Almost Similar Tests for Mediation Effects and other Hypotheses with Singularities *

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Abstract

Testing for mediation effects is empirically important and theoretically interesting. It is important in psychology, medicine, economics, accountancy, and marketing for instance, generating over 90,000 citations to a single key paper in the field. It also leads to a statistically interesting and long-standing problem that this paper solves. The no-mediation hypothesis, expressed as $H_0 : \theta_1 \theta_2 = 0$, defines a manifold that is non-regular in the origin where rejection probabilities of standard tests are extremely low. We propose a general method for obtaining near similar tests using a flexible $g$-function to bound the critical region. We prove that no similar test exists for mediation, but using our new varying $g$-method obtain a test that is all but similar and easy to use in practice. We derive tight upper bounds to similar and nonsimilar power envelopes and derive an optimal test. We extend the test to higher dimensions and illustrate the results in a trade union sentiment application.

Keywords: Varying $g$-method, Mediation, Indirect Effect, Power Envelope, Similar Tests, Invariant Tests, Optimal Tests

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1 Introduction

Testing for mediation effects is empirically extremely important in various scientific disciplines. A key paper in psychology, Baron and Kenny (1986) has more than 90,000 citations\(^1\) and is used in many other fields. Mediation testing is important in accounting, e.g. Coletti et al. (2005), marketing, e.g. MacKenzie et al. (1986), sociology, e.g. Alwin and Hauser (1975) who used the term indirect effect, in epidemiology, e.g. Freedman and Schatzkin (1992) who coined the term intermediate endpoint effect, and in econometrics e.g. Heckman and Pinto (2015a,b) on treatment effects and production technology. This minimal selection is hardly representative for the vast body of literature on mediation analysis. It only illustrates the breadth of its empirical relevance. Tests for mediation effects can have extremely low power, especially when the effect is small or estimated with large variance. The primary purpose of this paper is to provide a new and powerful test.

The aim of mediation testing is to discover if an independent variable \((X)\) causes a dependent variable \((Y)\) via an intervening or mediating variable \((M)\). The mediating variable is exogenous in the common experimental settings in psychology and other fields, but is also considered exogenous in other settings where assignments are random or constitute a natural experiment. The basic model is:

\[
\begin{align*}
    Y &= \tau X + \beta M + u, \\
    M &= \alpha X + v,
\end{align*}
\]

(1) (2)

where all variables are taken in deviation from their means or more generally after partialing out other exogenous effects. The disturbances \(u\) and \(v\) are assumed to be independent because of an experimental set up and more generally because no influence of \(Y\) on \(M\) is assumed in this type of model. This independence is a crucial identification condition. The parameter \(\beta\) cannot be estimated consistently if \(M\) is endogenous. We will further make the distributional assumption: \((u_i, v_i) \sim \mathcal{N}(0, \text{diag}(\sigma_{11}, \sigma_{22}))\), \(i = 1, \cdots, n\), with \(n\) the number of observations. This facilitates a likelihood analysis, but has no consequences for the asymptotic normality of the \(t\)-statistics that will be used.

MacKinnon et al. (2002) give a literature review and compare 14 different methods for testing the effects of a mediation variable. These methods are based on standardized measures of the product of two coefficients \((\alpha, \beta)\) or based on the difference of two related coefficients \((\tau^* - \tau)\) in equations (1) and (3)\(^2\):

\[Y = \tau^* X + w.\]

If there is a mediation effect then \(X\) influences \(M\), such that \(\alpha \neq 0\), and \(M\) influences \(Y\), such that \(\beta \neq 0\). If there is no mediation by \(M\) then the effect of \(X\) on \(Y\) is not altered by the inclusion of \(M\) such that \(\tau^* - \tau = 0\).

Since model (3) is a restricted version of (1) with \(\beta = 0\), it is straightforward to show that the OLS estimates for the three models satisfy \(\hat{\tau}^* = \hat{\tau} + \hat{\alpha} \hat{\beta}\) and the relation \(\tau^* - \tau = \alpha \beta\) also holds in model interpretation terms; see Appendix A.

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\(^1\)Cited by 90,147 on 15 January 2020 and 79,205 on 22 October 2018.

\(^2\)A simple extension is to add other explanatory variables to the model \(\sum_{k=1}^{K} \gamma_k Z_k\). If the covariates \(Z_k\) are added to all three models, then the degrees of freedom of the relevant \(t\)-tests are reduced by \(K\).
The best well known and commonly used test by Sobel (1982) is a Wald-type test of the form \( \sqrt{\hat{\alpha} \hat{\beta} / \hat{\sigma}_{\alpha \beta}} \), with \( \hat{\sigma}_{\alpha \beta} \) an estimate of the standard error of the product \( \hat{\alpha} \hat{\beta} \). It is available in standard statistical packages such as SPSS, SAS, and R. It has good properties when either \( \alpha \) or \( \beta \) is large and the standard errors of \( \hat{\alpha} \) and \( \hat{\beta} \) are small, but if the two \( t \)-tests for testing \( \alpha = 0 \) and \( \beta = 0 \) tend to be small, properties deteriorate. For parameter values under the null, the Null Rejection Probability (NRP) can be very close to zero and, under the alternative, power can fall far below the size (highest NRP) of 5% that we use throughout. Other tests considered in MacKinnon et al. (2002) suffer from the same problems.

The origin is exceptional even under the null. The null hypotheses \( \alpha \beta = 0 \) defines a manifold that is almost everywhere continuously differentiable, with the exception of the origin which is a singular point. The problematic behavior of the Wald test under the null with singularities is well known and as yet unresolved. Dufour et al. (2017) provide an extensive characterization of the asymptotic null distribution of Wald-type statistics for testing restrictions given by polynomial functions with local singularities. We refer to their comprehensive review of the literature on the problems of Wald tests with singularities. In the case of a single restriction, such as mediation testing, they provide limit distributions and bounds. This only shows the extent of the problems, but does not solve them or show how the Wald test can be salvaged. We will construct a new test that has good power properties uniformly, even in a neighborhood of the singularity.

The distributions of all test statistics considered in the literature depend on the value of the parameters under the null. As a consequence none of these tests is similar, meaning that it has rejection probability that is not constant on the boundary of the null hypothesis. In fact rejection probabilities under alternatives close to the origin can be much lower than the size. These tests are therefore seriously biased since power can be much less than the NRPs for certain parameter values. The Wald-type (Sobel) test’s dependence on the parameter value is extreme in the sense that the asymptotic critical value when both \( \alpha = 0 \) and \( \beta = 0 \) is \( \frac{1}{4} \chi^2_1 (0.95) \) and for any other value equals \( \chi^2_1 (0.95) \), i.e. the usual Chi-squared critical value for one restriction. This discontinuity in the asymptotic distribution and dependence on the parameter also invalidates bootstrap procedures.

Much effort in the literature has gone into improving well-known test statistics, such as the Wald statistic, without a satisfactory solution. The key step in our approach is to move away from test statistics and consider a flexible boundary of the critical region in the sample space which is then optimized.

We make three main contributions in this paper: one theoretical, one practical, and the introduction of a new general method for constructing near similar tests, i.e. tests that have rejection probabilities near 5% for all parameter values on the boundary of the null hypothesis. Our main theoretical contribution is to show that no similar mediation test exists in Theorem 1. In sharp contrast to this negative theoretical finding is our main practical contribution which is the construction of a new simple test that is almost similar based on a new general, so called varying-\( g \) method. It varies a function \( g \) that defines the boundary of the critical region to obtain a test that has NRPs as close as possible to 5%. This new method is not limited to the mediation hypothesis, but can be applied to many other testing problems with nuisance parameters to obtain near similar tests more generally.

For the mediation hypothesis we construct this boundary in the space of the two common
t-statistics for testing $\alpha = 0$ or $\beta = 0$. We develop a numerical method that does not use
simulations to determine this critical region which has NRPs that are extremely close to 5%
for all values of $\alpha$ and $\beta$ under the null. This requires some computing effort on our part
initially, but once completed our results can be easily implemented in practice using the
table or the computer code provided. This test has much better power properties than Wald
and LR tests, but a natural question is if one can do even better. To determine the quality
of the test in absolute terms requires an appropriate power envelope. The power envelope
cannot be constructed by point optimal invariant tests based on a simple application of
the Neyman-Pearson lemma because the null and alternative are composite. Andrews and
Ploberger (1994) address this issue by optimizing weighted power and recent econometric
contributions, including Andrews et al. (2006, 2008), Elliott et al. (2015), Guggenberger
et al. (2019), to name but a few, have considered null and/or alternative weighted mixture
distributions such that the Neyman-Pearson lemma can be applied to the resulting point
null and point alternative distributions. Within the specified class of mixture distributions,
the least favorable distribution is then constructed and a critical value calculated. Any other
test in this class has power no higher than the test constructed, resulting in an optimality
property. There is no guarantee however, that the resulting test is similar and it can still
be seriously biased as we have confirmed (but not reported) in the mediation context for a
variety of mixtures.

We take a more direct approach to constructing the power envelope and maximize power
for a grid of points in the alternative. We introduce a class of near similar invariant tests
$\Gamma_\epsilon$ with $0.05 - \epsilon \leq NRP \leq 0.05$ for a grid of points under $H_0$ and $\epsilon$ small. The algorithm
generates a different test for each grid point in the alternative that is an approximately
similar invariant test that maximizes power for that point alternative. This provides a
power envelope (upper bound) for similar tests. Using the same algorithm we can further
determine a power envelope for nonsimilar tests by discarding the near similarity restriction
$0.05 - \epsilon \leq NRP$.

We use the near similar power envelope to construct an optimal test within $\Gamma_\epsilon$ that
minimizes the total power difference from the envelope on a grid of points. It has power
that deviates less than 0.0062 from the upper bound anywhere on the grid of points. The maximal difference from the (higher) nonsimilar power envelope is 2% points when power
is around 40%, showing that the potential power loss due to the similarity requirement is small.

Andrews (2012) shows that an exact similar test exists in the related one-sided testing
problem $H_0 : \alpha \geq 0$ and $\beta \geq 0$ against at least one of them smaller than zero. It is a
randomized test and the result is of theoretical interest, but the main point of his paper
is that power is abysmal. This result is in line with Perlman and Wu (1999) who coined the term “Emperor’s New Tests” for similar tests with very poor properties. Insistence on
similarity can render Likelihood Ratio (LR) tests $\alpha$-inadmissible, cf. Lehmann and Romano
(2005, Section 6.7), but Perlman and Wu (1999) give examples where similar tests have
extremely undesirable properties yet inadmissible LR tests still provide reasonable answers.
In the mediation setting the LR test is much better than the Wald test as we will show,
but still suffers from poor power properties close to the origin and is inadmissible. Our test
is non-randomized and has good power properties uniformly superior to the Wald, LR, and
LM tests considered here and would please even statistically erudite emperors.
Moreira and Mourão (2016) consider random critical values. Such an interpretation can be given quite generally to any critical region in a higher dimensional space since the boundary of the critical region for one statistic can be expressed as a function of the remaining statistics. Our solution can be framed in terms of a random critical value for the minimum of the absolute $t$-values. This critical value is a function of the maximum of the absolute $t$-values. The critical region that we construct is fixed however and not at all random. Our approach appears to lend itself better to multivariate extensions.

Our empirical illustration requires such an extension to three dimensions and we consider general hypotheses of the form $H_0 : \theta_1 \cdots \theta_K = 0$. In order to derive the critical region for dimensions three and higher we exploit the symmetries of the testing problem further. The testing problem is invariant to ordering (permutations) of parameters and statistics and sign changes (reflections), giving rise to a finite group with eight transformations on the parameter and sample space in two dimensions and $K!2^2$ elements in $K$ dimensions. We give the relevant distribution of the maximal invariant and use it to derive a critical region explicitly in two and three dimensions. For three dimensions we use a method to obtain the solution in dimension $K$ from the preceding solution in dimension $K - 1$. These solutions are dimensionally coherent in the sense that for extremely large values of a number, $k$ say, of $t$-statistics, the solution reduces to the $K - k$ dimensional reduction since in such cases it is essentially known that $k$ parameters are non-zero and rejection of the null depends on the remaining $(K - k)$ $t$-statistics.

An empirical illustration on union sentiment among southern nonunion textile workers in Section 6 shows the practical implementation in two and three dimensions and leads to different conclusions than standard tests. For practitioners the major advantage of our test is that there is a better chance of formally showing that there is a mediation effect. Our test has better power, especially when the two channeling effects are small or less accurately estimated. Given the enormous interest in testing for mediation and the fact that our test can have close to 5% more power than existing tests, many unpublished examples will exist where it can now be concluded that there is a statistically significant mediation effect.

