Overview and Warmup Example for Perturbation Theory with Instantons

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Abstract

The large $k$ asymptotics (perturbation series) for integrals of the form $\int_{\mathcal{F}} \mu e^{ikS}$, where $\mu$ is a smooth top form and $S$ is a smooth function on a manifold $\mathcal{F}$, both of which are invariant under the action of a symmetry group $\mathcal{G}$, may be computed using the stationary phase approximation. This perturbation series can be expressed as the integral of a top form on the space $\mathcal{M}$ of critical points of $S$ mod the action of $\mathcal{G}$. In this paper we overview a formulation of the “Feynman rules” computing this top form and a proof that the perturbation series one obtains is independent of the choice of metric on $\mathcal{F}$ needed to define it. We also overview how this definition can be adapted to the context of 3-dimensional Chern–Simons quantum field theory where $\mathcal{F}$ is infinite dimensional. This results in a construction of new differential invariants depending on a closed, oriented 3-manifold $M$ together with a choice of smooth component of the moduli space of flat connections on $M$ with compact structure group $G$. To make this paper more accessible we warm up with a trivial example and only give an outline of the proof that one obtains invariants in the Chern–Simons case. Full details will appear elsewhere.

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1 Introduction

In previous joint work with I.M. Singer [3, 4], we defined Chern–Simons perturbation theory about an acyclic flat connection on some principle bundle \( P \) with compact structure group \( G \) over a closed, oriented 3-manifold \( M \). In my talks at the conference I explained how to generalize this result to perturbation theory about a smooth component \( \mathcal{M} \) of the moduli space of flat connections. Specifically, I proved the following theorem.

**Theorem 1.1** Let \( \mathcal{M} \) be a smooth component of the moduli space of flat connections on \( P \rightarrow M \). Then Chern-Simons perturbation theory about \( \mathcal{M} \) defines a differential invariant

\[
Z_{k}^{\text{pert}}(M, \mathcal{M}) \in PS(k)
\]  

depending on the choice of \( M \) and \( \mathcal{M} \).

*Remark 1.* Let \( \mathcal{S} \) be the set of all flat connections on \( P \) representing elements of \( \mathcal{M} \). Our smoothness assumption on \( \mathcal{M} \) includes the requirement that \( \mathcal{S} \rightarrow \mathcal{M} \) be a smooth bundle (with fibers a homogeneous space \( G/\mathcal{H} \)) and that the tangent space \( T_{A_{0}}\mathcal{S} \) to \( \mathcal{S} \) at a point \( A_{0} \) be equal to the deformation tangent space, that is, the kernel of the exterior derivative operator acting on the space \( \Omega^{1}(M; g) \) of 1-forms on \( M \) with values in the adjoint bundle associated to \( P \).

*Remark 2.* The ring, \( PS(k) \), of perturbation series is the ring generated by formal power series in \( k^{-1} \), fractional powers of \( k \), and oscillatory exponentials in \( k \) of the form \( e^{ikS_{0}} \). Since \( k \) only takes on integer values, \( S_{0} \) is an element of \( \mathbb{R}/2\pi\mathbb{Z} \).

Let us explain our strategy for arriving at a definition of “Chern–Simons perturbation theory about \( \mathcal{M} \)”. To begin, we consider a finite dimensional integral of the form

\[
Z_{k} = \frac{1}{\text{vol}(\mathcal{G})} \int_{A \in \mathcal{F}} \mu e^{ikS(A)}
\]

where \( S : \mathcal{F} \rightarrow \mathbb{R} \) is a Morse-Bott function, and \( \mu \) is a smooth measure on a manifold \( \mathcal{F} \), both of which are invariant under the action of a compact symmetry group \( \mathcal{G} \) with a bi-invariant volume form of total volume \( \text{vol}(\mathcal{G}) \).

Using a partition of unity, we can break up the above integral into a sum of contributions from the integral near each of the components of the critical
point set of $S$ and a contribution from the integral away from the critical point set. The large $k$ asymptotics of the latter vanish faster than any power of $k$. The large $k$ asymptotics of the integral (2) near a component $S$ of the critical point set is what we mean by “perturbation theory” about $S$. Since we can also think of the integral as an integral over $\mathcal{F}/\mathcal{G}$, we can equally well speak of this as perturbation theory about the quotient space $\mathcal{M} = S/\mathcal{G}$. For this finite dimensional integral, the stationary phase approximation yields an algorithm to calculate perturbation theory near $S$. For simplicity, we assume $\mathcal{F}$, $S$, and $\mathcal{M}$ are oriented so that a smooth measure on any of these spaces is just a smooth top form.

The challenge is to formulate the stationary phase approximation computing the large $k$ asymptotics of (2) in a way such that the result can be generalized to the case of Chern–Simons theory where the space $\mathcal{F}$ becomes the infinite dimensional space of connections on a principal bundle $P$, the group $\mathcal{G}$ becomes the group of base preserving automorphisms of $P$, the Morse-Bott function $S$ becomes the Chern–Simons functional, $\mathcal{M}$ is a smooth component of the moduli space of flat connections on $P$, and $k$ is a positive integer called the level of the theory. The generalization of the stationary phase algorithm to the field theory case involves integrals over multiple copies of the spacetime manifold $M$. In most quantum field theories these integrals diverge and one has to perform the procedure of perturbative regularization and renormalization. Fortunately, just as in [3], it is possible to package the finite dimensional answer in such a way that its generalization to the Chern-Simons case involves only convergent integrals. (For experts, there is a single point-splitting regularization one must introduce for the “tad-pole”.) To formulate a concrete algorithm to compute the perturbation theory in the case of a finite dimensional integral, we need to pick a metric $g$ in the space $\text{Met}$ of $\mathcal{G}$ invariant metrics on $\mathcal{F}$. The result, however, is a priori independent of $g$ because the integral (2) that one is calculating the asymptotics of does not depend on $g$. In the Chern–Simons case, the space $\text{Met}$ will denote the space of Riemannian metrics on $M$, each of which determines a $\mathcal{G}$ invariant metric on the space $\mathcal{F}$ of connections. The difficulty is that there is no a priori proof of the independence of $g$ in the Chern–Simons case since the functional integral is ill-defined. What we do is formulate not only the algorithm to compute the stationary phase approximation but also a proof of independence of $g$ in the finite dimensional case in a way that can be carried over to the quantum field theory problem.
To calculate the stationary phase approximation in the finite dimensional case what one has to do is to first integrate over the directions normal to the set $S$ of critical points of $S$ and then to subsequently integrate over $S$. The standard method of stationary phase about an isolated critical point (as for example captured in the Feynman rules described in the Appendix) applies to the integral in the normal directions. To do this properly however, one needs to carefully keep track of what happens to the differential forms as one makes the split between the normal directions and the directions tangent to $S$. We pick a metric $g \in \text{Met}$ in order to identify “normal directions” with directions orthogonal to the tangent space of $S$. To have a mechanism to keep track of the dependence on the metric, we introduce the evaluation map

$$E : N \times \text{Met} \rightarrow \mathcal{F}. \quad (3)$$

Here $N \rightarrow S$ is the normal bundle to $S$ in $\mathcal{F}$. The fiber of the bundle $N \times \text{Met} \rightarrow S \times \text{Met}$ above a point $(A_0, g)$ is just the orthocomplement of the subspace $T_{A_0}S$ in $T_{A_0}\mathcal{F}$ with respect to the metric $g$. The evaluation map takes an element $B$ of this space and sends it to the point determined by the exponential map coming from the geodesic flow from $g$, i.e.

$$E((A_0, g), B) = \exp_{A_0}(B). \quad (4)$$

For brevity, we will write $\hat{N}$ for $N \times \text{Met}$, $\hat{S}$ for $S \times \text{Met}$, and $\hat{\mathcal{M}}$ for $\mathcal{M} \times \text{Met}$. In general, objects adorned with a hat fiber over Met or depend on a choice of $g \in \text{Met}$.

In addition, to get a formula that will be well defined and not involve infinite dimensional integrals in the Chern–Simons example, we must make use of invariance under the group $G$ to replace the ill-defined integral over the infinite dimensional space $\mathcal{S}$ by a well-defined integral over the finite dimensional space $\mathcal{M} = S/G$ of gauge equivalence classes of flat connections.

