Growth-fragmentation process embedded in a planar Brownian excursion

by

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Summary.
The aim of this paper is to present a self-similar growth-fragmentation process linked to a Brownian excursion in the upper half-plane $\mathbb{H}$, obtained by cutting the excursion at horizontal levels. We prove that the associated growth-fragmentation is related to one of the growth-fragmentation processes introduced by Bertoin, Budd, Curien and Kortchemski in [5].

Keywords. Growth-fragmentation process, self-similar Markov process, planar Brownian motion, excursion theory.

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1 Introduction

We consider a Brownian excursion in the upper half-plane $\mathbb{H}$ from 0 to a positive real number $z_0$. For $a > 0$, if the excursion hits the set $\{ z \in \mathbb{C} : \Im(z) = a \}$ of points with imaginary part $a$, it will make a countable number of excursions above it, that we denote by $(e_i^{a,+}, i \geq 1)$. For any such excursion, we let $\Delta e_i^{a,+}$ be the difference between the endpoint of the excursion and its starting point, which we will refer to as the size or length of the excursion. Since both points have the same imaginary part, the collection $(\Delta e_i^{a,+}, i \geq 1)$ is a collection of real numbers and we suppose that they are ranked in decreasing order of their magnitude. Our main theorem describes the law of the process $(\Delta e_i^{a,+}, i \geq 1)_{a \geq 0}$ indexed by $a$ in terms of a self-similar growth-fragmentation. We refer to [4] and [5] for background on growth-fragmentations. Let us describe the growth-fragmentation process involved in our case.

Let $Z = (Z_a)_{0 \leq a < \xi}$ be the positive self-similar Markov process of index 1 whose Lamperti representation is

$$Z_a = z_0 \exp(\xi(\tau(z_0^{-1}a))),$$

where $\xi$ is the Lévy process with Laplace exponent

$$\Psi(q) = -\frac{4}{\pi}q + \frac{2}{\pi} \int_{y \geq -\ln(2)} (e^{qy} - 1 - q(e^y - 1)) \frac{e^{-y}dy}{(e^y - 1)^2}, \quad q < 3, \tag{1}$$

$\tau$ is the time change

$$\tau(a) = \inf \left\{ s \geq 0, \int_0^s e^{\xi(u)} du > a \right\},$$

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and \( \zeta = \inf \{ a \geq 0, \ Z_a = 0 \} \). The cell system driven by \( Z \) can be roughly constructed as follows. The size of the so-called Eve cell is \( z_0 \) at time 0 and evolves according to \( Z \). Then, conditionally on \( Z \), we start at times \( a \) when a jump \( \Delta Z_a = Z_a - Z_{a-} \) occurs independent processes starting from \( -\Delta Z_a \), distributed as \( Z \) when \( \Delta Z_a < 0 \) and as \(-Z \) when \( \Delta Z_a > 0 \). These processes represent the sizes of the daughters of the Eve particle. Then repeat the process for all the daughter cells: at each jump time of the cell process, start an independent copy of the process \( Z \) if the jump is negative, \(-Z \) if the jump is positive, with initial value the negative of the corresponding jump. This defines the sizes of the cells of the next generation and we proceed likewise. We then define, for \( a \geq 0 \), \( X(a) \) as the collection of sizes of cells alive at time \( a \), ranked in decreasing order of their magnitude.

Growth-fragmentation processes were introduced in [4]. Beware that the growth-fragmentation process we just defined is not included in the framework of [4] or [5] because we allow cells to be created at times corresponding to positive jumps, giving birth to cells with negative size. Therefore, the process \( X \) is not a true growth-fragmentation process. The formal construction of the process \( X \) is done in Section 4. The following theorem is the main result of the paper.

**Theorem 1.1.** The process \((\Delta e^{a,+}_i, i \geq 1)_{a \geq 0}\) is distributed as \( X \).

**Remarks.**

- The fact that there is no local explosion (in the sense that there is no compact of \( \mathbb{R} \backslash \{0\} \) with infinitely many elements of \( X \)) can be seen as a consequence of the theorem.

- From the skew-product representation of planar Brownian motion, this theorem has an analog in the radial setting. It can be stated as follows. Take a Brownian excursion in the unit disc from boundary to boundary, with continuous determination of its argument (i.e., its winding number around the origin) \( z_0 > 0 \). Then, for each \( a \geq 0 \), record for each excursion made in the disc of radius \( e^{-a} \) the corresponding winding number. The collection of these winding numbers, ranked in decreasing order of their magnitude and indexed by \( a \) is distributed as \( X \).

- One could finally look at the growth-fragmentation associated to the Brownian bubble measure in \( \mathbb{H} \). It would give an infinite measure on the space of (signed) growth-fragmentation processes starting from 0.

**Related works.** A pure fragmentation process was identified by Bertoin [3] in the case of the linear Brownian excursion where the size of an excursion was there its duration. Le Gall and Riera [10] identified a growth-fragmentation process in the Brownian motion indexed by the Brownian tree. We will follow the strategy of this paper, making use of excursion theory to prove our theorem.

When killing in \( X \) all cells with negative size (and their progeny), one recovers a genuine self-similar (positive) growth-fragmentation driven by \( Z \), call it \( X \). The process \( X \) appears in the work of Bertoin et al. [5], compare Proposition 5.2 in [5] with Proposition 4.2 below. In Section 3.3 of [5], the authors exhibit remarkable martingales associated to growth-fragmentation processes and describe the corresponding changes of measure. In the case of \( X \), the martingale consists in summing the sizes raised to the power \( 5/2 \) of all cells alive at time \( a \). Under the change of measure, the process \( X \) has a spinal decomposition: the size of the tagged particle is a Cauchy process conditioned...
on staying positive, while other cells behave normally. In the case of $\overline{X}$, where we also include cells with negative size, a similar martingale appears, substituting $2$ for $5/2$, while the tagged particle will now follow a Cauchy process (with no conditioning). It is the content of Section 3.3. This martingale is related to the one appearing in [1], where a change of measure was also specified. In that paper, the authors exhibit a martingale in the radial case, see Section 7.1 there. The martingale in our setting can be viewed as a limit case, where one conformally maps the unit disc to the upper half-plane, then sends the image of the origin towards infinity.

The paper is organized as follows. In Section 2, we recall some excursion theory for the planar Brownian motion. Among others, we will define the locally largest fragment, which will be our Eve particle. In Section 3, we show the branching property, identify the law of the Eve particle with that of $Z$ and exhibit the martingale in our context. Finally Theorem 1.1 will be proved in Section 4, where we also show the relation with [5].

2 Excursions of Brownian motion in $\mathbb{H}$

2.1 The excursion process of Brownian motion in $\mathbb{H}$

In this section, we recall some basic facts from excursion theory. Let $(X,Y)$ be a planar Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $(\mathcal{F}_t)_{t \geq 0}$ be the usual augmented filtration. In addition, we call $X$ the space of real-valued continuous functions $w$ defined on an interval $[0, R(w))] \subset [0, \infty)$, endowed with the usual $\sigma$-fields generated by the coordinate mappings $w \mapsto w(t \wedge R(w))$. Let also $\mathcal{X}_0$ be the subset of functions in $X$ vanishing at their endpoint $R(w)$. We set $U := \{u = (x,y) \in \mathcal{X} \times \mathcal{X}_0, u(0) = 0$ and $R(x) = R(y)\}$ and $U_\delta := U \cup \{\delta\}$, where $\delta$ is a cemetery function and write $U^\pm$ for the set of such functions in $U$ with nonnegative and nonpositive imaginary part respectively. These sets are endowed with the product $\sigma$-field denoted $\mathcal{U}_\delta$ and the filtration $(\mathcal{F}_t)_{t \geq 0}$ adapted to the coordinate process on $U$. For $u \in U$, we take the obvious notation $R(u) := R(x) = R(y)$. Finally, let $(L_s)_{s \geq 0} = (L^u_s)_{s \geq 0}$ denote the local time at 0 of $Y$ and $\tau_s = \tau_s^Y$ its inverse defined by $\tau_s := \inf\{r > 0, L_r > s\}$. Recall that the set of zeros of $Y$ is almost surely equal to the set of $\tau_s, \tau_s^-$; we refer to [11] for more details on local times.

Definition 2.1. The excursion process is the process $e = (e_s, s > 0)$ with values in $(U_\delta, \mathcal{U}_\delta)$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ by

(i) if $\tau_s - \tau_{s^-} > 0$, then

\[ e_s : r \mapsto \left( X_{r+\tau_{s^-}} - X_{\tau_{s^-}}, Y_{r+\tau_{s^-}} \right), \quad r \leq \tau_s - \tau_{s^-}, \]

(ii) if $\tau_s - \tau_{s^-} = 0$, then $e_s = \delta$.

Figure 1 is a (naive) drawing of such an excursion.
Figure 1: Drawing of an excursion in the upper half-plane $\mathbb{H}$.

The next proposition follows from the one-dimensional case.

**Proposition 2.2.** The excursion process $(e_s)_{s > 0}$ is a $(\mathcal{F}_s)_{s > 0}$-Poisson point process.

We write $\mathfrak{n}$ for the intensity measure of this Poisson point process. It is a measure on $U$, and we shall denote by $\mathfrak{n}_+$ and $\mathfrak{n}_-$ its restrictions to $U^+$ and $U^-$. We have the following expression for $\mathfrak{n}$.

**Proposition 2.3.** $\mathfrak{n}(dx, dy) = n(dy)P(X_{R(y)} \in dx)$, where $n$ denotes the one-dimensional Itô’s measure on $\mathcal{X}_0$ and $X_T := (X_t, t \in [0, T])$.

### 2.2 The Markov property under $\mathfrak{n}$

For any $u \in U$ and any $a > 0$, let $T_a := \inf\{0 \leq t \leq R(u), y(t) = a\}$ be the hitting time of $a$ by $y$. Then we have the following kind of Markov property under $\mathfrak{n}_+$.

**Lemma 2.4.** *(Markov property under $\mathfrak{n}_+$)*

Under $\mathfrak{n}_+$, on the event $\{T_a < \infty\}$, the process $(u(T_a + t) - u(T_a))_{0 \leq t \leq R(u) - T_a}$ is independent of $\mathcal{F}_{T_a}$ and has the law of a Brownian motion killed at the time $\rho$ when it reaches $\{\Im(z) = -a\}$.

**Proof.** This results from the fact that under the one-dimensional Itô’s measure $n_+$, the coordinate process $t \mapsto y(t)$ has the transition of a Brownian motion killed when it reaches $0$ (cf. Theorem 4.1, Chap. XII in [11]).

Let $f, g, h_1, h_2$ be nonnegative measurable functions defined on $\mathcal{X}$. For simplicity, write for $w \in \mathcal{X}$ or $w \in C([0, \infty))$, $w(\theta_r) = w(r + \cdot) - w(r)$ and for $T > 0$, $w_T := (w(t), t \in [0, T])$. We want to compute

\[
\int_U f(x(\theta_{T_a}))g(y(\theta_{T_a}))h_1(x^{T_a})h_2(y^{T_a})1_{\{T_a < \infty\}}\mathfrak{n}_+(dx, dy)
\]

\[
= \int_U f(x(\theta_{T_a}))g(y(\theta_{T_a}))h_1(x^{T_a})h_2(y^{T_a})1_{\{T_a < \infty\}}n_+(dy)P(X_{R(y)} \in dx)
\]

\[
= \int_{\mathcal{X}_0} g(y(\theta_{T_a}))h_2(y^{T_a})1_{\{T_a < \infty\}}E \left[ f \left( \overline{X}^{R(y)}_{-T_a(y)} \right) h_1 \left( X^{T_a(y)}_y \right) \right] n_+(dy)
\]
Then we can use the Markov property under 
the simple Markov property at time \( T_a(y) \) in the above expectation gives
\[
\int_{\mathcal{D}} f(x(\theta T_a)) \eta(y(\theta T_a)) h_1(x_T) h_2(y_T) \mathbb{1}_{\{T_a < \infty\}} n_+(dx, dy) = \int_{\mathcal{D}} g(y(\theta T_a)) h_2(y_T) \mathbb{1}_{\{T_a < \infty\}} \mathbb{E} \left[ f \left( X^{R(y-T_a)}(y) \right) \right] \mathbb{E} \left[ h_1 \left( X^{T_a}(y) \right) \right] n_+(dy).
\]
Then we can use the Markov property under \( n_+ \) stated in Theorem 4.1, Chap. XII in [11]:
\[
\int_{\mathcal{D}} f(x(\theta T_a)) \eta(y(\theta T_a)) h_1(x_T) h_2(y_T) \mathbb{1}_{\{T_a < \infty\}} n_+(dx, dy) = \int_{\mathcal{D}} \mathbb{E} \left[ h_1 \left( X^{T_a}(y) \right) \right] h_2(y_T) \mathbb{1}_{\{T_a < \infty\}} n_+(dy) \mathbb{E} \left[ g \left( Y^{T_a} \right) f \left( X^{T_a} \right) \right] = \int_{\mathcal{D}} h_1(x_T) h_2(y_T) \mathbb{1}_{\{T_a < \infty\}} n_+(dx, dy) \mathbb{E} \left[ g \left( Y^{T_a} \right) f \left( X^{T_a} \right) \right].
\]
This concludes the proof of Lemma 2.4.

2.3 Excursions above horizontal levels

We next set some notation for studying the excursions above a given level. Let \( a \geq 0 \) and \( u = (x, y) \in U^+ \). In the following list of definitions, one should think of \( u \) as a Brownian excursion in the sense of Definition 2.1.

Define
\[ \mathcal{I}(a) = \{ s \in [0, R(u)], y(s) > a \} . \]

Then by continuity \( \mathcal{I}(a) \) is a countable (possibly empty) union of disjoint open intervals \( I_1, I_2, \ldots \). For any such interval \( I = (i_-, i_+) \), take \( u_I(s) = u(i_+ - s) - u(i_-), 0 \leq s \leq i_+ - i_- \), for the restriction of \( u \) to \( I \), and \( \Delta u_I = x(i_+) - x(i_-) \) for the size or length of \( u_I \). Note that \( u_I \in U \).

If now \( z = u(t), 0 \leq t \leq R(u) \), is on the path of \( u \) and \( 0 \leq a < \exists(z), \) we define
\[ e_a^{(t)} = e_a^{(t)}(u) = u_I, \]
where \( I \) is the unique open interval in the above partition of \( \mathcal{I}(a) \) such that \( t \in I \) (note that this depends on \( t \) and not only on \( z \), which could be a double point). By convention, we also set for \( a = \exists(z) \), \( e_a^{(t)} = z \) and \( \Delta e_a^{(t)} = 0 \). This is represented in an excessively naive way in Figure 2 below.

