Torsors on Affine Varieties

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Abstract

Let $X$ be a smooth affine algebraic variety over the field of complex numbers which is contractible. Then every algebraic $G$-torsor on $X$ is algebraically trivial if $G$ is a semi-simple algebraic group. We also show that if $X$ is a smooth affine algebraic variety such that $\Omega^1_X$ is trivial and $X$ is topologically simply connected with $H^i(X, \mathbb{Z}) = 0$ for $i = 1, 2$ and $3$, then every algebraic $G$-torsor on $X$ is algebraically trivial for a semi-simple algebraic group $G$.

1 Introduction

Let $k$ be an algebraically closed field of characteristic $0$ contained in $\mathbb{C}$. Let $X$ be a smooth affine variety over $k$ and let $G$ be an affine algebraic semisimple group over $k$. We use the words torsor and principal $G$-bundle interchangeably. The question we are interested in is, when are $G$ bundles on $X$ trivial? We let $X_{\mathbb{C}}$ denote the base change of $X$ to $\mathbb{C}$, the field of complex numbers. In this article, we show the following

Main Theorem  Let the singular cohomology of $X_{\mathbb{C}}$ with integer coefficients be trivial, let $\Omega^1_X$ be algebraically trivial, and assume that the topological fundamental group of $X_{\mathbb{C}}$ is trivial. Then every algebraic $G$-bundle on
X is algebraically trivial. (See Theorem (6.1), Remark (6.2) and Remark (6.4))

Remark (1.1) For the definition and basic properties of torsors, see [8], Ch.3, section 4. In our case, G-torsors are locally isotrivial, i.e., every point has a finite etale neighborhood over which the torsor is trivial (this is a theorem of Grothendieck, see [11], Lemme XIV 1.4). The main theorem states that under the hypotheses mentioned in the theorem, every algebraic G-bundle is globally algebraically trivial in the Zariski topology.

Remark (1.2) The theorem of Quillen-Suslin (Serre’s conjecture) states that every projective module over a polynomial ring over C is free. The case where the module has trivial determinant is a case of the above theorem. It is not difficult to see that rank one projective modules are free on a polynomial ring over C.

2 A Reduction

In this short section, we will show that in order to prove the main theorem over an algebraically closed field k contained in the field of complex numbers, it is enough to consider the field of complex numbers. We have the following

Lemma (2.1) Let k be an algebraically closed field contained in C and let X be an affine variety defined over k. Let H be an affine algebraic group of finite type defined over k. Let $E_H \rightarrow X$ be a H-torsor on X such that $E_H \otimes \mathbf{C}$ is trivial over $X_\mathbf{C}$. Then $E_H$ is trivial on X over k.

Proof Since $E_H \otimes \mathbf{C}$ is trivial over $X_\mathbf{C}$, we can assume that there is a finitely generated k-algebra B such that $E_H \otimes B$ is trivial over $X_B$, where $X_B$ denotes
the base change of $X$ to $B$. Since $k$ is algebraically closed we can find a closed point of $\text{Spec } B$ which is rational over $k$, i.e., surjection

$$B \to k$$

such that the composite

$$k \to B \to k$$

is the identity. We get

$$X \xrightarrow{p} X \otimes_k B \xrightarrow{q} X$$

with the composite identity. Since $E_H \otimes B$ is trivial over $X_B$, it follows that $q^* E_H$ is trivial over $X_B$. Hence $p^* q^* E_H$ is trivial over $X$. Therefore $E_H$ is trivial on $X$ over $k$. Q.E.D.

3 The Differential Complex

Let $k$ be an algebraically closed field of characteristic 0. Let $X$ be a smooth, affine variety over $k$. Let $G$ be an affine algebraic group over $k$ of finite type. Let $\pi : E \to X$ be a principal $G$-bundle.

Let $\Omega^i_E$ denote the bundle of $i$-forms on $E$ and let $\Omega^i_X$ denote the bundle of $i$-forms on $X$. We have the natural morphism.

$$\pi^* \Omega^i_X \to \Omega^i_E \quad i = 1, \ldots, n.$$  

Since the group $G$ acts on $E$ (on the right, by convention), $G$ also acts on $\Omega^i_E$. We can consider the sheaf of germs of invariant $i$-forms for the $G$-action. This sheaf descends to a vector bundle $A^i(E)$ on $X$.

Further, the canonical map

$$\pi^* \Omega^i_X \to \Omega^i_E$$
induces an inclusion of bundles

$$0 \to \Omega^i_X \to A^i(E).$$

We now consider the exterior differential

$$d_E : \Omega^i_E \to \Omega^{i+1}_E, \ i = 0, 1, \ldots$$

on $E$. We observe that $d_E$ is equivariant for the $G$-action and hence descends to an operator (also denoted by $d_E$)

$$d_E : A^i(E) \to A^{i+1}(E) \ i = 0, 1, 2, 3, \ldots$$

We further obtain a commutative diagram

$$\begin{array}{ccc}
A^i(E) & \xrightarrow{d_E} & A^{i+1}(E) \\
\downarrow & & \downarrow \\
\Omega^i_X & \xrightarrow{d_X} & \Omega^{i+1}_X
\end{array}$$

where $d_X$ denotes the exterior differential on $X$, for $i = 0, 1, 2, \ldots, n$ ($\Omega^i_X$ is taken to be $\mathcal{O}_X$, the structure sheaf, for $i = 0$).

Since $X$ is affine, there is a splitting of the inclusion of bundles

$$\Omega^i_X \subset A^i(E)$$

for $i = 1, \ldots, n$. This follows from the fact that on an affine variety, a short exact sequence of vector bundles splits as $H^1(X, F)$ vanishes for any vector bundle $F$.

We now recall Grothendieck’s algebraic de Rham theorem (see [4], page 453); it says that for any smooth affine algebraic variety over the field $\mathbb{C}$ of complex numbers, the cohomology of the algebraic de Rham complex of the variety is isomorphic to the singular cohomology of the variety with complex coefficients. In particular, if we assume that the singular cohomology with
complex coefficients of the variety is trivial, it implies that the algebraic de
Rham complex is globally exact.

We now have

**Proposition (3.1)** Let $X$ be a smooth affine variety over $\mathbb{C}$ such that $X_{\mathbb{C}}$
has trivial singular cohomology with complex coefficients, and let $\Omega^1_X$ be alge-
braically trivial. Let $G$ be semisimple and simply connected. Then there is
a system of splittings

$$s_i : A^i(E) \to \Omega^i_X, \ i = 1, 2,$$

such that

$$d_X \circ s_1 = s_2 \circ d_E.$$

**Proof** By Grothendieck’s algebraic de Rham theorem (see paragraph above),
the algebraic de Rham complex of $X_{\mathbb{C}}$ is exact globally. However, we observe
that if a $i$-form on $X$ defined over $k$ is exact over $\mathbb{C}$, then it is exact over
$k$. It follows that the algebraic de Rham complex of $X$ over $k$ is globally exact.

