On The Decoding Error Weight of One or Two Deletion Channels

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Abstract

This paper tackles two problems that fall under the study of coding for insertions and deletions. These problems are motivated by several applications, among them is reconstructing strands in DNA-based storage systems. Under this paradigm, a word is transmitted over some fixed number of identical independent channels and the goal of the decoder is to output the transmitted word or some close approximation of it. The first part of this paper studies the deletion channel that deletes a symbol with some fixed probability $p$, while focusing on two instances of this channel. Since operating the maximum likelihood (ML) decoder in this case is computationally unfeasible, we study a slightly degraded version of this decoder for two channels and study its expected normalized distance. We observe that the dominant error patterns are deletions in the same run or errors resulting from alternating sequences. Based on these observations, it is derived that the expected normalized distance of the degraded ML decoder is roughly $\frac{1}{2} - p^2$, when the transmitted word is any $q$-ary sequence and $p$ is the channel’s deletion probability. We also study the cases when the transmitted word belongs to the Varshamov Tenengolts (VT) code or the shifted VT code. Additionally, the insertion channel is studied as well as the case of two insertion channels. These theoretical results are verified by corresponding simulations. The second part of the paper studies optimal decoding for a special case of the deletion channel, referred by the $k$-deletion channel, which deletes exactly $k$ symbols of the transmitted word uniformly at random. In this part, the goal is to understand how an optimal decoder operates in order to minimize the expected normalized distance. A full characterization of an efficient optimal decoder for this setup, referred to as the maximum likelihood* (ML*) decoder, is given for a channel that deletes one or two symbols. For $k = 1$ it is shown that when the code is the entire space, the decoder is the lazy decoder which simply returns the channel output. Similarly, for $k = 2$ it is shown that the decoder acts as the lazy decoder in almost all cases and when the longest run is significantly long (roughly $(2 - \sqrt{2})n$ when $n$ is the word length), it prolongs the longest run by one symbol.

Index Terms

Deletion channel, insertion channel, sequence reconstruction.

I. INTRODUCTION

Codes correcting insertions/deletions recently attracted considerable attention due to their relevance to the special error behavior in DNA-based data storage [8], [40], [51], [64], [67], [69], [90], [91]. These codes are relevant for other applications in communications models. For example, insertions/deletions happen during the synchronization of files and symbols of data streams [70] or due to over-sampling and under-sampling at the receiver side [25]. The algebraic concepts of codes correcting insertions/deletions date back to the 1960s when Varshamov and Tenengolts designed a class of binary codes, nowadays called VT codes [85]. These codes were originally designed to correct a single asymmetric error and later were proven to correct a single insertion/deletion [52]. Extensions for multiple deletions were recently proposed in several studies; see e.g. [10], [30], [74], [75]. However, while codes correcting substitution errors were widely studied and efficient capacity achieving codes both for small and large block lengths are used conventionally, much less is known for codes correcting insertions/deletions. More than that, even the deletion channel capacity is far from being solved [13]–[15], [21], [60], [61], [66].

In the same context, reconstruction of sequences refers to a large class of problems in which there are several noisy copies of the information and the goal is to decode the information, either with small or zero error probability. The first example is the sequence reconstruction problem which was first studied by Levenshtein and others [51], [53]–[56], [71], [88], [89]. Another example, which is also one of the more relevant models to the discussion in the first part of this paper, is the trace reconstruction problem [7], [43], [44], [63], [65], where it is assumed that a sequence is transmitted through multiple deletion channels, and each bit is deleted with some fixed probability $p$. Under this setup, the goal is to determine the minimum number of traces, i.e., channels, required to reconstruct the sequence with high probability. One of the dominant motivating applications of the sequence reconstruction problems is DNA storage [2], [5], [19], [33], [64], [90], where every DNA strand has several noisy copies. Several new results on the trace reconstruction problem have been recently studied in [12], [16], [22], [34], [78].

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Many of the reconstruction problems are focused on studying the minimum number of channels required for successful decoding. However, in many cases, the number of channels is fixed and then the goal is to find the best code construction that is suitable for this channel setup. Motivated by this important observation, the first part of this paper also studies the error probability of maximum-likelihood decoding when a word is transmitted over two deletion or insertion channels. We should note that we study a degraded version of the maximum likelihood decoder, which allows the decoder to output words of shorter length than the code length. This flexibility of the decoder is useful especially in cases where the same symbol is deleted in both of the channels, or when the code does not have deletion-correcting capabilities. This study is also motivated by the recent works of Srinivasavaradhan et al. [76], [77], where reconstruction algorithms that are based on the maximum-likelihood approach have been studied. Abroshan et al. presented in [1] a new coding scheme for sequence reconstruction which is based on the Varshamov Tenengolts (VT) code [85] and in [49] it was studied how to design codes for the worst case, when the number of channels is given.

When a word is transmitted over the deletion channel, the channel output is necessarily a subsequence of the transmitted word. Hence, when transmitting the same word over multiple deletion channels, the possible candidate words for decoding are the so-called common supersequences of all of the channels' outputs. Hence, an important part of the decoding process is to find the set of all possible common supersequences and in particular the shortest common supersequences (SCS) [47]. Even though this problem is in general NP hard [9] for an arbitrary number of sequences, for two words a dynamic programming algorithm exists with quadratic complexity; see [47] for more details and further improvements and approximations for two or more sequences [41], [46], [83], [84]. The case of finding the longest common subsequences (LCS) is no less interesting and has been extensively studied in several previous works; see e.g. [3], [18], [42], [45], [58], [72]. Most of these works focused on improving the complexity of the dynamic programming algorithm suggested in [3] and presented heuristics and approximations for the LCS.

Back to a single instance of a channel with deletion errors, there are two main models which are studied for this type of errors. While in the first one, the goal is to correct a fixed number of deletions in the worst case, for the second one, which corresponds to the channel capacity of the deletion channel, one seeks to construct codes which correct a fraction $p$ of deletions with high probability [11], [14], [20], [24], [26], [29], [48], [50], [61], [80], [86]. The second part of this paper considers a combination of these two models. In this channel, referred as the $k$-deletion channel, $k$ symbols of the length-$n$ transmitted word are deleted uniformly at random; see e.g. [3], [82]. Consider for example the case of $k = 1$, i.e., one of the $n$ transmitted symbols is deleted, each with the same probability. In case the transmitted word belongs to a single-deletion-correcting code then clearly it is possible to successfully decode the transmitted word. However, if such error correction capability is not guaranteed in the worst case, two approaches can be of interest. In the first, one may output a list of all possible transmitted words, that is, list decoding for deletion errors as was studied recently in several works; see e.g. [35], [86], [38], [39], [48], [57], [87]. The second one, which is taken in the present work, seeks to output a word that minimizes the expected normalized distance between the decoder’s output and the transmitted word. This channel was also studied in several previous works. In [32], the author studied the maximal length of words that can be uniquely reconstructed using a sufficient number of channel outputs of the $k$-deletion channel and calculated this maximal length explicitly for $n − k ≤ 6$. In [4], the goal was to study the entropy of the set of the potentially channel input words given a corrupted word, which is the output of a channel that deletes either one or two symbols. The minimum and maximum values of this entropy were explored. In [80], [82], the authors presented a polar coding solution in order to correct deletions in the $k$-deletion channel.

Mathematically speaking, assume $S$ is a channel that is characterized by a conditional probability $P_{S|\mathcal{C}} \{ y \ rec. \mid x \ trans. \}$, for every pair $(x, y) \in (\Sigma_q^*)^2$. A decoder for a code $\mathcal{C}$ with respect to the channel $S$ is a function $D : \Sigma_q^* \rightarrow \mathcal{C}$. Its average decoding failure probability is the probability that the decoder output is not the transmitted word. The maximum-likelihood (ML) decoder for $\mathcal{C}$ with respect to $S$, denoted by $D_{\text{ML}}$, outputs a codeword $c \in \mathcal{C}$ that maximizes the probability $P_{S|\mathcal{C}} \{ y \ rec. \mid c \ trans. \}$. This decoder minimizes the average decoding failure probability and thus it outputs only codewords. However, if one seeks to minimize the expected normalized distance, then the decoder should consider non-codewords as well. The expected normalized distance is the average normalized distance between the transmitted word and the decoder’s output, where the distance function depends upon the channel of interest. In this work we study the ML* decoder, which outputs words that minimize the expected normalized distance.

The rest of the paper is organized as follows. Section II presents the formal definition of channel transmission and maximum likelihood decoding in order to minimize the expected normalized distance. Section III introduces the deletion channel, the insertion channel, and the $k$-deletion channel. In Section IV we present our main results for the case of two deletion channels. We consider the average decoding failure probability of a degraded version of the ML decoder and its expected normalized distance when the code is the entire space, the VT code, and the shifted VT code. Among our results, it is shown that when the code is the entire space the expected normalized distance is roughly $\frac{3q-1}{q-1} p^2$, when $q$ is the alphabet size and $p$ is the channel's deletion probability. We observe that the dominant error patterns are deletions from the same run or errors resulting from alternating sequences. Lastly, the insertion channel is studied as well as the case of two insertion channels. These theoretical results are verified by corresponding simulations. We then continue in Section V to study the equivalent problem of two insertion channels. Section VI studies the 1-deletion channel. It introduces two types of decoders. The first one, referred as the embedding number decoder, maximizes the so-called embedding number between the channel output and all possible
codewords. The second one is called the lazy decoder which simply returns the channel output. The main result of this section states that if the code is the entire space then the ML* decoder is the lazy decoder. Similarly, Section VII studies the 2-deletion channel where it is shown that in almost all cases the ML* decoder should act as the lazy decoder and in the rest of the cases it returns a length-(n − 1) word which maximizes the embedding number. Section VIII concludes the paper and discusses open problems.

II. DEFINITIONS AND PRELIMINARIES

We denote by $\Sigma_q = \{0, \ldots, q - 1\}$ the alphabet of size $q$ and $\Sigma_q^* \triangleq \bigcup_{n=0}^{\infty} \Sigma_q^n$. $\Sigma_q^n$ denotes the length-$n$ words. The Levenshtein distance between two words $x, y \in \Sigma_q^n$, denoted by $d_L(x, y)$, is the minimum number of insertions and deletions required to transform $x$ into $y$, and $d_H(x, y)$ denotes the Hamming distance between $x$ and $y$, when $|x| = |y|$. A word $x \in \Sigma_q^n$ will be referred to as an alternating sequence if it cyclically repeats all symbols in $\Sigma_q$ in the same order. For example, for $\Sigma_2 = \{0, 1\}$, the two alternating sequences are 010101 · · · and 101010 · · · , and in general there are $q!$ alternating sequences. For $n \geq 1$, the set $\{1, \ldots, n\}$ is abbreviated by $[n]$.

For a word $x \in \Sigma_q^n$ and a set of indices $I \subseteq [|x|]$, the word $x_I$ is the projection of $x$ on the indices $I$ which is the subsequence of $x$ received by the symbols in the entries of $I$. A word $x \in \Sigma^*$ is called a supersequence of $y \in \Sigma^*$, if $y$ can be obtained by deleting symbols from $x$, that is, there exists a set of indices $I \subseteq [|x|]$ such that $y = x_I$. In this case, it is also said that $y$ is a subsequence of $x$. Furthermore, $x$ is called a common supersequence (subsequence) of some words $y_1, \ldots, y_t$, if $x$ is a supersequence (subsequence) of each one of these $t$ words. The set of all common supersequences of $y_1, \ldots, y_t \in \Sigma_q^*$ is denoted by $\text{SCS}(y_1, \ldots, y_t)$ and $\text{SCS}(y_1, \ldots, y_t)$ is the length of the shortest common supersequence (SCS) of $y_1, \ldots, y_t$, that is, $\text{SCS}(y_1, \ldots, y_t) \triangleq \min_{\sigma \in \text{SCS}(y_1, \ldots, y_t)} |\sigma|$. Similarly, $\text{LCS}(y_1, \ldots, y_t)$ is the set of all subsequences of $y_1, \ldots, y_t$ and $\text{LCS}(y_1, \ldots, y_t)$ is the length of the longest common subsequence (LCS) of $y_1, \ldots, y_t$, that is, $\text{LCS}(y_1, \ldots, y_t) \triangleq \max_{\sigma \in \text{LCS}(y_1, \ldots, y_t)} |\sigma|$. The radius-$r$ insertion ball of a word $x \in \Sigma_q^*$, denoted by $I_r(x)$, is the set of all supersequences of $x$ of length $|x| + r$. From [52] it is known that $|I_r(x)| = \sum_{t=0}^{(|x|+r)} (q−1)^t$. Similarly, the radius-$r$ deletion ball of a word $x \in \Sigma_q^*$, denoted by $D_r(x)$, is the set of all subsequences of $x$ of length $|x| - r$.

We consider a channel $S$ that is characterized by a conditional probability $P_{S|y}$, and is defined by

$$P_{S|y} \{ y \text{ rec.} | x \text{ trans.} \},$$

for every pair $(x, y) \in (\Sigma_q^*)^2$. Note that it is not assumed that the lengths of the input and output words are the same as we consider also deletions and insertions of symbols, which are the main topic of this work. As an example, it is well known that if $S$ is the binary symmetric channel (BSC) with crossover probability $0 \leq p \leq 1/2$, denoted by BSC($p$), it holds that

$$P_{\text{BSC}(p)} \{ y \text{ rec.} | x \text{ trans.} \} = p^{d_H(y, x)}(1 - p)^{|x| - d_H(y, x)},$$

for all $(x, y) \in (\Sigma_q^*)^2$, and otherwise (the lengths of $x$ and $y$ is not the same) this probability equals 0. Similarly, for the Z-channel, denoted by Z($p$), it is assumed that only a 0 can change to a 1 with probability $p$ and so

$$P_{\text{Z}(p)} \{ y \text{ rec.} | x \text{ trans.} \} = p^{d_R(y, x)}(1 - p)^{|x| - d_R(y, x)},$$

for all $(x, y) \in (\Sigma_q^*)^2$ such that for any $1 \leq i \leq n$, $x_i \neq y_i$, and otherwise this probability equals 0.

In the deletion channel with deletion probability $p$, denoted by Del($p$), every symbol of the word $x$ is deleted with probability $p$. Similarly, in the insertion channel with insertion probability $p$, denoted by Ins($p$), a symbol is inserted in each of the possible $|x| + 1$ positions of the word $x$ with probability $p$, while the probability to insert each of the symbols in $\Sigma_q$ is the same and equals $p/q$. Another variation of the deletion channel, studied in this work in Sections VII and VIII, is the $k$-deletion channel, denoted by $k$-Del, where exactly $k$ symbols are deleted from the transmitted word. The $k$ symbols are selected randomly from the $(\binom{n}{k})$ options. This channel was studied in [4], where the authors studied the words that maximize and minimize the entropy of the set of the possible transmitted words, given a channel output. In [82], a polar codes based coding solution that corrects deletions from the $k$-deletion channel was presented.

A decoder for a code $C$ with respect to the channel $S$ is a function $D : \Sigma_q^* \rightarrow C$. Its average decoding failure probability is defined by $P_{\text{fail}}(S, C, D) \triangleq \sum_{c \in C} P_{\text{fail}}(c) / |C|$, where

$$P_{\text{fail}}(c) \triangleq \sum_{y \in D(y) \neq c} P_{S|y} \{ y \text{ rec.} | c \text{ trans.} \}.$$
consider the Hamming distance, while for the deletion and insertion channels, the Levenshtein distance will be of interest. Hence, for a channel $S$, distance function $d$, and a decoder $D$, we let $P_{\text{err}}(S, C, D, d) = \sum_{c \in C} P_{\text{err}}(c, d)$, where

$$P_{\text{err}}(c, d) \triangleq \sum_{y \in D(y) \neq c} \frac{d(D(y), c)}{|c|} \cdot P_{S}(y \text{ rec. } | c \text{ trans.}) \cdot .$$

The maximum-likelihood (ML) decoder for a code $C$ with respect to a channel $S$, denoted by $D_{\text{ML}}$, outputs a codeword $c \in C$ that maximizes the probability $P_{S}(y \text{ rec. } | c \text{ trans.})$. That is, for $y \in \Sigma^{*}$,

$$D_{\text{ML}}(y) \triangleq \arg \max_{c \in C} \{P_{S}(y \text{ rec. } | c \text{ trans.})\} .$$

It is well known that for the BSC, the ML decoder simply chooses the closest codeword with respect to the Hamming distance. The channel capacity is referred to as the maximum information rate that can be reliably transmitted over the channel $S$ and is denoted by $\text{Cap}(S)$. For example, $\text{Cap}(\text{BSC}(p)) = 1 - H(p)$, where $H(p) = -p \log(p) - (1 - p) \log(1 - p)$ is the binary entropy function.

The conventional setup of channel transmission is extended to the case of more than a single instance of the channel. Assume a word $x$ is transmitted over some $t$ identical channels of $S$ and the decoder receives all channel outputs $y_{1}, \ldots, y_{t}$. Unless stated otherwise, it is assumed that all channels are independent and thus this setup is characterized by the conditional probability

$$P_{r}(S, t) \{y_{1}, \ldots, y_{t} \text{ rec. } | x \text{ trans.} \} = \prod_{i=1}^{t} P_{S}(y_{i} \text{ rec. } | x \text{ trans.}) .$$

The definitions of a decoder, the ML decoder, and the error probabilities are extended similarly. The input to the ML decoder is the words $y_{1}, \ldots, y_{t}$ and the output is the codeword $c$ which maximizes the probability $P_{r}(S, t) \{y_{1}, \ldots, y_{t} \text{ rec. } | c \text{ trans.} \}$. That is,

$$D_{\text{ML}}(y_{1}, \ldots, y_{t}) \triangleq \arg \max_{c \in C} \{P_{r}(S, t) \{y_{1}, \ldots, y_{t} \text{ rec. } | c \text{ trans.} \}\} .$$

Since the outputs of all channels are independent, the output of the ML decoder is defined to be,

$$D_{\text{ML}}(y_{1}, \ldots, y_{t}) \triangleq \arg \max_{c \in C} \left\{ \prod_{i=1}^{t} P_{S}(y_{i} \text{ rec. } | c \text{ trans.}) \right\} .$$

The average decoding failure probability, the expected normalized distance is generalized in the same way and is denoted by $P_{\text{fail}}(S, t, C, D)$, $P_{\text{err}}(S, t, C, D, d)$, respectively. The capacity of this channel is denoted by $\text{Cap}(S, t)$, so $\text{Cap}(S, 1) = \text{Cap}(S)$.

The case of the BSC was studied by Mitzenmacher in [59], where he showed that

$$\text{Cap}(\text{BSC}(p), t) = 1 + \log \left( \frac{1}{p(1 - p)} \right)^{t - 1} \log \left( \frac{p(1 - p)}{p - (1 - p)^{t - 1}} \right) .$$

On the other hand, the $Z$ channel is significantly easier to solve and it is possible to verify that $\text{Cap}(\text{Z}(p), t) = \text{Cap}(\text{Z}(p'))$. It is also possible to calculate the expected normalized distance and the average decoding failure probability for the BSC and $Z$ channels. For example, when $C = \Sigma^{n}$, one can verify that

$$P_{\text{err}}(\text{Z}(p), t, \Sigma^{n}, D_{\text{ML}}, d_{H}) = p^{t} ,$$

and if $t$ is odd then

$$P_{\text{err}}(\text{BSC}(p), t, \Sigma^{n}, D_{\text{ML}}, d_{H}) = \sum_{i=0}^{\frac{t-1}{2}} \binom{t}{i} p^{t-i}(1 - p)^{i} .$$

Similarly, $P_{\text{fail}}(\text{Z}(p), t, \Sigma^{n}, D_{\text{ML}}) = 1 - (1 - p)^{n}$ for odd $t$, and $P_{\text{fail}}(\text{BSC}(p), t, \Sigma^{n}, D_{\text{ML}}) = 1 - (1 - \sum_{i=0}^{\frac{t-1}{2}} \binom{t}{i} p^{t-i}(1 - p))^{n}$. However, calculating these probabilities for the deletion and insertion channels is a far more challenging task.

