Elementary quantum cloning machines

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Abstract

The task of reception of a copy of an arbitrary quantum state with use of a minimum quantity of quantum operations is considered.

1 Introduction

It is known, that an arbitrary quantum state cannot be copied perfectly [1]. However V.Buzek and M.Hillery in work [2] offered a universal quantum cloning machine (UQCM), allowing to create 2 identical qubits from 1 qubit. Two output qubits are a copy of each other, but they are not a copy of an initial quantum state, and are similar to it only in $5/6 \approx 0.83$. Universality of the UQCM is that it clones any quantum state with identical accuracy. In the present work some variants of not universal QCM are considered. Their lack is that they clone qubits non-uniformly depending on their state. In a final section the calculation scheme of UQCM from the general principles is submitted which includes known results [3,4] as special cases. It is shown, how one can choose optimum by quantity of used quantum operations UQCM from given ones.

1.1 Brief theoretical data

Any quantum state

$$|\psi_0\rangle = \alpha |0\rangle + \beta |1\rangle$$

(1)

is represented by a point on Bloch sphere and changes by means of rotation operator [5]:

$$R(\theta, \varphi) = \left( \begin{array}{cc} \cos \theta & -i e^{-i \varphi} \sin \theta \\ -i e^{i \varphi} \sin \theta & \cos \theta \end{array} \right).$$

For simplicity we shall work with equatorial qubits, then $\varphi = \pi/2$ and

$$R(\theta) |0\rangle = \cos \theta |0\rangle + \sin \theta |1\rangle,$$

$$R(\theta) |1\rangle = -\sin \theta |0\rangle + \cos \theta |1\rangle.$$
Pauli matrices can be written down as projectors
\[ \hat{I} = \sigma_0 = |0\rangle \langle 0| + |1\rangle \langle 1|, \quad \sigma_1 = |1\rangle \langle 0| + |0\rangle \langle 1|, \]
\[ \sigma_2 = i (|1\rangle \langle 0| - |0\rangle \langle 1|), \quad \sigma_3 = |0\rangle \langle 0| - |1\rangle \langle 1|. \]

Action of Pauli matrixes on an input \( |\psi_0\rangle \) is given by expression
\[ |\psi_i\rangle = \sigma_i |\psi_0\rangle. \]  
We shall note at once, that scalar product \( \langle \psi_0 | \psi_2 \rangle = 0 \). It means, that for any \( |\psi_0\rangle \) with the help of the universal operator \( \text{NOT} = -i\sigma_2 \) it is possible to create an orthogonal state \( |\psi_2\rangle \). For corresponding matrixes of density we shall receive
\[ \rho_i^{\text{in}} = |\psi_i\rangle \langle \psi_i|. \]

Similarity of two quantum states \( |\psi\rangle \) and \( |\chi\rangle \) is determined by overlapping of their wave functions [6]:
\[ F = \langle \psi | \rho_{\chi} | \psi \rangle, \]
where \( \rho_{\chi} = |\chi\rangle \langle \chi| \).

The basic two-qubit gate is operation of controlled-NOT (CNOT) [5]:
\[ P_{12} |x, y\rangle = |x, x \oplus y\rangle \]
where \( \oplus \) is the modulus-2 summation. The first qubit in this expression refers to as control. It does not vary at CNOT transformation. The second qubit is a controllable or the target-qubit of CNOT operator. If indexes of the operator to change places: \( P_{12} = P_{21} \), the second qubit becomes control, and the first becomes a target qubit. CNOT operator allows to create the entangled state of two qubits. Studying and use of the entangled states is one of the basic problems of quantum calculations.

