SMOOTH NUMBERS AND THE DICKMAN $\rho$ FUNCTION

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To Professor Peter Sarnak on his 70th birthday

Abstract. We establish an asymptotic formula for $\Psi(x, y)$ whose shape is $x\rho(\log x / \log y)$ times correction factors. These factors take into account the contributions of zeta zeros and prime powers and the formula can be regarded as an (approximate) explicit formula for $\Psi(x, y)$. With this formula at hand we prove oscillation results for $\Psi(x, y)$, which resolve a question of Hildebrand on the range of validity of $\Psi(x, y) \asymp x\rho(\log x / \log y)$. We also address a question of Pomerance on the range of validity of $\Psi(x, y) \geq x\rho(\log x / \log y)$.

Along the way we improve classical estimates for $\Psi(x, y)$ and, on the Riemann Hypothesis, uncover an unexpected phase transition of $\Psi(x, y)$ at $y = (\log x)^{3/2+o(1)}$.

1. Introduction

A positive integer is called $y$-smooth if each of its prime factors does not exceed $y$. We denote the number of $y$-smooth integers not exceeding $x$ by $\Psi(x, y)$. We assume throughout $x \geq y \geq 2$. Let $\rho: [0, \infty) \to (0, \infty)$ be the Dickman function, defined as $\rho(t) = 1$ for $t \leq 1$ and via the delay differential equation $t\rho'(t) = -\rho(t - 1)$ for $t > 1$. Dickman [8] showed that

$$\Psi(x, y) \sim x\rho(\log x / \log y) \quad (x \to \infty)$$

holds when $y \geq x^\varepsilon$. For this reason, it is useful to introduce the parameter

$$u := \log x / \log y.$$ 

The range of validity of (1.1) was considerably improved by de Bruijn [5] and H. Maier (unpublished), and the state of the art is due to Hildebrand who showed that

$$\Psi(x, y) = x\rho(u) \left(1 + O_\varepsilon \left(\frac{\log(u + 1)}{\log y}\right)\right)$$

holds when $\log y \geq (\log \log x)^{\frac{5}{2}+\varepsilon}$. In [17], Hildebrand showed that the Riemann Hypothesis (RH) implies that (1.2) holds in

$$y \geq (\log x)^{2+\varepsilon}.$$ 

In fact, in this range he proves the slightly stronger estimate

$$\Psi(x, y) = x\rho(u) \exp\left(O_\varepsilon \left(\frac{\log(u + 1)}{\log y}\right)\right).$$

The reverse implication is also true: if even the weaker estimate $\Psi(x, y) = x\rho(u) \exp(O_\varepsilon(y^\varepsilon))$ holds in the range (1.3) for every $\varepsilon > 0$ then RH must be true. In [19, p. 290], Hildebrand speculates that $\Psi(x, y) \asymp x\rho(u)$ does not hold for $y \leq (\log x)^{2-\varepsilon}$. He writes

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If the Riemann hypothesis is assumed, the range for $u$ can be further extended to $1 \leq u \leq \log x/(2 + \varepsilon)\log \log x$, but it seems likely that then the critical limit is attained: it may be conjectured that for $u > \log x/(2 - \varepsilon)\log \log x$, the relation $\Psi(x, x^{1/u}) \sim \varepsilon x(u)$ no longer holds.

This conjecture is repeated in [18] and by Granville in [14, 15]. We confirm it in stronger form. To state our result we recall the function

$$K: (-1, 0) \to \mathbb{R}, \quad K(t) = t\zeta(t+1)/(t+1), \quad K(0) = 1,$$

was introduced by de Bruijn [5, Eq. (2.8)]. Evidently $\lim_{t \to -1^+} K(t) = \infty$. It is strictly decreasing; the identity $\zeta(s) = s/(s-1) - s\int_1^\infty \{x\}x^{-1-s}\,dx$ for $s > 0$ [23, Eq. (1.24)] implies

$K(t) = 1 - t\int_1^\infty \{x\}x^{-2-t}\,dx$ and $K'(t) = -\int_1^\infty \{x\}(1 - t \log x)x^{-2-t}\,dx < 0$.

**Theorem 1.1.** Let $\vartheta := \sup_{\zeta(\rho)=0} \mathbb{R} \in [1/2, 1]$ be supremum of the real parts of the zeros of the Riemann zeta function. Fix $\varepsilon > 0$.

1. Suppose $\vartheta \neq 1$. If $x \geq y \geq (\log x)^{1/(1-\vartheta)+\varepsilon}$ then

$$\Psi(x, y) \sim \varepsilon x(u)K(-\log \log x/\log y), \quad x \to \infty.$$

2. Given $A \in (1/\vartheta, 1/(1-\vartheta))$ and $\operatorname{sgn} \in \{+, -\}$ there exists an explicit function

$$x = x(y)$$

of $y$ satisfying $y = (\log x)^{A+o(1)}$ and

$$(1.4) \quad \Psi(x, y) = \varepsilon x(u)\exp(\Omega_{\text{sgn}}(y^{\vartheta+A-1-\varepsilon})).$$

3. Suppose $\vartheta \neq 1$. If $2 \log x \leq y \leq (\log x)^{1/\vartheta - \varepsilon}$ and $x \geq C_\varepsilon$ then

$$\Psi(x, y) = \varepsilon x(u)\exp \left(\Theta_{\varepsilon} \left(\frac{\log^2 x}{y \log y}\right)\right).$$

The key input to (1.4) is Landau’s Oscillation Theorem. See Remark 2 for a refinement.

In [16, 22], Pomerance asked whether $\Psi(x, y) \geq \varepsilon x(u)$ holds for all $x/2 \geq y \geq 1$. In [16, p. 274], Granville proved that $\Psi(x, y) \geq 2\varepsilon x(u)$ holds for $y \leq c(\log x)(\log \log x)/\log \log \log x$ if $x \geq C$. Below we extend Granville’s range considerably. Moreover, if RH is true, we show $\Psi(x, y) \geq \varepsilon x(u)$ holds when $y \not\in ((\log x)^{2-\varepsilon}, (\log x)^{2+\varepsilon})$. For $y$ near $(\log x)^2$, the question lies beyond RH in a precise sense, but we indicate that a positive answer follows from a conjecture of Montgomery and Vaughan on the size of the remainder term in the PNT. If RH is false, the $\Omega_-$ result in (1.4) already implies the inequality fails infinitely often.

Let

$$L := \max_{v \in \mathbb{R}} e^v \left(-\log(-\zeta(1/2)) - \frac{1}{2} \int_v^{2v} e^{-r^{-1}}\,dr\right) \approx -0.666217.$$

**Theorem 1.2.** Fix $\varepsilon > 0$ and suppose $x \geq C_\varepsilon$.

1. For $y \in [\exp((\log \log x)^{5/3+\varepsilon}), (1-\varepsilon)x]$ we have

$$(1.5) \quad \Psi(x, y) = \varepsilon x(u)(1 + \Theta_{\varepsilon}(\log(u+1)/\log y)) \geq \varepsilon x(u)$$

and for $y \in [\varepsilon \log x, \exp((\log \log x)^{3/5-\varepsilon})(\log x)]$

$$(1.6) \quad \Psi(x, y) \geq \varepsilon x(u)\exp \left(\Theta_{\varepsilon} \left(\frac{\log^2 x}{y \log y}\right)\right) \geq \varepsilon x(u).$$

1If $\vartheta = 1$ we define $1/(1-\vartheta) := \infty$. 
Suppose RH is true. Then (1.5) holds for \( y \in [(\log x)^{2+\varepsilon}, (1-\varepsilon)x] \) and (1.6) holds for \( y \in [\varepsilon \log x, (\log x)^{2-\varepsilon}] \). If \( \Psi(x, y) \geq x \rho(u) \) holds for \( y \in [(\log x)^{3/2}, (\log x)^3] \) then

\[
(1.7) \quad \liminf_{y \to \infty} \frac{\psi(y) - y}{\sqrt{y} \log y} \geq L.
\]

If (1.7) holds with strict inequality then \( \Psi(x, y) \geq x \rho(u) \) holds for \( y \in [2, (1-\varepsilon)x] \).

RH implies \( \psi(y) - y \ll \sqrt{y}(\log y)^2 \) \cite{23} Thm. 13.1]. It is believed that

\[
(1.8) \quad \liminf_{y \to \infty} \frac{\psi(y) - y}{\sqrt{y} \log \log \log y^2} = -\frac{1}{2\pi},
\]

see \cite{23} p. 484]; (1.8) implies that the limit considered in (1.7) is 0.

**Conventions and notation.** The letters \( C, c \) denote absolute positive constants that may change between different occurrences. The notation \( A \ll B \) means \( |A| \leq CB \) for some absolute constant \( C \), and \( A \ll_{a,b,...} B \) means \( |A| \leq C_{a,b,...} B \) for \( C_{a,b,...} \) that may depend on the subscripts. We write \( A \asymp B \) to mean \( C_1 B \leq A \leq C_2 B \) for some absolute positive constants \( C_i \), and \( A \asymp_{a,b,...} B \) means \( C_1 \) may depend on \( a, b, \ldots \). We write \( \Theta(B) \) and \( \Theta_{a,b,...}(B) \) to indicate a quantity \( A \) with \( A \asymp B \) and \( A \asymp_{a,b,...} B \), respectively. We write \( \Omega_+(g(x)) \) (resp. \( \Omega_-(g(x)) \)) to indicate a function \( f(x) \) with \( \limsup_{x \to \infty} f(x)/g(x) > 0 \) (resp. \( \liminf_{x \to \infty} f(x)/g(x) < 0 \)). A function \( f \) is \( \Omega_{+,-}(g(x)) \) if \( \limsup f/g > 0 > \liminf f/g \). Throughout

\[
L(x) = \exp((\log x)^{1/2} (\log \log (x+1))^{-1/2}).
\]

We denote

\[
\vartheta = \sup_{\zeta(\rho)=0} \Re \rho \in [1/2, 1].
\]

For \( y \geq 2 \) and \( \Re s > 0 \) we define the partial zeta function

\[
\zeta(s, y) = \prod_{p \leq y} (1 - p^{-s})^{-1} = \sum_{n \text{ is } y\text{-smooth}} n^{-s}.
\]

We define \( \xi : [1, \infty) \to [0, \infty) \) via

\[
e^{\xi(v)} = 1 + v \xi(v).
\]

We define the entire function

\[
I(s) = \int_0^s e^v - \frac{1}{v} \, dv = \sum_{i \geq 1} s^i / i!.
\]

We denote the Euler–Mascheroni constant by \( \gamma \). The Laplace transform of \( \rho \) is given by [4 Eq. (1.9)]

\[
\hat{\rho}(s) := \int_0^\infty \rho(v)e^{-sv} \, dv = \exp(\gamma + I(-s))
\]

for \( s \in \mathbb{C} \). We define

\[
F'(s, y) = \zeta(s)(s-1)F_2(s, y)
\]

for \( s \in \mathbb{C} \) where

\[
F_2(s, y) = \hat{\rho}((s-1) \log y) \log y.
\]

We write \( \psi(x) = \sum_{n \leq x} \Lambda(n) \) for the Chebyshev function. We set

\[
\bar{u} = \min\{y/\log y, u\}.
\]
If a meromorphic function has a removable singularity (e.g. $\zeta(s)(s-1)$ at $s=1$), we identify its value there with its limit. When we differentiate a bivariate function (e.g. $\zeta(s, y)$) we always do so with respect to the first variable. In sums and products over $p$, $p$ is understood to be prime. Throughout, $\varepsilon > 0$ is an arbitrary fixed constant.

