HIGHER INTEGRABILITY AND STABILITY OF \((p,q)\)-QUASIMINIMIZERS

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Abstract. Using purely variational methods, we prove local and global higher integrability results for upper gradients of quasiminimizers of a \((p,q)\)-Dirichlet integral with fixed boundary data, assuming it belongs to a slightly better Newtonian space. We also obtain a stability property with respect to the varying exponents \(p\) and \(q\). The setting is a doubling metric measure space supporting a Poincaré inequality.

1. Introduction

We study regularity for the gradient of quasiminimizers to the following \((p,q)\)-Dirichlet integral

\[
\int_{\Omega} (ag_u^p + bg_u^q) \, d\mu,
\]

in the context of metric measure spaces, where \(1 < p < q\) and \(g_u\) is the minimal \(q\)-weak upper gradient of \(u\). Here, \((X, d, \mu)\) is a complete metric measure space endowed with a metric \(d\) and a doubling measure \(\mu\), supporting a weak \((1,p)\)-Poincaré inequality and \(\Omega \subset X\) is an open bounded set, whose complement satisfies a uniform \(p\)-fatness condition. Furthermore, we consider some coefficient functions \(a\) and \(b\) to be measurable and satisfying \(0 \leq \alpha \leq a, b \leq \beta\), for some positive constants \(\alpha\) and \(\beta\). This study extends the work in [32] since this only concerns \(p\)-quasiminimizers.

We give not only qualitative properties but also quantitative ones providing an explicit analysis on the dependencies of the constants.

Quasiminimizers were originally introduced by Giaquinta and Giusti [14, 15] as a unifying tool. The main advantage is that the definition of quasiminimizers in \(\mathbb{R}^n\) depends only on moduli of gradients, therefore we can substitute them by upper gradients. This way, the theory of partial differential equations is generalized to metric measure spaces. Quasiminimizers have been an active research topic for several years in the setting of doubling metric measure spaces supporting a Poincaré inequality. For instance, the boundary continuity for \((p,q)\)-quasiminimizers on a bounded set \(\Omega\) with fixed boundary data has been examined in [40], furthermore, it was proven that \((p,q)\)-quasiminimizers are (locally) Hölder continuous. In [24], Kinnunen, Marola, and Martio proved that an increasing sequence of quasiminimizers converges locally uniformly to a quasiminimizer, provided that the limit function is finite at some point.

In this manuscript, we first show a global higher integrability for upper gradients of \((p,q)\)-quasiminimizers of the Dirichlet integral (1.1) with fixed boundary data, see Theorem 4.2. In the Euclidean setting, the first results concerning local higher integrability are by Bojarski [2] and Elcrat and Meyers [9]. For more local results, see also [6, 16, 17, 43]. In [18], Granlund showed that if the complement of the domain satisfies a certain measure density condition, then minimizers have a global higher integrability property. Later, Kilpeläinen and Koskela [22] generalized this result to a uniform capacity density condition. In order to prove global higher integrability, we also require a regularity condition for the complement of the domain, more specifically, \(X \setminus \Omega\) is assumed to be uniformly \(p\)-fat. The main steps in our proof are showing that minimal upper gradients satisfy a reverse type Hölder inequality, applying Gehring’s lemma [31, 42] and finally, generalizing the
resulting local higher integrability to the entire $\Omega$. Therefore, an appropriate covering argument is needed. Since we are considering quasiminimizers with fixed boundary data, we are able to work near and also on the boundary. Consequently, we can cover $\Omega$ by balls that are inside the set, together with those that intersect the complement. On the one hand, when working inside $\Omega$, a De Giorgi type inequality (3.2) implies that the minimal upper gradient satisfies a reverse Hölder inequality. On the other hand, we have to be careful when working up to the boundary and use some delicate extra tools. Here the $p$-fatness of the complement plays an important role. Furthermore, we need self-improving properties of the Poincaré inequality (see [20]) and the $p$-fatness condition (see [5]). These results allow us to use a capacity version of a Sobolev-Poincaré type inequality, i.e., a Maz'ya type estimate (see [3]) and so, obtain the desired reverse Hölder inequality. Finally, since $\Omega$ is by hypothesis a bounded set, it is possible to obtain a finite covering using balls that either are contained in $\Omega$ or intersect its complement. There is a rich literature concerning higher integrability results in the Euclidean setting regarding both elliptic and parabolic cases, see for example [1, 7, 10, 12, 28, 39]. In particular, Colombo and Mingione [7] prove local regularity of the gradient for minimizers for double phase variational problems. Instead, for the metric setting we refer the reader to [32] for the elliptic case and to [11, 37, 38] for the parabolic case.

The other result obtained in our study is a stability property, see Theorem 5.2. We consider a sequence $(u_i)$ of $(p_i, q_i)$-quasiminimizers of the corresponding integral (1.1). We assume that all functions $u_i$ have the same boundary data and quasiminimizing constant. Unlike the Euclidean case studied in [23], where the authors prove that minimizers with varying exponent converge to the solution of the limit problem in the Sobolev space, we need to assume a priori that the sequence of $(p_i, q_i)$-quasiminimizers converges to a certain $u$. Roughly speaking, we need this convergence assumption from the beginning because when working with quasiminimizers, instead of minimizers, we lose uniqueness and thus, the uniqueness of the limit. We prove that if the sequences $(p_i)$ and $(q_i)$ converge respectively to $p$ and $q$, then there exists a $(p, q)$-quasiminimizer $u$ of (1.1) with the same boundary data and, furthermore, the sequence $(u_i)$ converges to $u$. To be able to prove this convergence, we use the global higher integrability. Li and Martio [29] examined a quasilinear elliptic operator and proved a corresponding convergence result for solutions of an obstacle problem in a bounded subset of $\mathbb{R}^n$. Later, they proved a similar result for a double obstacle problem [28]. For more references concerning stability results in the Euclidean setting, we refer the reader to [26, 29, 30] and the references therein.

This paper is motivated by the work of Maasalo and Zatorska-Goldstein [32] and is a continuation of [10]. The novelty is that, as already anticipated, we include both $p$-Laplace and $q$-Laplace operators, involving also some measurable coefficient functions $a$ and $b$, assuming only they are bounded away from zero and infinity. This condition over the coefficients is essential for our approach, specifically when using Maz’ya type estimate for the higher integrability up to the boundary part, however it is an open question if it could be relaxed. Already in the Euclidean setting, it would be interesting to find out if one can establish global higher integrability with assumptions as in [7]. To see more about non standard growth conditions, see [8, 33, 34, 35, 36].

The present work is organized as follows: in Section 2 we fix the general setup and we present basic facts about analytic tools used in metric setting. The results are stated without proofs. The reader familiar with metric measure spaces may omit this part. In particular, in Section 2.3, we present Newtonian spaces, results related to the Poincaré inequality and also some useful properties concerning the space with zero boundary values. Section 3 is devoted to introduce the concept of $(p, q)$-quasiminimizers and report the De Giorgi type inequality, as proven in [10]. Section 4 deals with the higher integrability problem for quasiminimizers and Section 5 contains the proof of the stability result.
HIGHER INTEGRABILITY AND STABILITY OF \((p,q)\)-QUASIMINIMIZERS

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2. Mathematical background

Let \((X,d,\mu)\) be a complete metric measure space, where \(\mu\) is a Borel regular measure with \(\mu(E) > 0\) for every \(E \subset X\) nonempty open set and \(\mu(A) < \infty\) for every \(A \subset X\) bounded set. Let \(B = B(y,r) \subset X\) be a ball with center \(y \in X\) and radius \(r > 0\). When there is no possibility of confusion, we denote by \(\lambda B\) a ball with the same centre as \(B\) but \(\lambda\) times its radius.

We set

\[
u_S = \frac{1}{\mu(S)} \int_S u \, d\mu = \int_S u \, d\mu,
\]

where \(S \subset X\) is a measurable set of finite positive measure and \(u : S \to \mathbb{R}\) is a measurable function. Throughout this paper, we will indicate with \(C\) all positive constants, even if they assume different values, unless otherwise specified.

**Definition 2.1** ([3], Section 3.1). A measure \(\mu\) on \(X\) is said to be doubling if there exists a constant \(C_d \geq 1\), called the doubling constant, such that for every ball \(B\) in \(X\)

\[
0 < \mu(2B) \leq C_d \mu(B) < \infty.
\]

**Lemma 2.2** ([3], Lemma 3.3). Let \((X,d,\mu)\) be a metric measure space with \(\mu\) doubling. Then there is \(Q > 0\) such that

\[
\frac{\mu(B(y,\rho))}{\mu(B(x,R))} \geq C \left(\frac{\rho}{R}\right)^Q,
\]

for all \(\rho \in [0,R]\), \(x \in \Omega\), \(y \in B(x,R)\), where constants \(Q\) and \(C\) depend only on \(C_d\).

**Definition 2.3** ([32], Section 2.1.7). A metric space \(X\) is said to be linearly locally connected, denoted as LLC, if there exist constants \(C \geq 1\) and \(r_0 > 0\) such that for all balls \(B\) in \(X\) with radius at most \(r_0\), every pair of distinct points in the annulus \(2B \setminus B\) can be connected by a curve lying in the annulus \(2B \setminus C^{-1}B\).

2.1. Upper gradients. We introduce the notion of upper gradient as a way to overcome the lack of a differentiable structure in the metric setting. Upper gradients are a generalization of the modulus of the gradient in the Euclidean case. For further details, we refer the reader to the book by Björn and Björn [4].

**Definition 2.4** ([3], Definition 1.13). A non negative Borel measurable function \(g\) is said to be an upper gradient of function \(u : X \to [-\infty, +\infty]\) if, for all compact rectifiable arc length parametrized paths \(\gamma\) connecting \(x\) and \(y\), we have

\[
|u(x) - u(y)| \leq \int_\gamma g \, ds,
\]

whenever \(u(x)\) and \(u(y)\) are both finite and \(\int_\gamma g \, ds = \infty\) otherwise.

Notice that, as a consequence of the last definition, if \(g\) is an upper gradient of \(u\) and \(\phi\) is any non negative Borel measurable function, then \(g + \phi\) is still an upper gradient of \(u\). To avoid this lack of uniqueness, we introduce \(q\)-weak upper gradients. More specifically, if \(g\) satisfies (2.3) for \(q\)-almost all paths, then \(g\) is a \(q\)-weak upper gradient of \(u\). The next result gives us the existence of a unique minimal \(q\)-weak upper gradient of \(u\).

