On the spectrum of the tridiagonal matrices with two-periodic main diagonal

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Abstract

We find the spectrum and eigenvectors of an arbitrary irreducible complex tridiagonal matrix with two-periodic main diagonal provided that the spectrum and eigenvectors of the matrix with the same sub- and superdiagonals and zero main diagonal is known. Our result substantially generalises some recent results on the Sylvester-Kac matrix and its certain main principal submatrices.

Key words. Tridiagonal matrices, spectrum, eigenvectors, two-periodic perturbation

AMS subject classification. 15A18, 15B05, 15A15

1 Introduction

The present note is intended to simplify and generalise some recent findings on certain modifications of the so-called Sylvester-Kac matrix

\[ K_N := \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ N & 0 & 2 & \ldots & 0 & 0 & 0 \\ 0 & N-1 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & N-1 & 0 \\ 0 & 0 & 0 & \ldots & 2 & 0 & N \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \end{pmatrix} . \]

This matrix seemingly appeared for the first time in a work of J.J. Sylvester [11] and later was studied by a number of mathematicians in XIX century, see [10] for references. In XX century, Mark Kac rediscovered this matrix in his famous work on the Brownian motion [7]. A good survey on the Sylvester-Kac matrix was presented by O. Taussky and J. Todd in [12]. References to some recent results on the Sylvester-Kac matrix are given, e.g., in [2]. Here we avoid reviewing this topic: instead, we show that some recent results around Sylvester-Kac-like matrices can be extended to an interesting property of a much larger class of tridiagonal matrices.

In the paper [8], the author added to \( K_N \) the main diagonal where the entries with even and odd indices have the same absolute value but opposite signs, and found the determinant of that matrix. The authors of [9] calculated the determinant of a matrix obtained from \( K_N \) by adding a non-zero two-periodic main diagonal. Note that a shorter proof of the result of [9] was given in [4]. From our Section 4 it is clear that the results of [4, 9] immediately follow from the result of [8].

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In the paper [5], the authors dealt with a certain submatrix of the Sylvester-Kac matrix $K_N$ (considered much earlier by A. Caley [1]). Among other results, they found the determinant and the eigenvalues of this submatrix with an added two-periodic main diagonal.

In this paper, we generalise the aforementioned results of the works [4, 5, 8, 9] to the class of arbitrary (irreducible) complex tridiagonal matrices with two-periodic main diagonal. We show that the findings of [8] can be easily extended (we give an independent proof) to such a class of matrices\footnote{It is clear that the Sylvester-Kac matrix and the matrix considered in [5] belong to this class, since they are irreducible tridiagonal and have zero main diagonal.}. This result allows us to generalise [9] and (partially) [5]. Namely, we determine the eigenvalues and the determinant of a tridiagonal matrix with two-periodic main diagonal via the eigenvalues of the same matrix but with zero main diagonal – treating the former (perturbed) matrix as a two-periodic perturbation of the latter (unperturbed) matrix.

The paper is organised as follows. Section 2 is devoted to a brief review of spectral properties of tridiagonal matrices with zero main diagonal. In Section 3, we express the eigenvalues, eigenvectors, and first generalised eigenvectors of the perturbed matrix through those of the corresponding unperturbed matrix. In Section 4, we extend the results of Section 3 to tridiagonal matrices with two-periodic main diagonal.

Note that irreducible tridiagonal matrices are closely related to orthogonal polynomials. In our further work [3] we present alternative proofs of the results of the present paper and exact formulas for eigenvectors’ interrelations.

## 2 Tridiagonal matrices with zero main diagonal

Consider an $n \times n$ complex irreducible tridiagonal matrix whose main diagonal only contains zero entries

$$J_n = \begin{pmatrix} 0 & c_1 & 0 & \ldots & 0 & 0 \\ a_1 & 0 & c_2 & \ldots & 0 & 0 \\ 0 & a_2 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & c_{n-1} \\ 0 & 0 & 0 & \ldots & a_{n-1} & 0 \end{pmatrix}, \quad a_k, c_k \in \mathbb{C} \setminus \{0\}. \quad (2.1)$$

Since the main diagonal of $J_n$ is zero, the spectrum of $J_n$ is symmetric w.r.t. zero.

