Riemann-Finsler Geometry and Lorentz-Violating Scalar Fields

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Abstract

The correspondence between Riemann-Finsler geometries and effective field theories with spin-independent Lorentz violation is explored. We obtain the general quadratic action for effective scalar field theories in any spacetime dimension with Lorentz-violating operators of arbitrary mass dimension. Classical relativistic point-particle lagrangians are derived that reproduce the momentum-velocity and dispersion relations of quantum wave packets. The correspondence to Finsler structures is established, and some properties of the resulting Riemann-Finsler spaces are investigated. The results provide support for open conjectures about Riemann-Finsler geometries associated with Lorentz-violating field theories.

1. Introduction

A correspondence between a large class of Riemann-Finsler geometries and realistic effective field theories with explicit Lorentz violation has recently been identified [3]. The underlying idea is that the classical trajectory of a relativistic wave packet in the presence of perturbative Lorentz violation can be mapped via a suitable continuation to a geodesic in a Riemann-Finsler space. The correspondence is of both mathematical and physical interest. On the mathematics side, it provides a rich source of examples of Riemann-Finsler geometries that are perturbatively close to Riemann geometry. One example uncovered in this way is a calculable geometry, called Randers geometry [4]. The known classification and enumeration of Lorentz-violating effects may also permit a parallel classification of the corresponding Riemann-Finsler spaces. On the physics side, the correspondence is expected to shed light on the poorly understood geometric structure of theories of gravitation with explicit Lorentz breaking [3]. Also, in analogy with the geometric interpretation of Zermelo navigation [6] in terms of Randers geometry [4], the correspondence can be applied to geometric descriptions of physical systems [8]. Related concepts are explored in various contexts in a broad recent literature [9–24].

In nature, Lorentz and CPT violation could arise from an underlying theory combining gravity with quantum physics such as strings [25]. Observable effects on the behavior of known fundamental particles can be inferred from the comprehensive realistic effective field theory for Lorentz violation incorporating the Standard Model of particle physics and General Relativity, called the Standard-Model Extension (SME) [5, 26]. Most of the known fundamental particles have spin, with only the Higgs boson being a spinless field in the Standard Model. A nonzero spin complicates the particle trajectory in part because it involves intrinsically quartic dispersion relations rather than intrinsically quadratic ones [27]. However, even for a particle with nonzero spin, a subset of Lorentz-violating effects are spin independent and hence can be handled as though the particle had zero spin. The combination of relevance and comparative simplicity enhances interest in the correspondence between Riemann-Finsler geometries and the trajectories of particles experiencing spin-independent Lorentz violation.

In this work, we construct the general effective scalar field theories in any spacetime dimension that contain explicit perturbative spin-independent Lorentz-violating operators of arbitrary mass dimension. The results are used to obtain the general classical lagrangian describing the propagation of a relativistic spinless point particle in the presence of Lorentz violation. The correspondence between the classical lagrangian and Riemann-Finsler geometries is established, and some properties of the latter are studied. Among the results is a set of calculable global Riemann-Finsler geometries that are perturbatively close to Riemann geometries. The properties of these spaces offer support for some unresolved conjectures about Riemann-Finsler geometries associated with Lorentz-violating field theories.

2. Scalar field theory

Consider a complex scalar field $\phi(x^\mu)$ of mass $m$ in $n$-dimensional spacetime with Minkowski metric $\eta_{\mu\nu}$ of negative signature for $n > 2$. The effective quadratic Lagrange density describing the propagation of $\phi$ in the presence of arbitrary Lorentz-violating effects can be written in the form

$$
\mathcal{L}(\phi, \phi^\dagger) = \partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi - \frac{1}{2} \left( \phi^\dagger (\hat{k}_\alpha)^{\mu\nu} \partial_\mu \phi + \text{h.c.} \right) + \partial_\mu \phi^\dagger (\hat{k}_\alpha)^{\mu\nu} \partial_\nu \phi, \tag{1}
$$

where $(\hat{k}_\alpha)^{\mu\nu}$ are operators constructed as series of even powers of the partial spacetime derivatives $\partial_\alpha$. Since Lorentz violation is expected to be small in nature and perhaps...
even Planck suppressed, both $(\hat{k}_s)^\rho$ and $(\hat{k}_y)^\mu$ can be assumed to introduce only perturbations to conventional physics. For some considerations, it is convenient also to assume that $(\hat{k}_s)^\rho$ and $(\hat{k}_y)^\mu$ are independent of spacetime position. This implies translation invariance and hence guarantees conservation of energy and momentum, thereby permitting a focus on Lorentz-violating effects. The hermiticity of $\mathcal{L}$ then implies that $(\hat{k}_s)^\rho$ and $(\hat{k}_y)^\mu$ can be taken as hermitian without loss of generality.

In the limiting scenario in which $\phi$ is a hermitian scalar field, $\phi^\dagger \equiv \phi$, the term involving $(\hat{k}_s)^\rho$ becomes proportional to $i\hbar(\hat{k}_s)^\rho \phi \text{d}\phi + c.c$. However, all spacetime-constant terms of this type reduce to total derivatives up to surface terms and so in the absence of topological effects contribute nothing to the classical action. Note that in the special case of four spacetime dimensions the term involving $(\hat{k}_s)^\rho$ in the theory (1) is odd, while the one involving $(\hat{k}_y)^\mu$ is CPT even. It therefore follows that CPT invariance becomes an automatic property of the propagation of a hermitian scalar field in the presence of spacetime-constant Lorentz violation.

The freedom to redefine the canonical variables in a field theory can imply that certain Lorentz-violating terms in a Lagrange density are unobservable \cite{5,22,30}. In the present case, one useful field redefinition takes the form $\phi \to \phi' = (1 + Z)\phi$, where $Z$ is a Lorentz-violating spacetime-constant operator formed as a series of powers of derivatives $\partial_{\mu}$. To preserve the physics of the original theory (1), which is a perturbation of the free complex scalar field, the redefinition itself and hence $Z$ must be perturbative. Applying the redefinition to the free field theory for $\phi'$ generates perturbative terms proportional to $\phi'Z(\partial^\alpha \partial_{\alpha} + m^2)\phi + c.c$, thereby showing that terms of this form occurring in the Lagrange density (1) describe Lorentz-invariant physics despite their apparent Lorentz-violating form. It follows that any term in $(\hat{k}_s)^\rho$ or $(\hat{k}_y)^\mu$ that involves contracted derivatives can be converted to one with fewer derivatives and hence can be absorbed in other terms in the theory (1).