2 Two qubit QCM

The ideal cloning of the qubit is forbidden. Really, as Wooters and Zurek have shown [1], if there is such a linear unitary operator, that
\[ U |\alpha\rangle |\nu\rangle = |\alpha\rangle |\alpha\rangle, \]
than for orthogonal states \( \langle \alpha | \beta \rangle = 0 \) we have
\[ U |\alpha + \beta\rangle |\nu\rangle = |\alpha + \beta\rangle |\alpha + \beta\rangle = |\alpha\rangle |\alpha\rangle + |\beta\rangle |\alpha\rangle + |\alpha\rangle |\beta\rangle + |\beta\rangle |\beta\rangle. \]
On the other hand, owing to linearity \( U \):
\[ U |\alpha + \beta\rangle |\nu\rangle = U |\alpha\rangle |\nu\rangle + U |\beta\rangle |\nu\rangle = |\alpha\rangle |\alpha\rangle + |\beta\rangle |\beta\rangle. \]
We came to various expressions that proves impossibility of existence of such operation. In the present work the less complicated task is considered: whether it is possible to receive, even nonideal copy of any quantum state, and what quantum operations are necessary for this purpose?

2.1 One-operational QCM

The elementary QCM can be constructed from a single CNOT operator (Fig.1). Submitting on one input of the QCM an arbitrary quantum state $|\psi_0\rangle$, and on other - a qubit in state $|0\rangle$ we shall receive

$$|\Psi^\text{in}\rangle = |\psi_0\rangle |0\rangle = \alpha |00\rangle + \beta |10\rangle .$$

![Figure 1: One-operational QCM](image)

The work of the QCM is reduced to action by CNOT operator on input qubits

$$|\Psi^\text{out}\rangle = P_{12} |\Psi^\text{in}\rangle = \alpha |00\rangle + \beta |11\rangle ,$$

and on an output we receive the entangled state (ebit). The density matrix of the output ebit is given by expression $\rho^\text{out} = |\Psi^\text{out}\rangle \langle \Psi^\text{out}|$. The reduced density operators of the output qubits look like

$$\rho^\text{out}_{1,2} = \frac{\alpha^2}{2} |0\rangle \langle 0| + \frac{\beta^2}{2} |1\rangle \langle 1|. \quad (6)$$

At the output of the QCM it is received two identical states. It is necessary to determine, how output states differ from the input qubit $|\psi_0\rangle$. Comparing (6) and (3) it is easy to see, that the output density operator is expressed through the input density operator as follows:

$$\rho^\text{out}_{1,2} = \frac{1}{2} \rho^\text{in}_0 + \frac{1}{2} \rho^\text{in}_3 .$$

Looking at this expression it is possible to make the assumption, that the target state on 50% coincides with input state and has 50% of an impurity. However more detailed consideration results in other conclusion. Really, scalar product

$$\langle \psi_0 | \psi_3 \rangle = \alpha^2 - \beta^2 \neq 0,$$

i.e. states are not orthogonal, and it means, that wave functions $|\psi_0\rangle$ and $|\psi_3\rangle$ are overlapped and part of the information concerning $|\psi_0\rangle$ is contained in a density
matrix $\rho_3^{in}$. Therefore for calculation of copying accuracy we shall take advantage of the formula (4)

$$F = \langle \psi_0 | \rho_{1,2}^{out} | \psi_0 \rangle = \alpha^4 + \beta^4. \quad (7)$$

From last expression it is visible, that accuracy of cloning depends on an input state of the original, so received QCM is not universal. It cannot cloning any, beforehand unknown states with identical accuracy. Averages (7) on Bloch sphere we shall receive:

$$\bar{F} = \frac{1}{2\pi} \int_0^{2\pi} F d\theta = \frac{3}{4}.$$

2.2 Two-operator QCM

Now we shall rotate input $|0\rangle$ qubit for increase of cloning fidelity (Fig.2). Rotation of the $|0\rangle$ qubit on angle $\phi$ gives

$$|\psi_R\rangle = R(\phi) |0\rangle = \cos \phi |0\rangle + \sin \phi |1\rangle.$$

![Figure 2: Two-operator QCM](image)