**Theorem 2.1.** The spectrum of the matrix $J_n$ has the form

$$\sigma(J_{2l}) = \{ \pm \lambda_1, \pm \lambda_2, \ldots, \pm \lambda_l \} \quad \text{and} \quad \sigma(J_{2l+1}) = \{ 0, \pm \lambda_1, \pm \lambda_2, \ldots, \pm \lambda_l \}. \quad (2.2)$$

Here the numbers $\lambda_i$ are not necessary distinct: each eigenvalue of $J_n$ appears as many times as its algebraic multiplicity. For even $n$ all $\lambda_i$ are non-zero.

**Proof.** Let $\chi_k(z), k = 1, \ldots, n$, be the characteristic polynomial of the $k^{th}$ leading principal submatrix of $J_n$. Then the following three-term recurrence relations hold

$$\chi_{k+1}(z) = z\chi_k(z) - a_k c_k \chi_{k-1}(z), \quad k = 0, 1, \ldots, n-1, \quad (2.3)$$

with $\chi_{-1}(z) \equiv 0, \chi_0(z) \equiv 1$. From (2.3) it follows that the polynomials $\chi_k(z)$ do not depend on $a_k$ and $c_k$ separately – only on the product $a_k c_k$. Therefore, the matrices $J_n$ and $-J_n$ have the same eigenvalues, and hence the spectrum of $J_n$ is symmetric w.r.t. 0. In particular, if $n$ is odd, the matrix $J_n$ is singular.

However,

$$\det(J_{2l}) = (-1)^l \prod_{k=1}^{l} a_{2k-1} c_{2k-1} \neq 0,$$

so for even $n$, the matrix $J_n$ is non-singular. \qed

At the same time, the matrix $J_n$ may have the zero eigenvalue of any odd multiplicity.
Example 2.2. When \( n \) is odd, the matrix (2.1) can even be nilpotent. For instance, zero is the only eigenvalue of the matrix

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & -4 & 0 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}.
\] (2.4)

It is a folklore that any irreducible\(^2\) tridiagonal matrix \( J \) is non-derogatory, that is, all its eigenvalues are geometrically simple. Indeed, if \( \lambda \) is a an eigenvalue of \( J \), then \( J - \lambda I_n \), where \( I_n \) is the \( n \times n \) identity matrix, is singular but its submatrix obtained from \( J - \lambda I_n \) by deleting, say, the first column and the last row, is regular. Thus, every eigenvalue has only one eigenvector.

From the form of the matrix (2.1) and its spectrum, it follows that the eigenvectors and the generalised eigenvectors of the eigenvalues \( \lambda_i \) and \( -\lambda_i \) are closely related.

Theorem 2.3. Let \( \lambda \) and \( -\lambda \) be eigenvalues of the matrix \( J_n \) of multiplicity \( k \geq 1 \). If \( u^{(\lambda)}_0 \) is the eigenvector and \( u^{(\lambda)}_j \), \( j = 1, \ldots, k - 1 \), are the generalised eigenvectors of \( J_n \) corresponding to \( \lambda \), then

\[
u^{(-\lambda)}_0 = E_n u^{(\lambda)}_0,
\]

\[
u^{(-\lambda)}_j = (-1)^j E_n u^{(\lambda)}_j, \quad i = 1, \ldots, k - 1,
\] (2.5)

are the eigenvector and the generalised eigenvectors of \( J_n \) corresponding to \( -\lambda \). Here the matrix \( E_n = \{ e_{ij} \}^{n}_{i,j=1} \) is defined as follows

\[
e_{ij} = \begin{cases} (-1)^{i-j} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}
\] (2.6)

Remark 2.4. Substitution of \( \lambda = 0 \) shows that the eigenvector \( u^{(0)}_0 \) only has zero odd components; more generally, for \( j = 0, 1, \ldots \) all odd components of \( u^{(0)}_j \) and even components of \( u^{(0)}_{j+1} \) are equal to zero.