2. A FORMULA AND ITS INVESTIGATION

2.1. On two saddle points. Let $f: \mathbb{R} \to \mathbb{R}$ be given by

$$f(t) := t \log x + \log(F_2(t, y)).$$

It satisfies

$$f'(t) = \log x - (\log y)I'((1-t) \log y), \quad f''(t) = (\log y)^2 I''((1-t) \log y).$$

The function $f$ is convex since $I''(-r) = r^{-2}e^{-r}(e^r - (r + 1)) > 0$. We define

$$\beta = \beta(x, y) = 1 - \frac{\xi(u)}{\log y}$$

for $u = \log x/\log y$. We often shorten $\xi(u)$ to $\xi$ when no confusion may arise. The function $f$ attains its global minimum at $\beta$ because it is readily verified that

$$f'(\beta) = 0.$$  \hspace{1cm} (2.1)

The asymptotics of $\beta$ are well understood thanks to the next lemma.

Lemma 2.1. \cite{20} Lem. 1|\cite{26} Lem. 4.5] For $v \geq 3$ we have

$$\xi(v) = \log v + \log \log v + O(\log \log v / \log v), \hspace{1cm} (2.2)$$

$$I''(\xi(v)) = v(1 + O(1 / \log v)). \hspace{1cm} (2.3)$$

Corollary 2.2. Fix $\varepsilon > 0$.

1. If $x \geq y \geq (1 + \varepsilon) \log x$ then

$$\beta = \frac{\log \left( \frac{y}{\log x} \right) \left( 1 + O_\varepsilon \left( \frac{\log \log(y + 1)}{\log y} \right) \right)}{\log y}. \hspace{1cm} (2.4)$$

2. If $2 \log x \geq y \geq \varepsilon \log x$ then $\beta = O_\varepsilon(1 / \log y)$.

3. We have $\beta \leq 1$. Equality occurs if and only if $y = x$.

4. We have $\beta \geq 0$ if and only if $y \geq 1 + \log x$, and $\beta = 0$ if and only if $y = 1 + \log x$.

Proof. If $u < 3$ then (2.4) means $\beta = 1 + O(\log \log(x + 1) / \log x)$ which follows from $\xi(u) = O(1)$. If $u \geq 3$ we use (2.2) to write

$$\beta = \frac{\log(y/(u \log u)) + O(\log \log(u + 1) / \log u)}{\log y}$$

and reduce (2.4) to

$$\log \left( \frac{\log y}{\log u} \right) + O\left( \frac{\log \log(u + 1)}{\log u} \right) \ll \varepsilon \log \left( \frac{y}{\log x} \right) \log \log(y + 1) / \log y.$$  \hspace{1cm} (2.5)
If $x \ll \varepsilon$ then $y \ll \varepsilon$ and (2.4) is trivial, so we may assume $x \geq C \varepsilon$. If $y \geq \log^{3/2} x$ then the right-hand side of (2.5) is $\gg \log \log (y + 1)$ and it suffices to show $\log \log u \ll \varepsilon \log \log (y + 1)$, which is clear. If $(1 + \varepsilon) \log x \leq y < \log^{3/2} x$, the right-hand side of (2.5) is $\gg \varepsilon \log \log \log x \log \log x$.

Since $\log \log (u + 1)/\log u \ll \varepsilon \log \log x/\log x$, (2.6) reduces further to

\[
\log \left( \frac{\log y}{\log u} \right) \ll \varepsilon \log \log \log x \log \log x.
\]

We write

$$\log \frac{y}{\log x} = \log y \log \log x \left(1 - \frac{\log \log y}{\log \log x}\right) - 1 = \left(1 + \frac{\log(y/\log x)}{\log \log x}\right)\left(1 - \frac{\log \log y}{\log \log x}\right) - 1$$

\[\text{and (2.6) follows by using } \log(1 + t) \ll |t| \text{ for } |t| \leq 1/2. \text{ The second part of the lemma is similar to the first and is left to the reader. The third part follows from } \xi \text{ being } 0 \text{ at } v = 1 \text{ and being strictly increasing. For the last part we need to solve } \beta \geq 0, \text{ or } \log y \geq \xi(u). \text{ Since } \xi \text{ is strictly increasing, it suffices to solve } \log y = \xi(u). \text{ Exponentiating, this implies } y = e^{\xi(u)} = 1 + u\xi(u) = 1 + \log x. \square
\]

Let $g: (0, \infty) \to \mathbb{R}$ be given by

$$g(t) := t \log x + \log \zeta(t, y).$$

It satisfies

$$g'(t) = \log x - \sum_{p \leq y} \frac{\log p}{p^t - 1}, \quad g''(t) = \sum_{p \leq y} \frac{p^t (\log p)^2}{(p^t - 1)^2}.$$ 

The function $g$ is convex because $g''(t) > 0$ for $t > 0$. Since $\lim_{t \to 0^+} g(t) = \lim_{t \to \infty} g(t) = \infty$ it follows that $g$ has a global minimum, attained at a point

$$\alpha = \alpha(x, y)$$

which must satisfy

(2.7) \quad $g'(\alpha) = 0$.\]

The following theorem of Hildebrand and Tenenbaum expresses the asymptotics of $\Psi(x, y)$ in terms of $\alpha$.

**Theorem 2.3.** [20 Thms. 1, 2] For $x \geq y \geq 2$ we have

(2.8) \quad $\Psi(x, y) = \frac{x^\alpha \zeta(\alpha, y)}{\alpha \sqrt{2\pi \phi_2(\alpha, y)}} \left(1 + O(\bar{u}^{-1})\right)$

where

(2.9) \quad $\phi_2(\alpha, y) := \sum_{p \leq y} \frac{p^\alpha (\log p)^2}{(p^\alpha - 1)^2} = \left(1 + \frac{\log x}{y}\right) (\log x)(\log y)(1 + O(\log(1 + \bar{u})^{-1})).$

Additionally,

(2.10) \quad $\alpha = \frac{\log \left(1 + \frac{y}{\log y}\right)}{\log y} \left(1 + O \left(\frac{\log \log(y + 1)}{\log y}\right)\right).$

The points $\alpha$ and $\beta$ are known to be close:
Lemma 2.4. For \( x \geq y \geq \varepsilon \log x \) we have \( \beta - \alpha = O_{\varepsilon}(1/\log y) \).

Proof. If \( x \geq y > \log x \) this is in [20, Eq. (3.5)]. If \( \log x \geq y \geq \varepsilon \log x \) both \( \beta \) and \( \alpha \) are \( O_{\varepsilon}(1/\log y) \) by Corollary 2.2 and (2.10).

The following lemma is useful in simplifying the order of magnitude of \( y^\alpha \) and \( y^\beta \).

Lemma 2.5. For \( x \geq y \geq 2 \) we have \( y^{1-\beta} \leq u \log(u+1) \). For \( x \geq y \geq \varepsilon \log x \) we have \( y^{1-\alpha} \leq u \log(u+1) \).

Proof. The estimate for \( y^{1-\beta} \) follows from the definition of \( \beta \) and Lemma 2.4. The estimate for \( y^{1-\alpha} \) follows from the estimate for \( y^{1-\beta} \) by Lemma 2.4. \( \square \)

2.2. The function \( G \). We introduce

\[
G(s, y) := \frac{\zeta(s, y)}{F(s, y)}.
\]

The following identities are tautological in their domain of definition:

\[
\frac{x^\alpha \zeta(\alpha, y)}{x^\beta F_2(\beta, y)} = \zeta(\alpha)(\alpha - 1)G(\alpha, y) \exp(f(\alpha) - f(\beta))
\]

\[
= \zeta(\beta)(\beta - 1)G(\beta, y) \exp(g(\alpha) - g(\beta)).
\]

Let

\[
B(x, y) := \sqrt{\frac{(\log y)^2 I''(\xi(u))}{\phi_2(\alpha, y)}}.
\]

Alladi proved [2, Eq. (3.9)]

\[
\rho(u) = \frac{e^{-u \xi} \hat{\phi}(-\xi)}{\sqrt{2\pi I''(\xi(u))}} (1 + O(u^{-1}))
\]

which can be phrased as

\[
x \rho(u) = \frac{x^\beta F_2(\beta, y)}{\sqrt{2\pi (\log y)^2 I''(\xi(u))}} (1 + O(u^{-1})).
\]

Dividing (2.8) by (2.12) and employing (2.11) we obtain

Lemma 2.6 (Asymptotic formula). If \( x \geq y \geq 2 \) then \( f(\alpha) \geq f(\beta) \) and

\[
\frac{\Psi(x, y)}{x \rho(u)} = K(\alpha - 1)G(\alpha, y) \exp(f(\alpha) - f(\beta)) B(x, y)(1 + O(u^{-1})).
\]

If \( x \geq y > 1 + \log x \) then \( g(\beta) \geq g(\alpha) \) and

\[
\frac{\Psi(x, y)}{x \rho(u)} = K(\beta - 1)G(\beta, y) \exp(g(\alpha) - g(\beta)) \frac{\beta}{\alpha} B(x, y)(1 + O(u^{-1})).
\]

When \( y/\log x \to \infty \), (2.11) and (2.10) show \( \alpha/\beta \sim 1 \) and \( K(\alpha - 1) \sim K(\beta - 1) \sim K(-\log \log x/\log y) \). If further \( u \to \infty \) then \( B(x, y) \sim 1 \) by (2.9) and (2.3). Lemma 2.6 then implies

\[
(\beta, y)(1 + o(1)) \leq \frac{\Psi(x, y)}{x \rho(u)} K(-\log \log x/\log y) \leq G(\beta, y)(1 + o(1))
\]

as long as \( u \) and \( y/\log x \) both tend to \( \infty \).

The savings in (2.8) and (2.12) are sharp. We can show that the lower order terms within the error terms are close, which allows us to prove in §7.4 the following
Proposition 2.7. If $y > 1 + \log x$ then the error terms in Lemma 2.6 are $O(1/(\alpha \log x))$.

Remark 1. With more work, $O(1/(\alpha \log x))$ may be improved to an explicit term of size $\asymp 1/(\alpha \log x)$ and an error of size $O(1/(\alpha^2 (\log x) (\log y)))$.

Working in a zero-free region of $\zeta$ we choose the following logarithms of $\zeta(s,y)$ and $F(s,y)$:

$$\log \zeta(s,y) = \sum_{n \leq y} \Lambda(n)/(n^s \log n),$$

$$\log F(s,y) = \log \log y + \gamma + I((1-s) \log y) + \log(\zeta(s)(s-1))$$

where $\log(\zeta(s)(s-1))$ is chosen to be real-valued for $s > 1$. In further using Lemma 2.6 it will be of crucial importance to split $G$ as $G_1 G_2$ where

$$\log G_1(s,y) = \sum_{n \leq y} \frac{\Lambda(n)}{n^s \log n} - (\log(\zeta(s)(s-1)) + \log \log y + \gamma + I((1-s) \log y)),$$

$$\log G_2(s,y) = \sum_{k \geq 2} \frac{p_{k,s}^{-s}}{k}.$$ 

We compute the Mellin transform of $\log G_1(s,y)$ (Proposition 5.1) and then, via Landau’s Oscillation Theorem, obtain the following lemma, essential to the proof of Theorem 1.1.

