A SHARP INCLUSION THEOREM FOR A CERTAIN CLASS OF ANALYTIC FUNCTIONS DEFINED BY THE SALAGEAN DERIVATIVE

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Abstract. In this short note we employ the Briot-Bouquet differential subordination to determine the best possible inclusion relation within a certain family of analytic functions defined by the Salagean derivative.

1. Introduction

Let $A$ denote the class of functions:

$$f(z) = z + a_2 z^2 + ...$$

which are analytic in the unit disk $E = \{ z \in \mathbb{C} : |z| < 1 \}$. Also let $P$ be the class of functions:

$$(1) \quad p(z) = 1 + p_1 z + ...$$

which are also analytic in the unit disk $E$ and have positive real part. In [3], Opoola introduced the subclass $T_{\alpha}^{\alpha}(\beta)$ consisting of functions $f \in A$ which satisfy:

$$(2) \quad \Re \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} > \beta$$

where $\alpha > 0$ is real, $0 \leq \beta < 1$, $D^n (n \in \mathbb{N}_0 = \{0, 1, 2, ...\})$ is the Salagean derivative operator defined as: $D^n f(z) = D(D^{n-1} f(z)) = z[D^{n-1} f(z)]'$ with $D^0 f(z) = f(z)$ and powers in $D^n$ meaning principal determinations only. The geometric condition (2) slightly modifies the one given originally in [3] (see [1]).

In earlier works [1, 3], inclusion relations have been discussed for the family $T_{\alpha}^{\alpha}(\beta)$. In particular, it has been shown that members of the family are related by the inclusion.

Theorem 1 ([1, 3]).

$$T_{n+1}^{\alpha}(\beta) \subset T_{n}^{\alpha}(\beta), \quad n \in \mathbb{N}_0.$$

The object of the present paper is to sharpen the above result. Our result is the following:

Theorem 2.

$$T_{n+1}^{\alpha}(\beta) \subset T_{n}^{\alpha}(\delta(\alpha, \beta)), \quad n \in \mathbb{N}_0$$
where

\[
\delta(\alpha, \beta) = 1 + 2(1 - \beta) \sum_{k=1}^{\infty} \frac{\alpha}{\alpha + k} (-1)^k.
\]

The result is sharp.

We will make use of the powerful technique of Briot-Bouquet differential subordination to prove the above result. This technique has been employed frequently in recent times to sharpen and improve many results in geometric function theory. A function \(p(z)\) given by (1) is said to satisfy the Briot-Bouquet differential subordination if

\[
p(z) + \frac{zp'(z)}{\eta p(z) + \gamma} \prec h(z), \quad z \in E
\]

where \(\eta\) and \(\gamma\) are complex constants and \(h(z)\) a complex function satisfying \(h(0) = 1\), and \(\text{Re}(\eta h(z) + \gamma) > 0\) in \(E\). It is well known that if \(p(z)\) given by (1) satisfies the Briot-Bouquet differential subordination, then \(p(z) \prec h(z)\) [2].

A univalent function \(q(z)\) is said to be a dominant of (4) if \(p(z) \prec q(z)\) for all \(p(z)\) satisfying (4). If \(\tilde{q}(z)\) is a dominant of (4) and \(\tilde{q}(z) \prec q(z)\) for all dominants \(q(z)\) of (4), then \(\tilde{q}(z)\) is said to be the best dominant of (4). The best dominant is unique up to rotation. Furthermore, it is known [2] that if the differential equation

\[
q(z) + \frac{zq'(z)}{\eta q(z) + \gamma} = h(z), \quad q(0) = 1
\]

has univalent solution \(q(z)\) in \(E\), then \(p(z) \prec q(z) \prec h(z)\) and \(q(z)\) is the best dominant.

