Difference Nullstellensatz in the case of finite group

D. V. Trushin

Abstract

We consider commutative rings on which a finite group is acting by automorphisms. Our purpose is to develop geometrical theory for difference equations with a given group of automorphisms. To solve this problem we extend the class of difference fields to a class of absolutely flat simple difference rings called pseudofields. We prove Nullstellensatz over pseudofields and investigate geometrical properties of pseudovarieties.

1 Introduction

Solving the following problem this text appeared. Let $K$ be a difference field, $R_n = K\{y_1, \ldots, y_n\}$ be a ring of difference polynomials over $K$. Then there are two mappings. For every set of difference polynomial $E$ the set $V(E) \subseteq K^n$ is defined as the set of all common zeros for $E$. Conversely, for every subset $X \subseteq K^n$ there is a difference ideal $I(X)$ consisting of all polynomials vanishing on $X$. The classical result says that the ideals $I(V(E))$ are perfect difference ideals. We want to find a difference field such that for every radical difference ideal $I$ there is the equality $t = I(V(t))$. However, there is no such fields. We consider all radical difference ideals because we want to guaranty that for every proper difference ideal $a \subseteq R_n$ there is a common zero. Thus we need to extend the class of all difference fields to a more wide class. The appropriate class is the class of all absolutely flat simple difference rings. We shall call such rings pseudofields. Changing field onto a pseudofields we are able to prove existence and uniqueness of difference closure (compare with [3 chapter 2, sec. 2.6]). Particularly, every difference field can be embedded in differencly closed pseudofield. We are able to choose the minimal such pseudofield and it will be unique up to isomorphism. For every differently closed pseudofield we develop geometrical theory of difference equations. We have only one essential lack in our theory, the group of automorphisms must be finite. For example, difference rings with the mentioned condition appears from difference equation with periodic functions.

Our purpose is to develop a theory providing needs of the geometrical theory of difference equations. It should be noted that we not only prove technical results about pseudofields but investigate geometrical properties of solutions.

*Xy-pic package is used
All necessary terms and notation are introduced in section 2. Section 3 is devoted to basic technique used in further sections. In subsection 3.1 we introduce the notion of pseudoprime ideal and investigate their properties. In the next subsection 3.2 we deal with pseudospectrum and introduce a topology on it. In subsection 3.3 the most important class of difference ring is presented. We prove the theorem of the Taylor homomorphism for this class of rings (statement 10). The most interesting case for us is the case of finite group of automorphisms.

Section 4 is devoted to this case. In its first subsection 4.1 we improve basic technical results obtained in section 3. Subsection 4.2 is devoted to relation between commutative structure of the ring and its difference structure. Since in difference algebra we are not able to produce fraction rings with respect to arbitrary multiplicatively closed sets, we need an alternative technique. This alternative technique is based on inheriting of properties. The main technical result is statement 11 allowing to avoid localization. To solve our main problem we need to extend the class of all difference fields to the class of absolutely flat simple difference rings. Such rings are called pseudofields. Structure of pseudo-fields is scrutinized in subsection 4.3. We introduce the notion of differenly closed pseudo-field and classify them up to isomorphism (statement 17). We prove that every pseudo-field (so thus every field) can be embedded to differenly closed pseudo-field (statements 19 and 20). Our technique is illustrated on the sequence of examples. Subsection 4.4 plays an auxiliary role. Its results has special geometric interpretation. The most important statements are statement 29 and its corollaries 30 and 31. The notion of differenly closed pseudo-field allows to produce geometric theory of difference equations with finite group of automorphisms. Section 4.5 is devoted to this geometrical theory.

The section is devoted to basic notions and objects used further. We shall define the interesting for us class of rings and the notion of pseudospectrum.

Let $\Sigma$ be an arbitrary group. A ring $A$ will be said to be difference ring if $A$ is an associative commutative ring with an identity element such that the group $\Sigma$ is acting on $A$ by ring automorphisms. A difference homomorphism of difference rings is a homomorphism preserving the identity element and commuting with $\Sigma$ action. A difference ideal is an ideal stable under the action of the group $\Sigma$.

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2 Terms and notation

The section is devoted to basic notions and objects used further. We shall define the interesting for us class of rings and the notion of pseudospectrum.
We shall write \( \Sigma \) instead of the word difference. A simple difference ring is a ring with no nontrivial difference ideals. The set of all \( \Sigma \)-ideals of \( A \) will be denoted by \( \text{Id}^\Sigma A \). For every ideal \( a \subseteq A \) and every element \( \sigma \in \Sigma \) the image of \( a \) under \( \sigma \) will be denoted by \( a^\sigma \).

The set of all, radical, prime, maximal ideals of \( A \) will be denoted by \( \text{Id} A \), \( \text{Rad} A \), \( \text{Spec} A \), \( \text{Max} A \), respectively. The set of all prime difference ideals of \( A \) will be denoted by \( \text{Spec}^\Sigma A \). For every ideal \( a \subseteq A \) the largest \( \Sigma \)-ideal laying in \( a \) will be denoted by \( a^\Sigma \). Such an ideal exists because it coincides with the sum of all difference ideals contained in \( a \). Note that

\[
a^\Sigma = \{ a \in a \mid \forall \sigma \in \Sigma : \sigma(a) \in a \}.
\]

So we have the mapping

\[
\pi : \text{Id} A \to \text{Id}^\Sigma A
\]
defined by the rule \( a \mapsto a^\Sigma \). Straightforward calculation shows that for every family of ideals \( a_\alpha \) we have

\[
\pi(\bigcap_\alpha a_\alpha) = \bigcap_\alpha \pi(a_\alpha).
\]

It is easy to see that for any ideal \( a \) there is the equality

\[
a^\Sigma = \bigcap_{\sigma \in \Sigma} a^\sigma.
\]

We shall define the notion of pseudoprime ideal of \( \Sigma \)-ring \( A \). Let \( S \subseteq A \) be a multiplicatively closed subset containing the identity element, and let \( q \) be a maximal \( \Sigma \)-ideal not meeting \( S \). Then the ideal \( q \) will be called pseudoprime. The set of all pseudoprime ideals will be denoted by \( \text{PSpec} A \) and called pseudospectrum.

Note that the restriction of \( \pi \) onto spectrum gives the mapping

\[
\pi : \text{Spec} A \to \text{PSpec} A.
\]

The ideal \( p \) will be called \( \Sigma \)-associated with pseudoprime \( q \) if \( \pi(p) = q \). Let \( q \) be a pseudoprime ideal, and let \( S \) be multiplicatively closed set from the definition of \( q \), then every prime ideal containing \( q \) and not meeting \( S \) is \( \Sigma \)-associated with \( q \). So, the mapping \( \pi : \text{Spec} A \to \text{PSpec} A \) is surjective.

Let \( S \) be multiplicatively closed set, and \( a \) be an ideal of \( A \). Then saturation of \( a \) with respect to \( S \) will be the following ideal

\[
S(a) = \bigcup_{s \in S} (a : s).
\]

If \( s \) is an element of \( A \), then saturation of \( a \) with respect to \( \{s^n\} \) will be denoted by \( a : s^\infty \).

If \( S \) is multiplicatively closed subset of \( A \), the ring of fraction of \( A \) with respect to \( S \) will be denoted by \( S^{-1} A \). If \( S = \{t^n\}_{n=0}^{\infty} \), then the ring \( S^{-1} A \) will
be denoted by $A_t$. If $p$ is a prime ideal of $A$ and $S = A \setminus p$, then the ring $S^{-1}A$ will be denoted by $A_p$.

For any subset $X \subseteq A$ the smallest difference ideal containing $X$ will be denoted by $[X]$. The smallest radical difference ideal containing $X$ will be denoted by $\{X\}$. The radical of an ideal $a$ will be denoted by $\tau(a)$. So, we have that $\{X\} = \tau([X])$.

Let $f: A \to B$ be a homomorphism of rings and let $a$ and $b$ be the ideals of $A$ and $B$, respectively. Then we define the extension $a^e$ to be the ideal $f(a)B$ generated by $f(a)$. The contraction $b^c$ is the ideal $f^*(b) = f^{-1}(b)$. If homomorphism $f: A \to B$ is difference homomorphism, then both extension and contraction of difference ideals are difference ones.

Let $f: A \to B$ be a $\Sigma$-homomorphism of difference rings, and let $q$ is a pseudoprime ideal of $B$. It is clear, that contraction $q^c$ is pseudoprime. So we have a mapping from $\text{PSpec } B$ to $\text{PSpec } A$. This mapping will be denoted by $f^*_{\Sigma}$. From the definition it follows that the following diagram is commutative

\[
\begin{array}{ccc}
\text{Spec } B & \xrightarrow{f^*_{\Sigma}} & \text{Spec } A \\
\downarrow \pi & & \downarrow \pi \\
\text{PSpec } B & \xrightarrow{f^*_{\Sigma}} & \text{PSpec } A
\end{array}
\]

The set of all radical $\Sigma$-ideals of $A$ will be denoted by $\text{Rad}^\Sigma A$. For the convenience maximal difference ideals will be called pseudomaximal. This set will be denoted by $\text{PMax } A$. It is clear, that every pseudomaximal ideal is pseudoprime ($S = \{1\}$). It is easy to see that radical difference ideal can be presented as an intersection of pseudoprime ideals. So, the objects with prefix pseudo have the same behavior as the objects without it.

The ring of difference polynomials $A\{Y\}$ is a ring $A[\Sigma Y]$, where $\Sigma$ acts in natural way. A difference ring $B$ will be called an $A$-algebra if there is a difference homomorphism $A \to B$. It is clear, that every $A$-algebra can be presented as a quotient ring of some polynomial ring $A\{Y\}$.

## 3 Basic technique

In this section we shall prove basic results about introduced set of difference ideals.

### 3.1 Pseudoprime ideals

**Statement 1.** Let $q$ and $q'$ be pseudoprime ideals of a difference ring $A$. Then

1. Ideal $q$ is radical.

2. For every ideal $p$ $\Sigma$-associated with $q$ there is the equality

$$q = \bigcap_{\sigma \in \Sigma} p^\sigma.$$


3. For every element \( s \notin q \) there is the equality
\[
(q : s^{\infty})\Sigma = q.
\]

4. For every element \( s \) from the equality \((q : s^{\infty})\Sigma = (q' : s^{\infty})\Sigma\) it follows that either \( s \) belongs to \( q \) and \( q' \), or \( q = q' \).

