EXPONENTIAL CONCENTRATION FOR ZEROES OF STATIONARY GAUSSIAN PROCESSES

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Abstract. We show that for any centered stationary Gaussian process of integrable covariance, whose spectral measure has compact support, or finite exponential moments (and some additional regularity), the number of zeroes of the process in $[0, T]$ is within $\eta T$ of its mean value, up to an exponentially small in $T$ probability.

1. Introduction

The study of zeroes of Gaussian stationary processes goes back at least to Kac [10] and Rice [19]. Since then, much work on this topic appeared in the statistics, physics, mathematics and engineering literature. One of the earliest and most fundamental results in this area is the Kac-Rice formula, which calculates the mean number of zeroes in any interval. A similar formula may be written for the variance of the number of zeroes, but it is much harder to analyze. It was only many years after Kac and Rice that fluctuations and central limit theorems were better understood, with works by Cuzick [5], Slud [21], Azaïs-León [2] and others. Questions of large deviations, that is, estimation of the rare event of having many more or much less zeroes than expected in a long interval, remained almost unexplored. One particular such event is that of having no zeroes at all in a long interval, which is also known by the name of “persistence”. Results (and speculations) about this event were initiated by Slepian [20] and were better understood only recently [8].

In the meantime, complex zeroes of certain Gaussian analytic functions received much attention. Most notably, zeroes of the Fock-Bargmann model were introduced by Sodin-Tsirelson [23] and extensively studied by many authors since then. This model has a remarkable property: its zeroes form a point process in the plane with quadratic repulsion, and invariance of distribution under all planar isometries. Sodin-Tsirelson proved the asymptotic normality of these zeroes in [23], and moreover, an exponential concentration of the zeroes around the mean in [24] (See also [12, 17] for more about concentration, and [16] for finer results on asymptotic normality). Exponential concentration was proved for other related models, such as nodal lines of spherical harmonics in [15]. Inspired by these works, the question of concentration for real zeroes got some attention [22], [26, Thm. 2c1], but, until now, was not settled even for a single particular example.

The aim of this paper is to prove exponential concentration for real zeroes of certain Gaussian stationary functions on $\mathbb{R}$, which have an analytic extension to a strip in the complex plane and smooth spectral density. These conditions allow us to use tools from complex analysis, thus generalizing the mechanics of the aforementioned works on the Fock-Bargmann model.

We consider here centered Gaussian stationary processes $\{X(t) : t \in \mathbb{R}\}$ having a.s. absolutely continuous sample path. That is, random absolutely continuous functions $f : \mathbb{R} \to \mathbb{R}$, whose finite
marginal distributions are mean-zero multi-normal, invariant to real shifts. Normalizing wlog the
process \( X \) to have variance one, its joint law is determined by \( r(t - s) := \text{Cov}(X(t), X(s)) \). Here
\( r : \mathbb{R} \to \mathbb{R} \) is a continuous, positive semi-definite function (with \( r(0) = 1 \)). By Bochner’s theorem,
this yields the existence of a probability measure \( \rho \) on \( \mathbb{R} \), called the spectral measure, such that
\[
    r(t) = \int_{\mathbb{R}} e^{-it\lambda} \, d\rho(\lambda) .
\]  

We further assume throughout that \( \int \lambda^2 \, d\rho < \infty \), or equivalently that \( r(t) \) has finite second deriva-
tive at \( t = 0 \) (in which case \( r(t) \) is twice continuously differentiable and \( - r''(0) = \int \lambda^2 \, d\rho \)), and
let \( N_X(I) = |\{ t \in I : f(t) = 0 \}| \) count the number of zeroes, possibly infinite, of such a process in
the interval \( I \subset \mathbb{R} \). Since \( X \) is stationary, \( \mathbb{E} N_X([0,T]) = \alpha T \) for \( \alpha := \mathbb{E} N_X([0,1]) \) and from the
Kac-Rice formula we have that in this case
\[
    \alpha = \frac{1}{\pi} \left( \int \lambda^2 \, d\rho \right)^{\frac{1}{2}} < \infty .
\]  
Indeed, \( \alpha/2 \) is the expected number of 0-upcrossings by \( X([0,1]) \), all of which are strict (see [13, Theorem 7.3.2]), and \( X^{-1}\{ 0 \} \cap [0,1] \) has expected size \( \alpha \) since it a.s. consists of only the (strict) 0-upcrossings and 0-downcrossings of \( X([0,1]) \) (see [13, Theorem 7.2.5]). Our objective is
to prove the following exponential concentration of \( N_X([0,T]) \).

**Theorem 1.1.** Suppose the centered stationary Gaussian process \( \{ X(t) : t \in \mathbb{R} \} \) of a compactly
supported spectral measure has an integrable covariance (ie. \( \int |r(t)| \, dt < \infty \)). Then, for some
\( C < \infty \) and \( c(\cdot) > 0 \),
\[
    \mathbb{P} \left( |N_X([0,T]) - \alpha T| \geq \eta T \right) \leq C e^{-c(\eta) T}, \quad \forall \eta > 0, \ T < \infty .
\]  
Further, if the spectral measure has only a finite exponential moment, namely, for some \( \kappa > 0 \)
\[
    \int_{\mathbb{R}} e^{\lambda|\kappa|} \, d\rho(\lambda) < \infty ,
\]  
then (1.3) holds whenever \( \int |r(t; \kappa_0)| \, dt < \infty \) for some \( \kappa_0 \in (0, \kappa/2) \) and
\[
    r(t; \kappa_0) := \int_{\mathbb{R}} \cos(t\lambda) \cosh(2\kappa_0 \lambda) \, d\rho(\lambda) .
\]  

**Remark 1.2.** Theorem 1.1 applies for example to any spectral measure whose compactly supported
density is in \( W^{1,2}(\mathbb{R}) \), as well as covariances such as \( r(t) = e^{-t^2/2} \) or \( r(t) = 1/(1 + t^2) \) (of spectral
densities \( p(\lambda) = \frac{1}{\sqrt{2\pi}} e^{-\lambda^2/2} \), and \( p(\lambda) = \frac{1}{2} e^{-|\lambda|} \), respectively), for which (1.4) holds and the LHS of
(1.5) is integrable.

As shown next, the lower tail in (1.3) holds under much weaker regularity assumptions.

**Proposition 1.3.** Suppose the centered stationary Gaussian process \( \{ X(t) : t \in \mathbb{R} \} \) has an abso-
lutely continuous sample path and bounded, continuous spectral density \( p(\lambda) \) with \( \int \lambda^2 p(\lambda) \, d\lambda < \infty \). Then, for some \( C < \infty \) and \( c(\cdot) > 0 \) we have:
\[
    \mathbb{P} \left( N_X([0,T]) - \alpha T \leq -\eta T \right) \leq C e^{-c(\eta) T}, \quad \forall \eta > 0, \ T < \infty .
\]  

Proposition 1.3 is a consequence of our next result, on exponential concentration of the number of
sign changes in \([0,T] \), for any discrete-time centered stationary Gaussian process \( \{ Y_k : k \in \mathbb{Z} \} \) of
continuous spectral density.
Theorem 1.4. Suppose \( \{Y_k : k \in \mathbb{Z}\} \) is a centered stationary Gaussian process whose spectral measure \( \rho_Y \) has a continuous density \( p_Y(\lambda) \) (supported within \([-\pi, \pi]\)). Then, for
\[
N^+(T) := \sum_{k=0}^{T-1} 1_{\{Y_k Y_{k+1} < 0\}},
\]
some \( C < \infty \) and \( c(\cdot) > 0 \),
\[
\mathbb{P}\left( |N^+(T) - \mathbb{E}N^+(T)| \geq \eta T \right) \leq Ce^{-c(\eta)T}, \quad \forall \eta > 0, \ T \in \mathbb{N}.
\] (1.8)

Remark 1.5. In the setting of Theorem 1.4 one has that \( T^{-1} \sum_{k=0}^{T-1} h(Y_k, Y_{k+1}) \) satisfies the LDP for any fixed \( h \in C_b(\mathbb{R}^2) \), with a convex, good rate function (see [3, Theorem 4.25]). While this may be extended to \( N^+(T) \) by suitable approximations, we do not follow this route since the rate function is in any case not easily identifiable (see [3, Sect. 7(a)]).

Proposition 1.3 will follow from Theorem 1.4, using the key observation that having few zeroes of a continuous time process implies few sign changes of its restriction to a lattice. This approach does not work for the more challenging upper tail in (1.3), since having many zeroes does not imply having many sign-changes on a lattice, and to establish the upper tail we require the following decay and regularity assumptions about the spectral density of \( X \).

Assumption A: The spectral measure is non-atomic and has finite exponential moment as in (1.4). Further, the covariance functions
\[
r_\ell(x; y) := \int_\mathbb{R} e^{-i\lambda x} \varphi(\lambda y) d\rho(\lambda), \quad \ell = 1, 2, \quad \varphi(\lambda) := \sinh(\lambda)/\lambda,
\] (1.9)
and their \( x \)-derivatives, satisfy for some \( \kappa' \in (0, \kappa/2) \) and finite \( x_\star \),
\[
\omega_\star(k) := \sum_{j \geq k} |r(jx_\star)| + \sup_{|y| < \kappa'} \left\{ \sum_{j \geq k} |r_1'(jx_\star; 2y)| + \sum_{j \geq k} |r_2''(jx_\star; y)| \right\} \to 0, \quad \text{when } k \to \infty.
\] (1.10)

Equipped with Assumption A, we state our main (technical) result.

Theorem 1.6. Subject to Assumption A we have for some \( C < \infty \) and \( c(\cdot) > 0 \), the exponential upper tail
\[
\mathbb{P}\left( N_X([0, T]) - \alpha T \geq 3\eta T \right) \leq Ce^{-c(\eta)T}, \quad \forall \eta > 0, \ T < \infty.
\] (1.11)

In particular, we recover Theorem 1.1 from Proposition 1.3 and Theorem 1.6, thanks to the following explicit sufficient condition for Assumption A.