![Figure 2: Excursions above the level t.](image-url)
For \( z = u(t) \), let \( F^{(t)} : a \in [0, \mathcal{S}(z)] \mapsto \Delta e_{a}^{(t)} \). Define
\[
\begin{align*}
    u_{t}^{l-} & := (u(t) - u(t))_{0 \leq s \leq t}, \quad (2) \\
    u_{t}^{t-} & := (u(t) - u(t))_{0 \leq s \leq R(u) - t}. \quad (3)
\end{align*}
\]
If we set for \( a \in [0, y(t)] \),
\[
\begin{align*}
    T_{a}^{t-} & := \inf \{ s \geq 0, y(t - s) = a \}, \quad (4) \\
    T_{a}^{t+} & := \inf \{ s \geq 0, y(t + s) = a \}, \quad (5)
\end{align*}
\]
we can write \( F^{(t)}(a) = u_{t}^{t-}(T_{a}^{t-}) - u_{t}^{l-}(T_{a}^{l-}) \).

**Lemma 2.5.** For any \( u \in U^{+} \), for all \( 0 \leq t \leq R(u) \), the function \( F^{(t)} \) is càdlàg.

**Proof.** Fix \( t \in [0, R(u)] \). We want to show that \( F^{(t)} \) is càdlàg on \([0, y(t)]\). By usual properties of inverse of continuous functions (see Lemma 4.8 and the remark following it in Chapter 0 of Revuz-Yor [11]), \( a \mapsto T_{a}^{t-} \) and \( a \mapsto T_{a}^{l-} \) are càdlàg (in \( a \)). Hence \( F^{(t)} \) is càdlàg since \( u \) is continuous. \( \square \)

### 2.4 Bismut’s description of Itô’s measure in \( \mathbb{H} \)

In the case of one-dimensional Itô’s measure \( n_{+} \), Bismut’s description roughly states that if we pick an excursion \( u \) at random according to \( n_{+} \), and some time \( 0 \leq t \leq R(u) \) according to the Lebesgue measure, then the "law" of \( u(t) \) is the Lebesgue measure and conditionally on \( u(t) = \alpha \), the left and right parts of \( u \) (seen from \( u(t) \)) are independent Brownian motions killed at \(-\alpha\) (see Theorem 4.7, Chap. XII in [11]). We deduce an analogous result in the case of Itô’s measure in \( \mathbb{H} \) and we apply it to show that for \( n_{+} \)–almost every excursion, there is no loop remaining above any horizontal level.

**Proposition 2.6.** *(Bismut’s description of Itô’s measure in \( \mathbb{H} \))*

Let \( \pi_{+} \) be the measure defined on \( \mathbb{R}^{+} \times U^{+} \) by
\[
\pi_{+}(dt, du) = \mathbb{1}_{\{0 \leq t \leq R(u) \}} dt \, \pi_{+}(du).
\]
Then under \( \pi_{+} \) the "law" of \( (t, (x, y)) \mapsto y(t) \) is the Lebesgue measure \( \, \text{d} \alpha \) and conditionally on \( y(t) = \alpha \), \( u_{t}^{l-} = (u(t - s) - u(t))_{0 \leq s \leq t} \) and \( u_{t}^{t-} = (u(t + s) - u(t))_{0 \leq s \leq R(u) - t} \) are independent Brownian motions killed when reaching \( \{ \mathcal{S}(z) = -\alpha \} \).

See Figure 3. Proposition 2.6 is a direct consequence of the one-dimensional analogous result, for which we refer to [11] (see Theorem 4.7, Chap. XII).

The next proposition ensures that for almost every excursion under \( n_{+} \), there is no loop growing above any horizontal level. Let
\[
\mathcal{L} := \{ u \in U^{+}, \exists 0 \leq t \leq R(u), \exists 0 \leq a < y(t), \Delta e_{a}^{(t)}(u) = 0 \},
\]
be the set of excursions \( u \) having a loop remaining above some level \( a \). Then we have :

**Proposition 2.7.**
\[
n_{+}(\mathcal{L}) = 0.
\]

**Proof.** We first prove the result under \( \bar{n}_{+} \), namely
\[
\bar{n}_{+}\left(\{(t, u) \in \mathbb{R}^{+} \times U^{+}, \exists 0 \leq a < y(t), \Delta e_{a}^{(t)}(u) = 0\}\right) = 0.
\]
Recall the notation (2)-(5). From Bismut’s description of $\pi_+^{(n)}$ we get
\[
\pi_+^{(n)} \left( \{(t, u) \in \mathbb{R}^+ \times U^+, \ 30 \leq a < y(t), \Delta e_a^{(t)}(u) = 0 \} \right)
= \pi_+^{(n)} \left( \{(t, u) \in \mathbb{R}^+ \times U^+, \ 30 \leq a < y(t), \ u^{\leftarrow}(T_a^{4,\rightarrow}) = u^{\rightarrow}(T_a^{\rightarrow,4}) \} \right)
= \int_0^\infty \, da \, \mathbb{P} \left( 30 < a \leq \alpha, \ X_{T_a} = X_{T_a'} \right),
\]
where $X$ and $X'$ are independent linear Brownian motions, and $T_a$ and $T_a'$ are hitting times of $a$ of other independent Brownian motions (corresponding to the imaginary parts). Now, $X_{T_a}$ and $X_{T_a'}$ are independent symmetric Cauchy processes, and therefore $X_{T_a} - X_{T_a'}$ is again a Cauchy process (see Section 4, Chap. III of [11]). Since points are polar for the symmetric Cauchy process (see [2], Chap. II, Section 5), we obtain
\[
\mathbb{P} \left( \exists 0 < a \leq \alpha, \ X_{T_a} = X_{T_a'} \right) = 0 \quad \text{and under} \quad \pi_+^{(n)} \quad \text{the result is proved.}
\]

To extend the result to $\pi_+^{(n)}$, we notice that if $u \in \mathcal{L}$, then the set of $t$’s satisfying the definition of $\mathcal{L}$ has positive Lebesgue measure: namely, it contains all the times until the loop comes back to itself. This translates into
\[
\mathcal{L} \subset \left\{ u \in U^+, \ \int_0^{R(u)} \mathbb{I}_{\{30 \leq a < y(t), \Delta e_a^{(t)}(u) = 0\}} \, dt > 0 \right\}.
\]
But
\[
n_+^{(n)} \left( \int_0^{R(u)} \mathbb{I}_{\{30 \leq a < y(t), \Delta e_a^{(t)}(u) = 0\}} \, dt \right)
= \int_{U^+} \int_0^{R(u)} \mathbb{I}_{\{30 \leq a < y(t), \Delta e_a^{(t)}(u) = 0\}} \, dt \, n_+^{(d)}(du)
= \pi_+^{(n)} \left( \{(t, u) \in \mathbb{R}^+ \times U^+, \ 30 \leq a < y(t), \Delta e_a^{(t)}(u) = 0 \} \right).
\]
Hence, by the first step of the proof,
\[ n_+ \left( \int_0^{R(u)} 1_{\{0 \leq a < y(t), \Delta e_a(t)(u) = 0\}} dt \right) = 0, \]
which gives \[ \int_0^{R(u)} 1_{\{0 \leq a < y(t), \Delta e_a(t)(u) = 0\}} dt = 0 \] for \( n_+ \)-almost every excursion, and the desired result. □

2.5 The locally largest excursion

In [5], the authors give a canonical way to construct the growth-fragmentation, through the so-called locally largest fragment. We want to mimic this construction in our case.

In order to define the locally largest excursion, we set for \( u \in U^+ \) and \( 0 \leq t \leq R(u) \),
\[ S(t) := \sup \{ a \in [0, y(t)], \forall 0 \leq a' \leq a, \ |F(t)(a')| \geq |F(t)(a) - F(t)(a')| \}. \]

Observe that the supremum is taken over a non-empty set by Lemma 2.5 as soon as \( y(t) > 0 \) and \( u(R(u)) \neq 0 \). Let
\[ S := \sup_{0 \leq t \leq R(u)} S(t). \]

In the case of Brownian excursions, the following proposition holds.

**Proposition 2.8.** For almost every \( u \) under \( n_+ \), there exists a unique \( 0 \leq t^* \leq R(u) \) such that \( S(t^*) = S \). Moreover, \( S = \Xi(z^*) \) where \( z^* = u(t^*) \).

We call \( (e_a(t^*))_{0 \leq a \leq \Xi(z^*)} \) the locally largest excursion and \( (\Xi(a) = \Delta e_a(t^*))_{0 \leq a \leq \Xi(z^*)} \) the locally largest fragment.

Thus \( \Xi \) is the length of the excursion which is locally the largest, meaning that at any level \( a \) where the locally largest excursion splits, \( \Xi \) is larger (in absolute value) than the length of the other excursion. See Figure 4 for a picture of \( z^* \). Following [5], we will see it as the Eve particle of our growth-fragmentation process.

![Figure 4: The locally largest excursion.](image-url)

**Proof.** Existence. We deal with the excursions \( u \) satisfying the following properties, which happen \( n_+ \)-almost everywhere: \( u \) has no loop above any horizontal level (see
Proposition 2.7) and \( y \) has distinct local minima. Take a convergent sequence \((t_n, n \geq 1)\) such that \( S(t_n) \) converges to \( S \), and denote by \( t^* \) the limit of \( t_n \). We have necessarily, by definition of \( S(t) \), that \( y(t_n) \geq S(t_n) \). By continuity of \( y \), we get that \( y(t^*) \geq S \).

Take \( a < S \). For \( n \) large enough, since \( a < y(t^*) \), we observe that \( t_n \) and \( t^* \) are in the same excursion above \( a \), i.e. \( e_a(t_n) = e_a(t^*) \). For such \( n \), \( F(t_n)(a') = F(t^*)(a') \) for all \( a' \leq a \). Moreover, for \( n \) large enough, \( S(t_n) > a \) hence for all \( a' \leq a \), \(|F(t^*)(a')| = |F(t_n)(a')| \geq |F(t_n)(a') - F(t_n)(a)| = |F(t^*)(a') - F(t^*)(a)|\). It implies that \( S(t^*) \geq a \), hence \( S(t^*) \geq S \) by taking \( a \) arbitrarily close to \( S \). We found \( t^* \) such that \( S(t^*) = S \).

We show that \( y(t^*) = S \). Notice that, for all \( 0 \leq t \leq R(u) \), by right-continuity of \( F(t) \), the set

\[
A(t) := \left\{ 0 \leq a \leq y(t), \; \forall 0 \leq a' \leq a, \; |F(t)(a')| \geq |F(t)(a') - F(t)(a')| \right\}
\]

is open in \([0, y(t)]\). Indeed, for \( a < y(t) \), \( e_a(t) \) cannot be an excursion with size 0 by assumption, and so by right-continuity, we can take \( \delta > 0 \) such that on \([a, a + \delta] \), \( F(t) \) takes values in \( (\frac{3}{4} F(t)(a), \frac{3}{2} F(t)(a)) \) (in the case \( F(t)(a) > 0 \), without loss of generality).

For such a \( \delta \), and for any \( a' \in [a, a + \delta] \), \( F(t)(a') > \frac{3}{4} F(t)(a) > \frac{3}{2} F(t)(a) = \frac{1}{2} F(t)(a') \), and \( F(t)(a') \geq 0 \). These two inequalities imply that \( |F(t)(a')| \geq |F(t)(a') - F(t)(a')| \).

Now suppose that \( S < y(t^*) \) and let us find a contradiction. We have \( A(t^*) = [0, S] \), hence \( |F(t^*)(S)| < |F(t^*)(S) - F(t^*)(S)| \). Write \( e_a(t^*) = u_I \) with \( I = (i_{a,-}, i_{a,+}) \), so that \( F(t^*)(a) = \frac{1}{2} F(t^*)(a) \). Since \( F(t^*) \) jumps at \( S \), either \( i_{-} \) or \( i_{+} \) jumps at \( S \). Both cases cannot happen at the same time because local minima of \( y \) are all distinct.

Suppose for example that \( is_{-} < is_{-} \). Take \( t \in (is_{-}, is_{-}) \) (see Figure 5). We have \( F(t)(a) = F(t^*)(a) \) for all \( a < S \) and

\[
F(t)(S) = x(is_{-}) - x(is_{-}) = x(is_{-}) - x(is_{-}) - (x(is_{+}) - x(is_{-})) = F(t^*)(S) - F(t^*)(S) = F(t^*)(S) - F(t^*)(S).
\]

\[\text{Figure 5: Construction of the locally largest excursion.}\]
We deduce that \( |F(t)(S)| = |F(t^*)(S^-) - F(t^*)(S^+)| > |F(t^*)(S)| = |F(t)(S^-) - F(t)(S)| \). Then \( A(t) \) is open in \([0, y(t)]\), contains \( S \), and we have \( y(t) > S \). Hence \( \sup A(t) > S \) which gives the desired contradiction.

**Uniqueness.** Suppose that \( S(t) = S(t') = S \) with \( t < t' \) and let us find again a contradiction. We showed that necessarily, \( y(t) = y(t') = S \). Let \( t_m \in [t, t'] \) such that \( y(t_m) = \min \{y(r), r \in [t, t']\} \). Set \( a_m := y(t_m) \). Observe that \( t \) and \( t' \) cannot be starting times or ending times of an excursion of \( y \) (otherwise we could have extended the locally largest fragment inside this excursion for some positive height). Hence \( a_m < S \). At level \( a_m \), there must be a splitting into two excursions (one straddling time \( t \), the other \( t' \)) with equal size. It happens on a negligible set under \( \mathbb{P}_r \). To see it, we can restrict to \( t < t' \) rationals and use the Markov property at time \( t' \).

\[
\square
\]

### 2.6 Disintegration of Itô’s measure over the size of the excursions

We are interested in conditioning Itô’s measure of excursions in \( \mathbb{H} \) on their initial size, i.e. in fixing the value of \( x(R(u)) = z \). This will allow us to define probability measures \( \gamma_z \) which disintegrate \( \mathbb{P}_+ \) over the value of the endpoint \( z \). Properties will simply transfer from \( \mathbb{P}_+ \) to \( \gamma_z \) via the disintegration formula. Define \( P_r^{a-b} \) as the law of the one-dimensional Brownian bridge of length \( r \) between \( a \) and \( b \), and \( \Pi_r \) as the law of a three-dimensional Bessel \((BES^3)\) bridge of length \( r \) from \( 0 \) to \( 0 \).

