Mixed Nash equilibria in Eisert-Lewenstein-Wilkens (ELW) games

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Abstract. The classification of all mixed Nash equilibria for the original ELW game is presented. It is based on the quaternionic form of the game proposed by Landsburg (Proc. Am. Math. Soc. 139 (2011), 4423; Rochester Working Paper No 524 (2006); Wiley Encyclopedia of Operations Research and Management Science (Wiley and Sons, New York, 2011)). This approach allows to reduce the problem of finding the Nash equilibria to relatively simple analysis of the extrema of certain quadratic forms.

1. Introduction

We present here the review of results obtained in our paper [1]. Many papers concerning quantum game theory have appeared in recent decades [2],[3],[4],[5],[6],[7],[8],[9],[10],[11],[12],[13],[14],[15],[16],[17],[18],[19],[20],[21],[22],[23],[24],[25],[26],[27],[28],[29],[30],[31],[32],[33],[34],[35],[36],[37],[38],[39],[40],[41],[42],[43],[44],[45],[46],[47],[48],[49],[50],[51],[52],[53]. One of the starting points of the research was a quantum game defined by Eisert-Lewenstein-Wilkens (ELW) [2],[3]. They considered classical two-player (Alice and Bob) two-strategies noncooperative game defined by the relevant payoff matrix given in Table 1; here C and D are the possible strategies of players and $X_0$, $X_1$, $X_2$, $X_3$ are the corresponding outcomes. We will be dealing with the games which obey the conditions: (i) all $X_\alpha$ are distinct, (ii) all twofold sums $X_\alpha + X_\beta$ are distinct as well. The games defined by these two conditions are called generic in Landsburg’s terminology.

Table 1. Payoff matrix of classical game

|       | C    | D    |
|-------|------|------|
| Bob   |      |      |
| C     | $(X_0, X_0)$ | $(X_2, X_1)$ |
| D     | $(X_1, X_2)$ | $(X_3, X_3)$ |

The quantization of classical game begins by ascribing to each player a two-dimensional complex Hilbert space $H$; then the total space of the game is the tensor product $H \otimes H$. The basic vectors in $H$ correspond to two classical strategies

$$|C\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |D\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(1)
Next, we define the initial state of the game
\[ |\Psi_f \rangle \equiv J \left( |C \rangle \otimes |C \rangle \right) \] (2)
where \( J \) is the gate operator (entangler) introducing quantum correlations. The operator \( J \) has the following form
\[ J = \exp \left( -\frac{i\gamma}{2} \sigma_2 \otimes \sigma_2 \right) \] (3)
with \( \gamma \in \langle 0, \frac{\pi}{2} \rangle \) and \( \sigma_2 \) being the Pauli matrix. The form of gate operator is determined by two conditions: the quantum game is symmetric with respect to the exchange of players and the classical pure strategies are contained in the set of pure quantum ones.

Quantum strategies of both players are associated with unitary \( 2 \times 2 \) operators \( U_A \) and \( U_B \) belonging to \( SU(2) \). Defining the final state of the game as
\[ |\Psi_f \rangle = J^+ \left( U_A \otimes U_B \right) J \left( |C \rangle \otimes |C \rangle \right). \] (4)
we can compute Alice and Bob payoffs
\[ \$_A = X_0 \left( |\langle C \rangle \otimes \langle C \rangle | \right)^2 + X_1 \left( |\langle D \rangle \otimes \langle C \rangle | \right)^2 + X_2 \left( |\langle C \rangle \otimes \langle D \rangle | \right)^2 \] (5)
\[ \$_B = X_0 \left( |\langle C \rangle \otimes \langle C \rangle | \right)^2 + X_1 \left( |\langle C \rangle \otimes \langle D \rangle | \right)^2 + X_2 \left( |\langle D \rangle \otimes \langle C \rangle | \right)^2 \] (6)

Summarizing, the properties of the game depend on three elements:

- the choice of payoff matrix; for example, if the classical values of outcomes satisfy \( X_1 > X_0 > X_3 > X_2 \) we obtain the game named Prisoner Dilemma. This game provides a paradox where both players choose to protect themselves at the expense of the other player; as a result, both players find themselves in a worse state then if they had cooperated with each other.