5. For every two difference ideals \( a \) and \( b \) the inclusion \( ab \subseteq q \) implies either \( a \subseteq q \), or \( b \subseteq q \).

**Proof.** (1). Let \( S \) be multiplicatively closed subset of \( A \) such that \( q \) is a maximal difference ideal not meeting \( S \). Then \( r(q) \) is a difference ideal containing \( q \) and not meeting \( S \). Consequently, \( r(q) = q \).

(2). The following is always true \( p \Sigma = \cap p^\sigma \). But from the definition we have \( q = p\Sigma \).

(3). Let \( p \) be a \( \Sigma \)-associated with \( q \) prime ideal. Then from (2) it follows that there exists a \( \sigma \in \Sigma \) such that \( s \notin p^\sigma \). Therefore there is the inclusion
\[
(q : s^{\infty}) \subseteq (p^\sigma : s^{\infty}) = p^\sigma,
\]
and consequently
\[
(q : s^{\infty}) \Sigma \subseteq p^\sigma \Sigma = q.
\]

The other inclusion is obvious.

Note that for every ideal \( a \) the equality \( a : s^{\infty} = A \) holds if and only if \( s \in a \). Therefore, we need to consider the case \( s \notin q \) and \( x \notin q' \). From the previous item we have
\[
q = (q : s^{\infty})\Sigma = (q' : s^{\infty})\Sigma = q'.
\]

(5). Let \( p \) be a \( \Sigma \)-associated with \( q \) prime ideal. Then either \( a \subseteq p \), or \( b \subseteq p \).

Suppose that the first one holds. Then
\[
a = a\Sigma \subseteq p\Sigma = q.
\]

We shall show that condition (3) does not hold for an arbitrary multiplicatively closed subset \( S \).

**Example 2.** Let \( \Sigma = \mathbb{Z} \), consider the ring \( A = K\Sigma \), where \( K \) is a field. Then this ring is of Krull dimension zero. So, every prime ideal is maximal. It is well-known fact that maximal ideals of \( A \) can be described in terms of maximal filters on \( \Sigma \). Namely, for an arbitrary filter \( \mathcal{F} \) of \( \Sigma \) we define the ideal
\[
m_{\mathcal{F}} = \{ x \in A \mid \{ n \mid x_n = 0 \} \in \mathcal{F} \}.
\]

There are two different types of maximal ideals. The first type corresponds to principal maximal filters
\[
m_k = \{ x \in A \mid x_k = 0 \}
\]
and the second type corresponds to ultrafilters $m_F$. It is clear, that for all ideals of the first type we have $(m_k)_\Sigma = 0$. But for any ultrafilter $\mathcal{F}$ the ideal $m_{\mathcal{F}}$ contains the ideal $K^\oplus \Sigma$ consisting of all finite sequences. Therefore, $(m_{\mathcal{F}})_\Sigma \neq 0$. As we can see not every minimal prime ideal containing zero ideal is $\Sigma$-associated with it. Additionally, set $S = A \setminus m_{\mathcal{F}}$, where $\mathcal{F}$ is an ultrafilter. Then

$$(S(0))_\Sigma = (m_{\mathcal{F}})_\Sigma \neq 0.$$ 

Let us note one peculiarity of radical difference ideals.

**Example 3.** Let $\Sigma = \mathbb{Z}$. Consider the ring $A = K \times K$, where $\Sigma$ acts as permutation of factors. Then

$$\{(1, 0)\} \{ (0, 1) \} \nsubseteq \{(1, 0) (0, 1)\},$$

because the left part is $A$ and the right part is $0$. So, the condition $\{X\} \{Y\} \subseteq \{XY\}$ does not hold.

**3.2 Pseudospectrum**

We shall provide a pseudospectrum by a structure of topological space such that the mapping $\pi$ will be continuous.

Let $A$ be an arbitrary difference ring, $X$ be a set of all its pseudoprime ideals. For every subset $E \subseteq A$ we define $V(E)$ as the set of all pseudoprime ideals containing $E$.

**Statement 4.** Using above notation the following holds:

1. If $a$ is a difference ideal generated by $E$, then

$$V(E) = V(a) = V(r(a)).$$

2. $V(0) = X$, $V(1) = \emptyset$.

3. Let $(E_i)_{i \in I}$ be a family of subsets of $A$. Then

$$V \left( \bigcup_{i \in I} E_i \right) = \bigcap_{i \in I} V(E_i).$$

4. For any difference ideals $a, b$ in $A$ the following holds

$$V(a \cap b) = V(ab) = V(a) \cup V(b).$$

**Proof.** Condition (1) immediately follows from the definition of $V(E)$ and the fact that pseudoprime ideal is radical. Conditions (2) and (3) are obvious. The last statement immediately follows from condition (5) of statement 1.
So, we see that the sets $V(E)$ satisfy the axioms for closed sets in topological space. We shall fix this topology on pseudospectrum. Consider the mapping

$$\pi: \text{Spec } A \rightarrow \text{PSpec } A.$$  

For every difference ideal $a$ we have

$$\pi^{-1}(V(a)) = V(a),$$

i.e., the mapping $\pi$ is continuous. Let us recall that $\pi$ is always surjective.

Denote the pseudospectrum of the difference ring $A$ by $X$. Then for every element $s \in A$ define the set $X_s$ as the complement of $V(s)$. From the definition we have that every open set can be presented as a union of the sets of the form $X_s$. In other words the family $\{X_s \mid s \in A\}$ form a basis of topology. It should be noted that the intersection $X_s \cap X_t$ is not necessarily of the form $X_u$.

**Statement 5.** Using above notation we have

1. $X_s \cap X_t = \cup_{\sigma, \tau \in \Sigma} X_{\sigma(t)}.$
2. $X_s = \emptyset$ iff $s$ is nilpotent.
3. $X$ is quasi-compact (that is, every open covering of $X$ has a finite subcovering).
4. There is one to one correspondence between the set of all closed subsets of pseudospectrum and the set of all radical difference ideals:

$$t \mapsto V(t) \quad V(E) \mapsto \bigcap_{q \in V(E)} q.$$  

**Proof.** Condition (1) is proved by straightforward calculation.

(2). Note that $X_s$ is not empty if and only if the set of all prime ideals not containing $s$ is not empty. The last one is equivalent to the condition: $s$ is not nilpotent.

(3). Let $\{V(a_i)\}$ be a centered family of closed subsets (that is, every intersection of finitely many elements is not empty), where $a_i$ are difference ideals. We need to show that $\cap_i V(a_i)$ is not empty. Suppose that contrary holds $\cap_i V(a_i) = \emptyset$. But

$$\bigcap_i V(a_i) = V(\sum_i a_i) = \emptyset.$$

The last equality is equivalent to condition that 1 belongs to $\sum_i a_i$. But in this situation 1 belongs to a finite sum. Therefore, corresponding intersection of finitely many closed subsets is empty, contradiction.

(4). The statement is immediately follows from the equality

$$\tau([E]) = \bigcap_{q \in V(E)} q.$$  

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Let us show that this equality holds. The inclusion $\subseteq$ is obvious. Let us show the other one. Let $g$ belong to radical of $[E]$, then consider the set of difference ideals containing $E$ and not meeting $\{g^n\}_{n=0}^\infty$. This set is not empty, since $[E]$ is in it. From Zorn’s lemma there is a maximal difference ideal with that property. From the definition this ideal is pseudoprime.

3.3 Functions on the group

For every commutative ring $B$ the set of all functions from $\Sigma$ to $B$ will be denoted by $F_B$. As a commutative ring it coincides with product $\prod_{\sigma \in \Sigma} B$. Let us provide $F_B$ by the structure of difference ring. We define $\sigma(f)(\tau) = f(\sigma^{-1}\tau)$.

For every element $\sigma$ of the group $\Sigma$ there is a homomorphism $\gamma_{\sigma}: F_B \rightarrow B$

$$f \mapsto f(\sigma)$$

It is clear that $\gamma_{\tau}(\sigma f) = \gamma_{\sigma^{-1}\tau}(f)$.

**Statement 6.** Let $A$ be a difference ring, and let $\varphi: A \rightarrow B$ be a homomorphism of rings. Then for every element $\sigma \in \Sigma$ there is a unique difference homomorphism $\Phi_{\sigma}: A \rightarrow F_B$ such that the following diagram is commutative

$$\begin{array}{ccc}
F_B & \xrightarrow{\Phi_{\sigma}} & B \\
\gamma_{\sigma} \downarrow & & \downarrow \\
A & \xrightarrow{\varphi} & B
\end{array}$$

**Proof.** By the data the homomorphism $\Phi_{\sigma}$ satisfies the property

$$\Phi_{\sigma}(a)(\tau^{-1}\sigma) = (\tau\Phi_{\sigma}(a))(\sigma) = \varphi(\tau a)$$

whenever $a \in A$ and $\tau \in \Sigma$. Consequently, if $\Phi_{\sigma}$ exists, then it is unique. Define the mapping $\Phi_{\sigma}$ by the following rule

$$\Phi_{\sigma}(a)(\tau) = \varphi(\sigma \tau^{-1} a).$$

It is clear that this mapping is a homomorphism. The following calculation shows that this homomorphism is a difference one.

$$(\nu\Phi_{\sigma}(a))(\tau) = \Phi_{\sigma}(a)(\nu^{-1}\tau) = \varphi(\sigma \tau^{-1} \nu a) = \Phi_{\sigma}(\nu a)(\tau).$$

The ring $F_B$ is an essential analogue of the Hurwitz series ring. The elements of $F_B$ are the analogues of the Taylor series. Homomorphisms $\Phi_{\sigma}$ are analogues of the Taylor homomorphism for the Hurwitz series ring. Therefore we shall call these homomorphisms the Taylor homomorphism at $\sigma$. The Taylor homomorphism at the identity of the group will be called simpler the Taylor homomorphism.

