = \min(T_0, \frac{\alpha-\theta}{\lambda}) \]  

and \(f \in L_{-\theta+\beta\log, \lambda}(T_0)\), problem \((1)\) with \(u_0 \in H^{1-\theta+\beta\log}\) and \(u_1 \in H^{-\theta+\beta\log}\) admits a unique weak solution \(u \in C_{-\theta+\beta\log, \lambda}(T^*) \cap H_{-\theta+\beta\log, \lambda}(T^*)\) satisfying \(\partial_t u \in C_{-\theta+\beta\log, \lambda}(T^*) \cap H_{-\theta+\beta\log, \lambda}(T^*)\). Moreover, we have that for \(0 \leq t \leq T^*,\)

\[ \sup_{\tau \in [0, t]} \|u(\tau)\|_{H^{-\theta-\lambda t+\beta\log}}^2 + \sup_{\tau \in [0, t]} \|\partial_t u(\tau)\|_{H^{-\theta-\lambda t+\beta\log}}^2 \]

\[ + \int_0^t \left(\|u(\tau)\|_{H^{-\theta-\lambda t+\beta\log}}^2 + \|\partial_t u(\tau)\|_{H^{-\theta-\lambda t+\beta\log}}^2\right) d\tau \]

\[ \leq C \{ \|u_0\|_{H^{1-\theta+\beta\log}}^2 + \|u_1\|_{H^{-\theta+\beta\log}}^2 + \left(\int_0^t \|f(\tau)\|_{H^{-\theta-\lambda t+\beta\log}} d\tau\right)^2 \}. \]

**Remark 1.** Note that \(-\lambda t - \theta \geq -\alpha\) holds for \(t \in [0, T^*]\), and \(f \in L_{-\theta+\beta\log, \lambda}(T_0)\), then \(f \in L^1(0, T^*; H^{-\alpha+\beta\log})\). In addition, \(u \in L^2(0, T^*; H^{-\alpha+\beta\log})\) and \(\partial_t u \in L^2(0, T^*; H^{-\alpha+\beta\log})(0, T^*) \times \mathbb{R}^n\). By Lemma A.1 in Appendix, we know that problem \((1)\) makes sense in the distribution and further \(u\) is a weak solution of \((1)\).
Theorem 1.3. Let $1 - \alpha \leq \theta < \theta' < \alpha$ and $\beta < -\frac{1}{2}$. Then there exists a constant $\lambda > 0$ which depends on the LL-norms of $a_l$ $(0 \leq l \leq n)$, $a_{jk}$ $(1 \leq j, k \leq n)$ in (2), the positive numbers $\delta_0, \delta_1$ in (3), and $\alpha, \beta$, such that for

$$T^* = \min(T_0, \frac{\theta' - \theta}{\lambda})$$

(21)

and $f = f_1 + f_2$ with $f_1 \in \mathcal{L}_{\theta, \beta \log}(T_0)$ and $f_2 \in \mathcal{H}_{-\theta + (\beta + \frac{1}{2}) \log, \lambda}(T_0)$, problem (1) with $u_0 \in H^{1-\theta + \beta \log}$ and $u_1 \in H^{-\theta + \beta \log}$ admits a unique weak solution $u \in \mathcal{C}_{1-\theta + \beta \log, \lambda}(T^*) \cap \mathcal{H}_{-\theta + (\beta + \frac{1}{2}) \log, \lambda}(T^*)$ satisfying $\partial_t u \in \mathcal{C}_{-\theta + \beta \log, \lambda}(T^*) \cap \mathcal{H}_{-\theta + (\beta + \frac{1}{2}) \log, \lambda}(T^*)$. Moreover, we have that for $0 \leq t \leq T^*$,

$$\sup_{\tau \in [0,t]}||u(\tau)||^2_{H^{1-\theta - \lambda \tau + \beta \log}} + \sup_{\tau \in [0,t]}||\partial_t u(\tau)||^2_{H^{-\theta - \lambda \tau + \beta \log}}$$

$$+ \int_0^t (||u(\tau)||^2_{H^{1-\theta - \lambda \tau + (\beta + \frac{1}{2}) \log}} + ||\partial_t u(\tau)||^2_{H^{-\theta - \lambda \tau + (\beta + \frac{1}{2}) \log})d\tau$$

$$\leq C(||u_0||^2_{H^{1-\theta + \beta \log}} + ||u_1||^2_{H^{-\theta + \beta \log}} + (\int_0^t ||f_1(\tau)||_{H^{-\theta + (\beta + \frac{1}{2}) \log}}d\tau)^2$$

$$+ \int_0^t ||f_2(\tau)||^2_{H^{-\theta - \lambda \tau + (\beta + \frac{1}{2}) \log}}d\tau$$

(22)

In addition, if $\beta \in \mathbb{R}$, then for $1 - \alpha < \theta < \theta' < \alpha$, we have the same result as in (22).

Remark 2. From Theorem 1.3, we know that if $\beta < -\frac{1}{2}$, then (22) holds for $\theta \in (1 - \alpha, \alpha)$, which includes the case of left endpoint $\theta = 1 - \alpha$.

Remark 3. As explained in Remark 1, problem (1) makes sense in the distribution under the assumptions of Theorem 1.3 and further $u$ is a weak solution of (1).

Next we give the comments on the proofs of Theorem 1.2 and Theorem 1.3. To prove Theorem 1.2-1.3, one of the key techniques is that the multiplication $(b, v) \mapsto bv$ is continuous from $C^\alpha \times H^{s+\beta \log}$ to $H^{s+\beta \log}$, when $s \in [-\alpha, \alpha]$ and $\beta > \frac{1}{2}$; or $s \in (-\alpha, \alpha)$ and $\beta \in \mathbb{R}$; or $s \in (-\alpha, \alpha]$ and $\beta < -\frac{1}{2}$, which will be shown in Proposition 2 of Section 2. Based on this basic observation together with some careful analysis, for the first case of $s \in [-\alpha, \alpha)$ and $\beta > \frac{1}{2}$, we can extend the lifespan of the weak solution $u$ of (1) up to $T^* = \frac{\alpha - \theta}{\lambda}$, where the constant $\lambda > 0$ is independent of $\theta$, and solution $u$ satisfies

$$u(t, \cdot) \in H^{1 - \lambda \theta + (\beta + \frac{1}{2}) \log}, \quad \partial_t u(t, \cdot) \in H^{-\lambda \theta + (\beta + \frac{1}{2}) \log}$$

when $(u(0, x), \partial_t u(0, x)) \in H^{1-\theta + \beta \log} \times H^{-\theta + \beta \log}$; for the third case of $s \in (-\alpha, \alpha]$ and $\beta < -\frac{1}{2}$, we can extend the well-posed range of $\theta$ for (1) to $|1 - \alpha, \alpha|$ including the left endpoint of the interval; meanwhile for the second case of $s \in (-\alpha, \alpha)$ (excluding the two points $s = -\alpha$ and $s = \alpha$) and $\beta \in \mathbb{R}$, we prove the well-posedness of (1). On the other hand, in order to avoid the involved analysis on the existence and energy estimates of the solution to the duality problem of (1) (in fact, the space $\mathcal{C}_{1-\theta + \beta \log, \lambda}(T) \cap \mathcal{H}_{-\theta + (\beta + \frac{1}{2}) \log, \lambda}(T)$ appeared in Theorem 1.2-1.3 is not reflexive), instead of the duality argument used in [7], we split the total energy in Theorem 1.2-1.3 into the localized components associated to the dyadic block $\Delta_k u$, meanwhile it will be clearer to understand the mechanism of the regularity loss of solutions in the microlocal version. By establishing the related estimates on
the dyadic block $\Delta_ku$ and then technically putting all of these pieces together, we derive the needed energy estimates of weak solution $u$ in Theorem 1.2-1.3.

This paper is organized as follows. Some basic results including paradifferential calculus and positivity estimates are listed or established in Section 2. In Section 3, we rewrite the equation in (1) as a new system on $(u, v)$ with $v = Xu + c_0u$, and some a-priori estimates for smooth solution $(u, v)$ is obtained. In Section 4, with the aid of a-priori estimates derived in Section 3, together with the mollifier argument, we establish the energy estimate for the weak solution of (1) and further complete the proofs of Theorem 1.2 and Theorem 1.3. In addition, Lemma A.1 in Appendix gives a validity illustration on the weak solution $u$ to problem (1) under the suitable regularity assumptions in Theorem 1.2 and Theorem 1.3.

2. Paradifferential calculus. In this section, some useful results on paradifferential calculus will be listed or established. These results, especially Proposition 2, will play a key role in proving Theorem 1.2 and Theorem 1.3.

2.1. Dyadic partition of unity. As in [2] (page 212) or [1] (page 59), one can introduce the following dyadic partition of unity:

**Lemma 2.1.** Let $0 < r_1 < r_2 < \infty$, $C_0$ be the ball $\{\xi \in \mathbb{R}^n : |\xi| \leq r_1\}$, and $C_k$ be the annulus $\{\xi \in \mathbb{R}^n : 2^{k-1}r_1 \leq |\xi| \leq 2^kr_2\}$ for $k \geq 1$. There exist radial smooth functions $\chi$ and $\varphi$, valued in the interval $[0, 1]$, belonging to $C_0^\infty(\mathbb{R}^n)$, and such that

1. $\text{Supp} \chi \subset C_0, \text{Supp} \varphi \subset \{\xi \in \mathbb{R}^n : \frac{1}{2}r_1 \leq |\xi| \leq r_2\}$.
2. $\chi(\xi) + \sum_{k \geq 1} \varphi(2^{-k}\xi) = 1, \forall \xi \in \mathbb{R}^n$.
3. $\chi(\xi) + \sum_{k=1}^{l} \varphi(2^{-k}\xi) = \chi(2^{-l}\xi), \forall \xi \in \mathbb{R}^n$.

**Remark 4.** Let $N_0 := 2 + \left[\frac{\log 2 - \log r_1}{\log 2}\right]$. Then for all nonnegative integers $k$ and $l$ satisfying $|k - l| > N_0$, one has $C_k \cap C_l = \emptyset$.

From now on, we fix two functions $\chi$ and $\varphi$ satisfying Lemma 2.1. The Fourier transform of an $L^1$ function $f$ is defined by

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx$$

and the inverse Fourier transform is given by

$$\mathcal{F}^{-1}(f)(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi) d\xi,$$

where $x \cdot \xi = x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n$. In general, for $f \in \mathcal{S}'$, its Fourier transform is defined by

$$\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle$$

and its inverse Fourier transform is given by

$$\langle \mathcal{F}^{-1}f, \phi \rangle = \langle f, \mathcal{F}^{-1}\phi \rangle,$$

where $\phi \in \mathcal{S}$, and $\mathcal{S}$ stands for the Schwartz space.

The nonhomogeneous dyadic blocks $\Delta_k$ are given by

$$\Delta_0u = \mathcal{F}^{-1}(\chi(\xi)\hat{u}),$$
$$\Delta_ku = \mathcal{F}^{-1}(\varphi(2^{-k}\xi)\hat{u}), \quad k \geq 1.$$

The nonhomogeneous low-frequency cut-off operator are defined by

$$S_ku = \mathcal{F}^{-1}(\chi(2^{-k}\xi)\hat{u}), \quad k \geq 0.$$
For \( u \in S' \), one then has
\[
    u = \sum_{k \geq 0} \Delta_k u.
\]

Next proposition shows some properties of \( \Delta_k a \) and \( S_k a \) when \( a \in LL \) (one can see Proposition 3.3 of [7]).

**Proposition 1.** (1) There is a constant \( C > 0 \) such that for all \( a \in LL(\mathbb{R}^n) \),
\[
    \|\Delta_k a\|_{L^\infty} \leq C(1+k)2^{-k|a|_{LL}}, \quad k > 0.
\]
\[
    \|a - S_k a\|_{L^\infty} \leq C(1+k)2^{-k|a|_{LL}}, \quad k \geq 0.
\]
\[
    \|\nabla S_k a\|_{L^\infty} \leq C(1+k)|a|_{LL}, \quad k \geq 0.
\]
(2) If \( \alpha \in (0, 1) \) and \( a \in C^\alpha(\mathbb{R}^n) \), then
\[
    \|\Delta_k a\|_{L^\infty} \leq C2^{-k\alpha}\|a\|_{C^\alpha}, \quad k \geq 0.
\]

2.2. **Paraproducts.** For some positive integer \( N \), the nonhomogeneous paraproduct of \( a \) and \( u \) is defined by
\[
    T_a u := \sum_{k \geq N} S_k - N a \Delta_k u,
\]
and
\[
    T_a a := \sum_{k \geq N} S_k - N u \Delta_k a.
\]

The remainder \( R(a, u) \) is defined by
\[
    R(a, u) := \sum_{|k-j| \leq N-1} \Delta_k u \Delta_j a
\]
and
\[
    R_a u := T_a a + R(a, u).
\]

**Lemma 2.2.** Let \( s, \beta \in \mathbb{R} \), \( u \in S' \) belongs to \( H^{s+\beta\log}(\mathbb{R}^n) \) if and only if
(1) \( \Delta_k u \in L^2(\mathbb{R}^n) \), \( \forall k \geq 0 \);
(2) The sequence \( \{2^{ks}(1+k)^\beta\|\Delta_k u\|_{L^2(\mathbb{R}^n)}\} \) belongs to \( l^2(\mathbb{N}) \), and
\[
    \|u\|_{H^{s+\beta\log}(\mathbb{R}^n)} \sim \sum_{k=0}^{\infty} \left[ 2^{ks}(1+k)^\beta\|\Delta_k u\|_{L^2(\mathbb{R}^n)} \right]^2.
\]

**Proof.** Since the proof is basically similar to that for \( u \in H^s \) (see page 212 of [2]), we omit it here.

The next proposition illustrates some key properties of \( R_a u \) and \( au \) for \( a \in LL \) and \( u \in H^{s+\beta\log} \).

**Proposition 2.** (1) If \( a \in L^\infty \cap LL \), \( s \in (0, 1) \) and \( \beta \in \mathbb{R} \), then \( R_a \) maps \( H^{(s-1)+\beta+1\log} \) into \( H^{s+\beta\log} \), and
\[
    \|R_a u\|_{H^{s+\beta\log}} \leq C(\beta, s)|a|_{LL}\|u\|_{H^{(s-1)+\beta+1\log}}, \quad \tag{23}
\]
where the constant \( C(\beta, s) > 0 \) depends on \( \beta \) and \( s \), which is uniformly bounded when \( (\beta, s) \) lies in a compact subset of \( \mathbb{R} \times (0, 1) \).