Proof of Theorem 2.3. Indeed, it is easy to see that

\[
J_n - \lambda I_n = -E_n (J_n + \lambda I_n) E_n,
\]

where \( I_n \) is the \( n \times n \) identity matrix. By definition,

\[
(J_n - \lambda I_n) u^{(\lambda)}_0 = 0, \quad (J_n - \lambda I_n) u^{(\lambda)}_j = u^{(\lambda)}_{j-1}, \quad j = 1, \ldots, k - 1,
\]

so we have

\[-E_n (J_n + \lambda I_n) E_n u^{(\lambda)}_0 = 0, \quad -E_n (J_n + \lambda I_n) E_n u^{(\lambda)}_j = u^{(\lambda)}_{j-1}, \quad j = 1, \ldots, k - 1,
\]

or

\[
(J_n + \lambda I_n) u^{(-\lambda)}_0 = 0, \quad (J_n + \lambda I_n) u^{(-\lambda)}_j = u^{(-\lambda)}_{j-1}, \quad j = 1, \ldots, k - 1,
\]

where \( u^{(-\lambda)}_j \) are defined in (2.5), as required.

3 Tridiagonal matrices with alternating signs main diagonal

Consider the matrix

\[
A_n = J_n + x E_n,
\] (3.1)

where the matrices \( J_n \) and \( E_n = \{ e_{ij} \}^{n}_{i,j=1} \) are defined in (2.1) and (2.6), respectively, so the main diagonal of the matrix \( A_n \) contains entries of the same non-zero absolute value and of alternating signs. The following fact holds.

\(^2\)A tridiagonal matrix is irreducible whenever its sub- and superdiagonals contain no zeroes.
Theorem 3.1. The spectrum of the matrix $A_n$ defined in (3.1) has the form
\[ \sigma(A_{2l}) = \{ \pm \sqrt{\lambda_k^2 + x^2}, \pm \sqrt{\lambda_k^2 + x^2}, \ldots, \pm \sqrt{\lambda_k^2 + x^2} \}, \]
and
\[ \sigma(A_{n+1}) = \{ x, \pm \sqrt{\lambda_k^2 + x^2}, \pm \sqrt{\lambda_k^2 + x^2}, \ldots, \pm \sqrt{\lambda_k^2 + x^2} \} \]
for $l = \left[ \frac{n}{2} \right]$, where $\{ \lambda_i \}_{i=1}^l$ belong to the spectrum of $J_n$, see (2.2).

In particular, if $x^2 = -\lambda_k^2 \neq 0$ for some $j$, then 0 is an eigenvalue of $A_n$ of even multiplicity. Moreover, for $n = 2l + 1$ some $\lambda_k$ may vanish, in which case formulae (3.2)–(3.3) show the existence of eigenvalues $\pm x$: the eigenvalue $x$ of $A_{2l+1}$ has odd multiplicity, while $-x$ is its eigenvalue of even multiplicity.

Proof. Note that the easily verifiable identity $J_n E_n + E_n J_n = 0$ implies that
\[ A_n^2 = (J_n + x E_n)^2 = J_n^2 + x J_n E_n + x E_n J_n + x^2 I_n = J_n^2 + x^2 I_n^2, \]
where $I_n$ is the $n \times n$ identity matrix.

Let $n = 2l$. In this case, $\lambda_k \neq 0$ for any $k = 1, \ldots, l$. Suppose first that all the $\lambda_k$ are distinct. From (3.3), it follows that all the eigenvalues of $A_{2l}^2$ are double and equal to $\lambda_k^2 + x^2$ for some $k = 1, \ldots, l$. According to [6], Chapter VIII, §6–7, for each $k$ the numbers $\sqrt{\lambda_k^2 + x^2}$ and/or $-\sqrt{\lambda_k^2 + x^2}$ (and only they) are in the spectrum of the matrix $A_{2l}$. Our aim is to show that the spectrum of $A_{2l}$ is indeed given by (3.2), where all eigenvalues are simple possibly excluding the double eigenvalue 0 corresponding to $x^2 = -\lambda_k^2$. We do this by finding an explicit expression for the eigenvector of $A_{2l}$ for each eigenvalue.

Let $\mu = \sqrt{\lambda_k^2 + x^2} \neq 0$ for some $k$ and some fixed branch of the square root. If $\mu$ is an eigenvalue of $A_{2l}$, then there exists a corresponding eigenvector $v_0^{(\mu)}$ satisfying
\[ A_{2l}^2 v_0^{(\mu)} = \mu^2 v_0^{(\mu)} = (\lambda_k^2 + x^2) v_0^{(\mu)}. \]