We note that the capacity of several deletion channels has been studied in [37], where it was shown that for some $t > 0$ deletion channels with deletion probability $p$, the capacity under a random codebook satisfies

$$\text{Cap}(\text{Del}(p), t) = 1 - A(t) \cdot p^{t} \log(1/p) - O(p^{t}) ,$$

where $A(t) = \sum_{j=1}^{\infty} 2^{-j-1} t^{j}$. For example, when $t = 2$, the capacity is $1 - 6p^{2} \log(1/p) - O(p^{3})$. One of the goals of this paper, which is discussed in Section IV, is to study in depth the special case of $t = 2$ and estimate the average error and failure probabilities, when the code is the entire space, the Varshamov Tenengolts (VT) code [83], and the shifted VT (SVT) code [73].
III. THE DELETION AND INSERTION CHANNELS

In this section, we establish several basic results for the deletion channels with one or multiple instances. We start with several useful definitions. For two words \( x, y \in \Sigma^* \), the number of different ways in which \( y \) can be received as a subsequence of \( x \) is called the embedding number of \( y \) in \( x \) and is defined by

\[
\text{Emb}(x; y) \triangleq |\{ I \subseteq [|x|] \mid x_I = y \}|.
\]

Note that if \( y \) is not a subsequence of \( x \) then \( \text{Emb}(x; y) = 0 \). The embedding number has been studied in several previous works; see e.g. [4], [28] and in [76], it was referred to as the binomial coefficient. In particular, this value can be computed with quadratic complexity [28].

While the calculation of the conditional probability \( \Pr_S \{ y \text{ rec.} \mid x \text{ trans.} \} \) is a rather simple task for many of the known channels, it is not straightforward for channels that introduce insertions or deletions. The following basic claim is well known and was also stated in [76]. It will be used in our derivations to follow.

**Claim 1.** For all \( (x, y) \in (\Sigma_q^n)^2 \), it holds that

\[
\text{Pr}_{\text{Del}(p)} \{ y \text{ rec.} \mid x \text{ trans.} \} = p^{|x| - |y|} (1 - p)^{|y|} \cdot \text{Emb}(x; y),
\]

\[
\text{Pr}_{\text{Ins}(p)} \{ y \text{ rec.} \mid x \text{ trans.} \} = \left( \frac{p}{q} \right)^{|y| - |x|} (1 - p)^{|x| + 1 - (|y| - |x|)} \cdot \text{Emb}(y; x).
\]

According to Claim 1, it is possible to explicitly characterize the ML decoder for the deletion and insertion channels as described also in [76]. The proof is added for completeness.

**Claim 2.** Assume \( c \in \mathcal{C} \subseteq \Sigma_q^n \) is the transmitted word and \( y \in \Sigma_q^{\leq n} \) is the output of the deletion channel \( \text{Del}(p) \), then

\[
\mathcal{D}_{\text{ML}}(y) = \arg \max_{c \in \mathcal{C}} \{ \text{Emb}(c; y) \}.
\]

Similarly, for the insertion channel \( \text{Ins}(p) \), and \( y \in \Sigma_q^{\geq n} \),

\[
\mathcal{D}_{\text{ML}}(y) = \arg \max_{c \in \mathcal{C}} \{ \text{Emb}(y; c) \}.
\]

**Proof:** It is verified that

\[
\mathcal{D}_{\text{ML}}(y) \overset{(a)}{=} \arg \max_{c \in \mathcal{C}} \{ \Pr_S \{ y \text{ rec.} \mid c \text{ trans.} \} \}
\]

\[
\overset{(b)}{=} \arg \max_{c \in \mathcal{C}} \left\{ p^{|c| - |y|} (1 - p)^{|y|} \cdot \text{Emb}(c; y) \right\}
\]

\[
\overset{(c)}{=} \arg \max_{c \in \mathcal{C}} \{ \text{Emb}(c; y) \},
\]

where (a) is the definition of the ML decoder, (b) follows from Claim 1 and (c) holds since the value \( p^{|c| - |y|} (1 - p)^{|y|} \) is the same for every codeword in \( \mathcal{C} \). The proof for the insertion channel is similar.

In case there is more than a single instance of the deletion/insertion channel, the following claim follows.

**Claim 3.** Assume \( c \in \mathcal{C} \subseteq \Sigma_q^n \) is the transmitted word and \( y_1, \ldots, y_t \in \Sigma_q^{\leq n} \) are the output words from \( t \) instances of the deletion channel \( \text{Del}(p) \), then

\[
\mathcal{D}_{\text{ML}}(y_1, \ldots, y_t) = \arg \max_{c \in \mathcal{C}} \left\{ \prod_{i=1}^t \text{Emb}(c; y_i) \right\},
\]

and for the insertion channel \( \text{Ins}(p) \), and \( y_1, \ldots, y_t \in \Sigma_q^{\geq n} \),

\[
\mathcal{D}_{\text{ML}}(y_1, \ldots, y_t) = \arg \max_{c \in \mathcal{C}} \left\{ \prod_{i=1}^t \text{Emb}(y_i; c) \right\}.
\]
Proof: It holds that

\[ D_{\text{ML}}(y_1, \ldots, y_t) \overset{(a)}{=} \arg\max_{c \in \mathcal{C}} \left\{ \Pr_{(S,t)}(y_1, \ldots, y_t \mid \text{rec.} \ c \ \text{trans.}) \right\} \]

\[ \overset{(b)}{=} \arg\max_{c \in \mathcal{C}} \left\{ \prod_{i=1}^{t} \Pr_{S}(y_i \mid \text{rec.} \ c \ \text{trans.}) \right\} \]

\[ \overset{(c)}{=} \arg\max_{c \in \mathcal{C}} \left\{ \prod_{i=1}^{t} \Pr_{S}(y_i \mid \text{rec.} \ c \ \text{trans.}) \right\} \]

\[ \overset{(d)}{=} \arg\max_{c \in \mathcal{C}} \left\{ \prod_{i=1}^{t} \Pr_{S}(y_i \mid \text{rec.} \ c \ \text{trans.}) \right\} \]

where (a) is the definition of the ML decoder, (b) holds since the channels’ outputs are independent, (c) follows from the fact that the conditional probability \( \Pr_{S}(y_i \mid \text{rec.} \ c \ \text{trans.}) \) equals 0 when \( c \) is not a supersequence of \( y_i \) for \( 1 \leq i \leq t \). Lastly, (d) holds from Claim 1 and from the fact that the value \( \prod_{i=1}^{t} p^{|c|-|y_i|} (1-p)^{|y_i|} \) is the same for every codeword in \( \mathcal{C} \). The proof for the insertion channel is similar.

Since the deletion (insertion) channel affects the length of its output, it is possible that the length of the shortest (longest) common supersequence (subsequence) of a given channels’ outputs will be smaller (larger) than the code length. If the goal is to minimize the average decoding failure probability then clearly the decoder’s output should be a codeword as there is no point in outputting a non-codeword. However, if one seeks to minimize the expected normalized distance, then the decoder should consider non-codewords as well. Therefore, we present here the ML* decoder, which is an alternative definition of the ML decoder that takes into account non-codewords and in particular words with different length than the code length. The maximum-likelihood* (ML*) decoder for \( \mathcal{C} \) with respect to a channel \( S \), denoted by \( D_{\text{ML}*} \), should output words that minimize the expected normalized distance \( \Pr_{\text{err}}(S, \mathcal{C}, D, d) \):

\[ \Pr_{\text{err}}(S, \mathcal{C}, D, d) = \frac{1}{|\mathcal{C}|} \sum_{c \in \mathcal{C}} \Pr_{\text{err}}(c, d) \]

\[ \overset{(a)}{=} \frac{1}{|\mathcal{C}|} \sum_{c \in \mathcal{C}} \sum_{y:D(y) \neq c} \frac{d(D(y), c)}{|c|} \Pr_{S}(y \mid \text{rec.} \ c \ \text{trans.}) \]

\[ \overset{(b)}{=} \frac{1}{|\mathcal{C}|} \sum_{c \in \mathcal{C}} \sum_{y:D(y) \neq c} \frac{d(D(y), c)}{|c|} \Pr_{S}(y \mid \text{rec.} \ c \ \text{trans.}) \]

where (a) is the definition of the expected normalized distance and in (b) we changed the order of summation, while taking into account all possible channel’s outputs. For every \( y \in \Sigma_q^n \), denote the value \( \sum_{c:D(y) \neq c} \frac{d(D(y), c)}{|c|} \Pr_{S}(y \mid \text{rec.} \ c \ \text{trans.}) \) by \( f_y(D(y)) \) and if \( D(y) \) is some arbitrary value \( x \) then this value is denoted by \( f_y(x) \). Hence, the ML* decoder is defined to be

\[ D_{\text{ML}*}(y) \triangleq \arg\min_{x \in \Sigma_q^n} \{ f_y(x) \}. \]

For the deletion and insertion channels, the ML* decoder can be characterized as follows.

Claim 4. Assume \( c \in \mathcal{C} \subseteq \Sigma_q^n \) is the transmitted word and \( y \in \Sigma_q^{\leq n} \) is the output word from the deletion channel \( \text{Del}(p) \), then

\[ D_{\text{ML}*}(y) = \arg\min_{x \in \Sigma_q^n} \left\{ \sum_{c \in \mathcal{C}} d_L(x, c) \text{Emb}(c; y) \right\} \]

and for the insertion channel \( \text{Ins}(p) \), and \( y \in \Sigma_q^{\geq n} \),

\[ D_{\text{ML}*}(y) = \arg\min_{x \in \Sigma_q^n} \left\{ \sum_{c \in \mathcal{C}} d_L(x, c) \text{Emb}(y; c) \right\} \]

where (a) is the definition of the ML decoder, (b) holds since the channels’ outputs are independent, (c) follows from the fact that the conditional probability \( \Pr_{S}(y_i \mid \text{rec.} \ c \ \text{trans.}) \) equals 0 when \( c \) is not a supersequence of \( y_i \) for \( 1 \leq i \leq t \). Lastly, (d) holds from Claim 1 and from the fact that the value \( \prod_{i=1}^{t} p^{|c|-|y_i|} (1-p)^{|y_i|} \) is the same for every codeword in \( \mathcal{C} \). The proof for the insertion channel is similar.
Proof: The following equations hold

\[ \mathcal{D}_{ML^*}(y) = \arg\min_{x \in \Sigma_q^n} \{ f_y(x) \} \]
\[ \overset{(a)}{=} \arg\min_{x \in \Sigma_q^n} \left\{ \sum_{c_{xy} \neq e} d_L(x, c) \Pr_S \{ y \text{ rec.} | c \text{ trans.} \} \right\} \]
\[ \overset{(b)}{=} \arg\min_{x \in \Sigma_q^n} \left\{ \sum_{c_{xy} \neq e} d_L(x, c) p^{(|c|-|y|)}(1-p)^{|y|} \Pr_S \{ y \text{ rec.} | c \text{ trans.} \} \right\} \]
\[ \overset{(c)}{=} \arg\min_{x \in \Sigma_q^n} \left\{ \sum_{c \in C} d_L(x, c) \Pr_S \{ y \text{ rec.} | c \text{ trans.} \} \right\} , \]

where (a) follows from the definition of the ML* decoder, (b) follows from Claim 1 and (c) holds since for every \( x \in \Sigma_q^n \), the values of \(|c|, |y|, \) and \( p \) are fixed. The proof for the insertion channel is similar.

The definition of the ML* decoder can be easily generalized to the case of multiple channel outputs. Recall that the definition of the expected normalized distance \( P_{err}(S, t, C, D, d) \) for multiple channels states that

\[ P_{err}(S, t, C, D, d) = \frac{1}{|C|} \sum_{c \in C} \sum_{y_1, \ldots, y_t \in \Sigma_q} d(D(y_1, \ldots, y_t), c) \Pr_S \{ y_1, \ldots, y_t \text{ rec.} | c \text{ trans.} \} \]
\[ = \frac{1}{|C|} \sum_{y_1, \ldots, y_t \in \Sigma_q} \sum_{c: D(y_1, \ldots, y_t) \neq e} d(D(y_1, \ldots, y_t), c) \prod_{i=1}^t \Pr_S \{ y_i \text{ rec.} | c \text{ trans.} \} . \]

In this case, we let

\[ f_{y_1, \ldots, y_t}(D(y_1, \ldots, y_t)) \triangleq \sum_{c: D(y_1, \ldots, y_t) \neq e} d(D(y_1, \ldots, y_t), c) \prod_{i=1}^t \Pr_S \{ y_i \text{ rec.} | c \text{ trans.} \} , \]

where \( y_1, \ldots, y_t \) are the \( t \) channel outputs. Then, the ML* decoder is defined to be

\[ \mathcal{D}_{ML^*}(y_1, \ldots, y_t) \triangleq \arg\min_{x \in \Sigma_q^n} \{ f_{y_1, \ldots, y_t}(x) \} . \]

The following claim solves this setup for the case of deletions or insertions.

Claim 5. Assume \( c \in C \subseteq \Sigma_q^n \) is the transmitted word and \( y_1, \ldots, y_t \in \Sigma_q^n \) are the output words from \( t \) deletion channels \( \text{Del}(p) \). Then,

\[ \mathcal{D}_{ML^*}(y_1, \ldots, y_t) = \arg\min_{x \in \Sigma_q^n} \left\{ \sum_{c \in C} d_L(x, c) \prod_{i=1}^t \text{Emb}(c; y_i) \right\} \]

and for the insertion channel \( \text{Ins}(p) \), for \( y_1, \ldots, y_t \in \Sigma_q^n \),

\[ \mathcal{D}_{ML^*}(y_1, \ldots, y_t) = \arg\min_{x \in \Sigma_q^n} \left\{ \sum_{c \in C} d_L(x, c) \prod_{i=1}^t \text{Emb}(y_i; c) \right\} . \]
Proof: The following equations hold

\[ D_{\text{ML}}^*(y_1, \ldots, y_l) = \underset{x \in \Sigma^*}{\arg\min}\{f_{y_1, \ldots, y_l}(x)\} \]

\[ = \underset{x \in \Sigma^*}{\arg\min}\left\{ \sum_{c \notin c} \frac{d_L(x, c)}{|c|} \prod_{i=1}^l \Pr_{\Sigma}\{y_i \text{ rec. } | c \text{ trans.}\} \right\} \]

\[ = \underset{x \in \Sigma^*}{\arg\min}\left\{ \sum_{c \notin c} \frac{d_L(x, c)}{|c|} \prod_{i=1}^l p(|c|-|y_i|)(1-p)^{|y_i|}\text{Emb}(c; y_i) \right\} \]

\[ = \underset{x \in \Sigma^*}{\arg\min}\left\{ \sum_{c \notin c} d_L(x, c) \prod_{i=1}^l \text{Emb}(c; y_i) \right\} \]

\[ = \underset{x \in \Sigma^*}{\arg\min}\left\{ \sum_{c \notin \text{SCS}(y_1, \ldots, y_l)} d_L(x, c) \prod_{i=1}^l \text{Emb}(c; y_i) \right\}, \]

where (a) follows from the definition of the ML* decoder, (b) follows from Claim 1, (c) holds since for every \( x \in \Sigma^*_t \), the values of \(|c|\), \(|y_i|\), and \( p \) are fixed, and (d) holds since \( \prod_{i=1}^l \text{Emb}(c; y_i) = 0 \) for every \( c \in C \) such that \( c \notin \text{SCS}(y_1, \ldots, y_l) \). The proof for the insertion channel is similar.

In the rest of the paper we primarily focus on two versions of the deletion channel. While the first one is Del(\( p \)) which was already discussed so far in the paper, the second will be studied in Section VII and Section VIII. This channel, denoted by \( k\)-Del and referred as the \( k \)-deletion channel, deletes exactly \( k \) symbols of the transmitted word, while the choice of the \( k \) deleted symbol is identical among all \( \binom{n}{k} \) options, where \( n \) is the word length. This channel is formally introduced in Section VII and our goal is to analyze the ML* decoder for a single instance of this channel. In Section IV we focus on the deletion channel Del(\( p \)) and study the case of two channels of this setup. While computing the ML* decoder in this case is computationally impractical we instead analyze and study a degraded version of this decoder and study its expected normalized distance. The same task is carried also for two insertion channels in Section V. Then, Section VI is dedicated to the \( k \)-deletion channel for \( k = 1 \), while in Section VII we solve the \( k = 2 \) case. In both cases we give a full characterization of the ML* decoder and its expected normalized distance.

IV. Two Deletion Channels

In this section we study the case of two instances of the deletion channel, Del(\( p \)), where every symbol is deleted with probability \( p \). Recall that for a given codeword \( c \in C \) and two channel outputs \( y_1, y_2 \in (\Sigma_q)^{|c|} \), by Claim 5 the output of the ML* decoder is

\[ D_{\text{ML}}(y_1, y_2) = \underset{x \in \Sigma^*}{\arg\min}\left\{ \sum_{c \notin C \cap \text{SCS}(y_1, y_2)} d_L(x, c) \prod_{i=1}^2 \text{Emb}(c; y_i) \right\}. \]

Since the number of shortest common supersequences of \( y_1 \) and \( y_2 \) can grow exponentially with their lengths [47], a direct computation of the ML* decoder might be impractical in this case. Hence, not only that the number of candidates \( x \) is infinite, the number of codewords \( c \in C \cap \text{SCS}(y_1, y_2) \) that are evaluated in the summation can be exponential. Therefore, we suggest a suboptimal approach, which is yet very practical. Instead of using the formal definition of the ML* decoder, in this section a degraded version of the ML* decoder is used. This decoder, denoted by \( D_{\text{MLD}} \) and referred as the ML\(^D\) decoder, is defined as follows

\[ D_{\text{MLD}}(y_1, y_2) = \underset{x \in \text{SCS}(y_1, y_2)}{\arg\max}\{\text{Emb}(x; y_1)\text{Emb}(x; y_2)\}. \]

Assume \( C \in \Sigma^*_t \). The average decoding failure probability of the ML\(^D\) decoder over two deletion channels Del(\( p \)) is denoted by \( P_{\text{fail}}(q, p) \). Similarly, the expected normalized distance is denoted by \( P_{\text{err}}(q, p) \). These values provide upper bounds on the corresponding error probabilities of the ML* decoder. Our main goal in this section is to calculate close approximations for \( P_{\text{fail}}(q, p) \) and \( P_{\text{err}}(q, p) \). Note that a lower bound on these probabilities is \( p^2 \) (and more generally \( p^t \) for \( t \) channels) since if the same symbol is deleted in all of the channels, then it is not possible to recover its value and thus it will be deleted also in the output of the ML\(^D\) decoder. This was already observed in [76] and in their simulation results. We will also analyze these failure and error probabilities for the VT code [85] and the SVT code [73].

