ON THE STABILITY OF SHOCKS WITH PARTICLE PRESSURE

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ABSTRACT

We perform a linear stability analysis for corrugations of a Newtonian shock, with particle pressure included, for an arbitrary diffusion coefficient. We study first the dispersion relation for homogeneous media, showing that, besides the conventional pressure waves and entropy/vorticity disturbances, two new turburation modes exist, dominated by the particles’ pressure and damped by diffusion. We show that, due to particle diffusion into the upstream region, the fluid will be perturbed also upstream; we treat these perturbation in the short-wavelength (WKBJ) regime. We then show how to construct a corrugational mode for the shock itself, one that is, where the shock executes free oscillations (possibly damped or growing) and sheds perturbations away from itself; this global mode requires the new modes. Then, using the perturbed Rankine-Hugoniot conditions, we show that this leads to the determination of the corrugational eigenfrequency. We solve numerically the equations for the eigenfrequency in the WKBJ regime for the models of Amato & Blasi, showing that they are stable. We then discuss the differences between our treatment and previous work.

Subject headings: cosmic rays — shock waves

1. INTRODUCTION

One of the major uncertainties still surrounding particle acceleration around shocks is the level at which this saturates, i.e., the total energy content in nonthermal particles, per unit volume, \( R \), in units of the incoming fluid kinetic energy density. This saturation level plays a very important role, for instance, in discussions of the origin of cosmic rays as observed at Earth; it has to be rather large (\( \geq 0.1 \)) to allow supernovae (SNe) to provide the observed flux. Also, discussions on the origin of ultra-high-energy cosmic rays (UHECRs) are influenced by similar considerations; the idea that UHECRs are accelerated by gamma-ray bursts (GRBs; Vietri 1995; Waxman 1995) draws some measure of support by the coincidence between the energy release rate of GRBs in gamma-ray photons and that required to account for observations at Earth (Vietri et al. 2003; Waxman 2004).

It is certainly possible that this level of saturation is somehow connected with the very uncertain particle injection mechanism, and that saturation occurs, in many astrophysically important situations, at very low levels. Yet, given the arguments concerning SNe and GRBs mentioned above, it appears that at least occasionally saturation must occur at rather large levels. This is especially so in the case of Galactic cosmic rays; if in fact we were to discard SNe as sources of cosmic rays, all other possible classes of sources, being both less numerous and less energetic, would force us to require saturation at even larger levels.\(^2\)

A distinct possibility is that the saturation level is determined not by the injection mechanisms, but by modifications which the particles’ pressure induces in the shock properties. After all, in the test particle limit the particles’ spectrum is ultraviolet divergent (although marginally so, of course), and one may hope that, since the particles’ back-reaction on the fluid makes acceleration less efficient (by reducing the velocity jump around the gas sub-shock), a convergent spectrum will be obtained, with a definite value for the parameter \( R \). After Malkov’s seminal papers (Malkov 1997; Malkov et al. 2000; Malkov & O’C. Drury 2001), fully self-consistent solutions with the particles’ pressure properly included have been obtained by Amato & Blasi (2005). In these models, it appears that particles can carry away an arbitrarily large fraction of the incoming energy flux, for sufficiently large Mach number at upstream infinity. Even more intriguing is the fact that these solutions have particles’ spectra which are more, not less, ultraviolet divergent than those in the test particle limit. In fact, this divergence is arrested only by arbitrarily limiting the largest individual energy to some fiducial value (Amato & Blasi 2005).

These solutions then seem to beckon for a stability analysis, in the hope that they are found to be unstable once \( R \) exceeds some critical value. The instability we envision is of course the corrugational instability for shocks, whereby ripples on the shock surface become of larger and larger amplitude.\(^3\) If this instability were to exist, we might easily imagine that the shock is substituted by a region of strong fluid turbulence, where particles moving a few Larmor radii perceive only a small velocity difference at the two ends of their free wandering, and thus, acceleration to high energies is made somewhat less efficient.

However, it is well known (Landau & Lifshitz 1987) that shocks in polytropic fluids are stable against this kind of perturbation and against the spontaneous emission of sound waves as well. The hope for the existence of an instability is based on a rather more subtle argument. When the shock surface flaps, it sheds in the downstream region pressure waves as well as entropy and vorticity perturbations. We show below that the last two do not couple to particle perturbations, but pressure waves do, thus generating small perturbations in the particles’ distribution function, \( \delta f \). These particles will however return upstream by means of diffusion and here generate, by their shear perturbed pressure, some more fluid perturbations. Thus, not all energy shed by the shock is lost forever toward upstream infinity, as is

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\(^2\) The situation is much less clear for UHECRs, because many of the proposed classes of sources are purely hypothetical, so that there are no constraints on their properties.

\(^3\) In all of this paper, the term corrugation instability is taken to include also the spontaneous emission of sound waves by the shock.
the case in the purely hydrodynamical case, but some fraction of it returns to the shock to generate more flapping. In fact, since pressure waves are strongly damped by diffusion, and nearly all particles return to the upstream section since the shock is Newtonian, we may conjecture that most energy shed by the shock flapping makes it back to feed more flapping, even though account must be taken of diffusion and phase mixing. Is it possible that this sets up some strong reinforcement, making the whole system unstable? This is the question we address in this paper.

This question has of course been studied before, under what different conditions. The diffusion coefficient of nonthermal particles by the fluid (=D) has been taken (by other authors) as independent of the particle impulse; this is conventionally referred to as the two-fluid approximation, because the Boltzmann equation for the nonthermal particles can then easily be recast into an equation for their pressure, thusly erasing all references to the underlying distribution function. Besides making the two-fluid approximation, Mond & O’C. Drury (1998) also neglected diffusion altogether. A more complex analysis has been presented by Toptygin (1999), who included energy transport and particle injection at the shock, but still in the two-fluid approximation. In the following, we make no such approximations; we consider a finite diffusion coefficient, D, and it is allowed to be an arbitrary function of both p, the particle momentum, and φ, the local fluid density, so as to at least mimic the increasing of magnetic field strength due to flux freezing. The arbitrary nature of the dependence of D on φ automatically prevents the use of the two-fluid approximation and forces us to use the full Boltzmann-Skilling equation.

A word of caution is in order about some assumptions. We neglect all energy in the form of magnetic field and Alfvén waves; of course, this is necessary because the zeroth-order solutions of Blasi and collaborators do not include these effects, but in our case, this neglect requires one extra assumption, i.e., that the timescales for energy to accumulate into any of these energy sinks, T_in, and to flow out of each of them, T_out, be ordered like T_out ≤ T_in. If this inequality were severely violated, the magnetic field or Alfvén waves might acquire a significant fraction of the total energy, despite their negligibility in the zeroth-order solution, and make our treatment completely irrelevant.

In the following, we closely follow the treatment of the shock corrugational instability given by Landau & Lifshitz (1987). In particular, we consider a steady state shock in its own frame, located at z = 0, with fluid coming from the left and exiting from the right, so that all speeds are >0. Exactly like Landau & Lifshitz, we consider perturbations generated by the shock flapping, so that there can be no incoming waves, from either upstream or downstream infinity. In §2 we consider perturbations in a homogeneous medium; this is perhaps a tad boring, but it contains a number of new results which are of the utmost importance further below. In §3 we briefly discuss perturbations upstream, i.e., where the flow is inhomogeneous; we show here that we can easily obtain the perturbations in the WKBJ limit, k_y → ∞, which restricts our analysis to the regime z ≤ L, where z is the perturbation wave-length perpendicular to the shock and L is the typical size of the region of inhomogeneity upstream. In §4 we discuss the boundary conditions on the perturbed particle distribution function and derive how it relates to the amplitude of the modes to which particle perturbations couple. We present in §5 the perturbed Rankine-Hugoniot conditions and in §6 what fixes the global corrugation mode eigenfrequency. In §7 we present our numerical computations for the stability of the exact solutions of the zeroth-order problem by Amato & Blasi (2005). In §8 we compare our results with other works in the literature and briefly summarize our work.

2. PERTURBATIONS IN A HOMOGENEOUS MEDIUM

We give here, for future reference, our basic equations. They are the conventional hydrodynamic equations

\[ \frac{\partial p}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \]

\[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla (P + P_c), \]

\[ \frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s = 0, \]

which contain a term for the momentum exchange between the fluid and the nonthermal particles represented by the gradient of the particle pressure P_c, plus the usual Boltzmann equation in Skilling (1975),

\[ \frac{\partial f}{\partial t} = \nabla \cdot (D \nabla f) - \mathbf{v} \cdot \nabla f + \frac{1}{3} \nabla \cdot \mathbf{v} \frac{\partial f}{\partial p}, \]

We assume D = D(p, ρ) to be a given function of ρ and p.