Proposition 1.7. Assumption A is satisfied, the spectral measure \( \rho(\cdot) \) has a bounded, continuous density, and a.s. the sample path \( t \mapsto X(t) \in C^\infty(\mathbb{R}) \), when either of the following holds:
(a). The support of \( \rho(\cdot) \) is compact and \( \int |r(t)| dt < \infty \) for the covariance \( r(t) \) of (1.1).
(b). Condition (1.4) holds and \( \int |r(t; \kappa_\alpha)| dt < \infty \) for the covariance \( r(t; \kappa_\alpha) \) of (1.5).

It is reasonable when seeking the exponential concentration of \( N_X([0, T]) \), to require smoothness of the covariance \( r(\cdot) \), such as having all spectral moments finite (or the stronger condition (1.4)). Indeed, such exponential concentration implies the finiteness of all moments \( m_k := \mathbb{E}[N_X([0, T])^k] \), with \( m_k = O((T \vee k)^k) \), and such conditions appear in previous studies concerning the finiteness of \( \{m_k\} \). For instance, Nualart and Wschebor [18] show that \( m_k < \infty \) for all \( k \) when \( t \mapsto r(t) \) is real-analytic (hence all spectral moments are finite), while when \( \int \lambda^4 d\rho = \infty \), Cuzick [6] can prove only the finiteness of \( m_k \) up to a certain order \( \kappa_\alpha \). Similarly, Longuett-Higgins [14] shows that for \( r(t) \) real-analytic, \( q_k(\tau) := \mathbb{P}(N_X(0, \tau) \geq k) \) decays, for \( \tau \to 0 \), as \( c(k)\tau^{\frac{1}{2}k^2 + O(k)} \) (indicative of the mutual repulsion of zeroes), while with a discontinuity of \( r(3) \) at the origin the decay of \( q_k(\tau) \) is
merely $c(k)r^2$ for all $k$ (so having a pair of nearby zeroes, the probability of $k$ extra zeroes within the same short interval is $O(k(1))$).

A natural path towards proving the upper tail in our concentration result is to improve Cuzick’s results on moments $m_k$ [5] or Longuett-Higgins estimates on the tail of the number of zeroes $q_k$ [14], so as to get accurate asymptotics of those quantities in $k$. Efforts in this direction were made by many authors (e.g. [2] and the references within). However, in our context it requires a lower bound on the determinant of nearly singular matrices (specifically, the covariance matrices for values of $X(t)$ at a short range), at a level of accuracy which seems out of reach. We bypass this difficulty by relating $N_X([0,T])$ to the count of zeroes within a suitable cover of $[0,T]$, for certain random analytic function $f : S \to \mathbb{C}$ on a thin strip. Thereby, complex analytic tools allow us to replace exponential moments of zero counts by more regular integrals of log $|f(z)|$. After this reduction, the core challenge of our strategy remains in the need to sharply estimate fractional moments of products of many dependent Gaussian variables. This highly non-trivial task (even for integer moments, see [25] and the references within), requires our assumption (1.10), in order to get suitable diagonally dominant covariance matrices.

The paper is organized as follows. In Section 2 we prove Proposition 1.3, Theorem 1.4 and Proposition 1.7. The remainder of the paper is devoted to the proof of Theorem 1.6. In Section 3 this theorem is reduced to the key Proposition 3.6, concerning fractional moments of products of C-valued Gaussian random variables. Proposition 3.6 is proved in Section 5, building on the auxiliary results about weakly-correlated Gaussian variables that we establish in Section 4.

2. PROOFS OF PROPOSITION 1.3, THEOREM 1.4 AND PROPOSITION 1.7

2.1. PROOF OF PROPOSITION 1.3. Assume wlog that $c(\eta) \leq 1$ and $C \geq e$. It suffices to consider $T \geq 1$. Fixing small $\delta > 0$, by the mean-value theorem we have that $N_X([0,T]) \geq N^x_\delta([T/\delta] - 1)$ for the stationary centered Gaussian sequence $Y_k := \delta^{-1} \int_0^\delta X(\delta k + t)dt$. It is further easy to check that

$$\gamma_k := \mathbb{E}Y_0Y_k = \delta^{-2} \int_0^\delta \int_0^\delta r(\delta k + t - s)dsdt,$$

(2.1)

corresponds to the spectral density

$$p_Y(\lambda) = \delta^{-1} \sum_{n \in \mathbb{Z}} p_0(\lambda + 2\pi \delta) \sin^2\left(\frac{\lambda}{2} + \pi n\right) \quad \lambda \in [-\pi, \pi],$$

where $\sin^2(\lambda) := \frac{\sin\lambda}{\lambda}$. Note that for $p(\cdot)$ bounded and continuous, $p_Y$ is also continuous (by dominated convergence). Here $r(0) = 1$, $r'(0) = 0$ and $-r''(t)$, being the characteristic function of the finite measure $\lambda^2p(\lambda)d\lambda$, is continuous at $t \to 0$. We thereby get from (2.1) that $\gamma_0 \to 1$ and $2\delta^{-2}(\gamma_1 - \gamma_0) \to r''(0)$ when $\delta \downarrow 0$. By a Gaussian computation $\mathbb{P}(Y_kY_{k+1} < 0) = \frac{1}{\pi} \arccos(\gamma_1/\gamma_0)$ and it follows that

$$\inf_{T \geq 1} \frac{1}{T} \mathbb{E}N^x_\delta([T/\delta] - 1) \geq (\delta^{-1} - 2)\mathbb{P}(Y_0Y_1 < 0) \to \alpha.$$

As a result of the preceding, we get (1.6) by considering (1.8) for $\delta = \delta(\eta) > 0$ small enough. □

2.2. PROOF OF THEOREM 1.4. We shall use the following easy consequence of weak convergence.

Lemma 2.1. Let $(Y_0, Y_1)$ be a zero-mean jointly Gaussian, having $\mathbb{E}[Y_0^2] > 0$, $\mathbb{E}[Y_1^2] > 0$ and $\alpha := \mathbb{P}(Y_0Y_1 < \xi)$. If the covariance matrices $\Sigma^{(m)}$ of the zero-mean Gaussian vectors $(W_0^{(m)}, W_1^{(m)})$ converge to the covariance matrix $\Sigma$ of $(Y_0, Y_1)$, then

$$\lim_{\xi \to 0} \lim_{m \to \infty} \alpha^{(m)} = \alpha_0,$$

$$\alpha^{(m)} := \mathbb{P}(W_0^{(m)}W_1^{(m)} < \xi).$$

(2.2)
Proof. Since $Y_0$ and $Y_1$ have positive variances, the CDF of $Y_0 Y_1$ is continuous, so the weak convergence of $(W_0^{(m)}, W_1^{(m)})$ to $(Y_0, Y_1)$ implies that for any $\xi$ fixed, $\alpha_\xi^{(m)} \to \alpha_\xi$ as $m \to \infty$. Further, the monotone function $\xi \mapsto \alpha_\xi$ is then continuous and (2.2) holds. \qed

We turn to the proof of Theorem 1.4. wlog normalize to have $\mathbb{E}Y_0^2 = 1$. Let $\eta > 0$ be given. For any $m \geq 2$ we approximate $Y$ by an $(m-1)$-dependent process using the following construction (see e.g. [4, Proof of Theorem A]). Let \{a_k : k \in \mathbb{Z}\} denote the Fourier coefficients of the continuous function $\sqrt{p_Y(\lambda)}$ on $[-\pi, \pi]$, and define $a_k^{(m)} := \left(1 - \frac{|k|}{m-1}\right) + a_k$. Then $Y_k = W_k^{(m)} + Z_k^{(m)}$, where \{\{W_k^{(m)} : k \in \mathbb{Z}\} is an $(m-1)$-dependent, centered, stationary Gaussian sequence with covariance $\mathbb{E}[W_0^{(m)} W_n^{(m)}] = \sum_k a_k^{(m)} a_{k+n}$, while by Fejer’s theorem, the spectral density $p_Z^{(m)}(\lambda) = \left(\sqrt{p_Y(\lambda)} - \sqrt{p_W(\lambda)}\right)^2$ of the centered, stationary Gaussian sequence \{\{Z_k^{(m)} : k \in \mathbb{Z}\} converges to zero as $m \to \infty$, uniformly on $[-\pi, \pi]$. Namely, $\varepsilon_m := \sup_{\lambda} \{p_Z^{(m)}(\lambda)\} \to 0$ as $m \to \infty$.