Lemma 2.8. Fix $s > -2$. Then, as $x \to \infty$, $\log G_1(s,x) = \Omega_{\pm}(x^{\theta - s - \epsilon})$.

Lemma 2.8 is proved in §5.1. For $s = 0$ it goes back to Landau. For $s = 1$ it is due to Diamond and Pintz [7] and our standard proof follows theirs.

The function $\log G_1(s,y)$ and its derivatives with respect to $s$ can be expressed as sums over zeros of $\zeta$ (Corollaries 5.4–5.5). This allows us to prove in §5.3 the following

Lemma 2.9. Fix $i \geq 0$. For $\epsilon - 2 \leq s \leq 1/\epsilon$ and $x \geq 4$ we have

$$\log G_1^{(i)}(s,x) \ll_{i, \epsilon} (\log x)^i x^{1-s} L(x)^{-c} \ll_{i, \epsilon} x^{1-s} L(x)^{-c_i},$$

$$\log G_1^{(i)}(s,x) = (-1)^i (\log x)^{i-1} \frac{\psi(x) - x + O_{i, \epsilon}(x^\theta)}{x^s} \ll_{i, \epsilon} x^{\theta - s} (\log x)^{i+1}.$$ 

Littlewood proved that $\psi(x) - x = \Omega_{\pm}(\sqrt{x \log \log x})$ [23 Thm. 15.11]. Applying (2.15) if $\theta = 1/2$ and Lemma 2.8 otherwise, we get

Corollary 2.10. Fix $s > -2$. Then, as $x \to \infty$, $\log G_1(s,x) = \Omega_{\pm}(x^{\theta - s} \log \log x / \log x)$.

In §5.4 we prove the following illuminating representations of $\log G_1$.

Lemma 2.11. For $x > 2$ and $s > \theta$,

$$\log G_1(s,x) = - \int_x^\infty \frac{d(t) - t}{t^s \log t}. $$

For $x > 2$ and $s > -2$,

$$\log G_1(s,x) = \int_2^x \frac{d(t) - t}{t^s \log t} + \int_1^2 \frac{t^{-s} - t^{-s}}{\log t} dt + \int_2^\infty \frac{dt}{t^2 \log t} - \log(\zeta(s)(s-1)).$$

One can show $\log G_2(s,x) \to 0$ when $s - 1/2 \geq \epsilon$ and $x \to \infty$. In §6 we prove more:
Proposition 2.12. Fix \( i \geq 0 \). For \( x \geq 2 \) and \( 1 \geq s \geq \varepsilon / \log x \) we have
\[
(2.17) \quad (\log G_2)^{(i)}(s, x) = (1 + O_{x,i}(L(x)^{-c} + x^{-s})) \frac{(-2)^i}{2} \int_{\sqrt{x}}^{x} (\log t)^{i-1} t^{-2s} dt
\]
\[
\ll_{x,i} (-\log x)^i x^{\max(1-2s, \frac{1-s}{2})} \max\{1, |s-1/2| \log x\}.
\]
For \( 1/4 \geq s \geq \varepsilon / \log x \) we have
\[
(2.18) \quad \log G_2(s, x) = (1 + O_x(L(x)^{-c})) \int_{\sqrt{x}}^{x} (-\log(1-t^{-s}) - t^{-s}) \frac{dt}{\log t}.
\]

2.3. Proof of Theorem 1.1

2.3.1. First part. We may assume \( u \to \infty \) due to (1.1). By (2.13) it suffices to show \( G(\alpha, y), G(\beta, y) \sim 1 \). When \( y \geq (\log x)^{1/(1-\vartheta)} + \varepsilon \), the bounds for \( \log G_1 \) and \( \log G_2 \) given in (2.15) and (2.17) imply this (we simplify \( y^{-\alpha} \) and \( y^{-\beta} \) using Lemma 2.5).

2.3.2. Second part. Fix \( \beta_0 \in (0, 1) \). Given \( y \geq 2 \) there is a unique \( x = x(y) \) with \( \beta(x, y) = \beta_0 \). Using (2.1) we see that \( x \) is determined by
\[
(y^{1-\beta_0} - 1)/(1 - \beta_0) = \log x,
\]
and in particular \( y = (\log x)^{1/(1-\beta_0)} + o(1) \). Applying (2.13) with our \( x = x(y) \) (so that \( \beta = \beta_0 \)),
\[
(2.19) \quad \Psi(x, y) \ll x \rho(u) \exp(\log G_1(\beta_0, y) + \log G_2(\beta_0, y)).
\]
For our fixed \( \beta_0 \),
\[
(2.20) \quad \log G_1(\beta_0, y) = \Omega_-(y^{\beta - \beta_0 - \varepsilon})
\]
by Lemma 2.8. If \( \beta_0 \in (1 - \vartheta, \vartheta) \), the \( \Omega_- \) result follows from (2.19), (2.20) and the bound for \( \log G_2 \) given in (2.17). We now prove the \( \Omega_+ \) result. Given \( y \geq 2 \) and fixed \( \alpha_0 \in (0, 1) \)
there is a unique \( x = x(y) \) with \( \alpha(x, y) = \alpha_0 \). Using (2.7) we see that \( x \) is determined by
\[
-\zeta'(\alpha_0, y) / \zeta(\alpha_0, y) = \sum_{p \leq y} \rho(p) / (p^{\alpha_0} - 1) = \log x.
\]
By (2.10), \( y = (\log x)^{1/(1-\alpha_0)} + o(1) \). Applying (2.13) with our \( x = x(y) \) (so that \( \alpha = \alpha_0 \)),
\[
(2.21) \quad \Psi(x, y) \gg x \rho(u) \exp(\log G_1(\alpha_0, y) + \log G_2(\alpha_0, y)) \geq \exp(\log G_1(\alpha_0, y)).
\]
For our fixed \( \alpha_0 \),
\[
(2.22) \quad \log G_1(\alpha_0, y) = \Omega_+(y^{\beta - \alpha_0 - \varepsilon})
\]
by Lemma 2.8. If \( \alpha_0 \in (0, \vartheta) \), the \( \Omega_+ \) result follows from (2.21) and (2.22).

2.3.3. Third part. Let \( s \in \{\alpha, \beta\} \). We use (2.15) and (2.17) to see that \( \log G_1(s, y) \ll y^{\beta - s} \log y \) and \( \log G_2(s, y) \gg \varepsilon y^{1-2s} / \log y \gg \varepsilon \log^2 x / (y \log y) \) whenever \( (\log x)^{1/(1-\vartheta)} \geq y \geq 2 \log x \) and \( x \geq C \). We apply these estimates to
\[
(1 + o(1)) K(\alpha - 1) G(\alpha, y) B(x, y) \leq \frac{\Psi(x, y)}{x \rho(u)} \leq (1 + o(1)) K(\beta - 1) G(\beta, y) B(x, y) \frac{\beta}{\alpha},
\]
which follows from Lemma 2.6. The logarithms of \( B(x, y) \), \( K(s - 1) \) and \( \beta / \alpha \) are negligible: they are \( \ll \varepsilon \log \log y \) as seen from (2.3), (2.3) (for \( B \)) and (2.10), (2.4) (for \( K \) and \( \beta / \alpha \)).
Corollary 3.1. Suppose \( \vartheta < 1 \). If \( x \geq y \geq (\log x)^{\frac{1}{2} \max\{3, (1-\vartheta)^{-1}\}+\varepsilon} \) then, as \( x \to \infty \),

\[
\Psi(x, y) \sim x\rho(u)K(\beta - 1)G(\beta, y) \sim x\rho(u)K(\alpha - 1)G(\alpha, y).
\]

If \( \vartheta = 3/4 \) then Corollary 3.1 implies that, as \( x \to \infty \), (3.1) holds for \( y \geq (\log x)^{2+\varepsilon} \).

Under RH, (3.1) holds for \( x \geq y \geq (\log x)^{3/2+\varepsilon} \). A different behavior emerges once \( y \geq (\log x)^{3/2}(\log \log x)^{-1/2} \).

Corollary 3.2. Assume RH. Suppose \( x \geq C_\varepsilon \). If \( \varepsilon \log x \leq y \leq (\log x)^{2-\varepsilon} \) then

\[
\Psi(x, y) = x\rho(u)K(\alpha - 1)G(\alpha, y) \exp\left(\Theta_\varepsilon\left(\frac{\log^3 x}{y^2 \log y}\right)\right).
\]

If \( (1+\varepsilon) \log x \leq y \leq (\log x)^{2-\varepsilon} \) then

\[
\Psi(x, y) = x\rho(u)K(\beta - 1)G(\beta, y) \exp\left(-\Theta_\varepsilon\left(\frac{\log^3 x}{y^2 \log y}\right)\right).
\]

We do not explore this, but the contribution of \( k = 3 \) term (i.e. cubes) to \( \log G_2(\alpha, y) \) and \( \log G_2(\beta, y) \) also undergoes a phase transition and is of size \( \asymp \varepsilon \log x^3/(y^2 \log y) \) once \( y \leq (\log x)^{3/2-\varepsilon} \).

A classical estimate of Hildebrand and Tenenbaum [20, Thm. 2] states that

\[
\log\left(\frac{\Psi(x, y)}{x}\right) = (1 + O_\varepsilon(\exp(-(\log y)^{\frac{3}{5} - \varepsilon}))) \log \rho(u) + O_\varepsilon\left(\frac{\log(u + 1)}{\log y}\right)
\]

holds for \( x \geq y \geq (\log x)^{1+\varepsilon} \) (cf. [27, Thm. III.5.21]). We offer an improvement in terms of range and error.

Corollary 3.3. Suppose \( x \geq C_\varepsilon \). For \( x \geq y \geq \log x \cdot L(\log x)^{\varepsilon} \),

\[
\log\left(\frac{\Psi(x, y)}{xK(\alpha - 1)}\right) = \left(1 + O_\varepsilon(L(y)^{-c_\varepsilon})\right) \log \rho(u) + O_\varepsilon\left(\frac{1}{\log x}\right),
\]

and the same holds with \( K(\alpha - 1) \) replaced by \( K(\beta - 1) \). For \( \log x \cdot L(\log x)^{c} \geq y \geq \varepsilon \log x \),

\[
\log\left(\frac{\Psi(x, y)}{x\rho(u)}\right) = \left(1 + \Theta_\varepsilon\left(\frac{\log x}{y}\right)\right) \log G_2(\alpha, y).
\]

We zoom in on the behavior at \( y \asymp \log x \), which was considered by Erdős [9] (who studied \( y = \log x \)), Erdős and van Lint [10], de Bruijn [6] and Granville [14]. In [14] it is explained why “the real difficulty lies in this range”.

