A No-Go Theorem for the Consistent Quantization of Spin 3/2 Fields on General Curved Spacetimes

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Abstract. We first introduce a set of conditions which assure that a free spin $\frac{3}{2}$ field with $m \geq 0$ can be consistently (‘unitarily’) quantized on all four-dimensional curved spacetimes, i.e. also on spacetimes which are not assumed to be solutions of the Einstein equations. We discuss a large – and, as we argue, exhaustive – class of spin $\frac{3}{2}$ field equations obtained from the Rarita-Schwinger equation by the addition of non-minimal couplings and prove that no equation in this class fulfills all sufficient conditions.

In supergravity theories, the curved background is usually assumed to satisfy the Einstein equations and thus detailed knowledge on the spacetime curvature is available. Hence, our no-go theorem does not cover supergravity theories, but rather complements previous results indicating that they may be the only consistent field-theoretic models which contain spin $\frac{3}{2}$ fields. Particularly, our no-go theorem seems to imply that composite systems with spin $\frac{3}{2}$ can not be stable in curved spacetimes.

1 Introduction

As our universe is modelled well by a curved spacetime, and quantum field theories seem to be the correct theoretical underpinning of particle physics, quantum field theory (QFT) on curved spacetime (see, e.g. \textsuperscript{[BiDa82, Wa95]}) is the natural theoretical framework to describe matter in our universe at a fundamental level, at least in regimes where quantum gravity effects are negligible. In fact, one could claim that QFT on curved spacetime is the most fundamental theoretical model available to date which has passed all experimental tests so far and which is the minimal model to describe non-trivial phenomena such as the expansion history of our universe, the generation of primordial inhomogeneities in the homogeneous early universe, the synthesis of primordial elements (H, D, He, Li) in the early universe, the cosmic microwave background (CMB) radiation / the cooling of the universe and black hole radiation. Of course some of these phenomena are usually described in more “effective” terms without recourse to QFT on curved spacetime, but a description in terms of the latter is usually tacitly assumed; for example, the cooling of the universe is usually described by the redshift of electromagnetic radiation, but of course photons are excitations of quantum fields.
In the modern approach to QFT on curved spacetimes (see e.g. [Hac10] for a review), one usually tries to quantize a field theory “on all spacetimes at once”, simply because we have only a rough knowledge of our universe and it is unsatisfactory to base a model on unknown ingredients. Moreover, in cosmological applications of QFT on curved spacetimes one would like to gain detailed knowledge on the spacetime curvature by comparison with observations, that is of course only possible if the framework is sufficiently flexible regarding the background spacetime. From this point of view, it is natural to say that a QFT model is consistent only if it is a consistent QFT on all spacetimes. Hence, it seems advisable to ultimately check all QFT models discussed in particle physics on flat spacetime for their consistency on arbitrary curved spacetimes. This has been achieved already for fields of spin $\leq 1$ with renormalisation interactions, cf. [BFV03, DHP09, DaLa11, FrRe11, Ho07, HoWa01, HoWa03, San09] and references therein. However, in spite of some partial results in spacetimes of constant curvature [DeWa01], the general situation has been yet unclear for fields of spin $\frac{3}{2}$, even without any interaction.

One of the main reasons for this situation is that non-trivial problems appear already on the classical level. Namely, it was realised quite early that the spin $\frac{3}{2}$ field equation in flat spacetime – the Rarita-Schwinger equation [RaSch41] – does not make sense as a minimally coupled equation on arbitrary curved spacetimes [Buc58]. As we review in more detail in the main body of this work, this equation leads to integrability conditions which are only satisfied on Einstein spacetimes where the Ricci curvature tensor is proportional to the metric. Hence, the first problem to solve was to find the “correct” field equation for a field of spin $\frac{3}{2}$ in curved spacetimes, see [IlSch99] for a nice review of the issue. Various options have been proposed, and we shall review many of them in this work.

The probably most prominent solution of this classical problem is supergravity [DeZu76, FNF76], moreover, this solution is unique if one is interested in a consistent interacting gauge theory [BoEs07]. However, this theory, already on the classical level, only “works” on background spacetimes which solve the Einstein equations sourced by the energy momentum tensor of all fields in the supergravity theory [DeZu76, KKL00], so we can not expect the spin $\frac{3}{2}$ field in these theories – the gravitino – to be a consistent quantum field on all spacetimes. Although this does not imply that supergravity should be regarded as being unsatisfactory – since after all consistency on all spacetimes is not necessary for consistency in our spacetime – it puts severe constraints on and makes it very difficult to check the ultimate consistency of supergravity as a QFT in our curved spacetime, since the “true” stress-energy tensor in our universe is certainly more than just the one of the classical background fields in the supergravity theory, but also contains “quantum contributions”, e.g. the stress-energy of “particles” and radiation. We shall not analyse this problem in detail in this work, but we will comment on the current status of this issue in our final conclusions.

Instead, we shall investigate if any field equation describing a free spin $\frac{3}{2}$ field in curved spacetimes can lead to a consistent QFT on all curved spacetimes. To this avail, we first collect four conditions and prove that they are sufficient to guarantee that a free
field equation leads to a consistent spin $\frac{3}{2}$ on all topologically trivial curved spacetimes; we briefly discuss whether these conditions are sufficient on spacetimes of non-trivial topology as well and whether they are even necessary in general. In short, the conditions are

- **(Irreducibility)** The field equation describes an elementary spin $\frac{3}{2}$ particle, i.e. propagates the correct number of degrees of freedom.
- **(Covariance)** The number of degrees of freedom is independent of the background spacetime curvature.
- **(Causality)** The solutions of the field equation propagate causally.
- **(Selfadjointness)** The adjoint field equation can be obtained by partial integration.

Given these conditions, we consider a large class of linear spin $\frac{3}{2}$ field equations obtained from the minimally coupled Rarita-Schwinger equation by the addition of non-minimal couplings, and prove that no field equation in this class satisfies all four sufficient conditions. While proving this it becomes apparent that an enlargement of this class of field operators can hardly improve the result, such that, on practical grounds, one could claim that no modified Rarita-Schwinger operator can satisfy all these conditions.

As we consider modified Rarita-Schwinger operators, our no-go theorem only covers field equations in the Rarita-Schwinger representation $(1, \frac{1}{2}) \oplus (1, \frac{1}{2})$ of $SL(2, \mathbb{C})$, where the spinor is written with one Dirac index and one Lorentz index, but does not encompass field equations in the other possible spin $\frac{3}{2}$ representation, the Buchdahl representation $(\frac{3}{2}, 0) \oplus (1, \frac{1}{2})$, where the equations are written in terms of two-spinors [Buc58, Buc82]. These two representations are equivalent in Minkowski spacetime and for free fields, but fail to be so in curved spacetimes, see e.g. [IlSch99]. Indeed, Buchdahl has written down a set of equations in the Buchdahl representation [Buc82, Wun85] which have the advantage that they solve the consistency problem for higher spin field equations [Buc58] simultaneously for all spins. These equations have been analysed in great detail, see [IlSch99] for a review, and the possibility to obtain a consistent quantum theory for these equations has been explored [Mih07, Mak11]. However, the results to date have not been promising and, although not proving a no-go theorem for the Buchdahl equations and modifications thereof, we shall argue why a consistent spin $\frac{3}{2}$ quantum field theory on the basis of the Buchdahl equations seems unlikely to exist.

Our paper is organised as follows. In section 2 we recall the quantization of the free Rarita-Schwinger field in Minkowski spacetime, and discuss why it is consistent, i.e. unitary. Afterwards, we discuss the issue of consistency of spin $\frac{3}{2}$ quantum theories in curved spacetimes in section 3, and introduce four conditions to assure this consistency. In section 4, we discuss various spin $\frac{3}{2}$ field equations present in the literature and argue why they fail to satisfy the consistency conditions. Finally, we consider a class of modified Rarita-Schwinger equations for spin $\frac{3}{2}$ fields in section 5 and prove a no-go theorem for their consistent quantization. The appendices A to F contain background material and the proof of a few fiducial technical results.
In this work, “spacetime” always means “globally hyperbolic smooth four-dimensional spacetime”\(^1\). We shall denote Lorentz/spacetime indices by small Greek letters, whereas Dirac spinor indices will be suppressed throughout. All expressions are valid in an arbitrary basis of the considered vector bundles unless otherwise noted. We work with spacetime signature \((+,-,-,-)\) and our conventions and notations regarding Dirac spinors and curvature tensors are collected in the appendices \(\text{A} \text{ and } \text{B}\).

2 The free Rarita-Schwinger field on flat spacetime and conditions for a consistent quantization

2.1 The classical free Rarita-Schwinger field on flat spacetime

We briefly review the classical theory of the free Rarita-Schwinger field \(\psi^\alpha\) in flat spacetime, and already present it in a form suitable for our analysis in this work. The Rarita-Schwinger field is a function on Minkowski spacetime which carries one spacetime index and one Dirac index, i.e. mathematically speaking, \(\psi^\alpha \in \Gamma(RM)\) is a smooth (i.e. infinitely often differentiable) section of the Rarita-Schwinger bundle \(RM := DM \otimes TM\) over \(M\), where \(TM\) and \(DM\) denote the tangent and Dirac bundles over \(M\) respectively, and \(\Gamma(B) (\Gamma_0(B))\) denotes the smooth sections (smooth sections with compact support, i.e. ‘test sections’) of a bundle \(B\). To endow this field with dynamics, one imposes the Rarita-Schwinger equation\(^2\) [RaSch41]

\[
\mathcal{R}_0 \psi^\alpha := (-i\partial + m)\psi^\alpha = 0
\]

and the constraint

\[
\psi := \gamma_\alpha \psi^\alpha = 0.
\]

The latter constraint corresponds to removing the spin \(\frac{1}{2}\) piece \((\frac{1}{2},0) \oplus (0,\frac{1}{2})\) from the reducible representation

\[
\left( \left( \left( \frac{1}{2},0 \right) \oplus \left( 0,\frac{1}{2} \right) \right) \otimes \left( \frac{1}{2},\frac{1}{2} \right) \right) = \left( \frac{1}{2}\cdot\frac{1}{2} \right) \oplus \left( 1,\frac{1}{2} \right) \oplus \left( \frac{1}{2},1 \right) \oplus \left( \frac{1}{2},0 \right) \oplus \left( 0,\frac{1}{2} \right)
\]

of \(SL(2,\mathbb{C})\) corresponding to \(DM \otimes TM\). Upon contracting the Rarita-Schwinger equation with \(\gamma^\alpha\) and applying the constraint,

\[
\partial_\alpha \psi^\alpha = 0
\]

\(^1\)For a full definition of such spacetimes, see e.g. [BGP07, Wa84]. Loosely speaking, on such spacetimes there always exist unique solutions of hyperbolic partial differential equations with suitable initial data on equal-time surfaces.

