FINITE-DIMENSIONAL REPRESENTATIONS
OF THE QUANTUM SUPERALGEBRA $U_q[gl(2/2)]$:
I. Typical representations at generic $q$

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Abstract

In the present paper we construct all typical finite-dimensional representations of the quantum Lie superalgebra $U_q[gl(2/2)]$ at generic deformation parameter $q$. As in the non-deformed case the finite-dimensional $U_q[gl(2/2)]$-module $W^q$ obtained is irreducible and can be decomposed into finite-dimensional irreducible $U_q[gl(2) \oplus gl(2)]$-submodules $V^q_k$.

PACS numbers: 02.20Tw, 11.30Pb.

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1. Introduction

Since the quantum deformations\(^1\)\(^−\)\(^^5\) became a subject of intensive investigations many (algebraic and geometric) structures and different representations of quantum (super-) groups have been obtained and understood. For instance, the quantum algebra \(U_q[sl(2)]\) is very well studied\(^6\)\(^−\)\(^^8\). Originated from intensive investigations on the quantum inverse scattering method and the Yang-Baxter equations, the quantum groups have found various applications in theoretical physics and mathematics (see in this context, for example, Refs.\(^^9\)\(^−\)\(^^12\)). As in the non-deformed case for applications of quantum groups we often need their explicit representations. Being a subject of many investigations, representations of quantum groups, especially representations of quantum superalgebras are presently under development. However, although the progress in this direction is remarkable the problem is still far from being satisfactorily solved. Explicit representations are known only for quantum Lie superalgebras of lower ranks and of particular types like \(U_q[osp(1/2)]\)\(^13\), \(U_q[gl(1/n)]\)\(^14\), etc.. For higher rank quantum Lie superalgebras\(^15\)\(^−\)\(^18\), besides some \(q\)-oscillator representations which are most popular among those constructed, we do not know so much about other representations, in particular the finite-dimensional ones. Some general aspects and module structures of finite-dimensional representations of \(U_q[gl(m/n)]\) are considered in Ref.\(^18\) (see also Ref.\(^14\)) but without their explicit constructions. So the question concerning an explicit construction of finite-dimensional representations of \(U_q[gl(m/n)]\) is still unsolved for \(m\) and \(n \geq 2\).

Here, extending the method developed by Kac\(^19\) in the case of Lie superalgebras (from now on, only superalgebras) we shall construct all finite-dimensional representations of the quantum Lie superalgebra \(U_q[gl(2/2)]\) at generic \(q\), i.e. \(q\) is not a root of unity. It turns out that the finite-dimensional \(U_q[gl(2/2)]\)-modules have similar structures to that of the non-deformed ones\(^20\)\(^,21\) and are decomposed into finite-dimensional irreducible \(U_q[gl(2) \oplus gl(2)]\)-modules. Finite-dimensional \(U_q[gl(2/2)]\)-modules can be classified again as typical or atypical ones (see the Proposition 2). In the frame-work of this paper for the sake of simplicity we shall consider only the typical representations at generic \(q\). When \(q\) is a root of unity, as emphasized also
in 18, the structures of $U_q[gl(2/2)]$-modules are drastically different in comparison with the structures of $gl(2/2)$-modules 20,21. The present investigation on typical representations at generic $q$ is easily extended on atypical representations at generic $q$ 22 and finite-dimensional representations at $q$ being a root of unity 23.

The paper is organized as follows. In order to make the present construction clear, in the section 2 we expose some introductory concepts and basic definitions of quantum superalgebras, especially $U_q[gl(m/n)]$. We also describe briefly the procedure used for constructing finite-dimensional representations of $U_q[gl(m/n)]$. The quantum superalgebra $U_q[gl(2/2)]$ is defined in section 3. The section 4 is devoted to construction of finite-dimensional representations of $U_q[gl(2/2)]$. Some comments and the conclusion are made in the section 5, while the references are given in the last section 6.

For a convenient reading we shall keep as many as possible the abbreviations and notations used in Ref. 20 among the following ones:

- fidirmod(s) - finite-dimensional irreducible module(s),
- GZ basis - Gel’fand-Zetlin basis,
- lin.env.\{X\} - linear envelope of X,
- $q$ - the deformation parameter,
- $V^q_l \otimes V^q_r$ - tensor product between two linear spaces $V^q_l$ and $V^q_r$
  or a tensor product between a $U_q[gl(2)]$-module $V^q_l$
  and a $U_q[gl(2)_r]$-module $V^q_r$,
- $T^q \otimes V^q_0$ - tensor product between two $U_q[gl(2) \oplus gl(2)]$-modules $T^q$ and $V^q_0$,
- $[x]_q = \frac{q^x-q^{-x}}{q-q^{-1}}$, where $x$ is some number or operator,
- $[x] \equiv [x]_q^2$,
- $\{E, F\}$ - supercommutator between $E$ and $F$,
- $[E, F]_q \equiv EF - qFE$ - q-deformed commutator between $E$ and $F$, 
$a_{ij}$ - an element of the Cartan matrix $(a_{ij})$,

$q_i = q^{d_i}$, where $d_i$ are rational numbers such that

$d_i a_{ij} = d_j a_{ji}$, $i, j = 1, 2, ..., r$,

$\mathcal{E}_i = e_i q_i^{-h_i} \equiv e_i k_i^{-1} k_{i+1}$,

$\mathcal{F}_i = f_i q_i^{-h_i} \equiv f_i k_i^{-1} k_{i+1}$.

Note that we must not confuse the quantum deformation $[x] \equiv [x]_{q^2}$ of $x$ with the highest weight (signature) $[m]$ in the GZ basis $(m)$ or with the notation $[\ , \ ]$ for commutators.

2. Some introductory concepts of quantum superalgebras

Let $g$ be a rank $r$ (semi-) simple superalgebra, for example, $sl(m/n)$ or $osp(m/n)$. The quantum superalgebra $U_q(g)$ as a quantum deformation (q-deformation) of the universal enveloping algebra $U(g)$ of $g$, is completely defined by the Cartan-Chevalley canonical generators $h_i$, $e_i$ and $f_i$, $i = 1, 2, ..., r$ which satisfy

a) the quantum Cartan-Kac supercommutation relations

$$[h_i, h_j] = 0,$$

$$[h_i, e_j] = a_{ij} e_j,$$

$$[h_i, f_j] = -a_{ij} f_j,$$

$$[e_i, f_j] = \delta_{ij} [h_i]_{q^2},$$

$$\quad (2.1)$$

b) the quantum Serre relations

$$(ad_q \mathcal{E}_i)^{1-\tilde{a}_{ij}} \mathcal{E}_j = 0,$$

$$(ad_q \mathcal{F}_i)^{1-\tilde{a}_{ij}} \mathcal{F}_j = 0$$

where $(\tilde{a}_{ij})$ is a matrix obtained from the non-symmetric Cartan matrix $(a_{ij})$ by replacing the strictly positive elements in the rows with 0 on the diagonal entry by $-1$, while $ad_q$ is the $q$-deformed adjoint operator given by the formula (2.8) and
c) the quantum extra-Serre relations $^{24-26}$ (for $g$ being $sl(m/n)$ or $osp(m/n)$)

\[
\{[e_{m-1}, e_{m}]_{q^2}, [e_{m}, e_{m+1}]_{q^2}\} = 0
\]

\[
\{[f_{m-1}, f_{m}]_{q^2}, [f_{m}, f_{m+1}]_{q^2}\} = 0
\]

being additional constraints on the unique odd Chevalley generators $e_m$ and $f_m$. In the above formulas we denoted $q_i = q^{d_i}$ where $d_i$ are rational numbers symmetrizing the Cartan matrix $d_i a_{ij} = d_j a_{ji}$, $1 \leq i, j \leq r$. For example, in case $g = sl(m/n)$ we have

\[
d_i = \begin{cases} 
1 & \text{if } 1 \leq i \leq m, \\
-1 & \text{if } m + 1 \leq i \leq r = m + n - 1.
\end{cases}
\]

The above-defined quantum superalgebras form a subclass of a special class of Hopf algebras called by Drinfel’d quasitriangular Hopf algebras $^2$. They are endowed with a Hopf algebra structure given by the following additional maps:

a) coproduct $\Delta : U \rightarrow U \otimes U$

\[
\Delta(1) = 1 \otimes 1,
\]

\[
\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i,
\]

\[
\Delta(e_i) = e_i \otimes q_i^{h_i} + q_i^{-h_i} \otimes e_i,
\]

\[
\Delta(f_i) = f_i \otimes q_i^{h_i} + q_i^{-h_i} \otimes f_i,
\]

b) antipode $S : U \rightarrow U$

\[
S(1) = 1,
\]

\[
S(h_i) = -h_i,
\]

\[
S(e_i) = -q_i^{a_{ii}} e_i,
\]

\[
S(f_i) = -q_i^{-a_{ii}} f_i
\]

and

c) counit $\varepsilon : U \rightarrow C$
ε(1) = 1,
ε(h_i) = ε(e_i) = ε(f_i) = 0, \hspace{1cm} (2.7)

Then the quantum adjoint operator \( ad_q \) has the following form \(^{16,27}\)

\[
ad_q = (\mu_L \otimes \mu_R)(id \otimes S)\Delta
\hspace{1cm} (2.8)
\]

with \( \mu_L \) (respectively, \( \mu_R \)) being the left (respectively, right) multiplication:
\[
\mu_L(x)y = xy \hspace{1cm} \text{(respectively),} \hspace{1cm} \mu_R(x)y = (-1)^{\text{deg}_x \cdot \text{deg}_y}yx.
\]

A quantum superalgebra \( U_q[gl(m/n)] \) is generated by the generators
\[
k_i^{\pm 1} \equiv q_i^{\pm E_{ii}}, \hspace{1cm} e_j \equiv E_{j+1,j}, \hspace{1cm} i = 1, 2, ..., m+n, \hspace{1cm} j = 1, 2, ..., m+n-1
\]
such that the following relations hold (cf. Refs. \(^{14,18}\))

\begin{enumerate}
  \item \text{a) the super-commutation relations}
  \[
  \begin{aligned}
  k_i k_j &= k_j k_i, & k_i k_i^{-1} &= k_i^{-1} k_i = 1, \\
  k_i e_j k_i^{-1} &= q_i^{(\delta_{i,j-\delta_{i,j+1}})} e_j, & k_i f_j k_i^{-1} &= q_i^{(\delta_{i,j+1-\delta_{i,j}})} f_j, \\
  [e_i, f_j] &= \delta_{ij} [h_i]_{q^2}, \hspace{1cm} \text{where} \hspace{1cm} q_i^{h_i} = k_i k_i^{-1}, \hspace{1cm} (2.9)
  \end{aligned}
  \]
  
  \item \text{b) the Serre relations} \(^{2.2}\) taking now the explicit forms
  \[
  \begin{aligned}
  [e_i, e_j] &= [f_i, f_j] = 0, \hspace{1cm} \text{if} \hspace{1cm} |i-j| \neq 1, \\
  e_i^2 &= f_i^2 = 0, \\
  [e_i, [e_i, e_j]_{q^{\pm 2}}]_{q^{\mp 2}} &= [f_i, [f_i, f_j]_{q^{\pm 2}}]_{q^{\mp 2}} = 0, \hspace{1cm} \text{if} \hspace{1cm} |i-j| = 1 \hspace{1cm} (2.10)
  \end{aligned}
  \]
  
  \item \text{c) the extra-Serre relations} \(^{2.3}\)
  \[
  \begin{aligned}
  \{[e_{m-1}, e_m]_{q^2}, [e_m, e_{m+1}]_{q^2}\} &= 0, \\
  \{[f_{m-1}, f_m]_{q^2}, [f_m, f_{m+1}]_{q^2}\} &= 0. \hspace{1cm} (2.11)
  \end{aligned}
  \]
\end{enumerate}

Here, besides \( d_i, 1 \leq i \leq r = m + n - 1 \) given in (2.4) we introduced \( d_{m+n} = -1 \).

