LYAPUNOV CRITERIA FOR BOUNDEDNESS OF REACHABILITY SETS OF DISTRIBUTED PARAMETER SYSTEMS

ANDRII MIRONCHENKO

Abstract. We show that boundedness of reachability sets for distributed parameter systems is equivalent to existence of a corresponding Lyapunov function, that increases at most exponentially along the trajectories. Next we show a similar characterization for the robust forward completeness property.

1. Introduction

A control system is called forward complete if for any initial condition $x$, and any input $u$, the corresponding trajectory $\phi(\cdot, x, u)$ is well-defined on the whole nonnegative time axis. If additionally, for any magnitude $R > 0$ and any time $\tau > 0$

$$\sup_{\|x\| \leq R, \|u\| \leq R, \ t \in [0, \tau]} \|\phi(t, x, u)\| < +\infty,$$

then a control system is said to have bounded (finite-time) reachability sets.

Boundedness of reachability sets is important in many contexts, such as derivation of converse Lyapunov theorems for global asymptotic stability [10], characterization of input-to-state stability for nonlinear systems [11], non-coercive Lyapunov methods [12, 11, 5], characterization of global asymptotic stability for retarded systems [8], to name a few.

Sufficient conditions for the global existence of solutions for ordinary differential equations (ODEs) and other classes of control systems are a classic subject [16, 2, 6, 14, 4]. For example, Wintner’s theorem [16] shows that an ODE

$$\dot{x} = f(x)$$

with locally Lipschitz $f$ has unique global solutions provided that $|f(x)| \leq L(|x|)$ with $L$ satisfying

$$\int_0^\infty \frac{1}{L(s)} ds = +\infty.$$

In particular, if $f$ is globally Lipschitz or linearly bounded, then the solutions for the above ODE exist globally, and the reachability sets are bounded. This result can be extended to evolution equations in Banach spaces and other system classes, e.g., [13 Theorem 3.3, p. 199].

Much less is known in necessary conditions for forward completeness. Necessary and sufficient conditions of Lyapunov type for forward completeness of ODEs without inputs have been proposed in [9]. However, Lyapunov functions constructed in [9] are time-variant even for time-invariant ODEs.
In [10][1] for systems
\[ \dot{x} = f(x,u) \]
with Lipschitz continuous \( f \), it was shown that: forward completeness, boundedness of reachability sets for ODEs with inputs, and the existence of a Lyapunov function that increases at most exponentially, are equivalent properties.

For distributed parameter systems, the situation is more complex. Linear forward complete infinite-dimensional systems have always bounded reachability sets [15 Proposition 2.5]. However, nonlinear forward complete infinite-dimensional systems with Lipschitz continuous right-hand sides do not necessarily have bounded reachability sets, as demonstrated in [11 Example 2]. This fact indicates that the BRS property (establishing uniform bounds for solutions on finite time intervals) is a bridge between the pure well-posedness theory (that studies existence and uniqueness, but does not care much about the bounds for solutions), and the stability theory (which is interested in establishing certain bounds for solutions for all nonnegative times, as well as their convergence).

In this work, we consider a broad class of control systems satisfying the so-called boundedness-implies-continuation property and having flows that are Lipschitz continuous on compact intervals. We show that for this class of systems boundedness of reachability sets is equivalent to the existence of a Lyapunov function that increases at most exponentially along the trajectories.

Our proof is different from that of [1], where a finite-dimensional version of this result was shown. Namely, for ODEs with Lipschitz right-hand sides, local solutions exist not only in a positive direction but also in a negative direction. This fact was used for construction of “BRS Lyapunov functions” in [1]. At the same time, for the class of systems that we consider, the solutions backwards in time do not necessarily exist, and if they do, then they do not need to be unique. To overcome this challenge, we propose a different proof scheme motivated by the converse Lyapunov results for the UGAS property, e.g., [3, Theorem 4.2.1].

**Notation.** We write \( \mathbb{N}, \mathbb{R}, \) and \( \mathbb{R}_+ \) for the sets of positive integers, real numbers, and nonnegative real numbers, respectively. We say that \( \gamma : \mathbb{R}_+ \to \mathbb{R}_+ \) belongs to the class \( \mathcal{K} \) if \( \gamma \) is continuous, \( \gamma(0) = 0 \), and \( \gamma \) is increasing. \( \gamma \in \mathcal{K}_\infty \) if \( \gamma \) is unbounded.

## 2. General class of systems

We start with a general definition of a control system.

**Definition 2.1.** Consider the triple \( \Sigma = (X, U, \phi) \) consisting of

(i) A normed vector space \( (X, \| \cdot \|_X) \), called the state space, endowed with the norm \( \| \cdot \|_X \).

(ii) A normed vector space of inputs \( U \subset \{ u : \mathbb{R}_+ \to U \} \) endowed with a norm \( \| \cdot \|_U \), where \( U \) is a normed vector space of input values. We assume that the following two axioms hold:

The axiom of shift invariance: for all \( u \in U \) and all \( \tau \geq 0 \) the time shift \( u(\cdot + \tau) \) belongs to \( U \) with \( \| u \|_U \geq \| u(\cdot + \tau) \|_U \).