\(^2\)In works on supergravity the term “Rarita-Schwinger equation” refers to a different equation which we discuss at the end of this subsection and in subsection 4.3.
follows, and, applying the operator $i\partial + m_1$ to the Rarita-Schwinger equation yields the Klein-Gordon equation
\[(\Box + m^2)\psi^\alpha = 0.\]
If $m = 0$, there is a gauge freedom present. Indeed, 
\[\psi^\alpha \quad \text{and} \quad \psi^\alpha + \partial^\alpha \chi, \chi \in \Gamma(DM) \& \partial \chi = 0\]
are gauge-equivalent \cite{RaSch41}; we shall see in the discussion of the quantization why “gauge-solutions” $\psi^\alpha = \partial^\alpha \chi$ represents “unphysical” degrees of freedom. In analogy to the Dirac field case, one can show that the Dirac-conjugated Rarita-Schwinger field $\bar{\psi}$ (cf. appendix A for the exact definition) solves the conjugated Rarita-Schwinger equation
\[(i\partial + m)\bar{\psi} = 0 \iff R_0\psi^\alpha = 0.\]
As $(i\partial + m)R_0$ is the unit matrix times the Klein-Gordon operator, i.e. a hyperbolic differential operator, one can prove that unique advanced and retarded Green’s operators/functions for the Rarita-Schwinger operator exist \cite{BGP07, Muh10}. Denoting them by $G^\pm_{\alpha \beta}$, their operator versions fulfil 
\[R_0 G^\pm_{\alpha \beta} f^\beta = G^\pm_{\alpha \beta} R_0 f^\beta = f^\alpha \quad \forall f^\alpha \in \Gamma_0(RM)\]
and have the usual causal support properties, i.e. $G^+_{\alpha \beta} f^\beta (G^-_{\alpha \beta} f^\beta)$ is only non-vanishing in the forward (backward) lightcone of the support of $f^\beta$. By defining 
\[G^\pm_{\alpha \beta}(\bar{f}_{\beta}, g^\alpha) := \langle f^\beta, G^\pm_{\alpha \beta} g^\alpha \rangle := \int_M dx \bar{f}_\beta(x) \left[ G^\pm_{\alpha \beta} g^\alpha \right](x)\]
and interpreting $G^\pm_{\alpha \beta}(\bar{f}_{\beta}, g^\alpha)$ as 
\[G^\pm_{\alpha \beta}(\bar{f}_{\beta}, g^\alpha) = \int_{M \times M} dxdy \bar{f}_\beta(x) G^\pm_{\alpha \beta}(x, y) g^\alpha(y),\]
one obtains the Green’s functions $G^\pm_{\alpha}(x, y)$ associated to the Green’s operators $G^\pm_{\alpha}$. These fulfil $R_0 G^\pm_{\alpha}(x, y) = \delta^\alpha_\beta \delta(x, y)$, which is equivalent to $R_0 G^\pm_{\alpha} f^\beta = f^\alpha$. For our discussion, viewing the Green’s solutions of the Rarita-Schwinger operator as operators rather than functions (bi-distributions) is more convenient since it allows for a concise notation. The difference 
\[G^\alpha_{\beta} := G^-_{\alpha \beta} - G^+_{\alpha \beta}\]
defines the causal propagator $G^\alpha_\beta$, a surjective map \cite[thm. 3.4.7]{BGP07} from $\Gamma_0(RM)$ to the solutions of the Rarita-Schwinger equation with compactly supported initial conditions, whose space we denote by $S(R_0, M)$. In other words, every solution of $R_0\psi^\alpha = 0$
\footnote{Theorem 3.4.7 in \cite{BGP07} does not prove the surjectivity of the causal propagator for Rarita-Schwinger operators, but the proof holds analogously for all hyperbolic operators.}
with compactly supported initial conditions on a Cauchy surface/equal-time surface, i.e. every “Rarita-Schwinger wave packet”, is of the form $\psi^\alpha = G_\beta^\alpha f^\beta$ with some $f^\beta \in \Gamma_0(RM)$. Such $f^\beta$ is non-unique, as the kernel of $G_\beta^\alpha$ is non-trivial. Indeed, $G_\beta^\alpha g^\beta = 0$ for all $g^\beta$ of the form $g^\beta = R_0 \tilde{g}^\beta$ for some $\tilde{g}^\beta \in \Gamma_0(RM)$. In the quantized Rarita-Schwinger field theory, the causal propagator is employed to define covariant canonical anticommutation relations (CAR) as we shall discuss in the next subsection, which is why $G_\beta^\alpha(x, y)$ is often called “anticommutator function”. Note that $G_\beta^\alpha(x, y) = 0$ if $x$ and $y$ are spacelike separated, which is why $G_\beta^\alpha(x, y) = 0$ is indeed a physically sensible choice of anticommutator function.

The above synopsis of the solution theory of $R_0 \psi^\alpha = 0$ has been independent of the constraint $\psi = 0$. In fact, it is important to clearly fix the convention that $\psi = 0$ is considered as an additional constraint on solutions, i.e. elements of $S(R_0, M)$. To distinguish constrained and unconstrained solutions, we define

$$S(R_0, M) := \{ \psi^\alpha \in S(R_0, M) | \psi = 0 \}$$

$$\Gamma_0(RM) := \{ f^\alpha \in \Gamma_0(RM) | G_\beta^\alpha f^\alpha \in S(R_0, M) \} .$$

To make sure that the classical Rarita-Schwinger field theory is non-trivial, one has to check that $\psi$ and its derivatives are vanishing on a Cauchy surface is enough to assure that $\psi = 0$ on the full spacetime without further restrictions on $\psi^\alpha$ itself. With other words, $\psi = 0$ can be regarded as a constraint on the initial conditions of elements of $S(R_0, M)$ and one can check that the constrained space of initial conditions is non-trivial; hence, $S(R_0, M)$ is non-trivial either.

Our way to introduce and define the Rarita-Schwinger equation mimics the original definition by Rarita and Schwinger [RaSch41], whereas in modern treatments of the subject – usually in the context of supergravity theories –, a slightly different approach is taken. Namely, instead of specifying the differential equation (1) plus the constraint $\gamma^\alpha \psi^\alpha = 0$, one specifies only a differential equation which is such that its solutions automatically fulfi this constraint. Since these two approaches yield different results upon minimal coupling to spacetime curvature, we briefly review the modern definition. To this avail, let $\psi^\alpha \in \Gamma(RM)$ fulfil

$$R_1 \psi^\alpha := -(i\gamma^{[\alpha} \gamma^\beta \gamma^\gamma) \partial_{\gamma} + m\gamma^{[\alpha} \gamma^\beta) \psi^\beta$$

$$= -i\psi^\beta + i\partial^\alpha \phi + i\gamma^\beta \partial_{\gamma} \psi^\gamma - i\gamma^\alpha \partial^\beta \psi^\beta + m\psi^\alpha - m\gamma^\alpha \psi = 0 ,$$

where $[ ]$ denotes idempotent antisymmetrisation. Let now $m > 0$. Contracting $R_1 \psi^\alpha = 0$ with $\partial_{\alpha}$ yields $\partial_{\alpha} \psi^\beta = \partial_{\nu} \psi^\nu$ for all solutions of (1). Inserting this into (1) and contracting the
result with $\gamma_\alpha$ leads to $\psi = 0$ on shell and, hence, to $\partial_\nu \psi^- = 0$ on shell. Finally, inserting these identities into (4), one finds that, on solutions, $R_1 \equiv R_0$, hence, $S(R_0, M) = S(R_1, M)$. In the massless case, one can use part of the gauge equivalence of $\psi^\alpha$ and $\psi^\alpha + \partial^\alpha \chi$ with arbitrary $\chi \in \Gamma(D M)$ to require $\psi = 0$ (see e.g. [VNi81]), which, once more, leads to the on-shell identities $\partial_\mu \psi^\mu = 0$, $R_1 \equiv R_0$.

2.2 The consistency of the free Rarita-Schwinger quantum field in flat spacetime

To canonically quantize a classical field theory, one imposes canonical (anti)commutation relations. In the case at hand, the covariant CAR of the Rarita-Schwinger field are specified by the causal propagator $G_\beta^\alpha$, i.e.

$$\{ \psi_\beta(x), \overline{\psi}_\alpha(y) \} := \psi_\beta(x) \overline{\psi}_\alpha(y) + \overline{\psi}_\alpha(y) \psi_\beta(x) = i G_\alpha^\beta(x, y) 1,$$

where we shall denote the (anticommuting) quantum field $\psi^\alpha$ with a bold-faced letter here and in the following, in order to distinguish it from the (c-number) classical solutions $\psi^\alpha$ of the field equation. By means of the results discussed in appendix D, these covariant CAR can be seen to be equivalent to the often imposed equal-time CAR. To understand this, we first recall that a quantum field $\psi^\alpha(x)$ at a spacetime point $x$ is too singular to be a well-defined operator on some Hilbert space. In other words, if $|\Omega\rangle$ is a normalised state, i.e. a Hilbert space vector of unit norm, $\psi^\alpha(x)|\Omega\rangle$ is not normalisable any more, which is related to the fact that the anticommutator function $i G_\alpha^\beta(x, y)$ is singular if $x = y$ (in fact, if $x - y$ is lightlike). To cure this, one can “smear” the covariant field $\psi^\alpha(x)$, i.e. integrate it with a test section $\overline{\alpha}(x)$. In contrast, in the equal-time formalism, the quantum field $\Psi^\alpha(t, \vec{x})$ is a well-defined operator once integrated with a solution of the Rarita-Schwinger equation rather than a test section, hence, we employ the capitalised notation to distinguish the equal-time quantum field $\Psi^\alpha$ from the covariant quantum field $\psi^\alpha$. In more detail, the smeared fields

$$\psi^\alpha(\alpha) := \int_M dx \overline{\alpha}(x) \psi^\alpha(x) \quad \text{and} \quad \Psi^\alpha(G_\alpha f_\beta) := \int d\vec{x} \left[ G_\alpha^\beta f_\beta \right](t, \vec{x}) \gamma^0 \Psi^\alpha(t, \vec{x})$$

are considered to represent the same operators, such that the discussion in appendix D can be subsumed as

$$i \int_{M \times M} dx dy \overline{\alpha}(x) G_\alpha^\beta(x, y) g^\alpha(y) = i G_\alpha^\beta(\overline{\alpha}, g^\alpha) = \{ \psi^\beta(\overline{\alpha}), \overline{\psi}_\alpha(g^\alpha) \} = \{ \Psi^\beta(G_\beta^\alpha f_\mu), \overline{\Psi}_\alpha(G_\alpha^\nu g^\nu) \} = - \int d\vec{x} \left[ G_\beta^\alpha f_\mu \right](t, \vec{x}) \gamma^0 [G_\alpha^\nu g^\nu](t, \vec{x}).$$

Hence, the covariant CAR (4) are equivalent to the equal-time CAR

$$\{ \Psi^\alpha(t, \vec{x}_1), \Pi_\beta(t, \vec{x}_2) \} = i \delta^\alpha_\beta \delta(\vec{x}_1, \vec{x}_2) 1,$$
where $\Pi_{\alpha} := i\Psi_{\alpha}\gamma^0$ is the momentum canonically conjugated to $\Psi_{\alpha}$.

We shall now discuss a non-trivial condition – already pointed out in the 60es [JoSu60] – for the above canonical quantum theory of the free Rarita-Schwinger field in flat spacetime to be consistent, i.e. ‘unitary’. To this avail, let us note that the proposed anticommutator of a smeared Rarita-Schwinger field and its adjoint

$$\{\bar{\psi}_{\alpha}(f^\alpha), \psi^\beta(\bar{f}_\beta)\}$$

is bound to be a positive operator, i.e. to have positive or vanishing expectation value in any quantum state $|\Omega\rangle$. Indeed, Hermitean conjugation (i.e. conjugation with respect to the Hilbert space scalar product) acts on the quantized Rarita-Schwinger fields as

$$\left(\psi_{\alpha}(\bar{f}_\alpha)\right)^\dagger := \bar{\psi}_{\alpha}(f^\alpha) \quad \left(\bar{\psi}_{\alpha}(f^\alpha)\right)^\dagger := \psi_{\alpha}(\bar{f}_\alpha) \quad \forall f^\alpha \in \Gamma_0(RM),$$

which corresponds to the classical identity

$$\langle f^\alpha, \psi^\beta \rangle^* = \left(\int_M dx \bar{f}_\alpha(x)\psi^\alpha(x)\right)^* = \langle \psi^\beta, f^\alpha \rangle,$$

with * denoting complex conjugation; hence, the considered anticommutator can be written as

$$\{\bar{\psi}_{\alpha}(f^\alpha), \psi^\beta(\bar{f}_\beta)\} = A^\dagger A + AA^\dagger \quad A := \psi^\beta(\bar{f}_\beta),$$

and this expression fulfills

$$\langle \Omega | (A^\dagger A + AA^\dagger) | \Omega \rangle = \langle A\Omega | A\Omega \rangle + \langle A^\dagger \Omega | A^\dagger \Omega \rangle \geq 0.$$

Since $\{\bar{\psi}_{\alpha}(f^\alpha), \psi^\beta(\bar{f}_\beta)\}$ is equal to the smeared causal propagator times the unit operator 1, and $c1$ has positive expectation values iff $c > 0$, the quantization of the free Rarita-Schwinger field on Minkowski spacetime is only consistent if $i$ times the causal propagator is a positive semidefinite distribution on test sections corresponding to “physical degrees of freedom”, i.e.

$$iG^\beta_\alpha(\bar{f}_\beta, f^\alpha) \geq 0 \quad \forall f^\alpha \in V_0(RM).$$

It is important to require this condition for all test sections $f^\alpha$, since $\psi^\beta(\bar{f}_\beta)$ for different $f^\alpha$ represent different operators in general. In more detail, if $|\Omega\rangle$ is the vacuum state, $\psi^\beta(\bar{f}_\beta)|\Omega\rangle$ corresponds to a single particle state associated to the classical wave packet $\psi^\alpha = G^\beta_\alpha f^\beta$. Hence, the consistency condition discussed above is loosely equivalent to demanding that all wave packet quantum states have positive norm (they may have zero norm if the corresponding classical wave packet $\psi^\alpha$ is vanishing).

