Local angular fractal and galaxy distribution

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Abstract

The power-law dependence of the angle in the angular projection of galaxy distribution is explained by assuming that in the spherical shells within a small angle the distributions are also fractal. If this local angular fractal is possessed, a fractal structure is angularly-isotropic at each occupied point though inhomogeneous, and is compatible with the present evidence claimed to be of homogeneity for galaxy distribution. Further, it is most likely to be isotropic rather than only angularly-isotropic. Several related issues are discussed.

PACS: 98.62.Py, 98.80.Es, 05.90.+m, 61.43.Hv

Keywords: Fractal; Galaxy distribution; Universe

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The past two decades witnessed an explosive activity on the scale-invariant but non-homogeneous objects, fractals [1], in almost all branches of science. Along with the findings of more structures by redshift surveys, it is interesting and important to know whether the large scale structure of the universe is a fractal, a question which was actually a stimulation in developing fractal geometry [1]. While it has been a consensus that the galaxy distribution approximates a fractal over a considerable scales [2], it is still under debate whether it crossovers to homogeneity on a scale about, say, $20 h^{-1} \text{Mpc}$ [3] [4], or extends up to the present limit of observation as claimed by Pietronero and his collaborators (hereafter P) [5] [6]. A crucial problem in this context concerns the angular projection of the galaxy distribution since before the extensive redshift survey the galaxy catalogs were for angular coordinates and now there has been much angular information [4]. Based on a numerical simulation on a fractal structure generated by a Levy flight in 3-dimensional space, P stated that the angular projection shows fractal correlation at small angles but becomes homogeneous at large angles [5], a so-called local isotropy is claimed to exist and realize the Cosmological Principle in the fractal structure [5]. As we point out later, however, their approach contains an implicit assumption of angular homogeneity in distribution. Note that angular dis-
tribution and projection are two different concepts. It will be shown in a consistent way that power laws for angular projection at small angles are manifestations of fractal of angular distribution. The behavior at large angles deserves further study. We distinguish homogeneity from isotropy, which is also different from only angular-isotropy. A fractal structure with local angular fractal is inhomogeneous but angularly-isotropic, hence can satisfy much evidence calimed to be of homogeneity but actually of angular-isotropy. Generally a fractal can be neither homogeneous nor angularly-isotropic, consequently, as we will see, the evidence of fractal on scales larger than $50h^{-1}Mpc$ is quite problematic. However, they will be more evident on the contrary if some basis of opposition arguments is established.

In the traditional analysis, one usually calculates the two-point correlation function

$$\xi(r) = \frac{\langle n(\vec{r}_0) \cdot n(\vec{r}_0 + \vec{r}) >_{\vec{r}_0} \rangle}{\langle n >^{2}} - 1$$

and the the angular correlation function

$$\omega(\theta) = \frac{\langle n(\theta_0) \cdot n(\theta_0 + \theta) >_{\theta_0} \rangle}{\langle n >^{2}} - 1$$
where $n(\vec{r}_0 + \vec{r})$ is the number density, $\langle \rangle_{\vec{r}_0}$ means average over settings of the origin $\vec{r}_0$ over the sample. $\langle n \rangle$ is the average density within the sample. The angular correlation function is defined in the similar way for angular coordinates, but note that densities are defined for the solid angle $2\pi(1-\cos \theta)$ instead of $\theta$ since the sample is a conic part of a sphere. This makes the angular projection more complicated than the original distribution. Since within the crossover scales for homogeneity, the average density and therefore the correlation functions are dependent on the sample depth, P suggested to study particularly the conditional densities $\Gamma(r)$ and $\Gamma(\theta)$, which are just the numerators of the first terms in rhs. of Eqs. (1) and (2), respectively.