**Theorem 2.5** ([4], Theorem 2.5). Let \(q \in ]1,\infty[\). Suppose that \(u \in L^q(X)\) has an \(L^q(X)\) integrable \(q\)-weak upper gradient. Then there exists a \(q\)-weak upper gradient, denoted with \(g_u\), such that \(g_u \leq g \, \mu\text{-a.e. in } X\), for each \(q\)-weak upper gradient \(g\) of \(u\). This \(g_u\) is called the minimal \(q\)-weak upper gradient of \(u\).
2.2. Poincaré inequalities. In general, the upper gradients of a function do not necessarily give us a control over it. In order to gain such control, one standard hypothesis when working in the metric setting is to assume that the space supports a Poincaré inequality.

**Definition 2.6.** Let $p \in [1, \infty]$. A metric measure space $X$ supports a weak $(1,p)$-Poincaré inequality if there exist $C_{PI}$ and a dilation factor $\lambda \geq 1$ such that

$$\int_B |u - u_B| \, d\mu \leq C_{PI} \left( \int_{\lambda B} g_u^p \, d\mu \right)^{\frac{1}{p}},$$

for all balls $B = B(y,r) \subset X$ and for all $u \in L^1_{loc}(X)$.

We note that a standard, yet non-trivial, assumption in the metric setting is that the space supports a Poincaré inequality.

**Theorem 2.7** ([20], Theorem 1.0.1). Let $(X,d,\mu)$ be a complete metric measure space with $\mu$ Borel and doubling, supporting a weak $(1,p)$-Poincaré inequality for $p > 1$, then there exists $\epsilon > 0$ such that $X$ supports a weak $(1,s)$-Poincaré inequality for every $s > p - \epsilon$. Here, $\epsilon$ and the constants associated with the $(1,s)$-Poincaré inequality depend only on $p$, the doubling constant $C_d$ and $C_{PI}$.

The following results show some further self-improving properties of the weak $(1,s)$-Poincaré inequality.

**Theorem 2.8** ([4], Theorem 4.21). Assume that $X$ supports a weak $(1,s)$-Poincaré inequality and that $Q$ in (2.2) satisfies $Q > s$. Then $X$ supports a weak $(s^*, s)$-Poincaré inequality with $s^* = \frac{Qs}{Q-s}$. More precisely, there are constants $C$ and a dilation factor $\lambda' > 1$ such that

$$\left( \int_B |u - u_B|^{s^*} \, d\mu \right)^{\frac{1}{s^*}} \leq C r \left( \int_{\lambda'B} g_u^s \, d\mu \right)^{\frac{1}{s}},$$

(2.4)

for all balls $B = B(y,r) \subset X$ and all integrable functions $u$ in $B(y,r)$. The constant $C$ depends on $C_{PI}$ and the dilation factor $\lambda'$ depends on $\lambda$ from Definition 2.7.

**Corollary 2.9** ([4], Corollary 4.26). If $X$ supports a weak $(1,s)$-Poincaré inequality and $Q$ in (2.2) satisfies $Q \leq s$, then $X$ supports a weak $(t,s)$-Poincaré inequality for all $1 \leq t < \infty$.

**Remark 2.10.** By the Hölder inequality we see that a weak $(s^*, s)$-Poincaré inequality implies the same inequality for smaller values of $s^*$. Meaning that $X$ will then support a weak $(t,s)$-Poincaré inequality for all $1 < t < s^*$.

**Remark 2.11.** The exponent $Q$ in (2.2) is not uniquely determined, in particular, since $p < R$, we can always make $Q$ larger. Thus, the assumption $Q > s$ in Theorem 2.8 can always be fulfilled.

2.3. Newtonian spaces. When working in the general metric setting we need to be careful about defining the suitable working space. That is why, before introducing the $(p,q)$-Dirichlet boundary value problem, we devote this section to collect some detailed properties and results concerning these function spaces. We define $\tilde{N}^{1,q}(X)$ to be the space of all $q$-integrable functions $u$ on $X$ that have a $q$-integrable $q$-weak upper gradient $g$ on $X$. We equip this space with the seminorm $\|u\|_{\tilde{N}^{1,q}(X)} = \|u\|_{L^q(X)} + \inf \|g\|_{L^q(X)}$, where the infimum is taken over all $q$-weak upper gradients of $u$. We define the equivalence relation in $\tilde{N}^{1,q}(X)$ by saying that $u \sim v$ if $\|u - v\|_{\tilde{N}^{1,q}(X)} = 0$. The Newtonian space $N^{1,q}(X)$ is then defined by $\tilde{N}^{1,q}(X)/\sim$, with the norm $\|u\|_{N^{1,q}(X)} = \|u\|_{\tilde{N}^{1,q}(X)}$.

**Remark 2.12** ([4], Corollary A.9). Let $1 < p < q$, $u \in N^{1,q}(X)$. If $(X,d,\mu)$ is a complete doubling $(1,p)$-Poincaré space, then the minimal $p$-weak upper gradient and the minimal $q$-weak upper gradient of $u$ coincide $\mu$-a.e.
In virtue of Remark 2.12 we consider $q$-weak upper gradients rather than $p$-weak upper gradients.

2.4. Capacities.

Definition 2.13 ([3], Definition 1.24). Let $E \subset X$ be a Borel set. We define the $p$-capacity of $E$ as

$$C_p(E) = \inf_u \left( \int_X |u|^p \, d\mu + \inf_{S,B} \int_{\lambda B} g_u^p \, d\mu \right),$$

where the infimum is taken over all $u \in N^{1,p}(X)$.

We say that a property holds $p$-quasieverywhere $(p\text{-q.e.})$ if the set of points for which it does not hold has $p$-capacity zero.

Definition 2.14 ([3], Definition 6.13). Let $B \subset X$ be a ball and $E \subset B$. We define the variational capacity

$$\text{cap}_p(E; 2B) = \inf_u \int_{2B} g_u^p \, d\mu,$$

where the infimum is taken over all $u \in N^{1,p}(2B)$ such that $u \geq 1$ on $E$ and $u = 0$ on $X \setminus 2B$ $p$-q.e..

The following lemma compares the capacities $\text{cap}_p$ and $C_p$ and shows that they are in many cases equivalent.

Lemma 2.15 ([5], Lemma 2.6). Let $B$ be a ball in $X$ with radius $r$ and $E \subset B$ be a Borel set. Then for each $\lambda > 1$ with $\lambda r < \frac{1}{4}\text{diam}X$, there exists $C_\lambda > 0$ such that

$$\frac{\mu(E)}{C_\lambda r^p} \leq \text{cap}_p(E; \lambda B) \leq \frac{C_\lambda \mu(B)}{r^p}$$

and

$$\frac{C_p(E)}{C_\lambda (1 + r^p)} \leq \text{cap}_p(E; \lambda B) \leq C_\lambda \left( 1 + \frac{1}{r^p} \right) C_p(E).$$

In particular, $C_p(E) = 0$ if and only if $\text{cap}_p(E \cap B; \lambda B) = 0$ for all balls $B \subset X$ and some $\lambda > 1$; and $\text{cap}_p(E; \lambda B)$ is comparable to $r^{-p}\mu(B)$, where the comparison constant depends only on the data of $X$ and on $\lambda$.

Definition 2.16. We say that the set $E \subset X$ is uniformly $p$-fat if there exist constant $C_f > 0$ and $r_0 > 0$ such that for all $x \in E$ and $0 < r < r_0$, we have

$$\text{cap}_p(E \cap B(x, r); B(x, 2r)) \geq C_f \text{ cap}_p(B(x, r); B(x, 2r)).$$

We point out that, as it happens for Poincaré inequalities, we have a self-improving property for $p$-fatness condition. For a complete proof of the next result, we refer the reader to [5].

Proposition 2.17 ([5], Theorem 1.2). Let $X$ be a proper, LLC, doubling metric measure space supporting a $(1, s)$-Poincaré inequality for some $s$ with $1 \leq s < \infty$. Let $p > s$ and suppose that $E$ is uniformly $p$-fat. Then there exists $p_0 < p$ so that $E$ is uniformly $p_0$-fat.

The following proposition is a capacity version of the Sobolev-Poincaré inequality in Remark 2.19 also referred as Maz’ya type estimate. The proof is a straightforward generalization of the Euclidean case and it can be found in [3].

Proposition 2.18 ([3], Proposition 3.2). Let $X$ be a doubling metric measure space supporting a weak $(1, s)$-Poincaré inequality. Then there exists $C$ and $\lambda \geq 1$ such that for all balls $B$ in $X$, $u \in N^{1,s}(X)$ and $S = \{ x \in \frac{1}{2}B : u(x) = 0 \}$, then

$$\left( \int_B |u|^t \, d\mu \right)^{\frac{1}{t}} \leq \left( \frac{C}{\text{cap}_p(S, B)} \int_{\lambda B} g^s \, d\mu \right)^{\frac{1}{t}},$$

for $C$ depending on $C_{P1}$ and $t$ as in Corollary 2.19.
2.5. Newtonian spaces with zero boundary values. Let \( \Omega \) be an open and bounded subset of \( X \). We define \( N^{1,q}_0(\Omega) \) to be the set of functions \( u \in N^{1,q}(X) \) that are zero on \( X \setminus \Omega \) q.a.e. The space \( N^{1,q}_0(\Omega) \) is equipped with the norm \( \| \cdot \|_{N^{1,q}} \). Note also that if \( C_q(X \setminus \Omega) = 0 \), then \( N^{1,q}_0(\Omega) = N^{1,q}(X) \). We shall therefore always assume that \( C_q(X \setminus \Omega) > 0 \).

The following remark is an important consequence of the \((1,s)\)-Poincaré inequality, it gives a useful Sobolev inequality for functions vanishing outside a ball \( B \), see [27].