By stationarity, $\mathbb{E}N^*_Y(T) = \alpha_0 T$ for $\alpha_0 := \mathbb{P}(Y_0 Y_1 < 0)$. Our assumption that the spectral measure $\rho_Y$ has a continuous density implies that $|r_Y(1)| < 1$, hence the covariance matrix $\Sigma$ of $(Y_0, Y_1)$ is positive-definite. Further, by construction, the covariances matrices $\Sigma^{(m)}$ of $(W_0^{(m)}, W_1^{(m)})$ converge to $\Sigma$ when $m \to \infty$ and Lemma 2.1 applies. In particular, there exist $\xi \in (0, 1]$ and $m_* < \infty$ so $\alpha_\xi^{(m)} \leq \alpha_0 + \eta$ whenever $m \geq m_*$. Further, for such $\xi$, $m \geq m_*$ and any $R := \xi/\delta \geq 1$,\n
\[
\{Y_k Y_{k+1} < 0 \} \subseteq \{W_k^{(m)} W_{k+1}^{(m)} < 3\xi\} \cup \{|W_k^{(m)}| \geq R\} \cup \{|W_{k+1}^{(m)}| \geq R\} \cup \{|Z_k^{(m)}| \geq \delta\} \cup \{|Z_{k+1}^{(m)}| \geq \delta\}.
\]

Thus, the following bound applies for the upper tail of $N^*_Y(T)$,

\[
\mathbb{P}(N^*_Y(T) - \alpha_0 T \geq 8\eta T) \leq \mathbb{P}(N_{m,\xi}(T) - \alpha_\xi^{(m)} T \geq \eta T) + 2\mathbb{P}(N^R_m(T) \geq 2\eta T) + 2\mathbb{P}(N^Z_{m}(T) \geq \eta T),
\]

where

\[
N_{m,\xi}(T) := \sum_{k=0}^{T-1} \mathbb{1}_{\{|W_k^{(m)}| < 3\xi\}}, \quad N^R_m(T) := \sum_{k=0}^{T-1} \mathbb{1}_{\{|W_k^{(m)}| \geq R\}}, \quad N^Z(T) := \sum_{k=0}^{T-1} \mathbb{1}_{\{|Z_k| \geq \delta\}}.
\]

(2.3)

The zero-mean, $[-1, 1]$-valued variables $I_k := \mathbb{1}_{\{|W_k^{(m)}| < 3\xi\}} - \alpha_\xi^{(m)}$ are $m$-dependent. Hence, setting $n_T := \lfloor T/m \rfloor$ we get by stationarity, followed by Hoeffding’s inequality for the i.i.d. variables \{\{I_{jm}\}_{j}\}, that for $m \geq m_*$

\[
\mathbb{P}(N_{m,\xi}(T) - (\alpha_\xi^{(m)} + \eta) T) \leq m \max_{n \in \{n_T, n_T+1\}} \mathbb{P}(\sum_{j=0}^{n-1} I_{jm} \geq \eta m) \leq me^{-nT\eta^2/2}.
\]

(2.4)

Since $\mathbb{E}[(W_0^{(m)})^2] \leq 1$, fixing $R < \infty$ with $\mathbb{P}(|Y_0| \geq R) \leq \eta$, we have that $\hat{\alpha}_R^{(m)} := \mathbb{P}(|W_0^{(m)}| \geq R) \leq \eta$ for all $m \geq 1$. Hence, by stationarity and the $m$-dependence of the $[-1, 1]$-valued zero-mean $\hat{I}_k := \mathbb{1}_{\{|W_k^{(m)}| > R\}} - \hat{\alpha}_R^{(m)}$, applying once more Hoeffding’s inequality, we get that

\[
\mathbb{P}\left(N^R_m(T) \geq 2\eta T\right) \leq \mathbb{P}\left(\sum_{k=0}^{T-1} \hat{I}_k \geq \eta T\right) \leq m \max_{n \in \{n_T, n_T+1\}} \mathbb{P}(\sum_{j=0}^{n-1} \hat{I}_{jm} \geq \eta m) \leq me^{-nT\eta^2/2}.
\]

(2.5)

Finally, with $\mathbb{E}[(Z_0^{(m)})^2] \leq 2\pi \varepsilon_m$, from Markov’s inequality and [4, identity (7)] at $\theta_m = \varepsilon_m^{-1/2}$, we deduce that for all $T$ large enough

\[
\mathbb{P}(N^Z_{m}(T) \geq \eta T) \leq e^{-\theta_m \eta^2 T} \mathbb{E}\left[e^{\theta_m \sum_{k=0}^{T-1} |Z_k^{(m)}|}\right] \leq e^{-(\theta_m \delta_\eta - 2\eta) T}.
\]

(2.6)
To complete the proof of the upper tail, combine (2.4)–(2.6) taking \( m \geq m_* \) so large that \( \theta_m \delta \eta \geq 28 \).

Turning to prove the lower tail, set \( \xi \in (0,1] \) and \( m_* < \infty \) so \( \alpha_{-\xi}^{(m)} \geq \alpha - \eta \) whenever \( m \geq m_* \) and note that for any \( \eta > 0 \) \( \|Z_{k+1}^{(m)}\| \geq \delta \). Thus, recalling from (1.7) and (2.3) that

\[
N_\ell^\star(T) = T - \sum_{k=0}^{T-1} 1\{Y_k Y_{k+1} \geq 0\}, \quad N_{m,-\xi}(T) = T - \sum_{k=0}^{T-1} 1\{W_k^{(m)} W_{k+1}^{(m)} \geq \delta \},
\]

we have for any \( \eta > 0 \), the bound

\[
\mathbb{P}(\alpha_0 T - N_\ell^\star(T) \geq 8\eta T) \leq \mathbb{P}(\alpha_{-\delta}^{(m)} T - N_{m,-\xi}(T) \geq \eta T) + 2\mathbb{P}(N_m^{R}(T) \geq \eta T) + 2\mathbb{P}(N_{Z_{k+1}}^{(m)}(T) \geq \eta T).
\]

We have already established exponentially small in \( T \) upper bounds on the two left-most terms (in (2.5) and (2.6)), and re-running the derivation of (2.4) for \( I_k := \alpha_{-\xi}^{(m)} - 1\{W_k^{(m)} W_{k+1}^{(m)} < -\delta\} \) yields such a bound on \( \mathbb{P}(\alpha_{-\delta}^{(m)} T - N_{m,-\xi}(T) \geq \eta T) \).

2.3. Proof of Proposition 1.7. (a) Recall that \( \int |r(t)| dt < \infty \) for \( r(\cdot) \) of (1.1) implies that \( \rho \) has a continuous, bounded density \( p(\lambda) \). Assuming \( \rho \) (hence \( p(\lambda) \)) is supported on \([-K,K]\), fix a compactly supported even function \( \psi(\cdot) \) such that \( \psi(\lambda) \equiv 1 \) on \([-K,K]\) and \( \sum_{j=0}^{2} \|\psi(t)\|_\infty \leq 1 \).

Setting \( h_{\ell,y}(\lambda) := [\lambda \varphi(\lambda y)]^k \psi(\lambda) \) and \( r_{\ell,\psi}(x;y) \) via (1.9) but with \( \psi(\cdot) \) replacing \( p(\cdot) \), we find that

\[
r'_1(x;y) = \int \sin(\lambda x) h_{1,y}(\lambda) d\lambda = \frac{1}{x^2} \int \sin(\lambda x) h''_{1,y}(\lambda) d\lambda \quad (2.7)
\]

(getting the RHS for \( x \neq 0 \) upon twice integrating by parts). One easily verifies that

\[
c_\psi := 2 \max_{\ell=0,1,2} \sup_{|\psi| \leq 2\kappa} \{\|h''_{\ell,y}\|_1 + \|h_{\ell,y}\|_1\} < \infty \quad (2.8)
\]

hence \( |r'_1(x;2y)| \leq c_\psi/(1 + x^2) \) for any \( |y| < \kappa' \) and all \( x \in \mathbb{R} \). Since \( \psi(\lambda)p(\lambda) = p(\lambda) \), it follows that for \( g(x) := c_\psi \int dt |r(t)|/[1 + (x - t)^2] \), any \( |y| < \kappa' \) and \( x \in \mathbb{R} \),

\[
|r'_1(x;2y)| = |\int r'_1(x-t;2y) r(t) dt| \leq g(x). \quad (2.9)
\]

Likewise,

\[
r''_2(x;y) = \int \cos(\lambda x) h_{2,y}(\lambda) d\lambda = \frac{1}{x^2} \int \cos(\lambda x) h''_{2,y}(\lambda) d\lambda, \quad (2.10)
\]

hence \( |r''_2(x;y)| \leq c_\psi/(1 + x^2) \) for all \( |y| < \kappa' \), yielding similarly to (2.9) that

\[
|r''_2(x;y)| = |\int r''_2(x-t;y) r(t) dt| \leq g(x). \quad (2.11)
\]

The same argument shows that

\[
|r(x)| = |\int r_0(x-t) r(t) dt| \leq g(x). \quad (2.12)
\]

Taking \( \kappa = 1 \) we thus find, in view of (2.9) and (2.11), that \( \omega_\star(k) \leq 3 \sum_{j \geq k} g(j) \) and (1.10) follows from the finiteness of

\[
\sum_{j=1}^{\infty} g(j) = c_\psi \int dt |r(t)| \left[ \sum_{j=1}^{\infty} \frac{1}{1 + (j-t)^2} \right] \leq 2 c_\psi \left[ \sum_{j \geq 0} \frac{1}{1 + j^2} \right] \int |r(t)| dt.
\]

(b) If \( \rho(\cdot) \) of unbounded support satisfies (1.4), then the covariance \( r(\cdot) \) of (1.1) is real-analytic and a.s. the sample path \( t \mapsto X(t) \) is in \( C^\infty(\mathbb{R}) \). Suppose also that for some \( \kappa_0 \in (0, \kappa/2) \) the covariance...
Having \((\psi; \kappa_0)\) of (1.5) is integrable. The latter implies that the measure \(\cosh(2\kappa_0 \lambda) d\rho(\lambda)\) has a continuous, bounded density \(p_{\kappa_0}(\lambda)\), hence \(\rho(\cdot)\) has the continuous, bounded density \(\rho(\lambda) = \psi(\lambda) p_{\kappa_0}(\lambda)\) for the even, \((0, 1)\)-valued, integrable function \(\psi(\lambda) := 1/\cosh(2\kappa_0 \lambda)\). It is easy to verify that (2.8) remains valid for such choice of \(\psi(\lambda)\), provided \(\kappa^\prime < \kappa_0\). Further, in this case \(|h_t(\lambda)| \to 0\) and \(|h_t'(\lambda)| \to 0\) as \(|\lambda| \to \infty\), whenever \(|y| < 2\kappa^\prime\), justifying the integration by parts that lead to the right-most equality in both (2.7) and (2.10). The convolution identities (2.9), (2.11) and (2.12) apply upon replacing \(r(t)\) by \(r(t; \kappa_0)\), as do the corresponding bounds, albeit with \(g(x) := c_\psi \int dt |r(t; \kappa_0)|/[1 + (x - t)^2]\), so the integrability of \(|r(\cdot; \kappa_0)|\) indeed suffices for (1.10). \(\quad \square\)

3. Analytic Extension, Jensen’s Formula and De-correlation

3.1. An analytic extension and its properties. Under (1.4) the covariance kernel \(r : \mathbb{R} \to \mathbb{R}\) of the process \(X(t)\) analytically extends to the strip \(S_\kappa = \{z \in \mathbb{C} : |\operatorname{Im}(z)| < \kappa\}\), by plugging \(t = z\) in (1.1). Utilizing this fact, we next construct a complex analytic, mean zero, Gaussian function \(f : \mathbb{S} = S_{\kappa/2} \to \mathbb{C}\) which is at the center of our proof of Theorem 1.6.