The Limber scheme gives $\omega(\theta) \propto (r_0/L)^\gamma \theta^{1-\gamma}$, where $L$ is the sample depth, the length of the cone, $r_0$ is defined by $\xi(r_0) = 1$ therefore is proportional to $L$. As pointed out by P in their pioneering work, the homogeneity assumption was taken in Limber scheme, as well as in the manipulation of data, without considering the possible dependence on the sample angle $2\theta_M$. Therefore the consistency of analysis with homogeneity claiming that $\omega(\theta)$ for galaxy distribution is dependent on sample depth is questionable, also we cannot deduce the functions for the case of fractal just by setting $r_0 \propto L$ in that obtained through Limber scheme. According to P,
for both the angular data from galaxy catalogs and the fractal generated by a Levy flight, \( \omega(\theta) \) increases with \( \theta_M \) while independent of L. From Fig. 36 in Ref. [3] we may observe that the feature of \( \omega(\theta) \) is almost the same as \( \xi(r) \) (Figs. 5 and 22 there). As exposed later, this is due to the local angular fractality. \( \Gamma(\theta) \) for CfA was found to increase with the depth. Both \( \omega(\theta) + 1 \) for simulated fractal and \( \Gamma(\theta) \) for CfA depend on \( \theta \) with a power law when \( \theta \) is small.

For a fractal distribution, the number of points within a sphere of a radius \( r \) is \( N(r) = Br^D \), where \( D \) is the fractal dimension, \( B \) is a coefficient. A crucial concept throughout the work of P is the conditional density from an occupied point defined as [3]

\[
\Gamma(r) = \frac{1}{S(r)} \frac{dN(r)}{dr} = \left( \frac{DB}{4\pi} \right) r^{D-3},
\]

where \( S(r) \) is the area of a spherical shell of radius \( r \). We note that this is a quantity averaged over all the angular directions, hence loses information on angular distribution. This issue is irrelevant when one only determine \( D \) sampling with sphere. But there are two approaches where \( \Gamma(r) \) was inappropriately used, taking the implicit assumption that it is homogeneous on the sphere, i.e., angular homogeneity. One is that to the angular correlation,
explaining the dependence of $\theta^*$ on $\theta_M$ \[6\], here $\theta^*$ is defined by $\omega(\theta^*) = 1$.

The number of points on a spherical shell within $\theta$ was taken as $\Gamma(r)\Omega(\theta)r^2$.

So it is inconsistent in a way similar to what they criticized. In fact, the dependence of $\Gamma(\theta)$ and $\omega(\theta)$ on $\theta$ is just a signature of angular inhomogeneity in distribution. In the following the behavior of angular projection is explained in a simple and natural way by the so-called local angular fractal. At first it is an assumption and approximation, nevertheless it is much better than assuming angular homogeneity, its validity is proved by the results and is expected to be established in a more rigorous formulation under way, on the other hand it may be tested on the simulated fractal and the data of galaxy distribution.

Consider a small conic part of a sphere defined by $L$ and $2\theta$. The local angular fractal refers to that when $\theta$ is small enough, the points on each spherical shell within this conic part have fractal distribution while the spherical shells also distribute fractally. So actually, it is a sort of bifractal \[7\], a product of two subfractals, locally. In many cases such as galaxy distribution, it should be, of course, in a statistical way. The number of points in
the considered volume is

\[ N(L, \theta) = AD \int (r \theta)^{D_\Omega} r^{D_r-1} dr = AL^D \theta^{D_\Omega}, \quad (4) \]

where \( A \) is a coefficient, \( D_r \) and \( D_\Omega \) are the radial and angular dimensions respectively, with \( D = D_r + D_\Omega \). If the sample size is characterized by the angle \( 2\theta_M \) and the radial depth \( L \), then the solid angle is \( \Omega(\theta_M) = 2\pi(1 - \cos \theta_M) \approx \pi \theta_M^2 \), the average angular density is thus \( \langle n \rangle_M = N(\theta_M)/\Omega(\theta_M) \approx (A/\pi)L^D \theta_M^{D_\Omega-2} \). The density defined at the origin is \( n(\theta) = dN/d\Omega(\theta) = (dN/d\theta)/(d\Omega/d\theta) \approx (AD_\Omega/2\pi)L^D \theta^{D_\Omega-2} \), theoretically it is just \( \Gamma(\theta) \). The angular correlation function is \( \omega(\theta) = \Gamma(\theta)/\langle n \rangle_M - 1 \approx (D_\Omega/2)\theta/\theta_M) - 1 \). Similar to the conditional average density \( \tilde{\Gamma} \), we may also obtain a conditional average angular density \( \Gamma^*(\theta) = N(\theta)/\Omega(\theta) \approx (A/\pi)L^D \theta^{D_\Omega-2} \). For \( D_\Omega < 2 \), it is obvious that \( \omega(\theta) \) increases with \( \theta_M \) and dependent of \( L \), while \( \Gamma(\theta) \) is independent of \( \theta_M \) and increases with \( L \). \( \theta^* \) defined by \( \omega(\theta^*) = 1 \) is given by \( \theta^* \approx (D_\Omega/4)^{1/(2-D_\Omega)} \theta_M \), it is as spurious as \( \theta_0 \), since we may see that \( \omega(\theta) \) is not a well defined function for angular inhomogeneity. To see this, compare (a) for \( \omega(\theta) \) and (b) for \( \Gamma(\theta) \) in Fig. 1. Since \( \Gamma(\theta) \) and \( \Gamma^*(\theta) \) are independent of \( \theta_M \), they can be used to study angular distribution at small angles. For large angles, unlike the correspondence
in the full distribution, the solid angle is not simply a power function of $\theta$, things become artificially complicated as observed. The best way is just to study the scaling of the number of points (galaxies) with the angle.