The axiom of concatenation: for all \( u_1, u_2 \in U \) and for all \( t > 0 \) the concatenation of \( u_1 \) and \( u_2 \) at time \( t \), defined by

\[
\begin{align*}
 u_1 \hat{\otimes}_t u_2(\tau) &:=
 \begin{cases}
 u_1(\tau), & \text{if } \tau \in [0,t], \\
 u_2(\tau - t), & \text{otherwise},
 \end{cases}
\end{align*}
\]

belongs to \( U \).
Lyapunov criteria for boundedness of reachability sets of distributed parameter systems

(iii) A map \( \phi : D_\phi \to X, \ D_\phi \subseteq \mathbb{R}_+ \times X \times \mathcal{U} \) (called transition map), such that for all \((x,u) \in X \times \mathcal{U}\) it holds that \(D_\phi \cap ([0,t_m] \times \{(x,u)\}) \subset D_\phi\), for a certain \(t_m = t_m(x,u) \in (0, +\infty)\).

The corresponding interval \([0,t_m]\) is called the maximal domain of definition of \(t \mapsto \phi(t,x,u)\).

The triple \(\Sigma\) is called a (control) system, if the following properties hold:

(Σ1) The identity property: for every \((x,u) \in X \times \mathcal{U}\) it holds that \(\phi(0,x,u) = x\).

(Σ2) Causality: for every \((t,x,u) \in D_\phi\), for every \(\bar{u} \in \mathcal{U}\), such that \(u(s) = \bar{u}(s)\) for all \(s \in [0,t]\) it holds that \([0,t] \times \{(x,\bar{u})\} \subset D_\phi\) and \(\phi(t,x,u) = \phi(t,x,\bar{u})\).

(Σ3) Continuity: for each \((x,u) \in X \times \mathcal{U}\) the map \(t \mapsto \phi(t,x,u)\) is continuous on its maximal domain of definition.

(Σ4) The cocycle property: for all \(x \in X\), \(u \in \mathcal{U}\), for all \(t,h \geq 0\) so that \([0,t+h] \times \{(x,u)\} \subset D_\phi\), we have

\[\phi(h,\phi(t,x,u),u(t+\cdot)) = \phi(t+h,x,u)\]

Definition [23] can be viewed as a direct generalization and a unification of the concepts of strongly continuous nonlinear semigroups with abstract linear control systems [15]. This class of systems encompasses control systems generated by ODEs, switched systems, time-delay systems, evolution partial differential equations, differential equations in Banach spaces and many others [4] Chapter 1.

Definition 2.2. We say that a control system \(\Sigma = (X, \mathcal{U}, \phi)\) (as introduced in Definition [24]) is forward complete, if \(D_\phi = \mathbb{R}_+ \times X \times \mathcal{U}\), that is for every \((x,u) \in X \times \mathcal{U}\) and for all \(t \geq 0\) the value \(\phi(t,x,u) \in X\) is well-defined.

Forward completeness alone does not imply, in general, the existence of any uniform bounds on the trajectories emanating from bounded balls that are subject to uniformly bounded inputs [11] Example 2. Systems exhibiting such bounds deserve a special name.

Definition 2.3. We say that \(\Sigma = (X, \mathcal{U}, \phi)\) has bounded reachability sets (BRS) if it is forward complete and for any \(C > 0\) and any \(\tau > 0\), it holds that

\[\sup \{ \|\phi(t,x,u)\|_X : \|x\|_X \leq C, \|u\|_\mathcal{U} \leq C, \ t \in [0,\tau] \} < \infty\]

For a wide class of control systems, the boundedness of a solution implies the possibility of prolonging it to a larger interval, see [7] Chapter 1. Next, we formulate this property for abstract systems:

Definition 2.4. We say that a system \(\Sigma\) satisfies the boundedness-implies-continuation (BIC) property if for each \((x,u) \in X \times \mathcal{U}\) such that the maximal existence time \(t_m(x,u)\) is finite, and for all \(M > 0\), there exists \(t \in [0,t_m(x,u)]\) with \(\|\phi(t,x,u)\|_X > M\).

3. Criteria for boundedness of reachability sets

We call a function \(h : \mathbb{R}_+^3 \to \mathbb{R}_+\) increasing, if \((r_1, r_2, r_3) \leq (R_1, R_2, R_3)\) implies that \(h(r_1, r_2, r_3) \leq h(R_1, R_2, R_3)\), where we use the component-wise partial order on \(\mathbb{R}_+^3\). We call \(h\) strictly increasing if \((r_1, r_2, r_3) \leq (R_1, R_2, R_3)\) and \((r_1, r_2, r_3) \neq (R_1, R_2, R_3)\) imply \(h(r_1, r_2, r_3) < h(R_1, R_2, R_3)\).