It should be noted that the set of all invariant points of $F_B$ can be identified with $B$. Namely, $B$ coincides with the set of all constant functions. So, we suppose that $B$ is embedded in $F_B$.
4 The case of finite group

4.1 Basic technique

From now we shall suppose that the group Σ is finite. First of all we shall prove more delicate technical results for the finite group.

**Statement 7.** Let $A$ be a difference ring, $q$ be a pseudoprime ideal of $A$, and $S$ be a multiplicatively closed subset in $A$. Then

1. Every minimal prime ideal containing $q$ is $Σ$-associated with $q$.
2. Restriction of $π$ onto $\text{Max } A$ is a well-defined mapping $π : \text{Max } A → \text{PMax } A$.
3. If $S ∩ q = \emptyset$, then $(S(q))_Σ = q$.

**Proof.** (1). Let $p'$ be a $Σ$-associated with $q$ ideal. Let $p$ be a prime ideal such that $q ⊆ p ⊆ p'$. Then $p$ is $Σ$-associated with $q$. Therefore $q = \cap_σ p^σ$. Let now $p''$ be an arbitrary minimal prime ideal containing $q$. Then $\cap_σ p^σ = q ⊆ p''$. Consequently, $p^σ ⊆ p''$ for some $σ$, and thus $p^σ = p''$.

(2). Let $m$ be a maximal ideal, and let $q = m_Σ$. We shall show that $q$ is a maximal difference ideal. Since the mapping $\text{Spec } A → \text{PSpec } A$ is surjective, it suffices to show that every prime ideal containing $q$ coincides with $m_σ$ for some $σ$. Indeed, Let $q ⊆ p$. Then since $q = \cap_σ m_σ$, we have $m_σ ⊆ p$ for some $σ$. The desired result holds, because $m$ is maximal.

(3). By the data there is a prime ideal $p$ such that $q ⊆ p$ and $S ∩ p = \emptyset$. Then there exists a minimal prime ideal $p'$ with the same condition. From the definition we have $S(q) ⊆ S(p') = p'$. Thus the equality $q = p'_Σ$ follows from condition (1).

Let $A$ be a difference ring, and $X$ be a pseudospectrum of $A$. For any radical difference ideal $t$ we set the closed subset $V(t)$ in $X$. Conversely, for every closed subset $Z$ we define the radical difference ideal $\cap_{q ∈ Z} q$.

**Statement 8.** The mentioned mappings are inverse to each other bijections between $\text{Rad}_Σ A$ and $\{ Z ⊆ X \mid Z = V(E) \}$. Suppose additionally that any radical difference ideal in $A$ is an intersection of finitely many prime ideals (for example $A$ is noetherian). Then the closed set is irreducible if and only if it corresponds to pseudoprime ideal.

**Proof.** The first statement follows from statement 5 item (4). Let us show that irreducible sets corresponds to pseudoprime ideals.

Let $q$ be a pseudoprime ideal and let $V(q) = V(a) ∪ V(b) = (a ∩ b)$. Then $q ⊇ a ∩ b$. Then either $q ⊇ a$, or $q ⊇ b$ (see statement 3 (4)). Suppose that the first condition holds. Then $V(q) ⊆ V(a)$. The other inclusion holds because of the choice of $V(a)$.

Conversely, let $t$ be a radical difference ideal. Suppose that $t$ is not pseudoprime. Let $p_1, \ldots, p_n$ be all minimal prime ideals containing $t$. Then the action
of $\Sigma$ on this set is not transitive. Let $p_1, \ldots, p_k$ be an orbit. Then form \[1\] chapter 1, sec. 6, prop. 1.11(1) it follows that $p_n$ is not contained in $\cup_{i=1}^k p_i$. Let $s$ be an element belonging to $p_n$ and not belonging to the mentioned union. Then $\sigma s$ does not belong to $\cup_{i=1}^k p_i$ for every $\sigma$. Then the element $t = \prod_{\sigma} \sigma(s)$ is invariant under the action of $\Sigma$ and do not belong to $\cup_{i=1}^k p_i$. So, ideals $(t: t)$ and $t + (t)$ are difference. Moreover, since $s$ divides $t$, $t$ is a zero divisor modulo $t$. Therefore both ideals strictly larger than $t$. We shall show that $t = (t: t) \cap (t + (t))$. The inclusion $\subseteq$ is obvious. Let us show the other one. Let $x \in (t: t) \cap (t + (t))$, then $x = a + rt$, where $a \in t$, $r \in A$. Then $x^2 = x(a + rt) = ax + rtx \in t$. Since ideal $t$ is radical, we have $x \in t$.

4.2 Inheriting of properties

Let $f: A \to B$ be a difference homomorphism of difference rings. We shall consider the following pairs of properties:

(A1): is a property of $f$, where $f$ is considered as a homomorphism

(A2): is a property of $f$, where $f$ is considered as a difference homomorphism

such that (A1) implies (A2). The idea is the following: finding such pair of properties, we shall reduce the difference problem to a non difference one.

The homomorphism $f: A \to B$ is said to have the going-up property if for every chain of prime ideals $p_1 \subseteq p_2 \subseteq \ldots \subseteq p_n$ in $A$ and every chain of prime ideals $q_1 \subseteq q_2 \subseteq \ldots \subseteq q_m$ in $B$ such that $0 < m < n$ and $q_i = p_i$ $(1 \leq i \leq m)$ the second chain can be extended to a chain $q_1 \subseteq q_2 \subseteq \ldots \subseteq q_n$ with condition $q_i = p_i$ $(1 \leq i \leq n)$.

The homomorphism $f: A \to B$ is said to have the going-down property if for every chain of prime ideals $p_1 \supseteq p_2 \supseteq \ldots \supseteq p_n$ in $A$ and every chain of prime ideals $q_1 \supseteq q_2 \supseteq \ldots \supseteq q_m$ in $B$ such that $0 < m < n$ and $q_i \supseteq p_i$ $(1 \leq i \leq m)$, the second chain can be extended to a chain $q_1 \supseteq q_2 \supseteq \ldots \supseteq q_n$ with condition $q_i = p_i$ $(1 \leq i \leq n)$.

Let $f: A \to B$ be a difference homomorphism. This homomorphism is said to have going-up (going-down) property for difference ideals if the mentioned above properties hold for the chains of pseudoprime ideals.

**Statement 9.** For every difference homomorphism $f: A \to B$ the following holds

1. In the following diagram

$$
\begin{array}{ccc}
\text{Spec } B & \xrightarrow{f^*} & \text{Spec } A \\
\downarrow \pi & & \downarrow \pi \\
\text{PSpec } B & \xrightarrow{f^*_\Sigma} & \text{PSpec } A
\end{array}
$$

if $f^*$ is surjective, then $f^*_\Sigma$ is surjective.
2. \( f \) has the going-up property \( \Rightarrow \) \( f \) has the going-up property for difference ideals.

3. \( f \) has the going-down property \( \Rightarrow \) \( f \) has the going-down property for difference ideals.

Proof. (1) This property follows from the fact that \( \pi \) is surjective.

(2). Let \( q_1 \subseteq q_2 \) be a chain of pseudoprime ideals of \( A \) and let \( q'_1 \) be a pseudoprime ideal in \( B \) contracting to \( q_1 \). Consider a prime ideal \( p'_1 \) \( \Sigma \)-associated with \( q'_1 \). The contraction of \( p'_1 \) to \( A \) will be denoted by \( p_1 \). Then \( p_1 \) will be \( \Sigma \)-associated with \( q_1 \). Let \( p_2 \) be a prime ideal \( \Sigma \)-associated with \( q_2 \). Then \( \cap_{p} p''_1 = q_1 \subseteq p_1 \). Thus, from [1 chapter 1, sec. 6, prop. 1.11(2)] it follows that for some \( \sigma \) we have \( p''_1 \subseteq p_2 \). Consider two chains of prime ideals \( p''_1 \subseteq p_2 \) in \( A \) and \( (p''_1)^\sigma \) in \( B \). From the going-up property there exists a prime ideal \( p''_2 \) such that \( (p''_1)^\sigma \subseteq p''_2 \) and \( (p''_2)^\sigma = p_2 \). Therefore the ideal \( (p''_2)^\Sigma \) is the desired pseudoprime ideal.

(3). Let \( q_1 \supseteq q_2 \) be a chain of pseudoprime ideals in \( A \), and let \( q'_1 \) be a pseudoprime ideal in \( B \) contracting to \( q_1 \). Let \( p'_1 \) be a prime ideal \( \Sigma \)-associated with \( q'_1 \). Its contraction to \( A \) will be denoted by \( p_1 \). Then \( p_1 \) is \( \Sigma \)-associated with \( q_1 \). Let \( p \) be a prime ideal \( \Sigma \)-associated with \( q_2 \). Then \( \cap_{p} p''_1 = q_2 \subseteq p_1 \). Consequently, for some \( \sigma \) we have \( p''_1 \subseteq p \) (see. [1 chapter 1, sec. 6, prop. 1.11(2)]).

The going-down property guaranties that there exists a prime ideal \( p'_2 \) with conditions \( p''_2 \subseteq p'_1 \) and \( (p'_2)^\sigma = p''_1 \). Then the ideal \( (p'_2)^\Sigma \) is the desired one. \( \square \)

Since not for every multiplicatively closed set \( S \) the fraction ring is a difference ring, we need to generalize the previous statement. Let \( f: A \to B \) be a difference homomorphism, and let \( X \) and \( Y \) be subsets of pseudospectra of \( A \) and \( B \), respectively, such that \( f''_Y (Y) \subseteq X \). We shall say that that the going-up property holds for \( f''_Y: Y \to X \) if for every chain of pseudoprime ideals \( p_1 \subseteq p_2 \subseteq \ldots \subseteq p_n \) in \( X \) and every chain of pseudoprime ideals \( q_1 \subseteq q_2 \subseteq \ldots \subseteq q_m \) in \( Y \) such that \( 0 < m < n \) and \( q_i'' = p_i \) \( (1 \leq i \leq m) \), the second chain can be extended to a chain \( q_1 \subseteq q_2 \subseteq \ldots \subseteq q_n \) in \( Y \) with condition \( q_i'' = p_i \) \( (1 \leq i \leq n) \).

