ON THE EXTENSION OF HÖLDER MAPS WITH VALUES IN SPACES OF CONTINUOUS FUNCTIONS

GILLES LANCIEN AND BEATA RANDRIANANTOANINA

Abstract. We study the isometric extension problem for Hölder maps from subsets of any Banach space into $c_0$ or into a space of continuous functions. For a Banach space $X$, we prove that any $\alpha$-Hölder map, with $0 < \alpha \leq 1$, from a subset of $X$ into $c_0$ can be isometrically extended to $X$ if and only if $X$ is finite dimensional. For a finite dimensional normed space $X$ and for a compact metric space $K$, we prove that the set of $\alpha$'s for which all $\alpha$-Hölder maps from a subset of $X$ into $C(K)$ can be extended isometrically is either $(0,1]$ or $(0,1)$ and we give examples of both occurrences. We also prove that for any metric space $X$, the described above set of $\alpha$'s does not depend on $K$, but only on finiteness of $K$.

1. Introduction - Notation

If $(X,d)$ and $(Y,\rho)$ are metric spaces, $\alpha \in (0,1]$ and $K > 0$, we will say that a map $f : X \to Y$ is $\alpha$-Hölder with constant $K$ (or in short $(K,\alpha)$-Hölder) if
\[ \forall x, y \in X, \quad \rho(f(x), f(y)) \leq Kd(x,y)^\alpha. \]

Let us now recall and extend the notation introduced by Naor in [13]. For $C \geq 1$, $B_C(X,Y)$ will denote the set of all $\alpha \in (0,1]$ such that any $(K,\alpha)$-Hölder function $f$ from a subset of $X$ into $Y$ can be extended to a $(CK,\alpha)$-Hölder function from $X$ into $Y$. If $C = 1$, such an extension is called an isometric extension. When $C > 1$, it is called an isomorphic extension. If a $(CK,\alpha)$-Hölder extension exists for all $C > 1$, we will say that $f$ can be almost isometrically extended. So, let us define:
\[
A(X,Y) = B_1(X,Y), \quad B(X,Y) = \bigcup_{C \geq 1} B_C(X,Y), \quad \text{and} \quad \tilde{A}(X,Y) = \bigcap_{C > 1} B_C(X,Y).
\]

The study of these sets goes back to a classical result of Kirszbraun [10] asserting that if $H$ is a Hilbert space, then $1 \in A(H,H)$. This was extended by Grünbaum and Zarantonello [5] who showed that $A(H,H) = (0,1]$. Then the complete description of $A(L^p,L^q)$ for $1 < p, q < \infty$ relies on works by Minty [12] and Hayden, Wells and Williams [6] (see also the book of Wells and Williams [14] for a very nice exposition of the subject). More recently, K. Ball [1] introduced a very important notion of non linear type or cotype and used it to prove a general extension theorem for Lipschitz maps. Building on this work, Naor [13] described completely the sets $B(L^p,L^q)$ for $1 < p, q < \infty$.

2000 Mathematics Subject Classification. 46B20 (46T99, 54C20, 54E35).

*Partially funded by a CFR Grant from Miami University.
In this paper, we concentrate on the study of $A(X,Y)$ and $\tilde{A}(X,Y)$, when $X$ is a Banach space and $Y$ is a space of converging sequences or, more generally, a space of continuous functions on a compact metric space. This can be viewed as an attempt to obtain a non-linear version of the results of Lindenstrauss and Pełczyński [11] and later of Johnson and Zippin ([2] and [3]) on the extension of linear operators with values in $C(K)$ spaces.

So let us denote by $c$ the space of all real converging sequences equipped with the supremum norm and by $c_0$ the subspace of $c$ consisting of all sequences converging to 0. If $K$ is a compact space, $C(K)$ denotes the space of all real valued continuous functions on $K$, equipped again with the supremum norm.

In section 2, we show that if $X$ is infinite dimensional and $Y$ is any separable Banach space containing an isomorphic copy of $c_0$, then $\tilde{A}(X,Y)$ is empty. On the other hand, we prove that $A(X,c_0) = (0,1]$, whenever $X$ is finite dimensional.

In section 3, we show that for any finite dimensional space $X$, $\tilde{A}(X,c) = (0,1]$ and $A(X,c)$ contains $(0,1)$. Then the study of the isometric extension for Lipschitz maps turns out to be a bit more surprising. Indeed, we give an example of a 4-dimensional space $X$ such that $A(X,c) = (0,1)$. To our knowledge, this provides the first example of Banach spaces $X$ and $Y$ such that $A(X,Y)$ is not closed in $(0,1]$ and also such that $A(X,Y) \neq \tilde{A}(X,Y)$. On the other hand, we show that if the unit ball of a finite dimensional Banach space is a polytope, then $A(X,c) = (0,1]$.

Finally, we prove in section 4, that $c$ is the only $C(K)$ space that one needs to consider as the image space in the study of the isometric extension problem. More precisely, we show that for every infinite compact metric space $K$ and every metric space $X$, $A(X,c) = A(X,C(K))$.

Acknowledgments. The research on this paper started during a sabbatical visit of the second named author at the Département de Mathématiques, Université de Franche-Comté in Besançon, France. She wishes to thank all members of the Functional Analysis Group, and especially Prof. F. Lancien, for their hospitality during that visit.

2. Maps into $c_0$

It is well known that for any metric space $(X,d)$, $A(X,\mathbb{R}) = (0,1]$. Indeed, if $M$ is a subset of $X$ and $f : M \to \mathbb{R}$ is a $(K,\alpha)$-Hölder function, then a $(K,\alpha)$-Hölder extension $g$ of $f$ on $X$ is given for instance by the inf-convolution formula:

$$\forall x \in X, \quad g(x) = \inf \{ f(u) + K(d(u,x))^{\alpha} \mid u \in M \}.$$ 

It follows immediately that $A(X,\ell_\infty) = (0,1]$, where $\ell_\infty$ is the space of all real bounded sequences equipped with the supremum norm. Now, since there is a 2-Lipschitz retraction from $\ell_\infty$ onto $c_0$ (see for instance [2] page 14), it is clear that for any metric space $X$, $B_2(X,c_0) = (0,1]$. Our first result shows that the difference between the isometric and isomorphic extension problems which is revealed in [13] is extreme when $c_0$ is the image space. More precisely:
Theorem 2.1. Let $X$ be an infinite dimensional normed vector space and $Y$ be a separable Banach space containing an isomorphic copy of $c_0$. Then

$$\tilde{A}(X,Y) = \emptyset.$$ 

Proof. By a theorem of R.C. James [7], $Y$ contains almost isometric copies of $c_0$. So, since we are studying the almost isometric extension problem, we may as well assume that there is a closed subspace $Z$ of $Y$ which is isometric to $c_0$. Let $(e_n)$ be the isometric image in $Z$ of the canonical basis of $c_0$ and $(e_n^*)$ be the Hahn-Banach extensions to $Y$ of the corresponding coordinate functionals (this sequence is included in the unit sphere of $Y^*$). Since $Y$ is separable, there is a subsequence $(e^*_n)_{k \geq 1}$ which is weak*-converging to some $y^*$ in the unit ball of $Y^*$.