Under the above assumptions, the nonderivative pieces of $(\hat{k}_s)^\rho$ and $(\hat{k}_y)^\mu$ can in principle also be removed from the theory (1). A spacetime-constant nonderivative component of $(\hat{k}_s)^\rho$ is unobservable because it can be generated from a conventional free field theory using a field redefinition with a nonderivative $(\hat{k}_s)^\rho$ of the form $\phi' = \exp[i(\hat{k}_s)^\rho \phi^2/2]\phi$, which amounts to a position-dependent redefinition of the field phase. Also, if $(\hat{k}_y)^\mu$ has a nonderivative piece, it can be absorbed into the metric by a suitable change of coordinates. However, in realistic scenarios involving multiple interacting fields with distinct nonderivative pieces, only one combination of pieces can be removed via each of the above methods. For generality in what follows, we therefore disregard these options and instead keep explicitly any nonderivative pieces of $(\hat{k}_s)^\rho$ and $(\hat{k}_y)^\mu$.

The Euler-Lagrange equations of motion for the theory (1) are

\begin{equation}
(\partial^\rho \partial_\rho + m^2 + i(\hat{k}_s)^\rho \partial_\rho + (\hat{k}_y)^\mu \partial_\rho \partial_\mu) \phi = 0.
\end{equation}

Performing a Fourier transform to momentum space with the correspondence $p_{\mu} \leftrightarrow i\partial_{\mu}$ yields the exact dispersion relation for the theory (1) in the compact form

\begin{equation}
p^2 - m^2 - (\hat{k}_s)^\rho p_\rho + (\hat{k}_y)^\mu p_\mu p_\nu = 0.
\end{equation}

The operators $(\hat{k}_s)^\rho$ and $(\hat{k}_y)^\mu$ can conveniently be expressed as expansions in even powers of the $n$-momentum $p_\mu$ of the form

\begin{equation}
(\hat{k}_s)^\rho = \sum_{d=2} a_d p_\alpha p_\beta \cdots p_{\alpha+d-1},
\end{equation}

\begin{equation}
(\hat{k}_y)^\mu = \sum_{d=2} c_d p_\alpha p_\beta \cdots p_{\alpha+d-1},
\end{equation}

where each sum is over either even or odd values of $d$. The definition of the effective field theory (1) for infinite sums over $d$ may be problematic, so where necessary in what follows we can assume the number of Lorentz-violating terms is arbitrary but finite \cite{29}.

In Eq. (4), the quantities $(k_d)^{\mu_1 \cdots \mu_{d-1}}$ and $(k_{e_\mu})^{\alpha_1 \cdots \alpha_{d-2}}$ are termed coefficients for Lorentz violation. They control deviations from conventional propagation governed by Lorentz-violating operators of mass dimension $d$, and in physical applications they are the target of experiments \cite{31}. The coefficients have mass dimension $n - d$, and hermiticity of $\mathcal{L}$ implies they are real. The assumption of translation invariance assures they have constant cartesian components. Furthermore, the commutativity of partial derivatives and the elimination of contracted derivatives via field redefinitions means that the coefficients can be taken as symmetric and traceless without loss of generality. The number $N_n$ of independent components of $(k_d)^{\mu_1 \cdots \mu_{d-1}}$ or $(k_{e_\mu})^{\alpha_1 \cdots \alpha_{d-2}}$ is then found to be

\begin{equation}
N_n = (2d - n + 2)(d - 1)!/(d - n + 2)!.
\end{equation}

For the special case $n = 4$, this reduces to the standard counting $N_n^d = (d - 1)^2$ in four spacetime dimensions.

Various limits of the theory (1) can be considered. For example, restricting attention to a single nonzero coefficient at a time can simplify calculations and provide insight. The coefficients with $d = 3$ and $4$ in $n = 4$ spacetime were introduced in Refs. \cite{13,22} in the context of the Higgs-boson sector of the SME. The properties of various scalar field theories containing these coefficients have been widely explored in the literature \cite{12}. A model with a vector coefficient contributing to a $d = 6$ term in $n = 4$ spacetime has recently been considered in Ref. \cite{33}, but other scenarios with $n \neq 4$ or $d > 6$ appear unexplored to date.

Another unexplored limit of potential interest allows only coefficients with timelike indices to be nonzero. This model is spatially isotropic in the defining inertial frame, although spatial anisotropies arise in most other frames. Note that spatial isotropy could in principle also be achieved by tracing over spatial components, but the requirement that all coefficients are traceless in any pair of spacetime indices implies that a pair of traced spatial indices can be replaced with a pair of timelike indices without loss of generality. At each value of $d$, the spatially isotropic limit therefore allows only one coefficient, denoted by $k_d$. The dispersion relation for this model then takes the comparatively simple form

\begin{equation}
E^2 - |p|^2 - m^2 + \sum_{d=2} (-1)^{d-n}k_d E^{d-n+2} = 0,
\end{equation}

where $E$ is the energy of the particle of spatial momentum $p$ and where the sum is over all values of $d \geq n - 1$. 


3. Classical kinematics

The behavior of a wave packet obeying the equations of motion (3) is controlled by the dispersion relation (3), which describes the effects of Lorentz violation on the energies of plane waves of different momenta. The dispersion relation can alternatively be interpreted as the energy-momentum relation for an analogue classical point particle. The motion of this analogue particle is determined by a lagrangian $\mathcal{L}$, which in turn can be related to Finsler geometry. The construction of $\mathcal{L}$ for a given dispersion relation is therefore of definite interest. Although obtaining an explicit result for $\mathcal{L}$ can be challenging, a formal procedure to achieve this has been given in Ref. [30]. Here, we extend this procedure to $n$ spacetime dimensions and develop an iterative method to calculate $\mathcal{L}$ explicitly. For definiteness in what follows, we assume $m \neq 0$. Also, where appropriate and convenient we write $(k^{(d)})^{0a_1\ldots a_{d-1}}$ for either $-(k^{(d)})^{0a_1\ldots a_{d-2}a_d}$ or $(k^{(d)})^{0a_1\ldots a_{d-2}a_{d-1}}$, which simplifies expressions that contain coefficients with both $a$ and $c$ subscripts or that are valid for either alone.