2 Theory

The joint density of $(Y, M)$ given $X$ can be written as $f(Y, M|X; \lambda) = f(Y|M, X; \lambda_1)f(M|X; \lambda_2)$ with $\lambda_1 = (\tau, \beta, \sigma_{11})'$ and $\lambda_2 = (\alpha, \sigma_{22})'$. The parameters $\lambda_1$ and $\lambda_2$ vary freely as a result of the triangular structure of the model. The mediation variable is the endogenous variable in (2) but is strongly exogenous for $\beta$ in (1) since $Y$ is not causal for $M$. For a sample of $n$ independent observations the loglikelihood equals the sum of two normal loglikelihoods corresponding to (1) and (2)$^3$:

$$\ell(\lambda) \propto -\frac{1}{2\sigma_{11}} \sum_{i=1}^{n} (y_i - \tau x_i - \beta m_i)^2 - \frac{n}{2} \log(\sigma_{11}) - \frac{1}{2\sigma_{22}} \sum_{i=1}^{n} (m_i - \alpha x_i)^2 - \frac{n}{2} \log(\sigma_{22}). \quad (4)$$

$^3$This can easily be extended to include more regressors/covariates. Instrumental variables can also be used, but note that $X$ and $M$ appear in both equations and in the standard setup $u$ and $v$ are independent because of the experimental interpretation of $M$. 5
As a consequence the Maximum Likelihood Estimators (MLEs) for $\alpha$ and $\beta$ are the usual OLS estimators for the two equations separately. Furthermore, both observed and expected Fisher information matrices will be block diagonal in terms of $\lambda_1$ and $\lambda_2$ as well as in $(\tau, \beta)'$, $\sigma_{11}$, $\alpha$, and $\sigma_{22}$. As a result the standard $t$-statistics $T_1$ and $T_2$ for $\alpha$ and $\beta$ respectively are asymptotically independent and normally distributed with means $\mu_1 \equiv \alpha_0/\sigma_\alpha$, $\mu_2 \equiv \beta_0/\sigma_\beta$ where $\alpha_0$, $\beta_0$ denote the true parameter values and $\sigma_\alpha$, $\sigma_\beta$ the standard deviations of the OLS estimators: $(T - \mu) \overset{d}{\rightarrow} N(0, I_2)$. Throughout the rest of the paper we will use the normal distribution for the $t$-statistics,

$$(T - \mu) \equiv \begin{pmatrix} T_1 - \mu_1 \\ T_2 - \mu_2 \end{pmatrix} \sim N(0, I_2),$$

with the understanding that this is an asymptotic approximation, but exploited as if it is the exact distribution. This is analogous to the assumption in the weak instruments literature that the covariance matrix is known (e.g. Andrews et al. (2006)).

The finite-sample distribution involves $t$-distributions with different degrees of freedom and is complicated by the fact that $\sigma_\beta$ depends on $M$. A strong justification for restricting attention to $T$, even in finite samples is the exact result that $T$ is maximal invariant with respect to an appropriate group of (location-scale) transformations that leave the testing problem invariant. Appendix C proves this result and provides further distributional details relevant for the model.

2.1 The Problem with Standard Test Statistics

Standard test statistics used in practice have distributions that depend on the parameter values under the null. The rejection probabilities are therefore not constant and the tests biased with power dropping below the size of the test, especially in a neighborhood of the origin. We illustrate the issue for the classic trinity of Wald, LR, and LM methods for constructing test statistics.

The Wald test for testing $H_0 : \alpha \beta = 0$ together with its asymptotic distribution is given by Glonek (1993) and further analyzed in Drton and Xiao (2016):

$$W = \frac{T_1^2 T_2^2}{T_1^2 + T_2^2} \overset{d}{\rightarrow} \begin{cases} \chi_1^2 & \text{if } \alpha = 0 \text{ or } \beta = 0, \text{ but not both,} \\ \frac{1}{4} \chi_1^2 & \text{if } \alpha = \beta = 0. \end{cases} (5)$$

The widely used Sobel (1982) test equals $\sqrt{W}$. The discrete jump in the asymptotic distribution from the origin to any other parameter value that is fixed is remarkable and shows explicitly that the distribution depends heavily on the parameter values under the null. The critical value is the usual 3.84 in all cases other than the origin. For an NRP of 5% at the origin the critical value should be 0.96 but this would lead to over-rejection for other values under the null and the test would be oversized (size > 30%) and invalid. One could consider drifting sequences of parameter values to investigate the behavior of the Wald statistic near the origin, but that in itself does not solve the problem. In fact no satisfactory solution has been found in preceding decades to salvage the Wald statistic, see e.g. Dufour et al. (2017). This prompted our investigation and to propose an alternative solution.
The LR test was shown by van Giersbergen (2014) to equal:

\[ LR = \min \{|T_1|, |T_2|\}, \]  

(6)

and rejects when both \( H_{0\alpha} : \alpha = 0 \) and \( H_{0\beta} : \beta = 0 \) are rejected. In MacKinnon et al. (2002) this is referred to as the test for joint significance, but not identified as the LR test. The rejection probability is:

\[ P[L_R \geq cv] = P[|T_1| \geq cv \cap |T_2| \geq cv] = P[|T_1| \geq cv] \cdot P[|T_2| \geq cv] \]

by independence of \( T_1 \) and \( T_2 \). These rejection probabilities are monotonically increasing in the absolute values of \( \alpha \) and \( \beta \). Correct size is therefore obtained by choosing the critical value of the test by letting \( \alpha \to \infty \) when \( \beta = 0 \), or \( \beta \to \infty \) if \( \alpha = 0 \), to guarantee that the rejection probability under the null is always smaller than or equal to the nominal size. The asymptotic 5% critical value is therefore the usual 1.96. The NRP will depend on the values of \( \alpha \) and \( \beta \) and vary between the following two extremes:

\[ P[L_R \geq z_{0.025}] = \begin{cases} 
0.05 : & \text{if } \alpha \to \infty \land \beta = 0, \text{ or } \beta \to \infty \land \alpha = 0, \\
0.05^2 = 0.0025 : & \text{if } \alpha = 0 \land \beta = 0,
\end{cases} \]

where \( z_{0.025} \) is the upper 2.5% percentile of the standard normal distribution. For an NRP of 5% at the origin \((\alpha, \beta) = (0, 0)\), the critical value should equal \( cv_{LR00} = 1.21699 \). This leads to massive over-rejection if only one parameter is zero and the other much larger. The test with this critical value is oversized (size > 20%) and invalid.

The definition of the LM test itself depends on parameter values under the null. There are three different versions depending on which one of the three constituent part hypotheses is considered: \( H_{0\alpha\beta} : \alpha = 0 \land \beta \neq 0 \), \( H_{0\alpha} : \beta = 0 \land \alpha \neq 0 \), or \( H_{0\beta} : \alpha = 0 \land \beta = 0 \). Explicit expressions for the three LM tests are given in Appendix A. These LM tests are essentially squared \( t \)-tests with restricted variance estimates. The three versions can be combined into a single statistic, but its distribution will depend on the true parameter values under the null.

All these classic tests are functions of two \( t \)-statistics and their distributions, as well as the NRPs, clearly depend on the parameter values under the null and the tests are not similar. A test is called similar on the boundary of \( H_0 \) if the probability of rejection of the null is constant for all parameter values on the boundary of \( H_0 \) and \( H_1 \):

**Definition 1** Similar test. Let \( \omega \subset \Theta \) be the boundary between \( H_0 : \theta \in \Theta_0 \) and \( H_1 : \theta \in \Theta \setminus \Theta_0 \). A test is similar on the boundary \( \omega \) if the null rejection probability does not depend on \( \theta \in \omega \).

For the null hypothesis of no mediation, the boundary consists of the horizontal and vertical axes of the \((\alpha, \beta)\) space and is equal to \( H_0 \) itself. None of the classic tests is similar and in a neighborhood of the origin the NRPs are close to zero. As a result the power in a neighborhood of the origin is also close to zero and far below the size of the test and the tests are biased since there are parameter values with probability of rejection under the alternative lower than under the null.
2.2 Critical Regions

The behavior and construction of the classic test statistics is problematic. Given that no satisfactory adjustments of classic test statistics have been found, despite considerable efforts over recent decades, a different approach is required.

In order to derive an alternative test procedure we shift the focus from the test statistic to the critical region. A critical region defines a test statistic of course, but choosing a class of tests, such as Wald, LR, or LM tests, restricts the shape of the critical region. For the same reason the tests focusing on improving the standard error of $\hat{\alpha}$ or $(\hat{\tau}^* - \hat{\tau})$ analyzed in MacKinnon et al. (2002) limit the shape.

We construct a new test procedure by constructing the critical region directly in the two-dimensional sample space of the $t$-statistics used in the construction of the tests. We consider critical regions that are bounded by a measurable function $g(\cdot)$ and give the following definition.

**Definition 2** Boundary function (of the critical region). $g: \mathbb{R} \rightarrow \mathbb{R}$ defines the:

- Critical Region $CR_g = \{ (T_1, T_2) \in \mathbb{R}^2 \mid |T_1| \geq g(|T_2|) \cap |T_2| \geq g(|T_1|) \}$,
- Acceptance Region $AR_g = \{ (T_1, T_2) \in \mathbb{R}^2 \mid |T_1| < g(|T_2|) \cup |T_2| < g(|T_1|) \}$.

The justification for considering the $t$-statistics is threefold. First, the MLE $\hat{\lambda} = (\hat{\tau}, \hat{\beta}, \hat{\sigma}_{11}, \hat{\alpha}, \hat{\sigma}_{22})'$ is a complete minimal sufficient statistic because the model constitutes a full exponential model given the dimensional equality of the minimal sufficient statistic and the parameter space; see van Garderen (1997). Second, $T_1$ and $T_2$ have distributions under the null that are independent of the nuisance parameters $\tau$, $\sigma_{11}$, and $\sigma_{22}$. Finally, $T = (T_1, T_2)'$ is a maximal invariant under an appropriate group of transformations generalizing the scale invariance of the $t$-statistics.\(^4\) In the mediation problem there are further symmetries and invariances. The null hypothesis is not changed if $\alpha$ and $\beta$ are permuted or their signs changed. Consequently, we can permute the $t$-statistics and change their sign without affecting the problem. As a consequence only $1/8^{th}$ of the two-dimensional sample space of $T$ needs consideration and we define the critical region in the first octant (east to northeast). The other seven parts follow by symmetry. The test defined by $CR_g$ is indeed invariant to permutations, reflections, and scale transformations. The domain of $g(\cdot)$ can therefore be restricted to the non-negative real line and bounded by the $45^\circ$ line: $g(x) \leq x$. In Section 5 we show that the ordered absolute $t$-statistic is a maximal invariant not only for this testing problem but also its generalization to higher dimensions.

We can put the general definition of a similar test in terms of the boundary function $g(\cdot)$, noting that $H_0$ is itself the boundary $\omega$ of $H_0$ and $H_1$:

**Definition 3** $g(\cdot)$ is said to be a similar boundary function if the probability of the critical region $CR_g$ defined by $g$ is constant under $H_0$:

$$P [T \in CR_g \mid H_0] = \text{constant} \quad \forall (\alpha, \beta) \in \mathbb{R}^2 \quad \text{with} \quad \alpha\beta = 0.$$\(^4\)Appendix C shows that this also holds exactly in finite samples.
The boundary functions defined by the Wald, LR, and LM test are not similar. Figures 1a and 1b show the boundary functions that define the critical region in terms of \((T_1, T_2)\) for the Wald and LR test. The LM test is not illustrated because it is not properly and uniquely defined. We show the boundaries for two critical values: one such that for large \(|\alpha|\) or \(|\beta|\) the NRP is 5% asymptotically. This value is the usual 3.84 for the Wald test and 1.96 for the LR test. The second, smaller critical value is such that the NRP is 5% when \(\alpha = \beta = 0\). This value is 0.96 for the Wald test and 1.22 for the LR test. The rejection probabilities are shown as a function of the noncentrality parameter \(\mu_1 = \alpha/\sigma_\alpha\) for given \(\mu_2 = 0\), such that \(H_0\) holds. For the LR test with critical value 1.96 the NRP goes to 0.0025 = 0.05 when \(\alpha = \beta = 0\). In the second case, with the smaller critical value 1.22, the NRP is 5% by construction when \(\alpha = \beta = 0\), but for other values the NRPs are much higher than the nominal size of the test and hence they are not valid 5% tests. The same situation will occur when constructing point optimal invariant tests. The Wald test is considerably worse with lower NRP over a wider range of \(\mu_1\), see figures 1c and 1d.

The trinity of classic tests is clearly nonsimilar. The question is if we can do much better. Does there exist a similar test or is this a problem that is intrinsically unsolvable? Our main theoretical contribution, Theorem 4, states that no similar test exists. The practical answer however is that we can get very close to similarity and can do much better in terms of power than existing tests.

**Theorem 4** No similar boundary function \(g(\cdot)\) exists for testing \(H_0 : \alpha \beta = 0\).

The proof of this main theorem is given in Appendix B and exploits the symmetries of the problem and the completeness of the normal distribution. We use 5% significance throughout but it is immediate from the proof that there is no significance level for which there exists a similar boundary function, apart from two trivial exceptions. A size of 0% would yield \(g(t) = t\) and \(ARg = \mathbb{R}^2\) such that the test would never reject. The other trivial solution is \(g(t) = 0\) and defining \(g^{-1}(0) = \infty\) accordingly. This test always rejects, leading to an NRP of 100% for all parameter values.

Andrews (2012) proves constructively that a similar test exists for the “one-sided” testing problem \(H_0 : \mu \geq 0\) versus \(H_1 : \mu \not< 0\), but his test is randomized and he makes the point that it has very low power. Andrews shows that on the negative parts of the axes the NRP is 5%, which is (trivial) power in his setting and correct size in ours. One-sided alternatives destroy the symmetry of the problem that we exploit and use to prove the non-existence. We do not consider randomized tests and our new test below has very good power properties close to the power envelope.

Despite the negative non-existence result of Theorem 4, we construct a critical region that is all but similar with NRPs that do not differ from 5% in practical terms for any parameter value under the null and has good power. This new test is easy to implement using Table 1. We also provide R-code in Appendix E.

The new test is obtained in two steps. We propose a new general method for constructing near similar tests. In the first step we use this method to derive a near similar test for the mediation hypothesis. We derive the power envelope for near similar tests, which can be used to show that the test has good power properties. In the second step, however, we use this power envelope to optimize the test and maximize power within the class of near similar
Figure 1: Critical regions for Wald (Sobel) and LR tests and their rejection probabilities. White areas are the critical regions for the valid 5% tests. For 5% rejection in the origin \((\alpha, \beta) = (0, 0)\), the dark areas are added to the critical regions which leads to over-sized tests. The LR test is poor, the Sobel test is worse.
tests that we define. The new test minimizes the difference between the power surface and the power envelope and is therefore optimal in this sense.