Now let \(f \in A\). If we let

\[
p(z) = \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}
\]

Then we find that

\[
p(z) + \frac{zp'(z)}{\alpha} = \frac{D^{n+1} f(z)^\alpha}{\alpha^{n+1} z^\alpha}
\]

Thus it follows that

(a) if \(\frac{D^{n+1} f(z)^\alpha}{\alpha^{n+1} z^\alpha} \prec h(z)\), then \(\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} \prec h(z)\).

(b) if the differential equation \(q(z) + \frac{zq'(z)}{\eta q(z) + \gamma} = h(z), \quad q(0) = 1\), has univalent solution \(q(z)\) in \(E\), then \(\frac{D^{n+1} f(z)^\alpha}{\alpha^{n+1} z^\alpha} \prec h(z)\) implies \(\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} \prec q(z) \prec h(z)\) and \(q(z)\) is the best dominant.

In view of the above expositions we proceed to the proof of our result in the next section.
2. Proof of Theorem

Let \( f \in T_{n+1}^\alpha(\beta) \). Define
\[
h(z) = h_\beta(z) = \frac{1 + (1 - 2\beta)z}{1 - z}.
\]
It is obvious that \( h_\beta(z) \) maps the unit disk onto the plane \( \Re \omega > \beta \). Hence
\[
f \in T_{n+1}^\alpha(\beta) \iff D_n^{n+1}f(z)^\alpha < h_\beta(z)
\]
and the differential equation
\[
q(z) + \frac{zq'(z)}{\alpha} = h_\beta(z), \quad q(0) = 1
\]
has univalent solution
\[
q_\beta(z) = \frac{\alpha}{z^\alpha} \int_0^z t^{\alpha-1} h_\beta(t) dt.
\]
This yields
\[
q_\beta(z) = 1 + 2(1 - \beta) \sum_{k=1}^{\infty} \frac{\alpha}{\alpha + k} z^k.
\]
That is,
\[
q_\beta(z) = \beta + (1 - \beta) \left\{ 1 + 2 \sum_{k=1}^{\infty} \frac{\alpha}{\alpha + k} z^k \right\}.
\]
In [1] we have shown that
\[
\Re \left\{ 1 + 2 \sum_{k=1}^{\infty} \frac{\alpha}{\alpha + k} z^k \right\} \geq 1 + 2 \sum_{k=1}^{\infty} \frac{\alpha}{\alpha + k} (-r)^k, \quad |z| = r.
\]
Hence taking limit as \( r \to 1^- \), we have where \( \min_{|z| \leq 1} \Re q_\beta(z) = q_\beta(-1) = \delta(\alpha, \beta) \) where \( \delta(\alpha, \beta) \) is given by [3]. Hence we have
\[
f \in T_{n+1}^\alpha(\beta) \Rightarrow D_n^{n+1}f(z)^\alpha < q_\beta(z) < h_\beta(z)
\]
That is,
\[
f \in T_{n+1}^\alpha(\beta) \Rightarrow \Re \frac{D_n^{n+1}f(z)^\alpha}{\alpha^n z^\alpha} \geq \delta(\alpha, \beta)
\]
i.e. \( f \in T_n^\alpha(\delta(\alpha, \beta)) \). The inclusion is sharp since \( q_\beta(z) \) is the best dominant. This completes the proof.

If we choose \( \alpha = 1 \), we have

**Corollary 1.**
\[
T_{n+1}^1(\beta) \subset T_n^1(2(1 - \beta) \ln 2 + 2\beta - 1), \quad n \in \mathbb{N}_0.
\]
In particular for \( n = 0 \), we have
Corollary 2. Let \( f \in A \). Then

\[
Re f'(z) > \beta \Rightarrow Re \frac{f(z)}{z} > 2(1 - \beta) \ln 2 + 2\beta - 1.
\]

The result is sharp.

Lastly we remark that the above result improves an earlier one due to Owa and Obradovic [4] in which they proved that if \( f \in A \) satisfies \( Re f'(z) > \beta \) for \( 0 \leq \beta < 1 \) and \( z \in E \), then

\[
Re \frac{f(z)}{z} > \frac{1 + 2\beta}{3}.
\]

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