Corollary 3.4. Suppose \( x \geq C_\varepsilon \). For \( \varepsilon \log x \leq y \leq \varepsilon^{-1} \log x \) we have

\[
\log\left(\frac{\Psi(x, y)}{x\rho(u)}\right) = \left(1 + O_\varepsilon(L(y)^{-c_\varepsilon})\right) \int_{\sqrt{y}}^{y} (-\log(1 - v^{-\alpha}) - v^{-\alpha}) \frac{dv}{\log v} + f(\alpha) - f(\beta).
\]
Setting \( t := \frac{y}{\log x} \),
\[
0 < f(\alpha) - f(\beta) = u(\log(1 + t^{-1}) - (1 + t^{-1})^{-1}) + O_\varepsilon\left(\frac{y}{\log y}\right),
\]
\[
0 < \int_{\sqrt{\pi}}^{y} \left( -\log(1 - v^{-\alpha}) - v^{-\alpha} \right) \frac{dv}{\log v} = \frac{y}{\log y} \left( (1 + t^{-1}) - (1 + t^{-1})^{-1} \right) + O_\varepsilon\left(\frac{y}{\log y}\right).
\]

For example, if \( y = \log x \) then both \( f(\alpha) - f(\beta) \) and the integral in (3.4) are \( \sim \log(2) - \frac{1}{2} \).

**Remark 3.** If \( y \leq \log x \) then \( \Psi(x,y) \leq x^{1/\log \zeta(1/\log y,y)} = e^{O(u)} \). Since \( \rho(u) = e^{-u \log(u \log u) + O(u)} \), \( \Psi(x,y)/(x \rho(u)) = \log(x)/(\log x) + O(1) \) for \( y \leq \log x \). In particular \( \log(\Psi(x,y)/(x \rho(u))) \sim \log(\log x)/\log x \) as \( y/\log x \to 0 \).

Let
\[
A(s) := (\log(\zeta(s)(s - 1)))' \approx 1
\]
for \( s \in [0,1] \). Recall \( f''(\beta) = (\log y)^2 f''(\xi(u)) \) and \( g''(\alpha) = \phi_3(\alpha,y) \) are estimated in (2.3) and (2.2), respectively, and are \( \approx_{\varepsilon} (\log x)(\log y) \) when \( y \geq \varepsilon \log x \). The following technical lemma is proved in [7] and is used in the proofs of Corollaries 3.1 and 3.4.

**Lemma 3.5.** Suppose \( x \geq C\varepsilon \).

1. Define quantities \( E \) and \( E' \) via
\[
\beta - \alpha = \frac{f''(\alpha,y) + A(\alpha)}{f''(\beta)} (1 + E) = \frac{f''(\beta,y) + A(\beta)}{g''(\alpha)} (1 + E').
\]
When \( x \geq y \geq \varepsilon \log x \) we have \( E \ll_{\varepsilon} (|G''(\alpha,y)| + 1)/\log x \) and \( 1 + E \approx_{\varepsilon} 1 \). When \( x \geq y \geq (1 + \varepsilon) \log x \) we have \( E' \ll_{\varepsilon} (|G''(\beta,y)| + 1)/\log x \) and \( 1 + E' \approx_{\varepsilon} 1 \).

2. Define quantities \( E_1 \) and \( E_2 \) via
\[
g(\beta) - g(\alpha) = g''(\alpha)(\beta - \alpha)^2(1 + E_1)/2 \quad \text{if} \quad x \geq y \geq (1 + \varepsilon) \log x,
\]
\[
f(\alpha) - f(\beta) = f''(\beta)(\beta - \alpha)^2(1 + E_2)/2 \quad \text{if} \quad x \geq y \geq \varepsilon \log x.
\]
Then \( E_1, E_2 \ll_{\varepsilon} (|G''(\alpha,y)| + 1)/\log x \) and \( 1 + E_1, 1 + E_2 \approx_{\varepsilon} 1 \).

3. If \( x \geq y \geq \varepsilon \log x \) then
\[
B(x,y) = 1 + O_\varepsilon\left(\frac{|G''(\alpha,y)| + (\log G)(\alpha,y)(\log y)^{-1} + 1}{\log x}\right).
\]\nLet \( h(t) := (1 + t^{-1})^{-1/2} \). For \( x \geq y \geq 2 \),
\[
B(x,y) = h(y/\log x)(1 + O((\log(1 + y))^{-1})).
\]

**Remark 4.** Suppose \( x \geq y \geq (\log x)^{1/(1-\vartheta)^+} \) and \( x \geq C\varepsilon \). From Lemma 3.5, (2.15) and (2.17), the logarithms of \( G(\alpha,y), G(\beta,y), B(x,y) \) and \( \beta/\alpha \) are
\[
\ll_{\varepsilon} \frac{1}{\log x} + \frac{(\log x)(\log \log x)}{y^{1-\vartheta}}.
\]
Thus, from Lemma 2.6 with the error term supplied by Proposition 2.7,
\[
\Psi(x,y) = x \rho(u) K(s - 1) \left(1 + O_{\varepsilon}\left(\frac{1}{\log x} + \frac{(\log x)(\log \log x)}{y^{1-\vartheta}}\right)\right)
\]
for \( s \in \{\alpha, \beta\} \). This gives a stronger version of the first part of Theorem 1.1.
3.1. **Proof of Corollary 3.1** Our starting point is Lemma 2.6 with the error term supplied by Proposition 2.7. Since \( B(x, y) \sim 1 \) by (3.6) and \( \beta/\alpha \sim 1 \) when \( y \geq (\log x)^{1+\varepsilon} \), it suffices to show that \( g(\alpha) - g(\beta) \) and \( f(\alpha) - f(\beta) \) are \( o(1) \) as \( x \to \infty \) in the considered range. By Lemma 3.5,\n
\[
(3.9) \quad g(\beta) - g(\alpha), f(\alpha) - f(\beta) \approx (\log x)^{-1}(\log y)^{-1}\left(\frac{G'_1}{G_1}(\alpha, y) + A(\alpha)\right)^2.
\]

By (2.15) and (2.17) with \( i = 1 \), \( G'_1(\alpha, y)/G_1(\alpha, y) \ll y^{1-\alpha}(\log y)^2 \), \( G'_2(\alpha, y)/G_2(\alpha, y) \approx \int_y^{\infty} t^{-2\alpha} dt \approx \frac{\log y}{\max\{1, |\alpha - 1/2|\log y\}} \ll (y^{1-2\alpha} + 1) \log y \).

By Lemma 2.5, these estimates give the result. The exponents \((1 - \vartheta)^{-1}/2 \) and \( 3/2 \) in the corollary arise from our bounds for \( G'_1/G_1 \) and \( G'_2/G_2 \), respectively.

3.2. **Proof of Corollary 3.2**. Our starting point is Lemma 2.6. In the considered ranges, \( \alpha, \beta \leq 1/2 - c_\varepsilon \) by (2.10) and Lemma 2.4. According to Lemma 3.5 and (2.15) and (2.17) with \( i = 1, 2 \), RH implies in our range that \( \frac{\beta}{\alpha} - 1, B(x, y) - 1 \ll \log^2 y + \log x \ll \left(\frac{\log^3 x}{y^2 \log y}\right) \).

By (3.9) and (3.10), \( g(\beta) - g(\alpha) \) and \( f(\alpha) - f(\beta) \) are \( \approx \log^3 x/(y^2 \log y) \).

3.3. **Proof of Corollary 3.3**. Suppose \( y \geq \varepsilon \log x \). Taking logarithms in the first part of Lemma 2.6 and using the error term supplied by Proposition 2.7 we see \[
\log \left(\frac{\Psi(x, y)}{x \rho(u) K'(\alpha - 1)}\right) = \log G(\alpha, y) + f(\alpha) - f(\beta) + \log B(x, y) + O\left(\frac{1}{\alpha \log x}\right).
\]

Recall \( G = G_1 G_2 \). By (2.14) with \( i = 0 \) and Lemma 2.5, \( \log G_1(\alpha, y) \ll y^{1-\alpha} L(y)^{-c} \approx \varepsilon u \log(u + 1) L(y)^{-c} \).

The term \( \log G_2(\alpha, y) \) is studied in (2.17). The terms \( \log B(x, y) \) and \( f(\alpha) - f(\beta) \) are estimated in Lemma 3.5 in terms of \( G \). By (2.14) and (2.17) we find \[
\log B(x, y) \ll \varepsilon \log G_2(\alpha, y) + L(y)^{-c} + \frac{1}{\log x},
\]
\[
f(\alpha) - f(\beta) = \Theta(\log^2 G(\alpha, y) + O\left(\frac{\log G_2(\alpha, y)}{\min\{\log x, L(y)^{c}\}} + \frac{\log x}{L(y)^c} + \frac{1}{\log x (\log y)}\right).
\]

Using (2.17) this gives (3.2) and (3.3). By (3.5), (3.2) holds with \( \beta \) instead of \( \alpha \) too.

3.4. **Proof of Corollary 3.4**. We take logarithms in the first part of Lemma 2.6. We have \( K(\alpha - 1) \approx \alpha^{-1} \approx \varepsilon \log y \)

by (2.10) and \( \log B(x, y) = O(1) \) by (3.7). Hence
\[
\log \left(\frac{\Psi(x, y)}{x \rho(u)}\right) = \log G_1(\alpha, y) + \log G_2(\alpha, y) + f(\alpha) - f(\beta) + O\left(\log \log y\right).
\]

By (2.14) we have \( \log G_1(\alpha, y) \ll y^{1-\alpha} L(y)^{-c} \approx \varepsilon y L(y)^{-c} \).
By (2.18), the quantity \( \log G_2(\alpha, y) \) is equal to the integral in (3.4), finishing the proof of (3.4). To study \( f(\alpha) - f(\beta) \) we use [20, Eq. (7.6)] and [3, p. 88], which say

\[
\alpha = \frac{\log (1 + \frac{x}{\log x})}{\log y} \left( 1 + O\left( \frac{1}{\log y} \right) \right), \quad \xi = \log (u \log u) + \frac{\log \log u}{\log u} + O\left( \frac{1}{\log u} \right),
\]
to deduce that

\[
\alpha - \beta = \frac{\log (1 + \frac{x}{\log y})}{\log y} + O_{\epsilon}\left( \frac{1}{\log^2 y} \right) > 0.
\]

By the definition of \( f \),

\[
f(\alpha) - f(\beta) = (\alpha - \beta) \log x + \int_{\xi}^{(1-\alpha) \log y} e^t - 1 \frac{dt}{t}.
\]

The term \( \alpha - \beta \) was just estimated. We estimate the integral in (3.13) using (3.11)–(3.12):

\[
\int_{\xi}^{(1-\alpha) \log y} e^t - 1 \frac{dt}{t} = \frac{1}{(1-\alpha) \log y} \left( 1 + O_{\epsilon}\left( \frac{1}{\log y} \right) \right) \int_{\xi}^{(1-\alpha) \log y} (e^t - 1) dt
\]

\[
= \frac{1}{\log y} \left( 1 + O_{\epsilon}\left( \frac{1}{\log y} \right) \right) (e^t - t) \bigg|_{t=\xi}^{t=(1-\alpha) \log y}
\]

\[
= \frac{y^{1-\alpha} - e^\xi}{\log y} \left( 1 + O_{\epsilon}\left( \frac{1}{\log y} \right) \right)
\]

and

\[
y^{1-\alpha} - e^\xi = e^\xi (e^{\log (y^{1-\alpha})} - 1) = -e^\xi \left( 1 + \frac{\log x}{y} \right)^{-1} \left( 1 + O_{\epsilon}\left( \frac{1}{\log y} \right) \right)
\]

\[
= -\log x \left( 1 + \frac{\log x}{y} \right)^{-1} \left( 1 + O_{\epsilon}\left( \frac{1}{\log y} \right) \right).
\]

To estimate the integral in (3.4) we integrate by parts, obtaining it equals

\[
\frac{y}{\log y} (-\log (1 - y^{-\alpha}) - y^{-\alpha}) \left( 1 + O_{\epsilon}\left( \frac{1}{\log y} \right) \right).
\]

We use (3.11) to write

\[
y^{-\alpha} = \left( 1 + \frac{y}{\log x} \right)^{-1} \left( 1 + O_{\epsilon}\left( \frac{1}{\log y} \right) \right)
\]

which gives the desired approximation for the integral.

