The mutual affinity of random measures

M. Fannes and P. Spincemaille

Instituut voor Theoretische Fysica
Katholieke Universiteit Leuven
Celestijnenlaan 200D
B-3001 Heverlee, Belgium

Abstract

We consider a set of probability measures on a finite event space \( \Omega \). The mutual affinity is introduced in terms of the spectrum of the associated Gram matrix. We show that, for randomly chosen measures, the empirical eigenvalue distribution of the Gram matrix converges to a fixed distribution in the limit where the number of measures, together with the cardinality of \( \Omega \), goes to infinity.
1 Introduction

Given two probability measures, there are several ways to define their distance. This is, e.g., important in problems where a sequence of measures converges and the nature of this convergence has to be dealt with in a quantitative way. Common examples are the relative entropy and the total variation distance.

Here we shall focus on the Hellinger distance. As we shall only consider finite event spaces, say $\#\Omega = N$ with $N \in \mathbb{N}_0$, probability measures $\mu$ on $\Omega$ are $N$-tuples $(\mu_1, \mu_2, \ldots, \mu_N)$ which satisfy $\mu_\alpha \geq 0$ and $\sum_\alpha \mu_\alpha = 1$. The Hellinger distance is:

$$d^2_H(\mu_1, \mu_2) := \frac{1}{2} \sum_{\alpha=1}^N \left( \sqrt{\mu_1\alpha} - \sqrt{\mu_2\alpha} \right)^2,$$

sometimes, the factor $\frac{1}{2}$ is left out. The Hellinger distance is a real number between 0 and 1. A related notion is the affinity between two probability measures, defined as

$$A(\mu_1, \mu_2) := 1 - d^2_H(\mu_1, \mu_2) = \sum_{\alpha=1}^N \sqrt{\mu_1\alpha \mu_2\alpha} = \langle \mu_1^{1/2}, \mu_2^{1/2} \rangle.$$

In the last term, we have used the short-handed notation $\mu^{1/2}$ for the $N$-dimensional vector $(\mu_1^{1/2}, \mu_2^{1/2}, \ldots, \mu_N^{1/2})$. Two probability measures have affinity one only when they are equal. Two different degenerate probability measures have affinity zero.

Given several measures $\mu_i$, $i = 1, \ldots, K$, one can ask for a generalisation of the notion of affinity. The problem is to find a way of measuring how many of those measures are close to each other. Here we propose to use the concept of Gram matrix

$$G := [A(\mu_i, \mu_j)]_{i,j=1,\ldots,K}.$$

$G$ is positive semi-definite and its spectrum is independent of the order of the $\mu_i$’s.

A lot of information about the mutual affinities of the probability measures is encoded in the spectrum of $G$. To appreciate this fact, let us for a moment consider degenerate probability measures. The affinity between any two of
these can only be one or zero. In the case all $K$ probability measures are equal, all entries of $G$ are equal to 1 and, therefore, its eigenvalues are $K$ and 0 with respective multiplicities 1 and $K - 1$. The other extreme situation is $K$ different degenerate probability measures, in which case $G$ is the $K$-dimensional identity matrix with eigenvalue 1 occurring $K$ times. For an arbitrary set of degenerate measures, i.e., for a set of symbols in $\Omega$, the spectrum of $G$ determines the relative frequencies of the different symbols appearing in the set.

Of course, allowing general probability measures, any positive number can be an eigenvalue of $G$, but the general picture remains and can be described as follows. An eigenvalue distribution which puts a lot of weight on eigenvalues close to zero indicates that a large group of probability measures are close to each other (have large mutual affinities). If, on the other hand, a sizeable portion of the eigenvalues occur relatively far away from zero, the probability measures have in general low mutual affinities.

Here we shall study the Gram matrix for independently and randomly chosen probability measures with respect to the uniform distribution on the simplex $\Lambda_N = \{ \mu = (\mu_1, \ldots, \mu_N) \mid \sum_{\alpha} \mu_{\alpha} = 1 \text{ and } \mu_{\alpha} \geq 0 \}$. The Gram matrix and its spectrum are now random objects. We want to study these objects when both the number of measures and the cardinality of the event space become large. More specifically, we study the spectrum of the random Gram matrix in the limit $N = \#\Omega \to \infty$, the number of measures $K(N) \to \infty$ and $K(N)/N \to \tau$ where $\tau$ is a given positive number. We shall explicitly calculate the limiting expectation value of the empirical eigenvalue distribution

$$
\rho_K(x) := \frac{1}{K} \sum_{i=1}^{K} \delta(x - \lambda_i),
$$

where $\lambda_1, \ldots, \lambda_K$ are the (random) eigenvalues of the Gram matrix. We shall, moreover, prove that the convergence occurs with probability 1.

The setting of this problem is similar to that of the Wishart matrices: let $A$ be a real random $N \times K$ matrix with $N(0, 1)$ i.i.d. entries, let $K = \tau N$ for $\tau \geq 0$ and consider the limit $N \to \infty$. It is known that the empirical eigenvalue distribution of the random matrix $A^*A/K$ converges to the distribution

$$
\rho_{MP}(x, \tau) = \begin{cases} 
\delta(x - 1) & \text{if } \tau = 0 \\
\sigma(x, \tau) & \text{if } 0 < \tau \leq 1 \\
\frac{\tau - 1}{\tau} \delta(x) + \sigma(x, \tau) & \text{if } \tau > 1
\end{cases}
$$

(1)
with
\[
\sigma(x, \tau) = \begin{cases} 
\frac{\sqrt{4\tau x - (x + \tau - 1)^2}}{2\pi \tau x} & (1 - \sqrt{\tau})^2 \leq x \leq (1 + \sqrt{\tau})^2 \\
0 & \text{otherwise}.
\end{cases}
\]
This distribution is known as the Marchenko-Pastur distribution \( [5] \) and we shall obtain it in Theorem \( \[4\] \). In \( \[1\] \), the same distribution arose in the context of Gram matrices associated to random vectors.

The paper consists of two more parts. In Section \( \[2\] \) we discuss some general features of the spectrum of the random Gram matrices and calculate the limiting expectation of the empirical eigenvalue distribution using the Stieltjes transform. The main theorem in this section is Theorem \( \[1\] \). In Section \( \[3\] \) we prove that the convergence of the empirical eigenvalue distribution occurs almost surely. This is the contents of Theorem \( \[4\] \).

## 2 Convergence in expectation

Denote by \( \Lambda_N \) the simplex \( \{ \mu = (\mu_1, \ldots, \mu_N) \in \mathbb{R}^N \mid \mu_\alpha \geq 0 \text{ and } \sum_{\alpha=1}^N \mu_\alpha = 1 \} \). It is the space of probability measures on an event space \( \Omega \) with \( N \) elements. On this space, a uniform measure \( \sigma \) can be put in the sense that
\[
\int_{\Lambda_N} f(\mu) \, d\sigma(\mu) = \frac{1}{|\det(A)|} \int_{\Lambda_N} f(A \mu) \, d\sigma(\mu),
\]
for every integrable function \( f \) on \( \Lambda_N \) supported in \( A \Lambda_N \) and for every invertible stochastic matrix \( A \). (\( A \) is stochastic if \( A_{\alpha\beta} \geq 0 \) and \( \sum_{\beta} A_{\alpha\beta} = 1 \). This uniform measure is just the Lebesgue measure on \( \Lambda_N \). We can also obtain this measure in terms of the larger space \( (\mathbb{R}^+)^N \) of which \( \Lambda_N \) is a subset. If we choose \( N \) independent random variables \( x_i \), all distributed according to the exponential distribution with some fixed mean, then \( \mu := (x_1, \ldots, x_N)/(x_1 + \cdots + x_N) \) is uniformly distributed on \( \Lambda_N \), a fact which is, e.g., proven in \( \[7\] \).