We shall say that that the going-down property holds for \( f''_Y: Y \to X \) if for every chain of pseudoprime ideals \( q_1 \supseteq q_2 \supseteq \ldots \supseteq q_m \) in \( X \) and every chain of pseudoprime ideals \( q_1 \supseteq q_2 \supseteq \ldots \supseteq q_m \) in \( Y \) such that \( 0 < m < n \) and \( q_i'' = p_i \) \( (1 \leq i \leq m) \), the second chain can be extended to a chain \( q_1 \supseteq q_2 \supseteq \ldots \supseteq q_n \) in \( Y \) with condition \( q_i'' = p_i \) \( (1 \leq i \leq n) \).

**Statement 10.** Let \( f: A \to B \) be a difference homomorphism of difference rings. Pseudospectrum of \( A \) will be denoted by \( X \), pseudospectrum of \( B \) by \( Y \). Then the following holds:

1. Let for some \( s \in A \) \( u \in B \) the mapping

\[
    f^*: \text{Spec } B_{su} \to \text{Spec } A_u,
\]

is surjective. Then the mapping \( f''_Y: Y_{f(s)u} \to X_u \) is surjective.
2. Let for some \( s \in A \) the mapping

\[
f^*_s : \text{Spec } B_s \to \text{Spec } A_s
\]

has the going-up property. Then the mapping \( f^{*_s}_\Sigma : Y_{f(s)} \to X_s \) has the going-up property.

3. Let for some \( s \in A \) \( u \in B \) the mapping

\[
f^* : \text{Spec } B_{su} \to \text{Spec } A_s
\]

has the going-down property. Then the mapping \( f^{*_s}_\Sigma : Y_{f(s)u} \to X_s \) has the going-down property.

Proof. (1) Let \( q \in X_s \). Since \( s \notin q \), then there exists a prime ideal \( p' \) such that \( q \subseteq p' \) and \( s \notin p' \). Then there exists a minimal prime ideal \( p \) with this property. Consequently, \( p \) is \( \Sigma \)-associated with \( q \). By the data there is a prime ideal \( p_1 \) in \( B \) not containing \( f(s)u \) such that \( p'_1 = p \). Therefore the ideal \( (p_1)_\Sigma \) is the desired one.

(2) Let \( q_1 \subseteq q_2 \) be a chain of pseudoprime ideals of \( A \) not containing \( s \), and \( q'_1 \) be a pseudoprime ideal of \( B \) not containing \( f(s) \). Let \( p'_1 \) be a prime ideal \( \Sigma \)-associated with \( q'_1 \), and \( p_1 \) is its contraction to \( A \). As in item (1) we shall find a prime ideal \( p_2 \) \( \Sigma \)-associated with \( q_2 \) and not containing \( s \). Then \( \cap \sigma p'_{\sigma} = 0 \subseteq p_2 \). Thus for some \( \sigma \) we have \( p'_{\sigma} \subseteq p_2 \). Consider the sequence of ideals \( p'_{\sigma} \subseteq p_2 \) and ideal \( (p'_1)^{\sigma} \) contracting to \( p'_{\sigma} \). By the data there exists a prime ideal \( p_2 \) containing \( (p'_1)^{\sigma} \) and contracting to \( p_2 \). Then the ideal \( (p_2)_\Sigma \) is the desired one.

(3) Let \( q_1 \supseteq q_2 \) be a chain of pseudoprime ideals in \( A \) not containing \( s \) and \( q'_1 \) is a pseudoprime ideal in \( B \) contracting to \( q_1 \). As in item (2) we shall find a prime ideal \( p'_1 \) \( \sigma \)-associated with \( q'_1 \) and not containing \( f(s)u \). Its contraction will be denoted by \( p_1 \). Let \( p_2 \) be a prime ideal \( \Sigma \)-associated with \( q_2 \). Then \( \cap \sigma p'_{\sigma} = 0 \subseteq p_1 \). Thus for some \( \sigma \) we have \( p'_{\sigma} \subseteq p_1 \). By the data for the chain \( p_1 \supseteq p'_{\sigma} \) the ideal \( p'_1 \) there is a prime ideal \( p'_2 \) lying in \( p'_1 \) and contracting to \( p'_2 \). Then the ideal \( (p'_2)_\Sigma \) is the desired one.

\[ \square \]

Example 11. Let \( \Sigma = \mathbb{Z}/2\mathbb{Z} \), where \( \sigma = 1 \) is the nonzero element of the group, and let \( C \) be an algebraically closed field. Define \( A = C[x] \), where \( \sigma \) coincides with the identity mapping on \( A \). Now consider the ring \( B = C[t] \), where \( \sigma(t) = -t \). There is the embedding \( \varphi : A \to B \) such that \( x \mapsto t^2 \). Then this mapping is a difference homomorphism. So, we can identify \( A \) with the subring \( C[t^2] \) in \( B \). Now consider the set of prime difference ideals of the rings \( A \) and \( B \). It is clear that \( \text{Spec}^\Sigma B = \{0\} \) consists of one single point and \( \text{Spec}^\Sigma A = \text{Spec } A \). The contraction mapping \( \varphi^* : \text{Spec}^\Sigma B \to \text{Spec}^\Sigma A \) maps zero ideal to a zero ideal. It is clear that \( \text{Spec}^\Sigma B \) is dense in \( \text{Spec}^\Sigma A \) but does not contain an open in its closure. So \( \text{Spec}^\Sigma B \) is very poor. Now let us show what will happen if we use pseudoprime ideals instead of prime ones. It is clear that \( \text{PSpec } A = \text{Spec } A \). Let us describe \( \text{PSpec } B \). Consider the mapping \( \pi : \text{Spec } B \to \text{PSpec } B \). Every maximal ideal of \( B \) is of the form \((t - a)\), then
\((t - a)_{\Sigma} = (t^2 - a^2)\) is pseudoprime. Therefore the set of all pseudoprime ideals is the following \(\text{PSpec } B = \{0\} \cup \{(t^2 - a^2) \mid a \in C\}\). We can identify pseudomaximal spectrum with an affine line \(C\) by the rule \((t^2 - a^2) \mapsto a^2\).

Now consider the contraction mapping \(\varphi^*_\Sigma: \text{PSpec } B \to \text{PSpec } A\). As we can see \(\varphi^*_\Sigma(t^2 - a^2) = (x - a^2)\). Identifying pseudomaximal spectrum of \(A\) with \(C\) by the rule \((x - a^2) \mapsto -a\), we see that the mapping \(\varphi^*_\Sigma: \text{PMax } B \to \text{PMax } A\) coincides with the identity mapping. It is easy to see that the homomorphism \(\varphi: A \to B\) has the going-up and going-down properties. Therefore it has the going-up and going-down properties for difference ideals. But this is obvious from the discussion above. Consequently, the mapping \(\varphi^*_\Sigma\) is a homeomorphism between \(\text{PSpec } A\) and \(\text{PSpec } B\).

### 4.3 Pseudofields

A absolutely flat simple difference ring will be called a pseudofield.

**Statement 12.** For every pseudofield \(A\) the group \(\Sigma\) is transitively acting on \(\text{Max } A\). Moreover, as a commutative ring \(A\) is isomorphic to \(K^n\) where \(n\) is the number of maximal ideals in \(A\) and \(K\) is isomorphic to \(A/\mathfrak{m}\), where \(\mathfrak{m}\) is a maximal ideal in \(A\).

**Proof.** Let \(\mathfrak{m}\) be a prime ideal of \(A\). By the data this ideal is simultaneously maximal and minimal (see [1], chapter 3, ex. 11). Then \(\cap_\Sigma \mathfrak{m}\) is a difference ideal, and thus equals zero. Let \(\mathfrak{n}\) be arbitrary prime ideal of \(A\), then \(\cap_\Sigma \mathfrak{m}\mathfrak{n} = 0 \subseteq \mathfrak{n}\). Consequently, \(\mathfrak{n} = \mathfrak{m}\sigma\) for some \(\sigma\), i.e., \(\Sigma\) transitively acts on \(\text{Max } A\). Let \(\mathfrak{m}_1, \ldots, \mathfrak{m}_n\) be the set of all maximal ideals of \(A\). Then from [1], chapter 1, sec. 6, prop. 1.10 it follows that \(A\) is isomorphic to \(\prod_i A/\mathfrak{m}_i\). Since every element of \(\Sigma\) is an isomorphism, for every \(\sigma\) the field \(A/\mathfrak{m}\) is isomorphic to \(A/\mathfrak{m}\sigma\).

**Statement 13.** Let \(A\) be a difference ring, \(\mathfrak{q}\) be its difference ideal. The ideal is pseudomaximal if and only if \(A/\mathfrak{q}\) is pseudofield. In other words every simple difference ring is absolutely flat.

**Proof.** If \(A/\mathfrak{q}\) is a pseudofield, then \(\mathfrak{q}\) is a maximal difference ideal, and hence, pseudomaximal. Conversely, let \(\mathfrak{q}\) is pseudomaximal, and let \(\mathfrak{m}\) be a maximal ideal containing \(\mathfrak{q}\). Since \(\mathfrak{q}\) is maximal difference ideal, then \(\mathfrak{m}\) is \(\Sigma\)-associated with \(\mathfrak{q}\). Hence \(\mathfrak{q} = \cap_\Sigma \mathfrak{m}\). And from [1], chapter 1, sec. 6, prop. 1.10 it follows that \(A/\mathfrak{q} = \prod_\Sigma A/\mathfrak{m}\sigma\).}

As we see a simple difference ring and a pseudofield are the same notions. Note that the ring \(K A\) is a pseudofield if and only if \(A\) is a field. We shall introduce the notion of differenly closed pseudofield. Let \(A\) be a pseudofield. Consider the ring of difference polynomials \(\mathfrak{A}\{y_1, \ldots, y_n\}\). Let \(E \subseteq A\{y_1, \ldots, y_n\}\) be an arbitrary subset. The set of all common zeros of \(E\) in \(A^n\) will be denoted by \(V(E)\). Conversely, let \(X \subseteq A^n\) be an arbitrary subset. The set of all polynomials vanishing on \(X\) will be denoted by \(I(X)\). It is clear, that for any difference ideal \(\mathfrak{a} \subseteq A\{y_1, \ldots, y_n\}\) the following holds: \(\mathfrak{r} = I(V(\mathfrak{a}))\). A
pseudofield $A$ will be said to be a differencly closed pseudofield if for every $n$ and every difference ideal $a \subseteq A\{y_1, \ldots, y_n\}$ there is the equality $\tau(a) = I(V(a))$.