We consider small-amplitude perturbations around a homogeneous solution where the particles are supposed to exert a nonnegligible pressure. First, we consider entropy perturbations. Perturbations can be taken in the form

\[ \delta s \propto \exp(\omega t - i k \cdot r), \]

so as to obtain, from the equation of entropy conservation (eq. [3]),

\[ (\omega - u_k) \delta s = 0 \rightarrow \omega - u_k = 0, \]

where u is the unperturbed fluid velocity in the x-direction. Perturbation of the mass conservation equation yields

\[ (\omega - u_k) \delta P - \rho k \cdot \delta \mathbf{v} = 0 \rightarrow \delta \mathbf{P} = -u_k \delta \mathbf{v} = 0. \]

Perturbation of the momentum equation yields

\[ (u_k - \omega) \delta \mathbf{v} + \frac{k}{\rho} (\delta P + \delta P_c) = 0 \rightarrow \delta P = -\delta P_c. \]

Now, the perturbation of the Boltzmann equation yields

\[ \omega \delta f = -k^2 D \delta f + u_k \delta f - \frac{1}{3} k \cdot \mathbf{v} \frac{\partial f}{\partial p}, \]

which implies δf = 0, because, for entropy perturbations, \( \omega = u_k \) and \( k \cdot \delta \mathbf{v} = 0 \), as deduced above. Furthermore, since δf = 0, necessarily δP_c = 0, and thus δP = 0. It follows that entropy perturbations and particle perturbations are completely decoupled; in fact, entropy perturbations are just advected by the zeroth-order flow.

In summary, entropy perturbations have the following characteristics, where we define, for future convenience, \( V = 1/\rho \),

\[ \delta \mathbf{s} = \delta \mathbf{s}_0 e^{iωt - sk_x - s^2 k_y}, \]

\[ \omega - u_k = 0, \]

\[ \delta P = \delta P_c = \delta f = \delta \mathbf{v} = 0, \]

\[ \frac{\delta V}{V} = \frac{\gamma - 1}{\gamma} \frac{m \delta s}{k_B}, \]
where the last equation applies to ideal fluids; \( m \) is the average particle mass and \( k_B \) is Boltzmann’s constant.

We turn now to isentropic perturbations, which automatically satisfy equation (3). Mass conservation implies

\[
(\omega - uk_x)\delta \rho = \rho k \cdot \delta \mathbf{v},
\]

while momentum conservation implies

\[
(\omega - uk_x)\delta \mathbf{v} = \frac{k}{\rho} (\delta P + \delta P_c).
\]

Let us begin by assuming that the left-hand side of equation (11) vanishes; the same is then true for \( \nabla \cdot \delta \mathbf{v} \). Multiplying equation (12) by \( k \land \), we see that \( \nabla \land \delta \mathbf{v} = 0 \), unless \( \omega = uk_x \). If the curl vanishes, then so does \( \delta \mathbf{v} \), because any vector with vanishing divergence and curl is a constant, which can always be set to zero by a suitable choice of reference system. So we must have \( \omega = uk_x \) for perturbations with nonzero vorticity; we thus find that \( \delta P + \delta P_c = 0 \), and then, again from equation (9), that \( \delta \mathbf{v} = 0 \), implying \( \delta P = \delta P_c = 0 \). Vorticity perturbations do not couple to particles either. We have for them

\[
\begin{align*}
\delta s &= \delta \rho = \delta P = \delta f = \delta P_c = 0, \\
\omega - uk_x &= 0, \\
k \cdot \delta \mathbf{v} &= 0.
\end{align*}
\]

(13)

We consider now those perturbations where \( \delta \mathbf{v} \) does not vanish. We can show that here too there are two distinct classes of modes; in the first one, \( \delta \mathbf{v} \) is not coupled to the fluid quantities, while in the second one, it is through the term \( k \cdot \delta \mathbf{v} \neq 0 \). The first class of modes, which we call \( d \)-mode, cannot be obtained directly from equation (9), for reasons to be made clear shortly. We use instead the correct form

\[
\frac{\partial \delta f}{\partial t} = \nabla \cdot (D \nabla \delta f) - u \frac{\partial \delta f}{\partial x},
\]

where we dropped the term \( \omega k \cdot \delta \mathbf{v} \) in keeping with our desire to find a solution totally uncoupled from the fluid. The solution of this equation is known from elementary courses; if \( \phi_0(x, y, p) \) is the initial distribution function at time \( t = 0 \) (possibly dependent on \( p \)), the solution at later times for \( u = 0 \) is

\[
\delta f(x, y, t, p) = \int dx_0 \int dy_0 \int dp_0 \phi_0(x_0, y_0, p_0)
\]

\[
\times \frac{1}{4 \pi Dt} e^{-(x-x_0)^2/4Dt} e^{-(y-y_0)^2/4Dt},
\]

(15)

and the solution for \( u \neq 0 \) is just \( \delta f(x - ut, y, p) \). At the same time, we must make sure that this solution does not ruffle the fluid; this obviously requires \( \delta P_c = 0 \) at all times. Now integrating equation (14) over \( 4\pi p^3 dp/3 \), we find

\[
\frac{\partial \delta P_c}{\partial t} = \frac{4\pi}{3} \frac{\partial^2}{\partial x^2} \int D(p) \delta f p^3 dp + u \frac{\partial \delta P_c}{\partial x},
\]

(16)

which clearly shows that, in order to have \( \delta P_c = 0 \) everywhere at all times, we need \( \delta P_c = \int D \delta f p^3 dp = 0 \) everywhere at the initial time. If we now Fourier-expand the initial condition \( \phi_0 \) with respect to \( x \) and \( y \), the above conditions become

\[
\phi_0(x, y) = \int dk_x \int dk_y f(p, k) e^{-ik_x x - ik_y y},
\]

\[
\int g(p, k) p^3 dp = \int g(p, k) D(p) p^3 dp = 0.
\]

(17)

This completes the derivation of this purely damped d-mode, which will not perturb the fluid. Although it may look at this stage like a mathematical oddity, this mode plays a key role in the matching of boundary conditions between the upstream and downstream regions. It is also worth remarking why it cannot be derived from its Fourier-analyzed counterpart: the solution in question contains a term \( \sim e^{-t/\bar{c}} \), which does not have a Fourier transform with respect to \( t \).

When \( \delta P_c \neq 0 \), velocity perturbations \( \delta \mathbf{v} \neq 0 \) are induced in the fluid by the nonvanishing particles’ pressure gradient, and equation (9) can be solved for \( \delta f \) as

\[
\delta f = - \frac{\delta \rho}{\rho} \frac{\partial}{\partial p} \frac{1}{3} \frac{(\omega - uk_x)}{\rho} + \frac{\bar{c}^2 D}{3}. 
\]

(18)

If we integrate this over \( 4\pi p^3 dp/3 \), we find

\[
\delta P_c = \frac{\delta \rho}{\rho} \int p_{	ext{u}} \frac{4\pi}{9} dp \int p^4 \frac{\partial f}{\partial p} - \frac{1}{3} \frac{(\omega - uk_x)}{\rho} + \frac{\bar{c}^2 D}{3},
\]

(19)

where we assumed the nonthermal particles to have a minimum \( (p_m) \) and a maximum \( (p_M) \) momentum. It is convenient to introduce a weighted diffusion coefficient \( \bar{D} \), defined as

\[
\frac{1}{1 - z \bar{D}(z)} \equiv \frac{\int_{p_m}^{p_u} (4\pi/9) dp \int p^4 \frac{\partial f}{\partial p} \{1/[1 - z \bar{D}(p)] \}}{\int_{p_m}^{p_u} (4\pi/9) dp \int p^4 \frac{\partial f}{\partial p} \}} ,
\]

(20)

which shows \( \bar{D} \) to be a function of \( z \) only.