Then, due to
\[ A_{2l}^2 u_0^{(\lambda_k)} = (\lambda_k^2 + x^2) u_0^{(\lambda_k)} \quad \text{and} \quad A_{2l}^2 u_0^{(-\lambda_k)} = (\lambda_k^2 + x^2) u_0^{(-\lambda_k)}, \]
where the eigenvectors $u_0^{(\lambda_k)}$ and $u_0^{(-\lambda_k)}$ of $J_{2l}$ correspond to $\lambda_k$ and $-\lambda_k$ respectively, we have
\[ v_0^{(\mu)} = \alpha u_0^{(\lambda_k)} + \beta u_0^{(-\lambda_k)}, \]
for a certain choice of the coefficients $\alpha$ and $\beta$. Therefore,
\[ \mu (\alpha u_0^{(\lambda_k)} + \beta u_0^{(-\lambda_k)}) = \mu v_0^{(\mu)} = A_{2l} u_0^{(\mu)} = (J_{2l} + x E_{2l})(\alpha u_0^{(\lambda_k)} + \beta u_0^{(-\lambda_k)}) = \alpha J_{2l} u_0^{(\lambda_k)} + \alpha x E_{2l} u_0^{(\lambda_k)} + \beta J_{2l} u_0^{(-\lambda_k)} + \beta x E_{2l} u_0^{(-\lambda_k)} = (\alpha \lambda + \beta x) u_0^{(\lambda_k)} + (\alpha x - \beta \lambda_k) u_0^{(-\lambda_k)}, \]
whence on choosing $\alpha = 1$ we obtain $\beta = \frac{\mu - \lambda_k}{x + \mu}$. So,
\[ v_0^{(\mu)} = u_0^{(\lambda_k)} + \frac{x}{\lambda + \mu} u_0^{(-\lambda_k)}, \]
and, on taking another branch of the square root ($-\mu$ instead of $\mu$), it follows that $u_0^{(-\mu)}$ may be set to be
\[ u_0^{(\lambda_k)} + \frac{x}{\lambda_k + \mu} u_0^{(-\lambda_k)} = u_0^{(\lambda_k)} - \frac{\lambda_k + \mu}{x} u_0^{(-\lambda_k)}. \]
The last expression for $v_0^{(-\mu)}$ degenerates as $x \to 0$, so stretching it by the factor $-\frac{x}{\lambda_k + \mu}$ leads to a more convenient renormalisation
\[ v_0^{(-\mu)} = u_0^{(-\lambda_k)} - \frac{x}{\lambda + \mu} u_0^{(\lambda_k)}. \]
Thus, both numbers $\mu \neq 0$ and $-\mu$ belong to the spectrum $A_{2l}$ provided $J_{2l}$ has only simple eigenvalues.

4
Moreover, in this case

Suppose now that

are, respectively, the eigenvector and generalised eigenvector of the zero eigenvalue of the matrix \( A_{2l} \).

Thus, if all eigenvalues \( \pm \lambda_k \) of \( J_{2l} \) are simple, then \( A_{2l} \) has simple non-zero eigenvalues of the form \( \pm \sqrt{\lambda_k^2 + x^2} \) (and, in the case \( x = \pm \lambda_j \), a double zero eigenvalue). Now, if some \( \lambda_k \) is an eigenvalue of \( J_{2l} \) of multiplicity \( r \) and \( x^2 \neq -\lambda_k^2 \), then due to continuous dependence of the characteristic polynomial on its roots, the eigenvalues \( \pm \sqrt{\lambda_k^2 + x^2} \) of \( A_{2l} \) are also of multiplicity \( r \). Analogously, if \( x^2 = -\lambda_k^2 \) and \( \lambda_j \) is an eigenvalue of \( J_{2l} \) of multiplicity \( r \), then by continuity the eigenvalue \( \mu = 0 \) of \( A_{2l} \) is of multiplicity \( 2r \).

Suppose now that \( n = 2l + 1 \). From [3.4] and [8] Chapter VIII, §§6–7, it follows that the set

contains all possible eigenvalues of the matrix \( A_{2l+1} \). Similarly to the case of \( n = 2l \), if non-zero eigenvalues \( \pm \lambda_k \) of \( J_{2l+1} \) have multiplicity \( r \), one can show that the matrix \( A_{2l+1} \) has eigenvalues \( \pm \sqrt{\lambda_k^2 + x^2} \) of multiplicity \( r \) for \( x^2 \neq -\lambda_k^2 \), or the zero eigenvalue of multiplicity \( 2r \) for \( x^2 = -\lambda_k^2 \). (In fact, the relations between the corresponding eigenvectors of \( A_n \) and \( J_n \) for \( n = 2l + 1 \) remain the same as for \( n = 2l \).)

