Algorithms for computing sums of powers of arithmetic progressions by using Eulerian numbers

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Abstract: The sums of powers of arithmetic progressions is of the form \( a^p + (a + d)^p + (a + 2d)^p + \cdots + (a + (n - 1)d)^p \), where \( n \geq 1, p \) is a non-negative integer, and \( a \) and \( d \) are complex numbers with \( d \neq 0 \). This sum can be computed using classical Eulerian numbers [11] and general Eulerian numbers [12]. This paper gives a new method using classical Eulerian numbers to compute this sum. The existing formula that uses general Eulerian numbers are more algorithmically complex due to more numbers to compute. This paper presents and focuses on two novel algorithms involving both types of Eulerian numbers. This paper gives a comparison to Xiong et al.’s result with general Eulerian numbers [12]. Moreover, an analysis of theoretical time complexities is presented to show our algorithm is less complex. Various values of \( p \) are analyzed in the proposed algorithms to add significance to the results. The experimental results show the proposed algorithm remains around 70% faster as \( p \) increases.

Keywords: Analysis of algorithms, Sums of powers of arithmetic progressions, Dynamic programming, Classical Eulerian numbers, General Eulerian numbers.

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1 Introduction

This paper is a continuation of the work on sums of powers of arithmetic progressions [9] with classical Eulerian numbers. In [9], the authors used Stirling numbers of the second kind to compute the sums of powers of arithmetic progressions. Our purpose is to discover different ways to compute the sum and compare algorithmically with other formulas of similar type. The sums of powers of arithmetic progressions are of the form

\[ S_{p,a,d}(n) = a^p + (a + d)^p + \cdots + (a + (n - 1)d)^p = \sum_{j=0}^{n-1} (a + jd)^p, \]  

(1)

where \( n \geq 1 \), \( p \) is a non-negative integer, and \( a \) and \( d \) are complex numbers with \( d \neq 0 \). Several formulae and algorithms using different special functions and numbers have been developed to compute this sum [1, 6, 9, 12].

Letting \( a = d = 1 \) in (1) gives the following famous formula:

\[ S_{p,1,1}(n) = \sum_{j=1}^{n} j^p = 1^p + 2^p + 3^p + \cdots + n^p. \]  

(2)

For example, letting \( p = 1 \) gives the identity

\[ S_{1,1,1}(n) = \sum_{j=1}^{n} j = \frac{n(n + 1)}{2}. \]

In 2013, Xiong et al. [12] presented a method to compute such a sum based on general Eulerian numbers. However, generalized Eulerian numbers may not be good because it takes longer to compute. Other authors have also published results using generalized Eulerian numbers [7, 8]. This paper aims to use a new method with classical Eulerian numbers as a comparison with Xiong et al.’s method with general Eulerian numbers. More specifically, we algorithmically analyze their method to obtain theoretical time complexity.

Classical Eulerian numbers are defined as the coefficients of \( \left( \frac{x + p - j}{p} \right) \), \( j = 0, 1, \ldots, n \) in the factorial expansion of \( x^n \), namely,

\[ x^p = \sum_{j=0}^{p} \binom{p}{j} \left( \frac{x + j - 1}{p} \right), \quad p = 0, 1, \ldots, \]  

(3)

which is called Worpitzky’s identity [2, 3, 11]. For example, the values of \( \binom{p}{j} \) for \( 1 \leq p \leq 5 \) are given as follows [3]:

| \( p \) | \( j \) | 0 | 1 | 2 | 3 | 4 |
|-------|-----|---|---|---|---|---|
| 1     |     | 1 |   |   |   |   |
| 2     |     | 1 | 1 |   |   |   |
| 3     |     | 1 | 4 | 1 |   |   |
| 4     |     | 1 | 11| 11| 1 |   |
| 5     |     | 1 | 26| 66| 26| 1 |

Table 1. Values of Eulerian numbers for \( 1 \leq p \leq 5 \).
Xiong et al. [12] defined the general Eulerian numbers \( A_{p,j}(a, d) \) with the following recurrence relation as \( A_{0,-1} = 1, A_{p,j} = 0 \) \((j \geq p \text{ or } j \leq -2)\) and
\[
A_{p,j}(a, d) = (-a + (j + 2)d)A_{p-1,j}(a, d) + (a + (p - j - 1)d)A_{p-1,j-1}(a, d).
\]
\((0 \leq j \leq p - 1).\)

Two lemmas are presented in the next section to give explicit forms of the two special numbers discussed. In addition, the new method with classical Eulerian numbers is given. In Section 3, we analyze both Xiong et al.’s formula and our formula algorithmically and present their corresponding theoretical time complexities. In Section 4, experiments are conducted to compare both algorithms.