3 Near Similar Test Construction: Varying $g$-Method

The new general method for the construction of near similar tests is easily described by three generic steps:

1. Define a flexible boundary $g$ for the critical region in the relevant sample space.

2. Define a criterion function $Q(g)$ that penalizes the deviation of the NRP from 5% for a grid of parameter values under the null (and possibly restrictions on $g$ and other aspects deemed relevant).

3. Systematically vary and determine $g$ such that it minimizes the criterion function and is therefore as close to similarity as possible in the metric defined by $Q$.

The relevant sample space is determined by the particular testing problem at hand and may have been reduced by sufficiency, invariance, or other principles, to dimension $k$, say. The boundary $g$ of the critical and acceptance region is then of dimension $(k - 1)$, but may consist of disjoint parts if the critical and/or acceptance region are not simply connected in a topological sense. There are various possibilities to define $g$ flexibly, but we will use splines. The criterion function may include aspects other than similarity, for instance smoothness and monotonicity of $g$, convexity of the critical or acceptance regions, or even rejection probabilities under alternatives. Consequently, Step 3 will generally be a constraint optimization problem. The systematic variation of $g$ is intended to be in line with the optimization routine used to minimize $Q$ as in, e.g. a Newton-Raphson-type procedure.

An explicit implementation of the varying $g$-method to the mediation problem is given next.

3.1 Near Similar Mediation $g$-Test

The initial step in the varying $g$-method is to determine the relevant sample space for the testing problem. The mediation model allows a reduction of the sample space by sufficiency to the MLE of the five parameters. A further reduction to $(T_1, T_2)$ in two dimensions follows from location-scale invariance as proved in Appendix C. Permutation and reflection symmetries reduce the sample space to one octant, because the absolute order statistic defined as:

\[
\left( |T|_{(1)}, |T|_{(2)} \right) = \left( \min \left( |T_1|, |T_2| \right), \max \left( |T_1|, |T_2| \right) \right),
\]

is a maximal invariant. It has a distribution that depends only on the ordered absolute noncentrality parameter which is the corresponding maximal invariant in the parameter space:

\[
\left( |\mu|_{(1)}, |\mu|_{(2)} \right) = \left( \min \left( |\mu_1|, |\mu_2| \right), \max \left( |\mu_1|, |\mu_2| \right) \right) \text{ and } (\mu_1, \mu_2) = \left( \alpha/\sigma_\alpha, \beta/\sigma_\beta \right).
\]
Figure 2: Construction of the basic $g$-function with $J = 6$, so 8 knots in all and the resulting CR boundary in $(T_1, T_2)$ space.

The $g$-boundary is generally determined by an algorithm. Appendix D shows the basic implementation of the varying $g$-method using linear splines with $J + 2$ knots, with the first and last knots fixed. In spite of its simplicity, it leads to big improvements even for small values of $J$. Figure 2 illustrates the construction of the $g$-function for a fixed number of grid points $J = 6$ and the resulting $CR_g$ in the sample space of $(T_1, T_2)$.

Figure 3 shows the NRPs of the test in comparison to the LR and Wald (Sobel) tests. There is a remarkable gain in the lowest NRP, and therefore local power, from 0.25% to 4.8%, even for $J = 2$. We started with $J = 2$, after $J = 16$ the improvements were very small, and with $J = 32$ there was essentially no improvement. The figure shows a slight over-rejection for some parameter values under $H_0$ and hence the test is not a valid 5% test. We will correct for this subsequently by imposing strict side conditions on the NRP, because simply increasing penalties on over-rejection will not solve the issue.

### 3.2 Power

Insistence on similarity can have negative consequences for the power in general, but not here. Even the basic test with $J = 32$ has good power, especially in comparison with the Sobel and LR tests. It is uniformly better for all values of the noncentrality parameter $\mu$ and in a neighborhood of the origin with $\mu = 0$ it is essentially 5% points higher.
Denote the rejection probability of the \( g \)-test as a function of the noncentrality parameters \((\mu_1, \mu_2) = (\alpha/\sigma_\alpha, \beta/\sigma_\beta)\) by:

\[
\pi_g(\mu_1, \mu_2) = P \left[ T \in CR_g \mid (\mu_1, \mu_2) \right].
\]

(7)

If \( \mu_1 \) and/or \( \mu_2 \) equal 0 then the null hypothesis is true and \( \pi_g \) is the NRP. When both are non-zero, \( H_0 \) is false and \( \pi_g \) is the power of the test defined by \( CR_g \). Figure 4 illustrates the power in the 45° direction \( \mu_1 = \mu_2 \), but in other directions power is also superior to the Wald (Sobel) and LR tests.

There is a straightforward explanation for the additional power. The Wald and LR test both reject much less than 5% near the origin. The critical region can be extended and the power increased without failing the size condition. In the origin the NRPs are close to 0% for the LR and Wald (Sobel) tests. By extending the critical region we can therefore gain almost 5% power without violating the size condition. Nevertheless, the LR test has some attractive features including that rejection for a particular \((t_1, t_2)\) implies rejection for larger values of \( t_1 \) and/or \( t_2 \) which is intuitive since the evidence against the null is increasing.

Figure 3: NRPs basic \( g \)-tests with \( J = 2, 5, 8 \) versus LR and Wald (Sobel) tests.

Figure 4: Power comparison basic \( g \)-tests, LR- and W (Sobel) tests along \( \mu_1 = \mu_2 \) (= \( \mu \)).
disadvantage is however that one never rejects when either $t_1$ or $t_2$ is smaller than 1.96 and this causes conservativeness that can be resolved by adding area to the critical region. We elaborate on this after deriving the optimal $g$-test.

### 3.3 Power Envelope

Comparison to the Sobel and LR tests is limited because they are very poor for small values of $\mu$. The absolute quality, or even near optimality, of the new $g$-test can only be assessed by comparing the power surface of the test to the power envelope, or a tight upper bound thereof, for a class of tests that satisfy appropriate invariance-, size and almost similarity restrictions. Since no exact similar invariant test exists, we introduce a class $\Gamma_\epsilon$ of near similar tests with NRPs that deviate less than $\epsilon$ from the 5% level and an operational (super)class $\Gamma_\epsilon^{M_0} \supseteq \Gamma_\epsilon$ as follows.

**Definition 5** The class $\Gamma_\epsilon$ of near similar boundary functions with $\epsilon > 0$ is defined by:

$$\Gamma_\epsilon = \left\{ g : \mathbb{R} \rightarrow \mathbb{R} \mid \sup_{\mu_0 \geq 0} P[CR_g \mid (0, \mu_0)] \leq 0.05 \text{ and } \inf_{\mu_0 \geq 0} P[CR_g \mid (0, \mu_0)] \geq 0.05 - \epsilon \right\}$$

The class $\Gamma_\epsilon^{M_0}$ with $M_0 = \left\{ \left(0, \mu_0^{(i)}\right) \right\}_{i=1}^{\Upsilon_0}$ a set containing $\Upsilon_0$ points under $H_0$, is defined by:

$$\Gamma_\epsilon^{M_0} = \left\{ g : \mathbb{R} \rightarrow \mathbb{R} \mid \sup_{(0, \mu_0) \in M_0} P[CR_g \mid (0, \mu_0)] \leq 0.05 \text{ and } \inf_{(0, \mu_0) \in M_0} P[CR_g \mid (0, \mu_0)] \geq 0.05 - \epsilon \right\}.$$  

For $\epsilon = 0$ the boundary functions in $\Gamma_0$ would be similar and, since no such boundary exists by Theorem 1, $\Gamma_0$ would be empty. For $\epsilon = 0.05$, on the other hand, $\Gamma_{0.05}$ contains all tests that satisfy the size condition. For $\epsilon$ close to 0, $\Gamma_\epsilon$ contains boundaries that are almost similar. The minimum value of $\epsilon$ for which $\Gamma_\epsilon$ are not empty depends in general on the testing problem.

The class $\Gamma_\epsilon^{M_0}$ can be thought of as a discretization of $\Gamma_\epsilon$ in the sense that a grid of points under the null is considered. It imposes less restrictions and enforces near similarity conditions on a finite number of points only. As a consequence it may contain boundaries that do not satisfy the size condition for points that are not in $M_0$. Obviously $\Gamma_\epsilon \subseteq \Gamma_\epsilon^{M_0}$ since the size and NRP conditions also hold for the points in $M_0$.

Within the class $\Gamma_\epsilon$ there is no unique solution. As a consequence one is forced to choose a boundary function from $\Gamma_\epsilon$, or in practice from $\Gamma_\epsilon^{M_0}$, to obtain an operational test. For the construction of the power envelope we can select the test that maximizes the power against a particular point $(\mu_1, \mu_2)$ in the alternative. This test is a Point Optimal Invariant Near Similar (POINS) test. The critical region of this test varies with $(\mu_1, \mu_2)$ and no uniformly most powerful test exists within the class $\Gamma_\epsilon$. It can be used however, to construct an upper bound for the power envelope.

**Definition 6** The power envelope of a near similar invariant test with $\epsilon > 0$ is defined as:

$$\pi(\mu_1, \mu_2) = \max_{g \in \Gamma_\epsilon} P[CR_g \mid (\mu_1, \mu_2)].$$
For a given set of points \( \mathbb{M}_0 = \{(0, \mu_0^{(i)})\}_{i=1}^{T_0} \) define an upper bound to the power envelope by:

\[
\bar{\pi}(\mu_1, \mu_2) = \max_{g \in \Gamma_{\mathbb{M}_0}} P[CR_g | (\mu_1, \mu_2)].
\]

For notational simplicity we have suppressed the dependence on \( \epsilon \) and \( \mathbb{M}_0 \). Since \( \Gamma_{\epsilon} \subseteq \Gamma_{\mathbb{M}_0} \) and elements of \( \Gamma_{\mathbb{M}_0} \) do not necessarily satisfy the size condition for all parameter values it follows that \( \bar{\pi}(\mu_1, \mu_2) \geq \pi(\mu_1, \mu_2) \), because fewer conditions are imposed. Choosing a finer grid for \( \mathbb{M}_0 \) will force \( \bar{\pi}(\mu_1, \mu_2) \) closer to \( \pi(\mu_1, \mu_2) \), at least in the additional points in \( \mathbb{M}_0 \) where the size condition is now required to hold. Also note that the “point” optimal \( g \) that maximizes power for the point \( (\mu_1, \mu_2) \), may have undesirable features such as including parts of the axes in the critical region, even though such observations are perfectly in line with the null hypothesis.

We determine \( \bar{\pi}(\mu_1, \mu_2) \) numerically by maximizing the power directly by selecting critical region points in the sample space that maximize the probability of rejection when the true density has parameter \( (\mu_1, \mu_2) \), under the side conditions that the NRP \( \in [0.05 - \epsilon, 0.05] \) for all parameters \( (0, \mu_0) \in \mathbb{M}_0 \). The sample space is decomposed into 285,150 squares and for each square it is determined whether it should be included in the critical or acceptance region in order to maximize the power while at the same time satisfying the approximate similarity condition. This is repeated for a grid of \( (\mu_1, \mu_2) \) points. So for each point on the grid the POINS critical region is determined and the power recorded. Appendix D gives details of the algorithm and the optimization routine that can deal with a large number of variables and side conditions.

By dropping the near similarity restriction \( 0.05 - \epsilon \leq NRP \) in the same algorithm, we can construct a power envelope for nonsimilar tests. The maximal difference from the (higher) nonsimilar power surface is 2% points when power is around 40%, showing that the power loss due to the similarity requirement is small.

The power surface of the basic test based on 32 knots, \( g_{J=32} \), can now be compared to the power envelope upper bound. They are very close over the whole of the parameter space. The test \( g_{J=32} \) is oversized, however, and even though the over-rejection seems practically irrelevant, theoretically the test is not a valid 5% test.

The power envelope enables us to construct a correctly sized optimal test derived next and will show that the upper bound is tight.

4 Optimal \( g \)-test

Having determined an upper bound to the power envelope, we can determine a \( g \)-boundary function with a power surface as close as possible to this upper bound. This optimal test is found using the algorithm given in Appendix D. We parsimoniously simplify the \( g \)-function to just three clamped splines joined by three linear parts. This function is given in Appendix E and R-code is also provided there. For ease of implementation we give values of \( g(\tau) \) in Table 1. Figure 5 shows the optimal \( g \)-boundary test for the mediation problem.

The optimal \( CR_g \) includes a narrow region close to the 45° line where both \( t \)-statistics are of the same magnitude. This is expedient because mediation requires both \( \alpha \) and \( \beta \) to
Figure 5: Optimal $g$-boundary function. The dashed line is the LR boundary.