The motion of the analogue particle follows a worldline in $n$ dimensions. The worldline can be parametrized by $\lambda$ and specified by the $n$ equations $x^i = \chi^i(\lambda)$, and the $n$-velocity $u^i$ of the particle is then given by $u^i = dx^i/d\lambda$. In the general case $L$ depends on both the position and the velocity of the particle, but the assumption of translation invariance of $\mathcal{L}$ requires that $L = L(u, k)$ is independent of the position and that the canonical $n$-momentum $p_\mu = -\partial L/\partial u^\mu$ is conserved. Invariance of the action under reparametrizations of $\lambda$ requires that $L$ be homogeneous of degree 1 in $u^\mu$. Applying Euler’s theorem then reveals that $L$ can be written implicitly as $L = -u^\mu p_\mu$. The relation between the $n$-momentum and the $n$-velocity is fixed by matching the spatial velocity of the analogue particle to the group velocity of the wave packet in the field theory, $-u^\mu/p_\mu = \partial \mathcal{L}/\partial p_\mu$. The challenge of constructing an explicit expression for $L(u, k)$ such that the Euler-Lagrange equations reproduce the dispersion relation (3) then reduces to solving simultaneously the $n-1$ matching equations and the dispersion relation to obtain $p_\mu$ in terms of $u^\mu$.

For simple cases, an analytical solution for $L = L(u, k)$ can be found. Consider, for example, the field theory with only one particular nonvanishing coefficient, $(k^{(n)})^{\mu\nu} \neq 0$. The dispersion relation can then be written as $p_\mu \Omega^{\mu\nu} p_\nu = m^2$, where $\Omega^{\mu\nu} = \eta^{\mu\nu} + (k^{(n)})^{\mu\nu}$. Taking the derivative of the dispersion relation with respect to $p_\mu$, yields $(d^\mu \Omega^{-\mu\nu} - u^\mu \Omega^{\mu\nu}) p_\nu = 0$. Multiplying by $p_\mu$ and some manipulation of the result provides an implicit expression for the $n$-velocity, $u^\mu = -L^{(n)} \Omega^{\mu\nu} p_\nu/m^2$. Since the coefficients $(k^{(n)})^{\mu\nu}$ are assumed perturbative, the inverse of the matrix $\Omega$ exists. Left multiplication of the implicit expression for $u^\mu$ with $u^\mu(\Omega^{-1})_{\mu\nu}$ then yields an expression for $(L^{(n)})^2$. Identifying the physical root by requiring that the usual result is recovered in the limit $(k^{(n)})^{\mu\nu} \to 0$ reveals that

$$L^{(n)}(u, k^{(n)}) = -m \sqrt{\eta^u(\Omega^{-1})_{\mu\nu} u^\mu}. \quad (7)$$

For the special case $n = 4$, this matches the result in Ref. [30]. Notice that smoothness of $L^{(n)}$ fails for any $n$-velocity for which $u^\mu(\Omega^{-1})_{\mu\nu} u^\nu = 0$. This reflects the deformation of the light cone introduced by the coefficients $(k^{(n)})^{\mu\nu}$ and it parallels the failure of smoothness of the standard free-particle lagrangian $L(u) = -m \sqrt{\eta^u u^\mu u_\mu}$ when $u^\mu u_\mu = 0$.

For field theories having coefficients with $d \geq n + 1$, the dispersion relation can still be written in the form $p_\mu \Omega^{\mu\nu} p_\nu = m^2$, but with $\Omega^{\mu\nu} = \eta^{\mu\nu}(p)$ now a function of the $n$-momentum. Following the above procedure then leads to a higher-order polynomial in $\mathcal{L}$ for which explicit solution is typically impossible. Nonetheless, the intermediate steps provide useful implicit expressions. Incorporating arbitrary $k$ coefficients for $d \geq n + 1$, we find an implicit expression for the $n$-velocity to be

$$u^\mu = -L p_\mu \eta^{\mu\nu} \left[ \frac{1}{2} \sum_{d} (d-n+2) p_{a_1} \ldots p_{a_{d-2}} (k^{(d)})^{a_{d-1} a_{d-2} a_1 \ldots a_{d-3} \nu} \right] \times \left[ \eta^{\mu\nu} + \frac{1}{2} \sum_{d} (d-n) p_{a_1} \ldots p_{a_{d-2}} (k^{(d)})^{a_{d-1} a_{d-2} a_1 \ldots a_{d-3} \nu} \right]^{-1}, \quad (8)$$

where the sums are over all values of $d \geq n+1$. Contraction with $u_\mu$ yields a quadratic polynomial for $L$, the solution of which gives the implicit expression for the lagrangian as

$$L = -m^2 \bar{a}^2 u^\mu \eta^{\mu\nu} \left[ \frac{1}{2} \sum_{d} (d-n+2) p_{a_1} \ldots p_{a_{d-2}} (k^{(d)})^{a_{d-1} a_{d-2} a_1 \ldots a_{d-3} \nu} \right]^2 \times \left[ \eta^{\mu\nu} + \frac{1}{2} \sum_{d} (d-n) p_{a_1} \ldots p_{a_{d-2}} (k^{(d)})^{a_{d-1} a_{d-2} a_1 \ldots a_{d-3} \nu} \right]^{-1}, \quad (9)$$

where $\bar{a} = \sqrt{\frac{\eta^{\mu\nu} u_\mu u_\nu}{m^2}}$.

Direct manipulation of the results (8) and (9) to extract $L(u, k)$ is infeasible in many cases. However, we can develop an iterative method that generates the solution as a series in powers of the coefficients for Lorentz violation. The idea is to expand both implicit expressions for $L(u, k)$ and $u^\mu$ as power series in $k$ and then to perform successive substitutions to derive an expression for $L(u, k)$ valid at the chosen order in $k$. For simplicity, we illustrate the method in the special case of a model with only one nonzero coefficient $(k^{(d)})^{a_1 \ldots a_{d-2} a_1 \ldots a_{d-3} \nu}$ of mass dimension $n-d$, denoting the resulting lagrangian by $L^{(d)}$. However, the results presented below can be generalized to more complicated scenarios as desired.

