ABS Methods and ABSPACK for Linear Systems and Optimization, a Review

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Running title: A review of ABS methods
Abstract

ABS methods are a large class of methods, based upon the Egervary rank reducing algebraic process, first introduced in 1984 by Abaffy, Broyden and Spedicato for solving linear algebraic systems, and later extended to nonlinear algebraic equations, to optimization problems and other fields; software based upon ABS methods is now under development. Current ABS literature consists of about 400 papers. ABS methods provide a unification of several classes of classical algorithms and more efficient new solvers for a number of problems. In this paper we review ABS methods for linear systems and optimization, from both the point of view of theory and the numerical performance of ABSPACK.

Key Words ABS methods, linear algebraic systems, feasible direction methods, simplex method, Diophantine equations, ABSPACK.

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1 Introduction

ABS algorithms were introduced by Abaffy, Broyden and Spedicato (1984), to solve linear equations first in the form of the basic ABS class, later generalized as the scaled ABS class. They were then applied to linear least squares, nonlinear equations and optimization problems, see e.g. the monographs by Abaffy and Spedicato (1989) and Feng et al. (1998), or the bibliography by Spedicato et al. (2000) listing 350 ABS papers. In this paper we review some of the main results obtained in the field of ABS methods and we provide some results on the performance of ABSPACK, a FORTRAN package based on ABS methods presently under development.

The main results obtained in almost twenty years of research in the ABS field can be summarized as follows:

- ABS methods provide a unification of the field of finitely terminating methods for linear systems and for feasible direction methods for linearly constrained optimization, see sections 2 and 6; due to the several alternative formulations of their algebra they lead to different computational implementations of the algorithms, each one with a special feature that may be of advantage

- ABS methods provide some new methods that are better than classical ones under several respects. For instance the implicit LX method requires the same number of multiplications as Gaussian elimination (which is optimal under very general conditions) but requires less memory and does not need pivoting in general; moreover its application to the simplex method has not only a lower memory requirement, but is cheaper in multiplications up to one order with respect to e.g. the Forrest-Goldfarb implementation of the simplex method via
the LU factorization, see section 5. Also there are ABS methods for nonlinear problems that are better than Newton method in terms of convergence speed (some ABS algorithms keep quadratic rate of convergence even if the Jacobian is singular at the solution) or in terms of required information (about \( n^2/2 \) Jacobian components against the full \( n^2 \))

- on some classes of significant problems ABSPACK codes have a better performance in both accuracy and speed that the codes in LAPACK (usually considered the best package now on the market); see section 7.

- ABS methods have allowed to solve open problems in the literature (e.g. the explicit determination of Quasi-Newton updates for the sparse symmetric case, see Spedicato and Zhao (1993))

- for linear Diophantine equations ABS methods have led to a new solution method, that also provides the first generalization of the classical existence theorem of Euclides, Diophantus and Euler from a single \( n \)-dimensional equation to a general \( m \)-equations system in \( n \) variables, see section 4.

2 The scaled ABS class: definition and main properties

Let us consider the following determined or underdetermined linear system, where \( \text{rank}(A) \) is arbitrary and \( A^T = (a_1, \ldots, a_m) \)

\[
Ax = b \quad x \in \mathbb{R}^n, \quad b \in \mathbb{R}^m, \quad m \leq n
\]  

or

\[
a_i^T x - b_i = 0, \quad i = 1, \ldots, m
\]

The above system can be solved by the following scaled ABS class of algorithms, which can be shown to provide a complete realization of the general class of methods implementing the Petrov-Galerkin criterion (stating that the residual vector at the \( i \)-th iteration is orthogonal to a given arbitrary set of \( i - 1 \) linearly independent vectors).

The scaled ABS algorithm

(A) Give \( x_1 \in \mathbb{R}^n \) arbitrary, \( H_1 \in \mathbb{R}^{m \times n} \) nonsingular arbitrary, \( v_1 \in \mathbb{R}^m \) arbitrary nonzero. Set \( i = 1 \).

(B) Compute the residual \( r_i = Ax_i - b \). If \( r_i = 0 \) stop (\( x_i \) solves the problem.) Otherwise compute \( s_i = H_i A^T v_i \). If \( s_i \neq 0 \), then go to (C). If \( s_i = 0 \) and \( \tau = v_i^T r_i = 0 \), then set \( x_{i+1} = x_i, \quad H_{i+1} = H_i \) (the \( i \)-th equation is redundant) and go to (F). Otherwise stop (the system has no solution).
(C) Compute the search vector $p_i$ by 

$$p_i = H_i^T z_i$$

where $z_i \in \mathbb{R}^n$ is arbitrary save for the condition 

$$v_i^T A H_i^T z_i \neq 0$$

(D) Update the estimate of the solution by 

$$x_{i+1} = x_i - \alpha_i p_i, \quad \alpha_i = v_i^T r_i / v_i^T A p_i.$$ 

(E) Update the matrix $H_i$ by 

$$H_{i+1} = H_i - H_i A^T v_i w_i^T H_i / w_i^T H_i A^T v_i$$

where $w_i \in \mathbb{R}^n$ is arbitrary save for the condition 

$$w_i^T H_i A^T v_i \neq 0.$$

(F) If $i = m$, stop ($x_{m+1}$ solves the system). Otherwise give $v_{i+1} \in \mathbb{R}^m$ arbitrary linearly independent from $v_1, \ldots, v_i$. Increment $i$ by one and go to (B).

