Ruled minimal surfaces in $\mathbb{R}^3$ with density $e^z$

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Abstract

We classify ruled minimal surfaces in $\mathbb{R}^3$ with density $e^z$. It is showed that there is no none cylindric ruled minimal surface and there is a family of cylindrical ruled minimal surfaces in $\mathbb{R}^3$ with density $e^z$. It is also proved that all translation minimal surfaces are ruled.

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1 Introduction

Manifolds with density, a new category in geometry, appeared in many ways in mathematics, such as quotients of Riemannian manifolds or as Gauss space and, and can be viewed as smooth case of Gromov’s $\textit{mm}$-spaces. A density on a Riemannian manifold $M^n$ is a positive function $e^{\varphi(x)}$ used to weight volume and hypersurface area. Gauss space $G^n$ is a Euclidean space with Gaussian probability density $(2\pi)^{-\frac{n}{2}}e^{-\frac{r^2}{2}}$ that is very interesting to probabilists. For more details about manifolds with density and some first results on Morgan’s grand goal to “generalize all of Riemannian geometry to manifolds with density” we refer the reader to [5], [6], [7], [3]. We refer especially the reader to Chapter 18 of Morgan’s book Geometric Measure Theory ([6]), in which the author described general manifolds with density and their relationship to Perelman’s proof of the Poincaré conjecture. Following Gromov ([4, p. 213]) the natural generalization of the mean curvature of hypersurfac e on a manifold with density $e^\varphi$ is defined as

$$H_\varphi = H - \frac{1}{n-1} \frac{d\varphi}{dn}$$

(1)
and therefore, the mean curvature of a surface in $\mathbb{R}^3$ with density $e^\varphi$ is

$$H_\varphi = H - \frac{1}{2} \cdot \frac{d\varphi}{dn},$$

where $H$ is the Euclidean mean curvature and $n$ is the normal vector field of the surface. We call $H_\varphi$ the mean curvature with density or mean $\varphi$-curvature of the surface.

The literature of minimal surfaces began with Lagrange’s work in 1760. Lagrange established a PDE, named after him, for the graph of a $C^2$-function of two variables to be minimal. At that time, the only known solution of Lagrange’s equation was planes. In 1776, Meusnier solved that equation with an additional assumption that the level curves were straight lines and obtained the solution of ruled minimal surface Helicoid. It is well known that, beside the trivial case of planes, helicoid is the unique (noncylindrical) ruled minimal surface (see [2], [1]). In 1835, Scherk solved Lagrange’s equation for translation functions, i.e. functions of the type $f(x, y) = g(x) + h(y)$ and discovered Scherk’s minimal surfaces.

Often, locally a regular surface can be considered as the graph of the function

$$X : U \longrightarrow \mathbb{R},$$

where $U$ is a domain in $\mathbb{R}^2$, in product space $\mathbb{R}^2 \times \mathbb{R}$. In this paper, we consider ruled minimal surfaces in space with log-linear density $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}_\varphi$, where $\mathbb{R}_\varphi$ is the real line with log-linear density $e^\varphi$. The space with log-linear density $e^\varphi$ is the first non-trivial case of the new category “manifolds with density” and is equivalent with the case of space with density $e^z$ as it is showed in section 2. All ruled minimal surfaces in $\mathbb{R}^3$ with density $e^z$ are classified. Quite different as the classical case, it is showed that there is no noncylindrical ruled minimal surfaces and there is a family of cylindrical ruled minimal surfaces. We also consider the case of translation minimal surfaces and it is proved that, all translation minimal surfaces are rule.

All functions in this paper are assumed belong to the class $C^2$.

