A STRUCTURE-PRESERVING SCHEME FOR THE ALLEN–CAHN EQUATION WITH A DYNAMIC BOUNDARY CONDITION

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Abstract. We propose a structure-preserving finite difference scheme for the Allen–Cahn equation with a dynamic boundary condition using the discrete variational derivative method [9]. In this method, how to discretize the energy which characterizes the equation is essential. Modifying the conventional manner and using an appropriate summation-by-parts formula, we can use a central difference operator as an approximation of an outward normal derivative on the boundary condition in the scheme. We show the stability and the existence and uniqueness of the solution for the proposed scheme. Also, we give the error estimate for the scheme. Numerical experiments demonstrate the effectiveness of the proposed scheme. Besides, through numerical experiments, we confirm that the long-time behavior of the solution under a dynamic boundary condition may differ from that under the Neumann boundary condition.

1. Introduction. Let $T > 0$ be a finite time and let $L > 0$ be the length of the one-dimensional material. In this paper, we study the following Allen–Cahn equation [1]:

$$\partial_t u = \partial_x^2 u - W'(u) \quad \text{in } (0, L) \times (0, T]$$

(1)

under the following dynamic boundary condition:

$$\partial_t u(0, t) = \partial_x u(x, t)|_{x=0} - W'(u(0, t)) \quad \text{in } (0, T],$$

(2)

$$\partial_t u(L, t) = -\partial_x u(x, t)|_{x=L} - W'(u(L, t)) \quad \text{in } (0, T].$$

(3)

The unknown function $u$ is the order parameter, which represents the concentration of one of two components in a binary mixture. Moreover, $W$ is the potential, and $W'$ is its derivative. For example, $W$ can be a double-well potential, i.e.,...
$W(s) = (1/4)(s^2 - 1)^2$ for all $s \in \mathbb{R}$. In this article, we assume that the potential $W$ is in $C^4(\mathbb{R})$ and satisfies the following properties:

$$W'(0) = 0, \quad W(s) \geq \mu s^2 - c \quad \text{for all } s \in \mathbb{R},$$

where $\mu$ is a positive constant and $c$ is a non-negative constant. Let us define the “local energy” $G$ and the “global energy” $J$, which characterize the equation (1):

$$G(u, \partial_x u) := \frac{|\partial_x u|^2}{2} + W(u), \quad J(u) := \int_0^L G(u, \partial_x u) dx.$$ 

We remark that the above words “local energy” and “global energy” are ones for space, not for time, and that these “local” and “global” are different from ones of the words “local existence” and “global existence,” which appear later. Then, the solution of the equation (1) satisfies the following inequality:

$$\frac{d}{dt} \{ J(u(t)) + W(u(0, t)) + W(u(L, t)) \} \leq 0 \quad \text{(5)}$$

under the boundary conditions (2) and (3).

From a mathematical point of view, the above problem (1)–(3) with an initial value has been studied in [2, 5, 6, 10, 15, 19]. Here, we remark that the original problem was considered in the two-dimensional or three-dimensional case, where the boundary condition (2) and (3) includes the Laplace–Beltrami operator which plays the role of diffusion on the boundary. Calatroni and Colli proved the existence and uniqueness of the solution of the problem (1)–(3) with an initial value, where the Laplace–Beltrami operator disappears on the boundary [2].

From a numerical point of view, there is a lot of study of a structure-preserving scheme for the Allen–Cahn equation with classical (non-dynamical) boundary conditions, for example, Dirichlet or Neumann boundary conditions (see, for instance, [7, 9, 11, 18]). Also, the results of a structure-preserving scheme for a non-local Allen–Cahn equation with Neumann or periodic boundary conditions can be found in [14, 17, 20, 25]. In this article, structure-preserving means that the scheme inherits the conservative property such as mass conservation or the dissipative property such as energy dissipation. In [23], Yoshikawa mentioned that the merit of the structure-preserving scheme is that we obtain the stability of numerical solutions automatically. Besides that, he mentioned that the advantage of the structure-preserving scheme is that various strategies for the continuous case such as the energy method can be applied to the scheme similarly. Actually, Yoshikawa and co-authors applied the energy method to show the existence and uniqueness of the solution and the error estimate for the scheme (see [8, 21, 22, 23, 24]).

Here, we remark that there are few results for the Allen–Cahn equation with dynamic boundary conditions. These are different from the more studied Neumann boundary conditions, and such give a different long-time behavior of the solution (see Appendix for an example). In [13], a numerical scheme for semilinear problems with the dynamic boundary condition (2) and (3) has been considered in a finite element approach, and the error estimate has been obtained. However, there are no results of a structure-preserving scheme for the above problem (1)–(3) in a finite difference approach to the best of our knowledge. Meanwhile, there are some numerical studies of the Cahn–Hilliard equation with different dynamic boundary conditions (see, for example, [3, 4] for the finite element method, [16] for the finite volume method, and [8] for the finite difference method). In [8], Fukao, Yoshikawa, and Wada proposed structure-preserving schemes for the Cahn–Hilliard equation.
with two different dynamic boundary conditions in the one-dimensional case, respectively, based on the discrete variational derivative method (DVDM) proposed by Furihata and Matsuo [9]. We remark that they use a forward difference operator as an approximation of an outward normal derivative on the discrete boundary condition of the structure-preserving scheme.

The rest of this paper proceeds as follows. In section 2, we propose a structure-preserving scheme for (1)–(3), whose solution satisfies the discrete version of the dissipative property (5). In section 3, we prove that the solution of the proposed scheme satisfies the global boundedness. In section 4, we prove that the scheme has a unique solution under a specific condition. In section 5, we prove the error estimate for the scheme. In section 6, we show that the numerical experiments demonstrate the effectiveness of the scheme.

2. Proposed scheme. In this section, we propose a scheme for (1)–(3) and show that it has a property corresponding to (5).

2.1. Preparation. We define \( u_k^{(m)} \) \((k = −1, 0, 1, \ldots, K, K + 1, m = 0, 1, \ldots)\) to be the approximation to \( u(x, t) \) at location \( x = k\Delta x \) and time \( t = m\Delta t \), where \( \Delta x \) is a space mesh size, i.e., \( \Delta x := L/K \) and \( \Delta t \) is a time mesh size. They are also written in vector as

\[
U^{(m)} := (U_0^{(m)}, U_1^{(m)}, \ldots, U_K^{(m)}, U_{K+1}^{(m)})^T \quad \text{or} \quad U^{(m)} := (U_0^{(m)}, U_1^{(m)}, \ldots, U_{K-1}^{(m)}, U_K^{(m)})^T.
\]

The superscript \((m)\) is omitted when no confusion occurs. Guess the meaning of \( U \) from the context.

Remark 1. \( U_{K+1}^{(m)} \) and \( U_K^{(m)} \) are the artificial quantity and determined by the imposed discrete boundary condition.

Let us define the difference operators \( \delta^+_k, \delta^-_k, \delta^{(1)}_k, \) and \( \delta^{(2)}_k \) concerning subscript \( k \) by

\[
\begin{align*}
\delta^+_k f_k &:= \frac{f_{k+1} - f_k}{\Delta x}, \\
\delta^-_k f_k &:= \frac{f_k - f_{k-1}}{\Delta x}, \\
\delta^{(1)}_k f_k &:= \frac{f_{k+1} - f_{k-1}}{2\Delta x}, \\
\delta^{(2)}_k f_k &:= \frac{f_{k+1} - 2f_k + f_{k-1}}{(\Delta x)^2} \quad (k = 0, 1, \ldots, K).
\end{align*}
\]

for all \( \{f_k\}_{k=-1}^{K+1} \in \mathbb{R}^{K+2} \). Similarly, we define the difference operator \( \delta^+_m \) corresponding superscript \((m)\) by

\[
\delta^+_m f^{(m)} := \frac{f^{(m+1)} - f^{(m)}}{\Delta t}.
\]

As a discretization of the integral, we adopt the summation operator \( \sum_{k=0}^{K} \) : \( \mathbb{R}^{K+1} \rightarrow \mathbb{R} \) defined by

\[
\sum_{k=0}^{K} f_k := \frac{1}{2} f_0 + \sum_{k=1}^{K-1} f_k + \frac{1}{2} f_K \quad \text{for all} \quad \{f_k\}_{k=0}^{K} \in \mathbb{R}^{K+1}.
\]

For use later, we define the difference quotient. Let \( \Omega \) be a domain in \( \mathbb{R} \). For a function \( F \in C^1(\Omega) \), and \( \xi, \eta \in \Omega \), the difference quotient \( dF/d(\xi, \eta) \) of \( F \) at \((\xi, \eta)\) is defined as follows:

\[
dF
d(\xi, \eta) := \begin{cases} 
F(\xi) - F(\eta) \over \xi - \eta, & (\xi \neq \eta), \\
F'(\xi), & (\xi = \eta).
\end{cases}
\]
Here, let us define two discrete local energies $G_{d,k}^\pm: \mathbb{R}^{K+3} \to \mathbb{R}$ by

$$G_{d,k}^+(U) := \frac{\langle \delta^+_k U_k \rangle^2}{2} + W(U_k) \quad (k = 0, \ldots, K-1),$$

$$G_{d,k}^-(U) := \frac{\langle \delta^-_k U_k \rangle^2}{2} + W(U_k) \quad (k = 1, \ldots, K),$$

for all $U \in \mathbb{R}^{K+3}$. Furthermore, we define discrete global energy $J_d: \mathbb{R}^{K+3} \to \mathbb{R}$ as follows:

$$J_d(U) := \frac{1}{2} \left\{ \sum_{k=0}^{K-1} G_{d,k}^+(U) \Delta x + \sum_{k=1}^K G_{d,k}^-(U) \Delta x \right\}. \quad (6)$$

From the idea in DVDM[9], we take a discrete variation to derive a structure-preserving scheme for (1)–(3). That is, we calculate the difference $J_d(U) - J_d(V)$ for all $U, V \in \mathbb{R}^{K+3}$. For the purpose, we use the following lemmas. All the proofs can be obtained by direct calculation and are here omitted.

**Lemma 2.1.** The following identity holds:

$$\frac{1}{2} \left( \sum_{k=0}^{K-1} f_k \Delta x + \sum_{k=1}^K f_k \Delta x \right) = \sum_{k=0}^K \langle f_k \rangle \Delta x \quad \text{for all } \{f_k\}_{k=0}^K \in \mathbb{R}^{K+1}. \quad (7)$$

**Lemma 2.2** (summation by parts formula). Let us denote $f_K - f_0$ by $[f_k]_0^K$. The following summation by parts formulas hold:

$$\sum_{k=0}^{K-1} (\delta^+_k f_k) (\delta^+_k g_k) \Delta x = -\sum_{k=0}^{K-1} (\delta^{(2)}_k f_k) g_k \Delta x + [(\delta^-_k f_k) g_k]_0^K, \quad (7)$$

$$\sum_{k=1}^K (\delta^-_k f_k) (\delta^-_k g_k) \Delta x = -\sum_{k=1}^K (\delta^{(2)}_k f_k) g_k \Delta x + [(\delta^+_k f_k) g_k]_0^K, \quad (8)$$

for all $\{f_k\}_{k=-1}^{K+1}, \{g_k\}_{k=-1}^{K+1} \in \mathbb{R}^{K+3}$.

**Corollary 1** (summation by parts formula). The summation by parts formula holds as follows:

$$\sum_{k=0}^{K-1} (\delta^+_k f_k) (\delta^-_k g_k) \Delta x = -\sum_{k=0}^K \langle f_k \rangle \Delta x + \left[ \langle (\delta^{(1)}_k f_k) g_k \rangle \right]_0^K \quad (9)$$

for all $\{f_k\}_{k=-1}^{K+1}, \{g_k\}_{k=-1}^{K+1} \in \mathbb{R}^{K+3}$.

Using Lemma 2.1, we have the following lemma.

**Lemma 2.3.** The definition (6) of $J_d$ is rewritten as follows:

$$J_d(U) = \sum_{k=0}^{K-1} \langle \delta^+_k U_k \rangle \Delta x + \sum_{k=0}^K W(U_k) \Delta x. \quad (10)$$

By using Lemma 2.3 and Corollary 1, we have the following lemma.

**Lemma 2.4.** The following equality holds: for all $U, V \in \mathbb{R}^{K+3}$,

$$J_d(U) - J_d(V) = \sum_{k=0}^K \langle \delta^{(2)}_k \left( \frac{U_k + V_k}{2} \right) \rangle \Delta x + \left[ \langle \delta^{(1)}_k \left( \frac{U_k + V_k}{2} \right) \rangle \right]_0^K (U_k - V_k) \Delta x.$$  

(10)
This equality (10) is essential for the discrete dissipation of energy (Theorem 2.5).

2.2. Proposed scheme. The concrete form of our scheme for (1) with (2) and (3) is, for \( m = 0, 1, \ldots, \)

\[
\frac{U_k^{(m+1)} - U_k^{(m)}}{\Delta t} = -\frac{\delta G_d}{\delta \left( U^{(m+1)}, U^{(m)} \right)_k} (k = 0, \ldots, K), \tag{11}
\]

\[
\frac{U_0^{(m+1)} - U_0^{(m)}}{\Delta t} = \delta_k^{(1)} \left( \frac{U_k^{(m+1)} + U_k^{(m)}}{2} \right) \bigg|_{k=0} - \frac{dW}{d(U_0^{(m+1)}, U_0^{(m)})}, \tag{12}
\]

\[
\frac{U_K^{(m+1)} - U_K^{(m)}}{\Delta t} = -\delta_k^{(1)} \left( \frac{U_k^{(m+1)} + U_k^{(m)}}{2} \right) \bigg|_{k=K} - \frac{dW}{d(U_K^{(m+1)}, U_K^{(m)})}, \tag{13}
\]

where

\[
\frac{\delta G_d}{\delta \left( U^{(m+1)}, U^{(m)} \right)_k} = -\delta_k^{(2)} \left( \frac{U_k^{(m+1)} + U_k^{(m)}}{2} \right) + \frac{dW}{d(U_k^{(m+1)}, U_k^{(m)})} (k = 0, \ldots, K). \tag{14}
\]

Then the proposed scheme (11)–(13) has the following property corresponding to (5), i.e.,

**Theorem 2.5.** The solution to the scheme (11)–(13) satisfies

\[
\delta_m^+ \left\{ J_d(U^{(m)}) + W(U_0^{(m)}) + W(U_K^{(m)}) \right\} \leq 0 \quad (m = 0, 1, \ldots). \tag{15}
\]

**Proof.** From Lemma 2.4, we have

\[
\frac{1}{\Delta t} \left\{ J_d(U^{(m+1)}) - J_d(U^{(m)}) \right\} = \sum_{k=0}^K \frac{\delta G_d}{\delta \left( U^{(m+1)}, U^{(m)} \right)_k} \frac{U_k^{(m+1)} - U_k^{(m)}}{\Delta t} \Delta x + \left[ \delta_k^{(1)} \left( \frac{U_k^{(m+1)} + U_k^{(m)}}{2} \right) \right]_{k=0}^K \frac{U_k^{(m+1)} - U_k^{(m)}}{\Delta t}, \tag{16}
\]

for all \( m = 0, 1, \ldots. \) Using (12) and (13), we calculate the boundary term on the right-hand side of (16) as follows:

\[
\left[ \left\{ \delta_k^{(1)} \left( \frac{U_k^{(m+1)} + U_k^{(m)}}{2} \right) \right\} \right]_{k=0}^K \frac{U_k^{(m+1)} - U_k^{(m)}}{\Delta t}
\]

\[
= \left( \delta_m^+ U_K^{(m)} \right)^2 - \frac{dW}{d(U_K^{(m+1)}, U_K^{(m)})} \delta_m^+ U_K^{(m)} - \left( \delta_m^+ U_0^{(m)} \right)^2 - \frac{dW}{d(U_0^{(m+1)}, U_0^{(m)})} \delta_m^+ U_0^{(m)}
\]

\[
= - \left( \delta_m^+ U_K^{(m)} \right)^2 - \left( \delta_m^+ U_K^{(m)} \right)^2 - \delta_m^+ W(U_0^{(m)}) - \delta_m^+ W(U_K^{(m)}).
\tag{17}
\]

Applying (11) and (17) to (16), we obtain

\[
\delta_m^+ J_d(U^{(m)}) + W(U_0^{(m)}) + W(U_K^{(m)}) = -\sum_{k=0}^K \frac{\delta G_d}{\delta \left( U^{(m+1)}, U^{(m)} \right)_k} \Delta x - \left| \delta_m^+ U_0^{(m)} \right|^2 - \left| \delta_m^+ U_K^{(m)} \right|^2.
\]

Therefore, the inequality (15) holds.
3. **Stability of the proposed scheme.** In this section, we show that, if the proposed scheme has a solution, then it satisfies the global boundedness. Firstly, let us define the discrete $L^\infty$-norm, the discrete $L^2$-norm, the discrete Dirichlet semi-norm, and discrete Sobolev norm.