**Proposition 2.9.** We have the following disintegration formula

\[
\mathbb{P}_+ = \int_{\mathbb{R}} \frac{dz}{2\pi z^2} \gamma_z,
\]

where for \( z \neq 0 \),

\[
\gamma_z = \int_{\mathbb{R}^+} dv \frac{e^{-1/2v}}{2v^2} P_{vz^2}^{0-z} \otimes \Pi_{vz^2}.
\]

**Proof.** Let \( f \) and \( g \) be two nonnegative measurable functions defined on \( \mathcal{F} \) and \( \mathcal{F}_0 \) respectively. Thanks to Itô’s description of \( \mathbb{P}_+ \) (see [11], Chap. XII, Theorem 4.2), we have

\[
\int_U f(x)g(y) \, n_+(dx, dy) = \int_U f(x)g(y) \, n_+(dy) \mathbb{P}(X^R(y) \in dx)
\]

\[
= \int_{\mathbb{R}^+} \frac{dr}{2\pi r^3} \int_{\mathcal{F}} f(x) \, \Pi_r[g] \, \mathbb{P}(X^r \in dx).
\]

Now, decomposing on the value of the Gaussian r.v. \( X_r \) yields

\[
\int_U f(x)g(y) \, n_+(dx, dy) = \int_{\mathbb{R}^+} \frac{dr}{2\pi r^3} \int_{\mathbb{R}} \frac{dz}{\sqrt{2\pi r}} \Pi_r[g] \, E_r^{0-z} [f].
\]

We finally perform the change of variables \( v(r) = r/z^2 \) to get

\[
\int_U f(x)g(y) \, n_+(dx, dy) = \int_{\mathbb{R}} \frac{dz}{2\pi z^2} \int_{\mathbb{R}^+} \frac{dv}{2v^2} \frac{e^{-1/2v}}{2v^2} E_{vz^2}^{0-z} [f] \, \Pi_{vz^2}[g].
\]

\[
\square
\]

**Lemma 2.10.** Let \( z \) be a nonzero real number. The image measure of \( \gamma_z \) by the function which sends \((x, y)\) to

\[
\left( \frac{x(tz^2)}{z}, \frac{y(tz^2)}{z} \right), \quad 0 \leq t \leq \frac{R(u)}{z^2},
\]

is \( \gamma_1 \).

**Proof.** It comes from the definition of \( \gamma_z \) and the scaling property of \( BES^3 \) bridge and Brownian bridge.

\[
\square
\]
2.7 The metric space of excursions in $\mathbb{H}$

Very often, results under $\gamma_z$ can be obtained by proving the analog under the Itô's measure $\mathcal{I}(a)$, and then disintegrating over $z = x(R(u))$. This usually provides results under $\gamma_z$ for Lebesgue-almost every $z > 0$, and so we would like to study the continuity of $z \mapsto \gamma_z$. This requires to define a topology on the space of excursions $U^+$. All these results will be stated for $z > 0$ because the scaling depends on the sign of the endpoint (Lemma 2.10), but they all extend to the general case.

We therefore introduce the usual distance

$$d(u, v) = |R(u) - R(v)| + \sup_{t \geq 0} |u(t \wedge R(u)) - v(t \wedge R(v))|,$$

where we identified $\delta$ with the excursion with lifetime 0. The distance $d$ makes $U^+$ into a Polish space. The following lemmas may come in useful.

**Lemma 2.11.** The map $\Delta : u \in U^+ \mapsto \Delta u = x(R(u))$ is continuous.

**Proof.** This is straightforward since $|x(R(u)) - x'(R(u'))| = |u(R(u)) - u'(R(u'))| \leq d(u, u')$ for $u = (x, y)$ and $u' = (x', y')$. \hfill $\square$

**Lemma 2.12.** Let $u \in U^+$. Then $z \in \mathbb{R}_+^* \mapsto u^{(z)} := zu(\cdot/z^2) = (zu(t/z^2), 0 \leq t \leq R(u)z^2)$ is a continuous function.

**Proof.** Let $z_0 > 0$. Then for all $z > 0$

$$d(u^{(z)}, u^{(z_0)}) = R(u)|z^2 - z_0^2| + \sup_{t \geq 0} \left|zu\left(\frac{t}{z^2} \wedge R(u)\right) - z_0u\left(\frac{t}{z_0^2} \wedge R(u)\right)\right|.

The second term is

$$\sup_{t \geq 0} \left|zu\left(\frac{t}{z^2} \wedge R(u)\right) - z_0u\left(\frac{t}{z_0^2} \wedge R(u)\right)\right| \leq z \sup_{t \geq 0} \left|u\left(\frac{t}{z^2} \wedge R(u)\right) - u\left(\frac{t}{z_0^2} \wedge R(u)\right)\right| + \sup_{t \geq 0} \left|(z - z_0)u\left(\frac{t}{z_0^2} \wedge R(u)\right)\right|

\leq z \sup_{t \geq 0} \left|u\left(\frac{t}{z^2} \wedge R(u)\right) - u\left(\frac{t}{z_0^2} \wedge R(u)\right)\right| + |z - z_0| \sup_{t \geq 0} |u(t)|.

We conclude by using the uniform continuity of $u$. \hfill $\square$

If we equip the set $\mathcal{P}(U^+)$ of probability measures on $U^+$ with the topology of weak convergence, we have the following result.

**Proposition 2.13.** The map $z \in \mathbb{R}_+^* \mapsto \gamma_z$ is continuous.

**Proof.** Let $G$ be a continuous bounded function on $U^+$. Then by scaling (Lemma 2.10), for all $z > 0$,

$$\gamma_z(G) = \gamma_1\left[G(u^{(z)})\right].

Applying Lemma 2.12 together with the dominated convergence theorem yields the desired result. \hfill $\square$

Also, we will use the continuity of the excursions cut at horizontal levels. Recall from Section 2.3 that $\mathcal{I}(a)$ is the set of times when the excursion $u \in U^+$ lies above $a$, and for each connected component $I$ of $\mathcal{I}(a)$, $u_I$ denotes the associated excursion above $a$. The path $u_I$ is an excursion above $a$, $I$ is the time interval of $u_I$, and the size or length of $u_I$ is the difference between its endpoint and its starting point.
On \( \{ T_a < \infty \} \), we rank the excursions above \( a \) according to the absolute value of their size. Write \( z_1^{a,+} = z_2^{a,+}(u), z_2^{a,+} = z_2^{a,+}(u), \ldots \) for the sizes, ranked in descending order of their absolute value, and \( e_1^{a,+} = e_1^{a,+}(u), e_2^{a,+} = e_2^{a,+}(u), \ldots \) for the corresponding excursions. This is possible since for any fixed \( \varepsilon > 0 \) there are only finitely many excursions with length larger than \( \varepsilon \) in absolute value.

**Proposition 2.14.** Let \( a > 0 \) and \( z > 0 \). For any \( i \geq 1 \), the function \( e_i^{a,+} \) is continuous on \( U^+ \) on the event \( \{ T_a < \infty \} \) outside a \( \gamma_z \)-negligible set.

**Proof.** We consider the set \( \mathcal{E} \) of trajectories \( u = (x, y) \) such that \( T_a < \infty \) and satisfying the following conditions, which occur with \( \gamma_z \)-probability one when conditioned on touching \( a \): the level \( a \) is not a local minimum for \( y \), there exist infinitely many excursions above \( a \), all excursions touch \( a \) only at their starting point and endpoint, the sizes \( (z_i^{a,+}, i \geq 1) \) of the excursions are all distinct. Let \( i \geq 1 \) and \( u = (x, y) \in \mathcal{E} \).

We want to show that \( e_i^{(a,+)} \) is continuous at \( u \).

Let \( t \) be a time in the excursion \( e_i^{a,+} \), i.e. such that \( y(t) > a \) and \( e_a(t) = e_i^{a,+} \). We restrict our attention to \( u' = (x', y') \in \mathcal{E} \) close enough to \( u \) so that \( y'(t) > a \) and we will write \( e_n^{u(t)} \) for the excursion of \( u' \) corresponding to \( t \). Let \( \varepsilon > 0 \).

- **First**, we want to find \( \delta > 0 \) such that, whenever \( d(u, u') < \delta \), the durations of the excursions \( e_a(t) \) and \( e_n^{u(t)} \) are close, namely \( |R(e_a(t)) - R(e_n^{u(t)})| < \varepsilon \). Write \( (i_-(a), i_+(a)) \), and \( (i'_-(a), i'_+(a)) \), for the excursion time intervals corresponding to \( e_a(t) \) and \( e_n^{u(t)} \) respectively. For simplicity, we take the notation \( R = R(e_a(t)) \) and \( R' = R(e_n^{u(t)}) \). Since \( a \) is not a local minimum for \( y \), there exist times \( t_1 \in (i_-(a) - \delta, i_-(a)) \) and \( t_2 \in (i_+(a), i_+(a) + \delta) \) when \( y \) is strictly below \( a \). Take \( \delta_1 \in (0, a) \) such that \( y(t_1) \) and \( y(t_2) \) are in \( (0, a - \delta_1) \). Let \( u' = (x', y') \in \mathcal{E} \) such that \( d(u, u') < \frac{\delta_2}{2} \). We deduce that \( y'(t_1) < t_1 + \frac{\delta_2}{2} < a \) and similarly \( y'(t_2) < a \). This implies that \( i'_-(a) \geq t_1 > i_-(a) - \frac{\varepsilon}{2} \) and \( i'_+(a) \leq t_2 < i_+(a) + \frac{\varepsilon}{2} \). Likewise, pick two times \( t_3 \in (i_-(a), i_-(a) + \frac{\varepsilon}{2}) \) and \( t_4 \in (i_+(a) - \frac{\varepsilon}{2}, i_+(a)) \) such that \( t_3 < t < t_4 \). Since the excursion \( e_n^{u(t)} \) touches level \( a \) only at its extremities, the distance between the compact \( u[t_3, t_4] \) and the closed set \( \{ \exists z = a \} \) is positive, and so, on the interval \( [t_3, t_4] \), \( y \) remains above, say, \( a + \delta_2 \) where \( \delta_2 > 0 \). Then when \( d(u, u') < \frac{\delta_2}{2} \), the excursion \( e_n^{u(t)} \) will satisfy \( i'_-(a) < t_3 < i_-(a) + \frac{\varepsilon}{2} \) and \( i'_+(a) > t_4 > i_+(a) - \frac{\varepsilon}{2} \). Therefore, when \( d(u, u') < \delta = \min \left( \frac{\delta_1}{\varepsilon}, \frac{\delta_2}{2} \right) \), we get that \( |i'_-(a) - i_-(a)| < \frac{\varepsilon}{2} \) and \( |i'_+(a) - i_+(a)| < \frac{\varepsilon}{2} \), so in particular \( |R - R'| < \varepsilon \). Observe that we not only proved that the durations are close, but also that the times \( i_-, i'_- \) (and \( i_+, i'_+ \)) are close, and this will be useful in the remainder of the proof.

- **Secondly**, we show that we can take \( \delta' > 0 \) small enough so that

\[
\sup_{s \geq 0} |e_a^{u(t)}(s \wedge R) - e_n^{u(t)}(s \wedge R')| < \varepsilon,
\]

whenever \( d(u, u') < \delta' \).

Take \( \eta = \eta(\varepsilon) > 0 \) some modulus of uniform continuity of \( u \) with respect to \( \varepsilon \). The previous paragraph gives the existence of \( \delta > 0 \) such that when \( u' \in \mathcal{E} \) and \( d(u, u') < \delta \), \( |i'_-(a) - i_-(a)| < \eta/3 \) and \( |i'_+(a) - i_+(a)| < \eta/3 \). Without loss of generality, we can assume that \( \delta < \varepsilon \). Define \( \delta' = \min(\delta, \eta) \), and let \( u' \in \mathcal{E} \) such that \( d(u, u') < \delta' \). For all \( s \geq 0 \), we have

\[
|e_a^{u(t)}(s \wedge R) - e_n^{u(t)}(s \wedge R')| \\
= |u(i_-(a) + (s \wedge R)) - u(i_-(a)) - u'(i'_-(a) + (s \wedge R')) + u'(i'_-(a))| \\
\leq |u(i_-(a)) - u'(i'_-(a))| + |u(i_-(a) + (s \wedge R)) - u(i_-(a) + (s \wedge R'))|.
\]
Now, 
\[ |u(i_-(a)) - u'(i'_-(a))| \leq |u(i_-(a)) - u(i'_-(a))| + |u(i'_-(a)) - u'(i'_-(a))|, \]
and so by uniform continuity of \( u \) and because \( d(u, u') < \delta' < \varepsilon \), we obtain
\[ |u(i_-(a)) - u'(i'_-(a))| \leq 2\varepsilon. \tag{9} \]

Similarly, the second term of (8) is
\[
|u(i_-(a) + (s \wedge R)) - u'(i'_-(a) + (s \wedge R'))| \\
\leq |u(i_-(a) + (s \wedge R)) - u(i'_-(a) + (s \wedge R'))| \\
+ |u(i'_-(a) + (s \wedge R')) - u'(i'_-(a) + (s \wedge R'))|,
\]
and since \( |i_-(a) + (s \wedge R) - i'_-(a) - (s \wedge R')| < \eta \), we can conclude in the same way that
\[ |u(i_-(a) + (s \wedge R)) - u'(i'_-(a) + (s \wedge R'))| \leq 2\varepsilon. \tag{10} \]

Inequalities (8), (9) and (10) give
\[ |e_a^{(t)}(s \wedge R) - e_a^{(t)}(s \wedge R')| \leq 4\varepsilon, \]
which is the desired result.

So far, we proved that \( e_a^{(t)} \) is continuous at \( u \). To conclude, we need an argument to say that this is the \( i \)-th excursion above \( a \) for \( u' \) sufficiently close to \( u \).

- Finally, we show that we can take \( \delta'' > 0 \) small enough so that \( e_{i_0 + }^{(t)} = e_{a}^{(t)} \) whenever \( d(u, u') < \delta'' \).

This is derived in two steps.