**Statement 14.** If $A$ is a differencly closed pseudofield, then every differencly finitely generated over $A$ pseudofield coincides with $A$.

**Proof.** Every differencly finitely generated over $A$ pseudofield can be presented as $A\{y_1, \ldots, y_n\}/q$, where $q$ is a pseudomaximal ideal. It is easy to see that the ideal $q$ is of the form $I(x)$ for some $x \in A^n$. Hence $q = [y_1 - a_1, \ldots, y_n - a_n]$. Therefore $A\{y_1, \ldots, y_n\}/q$ coincides with $A$.

**Statement 15.** A pseudofield $F_K$ is differencly closed if and only if $K$ is algebraically closed.

**Proof.** Let $F_K$ be differencly closed. We recall that $K$ can be embedded into $F_K$. Consider the ring

$$R = F_K\{y\}/(\ldots, \sigma y - y, \ldots)_{\sigma \in \Sigma}.$$  

As a commutative ring it isomorphic to $F_K[y]$. Let $f$ be a polynomial in one variable with coefficients in $K$. The ideal $(f(y))$ is a nontrivial ideal in $F_K[y]$. Moreover, since $f(y)$ is an invariant element, the mentioned ideal is difference. Consequently, $B = R/(f(y))$ is nontrivial difference ring. Let $m$ be a pseudomaximal ideal in $B$. Then pseudofield $B/m$ coincides with $F_K$. Denote the image of the element $y$ in $F_K$ by $t$. By the definition $f(t) = 0$ and $t$ is invariant. Thus $t$ is in $K$. So, $K$ is algebraically closed.

Conversely, let $K$ be an algebraically closed field. Let $a$ be an arbitrary difference ideal in $F_A\{y_1, \ldots, y_n\}$. Consider the algebra

$$B = F_A\{y_1, \ldots, y_n\}/a.$$  

We shall show that for every element $s \in B$ not belonging to nilradical there is a difference homomorphism $f : B \to F_K$ such that $f(s) \neq 0$. From statement 6 it follows that it suffices to show that there is a homomorphism $\psi : B \to K$ such that for some $\sigma$ the following diagram is commutative

$$\begin{array}{ccc}
B & \overset{\psi}{\longrightarrow} & K \\
\downarrow & & \\
FA & \overset{\gamma_a}{\longrightarrow} & K
\end{array}$$

Indeed, consider the ring $B_s$ and let $n$ be a maximal ideal of $B_s$. Then $B_s/n$ is a finitely generated algebra over $K$ and is a field. Therefore $B_s/n$ coincides with $K$ (see the Hilbert Nullstellensatz). It is easy to see that this mapping satisfies all desired properties.

**Statement 16.** Let $A$ be a pseudofield. Suppose that every differencly generated over $A$ by one single element pseudofield coincides with $A$. Then the Taylor homomorphism is an isomorphism between $A$ and $F_K$, where $K = A/m$ for every maximal ideal $m$ of $A$. 

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Proof. Let $m$ be a maximal ideal of $A$. Consider a field $K = A/m$ and define the ring $F K$. From statement 6 it follows that there exists a difference homomorphism $\Phi: A \to F K$ for a quotient homomorphism $\pi: A \to K$.

Since $A$ is a simple difference ring, $\Phi$ is injective. Let us show that $\Phi$ is surjective. Assume that contrary holds, and there is an element $\eta \in F K \setminus A$. Consider the ring $A\{y\} = A[\ldots, \sigma y, \ldots]$ and its quotient ring $K[\ldots, \sigma y, \ldots]$. In the last ring there is a maximal ideal $(\ldots, \sigma y - \eta(\sigma), \ldots)$. This ideal contracts to the maximal ideal $m'$ in $A\{y\}$. From statement 7 it follows that the ideal $n = m'_2$ is pseudomaximal. So, $A\{y\}/n$ is a pseudofield differently generated over $A$ by one singly element. Thus $A\{y\}/n$ coincides with $A$. From the other hand, there is a homomorphism

$$
\varphi: A\{y\}/n \to A\{y\}/m' \to K[\ldots, \sigma y, \ldots]/(\ldots, \sigma y - \eta(\sigma), \ldots) = K.
$$

The restriction of this homomorphism to $A$ coincides with a quotient homomorphism. Statement 8 guaranties that there is the embedding $\Psi: A\{y\}/n \to F K$. From the uniqueness of the Taylor homomorphism it follows that the restriction of the last mapping to $A$ coincides with $\Phi$.

From the definition we have $\Psi(y)(\sigma) = \eta(\sigma)$. Consequently, $\Psi(y) = \eta$, and thus the image of pseudofield $A\{y\}/n$ contains $\eta$. Form the other hand the image coincides with $A$, contradiction.

The following theorem is a corollary of the previous statements.

**Theorem 17.** Let $A$ be a pseudofield, then the following conditions are equivalent:

1. $A$ is differencly closed.
2. Every differencly finitely generated over $A$ pseudofield coincides with $A$.
3. Every pseudofield generated over $A$ by one single element coincides with $A$.
4. Pseudofield $A$ is isomorphic to $F K$, where $K$ is algebraically closed field.
Moreover, since $m$ is acting as follows $\sigma$ is a group $\Sigma = {\sigma} \subseteq \Sigma$. As a commutative ring it is isomorphic to $F K$. We only need to show that $K$ is algebraically closed (see statement 14). For that we shall repeat the first half of the proof of statement 15.

We know that every pseudofield differencly generated by one single element over $F K$ coincides with $F K$. Let us recall that $K$ can be embedded into $F K$. Consider the ring 

$$R = F K\{y\}/(\ldots, \sigma y - y, \ldots)_{\sigma \in \Sigma}.$$  

As a commutative ring it is isomorphic to $F K[y]$. Let $f$ be a polynomial in one variable with coefficients in $K$. The ideal $(f(y))$ is a nontrivial ideal in $F K[y]$. Moreover, since $f(y)$ is an invariant element, the mentioned ideal is difference. Let $m$ be a pseudomaximal ideal in $B$, then the pseudofield $B/m$ coincides with $F K$. The image of $y$ in $F K$ will be denoted by $t$. By the definition we have $f(t) = 0$ and $t$ is an invariant element. Thus $t$ is in $K$. Therefore the field $K$ is algebraically closed.

$(4) \Rightarrow (1)$. It follows from statement 15.  

**Example 18.** Consider the field of complex numbers $C$ and its automorphism $\sigma$ (the complex conjugation). This pair can be regarded as a difference ring with a group $\Sigma = \mathbb{Z}/2\mathbb{Z}$. Let $C[x]$ be a ring of polynomials over $C$ and automorphism $\sigma$ is acting as follows $\sigma(f(x)) = (-x)$. Then the ideal $(x^2 - 1)$ is a difference ideal. Consider the ring $A = \mathbb{C}[x]/(x^2 - 1)$. As a commutative ring it can be presented as follows

$$\mathbb{C}[x]/(x^2 - 1) = \mathbb{C}[x]/(x - 1) \times \mathbb{C}[x]/(x + 1) = \mathbb{C} \times \mathbb{C}.$$  

Under this mapping the element of the field $c$ maps to $(c, c)$ and $x$ maps to $(1, -1)$. The automorphism acts as follows $(a, b) \mapsto (\overline{a}, \overline{b})$. Consider the projection of $A$ on its first factor. For this homomorphism there is the Taylor homomorphism $A \to \mathbb{F} C$. As a commutative ring the ring $\mathbb{F} C$ coincides with $\mathbb{C} \times \mathbb{C}$. Automorphism acts as follows $(a, b) \mapsto (b, a)$. The Taylor homomorphism is defined by the following rule $a + bx \mapsto (a + b, \overline{a} - \overline{b})$.

As we can see there are two homomorphisms the first one is $f: A \to \mathbb{C} \times \mathbb{C}$ and is defines by the rule $a + bx \mapsto (a + b, a - b)$ and the second one is $g: A \to \mathbb{C} \times \mathbb{C}$ and is defined by the rule $a + bx \mapsto (a + b, \overline{a} - \overline{b})$. Then composition $g \circ f^{-1}$ acts as follows $(a, b) \mapsto (a, \overline{b})$.

So, pseudofield $A$ is differencly closed. Moreover, the homomorphism $g \circ f^{-1}$ transforms the initial action of $\sigma$ into more simple one.

Let $A$ be a pseudofield, and let $m$ be its maximal ideal. Then the residue field of $m$ will be denoted by $K$, i.e., $K = A/m$. Let $L$ be the algebraical closure of $K$. The pseudofield $F L$ will be denoted by $\overline{A}$. Let $\varphi: A \to L$ is a composition of the quotient morphism and a natural embedding of $K$ to $L$. Let $\Phi: A \to \overline{A}$ be the Taylor homomorphism corresponding to $\varphi$. We know that $\overline{A}$ is differencly closed. Let us show that $\overline{A}$ is a minimal differencly closed pseudofield containing $A$. 

**Proof.** $(1) \Rightarrow (2)$. It follows from statement 14.  

$(2) \Rightarrow (3)$. Is trivial.  

$(3) \Rightarrow (4)$. From statement 16 it follows that the ring $A$ is isomorphic to $F K$.
Statement 19. Let $D$ be a differentially closed pseudofield such that $A \subseteq D \subseteq \overline{A}$. Then $D = \overline{A}$.