We need the following two simple lemmas

Lemma 3.1. Let \(f, g : D \to \mathbb{R}_+\) be any functions for which \(\sup_{d \in D} f(d)\) is finite. Then

\[\sup_{d \in D} f(d) - \sup_{d \in D} g(d) \leq \sup_{d \in D} (f(d) - g(d)).\]
Lemma 3.2. For any \( k \in \mathbb{N} \) the functions \( G_k(r) := \max\{r - \frac{1}{k}, 0\} \) are Lipschitz continuous with a unit Lipschitz constant, i.e., for all \( r_1, r_2 \geq 0 \) it holds that
\[
|G_k(r_1) - G_k(r_2)| \leq |r_1 - r_2|.
\]

Proof. This holds as each \( G_k \) is the maximum of two Lipschitz continuous functions with Lipschitz constant at most 1. \( \square \)

The following lemma is taken from [4, p.130] (where it was stated informally).

Lemma 3.3. For any \( \alpha \in \mathcal{K}_\infty \) there exist \( \rho \in \mathcal{K}_\infty \) so that \( \rho(s) \leq \alpha(s) \) for all \( s \in \mathbb{R}_+ \) and \( \rho \) is globally Lipschitz with a unit Lipschitz constant, i.e., for any \( s_1, s_2 \geq 0 \) it holds that
\[
|\rho(s_1) - \rho(s_2)| \leq |s_1 - s_2|.
\]

The regularity of Lyapunov functions, constructed via converse Lyapunov techniques, depends on the regularity of the flow map.

Definition 3.4. We say that the flow of a control system \( \Sigma = (X, \mathcal{U}, \phi) \) is Lipschitz continuous on compact intervals, if for any \( \tau > 0 \) and any \( r > 0 \), there exists \( L > 0 \) so that
\[
x, y \in B_r, \ t \in [0, \tau], \ u \in B_r \mathcal{U} \Rightarrow \|\phi(t, x, u) - \phi(t, y, u)\|_X \leq L\|x - y\|_X.
\]

Next, we show a Lyapunov criterion for the BRS property for general control systems satisfying the BIC property. It states that such systems have BRS property if and only if there is a “BRS Lyapunov function” that grows at most exponentially for states that are large enough in comparison to inputs.

Theorem 3.5 (Criteria for BRS property). Consider a control system \( \Sigma = (X, \mathcal{U}, \phi) \) with \( \mathcal{U} := L^\infty(\mathbb{R}_+, U) \) for a certain normed linear space \( U \). Let \( \Sigma \) satisfy the BIC property and have a flow that is Lipschitz continuous on compact intervals.

The following statements are equivalent:

(i) \( \Sigma \) has BRS property.

(ii) There exists a continuous, increasing function \( \mu : \mathbb{R}_+^3 \to \mathbb{R}_+ \), such that for all \( x \in X, u \in \mathcal{U} \) and all \( t \geq 0 \) we have
\[
\|\phi(t, x, u)\|_X \leq \mu(\|x\|_X, \|u\|_U, t).
\]

(iii) There exists a continuous function \( \mu : \mathbb{R}_+^3 \to \mathbb{R}_+ \), such that for all \( x \in X, u \in \mathcal{U} \) and all \( t \geq 0 \) the inequality [10] holds.

(iv) There are \( \xi \in \mathcal{K}_\infty \) \( \) and \( c > 0 \), such that for all \( t \geq 0, \ x \in X, \ u \in \mathcal{U} \)
\[
\|\phi(t, x, u)\|_X \leq \xi(\|x\|_X) + \xi(\|u\|_U) + \xi(t) + c.
\]

(v) There are a Lipschitz continuous map (“BRS Lyapunov function”) \( V : X \to \mathbb{R}_+ \), maps \( \psi_1, \psi_2 \in \mathcal{K}_\infty \), and \( c > 0 \) such that
\[
\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X) + c, \ x \in X,
\]
and there are \( a > 0 \) and \( \gamma \in \mathcal{K}_\infty \), such that for all \( x \in X \) and \( u \in \mathcal{U} \) the following holds:
\[
\|x\|_X \geq \gamma(\|u\|_U) \Rightarrow \dot{V}_u(x) \leq aV(x).
\]
Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). The proof is analogous to the proof of [12, Lemma 2.12] and is omitted.

(ii) $\Rightarrow$ (iv). Define $\zeta(r) := \mu(r, r, r)$, $r \geq 0$. As $\mu$ is increasing and continuous, $\zeta$ is increasing and continuous as well. In view of $\|x\| \leq \mu(\|x\|, \|u\|_t, 0)$, it is clear, that $\zeta$ is unbounded. Now, for all $x, u, t, w$ we have that

$$\mu(\|x\|, \|u\|_t, t) \leq \zeta(\|x\|) + \zeta(\|u\|) + \zeta(t).$$

Define $\zeta(r) := \zeta(r) - \lim_{r \to +0} \zeta(r)$. Then $\zeta \in K_\infty$, and (7) holds with this $\zeta$ and $c := 3 \lim_{r \to +0} \zeta(r)$.

(v) $\Rightarrow$ (i). Pick any $x \in X$ and any $u \in U$. As $\Sigma$ is a well-defined control system, there is a maximal time $t_m(x, u)$ such that $\phi(\cdot, x, u)$ is well-defined on $[0, t_m(x, u))$.