**Proposition 3.1.** For a real, stationary mean-zero, Gaussian process \(X\) that satisfies (1.4) there exist a complex analytic, zero mean, Gaussian \(f : \mathbb{S} := S_{\kappa/2} \to \mathbb{C}\) such that:

(a) The function \(f(\cdot)\) is conjugation equivariant, namely \(f(\bar{z}) = \overline{f(z)}\).

(b) The covariances of \(f(\cdot)\) are given by the formulae

\[
K(z, w) := \mathbb{E}[f(z)f(w)] = r(z - \bar{w}); \quad \mathbb{E}[f(z)f(w)] = r(z - w) \quad \forall z, w \in \mathbb{S}. \tag{3.1}
\]

(c) The law of \(z \mapsto f(z)\) is stationary under real translations and \(\{f(t + i0)\}_{t \in \mathbb{R}} \overset{d}{=} \{X(t)\}_{t \in \mathbb{R}}\).

**Proof.** As \(\rho\) an even real-valued measure, there exists an orthonormal basis (ONB) for \(\mathcal{L}^2_{\rho}(\mathbb{R})\) composed of Hermitian functions \(\{\varphi_n\}\) (ie. with \(\varphi_n(-\lambda) = \overline{\varphi_n(\lambda)}\)). For such a basis and \(e_z(\lambda) := e^{i\lambda z}, z \in S_{\kappa/2}\) let

\[
\psi_n(z) := \langle \varphi_n, e_z \rangle_{\mathcal{L}^2_{\rho}(\mathbb{R})} = \int_{\mathbb{R}} \varphi_n(\lambda)e_z(\lambda)d\rho(\lambda) = \int_{\mathbb{R}} e^{-i\lambda z}\varphi_n(\lambda)d\rho(\lambda), \tag{3.2}
\]

and for i.i.d. coefficients \(\zeta_n \sim \mathcal{N}_{\mathbb{R}}(0, 1)\) consider the random series

\[
f(z) := \sum_n \zeta_n \psi_n(z).
\]

Having (1.4) hold for \(\kappa\), standard arguments (see [11, Chapter 3, Thm. 2] or [9, Lemma 2.2.3]) yield that the series defining \(f(\cdot)\) converges almost surely to a zero-mean, complex analytic Gaussian function on \(\mathbb{S} = S_{\kappa/2}\), having there the covariance

\[
K(z, w) = \mathbb{E}[f(z)f(w)] = \sum_n \psi_n(z) \overline{\psi_n(w)}.
\]

Since \(\{\varphi_n\}\) are Hermitian and \(\rho\) is even and real-valued, it follows that \(\overline{\psi_n(z)} = \psi_n(\bar{z})\) and we get part (a) upon taking the conjugate in the defining series for \(f(z)\). Further, since \(\{\varphi_n\}\) is an ONB in \(\mathcal{L}^2_{\rho}(\mathbb{R})\)

\[
K(z, w) = \sum_n \langle \varphi_n, e_z \rangle_{\mathcal{L}^2_{\rho}(\mathbb{R})} \langle e_w, \varphi_n \rangle_{\mathcal{L}^2_{\rho}(\mathbb{R})} = \langle e_w, e_z \rangle_{\mathcal{L}^2_{\rho}(\mathbb{R})} = r(z - \bar{w}),
\]

as stated in (3.1) (and the RHS of (3.1) then follows from part (a)). The formulas (3.1) are invariant to real shifts \((z, w) \mapsto (z + t, w + t), t \in \mathbb{R}\) hence the Gaussian function \(f(\cdot)\) is stationary with respect to such real shifts. To complete the proof of part (c), note that by part (a) the function \(f(z)\) is real-valued when \(z \in \mathbb{R}\) and the covariance kernel of (3.1) coincides for \(z, w \in \mathbb{R}\) with the original covariance \(r : \mathbb{R} \to \mathbb{R}\) of the given real Gaussian process \(X\). \(\quad \square\)
Remark 3.2. Recall that \( \text{Re}(z) = \frac{f(z) + \overline{f(z)}}{2} \), \( \text{Im}(z) = \frac{f(z) - \overline{f(z)}}{2i} \) with (3.1) determining the covariance between the real and imaginary parts of \( f(z) \) and \( f(w) \), \( z, w \in \mathbb{S} \). By Proposition 3.1(c), when \( \text{Im}(z) = \text{Im}(w) \) the latter depend only on \( w - z \) so WLOG we may set \( \text{Re}(z) = 0 \). Specifically, for \( |y| < \kappa/2 \) and \( x \in \mathbb{R} \) we have

\[
\mathbb{E}[\text{Re}(fy)\text{Re}(fx + iy)] = \frac{1}{2} \left[ \text{Re}(r(x + 2iy)) + r(x) \right] = \int_{\mathbb{R}} \cos(\lambda x) \cos^2(\lambda y) d\rho(\lambda), \quad (3.3)
\]

\[
\mathbb{E}[\text{Im}(fy)\text{Im}(fx + iy)] = \frac{1}{2} \left[ \text{Re}(r(x + 2iy)) - r(x) \right] = y^2 \int_{\mathbb{R}} \cos(\lambda x) \lambda^2 \varphi^2(\lambda y) d\rho(\lambda), \quad (3.4)
\]

\[
\mathbb{E}[\text{Re}(fy)\text{Im}(fx + iy)] = \frac{1}{2} \text{Im}(r(x + 2iy)) = -y \int_{\mathbb{R}} \sin(\lambda x) \lambda \varphi(2\lambda y) d\rho(\lambda), \quad (3.5)
\]

where \( \varphi(\lambda) := \sinh(\lambda)/\lambda \) and the RHS of (3.3)–(3.5) follows from (1.1) and the even symmetry of the spectral measure \( \rho(\cdot) \).

We next utilize (3.3)–(3.5) to deduce from Assumption A the absolute summability of the corresponding correlations, uniformly in \( \mathbb{S}_{\kappa'} \).

Lemma 3.3. For \( f(\cdot) \) of Proposition 3.1, consider the vector \( \hat{\mathbf{r}}(z) \in [-1, 1]^4 \) of correlations between \( [\text{Re}(f(z)), \text{Im}(f(z))] \) and \( [\text{Re}(f(y)), \text{Im}(f(y))] \) when \( y = \text{Im}(z) \). Then, for \( x_* \) and \( 0 < \kappa' < \kappa/2 \) of Assumption A,

\[
\omega(k) := 4 \sup_{|y| < \kappa'} \left\{ \sum_{j \in \mathbb{Z}} \| \hat{\mathbf{r}}(jx_* + iy) \| \right\} \to 0 \quad \text{for} \quad k \to \infty \quad (3.6)
\]

and in particular

\[
\sup_{|y| < \kappa'} \left\{ \sum_{j \in \mathbb{Z}} |r(jx_* + 2iy)| \right\} < \infty. \quad (3.7)
\]

Proof. In view of (3.3) and (3.4), for any \( |y| < \kappa/2 \) and \( x \in \mathbb{R} \),

\[
v_1(y) := y^{-2} \text{Var} (\text{Im}(f + iy)) = \int_{\mathbb{R}} \lambda^2 \varphi^2(\lambda y) d\rho(\lambda), \quad (3.8)
\]

\[
v_\text{R}(y) := \text{Var} (\text{Re}(f + iy)) = \int_{\mathbb{R}} \cos^2(\lambda y) d\rho(\lambda), \quad (3.9)
\]

are uniformly in \( y \), bounded away from zero and thanks to (1.4),

\[
\text{Var} (\text{Im}(f + iy)) \leq \text{Var} (\text{Re}(f + iy)) \leq r(2iy)
\]

is uniformly bounded over \( |y| \leq \kappa' < \kappa/2 \). Further, by parts (b) and (c) of Proposition 3.1,

\[
|r(x + 2iy)| = \| \mathbb{E}[\overline{f(y)}f(x + iy)] \| \leq r(2iy) \| \hat{\mathbf{r}}(|x| + iy) \|.
\]

Thus, if (3.6) holds, then necessarily \( \omega(1) \) is finite and (3.7) must hold as well. Turning to show (3.6), we have from (3.3)–(3.5), that the coordinates of the vector \( \hat{\mathbf{r}}(x + iy) \) satisfy the relations

\[
\hat{\mathbf{r}}_{\text{RR}}(x + iy) = (1 - \beta_y) \hat{\mathbf{r}}_{\text{H}}(x + iy) + \beta_y r(x), \quad \hat{\mathbf{r}}_{\text{RR}}(x + iy) = -\hat{\mathbf{r}}_{\text{R}}(x + iy), \quad (3.10)
\]

\[
v_1(y) \hat{\mathbf{r}}_{\text{H}}(x + iy) = -r''(x; y), \quad \sqrt{v_\text{R}(y)v_1(y)} \hat{\mathbf{r}}_{\text{R}}(x + iy) = r'(x; 2y), \quad (3.11)
\]

for \( \beta_y := 2r(0)/(r(2iy) + r(0)) \in (0, 1) \) and the \( x \)-derivatives of \( r_\ell(\cdot) \) of (1.9). Recalling that \( \inf_{y} (\eta(y) \wedge v_\text{R}(y)) \geq c^{-1} \) we deduce from (3.10)–(3.11) that

\[
\| \hat{\mathbf{r}}(jx_* + iy) \| \leq r(jx_*) + 2c|r''(jx_*; y)| + 2c|r'(jx_*; 2y)|,
\]