The first step is to expand the implicit lagrangian (9) in powers of $k$. For the chosen model, we find

$$L^{(d)} = \frac{1}{2} (d-n+2) u_{a_1} p_{a_2} \ldots p_{a_{d-2}} (k^{(d)})^{a_{d-1} a_{d-2} a_1 \ldots a_{d-3} \nu} \times m \bar{a} \sum_{q=1}^{d} \sum_{q=1}^{d} (-1)^q a_{q_1} \times [(d-n+2) u_{a_1} p_{a_2} \ldots p_{a_{d-2}} (k^{(d)})^{a_{d-1} a_{d-2} a_1 \ldots a_{d-3} \nu}]^{q-1} \times [(d-n) u_{a_1} p_{a_2} \ldots p_{a_{d-2}} (k^{(d)})^{a_{d-1} a_{d-2} a_1 \ldots a_{d-3} \nu}]^{q-1}, \quad (10)$$

where

$$a_{q_1} = \frac{(2q)!}{m^2 \bar{a}^2 (2q-1) 8^{q} q! s!(q-s)!}. \quad (11)$$
Given \( L^{(d)} = L^{(d)}(u, p, k^{(d)}) \) expressed as Eq. (10), the iteration then proceeds as follows. The zeroth-order lagrangian \( L_0^{(d)} \equiv L^{(d)}(u, p, 0) = -m\overline{u}u \) is defined as the limit of vanishing \( k^{(d)} \). The corresponding zeroth-order momentum is \( (p_0)_\mu = -\partial L_0^{(d)}/\partial \overline{u}^\mu = m\overline{u}\overline{u} \). The \( q \)-th order lagrangian is then defined by inserting the \((q-1)\)th-order momentum into Eq. (10), \( L^{(d)} = L^{(d)}(u, p_{q-1}(u), k^{(d)}) \), keeping only terms up to the \( q \)th power of \( k^{(d)} \). The \( q \)-th order momentum is obtained in the canonical way by differentiation, \( (p_q)_\mu = -\partial L^{(d)}_q/\partial \overline{u}^\mu \).

This iteration method shows that \( L^{(d)}(u, k^{(d)}) \) can be determined to any order in \( k^{(d)} \) and that the explicit relationship between the \( n \)-momentum and the \( n \)-velocity is obtained at each step. Smoothness of the lagrangian \( L^{(d)}(u, k^{(d)}) \) outside the usual slit \( S_0 \equiv \{ u^\mu | \overline{u} = 0 \} \) is then insured at any order in Lorentz violation. Note that the derivation of the result (10) involves expanding the radical in the implicit expression (9), which implies the allowed values of \( k^{(d)} \) are constrained. A first-order form of the constraint is that the magnitude of the ratio of the summands in the radical is bounded above by unity and inserting the zeroth-order momentum \((p_0)_\mu\), giving \([k^{(d)}]_{\alpha_1...\alpha_{q-2}}\overline{u}_{\alpha_1}...\overline{u}_{\alpha_{q-1}} < 2/(d - n) \) for \( d > n \). Potential convergence issues arising in the limit of an infinite sum over \( d \) are tied to the corresponding definition of the effective field theory in that limit and hence are moot in the present context.

As an illustration in the context of the chosen model, we present here the results of a calculation using this iterative method applied to third order in the coefficient for Lorentz violation. The third-order lagrangian is found to be

\[
L^{(d)}_3 = L_0^{(d)} \left[ 1 - \frac{1}{8!} \left( \overline{u}^2 \right)^2 \right] + \frac{1}{12} \left( d - n + 2 \right) \left( d - n + 1 \right) \left( k^{(d)} \right)^3 \\
+ \frac{1}{16} \left( d - n + 1 \right) \left( d - n + 2 \right) \left( \overline{u}^2 \right)^2 \left( k^{(d)} \right) \left( \overline{u}^2 \right)^2 \\
- \frac{1}{16} \left( d - n + 1 \right) \left( d - n + 2 \right) \left( k^{(d)} \right)^2 \left( \overline{u}^2 \right)^2 \left( \overline{u}^2 \right)^2 \
\]

(12)

where we have introduced the dimensionless quantities

\[
\overline{u}^{\alpha_1...\alpha_{q-2}} = \eta^{\alpha_1...\alpha_{q-2}} (k^{(d)})_{\alpha_1...\alpha_{q-2}} \overline{u}_{\alpha_1}...\overline{u}_{\alpha_{q-2}} \\
\]

(13)

with \( \overline{u}^\mu \equiv \overline{u}^\mu / \overline{u} \). This expression is indeed smooth away from \( \overline{u} = 0 \), as expected. Note that although the derivation assumes \( d \geq n + 1 \), the results also hold for \( d = n - 1 \) and \( d = n \). For the former case, the expression (12) directly matches the analytical result. For the latter case, all the quantities \( a_{\alpha_1...\alpha_{q-2}} \) given in Eq. (11) vanish except when \( s = q \), leaving the expected third-order approximation to the exact result (11).

In the context of the general field theory for the propagation of Dirac fermions in \( n = 4 \) spacetime dimensions in the presence of arbitrary Lorentz violation [34], Reis and Schreck used an ansatz-based technique to obtain the corresponding classical lagrangian for the analogue particle at leading order in coefficients for Lorentz violation [35]. The resulting effects of spin-independent Lorentz violation can be expected to match those of the theory (11) because the latter contains all possible spin-independent effects for a propagating particle. Indeed, we can confirm that a match exists to the first-order part of the expression (12) with \( n = 4 \), via the correspondences \( \delta^e \leftrightarrow \delta^{(d)}(\alpha_1...\alpha_{q-2})u^{\alpha_1}...u^{\alpha_{q-2}} \) and \( \overline{\delta}^e \leftrightarrow \frac{1}{2} \left( k^{(d)} \right)_{\alpha_1...\alpha_{q-2}} u^{\alpha_1}...u^{\alpha_{q-2}} \).

Substituting these correspondences into the full expression (12) is therefore expected to generate the third-order lagrangian describing spin-independent Lorentz-violating effects on the propagation of a Dirac particle. Similar results can be anticipated for spin-independent Lorentz-violating effects on photon [29] and neutrino [30] propagation as well.

4. Finsler geometry

A classical reparametrization-invariant point-particle lagrangian is a smooth real-valued function on the slit tangent bundle that is 1-homogeneous in the velocity and that yields the equation of motion via a variational principle. Its features have parallels with those of a Finsler structure underlying a Riemann-Finsler geometry, with key differences being the signature of the metric and the requirement of positivity. These differences could conceivably be obviated via a suitable definition of Lorentz-Finsler geometry, producing a relationship to Riemann-Finsler geometry geometry that to between Lorentz and Riemann geometry.

To date, no completely satisfactory and widely accepted definition of Lorentz-Finsler geometry exists. Various approaches have been suggested including, for example, those in Refs. [31, 42, 43]. However, relaxing the positivity requirement in a consistent way while including all natural physical examples of point-particle lagrangians remains an elusive goal. For instance, a sophisticated recent effort is the causality-based construction of Javaloyes and Sánchez [43], which succeeds in incorporating special cases of the \( a \) and \( b \) lagrangians derived from effective field theory with Lorentz violation [36] and also exposes sharply the challenge of finding a definition that includes other related physical examples.