- the value of parameter \( \gamma \) which determines quantum entanglement; for \( \gamma = 0 \) we have classical game whereas for \( \gamma = \frac{\pi}{2} \) the initial state of the game is maximally entangled.

- the choice of the manifold \( S \subset SU(2) \) of allowed strategies. Eisert et al. considered the set of allowed strategies belonging to the twodimensional submanifold of \( SU(2) \) which itself is not a group. In our paper we assume that the set of allowed strategies is the whole \( SU(2) \) group as indicated above.

One of the most important elements of the game is the notion of Nash equilibrium determining the optimal solution of the game. A pair of strategies \( (\mu_0, \nu_0) \) defines a Nash equilibrium iff
\[ \$_A (\mu_0, \nu_0) \geq \$_A (\mu, \nu_0) \] (7)
\[ \$_B (\mu_0, \nu_0) \geq \$_B (\mu_0, \nu) \] (8)
for all strategies \( \mu \) and \( \nu \). Landsburg described the general structure of possible, in general mixed, Nash equilibria for maximally entangled ELW game in the papers \[25\], [37]; [40]. In the present paper, we classify all Nash equilibria for the original ELW game. We use the method proposed by Landsburg with some modifications.

Landsburg proved that all Nash equilibria have a particularly simple form. He showed that any mixed strategy is equivalent to the one supported on at most four orthogonal quaternions and what’s more the analysis of possible mixed strategies reduces to that of degeneracies of highest
eigenvalues of the matrices determining the players payoffs. If we combine the symmetries of
the game with this principle the method of finding and classifying Nash equilibria becomes
transparent.

The paper is organized as follows. In Sec. 2 we present ELW game in terms of quaternion
algebra. Then in Sec. 3 Landsburg’s method of classifying Nash equilibria is described. The
classification of Nash equilibria for the original ELW game is considered in Sec. 4. The last
Section contains conclusions.

2. ELW game in terms of quaternion algebra

The $SU(2)$ group is isomorphic to the group of unit quaternions; therefore, we can rewrite
the ELW game using quaternions. We start with the 4D real vector space spanned by the
orthonormal vectors $e_\alpha$, $\alpha = 0, 1, 2, 3$, $(e_\alpha, e_\beta) = \delta_{\alpha,\beta}$. We have the following multiplication law:

\[ e_0 = 1, \quad e_i^2 = -1, \quad e_i e_j = \varepsilon_{ijk} e_k, \quad i, j, k = 1, 2, 3. \]  

(9)

The product of two quaternions $p = p_\alpha e_\alpha$, $q = q_\alpha e_\alpha$ has form

\[
(pc)_0 = p_0 q_0 - p_i q_i \\
(pc)_i = p_0 q_i + q_0 p_i + \varepsilon_{ijk} p_j q_k
\]

(10)

(summations over latin indices run from 1 to 3).

The group isomorphism is defined by the mapping

\[ U \leftrightarrow p = p_\alpha e_\alpha \]  

(11)

and any matrix $U \in SU(2)$ can be written as

\[ U = p_0 1 - i p_k \sigma_k, \quad p_0^2 + \sum_{k=1}^{3} p_k^2 = 1. \]  

(12)

In what follows we consider the particular case $\gamma = \frac{\pi}{2}$. Then the initial state of the game is
maximally entangled and the role of quantum correlations is the most significant. The strategies
of Alice and Bob in terms of $SU(2)$ matrices have the form

\[ U_A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad |a|^2 + |b|^2 = 1 \]  

(13)

\[ U_B = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1. \]  

(14)

Using the isomorphism (11) the Alice strategy is represented by the quaternion

\[ p = \text{Re}(a) e_0 - \text{Im}(b) e_1 - \text{Re}(b) e_2 - \text{Im}(a) e_3 \]  

(15)

The quaternion corresponding to Bob strategy reads

\[ q = \text{Re}(\alpha) e_0 + \text{Re}(\beta) e_1 + \text{Im}(\beta) e_2 + \text{Im}(\alpha) e_3 \]  

(16)

which corresponds to the alternative isomorphism of $SU(2)$ and unit quaternions

\[ e_1 \leftrightarrow i \sigma_2, \quad e_2 \leftrightarrow i \sigma_1, \quad e_3 \leftrightarrow i \sigma_3. \]  

