Scaling limit of vicious walks and two-matrix model

Makoto Katori

University of Oxford, Department of Physics—Theoretical Physics,
1 Keble Road, Oxford OX1 3NP, United Kingdom

Hideki Tanemura

Department of Mathematics and Informatics, Faculty of Science,
Chiba University, 1-33 Yayoi-cho, Inage-ku, Chiba 263-8522, Japan
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We consider the diffusion scaling limit of the one-dimensional vicious walker model of Fisher and derive a system of nonintersecting Brownian motions. The spatial distribution of N particles is studied and it is described by use of the probability density function of eigenvalues of \(N \times N\) Gaussian random matrices. The particle distribution depends on the ratio of the observation time \(t\) and the time interval \(T\) in which the nonintersecting condition is imposed. As \(t/T\) is going on from 0 to 1, there occurs a transition of distribution, which is identified with the transition observed in the two-matrix model of Pandey and Mehta. Despite of the absence of matrix structure in the original vicious walker model, in the diffusion scaling limit, accumulation of contact repulsive interactions realizes the correlated distribution of eigenvalues in the multimatrix model as the particle distribution.

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I. INTRODUCTION

The vicious walker models, in which random walkers walk without intersecting with any others in a given time interval, were introduced by Michael Fisher and applications of the models to various wetting and melting phenomena were described in his Boltzmann medal lecture [1]. Recently, using the standard one-to-one correspondence between walks and Young tableaux, Guttmann [1] showed that exact formulae for total numbers of one-dimensional vicious walks, some of which were conjectured in previous papers [2, 3, +], are derived following the theory of symmetric functions associated with Young diagrams [5, 6, 7] or the representation theory of classical groups [13, 14]. Important analogies between the ensembles of Young tableaux and those of Gaussian random matrices were reported by Johansson [11], and then Baik [12] and Nagao and Forrester [13, 14] studied the vicious walker models using the random matrix theory [11, 14].

The purpose of the present paper is to demonstrate more explicit relations among the vicious walker model, the symmetric function called the Schur function and the Gaussian ensembles of random matrices by considering the diffusion scaling limit of the one-dimensional vicious walks. Since each random walk converges to a Brownian motion in the scaling limit, the limit process of N vicious walkers will be a system of \(N\) nonintersecting Brownian motions [17]. In order to enumerate all possible nonintersecting paths of walkers realized on a spatio-temporal plane, we use the so-called Lindström-Gessel-Viennot formula [15, 16, 20], which leads us to a useful determinantal expression for the transition probability density of nonintersecting Brownian motions. We found that its initial-configuration dependence can be generally described by using the Schur function and the well-known properties of this function enable us to define the nonintersecting Brownian motions in which all particles start from a single position. Because the nonintersecting condition will be imposed for a given time interval, say \(T\), all the particles are immediately disunited from the initial point, and then they walk randomly keeping the nonintersecting condition. We have studied the time dependence of the spatial distribution of particle positions. We report in this paper that the position distribution of \(N\) nonintersecting Brownian motions can be identified with the distribution of eigenvalues of \(N \times N\) complex hermitian matrix \(H\) coupled to a real symmetric matrix \(A\), in which \(H\) and \(A\) are randomly chosen from the Gaussian ensembles. Such a two-matrix model was studied by Pandey and Mehta [21, 22], in which one parameter was introduced to control the coupling strength between two matrices. We will show that the time dependence of our process can be expressed by the parameter dependence of Pandey-Mehta’s two-matrix model.

Here we consider the probability density function of \(N\) real variables \(\{x_1, \cdots, x_N\}\) with a real parameter \(\beta \geq 0\),

\[
P_\beta(x_1, \cdots, x_N) = Ce^{-\beta \sum_j x_j^2/2} \prod_{1 \leq j < k \leq N} |x_j - x_k|^\beta
\]

with

\[
W(\{x_j\}) = \frac{1}{2} \sum_{j=1}^{N} x_j^2 - \sum_{1 \leq j < k \leq N} \ln |x_j - x_k|
\]

where \(C\) is a normalization constant. It is known that [1]...
with $\beta = 1, 2$ and 4 describe the distributions of eigenvalues of random matrices in the Gaussian orthogonal, unitary and symplectic ensembles, respectively (abbreviated as GOE, GUE and GSE, respectively) \cite{[15]}. For the one-dimensional $N$ Brownian motions, in which all particles start from the origin and the nonintersecting condition is imposed in the time interval $(0, T)$, we will show that (i) at the very early stage, i.e. $t/T \ll 1$, the particle distribution is described by using GUE, (ii) as time $t$ is going on, a transition from GUE to GOE is observed, and (iii) at the final stage $t = T$ the particle distribution can be identified with GOE. As shown by the second equality of \cite{[14]}, the Gaussian ensemble of random matrices can be regarded as the thermodynamical equilibrium of one dimensional gas system with (two-dimensional) Coulomb repulsive potential \cite{[2]} at the inverse temperature $\beta$. Here it should be noted that the vicious walks on a lattice have only contact repulsive interactions to satisfy the nonintersecting condition. The global effective interactions among walkers are accumulated by taking the diffusion scaling limit and as its result a long-ranged Coulomb gas system is constructed. Such emergence of long-range effects in macroscopic scales from systems having only short-ranged microscopic interactions is found only at critical points in thermodynamical equilibrium systems, but it is a typical phenomenon observed in a various interacting particle systems in far from equilibrium.

In particular, in the limit $T \to \infty$, that is, when the nonintersecting condition will be imposed forever, we can derive a system of stochastic differential equations for the process with the drift terms which act as the repulsive two-body forces proportional to the inverse of distances between particles. In other words, the scaling limit of vicious walks with $T \to \infty$ can realize Dyson’s Brownian motion model at $\beta = 2$ \cite{[23]}. It is reasonable to obtain such a stochastic process from the vicious walker model, since it is known that Dyson’s Brownian motion model at $\beta = 2$ can be mapped to the free fermion model \cite{[17]}. The transition from GUE to GOE is, however, first reported for vicious walkers with $T < \infty$ and explained using the two-matrix model in the present paper.

II. MODEL AND LINDSTROM-GESSEL-VIENNOT DETERMINANT

One-dimensional vicious walks are defined as a subset of simple random walks as follows. Let $\{R^x_j\}_{k \geq 0}, j = 1, 2, \cdots, N$, be the $N$ independent symmetric simple random walks on $Z = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ started from $N$ distinct positions, $2s_1 < 2s_2 < \cdots < 2s_N$, $s_j \in Z$. That is,

$$R^x_0 = 2s_j, \quad R^x_{k+1} = R^x_k - 1 \quad \text{or} \quad R^x_k + 1,$$

for $j = 1, 2, \cdots, N, k = 0, 1, 2, \cdots$. Fix the time interval $K$ as a positive even number. The total number of walks is $2^N K$, all of which are assumed to be realized with equal probability $2^{-K}$. We consider a subset of walks such that any of walkers does not meet other walkers up to time $K$. In other words, the condition

$$R^x_k < R^x_{k+1} < \cdots < R^x_{k+K}, \quad k = 1, 2, \cdots, K,$$  \hspace{1cm} (3)

is imposed. Such a subset of walks is called the vicious walks (up to time $K$) \cite{[3]}. Let $N_N(K; \{e_j\}|\{s_j\})$ be the total number of the vicious walks, in which the $N$ walkers starting from $2s_1 < 2s_2 < \cdots < 2s_N$ arrive at the positions $2e_1 < 2e_2 < \cdots < 2e_N$ at time $K$. Then the probability that such vicious walks with those fixed end-points are realized in all possible random walks started from the given initial positions, which is denoted as $V_N(\{R^x_k\}_{k=0}^K; R^x_K = 2e_j)$, is

$$V_N(\{R^x_k\}_{k=0}^K; R^x_K = 2e_j) = \frac{N_N(K; \{e_j\}|\{s_j\})}{2^N K}.$$  \hspace{1cm} (4)