The results of Sec. 3 above play two primary roles in the context of Finsler geometry [36]. First, we can promote the Minkowski metric \( \eta_{\mu\nu} \) to a spacetime metric \( g_{\mu\nu}(x) \) and allow position dependence of the coefficients. This procedure generates all the classical lagrangians controlling dominant effects on the spin-independent propagation of a particle in a general spacetime background perturbed by arbitrary Lorentz violation. It thereby substantially increases the known physical examples offering potential guidance in the search for a suitable definition of Lorentz-Finsler geometry.

Second, independently of the definition of Lorentz-Finsler geometry, we can focus instead on the issue of generating a Finsler structure for a Riemann-Finsler geometry from a classical point-particle lagrangian with Lorentz violation. The existence of this relationship is of direct interest in its own right, particularly since a subset of the mathematical properties derived for the lagrangian formulation can be expected to transfer to the Riemann-Finsler geometry. In the present context, the results obtained in Sec. 3 can be used to generate all Riemann-Finsler geometries associated with spin-independent Lorentz violation that are perturbations of conventional Riemann geometry. The classification and enumeration of these lagrangians is therefore expected to establish a corresponding classification of Riemann-Finsler spaces that are perturbatively related to a Riemann space.
Several methods can be countenanced to establish the desired relationship. The most direct procedure amounts to defining a suitable analytic continuation of the spacetime coordinates and derivatives, the coefficients for Lorentz violation, and the lagrangian, thereby yielding directly a Finsler structure for a Riemann-Finsler geometry. This method has some features in common with a Wick rotation in quantum field theory, and we adopt it in what follows. Other possible approaches could include converting the original quantum field theory to its euclidean counterpart via analytic continuation and then performing an analysis in parallel with that in Sec. 3 above, or implementing a projection or truncation of the spacetime to the purely spatial subspace and suitably adapting the classical lagrangian. Investigation of these alternative options and of their uniqueness and potential equivalence would be of interest but lies outside our present scope.

Starting with the results presented in Sec. 3, the continuation is implemented via the mappings \( u^i \to \hat{p}^i y^j, p_\mu \to (-i)^{\mu} p_j, (k^{(d)} y^i)^{-1} \to \hat{p}^i (k^{(d)} y^j)^{-1} \), and \( L \to -F = -y \cdot p \). For convenience and to match conventions in the Riemann-Finsler literature, we also impose \( m \to 1 \). In these expressions, the \( n \) spacetime dimensions \( x^\mu \) labeled with Greek indices are replaced with \( n \) spatial dimensions \( x^i \) labeled with Latin indices. Also, \( N \) is a generic symbol representing the number of spacelike indices present in a quantity prior to its continuation. For instance, the timelike component of \( u^i \) acquires a factor of 1, while each spacelike component acquires a factor of \( i \). The resulting expressions can naturally be rewritten using the euclidean metric. We can then promote this metric to a Riemann metric \( \hat{r}_{jk}(x) \) and allow spacetime dependence of the coefficients, in parallel with the procedure discussed above for the spacetime case. As an example, this produces the map \( \sqrt{\hat{w}^{\mu} \eta_{\mu \nu} u^\nu} \to \sqrt{y^j \hat{r}_{jk}(x) y^k} \).

When implemented on the broad set of classical lagrangians associated to effective field theories with Lorentz violation, the above procedure yields Finsler structures for Riemann-Finsler geometries that are perturbatively related to a Riemann space \([1]\). Note that the original theory \([1]\) is defined using components \((k^{(d)} y^i)^{-1}\) of tensors in \( \otimes TM \) and hence generates a Finsler structure in terms of tensor components \((k^{(d)} y^i)^{-1}\). Starting instead with cotensor or mixed-tensor components generates a family of distinct Finsler structures related by factors of \( r_{jk} \). However, the \( y \) dependence is unaffected, so the results obtained below for \((k^{(d)} y^i)^{-1}\) can be directly transcribed to any other desired member of the family.

Using this technique, the results in Sec. 3 for the velocity-momentum relation \([8]\) and the classical lagrangian \([9]\) become implicit expressions for the Finsler structure of a Riemann-Finsler geometry,

\[
y^j = \frac{F p_\mu \left( p^\mu + \frac{1}{2} \sum_{d} (d - n + 2) p_{i_1} \ldots p_{i_{d-2}} (k^{(d)} y^i)^{-1} \right)}{1 + \frac{1}{2} \sum_{d} (d - n) p_{i_1} \ldots p_{i_{d-2}} (k^{(d)} y^i)^{-1} \right)}
\]

and

\[
F = \left( \sqrt{y^j} \right)^2 + \frac{1}{2} \left( \sum_{d} (d - n + 2) p_{i_1} \ldots p_{i_{d-2}} (k^{(d)} y^i)^{-1} \right)^2 + \frac{1}{2} \left( \sum_{d} (d - n) p_{i_1} \ldots p_{i_{d-2}} (k^{(d)} y^i)^{-1} \right)^2
\]

\[
-\frac{1}{2} \sum_{d} (d - n + 2) p_{i_1} \ldots p_{i_{d-2}} (k^{(d)} y^i)^{-1} \right)
\]

where \( \sqrt{y^j} \) is smooth on the slit bundle \( TM \setminus S_0 \), where \( S = S_0 + S_1 \) contains the usual slit \( S_0 \) containing \( y^j = 0 \) but is extended to include other roots of the expression in the radical above. This geometry is therefore genuinely \( y \) local, although for certain restrictions the geometry may be resolvable along \( S_1 \).

Direct solution of the above implicit results to yield an explicit expression for \( F \) is typically impractical. To extract an explicit result, we can instead parallel the iteration procedure described in Sec. 3 and thereby generate the \( q \)-th order Finsler structure \( F_q \). In this context, it is natural to define

\[
\hat{k}_{\mu \nu} = (k^{(d)} y^i)^{-1} \hat{r}_{\mu \nu}(x) y^i
\]

with \( \hat{y} \equiv y/\sqrt{y} \), as the iteration introduces these combinations. Note that the indices on \((k^{(d)} y^i)^{-1}\) are lowered using the Riemann metric \( \hat{r}_{jk} \). Also, the number of indices on \( \hat{k}_{\mu \nu} \) reveals the number of contractions with \( \hat{y} \). For example, \( \hat{k}_{\mu \nu} \) denotes contraction of all indices. As before, the iteration procedure involves an expansion in powers of the coefficients of the radical in Eq. \([13]\). Requiring the magnitude of the ratio of the summands in the radical to be bounded above by unity at first order and inserting the unperturbed momentum \( p_j = \hat{y}_j = y^j/\sqrt{y} \) yields the constraint \( |k^{(d)} y^i| < 2/(d - n) \).