We first note that $A^0(E) = \mathcal{O}_X$ and $\Omega^0_X = \mathcal{O}_X$. So to find the system
of splittings, we start with $A^1(E)$. Since $X$ is affine, there exists a splitting
$s_1 : A^1(E) \to \Omega^1_X$ of the canonical inclusion, $\Omega^1_X \subset A^1(E)$ (since $H^1(X, F)$
vanishes for every vector bundle $F$ since $X$ is affine, it follows that every
short exact sequence of vector bundles on $X$ is split). This gives a direct
sum decomposition

$$A^1(E) = \Omega^1_X \oplus B^1_E$$

where $B^1_E$ is isomorphic to the quotient of $A^1(E)$ by $\Omega^1_X$. 

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To find $s_2$, we proceed as follows:

Let the splitting $s_1$ define the direct same decomposition

$$A^1(E) = \Omega_X^1 \oplus B_E^1.$$  

We now consider $d_E(B_E^1) \subset A^2(E)$.

We now claim that

$$d_E(B_E^1) \cap \Omega_X^2 = 0.$$  

For otherwise, let $\omega$ be a 2 form on $X$ in $d_E(B_E^1)$. We know that $d_X \omega = d_E \omega$ if $\omega$ is considered as an element of $A^2(E)$. Since $\omega \in d_E(B_E^1)$, and $d^2_E = 0$, it follows that $d_X \omega = 0$. By the exactness of the algebraic de Rham complex, it follows that there is a 1-form $\eta$ on $X$ such that $\omega = d_X \eta$(recall that $\omega \in d_E(B_E^1)$). It follows that there exists $\eta' \in B_E^1$ such that $d_E \eta' = d_E \eta$.

We need the following two lemmas. We will resume the proof of the proposition after the lemmas:

**Lemma(3.2)**  Let $p : Y_1 \to Y_2$ be a finite Galois etale morphism of schemes over a field of characteristic zero such that $Y_1$ is affine. Then $Y_2$ is affine.

**Proof**  Let $\Gamma$ be the Galois group of $p : Y_1 \to Y_2$. Let $\mathcal{F}$ be a coherent sheaf on $Y_2$. Then $H^i(Y_1, p^* \mathcal{F}) = 0$ for $i \geq 1$ since $Y_1$ is affine. This implies that $H^i(Y_2, p_* p^* \mathcal{F}) = 0$ for $i \geq 1$. We now observe that $p_* p^* \mathcal{F} = \mathcal{F} \otimes p_* \mathcal{O}_{Y_1}$ (by the projection formula). Also, $p_* \mathcal{O}_{Y_1}$ is associated to the $\Gamma$ torsor $p : Y_1 \to Y_2$ by the regular representation of $\Gamma$ on its coordinate ring $k[\Gamma]$ over $k$. Since $k$ is a field of characteristic zero, $k[\Gamma]$ is a direct sum of irreducible modules; in particular, the trivial one dimensional representation is a direct summand of $k[\Gamma]$. It follows that $\mathcal{O}_{Y_2}$ is a direct summand of $p_* \mathcal{O}_{Y_1}$. Hence $\mathcal{F}$ is a direct summand of $\mathcal{F} \otimes p_* \mathcal{O}_{Y_1}$. Since $H^i(Y_2, \mathcal{F} \otimes p_* \mathcal{O}_{Y_1}) = 0$ for $i \geq 1$, it follows that $H^i(Y_2, \mathcal{F}) = 0$ for $i \geq 1$. Hence $Y_2$ is affine. QED.
Remark  It is a theorem of Chevalley that the image of a finite surjective morphism from an affine variety is affine.

Lemma (3.3)  The total space $E$ is affine.

Proof  We know that $\pi : E \to X$ is locally trivial in the etale topology on $X$. Since the structure group $G$ is affine, this implies by the above lemma (Lemma(3.2)), that every point $x$ in $X$ has a Zariski neighbourhood $U$ such that $\pi^{-1}(U)$ is affine.

Let $\mathcal{F}$ be any coherent sheaf on the total space $E$. By Proposition(8.1), page (250) in Hartshone’s Algebraic Geometry (see [6]), it follows that the higher direct images $R^i\pi_*\mathcal{F}$, $i \geq 1$ are zero (by the remark in the above paragraph). This implies that $H^i(E, \mathcal{F}) \cong H^i(X, \pi_*\mathcal{F})$ for $i \geq 0$. However, $X$ is affine, and $\pi_*\mathcal{F}$ is quasi coherent on $X$, so

$$H^i(X, \pi_*\mathcal{F}) = 0 \text{ for } i \geq 1.$$ 

Hence $H^i(E, \mathcal{F}) = 0$ for $i \geq 1$ and hence $E$ is affine. QED.

Remark  We observe that to show that $E$ is affine, we can argue as follows: since $E \to X$ is a $G$-torsor, there is a free action of $G$ on $E$, whose quotient map is the given map $E \to X$. By Proposition 0.7, page 14 in [9], the map $E \to X$ is affine. Since $X$ is affine, $E$ is affine.

We now resume our proof of Proposition (3.1). If $G$ is simply connected, and under our hypothesis that $H^1(X, \mathbb{Z}) = 0$, it follows that the first singular cohomology of $E$ is zero. This can be seen as follows: the homotopy exact sequence and the hypothesis that $G$ is simply connected implies that the
topological fundamental group of $E$ is isomorphic to that of $X$; therefore their abelianisations are isomorphic and so the vanishing of $H^1(X,\mathbb{Z})$ implies the vanishing of $H^1(E,\mathbb{Z})$ (a proof of the justification of using the homotopy exact sequence for an algebraic torsor is given in the proof of Theorem (5.3) below).

Since $E$ is affine (Lemma (3.3) above), by Grothendieck’s algebraic de Rham theorem it follows that if $\psi$ is a 1-form on $E$ such that $d_E \psi = 0$, then there is a function $\varphi$ on $E$ such that $d_E \varphi = \psi$. Now $d_E \eta' = d_E \eta \Rightarrow d_E (\eta - \eta') = 0$, and hence there is a function $\varphi$ on $E$ such that $d_E \varphi = \eta - \eta'$. However, $\eta - \eta'$ is $G$-invariant, and hence $d_E \varphi$ is $G$-invariant. For $g \in G$, $g^* d_E \varphi = d_E g^* \varphi = d_E \varphi$ and hence

$$d_E (g^* \varphi - \varphi) = 0$$

$\Rightarrow$ $g^* \varphi - \varphi = \text{constant } C_g \in k$ ($C_g$ is constant since $E$ is smooth and connected and so the zeroth de Rham cohomology of $E$ is $k$ as $X$ and $G$ are smooth and connected).