The lower bound \( p^2 \) on \( P_{\text{err}}(q, p) \) is not tight since if symbols from the same run are deleted then the outputs of the two channels of this run are the same and it is impossible to detect that this run experienced a deletion in both of its copies. The error probability due to deletions within runs is denoted by \( P_{\text{run}}(q, p) \) and the next lemma approximates this probability.
Lemma 6. For the deletion channel \( \text{Del}(p) \), it holds that

\[ P_{\text{run}}(q,p) \approx \frac{q+1}{q-1} p^2. \]

**Proof:** Given a run of length \( r \), the probability that both of its copies have experienced a deletion is roughly \((rp)^2\). Furthermore, the occurrence probability of a run of length exactly \( r \) is \( \frac{q-1}{q} \left( \frac{1}{q} \right)^{r-1} \). Thus, for \( n \) large enough, the error probability is approximated by

\[
\sum_{r=1}^{\infty} (rp)^2 \cdot \frac{q-1}{q} \cdot \left( \frac{1}{q} \right)^{r-1} = p^2 \cdot \frac{q-1}{q} \sum_{r=1}^{\infty} r^2 \left( \frac{1}{q} \right)^{r-1} = p^2 \cdot \frac{q-1}{q} \cdot \frac{1+\frac{1}{q}}{(1-\frac{1}{q})^3} = p^2 \cdot \frac{q(q+1)}{(q-1)^2}.
\]

The expected length of a run is given by

\[
\sum_{r=1}^{\infty} r \left( \frac{1}{q} \right)^{r-1} = \frac{q-1}{q} \sum_{r=1}^{\infty} r \left( \frac{1}{q} \right)^{r-1} = \frac{q}{q-1}.
\]

Hence, the expected number of runs in a vector of length \( n \) is \( n \cdot \frac{2-1}{q} \) and thus the approximated number of deletions in the decoder’s output due to runs is

\[
n \cdot \frac{q-1}{q} \cdot p^2 \cdot \frac{q(q+1)}{(q-1)^2} = np^2 \cdot \frac{q+1}{q-1},
\]

which verifies the statement in the lemma. □

However, runs are not the only source of errors in the output of the ML\(^D\) decoder. For example, assume the \( i \)-th and the \((i+1)\)-st symbols are deleted from the first and the second channel output, respectively. If the transmitted word \( x \) is of the form \( x = (x_1, \ldots, x_i-1, 0, 1, x_{i+2}, \ldots, x_n) \), then the two channels’ outputs are \( y_1 = (x_1, \ldots, x_{i-1}, 0, x_{i+2}, \ldots, x_n) \) and \( y_2 = (x_1, \ldots, x_{i-1}, 1, x_{i+2}, \ldots, x_n) \). However, these two outputs could also be received upon deletions exactly in the same positions if the transmitted word was \( x' = (x_1, \ldots, x_{i-1}, 1, 0, x_{i+2}, \ldots, x_n) \). Hence, the ML\(^D\) decoder can output the correct word only in one of these two cases. Longer alternating sequences cause the same problem as well and the occurrence probability of this event, denoted by \( P_{\text{alt}}(q,p) \), will be estimated in the next lemma.

Lemma 7. For the deletion channel \( \text{Del}(p) \), it holds that

\[ P_{\text{alt}}(q,p) \approx 2p^2. \]

**Proof:** Assume there is a deletion in the first channel in the \( i \)-th position and the closest deletion in the second channel is \( j > 0 \) positions apart, i.e., either in position \( i-j \) or \( i+j \). W.l.o.g. assume it is in the \((i+j)\)-th position and \( x_{[i:j]} \) is an alternating sequence \( ABAB \cdots \). Then, the same outputs from the two channels could be received if the transmitted word was the same as \( x \) but with the opposite order of the symbols of the alternating sequence, that is, the symbols of the word in the positions of \([i:j]\) are \( BABA \cdots \). Therefore, the occurrence probability of this event can be approximated by

\[
2p^2 \cdot \sum_{j=1}^{\infty} \frac{q-1}{q} \cdot \frac{1}{q^{j-1}} = 2p^2,
\]

where \( \frac{q-1}{q} \cdot \frac{1}{q^{j-1}} \) is the probability that \( x_{[i:j]} \) is any alternating sequence and the multiplication by 2 takes into account the cases of deletion in either position \( i-j \) or \( i+j \). □

It should be noted that the approximation of \( P_{\text{run}}(q,p) \), \( P_{\text{alt}}(q,p) \) that was proved in Lemma 6, Lemma 7 is in fact an upper bound on this value at our simulations, respectively. This can be explained due to the fact that in these approximations we consider runs and alternating sequences of any length. However, when the code is \( C = \Sigma_q^u \) almost all of the codewords do not have long runs or long alternating sequences (for example of length that is \( \Theta(n) \)), and hence some of these error events are extremely rare and most likely were not generated in our simulations. Moreover, the probability to have long runs and long alternating sequences decreases as \( q \) increases. This observation can also be seen in our simulations in Figures 1 and 2. On the other hand, these lemmas neglect any error event that involves more than a single deletion in a specific run or a specific alternating sequence in one of the channel outputs. These error events do increase the error probability of the decoder, but their probability is \( O(p^3) \). As a conclusion from Lemma 6 and Lemma 7 we can approximate the expected normalized distance for the case of two deletion channels to be the sum of the Levenshtein error rate due to errors in runs (Lemma 6), the Levenshtein error rate due to errors in alternating sequences (Lemma 7), and the Levenshtein error rate due to errors in runs and alternating sequences (Lemma 6 and Lemma 7).
error rate due to alternating sequence errors (Lemma 7), and additional errors which are in the order of $p^3$. Hence, the expected normalized distance of the ML\textsuperscript{D} decoder for the case of two deletion channels is approximated by

$$ P_{\text{err}}(q, p) \lesssim P_{\text{run}}(q, p) + P_{\text{alt}}(q, p) + O(p^3) = \frac{3q - 1}{q - 1} p^2 + O(p^3). $$

Using these observations, we are also able to approximate the average decoding failure probability.

**Theorem 8.** The average decoding failure probability is

$$ P_{\text{fail}}(q, p) \gtrsim e^{-\frac{3q - 1}{q - 1} p^2 n}. $$

**Proof:** This probability is bounded from below by the probability that there was neither a deletion error because of the runs nor errors because of the alternating sequences. Hence, a lower bound on this probability is

$$(1 - P_{\text{run}}(q, p))^n \cdot (1 - P_{\text{alt}}(q, p))^n \approx (1 - (P_{\text{run}}(q, p) + P_{\text{alt}}(q, p)))^n = \left(1 - \frac{3q - 1}{q - 1} p^2\right)^n \approx e^{-\frac{3q - 1}{q - 1} p^2 n}. $$

So far we have discussed only the case in which the code $C$ is the entire space. However, the most popular deletion-correcting code is the VT code \[85\]. Recently, SVT, an extension of the VT code, has been proposed in [73] for the correction of burst deletions. The goal of the VT code is to correct a deletion error while its position is known up to some roughly $\log(n)$ consecutive errors. However, this construction has been recently used in [17] to build a code that is specifically targeted for the reconstruction of a word that is transmitted through two single-deletion channels. Due to the relevance of correcting deletion and alternating errors, the decoding failure probabilities of these two codes are investigated in this work. We abbreviate the notation of deletion and alternating errors, the decoding failure probabilities of these two codes are investigated in this work. We abbreviate the notation of the VT code was taken from [81], and we modified this implementation for SVT codes.

Complexity wise, it is well known that the time complexity to calculate the SCS length and the embedding numbers of two sequences are both quadratic with the sequences’ lengths. However, the number of SCSs can grow exponentially [28], [47]. Thus, given a set of SCSs of size $L$, the complexity of the ML\textsuperscript{D} decoder for $t = 2$ will be $O(Ln^2)$. The main idea behind these algorithms uses dynamic programming in order to calculate the SCS length and the embedding numbers for all prefixes of the given words. However, when calculating for example the SCS for $y_1$ and $y_2$, it is already known that $\text{SCS}(y_1, y_2) \leq n$. Hence, it is not hard to observe that (see e.g. [3]) many paths corresponding to prefixes which their length difference is greater than $d_1 + d_2$ can be eliminated, when $d_1, d_2$ is the number of deletions in $y_1, y_2$, respectively. In particular, when $d_1$ and $d_2$ are fixed, then the time complexity is linear. In our simulations we used this improvement when implementing the ML\textsuperscript{D} decoder. Other improvements and algorithms of the ML decoder are discussed in [76], [77].
V. TWO INSERTION CHANNELS

This section continues the two-channel study but for the insertion case. In a similar manner to the deletion case, the ML^D decoder is defined as
\[ D_{ML^D}(y_1, y_2) \triangleq \arg \min_{x \in LCS(y_1, y_2)} \{ \text{Emb}(x; y_1) \text{Emb}(x; y_2) \} . \]

This section proves that, similarly to the deletion case, the dominant errors resulting from increasing the length of a run in both of the channel outputs and error that results from the occurrence of an alternating sequence. We denote by \( P_{\text{err}}(q, p) \) the approximated expected normalized distance of the ML^D decoder upon two instances of the insertion channel \( \text{Ins}(p) \). Similarly, \( P_{\text{fail}}(q, p) \) is the average decoding failure probability and lastly \( P_{\text{run}}(q, p), P_{\text{alt}}(q, p) \) is the insertion probability due to runs, occurrence probability due to alternating sequences, respectively. The following theorem summarizes the results of this section.
Theorem 10. For the insertion channel $\text{ins}(p)$, it holds that

$$P_{\text{run}}^{\text{ins}}(q, p) \approx \frac{q+1}{q(q-1)}p^2, \quad P_{\text{alt}}^{\text{ins}}(q, p) \approx \frac{2}{q}p^2.$$ 

The proof of this theorem repeats the same ideas as the ones for the deletion case. However, we list here the relevant lemmas and their proofs for the completeness of the results in this section.

Lemma 11. For the insertion channel $\text{ins}(p)$, it holds that

$$P_{\text{run}}^{\text{ins}}(q, p) \approx \frac{q+1}{q(q-1)}p^2.$$ 

Proof: The probability that both channels have experienced an insertion in a given run of length $r$ is roughly $\left((r+1) \cdot \frac{p}{q}\right)^2$. Similarly to our proof of Lemma 6, the occurrence probability of a run of length exactly $r$ is $\approx \left(\frac{r}{q}\right)^{r-1}$.

The expected number of runs in a vector of length $n$ is $n \cdot \frac{q-1}{q}$, and thus the approximated number of insertions to the decoder output due to runs is

$$n \cdot \frac{q-1}{q} \cdot p^2 \cdot \frac{(q+1)}{(q-1)^2} = np^2 \cdot \frac{q+1}{q(q-1)},$$

which verifies the statement in the lemma.

Lemma 12. For the insertion channel $\text{ins}(p)$, it holds that

$$P_{\text{alt}}^{\text{ins}}(q, p) \approx \frac{2}{q}p^2.$$ 

Proof: Assume there are two close insertions. The first insertion is in the $i$-th position of the first channel, and the closest insertion in the second channel is in position $i+j$. If $x_{[i:i+j]}$ is an alternating sequence $ABAB\cdots$, the same output from the two channels could be received if the transmitted word is the same as $x$ but with the opposite order for the symbols of the alternating sequence, that is, the symbols of the word in the positions of $[i:j]$ are $BABA\cdots$. The same holds if the closest insertion in the second channel is in position $i-j$. Therefore, the occurrence probability of this event can be approximated by

$$\frac{2}{q}p^2 \cdot \sum_{j=0}^{\infty} \frac{q-1}{q} \cdot \frac{1}{q^{j-1}} = \frac{2}{q}p^2,$$

where $\frac{q-1}{q} \cdot \frac{1}{q^{j-1}}$ is the probability that $x_{[i:i+j]}$ is any alternating sequence and the multiplication by 2 takes into account the cases of deletion in either position $i-j$ or $i+j$.

Similarly to our discussion in Section IV, it should be noted that the approximations that were calculated in Lemma 11 and in Lemma 12 are in fact upper bounds of $P_{\text{run}}^{\text{ins}}(q, p)$ and $P_{\text{alt}}^{\text{ins}}(q, p)$ in our simulations. Hence the Levenshtein error rate for two insertion channels is approximated by

$$P_{\text{err}}(q, p) \leq P_{\text{run}}^{\text{ins}}(q, p) + P_{\text{alt}}^{\text{ins}}(q, p) + O(p^3) = \frac{3q-1}{q(q-1)}p^2 + O(p^3).$$

The theoretical results of Theorem 10 have also been verified by simulation results over words of length $n = 500$, which were used to create two noisy copies with a given fixed insertion probability $p \in [0.005, 0.05]$. Then, the two copies were decoded with the ML^D decoder. Lastly, we calculated and plotted in Fig. 4 the Levenshtein error rate as well as the error rates from runs and alternating sequences.
The expected normalized distance of the lazy decoder is the radius of the lazy decoder’s output $y$ least as good as any other decoder, and hence, follows immediately.

**Example 1.** Assume the word $x = 01001$ is transmitted through the $k$-deletion channel, for $k = 2$. Then, the set of all possible outputs is the radius-2 deletion ball of $x$, which is $D_2(x) = \{000, 001, 010, 011, 100, 101\}$. We denote the word 000 by $y_1$, and 001 by $y_2$. Note that $\text{Emb}(x; y_1) = 1$ and $\text{Emb}(x; y_2) = 3$, and hence, $\Pr_{k:\text{-Del}}\{y_1 \text{ rec. } | x \text{ trans.}\} = \frac{1}{\binom{5}{2}} \Pr_{2:\text{-Del}}\{y_2 \text{ rec. } | x \text{ trans.}\} = \frac{3}{\binom{5}{2}}$.

In the rest of the section the 1-deletion channel which deletes one symbol randomly is considered. Note that this is a special case of the $k$-deletion channel where $k = 1$. Given a single-deletion-correcting code, any channel output can be easily decoded, and therefore for the rest of this section we assume that the given code is not a single-deletion-correcting code. We start by examining two types of decoders for this channel. The first decoder, referred as the embedding number decoder and denoted by $D_{EN}$, returns for a channel output $y$ the word $D_{EN}(y)$ which is a codeword in the code $\mathcal{C}$ that maximizes the embedding number of $y$ in $D_{EN}(y)$. That is, $D_{EN}(y) = \arg\max_{c \in \mathcal{C}} \{\text{Emb}(c; y)\}$, where, for now, if there is more than one such a word the decoder chooses one of them arbitrarily. The second decoder, referred as the lazy decoder, is denoted by $D_{Lazy}$. For a channel output $y$, $D_{Lazy}$ simply returns $y$ as the output, i.e., $D_{Lazy}(y) = y$. Note that the lazy decoder does not return a codeword. Additionally, $d_L(D_{Lazy}(y), c) = 1$ since $y \in D_1(c)$ and hence, the expected normalized distance of the lazy decoder is $\frac{1}{n}$, when $n$ is the code length (see Lemma 13).

In the main result of this section, presented in Theorem 23, we prove for $S = 1$-Del and $\mathcal{C} = \Sigma_2^n$, that $D_{Lazy}$ performs at least as good as any other decoder, and hence $D_{Lazy} = D_{ML}^*$. For the rest of this section it is assumed that $\mathcal{C} \subseteq \Sigma_2^n$ and $S = 1$-Del. Under this setup, the Levenshtein distance between the lazy decoder’s output $y$ and the transmitted word $c$ is always $d_L(y, c) = 1$, since $y \in D_1(c)$. Hence, the following lemma follows immediately.

**Lemma 13.** The expected normalized distance of the lazy decoder $D_{Lazy}$ under the 1-deletion channel 1-Del is $P_{\text{err}}(1\text{-Del}, \mathcal{C}, D_{Lazy}, d_L) = \frac{1}{n}$.
Proof: The expected normalized distance of the lazy decoder for each codeword \( c \) is calculated as follows.

\[
P_{\text{err}}(c, d_L) = \sum_{y : D_{\text{Lazy}}(y) \neq c} \frac{d_L(D_{\text{Lazy}}(y), c)}{|c|} p(y|c)
\]

\[
= \sum_{y \in D_1(c)} \frac{1}{n} p(y|c) = \frac{1}{n}.
\]

Since this is true for every \( c \in C \), we get that

\[
P_{\text{err}}(1-\text{Del}, C, D_{\text{Lazy}}, d_L) = \frac{1}{n} \cdot |C| \cdot \frac{1}{|C|} = \frac{1}{n}.
\]

We can now show that the lazy decoder is preferable, with respect to the expected normalized distance, over any decoder that outputs a word of the same length as its input.

Lemma 14. Let \( D : \Sigma_2^{n-1} \rightarrow \Sigma_2^{n-1} \) be a general decoder that preserves the length of the channel’s output. It follows that

\[
P_{\text{err}}(1-\text{Del}, C, D, d_L) \geq P_{\text{err}}(1-\text{Del}, C, D_{\text{Lazy}}, d_L),
\]

and for \( C = \Sigma_2^n \) equality is obtained if and only if \( D = D_{\text{Lazy}} \).

Proof: Equality is trivial when \( D = D_{\text{Lazy}} \). Furthermore, since for every \( y \in C \) it holds that \(|D(y)| = n - 1\), it is deduced that \( d_L(c, D(y)) \geq 1 \). Hence, similarly to the proof of Lemma 13 it is easy to verify that

\[
P_{\text{err}}(1-\text{Del}, C, D, d_L) \geq \frac{1}{n} = P_{\text{err}}(1-\text{Del}, C, D_{\text{Lazy}}, d_L),
\]

where the last equality follows from Lemma 13.

Let us now assume that \( D \neq D_{\text{Lazy}} \), i.e., there exists \( z \in \Sigma_2^{n-1} \) such that \( D(z) = z' \neq z \). Since \( z' \neq z \) we have that \( I_1(z') \neq I_1(z) \), i.e., there exists a word \( c \in \Sigma_2^n \) such that \( c \in I_1(z) \) and \( c \notin I_1(z') \). Equivalently, \( z \in D_1(c) \) and \( z' \notin D_1(c) \), and so \( d_L(c, z') \geq 3 \) (at least one more deletion and one more insertion are needed in addition to the insertion needed for every word in the deletion ball).

Hence, it is derived that

\[
P_{\text{err}}(c, d_L) = \sum_{y \in D_1(c)} \frac{d_L(D(y), c)}{n} p(y|c)
\]

\[
\geq \sum_{y \in D_1(c) \setminus \{z\}} \frac{1}{n} p(y|c) + \frac{d_L(D(z) = z', c)}{n} \cdot p(y|c)
\]

\[
> \sum_{y \in D_1(c)} \frac{1}{n} p(y|c) = \frac{1}{n}.
\]

If \( C = \Sigma_2^n \) it must hold that \( c \in C \), and so

\[
P_{\text{err}}(1-\text{Del}, (\Sigma_2)^n, D, d_L) \geq \frac{|C| - 1}{|C|} \cdot \frac{1}{n} + \frac{1}{|C|} P_{\text{err}}(c, d_L) > \frac{1}{n}.
\]

Combining with Lemma 13 again completes the proof.

Before examining the performance of the embedding number decoder, we first discuss its properties over the 1-deletion channel. It is first shown that a decoder that prolongs an arbitrary run of maximal length within the input word is equivalent to the embedding number decoder.

Lemma 15. Given \( y \in \Sigma_2^{n-1} \), the word \( \hat{x} \in \Sigma_2^n \) obtained by prolonging a run of maximal length in \( y \) satisfies

\[
\text{Emb}(\hat{x}; y) = \max_{x \in \Sigma_2^n} \{\text{Emb}(x; y)\}.
\]

Proof: Let \( y \) be a word with \( n_r \) runs of lengths \( r_1, r_2, \ldots, r_{n_r} \). Let \( x_0 \) be any word obtained from \( y \) by creating a new run of length one, and so \( \text{Emb}(x_0; y) = 1 \). Let \( x_i, 1 \leq i \leq n_r \) be the word obtained from \( y \) by prolonging the \( i \)-th run by one, and so \( \text{Emb}(x_i; y) = r_i + 1 \). Hence, it follows that

\[
\arg\max_{0 \leq i \leq n_r} \{\text{Emb}(x_i; y)\} = \arg\max_{0 \leq i \leq n_r} \{r_i + 1\},
\]
where by definition \( r_0 = 0 \). It should note that the union of the words \( x_i, 1 \leq i \leq n_r \) is all of the words that can be obtained from \( x \) by one deletion, and hence are the only words of length \( n - 1 \) with embedding number larger than 0.

According to Lemma 15, we can arbitrarily choose the decoder that prolongs the first run of maximal length as the embedding number decoder.

**Definition 16.** The embedding number decoder \( D_{EN} \) prolongs the first run of maximal length in \( y \) by one symbol. A decoder \( D \) that prolongs one of the runs of maximal length in \( y \) by one symbol is said to be equivalent to the embedding number decoder, and is denoted by \( D \equiv D_{EN} \).

The rest of this section will focus on the case for which \( C = \Sigma_2^n \). The following lemmas will be stated for the embedding number decoder for the simplicity of the proofs, but unless stated otherwise they hold for any decoder \( D \) for which \( D \equiv D_{EN} \).