(2) If \( a \in L^\infty \cap LL \), \( s \in (-1, 1) \) and \( \beta \in \mathbb{R} \), \( u \in H^{s+\beta\log} \), then \( au \in H^{s+\beta\log} \), and
\[
    \|au\|_{H^{s+\beta\log}} \leq C(\beta, s)|a|_{LL}\|u\|_{H^{s+\beta\log}}, \quad \tag{24}
\]
where the constant \( C(\beta, s) > 0 \) depends on \( \beta \) and \( s \), which is uniformly bounded when \((\beta, s)\) lies in a compact subset of \( \mathbb{R} \times (-1, 1) \).

(3) If \( a \in L^\infty \cap C^\alpha \), \( s \in (-\alpha, \alpha) \) and \( \beta \in \mathbb{R} \), \( u \in H^{s+\beta \log} \), then \( au \in H^{s+\beta \log} \), and

\[
||au||_{H^{s+\beta \log}} \leq C(\beta, s)||a||_{C^\alpha} ||u||_{H^{s+\beta \log}},
\]

(25)

where the constant \( C(\beta, s) > 0 \) depends on \( \beta \) and \( s \), which is uniformly bounded when \((\beta, s)\) lies in a compact subset of \( \mathbb{R} \times (-\alpha, \alpha) \).

(4) If \( a \in L^\infty \cap C^\alpha \), \( s \in [-\alpha, \alpha) \) and \( \beta > \frac{1}{2} \), \( u \in H^{s+\beta \log} \), then \( au \in H^{s+\beta \log} \), and

\[
||au||_{H^{s+\beta \log}} \leq C(\beta, s)||a||_{C^\alpha} ||u||_{H^{s+\beta \log}},
\]

(26)

where the constant \( C(\beta, s) > 0 \) depends on \( \beta \) and \( s \), which is uniformly bounded when \((\beta, s)\) lies in a compact subset of \((\frac{1}{2}, +\infty) \times [-\alpha, \alpha) \).

(5) If \( a \in L^\infty \cap C^\alpha \), \( s \in (-\alpha, \alpha] \) and \( \beta < -\frac{1}{2} \), \( u \in H^{s+\beta \log} \), then \( au \in H^{s+\beta \log} \), and

\[
||au||_{H^{s+\beta \log}} \leq C(\beta, s)||a||_{C^\alpha} ||u||_{H^{s+\beta \log}},
\]

(27)

where the constant \( C(\beta, s) > 0 \) depends on \( \beta \) and \( s \), which is uniformly bounded when \((\beta, s)\) lies in a compact subset of \((-\infty, -\frac{1}{2}) \times (-\alpha, \alpha] \).

**Remark 5.** In the general case, one can not get \( au \in H^{-\alpha} \) for \( a \in C^\alpha \) and \( u \in H^{-\alpha} \) \((\alpha > 0)\). On the other hand, by (4) of Proposition 2, we know that \( au \in H^{-\alpha+\beta \log} \) for \( a \in C^\alpha \) and \( u \in H^{-\alpha+\beta \log} \) with \( \alpha > 0 \) and \( \beta > \frac{1}{2} \).

**Proof.** Note that

\[
\Delta_k(T_a u) = \sum_{m \geq N \atop |m-k| \leq N_0} \Delta_k(S_{m-N} a \Delta_m u),
\]

\[
\Delta_k(T_a u) = \sum_{m \geq N \atop |m-k| \leq N_0} \Delta_k(S_{m-N} u \Delta_m a),
\]

\[
\Delta_k R(a, u) = \sum_{m \geq N \atop |m-k| \leq N_0} \Delta_k(\Delta_m u \Delta_a)
\]

with \( N_1 := N + N_0 \). By Proposition 1, one then has

\[
||\Delta_k(T_a u)||_{L^2} \leq C||a||_{L^\infty} \sum_{m \geq N \atop |m-k| \leq N_0} ||\Delta_m u||_{L^2}, \quad a \in L^\infty,
\]

(28)

\[
||\Delta_k(T_a u)||_{L^2} \leq \begin{cases} C||a||_{L^\infty} 2^{-k}(1 + k) \sum_{0 \leq m \leq k+N_0} ||\Delta_m u||_{L^2}, \quad a \in L^\infty \cap LL, \\ C||a||_{C^\alpha} 2^{-k \alpha} \sum_{0 \leq m \leq k+N_0} ||\Delta_m u||_{L^2}, \quad a \in L^\infty \cap C^\alpha, \end{cases}
\]

(29)

\[
||\Delta_k R(a, u)||_{L^2} \leq \begin{cases} C||a||_{LL} \sum_{m \geq \max(k-N_1, 0)} 2^{-m}(1 + m)||\Delta_m u||_{L^2}, \quad a \in L^\infty \cap LL, \\ C||a||_{C^\alpha} \sum_{m \geq \max(k-N_1, 0)} 2^{-m \alpha}||\Delta_m u||_{L^2}, \quad a \in L^\infty \cap C^\alpha. \end{cases}
\]

(30)
By (2.6), for $s, \beta \in \mathbb{R}$, $a \in L^\infty$ and $u \in H^{s+\beta \log}$, one has $T_n u \in H^{s+\beta \log}$, and
$$||T_n u||_{H^{s+\beta \log}} \leq C||a||_{L^\infty} ||u||_{H^{s+\beta \log}}. \quad (31)$$

(1) Note that $s < 1$. Then, by (29), Lemma 2.2 and Young’s inequality,
$$||T_n a||_{H^{s+\beta \log}} \leq C(\beta, s)a_{LL}\left[\sum_{k \geq 0} \left( \sum_{0 \leq m \leq k+N_0} 2^{m(s-1)(1+m)\beta+1} ||\Delta_m u||_{L^2} 2^{(k-m+N_0)\beta \frac{s-1}{2}} \right)^{\frac{1}{2}} \right] \leq C(\beta, s)a_{LL}||u||_{H^{(s-1)+\beta \log}}. \quad (32)$$

Similarly, because of $s > 0$, (30) and Lemma 2.2 imply
$$||R(a, u)||_{H^{s+\beta \log}} \leq C(\beta, s)a_{LL}||u||_{H^{(s-1)+\beta \log}}. \quad (33)$$

Thus $R_n u$ belongs to $H^{s+\beta \log}$, and estimate (23) follows.

(2) By Proposition 1 and Lemma 2.2, for $s > -1$ we have
$$||R(a, u)||_{H^{s+\beta \log}} \leq C(\beta, s)a_{LL}\left[\sum_{k \geq 0} \left( \sum_{m \geq \max(k-N_1, 0)} \delta_m 2^{(k-m-N_1)\beta \frac{1}{2}} \right)^{\frac{1}{2}} \right] \leq C(\beta, s)a_{LL}||u||_{H^{s+\beta \log}}. \quad (34)$$

Together with (31) and (32), this yields $au \in H^{s+\beta \log}$.

(3) (4) and (5) It follows from (32) and Lemma 2.2 that
$$||T_n a||_{H^{s+\beta \log}} \leq C||a||_{C^\alpha}\left[\sum_{k \geq 0} \left( \sum_{0 \leq m \leq k+N_0} \delta_m 2^{(k-m-N_0)(s-\alpha)} \frac{1}{2^{m(s-1)(1+m)\beta}} \right)^{\frac{1}{2}} \right].$$

Thus if $s < \alpha$ and $\beta \in \mathbb{R}$ or $s \leq \alpha$ and $\beta < -\frac{1}{2}$, by Young’s inequality and Hölder’s inequality respectively, one has
$$||T_n a||_{H^{s+\beta \log}} \leq C(\beta, s)||a||_{C^\alpha}||u||_{H^{s+\beta \log}}. \quad (34)$$

Similarly, (30) and Lemma 2.2 yield
$$||R(a, u)||_{H^{s+\beta \log}} \leq C||a||_{C^\alpha}\left[\sum_{k \geq 0} \left( \sum_{m \geq \max(k-N_1, 0)} \delta_m 2^{(k-m-N_1)(s+\alpha)} \frac{1}{2^{m(s-1)(1+m)\beta}} \right)^{\frac{1}{2}} \right].$$

If $s > -\alpha$ and $\beta > \frac{1}{2}$ or $s > -\alpha$ and $\beta \in \mathbb{R}$, then by Hölder’s inequality and Young’s inequality respectively, we have
$$||R(a, u)||_{H^{s+\beta \log}} \leq C(\beta, s)||a||_{C^\alpha}||u||_{H^{s+\beta \log}}.$$

This, together with (34) and (31), completes the proof of (3), (4) and (5).

The following commutator estimate can be found in [1] (see Lemma 2.97 of page 110).

**Lemma 2.3.** Let $\theta$ be a $C^1(\mathbb{R}^n)$ function such that $(1 + |\xi|)\hat{\theta} \in L^1$. Then for $a \in Lip$ with $\nabla a \in L^p$ and $b \in L^q$ ($p, q \geq 1$), we have that for any $\lambda > 0$,
$$||[\theta(\lambda^{-1}D), a]b||_{L^r} \leq \lambda^{-1}||k||_{L^1}||\nabla a||_{L^p}||b||_{L^q}.$$
where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and $k = \|z\|((F^{-1}\theta)(z))$.

**Proposition 3.** If $a \in L^\infty \cap LL$, and $\alpha_1, \alpha_2 \in \{0, 1\}$, then for $s, \beta \in \mathbb{R}$, we have

$$\left[ \sum_{k \geq 0} \| \Lambda^{-\frac{1}{2}} [\partial_j, T_a] \Delta_k u \|_{L^2}^2 \right]^{\frac{1}{2}} \leq C |a|_{LL} \| u \|_{H^{\alpha_1 + \alpha_2 - s - 1 + \beta + \frac{1}{2}}}, \quad (35)$$

$$\left[ \sum_{k \geq 0} \| \Lambda^{-\frac{1}{2}} [\partial_j^{\alpha_1} [\partial_j^{\alpha_2} Q^{-s} \beta] \Delta_k, T_a] u \|_{L^2}^2 \right]^{\frac{1}{2}} \leq C |a|_{LL} \| u \|_{H^{\alpha_1 + \alpha_2 - s - 1 + \beta + \frac{1}{2}}}, \quad (36)$$

where $Q^{-s} = (2 - \Delta_s)^{-s/2} \Lambda^s$.

**Proof.** Since the operator $\partial_j^{\alpha_1} [\partial_j^{\alpha_2} Q^{-s} \beta] \Delta_k$ commutes with $\Delta_l$ and the spectra of both $S_{l-N} a \Delta_l u$ and $\Delta_l u$ are contained in the annulus of size $\approx 2^l$ for $l \geq N$, one immediately has

$$\Lambda^{-\frac{1}{2}} [\partial_j, T_a] \Delta_k u = \sum_{l \geq N} \Lambda^{-\frac{1}{2}} [\partial_j, S_{l-N} a \Delta_l u] - \sum_{l \geq N} \Lambda^{-\frac{1}{2}} S_{l-N} a (\partial_j \Delta_l u)$$

$$= \sum_{l \geq N} \Lambda^{-\frac{1}{2}} [\partial_j, S_{l-N} a \Delta_l u] - \sum_{l \geq N} \Lambda^{-\frac{1}{2}} S_{l-N} a (\partial_j \Phi_l \Delta_l u)$$

$$= \sum_{l \geq N} \Lambda^{-\frac{1}{2}} [\partial_j \Phi_l, S_{l-N} a] \Delta_l u,$$

where $\Phi_l = \phi(2^{-1}D)$, and $\phi \in C^\infty_0$ is supported in some fixed annulus. In addition,

$$\Lambda^{-\frac{1}{2}} \partial_j^{\alpha_1} [\partial_j^{\alpha_2} Q^{-s} \beta] \Delta_k, T_a] u = \sum_{l \geq N} \Lambda^{-\frac{1}{2}} \partial_j^{\alpha_1} [\partial_j^{\alpha_2} Q^{-s} \beta] \Delta_k, S_{l-N} a] \Delta_l u.$$ 

Note that the spectra of $[\partial_j \Phi_l, S_{l-N} a] \Delta_l u$ and $[\partial_j^{\alpha_1} [\partial_j^{\alpha_2} Q^{-s} \beta] \Delta_k, S_{l-N} a] \Delta_l u$ are contained in some annulus. This, together with Lemma 2.3 and Proposition 1, yields

$$\| \Lambda^{-\frac{1}{2}} [\partial_j \Phi_l, S_{l-N} a] \Delta_k u \|_{L^2} \leq C(1 + l)^{-\frac{1}{2}} \| \nabla S_{l-N} a \|_{L^\infty} \| \Delta_l u \|_{L^2}$$

$$\leq C(1 + l)^{\frac{1}{2}} |a|_{LL} \| \Delta_l u \|_{L^2}$$

and

$$\| \Lambda^{-\frac{1}{2}} \partial_j^{\alpha_1} [\partial_j^{\alpha_2} Q^{-s} \beta] \Delta_k, S_{l-N} a] \Delta_l u \|_{L^2} \leq C(1 + l)^{\frac{1}{2} + \beta} 2^{(\alpha_1 + \alpha_2 - s - 1)} |a|_{LL} \| \Delta_l u \|_{L^2}.$$ 

Then (35) and (36) are proved. $\square$

In particularly, we have the following corollary.