The Hopf structure on \( k_i \) looks as
\[ \Delta(k_i) = k_i \otimes k_i, \]
\[ S(k_i) = k_i^{-1}, \]
\[ \varepsilon(k_i) = 1. \] (2.12)

The generators \( E_{ii}, E_{i,i+1} \) and \( E_{i+1,i} \) together with the generators defined in the following way
\[ E_{ij} := [E_{ik}E_{kj}]_{q^{-2}} \equiv E_{ik}E_{kj} - q^{-2}E_{kj}E_{ik}, \quad i < k < j, \]
\[ E_{ji} := [E_{jk}E_{ki}]_{q^2} \equiv E_{jk}E_{ki} - q^2E_{ki}E_{jk}, \quad i < k < j, \] (2.13)
play an analogous role as the Weyl generators \( e_{ij} \),
\[ (e_{ij})_{kl} = \delta_{ik}\delta_{jl}, \] (2.14)
of the superalgebra \( gl(m/n) \) whose universal enveloping algebra \( U[gl(m/n)] \) represents a classical limit of \( U_q[gl(m/n)] \) when \( q \to 1 \).

The quantum algebra \( U_q[gl(m/n)] \cong U_q[gl(m) \oplus gl(n)] \) generated by \( k_i, e_j \) and \( f_j, i = 1, 2, \ldots, m+n, m \neq j = 1, 2, \ldots, m+n-1, \)
\[ U_q[gl(m/n)] = \text{lin.env.}\{E_{ij} \mid 1 \leq i, j \leq m \text{ and } m+1 \leq i, j \leq m+n\} \] (2.15)
is an even subalgebra of \( U_q[gl(m/n)] \). Note that \( U_q[gl(m/n)] \) is included in the largest even subalgebra \( U_q[gl(m/n)]_0 \) containing all elements of \( U_q[gl(m/n)] \) with even powers of the odd generators.

As is shown by M. Rosso \cite{Rosso28} and C. Lusztig \cite{Lusztig29}, a finite-dimensional representation of a Lie algebra \( g \) can be deformed to a finite-dimensional representation of its quantum analogue \( U_q(g) \). In particular, finite-dimensional representations of \( U_q[gl(m) \oplus gl(n)] \) are quantum deformations of those of \( gl(m) \oplus gl(n) \). Hence, a finite-dimensional irreducible representation of \( U_q[gl(m) \oplus gl(n)] \) is again highest weight. Following the classical procedure \cite{Ganley19,Ganley20} we can construct representations of \( U_q[gl(m/n)] \) induced from finite-dimensional irreducible representations of \( U_q[gl(m) \oplus gl(n)] \) which, as we can see from (2.9-11) and (2.15), is the stability subalgebra of \( U_q[gl(m/n)] \). Let \( V_0^q(\Lambda) \) be a \( U_q[gl(m) \oplus gl(n)] \)-fidirdmod characterized
by some highest weight Λ. For a basis of \( V_0^q(Λ) \) we can choose the Gel’fand-Zetlin (GZ) tableaux \(^30\), since the latter are invariant under the quantum deformations \(^28,29,31,32\). Therefore, the highest weight Λ is described again by the first row of the GZ tableaux called from now on as the GZ (basis) vectors.

Demanding

\[
E_{m,m+1}V_0^q(Λ) \equiv e_{m}V_0^q(Λ) = 0
\]  

(2.16)

i.e.

\[
U_q(A_+)V_0^q(Λ) = 0
\]  

(2.17)

we turn \( V_0^q(Λ) \) into a \( U_q(B) \)-module, where

\[
A_+ = \{ E_{ij} \mid 1 \leq i \leq m < j \leq m + n \}
\]  

(2.18)

\[
B = A_+ \oplus gl(m) \oplus gl(n)
\]  

(2.19)

The \( U_q[gl(m/n)] \)-module \( W^q \) induced from the \( U_q[gl(m) \oplus gl(n)] \)-module \( V_0^q(Λ) \) is the factor-space

\[
W^q = W^q(Λ) = [U_q \otimes V_0^q(Λ)]/I^q(Λ)
\]  

(2.20)

where \( U_q \equiv U_q[gl(m/n)] \), while \( I^q(Λ) \) is the subspace

\[
I^q(Λ) = lin.env.\{ uv \otimes v - u \otimes bv \mid u \in U_q, \ b \in U_q(B) \subset U_q, \ v \in V_0^q(Λ) \}
\]  

(2.21)

In order to complete the present section let us note that the modules \( W^q(Λ) \) and \( V_0^q(Λ) \) have one and the same highest vector. Therefore, they are characterized by one and the same highest weight Λ.

3. \( U[gl(2/2)]U[gl(2/2)] \) The quantum superalgebra \( U_q[gl(2/2)] \)

The quantum superalgebra \( U_q \equiv U_q[gl(2/2)] \) is generated by the generators \( E_{ii}, i = 1,2,3,4, E_{12} \equiv e_1, E_{23} \equiv e_2, E_{34} \equiv e_3, E_{21} \equiv f_1, E_{32} \equiv f_2 \) and \( E_{43} \equiv f_3 \) satisfying the relations (2.9-11) which now read

a) the super-commutation relations \((1 \leq i, i + 1, j, j + 1 \leq 4)\):  

\[
[E_{ii}, E_{jj}] = 0,
\]
\[
\begin{align*}
[E_{ij}, E_{j,j+1}] &= (\delta_{ij} - \delta_{i,j+1})E_{j,j+1}, \\
[E_{ii}, E_{j+1,j}] &= (\delta_{i,j+1} - \delta_{ij})E_{j+1,j}, \\
[E_{i,i+1}, E_{j+1,j}] &= \delta_{ij}[h_i]_{q^2}, \quad h_i = (E_{ii} - \frac{d_{i+1}}{d_i}E_{i+1,i+1}), \quad (3.1)
\end{align*}
\]

with \(d_1 = d_2 = -d_3 = -d_4 = 1,\)

b) the Serre-relations:
\[
\begin{align*}
[E_{12}, E_{34}] &= [E_{21}, E_{43}] = 0, \\
E_{23}^2 &= E_{32}^2 = 0, \\
[E_{12}, E_{13}]_{q^2} &= [E_{24}, E_{34}]_{q^2} = 0, \\
[E_{21}, E_{31}]_{q^2} &= [E_{42}, E_{43}]_{q^2} = 0, \quad (3.2)
\end{align*}
\]

and

c) the extra-Serre relations:
\[
\begin{align*}
\{E_{13}, E_{24}\} &= 0, \\
\{E_{31}, E_{42}\} &= 0, \quad (3.3)
\end{align*}
\]

respectively. Here, for a further convenience, the operators
\[
\begin{align*}
E_{13} &:= [E_{12}, E_{23}]_{q^{-2}}, \\
E_{24} &:= [E_{23}, E_{34}]_{q^{-2}}, \\
E_{31} &:= -[E_{21}, E_{32}]_{q^{-2}}, \\
E_{42} &:= -[E_{32}, E_{43}]_{q^{-2}}. \quad (3.4)
\end{align*}
\]

and the operators composed in the following way
\[
\begin{align*}
E_{14} &:= [E_{12}, [E_{23}, E_{34}]_{q^{-2}}]_{q^{-2}} \equiv [E_{12}, E_{24}]_{q^{-2}}, \\
E_{41} &:= [E_{21}, [E_{32}, E_{43}]_{q^{-2}}]_{q^{-2}} \equiv -[E_{21}, E_{42}]_{q^{-2}} \quad (3.5)
\end{align*}
\]

are defined as new generators. The latter are odd and have vanishing squares. They, together with the Cartan-Chevalley generators, form a full system of \(q\)-analogues of the Weyl generators \(e_{ij}, 1 \leq i, j \leq 4,\) of the superalgebra \(gl(2/2)\) whose universal enveloping algebra \(U[gl(2/2)]\) is a classical limit of \(U_q[gl(2/2)]\) when \(q \to 1.\) Other commutation relations between \(E_{ij}\) follow from the relations (3.1-3) and the defini-
The subalgebra $U_q[gl(2/2)_0] \subset U_q[gl(2/2)] \subset U_q[gl(2/2)]$ is even and isomorphic to $U_q[gl(2) \oplus gl(2)] \equiv U_q[gl(2)] \oplus U_q[gl(2)]$ which is completely defined by $E_{ii}, 1 \leq i \leq 4, E_{12}, E_{34}, E_{21}$ and $E_{43}$.

$$U_q[gl(2/2)_0] = \text{lin.env.}\{E_{ij} || i, j = 1, 2 \text{ and } i, j = 3, 4\} \quad (3.6)$$

In order to distinguish two components of $U_q[gl(2/2)_0]$ we set

$$\text{left } U_q[gl(2)] \equiv U_q[gl(2)]_l := \text{lin.env.}\{E_{ij} || i, j = 1, 2\}, \quad (3.7)$$

$$\text{right } U_q[gl(2)] \equiv U_q[gl(2)]_r := \text{lin.env.}\{E_{ij} || i, j = 3, 4\}. \quad (3.8)$$

That means

$$U_q[gl(2/2)_0] = U_q[gl(2)]_l \oplus gl(2)]_r. \quad (3.9)$$

Let $V^q(\Lambda)$ be a $U_q[gl(2/2)_0]$-fidirdmod of the highest weight $\Lambda$. Thus $V^q$ can be decomposed into a tensor product

$$V^q(\Lambda) = V^q_l(\Lambda_l) \otimes V^q_r(\Lambda_r), \quad (3.10)$$

between a $U_q[gl(2)]_l$-fidirdmod $V^q_l(\Lambda_l)$ of a highest weight $\Lambda_l$ and a $U_q[gl(2)]_r$-fidirdmod $V^q_r(\Lambda_r)$ of a highest weight $\Lambda_r$, where $\Lambda_l$ and $\Lambda_r$ are defined respectively as the left and right components of $\Lambda$:

$$\Lambda = [\Lambda_l, \Lambda_r]. \quad (3.11)$$

### 4. Finite-dimensional representations of $U_q[gl(2/2)]$

Here, we shall construct finite-dimensional representations of $U_q[gl(2/2)]$ induced from finite-dimensional irreducible representations of $U_q[gl(2/2)_0]$. In the framework of the present paper we consider only typical representations of $U_q[gl(2/2)]$ at generic $q$. Atypical representations at generic $q$ and finite-dimensional representations of $U_q[gl(2/2)]$ at roots of unity are subjects of later publications $^{22,23}$. 
As mentioned earlier a fidirmod $V_0^q(\Lambda)$ of the quantum algebra $U_q[gl(2/2)_0]$ represents a quantum deformation (q-deformation) of some fidirmod $V_0(\Lambda)$ of the algebra $gl(2/2)_0$. Moreover, following the classical procedure we can construct $U_q[gl(2/2)]$-fidirmods induced from $U_q[gl(2/2)_0]$-fidirmods. Setting

$$E_{23}V_0^q = 0,$$

(4.1)

a $U_q[gl(2/2)]$-module $W^q$ induced from the $U_q[gl(2/2)_0]$-fidirmod $V_0^q$, by the construction, is the factor-space (2.20) with $m = n = 2$:

$$W^q(\Lambda) = [U_q \otimes V_0^q(\Lambda)]/I^q(\Lambda),$$

(4.2)

where

$$I^q(\Lambda) = \text{lin.env.}\{ub \otimes v - u \otimes bv\| u \in U_q, b \in U_q(B) \subseteq U_q, v \in V_0^q(\Lambda)\}$$

$$U_q(B) = \text{lin.env.}\{E_{ij}, E_{23}\| i, j = 1, 2 \text{ and } i, j = 3, 4\}$$

(4.3)

Any vector $w$ from the module $W^q$ has the form

$$w = u \otimes v, \quad u \in U_q, \quad v \in V_0^q$$

(4.4)

Then $W^q$ is a $U_q[gl(2/2)]$-module in the sense

$$gw \equiv g(u \otimes v) = gu \otimes v \in W^q$$

(4.5)

for $g, u \in U_q, w \in W^q$ and $v \in V_0^q$.