The above can be simplified a bit by integration by parts. If the integral were to extend from 0 to \( +\infty \), we would have

\[
- \int_0^{\infty} \frac{4\pi}{9} dp \int p^4 \frac{\partial f}{\partial p} = \int_0^{\infty} \frac{4\pi}{9} dp \left( \frac{d}{dp} \int p^4 \right) f = \frac{4\pi}{3} \int_0^{\infty} dp \int p^3 \left( \frac{4}{3} + \frac{m^2}{3E^2} \right) \equiv \gamma_c P_c,
\]

(21)

where \( P_c \) is the particle pressure in the unperturbed fluid, and \( \gamma_c \), the effective particle polytropic index, satisfies \( 4/3 \leq \gamma_c \leq 5/3 \). Since however the integral extends only from \( p_m \) to \( p_M \), we define

\[
\delta f = \gamma_c \int_{p_m}^{p_u} \frac{4\pi}{9} dp \int p^4 \frac{\partial f}{\partial p} \equiv \gamma_c P_c.
\]

(22)

Thus, equation (19) can now be rewritten as

\[
\delta P_c = \gamma_c P_c \frac{\delta \rho}{\rho} \frac{1}{1 - [ik^2/(\omega - uk_x)] \bar{D}(ik^2/(\omega - uk_x))}.
\]

(23)

We remark that both \( \gamma_c \) and \( \bar{D} \) are obtained by weighing the zeroth-order solution, so that they can be immediately computed as soon as this solution is available.
We can now eliminate $k \cdot \delta r$ between equations (11) and (12) and then use $\delta p = c_s^2 \delta \rho$ (with $c_s$ the sound speed because we are considering isentropic perturbations) and the equation above to obtain

$$\Omega^2 = 1 + \frac{\gamma \rho}{\gamma P} \frac{1}{1 - [ik^2/(\omega - uk_s)]D(ik^2/(\omega - uk_s))},$$  \hspace{1cm} (24)

where we have called

$$\Omega \equiv \frac{\omega - uk_s}{kc_s}$$  \hspace{1cm} (25)

the comoving eigenfrequency, in suitably scaled units. Equation (24), together with the definition of $D$ (eq. [20]), is the sought-after dispersion relation for the coupled small perturbations in a homogeneous medium.

2.1. Properties of Coupled Modes

The case where $D$ is independent of $p$ has been derived before (Ptuskin 1981) and coincides with the above equation. Surprisingly, the existence of this mode was not noticed by Toptygin (1999) even though a careful treatment of his equations yields precisely the same dispersion relation as above; this neglect of this mode has important consequences to be discussed below.

It is best to begin our discussion with the case when $D$, and thus $\tilde{D}$, is a constant, independent of $p$. Equation (24) then reduces to

$$\left(\Omega^2 - 1\right) = \Omega \frac{\gamma \rho}{\gamma P},$$  \hspace{1cm} (26)

which is a simple polynomial equation of the third order. In this case two modes reduce to pressure waves, as is most easily seen in the test particle regime $P_e = 0$. There is however a third solution which is only slightly more mysterious; in the test particle regime these modes represent a local overdensity of particles dissipated by diffusion. When the test particle regime does not apply, a particle contribution to the sound speed is introduced by the term $\propto P_e$. This new mode (which we call the third mode) is the equivalent of the d-mode when however the conditions from equation (17) are not satisfied; the basic idea is still the same, the particle overdensity is dissipated, but since the particle pressure does not vanish, the fluid is consequently ruffled. Notice also that there are two third modes, traveling in opposite directions.

The situation is slightly more complex when $D = D(p)$, because one must solve simultaneously equations (24) and (20). We begin by considering the limit $k \to 0$. In this case, and assuming $\Omega \to \text{const}$, we easily find, to leading order in $k$, $\Omega^2 = 1 + \frac{\gamma \rho}{\gamma P}$; as it must, the dispersion relation allows pure pressure waves, with the particle pressure providing a correction to the sound speed because we are considering isentropic perturbations) and the equation above to obtain

$$\Omega^2 = 1 + \frac{\gamma \rho}{\gamma P},$$  \hspace{1cm} (27)

Now we use the definition $z \equiv ik^2/(\omega - uk_s) = ik/(\Omega c_s)$ to simplify the above to

$$1 - z\tilde{D}(z) = -\frac{\gamma \rho}{\gamma P},$$  \hspace{1cm} (28)

which can be now used with equations (20) and (22) to obtain

$$\gamma P = \int_{p_m}^{p_M} \frac{4 \pi}{9} dp v^2 \tilde{D}(z) \frac{1}{\partial p} - zD(p),$$  \hspace{1cm} (29)

where of course $\partial f/\partial p < 0$. As a function of real $z$, the right-hand side above (where it exists) is easily seen to be a monotonically decreasing function of $z$, vanishing for $z \to \pm \infty$. In any realistic physical problem, the integral must extend from a minimum $p_m$ to a maximum momentum $p_M$; since $D(p)$ is expected to be a monotonically increasing function of $p$, we see that the integral above always exists for $z < 1/D(p_m) \equiv 1/D_m$ and $z > 1/D(p_M) \equiv 1/D_m$, and it diverges exactly at $z = 1/D_m$ and $z = 1/D_M$. Thus, the integral on the right-hand side of the equation above spans the whole range from 0 to $-\infty$ as $z$ varies between $-\infty$ and $+1/D(p_m)$, and the range $+\infty$ to 0 as $z$ varies between $1/D_m$ and $+\infty$. Somewhere in the range $1/D_m < z < +\infty$ there is the one and only solution of the above equation. An illustration of the integral on the right-hand side of the previous equation is shown in Figure 1, for a specific distribution function from Amato & Blasi (2005); the qualitative features of this plot are generic to all distribution functions we have tried.

Furthermore, since $z = ik^2/(\omega - uk_s)$ and the small perturbations were assumed to vary as $e^{i(k\omega - uk_s)t}$, the result that $z > 0$ implies that all modes are damped by diffusion, as physical intuition obviously suggests. The discussion in the opposite limit, $k \to +\infty$, is similar. If we assume $\Omega \to \text{const}$, we find the solution

$$\Omega^2 = 1,$$  \hspace{1cm} (30)

without the correction to the sound speed due to the presence of the particles’ pressure; in the limit $k \to +\infty$, particles escape by free streaming and do not contribute to the sound speed.
Again, we lost a solution, so we now search for the third mode as \( \Omega = \alpha k + \) lower order terms in \( k \). We obtain now
\[
\Omega = \frac{ikD(\omega - uk_y)}{c_s},
\]
which can be rewritten as
\[
zD(z) = 1.
\]
Comparing this with equation (20), we see that the value of \( z \) we are searching for is the one that makes the integral on the numerator of the right-hand side of equation (20) diverge. Following the previous discussion, we see that this is
\[
z = \frac{1}{D(p_m)}.
\]
This is always positive, so that the solution is always damped by diffusion. This result has a simple physical explanation: when a small overdensity of particles is generated locally, the timescale for damping of this overdensity is dictated by diffusion of the slowest particles.

Again for illustrative purposes, the real and imaginary parts of sonic and third modes are displayed in Figure 2 for a specific distribution function from Amato & Blasi (2005). Again, the qualitative features are generic to all models we investigated. In summary, we have seen that the introduction of particles modifies the modes of a homogeneous medium by adding two new modes, one coupled and one uncoupled to the fluid, both strongly damped by diffusion.

For future reference, we give the expressions for all small quantities, in units of \( \delta P_c \); the link between \( \delta P_c \) and \( \delta P \) is obtained from equations (19), (20), and (22), and the others follow easily. For later convenience, we use again \( V \equiv 1/\rho \) rather than \( \rho \),
\[
\delta P_c = \delta P_c, v e^{(at - ak_y, y - k_z)},
\]
\[
\frac{\delta V}{V} = -\frac{1}{\gamma^*} \frac{\delta P_c}{P_c} \left( 1 - \frac{\gamma^*}{\gamma^* \Omega} \right) = -\frac{\delta P_c}{\gamma^* (\Omega^2 - 1)} \equiv -q \frac{\delta P_c}{P_c},
\]
\[
\frac{\delta v}{V} = -\frac{k}{\kappa} _c \frac{\delta V}{V} \left[ 1 + \frac{\gamma^* P_c V}{c_s^2} \left( 1 - \frac{1}{\gamma^* \Omega} \right) \right]
\]
\[
= -\frac{k}{\kappa} _c \frac{\delta V}{V} \equiv z \frac{\delta V}{V},
\]
\[
\frac{\delta P}{P} = -\gamma \frac{\delta V}{V},
\]
(34)
where the definitions of $z$ and $q$ will come in handy below. In these equations, $k_i$ must be regarded as a known function of $\omega$ and $k_y$, specified by equation (24).