On the other hand, by a theorem of Elton and Odell [4], there exists $\varepsilon > 0$ and a sequence $(x_k)_{k \geq 1}$ in $X$ such that:

$$\forall k \quad \|x_k\| = 1 - \varepsilon \quad \text{and} \quad \forall k \neq l \quad \|x_k - x_l\| \geq 1.$$ 

Let now $f$ be defined by $f(x_k) = (-1)^k e^*_n$. This is clearly a $(1, \alpha)$-Hölder function for any $\alpha$ in $(0, 1]$. Let $\delta > 0$ such that $(1 + \delta)(1 - \varepsilon)\alpha < 1$ and $\eta = 1 - (1 + \delta)(1 - \varepsilon)\alpha > 0$. Assume that $f$ can be extended at 0 into a $(1 + \delta, \alpha)$-Hölder function $g$ with $g(0) = y$. Then, for any even $k$, $e^*_n(y) \geq \eta$ and for any odd $k$, $e^*_n(y) \leq -\eta$. This is in contradiction with the fact that $(e^*_n)$ is weak*-converging.

□

We will now solve the extension problem for Hölder maps from a finite dimensional space into $c_0$. First, we need the following elementary Lemma.

Lemma 2.2. Let $X$ be a finite dimensional Banach space and $\delta > 0$. Then there exist $C_1, \ldots, C_n$ subsets of $X$ such that

$$X \setminus \{0\} = \bigcup_{i=1}^n C_i$$ 

and

$$\forall 1 \leq i \leq n, \quad \forall x, y \in C_i \quad \text{so that} \quad \|x\| \geq \|y\| : \quad \|x - y\| \leq \|x\| - (1 - \delta)\|y\|.$$ 

Proof. Since $X$ is finite dimensional, we can cover the unit sphere of $X$ with $B_1, \ldots, B_n$, balls of radius $\delta/2$ and define

$$C_i = \{y \in X \setminus \{0\} : \quad \frac{y}{\|y\|} \in B_i\}.$$ 

Let now $x, y \in C_i$ so that $\|x\| \geq \|y\|$. We have

$$\|x - y\| \leq \|x - x\| \frac{y}{\|x\|} \| + \|y\| \| x\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \| \leq \|x\| - (1 - \delta)\|y\|.$$ 

□
Then our result is

**Theorem 2.3.** If $X$ is a finite dimensional normed vector space, then

$$\mathcal{A}(X, c_0) = (0, 1].$$

**Proof.** Let $\alpha \in (0, 1]$, $M \subset X$ and $f : M \to c_0$ be a $(K, \alpha)$-Hölder function. We may assume that $K = 1$ and that $M$ is closed. It is enough to show that for any $x_0 \in X \setminus M$, $f$ can be extended into a $(1, \alpha)$-Hölder function $g$ on $M \cup \{x_0\}$ and we will assume that $x_0 = 0$.

For $\delta = 1/2$, let $C_1, \ldots, C_n$ be given by Lemma 2.2 and $I = \{i, \ 1 \leq i \leq n \text{ and } C_i \cap M \neq \emptyset\}$. Since $X$ is finite dimensional, for each $i$ in $I$, we can pick $x_i$ in $C_i \cap M$ such that for any $x \in C_i \cap M$, $\|x\| \geq \|x_i\|$. Then, by Lemma 2.2 we have that

$$\forall x \in C_i \cap M, \|x - x_i\| \leq \|x\| - \frac{1}{2}\|x_i\| \leq \|x\|.$$

Let us now pick $\varepsilon > 0$ such that $\varepsilon < \frac{1}{2} \text{dist}(0, M)^\alpha$. Then

$$\exists N \in \mathbb{N} \ \forall n > N \ \forall i \in \{1, \ldots, n\} : |f(x_i)(n)| < \varepsilon.$$

We will now choose $g(0) = (u(n))_{n \geq 1}$.

Since $\mathbb{R}$-valued contractions can be extended into contractions, we can pick $(\eta_n)_{n \geq 1}$ in $\ell_\infty$ so that

$$\forall n \in \mathbb{N} \ \forall x \in M, \ |f(x)(n) - \eta_n| \leq \|x\|^\alpha.$$

For $n \leq N$, we set $u(n) = \eta_n$.

For $n > N$, let $\delta_n \in \{-1, 1\}$ be the sign of $\eta_n$. Now we set

$$u(n) = \delta_n \min\{|\eta_n|, \max_{i \in I}|f(x_i)(n)|\}.$$

Note that since $I$ is finite and each $f(x_i) \in c_0$, we have that $g(0) = (u(n))_{n \geq 1} \in c_0$.

Next we check that for all $x \in M$ and all $n > N$, $|f(x)(n) - u(n)| \leq \|x\|^\alpha$. So let $x \in M$ and $i_0 \in I$ such that $x \in C_{i_0} \cap M$. We have four cases:

1) If $|f(x)(n)| \leq |u(n)|$, then

$$|f(x)(n) - u(n)| \leq 2\varepsilon \leq \|x\|^\alpha.$$

2) If $|f(x)(n)| > |u(n)|$, $\text{sgn}(f(x)(n)) = \delta_n$, and $|u(n)| = |\eta_n|$ then, by the definition of $\eta_n$:

$$|f(x)(n) - u(n)| \leq \|x\|^\alpha.$$

3) If $|f(x)(n)| > |u(n)|$, $\text{sgn}(f(x)(n)) = \delta_n$, and $|u(n)| = \max_{i \in I}|f(x_i)(n)| \geq |f(x_{i_0})(n)|$, then

$$|f(x)(n) - u(n)| = |f(x)(n)| - |u(n)| \leq |f(x)(n) - f(x_{i_0})(n)| + |f(x_{i_0})(n)| - |u(n)|$$