We shall now discuss why the causal propagator bears the positivity property required by consistency in Minkowski spacetime and at the same time obtain a more hands-on
understanding of why the smeared anticommutator of a Fermionic quantum field should have positive expectation values. By (6), we have to check if

\[ -\int_{\mathbb{R}^3} d\vec{x} \left[ G^\mu_{\beta f} \right] (t, \vec{x}) \gamma^0 \left[ G^\alpha_{\nu f'} \right] (t, \vec{x}) \geq 0 \quad \forall f^\alpha \in V_0(RM), \]

i.e. if

\[ -\int_{\mathbb{R}^3} d\vec{x} \bar{\psi}_\alpha(t, \vec{x}) \gamma^0 \psi^\alpha(t, \vec{x}) \geq 0 \]

for all wave-packet solutions \( \psi_\alpha \) of \( \mathcal{R}_0 \psi^\alpha = 0 \) which fulfil \( \psi = 0 \). Since the Rarita-Schwinger equation is diagonal in the Lorentz-index, every \( \psi^\alpha \in S(\mathcal{R}_0, RM) \) can be expanded as

\[ \psi^\alpha(t, \vec{x}) = \int_{\mathbb{R}^3} d\vec{k} \hat{\psi}_{\alpha,i,\vec{k}}^0 \psi^\alpha_{i,\vec{k}}(t, \vec{x}), \]

where \( \hat{\psi}_{\alpha,i,\vec{k}}^0 \) are coefficients rapidly decreasing in \( \vec{k} \) for large \( \vec{k} \) and \( \psi^\alpha_{i,\vec{k}}(t, \vec{x}) \) are orthonormal and complete modes of the Dirac field given in appendix [C]. These properties of \( \psi^\alpha_{i,\vec{k}}(t, \vec{x}) \) imply that

\[ -\int_{\mathbb{R}^3} d\vec{x} \bar{\psi}_\alpha(t, \vec{x}) \gamma^0 \psi^\alpha(t, \vec{x}) = -\int_{\mathbb{R}^3} d\vec{k} \sum_i (\hat{\psi}_{\alpha,i,\vec{k}}) \dagger \hat{\psi}_{\alpha,i,\vec{k}}. \]

We now recall that \( \psi = 0 \) implies \( \partial_\alpha \psi^\alpha = 0 \). Hence, the mode expansion coefficients of \( \psi^\alpha \) fulfil

\[ k^+_\alpha \hat{\psi}_{1,\vec{k}}^\alpha = k^+_\alpha \hat{\psi}_{2,\vec{k}}^\alpha = k^-\alpha \hat{\psi}_{3,\vec{k}}^\alpha = k^-\alpha \hat{\psi}_{4,\vec{k}}^\alpha = 0, \]

where

\[ (k^\pm)^\alpha = \left( \frac{\pm \omega}{\vec{k}} \right), \quad \omega = \sqrt{\vec{k}^2 + m^2}. \]

Let now \( m > 0 \), then \( (k^\pm)^\alpha \) is timelike, hence, \( \hat{\psi}_{i,\vec{k}}^\alpha \) must be spacelike for all \( i \) and \( \vec{k} \) on account of the linear independence of the Dirac modes. Consequently,

\[ -\int_{\mathbb{R}^3} d\vec{k} \sum_i (\hat{\psi}_{\alpha,i,\vec{k}}) \dagger \hat{\psi}_{\alpha,i,\vec{k}} > 0 \]

and we find that \( iG^\alpha_{\beta} \) is strictly positive on the constrained test sections. If \( m = 0 \), \( (k^\pm)^\alpha \) is lightlike and the coefficients \( \hat{\psi}_{i,\vec{k}}^\alpha \) are thus either spacelike or proportional to \( (k^\pm)^\alpha \). The constrained solutions \( \psi^\alpha \) with the latter property are precisely the “gauge solutions” \( \psi^\alpha = \partial^\alpha \chi \) briefly mentioned in the previous subsection. If one divides them out from the allowed solution space (which heuristically corresponds to removing “zero norm states”), \( iG^\alpha_{\beta} \) is strictly positive on the resulting space of test sections corresponding to gauge-equivalence classes of constrained solutions.
3 Conditions for a consistent canonical quantization in curved spacetimes

The discussion in the previous section on what canonical quantization of an elementary spin $\frac{3}{2}$ field means in flat spacetime can be directly taken over to curved spacetimes $(M, g_{\mu\nu})$, where $M$ is a smooth manifold endowed with a smooth Lorentzian metric $g_{\mu\nu}$. We consider a first order differential operator $\mathcal{R}$ on smooth sections $\psi^\alpha$ of $RM = DM \otimes TM$, where $TM$ and $DM$ are the tangent and Dirac spinor bundles of $(M, g_{\mu\nu})$ and a supplementary constraint on the contraction $\gamma^\alpha \psi^\alpha$ of solutions $\psi^\alpha$ of $\mathcal{R} \psi^\alpha = 0$, i.e. $\psi^\alpha$ is (locally) a function on $M$ carrying a tangent space index and a (suppressed) Dirac index. To canonically quantize the field theory related to $\mathcal{R}$ and the supplementary constraint, we impose canonical anticommutation relations

$$\{ \psi^\beta(x), \overline{\psi}_\alpha(y) \} = i G^\beta_\alpha(x, y)$$

specified by the causal propagator $G^\alpha_\beta = G^{-\alpha}_\beta - G^{+\alpha}_\beta$ of $\mathcal{R}$, provided that unique advanced and retarded Green’s operators $G^{\pm\alpha}_\beta$ exist for $\mathcal{R}$. Again, a quantum field $\psi^\beta(x)$ at a point $x$ is too singular to be a well-defined operator on some Hilbert space, and one rather has to consider fields integrated with test sections (i.e. compactly supported smooth sections)

$$\psi^\alpha(f_\alpha) := \int_M d_gx \tilde{f}_\alpha(x) \psi^\alpha(x), \quad \overline{\psi}_\alpha(f^\alpha) := \int_M d_gx \overline{\psi}_\alpha(x)f^\alpha(x) \quad f^\alpha \in \Gamma_0(RM),$$

where $d_gx$ denotes the metric-induced covariant measure on $M$, as candidates for proper Hilbert space operators. As the causal propagator $G^\alpha_\beta$ of $\mathcal{R}$ maps test sections $f^\alpha$ to wave-packet solutions of $\mathcal{R} \psi^\alpha = 0$ in a surjective manner, $\psi^\alpha(f_\alpha), \overline{\psi}_\alpha(f^\alpha)$ can be considered as the quantum field operators corresponding to the classical wave packet $\psi^\alpha = G^\beta_\alpha f^\beta$. These operators should be interrelated by Hermitean conjugation $\dagger$ as

$$(\psi^\alpha(f_\alpha))^\dagger := \overline{\psi}_\alpha(f^\alpha) \quad (\overline{\psi}_\alpha(f^\alpha))^\dagger := \psi^\alpha(f_\alpha),$$

thus consistency requires that their anticommutator has a positive expectation value; this, i.e. “unitarity of the quantum theory”, in turn is fulfilled only if

$$i G^\beta_\alpha(f_\beta, f^\alpha) = i (f^\beta, G^\beta_\alpha f^\alpha) = i \int_M d_gx f_\beta(x) \left[ G^\beta_\alpha f \right](x) \geq 0.$$

This condition on $i G^\beta_\alpha(f_\beta, f^\alpha)$ is, however, only required for $f^\alpha$ which are such that $G^\alpha_\beta f^\beta$ solves the supplementary constraint on $\gamma^\alpha G^\alpha_\beta f^\beta$, as we are only interested in quantizing the degrees of freedom corresponding to an elementary spin $\frac{3}{2}$ field, which in flat spacetimes corresponds to an irreducible representation of the Poincaré group specified by $\mathcal{R} = \mathcal{R}_0$.

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4It is well-known that a spin structure exists on all four-dimensional, globally hyperbolic curved spacetimes [Ger68, Ger70], such that $DM$ can always be constructed for such manifolds.
and the constraint $\psi = 0$. Note that $iG^{\beta}_{\alpha}(\bar{f}_\beta, g^\alpha) \geq 0$ on constrained test sections implies that $(\cdot, \cdot)$ defined as
\[
(G^{\alpha}_{\mu} f^\mu, G^{\beta}_{\nu} g^\nu) := iG^{\beta}_{\alpha}(\bar{f}_\beta, g^\alpha)
\]
gives a positive semidefinite product on constrained wave packet solutions of $\mathcal{R}\psi^\alpha = 0$.

From the above outline of the canonical quantization of a free Rarita-Schwinger field on general curved spacetimes we can already read off a few necessary conditions for such quantization to be well-defined. In the following, we complete and/or transform these conditions to a – as we will argue – complete set of sufficient conditions. Note that, strictly speaking, we are only dealing with algebraic conditions on the quantum field and it does not matter how the Hilbert space on which quantization to be well-defined. In the following, we complete and/or transform these conditions to a – as we will argue – complete set of sufficient conditions. Note that, strictly speaking, we are only dealing with algebraic conditions on the quantum field and it does not matter how the Hilbert space on which quantization to be well-defined.

We now state the anticipated conditions and comment on their individual relevance as well as their sufficiency afterwards.

**Definition 1** Let $\mathcal{R}$ be a first order differential operator on smooth sections $\Gamma(RM) \ni \psi^\alpha$ of $RM = DM \otimes TM$. Moreover, let $A_\alpha$ be a differential operator which maps smooth sections of $RM$ to smooth sections of $DM$ and let $S(\mathcal{R}, M)$ and $\mathcal{I}(\mathcal{R}, M)$ be defined as
\[
S(\mathcal{R}, M) := \{\psi^\alpha \in \Gamma(RM) \mid \mathcal{R}\psi^\alpha = 0 \text{ and } \psi^\alpha \text{ has compact support on any Cauchy surface}\}
\]
\[
\mathcal{I}(\mathcal{R}, M) := \{\psi^\alpha \in S(\mathcal{R}, M) \mid \psi = A_\alpha \psi^\alpha\}.
\]
We say that $\mathcal{R}$ and $A_\alpha$ lead to a consistent canonical quantum field theory of an elementary spin $\frac{3}{2}$ field on $(M, g_{\mu\nu})$, if the following conditions are satisfied.

**(Irreducibility)** In Minkowski spacetime, $A_\alpha \equiv 0$ and $\mathcal{I}(\mathcal{R}, M) \subset \mathcal{I}(\mathcal{R}_0, M)$.

**(Covariance)** $\mathcal{I}(\mathcal{R}, M)$ is locally covariant, i.e. for every globally hyperbolic region $(M', g_{\mu\nu}|_{M'}) \subset (M, g_{\mu\nu})$ whose causal structure is independent of $(M, g_{\mu\nu})$ outside of $(M', g_{\mu\nu}|_{M'})$, $\mathcal{I}(\mathcal{R}, M')$ is independent of $(M, g_{\mu\nu})$ outside of $(M', g_{\mu\nu}|_{M'})$, i.e. $\mathcal{I}(\mathcal{R}, M') = \mathcal{I}(\mathcal{R}, M)|_{M'}$. Moreover, either $A_\alpha \equiv 0$ on all spacetimes, or $\psi = A_\alpha \psi^\alpha$ is automatically fulfilled for all solutions, viz. $\mathcal{I}(\mathcal{R}, M) = S(\mathcal{R}, M)$.

