Topology and chiral symmetry in finite temperature QCD

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We investigate the realization of chiral symmetry in the vicinity of the deconfinement transition in quenched QCD using overlap fermions. Via the index theorem obeyed by the overlap fermions, we gain insight into the behavior of topology at finite temperature. We find small eigenvalues, clearly separated from the bulk of the eigenvalues, and study the properties of their distribution. We compare the distribution with a model of a dilute gas of instantons and anti-instantons and find good agreement.

1. Some properties of the overlap Dirac operator

The overlap Dirac operator is constructed out of a Wilson-like Dirac operator (the usual Wilson Dirac operator is used for this talk) with a large negative mass \cite{1,2}:

\[ D(\mu) = \frac{1}{2} \left[ 1 + \mu + (1 - \mu) \gamma_5 \epsilon(H_w(m)) \right]. \] (1)

Here \( H_w(m) = \gamma_5 D_{\text{Wilson}}(-m) \). This form insures that topology is seen by the overlap fermions, since \( Q = \text{tr}(H_w)/2 \). The quark mass is proportional to \( \mu \) for small \( \mu \).

The propagator for external fermions is given by

\[ \tilde{D}^{-1}(\mu) = (1 - \mu)^{-1} \left[ D^{-1}(\mu) - 1 \right], \] (2)

i.e. it has a contact term subtracted. The propagator is then chiral in the massless case.

In many cases, it is more convenient to use the hermitian version \( H_o(\mu) = \gamma_5 D(\mu) \). The massless version satisfies,

\[ \{ H_o(0), \gamma_5 \} = 2H_o^2(0). \] (3)

This is the Ginsparg-Wilson relation, which is often used these days to prove good chiral properties of the overlap Dirac operator.

It follows that \( [H_o^2(0), \gamma_5] = 0 \), i.e. the eigenvectors of \( H_o^2(0) \) can be chosen as chiral. Since

\[ H_o^2(\mu) = (1 - \mu^2)H_o^2(0) + \mu^2, \] (4)

this holds also for the massive case.

Each eigenvalue \( 0 < \lambda^2 < 1 \) of \( H_o^2(0) \) is doubly degenerate with opposite chirality eigenvectors. In this basis, \( H_o(\mu) \) and \( D(\mu) \) are \( 2 \times 2 \) block diagonal. In \( D(\mu) \) the blocks are

\[ \begin{pmatrix} (1 - \mu)\lambda^2 + \mu & (1 - \mu)\sqrt{1 - \lambda^2} \\ -(1 - \mu)\lambda\sqrt{1 - \lambda^2} & (1 - \mu)\lambda^2 + \mu \end{pmatrix} \] (5)

where

\[ \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \] (6)

For a gauge field with topological charge \( Q \neq 0 \), there are, in addition, \( |Q| \) exact zero modes with chirality sign(\( Q \)) paired with eigenvectors of opposite chirality and eigenvalue 1. These are also eigenvectors of \( H_o(\mu) \) and \( D(\mu) \):

\[ D(\mu)_{\text{zero sector}} : \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix} \] (7)

depending on the sign of \( Q \).

2. Small eigenvalues and the chiral condensate

In the chiral eigenbasis of \( H_o^2(0) \), the external propagator takes the block diagonal form with
\[ D^{-1}(\mu) : (\lambda^2(1 - \mu^2) + \mu^2)^{-1} \times \begin{pmatrix} \mu(1 - \lambda^2) & -\lambda \sqrt{1 - \lambda^2} \\ \lambda \sqrt{1 - \lambda^2} & \mu(1 - \lambda^2) \end{pmatrix} \, , \] (8)

In topologically non-trivial background fields, there are \(|Q|\) additional blocks
\[
\begin{pmatrix} \frac{1}{\mu} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\mu} \end{pmatrix}
\] (9)
depending on the sign of \(Q\).

We thus find in a fixed gauge field background,

\[
\langle \overline{\psi} \psi \rangle(\{U\}) = \frac{|Q|}{\mu V} + \frac{1}{V} \sum_{\lambda > 0} \frac{2\mu(1 - \lambda^2)}{\lambda^2(1 - \mu^2) + \mu^2} , \quad (10)
\]

and averaged over gauge fields, we get the condensate. It is dominated by the small (non-zero) eigenvalues, and in the thermodynamic limit where the first term vanishes, it is given by the density of eigenvalues at zero \(\rho(0^+)\).