hence \( \omega(k) \leq 8(c \vee 1)\omega_\star(k) \) and (3.6) follows from our assumption (1.10). \( \square \)
3.2. Relating real and complex zeroes. Thanks to the second part of Proposition 3.1(c), for any $\mathbb{D} \subseteq \mathbb{S}$ containing $[0, T]$ we have
\[
N_X([0, T]) \leq N_f(\mathbb{D}) = |\{z \in \mathbb{D} : f(z) = 0\}|. \tag{3.12}
\]
For $\kappa'$ of Assumption A and $\delta \in (0, \kappa'/2)$, let $\mathbb{B}_j(r)$ denote the ball of radius $r$ centered at $x_j := (2j - 1)\delta$. We shall use the bound (3.12) with the disjoint union of $n := \lceil T/(2\delta) \rceil$ balls
\[
\mathbb{D} = \mathbb{D}_{n, \delta} := \bigcup_{j=1}^{n} \mathbb{B}_j(\delta),
\]

further estimating the value of $N_f(\mathbb{B}_j(\delta))$ by Jensen’s enumeration formula for the zeroes of a complex analytic function (see [1, Section 5.3.1]). Specifically, for $\beta \in [0, \log 2]$ define the integral
\[
\Gamma_j(\beta) := \int_{-1/2}^{1/2} \log |f(x_j + \delta e^{\beta} e^{i2\pi\theta})| d\theta . \tag{3.13}
\]
With such choices $\delta e^{\beta} < \kappa' < \kappa/2$, so $\mathbb{B}_j(\delta e^{\beta}) \subset \mathbb{S}$ and Jensen’s formula tells us that for each $j$
\[
\int_{\delta}^{\delta e^{\beta}} N_f(\mathbb{B}_j(r)) \frac{dr}{r} = \Gamma_j(\beta) - \Gamma_j(0). \tag{3.14}
\]
Since $r \mapsto N_f(\mathbb{B}_j(r))$ is non-decreasing, from (3.14) we deduce that
\[
N_f(\mathbb{B}_j(\delta)) \leq \frac{1}{\beta} \left[ \Gamma_j(\beta) - \Gamma_j(0) \right] \leq N_f(\mathbb{B}_j(\delta e^{\beta})). \tag{3.15}
\]
Sum (3.15) over $j$ to get
\[
N_f(\mathbb{D}_{n, \delta}) \leq \frac{1}{\beta} \sum_{j=1}^{n} \hat{\Gamma}_j(\beta) - \frac{1}{\beta} \sum_{j=1}^{n} \hat{\Gamma}_j(0) + \sum_{j=1}^{n} \mathbb{E}[N_f(\mathbb{B}_j(\delta e^{\beta}))], \tag{3.16}
\]
where $\hat{\Gamma}_j(\cdot) := \Gamma_j(\cdot) - \beta \gamma_j(\cdot)$. The next lemma shows that for small positive $\delta$ and $\beta$ the right-most (non-random) term in (3.16) is at most $(\alpha + \eta)T$.

Lemma 3.4. Suppose that (1.4) holds and the spectral measure $\rho(\cdot)$ is non-atomic. There exist $\delta_*(\eta)$ and $\beta_*(\eta)$ positive, such that for any $\delta \leq \delta_*(\eta)$, $\beta \leq \beta_*(\eta)$ and all $T \geq 1$,
\[
\frac{1}{T} \sum_{j=1}^{n} \mathbb{E}[N_f(\mathbb{B}_j(\delta e^{\beta}))] \leq \alpha + \eta . \tag{3.17}
\]

Proof. Since the Gaussian function $f(z)$ has non-atomic spectral measure,
\[
\mathbb{E}[N_f([0, 1] \times [-r, r])] = \alpha + \mu_f([-r, r]), \quad \forall r \geq 0 ,
\]
where $\mu_f(\cdot)$ is some absolutely continuous, non-negative measure on $\mathbb{R}$ (see [7, Theorem 1]). Further, $z \mapsto f(z)$ is stationary under real translations, hence for any $x_j \in \mathbb{R}$ and $r \in [\delta e^{\beta}, \frac{1}{2}] \cap \mathbb{Q}$
\[
\mathbb{E}[N_f(\mathbb{B}_j(\delta e^{\beta}))] \leq \mathbb{E}[N_f([-r, r]^2)] = 2r(\alpha + \mu_f([-r, r])).
\]
With $n \leq \frac{T}{2\delta} + 1$, the LHS of (3.17) is thus for $T \geq 1$ and $\delta < \frac{1}{T}$, at most
\[
h(\delta, \beta) := (1 + 2\delta)e^{\beta}(\alpha + \mu_f([-\delta e^{\beta}, \delta e^{\beta}]))
\]
and we are done, since $h(\cdot, \cdot)$ is continuous with $h(0, 0) = \alpha$. \qed
3.3. Reducing Theorem 1.6 to the decorrelation of moments. Fixing $\eta > 0$, in view of (3.12) and (3.16) it suffices for (1.11) to show that for $\beta = \beta^*$ and $\delta^*$ as in Lemma 3.4, there exist $\delta \in (0, \delta^*)$ and $c = c(\eta, \beta, \delta) > 0$ so that for all $n$ large enough

$$
P\left( \sum_{j=1}^{n} \hat{\Gamma}_j(\beta) \geq n\beta \eta n \right) + P\left( \sum_{j=1}^{n} \hat{\Gamma}_j(0) \leq -n\beta \eta n \right) \leq e^{cn}.
$$

(3.18)

To this end, let $x_*$ be as in Assumption A and consider $\delta \in (0, \delta^*)$ such that $x_*/(2\delta) := \ell_* \in \mathbb{N}$. Then, to utilize the decay of correlations in Lemma 3.3, fix $\ell = k\ell_*$ for some $k \in \mathbb{N}$ and cover $\{1, \ldots, n\}$ by the disjoint union of $\ell$ sets $S_{\tau} := \{\ell - \tau, 2\ell - \tau, \ldots, m\ell - \tau\}$ (namely $\tau = 0, \ldots, \ell - 1$), with $m = \lceil n/\ell \rceil \geq 2$. By stationarity of $f(\cdot)$ under real translations, the law of $\sum_{j \in S_{\tau}} \hat{\Gamma}_j(\cdot)$ is independent of $\tau$. Setting $\xi \equiv \eta \beta/10$, by a union bound on the $\ell$ choices of $\tau$, it suffices for (3.18) to show that some $c = c(\xi, \delta, \ell) > 0$ and all $m$ large enough

$$
P\left( \sum_{j \in S_{\tau}} \hat{\Gamma}_j(\beta) \geq 5\xi \delta m \right) + P\left( \sum_{j \in S_{\tau}} \hat{\Gamma}_j(0) \leq -5\xi \delta m \right) \leq 2e^{-2c\xi m}.
$$

(3.19)

A standard application of the exponential Markov inequality reduces this task for $c = \varepsilon \xi \delta (2\ell)$, into showing that for some $\varepsilon = \varepsilon(\xi, \delta, \ell) > 0$ and all large enough $m$,

$$
E\left[ \exp(\varepsilon \sum_{j=1}^{m} \hat{\Gamma}_j(\beta)) \right] \leq e^{4\xi \delta m} \quad \& \quad E\left[ \exp(-\varepsilon \sum_{j=1}^{m} \hat{\Gamma}_j(0)) \right] \leq e^{4\xi \delta m}.
$$

(3.20)

Upon setting $z_\beta(\theta) := \delta e^{\beta} e^{i\pi \theta} - \delta$, we get from (3.13) that

$$
\sum_{j=1}^{m} \hat{\Gamma}_j(\beta) = \frac{1}{2} \int_{-1}^{1} S_m(z_\beta(\theta)) d\theta
$$

where, $2\delta \ell = k x_*$ thanks to our choice of $\ell$, so

$$
S_m(z) := \sum_{j=1}^{m} \left\{ \log |f(jkx_* + z)| - E[\log |f(z)|] \right\}.
$$

(3.21)

Thus, applying Jensen’s inequality for the convex functions $\exp(\pm \varepsilon \cdot)$, further reduces the task of proving (3.20) into showing that

$$
\sup_{|\theta| \leq 1} E\left[ e^{\varepsilon S_m(z_\beta(\theta))} \right] \leq e^{4\xi \delta m} \quad \& \quad \sup_{|\theta| \leq 1} E\left[ e^{-\varepsilon S_m(z_\theta(\theta))} \right] \leq e^{4\xi \delta m}.
$$

(3.22)

In view of the stationarity of $f(\cdot)$ under real translations, the law of $S_m(z)$ of (3.21) depends only on $\text{Im}(z)$, hence in (3.22) we can WLOG re-set $z_\beta(\theta) = iy$ for $y = \sin(\pi \theta) \delta e^\beta$. Doing so, we consider for $|y| \leq 2\delta$, the mean-zero, Gaussian variables

$$
G_j(y) := f(jx_* + iy),
$$

(3.23)

and first relate $E[\log |G_0(y)|]$ which is part of $S_m(iy)$ to $E[|G_0(y)|^{\pm \varepsilon}]$.