2 Main results

In this section, a new method using classical Eulerian numbers, as opposed to using general Eulerian numbers, is given. As a result, a new formula to compute sums of powers of arithmetic progression using classical Eulerian numbers is presented. A detailed analysis of this algorithm is given in the next section.

2.1 Preliminaries

The classical Eulerian number \( \langle p \rangle \) is the number of permutations of the numbers 0 to \( p \) in which exactly \( j \) elements are greater than the previous element.

The binomial expansion of \( (x)_{p} \) can be rewritten using the Pochhammer symbol \((x)_{p}\), or falling factorial, as follows:
\[
\binom{x}{p} = \frac{(x)_{p}}{p!},
\]
where \((x)_{p} = x(x - 1) \cdots (x - p + 1)\). Next, two lemmas are presented to give the explicit forms of Eulerian numbers and general Eulerian numbers, which we will use in the experiments.

**Lemma 2.1.** The classical Eulerian numbers \( \langle \frac{p}{j} \rangle \) are given explicitly by the sum
\[
\langle \frac{p}{j} \rangle = \sum_{i=0}^{j} (-1)^{i} \binom{p + 1}{i} (j - i)^{p}.
\]
*This lemma is proved in many manuscripts [3–5].*

**Lemma 2.2.** The general Eulerian numbers \( A_{p,j}(a, d) \) are given explicitly by the sum
\[
A_{p,j}(a, d) = \sum_{i=0}^{j+1} (-1)^{i} \binom{p + 1}{i} [(j + 2 - i)d - a]^{p}.
\]
*This lemma is proved by Xiong et al. [12].*
2.2 New method with classical Eulerian numbers

Given

\[ x^p = \sum_{j=0}^{p} \binom{p}{j} \left( x + j - 1 \right), \]  

(8)

the substitution \( x = (k - 1) + \frac{a}{d} \) results in

\[ \left[ (k - 1) + \frac{a}{d} \right]^p = \sum_{j=0}^{p} \binom{p}{j} \left( (k - 1) + \frac{a}{d} + j - 1 \right). \]  

(9)

Multiplying both sides by \( dp \) and summing over \( k \) leads to

\[ \sum_{k=1}^{n} \left[a + d(k - 1)\right]^p = dp \sum_{j=0}^{p} \binom{p}{j} \sum_{k=1}^{n} \left( (k - 1) + \frac{a}{d} + j - 1 \right). \]  

(10)

Now, consider just \( \sum_{k=1}^{n} \binom{(k - 1) + \frac{a}{d} + j - 1}{p} \). Expanding the right-hand side of (10) results in

\[ \sum_{k=1}^{n} \binom{(k - 1) + \frac{a}{d} + j - 1}{p} = \binom{a + j - 1}{p} + \binom{a + j}{p} + \cdots + \binom{n - 2 + \frac{a}{d} + j}{p}. \]

Now using Pascal’s identity \( \binom{n}{p} = \binom{n-1}{p-1} + \binom{n-1}{p} \), the right-hand side becomes

\[ \binom{a + j}{p + 1} - \binom{a + j - 1}{p + 1} + \binom{1 + \frac{a}{d} + j}{p + 1} - \binom{\frac{a}{d} + j}{p + 1} + \cdots + \binom{n - 1 + \frac{a}{d} + j}{p + 1} - \binom{n - 2 + \frac{a}{d} + j}{p + 1}. \]

We see that the sum telescopes, so we have the following result

\[ \sum_{k=1}^{n} \binom{(k - 1) + \frac{a}{d} + j}{p} = \binom{n - 1 + \frac{a}{d} + j}{p + 1} - \binom{\frac{a}{d} + j - 1}{p + 1}. \]  

(11)

Utilizing the identity (5) we arrive at the following

\[ \binom{n - 1 + \frac{a}{d} + j}{p + 1} - \binom{\frac{a}{d} + j - 1}{p + 1} = \frac{(\frac{a}{d} + j + n - 1)_{p+1} - (\frac{a}{d} + j - 1)_{p+1}}{(p + 1)!}. \]  

(12)

Plugging (12) back into (10) gives us

\[ \sum_{k=1}^{n} \left[a + d(k - 1)\right]^p = \frac{dp}{(p + 1)!} \sum_{j=0}^{p} \binom{p}{j} \left[ \binom{a}{d} + j + n - 1 \right]_{p+1} - \binom{\frac{a}{d} + j - 1}{p+1}. \]  

(13)

3 Analysis of algorithms

A comparison between the general Eulerian algorithm and the proposed classical Eulerian algorithm is provided.