**Definition 3.1.** We define the discrete $L^\infty$-norm $\| \cdot \|_{L^\infty}$ and the discrete $L^2$-norm $\| \cdot \|_{L^2}$ by

$$\| f \|_{L^\infty} := \max_{0 \leq k \leq K} | f_k |, \quad \| f \|_{L^2} := \sqrt{\sum_{k=0}^{K} \| f_k \|^2 \Delta x}$$

for all $f = \{ f_k \}_{k=0}^{K} \in \mathbb{R}^{K+1}$. Furthermore, for all $f = \{ f_k \}_{k=0}^{K} \in \mathbb{R}^{K+1}$, we define the discrete Dirichlet semi-norm $\| Df \|$ of $f$ by

$$\| Df \| := \sqrt{\sum_{k=0}^{K} \| \delta_k^+ f_k \|^2 \Delta x},$$

where $Df$ is denoted by $Df = \{ \delta_k^+ f_k \}_{k=0}^{K-1} \in \mathbb{R}^K$. Also, define the discrete Sobolev norm $\| \cdot \|_{H^1}$ by

$$\| f \|_{H^1} := \sqrt{\| f \|^2_{L^2} + \| Df \|^2}$$

for all $f = \{ f_k \}_{k=0}^{K} \in \mathbb{R}^{K+1}$.

For the proof of the global boundedness of the numerical solution, we use the following lemmas.

**Lemma 3.2.** The solution of the scheme (11)–(13) satisfies the following inequality. For $m = 0, 1, \ldots$, it holds that

$$\| U^{(m)} \|^2_{H^1} + | U_0^{(m)} |^2 + | U_K^{(m)} |^2 \leq \frac{1}{\min \{ \frac{1}{2}, \mu \}} \left\{ J_d(U^{(0)}) + W(U_0^{(0)}) + W(U_K^{(0)}) + c(L+2) \right\}.$$  

(18)

**Proof.** From the discrete dissipative property (Theorem 2.5) and the assumption (4) for the potential $W$, we can show

\[
J_d(U^{(0)}) + W(U_0^{(0)}) + W(U_K^{(0)}) \\
\geq J_d(U^{(m)}) + W(U_0^{(m)}) + W(U_K^{(m)}) \\
= \sum_{k=0}^{K-1} \frac{1}{2} \| \delta_k^+ U_k^{(m)} \|^2 \Delta x + \sum_{k=0}^{K} \| W(U_k^{(m)}) \| \Delta x + W(U_0^{(m)}) + W(U_K^{(m)}) \\
\geq \frac{1}{2} \sum_{k=0}^{K-1} \| \delta_k^+ U_k^{(m)} \|^2 \Delta x + \sum_{k=0}^{K} \mu \| U_k^{(m)} \|^2 - c \| \Delta x + \mu \| U_0^{(m)} \|^2 - c + \mu \| U_K^{(m)} \|^2 - c \\
\geq \min \left\{ \frac{1}{2}, \mu \right\} \left\{ \| U^{(m)} \|^2_{H^1} + \| U_0^{(m)} \|^2 + \| U_K^{(m)} \|^2 \right\} - c(L+2)
\]

for $m = 0, 1, \ldots$. Therefore, the inequality (18) holds.

**Lemma 3.3** (Sobolev type inequality [23, Proposition 2.2]). We define a constant $C_L$ as follows:

$$C_L := \sqrt{\frac{1 + 4L^2}{2L}}.$$
Then, the following inequality holds:
\[
\|f\|_{L^\infty} \leq \bar{C}_L \|f\|_{H_1} \quad \text{for all } \{f_k\}_{k=0}^K \in \mathbb{R}^{K+1}.
\] (19)

**Proof.** We can obtain (19) from the proof of Yoshikawa [23]. □

Applying Lemma 3.3 to (18), we can obtain the following global boundedness.

**Theorem 3.4** (global boundedness). The solution of the scheme (11) under the discrete boundary conditions (12) and (13) satisfies the following inequality:
\[
\|U^{(m)}\|_{L^\infty} \leq \bar{C}_L \left[ \min \left\{ \frac{1}{2}, \mu \right\} \left( \frac{1}{4} \right) \right] \|J_4(U^{(0)}) + W(U_0^{(0)}) + W(U_K^{(0)}) + c(L + 2)\|^{\frac{1}{2}}
\]
for \(m = 0, 1, \ldots\).

4. **Existence and uniqueness of the solution to the proposed scheme.** In this section, using the energy method in [8, 21, 22, 23, 24], we prove that the proposed scheme (11)–(13) has a unique solution under a specific condition on \(\Delta t\).

4.1. **Preparation.** We define \(\bar{F}''\) for \(F \in C^2\) and give several lemmas necessary for the proof of the existence and uniqueness of the solution.

**Definition 4.1.** Let \(\Omega\) be a domain in \(\mathbb{R}\). For a function \(F \in C^2(\Omega)\), let us define \(\bar{F}''\): \(\Omega^4 \to \mathbb{R}\) by
\[
\bar{F}''(\xi, \hat{\xi}; \eta, \hat{\eta}) := \begin{cases} \frac{1}{\xi - \hat{\xi}} \left( \frac{dF}{d(\xi, \eta)} + \frac{dF}{d(\xi, \hat{\eta})} \right) - \left( \frac{dF}{d(\hat{\xi}, \eta)} + \frac{dF}{d(\hat{\xi}, \hat{\eta})} \right), & (\xi \neq \hat{\xi}), \\ \partial_\xi \left( \frac{dF}{d(\xi, \eta)} + \frac{dF}{d(\xi, \hat{\eta})} \right) |_{\xi = \hat{\xi}}, & (\xi = \hat{\xi}) \end{cases}
\]
for all \((\xi, \hat{\xi}, \eta, \hat{\eta}) \in \Omega^4\).

Since proofs of the following lemmas can be found in [23], we omit them.

**Lemma 4.2** ([23, Lemma 2.4]). Let \(\Omega\) be a domain in \(\mathbb{R}\). If \(F \in C^2(\Omega)\), then \(\bar{F}'' \in C(\Omega^4)\). Moreover, we have
\[
\left| \bar{F}''(\xi, \hat{\xi}; \eta, \hat{\eta}) \right| \leq \sup_{\eta, \hat{\eta} \in \Omega} \left| \frac{\partial}{\partial_\xi} \left( \frac{dF}{d(\xi, \eta)} + \frac{dF}{d(\xi, \hat{\eta})} \right) \right| \leq \sup_{\xi \in \Omega} |\bar{F}''(\xi)|
\]
for all \(\xi, \hat{\xi}, \eta, \hat{\eta} \in \Omega\).

**Lemma 4.3** ([23, Proposition 2.5]). Let \(\Omega\) be a domain in \(\mathbb{R}\) and assume that \(F \in C^2(\Omega)\). For all \(\xi, \hat{\xi} \) and \(\eta, \hat{\eta} \in \Omega\), we have
\[
\frac{dF}{d(\xi, \eta)} - \frac{dF}{d(\xi, \hat{\eta})} = \frac{1}{2} \bar{F}''(\xi, \hat{\xi}; \eta, \hat{\eta}) (\xi - \hat{\xi}) + \frac{1}{2} \bar{F}''(\eta, \hat{\eta}; \xi, \hat{\xi}) (\eta - \hat{\eta}).
\]

**Lemma 4.4** ([23, Lemma 2.3]). The following inequality holds:
\[
\|D(fg)\| \leq \|f\|_{L^\infty} \|Dg\| + \|g\|_{L^\infty} \|Df\| \quad \text{for all } f = \{f_k\}_{k=0}^K, g = \{g_k\}_{k=0}^K \in \mathbb{R}^{K+1},
\]
where \(fg := \{f_kg_k\}_{k=0}^K \in \mathbb{R}^{K+1}\).

The following lemma follows from the same argument as Lemma 2.6 in [23].
Lemma 4.5 ([8, Lemma 3.3 (2)]). Let $\Omega$ be a domain in $\mathbb{R}$ and assume that $F \in C^3(\Omega)$. For any $f_1 = \{f_{1,k}\}_{k=0}^K, f_2 = \{f_{2,k}\}_{k=0}^K, f_3 = \{f_{3,k}\}_{k=0}^K, f_4 = \{f_{4,k}\}_{k=0}^K \in \mathbb{R}^{K+1}$ all the elements of which are in $\Omega$, we have
\[
\|D\tilde{F}'(f_1, f_2, f_3, f_4)\| \leq \frac{1}{6} \sup_{\xi \in \Omega} |F'''(\xi)| (2 \|DF_1\| + 2 \|DF_2\| + \|DF_3\| + \|DF_4\|).
\]

4.2. Existence and uniqueness of the solution.

Theorem 4.6 (local existence and uniqueness). Let
\[
R(\rho) := \max \left\{ 5 \max_{|\xi| \leq 2\rho} |W'''(\xi)|^2, \frac{1}{2} \max_{|\xi| \leq 2\rho} |W'''(\xi)|^2 + \frac{25}{18} \rho^2 \max_{|\xi| \leq 2\rho} |W'''(\xi)|^2 \right\}
\]
for all $\rho \geq 0$. For any given $U^{(m)} = \{U_k^{(m)}\}_{k=-1}^{K+1} \in \mathbb{R}^{K+3}$, if $\Delta t$ satisfies
\[
(\Delta t)^2 R \left( C_L \sqrt{\|U^{(m)}\|^2_{\mathcal{H}_0^1} + \|U_0^{(m)}\|^2 + \|U_K^{(m)}\|^2} \right) < 1, \tag{20}
\]
then there exists a unique solution $\{U_k^{(m+1)}\}_{k=-1}^{K+1} \in \mathbb{R}^{K+3}$ satisfying (11)-(13).

Proof. For any given $U^{(m)} = \{U_k^{(m)}\}_{k=-1}^{K+1} \in \mathbb{R}^{K+3}$, we define the mapping $\Psi$: \{\{U_k\}_{k=0}^K \mapsto \{\tilde{U}_k\}_{k=-1}^{K+1}\} by
\[
\frac{\tilde{U}_k - U_k^{(m)}}{\Delta t} = \delta_k^{(2)} \left( \tilde{U}_k + \frac{U_k^{(m)}}{2} \right) - \frac{dW}{d(U_k, U_k^{(m)})} \quad (k = 0, \ldots, K), \tag{21}
\]
\[
\frac{\tilde{U}_0 - U_0^{(m)}}{\Delta t} = \delta_0^{(1)} \left( \tilde{U}_0 + \frac{U_0^{(m)}}{2} \right) \bigg|_{k=0} - \frac{dW}{d(U_0, U_0^{(m)})}, \tag{22}
\]
\[
\frac{\tilde{U}_K - U_K^{(m)}}{\Delta t} = -\delta_k^{(1)} \left( \tilde{U}_k + \frac{U_k^{(m)}}{2} \right) \bigg|_{k=K} - \frac{dW}{d(U_K, U_K^{(m)})}. \tag{23}
\]
Firstly, we show that the mapping $\Psi$ is well-defined. Here, let $\alpha := \Delta t/4(\Delta x)$ and $\beta := \Delta t/(2(\Delta x)^2)$. For the purpose, we give the following matrix expression of $\Psi$:
\[
A\tilde{U} = f(U, U^{(m)}). \tag{24}
\]

Then the $(K+3) \times (K+3)$ matrix $A$ is defined by
\[
A := \begin{pmatrix}
\alpha & 1 & -\alpha \\
-\beta & 1 + 2\beta & -\beta \\
-\beta & 1 + 2\beta & -\beta \\
& & & \ddots & \ddots & \ddots \\
& & & & -\beta & 1 + 2\beta & -\beta \\
& & & & -\beta & 1 + 2\beta & -\beta \\
& & & & & & -\alpha & 1 & \alpha
\end{pmatrix}.
\]

If the matrix $A$ is nonsingular, then the mapping $\Psi$ is well-defined. We can show that $A$ is nonsingular by direct calculation of the determinant of $A$ and skip it.

Next, we prove the existence and uniqueness of the solution of the proposed scheme by the fixed-point theorem for a contraction mapping. From the definition
Hence, it is sufficient to show the existence of a $K + 1$-vector $U$ that satisfies $\tilde{U}_k = U_k$ ($k = 0, \ldots, K$). Here, we define the mapping $\Phi : \mathbb{R}^{K+1} \to \mathbb{R}^{K+1}$ by

$$
\Phi(U) := \{\tilde{U}_k\}_{k=0}^K = \{\Psi_k(U)\}_{k=0}^K \quad \text{for all } U \in \mathbb{R}^{K+1},
$$

where $\Psi_k(U)$ is the $k$th element of $\Psi(U)$. Also, let $X := \{f \in \mathbb{R}^{K+1}; \|f\|_X \leq 4M^2\}$, where $M := \|U^{(m)}\|_X$ and $\|f\|_X := \sqrt{\|f\|_{H^1}^2 + |f|_0^2 + |\tilde{f}K|}$ for all $f \in \mathbb{R}^{K+1}$. We show that the mapping $\Phi$ is a contraction mapping on $X$. If $\Phi$ is a contraction mapping, $\Phi$ has a unique fixed-point $V^*$ in the closed ball $X$ from the fixed-point theorem for a contraction mapping. This $V^*$ is the solution $U^{(m+1)}$ to the scheme (11)–(13). Firstly, we show $\Phi(X) \subset X$. For any fixed $U \in X$, we have

$$
\frac{1}{2\Delta t} \left( \|\tilde{U}\|_{L^2}^2 - \|U^{(m)}\|_{L^2}^2 \right) = \sum_{k=0}^{K} \frac{1}{2\Delta t} \left( \tilde{U}_k - U_k \right) \frac{dW}{d(u_k, U_k^{(m)})} - \frac{1}{2\Delta t} \delta_k \left( \tilde{U}_k + U_k \right) \frac{dW}{d(u_k, U_k^{(m)})} \leq \frac{1}{2\Delta t} \left( \|\tilde{U}_0\|_{L^2}^2 - \|U_0\|_{L^2}^2 \right) \frac{dW}{d(u_0, U_0^{(m)})}.\n$$

from Corollary 1 (summation by parts formula), (21)–(23). Moreover, using the Young inequality: $ab \leq (\varepsilon/2)a^2 + (1/(2\varepsilon))b^2$ and the following inequality: $(a+b)^2 \leq 2(a^2 + b^2)$, we obtain

$$
\frac{1}{2\Delta t} \left( \|\tilde{U}\|_{L^2}^2 - \|U^{(m)}\|_{L^2}^2 \right) \leq \frac{1}{2\Delta t} \left( \|\tilde{U}_0\|_{L^2}^2 - \|U_0\|_{L^2}^2 \right) + \frac{\Delta t}{2} \left( \frac{dW}{d(u_0, U_0^{(m)})} \right) + \frac{1}{2\Delta t} \left( \frac{dW}{d(u_k, U_k^{(m)})} \right) + \frac{\Delta t}{2} \left( \frac{dW}{d(U_k, U_k^{(m)})} \right).\n$$
Therefore, multiplying both sides of (25) by $4\Delta t$, we get

$$
\frac{1}{2\Delta t} \left( \left\| D\tilde{U} \right\|^2 - \left\| D\tilde{U}^{(m)} \right\|^2 \right) = \sum_{k=0}^{K-1} \delta_k^+ \left( \frac{\tilde{U}_k + \tilde{U}_k^{(m)}}{2} \right) \delta_k + \sum_{k=0}^{K-1} \left\{ \delta_k^+ \left( \frac{\tilde{U}_k - \tilde{U}_k^{(m)}}{\Delta t} \right) \right\} D\tilde{U}_k - D\tilde{U}_k^{(m)} \Delta x
$$