After that, the state $|\Psi^{in}\rangle = |\psi_0\psi_R\rangle$ comes to the input channel of the CNOT. At the output of the QCM we’ll receive $|\Psi^{out}\rangle = P_{12} |\Psi^{in}\rangle$ with the reduced density matrixes

$$\rho_1^{out} = \alpha^2 |0\rangle \langle 0| + 2\alpha \beta \cos \phi \sin \phi (|0\rangle \langle 1| + |1\rangle \langle 0|) + \beta^2 |1\rangle \langle 1|,$$

$$\rho_2^{out} = \left( \alpha^2 \cos^2 \phi + \beta^2 \sin^2 \phi \right) |0\rangle \langle 0| + \cos \phi \sin \phi (|0\rangle \langle 1| + |1\rangle \langle 0|) + \left( \alpha^2 \sin^2 \phi + \beta^2 \cos^2 \phi \right) |1\rangle \langle 1|$$

and average fidelity

$$\bar{F}_1 = \frac{2}{3} (\cos \phi \sin \phi + 1), \quad \bar{F}_2 = \frac{\pi}{4} \cos \phi \sin \phi + \frac{2}{3} \cos^2 \phi + \frac{1}{3} \sin^2 \phi.$$
As well as in previous case, cloning fidelity depends on the initial state of the qubit-original. In addition, accuracy depends on a angle $\phi$ of the rotation $|0\rangle$ qubit. Here it is possible to specify some cases (Fig.3):

1. at $\phi = 0^\circ$ we shall receive two copies with identical average fidelity: $\bar{F}_{1,2} = 3/4$.
2. at $\phi = \pi/4$ the accuracy dispersion of the first qubit is equal to zero: $F_2^1 - \bar{F}_2^1 = 0$ and the states are separable, i.e.

$$|\Psi^{\text{out}}\rangle = P_{12} |\psi_0\rangle |\psi_R\rangle = |\psi_0\rangle |\psi_R\rangle = |\Psi^{\text{in}}\rangle.$$  

3. at $\phi = \pi/2$ measurement results of output states are absolutely anticorrelative:

$$\frac{\sqrt{F_1^1 F_2^2 - F_1^1 F_2^2}}{\sqrt{F_1^1 F_2^2 - F_1^1 F_2^2}} = -1$$

4. at $\phi = 3\pi/2$ average fidelity of output states again coincides, but we have received "no" results: $\bar{F}_1 = \bar{F}_2 = 1/2$.

### 3 Tree-qubit QCM

More complex case of quantum cloning is carried out by Buzek and Hillery’s universal QCM (UQCM BH) [2]. Universality of this machine is that its cloning accuracy does
not depend on a state of the qubit-original. We’ll search for the reduced density matrixes of output as decomposition on projectors of orthogonal wave functions $|\psi_0\rangle$ and $|\psi_2\rangle$:

$$\rho_{1,2}^{\text{out}} = f_0^2 \rho_0^{\text{in}} + f_2^2 \rho_2^{\text{in}},$$

(8)

where $f_0^2$ and $f_2^2$ are so far unknown factors. Substituting in (8) decomposition $\rho_0^{\text{in}}$ and $\rho_2^{\text{in}}$ from (3) we shall receive

$$\rho_{0,1}^{\text{out}} = (\alpha^2 f_0^2 + \beta^2 f_2^2) |0\rangle \langle 0| + \alpha \beta (f_0^2 - f_2^2) (|0\rangle \langle 1| + |1\rangle \langle 0|) + (\alpha^2 f_2^2 + \beta^2 f_0^2) |1\rangle \langle 1|.$$  

(9)

It is easy to show, that reduced density (9) can be received from three qubits mix

$$|\Psi^{\text{out}}\rangle = |\Phi_0\rangle_0 |0\rangle_2 + |\Phi_1\rangle_0 |1\rangle_2,$$

(10)

where

$$|\Phi_0\rangle = \alpha \sqrt{f_0^2 - f_2^2} |00\rangle + \beta f_2 |01\rangle + \beta f_2 |10\rangle,$$

$$|\Phi_1\rangle = \beta \sqrt{f_0^2 - f_2^2} |00\rangle + \alpha f_2 |01\rangle + \alpha f_2 |10\rangle.$$

Then,

$$\rho_{0,1}^{\text{out}} = (\alpha^2 f_0^2 + \beta^2 f_2^2) |0\rangle \langle 0|$$

$$+ 2\alpha \beta f_2 \sqrt{f_0^2 - f_2^2} (|0\rangle \langle 1| + |1\rangle \langle 0|) + (\alpha^2 f_2^2 + \beta^2 f_0^2) |1\rangle \langle 1|,$$