Proof. Consider the sequence of rings $A \subseteq D \subseteq \overline{A}$. Let $m$ be a maximal ideal of $\overline{A}$. Then we have the following sequence of fields $A/A \cap m \subseteq D/D \cap m \subseteq \overline{A}/m$. Since $D$ is differentially closed from theorem 17 it follows that the field $D/D \cap m$ coincides with $L = \overline{A}/m$. Now consider the composition of $D \rightarrow D/D \cap m$ and $D/D \cap m \rightarrow L$. Let $\Psi : D \rightarrow L$ be the corresponding Taylor homomorphism. From the uniqueness of the Taylor homomorphism it follows that $\Psi$ coincides with the initial embedding of $D$ to $\overline{A}$. So $D$ satisfies the condition of statement 16. □

Statement 20. Let $B$ be a differentially closed pseudofield containing $A$. Then there exists an embedding of $\overline{A}$ to $B$ over $A$.

Proof. On the following diagram arrows present the embeddings of $A$ to $\overline{A}$ and to $B$ respectively:

\[
\begin{array}{ccc}
  \overline{A} & \xleftarrow{A} & B \\
  A & \xrightarrow{B} & \\
  \end{array}
\]

Let $m$ be a maximal ideal in $B$. Then it contracts to maximal ideal $m^c$ in $A$. Since $A$ absolutely flat ring there exists an ideal $n$ in $\overline{A}$ contracting to $m^c$ (see [1], chapter 3, ex. 29, ex. 30). So, we have

\[
\begin{array}{ccc}
  \overline{A}/n & \xleftarrow{A/m^c} & B/m \\
  A/m^c & \xrightarrow{B/m} & \\
  \end{array}
\]

By the definition the field $\overline{A}/n$ is the algebraic closure of $A/m^c$ and $B/m$ is algebraically closed (theorem 17). Therefore there exists an embedding of $\overline{A}/n$ to $B/m$.

\[
\begin{array}{ccc}
  \overline{A} & \xrightarrow{B} & \\
  \overline{A}/n & \xleftarrow{A/m^c} & B/m \\
  \end{array}
\]

So, there is a homomorphism $\overline{A} \rightarrow B/m$. Then statement 19 guaranties that there is a difference homomorphism $\varphi$ such that the following diagram is commutative

\[
\begin{array}{ccc}
  \overline{A} & \xrightarrow{\varphi} & B \\
  \overline{A}/n & \xleftarrow{A/m^c} & B/m \\
  \end{array}
\]

The restriction of $\varphi$ to $A$ coincides with the Taylor homomorphism for the mapping $A \rightarrow B/m$. From the uniqueness it follows that the Taylor homomorphism coincides with the initial embedding of $A$ to $B$. □
Example 21. Let $\Sigma = \mathbb{Z}/2\mathbb{Z}$, and let $\sigma = 1$ be a nonzero element of $\Sigma$. Consider the field $\mathbb{C}(t)$, where $t$ is a transcendental element over $\mathbb{C}$. We assume that action of $\Sigma$ is trivial on $\mathbb{C}(t)$. Consider the following system of difference equations

$$\begin{cases}
\sigma x = -x, \\
x^2 = t.
\end{cases}$$

Let $L$ be the algebraical closure of the field $\mathbb{C}(t)$. Then difference closure of $\mathbb{C}(t)$ coincides with $F_L$. From the definition we have $F_L = L \times L$, where the first factor corresponds to 0 and the second one to 1 in $\mathbb{Z}/2\mathbb{Z}$. Then our system has two solutions $(\sqrt{t}, -\sqrt{t})$ and $(-\sqrt{t}, \sqrt{t})$.

Moreover, we are able to construct the field containing solutions of this system. Consider the ring of polynomials $\mathbb{C}(t)[x]$, where $\sigma x = -x$. Then the ideal $(x^2 - t)$ is maximal difference ideal. Define $D = \mathbb{C}(t)[x]/(x^2 - t)$.

By the definition $D$ is a minimal field containing solutions of the system. From the other hand, statement 6 guaranties that $D$ can be embedded to difference closure of $\mathbb{C}(t)$.

Example 22. Consider a ring $A = \mathbb{C} \times \mathbb{C}$ and a group $\Sigma = \mathbb{Z}/4\mathbb{Z}$. Let $\sigma = 1$ be a generator of $\Sigma$. Let $\Sigma$ act on $A$ by the following rule $\sigma(a, b) = (b, a)$. Then $\sigma$ is an automorphism of fourth order. Consider the projection of $A$ onto the first factor. Then there exists a homomorphism $\Phi: A \to \mathbb{F}_C$ such that the following diagram is commutative

$$
\begin{array}{ccc}
A & \xrightarrow{\pi} & \mathbb{C} \\
| \downarrow & & \downarrow \\
\Phi \quad & & \gamma_e \\
\end{array}
$$

where $\pi$ is the projection onto the first factor of $A$. Pseudofield $\mathbb{F}_C$ is of the following form $\mathbb{C}_0 \times \mathbb{C}_1 \times \mathbb{C}_2 \times \mathbb{C}_3$, where $\mathbb{C}_i$ is a field $\mathbb{C}$ over point $i$ of $\Sigma$. Using this notation, the homomorphism $\gamma_e$ coincides with the projection onto the first factor. The element $\sigma$ acts on $\mathbb{F}_C$ by the right transaction. The Taylor homomorphism is defined by the rule $(a, b) \mapsto (a, \overline{b}, \overline{a}, b)$. Consider the embedding of $\mathbb{C}$ into $A$ by the rule $c \mapsto (c, c)$ and the embedding into $\mathbb{F}_C$ by the rule $(c, \overline{c}, \overline{c}, c)$. These both embeddings induce the structure of $\mathbb{C}$-algebra. Since dimensions of $A$ and $\mathbb{F}_C$ equals 2 and 4, respectively, $\mathbb{F}_C$ is generated by one single element over $A$. We shall find this element explicitly. Consider the element $x = (i, i, i, i)$ of $FA$. This element does not belong to $A$, therefore $\mathbb{F}_C = A\{x\}$. We have the following relations on the element $x$: $\sigma x = x$ and $x^2 + 1 = 0$. Comparing the dimensions, we get $\mathbb{F}_C = A\{y\}/[\sigma x - x, x^2 + 1]$.

Example 23. Let $\mathbb{C}$ be a field of complex numbers considered as a difference ring over $\Sigma = \mathbb{Z}/2\mathbb{Z}$, and let $\sigma = 1$ be a nonzero element of the group. Then the system of equations

$$\begin{cases}
x\sigma x = 0, \\
x + \sigma x = 1
\end{cases}$$
has no solutions in every difference overfield containing \( C \). But the ideal \([x \sigma x, x + \sigma x - 1]\) of the ring \( C\{x\} \) is not trivial. Therefore the system has solutions in difference closure of \( C \). The closure coincides with \( F \, C \). Namely, \( F \, C = C \times C \), where the first factor corresponds to zero and the second one to the element \( \sigma \). Then the solutions are \((1, 0)\) and \((0, 1)\).

**Example 24.** Let \( U \) be an open subset in complex plane \( C \), and let \( \Sigma \) be a finite group of automorphisms of \( U \). The ring of all holomorphic functions in \( U \) will be denoted by \( A \). Then \( A \) is a \( \Sigma \)-algebra with respect to the action \((\sigma \varphi)(z) = \varphi(\sigma^{-1}z)\). Difference closure of \( C \) is \( F \, C \). Namely, \( F \, C = C \times C \), where the first factor corresponds to zero and the second one to the element \( \sigma \).

Then the solutions are \((1, 0)\) and \((0, 1)\).

**Statement 25.** Let \( A \) be a pseudofield. The following conditions are equivalent:

1. \( A \) is differentially closed
2. For every \( n \) and every set \( E \subseteq A\{y_1, \ldots, y_n\} \) if there is a common zero for \( E \) in \( B^n \), where \( B \) is a pseudofield containing \( A \), then there is a common zero in \( A^n \).

**Proof.** (1)\(\Rightarrow\)(2). Let \((b_1, \ldots, b_n)\) be a common zero for \( E \) in \( B^n \). Consider a substitution homomorphism \( A\{y_1, \ldots, y_n\} \rightarrow B \) by the rule

\[
 f(y_1, \ldots, y_n) \mapsto f(b_1, \ldots, b_n).
\]

Then its kernel contains \( E \) and is not trivial. Consequently, there exists a pseudomaximal ideal \( m \) containing \( E \). Then pseudofield \( A\{y_1, \ldots, y_n\}/m \) coincides with \( A \). Denote the images of \( y_i \) in \( A \) by \( a_i \). Then \((a_1, \ldots, a_n)\) is the desired common zero in \( A^n \).

(2)\(\Rightarrow\)(1). Let us show that every pseudofield differentially finitely generated over \( A \) coincides with \( A \). Let \( B \) be a pseudofield differentially finitely generated over \( A \). Then it can be presented in the following form

\[
 B = A\{y_1, \ldots, y_n\}/m
\]

where \( m \) is a pseudomaximal ideal. Then this ideal has a common zero in \( B^n \), namely, \((y_1, \ldots, y_n)\). Consequently, there is a common zero

\[
 (a_1, \ldots, a_n) \in A^n.
\]

Consider a substitution homomorphism \( A\{y_1, \ldots, y_n\} \rightarrow A \) by the rule \( y_i \mapsto a_i \). Then all elements of \( m \) maps to zero. So there is a difference homomorphism \( B \rightarrow A \). Thus \( B \) coincides with \( A \). \(\Box\)
4.4 Differently finitely generated algebras

The section is devoted to different technical conditions on differently finitely generated algebras.

Lemma 26. Let \( A \) be a ring with finitely many minimal prime ideals. Then there exists an element \( s \in A \) such that there is only one minimal prime ideal in \( A_s \).

Proof. Let \( p_1, \ldots, p_n \) are all minimal prime ideal of \( A \). Then from [1, chapter 1, sec. 6, prop. 1.11(II)] it follows that there exists an element \( s \) such that

\[
s \in \bigcap_{i=2}^{n} p_i \setminus p_1.
\]

Then there is only one minimal prime ideal in \( A_s \) and this ideal corresponds to \( p_1 \).

Lemma 27. Let \( A \subseteq B \) be rings such that \( A \) is an integral domain and \( B \) is finitely generated over \( A \). Then there exists an element \( s \in A \) with the following property. For any algebraically closed field \( L \) every homomorphism \( A_s \to L \) can be extended to a homomorphism \( B_s \to L \).