(17)
The gate operator, defined by eq. (3), takes now the following form
\[ J = \cos\left(\frac{\gamma}{2}\right) e_0 \otimes e_0 + i \sin\left(\frac{\gamma}{2}\right) e_2 \otimes e_2 = \frac{1}{\sqrt{2}} (e_0 \otimes e_0 + i e_2 \otimes e_2). \] (18)

Finally, we can check that the payoff functions of both players, eqs. (5) and (6), take the form
\[ S_A(p, q) = X_0 (pq^{-1})_0^2 + X_1 (pq^{-1})_1^2 + X_2 (pq^{-1})_2^2 + X_3 (pq^{-1})_3^2 \equiv \sum_{\alpha=0}^{3} X_{\alpha} (pq^{-1})_{\alpha}^2 \] (19)
\[ S_B(p, q) = X_0 (pq^{-1})_0^2 + X_2 (pq^{-1})_2^2 + X_1 (pq^{-1})_1^2 + X_3 (pq^{-1})_3^2 \equiv \sum_{\alpha=0}^{3} \tilde{X}_{\alpha} (pq^{-1})_{\alpha}^2 \] (20)
with \( \tilde{X}_{0,3} = X_{0,3} \), \( \tilde{X}_{1,2} = X_{2,1} \).

We see that the payoff functions depend only on the product \( pq^{-1} \). The reason for such a simple form of payoff functions is the form of the stability subgroup which is maximal for the maximally entangled initial state. The stability subgroup \( G_s \) is a set of elements \( g \) belonging to \( SU(2) \times SU(2) \) which do not change the initial state. Any element \( g \in G_s \) has the form \( g = (U, \tilde{U}) \), \( U \in SU(2) \). Therefore, two pairs of strategies, \((U_A, U_B)\) and \((U'_A, U'_B)\), differing only by element \( g \) belonging to the stability subgroup
\[ (U_A, U_B) = (U'_A, U'_B) \cdot g \] (21)
provide to the same final result. The maximal entanglement allows us to write the following decomposition
\[ (U_A, U_B) = (U_A U_B^T, I) \cdot (U_B, U_B). \] (22)

Eqs. (21) and (22) lead us to the conclusion that the outcomes of both players depend only on the product \( U_A U_B^T \).

3. Landsburg’s method of classifying Nash equilibria

A mixed strategy is a probability distribution on the space of unit quaternions representing pure strategies. If both players play mixed strategies the expected payoffs read
\[ S_A(\mu, \nu) = \int_{S^3 \times S^3} S_A(p, q) \, d\mu(p) \, d\nu(q) \] (23)
\[ S_B(\mu, \nu) = \int_{S^3 \times S^3} S_B(p, q) \, d\mu(p) \, d\nu(q). \] (24)

Landsburg’s method starts from the observation that any mixed strategy is equivalent to the one supported on at most four orthogonal quaternions. To see this we write two definitions concerning equivalence. Two mixed strategies \( \nu \) and \( \nu' \) are left equivalent iff
\[ \int_{S^3} (pq)_{\alpha}^2 \, d\nu(q) = \int_{S^3} (pq)_{\alpha}^2 \, d\nu'(q) \] (25)
for any unit quaternion $p$ and $\alpha = 0, 1, 2, 3$. Analogously, two mixed strategies $\mu$ and $\mu'$, are right equivalent iff

$$\int_{S^3} (pq)^2_\alpha \, d\mu(p) = \int_{S^3} (pq)^2_\alpha \, d\mu'(p)$$

(26)

for any unit quaternion $q$ and $\alpha = 0, 1, 2, 3$. One can show that (see [1]) any mixed strategy $\nu$ is left equivalent to the one supported on at most four orthonormal quaternions. The analogous principle is true for right equivalence.