(11)

$$\rho_2^{\text{out}} = \frac{1}{2} (f_0^2 - f_2^2) \rho_0^{\text{in}} + 2 f_2^2 \rho_2^{\text{in}} + \frac{1}{2} (f_0^2 - f_2^2) \rho_0^{\text{in}}.$$  

(12)

In order cross components in (9) and (11) coincided, it is necessary to performe the condition

$$2 f_2 \sqrt{f_0^2 - f_2^2} = f_0^2 - f_2^2$$

or

$$f_0^2 = \frac{5}{6}, f_2^2 = \frac{1}{6}.$$  

(13)

From orthogonality $|\psi_0\rangle$ and $|\psi_2\rangle$ follows, that the factor of decomposition $f_0^2$ and is a cloning fidelity which coincides with average fidelity.

Sometimes the criterion of universality of quantum cloning is determined by an opportunity of representation of the output state as

$$\rho^{\text{out}} = s \rho^{\text{in}} + \frac{1-s}{2} \mathbb{1},$$

where

$$s = \frac{9}{16}.$$
where $s$ is scaling factor. It is easy to show, that the latter expression is a direct consequence from orthogonal decomposition (8):

$$\rho_{out} = (f_0^2 - f_2^2) \rho_{in} + f_2^2 \mathbb{I}.$$ 

Hence there follows connection between the dimensional factor $s$ and factors of decomposition $f_0^2$ and $f_2^2$.

Small increase of fidelity can be obtained, using phase-covariant (PC) UQCM [3] in which the output state is reduced from three qubit mix such as:

$$|\Psi^{out}\rangle = |0\rangle_0 |\Phi_0\rangle_{12} + |1\rangle_0 |\Phi_1\rangle_{12},$$  \hspace{1cm} (14)

where

$$|\Phi_0\rangle = \alpha x |00\rangle + \beta y (|01\rangle + |10\rangle) + \alpha z |11\rangle,$$

$$|\Phi_1\rangle = \beta z |00\rangle + \alpha y (|01\rangle + |10\rangle) + \beta x |11\rangle.$$  

Then,

$$\rho_{out}^{1,2} = \left( \alpha^2 (x^2 + y^2) + \beta^2 (y^2 + z^2) \right) |0\rangle \langle 0| + 2\alpha\beta (xy + yz) (|0\rangle \langle 1| + |1\rangle \langle 0|) + \left( \alpha^2 (y^2 + z^2) + \beta^2 (x^2 + y^2) \right) |1\rangle \langle 1|. $$  \hspace{1cm} (15)

Comparing (15) and (9) we see, that in order to obtain the reduced output mix such as (8) it is necessary to perform the following conditions:

$$x^2 + y^2 = f_0^2,$$  \hspace{1cm} (16)

$$y^2 + z^2 = f_2^2,$$  \hspace{1cm} (17)

$$2(xy + yz) = f_0^2 - f_2^2,$$  \hspace{1cm} (18)

$$f_0^2 + f_2^2 = 1.$$  \hspace{1cm} (19)

Obtained optimal task with criterion function (10) $f_0^2 \to \text{max}$ and restrictions (17)-(19) has the solution

$$f_0^2 (\text{max}) = \frac{1}{2} + \frac{1}{\sqrt{8}}, \text{ at } x = \frac{1}{2} + \frac{1}{\sqrt{8}}, \hspace{0.5cm} y = \frac{1}{\sqrt{8}}, \hspace{0.5cm} z = \frac{1}{2} - \frac{1}{\sqrt{8}}.$$  

Let’s notice, that if in system (16)-(19) we take $z = 0$, we shall obtain UQCM BH (10).

It is obvious, that for the work of the UQCM it is necessary to have in addition to the qubit-original two more $|00\rangle$-qubits. It is necessary to find such transformations of input system of qubits $|\psi_0\rangle \otimes |0\rangle \otimes |0\rangle$, which will result to $|\Psi^{out}\rangle$ (10) or (14).
It is convenient to divide the process of cloning into two stages (Fig.4). At the first stage the entangled state is made from two \(|00\rangle\)-qubits with the help of the rotate and CNOT operators:

\[
|00\rangle_{12} \rightarrow |\Psi^{\text{prep}}\rangle = C_1 |00\rangle + C_2 |01\rangle + C_3 |10\rangle + C_4 |11\rangle.
\]  \hspace{1cm} (20)