**Lemma 17.** For every codeword \( c \in C \), the embedding number decoder satisfies

\[
P_{\text{err}}(c, d_L) = \frac{2}{n} \sum_{y \in D_1(c), c \neq D_{EN}(y)} \frac{\text{Emb}(c; y)}{n}.
\]

**Proof:** Let \( c \in C \) be a codeword and let \( y \in D_1(c) \) be a channel output such that \( D_{EN}(y) \neq c \). Since \( D_{EN}(y) \) can be obtained from a word in \( D_1(c) \) by one insertion, it follows that \( d_L(D_{EN}(y), c) = 2 \). Thus,

\[
P_{\text{err}}(c, d_L) = \sum_{y : D_{EN}(y) \neq c} \frac{d_L(D_{EN}(y), c)}{|c|} p(y|c)
\]

\[
= \frac{2}{n} \sum_{y \in D_1(c)} p(y|c) \cdot 1\{D_{EN}(y) \neq c\}
\]

\[
= \frac{2}{n} \sum_{y \in D_1(c), c \neq D_{EN}(y)} \frac{\text{Emb}(c; y)}{n}.
\]

For \( y \in D_1(c) \), we have that \( D_{EN}(y) = c \) if and only if the deletion occurred within the run corresponding to the first run of maximal length in \( y \). Hence, the embedding number decoder will fail at least for any deletion occurring outside of the first run of maximal length in \( c \). This observation will be used in the proof of Lemma 18. Before presenting this Lemma, one more definition is introduced. For a word \( x \in \Sigma_2^n \), we denote by \( \tau(x) \) the length of its maximal run. For example \( \tau(00111010) = 3 \) and \( \tau(01010101) = 1 \). For a code \( C \subseteq \Sigma_2^n \), we denote by \( \tau(C) \) the average length of the maximal runs of its codewords. That is,

\[
\tau(C) = \frac{\sum_{c \in C} \tau(c)}{|C|}.
\]

Furthermore, if \( N(r) \), for \( 1 \leq r \leq n \) denotes the number of codewords in \( C \) in which the length of their maximal run is \( r \), then \( \tau(C) = \frac{\sum_{r=1}^{n} r N(r)}{|C|} \). We are now ready to present a lower bound on the expected normalized distance of the embedding number decoder.

**Lemma 18.** The expected normalized distance of the embedding number decoder \( D_{EN} \) satisfies

\[
P_{\text{err}}(1\text{-Del}, C, D_{EN}, d_L) \geq \frac{2}{n} \left( 1 - \frac{\tau(C)}{n} \right).
\]

**Proof:** Let \( C_r \subseteq C \) be the subset of codewords with maximal run length of \( r \), and let its size be denoted by \( N(r) \). For any codeword \( c \), any deletion outside of the first run of maximal length will result in a decoding failure. Since the sum

\[
\sum_{y \in D_1(c), c \neq D_{EN}(y)} \frac{\text{Emb}(c; y)}{n}
\]

is equivalent to counting the indices in \( c \) in which a deletion will result in a decoding failure (and normalizing it by \( n \)), using Lemma 17 we get that for every \( c \in C_r \),

\[
P_{\text{err}}(c, d_L) \geq \frac{2}{n} \cdot \frac{n - r}{n},
\]
and the expected normalized distance becomes
\[ P_{\text{err}}(1-\text{Del}, \mathcal{C}, \mathcal{D}_{\text{EN}}, d_L) = \frac{1}{|\mathcal{C}|} \sum_{c \in \mathcal{C}} P_{\text{err}}(c, d_L) \]
\[ = \frac{1}{|\mathcal{C}|} \sum_{r=1}^{n} \sum_{c \in \mathcal{C}} P_{\text{err}}(c, d_L) \geq \frac{1}{|\mathcal{C}|} \sum_{r=1}^{n} \sum_{c \in \mathcal{C}} \frac{2}{n} \left( 1 - \frac{r}{n} \right) \]
\[ = \frac{1}{|\mathcal{C}|} \sum_{r=1}^{n} N(r) \left( 1 - \frac{r}{n} \right) \]
\[ = \frac{2}{n} \left( 1 - \frac{1}{n} \sum_{r=1}^{n} r \cdot N(r) \right) \]
\[ = \frac{2}{n} \left( 1 - \frac{\tau(|\mathcal{C}|)}{n} \right). \]

For the special case of \( \mathcal{C} = \Sigma_2^{|\mathcal{C}|} \), the next claim is proved in Appendix A.

**Claim 19.** For all \( n \geq 1 \) it holds that \( \tau(|\mathcal{C}|) \leq 2 \log_2(n) \).

We will now show that the embedding number decoder is preferable over any other decoder that outputs a word of the original codeword length.

**Lemma 20.** Let \( \mathcal{D} : \Sigma_2^{n-1} \rightarrow \Sigma_2^n \) be a general decoder that prolongs the input length by one. It follows that
\[ P_{\text{err}}(1-\text{Del}, \mathcal{C}, \mathcal{D}, d_L) \geq P_{\text{err}}(1-\text{Del}, \mathcal{C}, \mathcal{D}_{\text{EN}}, d_L). \]

and equality is obtained if and only if \( \mathcal{D} \equiv \mathcal{D}_{\text{EN}} \).

**Proof:** We have the following sequence of equalities and inequalities
\[ P_{\text{err}}(1-\text{Del}, \mathcal{C}, \mathcal{D}, d_L) = \frac{1}{|\mathcal{C}|} \sum_{c \in \mathcal{C}} \sum_{y : \mathcal{D}(y) \neq c} \frac{d_L(\mathcal{D}(y), c)}{|c|} p(y|c) \]
\[ = \frac{1}{|\mathcal{C}|} \sum_{y \in \Sigma_2^{n-1}} \sum_{c \in \mathcal{C}} \frac{d_L(\mathcal{D}(y), c)}{|c|} p(y|c) \]
\[ \geq \frac{1}{|\mathcal{C}|} \sum_{y \in \Sigma_2^{n-1}} \frac{2}{n |\mathcal{C}|} \left( \left( \sum_{c \in \mathcal{C}} p(y|c) \right) - p(y|\mathcal{D}(y)) \right) \]
\[ = \frac{2}{n |\mathcal{C}|} \sum_{y \in \Sigma_2^{n-1}} \sum_{c \in \mathcal{C}} p(y|c) - \frac{2}{n |\mathcal{C}|} \sum_{y \in \Sigma_2^{n-1}} p(y|\mathcal{D}(y)) \]
\[ \geq \frac{2}{n |\mathcal{C}|} \sum_{y \in \Sigma_2^{n-1}} \sum_{c \in \mathcal{C}} p(y|c) - \frac{2}{n^2 |\mathcal{C}|} \sum_{y \in \Sigma_2^{n-1}} \sum_{c \in \mathcal{C}} \max \{ \text{Emb}(c; y) \} \]
\[ \geq \frac{2}{n |\mathcal{C}|} \sum_{y \in \Sigma_2^{n-1}} \sum_{c \in \mathcal{C}} p(y|c) - \frac{2}{n^2 |\mathcal{C}|} \sum_{y \in \Sigma_2^{n-1}} \sum_{c \in \mathcal{C}} \text{Emb}(\mathcal{D}_{\text{EN}}(y); y) \]
\[ = P_{\text{err}}(1-\text{Del}, \mathcal{C}, \mathcal{D}_{\text{EN}}, d_L). \]

where (a) is a result of replacing the order of summation, (b) holds since for every \( c \) such that \( \mathcal{D}(y) \neq c \) we have that \( d_L(\mathcal{D}(y), c) \geq 2 \), and for \( c^* = \mathcal{D}(y) \) \( d_L(\mathcal{D}(y), c^*) = 0 \). The equality (c) is obtained by the definition of the 1-deletion channel, and in (d) we simply choose the word that maximizes the value of \( \text{Emb}(c; y) \), which is the definition of the ML decoder as derived in step (e). From steps (b) and (e) it also follows that equality is obtained if and only if \( \mathcal{D} \equiv \mathcal{D}_{\text{EN}} \).

It can now be shown that in this case, the lazy decoder is preferable over the embedding number decoder.

**Lemma 21.** For every \( n \geq 17 \) it holds that
\[ P_{\text{err}}(1-\text{Del}, \Sigma_2^n, \mathcal{D}_{\text{EN}}, d_L) > P_{\text{err}}(1-\text{Del}, \Sigma_2^n, \mathcal{D}_{\text{Lazy}}, d_L). \]
Proof: For \( \mathcal{C} = \Sigma_L^n \), from Claim [19] we have that \( \tau(\mathcal{C}) \leq 2 \log_2 n \). Since for every \( n \geq 17 \) it follows that \( 2 \log_2(n) < n/2 \), using Lemma [18] we have that

\[
P_{\text{err}}(\text{1-Del, } (\Sigma_2)^n, \mathcal{D}_{\text{EN}, d_L}) \geq \frac{2}{n} \cdot \left( 1 - \frac{2 \log_2(n)}{n} \right) > \frac{1}{n}.
\]

For the rest of this paper we assume \( n \geq 17 \). Next, we examine a hybrid decoder which returns words of length either \( n - 1 \) or \( n \) and it will be shown that the lazy decoder is also preferable over any hybrid decoder.

**Lemma 22.** Let \( \mathcal{D} : \Sigma_2^{n-1} \to \Sigma_2^{n-1} \cup \Sigma_2^n \) be a general decoder that either preserves the channel outputs’ length or prolongs it by one. Then, it holds that

\[
P_{\text{err}}(\text{1-Del, } \Sigma_2^n, \mathcal{D}, d_L) \geq P_{\text{err}}(\text{1-Del, } \Sigma_2^n, \mathcal{D}_{\text{Lazy}, d_L}).
\]

**Proof:** Let \( \mathcal{D} \) be a decoder as defined in the lemma. Similarly to the proof of Lemma [20] by definition,

\[
P_{\text{err}}(\text{1-Del, } (\Sigma_2)^n, \mathcal{D}, d_L) = \frac{1}{|\mathcal{C}|} \sum_{c \in \mathcal{C}} \sum_{y \in \Sigma_2^{n-1} \setminus \mathcal{D}(y) \neq c} \frac{d_L(\mathcal{D}(y), c)}{|c|} p(y|c)
\]

\[
= \frac{1}{|\mathcal{C}|} \sum_{y \in \Sigma_2^{n-1}} \sum_{c \in I_1(y)} \frac{d_L(\mathcal{D}(y), c)}{|c|} p(y|c)
\]

\[
\geq \frac{2}{n|\mathcal{C}|} \sum_{y \in \Sigma_2^{n-1}} \left( \sum_{c \in I_1(y)} p(y|c) - p(y|\mathcal{D}(y)) \right)
\]

\[
+ \frac{1}{n|\mathcal{C}|} \sum_{y \in \Sigma_2^{n-1}} \sum_{c \in I_1(y)} p(y|c).
\]

(2)

We first show that for each \( y \in \Sigma_2^{n-1} \) such that \( |\mathcal{D}(y)| = n \) it holds that

\[
2 \sum_{c \in I_1(y)} p(y|c) - 2p(y|\mathcal{D}(y)) \geq \sum_{c \in I_1(y)} p(y|c).
\]

(3)

This is proved by verifying that

\[
2 \sum_{c \in I_1(y)} p(y|c) - 2p(y|\mathcal{D}(y)) - \sum_{c \in I_1(y)} p(y|c)
\]

\[
= \sum_{c \in I_1(y)} p(y|c) - 2p(y|\mathcal{D}(y))
\]

\[
\geq \sum_{c \in I_1(y)} \frac{1}{n} - p(y|\mathcal{D}(y))
\]

\[
\geq 1 - p(y|\mathcal{D}(y)) \geq 0,
\]

where in (a) we split the summation of \( c \in I_1(y) \) into two parts when \( \mathcal{D}(y) \in I_1(y) \) and note that this equality holds also when \( \mathcal{D}(y) \notin I_1(y) \). In (b) we used the inequality \( p(y|c) \geq 1/n \) when \( c \in I_1(y) \) and lastly, (c) follows since \( |I_1(y)| = n + 1 \) and hence the size of the set \( I_1(y) \setminus \{ \mathcal{D}(y) \} \) is at least \( n \).
Lastly, combining (2) and (3) and remembering that \( d_L(c, D_{\text{Lazy}}(y)) = 1 \) we have that
\[
P_{\text{err}}(1-\text{Del}, \Sigma^n_2, D, d_L) \geq \frac{1}{n|C|} \left( \sum_{y \in \Sigma_2^{n-1}} \sum_{c \in I_1(y)} p(y|c) + \sum_{y \in \Sigma_2^{n-1}} \sum_{c \in I_1(y)} p(y|c) \right) \]
\[
= \frac{1}{n|C|} \sum_{y \in \Sigma_2^{n-1}} \sum_{c \in I_1(y)} p(y|c) \]
\[
= \frac{1}{|C|} \sum_{y \in \Sigma_2^{n-1}} \sum_{c \in I_1(y)} d_L(c, D_{\text{Lazy}}(y)) \frac{d_L(D(y), c)}{|c|} p(y|c) \]
\[
= \frac{1}{|C|} \sum_{c \in C} \sum_{y \in \Sigma_2^{n-1}} d_L(c, D_{\text{Lazy}}(y)) \frac{d_L(D(y), c)}{|c|} p(y|c) \]
\[
= P_{\text{err}}(1-\text{Del}, \Sigma^n_2, D_{\text{Lazy}}, d_L). \]

Finally, it is shown that the lazy decoder is preferable over any other type of decoder that returns words of any length.

**Theorem 23.** For any decoder \( D : \Sigma_2^n \rightarrow \Sigma^*_2 \),
\[
P_{\text{err}}(1-\text{Del}, \Sigma^n_2, D, d_L) \geq P_{\text{err}}(1-\text{Del}, \Sigma^n_2, D_{\text{Lazy}}, d_L). \]

**Proof:** Let \( D \) be a decoder. By Lemma 22, the theorem holds for any hybrid decoder and therefore we can assume that \( D \) is not a hybrid decoder. Hence, there exists at least one channel output \( y' \), such that, \( D(y') \) is neither of length \( n \), nor of length \( n - 1 \). We consider the following two cases.

**Case 1:** \(|D(y')| \notin \{n-1, n, n+1\} \), and thus \( d_L(D(y'), c) \geq 2 \). Similarly to the proof of Lemma 20 by definition we have the following
\[
P_{\text{err}}(1-\text{Del}, \Sigma^n_2, D, d_L) \]
\[
= \frac{1}{|C|} \sum_{y \in \Sigma_2^{n-1}} \sum_{c \in I_1(y)} \frac{d_L(D(y), c)}{|c|} p(y|c) \]
\[
\geq \frac{1}{n|C|} \sum_{y \in \Sigma_2^{n-1}} \sum_{c \in I_1(y)} p(y|c) \]
\[
+ \frac{1}{|C|} \sum_{y \in \Sigma_2^{n-1}} \sum_{c \in I_1(y)} \frac{d_L(D(y), c)}{|c|} p(y|c) \]
\[
+ \frac{1}{|C|} \sum_{y \in \Sigma_2^{n-1}} \sum_{c \in I_1(y)} \frac{d_L(D(y), c)}{|c|} p(y|c) \]
\[
\geq \frac{1}{n|C|} \sum_{y \in \Sigma_2^{n-1}} \sum_{c \in I_1(y)} p(y|c) \]
\[
+ \frac{1}{|C|} \sum_{y \in \Sigma_2^{n-1}} \sum_{c \in I_1(y)} \frac{d_L(D(y), c)}{|c|} p(y|c) \]
\[
+ \frac{1}{|C|} \sum_{y \in \Sigma_2^{n-1}} \sum_{c \in I_1(y)} \frac{d_L(D(y), c)}{|c|} p(y|c) \]
\[
= \frac{1}{n|C|} \sum_{y \in \Sigma_2^{n-1}} \sum_{c \in I_1(y)} p(y|c) = \frac{1}{n}, \]
where (a) follows from the fact that if \(|D(y)| = n \), then \( d_L(D(y), c) \geq 1 \) for each \( c \in I_1(y) \), (b) is obtained using the inequalities in (2) and (3), and (c) is obtained from the fact that whenever \(|D(y)| \notin \{n, n+1\} \), we have that \( d_L(D(y), c) \geq 2 \) for each \( c \in I_1(y) \) and this summation is not empty. The last equality results from the summing over all probabilities.
Case 2: $|D(y')| = n + 1$. If $D(y')$ is not the alternating word, then $|D_1(D(y'))| \leq n$, i.e., there are at most $n$ words of length $n$ and distance 1 from $D(y')$. Since $|I_1(y')| = n + 1$, there is at least one word $c \in I_1(y')$ such that $d_L(D(y'), c) > 1$. Using this observation and as was done in the first case of this proof we derive that

$$P_{err}(1-\text{Del}, \Sigma^n_2, D, d_L) = \frac{1}{|C|} \sum_{y \in \Sigma^{n-1}_2} \sum_{c \in I_1(y)} \frac{d_L(D(y), c)}{|c|} p(y|c)$$

$$\geq \frac{1}{n|C|} \sum_{y \in \Sigma^{n-1}_2} \sum_{c \in I_1(y)} \frac{d_L(D(y), c)}{|c|} p(y|c)$$

$$+ \frac{1}{|C|} \sum_{y \in \Sigma^{n-1}_2} \sum_{c \in I_1(y)} \frac{d_L(D(y), c)}{|c|} p(y|c)$$

$$+ \frac{1}{|C|} \sum_{y \in \Sigma^{n-1}_2} \sum_{c \in I_1(y)} \frac{d_L(D(y), c)}{|c|} p(y|c)$$

$$> \frac{1}{n|C|} \sum_{y \in \Sigma^{n-1}_2} \sum_{c \in I_1(y)} p(y|c) = \frac{1}{n}$$

where the last inequality results from the words $y', c$ which satisfy $d_L(D(y'), c) > 1$ That is, it is concluded that

$$P_{err}(1-\text{Del}, \Sigma^n_2, D, d_L) > \frac{1}{n} = P_{err}(1-\text{Del}, \Sigma^n_2, D_{\text{Lazy}}, d_L).$$

Note that, for the special case where $D(y')$ is the alternating sequence of length $n + 1$, $|I_1(y')| = |D_1(D(y'))| = n + 1$, which implies that inequality is weak.

Theorem 23 verifies that $D_{\text{Lazy}}$ minimizes the expected normalized distance for the case when $C = \Sigma^n_2$, which implies that $D_{\text{Lazy}}$ is the ML* decoder for the 1-deletion channel.

VII. THE 2-DELETION CHANNEL

In this section we consider the case of a single 2-deletion channel over a code which is the entire space, i.e., $C = \Sigma^n_2$. In this setup, a word $x \in \Sigma^n_2$ is transmitted over the channel 2-Del, where exactly 2 symbols from $x$ are selected and deleted, resulting in the channel output $y \in \Sigma^{n-2}_2$. We construct a decoder that is based on the lazy decoder and on a variant of the embedding number decoder and prove that it minimizes the expected normalized distance, that is, we explicitly find the ML* decoder for the 2-Del channel.

Recall that the expected normalized distance of a decoder $D$ over a single 2-deletion channel is defined as

$$P_{err}(D) = \frac{1}{|C|} \sum_{c \in C} \sum_{y \in \Sigma^{n-2}_2} p_{err}(c) = \frac{1}{|C|} \sum_{c \in C} \sum_{y \in \Sigma^{n-2}_2, y \neq c} d_L(D(y), c) \cdot p(y|c).$$

We can rearrange the sum as follows

$$P_{err}(D) = \frac{1}{|C|} \sum_{c \in C} \sum_{y \in \Sigma^{n-2}_2} d_L(D(y), c) \cdot p(y|c).$$

As mentioned before we denote $\sum_{c \in C} \sum_{y \in \Sigma^{n-2}_2} d_L(D(y), c) \cdot p(y|c)$ by $f_y(D(y))$. Recall that, a decoder that minimizes $f_y(D(y))$ for any channel output $y \in \Sigma^{n-2}_2$, also minimizes the expected normalized distance. Hence, when comparing two decoders, it is enough to compare $f_y(D(y))$ for each channel output $y$.