**(Causality)** $\mathcal{R}$ is hyperbolic and the constraint $\psi = A_\alpha \psi^\alpha$ is compatible with the equation of motion $\mathcal{R}\psi^\alpha = 0$ i.e. if initial conditions for $\mathcal{R}\psi^\alpha = 0$ fulfil the constraint, then the corresponding solution does as well.

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The condition has been emphasised and elevated to a principle in [HoWa01, BFV03].

By this we mean that every causal curve between two points in $(M', g_{\mu\nu}|_{M'})$ as a subspace of $(M, g_{\mu\nu})$ lies completely in $M'$, such that the embedding of $(M', g_{\mu\nu}|_{M'})$ into $(M, g_{\mu\nu})$ does not add new causal relations between points in $M'$.
(Selfadjointness) \( \mathcal{R} \) is formally selfadjoint w.r.t. \( \langle \cdot, \cdot \rangle \), i.e. \( \langle \mathcal{R}^\dagger g^\beta, f^\alpha \rangle = \langle \mathcal{R} g^\beta, f^\alpha \rangle \) with \( \mathcal{R}^\dagger \) defined as
\[
\langle \mathcal{R}^\dagger g^\beta, f^\alpha \rangle = \int_M d^g x \overline{g}_\alpha(x) \mathcal{R} f^\alpha(x) = \langle g^\beta, \mathcal{R} f^\alpha \rangle
\]
for all \( f^\alpha, g^\beta \) in \( \Gamma_0(\mathcal{R}M) \).

Let us first comment on (Causality). This condition avoids that solutions of \( \mathcal{R} \) propagate acausally or have no sensible propagation behaviour at all. Additionally, it assures the existence of unique advanced and retarded propagators \( G^{\pm \alpha}_\beta \) of \( \mathcal{R} \), such that covariant canonical anticommutation relations can be formulated at all by means of \( G^\alpha_\beta = G^{-\alpha}_\beta - G^{+ \alpha}_\beta \). (Causality) is in particular fulfilled if there exists a fiducial operator \( \bar{\mathcal{R}} \) such that \( \bar{\mathcal{R}} \mathcal{R} \) is a normally hyperbolic operator, i.e. a wave operator \cite{BGP07, Muh10}, but the more general condition that \( \mathcal{R} \) be hyperbolic is sufficient for \( \mathcal{R} \) to have a good solution theory, see e.g. \cite{CoHi89, CDD96, Hör94}. Without going into details, we briefly mention that hyperbolicity is a condition on the principal symbol – the coefficient of the highest derivative – of a differential operator.

Additionally, the first two of the above conditions assure that the field theory defined by \( \mathcal{R} \) and \( A_\alpha \) is a covariant generalisation of the free Rarita-Schwinger field theory on Minkowski spacetime. Particularly, we want to assure that the considered field theory does not contain more (or less!) physical degrees of freedom than those possessed by a free Rarita-Schwinger field theory (with \( m \geq 0 \)) on flat spacetime, and that we can analyse these physical degrees of freedom in an arbitrarily small region of a spacetime without knowing what the spacetime looks like far away from this arbitrarily small region. One may think that simply writing down the Rarita-Schwinger equation and the constraint as covariant tensor equations is enough to achieve this. That this is not the case and what can go wrong will become clear when we discuss the minimally coupled Rarita-Schwinger equation in the next section.

Actually, one can prove that (Causality) (if it holds on all spacetimes) already implies (Covariance). Nonetheless, we have asked that (Covariance) holds explicitly in order to emphasize its physical meaning and because it turned out that it is easier to disprove the part of (Causality) which demands the compatibility of constraint and equation of motion indirectly by disproving (Covariance) instead of disproving that compatibility directly. In a similar spirit, the additional requirement posed in (Covariance), namely, that \( A_\alpha \equiv 0 \) or \( \S(\mathcal{R}, M) = S(\mathcal{R}, M) \), is in principle stronger than the local covariance we would like to achieve, but we have not been able to prove that the constraint \( \psi = A_\alpha \psi^\alpha \) fulfils local covariance except in these two special cases. Indeed, if \( \mathcal{R} \) is a covariant differential operator and (Causality) holds, then \( S(\mathcal{R}, M) \) is locally covariant, because initial conditions in \( M' \subset M \) completely determine elements of \( S(\mathcal{R}, M') \); given that \( S(\mathcal{R}, M) \) is locally covariant, \( \S(\mathcal{R}, M) \) is locally covariant as well if one of the two required

\footnote{See \cite{BFV03} for an in-depth discussion of the physical idea behind local covariance.}
conditions on $A_{\alpha}$ is fulfilled, as both are trivially “spacetime-independent”. Hence, with our current understanding, the additional conditions on $A_{\alpha}$ posed in (Covariance) are only sufficient, but not necessary. However, it is probably not possible to construct a different covariant $A_{\alpha}$ which is such that the dimension of the solution space of $\psi = A_{\alpha}\psi^\alpha$ is spacetime-independent; it is certainly not possible to choose e.g. $\nabla_{\alpha}R$ with $R$ the scalar curvature.

One might hope that abandoning (Irreducibility) and allowing for a non-trivial coupling of the components with spin $\frac{3}{2}$ and spin $\frac{1}{2}$ of $\psi^\alpha \in \Gamma(DM \otimes TM)$ simplifies the situation. However, we will see in the next section that two examples of operators which fulfil (Covariance), (Causality), and (Selfadjointness), but not (Irreducibility), do not lead to a consistent quantum field theory. Of course, a composite field of spin $\frac{3}{2}$, e.g. a tensor product of a Dirac field and a vector field, can be quantized consistently. But this composite field fulfils a differential equation of mixed (second and first) order, and we are only considering first order operators on $\Gamma(DM \otimes TM)$ in this treatment (this also rules out a tensor product of three Dirac fields, which can certainly also be quantized in a consistent manner).

We finally comment on (Selfadjointness). As we have seen in the last subsection, the relation between equal-time and covariant CAR is essential for the proof that the free Rarita-Schwinger field in Minkowski spacetime can be canonically quantized in a consistent way. This relation in turn relies on the fact that $R_0$ is formally selfadjoint. Indeed, as discussed in appendix [D] formal selfadjointness of a first order differential operator $R$ implies that the principal symbol $\sigma^\mu$ of $R$ defines a covariantly conserved current

$$j^\mu \left[ \psi^\alpha_1, \psi^\beta_2 \right] = \overline{\psi}_{\alpha,1}^\sigma \psi_{2}^\sigma$$

and that the smeared causal propagator $iG^\alpha_\beta(f^\beta, g^\alpha)$ is just the “charge” corresponding to this current, i.e. the time-component of $j^\mu$ integrated over an equal-time surface. We are not aware of any way to assure the duality between equal-time and covariant CAR without using formal selfadjointness of the considered first order differential operator $R$, but, although it may be awkward to abandon this duality, one might consider the possibility that only covariant CAR can be implemented and $iG^\alpha_\beta(f^\beta, g^\alpha) \geq 0$ can be proven even if $R$ is not formally selfadjoint. However, our results in appendix [E] demonstrate the tight relation between positivity of $iG^\alpha_\beta(f^\beta, f^\alpha)$ and formal selfadjointness of $R$. On the one hand, we are able to prove that selfadjointness implies positivity on topologically trivial curved spacetimes, if positivity on Minkowski spacetime is already known. On the other hand, we prove a weak converse of this: if positivity holds, then the Green’s operators of $R$ and its formal adjoint $R^\dagger$ coincide on “constrained test sections”, i.e. test sections $f^\alpha$ which via $G^\alpha_\beta$ correspond to solutions $\psi^\alpha$ of $R\psi^\alpha = 0$ that fulfil the constraint $\psi = A_{\alpha}\psi^\alpha$. Moreover, we can prove that, if positivity holds and the principal symbol $\sigma^\mu$ is covariantly conserved, then the current $j^\mu$ constructed out of the principal symbol of $R$ must be conserved. Hence, although we don’t have a full proof that selfadjointness of $R$ is necessary for the unitarity of the canonical quantum theory associated to $R$, we
altogether consider selfadjointness to be an essential ingredient.

Note that, in principle, it is sufficient to prove selfadjointness of $\mathcal{R}$ only on the constrained test sections mentioned above. However, as we have not been able to characterise these explicitly and without knowing selfadjointness \textit{a priori}, we require the sufficient condition that selfadjointness holds on all test sections. This holds for e.g. the Rarita-Schwinger-operator $\mathcal{R}_0$ on Minkowski spacetime and for the Dirac operator on all curved spacetimes. Moreover, if $\mathcal{I}(\mathcal{R}, M) = S(\mathcal{R}, M)$, as optionally required in (Covariance), then all test sections trivially belong to the class of constrained test sections.

To establish that the four introduced conditions are also sufficient for a consistent quantization on spacetimes of arbitrary topology one could maybe employ a partition-of-unity argument similar to the one used in \cite{DaLa11} to analyse the commutator of the vector potential in topologically non-trivial spacetimes.

4 (Modified) Rarita-Schwinger equations present in the literature and their drawbacks

We shall now comment on various versions of the Rarita-Schwinger equation on curved spacetimes present in the literature, and remark why they fail to satisfy one or several consistency conditions formulated in definition 1.

4.1 The original Rarita-Schwinger operator with $\psi = 0$

The minimally coupled original Rarita-Schwinger operator $\mathcal{R}_0$ on $RM$ is defined as

$$\mathcal{R}_0 := -i\nabla + m \mathbb{1}$$

and supplemented by the constraint

$$\psi = 0,$$

where $\nabla$ is the covariant derivative associated to the Levi-Civita connection on $DM \otimes TM$. $\mathcal{R}_0$ and $A_\alpha \equiv 0$ manifestly fulfils (Irreducibility). Moreover, the hyperbolicity bit of (Causality) is fulfilled as $(i\nabla + m)\mathcal{R}_0$ is a wave operator, and one can easily verify that (Selfadjointness) holds as well. However, (Covariance) does not hold, as one can check by a direct computation. Namely, using $\psi = 0 \land \mathcal{R}_0\psi^\alpha = 0 \Rightarrow \nabla_\alpha\psi^\alpha = 0$ and the curvature identities listed in section \cite{Buc58} we can compute

$$\nabla_\alpha(-i\nabla + m)\psi^\alpha = 0 \Rightarrow \nabla_\alpha\nabla\psi^\alpha = 0 \Rightarrow [\nabla_\alpha, \nabla]\psi^\alpha = 0 \Rightarrow R_{\alpha\beta\gamma}^\beta\psi^\alpha = 0.$$

Unless the spacetime $(M, g_{\mu\nu})$ is an Einstein manifold, i.e. unless $R_{\mu\nu} \equiv \frac{1}{4}R g_{\mu\nu}$, the latter identity can only hold if $\psi^\alpha = 0$, hence, $\mathcal{R}_0\psi^\alpha = 0 \land \psi = 0$ has only the trivial solution on general spacetimes $(M, g_{\mu\nu})$. This phenomenon is well-known since the works of Buchdahl \cite{Buc58} and usually called ‘inconsistency of the classical field equation’. Moreover, this is in conflict with (Covariance) as one can see by considering the following gedankenexperiment: we image that we are living in a region of spacetime, where $R_{\mu\nu}$ =
\( \frac{1}{4} R_{\mu\nu} \) holds, e.g. in a locally Ricci-flat region, and we ask ourselves if non-trivial solutions of \( R_\alpha \psi^\alpha = 0 \wedge \psi = 0 \) exist. There is no way to unambiguously answer this question by knowing only our local Ricci-flat spacetime region. If there is an arbitrarily small and arbitrarily distant region in spacetime, where \( R_{\mu\nu} \neq \frac{1}{4} R g_{\mu\nu} \), then \( S(\mathcal{R}_0, M) \) contains only the trivial element, but if \( R_{\mu\nu} \equiv \frac{1}{4} R g_{\mu\nu} \) on the full spacetime we are living in, then \( S(\mathcal{R}_0, M) \) is as large as it is suitable for the solution space of the equations describing an elementary field of spin \( \frac{3}{2} \).