For UQCM BH it corresponds to the first part (10):

\[
|00\rangle_{12} \rightarrow |\Psi^{\text{prep}}\rangle = \sqrt{f_0^2 - f_2^2} |00\rangle + f_2 |01\rangle + f_2 |10\rangle.
\]  \hspace{1cm} (21)

For UQCM PC, in accordance with (14) we’ll receive

\[
|00\rangle_{12} \rightarrow |\Psi^{\text{prep}}\rangle = f_0 |00\rangle + \sqrt{f_0^2 - f_2^2} (|10\rangle + |01\rangle) + f_2 |11\rangle.
\]  \hspace{1cm} (22)

It is a stage of preparation UQCM for work. At the second stage entangling of input \(|\psi_0\rangle\) and prepared \(|\Psi^{\text{prep}}\rangle\) qubits by the CNOT operators is made: \(|\psi_0\rangle \otimes |\Psi^{\text{prep}}\rangle \rightarrow |\Psi^{\text{out}}\rangle\). We shall examine more attentively each of the stages.

### 3.1 Preparation of the UQCM for work

Process of preparation of the UQCM for work consists in obtaining of two \(|00\rangle\)-qubits of the entangled state (20) with the help of rotation operators \(R(\theta)\) and CNOT:

\[
|\Psi^{\text{prep}}\rangle = R_1(\theta_3) P_2 R_2(\theta_2) P_1 R_1(\theta_1) |00\rangle_{12}
\]  \hspace{1cm} (23)

\[
= (\cos \theta_1 \cos \theta_2 \cos \theta_3 + \sin \theta_1 \sin \theta_2 \sin \theta_3) |00\rangle + (\sin \theta_1 \cos \theta_2 \cos \theta_3 - \cos \theta_1 \sin \theta_2 \sin \theta_3) |01\rangle
\]

\[
+ (\cos \theta_1 \cos \theta_2 \sin \theta_3 - \sin \theta_1 \sin \theta_2 \cos \theta_3) |10\rangle
\]

\[
+ (\cos \theta_1 \sin \theta_2 \cos \theta_3 + \sin \theta_1 \cos \theta_2 \sin \theta_3) |11\rangle.
\]

Comparing (20) and (23) we’ll receive the system

\[
\begin{align*}
\cos \theta_1 \cos \theta_2 \cos \theta_3 + \sin \theta_1 \sin \theta_2 \sin \theta_3 &= C_1, \\
\sin \theta_1 \cos \theta_2 \cos \theta_3 - \cos \theta_1 \sin \theta_2 \sin \theta_3 &= C_2, \\
\cos \theta_1 \cos \theta_2 \sin \theta_3 - \sin \theta_1 \sin \theta_2 \cos \theta_3 &= C_3, \\
\cos \theta_1 \sin \theta_2 \cos \theta_3 + \sin \theta_1 \cos \theta_2 \sin \theta_3 &= C_4,
\end{align*}
\]  \hspace{1cm} (24)

which has the solution

\[
\begin{align*}
cos^2 \theta_1 &= \frac{C_2^2 - C_3^2}{1 - 2C_3^2 - 2C_4^2} + \cos^2 \theta_3 \frac{1 - 2C_2^2 - 2C_4^2}{1 - 2C_3^2 - 2C_4^2}, \\
cos^2 \theta_2 &= \frac{C_3^2 + C_4^2 - \cos^2 \theta_3}{1 - 2\cos^2 \theta_3}.
\end{align*}
\]
\[ \cos^2 \theta_3 = \frac{1}{2} \left( 1 \pm \frac{1 - 2C_3^2 - 2C_4^2}{1 - 4(C_1^2C_4^2 + C_2^2C_3^2)} \sqrt{1 - 4(C_1^2C_4^2 + C_2^2C_3^2) + 8C_1C_2C_3C_4} \right). \]