Figure 6: NRP as a function of the non-centrality parameter $\mu$. The solid line is the optimal test and is strictly between 0.04999 and 0.05. The dotted horizontal line is uniformly 5% of the unattainable similar test. The dashed wave is the NRP of the basic $g_{J=32}$ which is marginally over-sized.
| $t$ | 0   | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|-----|-----|------|------|------|------|------|------|------|------|------|
| 0   | 0   | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
| 0.1 | 0.08829 | 0.09415 | 0.0998 | 0.10542 | 0.11117 | 0.11723 | 0.12376 | 0.13091 | 0.13878 | 0.14745 |
| 0.2 | 0.15699 | 0.16699 | 0.17699 | 0.18699 | 0.19699 | 0.20699 | 0.21699 | 0.22699 | 0.23699 | 0.24699 |
| 0.3 | 0.25699 | 0.26699 | 0.27699 | 0.28699 | 0.29699 | 0.30699 | 0.31699 | 0.32699 | 0.33699 | 0.34699 |
| 0.4 | 0.35699 | 0.36699 | 0.37699 | 0.38699 | 0.39699 | 0.40699 | 0.41699 | 0.42699 | 0.43699 | 0.44699 |
| 0.5 | 0.45699 | 0.46699 | 0.47699 | 0.48699 | 0.49699 | 0.50699 | 0.51699 | 0.52699 | 0.53699 | 0.54699 |
| 0.6 | 0.55699 | 0.56699 | 0.57699 | 0.58699 | 0.59699 | 0.60699 | 0.61699 | 0.62699 | 0.63699 | 0.64699 |
| 0.7 | 0.65699 | 0.66699 | 0.67699 | 0.68699 | 0.69699 | 0.70699 | 0.71699 | 0.72699 | 0.73699 | 0.74699 |
| 0.8 | 0.75699 | 0.76699 | 0.77699 | 0.78699 | 0.79699 | 0.80699 | 0.81699 | 0.82699 | 0.83699 | 0.84699 |
| 0.9 | 0.85699 | 0.86699 | 0.87699 | 0.88699 | 0.89699 | 0.90699 | 0.91699 | 0.92699 | 0.93699 | 0.94699 |
| 1.0 | 0.95699 | 0.96699 | 0.97699 | 0.98699 | 0.99699 | 1.00699 | 1.01699 | 1.02699 | 1.03699 | 1.04699 |
| 1.1 | 1.05699 | 1.06699 | 1.07699 | 1.08699 | 1.09699 | 1.10699 | 1.11699 | 1.12699 | 1.13699 | 1.14699 |
| 1.2 | 1.15699 | 1.16699 | 1.17694 | 1.18678 | 1.19646 | 1.20592 | 1.2151 | 1.22393 | 1.23237 | 1.24034 |
| 1.3 | 1.24779 | 1.25469 | 1.26108 | 1.26704 | 1.27264 | 1.27794 | 1.28303 | 1.28797 | 1.29284 | 1.2977 |
| 1.4 | 1.30263 | 1.30771 | 1.31299 | 1.31857 | 1.32449 | 1.33085 | 1.33771 | 1.34514 | 1.35322 | 1.36201 |
| 1.5 | 1.37159 | 1.38159 | 1.39159 | 1.40159 | 1.41159 | 1.42159 | 1.43159 | 1.44159 | 1.45159 | 1.46159 |
| 1.6 | 1.47159 | 1.48159 | 1.49159 | 1.50159 | 1.51159 | 1.52159 | 1.53159 | 1.54159 | 1.55159 | 1.56159 |
| 1.7 | 1.57159 | 1.58159 | 1.59159 | 1.60159 | 1.61159 | 1.62159 | 1.63159 | 1.64159 | 1.65159 | 1.66159 |
| 1.8 | 1.67159 | 1.68159 | 1.69159 | 1.70159 | 1.71159 | 1.72159 | 1.73159 | 1.74159 | 1.75159 | 1.76159 |
| 1.9 | 1.77159 | 1.78159 | 1.79159 | 1.80159 | 1.81159 | 1.82159 | 1.83159 | 1.84159 | 1.85159 | 1.86159 |
| 2.0 | 1.87159 | 1.88135 | 1.89061 | 1.89934 | 1.90752 | 1.91515 | 1.92218 | 1.92861 | 1.9344 | 1.93955 |
| 2.1 | 1.94402 | 1.94781 | 1.95096 | 1.95353 | 1.95557 | 1.95715 | 1.95831 | 1.95912 | 1.95963 | 1.95989 |
| 2.2 | 1.95996 | 1.95996 | 1.95996 | 1.95996 | 1.95996 | 1.95996 | 1.95996 | 1.95996 | 1.95996 | 1.95996 |

Table 1: $g$-function: table entries are $g(t)$ values for corresponding $t$-value in first column + value first row. e.g. $g(1.09) = 1.04699$. Compare smallest absolute $t$-statistic with $g$ (largest absolute $t$-statistic). Linear interpolation results in $|NRP - 5\%| < 0.00001$ uniformly.
be non-zero. The best possibility of detecting this is along the 45° line as illustrated by the power surface in Figure 7 showing highest power on the diagonal. Second, the near similarity condition requires additional critical region area in the left corner of the octant because NRPs are particularly low for small parameter values. The increased power is naturally linked to the increase in Type I error, but correct size of a test by definition merely requires that this is not larger than 5%. Nevertheless, size (NRP)/power trade-off exists as well as other compromises that can be assessed using critical region analysis. For instance, it may seem less intuitive that rejection is not monotonic in $t_1$ and $t_2$ since an increase in both $t_1$ and $t_2$ represents increased evidence against the null. The LR and Wald tests are monotonic in this sense, but lead to a reduction in power to nearly zero for small parameter values. No observed value $t$ of $T$ will ever lie on the horizontal or vertical axis and any observed $t$ is therefore more likely given an alternative parameter value than a value under the null. It is therefore desirable to add area to the LR critical region even if this results in a non-convex critical region or acceptance region. One could cogitate about the very narrow region close to the diagonal and whether the acceptance should not continue along the 45° line until e.g. 1.2, but the new $g$-boundary is the optimal solution to a well-defined problem.

The narrow region of the optimal $CR_g$ is a strict extension of the $CR_{LR}$, which itself is strictly larger than the Sobel (Wald) $CR_W$. Since the new test is constructed to satisfy the size condition we have the following:

\textbf{Theorem 7} The Sobel/Wald test and the LR test are inadmissible.

\textbf{Proof.} $CR_W \subset CR_{LR} \subset CR_g$ hence $P[CR_W] < P[CR_{LR}] < P[CR_g] \leq 0.05$. The optimal $g$-test has uniformly higher power and is correctly sized by construction. \hfill \Box

The NRP as a function of the noncentrality parameter $\mu$ is shown in Figure 6. The difference from 5% is less than $10^{-5}$ and so small that the scale had to be magnified greatly and to an extend that prevents comparison with the LR and Sobel tests in the same graph. We include the NRPs for the basic $g_{J=32}$ which shows the over-rejection less than 0.01% points.

The power of the new $g$-test is very close to the power envelope (upper bound). The maximal difference is 0.00617. This has important implications. First, the upper bound is tight as claimed earlier. Second, the upper bound of the power envelope and power surface of the $g$-test look almost identical when graphed. Figure 7 therefore shows only the power surface of the optimal $g$-test. Finally, the new $g$-test is optimal for all intents and purposes in a larger class of tests. It is optimal by construction within the class of near similar tests $\Gamma_{\epsilon_{\mu_0}}$, but given the closeness of its power surface to the (non)similar power envelope, there cannot exist any near similar test that has additional power more than 0.00617, even if construction is based on a different method. More generally, nonsimilar tests can have 2% points more power, but at the possible cost of odd rejection regions and low power for other parameter values that are not used in the construction of the test. The optimal $g$-test has good properties for all parameter values.

The power surface in Figure 7 shows only the first quadrant of the parameter space of $(\mu_1, \mu_2)$. The other four quadrants follow by simple permutations and reflections of the parameters.
5 Higher Dimensions

If mediation is through a chain of effects where $X \rightarrow M^{(0)} \rightarrow \cdots \rightarrow M^{(K-1)} \rightarrow Y$ then $K$ parameters are required to be non-zero for this channel to operate. The empirical example in the next section requires an extension to three dimensions, but there are many other problems in econometrics that involve restrictions that at least one parameter is zero, see e.g. Dufour et al. (2017). In $K$ dimensions the null hypothesis that at least one parameter is zero and the alternative is all $K$ parameters non-zero:

$$
H_0 : \theta_1 \theta_2 \ldots \theta_K = 0
$$

$$
H_1 : \theta_1 \theta_2 \ldots \theta_K \neq 0
$$

As before we assume that the estimator $\hat{\theta}$ is normally distributed with covariance matrix $\Omega = \text{diag}(\sigma_1^2, \ldots, \sigma_K^2)$ and known such that $\Omega^{-1/2} \left( \hat{\theta} - \theta_0 \right) \sim N(0, I_K)$ with elements $T_k = \hat{\theta}_k / \sigma_k$ and noncentrality parameters $\mu_k = \theta_k / \sigma_k$.

In higher dimensions it is even more important to exploit invariance and symmetry properties of the problem because it can reduce the domain of integration by a factor $K!2^K$. The testing problem is invariant to reordering the parameters (permutations) and sign changes (reflections) of the $K$ parameters $\{\theta_i\}_{i=1}^K$. There is an associated group of transformations on $T$ that leaves the distribution invariant. It can be decomposed into two proper subgroups: the group of permutations, $G_1$ say, with $K!$ elements, and the group of sign changes, $G_2$ say, with $2^K$ elements (two possible signs for each element). The groups $G_1$ and $G_2$ have only the identity element in common, but are otherwise non-overlapping. The full group $G$ generated by $G_1$ and $G_2$ therefore has $K!2^K$ elements. In two dimensions this equals eight, in three dimensions 48, in four dimensions 384 etc. with a multiplicative factor $2^K$ to obtain dimension $K$ from the one before.

The density after a sign change in $T_k$ is obtained by a corresponding sign change in $\mu_k$ and for a permutation of $T$ also $\mu$ permutes accordingly. Hence for any element $h \in G$ we
have $h \cdot T \sim N(h \cdot \mu, I_K)$ or $P_{h\mu} [hT \in A] = P_\mu [T \in A]$ so the distribution is invariant; see Lehmann and Romano (2005).

Define the absolute order statistic $\{ |T|_{(1)} , ..., |T|_{(K)} \}$ as the reordered absolute values of the $t$-statistics such that $|T|_{(1)} < |T|_{(2)} < \ldots < |T|_{(K)}$ and the absolute order parameter $\{ |\mu|_{(1)} , ..., |\mu|_{(K)} \}$ as the reordered absolute values of the parameters $\mu_k$.

**Lemma 8** If $T \sim N(\mu, I_K)$ then the absolute order statistic $\{ |T|_{(1)} , ..., |T|_{(K)} \}$ is a maximal invariant statistic and the absolute order parameter $\{ |\mu|_{(1)} , ..., |\mu|_{(K)} \}$ is a maximal invariant parameter under the group of transformations $G = G_1 \times G_2$. The distribution of $\{ |T|_{(1)} , ..., |T|_{(K)} \}$ depends only on $\{ |\mu|_{(1)} , ..., |\mu|_{(K)} \}$.

**Lemma 9** The probability density function of the absolute order statistic is given by:

$$ f_{\{ |T|_{(1)} , ..., |T|_{(K)} \}} (|t|_{(1)} , ..., |t|_{(K)}) = \text{perm} \begin{pmatrix} \chi (|t|_{(1)} , |\mu|_{(1)}) & \cdots & \chi (|t|_{(1)} , |\mu|_{(K)}) \\ \vdots & \ddots & \vdots \\ \chi (|t|_{(K)} , |\mu|_{(1)}) & \cdots & \chi (|t|_{(K)} , |\mu|_{(K)}) \end{pmatrix} , \quad (8) $$

where $\text{perm} (A)$ the permanent$^5$ of the square matrix $A$ and $\chi (x, \lambda)$ the noncentral Chi-distribution with one degree of freedom and noncentrality parameter $\lambda > 0$.$^6$

Note that the null hypothesis implies $|\mu|_{(1)} = \cdots = |\mu|_{(k)} = 0$ for some $k \geq 1$. We will use density (8) on the relevant domain $0 \leq |T|_{(1)} \leq \cdots \leq |T|_{(K)} \in [0, \infty)$ to calculate rejection probabilities based on the critical region defined by the boundary function $g \left( |t|_{(2)} , \ldots , |t|_{(K)} \right)$.

**Dimensional Coherency.**

If it is known that $\theta_K \neq 0$, then the null hypothesis reduces to $H_0 : \theta_1 \theta_2 \ldots \theta_{K-1} = 0$. This implies that the critical region for the $K - 1$ corresponding $t$-statistics must reduce to the solution found for $K - 1$ dimensions when $|\mu_K|$ is large. For large values of $|T_K|$ (p-values very small) it is essentially known that $\mu_K$ and $\theta_K$ are non-zero. The probability of rejection will effectively depend only on the $K - 1$ other $t$-values. In two dimensions this means that as $t \to \infty$ the boundary function $g (t) \to 1.96$, which is the one-dimensional solution for testing $H_0 : \theta_1 = 0$. In three dimensions it means that the solution must reduce to the $g$-test derived in Section 4. For three dimensions we have used a multivariate spline generalization using barycentric coordinates of the basic spline version of the varying $g$-method in two dimensions.

$^5$The permanent is defined as $\text{perm} (A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i, \sigma(i)}$ with the sum over all permutations $\sigma$ of the numbers $1, \ldots, n$, akin the determinant but without the $\pm$ signature of the permutation.

$^6$The noncentral Chi-distribution with $k$ degrees of freedom and noncentrality parameter $\lambda$ has density:

$$ f (x, k, \lambda) = e^{-(x^2+\lambda^2)/2} x^{k/2} \lambda^{(2-k)/2} I_{k/2-1} (\lambda x) , \quad \lambda > 0, x > 0 $$

with $I_{k/2-1} (\cdot)$ the modified Bessel function of the first kind.
and imposed dimensional coherency. This resulted in a maximum of 0.2% points difference from 5%, and hence not as close to similarity as in two dimensions.

With increasing $K$, the dimension of the integral and the number of knots needed to define the function $g$ increases. The problem becomes progressively involved and suffers from the curse of dimensionality. We can determine the solution in dimension $K$ using the solution in dimension $K - 1$ using an one-dimensional weight function. For $K = 3$ the solution is given in Figure 8. It uses the solution in two dimensions and an optimized weight function which leads to a maximum of 0.13% points difference from 5%. In four dimensions we also determined a solution using this method based on optimized weights and dimensional coherency, but do not report it. Employing this method one could, in principle, recursively determine the $K + 1$ dimensional solution on the basis of the $K$-dimensional solution, but this is left for future research.