In the next two subsections we shall construct the bases of the module $W^q$ and find the explicit matrix elements for the typical representations of $U_q[gl(2/2)]$.

### 4.1 The bases

Since the GZ basis is invariant under the q-deformation, for a basis of a $U_q[gl(2)]$-fidirmod $V_0$ we can choose

$$\begin{bmatrix} m_{12} & m_{22} \\ m_{11} & m_{11} \end{bmatrix} \equiv \begin{bmatrix} [m] \\ m_{11} \end{bmatrix}$$

(4.6)
where \( m_{ij} \) are complex numbers such that \( m_{12} - m_{11} \in \mathbb{Z}_+ \) and \( m_{11} - m_{22} \in \mathbb{Z}_+ \). Under the actions of the \( U_q[gl(2)] \)-generators \( E_{ij}, i, j = 1, 2 \) the basis (4.6) transforms as follows:

\[
E_{11} \begin{bmatrix} m_{12} & m_{22} \\ m_{11} & \end{bmatrix} = (l_{11} + 1) \begin{bmatrix} m_{12} & m_{22} \\ m_{11} & \end{bmatrix},
\]

\[
E_{22} \begin{bmatrix} m_{12} & m_{22} \\ m_{11} & \end{bmatrix} = (l_{12} + l_{22} - l_{11} + 2) \begin{bmatrix} m_{12} & m_{22} \\ m_{11} & \end{bmatrix},
\]

\[
E_{12} \begin{bmatrix} m_{12} & m_{22} \\ m_{11} & \end{bmatrix} = (l_{12} - l_{11}) (l_{11} - l_{22})^{1/2} \begin{bmatrix} m_{12} & m_{22} \\ m_{11} + 1 & \end{bmatrix},
\]

\[
E_{21} \begin{bmatrix} m_{12} & m_{22} \\ m_{11} & \end{bmatrix} = (l_{12} - l_{11} + 1) (l_{11} - l_{22} - 1)^{1/2} \begin{bmatrix} m_{12} & m_{22} \\ m_{11} - 1 & \end{bmatrix},
\]

(4.7)

where \( l_{ij} = m_{ij} - i \) for \( i = 1, 2 \) and \( l_{ij} = m_{ij} - i + 2 \) for \( i = 3, 4 \).

On the other hands, \( V_0^q \) is decomposed into the tensor product

\[
V_0^q = V_{0,l}^q \otimes V_{0,r}^q,
\]

(4.8)

where \( V_{0,l}^q \) and \( V_{0,r}^q \) are a \( U_q[gl(2)] \)- and a \( U_q[gl(2),r] \)-fidirmods, respectively. Therefore, the GZ basis of \( V_0^q \) is the tensor product

\[
\begin{bmatrix} m_{13} & m_{23} \\ m_{11} & \end{bmatrix} \otimes \begin{bmatrix} m_{33} & m_{43} \\ m_{31} & \end{bmatrix} \equiv \begin{bmatrix} [m]_l \\ m_{11} & \end{bmatrix} \otimes \begin{bmatrix} [m]_r \\ m_{31} & \end{bmatrix} \equiv (m)_l \otimes (m)_r \equiv (m)
\]

(4.9)

between the GZ basis of \( V_{0,l}^q \) spanned on the vectors \( (m)_l \) and the GZ basis of \( V_{0,r}^q \) spanned on the vectors \( (m)_r \). Following the approach of Ref. 20 and keeping the notations used there, we can represent the GZ basis (4.9) of \( V_0^q \) in the form

\[
\begin{bmatrix} m_{13} & m_{23} & m_{33} & m_{43} \\ m_{11} & m_{31} & \end{bmatrix} \equiv \begin{bmatrix} [m]_l \\ [m]_r \\ m_{11} & m_{31} & \end{bmatrix} \equiv (m)
\]

(4.10)

Then, the highest weight \( \Lambda \) is given by the first row (signature) \([m]_{13}, [m]_{23}, [m]_{33}, [m]_{43} \equiv [[m]_l, [m]_r] \equiv [m]\) common for all the GZ basis vectors (4.10) of \( V_0^q \):

\[
V_0^q \equiv V_0^q(\Lambda) = V_0^q([m]) = V_{0,l}^q([m]_l) \otimes V_{0,r}^q([m]_r)
\]

(4.11)
The explicit action of $U_q[gl(2/2)_0]$ on $V_0^q([m])$ follows directly from (4.7) and:

$$g_0(m) = g_{0,l}(m)_i \otimes (m)_r + (m)_i \otimes g_{0,r}(m)_r$$

(4.12)

for $g_0 \equiv g_{0,l} \oplus g_{0,r} \in U_q[gl(2/2)_0]$ and $(m) \in V_0^q([m])$.

The GZ basis vector

$$\begin{bmatrix}
  m_{13} & m_{23} & m_{33} & m_{43} \\
  m_{13} & m_{33}
\end{bmatrix}
\equiv
\begin{bmatrix}
  [m]_i & [m]_r \\
  m_{13} & m_{33}
\end{bmatrix}
\equiv (M)$$

(4.13)

satisfying the conditions

$$E_{ii}(M) = m_{i;i}(M), \quad i = 1, 2, 3, 4,$$

$$E_{12}(M) = E_{34}(M) = 0$$

(4.14)

by definition, is the highest weight vector in $V_0^q([m])$. Therefore, as in the classical case ($q = 1$) the highest weight $[m]$ is nothing but an ordered set of the eigen values of the Cartan generators $E_{ii}$ on the highest weight vector $(M)$. The latter is also highest weight vector in $W^q([m])$ because of the condition (4.1). All other, i.e. lower weight, basis vectors of $V_0^q$ can be obtained from the highest weight vector $(M)$ through acting on the latter by monomials of definite powers of the lowering generators $E_{21}$ and $E_{43}$:

$$(m) = \left(\frac{[m_{11} - m_{23}]! [m_{31} - m_{43}]!}{[m_{13} - m_{23}]! [m_{13} - m_{11}]! [m_{33} - m_{43}]! [m_{33} - m_{31}]!}\right)^{1/2}
\times (E_{21})^{m_{13} - m_{11}} (E_{43})^{m_{33} - m_{31}}(M),$$

(4.15)

where $[n]'s$ are short hands of

$$\frac{q^{2n} - q^{-2n}}{q^2 - q^{-2}} \equiv [n]_q^2 \equiv [n],$$

(4.16)

while

$$[n]! = [1][2]...[n-1][n].$$

(4.17)

Using (3.1-5) we can show that a q-analogue of the Poincaré-Birkhoff-Witt theorem holds

$$g = (E_{31})^{\theta_1} (E_{32})^{\theta_2} (E_{41})^{\theta_3} (E_{42})^{\theta_4} b, \quad b \in U_q(B), \quad \theta_i = 0, 1, \quad i = 1, 2, 3, 4.$$
Indeed, a right-ordered basis vector of $U_q$ like

$$h = (E_{23})^{\eta_1}(E_{24})^{\eta_2}(E_{13})^{\eta_3}(E_{14})^{\eta_4}(E_{31})^{\theta_1}(E_{32})^{\theta_2}(E_{41})^{\theta_3}(E_{42})^{\theta_4}h_0,$$

where $\eta_i, \theta_i = 0, 1$, $h_0 \in U_q[gl(2/2)_0]$, can be re-ordered and expressed through the vectors (4.18) which are more convenient for consulting the classical case in Refs. 20,21. Taking into account the fact that $V^q_0([m])$ is a $U_q(B)$-module we have

$$W^q([m]) = \text{lin.env.}\{(E_{31})^{\theta_1}(E_{32})^{\theta_2}(E_{41})^{\theta_3}(E_{42})^{\theta_4} \otimes v \| v \in V^q_0, \theta_1, \ldots, \theta_4 = 0, 1\}.$$  

(4.19)

Consequently, the vectors

$$|\theta_1, \theta_2, \theta_3, \theta_4; (m)\rangle := (E_{31})^{\theta_1}(E_{32})^{\theta_2}(E_{41})^{\theta_3}(E_{42})^{\theta_4} \otimes (m),$$

all together span a basis of the module $W^q([m])$. We shall call this basis induced in order to distinguish it from the introduced later reduced basis and is more convenient for us to investigate the reducibleness of $W^q([m])$.

The subspace $T^q$ consisting of

$$|\theta_1, \theta_2, \theta_3, \theta_4\rangle := (E_{31})^{\theta_1}(E_{32})^{\theta_2}(E_{41})^{\theta_3}(E_{42})^{\theta_4}$$

(4.20)

can be considered as a $U_q[gl(2/2)_0]$-adjoint module (upto rescaling by $k_i$ in definite powers). The latter is 16-dimensional as begins from $|0, 0, 0, 0\rangle$ when $\forall \theta_i = 0$ and ends at $|1, 1, 1, 1\rangle$ when $\forall \theta_i = 1$. Therefore, $W^q([m])$, as a $U_q[gl(2/2)_0]$-module

$$W^q([m]) = T^q \circ V^q_0([m]),$$

(4.19')
is reducible and can be decomposed into 16 finite-dimensional irreducible $U_q[gl(2/2)_0]$-submodules $V^q_k([m])_k$, $k = 0, 1, \ldots, 15$:

$$W^q([m]) = \bigoplus_{k=0}^{15} V^q_k([m])_k.$$  

(4.22)

Here, $[m]_k \equiv [m_{12}, m_{22}, m_{32}, m_{42}]_k$ are the local highest weights of the submodules $V^q_k$ in their GZ bases denoted now as

$$\begin{bmatrix} m_{12} & m_{22} & m_{32} & m_{42} \\ m_{11} & m_{31} \end{bmatrix}_k \equiv (m)_k.$$  

(4.23)
The highest weight \([m]_0 \equiv [m]\) of \(V_0^q\) being also the highest weight of \(W^q\) is referred as a global highest weight. We call \([m]_k, k \geq 1\) the local highest weights in the sense that they characterize only the submodules \(V_k^q \subset W^q\) as \(U_q[gl(2/2)_0]\)-fidiirmods, while the global highest weight \([m]\) characterizes the \(U_q[gl(2/2)]\)-module \(W^q\) as the whole. In the same way we define the local highest weight vectors \((M)_k\) in \(V_k^q\) as those \((m)_k\) satisfying the conditions (cf. (4.14))

\[
E_{ii}(M)_k = m_{ij}(M)_k, \quad i = 1, 2, 3, 4, \\
E_{12}(M)_k = E_{34}(M)_k = 0.
\] (4.24)

The highest weight vector \((M)\) of \(V_0^q\) is also the global highest weight vector in \(W^q\) for which the condition (see (4.1))

\[
E_{23}(M) = 0
\] (4.25)

and the conditions (4.24) simultaneously hold.