4.2 The original Rarita-Schwinger operator without \( \psi = 0 \)

Since in the previous example, the constraint \( \psi = 0 \) has been used in the derivation of \( R_\alpha \gamma^\beta \psi^\alpha = 0 \) for all solutions of \( R_\alpha \psi^\alpha = 0 \), abandoning this constraint immediately avoids this very strong restriction on solutions or on the background spacetime [Müh11]. Indeed this setup fulfills \((\text{Covariance}), (\text{Causality}), \text{and} (\text{Selfadjointness}) \) [Müh11], but obviously fails to satisfy \((\text{Irreducibility}) \). Hence, looking at the proof of unitarity in Minkowski, one can easily find elements \( \psi^\alpha \) of \( S(\mathcal{R}_0, M) \) which fail to fulfil

\[
- \int d\bar{x} \bar{\psi}_\alpha(t, \bar{x}) \gamma^0 \psi^\alpha(t, \bar{x}) \geq 0 ,
\]
e.g. all \( \psi^\alpha \) whose mode expansion coefficients \( \hat{\psi}^\alpha_{i,\vec{k}} \) are timelike for all \( i, \vec{k} \) would do.

4.3 The supergravity Rarita-Schwinger operator

Let us now consider the minimally coupled version of \( \mathcal{R}_1 \), i.e. the equation

\[
\mathcal{R}_1 \psi^\alpha := - (i \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta \nabla_\gamma + m \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta ) \psi^\beta = 0 ,
\]

Following the route taken in the analysis of this operator on Minkowski spacetime, one can contract the above equation with both \( \nabla_\mu \) and \( \gamma_\mu \) to obtain derived identities satisfied on shell. Upon doing so, one finds that this differential equation can be equivalently expressed as

\[
\tilde{\mathcal{R}}_1 \psi^\alpha := ( - i \nabla + m ) \psi^\alpha + \left( i \nabla^\alpha + m \frac{2}{2} \gamma^\alpha \right) \psi = 0 ,
\]

and that solutions of this equation satisfy

\[
3m^2 \psi = G_{\mu\nu} \gamma^\mu \psi^\nu \quad \nabla_\mu \psi^\mu = \left( \nabla - \frac{3im}{2} \right) \psi ,
\]

with \( G_{\mu\nu} \) denoting the Einstein tensor \( R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} \). From this we can immediately infer that this equation does not satisfy \((\text{Covariance}) \), since \( 3m^2 \psi = G_{\mu\nu} \gamma^\mu \psi^\nu \) is identically fulfilled if \( G_{\mu\nu} \equiv 3m^2 g_{\mu\nu} \), but restricts the solution space otherwise [DeZu76, Tow77].
Furthermore, the above equation satisfies (Irreducibility) only if \( m > 0 \), but \( R_1 \) satisfies (Selfadjointness) although \( \tilde{R}_1 \) does not.\(^8\)

However, there are difficulties with (Causality), as already pointed out by Velo and Zwanziger in [VeZw69]. They have analysed the equation \( R_1 \psi^\mu = 0 \) in the context of a classical electromagnetic background field rather than on a classical curved background, but their results also hold in the latter case. We will not perform the computations necessary to see the failure of (Causality) for \( R_1 \) at this point since we will present the same calculations in a more general context in the next section. Within supergravity, the failure of \( R_1 \) to satisfy (Causality) can be overcome by choosing a gauge differing from the usually employed “unitary gauge” [HMS12].

### 4.4 The projected Rarita-Schwinger operator on \( RM \) mod \( \psi = 0 \)

This operator is the version of the Rarita-Schwinger operator which is usually discussed in the rather mathematically oriented literature, e.g. [BäGi11], see also [FrSp95] for a generalisation to arbitrary half-integral spin. It can be constructed by implementing the condition \( \psi = 0 \) directly into the underlying vector bundle. This is achieved by defining the (pointwise) projector \( P \) as

\[
P\psi^\alpha = \psi^\alpha - \frac{1}{4} \gamma^\alpha \psi,
\]

and considering the (with obvious notation) bundle \( PRM := P(DM \otimes TM) \) rather than \( RM = DM \otimes TM \). In order to obtain a well-defined field theory on \( PRM \), one has to make sure that the considered differential operator \( R \) maps \( \Gamma(PRM) \) into itself rather than into \( \Gamma(RM) \). The latter is indeed the case for \( R_0 \), but this can be easily cured by considering the operator \( R_P := P \mathcal{R}_0 \) instead. Hence,

\[
R_P \psi^\alpha = (-i \nabla + m) \psi^\alpha + \frac{i}{2} \gamma^\alpha \nabla_\mu \psi^\mu = 0
\]

is in principle a well-defined field equation for \( \psi^\alpha \in \Gamma(PRM) \). Moreover, the authors of [BäGi11, FrSp95] verify that \( R_P \) fulfils (Causality) and in [BäGi11] it is remarked that \( R_P \) fulfils (Selfadjointness), but that the current \( j^\mu[\psi^\alpha, \psi^\alpha] \) built out of the principal symbol \( \sigma^\mu \) of \( R_P \) does not give a positive result once integrated over a Cauchy surface. However, the authors of [BäGi11] require that such positivity holds for arbitrary \( \psi^\alpha \in \Gamma(PRM) \), whereas for a unitary quantum theory it only has to hold for solutions of \( R_P \psi^\alpha = 0 \). Notwithstanding, as \( R_P \) does not fulfil (Irreducibility), one can not repeat the positivity proof obtained for \( R_0 \) in Minkowski spacetime. Although one can very well repeat it

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\(^8\)In a previous version of this work we have argued that supergravity fails to satisfy unitarity on the basis of the derived operator \( \tilde{R}_1 \) and proposition\[^6\]. However, as the “true” supergravity Rarita-Schwinger operator \( R_1 \) does satisfy (Selfadjointness), that statement was physically void albeit being mathematically correct. Notwithstanding, the unitarity of supergravity is – to our knowledge – yet unproved, see the comments in section\[^6\].
for the solutions of $\mathcal{R}_P\psi^\alpha = 0$ which satisfy $\partial_\alpha \psi^\alpha = 0$ in addition, one can employ the mode expansion of solutions of the Dirac equation in Minkowski spacetime to explicitly construct a mode expansion for solutions of $\mathcal{R}_P\psi^\alpha = 0$ which do not satisfy $\partial_\alpha \psi^\alpha = 0$ and find examples for which $j^\mu[\psi^\alpha, \psi^\alpha]$ integrated over an equal-time surface gives a negative result.

4.5 The Buchdahl-Rarita-Schwinger operator with $\psi = 0$

This operator has been constructed by Buchdahl in [Buc82] and is in fact the first operator in the enumeration of this section which contains a non-minimal coupling to the background curvature. Buchdahl’s idea was to modify the minimally coupled operator $\mathcal{R}_0$ in such a way that the equation $\mathcal{R}_{\mu\nu}\gamma^\mu\psi^\nu = 0$ resulting from $\mathcal{R}_0\psi^\alpha = 0$ obtains a suitable ‘right hand side’ which assures that it is identically fulfilled on all curved spacetimes. To this avail, one starts with the ansatz

$$\mathcal{R}_B\psi^\alpha := (-i\nabla + m)\psi^\alpha + \tilde{\psi}^\alpha = 0 \quad \psi^\alpha \in \Gamma(RM),$$

where $\tilde{\psi}^\alpha$ is the sought-for non-minimal coupling term. Contracting this equation with $\nabla_\alpha$ and $\gamma_\alpha$, combining the results, and using the spin curvature identities in section 3, one obtains the following differential equation for $\tilde{\psi}^\alpha$

$$(-i\nabla + m)\tilde{\psi}^\alpha + 2i\nabla_\alpha \tilde{\psi}^\alpha = \mathcal{R}_{\mu\nu}\gamma^\mu\psi^\nu.$$

Realising that this is a $DM$-valued equation, one makes the ansatz that $\tilde{\psi}^\alpha$ is completely determined by a Dirac spinor $B$, i.e.

$$\tilde{\psi}^\alpha := (a\nabla^\alpha + 2i\gamma^\alpha\nabla + cm\gamma^\alpha) B.$$

Inserting this into the equation for $\tilde{\psi}^\alpha$, one obtains

$$\left\{ (2b - a)\Box + \frac{(2b + a)}{4} R + 4cm^2 + (a + 4b - 2c)im\nabla \right\} B = \mathcal{R}_{\mu\nu}\gamma^\mu\psi^\nu.$$

If this equation is not an algebraic equation for $B$, it will not assure that $B = 0$ in Minkowski spacetime, and the resulting operator would not fulfil (Irreducibility). Hence, one chooses $a = 2b, c = 3b$ and finds that

$$\tilde{\psi}^\alpha = (2i\nabla^\alpha + i\gamma^\alpha\nabla + 3m\gamma^\alpha) \frac{\mathcal{R}_{\mu\nu}\gamma^\mu\psi^\nu}{12m^2 + R}$$

is the only solution compatible with (Irreducibility). Note that the resulting operator is well-defined also for $m = 0$, although the original derivation in [Buc82] has assumed $m > 0$. However, if $m = 0$, $\mathcal{R}_B$ is not analytic in the spacetime curvature. $\mathcal{R}_B$ fulfils (Irreducibility) and (Covariance) by construction, but one can compute that (Selfadjointness) and (Causality) do not hold on general spacetimes. Again, we refer the reader interested in computational details to the next section.
4.6 The Buchdahl-Wünsch operator on BM

The Buchdahl-Wünsch operator $\mathcal{R}_{BW}$ is constructed on a bundle differing from $RM$, the Buchdahl-bundle $BM$. We shall not go into details here, but only mention that this bundle corresponds to the representation $(\frac{3}{2},0) \oplus (1,\frac{1}{2})$ of $SL(2,\mathbb{C})$ – we refer the reader interested in details to [Buc82, Wün85] and [Müh07, Mak11, BäGi11] for a recent review.

Much like in the case of the projected Rarita-Schwinger operator on $RM$ mod $\psi = 0$, $BM$ has an irreducibility condition similar to the constraint $\psi = 0$ built in and $\mathcal{R}_{BW}$ can be understood as the projection of a fiducial operator to $\Gamma(BM)$; this projection is only possible for $m > 0$. $\mathcal{R}_{BW}$ on $\Gamma(BM)$ fulfils (Covariance) and (Irreducibility) (as discussed above, in a generalised sense) and can be shown to fulfil (Causality) as well [IlSch99, Müh07, Müh10]. However, it does not fulfil (Selfadjointness) as observed in [Mak11]. The problem here is that any canonical product on $\Gamma_0(BM)$ which corresponds to

$$\langle f^\alpha, g^\beta \rangle = \int_M d_g \bar{f}_\alpha g^\beta$$

on $\Gamma_0(RM)$ has to include two covariant derivatives in order to make sure that “all free indices are contracted in a covariant way”. Consequently, there are three canonical products on $\Gamma_0(RM)$, corresponding to the three possibilities to distribute two covariant derivatives among two sections. All, this does not pose a problem in flat spacetime, where partial derivatives commute, hence, all canonical products on $\Gamma_0(BM)$ and the one on $\Gamma_0(RM)$ are equivalent in this case [Mak11]. However, in curved spacetimes, covariant derivatives do not commute, consequently, the operator $\mathcal{R}_{BW}$ is likely not to satisfy (Selfadjointness) on $\Gamma_0(BM)$ endowed with any of the canonical products. Indeed, in [Mak11] it has been shown that $\mathcal{R}_{BW}$ is not formally selfadjoint with respect to the canonical product where the field on the right is differentiated twice on $\Gamma_0(BM)$ by proving that a necessary condition for (Selfadjointness) to hold, the covariant conservation of the canonical current associated to $\mathcal{R}_{BW}$ and the chosen product on $\Gamma_0(BM)$, is only met in spacetimes of constant curvature.

5 A large class of modified Rarita-Schwinger equations and a no-go theorem for their consistent quantization

After posing sufficient conditions for a consistent canonical quantization in definition 11 and discussing several counterexamples, we proceed to the main goal of this paper, i.e. proving that a large class of first order differential operators $\mathcal{R}$ on $\Gamma(RM)$ fails to satisfy all four conditions (Covariance), (Irreducibility), (Selfadjointness), and (Causality). In the course of proving this no-go theorem, it will become clear that the proof can be extended to any larger class of operators without much effort, such that the class we shall consider can be safely regarded as effectively exhausting all possible covariant first order differential operators on $\Gamma(RM)$. Our proof does not cover operators on the projected bundle $PRM$ (cf. subsection 4.4), since any such operator must be equal to $\mathcal{R}_P$ on flat
spacetime due to the requirement that it maps $\Gamma(RP\mathbb{M})$ to itself, and we have seen that already $R_P$ itself does not fulfill (Irreducibility) in subsection 4.4.