2.2. Initial Value Problem

As a preparation for later work, we discuss the initial value problem. This has some interest because perturbations in $A\delta f$ belonging to the various modes are not mutually orthogonal, and thus, it may appear that initial conditions, especially when given only in terms of $A\delta f$, cannot be decomposed into mutually independent modes. To fix ideas, let us consider a homogeneous zeroth-order solution where all fluid quantities are unperturbed, but a small perturbation $A\delta f_0(x,y,p)$ at time $t = 0$ is given; what are the amplitudes of the four modes that will be excited (two pressure waves, the third mode, and the d-mode)? First, of course, we Fourier-analyze $A\delta f_0$, calling $a(p)$ its amplitude. We must then have

$$a(p) = A_i\delta f_i + g(p),$$

where we have introduced some notation that will be useful in the following: a summation convention over $i$ is understood, the $A_i$ are the amplitudes of the pressure waves and third modes for $i = 1, 2, 4$, respectively; $g(p)$ is the amplitude of the d-mode, as in equation (17), and the $\delta f_i$ are

$$\delta f_i = -\frac{1}{3P} \frac{\partial f}{\partial p} \left( \omega_i - uk_{ii} \right) + k_i^2 D(p).$$

Comparing this with equation (18), it becomes clear that the mode amplitudes $A_i$ are simply $\delta p_i/\rho$. Here the quantities $\omega_i$, $k_{ii}$, and $k_i^2$ are supposed to be linked by the appropriate branch of the dispersion relation. For $g(p)$ to represent a proper d-mode, we know that it must satisfy the two constraints in equation (17); thus, we derive two conditions on the mode amplitudes,

$$\int p^3 v_a(p) dp = A_i \int p^3 v_\delta f_i dp,$$

$$\int D(p)p^3 v_a(p) dp = A_i \int D(p)p^3 v_\delta f_i dp.$$

The last condition can be obtained by satisfying the requirement that the perturbations to both density and velocity vanish at the initial time. With reference to equation (34), we see that the two conditions, the vanishing of the density and of the velocity, imply

$$\sum_i A_i = 0, \quad \Omega a A_i = 0,$$

which are two more linear relations which, together with equations (37) and (38), determine the $A_i$. This simple example illustrates the importance of the d-mode, a kind of elephants’ graveyard, because once particles join it, the perturbations they generate can only be dissipated away without ruffling the fluid ever again.

3. PERTURBATIONS UPSTREAM

It is useful to remark that, upstream, entropy perturbations are not allowed for arbitrarily inhomogeneous flows, while the same is not true for vorticity perturbations. In fact, the perturbation of the entropy equation is

$$\frac{\partial \delta s}{\partial t} + u(x)\frac{\partial \delta s}{\partial x} = 0,$$

where we could neglect the term $\delta s$, $ds/dx$ because the fluid is isentropic in the unperturbed state. From the above, we see that entropy perturbations are advected by the background flow all the way from upstream infinity to the shock; we cannot however accept this, since in our problem all perturbations must have as a source the shock flapping. Perturbations riding all the way from upstream infinity belong to perturbations in the boundary conditions and are thus irrelevant. Thus, the perturbed fluid will be assumed adiabatic, from now on.

The same argument does not apply to vorticity perturbations when the fluid is stratified, because they do couple to particle perturbations; in fact, we easily obtain from equation (2) that the equation for the vorticity, $\vec{\eta}$, is

$$(\frac{\partial}{\partial t} + v \cdot \nabla) \vec{\eta} = \left( \frac{\partial}{\partial \rho} + \frac{1}{\rho^3} \nabla \rho \nabla(P + P_\rho) \right).$$

In the absence of particles, this equation tells us that vorticity is exactly (i.e., not just to zeroth or first order) Lie-advected by the flow, because for adiabatic flows, $\nabla \rho \cdot \nabla v = 0$. But in the presence of particles of and of spatial gradients, it is easily seen that the particle pressure $P_\rho$ acts as the source of vorticity perturbations. In order to make progress, we compute the spatial dependence of the various modes upstream in the WKBJ approximation, i.e., in the limit $k_y \to \infty$, and in particular $k_y^{-1} \ll l$, the size of the upstream precursor. In other words, we take all perturbed physical quantities to be of the form

$$\delta X \approx Q_x(x)e^{i\omega t + \int_0^x \frac{W_x(y)}{dx}dy - ik_y y},$$

and all derivatives in the $x$-direction (i.e., perpendicular to the shock) are considered small when compared with terms proportional to $k_y$. This is a perturbation analysis in which the transverse wavenumber $k_y$ is assumed large, and as a consequence, the longitudinal wavenumber $l = W_x$ is also large. The presence of a nonconstant amplitude $Q_x(x)$ is equivalent to keeping the first two terms in an asymptotic expansion in the small parameter $(k_yL)^{-1}$. This is often called the physical optics approximation (Bender & Orszag 1978).

This analysis is quite standard, but the amusing thing is that we do not even need to carry it through. In fact, we show below that the stability analysis requires knowledge of the physical quantities immediately before the shock, while knowledge of the perturbations run with $x$ further from the shock is immaterial. We see from the above that all physical quantities close to the shock satisfy

$$\delta X \approx Q_x(0)e^{i\omega t - \int_0^x W_x(0)dx - ik_y x}.$$
approximation. It clearly breaks down when the WKBJ analysis does, which occurs for $k_x^{-1} \approx L$, the size of the upstream precursor.

4. BOUNDARY CONDITIONS FOR THE PARTICLE DISTRIBUTION FUNCTION

Because of diffusion, particles are not restricted to the downstream part of the flow; there will be a $g \neq 0$ also upstream, so that we need to consider appropriate conditions to match $g$ in the two regions. The first condition is obviously the continuity across the shock,

$$\delta f_1 = \delta f_2.$$ (44)

It is also well known that the spatial gradient of $\delta f$ needs to satisfy a boundary condition at the shock; this is derived by integrating equation (4) on an infinitesimal interval straddling the shock. The unit vector normal to the surface of the flapping shock is $\hat{n} = (1, i k_x \zeta)$, where $\zeta$ is the shock corrugation amplitude. So, we obtain

$$\hat{n} \cdot (D \nabla f)_2 - \hat{n} \cdot (D \nabla f)_1 + \frac{1}{3} \frac{\partial}{\partial p} \hat{n} \cdot (v_2 - v_1) = 0.$$ (45)

Writing its first-order linearization and using equation (44), we find

$$D \left( \frac{\partial \delta f}{\partial x} \right)_2 - \frac{\partial \delta f}{\partial x} \right)_1 + i k_x \zeta D (\delta f / \partial y)_1 \right)_2 - \frac{\partial \delta f}{\partial y} \right)_1 \right)_2 + \frac{1}{3} \frac{\partial \delta f}{\partial p} (u_2 - u_1) = 0.$$ (46)

We note that $(\partial \delta f/\partial y)_2 = (\partial \delta f/\partial y)_1 = 0$, because at zeroth order, the medium is uniform in coordinates parallel to the shock surface. The result is

$$D \left( \frac{\partial \delta f}{\partial x} \right)_2 - \frac{\partial \delta f}{\partial x} \right)_1 + \frac{1}{3} \frac{\partial \delta f}{\partial p} (u_2 - u_1) = 0.$$ (47)

Equations (44) and (47) are the appropriate boundary conditions for our problem.

We now show how to satisfy the boundary conditions (eqs. [44] and [47]) at the shock. Using the notation introduced in § 2.2, we know that $\delta f_+$ on the downstream side of the shock (the factor $e^{I_d - i k_x y}$ is omitted for simplicity in this subsection) satisfies

$$\delta f_+ = A_{id} \delta f_{id} + g_d.$$ (48)

Please notice that both downstream and upstream the summation is over two modes (a pressure wave and a third mode). In fact, we know on the one hand that particle perturbations are not coupled to entropy and vorticity perturbations, while on the other hand, we know that pressure and third modes, for given values of $k_y$ and $\omega$, exist for opposite values of $k_x$ (see eq. [24]), and thus, at least one pressure mode and one third mode will diverge exponentially toward infinity. This is unacceptable because we are studying flow instabilities, not variations in the boundary conditions at infinity. This discussion will be completed in § 6.