Moreover, \( \mu = x \) is always an eigenvalue of \( A_{2l+1} \). Indeed, if \( J_{2l+1} u_0^{(0)} = 0 \), then \( E_{2l+1} u_0^{(0)} = u_0^{(0)} \) according to Remark [2.4] and hence

Suppose now that \( J_{2l+1} \) has a zero eigenvalue of multiplicity \( 3 \), and

In this case, \( -x \) is also an eigenvalue of the matrix \( A_{2l+1} \) with the eigenvector \( v_0^{(-x)} = u_0^{(0)} - 2x u_1^{(0)} \). Indeed,

Moreover, in this case \( x \) is an eigenvalue of \( A_{2l+1} \) of multiplicity at least 2, and by [2.5] the vector \( v_1^{(x)} = u_1^{(0)} + 2x u_2^{(0)} \) satisfies

Hence, if 0 is a triple eigenvalue of \( J_{2l+1} \), then \( x \) is double and \( -x \) is simple due to multiplicities of the other eigenvalues of \( A_{2l+1} \).

Now by continuity we get that if 0 is an eigenvalue of \( J_{2l+1} \) of multiplicity \( 2r + 1 \), \( r \geq 0 \), then \( x \) is an eigenvalue of \( A_{2l+1} \) of multiplicity \( r + 1 \) while \( -x \) is of multiplicity \( r \). Consequently, formulæ [3.2]–[3.3] completely describe the spectrum of \( A_n \).

As a consequence of this theorem one gets the following formulae generalising the result of [8].

**Corollary 3.2.** For the matrix \( A_n \) defined in [3.1],

where some of the numbers \( \lambda_k \) in the second product can be zero.
3.1 Eigenvectors and generalised eigenvectors

Let us list the explicit expressions for eigenvectors and first generalised eigenvectors for all possible eigenvalues of $A_n$.

1) $\mu = x$. According to Theorem 3.1, if $\lambda = 0$ is an eigenvalue of $J_n$ of multiplicity $2r + 1$, then $\mu = x$ is an eigenvalue of $A_n$ of multiplicity $r + 1$. In the proof of that theorem we showed that the eigenvector and first generalised eigenvector of $A_n$ corresponding to $\mu = x$ are

$$v_0^{(x)} = u_0^{(0)},$$
$$v_1^{(x)} = u_1^{(0)} + 2xu_2^{(0)},$$

where $u_k^{(0)}$, $k = 0, 1, 2$, are the eigenvector and generalised eigenvectors of $J_n$ corresponding to the eigenvalue $\lambda = 0$.

2) $\mu = -x$. By Theorem 3.1 it is an eigenvalue of $A_n$ only if $\lambda = 0$ is an eigenvalue of $J_n$ of multiplicity at least 3; if multiplicity of $\lambda = 0$ is at least 5, then $\mu = -x$ is a multiple eigenvalue of $A_n$. In the proof of Theorem 3.1 we showed that the eigenvector of $A_n$ corresponding to $\mu = -x$ has the form

$$v_0^{(-x)} = u_0^{(0)} - 2xu_1^{(0)}.$$

Let us find $v_1^{(-x)}$. Since by definition

$$(A_n + xI_n) v_1^{(-x)} = v_0^{(-x)},$$

with use of (3.4) one has

$$0 = (A_n + xI_n)^2 v_1^{(-x)} = (J_n^2 + x^2I_n) v_1^{(-x)} = 2x^2 v_1^{(-x)} + 2xv_0^{(-x)} + x^2 v_1^{(-x)} = J_n^2 v_1^{(-x)} + 2xv_0^{(-x)},$$

so that

$$J_n^2 v_1^{(-x)} = -2xv_0^{(-x)} = -2xu_0^{(0)} + 4x^2 u_1^{(0)}.$$

Therefore, the vector $v_1^{(-x)}$ is a linear combination of the vectors $u_k^{(0)}$, $k = 0, \ldots, 3$,

$$v_1^{(-x)} = \alpha u_0^{(0)} + \beta u_1^{(0)} - 2xu_2^{(0)} + 4x^2 u_3^{(0)}.$$