**Theorem 2** Let $\mathcal{R}$ be a differential operator on $\Gamma(RM) \ni \psi^\alpha$ of the form

$$\mathcal{R}\psi^\alpha := (-i\nabla + m)\psi^\alpha + a_0 m\gamma^\alpha \psi + a_1 i\nabla^\alpha \psi + a_2 i\gamma^\alpha \nabla_\mu \psi^\mu + a_3 i\gamma^\alpha \nabla \psi + \tilde{\psi}^\alpha$$

where $a_i \in \mathbb{C}$ are arbitrary constants and $\tilde{\psi}^\alpha$ contains the following explicit non-minimal curvature coupling:

$$\tilde{\psi}^\alpha := m\gamma^\alpha B + mC^\alpha + iD^\alpha + i\gamma^\alpha E$$

$$B := b_1 R_{\mu\nu}\gamma^\mu \psi^\nu + b_2 R\psi$$

$$C^\alpha := c_1 R^\alpha_{\mu\nu}\psi^\nu + c_2 R^\alpha_{\nu}\gamma^\nu \psi^\nu + c_3 R\psi^\nu + c_4 R^\alpha_{\nu}\psi^\nu$$

$$D^\alpha := d_1 R^\alpha_{\nu}\psi^\nu + d_2 (\nabla^2 R) \psi^\nu + d_3 R^\alpha_{\nu}\gamma^\nu \psi^\nu + d_4 (\nabla^2 R) \psi^\nu + d_5 \nabla^\alpha R^\mu_{\nu} \psi^\mu + d_6 (\nabla^2 R) \psi^\nu + d_7 \nabla^\alpha R^\mu_{\nu} \psi^\mu$$

$$E := e_1 R^\mu_{\nu}\gamma^\nu \nabla \psi^\mu + e_2 R^\mu_{\nu} \gamma^\nu \psi^\mu + e_3 R^\mu_{\nu} \nabla \psi^\mu + e_4 (\nabla^2 R) \psi^\nu + e_5 (\nabla^2 R) \psi^\nu + e_6 R^\mu_{\nu} \nabla \psi^\nu$$

Here, derivatives in brackets are meant to act only on the curvature tensors in the brackets, and $b_1, c_1, d_1, e_1$ are arbitrary complex-valued functions of curvature invariants and $m$ of mass dimension $-2$. No such $\mathcal{R}$ fulfils all four conditions (Irreducibility), (Covariance), (Causality), and (Selfadjointness).

**Proof.** We start by checking (Selfadjointness), since this turns out to be the strongest condition. Indeed, as one can check by direct computation, (Selfadjointness) is fulfilled on arbitrary curved spacetimes iff the following conditions hold.

$$a_0^* = a_0 \quad a_2 = a_1^* \quad a_3^* = a_3 \quad b_1 = c_2^* \quad b_2 = b_2 \quad c_1^* = c_1 \quad c_3 = c_3 \quad c_4^* = c_4$$

$$d_1 = d_3 = d_5 = d_7 = d_9 = d_{10} = d_{12} = d_{15} = e_1 = e_3 = e_6 = e_8 = e_9 = 0$$

$$d_2^* = d_2 \quad d_4^* = e_2 \quad d_6^* = d_6 \quad d_8 = e_5^* \quad d_{11} = e_7^* \quad d_{13}^* = d_{14} \quad e_4^* = e_4$$

Here, $\ast$ denotes complex conjugation. In essence, requiring (Selfadjointness) rules out terms where a curvature tensor multiplies a derivative of $\psi^\alpha$, because such terms generate derivatives of curvature tensors by the partial integration involved in the definition of the formal adjoint of $\mathcal{R}$. These curvature tensor derivatives can not be cured by explicitly adding couplings of $\psi^\alpha$ to curvature derivatives, as such terms must be present both in $\mathcal{R}$ and in $\mathcal{R}$. Hence, (Selfadjointness) rules out arbitrary terms where a curvature

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9Note that all couplings containing the Riemann tensor $R_{\alpha\beta\mu\nu}$ can be expressed via the spin curvature tensor $\mathfrak{R}_{\alpha\beta}$. Furthermore, we have omitted all couplings which would be linearly dependent by means of Bianchi identities.
tensor multiplies a derivative of \(\psi^\alpha\), extending the validity of this proof to a larger class of \(\mathcal{R}\) containing all possible such terms. Moreover, the remaining terms in \(\mathcal{R}\) allowed by (Selfadjointness) must be either “symmetric” themselves or appear in a “symmetrised” fashion. Altogether one sees why the Buchdahl-Rarita-Schwinger-operator \(\sigma\) e.g. \(\text{BaeGi11, remark 2.27}\) that the principal symbol of \(\mathcal{R}\) is an invertible linear map if contracted with timelike or spacelike \(k^\mu\), but is non-invertible once contracted with a lightlike \(k^\mu\). Let \(k^\mu\) be timelike or spacelike and let \(\psi^\alpha \in \Gamma_0(RM)\) fulfill
\[
 i k^\mu\sigma^\mu\psi^\alpha = k^\mu\psi^\alpha - a_1k^\alpha\psi - a_2\gamma^\alpha k^\mu\psi^\mu - a_3\gamma^\alpha k^\mu\psi^\mu = 0,
\]
where we have already taken into account that the allowed principal symbols are reduced by (Selfadjointness). We have to check for which \(a_i\) the above equation implies \(\psi^\alpha \equiv 0\). By multiplying the above equation with \(k^\mu\) and \(k^\alpha\), we can obtain the following derived equations
\[
 (1 - a_2)k^\mu\psi^\mu = (a_1 + a_3)k^2\psi, \quad (1 - 3a_2)k^\mu\psi^\mu = (1 + 3a_3)k^2\psi,
\]
which we can rewrite as
\[
 \begin{pmatrix}
 (1 - a_2)k^2\psi & -(a_1 + a_3)k^2\psi \\
 (1 - 3a_2)k^2\psi & -(1 + 3a_3)k^2\psi
\end{pmatrix}
 = 0.
\]
As \(k^\mu\) is timelike or spacelike, this equation together with \(ik^\mu\sigma^\mu\psi^\alpha = 0\) implies \(\psi^\alpha \equiv 0\) iff the determinant of the appearing \(8 \times 8\) matrix is non-zero; this in turn is the case iff
\[
 -3a_1a_2 + a_1 + a_2 - 2a_3 - 1 \neq 0.
\]
We do not discuss lightlike \(k^\mu\), as \(8\) will be sufficient to prove the theorem\(^{10}\).

Finally, we check (Covariance) and (Irreducibility). To this avail, we contract \(\mathcal{R}\psi^\alpha = 0\) with both \(\gamma^\alpha\) and \(\nabla^\alpha\), combine the results and use the spin curvature identities listed in appendix \(\text{B}\) to obtain the following equation for \(\psi\).
\[
 - \left( \frac{(a_2 - 1)(1 + a_2 + 4a_3)}{2 - 4a_2} + a_1 + a_3 \right) \Box \psi
 + \left( \frac{(a_2 - 1)(1 + 4a_0)}{2 - 4a_2} + \frac{1 + a_1 + 4a_3}{2 - 4a_2} + a_0 \right) i m \nabla^\mu \psi
 + \left( \frac{(a_2 - 1)(1 + a_2 + 4a_3)}{2 - 4a_2} + a_3 \right) \frac{R}{4} \psi
 + \frac{1 + 4a_0}{2 - 4a_2} m^2 \psi - \frac{1}{2} R_{\mu \nu} \gamma^\mu \psi^\nu
 + \frac{a_2 - 1}{2 - 4a_2} i \nabla^\mu \psi^\mu + \frac{m}{2 - 4a_2} \tilde{\psi} = 0.
\]
\(^{10}\)From the mathematical point of view, hyperbolicity does not require that the notions of “timelike”, “spacelike” and “lightlike” must be the ones inferred from \(g_{\mu \nu}\), but they could be related to any Lorentzian metric \(g'_{\mu \nu}\) on \(M\). However, our discussion of \(k^\mu\sigma^\mu\) for \(k^\mu\) spacelike or timelike w.r.t. \(g_{\mu \nu}\) implies that \(8\) is a necessary condition for \(\mathcal{R}\) to be hyperbolic w.r.t. to any Lorentzian metric on \(M\).
Here, requiring (Irreducibility) assures that $2 - 4a_2 \neq 0$. To see this, note that contracting $\mathcal{R}_\psi^\alpha = 0$ with $\gamma_\alpha$ yields an equation which can be rewritten as

$$(2 - 4a_2)i\nabla_\mu \psi^\mu = (1 + a_1 + 4a_3)i\slashed{\nabla}\psi + (1 + 4a_6)m\psi + \tilde{\psi}.$$ \hspace{1cm} (10)

If $2 - 4a_2 = 0$, then $i\nabla_\mu \psi^\mu = 0$ would not follow from $\mathcal{R}_\psi^\alpha = 0$ and $\tilde{\psi} = 0$ on Minkowski spacetime, hence $\mathcal{S}(\mathcal{R}, \mathbb{M}) \subset \mathcal{S}(\mathcal{R}_0, \mathbb{M})$ would not hold.

To assure that (Covariance) holds, we have to either guarantee that $\psi^\alpha = A_\mu \psi^\mu$ holds automatically for solutions of $\mathcal{R}_\psi^\alpha = 0$ or that $A_\alpha \equiv 0$ on all spacetimes. Let us check if the first of these conditions can be fulfilled. Without specifying $A_\mu$ explicitly, we know that, in Minkowski spacetime, $A_\mu \equiv 0$ must hold on account of (Irreducibility). However, in flat spacetime, (9) is a hyperbolic partial differential equation for $\psi$, as the coefficient of $\Box \psi$ is non-zero if we apply the condition (8) derived from (Causality) and (Selfadjointness). Such a differential equation has certainly more possible solutions than just $\psi \equiv 0$, hence, by combining (Causality), (Selfadjointness), and (Irreducibility), we find that only the optional condition in (Covariance) that $A_\mu$ be identically vanishing on all spacetimes can be fulfilled. Inserting this into (9), we are left with

$$-\frac{1}{2} R_{\mu\nu} \gamma^\mu \psi^\nu - \frac{a_2 - 1}{2 - 4a_2} i\nabla_\mu \tilde{\psi} + i\nabla_\mu \psi^\mu + \frac{m}{2 - 4a_2} \tilde{\psi} = 0.$$ \hspace{1cm} (11)

In Minkowski spacetime, this equation is identically fulfilled and, hence, poses no additional constraints on solutions of $\mathcal{R}_\psi^\alpha = 0$ and $\tilde{\psi} = 0$. To check if (Covariance) holds, we have to make sure that (11) is identically fulfilled on all spacetimes once $\mathcal{R}_\psi^\alpha = 0$ and $\tilde{\psi} = 0$ hold. To this avail, we insert $\psi = 0$ into (10), and both $\psi = 0$ and (11) into $\mathcal{R}_\psi^\alpha = 0$ to obtain

$$i\nabla_\mu \psi^\mu = \frac{1}{2 - 4a_2} \tilde{\psi}, \quad (-i\slashed{\nabla} + m)\psi^\alpha + \frac{a_2}{2 - 4a_2} \gamma^\alpha \tilde{\psi} + \tilde{\psi}^\alpha = 0.$$ \hspace{1cm} (12)

