Reset control systems: the zero-crossing resetting law

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Abstract

A novel representation of reset control systems with a zero-crossing resetting law, in the framework of hybrid inclusions, is postulated. The problems of well-posedness and stability of the resulting hybrid dynamical system are investigated, with a strong motivation in how non-deterministic behavior is accomplished in control practice. Several stability conditions, based on the eigenstructure of matrices related with periods of the reset interval sequences, and on Lyapunov function-based conditions, are developed.

Keywords: Hybrid dynamical systems, Hybrid control systems, Reset control systems, Robustness to measurement noise, Robustness, stability

1. Introduction

Informally speaking, a reset controller is any controller, usually referred to as the base controller, that is equipped with a mechanism for zeroing some of its states, when some event occurs in the control system. Although the term was coined in the late 90s by Hollot, Chait and coworkers ([11]), specifically to describe "a linear and time invariant system with mechanisms and laws to reset their states to zero", the concept was devised much earlier, in the seminal works of Clegg ([14]) and Horowitz and coworkers ([24, 22]). Since then, reset control has considerably evolved by using different resetting laws: the original zero crossing of the error ([11, 6, 8, 23, 9, 32]), sector-based resetting ([25, 11, 27, 35, 36]), error band ([10, 7]), reset at fixed instants ([21, 20]), Lyapunov function-based resetting ([30]), and somehow relaxing the original concept, both including nonlinear base systems and reset to non-zero values in some cases. This has lead to a fecund research area that

$\ast$This work has been supported by FEDER-EU and Ministerio de Ciencia e Innovación (Gobierno de España) under project DPI2016-79278-C2-1-R, and Fundación Séneca (Comunidad Autónoma de la Región de Murcia) under project 20842/PI/18.

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has been successfully applied in many practical applications, and that has opened many relevant topics in control theory and practice.

In this work, the focus is on reset control with emphasis on the original concept, using a linear and time-invariant controller that zeroes its state (fully or partially) when the closed-loop error signal is zero. The main motivation has been to update and formalize previous work by the authors, that was developed in the framework of impulsive dynamical systems (IDS), by using the hybrid dynamical systems framework (HI) of [17, 18]. Note that in the IDS framework, resetting laws are based on the exact crossing of the zero error hypersurface, and there is some fragility in detecting a zero crossing specially if measurement noise is present. Although this robustness problem has been alleviated for specific class of exogenous signals ([2]), it is acknowledged the HI framework is more conclusive for equipping reset control systems with good structural properties (specially when considering exogenous signals with jump discontinuities) such as continuous-dependence on initial conditions, closeness of perturbed (due for example to measurement noise) and unperturbed solutions, asymptotic stability is preserved under small perturbations, etc.

There exist already several relevant works about reset control in the HI framework, most of them based on a sector-based resetting law (see for example [28][1][27][35][36]). It is important to emphasize that, in general, this resetting law produces different control solutions in comparison with the zero-crossing resetting law (see Example 3.4 of this manuscript). Here, it is not argued that one resetting law is superior to the other, as well as in comparison with other resetting laws, they simply are different solutions that may properly work in control practice.

1.1. Background: Hybrid dynamical systems

This work follows the hybrid system framework developed in [18] (and references therein), that following [16], has been referred to as Hybrid Inclusions (HI) framework, and the reader is referred to [18][17] for a detailed exposition of it (see also [13] where hybrid systems with inputs are explicitly analyzed). A hybrid system \( \mathcal{H}_w \), with state \( x \in \mathbb{R}^n \) and input \( w \in \mathbb{R}^m \), is given by

\[
\begin{align*}
\mathcal{H}_w : \begin{cases}
\dot{x} &= f(x, w), \quad (x, w) \in \mathcal{C}, \\
x^+ &= g(x, w), \quad (x, w) \in \mathcal{D}.
\end{cases}
\end{align*}
\]

and is defined by the following data: i) the flow set \( \mathcal{C} \subset \mathbb{R}^n \times \mathbb{R}^m \), ii) the flow mapping \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \), iii) the jump set \( \mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^m \), and iv) the jump mapping \( g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \).

Hybrid signals are defined as functions on hybrid time domains ([19],[18]). A hybrid arc \( x : \text{dom } x \mapsto \mathbb{R}^n \) is a hybrid signal in which \( x(\cdot, j) \) is locally absolutely continuous for each \( j \). A hybrid input \( w : \text{dom } w \mapsto \mathbb{R}^m \) is a hybrid signal in which \( w(\cdot, j) \) is Lebesgue measurable and locally essentially bounded for each \( j \). A solution to (1) is defined as a pair \( (x, w) \), consisting of a hybrid arc and a hybrid input with \( \text{dom } x = \text{dom } w \), that satisfy the dynamics of the hybrid system \( \mathcal{H}_w \) (see [13], [35] for details about solution pairs to hybrid systems with inputs). Note that the jump set \( \mathcal{D} \) enables jumps but does not force them if there are points in which it is also possible to flow (a similar argument applies to the flow set \( \mathcal{C} \)); and thus if \( \mathcal{C} \) are \( \mathcal{D} \) are not disjoint then for a point \( (\xi, \psi) \in \mathcal{C} \cap \mathcal{D} \) there may exist several solution pairs \( (x, w) \) to \( \mathcal{H}_w \) with \( x(0, 0) = \xi \), for any hybrid input \( w \) with \( w(0, 0) = \psi \).

For \( \mathcal{H}_w \), the so-called hybrid basic conditions defined in [18] (see also regularity conditions in [13]) are satisfied if \( \mathcal{C} \) and \( \mathcal{D} \) are both closed subsets of \( \mathbb{R}^n \times \mathbb{R}^m \), and if \( f \) and \( g \) are continuous functions. These hybrid basic conditions guarantee that \( \mathcal{H}_w \) (without inputs, that is with \( w = 0 \)) is well posed in the sense that their solution sets inherit several good structural properties:
upper-semicontinuous dependence with respect to initial conditions, closeness of perturbed (due for example to measurement noise) and unperturbed solutions, asymptotic stability is preserved under small enough perturbations (18).

1.2. Organization of the manuscript

This work is mainly devoted to develop a representation of reset control systems, with a zero-crossing resetting law and a linear and time invariant base system, in the HI framework. The focus is about well-posedness and stability, with a strong motivation in obtaining HI models that capture key properties in control practice. In Section 2, starting with a new Clegg integrator model equipped with an input zero-crossing detection (ZCD) mechanism, it is postulated a reset controller model based on it. Section 3 analyzes closed-loop reset control systems, resulting of the feedback connection of a reset controller (with the ZCD mechanism) and a linear and time invariant plant (using plant output measurement). Some basic properties of closed-loop hybrid system like well-posedness, existence of solutions and flow persistence (a concept introduced to guaranty the existence of solutions that are unbounded in the t-direction) are investigated. Also, a deep analysis of how solutions to the closed-loop hybrid system are related to operation in control practice. To avoid existence of defective solutions, a standard approach based on time regularization is used. Finally, a reset control system in the HI framework, with a zero-crossing resetting law and time-regularized, is postulated. An example, based on a classical case analyzed by Horowitz, is investigated with the proposed model; also a comparison with a time-regularized reset control system with a sector-based resetting law is performed. In Section 4, stability of the proposed reset control system is investigated. A basic result will be a reformulation of previous stability result in the new HI framework, relating stability of the closed-loop hybrid system with the stability of a discrete-time system obtained as a Poincaré-like map. Two different stability approaches are then investigated: one based on the analysis of the reset interval sequences periods, that result in analyzing eigenvalues of different matrices associated with those periods; and another based on the use of Lyapunov functions that finally results in LMIs conditions whose feasibility determine the stability of the reset control system. Moreover, several examples, that illustrates the applicability of the proposed results, are developed.

2. From the Clegg integrator to reset controllers

In this work, the main focus is on reset control systems in which a continuous-time plant is controlled by a reset controller with plant output feedback (see Fig. 1). This feedback control system, that uses plant output measurement, will be modeled in the HI framework by using (1). More specifically, the plant is linear and time invariant (LTI) and single-input single-output, and described by the differential equation:

\begin{align}
P: \begin{cases}
\dot{x}_p &= A_p x_p + B_p u \\
y_p &= C_p x_p
\end{cases}
\end{align}

where \(x_p \in \mathbb{R}^{n_p}\) is the plant state, \(u \in \mathbb{R}\) the control input, \(y_p \in \mathbb{R}\) is the plant output, and \(A_p\), \(B_p\) and \(C_p\) are matrices of appropriate dimensions. The reset controller, with continuous state \(x_r \in \mathbb{R}^{n_r}\), will be endowed with a zero-crossing detection mechanism based on a discrete state \(q \in \{1, -1\}\), being finally \((x_r, q) \in O^{n_r} := \mathbb{R}^{n_r} \times \{1, -1\}\) the controller state. In the following, the proposed reset controller will be analyzed in detail; since the controller setup will be based on a modification of the Clegg integrator (14), this is first described.
2.1. A Clegg integrator with a zero-crossing detection mechanism

A basic and well-known reset controller is the Clegg integrator ([14],[24]), that will be adapted in this work attaching a zero-crossing detection procedure based on a the discrete state $q \in \{1, -1\}$, and also adding an extra input. Besides the trigger input $e \in \mathbb{R}$ (usually the error signal in the case of an output feedback control system) the input signal $e_{CI} \in \mathbb{R}$ is proposed. Using (1), the result is a new model of the Clegg integrator in the HI framework. It is given by:

$$
\dot{x}_r = e_{CI}, \quad (x_r, q, e_{CI}, e) \in \mathcal{C}
$$

where $(x_r, q) \in \mathcal{O}$ is the CI state, $(e_{CI}, e) \in \mathbb{R}^2$ is its input, and $v = x_r$ is its output, and the flow set $\mathcal{C}$ and the jump set $\mathcal{D}$ are given by

$$
\mathcal{C} = \{(x_r, q, e_{CI}, e) \in \mathcal{O} \times \mathbb{R}^2 : qe \leq 0\}
$$

and

$$
\mathcal{D} = \{(x_r, q, e_{CI}, e) \in \mathcal{O} \times \mathbb{R}^2 : qe \geq 0\},
$$

respectively. Note that since the CI discrete state $q$ is constant when flowing, its flow equation is not explicitly shown by simplicity. The two input signals $e_{CI}$ and $e$ are useful for modeling more complex reset controllers by using CI as a building block (see Fig. 2). This capability will be fully exploited by higher order reset controller in the next Section.