**Corollary 1.** Under the assumptions of Proposition 3, one has

$$\left[ \sum_{k \geq 0} \| \Lambda^{-\frac{1}{2}} \partial_j [Q^{-s} \beta] \Delta_k, T_a] u \|_{L^2}^2 \right]^{\frac{1}{2}} \leq C |a|_{LL} \| u \|_{H^{\alpha_1 + \alpha_2 + \frac{1}{2}}},$$

$$\left[ \sum_{k \geq 0} \| \Lambda^{-\frac{1}{2}} [\partial_j Q^{-s} \beta] \Delta_k, T_a] u \|_{L^2}^2 \right]^{\frac{1}{2}} \leq C |a|_{LL} \| u \|_{H^{\alpha_1 + \alpha_2 + \frac{1}{2}}},$$

**Proposition 4.** If $a \in L^\infty \cap LL$, and $b \in L^\infty$, then for $u \in H^{\alpha_1 + \frac{1}{2}}$,

$$\left[ \sum_{k \geq 0} \| \Lambda^{-\frac{1}{2}} \partial_j (T_a - T_{a^b}) \Delta_k u \|_{L^2}^2 \right]^{\frac{1}{2}} \leq C |a|_{LL} \| u \|_{H^{\alpha_1 + \frac{1}{2}}}, \quad (37)$$
\[
\left[ \sum_{k \geq 0} \| \Lambda^{-\frac{1}{2}}(T_a - T_a^*) \partial_j \Delta_k u \|_{L^2}^2 \right]^{\frac{1}{2}} \leq C|a|_{LL} \| u \|_{L^{0+\frac{1}{2}log}}, \quad (38)
\]
\[
\left[ \sum_{k \geq 0} \| \Lambda^{-\frac{1}{2}} \partial_j (T_a - T_a^*) \Delta_k u \|_{L^2}^2 \right]^{\frac{1}{2}} \leq C|a|_{LL} \| b \|_{L^{\infty}} \| u \|_{H^{0+\frac{1}{2}log}}, \quad (39)
\]

**Proof.** Since \( \chi \) is a real radial function, we get that for \( l \geq N \),
\[
\mathcal{F}^{-1}(\chi(2^{-l+N} \xi)) = (2\pi)^{-n} \int e^{i\xi \cdot \xi}(2^{-l+N} \xi) d\xi
\]
\[
= (2\pi)^{-n} \int e^{i\xi \cdot \xi}(2^{-l+N} \xi) d\xi
\]
\[
= \mathcal{F}^{-1}(\chi(2^{-l+N} \xi)).
\]
This means that \( \mathcal{F}^{-1}(\chi(2^{-l+N} \xi)) \) is a real value function. Note that \( a \) is also real, then \( T_a^* u = \sum_{l \geq N} \Delta_l ((S_{l-N} a) u) = \sum_{l \geq N} \Delta_l ((S_{l-N} a) \Phi_l u) \),
where \( \Phi_l = \phi(2^{-l} D), \phi \in C_0^\infty \) is supported in some fixed annulus. Therefore,
\[
\Lambda^{-\frac{1}{2}} \partial_j (T_a - T_a^*) \Delta_k u = \sum_{l \geq N} \Delta_l ((S_{l-N} a) \Phi_l \Delta_k u),
\]
\[
\Lambda^{-\frac{1}{2}} \partial_j (T_a - T_a^*) \Delta_k u = \sum_{l,m \geq N} \Delta_l ((S_{l-N} a) \Phi_l (S_{m-N} b(\Delta_m \Delta_k u))).
\]
By Lemma 2.3 and Proposition 1, it follows direct computation that
\[
\| \Lambda^{-\frac{1}{2}} \partial_j [S_{l-N} a, \Delta_l] \Phi_l \Delta_k u \|_{L^2} \leq C|a|_{LL} (1 + l)^{\frac{1}{2}} \| \Delta_k u \|_{L^2}
\]
and
\[
\| \Lambda^{-\frac{1}{2}} \partial_j [S_{l-N} a, \Delta_l] \Phi_l (S_{m-N} b(\Delta_m \Delta_k u)) \|_{L^2} \leq C|a|_{LL} \| b \|_{L^{\infty}} (1 + l)^{\frac{1}{2}} \| \Delta_k u \|_{L^2}.
\]
Then (37) and (39) are shown. On the other hand, the proof of (38) is similar to that of (37), we omit it here. \( \square \)

**Proposition 5.** If \( a, b \in L^\infty \cap LL \), then for \( u \in H^{0+\frac{1}{2}log} \),
\[
\left[ \sum_{k \geq 0} \| \Lambda^{-\frac{1}{2}} \partial_j (T_a T_b - T_{ab}) \Delta_k u \|_{L^2}^2 \right]^{\frac{1}{2}} \leq C\| ab \|_{LL} \| u \|_{H^{0+\frac{1}{2}log}} \quad (40)
\]

**Proof.** Note that
\[
(T_a T_b - T_{ab}) \Delta_k u = \sum_{l,m \geq N} [S_{l-N} a S_{m-N} b - S_{l-N} (ab)] \Delta_l \Delta_m \Delta_k u
\]
\[+ \sum_{l,m \geq N} S_{l-N} a [\Delta_l, S_{m-N} b] \Delta_m \Delta_k u - \sum_{l \geq N, 0 \leq m \leq N-1} S_{l-N} (ab) \Delta_l \Delta_m \Delta_k u.
\]

By Proposition 1 and Lemma 2.3, one has that for large fixed \( N \in \mathbb{N} \),
\[
||\Lambda^{-\frac{1}{2}} \partial_j (T_a T_b - T_{ab}) \Delta_k u ||_{L^2}
\leq C(1 + k)^{-\frac{1}{2}} 2^k \sum_{l,m \geq N} \frac{||S_{l-N} a S_{m-N} b - S_{l-N} (ab)||_{L^\infty} ||\Delta_k u||_{L^2}}{|l-m| \leq N_0 |k-m| \leq N_0}
+ C||a||_{L^\infty} ||b||_{L^2} (1 + k)^{\frac{1}{2}} ||\Delta_k u||_{L^2} + C||ab||_{L^\infty} \sum_{0 \leq m \leq N-1 |k-m| \leq N_0} ||\Delta_m u||_{L^2}.
\] (41)

In addition,
\[
||S_{l-N} a S_{m-N} b - S_{l-N} (ab)||_{L^\infty}
\leq ||S_{l-N} a - a||_{L^\infty} ||S_{m-N} b||_{L^\infty} + ||a||_{L^\infty} ||S_{m-N} b - b||_{L^\infty} + ||ab - S_{l-N} (ab)||_{L^\infty}.
\]
Together with Proposition 1 and (41), this yields
\[
||\Lambda^{-\frac{1}{2}} \partial_j (T_a T_b - T_{ab}) \Delta_k u ||_{L^2}^2 \leq C(||a||_{L^2} ||b||_{L^\infty} + ||a||_{L^\infty} ||b||_{L^2}) (1 + k)^{\frac{1}{2}} ||\Delta_k u||_{L^2}^2 + C||ab||_{L^\infty} \sum_{0 \leq m \leq N-1 |k-m| \leq N_0} ||\Delta_m u||_{L^2}.
\]

Then (40) is proved.

\[ \square \]

2.3. **Positivity estimates.** For \( \nu \in \mathbb{N}_0 \), we define the modified paraproduct by
\[ T_\nu a = S_\nu a S_{\nu+N-1} u + \sum_{k \geq \nu} S_k a \Delta_{k+N} u. \]

Direct computation yields
\[ T_\nu u - T_a u = \sum_{k=0}^{\nu+N-1} \sum_{j=\max\{0,k-N+1\}}^{\nu} \Delta_j a \Delta_k u \] (42)
and
\[ a u - T_a u = \sum_{j \geq \nu + 1} \Delta_j a S_{j+N-1} u. \]

If \( a \) is smooth, one then has
\[ [\partial_t, T_a] = T_{\partial_t a}. \]

However, this is not true for \( a \in LL \) since \( \partial_t a \) makes no sense as a function.

This difficulty and the absence of positivity results of \( T_a \) are overcome in [7] (see Definition 3.13 of page 193) by introducing the modified paraproduct \( \overline{T}_a u \).

**Definition 2.4.** Let \( a \in LL([0,T_0] \times \mathbb{R}^n) \), for any fixed \( t \in [0,T_0] \), we define
\[ \overline{T}_a u = S_\nu a S_{\nu+N-1} u + \sum_{k \geq \nu} S_k a \Delta_{k+N} u, \] (43)
where \( a_k = \rho_k * a \) denotes the convolution of \( \rho_k \) and \( a \) with respect to the \( t \) variable, \( a \) is the extension of \( a \) given by
\[
\tilde{a}(t,x) = a(0,x), t < 0; \quad \tilde{a}(t,x) = a(t,x), 0 \leq t \leq T_0; \quad \tilde{a}(t,x) = a(T_0,x), t > T_0,
\]
and \( \rho_k := 2^k \rho(2^k t) \) is a one-dimensional mollifier.
Proposition 6. (Positivity of paraproduct) There is a constant $C > 0$ such that for $a \in LL([0, T_0] \times \mathbb{R}^n)$ with $\min a = \delta > 0$, and $u \in H^0$,
\[
Re \sum_{k \geq 0} (\tilde{T}_a \Delta_k u, \Delta_k u) \geq \frac{\delta}{2} ||u||_{H^0}^2, \tag{44}
\]
where the number $\nu$ chosen in (43) satisfies $\nu 2^{-\nu} \leq C\delta/|a|_{LL}$.

Proof. Note that
\[
a \Delta_k u - \tilde{T}_a \Delta_k u = (a - S_\nu a) S_{\nu + N-1} (\Delta_k u) + \sum_{m \geq \nu \atop |k-m| \leq N_0} (a - S_m a_m) \Delta_m + N (\Delta_k u)
\]
By (1.24) of [6], one has,
\[
||a - S_m a||_{L^\infty} \leq ||a - S_m a||_{L^\infty} + ||S_m(a - a_m)||_{L^\infty} \leq C|a|_{LL}(1 + m)2^{-m},
\]
which implies
\[
||a \Delta_k u - \tilde{T}_a \Delta_k u||_{L^2} \leq C|a|_{LL} \left( (1 + \nu)2^{-\nu} ||\Delta_k u||_{L^2} + \sum_{m \geq \nu \atop |k-m| \leq N_0} (1+m)2^{-m}||\Delta_k u||_{L^2} \right)
\]
\[
\leq C|a|_{LL}(1 + \nu)2^{-\nu} ||\Delta_k u||_{L^2}.
\]
Thus,
\[
\sum_{k \geq 0} ||a \Delta_k u - \tilde{T}_a \Delta_k u||_{L^2}^2 \leq C|a|_{LL}^2 (1 + \nu)^2 2^{2\nu} ||u||_{H^0}^2
\]
and
\[
Re \sum_{k \geq 0} (\tilde{T}_a \Delta_k u, \Delta_k u) \geq \sum_{k \geq 0} ||a \Delta_k u - \tilde{T}_a \Delta_k u, \Delta_k u||_{L^2} - \sum_{k \geq 0} ||(\tilde{T}_a \Delta_k u - a \Delta_k u, \Delta_k u)||_{L^2} \geq (\delta - C|a|_{LL}(1 + \nu)2^{-\nu}) ||u||_{H^0}^2.
\]
By the assumption on the number $\nu$, we finish the proof of (44). \qed

Proposition 7. Suppose $a \in L^\infty \cap LL([0, T_0] \times \mathbb{R}^n)$, then for $u \in H^{1 + \frac{1}{2} \log}$ and $t \in [0, T_0]$, we have
\[
\sum_{k \geq 0} ||\Lambda^{-\frac{1}{2}} \partial_j(T_a - \tilde{T}_a) \partial_l \Delta_k u||_{L^2}^2 \leq C||a||_{LL}(1 + \nu)^2 ||u||_{H^{1 + \frac{1}{2} \log}}^2. \tag{45}
\]

Proof. By (42) and Definition 2.4, one has
\[
\Lambda^{-\frac{1}{2}} \partial_j(T_a - \tilde{T}_a) \partial_l \Delta_k u = \Lambda^{-\frac{1}{2}} \partial_j(T_a - T_a) \partial_l \Delta_k u + \Lambda^{-\frac{1}{2}} \partial_j(T_a - \tilde{T}_a) \partial_l \Delta_k u
\]
\[
= \sum_{m=0}^{\nu} \sum_{i=\max(0, m-N+1)}^{\nu} \left\{ \Lambda^{-\frac{1}{2}} \partial_j(\Delta_i a(\Delta_m \partial_l \Delta_k u)) \right\}
\]
\[
+ \sum_{m \geq \nu} \left\{ \Lambda^{-\frac{1}{2}} \partial_j(S_\nu(a-a_\nu) S_{\nu + N-1} \partial_l \Delta_k u) \right\}
\]
\[
+ \sum_{m \geq \nu} \left\{ \Lambda^{-\frac{1}{2}} \partial_j(S_m(a-a_m) \Delta_{m+N} \partial_l \Delta_k u) \right\}.
\]
Since the spectra of $\Delta_s a \Delta_m u$ and $S_{\nu} a S_{\nu+N-1} u$ are contained in a ball, and the spectrum of $S_{m} a \Delta_{m+N} u$ lies in an annulus, one has

$$
\left\| \sum_{m=0}^{\nu+N-1} \sum_{i=\max\{0, m-N+1\}}^{\nu} \Lambda^{-\frac{i}{2}} \partial_j (\Delta_i a (\Delta_m \partial_i \Delta_k u)) \right\|_{L^2} \leq C(\|a\|_{LL} + \|a\|_{L^\infty})(1 + \nu)^2 \|\Delta_k u\|_{L^2}.
$$

(46)

By (1.24) of [6], $\|a - a_m\|_{L^\infty} \leq C(1 + m)2^{-m}\|a\|_{LL}$, we arrive at

$$
\left\| \Lambda^{-\frac{i}{2}} \partial_j (S_m (a - a_v) S_{\nu+N-1} \partial_i \Delta_k u) \right\|_{L^2} \leq C\|a\|_{LL} (1 + \nu)^2 \|\Delta_k u\|_{L^2}.
$$

(47)

Similiarly,

$$
\left\| \sum_{m \geq \nu} \Lambda^{-\frac{i}{2}} \partial_j (S_m (a - a_m) \Delta_{m+N} \partial_i \Delta_k u) \right\|_{L^2} \leq C\|a\|_{LL} (1 + k)^{\frac{3}{2}} \|\Delta_k u\|_{L^2}.
$$

(48)

This, together with (46) and (47), yields (45).