We also consider the probability

$$V_N(\{R^x_k\}_{k=0}^K) = \sum_{e_1 < e_2 < \cdots < e_N} V_N(\{R^x_k\}_{k=0}^K; R^x_K = 2e_j).$$  \hspace{1cm} (5)

Consider a subset of the square lattice $Z^2$, and the set $\mathcal{E}_K$ of all edges which connect the nearest-neighbor pairs of vertices in $\mathcal{L}_K$. The lattice $(\mathcal{L}_K, \mathcal{E}_K)$ provides the spatio-temporal plane and each walk of the $j$-th walker, $j = 1, 2, \cdots, N$, can be represented as a sequence of successive edges connecting vertices $S_j = (2s_j, 0)$ and $E_j = (2e_j, K)$ on it, which we call the lattice path running from $S_j$ to $E_j$. If such lattice paths share a common vertex, they are said to intersect. Under the vicious walk condition \cite{[3]}, what we consider is a set of all $N$-tuples of nonintersecting paths \cite{[2]}. Let $\pi(S \to E)$ be the set of all lattice paths from $S$ to $E$, and $\pi_0(\{S_j\}_{j=1}^N) = \{E_j\}_{j=1}^N$ be the set of all $N$-tuples $(\pi_1, \cdots, \pi_N)$ of nonintersecting lattice paths, in which $\pi_j$ runs from $S_j$ to $E_j$, $j = 1, 2, \cdots, N$. If we write the number of elements in a set $A$ as $|A|$, then $N_N(K; \{e_j\}|\{s_j\}) = |\pi_0(\{S_j\}_{j=1}^N)\}$.

The Lindström-Gessel-Viennot theorem gives \cite{[13], [14]},

$$N_N(K; \{e_j\}|\{s_j\}) = \det_{1 \leq j, k \leq N} (|\pi(S_k \to E_j)|).$$

Since $|\pi(S_k \to E_j)| = (K/2 + s_k - e_j)$, we have the following binomial determinantal expressions

$$V_N(\{R^x_k\}_{k=0}^K; R^x_K = 2e_j) = 2^{-NK} \det_{1 \leq j, k \leq N} \left( \binom{K}{K/2 + s_k - e_j} \right),$$  \hspace{1cm} (4)

and

$$V_N(\{R^x_k\}_{k=0}^K) = 2^{-NK} \sum_{e_1 < e_2 < \cdots < e_N} \det_{1 \leq j, k \leq N} \left( \binom{K}{K/2 + s_k - e_j} \right).$$  \hspace{1cm} (5)
III. SCALING LIMIT OF VICIOUS WALKS

Recently Krattenthaler et al. \cite{1} evaluated the asymptotes of (3) for large $K$ in the two special initial configurations, (i) $s_j = j-1$ and (ii) $s_j = 2(j-1)$, as
\[
V_N(\{R_k^{(j)}\}_{k=0}^{K}) = a_N b_N(\{s_j\}) K^{-N(N-1)/4} (1 + O(1/K)),
\]
where
\[
a_N = \begin{cases} 
(2^N/\pi)^{N/4} \prod_{j=1}^{N/2} (2j-2)! & \text{if } N = \text{even} \\
(2^{N+1}/\pi)^{(N-1)/4} \prod_{j=1}^{(N-1)/2} (2j-1)! & \text{if } N = \text{odd},
\end{cases}
\]
and
\[
b_N(\{j-1\}) = 1, \quad b_N(\{2(j-1)\}) = 2^{N(N-1)/2}. \quad (8)
\]

We found that their result can be immediately generalized as
\[
b_N(\{s(j-1)\}) = s^{N(N-1)/2}
\]
for $s_j = s(j-1), s = 1, 2, 3, \cdots$. This observation suggests that we can take the scaling limit $L \to \infty$, where the time interval $K \propto L$ and the initial spacing of walkers $s \propto \sqrt{L}$.

A. Schur function

In order to describe the scaling limit of the vicious walks, the symmetric function called the Schur function is useful. Here we give some of the fundamental properties of Schur function \cite{4,5,6,7,8}, which will be used below.

A partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N)$ is a non-increasing series of non-negative integers, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$. Let $V$ be the $N$-dimensional complex vector space. Then the Schur function $s_{\lambda}(z_1, \cdots, z_N)$ associated with $\lambda$ is a function of $(z_1, \cdots, z_N) \in V$ defined by
\[
s_{\lambda}(z_1, \cdots, z_N) = \det_{1 \leq j, k \leq N} \frac{(z_j + N - k)}{(z_j)}, \quad (9)
\]
Let $\Delta_{\lambda}(\{z_j\})$ be the numerator of (9), which is an $N \times N$ determinant. If we set $z_{\ell_1} = z_{\ell_2}$ for $1 \leq \ell_1 < \ell_2 \leq N$, then $\Delta_{\lambda}(\{z_j\}) = 0$, since the $\ell_2$-th row is equal to the $\ell_2$-th row. Then it is divisible by each of the differences $z_{\ell_1} - z_{\ell_2}, 1 \leq \ell_1 < \ell_2 \leq N$, and hence by their product $\prod_{1 \leq \ell_1 < \ell_2 \leq N} (z_{\ell_1} - z_{\ell_2})$. This product of all differences is known as the Vandermonde determinant, which is nothing but the denominator of (9);
\[
\Delta_0(\{z_j\}) \equiv \det_{1 \leq j, k \leq N} (z_j^{N-k}) = \prod_{1 \leq j < k \leq N} (z_j - z_k). \quad (10)
\]
Therefore it is concluded that the ratio of two determinants $\Delta_{\lambda}/\Delta_0$ is a polynomial in $z_1, \cdots, z_N$. Moreover, it can be readily seen from (9) that the Schur function is a homogeneous polynomial of degree $\sum_{j=1}^{N} \lambda_j$ in $z_1, \cdots, z_N$.

Let $q$ be a complex variable and set $z_j = q^{j-1}$ in (9). Then we have
\[
s_{\lambda}(1, q, q^2, \cdots, q^{N-1}) = \frac{\det_{1 \leq j, k \leq N} \left(q^{(j-1)(\lambda_k + N - k)}\right)}{\det_{1 \leq j, k \leq N} \left(q^{(j-1)(N - k)}\right)}.
\]

Appropriate application of the formula of Vandermonde determinant \cite{11} gives the product form
\[
s_{\lambda}(1, q, q^2, \cdots, q^{N-1}) = q^{\sum_{j=1}^{N} \lambda_j} \prod_{1 \leq j < k \leq N} \frac{q^{\lambda_j - \lambda_k + k - j} - 1}{q^{k - j} - 1}.
\]
Taking the limit $q \to 1$, we have the formula
\[
s_{\lambda}(1, 1, \cdots, 1) = \prod_{1 \leq j < k \leq N} \frac{\lambda_j - \lambda_k + k - j}{k - j}. \quad (11)
\]
The Schur function is a character of the irreducible representation specified by $\lambda$ of the group $GL(V)$ and (11) gives the dimension of the representation.