Since it takes \( n - 1 \) operations to compute \( n! \) and \((x)_n\), both operations have a time complexity of \( O(n) \). Therefore, the binomial coefficients \( \binom{n}{k} \) can be computed with \( n - 1 + k - 1 + (n - k) - 1 + 2 = 2n - 1 \) operations and has a time complexity of \( O(n) \).
3.1 Xiong et al.’s formula

Xiong et al. gave a method to compute the sums of powers of arithmetic progressions using general Eulerian numbers [12]. In this section, we will break down their method and present an analysis on the following:

\[
\sum_{k=1}^{n} (a + d(k - 1))^p = \sum_{j=-1}^{p-1} A_{p,j}(a, d) \binom{n+j+1}{p+1}.
\]  

(14)

To calculate the time complexity of Xiong et al.’s algorithm, we only need to consider the time complexity of the binomial coefficient and of the general Eulerian numbers. It is clear that from the binomial coefficient we get the time complexity in terms of \( n \) and \( j \).

From (14), the portion of the algorithm which computes \( \binom{n+j+1}{p+1} \), takes \( O(2(n+j+1) - 1) = O(2n+2j+1) = O(n+j) \) operations to compute.

Lemma 2.2 can be used to compute the general Eulerian numbers. Examining (7), we get a time complexity of \( O(i+2p+2+5+p) = O(i+p) \). Summing from \( i = 0 \) to \( j + 1 \) gives an upper bound for the time complexity of \( O(ji+jp) = O(j^2+jp) \). Using this calculation of \( A_{p,j}(a, d) \) in (14) and summing from \( k = 0 \) to \( p \) gives an upper bound for the time complexity of the first algorithm as

\[
O(pj^2+jp^2+2pn+2pj+3p) = O(2p^3+2p^2+2pn+p) = O(p^3+pn)
\]  

(15)
in terms of the input variables \( p, n \).

3.2 Our algorithm

An analysis on the new method presented in Subsection 2.2 is provided using Lemma 2.1 for the classical Eulerian numbers.

\[
\sum_{k=1}^{n} [a + d(k - 1)]^p = \frac{dp}{(p+1)!} \sum_{j=0}^{p} \binom{p}{j} \left[ \left( \frac{a}{d} + j + n - 1 \right)_{p+1} - \left( \frac{a}{d} + j - 1 \right)_{p+1} \right].
\]  

(16)

To calculate the time complexity of our improved formula (16), we begin with the term outside the sum, \( \frac{dp}{(p+1)!} \), which takes \( 2p+1 \) operations to compute and, consequently, has a \( O(p) \) time complexity. In addition, we see that each falling factorial terms will have a time complexity of \( O(p) \). Note that the complexity no longer depends directly on the value of \( n \).

From (6), we get a time complexity of \( O(i+2p+2+3+p) = O(i+p) \) for the classical Eulerian numbers. Summing from \( i = 0 \) to \( j + 1 \) gives an upper bound for the time complexity of \( O(ji+jp) = O(j^2+jp) \). Using this calculation of \( \binom{p}{j} \) in (16) and summing from \( k = 0 \) to \( p \) gives an upper bound for the time complexity of the first algorithm as

\[
O(pj^2+jp^2+2p^2) = O(2p^3+2p^2) = O(p^3)
\]  

(17)
in terms of the input variables $p$. Note that the complexity no longer depends directly on the value of $n$; therefore, when comparing (14) and (16), this gives evidence to suggest that the improved algorithm will run faster than the original algorithm on average. This expectation is realized in the following section, where we conduct extensive experiments.

4 Experimental results

In this section, experimental results for evaluating the sums of powers of arithmetic progressions are presented. The specifications of the testing environment are given, and the details of the experiments are described. Thereafter, the results of the original method compared to the simplified method are to be discussed.

4.1 Specifications

The computer used to run the tests consists of an Intel i7-8700K running at 3.7GHz with 6 physical cores and 16 gigabytes of RAM. The operating system used is Windows 10 Pro 20H2, 64 bit version. All of the experiments are implemented using Python 3.7 with Jupyter Notebook 6.0.3. The libraries include Sympy and time. Sympy is a library that does symbolic mathematics. The time library stores start-time and end-time, which are subtracted to obtain actual computation time. Mathematica is used to perform data visualizations.

4.2 Description of experiments

In the first set of experiments, the value of the power $p$ varies from $p = 10$ to $p = 100$ with $\Delta p = 10$ as the increments. This aims to give the sums of powers of arithmetic progressions in the general form. The results are simplified when comparing the results.

In the second set of experiments, the values of $a, d,$ and $n$ are fixed to randomly generated 20-digit numbers. The value of the power $p$ varies from $p = 200$ to $p = 2000$ with $\Delta p = 200$ as the increments. This aims to better compare the two algorithms as lower values of $p$ may not give significant results. In both sets of experiments, the run-times are collected and averaged over 100 runs.