Now, substituting as the right part (24) values (21), (22) we shall have:

for the UQCM BH

\[ \cos^2 \theta_1 = \cos^2 \theta_3 = \frac{1}{2} \left( 1 + \frac{1}{\sqrt{2}} \right), \]
\[ \cos^2 \theta_2 = \frac{1}{2} + \frac{\sqrt{2}}{3}, \]

for the UQCM PC

\[ \cos^2 \theta_1 = \cos^2 \theta_3 = \frac{1}{2} \left( 1 + \frac{1}{\sqrt{2}} \right), \]
\[ \cos^2 \theta_2 = 1. \]

Thus, at the first stage we have received angles of rotation operators

\[ R_1 (\theta_1), R_2 (\theta_2), R_3 (\theta_3) \]

which together with action of CNOT operators allow to create state \(|\Psi_{\text{prep}}\rangle\) (21), (22) from two \(|00\rangle\)-qubits. We have prepared UQCM to work. Actually, the cloning process is carried out at the second stage.

### 3.2 A stage of cloning

Let’s choose such combination of CNOT operators which will make an output state \(|\Psi_{\text{out}}\rangle\) from input \(|\psi_0\rangle\) and prepared \(|\Psi_{\text{prep}}\rangle\) qubits. We’ll consider more in detail the construction sequence of such operators on the example of UQCM PC. From state \(|\psi_0\rangle\) and (22)

\[ |\psi_0\rangle \otimes |\Psi_{\text{prep}}\rangle = \alpha \left( f_0^2 |000\rangle + \frac{f_0^2 - f_2^2}{2} (|001\rangle + |010\rangle) + f_2^2 |011\rangle \right) \]

\[ + \ \beta \left( f_0^2 |100\rangle + \frac{f_0^2 - f_2^2}{2} (|101\rangle + |110\rangle) + f_2^2 |111\rangle \right) \]  

(26)

we are to obtain the output state (14):

\[ |\Psi_{\text{out}}\rangle = \alpha \left( f_0^2 |000\rangle + \frac{f_0^2 - f_2^2}{2} (|101\rangle + |110\rangle) + f_2^2 |011\rangle \right) \]

\[ + \ \beta \left( f_0^2 |100\rangle + \frac{f_0^2 - f_2^2}{2} (|001\rangle + |010\rangle) + f_2^2 |111\rangle \right). \]  

(27)
Let’s present a state (26) as
\[ |\psi_0\rangle \otimes |\Psi_{\text{prep}}\rangle = C_{xyz} |x, y, z\rangle = C_{xyz} |\Psi_{\text{in}}^{\text{xyz}}\rangle, \]
and output state (27) as
\[ |\Psi_{\text{out}}^{pqr}\rangle = C_{pqr} |p, q, r\rangle = C_{pqr} |\Psi_{\text{out}}^{pqr}\rangle, \]
where \( x, y, z, p, q, r \) are 0, 1.

Comparing values of wave function at identical coefficients \( C_i \), we shall reduce transition from (26) to (27) in the table (see Tab.1.)

**Table 1. Truth table for Boolean functions** \( p(x, y, z), q(x, y, z), r(x, y, z) \).

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| x | y | z | p | q | r |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |

We have received the table of the validity for Boolean functions of three variables \( p(x, y, z), q(x, y, z), r(x, y, z) \). According to the given table we shall write down set of disjunctive normal forms and present them as Zhegalkin polynomials:

\[ p = \bar{x} \& \bar{y} \& z \lor \bar{x} \& y \& \bar{z} \lor x \& y \& z = x \oplus y \oplus z, \]
\[ q = \bar{x} \& y \& \bar{z} \lor \bar{x} \& \bar{y} \& \bar{z} \lor x \& \bar{y} \& z \lor x \& y \& z = y, \]
\[ r = \bar{x} \& \bar{y} \& \bar{z} \lor \bar{x} \& y \& \bar{z} \lor x \& \bar{y} \& z \lor x \& y \& z = z. \]

Using the definition of CNOT operator (5) from (30) we’ll obtain
\[ |\Psi_{\text{out}}^{pqr}\rangle = |p, q, r\rangle = |x \oplus y \oplus z, y, z\rangle = P_{21} P_{31} |x, y, z\rangle = P_{21} P_{31} |\Psi_{\text{in}}^{xyz}\rangle. \]