B. Diffusion scaling limit

We set
\[
K = Lt, \quad s_j = \frac{\sqrt{L}}{2} x_j, \quad e_j = \frac{\sqrt{L}}{2} y_j, \quad (12)
\]
for $j = 1, 2, \cdots, N$, and take the limit $L \to \infty$. Since in this limit each random walk $R_{Lt}^j$ converges to a Brownian motion, whose distribution function solves the diffusion equation, this scaling limit is especially called diffusion scaling limit. First we remark that, for each strictly increasing series of integers $y_1 < y_2 < \cdots < y_N$, a weakly decreasing series of integers $\xi_j(y) = (\xi_1(y), \cdots, \xi_N(y))$ can be assigned by setting
\[
\xi_j(y) = y_{N-j+1} - (N-j), \quad j = 1, 2, \cdots, N. \quad (13)
\]
Then we can prove that, for given $t > 0$, $x_1 < x_2 < \cdots < x_N$, and $y_1 < y_2 < \cdots < y_N$,
\[
\lim_{L \to \infty} \left(\frac{\sqrt{t}}{2}\right)^N V_N \left(\{R_{Lt}^{\sqrt{t}x_j/2}\}_{k=0}^{Lt} \right) = \left(2\pi t\right)^{-N/2} \det_{1 \leq j, k \leq N} \left[\exp\left(-\frac{1}{2t}(x_j - y_j)^2\right)\right]
\]
\[
= \left(2\pi t\right)^{-N/2} \exp\left(-\frac{1}{2t} \sum_{j=1}^{N} (x_j^2 + y_j^2)\right) h_N(\{e^{x_j/2t}\}), \quad (14)
\]
where $s\xi(y)(z_1, \ldots, z_N)$ is the Schur function associated with $\xi(y)$, defined by (9) with $\lambda = \xi(y)$, and
\[
\begin{align*}
h_N(\{z_j\}) &= \det_{1 \leq j, k \leq N} (z_k^{j-1}) \\
&= (-1)^{N(N-1)/2} \Delta_0(\{z_j\}) \\
&= \prod_{1 \leq j < k \leq N} (z_k - z_j). \tag{15}
\end{align*}
\]

The proof is given as follows. Setting (12), we apply Stirling’s formula to the RHS of (11) multiplied by $(\sqrt{L}/2)^N$,
\[
\begin{align*}
\lim_{L \to \infty} 2^{-L} \left( \frac{\sqrt{L}}{2} \right)^L \det_{1 \leq j, k \leq N} \left( \frac{Lt}{2} + \frac{Lt(x_k - y_j)}{2} \right) \\
= \det_{1 \leq j, k \leq N} \left[ \frac{1}{\sqrt{2\pi t}} e^{-(x_k - y_j)^2/2t} \right], \tag{16}
\end{align*}
\]
which gives the first equality of (14). For the second equality, we rewrite (13) as
\[
\begin{align*}
(2\pi t)^{-N/2} e^{-\sum (x_j^2 + y_j^2)/2t} \det_{1 \leq j, k \leq N} (e^{x_k y_j/t}). \\
\end{align*}
\]
The determinant is written as
\[
\begin{align*}
\det_{1 \leq j, k \leq N} (e^{x_k y_j/t}) \\
= \det_{1 \leq j, k \leq N} ((e^{x_j/t})^N)^\times h_N(\{e^{x_j/t}\}).
\end{align*}
\]
Using (13) and the definition of Schur function (9), the second equality of (14) is obtained.

Before studying the stochastic process defined by the transition probability density (13), we assume $|x| = \sum_{j=1}^N |x_j| < \infty$ and evaluate the $t \to \infty$ asymptote of $N_N(t; \{x_j\})$. In order to do that, the second expression of $f_N(t; \{y_j\}|\{x_j\})$ in (17) will be useful,
\[
\begin{align*}
N_N(t; \{x_j\}) &= e^{-\sum x_j^2/2t} \left(2\pi t\right)^{-N/2} h_N(\{e^{x_j/t}\}) \\
&\times \int_{y_1 < \cdots < y_N} d^{Ny} s_N(y_1, \cdots, y_N) e^{-\sum y_j^2/2t}.
\end{align*}
\]
By (11), (13) and (15),
\[
\begin{align*}
\lim_{t \to \infty} s_N(y_1, \cdots, e^{x_N/t}) &= s_N(y_1, 1, \cdots, 1) \\
= h_N(\{y_j\})/\prod_{1 \leq j < k \leq N} (k - j),
\end{align*}
\]
and
\[
\begin{align*}
\lim_{t \to \infty} t^{N(N-1)/2} h_N(\{e^{x_j/t}\}) &= h_N(\{x_j\}).
\end{align*}
\]
We define
\[
\begin{align*}
b_N(\{x_j\}) = h_N(\{x_j\})/\prod_{1 \leq j < k \leq N} (k - j).
\end{align*}
\]

C. $t \to \infty$ asymptote of $N_N(t; \{x_j\})$

It should be noted that, since $N_N(t; \{y_j\})$ is the integral of $f_N(t; \{y_j\}|\{x_j\})$ over all possible end-positions $(y_j)$, it is the probability that $N$ Brownian motions starting from $\{x_j\}$ do not intersect up to time $t$. Before studying the stochastic process defined by the transition probability density (13), we assume $|x| = \sum_{j=1}^N |x_j| < \infty$ and evaluate the $t \to \infty$ asymptote of $N_N(t; \{x_j\})$. In order to do that, the second expression of $f_N(t; \{y_j\}|\{x_j\})$ in (17) will be useful,
Remark that this definition of \( b_N (\{ x \} ) \) is consistent with [8]. Then we have

\[
\mathcal{N}_N (t; \{ x \} ) = t^{-N^2/2} b_N (\{ x \} ) \\
\times \int d^N y \ e^{-\sum y_j^2/2t} | h_N (\{ y \} ) | \times (1 + \mathcal{O}(1/t))
\]

\[
= t^{-N(N-1)/4} b_N (\{ x \} ) \\
\times \int d^N u \ e^{-\sum u_j^2/2t} | h_N (\{ u \} ) | \times (1 + \mathcal{O}(1/t)),
\]

as \( t \) tends to infinity, where we have used the facts that with the absolute values the product of differences \( | h_N (\{ y \} ) | \) is invariant under permutation of \( y_j \), and \( u_j = y_j / \sqrt{T} \). The last integral is the special case \( \gamma = 1/2 \) and \( a = 1/2 \) of

\[
\int d^N u \ e^{-u} \prod_{1 \leq j < k \leq N} | u_k - u_j |^{2\gamma}
\]

\[
= (2\pi)^{N/2} (2\alpha)^{-N(\gamma(N-1)/2)+1} \prod_{j=1}^N \Gamma(1+j/\gamma) / \Gamma(1+\gamma), (19)
\]

which is found in Mehta [15, eq.(17.6.7) on page 354], whose proof was given in [14] by use of Selberg’s integral [24]. Here \( \Gamma(x) \) is the Gamma function with the values \( \Gamma(3/2) = \sqrt{\pi} / 2 \) and \( \prod_{j=1}^N \Gamma(1+j/2) = 2^{-N(N-1)/4} (\sqrt{\pi} / 2)^N N! \) a\_N, where a\_N is given by [6]. Then we have

\[
\mathcal{N}_N (t; \{ x \} ) = t^{-\psi_N} 2^{-2\psi_N} a_N b_N (\{ x \} ) (1 + \mathcal{O}(1/t)) (20)
\]

with

\[
\psi_N = \frac{1}{4} N (N - 1),
\]

as \( t \) tends to infinity, where \( \psi_N \) is known as the critical exponent of survival probability of vicious walkers [4, 27, 28]. Since

\[
t^{-\psi_N} 2^{-2\psi_N} b_N (\{ x \} ) = (Lt)^{-\psi_N} b_N (\{ \sqrt{T} x_j / 2 \}),
\]

[24] suggests that the result [4] with [3] and [8] of Krattenthaler et al. shall be generalized for arbitrary initial positions of vicious walkers on the lattice.