**Remark 2.19.** There exist \( C \) and \( t > 1 \) such that for all balls \( B \) in \( X \) with radius \( r < \frac{\text{diam} X}{3} \) and all \( u \in N^{1,s}_0(B) \) we have

\[
\left( \int_B |u|^t \, d\mu \right)^{\frac{1}{t}} \leq Cr \left( \int_B g_u^q \, d\mu \right)^{\frac{1}{q}},
\]

where \( C \) depends on \( C_{pl} \) and \( t \) as in Remark 2.10.

In order to avoid clumsy notation, we assume that \( \text{diam} X = \infty \), i.e., that the above Sobolev inequality holds for all balls. In the opposite case, some of the results in this paper only hold for small balls whose radius depends on \( \text{diam} X \).

Next, we present some useful results concerning Newtonian spaces with zero boundary values. Proposition 2.20 provides a characterization for \( N^{1,q}_0 \)-functions by means of the Hardy inequality. Lemma 2.22 gives a sufficient condition for a sequence of \( N^{1,q}_0 \)-functions to converge to a \( N^{1,q}_0 \)-function. Finally, Proposition 2.23 shows that \( N^{1,q}_0 \) can be presented as an intersection of \( N^{1,q}_0 \) and of zero Newtonian spaces with lower exponents. For further details and the proofs of the next results we refer the reader to [32].

**Proposition 2.20** ([32], Proposition 2.5). Let \( X \) be a proper, doubling, LLC metric measure space supporting a weak \((1,s)\)-Poincaré inequality for some \( 1 < s < q \), and suppose that \( \Omega \) is a bounded domain in \( X \) such that \( X \setminus \Omega \) is uniformly \( q \)-fat. Then there is a constant \( C \), depending only on \( q \) and \( s \), such that \( u \in N^{1,q}(X) \) is in \( N^{1,q}_0(\Omega) \) if and only if

\[
\int_\Omega \left( \frac{|u(x)|}{\text{dist}(x,X \setminus \Omega)} \right)^q \, d\mu \leq C \int_\Omega g_u(x)^q \, d\mu.
\]

**Remark 2.21.** The constant \( C \) in the above proposition formally depends on \( q \). However, if \( q \) varies inside a bounded interval, then the arguments in the proof of Proposition 2.20 show that the appropriate constants are uniformly bounded. For this reason, since in our case all exponents vary inside a bounded interval \((s,s')\) we omit the dependence of the constant on \( q \).

**Lemma 2.22** ([32], Lemma 2.6). In the setting of Proposition 2.20, let \( u_i \in N^{1,q}_0(\Omega) \) be a sequence that is bounded in \( N^{1,q}_0(\Omega) \). If \( u_i \rightharpoonup u \) \( \mu \)-a.e., then \( u \in N^{1,q}_0(\Omega) \).

We note that Lemma 2.22 was originally formulated in [21] for \((X,d,\mu)\) doubling and for \( \Omega \) open such that \( X \setminus \Omega \) satisfies a measure thickness assumption. Even though a measure thickness condition is stronger than a fatness assumption, the lemma still follows from Proposition 2.20 (for further details see [32] and the references therein). The assertion of the next proposition depends on the set \( \Omega \). Even in \( \mathbb{R}^n \) some type of thickness assumption on the domain is needed (see [19]).

**Proposition 2.23** ([32], Proposition 2.7). Let \( X \) be a proper, doubling, LLC metric measure space supporting a weak \((1,s)\)-Poincaré inequality for some \( 1 < s < q \) and suppose that \( \Omega \) is a bounded domain in \( X \) such that \( X \setminus \Omega \) is uniformly \( q \)-fat. Then

\[
N^{1,q}_0(\Omega) = N^{1,q}(\Omega) \cap \bigcap_{\epsilon > 0} N^{1,q-\epsilon}_0(\Omega).
\]

Throughout this paper, we suppose that \((X,d,\mu)\) is a complete, locally linearly connected (LLC), metric measure space with metric \( d \) and a doubling Borel regular measure \( \mu \). We work on \( \Omega \subset X \), an open and bounded subset such that \( X \setminus \Omega \) is of positive \( q \)-capacity and uniformly \( p \)-fat, with
1 \leq p < q. Moreover, we assume that $X$ supports a weak $(1,p)$-Poincaré inequality. From now on and without further notice, we fix $1 < s < p < q < s^*$ for which $X$ also admits a weak $(1,s)$-Poincaré inequality. Such $s$ is given by Theorem 2.7 and will be used in various of our results.

3. $(p,q)$-Quasiminimizers

In this note, we are interested in the $(p,q)$-Dirichlet integral given by (1.1), for some exponents $1 < p < q$. As aforementioned, since we work in metric measure spaces and due to the methods we use, we treat it under sharp assumptions. That is, we assume that the coefficient functions $a, b : X \to \mathbb{R}$ are bounded and measurable with $0 < a \leq b \leq \beta$, for some positive constants $\alpha, \beta$. Now, we introduce the definition of $(p,q)$-quasiminimizers of integral (1.1).

**Definition 3.1.** A function $u \in N^{1,q}(\Omega)$ is a $(p,q)$-quasiminimizer on $\Omega$ if there exists $K > 0$, called quasiminimizing constant, such that for every open $\Omega' \subset \Omega$ and every test function $v \in N^{1,q}(\Omega')$ with $u - v \in N^{1,q}_0(\Omega')$ the inequality

$$
\int_{\Omega'} (ag_u^n + bg_u^n) \, d\mu \leq K \int_{\Omega'} (ag_v^n + bg_v^n) \, d\mu
$$

(3.1)

holds, where $g_u, g_v$ are the minimal $q$-weak upper gradients of $u$ and $v$ in $\Omega$, respectively. Furthermore, a function $u \in N^{1,q}(\Omega)$ is a global $(p,q)$-quasiminimizer on $\Omega$ if (3.1) is satisfied with $\Omega'$, for all $v \in N^{1,q}(\Omega)$, with $u - v \in N^{1,q}_0(\Omega)$.

From now on, to simplify notation we refer to global $(p,q)$-quasiminimizers by just writing $(p,q)$-quasiminimizers.

Next, we report the De Giorgi Lemma, which has a key role in our paper and whose proof can be found in [40]. In order to do so, we start introducing some notations. We denote $S_{k,r} = \{ x \in B(y,r) \cap \Omega : u(x) > k \}$, where $k \in \mathbb{R}$ and $r > 0$. Also, for every $y \in \Omega$, we define $R(y) = \frac{d(y,X\setminus\Omega)}{2}$.

**Lemma 3.2 ([40], Lemma 3.1).** Let $u \in N^{1,q}(\Omega)$ be a $(p,q)$-quasiminimizer. If $0 < \rho < R < R(y)$, then there exists $C$ such that the following De Giorgi type inequality

$$
\int_{S_{u,\rho}} (ag_u^n + bg_u^n) \, d\mu \leq C \left( \frac{1}{(R-\rho)^p} \int_{S_{k,R}} a(u-k)^p \, d\mu + \frac{1}{(R-\rho)^q} \int_{S_{k,R}} b(u-k)^q \, d\mu \right),
$$

(3.2)

is satisfied. The constant $C$ depends on $K$, given by Definition 3.1 and $q$.

**Remark 3.3.** We note that inequality (3.2) is equivalent to the following

$$
\int_{B(y,\rho)} (ag_u^n + bg_u^n) \, d\mu \leq C \left( \frac{1}{(R-\rho)^p} \int_{B(y,R)} a(u-k)^p \, d\mu + \frac{1}{(R-\rho)^q} \int_{B(y,R)} b(u-k)^q \, d\mu \right),
$$

(3.3)

where $(u-k)_+ = \max\{u-k,0\}$.

4. Higher integrability property

This section is devoted to prove global higher integrability for upper gradients of $(p,q)$-quasiminimizers with fixed boundary data belonging to a slightly better Newtonian space. Let $u \in N^{1,q}(\Omega)$, we say that $u$ is a $(p,q)$-quasiminimizer with boundary data $w \in N^{1,q}(\Omega)$, if $w - u \in N^{1,q}_0(\Omega)$. A complete survey on global higher integrability of gradients in the Euclidean case can be found in the book by Kinnunen, Lehrbäck and Välimäki in [23] for weak solutions of p-Laplace equation. In the general metric setting, the improvement of integrability is obtained by using a metric version of the Gehring Lemma, whose proof can be found, for example, in [31] or [42]. We remark that this lemma holds in all doubling metric measure spaces.
Thus, the H"older inequality implies (moreover we can assume such that $p$),

\[ \left( \int_{\lambda B} g^\sigma d\mu \right)^{\frac{1}{\sigma}} \leq C \left( \int_{\lambda B} g d\mu \right)^{\beta} + \left( \int_{\lambda B} f^\sigma d\mu \right)^{\frac{1}{\sigma}} \]

holds for some $\sigma > 1$. Then there exists $\epsilon_0 > 0$ such that $g \in L^{\tilde{s}}_{\text{loc}}(\mu)$ for $\tilde{s} \in [\sigma, \sigma + \epsilon_0]$ and moreover

\[ \left( \int_{\lambda B} \tilde{g}^{\tilde{s}} d\mu \right)^{\frac{1}{\tilde{s}}} \leq C \left( \int_{\lambda B} g d\mu \right)^\sigma + \left( \int_{\lambda B} f^\sigma d\mu \right)^{\frac{1}{\sigma}} \]

for $\epsilon_0$ and $C$ depending on $s_0, s_1, \lambda, C_d$ and $C_G$.

Now, we state the global higher integrability result for the minimal weak upper gradient of a $(p,q)$-quasiminimizer. As in [22], the proof is based on the capacity version of the Sobolev-Poincaré inequality in Proposition 2.18 which implies a reverse H"older inequality for the minimal weak upper gradient. Global higher integrability then follows from Lemma 4.1.

**Theorem 4.2.** Let $w \in N^{1,\tilde{q}}(\Omega)$ for some $\tilde{q} > q$. If $u \in N^{1,\tilde{q}}(\Omega)$ is a $(p,q)$-quasiminimizer with boundary data $w$, then there exists $\delta_0 \in [0, q - \tilde{q}]$ such that $g_u \in L^{q+\delta}(\Omega)$ for all $\delta \in [0, \delta_0]$ and

\[ \left( \int_{\Omega} g_u^{q+\delta} d\mu \right)^{\frac{1}{q+\delta}} \leq C \left( \int_{\Omega} g_u^q d\mu \right)^{\frac{1}{q}} + \left( \int_{\Omega} g_u^{q+\delta} d\mu \right)^{\frac{1}{q+\delta}} + 1 \]

where $\delta_0$ and $C$ depend on $p$ and $q$.