On the upstream side we have analogously

$$\delta f_- = A_{iu} \delta f_{iu} + g_u.$$ (49)

The continuity of $\delta f$ at the shock allows us to derive a relationship between the $g$ variables,

$$g_d = g_u + A_{iu} \delta f_{iu} - A_{id} \delta f_{id}.$$ (50)

Using

$$\frac{\partial \delta f_+}{\partial x} = - \nu A_{id} k_{sid} \delta f_{id} + k_{sd} g_d$$

$$\frac{\partial \delta f_-}{\partial x} = - \nu A_{id} k_{sid} \delta f_{id} + k_{sd} g_u + A_{iu} k_{sid} \delta f_{iu} - A_{id} k_{sd} \delta f_{id},$$ (51) (52)

in equation (47), we find

$$D(k_{sid} - k_{su}) g_u + \frac{u_2 - u_1}{3} \frac{\partial g_u}{\partial \ln p}$$

$$- \frac{u_2 - u_1}{3} A_{iu} \frac{\partial \delta f_{iu}}{\partial \ln p} + \frac{1}{3} \frac{\partial \delta f}{\partial p} (\delta v_{2x} - \delta v_{1x})$$

$$- DA_{id} \delta f_{id}(k_{sid} + k_{sd}) + DA_{iu} \delta f_{iu}(k_{sid} + k_{sd}),$$ (53)

which we regard as an equation for $g_d(p)$, whose solution can be written as

$$g_d(p) = Cw_C(p) + A_{iu} w_u(p)$$

$$+ A_{id} w_d(p) + (\delta v_{2x} - \delta v_{1x}) w_0(p),$$ (54)

where the functions $w$ are derived in Appendix B.

We can now impose the conditions from equation (17) on $g_u$, obtaining

$$C \int p^3 v w_C D p + A_{iu} \int p^3 v w_u D p + A_{id} \int p^3 v w_d D p$$

$$+ (\delta v_{2x} - \delta v_{1x}) \int p^3 v w_0 D p = 0,$$ (55)

$$C \int p^3 v w_C D p + A_{iu} \int p^3 v w_u D p + A_{id} \int p^3 v w_d D p$$

$$+ (\delta v_{2x} - \delta v_{1x}) \int p^3 v D w_0 D p = 0.$$ (56)

And now we can impose the very same conditions on $g_d$ (eq. [50]) to obtain

$$A_{id} \int p^3 v \delta f_{id} D p - A_{iu} \int p^3 v \delta f_{iu} D p = 0,$$ (57)

$$A_{id} \int p^3 v \delta f_{id} D p - A_{iu} \int p^3 v \delta f_{iu} D p = 0.$$ (58)

This set of four linear equations in five unknowns ($C$ and the $A_i$) can be solved in terms of one of them, say $A_{id}$, the pressure wave downstream.

We thus see that the conditions at the shock, plus knowledge of the modes, allows us to determine the amplitude of all modes (except the vorticity and entropy modes downstream) in terms of the amplitude of the pressure wave downstream. Remembering equation (34), we now see that all fluid quantities at the shock upstream are determined in terms of the amplitude of this very same wave; as for the two d-modes, their explicit space and time dependence is not needed, but is given for completeness’ sake in Appendix A. Now, before discussing how to fix $\omega$, the shock
eigenfrequency, we briefly summarize how to perturb the standard Rankine-Hugoniot conditions.

5. THE FLUID CONDITIONS AT THE SHOCK

We follow closely the discussion in Landau & Lifshitz (1987, § 90), which requires only a small adaptation to our problem. In their consideration of the corrugational instability, in fact, there were no perturbations on the upstream side of the shock; deviations from the unperturbed state were generated by the corrugation of the shock surface (and its motion) and led to nonzero perturbations only downstream. In our problem, instead, the particles crossing the shock manage to generate new perturbations on the upstream side, leading to a slight modification to the Rankine-Hugoniot conditions. From now on, we indicate with the subscript 1 the quantities on the upstream side of the shock and with the subscript 2 those on the downstream side. Also, to make contact with Landau & Lifshitz’s work easier, we use as a variable \( V \equiv 1/\rho \) rather than the density \( \rho \) directly.

We consider a small-amplitude corrugation of the shock surface, away from the \( z = 0 \) plane, by a small displacement of the form

\[
\zeta = \zeta(x-t) = \zeta_0 e^{i\omega t-k_y y},
\]

with respect to which the unit vectors parallel \( \hat{i} \) and normal \( \hat{n} \) to the surface have components in the \( x-y \) plane,

\[
\hat{i} = (-ik_y \zeta, 1), \quad \hat{n} = (1, ik_y \zeta),
\]

while the surface speed in the direction normal to the surface, with respect to the reference frame of the unperturbed shock, is

\[
q \cdot \hat{n} = i\omega \zeta.
\]

All quantities are, of course, accurate to first order only.

The first two Rankine-Hugoniot conditions to be perturbed involve the fluid speed, and they are the continuity of the fluid speed parallel to the shock surface and the discontinuity of the perpendicular component in terms of perturbed pressure and density (eq. [85.7]) of Landau & Lifshitz 1987). We have

\[
(v_1 + \delta v_1) \cdot \hat{i} = (v_2 + \delta v_2) \cdot \hat{i},
\]

\[
(v_1 + \delta v_1) \cdot \hat{n} = (v_2 + \delta v_2) \cdot \hat{n},
\]

\[
= \sqrt{(P_2 + \delta P_2 - P_1 - \delta P_1)(V_1 + \delta V_1 - V_2 - \delta V_2)},
\]

whose first-order linearizations are

\[
\delta v_2 = \delta v_1 = ik_y \zeta(v_2 - v_1),
\]

\[
\delta v_{2x} - \delta v_{1x} = \frac{1}{2} (v_2 - v_1) \left( \frac{\delta P_2 - \delta P_1}{P_2 - P_1} - \frac{\delta V_2 - \delta V_1}{V_2 - V_1} \right).
\]

The next equation to be perturbed is the shock adiabat, which is convenient because it is independent of all speeds involved. For a polytropic gas like the one we are considering, the unperturbed Hugoniot adiabat is given by (Landau & Lifshitz 1987, eq. [89.1])

\[
\frac{V_2}{V_1} = \frac{(\gamma + 1)P_2 + (\gamma - 1)P_1}{(\gamma - 1)P_1 + (\gamma + 1)P_2},
\]

whose perturbation yields

\[
\frac{\delta V_2}{V_2} = \frac{\delta V_1}{V_1} + h \left( \frac{\delta P_1}{P_1} - \frac{\delta P_2}{P_2} \right),
\]

where we have defined

\[
h = \frac{4\gamma}{(\gamma + 1) + (\gamma - 1)\frac{P_2}{P_1} + [(\gamma - 1)\frac{P_1}{P_2} + (\gamma + 1)]},
\]

while the ratio \( P_2/P_1 \) can be expressed as

\[
\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - (\gamma - 1)}{\gamma + 1},
\]

with \( M_1 \) the Mach number of the upstream fluid.

We need one more equation; we can use the equation relating the mass flux to the discontinuities in pressure and density. Equation (85.6) of Landau & Lifshitz (1987) gives

\[
j^2 = \frac{P_2 - P_1}{V_1 - V_2},
\]

where \( j \) is the mass flux. When the shock surface is perturbed, the above becomes

\[
\frac{2\delta v_{1x}}{v_1} - \frac{2\omega \zeta}{v_1} - \frac{2\delta v_1}{V_1} = \frac{\delta P_2 - \delta P_1}{P_2 - P_1} - \frac{\delta V_1 - \delta V_2}{V_1 - V_2}.
\]

Lastly, the condition that \( \delta f \) be continuous across the shock has already been imposed (see eq. [44]).

6. THE EQUATION FOR THE PERTURBATION EIGENFREQUENCY

We discuss here first how this problem differs from the one without particles, keeping in mind that we are interested in the stability of the shock flapping, so that all modes present must have this flapping as their source. We cannot allow modes to come in from spatial infinity, because this amounts to perturbation of the boundary conditions, not of the shock geometry.