6 Empirical Illustration

For a numerical illustration, we consider the recursive model of union sentiment among southern nonunion textile workers as used by Bollen and Stine (1990). The model:

$$
\begin{bmatrix}
y \\
m_2 \\
m_1
\end{bmatrix}
= \begin{bmatrix} 0 & \beta_{12} & \beta_{13} \\
\beta_{23} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
y \\
m_2 \\
m_1
\end{bmatrix}
+ \begin{bmatrix} \tau_{11} & 0 & 0 \\
0 & \alpha_{22} & 0 \\
0 & 0 & \alpha_{32}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
v_1
\end{bmatrix}
+ \begin{bmatrix} u \\
v_1 \\
v_2
\end{bmatrix},
$$

(9)
is a simplified version of McDonald and Clelland (1984) and discussed in some detail by Bollen (1989, p. 82–93). It analyses the direct and indirect effects of tenure and age on union sentiment via deference and/or labor activism. Tenure $x_1$ is measured in log of years working in a particular textile mill and age $x_2$ is measured in years. The variables sentiment towards unions $y$, deference/submissiveness to managers $m_1$, and support for labor activism $m_2$, are measures based on 7, 4, and 9 survey questions respectively. The disturbances ($u, v_1$ and $v_2$) are assumed to be uncorrelated across equations and individuals. When they are normally distributed, ML estimation of the system reduces to OLS applied to each equation separately due to the recursive structure.

We use a selection of 100 observations out of the original 173 and focus on three alternative theories of the indirect effects from age to union sentiment: two competing parallel effects that the age effect is mediated by increased deference in which case $i_1 = \alpha_{32} \beta_{13}$ quantifies the indirect effect. The alternative mediation channel is that activism mediates such that $i_2 = \alpha_{22} \beta_{12}$ is the indirect effect. The third channel is a serial effect that age affects deference, which in turn affects activism, which in turn affects union sentiment such that $i_3 = \alpha_{32} \beta_{12} \beta_{23}$ measures the indirect effect. Figure 9 illustrates the three mediation channels. The OLS estimates of the coefficients of the structural equations and their $t$-statistics are shown in Table 2.

The point estimates of the indirect effects and their $t$-statistics based on the delta method are shown in Table 3. For the $g$-test we need the absolute order statistics, evaluate $g$, and compare. For $H_0 : i_1 = 0$ we observe $|t(\hat{\beta}_{13})| = 1.838 > 1.774 = g(1.902) = g(|t(\hat{\alpha}_{32})|)$, and hence reject. For $H_0 : i_2 = 0$ we have $|t(\hat{\alpha}_{22})| = 2.709 > 1.960 = g(7.120) = g\left(\left|t\left(\hat{\beta}_{12}\right)\right|\right)$ and also reject. Testing the last null hypothesis $H_0 : i_3 = 0$ requires the three-dimensional solution given in Figure 8. We have $|t(\hat{\alpha}_{32})| = 1.902 < 1.96 = g_2(3.582, 7.120) = g_2\left(\left|t\left(\hat{\beta}_{23}\right)\right|, \left|t\left(\hat{\beta}_{12}\right)\right|\right)$ and do not reject.

Table 2: OLS Estimates and $t$-statistics for Union Sentiment Model ($N = 100$)

|        | $\alpha_{32}$ | $\alpha_{22}$ | $\beta_{23}$ | $\beta_{12}$ | $\beta_{13}$ | $\tau_{11}$ |
|--------|---------------|---------------|--------------|--------------|--------------|-------------|
| Estimate | -0.050        | 0.057         | -0.283       | 0.987        | -0.215       | 0.720       |
| $t$-statistic | -1.902        | 2.709         | -3.582       | 7.120        | -1.838       | 1.777       |

Figure 9: Union sentiment mediation graph
| Estimate | Sobel t-statistic | g-test |
|----------|------------------|-------|
| $i_1 = \alpha_{32}\beta_{13}$ | 0.0108 | 1.322 | \(1.838^* > 1.774 = g(|-1.902|)\) |
| $i_2 = \alpha_{22}\beta_{12}$ | 0.0561 | 2.532* | \(2.709^* > 1.960 = g(7.120)\) |
| $i_3 = \alpha_{32}\beta_{12}\beta_{23}$ | 0.0140 | 1.635 | \(1.902 \neq 1.96 = g_2(3.582, 7.120)\) |

Table 3: Estimates, Sobel t-statistics and g-test. * indicates significance at 5%.

The Sobel test with critical value 1.96 concludes that $i_2$ is significant but does not find enough evidence for the $i_1$ mediation channel. The new g-test in contrast, concludes that $i_1$ is also significant. Both $t$-values in this case are smaller than 1.96, so the LR test would not reject either. The two $t$-values are of comparable magnitude and the $g$-test finds a significant mediation effect. For implementation of the $g$-test only the relevant $t$-statistics are required. The absolute values are ordered and the smallest value compared with the value of the $g$-function evaluated at the largest absolute $t$-value. This can be looked up in Table 1 (possibly using linear interpolation) or one can use the spline function detailed in Table 4 and coded in R provided in Appendix E.7

For $i_3$ both tests draw the same conclusion. The three $t$-values involved are not of comparable magnitude and the $t$-statistics for $\beta_{12}$ and $\beta_{23}$ are so large that rejecting the null $H_0 : \alpha_{32}\beta_{12}\beta_{23} = 0$ essentially depends on whether $\alpha_{32}$ is zero. The corresponding absolute $t$-value of 1.90 is too small to warrant such conclusion.

7 Conclusion

This paper has addressed the mediation problem that is empirically extremely important with thousands of applications per year in many different fields including economics, business, marketing, and accounting and has resulted in more than 90,000 citations to a key reference. Theoretically it is an interesting statistical problem which has generated results dating back to Craig (1936) and still continues today with contributions on poor performance of the Wald statistic, construction of similar tests, involving many different hypotheses with singularities in econometrics and elsewhere.

We have proposed a new general method for constructing tests that are as near as possible to similarity. This varying-$g$ method proposes a flexible critical region boundary and minimizes the difference from 5% of the rejection probabilities at a number of points on the boundary of the null hypothesis. Conceptually and practically this is very simple and straightforward to implement. It does not require a choice of mixture distribution, nor the construction of least favorable distributions which may lead to nonsimilar solutions. Numerically it is also attractive in terms of convergence properties and avoids the need for

---

7 The bootstrap is a popular alternative for testing mediation. Because of the asymmetry of the distribution involved this is carried out through alternative confidence intervals of the indirect effect. See e.g. MacKinnon et al. (2004) and Preacher and Hayes (2008). It is well known however that the bootstrap is not valid. Simulations we carried out showed that bootstrap tests for mediation based on generally preferred BCa confidence intervals can have sizes of 8% when $n = 100$ and higher for $n$ smaller.
simulations when the densities are available, as they are in the mediation case.

The new method is applicable to many other testing problems with nuisance parameters. It is remarkable that such a simple method works so well and can deliver big improvements. Even the simplest linear interpolation implementation for the mediation hypothesis delivers big reductions in test bias and increases power by almost 5% points when mediation effects are small.

We have derived a power envelope upper bound for the mediation testing problem that is very tight. Using this result, we were able to construct a test that is optimal within the class of near similar tests. It minimizes the total difference between its power surface and the power envelope bound and results in a point wise difference less than 0.0062 for all alternative parameter points considered. This implies that the test is practically optimal even if a more general class of possible test construction is considered. A power envelope for nonsimilar tests showed that power loss due to the similarity requirement is minimal since the maximum power loss is less than 2% points when power is around 40%.

The optimal \( g \)-test satisfies the size condition. The critical region is strictly larger than the LR and Wald critical regions and is therefore strictly and uniformly more powerful. The classic tests are therefore not admissible. For large values of the coefficients the power difference becomes negligible, but when mediation effects are small or have relatively big standard errors, the power can be close to 5% points higher than these classic tests. This has important consequences for empirical work. It enables researcher to prove mediation effects earlier in circumstances that one could not show mediation before due to extreme conservativeness of standard tests near the origin.

### Appendix A Theory

#### A.1 Elementary Relation

Let \( y = (y_1, \ldots, y_n)' \), \( m = (m_1, \ldots, m_n)' \), \( x = (x_1, \ldots, x_n)' \), be vectors of observables in deviations from their means such that \( \bar{y} = 0 \), \( \bar{m} = 0 \), \( \bar{x} = 0 \) and disturbance vectors \( u = (u_1, \ldots, u_n)' \), \( v = (v_1, \ldots, v_n)' \). The model is then:

\[
y_i = \tau x_i + \beta m_i + u_i, \quad (10)
\]

\[
m_i = \alpha x_i + v_i, \quad (11)
\]

and the restricted version of equation (10) with \( \beta = 0 \) equals:

\[
y_i = \tau^* x_i + w_i. \quad (12)
\]

The claim \( \hat{\tau}^* = \hat{\tau} + \hat{\alpha} \hat{\beta} \) follows from a standard exercise to relate restricted and unrestricted OLS estimators:

\[
\hat{\tau}^* = (x'x)^{-1} x'y = (x'x)^{-1} x' \left( x\hat{\tau} + m\hat{\beta} + \hat{u} \right) = \hat{\tau} + (x'x)^{-1} x'm\hat{\beta} + (x'x)^{-1} x'\hat{u},
\]

and \( \hat{\alpha} = (x'x)^{-1} x'm \) is the OLS estimator in equation (11) and \( x'\hat{u} = 0 \) since \( \hat{u} \) are the OLS residuals from equation (10) and orthogonal to \( x \).
The parameter relation \( \tau^* - \tau = \alpha \beta \) follows by substituting (11) in (10):

\[
y_i = \tau x_i + \beta m_i + u_i = (\tau + \beta \alpha) x_i + (\beta v_i + u_i) = \tau^* x_i + w_i.
\]

It follows that \( H_0 : \alpha \beta = 0 \Leftrightarrow H_0 : \tau^* = \tau \)

### A.2 Likelihood

The joint density of \((y, m)\) given \(x\) is \(f(y, m| x) = f(y| m, x) f(m, x)\), and according to the model:

\[
y_i|m_i, x_i \sim N(\tau x_i + \beta m_i, \sigma_{11}), \\
m_i|x_i \sim N(\alpha x_i, \sigma_{22}).
\]

Hence the log-likelihood:

\[
\ell = \ell(\tau, \beta, \sigma_{11}, \alpha, \sigma_{22}) = \log f(y, m| x; \tau, \beta, \sigma_{11}, \alpha, \sigma_{22}) = \log f(y| m, x; \tau, \beta, \sigma_{11}) + \log f(m| x; \alpha, \sigma_{22}),
\]

for \(n\) independent observations equals equation (4) which can be written as:

\[
\ell \propto -\frac{1}{2\sigma_{11}}y'y + \frac{\tau}{\sigma_{11}}y'x + \frac{\beta}{\sigma_{11}}y'm - \left(\frac{\tau \beta}{\sigma_{11}} - \frac{\alpha}{\sigma_{22}}\right)x'm - \frac{1}{2}\left(\frac{\beta^2}{\sigma_{11}} + \frac{1}{\sigma_{22}}\right)m'm + \\
- \left(\frac{\tau^2}{2\sigma_{11}} - \frac{\alpha^2}{2\sigma_{22}}\right)x'x - \frac{n}{2}\log(\sigma_{11}\sigma_{22})
\]

\[
= \eta|r - \kappa(\eta), \quad \text{with:}
\]

\[
\eta = \left(-\frac{1}{2\sigma_{11}}, \frac{\tau}{\sigma_{11}}, \frac{\beta}{\sigma_{11}}, -\left(\frac{\tau \beta}{\sigma_{11}} - \frac{\alpha}{\sigma_{22}}\right), -\frac{1}{2}\left(\frac{\beta^2}{\sigma_{11}} + \frac{1}{\sigma_{22}}\right)\right)',
\]

\[
r = (y' y, y' x, y' m, x' m, m'm)',
\]

and \(\kappa\) some function of \(\eta\) and \(x' x\) which is fixed. Since \(\dim(\eta) = \dim(r)\) the model is a full exponential model of dimension five following the Koopman-Fisher-Darmois theorem (see van Garderen (1997)) and \(r\) is a complete sufficient statistic. The score \(s(\alpha, \beta, \delta, \sigma_{11}, \sigma_{22}) = s = (s_1', s_2')'\) is analogous to the scores of the two separate regression models since \((\tau, \beta, \sigma_{11})\) appears in the first equation only, and \((\alpha, \sigma_{22})\) appears in the second equation only. So:

\[
s = \begin{pmatrix}
(y-\tau x-\beta m)'x \\
(y-\tau x-\beta m)'m
\end{pmatrix}_{\sigma_{11}}^{-1}
\begin{pmatrix}
(y-\tau x-\beta m)'y \\
(y-\tau x-\beta m)'(y-\tau x-\beta m)
\end{pmatrix}_{\sigma_{11}}^{-1}
- \frac{n}{2\sigma_{11}} \begin{pmatrix}
(m-\alpha x)'x \\
(m-\alpha x)'(m-\alpha x)
\end{pmatrix}_{\sigma_{22}}^{-1} - \frac{n}{2\sigma_{22}}
\]

and the Maximum Likelihood Estimator (MLE) equals the MLE for the two equations separately:

\[
\begin{pmatrix}
\hat{\tau} \\
\hat{\beta}
\end{pmatrix} = ((x : m)'(x : m))^{-1}(x : m)' y; \quad \hat{\sigma}_{11} = \frac{1}{n} y' M_X y; \\
\hat{\sigma}_{22} = \frac{1}{n} m'M_x m,
\]

with \(M_A = I - A (A'A)^{-1} A'\) and \(X = [x : m]\) an \(n \times 2\) matrix. The MLE is minimal sufficient and complete because it is a bijective transformation of \(r\) which is a minimal sufficient and complete statistic.
A.3 Classic Tests

Wald test. Under $H_0 : \alpha \beta = r(\alpha, \beta) = 0$. Then $R(\alpha, \beta) = \frac{\partial r(\alpha, \beta)}{\partial (\alpha, \beta)} = (\beta, \alpha)'$ and evaluated at the (unrestricted) MLE equals $R(\hat{\alpha}, \hat{\beta}) = (\hat{\beta}, \hat{\alpha})'$. The Wald test therefore becomes:

$$W = \hat{\alpha} \hat{\beta} \left( \begin{pmatrix} \hat{\beta} \\ \hat{\alpha} \end{pmatrix} \right)' \left( \begin{pmatrix} \sigma^2_{\hat{\beta}} & 0 \\ 0 & \sigma^2_{\hat{\alpha}} \end{pmatrix} \right)^{-1} \left( \begin{pmatrix} \hat{\beta} \\ \hat{\alpha} \end{pmatrix} \right) \hat{\alpha} \hat{\beta}$$

$$= \frac{\hat{\alpha}^2 \hat{\beta}^2}{\hat{\alpha}^2 \sigma^2_{\hat{\beta}} + \hat{\beta}^2 \sigma^2_{\hat{\alpha}}} (\sigma_{\hat{\alpha}} \sigma_{\hat{\beta}})^{-1}$$

$$= \frac{T_1^2 T_2^2}{T_1^2 + T_2^2}$$

The Sobel test equals $\sqrt{W}$ and is usually expressed as the square root of the first term in the second line: $\sqrt{\frac{\hat{\alpha} \hat{\beta}}{\hat{\alpha}^2 \sigma^2_{\hat{\beta}} + \hat{\beta}^2 \sigma^2_{\hat{\alpha}}}}$.