**Proof.** We note that, by Remark 2.19, the capacity version of the Sobolev-Poincaré inequality with $1 < s < p < q < s^*$ implies both a $(p, s)$ and a $(q, s)$-Poincaré inequalities. Moreover, we remark that $\Omega \setminus \Omega$ is uniformly $p$-fat. By Proposition 2.17, then $X \setminus \Omega$ is also uniformly $p_0$-fat for some $p_0 < p$. Without loss of generality, we can assume $p_0 \geq s$. Indeed, if $p_0 < s$ we can choose a bigger $p_0$ so that $p_0 = s$. Thus, the H"older inequality implies $(p, p_0)$ and $(q, p_0)$-Poincaré inequalities. Let $B_0$ be a ball in $X$ such that $\lambda \subset B_0 \subset 2B_0$. Now we consider a ball $B = B(\lambda, r)$ with fixed radius $r > 0$ satisfying $2\lambda \subset 2B_0$, where $\lambda$ is the dilution factor in Definition 2.6.

First, let $2\lambda \subset \Omega$. By (3.3) with $\alpha = u_{2 \lambda}, (2.1)$, $(p, p_0)$ and $(q, p_0)$-Poincaré inequalities we obtain

\[ \alpha \left( g_u^\alpha + g_u^\beta \right) d\mu \leq \left( ag_u^p + bg_u^q \right) d\mu \]

\[ \leq C \left( \frac{1}{r^p} \int_{2B} a|u - u_{2B}|^p d\mu + \frac{1}{r^q} \int_{2B} b|u - u_{2B}|^q d\mu \right) \]

\[ \leq C \left( \frac{1}{r^p} \int_{2B} \beta|u - u_{2B}|^p d\mu + \frac{1}{r^q} \int_{2B} \beta|u - u_{2B}|^q d\mu \right) \]

\[ = C \left( \frac{1}{r^p} \int_{2B} |u - u_{2B}|^p d\mu + \frac{1}{r^q} \int_{2B} |u - u_{2B}|^q d\mu \right) \]

\[ \leq C \left( \left( \int_{2\lambda} g_u^{p_0} d\mu \right)^{\frac{1}{p_0}} + \left( \int_{2\lambda} g_u^{q_0} d\mu \right)^{\frac{1}{q_0}} \right). \]

Therefore,

\[ \left( \int_{2\lambda} g_u^{p_0} d\mu \right)^{\frac{1}{p_0}} + \left( \int_{2\lambda} g_u^{q_0} d\mu \right)^{\frac{1}{q_0}}, \]

where $C$ depends on $q, \alpha, \beta, K, C_d$ and $C_{p_1}$. 

We recall Young’s inequality $|z|^q \leq |z|^p + |z|^q \leq C(1 + |z|^q)$, with $C = \frac{p+q}{q} < 2$. Thus,
\[
\int_B g_u^q \, d\mu \leq C \left( 1 + \left( \int_{2B} g_u^{p_0} \, d\mu \right)^{\frac{q}{p_0}} \right),
\] (4.1)
where $C$ depends on $q, \alpha, \beta, K, C_d$ and $C_{FI}$.

Secondly, let $2\lambda B \setminus \Omega \neq \emptyset$. We consider a Lipschitz cut-off function such that $0 \leq \eta \leq 1$, $\eta = 1$ on $B$, supp $\eta \subset 2B$ and $g_\eta \leq \frac{C}{\eta}$. Therefore, $\eta(u - w) \in N^1_{\eta}(2B \cap \Omega)$ and so we can plug it in (3.1). We get
\[
\int_{2B \cap \Omega} (a g_u^p + bg_u^q) \, d\mu \leq K \int_{2B \cap \Omega} (a g_u^p + bg_u^q) \, d\mu,
\]
where $v = u + \eta(w - u)$ and $g_v \leq (1 - \eta) g_u + \eta g_w + |u - w| g_\eta$. Thus, we have
\[
\int_{2B \cap \Omega} (a g_u^p + bg_u^q) \, d\mu \leq C \int_{2B \cap \Omega} a(1 - \eta)^p g_u^p \, d\mu + C \int_{2B \cap \Omega} b(1 - \eta)^q g_u^q \, d\mu
\]
\[
+ C \int_{2B \cap \Omega} b|u - w|^q g_u^q \, d\mu + C \int_{2B \cap \Omega} b|u - w|^q g_\eta^q \, d\mu.
\]
Now, we add $C \int_{2B \cap \Omega} (a g_w^p + bg_w^q) \, d\mu$ to both sides of the previous inequality and then divide by $(1 + C)$, obtaining
\[
\int_{2B \cap \Omega} (a g_u^p + bg_u^q) \, d\mu \leq \theta \int_{2B \cap \Omega} (a g_u^p + bg_u^q) \, d\mu + \frac{\theta}{r^p} \int_{2B \cap \Omega} a|u - w|^p \, d\mu
\]
\[
+ \frac{\theta}{r^q} \int_{2B \cap \Omega} b|u - w|^q \, d\mu + \theta \int_{2B \cap \Omega} (a g_w^p + bg_w^q) \, d\mu,
\]
where $\theta = \frac{C}{1+C} < 1$.

By Lemma 6.1 of [17], we get
\[
\int_{2B \cap \Omega} (a g_u^p + bg_u^q) \, d\mu \leq \frac{C}{r^p} \int_{2B \cap \Omega} a|u - w|^p \, d\mu + \frac{C}{r^q} \int_{2B \cap \Omega} b|u - w|^q \, d\mu
\]
\[
+ C \int_{2B \cap \Omega} (a g_w^p + bg_w^q) \, d\mu.
\]
Notice that the last inequality can be rewritten as
\[
\int_{2B \cap \Omega} (g_u^p + g_u^q) \, d\mu \leq \frac{C}{r^p} \int_{2B \cap \Omega} |u - w|^p \, d\mu + \frac{C}{r^q} \int_{2B \cap \Omega} |u - w|^q \, d\mu
\]
\[
+ C \int_{2B \cap \Omega} (g_w^p + g_w^q) \, d\mu,
\] (4.2)
where $C$ depends on $q, \alpha, \beta$ and $K$. From now on, for the integrals on the right-hand side of the previous inequality we will work on the bigger ball $4B$. By Proposition 2.18 with $s = p_0$ and the doubling condition, for $l = p, q$, we deduce
\[
\left( \frac{C}{r^l} \int_{4B} |u - w|^l \, d\mu \right)^{\frac{1}{l}} \leq \left( \frac{1}{\text{cap}_{p_0}(S; 4B)} \int_{4\lambda B} g_{u-w}^{p_0} \, d\mu \right)^{\frac{1}{p_0}}
\]
\[
\leq C \left( \frac{\mu(2B)^{1-p_0}}{\text{cap}_{p_0}(S; 4B)} \int_{4\lambda B} g_{u-w}^{p_0} \, d\mu \right)^{\frac{1}{p_0}}.
\]
where $S = \{x \in 2B : u(x) = w(x)\}$. We note that $u - w = 0$ q.e., therefore $u - w = 0$ p$_0$-fat, in $X \setminus \Omega$. Also, since $X \setminus \Omega$ is uniformly p$_0$-fat, we obtain the following chain of inequalities
\[
\text{cap}_{p_0}(S; 4B) \geq \text{cap}_{p_0}(2B \setminus \Omega; 4B) \geq C_f \text{ cap}_{p_0}(2B; 4B) \geq C \mu(2B)^{1-p_0}.
\]
Since \( u - w = 0 \) \( q \)-q.e, then \( \mu \)-a.e. in \( X \setminus \Omega \) and so \( g_{u-w} = 0 \) \( \mu \)-a.e. in \( X \setminus \Omega \). Thus, for \( l = p, q \),
\[
\left( \frac{1}{\mu(4B)} \int_{4B \cap \Omega} |u - w|^l \, d\mu \right)^{\frac{1}{l}} = \frac{1}{\mu(4B)} \int_{4B \cap \Omega} |u - w|^l \, d\mu
\]
\[
\leq C \left( \int_{4B} g_{u-w}^{p_0} \, d\mu \right)^{\frac{1}{p_0}}
\]
\[
= C \left( \frac{1}{\mu(4B)} \int_{4B \cap \Omega} g_{u-w}^{p_0} \, d\mu \right)^{\frac{1}{p_0}}
\]
\[
\leq C \left( \frac{1}{\mu(4B)} \int_{4B} g_{u}^{p_0} \, d\mu \right)^{\frac{1}{p_0}} + C \left( \frac{1}{\mu(4B)} \int_{4B} g_{w}^{p_0} \, d\mu \right)^{\frac{1}{p_0}}.
\]
(4.3)

Now, Hölder inequality implies
\[
\left( \frac{1}{\mu(4B)} \int_{4B \cap \Omega} g_{u}^{p_0} \, d\mu \right)^{\frac{1}{p_0}} \leq \left( \frac{1}{\mu(4B)} \int_{4B \cap \Omega} g_{u}^{l} \, d\mu \right)^{\frac{1}{l}},
\]
for \( l = p, q \). Using (4.2), (4.3) and (4.4), we have
\[
\frac{1}{\mu(B)} \int_{B \cap \Omega} (g_u^p + g_u^q) \, d\mu \leq C \left( \frac{1}{\mu(4B)} \int_{4B \cap \Omega} (g_u^p + g_u^q) \, d\mu \right)^{\frac{1}{p_0}} + \left( \frac{1}{\mu(4B)} \int_{4B \cap \Omega} g_u^{p_0} \, d\mu \right)^{\frac{1}{p_0}}
\]
\[
+ \left( \frac{1}{\mu(4B)} \int_{4B \cap \Omega} g_u^{q_0} \, d\mu \right)^{\frac{1}{p_0}}.
\]
(4.5)

By the doubling property, we get
\[
\frac{1}{\mu(B)} \int_{B \cap \Omega} (g_u^p + g_u^q) \, d\mu \leq C \left( \frac{1}{\mu(4B)} \int_{4B \cap \Omega} (g_u^p + g_u^q) \, d\mu \right)^{\frac{1}{p_0}} + \left( \frac{1}{\mu(4B)} \int_{4B \cap \Omega} g_u^{p_0} \, d\mu \right)^{\frac{1}{p_0}}
\]
\[
+ \left( \frac{1}{\mu(4B)} \int_{4B \cap \Omega} g_u^{q_0} \, d\mu \right)^{\frac{1}{p_0}}.
\]
where \( C \) depends on \( q, \alpha, \beta, K, C_d \) and \( C_f \).