Moreover, subsets $\mathcal{C}_1 \subset \mathcal{C}$ and $\mathcal{D}_1 \subset \mathcal{D}$ are defined as $\mathcal{C}_1 = \{(x_r, q, e_{CI}, e) \in \mathcal{C} : q = 1\}$ and $\mathcal{D}_1 = \{(x_r, q, e_{CI}, e) \in \mathcal{D} : q = 1\}$; the subsets $\mathcal{C}_{-1} \subset \mathcal{C}$ and $\mathcal{D}_{-1} \subset \mathcal{D}$ are defined accordingly. When $(x_r, q, e_{CI}, e)$ goes from $\mathcal{C}_{\pm 1}$ to $\mathcal{D}_{\pm 1}$ either crossing or jumping through their boundary, a jump of the CI state may be performed. This guarantees the detection of a zero-crossing even if the signal $e$ has jump discontinuities, for example due to some noise measurement $n$ (see Fig. 3).

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1Note that the original Clegg integrator is recovered from CI by removing the discrete state $q$ and doing $e_{CI} = e$. 

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2.2. A reset controller with zero-crossing resetting law

In this work, it is proposed a reset controller inspired in [11, 3, 9]. Here, the new CI is used as a building block, and thus the reset controller has also attached a zero-crossing detection mechanism.

It has a state \((x_r, q) \in \mathbb{O}^{n_r}\) and a scalar input \(e\). Using again (1), it is given by

\[
\dot{x}_r = A_r x_r + B_r e, \quad (x_r, q, e) \in \mathcal{C}
\]

where the output and control signal is the scalar \(v = C_r x_r + D_r e\), and now the flow and jump sets, \(\mathcal{C}\) and \(\mathcal{D}\), are

\[
\mathcal{C} = \{(x_r, q, e) \in \mathbb{O}^{n_r} \times \mathbb{R} : qe \leq 0\}
\]

and

\[
\mathcal{D} = \{(x_r, q, e) \in \mathbb{O}^{n_r} \times \mathbb{R} : qe \geq 0\},
\]

respectively. Here \(A_r, B_r, C_r,\) and \(D_r\) are real matrices with appropriate dimensions. And \(A_\rho\) is a matrix that set to zero some of the controller states when a zero-crossing has been detected (by convention, the last \(n_\rho\) states of \(x_r\) are set to zero, while the first \(n_\bar{\rho} = n_r - n_\rho\) states are kept without change). It is given by

\[
A_\rho = \begin{pmatrix}
I_{n_\rho \times n_\rho} & 0_{n_\rho \times n_\bar{\rho}} \\
0_{n_\bar{\rho} \times n_\rho} & 0_{n_\bar{\rho} \times n_\bar{\rho}}
\end{pmatrix},
\]

In the case of a full reset controller \(n_\rho = n_r\), while if \(n_\rho < n_r\) then \(R\) is a partial reset controller. In addition, \(A_r, B_r,\) and \(C_r\) are partitioned into blocks with appropriate block dimensions:

\[
A_r = \begin{pmatrix}
A_{r_{11}} & A_{r_{12}} \\
A_{r_{21}} & A_{r_{22}}
\end{pmatrix}, \quad B_r = \begin{pmatrix}
B_{r_1} \\
B_{r_2}
\end{pmatrix}, \quad C_r = \begin{pmatrix}
C_{r_1} \\
C_{r_2}
\end{pmatrix}
\]

For a block diagram representation of the reset controller \(R\), besides integrator blocks it is sufficient to use the modified Clegg integrator CI given by (3) as a basic block. A block diagram of \(R\) that allows a direct implementation is given in Fig. 4. On the other hand, if \(A_{r_{21}} = O (A_{r_{12}} = O)\)
Figure 3: Zero-crossing detection mechanism: (left) from the initial point A (up), the system flows until a zero-crossing is detected \((q_e = 0)\), jumping from \(C_1 \cap D_1\) to \((e, v, q) = (0, 0, -1)\) (bottom); (right) a perturbation of the error signal at point B makes the system jump from \(B \in C_1\) to \(C \in D_1\) (up), then a zero-crossing is also detected \((q_e > 0)\) and the system jumps to \(D \in C_{-1} \setminus D_{-1}\) (bottom); from D the system flows again and finally jumps to \((0, 0, 1)\) (up).

Then \(R\) will be referred to as a right reset controller (left reset controller); the name is related with the right (left) triangular block structure of the matrix \(A_r\). Informally speaking, for a right reset controller the inputs of the CI blocks are not influenced by the outputs of the integrator blocks. It is worthwhile to mention that some of the reset controller with partial reset (see Fig. 5) that has been found useful in practice are right reset controllers (3).

3. The closed-loop reset control system

Once the plant and the reset controller has been defined, now the feedback control system \(\mathcal{H}_{cl}^d\) is obtained (see Fig. 1) as a hybrid control system, where the plant output \(y_p\) is the feedback signal. Its state is \((x_p, x_r, q) \in \mathcal{O}^n\), and is given by

\[
\mathcal{H}_{cl}^d : \begin{cases} 
\dot{\mathbf{x}}_p = (A_p - B_pD_rC_p \quad B_pC_r \quad A_r) \begin{pmatrix} x_p \\ x_r \end{pmatrix} + \begin{pmatrix} B_pD_r \\ B_p \quad O \end{pmatrix} w, & (x, q, w) \in \mathcal{C}_{cl}^d \\
\begin{pmatrix} x_r^+ \\ q^+ \end{pmatrix} = \begin{pmatrix} A_r & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_r \\ q \end{pmatrix}, & (x, q, w) \in \mathcal{D}_{cl}^d
\end{cases}
\]

where the exogenous input is \(w = (w_1, w_2) = (r + n, d)\), and the flow and jump sets are given by

\[
\mathcal{C}_{cl}^d = \{(x_p, x_r, q, w) \in \mathcal{O}^{n_p+n_r} \times \mathbb{R}^2 : q(w_1 - C_p x_p) \leq 0\}
\]

and

\[
\mathcal{D}_{cl}^d = \{(x_p, x_r, q, w) \in \mathcal{O}^{n_p+n_r} \times \mathbb{R}^2 : q(w_1 - C_p x_p) \geq 0\},
\]

respectively.
3.1. Closed-loop solutions and properties

Following Section 1.2, solutions to $\mathcal{H}_w^{cl}$ are defined as pairs $(x, w)$ satisfying (11), where $x$ is a hybrid arc and $w$ is a hybrid input. Since $\mathcal{H}_w^{cl}$ satisfies the hybrid basic conditions (the flow and jump maps are continuous, and the flow and the jump sets are closed), it directly follows that for the case of no exogenous inputs, that is for $w = 0$, the system $\mathcal{H}_w^{cl}$ is well posed and the property of asymptotic stability is robust (see [18] for precise definitions and results).

Now, a key question is to analyze if $\mathcal{H}_w^{cl}$ has also good properties for relevant sets of exogenous inputs. Firstly, the problem of modeling relevant exogenous inputs arises; since the plant is originally continuous-time, it is natural to consider that exogenous signals only depend on time $t$ and not on $j$, and thus hybrid inputs $w$ can be considered as given by setting $w(t, j) = w'(t)$ for all $(t, j) \in E$, for some continuous time signal $w'$ and any arbitrary time domain $E$. Moreover, since exogenous inputs should be relevant in control practice, it will be assumed that $w'$ is generated by an exosystem $\Sigma$ with state $x_\Sigma$, that is
These exosystems will allow to generate signals like steps, ramps, sinusoids, and in general Bohl functions. Finally, these exosystems are embedded in the reset control system (11) resulting in the (autonomous) reset control system $H^{cl}$ with state $(x, q) = (x_w, x_p, x_r, q)$:

$$
\Sigma : \begin{cases}
\dot{x}_w = A_w x_w \\
w = \left( \begin{array}{c} C_{w_1} \\ C_{w_2} \end{array} \right) x_w
\end{cases}
$$

These exosystems will allow to generate signals like steps, ramps, sinusoids, and in general Bohl functions. Finally, these exosystems are embedded in the reset control system (11) resulting in the (autonomous) reset control system $H^{cl}$ with state $(x, q) = (x_w, x_p, x_r, q)$:

$$
H^{cl} : \begin{cases}
\dot{x} = Ax, \ (x, q) \in C^{cl} \\
(x^+, q^+) = \left( \begin{array}{cc} A_R & 0 \\ 0 & -1 \end{array} \right) \left( \begin{array}{c} x \\ q \end{array} \right), \ (x, q) \in D^{cl}
\end{cases}
$$

where the matrices $A$ and $A_R$ are given by

$$
A = \left( \begin{array}{cccc} A_w & O & O \\ B_p(D_r C_{w_1} + C_{w_2}) & A_p - B_p D_r C_p & B_p C_r \\ B_r C_{w_1} & -B_r C_p & A_r \end{array} \right), \quad A_R = \left( \begin{array}{ccc} I & O & O \\ O & I & O \\ O & O & A_{\rho} \end{array} \right),
$$

and the flow and jump sets are given by

$$
C^{cl} = \{ (x, q) \in O^n : qC x \leq 0 \}
$$

and

$$
D^{cl} = \{ (x, q) \in O^n : qC x \geq 0 \},
$$

respectively, where $n = n_w + n_p + n_r$, and $C$ is given by

$$
C = \left( \begin{array}{ccc} C_{w_1} & -C_p & O \end{array} \right)
$$
Proposition 3.1. Consider the reset control system \( \mathcal{H}^d \), and a point \( \xi = (x_0, q_0) \in \mathcal{O}^n \), then:

1. (Well-posedness) \( \mathcal{H}^d \) is well-posed.
2. (Existence of solutions) There exist nontrivial solutions \( \phi \) to \( \mathcal{H}^d \) with \( \phi(0,0) = \xi \), and if \( \phi \in S_{\mathcal{H}^d}(\xi) \) then it is complete, that is \( \mathcal{H}^d \) is forward complete from \( \mathcal{O}^n \).
3. (Flow persistence) \( \mathcal{H}^d \) is flow persistent.

Proof. 1. It directly follows since \( \mathcal{H}^d \) satisfies the basic hybrid conditions. Note that \( \mathcal{C} \) and \( \mathcal{D} \) are closed, and the functions \( f, g : \mathcal{O}^n \to \mathcal{O}^n \), defining the flowing and jumping dynamics, respectively, and given by \( f((x, q)) = (Ax, 0) \) and \( g((x, q)) = (ARx, -q), \) are continuous.

2. If \( \xi \in \mathcal{C} \setminus \mathcal{D} \) then there exists a solution \( \phi \) with \( \phi(t,0) = (e^{At}x_0, q_0) \in \mathcal{C} \) for \( t \in [0, \epsilon] \) and \( \epsilon = \min\{t \in \mathbb{R}^+: Ce^{At}x_0 = 0\} \); in the case that \( Ce^{At}x_0 \neq 0 \) for any \( t > 0 \) then \( \phi(t,0) = (e^{At}x_0, q_0) \in \mathcal{C} \) for \( t \in (0, \infty) \). Also, if \( \xi \in \mathcal{D} \) then there exists solutions \( \phi \) with \( \phi(0,1) = (ARx_0, -q_0) \) and \( \phi(0,0) = \xi \). Thus there exists a nontrivial solution to \( \mathcal{H}^d \) starting from \( \xi \). Moreover, for \( (x, q) \in \mathcal{D}^d \) it is true that \( qCx \geq 0 \), and thus after a jump \( q^+Cx^+ = -qCAx \leq 0 \), that is \( (x^+, q^+) \in \mathcal{C}^d \) and thus \( g(\mathcal{D}^d) \subset \mathcal{C}^d \); since, in addition, any solution to the flow equation \( \dot{x} = Ax \) defined on an interval, open to the right, can be trivially extended to an interval including the right endpoint, it is concluded that any maximal solution is complete \([18] \text{ Prop. 2.10} \).