Before we continue, two more families of decoders are introduced. The maximum likelihood* decoder of length $m$, denoted by $D^m_{\text{ML}}$, is the decoder that for any given channel output $y$ returns a word $x$ of length $m$ that minimizes $f_y(x)$. That is,

$$D^m_{\text{ML}*}(y) = \arg\min_{x \in \Sigma^n_2} \{f_y(x)\}.$$ 

The embedding number decoder of length $m$, denoted by $D^m_{\text{EN}}$, is the decoder that for any given channel output $y$ returns a word $x$ of length $m$ that maximizes the embedding number of $y$ in $x$. That is,

$$D^m_{\text{EN}}(y) = \arg\max_{x \in \Sigma^n_2} \{\text{Emb}(x; y)\}.$$
In these decoders’ definitions, and unless stated otherwise, if there is more than one word $x$ that optimizes these expressions, the decoder chooses one of them arbitrarily.

Similarly to the analysis of the 1-Del channel in Section VI any embedding number decoder prolongs existing runs in the word $y$. The following lemma proves that any embedding number decoder of length $m > |y|$ prolongs at least one of the longest runs in $y$ by at least one symbol.

**Lemma 24.** Let $y \in \Sigma_{2}^{n-2}$ be a channel output. For any $m > |y|$, the decoder $D_{EN}^{m}$ prolongs one of the longest runs of $y$ by at least one symbol.

**Proof:** Assume that the number of runs in $y$ is $\rho(y) = t$ and let $r_{j}$ denote the length of the $j$-th run for $1 \leq j \leq t$. We further assume that the $i$-th run is of longest length in the word $y$, and that its length is denoted by $r_{i}$. Assume to the contrary that none of the longest runs in $y$ was prolonged and let $i'$ be one of the indices of the longest runs in $y$ that was prolonged by the decoder $D_{EN}^{m}$ and note that $r_{i} > r_{i'}$. However, if the decoder will instead prolong the $i$-th run in $y$ with the same number of symbols as the $i'$-th run was prolonged, we will get a word that strictly increases the embedding number, in contradiction.

For simplicity, we assume that in the case where there are two or more longest runs in $y$, the embedding number decoder $D_{EN}^{m}$ for $m > |y|$ necessarily chooses to prolong the first ones. Moreover, if there is more than one option that maximize the embedding number, the embedding number decoder $D_{EN}^{m}$ will choose the one that prolongs the least number of runs.

In the following lemma a useful property about $D_{ML}^{n}$, the embedding number decoder of length $n$, is given.

**Lemma 25.** Let $y \in \Sigma_{2}^{n-2}$ be a channel output. Assume that the number of runs in $y$ is $\rho(y) = r$ and let $r_{i}$ denote the length of the $i$-th run for $1 \leq i \leq t$. In addition, let the $i$-th and the $j$-th runs be the two longest runs in $y$, such that $r_{i} \geq r_{j}$. The decoder $D_{EN}^{n}$ operates as follows.

1) If $r_{i} \geq 2r_{j}$, the decoder prolongs the $i$-th run by two symbols.

2) If $r_{i} < 2r_{j}$, the decoder prolongs the $i$-th and the $j$-th runs, each by one symbol.

**Proof:** The embedding number decoder in this case has two options. The first one is to prolong one of the runs in $y$ by two symbols and the second is to prolong two runs in $y$ each by one symbol. We ignore the option of creating new runs since it won’t increase the embedding number. Thus, the maximum embedding number value is given by

$$\max \left\{ \max_{1 \leq s < \ell \leq r} \left\{ \left( \frac{r_{s} + 1}{1} \right) \cdot \left( \frac{r_{\ell} + 1}{1} \right), \max_{1 \leq s < \ell \leq r} \frac{r_{s} + 2}{2} \right\} \right\}$$

$$= \max \left\{ \max_{1 \leq s < \ell \leq r} \left\{ (r_{s} + 1)(r_{\ell} + 1), \max_{1 \leq s < \ell \leq r} \frac{r_{s} + 1}{2} \right\} \right\}$$

$$= \max \left\{ (r_{i} + 1)(r_{j} + 1), \frac{(r_{i} + 1)(r_{j} + 2)}{2} \right\}.$$

Finally, in order to determine the option which maximizes the embedding number, it is left to compare between $(r_{i} + 1)(r_{j} + 1)$ and $(r_{i} + 1)(r_{j} + 2)$ and $(r_{i} + 1)(r_{j} + 2)$. Thus, the decoder $D_{EN}^{n}$ chooses the first option, i.e., prolonging the longest run with two symbols, if and only if $(r_{i} + 1) \geq (r_{j} + 1)$ which is equivalent to $r_{i} \geq 2r_{j}$.

In the rest of this section we prove several properties on $D_{ML}^{n}$, the ML* decoder for a single 2-deletion channel and lastly in Theorem 37 we construct this decoder explicitly. Unless specified otherwise, we assume that $D_{ML}^{n}$ returns a word with minimum length that minimizes $f(y(D(y)))$.

**Lemma 26.** For a channel output $y \in \Sigma_{2}^{n-2}$, it holds that

$$n - 2 \leq |D_{ML}^{*}(y)| \leq n + 1.$$

**Proof:** Let $y \in \Sigma_{2}^{n-2}$ be a channel output and assume to the contrary that $|D_{ML}^{*}(y)| \geq n + 2$ or $|D_{ML}^{*}(y)| \leq n - 3$. In order to show a contradiction, we prove that

$$\sum_{c \in I_{2}(y)} \frac{d_{L}(D_{ML}^{*}(y), c)}{|c|} \cdot p(y|c) \geq \sum_{c \in I_{2}(y)} \frac{d_{L}(D_{Laz}(y), c)}{|c|} \cdot p(y|c),$$

and equality can be obtained only in the case $|D_{ML}^{*}(y)| = n + 2$. If $|D_{ML}^{*}(y)| \leq n - 3$ or $|D_{ML}^{*}(y)| \geq n + 3$, then $d_{L}(D_{ML}^{*}(y), c) \geq 3$ and since $d_{L}(D_{Laz}(y), c) = 2$ a strict inequality holds for each $y$. In case $|D_{ML}^{*}(y)| = n + 2$, $d_{L}(D_{ML}^{*}(y), c) \geq 2$ and the inequality holds. Recall that $D_{ML}^{*}(y)$ returns a word with minimum length which implies that $|D_{ML}^{*}(y)| \leq n + 1$. 

"
For $y \in \Sigma_2^{n-2}$, Lemma 26 implies that $m = |D_{ML^*}(y)| \in \{n-2, n-1, n, n+1\}$. In the following lemmas, we show that for any $m \in \{n-2, n-1, n\}$,

$$D_m^{n-2} = D_{EN}^{n-2} = D_{ML^*}.$$ 

**Lemma 27.** It holds that

$$D_m^{n-2} = D_{ML^*} = D_{EN}^{n-2} = D_{Lazy}.$$ 

**Proof:** Let $y \in \Sigma_2^{n-2}$ be a channel output. Each $y' \in \Sigma_2^{n-2}$ such that $y' \neq y$ satisfies $\text{Emb}(y'; y) = 0$. Hence $D_{EN}^{n-2}(y) = y$, which implies that $D_{EN}^{n-2} = D_{Lazy}$.

In order to show that $D_{Lazy} = D_{ML^*}$, let us consider any decoder $D$ that outputs words of length $n-2$ such that $D \neq D_{Lazy}$, i.e., there exists $y \in \Sigma_2^{n-2}$ such that $D(y) = y' \neq y$. Since $y' \neq y$ it holds that $I_2(y') \neq I_2(y)$ and hence, there exists a codeword $c \in \Sigma_2^n$ such that $c \in I_2(y)$ and $c \notin I_2(y')$. Equivalently, $y \in D_2(c)$, $y' \notin D_2(c)$ and therefore $d_L(c, y') \geq 4$ (at least one more deletion and one more insertion are needed in addition to the two insertions needed for every word in the deletion ball). Hence,

$$f_y(D(y)) = \sum_{c' \in \Sigma_2^n} \frac{d_L(D(y), c')}{|c'|} p(y|c')$$

$$= \sum_{c' \in \Sigma_2^n} \frac{d_L_D(y, c')}{|c'|} p(y|c') + \frac{d_L(I_L(y), c)}{|c|} p(y|c)$$

$$\geq \sum_{c' \in \Sigma_2^n} \frac{2}{|c'|} p(y|c') + \frac{d_L(I_L(y), c)}{|c|} p(y|c)$$

$$\geq \sum_{c' \in \Sigma_2^n} \frac{2}{|c'|} p(y|c') + \frac{4}{|c|} p(y|c)$$

$$> \sum_{c' \in \Sigma_2^n} \frac{2}{|c'|} p(y|c') = f_y(D_{Lazy}(y)) = f_y(D_{ML^*}^{n-2}(y)).$$

These inequalities state that $D_{Lazy}$ is the decoder that minimizes $f_y(D(y))$ for any $y \in \Sigma_2^{n-2}$ among all decoders that return words of length $n-2$. Hence, we deduce that the ML* decoder of length $n-2$ is $D_{Lazy}$.

For the rest of this section we use the following observation, given two decoder $D_1$ and $D_2$,

$$f_y(D_1(y)) - f_y(D_2(y))$$

$$= \sum_{c \in D_1(y) \neq c} \frac{d_L(D_1(y), c)}{|c|} p(y|c) - \sum_{c \in D_2(y) \neq c} \frac{d_L(D_2(y), c)}{|c|} p(y|c)$$

$$= \frac{1}{|c|} \left( \sum_{c \in \Sigma_2^n} d_L(D_1(y), c) p(y|c) - \sum_{c \in \Sigma_2^n} d_L(D_2(y), c) p(y|c) \right)$$

$$= \frac{1}{|c|} \sum_{c \in \Sigma_2^n} p(y|c) \left( d_L(D_1(y), c) - d_L(D_2(y), c) \right)$$

$$= \frac{1}{|c|} \sum_{c \in \Sigma_2^n} \text{Emb}(c; y) \left( d_L(D_1(y), c) - d_L(D_2(y), c) \right)$$

$$= \frac{1}{|c|} \sum_{c \in \Sigma_2^n} \text{Emb}(c; y) \left( d_L(D_1(y), c) - d_L(D_2(y), c) \right),$$

where the last equality holds since for any $c \in \Sigma_2^n$ such that $c \notin I_2(y)$ it holds that $\text{Emb}(c; y) = 0$. Hence when comparing the expected normalized distance of two decoders $D_1$ and $D_2$, it holds that,

$$f_y(D_1(y)) \geq f_y(D_2(y))$$

if and only if
\[ \sum_{c \in I_2(y)} \text{Emb}(c; y) \left( d_L(D_1(y), c) - d_L(D_2(y), c) \right) \geq 0. \] 

(4)

**Lemma 28.** It holds that 
\[ D_{\text{ML}}^{n-1} = D_{\text{EN}}^{n-1}. \]

**Proof:** By similar arguments to those presented in Lemma 24 for any channel output \( y \), \( D_{\text{EN}}^{n-1}(y) \) is obtained from \( y \) by prolonging the first longest run of \( y \) by one symbol. Let \( y \) be the channel output and let \( D \) be a decoder such that \( |D| = n - 1 \). Our goal is to prove that the inequality stated in (4) holds when \( D_1 = D \) and \( D_2 = D_{\text{EN}}^{n-1} \). This completes the lemma’s proof. The latter will be verified in the following claims.

**Claim 29.** For any decoder \( D \) such that \( D(y) \neq D_{\text{EN}}^{n-1}(y) \) and \( |D(y)| = n - 1 \), where \( D(y) \) is obtained from \( y \) by prolonging one of the runs in \( y \), the inequality stated in (4) holds.

**Proof:** Assume that the number of runs in \( y \) is \( \rho(y) = r \), let \( r_j \) denote the length of the \( j \)-th run for \( 1 \leq j \leq r \), and let the \( i \)-th run of \( y \) be the first longest run of \( y \). Assume that \( D(y) \) is obtained by prolonging the \( j \)-th run of \( y \) by one symbol. Since \( D(y) \neq D_{\text{EN}}^{n-1}(y) \) it holds that \( j \neq i \). Note that 
\[ |I_1(D(y)) \cap I_1(D_{\text{EN}}^{n-1}(y))| = 1 \]

since the only word \( c \) in this set is the word that is obtained by prolonging the \( i \)-th and \( j \)-th runs of \( y \). It holds that 
\[ d_L(D(y), c) = d_L(D_{\text{EN}}^{n-1}(y), c) = 1 \]

and hence this word can be eliminated from inequality (4). Similarly for words \( c \) such that \( c \notin I_1(D(y)) \) and \( c \notin I_1(D_{\text{EN}}^{n-1}(y)) \), we get that 
\[ d_L(D(y), c) = d_L(D_{\text{EN}}^{n-1}(y), c) = 3 \]

and therefore these words can also be eliminated from inequality (4). Note the number of such words is 
\[
|I_2(y)| - |I_1(D_{\text{EN}}^{n-1}(y))| - |I_1(D(y))| + |I_1(D(y)) \cap I_1(D_{\text{EN}}^{n-1}(y))| = \left( \begin{array}{c} n \\ 2 \end{array} \right) + n + 1 - 2(n + 1) + 1 = \left( \begin{array}{c} n \\ 2 \end{array} \right) - n.
\]

Let us consider the remaining \( 2n \) words in \( I_2(y) \).

1) \( c \in I_1(D_{\text{EN}}^{n-1}(y)) \) and \( c \notin I_1(D(y)) \): Since the embedding number decoder prolongs a run in \( y \), \( I_1(D_{\text{EN}}^{n-1}(y)) \subseteq I_2(y) \). Therefore, there are \( n + 1 - 1 = n \) such words and for each one of them, 
\[ d_L(D(y), c) = 3 \] and \( d_L(D_{\text{EN}}^{n-1}(y), c) = 1 \).

We consider three possible options for the word \( c \) in this case. If \( c \) is the word obtained by prolonging the \( i \)-th run of \( y \) by two symbols, then \( \text{Emb}(c; y) = (r_i + 2) \). Let \( c = c_h \) be the word obtained by prolonging the \( i \)-th and the \( h \)-th run for \( h \neq i, j \). Since there are \( t - 2 \) runs other than the \( i \)-th and the \( j \)-th run, the number of such words is \( t - 2 \), while \( \text{Emb}(c_h; y) = (r_i + 1)(r_h + 1) \). Lastly, if \( c \) is obtained by prolonging the \( i \)-th run and creating a new run in \( y \) then \( \text{Emb}(c; y) = r_i + 1 \), and the number of such words is \( n - t + 1 \). Thus, 
\[
\sum_{c \in I_1(D_{\text{EN}}^{n-1}(y)) \atop c \notin I_1(D(y))} \text{Emb}(c; y) \left( d_L(D(y), c) - d_L(D_{\text{EN}}^{n-1}(y), c) \right)
\]
\[= 2 \left( \begin{array}{c} r_i + 2 \\ 2 \end{array} \right) + \sum_{h=1}^{r} (r_h + 1)(r_i + 1) + (n - r + 1)(r_i + 1) + \left( \begin{array}{c} r_i + 2 \\ 2 \end{array} \right) + (r_i + 1)(n - r - r_i + r - 2) + (n - r + 1)(r_i + 1) + \left( \begin{array}{c} r_i + 2 \\ 2 \end{array} \right) + (r_i + 1)(n - r - r_i + r - 4) + (n - r + 1)(r_i + 1).
\]
2) \( c \notin I_1(D_{EN}^{n-1}(y)) \) and \( c \in I_1(D(y)) \): The decoder \( D \) prolongs a run in \( y \), and therefore \( I_1(D(y)) \subseteq I_2(y) \). Similarly to Case 1, there are \( n \) such words, and

\[
\sum_{c \in I_1(D_{EN}^{n-1}(y))} \text{Emb}(c; y) \left( d_L(D(y), c) - d_L(D_{EN}^{n-1}(y), c) \right) \\
= 2 \left( \binom{r_j + 2}{2} + \sum_{h=1}^{r} \binom{h+1}{2} (r_{h+1} + (n-r-1)(r_{h+1}) \right) \\
= 2 \left( \binom{r_j + 2}{2} + (r_j + 1)(n-r-j+r-2) + (n-r-1)(r_{j+1}) \right) \\
= 2 \left( \binom{r_j + 2}{2} + (r_j + 1)(n-r-j+r-4) + (n-r-1)(r_{j+1}) \right).
\]

Thus,

\[
\sum_{c \in I_2(y)} \text{Emb}(c; y) \left( d_L(D_1(y), c) - d_L(D_2(y), c) \right) \\
= 2 \left( \binom{r_i + 2}{2} + (r_i + 1)(n-r-i+r-4) + (n-r-1)(r_{i+1}) \right) \\
- 2 \left( \binom{r_j + 2}{2} + (r_j + 1)(n-r-j+r-4) + (n-r-1)(r_{j+1}) \right) \\
\geq 0,
\]

where the last inequality holds since \( r_i \geq r_j \).

---

Claim 30. For any decoder \( D \) such that \( D(y) \neq D_{EN}^{n-1}(y) \) and \( |D(y)| = n - 1 \), where \( D(y) \) is obtained from \( y \) by creating a new run of one symbol in \( y \), the inequality stated in (4) holds.

**Proof:** Assume that the number of runs in \( y \) is \( \rho(y) = r \), let \( r_j \) denote the length of the \( j \)-th run for \( 1 \leq j \leq r \), and let the \( i \)-th run of \( y \) be the first longest run of \( y \). As in Claim 29 if \( c \in \left( I_1(D(y)) \cap I_1(D_{EN}^{n-1}(y)) \right) \), then \( c \) can be eliminated from (4). Similarly, any word \( c \) such that \( c \notin I_1(D(y)) \) and \( c \notin I_1(D_{EN}^{n-1}(y)) \) can be eliminated from (4). Let us consider the remaining \( 2n \) words in \( I_2(y) \):

1) \( c \in I_1(D_{EN}^{n-1}(y)) \) and \( c \notin I_1(D(y)) \): From arguments similar to those presented in Claim 29, there are \( n \) such words and

\[
\sum_{c \in I_1(D_{EN}^{n-1}(y))} \text{Emb}(c; y) \left( d_L(D(y), c) - d_L(D_{EN}^{n-1}(y), c) \right) \\
= 2 \left( \binom{r_i + 2}{2} + (r_i + 1)(n-r-i+r-3) + (n-r-1)(r_{i+1}) \right).
\]

2) \( c \notin I_1(D_{EN}^{n-1}(y)) \) and \( c \in I_1(D(y)) \): As in Claim 29, the number of such words is \( n \), and for each of these words,

\[ d_L(D(y), c) = 1 \] and \( d_L(D_{EN}^{n-1}(y), c) = 3 \).

We consider three possible options for the word \( c \) in this case. If \( c \) is the word obtained by prolonging the new run of \( D(y) \) by additional symbol then \( \text{Emb}(c; y) = 1 \). Let \( c = c_h \) be the word obtained by prolonging the \( h \)-th run of \( y \) for \( h \neq i \) and creating the same new run of one symbol as in \( D(y) \). Since there are \( r-1 \) runs other than the \( i \)-th run, the number of such words is \( r - 1 \), while \( \text{Emb}(c_h, y) = (r_h + 1) \). Lastly, if \( c \) is obtained by creating additional new run in
Claim 31. For any decoder \( \mathcal{D} \) such that \( \mathcal{D}(y) \neq \mathcal{D}_{EN}^{n-1}(y) \) and \( |\mathcal{D}(y)| = n-1 \), where \( \mathcal{D}(y) \) is not a supersequence of \( y \), the inequality stated in (4) holds.