Now we choose \( K \) measures \( \mu_j \in \Lambda_N \), independently and uniformly distributed, and associate with them the Gram matrix \( G \):
\[
G = [A(\mu_i, \mu_j)]_{i,j=1,\ldots,K}.
\]
We shall study the spectrum of $G$ in the limit $K, N \to \infty$, keeping the ratio $K/N =: \tau$ fixed. The Gram matrix is of course a random object but its spectrum has typical properties. The first characteristic of the spectrum is the presence of one eigenvalue much larger than the others. This eigenvalue is the norm of $G$ as $G$ is positive definite and it grows, as we shall show, linearly with $N$. The remaining eigenvalues are typically concentrated on an interval close to zero. In fact we prove:

**Theorem 1** The empirical eigenvalue distribution $\rho_K(x)$ converges weakly in expectation to the Marchenko-Pastur distribution (1) scaled with the factor $a = 1 - \frac{1}{4\pi}$, i.e.

$$ E(\rho_K(x)) \to \frac{1}{a} \rho_{MP}(\frac{x}{a}) $$

We first prove some lemmas and comment on the (expectation of the) norm of the random Gram matrix. First, we need expectations of arbitrary moments of the components of random probability measures.

**Lemma 1** Let $\mu = (\mu_1, \ldots, \mu_N)$ be a uniformly random probability measure from $\Lambda_N$ and let $\alpha_1, \ldots, \alpha_N \geq 0$; then

$$ E(\mu_1^{\alpha_1} \cdots \mu_N^{\alpha_N}) = \frac{(N - 1)! \prod_{i=1}^N \Gamma(\alpha_i + 1)}{\Gamma(\alpha_1 + \cdots + \alpha_N + N)}. $$

**Proof:** Using the representation of the uniform measure on $\Lambda_N$ in terms of the exponential distribution with mean 1, we write the expectation as

$$ E(\mu_1^{\alpha_1} \cdots \mu_N^{\alpha_N}) = \int_0^\infty dx_1 \cdots \int_0^\infty dx_N e^{-(x_1 + \cdots + x_N)} \frac{x_1^{\alpha_1} \cdots x_N^{\alpha_N}}{(x_1 + \cdots + x_N)^{\alpha_1 + \cdots + \alpha_N}}. $$

The change of coordinates

$$ y_i := \frac{x_i}{x_1 + \cdots + x_N}, \quad i = 1, \ldots, N - 1, \quad y_N := x_N $$

transforms the integral into

$$ \int_0^1 dy_1 \int_0^{1-y_1} dy_2 \cdots \int_0^{1-y_1-\cdots-y_{N-2}} dy_{N-1} \int_0^\infty dy_N y_1^{\alpha_1} \cdots y_{N-1}^{\alpha_{N-1}} y_N^{\alpha_N} (1 - y_1 - \cdots - y_{N-1})^{\alpha_N-N} \exp\left(-\frac{y_N}{1 - y_1 - \cdots - y_{N-1}}\right). $$

5
Integrating with respect to $y_N$ yields
\[
\int_0^\infty dy_N y_N^{N-1} \exp \left( -\frac{y_N}{1 - y_1 - \cdots - y_{N-1}} \right) = (1 - y_1 - \cdots - y_{N-1})^N (N-1)!.
\]

After this step, the successive calculation of the integrals over $y_{N-1}, \ldots, y_1$ can be completed using
\[
\int_0^x dy y^p (x - y)^q = x^{1+p+q} B(p+1,q+1) = x^{1+p+q} \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)},
\]
with $B$ the Beta Function.

As a first application of this lemma, we compute the expectation of a single entry in the Gram matrix
\[
\mathbb{E}(A(\mu_1, \mu_2)) = N \mathbb{E}(\sqrt{\mu_1})^2 = N \left( \frac{(N-1)! \Gamma(\frac{3}{2})}{\Gamma(N + \frac{1}{2})} \right)^2 = \frac{\pi}{4} + \frac{1}{16N} + \frac{1}{128N^2} - \frac{1}{512N^3} + \cdots
\]

This means that in the $N \to \infty$ limit, every matrix element has a non-zero mean and, therefore, the norm of the Gram matrix will grow linearly with $N$; see e.g. [8]. It turns out that an expression for $\mathbb{E}(\|G\|)$ can be given in terms of the $R$-transform, a basic notion from free probability; see [9, 4]. To state the result, we need some terminology.

In non-commutative probability, a random variable is an element from a unital algebra and expectation values are given by unital linear functionals $\Phi$ on this algebra. The moments of the random variable $A$ are $m_n := \Phi(A^n)$ with $n \in \mathbb{N}$. Another sequence of numbers associated with a random variable are its free cumulants $(k_n)_{n \in \mathbb{N}}$. These are defined in terms of non-crossing partitions. A partition $\pi = \{V_1, \ldots, V_s\}$ of the set $\{1, \ldots, n\}$ is called crossing when there exist numbers $1 \leq p_1 < q_1 < p_2 < q_2 \leq n$ and $1 \leq i < j \leq s$ for which $p_1, p_2 \in V_i$ and $q_1, q_2 \in V_j$. A partition in which no crossing occurs is called non-crossing. Denote by $\textbf{NC}(n)$ the set of all non-crossing partitions on $\{1, \ldots, n\}$. The free cumulants are defined recursively by the equations
\[
m_n = \sum_{\pi \in \textbf{NC}(n)} k_{\pi},
\]
with \( k_\pi = k_{\#V_1} \cdots k_{\#V_s} \) for \( \pi = \{V_1, \ldots, V_s\} \). For \( n = 1, 2, 3 \) the free cumulants are equal to the usual cumulants of probability theory, where no restriction on the partitions occurs. Only starting from \( k_4 \), there is a difference due to the fact that at least 4 different indices are needed to have a crossing. E.g., for a centred \( A \), which means that \( \Phi(A) = 0 \), we find \( \Phi(A^4) - 3\Phi(A^2)^2 \) for the usual fourth cumulant, while \( k_4 = \Phi(A^4) - 2\Phi(A^2)^2 \). The relation between the \( (k_n)_n \) and \( (m_n)_n \) can be formulated elegantly using formal power series. The first one is the Cauchy transform

\[
C_A(z) := \sum_{n=0}^{\infty} \frac{m_n}{z^{n+1}} = \Phi((z - A)^{-1})
\]

and the second the \( R \)-transform

\[
R_A(z) := \sum_{n=0}^{\infty} k_{n+1}z^n.
\]

The relation between these two transforms is then given by Voiculescu’s formula

\[
C_A \left( R_A(z) + \frac{1}{z} \right) = z.
\]

**Lemma 2** Let \( \varphi \) be a normalised vector in a Hilbert space \( \mathcal{H} \), let \( X \) be a bounded, linear, self-adjoint operator on \( \mathcal{H} \) and let \( |\varphi\rangle\langle\varphi| \) denote the operator \( \psi \mapsto \langle\varphi, \psi\rangle\varphi \). The norm of

\[
A(\epsilon) := |\varphi\rangle\langle\varphi| + \epsilon X, \quad \epsilon \in \mathbb{R}
\]

is given by the asymptotic series

\[
\|A(\epsilon)\| = \sum_{n=0}^{\infty} k_{n+1} \epsilon^n = 1 + \epsilon R_X(\epsilon),
\]

i.e., for any \( n_0 \in \mathbb{N} \)

\[
\|A(\epsilon)\| = \sum_{n=0}^{n_0} k_{n+1} \epsilon^n + o(\epsilon^{n_0}).
\]

The \( (k_n)_n \) are the non-crossing cumulants of \( X \) with respect to the expectation \( \Phi(\cdot) := \langle\varphi, \cdot\varphi\rangle \).
**Proof:** For $\epsilon$ sufficiently small, $A(\epsilon)$ will have an eigenvalue coinciding with its norm. Let $\psi(\epsilon)$ be the corresponding eigenvector, then

$$(|\varphi\rangle \langle \varphi| + \epsilon X)\psi(\epsilon) = \|A(\epsilon)\| \psi(\epsilon).$$