$$\leq \|x - x_{i_0}\|^\alpha \leq \|x\|^\alpha.$$

4) If $|f(x)(n)| > |u(n)|$ and $\text{sgn}(f(x)(n)) \neq \delta_n$, then

$$|f(x)(n) - u(n)| = |f(x)(n)| + |u(n)| \leq |f(x)(n)| + |\eta_n| = |f(x)(n) - \eta_n| \leq \|x\|^\alpha.$$
Remark 2.4. The proof is much simpler in the case $\alpha = 1$. Indeed it is enough to set $u(n) = 0$ for $n > N$. Then, for $x \in M$, pick $i_0 \in I$ such that $x \in C_{i_0} \cap M$. Thus, for all $n > N$:

$$|f(x)(n) - u(n)| = |f(x)(n)| \leq |f(x)(n) - f(x_{i_0})(n)| + \varepsilon \leq \|x - x_{i_0}\| + \varepsilon$$

$$\leq \|x\| - \frac{1}{2}\|x_{i_0}\| + \varepsilon \leq \|x\|.$$ 

3. Maps into $c$

We now consider the isometric and almost isometric extension problem for Hölder maps from a normed vector space into $c$. If $X$ is infinite dimensional, this question is settled by Theorem 2.1. Therefore, throughout this section, $X$ will denote a finite dimensional normed vector space. The study of the almost isometric extensions is then rather simple. For this purpose, we recall that, for $\lambda > 1$, a Banach space $Y$ is said to be a $\ell_\infty^\dim F$ space if every finite dimensional subspace of $Y$ is contained in a finite dimensional subspace $F$ of $Y$ which is $\lambda$-isomorphic to $\ell_\infty^{\dim F}$ (namely, there is an isomorphism $T$ from $F$ onto $\ell_\infty^{\dim F}$ such that $\|T\| \|T^{-1}\| \leq \lambda$).

Proposition 3.1. Let $X$ be a finite dimensional normed vector space and $Y$ be a Banach space which is a $\ell_\infty^\dim F$ space for any $\lambda > 1$. Then

$$\tilde{A}(X,Y) = (0,1].$$

In particular, for every compact space $K$,

$$\tilde{A}(X,C(K)) = (0,1].$$

Proof. Let $M$ be a closed subset of $X$ and $f : M \to Y$ be a $(1,\alpha)$-Hölder map. We start with the following Lemma.

Lemma 3.2. For any $x \in X \setminus M$ and any $\varepsilon > 0$, $f$ admits a $(1+\varepsilon,\alpha)$-Hölder extension to $M \cup \{x\}$.

Proof. If $M$ is compact and $\delta > 0$, we pick a $\delta$-net $\{x_1, \ldots, x_n\}$ of $M$ and a finite dimensional subspace $F$ of $Y$, containing $f(x_1), \ldots, f(x_n)$ such that $F$ is $(1+\delta)$-isomorphic to some $\ell_\infty^m$. Then, there is $y \in F$ such that for all $1 \leq i \leq n$, $\|f(x_i) - y\| \leq (1+\delta)\|x_i - x\|^\alpha$. If $\delta$ was chosen small enough, then for any $z \in M$, $\|f(z) - y\| \leq (1+\varepsilon)\|z - x\|^\alpha$.

For a general $M$ and a fixed $x \in X \setminus M$, we apply the compact case to the restriction of $f$ to $M \cap KB_X$, for $K$ big enough and where $B_X$ denotes the closed unit ball of $X$.

We now finish the proof of Proposition 3.1. Let $(x_n)_{n\geq 1}$ be a dense sequence in $X \setminus M$. for a given $\varepsilon > 0$, we pick $(\varepsilon_n)_{n\geq 1}$ in $(0,1)$ so that $\prod_{n\geq 1}(1+\varepsilon_n) < 1 + \varepsilon$. It follows from the
above Lemma and an easy induction that $f$ can be extended to a $(1 + \varepsilon, \alpha)$-Hölder function on $M \cup \{x_n, \ n \geq 1\}$, which in turn can be extended by density to $X$. \hfill \square

**Remark 3.3.** For $Y = C(K)$, there is a more concrete argument, which even allows to extend $f$ isometrically when $M$ is compact. We use the Inf-convolution formula and define:

$$\forall t \in K \ f(x)(t) = \inf_{y \in M} [f(y)(t) + \|x - y\|^\alpha].$$

Clearly, $\|f(x) - f(y)\|_{\infty} \leq \|x - y\|^\alpha$. Since $f(M)$ is compact in $C(K)$, $f(x)$ is the infimum of an equicontinuous family of functions and therefore is continuous on $K$.

Let us now concentrate on the isometric extension problem. We will need the following characterization.

**Lemma 3.4.** Let $(X, d)$ be a metric space, $M$ a subset of $X$, $f : M \to c$ a contraction and $x \in X \setminus M$. Then, the following statements are equivalent:

1. $f$ can be extended to a contraction $g : M \cup \{x\} \to c$.
2. $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n, m > N \ \forall y, z \in M$
   $$|f(y)(n) - f(z)(m)| \leq d(y, x) + d(z, x) + \varepsilon.$$

**Proof.** Suppose that (1) holds. Then $(g(x)(n))_n$ is a Cauchy sequence. Thus

$$\forall \varepsilon > 0 \ \exists N \ \forall n, m > N \ |g(x)(n) - g(x)(m)| < \varepsilon.$$ 

Since $g$ is a contractive extension of $f$, we have that for all $n, m > N$ and all $y, z \in M$

$$|f(y)(n) - f(z)(m)| \leq |f(y)(n) - g(x)(n)| + |g(x)(n) - g(x)(m)| + |g(x)(m) - f(z)(m)|$$

$$\leq d(y, x) + d(z, x) + \varepsilon.$$

Suppose now that (2) holds. Define

$$s(j) = \sup_{m \geq j} \sup_{z \in M} (f(z)(m) - d(z, x)).$$

Let us fix $z_0 \in M$. Then, it is easily seen that

$$\forall j \in \mathbb{N}, \ |s(j)| \leq \|f(z_0)\|_{\infty} + d(x, z_0).$$

On the other hand $\{s(j)\}_{j \in \mathbb{N}}$ is a decreasing sequence and therefore converges. We will denote by $s(\infty)$ its limit.

In order to define $(g(x)(n))_{n \geq 1}$, we pick a sequence $(N_k)_{k \geq 1}$ of integers such that

(i) $(2)$ holds with $\varepsilon = 2^{-k}$ and $N = N_k$;
(ii) $\forall j > N_k \ s(j) \leq s(\infty) + 2^{-k}$;
(iii) $\forall k \in \mathbb{N} \ N_{k+1} > N_k$. 