## 2 Minimal surfaces in spaces with densities

From the formula of the $\varphi$-curvature, it is clear that if we understand the geometric meaning of $\frac{d\varphi}{dn}$ we can discover some simple minimal surfaces in space with density. For example, in Gauss space $G^3$, $\frac{d\varphi}{dn}$ is the distance from the origin to the tangent hyperplane at the corresponding point of the surface. So it is easy to see that in Gauss space $G^3$ (see also [3])

1. planes have constant mean curvarture and planes passing through the origin are minimal;
2. spheres about the origin have constant mean curvature and the one with radius \( \frac{1}{\sqrt{2}} \) is minimal;
3. circular cylinders with the axis passing through the origin have constant curvature and the one with radius 1 is minimal.

Let \( \varphi(x_1, x_2, \ldots, x_n) \) be a linear function on Euclidean space \( \mathbb{R}^n \)
\[
\varphi(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i.
\]
A linear density on \( \mathbb{R}^n \) is the positive function \( e^{\varphi(x)} \), where \( x = (x_1, x_2, \ldots, x_n) \). It is easy to see that, the sets of points in \( \mathbb{R}^n \) with the linear density \( e^\varphi \) that have the same density are hyperplanes. By a suitable changing the coordinate system we can assume that the density has the form \( e^{x_n} \) and therefore we can view the space \( \mathbb{R}^n \) with density \( e^{\varphi(x)} \) as product \( \mathbb{R}^{n-1} \oplus \mathbb{R}_\varphi \), where \( \mathbb{R}^{n-1} \) is nothing but Euclidean \((n - 1)\)-space and \( \mathbb{R}_\varphi \) is a real line with density \( e^{x_n} \).

Since \( \nabla \varphi = (0, 0, \ldots, 1) \), \( \frac{d\varphi}{dn} = \langle \nabla \varphi, n \rangle \) is the cosine of the angle between \( n \) and \( z \)-axis. By the definition of the mean \( \varphi \)-curvature, it is easy to see that \( H_\varphi \) does not change under a translation or a rotation about \( z \)-axis and moreover, we have

1. hyperplanes in \( \mathbb{R}^n \) with density \( e^{x_n} \) have constant mean curvature.
2. hyperplanes in \( \mathbb{R}^n \) with density \( e^{x_n} \) that are parallel to the \( x_n \)-axis have zero mean curvature.
3. a circular hypercylinder with rulings parallel to \( x_n \)-axis have constant mean curvature.

### 3 Ruled minimal surfaces in \( \mathbb{R}^3 \) with linear density \( e^z \)

Now we consider the problem of classifying all ruled minimal surfaces in \( \mathbb{R}^3 \) with a linear density. Coordinates in \( \mathbb{R}^3 \) are denoted by \((x, y, z)\). Without loss of generality we can assume that the density is \( e^z \).

Locally, a ruled surface is given by the equation
\[
X(u, v) = \alpha(u) + v\beta(u), \quad u \in (a, b), \quad v \in (c, d).
\]
(3)

We can assume that \( |\alpha'| = 1, \quad |\beta| = 1, \quad \langle \alpha', \beta \rangle = 0 \).

We will focus on two cases: cylindrical ruled surface \( (\beta = \text{const.}, \, \text{for all } v \in (c, d)) \) and noncylindrical ruled surfaces \( (\beta' \neq 0 \text{ for all } v \in (c, d)) \). Our results show that it is no need to consider at the isolated points where \( \beta'(v) = 0 \).
Denote $E, F, G$ the coefficients of the first fundamental form and $e, f, g$ the coefficients of the second fundamental form, a direct computation yields

$$N = \frac{(\alpha' + v\beta') \wedge \beta}{|(\alpha' + v\beta') \wedge \beta|};$$

$$E = 1 + 2v\langle \alpha', \beta' \rangle + v^2|\beta'|^2, \quad F = 0, \quad G = 1;$$

$$e = \langle N, \alpha'' + v\beta'' \rangle, \quad f = \langle N, \beta' \rangle, \quad g = 0;$$

$$H_\varphi = \frac{1}{2} \left[ \frac{\langle N, \alpha'' + v\beta'' \rangle}{1 + 2v\langle \alpha', \beta' \rangle + v^2|\beta'|^2} - \langle N, \nabla \varphi \rangle \right]. \quad (4)$$