The vector $\psi(\epsilon)$ depends continuously on $\epsilon$ and tends to $\varphi$ when $\epsilon \to 0$. Moreover, $\lim_{\epsilon} \|A(\epsilon)\| = 1 \ [3]$. We can rewrite the eigenvalue equation as

$$\psi(\epsilon) = \langle \varphi, \psi(\epsilon) \rangle (\|A(\epsilon)\| - \epsilon X)^{-1} \varphi.$$ 

Multiplying with $\varphi$ and using $\langle \varphi, \psi \rangle \neq 0$ for sufficiently small $\epsilon$ yields

$$\langle \varphi, (\|A(\epsilon)\| - \epsilon X)^{-1} \varphi \rangle = 1. \quad (3)$$

Then (3) gives

$$C_X \left( R_X (\epsilon) + \frac{1}{\epsilon} \right) = \epsilon = C_X \left( \frac{\|A(\epsilon)\|}{\epsilon} \right),$$

which is valid for arbitrary small $\epsilon$ and so

$$\|A(\epsilon)\| = 1 + \epsilon R_X (\epsilon).$$

Using Lemma 2, we can compute the asymptotic series for $E(\|G\|^n)$. Set $\varphi = 1 := \left( \frac{1}{\sqrt{K}}, \ldots, \frac{1}{\sqrt{K}} \right)$ and $\epsilon = 4/K\pi$; then

$$\epsilon G = |\varphi\rangle \langle \varphi| + \epsilon X \quad \text{with} \quad X = G - \frac{1}{\epsilon} |\varphi\rangle \langle \varphi|.$$ 

E.g., for $E(\|G\|)$, we get

$$E(\|G\|) = \frac{K\pi}{4} + E(\langle 1, X 1 \rangle) + \frac{4}{K\pi} E \left( \langle 1, X^2 1 \rangle - \langle 1, X 1 \rangle^2 \right) + O \left( \frac{1}{N} \right).$$

Using (2) and putting as before $\tau = K/N$

$$E(\langle 1, X 1 \rangle) = 1 - \frac{\pi}{4} + \tau \frac{\pi}{16} + O \left( \frac{1}{N} \right),$$

$$E(\langle 1, X^2 1 \rangle) = K\tau \left( \frac{\pi}{4} - \frac{5\pi^2}{64} \right) + O (1), \quad \text{and}$$

$$E(\langle 1, X 1 \rangle^2) = O \left( \frac{1}{N} \right).$$
We can repeat this procedure to get with arbitrary accuracy the expectation of any power of the norm of the Gram matrix.

To study the asymptotic eigenvalue distribution, we could, as a first step, try to obtain the moments of the limiting distribution as limits of the expectations of moments of $\rho_K$. The largest eigenvalue of $G$ contributes with a weight $1/K$ in $\rho_K$, but, as this largest eigenvalue is essentially located around $K \pi/4$, its contribution to the expectation value of the $n$th moment of $\rho_K$ is of the order $K^{n-1}$ which leads to a divergence. We must therefore remove that contribution and study the expectations of the moments of the non-normalised distribution

$$\rho'_K(x) := \frac{1}{K} \sum_{\lambda_i \neq \|G\|} \delta(x - \lambda_i). \quad (4)$$

In the limit, the weight of the largest eigenvalue will become negligible and we recover a normalised distribution. It turns out that, in principle, the moments of the limiting distribution can be obtained by calculating the limit of the expectations of the moments of (4) using (2). The first two moments yield

$$m^K_1 := \int x d\rho'_K(x) = E \left( \frac{1}{K} (\text{Tr } G - \|G\|) \right)$$

$$= \frac{1}{K} \left( K - \frac{K \pi}{4} + O(1) \right) = 1 - \frac{\pi}{4} + O \left( \frac{1}{N} \right)$$

$$m^K_2 := \int x^2 d\rho'_K(x) = E \left( \frac{1}{K} (\text{Tr } G^2 - \|G\|^2) \right)$$

$$= \frac{1}{K} \left( K(\tau + 1)(1 - \frac{1}{16}) - (\tau + 1)(\frac{\pi}{2} - \frac{\pi^2}{8}) + O \left( \frac{1}{N} \right) \right)$$

$$= (1 + \tau)(1 - \frac{\pi}{4})^2 + O \left( \frac{1}{N} \right)$$

These moments coincide of course with those from Theorem 1. The computation is however very hard, as the $n$th moment of $\rho'_K$ requires $n$ terms in the series expansion of $E (\|G\|)$. A quite complicated combinatorial argument is already required just to cancel the orders of $N$ larger than one in the traces of $G$. A much more convenient function of the spectrum of the Gram matrices is the normalised trace of its resolvent. The following proof bears some resemblance to the approach presented in [6], but is technically rather different.
Proof of Theorem 1: Denote by $\sigma(G)$ the spectrum of $G$ and define for $z \in \mathbb{C} \setminus \sigma(G)$

$$C_K(z) := \frac{1}{K} \Tr \frac{1}{G - z} = \int_0^\infty \frac{1}{x - z} \, d\rho_K(x).$$

The last equality shows that $C_K$ is the Stieltjes transform of the empirical eigenvalue distribution.

Let $(e_j)_{j=1}^K$ be the standard orthonormal basis of $\mathbb{C}^K$ and $z \in \mathbb{C} \setminus \sigma(G)$; then

$$\frac{1}{K} \Tr \frac{1}{G - z} = \frac{1}{K} \sum_{j=1}^K \langle e_j, \frac{1}{G - z} e_j \rangle.$$

Now, for every $j$ in the sum, we peel off the $j$th row and column:

$$G = \left( \begin{array}{c} G^{(j)} \\ \langle \varphi^{(j)}, \cdot \rangle \\ 1 \end{array} \right) \text{ with } \varphi^{(j)} := \left( \langle \mu_k^{1/2}, \mu_j^{1/2} \rangle \right)_{k \neq j}.$$

This means that we write $\mathbb{C}^K$ as $\mathbb{C}^{K-1} \oplus \mathbb{C} e_j$. The corresponding form for the resolvent is:

$$\frac{1}{G - z} = \left( \begin{array}{ccc} \frac{1}{G^{(j)} - z} + (G^{(j)} - z)^{-1} & (G^{(j)} - z)^{-1} \varphi^{(j)} \langle \varphi^{(j)}(G^{(j)} - z)^{-1} \rangle \\ -(G^{(j)} - z)^{-1} \varphi^{(j)} & \frac{1}{1 - z - \alpha^{(j)}} \end{array} \right),$$

with

$$\alpha^{(j)} := \langle \varphi^{(j)}, (G^{(j)} - z)^{-1} \varphi^{(j)} \rangle.$$ (5)

Note that in $\alpha^{(j)}$ the vectors $\varphi^{(j)}$ are the only place where random variables of the $j$th measure occur. The Stieltjes transform can then be written as

$$C_K(z) = \frac{1}{K} \sum_{j=1}^K \frac{1}{1 - z - \alpha^{(j)}}.$$ (6)

We shall now take the limit of the expectation value of (6). Therefore, we fix a compact $A \subset \mathbb{C} \setminus \mathbb{R}^+$ and $z \in A$. We first calculate $E_j(\alpha^{(j)})$, where the subscript $j$ means that only the random variables appearing in the $j$th vector
will be averaged out. Let \( X = 1/(G^{(j)} - z) \) and use \( \mathbb{E}\left(\mu_{\alpha}^{1/2} \mu_{\beta}^{1/2}\right) = \pi/4N \) with \( \alpha \neq \beta \) and \( \mathbb{E}(\mu_\alpha) = 1/N \).