3. The only obstacle for the existence of solutions that are unbounded in the \( t \)-direction is that when the system jumps from \( \mathcal{C}^d \cap \mathcal{D}^d \) to \( \mathcal{C}^d \setminus \mathcal{D}^d \), and flows again to \( \mathcal{C}^d \cap \mathcal{D}^d \) repeating the sequence, the result is a jump instants sequence that is convergent to a finite value. In this case, the system solution \( \phi \) would be a Zeno solution with dom \( \phi \) not a finite time. Also, this is not possible since sequences of jumps \( \{t_i\}_{i=0}^{\infty} \) can not have subsequences of decreasing jump instants of length greater than \( n - n_p \) (see \[3 \text{ Prop. 2.4, also } 9 \]). Thus, there always exists a system solution that is unbounded in the \( t \)-direction.

\[ \Box \]

3.2. Analysis of defective solutions and time-regularized reset control system

In a reset control system formulated as \([13] \), in which the hybrid dynamics is due to the controller (the plant is a LTI continuous-time system), \( I \text{ it is important to analyze how hybrid time domains of solutions (see Appendix A) and the non-deterministic behavior of the system are related with the final operation in control practice. For example, for a hybrid time domain that consists of the union of intervals } I_j \times j = [t_j, t_{j+1}] \times j, \text{ with } 0 = t_0 < t_1 < t_2 = t_3 = t_4 = t_5 < t_6 < \cdots, \text{ the solution flows in the time interval } [t_0, t_1], \text{ jumps at } t_1, \text{ flows in } [t_1, t_2], \text{ then it performs three consecutive jumps, keeps flowing in } [t_2, t_6], \text{ performs again a jump at } t_{10}, \text{ etc. From a practical point of view, for solutions to be implemented in a controller, it is necessary to assume that the controller is able to perform a finite number of consecutive jumps instantaneously. In addition, it is compulsory that from any point there always exist solutions that are unbounded in the } t \text{-direction. Otherwise, the} \]


control system only would present Zeno solutions (genuinely or eventually discrete), that simply can not be implemented in practice, and are considered as defective solutions.

It has been introduced a property, flow persistence, that is useful to analyze wether a reset control system may be effectively used in control practice regarding the existence of non-defective solutions. Note that if a control system is flow persistent then there always exists a solution which is unbounded in the t-direction; on the contrary, if it is not flow persistent then there may exist points from where all the solutions are bounded in the t-direction, that is there would exist only defective solutions. Thus, although flow persistence is a necessary property in control practice, it is less obvious whether it is a sufficient property, that is, (for a given initial point) is it a problem the existence of Zeno solutions besides solutions that are unbounded in the t-direction?. Note that there is always infinite Zeno solutions starting at the points \((0,1), (0,-1) \in C^{cl} \cap D^{cl}\), besides an infinite number of solutions unbounded in the t-direction.

Another important aspect regarding the final implementation of the hybrid controller is related with its non-deterministic behavior. In principle, the above formulation allows a (finite or infinite) number of different solutions from some initial points. In practice, it is clear that any realistic controller implementation entails a decision such as a solution is selected within all the existing solutions. At this point, a possible answer to the above question is that there is no problem once it is assumed that the controller is able to choose only the solutions that are unbounded in the t-direction. However, this type of implementation would require some procedure to properly select the implementable solutions.

A more simple and common approach to implement the non-deterministic behavior is assume that it is irrelevant the solution that the controller selects, and thus the reset control system would correctly performs for any chosen solution. This is the approach to be followed in this work, and thus it is necessary to remove all the defective solutions. A standard way to avoid the existence of defective solutions is to perform a time regularization of \(H^{cl}\), introducing a timer \(\tau \in [0, \infty)\) that prevent the system to perform two o more consecutive jumps, simply by initializing \(\tau\) to 0 after a jump, and avoiding to perform a new jump until \(\tau \geq \tau_m\), where \(\tau_m > 0\) is a design parameter (usually referred to as the minimum dwell-time).

A time-regularized reset control system \(H^{cl}_\tau\) is given by:

\[
H^{cl}_\tau : \begin{cases}
\dot{\tau} = 1, & (x, q, \tau) \in C^{cl}_\tau \\
\tau^+ = 0, & (x^+, q^+, \tau^+) = \left( A_R x, q, 0 \right), (x, q, \tau) \in D^{cl}_\tau
\end{cases}
\] (20)

where

\[
C^{cl}_\tau = C^{cl} \times [0, \infty) \cup D^{cl} \times [0, \tau_m]
\] (21)

and

\[
D^{cl}_\tau = D^{cl} \times [\tau_m, \infty).
\] (22)

The following property of of \(H^{cl}_\tau\) easily follows.

**Corollary 3.2.** For any \(\tau_m > 0\), \(H^{cl}_\tau\) is flow persistent and does not have Zeno solutions.

**Proof.** It trivially follows since for any \((x, q, \tau) \in D^{cl}_\tau\) it results \((x^+, q^+, \tau^+) = (A_R x, -q, 0) \in C^{cl}_\tau \setminus D^{cl}_\tau\), that is \(H^{cl}_\tau\) always jumps from \(D^{cl}_\tau\) to the interior of \(C^{cl}_\tau\), and then it flows for at least a time \(\tau_m > 0\). □
In principle, an election of a small value of the minimum dwell-time $\tau_m$ is all what is needed to prevent the existence of defective solutions. Note, however, that this does not avoid the existence of multiple solutions for some initial conditions, this is for example the case of points $(0,1,0)$ and $(0,-1,0)$. This non-deterministic behavior will be explored in the next example.

**Example 3.3.** Consider the reset control system of Fig. 6 composed by a Horowitz reset controller $R$ and a first-order plant. It will be analyzed its flow persistence for a exogenous input $w = r$ corresponding to a step reference (no disturbances are present), as well as the influence of $\tau_m$ on the reset control system’s solutions. For some $\tau_m > 0$, the time regularized reset control system is given by (20), with state $\mathbf{x} = (x_w, x_p, x_{r_1}, x_{r_2})$, and

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 4 & -4 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \quad A_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 & 0 & 0 \end{pmatrix}, \quad (23)$$

Here the reset controller output is $v = x_{r_1} + x_{r_2}$ and the error signal is $e = x_w - x_p$. The sets $C^{cl}$ and $D^{cl}$ are shown in Fig. 6 in the e-v planes corresponding to $q = 1$ and $q = -1$, as green and red regions, respectively. Note that the flow and jump sets are given by (21)–(22), and thus flowing is, in principle, possible in the set $D^{cl}$. In this example, an initial point $\xi = (x_0, 1, \tau_m)$ is considered, and thus the timer allows jumps at the initial instant.

Consider $\xi$ with $x_0 = (1,0,0,0)$. This corresponds to a unit step reference, and the controller and plant initially at rest. The fact that the reset control system is flow persistent means that there exist a solution that is unbounded in the $t$-direction; this solution is shown in Fig. 6 where jumps from $A$ to $B$, from $C$ to $D$, \cdots are visible (it is the unique solution for the initial point $A$). In this case,
the solution is not flowing in the set $D^{cl}$ since it always flows during a time larger than $\tau_m$ before jumps are enabled. This is true as far as $\tau_m < \tau_m^* \approx 1.4416$; in fact, without time regularization, the reset instants sequence is periodic with fundamental period $\tau_m^*$ after the second jump. This simple structure of the reset instants sequence is common in low-order reset control systems and in these cases time regularization would not be necessary for most initial points. However, there may exist initial points in which time regularization must be used to remove defective solutions, as it is discussed in the following.

For an initial point $\xi$ with $x_0 = (1,1,1,0)$, corresponding again to a unit step reference but now the controller output is initially $v = 1$ (with the CI initially at rest), it can be easily checked that $\xi \in C_{1r} \cap D_{r}^{cl}$; moreover, since $A x_0 = 0$ and $A_{1r} x_0 = x_0$ it directly follows that there exists an infinite number of solutions having one of the following hybrid time domains: $[0,\infty) \times \{0\}, [0,t_1) \times \{0\} \cup [t_1,\infty) \times \{1\}, [0,t_1) \times \{0\} \cup [t_1,t_2] \times \{1\} \cup [t_2,\infty) \times \{2\}, \cdots$, where $t_1 \in [0,\infty)$ and $t_{j+1} \in [t_j + \tau_m,\infty)$ for $j = 1,2,\cdots$. That is, there exists an only-flowing solution, and an infinite number of solutions that jumps a finite or infinite number of times. Note that all the solutions are unbounded in the $t$-direction and no Zeno solutions do exist, as far as $\tau_m > 0$. Finally, note that any solution $\phi$ satisfies $\phi(t,j) = \xi$ for any $(t,j) \in \text{dom} \, \phi$; informly speaking, all the solutions produce the same values of controller output and error.

### 3.3. Reset controllers with a sector resetting law

Although this work is focused on reset control systems with a zero crossing resetting law, it is instructive to analyze other resetting laws that has been developed in the literature. The sector resetting law was introduced in [37], and has been the main approach within the framework of hybrid inclusions, followed and also extended in several works ([38, 35, 39], · · ·).

A basic reset controller with a sector resetting law, and with a state $x_r$ and input $e$, is given by

$$
R : \begin{cases}
\dot{x}_r = A_r x_r + B_r e, & (x_r,e) \in C \\
x_r^+ = A_r x_r, & (x_r,e) \in D
\end{cases}
$$

(24)

where $C = \{(x_r,e) \in \mathbb{R}^{n_r+1} : ev \geq 0\}$ and $D = \{(x_r,e) \in \mathbb{R}^{n_r+1} : ev \leq 0\}$, being $v = C_r x_r$ the controller output. Thus, the basic jump set $D$ is a sector in the $e$-$v$ plane, consisting of its second and fourth quadrants. In combination with the plant (2) and the exosystems (14), and also including time-regularization, the resulting reset control system, with a sector resetting law, is given by

$$
\begin{cases}
\dot{\tau} = 1, & x = A x \\
\tau^+ = 0, & x^+ = A_{1r} x
\end{cases}
$$

(25)

where the closed-loop state is now $(x,\tau) = (x_w, x_p, x_r, \tau)$, and the flow and jump sets are also $C_{cl} = C^{cl} \times [0,\infty) \cup D^{cl} \times [0,\tau_m]$ and $D_{cl} = D^{cl} \times [\tau_m,\infty)$, respectively. But now the sets $C^{cl}$ and $D^{cl}$ defined by the sector resetting law, are given by

$$
C^{cl} = \{x \in \mathbb{R}^n : x^T M x \geq 0\}
$$

(26)

and

$$
D^{cl} = \{x \in \mathbb{R}^n : x^T M x \leq 0\}
$$

(27)

respectively, where

$$
M = \begin{pmatrix}
O & O & C^T_w C_r \\
O & O & -C^T_p C_r \\
C_r^T C_{w_1} & -C_r^T C_p & O
\end{pmatrix}
$$

(28)
Note that time-regularization may force solutions to flow in the jump set \( D^c \), or in the \( e-v \) plane to flow in the sector \( D \). It easily follows that this reset control system is flow persistent and that does not have defective solutions.