(27)

Next, using Corollary 1 (summation by parts formula), we have

$$
-\sum_{k=0}^{K} \left\{ \delta_k^+ \left( \frac{\tilde{U}_k + \tilde{U}_k^{(m)}}{2} \right) \right\} \frac{\tilde{U}_k - \tilde{U}_k^{(m)}}{\Delta t} \Delta x
$$

$$
= -\sum_{k=0}^{K} \left\{ \delta_k^+ \left( \frac{\tilde{U}_k + \tilde{U}_k^{(m)}}{2} \right) \right\} \frac{\tilde{U}_k - \tilde{U}_k^{(m)}}{\Delta t} \Delta x + \sum_{k=0}^{K} \left\{ \delta_k^+ \left( \frac{\tilde{U}_k + \tilde{U}_k^{(m)}}{2} \right) \right\} \frac{dW}{d(U_k, U_k^{(m)})} \Delta x
$$

$$
\leq \sum_{k=0}^{K-1} \left\{ \frac{1}{2\Delta t} \left\{ \delta_k^+ \left( \frac{\tilde{U}_k + \tilde{U}_k^{(m)}}{2} \right) \right\} \frac{\tilde{U}_k - \tilde{U}_k^{(m)}}{\Delta t} \Delta x + \frac{\Delta t}{2} \left\{ \delta_k^+ \left( \frac{\tilde{U}_k + \tilde{U}_k^{(m)}}{2} \right) \right\} \frac{dW}{d(U_k, U_k^{(m)})} \Delta x
$$

$$
\leq \left\| D\tilde{U} \right\|^2 + \frac{1}{4\Delta t} \left( \left\| D\tilde{U}^{(m)} \right\|^2 \right) + \frac{\Delta t}{2} \left\{ \delta_k^+ \left( \frac{\tilde{U}_k + \tilde{U}_k^{(m)}}{2} \right) \right\} \frac{dW}{d(U_k, U_k^{(m)})} \Delta x
$$

(26)
Hence, we obtain
\[
\frac{1}{2\Delta t} \left( \| D\tilde{U} \|^2 - \| DU^{(m)} \|^2 \right) \leq \frac{1}{4\Delta t} \left( \| D\tilde{U} \|^2 + \| DU^{(m)} \|^2 \right) + \frac{\Delta t}{2} \left\| D \left( \frac{dW}{d(U, U^{(m)})} \right) \right\|^2 + \left[ \left\langle \delta_k^{(1)} \left( \frac{\tilde{U}_k + U_k^{(m)}}{2} \right) \right\rangle \left( \frac{\tilde{U}_k - U_k^{(m)}}{\Delta t} \right) + \frac{dW}{d(U, U^{(m)})} \right]_0^K .
\]

Furthermore, we see from (22) and (23) that
\[
\left\{ \delta_k^{(i)} \left( \frac{\tilde{U}_k + U_k^{(m)}}{2} \right) \right\} \left( \frac{\tilde{U}_k - U_k^{(m)}}{\Delta t} \right) + \left\{ \delta_k^{(1)} \left( \frac{\tilde{U}_k + U_k^{(m)}}{2} \right) \right\}^2 \leq 0.
\]
Namely, we have
\[
\| D\tilde{U} \|^2 \leq 3 \| DU^{(m)} \|^2 + 2(\Delta t)^2 \left\| D \left( \frac{dW}{d(U, U^{(m)})} \right) \right\|^2 .
\]
Combining (26) and (28), we obtain
\[
\| \tilde{U} \|^2 \leq 3 \| U^{(m)} \|^2 + 2(\Delta t)^2 \left\| \frac{dW}{d(U, U^{(m)})} \right\|^2 .
\]
All that is left to show that the right-hand side of (29) is no more than \(4M^2\). Using Lemma 4.3 and the assumption (4) for \(W\), the following equality holds:
\[
\frac{dW}{d(U_k, U_k^{(m)})} = \frac{dW}{d(U_k, U_k^{(m)})} - \frac{dW}{d(0, 0)} = \frac{1}{2} \bar{W}''(U_k, 0; U_k^{(m)}, 0)U_k + \frac{1}{2} \bar{W}''(U_k^{(m)}, 0; U_k, 0)U_k^{(m)}
\]
for \(k = 0, \ldots, K\). Moreover, from Lemma 3.3 and definitions of \(X\) and \(M\), it holds that
\[
\| U^{(m)} \|_{L^\infty} \leq \bar{C}_L \| U^{(m)} \|_{\bar{B}_L^4} \leq \bar{C}_L M, \quad \| U \|_{L^\infty} \leq \bar{C}_L \| U \|_{\bar{B}_L^4} \leq 2\bar{C}_LM.
\]
Hence, from Lemma 4.2, we have
\[
\left\| \bar{W}''(U_k, 0; U_k^{(m)}, 0) \right\| \leq \max_{|\xi| \leq 2\bar{C}_LM} |W''(\xi)| \quad (k = 0, \ldots, K),
\]
\[
\left\| \bar{W}''(U_k^{(m)}, 0; U_k, 0) \right\| \leq \max_{|\xi| \leq 2\bar{C}_LM} |W''(\xi)| \quad (k = 0, \ldots, K).
\]
Therefore, we obtain
\[
\left\| \frac{dW}{d(U_k, U_k^{(m)})} \right\|^2 \leq \frac{1}{2} \left\{ \frac{1}{4} \left| \bar{W}''(U_k, 0; U_k^{(m)}, 0) \right|^2 |U_k|^2 + \frac{1}{4} \left| \bar{W}''(U_k^{(m)}, 0; U_k, 0) \right|^2 |U_k^{(m)}|^2 \right\} \leq \frac{1}{2} \max_{|\xi| \leq 2\bar{C}_LM} |W''(\xi)|^2 \left\{ |U_k|^2 + |U_k^{(m)}|^2 \right\} \quad (k = 0, \ldots, K) .
\]
Thus, the following inequality holds:
\[
\left\| \frac{dW}{d(U, U^{(m)})} \right\|^2 \leq \sum_{k=0}^K \left\| \frac{dW}{d(U_k, U_k^{(m)})} \right\|^2 \Delta x \leq \frac{1}{2} \max_{|\xi| \leq 2\bar{C}_LM} |W''(\xi)|^2 \left( \| U \|^2_{L^2} + \| U^{(m)} \|^2_{L^2} \right) .
\]
From Lemma 4.3, we also have

\[\delta_k^+ \left( \frac{dW}{d(U_k, U_k^{(m)})} \right) = \frac{1}{2} W''(U_{k+1}, U_k; U_k^{(m)} U_k^{(m)}) \delta_k U_k + \frac{1}{2} W''(U_{k+1}, U_k^{(m)} U_k^{(m)}) \delta_k U_k \]

for \( k = 0, \ldots, K - 1 \). Hence, it follows from the same argument as (30) and (31) that

\[\left\| D \left( \frac{dW}{d(U, U^{(m)})} \right) \right\|^2 \leq \frac{1}{2} \max_{|\xi| \leq 2C_L M} |W''(\xi)|^2 \left( \| D(U) \|^2 + \| D(U^{(m)}) \|^2 \right). \quad (32)\]

Therefore, using (30)–(32), the following estimate holds:

\[\left\| \frac{dW}{d(U, U^{(m)})} \right\|_X^2 \leq \frac{1}{2} \max_{|\xi| \leq 2C_L M} |W''(\xi)|^2 \left( \| U \|^2_X + \| U^{(m)} \|^2_X \right) \leq \frac{5}{2} \max_{|\xi| \leq 2C_L M} |W''(\xi)|^2 M^2. \quad (33)\]

Thus, from (29), (33), and the assumption (20), we conclude that

\[\left\| \tilde{U}_1 \right\|_X^2 \leq 3M^2 + 5(\Delta t)^2 \max_{|\xi| \leq 2C_L M} |W''(\xi)|^2 M^2 \leq \left\{ 3 + (\Delta t)^2 R \left( C_L M \right) \right\} M^2 \leq 4M^2.\]

Namely, \( \Phi(U) = \{ \tilde{U}_k \}_{k=0}^{K} \in X \).

Next, we show that \( \Phi \) is contractive. For any fixed \( U_1, U_2 \in X \), the vector \( \{ \tilde{U}_{i,k} \}_{k=0}^{K} = \{ \Psi_k(U_i) \}_{k=0}^{K} \) satisfies (21)–(23) \((i = 1, 2)\) from the definition of \( \Psi \). Subtracting these relations, we obtain

\[\frac{\tilde{U}_{1,k} - \tilde{U}_{2,k}}{\Delta t} = \delta_k^+ \left( \frac{\tilde{U}_{1,k} - \tilde{U}_{2,k}}{2} \right) - \left( \frac{dW}{d(U_{1,k}, U_k^{(m)})} - \frac{dW}{d(U_{2,k}, U_k^{(m)})} \right) (k = 0, \ldots, K), \quad (34)\]

\[\frac{\tilde{U}_{1,0} - \tilde{U}_{2,0}}{\Delta t} = \delta_k^+ \left( \frac{\tilde{U}_{1,k} - \tilde{U}_{2,k}}{2} \right) - \left( \frac{dW}{d(U_{1,0}, U_{0}^{(m)})} - \frac{dW}{d(U_{2,0}, U_{0}^{(m)})} \right), \quad (35)\]

\[\frac{\tilde{U}_{1,K} - \tilde{U}_{2,K}}{\Delta t} = -\delta_k^+ \left( \frac{\tilde{U}_{1,k} - \tilde{U}_{2,k}}{2} \right) - \left( \frac{dW}{d(U_{1,K}, U_{K}^{(m)})} - \frac{dW}{d(U_{2,K}, U_{K}^{(m)})} \right). \quad (36)\]

From (34), we have

\[\frac{1}{\Delta t} \left\| \tilde{U}_1 - \tilde{U}_2 \right\|_{L^2_t}^2 = \frac{1}{2} \sum_{k=0}^{K} \left\{ \delta_k^+ \left( \tilde{U}_{1,k} - \tilde{U}_{2,k} \right) \right\} (\tilde{U}_{1,k} - \tilde{U}_{2,k}) \Delta x \]

\[= -\frac{1}{2} \sum_{k=0}^{K} \left\{ \delta_k^+ \left( \tilde{U}_{1,k} - \tilde{U}_{2,k} \right) \right\}^2 - \sum_{k=0}^{K} \left( \frac{dW}{d(U_{1,k}, U_k^{(m)})} - \frac{dW}{d(U_{2,k}, U_k^{(m)})} \right) (\tilde{U}_{1,k} - \tilde{U}_{2,k}) \Delta x. \quad (37)\]

Using Corollary 1 (summation by parts formula), (35), and (36), we estimate the first term on the right-hand side of (37) as follows:

\[\frac{1}{2} \sum_{k=0}^{K} \left\{ \delta_k^+ \left( \tilde{U}_{1,k} - \tilde{U}_{2,k} \right) \right\} (\tilde{U}_{1,k} - \tilde{U}_{2,k}) \Delta x \]

\[= -\frac{1}{2} \sum_{k=0}^{K} \left\{ \delta_k^+ \left( \tilde{U}_{1,k} - \tilde{U}_{2,k} \right) \right\}^2 - \sum_{k=0}^{K} \left( \delta_k^+ \left( \tilde{U}_{1,k} - \tilde{U}_{2,k} \right) \right) (\tilde{U}_{1,k} - \tilde{U}_{2,k}) \Delta x. \]
Therefore, multiplying both sides of (38) by \( \frac{2 \Delta t}{\Delta t} \), we obtain
\[
\frac{1}{\Delta t} \left\| \mathbf{U}_t - \mathbf{U}_0 \right\|_{L_2}^2 \leq \left( \frac{\Delta t}{2} \right)^2 \left( \frac{dW}{d(U_{1,0}, U_{0}^{(m)})} - \frac{dW}{d(U_{2,0}, U_{0}^{(m)})} \right) (\mathbf{U}_t - \mathbf{U}_0)
- \left( \frac{\Delta t}{2} \right)^2 \left( \frac{dW}{d(U_{1,1}, U_{1}^{(m)})} - \frac{dW}{d(U_{2,1}, U_{1}^{(m)})} \right) (\mathbf{U}_{t+1} - \mathbf{U}_{t+2})
- \sum_{k=0}^{K} \left( \frac{dW}{d(U_{1,k}, U_{k}^{(m)})} - \frac{dW}{d(U_{2,k}, U_{k}^{(m)})} \right) (\mathbf{U}_{t+k} - \mathbf{U}_{t+k+1}) \Delta x.
\]

Hence, using the Young inequality, we obtain
\[
\frac{1}{\Delta t} \left\| \mathbf{U}_t - \mathbf{U}_0 \right\|_{L_2}^2 \leq \left( \frac{\Delta t}{2} \right)^2 \left( \frac{dW}{d(U_{1,0}, U_{0}^{(m)})} - \frac{dW}{d(U_{2,0}, U_{0}^{(m)})} \right) (\mathbf{U}_t - \mathbf{U}_0)
- \left( \frac{\Delta t}{2} \right)^2 \left( \frac{dW}{d(U_{1,1}, U_{1}^{(m)})} - \frac{dW}{d(U_{2,1}, U_{1}^{(m)})} \right) (\mathbf{U}_{t+1} - \mathbf{U}_{t+2})
- \sum_{k=0}^{K} \left( \frac{dW}{d(U_{1,k}, U_{k}^{(m)})} - \frac{dW}{d(U_{2,k}, U_{k}^{(m)})} \right) (\mathbf{U}_{t+k} - \mathbf{U}_{t+k+1}) \Delta x
\]
\[
\leq \left( \frac{\Delta t}{2} \right)^2 \left( \frac{dW}{d(U_{1,0}, U_{0}^{(m)})} - \frac{dW}{d(U_{2,0}, U_{0}^{(m)})} \right) (\mathbf{U}_t - \mathbf{U}_0)
- \left( \frac{\Delta t}{2} \right)^2 \left( \frac{dW}{d(U_{1,1}, U_{1}^{(m)})} - \frac{dW}{d(U_{2,1}, U_{1}^{(m)})} \right) (\mathbf{U}_{t+1} - \mathbf{U}_{t+2})
- \sum_{k=0}^{K} \left( \frac{dW}{d(U_{1,k}, U_{k}^{(m)})} - \frac{dW}{d(U_{2,k}, U_{k}^{(m)})} \right) (\mathbf{U}_{t+k} - \mathbf{U}_{t+k+1}) \Delta x.
\]