Let us denote by \(\Gamma^q_k\) the basis system spanned on the basis vectors \((m)_k\) (4.23) in each \(V_k^q([m])\). For a basis of \(W^q\) we can choose the union \(\Gamma^q = \bigcup_{k=0}^{15} \Gamma^q_k\) of all the bases \(\Gamma^q_k\), namely, a basis vector of \(W^q\) has to be identified with one of the vectors \((m)_k, k = 0, 1, ..., 15\). The basis \(\Gamma^q\) is referred as a \((U_q[gl(2/2)_0]\)-) reduced basis. It is clear that every basis \(\Gamma^q_k = \Gamma_k([m]_k)^q\) is labelled by a local highest weight \([m]_k\), while the basis \(\Gamma^q = \Gamma^q([m])\) is labelled by the global highest weight \([m]\). Going ahead, we modify the notation (4.23) for the basis vectors in \(\Gamma^q\) as follows (see (3.54) in Ref. 20)

\[
\begin{bmatrix}
  m_{13} & m_{23} & m_{33} & m_{43} \\
  m_{12} & m_{22} & m_{32} & m_{42} \\
  m_{11} & 0 & m_{31} & 0 
\end{bmatrix}_{k} \equiv \begin{bmatrix}
  m_{12} & m_{22} \; ; \; m_{32} & m_{42} \\
  m_{11} \; ; \; m_{31} & m_{31} 
\end{bmatrix}_{k} \equiv (m)_k, \quad (4.26)
\]

with \(k\) running from 0 to 15 as for \(k = 0\) we have to take into account \(m_{i2} = m_{i3}, \quad i = 1, 2, 3, 4, \) i.e.

\[
(m)_0 \equiv (m) \equiv \begin{bmatrix}
  m_{13} & m_{23} & m_{33} & m_{43} \\
  m_{13} & m_{23} & m_{33} & m_{43} \\
  m_{11} & 0 & m_{31} & 0 
\end{bmatrix}. \quad (4.27)
\]
In (4.26) the first row \([m] = [m_{13}, m_{23}, m_{33}, m_{43}]\) being the (global) highest weight of \(W^q\) is fixed for all the vectors in the whole \(W^q\) and characterizes the module itself, while the second row is a (local) highest weight of some submodule \(V_k^q\) and tells us that the considered basis vector \((m)_k\) of \(W^q\) belongs to this submodule in the decomposition (4.22) corresponding to the branching rule \(U_q[gl(2/2)] \supset U_q[gl(2/2)_0]\).

It is easy to see that the highest vectors \((M)_k\) in the notation (4.26) are

\[
(M)_k = \begin{bmatrix}
m_{13} & m_{23} & m_{33} & m_{43} \\
m_{12} & m_{22} & m_{32} & m_{42} \\
m_{12} & 0 & m_{32} & 0
\end{bmatrix}
\]

The (global) highest weight vector \((M)\) (4.13) is given now by

\[
(M) = \begin{bmatrix}
m_{13} & m_{23} & m_{33} & m_{43} \\
m_{13} & m_{23} & m_{33} & m_{43} \\
m_{13} & 0 & m_{33} & 0
\end{bmatrix}
\]

Let us denote by \((m)^{\pm ij}_k\) a GZ vector obtained from \((m)_k\) by replacing the element \(m_{ij}\) of the latter by \(m_{ij} \pm 1\). We can prove that the highest weight vectors \((M)_k\) expressed in terms of the induced basis (4.20) have the following explicit forms

\[
(M)_0 = a_0 [0, 0, 0, 0; (M)], \quad a_0 \equiv 1,

(M)_1 = a_1 [0, 1, 0, 0; (M)],

(M)_2 = a_2 \left\{ [1, 0, 0, 0; (M)] + q^{4t}[2l]^{-1/2} [0, 1, 0, 0; (M)^{-11}] \right\},

(M)_3 = a_3 \left\{ [0, 0, 0, 1; (M)] - q^{-4t-2}[2l']^{-1/2} [0, 1, 0, 0; (M)^{-31}] \right\},

(M)_4 = a_4 \left\{ [0, 0, 1, 0; (M)] + q^{4t}[2l]^{-1/2} [0, 0, 0, 1; (M)^{-11}] \\
- q^{-4t-2}[2l']^{-1/2} [1, 0, 0, 0; (M)^{-31}] \\
- q^{4t-4t'} (2l')(2l')^{-1/2} [0, 1, 0, 0; (M)^{-11-31}] \right\},

(M)_5 = a_5 [0, 1, 0, 1; (M)],

(M)_6 = a_6 \left\{ [1, 0, 0, 1; (M)] + q^{2} [0, 1, 1, 0; (M)] \right\}.
\[ + q^{4l}(q^2 + q^{-2})[2l]^{-1/2} \left| 0, 1, 0, 1; (M)^{-11} \right\rangle, \]

\[(M)_7 = a_7 \left\{ [1, 0, 1, 0; (M)) + q^{4l}[2l]^{-1/2} \left| 1, 0, 0, 1; (M)^{-11} \right\rangle \right. \]
\[+ q^{-4l} [2l]^{-1/2} \left| 0, 1, 1, 0; (M)^{-11} \right\rangle \]
\[+ q^{8l-4} \left( q^2 + q^{-2} \right)^{1/2} \left( [2l][2l - 1] \right)^{-1/2} \left| 0, 1, 0, 1; (M)^{-11-11} \right\rangle \}, \]

\[(M)_8 = a_8 \left| 1, 1, 0, 0; (M) \right\rangle, \]

\[(M)_9 = a_9 \left\{ [1, 0, 0, 1; (M)) - q^2 \left| 0, 1, 1, 0; (M) \right\rangle \right. \]
\[+ \left. -q^{-4l} [q^2 + q^{-2}] [2l']^{-1/2} \left| 1, 0, 0, 1; (M)^{-31} \right\rangle \right\} , \]

\[(M)_{10} = a_{10} \left\{ [0, 0, 1, 1; (M)) - q^{-4l-4} [2l']^{-1/2} \left| 1, 0, 0, 1; (M)^{-31} \right\rangle \right. \]
\[+ q^{-4l-2} [2l']^{-1/2} \left| 0, 1, 1, 0; (M)^{-31} \right\rangle \]
\[+ q^{-8l'} \left( q^2 + q^{-2} \right)^{1/2} \left( [2l'] [2l' - 1] \right)^{-1/2} \left| 1, 1, 0, 0; (M)^{-31-31} \right\rangle \}, \]

\[(M)_{11} = a_{11} \left| 1, 1, 0, 1; (M) \right\rangle, \]

\[(M)_{12} = a_{12} \left\{ [1, 1, 1, 0; (M)) + q^{4l}[2l]^{-1/2} \left| 1, 1, 0, 1; (M)^{-11} \right\rangle \right. \}
\[+ q^{-4l} [2l]^{-1/2} \left| 1, 0, 1, 1; (M)^{-11} \right\rangle \]
\[+ q^{-4l-2} [2l']^{-1/2} \left| 1, 1, 0, 0; (M)^{-31} \right\rangle \]
\[+ q^{-4l-2} \left( [2l][2l'] \right)^{-1/2} \left| 1, 1, 0, 1; (M)^{-11-31} \right\rangle \}, \]

\[(M)_{13} = a_{13} \left| 0, 1, 1, 1; (M) \right\rangle, \]

\[(M)_{14} = a_{14} \left\{ [1, 0, 1, 1; (M)) + q^{4l}[2l]^{-1/2} \left| 0, 1, 1, 1; (M)^{-11} \right\rangle \right. \]
\[+ q^{-4l-2} [2l']^{-1/2} \left| 1, 1, 0, 0; (M)^{-31} \right\rangle \]
\[+ q^{-4l-2} \left( [2l][2l'] \right)^{-1/2} \left| 1, 1, 0, 1; (M)^{-11-31} \right\rangle \}, \]

\[(M)_{15} = a_{15} \left| 1, 1, 1, 1; (M) \right\rangle, \]

(4.30)

where \( l = \frac{1}{2}(m_{13} - m_{23}) \) and \( l' = \frac{1}{2}(m_{33} - m_{43}) \), while \( a_k = a_k(q) \) are coefficients depending on \( q \). Indeed, \( (M)_k \) given in (4.30) form a set of all linear independent vectors satisfying the conditions (4.24). For a further convenience, let us rescale the coefficients \( a_k \) as follows

\[ a_0 = c_0 \equiv 1, \]
\[ a_1 = -q^{-2}c_1, \]
\[ a_8 = q^{-2}c_8, \]
\[ a_9 = -q^{-2} \left( \frac{[2l']}{[2][2l' + 2]} \right)^{1/2} c_9, \]
\[ a_2 = -q^{-2} \left( \frac{[2l]}{[2l+1]} \right)^{1/2} c_2 , \quad a_{10} = q^2 \left( \frac{[2l'-1]}{[2l'+1]} \right)^{1/2} c_{10} , \]
\[ a_3 = \left( \frac{[2l']}{[2l'+1]} \right)^{1/2} c_3 , \quad a_{11} = q^{-2} c_{11} , \]
\[ a_4 = \left( \frac{[2l][2l']}{[2l+1][2l'+1]} \right)^{1/2} c_4 , \quad a_{12} = q^{-2} \left( \frac{[2l]}{[2l+1]} \right)^{1/2} c_{12} , \]
\[ a_5 = q^{-2} c_5 , \quad a_{13} = \left( \frac{[2l']}{[2l'+1]} \right)^{1/2} c_{13} , \]
\[ a_6 = q^{-2} \left( \frac{[2l]}{[2l+2]} \right)^{1/2} c_6 , \quad a_{14} = \left( \frac{[2l][2l']}{[2l+1][2l'+1]} \right)^{1/2} c_{14} , \]
\[ a_7 = q^{-2} \left( \frac{[2l-1]}{[2l+1]} \right)^{1/2} c_7 , \quad a_{15} = c_{15} , \] (4.31)

where \( c_k = c_k(q) \) are some other constants which may depend on \( q \). Looking at (4.30) we easily identify the highest weights \([m]_k\)

\[
[m]_0 = [m_{13}, m_{23}, m_{33}, m_{43}], \\
[m]_1 = [m_{13}, m_{23} - 1, m_{33} + 1, m_{43}], \\
[m]_2 = [m_{13} - 1, m_{23}, m_{33} + 1, m_{43}], \\
[m]_3 = [m_{13}, m_{23} - 1, m_{33} + 1, m_{43} + 1], \\
[m]_4 = [m_{13} - 1, m_{23}, m_{33} + 1, m_{43} + 1], \\
[m]_5 = [m_{13}, m_{23} - 2, m_{33} + 1, m_{43} + 1], \\
[m]_6 = [m_{13} - 1, m_{23} - 1, m_{33} + 1, m_{43} + 1]_6, \\
[m]_7 = [m_{13} - 2, m_{23}, m_{33} + 1, m_{43} + 1], \\
[m]_8 = [m_{13} - 1, m_{23} - 1, m_{33} + 2, m_{43}], \\
[m]_9 = [m_{13} - 1, m_{23} - 1, m_{33} + 1, m_{43} + 1]_9, \\
[m]_{10} = [m_{13} - 1, m_{23} - 1, m_{33} + 1, m_{43} + 2], \\
[m]_{11} = [m_{13} - 1, m_{23} - 2, m_{33} + 2, m_{43} + 1], \\
[m]_{12} = [m_{13} - 2, m_{23} - 1, m_{33} + 2, m_{43} + 1], \\
[m]_{13} = [m_{13} - 1, m_{23} - 2, m_{33} + 1, m_{43} + 2], \\
[m]_{14} = [m_{13} - 2, m_{23} - 1, m_{33} + 1, m_{43} + 2],
\]
\[ [m]_{15} = [m_{13} - 2, m_{23} - 2, m_{33} + 2, m_{43} + 2]. \] (4.32)