\[
\mathbb{E}_j\left(\alpha^{(j)}\right) = \mathbb{E}_j\left(\sum_{k,l}^{N} \sum_{\alpha,\beta}^{K} \mu_{j_\alpha}^{1/2} \mu_{k_\alpha}^{1/2} X_{kl} \mu_{j_\beta}^{1/2} \mu_{k_\beta}^{1/2}\right)
= \sum_{k,l} \left(\frac{1}{N} \sum_{\alpha}^{N} \mu_{k_\alpha}^{1/2} X_{kl} \mu_{k_\alpha}^{1/2} + \frac{\pi}{4N} \sum_{\alpha \neq \beta}^{N} \mu_{k_\alpha}^{1/2} X_{kl} \mu_{k_\beta}^{1/2}\right)
= \frac{1}{N} (1 - \frac{\pi}{4}) \text{Tr} G^{(j)} X + \frac{\pi}{4N} \langle \gamma^{(j)}, X \gamma^{(j)} \rangle,
\]
with \( \gamma^{(j)} := \left(\sum_{k}^{N} \mu_{k_\alpha}^{1/2}\right)_{k \neq j} \). In Lemma 3, we prove that the expectation (now averaging over all random variables) of the second term converges to \( \pi/4 \), uniformly on \( A \). Setting

\[
f_K(z) := \mathbb{E}\left(\frac{1}{K} \text{Tr} \frac{1}{G - z}\right) \quad \text{and} \quad f_K^{(j)}(z) := \mathbb{E}\left(\frac{1}{K} \text{Tr} \frac{1}{G^{(j)} - z}\right),
\]
we get that

\[
\mathbb{E}\left(\alpha^{(j)}\right) = \frac{1}{N} \left(1 - \frac{\pi}{4}\right) \text{Tr} \left(\mathbb{I} + z \frac{1}{G^{(j)} - z}\right) + \frac{\pi}{4} + Q(N, z)
= \left(1 - \frac{\pi}{4}\right) \left(\tau + z \tau f_K^{(j)}\right) + \frac{\pi}{4} + Q(N, z),
\]
with \( Q(N, z) \) converging to zero, uniformly on \( A \). In Lemma 4, we show that

\[
\mathbb{E}_j\left(\alpha^{(j)}\right)^2 = \left(\mathbb{E}_j\left(\alpha^{(j)}\right)\right)^2 + R(N, z),
\]
where \( \mathbb{E}(R(N, z)) \) (averaging over all remaining random variables) converges to zero, uniformly on \( A \). This allows us to write

\[
\left|\mathbb{E}\left(\frac{1}{1 - z - \alpha^{(j)}}\right) - \frac{1}{1 - z - \mathbb{E}\left(\alpha^{(j)}\right)}\right| \leq \mathbb{E}\left(\frac{|\mathbb{E}\left(\alpha^{(j)}\right) - \alpha^{(j)}|}{|1 - z - \alpha^{(j)}||1 - z - \mathbb{E}(\alpha^{(j)})|}\right)
\leq \frac{1}{|\mathbb{E}(R(N, z))|^2} \sqrt{\mathbb{E}\left((\alpha^{(j)} - \mathbb{E}(\alpha^{(j)}))^2\right)},
\]
which goes to zero, uniformly on \( A \). We get

\[
f_K(z) = \mathbb{E}\left(C_K(z)\right) = \frac{1}{K} \sum_{j=1}^{K} \mathbb{E}\left(\frac{1}{1 - z - \alpha^{(j)}}\right) = \frac{1}{K} \sum_{j=1}^{K} \frac{1}{1 - z - \mathbb{E}(\alpha^{(j)})} + \mathcal{O}\left(\frac{1}{N}\right)
= \frac{1}{1 - z - \left(1 - \frac{\pi}{4}\right)(\tau + z \tau f_K^{(j)}(z)) - \frac{\pi}{4}} + \mathcal{O}\left(\frac{1}{N}\right).
\]
Consider for a fixed \( z \in A \) the sequence \( f_1(z), f_2(z), \ldots \). This sequence of complex numbers lies in a compact set, so it must have a convergent subsequence. Moreover, every convergent subsequence has the same limit because there is only one number \( f(z) \) that satisfies both the equation

\[
f(z) = \frac{1}{a - z - a\tau - azf(z)},
\]

with \( a = 1 - \frac{1}{4} \pi \) and the condition \( \Im(z)\Im(f(z)) > 0 \). From this it is immediately clear that

\[
\lim_{n \to \infty} f_n(z) = \frac{1}{a} f_{\text{MP}}(\frac{z}{a}),
\]

where \( f_{\text{MP}}(z) \) is the Stieltjes transform of the Marchenko-Pastur distribution. Because the convergence in expectation of \( f_K(z) \) to \( f(z) \) is uniform on compact subsets of \( \mathbb{C} \setminus \mathbb{R}^+ \), it follows that \( \rho_K(x) \) converges in expectation to \( \rho_{\text{MP}}(x) \).

In the proof of Theorem 1 we used Lemmas 3 and 4. The idea behind their proofs is the following. Each of the entries in the random Gram matrices has approximately the same value. The eigenvector belonging to the largest eigenvalue, i.e. the norm, of such a matrix, has also nearly constant entries. Vectors like \( \gamma^{(j)} \) defined in Lemma 3 are of this kind. This means that an expression like \( \langle \gamma^{(j)}, f(G^{(j)})\gamma^{(j)} \rangle \) is approximately equal to \( f(\|G^{(j)}\|) \|\gamma^{(j)}\|^2 \). In the sequel, we shall drop the superscript \( (j) \) in \( \gamma \) and in \( G \) as well, moreover, we shall replace \( K - 1 \) by \( K \) wherever it is not relevant for the result, e.g., wherever we need quantities estimated up to order 1 in \( K \).

**Lemma 3** Let \( \gamma = \left( \sum_{\alpha=1}^N \mu_{k\alpha}^{1/2} \right)_{k=1,2,\ldots,K} \); then

\[
\lim_{N \to \infty} \mathbb{E} \left( \frac{1}{N} \langle \gamma, \frac{1}{G - z} \gamma \rangle \right) = 1,
\]

uniformly on compact subsets of \( \mathbb{C} \setminus \mathbb{R}^+ \).

**Proof:** We start with the calculation of some useful expectations, all of them just applications of (2). First,

\[
\mathbb{E} \left( \|\gamma\|^2 \right) = \frac{\pi \tau}{4} N^2 + \left( 1 - \frac{\pi}{4} \right) \tau N.
\]
Setting
\[ \eta := \frac{1}{\sqrt{\mathbb{E}(\|\gamma\|^2)}} \gamma, \]
implies \( \mathbb{E}(\|\eta\|^2) = 1 \) and
\[
\mathbb{E}(\langle \eta, G\eta \rangle) = \frac{\pi \tau}{4} N + \frac{1}{4}(4 - \pi)(1 + \tau) + O\left(\frac{1}{N}\right) \tag{7}
\]
\[
\mathbb{E}(\langle \eta, G^2\eta \rangle) = \frac{\pi^2 \tau^2}{16} N^2 + \frac{\pi}{8}(4 - \pi)\tau(1 + \tau) + O(1). \tag{8}
\]
Next, we use the spectral theorem for selfadjoint matrices
\[ G = \int \lambda \ dE(\lambda). \]
Set \( \lambda_0 := \mathbb{E}(\langle \eta, G\eta \rangle) \); the spectral measure \( d\|E(\lambda)\eta\|^2 \) is, in expectation, very much concentrated around \( \lambda_0 \):
\[
\mathbb{E} \left( \int_0^\infty (\lambda - \lambda_0)^2 d\|E(\lambda)\eta\|^2 \right) = \mathbb{E}(\langle \eta, G^2\eta \rangle) - \mathbb{E}(\langle \eta, G\eta \rangle)^2
\]
\[ =: C = O(1), \]
by (7) and (8). A consequence, using Tchebyshev's inequality, is
\[
\int_{|\lambda - \lambda_0| > \lambda_0/2} d\|E(\lambda)\eta\|^2 \leq \frac{4}{\lambda_0^2} \mathbb{E} \left( \int_0^\infty (\lambda - \lambda_0)^2 d\|E(\lambda)\eta\|^2 \right) = \frac{4C}{\lambda_0^2}. \tag{9}
\]
Now, we are able to prove the lemma. Consider a compact subset \( A \subset \mathbb{C} \setminus \mathbb{R}^+ \) and choose \( z \in A \); then
\[
\left| \mathbb{E} \left( \frac{1}{N} \langle \gamma, \frac{1}{G - z} \gamma \rangle \right) - 1 \right| \leq \left| \mathbb{E} \left( \frac{1}{N} \langle \gamma, \frac{1}{G - z} \gamma \rangle \right) - \frac{1}{N} \mathbb{E}(\|\gamma\|^2) \right| + \frac{1}{N} \mathbb{E}(\langle \eta, G\eta \rangle) - z - 1 \right| \tag{10}
\]
The second term is equal to
\[
\left| \frac{1}{4}(4 - \pi)(1 + \tau) + z \right| - \frac{1}{4N\pi\tau - z + O(1)},
\]
13
which goes to zero uniformly on \( A \). The first term of (10) gives