The second important element of Landsburg’s analysis concerns the search for Nash equilibria. It reduces to finding the strategies corresponding to maximal eigenvalues of quadratic forms defined by payoff functions. In more detail, the payoff functions can be always written in the form

$$A,B \left( \mu, \nu \right) = \sum_{a,b=1}^{4} \sigma_a \rho_b A_B \left( p^{(a)}, q^{(b)} \right)$$

(27)

with $\rho_a \geq 0$, $\sigma_a \geq 0$, $\sum_{a=1}^{4} \rho_a = \sum_{a=1}^{4} \sigma_a = 1$. Our aim is to find all sets $\left( \sigma_a, p^{(a)} \right)$ and $\left( \rho_b, q^{(b)} \right)$ such that $A$ as a function of first strategy maximizes on $\left( \sigma_a, p^{(a)} \right)$ while $B$ as a function of second strategy maximizes on $\left( \rho_b, q^{(b)} \right)$. Let $\Lambda \subset \{1, 2, 3, 4\}$ be the set of indices $a$ for which $\sigma_a > 0$.

Then the payoff function of Alice is a convex combination of the quantities $\sum_{b=1}^{4} \rho_b A_A \left( p^{(a)}, q^{(b)} \right)$ and acquires a maximal value iff these quantities take the same maximal value for all $a \in \Lambda$.

Considering the Alice payoff

$$A \left( p, \nu \right) = \sum_{b=1}^{4} \rho_b A_A \left( p, q^{(b)} \right)$$

(28)

which is a quadratic form in $p$, we conclude that the vectors $p^{(a)}$, $a \in \Lambda$, span the eigenspace corresponding to the maximal eigenvalue; therefore, the highest eigenvalue has $|\Lambda|$-fold degeneracy. The same conclusion concerns Bob payoff function. If $\Sigma \subset \{1, 2, 3, 4\}$ is the set of indices $b$ such that $\rho_b > 0$ then the eigenspace of the matrix defined by the quadratic form in $q$

$$B \left( \mu, q \right) = \sum_{a=1}^{4} \sigma_a B_B \left( p^{(a)}, q \right)$$

(29)

which corresponds to the maximal eigenvalue is spanned by the vectors $q^{(b)}$, $b \in \Sigma$.

Summarizing, in order to determine Nash equilibria we have to find two sets, $\mu = \{ \sigma_a, p^{(a)} \}$ and $\nu = \{ \rho_b, q^{(b)} \}$, such that $p^{(a)}$, $a \in \Lambda$ span the maximal eigenspace of the quadratic form $A \left( p, \nu \right)$, eq. (28), while $q^{(b)}$, $b \in \Sigma$ span the maximal eigenspace of quadratic form $B \left( \mu, q \right)$, eq. (29). As a result we can introduce the following:

**Definition 1.** The pair $\{ \{ \sigma_a, p^{(a)} \}, \{ \rho_b, q^{(b)} \} \}$ is called a Nash equilibrium of the $(M, N)$ type if $|\Lambda| = M$, $|\Sigma| = N$, $1 \leq M, N \leq 4$ (cf. [37]).

The problem of finding Nash equilibria becomes easier if we use symmetries of game. In our consideration two symmetries are useful:

- the structure of the game does not change, except the actual values of the payoffs, if one makes the substitution
    $$X_\alpha \rightarrow \lambda X_\alpha + \mu, \quad \mu \in \mathbb{R}, \quad \lambda \in \mathbb{R}_+.$$
• for any unit quaternion \( r \) eqs. (19) and (20) imply (with \( q \) replaced by \( q^{-1} \) which simplifies notation)

\[
\$_{A,B} (p, q) = \$_{A,B} (pr^{-1}, rq).
\]

• both classical and quantum games are symmetric with respect to the exchange of players. In order to see this on the quaternionic level, let

\[
r = \frac{1}{\sqrt{2}} (e_0 + e_3);
\]

then

\[
re_0r^{-1} = e_0, \quad re_1r^{-1} = e_2, \quad re_2r^{-1} = -e_1, \quad re_3r^{-1} = e_3.
\]

Therefore, the payoff functions of both players can be written as

\[
\$_A (p, q) = \$ (p, q)
\]

\[
\$_B (p, q) = \$ (r q r^{-1}, r p r^{-1})
\]

which express symmetric role of both players.