### IV. GAUSSIAN RANDOM MATRIX ENSEMBLES AND DYSON’S BROWNIAN MOTIONS

In this section we study two special choices of \( T ; T = t \) and \( T \to \infty \). We show that there is an interesting correspondence between these choices of \( T \) and the Gaussian ensembles of random matrices. In order to see it we consider the limit \( |x| \to 0 \), where \( |x| = \sum_{j=1}^N |x_j| \). It will be shown that the second expression of \( f_N (t; \{ y_j \} | \{ x_j \} ) \) in [17] is useful for taking this limit.

**A. \( T = t \) case and GOE**

Since the first expression in [17] gives

\[
limit_{t \to 0} f_N (t; \{ y_j \} | \{ x_j \} ) = \prod_{j=1}^N \delta (x_j - y_j) \]

with Dirac’s delta functions, \( \mathcal{N}_N (0; \{ x_j \} ) = 1 \) for any \( \{ x_j \} \). Then setting \( T = t \) makes [18] depend only on \( t - s \). Set \( s = 0 \) and use the second expression in [17] for \( f_N (t; \{ y_j \} | \{ x_j \} ) \) and \( \mathcal{N}_N (t; \{ x_j \} ) \). By virtue of the Schur function [11], for \( t > 0 \) and \( |x| < 1 \), we have

\[
f_N (t; \{ y_j \} | \{ x_j \} ) = (2\pi t)^{-N/2} h_N (\{ e^{x_j/t} \})
\]

\[
\times s_{\xi (y)} (1, \cdots, 1) e^{-\sum y_j^2/2t} (1 + \mathcal{O}(|x|))
\]

\[
= t^{-N/2} \prod_{1 \leq j < k \leq N} e^{x_k/t - x_j/t} (1 + \mathcal{O}(|x|)),
\]

and

\[
\mathcal{N}_N (t; \{ x_j \} ) = (2\pi t)^{-N/2} a_N (\{ e^{x_j/t} \})
\]

\[
\times \int d^N y \ s_{\xi (y)} (1, \cdots, 1) e^{-\sum y_j^2/2t} (1 + \mathcal{O}(|x|))
\]

\[
= t^N (N-1)/4 \prod_{1 \leq j < k \leq N} e^{x_k/t - x_j/t} (1 + \mathcal{O}(|x|)),
\]

where the integral [13] was used and

\[
c_N = 2^{N(N-2)/2} \pi^{N/2} a_N = \left( 2^{N/2} \prod_{j=1}^N \Gamma(j/2) \right)^{-1}.
\]

Then [18] gives

\[
g_N^f (0; \{ 0 \} ; t, \{ y_j \} ) = c_N t^{-\zeta_N} e^{-\sum y_j^2/2t} h_N (\{ y_j \})
\]

for \( y_1 < \cdots < y_N \) with

\[
\zeta_N = \frac{1}{4} N (N + 1).
\]

It means that

\[
g_N^f (0; \{ 0 \} ; t, \{ y_j \} ) = N! g_N^{GOE} (\{ y_j \} ; t)
\]

for \( y_1 < \cdots < y_N \), where

\[
g_N^{GOE} (\{ y_j \} ; \sigma^2) = \frac{c_N}{N!} \sigma^{-2N} \exp \left( -\frac{1}{2\sigma^2} \sum_{j=1}^N y_j^2 \right) h_N (\{ y_j \})
\]

is the probability density function of eigenvalues \( \{ y_j \} \) of random matrices in the Gaussian orthogonal ensemble with variance \( \sigma^2 \) [14].
B. \( T \to \infty \) limit and GUE

Let

\[ p_N(s, \{x_j\}; t, \{y_j\}) = \lim_{T \to \infty} g^T_N(s, \{x_j\}; t, \{y_j\}). \]

By use of (20) we can determine the explicit form for any initial configuration \( x_1 < x_2 < \cdots < x_N \), in this case as

\[ p_N(0, \{x_j\}; t, \{y_j\}) = \frac{h_N(\{y_j\})}{h_N(\{x_j\})} f_N(t; \{y_j\}|\{x_j\}), \quad (23) \]

where \( h_N \) is given by (15). Moreover, if we take the limit \( \|x\| \to 0 \), we have

\[ p_N(0, \{0\}; t, \{y_j\}) = c'_N t^{-\zeta'_N} e^{-\sum \frac{y_j^2}{2t}} h_N(\{y_j\})^2, \quad (24) \]

with

\[ \zeta'_N = \frac{N^2}{2} \quad \text{and} \quad c'_N = \left( (2\pi)^{N/2} \prod_{j=1}^N \Gamma(j) \right)^{-1}. \]

That is, we have the identity

\[ p_N(0, \{0\}; t, \{y_j\}) = N! g^\text{GUE}_N(\{y_j\}; t), \]

for \( y_1 < \cdots < y_N \), where

\[ g^\text{GUE}_N(\{y_j\}; \sigma^2) = \frac{c'_N}{N!} \sigma^{-2N} \exp \left( -\frac{1}{2\sigma^2} \sum_{j=1}^N y_j^2 \right) h_N(\{y_j\})^2 \quad (25) \]

is the probability density function of eigenvalues \( \{y_j\} \) of random matrices in the Gaussian unitary ensemble with variance \( \sigma^2 \) [13].

In the case \( T \to \infty \), the nonintersecting condition will be imposed forever, while in the case \( T = t \), there will be no condition in the future. The distributions of particles at present depend on the condition in the future.

By generalizing the calculation, which we did in the case \( T = t \), for arbitrary \( T \) and comparing the result with (24), we have

\[ \frac{g^T_N(0, \{0\}; t, \{y_j\})}{p_N(0, \{0\}; t, \{y_j\})} = \tilde{c}_N T^{\psi_N} \frac{N^N(T-t; \{y_j\})}{h_N(\{y_j\})} \quad (26) \]

for \( y_1 < \cdots < y_N \), with (21) and

\[ \tilde{c}_N = \frac{c'_N}{c_N} = \pi^{N/2} \prod_{j=1}^N \frac{\Gamma(j)}{\Gamma(j/2)}. \]

When \( N = 2 \), we can consider the process of one variable \( y = y_2 - y_1 \). In this case \( q^2_2 \) and \( p_2 \) define the Brownian meander and the Bessel process, respectively, both of which are stochastic processes well-studied in probability theory [23]. The equality (20) can be regarded as the multi-variable generalization of Imhof’s relation [30] between the Brownian meander and the Bessel process.