![Figure 7: Zero-crossing and sector resetting laws for Example 3.3: (top) Closed-loop outputs, (bottom) Control signals.](image)

**Example 3.4.** Consider again the reset control system of Fig. 6, with a zero crossing resetting law; and also a reset control system wit a sector resetting law with state \( (x, \tau) = (x_w, x_p, x_{r_1}, x_{r_2}, \tau) \), as given by (25)-(28), with \( A \) and \( A_R \) given by (23), and

\[
M = \begin{pmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & -1 & -1 \\
1 & -1 & 0 & 0 \\
1 & -1 & 0 & 0
\end{pmatrix}
\] (29)

Figures 7-8 show a simulation of both resetting laws. Fig. 7 shows the step responses, including closed-loop outputs and control signals. Note that there is an important difference in how both resetting laws performs jumps, specially in the case in which \( e < 0 \) and \( v > 0 \) and a jump is enabled. In this case, which corresponds for example to the first jump in Fig. 7, the zero crossing resetting law performs a jump to its flow set and then the solution flows until the next jump at \( t \approx 2.25 \), while the sector resetting law performs a jump to its jump set. This produces a chattering behavior of the sector resetting law, and in fact the obtained solution would be defective if time-regularization would not have been used. On the other hand, strictly speaking, time-regularization is not necessary for
this solution of the zero crossing resetting law, and the same solution is obtained with or without time-regularization (as far as $\tau_m < \tau_m^*$ -see Example 3.3). The control signals and its component are best analyzed in Fig. 8: note that after the first jump, in contrast with the zero crossing resetting law, the sector resetting law periodically reset its $x_{r2}$ state (with a fundamental period $\tau_m = 0.2$) until $t \approx 2.45$. This is the cause of its chattering, and of its bigger overshoot and undershoot in the step response. The undershoot is specially worst due to the fact that when the error signal changes its sign at $t \approx 2.45$, $x_{r2} \approx 0$ due to a recent reset. This example shows that the response of both resetting laws may be very different in general, and although in this case it is clear that the response of the zero-crossing resetting law is qualitatively better in terms of tracking error and control signal chattering, the situation may be different in another cases.

Figure 8: Zero-crossing and sector resetting laws for Example 3.3.

4. Stability analysis

The stability of the time-regularized reset control system $\mathcal{H}^c_T$ with no exogenous inputs is analyzed in the following. Reset times-dependent stability criteria will be developed, inspired from previous results developed in [3, 6]. The basic idea of this approach is to analyze stability by using a discrete-time system (a Poincaré-like map), that represents the sampling of $\mathcal{H}^c_T$ at the after-reset instants.

As it is usual in hybrid dynamical systems, stability is referred to sets instead of a single point. The following stability definitions are based on [17], note that they are applicable to continuous-time
or discrete-time systems as particular cases of hybrid systems. Consider a generic hybrid system $\mathcal{H}$ on $\mathbb{R}^n$. For a set $\mathcal{A} \subset \mathbb{R}^n$ and a vector $\phi \in \mathbb{R}^n$, the notation $\|\phi\|_\mathcal{A} = \min\{\|\phi - \psi\| : \psi \in \mathcal{A}\}$ indicates the distance of $\phi$ to $\mathcal{A}$. The set $\mathcal{A}$ is stable for $\mathcal{H}$ if for each $\epsilon > 0$ there exist $\delta > 0$ such that $\|\phi(0,0)\|_\mathcal{A} \leq \delta$ implies $\|\phi(t,j)\|_\mathcal{A} \leq \epsilon$ for all solutions $\phi$ to $\mathcal{H}$ and all $(t,j) \in \text{dom } x$. The set $\mathcal{A}$ is attractive if there exists a ball $\mathcal{B} \subset \mathbb{R}^n$ centered at the origin such that for any $\xi \in \mathcal{B}$ solutions $\phi$ to $\mathcal{H}^d$ with $\phi(0,0) = \xi$ converge to a set $\mathcal{A}$, that is $\|\phi(t,j)\|_\mathcal{A} \to 0$ as $t + j \to \infty$, where $(t,j) \in \text{dom } \phi$. The set $\mathcal{A}$ is asymptotically stable if it is stable and attractive, and its basin of attraction is $\mathcal{B}$. If the basin of attraction is $\mathbb{R}^n$ then $\mathcal{A}$ is globally asymptotically stable. In the case of the reset control $\mathcal{H}^d$, stability will be referred to the set $\mathcal{A}_0 = \{0\} \times \{1, -1\} \times [0, \infty)$.

For $\xi \in \mathbb{R}^n \times \{1, -1\} \times \{0\}$, and a solution $\phi = (x, q, \tau)$ to $\mathcal{H}^d$ with $\phi(0,0) = \xi$, and with $\text{dom } \phi = [0, t_1] \times \{0\} \cup [t_1, t_2] \times \{1\} \cup \cdots$, the reset intervals sequence $\tau_\phi$ is defined as $\tau_\phi = (\tau_1, \tau_2, \tau_3, \cdots)$, where $\tau_j = (t_j, j - 1)$ for $j = 1, 2, \cdots$, which corresponds to the values of the timer $\tau$ just before jumps are performed. Assume, by simplicity, that the timer $\tau$ in $\mathcal{H}^d$ is initially at rest and that $q(0,0) = 1$, otherwise jumps may always be performed at the initial instant to prepare the system for that state. Define by simplicity of notation the matrix $J$ as

$$J = \begin{pmatrix}
I_{(n-n_p) \times (n-n_p)} \\
O_{n_p \times (n-n_p)}
\end{pmatrix}$$

(30)

such as for any $z \in \mathbb{R}^{n-n_p}$, $Jz = \begin{pmatrix} z \\ 0 \end{pmatrix} \in \mathbb{R}^n$. It easily follows that $A_R = JJ^T$, $J = A_RJ$ and $J^T = J^T A_R$. Now, the mapping $I : \mathbb{R}^{n-n_p} \to \mathbb{R}_{\geq 0}$, is defined as

$$I(z) = \min\{\tau \geq \tau_m : Ce^{A\tau}Jz \geq 0\}$$

(31)

Moreover, the following standing assumption will operate in the rest of this work. It is assumed that there also exist an upper bound of $I$ in the cases in which the flow dynamics is unstable. Note that, otherwise, jumps would be unable to stabilize the flow dynamics.

**Assumption A:**

- The reset controller states to be reset and the timer are initially at rest, that is $x(0,0) = Jz_0$ for some $z_0 \in \mathbb{R}^{n-n_p}$, and $\tau(0,0) = 0$. In addition, $q(0,0) = 1$.

- Either the state matrix $A$ in $[20]$ is Hurwitz or the mapping $J$ is upper bounded (that is there exists $\Delta_M > 0$ such that $I(z) \leq \Delta_M$ for any $z \in \mathbb{R}^{n-n_p}$).

Here, a a Poincaré-like map, that gives the evolution from one after-jump state to the next one with a sign change, is postulated. By definition, the discrete-time system $\mathcal{D}\mathcal{H}^d$, with state $z \in \mathbb{R}^{n_p+n_p}$, is given by

$$z^+ = g(z) = -J^T e^{A\tau}Jz$$

(32)

Note that sign change allows, starting at $q = 1$, to obtain the successive reset intervals by using the mapping $I$, which does not explicitly depends on $q$. Also to obtain a simplified dynamic discrete system $\mathcal{D}\mathcal{H}^d$ in which the state $z$ only consists of the first $n - n_p$ values of $x = Jz$, and that will be perfectly valid to analyze the stability of the original hybrid control system as it will be seen in
the following. Some homogeneity properties of the maps $I$ and $g$ easily follow: for any $\lambda > 0$ it is true that

$$
\begin{align*}
(i) & \quad I(\lambda z) = I(z) \\
(ii) & \quad g(\lambda z) = \lambda g(z)
\end{align*}
$$

(33)

**Proposition 4.1.** The set $\mathcal{A}_0$ is (globally) asymptotically stable for the reset control system $\mathcal{H}_r^d$ if and only if the origin $\{0\}$ is (globally) asymptotically stable for the discrete-time system $\mathcal{D}\mathcal{H}_r^d$.

**Proof.** It is an adaptation of [3]-Prop. 3.1 to the hybrid formalism adopted in this work. According to Assumption A, and from an initial state $(x_0, q_0, \tau_0) \in \mathbb{R}^n \times \{1, -1\} \times \mathbb{R}$, the reset control system $\mathcal{H}_r^d$ is prepared (forcing jumps if necessary) to be in a state $\xi = (Jz_0, 1, 0) = ((z_0, 0), 1, 0)$, with $z_0 \in \mathbb{R}^{n_p+n_q}$, that is redefined to be the initial state. Also, consider a solution $\phi = (x, q, \tau)$ to $\mathcal{H}_r^d$ with $\phi(0, 0) = \xi$.

*(only if)* From the definition of $\mathcal{H}_r^d$ and its jump set, it follows that the values of $\phi$ at the after-reset instants (including the initial value) are given by

$$
\begin{align*}
\phi(0, 0) &= (Jz_0, 1, 0) \\
\phi(t_1, 1) &= (-Jz(1), -1, 0) \\
\phi(t_2, 2) &= (Jz(2), 1, 0) \\
&\vdots
\end{align*}
$$

(34)

If $\{0\}$ is not stable for $\mathcal{D}\mathcal{H}_r^d$ then there exists an $\varepsilon > 0$ such that $\|\phi(t_j, j)\| \in \mathcal{A}_0 = \|z(j)\| > \varepsilon$ for some $(t_j, j) \in \text{dom}\phi$. As a result, $\mathcal{A}_0$ is not stable for $\mathcal{H}_r^d$. On the other hand, if $\{0\}$ is not atractive for $\mathcal{D}\mathcal{H}_r^d$ then there will exist a sequence of values $\|\phi(t_j, j)\| \in \mathcal{A}_0 = \|z(j)\|$, for $j = 1, 2, \cdots$ which does not converge to zero, and thus $\mathcal{A}$ will not be atractive for $\mathcal{H}_r^d$.