At first iteration order and keeping coefficients of arbitrary \( d \), the iteration produces the compact expression

\[
F_1 = \sqrt{\hat{y}} - \frac{1}{2} \sum_{d} \hat{k}_{\mu \nu} d d \hat{y} \left( \hat{y} \right)^{1/2}
\]

This Finsler structure is smooth on the usual slit bundle \( TM \setminus S_0 \), so the corresponding geometry is \( y \) global. Indeed, the same is true for \( F_q \) at any finite \( q \) because the process generates a series of terms in powers of \( (k^{(d)} y^i)^{-1} \), and the latter is smooth away from \( y^j = 0 \). Note that \( F_q \) for any given \( q \) can be viewed either as generating an approximation to the full geometry implied by Eqs. \([14]\) and \([15]\) or as an independent Finsler structure yielding a \( y \) global geometry of interest in its own right. Note also that \( F_1 \) is reversible, \( F_1(y) = F_1(-y) \), iff \( (k^{(d)} y^i)^{-1} \) vanishes. This property holds at any iteration order as well. Reversibility of a Riemann-Finsler geometry corresponds to CPT invariance in effective field theories with \( n = 4 \). Imposing it eliminates half of the allowed values of \( d \) in Eq. \([17]\).

At higher iteration orders, the mixing of coefficients of different \( d \) makes \( F_q \) unwieldy. Also, the Finsler metric \( g_{jk} = (F^2)^{1/2}/2 \) involves derivatives of the square of \( F_q \), which introduces further mixing and yields burdensome expressions. To
gain insight through direct calculations, it is therefore useful to consider the special case with only one nonzero coefficient. For example, the explicit form of the third-order Finsler structure $F_3^{(d)}$ can be found immediately by continuation from Eq. (12).

However, for our purposes below it suffices to limit attention to the first-order Finsler structure

$$F_1^{(d)} = \nabla - \frac{1}{2} \tilde{\kappa}^{(d)}.$$  \hfill (18)

In the context of the above discussion, the derivation of this expression assumes $d > n$. However, for the case $d = n$ we can understand $F_1^{(d)}$ as implementing a linearized shift of a conventional Riemann metric, while for the case $d = n - 2$ we see that $F_1^{(d-2)}$ is merely the usual Riemann geometry with a scaled mass. Also, for the case $d = n - 1$, inspection reveals that $F_1^{(d-1)}$ is the standard Randers structure built with the 1-form $(k_n^{n-1}) y^i/2$. We can therefore extend the interpretation of Eqs. (17) and (18) to $d \geq n-2$ when desired. Note that from this perspective the Finsler structure $F_1^{(d)}$ with $d > n$ can be viewed as a natural generalization of the Randers structures, in which the 1-form is replaced by a symmetric $(d - n + 2)$-form. In a similar vein, $F_1^{(d)}$ can be viewed as a generalization of the Finsler structure for a geometry with an $(\alpha, \beta)$ metric, in which $\alpha \equiv \nabla$ and the 1-form $\beta$ is generalized to a symmetric $(d - n + 2)$-form.

To verify that $F_1^{(d)}$ is indeed a Finsler structure, certain conditions must be met [44]. One is positive homogeneity in $y^i$, which is evident by inspection. Another is smoothness on the usual slit bundle $TM \backslash S_0$, which holds as already noted above. A third is nonnegativity, which is achieved when $1 - \frac{1}{2} \tilde{\kappa}^{(d)} > 0$. This condition is automatically satisfied when $\tilde{\kappa}^{(d)} < 2/(d - n)$, which is the constraint obtained above from expanding the radical in Eq. (15).

Another condition is positivity of the Finsler metric. Imposing this can be expected to translate into an additional constraint on $\tilde{\kappa}^{(d)}_{j...j}$ in terms of $d$ and $n$. Here, we derive this constraint explicitly at leading order in $\tilde{\kappa}^{(d)}_{j...j}$. At this order, we find the Finsler metric $g_{(d)}^{(d)}$ is given by

$$g_{(d)}^{(d)} = r_{\tilde{k}}[1 + \frac{1}{2}(d - n)\tilde{\kappa}^{(d)}] - \frac{1}{2}(d - n + 1)(d - n + 2)\tilde{\kappa}^{(d)}_{j...k} + \frac{1}{2}(d - n)(d - n + 2)\tilde{\kappa}^{(d)}_{j...k} y^j y^k - \tilde{\kappa}^{(d)}_{j...k} y^j y^k).$$ \hfill (19)

As expected, for $d = n$ this result represents a simple scaling of $r_{\tilde{k}}$, while for $d = n - 1$ it reduces to the linearized Randers metric.

An argument for positivity of the metric (19) can be made in terms of the positivity of its determinant $g_{(d)}^{(d)}$, so we first consider the latter. It is convenient to define $\kappa \equiv \max[j \tilde{\kappa}^{(d)}_{j...j}]].$ We then find $[\tilde{\kappa}^{(d)}] = [(\kappa^{(d)}_{j...j}) \leq \tilde{y}^j \cdots \tilde{y}^j] \leq n^{d-n+2} \kappa$. Similarly, $[\tilde{\kappa}^{(d)/}] \leq [\tilde{y}^j] n^{d-n+1} \kappa$. Writing $g_{(d)}^{(d)} = r_{\tilde{k}} + h_{(d)}$ implies $\det g = (1 + h/r) \det r$ at first order, where the trace is with respect to $(r^{-1})_{(d)}$. The triangle inequality then yields the relation

$$|\det g| \leq \det (r_{(d)} + h_{(d)}) - (r_{(d)} + h_{(d)}) h_{(d)} - (d - n + 1)(d - n + 2)\tilde{\kappa}^{(d)}_{j...k}.$$

It follows that $\det g^{(d)} > 0$ at linear order when

$$\kappa < \frac{2}{[(d - n + 1)(d - n + 2) + (d - n)(d + 2)n^{d-n+1}].}$$ \hfill (21)