We have

**Proposition 1**

$$H_\varphi = 0 \iff \begin{cases} \langle \alpha' \wedge \beta, \alpha'' \rangle = \langle \alpha' \wedge \beta, \nabla \varphi \rangle \\ \langle \alpha' \wedge \beta, \beta'' \rangle + \langle \beta' \wedge \beta, \alpha'' \rangle = \langle \beta' \wedge \beta, \nabla \varphi \rangle + \langle \alpha' \wedge \beta, 2\langle \alpha', \beta' \rangle \nabla \varphi \rangle \\ \langle \beta' \wedge \beta, \beta'' \rangle = \langle \beta' \wedge \beta, 2\langle \alpha', \beta' \rangle \nabla \varphi \rangle + \langle \alpha' \wedge \beta, |\beta'|^2 \nabla \varphi \rangle \\ \langle \beta' \wedge \beta, |\beta'|^2 \nabla \varphi \rangle = 0 \end{cases}. \quad (5)$$

**Proof.** $H_\varphi = 0$ if and only if

$$\frac{\langle N, \alpha'' + v\beta'' \rangle}{1 + 2v\langle \alpha', \beta' \rangle + v^2|\beta'|^2} = \langle N, \nabla \varphi \rangle \quad (6)$$

Replace $N = \frac{(\alpha' + v\beta') \wedge \beta}{|(\alpha' + v\beta') \wedge \beta|}$ in (6) we get an equality with both RHS and LHS are polynomials on the variable $v$. Identifying the coefficients we obtain (5). \hfill \Box

### 3.1 The case of $\beta' \neq 0$

From the last equation of (5)

$$\langle \beta' \wedge \beta, |\beta'|^2 \nabla \varphi \rangle = 0, \quad (7)$$

we have $\langle \beta' \wedge \beta, \nabla \varphi \rangle = 0$. Note that $\nabla \varphi = (0, 0, 1)$, and $\beta \perp \beta'$, we conclude $\beta$ belongs to a plane containing the $z$-axis. After a rotation about the $z$-axis, we can assume that $\beta = (\cos t(u), 0, \sin t(u))$, with $t' \neq 0$ and therefore, the third equality of (5) becomes

$$\langle \alpha' \wedge \beta, |\beta'|^2 \nabla \varphi \rangle = 0. \quad (8)$$

From (8) we conclude that $\alpha'$ belongs to the plane $\{y = 0\}$ and the curve $\alpha$ lies on a plane parallel to $xz$-plane. It is clear that, $\alpha$ and $\beta$ satisfy system of equations (5). Thus,

**Proposition 2** If $\beta' \neq 0$, $\forall v \in (c, d)$, ruled minimal surfaces determined by (3) are planes parallel to the $z$-axis.
3.2 The case of $\beta' = 0$

Since $\beta' = 0$, $\beta = (a, b, c) = \text{const.}$ and $a^2 + b^2 + c^2 = 1$, system (5) becomes

$$\begin{aligned}
\langle \alpha' \wedge \beta, \alpha'' \rangle &= \langle \alpha' \wedge \beta, \nabla \varphi \rangle \\
\beta &= (a, b, c) = \text{const.}
\end{aligned} \tag{9}$$

The first equation of (9) implies

$$\alpha'' - \nabla \varphi = m \alpha' + n \beta,$$

and hence

$$\begin{aligned}
\langle \alpha'' - \nabla \varphi, \alpha' \rangle &= m = -\langle \nabla \varphi, \alpha' \rangle, \\
\langle \alpha'' - \nabla \varphi, \beta \rangle &= n = -\langle \nabla \varphi, \beta \rangle.
\end{aligned}$$