In the latest formula (4.32), with the exception of \([m]_6\) and \([m]_9\) where a degeneration is present, we skip the subscript \(k\) in the r.h.s.. The proofs of (4.30) and (4.32) follow from direct computations.

Using the rule (4.15) which now reads
\[
(m)_k = \left( \frac{[m_{11} - m_{22}] [m_{31} - m_{42}]!}{[m_{12} - m_{22}] [m_{11} - m_{32}] [m_{32} - m_{42}] [m_{32} - m_{31}]!} \right)^{1/2} 
\times (E_2)^{m_{12} - m_{11}} (E_3)^{m_{32} - m_{31}} (M)_k \]
(4.15')

we can find all the basis vectors \((m)_k:\)

\[
(m)_0 = |0, 0, 0, 0; (m)\rangle,
\]

\[
(m)_1 = c_1 \left\{ q^{2(l' - l')} \left( \frac{[l_{13} - l_{11}] [l_{31} - l_{43} - 1]}{[2l + 1][2l' + 1]} \right)^{1/2} \right| 1, 0, 0, 0; (m)^{+11-31} \rangle
- q^{2(-l + p + l' - l')} \left( \frac{[l_{11} - l_{23}] [l_{31} - l_{43} - 1]}{[2l + 1][2l' + 1]} \right)^{1/2} \right| 0, 1, 0, 0; (m)^{-31} \rangle
+ \left( \frac{[l_{13} - l_{11}] [l_{33} - l_{31} + 1]}{[2l + 1][2l' + 1]} \right)^{1/2} \right| 0, 0, 1, 0; (m)^{+11} \rangle
- q^{2(-l + p)} \left( \frac{[l_{11} - l_{23}] [l_{33} - l_{31} + 1]}{[2l + 1][2l' + 1]} \right)^{1/2} \right| 0, 0, 0, 1; (m) \rangle, \]

\[
(m)_2 = c_2 \left\{ -q^{2(l' - l')} \left( \frac{[l_{11} - l_{23}] [l_{31} - l_{43} - 1]}{[2l + 1][2l' + 1]} \right)^{1/2} \right| 1, 0, 0, 0; (m)^{+11-31} \rangle
- q^{2(l + p + l' - l' + 1)} \left( \frac{[l_{13} - l_{11}] [l_{31} - l_{43} - 1]}{[2l + 1][2l' + 1]} \right)^{1/2} \right| 0, 1, 0, 0; (m)^{-31} \rangle
- \left( \frac{[l_{11} - l_{23}] [l_{33} - l_{31} + 1]}{[2l + 1][2l' + 1]} \right)^{1/2} \right| 0, 0, 1, 0; (m)^{+11} \rangle
- q^{2(l + p + 1)} \left( \frac{[l_{13} - l_{11}] [l_{33} - l_{31} + 1]}{[2l + 1][2l' + 1]} \right)^{1/2} \right| 0, 0, 0, 1; (m) \rangle, \]

\[
(m)_3 = c_3 \left\{ q^{-2(l' + p + 1)} \left( \frac{[l_{13} - l_{11}] [l_{33} - l_{31} + 1]}{[2l + 1][2l' + 1]} \right)^{1/2} \right| 1, 0, 0, 0; (m)^{+11-31} \rangle
\]
\[-q^{2(t-p+t'-p'+1)} \left( \frac{[l_{11} - l_{23}][l_{33} - l_{31} + 1]}{[2l + 1][2l' + 1]} \right)^{1/2} \mid 0, 1, 0; (m)^{-31} \]

\[- \left( \frac{[l_{13} - l_{11}][l_{33} - l_{31} - 1]}{[2l + 1][2l' + 1]} \right)^{1/2} \mid 0, 0, 1, 0; (m)^{+11} \]

\[+q^{2(-t+p)} \left( \frac{[l_{11} - l_{23}][l_{31} - l_{43} - 1]}{[2l + 1][2l' + 1]} \right)^{1/2} \mid 0, 0, 0, 1; (m) \}

\[(m)_4 = c_4 \left\{ -q^{2(t' + p' + 1)} \left( \frac{[l_{11} - l_{23}][l_{33} - l_{31} + 1]}{[2l + 1][2l' + 1]} \right)^{1/2} \mid 1, 0, 0, 0; (m)^{+11-31} \right. \]

\[-q^{2(l+p'-p')} \left( \frac{[l_{13} - l_{11}][l_{33} - l_{31} + 1]}{[2l + 1][2l' + 1]} \right)^{1/2} \mid 0, 1, 0, 0; (m)^{-31} \]

\[+ \left( \frac{[l_{11} - l_{23}][l_{31} - l_{43} - 1]}{[2l + 1][2l' + 1]} \right)^{1/2} \mid 0, 0, 1, 0; (m)^{+11} \]

\[+q^{2(t+p+1)} \left( \frac{[l_{13} - l_{11}][l_{31} - l_{43} - 1]}{[2l + 1][2l' + 1]} \right)^{1/2} \mid 0, 0, 0, 1; (m) \}

\[(m)_5 = c_5 \left\{ q^{-2} \left( \frac{[l_{13} - l_{11}][l_{33} - l_{11} - 1]}{[2l + 1][2l + 2]} \right)^{1/2} \mid 1, 0, 1, 0; (m)^{+11+11-31} \right. \]

\[-q^{2(-t+p)} \left( \frac{[l_{13} - l_{11}][l_{11} - l_{23} + 1]}{[2l + 1][2l + 2]} \right)^{1/2} \mid 1, 0, 0, 1; (m)^{+11-31} \]

\[-q^{2(-t+p-1)} \left( \frac{[l_{13} - l_{11}][l_{11} - l_{23} + 1]}{[2l + 1][2l + 2]} \right)^{1/2} \mid 0, 1, 1, 0; (m)^{+11-31} \]

\[+q^{2(-2l+2p-1)} \left( \frac{[l_{11} - l_{23}][l_{11} - l_{23} + 1]}{[2l + 1][2l + 2]} \right)^{1/2} \mid 0, 1, 0, 1; (m)^{-31} \}

\[(m)_6 = c_6 \left\{ -q^{-2}(q^2 + q^{-2}) \left( \frac{[l_{13} - l_{11} - 1][l_{11} - l_{23} + 1]}{[2][2l][2l + 2]} \right)^{1/2} \mid 1, 0, 1, 0; (m)^{+11+11-31} \right. \]

\[+q^{2(-t+p)} \left( \frac{[l_{11} - l_{23}] - q^{4(t+1)}[l_{13} - l_{11} - 1]}{[2][2l][2l + 2]} \right)^{1/2} \mid 1, 0, 0, 1; (m)^{+11+11-31} \]

\[+q^{2(-t+p-1)} \left( \frac{[l_{11} - l_{23}] - q^{4(t+1)}[l_{13} - l_{11} - 1]}{[2][2l][2l + 2]} \right)^{1/2} \mid 0, 1, 1, 0; (m)^{+11-31} \]

\[+(q^2 + q^{-2})q^{2(t+p+1)} \left( \frac{[l_{11} - l_{23}][l_{13} - l_{11}]}{[2][2l][2l + 2]} \right)^{1/2} \mid 0, 1, 0, 1; (m)^{-31} \}