These two equations are the only information on first derivatives of $\psi^\alpha$ which one can obtain from $\mathcal{R}_\psi^\alpha = 0$ and $\tilde{\psi} = 0$. However, the summand $\nabla_\mu \psi^\mu$ in (11) contains first derivatives of $\psi^\alpha$ also in terms like e.g. $R_{\mu\nu} \psi^\nu$, on which $\mathcal{R}_\psi^\alpha = 0$ and $\tilde{\psi} = 0$ give no information in general curved spacetimes. Hence, these terms must identically vanish in $\nabla_\mu \tilde{\psi}^\mu$, which implies that the coefficients of all terms in $\tilde{\psi}^\alpha$ which survive after inserting $\psi = 0$ and whose free index $\alpha$ does not belong to $\gamma^\alpha$ or $\psi^\alpha$ must vanish. Moreover the coefficients of all terms where $\gamma^\alpha$ appears followed by other $\gamma$-matrices must vanish as well, as these terms also give rise to terms like e.g. $R_{\mu\nu} \psi^\nu$ if one considers them in $\nabla_\mu \psi^\mu$ and commutes the contracted covariant derivative $\slashed{\nabla}$ with the additional $\gamma$-matrices in order to use the available information on $\nabla_\mu \psi^\alpha$. Analogously, the terms in $\tilde{\psi}^\alpha$ where the free index $\alpha$ belongs to $\psi^\alpha$ but $\psi^\alpha$ is multiplied by $\gamma$-matrices are problematic in $\slashed{\nabla} \tilde{\psi}$ and have to vanish identically. Altogether, avoiding the appearance of in general undetermined $\psi^\alpha$-derivatives in (11) enforces

$$b_1 = c_1 = c_4 = d_2 = d_6 = d_{13} = d_{14} = e_2 = e_7 = 0.$$
hence, the remaining terms in $\tilde{\psi}^\alpha$ not yet ruled out by (Covariance) are

$$\tilde{\psi}^\alpha = mc R \psi^\alpha + e_5 \gamma^\alpha (\nabla_\nu R) \psi^\nu.$$ 

We can now explicitly compute the left hand side of (11) by inserting this expression for $\tilde{\psi}^\alpha$ and the knowledge on $\nabla_\mu \psi^\mu$ and $\nabla_\nu \psi^\alpha$ obtained from $R \psi^\alpha = 0$ and $\psi = 0$. The result does not contain any derivatives of $\psi^\alpha$, but is a sum various curvature tensors multiplying $\psi^\alpha$. In general spacetimes, some of these terms are linearly independent and, hence, have to vanish individually in order for (11) to be identically fulfilled on all spacetimes. Particularly, since the only term in the left hand side of (11) containing the Ricci tensor turns out to be the one explicitly visible in (11), we obtain

$$R_{\mu \nu} \gamma^\mu \psi^\nu = 0$$

as a necessary condition for (11) to hold on general spacetimes. However, this is in conflict with (Covariance), which closes the proof.

One can imagine that the steps taken in the last paragraph of this proof can be generalised to arbitrary couplings of the curvature to $\psi^\alpha$, and we have argued in the discussion of (Selfadjointness) that the same holds for arbitrary couplings of the curvature to derivatives of $\psi^\alpha$, hence, we presume that our proof effectively exhausts all possible covariant first order differential operators on $\Gamma(RM)$. Finally, we would like to emphasise that our proof covers both $m > 0$ and $m = 0$.

### 6 Conclusions & outlook

In this work, we have been concerned with the issue of whether free spin $\frac{3}{2}$ field theories can be consistently quantized on curved spacetimes. Following the general philosophy of quantum field theory in curved spacetimes, we have investigated whether this is possible on arbitrary curved spacetimes, particularly, also on spacetimes which are not required to solve the Einstein equations. To this avail, we have introduced four conditions on linear spin $\frac{3}{2}$ field equations in curved spacetimes and shown they are sufficient to guarantee that the associated quantum field theory is consistent. Subsequently, we have analysed a large class of field equations obtained from the Rarita-Schwinger equation by the addition of various non-minimal couplings and have proven that no equation in this class fulfils all sufficient conditions; in the course of this proof it became clear that it can be extended to arbitrary non-minimal couplings without much effort. While our conditions are only sufficient, we have been able to obtain partial results which seem to indicate that they might be necessary. Moreover, although our treatment does not treat field equations in the Buchdahl-representation which can only be written in two-spinor notation, we have indicated why such equations are unlikely to evade our no-go theorem. If no consistent quantum theory of a free spin $\frac{3}{2}$ field on arbitrary curved spacetimes exists, this might imply that composite systems of spin $\frac{3}{2}$ can not be stable in arbitrary curved spacetimes.\[11\]

\[11\]We are grateful to Klaus Fredenhagen for pointing out this potential interpretation of the no-go theorem to us.
it would certainly be interesting to analyse this issue in more detail.

Quantized supergravity theories are quantum field theories on very specific curved spacetimes, i.e., on spacetimes which are solutions of the Einstein equations sourced by the stress-energy of the Bosonic background fields of the model under consideration [DeZu76, KKLP00]. Hence, our no-go theorem does not cover quantized supergravity theories, even though these contain (at least) one spin $\tfrac{3}{2}$ field, the gravitino. Quite on the contrary, one could even interpret our result as a hint that supergravity theories on their associated spacetime backgrounds might be the only consistent quantum field theories in curved spacetimes containing spin $\tfrac{3}{2}$ fields. However, to our knowledge, it is yet unclear in full generality whether quantized supergravity theories are consistent. Although they are known to be consistent in the idealized case of a flat Minkowski background [VNï81], there are potential problems in more realistic spacetimes like our universe. Firstly, the supergravity Rarita-Schwinger operator $\mathcal{R}_1$ is not hyperbolic, so there is no general proof that quantum supergravity theories are causal in general, although there are partial results on models in Robertson-Walker spacetimes [KKLP00, SchUh11]. We will indicate how this problem can be overcome by choosing a gauge differing from the usually employed “unitary gauge” in a forthcoming paper [HMS12]. Secondly, it is not clear how to prove in general that (even at lowest order in perturbation theory) quantum supergravity theories are unitary. Our method of proving that the formal selfadjointness of the supergravity Rarita-Schwinger operator $\mathcal{R}_1$ is sufficient for this can not be directly applied to the supergravity setting, because all considered spacetimes have to solve the Einstein equations there. However, in [SchUh11] the unitarity of a specific model on Robertson-Walker spacetime has been established by a direct computation. Finally, the situation is even worse if the stress-energy tensor entering the Einstein equations for the background spacetime is not only given by classical Bosonic background fields of the supergravity model, but also by quantum contributions from the state of the quantized fields, which is the realistic situation in view of dark matter and the cosmic microwave background radiation. In this case, even the causality of the theory is unclear and a complete analysis is quite cumbersome since it requires to solve the coupled system of equations determining the state of all quantum fields and all classical background fields including the metric at once. Therefore, even if supergravity theories are certainly appealing, there is still quite a number of steps to be undertaken in order to prove the consistency of such theories in realistic situations.

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A Dirac and Majorana spinors

By Pauli’s theorem, two different representations of the Clifford algebra $\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\eta^{\alpha\beta}$ are related by a similarity transformation, hence, there are matrices $\beta, C$ which fulfil

$$(\gamma^\alpha)^\dagger = \beta \gamma^\alpha \beta^{-1}, \quad (\gamma^\alpha)^T = -C \gamma^\alpha C^{-1},$$

where $^\dagger$ ($^T$) denotes the Hermitean adjoint (transpose) of a matrix. One can additionally fix $\beta^\dagger = \beta$, and, in a standard representation of the Clifford algebra (e.g. in the Dirac-, Majorana-, or Weyl-representation) with $(\gamma^0)^\dagger = \gamma^0$, $(\gamma^i)^\dagger = -\gamma^i$, set $\beta = \gamma^0 = \beta^{-1}$. We shall work with a standard representation throughout this work. Given $\beta$ and $C$, one can define the Dirac adjoint $\overline{\psi}$ and charge conjugated $\psi^c$ version of a Dirac spinor as

$$\overline{\psi} := \psi^\dagger \beta, \quad \psi^c := \beta C^\dagger \psi^*,$$

where $^*$ denotes complex conjugation and define the same operations on cospinors in such a way that $\overline{\psi} = \psi, (\psi^c)^c = \psi, (\overline{\psi})^c = -\overline{\psi}$ and the same identities hold for cospinors.

The above definitions of $\gamma^\alpha$, $\beta$, $C$ and the associated conjugations can be extended from Minkowski spacetime to a curved spacetime $M$ by employing a frame/tetrad/vielbein-basis of the tangent bundle $TM$ and Dirac bundle $DM$ of $M$; the resulting section of $\gamma$-matrices is covariantly constant with respect to the Levi-Civita covariant derivative, see e.g. [San09, Hac10, Müh11]. Furthermore, the definitions of the Dirac and charge conjugation can be straightforwardly extended to Rarita-Schwinger spinors by applying them to the Dirac factor of the tensor product $DM \otimes TM$.

A spinor is defined to be Majorana if $\overline{\psi^c} = \psi$. In the (real-linear) Majorana representation, where all $\gamma$-matrices are imaginary, one can choose $C = \gamma^0$, such that the Majorana condition becomes $\psi^* = \psi$. Hence, one often says that Majorana spinors are real, but the Majorana condition and the statement that every Dirac spinor is a unique complex linear combination of two Majorana spinors is independent of the chosen representation of the Clifford algebra, see e.g. [San09].

B Spinor curvature tensor identities

The curvature tensor $\mathcal{R}_{\mu\nu}$ of the Levi-Civita connection on $DM$ fulfils the following identities, see e.g. [Hac10, sec. I.2.2] for a proof.

$$\mathcal{R}_{\mu\nu} = \frac{1}{4} R_{\mu\nu\rho\sigma} \gamma^\rho \gamma^\sigma - \gamma^\mu \mathcal{R}_{\mu\nu} = \frac{1}{2} \gamma^\mu R_{\mu\nu} \gamma^\nu = \frac{1}{2} \gamma^\mu \gamma^\nu R_{\mu\nu} = \mathcal{R}_{\mu\nu} \gamma^\mu \gamma^\nu = \frac{1}{2} R$$

Note that, defining the Riemann and Ricci tensor, as well as the Ricci scalar by the convention chosen in [Wa84], these identities are valid for both metric sign conventions.

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12 One can define $C$ via the complex-conjugated rather than transposed representation of the Clifford algebra and then define the charge-conjugated spinor in a different manner such that the overall definition is equivalent to the one used here.

13 If one chooses signature $(-,+,+,+)$, they are real.
C Mode solutions of the Dirac equation

A complete and orthonormal set of mode solutions of the Dirac equation on Minkowski spacetime and in the Dirac representation of the Clifford algebra is given by

\[ v_k^1(t, \vec{x}) = \frac{e^{i(\vec{k}\cdot\vec{x} - \omega t)}}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega(\omega + m)}} M^+ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \]

\[ v_k^2(t, \vec{x}) = \frac{e^{i(\vec{k}\cdot\vec{x} - \omega t)}}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega(\omega + m)}} M^+ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \]

\[ v_k^3(t, \vec{x}) = \frac{e^{-i(\vec{k}\cdot\vec{x} - \omega t)}}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega(\omega + m)}} M^- \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \]

\[ v_k^4(t, \vec{x}) = \frac{e^{-i(\vec{k}\cdot\vec{x} - \omega t)}}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega(\omega + m)}} M^- \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \]

where

\[ M^\pm := \begin{pmatrix} (\mp \omega - m)1 & \vec{k}\vec{\sigma} \\ \vec{k}\vec{\sigma} & (\mp \omega + m)1 \end{pmatrix}, \]

\[ \omega = \sqrt{\vec{k}^2 + m^2}, \text{ and } \vec{\sigma} \text{ is the vector of Pauli matrices.} \]

D The relation between equal-time and covariant canonical anticommutation relations

Let \( \mathcal{R} \) be a strictly hyperbolic first order differential operator on some Hermitian vector bundle \( VM \) over a globally hyperbolic spacetime \( (M, g_{\mu\nu}) \) with principal symbol \( \sigma^\mu \), retarded and advanced Green’s operators \( G^\pm \), and causal propagator \( G = G^- - G^+ \). One can define a Hermitian product \( \langle \cdot, \cdot \rangle \) on test sections of \( VM \) via

\[ \langle f, g \rangle := \int_M d_\Sigma x \bar{f}(x)g(x), \]

where \( \bar{\cdot} \) denotes the adjoint with respect to the Hermitian product on \( VM \). With this setup, one can prove (see e.g. [BaGi11]) that the smeared causal propagator is the “charge” of a conserved current built from the principal symbol of \( \mathcal{R} \).