LR test. The maximum value of the log-likelihood can be expressed in terms of the OLS residual sum of squares in the usual way for the first and second equation, $RSS_1$ and $RSS_2$ respectively:

$$\ell \left( \hat{\tau}, \hat{\beta}, \hat{\sigma}_{11}, \hat{\alpha}, \hat{\sigma}_{22} \right) \propto -\frac{1}{2 \sigma_{11}} \sum_{i=1}^{n} \left( y_i - \hat{\tau}x_i - \hat{\beta}m_i \right)^2 - \frac{n}{2} \log (\hat{\sigma}_{11}) +$$

$$\frac{1}{2 \sigma_{22}} \sum_{i=1}^{n} \left( m_i - \hat{\alpha}x_i \right)^2 - \frac{n}{2} \log (\hat{\sigma}_{22})$$

$$= -\frac{n}{2} - \frac{n}{2} \log (RSS_1/n) - \frac{n}{2} - \frac{n}{2} \log (RSS_2/n).$$

Denote the restricted residual sums of squares by $\tilde{RSS}_1$ when $\beta = 0$, $\tilde{RSS}_2$ when $\alpha = 0$, and the restricted maximized log-likelihoods by:

$$\ell_{\alpha=0} \left( \hat{\tau}, \hat{\beta}, \hat{\sigma}_{11}, 0, \hat{\sigma}_{22} \right) \propto -\frac{n}{2} - \frac{n}{2} \log (\tilde{RSS}_1/n) - \frac{n}{2} - \frac{n}{2} \log (\tilde{RSS}_2/n),$$

$$\ell_{\beta=0} \left( \hat{\tau}, 0, \hat{\sigma}_{11}, \hat{\alpha}, \hat{\sigma}_{22} \right) \propto -\frac{n}{2} - \frac{n}{2} \log (\tilde{RSS}_1/n) - \frac{n}{2} - \frac{n}{2} \log (\tilde{RSS}_2/n).$$

The LR test of the full model with five parameters, against the model with the single restriction $\beta = 0$ equals:

$$LR_{\beta=0} = 2 \left( -\frac{n}{2} \log (RSS_1/n) + \frac{n}{2} \log \left( \tilde{RSS}_1/n \right) \right) = n \log \left( 1 + \frac{T_2^2}{n} \right)$$

since

$$\tilde{RSS}_1 = \hat{\beta}' m (m' M_x m)^{-1} m \hat{\beta} + RSS_1$$ and

$$T_2^2 = \hat{\beta}' m (m' M_x m)^{-1} m \hat{\beta} / \hat{\sigma}_{11} = \left( \tilde{RSS}_1 - RSS_1 \right) / (RSS_1/n).$$
Analogously the LR test for $\alpha = 0$ equals:

$$LR_{\alpha=0} = n \log \left( 1 + \frac{1}{n} T_1^2 \right).$$

The likelihood ratio test for $H_0 : \alpha = 0$ and/or $\beta = 0$ uses the maximized log-likelihood under the alternative (the same in both cases) and under the null, which means minimizing over $LR_{\alpha=0}$ and $LR_{\beta=0}$ and hence:

$$LR = \min \{ LR_{\alpha=0}, LR_{\beta=0} \},$$

which is equivalent to rejecting for large values of:

$$\min \{ T_1^2, T_2^2 \} \text{ or } \min \{ |T_1|, |T_2| \}.$$

**LM tests.** The score version of the LM statistic:

$$LM = s \left( \tilde{\lambda} \right)' I^{-1}_\lambda s \left( \tilde{\lambda} \right)$$

requires the score vector evaluated under the null, but there are three cases: (i) $\alpha = 0 \land \beta \neq 0$ (ii) $\beta = 0 \land \alpha \neq 0$ (iii) $\alpha = 0 \land \beta = 0$

$$s \left( \tilde{\lambda}_{\alpha=0} \right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\tilde{v}' x}{\tilde{v}' \tilde{v}/n} \end{pmatrix};
\quad s \left( \tilde{\lambda}_{\beta=0} \right) = \begin{pmatrix} 0 \\ \tilde{u}' x_m / \tilde{u}' \tilde{u}/n \\ 0 \\ 0 \\ 0 \\ \tilde{v}' x/m \end{pmatrix};
\quad s \left( \tilde{\lambda}_{\alpha=0, \beta=0} \right) = \begin{pmatrix} 0 \\ \tilde{u}' x_m / \tilde{u}' \tilde{u}/n \\ 0 \\ \tilde{v}' x/m \tilde{v}' \tilde{v}/n \end{pmatrix}. $$

the inverse information matrix equals:

$$I^{-1}_\lambda = \begin{pmatrix}
\frac{1}{n} \sigma_{11} \left( 1 + \frac{n \sigma_{22}}{x' x} \right) & -\frac{1}{n} \sigma_{11} \sigma_{22} & 0 & 0 & 0 \\
-\frac{1}{n} \sigma_{22} & \frac{1}{n} \sigma_{11} & 0 & 0 & 0 \\
0 & 0 & \frac{2 \sigma^2_{11}}{n} & 0 & 0 \\
0 & 0 & 0 & \frac{\sigma_{22}}{x' x} & 0 \\
0 & 0 & 0 & 0 & \frac{2 \sigma^2_{22}}{n} \\
\end{pmatrix}. $$

Hence the three score versions of the LM test equal:

$$LM_{\alpha=0} = n^2 \left( \tilde{v}' x / \tilde{v}' \tilde{v} \right)^2 \left( \tilde{v}' \tilde{v}/n \right) = n \frac{x'_m x' x_m}{x' x x'_m x_m},$$

$$LM_{\beta=0} = n^2 \left( \tilde{u}' x_m / \tilde{u}' \tilde{u} \right)^2 \left( \tilde{u}' \tilde{u}/n \right) = n \frac{\tilde{u}' x_m x'_m \tilde{u}}{\tilde{u}' \tilde{u} \tilde{v}' \tilde{v}},$$

$$LM_{\alpha=0, \beta=0} = n \frac{x'_m x' x_m}{x' x x'_m x_m} + n \frac{\tilde{u}' x_m x'_m \tilde{u}}{\tilde{u}' \tilde{u} \tilde{v}' \tilde{v}}.$$
Appendix B  Proof of Theorem 1

The proof is slightly more transparent in terms of the acceptance region. The probability of not rejecting $H_0$ should equal 0.95 uniformly over $H_0$: $P[AR_g | \forall \alpha \in \mathbb{R} \land \beta = 0] = 0.95 = P[AR_g | \forall \beta \in \mathbb{R} \land \alpha = 0]$. Without loss of generality we set $\beta = 0$.

$$P[AR_g | \alpha] = P[|T_1| < g(|T_2|) \cup |T_2| < g(|T_1|) | \beta = 0 \land \alpha \in \mathbb{R}].$$

Under $H_0: \beta = 0$ and $\alpha \in \mathbb{R}: T_1 \sim N(\mu, 1)$ with $\mu = \frac{\alpha}{\sigma_\alpha}$ and $T_2 \sim N(0, 1)$. By independence and symmetry we have, with $\phi(\cdot)$ denoting the standard normal density (see also Figure 2 for the areas of integration):

$$0.95 = \int_{-\infty}^{+\infty} \phi(t_1 - \mu) \left[ \int_0^{g(t_1)} \phi(t_2) + \int_{g^{-1}(t_1)}^{+\infty} \phi(t_2) \right] dt_2 dt_1 =$$

$$0.95 = 2 \int_{-\infty}^{+\infty} \phi(t_1 - \mu) \left[ \Phi(g(t_1)) - \Phi(g^{-1}(t_1)) + \frac{1}{2} \right] dt_1.$$

This implies restrictions on $g(\cdot)$. Define $F(t) = 2 \cdot \left[ \Phi(g(t)) - \Phi(g^{-1}(t)) + \frac{1}{2} - 0.95/2 \right]$ then the restrictions become:

$$0 = \int_{-\infty}^{+\infty} \phi(t - \mu) F(t) dt \ \forall \mu \in \mathbb{R}.$$

The normal distribution $N(\mu, 1)$ is a one parameter full exponential family and therefore complete. Hence $F(T) \equiv 0$ is the only function with expectation 0 for all values of $\mu$. Consequently $g(t)$ must satisfy:

$$\Phi(g(t)) - \Phi(g^{-1}(t)) = -0.025 \ \forall t \in \mathbb{R}$$

But $g(0) = 0$ implies $g^{-1}(0) = 0$ and hence $\Phi(g(0)) - \Phi(g^{-1}(0)) = 0 \neq -0.025$ and no similar boundary exists.

Notes

1. The proof can be extended explicitly to a function $g(t)$ that is defined for $t \geq L$ only, and undefined on $[0, L)$. Implicitly this is already covered by defining $g(t) = 0 \ \forall t \in [0, L)$ since this horizontal line segment contains no probability to contribute to the NRP.

2. We use 5% significance throughout but it is immediate from the proof that there is no significance level for which there exists a similar boundary function, apart from two trivial exceptions. A size of 0% would yield $g(t) = t$ and $AR = \mathbb{R}^2$ such that the test would never reject. The other trivial solution is $g(t) = 0$ and defining $g^{-1}(0) = \infty$ accordingly, which is a test that always rejects and leading to an NRP of 100% for all parameter values.
Appendix C  Invariance

When testing the no-mediation hypothesis $H_0 : \alpha \beta = 0$, the parameters $\tau, \sigma_{11}, \sigma_{22}$ are nuisance parameters. Their values have no influence on whether the null is true or not and therefore we want a test that is invariant with respect to an appropriate group of transformations that leaves the relevant distributions and hypotheses invariant. All distributions are conditionally on $x$ since it is strictly exogenous, but $m$ is a random variable that depends on $x$, and $y$ depends on both $x$ and $m$. In the notation conditionality is implicit on $x$, but explicit if conditional on $m$. So conditional on $x$ and under the normality assumption of $(u, v)$ we have the common OLS results, with all variables in deviation from their means:

$$
\begin{pmatrix}
\hat{\tau} \\
\hat{\beta}
\end{pmatrix}
| m \sim N\left(\begin{pmatrix}
\tau \\
\beta
\end{pmatrix}, \sigma_{11} (X'X)^{-1}\right), \quad \text{and } s_{11}/\sigma_{11} \sim \chi^2_{n-3}
$$

\[
\hat{\alpha} = (x'x)^{-1} x'm \sim N(\alpha, \sigma_{22} (x'x)^{-1}) \quad \text{and } s_{22}/\sigma_{22} \sim \chi^2_{n-2},
\]

with $X = [x : m]$, $s_{11} = y'M_X y = n \hat{\sigma}_{11}$, $s_{22} = m'M_x m = n \hat{\sigma}_{22}$. The conditional variance of $\begin{pmatrix} \hat{\tau} \\ \hat{\beta} \end{pmatrix}'$ depends on $m$ only through $\hat{\alpha}$ and $s_{22}$ because:

$$
\begin{pmatrix} [x : m]' [x : m] \end{pmatrix}^{-1} = \frac{1}{s_{22}} (x'x)^{-1} \begin{bmatrix} m'm & -x'm \\ -m'x & x'x \end{bmatrix} = \begin{bmatrix} (x'x)^{-1} + \frac{\hat{\alpha}^2}{s_{22}} & -\frac{\hat{\alpha}}{s_{22}} \\ -\frac{\hat{\alpha}}{s_{22}} & \frac{1}{s_{22}} \end{bmatrix},
$$

using:

$$
|X'X| = x'xm'm - x'mm'x = x'(m'm - m'x (x'x)^{-1} x'm) = x'xm'M_x m = x'x \cdot s_{22},
$$

$$
m'm = s_{22} + m'x (x'x)^{-1} x'm = s_{22} + x'x (x'x)^{-1} x'm = s_{22} + x'x \hat{\alpha}^2,
$$

$$
x'm = x'x \hat{\alpha}.
$$

This implies that $s_{11}$ depends on $m$ only through $\hat{\alpha}$ and $s_{22}$ since $s_{11} = y'M_X y = y'y - (\hat{\beta}, m'x m'm)' (\hat{\beta}, m'x m'm)$. Hence $\frac{s_{11}}{\sigma_{11}} | m = \frac{s_{11}}{\sigma_{11}} | \hat{\alpha}, s_{22}, x \sim \chi^2_{n-3}$. Conditional on $m$, $s_{11}$ is also distributed independently of $\begin{pmatrix} \hat{\tau}, \hat{\beta} \end{pmatrix}$. Further note that conditional on $m$, or $\hat{\alpha}, s_{22}$, the distribution of $\hat{\beta} | \hat{\alpha}, s_{22} \sim N(\beta, \sigma_{11}/s_{22})$ and $\hat{\tau} | \hat{\alpha}, s_{22}, \hat{\beta} \sim N\left(\tau - \hat{\alpha} \left(\hat{\beta} - \beta\right), \sigma_{11} (x'x)^{-1}\right)$.

