GODBILLON-VEY INVARIANTS FOR FAMILIES OF FOLIATIONS

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Abstract. The classical Godbillon-Vey invariant is an odd degree cohomology class that is a cobordism invariant of a single foliation. Here we investigate cohomology classes of even degree that are cobordism invariants of (germs of) 1-parameter families of foliations.

In this paper we study an analogue of the classical Godbillon-Vey invariant \[ \mathcal{GV} \] that is an invariant not of a single foliation but of a family of foliations depending smoothly on a real parameter. For families of foliations of codimension \( q \), this invariant is a family of cohomology classes in degree \( 2q + 2 \). It is constructed and investigated here using explicit elementary calculations with differential forms in the spirit of the original work of Godbillon and Vey. We do not use Weil algebras or Gelfand-Fuks cohomology.

Recall that associated to a codimension \( q \) foliation \( \mathcal{F} \) with defining \( q \)-form \( \alpha \) one has a 1-form \( \beta \) such that \( d\alpha = \alpha \wedge \beta \). This has the remarkable property that \( \beta \wedge (d\beta)^q \) is closed, and that its cohomology class is independent of the particular choices made for \( \alpha \) and \( \beta \). This cohomology class is the Godbillon-Vey invariant of \( \mathcal{F} \). At least for codimension one, the calculations showing the independence of choices are quite concise, but non-trivial, see Section 1 below.

In the case of a 1-parameter family \( \mathcal{F}_t \), the defining form \( \alpha \) depends on \( t \), and so does \( \beta \). Denoting the time derivative by a dot, our invariant is the cohomology class \( TGV(\mathcal{F}_t) \) represented by the form \( \dot{\beta} \wedge \beta \wedge (d\beta)^q \). It is still easy to see that this form is closed. However, proving the independence of choices comes down to a highly non-trivial and quite miraculous calculation. This is carried out for \( q = 1 \) in Section 2, and for arbitrary \( q \) in Section 5.

Gelfand-Feigin-Fuks \[ \mathfrak{g} \] introduced certain characteristic classes of 1-parameter families of foliations from the point of view of the cohomology of the Lie algebra of formal vector fields. Elaborating on this,
Fuks [2] mentioned that $\dot{\beta} \wedge \beta \wedge d\beta$ represents a characteristic class of 1-parameter families of codimension one foliations, without giving a proof that it is well-defined. In a recent paper [12], Lodder tried to supply this proof, but his calculation was flawed, in that he disregarded several terms involving the time derivatives, see Remark 3. For $q > 1$, the forms $\dot{\beta} \wedge \beta \wedge (d\beta)^q$ have not been considered in the literature at all.

Our invariant of families admits a factorisation into a product of the Reeb class with a cohomology class $T(F_t)$ of degree $2q + 1$ in the ideal of the foliation. This is analogous to the factorisation of the classical Godbillon-Vey invariant discovered by Duminy. In the case of codimension one, the class $T(F_t)$ detects an infinitesimal rigidity of the Godbillon-Vey classes of transversely homographic foliations under arbitrary variations, which are allowed to go outside the subset of transversely homographic foliations. For variations only among transversely homographic foliations, rigidity of the Godbillon-Vey invariant was proved in [1].

In higher codimension, but not in codimension one, $T(F_t)$ coincides, up to a universal constant, with the time-derivative of the classical Godbillon-Vey invariant in a family. There is another difference between the case of codimension one and that of higher codimension: in codimension one the class $TGF(V_t)$ can be interpreted as an asymptotic linking number between certain foliations and vector fields [11], and this interpretation breaks down in higher codimension.

In the first section we recall the definition of the classical Godbillon-Vey invariant of a transversely oriented codimension one foliation. These well-known calculations will be used later in the paper and also serve to fix our notation. In Section 2 we introduce degree 4 cohomology classes that are invariants of families of codimension one foliations, substantiating the claims in [2, 12]. In Section 3 we prove several vanishing theorems; in particular we show that the invariants vanish for families generated by moving a fixed foliation by a flow. We also show that there are many foliations for which the invariant vanishes for all (infinitesimal) deformations. In Section 4 we show that the new invariants factorise through the Godbillon operator, and discuss the relationship with the time-derivative of the classical Godbillon-Vey invariant. In Section 5 we extend the whole discussion to higher codimension. Section 6 contains a few remarks.

We work with smooth foliations and smooth families, with smoothness of class $C^\infty$. For background on the Godbillon-Vey invariant we refer to [1].
1. The classical Godbillon-Vey invariant

First, recall the definition of the Godbillon-Vey invariant [6]. Let $\mathcal{F}$ be a smooth codimension 1 foliation on a smooth manifold $M$ (of arbitrary dimension). We shall assume throughout that $\mathcal{F}$ is transversely oriented, so that the normal bundle is trivialised. This situation can always be achieved by passing to a 2-fold covering.

We denote by $\mathcal{I}(\mathcal{F})$ the differential ideal of forms vanishing on $T\mathcal{F}$. The product of any two forms in $\mathcal{I}(\mathcal{F})$ vanishes identically. Let $\alpha \in \mathcal{I}(\mathcal{F})$ be a defining 1-form for $\mathcal{F}$; we assume that it is positive on positively oriented transversals. By the Frobenius theorem, we have

\begin{equation}
\label{frobenius}
d\alpha = \alpha \wedge \beta,
\end{equation}

for some $\beta$.

**Theorem 1.** The form $\beta \wedge d\beta$ is closed, and its cohomology class is an invariant of $\mathcal{F}$. In particular, it is independent of the choices made for $\alpha$ and $\beta$.

**Proof.** Differentiating (1) and substituting back from it, we find $\alpha \wedge d\beta = 0$, and so

\begin{equation}
\label{delta}
d\beta = \alpha \wedge \gamma,
\end{equation}

for some $\gamma$.

Now $d(\beta \wedge d\beta) = d\beta \wedge d\beta = 0$, as $d\beta \in \mathcal{I}(\mathcal{F})$ by (2). Thus $\beta \wedge d\beta$ is closed and defines a de Rham cohomology class $GV(F) \in H^3(M, \mathbb{R})$. To see that it is well-defined, suppose first that we fix $\alpha$, but make a different choice for $\beta$. We replace $\beta$ by $\beta + \tau$, for some $\tau \in \mathcal{I}(\mathcal{F})$. Then $d\tau, d\beta \in \mathcal{I}(\mathcal{F})$, and the product of any two forms in $\mathcal{I}(\mathcal{F})$ vanishes.