Therefore, multiplying both sides of (38) by \( 2 \Delta t \), we get
\[
\left\| \mathbf{U}_t - \mathbf{U}_0 \right\|_{L_2}^2 \leq \left( \frac{\Delta t}{2} \right)^2 \left( \frac{dW}{d(U_{1,0}, U_{0}^{(m)})} - \frac{dW}{d(U_{2,0}, U_{0}^{(m)})} \right) (\mathbf{U}_t - \mathbf{U}_0)
- \left( \frac{\Delta t}{2} \right)^2 \left( \frac{dW}{d(U_{1,1}, U_{1}^{(m)})} - \frac{dW}{d(U_{2,1}, U_{1}^{(m)})} \right) (\mathbf{U}_{t+1} - \mathbf{U}_{t+2})
- \sum_{k=0}^{K} \left( \frac{dW}{d(U_{1,k}, U_{k}^{(m)})} - \frac{dW}{d(U_{2,k}, U_{k}^{(m)})} \right) (\mathbf{U}_{t+k} - \mathbf{U}_{t+k+1}) \Delta x.
\]

Next, from Corollary 1 (summation by parts formula), we have
\[
\frac{1}{\Delta t} \left\| \mathbf{U}_t - \mathbf{U}_0 \right\|_{L_2}^2 \leq \left( \frac{\Delta t}{2} \right)^2 \left( \frac{dW}{d(U_{1,0}, U_{0}^{(m)})} - \frac{dW}{d(U_{2,0}, U_{0}^{(m)})} \right) (\mathbf{U}_t - \mathbf{U}_0)
- \left( \frac{\Delta t}{2} \right)^2 \left( \frac{dW}{d(U_{1,1}, U_{1}^{(m)})} - \frac{dW}{d(U_{2,1}, U_{1}^{(m)})} \right) (\mathbf{U}_{t+1} - \mathbf{U}_{t+2})
- \sum_{k=0}^{K} \left( \frac{dW}{d(U_{1,k}, U_{k}^{(m)})} - \frac{dW}{d(U_{2,k}, U_{k}^{(m)})} \right) (\mathbf{U}_{t+k} - \mathbf{U}_{t+k+1}) \Delta x.
\]

Using (34), Corollary 1 (summation by parts formula), and the Young inequality, we estimate the first term on the right-hand side of (40) as follows:
\[
- \frac{1}{\Delta t} \left\| \mathbf{U}_t - \mathbf{U}_0 \right\|_{L_2}^2 \leq \sum_{k=0}^{K} \left( \frac{dW}{d(U_{1,k}, U_{k}^{(m)})} - \frac{dW}{d(U_{2,k}, U_{k}^{(m)})} \right) (\mathbf{U}_{t+k} - \mathbf{U}_{t+k+1}) \Delta x.
\]
\[
\frac{1}{\Delta t} \left\| D(\bar{U}_1 - \bar{U}_2) \right\|^2 \\
\leq \frac{1}{2\Delta t} \left\| D(\bar{U}_1 - \bar{U}_2) \right\|^2 + \frac{\Delta t}{2} \left\| D \left( \frac{dW}{d(U_1, U^{(m)})} - \frac{dW}{d(U_2, U^{(m)})} \right) \right\|^2 \\
+ \left\{ \frac{\delta_k^{(1)}(\bar{U}_{1,k} - \bar{U}_{2,k})}{\Delta t} \right\} \left\{ \frac{\bar{U}_{1,k} - \bar{U}_{2,k}}{\Delta t} + \left( \frac{dW}{d(U_1, U^{(m)})} - \frac{dW}{d(U_2, U^{(m)})} \right) \right\}_0^K \\
= \frac{1}{2\Delta t} \left\| D(\bar{U}_1 - \bar{U}_2) \right\|^2 + \frac{\Delta t}{2} \left\| D \left( \frac{dW}{d(U_1, U^{(m)})} - \frac{dW}{d(U_2, U^{(m)})} \right) \right\|^2 \\
- \frac{1}{2} \left\{ \frac{\delta_k^{(1)}(\bar{U}_{1,k} - \bar{U}_{2,k})}{\Delta t} \right\}^2 - \frac{1}{2} \left\{ \delta_k^{(1)}(\bar{U}_{1,k} - \bar{U}_{2,k}) \right\}_{k=0}^K \right\|^2 \\
\leq \frac{1}{2\Delta t} \left\| D(\bar{U}_1 - \bar{U}_2) \right\|^2 + \frac{\Delta t}{2} \left\| D \left( \frac{dW}{d(U_1, U^{(m)})} - \frac{dW}{d(U_2, U^{(m)})} \right) \right\|^2 \\
\leq (\Delta t)^2 \left\| D \left( \frac{dW}{d(U_1, U^{(m)})} - \frac{dW}{d(U_2, U^{(m)})} \right) \right\|^2. \tag{41}
\]

That is,
\[
\left\| D(\bar{U}_1 - \bar{U}_2) \right\|^2 \leq (\Delta t)^2 \left\| D \left( \frac{dW}{d(U_1, U^{(m)})} - \frac{dW}{d(U_2, U^{(m)})} \right) \right\|^2.
\]

Therefore, using (39) and (41), we obtain
\[
\left\| \bar{U}_1 - \bar{U}_2 \right\|_X \leq (\Delta t)^2 \left\| \frac{dW}{d(U_1, U^{(m)})} - \frac{dW}{d(U_2, U^{(m)})} \right\|_X. \tag{42}
\]

Using Lemma 4.3, the following equality holds:
\[
\frac{dW}{d(U_1, k; U_k^{(m)'})} - \frac{dW}{d(U_2, k; U_k^{(m)'})} = \frac{1}{2} \tilde{W}''(U_1, k; U_2, k; U_k^{(m)'}, U_k^{(m)'}) (U_1, k, k) \tag{43}
\]\nfor \(k = 0, \ldots, K\). Since it holds that
\[
\| U_i \|_{L^\infty} \leq \tilde{C}_L \| U_i \|_{\tilde{H}^1} \leq 2\tilde{C}_LM \quad (i = 1, 2)
\]
from Lemma 3.3 and the definition of \(X\), we get
\[
\left| \tilde{W}''(U_1, k; U_2, k; U_k^{(m)'}, U_k^{(m)'}) \right| \leq \max_{|\xi| \leq 2\tilde{C}_LM} |W''(\xi)| \quad (k = 0, \ldots, K) \tag{44}
\]
using Lemma 4.2. From (43) and (44), we obtain

\[
\left| \frac{dW}{d(U_{1,k},U_{1,k}^{(m)})} - \frac{dW}{d(U_{2,k},U_{2,k}^{(m)})} \right|^2 \leq \frac{1}{4} \max_{|\xi| \leq 2 \hat{C}_L M} |W''(\xi)|^2 |U_{1,k} - U_{2,k}|^2 \quad (k=0, \ldots, K).
\]

Hence, the following inequality holds:

\[
\left\| \frac{dW}{d(U_1,U_1^{(m)})} - \frac{dW}{d(U_2,U_2^{(m)})} \right\|_{L^2_\lambda}^2 \leq \frac{1}{4} \max_{|\xi| \leq 2 \hat{C}_L M} |W''(\xi)|^2 \|U_1 - U_2\|_{L^2_\lambda}^2.
\] (46)

Furthermore, using (43), (44), Lemma 3.3, Lemma 4.4, and Lemma 4.5, we have

\[
\left\| D \left( \frac{dW}{d(U_1,U_1^{(m)})} - \frac{dW}{d(U_2,U_2^{(m)})} \right) \right\|^2 \leq \frac{1}{4} \max_{|\xi| \leq 2 \hat{C}_L M} |W''(\xi)|^2 \|D(U_1 - U_2)\|^2 \\
+ \|U_1 - U_2\|_{L^\infty} \left\| D W''(U_1,U_2;U_1^{(m)},U_2^{(m)}) \right\|^2 \leq \frac{1}{2} \max_{|\xi| \leq 2 \hat{C}_L M} |W''(\xi)|^2 \|D(U_1 - U_2)\|^2 \\
+ \tilde{C}_L^2 \max_{|\xi| \leq 2 \hat{C}_L M} |W''(\xi)|^2 \left( \|DU_1\| + \|DU_2\| + \|DU_1^{(m)}(\xi)\| \right)^2 \|U_1 - U_2\|_{L^2_\lambda}^2 \\
\leq \frac{1}{2} \max_{|\xi| \leq 2 \hat{C}_L M} |W''(\xi)|^2 \|D(U_1 - U_2)\|^2 + \frac{25}{18} \tilde{C}_L^2 M^2 \max_{|\xi| \leq 2 \hat{C}_L M} |W''(\xi)|^2 \|U_1 - U_2\|_{L^2_\lambda}^2.
\] (47)

Thus, from (45)–(47), we obtain

\[
\left\| \frac{dW}{d(U_1,U_1^{(m)})} - \frac{dW}{d(U_2,U_2^{(m)})} \right\|^2_X \leq \left\{ \frac{1}{2} \max_{|\xi| \leq 2 \hat{C}_L M} |W''(\xi)|^2 + \frac{25}{18} \tilde{C}_L^2 M^2 \max_{|\xi| \leq 2 \hat{C}_L M} |W''(\xi)|^2 \right\} \|U_1 - U_2\|_{L^2_\lambda}^2.
\] (48)

Applying (48) to (42), we conclude that

\[
\|\Phi(U_1) - \Phi(U_2)\|_X^2 = \|\tilde{U}_1 - \tilde{U}_2\|_X^2 \leq (\Delta t)^2 R \left( \hat{C}_L M \right) \|U_1 - U_2\|_{L^2_\lambda}^2.
\]

Since

\[
0 \leq (\Delta t)^2 R \left( \hat{C}_L M \right) < 1
\]

from the assumption (20) for \(\Delta t\), the mapping \(\Phi\) is contraction into \(X\). This completes the proof. \(\square\)

**Theorem 4.7** (global existence and uniqueness). Let \(C_0\) be the square root of the right-hand side of (18), i.e.,

\[
C_0 := \left[ \frac{1}{\min \left\{ \frac{1}{2} \hat{C}_L, \mu \right\}} \left\{ J_d(U^{(0)}) + W(U_0^{(0)}) + W(U_K^{(0)}) + c(L + 2) \right\} \right]^{\frac{1}{2}}.
\]
If $\Delta t$ satisfies $\Delta t^2 R(C_L C_0) < 1$, then there exists a unique solution $\{U^{(m)}_k\}_{k=-1}^{K+1} (m \in \mathbb{N})$ satisfying (11) with (12) and (13).

**Proof.** Since $\{\|U^{(0)}\|_{\dot{H}_1^1}^2 + \|U^{(0)}_0\|_{H_0^1}^2 + \|U^{(1)}_1\|_{H_1^1}^2\}^{1/2} \leq C_0$ from Lemma 3.2, there exists a unique solution $U^{(1)}$ satisfying $\{\|U^{(1)}\|_{\dot{H}_1^1}^2 + \|U^{(0)}_0\|_{H_0^1}^2 + \|U^{(1)}_1\|_{H_1^1}^2\}^{1/2} \leq C_0$. Repeating the procedure, we have completed the proof. \qed

5. **Error estimate.** In this section, we show an error estimate. We also use the energy method in [8, 21, 22, 23, 24]. Let $\Delta U$ satisfy

$$
\begin{align*}
&\text{Lemma 5.1.} \quad \text{Let} \quad \Delta U \quad \text{satisfies} \\
&\text{Appendix):}
\end{align*}
$$

for $t \in [0, T]$, where $\partial_x f(a)$ means $\partial_x f(x)|_{x=a}$. From direct calculation, we can check $\tilde{u} \in C^3((-L, 2L) \times [0, T])$. Furthermore, we can also check that if $u \in C^4([0, L] \times [0, T])$, then $\tilde{u} \in C^4((-L, 2L) \times [0, T])$. Moreover, we define the error as

$$
\varepsilon^{(m)}_k := U^{(m)}_k - \tilde{u}(k\Delta x, m\Delta t) \quad (k = -1, 0, \ldots, K, K+1, m = 0, 1, \ldots, M).
$$

For the sake of simplicity, let us use the expression $\varepsilon^{(m)}_k := \tilde{u}(k\Delta x, m\Delta t)$ from now on. Also, the expression $\delta^*_k f_{t_1}$ means $\delta^*_k f_{t_1}|_{t_{1}=t}$, where the symbol $*$ denotes $+$, (1) or (2). Then, the following lemma and theorem hold (the proofs can be found in Appendix):

**Lemma 5.1.** Let $u$ be the solution to the problem (1)–(3) with an initial value satisfying $u \in C^3([0, L] \times [0, T])$. Then, we obtain the following equations on the error:

$$
\begin{align*}
\frac{\varepsilon^{(m+1)}_k - \varepsilon^{(m)}_k}{\Delta t} &= \delta^{(2)}_k \left( \frac{(m+1)}{2} \varepsilon^{(m+1)}_k + \varepsilon^{(m)}_k \right) + \frac{dW}{d(U^{(m+1)}_k, U^{(m)}_k)} - \frac{dW}{d(u^{(m+1)}_k, u^{(m)}_k)} + \xi_1^{(m+1)} + \xi_2^{(m+1)} + \xi_3^{(m+1)} \quad (k = 0, \ldots, K), \tag{49} \\
\frac{\varepsilon^{(m+1)}_0 - \varepsilon^{(m)}_0}{\Delta t} &= \delta^{(1)}_k \left( \frac{(m+1)}{2} \varepsilon^{(m+1)}_0 + \varepsilon^{(m)}_0 \right) + \frac{dW}{d(U^{(m+1)}_0, U^{(m)}_0)} - \frac{dW}{d(u^{(m+1)}_0, u^{(m)}_0)} + \xi_1^{(m+1)} + \xi_3^{(m+1)} \quad (k = 0, \ldots, K), \tag{50} \\
\frac{\varepsilon^{(m+1)}_K - \varepsilon^{(m)}_K}{\Delta t} &= -\delta^{(1)}_K \left( \frac{(m+1)}{2} \varepsilon^{(m+1)}_K + \varepsilon^{(m)}_K \right) - \frac{dW}{d(U^{(m+1)}_K, U^{(m)}_K)} - \frac{dW}{d(u^{(m+1)}_K, u^{(m)}_K)} + \xi_1^{(m+1)} + \xi_3^{(m+1)} \quad (k = 0, \ldots, K). \tag{51}
\end{align*}
$$
for $m = 0, 1, \ldots, M - 1$, where $\xi_1$, $\xi_2$, $\xi_3$, and $\xi_4$ are defined as follows:

$$
\xi_{1,k}^{(m+\frac{1}{2})} := \partial_t u_k^{(m+\frac{1}{2})} - \frac{u_k^{(m+1)} - u_k^{(m)}}{\Delta t} \quad (k = 0, \ldots, K),
$$

$$
\xi_{2,k}^{(m+\frac{1}{2})} := \partial^{(2)}_t \left( \frac{u_k^{(m+1)} + u_k^{(m)}}{2} \right) - \partial^2_x u_k^{(m+\frac{1}{2})} \quad (k = 0, \ldots, K),
$$

$$
\xi_{3,k}^{(m+\frac{1}{2})} := W'(u_k^{(m+\frac{1}{2})}) - \frac{dW}{d(u_k^{(m+1)}, u_k^{(m)})} \quad (k = 0, \ldots, K),
$$

$$
\xi_{4,k}^{(m+\frac{1}{2})} := \delta_k^{(1)} \left( \frac{u_k^{(m+1)} + u_k^{(m)}}{2} \right) - \partial_x u_k^{(m+\frac{1}{2})} \quad (k = 0, K).
$$

Theorem 5.2. Assume that the problem (1)–(3) with an initial value has a smooth solution $u$ that satisfies $u \in C^3([0, L] \times [0, T])$. Denote the bounds by

$$
\max_{0 \leq m \leq M} \left\{ \left\| D U^{(m)} \right\|, \left\| D u^{(m)} \right\| \right\} \leq C_1, \quad \max_{0 \leq m \leq M} \left\{ \left\| U^{(m)} \right\|_{L^\infty}, \left\| u^{(m)} \right\|_{L^\infty} \right\} \leq C_2. \quad (52)
$$

