New Evidence of Discrete Scale Invariance in the Energy Dissipation of Three-Dimensional Turbulence: Correlation Approach and Direct Spectral Detection

Wei-Xing Zhou,1 Didier Sornette,1,2,3,* and Vladilen Pisarenko4
1Institute of Geophysics and Planetary Physics, University of California, Los Angeles, CA 90095
2Department of Earth and Space Sciences, University of California, Los Angeles, CA 90095
3Laboratoire de Physique de la Matière Condensée, CNRS UMR 6622 and Université de Nice-Sophia Antipolis, 06108 Nice Cedex 2, France
4International Institute of Earthquake Prediction Theory and Mathematical Geophysics, Russian Ac. Sci. Warshavskoye sh., 79, kor. 2, Moscow 113556, Russia

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We extend the analysis of [Zhou and Sornette, Physica D 165, 94-125, 2002] showing statistically significant log-periodic corrections to scaling in the moments of the energy dissipation rate in experiments at high Reynolds number (≈ 2500) of three-dimensional fully developed turbulence. First, we develop a simple variant of the canonical averaging method using a rephasing scheme between different samples based on pairwise correlations that confirms Zhou and Sornette’s previous results. The second analysis uses a simpler local spectral approach and then performs averages over many local spectra. This yields stronger evidence of the existence of underlying log-periodic undulations, with the detection of more than 20 harmonics of a fundamental logarithmic frequency $f = 1.434 \pm 0.007$ corresponding to the preferred scaling ratio $\gamma = 2.008 \pm 0.006$.

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I. INTRODUCTION

The small-scale physical quantity that has received the most attention in the context of a phenomenological description of hydrodynamical turbulence is the dissipation rate of kinetic energy $\eta$. Richardson’s picture of turbulent cascades [2], in which large eddies break down into smaller ones receiving a certain fraction of the flux of kinetic energy from larger scales, is a multiplicative process. The hypothesis that eddies of any generation are space filling led to the famous K41 theory [3, 4]. A conceptually appealing view, dating back to Obukhov [5] and Kolmogorov [6], visualizes the transfer of kinetic energy as a self-similar multiplicative process. This view is still the point of departure of many phenomenological intermittency models today. The thought of the general theory of multifractals was proposed by Mandelbrot [7] in the study of certain high-order moments with a general cascade model of random curling. A phenomenological model of intermittency, referred to as the $\beta$-model [8], analyzed the multiplicative cascade of the breakup of eddies without space filling leading to an energy transfer to a smaller and smaller fraction of the total space. Benzi, Paladin, Parisi & Vulpiani [9] and Frisch & Parisi [10] have argued for a singular structure and for a multifractal nature of the energy dissipation in three-dimensional fully-developed turbulence. Meneveau & Sreenivasan [11] presented a simple multifractal model, the binomial model for fully developed turbulence based on the two-scale Cantor set, which is known as the $p$-model. This model and the specific multifractal facets of turbulent energy dissipation were investigated experimentally by [12], using hot wire measurement technique. These works clarified the phenomenological nature of the multifractality of turbulent energy dissipation. In particular, the view of the energy cascade as belonging to the general class of multiplicative processes opens up the possibility that the scale invariant properties reported for turbulence flows might be broken in part into a discrete scale invariance, leading to the observation of log-periodic corrections to scaling [13].

Based on theoretical argument and experimental evidence, Sornette [14] conjectured that structure functions of turbulent time series may exhibit log-periodic modulations decorating their power law dependence. He stressed the need for novel methods of averaging and proposed a novel “canonical” averaging scheme for the analysis of structure factors of turbulent flows in order to provide convincing experimental evidence. The strategy was proposed to determine the sample-specific control parameter $r_c$ and translate it into a specific “phase” in the logarithm of the scale $\ln r$ which, when used as the origin, allows one to rephase the different measurements of a structure factor in different realizations.

*Electronic address: sornette@moho.ess.ucla.edu
A first systematic search of discrete scale invariance in turbulence was carried out in freely decaying two-dimensional turbulence \[13\]. The number of vortices, their radius and separation were found to display log-periodic oscillations as a function of time with an average log-frequency of $\sim 4 - 5$ corresponding to a preferred scaling ratio of $\sim 1.2 - 1.3$. Most recently, significant evidence for the presence of intermittent discretely self-similar cascades was obtained from the analysis of the moments of the energy dissipation rate in three-dimensional fully developed turbulence \[14\]. The results of spectral analyses of the canonically averaged local scaling exponents of high moments of the energy dissipation rate obtained in \[16\] are summarized in Fig. 1. The fundamental log-frequency is estimated to be $f \approx 1.44$ which suggests a preferred scaling ratio of $\gamma = e^{1/f} \approx 2$. In \[16\], a simple multifractal model was introduced to explain the fact that the log-frequency is independent of the order of the moments.