Thus $\beta \wedge d\beta$ is replaced by

\[(\beta + \tau) \wedge d(\beta + \tau) = \beta \wedge d\beta + d(\tau \wedge \beta),\]

which represents the same cohomology class. Thus, for fixed $\alpha$, we can use any $\beta$ satisfying (2) in the definition of $GV(F)$. It remains to be shown that $GV(F)$ is independent of the choice of $\alpha$.

The other forms defining $\mathcal{F}$ (which are also positive on positively oriented transversals) are of the form $f\alpha$, with $f$ a positive smooth function on $M$. We have

\[d(f\alpha) = df \wedge \alpha + f\alpha \wedge \beta = \alpha \wedge (-df + f\beta) = f\alpha \wedge (\beta - d\log f).
\]

Thus $\beta \wedge d\beta$ is replaced by

\[\beta \wedge d\beta - d((\log f)d\beta),\]

which represents the same cohomology class. $\square$
The Godbillon-Vey invariant $GV(\mathcal{F})$ is an invariant of cobordism classes of foliations, in the following sense:

**Proposition 2.** Given two oriented codimension one foliations $\mathcal{F}'$ and $\mathcal{F}''$ on closed oriented 3-manifolds $M'$ and $M''$, suppose that there is an oriented cobordism $W$ between $M'$ and $M''$, with an oriented codimension one foliation $\mathcal{F}$ which restricts to $\mathcal{F}'$ and $\mathcal{F}''$ on $M'$ and $M''$ respectively. Then $GV(\mathcal{F}') = GV(\mathcal{F}'') \in \mathbb{R}$.

This follows directly from Stokes’s theorem, compare the proof of Proposition 6 below.

**Remark 3.** The obvious naturality of the above calculations under pull-backs implies invariance properties also in the case where $M$ has larger dimension, so that the Godbillon-Vey invariant is not in the top dimension. Suppose for example that we have a foliation $\mathcal{F}$ on $M \times [0,1]$ which restricts to $\mathcal{F}'$ at one end and to $\mathcal{F}''$ at the other. Then $GV(\mathcal{F}') = GV(\mathcal{F}'') \in H^3(M, \mathbb{R})$.

2. A 4-dimensional Godbillon-Vey invariant

We now consider a smoothly varying 1-parameter family $\mathcal{F}_t$ of transversely oriented codimension one foliations on $M$. In this section all forms and functions are functions of the parameter $t \in \mathbb{R}$. Derivatives with respect to $t$ are denoted by a dot. For example, let $\alpha = \alpha(t)$ be a 1-parameter family of 1-forms with $\alpha(t)$ defining $\mathcal{F}_t$. Then equation (1) holds, with $\gamma$ also depending on $t$. Let $\dot{\gamma}$ be its time-derivative.

**Theorem 4.** For every $t \in \mathbb{R}$ the form $(\dot{\gamma} \wedge \gamma \wedge d\gamma)(t)$ is closed and its cohomology class is an invariant of the family $\mathcal{F}_t$. In particular, it is independent of the choices made for $\alpha$ and $\beta$.

**Proof.** Equation (2) holds as before, and $\gamma$ also depends on $t$. Differentiating (2) with respect to $t$ we obtain

$$d\dot{\gamma} = \dot{\alpha} \wedge \gamma + \alpha \wedge \dot{\gamma}.$$  

Combining this with (2), we obtain $d\dot{\gamma} \wedge d\gamma = 0$.

Now $d(\dot{\gamma} \wedge \gamma \wedge d\gamma) = d\dot{\gamma} \wedge \gamma \wedge d\gamma - \dot{\gamma} \wedge d(\dot{\gamma} \wedge \gamma \wedge d\gamma)$, with the first summand vanishing by the above argument. The second summand vanishes because $\dot{\gamma} \wedge d\gamma$ is closed as shown in the proof of Theorem 1. Thus $\dot{\gamma} \wedge \gamma \wedge d\gamma$ is also closed. To see that its cohomology class is independent of choices, we adapt the proof of Theorem 1.

Suppose that $A, B \in \mathcal{I}(\mathcal{F})$. Then $A \wedge B = 0$, and taking the time differential, we find

$$\dot{A} \wedge B + A \wedge \dot{B} = 0.$$  

(4)
If we replace $\beta$ by $\beta + \tau$ for some $\tau \in I(\mathcal{F})$, then $d\tau, d\beta \in I(\mathcal{F})$, and the product of any two forms in $I(\mathcal{F})$ vanishes. From the proof of Theorem 1 above we know that $\beta \wedge d\beta$ is replaced by $\beta \wedge d\beta + d(\tau \wedge \beta)$. Thus $\dot{\beta} \wedge \beta \wedge d\beta$ is replaced by

$$(\dot{\beta} + \dot{\tau}) \wedge (\beta \wedge d\beta + d(\tau \wedge \beta)) = \dot{\beta} \wedge (\beta \wedge d\beta + \dot{\tau} \wedge \beta) + \dot{\beta} \wedge d\tau \wedge (\beta + \dot{\tau} \wedge d\tau \wedge \beta).$$

Consider now the exact form

$$d(-\dot{\beta} \wedge \tau \wedge \beta - \frac{1}{2} \dot{\tau} \wedge \tau \wedge \beta) = -d\dot{\beta} \wedge \tau \wedge \beta + \dot{\beta} \wedge d\tau \wedge \beta - \frac{1}{2} d\dot{\tau} \wedge \tau \wedge \beta + \frac{1}{2} \dot{\tau} \wedge d\tau \wedge \beta,$$

where we have used again that the product of any two forms in $I(\mathcal{F})$ vanishes. The second summand on the right-hand side matches the third summand in the calculation above. Using (4), the last two summands are equal, and their sum equals $\dot{\tau} \wedge d\tau \wedge \beta$. Using (4) again, we have

$$\dot{\tau} \wedge d\beta + \tau \wedge d\dot{\beta} = 0,$$

and so $\dot{\tau} \wedge \beta \wedge d\beta = \beta \wedge \tau \wedge d\dot{\beta} = -d\dot{\beta} \wedge \tau \wedge \beta$.

Thus, for fixed $\alpha$, we can use any $\beta$ satisfying (1), and the cohomology class of $(\dot{\beta} \wedge \beta \wedge d\beta)(t)$ will be independent of this choice. It remains to be seen that it is also independent of the choice of $\alpha$.