We observe that this defines a homomorphism form $G$ to the additive group (which is abelian). If we assume that $G$ is semisimple, this homomorphism is trivial, and hence we can choose a $G$-invariant function $\varphi$ such that,

$$d_E \varphi = \eta - \eta'$$

Hence

$$\eta' = \eta - d_E \varphi$$

From the exact sequence

$$0 \rightarrow \Omega^1_X \rightarrow A^1(E) \rightarrow A^1(E)/\Omega^1_X \rightarrow 0$$

and since both $\eta$ and $d_E \varphi$ are in $\Omega^1_X$, it follows that the image of $\eta - d_E \varphi$ in $A^1(E)/\Omega^1_X$ is zero. Since $\eta' \in B^1_E$, it follows that $\eta' = 0$. Hence
\[ \omega = dq' = 0, \text{and we have proved the claim that} \]
\[ d_E(B^1_E) \cap \Omega^2_X = 0 \]

We observe that \( d_E(B^1_E) \) denotes the subbundle of \( A^2(E) \) generated by the sections \( d_E\phi \) of \( A^2(E) \) as \( \phi \) varies over \( H^0(X, B^1_E) \). This can be done as \( d_EH^0(X, B^1_E) \) is a \( k \)-vector subspace of \( H^0(X, A^2(E)) \) since \( d_E \) is \( k \)-linear (see Appendix 1). We now have, on \( X \)

\[ 0 \to \Omega^2_X \oplus d_E(B^1_E) \to A^2(E) \to Q^2 \to 0 \]

We choose a splitting of

\[ A^2(E) \to Q^2 \to 0 \]

(which exists since every short exact sequence of vector bundles on \( X \) is split, as observed before) to get a direct sum decomposition

\[ A^2(E) = \Omega^2_X \oplus d_E(B^1_E) \oplus Q^2 \]

and this defines a splitting

\[ s_2 : A^2(E) \to \Omega^2_X \]

satisfying

\[ d_X s_1 = s_2 d_E \]

QED

4 Connections on induced vector bundles.

Let \( X \) be a smooth affine variety over \( k \) such that \( X_C \), the base change of \( X \) to \( \mathbb{C} \), has trivial singular cohomology. Let \( \Omega^1_X \) be algebraically trivial. Let \( G \) be a semi-simple, simply connected group over \( \mathbb{C} \) and let \( \pi : E \to X \) be a \( G \) bundle on \( X \). Let \( V \) be a rational \( G \)-module over \( k \). Then we can form the
associated vector bundle $\tilde{V} = E(V)$ on $X$.

On $E$, we have the complex

$$V \otimes \mathcal{O}_E \xrightarrow{d_E} V \otimes \mathcal{O}_E \otimes \Omega^1_E \xrightarrow{d_E} V \otimes \mathcal{O}_E \otimes \Omega^2_E$$

where $V \otimes \mathcal{O}_E$ is the trivial vector bundle with fibre the vector space $V$ on which $G$ acts, and $d_E$ is the exterior differential operator on $E$. Since, as we have observed before, $d_E$ is equivariant for the $G$ action, we can go modulo the $G$ action, and we obtain,

$$\tilde{V} \xrightarrow{d_{\tilde{V}}} \tilde{V} \otimes \Lambda^1(E) \xrightarrow{d_{\tilde{V}}} \tilde{V} \otimes \Lambda^2(E)$$

on $X$. Since $d_{\tilde{V}}^2 = 0$, we also have $d_{\tilde{V}}^2 = 0$.

By Proposition (3.1), we have splittings

$$s_i : \Lambda^i(E) \to \Omega^i_X, \quad i = 1, 2$$

such that $d_X s_1 = s_2 d_E$. We thus obtain

$$\tilde{V} \xrightarrow{\nabla} \tilde{V} \otimes \Omega^1_X \xrightarrow{\nabla} \tilde{V} \otimes \Omega^2_X$$

on $X$, where $\nabla$ is a connection on $\tilde{V}$ such that $\nabla^2 = 0$. In other words, we have constructed a flat connection on $\tilde{V}$.

**Remark (4.1)** We observe that $s_1$ (as in Proposition (3.1)) defines a connection on $E$. However, $\nabla$ is not this connection, though $s_1$ is used to define $\nabla$.

## 5 The Monodromy

Let $X$ be a smooth affine variety over $k$. We assume that the singular cohomology of $X_C$ with integer coefficients is trivial (i.e., $H^0(X_C, \mathbb{Z}) = \mathbb{Z}$, $H^i(X_C, \mathbb{Z}) = \mathbb{Z}$, $i > 0$), then $\tilde{V}$ is a flat connection on $X$. Therefore, the monodromy representation is trivial. This implies that the singular homology groups $H_1(X, \mathbb{Z})$ and $H_2(X, \mathbb{Z})$ are trivial.

**Remark (4.2)** If $X$ is a smooth projective variety over $k$, then the singular cohomology groups $H^i(X, \mathbb{Z})$ and $H^i(X, \mathbb{Z})$ are finite-dimensional vector spaces over $\mathbb{Q}$, and we can define the monodromy representation as usual. In this case, the monodromy representation is a homomorphism from the fundamental group of $X$ to the general linear group of a finite-dimensional vector space over $\mathbb{Q}$.

**Remark (4.3)** If $X$ is a smooth affine variety over $k$ and $\mathcal{F}$ is a coherent sheaf on $X$, then the higher direct images $R^i f_* \mathcal{F}$ are finite-dimensional vector spaces over $\mathbb{Q}$, and we can define the monodromy representation as usual. In this case, the monodromy representation is a homomorphism from the fundamental group of $X$ to the general linear group of a finite-dimensional vector space over $\mathbb{Q}$.

**Remark (4.4)** If $X$ is a smooth projective variety over $k$ and $\mathcal{F}$ is a coherent sheaf on $X$, then the higher direct images $R^i f_* \mathcal{F}$ are finite-dimensional vector spaces over $\mathbb{Q}$, and we can define the monodromy representation as usual. In this case, the monodromy representation is a homomorphism from the fundamental group of $X$ to the general linear group of a finite-dimensional vector space over $\mathbb{Q}$.

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Let \( \Omega^1_X \) be algebraically trivial. Let \( G \) be a semi simple, simply connected algebraic group over \( k \). Let \( \pi : E \to X \) be a \( G \)-bundle on \( X \). By Proposition (3.1) above, there are splittings \( s_i : A^i(E) \to \Omega^1_X \) \( i = 1, 2 \), such that \( d_X \circ s_1 = s_2 \circ d_E \).