By Young’s inequality, we deduce
\[
\frac{1}{\mu(4B)} \int_{4B \cap \Omega} (g_u^p + g_u^q) \, d\mu \leq \frac{2}{\mu(4B)} \int_{4B \cap \Omega} (1 + g_u^p) \, d\mu
\]
\[
= 2 \left( \frac{\mu(4B \cap \Omega)}{\mu(4B)} \right) + \frac{1}{\mu(4B)} \int_{4B \cap \Omega} g_u^p \, d\mu
\]
\[
\leq 2 \left( 1 + \frac{1}{\mu(4B)} \int_{4B \cap \Omega} g_u^p \, d\mu \right)
\]
\[
\int_{4B} \left( \frac{1}{\mu(4B)} \int_{4B \cap \Omega} g_u^{p_0} \, d\mu \right)^{\frac{1}{p_0}} + \left( \frac{1}{\mu(4B)} \int_{4B \cap \Omega} g_u^{q_0} \, d\mu \right)^{\frac{1}{p_0}} \leq 2 \left( 1 + \left( \frac{1}{\mu(4B)} \int_{4B \cap \Omega} g_u^{p_0} \, d\mu \right)^{\frac{1}{p_0}} \right)
\]
(4.5)

Analogously, we get
\[
\left( \frac{1}{\mu(4B)} \int_{4B \cap \Omega} g_u^{p_0} \, d\mu \right)^{\frac{1}{p_0}} + \left( \frac{1}{\mu(4B)} \int_{4B \cap \Omega} g_u^{q_0} \, d\mu \right)^{\frac{1}{p_0}} \leq 2 \left( 1 + \left( \frac{1}{\mu(4B)} \int_{4B \cap \Omega} g_u^{p_0} \, d\mu \right)^{\frac{1}{p_0}} \right)
\]
(4.5)

By (4.5), we obtain
\[
\frac{1}{\mu(B)} \int_{B \cap \Omega} (g_u^p + g_u^q) \, d\mu \leq C \left( 1 + \frac{1}{\mu(4B)} \int_{4B \cap \Omega} g_u^p \, d\mu + \left( \frac{1}{\mu(4B)} \int_{4B \cap \Omega} g_u^{p_0} \, d\mu \right)^{\frac{1}{p_0}} \right)
\]
\[
= C \left( \int_{4B} (g_u^p + 1) \, d\mu + \left( \frac{1}{\mu(4B)} \int_{4B \cap \Omega} g_u^{p_0} \, d\mu \right)^{\frac{1}{p_0}} \right).
\]
Thus,
\[
\frac{1}{\mu(B)} \int_{B \cap \Omega} g^q \, d\mu \leq C \left( \int_{4\lambda B} \left( g^q_w \chi_{4\lambda B \cap \Omega} + 1 \right) \, d\mu \right) \leq C \left( \int_{4\lambda B} \left( g^{p_0} \chi_{4\lambda B \cap \Omega} + 1 \right)^{\frac{q}{p_0}} \, d\mu \right).
\]
And so,
\[
\frac{1}{\mu(B)} \int_{B \cap \Omega} g^q \, d\mu \leq C \left( \int_{4\lambda B} \left( g^{p_0} \chi_{4\lambda B \cap \Omega} + 1 \right)^{\frac{q}{p_0}} \, d\mu \right).
\]
We define \( \sigma = \frac{q}{p_0} > 1 \),
\[
g = \begin{cases} 
g^{p_0} & \text{in } \Omega, \\
0 & \text{otherwise,}
\end{cases}
\]
and
\[
f = \begin{cases} 
g^{p_0} \chi_{4\lambda B \cap \Omega} + 1 & \text{in } \Omega, \\
0 & \text{otherwise.}
\end{cases}
\]
If \( 4\lambda B \subset 2B_0 \), then by \((4.1)\) and \((4.4)\), the following reverse Hölder type inequality holds
\[
\left( \int_{4\lambda B} g^{\sigma} \, d\mu \right)^{\frac{1}{\sigma}} \leq C \left( \left( \int_{4\lambda B} g^{\tau} \, d\mu \right)^{\frac{1}{\tau}} + \left( \int_{4\lambda B} f^{\tau} \, d\mu \right)^{\frac{1}{\tau}} \right),
\]
where \( C \) depends on \( q, \alpha, \beta, K, C_d, C_f \) and \( C_{P_I} \). By applying Gehring Lemma \((4.1)\) we obtain the inequality
\[
\left( \int_{4\lambda B} g^{\tilde{\sigma}} \, d\mu \right)^{\frac{1}{\tilde{\sigma}}} \leq C \left( \left( \int_{4\lambda B} g^{\tilde{\sigma}} \, d\mu \right)^{\frac{1}{\tilde{\sigma}}} + \left( \int_{4\lambda B} f^{\tilde{\sigma}} \, d\mu \right)^{\frac{1}{\tilde{\sigma}}} \right),
\]
for \( \tilde{\sigma} \in [\sigma, \sigma + \epsilon_0] \), with \( C \) and \( \epsilon_0 \) both depending on \( p, q, \alpha, \beta, K, \lambda, C_d, C_f \) and \( C_{P_I} \). That is,
\[
\left( \int_{4\lambda B} g^{p_0 \tilde{\sigma}} \, d\mu \right)^{\frac{1}{p_0 \tilde{\sigma}}} \leq C \left( \left( \int_{4\lambda B} g^{p_0 \sigma} \, d\mu \right)^{\frac{1}{p_0 \sigma}} + \left( \int_{4\lambda B} \left( g^{p_0} \chi_{4\lambda B \cap \Omega} + 1 \right)^{\tilde{\sigma}} \, d\mu \right)^{\frac{1}{\tilde{\sigma}}} \right),
\]
where \( q \leq q + \delta_0 = p_0 \tilde{\sigma} \leq q + \epsilon_0 p_0 \), with \( \delta_0 = \epsilon_0 p_0 \), where \( \delta_0 \) has the same dependencies as \( \epsilon_0 \), and \( \delta \in [0, \delta_0] \). Therefore, we get
\[
\left( \int_{4\lambda B} g^{p_0 \tilde{\sigma}} \, d\mu \right)^{\frac{1}{p_0 \tilde{\sigma}}} \leq C \left( \left( \int_{4\lambda B} g^{p_0 \sigma} \, d\mu \right)^{\frac{1}{p_0 \sigma}} + \left( \int_{4\lambda B \cap \Omega} \left( g^{p_0} \chi_{4\lambda B \cap \Omega} + 1 \right)^{\tilde{\sigma}} \, d\mu \right)^{\frac{1}{\tilde{\sigma}}} \right).
\]
for all $\delta \in [0, \delta_0]$.

We recall that $\Omega$ is bounded and its diameter is finite, so we can cover it with a finite number of balls $B(x_j, r_j)$, $j = 1, 2, ..., N$, that is

$$B(x_j, 2\lambda r_j) \subset B_0 \quad \text{and} \quad \Omega \subset \bigcup_{j=1}^N B(x_j, r_j),$$

for a fixed $\lambda$. Now, we multiply (4.7) by $(4\lambda B)^{-\frac{1}{p}}$, we sum over $B(x_j, r_j)$ and divide by $\mu(\Omega)$, in order to complete the proof. We note that this last step may slightly modify the constant $C$, however this will only depend on $C_d$ and $\Omega$ itself. Finally, we point out that if $\alpha$, $\beta$, $K$, $\lambda$, $C_d$, $C_f$ and $C_P$ are fixed, then $C$ and $\delta_0$ depend essentially only on $p$ and $q$. $\square$

5. Stability property

In this section, we prove a stability result, with respect to the varying exponents $p_i, q_i$, for a family of quasiminimizers. We show that, under suitable assumptions, a sequence of $(p_i, q_i)$-quasiminimizers converge to a $(p, q)$-quasiminimizer of the limit problem. The global higher integrability in Theorem 4.2 serves as a starting point for the proof of the stability. Before stating our main theorem, we premise the following remark.

Remark 5.1. Under the assumptions of the Theorem 4.2, let $(p_i, q_i)$ be a sequence such that $\lim_{i \to \infty} p_i = p$ and $\lim_{i \to \infty} q_i = q$. Without loss of generality we can assume that $1 < \delta_i < q_i < s^*$ for every $i \in \mathbb{N}$. By Theorem 4.2, for every duple $(p_i, q_i)$ there exists $\delta_i$, such that if $u_i \in N^{1,q_i}(\Omega)$ is a $(p_i, q_i)$-quasiminimizer, then the minimal $q_i$-weak upper gradient $g_{u_i}$ belongs to the space $L^{q_i-\delta_i}(\Omega)$ and the following inequality is satisfied

$$\left(\frac{\int_{\Omega} g_{u_i}^{q_i+\delta_i} \, d\mu}{\mu(\Omega)}\right)^{\frac{1}{q_i+\delta_i}} \leq C_i \left(\left(\frac{\int_{\Omega} g_{u_i}^{q_i} \, d\mu}{\mu(\Omega)}\right)^{\frac{1}{q_i}} + \left(\int_{\Omega} g_{u_i}^{q_i+\delta_i} \, d\mu\right)^{\frac{1}{q_i+\delta_i}} + 1\right),$$

(5.1)

where $\delta_i$ and $C_i$ depend on $p_i$ and $q_i$. Since $1 < \delta_i < q_i < s^*$, we can find $C_i$, depending on $p$ and $q$, such that $C_i \leq C$. Moreover, in (5.1), $\delta_i$ is inverse proportional to $C_i$, therefore there exists $\delta_0$, with the same dependencies as $C$, such that $\delta_i \geq \delta_0$. For more details, we refer the reader to [12].

The main theorem of this section is the following stability result. Unlike the Euclidean case, we need to assume a priori that the sequence of $(p_i, q_i)$-quasiminimizers converges to a certain $u$, because when working with quasiminimizers, instead of minimizers, we lose the uniqueness of the limit.