So far we have stated the basic conceptual idea. There are quite a few technical details where we generalize some standard tricks of supermanifold theory and gauge fixing theory in order to come up with an explicit formula for the perturbation series. For the main result, the definition can be stated in terms of the structures on the deformation complex for the calculation of $\mathcal{M}$ that arise because we have metrics present and so may define Hodge theory and we have the Taylor series for the action $S$ near $\mathcal{M}$. Even though we expended a lot of effort in order to find the “right formulation” of the definition, it is rather technical. To make this paper accessible to non-experts, we
will focus our attention in §2 to a trivial example which illustrates some of the important points that one needs to understand in formulating the general perturbation theory. A more detailed overview of the general finite dimensional construction and the application to Chern–Simons theory is given in §3 and §4, but the complete details will appear elsewhere [2]. An appendix reviewing the derivation of the Feynman rules and a list of notations are included at the end for the reader’s convenience.

\section{Trivial Example}

For our trivial example we take $\mathcal{F}$ to be $\mathbb{R}^2$ with the usual $(x, y)$ coordinates, $\mathcal{G}$ to be the trivial group, $\mu$ to be the standard area form $dx \wedge dy$, and the Morse-Bott function $S$ to have the form

$$S(x, y) = S_0 + \frac{1}{2}H(x)y^2 + \frac{1}{3!}V_3(x)y^3 + \frac{1}{4!}V_4(x)y^4.$$  \hfill (5)

Here $S_0$ is a constant, $H(x)$ is any positive function of $x$ and we assume for simplicity that the Taylor series with respect to $y$ stops at the fourth order term. The positivity of $H(x)$ insures that the real axis $\mathcal{S} = \{(x_0, 0); x_0 \in \mathbb{R}\}$ is a smooth component of the critical point set. For our purposes, we will just assume that there are no other components. Since we are assuming $\mathcal{G}$ is trivial, the “moduli space” $\mathcal{M}$ is the same as $\mathcal{S}$. To be honest, we should note that the integral (2) does not actually converge. To make it converge we should replace $\mathcal{F}$ by $S^1 \times S^1$ or multiply $dx \wedge dy$ by a bump function supported near the origin, but we won’t introduce notation for this.

The stationary phase approximation for this example is of course just the integral over $\mathcal{S}$ of the result obtained by doing the stationary phase approximation to the integral over $y$. Abstractly, one can think of the integral over $y$ as the integral over the normal directions to $\mathcal{S}$ with respect to the standard metric on $\mathbb{R}^2$. Our goal is to see very explicitly why one gets the same result if one uses a different metric. To keep things simple, we take $\text{Met} = \mathbb{R}$ and identify $g \in \text{Met}$ with the translationally invariant metric represented by the matrix $\begin{bmatrix} 1 & -g \\ -g & 1 \end{bmatrix}$. The normal directions to $\mathcal{S}$ are then lines with slope $g$ relative to the $y$ axis. Note that the normal bundle to $\mathcal{S}$ is trivial (as one would expect in such a trivial example). We may identify the total space of $\tilde{\mathcal{N}}$ with $\mathbb{R}^3$, with coordinates $(x_0, g, t)$; $x_0$ determines a point
Figure 1: Setup for Trivial Example

\[ A_0 = (x_0, 0) \in S; \ g \text{ is a point in Met; and } t \text{ is the coordinate on the fibers of } \mathcal{N}. \text{ Since the exponential map is linear, we have} \]

\[ (x, y) = E(x_0, g, t) = (x_0, 0) + t(g, 1), \quad t \in \mathbb{R} \]  

\[ \text{going through } (x_0, 0) \text{ with slope } g. \text{ Our setup is depicted in Figure 1.} \]

We may now calculate.

\[ E^*(S)(x_0, g, t) = S(x_0 + tg, t) = S_0 + H(x_0) \frac{t^2}{2} \]

\[ + (3gH'(x_0) + V_3(x_0)) \frac{t^3}{3!} + \cdots \]  

\[ E^*(\mu)(x_0, g, t) = E^*(dx \wedge dy) = [dx_0 + t \, dg + g \, dt] \wedge dt \]

\[ = \left[ 1 + t \, dg \, I \left( \frac{\partial}{\partial x_0} \right) \right] \,(dx_0 \wedge dt). \]

Note that the quantity in brackets in the last line has both an operator of exterior product with \( dg \) and an interior product operator, \( I\left( \frac{\partial}{\partial x_0} \right) \), in the
direction of $\frac{\partial}{\partial x_0}$. If there were several interior product operators (as there would be in a higher dimensional example), we would write all of them on the right and view them as a multivector, i.e. as an element of $\Lambda^*(T\mathcal{S})$. Accordingly we view the quantity in brackets as a form with values in $\Lambda^*(T\mathcal{S})$.

The symbol $\|$ in the last line symbolizes that a form with values in multivectors is acting by a combination of wedge product and inner product.

To calculate the stationary phase approximation to $Z^{\text{pert}}_k$ we must first hold $g \in \text{Met}$ and $A_0 = (x_0, 0) \in \mathcal{S}$ fixed and calculate the stationary phase approximation to the integral over the normal directions (i.e. over $t$). This gives us a quantity which we write

$$\hat{Z}^{\text{pert}}_k = \int_{t \in \mathbb{R}} \exp(i k S)(x_0, g, t) \in \Omega^1_{\text{cl}}(\mathcal{S} \times \text{Met}) \otimes \text{PS}(k).$$

The symbol $\int^{\text{pert}}$ means that we are not to actually evaluate the integral, but merely to calculate its stationary phase approximation. This is given explicitly by the Feynman rules which are briefly recalled in the appendix. Notice that $\hat{Z}^{\text{pert}}_k$ is a closed form on $\mathcal{S} \times \text{Met}$ (which is denoted by the subscript $\text{cl}$ on $\Omega^1_{\text{cl}}$). This follows because it is a (perturbative) push-forward integral of a closed form. $Z^{\text{pert}}_k$ is the integral of this quantity over $\mathcal{S}$,

$$Z^{\text{pert}}_k = \int_{A_0 = (x_0, 0) \in \mathcal{S}} \hat{Z}^{\text{pert}}_k \in \Omega^0_{\text{cl}}(\text{Met}) \otimes \text{PS}(k) = \text{PS}(k).$$

Since $\hat{Z}^{\text{pert}}_k$ was a closed 1-form on $\mathcal{S} \times \text{Met}$, $Z^{\text{pert}}_k$ is a closed 0-form on Met, i.e. it is metric independent. Of course, in this finite dimensional example, $Z^{\text{pert}}_k$ actually is the asymptotic series for $Z_k$ and so is $a \text{ priori}$ independent of the metric. In the case of Chern–Simons field theory, however, to prove metric independence, we need to give a direct proof that $\hat{Z}^{\text{pert}}_k$ is closed. As a warmup, we will verify explicitly $\hat{Z}^{\text{pert}}_k$ is closed to subleading order for the trivial example.