4. Pomerance’s Question

4.1. Proof of first part of Theorem 1.2. Estimate (1.6) follows from (3.3) upon simplifying \( \log G_2 \) using (2.17). Estimate (1.5) is in (3.2) if \( x \geq C \) and \( x^\epsilon \geq y \) since, by Taylor-approximating \( K \) at 0,

\[
K(\beta - 1) = 1 + \Theta\left( \frac{\log(u + 1)}{\log y} \right)
\]
for \( x^\varepsilon \geq y \geq (\log x)^2 \). For \( x^\varepsilon \leq y \leq (1 - \varepsilon)x \), we recall de Bruijn’s approximation \( \Lambda(x, y) \) defined as

\[
\Lambda(x, y) := x \int_{\mathbb{R}} \rho(u - v)d([y^v]/y^v)
\]

for \( x \not\in \mathbb{Z} \). Integrating the definition by parts gives

\[
(4.2) \quad \Lambda(x, y) = x\rho(u) - \{x\} + x \int_{0}^{u-1} (-\rho'(u - v))\{y^v\}y^v dv.
\]

De Bruijn [5] proved \( \Psi(x, y) = \Lambda(x, y) + O(x \exp(-C' \log^{1/2} x)) \) for \( u = O(1) \). Suppose \( x^\varepsilon \leq y \leq (1 - \varepsilon)x \). Then the contribution of \( 0 \leq \varepsilon \leq c_\varepsilon/\log x \) to the integral in the right-hand side of (4.2) is \( \geq c_\varepsilon/\log x \), which yields (1.5) in the remaining range.

4.2. Proof of second part of Theorem 1.2. We assume RH holds. Estimate (1.6) in

\[
2 \log x \leq y \leq (2 + \varepsilon)(\log x)^{2-\varepsilon}
\]

follows from the third part of Theorem 1.1 for \( \varepsilon \log x \leq y < 2 \log x \). In the remaining range, Corollary 3.1 tells us

\[
\Psi(x, y) = (1 + o(1))xp(u)K(\beta - 1)G(\beta, y), \quad x \to \infty.
\]

The asymptotic estimates for \( \log G_1 \) and \( \log G_2 \) given in (2.13) and (2.17) respectively, yield

\[
\log G(\beta, y) = \frac{1 + o(1)}{2} \int_{\sqrt{y}}^{y} \frac{dt}{t^{2\beta} \log t} + \frac{\psi(y) - y}{y^{3} \log y} + O\left(\frac{y^{2-\beta}}{\log y}\right).
\]

We want

\[
(4.3) \quad \log K(\beta - 1) + \frac{y^{\frac{1}{2} - \beta}}{\log y} \left(\frac{\psi(y) - y}{\sqrt{y}} + O(1)\right) + \frac{1 + o(1)}{2} \int_{\sqrt{y}}^{y} \frac{dt}{t^{2\beta} \log t} + o(1)
\]

to be non-negative. We show that if

\[
(4.4) \quad \liminf_{y \to \infty} \frac{\psi(y) - y}{\sqrt{y} \log y} > L
\]

holds and \( x \geq C \) then (4.3) is non-negative for \( (\log x)^{3/2} \leq y \leq (\log x)^{3} \). We consider three cases. If \( (2\beta - 1) \log y \geq C \) then (4.4) implies that (4.3) is positive if \( x \geq C \). If \( (2\beta - 1) \log y \leq -C \) then (4.3) is positive by (4.4) and additionally invoking (2.17) to estimate the integral in (4.3). The most delicate range is \( (2\beta - 1) \log y = O(1) \). Here \( K(\beta - 1) \sim K(-1/2) \). Set

\[
\beta = \frac{1}{2} + \frac{v}{\log y}
\]

so that \( v \) is bounded. We express \( \log(\Psi(x, y)/(x\rho(u))) \) as a function of \( y \) and \( v \):

\[
(4.5) \quad \log\left(\frac{\Psi(x, y)}{x\rho(u)}\right) = \log K(-1/2) + e^{-v}\frac{\psi(y) - y}{\sqrt{y} \log y} + \frac{1}{2} \int_{v}^{2v} \frac{e^{-r}}{r} dr + o(1).
\]

If (4.4) holds, we find by the definition of \( L \) that the right-hand side of (4.5) is \( \geq c \) for some \( c > 0 \), if \( y \) is sufficiently large. If instead

\[
\liminf_{y \to \infty} \frac{\psi(y) - y}{\sqrt{y} \log y} < L
\]
then, by definition, we can find \( v \in \mathbb{R} \) such that if \( \beta = 1/2 + v/\log y \) then the right-hand side of (5.3) is \(-c\) for some \( c > 0 \), if \( y \) is sufficiently large. This finishes the proof.

5. Study of \( G_1 \)

5.1. Proof of Lemma [2.8] For our purposes, given a function \( A \) its Mellin transform is

\[
\{\mathcal{M}A\}(s) = \int_1^{\infty} A(x)x^{-s-1}dx.
\]

**Proposition 5.1.** Let \( T(x) := \int_x^{\infty} dt/(t^2 \log t) \), which decays like \( (x \log x)^{-1} \) as \( x \to \infty \) and blows up as \( x \to 1^+ \). Fix \( s_0 > -2 \) and \( a > 0 \). The function \( \log G_1(s_0, x) + T(x^a) \) is defined at \( x = 1^+ \) and

\[
\{\mathcal{M}(\log G_1(s_0, x) - T(x^a))\}(s) = \frac{1}{s} \log \frac{\zeta(s + s_0)(s + s_0 - 1)}{\zeta(s_0)(s_0 - 1)(1 + s/a)}
\]

holds for \( Rs > \max\{1 - s_0, 0\} \). This transform has analytic continuation to \( Rs > \max\{\vartheta - s_0, -a\} \).

**Proof.** We require the following identity [7, Lem. 2.2]:

\[
(5.1) \quad \log \log x + \gamma = \int_1^{x} \frac{1 - v^{-1}}{v \log v} dv - \int_{x}^{\infty} \frac{dv}{v^2 \log v}, \quad x > 1.
\]

Write \( \log G_1(s_0, x) \) as \( A_1(x) - (A_2(x) + A_3(x) + A_4(x)) \) where

\[
A_1(x) = \sum_{n \leq x} \frac{\Lambda(n)}{n^{s_0} \log n}, \quad A_2(x) = \log \log x + \gamma,
\]

\[
A_3(x) = I((1 - s_0) \log x), \quad A_4(x) = \log(\zeta(s_0)(s_0 - 1)).
\]

By [23, Thm. 1.3], the Mellin transform of \( x \mapsto \sum_{m \leq x} h(m) \) is \( s^{-1} \sum_{n=1}^{\infty} h(n)n^{-s} \), so

\[
\{\mathcal{M}A_1\}(s) = s^{-1} \log(\zeta(s + s_0))
\]

for \( Rs > \max\{1 - s_0, 0\} \). For any constant \( b \) we have \( \{\mathcal{M}b\}(s) = s^{-1}b \) and so

\[
\{\mathcal{M}A_4\}(s) = s^{-1} \log(\zeta(s_0)(s_0 - 1)).
\]

For \( A_3 \) first suppose \( s_0 < 1 \). We shall use the identity [7, Eq. (2.3)]

\[
(5.2) \quad \log \frac{z + 1}{z} = \int_1^{\infty} t^{-z} \frac{1 - t^{-1}}{t \log t} dt, \quad \Re z > 0.
\]

To verify (5.2) we check that both sides have the same derivative and tend to 0 when \( \Re z \to \infty \). Applying (5.2) with \( z = (s_0 + s - 1)/(1 - s_0) \) and substituting \( t = x^{-s_0} \) we obtain

\[
(5.3) \quad \log \frac{s}{s + s_0 - 1} = \int_1^{\infty} x^{-s} x^{1-s_0} - 1 \frac{dx}{x \log x} = s \int_1^{\infty} x^{-s-1} \int_1^{x} v^{1-s_0} - 1 dv dx
\]

for \( Rs > \max\{1 - s_0, 0\} \), where in the last equality we integrated by parts. Substituting \( v = e^{u/(1-s_0)} \) we find

\[
(5.4) \quad \{\mathcal{M}A_3\}(s) = s^{-1} \log \frac{s}{s + s_0 - 1}.
\]
If \( s_0 = 1 \) then \( A_3 \equiv 0 \) and (5.4) still holds. If \( s_0 > 1 \), the left-hand side of (5.3) still agrees with its right-hand side by the uniqueness principle and so (5.4) persists. For \( A_2 \), we apply (5.1) with \( x^a \) in place of \( x \), obtaining

\[
A_2(x) = \int_1^x \frac{1 - t^{-1}}{t \log t} dt - \int_{x^a}^\infty \frac{dt}{t^2 \log t} - \log a =: \tilde{A}_2(x) - T(x^a) - \log a
\]

where \( T \) is as in the statement of the proposition. We have \( \{ M \log a \} (s) = s^{-1} \log a \). By (5.3) with \( s_0 = 2 \), \( s/a \) in place of \( s \) and the substitution \( x = u^a \), we find

\[
\{ M \tilde{A}_2 \} (s) = s^{-1} \log \frac{s + a}{s}.
\]

We now sum the Mellin transforms of \( A_1, -A_2 - T(x^a) = \log a - \tilde{A}_2, -A_3 \) and \( -A_4 \). □

To establish Lemma 5.2 fix \( \varepsilon > 0 \) and \( s > -2 \). Suppose that \( \log G_1(s,x) < x^{\theta-s-\varepsilon} \) (resp. \( \log G(s,x) > x^{\theta-s-\varepsilon} \)) for \( x \geq C_{\varepsilon,s} \). Reach contradiction by applying Landau’s Oscillation Theorem [23, Lem. 15.1] to \( A(x) = 2x^{\theta-s-\varepsilon} - (\log G_1(s,x) - T(x^{s+2})) \) (resp. \( A(x) = 2x^{\theta-s-\varepsilon} + \log G_1(s,x) - T(x^{s+2}) \)), an eventually positive function.

5.2. Explicit formulas for \( G_1 \). Given \( x > 0 \) and \( s \in \mathbb{C} \) we let \( S_1(x,s) := \sum_{n\leq x} \Lambda(n)n^{-s} \), where the prime on the summation indicates that if \( x \) is a prime power, the last term of the sum should be multiplied by 1/2. Landau [21] established an explicit formula for \( S_1(x,s) \) when \( \zeta(s) \neq 0 \). In [A] we establish a truncated version of it, stated in the lemma below. We denote by \( x' \) the prime power closest to \( x \) not equal to \( x \), and set \( \langle x \rangle = |x-x'| \).