Matrices $H_i$, which are generalizations of (oblique) projection matrices, have been named Abaffians at the First International Conference on ABS methods (Luoyang, China, 1991). However we must recall that these matrices were used in several little known papers by Egervary, predating the ABS algorithms. There are alternative formulations of the scaled ABS algorithms, e.g. using vectors instead of the square matrix $H_i$, with possible advantages in storage and number of operations. For details see Abaffy and Spedicato (1989) and on how these formulations can be used to imbed in an ABS approach even iterative methods (including the Kaczmarz, Gauss-Seidel, ORTHOMIN, ORTHODIR etc. methods), see Spedicato and Li (2000).

The choice of the parameters $H_1$, $v_i$, $z_i$, $w_i$ determines particular methods. The basic ABS class is obtained by taking $v_i = e_i$, as the $i$-th unit vector in $\mathbb{R}^m$.

We recall some properties of the scaled ABS class, assuming that $A$ has full rank.

- Define $V_i = (v_1, \ldots, v_i)$ and $W_i = (w_1, \ldots, w_i)$. Then 

$$H_{i+1}^T V_i = 0, \quad H_{i+1}^T W_i = 0$$

- The vectors $H_i A^T v_i$, $H_i^T w_i$ are zero if and only if $a_i$, $w_i$ are respectively linearly dependent from $a_1, \ldots, a_{i-1}$, $w_1, \ldots, w_{i-1}$. 

4
• Define $P_i = (p_1, \ldots, p_i)$ and $A_i = (a_1, \ldots, a_i)$. Then the implicit factorization $V_i^T A_i^T P_i = L_i$ holds, where $L_i$ is nonsingular lower triangular. Hence, if $m = n$, one obtains a semiexplicit factorization of the inverse, with $P = P_n$, $V = V_n$, $L = L_n$

$$A^{-1} = PL^{-1}V^T.$$ (9)

For several choices of $V$ the matrix $L$ is diagonal, hence formula (8) gives a fully explicit factorization of the inverse as a byproduct of the ABS solution of a linear system.

• The general solution of system (1) can be written as follows, with $q \in \mathbb{R}^n$ arbitrary

$$x = x_{m+1} + H_{m+1}^T q$$ (10)

• The Abaffian can be written in terms of the parameter matrices as

$$H_{i+1} = H_1 - H_1 A^T V_i (W_i^T H_1 A^T V_i)^{-1} W_i^T H_1.$$ (11)

Letting $V = V_m$, $W = W_m$, one can show that the parameter matrices $H_1$, $V$, $W$ are admissible (i.e. condition (7) is satisfied) iff the matrix $Q = V^T A H_1^T W$ is strongly nonsingular (i.e. it is LU factorizable). This condition can be satisfied by exchanges of the columns of $V$ or $W$. If $Q$ is strongly nonsingular and we take, as is done in all algorithms so far considered, $z_i = w_i$, then condition (4) is also satisfied. Analysis of the conditions under which $Q$ is not strongly nonsingular leads, when dealing with Krylov space methods in their ABS formulation, to a characterization of the topology of the starting points that can produce a breakdown (either a division by zero or a vanishing search direction) and to several ways of curing it, including those considered in the literature.

Three subclasses of the scaled ABS class and particular algorithms are now recalled.

(a) The conjugate direction subclass. This class is obtained by setting $v_i = p_i$. It is well defined under the condition (sufficient but not necessary) that $A$ is symmetric and positive definite. It contains the ABS versions of the Choleski, the Hestenes-Stiefel and the Lanczos algorithms. This class generates all possible algorithms whose search directions are $A$-conjugate. If $x_1 = 0$, the vector $x_{i+1}$ minimizes the energy ($A$-weighted Euclidean) norm of the error over $\text{Span}(p_1, \ldots, p_i)$ and the solution is approached monotonically from below in the energy norm.

(b) The orthogonally scaled subclass. This class is obtained by setting $v_i = A p_i$. It is well defined if $A$ has full column rank and remains well defined even if $m$ is greater than $n$. It contains the ABS formulation of the QR algorithm (the so called implicit QR algorithm), the GMRES and the conjugate residual
algorithms. The scaling vectors are orthogonal and the search vectors are $A^TA$-conjugate. If $x_1 = 0$, the vector $x_{i+1}$ minimizes the Euclidean norm of the residual over $\text{Span}(p_1, \ldots, p_i)$ and the solution is monotonically approached from below in the residual norm. It can be shown that the methods in this class can be applied to overdetermined systems of $m > n$ equations, where in $n$ steps they obtain the solution in the least squares sense.