\[(m)_7 = c_7 \left\{ q^{-2} \left( \frac{[l_{11} - l_{23}][l_{11} - l_{23} + 1]}{[2l][2l + 1]} \right)^{1/2} \mid 1, 0, 1, 0; (m)^{+11+11-31} \right. \}
\begin{align*}
(m)_8 & = c_8 \left\{ q^2 \left( \frac{[l_{33} - l_{31} + 1][l_{33} - l_{31} + 2]}{[2l'][2l' + 2]} \right) \right\}^{1/2} \left| 0, 0, 1; (m)^{+11} \rightangle \\
 & \quad + q^{2(l'+p')} \left( \frac{[l_{33} - l_{31} + 2][l_{31} - l_{43} - 1]}{[2l']^2[2l' + 2]} \right)^{1/2} \left| 1, 0, 0; (m)^{+11-31} \rightangle \\
 & \quad - q^{2(l'-p') + 1} \left( \frac{[l_{31} - l_{43} - 2]}{[2l']^2[2l' + 2]} \right)^{1/2} \left| 0, 1, 1; (m)^{+11-31} \rightangle \\
 & \quad + q^{2(-2p') + 3} \left( \frac{[l_{31} - l_{43} - 2][l_{31} - l_{31} + 2]}{[2l']^2[2l' + 2]} \right)^{1/2} \left| 1, 1, 0; (m)^{+11-31-31} \rightangle \\
(m)_9 & = c_9 \left\{ -q^2 \left( q^2 + q^{-2} \right) \left( \frac{[l_{33} - l_{31} + 1][l_{31} - l_{43} - 1]}{[2l']^2[2l' + 2]} \right) \right\}^{1/2} \left| 0, 0, 1; (m)^{+11} \rightangle \\
 & \quad - q^{2(l'-p')} \left( \frac{[l_{31} - l_{43} - 2] - q^{-4(l'+1)}[l_{33} - l_{31} + 1]}{[2l']^2[2l' + 2]} \right)^{1/2} \left| 1, 0, 0; (m)^{+11-31} \rightangle \\
 & \quad + q^{2(l'-p') + 1} \left( \frac{[l_{31} - l_{43} - 2] - q^{-4(l'+1)}[l_{33} - l_{31} + 1]}{[2l']^2[2l' + 2]} \right)^{1/2} \left| 0, 1, 1; (m)^{+11-31} \rightangle \\
 & \quad + q^{2(-2p') + 1} \left( q^2 + q^{-2} \right) \left( \frac{[l_{31} - l_{43} - 2][l_{31} - l_{31} + 2]}{[2l']^2[2l' + 2]} \right)^{1/2} \left| 1, 1, 0; (m)^{+11-31-31} \rightangle \\
(m)_{10} & = c_{10} \left\{ q^2 \left( \frac{[l_{31} - l_{43} - 2][l_{31} - l_{43} - 1]}{[2l']^2[2l' + 1]} \right) \right\}^{1/2} \left| 0, 0, 1; (m)^{+11} \rightangle \\
 & \quad - q^{-2(l'+p') + 1} \left( \frac{[l_{33} - l_{31} + 1][l_{31} - l_{43} - 2]}{[2l']^2[2l' + 1]} \right)^{1/2} \left| 1, 0, 0; (m)^{+11-31} \rightangle \\
 & \quad + q^{-2(l'+p')} \left( \frac{[l_{33} - l_{31} + 1][l_{31} - l_{43} - 2]}{[2l']^2[2l' + 1]} \right)^{1/2} \left| 0, 1, 1; (m)^{+11-31} \rightangle \\
 & \quad + q^{-2(2l'+2p'-1)} \left( \frac{[l_{33} - l_{31} + 1][l_{33} - l_{31} + 2]}{[2l']^2[2l' + 1]} \right)^{1/2} \left| 1, 1, 0; (m)^{+11-31-31} \rightangle \\
(m)_{11} & = c_{11} \left\{ \left( \frac{[l_{31} - l_{11} + 1][l_{31} - l_{31} + 2]}{[2l + 1][2l' + 1]} \right)^{1/2} \right\} \left| 1, 0, 1, 1; (m)^{+11+11-31} \rightangle
\end{align*}
\[ -q^{2(-l+p+1)} \left( \frac{(l_{11} - l_{23} + 1)[l_{33} - l_{31} + 2]}{[2l + 1][2l' + 1]} \right)^{1/2} |1, 0, 1, 1; (m)^{+11-31} \]
\[ -q^{2(l'-p'+1)} \left( \frac{(l_{13} - l_{11} - 1)[l_{33} - l_{43} - 2]}{[2l + 1][2l' + 1]} \right)^{1/2} |1, 1, 1, 0; (m)^{+11+11-31-31} \]
\[ +q^{2(-l+p+l'-p'+2)} \left( \frac{(l_{11} - l_{23} + 1)[l_{33} - l_{43} - 2]}{[2l + 1][2l' + 1]} \right)^{1/2} |1, 1, 0, 1; (m)^{+11-31-31} \]
\[(m)_{12} = c_{12} \left\{ - \left( \frac{(l_{11} - l_{23} + 1)[l_{33} - l_{31} + 2]}{[2l + 1][2l' + 1]} \right)^{1/2} |1, 0, 1, 1; (m)^{+11+11-31} \right\}, \]
\[(m)_{13} = c_{13} \left\{ - \left( \frac{(l_{13} - l_{11} - 1)[l_{33} - l_{43} - 2]}{[2l + 1][2l' + 1]} \right)^{1/2} |1, 0, 1, 1; (m)^{+11+11-31} \right\}, \]
\[(m)_{14} = c_{14} \left\{ \left( \frac{(l_{11} - l_{23} + 1)[l_{33} - l_{43} - 2]}{[2l + 1][2l' + 1]} \right)^{1/2} |1, 0, 1, 1; (m)^{+11+11-31} \right\}, \]
\[(m)_{15} = c_{15} |1, 1, 1, 1; (m)\), \quad (4.33) \]

where \( l = \frac{1}{2}(m_{13} - m_{23}), \) \( p = m_{11} - \frac{1}{2}(m_{13} + m_{23}), \) \( l' = \frac{1}{2}(m_{33} - m_{43}) \) and \( p' = \)
$m_{31} - \frac{1}{2}(m_{33} + m_{43})$. The latest formula (4.33), in fact, represents a way in which the reduced basis is expressed in terms of the induced basis and vice versa it is not a problem for us to find the invert relation between these bases (see the Appendix).

Taking into account all results obtained above we have proved the following assertion

*Proposition 1*: The $U_q[gl(2/2)]$-module $W^q$ is decomposed as a direct sum (4.22) of sixteen $U_q[gl(2/2)_o]$-fidiroms $V^q_k$, $k = 0, 1, ..., 15$, every one of which is characterized by a highest weight $[m]_k$ given in (4.32) and is spanned by a GZ basis $(m)_k$ given in (4.33).

The decomposition (4.22) of $W^q([m]) \equiv W^q([m_{13}, m_{23}, m_{33}, m_{43}])$ can be rewritten in the form

\[
W^q([m]) = V^q_{(00)}([m_{13}, m_{23}, m_{33}, m_{43}])
\]

\[
\bigoplus_{i=0}^{\min(1,2l)} \bigoplus_{j=0}^{\min(1,2l')} V^q_{(10)}([m_{13} - i, m_{23} + i - 1, m_{33} - j + 1, m_{43} + j])
\]

\[
\bigoplus_{i=0}^{\min(2,2l)} V^q_{(11)}([m_{13} - i, m_{23} + i - 2, m_{33} + 1, m_{43} + 1])
\]

\[
\bigoplus_{j=0}^{\min(2,2l')} V^q_{(20)}([m_{13} - 1, m_{23} - 1, m_{33} - j + 2, m_{43} + j])
\]

\[
\bigoplus_{i=0}^{\min(1,2l')} \bigoplus_{j=0}^{\min(1,2l')} V^q_{(21)}([m_{13} - i - 1, m_{23} + i - 2, m_{33} - j + 2, m_{43} + j + 1])
\]

\[
\bigoplus V^q_{(22)}([m_{13} - 2, m_{23} - 2, m_{33} + 2, m_{43} + 2])
\] (4.34)

where $V^q_{(ab)}([m]_{(ab)})$ is an alternative notation of that submodule $V^q_k([m]_k)$ with $[m]_k = [m]_{(ab)}$:

\[
V^q_{(00)}([m]_{(00)}) \equiv V^q_{(00)}([m]) = V^q_0([m]),
\]

\[
V^q_{(10)}([m]_{(10)}) \equiv V^q_{(10)}([m]_k) = V^q_k([m]_k), \quad 1 \leq k \leq 4,
\]
\[ V_{(11)}^q([m]_{(11)}) \equiv V_{(20)}^q([m]_{(20)}) \equiv V_{(21)}^q([m]_{(21)}) \equiv V_{(22)}^q([m]_{(22)}) = \begin{bmatrix} m_{13} & m_{23} & m_{33} & m_{43} \\ m_{12} & m_{22} & m_{32} & m_{42} \\ m_{11} & 0 & m_{31} & 0 \end{bmatrix}_{(ab)}, \quad a, b \in \{0, 1, 2\} \] (4.36)

such that

\[
\begin{align*}
m_{12} &= m_{13} - r - \theta(a - 2) - \theta(b - 2) + 1, \\
m_{22} &= m_{23} + r - \theta(a - 1) - \theta(b - 1) - 1, \\
m_{32} &= m_{33} + a - s + 1, \\
m_{42} &= m_{43} + b + s - 1,
\end{align*}
\] (4.37)

where

\[
\begin{align*}
r &= 1, \ldots, 1 + \min(a - b, 2l'), \\
s &= 1, \ldots, 1 + \min(\langle a \rangle + \langle b \rangle, 2l),
\end{align*}
\] (4.38)

\[
\theta(x) = \begin{cases} 
1 & \text{if } x \geq 0, \\
0 & \text{if } x < 0
\end{cases}
\] (4.39)

and

\[
\langle i \rangle = \begin{cases} 
1 & \text{for odd } i, \\
0 & \text{for even } i.
\end{cases}
\] (4.40)

4.2. Typical representations
The $U_q[gl(2/2)]$-module $W^q$ constructed is either irreducible or indecomposable. We can verify that

**Proposition 2**: The induced $U_q[gl(2/2)]$-module $W^q$ is irreducible if and only if the following condition holds

$$[l_{13} + l_{33} + 3][l_{13} + l_{43} + 3][l_{23} + l_{33} + 3][l_{23} + l_{43} + 3] \neq 0. \quad (4.41)$$

In this case we say that the module $W^q$ is typical, otherwise it is called atypical.

The proof of the latter proposition follows the one for the classical case considered in Ref. 20 (cf. Ref. 18). Indeed, by the same argument we can conclude that $W^q$ is irreducible if and only if

$$E_{24}E_{14}E_{23}E_{31}E_{32}E_{41}E_{42} \otimes (M) \neq 0. \quad (4.42)$$

The latest condition (4.42) in turn can be proved, after some elementary calculations, to be equivalent to

$$[E_{11} + E_{33} + 1][E_{11} + E_{44}][E_{22} + E_{33}][E_{22} + E_{44} - 1](M) \neq 0, \quad (4.42')$$

which is nothing but the condition (4.41).

Since $U_q[gl(2/2)]$ is generated by the even generators and the odd Chevalley generators $E_{23}$ and $E_{32}$, any its representations in some basis is completely defined by the actions of these generators on the same basis. In the case of the typical representations the matrix elements of the generators in the reduced basis (4.33) can be obtained by keeping the conditions (4.1) and (4.41) valid and using the relations (3.1-5). For the even generators we readily have

$$E_{11}(m)_k = (l_{11} + 1)(m)_k,$$
$$E_{22}(m)_k = (l_{12} + l_{22} - l_{11} + 2)(m)_k,$$
$$E_{12}(m)_k = ([l_{12} - l_{11}][l_{11} - l_{22}])^{1/2} (m)_k^{+11},$$
$$E_{21}(m)_k = ([l_{12} - l_{11} + 1][l_{11} - l_{22} - 1])^{1/2} (m)_k^{-11},$$
$$E_{33}(m)_k = (l_{31} + 1)(m)_k,$$
\[
E_{43}(m)_k = (l_{32} + l_{42} - l_{31} + 2)(m)_k, \\
E_{34}(m)_k = ([l_{32} - l_{31}][l_{31} - l_{42}])^{1/2} (m)^{+31}_k, \\
E_{43}(m)_k = ([l_{32} - l_{31} + 1][l_{31} - l_{42} - 1])^{1/2} (m)^{-31}_k. \tag{4.43}
\]