4.3 Results and discussion

The results are visualized using Mathematica. The graphs are used to compare our algorithm (16) against Xiong et al.’s algorithm (14). Two tables are given to see the numerical results.

The main contribution to the drastic difference in times shown in Figure (a) is that Xiong et al.’s algorithm gives the result in polynomials in $n$, whereas our algorithm gives the result in an unsimplified form. Even though the theoretical results can be simplified to the identity, a computer check was also performed to verify that the answers are the same. This gives a more comparable result since the first set of experiments does not give the results in the same way.
Experiment One: The time is measured in seconds. Algorithms generate the general form of the sums of powers of arithmetic progressions. This table shows that our algorithm is far more efficient.

| Values of $p$ | 10   | 20   | 30   | 40   | 50   |
|--------------|------|------|------|------|------|
| **Our algorithm** | 0.4747 | 1.4272 | 3.1077 | 5.9152 | 10.2915 |
| **Xiong et al.’s algorithm** | 0.5635 | 4.5019 | 27.0317 | 101.6553 | 252.6017 |

| Values of $p$ | 60   | 70   | 80   | 90   | 100  |
|--------------|------|------|------|------|------|
| **Our algorithm** | 17.8694 | 25.0377 | 34.7735 | 46.7552 | 61.1408 |
| **Xiong et al.’s algorithm** | 500.5395 | 807.2444 | 1309.9709 | 2003.4787 | 2703.5214 |

Figure (a). A comparison of the two algorithms that calculates the general form of the sums of powers of arithmetic progressions by varying $p$ only shows that our algorithm is drastically more efficient.

Figure (b). A comparison of the two algorithms that calculate the result of the sums of powers of arithmetic progressions by fixing values of $a$, $d$, and $n$ and varying $p$. The plot includes the line of best fit for two data sets.

Experiment Two: The time is measured in seconds. The constants $a$, $d$, and $n$ are fixed to randomly generated 20-digit numbers, where the value of $p$ varies. This table shows that our algorithm is faster in generating the values of sums of powers of arithmetic progressions.

| Values of $p$ | 100  | 200  | 300  | 400  | 500  |
|--------------|------|------|------|------|------|
| **Our algorithm** | 0.4497 | 1.0876 | 2.1052 | 3.9748 | 5.9838 |
| **Xiong et al.’s’s algorithm** | 0.359 | 1.3106 | 4.1687 | 10.9749 | 22.7876 |

| Values of $p$ | 600  | 700  | 800  | 900  | 1000 |
|--------------|------|------|------|------|------|
| **Our algorithm** | 9.2827 | 13.244 | 17.6911 | 23.2921 | 34.1791 |
| **Xiong et al.’s’s algorithm** | 44.2563 | 70.7529 | 124.8384 | 188.3088 | 273.7294 |

| Values of $p$ | 1100 | 1200 | 1300 | 1400 | 1500 |
|--------------|------|------|------|------|------|
| **Our algorithm** | 82.4342 | 145.502 | 188.3088 | 290.617 | 346.7947 |
| **Xiong et al.’s’s algorithm** | 412.5691 | 604.4786 | 846.1521 | 1105.5296 | 1457.421 |

| Values of $p$ | 1600 | 1700 | 1800 | 1900 | 2000 |
|--------------|------|------|------|------|------|
| **Our algorithm** | 492.055 | 570.5803 | 681.9292 | 770.6358 | 979.0942 |
| **Xiong et al.’s’s algorithm** | 1822.8166 | 2221.2837 | 2681.4324 | 3336.1961 | 3840.6724 |
In the second set of experiments, the algorithms compute the value of the sums of powers of arithmetic progressions with $a$, $d$, and $n$ fixed and $p$ varied. The results show that our algorithm performs faster than Xiong et al.’s method since our algorithm does not depend on the value of $n$. As seen in Figure (b), both algorithms have a cubic polynomial time, with the general Eulerian number method using more time.

5 Conclusion

In this paper, two novel algorithms using the Eulerian numbers are given to compute the general form of sums of powers of arithmetic progressions $S_{p,a,d}(n)$. In addition, formula (13) is used to compare with Xiong et al.’s method, which uses the general Eulerian numbers.

In Section 3, extensive analyses of both algorithms are presented. These analyses suggest that our algorithm using classical Eulerian numbers will on average run faster than Xiong et al.’s algorithm using general Eulerian algorithm. The experimental results also support the analyses.

This paper focused on using a new method with classical Eulerian numbers. A future project may involve algorithms using dynamic programming for solving the sums of powers of arithmetic progressions using other special combinatorial numbers such as Bernoulli polynomials and numbers [1, 10].

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