Also, let

$$
C_W := 2 \left\{ C_1^2 \hat{C}_T^2 \max_{|\xi| \leq C_2} |W''(\xi)|^2 + \max_{|\xi| \leq C_2} |W'''(\xi)|^2 \right\}. \quad (53)
$$

Then, the following inequality holds:

$$
\{1 - (1 + C_W)\Delta t\} \left\{ \left\| e^{(m+1)} \right\|_{H^4}^2 + \left| e_0^{(m+1)} \right|^2 + \left| e_{K}^{(m+1)} \right|^2 \right\} \leq \{1 + (1 + C_W)\Delta t\} \left\{ \left\| e^{(m)} \right\|_{H^4}^2 + \left| e_0^{(m)} \right|^2 + \left| e_{K}^{(m)} \right|^2 \right\} + 2\Delta t \xi^{(m+\frac{1}{2})} \quad (m = 0, 1, \ldots, M - 1),
$$

where

$$
\xi^{(m+\frac{1}{2})} := \left\{ \left\| \xi_1^{(m+\frac{1}{2})} \right\|_{H^4}^2 + \left| \xi_{1,0}^{(m+\frac{1}{2})} \right|^2 + \left| \xi_{1,K}^{(m+\frac{1}{2})} \right|^2 \right\} + \left\{ \left\| \xi_2^{(m+\frac{1}{2})} \right\|_{H^4}^2 + \left| \xi_{2,0}^{(m+\frac{1}{2})} \right|^2 + \left| \xi_{2,K}^{(m+\frac{1}{2})} \right|^2 \right\} + \left\{ \left\| \xi_3^{(m+\frac{1}{2})} \right\|_{H^4}^2 + \left| \xi_{3,0}^{(m+\frac{1}{2})} \right|^2 + \left| \xi_{3,K}^{(m+\frac{1}{2})} \right|^2 \right\} + \left\{ \left\| \xi_4^{(m+\frac{1}{2})} \right\|_{H^4}^2 + \left| \xi_{4,0}^{(m+\frac{1}{2})} \right|^2 + \left| \xi_{4,K}^{(m+\frac{1}{2})} \right|^2 \right\}. \quad (54)
$$

Corollary 2. Assume that the problem (1)–(3) with an initial value has a smooth solution $u$ that satisfies $u \in C^4([0, L] \times [0, T])$. In the same manner, as Theorem 5.2, denote the bounds by (52). If $\Delta t$ satisfies

$$
\Delta t < \frac{1}{3(1 + C_W)}, \quad (55)
$$

then, there exists a constant $C$ independent of $k$ and $m$ such that

$$
\left\| e^{(m)} \right\|_{L^\infty} \leq C \left( (\Delta x)^2 + (\Delta t)^2 \right) \quad (m = 1, \ldots, M).
$$

Proof. For all $a > 0$, if $0 < \Delta t \leq 1/(3a)$, then two following inequalities hold:

$$
\frac{1 + a\Delta t}{1 - a\Delta t} \leq 1 + 3a\Delta t, \quad \frac{1}{1 - a\Delta t} \leq \frac{3}{2}. \quad (56)
$$
From (55), using Theorem 5.2 and (56), we obtain
\[ \left\| e^{(m+1)} \right\|^2_{B^2_h} + \left\| e^{(m+1)}_0 \right\|^2 + \left\| e^{(m+1)}_K \right\|^2 \leq \{1 + 3(1 + C_W)\Delta t\} \left\{ \left\| e^{(m)} \right\|^2_{B^2_h} + \left\| e^{(m)}_0 \right\|^2 + \left\| e^{(m)}_K \right\|^2 \right\} + 3\Delta t\xi^{(m \geq \frac{1}{2})} \] (57)
for \( m = 0, 1, \ldots, M - 1 \). Let \( C_3 := 1 + 3(1 + C_W)\Delta t \) and using (57) repeatedly, we obtain
\[ \left\| e^{(m)} \right\|^2_{B^2_h} + \left\| e^{(m)}_0 \right\|^2 + \left\| e^{(m)}_K \right\|^2 \leq C_3 \left\{ \left\| e^{(m-1)}_0 \right\|^2_{B^2_h} + \left\| e^{(m-1)}_0 \right\|^2 + \left\| e^{(m-1)}_K \right\|^2 \right\} + 3\Delta t\xi^{(m \geq \frac{1}{2})} \]
\[ \leq C_3^2 \left\{ \left\| e^{(m-2)}_0 \right\|^2_{B^2_h} + \left\| e^{(m-2)}_0 \right\|^2 + \left\| e^{(m-2)}_K \right\|^2 \right\} + 3\Delta tC_3\xi^{(m \geq \frac{1}{2})} + 3\Delta t\xi^{(m \geq \frac{1}{2})} \]
\[ \leq \cdots \]
\[ \leq C_3^m \left\{ \left\| e^{(0)} \right\|^2_{B^2_h} + \left\| e^{(0)}_0 \right\|^2 + \left\| e^{(0)}_K \right\|^2 \right\} + 3\Delta t\sum_{j=1}^{m} C_3^{j-1}\xi^{\left(m-j \geq \frac{1}{2}\right)} \]
\[ = 3\Delta t\sum_{j=1}^{m} C_3^{j-1}\xi^{\left(m-j \geq \frac{1}{2}\right)} \quad (m = 1, \ldots, M), \]
where the last equality holds from \( e^{(0)} = 0 \). Since it holds that \( 1 \leq C_3 \), using the following inequality: \( 1 + x \leq \exp(x) \) for all \( x > 0 \), we get
\[ C_3^{j-1} \leq C_3^M = \{1 + 3(1 + C_W)\Delta t\}^M \leq \exp \left\{ M \cdot 3(1 + C_W)\frac{T}{M} \right\} = \exp \{3(1 + C_W)T\} \]
for \( j = 1, \ldots, M \). Therefore, we obtain
\[ \left\| e^{(m)} \right\|^2_{B^2_h} + \left\| e^{(m)}_0 \right\|^2 + \left\| e^{(m)}_K \right\|^2 \leq 3\Delta t \exp \{3(1 + C_W)T\} \sum_{j=1}^{m} \xi^{\left(m-j \geq \frac{1}{2}\right)} \] (58)
for \( m = 1, \ldots, M \). Next, we estimate \( \xi \). Let us define
\[ M_{i,j}(v) := \max \left\{ \left| \frac{\partial^{i+j}v}{\partial x^i \partial t^j} \right| : (x, t) \in [0, L] \times [0, T] \right\} \quad \text{for all } i, j \in \mathbb{Z}, \]
\[ C_{W,i} := \max_{|\xi| \leq C_2} \left| W^{(i)}(\xi) \right| \quad (i = 2, 3). \] (59)
Firstly, we consider \( \xi_4 \). For any \( x \in [0, L] \), applying the Taylor theorem to \( \hat{u} \), there exists \( \theta_1 \in (0, 1) \) such that
\[ \hat{u}(x, (m \geq \frac{1}{2})\Delta t + \hat{u}(x, m\Delta t) \]
\[ = \hat{u} \left( x, \left(m + \frac{1}{2}\right)\Delta t \right) + \frac{(\Delta t)^2}{16} \left\{ \partial_x^2 \hat{u} \left( x, \left(m + \frac{1}{2}\theta_1\right)\Delta t \right) + \partial_x^2 \hat{u} \left( x, \left(m - \frac{1}{2}\theta_1\right)\Delta t \right) \right\}. \] (60)
Substituting \( k\Delta x \) (\( k = 0, K \)) into \( x \) in (60), we obtain
\[
\left| \delta_k^{(1)} \left( \frac{\tilde{u}_k^{(m+1)} + \tilde{u}_k^{(m)}}{2} \right) - \partial_x u_k^{(m+\frac{1}{2})} \right| = \left| \delta_k^{(1)} \tilde{u}_k^{(m+\frac{1}{2})} + \frac{(\Delta t)^2}{16} \delta_k^{(1)} \left( \partial^2_{xx}u_k^{(m+\frac{1}{2})} + \partial^2_{x}u_k^{(m+\frac{1}{2})} \right) - \partial_x u_k^{(m+\frac{1}{2})} \right|
\leq \left| \delta_k^{(1)} \tilde{u}_k^{(m+\frac{1}{2})} - \partial_x u_k^{(m+\frac{1}{2})} \right| + \left| \frac{(\Delta t)^2}{16} \left( \delta_k^{(1)} \left( \partial^2_{xx}u_k^{(m+\frac{1}{2})} \right) + \delta_k^{(1)} \left( \partial^2_{x}u_k^{(m+\frac{1}{2})} \right) \right) \right|
\]
for \( k = 0, K \). Here, we consider the case of \( k = 0 \). For any \( t \in [0, T] \), from the definition of \( \tilde{u} \), we have
\[
\tilde{u}(-\Delta x, t) = u(-\Delta x, t) - 2\Delta x\partial_x u(0, t) - \frac{(\Delta x)^3}{3} \partial^3_x u(0, t).
\]
Hence, substituting \( (m + 1/2)\Delta t \) into \( t \) in (61), we get
\[
\left| \delta_k^{(1)} \tilde{u}_0^{(m+\frac{1}{2})} - \partial_x u_0^{(m+\frac{1}{2})} \right| = \left| \frac{\tilde{u}_1^{(m+\frac{1}{2})} - \tilde{u}_1^{(m+\frac{1}{2})}}{2\Delta x} - \partial_x u_0^{(m+\frac{1}{2})} \right| - \partial_x u_0^{(m+\frac{1}{2})} \right| + \frac{(\Delta t)^3}{3} \partial^3_x u_0^{(m+\frac{1}{2})} \right|
\leq \frac{(\Delta x)^2}{6} M_{3,0}(u).
\]
Also, for any \( t \in [0, T] \), using the definition of \( \tilde{u} \) again, the following equality holds:
\[
\partial^2_t \tilde{u}(-\Delta x, t) = \partial^2_t u(-\Delta x, t) - 2\Delta x\partial^2_x \partial_x u(0, t) - \frac{(\Delta x)^3}{3} \partial^3_x \partial^3_x u(0, t).
\]
Hence, substituting \( (m + (1 \pm \theta_1)/2)\Delta t \) into \( t \) in (62), we obtain
\[
\delta_k^{(1)} \left( \partial^2_{xx}u_0^{(m+\frac{1}{2})} \right) = \frac{\partial^2_{xx}u_1^{(m+\frac{1}{2})} - \partial^2_{xx}u_1^{(m+\frac{1}{2})}}{2\Delta x} - \partial^2_{xx}u_0^{(m+\frac{1}{2})} + \frac{(\Delta t)^2}{6} \partial^2_x \partial^2_x u_0^{(m+\frac{1}{2})}.
\]
Therefore, we get
\[
\left| \delta_k^{(1)} \left( \partial^2_{xx}u_0^{(m+\frac{1}{2})} \right) \right| \leq M_{1,2}(u) + \frac{(\Delta x)^2}{6} M_{3,2}(u).
\]
Hence, we conclude that
\[
\left| \delta_k^{(1)} \left( \frac{\tilde{u}_k^{(m+1)} + \tilde{u}_k^{(m)}}{2} \right) - \partial_x u_k^{(m+\frac{1}{2})} \right| \leq \frac{(\Delta x)^2}{6} M_{3,0}(u) + \frac{(\Delta t)^2}{8} M_{1,2}(u) + \frac{(\Delta t)^2(\Delta x)^2}{48} M_{3,2}(u)
\]
for \( k = 0 \). In the same manner, the above equality holds in the case of \( k = K \), too. From the assumption (55) for \( \Delta t \) and the following inequality: \( 1 + C_W \geq 1 \), we get
\[
\Delta t < \frac{1}{3(1 + C_W)} \leq \frac{1}{3} < 1.
\]
Thus, we obtain the following estimate:

\[
\left| \xi_{4,k}^{(m+\frac{1}{2})} \right| = \left| \delta_k^{(1)} \left( \frac{\tilde{u}_{k}^{(m+1)} + \tilde{u}_{k}^{(m)}}{2} \right) - \partial_x u_k^{(m+\frac{1}{2})} \right| \\
\leq (\Delta x)^2 \left( \frac{1}{6} M_{3,0}(u) + \frac{1}{48} M_{3,2}(u) \right) + \frac{(\Delta t)^2}{8} M_{1,2}(u) \quad (k = 0, K).
\]

Next, we consider the \( \xi_2 \). Substituting \( k \Delta x \ (k = 0, \ldots, K) \) into \( x \) in (60), we obtain

\[
\delta_k^{(2)} \left( \frac{\tilde{u}_{k}^{(m+1)} + \tilde{u}_{k}^{(m)}}{2} \right) - \partial_x^2 u_k^{(m+\frac{1}{2})} = \delta_k^{(2)} \tilde{u}_{k}^{(m+\frac{1}{2})} - \partial_x^2 u_k^{(m+\frac{1}{2})} + \frac{(\Delta t)^2}{16} \{ \delta_k^{(2)} \left( \partial_x^4 u_k^{(m+\frac{1}{2})} + \partial_x^4 u_k^{(m+\frac{1}{2})} \right) \} + \delta_k^{(2)} \left( \partial_x^4 u_k^{(m+\frac{1}{2})} \right) \quad (k = 0, \ldots, K).
\]

for \( k = 0, \ldots, K \). For any \( t \in [0, T] \) and \( k = 0, \ldots, K \), applying the Taylor theorem to \( \tilde{u} \), there exists \( \theta_2 \in (0, 1) \) such that

\[
\tilde{u}(k+1)\Delta x, t - 2\tilde{u}(k\Delta x, t) + \tilde{u}((k-1)\Delta x, t) = \frac{(\Delta x)^2}{24} \left\{ \partial_x^4 \tilde{u}((k + \theta_2)\Delta x, t) + \partial_x^4 \tilde{u}((k - \theta_2)\Delta x, t) \right\}.
\]

(65)

From the definition of \( \tilde{u} \), it holds that \( \partial_x^4 \tilde{u}(x, t) = \partial_x^4 \tilde{u}(-x, t) \) for all \( x \in (-L, 0) \). Hence, we have \( \partial_x^4 \tilde{u}(-\theta_2\Delta x, t) = \partial_x^4 \tilde{u}(\theta_2\Delta x, t) \). From a similar observation, we obtain \( \partial_x^4 \tilde{u}((K + \theta_2)\Delta x, t) = \partial_x^4 \tilde{u}((K - \theta_2)\Delta x, t) \). Namely, substituting \( (m+1/2)\Delta t \) into \( t \) in (65), we get

\[
\delta_k^{(2)} \tilde{u}_{k}^{(m+\frac{1}{2})} - \partial_x^2 u_k^{(m+\frac{1}{2})} = \begin{cases} 
\frac{(\Delta x)^2}{12} \partial_x^4 u_{k-\theta_2}^{(m+\frac{1}{2})}, & (k = 0), \\
\frac{(\Delta x)^4}{24} \left( \partial_x^4 u_{k+\theta_2}^{(m+\frac{1}{2})} + \partial_x^4 u_{k-\theta_2}^{(m+\frac{1}{2})} \right), & (k = 1, \ldots, K - 1), \\
\frac{(\Delta x)^2}{12} \partial_x^4 u_{K-\theta_2}^{(m+\frac{1}{2})}, & (k = K).
\end{cases}
\]

Hence, we conclude that

\[
\left| \delta_k^{(2)} \tilde{u}_{k}^{(m+\frac{1}{2})} - \partial_x^2 u_k^{(m+\frac{1}{2})} \right| \leq \frac{(\Delta x)^2}{12} M_{4,0}(u) \quad (k = 0, \ldots, K).
\]