Applying the result in Appendix A, carrying all of the operators on forms along for the ride, we find

$$\hat{Z}^{\text{pert}}_k = \hat{Z}^{hl}_{\text{pert}} | \hat{Z}^{sc}_{\text{pert}}$$

$$\hat{Z}^{sc}_{\text{pert}} = \sqrt{\frac{2\pi i}{k}} \int_{x_0} \frac{e^{i k S}}{\sqrt{H(x_0)}} dx_0 \in \Omega^1_{\text{el}}(\mathcal{S}) \otimes \text{PS}(k) \subset \Omega^1_{\text{el}}(\hat{\mathcal{S}}) \otimes \text{PS}(k)$$

$$\hat{Z}^{hl}_{\text{pert}} = \sum_{\text{graph } \Gamma} \frac{1}{S(\Gamma)} I(\Gamma) \in \left[ \oplus \Omega^l(\hat{\mathcal{S}}; \Lambda^l(T\mathcal{S})) \right] \otimes \mathbb{C}[\tfrac{1}{k}].$$
The leading order “semi-classical” term $\hat{Z}_{k}^{sc}$ is not only closed, but manifestly metric independent. This property carries through (when appropriate counterterms are added) to the case of quantum field theory. The Feynman rules for the graphs contributing to $\hat{Z}_{k}^{hl}$ are as follows:

\begin{align}
-\frac{1}{ik}H(x_0)^{-1} \\
\left. ik \left( \frac{\partial^3 S}{\partial t^3} \right) \right|_{x_0} = ik \left[ 3H'(x_0)g + V_3(x_0) \right] \\
\left. ik \left( \frac{\partial^4 S}{\partial t^4} \right) \right|_{x_0} = ik \left[ 6H''(x_0)g^2 + 4V_3'(x_0)g + V_4(x_0) \right] \\
\vdots \\
\int dg I \left( \frac{\partial}{\partial x^0} \right)
\end{align}

We consider the vertex in (17) an external vertex and only allow it to appear at most once in any given graph because it comes from the term $t\, dg I \left( \frac{\partial}{\partial x^0} \right)$ in $E^*(\mu)$ (see (17)) rather than from a term in the Taylor expansion in $S$.

We may expand the “higher loop” term $\hat{Z}_{k}^{hl}$ in powers of $k$:

\begin{equation}
\hat{Z}_{k}^{hl} = 1 + \sum_{l>1} \hat{I}_l (-ik)^{1-l}
\end{equation}

where $(-ik)^{1-l}\hat{I}_l$ is a sum of $I(\Gamma)/S(\Gamma)$ over all graphs $\Gamma$ with

\begin{equation}
\text{number of edges} - \text{number of internal vertices} = l - 1.
\end{equation}

The graphs $\Gamma$ contributing to $\hat{I}_l$ are the graphs such that

\begin{equation}
b_1(\Gamma) + \text{number of external vertices} = l + (b_0(\Gamma) - 1),
\end{equation}

where $b_0(\Gamma)$ and $b_1(\Gamma)$ are the Betti numbers of $\Gamma$ (the number of components and the number of independent loops, resp.). Equivalently, a graph $\Gamma$ contributes to $\hat{I}_l$ if the graph obtained from $\Gamma$ by adding a loop at each external vertex has $l + (b_0(\Gamma) - 1)$ loops. We refer to $\hat{I}_l$ as the $l$’th order or $l$-loop piece of $\hat{Z}_{k}^{hl}$.
The subleading term $\hat{I}_2$ is given by

\[ (-ik)^{-1}\hat{I}_2 = \frac{1}{12} \quad + \quad \frac{1}{8} \quad + \quad \frac{1}{8} \quad + \quad \frac{1}{2} \quad \] (21)

Here the picture of a graph $\Gamma$ represents the term $I(\Gamma)$. Evaluating this, we obtain

\[ \hat{I}_2 = \left[ \frac{1}{12} + \frac{1}{8} \right] H^{-3} (3H'g + V_3)^2 - \frac{1}{8} H^{-2} (6H''g^2 + 4V'_3g + V_4) \]
\[ - \frac{1}{2} \left( H^{-2} (3H'g + V_3) dg I\left( \frac{\partial}{\partial x_0} \right) \right) \] (22)

Finally, the subleading contribution to $\hat{Z}_{pert}^k$ equals the $k$-dependent term $\frac{e^{ikS_0}}{ik} \sqrt{2\pi i k}$ times the following:

\[ \hat{I}_2 \left( \frac{dx_0}{\sqrt{H}} \right) = \left( \left[ \frac{1}{12} + \frac{1}{8} \right] H^{-\frac{7}{2}} V_3^2 - \frac{1}{8} H^{-\frac{5}{2}} V_4 \right) dx_0 \]
\[ + \frac{1}{8} \left[ dx_0 \frac{\partial}{\partial x_0} + dg \frac{\partial}{\partial g} \right] \left( -6 H^{-\frac{7}{2}} H'g^2 - 4 H^{-\frac{5}{2}} V_3g \right) \] (23)

The first term on the right in (23) is a closed form and the second term is exact on $S \times \text{Met}$. Thus the final result is a closed form on $S \times \text{Met}$ as desired. We could, of course, have obtained the first term (which is the only term which makes a contribution to $Z_{pert}^k$) much more easily by only considering the usual metric on $F = \mathbb{R}^2$ with slope $g$ equal to 0. The point to notice, however, is that the direct verification that $\hat{Z}_{pert}^k$ is closed, even at the lowest subleading order in this most trivial example, involves a delicate cancellation among several terms. In particular notice that it is only after summing over graphs with varying numbers of vertices that we obtain a closed form.

3 Outline for the General Finite Dimensional Case

In this section we give a very sketchy outline of our formulation of the perturbation series about a component $S$ of the critical point set of $S$ for a general integral of the form (2). We assume that $F$ is an affine space, that $G$ acts affinely, $S$ is a cubic function, and that $\text{Met}$ consists of translationally invariant metrics, as is the case for Chern–Simons theory.
3.1 Strategy in Language Using Differential Forms

The desired perturbation series may be expressed as an integral over $\mathcal{M}$,

$$Z^\text{pert}_k(\mathcal{M}) = \int_\mathcal{M} \hat{Z}^\text{pert}_k \in \Omega^0(\text{Met}) \otimes \text{PS}(k) = \text{PS}(k).$$

(24)

$Z^\text{pert}_k(\mathcal{M})$ is a closed 0-form on Met (and hence a constant) because the integrand here is a closed form on $\mathcal{M} \times \text{Met}$,

$$\hat{Z}^\text{pert}_k \in \Omega^{|\mathcal{M}|}(\mathcal{M} \times \text{Met}) \otimes \text{PS}(k).$$

(25)

Note that we denote the dimension of the manifold $X$ by $|X|$. Our goal is to find an explicit formula for $\hat{Z}^\text{pert}_k$ and a proof that it is closed which will carry over to the case of Chern–Simons quantum field theory.

$\hat{Z}^\text{pert}_k$ is the image of the integrand $\mu e^{ikS}$ of (2) under the following sequence of maps:

$$\Omega^{|\mathcal{F}|}_\text{cl}(\mathcal{F})^\mathcal{G} \xrightarrow{E^*} \Omega^{|\mathcal{F}|}(\mathcal{N})^\mathcal{G} \xrightarrow{I_{\nabla \mathcal{N} \rightarrow \mathcal{S}}} \Omega^{|\mathcal{S}|}_\text{cl}(\mathcal{S}; \Omega^{|\mathcal{F}|-|\mathcal{S}|}_\text{vert}(\mathcal{N}))^\mathcal{G}$$

$$\xrightarrow{\int^\text{pert}_{\mathcal{N} \rightarrow \mathcal{S}}} \Omega^{|\mathcal{S}|}_\text{cl}(\mathcal{S}) \otimes \text{PS}(k) \xrightarrow{\text{vol}(\mathcal{H})\mu_{\mathcal{G}/\mathcal{H}}}^{-1} \Omega^{|\mathcal{M}|}_\text{cl}(\mathcal{S})^\text{basic} \otimes \text{PS}(k)$$

$$\cong \Omega^{|\mathcal{M}|}_\text{cl}(\mathcal{M}) \otimes \text{PS}(k)$$

(26)

The first map, $E^*$, is just pull back under the evaluation map. To define the second map we must exhibit a connection on the vector bundle $\mathcal{N} \rightarrow \mathcal{S}$ (the normal bundle crossed with Met), which we view as a complement $T_{\text{hor}}\mathcal{N}$ to $T_{\text{vert}}\mathcal{N}$ within $T\mathcal{N}$. To handle the group theory in a way that will gives us a sensible answer in the field theory problem (which might be interpreted by physicists as a generalization of a familiar method of “gauge fixing”), we require that directions along the group orbits on $\mathcal{N}$ be horizontal. The remaining horizontal direction for the connection $\nabla_{\mathcal{N} \rightarrow \mathcal{S}}$ are defined using the fact that $\mathcal{N} \rightarrow \mathcal{S}$ is a subbundle of the Riemannian bundle with connection $T\mathcal{F}|_{\mathcal{S}} \times \text{Met} \rightarrow \mathcal{S}$. The map $I_{\nabla \mathcal{N} \rightarrow \mathcal{S}}$ comes from the identification

$$\Lambda^*(T^*\mathcal{N}) = \Lambda^*(T^*_\text{hor}\mathcal{N}) \otimes \Lambda^*(T^*_\text{vert}\mathcal{N}).$$