\[
\left| \frac{1}{N} \mathbb{E} (\| \eta \|^2) \mathbb{E} \left( \langle \eta, \left( \frac{1}{G - z} - \frac{1}{\mathbb{E} (\langle \eta, G \eta \rangle) - z} \right) \eta \rangle \right) \right|
\]

\[
= \frac{1}{N} \mathbb{E} (\| \eta \|^2) \mathbb{E} \left( \int_0^\infty \left( \frac{1}{\lambda - z} - \frac{1}{\lambda_0 - z} \right) d\| E(\lambda)\eta \|^2 \right)
\]

\[
\leq \frac{1}{N} \mathbb{E} (\| \eta \|^2) \mathbb{E} \left( \int_0^{\lambda_0/2} \frac{|\lambda_0 - \lambda|}{|\lambda - z||\lambda_0 - z|} d\| E(\lambda)\eta \|^2 \right)
\]

\[
+ \frac{1}{N} \mathbb{E} (\| \eta \|^2) \mathbb{E} \left( \int_0^\infty \frac{|\lambda_0 - \lambda|}{|\lambda - z||\lambda_0 - z|} d\| E(\lambda)\eta \|^2 \right).
\]

(11)

The first integral is bounded from above by

\[
\frac{\lambda_0}{|\Re(z)||\lambda_0 - z|} \mathbb{E} \left( \int_0^{\lambda_0/2} d\| E(\lambda)\eta \|^2 \right) \leq \frac{4C}{\lambda_0|\Re(z)||\lambda_0 - z|}.
\]

The second integral in (11) is, provided \( \Re(z) < \lambda_0/2 \), bounded by

\[
\frac{1}{|\lambda_0/2 - z|} \frac{1}{|\lambda_0 - z|} \mathbb{E} \left( \left( \int_0^{\lambda_0/2} |\lambda_0 - \lambda| d\| E(\lambda)\eta \|^2 \right)^{\frac{1}{2}} \| \eta \|\right)
\]

\[
\leq \frac{1}{|\lambda_0/2 - z|} \frac{1}{|\lambda_0 - z|} \left( \mathbb{E} \left( \left( \int_0^{\lambda_0/2} (\lambda_0 - \lambda)^2 d\| E(\lambda)\eta \|^2 \right)^{\frac{1}{2}} \| \eta \|^2 \right) \right)^{\frac{1}{2}}
\]

\[
= O \left( \frac{1}{N^2} \right).
\]

We conclude that (10) can be bounded from above by

\[
\frac{1}{N} \left( \frac{\pi \tau}{4} N^2 + \left( 1 - \frac{\pi}{4} \right) \tau N \right) \left( \frac{4C}{\lambda_0|\Re(z)||\lambda_0 - z|} + \frac{\sqrt{C}}{|\lambda_0/2 - z||\lambda_0 - z|} \right),
\]

which gives a uniform bound on \( A \), going to zero when \( N \to \infty \).
Lemma 4 With the notations introduced in the proof of Theorem 1

\[ \mathbb{E}_j \left( \alpha^{(j)^2} \right) = \left( \mathbb{E}_j \left( \alpha^{(j)} \right) \right)^2 + R(N, z), \]

where \( \mathbb{E}(R(N, z)) \) converges to zero, uniformly on compact subsets of \( \mathbb{C} \setminus \mathbb{R}^+ \).

Proof: Using the notation introduced in (5) and still denoting \( 1/(G^{(j)} - z) \) by \( X \), we compute the expectation with respect to the random variables appearing in the \( j \)th random probability measure by multiple applications of (2). We get

\[ \mathbb{E}_j \left( \langle \varphi^{(j)}, X \varphi^{(j)} \rangle^2 \right) = \mathbb{E}_j \left( \sum_{k,l,m,n} \sum_{\alpha,\beta,\gamma,\delta} \mu_{1/2}^k \mu_{1/2}^l \mu_{1/2}^m \mu_{1/2}^n \mu_{1/2}^{\alpha \beta \gamma \delta} \frac{\pi^2}{16N(N+1)} \right) \]

where the symbol \( \sum'_{\alpha,\beta,\gamma,\delta} \) means the sum over all \( r \)-tuples \( (\alpha, \ldots, \alpha_r) \) in which no two entries are equal. Denote the seven restricted sums in this expression \( X_1', \ldots, X_7' \). Rewriting the expression in terms of the unrestricted
sums, which we shall denote by $X_1, \ldots, X_7$, we get

\[
\frac{1}{N(N + 1)} \left[ \frac{\pi^2}{4} X_1 + \left( \frac{\pi}{2} - \frac{\pi^2}{8} \right) X_2 + \left( \pi - \frac{\pi^2}{4} \right) X_3 + \left( 1 - \frac{\pi}{2} - \frac{\pi^2}{16} \right) X_4 \\
+ \left( 2 - \pi + \frac{\pi^2}{8} \right) X_5 + \left( -3\pi/2 + \frac{\pi^2}{2} \right) X_6 + \left( -1 + 3\pi/2 - \frac{3\pi^2}{8} \right) X_7 \right].
\]

This can be written as

\[
\frac{1}{N(N + 1)} \left[ \left( 1 - \frac{\pi}{4} \right) \text{Tr} G^{(j)} X + \frac{\pi}{4} \langle \gamma^{(j)}, X \gamma^{(j)} \rangle \right]^2 \\
+ \left( \pi - \frac{\pi^2}{4} \right) \langle \gamma^{(j)}, X G^{(j)} \gamma^{(j)} \rangle + \left( 2 - \pi + \frac{\pi^2}{8} \right) \text{Tr} G^{(j)} X G^{(j)} X \\
+ \left( -3\pi/2 + \frac{\pi^2}{2} \right) \sum_{k,l,m,n} \sum_{\alpha,\beta} \mu_{k\alpha} \mu_{l\alpha} \mu_{m\beta} \mu_{n\beta} X_{kl} X_{mn} X_{mn}^{1/2} \\
+ \left( -1 + 3\pi/2 - \frac{3\pi^2}{8} \right) \sum_{k,l,m,n} \sum_{\alpha} \mu_{k\alpha} \mu_{l\alpha} \mu_{m\alpha} \mu_{n\alpha} X_{mn}^{1/2} \right].
\]

From this, the first statement of the lemma is clear. Now it has to be proven that the expectation of the remaining terms tends to zero.