In the same manner, as described above, we have

\[
\left| \delta_k^{(2)} \left( \partial_x^2 \tilde{u}_k^{(m+\frac{1}{2})} \right) \right| \leq M_{2,2}(u) + \frac{\Delta x}{3} M_{3,2}(u) \quad (k = 0, \ldots, K).
\]

The above estimate is checked in Appendix, Lemma 2. Hence, using the Young inequality and the following inequality: \( (\Delta t)^4 < (\Delta t)^2 \) obtained by (63), we obtain

\[
\left| \xi_{2,k}^{(m+\frac{1}{2})} \right| \leq \left( \delta_k^{(2)} \tilde{u}_{k}^{(m+\frac{1}{2})} - \partial_x^2 u_k^{(m+\frac{1}{2})} \right) + \frac{(\Delta t)^2}{16} \left\{ \delta_k^{(2)} \left( \partial_x^4 u_k^{(m+\frac{1}{2})} \right) + \delta_k^{(2)} \left( \partial_x^4 u_k^{(m+\frac{1}{2})} \right) \right\} \\
\leq \frac{(\Delta x)^2}{12} M_{4,0}(u) + \frac{\Delta x}{8} M_{2,2}(u) + \frac{\Delta x (\Delta t)^2}{24} M_{3,2}(u) \\
\leq \frac{(\Delta x)^2}{12} M_{4,0}(u) + \frac{(\Delta t)^2}{8} M_{2,2}(u) + \frac{(\Delta x)^2}{48} M_{3,2}(u) + \frac{(\Delta t)^4}{48} M_{3,2}(u) \\
\leq (\Delta x)^2 \left( \frac{1}{12} M_{4,0}(u) + \frac{1}{48} M_{3,2}(u) \right) + (\Delta t)^2 \left( \frac{1}{8} M_{2,2}(u) + \frac{1}{48} M_{3,2}(u) \right).
\]
Similarly, we see from the Taylor theorem that
\[ \left| \xi_{1,k}^{(m+\frac{1}{2})} \right| \leq C M_{0,3}(u)(\Delta t)^2 \quad (k = 0, \ldots, K), \]
\[ \left| \xi_{3,k}^{(m+\frac{1}{2})} \right| \leq C \left\{ C_{W,2} M_{0,2}(u) + C_{W,3} (M_{0,1}(u))^2 \right\} (\Delta t)^2 \quad (k = 0, \ldots, K). \]

From the regularity assumption of the solution \( u \) and the potential \( W \), we see that \( C_{W,i} \) (\( i = 2, 3 \)) and \( M_{i,j}(u) \) (\( i, j \in \mathbb{Z}, 0 \leq i + j \leq 5 \)) are bounded. Thus, we obtain the following estimates:
\[ \left| \xi_{i,k}^{(m+\frac{1}{2})} \right| \leq C_4 ((\Delta x)^2 + (\Delta t)^2) \quad (k = 0, \ldots, K, \ i = 1, 2, 3), \tag{66} \]
\[ \left| \xi_{4,k}^{(m+\frac{1}{2})} \right| \leq C_4 ((\Delta x)^2 + (\Delta t)^2) \quad (k = 0, K), \tag{67} \]
where \( C_4 \) is a constant. From a similar observation, we obtain
\[ \left| \delta_k^+ \xi_{i,k}^{(m+\frac{1}{2})} \right| \leq C_4 ((\Delta x)^2 + (\Delta t)^2) \quad (k = 0, \ldots, K - 1, \ i = 1, 2, 3). \tag{68} \]

Therefore, using (66) and (68), we have
\[ \left\| \xi_{i,k}^{(m+\frac{1}{2})} \right\|_{\tilde{H}_a^1}^2 = \sum_{k=0}^{K} \left\| \xi_{i,k}^{(m+\frac{1}{2})} \right\|_{\Delta x}^2 + \sum_{k=0}^{K-1} \left\| \delta_k^+ \xi_{i,k}^{(m+\frac{1}{2})} \right\|_{\Delta x}^2 \leq 2C_4^2 L ((\Delta x)^2 + (\Delta t)^2)^2 \quad (i = 1, 2, 3). \tag{69} \]

From the above estimates (66), (67), and (69), we have
\[ \xi^{(m+\frac{1}{2})} \leq 3 \cdot 2C_4^2 (L + 1)((\Delta x)^2 + (\Delta t)^2)^2 + 2 \cdot 2C_4^2 ((\Delta x)^2 + (\Delta t)^2)^2 \leq 10C_4^2 (L + 1)((\Delta x)^2 + (\Delta t)^2)^2 \quad (m = 0, 1, \ldots, M - 1). \]

Therefore, from (58), we obtain
\[ \left\| e^{(m)} \right\|_{\tilde{H}_a^1}^2 + \left\| e^{(m)}_0 \right\|_{\tilde{H}_a^1}^2 + \left\| e^{(m)}_K \right\|_{\tilde{H}_a^1}^2 \leq 3 \Delta t \left[ \exp \left\{ 3(1 + CW)T \right\} \right] \cdot 10C_4^2 (L + 1)((\Delta x)^2 + (\Delta t)^2)^2 \sum_{j=1}^{m} \leq 30C_4^2 (L + 1) \left[ \exp \left\{ 3(1 + CW)T \right\} \right] (\Delta x)^2 + (\Delta t)^2)^2. \]

for \( m = 1, \ldots, M \). That is,
\[ \sqrt{\left\| e^{(m)} \right\|_{\tilde{H}_a^1}^2 + \left\| e^{(m)}_0 \right\|_{\tilde{H}_a^1}^2 + \left\| e^{(m)}_K \right\|_{\tilde{H}_a^1}^2} \leq C \sqrt{30(L + 1)T} \left[ \exp \left\{ \frac{3}{2} (1 + CW)T \right\} \right] (\Delta x)^2 + (\Delta t)^2. \]

Here, let
\[ C := \tilde{C}_L C_4 \sqrt{30(L + 1)T} \left[ \exp \left\{ \frac{3}{2} (1 + CW)T \right\} \right]. \]

Hence, we conclude from Lemma 3.3 that
\[ \left\| e^{(m)} \right\|_{L_a^\infty} \leq \tilde{C}_L \left\| e^{(m)} \right\|_{\tilde{H}_a^1} \leq C ((\Delta x)^2 + (\Delta t)^2) \quad (m = 1, \ldots, M). \tag{70} \]

This completes the proof.
6. Numerical experiments. In this section, we demonstrate through numerical experiments that the numerical solution of the proposed scheme is efficient and that the scheme inherits the dissipative property from the original problem in a discrete sense. We consider the following dynamic boundary condition:

\[
\begin{align*}
\varepsilon_{\text{ex}} \partial_t u(0, t) &= \partial_x u(x, t)|_{x=0} - W'(u(0, t)), \\
\varepsilon_{\text{ex}} \partial_t u(L, t) &= -\partial_x u(x, t)|_{x=L} - W'(u(L, t)),
\end{align*}
\]  

(71)

where \(\varepsilon_{\text{ex}}\) is a positive constant. We choose \(K = 100\) and fix \(L = 1\) so that \(\Delta x = 1/100\). On the other hand, we choose \(T\) and \(\Delta t\) depending on the situation. We fix the parameter \(\varepsilon_{\text{ex}} = 10\). Also, we consider the nonlinear function \(W(s) = (\gamma/4)(s^2 - 1)^2\), where we fix the parameter \(\gamma = 100\).

6.1. Dynamic boundary condition.

Numerical experiment 1. As the initial value, we consider

\[
u(x, 0) = a_0 + a_1 \cos(5\pi x) + a_2 \sin(8\pi x) + a_3 \cos(2\pi x),
\]

where we choose \(a_0 = 0.02, a_1 = -0.05, a_2 = -0.008,\) and \(a_3 = 0.01\). Also, we choose \(M = 6000\) and fix \(T = 0.6\) so that \(\Delta t = 1/10000\). Figure 1 shows the time development of the solution for (1) with (71) from our scheme. Figure 2 shows the time development of total energy. This graph shows that the energy decreases numerically.

---

Numerical experiment 2. As the initial value, we choose

\[
u(x, 0) = \exp\{-500(x - 0.5)^2\}.
\]

Also, we choose \(M = 700\) and fix \(T = 0.7\) so that \(\Delta t = 1/1000\). Figure 3 shows the time development of the solution for (1) with (71) from our scheme. Figure 4 shows the time development of total energy. This graph shows that the energy decreases numerically.
As stated in the Introduction, our study for the dynamic boundary condition differs from previous studies for non-dynamical boundary conditions such as the Neumann boundary condition. Since there is a term of the time derivative on the boundary in (1) with (71), it is natural that the long-time behavior of the solution may differ from that to (1) with the Neumann boundary condition. In order to assure that the difference occurs, we present the numerical simulations of our structure-preserving scheme for (1) with the Neumann boundary condition (see Appendix for details).

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Appendix. We use the same notation as in Section 5.

Lemma 1. Let $u$ be the solution to the problem (1)–(3) with an initial value satisfying $u \in C^3([0, L] \times [0, T])$. Then, we have the following equations on the error:

$$
\frac{e^{(m+1)}_k - e^{(m)}_k}{\Delta t} = \delta_k^{(2)} \left( \frac{e^{(m+1)}_k + e^{(m)}_k}{2} \right) - \left( dW \left( d(U^{(m+1)}_k, U^{(m)}_k) \right) - dW \left( d(u^{(m+1)}_k, u^{(m)}_k) \right) \right) + \xi^{(m+\frac{1}{2})}_1 + \xi^{(m+\frac{1}{2})}_2 + \xi^{(m+\frac{1}{2})}_3 + \xi^{(m+\frac{1}{2})}_4 \quad (k = 0, \ldots, K), \tag{72}
$$

$$
\frac{e^{(m+1)}_0 - e^{(m)}_0}{\Delta t} = \delta_k^{(1)} \left( \frac{e^{(m+1)}_0 + e^{(m)}_0}{2} \right) - \left( dW \left( d(U_0^{(m+1)}), U_0^{(m)} \right) - dW \left( d(u_0^{(m+1)}, u_0^{(m)}) \right) \right) + \xi^{(m+\frac{1}{2})}_1 + \xi^{(m+\frac{1}{2})}_3 \tag{73}
$$

$$
\frac{e^{(m+1)}_K - e^{(m)}_K}{\Delta t} = -\delta_k^{(1)} \left( \frac{e^{(m+1)}_K + e^{(m)}_K}{2} \right) - \left( dW \left( d(U_K^{(m+1)}), U_K^{(m)} \right) - dW \left( d(u_K^{(m+1)}, u_K^{(m)}) \right) \right) + \xi^{(m+\frac{1}{2})}_1 + \xi^{(m+\frac{1}{2})}_3 - \xi^{(m+\frac{1}{2})}_4 \quad (k = 0, \ldots, M-1). \tag{74}
$$

for $m = 0, 1, \ldots, M - 1$. 

---

Figure 3. Numerical solution of (1) with (71) obtained by our scheme.

Figure 4. Time development of total energy.
Proof. For any fixed \( m = 0, 1, \ldots, M - 1 \), from the definition of \( e, (1), (11), \) and (14), we have

\[
\frac{e^{(m+1)}_k - e^{(m)}_k}{\Delta t} = U_k^{(m+1)} - U_k^{(m)} - \partial_t u_k^{(m+\frac{1}{2})} + \partial_t u_k^{(m+\frac{1}{2})} - \frac{u_k^{(m+1)} - u_k^{(m)}}{\Delta t}
\]

\[
= \delta_k^{(2)} \left( \frac{U_k^{(m+1)} + U_k^{(m)}}{2} \right) - \frac{dW}{d(U_k^{(m+1)}, U_k^{(m)})} - \frac{\partial_x^2 \left( u_k^{(m+\frac{1}{2})} \right) + W' \left( u_k^{(m+\frac{1}{2})} \right) + \xi^{(m+\frac{1}{2})}_{1,k}}{\Delta t}
\]

\[
= \frac{dW}{d(U_k^{(m+1)} - U_k^{(m)})} - \frac{dW}{d(U_k^{(m+1)}, U_k^{(m)})} - dW + W' \left( u_k^{(m+\frac{1}{2})} \right) + \xi^{(m+\frac{1}{2})}_{1,k}
\]

\[
= \frac{dW}{d(U_k^{(m+1)} - U_k^{(m)})} - \frac{dW}{d(U_k^{(m+1)}, U_k^{(m)})}
\]

\[
= \frac{dW}{d(U_k^{(m+1)} - U_k^{(m)})} + \frac{dW}{d(U_k^{(m+1)}, U_k^{(m)})} + \xi^{(m+\frac{1}{2})}_{1,k} + \xi^{(m+\frac{1}{2})}_{2,k} + \xi^{(m+\frac{1}{2})}_{3,k}
\]

(75)

for \( k = 1, \ldots, K - 1 \). We show that the above equality holds at \( k = 0, K \). We remark that the equation (1) holds in the interior of the domain \((0, L)\) only. Hence, we cannot apply the equation (1) directly in the calculation of (75) on the boundary. Therefore, we consider points slightly inside from the boundary of the domain, and we take the limit of them to show that (75) holds at \( k = 0, K \). For any \( \epsilon \in (0, 1) \), let

\[
e^{(m)}_{0,\epsilon} := U_0^{(m)} - u(\epsilon \Delta x, m \Delta t), \quad e^{(m)}_{K,\epsilon} := U_K^{(m)} - u((K-\epsilon) \Delta x, m \Delta t) \quad (m = 0, 1, \ldots, M).
\]

Furthermore, for \( m = 0, 1, \ldots, M - 1 \), let

\[
\xi^{(m+\frac{1}{2})}_{1,\epsilon} := \partial_t u_{\epsilon}^{(m+\frac{1}{2})} - \frac{u_{\epsilon}^{(m+1)} - u_{\epsilon}^{(m)}}{\Delta t}, \quad \xi^{(m+\frac{1}{2})}_{1,K-\epsilon} := \partial_t u_{K-\epsilon}^{(m+\frac{1}{2})} - \frac{u_{K-\epsilon}^{(m+1)} - u_{K-\epsilon}^{(m)}}{\Delta t}
\]

\[
\xi^{(m+\frac{1}{2})}_{2,\epsilon} := \partial^2_x u_{\epsilon}^{(m+\frac{1}{2})}, \quad \xi^{(m+\frac{1}{2})}_{2,K-\epsilon} := \partial^2_x u_{K-\epsilon}^{(m+\frac{1}{2})}
\]

\[
\xi^{(m+\frac{1}{2})}_{3,\epsilon} := W' \left( u_{\epsilon}^{(m+\frac{1}{2})} \right) - \frac{dW}{d(u_{\epsilon}^{(m+1)}, u_{\epsilon}^{(m)})}, \quad \xi^{(m+\frac{1}{2})}_{3,K-\epsilon} := W' \left( u_{K-\epsilon}^{(m+\frac{1}{2})} \right) - \frac{dW}{d(u_{K-\epsilon}^{(m+1)}, u_{K-\epsilon}^{(m)})}
\]

In the same manner, as (75), we have

\[
\frac{e^{(m+1)}_{0,\epsilon} - e^{(m)}_{0,\epsilon}}{\Delta t} = \delta_k^{(2)} \left( \frac{e^{(m+1)}_0 + e^{(m)}_0}{2} \right) - \frac{dW}{d(U_0^{(m+1)}, U_0^{(m)})} - \frac{dW}{d(u_{\epsilon}^{(m+1)}, u_{\epsilon}^{(m)})}
\]

(76)

\[
\frac{e^{(m+1)}_{K,\epsilon} - e^{(m)}_{K,\epsilon}}{\Delta t} = \delta_k^{(2)} \left( \frac{e^{(m+1)}_K + e^{(m)}_K}{2} \right) - \frac{dW}{d(U_K^{(m+1)}, U_K^{(m)})} - \frac{dW}{d(u_{K-\epsilon}^{(m+1)}, u_{K-\epsilon}^{(m)})}
\]