The other forms defining $\mathcal{F}_t$ (which are also positive on positively oriented transversals) are of the form $f\alpha$, with $f$ a positive smooth function on $M$, again depending on $t$. From the proof of Theorem 1 we know that replacing $\alpha$ by $f\alpha$ results in replacing $\beta \wedge d\beta$ by $\beta \wedge d\beta - d((\log f)d\beta)$. Now $\beta$ is replaced by $\dot{\beta} - d(\frac{d}{dt}\log f) = \dot{\beta} - d(\frac{d}{dt}f)$.

Putting these calculations together, $\dot{\beta} \wedge \beta \wedge d\beta$ is replaced by

$$(\dot{\beta} - d(\frac{1}{f} \dot{f})))) \wedge (\beta \wedge d\beta - d((\log f)d\beta)) = \dot{\beta} \wedge \beta \wedge d\beta - d(\frac{1}{f} \dot{f}) \wedge \beta \wedge d\beta - \dot{\beta} \wedge d((\log f)d\beta) + d(\frac{1}{f} \dot{f}) \wedge d((\log f)d\beta) = \dot{\beta} \wedge \beta \wedge d\beta - d(\frac{1}{f} \dot{f} \wedge \beta \wedge d\beta) + d(\dot{\beta} \wedge (\log f)d\beta) + d(\frac{1}{f} \dot{f} \wedge d((\log f)d\beta)),$$

which represents the same cohomology class. Here we have used that $\beta \wedge d\beta$ is closed in order to rewrite the second summand, and (3) to rewrite the third one.

Remark 5. A version of Theorem 4 appears in [12]. The proof of well-definedness of the cohomology class of $(\dot{\beta} \wedge \beta \wedge d\beta)(t)$ is incomplete in [12], because when replacing $\alpha$ by $f\alpha$ and $\beta$ by $\beta + \tau = \beta + g\alpha$, the author ignores the terms involving the time derivatives of $f$ and $g$. 

Because of Theorem 4, the family of cohomology classes $TGV(F_t) = [(\hat{\beta} \wedge \beta \wedge d\beta)(t)] \in H^4(M, \mathbb{R})$ is an invariant of the family $F_t$. This can be specialised to a single cohomology class either by integrating over $t$, say

$$IGV(F_t) = \int_0^1 TGV(F_t) dt,$$

or by evaluating at a specific value of $t$, say

$$DGV(F_t) = TGV(F_t)(0).$$

It is clear that $DGV(F_t)$ is an invariant of the germ of the foliation $F_t$ at $t = 0$, so it does not depend on the whole family of foliations $F_t$. In fact, any 1-form $\omega$ with

$$\omega \wedge d\alpha + \alpha \wedge d\omega = 0$$

can be considered as an infinitesimal variation of the foliation defined by $\alpha$, and in the case that this infinitesimal variation integrates to an actual variation with $\hat{\alpha} = \omega$, we have

$$\hat{\beta} \wedge \beta \wedge d\beta = \hat{\beta} \wedge \beta \wedge \alpha \wedge \gamma = d\omega \wedge \beta \wedge \gamma,$$

so that the right-hand side can be used to define $DGV$ for any infinitesimal variation of a codimension one foliation.

The family of 4-dimensional cohomology classes $TGV(F_t)$ is a cobordism invariant of families in the following sense, cf. Proposition 6:

**Proposition 6.** Given two smooth families of oriented codimension one foliations $F'_t$ and $F''_t$ on closed oriented 4-manifolds $M'$ and $M''$, suppose that there is an oriented cobordism $W$ between $M'$ and $M''$, with a smooth family of oriented codimension one foliations $F_t$ which restricts to $F'_t$ and $F''_t$ on $M'$ and $M''$ respectively. Then $TGV(F'_t) = TGV(F''_t) \in \mathbb{R}$.

**Proof.** Let $\alpha$ be a time-dependent 1-form defining $F_t$ on $W$, with $d\alpha = \alpha \wedge \beta$. Then

$$\langle TGV(F''_t), [M''] \rangle - \langle TGV(F'_t), [M'] \rangle$$

$$= \int_{M''} \hat{\beta} \wedge \beta \wedge d\beta(t) - \int_{M'} \hat{\beta} \wedge \beta \wedge d\beta(t)$$

$$= \int_W d(\hat{\beta} \wedge \beta \wedge d\beta(t)) = \int_W 0 = 0,$$

by Stokes's theorem, because $\hat{\beta} \wedge \beta \wedge d\beta(t)$ is closed. \hfill \Box

* A fortiori, $IGV$ and $DGV$ are also cobordism invariants.

* *Mutatis mutandis*, Remark 3 applies to $TGV(F_t)$ as well.
3. Vanishing theorems

It is clear from the definition that $TGV(F_t)$ vanishes if $F_t$ can be defined by a form $\alpha$ with $d\alpha = \alpha \wedge \beta$ and $\beta$ closed. In particular this holds if $\alpha$ can be chosen to be closed itself.

This observation can be generalised quite a bit. Recall equations (1) and (2). Applying $d$ to the latter we find $0 = \alpha \wedge (\beta \wedge \gamma - d\gamma)$. Therefore, there exists a 1-form $\delta$, such that

$$d\gamma = \beta \wedge \gamma + \alpha \wedge \delta.$$  \hfill (7)

Repeating the procedure, we also find that

$$d\delta = 2\beta \wedge \delta + \alpha \wedge \epsilon,$$  \hfill (8)

for some $\epsilon$.

With this notation, we have the following vanishing theorem:

**Theorem 7.** If $F_t$ can be defined by a 1-form $\alpha$ for which $\beta$ or $\gamma$ or $\delta$ vanishes, then $TGV(F_t) = 0$.

If $F_t$ can be defined by a 1-form $\alpha$ such that $\alpha$ or $\beta$ or $\gamma$ or $\delta$ is closed, then $TGV(F_t) = 0$.

**Proof.** We use the above equations and the time-differential of (1). First notice

$$\dot{\beta} \wedge \beta \wedge d\beta = \dot{\beta} \wedge \beta \wedge \alpha \wedge \gamma = d\dot{\alpha} \wedge \beta \wedge \gamma$$

$$= d(\dot{\alpha} \wedge \beta \wedge \gamma) - \dot{\alpha} \wedge \beta \wedge d\gamma = d(\dot{\alpha} \wedge \beta \wedge \gamma) - \dot{\alpha} \wedge \beta \wedge \alpha \wedge \delta.$$  \hfill (7)

Thus $TGV(F_t) = 0$ as soon as $\beta$ or $\gamma$ or $\delta$ vanishes identically. The vanishing of $\beta$, respectively $\gamma$, is equivalent to $\alpha$, respectively $\beta$, being closed. The above calculation also shows that $TGV(F_t)$ vanishes if $\gamma$ is closed.