Theorem 5.2. Let $w \in N^{1,\tilde{q}}(\Omega)$ for some $\tilde{q} > q$. Assume $p = \lim_{i \to \infty} p_i$, $q = \lim_{i \to \infty} q_i$, with $p_i, q_i \in ]s, s^*[\text{ and } p_i < q_i$ for every $i \in \mathbb{N}$. Let $(u_i)$ a sequence of $(p_i, q_i)$-quasiminimizers, where $u_i \in N^{1,\tilde{q}_i}(\Omega)$ with the same boundary data $w$ and equal quasinimizing constant $K$. If $u_i \to u$ $\mu$-a.e. in $\Omega$, then $u \in N^{1,\tilde{q}}(\Omega)$ is a $(p, q)$-quasinimimizer with boundary data $w$.

We note that functions $u_i$ are supposed to be not equal to the boundary data $w$, i.e. we assume that there is a set of positive measure where $u_i \neq w$ $\mu$-a.e., otherwise the result is trivial. Since the proof of Theorem 5.2 is considerably long, we have divided it into several lemmas, which are of independent interest.

Lemma 5.3. Let $u_i$ and $u$ be as in Theorem 5.2. Then there exists $\epsilon_0 > 0$ such that $u, u_i \in L^{q+\epsilon_0}(\Omega)$, $g_{u_i} \to g$ in $L^{q+\epsilon_0}(\Omega)$ and there is a subsequence such that $u_i \to u$ in $L^{q+\epsilon_0}(\Omega)$, $g_{u_i} \to g$ in $L^{q+\epsilon_0}(\Omega)$, where $g$ is a $q$-weak upper gradient of $u$. Here, $\epsilon_0$ depends on $p$ and $q$. 


We notice that in the previous lemma we have convergence to some \( q \)-weak upper gradient of \( u \) and not necessarily to the minimal \( q \)-weak upper gradient \( g_u \). It is not even known whether the sequence of minimal upper gradients converges weakly to the \( q \)-minimal upper gradient of \( u \), or not. Nevertheless, the lemma implies that

\[
\|g_u\|_{L^q(\Omega)} \leq \liminf_{i \to \infty} \|g_{u_i}\|_{L^q(\Omega)}.
\]

The proof of Lemma 5.3 makes use of the metric version of Rellich-Kondrachov Theorem, see [32], which we report below for the reader’s convenience.

**Theorem 5.4 ([32], Theorem 4.1).** Let \((X, d, \mu)\) be a metric space, where \( \mu \) is doubling. Suppose that all the pairs \((u_i, g_{u_i})\), satisfy a weak \((1, p)\)-Poincaré inequality. Fix a ball \( B \) and assume that the sequence \( \|u_i\|_{L^{1}(B)} + \|g_{i}\|_{L^{1}(\delta \tau B)} \) is bounded. Then there is a subsequence of \((u_i)\) that converges in \( L^q(B) \) for each \( 1 \leq q \leq \frac{Q_p}{Q-p} \), when \( p < Q \), and for each \( q \geq 1 \), when \( p \geq Q \). Here \( Q = \log C_d \) and \( C_d \) is the doubling constant of \( \mu \).

**Proof of Lemma 5.3.** By the quasiminimizing property (3.1) of \( u_i \) and using the boundary data \( w \) as a test function, we obtain

\[
\int_{\Omega} (ag_{u_i}^p + bg_{w}^p) \, d\mu \leq K \int_{\Omega} (ag_{w}^p + bg_{w}^p) \, d\mu.
\]

Therefore, \( u_i \rightarrow u \) and \( g_{u_i} \rightarrow g_u \) \( \mu \)-a.e. By the Rellich-Kondrachov Theorem, see [32], there exists \( g_{\Omega} \) such that

\[
\int_{\Omega} \left( \frac{p_u + q_u}{q U} \right) (1 + g_u^q) \, d\mu \leq 2(1 + g_u^q).
\]

Since, trivially \( g_u^q \leq g_{u_i}^q + g_w^q \), we get

\[
\int_{\Omega} g_{u_i}^q \, d\mu \leq C \int_{\Omega} (1 + g_w^q) \, d\mu = C \left( \int_{\Omega} g_w^q \, d\mu + 1 \right).
\]

Therefore,

\[
\left( \int_{\Omega} g_{u_i}^q \, d\mu \right)^{\frac{1}{q}} \leq C_i \left( \int_{\Omega} g_w^q \, d\mu + 1 \right)^{\frac{1}{q}} \leq C_i \left( \left( \int_{\Omega} g_w^q \, d\mu \right)^{\frac{1}{q}} + 1 \right),
\]

where now \( C_i \) depends also on \( q_i \). By Theorem 4.2 and the last inequality, we have

\[
\left( \int_{\Omega} g_{u_i}^{q_i + \delta_i} \, d\mu \right)^{\frac{1}{q_i + \delta_i}} \leq C_i \left( \left( \int_{\Omega} g_{u_i}^{q_i} \, d\mu \right)^{\frac{1}{q_i}} + \left( \int_{\Omega} g_{w}^{q_i + \delta_i} \, d\mu \right)^{\frac{1}{q_i + \delta_i}} + 1 \right) \leq C_i \left( \left( \int_{\Omega} g_{w}^{q_i} \, d\mu \right)^{\frac{1}{q_i}} + \left( \int_{\Omega} g_{w}^{q_i + \delta_i} \, d\mu \right)^{\frac{1}{q_i + \delta_i}} + 2 \right),
\]

(5.2)
with $C_i$ depending on $p_i$, $q_i$, $\alpha$, $\beta$ and $K$. Notice that, by Hölder inequality, we get

$$\left( \int_{\Omega} g_{u_i}^{q_i} \, d\mu \right)^{\frac{1}{q_i}} \leq \left( \int_{\Omega} g_{u_i}^{q_i + \delta_i} \, d\mu \right)^{\frac{1}{q_i + \delta_i}}.$$  

Therefore,

$$\left( \int_{\Omega} g_{u_i}^{q_i + \delta_i} \, d\mu \right)^{\frac{1}{q_i + \delta_i}} \leq C_i \left( 2 \left( \int_{\Omega} g_{u_i}^{q_i + \delta_i} \, d\mu \right)^{\frac{1}{q_i + \delta_i}} + 2 \right) \leq C_i \left( \int_{\Omega} g_{u_i}^{q_i + \delta_i} \, d\mu \right)^{\frac{1}{q_i + \delta_i}} + 1 \right).$$

By the last inequality we obtain

$$\left( \int_{\Omega} g_{u_i}^{q_i + \delta_i} \, d\mu \right)^{\frac{1}{q_i + \delta_i}} \leq C_i \left( \left( \int_{\Omega} g_{u_i}^{q_i} \, d\mu \right)^{\frac{1}{q_i + \delta_i}} + 1 \right).$$

(5.3)

Since $\lim_{i \to \infty} q_i = q$, there exists $N \in \mathbb{N}$ such that for every $i \geq N$ we have $q + \epsilon_0 \leq q_i + \delta \leq q_i + \delta_i \leq \tilde{q}$, where $\epsilon_0 = \frac{\delta}{\tilde{q}}$. By the uniform boundedness of $C_i$ given by Remark 6.1, the Hölder inequality and (5.3), we get

$$\left( \int_{\Omega} g_{u_i}^{q + \epsilon_0} \, d\mu \right)^{\frac{1}{q + \epsilon_0}} \leq C \left( \int_{\Omega} g_{u_i}^{q_i + \delta_i} \, d\mu \right)^{\frac{1}{q_i + \delta_i}} \leq C \left( \left( \int_{\Omega} g_{u_i}^{q_i} \, d\mu \right)^{\frac{1}{q_i}} + 1 \right) < \infty.$$  

As a consequence, we obtain

$$\sup_i \|g_{u_i - w}\|_{L^{q + \epsilon_0}(\Omega)} < \infty.$$  

(5.4)

We now consider $B_0 = B(x_0, r_0) \supset \Omega$. In $S = \{x \in B_0 : u(x) = w(x)\}$ we have $g_{u_i - w} = 0$ µ-a.e., where $g_{u_i - w}$ is the minimal $q_i$-weak upper gradient of $u_i - w$. Moreover, $u_i - w = 0$, $q_i$-q.e. on $X \setminus \Omega$ and therefore µ-a.e. on $X \setminus \Omega$. By decreasing $\epsilon_0$ if necessary, we can further assume that $q + \epsilon_0 < s^*$ and therefore, $X$ supports a $(q + \epsilon_0, s)$-Poincaré inequality. By Hölder inequality, we easily get that $X$ supports a $(q + \epsilon_0, q + \epsilon_0)$-Poincaré inequality as well, therefore we can use Proposition 2.18 for $u_i - w \in N^{1,q}(X)$. We also note that $p$-fatness implies $q + \epsilon_0$-fatness because $q + \epsilon_0 > p$. Thus, $\text{cap}_{q + \epsilon_0}(S, 2B_0) r^{q + \epsilon_0} \geq C \mu(B_0)$, and so

$$\left( \int_{\Omega} |u_i - w|^{q + \epsilon_0} \, d\mu \right)^{\frac{1}{q + \epsilon_0}} \leq \left( \mu(2B_0) \int_{2B_0} |u_i - w|^{q + \epsilon_0} \, d\mu \right)^{\frac{1}{q + \epsilon_0}} \leq \left( \frac{C \mu(B_0)}{\text{cap}_{q + \epsilon_0}(S, 2B_0)} \int_{2B_0} g_{u_i - w}^{q + \epsilon_0} \, d\mu \right)^{\frac{1}{q + \epsilon_0}} \leq C r_0 \left( \int_{B_0} g_{u_i - w}^{q + \epsilon_0} \, d\mu \right)^{\frac{1}{q + \epsilon_0}} \leq C r_0 \mu(\Omega)^{\frac{1}{q + \epsilon_0}} \left( \int_{\Omega} g_{u_i - w}^{q + \epsilon_0} \, d\mu \right)^{\frac{1}{q + \epsilon_0}} \leq C r_0 \mu(\Omega)^{\frac{1}{q + \epsilon_0}} \left( \int_{\Omega} (g_{u_i} + g_{w})^{q + \epsilon_0} \, d\mu \right)^{\frac{1}{q + \epsilon_0}} \leq C 2^{1 - \frac{1}{q + \epsilon_0}} r_0 \mu(\Omega)^{\frac{1}{q + \epsilon_0}} \left( \int_{\Omega} g_{u_i}^{q + \epsilon_0} \, d\mu + \int_{\Omega} g_{w}^{q + \epsilon_0} \, d\mu \right)^{\frac{1}{q + \epsilon_0}}$$
where in the latter step we applied Hölder inequality. So, by this last inequality and \(5.4\), we get \(\sup_{i} \| u_{i} - w \|_{N^{1,q+\epsilon}(\Omega)} < \infty\). Meaning, the sequence \((u_{i} - w)\) is uniformly bounded in \(N^{1,q+\epsilon}(\Omega)\). By extending \(u_{i} - w\) to zero in \(X \setminus \Omega\), we obtain that the sequence \((\| u_{i} - w \|_{L^{1}(B_{0})} + \| g_{u_{i}} - w \|_{L^{1}(\Sigma B_{0})})\) is bounded.