Lemma 3.5. Given $\zeta > 0$, for any $\varepsilon \leq \varepsilon_0(\zeta)$ positive and all $|y| \leq \kappa'$,

$$
E[|G_0(y)|^\varepsilon] \leq (1 + \varepsilon \zeta) \exp \left( \varepsilon E[\log |G_0(y)|] \right),
$$

(3.24)

$$
E[|G_0(y)|^{-\varepsilon}] \leq (1 + \varepsilon \zeta) \exp \left( -\varepsilon E[\log |G_0(y)|] \right).
$$

(3.25)

Proof. We re-write (3.24)–(3.25) in terms of $L(y) := \log |G_0(y)| - E[\log |G_0(y)|]$ and the non-negative function $g_{\varepsilon}(x) := |\varepsilon|^{-1} (e^{\varepsilon x} - \varepsilon x - 1)$, as $E[g_{\varepsilon}(L(y))] \leq \zeta$ and prove the lemma by showing that

$$
\lim_{\varepsilon \downarrow 0} \sup_{|y| \leq \kappa'} \left\{ E[g_{\varepsilon}(L(y))] \right\} = 0.
$$

(3.26)
Since \(|g_{\pm \epsilon}(x)| \leq \eta^{-1}e^{\eta|x|} := \tilde{g}_{\eta}(x)\) whenever \(|\epsilon| \leq \eta\) and \(g_{\pm \epsilon}(\cdot) \to 0\) uniformly on compact subsets of \(\mathbb{R}\), the uniform in \(y\) convergence (3.26), is a consequence of having for some \(\eta > 0\),
\[
\sup_{|y| \leq \kappa'} \left\{ \mathbb{E}[\tilde{g}_{\eta}(L(y))1_{\{|L(y)| \geq b\}}] \right\} \to 0 \quad \text{for} \quad b \to \infty.
\] (3.27)
Further, \(\tilde{g}_{\eta}(\cdot)\) diverges at infinity, so (3.27) follows from having \(\sup_{|y| \leq \kappa'} \left\{ \mathbb{E}[\tilde{g}_{\eta}^{2}(L(y))] \right\}\) finite, for which it suffices to verify that \(\sup_{|y| \leq \kappa'} \left\{ \mathbb{E}[|G_{0}(y)|^{\pm 2\eta}] \right\}\) is finite. For the latter, recall (3.1) that \(\mathbb{E}[|G_{0}(y)|^{2}] = r(2iy)\), which for \(|y| \leq \kappa'\) is uniformly bounded above (as \(\kappa' < \kappa/2\)), whereas \(\mathbb{E}[|G_{0}(y)|^{-1/2}] \leq Cr(0)^{-1/4}\) for some universal \(C < \infty\), since \(|G_{0}(y)|^{-1/2} \leq |X|^{-1/2}\) for the zero-mean \(\mathbb{R}\)-valued Gaussian \(X = \text{Re}(G_{0}(y))\) of \(\text{Var}(X) = v_{R}(y) \geq r(0)\) (see (3.9)).

The next proposition, which is our main technical statement, bounds small positive and negative fractional moments of the product of our Gaussian variables from (3.23), after a suitable dilution.

**Proposition 3.6.** For any \(\zeta > 0\) there is \(\epsilon_{\ast}(\zeta) > 0\) such that for \(\epsilon \leq \epsilon_{\ast}, k \geq k_{\ast}(\zeta, \epsilon) \in \mathbb{N}\), large enough \(m\), and all \(|y| < \kappa'\),
\[
M_{m}(\epsilon) := \mathbb{E} \left[ \prod_{j=1}^{m} |G_{jk}(y)|^{\epsilon} \right] \leq e^{2\epsilon \zeta m} \mathbb{E}[|G_{0}(y)|^{2\epsilon}]^{m/2},
\] (3.28)
\[
M_{m}(\epsilon) := \mathbb{E} \left[ \prod_{j=1}^{m} |G_{jk}(y)|^{-\epsilon} \right] \leq e^{2\epsilon \zeta m} \mathbb{E}[|G_{0}(y)|^{-2\epsilon}]^{m/2}.
\] (3.29)

We proceed to obtain (3.22) from Proposition 3.6. In view of (3.21) and (3.23), the LHS of (3.22) amounts (after setting \(z_{\beta} = iy\), to
\[
\mathbb{E} \left[ \prod_{j=1}^{m} |G_{jk}(y)|^{\epsilon} \right] \leq e^{4|\epsilon|\delta m} \exp \left( \epsilon m \mathbb{E} \log |G_{0}(y)| \right), \quad \forall |y| \leq 2\delta.
\] (3.30)
Proceeding to show (3.30), we set \(\zeta := \xi \delta > 0\) and a positive \(\epsilon \leq \epsilon_{\ast}(\zeta) \wedge \epsilon_{0}(\zeta)/2\), so Lemma 3.5 applies at \(2\epsilon\), then fix \(k \geq k_{\ast}(\zeta, \epsilon)\) large enough as needed for Proposition 3.6. Combining now the bound (3.28) with (3.24) at \(2\epsilon\) and the elementary inequality \((1 + 2\epsilon \zeta) \leq e^{2\epsilon \zeta}\), yields the bound (3.30). Similarly, the RHS of (3.22) amounts to the inequality (3.30) at \(-\epsilon < 0\), so having the control of (3.25) on the \(-\epsilon\)-moment of \(G_{0}(y)\) in terms of \(\mathbb{E} \log |G_{0}(y)|\), we deduce that the RHS of (3.22) follows from the bound (3.29).

In conclusion, we have by now reduced the proof of Theorem 1.6 to the de-correlated moment computations of Proposition 3.6, to which we devote Sections 4 and 5.

### 4. Diagonally dominant Gaussian laws

We establish here a few preparatory results about weakly correlated, centered, \(\mathbb{C}\)-valued Gaussian vectors. Our results are phrased in terms of
\[
f(jx_{\ast} + iy) := G_{j}(y) := \sqrt{v_{R}(y)}X_{j}(y) + iy|y|\sqrt{v_{R}(y)}Y_{j}(y),
\] (4.1)
for \(v_{l}(y)\) and \(v_{R}(y)\) of (3.8)–(3.9), standard, \(\mathbb{R}\)-valued Gaussian \(X_{j}(y), Y_{j}(y)\) which are independent of each other (see (3.5)), and all absolute constants are independent of \(y \in (-\kappa', \kappa')\). Such results apply whenever \(\mathbb{E}[|G_{j}|^{2}]\) are uniformly bounded above and below, provided the covariance matrix of the Gaussian \(G_{j}\) is diagonally dominant, in the sense that the correlations between \(\{X_{j}, Y_{j}\}\) and \(\{X_{j+k}, Y_{j+k}\}\) are absolutely summable (in \(k\)), with a uniform (in \(j\)), tail decay, as in (3.6).

Our first result (needed for proving (3.28)), is a uniform a-priori control on the second moment of the product of such Gaussian variables (assuming only that they have summable covariances, as in (3.7)).
Lemma 4.1. There exists finite $C_\ast \geq 1$ such that for all $|y| < \kappa'$ and any finite $J \subset \mathbb{N}$,
\begin{equation}
\mathbb{E} \left[ \prod_{j \in J} |G_j(y)|^2 \right] \leq C_\ast^{|J|}. \tag{4.2}
\end{equation}

Proof. For centered Gaussian $(Z_1, \ldots, Z_n) \in \mathbb{C}^n$ with $r(\ell, \ell') = \mathbb{E}[Z_{\ell} \overline{Z}_{\ell'}]$ one has that
\begin{equation}
M_n := \mathbb{E} \left[ \prod_{\ell=1}^n |Z_\ell|^2 \right] \leq \prod_{\ell=1}^n R_\ell \quad \text{where} \quad R_\ell := \sum_{\ell' = 1}^n |r(\ell, \ell')|. \tag{4.3}
\end{equation}

Indeed, by Wick’s formula (see [9, Lemma 2.1.7]),
\begin{equation}
M_n = \sum_{\pi} \prod_{j=1}^n r(j, \pi(j)), \tag{4.4}
\end{equation}
where we sum over permutations $\pi$ of $S := \{1, 2, \ldots, n\}$. To bound $M_n$ from above, replace each term $r(j, \pi(j))$ by $|r(j, \pi(j))|$, whereupon having only non-negative terms, further bound $M_n$ by summing over the larger collection of all functions $\pi : S \mapsto S$. The latter sum is precisely the product of $\{R_\ell : \ell \in S\}$, yielding (4.3).

Now apply (4.3) for the centered complex Gaussian $\{G_j(y), j \in J\}$ and bound $R_\ell$ by summing over all $\ell' \in \mathbb{Z}$. Setting $C_y := \sum_{j \in \mathbb{Z}} |r(jx_\ast + 2iy)|$, we thus get from Prop. 3.1(b) that for any $J$ and $|y| < \kappa'$,
\[ \sup_\ell \{R_\ell\} \leq C_y \leq \sup_{|y| < \kappa'} \{C_y\} := C_\ast^2 \]
which is finite by Lemma 3.3 (see (3.7)). \hfill \Box

Let $J_k$ denote the collection of all finite sets $\{j_1, j_2, \ldots, j_n\} \subset \mathbb{N}$, where $j_i \geq j_{i-1} + 1$ for $j_0 := 0$ and any $i \in [1, n]$. Note that for the sequence $\omega(k) \to 0$ of (3.6), and $J \in J_k$, the centered, $\mathbb{R}$-valued, Gaussian vector $Z = (X_0(y), Y_0(y), \{X_j(y), Y_j(y)\}_{j \in J})$ has covariance matrix $\Sigma := \mathbf{I} - S$ such that for all $|y| < \kappa'$,
\begin{equation}
\|S\|_{\infty \to \infty} := \sup_{x \neq 0} \left\{ \frac{\|Sx\|_\infty}{\|x\|_\infty} \right\} = \max_j \left\{ \sum_{j'} \|S_{jj'}\| \right\} \leq \omega(k). \tag{4.5}
\end{equation}

We next detail three elementary properties of Gaussian vectors having such a diagonally dominant covariance matrix.