It remains to see what happens when $\delta$ is closed. We continue with the last term in the above calculation and use (8):

$$\dot{\alpha} \wedge \beta \wedge \alpha \wedge \delta = -\dot{\alpha} \wedge \alpha \wedge \frac{1}{2}(d\delta - \alpha \wedge \epsilon) = -\frac{1}{2} \dot{\alpha} \wedge \alpha \wedge d\delta.$$  \hfill (8)

Thus $TGV(F_t)$ is represented by an exact form if $F_t$ can be defined by a form $\alpha$ for which $\delta$ can be taken to be closed. \hfill $\square$

Evaluating at $t = 0$ we obtain from the same arguments:

**Theorem 8.** If $F_0$ can be defined by a 1-form $\alpha$ for which $\beta$ or $\gamma$ or $\delta$ vanishes, then $DGV(F_t) = 0$ for all germs $F_t$ at $F_0$.

If $F_0$ can be defined by a 1-form $\alpha$ such that $\alpha$ or $\beta$ or $\gamma$ or $\delta$ is closed, then $DGV(F_t) = 0$ for all germs $F_t$ at $F_0$. 
Foliations with a defining form $\alpha$ for which $\delta$ can be taken to be zero have special transverse structures. These are the transversely homographic structures in the sense of [5], page 174. If $\omega$ is a 1-form considered as an infinitesimal variation of $F_0$, a calculation like the one in the proof of Theorem 7 shows that the derivative of $GV$ at $F_0$ in the direction of $\omega$ is the cohomology class of $-2\omega \wedge \alpha \wedge \delta$. Thus transversely homographic foliations are the critical points of $GV$, and for these Theorem 8 gives:

**Corollary 9.** Let $F_0$ be a codimension one foliation with a transversely homographic structure. Then for all germs $F_t$ based at $F_0$ we have $DGV(F_t) = 0$.

Another byproduct of the proof of Theorem 7 is the observation that variations in the direction of a closed 1-form, i.e. with $d\dot{\alpha} = 0$, always give $TGV(F_t) = 0$. A similar argument shows that $TGV(F_t)$ vanishes for a family of diffeomorphic foliations:

**Theorem 10.** Let $F$ be a smooth codimension one foliation on $M$, and $\Phi_t$ a smooth 1-parameter family of diffeomorphisms of $M$. If $F_t$ is the family generated by pulling back $F$ via $\Phi_t$, then $TGV(F_t) = 0$.

**Proof.** By definition, we can choose $\alpha$ so that $\dot{\alpha} = L_X\alpha$, with $X$ a (time-dependent) vector field. Now we calculate:

\[
\dot{\beta} \wedge \beta \wedge d\beta = \alpha \wedge \dot{\beta} \wedge \beta \wedge \gamma = (d\dot{\alpha}) \wedge \beta \wedge \gamma - \dot{\alpha} \wedge \beta \wedge \beta \wedge \gamma \\
= (dL_X\alpha) \wedge \beta \wedge \gamma = (di_Xd\alpha) \wedge \beta \wedge \gamma \\
= -d(\beta(X)) \wedge \alpha \wedge \beta \wedge \gamma = d(\beta(X)) \wedge \beta \wedge d\beta \\
= d(\beta(X)\beta \wedge d\beta).
\]

Here the step from the second line to the third is achieved by the following:

\[
di_Xd\alpha = di_X(\alpha \wedge \beta) = d(\alpha(X)\beta - \beta(X)\alpha) \\
= d(\alpha(X)) \wedge \beta + \alpha(X)d\beta - d(\beta(X)) \wedge \alpha - \beta(X)d\alpha \\
= d(\alpha(X)) \wedge \beta + \alpha(X)\alpha \wedge \gamma - d(\beta(X)) \wedge \alpha - \beta(X)\alpha \wedge \beta.
\]

Thus $TGV(F_t)$ is represented by an exact form. ~\(\square\)

**4. Factoring through the Reeb class**

There is a factorisation of the classical Godbillon-Vey invariant into two different invariants due to Duminy, see [4]. We shall explain in this section that there is an analogous factorisation of the invariant $TGV(F_t)$. 
Recall that $\mathcal{I}(\mathcal{F})$ is a graded differential ideal in the algebra $\Omega^*(M)$ of smooth differential forms on $M$. We shall denote its cohomology by $H^*(\mathcal{I}(\mathcal{F}))$.

The quotient $\Omega^*(\mathcal{F})$ of $\Omega^*(M)$ by $\mathcal{I}(\mathcal{F})$ can be thought of as differential forms on $M$ defined only along the leaves of $\mathcal{F}$, with the induced differential $d_{\mathcal{F}}$ being differentiation along the leaves. We shall write $H^*(\mathcal{F})$ for the cohomology of the complex $(\Omega^*(\mathcal{F}), d_{\mathcal{F}})$.

Note that $H^*(\mathcal{I}(\mathcal{F}))$ is a module over the de Rham cohomology of $M$. As the product of any two forms in $\mathcal{I}(\mathcal{F})$ vanishes, we obtain a bilinear map

$$H^i(\mathcal{I}(\mathcal{F})) \times H^j(\mathcal{F}) \longrightarrow H^{i+j}(\mathcal{I}(\mathcal{F}))$$

for all $i$ and $j$.

Suppose now that $\mathcal{F}$ is defined by $\alpha$ with $d\alpha = \alpha \wedge \beta$. Then $d\beta \in \mathcal{I}(\mathcal{F})$; equivalently the projection of $\beta$ to $\Omega^1(\mathcal{F})$ is $d_{\mathcal{F}}$-closed. Thus $\beta$ defines a cohomology class $[\beta] \in H^1(\mathcal{F})$. We saw in the proof of Theorem 1 that making a different choice for $\beta$, respectively for $\alpha$, will change $\beta$ by an element of $\mathcal{I}(\mathcal{F})$, respectively by an exact form. Thus the Reeb class $[\beta] \in H^1(\mathcal{F})$ is independent of these choices and is an invariant of the foliation $\mathcal{F}$.