For $d > n$, the smallest value of $d$ is $d = n + 1$, which gives $\kappa < 2/n^2(3 + n + 6)$. For example, if the Finsler structure is derived from a field theory in $(3 + 1)$ spacetime dimensions, then $n = 4$ and the smallest value $d = 5$ imposes $\kappa < 1/272$. More generally, this shows that for any case with $d > n$ sufficiently small coefficients can be found that ensure positivity of $\det g^{(d)}$ at linear order. A standard argument [44] then suffices to show positivity of the metric at linear order. Introducing $F_1^{(d)} = \nabla - \frac{1}{2} \tilde{\kappa}^{(d)}$, it follows from the above argument that $\det g^{(d)} > 0$, and so $g_{(d)}^{(d)}$ has no vanishing eigenvalues. Since $g_{(d)}^{(d)} \rightarrow r_{(d)}$ with positive eigenvalues when $\epsilon \rightarrow 0$, the eigenvalues must stay positive as $\epsilon \rightarrow 1$, and so $g_{(d)}^{(d)}$ must be positive definite at linear order. For sufficiently small $\kappa$, we expect positivity to hold at higher orders in $\tilde{\kappa}^{(d)}_{j...j}$ as well, but a formal proof of this remains open at present.

5. Some properties of $k$ spaces

Next, to gain insight about the various Riemann-Finsler spaces governed by $(\kappa^{(d)}_{j...j})$, we perform some explicit calculations for the Finsler structure (18). The expressions for key properties below are derived at first order in $\tilde{\kappa}^{(d)}_{j...j}$.

Consider first the Hilbert form $\omega \equiv F_{j...d} y^{x^j}$ for a given Finsler structure $F$. This is a section of the pullback bundle $\pi^* T^* M$ defined globally on the usual slit bundle $TM \backslash S_0$ [44]. The components $p_j = F_{j...d}$ are the Riemann-Finsler analogues of the components of the n-momentum per mass in the corresponding classical lagrangian. A short calculation for the Finsler structure $F_1^{(d)}$ reveals

$$p_{(d)}^{(d)} = [1 + \frac{1}{2}(d - n + 1)\tilde{\kappa}^{(d)}] \tilde{y}^j - \frac{1}{2}(d - n + 2)\tilde{\kappa}^{(d)}_{j...j}. \hfill (22)$$

In the Riemann limit with $\tilde{\kappa}^{(d)} \rightarrow 0$, $p_{(d)}^{(d)} \rightarrow \tilde{y}^j$, aligned with the velocity. The presence of nonzero $(\kappa^{(d)}_{j...j})$ scales this result in a direction-dependent way and shifts it by a direction-dependent covector, so that $p_j$ and $\tilde{y}^j$ generally become linearly independent.

For $d > n$, we can show explicitly that none of the Riemann-Finsler $k$ spaces with Finsler structure $F_1^{(d)}$ are Riemann geometries. The noneuclidean aspects of a Finsler structure $F$ interpreted as a Minkowski norm on any tangent space $T_x M$ are captured by the Cartan torsion $C_{jkl} \equiv (g_{jk} y^l) / \epsilon$, which according to Deicke’s theorem [45] vanishes only for Riemann geometries. We find that the first-order Cartan torsion is

$$C_{jkl}^{(d)} = \frac{1}{4\tilde{\gamma}} (d - n)(d + n + 2) \times \sum_{(d)} \left[ (\frac{1}{2}(d - n + 1)\tilde{\kappa}^{(d)}_{jkl} - (d - n)(d + 2)\tilde{\kappa}^{(d)}_{jkl}) y^l - \frac{1}{4}(d - n + 1)\tilde{\kappa}^{(d)}_{jkl} y^l + \hat{r}_{jkl} - (d - n + 2)\tilde{\kappa}^{(d)}_{jkl} y^l \right]. \hfill (23)$$
This Cartan torsion vanishes for \( d = n \) and \( d = n - 2 \), in agreement with our earlier identification of \( F_{1}^{(n)} \) and \( F_{1}^{(n-2)} \) as Finsler structures for Riemann geometries. Inspection reveals that the Cartan torsion also vanishes for \( n = 1 \), as is appropriate for a Riemann curve. However, the mean Cartan torsion is nonzero for other values of \( n \) and \( d \), indicating that in those cases the Finsler structures \( F_{1}^{(d)} \) cannot correspond to Riemann geometries. In the reversible scenario with \( (k_{ij}^{(d)})^r = 0 \) on a compact surface, this implies the Finsler metric has nonconstant flag curvature [46, 47]. In the nonreversible scenario for \( F_{1}^{(d)} \), a Finsler metric with constant positive flag curvature may exist and would be interesting to display [48].

The Cartan torsion (23) is nonvanishing for \( d = n - 1 \). The corresponding space is identified in Sec. 4 as a Randers geometry. According to the Matsumoto–Hójó theorem [49]. Randers spaces are distinguished by a nonvanishing Cartan torsion together with a vanishing Matsumoto torsion \( M_{jk} = C_{kl} - \sum_{(jl)} i_{jk} j_{l} (n + 1) \), where \( I_{jk} = e^{ijk} C_{kl} \) is the mean Cartan torsion and \( h_{jk} \equiv F_{jk} / \gamma_{j} \) is the angular metric. For the Finsler structure \( F_{1}^{(d)} \), the mean Cartan torsion at first order is

\[
\Gamma_{j}^{(d)} = \frac{1}{2} \left[ (d - n)(d - n + 2) \left( (d + 2) \tilde{k}_{ij}^{(d)} j_{i} - \tilde{k}_{ij}^{(d)} j_{i} \right) - (d - n + 1) \left( \tilde{k}_{ij}^{(d)} j_{i} - \tilde{k}_{ij}^{(d)} j_{i} \right) \right].
\] (24)

Calculation then yields the first-order Matsumoto torsion as

\[
M_{jk}^{(d)} = \frac{1}{2} \left[ (d - n)(d - n + 1)(d - n + 2) \right] \left[ \sum_{(jl)} (n - 2) \tilde{k}_{ij}^{(d)} j_{i} j_{l} j_{l} + r_{jk} (\tilde{k}_{ij}^{(d)} j_{i} - \tilde{k}_{ij}^{(d)} j_{i}) - (d - n + 1) \left( \tilde{k}_{ij}^{(d)} j_{i} - \tilde{k}_{ij}^{(d)} j_{i} \right) \right].
\] (25)

For the Randers value \( d = n - 1 \), this expression indeed vanishes. The Matsumoto torsion \( M_{jk}^{(d)} \) also vanishes for the Riemann values \( d = n \) and \( d = n - 2 \), as expected. However, it is nonvanishing for \( d > n \), which establishes that none of the corresponding spaces are Randers geometries. Note also that this result holds for \( n = 2 \), which implies the \( k \) spaces with \( d > 2 \) must be distinct from \( b \) space because the latter reduces for \( n = 2 \) to Randers geometry [5]. This distinction is consistent with the different nature of the \( k \)-space and \( b \)-space coefficients as bases for representations of the rotation group \( O(n) \) and might be anticipated because \( b \) space is related to spin-dependent Lorentz violation, unlike the \( k \) spaces. A similar argument suggests the \( k \) spaces differ from other Riemann–Finsler geometries related to spin-dependent Lorentz violation, including the various \( H \) spaces considered in Ref. [50].