C. Dyson’s Brownian motions

In the limit \( T \to \infty \) we have obtained the compact expression (24) for any \( x_1 < \cdots < x_N \) and \( y_1 < \cdots < y_N \). In this section, we show that a system of stochastic differential equations can be explicitly derived for (24). Using it we will explain why we have the GUE distribution.

Let

\[ E_k(\{x_j\}) = \sum_{j=1}^N \frac{1}{x_k - x_j} \quad \text{for} \quad k = 1, 2, \cdots N. \]

It is easy to verify that

\[ E_k(\{x_j\}) = \frac{\partial}{\partial x_k} \log h_N(\{x_j\}), \quad (27) \]

for \( k = 1, 2, \cdots, N \), and

\[ \sum_{k=1}^N \left[ \frac{\partial}{\partial x_k} E_k(\{x_j\}) + (E_k(\{x_j\}))^2 \right] = 0. \quad (28) \]

Using these equalities, we can prove that \( p_N(0, \{x_j\}; t, \{y_j\}) \) solves the equation

\[ \frac{\partial}{\partial t} u(t; \{x_j\}) = \frac{1}{2} \Delta u(t; \{x_j\}) + \sum_{k=1}^N E_k(\{x_j\}) \frac{\partial}{\partial x_k} u(t; \{x_j\}), \quad (29) \]

where \( \Delta = \sum_{k=1}^N \frac{\partial^2}{\partial x_k^2} \). The proof is the following.

First we remark that the first expression in (17) states that \( f_N \) is a finite summation of the products of Gaussian kernels and thus it satisfies the diffusion equation [31]. Therefore

\[ \frac{\partial}{\partial t} p_N(0, \{x_j\}; t, \{y_j\}) = \frac{1}{2} h_N(\{y_j\}) \frac{\partial}{\partial x_k} f_N(t; \{y_j\}|\{x_j\}). \]

Then we can find that, if \( \{E_k(\{x_j\})\} \) satisfy the equations

\[ \sum_{k=1}^N E_k(\{x_j\}) \frac{\partial}{\partial x_k} f_N(t; \{y_j\}|\{x_j\}) \]

\[ = - \sum_{k=1}^N \left\{ \frac{\partial}{\partial x_k} \frac{1}{h_N(\{x_j\})} \right\} \left\{ \frac{\partial}{\partial x_k} f_N(t; \{y_j\}|\{x_j\}) \right\} \quad (30) \]

and

\[ \sum_{k=1}^N E_k(\{x_j\}) \frac{\partial}{\partial x_k} \frac{1}{h_N(\{x_j\})} = -\frac{1}{2} \sum_{k=1}^N \frac{\partial^2}{\partial x_k^2} \frac{1}{h_N(\{x_j\})}. \quad (31) \]

(23) holds with \( u(t; \{x_j\}) = p_N(0, \{x_j\}; t, \{y_j\}) \). It is easy to see that (11) is satisfied if (27) holds for any \( k = 1, 2, \cdots, N \). Moreover, using (24), we can reduce (11) to (25). Then the proof is completed.

The above result implies that the process defined in the limit \( T \to \infty \) is the system of \( N \) particles with positions
where \( x_1(t), x_2(t), \ldots, x_N(t) \) at time \( t \) on the real axis, whose time evolution is governed by the stochastic differential equations

\[
dx_k(t) = E_k(\{x_j(t)\})dt + dB_k(t), \quad (32)
\]

for \( k = 1, 2, \ldots, N \), where \( \{B_k(t)\}_{k=1}^N \) are the independent standard Brownian motions;

\[
B_j(0) = 0, \quad \langle B_j(t) \rangle = 0,
\]

\[
\langle B_j(t) - B_j(s) | B_k(t) - B_k(s) \rangle = |t - s| \delta_{jk}
\]

for any \( t, s > 0, j, k = 1, 2, \ldots, N \). Because of the scaling property of Brownian motion, \( \sqrt{a}B_j(t) \) is equal to \( B_j(at) \) in distribution for arbitrary \( a > 0 \). Then, if we set \( t = 2t' \) and write \( x_k(t) = \tilde{x}_k(t') \), \( (32) \) are in the \( \alpha = 0, \beta = 2 \) case of the equations

\[
d\tilde{x}_k(t') = -\beta \frac{\partial}{\partial \tilde{x}_k} W^\alpha(\{\tilde{x}_j(t')\})dt' + \sqrt{2} dB_k(t'), \quad (33)
\]

for \( k = 1, 2, \ldots, N \), with

\[
W^\alpha(\{\tilde{x}_j\}) = \alpha \sum_{j=1}^n \frac{\tilde{x}_j^2}{2} - \sum_{1 \leq j < k \leq n} \log(\tilde{x}_k - \tilde{x}_j).
\]

When \( \alpha = 1, (33) \) is known as the stochastic differential equations for the Dyson Brownian motions at the inverse temperature \( \beta \) and the stationary distribution \( \propto \exp(-W^1(\{\tilde{x}_j\})) \). If \( \alpha = 0 \), the factor \( \exp(-\beta \alpha \sum \tilde{x}_j^2/2) \) will be replaced by \( \exp(-\sum \tilde{x}_j^2/4t') \) for finite \( t' \) and thus when \( t' = \sigma^2/\beta \) we may have the Gaussian distribution \( \propto \exp(-\beta/4\sigma^2) \sum \tilde{x}_j^2) h_N(\{\tilde{x}_j\})^{\beta/2} \). Setting \( \beta = 2 \) gives the form \( (25) \).

It should be noted that the system of diffusion equations describing the Dyson Brownian motions with \( \beta = 2 \) can be mapped to the free fermion model \([17, 24]\).

For general \( T < \infty \), we will have the stochastic differential equations

\[
dx_k(t) = E^T_k(\{x_j(t)\})dt + dB_k(t),
\]

for \( k = 1, 2, \ldots, N \), with

\[
E^T_k(\{x_j\}) = \frac{\partial}{\partial x_k} \ln N_N(T - t; \{x_j\}).
\]

V. TWO-MATRIX MODEL

In Section [11] we have constructed a system of nonintersecting Brownian motions in one dimension as the diffusion scaling limit of vicious walks. The obtained transition probability density (18) is temporally inhomogeneous and the particle distribution depends not only the observation time \( t - s \) but also on the time interval \( T \), in which nonintersecting condition is imposed. In the case that all particles start from the origin at time \( s = 0 \), it was shown in Section [V] that, (i) if \( T = t \), it can be identified with the eigenvalue distribution of random matrices in GOE, and (ii) if \( T \to \infty \), it becomes GUE.

For a fixed \( T < \infty \), the above results are summarized as follows. Consider the one dimensional \( N \) Brownian motions all starting from the origin at time \( t = 0 \). We impose the nonintersecting condition for the time interval \((0, T]\). As the ratio \( t/T \to 0 \), the particle distribution is asymptotically described by GUE. On the other hand, at \( t = T \), it can be identified with GOE. This implies that as time \( t \) is going on from 0 to \( T \), there occurs a transition of distribution from GUE to GOE. In this section, we study this transition.

GUE is the ensemble of complex hermitian matrices and GOE is that of real symmetric matrices. The degrees of freedom are, when the matrix sizes are \( N \), \( N^2 \) and \( N(N + 1)/2 \), respectively. If we change the variables from these independent matrix elements to the eigenvalues and other mutually independent variables, and then if we integrate the distribution functions over all variables other than eigenvalues, we will have the probability density functions for \( N \) real eigenvalues as \( (22) \) and \( (23) \).