Substituting this into (3.6) (with $\alpha = 0$) gives us

$$v_1^{(-x)} = u_1^{(0)} - 2xu_2^{(0)} + 4x^2 u_3^{(0)}.$$

3) $\mu = 0$. If $\lambda_j \neq 0$ and $x = \pm i\lambda_j$, then according to the proof of Theorem 3.1 the correspondent eigenvector and first generalised eigenvectors of $A_n$ are

$$v_0^{(0)} = u_0^{(\lambda_j)} \pm i u_0^{(-\lambda_j)}$$
and
$$v_1^{(0)} = \frac{i u_0^{(\lambda_j)} \mp i u_0^{(-\lambda_j)}}{2\lambda_j}.$$ (3.7)

4) $\mu = \sqrt{x^2 + \lambda_k^2}$ for some eigenvalue $\lambda_k \neq 0$ of the matrix $J_n$, where we choose any fixed branch of the complex square root. From the proof of Theorem 3.1 we know that

$$v_0^{(\mu)} = u_0^{(\lambda_k)} + \frac{x}{\lambda_k + \mu} u_0^{(-\lambda_k)}.$$

If $\mu$ is a multiple eigenvalue, the same approach we used to find the eigenvector allows to express the first generalised eigenvector of $A_n$ corresponding to $\mu$ as combinations of the eigenvectors and generalised eigenvectors of $J_n$ corresponding to $\lambda_k$. Namely, let

$$J_n u_1^{(\lambda_k)} = \lambda u_1^{(\lambda_k)} + u_0^{(\lambda_k)}.$$

If $v_1^{(\mu)}$ satisfying $A_n v_1^{(\mu)} = \mu v_1^{(\mu)} + v_0^{(\mu)}$ is sought in the form

$$v_1^{(\mu)} = \alpha u_1^{(\lambda_k)} + \beta u_1^{(-\lambda_k)} + \gamma u_0^{(\lambda_k)} + \delta u_0^{(-\lambda_k)},$$
then the same approach as above yields

$$v^{(\mu)}_1 = \frac{1}{2\lambda_k} \left( u_0^{(\lambda_k)} - \frac{x}{\lambda_k + \mu} u_0^{(-\lambda_k)} \right) + \frac{\mu}{\lambda_k} \left( u_1^{(\lambda_k)} - \frac{x}{\lambda_k + \mu} u_1^{(-\lambda_k)} \right). \quad (3.8)$$

Here the chosen values of \(\alpha, \beta, \gamma, \delta\) are natural in the sense that the \(v^{(\mu)}_0\) and \(v^{(\mu)}_1\) become \(3.7\) as \(\mu \to 0\) and do not degenerate as \(x \to 0\).

Analogously, one gets

$$v^{(-\mu)}_0 = u_0^{(-\lambda_k)} - \frac{x}{\lambda_k + \mu} u_0^{(\lambda_k)} \quad \text{and} \quad v^{(-\mu)}_1 = -\frac{1}{2\lambda_k} \left( \frac{x}{\lambda_k + \mu} u_0^{(\lambda_k)} + u_0^{(-\lambda_k)} \right) + \frac{\mu}{\lambda_k} \left( \frac{x}{\lambda_k + \mu} u_1^{(\lambda_k)} + u_1^{(-\lambda_k)} \right).$$

We note the choice of generalised eigenvectors is non-unique\(^3\) and the expression \(3.8\) may be replaced with a different (and in a sense more general) formula considered in our forthcoming publication [3]. That publication also gives a detailed description of the generalised eigenvectors corresponding to the eigenvalues \(\pm x\) of \(A_{2l+1}\) induced by the non-simple eigenvalue \(\lambda = 0\) of \(J_{2l+1}\).