If $\|\phi(t, x, u)\| \leq \gamma(\|u\|_t)$ for all $t \in [0, t_m(x, u))$, then the trajectory $\phi(\cdot, x, u)$ is uniformly bounded on $[0, t_m(x, u))$. By BIC property, $t_m(x, u) = +\infty$, and $\|\phi(t, x, u)\| \leq \gamma(\|u\|_t)$ for all $t \in \mathbb{R}_+$.

Otherwise, let there is some time $t \in [0, t_m(x, u))$ such that $\|\phi(t, x, u)\| \geq \gamma(\|u\|_t)$. Let $t_1$ be the smallest of such times. In view of the continuity of $\phi(\cdot, x, u)$ and due to the identity axiom of $\Sigma$, the following holds:

$$\|\phi(t_1, x, u)| \leq \max\{\|x\|, \gamma(\|u\|_t)\}.$$

Take the maximal time $t_2 \in [t_1, t_m(x, u)]$, such that

$$\|\phi(t, x, u)| \geq \gamma(\|u\|_t), \quad t \in [t_1, t_2).$$

In view of (ii), and thanks to the continuity of $V$, it holds that

$$V(\phi(t, x, u)) \leq e^{a(t-t_1)}V(\phi(t_1, x, u)), \quad t \in [t_1, t_2).$$

Thanks to the sandwich inequality (3), we have for all $t \in [t_1, t_2)$ that

$$\psi_1(\|\phi(t, x, u)| \leq e^{a(t-t_1)}(\psi_2(\|\phi(t_1, x, u)| + C).$$

Thus,

$$\|\phi(t, x, u)\| \leq \psi_1^{-1}(e^{a(t-t_1)}(\psi_2(\max\{\|x\|, \gamma(\|u\|_t)\}) + C))$$

(10)

Now either $t_2 = t_m(x, u)$, and then $\phi(\cdot, x, u)$ is uniformly bounded on the maximal interval of existence, and BIC property implies that $t_m(x, u) = +\infty$, and the bound (10) holds for $t \in \mathbb{R}_+$.

Otherwise, there is a time $t_3 \in (t_2, t_m(x, u))$, such that $\|\phi(t_3, x, u)\| \leq \gamma(\|u\|_t)$. Arguing again by cocycle property and BIC properties, we obtain that $t_m = +\infty$, and (10) holds for all $t \in \mathbb{R}_+, x \in X$ and $u \in U$. This shows the BRS property.

(iv) $\Rightarrow$ (v). This implication (reverse Lyapunov result) will be proved in several steps.

Construction of “pre-Lyapunov functions” $V_k$. Let $\xi \in K_\infty$ be as in (iv). Pick $\rho \in K_\infty$ such that $\rho \leq \xi^{-1}$ pointwise and $\rho$ is globally Lipschitz continuous with a unit Lipschitz constant. Such $\rho$ exists in view of Lemma 3.3.

For any $k \in \mathbb{N}$, consider $V_k : X \to \mathbb{R}_+$, defined for all $x \in X$ as follows:

$$V_k(x) := \sup_{\|u\|_t \leq \|x\|} \sup_{t \geq 0} G_k \left( e^{-t} \rho \left( \frac{1}{4} \|\phi(t, x, u)| \right) \right).$$


Thus, for all $v$ as in (13), we see that for all $\alpha \in \mathcal{K}_\infty$ and all $a, b, c, d \in \mathbb{R}_+$ it holds that
\begin{equation}
\alpha(a + b + c + d) \leq \alpha(4a) + \alpha(4b) + \alpha(4c) + \alpha(4d).
\end{equation}

Take any $x \in X$. Using in (11) the estimate (4), and the fact that $\rho \leq \xi^{-1}$, we have:
\begin{equation}
V_k(x) \\
\leq \sup_{\|u\|_\mathcal{U} \leq \|x\|_X} e^{-t} \xi^{-1} \left( \frac{1}{4} \left( \xi(\|x\|_X) + \xi(\|u\|_\mathcal{U}) + \xi(t) + \epsilon \right) \right).
\end{equation}

Applying (12) with $\alpha := \xi^{-1}$, we obtain that
\begin{equation}
V_k(x) \leq \sup_{\|u\|_\mathcal{U} \leq \|x\|_X} e^{-t} \left( \|x\|_X + \|u\|_\mathcal{U} + t + \xi^{-1}(c) \right)
\leq \sup_{t \geq 0} e^{-t} \left( 2\|x\|_X + t + \xi^{-1}(c) \right) \leq 2\|x\|_X + C,
\end{equation}
for a certain constant $C > 0$ and all $x \in X$.