Writing the joint density of the sufficient statistics as product of conditional and marginal distributions we obtain the representation:

\[
f(\hat{\tau}, \hat{\beta}, s_{11}, \hat{\alpha}, s_{22}) = N\left(\tau - \hat{\alpha} \left(\hat{\beta} - \beta\right), \sigma_{11} (x'x)^{-1}\right) \times N(\beta, \sigma_{11}/s_{22}) \times \left(\sigma_{11} \chi^2_{n-3}\right)
\times N(\alpha, \sigma_{22} (x'x)^{-1}) \times \left(\sigma_{22} \chi^2_{n-2}\right)
\]

which is equivalent to the likelihood which was given in logs in equation (4).

The transformations $s_{11} \rightarrow a_1 s_{11}$ and $s_{22} \rightarrow a_2 s_{22}$ with $a_1, a_2 > 0$ leave the joint density of $(s_{11}, s_{22})$ in the same family with $(\sigma_{11}, \sigma_{22})$ replaced by $(a_1 \sigma_{11}, a_2 \sigma_{22})$ and have no

---

[8]Most invariance results in this section are in collaboration and thanks to Hillier (2019).
bearing on the hypothesis under test. The same \( \sigma_{11} \) and \( \sigma_{22} \) are present in the other components and we need to transform the remaining variables accordingly as: \( \hat{\alpha} \to \sqrt{a_2} \hat{\alpha} \) and \( \hat{\beta} \to \sqrt{a_1/a_2} \hat{\beta} \) with densities: \( \sqrt{a_2} \hat{\alpha} \sim N(\sqrt{a_2} \alpha, a_2 \sigma_{22} (x'x)^{-1}) \) and \( \sqrt{a_1/a_2} \hat{\beta} \sim N(\sqrt{a_1/a_2} \beta/a_2 \sigma_{11} / (a_2 \sigma_{22}) (x'x)^{-1}) \) respectively. Finally, since \( \tau \) is not involved in the inference problem, we may transform \( \hat{\tau} \to \sqrt{a_1} (\hat{\tau} + a_0) \) so that:

\[
\sqrt{a_1} (\hat{\tau} + a_0) \mid \sqrt{a_2} \hat{\alpha}, \sqrt{a_1/a_2} \hat{\beta}, a_2 \sigma_{22} \sim N \left( \sqrt{a_1} (\tau + a_0) - \sqrt{a_2} \hat{\alpha}, \sqrt{a_1/a_2} \left( \hat{\beta} - \beta \right), a_1 \sigma_{11} (x'x)^{-1} \right),
\]

which has the same form as before the transformation.

These transformations preserve the family of distributions for the sufficient statistics (and MLEs), and transform the mediation effect as \( \alpha \beta \to \sqrt{a_1} \alpha \beta \). They do not change the null hypothesis being true or false, i.e. \( H_0 \) is true before iff it is true after the transformation. We may therefore state:

**Proposition 10** The testing problem is invariant under the group \( K \) of transformations acting on \((\hat{\tau}, \hat{\beta}, s_{11}, \hat{\alpha}, s_{22})\) defined by

\[
\left( \hat{\tau}, \hat{\beta}, s_{11}, \hat{\alpha}, s_{22} \right) \to \left( \sqrt{a_1} (\tau + a_0), \sqrt{a_1/a_2} \hat{\beta}, a_1 s_{11}, \sqrt{a_2} \hat{\alpha}, a_2 s_{22} \right),
\]

\[a_0 \in \mathbb{R}, \ a_1, a_2 \in \mathbb{R}^+.\]

The induced group of transformations \( \bar{K} \) acting on the parameter space is

\[
(\tau, \beta, \sigma_{11}, \alpha, \sigma_{22}) \to \left( \sqrt{a_1} (\tau + a_0), \sqrt{a_1/a_2} \beta, a_1 \sigma_{11}, \sqrt{a_2} \alpha, a_2 \sigma_{22} \right)
\]

**Proposition 11** A maximal invariant statistic under the group of transformations \( K \) is the vector of t-statistics:

\[
T = (T_1, T_2)' = \left( \hat{\alpha} / \sqrt{\frac{1}{n - 2} s_{22} (x'x)^{-1}}, \ \hat{\beta} / \sqrt{\frac{1}{n - 3} s_{11}/s_{22}} \right)'.
\]

A parameter-space maximal invariant under the induced group \( \bar{K} \) is

\[
\mu = (\mu_1, \mu_2)' = \left( \alpha / \sqrt{\sigma_{22} (x'x)^{-1}}, \ \beta / \sqrt{\frac{\sigma_{11}}{\sigma_{22}/n}} \right)'.
\]

The distribution of \((T_1, T_2)'\) depends only on \((\mu_1, \mu_2)'\).

**Proof.** The transformations on \( \hat{\tau} \) are transitive, so no invariant test can depend on \( \hat{\tau} \). We will therefore restrict further analysis to the four remaining statistics \((\hat{\beta}, s_{11}, \hat{\alpha}, s_{22})\) and use \( k \) and \( \bar{k} \) to denote the transformations restricted to these four statistics. Invariance of \((T_1, T_2)\) follows immediate upon substitution. Now \( T = (T_1, T_2) \) is a maximal invariant
if $T_1(\hat{\beta}, \hat{\alpha}, s_{11}, s_{22}) = T_1(\hat{\beta}, \alpha, s_{11}, s_{22})$ and $T_2(\hat{\beta}, \hat{\alpha}, s_{11}, s_{22}) = T_2(\hat{\beta}, \alpha, s_{11}, s_{22})$ implies that there exists a group element $k$ such that $(\hat{\beta}, \hat{\alpha}, s_{11}, s_{22}) = k(\beta, \alpha, s_{11}, s_{22})$:

$$T_1 : \frac{\hat{\alpha}}{\sqrt{\frac{1}{n-2} s_{22}}} = \frac{\alpha}{\sqrt{\frac{1}{n-2} s_{22}}} \Rightarrow \hat{\alpha} = \frac{\sqrt{s_{22}}}{s_{22}} \alpha = \sqrt{a_2} \alpha$$

$$T_2 : \frac{\hat{\beta}}{\sqrt{\frac{1}{n-3} s_{11} / s_{22}}} = \frac{\beta}{\sqrt{\frac{1}{n-3} s_{11} / s_{22}}} \Rightarrow \hat{\beta} = \sqrt{\frac{s_{11}}{s_{22}}} \beta = \sqrt{a_1 / a_2} \beta,$$

and therefore $a_1 = s_{11} / s_{11}$ and $a_2 = s_{22} / s_{22}$. The two values $a_1$ and $a_2$ give the correct transformation for $\hat{\alpha}$ and $\hat{\beta}$ and the same holds for $s_{11}$ and $s_{22}$: $\hat{s}_{11} = a_1 s_{11}$ and $\hat{s}_{22} = a_2 s_{22}$. So there is indeed a group element $k$ such that $(\hat{\beta}, \hat{\alpha}, \hat{s}_{11}, \hat{s}_{22}) = k(\beta, \alpha, s_{11}, s_{22})$. The same argument applies to the parameter space. The last statement is a well-known property of maximal invariants.

Note that $(T_1, T_2)$ are the basic $t$-statistics for testing $\alpha = 0$ and $\beta = 0$ when treating the two equations separately and estimating by OLS. The estimated standard error for $\hat{\alpha}$ is the standard formula $\sqrt{\frac{1}{n-2} s_{22} (x'x)^{-1}}$ and, using the Frisch-Waugh theorem, the estimated standard error for $\hat{\beta}$ conditional on $m$ and $x$ is $\sqrt{\frac{1}{n-3} s_{11} (m'Mx)^{-1}} = \sqrt{\frac{1}{n-3} s_{11} / s_{22}}$.

These exact invariance results provide a strong justification for restricting attention to the two $t$-statistics for any sample size, finite or asymptotically, since it is natural to restrict the problem to procedures that are scale invariant and do not depend on $\tau$. The testing problem has further symmetries. The problem is invariant to changing the signs (reflections) of $T_1$ and $T_2$ or permuting them. This leads to maximal invariants with a sample and parameter space that is only part of $\mathbb{R}^K$.

**Proof.** (of Lemma 8) $\begin{Bmatrix} |T|_{(1)}, \ldots, |T|_{(K)} \end{Bmatrix}$ is obviously invariant to changes in sign and permutation as a consequence of the absolute values and subsequent sorting. It is a maximal invariant because any two $T$ and $\tilde{T}$ such that $\begin{Bmatrix} |T|_{(1)}, \ldots, |T|_{(K)} \end{Bmatrix} = \begin{Bmatrix} |\tilde{T}|_{(1)}, \ldots, |\tilde{T}|_{(K)} \end{Bmatrix}$ can only hold if $\tilde{T}$ is a permutation of $T$ with a number of sign changes. Hence there will exist a transformation $h = h_1 \cdot h_2 \in G_1 \times G_2$ s.t. $\tilde{T} = h \cdot T$. The same argument holds for $\begin{Bmatrix} |\mu|_{(1)}, \ldots, |\mu|_{(K)} \end{Bmatrix}$ since the group of transformations on the parameter space is the same as on the sample space. That the distribution of $\begin{Bmatrix} |T|_{(1)}, \ldots, |T|_{(K)} \end{Bmatrix}$ depends only on $\begin{Bmatrix} |\mu|_{(1)}, \ldots, |\mu|_{(K)} \end{Bmatrix}$ is again a property of maximal invariant. Lemma 9 gives an explicit expression that further shows that the distribution is invariant under the $G_1 \times G_2$. □

**Proof.** (of Lemma 9) The absolute value of the normal variate $T_k$ with mean $\mu_k$ and variance 1 follows a noncentral Chi-distribution with one degree of freedom. The $K$ distributions $\chi \left( |t|_{(k)}, |\mu|_{(k)} \right)$ are independent. The result is then a direct application of Vaughan and Venables (1972, eq. 6). □
Appendix D Algorithms

The construction of the optimal g-test is in two steps. The first step is a basic implementation of the general varying-g method. This generates a near similar test that deviates less than 0.01% points from 5%. We use this $\epsilon$ as a starting value for determining an upper bound to the power envelope. The second step is using this upper bound to derive an optimal g-test that minimizes the distance between the power surface and the power envelope for tests in $\Gamma_{\epsilon}^{\infty}$. Implementation of the varying-g method:

**Basic g-function algorithm**

1. Define $g(\cdot)$ nonparametrically as a linear spline defined by $J+2$ knots $\{(t^{(j)}, g^{(j)})\}_{j=0}^{J+1}$, i.e. by $J+2$ values $g^{(j)}$ on a regular grid of points $t^{(j)}$. The first and last knots are fixed at $(0, 0)$ and $(2.5, z_{0.025})$ respectively, so there are $J$ knots to be chosen. For points $t$ not on the grid $g(t)$ is obtained by linear interpolation and $g(2.5) \approx 1.96$ for $t > 2.5$.

2. The criterion function $Q(g)$ is the accumulated NRP deviation from 5% as measured by the asymmetric loss function $q$ over a grid of points $\{\mu^{(i)}\}_{i=1}^{\Upsilon_0}$ with $\Upsilon_0 > J$ and $\mu^{(1)} = 0$:

$$Q(g) = \sum_{i=1}^{\Upsilon_0} q(NRP_g(\mu^{(i)}) - 0.05), \quad \text{with}$$

$$q(x) = \begin{cases} 
-5x & : x \leq 0 \\
100x & : x > 0 
\end{cases}$$

$$NRP_g(\mu) = P[T \in CR_g | \mu_1 = 0, \mu_2 = \mu \geq 0]$$

3. Minimize $Q(g)$ by varying $g(\cdot)$:

   (a) Initialize $g(\cdot)$ with knots $\{(0, 0), (0.98, 0.98), (1.96, 1.96), (2.5, 1.96)\}$ corresponding to the LR boundary. The first and last knot are fixed and the middle two are varied when optimizing $Q(g)$.

   (b) For given $g(\cdot)$ calculate the NRPs by numerical integration for the grid of $\Upsilon_0$ noncentrality parameter points $\{(0, \mu^{(i)})\}_{i=1}^{\Upsilon_0}$ under the null, with $\Upsilon_0 \geq J$ and calculate $Q(g)$.