Similarly, $d\beta$ is in the ideal $\mathcal{I}(\mathcal{F})$ and is obviously closed. The proof of Theorem 11 shows that making different choices changes $d\beta$ at most by the addition of $d\tau$ with $\tau \in \mathcal{I}(\mathcal{F})$. Thus the cohomology class $[d\beta] \in H^2(\mathcal{I}(\mathcal{F}))$ is well-defined. Taking the product of this class with the Reeb class according to (9) yields a class in $H^3(\mathcal{I}(\mathcal{F}))$ whose image in the de Rham cohomology is the classical Godbillon-Vey invariant $GV(\mathcal{F})$.

Here is the analogous statement for $TGV(\mathcal{F}_t)$.

**Theorem 11.** Let $\mathcal{F}_t$ be a smooth family of smooth codimension one foliations on $M$. Then for every $t$, the 3-form $(\dot{\beta} \wedge d\beta)(t)$ is in $\mathcal{I}(\mathcal{F}_t)$. It is closed, and its cohomology class $T(\mathcal{F}_t) \in H^3(\mathcal{I}(\mathcal{F}_t))$ is independent of choices. Its image in the de Rham cohomology is $\frac{d}{dt} GV(\mathcal{F}_t)$.

The product of $T(\mathcal{F}_t)$ with the Reeb class according to (9) is a class in $H^3(\mathcal{I}(\mathcal{F}))$ whose image in the de Rham cohomology equals $TGV(\mathcal{F}_t)$ up to sign.

**Proof.** As $d\beta$ is in the ideal, so is $\dot{\beta} \wedge d\beta$. We have $d(\dot{\beta} \wedge d\beta) = d\dot{\beta} \wedge d\beta$. We noted in the proof of Theorem 1 that this vanishes by the combination of (3) and (4).

That $[\beta \wedge d\beta] \in H^3(\mathcal{I}(\mathcal{F}_t))$ is independent of choices also follows from the proof of Theorem 3. In more detail, suppose that we replace $\beta$ by $\beta + \tau$ with $\tau \in \mathcal{I}(\mathcal{F}_t)$. Then $\dot{\beta} \wedge d\beta$ is replaced by $\dot{\beta} \wedge d\beta - d(\dot{\beta} \wedge \tau + \frac{1}{2} \dot{\tau} \wedge \tau)$,
where \( \dot{\beta} \wedge \tau \) and \( \dot{\tau} \wedge \tau \) are in \( \mathcal{I}(\mathcal{F}_t) \) because \( \tau \) is. If we replace \( \alpha \) by \( f\alpha \), then \( \dot{\beta} \wedge d\beta \) is replaced by \( \dot{\beta} \wedge d\beta - d(\frac{1}{2} \dot{f} d\beta) \). Thus \( T(\mathcal{F}_t) \) is well-defined in \( H^3(\mathcal{I}(\mathcal{F}_t)) \).

To calculate the image of \( T(\mathcal{F}_t) \) in the de Rham cohomology, consider the time-derivative of \( \beta \wedge d\beta \):

\[
\frac{d}{dt}(\beta \wedge d\beta) = \dot{\beta} \wedge d\beta + \beta \wedge d\dot{\beta} = 2 \dot{\beta} \wedge d\beta + d(\dot{\beta} \wedge \beta) .
\]

The left-hand side represents \( \frac{d}{dt}GV(\mathcal{F}_t) \) in the de Rham cohomology, and the right-hand side represents twice the image of \( T(\mathcal{F}_t) \).

Given the above, the last claim in the Theorem is obvious.

Note that there is no reason to expect the time-derivative of \( GV(\mathcal{F}_t) \) to define a class in \( H^3(\mathcal{I}(\mathcal{F}_t)) \). The formula (10) shows that, up to a factor of 2, \( T(\mathcal{F}_t) \) lifts \( \frac{d}{dt}GV(\mathcal{F}_t) \) to \( H^3(\mathcal{I}(\mathcal{F}_t)) \). But the difference term \( d(\dot{\beta} \wedge \beta) \) is not usually in the ideal \( \mathcal{I}(\mathcal{F}_t) \). Thus \( \dot{\beta} \wedge \beta \) does not define a cohomology class in \( H^2(\mathcal{F}_t) \), and the formula (10) does not have cohomological meaning, beyond expressing the relationship between the image of \( T(\mathcal{F}_t) \) in the de Rham cohomology and the time derivative of the classical Godbillon-Vey invariant. At this point there is a crucial difference between the codimension 1 case and that of higher codimension; see Theorem 16 below.

It is well-known that there are smooth families of foliations for which the classical Godbillon-Vey invariant is not constant. Every such family gives examples for the non-vanishing of \( T(\mathcal{F}_t) \).

This contrasts with the following vanishing result which sharpens Corollary 9.

**Theorem 12.** Let \( \mathcal{F}_0 \) be a codimension one foliation with a transversely homographic structure. Then for all germs \( \mathcal{F}_t \) based at \( \mathcal{F}_0 \) we have \( T(\mathcal{F}_t)(0) = 0 \in H^3(\mathcal{I}(\mathcal{F}_0)) \).

**Proof.** We calculate with the defining form of \( T(\mathcal{F}_t) \) using (2), the time differential of (1), and then (3) and (4):

\[
\dot{\beta} \wedge d\beta = \dot{\beta} \wedge \alpha \wedge \gamma = -(d\dot{\alpha}) \wedge \gamma + \dot{\alpha} \wedge \beta \wedge \gamma \\
= -d(\dot{\alpha} \wedge \gamma) - \dot{\alpha} \wedge d\gamma + \dot{\alpha} \wedge (d\gamma - \alpha \wedge \delta) \\
= d(\alpha \wedge \dot{\gamma}) + \alpha \wedge \dot{\alpha} \wedge \delta .
\]

This shows that \( T(\mathcal{F}_t) \in H^3(\mathcal{I}(\mathcal{F}_t)) \) is represented by the form \( \alpha \wedge \dot{\alpha} \wedge \delta \). By definition, this vanishes for a transversely homographic foliation, no matter what \( \dot{\alpha} \) is.

As a consequence of the above calculation, note that if a 1-form \( \omega \) is considered as an infinitesimal variation of a foliation \( \mathcal{F} \), we can define
the class \( T \in H^3(\mathcal{I}(\mathcal{F})) \) for this infinitesimal variation as the cohomology class of \( \alpha \wedge \omega \wedge \delta \), even if \( \omega \) does not integrate to an actual 1-parameter variation of \( \mathcal{F} \).