The first term of $R(N, z)$ is

\[
\frac{1}{N^2(N + 1)} \left( \left( 1 - \frac{\pi}{4} \right) \text{Tr} G^{(j)} X + \frac{\pi}{4} \langle \gamma^{(j)}, X \gamma^{(j)} \rangle \right)^2 \\
\leq \frac{2}{N^2(N + 1)} \left( \left( 1 - \frac{\pi}{4} \right)^2 \left( \text{Tr} G^{(j)} X \right)^2 + \frac{\pi^2}{16} \langle \gamma^{(j)}, X \gamma^{(j)} \rangle^2 \right).
\]

Now

\[
\mathbb{E} \left( \left( \text{Tr} G^{(j)} X \right)^2 \right) \leq \frac{1}{|3z|^2 (K - 1)^2},
\]

while also $\mathbb{E} \left( \langle \gamma^{(j)}, X \gamma^{(j)} \rangle^2 \right)$ is of order $N^2$. We shall show this using the methods of Lemma 3. Again we omit the superscript $(j)$, since this does not change the result in an essential way. We have

\[
\mathbb{E} \left( \| \gamma \otimes \gamma \|^2 \right) = \frac{\tau \pi^2}{16} N^4 + \frac{1}{2} \left( 1 - \frac{\pi}{4} \right) \tau^2 \pi N^3.
\]

Set

\[
\eta := \frac{1}{\mathbb{E} \left( \| \gamma \otimes \gamma \|^2 \right)^{1/2}} \gamma.
\]
This definition ensures that $\mathbb{E}(\|\eta \otimes \eta\|^2) = 1$. (Note that this definition differs slightly from the one given in the previous lemma. The previous definition would have given $1 + O(N^{-2})$ for the expectation of the square of the norm of $\gamma \otimes \gamma$.) Again using (3), we have
\[
\mathbb{E}(\langle \eta \otimes \eta, G \otimes \mathbb{I} \eta \otimes \eta \rangle) = \frac{\pi \tau^2}{4} N + \left(1 + \frac{\pi}{4}\right) (1 + \tau) + O\left(\frac{1}{N}\right)
\]
\[
\mathbb{E}(\langle \eta \otimes \eta, G^2 \otimes \mathbb{I} \eta \otimes \eta \rangle) = \frac{\pi^2 \tau^2}{16} N^2 + \frac{\pi \tau}{2} (1 - \frac{\pi}{4}) (1 + \tau) N + O(1).
\]
Denote by $E(\lambda_1, \lambda_2)$ the joint spectral family of the commuting operators $G \otimes \mathbb{I}$ and $\mathbb{I} \otimes G$ and put $\lambda_0 := \mathbb{E}(\langle \eta \otimes \eta, G \otimes \mathbb{I} \eta \otimes \eta \rangle)$. We then have
\[
\mathbb{E}\left(\int (\lambda_i - \lambda_0)^2 d\|E(\lambda_1, \lambda_2)\eta \otimes \eta\|^2\right) =: C' = O(1) \quad i = 1, 2
\]
\[
\mathbb{E}\left(\int A d\|E(\lambda_1, \lambda_2)\eta \otimes \eta\|^2\right) \leq \frac{8C'}{\lambda_0^2}
\]
with $A := \{(\lambda_1, \lambda_2) \mid (\lambda_1 - \lambda_0)^2 + (\lambda_2 - \lambda_0)^2 \leq \lambda_0^2/4\}$. We write
\[
\mathbb{E}(\langle \gamma, X \gamma \rangle^2) = \mathbb{E}(\|\gamma \otimes \gamma\|^2) \mathbb{E}\left(\langle \eta \otimes \eta, \frac{1}{G \otimes \mathbb{I} - z} \frac{1}{\mathbb{I} \otimes G - z} \eta \otimes \eta \rangle\right)
\]
\[
= \mathbb{E}(\|\gamma \otimes \gamma\|^2) \mathbb{E}\left(\int \frac{1}{(\lambda_1 - z)(\lambda_2 - z)} d\|E(\lambda_1, \lambda_2)\eta \otimes \eta\|^2\right)
\]
Then
\[
\left|\mathbb{E}\left(\int \frac{1}{(\lambda_1 - z)(\lambda_2 - z)} d\|E(\lambda_1, \lambda_2)\eta \otimes \eta\|^2\right)\right| \leq \frac{1}{|\lambda_0/2 - z||\lambda_0/2 - z|},
\]
and
\[
\left|\mathbb{E}\left(\int \frac{1}{(\lambda_1 - z)(\lambda_2 - z)} d\|E(\lambda_1, \lambda_2)\eta \otimes \eta\|^2\right)\right| \leq \frac{8C'}{|\Im(z)|^2 \lambda_0^2}.
\]
As $\lambda_0$ is of the order $N^2$, these last two inequalities show that the first term of $R(N, z)$ goes uniformly to zero on compact subsets of $\mathbb{C} \setminus \mathbb{R}^+$. The second term of $R(N, z)$ contains the matrix element $\langle \gamma^{(j)}, X G^{(j)} \gamma^{(j)} \rangle$. Again in the notation of Lemma 3, this gives
\[
|\mathbb{E}(\langle \gamma, X G \gamma \rangle)| = \mathbb{E}(\|\gamma\|^2) \left|\mathbb{E}\left(\int_0^\infty \frac{\lambda}{(\lambda - z)^2} d\|E(\lambda)\eta\|^2\right)\right|
\]
\[
\leq \mathbb{E}(\|\gamma\|^2) \mathbb{E}\left(\int_0^{\lambda_0/2} \frac{1}{|\lambda - z|^2} d\|E(\lambda)\eta\|^2\right)
\]
\[
+ \mathbb{E}(\|\gamma\|^2) \mathbb{E}\left(\int_{\lambda_0/2}^{\infty} \frac{1}{|\lambda - z|^2} d\|E(\lambda)\eta\|^2\right).
\]
The first term can be bounded from above by 
\[
\mathbb{E}(\|\gamma\|^2) \left( \frac{\lambda_0/|\Im(z)|}{4C/\lambda_0^2} \right)
\] 
by an application of formula (9). This gives a bound of order \(O(N)\). The second term has, provided that \(\Re(z) \leq \lambda_0/2\), a bound 
\[
E(\|\gamma\|^2) \left( \frac{\lambda_0/2}{\|\lambda_0/2 - z\|^2} \right) \leq E(\|\gamma\|^2) \left( \frac{\lambda_0/2}{\|\lambda_0/2 - z\|^2 (E(\lambda_1, \lambda_2) \xi_\alpha \otimes \xi_\alpha \|^2)} \right),
\] 
which is also, as a consequence of (7) and (8), of order \(O(N)\).