- Step 1: Let \( \eta > 0 \), and introduce, for \( u' \in \mathcal{E} \), the number \( N_\eta(u') \) of time intervals \( i_-, i_+ \) of excursions of \( u' \) above \( a \) such that \( i_+ - i_- > \eta \). Note that \( N_\eta(u') \leq \frac{R(u')}{\eta} < \infty \). We take \( \eta \) such that \( u \) has no excursion time interval above \( a \) satisfying \( i_+ - i_- = \eta \). The first step consists in proving that for \( u' \in \mathcal{E} \) sufficiently close to \( u \), \( N_\eta(u') = N_\eta(u) \). From the first point (applied \( N_\eta(u) \) times), we know that for \( \delta > 0 \) small enough, \( N_\eta(u') \geq N_\eta(u) \) whenever \( d(u, u') < \delta \). To prove that \( N_\eta(u') \leq N_\eta(u) \) holds as well when \( \delta \) is sufficiently small, we use an argument by contradiction and we consider a sequence \( (u_n)_{n \geq 1} \) of elements in \( \mathcal{E} \) such that \( d(u, u_n) \rightarrow 0 \) and \( N_\eta(u_n) \geq N_\eta(u) + 1 \). Consider \( N_\eta(u) + 1 \) distinct excursion time intervals \( (i_j^{(n)}) \), \( 1 \leq j \leq N_\eta(u) + 1 \), of \( u_n \) above \( a \) such that \( i_{j,+}^{(n)} - i_{j,-}^{(n)} > \eta \). We can write the corresponding excursions \( e_{a, t_j^{(n)}}^{(n)}(u_n) \) for some \( t_j^{(n)} \)'s. Moreover, we may take \( t_j^{(n)} \) such that \( |t_{j,+}^{(n)} - t_{j,-}^{(n)}| > \eta/2 \) and \( |i_{j,+}^{(n)} - i_{j,-}^{(n)}| > \eta/2 \). Since \( |R(u) - R(u_n)| \rightarrow 0 \), we can assume (up to some extraction) that when \( n \) goes to infinity, \( i_{j,+}^{(n)} \rightarrow i_{j,+}, i_{j,-}^{(n)} \rightarrow i_{j,-} \) and \( t_j^{(n)} \rightarrow t_j \) in \( [0, R(u)] \), for some \( i_{j,+}, i_{j,-}, t_j \in [0, R(u)] \). From \( u_n \rightarrow u \), we deduce that for all \( j \), \( y(i_{j,-}) = a \) and \( y(i_{j,+}) = a \). For \( n \) large enough, because \( i_{j,+}^{(n)} - i_{j,-}^{(n)} > \eta \) and \( |i_{j,+}^{(n)} - i_{j,-}^{(n)}| > \eta/2 \), we have \( e_{a, t_{j,+}^{(n)}}^{(n)}(u_n) = e_{a, t_{j,-}^{(n)}}^{(n)}(u_n) \). Now consider \( e_{a, t_j}^{(n)}(u_n) \). From the two previous points, \( e_{a, t_j}^{(n)}(u_n) \rightarrow e_{a, t_j}^{(n)}(u) \). For any time \( s \in (i_-, i_+), \) we have \( y(s) > a \) (otherwise \( a \) would be a local minimum of \( y \)). Hence \( (i_{j,-}, i_{j,+}) \) is an excursion time interval for \( u \) and \( i_{j,+} - i_{j,-} > \eta \). Therefore we constructed \( N_\eta(u) + 1 \) distinct excursion time intervals above \( a \) for \( u \), which gives the desired contradiction.
- Step 2: Suppose for example that $z_i^{a,+} > 0$. Take $\delta < \frac{z_i^{a,+}}{6}$ and $\eta = \eta(\delta) > 0$ some modulus of uniform continuity for $u$ with respect to $\delta$. We can assume in order to apply Step 1 that $\eta$ is such that $u$ has no excursion above $a$ satisfying $|i_+ - i_-| = \eta$. We look at the $N := N_i(u)$ excursions $e_1, \ldots, e_N$ of $u$ above $a$ (ranked by decreasing order of the absolute value of their sizes) such that $|i_+ - i_-| > \eta$, and denote their sizes by $z_1, \ldots, z_N$. Observe that the first $i$ excursions among these are the excursions $e_i^{a,+}, \ldots, e_i^{a,+}$. Indeed, if $|i_+ - i_-| \leq \eta$, then by uniform continuity,

$$|u(i_+) - u(i_-)| \leq \delta < z_i^{a,+}.$$ 

Let $\varepsilon' = \frac{1}{2} (\min_{1 \leq k \leq N-1} |z_{k+1} - z_k| \wedge z_i)$ (this is positive since all the sizes are assumed to be distinct in $\mathcal{E}$). Take times $t_1, \ldots, t_N$ in the excursion time intervals of $e_1, \ldots, e_N$. Thanks to Step 1 and the first point of the proof (applied $N$ times), there exists $\delta' > 0$ such that for $d(u, u') < \delta'$, if we denote by $(i_+(t), i_-(t), e_i^{(t)})$ the excursion time interval of $e_i^{(t)}$, $1 \leq k \leq N$, then

(i) $N_i(u') = N$,

(ii) the excursions $e_a^{(t)}(i), 1 \leq i \leq N$, are distinct,

(iii) $\forall 1 \leq i \leq N$, $|i_+(t) - i^{(t)}_i| > \eta$, $\forall 1 \leq i \leq N$, $|z_k^{a,+} - \Delta e_a^{(t)}| \leq \varepsilon'$.

An easy calculation shows that by our choice of $\varepsilon'$ and (iv), the $\Delta e_a^{(t)}, 1 \leq k \leq N$, are ranked in decreasing order, and that

$$\forall 1 \leq k \leq N, \quad \Delta e_a^{(t)} > \frac{z_k^{a,+}}{2}. \quad (11)$$

In addition, by (i), (ii) and (iii), the $e_a^{(t)}(i), 1 \leq i \leq N$, are the excursions of $u'$ above $a$ satisfying $|i_+ - i_-| > \eta$.

Now set $\delta'' = \min(\delta, \delta')$ and assume that $d(u, u') < \delta''$. Then for all $1 \leq k \leq i$, $e_a^{(t)}(i) = e_k^{a,+}(u')$. Indeed, if $(i_+, i_-)$ is an excursion time interval of $u'$ such that $|i_+ - i_-| \leq \eta$, then

$$|u'(i_+) - u'(i_-)| \leq |u'(i_+) - u(i_-)| + |u(i_+ - u(i_-)| + |u(i_-) - u'(i_-)| \leq 3\delta,$$

and so in particular $|u'(i_+) - u'(i_-)| < \frac{z_i^{a,+}}{2}$. This proves that the first $i$ excursions of $u'$ are among the $N$ previous excursions satisfying $|i_+ - i_-| > \eta$. Since these are ranked in decreasing order, necessarily $e_a^{(t)} = e_k^{a,+}(u')$ for all $1 \leq k \leq i$, which concludes the proof.

Putting these three points together, we proved that $e_i^{a,+}$ is continuous on $\delta$ which has full probability under $\gamma_z$, hence Proposition 2.14.

3 Markovian properties

In this section, we are interested in Markovian properties of excursions cut at horizontal levels. Time will therefore be indexed by the height $a$ of the cutting.
3.1 The branching property for excursions in $\mathbb{H}$

Consider an excursion under the measure $\gamma_z$. Then cutting it at some height $a > 0$ yields a family of excursions above $a$ as defined in Section 2.3. Our aim is to show that conditionally on what happens below $a$, these are independent and distributed according to the measures $\gamma_z$, where $z$ is the size of the corresponding excursion. We shall first consider the case when the original excursion is taken under the Itô’s measure $n_+ +$ in $\mathbb{H}$, and then transfer the property to $\gamma_z$ by the previous disintegration result (6).

Let $G_a^0$ be the $\sigma$–field containing all the information of the trajectory below level $a$ and $G_a$ be the completion of $G_a^0$ with the $n_+$-negligible sets. In other words, the $\sigma$–field $G_a^0$ is generated by the trajectory $u$ once you cut out the excursions above $a$, and close up the time gaps. A formal definition of this process is the process $u$ indexed by the generalized inverse of $t \mapsto \int_0^t 1_{\{u(s) \leq a\}} \, ds$.

![Figure 6: The excursion process above $a$.](image)

Recall from Section 2.7 that $z_1^{a,+}, z_2^{a,+}, \ldots$ are the sizes of the excursions above $a$, ranked in decreasing order of their absolute value, and $e_1^{a,+}, e_2^{a,+}, \ldots$ are the corresponding excursions.

**Proposition 3.1.** (*Branching property for excursions in $\mathbb{H}$ under $n_+$*)

For any $A \in G_a$, and for all nonnegative measurable functions $G_1, \ldots, G_k : U^+ \to \mathbb{R}_+$, $k \geq 1$,

$$n_+ \left( \mathbb{1}_{\{T_a < \infty\}} \mathbb{1}_A \prod_{i=1}^k G_i(e_i^{a,+}) \right) = n_+ \left( \mathbb{1}_{\{T_a < \infty\}} \mathbb{1}_A \prod_{i=1}^k \gamma_{z_i^{a,+}}[G_i] \right). \quad (12)$$

**Proof.** Lemma 2.4 ensures that on the event $\{T_a < \infty\}$, the trajectory $u$ after time $T_a$ has the law of a killed Brownian motion. Excursion theory tells us that given the excursions below $a$, the excursions above $a$ form a Poisson point process on $U^+$ with intensity $L n_+ (du)$, where $L$ is the total local time at level $a$, see Figure 6. Finally, conditionally on the sizes $(z_i^{a,+})_{i \geq 1}$ of the excursions above $a$, these excursions are independent with law $\gamma_{z_i^{a,+}}$. We deduce the proposition since the $\sigma$-field $G_a$ is generated by $\mathcal{F}_{T_a}$, the excursions below $a$, and the sizes $(z_i^{a,+})_{i \geq 1}$.

We can now transfer this property to the probability measures $\gamma_z$. 

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Proposition 3.2. (Branching property for excursions in \( H \) under \( \gamma_z \))

Let \( z \in \mathbb{R} \setminus \{0\} \). For any \( A \in G_a \), and for all nonnegative measurable functions \( G_1, \ldots, G_k : U^+ \to \mathbb{R}_+ \), \( k \geq 1 \),

\[
\gamma_z \left( \mathbb{1}_{\{T_n < \infty\}} \mathbb{1}_A \prod_{i=1}^{k} G_i(e_i^{a,+}) \right) = \gamma_z \left( \mathbb{1}_{\{T_n < \infty\}} \mathbb{1}_A \prod_{i=1}^{k} \gamma_z^{a,+}[G_i] \right).
\]

Proof. It suffices to prove the proposition for bounded continuous functions \( G_1, \ldots, G_k : U^+ \to \mathbb{R}_+ \), \( k \geq 1 \). Take a nonnegative measurable function \( f : \mathbb{R} \to \mathbb{R}_+ \) and a bounded continuous function \( h : U^+ \to \mathbb{R}_+ \) which is \( G_a \)-measurable. Observe that \( x(R(u)) \) is \( G_a \)-measurable as a function of \( u \). From Proposition 3.1, we know that

\[
n_+ \left( \mathbb{1}_{\{T_n < \infty\}} h(u)f(x(R(u))) \prod_{i=1}^{k} G_i(e_i^{a,+}) \right) = n_+ \left( \mathbb{1}_{\{T_n < \infty\}} h(u)f(x(R(u))) \prod_{i=1}^{k} \gamma_z^{a,+}[G_i] \right).
\]

Thanks to the disintegration formula (6), we can split \( n_+ \) over the size:

\[
\int_{\mathbb{R}} \frac{dz}{2\pi z^2} f(z) \gamma_z \left( \mathbb{1}_{\{T_n < \infty\}} h \prod_{i=1}^{k} G_i(e_i^{a,+}) \right) = \int_{\mathbb{R}} \frac{dz}{2\pi z^2} f(z) \gamma_z \left( \mathbb{1}_{\{T_n < \infty\}} h \prod_{i=1}^{k} \gamma_z^{a,+}[G_i] \right).
\]

Since this holds for any \( f \), it entails for Lebesgue-almost every \( z \in \mathbb{R} \),

\[
\gamma_z \left( \mathbb{1}_{\{T_n < \infty\}} h \prod_{i=1}^{k} G_i(e_i^{a,+}) \right) = \gamma_z \left( \mathbb{1}_{\{T_n < \infty\}} h \prod_{i=1}^{k} \gamma_z^{a,+}[G_i] \right). \tag{13}
\]

To prove that this holds for all \( z \), we need a continuity argument. We first treat the case \( z = 1 \). Using the scaling property 2.10 of the measures \( \gamma_z \), for \( z > 0 \) the left-hand side of (13) is

\[
\gamma_1 \left( \mathbb{1}_{\{T_n, z < \infty\}} h(u^{(z)}) \prod_{i=1}^{k} G_i(e_i^{a,+}(u^{(z)})) \right)
\]

where we recall from Lemma 2.12 that \( u^{(z)} = z u(\cdot / z^2) \). The right-hand side term, on the other hand, is

\[
\gamma_z \left( \mathbb{1}_{\{T_n, z < \infty\}} \prod_{i=1}^{k} \gamma_1^{a,+}[G_i] \right) = \gamma_1 \left( \mathbb{1}_{\{T_n, z < \infty\}} \prod_{i=1}^{k} \gamma_1^{a,+}(u^{(z)})[G_i] \right),
\]

and so (13) translates into

\[
\gamma_1 \left( \mathbb{1}_{\{T_n, z < \infty\}} h(u^{(z)}) \prod_{i=1}^{k} G_i(e_i^{a,+}(u^{(z)})) \right) = \gamma_1 \left( \mathbb{1}_{\{T_n, z < \infty\}} h(u^{(z)}) \prod_{i=1}^{k} \gamma_z^{a,+}(u^{(z)})[G_i] \right). \tag{14}
\]

for Lebesgue-almost every \( z > 0 \). In particular this is true for a dense set of \( z \). Taking \( z \searrow 1 \) along some decreasing sequence, we first get that \( u^{(z)} \to u \) by Lemma 2.12 and \( T_{n, z} \to T_n \) by left-continuity of the stopping times. In addition, for all \( 1 \leq i \leq k \), \( z_i^{a,+}(u^{(z)}) \to z_i^{a,+}(u) \) \( \gamma_1 \)-almost surely because \( z \to z_i^{a,+}(u^{(z)}) = \Delta e_i^{a,+}(u^{(z)}) \) is a continuous function (outside a negligible set) by Lemmas 2.11, 2.12 and Proposition 2.14. Finally, by continuity of \( z \to \gamma_z \) (Lemma 2.13), for all \( 1 \leq i \leq k \), \( \gamma_z^{a,+}(u^{(z)})[G_i] \to \gamma_z^{a,+}[G_i] \). Applying the dominated convergence theorem to both sides of equation (14) triggers

\[
\gamma_1 \left( \mathbb{1}_{\{T_n < \infty\}} \prod_{i=1}^{k} G_i(e_i^{a,+}) \right) = \gamma_1 \left( \mathbb{1}_{\{T_n < \infty\}} \prod_{i=1}^{k} \gamma_z^{a,+}[G_i] \right).
\]

and concludes the proof of Proposition 3.2 for \( z = 1 \). The general case follows by scaling.
3.2 The locally largest evolution

Recall that Proposition 2.8 gives a canonical choice of excursion at level $a > 0$, which is the locally largest excursion $e_a^{(\star)}$. One may wonder whether the locally largest fragment $\Xi(a) = \Delta e_a^{(\star)}$ still exhibits some kind of Markovian behavior. The following theorem answers this question.

**Theorem 3.3.** Let $z > 0$. Under $\gamma_z$, $(\Xi(a))_{0 \leq a < \zeta}$ is distributed as the positive self-similar Markov process $(Z_a)_{0 \leq a < \zeta}$ with index 1 starting from $z$ whose Lamperti representation is

$$Z_a = z \exp(\xi(\tau^{-1}a)),$$

where $\xi$ is the Lévy process with Laplace exponent $\Psi(q) := \log \gamma_z[e^{q\xi(1)}]$ given by

$$\Psi(q) = -\frac{4}{\pi}q + \frac{2}{\pi} \int_{y > -\ln(2)} (e^{qy} - 1 - q(e^{y} - 1)) \frac{e^{-y}dy}{(e^{y} - 1)^2}, \quad q < 3,$$

$\tau$ is the time change

$$\tau(a) = \inf \left\{ s \geq 0, \int_0^s e^{\xi(u)} du > a \right\},$$

and $\zeta = \inf \{a \geq 0, Z_a = 0\}$.

Recall the notation (2)-(5). We set

$$u_a^{t,=} := \left( (u^{t,=}^1(s + T_{a,b}^{t,=}), s \geq 0), (u^{t,=}^2(s + T_{a,b}^{t,=}), s \geq 0) \right) - \left( u^{t,=}^1(T_{a,b}^{t,=}), u^{t,=}^2(T_{a,b}^{t,=}) \right),$$

with the convention that $u_a^{t,=}$ is a cemetery function if $y(t) < a$.