**Proving the growth estimate (9)** for $V_k$. Now take any $v \in \mathcal{U}$ such that $\|v\|_\mathcal{U} \leq \|x\|_X$, and any $h > 0$. By the cocycle property, we obtain that
\begin{align*}
V_k(\phi(h, x, v)) &= \sup_{\|u\|_\mathcal{U} \leq \|x\|_X} \sup_{t \geq 0} G_k \left( e^{-t} \rho \left( \frac{1}{4} \|\phi(t, \phi(h, x, v), u)\|_X \right) \right) \\
&= \sup_{\|u\|_\mathcal{U} \leq \|x\|_X} \sup_{t \geq 0} G_k \left( e^{-t} \rho \left( \frac{1}{4} \|\phi(t + h, x, v)\|_X \right) \right).
\end{align*}
As $\mathcal{U} := L^\infty(\mathbb{R}_+, \mathcal{U})$, we have that $\|v \circ h\|_\mathcal{U} \leq \|x\|_X$. Thus, we only increase the rhs by taking the supremum over a larger space of inputs:
\begin{align*}
V_k(\phi(h, x, v)) &\leq \sup_{\|u\|_\mathcal{U} \leq \|x\|_X} \sup_{t \geq 0} G_k \left( e^{-t} \rho \left( \frac{1}{4} \|\phi(t + h, x, u)\|_X \right) \right) \\
&= e^h \sup_{\|u\|_\mathcal{U} \leq \|x\|_X} \sup_{t \geq 0} G_k \left( e^{-\rho(h, x, u)} \rho \left( \frac{1}{4} \|\phi(t + h, x, u)\|_X \right) \right) \\
&\leq e^h V_k(x).
\end{align*}
Thus, for all $v \in \mathcal{U}$ such that $\|v\|_\mathcal{U} \leq \|x\|_X$, we have
\begin{align*}
V_{k,v}(x) &= \limsup_{h \to 0} \frac{1}{h} \left( V_k(\phi(h, x, v)) - V_k(x) \right) \\
&\leq \limsup_{h \to 0} \frac{1}{h} (e^h - 1)V_k(x) \\
&= V_k(x).
\end{align*}

**Lipschitz continuity for $V_k$ on bounded balls.** Take any $R > 0$. Arguing as in (13), we see that for all $x \in B_R$, all $u \in B_{R, \mathcal{U}}$, and all $t \geq 0$, it holds that
\begin{equation}
e^{-t} \rho \left( \frac{1}{4} \|\phi(t, x, u)\|_X \right) \leq e^{-t}(2R + t + \xi^{-1}(c)).
\end{equation}
Now for any $k \in \mathbb{N}$, there is a time $T = T(R, k)$, such that
\begin{equation}
t \geq T(R, k) \quad \Rightarrow \quad \rho(\|\phi(t, x, u)\|_X) \leq \frac{1}{R}, \quad x \in B_R, \ u \in B_{R, \mathcal{U}}.
\end{equation}
Thus, the domain of maximization in the definition of $V_k$ has a finite length, i.e., for all $R > 0$ and all $x \in X$ with $\|x\|_X \leq R$, the function $V_k$ can be equivalently defined by

$$V_k(x) = \sup_{\|u\|_{u(t)} \leq \|x\|_X} \sup_{t \in [0,T(R,k)]} G_k\left(e^{-t} \rho\left(\frac{1}{4} \||\phi(t, x, u)||_X\right)\right).$$

Now pick any $x, y \in X$ such that $\|x\|_X \leq R$ and $\|y\|_X \leq R$, and consider $|V_k(x) - V_k(y)|$

$$= \left| \sup_{\|u\|_{u(t)} \leq \|x\|_X} \sup_{t \in [0,T(R,k)]} G_k\left(e^{-t} \rho\left(\frac{1}{4} \||\phi(t, x, u)||_X\right)\right) - \sup_{\|u\|_{u(t)} \leq \|y\|_X} \sup_{t \in [0,T(R,k)]} G_k\left(e^{-t} \rho\left(\frac{1}{4} \||\phi(t, y, u)||_X\right)\right) \right|.$$

The domains for taking the supremum in these two expressions are different. Assume that $V_k(x) > V_k(y)$. Then if $\|x\|_X \leq \|y\|_X$, we can upperestimate $V_k(x)$ as

$$V_k(x) \leq \sup_{\|u\|_{u(t)} \leq \|x\|_X} \sup_{t \in [0,T(R,k)]} G_k\left(e^{-t} \rho\left(\frac{1}{4} \||\phi(t, x, u)||_X\right)\right).$$

Otherwise, if $\|x\|_X \geq \|y\|_X$, then

$$V_k(y) \geq \sup_{\|u\|_{u(t)} \leq \|y\|_X} \sup_{t \in [0,T(R,k)]} G_k\left(e^{-t} \rho\left(\frac{1}{4} \||\phi(t, y, u)||_X\right)\right).$$

The case if $V_k(x) < V_k(y)$ can be treated similarly. In any case, we have that

$$|V_k(x) - V_k(y)| \leq \left| \sup_{\|u\|_{u(t)} \leq Q} \sup_{t \in [0,T(R,k)]} G_k\left(e^{-t} \rho\left(\frac{1}{4} \||\phi(t, x, u)||_X\right)\right) - \sup_{\|u\|_{u(t)} \leq Q} \sup_{t \in [0,T(R,k)]} G_k\left(e^{-t} \rho\left(\frac{1}{4} \||\phi(t, y, u)||_X\right)\right) \right|,$$

where either $Q = \|x\|_X$, or $Q = \|y\|_X$.