*(if)* From the flow equation in (20) it follows that for $(t, j) \in \text{dom}\phi$

$$
\dot{x}(t, j) = x(t, j) + \int_{t_j}^{t} A x(s, j) ds
$$

(35)

and applying the Gronwall inequality (observing that the induced norm $\|A\|$ is always bounded by some real number $\alpha$ and that $\|Jz\| = \|z\|$) it directly follows that

$$
\|x(t, j)\| \leq \|x(t_j, j)\| e^{\alpha(t-t_j)} = \|(-1)^{j+1} Jz(j)\| e^{\alpha(t-t_j)} = \|z(j)\| e^{\alpha(t-t_j)}
$$

(36)

and thus

$$
\|\phi(t, j)\| \in \mathcal{A}_0 \leq \|z(j)\| e^{\alpha(t-t_j)}
$$

(37)

Since $\{0\}$ is stable for $\mathcal{D}\mathcal{H}_r^d$ it is true that there exists $\gamma > 0$ such as $\|z(j)\| \leq \gamma \|z_0\| = \gamma \|\xi\| \in \mathcal{A}_0$ and thus (37) results in

$$
\|\phi(t, j)\| \in \mathcal{A}_0 \leq \gamma e^{\alpha(t-t_j)} \|\xi\| \in \mathcal{A}_0
$$

(38)

Now, from assumption $A$ it follows that either $\alpha < 0$ or $t - t_j < \Delta_M$ and thus stability of the set $\mathcal{A}_0$ for $\mathcal{H}_r^d$ directly follows. Asymptotic stability is obtained from the fact that for the discrete-time system $\mathcal{D}\mathcal{H}_r^d$ it is true that $\|z(j)\| < \gamma \lambda^j \|z_0\|$, for $j = 1, 2, \cdots$, and for any $z_0 \in \mathbb{B}$ and $0 \leq \lambda < 1$
Proposition 4.2. for the map $A_{\phi}$, analogously, for a periodic time $t$ and a point $g_k$ of $g$, denote as $\Pi_k$ the Jacobian matrix of $g_k$ at $g_k$. Assume that $g_k$ is all eigenvalues of $\Pi_k$ of $g_k$, and if $\Pi_k$ is the smaller such positive integer; and the orbit of $g_k$ is a periodic-$k$ orbit, is defined to be the result of applying $g_k$ to the point $g_k(p)$ and $g_k$ is differentiable in a neighborhood $U$ of $p$, the Jacobian matrix of $G$ at $p$, the fixed point $p$ is called a periodic-$k$ point if $\Pi_k(p) = p$ and if $k$ is the smaller such positive integer; the orbit of $p$ with $k$ points, that is $\{p, \Pi_k(p), \ldots, \Pi_k(p)\}$, is called a periodic-$k$ orbit. For $k = 1$, $p$ is referred to as a fixed point. Assume that $\Pi_g$ is differentiable in a neighborhood $U$ of a fixed point $p$ and let $D\Pi_g(p)$ be the Jacobian matrix of $G$ at $p$; the fixed point $p$ is called a sink if $D\Pi_g(p)$ is a Schur matrix, and a source is all eigenvalues of $D\Pi_g(p)$ has a magnitude greater than 1. The stable manifold of $p$, denoted as $S(p)$, is the set of points $s \in S^{n-n_p-1}$ such that $\|\Pi_g(s) - p\| \to 0$ as $k \to \infty$. Analogously, for a periodic-$k$ point $p$, its periodic-$k$ orbit is a sink (source) if $p$ is a sink (source) for the map $\Pi_g$.

Proposition 4.2. Assume that the angle map $\Pi_g$ has a periodic-$k$ point $p$, being $\Pi_g^k$ differentiable in a neighborhood $U$ of $p$, and that its periodic-$k$ orbit is a sink with stable manifold $S(p) = S^{n-n_p-1}$.
Then, $\mathbf{p}$ is an eigenvector of the matrix

$$M_{\mathbf{p}} := J^T e^{A \cdot (H_{\mathbf{g}}^{-1}(\mathbf{p}))} \cdots A_{R_\mathbf{e}} e^{A \cdot (H_{\mathbf{g}}(\mathbf{p}))} A_{R_{\mathbf{e}}} e^{A \cdot (\mathbf{p})} J \tag{41}$$

corresponding to a real eigenvalue $\lambda_{\mathbf{p}}$, and the set $A_0$ is asymptotically stable for the reset control system $\mathcal{H}^d_{\tau}$, if and only if $|\lambda_{\mathbf{p}}| < 1$. Moreover, the basin of attraction of $A_0$ is $\mathbb{R}^n \times \{1, -1\} \times [0, \infty)$ (stability is global).

**Proof.** Consider the discrete system $\mathcal{D}_{\tau}$ as given by (32), $z_0 \in \mathbb{R}^{n-n_p}$, $z_k = g^k(z_0)$, and also $s_0 = \frac{z_0}{\|z_0\|}$, and $s_k = H_{\mathbf{g}}(s_0)$, for $k = 1, 2, \ldots$. From the homogeneity property (33) and (40) it directly follows that

$$I(z_k) = I(s_k) \tag{42}$$

for $k = 0, 1, 2, \ldots$. The proof is particularized for the case in which $\mathbf{p}$ is a fixed point of the angle map $H_{\mathbf{g}}$ (that is $k = 1$); for the cases $k = 2, 3, \ldots$, the proof is similar, using $H_{\mathbf{g}}^{-1}$ instead of $H_{\mathbf{g}}$ in the following reasoning. Now, define the matrix functions $M, \delta M : \mathbb{R}^{n-n_p} \to \mathbb{R}^{(n-n_p) \times (n-n_p)}$, such as $M(z) = J^T e^{A \cdot (s)} J$, and thus $M(\mathbf{p}) = M_{\mathbf{p}}$ as given by (41), and $\delta M(z) = M(z) - M_{\mathbf{p}}$.

Firstly, it will be shown that $\mathbf{p}$ is an eigenvector of $M_{\mathbf{p}}$. Since $\mathbf{p}$ is a sink with stable manifold the whole sphere, then it is true that $\|s_k - \mathbf{p}\| \to 0$ as $k \to \infty$, for any $s_0 \in S^{n-n_p}$. Moreover, since the mapping $I$ is continuous at $\mathbf{p}$ (otherwise $H_{\mathbf{g}}$ would not be differentiable), and thus the mapping $M$ is also continuous at $\mathbf{p}$, it also follows that $\|I(s_k) - I(\mathbf{p})\| \to 0$ and finally

$$\|M(s_k) - M_{\mathbf{p}}\| \to 0 \tag{43}$$

as $k \to \infty$, for any $s_0 \in S^{n-n_p}$. Then, by directly using (32) and (10), two points $s_k$ and $s_{k+1} = H_{\mathbf{g}}(s_k)$ are related by

$$s_{k+1} = -\frac{M(s_k)s_k}{\|M(s_k)s_k\|} \tag{44}$$

or equivalently

$$M(s_k)s_k = -\|M(s_k)s_k\| s_{k+1} \tag{45}$$

and finally for $k \to \infty$ it results that

$$M_{\mathbf{p}} \mathbf{p} = -\|M_{\mathbf{p}} \mathbf{p}\| \mathbf{p} \tag{46}$$

that is $\mathbf{p}$ is an eigenvector of $M_{\mathbf{p}}$ with eigenvalue $\lambda_{\mathbf{p}} := -\|M_{\mathbf{p}} \mathbf{p}\|$, a (non-positive) real number.

Secondly, consider the (unique) orthogonal decomposition of $z_k$ as

$$z_k = \alpha_k \mathbf{p} + \alpha_k \delta_k \mathbf{p}^\perp \tag{47}$$

where $\mathbf{p}^\perp$ is a vector perpendicular to $\mathbf{p}$, and $\alpha_k$ and $\delta_k$ are real numbers, for any $k = 1, 2, \ldots$. Since it is true that $\|s_k - \mathbf{p}\| \to 0$ as $k \to \infty$, then it directly follows that $\delta_k \to 0$ as $k \to \infty$. In addition, consider two points $z_k$ and $z_{k+1} = g(z_k)$, for $k = 1, 2, \ldots$. It results that

$$\frac{\|z_{k+1}\|}{\|z_k\|} = \frac{\|M(\mathbf{p}) + \delta M_k(\alpha_k \mathbf{p} + \alpha_k \delta_k \mathbf{p}^\perp)\|}{\|\alpha_k \mathbf{p} + \alpha_k \delta_k \mathbf{p}^\perp\|} \to \|M_{\mathbf{p}} \mathbf{p}\| = |\lambda_{\mathbf{p}}| \tag{48}$$

for $k \to \infty$, where by definition $\delta M_k = \delta M(z_k)$, and it has been used the fact that $\|\delta M_k\| \to 0$ as $k \to \infty$ (it easily follows from (43)).
that is

\[ \|z_k\| \leq \|M(z_{k-1})\| \cdot \|M(z_{k-2})\| \cdots \|M(z_N)\| \|M(z_{N-1})\| \cdots \|M(z_0)\| \|z_0\| \leq \lambda^{k-N} (\epsilon \|A\| \Delta M)^{N} \|z_0\| \]  

(49)

that is

\[ \|z_k\| \leq \gamma \lambda^k \|z_0\| \]  

(50)

for \( k \geq N \), where \( \gamma = (\epsilon \|A\| \Delta M / \lambda)^N \) is a constant. It is also true that \( \|z_k\| \leq (\epsilon \|A\| \Delta M)^k \|z_0\| \), for \( k < N \). It easily follows that the origin is globally asymptotically stable for \( \mathcal{DH}^c_\tau \) and Prop. 4.1 certifies global asymptotic stability of \( A_0 \) for the reset control system \( \mathcal{H}^c_\tau \).

