Multi-modal character of the nonlinear dynamics of a vortex sheet in Rayleigh-Taylor and Richtmyer-Meshkov instabilities

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Abstract. Multi-modal character of interfaces in Rayleigh-Taylor (RT) and incompressible Richtmyer-Meshkov (RM) instabilities with planar and cylindrical geometries is studied numerically. An interface is treated as a vortex sheet and the vortex method which regularizes the singularity caused by the Cauchy integral in equation of motion is adapted for the calculations of interfacial motion. Successive profiles of interfaces and the temporal evolution of the strength of a vortex sheet are presented, and discussed the differences between pure cosine or sine mode and the mixed mode which includes both cosine and sine modes in the initial disturbance. It is shown that the sheet strength of the interface with multi-mode is considerably larger than that with single-mode for the RM instability, while the difference between those two modes is almost not found for the RT instability.

1. Introduction

Multi-modal character of interfaces in Rayleigh-Taylor (RT) or Richtmyer-Meshkov (RM) instability [1] is important for turbulence, especially for nonuniform and inhomogeneous turbulence in systems with different fluid densities. The RT instability is caused by the gravity, while the RM instability is caused due to instantaneous passage of a shock. Although the initial conditions are different, the fluid interfaces in these two instabilities are both possible to treat as vortex sheets [2]. In this article we numerically examine the nonlinear evolution of the interfaces as vortex sheets in the RM (with planar and cylindrical geometries) and RT instabilities with multi-modal initial conditions.

The velocity of the interface is given by the Birkhoff-Rott equation [2] which is derived from the Biot-Savart law

\[
Z^*_\beta(t) = \frac{1}{2\pi i} \text{PV} \int_{-\infty}^{\infty} \frac{\kappa(\beta', t) s_\beta(\beta') d\beta'}{Z(\beta, t) - Z'(\beta, t)} + \frac{\alpha \kappa(\beta, t) Z^*_\beta(\beta, t)}{2 s_\beta(\beta)},
\]

where \(Z^*\) is the complex conjugate of \(Z = X + iY\), \(x = X(\beta, t)\) and \(y = Y(\beta, t)\), the interface position, \(s\) is the arc length of the interface, the subscript denotes the differentiation with respect to the variable, \(\beta\) is a Lagrangian parameter which parameterizes the interface, and the notation PV denotes that the integral is the principal value integral. The parameter \(\alpha\) (\(|\alpha| \leq 1\)) is an artificial parameter such that \(\alpha \neq 0\) when the Atwood number \(A = (\rho_2 - \rho_1)/(\rho_1 + \rho_2) \neq 0\) [2],
where \( \rho_i \) \((i = 1, 2)\) is the density of fluid \( i \). The strength of a vortex sheet \( \kappa \), the absolute value of the velocity shear at the interface, is defined by the circulation \( \Gamma = \phi_2 - \phi_1, \phi_1, \phi_2 \), the velocity potential in fluid \( i \), as \( \kappa = \partial \Gamma / \partial s = \Gamma_\beta / s_\beta \) and the temporal evolution is given by [2]

\[
\kappa_t = - \frac{2A}{s_\beta} (X_\beta U_t + Y_\beta V_t) - \frac{(1 - \alpha A)\kappa}{s_\beta^2} (X_\beta U_\beta + Y_\beta V_\beta) - \frac{A - \alpha}{4s_\beta} (\kappa_\beta^2) - 2Ag_\beta \frac{y_\beta}{s_\beta}
\]

where \( U - iV \) is the vortex induced velocity which is given by the first term on the right hand side of Eq. (1) and \( g \) is the gravitational acceleration. Equations (1) and (2) are governing equations for solving the RT and planar RM instabilities.