So the power laws are direct consequences of local angular fractal. There was, of course, no direct investigation on $D_\Omega$. It can be estimated to be 1.3 for $\omega(\theta) \propto \theta^{-0.7}$. This is consistent with the finding that $D \approx 2$, which implies that $D_\Omega$ should be larger than 1 and less than 2. It should be pointed out that Eq. (4) cannot be generalized to an arbitrary angle. This can be understood by considering the case of homogeneity which takes $D_\Omega = 2$, the volume is proportional to $(1 - \cos \theta)$ rather than $\theta^2$. The property at large angles deserves further mathematical investigation, which is undertaken.

At this stage, we stress the difference between angular distribution and angular projection, as well as that among isotropy, angular-isotropy and angular homogeneity. The angular distribution refers to the number of points as a function of angle within a given radial depth. If it is homogeneous, then $N(\theta) \propto \theta^2$ at small angles. If it is angular fractal, then $N(\theta) \propto \theta^{D_\Omega}$ ($D_\Omega < 2$) at small angles. For a fractal in the whole space, there exist four possibilities. (i) Angular and radial irregular inhomogeneity. By irregular, we mean there is no fractal scaling. (ii)Radial fractal and local angular fractal.
(iii) Radial homogeneity and local angular fractal, this is a special case of (ii) with $D_r = 1$ and $D = D_\Omega + 1$. (iv) Radial fractal and angular homogeneity. In this case, one may see that not all points are statistically equivalents, there is one or more than one centers, the total dimension is larger than 2. For homogeneity in the whole space, it is, of course, homogeneous both radially and local-angularly. Angular-isotropy refers to that there are same number of points within same degree of angle, or say the angular distribution is isotropy. So not only angular homogeneity, but also local angular fractal is angularly-isotropic. Only if radial direction and angular direction is also equivalent, i.e., $D_r = D_\Omega/2$, the structure is not only angularly-isotropic but also isotropic. Now let us turn to the angular projection. It is defined through solid-angular densities represented as functions of $\theta$, so is more or less artificial. A homogeneous distribution in 3d space will, of course, lead to a homogeneous angular projection. But in principle, as being realized by P, there may be artificial homogeneity, which mean the value of the function does not change with $\theta$. But we are still uncertain about the angular projection of a fractal for an arbitrary angle. However, it is sure that for a fractal with local angular fractal, no matter whether the angular projection is homogeneous, the angular distribution, which is directly physical, is
angularly-isotropic though not homogeneous. To investigate angular distribution, as pointed out above, one had better just study the number of points (galaxies) within the corresponding angle.

It was claimed that local isotropy can exist in a fractal and realize the Cosmological Principle [5]. We see now this fractal should possess local angular fractal. If the universe is a fractal but not with local angular fractal, there cannot be local isotropy at every point, and the Cosmological Principle relaxes just to the statistical equivalence of each point. Similar situation is in the evidence of fractal above $\sim 150h^{-1}Mpc$ [5], which is another approach by appropriately using $\Gamma(r)$. To use thin deep catalogues up to the total depth, which extends to $900h^{-1}Mpc$, instead of sampling a complete sphere, the total sample within the small solid angle was used and $N(L, \Omega) = (\Omega/4\pi)BL^D$ was used to extract $D$. Now it is clear that this relation is valid only for angular homogeneity. The evidence of fractal within $150h^{-1}Mpc$ came from LEDA, which was claimed not to be suitable for statistical analysis [4]. Eliminating these from the results of analyses by P [5], the conclusion is that there is no evidence for fractal on scales larger than $50h^{-1}Mpc$. However, if there is local angular fractal, which is also angularly-isotropic, then $N(L, \Omega) \propto L^D$ is still valid though the rescaling is different in matching the results to data.
obtained by using full conditional density. So there can be evidence of fractal for galaxy distribution on scales above $50h^{-1} Mpc$ only if it is independently proved to be isotropic.