(only if) Consider an initial condition \( z_0 = \alpha_0 p \), where \( \alpha_0 \) is a non-zero arbitrary number. On the other hand, \( M(cp) = M(p) \) for any constant \( c \). Thus, \( z_1 = -M(z_0)z_0 = -M(\alpha_0 p) \alpha_0 = -\alpha_0 M(p) p = -\alpha_0 \lambda_p p \), and it is obtained that \( z_k = (-1)^k \alpha_0 \lambda_p^k p \) for \( k = 1, 2, \cdots \). Stability of \( A_0 \) for \( \mathcal{H}^c_\tau \) implies stability of the origin for \( \mathcal{DH}^c_\tau \), and this implies that \( |\lambda_p| < 1 \). \( \square \)

Note that, in general, the angle map \( \Pi_\theta \) may exhibit several periodic-\( k \) points, not necessarily sinks, with different integer values of \( k \geq 1 \), each periodic point having its own stable manifold. For example, in the case in which the point is a source the stable manifold is the point itself. The sinks, with different integer values of \( A \), here \( R \)

For the Example 3.3 without exogenous inputs, the reset control system \( \mathcal{H}^c_\tau \) is given by the matrices \( A, A_R \) and \( C \) obtained by removing the first column and the first row of the matrices in [23], that is

\[ A = \begin{pmatrix} -1 & 1 & 1 \\ -4 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 & 0 \end{pmatrix}, \]  

(53)

Here \( A \) has eigenvalues \( \lambda_{1,2} = -\frac{1}{2} \pm j \frac{\sqrt{15}}{2} \) and \( \lambda_3 = 0 \). After preparing \( \mathcal{H}^c_\tau \) according to Assumption A, an oscillatory error signal is always produced; moreover, it is not difficult to obtain that \( \Delta_M = \)
\[ \frac{2\pi}{\sqrt{19}} + \tau_m \] (it is the value of the minimum dwell-time added to half oscillation of the error signal). In this case \( n - n_p - 1 = 1 \), and thus \( \Pi_g \) maps values of \( s \) in the unit circle \( S^1 \). Moreover, by using the parametrization \( s = [\cos(\theta), \sin(\theta)] \), for \( \theta \in [-\pi, \pi] \), it is possible to redefine the angle map (with some abuse of notation) as \( \Pi_g : [-\pi, \pi] \to [-\pi, \pi] \), and \( \vartheta^* = \Pi_g(\theta) \). Fig. 9 shows the graph of \( \Pi_g \) for two values of \( \tau_m \).

For \( \tau_m = 0.25 \), \( \Pi_g \) has a jump discontinuity at the point \( \theta = d \); this point corresponds to the value of \( \theta \) such that \( C e^{A\tau_m}[\cos(\theta), \sin(\theta), 0] = 0 \), and it can be obtained is a closed-form after some computation, it is given by \( d = \pi + \frac{1}{2}(1 - \frac{\sqrt{70}}{\tan(\frac{\pi}{3})}) \). Moreover, \( \Pi_g \) has a unique fixed point at \( \theta = p = \pi/2 \), which is a sink with a basin of attraction \( S(\pi/2) = [-\pi, \pi] \). The direct application of Prop. 4.2 results in that the reset control system \( H^e \) is globally asymptotically stable, since \( I(p) = I(\pi/2) = 2\pi/\sqrt{19} \) and

\[
M_p = J^T e^{A-I(p)}J = \begin{pmatrix} -0.4863 & 0 \\ 0 & -0.1891 \end{pmatrix}
\] (54)

has an eigenvector \((1, 0)\) (corresponding to \( p = \pi/2 \)) with eigenvalue \(-0.4863\).

For \( \tau_m = 1.35 \), the mapping \( \Pi_g \) exhibits a more complex structure (Fig. 9). Besides a jump discontinuity at \( \theta = d = -\pi + \frac{1}{2}(1 - \frac{\sqrt{70}}{\tan(\frac{\pi}{3})}) \), \( \Pi_g \) has three fixed points: \( p_1 \approx 0.5322 \), \( p_2 \approx 1.4246 \), and \( p_3 = \pi/2 \). Both \( p_1 \) and \( p_3 \) are sinks, while \( p_2 \) is a source. Their basins of attraction are \( S(p_1) = [d, p_2), S(p_2) = \{p_2\} \) and \( S(p_3) = [-\pi, d) \cup (p_2, \pi] \). Finally, \( I(p_1) = I(p_2) = \tau_m = 1.35 \) and \( I(p_3) = 2\pi/\sqrt{19} \). As a result, since \( S(p_1) \cup S(p_2) \cup S(p_3) = [-\pi, \pi] \), and the three matrices

\[
M_{p_1} = J^T e^{A-I(p_1)}J = \begin{pmatrix} -0.5222 & 0.0463 \\ -0.1850 & -0.1808 \end{pmatrix},
\] (55)

\( M_{p_2} = M_{p_1} \), and \( M_{p_3} = A R e^{A-I(p_3)} = M_p \) (as given in (54)) are Schur matrices (and thus all eigenvalues are strictly inside the unit circle), it also follows that \( H^e \) is globally asymptotically stable for \( \tau_m = 1.35 \).

It turns out that the fixed points and periodic point patterns are heavily influenced by the minimum dwell-time \( \tau_m \) as it should be expected. Although, in principle, in control practice \( \tau_m \) is initially used to avoid defective solutions and a small value of it is all what is needed (and following the above analysis is not difficult to show that \( H^e \) is globally asymptotically stable for any \( \tau_m < 1.35 \), it is illustrative to analyze how the periodic point patterns change with it. Although an exhaustive analysis is out of scope of this work and it will be given elsewhere, there are several bifurcation points delimiting zones with one sink, with two sinks plus a source, with periodic-2 sinks, with periodic-3 sinks, etc. To conclude the example, a case with periodic-3 sinks is considered in the following.

For \( \tau_m = 2 \), there are three periodic-3 points (Fig. 10). They are \( p_1 \approx -1.5494, p_2 \approx -0.2162 \), and \( p_3 = \pi/2 \). For \( p_1 \), its periodic-3 orbit \( \{p_1, \Pi_g(p_1), \Pi_g^2(p_1)\} = \{p_1, p_3, p_2\} \) is a sink and, in addition, \( I(p_1) \approx 2.9042 \), and \( I(p_3) = I(p_2) = \tau_m = 2 \), and

\[
M_{p_1} = J^T e^{A-I(p_2)}A R e^{A-I(p_3)}A R e^{A-I(p_1)}J = 10^{-2} \begin{pmatrix} -2.2711 & 0.0337 \\ 0.0011 & -3.8597 \end{pmatrix},
\] (56)

is a Schur matrix. For \( p_2 \) and \( p_3 \), its periodic-3 orbits are \( \{p_2, p_1, p_3\} \) and \( \{p_3, p_2, p_1\} \), respectively. They are also sinks, and the matrices \( M_{p_2} = J^T e^{A-I(p_3)}A R e^{A-I(p_1)}A R e^{A-I(p_2)}J \) and \( M(p_3) = 20 \)
Figure 9: Graphs of the maps $\Pi_g$ (top) and $I$ (bottom) for the reset control system of Example 3.3: (left) $\tau_m = 0.25$, (right) $\tau_m = 1.35$.

$J^T e^{A \cdot I(p_1)} A_R e^{A \cdot I(p_2)} A_R e^{A \cdot I(p_3)} J$ can be easily checked to be Schur matrices. Finally, applying Corollary 4.3, using the fact that the union of the three basins of attraction is $[-\pi, \pi]$, it results that $H^{cl}_\tau$ is globally asymptotically stable for $\tau_m = 2$.

4.3. A case with a chaotic sequence of reset intervals

Obviously, Prop. 4.2-Corol. 4.3 are convenient in practice for those cases in which periodic orbits of $\Pi_g$ can be found with a reasonable effort (for a given value of $\tau_m$), like in the cases analyzed above. Although, in general, the result is useful in many practical cases, even some low-order reset control systems may exhibit extraordinarily complex interval patterns that makes elusive its application in those cases, motivating the investigation of alternative stability criteria. In the following, it is analyzed a reset control system consisting of a FORE and a third order plant, that produces chaotic sequences of reset intervals. Consider the time-regularized reset control system
Figure 10: Reset control system of Example 3.3 for $\tau_m = 2$: (top) Graph of the mapping $\Pi^3_g$ (left), showing the periodic-3 points $p_1$, $p_2$, and $p_3$, and graph of $\Pi_g$ (right); (bottom) graph of the map $I$ (left), and plot showing the periodicity of the reset intervals (right), where $I_3 = I(p_3) = 2$, $I_2 = I(p_2) = 2$, and $I_1 = I(p_1) \approx 2.9042$.

$H_r$, as given by (20), with no exogenous inputs and with $\tau_m = 0.1$, and with:

$$A = \begin{pmatrix} 0 & 0 & 3.5 & 5 \\ 1 & 0 & -4.3 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}, \quad A_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & -1 & 0 \end{pmatrix}, \quad (57)$$

This reset control system, with state $(x, q, \tau) = (x_p, x_r, q, \tau)$, is the feedback composition of a FORE with state $(x_r, q)$ and a third order plant with state $x_p = (x_1, x_2, x_3)$. Here, the mapping $\Pi_g : S^2 \to S^2$ has an invariant set $\mathcal{T} = \{(\frac{t}{\sqrt{1+t^2}}, \frac{1}{\sqrt{1+t^2}}, 0) : t \in [-3, 4]\}$, that is $\Pi_g(\mathcal{T}) \subset \mathcal{T}$.

And thus, $\Pi_g$ can be parameterized (with some abuse of notation) as $\Pi_g : [-3, 4] \to [-3, 4]$, where $t^+ = \Pi_g(t)$ (see Fig. 11).

For this case, $\Pi_g$ is a continuous mapping on $[-3, 4]$, its graph is shown in Fig. 11. Moreover, the mapping $I : [-3, 4] \to [0.1, \infty)$, such as $I(t)$ (also with some abuse of notation) defines the reset interval from the point $s = (\frac{t}{\sqrt{1+t^2}}, \frac{1}{\sqrt{1+t^2}}, 0)$. Since the mapping $\Pi_g$ is continuous on the interval $[-3, 4]$ and has periodic-3 points (see Fig. 11), then it turns out that there exists solutions.
Figure 11: An example of reset control system with a chaotic sequence of reset intervals: graphs of the mapps $\Pi_g$, $\Pi^2_g$, and $\Pi^3_g$ (with marks at the periodic points), and $I$.

with any possible integer period according to Sharkovskii theorem ([34], [2]), that is there exists periodic-$k$ points for $k = 1, 2, 3, \cdots$. For example, Fig. [11] shows the graphs of $\Pi_g$, $\Pi^2_g$, and $\Pi^3_g$, explicitly marking 1 periodic-1 point (a fixed point), 2 periodic-2 points, and 6 periodic-3 points. Note that all the periodic points are sources, and in fact this is the case for any periodic point since, as it is well known, period 3 implies chaos [25]. As a result, any initial point of $H_{cl}^\tau$, with $\xi \in \left(\frac{t}{\sqrt{1+t^2}}, \frac{1}{\sqrt{1+t^2}}, 0, 0\right) \times \{1\} \times \{0\}$, and $t \in [-3, 4]$, will produce solutions with chaotic sequences of reset intervals, making elusive to apply Prop. 4.2.