Proposition 8. Suppose $a \in L^\infty \cap LL([0, T_0] \times \mathbb{R}^n)$, then for $u \in H^{1+\frac{1}{2}\log}$ and $t \in [0, T_0]$, we have

$$
\left[ \sum_{k \geq 0} \left\| \Lambda^{-\frac{i}{2}} \partial_j (\tilde{T}_{a} - (\tilde{T}_{a})^*) \partial_i \Delta_k u \right\|_{L^2} \right]^\frac{2}{3} \leq C\|a\|_{LL} (1 + \nu)^2 \|u\|_{H^{1+\frac{1}{2}\log}},
$$

(49)

$$
\left[ \sum_{k \geq 0} \left\| \Lambda^{-\frac{i}{2}} [\tilde{T}_{a}, \partial_i] \Delta_k u \right\|_{L^2} \right]^\frac{2}{3} \leq C\|a\|_{LL} \|u\|_{H^{1+\frac{1}{2}\log}}.
$$

(50)

Proof. Note that,

$$
\Lambda^{-\frac{i}{2}} \partial_j (\tilde{T}_{a} - (\tilde{T}_{a})^*) \partial_i \Delta_k u = \Lambda^{-\frac{i}{2}} \partial_j (\tilde{T}_{a} - \tilde{T}_{a}) \partial_i \Delta_k u + \Lambda^{-\frac{i}{2}} \partial_j (\tilde{T}_{a} - \tilde{T}_{a})^* \partial_i \Delta_k u
$$

$$
\quad + \Lambda^{-\frac{i}{2}} \partial_j (\tilde{T}_{a}^* - (\tilde{T}_{a})^*) \partial_i \Delta_k u
$$

(51)

By (46)-(48),

$$
\left\| I \right\|_{L^2} \leq C(\|a\|_{LL} + \|a\|_{L^\infty})(1 + \nu)^2 (1 + k)^{\frac{3}{2}} \|\Delta_k u\|_{L^2}.
$$

(52)

Since $F^{-1}(\chi(2^{-k} \xi))$ and $a$ are real valued, we have

$$
II = \Lambda^{-\frac{i}{2}} \partial_j [S_{\nu} a, S_{\nu+N-1} S_{\nu-N}] \Phi_{\nu} \partial_i \Delta_k u
$$

$$
+ \sum_{m \geq \nu} \Lambda^{-\frac{i}{2}} \partial_j [S_m a, \Delta_{m+N}] \Phi_m \partial_i \Delta_k u,
$$

$$
III = \Lambda^{-\frac{i}{2}} [S_{\nu+N-1} S_{\nu}(a - a_{\nu}) \Psi_{\nu} \partial_i \Delta_k u]
$$

$$
+ \sum_{m \geq \nu} \Lambda^{-\frac{i}{2}} \partial_j S_{\nu+N-1} (S_m(a - a_m) \Phi_m \partial_i \Delta_k u),
$$

where $\Psi_{\nu} := \psi(2^{-\nu} D)$, $\Phi_m := \phi(2^{-m} D)$, $\psi \in C_0^\infty$ is supported in some fixed ball, and $\phi \in C_0^\infty$ in an annulus. It follows from Proposition 1 and Lemma 2.3 that

$$
\|II\|_{L^2}, \|III\|_{L^2} \leq C\|a\|_{LL}(1 + \nu)^2 \|\Delta_k u\|_{L^2} + C\|a\|_{LL} (1 + k)^{\frac{3}{2}} \|\Delta_k u\|_{L^2}.
$$

(53)
Substituting (52) and (53) into (51) yields (49). Since \( a_m \) is smooth in \( t \), direct computation yields
\[
\Lambda^{-\frac{1}{2}}[\overline{t}_a, \partial_t] \partial_t \Delta_k u = \Lambda^{-\frac{1}{2}} \left( S_{\nu}(\partial_t a_{\nu}) \Delta_{\nu+N-1}(\partial_t \Delta_k u) \right) + \sum_{m \geq \nu} \Lambda^{-\frac{1}{2}} \left( S_m(\partial_t a_m) \Delta_{m+N}(\partial_t \Delta_k u) \right).
\]
It follows from (1.24) of [6] that
\[
|\partial_t a_m| \leq C(1 + m)|a|_{LL}.
\]
Thus,
\[
||\Lambda^{-\frac{1}{2}}[\overline{t}_a, \partial_t] \partial_t \Delta_k u||_{L^2} \leq C|a|_{LL}(1 + k)^{\frac{1}{2}}||\Delta_k u||_{L^2},
\]
which yields the proof of (50). \( \square \)

**Proposition 9.** Suppose \( a \in L^\infty \cap LL([0, T_0] \times \mathbb{R}^n) \), then for \( u \in H^1 \) and \( t \in [0, T_0] \), we have
\[
\left[ \sum_{k \geq 0} ||\Lambda^\frac{1}{2} |\overline{t}_a| \partial_t \Delta_k u||_{L^2} \right]^\frac{1}{2} \leq C((1 + \nu)||a||_{L^\infty} + \frac{(1 + \nu)^2}{2\nu}|a|_{LL})||u||_{H^1}, \quad (54)
\]
\[
\left[ \sum_{k \geq 0} |||\Lambda^\frac{1}{2} |\overline{t}_a| \partial_t \Delta_k u||_{L^2} \right]^\frac{1}{2} \leq C(1 + \nu)||a||_{L^\infty} + \frac{(1 + \nu)^2}{2\nu}|a|_{LL})||u||_{H^1}. \quad (55)
\]
**Proof.** We only need to clarify (54) since (55) is analogous. Thanks to the spectral localization, direct computation yields
\[
\Lambda^\frac{1}{2} |\overline{t}_a| \partial_t \Delta_k u = \Lambda^\frac{1}{2} \left[ \overline{t}_a \psi_{\nu}, S_{\nu}a_{\nu} \right] S_{\nu+N-1}(\partial_t \Delta_k u) + \sum_{m \geq \nu} \Lambda^\frac{1}{2} \left[ \overline{t}_a \Phi_m, S_m a_m \right] \Delta_{m+N}(\partial_t \Delta_k u),
\]
(66)
where \( \psi_{\nu} =: \psi(2^{-\nu} D) \), \( \Phi_m =: \phi(2^{-m} D) \), \( \psi \in C_0^\infty \) is supported in a suitable fixed ball, and \( \phi \in C_0^\infty \) in an annulus. For the low-frequency term in (66), one has
\[
||\Lambda^\frac{1}{2} |\overline{t}_a \psi_{\nu}, S_{\nu}a_{\nu} | S_{\nu+N-1}(\partial_t \Delta_k u)||_{L^2} \leq ||\Lambda \psi_{\nu} | S_{\nu}a_{\nu} | S_{\nu+N-1}(\partial_t \Delta_k u)||_{L^2} + ||\Lambda^\frac{1}{2} |S_{\nu}a_{\nu} (\overline{t}_a \psi_{\nu} S_{\nu+N-1}(\partial_t \Delta_k u))||_{L^2} \leq C(1 + \nu)||a||_{L^\infty} 2^\nu \Delta_k u ||_{L^2}. \quad (57)
\]
In addition, by Lemma 2.3 and Proposition 1, we have
\[
||\Lambda^\frac{1}{2} |\overline{t}_a \Phi_m, S_m a_m | \Delta_{m+N}(\partial_t \Delta_k u)||_{L^2} \leq C|a|_{LL} (1 + m)^2 \Delta_k u ||_{L^2}. \quad (58)
\]
Since
\[
\sum_{m \geq \nu} (1 + m)^2 2^{-k} \leq C(1 + \nu)^2 2^{-\nu}, \quad (59)
\]
it follows from (66)-(59) that the proof of (54) is finished. \( \square \)
3. A-priori estimate. It is easy to know that the equation in (1) is equivalent to the following system of $g \equiv 0$

$$
\begin{align*}
L(u, v) &= \partial_t v - \sum_{j, k=1}^{n} \partial_j (\tilde{a}_{jk} \partial_k u) + \sum_{j=1}^{n} (\partial_j (\tilde{a}_j v + \tilde{c}_j u) + \tilde{b}_j \partial_j u) + \tilde{b}_0 v + \tilde{d} u, \\
v &= Xu + c_0 u + a_0 g,
\end{align*}
$$

(60)

where

$$
\tilde{a}_j = \frac{a_j}{a_0}, \quad \tilde{b}_j = \frac{b_j}{a_0}, \quad \tilde{a}_{jk} = a_{jk}, \quad \tilde{b}_j = b_j - \tilde{b}_0 a_j, \quad \tilde{c}_j = c_j - \tilde{a}_j c_0, \quad \tilde{d} = d - \tilde{b}_0 c_0.
$$

In this and the next section, we set

$$
T^* = \begin{cases}
\min (T_0, \frac{\alpha - \theta}{\lambda}), & \theta \in (1 - \alpha, \alpha), \beta > \frac{1}{2} \\
\min (T_0, \frac{\theta' - \theta}{\lambda}), & \theta \in [1 - \alpha, \theta'), \beta < -\frac{1}{2} \\
\min (T_0, \frac{\theta' - \theta}{\lambda}), & \theta \in (1 - \alpha, \theta'), \beta \in \mathbb{R}.
\end{cases}
$$

where $\theta' \in (\theta, \alpha)$ is a fixed number, and $\lambda > 0$ will be determined. We now establish some a-priori estimates for the smooth solution $(u, v)$ of system (60), which play a key role in proving Theorem 1.2 and Theorem 1.3.

**Theorem 3.1.** Let $\theta \in (1 - \alpha, \alpha)$ and $\beta > \frac{1}{2}$. Then there exists a positive number $\lambda_0$ depending on $\beta, \alpha, \delta_0, \delta_1$ and the LL-norms of $a_1 (0 \leq l \leq n)$, $a_{jk} (1 \leq j, k \leq n)$ in (2), such that for $\lambda > \lambda_0$, and for $u \in L^2(0, T^*; H^2)$ and $v \in L^2(0, T^*; H^1)$ with $\partial_t u \in L^2(0, T^*; H^1)$ and $\partial_v \in L^2(0, T^*; H^0)$, the following estimate holds that for $t \in [0, T^*]$,

$$
\sup_{\tau \in [0,t]} \|v\|^2_{H^{-\lambda \tau - \theta + \beta \log}} + \int_0^t \|v\|^2_{H^{-\lambda \tau - \theta + (\beta + \frac{1}{2}) \log}} d\tau \\
+ \sup_{\tau \in [0,t]} \|u\|^2_{H^{1 - \lambda \tau - \theta + \beta \log}} + \int_0^t \|u\|^2_{H^{1 - \lambda \tau - \theta + (\beta + \frac{1}{2}) \log}} d\tau \\
\leq \|v(0)\|^2_{L^{-\theta + \beta \log}} + \|u(0)\|^2_{H^{1 - \theta + \beta \log}} + \left( \int_0^t \|f_1\|_{H^{1 - \lambda \tau - \theta + (\beta + \frac{1}{2}) \log}} d\tau \right)^2 \\
+ \int_0^t \|f_2\|^2_{H^{-\lambda \tau - \theta + (\beta + \frac{1}{2}) \log}} d\tau + \int_0^t \|g\|^2_{H^{1 - \lambda \tau - \theta + (\beta + \frac{1}{2}) \log}} d\tau,
$$

(62)

where $L(u, v) = f = f_1 + f_2$, $f_1 \in \mathcal{C}_{-\theta + \beta \log, \lambda}(T^*)$ and $f_2 \in \mathcal{H}_{-\theta + (\beta + \frac{1}{2}) \log, \lambda}(T^*)$, $g \in \mathcal{H}_{1 - \theta + (\beta - \frac{1}{2}) \log, \lambda}(T^*)$.

**Remark 6.** If $\theta \in (1 - \alpha, \theta')$ and $\beta \in \mathbb{R}$ or $\theta \in [1 - \alpha, \theta')$ and $\beta < -\frac{1}{2}$, then all of the following calculations can go smoothly, thus (62) still holds in these two cases.

To prove Theorem 3.1, we require to define the following operators

$$
Q_1 = (2 - \Delta x)^{-s(t)/2}, \quad Q_2 = e^{-\gamma t} I^\beta Q,
$$

where $s(t) = \lambda t + \theta$.

Applying the operator $\Delta_k$ to $L(u, v)$ in (60), and performing the $L^2$ inner product with $Q^2_\gamma \Delta_k v$, we infer that

$$
2 \text{Re} \langle \Delta_k L(u, v), Q^2_\gamma \Delta_k v \rangle = 2 \text{Re} \langle \Delta_k f, Q^2_\gamma \Delta_k v \rangle.
$$

(63)
Namely,
\[
2\text{Re}(\partial_t \Delta_k v, Q_j^2 \Delta_k v) + \sum_{j=1}^{n} \langle \partial_j \Delta_k ( \tilde{a}_j v), Q_j^2 \Delta_k v \rangle + 2\text{Re}(\Delta_k (\tilde{b}_0 v), Q_j^2 \Delta_k v) = 2\text{Re} \sum_{j=1}^{n} \langle \Delta_k \partial_j (\tilde{a}_j v), Q_j^2 \Delta_k v \rangle - 2\text{Re} \sum_{j=1}^{n} \langle \Delta_k (\tilde{b}_j v), Q_j^2 \Delta_k v \rangle
- 2\text{Re} \sum_{j=1}^{n} \langle \Delta_k \partial_j (\tilde{c}_j u), Q_j^2 \Delta_k v \rangle - 2\text{Re} \langle \Delta_k (\tilde{d} u), Q_j^2 \Delta_k v \rangle + 2\text{Re} \langle \Delta_k f, Q_j^2 \Delta_k v \rangle.
\]

(64)

Next we treat each term in both sides of (64). This process will be divided into three subsections.

3.1. Treatment on the left hand side of (64). We now deal with each term in the left hand side (written as LHS) of (64).

I. Note that
\[
Q_j \partial_t (\Delta_k v) = \partial_t (\Delta_k v) + \gamma (\Delta_k v) + \lambda \Lambda (\Delta_k v),
\]
where \( v = Q_j v \). Then
\[
2\text{Re}(\partial_t \Delta_k v, Q_j^2 \Delta_k v) = \frac{d}{dt} ||\Delta_k v||_{L^2}^2 + 2\gamma ||\Delta_k v||_{L^2}^2 + 2\lambda ||\Lambda^{1/2} \Delta_k v||_{L^2}^2.
\]

(65)

II. By \( \tilde{a}_j v = T_{\tilde{a}_j} v + R_{\tilde{a}_j} v \), we get
\[
2\text{Re}(\partial_j \Delta_k (\tilde{a}_j v), Q_j^2 \Delta_k v)
= 2\text{Re}(\partial_j \Delta_k (T_{\tilde{a}_j} v), Q_j^2 \Delta_k v) + 2\text{Re}(\Lambda^{1/2} Q_j \partial_j \Delta_k (R_{\tilde{a}_j} v), \Lambda^{1/2} Q_j \Delta_k v)
= -E_{j, -1}^k.
\]

(66)

Note that
\[
2\text{Re}(\partial_j \Delta_k (T_{\tilde{a}_j} v), Q_j^2 \Delta_k v)
= 2\text{Re}(\partial_j \Delta_k (T_{\tilde{a}_j} v), \Delta_k v)
+ 2\text{Re}(\partial_j Q_j \Delta_k, T_{\tilde{a}_j} v, \Delta_k v)
= -2\text{Re}(\partial_j \Delta_k \partial_j v / \Lambda^{1/2} T_{\tilde{a}_j} v, \Delta_k v)
+ 2\text{Re}(\partial_j \Delta_k \partial_j v / \Lambda^{1/2} T_{\tilde{a}_j} v, \Delta_k v)
= -E_{j, -2}^k - E_{j, -3}^k - E_{j, -4}^k.
\]

Substituting this into (66) yields
\[
2\text{Re}(\partial_j \Delta_k (\tilde{a}_j v), Q_j^2 \Delta_k v) = - \sum_{i=-4}^{-1} E_{j,i}^k.
\]

(67)

III. Set
\[
-E_{-5}^k = 2\text{Re}(\Delta_k (\tilde{b}_0 v), Q_j^2 \Delta_k v).
\]

(68)

Substituting (65), (67) and (68) into (64) yields
\[
\text{LHS of (64)} = \frac{d}{dt} ||\Delta_k v||_{L^2}^2 + 2\gamma ||\Delta_k v||_{L^2}^2 + 2\lambda ||\Lambda^{1/2} \Delta_k v||_{L^2}^2 - \sum_{i=-5}^{-1} E_i^k,
\]

(69)
where $E^k = \sum_{j=1}^{n} E^k_{j,i}$ for $i = -1, \cdots, -4$.