Thus,

$$\alpha'' - \nabla \varphi = -\langle \nabla \varphi, \alpha' \rangle \alpha' - \langle \nabla \varphi, \beta \rangle \beta,$$

or

$$\alpha'' + \langle \nabla \varphi, \alpha' \rangle \alpha' = \nabla \varphi - \langle \nabla \varphi, \beta \rangle \beta \tag{10}$$

Since the mean $\varphi$-curvature does not change under a rotation about $z$-axis, we can assume $a = 0$. So we have

$$\nabla \varphi - \langle \nabla \varphi, \beta \rangle \beta = (0, -cb, 1 - c^2).$$

Because $b^2 = 1 - c^2$,

$$\nabla \varphi - \langle \nabla \varphi, \beta \rangle \beta = (0, -cb, b^2). \tag{11}$$

Suppose that $\alpha = (x(u), y(u), z(u))$, then (11) is equivalent to the following system of equations

$$\begin{aligned}
x'' + x' z' &= 0, \\
y'' + y' z' &= -cb, \\
z'' + z^2 &= b^2.
\end{aligned} \tag{12}$$

We treat the two special case $\beta = (0, 0, \pm 1)$ and $\beta = (0, \pm 1, 0)$ first.

If $\beta = (0, 0, \pm 1)$, the RHS of (10) equals zero. We conclude that $\alpha'$ and $\alpha''$ are parallel, and hence $\alpha'' = 0$. Thus, $\alpha$ is a straight line and we have

**Proposition 3** If $\beta = (0, 0, \pm 1)$, ruled minimal surfaces determined by equation (3) are planes parallel to the $z$-axis.

We can also treat this case by solving (12). In this case (12) becomes

$$\begin{aligned}
x'' + x' z' &= 0, \\
y'' + y' z' &= 0, \\
z'' + z^2 &= 0. \tag{13}
\end{aligned}$$
Since $\beta \perp \alpha'$, we get $z' = 0$, and hence $x'' = y'' = 0$. We conclude that $\alpha = (x, y, z)$ is a straight line lying on the plane $z = \text{const.}$ and hence the ruled surfaces is a plane parallel to the $z$-axis.

If $\beta = (0, \pm 1, 0)$, (12) becomes

\[
\begin{cases}
x'' + x'z' = 0, \\
y'' + y'z' = 0, \\
z'' + z'^2 = 1.
\end{cases}
\]

(14)

Since $\beta \perp \alpha'$, we get $y' = 0$ and conclude that $\alpha$ lies on the plane $y = \text{const.}$. The last equation of (14) gives us the solution

\[
z' = 1 - \frac{2}{1 + Ae^{2u}} = \frac{Ae^{2u} - 1}{Ae^{2u} + 1}, \quad A > 0
\]

\[z = \log(1 + Ae^{2u}) - u = \log(e^{-u} + Ae^{u}).\]

The first equation in (14) gives

\[x' = Be^{-z} \frac{B}{e^{-u} + Ae^{u}} = \frac{Be^u}{1 + Ae^{2u}},\]

and hence

\[x = \frac{B}{\sqrt{A}} \arctan \sqrt{Ae^{u}} + C.\]

Note that, since the mean $\varphi$-curvature does not change under a translation, we can take the constant in the expression of $x$ and $z$ to be zero.

Since $x'^2 + y'^2 + z'^2 = 1$, we have $4A = B^2$. Thus,

**Proposition 4** If $\beta = (0, \pm 1, 0)$, a ruled minimal surface determined by equation (3) has a parametric equation of the following form

\[
\begin{cases}
x = 2 \arctan \sqrt{A}e^u \\
y = \pm v \\
z = \log(e^{-u} + Ae^{u})
\end{cases}
\]

(15)

If $\beta = (0, b, c)$, $b, c \neq 0$ the system (12) becomes

\[
\begin{cases}
x'' + x'z' = 0, \\
y'' + y'z' = -cb, \\
z'' + z'^2 = b^2.
\end{cases}
\]

(16)

Since $\beta \perp \alpha'$, we get $by' = -cz'$ and conclude that $\alpha$ lies on the plane $by + cz + d = 0$. The last equation of (16) gives us the solution
Figure 1: Ruled minimal surface with $\beta' = (0, 1, 0)$.