(77)

From the smoothness assumption of \( u \), letting \( \epsilon \) tend to 0 in (76) and (77), we have

\[
\frac{e^{(m+1)}_k - e^{(m)}_k}{\Delta t} = \delta_k^{(2)} \left( \frac{e^{(m+1)}_k + e^{(m)}_k}{2} \right) - \frac{dW}{d(U_k^{(m+1)}, U_k^{(m)})} - \frac{dW}{d(u_{k}^{(m+1)}, u_{k}^{(m)})}
\]

\[
+ \xi^{(m+\frac{1}{2})}_{1,k} + \xi^{(m+\frac{1}{2})}_{2,k} + \xi^{(m+\frac{1}{2})}_{3,k} \quad (k = 0, K).
\]
Next, from (2) and (12), we obtain
\[
\frac{e_0^{(m+1)} - e_0^{(m)}}{\Delta t} = U_0^{(m+1)} - U_0^{(m)} - \partial_t u_0^{(m+\frac{1}{2})} + \partial_t u_0^{(m+\frac{1}{2})} - \frac{u_0^{(m+1)} - u_0^{(m)}}{\Delta t}
\]
\[
= \delta_k^{(1)} \left( \frac{e_0^{(m+1)} + U_0^{(m)}}{2} - \frac{dW}{d(U_0^{(m+1)}, U_0^{(m)})} - \partial_x u_0^{(m+\frac{1}{2})} + W'(u_0^{(m+\frac{1}{2})}) + \xi_{1,0} \right)
\]
\[
= \delta_k^{(1)} \left( \frac{e_0^{(m+1)} + u_0^{(m)}}{2} - \frac{dW}{d(u_0^{(m+1)}, u_0^{(m)})} + \partial_x u_0^{(m+\frac{1}{2})} - \partial_x u_0^{(m+\frac{1}{2})} \right)
\]
\[
+ \xi_{1,0} + \xi_{3,0} + \xi_{4,0}.
\]
Similarly, from (3) and (13), we have
\[
\frac{e_K^{(m+1)} - e_K^{(m)}}{\Delta t} = \delta_k^{(1)} \left( \frac{e_K^{(m+1)} + e_K^{(m)}}{2} - \frac{dW}{d(U_K^{(m+1)}, U_K^{(m)})} - \frac{dW}{d(u_K^{(m+1)}, u_K^{(m)})} \right)
\]
\[
+ \xi_{1,K} + \xi_{3,K} - \xi_{4,K}.
\]
Therefore, \((72)-(74)\) holds.

**Theorem 1.** Assume that the problem \((1)-(3)\) with an initial value has a smooth solution \(u\) that satisfies \(u \in C^3([0, L] \times [0, T])\). Denote the bounds by \((52)\). Also, let us define a constant \(C_W\) by \((53)\). Then, the following inequality holds:
\[
\{1-(1+C_W)\Delta t\} \left\{ \|e^{(m+1)}\|_{L^2}^2 + \|e_0^{(m+1)}\|_{L^2}^2 + \|e_K^{(m+1)}\|_{L^2}^2 \right\}
\leq \{1+(1+C_W)\Delta t\} \left\{ \|e^{(m)}\|_{L^2}^2 + \|e_0^{(m)}\|_{L^2}^2 + \|e_K^{(m)}\|_{L^2}^2 \right\} + 2\Delta t \xi^{(m+\frac{1}{2})} \quad (m = 0, 1, \ldots, M-1).
\]

**Proof.** For any fixed \(m = 0, 1, \ldots, M-1\), using \((72)\), we obtain
\[
\frac{1}{2\Delta t} \left( \|e^{(m+1)}\|_{L^2}^2 - \|e^{(m)}\|_{L^2}^2 \right)
\leq \sum_{k=0}^{K} \frac{\epsilon_k^{(m+1)} - \epsilon_k^{(m)}}{\Delta t} \Delta x
\leq \sum_{k=0}^{K} \left\{ \delta_k^{(2)} \left( \frac{\epsilon_k^{(m+1)} + \epsilon_k^{(m)}}{2} \right) - \frac{dW}{d(U_k^{(m+1)}, U_k^{(m)})} - \frac{dW}{d(u_k^{(m+1)}, u_k^{(m)})} \right\} + \xi_{1-k,-3,0} \Delta x,
\]
where \(\epsilon_{1-k,-3,0} := \xi_{1,k} + \xi_{2,k} + \xi_{3,k} + \xi_{4,k} \quad (k = 0, \ldots, K)\). From Corollary 1 (summation by parts formula), we calculate the first term on the right-hand side of
(78) as follows:

\[
\sum_{k=0}^{K} \left\{ \delta_k^{(2)} \left( \frac{e^{(m+1)}_k + e^{(m)}_k}{2} \right) - \frac{\Delta x}{2} \right\} \left( \frac{e^{(m+1)}_k + e^{(m)}_k}{2} \right) \Bigg|_{0}^{K}.
\]

(79)

From (73), (74), and the inequality: \((\sum_{i=1}^{n} a_i)^2 \leq n \sum_{i=1}^{n} a_i^2\), we estimate the boundary term on the right-hand side of (79) as follows:

\[
\leq - \left\{ \frac{\delta_0^{(1)} \left( \frac{e^{(m+1)}_0 + e^{(m)}_0}{2} \right)}{2} \right\}^2 + \frac{1}{4} \left\{ \frac{dW}{d(U^{(m+1)}_0, U^{(m)}_0)} - \frac{dW}{d(u^{(m+1)}_0, u^{(m)}_0)} \right\}^2 \xi_1,0 - \xi_{3,0} - \xi_{4,0}^2 + \frac{1}{4} \left\{ \frac{dW}{d(U^{(m+1)}_K, U^{(m)}_K)} - \frac{dW}{d(u^{(m+1)}_K, u^{(m)}_K)} \right\}^2 \xi_1,K - \xi_{3,K} + \xi_{4,K}^2.
\]

(80)

Similarly, we also estimate the second term on the right-hand side of (78) as follows:

\[
\sum_{k=0}^{K} \left\{ \frac{e^{(m+1)}_k + e^{(m)}_k}{2} \right\}^2 \Delta x
\]

\[
\sum_{k=0}^{K} \left\{ \frac{e^{(m+1)}_k + e^{(m)}_k}{2} \right\}^2 \Delta x
\]

\[
\leq \sum_{k=0}^{K} \left\{ \frac{\delta_k^{(1)} \left( \frac{e^{(m+1)}_k + e^{(m)}_k}{2} \right)}{2} \right\}^2 \Delta x
\]

\[
\leq \left\{ \frac{\xi_1^{(m+\frac{1}{2})}}{2} \right\}^2 + \left\{ \frac{\xi_2^{(m+\frac{1}{2})}}{2} \right\}^2 + \left\{ \frac{\xi_3^{(m+\frac{1}{2})}}{2} \right\}^2.
\]

(81)
Therefore, we see from (78)–(81) that
\[
\frac{1}{2\Delta t} \left( \|e^{(m+1)}\|_{L_2^d}^2 - \|e^{(m)}\|_{L_2^d}^2 \right) \leq - \frac{1}{2\Delta t} \left( \|D (e^{(m+1)} + e^{(m)})\|_{L_2^d}^2 - \|e^{(m)}\|_{L_2^d}^2 - \|e^{(m+1)}\|_{L_2^d}^2 - \|e^{(m)}\|_{L_2^d}^2 \right) \\
+ \frac{1}{2} \left\{ \left\| e^{(m+1)} \right\|_{L_2^d}^2 + \left\| e^{(m+1)} \right\|_{L_2^d}^2 \right\} - \frac{1}{2} \left\{ \left\| e^{(m)} \right\|_{L_2^d}^2 + \left\| e^{(m)} \right\|_{L_2^d}^2 \right\} \\
+ \left\{ \left\| \frac{dW}{d(L^{(m+1)}, U^{(m+1)})} - \frac{dW}{d(L^{(m+1)}, U^{(m)})} \right\|_{L_2^d}^2 + \left\| \frac{dW}{d(U^{(m+1)}, U^{(m+1)})} - \frac{dW}{d(U^{(m+1)}, U^{(m)})} \right\|_{L_2^d}^2 \right\} \\
+ \left\{ \left\| \xi^{(m+\frac{1}{2})} \right\|_{L_2^d}^2 + \left\| \xi^{(m+\frac{1}{2})} \right\|_{L_2^d}^2 \right\} + \left\{ \left\| \xi^{(m+\frac{1}{2})} \right\|_{L_2^d}^2 + \left\| \xi^{(m+\frac{1}{2})} \right\|_{L_2^d}^2 \right\}.
\]

Next, using Corollary 1 (summation by parts formula), we have
\[
\frac{1}{2\Delta t} \left( \|De^{(m+1)}\|_{L_2^d}^2 - \|De^{(m)}\|_{L_2^d}^2 \right) \\
= \sum_{k=0}^{K-1} \left\{ \delta^+ \left( e^{(m+1)}_k + e^{(m)}_k \right) \right\} \left\{ \delta^+ \left( e^{(m+1)}_k - e^{(m)}_k \right) \right\} \Delta x \\
= - \sum_{k=0}^{n} \left\{ \delta^{(2)} \left( \frac{e^{(m+1)}_k + e^{(m)}_k}{2} \right) \right\} \left( \frac{e^{(m+1)}_k - e^{(m)}_k}{\Delta t} \right) \Delta x + \left\{ \delta^{(1)} \left( \frac{e^{(m+1)}_k + e^{(m)}_k}{2} \right) \right\} \left( \frac{e^{(m+1)}_k - e^{(m)}_k}{\Delta t} \right) \Delta x.
\]

Applying (72) and Corollary 1 (summation by parts formula) to the first term on the right-hand side of (83), we obtain
\[
- \sum_{k=0}^{K-1} \left\{ \delta^{(2)} \left( \frac{e^{(m+1)}_k + e^{(m)}_k}{2} \right) \right\} \left( \frac{e^{(m+1)}_k - e^{(m)}_k}{\Delta t} \right) \Delta x \\
\leq \sum_{k=0}^{n} \left\{ \delta^{(2)} \left( \frac{e^{(m+1)}_k + e^{(m)}_k}{2} \right) \right\} \left( \frac{dW}{d(U^{(m+1)}_k, U^{(m+1)}_k)} - \frac{dW}{d(U^{(m+1)}_k, U^{(m+1)}_k)} \right) \xi^{(m+\frac{1}{2})}_{1-3,k} \Delta x \\
= - \sum_{k=0}^{K-1} \left\{ \delta^+ \left( \frac{e^{(m+1)}_k + e^{(m)}_k}{2} \right) \right\} \left\{ \delta^+ \left( \frac{dW}{d(U^{(m+1)}_k, U^{(m+1)}_k)} - \frac{dW}{d(U^{(m+1)}_k, U^{(m+1)}_k)} \right) \xi^{(m+\frac{1}{2})}_{1-3,k} \Delta x \\
+ \left\{ \delta^{(1)} \left( \frac{e^{(m+1)}_k + e^{(m)}_k}{2} \right) \right\} \left( \frac{dW}{d(U^{(m+1)}_k, U^{(m+1)}_k)} - \frac{dW}{d(U^{(m+1)}_k, U^{(m+1)}_k)} \right) \xi^{(m+\frac{1}{2})}_{1-3,k} \right\}^K.
\]

Using (84), the Young inequality, and the inequality: \((\sum_{i=1}^n a_i)^2 \leq n \sum_{i=1}^n a_i^2\), it holds that
Here, from (73), (74), and the Young inequality, we estimate the above boundary term as follows:

\[
- \sum_{k=0}^{K} \left\{ \delta_k \left( \frac{e_k^{(m+1)} + e_k^{(m)}}{2} \right) \right\} \left( \frac{e_k^{(m+1)} - e_k^{(m)}}{2\Delta t} \right) \Delta x
\]

\[
\leq \sum_{k=0}^{K-1} \left\{ \delta_k \left( \frac{e_k^{(m+1)} + e_k^{(m)}}{2} \right) \right\}^2 \Delta x
\]

\[
+ \frac{1}{4} \sum_{k=0}^{K-1} \left[ \delta_k \left\{ \left( \frac{dW}{d(U_k^{(m+1)}, U_k^{(m)})} - \frac{dW}{d(u_k^{(m+1)}, u_k^{(m)})} \right) - \xi_{k-1,3}^{(m+\frac{1}{2})} \right\}^2 \Delta x
\]

\[
+ \left\{ \left( \frac{\delta_k^{(1)} e_k^{(m+1)} + e_k^{(m)}}{2} \right) \right\} \left\{ \left( \frac{dW}{d(U_k^{(m+1)}, U_k^{(m)})} - \frac{dW}{d(u_k^{(m+1)}, u_k^{(m)})} \right) - \xi_{k-1,3}^{(m+\frac{1}{2})} \right\}^K_0
\]

Thus, we have

\[
\frac{1}{2\Delta t} \left( \| De^{(m+1)} \|^2 - \| De^{(m)} \|^2 \right)
\]

\[
\leq \left\| D \left( \frac{e^{(m+1)} + e^{(m)}}{2} \right) \right\|^2 + \left\| D \left( \frac{dW}{d(U^{(m+1)}, U^{(m)})} - \frac{dW}{d(u^{(m+1)}, u^{(m)})} \right) \right\|^2
\]

\[
+ \left\| D\xi_1^{(m+\frac{1}{2})} \right\|^2 + \left\| D\xi_2^{(m+\frac{1}{2})} \right\|^2 + \left\| D\xi_3^{(m+\frac{1}{2})} \right\|^2
\]

\[
+ \left\{ \delta_k^{(1)} \left( e_k^{(m+1)} + e_k^{(m)} \right) \right\} \left( \frac{\delta_k^{(1)} e_k^{(m+1)} + e_k^{(m)}}{2} \right) \left\{ \left( \frac{dW}{d(U_k^{(m+1)}, U_k^{(m)})} - \frac{dW}{d(u_k^{(m+1)}, u_k^{(m)})} \right) - \xi_{k-1,3}^{(m+\frac{1}{2})} \right\}^K_0
\]

Here, from (73), (74), and the Young inequality, we estimate the above boundary term as follows:

\[
\left\{ \delta_k^{(1)} \left( e_k^{(m+1)} + e_k^{(m)} \right) \right\} \left( \frac{\delta_k^{(1)} e_k^{(m+1)} + e_k^{(m)}}{2} \right) \left( \frac{dW}{d(U_k^{(m+1)}, U_k^{(m)})} - \frac{dW}{d(u_k^{(m+1)}, u_k^{(m)})} \right) - \xi_{k-1,3}^{(m+\frac{1}{2})} \right\}^K_0
\]

\[
= - \left\{ \delta_k^{(1)} \left( e_k^{(m+1)} + e_k^{(m)} \right) \right\}^2 - \left\{ \delta_k^{(1)} \left( e_k^{(m+1)} + e_k^{(m)} \right) \right\} \left( \xi_{k-1,3}^{(m+\frac{1}{2})} \right) \left( \xi_{k-1,3}^{(m+\frac{1}{2})} \right)
\]

\[
- \left\{ \delta_k^{(1)} \left( e_k^{(m+1)} + e_k^{(m)} \right) \right\}^2 - \left\{ \delta_k^{(1)} \left( e_k^{(m+1)} + e_k^{(m)} \right) \right\} \left( \xi_{k-1,3}^{(m+\frac{1}{2})} \right) \left( \xi_{k-1,3}^{(m+\frac{1}{2})} \right)
\]

\[
\leq \left\| \xi_{2,0}^{(m+\frac{1}{2})} \right\|^2 + \left\| \xi_{2,K}^{(m+\frac{1}{2})} \right\|^2 + \left\| \xi_{4,0}^{(m+\frac{1}{2})} \right\|^2 + \left\| \xi_{4,K}^{(m+\frac{1}{2})} \right\|^2
\]