As was explained in \[16\], the difficulty in detecting log-periodic oscillations (the hallmark of discrete scale invariance) in various functions such as moments of the energy dissipation rate stems from the fact that, while the physics of the cascade may be universal with an universal scaling factor $\gamma$, the nucleating scales (translating in phases of reference in the log-periodicity analysis) have to change from one realization to another as the cascade may be dynamically triggered from a variety of scales $\ell_a$ from the integral scale and deep in the inertial range. Therefore, the corresponding phases are expected to be non-universal. Averaging a signal over several such transient cascades will thus wash out the information on log-periodicity by “destructive interferences” of the log-periodic oscillations, giving them the appearance of noise.

The previous analyses of real 2-D and 3-D turbulence data addressed this problem by using two versions of the canonical averaging approach, devised precisely to rephase different realizations and thus alleviate the problem of the random phases. Using $n$ realizations of the data, the first version consists in performing a canonical averaging on the $n$ samples to get an average realization and then analyzes its spectral properties; the second version performs a spectral analysis of each of the $n$ samples and then averages the $n$ periodograms. In order to perform these averages, we need to have different realizations. For the turbulence data analyzed in \[16\] and revisited here, a long time series of velocity increments is divided in many samples, each lasting about $1/5$ of a turn-over time, so that we can avoid scrambling of the phases by a suitable canonical averaging as explained below.

The goal of the present note is to revisit these two schemes and propose two novel and simpler versions. By simplifying the analysis and by providing independent implementations, we thus confirm further the presence of log-periodicity in the moments of the energy dissipation rate, and thus the existence of discrete scale invariance in 3D fully-developed hydrodynamical turbulence.

II. REPHASING OF LOG-PERIODICITY BY THE CROSS-CORRELATION METHOD

In the first scheme of canonical averaging, the rephasing operation is the central part. In \[16\], the so-called central maxima approach was applied to the detrended logarithm of the moment $M_q(\ell/\eta)$ of the dissipation rate as a function of the logarithm $\ln(\ell/\eta)$ of the scale $\ell$ (in units of the Kolmogorov microscale $\eta = 0.195 \text{ mm}$ \[16\]): it consists in finding the maximum of the detrended logarithm of the moment closest to the middle point of a single sample and set the logarithm of the corresponding scale (its abscissa) as the origin of the logarithm of the scales. The new abscissa, called $\Delta$, becomes sample dependent and is used to superimpose and then average the detrended moments over the different samples. This approach was found to work satisfactorily except that the canonically averaged series present decreasing amplitudes with increasing $|\Delta|$.

The central maxima rephasing approach used in \[16\] determines an absolute sample-dependent phase which is specific to each single sample and then rephases all samples. The alternative idea explored here is based on the remark that the detection of phases can be performed in relative value, that is, by estimating the phase shift between all possible pairs of samples. The cross-correlation function is used to calculate the phase-shift between two samples \[17\]. This relative phase-shift approach thus uses the calculation of all phase-shifts between pairs of samples to rephase all samples by choosing arbitrarily one of them as the reference.

As in \[16\], we use the high-Reynolds turbulence longitudinal velocity data collected at the S1 ONERA wind tunnel by the Grenoble group from LEGI \[18\]. We use a set of 20 records, each containing 320 samples in the inertial range, each sample having $2^{11}$ points. These $2^{11}$ points correspond approximately to one-fifth of a turn-over time \[16\]. We calculate the local moment exponent $\tau(q, \ln(\ell/\eta))$ as the logarithmic derivative of $M_q(\ell/\eta)$ with respect to $\ell/\eta$:

$$\tau(q, \ln(\ell/\eta)) = \frac{d \ln[M_q(\ell/\eta)]}{d \ln(\ell/\eta)}.$$  (1)

We shall investigate in the sequel $\ell/\eta \in [230, 1843]$ which is well within the inertial range. To obtain $\tau(q, \ln(\ell/\eta))$ from \[14\], we used a smoothing low-pass digital filter, namely, the Savitzky-Golay filter. This filter approximates the underlying function in a moving window by means of polynomials of a low degree. The filter coefficients are
choosing so that, unlike other filters, several low-order moments of the filtered function are preserved, which is a very desirable property (see [13] for details). The Savitzky-Golay filters are applied to $\ln[M_{q}(\ell/\eta)]$ expressed as a function of $\ln(\ell/\eta)$ and the local derivative is given from the analytic derivative of the fitted polynomial [13]. There are two parameters in the Savitzky-Golay filters, i.e., the width of the sliding window $N_{L} + N_{R} + 1$ where $N_{L}$ and $N_{R}$ are the numbers of points to the left and to the right of the investigating point and the order $M$ of fitting polynomials. We used symmetric windows such that $N_{L} = N_{R}$. A total of 24 filters were applied to test the robustness of the results with respect to the choice of the filters: $N_{L} = 5, \ldots, 10$ and $M = 4, \ldots, 7$.