(27)

1 More precisely, if there is an isotropy group $\mathcal{H}$, we require the direction generated by elements of Lie($\mathcal{G}$) orthogonal to Lie($\mathcal{H}$) be horizontal.
The third map, \( \int_{\hat{N} \to \hat{S}}^{\text{pert}} \), in (26) is just the “perturbative integral”, or stationary phase approximation, to the integral over the fibers of \( \hat{N} \to \hat{S} \). This may be calculated using the Feynman rules as described in Appendix A since the Morse-Bott condition on \( S \) insures that the zero vector is an isolated critical point of the restriction of \( S \) to a given fiber of \( \hat{N} \to \hat{S} \). The fourth map, \( \left[ \text{vol}(\mathcal{H})\mu_{\mathcal{G}/\mathcal{H}} \right]^{-1} \), is just division by the volume form along the group orbits (which is normalized so that it’s integral over an orbit is \( \text{vol}(\mathcal{G}) \)). A precise formulation of what is meant by this division requires the use of the natural connection \( \nabla_{\hat{S} \to \hat{M}} \) on \( \hat{S} \to \hat{M} \). (Then the volume form \( \mu_{\mathcal{G}/\mathcal{H}} \) on the coset space \( \mathcal{G}/\mathcal{H} \), which is diffeomorphic to the group orbits, determines a vertical differential form on \( \hat{S} \) which is what we actually divide by.) The superscript \( \mathcal{G} \) throughout (26) indicates that we are working with \( \mathcal{G} \)-invariant subspaces. The subscript “basic” on \( \Omega^* \left( \hat{S} \right) \) in the space appearing before the final arrow indicates that the forms are both \( \mathcal{G} \)-invariant and annihilated by the interior product operator with any vector field along the group orbits. The isomorphism in the final map in (26) is the inverse of the pullback map associated to the projection from \( \hat{S} \) to \( \hat{M} \).

### 3.2 Explicit Formulas Obtained Using Supervariables

To obtain an explicit formula embodying the sequence (26) of maps, we introduce some supervariables. We let \( A \) denote a variable in \( F \), and introduce Fermionic variables \( \chi \) valued in \( T^*F_A \) and \( \delta A \) valued in \( TF_A \). We may identify a function of \( \delta A \) with an element of \( \Lambda^* \left( T^*F_A \right) \) and a function of \( \chi \) with an element of \( \Lambda^*(TF_A) \). We will not review the theory of supermanifolds\(^5\) for readers who are not familiar, but simply present the following Rosetta stone which will suffice for our purposes:

\[
\int_{A \in F} \mu e^{ikS} = \int_{A \in F} \int_{\chi \in [T^*F_A]-} \int_{\delta A \in [TF_A]-} e^{i<\varphi,\delta A>+ikS(A)} \tag{28}
\]

\[
\Omega^*(F; \Lambda^*(TF)) = C^\infty \left( \{A_0, \chi, \delta A\} \right) \tag{29}
\]

\[
dx^\mu \iff \delta A^\mu \tag{30}
\]

\[
I(\frac{\partial}{\partial A^\mu}) \iff \chi^\mu \tag{31}
\]

\[
\Omega^1(F; \Lambda^1(TF)) \ni \mathbb{1} \iff <\chi, \delta A> \tag{32}
\]

The minus subscripts on \([T^*F_A]-\) and \([TF_A]-\) are just there to remind us that the variables taking values in those space are Fermionic. This integral
is manifestly independent of the choice of $g \in \text{Met}$, but for later purposes we should consider this as one of our variables and also introduce a Fermionic variable $\delta g \in T_{\text{Met}}$, so that function of the pair $(g, \delta g)$ correspond to forms on Met.

The image of $\mu e^{ikS}$ under (26) is calculated using a sequence of changes of supervariables and (perturbative) fiber integrals. To define these, we first define the deformation chain complex associated to the “moduli problem” of calculating $\mathcal{M}$ and its tangent space. At a point $A_0 \in \mathcal{S}$, it is given by:

$$0 \longrightarrow \text{Lie}(\mathcal{G}) \overset{T_{A_0}}{\longrightarrow} T\mathcal{F}_{A_0} \overset{H(A_0)}{\longrightarrow} T^*\mathcal{F}_{A_0} \overset{-(T_{A_0})^T}{\longrightarrow} \text{Lie}(\mathcal{G})^* \longrightarrow 0$$

$$0 \longrightarrow \Omega^0_M \overset{D^0_{A_0}}{\longrightarrow} \Omega^1_M \overset{D^1_{A_0}}{\longrightarrow} \Omega^2_M \overset{D^2_{A_0}}{\longrightarrow} \Omega^3_M \longrightarrow 0.$$  \hspace{1cm} (33)

Here $T_{A_0}$ is the infinitesimal action of the group $\mathcal{G}$ at the point $A_0$ and $H(A_0)$ is the Hessian of $S$ at $A_0$. Diagram (33) is a definition of the complex $\Omega^*_M$ and the differential $D^*_M$. Note that the notation for the generic finite dimensional situation we are currently considering is also ideally suited for the Chern–Simons quantum field theory we will be considering. These complexes fit together to form a bundle $\hat{\Omega}^*_M$ of chain complexes over $\hat{S}$ (which is trivial in the Met directions). In fact, using the metric, the connections $\nabla^{\hat{S} \rightarrow \hat{M}}$ and $\nabla^{\hat{S} \rightarrow \hat{M}}$, and the Taylor expansion of the function $S$ near $\mathcal{S}$, $\hat{\Omega}^*_M$ becomes a bundle of exterior algebras with Hodge decomposition and connection, which are all invariant under an action of $\mathcal{G}$ which lifts the action of $\mathcal{G}$ on $\mathcal{S}$. (There is also a product structure which satisfies a variant of the rules for the product in a differential graded algebra) The Hodge structure means that we can decompose $\hat{\Omega}^*_M$ as a direct sum of a piece $\hat{\Omega}^*_d$ which equals the image of $D^*$, a piece $\hat{\Omega}^*_a$ which is the image of the adjoint of $D^*$, and a piece $\hat{\Omega}^*_h$ which is the orthocomplement of the other two:

$$\hat{\Omega}^*_M = \hat{\Omega}^*_a \oplus \hat{\Omega}^*_d \oplus \hat{\Omega}^*_h.$$  \hspace{1cm} (34)

The name $\Omega^*_M$ was chosen, because in the case of Chern–Simons theory

$$\Omega^*_M = \Omega^*(M; \mathfrak{g})$$  \hspace{1cm} (35)

and $D^*_{A_0}$ is just the exterior derivative twisted by $A_0$. The Hodge structure in the Chern–Simons case is the familiar one from Hodge theory of differential forms, and the product structure is a combination of wedge product and
Lie bracket. In general, we write the product as a bracket operation \([\cdot, \cdot] : \Omega^j_M \otimes \Omega^k_M \to \Omega^{j+k}_M\). The bracket with 0 forms comes from the Lie algebra action. The bracket on one forms \(B \in T\mathcal{F}_{A_0} = [\hat{\Omega}^1_M]_{(A_0, g)}\) is determined by the cubic term in \(S\) so that

\[
S(A_0 + B) = S(A_0) + \frac{1}{2} < B, H(A_0)B > + \frac{1}{6} < B, [B, B] > .
\]  

(36)

We remark that all of the above goes through even when \(\mathcal{F}\) is not affine, or \(S\) is not cubic, except then the fibers of \(\hat{\Omega}^*_M \to \hat{\mathcal{S}}\) would come equipped with even more structure that “keeps track” of the non-linearities. If we wanted to allow for this, we would have to add some extra terms below.