(c) The optimally stable subclass. This class is obtained by setting $v_i = (A^{-1})^T p_i$, the inverse disappearing in the actual recursions. The search vectors in this class are orthogonal. If $x_1 = 0$, then the vector $x_{i+1}$ is the vector of least Euclidean norm over $\text{Span}(p_1, \ldots, p_i)$ and the solution is approached monotonically from below in the Euclidean norm. The methods of Gram-Schmidt and of Craig belong to this subclass. The methods in this class have minimum error growth in the approximation to the solution according to a criterion by Broyden.

3 Special algorithms in the basic ABS class

In this section we consider in more detail three specially important algorithms, that belong to the basic ABS class (i.e. $V = I$).

(A) The Huang algorithm is obtained by the choices $H_1 = I$, $z_i = w_i = a_i$. A mathematically equivalent, but numerically more stable, formulation is the so called modified Huang algorithm ($p_i = H_i(a_i)$ and $H_{i+1} = H_i - p_i p_i^T / p_i^T p_i$). Huang algorithm generates search vectors that are orthogonal and identical with those obtained by the Gram-Schmidt procedure applied to the rows of $A$. If $x_1 = 0$, then $x_{i+1}$ is the solution with least Euclidean norm of the first $i$ equations. The solution $x^+$ with least Euclidean norm of the whole system is approached monotonically and from below by the sequence $x_i$. For arbitrary starting point the Huang algorithm generates that solution which is closest in Euclidean norm to the initial point.

(B) The implicit LU algorithm is given by the choices $H_1 = I$, $z_i = w_i = e_i$. It is well defined iff $A$ is regular (i.e. all principal submatrices are nonsingular). Otherwise column pivoting has to be performed (or, if $m = n$, equation pivoting). The Abaffian has the following structure, with $K_i \in \mathbb{R}^{n-i,i}$

$$H_{i+1} = \begin{bmatrix} 0 & 0 \\ \cdots & \cdots \\ 0 & 0 \\ K_i & I_{n-i} \end{bmatrix}.$$  (12)

implying that the matrix $P_i$ is unit upper triangular, so that the implicit factorization $A = LP^{-1}$ is of the LU type, with units on the diagonal. The
algorithm requires for \( m = n \), \( n^3/3 \) multiplications plus lower order terms, the same cost of classical LU factorization or Gaussian elimination. Storage requirement for \( K_i \) requires at most \( n^2/4 \) positions, i.e. half the storage needed by Gaussian elimination and a fourth that needed by the LU factorization algorithm (assuming that \( A \) is not overwritten). Hence the implicit LU algorithm has same arithmetic cost but uses less memory than the most efficient classical methods.

(C) The implicit LX algorithm, see Spedicato, Xia and Zhang (1997), is defined by the choices \( H_1 = I \), \( z_i = w_i = e_{k_i} \), where \( k_i \) is an integer, \( 1 \leq k_i \leq n \), such that \( e_{k_i}^T H_i a_i \neq 0 \). By a general property of the ABS class for \( A \) with full rank there is at least one index \( k_i \) such that \( e_{k_i}^T H_i a_i \neq 0 \). For stability reasons we select \( k_i \) such that \( |e_{k_i}^T H_i a_i| \) is maximized. This algorithm has the same overhead and memory requirement as the implicit LU algorithm, but does not require pivoting. Its computational performance is also superior and generally better than the performance of the classical LU factorization algorithm with row pivoting, as available for instance in LAPACK or MATLAB, see Mirnia (1996). Therefore this algorithm can be considered as the most efficient general purpose linear solver not of the Strassen type.

4 Solution of linear Diophantine equations

One of the main results in the ABS field has been the derivation of ABS methods for linear Diophantine equations. The ABS algorithm determines if the Diophantine system has an integer solution, computes a particular solution and provides a representation of all integer solutions. It is a generalization of a method proposed by Egervary (1955) for the particular case of a homogeneous system.

Let \( Z \) be the set of all integers and consider the Diophantine linear system of equations

\[
Ax = b, \quad x \in \mathbb{Z}^n, \quad A \in \mathbb{Z}^{m \times n}, \quad b \in \mathbb{Z}^m, \quad m \leq n. \quad (13)
\]

While thousands of papers have been written concerning nonlinear, usually polynomial, Diophantine equations in few variables, the general linear system has attracted much less attention. The single linear equation in \( n \) variables was first solved by Bertrand and Betti (1850). Egervary was probably the first author dealing with a system (albeit only the homogeneous one). Several methods for the nonhomogeneous system have recently been proposed based mainly on reduction to canonical forms.