Theorem \([5,4]\) implies the existence of \(\tilde{u} \in L^{q+\epsilon}(B_{0})\) and a subsequence \((u_{ik})\) such that \(u_{ik} - w \to \tilde{u} - w\) in \(L^{q+\epsilon}(B_{0})\). As \(u_{i} \to u\) \(\mu\)-a.e. in \(\Omega\), then \(\tilde{u} = u\) \(\mu\)-a.e. in \(\Omega\) and \(u \in L^{q+\epsilon}(\Omega)\). By the uniform boundedness of \(\| g_{u_{ik}} \|_{L^{q+\epsilon}(\Omega)}\), we find \(g \in L^{q+\epsilon}(\Omega)\) and a subsequence, still called \(g_{u_{ik}}\) to simplify notation, such that \(g_{u_{ik}} \to g\) weakly in \(L^{q+\epsilon}(\Omega)\). Finally, using Lemma 3.6 in \([41]\) we get that \(g\) is a weak upper gradient of \(u \in N^{1,q+\epsilon}(\Omega)\).

Now, we consider a compact set \(D \subset \Omega\) and define \(D(t) = \{ x \in \Omega \setminus \text{dist}(x, D) < t \}\) for every \(t > 0\). Then \(\overline{D(t)} \subset \Omega\) for \(t \in [0, t_{0}]\), where \(t_{0} = \text{dist}(D, X \setminus \Omega)\). We give a double phase version of Lemma 4.4 in \([32]\), which was already adapted from a result proven by Kinnunen and Martio \([23]\) concerning super-quasiminimizers. More specifically, we obtain a local uniform integrability estimate for the minimal upper gradients.

**Lemma 5.5.** Let \(u_{i}, u\) be as in Theorem \([5,4]\). Then

\[
\limsup_{i \to \infty} \int_{D(t)} (g_{u_{i}}^{p_{i}} + g_{u_{i}}^{q_{i}}) \, d\mu \leq C \int_{D(t)} (g_{u}^{p} + g_{u}^{q}) \, d\mu,
\]

for almost every \(t \in [0, t_{0}]\), with the constant \(C\) depending on \(\alpha, \beta\) and \(K\).

**Proof.** We define \(\phi_{i} = \eta(u - u_{i})\) where \(\eta\) is a Lipschitz cut-off function given by

\[
\eta = \begin{cases} 1 & \text{on } D(t'), \\ 0 & \text{on } \Omega \setminus D(t), \end{cases}
\]

with \(0 < t' < t < t_{0}\). Since \(\lim_{i \to \infty} q_{i} = q\), there exists \(N \in \mathbb{N}\) such that for every \(i \geq N\) we have \(p_{i} < q_{i} < q + \epsilon_{0}\). Notice that \(\phi_{i} \in N_{0}^{1,q_{i}}(D(t))\), because \(u_{i}\) and \(u\) belong to \(N^{1,q+\epsilon}(\Omega)\). As a consequence of the quasiminimizing property \((5.1)\) of \(u_{i}\), we obtain

\[
\int_{D(t')} (ag_{u_{i}}^{p_{i}} + bg_{u_{i}}^{q_{i}}) \, d\mu \leq \int_{D(t')} (ag_{u_{i}}^{p_{i}} + bg_{u_{i}}^{q_{i}}) \, d\mu \leq K \int_{D(t')} (ag_{u_{i} + \phi_{i}}^{p_{i}} + bg_{u_{i} + \phi_{i}}^{q_{i}}) \, d\mu.
\]

Now, using Lemma 2.1 of \([32]\), we get \(g_{u_{i} + \phi_{i}} \leq (1 - \eta)g_{u_{i}} + \eta g_{u} |u - u_{i}| + \eta g_{u}\), therefore

\[
\int_{D(t')} (ag_{u_{i}}^{p_{i}} + bg_{u_{i}}^{q_{i}}) \, d\mu \leq C \left( \int_{D(t')} a(1 - \eta) g_{u_{i}}^{p_{i}} \, d\mu + \int_{D(t')} a g_{u}^{p} |u - u_{i}|^{p_{i}} \, d\mu + \int_{D(t')} a \eta^{p_{i}} g_{u}^{q_{i}} \, d\mu \
+ \int_{D(t')} b(1 - \eta) g_{u_{i}}^{q_{i}} \, d\mu + \int_{D(t')} b g_{u}^{q} |u - u_{i}|^{q_{i}} \, d\mu + \int_{D(t')} b \eta^{q_{i}} g_{u}^{q_{i}} \, d\mu \right)
\leq C \left( \int_{D(t')} (1 - \eta) g_{u_{i}}^{p_{i}} \, d\mu + \int_{D(t')} g_{u}^{p_{i}} |u - u_{i}|^{p_{i}} \, d\mu + \int_{D(t')} \eta^{p_{i}} g_{u}^{q_{i}} \, d\mu \
+ \int_{D(t')} (1 - \eta) g_{u_{i}}^{q_{i}} \, d\mu + \int_{D(t')} g_{u}^{q_{i}} |u - u_{i}|^{q_{i}} \, d\mu + \int_{D(t')} \eta^{q_{i}} g_{u}^{q_{i}} \, d\mu \right),
\]
with $C$ depending on $\beta$ and $K$. Thus,
\[
\int_{D(t')} (g_{u_i}^p + g_{ui}^q) \, d\mu \leq C \left( \int_{D(t')} (1 - \eta)^p g_{u_i}^p \, d\mu + \int_{D(t')} g_{u}^p |u - u_i|^p \, d\mu + \int_{D(t')} \eta^p g_u^p \, d\mu \right.
\]
\[
+ \int_{D(t')} (1 - \eta)^q g_{ui}^q \, d\mu + \int_{D(t')} g_{u}^q |u - u_i|^q \, d\mu + \int_{D(t')} \eta^q g_u^q \, d\mu \right),
\]
where $C$ also depends on $\alpha$.
By definition $\eta = 1$ on $D(t')$, therefore by adding $C \int_{D(t')} (g_{u_i}^p + g_{ui}^q) \, d\mu$ to both sides of the inequality, we get
\[
(1 + C) \int_{D(t')} (g_{u_i}^p + g_{ui}^q) \, d\mu 
\]
\[
\leq C \left( \int_{D(t')} g_{u_i}^p \, d\mu + \int_{D(t')} g_{u}^p |u - u_i|^p \, d\mu + \int_{D(t')} \eta^p g_u^p \, d\mu \right.
\]
\[
+ \int_{D(t')} g_{u_i}^q \, d\mu + \int_{D(t')} g_{u}^q |u - u_i|^q \, d\mu + \int_{D(t')} \eta^q g_u^q \, d\mu \right). \tag{5.5}
\]
We consider a nondecreasing function $\psi$ defined on the interval $]0, t_0[$ as
\[
\psi(t) = \limsup_{i \to \infty} \int_{D(t')} (g_{u_i}^p + g_{ui}^q) \, d\mu < \infty,
\]
where the latter is given by the uniform higher integrability of $u_i$. As a consequence, $\psi$ is discontinuous in at most a countable number of points. Now, we consider a point $t$ where $\psi$ is continuous. Passing to superior limit on both sides of (5.5), it follows
\[
(1 + C)\psi(t') \leq C\psi(t) + C \int_{D(t')} (g_{u_i}^p + g_{ui}^q) \, d\mu
\]
\[
+ C \limsup_{i \to \infty} \int_{D(t')} |u - u_i|^p \, d\mu + C \limsup_{i \to \infty} \int_{D(t')} |u - u_i|^q \, d\mu.
\]
Using Hölder inequality and Lemma 5.3, we deduce that
\[
\limsup_{i \to \infty} \int_{D(t')} |u - u_i|^p \, d\mu = \limsup_{i \to \infty} \int_{D(t')} |u - u_i|^q \, d\mu = 0.
\]
Recalling that $\psi$ is continuous at $t$, we have
\[
(1 + C)\psi(t) \leq C\psi(t) + C \int_{D(t')} (g_{u_i}^p + g_{ui}^q) \, d\mu,
\]
that is
\[
\psi(t) \leq C \int_{D(t')} (g_{u_i}^p + g_{ui}^q) \, d\mu.
\]
This concludes the proof.

The next lemma will be needed as a first step in order to show that $u$ is a quasiminimizer of the $(p, q)$-energy integral with boundary data $w$.

**Lemma 5.6.** Under the assumptions in Theorem 5.2 and notation as in Lemma 5.3, we have $u - w \in N^{1,q}_0(\Omega)$. 
Proof. Let $0 < \epsilon < q - p$ and let $(u_i)$ be the subsequence given by Lemma 5.3. For sufficiently large $i \in \mathbb{N}$, we have $q_i > q - \epsilon$ and $w_i \in N_0^{1,q_i}(\Omega)$. Thus, $u_i - w \in N_0^{1,q-\epsilon}(\Omega)$ for every such $i \in \mathbb{N}$ and, eventually passing to a subsequence, we may assume that this holds for every $i \in \mathbb{N}$. As a consequence of the Sobolev inequality (see Remark 2.19), we deduce

\[ \|u_i - w\|_{N_0^{1,q-\epsilon}(\Omega)} \leq C\|u_i\|_{L^{q-\epsilon}(\Omega)} \leq C\|u_i\|_{L^q(\Omega)}. \]

As a consequence, the norms of $w_i$ depends on $\alpha, \beta$.