We may now define our change of variables. The variables \(g\) and \(\delta g\) stay as before. \(A\) is replaced by a point \(A_0 \in \mathcal{S}\) and a normal direction \(B \in N(A_0, g)\):

\[
A = A_0 + B.
\]  

(37)

Note that \(N_{(A_0, g)}\) just equals \(\hat{\Omega}^1_{h}(A_0, g)\) since the normal directions are orthogonal to \(T_{A_0} \mathcal{S}\) and the latter space is \(\text{Ker}(H(A_0)) = \text{Ker}(D^1_{A_0})\). \(\chi\) is replaced by its harmonic piece \(\chi_h \in \hat{\Omega}^2_{h}(A_0, g)\), its exact piece \(\chi_d \in \hat{\Omega}^2_{d}(A_0, g)\), and its coexact piece \(\chi_\delta \in \hat{\Omega}^2_{\delta}(A_0, g)\):

\[
\chi = \chi_h + \chi_d + \chi_\delta.
\]  

(38)

Finally, \(\delta A\) is replaced by the variables \(\delta A_{0,h}\) belonging to \(\hat{\Omega}^1_{h}(A_0, g)\), \(c\) belonging to \(\hat{\Omega}^0_{h}(A_0, g)\) (the subspace of \(\text{Lie}(\mathcal{G})\) orthogonal to the isotropy group at \(A_0\)), and \(\delta_{\text{vert}} B\) belonging to \(\hat{\Omega}^1_{\delta}(A_0, g)\). In order for these to combine into the element \(\delta A\) behaving as an element of \(T_A \mathcal{F}\), we set

\[
\delta A = \delta A_{0,h} + T_A(c) + \delta_{\text{vert}} B + (\delta(\delta A_{0,h}, \delta g) P_N) B.
\]  

(39)

Here \(P_N\) is the orthogonal projection operator from \(\hat{\Omega}^1 = T\mathcal{F}\) to \(\hat{\mathcal{N}} = \hat{\Omega}^1_{\delta}\), and the expression \(\delta(\delta A_{0,h}, \delta g)\) stands for the covariant derivative operator acting in the direction of \((\delta A_{0,h}, \delta g)\). The second term in (39) is in the direction of the orbit through \(A^\prime\),

\[
T_A(c) = T_{A_0}(c) + [c, B] = -(D^1(c) + [B, c]).
\]  

(40)

In summary, equations (37), (38), and (39) define a change of variables from the variables on the left below (with \(A\) bosonic and \(\chi\) and \(\delta A\) Fermionic) to
the variables on the right below (with the first group Bosonic and the last
two groups Fermionic):

\[ A, \chi, \delta A \longleftrightarrow (A_0, B), (\chi_h, \chi_d, \chi_\delta), (\delta A_{0h}, \delta_{\text{vert}} B, c). \]  

(41)

We write (28) in these new variables, trivially integrate out the variables \( \chi_d \) and \( \delta_{\text{vert}} B \), and do a perturbative integral over the combined variable \( 2A = c + B + \chi_\delta \in [\hat{\Omega}_M^*]_{(A_0, g)}. \) 

(42)

The result of these operations depends on the remaining variables \( A_0, \delta A_{0h}, g, \delta g, \) and \( \chi_h \). This result equals the integrand on the right hand side of the following formula

\[
\hat{Z}_{\text{pert}}^k(A_0, \delta A_{0h}, g, \delta g) = \int_{\chi_h} e^{i<\chi_h, \delta A_{0h}>} \hat{Z}_{\text{hl}}^k(A_0, \delta A_{0h}, g, \delta g, \chi_h) \hat{Z}_{\text{sc}}^k(A_0, \delta A_{0h}).
\]

(Note that the exponential term in the integrand is just the piece of \( e^{i<\chi, \delta A>} \) which does not involve any of the variables \( A, \chi_d, \) or \( \delta_{\text{vert}} B \) already integrated over.) \( \hat{Z}_{\text{sc}}^k \) is the semi-classical approximation obtained in doing the operations above. \( \hat{Z}_{\text{hl}}^k \) comes from the higher loop Feynman rules. It is possible to write it in the form

\[
\hat{Z}_{\text{hl}}^k(A_0, \delta A_{0h}, g, \delta g, \chi_h) = e^{ik\frac{1}{2} \int \chi_h \frac{\partial}{\partial t} J} |J = 0 e^{\frac{1}{2k} \Phi}. \]

(43)

Here \( J \in [\hat{\Omega}_M^*]_{(A_0, g)} \) and \( \Phi \) is bilinear in the pair \( (J, \chi_h) \).

### 3.3 Translating Back to Language of Differential Forms

When we translate back from the language of supermanifolds, we obtain explicit formulas for

\[
\hat{Z}_{\text{sc}}^k \in \Omega^{[M]}(S)_{\text{basic}} \otimes \text{PS}(k) = \Omega^{[M]}(\hat{M}) \otimes \text{PS}(k) \subset \Omega^{[\hat{M}]}(\hat{M}) \otimes \text{PS}(k),
\]

and

\[
\hat{Z}_{\text{hl}}^k \in \left[ \bigoplus_{l} \Omega^l(S; \Lambda^l(\hat{\Omega}_M^1))_{\text{basic}} \right] \otimes \mathbb{C}[\frac{1}{k}] = \left[ \bigoplus_{l} \Omega^l(\hat{M}; \Lambda^l(T\hat{M})) \right] \otimes \mathbb{C}[\frac{1}{k}].
\]

\(^2\) The variable \( A \) generalizes the variable of the same name appearing in [3].
The space $\Lambda^*(\hat{\Omega}^1_h)$ which the forms are valued in just corresponds to the space of smooth functions of the supervariable

$$\chi_h \in \hat{\Omega}^2_h = \left[\hat{\Omega}^1_h\right]^*$$

Equation (43) translates into the formula

$$\hat{Z}_{k}^{\text{pert}} = \hat{Z}_{k}^{hl} \mid \hat{Z}_{k}^{sc} \in \Omega_{|\mathcal{M}|}(\hat{\mathcal{S}})_{\text{basic}} \otimes \text{PS}(k) = \Omega_{|\mathcal{M}|}(\hat{\mathcal{M}}) \otimes \text{PS}(k). \quad (47)$$

The symbol $\mid$ indicates that the multi-vector valued form $\hat{Z}_{k}^{hl}$ is to act on the form $\hat{Z}_{k}^{sc}$ by a combination of inner product with multi-vectors and wedge products for forms. $\hat{Z}_{k}^{sc}$ is easily shown to be closed. $\hat{Z}_{k}^{\text{pert}}$ can be shown to be closed directly (i.e. without relying on the existence of the integral which is ill-defined in the case of quantum field theory) by showing that $\hat{Z}_{k}^{hl}$ is closed under an extension of the exterior derivative operator which acts on multi-vectors by taking exterior derivative in the Met directions and divergence with respect to the volume form $\hat{Z}_{k}^{sc}$ on $\mathcal{M}$ in the $\mathcal{M}$ directions so that

$$d\hat{\mathcal{M}}(\hat{Z}_{k}^{hl} \mid \hat{Z}_{k}^{sc}) = (d\hat{\mathcal{M}} \hat{Z}_{k}^{hl}) \mid \hat{Z}_{k}^{sc}. \quad (48)$$

That $\hat{Z}_{k}^{hl}$ is closed can be proved directly from the prescription (44).

This ends our overview of the formulation of the finite dimensional perturbation series and the proof of its metric independence in a way that does not involve any integrals that would become infinite dimensional in the field theory problem. Although we have been rather sketchy in this section and may have used some perhaps unfamiliar machinery, it is hoped that the reader can get a good feel for what’s going on here by comparing with the trivial example of the proceeding section.