Lemma 5.2. Suppose \( \zeta(s) \neq 0 \). For \( x \geq 4 \) and \( T \geq 2 + |\Im s| \) we define \( R_1(x,T,s) \) by

\[
S_1(x,s) = \frac{x^{1-s}}{1-s} - \frac{\zeta'(s)}{\zeta(s)} - \sum_{|\Im(s-\rho)| \leq T} \frac{x^{\rho-s}}{\rho-s} + \sum_{k=1}^{\infty} \frac{x^{-2k-s}}{2k+s} + R_1(x,T,s),
\]

where the first sum is over non-trivial zeros \( \rho \) of \( \zeta \). Then

\[
R_1(x,T,s) \ll (\log x)x^{1-\varepsilon\Re s} \min \left\{ 1, \frac{x}{T(x)} \right\} + \frac{\log^2 x T}{T} (2|x| \log x + 2^{-\varepsilon\Re s}).
\]

If \( s = 1 \) then the term \( x^{1-s}/(1-s) - \zeta'(s)/\zeta(s) \) should be interpreted as \( \log x - \gamma \).

Next we need the identity [11] p. 228

\[
I(-z) + \int_0^\infty \frac{e^{-z-t}}{z+t} dt + \gamma + \log z = 0, \quad z \in \mathbb{C} \setminus (-\infty, 0],
\]

where \( \log z \) is chosen to be real-valued for \( z > 0 \) (it is in fact equivalent to (5.1)).

Corollary 5.3. Let \( S_0(x,s) := \sum_{n\leq x} \Lambda(n)n^{-s}/\log n \). Suppose \( s \in \mathbb{C} \) has \( \zeta(s+t) \neq 0 \) for \( t \geq 0 \). For \( x \geq 4 \) and \( T \geq 2 + |\Im s| \) we define \( R_0(x,T,s) \) by

\[
S_0(x,s) = I((1-s) \log x) + \gamma + \log x + \log(\zeta(s)(s-1))
\]

\[
- \sum_{|\Im(s-\rho)| \leq T} \int_0^\infty \frac{x^{\rho-s-t}}{\rho-s-t} dt + \sum_{k=1}^{\infty} \int_0^\infty \frac{x^{-2k-s-t}}{2k+s+t} dt + R_0(x,T,s)
\]

where \( \log(\zeta(z)(z-1)) \) is real-valued for \( z > 1 \) and defined on \( \{ s+t : t \geq 0 \} \). Then

\[
R_0(x,T,s) \ll x^{1-\varepsilon\Re s} \min \left\{ 1, \frac{x}{T(x)} \right\} + \frac{\log^2 x T}{T \log x} (2|x| \log x + 2^{-\varepsilon\Re s}).
\]
Proof. We start with an integral identity (cf. [25, Prop. 1]):

\[ S_0(x, s) = \int_0^\infty \sum_{n \leq x} \frac{\Lambda(n)}{n^{s+t}} dt = \int_0^\infty S_1(x, s+t) dt. \]

We integrate both sides of (5.5) along \{s+t : t \geq 0\}. We may interchange sum and integral because the sum over \( \rho \) is finite, while the integral of the \( k \)-sum converges absolutely. It remains to show

\[ \lim_{A \to \infty} \int_0^A \left( \frac{x^{1-s-t} - 1}{1-s-t} - \frac{(\zeta(s+t)(s+t-1))'}{\zeta(s+t)(s+t-1)} \right) dt = I((1-s) \log x) + \gamma + \log(\zeta(s)(s-1) \log x). \]

The substitution \((1-s-t) \log x = v\) allows us to evaluate the integral as

\[ I((1-s) \log x) - I((1-s-A) \log x) - \log(\zeta(s+A)(s+A-1)) + \log(\zeta(s)(s-1)). \]

The required limit follows from (5.7) with \( z = (s+A-1) \log x \) \((A \to \infty)\). \(\square\)

From the definition of \( \log G'\), Lemma 5.2 and Corollary 5.3 we get

**Corollary 5.4.** Suppose \( \zeta(s) \neq 0 \). For \( x \geq 4 \) and \( T \geq 2 + |\Im s| \) we have, for \( R_1 \) estimated in (5.6),

\[-(\log G_1)'(s, x) = \frac{1_{x \in \mathbb{N}} \Lambda(x)}{2x^s} - \sum_{|\Im(\rho-s)| \leq T} \frac{x^{\rho-s} + \infty}{2k + s} + R_1(x, T, s). \]

Suppose further that \( \zeta(s+t) \neq 0 \) for \( t \geq 0 \). We have, for \( R_0 \) estimated in (5.8),

\[ \log G_1(x, s) = \frac{1_{x \in \mathbb{N}} \Lambda(x)}{2x^s \log x} - \sum_{|\Im(\rho-s)| \leq T} \int_0^\infty \frac{x^{\rho-s-t}}{\rho-s-t} dt + \sum_{k=1}^\infty \int_0^\infty \frac{x^{2k-s-t}}{2k + s + t} dt + R_0(x, T, s). \]

Applying Cauchy’s integral formula to the first part of Corollary 5.4 we get

**Corollary 5.5.** Fix \( i \geq 2 \) and \( a > 0 \). Let \( x \geq 4 \). Suppose that \( \zeta(z) \neq 0 \) for \( |z-s| \leq a/\log x \). Then for \( T \geq 2 + |\Im s| + a/\log x \) we have

\[-(\log G_1)^{(i)}(s, x) = \frac{1_{x \in \mathbb{N}} \Lambda(x)(-\log x)^{i-1}}{2x^s} - \sum_{|\Im(\rho-s)| \leq T} \frac{\partial^{i-1} x^{\rho-s}}{\partial s^{i-1} \rho-s} + \frac{\partial^{i-1}}{\partial s^{i-1}} \sum_{k=1}^\infty \frac{x^{2k-s}}{2k + s} + R_i \]

for \( R_i = R_i(x, T, s) \) satisfying

\[ R_i(x, T, s) \ll_{i,a} (\log x)^i x^{1-\Re s} \min \left\{ 1, \frac{x}{T(x)} \right\} + \frac{\log^2(xT)(\log x)^{i-1}}{T} (2^{|\Re s|} x^{1-\Re s} + \frac{2^{-\Re s}}{\log x}). \]

5.3. **Proof of Lemma 2.9.** We explain \( i = 0 \); general \( i \) is similar. Under our assumptions,

\[ -\int_0^\infty \frac{x^{\rho-s-t}}{\rho-s-t} dt = \frac{1}{\log x} \left( \frac{1}{\rho-s} \left( 1 + O_{\varepsilon}\left( \frac{1}{|\rho| \log x} \right) \right) \right) \]

for every zero of \( \zeta \). For the first estimate we apply Corollary 5.4 with \( T = L(x)^c \) and use the Vinogradov–Korobov zero-free region to bound the sum over the zeros. For the second estimate we apply Corollary 5.3 with \( T = x \) and write the sum over zeros as

\[ -\sum_{|\Im(\rho)| \leq x} \int_0^\infty \frac{x^{\rho-s-t}}{\rho-s-t} dt = -\frac{x^{-s}}{\log x} \sum_{|\rho| \leq x} \frac{x^{\rho}}{\rho} (1 + O_{\varepsilon}(|\rho|^{-1})). \]
Since $\sum_{\rho} 1/|\rho|^2$ converges \cite[Thm. 10.13]{23}, $\sum_{|\rho| \leq x} |x^\rho/\rho^2| \ll x^\vartheta$. By Lemma 5.2 with $s = 0$, $-\sum_{|\rho| \leq x} x^\rho/\rho = \psi(x) - x + O(\log^2 x)$ and so

$$- \sum_{|3\rho| \leq x} \frac{x^\rho - x - t}{\rho - s - t} dt = x^{-s}(\psi(x) - x + O_{\varepsilon}(x^\vartheta)) \ll_{\varepsilon} x^{-s} x^\vartheta \log^2 x$$

where the last inequality is Exercise 1 in \cite[p. 430]{23}.

5.4. **Proof of Lemma 2.11.** By definition,

$$\sum_{n \leq x} \frac{\Lambda(n)}{n^s \log n} = \int_{2^-}^x \frac{d\psi(t)}{t^s \log t}.$$  

Through the change of variables $v \mapsto (1 - s) \log t$ ($s \neq 0$ fixed),

$$I((1 - s) \log x) = \int_{1}^{x} \frac{t^{1-s} - 1}{t \log t} dt,$$

which is also true for $s = 1$. The second part follows from (5.9), (5.10) and (5.1). For (2.16) observe that both sides of are real-analytic functions for $s > \vartheta$ (since $\psi(x) = x + O(x^\vartheta \log^2 x)$ \cite[p. 430]{23}) so by the uniqueness principle it suffices to consider $s > 1$. We use (5.9), (5.10) and

$$\log \zeta(s) = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s \log n} = \int_{2^-}^{\infty} \frac{d\psi(t)}{t^s \log t}$$

to find that

$$\log G_1(s, x) = - \int_{x}^{\infty} \frac{d(\psi(t) - t)}{t^s \log y} + H(s, x),$$

$$H(s, x) := \int_{1}^{x} \frac{1 - t^{1-s} - 1}{t \log t} dt - \int_{x}^{\infty} \frac{1}{t^s \log t} - \log((s - 1) \log x) - \gamma.$$  

The derivative of $H$ with respect to $s$ is 0 and $H(2, x) \equiv 0$ by (5.1), so $H \equiv 0$.

6. **Study of $G_2$: proof of Proposition 2.12**

We have $\log G_2 = \log G_{2,1} + \log G_{2,2}$ for

$$\log G_{2,1}(s, x) = \sum_{\sqrt{x} < p \leq x} \frac{p^{2s}}{2}, \quad \log G_{2,2}(s, x) = \sum_{k \geq 1} \sum_{x^{1/k} < p \leq x} \frac{p^{-ks}}{k}.$$  

The PNT with error term shows, via integration by parts, that for $s \in [0, 1]$ we have

$$\log G_{2,1}(s, x) = \frac{1 + O(L(x)^{-\epsilon})}{2} \int_{\sqrt{x}}^{x} \frac{dt}{t^2 \log t}.$$  

For $x \neq 0$ let $Ei(x)$ be the exponential integral, to be understood in principal value sense:

$$Ei(x) = - \int_{-\infty}^{\infty} e^{-t} t^{-1} dt = \int_{-\infty}^{x} e^{t} t^{-1} dt = e^{x} x^{-1}(1 + O(x^{-1})).$$
By performing the change of variables \(v = (1 - 2s) \log t\) in (6.1) we see that
\[
\int_{\sqrt{x}}^{x} \frac{dt}{t^{2s} \log t} = \text{Ei}((1 - 2s) \log x) - \text{Ei}((\frac{1}{2} - s) \log x)
\]
\[
\sim \begin{cases} 
\frac{e^{1-s}}{(s-\frac{1}{2}) \log x} & \text{if } (2s - 1) \log x \to \infty \\
\frac{e^{x-(1-2s) \log x}}{(1-2s) \log x} & \text{if } (2s - 1) \log x \to -\infty
\end{cases} < \frac{x \max \{1-2s, \frac{1}{2}-s\}}{\max \{1, |s-1/2| \log x\}}
\]
for \(s \in [0,1] \setminus \{1/2\}\). When \(s = 1/2\) the integral is \(2\) since \((\log \log t)' = 1/(t \log t)\).