The third term of \(R(N, z)\) is also of order \(O(N)\) by
\[
|\text{Tr} \left( \frac{1}{G - z} G \frac{1}{G - z} \right) | = \left| \text{Tr} \left( \mathbb{I} + \frac{2z}{G - z} + \frac{z^2}{(G - z)^2} \right) \right| \leq K \left( 1 + \frac{2|z|}{|\Im(z)|} + \frac{|z|^2}{|\Im(z)|^2} \right) .
\] 
The fifth term admits the following estimate. Writing \(\xi_\alpha\) for the vector 
\[
\left( \frac{\mu_{k\alpha}}{2} \right)_{k=1,...,K},
\] 
we have
\[
\left| \mathbb{E} \left( \sum_{k,l,m,n} \sum_{\alpha} \mu_{k\alpha}^{1/2} X_{kl} \mu_{l\alpha}^{1/2} \mu_{mn}^{1/2} X_{mn} \mu_{n\alpha}^{1/2} \right) \right| 
\leq \sum_{\alpha} \left| \mathbb{E} \left( \langle \xi_\alpha \otimes \xi_\alpha, \frac{1}{G - z} \otimes \frac{1}{G - z} \xi_\alpha \otimes \xi_\alpha \rangle \right) \right| 
\leq \sum_{\alpha} \left| \mathbb{E} \left( \int_0^\infty \int_0^\infty \frac{1}{\lambda_1 - z} \frac{1}{\lambda_2 - z} \ d\|E(\lambda_1, \lambda_2) \xi_\alpha \otimes \xi_\alpha \|^2 \right) \right| 
\leq N \frac{1}{|\Im(z)|^2} \mathbb{E} \left( \langle \xi_\alpha \otimes \xi_\alpha \|^2 \right) .
\] 
(12)
The fourth term is estimated as
\[
\left| E \left( \sum_{k,l,m,n} \frac{\mu_{ka}}{2} X_{kl} \mu_{la}^{1/2} \mu_{ma}^{1/2} \mu_{n\beta}^{1/2} X_{mn} \right) \right|
\]
\[
\leq E \left( \sum_{\alpha} \left| \sum_{k,l} \frac{\mu_{ka}}{2} X_{kl} \mu_{la}^{1/2} \right| \left| \sum_{m,n} \frac{\lambda_{ma}^{1/2}}{2} \lambda_{n\beta}^{1/2} \right| \right)
\]
\[
\leq \left[ E \left( \sum_{k,l,m,n} \frac{\mu_{ka}}{2} X_{kl} \mu_{ma}^{1/2} \mu_{n\beta}^{1/2} \right) \right]^{1/2}
\cdot \left[ E \left( \sum_{k,l,m,n} \frac{\lambda_{ma}^{1/2}}{2} \lambda_{n\beta}^{1/2} \right) \right]^{1/2}.
\]

The first factor can be treated like \([\Sigma]\), while the second factor is just \(\langle \gamma^{(j)} X G^{(j)} X^\dagger \gamma^{(j)} \rangle\).

Hence, all terms contributing to \(R(N,z)\) are of \(O(N)\) divided by \(N(N+1)\). Therefore, the bound on \(R(N,z)\) tends to zero for large dimensions.

3 Almost sure convergence

In fact we can prove a stronger result. The empirical eigenvalue distributions are random measures. The randomness is described by the reference probability space \(\times \times (\mathbb{R}^+, e^{-x} dx)\) through the realization \(\mu_j = (x_j, \ldots, x_{jN})/(x_j + \cdots + x_{jN})\). We shall denote by \(\mathbb{P}\) expectations with respect to this reference probability space.

**Theorem 2** The convergence in Theorem 7 occurs with probability 1.

**Proof:** We essentially follow the proof in [2] for the almost sure convergence of the empirical eigenvalue distribution of the complex Wishart matrices, but use a different concentration–of–measure inequality. We need to show that
\[
\mathbb{P} \left( \lim_{N,K \to \infty} \frac{1}{K} \text{Tr} f(G_K) = \frac{1}{a} \int_0^\infty f(x) \rho_{MP} \left( \frac{x}{a} \right) dx \right) = 1
\]
with \( a = 1 - \frac{1}{4} \pi \) and \( f \) an arbitrary continuous function on \( \mathbb{R}^+ \) vanishing at infinity. We can further restrict ourselves to a dense subset of such functions, namely, we take for \( f \) a differentiable function on \( \mathbb{R}^+ \) with compact support. Define the function \( g \) by setting \( g(x) := f(x^2) \) for \( x \in \mathbb{R}^+ \). Then \( g \) is also differentiable with compact support and, like \( f \), a Lipschitz function with constant

\[
c_1 = \sup_{x \in \mathbb{R}^+} |g'(x)|.
\]

Let \( \mu = \{\mu_i\}_{i=1}^K, \sigma = \{\sigma_i\}_{i=1}^K \) denote two sets of \( K \) probability measures in \( \Lambda_N \). Define the \( N \times K \) matrix \( A_\mu \) by

\[
A_\mu = \begin{pmatrix} \sqrt{\mu_{11}} & \cdots & \sqrt{\mu_{1K}} \\ \sqrt{\mu_{12}} & \cdots & \sqrt{\mu_{2K}} \\ \vdots & \ddots & \vdots \\ \sqrt{\mu_{1N}} & \cdots & \sqrt{\mu_{KN}} \end{pmatrix},
\]

and analogously for \( A_\sigma \). Then \( A_\mu^* A_\mu \) is the Gram matrix associated with the set of measures \( \{\mu_i\}_{i=1}^K \). Define \( F : \Lambda_N \times \cdots \times \Lambda_N \to \mathbb{R} \) by

\[
F(\mu) := \frac{1}{K} \text{Tr} f(A_\mu^* A_\mu).
\]

We want to show that the function \( F \) satisfies a Lipschitz condition. Define \( \widetilde{A}_\mu, \widetilde{A}_\sigma \) by

\[
\widetilde{A}_\mu = \begin{pmatrix} 0 & A_\mu^* \\ A_\mu & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{A}_\sigma = \begin{pmatrix} 0 & A_\sigma^* \\ A_\sigma & 0 \end{pmatrix}.
\]

Now Lemma 3.5 in [2] is used to transport the Lipschitz property of \( g \) on \( \mathbb{R}^+ \) to \( M_{N+K}(\mathbb{C})_{sa} \), the set of \( (N+K) \)-dimensional complex selfadjoint matrices. This lemma implies

\[
\|g(\widetilde{A}_\mu) - g(\widetilde{A}_\sigma)\|_{\text{HS}} \leq c_1 \|\widetilde{A}_\mu - \widetilde{A}_\sigma\|_{\text{HS}}, \tag{13}
\]

where \( \|A\|_{\text{HS}} := \sqrt{\text{Tr} A^* A} \). Because

\[
\widetilde{A}_\mu^2 = \begin{pmatrix} A_\mu^* A_\mu & 0 \\ 0 & A_\mu A_\mu^* \end{pmatrix},
\]

we have

\[
g(\widetilde{A}_\mu) = \begin{pmatrix} f(A_\mu^* A_\mu) & 0 \\ 0 & f(A_\mu A_\mu^*) \end{pmatrix},
\]

and an analogous expression for \( g(\widetilde{A}_\sigma) \). Now (13) implies

\[
\|f(A_\mu^* A_\mu) - f(A_\mu A_\mu^*)\|_{\text{HS}}^2 + \|f(A_\mu A_\mu^*) - f(A_\mu A_\sigma^*)\|_{\text{HS}}^2 \leq c_1^2 (\|A_\mu - A_\sigma\|_{\text{HS}}^2 + \|A_\mu^* - A_\sigma^*\|_{\text{HS}}^2).
\]

20
Since \( \|A_\mu - A_\sigma\|_{\text{HS}} = \|A_\mu^* - A_\sigma^*\|_{\text{HS}} \) and because of the Cauchy-Schwarz inequality, we have

\[
|F(\mu) - F(\sigma)| \leq \frac{1}{\sqrt{K}} \| f(A_\mu^* A_\mu) - f(A_\sigma^* A_\sigma) \|_{\text{HS}} \leq c_1 \sqrt{\frac{2}{K}} \|A_\mu - A_\sigma\|_{\text{HS}}.
\]

Now

\[
\|A_\mu - A_\sigma\|_{\text{HS}}^2 = \sum_{i=1}^{K} \|\sqrt{\mu_i} - \sqrt{\sigma_i}\|^2 = \sum_{i=1}^{K} \sum_{\alpha=1}^{N} |\sqrt{\mu_{i\alpha}} - \sqrt{\sigma_{i\alpha}}|^2 \leq \sum_{i=1}^{K} \sum_{\alpha=1}^{N} |\mu_{i\alpha} - \sigma_{i\alpha}|^2 \leq \sum_{i=1}^{K} \sqrt{N} \sum_{\alpha=1}^{N} (\mu_{i\alpha} - \sigma_{i\alpha})^2 = \sqrt{N} \sum_{i=1}^{K} \|\mu_i - \sigma_i\|,
\]

with the notation \( \sqrt{\mu_i} = (\sqrt{\mu_{i1}}, \ldots, \sqrt{\mu_{iN}}) \). Now for arbitrary \( t > 0 \) we have, using this Lipschitz condition,

\[
\mathbb{P}(\|F(\mu) - F(\sigma)\|^2 > t^2) = \mathbb{P}(\|F(\mu) - F(\sigma)\| > t) \leq \mathbb{P} \left( \|\sum_{i=1}^{K} (\mu_{i\alpha} - \sigma_{i\alpha})^2\| > t^2 \right) \leq \mathbb{P} \left( \sum_{i=1}^{K} \|\mu_i - \sigma_i\|^2 > \frac{t^2 \sqrt{2K}}{2c_1^2} \right) \leq K \exp \left( -c_2 \frac{t^2 N}{2} \right)
\]