6
Then we define $g(x)$ as follows:

1. for $n \leq N_1$, let $g(x)(n)$ be any element of

$$\bigcap_{y \in M} [f(y)(n) - d(x, y), f(y)(n) + d(x, y)] = \left[\sup_{y \in M} (f(y)(n) - d(x, y)), \inf_{y \in M} (f(y)(n) + d(x, y))\right].$$

2. for $N_k < n \leq N_{k+1}$ we define

$$g(x)(n) = \max\{\sup_{y \in M} (f(y)(n) - d(x, y)), s(N_k) - 2^{-k}\}.$$

It follows from (i) that

$$\forall n > N_k \forall y \in M \quad s(N_k) - 2^{-k} \leq f(y)(n) + d(x, y).$$

So

$$\forall n \in \mathbb{N} \quad \sup_{y \in M} (f(y)(n) - d(x, y)) \leq g(x)(n) \leq \inf_{y \in M} (f(y)(n) + d(x, y)).$$

Thus $g(x) \in \ell_\infty$ and for all $y$ in $M$, $\|g(x) - f(y)\|_\infty \leq d(x, y)$.

Finally, note that

$$\forall n > N_k \quad \sup_{y \in M} (f(y)(n) - d(x, y)) \leq s(N_k).$$

Thus

$$\forall n \in (N_k, N_{k+1}] \quad s(N_k) - 2^{-k} \leq g(x)(n) \leq s(N_k).$$

It is now clear that $(g(x)(n))_{n \geq 1}$ converges to $s(\infty)$ and therefore belongs to $c$. \hfill \Box

As a first application we have

**Theorem 3.5.** For any finite dimensional normed vector space $X$

$$(0, 1) \subset \mathcal{A}(X, c).$$

*Proof.* Let $0 < \alpha < 1$, $M$ a closed subset of $X$ such that $0 \not\in M$ and $f : M \to c$ be a $(1, \alpha)$-Hölder function. It is enough to show that $f$ admits a $(1, \alpha)$-Hölder extension to $M \cup \{0\}$.

We fix $\varepsilon > 0$ and pick $x_0 \in M$. Since $\alpha < 1$,

$$\lim_{\|x\| \to \infty} \left[\left(\|x\| + \|x_0\|\right)^\alpha - \|x\|^\alpha\right] = 0.$$ 

So, there is $K > 0$ such that $\|x - x_0\|^\alpha \leq \|x\|^\alpha + \varepsilon/3$ for all $x$ so that $\|x\| > K$. Let us also choose $K$ such that $\|x_0\| \leq K$. Since $M_K = M \cap KB_X$ is compact,

$$\exists N \in \mathbb{N} \forall n, m > N \forall x \in M_K \quad |f(x)(n) - f(x)(m)| < \frac{\varepsilon}{3}.$$ 

Let now $x$ and $y$ in $M$.

If $x \in M_K$, then for all $n, m > N$:

$$|f(x)(n) - f(y)(m)| \leq \frac{\varepsilon}{3} + \|x - y\|^\alpha \leq \|x\|^\alpha + \|y\|^\alpha + \frac{\varepsilon}{3}.$$
If \( x \) and \( y \) belong to \( M \setminus M_K \), then for all \( n, m > N \):
\[
|f(x)(n) - f(y)(m)| \leq \|x - x_0\|^\alpha + \|y - x_0\|^\alpha + \frac{\varepsilon}{3} \leq \|x\|^\alpha + \|y\|^\alpha + \varepsilon.
\]

Then the conclusion follows directly from Lemma 3.4. \( \square \)

We will now see that the possibility of extending isometrically all Lipschitz maps from a finite dimensional space into \( c \) may depend on the geometry of the space \( X \). As a positive result, we have for instance

**Theorem 3.6.** For any \( n \in \mathbb{N} \)

\[ \mathcal{A}(\ell^n_\infty, c) = (0, 1]. \]

**Proof.** For \( j \in \{1, \ldots, n\} \), \( \delta \in \{-1, 1\} \), we denote by \( F_{j,\delta} \) the following \((n-1)\)–face of the unit ball of \( \ell^n_\infty \):
\[ F_{j,\delta} = \{x = (x_1, \ldots, x_n) : \|x\| = 1, x_j = \delta\}. \]

Let \( C_{j,\delta} \) denote the cone supported by \( F_{j,\delta} \):
\[ C_{j,\delta} = \{x \in \ell^n_\infty : x_j = \delta\|x\|\}. \]

For \( j, k \in \{1, \ldots, n\} \), \( j \neq k \), and \( \delta, \eta \in \{-1, 1\} \) we denote by \( F_{j,\delta,k,\eta} \) the \((n-2)\)–face of \( F_{j,\delta} \):
\[ F_{j,\delta,k,\eta} = F_{j,\delta} \cap F_{k,\eta}, \]
and by \( C_{j,\delta,k,\eta} \) the corresponding cone:
\[ C_{j,\delta,k,\eta} = C_{j,\delta} \cap C_{k,\eta}. \]

We also define a family of projections \( P_{j,\delta,k,\eta} : C_{j,\delta} \to C_{j,\delta,k,\eta} \) by
\[
P_{j,\delta,k,\eta}(x) = y, \text{ where } \begin{cases} y_k = \eta|x_j| \\ y_i = x_i, \text{ if } i \neq k. \end{cases}
\]

Note that for every \( x \in C_{j,\delta}, \eta|x_j| = \eta\delta x_j \), so \( P_{j,\delta,k,\eta} \) is linear on \( C_{j,\delta} \) and
\[
\forall x \in C_{j,\delta} \quad \|P_{j,\delta,k,\eta}(x)\| = \|x\|. \quad (3.1)
\]

Further, since for all \( x \in C_{j,\delta}, |x_j| \geq |x_k| \) we get
\[
\text{sgn}((P_{j,\delta,k,\eta}(x))_k - x_k) = \eta. \quad (3.2)
\]

We also introduce the projection \( Q_k : \mathbb{R}^n \to \mathbb{R}^{n-1} \) defined by
\[
Q_k(x_1, \ldots, x_n) = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n). \]

The following Lemma will provide us with a convenient finite covering of the space \( \ell^n_\infty \).
Lemma 3.7. For any $M \subset X = \ell_\infty^n$, any $\varepsilon > 0$ and any $j \in \{1, \ldots, n\}$, $\delta \in \{-1, 1\}$, such that $C_{j,\delta} \cap M \neq \emptyset$, there exist $A_1, \ldots, A_\mu$ subsets of $X$ such that

$$(C_{j,\delta} \cap M) \subset \bigcup_{i=1}^\mu A_i$$

and $\forall i \in \{1, \ldots, \mu\}$ $\exists x^i \in A_i \cap M$ satisfying

$$\forall x \in A_i \cap M \ |x| \geq |x^i| - \varepsilon \text{ and } |x - x^i| \leq |x| - |x^i| + \varepsilon.$$ 

Proof of Lemma 3.7. We will give a proof by induction on the dimension of $\ell_\infty^n$. If $n = 1$, the statement is clear, so let us now assume that it is satisfied for $n - 1$, where $n \geq 2$.