### 4 Application to Chern–Simons Theory

In the case of Chern–Simons theory, (44) may be translated to a definition of the following form, where the meaning of some of the symbols is explained below:

$$\hat{Z}_{k}^{hl} = e^{c(k)CS_{\text{grav}}(g,s)} \int_{\hat{\mathcal{M}}} \text{Tr}_k(e^{\frac{i}{2\pi k} \Phi}) \in \Omega^*(\hat{\mathcal{M}}; \Lambda^*(T\mathcal{M})) \otimes \mathbb{C}[[\frac{1}{k}]]. \quad (49)$$

3 For simplicity, and because it does not effect whether or not $Z_{k}^{\text{pert}}$ is an invariant, we have not included the normalization factors and the shift in $k$ discussed in needed for a conjectured agreement with the asymptotic expansion of Witten’s exact solution.
The term in front of the integral sign is the “counterterm”. In it, $CS_{\text{grav}}(g, s)$ is the Chern–Simons invariant of the metric connection defined relative to the canonical (bi-)framing. The quantity $c(k)$ is an expression depending only on $k$ and the structure group $G$ of the principle bundle $P$. For a direct translation from the finite dimensional case, the counterterm wouldn’t appear, i.e. $c(k)$ would be 0. We will see shortly the the counterterm is the same as was found necessary in the case of perturbation theory about an acyclic flat connection \cite{3, 4} and arises for the same reason.

The expression $e^M$ in (49) stands for the union

$$e^M = \bigcup_{V=0}^\infty M[V]/S_V,$$

where $M[V]$ is the compactification appearing in \cite{4} of the configuration space of $V$ distinct points in $M$ and $S_V$ is the action of the symmetric group on this space. Using some elliptic operator techniques and Hodge theory, $\Phi$ is defined in terms of the natural structures on the deformation complex discussed in the previous section. The restriction of $\Phi$ to $M[V]$ is a form\footnote{We use the notation in \cite{51} for simplicity. Strictly speaking $\Phi_V$ belongs to the $G$-invariant subspace of $\Omega^*(\widehat{\cal S}; \Lambda^*(\widehat{\cal O}^1_k)) \otimes \Omega^*(M[V]; A^*_V)^S_V$.}

$$\Phi_V \in \Omega^*(\widehat{\cal M}; \Lambda^*(TM)) \otimes \Omega^*(M[V]; A^*_V)^S_V$$

(51)

where

$$A^*_V = \Lambda^*(g_1 \oplus \ldots \oplus g_V)$$

(52)

with $g_i$ being the adjoint bundle to $P$ pulled back to $M[V]$ by the projection map from $M[V]$ onto the $i$’th copy of $M$. $\text{Tr}_k$ is a map (normalized by a power of $k$) from $A^*_V$ to $\mathbb{R}$, so that it makes sense to perform the integral in (49). It is convenient to define

$$\Omega^*(e^M) = \bigoplus_V \Omega^*(M[V]; A^*_V)^S_V.$$

(53)

Note that the first exponential in (44) written in language appropriate to Chern-Simons theory becomes the formal expression

$$e^{i k \frac{3}{2} \int_{x \in M} \text{Tr}(\omega_{\cal J}(x) \wedge (\omega_{\cal J}(x), \omega_{\cal J}(x)))}$$

where $\cal J \in \Omega^*(M; g)$ and $-\text{Tr}$ is an inner product on $\text{Lie}(G)$. It is not surprising that this can be given a sensible expression of the form (49).
In the case of perturbation theory about an acyclic connection considered in [4] (so that \( \mathcal{M} \) is a point), each of the forms \( \Phi_V \) is closed. This allowed us to apply Stokes theorem to calculate

\[
\begin{align*}
    d_{\text{Met}} \left( \int_{\mathcal{M}[V]} \Phi_V \right) &= \int_{\partial \mathcal{M}[V]} (d_{\text{Met}} + d_{\mathcal{M}[V]})\Phi_V - \int_{\partial \mathcal{M}[V]} \Phi_V = - \int_{\partial \mathcal{M}[V]} \Phi_V.
\end{align*}
\]

(54)

The right hand side of this equation is called an anomaly because it would vanish in the finite dimensional case. Using an explicition description of \( \partial \mathcal{M}[V] \) and \( \Phi_V|_{\partial \mathcal{M}[V]} \) to calculate the anomaly, we found that \( \hat{Z}_k^{hl} \) could be made closed (i.e. metric independent) by defining \( c(k) \) appropriately.

When \( \mathcal{M} \) consists of more than a point, the situation turns out to be quite a bit more complicated. Even in the trivial example of \( \mathcal{M} \), \( \hat{Z}_k^{hl} \) is only closed because of a cancellation among diagrams with different numbers of vertices. Since the only reason that the finite dimensional cancellation might not apply in the field theory case is due to the singularities that arise in the definition of \( \Phi \) and show up near points in \( \partial e^\mathcal{M} \); we expect there should be some way to reduce the anomaly calculation to some kind of boundary integral. In fact, with a bit of work, it turns out that we may apply a sort of “Stoke’s theorem on \( e^\mathcal{M} \)” so that the anomaly is reduced to a calculation on \( \partial e^\mathcal{M} \) as was true in the acyclic case.

To formulate this “Stoke’s theorem on \( e^\mathcal{M} \)”, we find an algebra \( \mathcal{A}^* \) (generated by a certain set of labelled graphs) with an operator \( \overline{D} : \mathcal{A}^* \to \mathcal{A}^{*+1} \) and an algebra homomorphism

\[
I : \mathcal{A}^* \to \Omega^*(\hat{\mathcal{M}}; \Lambda^*(T\mathcal{M})) \otimes \Omega^*(e^\mathcal{M})
\]

(55)

This algebra homomorphism embodies the Feynman rules and their variation under exterior derivative. \( \mathcal{A}^V \) is generated by graphs which have \( V \) vertices of a type that is considered internal and is mapped by \( I \) to

\[
\Omega^*(\hat{\mathcal{M}}; \Lambda^*(T\mathcal{M})) \otimes \Omega^*(M[V]; A^*_V)^{S_V}.
\]

(56)

The operator \( \overline{D} \) is a sum

\[
\overline{D} = D_0 + D_i
\]

(57)

where \( D_i \) increases the number of vertices by \( i \). Finally, we have

\[
\int_{e^\mathcal{M}} \text{Tr}_k(I(\overline{D}\omega)) = \int_{e^\mathcal{M}} \text{Tr}_k(d_{\mathcal{M} \times e^\mathcal{M}} I(\omega)) \quad \text{for} \quad \omega \in \mathcal{A}^*,
\]

(58)

17
where \( d_{\widehat{M} \times e^M} \) is the exterior derivative operator on \( \widehat{M} \times e^M \) (acting on sections of \( \Lambda^*(TM) \) by taking divergence with respect to the volume form \( \widehat{Z}_S \), see the comment above (48)).

To give a rudimentary idea of the meaning of \( A^*, I, \) and \( D \) let us begin by going back to the trivial example of §2. Recall that the Feynman rules in §2 associated to the edge of a graph a factor of \( -H(x_0)^{-1} \) (we set \( ik = 1 \) here for convenience). The exterior derivative on \( \widehat{S} \) of this term is

\[
d_{\widehat{S}}(-H(x_0)^{-1}) = (-H(x_0)^{-1})[H'(x_0)dx_0](-H'(x_0)^{-1}).
\]

(59)

Graphically, this can be represented as follows:

\[
\begin{array}{c}
\text{D}(
\begin{array}{c}
\text{---}
\end{array})
\end{array}
\end{array}
\]

(60)

The operator \( D \) acting on graphs correspond to the exterior derivative operator \( d_{\widehat{S}} \). In the graph on the right in (60), the top vertex is considered internal and the bottom vertex is considered external. The homomorphism \( I \), capturing the Feynman rules, associates a factor of \( -H(x_0)^{-1} \) with the solid edges as usual. The dashed edge (with an orientation pointing down indicated by the arrow) is given a factor of 1. The Feynman rule for the solid vertex is a factor of \( H'(x_0) \), and the Feynman rule for the bottom vertex is a factor of \( dx_0 \). Thus we see that graphically the exterior derivative of an edge, which has no vertices, is given by the labelled graph on the right in (60) which has one internal vertex. One can then proceed to find an algebra which is generated as a vector space by graphs with various kinds of edges and vertices which contains the orginal Feynman rules and has an operator \( D \) capturing exterior derivative. In fact, there is more than one way to do this. For example, rather than introduce the graph on the right in (60) and the Feynman rules above, we could simply have written the right hand side as a single dashed edge with Feynman rule given by the right hand side of (59). However such a definition would fail to capture the fact that the ingredients in the right hand side of (59) appear in other Feynman rules. Such relations are essential in proving that the exterior derivatives of various Feynman diagrams (such as the ones in (21)) cancel.