Proof. Let \( S = A \setminus 0 \), consider the ring \( S^{-1}B \). This ring is a finitely generated algebra over the field \( S^{-1}A \). Then there are finitely many minimal prime ideals in \( S^{-1}B \). These ideals corresponds to ideals in \( B \) contracting to 0. Let \( p \) be one of them. Consider the rings \( A \subseteq B/p \). From [1, chapter 5, ex. 21] it follows that there exists an element \( s \in A \) with the following property. For every algebraically closed field \( L \) every homomorphism \( A_s \to L \) can be extended to a homomorphism \( (B/p)_s \to L \). Considering the composition of the last one with \( B_s \to (B/p)_s \), we extend the initial homomorphism to the homomorphism \( B_s \to L \).

Lemma 28. Let \( A \subseteq B \) be rings such that \( A \) is an integral domain and \( B \) is finitely generated over \( A \). Then there exists an element \( s \in A \) such that the corresponding mapping \( \text{Spec} \, B_s \to \text{Spec} \, A_s \) is surjective.

Proof. From the previous lemma we find an element \( s \). Let \( p \) be a prime ideal in \( A \) not containing \( s \). The residue field of \( p \) will be denoted by \( K \). Let \( L \) denote the algebraic closure of \( K \). The composition of the mappings \( A \to K \) and \( K \to L \) will be denoted by \( \varphi: A \to L \). By the definition we have \( \varphi(s) \neq 0 \). Consequently there exists a homomorphism \( \overline{\varphi}: B \to L \) extending \( \varphi \). Then \( \ker \overline{\varphi} \) is the desired ideal laying over \( p \).

Statement 29. Let \( A \subseteq B \) be difference rings, \( B \) being differently finitely generated over \( A \) and there are finitely many minimal prime ideals in \( A \). Then there exists a nonnilpotent element \( u \) in \( A \) with the following property. For every differently closed pseudofield \( \Omega \) and every difference homomorphism \( \varphi: A \to \Omega \) such that \( \varphi(u) \neq 0 \) there exists a difference homomorphism \( \overline{\varphi}: B \to \Omega \) with condition \( \overline{\varphi}|_A = \varphi \).
Proof. From theorem 17 it follows that $\Omega$ is of the following form $FL$, where $L$ is algebraically closed field. Let $\gamma_\sigma \Omega \to L$ be corresponding substitution homomorphisms.

We shall reduce theorem to the case where $A$ and $B$ are reduced. Let assume that we have proved the theorem for rings without nilpotent elements. Let $a$ and $b$ be nilradicals of $A$ and $B$, respectively. Let $s' \in A/a$ be the desired element, denote by $s$ some primage of $s'$ in $A$. Let $\varphi: A \to \Omega$ be a difference homomorphism with condition $\varphi(s) \neq 0$. Since $\Omega$ do not contain nilpotent elements, $a$ is in kernel of $\varphi$. Consequently, there exists a homomorphism $\varphi': A/a \to \Omega$.

Since $\varphi(s') = \varphi(s) \neq 0$, from our hypothesis it follows that there is a difference homomorphism $\overline{\varphi}: B/b \to \Omega$.

The the desired homomorphism $\overline{\varphi}: B \to \Omega$ is a composition of quotient homomorphism and $\overline{\varphi}$.

Now we suppose that nilradicals of $A$ and $B$ are zero. From lemma 26 it follows that there exists an element $s \in A$ such that $A_s$ contains one minimal prime ideal. Since $A$ has no nilpotent elements, $A_s$ is an integral domain. Let us apply lemma 27 to the pair $A_s \subseteq B_s$. So, there exists an element $t \in A$ such that for any algebraically closed field $L$ every homomorphism of $A_{st} \to L$ can be extended to a homomorphism $B_{st} \to L$. Denote the element $st$ by $u$. Let us show that the desired property holds. Let $\varphi: A \to \Omega$ be a difference homomorphism such that $\varphi(u) \neq 0$. Then for some $\sigma$ we have $\gamma_\sigma \circ \varphi(u) \neq 0$. So, there is a homomorphism $\varphi_\sigma: A \to L$. We shall extend $\varphi_\sigma$ to a homomorphism $B \to L$ as it shown in the following diagram (numbers show the order).

Homomorphism 1 appears from condition $\varphi_\sigma(u) \neq 0$ and the universal property of localization. Homomorphism 2 exists because of definition of $u$. 

\[
\begin{array}{ccc}
A_u & B_u & A_u \\
\varphi_{\sigma} & L & \gamma_{\sigma} \\
A & \Omega & A
\end{array}
\begin{array}{ccc}
A & B_u & B_u \\
\varphi & L & \gamma_{\sigma} \\
A & \Omega & A
\end{array}
\begin{array}{ccc}
A_u & B_u & B_u \\
\varphi_{\sigma} & L & \gamma_{\sigma} \\
A & \Omega & A
\end{array}
\begin{array}{ccc}
A & B_u & B \\
\varphi & L & \gamma_{\sigma} \\
A & \Omega & A
\end{array}
\]
Homomorphism 3 is constructed as a composition. Then from statement 4 there exists a difference homomorphism \( \overline{\varphi} \). Since diagram is commutative, from the uniqueness of the Taylor homomorphism for \( A \) it follows that restriction of \( \overline{\varphi} \) onto \( A \) coincides with \( \varphi \).

There are two important particular cases of this statement.

**Corollary 30.** Let \( A \subseteq B \) be difference rings, \( B \) being differencly finitely generated over \( A \), and \( A \) is a pseudo integral domain. Then there exists a non nilpotent element \( u \) in \( A \) with the following property. For every differencly closed pseudofield \( \Omega \) and every difference homomorphism \( \varphi: A \to \Omega \) such that \( \varphi(u) \neq 0 \) there exists a difference homomorphism \( \overline{\varphi}: B \to \Omega \) with condition \( \overline{\varphi}|_A = \varphi \).

**Proof.** Since \( A \) is a pseudo integral domain, there are finitely many minimal prime ideals in \( A \). Indeed, let \( p \) be minimal prime ideal. Then it is \( \Sigma \)-associated with zero ideal. So \( \cap_{\sigma} p^\sigma = 0 \). Let \( q \) be an arbitrary minimal prime ideal of \( A \). Then \( \cap_{\sigma} p^\sigma \subseteq q \). Therefore for some \( \sigma \) we have \( p^\sigma \subseteq q \). But \( q \) is a minimal prime ideal, hence \( p^\sigma = q \). So, \( p^\sigma \) are all minimal prime ideals of \( A \).

**Corollary 31.** Let \( A \subseteq B \) be difference rings, \( B \) being differencly finitely generated over \( A \), and \( A \) is a differencly finitely generated algebra over a pseudofield. Then there exists a non nilpotent element \( u \) in \( A \) with the following property. For every differencly closed pseudofield \( \Omega \) and every difference homomorphism \( \varphi: A \to \Omega \) such that \( \varphi(u) \neq 0 \) there exists a difference homomorphism \( \overline{\varphi}: B \to \Omega \) with condition \( \overline{\varphi}|_A = \varphi \).

**Proof.** Every pseudofield is an Artin ring and thus is noetherian. If \( A \) is differencly finitely generated over a pseudofield, then \( A \) is finitely generated over it. Hence \( A \) is noetherian. Consequently, there are finitely many minimal prime ideals in \( A \).

The second technical condition is concerned with extensions of pseudoprime ideals.

**Statement 32.** Let \( A \subseteq B \) be difference rings, \( B \) being differencly finitely generated over \( A \), and there are finitely many minimal prime ideals in \( A \). Then there exists an element \( u \) in \( A \) such that the mapping

\[
(\text{PSpec } B)_u \to (\text{PSpec } A)_u
\]

is surjective.

**Proof.** We may suppose that nilradicals of the rings are zero. From lemma 20 it follows that there exists an element \( s \in A \) such that \( A_s \) is an integral domain. Further, as in lemma 27 there is an element \( t \) such that \( B_st \) is integral over \( A_st[x_1, \ldots, x_n] \) and elements \( x_1, \ldots, x_n \) are algebraically independent over \( A_st \). Let \( u = st \). From theorem 11 chapter 5, th. 5.10 it follows that the mapping

\[
\text{Spec } B_u \to \text{Spec } A_u[x_1, \ldots, x_n]
\]

is surjective. It is clear that the mapping

\[
\text{Spec } A_u[x_1, \ldots, x_n] \to \text{Spec } A_u
\]
is surjective too. So from statement 10 the mapping
\[(\text{PSpec } B)_u \rightarrow (\text{PSpec } A)_u\]
is surjective.

\[\square\]

**Statement 33.** Let \( A \subseteq B \) be differencly finitely generated algebras over a pseudofield. Then there exists an element \( u \) in \( A \) such that the mapping
\[(\text{PMax } B)_u \rightarrow (\text{PMax } A)_u\]
is surjective.

**Proof.** Since algebra \( A \) is differensly finitely generated over a pseudofield, \( A \) is noetherian. Consequently, there are finitely many minimal prime ideals in \( A \).

Following the proof of the previous statement, we are finding the element \( u \) such that the mapping \( \text{Spec } B_u \rightarrow \text{Spec } A_u \) is surjective. Since \( A_u \) and \( B_u \) are finitely generated over an Artin ring, the contraction of any maximal ideal is a maximal ideal. So, the mapping \( \text{Max } B_u \rightarrow \text{Max } A_u \) is well-defined and surjective. Then statement 7 (2) completes the proof. \( \square \)

### 4.5 Geometry

In this section we develop geometrical theory of difference equations with solutions in pseudofields. This theory is quite similar to the theory of polynomial equations.