We fix one such set of splittings to begin with. Given a rational representation \( V \) of \( G \), we can form the associated vector bundle \( E(V) \) on \( X \). By section 4 above, \( E(V) \) acquires a flat connection \( \nabla_V \). Further, given two \( G \)-modules \( V \) and \( W \), and a \( G \)-module homomorphism \( \varphi : V \to W \), we obtain a homomorphism of vector bundles \( \tilde{\varphi} : E(V) \to E(W) \). We note that \( \tilde{\varphi} \) is equivariant for the connections \( \nabla_V \) and \( \nabla_W \). In other words, the following diagram commutes

\[
\begin{array}{ccc}
E(V) & \xrightarrow{\nabla_V} & E(V) \otimes \Omega^1_X \\
\tilde{\varphi} \downarrow & & \downarrow \tilde{\varphi} \otimes id_X \\
E(W) & \xrightarrow{\nabla_W} & E(W) \otimes \Omega^1_X
\end{array}
\]

We now consider a faithful representation of \( G \) on a vector space \( V \) so that \( G \subset SL(V) \) (such a representation exists since \( G \) is assumed to be semisimple). We consider the induced vector bundle \( E(V) \) and the flat connection \( \nabla_V \) on \( E(V) \). We define the following category \( \mathcal{C}(E,V) = \{ \) vector bundles \( S \) on \( X \) with a flat connection such that there exist \( A \subset B \subset \oplus_i E(V)^{\otimes n_i} \otimes E(V)^{*\otimes m_i} \), with \( A \) and \( B \) preserved by the connection on \( \oplus E(V)^{\otimes n_i} \otimes E(V)^{*\otimes m_i} \), and \( S = B/A \) with the induced flat connection, where the direct sums are finite direct sums \( \} \). We now recall

**Theorem (5.1)** Let \( \mathcal{C} \) be a category with a distinguished object \( \mathcal{O} \) equipped with an operation

\[
\wedge : \mathcal{C} \times \mathcal{C} \to \mathcal{C}
\]
and let $T : \mathcal{C} \to k\text{-mod}$ be a functor satisfying the following eight conditions:

1. $\mathcal{C}$ is an abelian category with direct sums;
2. isomorphism classes of objects of $\mathcal{C}$ form a set;
3. $T$ is an additive, faithful and exact functor;
4. $\hat{\otimes}$ is $k$-linear in each variable, and $T \circ \hat{\otimes} = \otimes \circ (T \times T)$;
5. $\hat{\otimes}$ is associative preserving $T$;
6. $\hat{\otimes}$ is commutative preserving $T$;
7. the object $\mathcal{O}$ of $\mathcal{C}$ is equipped with an isomorphism $\phi : k \to T(\mathcal{O})$ such that $\mathcal{O}$ is an identity object of $\hat{\otimes}$ preserving $T$;
8. for every object $L$ of $\mathcal{C}$ such that $T(L)$ is one-dimensional, there is an object $L^{-1}$ of $\mathcal{C}$ such that $L \hat{\otimes} L^{-1}$ is isomorphic $\mathcal{O}$.

Then there exists a unique affine algebraic group scheme $H$ defined over $k$ such that the quadruple $(\mathcal{C}, \hat{\otimes}, T, \mathcal{O})$ is identified with the quadruple $(H - \text{mod}, \hat{\otimes}, T, \mathcal{O})$. A quadruple $(\mathcal{C}, \hat{\otimes}, T, \mathcal{O})$ as in the theorem satisfying the conditions is called a Tannakian category.

**Proof** See section 4, Chapter 2, pages 152-153 in [12] and pages 118-120 in [10]. QED

We now further recall the following. Let $H$ be an affine algebraic group scheme over $k$ and $S$ a scheme over $k$. Let $E_H \to S$ be a principal $H$-bundle on $S$. Then for every representation $\rho : H \to GL(V)$ in $H$-mod, we can construct the associated vector bundle $E_\rho = E_H(V)$. This defines a functor $\mathcal{F}_E$ from $H$-mod to the category Vect $(S)$ of vector bundles on $S$. We have
**Theorem (5.2)** Let \( \mathcal{F} : H\text{-mod} \rightarrow \text{Vect}(S) \) be a functor satisfying the following:

1. \( \mathcal{F} \) is a \( k \)-additive exact functor;
2. \( \mathcal{F} \circ \hat{\otimes} = \otimes \circ (\mathcal{F} \times \mathcal{F}) \);
3. \( \mathcal{F} \) preserves commutatively, in other words, if \( c \) is the canonical isomorphism of \( V \hat{\otimes} W \) with \( W \hat{\otimes} V \) in \( H\text{-mod} \), then \( \mathcal{F}(c) \) is the canonical isomorphism of the corresponding vector bundles;
4. \( \mathcal{F} \) preserves associatively, i.e., if \( a \) is the canonical isomorphism of \( U \hat{\otimes} (V \hat{\otimes} W) \) with \( (U \hat{\otimes} V) \hat{\otimes} W \) in \( H\text{-mod} \), then \( \mathcal{F}(a) \) is the canonical isomorphism of the corresponding vector bundles;
5. the vector bundle \( \mathcal{F}(\mathcal{O}) \) is the trivial line bundle \( \mathcal{O}_S \) on \( S \);
6. for any \( V \in H\text{-mod} \) of dimension \( r \), the vector bundle \( \mathcal{F}(V) \) is of rank \( r \).

Then there exists a unique principal \( H \)-bundle \( E \rightarrow S \) such that \( \mathcal{F} \) is identified with \( \mathcal{F}_E \).

**Proof** See [10], Lemma 2.3, Proposition 2.5. QED

We now observe that the category \( \mathcal{C}(E,V) \) defined above, with \( \hat{\otimes} \) as tensor product of vector bundles, \( T \) as the functor which assigns to a vector bundle \( W \in \mathcal{C}(E,V) \) the fibre \( W_x \) of \( W \) at a fixed point \( x \in X \), \( \mathcal{O} \) as the trivial line bundle \( \mathcal{O}_X \) on \( X \), and homomorphisms preserving connections, is a Tannakian category as in Theorem (5.1) above. Hence there is an affine group scheme \( M \) over \( k \), associated to \( \mathcal{C}(E,V) \), which we call the monodromy, by Theorem (5.1) above.