We recall some results from number theory. Let \( a \) and \( b \) be integers. If there is integer \( \gamma \) so that \( b = \gamma a \) then we say that \( a \) divides \( b \) and write \( a|b \), otherwise we write \( a \nmid b \). If \( a_1, \ldots, a_n \) are integers, not all being zero, then the greatest common divisor (gcd) of these numbers is the greatest positive integer \( \delta \) which divides all \( a_i, \ i = 1, \ldots, n \) and we write \( \delta = \gcd(a_1, \ldots, a_n) \). We note that \( \delta \geq 1 \) and that
\( \delta \) can be written as an integer linear combination of the \( a_i \), i.e. \( \delta = z^T a \) for some \( z \in \mathbb{R}^n \). One can show that \( \delta \) is the least positive integer for which the equation \( a_1 x_1 + \ldots + a_n x_n = \delta \) has an integer solution. Now \( \delta \) plays a main role in the following

**Fundamental Theorem of the Linear Diophantine Equation**

Let \( a_1, \ldots, a_n \) and \( b \) be integer numbers. Then the Diophantine linear equation \( a_1 x_1 + \ldots + a_n x_n = b \) has integer solutions if and only if \( \gcd(a_1, \ldots, a_n) | b \). In such a case if \( n > 1 \) then there is an infinite number of integer solutions.

In order to find the general integer solution of the Diophantine equation \( a_1 x_1 + \ldots + a_n x_n = b \), the main step is to solve \( a_1 x_1 + \ldots + a_n x_n = \delta \), where \( \delta = \gcd(a_1, \ldots, a_n) \), for a special integer solution. There exist several algorithms for this problem. The basic step is the computation of \( \delta \) and \( z \), often done using the algorithm of Rosser (1941), which avoids a too rapid growth of the intermediate integers, and which terminates in polynomial time, as shown by Schrijver (1986). The scaled ABS algorithm can be applied to Diophantine equations via a special choice of its parameters, originating from the following considerations and Theorems.

Suppose \( x_i \) is an integer vector. Since \( x_{i+1} = x_i - \alpha_i p_i \), then \( x_{i+1} \) is integer if \( \alpha_i \) and \( p_i \) are integers. If \( v_i^T A p_i | (v_i^T r_i) \), then \( \alpha_i \) is an integer. If \( H_i \) and \( z_i \) are respectively an integer matrix and an integer vector, then \( p_i = H_i^T z_i \) is also an integer vector. Assume \( H_i \) is an integer matrix. From (6), if \( v_i^T A H_i^T w_i \) divides all the components of \( H_i A^T v_i \), then \( H_{i+1} \) is an integer matrix.

Conditions for the existence of an integer solution and determination of all integer solutions of the Diophantine system are given in the following theorems, generalizing the Fundamental Theorem, see Esmaeili, Mahdavi-Amiri and Spedicato (2001a).

**Theorem 1** Let \( A \) be full rank and suppose that the Diophantine system (13) is integer solvable. Consider the Abaffians generated by the scaled ABS algorithm with the parameter choices: \( H_1 \) is unimodular (i.e. both \( H_1 \) and \( H_1^{-1} \) are integer matrices); for \( i = 1, \ldots, m \), \( w_i \) is such that \( w_i^T H_i A^T v_i = \delta_i \), \( \delta_i = \gcd(H_i A^T v_i) \). Then the following properties are true:

(a) the Abaffians generated by the algorithm are well-defined and are integer matrices

(b) if \( y \) is a special integer solution of the first \( i \) equations, then any integer solution \( x \) of such equations can be written as \( x = y + H_i^T q \) for some integer vector \( q \).

**Theorem 2** Let \( A \) be full rank and consider the sequence of matrices \( H_i \) generated by the scaled ABS algorithm with parameter choices as in Theorem 1. Let \( x_1 \) in the scaled ABS algorithm be an arbitrary integer vector and let \( z_i \) be such that \( z_i^T H_i A^T v_i = \gcd(H_i A^T v_i) \). Then system (13) has integer solutions iff \( \gcd(H_i A^T v_i) \) divides \( v_i^T r_i \) for \( i = 1, \ldots, m \).
From the above theorems we obtain the following scaled ABS algorithm for Diophantine equations.

The ABS Algorithm for Diophantine Linear Equations

(1) Choose $x_1 \in \mathbb{Z}^n$, arbitrary, $H_1 \in \mathbb{Z}^{n \times n}$, arbitrary unimodular. Let $i = 1$.

(2) Compute $\tau_i = v_i^T r_i$ and $s_i = H_i A^T v_i$.

(3) If $(s_i = 0$ and $\tau_i = 0)$ then let $x_{i+1} = x_i$, $H_{i+1} = H_i$, $r_{i+1} = r_i$ and go to step (5) (the $i$th equation is redundant). If $(s_i = 0$ and $\tau_i \neq 0)$ then Stop (the $i$th equation and hence the system is incompatible).