It is interesting to examine the other vanishing theorems in Section 3 in the light of the decomposition of \( TGV \). If \( \beta = 0 \), then both \( T(\mathcal{F}_t) \) and the Reeb class vanish. If \( \gamma = 0 \), then the Reeb class lifts to \( H^1(M, \mathbb{R}) \), and \( T(\mathcal{F}_t) = 0 \). If \( \delta = 0 \), we can say nothing about the Reeb class, but \( T(\mathcal{F}_t) \) vanishes by the Theorem above. In Theorem 7 we saw that \( TGV \) vanishes also when \( \gamma \) or \( \delta \) is closed. These results do not seem to come from either the vanishing of the Reeb class, or the vanishing of \( T(\mathcal{F}_t) \), but instead rely on the interplay between the two via (9).

Theorem 10 does in fact come from the vanishing of \( T(\mathcal{F}_t) \) in families generated from a fixed foliation by a flow. As a flow acts trivially on the de Rham cohomology, Theorem 11 shows that the image of \( T(\mathcal{F}_t) \) in the de Rham cohomology is trivial. It does not, however, show that \( T(\mathcal{F}_t) \) vanishes in \( H^3(\mathcal{I}(\mathcal{F})) \), which itself varies with \( t \). Nevertheless, an easy adaptation of the proof of Theorem 11 shows that \( T(\mathcal{F}_t) \) vanishes in the cohomology of the ideal. We shall give this argument for foliations of arbitrary codimension in the next section.

Dualising the decomposition in Theorem 11, we obtain:

**Theorem 13.** Let \( \mathcal{F}_t \) be a smooth family of smooth codimension one foliations on a closed oriented manifold \( M \). Then for every \( t \), the class \( TGV(\mathcal{F}_t) \in H^4(M) \), thought of as a linear functional on \( H^{n-4}(M) \), decomposes up to sign into the composition of the following two maps:

1. the map given by the product with \( T(\mathcal{F}_t) \):
   \[
   H^{n-4}(M) \longrightarrow H^{n-1}(\mathcal{I}(\mathcal{F}_t)) \quad x \mapsto x \cup T(\mathcal{F}_t) ,
   \]

2. the Godbillon operator given by the Reeb class according to (3):
   \[
   H^{n-1}(\mathcal{I}(\mathcal{F}_t)) \longrightarrow H^n(M, \mathbb{R}) = \mathbb{R} \quad y \mapsto y \cup [\beta] .
   \]

The definition of the Godbillon operator above uses (3) together with the surjection \( H^n(\mathcal{I}(\mathcal{F})) \rightarrow H^n(M, \mathbb{R}) \) in the top dimension.

This decomposition is very useful because the study of the classical Godbillon-Vey invariant has led to many vanishing theorems for the Godbillon operator in situations where the Reeb class need not vanish, cf. (4). These vanishing theorems arise from the localisation of the Godbillon operator on saturated sets discovered by Duminy. For example, it is known that the Godbillon operator vanishes for foliations
almost without holonomy and for those without resilient leaves. Thus, we obtain another vanishing theorem:

**Theorem 14.** Let $\mathcal{F}_0$ be a foliation whose Godbillon operator vanishes, for example a foliation almost without holonomy, or without resilient leaves. Then $DGV(\mathcal{F}_i)$ vanishes for all germs $\mathcal{F}_i$ at $\mathcal{F}_0$.

This leaves open the possibility that there may be a foliation $\mathcal{F}_0$ with $GV(\mathcal{F}_0) = 0$, but which admits an infinitesimal deformation with $DGV(\mathcal{F}_i)$ non-zero. The latter condition only implies the non-triviality of the Godbillon operator, which does not contradict the vanishing of the classical Godbillon-Vey invariant.

5. Higher codimension

There is a classical Godbillon-Vey invariant for foliations of higher codimension $q$ with oriented normal bundles. Such foliations are defined by locally decomposable $q$-forms $\alpha = \alpha_1 \wedge \ldots \wedge \alpha_q$ of maximal rank. Again we have

\[ d\alpha = \alpha \wedge \beta, \quad \text{(11)} \]

for some 1-form $\beta$.

**Theorem 15.** [6] The form $\beta \wedge (d\beta)^q$ is closed and its cohomology class $GV(\mathcal{F}) \in H^{2q+1}(M, \mathbb{R})$ is an invariant of $\mathcal{F}$. In particular, it is independent of the choices made for $\alpha$ and $\beta$.

Like Theorem 1, this is proved by a direct calculation. Instead of reproducing this lengthy calculation, we proceed directly to the decomposition into two invariants and prove that they are well-defined and that one recovers the Godbillon-Vey invariant by composition. We then generalise the argument to the case of families.

As in the codimension one case, denote by $\mathcal{I}(\mathcal{F})$ the graded differential ideal of forms on $M$ which vanish when evaluated on tuples of vectors all tangent to $\mathcal{F}$. These are locally linear combinations of decomposable forms for which each summand contains one of the $\alpha_i$ as a factor. Note that all $(q + 1)$-fold products of elements of $\mathcal{I}(\mathcal{F})$ vanish. We denote the cohomology of $\mathcal{I}(\mathcal{F})$ by $H^*(\mathcal{I}(\mathcal{F}))$ and use $H^*(\mathcal{F})$ for the cohomology of the quotient of $\Omega^*(M)$ by $\mathcal{I}(\mathcal{F})$.

Differentiating (11) and substituting back from it, we find $\alpha \wedge d\beta = 0$, and so

\[ d\beta = \sum_{i=1}^{q} \alpha_i \wedge \gamma_i \in \mathcal{I}(\mathcal{F}), \quad \text{(12)} \]
for some $\gamma_i$. Thus $\beta$ defines a cohomology class in $H^1(F)$, called the
Reeb class of $F$. As before, this is well-defined independent of choices.

The form $d\beta$ is in $\mathcal{I}(F)$ by (12). It is obviously closed and so defines
a cohomology class in $H^2(\mathcal{I}(F))$. Making different choices changes $d\beta$
at most by the addition of $d\tau$ for some $\tau \in \mathcal{I}(F))$. Thus the class
$[d\beta] \in H^2(\mathcal{I}(F))$ is well-defined.