   (c) Vary $g(\cdot)$ by changing $J$ knots and minimize the criterion function $Q(g)$, subject to:

      i. $0 \leq g^{(j+1)} - g^{(j)} < \delta$ : monotonicity and limited increase

      ii. $g(t) \leq t$ : logical restriction since maximal invariant is absolute order statistic

      iii. $g^{(J+1)} = z_{0.025}$ : dimensional coherence requires reduction to one dimensional solution (see Section 5)

   (d) Increase the number of knots $J$ and iterate until convergence.
Comments

1. The grid points \( \{ t^{(j)} \}_{j=1}^{J} \) are chosen equally spaced between 0 and \( t' \). The first and last knot, \( (0, 0) \) and \( (t^{(J+1)}), g^{(J+1)} = z_{0.025} \) remain fixed. For the illustration we have chosen \( t^{(1)} = 1.96 \) and \( t^{(J+1)} = 2.5 \). For large enough \( |T|_2 \) it is essentially known that \( \beta \neq 0 \) and the rejection depends only on whether \( \alpha = 0 \) is rejected. The corresponding 5% critical value for \( |T|_1 \) based on the normal distribution is the usual \( z_{0.025} \approx 1.96 \) as \( |T|_2 \rightarrow \infty \).

2. For \( J \) small there are big gains in reducing the deviation from 5% by varying the knots \( \{ t^{(j)}, g^{(j)} \}_{j=1}^{J} \) and also by increasing \( J \), see Figure 3.

3. The number \( \Upsilon_0 \) of \( \mu^{(i)} \) points to check similarity was chosen to be \( \Upsilon_0 = 76 > J \): 60 points equally spaced between 0 and 6, and 16 points equally spaced between 6 and 20. This imposes 152 side conditions. Step 3(c) imposes a further 3\( J \) restrictions approximately for every choice of \( J \), and about 100 when \( J = 32 \).

4. The loss function \( q \) was chosen such that it puts large penalty on positive values of \( (NRP - 0.05) \) that violate the size condition. Even extreme penalties still lead to NRPs that are over 5% for some values of \( \mu \) and therefore renders an invalid (oversized) test. Even though these NRP transgressions are very minor, we address this issue in the optimal test in Section 4 and use the following algorithm.

**Optimal \( g \)-function**

In order to find the optimal \( g \), we minimize the sum of differences between \( g \)'s power surface and the power envelope on a grid of points, subject to the size and \( \epsilon \) similarity conditions. The criterion function further includes a roughness penalty on \( g \) based on numerical second derivatives \( \Delta^2 g \left( t^{(i)} \right) \), and we impose monotonicity \( g \left( t^{(i+1)} \right) \geq g \left( t^{(i)} \right) \) and, since by definition of the absolute order statistic \( |T|_{(1)} \leq |T|_{(2)} \), we logically restrict \( g \) to \( 0 \leq g \left( t \right) \leq t \).

**Optimal \( g \)-function algorithm**

1. Define \( g \left( \cdot \right) \) nonparametrically as a linear spline defined above.

2. Define the criterion function \( Q^*_\epsilon \left( g \right) \) as the accumulated power difference over the triangular grid of points \( M_1 = \left\{ \left( \mu_1^{(\gamma, \kappa)}, \mu_2^{(\gamma, \kappa)} \right) \right\}_{1 \leq \gamma < \kappa \leq \Upsilon_1} : \)

\[
Q^*_\epsilon \left( g \right) = \sum_{\kappa=1}^{\Upsilon_1} \sum_{\gamma \leq \kappa} \pi \left( \mu_1^{(\gamma, \kappa)}, \mu_2^{(\gamma, \kappa)} \right) - P \left[ CR_g \left| \left( \mu_1^{(\gamma, \kappa)}, \mu_2^{(\gamma, \kappa)} \right) \right] \right] + \lambda \sum_{i=1}^{J} \Delta^2 g \left( t^{(i)} \right)
\]

3. Minimize \( Q^*_\epsilon \left( g \right) \) by varying \( g \left( \cdot \right) \):

   (a) Start with \( g \left( \cdot \right) \) equal to the previously determined basic \( g \)-function.

   (b) For given \( g \left( \cdot \right) \) calculate \( Q^*_\epsilon \left( g \right) \) by numerical integration.

   (c) Vary \( g \left( \cdot \right) \) by changing \( J \) knots and minimize the criterion function \( Q^*_\epsilon \left( g \right) \), subject to:

33
The regularization parameter was set to $\lambda = 0.01$.

The basic implementation algorithm solved the optimal $g$-boundary by minimizing $\epsilon$. Once $\epsilon$ is determined, the current algorithm is akin to solving a dual problem that uses $\epsilon$ for the inequality restrictions and maximizes power. It minimizes the total difference from the power envelope.

**Power Envelopes**

We calculate two power envelopes: one for near similar tests in $\Gamma_\epsilon$ and a second for nonsimilar tests. The algorithm for calculating the power envelope is related to Chiburis (2009) and implemented in Julia, see Bezanson et al. (2017), using Gurobi, an optimization package that can handle many side restrictions; see Gurobi Optimization (2019). We maximize power subject to size and near similarity restrictions on a grid of $\Upsilon_0$ parameter points under the null: $0.05 - \epsilon \leq \text{NRP} (\mu_0^i) \leq 0.05$ for $i = 1, \ldots, \Upsilon_0$. The upper bounds ensure correct size, at least for the points considered. The lower bounds constitute the near similarity restriction. The power envelope is obtained by repeating this maximization on a grid of points $(\mu_1, \mu_2)$ under the alternative.

For the nonsimilar power envelope we can discard the lower bound restrictions $0.05 - \epsilon \leq \text{NRP} (\mu_0^i)$. The power can only increase (or remain the same) and the difference between the two different power envelopes is the power loss one suffers from insisting on similarity. This turns out to be less than 2% points and it should be stressed that this overstates the loss since no single test achieves the power envelope.

Denote the parameter space for the ordered absolute noncentrality parameter

$$\Xi = \{ (\mu_1, \mu_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid 0 \leq \mu_1 \leq \mu_2 \} .$$

We will use a bounded (triangular) subset of this octant $\Xi$ defined as

$$\Xi = \{ (\mu_1, \mu_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid 0 \leq \mu_1 \leq \mu_2 \leq \mu_{\text{max}} \}$$

and partitioned it into a null and alternative parameter set

$$\Xi_0 = \{ (\mu_1, \mu_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid 0 = \mu_1 \leq \mu_2 \leq \mu_{\text{max}} \}$$

and

$$\Xi_1 = \{ (\mu_1, \mu_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid 0 < \mu_1 \leq \mu_2 \leq \mu_{\text{max}} \}$$

respectively.

Analogously define the sample space of the maximal invariant/absolute order statistic as

$$\Upsilon = \{ (t_1, t_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t_1 \leq t_2 \} .$$

Very large values of $t_1$ and $t_2$ are of limited interest and for computational purposes we can restrict ourselves to a bounded triangular subset of the sample space: $\overline{\Upsilon} = \{ (t_1, t_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid t_1 \leq t_2 \leq t_{\text{max}} \}$

**Power Envelope Algorithm**

1. Discretize $\Xi_0$ into $\Upsilon_0$ points under $H_0 : M_0 = \{ (0, \mu_0^i) \}^{\Upsilon_0}_{i=1}$.
2. Discretize $\Xi_1$ by choosing a triangular array of $\frac{1}{2} \Upsilon_1 (1 + \Upsilon_1)$ points under $H_1 : M_1 = \left\{ \left( \mu_1^{(\gamma, \kappa)}, \mu_2^{(\gamma, \kappa)} \right) \right\}_{1 \leq \gamma \leq \kappa \leq \Upsilon_1}$

3. Partition $\Upsilon$ into squares $s_{ij}$ with $1 \leq i \leq j \leq J$ such that $\bigcup_{1 \leq i \leq j \leq J} s_{ij} = \Upsilon$ and $s_{ij} \cap s_{kl} = \emptyset \ \forall (i, j) \neq (k, l)$.

4. Under $H_0$ for $\iota = 1, \ldots, \Upsilon_0$ calculate $p_{ij}^{\iota} = \Pr\left[\left( |T|_{(1)} , |T|_{(2)} \right) \in s_{ij} | \left( 0, \mu_0^{(\iota)} \right) \in \Xi_0 \right]

5. For each $1 \leq \gamma, \kappa \leq m$ choose $\mu^{(\gamma, \kappa)} = \left( \mu_1^{(\gamma, \kappa)}, \mu_2^{(\gamma, \kappa)} \right) \in M_1$ under the alternative. For this $\mu$:
   
   (a) Calculate $p_{ij}^{\gamma \kappa} = \Pr\left[\left( |T|_{(1)} , |T|_{(2)} \right) \in s_{ij} | \mu^{(\gamma, \kappa)} = \left( \mu_1^{(\gamma, \kappa)}, \mu_2^{(\gamma, \kappa)} \right) \right]$ for each $s_{ij} \in \Upsilon$
   
   (b) Determine the critical region to maximize the power
   
   $\max_{\left\{ \phi_{ij}^{\gamma \kappa} \right\}_{1 \leq i \leq j \leq J}} \sum_{1 \leq i \leq j \leq J} p_{ij}^{\gamma \kappa} \phi_{ij}^{\gamma \kappa}$

   by selecting indicators $\phi_{ij}^{\gamma \kappa} = 1_{CR}(s_{ij})$ equal to 1 if $s_{ij}$ is part of the critical region, or 0 if part of the acceptance region, subject to the near similarity and size restrictions on the NRPs:

   $0.05 - \epsilon \leq \sum_{1 \leq i \leq j \leq J} p_{ij}^{\mu} \phi_{ij}^{\mu} \leq 0.05$ for $\iota = 1, \ldots, \Upsilon_0$

   **Comments.** Optimizer: Gurobi: $t_{\text{max}} = 11$, each square $s_{ij}$ has lengths 0.01. Hence cardinality of $|\Upsilon| = 285150$. For power calculations we use for $M_1$ a regular grid with $\mu_2 \in \{0.2, 0.4, \ldots, 4\}$, $\mu_1 \in \{0.2, 0.4, \ldots, \mu_2\}$. For size and near similarity restrictions we use $\mu_0 \in \{0.0, 0.1, \ldots, 7.5\}$ and for near similarity $\epsilon = 10^{-5}$.

### Appendix E  \textit{g-Function R Code}

**R Code**

```r
# g <- function(y) {
  SplinePredict <- function(x,BP,coef) {
    if (x>=BP[2]) { idx <- 2 } else { idx <- 1 }
    h <- (x-BP[idx])
    return(coef[idx,1]+coef[idx,2]*h+coef[idx,3]*h^2+coef[idx,4]*h^3)
  }
  z0025 <- 1.9599639845400542355
  BP <- c(0.05, 0.16, 0.2, 1.2, 1.3, 1.5, 2.0, 2.1, 2.2)
  w <- c(0.05, 0.12375880123418455, 0.15698848058671158,
         1.1569884851127599, 1.247794136737431, 1.3715940041517778,
         1.8715940058845608, 1.9440175462611924, z0025)
```

35
coef1 <- rbind(c(0.05, 1.0, -6.0947848621912755, 28.178586075656632), c(0.12375880123418455, 0.6820300048642552, 3.204148542775413, 12.841273273689932, 3.204148542775413, 12.841273273689932))
coef2 <- rbind(c(1.1569884851127599, 1.0, 0.06613363957554252, -9.85568477108435), c(1.247794136737431, 0.7175561847825775, 3.204148542775413, 12.841273273689932, 3.204148542775413, 12.841273273689932))
coef3 <- rbind(c(1.8715940058845608, 1.0, -2.4006862861725504, -3.5695967616429662), c(1.9440175462611924, 0.41277483991620023, 3.204148542775413, 12.841273273689932, 3.204148542775413, 12.841273273689932))

x <- abs(y)
if (x <= BP[1]) { return(x) } else if (x <= BP[3]) { return(SplinePredict(x, BP[1:3], coef1)) } else if (x <= BP[6]) { return(w[3] + (x - BP[3]) / (BP[6] - BP[3]) * (w[4] - w[3])) } else if (x <= BP[7]) { return(SplinePredict(x, BP[4:6], coef2)) } else if (x <= BP[9]) { return(SplinePredict(x, BP[7:9], coef3)) } else { return(z0025) }

| \( t^{(i)} \) | \( t^{(i-1)} \) | \( l_i(t) \) | \( a_i \) | \( b_i \) | \( c_i \) | \( d_i \) |
|---|---|---|---|---|---|---|
| 0.00 | 0.05 | \( l_1(t) \) | 0.000000 | 1.000000 | 0.000000 | 1.000000 |
| 0.05 | 0.16 | \( s_2(t) \) | 0.050000 | 1.000000 | -6.094785 | 28.178586 |
| 0.16 | 0.20 | \( s_3(t) \) | 0.123759 | 0.682030 | 3.204149 | 12.841273 |
| 0.20 | 1.20 | \( l_4(t) \) | 0.156988 | 1.000000 | 0.000000 | 1.000000 |
| 1.20 | 1.30 | \( s_5(t) \) | 1.156988 | 1.000000 | 0.066134 | -9.855685 |
| 1.30 | 1.50 | \( s_6(t) \) | 1.247794 | 0.717556 | -2.890572 | 11.988938 |
| 1.50 | 2.00 | \( l_7(t) \) | 1.371594 | 1.000000 | 0.000000 | 1.000000 |
| 2.00 | 2.10 | \( s_8(t) \) | 1.871594 | 1.000000 | -2.400686 | -3.569597 |
| 2.10 | 2.20 | \( s_9(t) \) | 1.944018 | 0.412775 | -3.471565 | 9.384607 |
| 2.20 | ∞ | constant \( z_{0.025} \) | \[ 0.00 \] | \[ 0.75 \] | \[ 0.00 \] | \[ 0.75 \] |

Table 4: The optimal g function in spline representation.

Coefficients of the linear splines \( l_i(t) \) and clamped cubic splines \( s_i(t) \) :
\[
l_i(t) = a_i + b_i(t - t^{(i-1)})
\]
\[
s_i(t) = a_i + b_i(t - t^{(i-1)}) + c_i(t - t^{(i-1)})^2 + d_i(t - t^{(i-1)})^3
\]

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