2. Numerical results for the Rayleigh-Taylor and planar Richtmyer-Meshkov instabilities

Throughout this section, we choose the Atwood number \( A = 0.2 \), \( \alpha = -A^2 \), \( g = 0 \) for the RM instability and \( g = -1 \) for the RT instability, where the gravity is taken to be upward. The upper fluid is assumed to be lighter than the lower fluid in our calculations. We adapt the vortex method [3] which regularizes the Cauchy integral in Eq. (1) and the regularized parameter, called Krasny’s \( \delta \) [3], is chosen to \( \delta = 0.1 \) throughout this paper. For the normalization of variables and detailed numerical method, refer to Ref. [2].

Figure 1 shows the interfacial profiles and the sheet strength \( \kappa \) in the planar RM instability with initial conditions \( X = \beta, Y = 0 \) and \( \kappa = -2(\sin \beta - \sin 2\beta - 1.5\sin 3\beta) \), where \( t = 1.0 \) is the critical time in the computation and the calculation breaks down after a few steps of this time. The peak values in \( \kappa \) at \( t = 0.7 \) are larger than those at \( t = 1.0 \) in their absolute values, however, the concentration of \( \kappa \) is stronger at \( t = 1.0 \). This tendency is also found for the single-mode [2]. When we include higher modes in the initial condition, the finer roll-ups successively appear at the nonlinear stage as found in the experiment by Jacobs et al. [1].

Figure 2 shows the interfacial profiles and the sheet strength in the RT instability with initial conditions \( X = \beta, Y = 0.2(\cos \beta - 0.5\cos 2\beta - 0.5\cos 3\beta) \) and \( \kappa = 0 \). These initial conditions are taken so that the configuration of the interface is almost the same as Fig. 1. We see that the amplitude of the interface in the RM instability at the fully nonlinear stage is smaller than that in the RT instability, however, the sheet strength \( \kappa \) in the RM instability is far larger than the counterpart in the RT instability.

Figure 3 shows the interfacial profiles and the sheet strength in the planar RM and RT instabilities with mixed mode initial conditions \( X = \beta, Y = 0 \) and \( \kappa = -2 [(\sin \beta - 0.6\cos \beta) - 0.4(\sin 2\beta - 0.6\cos 2\beta) - 0.45(\sin 3\beta - 0.6\cos 3\beta)] \) for the planar RM instability, and \( X = \beta, Y = 0.2 [(\cos \beta + 0.6\sin \beta) - 0.2(\cos 2\beta + 0.6\sin 2\beta) - 0.15(\cos 3\beta + 0.6\sin 3\beta)] \) and \( \kappa = 0 \) for the RT instability, where \( t = 1.0 \) and \( t = 5.9 \) are the critical times in the computations for the RM and RT instabilities, respectively. The peak value in sheet strength \( \kappa \) is almost twice value in \( \kappa \) with single-mode initial condition in the RM instability for the same Atwood number, while the peak value in the RT instability is almost the same value as the single-mode [2].

The roll-up of the interface in Fig. 3 is weak and any singular behavior is not found in the interfacial shape and sheet strength, however, the curvature of the interface at \( t = 1.0 \) (\( t = 5.9 \)) is much larger than the one at \( t = 1.0 \) in Fig. 1 (\( t = 7.8 \) in Fig. 2) for the RM (RT) instability and it diverges just after the critical time. This phenomenon suggests the existence of the curvature singularity which usually appears when the regularized parameter \( \delta = 0 \) [2]. There may exist some effects in the mixed-mode initial conditions that disturb the regularization which we performed here. This instability for the mixed-mode initial condition is, however, not found clearly for the RM instability with cylindrical geometry below (see Fig. 4).
Figure 1. Multi-mode profiles [(a) and (b)] and the sheet strength $\kappa$ [(c) and (d)] in the RM instability at dimensionless time $t = (a)$ 0.7 and (b) 1.0, where (c) and (d) depict the strength at (a) and (b), respectively.

Figure 2. Multi-mode profiles [(a) and (b)] and the sheet strength $\kappa$ [(c) and (d)] in the RT instability at dimensionless time $t = (a)$ 6.2 and (b) 7.8, where (c) and (d) depict the strength at (a) and (b), respectively.