We shall use the following lemma. Note that the lemma does not disintegrate the law of $u_a^{t,=}$ on the measures $\gamma_z$, and one has to be careful not to confuse the $z$ appearing in the integral with the value of $x(R(u))$ (the reader should keep track of $x(R(u))$ in the proof).

**Lemma 3.4.** Let $(X,Y)$ and $(X',Y')$ be under $\mathbb{P}$ two independent planar Brownian motions starting from the origin, and for $b \leq 0$, $T_b$ and $T_b'$ their respective hitting times of $\{\Im(z) = b\}$, with $\overline{T}_b$ and $\overline{T}_b'$ denoting the hitting times of $\{\Im(z) < b\}$. For $a \geq a > 0$, and $z \in \mathbb{R}$ we set

$$E_{a,a,z} := \left\{ \left| z + (X'_{T_{b}'} - X_{T_{b}}) \right| \geq \left| (X'_{T_{b}'} - X_{T_{b}}) - (X'_{T_{b}'} - X_{T_{b}}) \right|, \forall b \in [-a,-a + a] \right\}.$$

Then, for any nonnegative measurable function $H$,

$$n_+[H(u_a^{t,=}) 1_{\{a < \zeta^{(\star)}\}}] = \int_z \frac{dz}{2\pi z^2} h(-a, z),$$

where $h$ is :

$$h(-a, z) := \mathbb{E} \left[ H \left( (X_s, Y_s)_{s \in [0,T_{-a}]} \right), (z + X'_{s}, Y'_s)_{s \in [0,T_{-a}')} \right], E_{a,a,z}.$$

**Remark.** Observe that the process $(\Xi(a'))$, $a' \leq a$ is measurable with respect to $u_a^{t,=}$. We denote by $\mathbb{D}$ the space of càdlàg real-valued paths with finite lifetime, endowed with the local Skorokhod topology. It results from the lemma that for any nonnegative measurable function on $\mathbb{D}$,

$$n_+[H(\Xi(b), b \in [0,a]) 1_{\{a < \zeta^{(\star)}\}}] = \int_z \frac{dz}{2\pi z^2} h(-a, z),$$

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where \( h \) is now:

\[
h(-a, z) := \mathbb{E} \left[ H \left( z + X'_{T_{-a}^+} - X_{T_{-a}^+}, b \in [0, a] \right), \mathcal{E}_{a,a,z} \right].
\]

**Proof.** Integrating over the duration of the excursion \( e_a^{(\ast)} \), we see that for \( n_+ \)-almost every \( u \in U^+ \),

\[
H(u_a^{\ast \ast}) \mathbb{I}_{\{a < \Im (z^\ast)\}} = \int_0^{R(u)} H(u_a^{\ast \ast}) \mathbb{I}_{\{g(t) > a, e_a^{(\ast \ast)} = e_a^{(u)}\}} \frac{1}{R(e_a^{(u)})} dt.
\]

With Bismut’s description of \( n_+ \) (Proposition 2.6, cf. Figure 3), we get

\[
n_+ [H(u_a^{\ast \ast}) \mathbb{I}_{\{a < \Im (z^\ast)\}}] = \int_{a > a} \alpha \mathbb{E} \left[ \frac{H \left( \left( (X, Y), (X', Y') \right) \right)}{T_{a + T_{-a}^+} + T_{-a}^+}, \mathcal{E}_{a,a,0} \right],
\]

where

\[
\begin{align*}
(X, Y) &= ((X, Y)(s + T_{-a}^+))_{s \in [0, T_{-a}^+ - T_{-a}^+] - (X, Y)(T_{-a}^+),} \\
(X', Y') &= ((X', Y')(s + T'_{-a}^+))_{s \in [0, T_{-a}^+ - T_{-a}^+] - (X, Y)(T_{-a}^+),}
\end{align*}
\]

with the notation \((X, Y)(s) = (X_s, Y_s).\) By the strong Markov property at times \( T_{-a}^+ \) and \( T'_{-a}^+, \) the former integral can be expressed as

\[
\int_{a > a} \alpha \mathbb{E} \left[ h(-a, X'_{T_{-a}^+} - X_{T_{-a}^+}) \frac{1}{T_{-a}^+ + T_{-a}^+} \right],
\]

for \( h \) defined as

\[
h(-a, z) := \mathbb{E} \left[ H \left( (X_s, Y_s)_{s \in [0, T_{-a}^+]}, (z + X'_s, Y'_s)_{s \in [0, T'_{-a}^+]}, \mathcal{E}_{a,a,z} \right) .
\]

See Figure 3. By a change of variables, the former integral is

\[
\int_{a > a} \alpha \mathbb{E} \left[ h(-a, X'_{T_{-a}^+} - X_{T_{-a}^+}) \frac{1}{T_{-a}^+ + T_{-a}^+} \right].
\]

Therefore, we proved that

\[
n_+ [H(u_a^{\ast \ast}) \mathbb{I}_{\{a < \Im (z^\ast)\}}] = \int_{a > a} \alpha \mathbb{E} \left[ h(-a, X'_{T_{-a}^+} - X_{T_{-a}^+}) \frac{1}{T_{-a}^+ + T_{-a}^+} \right].
\]

On the other hand, using again Bismut’s decomposition of \( n_+ \), we see that (actually for any \( h \),

\[
n_+ [h(-a, x(R(u)))] = n_+ \left( \int_0^{R(u)} h(-a, x(R(u)) \frac{1}{R(u)} dt \right)
\]

\[
= \int_{a > a} \alpha \mathbb{E} \left[ h(-a, X'_{T_{-a}^+} - X_{T_{-a}^+}) \frac{1}{T_{-a}^+ + T_{-a}^+} \right].
\]

Comparing the last two equations, we proved that

\[
n_+ [H(u_a^{\ast \ast}) \mathbb{I}_{\{a < \Im (z^\ast)\}}] = n_+ [h(-a, x(R(u)))] = \int_{\frac{d}{2\pi z^2}} h(-a, z),
\]

by Proposition 2.9.
We now come to the proof of Theorem 3.3. We closely follow the strategy of Le Gall and Riera in [10].

**Proof.** Let $H$ be a nonnegative bounded continuous function on $\mathbb{D}$. From the previous Lemma 3.4, or rather from the Remark following its statement, we know that

$$n_+[H(\Xi(b), b \in [0,a])1_{\{a<\Xi(x^*)\}}] = \int_\mathbb{R} \frac{dz}{2\pi z^2} h(-a,z),$$

where $h$ is:

$$h(-a,z) = E \left[ H \left( z + X^x_{T_{a+b}} - X^x_{T_{a+b}}, b \in [0,a] \right), \mathcal{E}_{a,a,z} \right].$$

Notice that, in the notation of Lemma 3.4, $b \mapsto X^x_{T_{a+b}} - X^x_{T_{a+b}}$ is a (càdlàg) symmetric Cauchy process of Laplace exponent $\psi(\lambda) = -2|\lambda|$ (for example, use that it is a Lévy process and Proposition 3.11 of [11], Chap. II). Denote by $\eta_b$ the double of the Cauchy process which under $P_z$, starts from $z$, and $\Delta \eta_b$ the jump at time $b$. Write $\hat{\eta}_b = \eta(a-b)-$ for the time-reversal of $\eta$, $\Delta \hat{\eta}_b$ being the jump of $\hat{\eta}$ at time $b$. Then by definition of $h$,

$$h(-a,z) = E_z \left[ H(\hat{\eta}_b, b \in [0,a])1_{\{\forall b \in [0,a], |\eta_b| \geq |\Delta \eta_b|\}} \right].$$

Now we want to reverse time in the function $h$. Conditioning on $\eta_a$,

$$h(-a,z) = \frac{1}{\pi} \int_\mathbb{R} \frac{2adx}{(2a)^2 + (x-z)^2} E_x[H(\hat{\eta}_b, b \in [0,a])1_{\{\forall b \in [0,a], |\eta_b| \geq |\Delta \eta_b|\}} | \eta_a = x].$$

By Corollary 3, Chap. II of [2]:

$$E_x[H(\hat{\eta}_b, b \in [0,a])1_{\{\forall b \in [0,a], |\eta_b| \geq |\Delta \eta_b|\}} | \eta_a = x] = E_x[H(\eta_b, b \in [0,a])1_{\{\forall b \in [0,a], |\eta_b| \geq |\Delta \eta_b|\}} | \eta_a = z].$$

Indeed, the Cauchy process $\eta$ is symmetric, hence is itself its dual. We obtain

$$\int_\mathbb{R} \frac{dz}{2\pi z^2} h(-a,z) = \int_\mathbb{R} \frac{dx}{2\pi x^2} \frac{1}{\pi} \int_\mathbb{R} \frac{2adz}{(2a)^2 + (x-z)^2} E_x[H(\eta_b, b \in [0,a])1_{\{\forall b \in [0,a], |\eta_b| \geq |\Delta \eta_b|\}} | \eta_a = z].$$

We can rewrite it as

$$\int_\mathbb{R} \frac{dx}{2\pi x^2} \frac{1}{\pi} \int_\mathbb{R} \frac{2adz}{(2a)^2 + (x-z)^2} E_x \left[ \frac{x^2}{2} H(\eta_b, b \in [0,a])1_{\{\forall b \in [0,a], |\eta_b| \geq |\Delta \eta_b|\}} | \eta_a = z \right],$$

which is

$$\int_\mathbb{R} \frac{dx}{2\pi x^2} E_x \left[ \frac{x^2}{2} H(\eta_b, b \in [0,a])1_{\{\forall b \in [0,a], |\eta_b| \geq |\Delta \eta_b|\}} \right].$$

Now this gives the law of $\Xi$ under the disintegration measures $\gamma_x$. Indeed, take instead of $H$ some nonnegative measurable function $f$ of the initial size $\Xi(0)$, multiplied by $H$. Then using the above expression, we find that

$$n_+[f(\Xi(0))H(\Xi(b), b \in [0,a])1_{\{a<\Xi(x^*)\}}] = \int_\mathbb{R} \frac{dx}{2\pi x^2} f(x) E_x \left[ \frac{x^2}{(\eta_b)^2} H(\eta_b, b \in [0,a])1_{\{\forall b \in [0,a], |\eta_b| \geq |\Delta \eta_b|\}} \right].$$
Hence for Lebesgue-almost every $x \in \mathbb{R}$,

$$\gamma_x[H(\Xi(b), b \in [0, a])1_{\{a < \Xi(x \bullet^*)\}}] = E_x \left[ \frac{x^2}{(\eta_b)^2} H(\eta_b, b \in [0, a])1_{\{\forall b \in [0, a], |\eta_b| \geq |\Delta \eta_b|\}} \right],$$

(16)

and by continuity this must hold for all $x \in \mathbb{R}$. Indeed, by scaling, the left-hand side of equation (16) is

$$\gamma_x[H(\Xi(b), b \in [0, a])1_{\{a < \Xi(x \bullet^*)\}}] = \gamma_1[H(x\Xi(x^{-1}b), b \in [0, a])1_{\{a < x\Xi(x \bullet^*)\}}].$$

The right-hand term can be put in the same form by using the scale invariance of the Cauchy process. Since (16) holds for almost every $x$, it must hold on a dense set of $x > 0$, and we may take $x \nearrow 1$ along a sequence. By dominated convergence, we get

$$\gamma_1[H(\Xi(b), b \in [0, a])1_{\{a < \Xi(x \bullet^*)\}}] = E_1 \left[ \frac{1}{(\eta_b)^2} H(\eta_b, b \in [0, a])1_{\{\forall b \in [0, a], |\eta_b| \geq |\Delta \eta_b|\}} \right],$$

and this proves that equation (16) holds for $x = 1$. The general case $x \in \mathbb{R}$ follows by scaling.

Notice that, almost surely, on the event $\{\forall b \in [0, a], |\eta_b| \geq |\Delta \eta_b|\}$, if $\eta_0 > 0$, then $\eta_b$ is positive for all $b \in [0, a]$. We know from [6] that a symmetric Cauchy process starting from $x > 0$ killed when entering the negative half-line can be written using its Lamperti representation as $xe^{\xi^0(\tau^0(a))}$ where

$$\tau^0(a) := \int_0^a \frac{ds}{\eta_s} = \inf \left\{ s \geq 0, \int_0^s x e^{\xi^0(u)}du \geq a \right\},$$

and $(\xi^0(a), a \geq 0)$ is under $P$ a Lévy process killed at an exponential time of parameter $\frac{q}{\pi}$, starting from 0 with Laplace exponent

$$\Psi^0(q) = \frac{2}{\pi} \int_{\mathbb{R}} (e^{qy} - 1 - q(e^y - 1))1_{|e^y - 1| < 1}e^y(e^y - 1)^{-2}dy - \frac{2}{\pi}, -1 < q < 1. \quad (17)$$

Let $\Delta \xi^0_b$ denote the jump of $\xi^0$ at time $b$, i.e. $\Delta \xi^0_b := \xi^0_b - \xi^0_{b-}$. The following lemma is the analog of Lemma 17 in [10].

**Lemma 3.5.** For every $a \geq 0$, set

$$M_a = e^{-2\xi^0_a}1_{\{\forall b \in [0, a], \Delta \xi^0_b > -\ln(2)\}}.$$ 

Then $(M_a)_{a \geq 0}$ is a martingale with respect to the canonical filtration of the process $\xi^0$. Under the tilted probability measure $e^{-2\xi^0_a}1_{\{\forall b \in [0, a], \Delta \xi^0_b > -\ln(2)\}} \cdot P$, the process $(\xi^0(b))_{b \in [0,a]}$ is a Lévy process with Laplace exponent $\Psi$ introduced in (15) in Theorem 3.3.