Namely, we have
Next, using Lemma 4.3, the following equality holds:
\[
\frac{1}{2\Delta t} \left( \left\| De^{(m+1)} \right\|^2 - \left\| De^{(m)} \right\|^2 \right)
\]
\[
\leq \left\| D \left( \frac{e^{(m+1)} + e^{(m)}}{2} \right) \right\|^2 +\left( \frac{dW}{d(U^{(m+1)}, U^{(m)})} - \frac{dW}{d(U^{(m)}, U^{(m)})} \right) \right\|^2 + \left\| D\xi_{1}^{(m+\frac{1}{2})} \right\|^2 
\]
\[
+ \left\| D\xi_{2}^{(m+\frac{1}{2})} \right\|^2 + \left\| D\xi_{3}^{(m+\frac{1}{2})} \right\|^2 + \left\| \xi_{2,0} \right\|^2 + \left\| \xi_{2,K} \right\|^2 + \left\| \xi_{4,0} \right\|^2 + \left\| \xi_{4,K} \right\|^2. \quad (85)
\]

Combining (82) and (85), we obtain
\[
\frac{1}{2\Delta t} \left( \left\| e^{(m+1)} \right\|^2_{\hat{H}^3_{t}} - \left\| e^{(m)} \right\|^2_{\hat{H}^3_{t}} \right)
\]
\[
\leq \frac{1}{2\Delta t} \left\{ \left\| e^{(m+1)} \right\|^2_{\hat{H}^3_{t}} + \left\| e^{(m)} \right\|^2_{\hat{H}^3_{t}} \right\} + \frac{1}{2\Delta t} \left\{ \left\| \left( e^{(m)} \right) \right\|^2_{\hat{H}^3_{t}} + \left\| \left( e^{(m)} \right) \right\|^2_{\hat{H}^3_{t}} \right\} 
\]
\[
+ \left\{ \left\| dW_{0} \right\|^2_{d(U^{(m+1)}, U^{(m)})} - \left\| dW_{0} \right\|^2_{d(U^{(m)}, U^{(m)})} \right\} + \left\| dW_{0} \right\|^2_{d(U^{(m+1)}, U^{(m)})} - \left\| dW_{0} \right\|^2_{d(U^{(m)}, U^{(m)})} \right\} \right\} + \xi_{1}^{(m+\frac{1}{2})}. \quad (86)
\]

Multiplying both sides of (86) by $2\Delta t$, we get
\[
(1 - \Delta t) \left\{ \left\| e^{(m+1)} \right\|^2_{\hat{H}^3_{t}} + \left\| e^{(m)} \right\|^2_{\hat{H}^3_{t}} \right\} 
\]
\[
\leq (1 + \Delta t) \left\{ \left\| e^{(m)} \right\|^2_{\hat{H}^3_{t}} + \left\| e^{(m)} \right\|^2_{\hat{H}^3_{t}} \right\} 
\]
\[
+ 2\Delta t \left\{ \left\| dW_{0} \right\|^2_{d(U^{(m+1)}, U^{(m)})} - \left\| dW_{0} \right\|^2_{d(U^{(m)}, U^{(m)})} \right\} \right\} + 2\Delta t \xi_{1}^{(m+\frac{1}{2})}. \quad (87)
\]

Next, using Lemma 4.3, the following equality holds:
\[
\frac{dW}{d(U^{(m+1)}, U^{(m)})} - \frac{dW}{d(U^{(m)}, U^{(m)})} = \frac{1}{2} W'' \left( U^{(m+1)}; u^{(m+1)}; U^{(m)}; u^{(m)} \right) e^{(m+1)} 
\]
\[
+ \frac{1}{2} W'' \left( U^{(m)}; u^{(m)}; U^{(m+1)}; u^{(m+1)} \right) e^{(m)} \quad (88)
\]

for $k = 0, \ldots, K$. From the assumption (52), using Lemma 4.2, we have
\[
\left\| W'' \left( U^{(m)}, u^{(m)}, U^{(m+1)}, u^{(m+1)} \right) \right\| L_{-3}^{\infty} \leq C_{W,2}, \quad (89)
\]
\[
\left\| W'' \left( U^{(m)}, u^{(m)}, U^{(m+1)}, u^{(m+1)} \right) \right\| L_{-3}^{\infty} \leq C_{W,2}, \quad (90)
\]

where $C_{W,i}$ ($i = 2, 3$) is defined by (59). Hence, we obtain
For $k = 0, \ldots , K$. Thus, we have

$$\left\| \frac{dW}{d(U^{(m+1)}, U^{(m)})} - \frac{dW}{d(U^{(m+1)}, u^{(m)})} \right\|_{L^2_3}^2 \leq \frac{C^2_{W,2}}{2} \left( \left\| e^{(m+1)} \right\|^2_{L^2_3} + \left\| e^{(m)} \right\|^2_{L^2_3} \right). \tag{92}$$

Next, using (88) and Lemma 4.4, we obtain

$$\left\| D \left( \frac{dW}{d(U^{(m+1)}, U^{(m)})} - \frac{dW}{d(U^{(m+1)}, u^{(m)})} \right) \right\| \leq \frac{1}{2} \left\| D \left( \bar{W}'' \left( U^{(m+1)}, u^{(m+1)}; U^{(m)}, u^{(m)} \right) \right) \right\| + \frac{1}{2} \left\| D \left( \bar{W}'' \left( U^{(m)}, u^{(m+1)}; U^{(m)}, u^{(m)} \right) \right) \right\| \left\| e^{(m+1)} \right\|_{L^\infty} \left\| D e^{(m+1)} \right\|_{L^\infty} \left\| e^{(m)} \right\|_{L^\infty} \left\| D e^{(m)} \right\|_{L^\infty} \tag{93}$$

From the assumption (52), using Lemma 4.5, we get

$$\left\| D \bar{W}'' \left( U^{(m+1)}, u^{(m+1)}; U^{(m)}, u^{(m)} \right) \right\| \leq \frac{C_{W,2}}{6} (2C_1 + 2C_1 + C_1 + C_1) = C_1 C_{W,3}. \tag{94}$$

Similarly,

$$\left\| D \bar{W}'' \left( U^{(m)}, u^{(m+1)}; U^{(m)}, u^{(m)} \right) \right\| \leq C_1 C_{W,3}. \tag{95}$$

Therefore, applying (89), (90), (94), and (95) to (93), we obtain

$$\left\| D \left( \frac{dW}{d(U^{(m+1)}, U^{(m)})} - \frac{dW}{d(U^{(m+1)}, u^{(m)})} \right) ^2 \right\| \leq C^2_{1} C^3_{W,3} \left( \left\| e^{(m+1)} \right\|^2_{L^\infty} + \left\| e^{(m)} \right\|^2_{L^\infty} \right) + C^2_{W,2} \left( \left\| De^{(m+1)} \right\|^2_{L^\infty} + \left\| De^{(m)} \right\|^2_{L^\infty} \right). \tag{96}$$

Combining (92) and (96), we have

$$\left\| \frac{dW}{d(U^{(m+1)}, U^{(m)})} - \frac{dW}{d(U^{(m+1)}, u^{(m)})} \right\|_{H^1_3}^2 \leq C^2_{1} C^3_{W,3} \left( \left\| e^{(m+1)} \right\|^2_{L^\infty} + \left\| e^{(m)} \right\|^2_{L^\infty} \right) + C^2_{W,2} \left( \left\| e^{(m+1)} \right\|^2_{H^1_3} + \left\| e^{(m)} \right\|^2_{H^1_3} \right).$$

Furthermore, using Lemma 3.3, we get the following estimate:

$$C^2_{1} C^3_{W,3} \left( \left\| e^{(m+1)} \right\|^2_{L^\infty} + \left\| e^{(m)} \right\|^2_{L^\infty} \right) \leq C^2_{1} C^3_{L} C^2_{W,3} \left( \left\| e^{(m+1)} \right\|^2_{H^1_3} + \left\| e^{(m)} \right\|^2_{H^1_3} \right).$$
Thus, we have
\[
\left\| \frac{dW}{d(U^{(m+1)}, u^{(m)})} - \frac{dW}{d(u^{(m+1)}, u^{(m)})} \right\|^2_{H^1} \leq \frac{C_W}{2} \left( \| e^{(m+1)} \|^2_{H^1} + \| e^{(m)} \|^2_{H^1} \right). \tag{97}
\]
Therefore, using (91) and (97), we obtain
\[
\left\| \frac{dW}{d(U^{(m+1)}, U^{(m)})} - \frac{dW}{d(U^{(m+1)}, u^{(m)})} \right\|^2_{H^1} + \left\| \frac{dW}{d(U^{(m+1)}, u^{(m)})} - \frac{dW}{d(u^{(m+1)}, u^{(m)})} \right\|^2_{H^1} \leq \frac{C_W}{2} \left\{ \left( \| e^{(m+1)} \|^2_{H^1} + \| e^{(m+1)} \|^2_{H^1} \right) + \left( \| e^{(m)} \|^2_{H^1} + \| e^{(m)} \|^2_{H^1} \right) \right\}. \tag{98}
\]
Applying (98) to (87), we obtain
\[
\{ 1 - (1 + C_W)\Delta t \} \left\{ \left\| e^{(m+1)} \right\|^2_{H^1} + \left\| e^{(m+1)} \right\|^2_{H^1} + \left\| e^{(m)]} \right\|^2_{H^1} \right\} \\
\leq \{ 1 + (1 + C_W)\Delta t \} \left\{ \left\| e^{(m)} \right\|^2_{H^1} + \left\| e^{(m)} \right\|^2_{H^1} + \left\| e^{(m)} \right\|^2_{H^1} \right\} + 2\Delta t \xi^{(m+\frac{1}{2})}.
\]
This completes the proof. \[\Box\]

**Lemma 2.** We impose the assumption in Corollary 2. Then, the following estimates hold:
\[
\left| \delta_k^{(2)} \left( \phi_{\theta}^{(m+\frac{1}{2})} \right) \right| \leq M_{2,2}(u) + \frac{\Delta x}{3} M_{3,2}(u) \quad (k = 0, \ldots, K).
\]

**Proof.** For any \( t \in [0, T] \) and \( k = 0, \ldots, K \), applying the Taylor theorem to \( \partial_t^2 \hat{u} \), there exists \( \theta_3 \in (0, 1) \) such that
\[
\partial_t^2 \hat{u}(k+1)\Delta x, t) - 2\partial_t^2 \hat{u}(k\Delta x, t) + \partial_t^2 \hat{u}(k-1)\Delta x, t) \\
(\Delta x)^2 \\
= \frac{1}{2} \left\{ \partial_x^2 \partial_t^2 \hat{u}(k + \theta_3)\Delta x, t) + \partial_x^2 \partial_t^2 \hat{u}(k - \theta_3)\Delta x, t) \right\}. \tag{99}
\]
From the definition of \( \hat{u} \), it holds that \( \partial_x^2 \partial_t^2 \hat{u}(x, t) = \partial_x^2 \partial_t^2 u(-x, t) + 2x \partial_x^2 \partial_t^2 u(0, t) \) for all \( x \in (-L, 0) \). Hence, we have
\[
\partial_x^2 \partial_t^2 \hat{u}(-\theta_3\Delta x, t) = \partial_x^2 \partial_t^2 u(\theta_3\Delta x, t) - 2\theta_3\Delta x \partial_x^2 \partial_t^2 u(0, t).
\]
In the same manner, we obtain
\[
\partial_x^2 \partial_t^2 \hat{u}(K + \theta_3)\Delta x, t) = \partial_x^2 \partial_t^2 u((K - \theta_3)\Delta x, t) + 2\theta_3\Delta x \partial_x^2 \partial_t^2 u(K\Delta x, t).
\]
Namely, substituting \((m + (1 \pm \theta_1)/2)\Delta t \) into \( t \) in (99), we get
\[
\delta_k^{(2)} \left( \phi_{\theta}^{(m+\frac{1}{2})} \right) = \begin{cases} \\
\partial_x^2 \partial_t^2 u_{\theta_3}^{(m+\frac{1}{2} \theta_1)} - \theta_3\Delta x \partial_x^2 \partial_t^2 u_{0}^{(m+\frac{1}{2} \theta_1)}, & (k = 0), \\
\frac{1}{2} \left( \partial_x^2 \partial_t^2 u_{K+\theta_3}^{(m+\frac{1}{2} \theta_1)} + \partial_x^2 \partial_t^2 u_{K-\theta_3}^{(m+\frac{1}{2} \theta_1)} \right), & (k = 1, \ldots, K-1), \\
\partial_x^2 \partial_t^2 u_{K-\theta_3}^{(m+\frac{1}{2} \theta_1)} + \theta_3\Delta x \partial_x^2 \partial_t^2 u_{K}^{(m+\frac{1}{2} \theta_1)}, & (k = K). \\
\end{cases}
\]
Therefore, we conclude that
\[
\left| \delta_k^{(2)} \left( \partial_t^2 \tilde{u}_k^{(m+1,\pm \theta)} \right) \right| \leq M_{2,2}(u) + \frac{\Delta x}{3} M_{3,2}(u) \quad (k = 1, \ldots, K - 1),
\]
and
\[
\left| \delta_k^{(2)} \left( \partial_t^2 \tilde{u}_k^{(m+1,\pm \theta)} \right) \right| \leq M_{2,2}(u) + \frac{\Delta x}{3} M_{3,2}(u) \quad (k = 0, K).
\]

\[\square\]

**Numerical results for the Neumann boundary condition.** As stated in Section 6, in order to verify that the difference in the long-time behavior of the solution occurs, we present the results of numerical experiments for (1) with the following inhomogeneous Neumann boundary condition:
\[
\begin{aligned}
\partial_x u(x,t) \bigg|_{x=0} - W'(u(0,t)) &= 0, \\
- \partial_x u(x,t) \bigg|_{x=L} - W'(u(L,t)) &= 0, \quad \text{in } (0,T],
\end{aligned}
\]
in the same setting as Section 6. We remark that the solution of (1) with (100) also satisfies the dissipative property (5). Since there are no results of the numerical experiment in the same setting as Section 6 in previous studies, we carry out the numerical experiment by the following structure-preserving scheme. The concrete form of our structure-preserving scheme for (1) with (100) is as follows: for \(m = 0, 1, \ldots,\)
\[
\frac{U_k^{(m+1)} - U_k^{(m)}}{\Delta t} = \delta_k^{(2)} \left( \frac{U_k^{(m+1)} + U_k^{(m)}}{2} \right) - \frac{dW}{d(U_k^{(m+1)}, U_k^{(m)})} \quad (k = 0, \ldots, K),
\]
and
\[
\delta_k^{(1)} \left( \frac{U_k^{(m+1)} + U_k^{(m)}}{2} \right) \bigg|_{k=0} - \frac{dW}{d(U_0^{(m+1)}, U_0^{(m)})} = 0,
\]
and
\[
\delta_k^{(1)} \left( \frac{U_k^{(m+1)} + U_k^{(m)}}{2} \right) \bigg|_{k=K} - \frac{dW}{d(U_K^{(m+1)}, U_K^{(m)})} = 0.
\]

**Numerical experiment 3.** The setting is the same as Numerical experiment 1. Figure 5 shows the time development of the solution for (1) with (100) from our scheme. Figure 6 shows the time development of total energy. This graph also shows that the energy decreases numerically.
Numerical experiment 4. The setting is the same as Numerical experiment 2. Figure 7 shows the time development of the solution for (1) with (100) from our scheme. Figure 8 shows the time development of total energy. This graph also shows that the energy decreases numerically.

As can be seen from Figure 1, Figure 3, Figure 5, and Figure 7, the solution for (1) with (100) arrives at the different state from that for (1) with (71). Thus, the results assure that the difference in the long-time behavior of the solution occurs.

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