Let’s sum up construction UQCM PC (Fig.4). The first stage is described by the expression (23). However, according to (25) \( R_2(\theta_2) = 1 \), therefore (23) can be copied as
\[ |\Psi_{\text{prep}}\rangle = R_1 P_{21} P_{12} R_1 |00\rangle, \]
where the rotation angle of the operator \( R_1 \) is determined in (25). At the second stage two CNOT operators entangle the qubit-original \( |\psi_0\rangle \) with prepared state \( |\Psi_{\text{prep}}\rangle \):
\[ |\Psi_{\text{out}}\rangle = P_{21} P_{31} |\psi_0\rangle \otimes |\Psi_{\text{prep}}\rangle. \]
Figure 4: Phase-covariant UQCM. After the preparation procedure it is enough to work on the qubit-original by two CNOT operators in order to obtain 2 copies with accuracy 0.854.

As result we shall receive an output three qubits density matrix $\rho_{012}^{\text{out}} = |\Psi^{\text{out}}\rangle \langle \Psi^{\text{out}}|$ from which with accuracy $f_0^2 = \frac{1}{2} + \frac{1}{\sqrt{8}} \approx 0.854$ two identical bunches $\rho_{0}^{\text{in}} = |\psi_0\rangle \langle \psi_0|$ are reduced.

The similar calculation algorithm at comparison of expressions (10), (21) result in sequence of actions of the UQCM BH [7]:

$$|\Psi^{\text{out}}\rangle = P_{21} P_{02} P_{10} |\psi_0\rangle \otimes |\Psi^{\text{prep}}\rangle.$$ 

We shall note the basic differences of two considered UQCM. For UQCM BH the pure input state $|\psi_0\rangle$ at the output is registered as a mix $\rho_{0}^{\text{in}}$. For UQCM PC the input state $|\psi_0\rangle$ after cloning procedure has the view

$$\rho_0^{\text{out}} = \frac{3}{4} \rho_0^{\text{in}} + \frac{1}{4} \rho_2^{\text{in}},$$

and has already 0.25 parts of the impurity.

The further generalization of calculation algorithms of the UQCM is connected to the ability of CNOT operator to interchange the position coefficients $C_{xyz}$ in decomposition (20). Then, in order to obtain an output state $|\Psi^{\text{out}}\rangle$ it is necessary to keep only a function $|\Psi^{\text{prep}}\rangle$ (20), and it becomes indifferent where is a factor $C_{xyz}$. As an example we shall examine UQCM PC (22). Calculation result are given in Table 2. The first line of the second column determines the initial combination of coefficients (22): $(C_1, C_2, C_3, C_4) = \left(\frac{1}{2}, \frac{1}{\sqrt{8}}, \frac{1}{\sqrt{8}}, \frac{1}{2} - \frac{1}{\sqrt{8}}\right)$. As $C_2 = C_3$, rearranging coefficients $C_i$ we shall receive 12 various combinations. For each combination on system (24) we shall find rotation angles $\theta_1, \theta_2, \theta_3$, and on algorithm (28)-(30) - operators of transition P: $|\psi_0\rangle \otimes |\Psi^{\text{prep}}\rangle \rightarrow |\Psi^{\text{out}}\rangle$. For simplification of record we shall enter the following approach:

$$\arccos \sqrt{\frac{1}{2} + \frac{1}{\sqrt{8}}} = \frac{\pi}{8} = 22^0.30', \quad \arccos \sqrt{\frac{1}{2} + \frac{1}{\sqrt{6}} = \frac{53\pi}{540} = 17^0.40'}.$$
\[
\arccos \frac{\sqrt{2} + \sqrt{3}}{2} = \frac{\pi}{12} = 15^0, \quad \arccos \sqrt{\frac{1}{2} \left( 1 + \frac{1}{\sqrt{3}} \right)} = \frac{41\pi}{270} = 27^020'
\]

For the description of CNOT operator with inversion the designation is accepted

\[
P_{12} |x, y\rangle = P_{12} |x, \bar{y}\rangle = |x, x \oplus \bar{y}\rangle = P_{12} |x, \sigma_1 y\rangle.
\]

From table 2 it is visible, that for each \(|\Psi_{prep}\rangle\) there are two ways to obtain \(|\Psi_{out}\rangle\). It is connected to symmetry of an output state with respect to rearrangement \(|\psi_1^{out}\rangle \leftrightarrow |\psi_2^{out}\rangle\). If we assump that for some combinations \((C_1, C_2, C_3, C_4)\) there are some decisions \(\theta_1, \theta_2, \theta_3\) it is quite possible to choose the most simple UQCM from the number of obtained UQCM (Fig.4).