3.2. Treatment on the right hand side of (64). This process is divided into two parts: Part I is devoted to treating the second-order term, Part II is to deal with the left terms, which are essentially remainders.

**Part I.** Note that $\tilde{a}_{j,l} \partial_{i} u = T_{\tilde{a}_{j,l}} \partial_{i} u + R_{\tilde{a}_{j,l}} \partial_{i} u$. Then

$$2Re \sum_{j,l=1}^{n} \langle \Delta_k \partial_{j}(\tilde{a}_{j,l} \partial_{i} u), Q_{k}^{2}\Delta_k v \rangle$$

$$= 2Re \sum_{j,l=1}^{n} \langle Q_{k}\Delta_k \partial_{j}(T_{\tilde{a}_{j,l}} \partial_{i} u), Q_{k}\Delta_k X u \rangle - 2Re \sum_{j,l=1}^{n} \langle Q_{k}\Delta_k (\tilde{a}_{j,l} \partial_{i} u), \partial_{j}Q_{k}\Delta_k (\tilde{c}_{0} u) \rangle$$

$$+ 2Re \sum_{j,l=1}^{n} \langle \partial_{j}\Lambda^{-\frac{1}{2}} Q_{k}\Delta_k (R_{\tilde{a}_{j,l}} \partial_{i} u), \Lambda^{\frac{1}{2}} Q_{k}\Delta_k X u \rangle$$

$$- 2Re \sum_{j,l=1}^{n} \langle \Lambda^{-\frac{1}{2}} Q_{k}\Delta_k (\tilde{a}_{j,l} \partial_{i} u), \Lambda^{\frac{1}{2}} \partial_{j}Q_{k}\Delta_k (a_{0} g) \rangle$$

$$=: I + E_1^k + E_2^k + E_3^k.$$  \hspace{1cm} (70)

Set $u_{\gamma} =: Q_{\gamma} u$ and

$$I = -2Re \sum_{j,l=1}^{n} \langle T_{\tilde{a}_{j,l}} \partial_{i} \Delta_k u_{\gamma}, \partial_{j}Q_{\gamma} \Delta_k X u \rangle + E_{3}^{k} =: II + E_{3}^{k}.$$ \hspace{1cm} (71)

where $E_{3}^{k} = 2Re \sum_{j,l=1}^{n} \langle \partial_{j} \Lambda^{-\frac{1}{2}} [Q_{k}\Delta_k, T_{\tilde{a}_{j,l}}] \partial_{i} u, \Lambda^{\frac{1}{2}} Q_{k}\Delta_k X u \rangle$.

By

$$X u = T_{a_{0}} \partial_{i} u + \sum_{i=1}^{n} T_{a_{i}} \partial_{i} u + R_{a_{0}} \partial_{i} u + \sum_{i=1}^{n} R_{a_{i}} \partial_{i} u,$$ \hspace{1cm} (72)

then substituting (72) into (71) yields

$$II = -2Re \sum_{j,l=1}^{n} \langle T_{\tilde{a}_{j,l}} \partial_{i} \Delta_k u_{\gamma}, \partial_{j}Q_{\gamma} \Delta_k (T_{a_{0}} \partial_{i} u) \rangle$$

$$- 2Re \sum_{i,j,l=1}^{n} \langle T_{\tilde{a}_{j,l}} \partial_{i} \Delta_k u_{\gamma}, \partial_{j}Q_{\gamma} \Delta_k (T_{a_{i}} \partial_{l} u) \rangle + E_{4}^{k}$$ \hspace{1cm} (73)

$$=: II_{1} + II_{2} + E_{4}^{k}.$$  

For the principal part $II_{1}$, by utilizing the composition and adjoint operator of para-product, one has

$$II_{1} = -2Re \sum_{j,l=1}^{n} \langle T_{a_{0}}^{*} T_{\tilde{a}_{j,l}} (\partial_{i} \Delta_k u_{\gamma}), \partial_{j}Q_{\gamma} \Delta_k \partial_{l} u \rangle$$

$$- 2Re \sum_{j,l=1}^{n} \langle \Lambda^{\frac{1}{2}} T_{\tilde{a}_{j,l}} (\partial_{i} \Delta_k u_{\gamma}), \Lambda^{-\frac{1}{2}} [\partial_{j}Q_{\gamma} \Delta_k, T_{a_{0}}] \partial_{l} u \rangle$$
\[-2Re \sum_{j,l=1}^{n} \langle T_{a_0} T_{\tilde{a}_{j,l}} \partial_t \Delta_k u_\gamma, \partial_j \Lambda_{\gamma} Q_{\gamma} \Delta_k \partial_l u \rangle + 2Re \sum_{j,l=1}^{n} \langle \partial_j \Lambda_{\gamma} T_{a_0} (T_{a_0} - T_{\tilde{a}_{j,l}}) \partial_t \Delta_k u_\gamma, \partial_j \partial_l Q_{\gamma} \Delta_k u_\gamma \rangle + E_k^6 \]

\[= -2Re \sum_{j,l=1}^{n} \langle T_{a_0 \tilde{a}_{j,l}} (\partial_t \Delta_k u_\gamma), \partial_j Q_{\gamma} \Delta_k \partial_l u \rangle + 2Re \sum_{j,l=1}^{n} \langle \partial_j \Lambda_{\gamma} T_{a_0} (T_{a_0} - T_{\tilde{a}_{j,l}}) \partial_t \Delta_k u_\gamma, \partial_j \Delta_k u_\gamma \rangle + E_k^6 + E_k^5 \]

\[=: III_1 + E_k^7 + E_k^6 + E_k^5. \quad (74) \]

Similarly to the analysis of $II_1$ in the above, by $\tilde{a}_{jk} = \tilde{a}_{kj}$, we have

\[II_2 = -2Re \sum_{i,j,l=1}^{n} \langle T_{a_0 \tilde{a}_{i,j,l}} (\partial_t \Delta_k u_\gamma), \partial_j \partial_l \Delta_k u_\gamma \rangle + E_k^6. \]

$II_2$ will be taken as a new error term. Indeed, direct computation yields

\[II_2 = Re \sum_{i,j,l=1}^{n} \langle \Lambda_{\gamma}^{-\frac{1}{2}} (T_{a_i \tilde{a}_{j,l}} - T^\ast_{a_i \tilde{a}_{j,l}}) (\partial_t \partial_l \Delta_k u_\gamma), \Lambda_{\gamma}^{-\frac{1}{2}} \partial_j \partial_l \Delta_k u_\gamma \rangle \]

\[+ Re \sum_{i,j,l=1}^{n} \langle \Lambda_{\gamma}^{-\frac{1}{2}} [\partial_t, T_{a_i \tilde{a}_{j,l}}] \partial_l \Delta_k u_\gamma, \Lambda_{\gamma}^{-\frac{1}{2}} \partial_j \partial_l \Delta_k u_\gamma \rangle + E_k^6 \]

\[=: E_{g,1}^k + E_{g,2}^k + E_k^6 \]

\[= E_k^6 + E_k^8. \]

To deal with $III_1$, we introduce the notation $b_{jl} = a_0 \tilde{a}_{jl}$. Note that the positivity result as in Proposition 6 does not hold for $\tilde{T}_{b_{jl}}$ due to the absence of some low-frequency terms in paraproduct. To overcome this difficulty and the low-regularity of $b_{jl}$ in $t$, we shall substitute $\tilde{T}_{b_{jl}}$ by $\tilde{T}_{b_{jl}}$. It follows from direct computation that

\[III_1 = -2Re \sum_{j,l=1}^{n} \langle (\tilde{T}_{b_{jl}} (\partial_t \Delta_k u_\gamma), \partial_j Q_{\gamma} \Delta_k \partial_l u) + E_k^1, \]

where $E_{10}^k = 2Re \sum_{j,l=1}^{n} \langle \partial_j \Lambda_{\gamma}^{-\frac{1}{2}} (\tilde{T}_{b_{jl}} - \tilde{T}_{b_{jl}}^\ast) \partial_t \Delta_k u_\gamma, \Lambda_{\gamma}^{-\frac{1}{2}} Q_{\gamma} \Delta_k \partial_l u \rangle$.

Note that $2 \tilde{T}_{b_{jl}} = (\tilde{T}_{b_{jl}} + \tilde{T}_{b_{jl}}^\ast) + (\tilde{T}_{b_{jl}} - \tilde{T}_{b_{jl}}^\ast)$ and $Q_{\gamma} \Delta_k \partial_l u = \partial_l (\Delta_k u_\gamma) + \gamma \Delta_k u_\gamma + \lambda \Delta_k u_\gamma$. Then we arrive at

\[III_1 = -Re \sum_{j,l=1}^{n} \langle (\tilde{T}_{b_{jl}} + \tilde{T}_{b_{jl}}^\ast) (\partial_t \Delta_k u_\gamma), \partial_j (\partial_j \Delta_k u_\gamma) \rangle \]

\[- \gamma Re \sum_{j,l=1}^{n} \langle (\tilde{T}_{b_{jl}} + \tilde{T}_{b_{jl}}^\ast) (\partial_t \Delta_k u_\gamma), \partial_j \Delta_k u_\gamma \rangle \]

\[- \lambda Re \sum_{j,l=1}^{n} \langle (\tilde{T}_{b_{jl}} + \tilde{T}_{b_{jl}}^\ast) (\partial_t \Delta_k u_\gamma), \partial_j \Delta_k u_\gamma \rangle \]
In terms of \(b_{ji} = b_{ij}\), then

\[
A_1 = -Re \sum_{j,l=1}^{n} \langle \bar{b}_{ji}, (\partial_j \Delta_k u_\gamma), (\partial_l \Delta_k u_\gamma) \rangle - Re \sum_{j,l=1}^{n} \langle \partial_i \bar{b}_{ji}, (\partial_j \Delta_k u_\gamma), (\partial_l \Delta_k u_\gamma) \rangle
\]

\[
- Re \sum_{j,l=1}^{n} \langle \Lambda^{-\frac{1}{2}} \bar{b}_{ji}, (\partial_i \Delta_k u_\gamma), \Lambda^{\frac{1}{2}} \partial_j \Delta_k u_\gamma \rangle
\]

\[
=: -\frac{d}{dt} Re \sum_{j,l=1}^{n} \langle \bar{b}_{ji}, (\partial_j \Delta_k u_\gamma), (\partial_l \Delta_k u_\gamma) \rangle + E_{12}^k
\]  

(77)

and

\[
A_3 = -2\lambda Re \sum_{j,l=1}^{n} \langle \bar{b}_{ji}, \Lambda^{\frac{1}{2}} (\partial_j \Delta_k u_\gamma), \Lambda^{\frac{1}{2}} \partial_j \Delta_k u_\gamma \rangle
\]

\[
- \lambda Re \sum_{j,l=1}^{n} \langle \Lambda^{\frac{1}{2}} \bar{b}_{ji}, (\partial_i \Delta_k u_\gamma), \partial_j \Delta_k u_\gamma \rangle
\]

\[
- \lambda Re \sum_{j,l=1}^{n} \langle \Lambda^{-\frac{1}{2}} \bar{b}_{ji}, (\partial_i \Delta_k u_\gamma), \partial_j \Delta_k u_\gamma \rangle
\]

\[
=: -2\lambda Re \sum_{j,l=1}^{n} \langle \bar{b}_{ji}, \Lambda^{\frac{1}{2}} (\partial_j \Delta_k u_\gamma), \Lambda^{\frac{1}{2}} \partial_j \Delta_k u_\gamma \rangle + E_{13}^k
\]  

(78)

and

\[
A_2 = -2\gamma Re \sum_{j,l=1}^{n} \langle \bar{b}_{ji}, (\partial_j \Delta_k u_\gamma), (\partial_l \Delta_k u_\gamma) \rangle.
\]  

(79)

Combining (71) and (73)-(79), we get

\[
2Re \sum_{j,l=1}^{n} \langle \Delta_k \partial_j (\bar{a}_{ji} \partial_l u), Q_\gamma^2 \Delta_k v \rangle
\]

\[
= -\frac{d}{dt} Re \sum_{j,l=1}^{n} \langle \bar{b}_{ji}, (\partial_j \Delta_k u_\gamma), (\partial_l \Delta_k u_\gamma) \rangle - 2\gamma Re \sum_{j,l=1}^{n} \langle \bar{b}_{ji}, (\partial_j \Delta_k u_\gamma), \partial_j \Delta_k u_\gamma \rangle - 2\lambda Re \sum_{j,l=1}^{n} \langle \bar{b}_{ji}, \Lambda^{\frac{1}{2}} (\partial_j \Delta_k u_\gamma), \Lambda^{\frac{1}{2}} \partial_j \Delta_k u_\gamma \rangle + \sum_{i=0}^{13} E_{i}^k.
\]  

(80)

\textbf{Part II.} Note that the rest terms are just remainders in the right hand side of (64). In addition, we set

\[
-2Re \sum_{j=1}^{n} \langle \Delta_k (\bar{b}_j \partial_j u), Q_\gamma^2 \Delta_k v \rangle = -2Re \sum_{j=1}^{n} \langle Q_\gamma \Delta_k (\bar{b}_j \partial_j u), \Delta_k v_\gamma \rangle =: E_{14}^k,
\]

\[
-2Re \sum_{j=1}^{n} \langle \partial_j \Delta_k (\bar{c}_j u), Q_\gamma^2 \Delta_k v \rangle = -2Re \sum_{j=1}^{n} \langle Q_\gamma \partial_j \Delta_k (\bar{c}_j u), \Delta_k v_\gamma \rangle =: E_{15}^k,
\]
\[-2\text{Re}(\Delta_k(\dot{d}u), Q \Delta_k v) = -2\text{Re}(Q \Delta_k(\dot{d}u), \Delta_k v\gamma) =: E_{16}^k.\]