We have an identification \( M\text{-mod} = \mathcal{C}(E,V) \) where \( M\text{-mod} \) denotes the
category of rational $M$-modules as before. Since every object of $\mathcal{C}(E, V)$ is a vector bundle on $X$, we obtain a natural functor

$$\mathcal{F} : M \text{ mod} \rightarrow \text{Vect}(X),$$

and it is easy to see that $\mathcal{F}$ satisfies all the conditions of Theorem (5.2) above. We thus obtain a $M$-bundle,

$$\pi_1 : E_M \rightarrow X \text{ on } X.$$

We now observe that every $G$-module is in a natural way an $M$-module as follows: Given a $G$-module $W$, it can be obtained as a subquotient of a finite direct sum $\bigoplus_i V^\otimes n_i \otimes V^* \otimes m_i$ where $V$ is the faithful $G$-module we have chosen at the outset. Hence the connection on $E(W)$ is preserved by the connection on $\bigoplus_i E(V)^{\otimes n_i} \otimes E(V)^{* \otimes m_i}$. Thus $E(W)$ is in the category $\mathcal{C}(E, V)$ and is hence an $M$-module. We thus obtain a natural functor

$$G \text{ mod} \rightarrow M \text{ mod}$$

and hence a homomorphism of group schemes $M \rightarrow G$. We now show that every $M$-module is a subquotient of a $G$-module, as follows: an $M$-module is an object of $\mathcal{C}(E, V)$, i.e., is a vector bundle $W_1$ on $X$ with a flat connection which is a subquotient of $\bigoplus_i E(V)^{\otimes n_i} \otimes E(V)^{* \otimes m_i}$ preserved by the flat connection. Since $\pi^* E(V)$ is trivial, with the pullback connection $\pi^*(\nabla)$ being the exterior differential (see Appendix 2), it follows that $\pi^*(W_1)$ is also trivial with the pullback of the connection being the exterior differential $d$. We thus obtain a vector space subquotient $W$ of $\bigoplus_i V^\otimes n_i \otimes V^* \otimes m_i$ with an $M$ action on $W$ induced by the $G$ action on $V$ through the already defined homomorphism $M \rightarrow G$. Since $\bigoplus_i V^\otimes n_i \otimes V^* \otimes m_i$ is a $G$ module, we have obtained an $M$-module $W$ as a subquotient of a $G$-module such that $W_1$ is isomorphic to the associated bundle $E_M(W)$ with the induced flat connection. This proves our claim.
Since every $M$ module is a subquotient of a $G$-module, by Proposition 2.21 (b), page 139, in Deligne-Milne's paper (see [3]), $M$ is a closed subgroup scheme of $G$. If $\mathcal{F}_E$ is the functor $\mathcal{F}_E : G \mod \to Vect(X)$ which associates to a rational $G$-module $W$ the vector bundle $E(W)$ on $X$, we obtain a commutative diagram

$$
\begin{array}{ccc}
G - \text{mod} & \longrightarrow & M - \text{mod} \\
\mathcal{F}_E & \swarrow & \mathcal{F}_{EM} \\
Vect(X) & \searrow & \\
\end{array}
$$

and this shows that the principal bundle $E_M$ induces the $G$-bundle $E$ on $X$, by Theorem (5.2) above. We have thus obtained a reduction of structure group $E_M \subset E$ of $E$ to $M$.

**Theorem (5.3)** Let $k$ be the field of complex numbers. Let the topological fundamental group of $X$ be trivial. Then $E_M$ is trivial.

**Proof** We first observe that since $X$ is simply connected, the universal cover of $X$ is algebraic, and the universal covering map is algebraic, viz, the identity map $X \to X$. If $\pi : E \to X$ is the $G$-torsor we started with, from our hypothesis that $G$ is simply connected, and the homotopy exact sequence for the topological fundamental group, we obtain that $E$ is also simply connected. We observe that the use of the homotopy exact sequence for an algebraic torsor can be justified as follows: we show that $E \to X$ is a Serre fibration (see Definition (6.2), Chapter 7 in [I]) and by Theorem (6.7), Chapter 7 in [II], we use the homotopy exact sequence. To see that $E \to X$ is a Serre fibration, we observe that by Theorem (6.11), Chapter 7, in [I], the property is local on the base. Now, $E \to X$ is locally isotrivial (i.e., every point has a finite etale neighbourhood over which the torsor is trivial, see Remark (1.1) above, and [III], Lemme XIV 1.4), and a finite etale cover is a Serre fibration since it
is a topological covering space, while a trivial bundle is clearly a fibration. It follows that $E \to X$ is a Serre fibration and we can apply the homotopy exact sequence. So the universal cover of $E$ is also algebraic, viz, the identity map $E \to E$. Let $E_{SL(V)}$ denote the principal frame bundle associated to the vector bundle $E(V)$. Since $\pi^*(E(V))$ is trivial on $E$, $\pi^*(E_{SL(V)})$ is the trivial $SL(V)$ bundle on $E$.

Let $\tilde{X}$ denote the universal cover of $X$ (since $X$ is simply connected, $\tilde{X} = X$). Since $E(V)$ carries a flat connection $\nabla$, $E_{SL(V)}$ is induced analytically by a map $\tilde{X} \to E_{SL(V)}$ inducing the flat structure. Similarly, if $\tilde{E}$ denotes the universal cover of $E$ ($\tilde{E} = E$ since $E$ is simply connected), the flat connection $\pi^*(\nabla)$ on $\pi^*E_{SL(V)}$ is induced analytically by a map $\tilde{E} \to \pi^*E_{SL(V)}$ inducing the flat structure on $\pi^*E_{SL(V)}$. Further, by construction (see section 4 above and Appendix 2), the connection $\nabla$ on $E(V)$ has the property that $\pi^*(\nabla) = d$, the exterior differential, on the trivial bundle $\pi^*(E(V))$. The map $\tilde{E} \to \pi^*E_{SL(V)}$ is defined by a linearly independent set of sections $s_i$ of $\pi^*(E(V))$ such that $\pi^*(\nabla)(s_i) = 0$. Since $\pi^*(\nabla) = d, ds_i = 0$, and so the $s_i$ are constant. Since constant sections are algebraic, it follows that the morphism $\tilde{E} \to \pi^*E_{SL(V)}$ given by the flat connection (viz, $d$) on $\pi^*(E_{SL(V)})$ is algebraic. From the commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\pi^*E_{SL(V)}} & \pi^*E_{SL(V)} \\
\pi \downarrow & & \downarrow \\
X & \xrightarrow{} & E_{SL(V)}
\end{array}
$$