Using Lemma 4.1, we proceed to

$$|V_k(x) - V_k(y)| \leq \sup_{\|u\|_{u(t)} \leq Q} \sup_{t \in [0,T(R,k)]} \left| G_k\left(e^{-t} \rho\left(\frac{1}{4} \||\phi(t, x, u)||_X\right)\right) - G_k\left(e^{-t} \rho\left(\frac{1}{4} \||\phi(t, y, u)||_X\right)\right) \right|.$$
Since \( \phi \) is Lipschitz continuous on compact intervals, there is some \( M = M(R, k) \), which we assume wlog to be increasing with respect to both arguments, such that:
\[
|V_k(x) - V_k(y)| \leq M(R, k)\|x - y\|_X, \quad x, y \in B_R.
\]

Defining “BRS Lyapunov function”. Setting in \((11)\) \( t := 0 \), and using the identity axiom of \( \Sigma \), we estimate \( V_k \) from below as
\[
V_k(x) \geq G_k \left( \rho \left( \frac{1}{4}\|x\|_X \right) \right) = G_k \circ \rho \left( \frac{1}{4}\|x\|_X \right).
\]
Thus,
\[
\rho \left( \frac{1}{4}\|x\|_X \right) > \frac{1}{k} \Rightarrow V_k(x) > 0.
\]
At the same time, if \( \rho \left( \frac{1}{4}\|x\|_X \right) < \frac{1}{k} \), we do not have a coercive estimate from below for \( V_k \). Hence, we define a Lyapunov candidate \( W : X \to \mathbb{R}_+ \) by
\[
W(x) := \sum_{k=1}^{\infty} \frac{2^{-k}}{1 + M(k, k)} V_k(x) \quad \forall x \in X.
\]
We have
\[
\psi_1(\|x\|_X) := \sum_{k=1}^{\infty} \frac{2^{-k}}{1 + M(k, k)} G_k \circ \rho \left( \frac{1}{4}\|x\|_X \right) \leq W(x) \leq 2\|x\|_X + C, \quad x \in X.
\]
Clearly, \( \psi_1(0) = 0 \). Since for each \( x \neq 0 \) there is some \( k \in \mathbb{N} \) such that \( \rho \left( \frac{1}{4}\|x\|_X \right) > \frac{1}{k} \), the condition \((15)\) ensures that \( \psi_1(r) > 0 \) for \( r > 0 \). Furthermore, for any \( r, s \geq 0 \) we have
\[
|\psi_1(r) - \psi_1(s)| \leq \sum_{k=1}^{\infty} \frac{2^{-k}}{1 + M(k, k)} G_k \circ \rho \left( \frac{1}{4}r \right) - G_k \circ \rho \left( \frac{1}{4}s \right) \leq \sum_{k=1}^{\infty} \frac{2^{-k}}{1 + M(k, k)} \left| G_k \circ \rho \left( \frac{1}{4}r \right) - G_k \circ \rho \left( \frac{1}{4}s \right) \right|.
\]
As both \( G_k, k \in \mathbb{N} \), and \( \rho \) are globally Lipschitz with unit Lipschitz constant, we proceed to
\[
|\psi_1(r) - \psi_1(s)| \leq \frac{1}{4} \sum_{k=1}^{\infty} \frac{2^{-k}}{1 + M(k, k)} |r - s| \leq \frac{1}{4} |r - s|,
\]
which shows Lipschitz continuity of \( \psi_1 \). Finally, as \( \rho \) is increasing to infinity, \( \psi_1 \) shares this property. Overall, \( \psi_1 \in K_\infty \).

Differentiating \( W \) along the trajectory, we obtain:
\[
\dot{W}(x) \leq \sum_{k=1}^{\infty} \frac{2^{-k}}{1 + M(k, k)} \dot{V}_k(x) \leq \sum_{k=1}^{\infty} \frac{2^{-k}}{1 + M(k, k)} V_k(x) = W(x).
\]
Now pick any $R > 0$ and $x, y \in X$ such that $\|x\|_X \leq R$ and $\|y\|_X \leq R$. Exploiting (14), we have
\[
|W(x) - W(y)| = \left| \sum_{k=1}^{\infty} \frac{2^{-k}}{1 + M(k,k)} (V_k(x) - V_k(y)) \right|
\leq \sum_{k=1}^{\infty} \frac{2^{-k}M(R,k)}{1 + M(k,k)} \|x - y\|_X
\leq \left( 1 + \sum_{k=1}^{\infty} \frac{2^{-k}M(R,k)}{1 + M(k,k)} \right) \|x - y\|_X.
\]
This shows that $W$ is a Lyapunov function for $\Sigma$, which is Lipschitz continuous on bounded balls. □

4. Robust forward completeness and its Lyapunov characterization

Take any $R \in \mathbb{R}_+ \cup \{\infty\}$, and assume that the inputs are restricted to a set
\[
D := \{ u \in U : \|u\|_U \leq R \}.
\]
The following variation of the BRS property is important in the context when $u$ plays a role of a disturbance rather than of an input, see [7, Definition 2.1]:

**Definition 4.1.** Consider a control system $\Sigma = (X, U, \phi)$, and let $D$ be the set of disturbances as defined by (17). We say that $\Sigma$ is robustly forward complete (RFC) if for any $C > 0$ and any $\tau > 0$, it holds that
\[
\sup \{ \|\phi(t,x,u)\|_X : \|x\|_X \leq C, \ u \in D, \ t \in [0,\tau] \} < \infty.
\]

Note that above the supremum is taken over all inputs in $D$. If $R$ is finite, then the BRS property clearly implies RFC. At the same time, if $R = +\infty$ (and thus $D = U$), RFC is essentially stronger than the BRS property. In particular, the scalar system $\dot{x} = xu$ with $X = \mathbb{R}$ and $D = U := L^\infty(\mathbb{R}_+, \mathbb{R})$ has bounded reachability sets, but it is not RFC. A simple example of an RFC system is given by the following scalar system with the same $X, U, D$:
\[
\dot{x} = \frac{1}{1 + |u(t)|} x.
\]

Characterization of the RFC property can be obtained similarly to that of the BRS property.

**Definition 4.2.** Let $D$ be the set of disturbances as defined by (17). We say that the flow of a control system $\Sigma = (X, U, \phi)$ is Lipschitz continuous on compact intervals uniformly in inputs from $D$, if for any $\tau > 0$ and any $r > 0$, there exists $L > 0$ so that
\[
x, y \in B_r, \ t \in [0,\tau], \ u \in D \quad \Rightarrow \quad \|\phi(t,x,u) - \phi(t,y,u)\|_X \leq L\|x - y\|_X.
\]

Next, we characterize the RFC property in Lyapunov terms in a fashion of Theorem 3.5.

**Theorem 4.3 (Criteria for RFC property).** Consider a control system $\Sigma = (X, U, \phi)$, with $U := L^\infty(\mathbb{R}_+, U)$ for a certain normed linear space $U$. Let $D$ be the set of disturbances as defined by (17). Let $\Sigma$ satisfy the BHC property and have a flow that is Lipschitz continuous on compact intervals uniformly in inputs from $D$.

The following statements are equivalent:

1. $\Sigma$ is RFC.
2. There exists a Lyapunov function $W$ for $\Sigma$.
3. The flow of $\Sigma$ is Lipschitz continuous on compact intervals uniformly in inputs from $D$. 

The proof of this theorem follows similar lines as in the proof of Theorem 3.5.
10 ANDRII MIRONCHENKO

(i) \( \Sigma \) is robustly forward complete.

(ii) There exists a continuous, increasing function \( \mu : \mathbb{R}_+^2 \to \mathbb{R}_+ \), such that for all \( x \in X, u \in D \) and all \( t \geq 0 \) we have

\[
\| \phi(t, x, u) \|_X \leq \mu(\|x\|_X, t).
\]

(iii) There exists a continuous function \( \mu : \mathbb{R}_+^2 \to \mathbb{R}_+ \) such that for all \( x \in X, u \in D \) and all \( t \geq 0 \) the inequality (19) holds.

(iv) There are \( \xi \in K_\infty \) and \( c > 0 \), such that for all \( t \geq 0, x \in X, u \in D \)

\[
\| \phi(t, x, u) \|_X \leq \xi(\|x\|_X) + \xi(t) + c.
\]

(v) There is a Lipschitz continuous function \( V : X \to \mathbb{R}_+ \), maps \( \psi_1, \psi_2 \in K_\infty \), and \( c > 0 \) such that

\[
\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X) + c, \quad x \in X,
\]

and there is a \( a > 0 \), such that for all \( x \in X \) and \( u \in D \), the following holds:

\[
\dot{V}_u(x) \leq aV(x).
\]

Proof. The only significant change is a construction of a Lyapunov function in the implication (iv) \( \Rightarrow \) (v).

Now we consider \( V_k : X \to \mathbb{R}_+ \), \( k \in \mathbb{N} \), defined for all \( x \in X \) as follows:

\[
V_k(x) := \sup_{u \in D} \sup_{t \geq 0} G_k \left( e^{-t} \rho \left( \frac{1}{3} \| \phi(t, x, u) \|_X \right) \right).
\]

In contrast to (11), we take supremum over all \( u \in D \) and use the weighting coefficient \( \frac{1}{3} \) as we have only three terms on the right-hand side of (iv). As an “RFC Lyapunov function” we can take again (10).

4.1. Relations to finite-dimensional results. In this final section, we show how Theorem 4.3 partially recovers [1, Theorem 2] in the special case of ODE systems.

Let \( \Sigma \) be an ODE system

\[
\dot{x} = f(x, u),
\]

where \( x(t) \in X := \mathbb{R}^n, u \in U := L^\infty(\mathbb{R}_+, \mathbb{R}^m) \), the following holds:

**Assumption 4.1.** \( f \) is continuous on \( \mathbb{R}^n \times \mathbb{R}^m \) and is Lipschitz continuous in \( x \) on bounded sets.

This assumption ensures that for any initial condition and any input, the corresponding mild solution (in the sense of Caratheodory) of (24) exists and is unique on a certain finite interval. Furthermore, the flow \( \phi \) is well-defined, and \( \Sigma = (X, U, \phi) \), is a well-defined control system with BIC property.