As the computations on finding the matrix elements of \( E_{23} \) and \( E_{32} \) are too cumbersome, we shall write down here only the final results. If we assume the formal notation

\[
\left| x_1...x_i \right| \left| y_1...y_j \right| := \left| x_1 \right|...\left| x_i \right| \left| y_1 \right|...\left| y_j \right| 
\tag{4.44}
\]

the generator \( E_{23} \) acts on the basis vectors \( (m)_{(ab)} \), i.e. on \( (m)_k \), as follows

\[
E_{23}(\tilde{m})_{(00)} = 0, \\
E_{23}(\tilde{m})_{(10)} = -q^{-2}[l_{3-r,3} + l_{s+2,3} + 3] \left| \frac{l_{3-r,3} - l_{11} + 1}{l_{12} - l_{22}} \frac{l_{7-j,2} - l_{31} + 1}{l_{32} - l_{42}} \right|^{1/2} \left| \frac{l_{2} - l_{3-i,2} + \langle r \rangle - 1}{2 - \langle r \rangle} \right|^{-b/2} \left( \tilde{m} \right)_{(00)}^{+i_2-j_2-31}, \\
E_{23}(\tilde{m})_{(ab)} = q^{-2} \sum_{i=\max(1,b-r+2)}^{\min(2,b-r+3)} \sum_{j=\max(3,b+s+1)}^{\min(4,b+s+2)} (-1)^{(b-1)i + b(j+1)} \times [l_{3} + l_{3} + \langle s \rangle - \langle r \rangle + 3] \\
\times \left| \frac{l_{2} - l_{11} + 1}{l_{12} - l_{22}} \frac{l_{7-j,2} - l_{31} + 1}{l_{32} - l_{42}} \right|^{1/2} \left| \frac{l_{2} - l_{3-i,2} + \langle r \rangle - 1}{2 - \langle r \rangle} \right|^{-b/2} \left( \tilde{m} \right)_{(10)}^{+i_2-j_2-31}, \quad a + b = 2, \\
E_{23}(\tilde{m})_{(21)} = -q^{-2} \sum_{i=1}^{2} \sum_{j=3}^{4} \sum_{(s+j)\leq k=0,1\leq i+r+i} (-1)^{(1-k)i + k} \times [l_{3} + l_{3} - (-1)^k (i + j + s + r) + 3] \\
\times \left| \frac{l_{2} - l_{11} + 1}{l_{12} - l_{22}} \frac{l_{7-j,2} - l_{31} + 1}{l_{32} - l_{42}} \right|^{1/2} \left| \frac{l_{2} - l_{3-i,2} + 2k - 2}{2 - k} \right|^{-i+r/2} \left( \tilde{m} \right)_{(1+k,1-k)}^{\langle i+r \rangle/2} \\
\times \left| \frac{l_{5-s,2} - l_{j} + 2k}{1 + k} \right|^{(s+j+1)/2} \left( \tilde{m} \right)_{(1+k,1-k)}^{+i_2-j_2-31}, \\
E_{23}(\tilde{m})_{(22)} = q^{-2} \sum_{i=1}^{2} \sum_{j=3}^{4} (-1)^{i+j} [l_{3} + l_{3} + 3] \\
\times \left| \frac{l_{2} - l_{11} + 1}{l_{12} - l_{22}} \frac{l_{7-j,2} - l_{31} + 1}{l_{32} - l_{42}} \right|^{1/2} \left| \frac{l_{2} - l_{3-i,2} + 2k - 2}{2 - k} \right|^{-i+r/2} \left( \tilde{m} \right)_{(1+k,1-k)}^{\langle i+r \rangle/2} \\
\times \left| \frac{l_{5-s,2} - l_{j} + 2k}{1 + k} \right|^{(s+j+1)/2} \left( \tilde{m} \right)_{(1+k,1-k)}^{+i_2-j_2-31}.
\]
while the generator $E_{32}$ has the following matrix elements

$$
E_{32}(\tilde{m})_{(00)} = -q^2 \sum_{i=1}^{2} \sum_{j=3}^{4} \left( \frac{l_{i3} - l_{11} - 1}{l_{13} - l_{23}} \right) \left( \frac{l_{i3} - l_{i3} - 3}{l_{33} - l_{43}} \right)^{1/2} (\tilde{m})_{(21)}^{i2-j^2+31},
$$

$$
E_{32}(\tilde{m})_{(10)} = q^2 \sum_{i=1}^{2} \sum_{j=3}^{4} \sum_{(r+i) \leq k \leq (s+j)} (-1)^{(1-k)i+k(j+1)} \left( \frac{l_{i2} - l_{11}}{l_{12} - l_{22}} \right) \left( \frac{l_{i3} - l_{i3} - 2k - 1}{l_{33} - l_{43}} \right)^{1/2} (\tilde{m})_{(21)}^{i2-j^2+31},
$$

$$
E_{32}(\tilde{m})_{(ab)} = -q^2 \sum_{i=\max(2, r-b+1)}^{\min(2, r-b+1)} \sum_{j=\max(6, 4-b-s)}^{\min(6, 4-b-s)} (-1)^{bi+(1-b)j} \left( \frac{l_{i2} - l_{11}}{l_{12} - l_{22}} \right) \left( \frac{l_{i3} - l_{i3} - (s+1) + 1}{l_{33} - l_{43}} \right)^{1/2} (\tilde{m})_{(21)}^{i2-j^2+31},
$$

$$
E_{32}(\tilde{m})_{(21)} = -q^2 (-1)^{r+s} \left( \frac{l_{i3} - l_{11} - 1}{l_{13} - l_{23}} \right) \left( \frac{l_{i3} - l_{i3} + 2}{l_{33} - l_{43}} \right)^{1/2} (\tilde{m})_{(21)}^{r2+j^2+31}, \quad j = 5 - s,
$$

$$
E_{32}(\tilde{m})_{(22)} = 0.
$$

where $(\tilde{m})_{(ab)}$ are obtained from $(m)_{(ab)}$ by rescaling

$$
(\tilde{m})_{k} = \frac{1}{c_k} (m)_{k}, \quad k = 0, 1, ..., 15
$$

5. Conclusion

In this work we have constructed all typical representations of the quantum superalgebras $U_q[gl(2/2)]$ at the generic $q$ leaving the coefficients $c_k$ (see (4.30-31)) as free parameters. The latter can be fixed by some additional conditions, for example the hermiticity condition. The quantum typical $U_q[gl(2/2)]$-module $W^q([m])$
obtained has the same structure as the classical $gl(2/2)$-module (the module $W$ in Ref. 20) and can be decomposed into sixteen $U_q[gl(2/2)_0]$-fiddirmods $V^q_k([m]_k)$, $k = 0, 1, ..., 15$. In general, the method used here is similar to that one of Ref. 20. It is not difficult for us to see that for

$$c_k = 1, \quad k = 0, 1, ..., 15,$$

we obtain at $q = 1$ the classical results given in Refs. 20 (see also Ref. 21). However, unlike the latter our approach in this paper avoids the use of the Clebsch-Gordan coefficients 33 which are not always known for higher rank (quantum and classical) algebras. We hope that the present approach can be applied for larger quantum superalgebras. Following the classical programme 20,21 we can construct atypical representations of $U_q[gl(2/2)]$ at generic $q$ 22. Moreover, an extension of the present investigations on the case when $q$ is a root of unity is also possible 23.

Acknowledgements

I am grateful to Prof. Tch. Palev for numerous discussions. It is a pleasure for me to thank Prof. E. Celeghini and Dr. M. Tarlini for the kind hospitality at the Florence University where the present investigations were reported. I am thankful to Profs. A. Barut, R. Floreanini and V. Rittenberg for useful discussions.

I would like to thank Prof. Abdus Salam, the International Atomic Energy Agency and UNESCO for the kind hospitality at the High Energy- and the Mathematical Section of the International Centre for Theoretical Physics, Trieste, Italy.

The present work also was partially supported by the National Scientific Foundation of the Bulgarian Ministry of Science and Higher Education under the contract F-33.
6. References

1. L. D. Faddeev, N. Yu. Reshetikhin and L. A. Takhtajan, *Algebra and Analys*, 1, 178 (1987).

2. V. D. Drinfel’d, *Quantum groups*, in *Proceedings of the International Congress of Mathematicians*, 1986, Berkeley, vol. 1, 798-820 (The American Mathematical Society, 1987).

3. Yu. I. Manin, *Quantum groups and non-commutative geometry*, Centre des Recherchers Mathématiques, Montréal (1988); *Topics in non-commutative geometry*, Princeton University Press, Princeton, New Jersey (1991).

4. M. Jimbo, *Lett. Math. Phys.* 10, 63 (1985), ibit 11, 247 (1986).

5. S. I. Woronowicz, *Comm. Math. Phys.*, 111, 613 (1987).

6. E. K. Sklyanin, *Funct. Anal. Appl.*, 16, 263 (1982).

7. P. P. Kulish and N. Yu. Reshetikhin, *Zapiski nauch. semin. LOMI* 101, 112(1980),(in Russian); English translation: *J. Soviet Math.* 23, 2436 (1983).

8. L. C. Biedenharn, *J. Phys. A* 22, L873 (1989); A. J. Macfarlane, *J. Phys. A* 22, 4581 (1989).

9. H. D. Doebner and J. D. Hennig eds., *Quantum groups*, *Lecture Notes in Physics* 370 (Springer - Verlag, Berlin 1990).

10. P. P. Kulish ed., *Quantum groups*, *Lecture Notes in Mathematics* 1510 (Springer - Verlag, Berlin 1992)

11. E. Celeghini and M. Tarlini eds., *Italian Workshop on quantum groups*, Florence, February 3-6, 1993, [hep-th/9304160]. E. Celeghini, *Quantum algebras and Lie groups*, contribution to the Symposium "*Symmetries in Science VI*" in honour of L. C. Biedenharn, Bregenz, Austria, August 2-7, 1992 - B. Grubered ed., Plenum, New York 1992.
12. C. N. Yang and M. L. Ge eds., *Braid groups, knot theory and statistical mechanics*, World Scientific, Singapore 1989.

13. P. P. Kulish and N. Yu. Reshetikhin, *Lett. Math. Phys.* 18, 143 (1989); E. Celeghini, Tch. Palev and M. Tarlini, *Mod. Phys. Lett.* B 5, 187 (1991).

14. Tch. D. Palev and V. N. Tolstoy, *Comm. Math. Phys.* 141, 549 (1991).

15. Yu. I. Manin, *Comm. Math. Phys.* 123, 163 (1989).

16. M. Chaichian and P. Kulish, *Phys. Lett.* B 234, 72 (1990).

17. R. Floreanini, V. Spiridonov and L. Vinet, *Comm. Math. Phys.* 137, 149 (1991); E. D’Hoker, R. Floreanini and L. Vinet, *J. Math. Phys.*, 32, 1427 (1991).

18. R. B. Zhang, *J. Math. Phys.*, 34, 1236 (1993).

19. V. Kac, *Comm. Math. Phys.*, 53, 31 (1977); *Adv. Math.* 26, 8 (1977); *Lecture Notes in Mathematics* 676, 597 (Springer - Verlag, Berlin 1978).

20. A. H. Kamupingene, Nguyen Anh Ky and Tch. D. Palev, *J. Math. Phys.* 30, 553 (1989).

21. Tch. Palev and N. Stoilova, *J. Math. Phys.*, 31, 953 (1990).

22. Nguyen Anh Ky, *Finite-dimensional representations of the quantum superalgebra U_q[gl(2/2)]: II. Atypical representations at generic q*, in preparation.

23. Nguyen Anh Ky, *Finite-dimensional representations of the quantum superalgebra U_q[gl(2/2)] at q being roots of unity*, in preparation.

24. R. Floreanini, D. Leites and L. Vinet, *Lett. Math. Phys.* 23, 127 (1991).

25. M. Scheunert, *Lett. Math. Phys.* 24, 173 (1992).

26. S. M. Khoroshkin and V. N. Tolstoy, *Comm. Math. Phys.* 141, 599 (1991).

27. M. Rosso, *Comm. Math. Phys.* 124, 307 (1989).
28. M. Rosso, *Comm. Math. Phys.* **117**, 581 (1987).

29. G. Lusztig, *Adv. in Math* **70**, 237 (1988).

30. I. M. Gel’fand and M. L. Zetlin, *Dokl. Akad. Nauk USSR*, **71**, 825 (1950), (in Russian); for a detailed description of the Gel’fand-Zetlin basis see also G. E. Baird and L. C. Biedenharn, *J. Math. Phys.*, **4**, 1449 (1963); A. O. Barut and R. Raczka, *Theory of Group Representations and Applications*, Polish Scientific Publishers, Warszawa, 1980.