it follows that the morphism $X \to E_{SL(V)}$ given by the flat connection $\nabla$ is also algebraic (we have to show that an algebraic function on $E_{SL(V)}$ pulls back to an algebraic function on $X$; however an algebraic function on $E_{SL(V)}$ pulls back to an algebraic function on $\pi^*(E_{SL(V)})$ since $\pi$ is algebraic, and an algebraic function on $\pi^*(E_{SL(V)})$ pulls back to an algebraic function on $E$ since $E \to \pi^*E_{SL(V)}$ is algebraic; since a function pulled back from $E_{SL(V)}$...
under \( \pi \) is \( G \)-invariant we obtain a \( G \)-invariant algebraic function on \( E \) which gives an algebraic function on \( X \). We observe that the connection on every object of \( \mathcal{C}(E,V) \) is induced from the connection on \( E(V) \), and hence by the construction of \( M \) and \( E_M \), the morphism \( X \to E_{SL(V)} \) factors through \( E_M \). We further observe that if \( \phi \) denotes the map \( \tilde{X} \to E_{SL(V)} \) defining the flat structure, and \( \tilde{\pi} \) denotes the projection \( E_{SL(V)} \to X \), then \( \tilde{\pi} \circ \phi \) is the universal covering map \( \tilde{X} \to X \) (this can be seen by noting that \( E_{SL(V)} \) is defined by a representation of the topological fundamental group of \( X \) since it carries a flat connection). Thus we obtain an algebraic section \( X \to E_M \). Since a principal bundle with a section is trivial, it follows that \( E_M \) is trivial (algebraically).

\textbf{QED}

\textbf{Remark (5.4)} We observe that \( \nabla \) has regular singularities by construction, since \( \pi^*(\nabla) = d \), and \( d \) has regular singularities. An alternative proof of the above theorem can be given using Deligne’s theorem (see [2] and [7]).

6 The Main Theorem

Let \( X \) be a smooth, affine variety defined over \( k \), where \( k \) is an algebraically closed field of characteristic 0 contained in \( \mathbb{C} \). Let \( G \) be a semisimple algebraic group over \( k \). In this section, we prove

\textbf{Theorem (6.1)} Let \( X_\mathbb{C} \) have trivial singular cohomology with integer coefficients (i.e., \( H^0(X_\mathbb{C}, \mathbb{Z}) = \mathbb{Z} \) and \( H^i(X_\mathbb{C}, \mathbb{Z}) = 0 \) for \( i \geq 1 \)), let \( \Omega^1_X \) be algebraically trivial, and assume that \( X_\mathbb{C} \) is topologically simply connected. Then any \( G \)-bundle \( \pi : E \to X \) on \( X \) over \( k \) is trivial. (See Remark (6.4))

\textbf{Proof} We first reduce to the simply connected case. So we assume (for the purpose of this step), that the theorem is true for every semisimple, simply
connected group over \( k \). We now consider a semisimple group \( G \). If \( G \) is not simply connected, then let \( \tilde{G} \to G \) be its simply connected cover. We observe that the kernel \( F = \text{Ker}(\tilde{G} \to G) \) is central in \( \tilde{G} \). We have

\[
1 \to F \to \tilde{G} \to G \to 1
\]

and the induced etale cohomology sequence

\[
H^1(X, \tilde{G}) \to H^1(X, G) \to H^2(X, F)
\]

We observe that we may assume without loss of generality that \( F \) is the group \( \mathbb{Z}/n \). Since we have assumed that \( H^i(X, \mathbb{Z}) \) vanishes for \( i = 2, 3 \), we obtain \( H^2(X, F) = 0 \) (we observe that etale cohomology and singular cohomology with cyclic coefficients are isomorphic by Theorem (3.12), Chapter 3 in [8]).

Thus \( H^1(X, \tilde{G}) \to H^1(X, G) \) is surjective. The \( G \)-bundle \( E \) on \( X \) can be lifted to a \( \tilde{G} \) bundle \( \tilde{E} \) on \( X \). Since \( \tilde{G} \) is semisimple and simply connected, if the bundle \( \tilde{E} \) is shown to be trivial, then \( E \) is trivial as well.

Thus we have reduced to the case when \( G \) is semisimple and simply connected.

For the rest of the proof, we assume that \( G \) is semisimple and simply connected.

As before, let \( \pi : E \to X \) be a \( G \)-bundle on \( X \). Then by Lemma (2.1), we can assume that \( k \) is the field of complex numbers \( \mathbb{C} \). By Theorem (5.3) and the paragraph preceding it, it follows that \( \pi : E \to X \) is the trivial \( G \)-torsor. QED

**Remark (6.2)** We observe that we have only used that \( H^i(X, \mathbb{Z}) = 0 \) for \( i = 1, 2 \), and \( 3, \Omega^1_X \) is algebraically trivial, and that \( X_\mathbb{C} \) is topologically simply connected.
Remark (6.3) There are contractible, affine, smooth surfaces over \( \mathbb{C} \) which are not isomorphic to affine two space (see [5]).

Remark (6.4) We note that when \( X_{\mathbb{C}} \) is contractible (as is the case when it is simply connected and the singular cohomology with integer coefficients is trivial), the tangent bundle of \( X_{\mathbb{C}} \) is topologically trivial, and since \( X_{\mathbb{C}} \) is Stein, this implies that the tangent bundle is holomorphically trivial. This is enough for the proof in appendix 1 to carry through, so Proposition (3.1) is valid under the hypothesis that \( X_{\mathbb{C}} \) is contractible. The main theorem then states that if \( X_{\mathbb{C}} \) is contractible, then every algebraic \( G \)-torsor on \( X \) is algebraically trivial, when \( G \) is a semisimple algebraic group.

A Appendix 1

We have the splitting
\[
A^1_E = \Omega^1_X \oplus B^1_E
\]
with \( B^1_E \) isomorphic to the Lie algebra bundle \( E(\mathfrak{g}) \).

There is natural inclusion of bundles
\[
\wedge^2 A^1_E \subset A^2_E
\]
and thus we get the subbundle
\[
\Omega^2_X \oplus (\Omega^1_X \otimes B^1_E) \oplus \wedge^2 B^1_E \subset A^2_E
\]
The differential operator \( d_E \) takes \( H^0(X, B^1_E) \) to a \( \mathbb{C} \)-vector subspace \( d_E H^0(X, B^1_E) \subset H^0(X, A^2_E) \).

We consider the \( \mathbb{C} \)-vector space spanned
\[
M = d_E H^0(X, B^1_E) + H^0(X, \Omega^1_X \otimes B^1_E)
\]
\( M \) is a \( \mathbb{C} \)-vector subspace of \( H^0(X, A^2_E) \).
For $f \in H^0(X, \mathcal{O}_X)$ and $\eta \in H^0(X, B_E^1)$, we have,

$$f \, d_E \eta = d_E(f \eta) - d \, f \otimes \eta$$

Hence $M$ is a $H^0(X, \mathcal{O}_X)$ submodule of $H^0(X, A^2_E)$.