In this case, we cannot compose the Reeb class with the class $[d\beta]$ to obtain a well-defined cohomology class of degree three. Though
$H^*(\mathcal{I}(F))$ is a module over the de Rham cohomology of $M$, if we choose
a form on all of $M$ representing a cohomology class in $H^*(F)$, then the
wedge product with a closed form in the ideal is not necessarily closed,
so we do not get a pairing of the form (9) in all cases. However, if the
form in the ideal is itself a $q$-fold product of forms in the ideal, then
the resulting wedge product is closed because $(q + 1)$-fold wedge products
of forms in the ideal vanish. Thus, the product of $[d\beta]^q \in H^{2q}(\mathcal{I}(F))$
with the Reeb class is a well-defined cohomology class in $H^{2q+1}(\mathcal{I}(F))$,
whose image in the de Rham cohomology is the Godbillon-Vey class of
Theorem 15. This of course proves Theorem 15.

Now we make the same extension to families as in the codimension
one case. Suppose $\alpha$ and $\beta$ above depend smoothly on a real parameter
t, and denote the differential with respect to $t$ by a dot.

**Theorem 16.** Let $\mathcal{F}_t$ be a smooth family of smooth codimension $q$ fol-
liations on $M$. Then for every $t$, the $(2q + 1)$-form $(\dot{\beta} \wedge (d\beta)^q)(t)$ is in
$\mathcal{I}(\mathcal{F}_t)$. It is closed, and its cohomology class $T(\mathcal{F}_t) \in H^{2q+1}(\mathcal{I}(\mathcal{F}_t))$ is
a well-defined invariant of the family $\mathcal{F}_t$.

If $q \geq 2$, then $T(\mathcal{F}_t) = \frac{1}{q+1} \frac{d}{dt} GV(\mathcal{F}_t) \in H^{2q+1}(\mathcal{I}(\mathcal{F}_t))$.

The product of $T(\mathcal{F}_t)$ with the Reeb class is a well-defined cohomology
class $TGV(\mathcal{F}_t) \in H^{2q+2}(\mathcal{I}(\mathcal{F}_t))$ represented by $(\dot{\beta} \wedge \beta \wedge (d\beta)^q)(t)$, up to
sign.

**Proof.** The form $\dot{\beta} \wedge (d\beta)^q$ is in the ideal $\mathcal{I}(\mathcal{F}_t)$ because $d\beta$ is.

Differentiating (12) with respect to $t$, we obtain

$$d\dot{\beta} = \sum_{i=1}^q (\dot{\alpha}_i \wedge \gamma_i + \alpha_i \wedge \dot{\gamma}_i).$$

We have $d(\dot{\beta} \wedge (d\beta)^q) = d\dot{\beta} \wedge (d\beta)^q$, which vanishes by the combination
of (12) and (13).
Now suppose that we replace $\beta$ by $\beta + \tau$, with $\tau \in I(F)$. Then we have

$$
(\dot{\beta} + \dot{\tau}) \wedge (d\beta + d\tau)^q - \dot{\beta} \wedge (d\beta)^q =
$$

$$
\dot{\tau} \wedge (d\beta)^q + \sum_{i=0}^{q-1} \left( \binom{q}{i} \dot{\beta} \wedge (d\beta)^i \wedge (d\tau)^{q-i} \right) = \dot{\tau} \wedge (d\tau)^q
$$

$$
+ \sum_{i=0}^{q-1} \left( \binom{q}{i} \dot{\beta} \wedge (d\beta)^i \wedge (d\tau)^{q-i} + \binom{q}{i+1} \dot{\tau} \wedge (d\beta)^{i+1} \wedge (d\tau)^{q-i-1} \right).
$$

We have to prove that this is the exterior differential of a form in $I(F)$. As $(q + 1)$-fold products of elements in $I(F)$ vanish, equation (13) now takes the form

$$
(14) \quad A_0 \wedge A_1 \wedge \ldots \wedge A_q + \ldots + A_0 \wedge \ldots \wedge A_{q-1} \wedge A_q = 0
$$

for all $A_0, \ldots, A_q \in I(F)$.

Applying this to $\tau \wedge (d\tau)^q = 0$, we find

$$
\dot{\tau} \wedge (d\tau)^q + q\tau \wedge d\dot{\tau} \wedge (d\tau)^{q-1} = 0.
$$

This implies

$$
d(\tau \wedge \dot{\tau} \wedge (d\tau)^{q-1}) = \dot{\tau} \wedge (d\tau)^q - \tau \wedge d\dot{\tau} \wedge (d\tau)^{q-1} = \frac{q + 1}{q} \dot{\tau} \wedge (d\tau)^q.
$$

Similarly, we find

$$
q\dot{\beta} \wedge (d\beta)^{q-1} \wedge d\tau + \dot{\tau} \wedge (d\beta)^q = qd(\tau \wedge \dot{\beta} \wedge (d\beta)^{q-1}) = 0.
$$

It remains to discuss the terms of the form

$$
\left( \binom{q}{i} \dot{\beta} \wedge (d\beta)^i \wedge (d\tau)^{q-i} \right) + \left( \binom{q}{i+1} \dot{\tau} \wedge (d\beta)^{i+1} \wedge (d\tau)^{q-i-1} \right)
$$

for $i \leq q - 2$. Using (14) in the form

$$
\dot{\tau} \wedge (d\beta)^{i+1} \wedge (d\tau)^{q-i-1} + (i + 1)\tau \wedge d\dot{\beta} \wedge (d\beta)^i \wedge (d\tau)^{q-i-1} + (q - i - 1)\tau \wedge d\dot{\tau} \wedge (d\beta)^{i+1} \wedge (d\tau)^{q-i-2} = 0
$$

one easily checks

$$
\dot{\beta} \wedge (d\beta)^i \wedge (d\tau)^{q-i} + \frac{q - i}{i + 1} \tau \wedge (d\beta)^{i+1} \wedge (d\tau)^{q-i-1} =
$$

$$
\frac{q - i - 1}{i + 1} d(\tau \wedge \dot{\tau} \wedge (d\beta)^{i+1} \wedge (d\tau)^{q-i-2}) - d(\dot{\beta} \wedge \tau \wedge (d\beta)^i \wedge (d\tau)^{q-i-1})
$$

where the right-hand side is obviously in $d(I(F))$. Because $\binom{q}{i+1} = \binom{q}{i}$, this finally proves the independence of the choice of $\beta$. Therefore, we are done.
If we replace \( \alpha \) by \( f \alpha \), then \( \beta \) can be replaced by \( \beta - d \log f \). Thus \( \beta \wedge (d \beta)^q \) is replaced by

\[
\dot{\beta} \wedge (d \beta)^q - d \left( \frac{1}{f} \dot{f}(d \beta)^q \right),
\]

which represents the same cohomology class in \( H^{2q+1}(\mathcal{I}(F_t)) \).