(4) $\{s_i \neq 0\}$ Compute $\delta_i = \text{gcd}(s_i)$ and $p_i = H_i^T z_i$, where $z_i \in \mathbb{Z}^n$ is an arbitrary integer vector satisfying $z_i^T s_i = \delta_i$. If $\delta_i \nmid \tau_i$ then Stop (the system is integerly inconsistent), else Compute $\alpha_i = \tau_i / \delta_i$, let $x_{i+1} = x_i - \alpha_i p_i$ and update $H_i$ by $H_{i+1} = H_i - \frac{H_i A^T v_i w_i^T H_i}{w_i^T H_i A^T v_i}$ where $w_i \in \mathbb{R}^n$ is an arbitrary integer vector satisfying $w_i^T s_i = \delta_i$.

(5) If $i = m$ then Stop ($x_{m+1}$ is a solution) else let $i = i + 1$ and go to step (2).

It follows from Theorem 1 that if there exists a solution for the system (13), then $x = x_{m+1} + H_{m+1}^T q$, with arbitrary $q \in \mathbb{Z}^n$, provides all solutions of (13).

Egerváry’s algorithm for homogeneous Diophantine systems corresponds to the choices $H_1 = I$, $x_1 = 0$ and $w_i = z_i$, for all $i$. Egerváry claimed, without proof, that any set of $n - m$ linearly independent rows of $H_{m+1}$ form an integer basis for the general solution of the system. Esmaeili et al. (2001a) have shown by a counterexample that Egerváry’s claim is not true in general; we have also provided an analysis of conditions under which $m$ rows in $H_{m+1}$ can be eliminated.

One can show that by special choices of the scaling parameters and by an inessential modification of the update of the Abaffian it is not necessary to solve in general $2m$ single $n$-dimensional Diophantine equations (to determine $z_i, w_i$), but just one single Diophantine equation, at the last step.

The ABS algorithm for linear Diophantine systems has been extended also to systems of $m$ linear inequalities ($m \leq n$), where it provides an almost explicit representation of all solutions, see Esmaeili, Mahdavi-Amiri and Spedicato (2001b). The used technique allows, in the case of $m \leq n$ real inequalities to find a fully explicit representation of all solutions, see Esmaeili, Mahdavi-Amiri and Spedicato (2000), thereby obtaining a generalization of formula (10). Finally, the ABS approach provides a new way of computing, given an integer matrix, its integer null space and, under some conditions, an integer basis of it, see Esmaeili, Mahdavi-Amiri and Spedicato (2001c).
5 Reformulation of the simplex method via the implicit LX method

The implicit LX algorithm has a natural application in reformulating the simplex method for the LP problem in standard form:

\[ \min c^T x \]

subject to

\[ Ax = b, \ x \geq 0. \]

The suitability of the implicit LX algorithm derives from the fact that the algorithm, when started with \( x_1 = 0 \), generates basic type vectors \( x_{i+1} \), which are vertices of the polytope defined by the constraints of the LP problem if the components not identically zero are nonnegative.

The basic step of the simplex method is moving from one vertex to another according to certain rules and reducing at each step the value of \( c^T x \). The direction of movement is obtained by solving a linear system, whose coefficient matrix \( A_B \), the basic matrix, is defined by \( m \) linearly independent columns of \( A \), called the basic columns. This system is usually solved via the LU factorization method or also via the QR factorization method. The new vertex is associated with a new basic matrix \( A'_B \), obtained by substituting one of the columns of \( A_B \) with a column of the matrix \( A_N \) that comprises the columns of \( A \) not belonging to \( A_B \). One has then to solve a new system where just one column has been changed. If the LU factorization method is used, then the most efficient way to recompute the modified factors is the steepest edge method of Forrest and Goldfarb (1992), that requires \( m^2 \) multiplications. This implementation of the simplex method has the following storage requirement: \( m^2 \) for the LU factors and \( mn \) for the matrix \( A \), that has to be kept to provide the columns needed for the exchanges.

The reformulation of the simplex method via the implicit LX method, see Zhang and Xia (1995) and Spedicato, Xia and Zhang (1997), exploits the fact that in the implicit LX method the exchange of the \( j \)-th column in \( A_B \) with the \( k \)-th column in \( A_N \) corresponds to exchanging previously chosen parameters \( z_j = w_j = e_{jB} \) with new parameters \( z_k = w_k = e_{kB} \). In terms of the implied modification of the Abaffian this operation is a special case of a general rank-one modification of the parameter matrix \( W = (w_{k1}, \ldots, w_{km}) \). The modified Abaffian can be efficiently evaluated using a general formula of Zhang (1995), without explicit use of the \( k \)-th column of \( A_N \). Moreover all information needed to build the search direction (the polytope edge) and to implement the (implicit) column exchange is contained in the Abaffian matrix. Thus there is no need to keep the matrix \( A \). Hence storage requirement is only that needed in the construction and keeping of the matrix \( H_{m+1} \), i.e. respectively \( n^2/4 \) and \( n(n-m) \). For \( m \) close to \( n \), such storage is about 8 times less than in the Forrest and Goldfarb implementation of the LU method. For small \( m \)
storage is higher, but there is an alternative formulation of the implicit LX method that has a similar storage, see Spedicato and Xia (1999).