We recall that, in the central maxima approach of [14], one identifies the sample-dependent reduced scale $\ell_{c}/\eta$ closest to the central point of the scale-interval at which $\tau(q, \ln(\ell/\eta))$ is maximum for each sample. In contrast, the correlation approach takes an arbitrary sample as the reference sample and evaluates the phase shift measured by the cross-correlation function of all other samples relative to this reference. We then rephase all functions $\tau(q, \ln(\ell/\eta))$ of the variable $\ln(\ell/\eta)$ so that the cross-correlation function of all sample pairs becomes peaked at zero lag, after the rephasing has been performed.

We then average these rephased functions $\tau(q, \Delta)$ over the 320 samples. This defines the local average exponent

$$D(\Delta) = \langle \tau(q, \Delta) \rangle,$$

where the average is performed over the different samples at fixed $\Delta = \ln(\ell/\eta) - \ln(\ell_{c}/\eta)$ (where $\ln(\ell_{c}/\eta)$ vary from sample to sample and $\ln(\ell/\eta)$ is adjusted accordingly to ensure the re-phasing). Fig. 2 shows four examples of $D(\Delta)$ as a function of $\Delta$ based on the correlation rephasing approach for four different filters with $N_{L} = 6$ and $M = 4, 5, 6, 7$ respectively. Notice that the amplitude of the oscillations decreases with decreasing $\Delta$. This is in line with the concept [16] that, the smaller is the scale $\ell$, the more complex is the superposition of intermittent cascades as the number of starting scales for the cascade proliferate when the nucleating scale departs from the integral scale. This should lead to a stronger self-averaging property, as we observe here when $\Delta$ decreases. Furthermore, since the samples have finite sizes, $D(\Delta)$ is well-defined only for not too large $|\Delta|$ because there are fewer samples involved in the averaging for larger $|\Delta|$. In practice, we will restrict our analysis to $|\Delta|$ smaller than a threshold $\Delta_{m}$ taken here approximately equal to 1.0, as suggested from a visual inspection of Fig. 2.

The next step in the analysis consists in performing a spectral analysis of $D(\Delta)$. We used a kind of spectral analysis known under the name of the Lomb spectrum (see [13] for details). It is designed to analyze a non-uniformly sampled functions, fitting them to a sine function of varying frequency on a non-uniform grid by means of the least square method. This provides the analog of a standard periodogram. We thus obtain 24 such Lomb periodograms for each record (of 320 samples), one for each filter. We then repeat the above steps for all the other 19 records of 320 samples each and we thus obtain a total of 480 Lomb periodograms. Alternatively, we have grouped the 20 records into a single large time series with 6400 samples on which the canonical averaging is performed directly. This increased statistics improves the significance level of the Lomb peaks in the resultant periodograms. The comparison of our method on different partitions of the data allows us to test the stability and robustness of the results.

Having computed the 480 Lomb periodograms (corresponding to 24 Lomb periodograms for each of the 20 records), we perform an average over all these 480 periodograms. This analysis was carried out for three different choices of the threshold $\Delta_{m} = 1.0, 0.8, 0.6$. The averaged Lomb powers are shown in Fig. 3. All the three curves have significant peaks at $f_{2} = 2/\ln 2$ and $f_{3} = 3/\ln 2$, which suggest a fundamental log-frequency $f = 1/\ln 2$. The periodogram with $\Delta_{m} = 1.0$ also possesses many spurious weaker peaks. The periodogram with $\Delta_{m} = 0.8$ exhibits the highest Lomb peaks and clearest harmonics. The difference in peak heights for $\Delta_{m} = 0.8$ compared with $\Delta_{m} = 0.6$ is partially explained by the fact that the former has more data points. It is indeed known that the Lomb power is approximately proportional to the number of analyzed points [20]. The results summarized in Figure 3 thus provide a strong confirmation of the analysis of [14] on the presence of discrete scale invariance.