4.4. Reset-times dependent stability conditions: Minimum dwell-time

An alternative approach to stability analysis of $H_{cl}^{\tau}$ will be based on the use of Lyapunov functions, with an explicit consideration of the reset interval sequences $\tau_\phi = \{\tau_1, \tau_2, \cdots\}$ corresponding to solutions $\phi = (x, q, \tau)$ to $H_{cl}^{\tau}$. This approach is based on the reset-times dependent stability criteria early developed in [6, 3]. The set of all possible reset interval sequences is defined as

$$S_{H_{cl}^{\tau}} = \{\tau_\phi = \{\tau_1, \tau_2, \cdots\} \subset \mathbb{R}_{\geq 0} : \tau_i = \tau(t_i, i), (t_i, i) \in \text{dom } \phi, \phi \text{ is a solution to } H_{cl}^{\tau}\}$$  \hspace{1cm} (58)
In the case in which $A$ is a Hurwitz matrix, a first strategy consists in embedding the set of reset intervals sequences in a larger set characterized by the minimum dwell-time associated to $\mathcal{H}_\tau$. It is defined the set $S_{\tau_m}$ as

$$S_{\tau_m} = \{\{\tau_1, \tau_2, \cdots\} \subset \mathbb{R}_\geq 0 : \tau_i \geq \tau_m\}$$

(59)

and then stability conditions are considered for any possible reset interval sequence in $S_{\tau_m}$. This approach will allow a direct application of computationally efficient methods imported from the impulsive systems literature. And although, in principle, results may be conservative due to the fact that $S_{\mathcal{H}_\tau}$ is a meager set compared to $S_{\tau_m} \supset S_{\mathcal{H}_\tau}$, in practice they allow to obtain a first value of $\tau_m$ for which stability is guaranteed.

**Proposition 4.4.** The set $A_0$ is globally asymptotically stable for $\mathcal{H}_\tau$ if there exist a sequence of positive definite matrices $\{P_1, P_2, \cdots\}$, such that

$$\eta I \leq P_k \leq \rho I$$

$$e^{AT_\tau_k} A_R P_{k+1} A_R e^{AT_\tau_k} - P_k \leq -\varepsilon I$$

(60)

hold for $k = 1, 2, \cdots$, for some positive constants $\eta$, $\rho$, and $\varepsilon$, and any $\{\tau_1, \tau_2, \cdots\} \in S_{\tau_m}$.

**Proof.** Firstly, it is a standard result ([31]) that if (60) hold then the time-dependent quadratic Lyapunov function $V(x, k) = x^T P_k z$ certifies that every discrete-time (time-varying) system

$$\begin{cases} 
 z^+ = -J^T A_R e^{AT_\tau_k} J z \\
 k^+ = k + 1
\end{cases}$$

(61)

with $\{\tau_1, \tau_2, \cdots\} \in S_{\tau_m}$, is globally asymptotically stable.

Now, since it is true that $I(z) > \tau_m$ for any $z \in \mathbb{R}^{n_p+n_\hat{p}}$ (it directly follows from [31]), and thus the solution $z$ to $D\mathcal{H}_\tau$ with $z(0) = z_0$ corresponds with the solution to a discrete-time system like (61) with $\{\tau_1, \tau_2, \tau_3, \cdots\} = \{I(z_0), I(g(z_0)), I((g^2(z_0)), \cdots\} \in S_{\tau_m}$, then it follows from (60) that (see [31], Th. 23.3)

$$||z(k)||^2 = ||g^k(z_0)||^2 \leq \frac{D}{\eta} \lambda^{2k} ||z_0||^2$$

(62)

for $k \geq 0$, where $\lambda < 1$. Since the constants $\rho$, $\eta$, and $\lambda$ do not depend on $z_0$ then it directly follows that the origin $\{0\}$ is globally asymptotically stable for the discrete time system $D\mathcal{H}_\tau$. Application of Prop. 4.1 ends the proof. □

Prop. 4.4 gives a nice and simple connection between stability of the reset control system $\mathcal{H}_\tau$ and stability of impulsive systems with impulses at fixed instants, since condition (60) can be easily linked with the stability of an impulsive system with impulses at instants $t_k = t_{k-1} + \tau_k$, $k = 1, 2, \cdots$. Moreover, the following Corollary directly follows.

**Corollary 4.5.** Assume that (60) hold for $k = 1, 2, \cdots$, for $\eta$, $\rho$, $\varepsilon > 0$, and for any $\{\tau_1, \tau_2, \cdots\} \in S_{\tau_m}$. Then the set $A_0$ is globally asymptotically stable for $\mathcal{H}_\tau$, for any $\tau_m \geq \tau^*_m$. 24
A particular simple instance of (60) is to consider a time independent Lyapunov function, that is $P_k = P$, for any $k = 1, 2, \ldots$. In this case, the procedure for solving (60) is reduced to searching for a matrix $P > 0$ such as

$$e^{AT \tau} A R P A R e^{A \tau} - P \leq -\varepsilon I$$

for some $\varepsilon > 0$ and any $\tau \geq \tau_m$ (or $\tau \geq \tau^*_m$). This problem is well-known in the literature and there exists several good methods for its resolution ([4, 6, 15], [12]). The next result is directly imported from [12].

**Corollary 4.6.** Assume that one of the following conditions applies:

- There exist a differentiable matrix function $R : [0, \tau^*_m] \rightarrow S^n$, $R(0) > 0$, and $\varepsilon > 0$ such that

$$\begin{align*}
AT(R(\theta) + R(\theta)A + \dot{R}(\theta)) &\leq 0, \\
AR(R(0)A - R(\tau^*_m)) &\leq -\varepsilon I,
\end{align*}$$

(64)

hold for any $\theta \in [0, \tau^*_m]$.

- There exist a differentiable matrix function $S : [0, \tau^*_m] \rightarrow S^n$, $S(0) > 0$, and $\varepsilon > 0$ such that

$$\begin{align*}
AT(S(\tau^*_m)) + S(\tau^*_m)A &< 0, \\
AT(S(\theta)) + S(\theta)A - \dot{S}(\theta) &\leq 0, \\
AR(S(\tau^*_m)A - S(0)) &\leq -\varepsilon I,
\end{align*}$$

(65)

hold for any $\theta \in [0, \tau^*_m]$.

then the set $A_0$ is globally asymptotically stable for $H_{cl}^{\tau}$, for any $\tau_m \geq \tau^*_m$.

**Proof.** It is a direct application of Th. 2.3 in [12], and Prop. 4.3-Corollary 4.4, for the case given by [63]. $\square$

This Corollary results in an efficient procedure for solving the $H_{cl}^{\tau}$ stability problem, specifically when the matrix function $R$ (or $S$) is searched in the set of matrix polynomials with a given degree $d_R$, that is $R(\theta) = \sum_{i=0}^{d_R} R_i \theta^i$, $R_i > 0$. In this case, (64) and (65) may be inserted in sum-of-squares conditions, and the problem can be solved by using some sum-of-squares programming package (see e.g. [29]).

**Example 4.7.** This is a classical example of reset control system, considered in early works about reset control like [11]. It consists of a feedback interconnection of a FORE and a second order plant. Here, it is defined as a time-regularized reset control system $H_{cl}^{\tau}$ with

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -0.2 & 1 \\ 0 & -1 & -1 \end{pmatrix}, \quad AR = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 & 0 \end{pmatrix}$$

(66)

and a minimum dwell-time $\tau_m$. Since $A$ is a Hurwitz matrix, in fact it has two complex dominant eigenvalues at $-\frac{1}{10} \pm j\frac{2\sqrt{11}}{10}$, stability of the set $\{0\} \times \{-1, -1\} \times [0, \infty)$ will be investigated using Corollary 4.6. After working with SOSTOOLS, a value of $\tau^*_m = 0.6145$ is obtained, and thus global asymptotic stability is guaranteed for any $\tau_m \geq 0.6145$. 

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As it is expected, result is somehow conservative. In this case, application of Prop. 4.2 for values \(0 < \tau_m \leq 0.6145\) results in that there exist only one fixed point \(p = (-1, 0)\) of the mapping \(\Pi_g\) (and no other periodic points), and the corresponding matrix \(M_p = J^T e^{A^T \beta} J\), with \(\beta = 3 \sqrt{11/10}\), is a Schur matrix. Thus, Prop. 4.2 certifies global asymptotic stability of \(H_{cl} \tau_m\) for smaller values of \(\tau_m\) in comparison to Corollary 4.6. Interestingly, for values \(\tau_m > \frac{3}{\beta} \approx 3.1574\), application of Prop. 4.2 is less direct. For increasing values of \(\tau_m\) they appear two fixed points (one sink and a source), periodic-2 points, etc. (details are omitted by brevity).

4.5. Reset-times dependent stability conditions: Ranged dwell-time

If \(A\) is not Hurwitz then it is necessary to include a ranged dwell-time \(\tau \in [\tau_m, \tau_M]\), enabling to flow only when \(\tau \leq \tau_M\). In this case, it is considered the reset control system \(\mathcal{H}^{\tau}_{cl}\), which is a modification of \(\mathcal{H}^{\tau}_{cl}\) including the maximum dwell-time \(\tau_M\). \(\mathcal{H}^{\tau}_{cl}\) is given by

\[
\mathcal{H}^{\tau}_{cl} : \begin{cases}
\dot{\tau} = 1, \\
\dot{x} = Ax
\end{cases}, \quad (x, q, \tau) \in \mathcal{C}^{\tau}_{cl}
\]

where

\[
\mathcal{C}^{\tau}_{cl} = \mathcal{C}^{cl} \times [0, \tau_M] \cup \mathcal{D}^{cl} \times [0, \tau_m]
\]

and

\[
\mathcal{D}^{\tau}_{cl} = \mathcal{C}^{cl} \times [\tau_M, \infty) \cup \mathcal{D}^{cl} \times [\tau_m, \infty).
\]

Now, it is defined the set of time sequences \(S_{[\tau_m, \tau_M]}\), given by

\[
S_{[\tau_m, \tau_M]} = \{\{\tau_1, \tau_2, \cdots \} \subset \mathbb{R}_0 : \tau_i \in [\tau_m, \tau_M]\}
\]

and it is clear that \(S_{\mathcal{H}^{\tau}_{cl}} \subset S_{[\tau_m, \tau_M]}\), being \(S_{\mathcal{H}^{\tau}_{cl}}\) defined in a way similar to (58). The next Proposition is a direct adaptation of Prop. 4.4 and Corollary 4.5 to \(\mathcal{H}^{\tau}_{cl}\).

**Proposition 4.8.** The set \(\mathcal{A}_0\) is globally asymptotically stable for \(\mathcal{H}^{\tau}_{cl}\), with \([\tau_m, \tau_M] \subset [\tau^*_m, \tau^*_M]\), if there exist a sequence of positive definite matrices \(\{P_1, P_2, \cdots \}\), such that

\[
\eta I \leq P_k \leq \rho I
\]

\[
e^{A^T \tau_k} A R P_{k+1} A R e^{A \tau_k} - P_k \leq -\varepsilon I
\]

hold for \(k = 1, 2, \cdots\), for some positive constants \(\eta, \rho, \varepsilon\), and any \(\{\tau_1, \tau_2, \cdots \} \in S_{[\tau_m, \tau_M]}\).