This completes the proof that \((\beta \wedge (d \beta)^q)(t)\) defines a cohomology class \( T(F_t) \in H^{2q+1}(\mathcal{I}(F_t)) \) that is independent of choices.

Now we calculate the time-derivative of \( GV(F_t) \):

\[
\frac{d}{dt} (\beta \wedge (d \beta)^q) = \dot{\beta} \wedge (d \beta)^q + q \beta \wedge d \dot{\beta} \wedge (d \beta)^{q-1}
\]

\[
= (q + 1) \dot{\beta} \wedge (d \beta)^q + q d(\dot{\beta} \wedge \beta \wedge (d \beta)^{q-1}).
\]

If \( q \geq 2 \), the right-hand side differs from the defining form of \((q + 1)T(F_t)\) by a form in \( d(\mathcal{I}(F_t)) \).

By definition, the defining form of \( T(F_t) \) is a \( q \)-fold product of elements in the ideal of \( F_t \). Thus, its product with the Reeb class is well-defined.

As in the case of codimension one we can define \( IGV(F_t) \) by integrating \( TGV(F_t) \) over \( t \) and \( DGV(F_t) \) by evaluation at \( t = 0 \).

We now prove the generalisation of Theorem 10 mentioned in Section 4:

**Theorem 17.** Let \( F \) be a smooth codimension \( q \) foliation on \( M \) and \( \Phi_t \) a smooth 1-parameter family of diffeomorphisms of \( M \). If \( F_t \) is the family generated by pulling back \( F \) via \( \Phi_t \), then \( T(F_t) = 0 \in H^{2q+1}(\mathcal{I}(F_t)) \).

**Proof.** In this situation the image of \( T(F_t) \) in the de Rham cohomology is trivial because the flow acts trivially on de Rham cohomology, and so \( T(F_t) \) is represented by an exact form. The point of the Theorem, and of this proof, is to see that the primitive can be chosen to be in the ideal \( \mathcal{I}(F_t) \).

Note that \((12)\) shows that \((d \beta)^q = \alpha \wedge \gamma \), where \( \gamma \) is a locally decomposable \( q \)-form. Under the assumption of the Theorem, we may choose \( \alpha \) so that \( \dot{\alpha} = L_X \alpha \), with \( X \) a (time-dependent) vector field.
Now we calculate:
\[
\dot{\beta} \wedge (d\beta)^q = \dot{\beta} \wedge \alpha \wedge \gamma = (-1)^q \alpha \wedge \dot{\beta} \wedge \gamma \\
= (-1)^q (d\dot{\alpha}) \wedge \gamma + (-1)^{q-1} \dot{\alpha} \wedge \beta \wedge \gamma \\
= (-1)^q (d_iX \alpha) \wedge \gamma + (-1)^{q-1} (d_iX \alpha) \wedge \beta \wedge \gamma \\
\quad + (-1)^{q-1} (i_X d\alpha) \wedge \beta \wedge \gamma \\
= (-1)^q (d_iX (\alpha \wedge \beta)) \wedge \gamma + (-1)^{q-1} (d_iX \alpha) \wedge \beta \wedge \gamma \\
\quad + (-1)^{q-1} (i_X (\alpha \wedge \beta)) \wedge \beta \wedge \gamma \\
= (-1)^q (d((i_X \alpha) \wedge \beta + (-1)^q \beta(X) \alpha)) \wedge \gamma \\
\quad + (-1)^{q-1} (d_iX \alpha) \wedge \beta \wedge \gamma - \beta(X) \alpha \wedge \beta \wedge \gamma \\
= (d(\beta(X))) (d(\beta))^q = d(\beta(X)) (d(\beta))^q,
\]
which is clearly in \(d(\mathcal{I}(\mathcal{F}_t))\). Here we have used \(d\beta \wedge \gamma = 0\), which follows from (12) and the definition of \(\gamma\) as the product (up to a constant) of the \(\gamma_i\).

If \(M\) is closed and oriented, we can think of \(TGV(\mathcal{F}_t)\) as a linear functional on \(H^{n-2q-2}(M, \mathbb{R})\). Dualising the above decomposition of \(TGV(\mathcal{F}_t)\), we see that this functional factors through the Godbillon operator defined by multiplication with the Reeb class as in Theorem 13. The vanishing theorems for the Godbillon operator have been extended to higher codimension by Hurder [10]. Combining his result with Theorem 16 we obtain:

**Theorem 18.** Let \(\mathcal{F}_0\) be a foliation almost all of whose leaves have subexponential growth. Then \(DGV(\mathcal{F}_t)\) vanishes for all germs \(\mathcal{F}_t\) at \(\mathcal{F}_0\).

6. **Final comments**

In this paper we have extended the Godbillon-Vey invariants to families of foliations, obtaining families of cohomology classes in even degrees. This can be done quite generally, for all the characteristic classes of foliations, and is the subject of joint work with M. Hoster and F. Kamber [9]. The general construction starts from the observation that a 1-parameter family \(\mathcal{F}_t\) of codimension \(q\) foliations on \(M\) can be thought of as a foliation of codimension \(q+1\) on \(M \times \mathbb{R}\), such that each cross-section \(M \times \{t\}\) is saturated. At the level of forms, or in the foliated cohomology, \(TGV(\mathcal{F}_t)\) is then obtained from the classical Godbillon-Vey invariant in codimension \(q+1\) by contraction with the vector field \(\dot{\beta}/\mathcal{F}\). Of course, this is not a cohomological calculation in de
Rham cohomology. In the general case, however, the explicit calculations with differential forms become very complicated, and we resort to the formalism of Weil algebras. Further generalisations to the case of multi-parameter families and to flags of foliations are discussed in Hoster’s thesis [8].

Heitsch [7] considered time-derivatives of characteristic classes, which are classes of the same degree as the classical characteristic classes of foliations. He showed that his classes vanish for all infinitesimal variations of the Roussarie example [6]. As this is transversely homographic, Heitsch’s result is a consequence of Theorem 12 above.

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