Let \( A \) be a differencly closed pseudofield. The ring of difference polynomial \( A\{y_1, \ldots, y_n\} \) will be denoted by \( R_n \). For every subset \( E \subseteq R_n \) we shall define the subset \( V(E) \) of \( A^n \) as follows
\[
V(E) = \{ a \in A^n \mid \forall f \in E : f(a) = 0 \}.
\]
This set will be called a pseudovariety. Conversely, let \( X \) be an arbitrary subset in \( A^n \), then we set
\[
I(X) = \{ f \in R_n \mid f|_X = 0 \}.
\]
This ideal is called the ideal of definition of \( X \). Let now \( \text{Hom}_A^\Sigma(R_n, A) \) denote the set of all difference homomorphisms from \( R_n \) to \( A \) over \( A \). Consider the mapping
\[
\varphi : A^n \rightarrow \text{Hom}_A^\Sigma(R_n, A)
\]
by the rule: every point \( a = (a_1, \ldots, a_n) \) maps to a homomorphism \( \xi_a \) such that \( \xi_a(f) = f(a) \). The mapping \( \psi : \text{Hom}_A^\Sigma(R_n, A) \rightarrow \text{PMax } A \) by the rule \( \xi \mapsto \ker \xi \) will be denoted by \( \psi \). So, we have the following sequence
\[
A^n \xrightarrow{\varphi} \text{Hom}_A^\Sigma(R_n, A) \xrightarrow{\psi} \text{PMax } A.
\]

**Statement 34.** The mappings \( \varphi \) and \( \psi \) are bijections.
Proof. The inverse mapping for \( \varphi \) is given by the rule
\[
\xi \mapsto (\xi(y_1), \ldots, \xi(y_n)).
\]
Since \( A \) is differencly closed, for every homomorphism \( \xi: R_n \to A \) its kernel is of the form \( \ker \xi = [y_1 - a_1, \ldots, y_n - a_n] \), where \( a_i = \xi(y_i) \). So, the mapping \( \psi \) injective and surjective.

It is clear that under the mapping \( \psi \circ \varphi \) the set \( V(E) \) of \( A^n \) maps to the set \( V(E) \) of PMax \( R_n \). So, the sets \( V(E) \) define a topology on \( A^n \) and the mentioned mapping is a homeomorphism. Therefore we can identify pseudomaximal spectrum of \( R_n \) with an affine space \( A^n \).

Corollary 35. The mappings \( \varphi \) and \( \psi \) are homeomorphisms.

Theorem 36. Let \( a \) be a difference ideal in \( R_n \). Then \( r(a) = I(V(a)) \).

Proof. Since \( A \) is an Artin ring, \( A \) is a Jacobson ring. \( R_n \) is finitely generated over \( A \), consequently \( R_n \) is a Jacobson ring too. Therefore every radical ideal in \( R_n \) can be presented as an intersection of maximal ideals. Hence every radical difference ideal can be presented as an intersection of pseudomaximal ideals (statement 7 item (2)). Now use the correspondence between points of \( V(a) \) and pseudomaximal ideals (statement 34).

Let \( X \) be a pseudovariety and let \( I(X) \) be its ideal of definition in the ring \( R_n = A\{y_1, \ldots, y_n\} \). Then the ring \( R_n/I(X) \) can be identified with the ring of polynomial functions on \( X \) and will be denoted by \( A(X) \). Let \( X \) and \( Y \) be two pseudovarieties laying in \( A^n \) and \( A^m \), respectively. Let the mapping \( f: A^n \to A^m \) be given such that \( f_i(x_1, \ldots, x_n) \) are difference polynomials and \( f(X) \subseteq Y \). Then the restriction of \( f \) onto \( X \) will be called a polynomial mapping from \( X \) to \( Y \). For every polynomial mapping \( f: X \to Y \) there is a difference homomorphism \( f^*: A\{Y\} \to A\{X\} \) by the rule \( f^*(\xi) = \xi \circ f \).

Conversely, for every difference homomorphism \( \varphi: A\{Y\} \to A\{X\} \) over \( A \) we shall define
\[
\varphi^*: \text{Hom}^\Sigma_{A}(A\{X\}, A) \to \text{Hom}^\Sigma_{A}(A\{Y\}, A)
\]
by the rule \( \varphi^*(\xi) = \xi \circ \varphi \). Let us recall that pseudovariety \( X \) can be identified with \( \text{Hom}^\Sigma_{A}(A\{X\}, A) \). Then we have the mapping
\[
\varphi^*: X \to Y.
\]

Statement 37. The constructed mappings are inverse to each other bijections between the set of all polynomial mappings from \( X \) to \( Y \) and the set of all difference homomorphisms from \( A\{Y\} \) to \( A\{X\} \).
Proof. The proof is routine.

Since every pseudofield is an Artin ring and differencly finitely generated algebra over pseudofield is finitely generated, every algebra finitely generated over a pseudofield is noetherian. So, we have the following.

**Statement 38.** Every pseudovariety is a noetherian topological space.

The following statements are devoted to the geometrical properties of polynomial mappings.

**Statement 39.** Let \( f: X \to Y \) be a polynomial mapping with dense image, and let \( Y \) be irreducible. Then the image of \( f \) contains an open in its closure.

**Proof.** Let \( A\{X\} \) and \( A\{Y\} \) be coordinate rings of pseudovarieties. Then the mapping \( f \) give us the mapping

\[ f^*: A\{Y\} \to A\{X\}. \]

Since the image of \( f \) is dense, the homomorphism \( f^* \) is injective. From statement \[33\] it follows that there exists an element \( s \in A\{Y\} \) such that the mapping \( \text{PMax} A\{X\}_s \to \text{PMax} A\{Y\}_s \) is surjective. But from statement \[34\] the last mapping coincides with \( f: X_s \to Y_s \). Since \( Y \) is irreducible, every open subset is dense.

**Statement 40.** Let \( f: Y \to X \) be a polynomial mapping with the dense image. Then there exists an element \( u \in A\{X\} \) such that the mapping \( f: Y_{f^*(u)} \to X_u \) is open.

**Proof.** Let the rings \( A\{X\} \) and \( A\{Y\} \) will be denoted by \( C \) and \( D \), respectively. Since the image of \( f \) is dense, \( f^* \) is injective. So, \( C \) can be identified with the subring in \( D \). From lemma \[26\] it follows that there exists an element \( s \in C \) such that \( C_s \) is an integral domain and \( C_s \subseteq D_s \). Since \( C_s \) is an integral domain and \( D_s \) is finitely generated over \( C_s \), there exists an element \( t \in C \) such that \( D_{st} \) is a free \( C_{st} \)-module (see \[2\] chapter 8, sec. 22, th. 52]). Let us denote \( st \) by \( u \). Then \( D_u \) is faithfully flat algebra over \( C_u \), thus \[1\] chapter 3, ex. 16] and \[1\] chapter 5, ex. 11] corresponding mapping \( \text{Spec} D_u \to \text{Spec} C_u \) has the going-down property and surjective. Statement \[10\] (1) and (3) guaranties that the mapping \( (\text{PSpec} D)_u \to (\text{PSpec} C)_u \) is surjective and has the going-down property. Let us show that \( f: Y_u \to X_u \) is open.

It suffices to show that the image of every principal open set is open. Let \( Y_t \) be a principal open set, then

\[ Y_u \cap Y_t = \cup_{\sigma \tau} Y_{\sigma(u)\tau(t)}. \]

It suffices to consider the set of the form \( Y_{\sigma(u)\tau(t)} \). Since \( Y_w = Y_{\sigma(w)} \), it suffices to consider the set of the form \( Y_{uw} \).

To show that the set \( f(Y_{uw}) \) is open we shall use the criterion \[1\] chapter 7, ex. 22]. Let \( Y' \) and \( X' \) be pseudospectra of \( D \) and \( C \), respectively. Not that
every irreducible closed subset in $X$ has the following form $X'_0 \cap X_u$, where $X'_0$ is an irreducible subset in $X'$. Consider the set $f(Y''_{uv})$ and let $X'_0$ be an irreducible closed subset in $X'$. Consider $f(Y''_{uv}) \cap X'_0$. Suppose that the last set is not empty. We have

$$f(Y''_{uv}) \cap X'_0 = f(Y''_{uv} \cap f^{-1}(X'_0)).$$

Let $X'_0 = V(q)$, where $q \in \text{PSpec } C$. Therefore

$$f(Y''_{uv}) \cap X'_0 = f(Y''_{uv} \cap V(q^e)).$$

The last set is not empty. Thus there exists a prime ideal $q'$ in $D$ such that $q^e \subseteq q'$ and $uv \notin q'$. Since $D_{uv}$ is a flat $C_u$-module, using the same arguments as above, we see that the mapping $\text{Spec } D_{uv} \to \text{Spec } C_u$ has the going-down property. Therefore the mapping $f: Y''_{uv} \to X_u$ has the going-down property. Now consider the chain of pseudoprime ideals $(q')^c \supseteq q$ in $C$ and $q'$ in $D$. Then there exists a pseudoprime ideal $q''$ in $D$ such that $(q'')^c = q$. Therefore homomorphism $C/q \to D/q^e$ is injective. Now consider the pair of rings

$$(C/q)_u \subseteq (D/q^e)_{uv}.$$

From lemma 26 it follows that there exists an element $s \in C/q$ such that $(C/q)_{us}$ is an integral domain. Then lemma 28 guaranties that for some element $t \in (C/q)_{st}$ the mapping

$$\text{Spec } (D/q^e)_{uvst} \to \text{Spec } (C/q)_{ust}$$

is surjective. Since rings in the last expression are finitely generated algebras over an Artin ring, the mapping

$$\text{Max } (D/q^e)_{uvst} \to \text{Max } (C/q)_{ust}$$

is surjective. From statement 7 (2) it follows that the mapping

$$(\text{PMax } D/q^e)_{uvst} \to (\text{PMax } C/q)_{ust}$$

is surjective. Thus $X_{ust} \cap (X'_0 \cap X)$ is contained in $f(Y_{uv})$. Now we are able to apply the criterion [1, chapter 7, ex. 22]. To complete the proof we need to remember that $\text{PMax } C$ can be identified with $X$ and $\text{PMax } D$ with $Y$. 

\[\square\]

**References**

[1] M.F. Atiyah, I.G. Macdonald. Introduction to commutative algebra. Addison-Wesley. 1969.

[2] H Matsumura. Commutative algebra. The Benjamin/cummings publishing company, 1980.

[3] A Levin. Difference algebra. Springer, 2008.