Lemma 4.2. Suppose $Z = (Z_1, Z_2)$ is centered, $n$-dimensional $\mathbb{R}$-valued Gaussian vector and $\text{Cov}(Z) := \mathbf{I} - S$ with $\|S\|_{\infty \to \infty} \leq \omega < 1$. Then, setting $\tilde{\omega}_i := \omega^i / (1 - \omega)$, $i = 0, 1, 2$, we have that:
(a) All entries of the PSD matrix $\text{Cov}(Z_1) - \text{Cov}(Z_1|Z_2)$ are within $[-\tilde{\omega}_2, \tilde{\omega}_2]$.
(b) The inequality $\|\mathbb{E}[Z_1|Z_2]\|_{\infty} \leq \tilde{\omega}_1\|Z_2\|_{\infty}$ holds.
(c) The density $f_Z(\cdot)$ of $Z$ with respect to i.i.d. standard Normal variables, is such that
\begin{equation}
f_Z(z) \leq (\tilde{\omega}_0)^{n/2} \exp(\tilde{\omega}_1\|z\|_2^2/2). \tag{4.6}
\end{equation}

Proof. (a). Our assumption that $\|S\| \leq \omega < 1$, implies that $\Sigma^{-1} = \sum_{n \geq 0} S^n$ satisfies
\begin{equation}
\|\mathbf{I} - \Sigma^{-1}\|_{\infty \to \infty} \leq \sum_{n=1}^\infty \omega^n = \tilde{\omega}_1, \quad \|\Sigma^{-1}\|_{\infty \to \infty} \leq \sum_{n=0}^\infty \omega^n = \tilde{\omega}_0. \tag{4.7}
\end{equation}

With $\Sigma_{11} := \text{Cov}(Z_1)$, $\Sigma_{22} := \text{Cov}(Z_2)$, $\Sigma_{12} = (\Sigma_{21})^* = \text{Cov}(Z_1, Z_2)$, and $\Sigma_{1|2} := \text{Cov}(Z_1|Z_2)$, recall that (see [9, Exer. 2.1.3]),
\begin{equation}
\Sigma_{11} - \Sigma_{1|2} = \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}. \tag{4.8}
\end{equation}
The $L_1$-norm of each column of $\Sigma_{21}$ is by assumption at most $\omega$. Further, the RHS of (4.7) applies to $\Sigma_{22}^{-1}$, which by (4.8) implies that all entries of $\Sigma_{11} - \Sigma_{12}$ are indeed within $[-\hat{\omega}, \hat{\omega}]$.

(b). Since $\mu := \mathbb{E}[Z_1 | Z_2] = \Sigma_{12} \Sigma_{22}^{-1} Z_2$ and the RHS of (4.7) applies for $\Sigma_{22}$, we deduce as in part (a), that necessarily $\| \mu \|_\infty \leq \omega \hat{\omega} || Z_2 ||_\infty$.

(c). The matrix norm of (4.5) dominates the spectral norm. In particular, from the LHS of (4.7) we deduce that

$$\langle z, (I - \Sigma^{-1})z \rangle \leq \hat{\omega} ||z||_2^2.$$ 

Further, the RHS of (4.7) implies that all eigenvalues of $\Sigma^{-1}$ are within $[0, \hat{\omega}]$, hence the density

$$f_z(z) = |\Sigma^{-1}|^{1/2} \exp \left( \frac{1}{2} \langle z, (I - \Sigma^{-1})z \rangle \right),$$

satisfies the bound (4.6), as claimed.

Relying on diagonal dominance to lower bound the conditional variances, as in Lemma 4.2(a), we get the following negative moment bound (which will later be useful when proving (3.29)).

**Lemma 4.3.** For some finite $k_o, C_o \geq 1$ and $\varepsilon_o > 0$, all $\varepsilon \leq \varepsilon_o$, $|y| < \kappa'$ and any $J \in \mathcal{J}_{k_o}$

$$\mathbb{E}\left[G_0(y)^{-4\varepsilon} \mid \{G_j(y) : j \in J\}\right] \leq C_o. \quad (4.9)$$

**Proof.** Since $|z| \geq |\text{Re}(z)|$ it suffices to show that (4.9) holds when $\text{Re}(G_0(y))$ replaces $|G_0(y)|$. Further, we only need to do so for say $\varepsilon_o = 1/8$, as it thereafter extends by Jensen’s inequality (and the convexity of $g(x) = x^p$ on $\mathbb{R}_+$ when $p = \varepsilon_o / \varepsilon \geq 1$), to all $\varepsilon \leq \varepsilon_o$. To this end, recall that the conditional law of $\text{Re}(G_0(y))$ given the finite $\mathbb{C}$-valued Gaussian collection $\{G_j(y), j \in J\}$, is Gaussian of some non-random (conditional) variance $v = v_R(y; J)$ and random mean $\mu \sqrt{v}$ (see [9, Exer. 2.1.3]). With $\phi(\cdot)$ denoting the standard normal density, we thus have by scaling, that the conditional expectation of $|\text{Re}(G_0(y))|^{-1/2}$ is at most

$$v^{-1/4} \sup_{\mu \in \mathbb{R}} \left\{ \int_{\mathbb{R}} (|x - \mu| \wedge 1)^{-1/2} \phi(x)dx \right\} := v^{-1/4} C_1,$$

for some finite constant $C_1 \leq 1 + \phi(0) \int_{-1}^1 |x|^{-1/2}dx$. With $v_R(y)$ uniformly bounded below and $\omega(k_o) \leq 1/3$ for some $k_o$ finite (see (3.6)), it follows that

$$u(k_o) := (1 - \omega(k_o)) \inf_{|y| < \kappa'} \{v_R(y)\} > 0$$

and we get (4.9) with $C_o = u(k_o)^{-1/4} C_1$, upon showing that

$$\inf_{J \in \mathcal{J}_{k_o}, |y| < \kappa'} \{v_R(y; J)\} \geq u(k_o). \quad (4.10)$$

To this end, as $\mathbb{E}[X_0(y)^2] = 1$ and $\omega(k_o) \leq 1/2$, from Lemma 4.2(a) we then have that

$$\frac{v_R(y; J)}{v_R(y)} = \mathbb{E}[X_0(y)^2 \mid \{X_j(y), Y_j(y) : j \in J\}] \geq \mathbb{E}[X_0(y)^2] - \frac{\omega(k_o)^2}{1 - \omega(k_o)} \geq 1 - \omega(k_o),$$

and (4.10) follows.

Next, for fixed $k \geq k_o$, $y$, $m$ and any threshold $\Delta > 0$, we define the collection

$$B_\Delta := \{1 \leq j \leq m : |X_{jk}(y)| \vee |Y_{jk}(y)| > \Delta\}, \quad (4.11)$$

of “bad” indices, and use diagonal dominance (specifically, Lemma 4.2(c)), to show that for large $\Delta$ it is exponentially highly unlikely to have many bad indices.

**Lemma 4.4.** There exists $c(\Delta) \to \infty$ as $\Delta \to \infty$ such that for any $k \geq k_o$, all $|y| < \kappa'$ and non-random $B \subset \{1, \ldots, m\}$,

$$\mathbb{P}(B \subseteq B_\Delta) \leq e^{-4c(\Delta)|B|}. \quad (4.12)$$
Proof. Since the event \( \{ |X_{jk}(y)| > \Delta, \forall j \in J \} \cap \{ |Y_{jk}(y)| > \Delta, \forall j \in B \setminus J \} \), over the \( 2^{|B|} \) possible \( J \subseteq B \), by Cauchy-Schwartz it suffices for (4.12) to show that for some \( b(\Delta) := 8c(\Delta) + \log 4 \to \infty \) as \( \Delta \to \infty \), both

\[
p_J(\Delta) := \mathbb{P}(|X_{jk}(y)| > \Delta, \forall j \in J) \leq e^{-b(\Delta)|J|},
\]

(4.13)

\[
q_J(\Delta) := \mathbb{P}(|Y_{jk}(y)| > \Delta, \forall j \in J) \leq e^{-b(\Delta)|J|}.
\]

(4.14)

To this end, recall that \( 1 - \omega(\kappa) \geq 2/3 \geq 2\omega(\kappa) \), whenever \( k \geq k_o \). Hence, from Lemma 4.2(c) we have the bound

\[
p_J(\Delta)^{1/|J|} \leq (1 - \omega(\kappa))^{-1/2} \mathbb{E}\left[ e^{\omega(k)X^2_2(\gamma)} 1_{|X_0|>\Delta} \right] \leq \frac{2\sqrt{3/2}}{\sqrt{2\pi}} \int_{\Delta}^\infty e^{-x^2/4} dx := e^{-b(\Delta)}
\]

for which (4.13) holds. Exactly the same argument applies for \( q_J(\Delta) \), yielding the bound (4.12). \( \square \)

We conclude the section by showing that, thanks to Lemma 4.2(b), for large \( k = k(\Delta, \varepsilon) \) and any \( J \in \mathcal{J}_k \), the conditional expectation of \( |G_0(y)|^{\pm \varepsilon} \) given a realization of \( \{ X_{jk}(y), Y_{jk}(y) : j \in J \} \), all of whom are in a specified range \([-\Delta, \Delta] \), is within error \((1 + o(\varepsilon))\) of the unconditional expectation.