Lemma 6.1. For \(1 \leq s \geq \varepsilon/\log x\) we have
\[
\log G_{2,2}(s, x) \ll_{\varepsilon} \frac{x \max \{1-3s, \frac{1}{2}-s\}}{\max \{1, |s-1/3| \log x\}}.
\]

Proof. If \(x \ll_{\varepsilon} 1\) then \(\log G_{2,2}(s, x) \ll_{\varepsilon} 1\) so we may assume \(x \geq C_{\varepsilon}\). In the same way we showed (6.1), we find that the contribution of \(k = 3\) to \(\log G_{2,2}(s, x)\) is acceptable, so we omit this case. We consider the contribution of \(k \geq \max \{2/s, \log_2 x\}\) (base-2 logarithm) to \(\log G_{2,2}\). For such \(k\),
\[
\sum_{x^{1/k} \leq p \leq x} p^{-ks} \leq 2^{-ks} + \sum_{p \geq 3} p^{-ks} \ll 2^{-ks} + \int_{2}^{x} t^{-ks} dt \ll 2^{-ks}.
\]

Hence
\[
(6.2) \quad \sum_{k \geq \max \{2/s, \log_2 x\}} \sum_{x^{1/k} \leq p \leq x} p^{-ks}/k \ll \sum_{k \geq \max \{2/s, \log_2 x\}} 2^{-ks}/k \ll x^{-s},
\]
which is negligible. It remains to consider the contribution of \(4 \leq k < \max \{2/s, \log_2 x\}\) to \(\log G_{2,2}\). We show that primes \(p \in (x^{1/4}, x]\) have an acceptable contribution. The assumption \(s \geq \varepsilon/\log x\) implies \((1/(1-t^{-s})) \ll_{\varepsilon} 1\) when \(t \geq x^{1/4}/2\), and so
\[
\sum_{\max \{2/s, \log_2 x\} > k \geq 4 x^{1/4} \leq p \leq x} p^{-ks}/k \ll \sum_{k \geq 4} \int_{x^{1/4}/2}^{x} t^{ks} k \log t dt \ll \sum_{x^{1/4}/2}^{x} t^{ks} \log t dt
\]
which is acceptable. For the primes \(p \in (x^{1/k}, x^{1/4}]\), the bound \(\pi(y) \ll y/\log y\) shows
\[
(6.3) \quad \sum_{\max \{2/s, \log_2 x\} > k \geq 4 x^{1/k} \leq p \leq x^{1/4}} p^{-ks}/k \ll \sum_{k \geq 4} x^{1-s} / \log x \ll_{\varepsilon} x^{1-s},
\]
where in the last inequality we used \(s \geq \varepsilon/\log x\). This is an acceptable bound. \(\square\)

Estimate (2.17) when \(i = 0\) is a direct consequence of (6.1) and Lemma 6.1. If \(i > 0\) is similar and so is omitted. We now assume \(1/4 \geq s \geq \varepsilon/\log x\) and establish (2.18). The contribution of \(k \geq \max \{2/s, \log_2 x\}\) to \(\log_2\) is \(\ll x^{-s}\) as in (6.2). We now consider \(2 \leq k < \max \{2/s, \log_2 x\}\). If \(x^{1/k} < p \leq \sqrt{x}\) we get a contribution of \(\ll_{\varepsilon} x^{1/2-s}\) similarly to (6.3). We handle \(2 \leq k < \max \{2/s, \log_2 x\}\) and \(\sqrt{x} < p \leq x\) by the PNT and integration by parts, obtaining a contribution of
\[
(1 + O_{\varepsilon}(L(x)^{-c})) \int_{\sqrt{x}}^{x} \sum_{2 \leq k < \max \{2/s, \log_2 x\}} \frac{t^{-ks}}{k \log t} dt.
\]
Since \(t^s - 1 \gg_{\varepsilon} 1\) when \(t \in [\sqrt{x}, x]\) we may extend the \(k\)-sum within the integral to the range \(k \geq 2\) at a cost of \(\ll_{\varepsilon} \int_{\sqrt{x}}^{x} t^{-2} dt \ll 1\).
7. Proofs of Lemma 3.35 and Proposition 2.7

Lemma 7.1. Fix $2 \leq k \leq 5$. Suppose $x \geq C_\varepsilon$. Let $I$ be the closed interval with endpoints $\alpha$ and $\beta$. For $t \in I$,

\begin{align}
(7.1) & \quad g^{(k)}(t) \asymp_\varepsilon (-1)^k (\log x)(\log y)^{k-1} \text{ if } x \geq y \geq (1 + \varepsilon) \log x, \\
(7.2) & \quad f^{(k)}(t) \asymp_\varepsilon (-1)^k (\log x)(\log y)^{k-1} \text{ if } x \geq y \geq \varepsilon \log x.
\end{align}

Proof. Suppose $x \geq y \geq (1 + \varepsilon) \log x$. As shown in Lemma 4 of [20],

\[ g^{(k)}(t) = (-1)^k \sum_{p \leq y} (\log p)(p^t - 1)^{-k} Q_{k-1}(p^t \log p) \]

for a polynomial $Q_{k-1}$ of degree $k-1$ and non-negative coefficients, so $(-1)^k g^{(k)}(t)$ is positive and monotone for $t > 0$. By the same lemma, $g^{(k)}(\alpha) \asymp (-1)^k (\log x)(\log y)^{k-1}$ for $x \geq y \geq \log x$. It remains to show $g^{(k)}(\beta)$ is also of order $(-1)^k (\log x)(\log y)^{k-1}$. Since $\beta \geq c_\varepsilon/\log y$ by Corollary 2.2, the same lemma shows that

\[ (\log y)^{k-1} y^{1-\beta} - 1 \ll_\varepsilon (-1)^k g^{(k)}(\beta) \ll_\varepsilon (\log y)^{k-1} \sum_{p \leq y} \log p / p^\beta - 1. \]

By definition of $\beta$, the left-hand side is $(\log x)(\log y)^{k-1}$. The sum in the right-hand side is upper bounded in [20, Eq. (7.1)] by

\[ \sum_{p \leq y} \log p / p^\beta - 1 \ll \frac{1}{1 - y^{-\beta}} \int_1^y t^{-\beta} dt + O(1) \]

which is $\ll_\varepsilon \log x$ by definition of $\beta$. This finishes the proof of (7.1). For $f^{(k)}$, \( f^{(k)}(t) = (-\log y)^k I^{(k)}((1-t) \log y). \)

Observe $I^{(k)}(v) \asymp e^v/(v+1)$ uniformly for $v \geq 0$ [26, Lem. 4.5] and $e^v/(v+1) \asymp u$ as long as $0 \leq v = \xi(u) + O(1)$. Hence, by monotonicity of $I^{(k)}$, it suffices to show $0 \leq (1-t) \log y = \xi(u) + O(\varepsilon)(1)$ holds for $t \in \{\alpha, \beta\}$. For $t = \beta$ it is trivial. For $t = \alpha$, $(1-\alpha) \log y = \xi(u) + O(\varepsilon)(1)$ follows from Lemma 2.4 so it is left to show $\alpha < 1$. By definition $\sum_{p \leq y} \log p / (p^\alpha - 1) = \log x$, and at $\alpha = 1$ the sum is $\log y - \gamma + o(1)$ [23, p. 182] which is $< \log x$ when $x \geq C$, so $\alpha < 1$. \( \square \)

Corollary 7.2. Suppose $x \geq C_\varepsilon$. Let $I$ be the closed interval with endpoints $\alpha$ and $\beta$. For $t \in I$,

\begin{align}
(7.3) & \quad g''(t) = g''(\alpha)(1 + O_\varepsilon(|\alpha - \beta| \log y)) \text{ if } x \geq y \geq (1 + \varepsilon) \log x, \\
& \quad f''(t) = f''(\beta)(1 + O_\varepsilon(|\alpha - \beta| \log y)) \text{ if } x \geq y \geq \varepsilon \log x.
\end{align}

Proof. For any $t \in I$, $g''(t) = g''(\alpha) + (t - \alpha) g''(t_2)$ for some $t_2 \in I$. The estimates for $g''$ and $g^{(3)}$ in Lemma 7.1 imply the result for $g''$. The proof of (7.3) is similar. \( \square \)

Lemma 7.3. Suppose $x \geq C_\varepsilon$. Let $2 \leq k \leq 4$ and $s \in \{\alpha, \beta\}$. We have

\[ g^{(k)}(\alpha) - f^{(k)}(\beta) = A^{(k-1)}(s) + O_\varepsilon(|\log G|^{(k)}(s, y)) + |\alpha - \beta| (\log x)(\log y)^k \]

in the range $x \geq y \geq (1 + \varepsilon) \log x$ if $s = \beta$ and in $x \geq y \geq \varepsilon \log x$ if $s = \alpha$.

Proof. When $s = \alpha$ we write $g^{(k)}(\alpha) - f^{(k)}(\beta) = f^{(k)}(\alpha) f^{(k)}(\beta) + (\log G)^{(k)}(\alpha, y) + A^{(k-1)}(\alpha)$ and replace $f^{(k)}(\alpha) - f^{(k)}(\beta)$ by $(\alpha - \beta) f^{(k+1)}(t)$ for $t$ between $\alpha$ and $\beta$. Lemma 7.1 bounds $f^{(k+1)}(t)$. The case $s = \beta$ is similar. \( \square \)
7.1. Proof of Lemma 3.5 – first part. The relations $g'(\alpha) = f'(\beta) = 0$ can be written as
\[ (\zeta'/\zeta)\alpha, y = (-F'_2/F_2)(\beta, y) = \log x. \]
Writing $\zeta(s, y)$ as $F_2(s, y)G(s, y)\zeta(s)(s - 1)$, (7.1) implies
\[ (-F'_2/F_2)(\alpha, y) + (F'_2/F_2)(\beta, y) = (G'/G)(\alpha, y) + A(\alpha). \]
By the mean value theorem, for some $t$ between $\alpha$ and $\beta$ we have
\[ (-F'_2/F_2)(\alpha, y) + (F'_2/F_2)(\beta, y) = (\beta - \alpha)g''(t). \]
We compare (7.3) with (7.4) to find $1 + E = f''(\beta)/f''(t)$ where $E$ is defined in (3.5).
By (7.2), $1 + E \approx \varepsilon$. To upper bound $|E|$ we use the estimate in Corollary 7.2 to find $E \ll \varepsilon |\alpha - \beta| \log y$. We simplify this using the bound for $|\alpha - \beta|$ we just derived: $1 + E \approx \varepsilon$ implies $|\alpha - \beta| \ll \varepsilon (|G'(\alpha, y)| + 1)/((\log x)(\log y))$. To study $E'$ we argue similarly using the relation
\[ (\beta - \alpha)g''(t') = -(\zeta'/\zeta)(\alpha, y) + (\zeta'/\zeta)(\beta, y) = (G'/G)(\beta, y) + A(\beta) \]
for some $t'$ between $\alpha$ and $\beta$. This shows $1 + E' = g''(\alpha)/g''(t')$.

7.2. Proof of Lemma 3.5 – second part. We approximate $g$ at $\alpha$ using a quadratic Taylor polynomial:
\[ g(\beta) - g(\alpha) = g''(\alpha)(\beta - \alpha)^2/2 \left( 1 + O(1/\log x/\log y) \right) \]
for some $t$ between $\alpha$ and $\beta$. By Lemma 7.1 $g''(3)(t) \ll \varepsilon (\log x)(\log y)^2$ and we bound $|\alpha - \beta|$ using (3.5). Alternatively, $g(\beta) - g(\alpha) = g''(t)(\beta - \alpha)^2/2$ for some $t'$ between $\alpha$ and $\beta$, and we appeal to Lemma 7.1 with $k = 2$. The same arguments work for $f(\alpha) - f(\beta)$.