We now show that $M$ generates a vector subbundle of $A^2_E$ disjoint from $\Omega^2_X$ as follows:

Let $Y = E$ and $p : Y \to X$ be the projection $p = \pi$. Then $p^* E = Y \times G$.

Let $\tilde{\pi} : p^* E \to Y$ so that

$$
\begin{array}{ccc}
\pi & \rightarrow & \pi \\
\tilde{\pi} \downarrow & & \downarrow \\
Y & \rightarrow & X
\end{array}
$$

is a commutative diagram (fibre product). Since $\pi^* B^1_E \subset \Omega^1_E$, we have

$$\tilde{\pi}^* p^* B^1_E \subset \Omega^1_{p^* E} = \Omega^1_Y \oplus \Omega^1_G$$

We also have

$$\pi^* d_E = d \text{ on } \Omega^1_E$$

so

$$\tilde{\pi}^* d_E = \tilde{\pi}^* \pi^* d_E = \tilde{\pi}^* d = d$$

Further since $\pi^* B^1_E$ is isomorphic to $E \times g^*$ (the trivial bundle).

We have

$$\tilde{\pi}^* p^* B^1_E$$

isomorphic to $Y \times G \times g^*$ (the trivial bundle with fibre $g^*$).

We have

$$\tilde{\pi}^* p^* B^1_E \subset \Omega^1_Y \oplus \Omega^1_G \oplus \Omega^2_G$$

Now, let $w_1, \ldots, w_n$ be $G$ invariant one forms on $G$ forming a global frame (trivialisation on $G$) of $\Omega^1_G$. 

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(We observe that a $G$-invariant differential form on $G$ is nowhere vanishing or identically zero).

We remark that $dw_i$ are $G$-invariant.

If $\sum_i \lambda_i dw_i(x) = 0$ for some $x \in G$ and $\lambda_i \in \mathbb{C}$.

then

$$\sum \lambda_i dw_i(x) = d(\sum \lambda_i w_i)(x) = 0$$

Since $\sum \lambda_i w_i$ is $G$-invariant, so is $d(\sum \lambda_i w_i)$ and since it vanishes at a point, it is identically zero.

Hence

$$d(\sum \lambda_i w_i) = 0 \text{ on } G.$$  

Since $G$ is simply connected, $H^1(G, \mathbb{C}) = 0$ and by Grothendieck’s algebraic de Rham theorem,

$$\sum \lambda_i w_i = df$$

for a function $f$ on $G$. Since $\sum \lambda_i w_i$ is $G$-invariant, for $g \in G$, $g^* df = df$

$$\Rightarrow d(g^* f) = df$$

$$\Rightarrow d(g^* f - f) = 0$$

$$\Rightarrow g^* f - f = C_g$$

where $C_g \in \mathbb{C}$ is a constant depending on $g \in G$.

We thus obtain a homomorphism $G \to G_a$ (where $G_a$ is the additive group) by sending $g \mapsto C_g$.

Since $G$ is semisimple and $G_a$ is abelian, and there is no nonconstant morphism from a semisimple algebraic group to an abelian group, we obtain $C_g = 0$ for all $g \in G$.

Hence $f$ is $G$-invariant. The only $G$-invariant function on $G$ is the constant function, so $f$ is constant. Therefore, $df = 0$, hence

$$\sum \lambda_i w_i = 0$$
contradicting the fact that the \( w_i \) form a frame at every point of \( G \).
Thus the \( dw_i \) span a constant rank subbundle (in fact trivial) of \( \Omega^2 p^* E \) disjoint from \( \tilde{\pi}^* p^* \Omega^2_X \) and \( \tilde{\pi}^* p^* (\Omega^1_X \otimes B^1_E) \).
The bundle \( \tilde{\pi}^* p^* B^1_E \) is trivial.
Since \( \Omega^1_X \) is algebraically trivial, there are forms \( \xi_1, \ldots, \xi_n \in H^0(X, \Omega^1_X) \) such that \( \xi_1 + w_1, \ldots, \xi_n + w_n \) form a global frame (trivialisation) for \( \tilde{\pi}^* p^* B^1_E \). A \( G \)-invariant section of \( \tilde{\pi}^* p^* B^1_E \) is of the form

\[
\sum_i f_i (\xi_i + w_i)
\]
where \( f_i \in H^0(Y, \mathcal{O}_Y) \). For a section \( \eta \) of \( B^1_E, \eta \in H^0(X, B^1_E) \), \( \tilde{\pi}^* p^* \eta \) is \( G \)-invariant so

\[
\tilde{\pi}^* p^* \eta = \sum_i f_i (\xi_i + w_i)
\]
where \( f_i \in H^0(Y, \mathcal{O}_Y) \).
We now have

\[
\tilde{\pi}^* p^* dE \eta = d\tilde{\pi}^* p^* \eta = d(\sum f_i (\xi_i + w_i)) = \sum_i df_i \otimes (\xi_i + w_i) + \sum_i f_i (d\xi_i + dw_i)
\]
If at some point \( p = (y_0, g_0) \in Y \times G \) we have a linear relation

\[
\sum \lambda_i (d\xi_i(p) + dw_i(p)) = 0
\]
for \( \lambda_i \in \mathbb{C} \),
then since \( p = (y_0, g_0) \) and \( d\xi_i \) is pulled back from \( Y \) while \( dw_i \) is pulled back from \( G \), we have

\[
\sum \lambda_i d\xi_i(y_0) = 0
\]
and

\[
\sum \lambda_i dw_i(g_0) = 0
\]
From the second relation above at \( g_0 \) and the earlier remark that the \( dw_i \) are linearly independent at every point of \( G \), it follows that that \( \lambda_i = 0 \ \forall i \). Hence \( d\xi_i + dw_i \) are linearly independent at every point of \( Y \times G \).

If
\[
\sum \lambda_i(df_i \otimes (\xi_i + w_i))(P) = \sum \mu_i f_i(d\xi_i + dw_i)(P)
\]
at some point \( P \), for some \( \lambda_i, \mu_i \in \mathbb{C} \), then we have
\[
\alpha_1 + \beta = \beta_2 + \gamma
\]
where
\[
\alpha_1 = \sum \lambda_i df_i \wedge \xi_i(P) \\
\in q_1^* \Omega^2_{Y,P}
\]
\[
\beta = \sum \lambda_i(df_i \otimes w_i)(P) \\
\in q_1^* \Omega^1_{Y,P} \otimes q_2^* \Omega^1_{G,P}
\]
\[
\alpha_2 = \sum \mu_i f_i d\xi_i(P) \\
\in q_1^* \Omega^2_{Y,P}
\]
and
\[
\gamma = \sum \mu_i f_i dw_i \\
\in q_2^* \Omega^2_{G,P}
\]
where \( q_1, q_2 \) are the two projections from \( Y \times G \) to \( Y \) and \( G \) respectively.