When we neglect the particles’ pressure, we know that there can be no perturbations upstream, since they are either generated at upstream infinity (in which case we would not be treating the case of shock instability) or, if generated at the shock, they cannot propagate away from it fast enough (the shock is supersonic). In the presence of particles, the situation changes because particles can diffuse back to the upstream region, so that a perturbation \( \delta f \) generated downstream can return to upstream and perturb the fluid quantities. The situation is even more remarkable when one notices that, in this way, there can be a generation of pressure waves (even though they are just sonic in a supersonic medium) in the upstream region. The reason is shown in equation (2); the gradient in particles’ pressure is a source of perturbations, and since particles scatter a finite (i.e., nonzero) distance from the shock, there is no obvious reason why even sonic perturbations should not be generated. The impossibility of having sonic perturbations in a supersonic medium arises only when the point of generation of the perturbations is the shock itself, not a finite distance from it.

We must also remark that the presence of particles shuffling between downstream and upstream, and vice versa, has important consequences also for the kind of waves present downstream. In fact, while in the absence of particles the only waves present can be those shed by the shock, i.e., entropy, vorticity, and pressure
disturbances propagating to downstream infinity, when particles are included in the picture we find, by complete analogy with the argument above for the upstream region, that they can seed the third mode and the d-mode.

We have seen in § 4 that the amplitudes of all modes (and thus of all physical quantities) upstream and of the third mode downstream can be expressed in terms of a single free parameter, the amplitude of pressure waves downstream. Thus, all amplitudes are fixed, except for three modes downstream (entropy, vorticity, pressure) and the shock displacement \( \zeta \); but we also have the four perturbed Rankine-Hugoniot conditions, which can be expressed in terms of these four quantities. The vanishing of the determinant, once substitution of amplitudes for the upstream modes and for the third mode downstream by means of the relations in § 4 has been made, thus fixes the eigenfrequency.

A word of caution is in order; we have tacitly assumed that the pressure wave propagating away from the shock in the downstream region is also the wave with the physically acceptable (i.e., decreasing) behavior at downstream infinity. This is not certain. In fact, even in the standard case with no particles, a proper mode exists when the mode explodes exponentially in time and the pressure wave propagating away from the shock dies down at downstream infinity (Landau & Lifshitz 1987). This condition can only be checked a posteriori, i.e., after having found numerically the value of \( \omega \).

We now explicitly derive the set of equations whose determinant must vanish, which is what fixes \( \omega \) for a given \( k_y \). In the four Rankine-Hugoniot equations, there appear the total perturbations of specific volume, pressure, and velocity. They can be written as a sum of the respective perturbation for each mode. Downstream, we have

\[
\delta V_2 = \delta V_{2s} + \delta V_{2p} + \delta V_{2t}, \\
\delta v_2 = \delta v_{2s} + \delta v_{2p} + \delta v_{2t}, \\
\delta P_2 = \delta P_{2p} + \delta P_{2t},
\]

(71) (72) (73)

where the subscripts \( s, v, p, \) and \( t \) label quantities for entropy, vorticity, pressure waves, and the third mode. Upstream, there are neither entropy nor vorticity perturbations,

\[
\delta V_1 = \delta V_{1p} + \delta V_{1t}, \\
\delta v_1 = \delta v_{1p} + \delta v_{1t}, \\
\delta P_1 = \delta P_{1p} + \delta P_{1t}.
\]

(74) (75) (76)
TABLE 1
Solutions of Equations (82) and (83) for $k_y = n_1 = 1$

| $\omega$  | $k_x$ |
|----------|-------|
| 1/2      | -1/2  |
| -1/2     | 1/2   |
| $\sqrt{5}/2$ | $-3\sqrt{5}/2$ |
| $\sqrt{3}/2$ | $3\sqrt{3}/2$ |
| $\sqrt{7}/4$ | $\sqrt{7}/4$ |
| -1/4     | -1    |

We must use these expressions in the perturbed Rankine-Hugoniot conditions. Furthermore, we can write one of the two components of $\delta v_{2x}$ in terms of the other one, through $k_{2y} + \delta v_{2y} = 0$, where $k_{2xy} \equiv k_y$ and $k_{2xt} = \omega v_2$ (see eqs. [13]). Then we can simplify our system by eliminating the shock displacement $\zeta$, the remaining component of the velocity perturbation due to vorticity, and $\delta v_{1x}$. One equation remains, linking only pressure waves and the third modes' perturbations,

$$\begin{align*}
\omega P_2(v_1 - v_2) V_1 (V_1 P_1 \left(\omega^2 - k_y^2 v_1 v_2\right) - \left\{\omega^2 \left[(1 + h)P_1 - hP_2\right] v_1 v_2 \right\} V_2) \\
- h P_2 + k_x^2 \left[(h - 1)P_1 - hP_2\right] v_1 v_2) V_2) (\delta P_{1p} + \delta P_{1r}) \\
- \omega P_1(v_1 - v_2) V_1 (V_1 P_2 \left(\omega^2 - k_y^2 v_1 v_2\right) - \left\{\omega^2 [hP_1 - (1 + h)P_2 v_1 v_2\right\} V_2) \\
- (h - 1)P_2 + k_x^2 [hP_1 - (1 + h)P_2 v_1 v_2\right\} V_2) \\
\times (\delta P_{2p} + \delta P_{2r}) + \omega P_1(P_1 - P_2) P_2 (v_1 - v_2) \left(\omega^2 - k_y^2 v_1 v_2\right) \\
\times (V_1 - V_2) (\delta V_{1p} + \delta V_{1r}) - 2\omega P_1(P_1 - P_2) P_2 \\
\times \left[\omega^2 + k_x^2 v_2(-v_1 + v_2)\right] V_1 (V_1 - V_2)(\delta v_{1px} + \delta v_{1rx}) \\
- 2\omega^2 k_x P_1(P_1 - P_2) P_2 v_1 V_1 (V_1 - V_2)(\delta v_{1py} + \delta v_{1ry}) \\
+ 2\omega^3 P_1(P_1 - P_2) P_2 V_1 (V_1 - V_2)(\delta v_{2px} + \delta v_{2rx}) + 2\omega^3 k_x \\
\times P_1(P_1 - P_2) P_2 v_2 V_1 (V_1 - V_2)(\delta v_{2py} + \delta v_{2ry}) = 0. \quad (77)
\end{align*}$$

![Fig. 4 — Solution of our equations as a function of $k_y$ for the shock structure with accelerated particles’ distribution plotted in the top panel of Fig. 3. Top: Absolute values of real part (left) and imaginary part (right) of the eigenfrequency $\omega$. Bottom: Absolute values of real parts (left) and imaginary parts (right) of the x-component of the wavevectors $k_{2px}$ (pressure mode upstream, dotted lines), $k_{2sx}$ (third mode upstream, solid lines), $k_{2py}$ (pressure mode downstream, long-dashed lines), and $k_{2sy}$ (third mode downstream, short-dashed lines). Signs of these quantities are reported in Table 2.](image-url)
Now, equations (34) hold both for pressure waves and third modes. We can use them to write the above equation in terms of volume perturbations. We remark that each mode has its own $z$ and its own $k_x$, but for given $k_y$ and $\omega$, they are fixed by the dispersion relation, equation (24). We obtain

$$X_i = \frac{p_i}{C_{14}} \frac{V_1}{V_2} + \frac{P_1}{C_{13}} \frac{V_1}{V_2} \frac{\omega^2 - k_y^2 v_1 v_2}{C_10} (V_1 - V_2) + \frac{P_2}{C_{13}} \frac{V_1}{V_2} \frac{k_y^2 v_1 v_2}{C_10} \frac{\omega^2 - k_y^2 v_1 v_2}{C_10} (V_1 - V_2),$$

where the sum is over the pressure mode and the third mode. Now, recalling the definition of $A_i$, we see that $A_i = -\delta V_i / V$ (see also the discussion following eq. [36]). Then we have the four equations (55), (56), (57), and (58), plus equation (78), for a total of five linear homogeneous equations in five unknowns, four $A_i$ and $C$. The system has a nontrivial solution only if its determinant vanishes. This condition determines the eigenfrequency of the system.

7. RESULTS

In this section we apply our stability analysis to two shock solutions, a single solution (Fig. 3, top) by Amato & Blasi (2005) and a multiple solution (Fig. 3, bottom) by Amato et al. (2008). As we stated above, instability takes place when perturbations exist and grow exponentially with time. Furthermore, they must decay away from the discontinuity surface. These conditions may be summarized as

$$\text{Im}(\omega) < 0, \quad \text{Im}(k_{xi}) < 0, \quad \text{Im}(k_{iu}) > 0, \quad (79)$$

where the subscript $i$ indicates that these conditions must hold for all waves.