Proof: By definition \( \mathcal{D}(y) \) is not a supersequence of \( y \) which implies that \( y \notin I_1(\mathcal{D}(y)) \). Note that for any word \( c \in I_2(y) \) such that \( c \notin I_1(\mathcal{D}(y)) \), it holds that \( d_L(\mathcal{D}(y), c) \geq 3 \), while \( d_L(\mathcal{D}_{EN}^{n-1}(y), c) \leq 3 \). Hence, if \( I_2(y) \cap I_1(\mathcal{D}(y)) = \emptyset \) then,

\[
\sum_{c \in I_2(y)} \operatorname{Emb}(c; y) \left( d_L(D_1(y), c) - d_L(D_2(y), c) \right) \geq 3 - 3 = 0.
\]

Otherwise, let \( c \) be a word such that \( c \in \left( I_2(y) \cap I_1(\mathcal{D}(y)) \right) \), let \( \rho(c) = r' \) be the number of runs in \( c \) and denote by \( r_j' \) the length of the \( j \)-th run in \( c \). Let the \( i \)-th run in \( c \) be the first longest run in \( c \). Note that \( y \in D_2(c) \) and \( D(y) \in D_1(c) \). Consider the following distinct cases.

1) There exists an index \( 1 \leq j \leq r' \) such that \( y \) is obtained from \( c \) by deleting two symbols from the \( j \)-th run of \( c \). In this case, since \( D(y) \) is not a supersequence of \( y \), \( D(y) \) must be obtained from \( c \) by deleting one symbol from the \( h \)-th run of \( c \) for some \( h \neq j \). Hence, \( c \) is the unique word that is obtained by inserting to \( y \) the two symbols that were deleted from the \( j \)-th run of \( c \), that is,

\[
I_2(y) \cap I_1(\mathcal{D}(y)) = \{ c \}.
\]

Note that, \( \operatorname{Emb}(c; y) = \binom{r_j'}{2} \leq \binom{r_j}{2} \) and \( d_L(D(y), c) = 1 \), while \( d_L(D_{EN}^{n-1}(y), c) \in \{1, 3\} \). If \( d_L(D_{EN}^{n-1}(y), c) = 1 \), (4) holds (since \( c \) is the only word in the intersection). Otherwise \( d_L(D_{EN}^{n-1}(y), c) = 3 \) and our goal is to find \( c' \in I_2(y) \) such that

\[
\sum_{w \in I_2(y)} \operatorname{Emb}(w; y) \left( d_L(D(y), w) - d_L(D_{EN}^{n-1}(y), w) \right) = \sum_{w \in I_2(y)} \operatorname{Emb}(w; y) \left( d_L(D(y), w) - d_L(D_{EN}^{n-1}(y), w) \right) + \operatorname{Emb}(c; y) \left( d_L(D(y), c) - d_L(D_{EN}^{n-1}(y), c) \right) + \operatorname{Emb}(c'; y) \left( d_L(D(y), c') - d_L(D_{EN}^{n-1}(y), c') \right) \geq 0.
\]
Since $d_L(D(y), w) - d_L(D_{EN}^{n-1}(y), w) \geq 0$ for every $w \neq c$, it is enough to find $c' \in I_2(y)$ such that,

$$\text{Emb}(c; y)\left(d_L(D(y), c) - d_L(D_{EN}^{n-1}(y), c)\right) + \text{Emb}(c'; y)\left(d_L(D(y), c') - d_L(D_{EN}^{n-1}(y), c')\right) \geq 0.$$ 

Recall that the embedding number decoder prolongs the first longest run in $y$. If the first longest run in $c$, which is the $i$-th run, satisfies $i \neq j$, this run is also the first longest run in $y$. In this case, let $c'$ be the word obtained from $y$ by prolonging this run by two symbols. It holds that, $d_L(D_{EN}^{n-1}(y), c') = 1$, $d_L(D(y), c') = 5$, and $\text{Emb}(c'; y) = \binom{r_{\ell} + 2}{2}$. Recall that $r_j^\prime \geq r_j^\ast$ and hence,

$$-2\left(\frac{r_j^\prime}{2}\right) + 4\left(\frac{r_j^\ast + 2}{2}\right) \geq 0.$$ 

Else, if the first longest run in $c$ is the $j$-th run (i.e., $i = j$) and all the other runs in $c$ are strictly shorter in more than two symbols from the $j$-th run. Then, the $j$-th run is also the first longest run in $y$. In this case $D(y) = D_{EN}^{n-1}(y)$ which is a contradiction to the definition of $D(y)$. Otherwise, the longest run in $c$ is the $j$-th run and there exists $s < j$ such that $r_s^\prime + 2 \geq r_j^\prime$, which implies that the $s$-th run is the first longest run in $y$. By Lemma 25, $D_{EN}^{n-1}$ prolongs the $s$-th run of $y$ by one symbol. Let $c'$ be the word that is obtained from $y$ by prolonging the $s$-th run by two symbols, it holds that $d_L(D_{EN}^{n-1}(y), c') = 1$, $d_L(D(y), c') = 5$ and

$$\text{Emb}(c'; y) = \binom{r_{s}^\prime + 2}{2} \geq \binom{r_{j}^\prime}{2} = \text{Emb}(c; y).$$

Which implies that,

$$-2\left(\frac{r_{j}^\prime}{2}\right) + 4\left(\frac{r_{s}^\prime + 2}{2}\right) \geq 0. $$

2) There exist $1 \leq j < j' \leq r'$ such that $y$ is obtained from $c$ by deleting one symbol from the $j$-th run and one symbol from the $j'$-th run. Similarly to the previous case, $D(y)$ must be obtained from $c$ by deleting one symbol from the $h$-th run for some $h \neq j, j'$. Hence, $c$ is the unique word that is obtained from $y$ by inserting one symbol to the $j$-th run, and one symbol to the $j'$-th run, that is,

$$I_2(y) \cap I_1(D(y)) = \{c\}.$$ 

Note that $\text{Emb}(c; y) = r_j^\prime r_{j'}^\prime$ and that $d_L(D(y), c) = 1$ and $d_L(D_{EN}^{n-1}(y), c) \in \{1, 3\}$. Similarly to the previous case we can assume that $d_L(D_{EN}^{n-1}(y), c) = 3$ and our goal is to find a word $c' \in I_2(y)$ such that,

$$\text{Emb}(c; y)\left(d_L(D(y), c) - d_L(D_{EN}^{n-1}(y), c)\right) + \text{Emb}(c'; y)\left(d_L(D(y), c') - d_L(D_{EN}^{n-1}(y), c')\right) \geq 0.$$ 

As in the previous case, if the $i$-th run, which is the first longest run in $c$ satisfies $i \neq j, j'$, the same run is also the first longest run in $y$. Let $c'$ be the word that is obtained from $y$ by prolonging this longest run by two symbols. It holds that $d_L(D_{EN}^{n-1}(y), c') = 1$, $d_L(D(y), c') = 5$ and $\text{Emb}(c'; y) = \binom{r_{i}^\ast + 2}{2}$, and since, $r_i^\prime \geq r_j^\ast r_{j'}^\ast$,

$$-2r_j^\prime r_{j'}^\prime + 4\binom{r_i^\ast + 2}{2} \geq 0.$$ 

Else, if the first longest run in $c$ is the $j$-th run, or the $j'$-th run (i.e., $i \in \{j, j'\}$), and the same run is also the first longest run in $y$. Then, similarly to the previous case $D(y) = D_{EN}^{n-1}(y)$ which contradicts the definition of $D(y)$. Otherwise, $i \in \{j, j'\}$, and there exists $s \neq j, j'$ such that $r_s^\prime + 1 \geq r_j^\ast r_{j'}^\prime$. In other words this run is the first longest run in $y$. By Lemma 25, $D_{EN}^{n-1}$ prolongs this run by one symbol. Assume w.l.o.g. that $r_j^\prime \geq r_{j'}^\prime$ and let $c'$ be the word obtained from $c$ by deleting one symbol from the $j$-th run and prolonging the $s$-th run by one symbol. In this case $d_L(D_{EN}^{n-1}(y), c') = 1$, $d_L(D(y), c') = 3$ and

$$\text{Emb}(c'; y) = r_j^\prime (r_s^\prime + 1) \geq r_j^\prime r_{j'}^\prime = \text{Emb}(c; y).$$

Therefore,

$$-2r_j^\prime r_{j'}^\prime + 2r_j^\prime (r_s^\prime + 1) \geq 0.$$

\[\square\]

**Lemma 32.** It holds that

$$D_{ML}^n = D_{EN}^n.$$


Proof: For any channel output \( y \in \Sigma_2^{n-2} \) and for any decoder \( D \), such that \( |D(y)| = n \), we have the following sequence of equalities and inequalities,

\[
f_y(D(y)) = \sum_{c \in \Sigma_2^n} \frac{d_L(D(y), c)}{|c|} p(y|c)
\]

\[
= \sum_{c \in I_2(y)} \frac{d_L(D(y), c)}{|c|} p(y|c)
\]

\[
\geq \frac{2}{n} \sum_{c \in I_2(y)} p(y|c) - \frac{2}{n} p(y|D(y))
\]

\[
= \frac{2}{n} \sum_{c \in I_2(y)} p(y|c) - \frac{2}{n} \text{Emb}(D(y); y)
\]

\[
\geq \frac{2}{n} \sum_{c \in I_2(y)} p(y|c) - \frac{2}{n} \max_{c \in \Sigma_2^n} \left\{ \frac{\text{Emb}(c; y)}{(n/2)} \right\}
\]

\[
= \frac{2}{n} \sum_{c \in I_2(y)} p(y|c) - \frac{2}{n} \text{Emb}(D_{EN}^n(y); y)
\]

\[
= f_y(D_{EN}^n(y)),
\]

where (a) holds since for every \( c \) such that \( D(y) \neq c \) it holds that \( d_L(D(y), c) \geq 2 \), and \( d_L(D(y), D(y)) = 0 \), (b) is obtained by the definition of the 2-deletion channel, and in (c) we simply choose the word that maximizes the value of \( \text{Emb}(c; y) \), which is the definition of the embedding number decoder of length \( n \) as derived in step (d). This verifies the lemma’s statement. ■

Lemma 33. Let \( y \in \Sigma_2^{n-2} \) be a channel output. It holds that

\[|D_{ML^*}(y)| \neq n.\]

Proof: Assume to the contrary that \( |D_{ML^*}(y)| = n \). We show that

\[f_y(D_{ML^*}(y)) \geq f_y(D_{Lazy}(y)),\]

which is a contradiction to the definition of the ML* decoder (since the ML* decoder is defined to return the shortest word that minimizes \( f_y(\cdot) \)). By Lemma 32, \( D_{EN}^n(y) \) is the decoder that minimizes \( f_y(D(y)) \) among all other decoders that return a word of length \( n \) for the channel output \( y \). Hence, it is enough to show that \( f_y(D_{EN}^n(y)) - f_y(D_{Lazy}(y)) \geq 0 \).

Note that since \( D_{EN}^n \) returns a word of length \( n \) that is a supersequence of \( y \) and therefore any possible output of \( D_{EN}^n \) is either of distance 0, 2, or 4 from the transmitted word \( c \). Hence,

\[
f_y(D_{EN}^n(y)) - f_y(D_{Lazy}(y))
\]

\[
= \sum_{c \in I_2(y)} \frac{p(y|c)}{|c|} (d_L(D_{EN}^n(y), c) - d_L(D_{Lazy}(y), c))
\]

\[
= \frac{2}{n} \left( \sum_{c \in I_2(y)} p(y|c) - \sum_{c \in I_2(y)} p(y|c) \right) + \frac{2}{n} \sum_{c \in I_2(y)} \frac{p(y|c)}{|c|} (2 - 2) + \frac{2}{n} \sum_{c \in I_2(y)} \frac{p(y|c)}{|c|} (0 - 2)
\]

where (a) holds since \( d_L(D_{Lazy}(y), c) = 2 \) for every \( c \in I_2(y) \) and (b) holds since \( |c| = n \).

Denote,

\[S_{sum4} \equiv \sum_{c \in I_2(y)} \frac{p(y|c)}{|c|} \]

\[P_0 \equiv \sum_{c \in I_2(y)} \frac{p(y|c)}{|c|} \]

\[P_0 = p(y|D_{EN}^n(y)) \]
From the above discussion, our objective is to prove that $\text{Sum}_4 \geq P_0$. Recall that $|I_2(y)| = \binom{n}{2} + n + 1$. Let the $i$-th, $i'$-th run be the first, second longest run of $y$, respectively, and denote their lengths by $r_i \geq r_{i'}$. We will bound the number of possible words $c \in I_2(y)$ such that $d_L(D_{EN}^n(y), c) = 4$.

**Case 1:** $D_{EN}^n$ prolongs the $i$-th run by two symbols. There is one word $c \in I_2(y)$ such that $d_L(D_{EN}^n(y), c) = 0$. Note that the set of words $c \in I_2(y)$ such that $d_L(D_{EN}^n(y), c) = 2$ consists of words $c$ that can be obtained from $y$ by prolonging the $i$-th run by exactly one symbol. Consider the word $y'$, which is the word obtained from $y$ by prolonging the $i$-th run by exactly one symbol. $y'$ is a word of length $n - 1$, and the words $c$, such that $d_L(D_{EN}^n(y), c) = 2$ are all the words in the radius-1 insertion ball centered at $y'$ expect to the word $D_{EN}^n(y)$. The number of such words is

$$I_1(y') - 1 = n + 1 - 1 = n.$$ 

Hence, there are $\binom{n}{2}$ words $c \in I_2(y)$ for which $d_L(D_{EN}^n(y), c) = 4$ and the conditional probability of each of these words is $p(y|c) \geq \frac{1}{\binom{n}{2}}$. Therefore,

$$\text{Sum}_4 = \sum_{c \in I_2(y), d_L(D_{EN}^n(y), c) = 4} p(y|c) \geq \frac{n}{\binom{n}{2}} \cdot \frac{1}{\binom{n}{2}} = 1.$$ 

On the other hand,

$$P_0 = \frac{(r_i + 2)(r_{i'} + 1)}{\binom{n}{2}} \leq 1,$$

which implies $\text{Sum}_4 \geq P_0$ for every $n > 0$ and thus,

$$f_y(D_{EN}^n(y)) - f_y(D_{Lazy}(y)) \geq 0.$$ 

**Case 2:** $D_{EN}^n(y)$ prolongs both the $i$-th run and the $i'$-th run, each by one symbol. By Lemma 25 we know that $D_{EN}^n(y)$ prolongs these two runs if and only if $\binom{r_i}{2} \leq r_{i'}/r_i$, and consequently $\frac{r_i - 1}{2} \leq r_{i'} < r_i$.

The only word $c \in I_2(y)$ that satisfies $d_L(D_{EN}^n(y), c) = 0$ is the word $c = D_{EN}^n(y)$. In addition the set of words $c \in I_2(y)$ such that $d_L(D_{EN}^n(y), c) = 2$ consists of words $c$ that can be obtained from $y$ by prolonging either the $i$-th run or the $i'$-run by exactly one symbol. Let $y'$ be the word obtained from $y$ by prolonging the $i$-th run by one symbol and let $y''$ be the word obtained from $y$ by prolonging the $i'$-run by one symbol. Similarly to the first case the number of such words is

$$I_1(y') - 1 + I_1(y'') - 1 = 2n,$$

which implies that the number of words $c \in I_2(y)$ such that $d_L(D_{EN}^n(y), c) = 4$ is $\binom{n}{2} - n$ and the conditional probabilities of these words satisfy $p(y|c) \geq \frac{1}{\binom{n}{2}}$. Hence,

$$\text{Sum}_4 = \sum_{c \in I_2(y), d_L(D_{EN}^n(y), c) = 4} p(y|c) \geq \frac{\binom{n}{2} - n}{\binom{n}{2}}.$$ 

On the other hand,

$$P_0 = \frac{(r_i + 1)(r_{i'} + 1)}{\binom{n}{2}} \leq \frac{(r_i + 1)(n - r_i - 1)}{\binom{n}{2}} \leq \frac{(\frac{n}{2} - 1)^2}{\binom{n}{2}} = \frac{n^2 - n + 1}{\binom{n}{2}},$$

where (a) holds since $r_i + r_{i'} \leq n - 2$ and (b) holds since the maximum of the function $f(x) = x(n - x)$ is achieved for $x = n/2$. Hence, $\text{Sum}_4 \geq P_0$ when $\frac{n^2}{4} - n + 1 \leq \binom{n}{2} - n$, which holds for any $n \geq 4$. Thus, for $n \geq 4$,

$$f_y(D_{EN}^n(y)) - f_y(D_{Lazy}(y)) \geq 0.$$ 

**Lemma 34.** Let $y \in \Sigma_2^{n-2}$ be a channel output. For any decoder $D$, such that $D(y)$ is not a supersequence of $y$ and $|D(y)| = n + 1$, it holds that

$$f_y(D(y)) \geq f_y(D_{EN}^{n+1}(y)).$$

**Proof:** Since $D(y)$ is not a supersequence of $y$, it is also not a supersequence of the transmitted word $c$. Therefore, for each $c \in I_2(y)$ it holds that $d_L(D(y), c) \geq 3$, while $d_L(D_{EN}^{n+1}(y), c) \leq 3$. Thus,
Lemma 35. Let \( y \in \Sigma^{n-2} \) be a channel output. For any decoder \( D \), such that \( D(y) \) is a supersequence of \( y \) and \( |D(y)| = n + 1 \), it holds that

\[
\text{f}_y(D(y)) \geq \text{f}_y(D_{EN}^{n-1}(y)).
\]

Proof: From similar arguments to those presented in Lemma 28, our goal is to prove that (4) holds for \( D(y) \) and \( D_{EN}^{n-1}(y) \), i.e., to prove that

\[
\sum_{c \in I_2(y)} \text{Emb}(c; y) \left( d_L(D(y), c) - d_L(D_{EN}^{n-1}(y), c) \right) \geq 0.
\]

Assume that the number of runs in \( y \) is \( \rho(y) = r \), let \( r_j \) denote the length of the \( j \)-th run for \( 1 \leq j \leq r \), and let the \( i \)-th run of \( y \) be the first longest run of \( y \). Note that the Levenshtein distance of \( D_{EN}^{n-1}(y) \) from the transmitted word \( c \) can be either 1 or 3. Similarly, \( D(y) \) can have distance of 1, 3 or 5 from \( c \). Recall that \( D_{EN}^{n-1} \) prolongs the \( i \)-th run by one symbol and that \( I_1(D_{EN}^{n-1}(y)) \subseteq I_2(y) \). \( D(y) \) is a supersequence of \( y \), and hence \( D(y) \) is obtained from \( y \) by prolonging existing runs or by creating new runs in \( y \). From the discussion above, for every word \( c \in I_2(y) \) such that

\[
c \notin I_1(D_{EN}^{n-1}(y)) \cup D_1(D(y)),
\]

it holds that \( d_L(D_{EN}^{n-1}(y), c) = 3 \) while \( d_L(D(y), c) \geq 3 \). Additionally, every word \( c \in I_2(y) \) such that

\[
c \in I_1(D_{EN}^{n-1}(y)) \cap D_1(D(y)),
\]

satisfies \( d_L(D_{EN}^{n-1}(y), c) = d_L(D(y), c) = 1 \). Hence, for these words it holds that \( d_L(D(y), c) - d_L(D_{EN}^{n-1}(y), c) \geq 0 \) and they can be eliminated from inequality (4). In order to complete the proof, the words \( c \in I_2(y) \) such that

\[
c \notin I_1(D_{EN}^{n-1}(y)) \text{ and } c \notin D_1(D(y))
\]

and the words \( c \in I_2(y) \) such that

\[
c \notin I_1(D_{EN}^{n-1}(y)) \text{ and } c \in D_1(D(y))
\]

should be considered. For words in the first case it holds that \( d_L(D_{EN}^{n-1}(y), c) = 1 \) and \( d_L(D(y), c) \geq 3 \), while for words in the latter case, \( d_L(D_{EN}^{n-1}(y), c) = 3 \) and \( d_L(D(y), c) \geq 1 \). Hence,

\[
\sum_{c \in I_2(y)} \text{Emb}(c; y) \left( d_L(D(y), c) - d_L(D_{EN}^{n-1}(y), c) \right)
\]

\[
\geq \sum_{c \in I_2(y)} \sum_{c \in I_1(D_{EN}^{n-1}(y)) \cap D_1(D(y))} \text{Emb}(c; y) \left( d_L(D(y), c) - d_L(D_{EN}^{n-1}(y), c) \right)
\]

\[
+ \sum_{c \in I_2(y)} \sum_{c \in I_1(D_{EN}^{n-1}(y)) \cap D_1(D(y))} \text{Emb}(c; y) \left( d_L(D(y), c) - d_L(D_{EN}^{n-1}(y), c) \right)
\]

\[
\geq 2 \sum_{c \in I_2(y)} \text{Emb}(c; y) - 2 \sum_{c \in I_2(y)} \text{Emb}(c; y).
\]
We first assume that $D(y)$ is obtained from $y$ by prolonging the $i$-th run by exactly one symbol. Let $c \in I_2(y)$ and consider the cases mentioned above.