4. Classification of Nash equilibria for the original ELW game

The original ELW game [2] is defined by the payoffs \( X_0 = 3, X_1 = 5, X_2 = 0, X_3 = 1 \). We will classify the Nash equilibria according to Definition 1. Due to the symmetry of the game we can assume \( M \geq N \). The procedure can be described as follows. Bob chooses the strategy \( q \). We insert this strategy to the payoff function of Alice and find its maximum. Then the strategy maximizing Alice’s payoff is inserted back to the payoff function of Bob and it is checked whether its maximum is achieved for the same quaternion \( q \) which we have chosen at the beginning. Considering all possible pairs \((M, N)\) we found [1] (cf. [37]–[40]) that the Nash equilibria exist only for \( N = 2 \); therefore, we present only this case in detail.

Due to the symmetry \( p \to pr, q \to r^{-1}q \) one can assume \( q^{(1)} = 1 = e_0, q^{(2)} = q, q = q_1 e_1 + q_2 e_2 + q_3 e_3, q^2 = -e_0 \). Denote \( \rho_1 = \rho, \rho_2 = 1 - \rho, 0 < \rho < 1 \). We find

\[
\$_A (p, \nu) = \rho \sum_{\alpha=0}^{3} X_\alpha p_\alpha^2 + \left(1 - \rho\right) \sum_{\alpha=0}^{3} (pq)_\alpha^2.
\]

Let \( X = \text{diag} (X_0, X_1, X_2, X_3) \); moreover, one can write

\[
(pq)_\alpha \equiv \bar{m}_{\alpha\beta} (q) p_\beta
\]

where

\[
\bar{m} (q) = \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & q_1 \\ q_3 & q_2 & q_1 & q_0 \end{pmatrix};
\]

\( q^2 = -e_0, \) i.e. \( q_0 = 0, \) implies \( \bar{m}^T (q) = -\bar{m} (q) \) and \( \bar{m}^2 (q) = -I \). The payoff function of Alice (36) can be rewritten as

\[
\$_A (p, \nu) = p^T \left( \rho X + (1 - \rho) \bar{m}^T (q) X \bar{m} (q) \right) p.
\]

The matrix

\[
Y (q, \rho, X) \equiv \rho X + (1 - \rho) \bar{m}^T (q) X \bar{m} (q)
\]
should possess $M$-fold ($M \geq 2$) degenerate highest eigenvalue. $Y(q, \rho, X)$ is a real symmetric matrix; explicitly

$$Y(q, \rho, X) = \begin{pmatrix}
Y_0 & (1-\rho)X_{32}q_2q_3 & (1-\rho)X_{13}q_1q_3 & (1-\rho)X_{21}q_1q_2 \\
(1-\rho)X_{12}q_2q_3 & Y_1 & (1-\rho)X_{03}q_1q_2 & (1-\rho)X_{02}q_1q_3 \\
(1-\rho)X_{23}q_1q_3 & (1-\rho)X_{03}q_2q_2 & Y_2 & (1-\rho)X_{01}q_2q_3 \\
(1-\rho)X_{21}q_1q_2 & (1-\rho)X_{02}q_1q_3 & (1-\rho)X_{01}q_2q_3 & Y_3
\end{pmatrix}$$  \hspace{1cm} (41)

where $X_{\alpha\beta} \equiv X_{\alpha} - X_{\beta}$ and

$$Y_0 = \rho X_0 + (1-\rho)(X_{11}q_1^2 + X_{22}q_2^2 + X_{33}q_3^2)$$  \hspace{1cm} (42)

$$Y_1 = \rho X_1 + (1-\rho)(X_{01}q_1^2 + X_{23}q_2^2 + X_{32}q_3^2)$$  \hspace{1cm} (43)

$$Y_2 = \rho X_2 + (1-\rho)(X_{02}q_1^2 + X_{13}q_2^2 + X_{31}q_3^2)$$  \hspace{1cm} (44)

$$Y_3 = \rho X_3 + (1-\rho)(X_{03}q_1^2 + X_{12}q_2^2 + X_{21}q_3^2).$$  \hspace{1cm} (45)

Assume first that only one component of $q$ is nonzero. Then the matrix $Y(q, \rho, X)$ is diagonal

$$Y(q, \rho, X) = \begin{pmatrix}
\rho X_0 + (1-\rho)X_1 & 0 & 0 & 0 \\
0 & \rho X_1 + (1-\rho)X_2 & 0 & 0 \\
0 & 0 & \rho X_2 + (1-\rho)X_1 & 0 \\
0 & 0 & 0 & \rho X_3 + (1-\rho)X_0
\end{pmatrix}.$$  \hspace{1cm} (46)

