About grid generation in constructions bounded by the surfaces of revolution

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Abstract. For constructions bounded by the surfaces of revolution, structured grid generation technique is presented. Its technology has been elaborated within the variational approach for constructing optimal grids satisfying optimality criteria: closeness of grids to uniform ones, closeness of grids to orthogonal ones and adaptation to a given function. Grid generation has been designed for numerical solution of the differential equations modeling the vortex processes of multi-component hydrodynamics. In the technology, the three-dimensional construction in which it is required to construct a grid is represented in the form of the curvilinear hexahedron defining its configuration. The specific feature of the required configurations is that some faces of a curvilinear hexahedron lie in one plane and along edges of adjoining faces grid cells degenerate into prisms. Grid generation in the considered constructions has started to be developed by the elaboration of algorithms for the volume of revolution which has become the basic construction. The basic construction is obtained by the rotation through 180\degree around the axis of a generatrix consisting of straight line segments, arcs of circles and ellipses. Then the deformed volumes of revolutions are considered along with the generalizations of the volume of revolution which represent constructions obtained by the surfaces of revolution with parallel axis of rotation. The aim of the further development of the technology is to consider more and more complicated constructions and elaborate the technology for them. In the presentation, the current state of the development of the technology is given. Examples of generated grids are supplied.

1. Introduction

Numerical simulation of the vortex processes of multi-component hydrodynamics \cite{1} with strong interface deformations is very important field of applied researches in many scientific areas, such as, high density matter and energy physics (inertial confinement fusion, explosive processes), astrophysics (birth and evolution of stars, supernew stars), physics of Earth’s atmosphere and hydrosphere (see \cite{1}). Such hydrodynamics processes are characterized by strong interface deformations, hydrodynamic instability, vortex and stream flows, a loss of an initial topological structure. A numerical solution of differential equations modeling such processes is a very difficult and actual problem (see \cite{1}) and it requires generation of grids of a special quality.

Physical processes described in \cite{1} often take place in constructions (volumes) bounded by the surfaces of revolutions. First of all for \cite{1}, it is desired to construct grids in the special volume of revolution which represents the half of the classical volume of revolution (see \cite{2}). This construction is called the basic construction. The basic construction is obtained by the rotation
through 180° around the axis of a generatrix consisting of straight line segments, arcs of circles and ellipses [3]. Then it is also required to construct grids in a deformed volume of revolution which represents the basic construction deformed by another basic construction. Besides a basic construction and a deformed basic construction the generalization of the basic construction is also considered. It is the volume formed by the surfaces of revolution with parallel axes of rotation. The rotation is also carried out through 180° around the axes of rotation. Finally, it is planned to generate grids in the volume obtained through the deformation of the basic construction by the generalization of the basic construction. Numerical modeling [1] needs the elaboration of tools for generating structured grids of a special configuration for constructions listed above. Under the configuration of a domain and a structured grid is understood the way of the representation of the domain in which the grid is constructed in the form of a curvilinear hexahedron. The specific features of configurations used for grid generation are that two opposite faces of the hexahedron are surfaces of revolutions or composed of several surfaces of revolution and other faces are planar or ruled surfaces or also can be surfaces of revolution. At least two planar faces lie in one plane and along common edges of adjoining faces lying in one plane there are cells degenerating into prisms [4]. Such configuration of grids is different from rotational grids obtained by the rotation of a two dimensional grid around an axis, so the generation of grids is fully three-dimensional and can not be reduced to a two-dimensional case. The description of the technology for generation of structured grid with such features in the volumes bounded by the surfaces of revolution has appeared in [5]. The technology has been designed for numerical modeling the vortex processes of multi-component hydrodynamics [1] and for solving other physical and engineering problems (see, for example, [6]). The technology includes several types of algorithms for above types of constructions: for definition of the geometry and generation of initial grids [3], realization of different types of boundary conditions for grid nodes arrangement and correction of grids to the boundaries of domains [7, 8, 9], deformation of grids and domains [8, 10, 11], optimization of grids [12] and testing of grids [4, 13, 14, 15, 16]. The technology has been developed within the variational approach for constructing optimal grids [13, 14] satisfying optimality criteria: closeness of grids to uniform ones by the distance between nodes (U), closeness of grids to orthogonal ones (O) and adaptation to a given function (A). First, the technology was developed in [3, 12] for basic constructions. Then, the technology was elaborated for generalizations of basic constructions and for deformed volumes [5]. The cases of deformation by simple basic constructions in the form of a conus, a sphere and a cylinder were considered there. Further development of the technology is described in [11]. In the present conference paper, another new developments for all types of constructions are given: new realizations of boundary nodes arrangement for the basic constructions, examples of computed grids for complicated cases of deformed volumes and generalizations of the basic constructions along with new realizations of the third optimality criterion of adaptation for deformed volumes. In this connection a brief description of the development of the technology is supplied for all types of constructions. Examples of computed grids are demonstrated. The paper starts with the presentation of the used variational technique including the description of approach [14] and method [7, 8, 9, 12, 17].