Thus, the right hand side of (64) reads

\[
\text{RHS of (64)} = -\frac{d}{dt} Re \sum_{j,l=1}^{n} \langle \tilde{T}_{b_{j,l}}(\partial_t \Delta_k u\gamma), \partial_j \Delta_k u\gamma \rangle + 2\text{Re}(Q \Delta_k f, \Delta_k v\gamma) \]

\[- 2\gamma Re \sum_{j,l=1}^{n} \langle \tilde{T}_{b_{j,l}}(\partial_t \Delta_k u\gamma), \partial_j \Delta_k u\gamma \rangle - 2\lambda Re \sum_{j,l=1}^{n} \langle \tilde{T}_{b_{j,l}} \Lambda^{\frac{1}{2}}(\partial_t \Delta_k u\gamma), \Lambda^{\frac{1}{2}} \partial_j \Delta_k u\gamma \rangle.\]

This, together with (69), yields

\[
\frac{d}{dt} ||\Delta_k v\gamma||_{L^2}^2 + 2\gamma ||\Delta_k v\gamma||_{L^2}^2 + 2\lambda ||\Lambda^{1/2} \Delta_k v\gamma||_{L^2}^2 \]

\[+ \frac{d}{dt} Re \sum_{j,l=1}^{n} \langle \tilde{T}_{b_{j,l}}(\partial_t \Delta_k u\gamma), \partial_j \Delta_k u\gamma \rangle + 2\gamma Re \sum_{j,l=1}^{n} \langle \tilde{T}_{b_{j,l}}(\partial_t \Delta_k u\gamma), \partial_j \Delta_k u\gamma \rangle \]

\[+ 2\lambda Re \sum_{j,l=1}^{n} \langle \tilde{T}_{b_{j,l}} \Lambda^{\frac{1}{2}}(\partial_t \Delta_k u\gamma), \Lambda^{\frac{1}{2}} \partial_j \Delta_k u\gamma \rangle \]

\[= 2\text{Re}(Q \Delta_k f, \Delta_k v\gamma) + \sum_{i=-5}^{16} E_i^k. \]  

(81)

Summing up over \(k\) in (81), we arrive at

\[
\frac{d}{dt} \sum_{k \geq 0} ||\Delta_k v\gamma||_{L^2}^2 + \frac{d}{dt} Re \sum_{j,l=1}^{n} \sum_{k \geq 0} \langle \tilde{T}_{b_{j,l}}(\partial_t \Delta_k u\gamma), \partial_j \Delta_k u\gamma \rangle + 2\gamma \sum_{k \geq 0} ||\Delta_k v\gamma||_{L^2}^2 \]

\[+ 2\lambda \sum_{k \geq 0} ||\Lambda^{1/2} \Delta_k v\gamma||_{L^2}^2 + 2\gamma Re \sum_{j,l=1}^{n} \sum_{k \geq 0} \langle \tilde{T}_{b_{j,l}}(\partial_t \Delta_k u\gamma), \partial_j \Delta_k u\gamma \rangle \]

\[+ 2\lambda Re \sum_{j,l=1}^{n} \sum_{k \geq 0} \langle \tilde{T}_{b_{j,l}} \Lambda^{\frac{1}{2}}(\partial_t \Delta_k u\gamma), \Lambda^{\frac{1}{2}} \partial_j \Delta_k u\gamma \rangle \]

\[= 2\text{Re} \sum_{k \geq 0} (Q \Delta_k f, \Delta_k v\gamma) + \sum_{i=-5}^{16} E_i^k. \]  

(82)

3.3. Estimates of remainders \(E_i = \sum_{k \geq 0} E_i^k\). Set \(E_i = \sum_{k \geq 0} E_i^k\) for \(-5 \leq i \leq 16\).

Next we treat each \(E_i\) respectively. For the simplicity of notations, we denote the constants which depend on \(\beta\) by \(D_1\), and \(D_2\) stands for the constants depending both on \(\alpha\) and \(\beta\).

I. Estimates of \(E_{-5}, E_{-1} - E_2, E_4, E_{14} - E_{16}\)

Since \(1 - \sigma(t) \in [1 - \alpha, \alpha]\), by (23) we immediately have

\[
|E_{-1}| \leq Ce^{-2\gamma t} \sum_{j=1}^{n} ||R_{a_j} v||_{H^{1-\sigma(t)+((\beta-\frac{1}{2})+\frac{1}{2})log}} ||v||_{H^{1-\sigma(t)+((\beta+\frac{1}{2})+\frac{1}{2})log}} 
\]

\[\leq D_2 e^{-2\gamma t} \sum_{j=1}^{n} ||\tilde{a}_j||_{L^2} ||v||_{H^{1-\sigma(t)+((\beta+\frac{1}{2})+\frac{1}{2})log}}. \]  

(83)
Analogously,
\[
|E_0| \leq D_2 e^{-2\gamma t} \sum_{j,l=1}^{n} |\bar{a}_{jl}| LL \|u\|_{H^{1-s(t)+(\beta+\frac{1}{2})\log}} \|Xu\|_{H^{1-s(t)+(\beta+\frac{1}{2})\log}},
\]
(84)
and
\[
|E_4| \leq D_2 |a_0| LL e^{-2\gamma t} \sum_{j,l=1}^{n} |\bar{a}_{jl}| L_{\infty} \|u\|_{H^{1-s(t)+(\beta+\frac{1}{2})\log}} \|\partial_t u\|_{H^{1-s(t)+(\beta+\frac{1}{2})\log}}
+ D_2 e^{-2\gamma t} \sum_{i,j,l=1}^{n} |a_i| LL \|\bar{a}_{jl}\|_{L_{\infty}} \|u\|_{H^{1-s(t)+(\beta+\frac{1}{2})\log}}^2.
\]
(85)
By (24), we have
\[
|E_2| \leq D_2 e^{-2\gamma t} |a_0| LL \sum_{j,l=1}^{n} |\bar{a}_{jl}| LL \|u\|_{H^{1-s(t)+(\beta+\frac{1}{2})\log}} \|\theta\|_{H^{1-s(t)+(\beta-\frac{1}{2})\log}}.
\]
(86)
By (24) and (25), we then arrive at
\[
|E_1| \leq D_2 e^{-2\gamma t} \|\tilde{c}_0\|_{C^0} \sum_{j,l=1}^{n} |\bar{a}_{jl}| LL \|u\|_{H^{1-s(t)+\beta\log}}^2,
\]
(87)
\[
|E_{15}| \leq D_2 e^{-2\gamma t} \sum_{j=1}^{n} |\tilde{c}_j|_{C^0} \|u\|_{H^{1-s(t)+\beta\log}} \|v\|_{H^{1-s(t)+\beta\log}}.
\]
(88)
Note that \(-s(t) \in [-\alpha, \alpha-1]\), thus by (26), for \(\beta > \frac{1}{2}\),
\[
|E_{-5}| \leq D_1 e^{-2\gamma t} |\tilde{c}_0|_{C^0} \|v\|_{H^{1-s(t)+\beta\log}} \|v\|_{H^{1-s(t)+(\beta+\frac{1}{2})\log}}
+ D_1 e^{-2\gamma t} \sum_{j=1}^{n} |\tilde{c}_j|_{C^0} \|u\|_{H^{1-s(t)+(\beta+\frac{1}{2})\log}} \|v\|_{H^{1-s(t)+(\beta\log)}}.
\]
(89)
Here we point out that for \(1-\alpha \leq \theta < \theta' < \alpha\) and \(\beta < -\frac{1}{2}\), or \(1-\alpha < \theta < \theta' < \alpha\) and \(\beta \in \mathbb{R}\), by (5) and (3) of Proposition 2 respectively, (87) and (88) still hold. On the other hand, for the zero-th order term \(E_{16}\), one has
\[
|E_{16}| \leq C e^{-2\gamma t} \|\tilde{d}\|_{L_{\infty}} \|u\|_{H^{1-s(t)+\beta\log}} \|v\|_{H^{1-s(t)+\beta\log}}.
\]
II. Estimates on the error terms \(E_{-3} - E_{-4}, E_{3}, E_5, E_{9.2},\) and \(E_{12} - E_{13}\) including commutators
By Proposition 3 and Corollary 1, we immediately get that
\[
|E_{-3}| + |E_{-4}| \leq C e^{-2\gamma t} \sum_{j=1}^{n} |\bar{a}_{jl}| LL \|v\|_{H^{1-s(t)+(\beta+\frac{1}{2})\log}}^2,
\]
\[
|E_3| \leq C e^{-2\gamma t} \sum_{j,l=1}^{n} |\bar{a}_{jl}| LL \|u\|_{H^{1-s(t)+(\beta+\frac{1}{2})\log}} \|Xu\|_{H^{1-s(t)+(\beta+\frac{1}{2})\log}}
+ C e^{-2\gamma t} |a_0| LL \sum_{j,l=1}^{n} |\bar{a}_{jl}| L_{\infty} \|u\|_{H^{1-s(t)+(\beta+\frac{1}{2})\log}} \|\partial_t u\|_{H^{1-s(t)+(\beta+\frac{1}{2})\log}}.
\]
(90)
\[
|E_5| \leq C e^{-2\gamma t} |a_0| LL \sum_{j,l=1}^{n} |\bar{a}_{jl}| L_{\infty} \|u\|_{H^{1-s(t)+(\beta+\frac{1}{2})\log}} \|\partial_t u\|_{H^{1-s(t)+(\beta+\frac{1}{2})\log}},
\]
\[
|E_{9.2}| = \sum_{k \geq 0} |E_{9.2}^k| \leq C e^{-2\gamma t} \sum_{i,j,l=1}^{n} |a_i| \bar{a}_{jl} |LL| \|u\|_{H^{1-s(t)+(\beta+\frac{1}{2})\log}}^2.
\]
Due to (50) and (54)-(55) that

\[ |E_{12}| \leq Ce^{-2\gamma t} \sum_{j,l=1}^{n} |b_{jl}| L_{L} ||u||_{H^{1-s(t)+\beta + \frac{1}{2}}}^{2}, \]

\[ |E_{13}| \leq CAe^{-2\gamma t} \sum_{j,l=1}^{n} ((1 + \nu)||b_{jl}||_{L_{\infty}} + \frac{(1 + \nu)^{2}}{2\nu}|b_{jl}|L_{L}) ||u||_{H^{1-s(t)+\beta \log}}^{2}. \]

III. Estimates of the error terms $E_{-2}, E_{6} - E_{8}, E_{9,1}$ and $E_{10} - E_{11}$

By (37), (39)-(40), (45) and (49) respectively, we arrive at

\[ |E_{-2}| \leq Ce^{-2\gamma t} \sum_{j=1}^{n} |\tilde{\alpha}_{j}| L_{L} ||v||_{H^{1-s(t)+\beta + \frac{1}{2}}}^{2}, \]

\[ |E_{6}| \leq Ce^{-2\gamma t} |a_{0}| L_{L} \sum_{j,l=1}^{n} ||\tilde{\alpha}_{jl}||_{L_{\infty}} ||u||_{H^{1-s(t)+\beta + \frac{1}{2}}} ||\partial_{t} u||_{H^{1-s(t)+\beta + \frac{1}{2}}}^{2}, \]

\[ |E_{7}| \leq Ce^{-2\gamma t} \sum_{j,l=1}^{n} ||a_{0} \tilde{\alpha}_{jl}||_{L_{L}} ||u||_{H^{1-s(t)+\beta + \frac{1}{2}}} ||\partial_{t} u||_{H^{1-s(t)+\beta + \frac{1}{2}}}^{2}, \]

\[ |E_{8}| \leq Ce^{-2\gamma t} \sum_{i,j,l=1}^{n} ||a_{i} \tilde{\alpha}_{jl}||_{L_{L}} ||u||_{H^{1-s(t)+\beta + \frac{1}{2}}}^{2}, \]

\[ |E_{10}| \leq Ce^{-2\gamma t} (1 + \nu)^{2} \sum_{j,l=1}^{n} ||b_{jl}|| L_{L} ||u||_{H^{1-s(t)+\beta + \frac{1}{2}}} ||\partial_{t} u||_{H^{1-s(t)+\beta + \frac{1}{2}}}^{2}, \]

\[ |E_{11}| \leq Ce^{-2\gamma t} (1 + \nu)^{2} \sum_{j,l=1}^{n} ||b_{jl}|| L_{L} ||u||_{H^{1-s(t)+\beta + \frac{1}{2}}} ||\partial_{t} u||_{H^{1-s(t)+\beta + \frac{1}{2}}}^{2}. \]

In addition, by (38), one has

\[ |E_{9,1}| = \sum_{k \geq 0} E_{9,1}^{k} \leq Ce^{-2\gamma t} \sum_{i,j,l=1}^{n} ||a_{i} \tilde{\alpha}_{jl}|| L_{L} ||u||_{H^{1-s(t)+\beta + \frac{1}{2}}}^{2}. \]

IV. Estimates of $Xu$ and $\partial_{t} u$

By $Xu = v - c_{0}u - a_{0}g$ and Proposition 2, for $\beta > 0$, we have

\[ ||Xu||_{H^{-s(t)+(\beta + \frac{1}{2})\log}} \leq ||v||_{H^{-s(t)+(\beta + \frac{1}{2})\log}} + D_{1} ||c_{0}||_{C^{\infty}} ||u||_{H^{-s(t)+(\beta + \frac{1}{2})\log}} + D_{2} ||a_{0}||_{L_{L}} ||g||_{H^{-s(t)+(\beta + \frac{1}{2})\log}} \]

\[ \leq ||v||_{H^{-s(t)+(\beta + \frac{1}{2})\log}} + D_{1} ||c_{0}||_{C^{\infty}} ||u||_{H^{-s(t)+\beta \log}} + D_{2} ||a_{0}||_{L_{L}} ||g||_{H^{-s(t)+(\beta + \frac{1}{2})\log}}. \]