Getting back to Chern–Simons theory, what is remarkable is that we can find \( A^*, D, \) and \( I \) satisfying (58) and such that there exists an element \( \Phi \in A^* \) which is taken by \( I \) to \( \Phi \) and satisfies the condition that \( \exp\left(\frac{1}{2\pi i} \Phi\right) \)
is annihilated by $D$. Applying (58) to $\omega = \exp(\frac{1}{12\pi} \Phi)$ which is annihilated by $D$, we obtain

$$0 = \int_{e^M} \text{Tr}_k(I(D e^{\frac{1}{2\pi} \Phi})) = \int_{e^M} \text{Tr}_k(d_{\tilde{\mathcal{M}} \times e^M} e^{\frac{1}{2\pi} \Phi}) = \int_{\partial e^M} \text{Tr}_k(e^{\frac{1}{2\pi} \Phi}).$$

(61)

We are thus left with a calculation of anomalies by a boundary integral as in the acyclic case. Although the definition of $\Phi$ is quite a bit more complicated than in the acyclic case, a simple calculation shows that the part of $\Phi|_{\partial e^M}$ which contributes to the anomaly calculation is the same as in the acyclic case. Thus the anomaly is the same as in the acyclic case, and $\tilde{Z}_{hl}^k$ is indeed closed, with the same counterterm as in the acyclic case. Hence $\tilde{Z}_{k}^{\text{pert}}$ is closed and the perturbation series $Z_{k}^{\text{pert}}$ is metric independent and therefore a diffeomorphism invariant.

## 5 Concluding Remarks

The result we have described here for Chern–Simons theory is a paradigm that we expect should generalize to other quantum field theories. Although complete details will appear elsewhere [2], we outlined how to formulate perturbation theory about a component of the moduli space of instantons (solutions to the equations of motion moduli gauge invariance) on an arbitrary compact spacetime manifold and how to rigorously determine the anomalies to all orders. The details of the calculation verify that the anomaly is in fact of a universal nature, independent of the geometry of the instanton moduli space. This is similar to a result obtained by Friedan [6] in studying sigma model perturbation theory and might be predicted based on naive physical arguments. A general formulation and proof of this statement would be extremely useful.

The meaning of the form $\tilde{Z}_{k}^{\text{pert}} \in \Omega^*(\mathcal{M} \times \text{Met}) \otimes \text{PS}(k)$ may be illuminated further by considering the case when our three manifold $M$ equals a product $\Sigma \times S^1$ and $\mathcal{M}$ is a smooth component of the moduli space of flat connections with structure group $G$ on a Riemann surface $\Sigma$, identified with a smooth component of the moduli space of flat connection on $M$. In this case, the path integral for the Chern–Simons theory formally gives and Witten’s exact solution [8] actually does give the result that $Z_k$ equals the dimension of the Hilbert space obtained by quantizing the moduli space of flat $G$-connections on $\Sigma$. The contribution of $\mathcal{M}$ to $Z_k$ then equals the dimension of the space of
holomorphic sections of $\mathcal{L}^\otimes k$, where $\mathcal{L}$ is a certain holomorphic line bundle on $\mathcal{M}$. (Recall that $\mathcal{M}$ receives a Kähler structure once a metric on $\Sigma$ is chosen and that $\mathcal{L}$ is a line bundle whose curvature 2-form equals the Kähler form of $\mathcal{M}$.) Using the Riemann-Roch index theorem and a vanishing theorem, we find the exact answer (for large enough $k$) is obtained by integrating the index density over $\mathcal{M}$. Assuming the general conjecture that the invariants we have defined here do indeed give the contributions of $\mathcal{M}$ to the large $k$ asymptotics of the exact solution, we find

$$\int_{\mathcal{M}} \text{Td}(T^{(1,0)}\mathcal{M}) \wedge \text{ch}(\mathcal{L}^k) = \int_{\mathcal{M}} \hat{Z}_{k \text{pert}}$$

(62)

Note that the we expect the asymptotic series on the right to converge to the function on the left since the latter is just a polynomial in $k$. It is natural to conjecture that the restriction of $\hat{Z}_{k \text{pert}}$ to $\mathcal{M} \times \text{Met}(\Sigma)$, with an appropriate embedding of $\text{Met}(\Sigma)$ in $\text{Met}(M)$, equals the index density above. In other words, this conjecture says that $\hat{Z}_{k \text{pert}}$ is a manifestly three-dimensionally invariant version of a quantity arising from a two-dimensional index theorem.

A Recap of the Feynman Rules

We now briefly recall the Feynman rules for calculating the stationary phase approximation near the origin for an integral of the form

$$Z_k = \int_{A \in \mathbb{R}^n} O(A) e^{ikS(A)}$$

(63)

where $S$ and $O$ are smooth function on $\mathbb{R}^n$, $S$ has a non-degenerate critical point at the origin, and $O$ has compact support (so that the integral converges). We will not give a proof of this formulation here, although it can be derived from the description of the stationary phase approximation given in [11], for example. We will simply recall the formal derivation due to Feynman which can be found in many textbooks on quantum field theory, for example [7].

We begin by writing down the Taylor expansion for $S$ about the origin,

$$S(A) = S(0) + \frac{1}{2} < A, H A > + V(A)$$

(64)

$$V(A) = \sum_{v=3}^{\infty} \frac{1}{v!} \frac{\partial^v S(0)}{\partial A^{i_1} \ldots \partial A^{i_v}} A^{i_1} \ldots A^{i_v}$$

(65)
where $H$ is the Hessian of $S$ at the origin, considered as a linear map from $\mathbb{R}^n$ to itself (or more precisely its dual space). Next, we introduce a variable $J$ in $\mathbb{R}^n$ (or again, more precisely its dual space) and interpret expressions such as $O\left( \frac{\partial}{\partial J} \right)$ as a formal power series obtained by plugging into the Taylor series for $O$,

\[
O\left( \frac{\partial}{\partial J} \right) = \sum_{v=0}^{\infty} \frac{1}{v!} \frac{\partial^v O(0)}{\partial A_{i_1} \cdots \partial A_{i_v}} \partial_{J_{i_1}} \cdots \partial_{J_{i_v}}
\]  

(66)

It is apparent formally and in fact true that we may write

\[
Z_k = e^{ikS(0)} \int O(A)e^{ikV(A)}e^{ik<A,H_A>}
\]

(67)

\[
Z_k^{\text{pert}} = e^{ikS(0)} \left[ O\left( \frac{\partial}{\partial J} \right)e^{ikV\left( \frac{\partial}{\partial J} \right)} \right] \int_{J=0} e^{ik<A,H_A> + <J,A>}
\]

(68)

The final integral is defined by adding a small imaginary part to $H$ so that the integral converges and then taking the limit as the imaginary part goes to zero. The result may be calculated by completing the square and recalling the value of a Gaussian integral. This yields

\[
Z_k^{\text{pert}} = Z_k^{\text{hl}} Z_k^{\text{sc}}
\]

(69)

\[
Z_k^{\text{hl}} = \left( \frac{2\pi}{k} \right)^{n/2} \frac{e^{\frac{\pi\text{sign}(H)}{4}}}{|\text{det}(H)|^2} e^{ikS(0)}
\]

(70)

\[
Z_k^{\text{sc}} = \left[ O\left( \frac{\partial}{\partial J} \right)e^{ikV\left( \frac{\partial}{\partial J} \right)} \right] \int_{J=0} e^{-\frac{1}{4ik} \frac{1}{2} <J,H^{-1}J>}
\]

(71)

\[
= \sum_{I=0}^{\infty} \sum_{V=0}^{\infty} \left[ O\left( \frac{\partial}{\partial J} \right) \left[ ikV\left( \frac{\partial}{\partial J} \right) \right]^V \right] \int_{J=0} \left[ \frac{-\frac{1}{4ik} \frac{1}{2} <J,H^{-1}J> }{I!} \right]^I
\]