4 Tridiagonal matrices with two-periodic main diagonal

Consider now the matrix

$$B_n = \begin{pmatrix} b_1 & c_1 & 0 & \ldots & 0 & 0 \\ a_1 & b_2 & c_2 & \ldots & 0 & 0 \\ 0 & a_2 & b_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b_{n-1} & c_{n-1} \\ 0 & 0 & 0 & \ldots & a_{n-1} & b_n \end{pmatrix}, \quad a_k, c_k \in \mathbb{C} \setminus \{0\}, \quad b_k = \begin{cases} x & \text{if } k \text{ is odd}, \\ y & \text{if } k \text{ is even}. \end{cases}$$

It is easy to see that

$$B_n = J_n + \frac{x-y}{2} E_n + \frac{x+y}{2} I_n,$$

where \(J_n\) is defined in [2.1], so from (3.2)–(3.3) we obtain

$$\sigma(B_{2l}) = \left\{ \frac{x+y}{2} \pm \frac{1}{2} \sqrt{4\lambda^2_k + (x-y)^2}, \frac{x+y}{2} \pm \frac{1}{2} \sqrt{4\lambda^2_k + (x-y)^2}, \ldots, \frac{x+y}{2} \pm \frac{1}{2} \sqrt{4\lambda^2_k + (x-y)^2} \right\},$$

and

$$\sigma(B_{2l+1}) = \left\{ x, \frac{x+y}{2} \pm \frac{1}{2} \sqrt{4\lambda^2_k + (x-y)^2}, \frac{x+y}{2} \pm \frac{1}{2} \sqrt{4\lambda^2_k + (x-y)^2}, \ldots, \frac{x+y}{2} \pm \frac{1}{2} \sqrt{4\lambda^2_k + (x-y)^2} \right\}. \quad (4.1)$$

These formulæ can be obtained from (3.2)–(3.3) by replacing \(x\) with \(\frac{x-y}{2}\) and then by adding \(\frac{x+y}{2}\) to all the eigenvalues.

From (4.1)–(4.2), one can easily obtain that the determinant of \(B_n\) has the form

$$\det B_{2l} = \prod_{k=1}^l (xy - \lambda^2_k), \quad \det B_{2l+1} = x \prod_{k=1}^l (xy - \lambda^2_k). \quad (4.3)$$

On letting \(J_n\) to be the Sylvester-Kac matrix or its main principal submatrix, the formulæ \(4.1\)–\(4.3\) generalise the results of the works [4]–[7], as well as an analogous transition in [8].

Observe that the eigenvectors and generalised eigenvalues of the matrices \(A_n\) and \(B_n\) are related through replacing \(x\) to \(\frac{x+y}{2}\).

The right eigenvectors of \(A_n\) and \(B_n\) may be obtained using the following remark.

\(^3\)For instance, for any constant \(\varepsilon \in \mathbb{C}\) the combination \(u_0^{(\lambda_k)} + \varepsilon u_0^{(-\lambda_k)}\) is a generalised eigenvector of \(J_n\) corresponding to \(\lambda\).
Remark 4.1. Note that if the (right) eigenvectors and generalised eigenvectors of some irreducible tridiagonal matrix are known, then it is easy to find its left eigenvectors: that is, the eigenvectors of a matrix

\[ J_n = \begin{pmatrix} b_1 & c_1 & 0 & \ldots & 0 & 0 \\ a_1 & b_2 & c_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b_{n-1} & c_{n-1} \\ 0 & 0 & 0 & \ldots & a_{n-1} & b_n \end{pmatrix}, \quad a_k, c_k \in \mathbb{C} \setminus \{0\} \]

are related to the eigenvectors of its transposed \( J_n^T \). It is clear that the spectra of \( J_n^T \) and \( J_n \) coincide. So if \( \tilde{u}_0^{(\lambda)} \) is the eigenvector of \( J_n^T \) corresponding to the eigenvalue \( \lambda \), then the obvious formula

\[ J_n^T = D_n^{-1} J_n D_n, \quad (4.4) \]

where the diagonal matrix \( D \) has the form

\[ D = \begin{pmatrix} d_1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & d_2 & 0 & \ldots & 0 & 0 \\ 0 & 0 & d_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & d_{n-1} & 0 \\ 0 & 0 & 0 & \ldots & 0 & d_n \end{pmatrix}, \quad \text{with} \quad d_1 = 1, \quad d_k = \frac{a_1 a_2 \cdots a_k}{c_1 c_2 \cdots c_k}, \quad k = 1, \ldots, n - 1. \]

implies by induction that

\[ \tilde{u}_0^{(\lambda)} = D_n^{-1} u_0^{(\lambda)}, \quad \tilde{u}_j^{(\lambda)} = D_n^{-1} u_j^{(\lambda)}, \quad j = 1, \ldots, k - 1, \]

where \( k \) is the multiplicity of the eigenvalue \( \lambda \).

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