Lemma 5.7. Under the assumptions in Theorem 5.2 and notation as in Lemma 5.3, we have

\[ \int_E g_u^q \, d\mu \leq \liminf_{i \to \infty} \int_E g_{u_i}^{q_i} \, d\mu, \quad (5.6) \]

for every $\mu$-measurable subset $E$ of $\Omega$.

Proof. Since $\lim_{i \to \infty} q_i = q$, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $i \geq N$ we have $1 < q_i < q - \epsilon_0$. As a consequence of Lemma 5.3, $(u_i)$ converges weakly to a weak upper gradient $g$ of $u$, therefore, for every $\mu$-measurable subset $E$ of $\Omega$, we get

\[ \int_E g_u^{q-\epsilon} \, d\mu \leq \int_E g^{q-\epsilon} \, d\mu \leq \liminf_{i \to \infty} \int_E g_{u_i}^{q_i-\epsilon} \, d\mu \]

\[ \leq \liminf_{i \to \infty} \left( \int_E g_{u_i}^{q_i} \, d\mu \right)^{\frac{q_i}{q_i - \epsilon}} \mu(E)^{1 - \frac{q_i}{q_i - \epsilon}} \]

\[ \leq \liminf_{i \to \infty} \left( \int_E g_{u_i}^{q_i} \, d\mu \right)^{\frac{q_i}{q_i - \epsilon}} \mu(E)^{\frac{q_i}{q_i - \epsilon}}. \]

Taking the limit as $\epsilon \to 0$, we conclude the proof. \qed

Remark 5.8. We note that Lemma 5.7 also holds for $p$ instead of $q$, meaning that, under the assumptions of Theorem 5.2, we get

\[ \int_E g_u^p \, d\mu \leq \liminf_{i \to \infty} \int_E g_{u_i}^{p_i} \, d\mu, \quad (5.7) \]

for every $\mu$-measurable subset $E$ of $\Omega$.

Finally, we are ready to show that the function $u$ is a quasiminimizer of the $(p, q)$-energy integral with boundary data $w$.

Lemma 5.9. Under the assumptions in Theorem 5.2 and notation as in Lemma 5.3, we have

\[ \int_{\Omega'} (ag_{u_i}^p + bg_{u_i}^q) \, d\mu \leq C \int_{\Omega'} (ag_{u+\phi}^p + bg_{u+\phi}^q) \, d\mu, \quad (5.8) \]

for every bounded open subset $\Omega'$ of $\Omega$ with $\Omega' \Subset \Omega$ and for all functions $\phi \in N_0^{1,q}(\Omega')$, where $C$ depends on $\alpha, \beta$ and $K$.
We consider a function \( \phi \) where \( C_\varepsilon > 0 \). Let \( t \) such that \( \| \phi_\varepsilon - \phi \|_{N, \varepsilon} < \varepsilon \), see Theorem 5.46 in [4].

Let \( \varepsilon > 0 \) and \( \Omega'' \) and \( \Omega_0 \) open sets with \( \Omega' \subseteq \Omega'' \subseteq \Omega_0 \subseteq \Omega \) and

\[
\int_{\Omega_0 \setminus \Omega'} (g_p^p + g_q^q) \, d\mu < \varepsilon.
\]

We consider a function \( \phi_\varepsilon \) defined as \( \phi_\varepsilon = \phi + \eta(u - u_\varepsilon) \), where \( \eta \) is a Lipschitz cut-off function defined as \( \eta = 1 \) in a neighbourhood of \( \Omega'' \) and \( \eta = 0 \) in \( \Omega \setminus \Omega'' \). Notice that, for \( i \) large enough, \( \phi_\varepsilon \in N_0^{1,\varepsilon} (\Omega'' \Omega') \), because \( \phi \in \text{Lip}_C(\Omega') \) and \( u_\varepsilon, u \in N^{1,q+\varepsilon}(\Omega) \). Using the quasiminimizing property (5.8) of \( u_\varepsilon \), with \( u_\varepsilon + \phi \) as test function, we have

\[
\int_{\Omega''} (ag_p^p + bg_q^q) \, d\mu \leq K \int_{\Omega''} (ag_p^p + bg_q^q) \, d\mu + \varepsilon \int_{\Omega' \setminus \Omega''} (ag_p^p + bg_q^q) \, d\mu.
\]

By the definition of \( \eta \) in a neighborhood of \( \Omega'' \), we obtain

\[
u_\varepsilon + \phi_\varepsilon = u + \phi.
\]

Moreover, \( \phi = 0 \) in \( \Omega'' \setminus \Omega' \), thus \( u_\varepsilon + \phi_\varepsilon = u_\varepsilon + \eta(u - u_\varepsilon) \) in \( \Omega'' \setminus \Omega' \). By applying Lemma 2.1 of [32], we get that \( g_{u_\varepsilon + \phi_\varepsilon} \leq (1 - \eta)g_{u_\varepsilon} + \eta g_{u_\varepsilon} + g_\eta |u - u_\varepsilon| \). Then, we have

\[
\int_{\Omega'' \setminus \Omega'} (ag_p^p + bg_q^q) \, d\mu \leq C \int_{\Omega'' \setminus \Omega'} (1 - \eta)^p g_p^p + (1 - \eta)^q g_q^q \, d\mu + \varepsilon \int_{\Omega' \setminus \Omega''} (ag_p^p + bg_q^q) \, d\mu.
\]

By the definition of \( \eta \), we can choose a compact set \( D \subset \Omega'' \) with \( D \cap \Omega' = \emptyset \) such that

\[
\int_{\Omega'' \setminus \Omega'} (1 - \eta)^p g_p^p + (1 - \eta)^q g_q^q \, d\mu \leq \int_D (g_p^p + g_q^q) \, d\mu.
\]

We choose \( t \) such that \( D(t) \subset \Omega_0 \setminus \Omega'' \) and for which we can use Lemma 5.5 to get

\[
\limsup_{i \to \infty} \int_{D(t)} (g_p^p + g_q^q) \, d\mu \leq C \int_{D(t)} (g_p^p + g_q^q) \, d\mu,
\]

where \( C \) depends also on \( \alpha \) and \( K \). By the choice of \( \Omega_0 \), this implies

\[
\limsup_{i \to \infty} \int_{\Omega'' \setminus \Omega'} (1 - \eta)^p g_p^p + (1 - \eta)^q g_q^q \, d\mu \leq \limsup_{i \to \infty} \int_D (g_p^p + g_q^q) \, d\mu \leq \limsup_{i \to \infty} \int_{D(t)} (g_p^p + g_q^q) \, d\mu \leq C \int_{D(t)} (g_p^p + g_q^q) \, d\mu \leq C \varepsilon.
\]
Now, we consider the second and third integrals on the right-hand side of (5.11). Analogously, because of the definition of $\Omega_0$, we get
\[
\limsup_{i \to \infty} \int_{\Omega \setminus \Omega'} (g_p^p g_u^p + \eta^q g_u^q) \, d\mu \leq \int_{\Omega \setminus \Omega'} (g_p^p + g_u^q) \, d\mu \\
\leq \int_{\Omega_0 \setminus \Omega'} (g_p^p + g_u^q) \, d\mu \leq \epsilon. \tag{5.13}
\]
We recall that the minimal $q$-weak upper gradient of a Lipschitz function is bounded by its Lipschitz constant $\mu$-a.e., thus, by Hölder inequality and Lemma 5.3 we have
\[
\limsup_{i \to \infty} \int_{\Omega \setminus \Omega'} (g_p^p |u - u_i|^p + g_u^q |u - u_i|^q) \, d\mu \\
\leq C \limsup_{i \to \infty} \int_{\Omega \setminus \Omega'} (|u - u_i|^p + |u - u_i|^q) \, d\mu \\
\leq C \limsup_{i \to \infty} \left( \int_{\Omega \setminus \Omega'} |u - u_i|^{q + \epsilon} \, d\mu \right)^{\frac{p}{q + \epsilon}} \\
\leq C \limsup_{i \to \infty} \left( \int_{\Omega \setminus \Omega'} |u - u_i|^{q + \epsilon} \, d\mu \right)^{\frac{q}{q + \epsilon}} = 0. \tag{5.14}
\]
From (5.11), together with the estimates (5.12), (5.13) and (5.14), we obtain
\[
\limsup_{i \to \infty} \int_{\Omega \setminus \Omega'} (ag_{u_0}^p + bg_{u_0}^q) \, d\mu \leq C \epsilon. \tag{5.15}
\]
Now, using (5.6), Remark 5.8, (5.9), (5.10) and (5.15), we deduce
\[
\int_{\Pi'} (ag_u^p + bg_u^q) \, d\mu \leq \liminf_{i \to \infty} \int_{\Omega'} (ag_{u_i}^p + bg_{u_i}^q) \, d\mu \\
\leq C \liminf_{i \to \infty} \int_{\Omega'} (ag_{u_i}^p + bg_{u_i}^q) \, d\mu \\
\leq C \liminf_{i \to \infty} \int_{\Pi'} ag_{u_i + \phi_i}^p \, d\mu + C \liminf_{i \to \infty} \int_{\Omega \setminus \Omega'} ag_{u_i + \phi_i}^p \, d\mu \\
+ C \liminf_{i \to \infty} \int_{\Pi'} bg_{u_i + \phi_i}^q \, d\mu + C \liminf_{i \to \infty} \int_{\Omega \setminus \Omega'} bg_{u_i + \phi_i}^q \, d\mu \\
\leq C \int_{\Pi'} (ag_{u+\phi}^p + bg_{u+\phi}^q) \, d\mu + C \epsilon. \tag{5.16}
\]
Taking the limit as $\epsilon \to 0$, we conclude the proof for every $\phi \in \text{Lip}_C(\Omega')$ and thus, by approximation, for every $\phi \in N^{1,q}_0(\Omega')$.

Now, the proof of the stability result stated in Theorem 5.2 follows immediately from the previous lemmata.

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HIGHER INTEGRABILITY AND STABILITY OF \((p,q)\)-QUASIMINIMIZERS

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