Let $M$, $\varepsilon$, $j$ and $\delta$ be as in the statement of Lemma 3.7. We pick an element $x_{j,\delta} \in C_{j,\delta} \cap M$ and we denote

$$B_{j,\delta} = x_{j,\delta} + C_{j,\delta}.$$ 

Note that

$$\forall x \in B_{j,\delta} \ |x - x_{j,\delta}| = |x| - |x_{j,\delta}|.$$ 

Denote $d_{j,\delta,k,\eta} = |(P_{j,\delta,k,\eta}(x_{j,\delta}))_k - (x_{j,\delta})_k|$. Let $x \in C_{j,\delta}$ such that for any $k \in \{1, \ldots, n\} \setminus \{j\}$, and any $\eta \in \{-1, 1\}$,

$$|(P_{j,\delta,k,\eta}(x))_k - x_k| \geq d_{j,\delta,k,\eta}$$ 

Then, we claim that $x \in B_{j,\delta}$.

Indeed, by (3.2)

$$|(P_{j,\delta,k,\eta}(x))_k - x_k| = \eta(P_{j,\delta,k,\eta}(x))_k - \eta x_k$$

and

$$|(P_{j,\delta,k,\eta}(x_{j,\delta}))_k - (x_{j,\delta})_k| = \eta(P_{j,\delta,k,\eta}(x_{j,\delta}))_k - \eta(x_{j,\delta})_k.$$ 

Thus (3.4) implies that

$$\eta(P_{j,\delta,k,\eta}(x))_k - \eta(P_{j,\delta,k,\eta}(x_{j,\delta}))_k \geq \eta x_k - \eta(x_{j,\delta})_k,$$

and hence

$$\eta \delta x_j - \eta \delta(x_{j,\delta})_j = \delta(x_j - (x_{j,\delta})_j) \geq \eta x_k - \eta(x_{j,\delta})_k.$$ 

Since this holds for all $\eta \in \{-1, 1\}$, we get that for all $k \in \{1, \ldots, n\} \setminus \{j\}$,

$$\delta(x - x_{j,\delta})_j \geq |(x - x_{j,\delta})_k|.$$ 

Thus $x - x_{j,\delta} \in C_{j,\delta}$ and $x \in B_{j,\delta}$.

Combining (3.2) and (3.4), we conclude that

$$C_{j,\delta} \setminus B_{j,\delta} \subset \bigcup_{k \in \{1, \ldots, n\} \setminus \{j\}, \eta \in \{-1, 1\}} B_{j,\delta,k,\eta}$$

where

$$B_{j,\delta,k,\eta} = \{x \in C_{j,\delta} : ((P_{j,\delta,k,\eta}(x))_k - x_k) \in \eta[0, d_{j,\delta,k,\eta}) \}. $$
Now, for each $k \in \{1, \ldots, n\} \setminus \{j\}$, and $\eta \in \{-1, 1\}$, we choose $N_{k, \eta} \in \mathbb{N}$ such that $$\frac{d_{j, \delta, k, \eta}}{N_{k, \eta}} < \frac{\varepsilon}{3}.$$ Then we set

$$\forall k \in \{1, \ldots, n\} \setminus \{j\} \ \forall \eta \in \{-1, 1\} \ \forall \nu \in \{1, \ldots, N_{k, \eta}\}:
$$

$$I_{j, \delta, k, \eta}^{\nu} = \left[ \frac{(\nu - 1)d_{j, \delta, k, \eta}}{N_{k, \eta}}, \frac{\nu d_{j, \delta, k, \eta}}{N_{k, \eta}} \right)$$

and $B_{j, \delta, k, \eta}^{\nu} = \{ x \in C_{j, \delta} : (P_{j, \delta, k, \eta}(x))_k - x_k \in \eta I_{j, \delta, k, \eta}^{\nu} \}$. 

So we have

$$(3.5) \quad C_{j, \delta} \setminus B_{j, \delta} = \bigcup_{k \in \{1, \ldots, n\} \setminus \{j\}} \bigcup_{\eta \in \{-1, 1\}} \bigcup_{\nu = 1}^{N_{k, \eta}} B_{j, \delta, k, \eta}^{\nu}.$$

We now fix $k \in \{1, \ldots, n\} \setminus \{j\}, \eta \in \{-1, 1\}$ and $\nu \leq N_{k, \eta}$ such that $B_{j, \delta, k, \eta}^{\nu} \cap M \neq \emptyset$ and denote for simplicity:

$$B = B_{j, \delta, k, \eta}^{\nu}, \quad I = \eta I_{j, \delta, k, \eta}^{\nu}, \quad \tilde{P} = P_{j, \delta, k, \eta}, \quad P = Q_k \tilde{P}, \quad M' = P(M \cap B)$$

and $C = P(C_{j, \delta}) = Q_k C_{j, \delta, k, \eta} = \{ x \in \ell_\infty^n : x_{\phi(j)} = \delta \|x\| \}$, where $\phi(j) = j$ if $k > j$ and $\phi(j) = j - 1$ if $k < j$.