For illustrative purposes, we study at first the solution of D'yakov's equation (see Landau & Lifshitz 1987, eq. [90.10]),

$$2\omega u_2 \frac{k_y^2 + \omega^2}{u_1 u_2} - \frac{\omega^2}{u_1 u_2} \frac{k_y^2}{u_2} (1 + h_1) = 0,$$

$$\frac{2\omega u_2}{u_1} \frac{k_y^2 + \omega^2}{u_2} - \frac{\omega^2}{u_1 u_2} \frac{k_y^2}{u_2} (1 + h_1) = 0,$$

Fig. 5.— Same as Fig. 4, but for the distribution plotted in the bottom panel of Fig. 3 with the solid line.
which fixes the shock eigenfrequency in the linear regime, with

\[ h_L \equiv j^2 \frac{\partial V_2}{\partial P_2} \bigg|_{V_1, P_1} \, . \]  

(81)

We apply D’yakov’s analysis to a test particle solution for a strong shock (upstream Mach number \( M_1 \to \infty \)) in a polytropic fluid with index \( \gamma = 5/3 \). From equations (89.6) and (89.9) in Landau & Lifshitz (1987), we obtain, respectively, the compression factor \( R = V_1/V_2 = 4 \) and the downstream Mach number \( 1/\sqrt{3} \). The downstream sound speed is \( c_s = u_2/M_2 = u_1/RM_2 = u_1 \sqrt{5}/4 \). In such a system, sound waves can propagate away from the shock only downstream. The dispersion relation for such perturbations is straightforward,

\[ \left( \omega - \frac{1}{4} u_1 k_x \right)^2 = \frac{5}{16} u_1^2 \left( k_y^2 + k_x^2 \right), \]  

(82)

where \( k_x \) is the \( x \)-component of the sound wave. Let us write the eigenfrequency equation for such a shock. We need to calculate \( h_L \). This task is straightforward for a strong shock because the upstream pressure vanishes. As a consequence, the derivative \( (\partial V_2/\partial P_2)_{V_1, P_1} \) vanishes too (see eq. [64]) and \( h_L = 0 \). Equation (80) becomes

\[ 4\omega^3 - \frac{1}{2} \omega u_1^2 k_y^2 + u_1 k_x \left( \omega^2 + \frac{1}{4} u_1^2 k_y^2 \right) = 0. \]  

(83)

Equations (82) and (83) form a system to be solved with respect to \( \omega \) and \( k_x \), for a given value of \( k_y \). Since these equations are third-degree homogeneous in \( \omega, k_x, \) and \( k_y \), if \( (\omega, k_x) \) is a solution for a given \( k_y \), then \( (i\omega, i k_x) \) is a solution for \( i k_y \). Thus, the problem is completely solved once all the solutions for a given \( k_y \) are found. Let \( k_y = 1 \). Also, the upstream fluid speed \( u_1 \) can be set to 1 by redefining the ratio between the units of measure of frequency and wavenumber. We calculated all the solutions of the above equations, for these values of \( u_1 \) and \( k_y \), and display them in Table 1.

The first four solutions must be discarded because they correspond to waves propagating from downstream infinity to the shock. The fifth solution must be discarded too because it diverges exponentially at downstream infinity. More intriguing is the last solution. It seems to satisfy all the requirements in order to be a real physical solution, and it actually satisfies the conditions from equation (79). Is it a real instability? No, because it represents a pressure perturbation advected by the fluid since...
From equation (90.5) in Landau & Lifshitz (1987), we see that such a perturbation should have $\delta P = 0$. This is clearly absurd. We remark that one assumed $k_x$ to be the wavevector of a pressure perturbation in order to obtain equation (80), but we have now found a solution with vanishing pressure perturbation. Note that the fifth solution is affected by this problem too.

Summarizing, we solved the equation for the shock eigenfrequency in the test particle regime. We found six solutions, but no one has physical meaning, since they correspond to sound waves either propagating from downstream infinity to the shock surface or with vanishing pressure perturbation.

Below, we apply our theory to two particular shock structures (one with a multiple solution) in order to seek for an instability. We do not want to carry out a systematic analysis of a set of solutions, but we shall just show how our machinery works. For the sake of completeness in Figure 3 we report the particle distributions we used in our calculations.

We adopted a system of units of measure so that the following three quantities equal 1: the speed of light, the proton mass, and the numerical constant of the Bohm diffusion coefficient, i.e., $D_p = \psi(p)p$. Below, everything is expressed in these units.

We proceeded in the following way. We calculated the determinant of the system of five equations (55), (56), (57), (58), and (79), and we set it to 0, obtaining an equation with five unknowns, $\omega$, $k_{xpu}$, $k_{xnu}$, $k_{xpd}$, and $k_{xnd}$, which are respectively the frequency and the $x$-components of the wavevectors of the upstream pressure wave, the upstream third mode, the downstream pressure wave, and the downstream third mode. This forms a system of five equations together with four dispersions relations as in equation (24), each one linking $\omega$ and $k_x$ with their respective $k_x$. This is exactly what we did above when we solved equations (82) and (83). In Figure 4 and in Figures 5, 6, and 7, we illustrate the solution of our system of equations as a function of $k_y$, respectively, for the single and the multiple solutions of the shock structure. Absolute values of the real parts (left panels) and imaginary parts (right panels) of $\omega$ (top panels) and $k_{xpu}$, $k_{xnu}$, $k_{xpd}$, and $k_{xnd}$ (bottom panels) are plotted. The signs of these quantities are reported in Table 2. These solutions must all be discarded because they have some waves diverging at infinity, just like the first four solutions in Table 1. We used as starting values in our searching algorithm each one of the six corresponding limit solutions in the linear case. All the solutions we have found cannot be accepted for the same reasons discussed above for the linear regime.

8. DISCUSSION

In essence, our method is exact, except for the short-wavelength (WKBJ) approximation necessary to analytically treat perturbations in the inhomogeneous upstream precursor. One may however...
wonder where our method differs from previous work (Mond & O’C. Drury 1998; Toptygin 1999), which has reported the existence of corrugational instabilities.

Mond & O’C. Drury (1998) have reported the existence of both genuine corrugational instabilities and the spontaneous emission of sound waves, for some (not all) of their models. There is of course a number of differences between this paper and theirs: we do not use the two-fluid approximations and are interested in small-wavelength perturbations, contrary to them; we also notice the existence of perturbations in the upstream fluid, which they do not discuss.

Toptygin (1999) included a number of novelties in his treatment, but he too did not notice that particles would diffuse upstream, so that he neglects the third and the d-modes altogether. Despite this, he does have perturbations upstream (pressure waves), because he remarks that, for sufficiently long wavelength perturbations, these become supersonic with respect to the fluid alone. This occurs because particles and the fluid are tightly coupled in long-wavelength perturbations by diffusion, which traps nonthermal particles, so that the restoring pressure is the sum of particles’ and fluid’s pressures, which is larger than the pure fluid speed. This is of course correct and exists in our computations as well.

Contrary to these authors, we have found that shocks with non-vanishing particle pressure are stable to corrugational instabilities, even in the region of parameter space where multiple solutions are possible. We believe that the reason for this discrepancy lies in our inclusion of diffusion and the abandonment of the two-fluid approach, as we now argue. We stated in §1 that the only possibility for the shock destabilization (polytropic shocks without particles are well known to be corrugationally stable) lies in setting up a loop whereby perturbations shed by the shock in the downstream region return to the upstream region via particle diffusion and excite more perturbations; thus, perturbation energy is not lost to downstream infinity, but returns to the shock to create more havoc. However, the very same mechanism that brings particles back upstream also causes perturbations’ dampening; we have seen that diffusion leads to damping both pressure waves and the so-called third modes. Also, we have established that a part of the perturbation occurs in the d-mode (where d stands for damping), where particles are perturbed in phase space, but the total perturbation to the pressure vanishes. Thus, some fraction of the perturbation goes into a totally useless form (the d-mode), the new mode (the third mode) is always damped, and even pressure waves acquire a damping which is altogether neglected in the two-fluid approximation. In the end, while diffusion brings particles (and perturbations) back to the shock, the simultaneous damping is so strong that it makes the excitation of an instability ineffective.