3. Numerical results for the Richtmyer-Meshkov instability with cylindrical geometry

The governing equations for the cylindrical RM instability corresponding to Eqs. (1) and (2) are as follows [4]:

\[
\begin{align*}
    r_t &= q_r + \frac{\alpha \kappa r \beta}{2s_\beta}, \\
    \theta_t &= q_\theta + \frac{\alpha \kappa \theta \beta}{2s_\beta}, \\
    \kappa_t &= -\frac{2A}{s_\beta} \left( r_\beta q_\theta^r + r \theta_\beta q_\theta^\theta \right) - \frac{(1 - \alpha A)}{s_\beta^2} \kappa \left[ (r_\beta q_\theta^r + r \theta_\beta q_\theta^\theta) + (rq_\theta^r \theta_\beta^2 - r_\beta \theta_\beta q_\theta^\theta) \right] - \frac{A - \alpha}{4s_\beta}(\kappa^2) \beta, 
\end{align*}
\]

where $r$ and $\theta$ are the components in the cylindrical coordinate system and $q_r$ and $q_\theta$ are the velocity components with respect to them which are given by the Biot-Savart law [4]. Throughout this section, we choose the Atwood number $A = 0.2$, where $A = 0.2$ indicates that the inner fluid is lighter than the outer one and $A = -0.2$ vice versa.

Figure 4 shows the interfacial profiles in the RM instability with cylindrical geometry, where the initial condition for $(r, \theta)$ is chosen that $(r, \theta) = (1, \beta)$ in all calculations and the sheet strength $\kappa$ is given by $\kappa = 2(\sin \beta - 1.5 \sin 3\beta)$ for (a) and (c), and $\kappa = 2((\sin 3\beta - \cos 3\beta) - 0.4(\sin 6\beta - \cos 6\beta) - 0.2(\sin 9\beta - \cos 9\beta))$ for (b) and (d), respectively. The base mode $n$ is taken that $n = 1$ for (a) and (c), while $n = 3$ for (b) and (d), respectively. When the initial disturbance is single-mode, the roll-up is not found for $n = 1$ case [4], however, it appears for pure sine multi-mode initial condition [(a) and (c)]. The breakdown of computations for the mixed mode conditions with base mode $n = 1$ occurs at earlier time than the one for the pure
sine mode condition and the roll-up also does not occur in the mixed mode conditions. However, when the base mode $n \geq 3$, the breakdown time of computations for the mixed-mode is almost the same as the one for the pure sine mode and the roll-up is also found [see (b) and (d)].

4. Concluding remarks
We have numerically investigated multi-modal character of interfaces in the RM and RT instabilities. The breakdown of computations with multi-modal initial condition occurs at earlier time than that with single-modal initial condition for all calculations throughout this article. When the higher mode is included initially, the sheet strength in the RM instability becomes larger than the one considered here, which may suggest that the RM instability is easier to shift to turbulence than the RT instability.

Figure 3. Mixed mode profiles in the RM (a) and RT (b) instabilities at dimensionless time $t = (a) 1.0$ and (b) 5.9, where (c) and (d) depict the strength at (a) and (b), respectively.

Figure 4. Multi-mode profiles in the RM instability with cylindrical geometry for the Atwood number $A = (a), (b) 0.2$ and (c), (d) $-0.2$, where all dashed lines depict $t = 0$. Solid lines in the figures depict (a) $t = 0.86$, (b) $t = 1.2$, (c) $t = 1.0$ and (d) $t = 2.0$, and the dashed and dotted line in (d) depicts $t = 1.2$. All box sizes in the figures are $|x| \leq 2.0$ and $|y| \leq 2.0$ and the letters ‘H’ and ‘L’ denote heavy and light fluids, respectively. The central dots indicate the origin. Here, (a) and (c) correspond to the pure multi-mode, while (b) and (d) correspond to the mixed multi-mode.

References
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