Since $M'$ is a non empty subset of $C$, our induction hypothesis yields the existence of $A_{l_1}, \ldots, A_{l_L} \subset C$ so that $M' \subset \bigcup_{l \leq L} A_l$ and $\forall l \in \{1, \ldots, L\} \ \exists y^l \in A_l \cap M' \text{ satisfying}$

$$\forall y \in A_l \cap M' \|y\| \geq \|y^l\| - \frac{\varepsilon}{3} \text{ and } \|y - y^l\| \leq \|y\| - \|y^l\| + \frac{\varepsilon}{3}.$$ 

Now let $A_l = \{ x = (x_i)_{i=1}^n \in C_{j, \delta}, \ P(x) \in A_l, x_k \in \delta \eta x_j - I \}$. We have that

$$B \cap M \subset \bigcup_{l \leq L} A_l.$$ 

Then, for any $l \leq L$, we pick $x^l \in A_l \cap M$ such that $P(x^l) = y^l$. Note that

$$\forall x \in A_l \cap M \|x\| = |x_j| = \|Px\| \geq \|y^l\| - \frac{\varepsilon}{3} = |x^l_j| - \frac{\varepsilon}{3} = \|x^l\| - \frac{\varepsilon}{3}.$$ 

Therefore

$$\forall x \in A_l \cap M \ |x_j - x^l_j| \leq |x_j| - |x^l_j| + \frac{2\varepsilon}{3}.$$ 

Now,

$$\|x - x^l\| = \max\{\|P(x) - P(x^l)\|, \ |x_k - x^l_k| \}.$$ 

We have

$$\|P(x) - P(x^l)\| \leq \|P(x)\| - \|P(x^l)\| + \frac{\varepsilon}{3} = \|x\| - \|x^l\| + \frac{\varepsilon}{3}.$$ 

Since the diameter of $I$ is less than $\frac{\varepsilon}{3}$, we get on the other hand that

$$|x_k - x^l_k| = |(x_k - \eta \delta x_j) - (x^l_k - \eta \delta x^l_j) + \eta \delta x_j - \eta \delta x^l_j|$$

$$\leq \frac{\varepsilon}{3} + |x_j - x^l_j| \leq \varepsilon + |x_j| - |x^l_j| = \varepsilon + \|x\| - \|x^l\|.$$ 

So the conclusion of the lemma follows from $(3.3)$ and $(3.5)$. \qed
We now proceed with the proof of Theorem 3.6. As usual, we consider a contraction $f : M \to c$, where $M$ is a closed subset of $\ell_\infty^n$ with $0 \notin M$. We will only show, as we may, that $f$ can be contractively extended to $M \cup \{0\}$.

Let $\varepsilon > 0$. It follows from Lemma 3.7 that there exist $A_1, \ldots, A_\mu$ subsets of $X$ such that

$$M \subset \bigcup_{i=1}^{\mu} A_i$$

and

$$\forall 1 \leq i \leq \mu \exists x^i \in A_i \cap M \text{ such that } \forall x \in A_i \cap M \quad \|x - x^i\| \leq \|x\| - \|x^i\| + \frac{\varepsilon}{2}.$$ 

There also exists $N \in \mathbb{N}$ such that

$$\forall n, m > N \forall i \in \{1, \ldots, \mu\} \quad |f(x^i)(n) - f(x^i)(m)| < \frac{\varepsilon}{2}.$$ 

Let now $x$ and $y$ in $M$. Then we pick $i$ such that $x \in A_i$. Thus, for all $n, m > N$

$$|f(x)(n) - f(y)(m)| \leq |f(x)(n) - f(x^i)(n)| + |f(x^i)(n) - f(x^i)(m)| + |f(x^i)(m) - f(y)(m)|$$

$$\leq \|x - x^i\| + \frac{\varepsilon}{2} + \|x^i - y\| \leq \|x\| - \|x^i\| + \frac{\varepsilon}{2} + \|y\| + \|x^i\|$$

$$\leq \|x\| + \|y\| + \varepsilon.$$

Then we can apply Lemma 3.4 to conclude our proof.

\[\square\]

Corollary 3.8. Let $X$ be any finite dimensional Banach space whose unit ball is a polytope. Then

$$\mathcal{A}(X, c) = (0, 1].$$

Proof. If $B_X$ is a polytope, we can find $f_1, \ldots, f_n$ in the unit sphere of the dual space of $X$ such that

$$B_X = \bigcap_{i=1}^{n} \{x \in X, \ |f_i(x)| \leq 1\}.$$ 

Then the map $T : X \to \ell_\infty^n$ defined by $Tx = (f(x_i))_{i=1}^{n}$ is clearly a linear isometry and the result follows immediately from Theorem 3.6. \[\square\]

We will finish this section with a counterexample in dimension 4. We denote by $\ell_2^2 \oplus_1 \ell_2^2$ the space $\mathbb{R}^4$ equipped with the norm:

$$\forall (s, t, u, v) \in \mathbb{R}^4, \quad \|(s, t, u, v)\| = (s^2 + t^2)^{1/2} + (u^2 + v^2)^{1/2}.$$ 

Then we have

Theorem 3.9.

$$\mathcal{A}(\ell_2^2 \oplus_1 \ell_2^2, c) = (0, 1].$$

Proof. First we pick $K > 1$ such that

$$\frac{1}{2} \left(\frac{K^2 - 1}{K^2}\right) \left(\frac{K}{K + 1}\right)^3 > \frac{3}{8}. \tag{3.7}$$
For \( n \in \mathbb{N} \), we define \( x_n = (K^{2n}, K^n, 0, 0) \) and \( y_n = (0, 0, K^{2n}, K^n) \). Note that

\[
\forall n \in \mathbb{N}, \quad \|x_n\| \leq K^{2n} + \frac{1}{2} \quad \text{and} \quad \|y_n\| \leq K^{2n} + \frac{1}{2}.
\]

On the other hand,

\[
\lim_{n \to \infty} (\|x_n\| - K^{2n}) = \lim_{n \to \infty} (\|y_n\| - K^{2n}) = \frac{1}{2}.
\]

So

\[
\exists n_0 \in \mathbb{N} \quad \forall n, m \geq n_0 \quad \|x_n - y_m\| \geq K^{2n} + K^{2m} + \frac{7}{8}.
\]

Now, for all \( n > m \), \( K^n + K^m \leq K^n\left(\frac{K+1}{K}\right) \) and \( K^{2n} - K^{2m} \geq K^{2n}\left(\frac{K^2-1}{K^2}\right) \).

Since

\[
\|x_n - x_m\| = (K^{2n} - K^{2m})\left[1 + \frac{1}{(K^n + K^m)^2}\right]^{1/2},
\]

we have

\[
\|x_n - x_m\| \geq (K^{2n} - K^{2m})\left[1 + \left(\frac{K}{K + 1}\right)^2 \frac{1}{K^{2n}}\right]^{1/2}.
\]

Therefore, there exists \( n_1 \geq n_0 \) such that for all \( n > m \geq n_1 \):

\[
\|x_n - x_m\| \geq (K^{2n} - K^{2m})\left[1 + \frac{1}{2}\left(\frac{K}{K+1}\right)^3 \frac{1}{K^{2n}}\right] \geq K^{2n} - K^{2m} + \frac{1}{2}\left(\frac{K^2-1}{K^2}\right)\left(\frac{K}{K + 1}\right)^3.
\]

Then, it follows from (3.7) that for all \( n > m \geq n_1 \):

\[
\|x_n - x_m\| \geq K^{2n} - K^{2m} + \frac{3}{8} \quad \text{and} \quad \|y_n - y_m\| \geq K^{2n} - K^{2m} + \frac{3}{8}.
\]

Let us denote \( M = \{x_n, \ n \geq n_1\} \cup \{y_n, \ n \geq n_1\} \). We will now construct \( u_n = f(x_n) \) and \( v_n = f(y_n) \) in \( c \) so that \( f : M \to c \) is 1-Lipschitz. So let \( n \geq n_1 \).