We should remark that damping of pressure waves is weakest for the largest wavelengths, a limit which is inaccessible to us because of our WKBJ approximation, but is exactly the limit investigated by Mond & O’C. Drury (1998). While we have not found a large wavelength beyond which the shock becomes unstable, we cannot exclude that a proper treatment of the perturbations in the space–dependent precursor may yield a transition to the unstable regime.

A further comment is in order; of all possible geometries, the planar one is probably the least likely to display instability. Consider in fact a spherically symmetric explosion like a supernova. In this case, the perturbations (except of course for entropy and vorticity) generated downstream do not escape to infinity, as they do in the planar case, but return to the shock because the downstream region is finite and because they are deflected by a spatially dependent refraction index; once they reach the shock, they may generate further perturbations. The situation is even more promising when the shock is due to an accretion flow or is stallvorticity) generated downstream do not escape to infinity, as they do in the planar case, but return to the shock because the downstream region is finite and because they are deflected by a spatially dependent refraction index; once they reach the shock, they may generate further perturbations. The situation is even more promising when the shock is due to an accretion flow or is stalled. In fact, except for the presence of particles, this is exactly the scenario proposed (see Laming 2007 for an analytic approach and discussion) for the generation of asymmetries in proto–neutron stars; in this case, the mechanism for the instability of a stalled accretion shock is the reflection by the hard star surface into outgoing pressure waves of advected entropy perturbations, which return to the shock to generate more mischief. In the problem with particles, there is the extra complication due to diffusion, to overcome to generate instability. Given the relative complexity of even a pure fluid analysis (Laming 2007), it is likely that this problem will require a numerical approach.

APPENDIX A

D-MODE IN A HOMOGENEOUS SEMI-INFINITE MEDIUM

We give here an explicit expression for the d-mode in a homogeneous but semi-infinite medium. The d-mode is the solution of the diffusion equation

$$\frac{\partial \delta f}{\partial t} = \nabla \cdot (D \nabla \delta f) - u \frac{\partial \delta f}{\partial x},$$

where the speed $u$ is assumed constant because of homogeneity. We solve first the equation with $u = 0$; to do so, we remark that suitable boundary conditions are that $\delta f \rightarrow 0$ as $x \rightarrow \pm \infty$, depending on whether we are considering downstream or upstream regions, respectively. The solution can be obtained by separation of variables, obtaining

$$\delta f = \alpha(p, k) e^{i\omega t} e^{ikx} e^{\pm i\lambda y},$$

TABLE 2

SIGNS OF FUNCTIONS PLOTTED IN FIGURES 4, 5, 6, AND 7

| Function | Single Solution | Multiple Solution |
|----------|-----------------|-------------------|
| $\text{Re}(\omega)$ | + | + |
| $\text{Im}(\omega)$ | - | - |
| $\text{Re}(k_{\mu})$ | + | + |
| $\text{Im}(k_{\mu})$ | - | - |
| $\text{Re}(k_{\nu})$ | + | + |
| $\text{Im}(k_{\nu})$ | - | - |
| $\text{Re}(k_{a})$ | - | - |
| $\text{Im}(k_{a})$ | + | + |
| $\text{Re}(k_{b})$ | + | + |

Notes.—None satisfy eq. (79). No instability has been found by this analysis.
subject to the constraint

$$\nu = D\left(k_x^2 - k_y^2\right).$$  \hspace{1cm} (A3)

The sign of $k_x$ is the one that allows the solution to remain finite at infinity. Solutions belonging to different values of $k_x$ and $p$ can obviously be superposed, but we know that there is another boundary condition (eqs. 48 and 49 and following discussion) to be satisfied: at the shock, $x = 0$,

$$\delta f = g(p)e^{\omega t}e^{-ik_y},$$  \hspace{1cm} (A4)

which obviously gives

$$\nu = \omega, \quad Dk_x^2 = \omega + Dk_y^2, \quad \alpha(p) = g(p).$$  \hspace{1cm} (A5)

The function $g(p)$ was chosen so that

$$\int p^3 v g(p) dp = \int D(p)p^3 v g(p) dp = 0.$$  \hspace{1cm} (A6)

We have already seen (see the discussion leading to eq. 17) that these are the conditions for the vanishing of $\delta P_e$ and its first time derivative at the initial time and, thus, at all times. The same property is of course acquired by $\alpha(p)$, so that the mode we just found is surely a d-mode for the upstream region, in the homogeneous approximation. When we assume $u \neq 0$, the above formulae remain correct except for the substitution $\nu \rightarrow \nu + uk_x$.

**APPENDIX B**

**EXPLICIT DERIVATION OF FUNCTIONS $w$**

We need to find the solution to

$$D(k_{ad} - k_{su})g_u + \frac{u_2 - u_1}{3} \frac{\partial g_u}{\partial \ln p} = - \frac{u_2 - u_1}{3} A_{iu} \frac{\partial \delta f_u}{\partial \ln p} - \frac{1}{3} \frac{\partial f}{\partial \ln p} (\delta v_{2x} - \delta v_{1x}) - DA_{iu} \delta f_u (uk_{xiu} + k_{id}) + DA_{id} \delta f_d (uk_{xid} + k_{ad}).$$  \hspace{1cm} (B1)

This is an equation for $g_u(p)$,

$$E(p)g_u(p) + F \frac{\partial g_u(p)}{\partial \ln p} = G(p),$$  \hspace{1cm} (B2)

$$E(p) \equiv D(p)[k_{ad}(p) - k_{su}(p)],$$  \hspace{1cm} (B3)

$$F \equiv \frac{u_2 - u_1}{3},$$  \hspace{1cm} (B4)

$$G(p) \equiv - \frac{u_2 - u_1}{3} A_{iu} \frac{\partial \delta f_u(p)}{\partial \ln p} - \frac{1}{3} \frac{\partial f(p)}{\partial \ln p} (\delta v_{2x} - \delta v_{1x}) - D(p)A_{iu} \delta f_u(p)[uk_{xiu} + k_{id}] + D(p)A_{id} \delta f_d(p)[uk_{xid} + k_{ad}].$$  \hspace{1cm} (B5)

We find a solution of the homogeneous form of equation (B2),

$$E(p)w_c(p) + F \frac{\partial w_c(p)}{\partial \ln p} = 0,$$  \hspace{1cm} (B6)

$$w_c(p) = \exp\left[- \frac{1}{F} \int_{p_w}^{p} E(p') \frac{dp'}{p'}\right] = \exp\left\{- \frac{3}{u_2 - u_1} \int_{p_w}^{p} [k_{ad}(p') - k_{su}(p')]D(p') \frac{dp'}{p'}\right\}.$$  \hspace{1cm} (B7)

Now we seek a solution of equation (B2) of the form $g_u(p) = \tilde{C}(p)w_c(p)$,

$$E(p)\tilde{C}(p)w_c(p) - \frac{E(p)}{F} F \tilde{C}(p)w_c(p) + Fw_c(p) \frac{\partial \tilde{C}(p)}{\partial \ln p} = G(p),$$  \hspace{1cm} (B8)

$$Fw_c(p) \frac{\partial \tilde{C}(p)}{\partial \ln p} = G(p),$$  \hspace{1cm} (B9)
whose solution is
\[
g_u(p) = Cw_C(p) + w_C(p) \frac{1}{F} \int_{p_u}^p \frac{G(p') \, dp'}{w_C(p')}.
\]

Let us define
\[
w_{iu}(p) \equiv -w_C(p) \left\{ \int_{p_u}^p \frac{1}{w_C(p')} \frac{\partial f_u(p') \, dp'}{\partial \ln p'} - \frac{3}{u_2 - u_1} \int_{p_u}^p \frac{[k_{uid} + k_{uid}(p')]D(p')\delta f_u(p') \, dp'}{w_C(p')} \right\},
\]
\[
w_{id}(p) \equiv w_C(p) \frac{3}{u_2 - u_1} \int_{p_u}^p \frac{[k_{uid} + k_{uid}(p')]D(p')\delta f_d(p') \, dp'}{w_C(p')},
\]
\[
w_0(p) \equiv -w_C(p) \frac{1}{u_2 - u_1} \int_{p_u}^p \frac{1}{w_C(p')} \frac{\partial f(p') \, dp'}{\partial \ln p'}.
\]
Therefore, we obtain
\[
g_u(p) = Cw_C(p) + A_{iu}w_{iu}(p) + A_{id}w_{id}(p) + (\delta v_{2x} - \delta v_{1x})w_0(p).
\]

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