1) $c \in I_1(D_{EN}^{n-1}(y))$ and $c \notin D_1(D(y))$: Recall that both decoders return supersequences of $y$. By the assumption $D(y)$ is obtained from $y$ by prolonging the $i$-th run by one symbol and then performing two more insertions to the obtained word. Since $c \in I_1(D_{EN}^{n-1}(y))$, $c$ must be obtained from $y$ by prolonging the $i$-th run and performing one more insertion. $c \notin D_1(D(y))$, and therefore the number of such words equals to

$$\|I_1(D_{EN}^{n-1}(y))\| - \left\{ c \in I_2(y) : c \in I_1(D_{EN}^{n-1}(y)) \cap D_1(D(y)) \right\}.$$

Note that

$$\left\{ c \in I_2(y) : c \in I_1(D_{EN}^{n-1}(y)) \cap D_1(D(y)) \right\} \leq 2$$

since the words in the latter intersection are the words that obtain from $y$ by prolonging the $i$-th run by one symbol and then performing one of the two other insertions performed to receive $D(y)$. Hence, there are at least $|I_1(D_{EN}^{n-1}(y))| - 2 = n - 1$ such words in this case and for each of them $\text{Emb}(c; y) \geq (r_i + 1)$. Recall that these words satisfy $d(D_{EN}^{n-1}(y), c) = 1$ and $d(D(y), c) \geq 3$.

2) $c \notin I_1(D_{EN}^{n-1}(y))$ and $c \in D_1(D(y))$: By the assumption, $D$ prolongs the $i$-th run by one symbol and performs two more insertions into the obtained word and $D_{EN}^{n-1}$ prolongs the $i$-th run by one symbol. Hence, the words $c \in I_2(y)$ such that $c \notin I_1(D_{EN}^{n-1}(y))$ and $c \in D_1(D(y))$ can not be obtained from $y$ by prolonging the $i$-th run. Therefore, it implies that $c$ is the unique word obtained from $D(y)$ by deleting the symbol that was inserted to the $i$-th run of $y$. It holds that $\text{Emb}(c; y) \leq (r_i + 1)^2$ and $d_1(D_{EN}^{n-1}(y), c) = 3$ and $d_1(D(y), c) = 1$.

Note that $r_i \leq n - 2$ since it is the length of the $i$-th run of $y \in \Sigma^*_2$. Thus,

$$2 \sum_{c \in I_2(y)} \text{Emb}(c; y) - 2 \sum_{c \in I_2(y)} \text{Emb}(c; y) \leq 2(n - 1)(r_i + 1) - 2(r_i + 1)^2 \geq 2(r_i + 1)^2 - 2(r_i + 1)^2 \geq 0.$$

Second we assume that $D(y)$ is obtained from $y$ by prolonging the $i$-th run by at least two symbols. In this case, it holds that $(D_1(D(y)) \cap I_2(y)) \subseteq I_1(D_{EN}^{n-1}(y))$, which implies that

$$\left\{ c \in I_2(y) : c \notin I_1(D_{EN}^{n-1}(y)) \text{ and } c \in D_1(D(y)) \right\} = 0,$$

and therefore,

$$2 \sum_{c \in I_2(y)} \text{Emb}(c; y) - 2 \sum_{c \in I_2(y)} \text{Emb}(c; y) \geq 0.$$

Lastly, we assume that $D(y)$ is obtained from $y$ by three insertions such that neither of these insertions prolongs the $i$-th run. It holds that

$$\left\{ c \in I_2(y) : c \in I_1(D_{EN}^{n-1}(y)) \cap D_1(D(y)) \right\} = 0.$$

Therefore the number of words $c \in I_2(y)$ such that $c \in I_1(D_{EN}^{n-1}(y))$ and $c \notin D_1(D(y))$ equals to $|I_1(D_{EN}^{n-1}(y))| = n + 1$. For any such word $c$ it holds that $\text{Emb}(c; y) \geq r_i + 1$. Furthermore, $|D_1(D(y))|$ equals to the number of runs in $D(y)$ and any $c \in D_1(D(y)) \cap I_2(y)$ is obtained from $D(y)$ by deleting one of the three symbols that were inserted into $y$ in order to obtain $D(y)$. Hence, there are at most three such words, and each is obtained by deleting one of the three inserted symbols. Let $c$ be one of those words. If the two remaining symbols belong to the same run, then $\text{Emb}(c; y) = \binom{m}{2}$ where $m$ is the length of this run in $c$ and $m \leq r_i + 2$. In this case consider the word $c'$ that is obtained by prolonging the $i$-th run of $y$ by two symbols. It holds that

$$\text{Emb}(c'; y) = \binom{r_i + 2}{2} \geq \binom{m}{2} = \text{Emb}(c; y).$$

Otherwise, $\text{Emb}(c; y) = m_1m_2$ where $m_1$ and $m_2$ are the lengths of the runs that include the remaining inserted symbols and $m_1, m_2 \leq r_i + 1$. Let $c'$ be the word that is obtained from $y$ by prolonging the $i$-th run and the run of length $\max\{m_1 - 1, m_2 - 1\}$ that is prolonged by $D$. In this case,

$$\text{Emb}(c'; y) = m_1(r_i + 1) \geq m_1m_2 = \text{Emb}(c; y).$$
Note that there is at most one such word $c$ that is obtained by prolonging the same run with two symbols, which implies that there is always a selection of words $c'$ such that,

$$2 \sum_{c \in I_2(y)} \text{Emb}(c; y) - 2 \sum_{c \in I_2(y)} \text{Emb}(c; y) \geq 0.$$ 

We proved that for any decoder $D$ such that $D(y)$ is a supersequence $y$ and $|D(y)| = n + 1$,

$$2 \sum_{c \in I_2(y)} \text{Emb}(c; y) - 2 \sum_{c \in I_2(y)} \text{Emb}(c; y) \geq 0.$$ 

Thus,

$$f_y(D(y)) - f_y(D_{EN}^{n-1}(y)) \geq 0.$$

From the previous lemmas it holds that for a given channel output $y \in \Sigma_2^{n-2}$, the length of $D_{ML^*}(y)$ is either $n - 1$ or $n$. Lemma 28 implies that if $|D_{ML^*}(y)| = n - 1$, then $D_{ML^*}(y) = D_{EN}^{n-1}(y)$. In the following result we define a condition on the length of the longest run in $y$ to decide whether prolonging it by one symbol can minimize the expected normalized distance. In other words, this result defines a criteria on a given channel output $y$ to define whether using the same output as $D_{Lazy}$ or using the same output as $D_{EN}^{n-1}$ is better in terms of minimizing $f_y(D(y))$ (and therefore minimizing the expected normalized distance). An immediate conclusion of this result is Theorem 57 which determines the ML* decoder for the case of a single 2-deletion channel.

**Lemma 36.** Let $y \in \Sigma_2^{n-2}$ be a channel output, such that the number of runs in $y$ is $\rho(y) = r$, and the first longest run in $y$ is the $i$-th run. Denote by $r_j$ the length of the $j$-th for $1 \leq j \leq r$. It holds that

$$f_y(D_{EN}^{n-1}(y)) - f_y(D_{Lazy}(y)) \geq 0$$

if and only if

$$2n^2 - 4nr_i - 6n + r_i^2 + 3r_i + r + 1 \geq 0.$$

**Proof:** By Lemma 24 $D_{EN}^{n-1}$ prolongs the $i$-th run of $y$ by one symbol. Therefore, the Levenshtein distance of $D_{EN}^{n-1}(y)$ from the transmitted word $c$ can be either 1 or 3. Hence,

$$f_y(D_{EN}^{n-1}(y)) - f_y(D_{Lazy}(y))$$

$$= \sum_{c \in I_2(y)} p(y|c) \left( d_L(D_{EN}^{n-1}(y), c) - d_L(D_{Lazy}(y), c) \right)$$

$$= \sum_{c \in I_2(y)} \frac{p(y|c)}{|c|} (3 - 2) + \sum_{c \in I_2(y)} \frac{p(y|c)}{|c|} (1 - 2)$$

$$= \frac{1}{n} \left( \sum_{d_L(D_{EN}^{n-1}(y), c) = 3} p(y|c) - \sum_{d_L(D_{EN}^{n-1}(y), c) = 1} p(y|c) \right).$$

Denote

$$\text{Sum}_3 \triangleq \sum_{d_L(D_{EN}^{n-1}(y), c) = 3} p(y|c),$$

$$\text{Sum}_1 \triangleq \sum_{d_L(D_{EN}^{n-1}(y), c) = 1} p(y|c).$$

Let us prove that

$$2n^2 - 4nr_i - 6n + r_i^2 + 3r_i + r + 1 \geq 0$$

is a necessary and sufficient condition for the inequality $\text{Sum}_3 \geq \text{Sum}_1$ to hold. First, we count the number of words $c \in I_2(y)$ such that $d_L(D_{EN}^{n-1}(y), c) = 1$. Each such $c$ is a supersequence of $D_{EN}^{n-1}(y)$ and therefore $c$ can be obtained from $y$ only by
one of the three following ways. The first way is by prolonging the \( i \)-th run and the \( j \)-th of \( y \) for \( j \neq i \), each by one symbol. The number of such words is \( r - 1 \). The second way is by prolonging the \( i \)-th run in \( y \) by one symbol and creating a new run in \( y \). The number of options to create a new run in \( y \) is \( n - r + 1 \) and therefore, there are \( n - r + 1 \) such words. The third way is by prolonging the \( i \)-th run by two symbols and there is only one such word. Hence, the total number of words \( c \in I_2(y) \) such that \( d_1(D_{EN}^{n-1}(y), c) = 1 \) is \( n + 1 = |I_1(D_{EN}^{n-1}(y))| \). Among them, the \( r - 1 \) words that are obtained by the first way has an embedding number of \( \text{Emb}(c; y) = (r_j + 1)(r_j + 1) \). Similarly, the \( n - r + 1 \) words that are obtained from \( y \) using the second way satisfy \( \text{Emb}(c; y) = r_j + 1 \). Lastly, for the word \( c \) that is obtained by prolonging the \( i \)-th run of \( y \) by two symbols it holds that \( \text{Emb}(c; y) = \binom{r_j + 2}{2} \). Hence,

\[
\text{Sum}_1 = \sum_{c \in I_2(y)} p(y|c) = \binom{r_j + 2}{2} + \sum_{1 \leq j \leq r \atop j \neq i} \binom{r_j + 1}{2} + \sum_{j=1}^{n-r+1} \binom{r_j + 1}{2}
\]

where (a) holds since \( \sum_{j \neq i} r_j = n - 2 - r_i \).

Next, let us evaluate the summation \( \text{Sum}_3 \). Note that if \( d_1(D_{EN}^{n-1}(y), c) = 3 \) then \( c \) is not in a supersequence of \( D_{EN}^{n-1}(y) \), and hence \( c \notin I_1(D_{EN}^{n-1}(y)) \). The words that contribute to the summation \( \text{Sum}_3 \) can be divided into three different types of words \( c \in I_2(y) \).

**Case 1:** Let \( C_1 \subseteq I_2(y) \) be the set of words \( c \in C_1 \), such that \( c \) includes additional run(s) that does not appear in \( y \). Such additional runs can be either one run of length 2, or two runs of length 1 each. The number of words such that the length of the new run is two is \( n - r \). And the number of words with two additional runs is \( \binom{n-r}{2} \). Additionally, for \( c \in C_1 \), \( \text{Emb}(c; y) = 1 \), which implies,

\[
\sum_{c \in C_1} p(y|c) = \sum_{c \in C_1} \frac{1}{\binom{n}{2}} = \frac{1}{\binom{n}{2}} \left( \binom{n-r}{2} + n - r \right) = \frac{2}{n(n-1)} \left( \frac{(n-r)(n-r-1)}{2} + n - r \right) = \frac{(n-r)(n-r+1)}{n(n-1)}.
\]

**Case 2:** Let \( C_2 \subseteq I_2(y) \) be the set of words \( c \in C_2 \), such that \( c \) is obtained from \( y \) by prolonging the \( j \)-th run and by creating a new run in \( y \). Note that the prolonged run cannot be the \( i \)-th run in order to ensure \( d_1(D_{EN}^{n-1}(y), c) = 3 \), i.e., \( j \neq i \). The number of words in \( C_2 \) is \( (r-1)(n-r+1) \), since there are \( r - 1 \) options for the index \( j \), and \( n - r + 1 \) ways to create a new run in the obtained word. For such a word \( c \in C_2 \), it holds that \( \text{Emb}(c; y) = r_j + 1 \) and hence,

\[
\sum_{c \in C_2} p(y|c) = \sum_{1 \leq j \leq r \atop j \neq i} \frac{r_{j} + 1}{\binom{n}{2}} \binom{r_{j} + 1}{2} \sum_{1 \leq j \leq r} \frac{r_{j} + 1}{\binom{n}{2}} = 2 \frac{(n-r+1)(n-r+3)}{n(n-1)}.\]

**Case 3:** Let \( C_3 \subseteq I_2(y) \) be the set of words \( c \in C_3 \), such that \( c \) is obtained from \( y \) by prolonging one or two existing runs in \( y \) (other than the \( i \)-th run). The number of words \( c \in C_3 \) obtained from \( y \) by prolonging a single run by two symbols is \( r - 1 \). If the \( j \)-th run is the prolonged run then \( \text{Emb}(c; y) = \binom{r_j + 2}{2} \). Additionally, there are \( \binom{r_j + 2}{2} \) words in \( C_3 \) that are obtained by prolonging the \( j \)-th and the \( j' \)-th runs of \( y \), each by one symbol. These words satisfy \( \text{Emb}(c; y) = (r_j + 1)(r_{j'} + 1) \).
Therefore,

\[
\sum_{c \in C_3} p(y|c) = \sum_{1 \leq j \leq r} \frac{r(r+1)}{2} + \sum_{1 \leq j < f \leq r} (r_j + 1)(r_j + 1)
\]

\[
= \frac{2}{n(n-1)} \left( \sum_{1 \leq j \leq r} \frac{(r_j + 2)(r_j + 1)}{2} + \frac{1}{2} \sum_{1 \leq j \leq r} \sum_{f \neq j} (r_j + 1)(r_j + 1) - \frac{1}{2} \sum_{1 \leq j \leq r} (r_j + 1)^2 \right)
\]

\[
= \frac{2}{n(n-1)} \left( \frac{1}{2} \sum_{1 \leq j \leq r} (r_j^2 + 3r_j + 2) + \frac{1}{2}(n - r_i + r - 3)^2 - \frac{1}{2} \sum_{1 \leq j \leq r} r_j^2 - \sum_{1 \leq j \leq r} r_j - r - 1 \right)
\]

\[
= \frac{(n - r_i + r - 3)(n - r_i + r - 2)}{n(n-1)}.
\]

Thus,

\[
\text{Sum}_3 = \sum_{c \in C_3} p(y|c) = \frac{d_4(D_{EN}^{n-1}(y), c)}{3} = \sum_{c \in C_1} p(y|c) + \sum_{c \in C_2} p(y|c) + \sum_{c \in C_3} p(y|c)
\]

\[
= \frac{(n - r)(n - r + 1)}{n(n-1)} + \frac{(n - r_i + r - 3)}{n(n-1)} \cdot (3n - r - r_i)
\]

\[
= \frac{1}{n(n-1)} \cdot (4n^2 - 4nr_i - 8n + r_i^2 + 3r_i + 2r).
\]

It holds that \(\text{Sum}_3 - \text{Sum}_1 \geq 0\) if and only if

\[
4n^2 - 4nr_i - 8n + r_i^2 + 3r_i + 2r \geq 4n(r_i + 1) - r_i^2 - 3r_i - 2
\]

\[
2n^2 - 4nr_i - 6n + r_i^2 + 3r_i + r + 1 \geq 0.
\]

(5)

Using this result we can explicitly define the ML* decoder \(D_{ML^*}\). This decoder works as follows. For each word \(y\) it calculates the number of runs \(r\) and the length of the longest run \(r_i\) and then checks if

\[
2n^2 - 4nr_i - 6n + r_i^2 + 3r_i + r + 1 \geq 0.
\]

If this condition holds, the decoder works as the lazy decoder and simply returns the word \(y\). Otherwise, it acts like the embedding number decoder of length \(n - 1\) and prolongs the first longest run by one. This result is summarized in the following theorem.

**Theorem 37.** The ML* decoder \(D_{ML^*}\) for a single 2-deletion channel is a decoder that performs as the lazy decoder if inequality (5) holds and otherwise it acts like the embedding number decoder of length \(n - 1\). i.e.,

\[
D_{ML^*}(y) = \begin{cases} 
D_{Lazy}(y) & \text{inequality (5) holds,} \\
D_{EN}^{n-1}(y) & \text{otherwise.}
\end{cases}
\]

**Proof:** Using the previous lemmas, one can verify that \(D_{ML^*}\) minimizes the expected normalized distance for any possible channel output \(y\) and hence it is the ML* decoder.

The result of Theorem 37 states that if the ML* decoder chooses the same output as the decoder \(D_{EN}^{n-1}\) then inequality (5) does not hold. It can be shown that this implies that \(r_i \geq (2 - \sqrt{2})n\) and thus, by Claim 19 in almost all cases the output of the ML* decoder is the lazy decoder’s output.

**VIII. Conclusion**

In this paper we first studied the problem of estimating the expected normalized distance of two deletion channels or two insertion channels when the code is the entire space, the VT code, and the shifted VT code. We then studied the ML* decoder of the 1-deletion and 2-deletion channels. It should be noted that, we also characterized the ML* decoder for the 1-Ins channel, where exactly 1 symbol is inserted to the transmitted word. When the code is the entire space, the ML* decoder of the 1-Ins channel in almost all of the cases simply returns the channel outputs. In cases where the channel outputs contain an extremely long run (more then half of the word), the ML* decoder shortens it by one symbol. These results were proved by Raïssa.
Nataf and Tomer Tsachor for alphabet of size \( q = 2 \) \(^{[53]}\), and for any \( q \geq 2 \) by Or Steiner and Michael Makhlevich \(^{[79]}\). While the results in the paper provide a significant contribution in the area of codes for insertions and deletions and sequence reconstruction, there are still several interesting problems which are left open. Some of them are summarized as follows:

1) Study the non-identical channels case. For example two deletion channels with different probabilities \( p_1 \) and \( p_2 \).
2) Study the expected normalized distance for more than two channels, both for insertions and deletions.
3) Study channels which introduce insertions, deletions, and substitutions.
4) Design coding schemes as well as complexity-efficient algorithms for the ML decoder in each case.

APPENDIX A

Claim \([19]\) For all \( n \geq 1 \) it holds that \( \tau((\Sigma_2)^n) \leq 2 \log(n) \).