7.3. Proof of Lemma 3.5 – third part. The square of $B$ can be written as
\[ B^2(x, y) = \frac{f''(\beta)}{g''(\alpha)} = 1 + \frac{f''(\beta) - g''(\alpha)}{g''(\alpha)}. \]
To prove (3.7) we estimate the numerator and denominator using (2.3) and (2.9), respectively. This also shows $B(x, y)$ is bounded when $y \geq \varepsilon \log x$. We turn to (3.6). The denominator is $\ll \varepsilon (\log x)(\log y)$ by Lemma 7.1 and the numerator is estimated in Lemma 7.3 in terms of $\alpha - \beta$ and $(\log G)^2$. Estimating $\alpha - \beta$ using (3.5) gives (3.6).

7.4. Proof of Proposition 2.7. The range $2\log x \geq y > 1 + \log x$ is already in Lemma 2.6 because $\alpha \approx 1/\log y$ in this range by (2.10). If $\log y \leq \sqrt{\log x}$ and $y \geq 2 \log x$, we make use of the Main Theorem of Saha, Sankaranarayanan and Suzuki [24], which in the current range gives
\[ \Psi(x, y) = \frac{x^\alpha \zeta(\alpha, y)}{\alpha \sqrt{2\pi \phi_2}(\alpha, y)} \left( 1 + \frac{g^{(4)}(\alpha)}{8g^{(2)}(\alpha)^2} - \frac{5g^{(3)}(\alpha)^2}{24g^{(2)}(\alpha)^3} + O\left( \frac{1}{\alpha \log x} \right) \right). \]
We also use Smida’s result [26, Thm. 1]
\[ \rho(u) = \frac{e^{\gamma - \psi'} - f(\xi)}{2\pi i u^{\xi}} \left( 1 + \frac{f^{(4)}(\beta)}{8f^{(2)}(\beta)^2} - \frac{5f^{(3)}(\beta)^2}{24f^{(2)}(\beta)^3} + O(u^{-2}) \right). \]
We divide these two estimates to get the formulas in Lemma \[\text{1.6}\] with the term $1 + O(u^{-1})$ replaced by
\[
1 + \frac{1}{8} \left( \frac{g^{(4)}(\alpha)}{g^{(2)}(\alpha)^2} - \frac{f^{(4)}(\beta)}{f^{(2)}(\beta)^2} \right) - \frac{5}{24} \left( \frac{g^{(3)}(\alpha)^2}{g^{(2)}(\alpha)^3} - \frac{f^{(3)}(\beta)^2}{f^{(2)}(\beta)^3} \right) + O\left(\frac{1}{\alpha \log x}\right).
\]
This is estimated in Lemma \[\text{7.3}\] as $1 + O(\log x)$ once we invoke \[\text{3.5}\], \[\text{2.14}\] and \[\text{2.17}\].

It remains to consider $\log y > \sqrt{\log x}$. By Lemma \[\text{3.5}\] \[\text{2.14}\] and \[\text{2.17}\], the quantities
\[
g(\alpha) - g(\beta), f(\alpha) - f(\beta), B(x, y) - 1 \text{ and } (\beta - \alpha)/\alpha \text{ are } O(1/\log x),
\]
so we need to show
\[
\frac{\Psi(x, y)}{\xi} = x \rho(u) K(\beta - 1)(1 + O(1/\log x)) = x \rho(u) K(\alpha - 1)(1 + O(1/\log x)).
\]
By \[\text{12 Prop. A.5}\],
\[
\frac{\Psi(x, y)}{\xi} = x \rho(u) K(\beta - 1) \left(1 + O\left(\frac{1}{(\log x)(\log y) + \frac{y}{x \log x}}\right)\right)
\]
for $\beta := 1 + \rho(u)/(\rho(u) \log y)$. This implies the first equality in \[\text{7.7}\] since $-\rho(u)/\rho(u) \in [\xi(u), \xi(u + 1)] = [\xi(u), \xi(u + 1)] \ [\text{11\, Lem. 2} \ [\text{17\, Lem. 1}]$ and so $\beta = \beta + O(1/\log x)$.

By \[\text{3.5}\], the first equality in \[\text{7.7}\] implies the second.

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**APPENDIX A. PROOF OF LEMMA 5.2**

We follow the proof of \[\text{23, Thm. 12.5}\]. We apply \[\text{23 Cor. 5.3}\] with $\sigma_0 = \max\{0, 1 - \Re s\} + 1/\log x$ and $a_n = \Lambda(n) n^{-s}$ to obtain
\[
S_1(x, s) = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} -\frac{\zeta'(s + w)}{\zeta(s + w)} x^w \frac{dw}{w} + E_s,
\]
\[
E_s \ll \sum_{x/2 < n < 2x \atop n \neq x} \frac{\Lambda(n)}{n^{\Re s}} \min\left\{1, \frac{x}{T|x - n|}\right\} + \frac{x^{\sigma_0}}{T} \left(-\frac{\zeta'}{\zeta}\right) (\sigma_0 + \Re s).
\]
In the sum over $(x/2, 2x)$ we consider separately $n = x'$ and $n \neq x'$. We find
\[
E_s \ll (\log x) x'^{-\Re s} \min\left\{1, \frac{x}{T(x)}\right\} + \frac{2^{\Re s} x^{1-\Re s} \log^2 x}{T} + \frac{x^{\sigma_0}}{T} \left(-\frac{\zeta'}{\zeta}\right) (\sigma_0 + \Re s).
\]
We use $-\zeta'/\zeta(t) \ll (t - 1)^{-1}$ for $t \in (1, 2]$ and $-\zeta'/\zeta(t) \ll 2^{-t}$ for $t \geq 2$ to find that $E_s$ can be absorbed in $R_1(x, T, s)$. Recall $T \geq 2 + |\Im s|$. By \[\text{23, Lem. 12.2}\], there are $T_1, T_2 \in [T, T + 1]$ such that
\[
\frac{\zeta'}{\zeta}(\sigma + i\Im s - iT_2), \frac{\zeta'}{\zeta}(\sigma + i\Im s + iT_1) \ll \log^2 T
\]
uniformly for $-1 \leq \sigma \leq 2$. We extend the range of integration in \[\text{A.1}\] from $|\Im w| \leq T$ to $-T_2 \leq \Im w \leq T_1$. The error we incur is at most
\[
\ll \frac{x^{\sigma_0}}{T} \left(-\frac{\zeta'}{\zeta}\right) (\sigma_0 + \Re s)
\]
which can be absorbed in our bound for $E_s$. Let $K > -\Re s$ denote an odd positive integer which will be taken to $\infty$, and let $C$ denote the contour consisting of three line segments,
connecting \( \sigma_0 - iT_2, -K - \Re s - iT_2, -K - \Re s + iT_1, \sigma_0 + iT_1 \), in this order. Cauchy’s residue theorem shows that
\[
\frac{1}{2\pi i} \int_{\sigma_0 - iT_2}^{\sigma_0 + iT_1} -\frac{\zeta'(s + w)}{\zeta(s + w)} x^w dw = \frac{x^{1-s} - \zeta'(s)}{\zeta(s)} - \frac{x^{\rho - s}}{\rho - s} + \sum_{1 \leq k < K} \frac{x^{-2k-s}}{k + s} + \frac{1}{2\pi i} \int_{C} -\frac{\zeta'(s + w)}{\zeta(s + w)} x^w dw \frac{dw}{w}
\]
if \( s \neq 1 \). If \( s = 1 \), the integrand has a double pole at \( w = 0 \) and \( x^{-1-s}/(1-s) - \zeta'(s)/\zeta(s) \) should be replaced with the residue \( \log x - \gamma \). We shorten the sum over \(-T_2 < \Im(\rho - s) < T_1\) to one over \(-T \leq \Im(\rho - s) \leq T\), and the incurred error is
\[
\ll \sum_{\Im(\rho - s) \in (T, T_1) \cup (-T_2, -T)} \frac{x^{1-\Re s}}{|\rho - s|} \ll \frac{x^{1-\Re s} \log T}{T},
\]
which is acceptable. It remains to bound the integral over \( C \). To bound its horizontal parts, we consider separately three ranges of \( \Re w \in [-K - \Re s, \sigma_0] \). The contribution of \( \Re w \in [-1 - \Re s, \min\{2 - \Re s, \sigma_0\}] \) can be bounded using (A.2):
\[
\frac{1}{2\pi i} \int_{-1 - \Re s + iT_1}^{\sigma_0 + iT_1} -\frac{\zeta'(s + w)}{\zeta(s + w)} x^w dw \frac{dw}{w} \ll \frac{\log \frac{T}{T} \cdot \min\{2 - \Re s, \sigma_0\}}{\log x},
\]
and the same bound holds if \( T_1 \) is replaced with \(-T_2\). This error is acceptable.

Next, the contribution of \( \Re w \in [2 - \Re s, \sigma_0] \) should only be considered if this is a non-empty interval, i.e. when \( \Re s > 2 - 1/\log x \). In this case, we use \(-\zeta'(t)/\zeta(t) \ll 2^{-t} (t \geq 2)\) to estimate the integral as
\[
\frac{1}{2\pi i} \int_{2 - \Re s + iT_1}^{\sigma_0 + iT_1} -\frac{\zeta'(s + w)}{\zeta(s + w)} x^w dw \frac{dw}{w} \ll \frac{2 - \Re s}{T} \cdot \frac{(x/2)^{\sigma_0}}{\log x} \ll \frac{2 - \Re s}{T}
\]
which is acceptable. The same bound holds if \( T_1 \) is replaced with \(-T_2\). To bound the contribution of \( \Re w \in [-K - \Re s, -1 - \Re s] \) we make use of [23, Lem. 12.4] which says that \( (\zeta'/\zeta)(z) \ll \log(|z| + 1) \) holds for all \( z \) with \( \Re z \leq -1 \) and \( \min_{k \geq 1} |z + 2k| \geq 1/4 \), and so
\[
\frac{1}{2\pi i} \int_{-K - \Re s + iT_1}^{-1 - \Re s + iT_1} -\frac{\zeta'(s + w)}{\zeta(s + w)} x^w dw \frac{dw}{w} \ll \int_{-K}^{-1} \log(T + |a|) \cdot x^{\rho - s} \frac{da}{T} \ll \frac{\log(T) \cdot x^{1-\Re s}}{T} \frac{\log x}{\log x}
\]
which is acceptable. The same bound holds if \( T_1 \) is replaced with \(-T_2\). The integral over the vertical part of \( C \) is bounded using \((\zeta'/\zeta)(z) \ll \log(|z| + 1)\) again:
\[
\frac{1}{2\pi i} \int_{-K - \Re s + iT_1}^{K - \Re s + iT_1} -\frac{\zeta'(s + w)}{\zeta(s + w)} x^w dw \ll \log(K) \cdot \frac{T}{K + \Re s} \int_{-T_2}^{T_1} \frac{dt}{t} \ll \frac{T \log(K) x^{-K - \Re s}}{K + \Re s},
\]
When we let \( K \) tend to \( \infty\), this bound tends to 0.

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