We now give the main formulas for the simplex method in the classical and in the ABS formulation. The column in $A_N$ that substitutes a column in $A_B$ is usually taken as the column with least relative cost. In the ABS approach this corresponds to minimize with respect to $i \in N_m$ the scalar $\eta_i = c^T H^T e_i$. Let $N^*$ be the index so chosen. The column in $A_B$ to be exchanged is often chosen with the criterion of the least edge displacement that keeps the basic variables nonnegative. Defining $\omega_i = x^T e_i / e_i^T H^T e_{N^*}$ with $x$ the current vertex, the above criterion is equivalent to minimize $\omega_i$ with respect to the set of indices $i \in B_m$ such that
\[ e_i^T H^T e_{N^*} > 0 \] (14)
Notice that $H^T e_{N^*} \neq 0$ and that an index $i$ such that (14) be satisfied always exists, unless $x$ is a solution.

The update of the Abaffian after the exchange of the obtained unit vectors is given by the following simple rank-one correction
\[ H' = H - (He_{B^*} - e_{B^*})e_{N^*}^T H / e_{N^*}^T H e_{B^*} \] (15)
The displacement vector $d$, classically obtained by solving system $A_B d = -A e_{N^*}$, is obtained at no cost by $d = H_{m+1}^T e_{N^*}$. The relative cost vector, classically given by $r = c - A^T A_{B}^{-1} c_B$, with $c_B$ comprising the components of $c$ with indices corresponding to those of the basic columns, is given by formula $r = H_{m+1} c$.

It is easily seen that update (15) requires no more than $m(n-m)$ multiplications. Cost is highest for $m = n/2$ and gets smaller as $m$ gets smaller or closer to $n$. In the method of Forrest and Goldfarb $m^2$ multiplications are needed to update the LU factors and again $m^2$ to solve the system. Hence the ABS approach via formula (15) is faster for $m > n/3$. For $m < n/3$ similar costs are obtained using the alternative formulation of the implicit LX algorithm described in Spedicato and Xia (1999). For $m$ very close to $n$ the advantage of the ABS formulation is essentially of one order and such that no need is seen to develop formulas for the sparse case.

There is an ABS generalization of the simplex method based upon a modification of the Huang algorithm which is started by a certain singular matrix, see Zhang (1997). In such generalization the solution is approached via a sequence of points lying on the faces of the polytope. If one of such points happens to be a vertex, then all successive iterates are vertices and the standard simplex method is reobtained.

6 ABS unification of feasible direction methods for linearly constrained minimization

ABS methods allow a unification of feasible direction methods for linearly constrained minimization, of which the LP problem is a special case. Let us first
consider the problem with only equality constraints

\[
\min f(x), \ x \in \mathbb{R}^n
\]

subject to

\[
Ax = b, \ A \in \mathbb{R}^{m,n}, \ m \leq n, \ \text{rank}(A) = m.
\]

Let \( x_1 \) be a feasible initial point. If we consider an iteration of the form \( x_{i+1} = x_i - \alpha_i d_i \), the sequence \( x_i \) consists of feasible points iff

\[
Ad_i = 0 \quad (16)
\]

The general solution of (16) can be written using the ABS formula (10)

\[
d_i = H_{m+1}^T q \quad (17)
\]

In (17) matrix \( H_{m+1} \) depends on parameters \( H_1, W \) and \( V \) and \( q \in \mathbb{R}^n \) can also be seen as a parameter. Hence the general iteration generating feasible points is

\[
x_{i+1} = x_i - \alpha_i H_{m+1}^T q \quad (18)
\]

The search vector is a descent direction if \( d_i^T \nabla f(x_i) = q^T H_{m+1} \nabla f(x) > 0 \). This condition can always be satisfied by a choice of \( q \) unless \( H_{m+1} \nabla f(x) = 0 \), implying, from the structure of \( \text{Null}(H_{m+1}) \), that \( \nabla f(x) = A^T \lambda \) for some \( \lambda \), hence that \( x_{i+1} \) is a KT point with \( \lambda \) the vector of Lagrange multipliers. If \( x_{i+1} \) is not a KT point, we can generate descent directions by taking

\[
q = QH_{m+1} \nabla f(x) \quad (19)
\]

with \( Q \) symmetric positive definite. We obtain therefore a large class of methods with four parameter matrices \( (H_1, W, V, Q) \).

Some well-known methods in the literature correspond to taking \( W = I \) in (19) and building the Abaffian as follows.

(a) The reduced gradient method of Wolfe. \( H_{m+1} \) is built via the implicit LU method.

(b) The projection method of Rosen. \( H_{m+1} \) is built via the Huang method.

(c) The Goldfarb and Idnani method. \( H_{m+1} \) is built via a modification of Huang method where \( H_1 \) is a symmetric positive definite approximation of the inverse Hessian of \( f(x) \).