**Lemma 3** Let \( \mathcal{R} \) be formally selfadjoint with respect to \( \langle \cdot, \cdot \rangle \), i.e. \( \mathcal{R}^\dagger = \mathcal{R} \) with \( \mathcal{R}^\dagger \) defined as \( \langle \mathcal{R}^\dagger f, g \rangle := \langle f, \mathcal{R} g \rangle \) and let \( \Sigma \) be an arbitrary (smooth) Cauchy surface of \( (M, g_{\mu\nu}) \) with volume measure \( d\Sigma \) and forward-pointing normal \( N^\mu \). Then \( G(\bar{f}, g) \) can be expressed as

\[ G(\bar{f}, g) = -\int_{\Sigma} d\Sigma \left[ Gf \right] \sigma^\mu N_\mu \left[ Gg \right]. \]
If $M$ is foliated as $\{t\} \times \Sigma$ and $\vec{x}$ are coordinates on $\Sigma$, this identity is formally equivalent to

$$G(t_1, \vec{x}_1, t_2, \vec{x}_2)|_{t_1=t_2} = -\sigma^\mu N_\mu \delta(\vec{x}_1, \vec{x}_2).$$

**Proof.** We split $M$ into the future ($\Sigma^+$) and past ($\Sigma^-$) of $\Sigma$ and compute

$$G(\bar{f}, g) = \int_{\Sigma^+} d_g x \bar{f} G g + \int_{\Sigma^-} d_g x \bar{f} G g = \int_{\Sigma^+} d_g x [\mathcal{R} G^{-} \bar{f}] G g + \int_{\Sigma^-} d_g x [\mathcal{R} G^{+} \bar{f}] G g$$

$$= \langle \mathcal{R} G^- f, G g \rangle_{\Sigma^+} + \langle \mathcal{R} G^+ f, G g \rangle_{\Sigma^-}$$

$$= \langle \mathcal{R} G^- f, G g \rangle_{\Sigma^+} - \langle G^- f, \mathcal{R} G g \rangle_{\Sigma^+} + \langle \mathcal{R} G^+ f, G g \rangle_{\Sigma^-} - \langle G^+ f, \mathcal{R} G g \rangle_{\Sigma^-},$$

where the definitions and support properties of $G^\pm$ and $G$ have been used and where the index $\Sigma^\pm$ means that the integration in the product $\langle \cdot, \cdot \rangle$ is restricted to $\Sigma^\pm$. Since $\mathcal{R}$ is formally selfadjoint, we can use Green’s identity

$$\langle \mathcal{R}^\dagger f, g \rangle_M - \langle f, \mathcal{R} g \rangle_M = \int_{\partial M} d\partial M \bar{f} N^\mu \sigma_\mu g$$

valid for a manifold $M$ with smooth boundary $\partial M$, boundary volume measure $d\partial M$, and outwards pointing boundary normal $N^\mu$. Due to the support properties of $G^\pm f$, the only relevant boundary of $\Sigma^\pm$ is $\Sigma$, with outwards pointing normal $\mp N^\mu$. Hence, applying Green’s identity and considering $G = G^- - G^+$ concludes the proof.

If one would like to associate to a hyperbolic first order differential operator $\mathcal{R}$ a Fermionic quantum field theory whose covariant CAR are specified by $G$, i.e.

$$\{\psi(\bar{f}), \overline{\psi}(g)\} = iG(\bar{f}, g)1 := i\langle f, G g \rangle 1,$$

the above lemma implies that one can deduce equal-time CAR from these covariant CAR if the differential operator $\mathcal{R}$ is formally selfadjoint, and we are not aware of any other way to do so if $\mathcal{R}$ does not bear this property. On the other hand, even if one imposes equal-time CAR at one time although $\mathcal{R}$ is not formally selfadjoint, there is, to the knowledge of the authors, no way to prove that these equal-time CAR are in fact time-independent and, hence, covariant.

**E** **On the relation between positivity/unitary of the quantum field theory related to $\mathcal{R}$ and the selfadjointness of $\mathcal{R}$**

In this section we would like to point out the strong relationship between the positivity of the product defined by the anticommutator function/causal propagator $G$ of a first order differential operator $\mathcal{R}$ and the selfadjointness of $\mathcal{R}$. The first of our results implies that positivity on general spacetimes follows from selfadjointness if positivity on Minkowski spacetime is already known.

\[14\] This is always possible on account of the results of [BeSa05, BeSa06].

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Proposition 4  With the notation and definitions of section [D] let $\Gamma(VM)$ ($\Gamma_0(VM)$) denote the smooth sections (compactly supported smooth sections) of $VM$ and let $S(\mathcal{R}, M)$ be the space of solutions $\psi$ of $R\psi = 0$ with compactly supported initial conditions. Moreover, let $C\psi = 0$ be a linear, local and covariant constraint on $\Gamma(VM)$ and let $\mathcal{V}_0(VM)$ be the subspace of $\mathcal{V}(\mathcal{R}, M)$ ($\mathcal{V}_0(VM)$) defined as

\[ \mathcal{V}(\mathcal{R}, M) := \{ \psi \in S(\mathcal{R}, M) | C\psi = 0 \} \]

\[ \mathcal{V}_0(VM) := \{ f \in \Gamma_0(VM) | Gf \notin S(\mathcal{R}, M) \} . \]

If $\mathcal{R}^\dagger f = \mathcal{R}f$ for all $f \in \mathcal{V}_0(VM)$ on arbitrary spacetimes and $iG(\bar{f}, f) \geq 0$ for all $f \in \mathcal{V}_0(VM)$ on Minkowski spacetime, then $iG(\bar{f}, f) \geq 0$ for all $f \in \mathcal{V}_0(VM)$ on all globally hyperbolic spacetimes with the topology of $\mathbb{R}^4$.

Proof. Let $\Sigma$ be a Cauchy surface of $M$. For definiteness, we can pick a Cauchy surface which lies to the past of the support of $f$. From lemma [3] we know that $iG(\bar{f}, f)$ can be computed on $\Sigma$ and that the result is independent of $\Sigma$. We can now use a result of [FNW81] to “deform” the past of $\Sigma$ in $(M, g_{\mu\nu})$ into a piece of Minkowski spacetime $(\mathbb{M}, \eta_{\mu\nu})$. In more detail, the authors of [FNW81] show that there exists a fiducial globally hyperbolic spacetime $(M', g'_{\mu\nu})$ which contains two Cauchy surfaces $\Sigma_1$ and $\Sigma_2$ such that $\Sigma_1$ lies to the future of $\Sigma_2$, the future of $\Sigma_1$ in $(M', g'_{\mu\nu})$ is isometric to the future of $\Sigma$ in $(M, g_{\mu\nu})$ and the past of $\Sigma_2$ in $(M', g'_{\mu\nu})$ is isometric to the past of a Cauchy surface $\Sigma_0$ in $(\mathbb{M}, \eta_{\mu\nu})$. The computation of $iG(\bar{f}, f) \geq 0$ on $\Sigma$ in $(M, g_{\mu\nu})$ is equivalent to the same computation on $\Sigma_1$ in $(M', g'_{\mu\nu})$, which, by lemma [3] gives the same result as a computation on $\Sigma_2$ in $(M', g'_{\mu\nu})$ and, hence, on $\Sigma_0$ in $(\mathbb{M}, \eta_{\mu\nu})$. By assumption, the latter gives a positive result; this proves $iG(\bar{f}, f) \geq 0$. \hfill \Box

The above proof may seem awkward, and one might think that it depends on the chosen deformation. However, this is not the case, and the reason for this is the covariance of all objects as well as the deterministic nature of solutions to hyperbolic partial differential equations. Given sufficient initial data at one “time”, the future and past of the solutions are completely determined, no matter how the background spacetime “looks like” at those times. The apparent strength of the above employed deformation argument is the reason for its ubiquity in modern works on quantum field theory in curved spacetimes, e.g. [FNW81, Köh95, San08, DHP09]. We presume that the proof can be extended to spacetimes of arbitrary topology by a partition of unity argument.

While proposition [4] gives a sufficient condition for the positivity of $iG(\bar{f}, f)$, we now prove a necessary condition. Unfortunately, we have not been able to prove that positivity implies selfadjointness, but only a weaker statement.

Proposition 5  With the notation and definitions of section [D] and proposition [4] let $iG(\bar{f}, f) \geq 0$ for all $f \in \mathcal{V}_0(RM)$. Then, the following statements hold.

a) The advanced/retarded Green’s operators of $\mathcal{R}^\dagger$ coincide with those of $\mathcal{R}$ on $\mathcal{V}_0(RM)$. 
b) If the principal symbol of $\mathcal{R}$ is covariantly conserved, i.e. $\nabla_\mu \sigma^\mu = \sigma^\mu \nabla_\mu$, then the current

$$j^\mu [\psi_1, \psi_2] := \bar{\psi}_1 \sigma^\mu \psi_2$$

is covariantly conserved on all constrained solutions, i.e.

$$\nabla_\mu j^\mu [\psi_1, \psi_2] = 0 \quad \forall \psi_1, \psi_2 \in \mathcal{S}(\mathcal{R}, M).$$

Proof. We define $(f, g) := iG(\bar{f}, g)$. Since $(f, f) \geq 0$ for all $f \in \mathcal{V}_0(RM)$, $(f, g)$ is a positive semidefinite sesquilinear form on $\mathcal{V}_0(RM)$ by the polarisation identity. This implies

$$iG(\bar{f}, g) = \langle f, iGg \rangle = \langle f, g \rangle = \langle g, iGf \rangle = \langle iGf, g \rangle,$$

i.e. that $iG$ is formally selfadjoint with respect to $\langle \cdot, \cdot \rangle$. As the formal adjoint of the causal propagator of $\mathcal{R}$ is the causal propagator of the formally adjoined operator $\mathcal{R}^\dagger$ and the advanced/retarded Green’s operators are the unique advanced/retarded pieces of the causal propagator, a) holds.

Remembering that causal propagators map test sections to solutions, a) implies

$$\mathcal{S}(\mathcal{R}, M) \subset \mathcal{S}(\mathcal{R}^\dagger, M).$$

Let now $\sigma_0$ be defined as

$$\sigma_0 = \mathcal{R} - \sigma^\mu \nabla_\mu,$$

i.e. as the “zeroth order part” of $\mathcal{R}$. Note that $\sigma_0$ is not the subprincipal symbol of $\mathcal{R}$ in the mathematical sense, as the latter is not covariantly defined, but $\sigma_0$ is. One can check that $\mathcal{R}^\dagger$ can be expressed in terms of $\sigma^\mu$ and $\sigma_0$ as

$$\mathcal{R}^\dagger = -\overline{\sigma^\mu} \nabla_\mu + \overline{\sigma_0},$$

where again $\overline{\cdot}$ denotes the (fibrewise) Hermitean conjugation in $VM$. Using this, $\mathcal{S}(\mathcal{R}, M) \subset \mathcal{S}(\mathcal{R}^\dagger, M)$, and $\nabla_\mu \sigma^\mu = \sigma^\mu \nabla_\mu$, one can straightforwardly compute that $\nabla_\mu (\bar{\psi}_1 \sigma^\mu \psi_2) = 0$ for all $\psi_1, \psi_2 \in \mathcal{S}(\mathcal{R}, M)$. □

Of course one would like to prove more, namely, that $\mathcal{R}$ is formally selfadjoint on $\mathcal{V}_0(RM)$ if $iG$ is positive on $\mathcal{V}_0(RM)$. We indicate why we have not been able to prove this on the basis of a) above. Let $\tilde{G}^\pm$ be the advanced/retarded Green’s operators of $\mathcal{R}^\dagger$. We know that $\tilde{G}^\pm f = G^\pm f$ for all $f \in \mathcal{V}_0(RM)$. It would be tempting to write $\mathcal{R}^\dagger f = \mathcal{R}^\dagger G^\pm \mathcal{R} f$ and to say that this is equal to $\mathcal{R} f$ for all $f \in \mathcal{V}_0(RM)$. However, we only know $\tilde{G}^\pm = G^\pm$ on $\Gamma_0(RM)$, but we do not know if $\mathcal{R}$ maps $\mathcal{V}_0(RM)$ to itself.

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