Dramatically, there is such evidence, which just came from what was thought to be for homogeneity [4], such as isotropic distributions of microwave background radiation and other radiation, and that in LCRS the distributions of number density as a function of redshift is the same in six separate slices. In our opinion, actually they refer to angular-isotropy rather than homogeneity. Angular-isotropy may be satisfied by local angular fractal or angular homogeneity. Even if angular homogeneity (in distribution) is proved, there is still the possibility that it is radially fractal, however, as discussed above, this model has total dimension larger than 2 and put us in a privileged position in the universe. Given angular-isotropy, the total fractal dimension can be measured using numbers of points in a finite solid angle.

The dimension of local angular fractal can be measured by the dependence of number on the angle. The radial distribution can be measured independently by pencil beam surveys, which was incorrectly interpreted by P as an intersection of 1 dimension with $D$ dimension in a 3$d$ substrate, thus possess a minus dimension [6]. Actually it gives a subset of $D$ dimension, or
say, an intersection of $D_r$ dimension with $D$ dimension in the $D$ dimensional substrate. Even if the reported periodicity of $128h^{-1}Mpc$ is confirmed to be due to the homogeneity of radial distribution on corresponding scales, there can still be angular fractal.

Before ending this letter, let us see what we can say about the controversy whether the boundness of cosmic fractal has been within observation. There has been much evidence for angular-isotropy. As exposed in this letter, this tells us the universe should be homogeneous or fractal with local angular fractal. The evidence for homogeneity came from the IRAS redshift survey. Since the IRAS has a certain degree of dilution, it was argued that the homogeneity behavior arises from diluteness, while the opposite argument stated that diluteness cannot change the picture. We think that it depends on how to dilute. If the diluteness is totally random with percentage $1 - p$, then a fractal is still a fractal after dilution, since the number of points within a scale $r$ only changes from $N(r)$ to $pN(r)$, without changing the fractal scaling. If the diluteness is related the distribution, a fractal can be changed toward homogeneity. Another evidence for homogeneity is the uniform distribution of $Ly - \alpha$ clouds. We think this only suggests the possibility of homogeneous distribution of intergalactic matter, without
conflicting a fractal model of galaxy distribution. Is the galaxy distribution isotropic, rather than only angularly-isotropic on the corresponding scales? Most likely yes, since from the observation \([5][6]\) it is obtained that \(D \approx 2\) while \(D_\Omega \approx 1.3\), thus \(D_r \approx D_\Omega / 2\).

In summary, the local angular fractal is claimed to cause the behavior of angular projection, in the way similar to that the behavior of two-point correlation function is caused by the fractal in the whole space. A fractal with local angular fractal is angularly-isotropic. As possessing a regularity between homogeneity and an ordinary fractal, and a reconciliation of evidence claimed to be of homogeneity and that of fractal, the local angular fractal may be adopted by Nature in distributing galaxies. Further it is most likely to be isotropic revealed by observation. Whether the fractal observed in the large scale structure of the universe crossovers to homogeneity on a scale much smaller than the present observational limit deserves further investigation.

I. Kanter is thanked for hospitality in BIU where this work is done.

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Figure Caption:

Fig. 1. (a) \( \omega(\theta) = \left( \frac{D_\Omega}{2} \right) \left( \frac{\theta}{\theta_M} \right)^{D_\Omega-2} - 1 \), and (b) \( \Gamma(\theta) = \left( AD_\Omega/2\pi \right) L^D \theta^{D_\Omega-2} \), setting \( A = 1, D = 2, D_\Omega = 1.3, \theta_M = 3.6^\circ = 0.02\pi \) and \( L = 20h^{-1} Mpc \). By comparing (a) and (b), it is clear that \( \theta^* \) where \( \omega(\theta^*) = 1 \) and the deviation from power-law begins is spurious.