Let \( R \) be finite, and let \( D \) be the set of disturbances as defined by (17).

Recall, that a map \( f : \mathbb{R}^n \to \mathbb{R}_+ \) is called proper if the preimage of any compact subset of \( \mathbb{R}_+ \) is compact in \( \mathbb{R}^n \).

For systems (24), Theorem 4.3 takes the form

**Proposition 4.4.** Let Assumption 4.1 hold. System (24) is robustly forward complete if and only if there exists a proper and Lipschitz continuous function \( V : \mathbb{R}^n \to \mathbb{R}_+ \) such that the following exponential growth condition holds:

\[
\dot{V}_u(x) \leq V(x), \quad x \in \mathbb{R}^n, u \in D.
\]
Proof. It is an easy exercise to show that $V \in C(\mathbb{R}^n, \mathbb{R}_+)$ is proper if and only if there is $\psi_1 \in K_\infty$, such that $V(x) \geq \psi_1(|x|)$ for all $x \in \mathbb{R}^n$. Furthermore, $V(x) \leq \omega(|x|)$, where $\omega : r \mapsto \sup_{|y| \leq r} V(y)$ is a continuous nondecreasing function. Setting $\psi_2(r) := r + \omega(r) - \lim_{s \to +0} \omega(s)$, we obtain $V(x) \leq \psi_2(|x|) + \lim_{s \to +0} \omega(s)$. Thus, $V$ is proper if and only if the sandwich bounds (21) hold.

"⇐". Follows from the above argument and Theorem 4.3.

"⇒". As (24) is robustly forward complete and Assumption 4.1 holds, one can show that the flow $\phi$ is Lipschitz continuous on compact intervals uniformly in inputs from $D$. The rest follows from Theorem 4.3. □

Proposition 4.4 is a version of [11, Theorem 2]. The difference is that in Proposition 4.4 our Lyapunov function $V$ is Lipschitz continuous, while in [11, Theorem 2] the existence of an infinitely differentiable Lyapunov function with the same properties is shown. First, the authors in [11, Theorem 2] construct a Lipschitz continuous Lyapunov functional (using a procedure different from that in this paper) and afterward apply the smoothification procedure motivated by [10]. At the same time, in Proposition 4.4 it is not required that $f$ is Lipschitz continuous with respect to inputs, which this is assumed in [10].

Acknowledgements. A. Mironchenko is supported by the German Research Foundation (DFG) via the grant MI 1886/2-2.

References

[1] D. Angeli and E. D. Sontag. Forward completeness, unboundedness observability, and their Lyapunov characterizations. Systems & Control Letters, 38(4-5):209–217, 1999.
[2] J. R. Haddock. Liapunov functions and boundedness and global existence of solutions. Applicable Analysis, 2(4):321–330, 1972.
[3] D. Henry. Geometric Theory of Semilinear Parabolic Equations. Springer, Berlin, 1981.
[4] T. Iwamiya. Global existence of solutions to nonautonomous differential equations in Banach spaces. Hiroshima Mathematical Journal, 13(1):65–81, 1983.
[5] B. Jacob, A. Mironchenko, J. R. Partington, and F. Wirth. Noncoercive Lyapunov functions for input-to-state stability of infinite-dimensional systems. SIAM Journal on Control and Optimization, 58(5):2952–2978, 2020.
[6] A. Juscenko. Necessary and sufficient conditions for the global existence of solutions of systems of differential equations. Dok. Akad. Nauk BSSR, 11:867–869, 1967.
[7] I. Karafyllis and Z.-P. Jiang. Stability and Stabilization of Nonlinear Systems. Springer, London, 2011.
[8] I. Karafyllis, P. Pepe, A. Chaillet, and Y. Wang. Is global asymptotic stability necessarily uniform for time-delay systems? arXiv preprint arXiv:2202.11298, 2022.
[9] J. Kato and A. Strauss. On the global existence of solutions and Liapunov functions. Annali di Matematica Pura ed Applicata, 77(1):303–316, 1967.
[10] Y. Lin, E. D. Sontag, and Y. Wang. A smooth converse Lyapunov theorem for robust stability. SIAM Journal on Control and Optimization, 34(1):124–160, 1996.
[11] A. Mironchenko and F. Wirth. Characterizations of input-to-state stability for infinite-dimensional systems. IEEE Transactions on Automatic Control, 63(6):1602–1617, 2018.
[12] A. Mironchenko and F. Wirth. Non-coercive Lyapunov functions for infinite-dimensional systems. Journal of Differential Equations, 105:7038–7072, 2019.
[13] A. Pazy. Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, New York, 1983.
[14] T. Taniguchi. Global existence of solutions of differential inclusions. Journal of Mathematical Analysis and Applications, 166(1):41–51, 1992.
[15] G. Weiss. Admissibility of unbounded control operators. SIAM Journal on Control and Optimization, 27(3):527–545, 1989.
[16] A. Wintner. The non-local existence problem of ordinary differential equations. American Journal of Mathematics, 67(2):277–284, 1945.
Faculty of Computer Science and Mathematics, University of Passau, Germany

Email address: andrii.mironchenko@uni-passau.de