Using the actual values of the payoffs $X_{\alpha}$, we find easily that the highest eigenvalue is degenerate only provided $\rho = \frac{1}{2}$. Then the eigenspace corresponding to the maximal eigenvalues is spanned by $e_1$ and $e_2$. Therefore, according to the general discussion presented above, Alice plays the strategies

$$p^{(1)} = e_1 \cos \theta + e_2 \sin \theta$$

$$p^{(2)} = -e_1 \sin \theta + e_2 \cos \theta$$  \hspace{1cm} (47)

with the probabilities $\sigma$ and $(1-\sigma)$, respectively. Then the quadratic matrix defining the Bob payoff reads

$$Z(p, \sigma, X) = \begin{pmatrix}
Z_0 & 0 & 0 & (2\sigma-1)cs(X_1-X_2) \\
0 & Z_1 & (2\sigma-1)cs(X_0-X_3) & 0 \\
0 & (2\sigma-1)cs(X_0-X_3) & Z_2 & 0 \\
(2\sigma-1)cs(X_1-X_2) & 0 & 0 & Z_3
\end{pmatrix}$$  \hspace{1cm} (48)

where $c \equiv \cos \theta$, $s \equiv \sin \theta$ and

$$Z_0 = \tilde{X}_1(s^2 + \sigma(c^2 - s^2)) + \tilde{X}_2(c^2 - \sigma(c^2 - s^2))$$  \hspace{1cm} (49)

$$Z_1 = \tilde{X}_0(s^2 + \sigma(c^2 - s^2)) + \tilde{X}_3(c^2 - \sigma(c^2 - s^2))$$  \hspace{1cm} (50)

$$Z_2 = \tilde{X}_0(c^2 - \sigma(c^2 - s^2)) + \tilde{X}_3(s^2 + \sigma(c^2 - s^2))$$  \hspace{1cm} (51)

$$Z_3 = \tilde{X}_1(c^2 - \sigma(c^2 - s^2)) + \tilde{X}_2(s^2 + \sigma(c^2 - s^2)).$$  \hspace{1cm} (52)

Now, $q = e_0$ should be an eigenvector corresponding to the highest eigenvalue. This implies

$$(2\sigma-1)\cos \theta \sin \theta = 0.$$  \hspace{1cm} (53)

If $\sigma = \frac{1}{2}$ we find double degeneracy of highest eigenvalue corresponding to $q^{(1)} = e_0$, $q^{(3)} = \pm e_3$. Therefore, we find the whole family of Nash equilibria parametrized by the angle $\theta$:
Alice:
The strategies
\[ p^{(1)} = e_1 \cos \theta + e_2 \sin \theta \]
\[ p^{(2)} = -e_1 \sin \theta + e_2 \cos \theta \]
played with the probabilities \( \frac{1}{2} \);

Bob:
The strategies
\[ q^{(1)} = e_0 \]
\[ q^{(2)} = \pm e_3 \]
played with the probabilities \( \frac{1}{2} \).

Another solution to eq. (53) is \( \sin 2\theta = 0 \); however, it does not lead to Nash equilibrium. In the case \( M = N = 2 \), one can check using MATHEMATICA that no solution exists unless two components of \( q \) vanish. We have also checked [1] that for the case \( M = 3, 4 \) there are no Nash equilibria. We conclude that all Nash equilibria for \( M = N = 2 \) are given by eqs. (31), (54) and (55).

5. Conclusion
We have sketched the way to classify all Nash equilibria for original ELW game. To this end we used, with some modifications, the method described by Landsburg. Landsburg’s method is based on two important pillars. First, any mixed strategy is equivalent to the one supported on at most four orthogonal quaternions; second, the condition that the strategy is supported on a given number of quaternions implies that the highest eigenvalues of the matrices determining the players payoffs must be degenerate with the multiplicities equal to the number of quaternions entering the supports of the corresponding measures. Additionally, we used the symmetries of the game what makes the classification easier.

For the case \( M = N = 2 \), where the strategies of both players are supported on two quaternions, we gave the complete description of Nash equilibria (for maximally entangled game).

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