III. DIRECT DETECTION OF LOG-PERIODICITY BY FAST FOURIER TRANSFORM

We now turn to the second method of implementation of the canonical averaging discussed in the introduction. In a nutshell, the methods consists in performing directly a spectral analysis of each sample and then average over the obtained spectra. In order to improve the signal-to-noise, we first filter the high-frequencies of the function $\ln[M_{q}(\ell/\eta)]$ by performing an interpolation of the slightly unevenly sampled third-order moment values $M_{3}(\ell/\eta)$ with cubic splines. This provides us with equally sampled points that are required for the use of a fast Fourier transform (FFT). We then define the first difference and subtract the mean which defines the function $\delta \ln[M_{3}(\ell/\eta)]$ The periodogram of $\delta \ln[M_{3}(\ell/\eta)]$ for each sample is determined via a fast Fourier transform. We then perform an averaging over all the periodograms.

Specifically, the calculation of the third-order moment $M_{3}(\ell/\eta)$ is done as in Sec. II with the same data sets. We use one record with 320 samples. The number of the evenly sampled points generated by the spline interpolation is...
257, such that we will have 256 points at which the function \( \delta \ln[M_3(\ell/\eta)] \) is sampled. Fig. 8 shows a realization of \( M_3(\ell) \) for one specific sample. For this, \( \delta \ln[M_3(\ell/\eta)] \) is obtained as

\[
\delta \ln[M_3(\ell/\eta)] = \ln[M_3(\ell_{i+1}/\eta)] - \ln[M_3(\ell_i/\eta)] - \mu,
\]

where \( i = 1, \cdots, 256 \) and \( \mu = (\ln[M_3(\ell_{j+1}/\eta)] - \ln[M_3(\ell_j/\eta)])/j \). The resulting detrended third-order moment is shown in Fig. 8 and its periodogram is depicted in Fig. 9. The vertical lines in Fig. 8 indicate log-frequencies equal to \( k/\ln 2 \) with \( k = 1, 2, \cdots, 21 \). One can observe well-defined peaks close to the frequencies corresponding to \( k = 1, 2, 3, 4, 5, 6, 9, 12, 15, 16, 17, 21 \). However, there are also peaks in between these vertical lines. Therefore, the signature of log-periodicity in Figs. 4-6 is ambiguous.

We then average over the 320 periodograms, one for each sample, and get Fig. 7. Most peaks at log-frequencies other than \( k/\ln 2 \) have been washed out by the averaging procedure. The vertical dashed lines correspond to log-frequencies \( f_\ell = k/\ln 2 \) with \( k = 1, \cdots, 21 \), which are the harmonics of a fundamental log-frequency \( 1/\ln 2 \) corresponding to a preferred scaling ratio equal to \( \gamma = 2 \). The local peaks defined preferred frequencies \( F_m \) that are shown as open circles. Most of the peaks \( (m = 3, \cdots, 18, 20, 22, 24, 27) \) in Fig. 7 are very close to the vertical dotted lines which can be interpreted as harmonics of \( f = 1/\ln 2 \). The background noise has a bell-shape. Comparing Fig. 7 with Fig. 8, the averaging of the spectra over the 320 samples has suppressed peaks that can be interpreted as noise and has improved remarkably the significance level of the other Lomb peaks.

In order to estimate as accurately as possible the numerical value of the fundamental log-frequency, Fig. 8 plots with open circles the log-frequencies \( F_m \) of all the peaks found in Fig. 7 as a function of the peak sequence number \( m \). One observes a straight line with slope \( f = 1.420 \) covering 16 harmonics. This value is very close to the theoretic ansatz \( f = 1/\ln 2 \approx 1.4427 \). An estimation of the uncertainty in the determination of \( f \) is obtained by removing one point in each data set, and redoing the whole procedure. Then, the standard deviation calculated over these samples with one removed point provides a quantification of the error in the determination of \( f \). In order to include more points in the fit, we collected peaks near the vertical lines in Fig. 8 with log-frequencies \( f_k \) and plotted \( f_k \) with open squares in Fig. 8 as well. Note that \( f_k \) with \( k = 1, \cdots, 20 \) corresponds to peaks enumerated \( m = 3, \cdots, 18, 20, 22, 24, 27 \) mentioned above. Again, a very good linear fit is obtained with slope \( f = 1.427 \), which provides a slightly better estimate of \( 1/\ln \gamma \). We estimated the slopes of 19 other experimental records which, together with the slope from the first record, give the estimate of the fundamental log-frequency \( f = 1.434 \pm 0.007 \). An alternative estimate of \( f \) is the mean of the ratios \( f_k/k \). This gives \( f = 1.455 \pm 0.023 \). In this estimation, the first point \((k = 1, f_k = 1.864)\) was discarded.