From the direct sum decomposition
\[
q_1^* \Omega^2_Y \oplus (q_1^* \Omega^1_Y \otimes q_2^* \Omega^1_G) \oplus q_2^* \Omega^2_G
\]
it follows that \( \beta(P) = 0 \) and \( \gamma(P) = 0 \).

Now,
\[
\sum \mu_i f_i(P) \ dw_i(P) = 0
\]
implies that
\( \mu_i = 0 \) whenever \( f_i(P) \neq 0 \).

Hence
\[
\sum \mu_i f_i(P) \, d \xi_i (P) = 0
\]
since for every \( i \), either
\[ f_i(P) = 0 \text{ or } \mu_i = 0 \]
Thus \( \alpha_2(P) \) is also 0. So the intersection is 0.

We have thus shown that \( M \) generates a vector subbundle of \( A^2_E \) isomorphic to
\[
(\Omega^1_X \otimes B^1_E) \oplus V
\]
where \( V \) is a rank \( n \) subbundle and this is disjoint from \( \Omega^2_X \).

B Appendix 2

On \( E \), we have the following commutative diagrams:

\[
\begin{array}{ccccccc}
0 & 0 & \\
\downarrow & \downarrow & \\
0 & \to & \pi^*\Omega^1_X & \to & \pi^*A^1_E & \to & Q_1 & \to & 0 \\
\downarrow & \downarrow & \parallel & \\
0 & \to & \Omega^1_E & \to & A^1_{\pi^*E} & \to & Q_1 & \to & 0 \\
\downarrow & \downarrow & \\
\Omega^1_{E/X} & = & \Omega^1_{E/X} \\
\downarrow & \downarrow & \\
0 & 0 & \\
\end{array}
\]

and
The splittings $s_1, s_2$ (see Proposition (3.1) in the manuscript),

$$
s_1 : A^1_E \to \Omega^1_X \quad s_2 : A^2_E \to \Omega^2_X
$$

chosen so that

$$d_X \circ s_1 = s_2 \circ d_E$$

are seen to induce splittings

$$\tilde{s}_i : A^i_{\pi^*E} \to \Omega^i_{\pi^*E} \quad i = 1, 2$$

such that

$$d \circ \tilde{s}_1 = \tilde{s}_2 \circ d_{\pi^*E}$$

where $d_{\pi^*E} : A^1_{\pi^*E} \to A^2_{\pi^*E}$ is the operator defined for the bundle $\pi^*E$ on $E$ as before, and such that

$$\tilde{s}_i \circ \pi^* A^i_E = \pi^* s_i \quad i = 1, 2$$

(This can be seen as follows: the splittings $s_i$ give splittings $Q_i \to \pi^* A^i_E$ and these give splittings $Q_i \to A^i_{\pi^*E}$, and these give the $\tilde{s}_i$.) Since $\pi^*E$ is trivial
on $E$, we have a trivialisation $\pi^*E(V) = E \times V$ on $E$. Let $\pi^*(\nabla) = d + \omega$ on $\pi^*E(V) = E \times V$ where $\omega$ is a matrix of 1-forms on $E$. We observe that $\pi^*(\nabla)(= d + \omega)$ on $E \times V$ is $s_1 \circ d_{\pi^*E}$ and $\pi^*(\nabla)(= d + \omega)$ on $E \times V \otimes \Omega^1_E$ is $\tilde{s}_2 \circ d_{\pi^*E} \circ i_1$, where $i_1$ is the inclusion

$$E \times V \otimes \Omega^1_E \subset E \times V \otimes A^1_{\pi^*E}$$

We have the commutative diagram

$$
\begin{array}{ccc}
E \times V & \xrightarrow{d_{\pi^*E}} & E \times V \otimes A^1_{\pi^*E} \\
\| & \downarrow{\tilde{s}_1} & \uparrow{i_1} \\
E \times V & \xrightarrow{\pi^*(\nabla)} & E \times V \otimes \Omega^1_E \\
\| & \downarrow{\tilde{s}_2} & \uparrow{i_2} \\
E \times V & \xrightarrow{\pi^*(\nabla)=d+\omega} & E \times V \otimes \Omega^2_E
\end{array}
$$

Since $\pi^*(\nabla) = d + \omega$ from $E \times V \otimes \Omega^1_E$ to $E \times V \otimes \Omega^2_E$ is equal to $\tilde{s}_2 \circ d_{\pi^*E} \circ i_1$, and by construction $d \circ \tilde{s}_1 = \tilde{s}_2 \circ d_{\pi^*E}$, it follows that $\pi^*(\nabla) = d$ on $E \times V \otimes \Omega^1_E$. In other words, $\omega = 0$ and $\pi^*(\nabla) = d$.

**References**

[1] G.E.Bredon, Topology and Geometry, Springer 2005.

[2] P.Deligne, Equations differentielles a points singuliers regulieres, Lecture Notes in Math. 163, Springer 1970.

[3] P. Deligne and J.S. Milne, Tannakian categories: in Hodge Cycles, Motives, and Shimura Varieties, Lecture Notes in Math. 900, Springer, Berlin, 101–228 (1982).

[4] P.Griffiths and J.Harris, Principles of Algebraic Geometry, Wiley-Interscience Publications, 1978.

[5] R.V.Gurjar and M.Miyanishi, Affine surfaces with $\pi \leq 1$, Algebraic Geometry and Commutative Algebra (in honor of M.Nagata), Academic Press, 1988.
[6] R. Hartshorne, Algebraic Geometry, Grad. Texts in Math. 52, Springer New York, (1977).

[7] Nicholas M. Katz, An overview of Deligne’s work on Hilbert’s twenty-first problem: in Mathematical developments arising from Hilbert problems, Proceedings of symposia in pure mathematics, Vol. 28, AMS 1976.

[8] J.S. Milne, Étale Cohomology, Princeton University Press, 1980.

[9] D. Mumford, Geometric Invariant Theory, second edition, Springer-Verlag.

[10] M.V. Nori, The fundamental group scheme, Proc. Ind. Acad. Sci. Math. Sci. 91, 73–122 (1982).

[11] M. Raynaud, Faisceaux amples sur les schemas en groupes et les espaces homogenes, Lecture Notes in Mathematics 119, Springer-Verlag, Berlin, (1970).

[12] N. Saavedra Rivano, Categories Tannakiens, Lecture Notes in Math. 265, Springer, Berlin, (1972).

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