2. A variational technique for generation of optimal grids

Generation of structured grids is carried out by the traditional mapping approach according to which a structured grid in the considered construction of a complicated geometrical form (a physical domain \(G\)) is defined by means of a continuous mapping \(x : P \rightarrow G\) of a rectangular parallelepiped (a computational domain or a parametric space)

\[ P = \xi = \{\xi_1, \xi_2, \xi_3 : 0 \leq \xi_l \leq I_l, l = 1, 2, 3\} \]
for integers $I_l$ specifying the number of grid nodes along each coordinate direction. According to this approach the values of the mapping $\mathbf{x} : P \rightarrow G$

$$\mathbf{x} = \mathbf{x}(\xi_1, \xi_2, \xi_3) = \{x_1(\xi_1, \xi_2, \xi_3), x_2(\xi_1, \xi_2, \xi_3), x_3(\xi_1, \xi_2, \xi_3)\}, \quad (1)$$

where $\xi_l = i_l, l = 1, 2, 3, i_l = 0, 1, ..., I_l$ determine the coordinates of nodes of the three-dimensional grid $\mathbf{x}_{i_1,i_2,i_3} = \mathbf{x}(i_1, i_2, i_3)$. The mapping (1) is searched only at nodes of the uniform and orthogonal grid of the computational domain $P$. The mapping is extended to other points by the trilinear mappings of the unit cubes of a grid in $P$ thus defining grid cells as hexahedral cells with ruled faces, see [18]. The constructions are also represented by the mapping (1) in the form of a curvilinear hexahedron which defines the configuration of a domain and a grid. Examples of different configurations for volumes of revolutions from [3] are given in figure 1 where numbers denote the correspondence of the vertices of the rectangular parallelepiped $P$ and the curvilinear hexahedron $G$. The mapping (1) defines a nondegenerate grid (the mapping is onto and “one-to-one” — homeomorphism). It has the positive Jacobian inside the domain $P$. On the boundary of the domain $P$ the Jacobian vanishes along common edges of joining of faces lying in one plane and cells along edges degenerate into prisms. Common criteria for homeomorphism of the mapping are obtained in [15], numerical criteria in [16, 18] and criteria of degeneration into prism along with other cases of degeneration in [4].

![Figure 1. Curvilinear hexahedrons, different configurations.](image)

For generation of grids of a desirable quality allowing to simulate physical processes with the necessary accuracy and effectively the criteria of uniformity, orthogonality and adaptation called as optimality criteria are usually utilized (see [13, 14]). To construct such grids we apply the variational technique for constructing optimal grids [13, 14] according to which grid nodes (1) are searched by the minimization of the special functional (see [5, 8, 12]) formalizing the criteria of optimality and representing the sum of measures of closeness of a curvilinear grid to a uniform one, an orthogonal one and an adaptive one:

$$D = A_U D_U + A_O D_O + A_A D_A \quad (2)$$

where $A_U > 0$, $A_O > 0$ and $A_A \geq 0$ are weight coefficients regulating the closeness of the grid to corresponding properties, $D_U$, $D_O$, $D_A$ are corresponding measures of closeness

$$D_U = \sum_{ijk} \left[r_{i+1,j,k} - r_{i-1,j,k}\right]^2 \left(\frac{1}{r_{i+1,j,k}^2} + \frac{1}{r_{i-1,j,k}^2}\right)$$
\[ [r_{i,j+1,k} - r_{i,j-1,k}]^2 \left( \frac{1}{r_{i,j,k}^2} + \frac{1}{r_{i,j,k+1}^2} \right) + [r_{i,j,k+1} - r_{i,j,k-1}]^2 \left( \frac{1}{r_{i,j,k+1}^2} + \frac{1}{r_{i,j,k-1}^2} \right), \]

\[ D_O = \sum_{ijk} \sum_{p=1}^4 \left( \frac{1}{\sin^2 \varphi_{ij}} + \frac{1}{\sin^2 \varphi_{ik}} + \frac{1}{\sin^2 \varphi_{jk}} \right), \]

\[ D_A = \sum_{ijk} r_{i+1,j,k}^2 [\Phi(H_{i+1,j,k}) - \Phi(H_{ijk})]^2 + r_{i,j+1,k}^2 [\Phi(H_{i,j+1,k}) - \Phi(H_{ijk})]^2 + r_{i,j,k+1}^2 [\Phi(H_{i,j,k+1}) - \Phi(H_{ijk})]^2. \]