To deal with linear inequality constraints there are two approaches in literature.
• The active set method. Here the set of equality constraints is augmented with some inequality constraints, whose selection varies in the course of the process till the final set of active constraints is determined. Adding or cancelling a single constraint corresponds to a rank-one correction to the matrix defining the active set. The corresponding change in the Abaffian can be performed in order two operations, see Zhang (1995).

• The standard form approach. Here one uses slack variables to put the problem in the following equivalent standard form

\[
\min f(x),
\]

with the constraints

\[
Ax = b, \quad x \geq 0.
\]

If \( x_1 \) satisfies the above constraints, then a sequence of feasible points is generated by the iteration

\[
x_{i+1} = x_i - \alpha_i \beta_i H_{m+1} \nabla f(x)
\]

where \( \alpha_i \) may be chosen by a line search along vector \( H_{m+1} \nabla f(x) \), while \( \beta_i > 0 \) is chosen to avoid violation of the nonnegativity constraints.

If \( f(x) \) is nonlinear, then \( H_{m+1} \) can usually be determined once for all at the first iteration, since generally \( \nabla f(x) \) changes from point to point, allowing the determination of a new search direction. However if \( f(x) = c^T x \) is linear, in which case we obtain the LP problem, then to get a new search direction we have to change \( H_{m+1} \).

We already observed in section 4 that the simplex method in the ABS formulation is obtained by constructing the matrix \( H_{m+1} \) via the implicit LX method and at each step modifying one of the unit vectors used to build the Abaffian. One can show that the Karmarkar method corresponds to Abaffians built via a variation of the Huang algorithm, where the initial matrix is \( H_1 = \text{Diag}(x_i) \) and is changed at every iteration (whether the update of the Abaffian can be performed in order two operations is still an open question). We expect that better methods may be obtained by exploiting all parameters available in the ABS class.

7 ABSPACK and its numerical performance

The ABSPACK project (based upon a collaboration between the University of Bergamo, the Dalian University of Technology and the Czech Academy of Sciences) aims at producing a mathematical package for solving linear and nonlinear systems and optimization problems using the ABS algorithms. The project will take several years for completion, in view of the substantial work needed to test the alternative ways ABS methods can be implemented (via different linear algebra formulations of the process, different possibilities of reprojections, different possible block formulations etc.) and of the necessity of comparing the produced software with the established
packages in the market (e.g. MATLAB, LINPACK, LAPACK, UFO ...). It is expected that the software will be documented in a forthcoming monograph and will be made available to general users.

Presently FORTRAN 77 implementations have been made of several versions of the following ABS algorithms for solving linear systems:

1. The Huang and the modified Huang algorithms in two different linear algebra versions of the process, for solving determined, underdetermined and overdetermined systems, for a solution of least Euclidean norm

2. The implicit LU and implicit LX algorithm for determined, underdetermined and overdetermined linear systems, for a solution of basic type

3. The implicit QR algorithm for determined, underdetermined and overdetermined linear systems, for a solution of basic type

4. The above algorithms for some structured problems, namely KT equations and banded matrices.

For a full presentation of the above methods and their comparison with NAG, LAPACK, LINPACK and UFO codes see Bodon, Spedicato and Luksan(2000a,b,c). Some results are presented at the end of this section. There the columns refer respectively to: the problem, the dimension, the algorithm, the relative solution error (in Euclidean norm), the relative residual error in Euclidean norm (i.e. ratio of residual norm over norm of right hand side), the computed rank and the time in seconds. Computations have been performed in double precision on a Digital Alpha workstation with machine zero about $10^{-17}$. All test problems have been generated with integer entries or powers of two such that all entries are exactly represented in the machine and the right hand side can be computed exactly, so that the given solution is an exact solution of the problem as it is represented in the machine. Comparison is given with some LAPACK codes, including those based upon singular value decomposition ($\text{svd}$) and rank revealing QR factorization ($\text{gqr}$).

Analysis of all obtained results indicates:

1. Modified Huang is generally the most accurate ABS algorithm and compares in accuracy with the best LAPACK solvers based upon singular value factorization and rank revealing QR factorization; also the estimated ranks are usually the same.

2. On problems where the numerical estimated rank is much less than the dimension, one of the versions of modified Huang is much faster than the LAPACK codes using SVD or rank revealing QR factorization, by even a factor more than 100. This is due to the fact that once an equation is recognized as dependent it does not contribute to the general overhead in ABS algorithms.
3. Modified Huang is more accurate than other ABS methods and faster than classical methods on KT equations.