Using Lemmas 5 and 6, we know that there exist constants \( T \geq 0 \) and \( c_2 > 0 \) such that for \( t > 2T/\sqrt{N} \)

\[
\mathbb{P} \left( \sum_{i=1}^{K} \|\mu_i\| > t \right) \leq K \exp \left( -c_2 \frac{tN}{2} \right).
\]

From this, it follows that if we choose \( N > 8Tc_1^2/\tau^2 \), the probability \( (14) \) can now be treated analogously as in the proof of Lemma 6, yielding:

\[
\mathbb{E}_\mu \left( \mathbb{P}_\sigma \left( \sum_{i=1}^{K} \|\sigma_i\| > \frac{t^2 \tau \sqrt{N}}{2c_1^2} - \sum_{i=1}^{K} \|\mu_i\| \right) \right) \leq K \exp - \left( \frac{c_2}{2} \frac{t^2 \tau \sqrt{N}}{2c_1^2} - \frac{c_2}{2} \frac{t^2 \tau \sqrt{N}}{2c_1^2} \right) N \exp \left( -c_2 \frac{t^2 \tau N^{3/2}}{4c_1^2} \right).
\]

21
Then

\[ P(|F(\mu) - \mathbb{E}(F)| > t) = P(\exp[\lambda^2|F(\mu) - \mathbb{E}(F)|^2] > \exp(\lambda^2 t^2)) \leq \frac{\mathbb{E}(\exp[\lambda^2|F(\mu) - \mathbb{E}(F)|^2])}{e^{\lambda^2 t^2}}. \]

Take now \( N > 32Tc_1^2/\tau^2 \). The function \( t \mapsto \exp[\lambda^2(F(\mu) - t)^2] \) is convex, so Jensen’s inequality implies

\[
\mathbb{E}_\mu(\exp[\lambda^2|F(\mu) - \mathbb{E}_\sigma(F(\sigma))|^2]) \leq \mathbb{E}_{\mu,\sigma}(\exp[\lambda^2|F(\mu) - F(\sigma)|^2]) \\
= \int_0^\infty 2\lambda^2 Ce^{\lambda^2 C^2} P(|F(\mu) - F(\sigma)| > C) \, dC \\
\leq \int_0^{t/2} 2\lambda^2 Ce^{\lambda^2 C^2} \, dC + 2\tau^2 N^2 \int_{t/2}^\infty 2\lambda^2 C \exp(\lambda^2 C^2 - \frac{c_2^2 \tau N^3/2}{4c_1^2}) \, dC.
\]

If we choose \( \lambda^2 = c_2 \tau N^3/2/8c_1^2 \), we get

\[
P(|F(\mu) - \mathbb{E}(F)| > t) \leq \frac{2\tau^2 N^2 + \exp \frac{c_2 \tau N^3/2}{8c_1^2} \frac{t^2}{4}}{\exp \frac{c_2 \tau N^3/2}{8c_1^2} \frac{t^2}{4}} \leq 2 \exp - \frac{3c_2 \tau N^3/2 t^2}{32c_1^2}.
\]

An application of the Borel-Cantelli lemma shows that this implies

\[
P \left( \lim_{N \to \infty} |F(\mu) - \mathbb{E}(F)| \leq t \right) = 1,
\]

for arbitrary \( t \). This completes the proof.

Lemma 5 There exist absolute constants \( T > 0 \) and \( c > 0 \) such that for \( N \) independent exponentially distributed random variables \( X_1, \ldots, X_N \) and all \( t > T/\sqrt{N} \) holds that, with \( X = (X_1, \ldots, X_N) \) and \( S = \sum_{i=1}^N X_i \),

\[
P \left( \frac{\|X\|}{S} > t \right) \leq e^{-ctN}.
\]

Proof: See Theorem 3 and Lemma 1 in [7].
**Lemma 6** Suppose that for a random variable \( X > 0 \) there exist constants \( c > 0 \) and \( T \geq 0 \) such that for all \( t > T \)

\[
P(X > t) \leq e^{-ct};
\]

then, for \( N \) identical independent copies \( X_1, \ldots, X_N \) of \( X \) and for \( t > 2T \)

\[
P\left( \sum_{i=1}^{N} X_i > t \right) \leq Ne^{-ct/2}.
\]

**Proof:** We prove this by induction on \( N \). The statement is obviously true for \( N = 1 \) because \( t > 2T \geq T \), so \( P(X_1 > t) \leq e^{-ct} \leq e^{-ct/2} \). Suppose now that the statement is true for \( N - 1 \) copies, then for \( t > 2T \)

\[
P\left( \sum_{i=1}^{N} X_i > t \right) = \mathbb{E}_{X_1, \ldots, X_{N-1}} \left( P_{X_N} \left( X_N > t - \sum_{i=1}^{N-1} X_i \right) \right)
\]

\[
= \mathbb{E}_{X_1, \ldots, X_{N-1}} \left( P_{X_N} \left( X_N > t - \sum_{i=1}^{N-1} X_i \right) I_{\left\{ \sum_{i=1}^{N-1} X_i \leq \frac{t}{2} \right\}} \right)
\]

\[
+ \mathbb{E}_{X_1, \ldots, X_{N-1}} \left( P_{X_N} \left( X_N > t - \sum_{i=1}^{N-1} X_i \right) I_{\left\{ \sum_{i=1}^{N-1} X_i > \frac{t}{2} \right\}} \right)
\]

\[
\leq e^{-ct/2} + (N - 1)e^{-ct/2} = Ne^{-ct/2}.
\]

\[\blacksquare\]

**References**

[1] **De Cock, M., Fannes, M. and Spincemaille, P.** (1999). On quantum dynamics and statistics of vectors. *J. Phys. A* **32**, 6547–6571.

[2] **Haagerup, U. and Thorbjørnsen, S.** (1998). Random matrices with complex Gaussian entries. Preprint, Odense University.

[3] **Kato, T.** (1984). *Perturbation Theory for Linear Operators*, 2nd edn. Springer, Berlin.

[4] **Hiai, F. and Petz, D.** (2000). *The Semicircle Law, Free Random Variables and Entropy*, Math. Surveys and Monographs, Vol. **77**, Americ. Math. Soc.
[5] Marchenko, V. and Pastur, L. (1967). The eigenvalue distribution in some ensembles of random matrices. *Math. USSR Sbornik* 1, 457–483.

[6] Pastur, L. (2000). A simple approach to the global regime of the random matrix theory, in *Mathematical Results in Statistical Mechanics*, Miracle–Sole, S., Ruiz, J., and Zagrebnov, V. (Eds), World Scientific, Singapore, 429–454.

[7] Schechtman, G. and Zinn, J. (1990). On the volume of the intersection of two $L^n_p$ balls. *Proc. A.M.S.* 110, 217-224.

[8] Silverstein, J.W. (1994). The spectral radii and norms of large dimensional non-central random matrices. *Commun. Statist.-Stochastic Models* 10, 525–532.

[9] Voiculescu, D.V., Dykema, K.J. and Nica, A. (1992). *Free Random Variables*, CRM Monograph Ser., Vol. 1, Americ. Math. Soc.