**Proof.** We compute

$$E[e^{(q-2)\xi^0_a}1_{\{\forall b \in [0, a], \Delta \xi^0_b > -\ln(2)\}}].$$

Indeed, that $(M_a)_{a \geq 0}$ is a martingale will come from the fact that $\xi^0$ is a Lévy process and that the expectation above is 1 when $q = 0$. To compute this expectation, we decompose $\xi^0$ into its small and large jumps parts:

$$\xi^0_a = \xi^0_a + \xi^0_a,$$
where \( \xi'' = \sum_{0 \leq b \leq a} \Delta \xi''_b \mathbb{1}_{\{\forall b \in [0,a], \Delta \xi''_b \leq -\ln(2)\}} \). Notice that and \( \xi' \) and \( \xi'' \) are independent. Then by independence, the above expectation is
\[
E[e^{(q-2)\xi''_a} \mathbb{1}_{\{\forall b \in [0,a], \Delta \xi''_b \leq -\ln(2)\}}] = E[E[\mathbb{1}_{\{\xi''_a = 0\}} e^{(q-2)\xi''_a} | \xi''_a = 0]] = P(\xi''_a = 0)E[e^{(q-2)\xi''_a}] = P(\xi''_a = 0)E[e^{(q-2)\xi''_a}] .
\]
Thus, we need to compute the Laplace exponents of \( \xi' \) and \( \xi'' \) (under \( P \)), that we denote respectively by \( \Psi' \) and \( \Psi'' \). Because \( \xi'' \) is the pure-jump process given by the jumps of \( \xi' \) smaller than \(-\ln(2)\), its Laplace exponent is given by the Lévy measure of \( \xi' \) restricted to \((-\infty, -\ln(2))\), namely
\[
\Psi''(q) = \frac{2}{\pi} \int_{y \leq -\ln(2)} (e^{qy} - 1)^2 \frac{e^y}{(e^y - 1)^2} dy .
\]
It results from the independence of \( \xi' \) and \( \xi'' \) that the Laplace exponent of \( \xi' \) is \( \Psi' = \Psi' - \Psi'' \), hence by equations (17) and (19), for all \(-1 < q < 1\),
\[
\Psi'(q) = \frac{2}{\pi} \int_{y > -\ln(2)} (e^{qy} - 1 - q(e^y - 1)\mathbb{1}_{\{e^y < 1\}}) e^y (e^y - 1)^{-2} dy - \frac{2}{\pi} \int_{y \leq -\ln(2)} (e^y - 1)\mathbb{1}_{\{e^y < 1\}} e^y (e^y - 1)^{-2} dy - \frac{2}{\pi} .
\]
The middle term in this expression (20) is
\[
\frac{2}{\pi} \int_{y \leq -\ln(2)} (e^{qy} - 1) e^y (e^y - 1)^{-2} dy = \frac{2}{\pi} q \int_{y \leq -\ln(2)} \frac{e^y}{1 - e^y} dy
\]
\[
= \frac{2}{\pi} q \int_{0}^{1/2} \frac{dx}{1 - x}
\]
\[
= \frac{2}{\pi} q \ln(2).
\]
Hence
\[
\Psi'(q) = \frac{2}{\pi} \int_{y > -\ln(2)} (e^{qy} - 1 - q(e^y - 1)\mathbb{1}_{\{e^y < 1\}}) e^y (e^y - 1)^{-2} dy + \frac{2}{\pi} q \ln(2) - \frac{2}{\pi}.
\]
This extends analytically to all \( q < 1 \). Let us come back to (18). We have for \( q < 3 \)
\[
E[e^{(q-2)\xi''_a} \mathbb{1}_{\{\forall b \in [0,a], \Delta \xi''_b \geq -\ln(2)\}}] = P(\xi''_a = 0)E[e^{(q-2)\xi''_a}]
\]
\[
= e^{q\Psi''(\infty)} e^{q\Psi'(q-2)}
\]
\[
= \exp\left(\frac{-2}{\pi} q \int_{y \leq -\ln(2)} \frac{e^y}{(e^y - 1)^2} dy\right) e^{q\Psi'(q-2)}
\]
\[
= e^{q(\Psi'(q-2) - \frac{2}{\pi})},
\]
by a change of variables.

This essentially concludes the calculation of the new Laplace exponent \( \tilde{\Psi} \) of \( \xi' \) under the tilted measure \( e^{-2\xi''_a} \mathbb{1}_{\{\forall b \in [0,a], \Delta \xi''_b \geq -\ln(2)\}} \cdot P \), which is simply
\[
\tilde{\Psi}(q) = \Psi'(q - 2) - \frac{2}{\pi}, \quad q < 3.
\]
Still we can put it in a Lévy-Khintchin form. Replacing \( q \) by \( q - 2 \) in the integral in (21), we get
\[
\int_{y > - \ln(2)} (e^{-2y}e^{qy} - 1 - (q - 2)(e^y - 1)\mathbb{1}_{|e^y - 1| < 1})e^y(e^y - 1)^{-2}dy
\]
\[
= \int_{y > - \ln(2)} (e^{qy} - e^{2y} - (q - 2)(e^{3y} - e^{2y})\mathbb{1}_{|e^y - 1| < 1})e^{qy}(e^y - 1)^{-2}dy
\]
\[
= \int_{y > - \ln(2)} (e^{qy} - 1 - q(e^y - 1)\mathbb{1}_{|e^y - 1| < 1})e^{-y}(e^y - 1)^{-2}dy
\]
\[
+ \int_{y > - \ln(2)} \left[ 1 - e^{2y} + (q(e^y - 1) - (q - 2)(e^{3y} - e^{2y})e^{-y}(e^y - 1)^{-2}\mathbb{1}_{|e^y - 1| < 1}\right] \frac{e^{-y}}{(e^y - 1)^2}dy.
\]
After simplifications, we find that the last integral is equal to
\[
\int_{y > - \ln(2)} \left[ 1 - e^{2y} + (q(e^y - 1) - (q - 2)(e^{3y} - e^{2y})e^{-y}(e^y - 1)^{-2}\mathbb{1}_{|e^y - 1| < 1}\right] \frac{e^{-y}}{(e^y - 1)^2}dy
\]
\[
= 2 + 2 \ln(2) - q \left( 2 \ln(2) + \frac{3}{2} \right). \tag{23}
\]
From equations (22), (20) and (23), we deduce
\[
\tilde{\Psi}(q) = -\frac{2}{\pi} \left( \ln(2) + \frac{3}{2} \right) q + \frac{2}{\pi} \int_{y > - \ln(2)} (e^{qy} - 1 - q(e^y - 1)\mathbb{1}_{|e^y - 1| < 1}) \frac{e^{-y}dy}{(e^y - 1)^2}. \tag{24}
\]
Finally, we can remove the indicator using simple calculations. One finds that
\[
\int_{y > - \ln(2)} (1 - e^y) \frac{e^{-y}}{(e^y - 1)^2} \mathbb{1}_{|e^y - 1| \geq 1}dy = \frac{1}{2} - \ln(2),
\]
and therefore
\[
\tilde{\Psi}(q) = -\frac{4}{\pi} q + \frac{2}{\pi} \int_{y > - \ln(2)} (e^{qy} - 1 - q(e^y - 1)) \frac{e^{-y}dy}{(e^y - 1)^2}, \quad q < 3.
\]
Hence we recovered the expression for \( \Psi \) in the statement of Theorem 3.3 and this gives both the martingale property and the law of \( \xi^0 \) under the change of measure.

We finish the proof of Theorem 3.3 with the arguments of [10] that we reproduce here to be self-contained. Let \( x > 0 \). Equation (16) reads
\[
\gamma_x H(\Xi(b), b \in [0, a])\mathbb{1}_{\{a < \Xi(x)\}} = E \left[ M_{\tau^0(a)} H \left( x \exp(\xi^0(\tau^0(b))), b \in [0, a] \right) \right].
\]
The optional stopping theorem implies that for any \( c > 0 \),
\[
E \left[ M_{\tau^0(a)} H \left( x \exp(\xi^0(\tau^0(b))), b \in [0, a] \right) \mathbb{1}_{\{c > \tau^0(a)\}} \right]
\]
\[
= E \left[ M_c H \left( x \exp(\xi^0(\tau^0(b))), b \in [0, a] \right) \mathbb{1}_{\{c > \tau(a)\}} \right].
\]
By the lemma, the right-hand side is, with the notation \( \xi \) of the theorem,
\[
E \left[ H \left( x \exp(\xi(\tau(b))), b \in [0, a] \right) \mathbb{1}_{\{c > \tau(a)\}} \right].
\]
Making \( c \to \infty \) and using dominated convergence completes the proof.
Theorem 3.6. Let $z \in \mathbb{R}\setminus\{0\}$. Under $\gamma_z$, conditionally on the jump sizes and jump times $(z_i, a_i)_{i \geq 1}$ of $\Xi$, the excursions $e_i, i \geq 1$, are independent and each $e_i$ has law $\gamma_z$.

**Proof.** By Lemma 3.4, we know that for all nonnegative measurable function $H$,

$$n_+ [H(u_a^{\ast, \ast}) 1_{\{a < \Im(z^{\ast})\}}] = \int \frac{dz}{2\pi z} h(-a, z),$$

where $h$ is :

$$h(-a, z) = \mathbb{E} \left[ H \left( (X_s, Y_s)_{s \in [0,T_{-a}]}; (z + X'_s, Y'_s)_{s \in [0,T_{-a}]} \right), \mathcal{E}_{a,a,z} \right].$$

Imagine that $H$ is some functional of the offspring of $\Xi$ below level $a$, say $H \left( u_a^{\ast, \ast} \right) = f_1(e_1^{(a)}) \cdots f_n(e_n^{(a)})$, where the $e_i^{(a)}$ denote the offspring of $\Xi$ created before $a$, ranked in descending order of the absolute value of their sizes $\Im(z_i^{(a)})$, and the $f_i$’s are taken continuous and bounded. For such a function $H$, $h$ is given by

$$h(-a, z) = \mathbb{E} \left[ f_1(e_1) \cdots f_n(e_n), \mathcal{E}_{a,a,z} \right],$$

where $e_1, \ldots, e_n$ are the $n$ largest excursions (before hitting $\{\Im(z) = -a\}$) of $(X, Y)$ and $(X', Y')$ above the past infimum of their imaginary parts. Consider the collection $\{(b, e_b^{\ast}), b \in [-a,0]\}$ where $e_b^{\ast}$ is an excursion of the Brownian motions $(X, Y)$ or $(X', Y')$ above the past infimum of their imaginary parts when the infimum is equal to $b$ (set $e_b^{\ast} = \delta$ if no such excursion exists). A consequence of Lévy’s Theorem, (Theorem 2.3, Chap. VI of [11]) is that the collection $\{(b, e_b^{\ast}), b \leq 0\}$, the excursions $e_b^{\ast}$ are distributed as independent excursions with law $\gamma_{z(e_b^{\ast})}$. Observe that $\mathcal{E}_{a,a,z}$ is measurable with respect to $\{(b, z(e_b^{\ast})), b \leq 0\}$. Therefore, conditioning on the sizes of the excursions yields

$$h(-a, z) = \mathbb{E} \left[ \gamma_{z(e_1)}(f_1) \cdots \gamma_{z(e_n)}(f_n), \mathcal{E}_{a,a,z} \right].$$

And so using Lemma 3.4 backwards, we get

$$n_+ \left[ f_1(e_1^{(a)}) \cdots f_n(e_n^{(a)}) 1_{\{a < \Im(z^{\ast})\}} \right] = n_+ \left[ \gamma_{z(e_1^{(a)})}(f_1) \cdots \gamma_{z(e_n^{(a)})}(f_n) 1_{\{a < \Im(z^{\ast})\}} \right].$$

Multiplying by a function of the endpoint $x(R(u))$ and disintegrating over it gives

$$\gamma_z \left[ f_1(e_1^{(a)}) \cdots f_n(e_n^{(a)}) 1_{\{a < \Im(z^{\ast})\}} \right] = \gamma_z \left[ \gamma_{z(e_1^{(a)})}(f_1) \cdots \gamma_{z(e_n^{(a)})}(f_n) 1_{\{a < \Im(z^{\ast})\}} \right],$$

for Lebesgue-almost every $z \in \mathbb{R}$. Let us prove that this holds for example when $z = 1$. By scaling (Lemma 2.10), for $z > 0$ this writes

$$\gamma_1 \left[ f_1(e_1^{(a)}(u(z))) \cdots f_n(e_n^{(a)}(u(z))) 1_{\{a < \Im(z^{\ast})\}} \right] = \gamma_1 \left[ \gamma_{z(e_1^{(a)}(u(z)))}(f_1) \cdots \gamma_{z(e_n^{(a)}(u(z)))}(f_n) 1_{\{a < \Im(z^{\ast})\}} \right].$$

We then condition on the birth times of these excursions. We can apply Proposition 2.14 at different levels and Lemma 2.12 to prove that $\gamma_1$-almost surely, for all $1 \leq i \leq n$, $e_i^{(a)}(u(z)) \xrightarrow{\gamma_1} e_i^{(a)}(u)$ in $U$. Besides, $z_i^{(a)}(u(z)) = \Delta \left( e_i^{(a)}(u(z)) \right)$, so by Lemma 2.11, $z_i^{(a)}(u(z)) \xrightarrow{\gamma_1} z_i^{(a)}(u)$, and by continuity of $z \mapsto \gamma_z$ (Proposition 2.13), $\gamma_{z_i^{(a)}(u(z))} \xrightarrow{\gamma_1}$
$\gamma_{z_1(a)}$ almost surely under $\gamma_1$. An application of the dominated convergence theorem finally gives

$$
\gamma_1 \left[ f_1(e_1^{(a)}) \cdots f_n(e_n^{(a)}) \mathds{1}_{\{a < \Im(z^*)\}} \right] = \gamma_1 \left[ \gamma_{z_1}^{(a)}(f_1) \cdots \gamma_{z_n}^{(a)}(f_n) \mathds{1}_{\{a < \Im(z^*)\}} \right].
$$

The statement follows.

Figure 7: Excursions of $B = (X, Y)$ and $B' = (X', Y')$ above their past infimum. The past infimum process is depicted in blue, and by Lévy’s theorem the excursions above it form a Poisson point process represented in red.

### 3.3 A change of measures

We begin by calling attention to a natural martingale associated to the growth-fragmentation process.

**Proposition 3.7.** Let $z \in \mathbb{R} \setminus \{0\}$. Under $\gamma_z$, the process

$$
\mathcal{M}_a = \mathds{1}_{\{T_a < \infty\}} \sum_{i \geq 1} |\Delta e_i^{a^+}|^2, \quad a \geq 0,
$$

is a $(\mathcal{G}_a)_{a \geq 0}$-martingale.

**Proof.** The branching property 3.1 shows that it is enough to prove that $\gamma_z[\mathcal{M}_a] = z^2$ for all $a \geq 0$.