31. M. Jimbo, *Lecture Notes in Physics* **246**, 335 (Springer-Verlag, Berlin 1985); I.V. Cherednik, *Duke Math. Jour.* **5**, 563 (1987); K. Ueno, T. Takebayashi and Y. Shibukawa, *Lett. Math. Phys.* **18**, 215 (1989).
32. V. N. Tolstoy, in ref. 9: *Quantum Groups, Lecture Notes in Physics 370* (Springer-Verlag, Berlin 1990), p. 118.

33. V. A. Groza, I. I. Kachurik and A. U. Klimyk, *J. Math. Phys.* 31, 2769 (1990).
The induced basis (4.20) is expressed in terms of the reduced basis through the following invert relation

\[
|1, 0, 0, 0; (m)\rangle = \left\{ \frac{1}{c_1} q^{2(l+p+1)} \left( \frac{[l_{13} - l_{11} + 1][l_{31} - l_{43}]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_1^{−11+31} \\
- \frac{1}{c_2} q^{2(l+p)} \left( \frac{[l_{11} - l_{23} - 1][l_{31} - l_{43}]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_2^{−11+31} \\
+ \frac{1}{c_3} q^{2(l+p+1)} \left( \frac{[l_{13} - l_{11} + 1][l_{33} - l_{31}]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_3^{−11+31} \\
- \frac{1}{c_4} q^{2(l+p)} \left( \frac{[l_{11} - l_{23} - 1][l_{33} - l_{31}]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_4^{−11+31} \right\},
\]

\[
|0, 1, 0, 0; (m)\rangle = \left\{ \frac{1}{c_1} q^2 \left( \frac{[l_{11} - l_{23}][l_{31} - l_{43}]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_1^{+31} \\
- \frac{1}{c_2} q^2 \left( \frac{[l_{13} - l_{11}][l_{31} - l_{43}]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_2^{+31} \\
- \frac{1}{c_3} q^2 \left( \frac{[l_{11} - l_{23}][l_{33} - l_{31}]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_3^{+31} \\
- \frac{1}{c_4} q^2 \left( \frac{[l_{13} - l_{11}][l_{33} - l_{31}]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_4^{+31} \right\},
\]

\[
|0, 0, 1, 0; (m)\rangle = \left\{ \frac{1}{c_1} q^{2(l+p'-p')} \left( \frac{[l_{13} - l_{11} + 1][l_{33} - l_{31} + 1]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_1^{−11} \\
- \frac{1}{c_2} q^{2(l+p-p'-1)} \left( \frac{[l_{11} - l_{23} - 1][l_{33} - l_{31} + 1]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_2^{−11} \\
- \frac{1}{c_3} q^{2(l+p'+p'-1)} \left( \frac{[l_{13} - l_{11} + 1][l_{31} - l_{43} - 1]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_3^{−11} \\
+ \frac{1}{c_4} q^{2(-l+p+p'-1)} \left( \frac{[l_{11} - l_{23} - 1][l_{31} - l_{43} - 1]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_4^{−11} \right\},
\]

\[
|0, 0, 1; (m)\rangle = \left\{ \frac{1}{c_1} q^{-2(l'+p')} \left( \frac{[l_{11} - l_{23}][l_{33} - l_{31} + 1]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_1 \\
- \frac{1}{c_2} q^{-2(l'+p')} \left( \frac{[l_{13} - l_{11}][l_{33} - l_{31} + 1]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_2 \right\},
\]
\begin{align*}
1, 0, 1, 0; (m) &= \left\{ \frac{1}{c_5} q^{2(2l+2p+1)} \left( \frac{[l_{13} - l_{11} + 1][l_{13} - l_{11} + 2]}{[2l + 1][2l + 2]} \right)^{1/2} (m)_{5-11+131} \\
&\quad - \frac{1}{c_6} q^{2(2p-1)} \left( \frac{[2][l_{11} - l_{11} + 1]}{[2l + 2]} \right)^{1/2} (m)_{6-11+131} \\
&\quad + \frac{1}{c_7} q^{2(-2l+2p-1)} \left( \frac{[l_{11} - l_{23} - 2][l_{11} - l_{23} - 1]}{[2l][2l + 1]} \right)^{1/2} (m)_{7-11+131} \right\}, \\
1, 0, 0, 1; (m) &= \left\{ \frac{1}{c_5} q^{2(l+p+2)} \left( \frac{[l_{11} - l_{11} + 1][l_{11} - l_{23}]}{[2l + 1][2l + 2]} \right)^{1/2} (m)_{5-11+31} \\
&\quad - \frac{1}{c_6} \left[ 2 \right]^{1/2} q^{2(p+1)} (q^{2l+2}[l_{13} - l_{11}] - q^{-2l-2}[l_{11} - l_{23} - 1]) (m)_{6-11+31} \\
&\quad + \frac{1}{c_7} q^{2(-l+p+1)} \left( \frac{[l_{11} - l_{11}][l_{11} - l_{23} - 1]}{[2l][2l + 1]} \right)^{1/2} (m)_{7-11+31} \\
&\quad + \frac{1}{c_8} q^{2(-l+p+1)} \left( \frac{[l_{33} - l_{31} + 1][l_{33} - l_{43}]}{[2l + 1][2l + 2]} \right)^{1/2} (m)_{8-11+31} \\
&\quad - \frac{1}{c_9} \left[ 2 \right]^{1/2} q^{2(-p'+1)} (q^{2l' + 2}[l_{33} - l_{31} - 1] - q^{-2l'-2}[l_{33} - l_{31}]) (m)_{9-11+31} \\
\left\{ \frac{1}{c_5} q^{2(l+p+1)} \left( \frac{[l_{13} - l_{11} + 1][l_{11} - l_{23}]}{[2l + 1][2l + 2]} \right)^{1/2} (m)_{5-11+31} \\
&\quad - \frac{1}{c_6} \left[ 2 \right]^{1/2} q^{2p}(q^{2l+2}[l_{13} - l_{11}] - q^{-2l-2}[l_{11} - l_{23} - 1]) (m)_{6-11+31} \\
&\quad + \frac{1}{c_7} q^{2(-l+p)} \left( \frac{[l_{11} - l_{11}][l_{11} - l_{23} - 1]}{[2l][2l + 1]} \right)^{1/2} (m)_{7-11+31} \\
&\quad - \frac{1}{c_8} q^{2(-l'-p+1)} \left( \frac{[l_{33} - l_{31} + 1][l_{33} - l_{43}]}{[2l + 1][2l + 2]} \right)^{1/2} (m)_{8-11+31} \\
&\quad + \frac{1}{c_9} \left[ 2 \right]^{1/2} q^{2(-p'+2)} (q^{2l' + 2}[l_{33} - l_{31} - 1] - q^{-2l'-2}[l_{33} - l_{31}]) (m)_{9-11+31} \right\}, \\
0, 1, 1, 0; (m) &= \left\{ \frac{1}{c_5} q^{2(l+p+1)} \left( \frac{[l_{13} - l_{11} + 1][l_{11} - l_{23}]}{[2l + 1][2l + 2]} \right)^{1/2} (m)_{5-11+31} \\
&\quad - \frac{1}{c_6} \left[ 2 \right]^{1/2} q^{2p}(q^{2l+2}[l_{13} - l_{11}] - q^{-2l-2}[l_{11} - l_{23} - 1]) (m)_{6-11+31} \\
&\quad + \frac{1}{c_7} q^{2(-l+p)} \left( \frac{[l_{11} - l_{11}][l_{11} - l_{23} - 1]}{[2l][2l + 1]} \right)^{1/2} (m)_{7-11+31} \\
&\quad + \frac{1}{c_8} q^{2(-l'-p+1)} \left( \frac{[l_{33} - l_{31} + 1][l_{33} - l_{43}]}{[2l + 1][2l + 2]} \right)^{1/2} (m)_{8-11+31} \\
&\quad + \frac{1}{c_9} \left[ 2 \right]^{1/2} q^{2(-p'+2)} (q^{2l' + 2}[l_{33} - l_{31} - 1] - q^{-2l'-2}[l_{33} - l_{31}]) (m)_{9-11+31} \right\}.
\end{align*}
\[ \frac{1}{c_{10}} q^{2(\nu - \nu') + 2} \left( \frac{[l_{33} - l_{31}][l_{31} - l_{43} - 1]}{[2l'][2l' + 1]} \right)^{1/2} (m)_{10}^{-11 + 31} \]
\[ + \frac{1}{c_{12}} q^2 \left( \frac{[l_{13} - l_{11}] [l_{33} - l_{43}]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_{12}^{11+31+31} \]
\[ + \frac{1}{c_{13}} q^2 \left( \frac{[l_{11} - l_{23}] [l_{33} - l_{31}]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_{3}^{11+31+31} \]
\[ + \frac{1}{c_{14}} q^2 \left( \frac{[l_{13} - l_{11}] [l_{33} - l_{31}]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_{14}^{11+31+31} \}

\[ |1, 0, 1, 1; (m)⟩ = \left\{ \frac{1}{c_{11}} q^{2(l+p-p'-p')} \left( \frac{[l_{13} - l_{11} + 1] [l_{33} - l_{31} + 1]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_{11}^{11-11+31} \]
\[ - \frac{1}{c_{12}} q^{2(-l+p-p'-p'-1)} \left( \frac{[l_{11} - l_{23} - 1] [l_{33} - l_{31} + 1]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_{12}^{11-11+31} \]
\[ - \frac{1}{c_{13}} q^{2(l+p+p'-p'+1)} \left( \frac{[l_{13} - l_{11} + 1] [l_{31} - l_{43} - 1]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_{13}^{11-11+31} \]
\[ + \frac{1}{c_{14}} q^{2(-l+p+p'-p')} \left( \frac{[l_{11} - l_{23} - 1] [l_{31} - l_{43} - 1]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_{14}^{11-11+31} \} \]

\[ |0, 1, 1, 1; (m)⟩ = \left\{ -\frac{1}{c_{11}} q^{-2(l'+p')} \left( \frac{[l_{11} - l_{23}] [l_{33} - l_{31} + 1]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_{11}^{11+31} \]
\[ - \frac{1}{c_{12}} q^{-2(l'+p')} \left( \frac{[l_{11} - l_{11}] [l_{33} - l_{31} + 1]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_{12}^{11+31} \]
\[ + \frac{1}{c_{13}} q^{2(l'-p'+1)} \left( \frac{[l_{11} - l_{23}] [l_{31} - l_{43} - 1]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_{13}^{11+31} \]
\[ + \frac{1}{c_{14}} q^{2(l'-p'+1)} \left( \frac{[l_{13} - l_{11}] [l_{31} - l_{43} - 1]}{[2l + 1][2l' + 1]} \right)^{1/2} (m)_{14}^{11+31} \} \]

\[ |1, 1, 1, 1; (m)⟩ = \frac{1}{c_{15}} (m)^{-11-11+31+31} \]