For \( k \) odd and \( k \leq n \), set \( u_n(k) = K^{2n} + \frac{5}{8} \) and \( u_n(k) = K^{2n} + \frac{1}{4} \) otherwise.

For \( k \) even and \( k \leq n \), set \( v_n(k) = -(K^{2n} + \frac{5}{8}) \) and \( v_n(k) = -(K^{2n} + \frac{1}{4}) \) otherwise.

We now check that \( f \) is 1-Lipschitz.

For all \( n > m \geq n_1 \), \( \|u_n - u_m\|_\infty \leq K^{2n} + \frac{5}{8} - (K^{2m} + \frac{1}{4}) = K^{2n} - K^{2m} + \frac{3}{8} \).

Therefore, by (3.10), \( \|u_n - u_m\|_\infty \leq \|x_n - x_m\| \).

We have, as well that \( \|v_n - v_m\|_\infty \leq \|y_n - y_m\| \).

We also have that for all \( n, m \geq n_1 \), \( \|u_n - v_m\|_\infty = K^{2n} + K^{2m} + \frac{7}{8} \).

Thus, (3.9) implies that \( \|u_n - u_m\|_\infty \leq \|x_n - y_m\| \).

We have shown that \( f \) is 1-Lipschitz.

Assume now that \( f \) can be extented at 0 into a 1-Lipschitz function \( g \) and let \( g(0) = w = (w(k))_{k \geq 1} \in c \). Then it follows from (3.8) that for all odd values of \( k \), \( w(k) \geq \frac{1}{8} \) and for all even values of \( k \), \( w(k) \leq -\frac{1}{8} \). This contradicts the fact that \( w \in c \).

\( \square \)
Remark 3.10. As we already mentioned in the introduction, this seems to be the first example of Banach spaces $X$ and $Y$ such that $\mathcal{A}(X,Y) \neq \tilde{\mathcal{A}}(X,Y)$ and also such that $\mathcal{A}(X,Y)$ is not closed in $(0,1]$.

4. Maps into $C(K)$ spaces

In this last section we show that if $K$ is an infinite compact metric space, then the study of the isometric extension for Lipschitz maps with values in $C(K)$ reduces to the results of the previous section. More precisely, we prove the following.

Theorem 4.1. Let $(X,d)$ be a metric space and $(K,\varrho)$ be an infinite compact metric space. Then

$$\mathcal{A}(X,C(K)) = \mathcal{A}(X,c).$$

The main step of the proof will be to establish the following generalization of Lemma 3.4.

Proposition 4.2. Let $M$ be a subset of $X$, $f : M \to C(K)$ a contraction and $x \in X \setminus M$. We denote by $D$ the diameter of $K$ for the distance $\varrho$. Then, the following statements are equivalent:

1. $f$ can be extended to a contraction $g : M \cup \{x\} \to C(K)$.
2. $\forall \varepsilon > 0 \ \exists \delta > 0$ such that $\forall t, s \in K$ with $\varrho(t, s) < \delta \ \forall y, z \in M$
   $$|f(y)(t) - f(z)(s)| \leq d(y, x) + d(z, x) + \varepsilon.$$
3. $\exists \varphi : [0, D] \longrightarrow [0, +\infty)$ such that $\varphi$ is continuous, $\varphi(0) = 0$ and
   $$\forall t, s \in K \ \forall y, z \in M \ |f(y)(t) - f(z)(s)| \leq d(y, x) + d(z, x) + \varphi(\varrho(t, s)).$$

Proof. Suppose that (1) holds. Then (2) follows from the triangle inequality and the fact that $g(x)$ is uniformly continuous on $K$.

Assume now that (2) holds. Let us define, for $\lambda \in (0, D]$: 

$$\xi(\lambda) = \sup_{y, z \in M} \sup_{\varrho(t, s) \leq \lambda} (|f(y)(t) - f(z)(s)| - d(x, y) - d(x, z)).$$

The function $\xi$ is clearly non decreasing and bounded below by $-2 \text{dist}(x, M)$. So we can set

$$\xi(0) = \lim_{\lambda \searrow 0} \xi(\lambda).$$

We have that

$$\forall t, s \in K \ \forall y, z \in M \ |f(y)(t) - f(z)(s)| \leq d(y, x) + d(z, x) + \xi(\varrho(t, s)).$$

It follows from (2) that $\xi(0) \leq 0$. So, if we set $\psi = \xi - \xi(0)$, we get that $\psi$ is non decreasing, $\psi(0) = 0$ and $\psi$ is continuous at 0. Since $\psi \geq \xi$, we still have

$$\forall t, s \in K \ \forall y, z \in M \ |f(y)(t) - f(z)(s)| \leq d(y, x) + d(z, x) + \psi(\varrho(t, s)).$$
We now define the function $\varphi$ in the following way: $\varphi(0) = 0$ and for $n \in \mathbb{N}$, $\varphi(\frac{P}{n+1}) = \psi(\frac{P}{n})$. We also ask $\varphi$ to be constant equal to $\psi(D)$ on $[\frac{P}{2}, D]$, and affine on each $[\frac{P}{n+2}, \frac{P}{n+1}]$ for $n \in \mathbb{N}$. It is now clear that $\varphi$ is non decreasing, continuous on $[0, D]$ and that $\psi \leq \varphi$ on $[0, D]$. So we have

$$\forall t, s \in K \forall y, z \in M \quad |f(y)(t) - f(z)(s)| \leq d(y, x) + d(z, x) + \varphi(g(t, s)).$$

This proves that (2) implies (3).

Suppose now that (3) holds and define, for $t \in K$,

$$g(x)(t) = \sup_{s \in K} \sup_{z \in M} (f(z)(s) - d(z, x) - \varphi(g(t, s))).$$

Fix $y_0 \in M$. Then, for all $z \in M$ and for all $s \in K$,

$$f(z)(s) - d(z, x) - \varphi(g(t, s)) \leq \|f(y_0)\|_{C(K)} + d(z, y_0) - d(z, x) \leq \|f(y_0)\|_{C(K)} + d(x, y_0).$$

So $g(x)(t)$ is well defined. Further, it follows from the uniform continuity of $\psi$ on $[0, D]$ that $g(x)$ is continuous on $K$.