Due to $\partial_{t} u = \frac{1}{a_{0}}v - \sum_{j=1}^{n} \tilde{a}_{j} \partial_{j} u - c_{0}u - g$, then it follows that

\[ ||\partial_{t} u||_{H^{-s(t)+(\beta + \frac{1}{2})\log}} \leq D_{2} \left\{ \frac{1}{a_{0}} ||L_{L}|| v ||H^{-s(t)+(\beta + \frac{1}{2})\log} \right. \]

\[ + \sum_{j=1}^{n} ||\tilde{a}_{j}|| L_{L} ||u||_{H^{1-s(t)+(\beta + \frac{1}{2})\log}} + ||g||_{H^{1-s(t)+(\beta - \frac{1}{2})\log}} \right\}. \]
We point out that for $1 - \alpha \leq \theta < \theta'$ and $\beta < -\frac{1}{2}$, or $1 - \alpha < \theta < \theta' < \alpha$ and $\beta \in \mathbb{R}$, by (5) and (3) of Proposition 2 respectively; (94) and (95) still hold. Combining the estimates in (83)-(95), then integrating (82) from 0 to $t$, and using Proposition 6 and Hölder’s inequality, we then arrive at

$$\frac{1}{2} \sup_{\tau \in [0,t]} e^{2\gamma \tau} |v|^2_{H^1 - \theta + \beta \log} + \frac{\delta_0}{2} \sup_{\tau \in [0,t]} e^{2\gamma \tau} |u|^2_{H^1 - \theta + \beta \log}$$

$$+ 2\gamma \int_0^t e^{2\gamma \tau} |v|^2_{H^1 - \theta + \beta \log} d\tau + 2\lambda \int_0^t e^{2\gamma \tau} |\|v\|_{H^1 - \theta + \beta \log}|^2 d\tau$$

$$+ \delta_0 \delta_1 (\gamma \int_0^t e^{2\gamma \tau} \|u\|^2_{H^1 - \theta + \beta \log} d\tau + \lambda \int_0^t e^{2\gamma \tau} \|u\|^2_{H^1 - \theta + \beta \log} d\tau)$$

$$\leq \|v(0)\|_{H^1 - \theta + \beta \log} + C_1 \|u(0)\|_{H^1 - \theta + \beta \log} + C_2 \int_0^t e^{2\gamma \tau} \|v\|^2_{H^1 - \theta + \beta \log} d\tau$$

$$+ C_1 \int_0^t e^{2\gamma \tau} \|u\|^2_{H^1 - \theta + \beta \log} d\tau + \lambda C_2 \int_0^t e^{2\gamma \tau} |\|u\|_{H^1 - \theta + \beta \log}|^2 d\tau$$

$$+ C_1 \int_0^t e^{2\gamma \tau} |\|u\|_{H^1 - \theta + \beta \log}|^2 d\tau + C_1 \int_0^t e^{2\gamma \tau} |g|^2_{H^1 - \theta + \beta \log} d\tau$$

$$+ 2(\int_0^t e^{2\gamma \tau} \|f_1\|_{H^1 - \theta + \beta \log} d\tau)^2 + \int_0^t e^{2\gamma \tau} \|f_2\|_{H^1 - \theta + \beta \log} d\tau$$

where $C_1 > 0$ depends on $\beta, \alpha, \delta_0$, and the $LL$-norms of $a_l \ (0 \leq l \leq n)$, $a_{jk} \ (1 \leq j, k \leq n)$ in (2), and $C_2 > 0$ depends on $\beta, \theta, \delta_0$, the $C^\alpha$-norms of $b_j, a_j \ (0 \leq j \leq n)$ and $|\|d\|_{L^\infty}$ in (2). Choosing $\lambda > C_1 \max\{\frac{1}{\delta_0 \delta_1}, \frac{1}{2}\}$ and $\gamma > C_2 \max\{\frac{\lambda}{\delta_0 \delta_1}, \frac{1}{2}\}$, we then know that the integrations involving $v$ and $u$ in the right hand side of (96) can be absorbed by corresponding terms in the left hand. Subsequently multiplying the inequality by $e^{2\gamma t}$ and noting $\frac{2}{X} - 2\gamma t \geq 0$ for $t \in [0, T^*)$, we can eliminate the factor $e^{2\gamma t}$ in the related integrands of (96). Therefore, the proof of Theorem 3.1 is completed.

4. Proof of Theorem 1.2 and Theorem 1.3. In this section, we start to prove Theorem 1.2. In addition, the proof of Theorem 1.3 is similar to that for Theorem 1.2 as long as one notices Remark 6. For this purpose, we rewrite equation (1) as system (60) (with $q \equiv 0$). Let $u$ be a solution of problem (1), and $u, u_0, u_1, f$ satisfy the following regularity assumptions:

$$u \in H_{-\theta + (\beta + \frac{1}{2}) \log, \lambda}(T^*), \quad \partial_t u \in H_{-\theta + (\beta + \frac{1}{2}) \log, \lambda}(T^*),$$

$$u_0 \in H_{1-\theta + \beta \log}, \quad u_1 \in H_{-\theta + \beta \log}, \quad f \in \mathcal{L}_{-\theta + \beta \log, \lambda}(T^*).$$

By $v = Xu + c_0u$ and Proposition 2, one can immediately see that

$$v \in H_{-\theta + (\beta + \frac{1}{2}) \log, \lambda}(T^*). \quad \quad (97)$$

Due to $\partial_t v = f - \sum_{j=1}^{n} \partial_j (a_j v) - b_{0j} v + \sum_{j,k=1}^{n} \partial_j (a_{jk} \partial_k u) - \sum_{j=1}^{n} b_{j} \partial_j u - \sum_{j=1}^{n} \partial_j (\tilde{c}_{j} u) - \tilde{d} u$, then

$$\partial_t v \in L^1(0, T^*; H^{-1-\alpha + (\beta + \frac{1}{2}) \log}). \quad \quad (98)$$

In addition, $v|_{t=0} = v_0 \in H^{-\theta + \beta \log}$ with $v_0 = u_1 + c_0|_{t=0} u_0$. To establish the energy estimate of $u$ and $v$ in (60) (with $q \equiv 0$), we need to mollify them since $(u, v)$ has
low regularities. For this end, we define \( Q_\varepsilon =: (1 - \varepsilon \Delta_x)^{-1} \) for \( \varepsilon > 0 \), and
\[
 u_\varepsilon = Q_\varepsilon u, \quad v_\varepsilon = Q_\varepsilon v, \quad f_\varepsilon = Q_\varepsilon f.
\]
From (97) and (98), we have
\[
 u_\varepsilon \in L^2(0, T^*; H^2), \quad \partial_t u_\varepsilon \in L^2(0, T^*; H^1),
\]
\[
 v_\varepsilon \in L^2(0, T^*; H^1), \quad \partial_t v_\varepsilon \in L^1(0, T^*; H^0),
\]
\[
f_\varepsilon \in L^1(0, T^*; H^1)
\]
and
\[
 u_\varepsilon \to u \quad \text{in} \quad H_{1-\theta+(\beta+\frac{1}{2})\log,\lambda}(T^*),
\]
\[
 v_\varepsilon \to v \quad \text{in} \quad H_{-\theta+(\beta+\frac{1}{2})\log,\lambda}(T^*),
\]
\[
f_\varepsilon \to f \quad \text{in} \quad \mathcal{L}_{-\theta+\beta\log,\lambda}(T^*).
\]
(100)
Acting the operator \( Q_\varepsilon \) on \( L(u, v) \) and \( v \) in (60) respectively, we infer that
\[
 L(u_\varepsilon, v_\varepsilon) = f_\varepsilon - r_\varepsilon,
\]
\[
v_\varepsilon = a_0 \partial_t u_\varepsilon + \sum_{j=1}^n a_j \partial_j u_\varepsilon + c_0 u_\varepsilon + g_\varepsilon,
\]
where
\[
r_\varepsilon = \sum_{j=1}^n \partial_j [Q_\varepsilon, \tilde{a}_j]v + [Q_\varepsilon, \tilde{b}_0]v - \sum_{j,k=1}^n \partial_j [Q_\varepsilon, \tilde{a}_{jk}] \partial_k u + \sum_{j=1}^n [Q_\varepsilon, \tilde{b}_j] \partial_j u
\]
\[
+ \sum_{j=1}^n \partial_j [Q_\varepsilon, \tilde{c}_j]u + [Q_\varepsilon, \tilde{d}]u,
\]
(102)
\[
g_\varepsilon = [Q_\varepsilon, a_0] \partial_t u + \sum_{j=1}^n [Q_\varepsilon, a_j] \partial_j u + [Q_\varepsilon, c_0] u.
\]

Next we establish the convergence of \( r_\varepsilon \) and \( g_\varepsilon \) as \( \varepsilon \to 0 \).

**Lemma 4.1.** \( r_\varepsilon \to 0 \) in \( H_{-\theta+(\beta-\frac{1}{2})\log,\lambda}(T^*) \) and \( g_\varepsilon \to 0 \) in \( H_{1-\theta+(\beta-\frac{1}{2})\log,\lambda}(T^*) \) as \( \varepsilon \to 0 \).

**Proof.** By (102), in order to prove Lemma 4.1, it suffices to prove
\[
 [Q_\varepsilon, a]u \to 0 \quad \text{in} \quad L^2([0, T^*] \times \mathbb{R}^n) \quad \text{for} \quad a \in L^\infty, \quad u \in L^2([0, T^*] \times \mathbb{R}^n),
\]
(103)
\[
 [Q_\varepsilon, b]u \to 0 \quad \text{in} \quad H_{1-\theta+(\beta-\frac{1}{2})\log,\lambda}(T^*) \quad \text{for} \quad b \in L^\infty \cap LL, \quad u \in H_{-\theta+(\beta+\frac{1}{2})\log,\lambda}(T^*),
\]
(104)
\[
 [Q_\varepsilon, c]u \to 0 \quad \text{in} \quad H_{-\theta+(\beta+\frac{1}{2})\log,\lambda}(T^*) \quad \text{for} \quad c \in L^\infty \cap C^{\alpha}, \quad u \in H_{-\theta+(\beta+\frac{1}{2})\log,\lambda}(T^*),
\]
(105)
\[
 [Q_\varepsilon, d]u \to 0 \quad \text{in} \quad H_{1-\theta+(\beta+\frac{1}{2})\log,\lambda}(T^*) \quad \text{for} \quad d \in L^\infty \cap C^{\alpha}, \quad u \in H_{1-\theta+(\beta+\frac{1}{2})\log,\lambda}(T^*),
\]
(106)

For (103), by Fourier transformation and Parseval’s equality, then it follows from the Dominated Convergence Theorem that \( Q_\varepsilon u \to u \) in \( L^2 \). Hence (103) is proved. For (104), one has \( [Q_\varepsilon, \tilde{b}]u = [Q_\varepsilon, T_b]u + Q_\varepsilon(R_b u) - R_b u_\varepsilon \). In addition, Proposition 2 implies that
\[
 ||Q_\varepsilon(R_b u)||_{H_{1-\theta+(\beta-\frac{1}{2})\log,\lambda}(T^*)}, \quad ||R_b u_\varepsilon||_{H_{1-\theta+(\beta-\frac{1}{2})\log,\lambda}(T^*)}
\]
where the constant $C > 0$ is independent of $\varepsilon$. The same conclusion is also true for the commutator $[Q_\varepsilon, T_b]u$ thanks to Lemma 2.2 and (26). On the other hand, if $u \in C_0^\infty$ and $|\sigma| < 1$, for $b \in L^\infty$, one immediately gets

$$[Q_\varepsilon, b]u = Q_\varepsilon(bu) - bu + bu - bu_\varepsilon \to 0 \quad \text{in } L^2(0, T^*; H^r).$$

So by the standard density argument, we can obtain (104). The proofs of (105) and (106) are similar, we omit them here. 

**Lemma 4.2.** For function $a(t, x)$ with $(t, x) \in [0, T_0] \times \mathbb{R}^n$, we define $a_\varepsilon = \rho_\varepsilon \ast \tilde{a}$, where $\tilde{a}$ is the extension of $a$ given by

$$\tilde{a} = a(0, x), \quad t < 0; \quad a = a(t, x), \quad 0 \leq t \leq T_0; \quad \tilde{a} = a(T_0, x), \quad t > T_0.$$  

(1) If $a \in LL \cap L^\infty([0, T_0] \times \mathbb{R}^n)$, then $a_\varepsilon \in LL \cap L^\infty(\mathbb{R} \times \mathbb{R}^n)$, moreover,

$$||a_\varepsilon||_{L^\infty} \lesssim ||a||_{L^\infty}, \quad ||a_\varepsilon||_{LL} \lesssim ||a||_{LL},$$

$$a_\varepsilon \to a \quad \text{in } L^\infty \cap C^{\alpha'}([0, T_0] \times \mathbb{R}^n) \quad \text{for any } \alpha' \in (0, 1).$$

(2) If $a \in L^\infty \cap C^\alpha([0, T_0] \times \mathbb{R}^n)$, then $a_\varepsilon \in L^\infty \cap C^\alpha(\mathbb{R} \times \mathbb{R}^n)$, moreover,

$$||a_\varepsilon||_{L^\infty} \lesssim ||a||_{L^\infty}, \quad ||a_\varepsilon||_{C^\alpha} \lesssim ||a||_{C^\alpha},$$

$$a_\varepsilon \to a \quad \text{in } L^\infty \cap C^{\alpha'}([0, T_0] \times \mathbb{R}^n) \quad \text{for } \alpha' \in (0, \alpha).$$

(3) If $a_0, a_j, a_{jk}, j, k = 1, ..., n$, satisfy (5), then so do $a_{0, \varepsilon}, a_{j, \varepsilon}, a_{jk, \varepsilon}$.

The proofs are just based upon direct computations and the definitions of the corresponding norms in Lemma 4.2, we omit them here.

**Lemma 4.3.** Let $a \in L^\infty([0, T_0] \times \mathbb{R}^n)$, define $a_\varepsilon = \rho_\varepsilon \ast \tilde{a}$, where $\tilde{a}$ is the extension of $a$ given by $\tilde{a} = ||a||_{L^\infty}$ for $t \notin [0, T_0]$, then $a_\varepsilon \to a$ in $L^2([0, T_0] \times \Omega)$ for any bounded open set $\Omega \subset \mathbb{R}^n$.

We now start to prove Theorem 1.2.