Here \( \Phi(H_{ijk}) \) is a value of the given function at the grid node \( H_{ijk} = x_{i,j,k}, r_{i\pm 1,j,k} = |H_{ijk}H_{i\pm 1,j,k}| = |h_{i\pm 1}|, r_{i,j\pm 1,k} = |h_{j\pm 1}|, r_{i,j,k\pm 1} = |h_{k\pm 1}|. \) Values \( \varphi_{ij} (p = 1, 2, 3, 4) \) denote angles between vectors \( h_{i\pm 1} \) and \( h_{j\pm 1} \) (see figure 2).

**Figure 2.** The elements of a three-dimensional grid.

For exception of the degeneracies, in the algorithms for minimization of the functional the variations of the function \( \Phi(x_1, x_2, x_3) \) are substituted by the values of the positive weight function \( w(x_1, x_2, x_3), \) see [14].

The use of the functional \( D_A \) allows us to condense nodes and, consequently, to make smaller the sizes of cells where the values of the derivatives of the given function \( \Phi(x_1, x_2, x_3) \) called a monitoring function are big and where they are small allows to make them bigger.

In [5], the case \( A_A = 0 \) is considered and the technology is developed for generation of geometrically optimal grids. Then in [11], the criterion of adaptation is added. It is applied there for generating grids in the volumes of revolution. In the present publication, it is utilized for the deformed volume, besides the adaption is carried out not only for the monitoring function \( \Phi(x_1, x_2, x_3) \) but also for its first derivative.

The algorithm of minimization of the functional \( D \) [12] is modified for the case of adaption (see [17]). The modification concerns the way of the computation of the functional and the construction of the set of points for its minimization. The construction of this set is based on the uniformity criterion in [12], and on the criterion of adaptation in [17].

For the algorithm of the minimization of the functional \( D \) [12] for the arrangement of nodes on the boundaries of constructions special algorithms [7, 8, 9] of corrections of nodes to the boundaries have been elaborated.

The distinctive feature of the functional of optimality (2) is a special way of a uniformity criterion formalization. It defines the type of Euler equations for constructing a grid (hyperbolic in the wide sense), permits to consider different types of boundary conditions in the variational problems for grid generation (fixed and free nodes, orthogonality of coordinate lines to the boundary and some other boundary conditions) and provides good computational properties of
grids (see [12, 14]). In the present paper, the new boundary conditions with free nodes possessing lines of rotation that should be “kept” are realized for the basic construction.

Generation of initial grids for the process of minimization of the discrete functional is characterized in [5].

3. A new development of the technology for volumes of revolution

The generatrix curve which is always a closed curve (see, for example, the curve $P_1P_2P_3P_4P_5P_6P_7P_8P_9P_1$ in figure 3 (a)) bounding a two-dimensional domain is given in the plane $x_1, x_3$ by the elements $P_iP_{i+1}$ which are straight line segments, arcs of circles and ellipses. It generates the volume of revolution $G$ under the rotation around the axis $x_3$ through the angle $\pi$.

Figure 3. A generatrix curve (a) and a part of a generatrix curve (b) with a point $P_i$ and a line of rotation that should be “kept”.

Figure 4. An initial grid (a), and optimal grids with the following boundary conditions: on the line that should be “kept”, nodes are fixed (b), closest nodes are put (c), closest within some given distance nodes are put (d). (A red color denotes nodes on the line that should be “kept”, blue color free nodes).

A description of the algorithm for a volume of revolution is given in [7, 12]. It is the process of optimization of the initial grid [3] by the minimization of the discrete functional (2). Realizations of the different boundary conditions (fixed and free nodes, orthogonality of coordinate lines to the boundary and some other conditions) are also given there. In the presentation, a new way of “keeping” lines of rotation in the construction is described. Among the initial data (see, for example [5]), the end of the element which should be “kept” (see figure 3 (b), a point $P_i$) is
given. After the rotation of this point the line of rotation is defined (by points $P'_i$). In the algorithm of an initial grid generation (see [3]), if some point is “kept”, boundary grid nodes are arranged on the corresponding line of rotation. If “keeping” a corresponding line of rotation is desirable, the fixing of grid nodes of an initial grid along this line during optimization can prevent the movement of grid nodes (see an example in [19] and figure 4 (b)) and create a limit in improving the grid quality. The idea of the algorithm in which “keeping” a point do not prevent the movement of nodes and improving the grid quality is the following. During optimization all nodes are allowed to move. The optimization process is carried out a definite number of iterations. Then all nodes closest along coordinates $x_3$, see figure 4 (c) (or closest within some given distance, see figure 4 (d)) to the line which should be “kept” are put on this line and optimization is carried out again a given number of iterations during which all nodes put on the line move only along it.