Figure 2: Ruled minimal surface with $\beta' = (0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. 
Figure 3: Ruled minimal surface with $\beta' = (0, 0.1, 0.99)$.

$$z' = b - \frac{2b}{1 + Ae^{2bu}} = \frac{bAe^{2bu} - b}{Ae^{2bu} + 1}, \quad A > 0$$

$$z = \log(1 + Ae^{2bu}) - bu = \log(e^{-bu} + Ae^{bu}).$$

The first equation in (16) gives

$$x' = Be^{-z} = \frac{B}{e^{-bu} + Ae^{bu}},$$

and hence

$$x = \frac{B}{b\sqrt{A}} \arctan(\sqrt{Ae^{bu}}) + C.$$  

Since $x'^2 + y'^2 + z'^2 = 1$, we have $4A = B^2$. Thus,

**Proposition 5** If $\beta = (0, b, c), \ b, c \neq 0$, a ruled minimal surface determined by equation (3) has a parametric equation of the following form

$$\begin{cases} 
  x = 2 \arctan \sqrt{A}e^{bu}, \\
  y = -\frac{c}{b} \log(e^{-bu} + Ae^{bu}) + bv, \\
  z = \log(e^{-bu} + Ae^{bu}) + cv.
\end{cases} \quad (17)$$
Combining these above results, we have

**Theorem 1** Beside planes parallel to the $z$-axis the only cylindrical ruled minimal surfaces in $\mathbb{R}^3$ with density $e^z$ are those given by (17).

### 4 Translation minimal surfaces in space with linear density $e^z$.

In this section we study translation minimal surfaces in $\mathbb{R}^3$ with density $e^z$. We prove that all translation surfaces that are minimal must be ruled.

**Theorem 2** A translation surface given by

$$X(u, v) = (u, v, g(u) + h(v))$$

is minimal if either $g(u) = au + b$ or $h(v) = cv + d$.

**Proof.** A straightforward computation shows that

$$H_\varphi = 0 \iff g''(1+h'^2) + h''(1+g'^2) = 1 + g'^2 + h'^2. \quad (18)$$
We fix $v = v_0$, and set $A = 1 - h''(v_0)$, $B = 1 + h'^2(v_0)$, $C = 1 + h'^2(v_0) - h''(v_0)$. Note that $B > 0$ and $C = B - A - 1$. Thus, $f$ must be satisfies the following equation

$$Ag'^2 + Bg'' = C,$$

and hence

$$g'' = \frac{C - Ag'^2}{B}.$$

Substitute $g''$ into (18), we get

$$g'^2 \left[ h'' - \frac{A}{B}(1 + h'^2) - 1 \right] = 1 - h'' + h'^2 - \frac{C}{B}(1 + h'^2). \quad (19)$$

From equation (19), unless $g' = \text{const.}$ we must have

$$h'' - \frac{A}{B}(1 + h'^2) - 1 = 0 \quad (20)$$

and

$$1 - h'' + h'^2 - \frac{C}{B}(1 + h'^2) = 0. \quad (21)$$

Substitute $h''$ from equation (20) into equation (21) we obtain

$$(1 + h'^2) \left( 1 - \frac{C}{B} - \frac{A}{B} \right) = 1.$$

Note that $C = B - A - 1$, we get

$$h'^2 = B - 1.$$

Thus, $h' = \text{const.}$ and the proof is completed. \qed

Since the role of $g$ and $h$ are the same, we only need to consider translation minimal surfaces of the following form

$$X(u, v) = (u, v, g(u) + cv + d).$$

A straightforward computation shows that $g$ must be of the form

$$-(1 + c^2) \log \left| \cos \frac{u + D}{\sqrt{1 + c^2}} \right|.$$

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