The representations of polyadic-like equality algebras

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Abstract

It is proven that Boolean set algebras with unit $V$ of the form $\bigcup_{k \in K} \alpha U_k$ are axiomatizable (i.e. if $V$ is a union of Cartesian products). The axiomatization coincides with that of cylindric polyadic equality algebras (class CPE$_\alpha$). This is an algebraic representation theorem for the class CPE$_\alpha$ by relativized polyadic set algebras in the class Gp$_\alpha$. Similar representation theorems are claimed for the classes strong cylindric polyadic equality algebras (CPES$_\alpha$) and cylindric $m$-quasi polyadic equality algebras ($m$CPE$_\alpha$). These are polyadic-like equality algebras with infinite substitution operators and single cylindrifications. They can be regarded also as infinite transformation systems equipped with diagonals and cylindrifications. No representation theorem or neat embedding theorem has proven for this class of algebras yet, except for the locally finite case. The theorems proven here answer some unsolved problems.

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This is a shortened version of the paper submitted, with the same title. In [7] it is shown that the class of Boolean set algebras with unit $V$, such that $V$ is a union of weak Cartesian spaces, i.e., $V = \bigcup_{k \in K} \alpha U_k^{(pk)}$ is axiomatizable (class Gwp$_\alpha$). The axiomatization thus obtained coincides with that of the transposition algebras (TA$_\alpha$). Transposition algebras are Boolean algebras extended by the abstract transposition operators ($p_{ij}$), single substitutions ($s'_i$), cylindrifications ($c_i$) and the diagonal constants ($d_{ij}$), where $i, j \in \alpha$. These algebras are definitionally equivalent to cylindric quasi polyadic equality algebras (CQE$_\alpha$), i.e., to Boolean algebras with finite substitutions ($s_\tau$, where $\tau$ is a finite transformation on $\alpha$), cylindrifications ($c_i$) and diagonal constants ($d_{ij}$), where $i, j \in \alpha$.

In this paper, it is shown, among others, that the class of Boolean set algebras with unit $V$, such that $V$ is a union of Cartesian spaces, i.e., $V = \bigcup_{k \in K} \alpha U_k$ is axiomatizable (class Gp$_\alpha$). The resulting axiomatization coincides with that of
the cylindric polyadic equality algebras (class CPE_α), see the main representation theorem, Theorem 3. These algebras are Boolean algebras extended by substitutions (s_τ, where τ ∈ α is arbitrary), cylindrifications (c_i) and the diagonal constants (d_{ij}), where i, j ∈ α. The attribute “cylindric” which qualifies the class CPE_α comes from the fact that only single cylindrifications are allowed here (in contrast with polyadic algebras where the operation “quantifier ∃” is defined for any subset of α). This is why, among others, the algebras occurring here are considered as polyadic-like algebras. The difference between the classes CQE_α and CPE_α is in the finite or infinite nature of the transformations occurring in the substitutions.

The techniques used in [7] and here are essentially different. While in [7] finite methods are used (which cannot be used here), now, the neat embedding technique is applied (which cannot be applied to the algebras in CQE_α).

Up until now, we have looked at our subject from the viewpoint of the axiomatizability of certain classes of set algebras (Gwp_α and Gp_α), i.e., from the viewpoint of set theory and geometry. Now, let us consider our subject from the viewpoint of the representation theory of algebraic logic.

The quasi polyadic (CQE_α) and the cylindric polyadic (CPE_α) cases (where the substitutions are finite and infinite, respectively) share several common features besides differing with respect to others.

As for the common properties, neither CQE_α nor CPE_α can be represented in the classical sense (i.e., as subdirect product of polyadic set algebras). Also, in both cases, the representability given here is achieved by relativized set algebras (Gwp_α and Gp_α, respectively), instead of ordinary set algebras. It is also important to emphasize that the commutativity of cylindrifications is not assumed in these classes, either.

Some words about the neat embedding technique applied in the paper. The proof of the main representation theorem (Theorem 3) follows a classical line of thought. The proof consists of two steps. The first step is based on a theorem which establishes that the algebra in question is neatly embeddable (now, the celebrated theorem of Daigneault, Monk and Keisler and its versions are used, see [10], II. Thm. 5.4.17). The second step of the proof is based on the neat embedding theorem Theorem 1 proven here. A neat embedding theorem for a given class says that neatly embeddability is equivalent to representability. With this kind of proofs of the representation theorems, the Henkin style proofs of completeness theorems can be associated in mathematical logic.

A few words about the history of the topic may be in order here. Daigneault, Monk and Keisler proved that polyadic algebras (without equality) are representable in the classical sense, but, polyadic equality algebras, in general, are not ([3], [14]). For polyadic equality algebras and cylindric algebras, the representability was proved only for special, but important classes, e.g., for the class of locally finite algebras (see [8], [9], [10]).

Relativized set algebras were used for representation first in the theory of cylindric algebras. Using an idea of Henkin, Resek and Thompson showed that extending the cylindric axioms by the so called merry-go-round property, cylindric algebras become to be representable by cylindric relativized set algebras.
First, AndrÁeka and Thompson published a proof for this theorem (see [2]), later, the present author gave a simpler axiomatization for the class in question (see [5] and [4]). The method used in AndrÁeka and Thompson’s proof is detailed in [12].