It should be noted that the performance of the considered ABS algorithms in term of time could be improved by developing block versions. ABS methods are moreover expected to be faster than many classical methods on vector and parallel computers, see Bodon (1993).
Numerical Results

RESULTS ON DETERMINED LINEAR SYSTEMS

Condition number: 0.21D+20
IDF2  2000  huang2  0.10D+01  0.69D-11  2000  262.00
IDF2  2000  mod.huang2  0.14D+01  0.96D-12  4  7.00
IDF2  2000  lu lapack  0.67D+04  0.18D-11  2000  53.00
IDF2  2000  qr lapack  0.34D+04  0.92D-12  2000  137.00
IDF2  2000  gqr lapack  0.10D+01  0.20D-14  3  226.00
IDF2  2000  lu linpack  0.67D+04  0.18D-11  2000  136.00

Condition number: 0.10D+61
IR50  1000  huang2  0.46D+00  0.33D-09  1000  36.00
IR50  1000  mod.huang2  0.46D+00  0.27D-14  772  61.00
IR50  1000  lu lapack  0.12D+04  0.12D+04  972  7.00
IR50  1000  qr lapack  0.63D+02  0.17D-12  1000  17.00
IR50  1000  gqr lapack  0.46D+00  0.42D-14  772  29.00
IR50  1000  lu linpack  --- break-down ---

RESULTS ON OVERDETERMINED SYSTEMS

Condition number: 0.16D+21
IDF3  1050  950  huang7  0.32D+04  0.52D-13  950  31.00
IDF3  1050  950  mod.huang7  0.14D+04  0.20D-09  2  0.00
IDF3  1050  950  qr lapack  0.37D+13  0.83D-02  950  17.00
IDF3  1050  950  svd lapack  0.10D+01  0.24D-14  2  145.00
IDF3  1050  950  gqr lapack  0.10D+01  0.22D-14  2  27.00

Condition number: 0.63D+19
IDF3  2000  400  huang7  0.38D+04  0.35D-12  400  9.00
IDF3  2000  400  mod.huang7  0.44D+03  0.67D-12  2  0.00
IDF3  2000  400  impl.qr5  0.44D+03  0.62D-16  2  0.00
IDF3  2000  400  expl.qr  0.10D+01  0.62D-03  2  0.00
IDF3  2000  400  qr lapack  0.45D+12  0.24D-02  400  8.00
IDF3  2000  400  svd lapack  0.10D+01  0.65D-15  2  17.00
IDF3  2000  400  gqr lapack  0.10D+01  0.19D-14  2  12.00
RESULTS ON UNDERDETERMINED LINEAR SYSTEMS

Condition number: 0.29D+18
| IDF2  | 400 2000 | huang2 | 0.12D-10 | 0.10D-12 | 400 | 12.00 |
| IDF2  | 400 2000 | mod.huang2 | 0.36D-08 | 0.61D-10 | 3 | 1.00 |
| IDF2  | 400 2000 | qr lapack | 0.29D+03 | 0.37D-14 | 400 | 9.00 |
| IDF2  | 400 2000 | svd lapack | 0.43D-13 | 0.22D-14 | 3 | 68.00 |
| IDF2  | 400 2000 | gqr lapack | 0.18D-13 | 0.24D-14 | 3 | 12.00 |

Condition number: 0.24D+19
| IDF3  | 950 1050 | huang2 | 0.00D+00 | 0.00D+00 | 950 | 33.00 |
| IDF3  | 950 1050 | mod.huang2 | 0.00D+00 | 0.00D+00 | 2 | 1.00 |
| IDF3  | 950 1050 | qr lapack | 0.24D+03 | 0.56D-14 | 950 | 17.00 |
| IDF3  | 950 1050 | svd lapack | 0.17D-14 | 0.92D-16 | 2 | 178.00 |
| IDF3  | 950 1050 | gqr lapack | 0.21D-14 | 0.55D-15 | 2 | 26.00 |

RESULTS ON KT SYSTEMS

Condition number: 0.26D+21
| IDF2  | 1000 900 | mod.huang | 0.55D+01 | 0.23D-14 | 16 | 24.00 |
| IDF2  | 1000 900 | impl.lu8 | 0.44D+13 | 0.21D-03 | 1900 | 18.00 |
| IDF2  | 1000 900 | impl.lu9 | 0.12D+15 | 0.80D-02 | 1900 | 21.00 |
| IDF2  | 1000 900 | lu lapack | 0.25D+03 | 0.31D-13 | 1900 | 62.00 |
| IDF2  | 1000 900 | range space | 0.16D+05 | 0.14D-11 | 1900 | 87.00 |
| IDF2  | 1000 900 | null space | 0.89D+03 | 0.15D-12 | 1900 | 93.00 |

Condition number: 0.70D+20
| IDF2  | 1200 600 | mod.huang | 0.62D+01 | 0.20D-14 | 17 | 36.00 |
| IDF2  | 1200 600 | impl.lu8 | 0.22D+07 | 0.10D-08 | 1800 | 44.00 |
| IDF2  | 1200 600 | impl.lu9 | 0.21D+06 | 0.56D-09 | 1800 | 33.00 |
| IDF2  | 1200 600 | lu lapack | 0.10D+03 | 0.79D-14 | 1800 | 47.00 |
| IDF2  | 1200 600 | range space | 0.11D+05 | 0.15D-11 | 1800 | 63.00 |
| IDF2  | 1200 600 | null space | 0.38D+04 | 0.13D-12 | 1800 | 105.00 |
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