For a Brownian excursion process $(e_s)_{s>0}$ in the sense of Definition 2.1, we use the shorthand $0 < s^+ \leq T$ to denote times $0 < s \leq T$ such that $e_s \in U^+$. Let $g : \mathbb{R} \to \mathbb{R}_+$ be a nonnegative measurable function. By the Markov property at time $T_a$, see Lemma 2.4,

$$
n_+ (\mathcal{M}_a g(x(R(u)))) = n_+ \left( \mathds{1}_{\{T_a < \infty\}} \mathbb{E} \left[ \sum_{s^+ \leq L_{T-a}} |\Delta e_s|^2 g(X(T-a)) \right] \right). \quad (25)
$$

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By the master formula,
\[
E \left[ \sum_{s^+ \leq L_{T,a}} |\Delta e_s|^2 g(X(T_{-a})) \right] = E \left[ \int_0^{T_a} dL_s \int_{-\infty}^{+\infty} \frac{dz}{2\pi} \int_{-\infty}^{+\infty} \frac{dz'}{2\pi} z^2 E[g(z' + X_{T_{-a}}) | z' = z + X_s] \right] = E \left[ \int_0^{T_a} dL_s \int_{-\infty}^{+\infty} \frac{dz'}{2\pi} E[g(z' + X_{T_{-a}})] \right] = E \left[ L_{T,a} \frac{1}{2\pi} \int_{-\infty}^{+\infty} g \right],
\]
since the Lebesgue measure is an invariant measure for the Brownian motion. Finally, the law of the Brownian local time \(L_{T,a}\) at the hitting time of \(-a\) is known to be exponential with mean \(2a\) (see for example Section 4, Chap. VI of [11]). Hence
\[
E \left[ L_{T,a} \frac{1}{2\pi} \int_{-\infty}^{+\infty} g \right] = 2a \times \frac{1}{2\pi} \int_{-\infty}^{+\infty} g. \quad \text{Coming back to (25), we get}
\]
\[
n_+ (M_\alpha g(x(R(u)))) = 2a \times \left( \frac{1}{2\pi} \int_{-\infty}^{+\infty} g \right) n_+ (T_a < \infty).
\]
But \(n_+ (T_a < \infty) = n_+ (\sup(y) \geq a) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g\) (see Proposition 3.6, Chapter XII, of [11]), so finally
\[
n_+ (M_\alpha g(x(R(u)))) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g.
\]
Disintegrating \(n_+\) over \(z\) as in Proposition 2.9 yields
\[
\int_{-\infty}^{+\infty} \frac{dz}{2\pi z^2} g(z) \gamma_z[M_\alpha] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g.
\]
This holds for all nonnegative measurable function \(g\), and thus for Lebesgue-almost every \(z \in \mathbb{R}\),
\[
\gamma_z[M_\alpha] = z^2.
\]
Recall the notation \(u(z) = zu(\cdot/z^2)\) for \(z > 0\) from Lemma 2.12. By scaling, this means for Lebesgue-almost every \(z > 0\),
\[
\gamma_1 \left[ \mathbb{I}_{\{z^2 T_{a/z} < \infty\}} \sum_{i \geq 1} |\Delta e_i^{a,z} (u(z))|^2 \right] = z^2,
\]
which yields
\[
\gamma_1 \left[ \mathbb{I}_{\{T_{a/z} < \infty\}} \sum_{i \geq 1} |\Delta e_i^{a,z} (u(z))|^2 \right] = z^2. \tag{26}
\]
Again, this must hold on a dense set of endpoints \(z\), and thus taking \(z\) according to some sequence, Lemma 2.12 and Proposition 2.14 together with Fatou’s lemma imply that \(\gamma_1[M_\alpha] \leq 1\). This holds for all \(a\), and so by scaling we deduce that for all \(z \neq 0\), \(\gamma_z[M_\alpha] \leq z^2\). On the other hand, notice that \(\Delta e_i^{a,z} (u(z)) = z \Delta e_i^{a/z^2,z} (u)\). By the branching property under \(\gamma_1\) (Proposition 3.2), for a \(z < 1\) such that equation (26) holds,
\[
1 = \gamma_1 \left[ \mathbb{I}_{\{T_{a/z} < \infty\}} \sum_{i \geq 1} |\Delta e_i^{a/z^2,z}|^2 \right] = \gamma_1 \left[ \mathbb{I}_{\{T_{a} < \infty\}} \sum_{i \geq 1} \gamma a \Delta e_i^{a,z} \left( \mathbb{I}_{\{T_{a/z} < \infty\}} \sum_{j \geq 1} |\Delta e_j^{a/z^2,a}|^2 \right) \right] \leq \gamma_1 \left[ \mathbb{I}_{\{T_{a} < \infty\}} \sum_{i \geq 1} |\Delta e_i^{a,z}|^2 \right].
\]
By the master formula, starting at $s_f, g$:

$$
\text{above is also } \int_{0}^{\infty} \text{d}z \gamma_{s} \text{ at time } a > 0.
$$

Theorem 3.8. Let $z \in \mathbb{R}\setminus\{0\}$. For any $a > 0$, under $\mu_z$, $(u(s))_{0 \leq s \leq T_a}$ and $(u(R(u) - s) - z)_{0 \leq s \leq S_a}$ are two independent $\mathbb{H}$--excursions stopped at the hitting time of $\{\Im(z) = a\}$.

Through the change of measures $\mu_z$, $u$ therefore splits into two independent $\mathbb{H}$--excursions starting at 0 and $z$ respectively.

**Proof.** The theorem follows from a similar application of the master formula. Let $f, g : U \rightarrow \mathbb{R}_+$ be two bounded continuous functions. Then

$$
n_+ \left( f(u(s), 0 \leq s \leq T_a)g(u(R(u) - s), 0 \leq s \leq S_a) \mathcal{M}_a \right)
$$

$$
= n_+ \left( f(u(s), 0 \leq s \leq T_a)I_{\{T_a < \infty\}} n_+ \left( g(u(R(u) - s), 0 \leq s \leq S_a) \mathcal{M}_a \big\vert \mathcal{F}_{T_a} \right) \right). 
$$

By the master formula,

$$
n_+ \left( g(u(R(u) - s), 0 \leq s \leq S_a) \mathcal{M}_a \big\vert \mathcal{F}_{T_a} \right)
$$

$$
= \mathbb{E} \left[ \int_{0}^{T_{T_a}} \int_{-\infty}^{+\infty} \frac{dx}{2\pi} \mathbb{E} \left[ g(x + x', X_{T_{T_a} - s}, a + Y_{T_{T_a} - s}, 0 \leq s \leq S) \right] \bigg\vert x = x' \right],
$$

where $S := \inf \{s > 0, Y_{T_{T_a} - s} = 0\}$. The change of variables $x + X_{T_a} \mapsto x$ provides

$$
n_+ \left( g(u(R(u) - s), 0 \leq s \leq S_a) \mathcal{M}_a \big\vert \mathcal{F}_{T_a} \right)
$$

$$
= \mathbb{E} \left[ L_{T_{T_a}} \int_{-\infty}^{+\infty} \frac{dx}{2\pi} \mathbb{E} \left[ g(x + X_{T_{T_a} - s}, a + Y_{T_{T_a} - s}, 0 \leq s \leq S) \right] \right]
$$

$$
= 2a \times \int_{-\infty}^{+\infty} \frac{dx}{2\pi} \mathbb{E} \left[ g(x + X_{T_{T_a} - s}, a + Y_{T_{T_a} - s}, 0 \leq s \leq S) \right].
$$

The path $(X_s, 0 \leq s \leq T_{T_a})$ is conditionally on $Y$ distributed as a linear Brownian motion stopped at time $T_{T_a}$ (recall that $T_{T_a}$ is a measurable function of $Y$). Since the Lebesgue measure is a reversible measure for the Brownian motion, by time-reversal, the "law" of $(X + X_{T_{T_a} - s}, 0 \leq s \leq S)$ for $x$ chosen with the Lebesgue measure is the "law" of a linear Brownian motion with initial measure the Lebesgue measure, stopped at time $S$ ($S$ is measurable with respect to $Y$). Therefore, the integral $\int_{-\infty}^{+\infty} \frac{dx}{2\pi} \mathbb{E} \left[ g(\ldots) \right]$ above is also

$$
\int_{-\infty}^{+\infty} \frac{dz}{2\pi} \mathbb{E} \left[ g(z + X_s, a + Y_{T_{T_a} - s}, 0 \leq s \leq S) \right].
$$
Now we use that \((a + Y_{T_{-a}} - s, 0 \leq s \leq S)\) has the law of a 3-dimensional Bessel process \(V\) starting from 0 and run until its hitting time of \(a\) (call this time \(T^V_a\)), see Corollary 4.6, Chap. VII of [11]. Hence the former integral is also

\[
\int_{-\infty}^{+\infty} \frac{dz}{2\pi} \mathbb{E} \left[ g(z + X_s, V_s, 0 \leq s \leq T^V_a) \right].
\]

Plugging this into equation (28) triggers

\[
n_+(f(u(s), 0 \leq s \leq T_a)g(u(R(u) - s), 0 \leq s \leq S_a)M_a) = 2a \times \int_{-\infty}^{+\infty} \frac{dz}{2\pi} \mathbb{E} \left[ g(z + X_s, V_s, 0 \leq s \leq T^V_a) \right] n_+(f(u(s), 0 \leq s \leq T_a)1_{\{T_a < \infty\}}) = \int_{-\infty}^{+\infty} \frac{dz}{2\pi} \mathbb{E} \left[ g(z + X_s, V_s, 0 \leq s \leq T^V_a) \right] \times n_+(f(u(s), 0 \leq s \leq T_a)1_{\{T_a < \infty\}}).
\]

Moreover, using for example Williams’ description of the Itô's measure \(n_+\) (Theorem 4.5, Chap. XII, in [11]), the law of \((u(s), 0 \leq s \leq T_a)\) conditionally on \(\{T_a < \infty\}\) is the one of \((X_s, V_s, 0 \leq s \leq T^V_a)\). We get eventually

\[
n_+(f(u(s), 0 \leq s \leq T_a)g(u(R(u) - s), 0 \leq s \leq S_a)M_a) = \int_{-\infty}^{+\infty} \frac{dz}{2\pi} \mathbb{E} \left[ g(z + X_s, V_s, 0 \leq s \leq T^V_a) \right] \times \mathbb{E} \left[ f(X_s, V_s, 0 \leq s \leq T^V_a) \right].
\]

Finally, we disintegrate \(n_+\) over \(x(R(u))\) to get

\[
\int_{-\infty}^{+\infty} \frac{dz}{2\pi} \gamma_z \left[ f(u(s), 0 \leq s \leq T_a)g(u(R(u) - s), 0 \leq s \leq S_a) \frac{M_a}{z^2} \right] = \int_{-\infty}^{+\infty} \frac{dz}{2\pi} \mathbb{E} \left[ g(z + X_s, V_s, 0 \leq s \leq T^V_a) \right] \times \mathbb{E} \left[ f(X_s, V_s, 0 \leq s \leq T^V_a) \right].
\]

Now multiply \(g\) by any measurable function \(h : \mathbb{R} \to \mathbb{R}_+\) of \(x(R(u))\) to see that for Lebesgue-almost every \(z \in \mathbb{R},\)

\[
\gamma_z \left[ f(u(s), 0 \leq s \leq T_a)g(u(R(u) - s), 0 \leq s \leq S_a) \frac{M_a}{z^2} \right] = \mathbb{E} \left[ g(z + X_s, V_s, 0 \leq s \leq T^V_a) \right] \times \mathbb{E} \left[ f(X_s, V_s, 0 \leq s \leq T^V_a) \right].
\]

The right-hand side of this equation is a continuous function of \(z\). Moreover, by scaling (Lemma 2.10), for \(z > 0\) the left-hand term can be written

\[
\gamma_z \left[ f(u(s), 0 \leq s \leq T_a)g(u(R(u) - s), 0 \leq s \leq S_a) \frac{M_a}{z^2} \right] = \gamma_1 \left[ f(zu(s/z^2), 0 \leq s \leq T_{a/z^2}u)g(zu(R(u) - \frac{s}{z^2}), 0 \leq s \leq S_{a/z^2})M_{a/z^2} \right].
\]

Since equality (29)-(30) holds almost everywhere, it must hold for a dense set of \(z > 0\). Take \(z \downarrow 1\) along such a subsequence. By Lemma 2.12 and the observation that \(T_{a/z} \to T_a, S_{a/z} \to S_a\), we get the convergences \((zu(s/z^2), 0 \leq s \leq T_{a/z^2}) \to (u(s), 0 \leq s \leq T_a)\) and \((zu(R(u) - \frac{s}{z^2}), 0 \leq s \leq S_{a/z^2}) \to (u(R(u) - s), 0 \leq s \leq S_a)\) in \(U\). In addition, we know that \(M_{a/z} \to M_a\) almost surely and \(\gamma_1 \left[ M_{a/z} \right] \to \gamma_1 [M_a] \) (both
these expressions are equal to 1 by Proposition 25. By Scheffé’s lemma, $\mathcal{M}_{a/z} \rightarrow \mathcal{M}_a$ in $L^1$. When $z \searrow 1$, this turns (32) into
\[
\gamma_z \left[ f(u(s), 0 \leq s \leq T_a)g(u(R(u) - s), 0 \leq s \leq S_a) \frac{\mathcal{M}_a}{z^2} \right] 
\xrightarrow{z \searrow 1} \gamma_1 \left[ f(u(s), 0 \leq s \leq T_a)g(u(R(u) - s), 0 \leq s \leq S_a) \mathcal{M}_a \right].
\]
Therefore (29)-(30) holds for $z = 1$, and then for any $z$ by scaling. So we proved that for all $z \in \mathbb{R}\setminus\{0\}$,
\[
\gamma_z \left[ f(u(s), 0 \leq s \leq T_a)g(u(R(u) - s), 0 \leq s \leq S_a) \frac{\mathcal{M}_a}{z^2} \right] = \mathbb{E} \left[ g(z + X_a, V_a, 0 \leq s \leq T_a^V) \right] \times \mathbb{E} \left[ f(X_a, V_a, 0 \leq s \leq T_a^V) \right],
\]
which is simply
\[
\mu_z \left[ f(u(s), 0 \leq s \leq T_a)g(u(R(u) - s), 0 \leq s \leq S_a) \right] = \mathbb{E} \left[ g(z + X_a, V_a, 0 \leq s \leq T_a^V) \right] \times \mathbb{E} \left[ f(X_a, V_a, 0 \leq s \leq T_a^V) \right].
\]
This proves that under $\mu_z$, the processes $(u(s))_{0 \leq s \leq T_a}$ and $(u(R(u) - s) - z)_{0 \leq s \leq S_a}$ are independent $\mathbb{H}$-excursions stopped at the hitting time of $\{\exists(z) = a\}$. \hfill \Box

**Remark.** This gives a new insight on why the Cauchy process should be hidden in some sense in the law of $\Xi$: under the tilted measure, $u$ splits into two independent $\mathbb{H}$-excursions and so the size at some level $a$ of the spine going to infinity is just the difference of two Brownian motions started from infinity taken at their hitting time of $\{\exists(z) = a\}$. 

4 The growth-fragmentation process of excursions in $\mathbb{H}$

In this section, we summarize the previous results in the language of the self-similar growth-fragmentations introduced by Bertoin in [4]. The main reference here is [5], but for the sake of completeness we shall recall in the first paragraph the bulk of the construction of such processes. At the heart of this section lies the calculation of the cumulant function. We recover the cumulant function of [5], formula (19), in the specific case when $\theta = 1$. Recall the definition of $Z$ in Theorem 3.3. The process $Z$ starting at $z < 0$ is defined to be the negative of the process $Z$ starting at $-z$.

4.1 Construction of $X$

We explain how one can define the *cell system* driven by $Z$. We use the Ulam tree $U = \bigcup_{n \geq 0} \mathbb{N}^n$, where $\mathbb{N} = \{1, 2, \ldots\}$, to encode the genealogy of the cells (we write $\mathbb{N}^0 = \{\emptyset\}$, and $\emptyset$ is called the Eve cell). A node $u \in U$ is a list $(u_1, \ldots, u_k)$ of positive integers where $|u| = i$ is the *generation* of $u$. The children of $u$ are the lists in $\mathbb{N}^{i+1}$ of the form $(u_1, \ldots, u_i, k)$, with $k \in \mathbb{N}$. A *cell system* is a family $\mathcal{X} = (\mathcal{X}_u, u \in U)$ indexed by $U$, where $\mathcal{X}_u = (\mathcal{X}_u(a))_{a \geq 0}$ is meant to describe the evolution of the size or mass of the cell $u$ with its age $a$.