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Table 2. Phase-covariant universal quantum cloning machines.

| N | \((C_1, C_2, C_3, C_4)\) | \(\theta_1\) | \(\theta_2\) | \(\theta_3\) | \(|\Psi^{out}\rangle_{xyz}\) | \(P : |\psi_0\rangle |\Psi^{prep}\rangle \rightarrow |\Psi^{out}\rangle\) |
|---|---|---|---|---|---|---|
| 1 | \((C_1, C_2, C_2, C_4)\) | 22°30′ | 0° | 22°30′ | \(|x \oplus y \oplus z, y, z\rangle\) | \(P_{10}P_{20}\) \(P_{12}P_{21}P_{10}P_{20}\) |
| 2 | \((C_1, C_2, C_4, C_2)\) | 27°20′ | 15° | 17°40′ | \(|x \oplus z, y, y \oplus z\rangle\) | \(P_{12}P_{20}\) \(P_{12}P_{21}P_{20}\) |
| 3 | \((C_1, C_4, C_2, C_2)\) | 17°40′ | 15° | 27°20′ | \(|x \oplus y, z, y \oplus z\rangle\) | \(P_{21}P_{12}P_{10}\) \(P_{21}P_{10}\) |
| 4 | \((C_2, C_1, C_2, C_4)\) | 62°40′ | −15° | 17°40′ | \(|x \oplus \overline{z}, y, y \oplus \overline{z}\rangle\) | \(P_{12}P_{20}\) \(P_{12}P_{21}\) |
| 5 | \((C_2, C_1, C_4, C_2)\) | 67°30′ | 0° | 22°30′ | \(|x \oplus y \oplus \overline{z}, y, \overline{z}\rangle\) | \(P_{10}P_{20}\) \(P_{21}P_{12}P_{10}P_{20}\) |
| 6 | \((C_2, C_2, C_1, C_4)\) | 17°40′ | −15° | 62°40′ | \(|x \oplus \overline{y}, z, \overline{y} \oplus z\rangle\) | \(P_{21}P_{12}P_{10}\) \(P_{10}P_{20}\) |
| 7 | \((C_2, C_2, C_4, C_1)\) | −17°40′ | 75° | −27°20′ | \(|x \oplus \overline{y}, y, y \oplus \overline{y}\rangle\) | \(P_{21}P_{12}P_{10}\) \(P_{21}P_{10}\) |
| 8 | \((C_2, C_4, C_1, C_2)\) | 22°30′ | 0° | 67°30′ | \(|x \oplus \overline{y} \oplus z, z, \overline{y}\rangle\) | \(P_{12}P_{21}P_{10}P_{20}\) \(P_{10}P_{20}\) |
| 9 | \((C_2, C_4, C_2, C_1)\) | −27°20′ | 75° | −17°40′ | \(|x \oplus \overline{y}, y \oplus z, \overline{y}\rangle\) | \(P_{12}P_{20}\) \(P_{21}\) \(P_{20}\) |
| 10 | \((C_4, C_2, C_2, C_1)\) | 67°30′ | 0° | 67°30′ | \(|x \oplus y \oplus \overline{y}, y, \overline{y}\rangle\) | \(P_{12}P_{21}P_{10}P_{20}\) \(P_{10}P_{20}\) |
| 11 | \((C_4, C_2, C_1, C_2)\) | 27°20′ | −15° | 72°20′ | \(|x \oplus z, \overline{y} \oplus z, \overline{y}\rangle\) | \(P_{21}P_{20}\) \(P_{21}P_{20}\) |
| 12 | \((C_4, C_1, C_2, C_2)\) | 72°20′ | −15° | 27°20′ | \(|x \oplus y, y \oplus \overline{z}, \overline{z}\rangle\) | \(P_{21}P_{10}\) \(P_{21}P_{12}P_{10}\) |