Although the vicious walker model has no matrix structure at all, here we show that its diffusion scaling limit, nonintersecting Brownian motions, can be regarded as the reduction of a one-parameter family of ensembles of matrix structures to a variable space of eigenvalues. The “hidden structure” is not a single matrix but a two-matrix model, in which a complex hermitian matrix is coupled with a real symmetric matrix.

In the first subsection we will derive the two-matrix model from the nonintersecting Brownian motions and the transition from GUE to GOE will be discussed in the second subsection. In the third subsection we will show that the obtained two-matrix model can be identified with the two-matrix model of Pandey and Mehta \([21, 22]\) by appropriate scale transformation of matrix elements.

A. From vicious walker model to two-matrix model

The generalized Imhof relation \( (28) \) with \( (24) \) gives

\[
g_N^T(0, \{0\}; t, \{y_j\}) \propto e^{-\sum y_j^2/2t} h_N(\{y_j\}) \int d^N z \, \text{sgn}(h_N(\{z_j\}))
\]

\[
\times \prod_{1 \leq j, k \leq N} \left[ \exp \left( -\frac{1}{2(T - t)} (y_j - z_k)^2 \right) \right], \quad (34)
\]

where \( \text{sgn}(x) = x/|x| \). The RHS is rewritten as

\[
h_N(\{y_j\}) \int d^N z \, \text{sgn}(h_N(\{z_j\}))
\]

\[
\times \prod_{1 \leq j, k \leq N} \left[ \exp \left( -\frac{1}{2T} y_j^2 - \frac{1}{2(T - t)} (y_j - z_k)^2 \right) \right]
\]

\[
= h_N(\{y_j\}) \int d^N z \, \text{sgn}(h_N(\{z_j\})) e^{-\sum z_j^2/(2T)}
\]
\[
\times \det_{1 \leq j, k \leq N} \left[ \exp \left( -\frac{T}{2t(T-t)} (y_j - \frac{t}{T})^2 \right) \right].
\]

Setting \((t/T)z_j = a_j, j = 1, 2, \cdots, N\), we have
\[
g_N^T(0, \{0\}; t, \{y_j\}) \propto h_N(\{y_j\}) \int dN a \text{ sgn}(h_N(\{a_j\})) \exp \left( -\frac{T}{2t^2} \sum_{j=1}^{N} a_j^2 \right) \times \det_{1 \leq j, k \leq N} \left[ \exp \left( -\frac{T}{2t(T-t)} (y_j - a_k)^2 \right) \right]. \tag{35}
\]

Consider an ensemble of \(N \times N\) real symmetric matrices \(\{A\}\) with an integration measure
\[
dA \equiv \prod_{1 \leq j \leq k \leq N} dA_{jk}.
\]

Let \(\{a_1, a_2, \cdots, a_N\}\) be the eigenvalues of the matrix \(A\) and \(\{p_1, p_2, \cdots, p_{N(N-1)/2}\}\) be other mutually independent variables. Then
\[
dA = J(\{a_j\}, \{p_j\}) \prod_{j=1}^{N} \frac{dA_j}{dA_j} \prod_{k=1}^{N(N-1)/2} dp_k,
\]
where \(J(\{a_j\}, \{p_j\})\) is the Jacobian
\[
J(\{a_j\}, \{p_j\}) = \left| \frac{\partial(A_{11}, A_{12}, \cdots, A_{NN})}{\partial(a_1, \cdots, a_N, p_1, \cdots, p_{N(N-1)/2})} \right|.
\]

It is known that we can write
\[
J(\{a_j\}, \{p_k\}) = |h_N(\{a_j\})| f(\{p_k\}),
\]
where \(f(\{p_k\})\) is independent of \(a_j\)'s \([5]\). Therefore, for any function \(G(\{a_j\})\) of \(\{a_1, \cdots, a_N\}\), we have the identity
\[
\int dA \ G(\{a_j\}) = c \int dA_j \ h_N(\{a_j\}) G(\{a_j\}) \tag{36}
\]
with
\[
c = \int \prod_{k=1}^{N(N-1)/2} dp_k \ f(\{p_j\}).
\]

Set
\[
G(\{a_j\}) = \frac{1}{h_N(\{a_j\})} \text{ sgn}(h_N(\{a_j\})) \exp \left( -\frac{T}{2t^2} \sum_{j=1}^{N} a_j^2 \right) \times \det_{1 \leq j, k \leq N} \left[ \exp \left( -\frac{T}{2t(T-t)} (y_j - a_k)^2 \right) \right].
\]

Then using the formula \([38], [35]\), \((35)\) becomes
\[
g_N^T(0, \{0\}; t, \{y_j\}) \propto h_N(\{y_j\}) \int dA \ h_N(\{a_j\}) \exp \left( -\frac{T}{2t^2} \sum_{j=1}^{N} a_j^2 \right) \times \det_{1 \leq j, k \leq N} \left[ \exp \left( -\frac{T}{2t(T-t)} (y_j - a_k)^2 \right) \right]. \tag{37}
\]

Next we use the following integral formula \([24, 33, 34]\):

for \(N \times N\) hermitian matrices \(A\) and \(B\) having eigenvalues \(\{a_1, \cdots, a_N\}\) and \(\{b_1, \cdots, b_N\}\), respectively, and for any constant \(\gamma\),
\[
\int dU \ \exp \left[ \gamma \text{ tr}(A - U^\dagger BU) \right] \propto \frac{1}{h_N(\{a_j\})h_N(\{b_j\})} \det_{1 \leq j, k \leq N} \left[ \exp \left( \gamma (a_j - b_k)^2 \right) \right],
\]
where the integral is taken over the group of unitary matrices \(U\). Then \((37)\) can be written as
\[
g_N^T(0, \{0\}; t, \{y_j\}) \propto h_N(\{y_j\}) \int dU \int dA \ \exp \left( -\frac{T}{2t^2} \text{ tr} A^2 \right) \times \exp \left( -\frac{T}{2t(T-t)} t \text{ tr}(U^\dagger YU - A)^2 \right), \tag{38}
\]
where \(Y\) is the \(N \times N\) diagonal matrix such that \(Y_{jk} = y_j \delta_{jk}\). Since \(U\) is a unitary matrix, \(H = U^\dagger YU\) is an \(N \times N\) complex hermitian matrix. Then the integrand of \((38)\) can be regarded as a weight for two matrices \(H\) and \(A\) given as
\[
\exp \left( -\text{ tr } (\gamma H H^2 - \gamma_H A^2) \right)
\]
with
\[
\gamma_H = \frac{T}{2t(T-t)}, \quad \gamma_H = \frac{T}{2t^2}, \quad \gamma_A = \frac{T^2}{2t^2(T-t)} \tag{39}
\]

Consider an ensemble of \(N \times N\) complex hermitian matrices \(\{H\}\) with the integration measure
\[
dH = \prod_{1 \leq j \leq k \leq N} d\text{Re}(H_{jk}) \prod_{1 \leq j < k \leq N} d\text{Im}(H_{jk}).
\]