(72)

With a little thought one can see that the term inside the final sum can be expressed in the form $\sum_{\Gamma} \frac{I(\Gamma)}{S(\Gamma)}$, where the sum runs over all graphs with $I$ edges, $V$ unmarked vertices all of which have valency at least three, and 1 marked vertex. The factor $S(\Gamma)$ is the order of the automorphism group of $\Gamma$. The “Feynman amplitude” $I(\Gamma)$ is obtained by the following rules. First, one labels each end of each edge by the name of an index running from 1 to $n$. Then one writes down the product of the following factors: a factor of $-\frac{1}{i k}(H^{-1})^{jk}$ associated with each edge, where $j$ and $k$ are the index names labelling the ends of the edge; a factor of $ik \frac{\partial^v S(0)}{\partial A_{i_1} \cdots \partial A_{i_v}}$ associated
to each unmarked vertex, where \( v \) is the valency of the vertex and \( i_1, \ldots, i_v \) are the index names labelling the edge ends incident on the vertex; and a factor of \( \frac{\partial^n O(0)}{\partial A^{i_1} \cdots \partial A^{i_v}} \) associated to the marked vertex, where \( v \) is its valency and \( i_1, \ldots, i_v \) are the incident edge end labels. Finally, one obtains the “Feynman amplitude” \( I(\Gamma) \) by summing the product of all these factors as each of the indices runs from 1 to \( n \). The empty graph has \( I(\Gamma) = 1 \) and \( S(\Gamma) = 1 \). All other graphs contributing to \( Z^{hl}_k \) have more than 1 loop.

For example, here are some pictures and their associated factors. The third picture, for example, represents a marked vertex of valency one with incident edge end label \( j \).

\[
\begin{align*}
\begin{array}{c}
\includegraphics{picture1.png} \\
\includegraphics{picture2.png} \\
\includegraphics{picture3.png}
\end{array}
\end{align*}
\]

As a final example, the graph

\[
\begin{align*}
\includegraphics{picture4.png}
\end{align*}
\]

has Feynman amplitude

\[
\sum_{j,k,l,m} \left[ \frac{1}{ik} (H^{-1})^{jk} \right] \left[ \frac{1}{ik} (H^{-1})^{lm} \right] \left[ \frac{\partial O(0)}{\partial A^j} \right] \left[ \frac{\partial^3 S(0)}{\partial A^j \partial A^k \partial A^l} \right]
\]

Finally, we remark that when \( O \) equals the constant function 1, then we may just as well delete the marked vertex and sum over ordinary unmarked graphs. In the trivial example of §2, we refer to the unmarked vertices as “internal vertices” and to the marked vertex as an “external vertex”. In that example, the indices are not even necessary since \( n = 1 \).
References

[1] V.I. Arnold, S.M. Gusein-Zade, A.N. Varchenko, *Singularities of Differential Maps, Volume II*, Birkhauser, Boston, 1988.

[2] S. Axelrod, to appear.

[3] S. Axelrod and I. M. Singer, *Chern–Simons Perturbation Theory*, Proc. XXth DGM Conference (New York, 1991) (S. Catto and A. Rocha, eds) World Scientific, 1992, 3–45.

[4] S. Axelrod and I. M. Singer, *Chern–Simons Perturbation Theory, II*, J. Diff. Geom 39 (1994) 173-213.

[5] B.S. DeWitt, *Supermanifolds*, Cambridge University Press, Cambridge, 1984.

[6] D. Friedan, *NonLinear Sigma Models in 2 + ε dimensions*, Annals of Physics, Vol. 163, No. 2, Sept. 1985

[7] P. Ramond, *A Primer on Quantum Field Theory*, Addison-Wesley, New York, 1990

[8] E. Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. 121 (1989) 351-399.
List of Notations

In the list below, the general description is for the finite dimensional context. Expression in square brackets apply in the Chern-Simons theory context. Expression in angle brackets apply to a general Euclidean quantum field theory.

\[ M \] [closed, oriented 3-manifold]
\[ G \] [compact Lie group]
\[ P \rightarrow M \] [principal \( G \) bundle]
\[ \mathcal{F} \] manifold integrated over \(<\text{space of fields}>\) [connections on \( P \)]
\[ S \] Morse-Bott function on \( \mathcal{F} \) \(<\text{action}>\) [Chern-Simons invariant]
\[ k \] real parameter\(<\text{inverse Planck constant}>\) [positive integer level]
\[ \mathcal{G} \] symmetry group acting on \( \mathcal{F} \) [gauge transformations of \( P \)]
\[ \mu \] volume form on \( \mathcal{F} \) \(<\text{formal path integral measure}>\)
\[ S \] set of critical points of \( S \) \(<\text{instantons}>\) [flat connections on \( P \)]
\[ \mathcal{M} \] \( \mathcal{S}/\mathcal{G} \) \(<\text{instanton moduli space}>\)
\[ \mathcal{H} \] subgroup of \( \mathcal{G} \) to which all isotropy subgroup are conjugate
\[ A_0 \] point in \( S \) [flat connection on \( P \)]
\[ [A_0] \] point in \( \mathcal{M} \) [gauge equivalence class of \( A_0 \)]
\[ \text{Met} \] space of \( \mathcal{G} \)-invariant metrics on \( \mathcal{F} \) [Riemannian metrics on \( M \)]
\[ g \] element of \( \text{Met} \)
\[ N \rightarrow S \] normal bundle to \( S \) in \( \mathcal{F} \)
\[ \hat{S} \] \( S \times \text{Met} \)
\[ \hat{N} \] \( N \times \text{Met} \)
\[ E : \hat{N} \rightarrow \mathcal{F} \] evaluation map (exponential map with variable \( g \))
\[ \int_{\text{pert}} \] perturbation series (stationary phase approx.) for integral
\[ \text{PS}(k) \] ring of perturbation series
\[ I(\Gamma) \] Feynman amplitude associated to a graph \( \Gamma \)
\[ S(\Gamma) \] symmetry factor of a graph \( \Gamma \)
\[ Z_k \] basic integral being considered \(<\text{partition function}>\)
\[ Z_k^{\text{pert}} \] stationary phase approximation to \( Z_k \) [invariant we define]
\[ \hat{Z}_k^{\text{pert}} \] form on \( \hat{\mathcal{M}} \); integrated over \( \mathcal{M} \) yields \( Z_k^{\text{pert}} \)
\[ \hat{Z}_k^{\text{sc}} \] semi-classical part of \( \hat{Z}_k \)
\[ \hat{Z}_k^{hl} \] higher-loop contribution to \( \hat{Z}_k \)
\[ \hat{I}_l \] \( l \)-loop piece of \( \hat{Z}_k^{hl} \)
combination of wedge product and interior product operators
\[ \Omega^*_M \]

deformation complex for \( M \) [\( \Omega^*(M, g) \)]

\[ D^*_{A_0} \]
differential on \( \Omega^*_M \) at \( A_0 \in S \) [twisted exterior derivative]

\[ T_{A_0} \]
infinitesimal action of \( G \) at \( A_0 \) (equals \(-D^0_{A_0}\))

\[ H(A_0) \]
Hessian of \( S \) at \( A_0 \) (equals \( D^1_{A_0} \))

\[ \hat{\Omega}^*_M \]
bundle over \( \hat{S} \) of deformation complexes

\[ \hat{\Omega}^*_H, \hat{\Omega}^*_d, \hat{\Omega}^*_\delta \]
harmonic, exact, and coexact subbundles of \( \hat{\Omega}^*_M \)

\[ \mathcal{A}^* \]
algebra generated by labelled graphs

\[ I \]
[Feynman rule homomorphism from \( \mathcal{A}^* \) to differential forms]

\[ D \]
[analogue of exterior derivative acting on \( \mathcal{A}^* \)]

\[ M[V] \]
[compactification of \( M^V \setminus \{\text{all diagonals}\} \)]

\[ e^M \]
[\( \bigcup_{V=0}^{\infty} M[V] / S^V \), a closure of the set of finite subsets of \( M \)]