Overall, we conclude that this new analysis agrees with and strengthens strongly previous results obtained in Sec. II and in [14] on the observation of log-periodicity. It is also quite surprising and somewhat gratifying to observe so many harmonics.

IV. CONCLUSION

In this paper, we have provided two novel analyses for the search of log-periodic corrections to scaling in the moments of energy dissipation of three-dimensional fully developed turbulence. The first analysis introduces a pairwise cross-correlation approach for the determination of a phase-shift, which is an alternative of the central maxima rephasing approach developed in [16] and is more natural and straightforward to implement. The resulting averaged Lomb periodogram have two significant peaks near log-frequencies equal to \( 2/\ln 2 \) and \( 3/\ln 2 \) and low peaks near other \( k/\ln 2 \) with \( k = 1, 4, 5 \). We have checked that these results are robust when varying the range of scales over which the analysis is performed.

The second method uses a more straightforward sample-by-sample spectral analysis, with a simple cubic spline interpolation (rather than the more delicate Savitzky-Golay filters) and a standard FFT. Averaging over the hundreds of spectra provides a strong log-periodic signals with more than 20 harmonics of what appears to be a fundamental logarithmic frequency \( f = 1.434 \pm 0.007 \). The corresponding preferred scaling ratio is thus estimated to be \( \gamma = 2.008 \pm 0.006 \). This novel analysis lifts possible remaining worries that one could keep that the results of [16] could be somehow due to the use of the Savitzky-Golay filters.

The present work further confirms the presence of log-periodicity in the energy transfer of fully developed turbulence, as well as provide additional evidence of the presence of multifractality in the energy dissipation [12]. However, the preferred scaling ratio \( \gamma \approx 2 \) found here does not necessarily imply that the correct model for turbulence is the binomial multifractal cascade [21]. We ascribe our results as an imprint of intermittent cascades from discrete hierarchical dissipation in turbulence [14].
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FIG. 1: Average of 2400 (continuous line) (resp. 480 (dashed line)) Lomb periodograms corresponding to 24 Savitzky-Golay filters per sample. The vertical dashed lines correspond to log-frequencies equal respectively to $f_k = k/\ln 2$ with $k = 1, \cdots, 5$. These values correspond to increasing harmonics of a putative fundamental frequency $f_1 = 1/\ln \gamma$ associated with the scaling ratio $\gamma = 2$. (See Ref. [16] for details.)

FIG. 2: Four examples of the canonically averaged variable $D(\Delta)$ as a function of the rephased log-scale $\Delta$ based on the correlation rephasing approach. The amplitude of the oscillations increases with increasing $\Delta$. See text for explanations.
FIG. 3: Average over all 480 Lomb periodograms corresponding to 24 filters per sample. The vertical dashed lines correspond to log-frequencies equal to $f_k = k/\ln 2$ with $k = 1, \cdots, 5$. These values correspond to increasing harmonics of a fundamental frequency $f_1 = 1/\ln \gamma$ associated with the scaling ratio close to $\gamma = 2$.

FIG. 4: $\ln(M_3)$ as a function of $\ln(\ell/\eta)$ for a sample (discrete open squares) and its corresponding interpolation with cubic splines (solid line).
FIG. 5: The zero-mean detrended $\delta \ln(M_3)$ as a function of $\ln(\ell/\eta)$ for the spline interpolation shown in Fig. 4.

FIG. 6: Fourier periodogram of $\delta \ln(M_3)$ shown in Fig. 5. The vertical lines indicate log-frequencies $k/\ln 2$ with $k = 1, \cdots, 21$. There are peaks close to the vertical lines with $k = 1, 2, 3, 4, 5, 6, 9, 12, 15, \cdots$ and also peaks that can not be integrated as integer multiples of $1/\ln 2$. 
FIG. 7: Average of all Fourier periodograms over 320 samples. The vertical dashed lines correspond to log-frequencies equal to $f_k = k/\ln 2$ with $k = 1, \ldots, 21$. The peaks are indicated with open circles. Most of the peaks are very close to the vertical dotted lines which can be interpreted as harmonics of $f = 1/\ln 2$.

FIG. 8: Determination of the fundamental log-frequency with a linear fit: the open circles are all peaks indicated in Fig. 7 and the open squares are those close to the vertical lines in Fig. 7. Excellent linear fits are obtained with the slopes 1.420 (○) and 1.427 (□), close to the value $1/\ln 2 = 1.4427$. 