Again, if it is considered a sequence of constants matrices \(P_k = P > 0\) in Prop. 4.8, then it is possible to apply some efficient methods already developed in the literature. For example, in [3] a method for obtaining a set of intervals \([\tau_m, \tau_M]\) is developed. Also in [12] a method based on sum-of-squares conditions is given for this case of ranged dwell-time; the following Corollary is directly based on it.
Corollary 4.9. Assume that there exist a differentiable matrix function $R : [0, \tau_M] \to \mathbb{S}^n$, $R(0) > 0$, and $\varepsilon > 0$ such that
\begin{align}
A^T R(\theta) + R(\theta) A + \dot{R}(\theta) & \leq 0, \\
AR(0)AR - R(\tau) & \leq -\varepsilon I,
\end{align}
(72)
hold for any $\theta \in [0, \tau^*_M]$ and any $\tau \in [\tau^*_m, \tau^*_M]$. Then the set $A_0$ is globally asymptotically stable for \(\mathcal{H}^{cl}_\tau\) and for any $[\tau_m, \tau_M] \subset [\tau^*_m, \tau^*_M]$.

Example 4.10. Consider again the reset control system with the Horowitz reset controller (Fig. 6), described in section 4.2. Note that the matrix $A$ is not Hurwitz, since it has an eigenvalue in the closed right half plane. Thus, the reset control system $\mathcal{H}^{cl}_\tau$ is used, forcing jumps when $\tau > \tau_M$. Here, it is found that (72) is feasible for $\tau^*_m = 0.1$ and $\tau^*_M = 37.5$ (the sum of squares tool SOSTOOLS, with a polynomial matrix function $R(\theta) = \sum_{i=0}^6 R_i \theta^i$ of degree 6, has been used). Note that the result is somehow conservative: firstly it can be easily shown that stability is also obtained for $\tau_m$ arbitrarily small (see section 4.2); secondly, it is necessary to include a maximum-dwell time $\tau_M < \tau^*_M = 37.5$ which is not necessary in the original reset control system $\mathcal{H}^{cl}_\tau$. In spite of that, note that in this case the reset control system $\mathcal{H}^{cl}_\tau$ performs jumps with reset intervals upper bounded by $\Delta_M = \frac{2\pi}{\sqrt{19}} + \tau_m$, that for $\tau_m = 0.1$, a reasonable value in practice to avoid defective solutions, results in $\Delta_M \approx 1.54$, far below the limit given by $\tau^*_M = 37.5$. In other words, this result guarantees not only that $\mathcal{H}^{cl}_\tau$ is globally asymptotically stable for $[\tau_m, \tau_M] = [0.1, 37.5]$, but also that $\mathcal{H}^{cl}_\tau$ is globally asymptotically stable for $\tau_m = 0.1$.

5. Conclusions

A new model of Clegg integrator, with an attached error zero-crossing mechanism, has been developed. This results in a reset controller model in the hybrid inclusions framework, enabling to equip the resulting reset control system with good structural properties like robustness against measurement noise and robustness in the stability. The manuscript has been focused on analysis of well-posedness and stability, adapting and extending previous work of the authors to the new reset model. More specifically, stability has been approached by analyzing stability of a Poincaré-like map, using two paths: a test based on the eigenvalues of matrices related with periods of reset interval sequences, and quadratic Lyapunov functions-based sufficient conditions. Both approaches have been analyzed in detail, including several examples. Although checking eigenvalues is a simple and efficient test to analyze stability, its applicability depends on the computation of a finite number of periodic points; as an interesting result, it has been formally shown that this is not always possible since in some cases reset intervals may produce chaotic sequences. As an alternative, Lyapunov function-based results may be applied: different results has been obtained for the case in which the base control system is stable or unstable. As a final conclusion, it is believed that the manuscript gives a solid framework for reset control systems with a zero-crossing resetting law, that may serve as a basis for new theoretical and practical advances.

Acknowledgments

It is gratefully acknowledged the helpful comments of Andrew R. Teel on an early version of the reset controller model developed in this work.
References

[1] Aangenent, W., Witvoet, G., Heemels, W., van de Molengraft, M., Steinbuch, M., 2009. Performance analysis of reset control systems. International Journal of Robust and Nonlinear Control 20, 1213–1233.

[2] Alligood, K. T., Sauer, T. D., Yorke, J. A., 2000. Chaos: an introduction to dynamical systems. Springer.

[3] Baños, A., Barreiro, A., 2012. Reset Control Systems. Springer, London.

[4] Baños, A., Barreiro, A., 2012. Reset Control Systems. AIC Series. Springer, London.

[5] Baños, A., Carrasco, J., Barreiro, A., 2007. Reset-times dependent stability of reset control systems with unstable base systems. In: IEEE (Ed.), Proc. IEEE International Symposium on Industrial Electronics. pp. 163–168.

[6] Baños, A., Carrasco, J., Barreiro, A., 2011. Reset times-dependent stability of reset control systems. IEEE Transactions on Automatic Control 56, 217–223.

[7] Baños, A., Davó, M. A., 2014. Tuning of reset proportional integral compensators with a variable reset ratio and reset band. IET Control Theory and Applications 8 (1949-1962).

[8] Baños, A., Mulero, J. I., 2012. Well-posedness of reset control systems as state-dependent impulsive dynamic systems. Abstract and Applied Analysis 2012, 1–16.

[9] Baños, A., Mulero, J. I., Barreiro, A., Davó, M. A., 2016. An impulsive dynamical systems framework for reset control systems. International Journal of Control 89 (10), 1985–2007.

[10] Barreiro, A., Baños, A., Dormido, S., González-Prieto, J. A., 2014. Reset control systems with reset band: well-posedness, limit cycles and stability analysis. Systems and Control Letters 63, 1–11.

[11] Beker, O., Hollot, C. V., Chait, Y., Han, H., 2004. Fundamental properties of reset control systems. Automatica 40, 905–915.

[12] Briat, C., 2013. Convex conditions for robust stability analysis and stabilization of linear aperiodic impulsive and sampled-data systems under dwell-time constraints. Automatica 49, 2449–2457.

[13] Cai, C., Teel, A. R., 2009. Characterizations of input-to-state stability for hybrid systems. Systems and Control Letters 58, 47–53.

[14] Clegg, J. C., 1958. A nonlinear integrator for servomechanisms. AIEE Transactions, Applications and Industry 77, 41–42.

[15] Dashkovskiy, S., Mironchenko, A., 2013. Input-to-state stability of nonlinear impulsive systems. SIAM Journal on Control and Optimization 51 (3), 1962–1987.

[16] de Schutter, B., Heemels, W. P. M. H., Lunze, J., Prieur, C., 2009. Survey of modeling, analysis, and control of hybrid systems. In: Lunze, J., Lamnabhi-Lagarrigue, F. (Eds.), Handbook of Hybrid Systems Control. Cambridge University Press, Cambridge, pp. 31–55.

[17] Goebel, R., Sanfelice, R. G., Teel, A. R., 2009. Hybrid dynamical systems. IEE Control Systems Magazine 29, 28–93.

[18] Goebel, R., Sanfelice, R. G., Teel, A. R., 2012. Hybrid Dynamical Systems: Modeling, Stability, and Robustness. Princeton University Press.

[19] Goebel, R., Teel, A. R., 2006. Solutions to hybrid inclusions via set and graphical convergence with stability theory applications. Automatica 42 (4), 573–587.

[20] Guo, Y., Wang, Y., Xie, L., Li, H., Gi, W., 2011. Optimal reset law design and its application to transient response improvement of HDD systems. IEEE Transactions Control Systems Technology 19, 1160–1167.

[21] Guo, Y., Wang, Y., Xie, L., Zheng, J., 2008. Stability analysis and design of reset systems: theory and application. Automatica 45, 492–497.

[22] Horowitz, I. M., Rosenbaum, P., 1975. Nonlinear design for cost of feedback reduction in systems with large parameter uncertainty. International Journal of Control 24, 977–1001.

[23] Hosseinnia, S. H., Tejado, I., Vinagre, B., 2013. Fractional-order reset control: application to a servomotor. Mechatronics 23 (7), 781–788.

[24] Krishman, K. R., Horowitz, I. M., 1974. Synthesis of a nonlinear feedback system with significant plant-ignorance for prescribed system tolerances. International Journal of Control 19, 689–706.

[25] Li, T. Y., Yorke, J. A., 1975. Period three implies chaos. American Mathematical Monthly 82 (10), 985–992.

[26] Luo, A. C. J., 2012. Regularity and Complexity in Dynamical Systems. Springer.

[27] Nesic, D., Teel, A. R., Zaccarian, L., 2011. Stability and performance of siso control systems with first order reset elements. IEEE Transactions on Automatic Control 56, 2567–2582.

[28] Nesic, D., Zaccarian, L., Teel, A. R., 2008. Stability properties of reset systems. Automatica 44, 2019–2026.

[29] Prajna, S., Papachristodoulou, A., Seiler, P., Parrilo, P. A., 2004. SOSTOOLS: sum of squares optimization toolbox for MATLAB.

[30] Prieur, C., Tarbouriech, S., Zaccarian, L., 2013. Lyapunov-based hybrid loops for stability and performance of continuous-time control systems. Automatica 49 (2), 577–584.
Rugh, W. J., 1996. Linear Systems Theory. Prentice-Hall, New Jersey.

Saikumar, N., 2021. Loop-shaping for reset control systems: a higher-order sinusoidal describing function approach. Control Engineering Practice 111.

Sanfelice, R., 2013. On the existence of control lyapunov functions and state-feedback laws for hybrid systems. IEEE Transactions on Automatic Control 58 (12), 3242–3248.

Sharkovskii, A. N., 1964. Co-existence of cycles of a continuous mapping of the line into itself. Ukrainian Math. Journal 16, 61–71.

Tarbouriech, S., Loquen, T., Prieur, C., 2011. Anti-windup strategy for reset control systems. International Journal of Robust and Nonlinear Control 21 (10), 1159–1177.

van Loon, S. J. L. M., Gruntjens, K. G. J., Heertjes, M. F., van de Wouw, N., Heemels, W. P. M. H., 2017. Frequency-domain tools for stability analysis of reset control systems. Automatica 82, 101–108.

Zaccarian, L., Nesic, D., Teel, A. R., 2005. First order reser element and the clegg integrator revisited. In: Proceedings of the American Control Conference. Vol. 1. pp. 563–568.

Zaccarian, L., Nesic, D., Teel, A. R., 2011. Analytical and numerical Lyapunov functions for SISO linear control systems with first-order reset elements. International Journal of Robust and Nonlinear Control 21, 71–76.

Zhao, G., Hua, C., 2017. Improved high-order reset element model based on circuit analysis. IEEE Transactions on Circuits and Systems-II: Express Briefs 64 (4), 432–436.