**Lemma 4.5.** Let \( H_J(y) := \max_{j \in J} \{|X_{jk}(y)|, |Y_{jk}(y)|\} \). There exist \( k_*(\Delta, \zeta, \varepsilon) : \mathbb{R}^3 \to [k_o, \infty) \) and \( \varepsilon_* > 0 \), that for any \( \Delta, \zeta > 0, \varepsilon \leq \varepsilon_* \), \( J \in \mathcal{J}_k \), and \( |y| < \kappa' \)

\[
\mathbb{E}\left[ |G_0(y)|^{\mp \varepsilon} \mid \{ G_j(y) \}_{j \in J} \right] 1_{|H_J(y)| \leq \Delta} \leq e^{\varepsilon\zeta} \mathbb{E}[|G_0(y)|^{\mp \varepsilon}],
\]

(4.15)

\[
\mathbb{E}\left[ |G_0(y)|^{\pm \varepsilon} \mid \{ G_j(y) \}_{j \in J} \right] 1_{|H_J(y)| \leq \Delta} \leq e^{\varepsilon\zeta} \mathbb{E}[|G_0(y)|^{\pm \varepsilon}].
\]

(4.16)

**Proof.** We use the representation (4.1), dividing (4.15) and (4.16) by \( v_R(y)^{\pm \varepsilon} \), respectively. Then, with \( u := y^2v_l(y)/v_R(y) \in [0, 1] \) and \( g_{u, \pm \varepsilon}(x) := \left( x_1^2 + u x_2^2 \right)^{\pm \varepsilon/2} \), the stated inequalities amount to

\[
1_{|H_J(y)| \leq \Delta} \int_{\mathbb{R}^2} g_{u, \pm \varepsilon}(x) f_J(x) d\gamma(x) \leq e^{\varepsilon\zeta} \int_{\mathbb{R}^2} g_{u, \pm \varepsilon}(x) d\gamma(x),
\]

(4.17)

where \( f_J(\cdot) \) is the Radon-Nikodym density of the conditional law of \((X_0(y), Y_0(y))\) with respect to the standard two-dimensional Gaussian law \( \gamma \). Recall Lemma 4.2(a) that for any \( J \in \mathcal{J}_k \), \( k \geq k_o \), the two-dimensional covariance matrix \( \Sigma_{|J|} := I_2 - \tilde{S} \) of \((X_0(y), Y_0(y))\) given \( \{ G_j(y) \}_{j \in J} \), satisfies \( \| S \|_{\infty \to \infty} \leq \omega(k) f/(1 - \omega(k)) \leq \omega(k) \). Further, by Lemma 4.2(b), the conditional mean \( \mu \) of \((X_0(y), Y_0(y))\) must satisfy \( \| \mu \|_\infty \leq \omega(k) H_J(y) \). Here \( \omega = \omega(k) \leq 1/3 \), so similarly to the derivation of (4.6), we have for the (random) two-dimensional Radon-Nikodym density \( f_J(\cdot) \) that

\[
f_J(x) = \left| \Sigma_{|J|2} \right|^{-1/2} \exp \left\{ \frac{1}{2} \left( \langle x, x \rangle - \langle x - \mu, \Sigma_{|J|2}^{-1}(x - \mu) \rangle \right) \right\} \leq \tilde{f}_{\omega, H_J}(x), \]

(4.18)

where for any fixed \( \Delta < \infty \),

\[
\tilde{f}_{\omega, \Delta}(x) := (1 - \omega)^{-1} \exp \left\{ \omega(x_1^2 + x_2^2 + 3|\Delta|x_1 + 3|\Delta|x_2) \right\} \chi_1, \quad \text{when } \omega \chi_0.
\]

Note that \( g_{u, \varepsilon} \leq g_{u, \varepsilon} \) and \( g_{u, -\varepsilon} \leq g_{u, -\varepsilon} \). Further, both \( g_{u, \varepsilon} \cdot (1 + \tilde{f}_{\omega, \Delta}) \) and \( g_{u, -\varepsilon} \cdot (1 + \tilde{f}_{\omega, \Delta}) \) are in \( L^1_\gamma(\mathbb{R}^2) \) as soon as \( \varepsilon \leq \varepsilon_* < 1 \) and \( \omega < 1/2 \). Consequently, per \( \varepsilon \leq \varepsilon_* \) and \( \Delta < \infty \), the functions \( x \mapsto g_{u, \pm \varepsilon}(x) [\tilde{f}_{\omega, \Delta}(x) - 1] \) are uniformly in \( u \) (and \( \omega \leq 1/3 \)), integrable with respect to \( \gamma \), and converge pointwise to zero as \( \omega \chi_0 \). Thus,

\[
\lim_{\omega \chi_0} \sup_{u \in [0, 1]} \int_{\mathbb{R}^2} g_{u, \pm \varepsilon}(x) \tilde{f}_{\omega, \Delta}(x) d\gamma(x) - \int_{\mathbb{R}^2} g_{u, \pm \varepsilon}(x) d\gamma(x) = 0,
\]

which together with (4.18) and (3.6) imply the existence of finite \( k_*(\Delta, \zeta, \varepsilon) \geq k_o \) such that (4.17) holds whenever \( J \in \mathcal{J}_k \) and \( |y| < \kappa' \). \( \square \)
5. Moment computations: Proof of Proposition 3.6

5.1. Proof of (3.28). Since $c(\Delta)$ of Lemma 4.4 is unbounded, for any $\zeta > 0$ and $\varepsilon \leq \varepsilon_* < 1$ (where $\varepsilon_*$ is from Lemma 4.5), we can take $\Delta = \Delta(\zeta, \varepsilon)$ so large that

$$C_* e^{-c(\Delta)} \leq \varepsilon \zeta e^{\varepsilon \zeta} \mathbb{E}[|G_0(y)|^{2\varepsilon}]^{1/2},$$

(5.1)

where $C_*$ is the finite constant from Lemma 4.1. Given such $\Delta$, let $h_{\varepsilon, \Delta}(G) := |G|^\varepsilon 1_{(|X|, |Y| \leq \Delta)}$ (for $G(y), X(y), Y(y)$ related as in (4.11)). Then, fix $k \geq k_*$ (also from Lemma 4.5), and partition the expression $M_m(\varepsilon)$ of (3.28) according to $B_\Delta$ of (4.11), to get that

$$M_m(\varepsilon) = \sum_B \mathbb{E} \left[ \prod_{j \in B} |G_{jk}(y)|^\varepsilon 1_{(B \subseteq B_\Delta)} \prod_{j \in B^c} h_{\varepsilon, \Delta}(G_{jk}(y)) \right],$$

(5.2)

where the sum is over all $B \subseteq \{1, \ldots, m\}$. For $(1 - \varepsilon)/2 \geq 1/4$, using Hölder's inequality we bound the generic summand on the RHS of (5.2) by

$$\mathbb{E} \left[ \prod_{j \in B} |G_{jk}(y)|^\varepsilon \right]^{1/2} \mathbb{P}(B \subseteq B_\Delta)^{1/4} \mathbb{E} \left[ \prod_{j \in B^c} h_{2\varepsilon, \Delta}(G_{jk}(y)) \right]^{1/2}.$$  

(5.3)

Enumerating $B^c = \{j_1, j_2, \ldots\}$ with $1 \leq j_1 < j_2 < \cdots < m$ and utilizing the stationarity of $\{G_j(y)\}_{j}$, we appeal sequentially for $s = 1, \ldots, \jmath$ (with $k^{-1}J = B^c \setminus \{j_1, \ldots, j_s\}$ shifted backward by $j_s$, to deduce that (since $k \geq k_*$),

$$\mathbb{E} \left[ \prod_{j \in B^c} h_{2\varepsilon, \Delta}(G_{jk}(y)) \right] \leq \left( e^{2\varepsilon \zeta} \mathbb{E}[|G_0(y)|^{2\varepsilon}] \right)^{|B^c|}. $$

(5.4)

Further bounding the left term of (5.3) via Lemma 4.1 and the middle one via Lemma 4.4, we complete the proof by deducing from (5.2) and (5.4) that

$$M_m(\varepsilon) \leq \sum_B C_* |B|^{\varepsilon - c(\Delta)} \left( e^{2\varepsilon \zeta} \mathbb{E}[|G_0(y)|^{2\varepsilon}] \right)^{|B^c|/2}$$

$$= \left\{ C_* e^{-c(\Delta)} + e^{\varepsilon \zeta} \mathbb{E}[|G_0(y)|^{2\varepsilon}]^{1/2} \right\}^m \leq e^{2\varepsilon \zeta} \mathbb{E}[|G_0(y)|^{2\varepsilon}]^{m/2}$$

(5.5)

(with the last inequality holding thanks to having chosen $\Delta$ that satisfies (5.1)). \qed

5.2. Proof of (3.29). Here Lemma 4.3 replaces Lemma 4.1, so upon further reducing $\varepsilon$ to satisfy $\varepsilon \leq \varepsilon_o$ we set $\Delta = \Delta(\zeta, \varepsilon)$ so large that

$$C_oe^{-c(\Delta)} \leq \varepsilon \zeta e^{\varepsilon \zeta} \mathbb{E}[|G_0(y)|^{-2\varepsilon}]^{1/2},$$

(5.6)

where $C_o$ and $\varepsilon_o$ are the finite constants from Lemma 4.3. Proceeding as in the proof of (3.28), for $k \geq k_* \geq k_o$ we partition the expression $M_m(-\varepsilon)$ of (3.29) according to $B_\Delta$ to get the identity (5.2) at $-\varepsilon$. Then, analogously to (5.3), we apply Hölder’s inequality to bound the generic summand on the RHS of that identity (now at $-\varepsilon$), by

$$\mathbb{E} \left[ \prod_{j \in B} |G_{jk}(y)|^{-4\varepsilon} \right]^{1/4} \mathbb{P}(B \subseteq B_\Delta)^{1/4} \mathbb{E} \left[ \prod_{j \in B^c} h_{2\varepsilon, \Delta}(G_{jk}(y)) \right]^{1/2}.$$  

(5.7)

The only difference wrt (5.3) is the first exponent $-4\varepsilon$ instead of $-2$ (as $\mathbb{E}[|G_0(y)|^{-2}] = \infty$). The middle and last term of (5.7) are handled precisely as in the proof of (3.28), upon appealing to Lemma 4.4 and (4.16), respectively. With $\Delta$ satisfying (5.6), the proof of (3.29) is thus complete upon establishing that

$$\mathbb{E} \left[ \prod_{j \in B} |G_{jk}(y)|^{-4\varepsilon} \right] \leq C_o^{|B|}.$$  

(5.8)
Similarly to the derivation of (5.4), upon enumerating $B = \{j_1 < j_2 < \ldots \}$ we get the bound (5.8) by repeated conditioning and using (4.9) for $s = 1, 2, \ldots$ with $k^{-1}J = B \setminus \{j_1, \ldots, j_s\}$ shifted backward by $j_s$.

**Open problem:** Does the exponential upper tail of (1.11) hold in case of covariance $r(t) = \text{sinc}(t)$ (with spectral density $p(\lambda) = \frac{1}{2}1_{[-1,1]}(\lambda)$)? Note that this covariance satisfies Assumption A (for $x_* = 2\pi$), apart from the lack of summability of the $r_1'(\cdot; 2y)$ term in (1.10).

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