To define the cell system driven by $Z$, we first define $\mathcal{X}_\emptyset$ as $Z$, started from some initial mass $z \neq 0$, and set $b_\emptyset = 0$. Observe the realization of $\mathcal{X}_\emptyset$ and its jumps. Since $Z$ hits 0 in finite time, we may rank the sequence of jump sizes and times $(x_1, \beta_1), (x_2, \beta_2), \ldots$ of $-\mathcal{X}_\emptyset$ by decreasing order of the $|x_i|$’s. Conditionally on these jump sizes and times, we define the first generation of our cell system $\mathcal{X}_i$, $i \in \mathbb{N}$, to
be independent with $X_i$ distributed at $Z$, starting from $x_i$. We also set $b_i = b_0 + \beta_i$ for the birth time of the particle $i \in \mathbb{N}$. By recursion, one defines the law of the $n$-th generation given generations $1, \ldots, n-1$ in the same way. Hence the cell labelled by $u = (u_1, \ldots, u_n) \in \mathbb{N}^n$ is born from $u' = (u_1, \ldots, u_{n-1}) \in \mathbb{N}^{n-1}$ at time $b_u = b_{u'} + \beta_{u_n}$, where $\beta_{u_n}$ is the time of the $u_n$-th largest jump of $X_{u'}$, and conditionally on $X_{u'}(\beta_{u_n}) - X_{u'}(\beta_{u_n}^-) = -y$, $X_u$ has the law of $Z$ with initial value $y$ and is independent of the other daughter cells at generation $n$. We write $\zeta_u$ for the lifetime of the particle $u$. We may then define, for $a \geq 0$,

$$\mathbf{X}(a) := (X_u(a - b_u), \ u \in \mathcal{U} \text{ and } b_u \leq a < b_u + \zeta_u), \quad (34)$$

as the family of the sizes of all the cells alive at time $a$. We arrange the elements in $\mathbf{X}(a)$ in descending order of their absolute values.

4.2 The growth-fragmentation process of excursions in $\mathbb{H}$

We restate Theorem 1.1. Beware that the signed growth-fragmentation $\mathbf{X}$ in this section starts from $z$.

**Theorem 4.1.** Let $z \in \mathbb{R}\setminus\{0\}$. Under $\gamma_z$,

$$(\mathbf{X}(a), a \geq 0) \overset{law}{=} \left( (\Delta_{e_i^a}^{+}, \ i \geq 1), a \geq 0 \right).$$

**Proof.** Let $u \in U^+$ be such that the locally largest excursion described in Subsection 3.2 is well-defined, i.e. $u$ has no loop above any level, has distinct local minima, and no splitting in two equal sizes (this set of excursions has full probability under $\gamma_z$). This gives our Eve cell process. The independence of the daughter excursions given their size at birth has already been proved in Theorem 3.6, and we have taken $Z$ according to the law of the largest fragment in Theorem 3.3, so it remains to prove that every excursion can be found in the genealogy of $\mathbf{X}$ as constructed in the former section.

For $a \geq 0$, we denote by $\mathbf{X}^{exc}(a)$ the set of all excursions associated to the sizes in $\mathbf{X}(a)$. Let $0 \leq t \leq R(u)$ such that $\exists (u(t)) > a$. We want to show that $e^{(t)}_a \in \mathbf{X}^{exc}(a)$. Set

$$\mathcal{A} = \{ a' \in [0,a], e^{(t)}_{a'} \in \mathbf{X}^{exc}(a') \}.$$ 

Then $\mathcal{A}$ is an interval containing 0.

- $\mathcal{A}$ is open in $[0,a]$. Let $a' \in \mathcal{A}$ with $a' < a$. Write $e^{(t)}_b, b \geq a'$, for the locally largest excursion inside $e^{(t)}_{a'}$. Then for small enough $\varepsilon > 0$, $e^{(t)}_{a'+\varepsilon} = e^{(t)}_{a'+\varepsilon}$. Indeed, the first height $b \geq a'$ when $e^{(t)}_b \neq e^{(t)}_{a'}$ is equal to the minimum of $y(s)$ for $s$ between $t$ and $\tau^*$, and so it is strictly above $a'$. This implies that $a' + \varepsilon \in \mathcal{A}$ since $e^{(t)}_{a'+\varepsilon} \in \mathbf{X}^{exc}(a' + \varepsilon)$ as the locally largest excursions are in the genealogy.

- $\mathcal{A}$ is closed in $[0,a]$. Let $a_\ell$ be a sequence of elements in $\mathcal{A}$ increasing to $a_\infty$. For all $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\forall a' \in (a_\infty - \delta, a_\infty), \quad |\Delta_{e^{(t)}_a}^{(t)} - \Delta_{e^{(t)}_{a_\infty}}^{(t)}| < \varepsilon.$$ 

Then for all $a_1, a_2 \in (a_\infty - \delta, a_\infty)$,

$$|\Delta_{e^{(t)}_{a_1}}^{(t)} - \Delta_{e^{(t)}_{a_2}}^{(t)}| \leq |\Delta_{e^{(t)}_{a_1}}^{(t)} - \Delta_{e^{(t)}_{a_\infty}}^{(t)}| + |\Delta_{e^{(t)}_{a_2}}^{(t)} - \Delta_{e^{(t)}_{a_\infty}}^{(t)}| < 2\varepsilon.$$
Take \( \varepsilon = |\Delta e_{a_\infty}^{(t)}|/4 \) and \( N \) large enough so that \( a_N \in (a_\infty - \delta, a_\infty) \). Then the excursion \( e_{a_N}^{(t)} \) is such that for all \( a' \in [a_N, a_\infty) \), \( e_{a'}^{(t)} \) is taken along the locally largest excursion inside \( e_{a_N}^{(t)} \). Indeed, it follows from these inequalities that for all \( a_1, a_2 \in (a_\infty - \delta, a_\infty) \), \(|\Delta e_{a_1}^{(t)} - \Delta e_{a_2}^{(t)}| \leq \frac{1}{2} |\Delta e_{a_\infty}^{(t)}| < |\Delta e_{a_1}^{(t)}| \), then take \( a_1 = a' \) and \( a_2 \not> a' \). This entails that \( a_\infty \in \mathcal{A} \).

By connectedness \( \mathcal{A} \) must be \([0, a]\). This concludes the proof. \( \square \)

### 4.3 The cumulant function

The process \( \mathbf{X} \) is not a growth-fragmentation in the sense of \([5]\) because it carries negative masses. We show in this section that if one discards all cells with negative masses together with their progeny, one obtains one of the growth-fragmentation processes studied in \([5]\).

Formally, let \( \mathbf{X} \) defined by (34) where we only consider the \( u \)'s such that \( \mathcal{X}_v(b_u) > 0 \) for all ancestors \( v \) of \( u \) (including itself) in the Ulam tree. The process \( \mathbf{X} \) is a growth-fragmentation in the sense of \([5]\). It is characterized by its self-similarity index \( \alpha = -1 \) and its cumulant function defined for \( q \geq 0 \), by

\[
\kappa(q) := \Psi(q) + \int_{-\infty}^{0} (1 - e^y)^q \Lambda(dy).
\]

The following proposition is Proposition 5.2 of \([5]\) in the case \( \theta = 1 \), \( \hat{\beta} = 1 \) and \( \gamma = \hat{\gamma} = 1/2 \) with the additional factor 2 (corresponding to a time change).

**Proposition 4.2.** Let \( \omega_+ = \omega = 5/2 \), and \( \Phi^+(q) = \kappa(q + \omega_+) \) for \( q \geq 0 \). Then \( \Phi^+ \) is the Laplace exponent of a symmetric Cauchy process conditioned to stay positive, namely

\[
\Phi^+(q) = -2 \frac{\Gamma(\frac{1}{2} - q) \Gamma(\frac{3}{2} + q)}{\Gamma(-q) \Gamma(1 + q)}, \quad -\frac{3}{2} < q < \frac{1}{2}.
\]

Furthermore, the associated growth-fragmentation \( \mathbf{X} \) has no killing and its cumulant function is

\[
\kappa(q) = -2 \frac{\cos(\pi q)}{\pi} \Gamma(q - 1) \Gamma(3 - q), \quad 1 < q < 3.
\]

**Remark.** In \([5]\), the roots of \( \kappa \) pave the way to remarkable martingales. It should not come as a surprise that in our case these roots happen to be \( \omega_- = \frac{3}{2} \) and \( \omega_+ = \frac{5}{2} \). Indeed, the \( h \)-transform for the symmetric Cauchy process conditioned to stay positive (resp. conditioned to hit 0 continuously) is given by \( x \mapsto x^{1/2} \) (resp. \( x \mapsto x^{-1/2} \)). This turns the martingale in Proposition 3.7 into the sum over all masses in \( \mathbf{X} \) to the power \( \omega_+ = 2 + \frac{1}{2} \), and \( \omega_- = 2 - \frac{1}{2} \) respectively, which are exactly the quantities considered in \([5]\).

**Proof.** The strategy is as follows. In view of Theorem 5.1 in \([5]\), we first compute \( \kappa(q + \omega) - \kappa(\omega) \) and we put it in a Lévy-Khintchin form so as to retrieve the Laplace exponent of the Lévy process involved in the Lamperti representation of a Cauchy process conditioned to stay positive, which is known from \([6]\). We then show that \( \kappa(\omega) = 0 \), and therefore deduce the expression of \( \kappa \).

Recall first that by definition

\[
\kappa(q) = \Psi(q) + \int_{-\infty}^{0} (1 - e^y)^q \Lambda(dy),
\]
with \( \Psi \) given by (15). In fact, we rather use formula (24), which is closer to [6]:

\[
\Psi(q) = -\frac{2}{\pi} \left( \ln(2) + \frac{3}{2} \right) q + \frac{2}{\pi} \int_{y > -\ln(2)} \left( e^{qy} - 1 - q(e^y - 1) \mathbb{1}_{|e^y - 1| < 1} \right) \frac{e^{-y} dy}{(e^y - 1)^2}.
\]

Let \(-\frac{3}{2} < q < \frac{1}{2}\). Then

\[
\frac{\pi}{2} (\kappa(q + \omega) - \kappa(\omega))
\]

\[
= -\left( \ln(2) + \frac{3}{2} \right) q + \int_{y > -\ln(2)} \left( e^{(q+\omega)y} - e^{\omega y} - q(e^y - 1) \mathbb{1}_{|e^y - 1| < 1} \right) \frac{e^{-y} dy}{(e^y - 1)^2}
\]

\[
+ \int_{-\ln(2)}^{0} \left( (1 - e^y)q^+ - (1 - e^y)\omega \right) \frac{e^{-y} dy}{(e^y - 1)^2}.
\]

Performing the change of variables \( e^x = 1 - e^y \) in the second integral entails

\[
\frac{\pi}{2} (\kappa(q + \omega) - \kappa(\omega))
\]

\[
= -\left( \ln(2) + \frac{3}{2} \right) q + \int_{-\ln(2)}^{+\infty} \left( e^{(q+\omega)y} - e^{\omega y} - qe^y(e^y - 1) \mathbb{1}_{|e^y - 1| < 1} \right) \frac{e^{-y} dy}{(e^y - 1)^2}
\]

\[
+ q \int_{y > -\ln(2)} \left( e^{\omega y} - (e^y - 1) \right) \mathbb{1}_{|e^y - 1| < 1} \frac{e^{-y} dy}{(e^y - 1)^2}
\]

\[
+ q \int_{-\ln(2)}^{-\infty} e^{\omega y} \mathbb{1}_{|e^y - 1| < 1} \frac{e^{-y} dy}{(e^y - 1)^2}
\]

\[
= -\left( \ln(2) + \frac{3}{2} \right) q + \int_{-\ln(2)}^{+\infty} \left( e^{qy} - 1 - q(e^y - 1) \mathbb{1}_{|e^y - 1| < 1} \right) \frac{e^{(\omega - 1)y} dy}{(e^y - 1)^2}
\]

\[
+ q \int_{-\ln(2)}^{+\infty} e^{\omega y} \mathbb{1}_{|e^y - 1| < 1} \frac{e^{-y} dy}{(e^y - 1)^2}
\]

Because \( \omega = 5/2 \), this has the form of \( \Phi^\dagger \) of Corollary 2 in [6] for the symmetric Cauchy process \((\alpha = 1 \text{ and } \rho = 1/2)\), apart from a possible extra drift. We now show that the drifts do in fact coincide. Let \( I \) and \( J \) denote the last two integrals in the above expression. Using the change of variables \( x = e^y \), we get

\[
I = \int_{1/2}^{1/2} \frac{x^{5/2} - 1}{x^2(x - 1)} dx, \quad J = \int_{0}^{1/2} \frac{\sqrt{x}}{x - 1} dx.
\]

Now

\[
I = \int_{1/2}^{1/2} \frac{x^{5/2} - x}{x^2(x - 1)} dx + \int_{1/2}^{1/2} \frac{x^2 - 1}{x^2(x - 1)} dx = \int_{1/2}^{1/2} \frac{\sqrt{x} - 1}{x - 1} dx + \int_{1/2}^{1/2} \frac{x + 1}{x^2} dx,
\]

and

\[
J = \int_{0}^{1/2} \frac{\sqrt{x} - 1}{x - 1} dx + \int_{0}^{1/2} \frac{1}{x - 1} dx.
\]
One can check that \( I_1 + J_1 = \ln(2) + \frac{5}{2} \). Therefore the linear term in the above expression of \( \kappa(q + \omega) - \kappa(\omega) \) is precisely

\[
a_+ = \frac{2}{\pi} \int_0^2 \frac{\sqrt{x} - 1}{x - 1} \, dx = \frac{2}{\pi} \int_1^1 \frac{\sqrt{1+u} - 1}{u} \, du - \frac{2}{\pi} \int_1^1 \frac{\sqrt{1-u} - 1}{u} \, du,
\]

which is exactly \( a_+ = 2a^+ \) as defined in Corollary 2 of [6] for the symmetric Cauchy process. Note that there is a sign error in formula (17) of the latter paper. Hence Corollary 2 of [6] triggers that \( \kappa(q + \omega) - \kappa(\omega) \) is twice the Laplace exponent of a Cauchy process conditioned to stay positive, and now by [8], we deduce

\[
\kappa(q + \omega) - \kappa(\omega) = -2 \frac{\Gamma(\frac{1}{2} - q)\Gamma(\frac{3}{2} + q)}{\Gamma(-q)\Gamma(1 + q)}, \quad -\frac{3}{2} < q < \frac{1}{2}.
\]

Taking \( q = -1/2 \) in this formula, one sees that \( \kappa(2) - \kappa(5/2) = -\frac{3}{\pi} \). Yet one can easily compute \( \kappa(2) \) from the definition of \( \kappa \). Simple calculations left to the reader actually lead to \( \kappa(2) = -\frac{2}{\pi} \), and thus \( \kappa(5/2) = 0 \). Finally, we recovered the expression of \( \Phi^+ \), and using Euler’s reflection formula

\[
\kappa(q) = -2 \frac{\cos(\pi q)}{\pi} \Gamma(q-1)\Gamma(3-q), \quad 1 < q < 3.
\]

\[\square\]

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