For each complex hermitian matrix \(H\), let \(\{y_1, \cdots, y_N\}\) be a set of eigenvalues and \(U\) be the \(N \times N\) unitary matrix such that \(H = U^\dagger YU\) with \(Y_{jk} = y_j \delta_{jk}\). Then it is known that the integration measure \(dH\) can be factorized into the product of the Haar measure for unitary matrices \(dU\) and an integration measure for eigenvalues \([34, 14]\)
\[
dH \propto dU \times h_N(\{y_j\})^2 \prod_{j=1}^{N} dy_j.
\]

Now we introduce a two-matrix model, which consists of \(N \times N\) real symmetric matrix \(A\) and an \(N \times N\) complex hermitian matrix \(H\), with a probability density function
\[
\mu_N(H, A) = \frac{1}{Z_N} \exp \left( -\text{ tr } (\gamma_H H^2 - \gamma_H A^2) \right).
\]

Here \(\gamma_H, \gamma_H, \gamma_A\) are given as \([33]\) and \(Z_N\) is the partition function of the two-matrix model,
\[
Z_N = \int dH \int dA \exp \left( -\text{ tr } (\gamma_H H^2 - \gamma_H A^2 + \gamma_A A^2) \right).
\]
Then the relation
\[
g_T^N(0, \{0\}; t, \{y_j\}) \propto h_N(\{y_j\})^2 \int dU \int dA \mu_N(U^\dagger YU, A)
\]
is established.

**B. Transition from GUE to GOE**

Consider the Gaussian ensembles of real symmetric matrices \{A\} and complex hermitian matrices \{H\} with sizes \(N\) with the probability density functions

\[
\nu_N(A) = C_A \exp \left( -\frac{1}{2\sigma_A^2} \text{tr} A^2 \right),
\]
and

\[
\tilde{\nu}_N(H) = C_H \exp \left( -\frac{1}{2\sigma_H^2} \text{tr} H^2 \right),
\]
respectively, where

\[
\sigma_A^2 = \frac{t^2}{T}, \quad \sigma_H^2 = t \left( 1 - \frac{t}{T} \right),
\]
and \(C_A = 2^{-N^2/2}(\pi\sigma_A^2)^{-\zeta_N}, C_H = 2^{-N^2/2}(\pi\sigma_H^2)^{-\zeta_N}\). Then consider the convolution

\[
\tilde{\mu}_N(H) = \int dA \nu_N(A) \tilde{\nu}_N(H - A).
\]

Since, for \(1 \leq j, k \leq N\),

\[
H_{jk} = \text{Re}(H_{jk}) + i \text{Im}(H_{jk})
\]
with \(i = \sqrt{-1}\), and

\[
\text{Re}(A_{jk}) = A_{jk}, \quad \text{Im}(A_{jk}) = 0,
\]
the convolution is also Gaussian distribution in the form

\[
\tilde{\mu}_N(H) \propto \exp \left( -\sum_{j,k} \left\{ \frac{(\text{Re}(H_{jk}))^2}{2(\sigma_{Re}^2 + \sigma_A^2)} + \frac{(\text{Im}(H_{jk}))^2}{2\sigma_{Im}^2} \right\} \right).
\]

Then (40) gives

\[
g_T^N(0, \{0\}; t, \{y_j\}) \propto h_N(\{y_j\})^2 \int dU \tilde{\mu}_N(H)
\]
\[
\propto h_N(\{y_j\})^2 \times \int dU \exp \left( -\sum_{j,k} \left\{ \frac{(\text{Re}(H_{jk}))^2}{2\sigma_{Re}^2} + \frac{(\text{Im}(H_{jk}))^2}{2\sigma_{Im}^2} \right\} \right),
\]
where \(H = U^\dagger YU\) and

\[
\sigma_{Re}^2 = t, \quad \sigma_{Im}^2 = t \left( 1 - \frac{t}{T} \right).
\]

Now the transition from GUE to GOE is explicitly represented by the time-dependent variances (41). With a fixed finite \(T\), if \(0 < t \ll T\), \(\sigma_{Re}^2 = t \approx \sigma_{Im}^2\). Then the real and imaginary parts of complex hermitian matrix elements are equally distributed as in GUE. While \(\sigma_{Re}^2\) increases linearly in time \(t\), \(\sigma_{Im}^2\) increases in time \(t\) only up to time \(t = T/2\) and then decreases in time. At time \(t = T\), \(\sigma_{Im}^2 = 0\), which implies that the imaginary parts of matrix elements are zeros with probability one. Then the distribution is identified with GOE.

**C. Pandey-Mehta’s two-matrix model**

As an interpolation between GUE and GOE, Pandey and Mehta introduced a family of Gaussian ensembles of hermitian matrices \{\(H\)\} with one parameter \(\alpha \in [0, 1]\) [21, 22],

\[
\mu_{PM}^N(\alpha, H) = C_{PM} \exp \left( -\sum_{j,k} \left\{ \frac{(\text{Re}(H_{jk}))^2}{4v^2} + \frac{(\text{Im}(H_{jk}))^2}{4v^2\alpha^2} \right\} \right),
\]
where \(v^2 = \{2(1 + \alpha^2)\}^{-1}\) and \(C_{PM} = 2^{-N/2}\alpha - N(N-1)/2(2\pi v^2)^{-N^2/2}\). Set

\[
\kappa = \sqrt{\frac{t(2T - t)}{T}}.
\]

Then, it is easy to see that, if

\[
\alpha^2 = 1 - \frac{t}{T},
\]
the equality

\[
\kappa^N \tilde{\mu}_N(\kappa H) = \tilde{\mu}_{PM}^N(H, \alpha)
\]
is established.

For an even integer \(N\) and an antisymmetric \(N \times N\) matrix \(B = (b_{jk})\) we put

\[
Pf_{1 \leq j < k \leq N}(b_{jk}) = \frac{1}{(N/2)!} \sum_\sigma \text{sgn}(\sigma) b_{\sigma(1)\sigma(2)} b_{\sigma(3)\sigma(4)} \cdots b_{\sigma(N-1)\sigma(N)},
\]
where the summation is extended over all permutations \(\sigma\) of \((1, 2, \ldots, N)\) with restriction \(\sigma(2k-1) < \sigma(2k), k = 1, 2, \ldots, N/2\). This expression is known as the Pfaffian [20]. Pandey and Mehta showed that the probability density function of eigenvalues \(\{y_j\}\) of the complex hermitian matrices, which are distributed following (42), is given by

\[
g_{PM}^N(\{y_j\}, \alpha) = C_N(\alpha) \exp \left( -\frac{1}{2}(1 + \alpha^2) \sum_{j=1}^N y_j^2 \right)
\]
\[
\times h_N(\{y_j\}) Pf_{1 \leq j < k \leq N}(F_{jk}),
\]
where
with another proof of the equality (45). It should be noted that using it (47) can be the observation time to represent the transition probability density. We have shown the limit of vicious walker model in one dimension and construction (40) and the equality (45) with (43) and (44) imply the expression

\[
F_j(t, \{y_k\}) = \begin{cases} 
\frac{2}{\sqrt{T}} \text{Erf} \left( \frac{y_k - y_j}{2 \sqrt{T}} \right) & \text{if } 1 \leq j, k \leq N, \\
1 & \text{if } 1 \leq j \leq N, k = N + 1, \\
-1 & \text{if } j = N + 1, 1 \leq k \leq N, \\
0 & \text{if } j = k = N + 1
\end{cases}
\]

with

\[
\text{Erf}(x) = \int_0^x du \ e^{-u^2}.
\]