Another approach to Riemann–Finsler geometry is through geodesic sprays [51]. For any choice of speed or diffeomorphism gauge, a Riemann–Finsler geodesic is a solution of the equation

\[
F \frac{d}{d\lambda} \left( \gamma_{j}^{i} \right) + G_{j}^{i} = 0,
\] (26)

where \( G_{j}^{i} = g^{k} \left( \partial_{i} \gamma_{j}^{k} + \partial_{k} \gamma_{i}^{k} - \partial_{k} \gamma_{i}^{k} \right) \gamma_{j}^{k} / 2 \) are the spray coefficients. Denoting the Christoffel symbol for \( r_{jk} \) by \( \gamma_{j}^{i} \) = \( 1 \Gamma_{j}^{i} / F \) and the covariant derivative with respect to \( r_{jk} \) by \( \tilde{\partial} \), some calculation reveals that the first-order spray coefficients for the Finsler structure \( F_{1}^{(d)} \) can be expressed as

\[
\frac{1}{\gamma_{j}^{i}} G_{j}^{i} + \frac{1}{2} \tilde{D}_{k}^{(d)} + \frac{1}{2} (d - n) \gamma_{j}^{i} \tilde{D}_{k}^{(d)}
\] (27)

Here, a bullet \( \bullet \) indicates contraction of a lower index \( j \) with \( \gamma_{j} \), and all contractions with \( \gamma_{j} \) are understood to be taken outside any derivatives. Note that the spray coefficients are homogeneous of degree two in \( \gamma_{j} \).

The expression (27) reveals the noteworthy result that if the coefficients are \( r \)-parallel, \( \tilde{D}_{k}^{(d)} \gamma_{j}^{i} = 0 \), then the first-order spray coefficients reduce to Riemann ones and hence the presence of \( r \)-parallel \( (k_{ij}^{(d)})^{r} = 0 \) leaves unaffected the geodesic curves. It turns out that the analogous result also holds for \( ab \) and \( f \) space [5]. Taken together, these results support the conjecture that any \( r \)-parallel coefficient leaves Riemann geodesics unaffected. Local conditions along a geodesic appear sufficiently uniform in a geometry with \( r \)-parallel coefficients that nonzero \( (k_{ij}^{(d)})^{r} = 0 \) cannot be observed. At the level of the effective field theory, these results lend weight to the open possibility of removing \( r \)-parallel coefficients using field redefinitions and coordinate choices similar to those already used to remove unphysical coefficients in certain limits of the SME [5, 26, 50].

The spray coefficients can be used to derive many useful quantities in Riemann–Finsler geometry [44, 57]. One is the nonlinear connection, which can be defined as \( N_{jk}^{i} \equiv (G_{j}^{i} / F)^{1/2} \). Using this definition and the homogeneity properties of the spray coefficients, we can write \( \gamma_{j}^{i} = \gamma_{j}^{i} / F \). For the Finsler structure \( F_{1}^{(d)} \), a calculation reveals that the first-order nonlinear connection \( N_{jk}^{i} \) takes the form

\[
\frac{1}{\gamma_{j}^{i}} N_{jk}^{i} = \gamma_{j}^{i} \gamma_{j}^{k} + \frac{1}{2} (d - n)(\gamma_{j}^{i} \tilde{D}_{k}^{(d)} + \gamma_{k}^{i} \tilde{D}_{j}^{(d)})
\] (28)

Note that this expression reduces to its Riemann equivalent for \( r \)-parallel coefficients \( (k_{ij}^{(d)})^{r} = 0 \).

Various connections for Riemann–Finsler geometry can be derived from the nonlinear connection. One is the Berwald connection \( B_{ij}^{kl} = (N_{ij}^{k} - N_{ij}^{l}) \). Its explicit form for the Finsler structure \( F_{1}^{(d)} \) is somewhat cumbersome, so we omit it here. However, the expressions given above imply that \( B_{ij}^{kl} = \gamma_{j}^{i} \gamma_{k}^{l} \) for \( r \)-parallel coefficients \( (k_{ij}^{(d)})^{r} = 0 \). It follows that the Berwald h-v curvature defined as \( B P_{j}^{i} = -F(B_{j}^{ik} \gamma_{k}^{l}) \) vanishes in this case. The \( r \)-parallel \( k \) spaces of this type are therefore Berwald spaces. This result adds further support to the open conjecture that any SME-based Riemann–Finsler space is a Berwald space if it has \( r \)-parallel coefficients [5], which was previously proved for Randers space [52, 55].
Another quantity of importance is the Chern connection \( [56] \), which is defined as \( \Gamma_{ijkl} = \partial_i \chi_j + \partial_j \chi_i + \partial_k \chi_l - \partial_k \chi_i - \partial_l \chi_j \), where \( \chi \) is a connection. For the Finsler structure \( F^{(d)} \), some calculation reveals the first-order expression
\[
\Gamma_{ijkl} = \frac{1}{r^d(d - n)(d - n + 2)} \sum_{(jkl)} \left[ (r_{ij} - (d - n + 2)\delta_{ij}\tilde{D}^{\tilde{k}}_{mij} \right]
\]
where \( r_{ij} = -\frac{1}{\sqrt{g}} \delta_{ij} \). This reduces to the usual Levi-Civita connection for \( r \)-parallel coefficients.

As a final remark, we note that other widely used connections may also be calculated for the \( C^j_{ml} \delta^d_k \) case. Another example is the Hashiguchi connection, which can be defined as \( \partial^d_{ij} = \partial_i \chi_j + \partial_j \chi_i + \partial_k \chi_l - \partial_k \chi_i - \partial_l \chi_j \). However, these connections fail to reduce to the Riemann result even for \( r \)-parallel coefficients due to the nonvanishing Cartan tensor.

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