Since $\varphi(0) = 0$, we have, by definition of $g(x)$, that for all $y \in M$ and all $t \in K$

$$f(y)(t) - g(x)(t) \leq d(x, y). \quad (4.1)$$

By (3), we get that for all $y, z \in M$ and for all $t, s \in K$

$$|f(z)(s) - f(y)(t)| \leq d(y, x) + d(z, x) + \varphi(g(t, s)),$$

so

$$f(z)(s) - d(z, x) - \varphi(g(t, s)) \leq f(y)(t) + d(y, x),$$

and by taking the supremum over $z$ and $s$ we obtain

$$g(x)(t) - f(y)(t) \leq d(x, y). \quad (4.2)$$

Combining (4.1) and (4.2), we get that for all $y \in M$ $\|g(x) - f(y)\|_{C(K)} \leq d(x, y)$. Thus (3) implies (1) and this ends the proof of Proposition 4.2.

Proof of Theorem 4.1

Since $K$ is an infinite compact metric space, it contains a closed subset $F$ which is homeomorphic to the one point compactification of $\mathbb{N}$. Then, $C(F)$ is clearly isometric to $c$. On the other hand, by the linear version of Tietze extension theorem due to K. Borsuk [3], there is a linear isometry $T : C(F) \rightarrow C(K)$ such that for any $f$ in $C(F)$, $Tf$ is an extension of $f$ to $K$. Let now $R$ be the restriction operator from $C(K)$ onto $C(F)$. Then $P = TR$ is a projection of norm 1 from $C(K)$ onto an isometric copy of $c$. Therefore, it is clear that for any metric space $X$, $A(X, C(K)) \subset A(X, c)$.

For the other inclusion, it is enough to show that if $1 \notin A(X, C(K))$, then $1 \notin A(X, c)$. So let us assume that $1 \notin A(X, C(K))$. Then there exist $M \subset X$, a contraction $f : M \rightarrow C(K)$ and $x \in X \setminus M$ such that $f$ can not be contractively extended to $M \cup \{x\}$. Thus, by
Proposition 4.2 there exists \( \varepsilon > 0 \) so that for all \( n \in \mathbb{N} \) there exist \( t_n, s_n \in K \) with \( \varrho(t_n, s_n) < 1/n \) and \( y_n, z_n \in M \) so that

\[
(4.3) \quad |f(y_n)(t_n) - f(z_n)(s_n)| > d(y_n, x) + d(z_n, x) + \varepsilon.
\]

Since \( K \) is compact, we may assume that the sequence \((t_n)_{n \in \mathbb{N}}\) is convergent. Define now a sequence \((w_n)_{n \in \mathbb{N}}\) in \( K \) by setting, for \( n \in \mathbb{N} \),
\[
w_{2n} - 1 = t_n \quad \text{and} \quad w_{2n} = s_n.
\]
Then the sequence \((w_n)_{n \in \mathbb{N}}\) is convergent. So we can define a 1-Lipschitz map \( h : M \to c \) by

\[
\forall y \in M \ h(y) = (h(y)(n))_{n \in \mathbb{N}} = (f(y)(w_n))_{n \in \mathbb{N}}.
\]

It now clearly follows from (4.3) and Lemma 3.4 that \( h \) does not have any extension to a 1-Lipschitz map from \( M \cup \{x\} \) into \( c \). Therefore \( 1 \notin \mathcal{A}(X, c) \).

\[\square\]

References

[1] K. Ball, Markov chains, Riesz transforms and Lipschitz maps, Geometric and Functional Analysis, 2 (1992), 137-172.

[2] Y. Benyamini and J. Lindenstrauss, Geometric Nonlinear Functional Analysis, Colloquium Publications, 48 (2000).

[3] K. Borsuk, Über Isomorphie der Funktionalräume, Bull. Int. Acad. Polon. Sci. A, 1/3 (1933), 1-10.

[4] J. Elton, E. Odell, The unit ball of every infinite dimensional normed linear space contains a \((1 + \varepsilon)\)-separated sequence, Colloq. Math. 44 (1981), no 1, 105-109.

[5] F. Grünbaum, E.H. Zarantonello, On the extension of uniformly continuous mappings, Michigan Math. J. 15 (1968), 65-74.

[6] T. Hayden, J.H. Wells, L.R. Williams, On the extension of Lipschitz-Hölder maps on \( L^p \) spaces, Studia Math. 39 (1971), 29-38.

[7] R.C. James, Uniformly non square Banach spaces, Ann. of Math. 80 (1964), no 2, 542-550.

[8] W.B. Johnson, M. Zippin, Extension of operators from subspaces of \( c_0(\Gamma) \) into \( C(K) \) spaces, Proc. Amer. Math. Soc. 107 (1989), no 3, 751-754.

[9] W.B. Johnson, M. Zippin, Extension of operators from weak*-closed subspaces of \( \ell_1 \) into \( C(K) \) spaces, Studia. Math. 117 (1995), no 1, 43-55.

[10] M.D. Kirszbraun, Über die zusammenziehende und Lipschitze Transformationen, Fund. Math., 22 (1934), 77-108.

[11] J. Lindenstrauss, A. Pelczyński, Contributions to the theory of the classical Banach spaces, J. Funct. Anal. 8 (1971), 225-249.

[12] G. Minty, On the extension of Lipschitz, Lipschitz-Hölder and monotone functions, Bull. Amer. Math. Soc., 76 (1970), 334-339.

[13] A. Naor, A phase transition phenomenon between the isometric and isomorphic extension problems for Hölder functions between \( L^p \) spaces, Mathematika 48 (2001), no 1-2, 253-271.

[14] J.H. Wells and L.R. Williams, Embeddings and Extensions in Analysis, Ergebnisse 84, Springer-Verlag (1975).
DÉPARTEMENT DE MATHEMATIQUES, UNIVERSITÉ DE FRANCHE-COMTÉ, 16 ROUTE DE GRAY, 25030
BESANÇON, FRANCE

E-mail address: glancien@math.univ-fcomte.fr

DEPARTMENT OF MATHEMATICS AND STATISTICS, MIAMI UNIVERSITY, OXFORD, OH 45056, USA

E-mail address: randrib@muohio.edu