**Proof of Theorem 1.2. Step 1. Energy estimate.**

For $\lambda > \lambda_0$, by estimate (62) of $(u_\varepsilon, v_\varepsilon)$ with $f_1 = f_\varepsilon$ and $f_2 = r_\varepsilon$, we have

$$\sup_{\tau \in [0, t]} ||v_\varepsilon||^2_{H^{-\theta}(\tau)} + \int_0^t ||v_\varepsilon||^2_{H^{-\theta}(\tau)} d\tau$$

$$\leq C ||u||_{H_1^{-\theta}(\beta + \frac{1}{2})} \int_0^t ||u_\varepsilon||^2_{H_1^{-\theta}(\beta + \frac{1}{2})} d\tau$$

$$+ C \sup_{\tau \in [0, t]} ||u_\varepsilon||^2_{H_1^{-\theta}(\beta + \frac{1}{2})} \int_0^t ||u_\varepsilon||^2_{H_1^{-\theta}(\beta + \frac{1}{2})} d\tau$$

$$\lesssim ||v_\varepsilon(0)||^2_{H^{-\theta}} + ||u_\varepsilon(0)||^2_{H_1^{-\theta}} + \left( \int_0^t ||f_\varepsilon||^2_{H^{-\theta}} d\tau \right)^2$$

$$+ \int_0^t ||r_\varepsilon||^2_{H^{-\theta}} d\tau + \int_0^t ||g_\varepsilon||^2_{H^{-\theta}} d\tau.$$  

Since $u_\varepsilon(0) \to u_0$ in $H_1^{-\theta}$ and $v_\varepsilon(0) \to v_0$ in $H^{-\theta}$, by (109), we derive that $u_\varepsilon$ and $v_\varepsilon$ are Cauchy sequences in $H_1^{-\theta}(\beta + \frac{1}{2}) \cap C([0, t]; H^{-\theta}(\beta + \frac{1}{2}))$ and $H^{-\theta}(\beta + \frac{1}{2})$. Therefore, we can pass to the limit in (101) and get the solution $(u, v)$ of (60) (with $g \equiv 0$). In addition, by
\[ \partial_t u = \frac{1}{a_0} v - \sum_{j=1}^{n} \tilde{a}_j \partial_j u - \tilde{c}_0 u, \]

one has \( \partial_t u \in C_{-\theta+\beta \log, \lambda}(T^*) \cap \mathcal{H}_{-\theta+\beta+\frac{1}{2} \log, \lambda}(T^*) \), moreover, for \( t \in [0, T^*] \)

\[ ||\partial_t u||_{H^{-\theta+\beta+\frac{1}{2} \log, \lambda}(T^*)} \leq ||v||_{H^{-\theta+\beta+\frac{1}{2} \log, \lambda}(T^*)} + ||u||_{H^{-\theta+\beta+\frac{1}{2} \log, \lambda}(T^*)}, \]

where \( \delta = 0 \) or \( \frac{1}{2} \). Replacing \( v \) by \( \partial_t u \), then the energy estimate (20) follows.

**Step 2. Existence and uniqueness.**

Mollifying the coefficients of operator \( L \) just as we have done in Lemma 4.2 and 4.3, and denoting the corresponding operator by \( L_\varepsilon \), we obtain a second order strictly hyperbolic equation whose coefficients are smooth. Thanks to the density properties, the corresponding initial data and source term can be approximated by smooth functions compactly supported in \( x \), which are denoted by \( u_0^\varepsilon, u_1^\varepsilon \) and \( f^\varepsilon \) respectively. That is, we obtain the following problem

\[
\begin{align*}
L_\varepsilon u^\varepsilon &= f^\varepsilon, \\
u^\varepsilon|_{t=0} &= u_0^\varepsilon, X_\varepsilon u^\varepsilon|_{t=0} = u_1^\varepsilon,
\end{align*}
\]

where \( u^\varepsilon \) is the unique smooth solution of problem (110) and admits a compact support in \( x \). On the other hand, by the energy estimate (20) one has that

\[
\begin{align*}
||u^\varepsilon||_{H^{-\theta+\beta+\frac{1}{2} \log, \lambda}(T^*)} &\leq C \\
||\partial_t u^\varepsilon||_{H^{-\theta+\beta+\frac{1}{2} \log, \lambda}(T^*)} &\leq C \\
||X_\varepsilon u^\varepsilon||_{H^{-\theta+\beta+\frac{1}{2} \log, \lambda}(T^*)} &\leq C
\end{align*}
\]

where \( C > 0 \) is independent of \( \varepsilon \). Therefore, by the weak compactness of the corresponding function spaces, one can extract a subsequence if necessary, we know that \( u^\varepsilon \) weakly converges to some \( u \) in \( L^2(0, T^*; H^{1-\alpha+\beta+\frac{1}{2} \log}) \cap H^1(0, T^*; H^{-\alpha+\beta+\frac{1}{2} \log}) \) and \( u \in H^{-\theta+\beta+\frac{1}{2} \log, \lambda}(T^*) \), meanwhile \( \partial_t u \in H^{-\theta+\beta+\frac{1}{2} \log, \lambda}(T^*) \). Let \( \varepsilon \to 0 \) in (110), by the weak convergence together with 4.2 and 4.3, we infer that \( Lu = f \) holds in the sense of distribution. Next we concern with the initial data. At first, thanks to the weak convergence of \( u^\varepsilon \) in \( L^2(0, T^*; H^{1-\alpha+\beta+\frac{1}{2} \log}) \cap H^1(0, T^*; H^{-\alpha+\beta+\frac{1}{2} \log}) \), \( u^\varepsilon \) converges to \( u \) in \( C([0, T^*]; H^{\alpha,c}) \). Thus \( u^\varepsilon|_{t=0} \) converges to \( u_0 \) in \( H^{\alpha,c} \), and then \( u|_{t=0} = u_0 \) follows. Let \( v^\varepsilon = X_\varepsilon u^\varepsilon + c_0 v, \) as we have done in the proof of Lemma A.1, we can get that \( v^\varepsilon|_{t=0} \) converges to \( v|_{t=0} = u_1 + c_0|_{t=0} u_0 \) in \( H^{\alpha,c,-1} \), thus \( X_\varepsilon u|_{t=0} = u_1 \) holds. From the argument in Step 1, one also obtains that \( u \in C_{1-\theta+\beta \log, \lambda}(T^*) \) and \( \partial_t u \in C_{-\theta+\beta \log, \lambda}(T^*) \).

Finally, we show the uniqueness of solution \( u \) to problem (1). Set \( w = w_1 - w_2 \), where \( w_1 \) and \( w_2 \) are two solutions to (1.1) and satisfy the same initial data. Note that the error solution \( w \) satisfies the energy inequality (20) whose right hand side vanishes. This derives \( w = 0 \) and then the proof of Theorem 1.2 is finished.

**Appendix A. Illustration on the weak solution to problem (1) and (60) under the suitable regularity assumptions on the coefficients and solutions.**

Set \( v = X_\varepsilon u + c_0 u \). Then the equation in (1) can be rewritten as

\[
\partial_t v + \sum_{j=1}^{n} \partial_j (\tilde{a}_j v) + \tilde{b}_0 v = \tilde{L}_2 u - \tilde{L}_1 u - \tilde{d} u + f,
\]

(111)
where
\[ \tilde{L}_2 u = \sum_{j,k=1}^{n} \partial_j (\tilde{a}_{jk} \partial_k u), \quad \tilde{L}_1 u = \sum_{j=1}^{n} \tilde{b}_j \partial_j u + \sum_{j=1}^{n} \partial_j (\tilde{c}_j u), \]
and
\[ \tilde{a}_j = \frac{a_j}{a_0}, \quad \tilde{b}_0 = \frac{b_0}{a_0}, \quad \tilde{a}_{jk} = a_{jk} + \frac{a_j a_k}{a_0}, \quad \tilde{b}_j = b_j - \tilde{b}_0 a_j, \quad \tilde{c}_j = c_j - \tilde{a}_j c_0, \quad \tilde{d} = d - \tilde{b}_0 c_0. \]
Under assumptions (5)-(8), we have

**Lemma A.1.** Let \( s \in (1 - \alpha, \alpha) \) with \( \alpha > \frac{1}{2} \), \( \delta > 1 \), \( u \in H^{s+\delta \log}((0, T) \times \mathbb{R}^n) \) and \( Lu \in L^1(0, T; H^{s-1+\frac{\delta}{2} \log}) \). Then

1. \( X u, d u, L_1 u \in H^{s+1+\delta \log}((0, T) \times \mathbb{R}^n) \) and \( L_2 u \in H^{s+2+\delta \log}((0, T) \times \mathbb{R}^n) \), which means that equation (1) is well-defined in the sense of distribution.
2. \( u \in L^2(0, T; H^{s+\delta \log}) \) and \( \partial_t u \in L^2(0, T; H^{s+1+\delta \frac{1}{2} \log}) \), which derives that \( u \in C((0, T]; H^{s-\frac{1}{2} + \delta \log}) \) and \( u|_{t=0} \) is well-defined. Moreover, the trace \( X u|_{t=0} \) makes sense, since \( X u \in L^2(0, T; H^{s+1+\delta \frac{1}{2} \log}) \cap C((0, T]; H^{s-\frac{1}{2} + \delta \log}) \).

**Proof.** (1) Since \( \partial_t u \) and \( \partial_j u \) belong to \( H^{s-1+\delta \log} \), by \( s - 1 > -\alpha \) and Proposition 2, we have \( a d u \in H^{s-1+\delta \log} \) for \( a \in L^1(0, T; H^{s+\delta \log}) \). This immediately implies that \( X u \in H^{s+1+\delta \log} \), and meanwhile \( L_1 u \in H^{s+1+\delta \log} \) holds as long as one notes that for \( s \geq 1 - \alpha > -\alpha \) and by Proposition 2, \( a u \in H^{s+\delta \log} \) for \( a \in C^\infty \). Analogously, one has \( L_2 u \in H^{s+2+\delta \log} \). It follows from \( H^{s+\delta \log} \hookrightarrow H^0 \) that \( d u \in H^0 \hookrightarrow H^{s-1+\delta \log} \) due to \( s - 1 < 0 \). Hence equation (1) is well-defined in the sense of distribution.

(2) At first, we define a space \( H^{s+\gamma \log, \rho' + \gamma \log}((0, T) \times \mathbb{R}^n) \) as follows
\[ u \in H^{s+\gamma \log, \rho' + \gamma \log}((0, T) \times \mathbb{R}^n) \quad \text{iff} \]
\[ (1 + \tau^2 + |\xi|^2)^{\frac{\delta}{2}} (1 + |\xi|^2)^{\frac{\delta}{2}} \log^\gamma (2 + |\tau| + |\xi|) \log^{\gamma'} (2 + |\xi|) \hat{u}(\tau, \xi) \in L^2((0, T) \times \mathbb{R}^n). \]
In addition, the space \( H^{s+\gamma \log, \rho' + \gamma \log}((0, T) \times \mathbb{R}^n) \) is defined by the restriction of \( H^{s+\gamma \log, \rho' + \gamma \log}((0, T) \times \mathbb{R}^n) \) on the domain \((0, T) \times \mathbb{R}^n\). Here we point out that the space \( H^{s+\gamma \log, \rho' + \gamma \log} \) without Log is introduced by Hörmander in [11] (see Definition 2.5.1 of page 51). Direct computation yields
\[ H^{0, \rho' + \gamma \log}((0, T) \times \mathbb{R}^n) = L^2(0, T; H^{\rho' + \gamma \log}), \]
\[ H^{0, \rho' + \gamma \log}((0, T) \times \mathbb{R}^n) \hookrightarrow H^{\rho' + \gamma \log, 0}((0, T) \times \mathbb{R}^n) \quad \text{for} \quad \rho' < 0; \]
\[ H^{0, \rho' + \gamma \log}((0, T) \times \mathbb{R}^n) \hookrightarrow H^{0, \rho' + \gamma \log}((0, T) \times \mathbb{R}^n) \quad \text{for} \quad \rho > 0, \]
\[ u \in H^{s+\gamma \log, \rho' + \gamma \log}((0, T) \times \mathbb{R}^n), \quad \partial_t u \in H^{s+\gamma \log, \rho' + 1+\gamma \log}((0, T) \times \mathbb{R}^n) \]
\[ \Rightarrow \quad u \in H^{s+\gamma \log, \rho' + 1+\gamma \log}((0, T) \times \mathbb{R}^n). \]
We claim that \( v \in H^{0, s-1+\frac{\delta}{2} \log} \). Indeed, by \( \partial_t u \in H^{s-1+\delta \log} \), similar to the analysis in (1), one has that \( v, \tilde{L}_1 u, \tilde{d} u \in H^{s-1+\delta \log} \) and\( \sum_{j=1}^{n} \partial_j (\tilde{a}_j v) \)
\[ \tilde{L}_2 u \in H^{s-1+\delta \log}. \]
This derives that \( h = -\sum_{j=1}^{n} \partial_j (\tilde{a}_j v) - \tilde{b}_0 v + \tilde{L}_2 u - \tilde{L}_1 u - \tilde{d} u \in H^{s-1+\delta \log}. \]
Let \( F(t) = \int_0^t f(\tau) d\tau \), then \( \partial_t (v - F) = \dot{h} \in H^{s-1+\delta \log}. \) By \( s - 1 < 0 \) and (112), one has \( F \in L^2(0, T; H^{s-1+\delta \log}) \). Hence \( F \in H^{0, s-1+\delta \log} \hookrightarrow H^{s-1+\delta \log} \), which means \( v - F \in H^{s-1+\delta \log} \). Together with (112) and
(113), this yields \( v - F \in H^{s+\frac{1}{2} \log -1} \hookrightarrow H^{0, s-1+\frac{1}{2} \log} \) for \( s > 0 \). Thus \( v \in H^{0, s-1+\frac{1}{2} \log} \). Due to \( \partial_t u = \frac{1}{a_0} v - \sum_{j=1}^{n} \tilde{a}_j \partial_j u - \tilde{c}_0 u \in H^{0, s-1+\frac{1}{2} \log} \) and \( u \in H^{s+\delta \log, 0} \hookrightarrow H^{0, s+\delta \log} \), we complete the proof for the first part of (2). By \( v \in H^{0, s-1+\frac{1}{2} \log} \), then \( -\sum_{j=1}^{n} \partial_j (\tilde{a}_j u) - \tilde{b}_0 v \in H^{0, s-2+\frac{1}{2} \log} \) since \( \delta > \frac{1}{2} > \frac{1}{2} \). Note that \( \partial_t u \in H^{s+\delta \log, -1} \hookrightarrow H^{0, s-1+\delta \log} \), this derives \( \hat{L}_2 u - \hat{L}_1 u - \hat{d} u \in H^{0, s-2+\delta \log} \) and \( \partial_t (v - F) = h \in H^{0, s-1+\delta \log} \). Together with \( v - F \in H^{0, s-1+\delta \log} \) and (113), one has \( v - F \in H^{s-2+(\delta-\frac{1}{2}) \log} \subset C([0, T]; H^{s-\frac{1}{2}+(\delta-\frac{1}{2}) \log}) \). Recall \( F \in C([0, T]; H^{s-1+(\delta-\frac{1}{2}) \log}) \), then \( v \in H^{0, s-1+\delta \log} \cap C([0, T]; H^{s-\frac{1}{2}+(\delta-\frac{1}{2}) \log}) \) \( \cap L^2(0, T; H^{s+\delta \log}) \), then the proof for the second part of (2) is completed. 

\[
\begin{align*}
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\end{align*}
\]