The \( r \)-representation of polyadic equality algebras by relativized set algebras was investigated in [6], [7] and [1]. In [6] it is shown that the background of the merry-go-round property is an axiom of transposition algebras (and it is also a property of polyadic algebras) and Resek and Thompson’s theorem is closely related to transposition algebras. The direct predecessor of the research here is the investigation of the representability of transposition algebras in [7].

We only deal with infinite dimensional algebras because the finite case is an instance of the quasi polyadic case (see [7]). Theorem 1 is a neat embedding theorem, it is the key result to Theorem 3. Theorem 3 states the \( r \)-representability of the classes cylindric polyadic equality algebras (for \( \text{CPE}_\alpha \)) and strong cylindric polyadic equality algebras (class \( \text{CPES}_\alpha \)). This theorem answers affirmatively two classical problems raised in the literature. The one is the representability of infinite transformation systems equipped with diagonals and cylindrifications (see [7] and [3]), and the other is whether the class \( \text{Gp}_\alpha \) is a variety. Theorem 4 is a consequence the representation theorems.

Now we list the important results with the necessary concepts:

The concept of polyadic equality algebra is assumed to be known (see [10], 5.4.1)

**Definition 1** Assume that \( m < \alpha \) and \( m \) is infinite. Given a set \( U \) and a fixed sequence \( p \in {}^\alpha U \), the set

\[
{}^m U(p) = \{ x \in {}^\alpha U : x \text{ and } p \text{ are different at most in } m \text{ many places} \}
\]

is called the \( m \)-weak space (or \( m \)-weak Cartesian space) determined by \( p \) and the base \( U \).

Recall that the finite version (\( m \) is finite) of the above definition is the concept of the weak space, in notation \( {}^\alpha U(p) \) ([10], II. 3.1.2). A unit \( V \) of some \( \mathcal{A} \in \text{Cprs}_\alpha \) can be composed as a disjoint union of certain subsets of \( m \)-weak spaces. These latter subsets are called \( m \)-subunits of \( \mathcal{A} \), the bases of these \( m \)-subunits are called \( m \)-subbases of \( \mathcal{A} \).

**Definition 2** (class \( m\text{Gwp}_\alpha \)) A set algebra in \( \text{Cprs}_\alpha \) is called a generalized \( m \)-weak polyadic relativized set algebra if there are sets \( U_k \) and \( p_k \in {}^\alpha U_k \) such that \( V = \bigcup_{k \in K} {}^m U_k(p_k) \), where \( V \) is the unit. The subclass of \( m\text{Gwp}_\alpha \) such that the disjointness of the \( U_k \)’s is assumed is denoted by \( m\text{Gwp}_{\alpha}^* \).
Definition 3 (class Gpα) A set algebra in Cprsα is called a generalized polyadic relativized set algebra if there are sets U_k such that \( V = \bigcup_{k \in K} U_k \), where V is the unit. The subclass of Gpα such that the disjointness of the U_k’s is assumed is denoted by Gpα.

Definition 4 (CPEα) A cylindric polyadic equality algebra is a polyadic algebra such that instead of the cylindrifications c_\Gamma only single cylindrifications c_i are defined, the cylindrifications are non-commutative and the following weakening of the axiom (P11) is assumed: (P11)' : c_is_\sigma x \leq s_\sigma c_j x, if \{j\} = \sigma^{-1} \{i\}, \{j\} \neq \emptyset, and c_is_\sigma x = s_\sigma x else.

Definition 5 (CPESα) The class of strong cylindric polyadic algebras is a subclass of CPEα such that the cylindrifications are commutative and (P11) is assumed for the single cylindrifications.

A transformation \( \tau \) defined on \( \alpha \) is said to be an \( m \)-transformation if \( \tau i = i \) except for \( m \)-many \( i \in \alpha \). The class of \( m \)-transformations is denoted by \( mT_\alpha \).

Definition 6 (class \( m \)CPEα) A cylindric \( m \)-quasi polyadic equality algebra of dimension \( \alpha \) is an algebra in CPEα such that the transformations \( \tau \) and \( \sigma \) are \( m \)-transformation in the definition of CPEα, i.e., \( \tau, \sigma \in mT_\alpha \).

The following one is a neat embedding theorem for the class \( m \)CPEα \( \cap \) Lm_\alpha:

Theorem 1 If \( A \in m \)CPEα \( \cap \) Lm_\alpha, where \( m \) is infinite, \( m < \alpha \), then \( A \in SNr_\alpha B \) for some \( B \in m \)CPEα+, where \( \varepsilon \) is infinite, if and only if \( A \in Is_{m}Gwp_\alpha \).

The following theorem is the main representation theorem of the paper:

Theorem 3 If \( A \in CPE_\alpha \cup CPES_\alpha \), then \( A \in IsGp_\alpha \).

For the class CPEα, the converse of the theorem also holds, i.e. \( A \in CPE_\alpha \) if and only if \( A \in IsGp_\alpha \).

Now, we state again a neat embedding theorem, a theorem for the class CPESα.

Theorem 4 Assume that \( A \in CPES_\alpha \) and \( \alpha \) is infinite. Then, the following propositions (i) and (ii) are equivalent:

(i) \( A \in SNr_\alpha CPES_{\alpha + \varepsilon} \) for some infinite \( \varepsilon \)

(ii) \( A \in Is(Gp_\alpha \cap Mod\{(C4), (P11)^*\}) \)
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