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**APPENDIX: INTEGRATION VERSION OF OKADA’S MINOR-SUMMATION FORMULA**

Assume $1 \leq n \leq m$ and let $z(j, k), 1 \leq j \leq n, 1 \leq k \leq m$, be indeterminates and $Z(z(j, k))$ be the $n \times m$ matrix with $(j, k)$-element $z(j, k)$. We consider the sum of all minors of $Z$ with a given size $r$. That is, we define

\[
d(a_1, \cdots, a_r) = \sum_{1 \leq b_1 < b_2 < \cdots < b_r \leq m} \det \left( z(a_j, b_k) \right)
\]

for $r = 1, 2, \cdots, m$. Okada proved the following equalities known as the minor-summation formula [23]: if $r$ is odd,
then
\[
d(a_1, \ldots, a_r) = \\
Pf \begin{pmatrix} 0 & d(a_1) & d(a_2) & \cdots & d(a_r) \\ -d(a_1) & 0 & d(a_1, a_2) & \cdots & d(a_1, a_r) \\ -d(a_2) & -d(a_1, a_2) & 0 & \cdots & d(a_2, a_r) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -d(a_r) & -d(a_1, a_r) & -d(a_2, a_r) & \cdots & 0 \\
\end{pmatrix}
\]

if \( r \) is even, then
\[
d(a_1, \ldots, a_r) = \\
Pf \begin{pmatrix} 0 & d(a_1, a_2) & d(a_1, a_3) & \cdots & d(a_1, a_r) \\ -d(a_1, a_2) & 0 & d(a_2, a_3) & \cdots & d(a_2, a_r) \\ -d(a_1, a_3) & -d(a_2, a_3) & 0 & \cdots & d(a_3, a_r) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -d(a_1, a_r) & -d(a_2, a_r) & -d(a_3, a_r) & \cdots & 0 \\
\end{pmatrix}
\]

Now we give the integration version of Okada’s formula. Let \( z(x, y) \) be a square integrable continuous function of real variables \( x, y \). Then
\[
\int_{-\infty}^{\infty} d^n y \det_{1 \leq j, k \leq n} (z(x_j, y_k)) = \text{Pf}_{1 \leq j < k \leq n}(F_{jk}(\{x_{\ell}\})) \tag{A.1}
\]

where
\[
\hat{n} = \begin{cases} n & \text{if } n \text{ is even} \\ n + 1 & \text{if } n \text{ is odd} \end{cases}
\]

and
\[
F_{jk}(\{x_{\ell}\}) = \begin{cases} I_z(x_j, x_k) & \text{if } 1 \leq j < k \leq n, \\
-I_z(x_j, x_k) & \text{if } 1 \leq k < j \leq n, \\
I_z(x_k) & \text{if } 1 \leq j, k \leq n, \\
-I_z(x_k) & \text{if } j = n + 1, 1 \leq k \leq n, \\
0 & \text{if } 1 \leq j = k \leq n + 1 \\
\end{cases}
\]

with
\[
I_z(x_j) = \int_{-\infty}^{\infty} z(x_j, y) dy,
I_z(x_j, x_k) = \int_{-\infty}^{\infty} \frac{z(x_j, y_1) z(x_j, y_2)}{z(x_k, y_1) z(x_k, y_2)} dy_1 dy_2.
\]

The proof is the following. We write the integral in LHS of (A.1) as a limit of summation;
\[
\int_{-\infty}^{\infty} d^n y \det_{1 \leq j, k \leq n} (z(x_j, y_k)) = \lim_{M \to \infty} \int_{-M/2}^{M/2} d^n y \det_{1 \leq j, k \leq n} (z(x_j, y_k)) = \lim_{M \to \infty} \lim_{\delta \to 0} \delta^n \sum_{1 \leq b_1 < \ldots < b_N \leq m(M, \delta)} \det_{1 \leq j, k \leq n} (z(x_j, \hat{y}(b_k))),
\]

where \( m(M, \delta) = \lfloor M/\delta \rfloor \), the greatest integer not greater than \( M/\delta \), and
\[
\hat{y}(b) = \left\{ \frac{M - \delta}{m(M, \delta) - 1} b - \left\{ \frac{M - \delta}{m(M, \delta) - 1} + \frac{M}{2} \right\} \right\}.
\]

Let
\[
\tilde{d}(x_j) = \sum_{1 \leq b \leq m(M, \delta)} z(x_j, \hat{y}(b)),
\]
\[
\tilde{d}(x_j, x_k) = \sum_{1 \leq b_1 < b_2 \leq m(M, \delta)} \begin{vmatrix} z(x_j, \hat{y}(b_1)) & z(x_j, \hat{y}(b_2)) \\ z(x_k, \hat{y}(b_1)) & z(x_k, \hat{y}(b_2)) \end{vmatrix},
\]

and set
\[
S_{jk}(\{x_{\ell}\}) = \begin{cases} \tilde{d}(x_j, x_k) & \text{if } 1 \leq j < k \leq n, \\
-\tilde{d}(x_j, x_k) & \text{if } 1 \leq k < j \leq n, \\
\tilde{d}(x_j) & \text{if } 1 \leq j, k \leq n, \\
-\tilde{d}(x_j) & \text{if } j = n + 1, 1 \leq k \leq n, \\
0 & \text{if } 1 \leq j = k \leq n + 1. \\
\end{cases}
\]

Then Okada’s formula gives
\[
\int_{-\infty}^{\infty} d^n y \det_{1 \leq j, k \leq n} (z(x_j, y_k)) = \lim_{M \to \infty} \lim_{\delta \to 0} \delta^n \text{Pf}_{1 \leq j < k \leq n}(S_{jk}(\{x_{\ell}\})). \tag{A.2}
\]

Since the Pfaffian in (A.2) is a finite summation of \( n/2 \) products of \( \tilde{d}(x_j, x_k) \)’s if \( n \) is even, and it is a finite summation of \((n-1)/2 \) products of \( \tilde{d}(x_j, x_k) \)’s multiplied by \( \tilde{d}(x_j) \) if \( n \) is odd, we may have
\[
\int_{-\infty}^{\infty} d^n y \det_{1 \leq j, k \leq n} (z(x_j, y_k)) = \text{Pf}_{1 \leq j < k \leq n} \left( \lim_{M \to \infty} \lim_{\delta \to 0} \delta^\alpha(j, k) S_{jk}(\{x_{\ell}\}) \right),
\]

where
\[
\alpha(j, k) = 2 \quad \text{for } 1 \leq j < k \leq n,
\]
\[
\alpha(j, n + 1) = 1 \quad \text{for } 1 \leq j \leq n.
\]

Since \( z(x, y) \) is assumed to be square integrable and continuous,
\[
\lim_{M \to \infty} \lim_{\delta \to 0} \delta S_{jk+1}(\{x_{\ell}\}) = I_z(x_j),
\]
\[
\lim_{M \to \infty} \lim_{\delta \to 0} \delta^2 S_{jk}(\{x_{\ell}\}) = I_z(x_j, x_k),
\]

for \( 1 \leq j, k \leq n \). Then the proof is completed.

By elementary calculation we can show that
\[
I_z(x) = 1,
I_z(x, y) = \frac{2}{\sqrt{\pi}} \text{Erf} \left( \frac{y - x}{2\sqrt{t}} \right)
\]
for \( z(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t} \). Applying (A.1) with \( n = N \), the expression (17) is obtained from (34).
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