The close connection between the distribution of small eigenvalues and the condensate motivated us to study the small eigenvalues of the overlap Dirac operator in quenched QCD as the temperature is raised above the deconfining transition temperature \(T_c\). A similar study for \(T = 0\) can be found in [3].

3. Small eigenvalue distribution in quenched QCD above \(T_c\)

We focus on the non-zero eigenvalues and display their distributions for SU(2) and SU(3) at various temperatures and spatial volumes in Figs. 1–3. All figures show a region around \(\lambda = 0.05\) with very few eigenvalues, or even none. There is also a concentration of small eigenvalues below this value except in the cases with the smallest volumes and the largest \(\beta\)'s. Fig. 3 shows that these very small non-zero modes \(\lambda < 0.05\). We denote the number of zero and small nonzero eigenvalues with chirality \(\pm\) by \(n_{\pm}\), and the total number by \(n = n_+ + n_-\). The topological charge is given by \(Q = n_+ - n_-\). Our results for the various ensembles are given in Tables 1 and 2. We see that for fixed \(\beta\), \(\langle n \rangle / V\)
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Figure 3. Small eigenvalue distribution in various topological charge sectors for SU(3) with $\beta = 5.75$ on a $16^3 \times 4$ lattice.

and $\langle Q^2 \rangle / V$ seem to remain finite and nonzero in the large volume limit, but they drop quickly as $\beta$, and hence the temperature, is increased.

4. Modeling by a dilute instanton – anti-instanton gas

Looking in more detailed at the small modes, we find that their number $n$ is roughly Poisson distributed $P(n, \langle n \rangle) = \langle n \rangle^n e^{-\langle n \rangle} / n!$, and in particular, the average and variance are approximately equal. Also for fixed $n$, $n_+$ and $n_-$ are roughly binomially distributed. These observations are consistent with interpreting the small modes to be due to a dilute gas of instantons and anti-instantons with $n_+$ and $n_-$ their numbers. $n - |Q|$ of the would-be zero modes overlap and mix to give small eigenvalues, while $|Q|$ exact zero modes remain.

At finite temperature, instantons fall off exponentially, and so do the fermionic zero modes associated with them. We consider a toy model of randomly (Poisson and binomially) distributed instantons and anti-instantons, inducing interactions of the form $h_0 e^{-d(i,j)/D}$ between the would-be zero modes of every instanton – anti-instanton pair $(i,j)$ with separation $d(i,j)$. Like sign pairs are assumed to have no interactions. This toy model reproduces the qualitative features of the small eigenvalue distributions. Preliminary estimates give $D \approx 1/(2T)$.

In conclusion, we find that in quenched QCD above the deconfining transition temperature topology, as manifested by exact zero modes, persists. Furthermore, a finite density of small eigenvalues, separated from the bulk of the eigenvalues, remains. The properties of their distribution is well modeled by a dilute gas of instantons and anti-instantons. Our observation might be a quenched artefact, since in full QCD the finite temperature transition is believed to be driven by chiral symmetry restoration.

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REFERENCES

1. H. Neuberger, Phys. Lett. B417 (1998) 141.
2. R.G. Edwards, U.M. Heller and R. Narayanan, Phys. Rev. D59 (1999) 094510.
3. R.G. Edwards, U.M. Heller, J. Kiskis and R. Narayanan, Phys. Rev. Lett. 82 (1999) 4188; and these proceedings.
4. R.G. Edwards, U.M. Heller, J. Kiskis and R. Narayanan, in preparation.

| $\langle n \rangle / V$ | 8$^3 \times 4$ | 16$^3 \times 4$ |
|----------------------|-------------|-------------|
| $\langle Q^2 \rangle / V$ | 8$^3 \times 4$ | 16$^3 \times 4$ |
| $\langle n \rangle / \sigma_n$ | 8$^3 \times 4$ | 16$^3 \times 4$ |

Table 1

| $\langle n \rangle / V$ | 8$^3 \times 4$ | 16$^3 \times 4$ |
|----------------------|-------------|-------------|
| $\langle Q^2 \rangle / V$ | 8$^3 \times 4$ | 16$^3 \times 4$ |
| $\langle n \rangle / \sigma_n$ | 8$^3 \times 4$ | 16$^3 \times 4$ |

Table 2

Same as Table 1 but for SU(3).