Proof: For \( 1 \leq r \leq n \), let \( N(r) \) denote the number of words in \( \Sigma_n^r \) which the length of their maximal run is \( r \). Note that \( N(r) \leq n^{2n-1} \). This holds since we can set the location of the maximal run to start at some index \( i \), which has less than \( n \) options. There are two options for the bit value in the maximal run, the two bits before and after the run are fixed and have to opposite to the bit value in the run, and the rest of the bits can be arbitrary. Then, it holds that

\[
\tau((\Sigma_2)^n) = \sum_{r=1}^{n} r N(r) \leq \sum_{r=1}^{\ell(n)} r N(r) + \sum_{r=\ell(n)+1}^{n} r N(r)
\]

\[
\leq \frac{\ell(n)}{2^n} N(r) + \frac{n^{2n-1}}{2^n} \ell(n) + \frac{n^{2n-1}}{2^n} \ell(n) + \frac{n^2}{2^{\ell(n)+2}}
\]

Finally, by setting \( \ell(n) = [2 \log(n)] - 2 \) we get that

\[
\tau((\Sigma_2)^n) \leq [2 \log(n)] - 2 + \frac{n^2}{2^{\left[\log(n)\right]}}
\]

\[
\leq [2 \log(n)] - 1 \leq 2 \log(n).
\]

REFERENCES

[1] M. Abroshan, R. Venkataramanan, L. Dolecek, and A. G. i Fàbregas, Coding for deletion channels with multiple traces, International Symposium on Information Theory (ISIT), pp. 1372-1376, 2019.

[2] L. Anavy, I. Vaknin, O. Atar, R. Amit, and Z. Yakhini, Data storage in DNA with fewer synthesis cycles using composite DNA letters, Nature biotechnology, vol. 37, no. 10, pp. 1229-1236, 2019.

[3] A. Apostolico, S. Browne, and C. Guerra, Fast linear-space computations of longest common subsequences, Theoretical Computer Science, vol. 92, no. 1, pp. 3-17, 1992.

[4] A. Atashpazgarad, M. Beunardeau, A. Connolly, R. Géraud, D. Mestel, A. W. Roscoe, and P. Y. A. Ryan, From clustering supersequences to entropy minimizing subsequences for single and double deletions, arXiv:1802.00703, 2018.

[5] D. Bar-Lev, I. Orr, O. Sabary, T. Etzion, and E. Yaakobi, Deep DNA Storage: Scalable and Robust DNA Storage via Coding Theory and Deep Learning, arXiv preprint arXiv:2109.00031, 2021.

[6] D. Bar-Lev, Y. Gershon, O. Sabary and E. Yaakobi, Decoding for optimal expected normalized distance over the t-deletion channel, International Symposium on Information Theory (ISIT), pp. 1847-1852, 2021.

[7] T. Batu, S. Kannan, S. Khanna, and A. McGregor, Reconstructing strings from random traces, ACM-SIAM symposium on Discrete algorithms, pp. 910–918. Society for Industrial and Applied Mathematics, 2004.

[8] M. Blawat, K. Gaedke, I. Hutter, X.-M. Chen, B. Turczyk, S. Inverso, B.W. Pruitt, and G.M. Church, Forward error correction for DNA data storage, International Conference on Computational Science, vol. 80, pp. 1011–1022, 2016.

[9] A. Blum, T. Jiang, M. Li, J. Tromp, and M. Yannakakis, Linear approximation of shortest superstrings, Journal of the ACM, vol. 41, no. 4, pp. 630-647, 1993.

[10] J. Brakensiek, V. Guruswami, and S. Zbarsky, Efficient low-redundancy codes for correcting multiple deletions, Annual ACM-SIAM Symposium on Discrete Algorithms, pp. 1884–1892. Philadelphia, PA, USA, 2016.

[11] J.A. Brifka, V. Buttigieg, and S. Wesemeyer, Time-varying block codes for synchronization errors: MAP decoder and practical issues, The Journal of Engineering, vol. 6, pp. 340-351, 2018.

[12] J. Sima and J. Bruck, Trace Reconstruction with Bounded Edit Distance, IEEE International Symposium on Information Theory (ISIT), pp. 2519–2524, 2021.

[13] B. Bukh, and V. Guruswami and J. Håstad, An improved bound on the fraction of correctable deletions, IEEE Trans. on Infor. Theory, vol. 63, no. 1, pp. 93–103, 2017.

[14] J. Castiglione and A. Kavcic, Trellis based lower bounds on capacities of channels with synchronization errors, Information Theory Workshop, pp. 24–28, Jeju, South Korea, 2015.

[15] M. Cheraghchi, Capacity upper bounds for deletion-type channels, Journal of the ACM, vol. 66, no. 2, p. 9, 2019.

[16] M. Cheraghchi, J. Downs, J. Ribeiro and A. Veliche, Mean-Based Trace Reconstruction over Practically any Replication-Insertion Channel, IEEE International Symposium on Information Theory (ISIT), pp. 2459–2464, 2021.
[17] Y. M. Chee, H. M. Kiah, A. Vardy, V. K. Vu, and E. Yaakobi, Coding for racetrack memories, *IEEE Transactions on Information Theory*, vol. 64, no. 11, pp. 7094–7112, 2018.

[18] Y. Chen, A. Wan, and W. Liu, A fast parallel algorithm for finding the longest common sequence of multiple biosequences, *BMC bioinformatics*, vol. 7, no. 4, pp. 4, 2006.

[19] G. M. Church, Y. Gao, and S. Kosuri, Next-generation digital information storage in DNA, *Science*, vol. 337, no. 6102, pp. 1628–1628, 2012.

[20] R. Con and A. Shpilka, Explicit and efficient constructions of coding schemes for the binary deletion channel and the Poisson repeat channel, *International Symposium on Information Theory (ISIT)*, pp. 84–89, 2020.

[21] M. Dalai, A new bound on the capacity of the binary deletion channel with high deletion probabilities, *International Symposium on Information Theory (ISIT)*, pp. 499–502, St. Petersburg, Russia, 2011.

[22] S. Davies, M. Z. Rácz, B. G. Schiffer and C. Rashtchian, Approximate Trace Reconstruction: Algorithms, *IEEE International Symposium on Information Theory (ISIT)*, pp. 2525–2530, 2021.

[23] V. I. Levenshtein and J. Siemons, Error graphs and the reconstruction of elements in groups, *Journal of Combinatorial Theory, Series A*, vol. 10, no. 8, pp. 707–710, 1966.

[24] S. Diggavi and M. Grossglauser, On information transmission over a finite buffer channel, *IEEE Transactions on Information Theory*, vol. 52, no. 3, pp. 1226–1237, 2006.

[25] L. Doleck and V. Anantharam, Using Reed Muller RM (1, m) codes over channels with synchronization and substitution errors, *IEEE Trans. on Inform. Theory*, vol. 53, no. 4, pp. 1430–1443, 2007.

[26] E. Drinea and M. Mitzenmacher, Improved lower bounds for the capacity of iid deletion and duplication channels, *IEEE Transactions on Information Theory*, vol. 53, no. 8, pp. 2693–2714, 2007.

[27] J. Duda, W. Szpankowski, and A. Gama, Fundamental bounds and approaches to sequence reconstruction from nanopore sequencers, arXiv preprint arXiv:1601.02420, 2016.

[28] C. Elzinga, S. Rahmann, and H. Wang, Algorithms for subsequence combinatorics, *Theoretical Computer Science*, vol. 409, no. 3, pp. 394–404, 2008.

[29] D. Ferronati and T. M. Duman, Novel bounds on the capacity of the binary deletion channel, *IEEE Transactions on Information Theory*, vol. 56, no. 6, pp. 2753–2765, 2010.

[30] R. Gabrys and E. Sala, Codes correcting two deletions, *IEEE Trans. on Inform. Theory*, vol. 65, no. 2, pp. 965–974, 2018.

[31] R. Gabrys and E. Yaakobi, Sequence reconstruction over the deletion channel, *IEEE Transactions on Information Theory*, vol. 64, no. 4, pp.2924–2931, 2018.

[32] B. Graham, A Binary Deletion Channel With a Fixed Number of Deletions, *Combinatorics, Probability and Computing*, vol. 24, no. 3, pp. 486-489, 2018.

[33] R. N. Grass, R. Heckel, M. Pudl, D. Pumescu, and W. J. Stark, Robust chemical preservation of digital information on DNA in silica with error-correcting codes, *Angewandte Chemie International Edition*, vol. 54, no. 8, pp. 2552–2555, 2015.

[34] E. Grigorescu, M. Sudant and M. Zhu, Limitations of Mean-Based Algorithms for Trace Reconstruction at Small Distance, *IEEE International Symposium on Information Theory (ISIT)*, pp. 2525–2530, 2021.

[35] V. Gurusswami, B. Haeupler, and A. Shahrasbi, Optimally resilient codes for list-decoding from insertions and deletions, *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing*, pp. 524–537, 2020.

[36] V. Gurusswami and C. Wang, Deletion codes in the high-noise and high-rate regimes, *IEEE Trans. Inf. Theory*, vol. 63, no. 4, pp. 1961–1970, 2017.

[37] B. Haeupler and M. Mitzenmacher, Repeated deletion channels, *2014 IEEE Information Theory Workshop (ITW 2014)*, pp. 152–156, 2014.

[38] T. Hayashi and K. Yasunaga, On the list decodability of insertions and deletions, *Int. Symp. Inform. Theory*, pp. 86–90, 2018.

[39] R. Heckel, G. Mikutis, and R.N. Grass, A characterization of the DNA data storage channel, arxiv.org/pdf/1803.03322.pdf, 2018.

[40] D. S. Hirschberg, A linear space algorithm for computing maximal common subsequences, *Communications of the ACM*, vol. 18, no. 6, pp. 341–343, 1975.

[41] D. S. Hirschberg, Algorithms for the longest common subsequence problem, *Journal of the ACM (JACM)*, vol. 24, no. 4, pp. 664–675, 1977.

[42] N. Holden, R. Pemantle, and Y. Peres, Subpolynomial trace reconstruction for random strings and arbitrary deletion probability, arXiv preprint arXiv:1801.04783, 2018.

[43] T. Holenstein, M. Mitzenmacher, R. Panigrahy, and U. Wieder, Trace reconstruction with constant deletion probability and related results, *Proceedings of the nineteenth annual ACM-SIAM symposium on Discrete algorithms*, pp. 389–398, 2008.

[44] W. Hsu and M. Du, Computing a longest common subsequence for a set of strings, *BIT Numerical Mathematics*, vol. 24, no.1, pp.45–59, 1984.

[45] R. W. Irving and C. B. Fraser, Maximal common subsequences and minimal common supersequences, *In M. Crochemore and D. Gusfield, editors, Combinatorial Pattern Matching*, pp.173–183, Berlin, Heidelberg, 1994.

[46] S. Y. Ito, The string merging problem, *BIT Numerical Mathematics*, vol. 21. no. 1, pp. 20–30, 1981.

[47] S. Kas Hanna and S. El Rouayheb, List decoding of deletions using guess & check codes, *Int. Symp. Inform. Theory*, pp. 2374–2378, 2019.

[48] K. Cai, H. M. Kiah, T. T. Nguyen, and E. Yaakobi, Coding for sequence reconstruction for single edits, *IEEE Transactions on Information Theory*, 2021.

[49] A. Kirsch and E. Drinea, Directly lower bounding the information capacity for channels with i.i.d. deletions and duplications, *IEEE Transactions on Information Theory*, vol. 56, no. 1, pp. 86–102, 2010.

[50] S. Kosuri and G.M. Church, Large-scale de novo DNA synthesis: technologies and applications, *Nature Methods*, vol. 11, no. 5, pp. 499–507, 2014.

[51] V. I. Levenshtein, Binary codes capable of correcting deletions, insertions, and reversals, *Soviet Physics Doklady*, vol. 10, no. 8, pp. 707–710, 1966.

[52] V. Levenshtein, E. Konstantinova, E. Konstantinov, and S. Molodtsov, Reconstruction of a graph from 2- vicinities of its vertices, *Discrete Applied Mathematics*, vol. 156, no. 9, pp.1399–1406, 2008.

[53] V. I. Levenshtein, Efficient reconstruction of sequences, *IEEE Transactions on Information Theory*, vol. 47, no. 1, pp. 2–22, 2001.

[54] V. I. Levenshtein, Efficient reconstruction of sequences from their subsequences or supersequences, *Journal of Combinatorial Theory, Series A*, vol. 93, no. 2, pp.310–332, 2001.

[55] V. I. Levenshtein and J. Siemons, Error graphs and the reconstruction of elements in groups, *Journal of Combinatorial Theory, Series A*, vol. 116, no. 4, pp. 795–815, 2009.

[56] S. Liu, I. Tjiauwinata, and C. Xing, On list decoding of insertion and deletion errors, https://arxiv.org/abs/1906.09705, 2019.

[57] W. J. MacKay and M. S. Paterson, A faster algorithm computing string edit distances, *Journal of Computer and System sciences*, vol. 20, no. 1, pp. 18–31, 1980.

[58] M. Mitzenmacher, On the theory and practice of data recovery with multiple versions, *IEEE International Symposium on Information Theory*, pp. 982–986, 2006.

[59] M. Mitzenmacher, A survey of results for deletion channels and related synchronization channels, *Probability Surveys*, vol. 6, pp. 1–33, 2009.

[60] M. Mitzenmacher and E. Drinea, A simple lower bound for the capacity of the deletion channel, *IEEE Transactions on Information Theory*, vol. 52, no. 10, pp. 4657–4660, 2006.

[61] Raissa Nataf and Tomer Tsachor, Coding and Algorithms for Memories Course – Final Project, https://www.omersabary.com/files/Raissa_Tomerdraft.pdf, 2021.

[62] F. Nazarov and Y. Peres, Trace reconstruction with exp(o(n^{1/3})) samples, *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing*, pp. 1042–1046. ACM, 2017.
[64] L. Organick, S. D. Ang, Y.-J. Chen, R. Lopez, S. Yekhanin, K. Makarychev, M. Z. Racz, G. Kamath, P. Gopalan, B. Nguyen, C. N. Takahashi, S. Newman, H.-Y. Parker, C. Rashtchian, K. Stewart, G. Gupta, R. Carlson, J. Mulligan, D. Carmean, G. Seelig, L. Ceze, and K. Strauss, Random access in large-scale DNA data storage, *Nature Biotechnology*, vol. 36, no. 3, pp. 242–248, 2018.

[65] Y. Peres and A. Zhai, Average-case reconstruction for the deletion channel: subpolynomially many traces suffice, *IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*, pp. 228–239, 2017.

[66] M. Rahmati and T. M. Duman, Upper bounds on the capacity of deletion channels using channel fragmentation, *IEEE Transaction of Information Theory*, vol. 61, no. 1, pp. 146–156, 2015.

[67] M.G. Ross, C. Russ, M. Costello, A. Hollinger, N.J. Lennon, R. Hegarty, N. Nusbaum, and D.B. Jaffe, Characterizing and measuring bias in sequence data, *Genome Biology*, vol. 14, no. 5, pp. 1–20, 2013.

[68] O. Sabary, Y. Yucovich, and E. Yaakobi, The error probability of maximum-likelihood decoding over two deletion/insertion channels, *International Symposium on Information Theory (ISIT)*, pp. 763–768, 2020.

[69] O. Sabary, Y. Orlev, R. Shafir, L. Anavy, E. Yaakobi, and Z. Yakhini, SOLQC: Synthetic oligo library quality control Tool, *Bioinformatics*, vol. 37, no. 5, pp. 720–722, 2021.

[70] F. Sala, C. Schoeny, N. Bitouzé, and L. Dolecek, Synchronizing files from a large number of insertions and deletions, *IEEE Transaction of Communications*, vol. 64, no. 6, pp. 2258–2273, 2016.

[71] F. Sala, R. Gabrys, C. Schoeny, and L. Dolecek, Three novel combinatorial theorems for the insertion/deletion channel, *IEEE International Symposium on Information Theory (ISIT)*, pp. 2702–2706, 2015.

[72] D. Sankoff, Matching sequences under deletion/insertion constraints, *Proceedings of the National Academy of Sciences*, vol. 69, no. 1, pp. 4–6, 1972.

[73] C. Schoeny, A. Wachter-Zeh, R. Gabrys, and E. Yaakobi, Codes correcting a burst of deletions or insertions, *IEEE Transactions on Information Theory*, vol. 63, no. 4, pp. 1971–1985, 2017.

[74] J. Sima and J. Bruck, Optimal k-deletion correcting codes, *IEEE International Symposium of Information Theory*, pp. 847–851, 2019.

[75] J. Sima, N. Raviv, and J. Bruck, On coding over sliced information, *IEEE Transactions on Information Theory*, vol. 67, no. 5, pp. 2793-2807, 2021.

[76] S. R. Srinivasavaradhan, M. Du, S. Diggavi, and C. Fragouli, On maximum likelihood reconstruction over multiple deletion channels, *IEEE International Symposium on Information Theory (ISIT)*, pp. 436–440, 2018.

[77] S. R. Srinivasavaradhan, M. Du, S. Diggavi, and C. Fragouli, Symbolwise map for multiple deletion channels, *IEEE International Symposium on Information Theory (ISIT)*, pp. 181–185, 2019.

[78] S. R. Srinivasavaradhan, S. Gopi, H. Pfister, S. Yekhanin, Trellis BMA: Coded Trace Reconstruction on IDS Channels for DNA Storage, *IEEE International Symposium on Information Theory (ISIT)*, pp. 2453–2458, 2021.

[79] O. Steiner, M. Makhlevich, and S. Yekhanin, Coding and Algorithms for Memories Course – Final Project, [https://www.omersabay.com/files/Or.pdf](https://www.omersabay.com/files/Or.pdf), 2021.

[80] I. Tal, H. D. Pfister, A. Fazeli and A. Vardy, Polar codes for the deletion channel: weak and strong polarization, *IEEE Transactions on Information Theory*, 2021.

[81] K. Tatwawadi and S. Chandak, Tutorial on algebraic deletion correction codes, [arXiv:1906.07887](https://arxiv.org/abs/1906.07887), 2019.

[82] K. Tian, A. Fazeli, A. Vardy and R. Liu, Polar codes for channels with deletions, *55th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pp. 572-579, 2017.

[83] Z. Tronicek, Problems related to subsequences and supersequences, *International Symposium on String Processing and Information Retrieval*. 5th International Workshop on Groupware (Cat. No. PR00268), pp. 199–205, 1999.

[84] E. Ukkonen, A linear-time algorithm for finding approximate shortest common superstrings, *Algorithmica*, vol. 5, no.1, pp. 313–323, 1990.

[85] R. R. Varshamov and G. M. Tenenholtz, A code for correcting a single asymmetric error, *Automatica i Telemekhanika*, vol. 26, no. 2, pp. 288–292, 1965.

[86] R. Venkataramanan, S. Tatwawadi, and K. Ramachandran, Achievable rates for channels with deletions and insertions, *IEEE Transactions on Information Theory*, vol. 59, no. 11, pp. 6990–7013, 2013.

[87] A. Wachter-Zeh, List decoding of insertions and deletions, *IEEE Transaction of Information Theory*, vol. 64, no. 9, pp. 6297–6304, 2017.

[88] E. Yaakobi and J. Bruck, On the uncertainty of information retrieval in associative memories, *IEEE International Symposium on Information Theory*, pp. 106–110, 2012.

[89] E. Yaakobi, M. Schwartz, M. Langberg, and J. Bruck, Sequence reconstruction for grassmann graphs and permutations, *IEEE International Symposium on Information Theory*, pp. 874–878, 2013.

[90] S. H. T. Yazdi, R. Gabrys, and O. Milenkovic, Portable and error-free DNA-based data storage, *Scientific Reports*, vol. 7, no. 1, pp. 1–6, 2017.

[91] A.-K.-Y. Yim, A.C.-S. Yu, J.-W. Li, A.I.-C. Wong, J.-F.C. Loo, K.M. Chan, S.K. Kong, and T.-F. Chan, The Essential component in DNA-based information storage system: Robust error-tolerating module, *Frontiers in Bioengineering and Biotechnology* vol. 2, pp. 1–5, 2014.