4. A new development of the technology for deformed volumes of revolution

First, we describe the way how the deformed volume of revolution is obtained. It is obtained though the deformation of the basic construction (main or deformed body) by another basic construction (auxiliary or deforming body). Since the grid generation technology for the basic constructions has been already elaborated (see [3, 7, 12]) the chosen approach is viewed as the most natural one. In the process of a deformation, an auxiliary body moves into the main body up to the necessary degree in the direction given by the vector of deformation. As a result we have the deformed main body: its boundary in the region of deformation moves inside and takes the form of the auxiliary body while the form of the other part of the main body which is not subject to the deformation is not changed.

In the technology for generating a grid, the fixed (stationary) geometry of the deformed body is given and the mapping defining the grid in the main body is found by the non-stationary algorithm. The used way of grid generation in the considered domains is referred to the moving grid technique [20] (mesh morphing or mesh warping technique [21]). In our case, the grid moves adjusting a non-stationary process of a changing geometry. This process is organized artificially (without connection with some physical problem) in which the geometry of the main body changes and the grid automatically updates to conform the modified geometry in such a way that finally the given geometry of the deformed body fixed beforehand and the grid in it are obtained. The process starts from the basic construction of the main body. In the iterative process during each iteration, the boundary of the main body changes slightly. It is deformed in such a way that the grid does not contain degeneracies: the deforming body moves into the main body and nodes of the grid for the main body which get inside the deforming body (deformed boundary nodes) are projected on it, in other words, deformed boundary nodes are placed on the deforming surface. Then optimization of a grid is performed by the variational method [8, 12]. Such process is carried out up to that moment when deformation of the main body reaches the necessary degree and the deformed body in which it is required to construct the grid is obtained. Note that analogous technique can be organize for generation of unstructured grids in deformed volumes of revolution.

The cases of a deformation by a conus, a cylinder and a sphere (see [5]) composed the basis for the development of the technology in more complicated cases of deformation [11]. In the paper, another complicated examples of deformation are considered. Then the criterion of adaptation is applied for the deformed volumes while in [11] it is utilized for the volume of revolution. Besides in present investigation the adaptation is realized not only for the given monitoring function but also for its first derivative.

For completeness, we start with the formulation of the problem and brief description of the algorithm presented in [11].
4.1. Formulation of the problem

In the Cartesian coordinate systems $X$ (main) with axes $\{x_1, x_2, x_3\}$ and $\Xi$ (auxiliary) with the axes $\{\xi_1, \xi_2, \xi_3\}$, the main body of revolution $G_x$ (figure 5 (a), (b)) and the auxiliary body $Q_\xi$ (figure 5(c)) are given, correspondingly. In the notation of the bodies, low indices denote coordinate systems in which they are considered. Each of the bodies is obtained by the rotation of the plane generatrix curve consisting of straight line segments and arcs of circles (called elements) through the angle $\varphi = \pi$ around the axis, see [3, 7, 8]. The generatrix curves of the main and auxiliary bodies are given in the planes $x_1, x_3$ and $\xi_1, \xi_3$ where the rotation is carried out around the axes $x_3$ and $\xi_3$, correspondingly. To define the deformed body the origin $O_{\Xi X}$ of the coordinate system $\Xi$ in the coordinate system $X$ and the matrix $C$ of the transformation from the coordinate system $\Xi$ to the coordinate system $X$ are given so that after the transformation of coordinates (to the coordinate system $X$) the deforming body $Q_x$ defines its final position inside the main body (figure 5 (a)). For the algorithm of deformation the vector of deformation $\vec{V}_X$ in the coordinate system $X$ which shows the direction of the deforming body pressure on to the main body is also given. The structured three-dimensional grid is constructed in the domain $G_x$ by the algorithms [3, 12]. The main body corresponding to the domain $G_x$ is deformed by pressure of the additional body. The problem consists in constructing the structured grid in the deformed domain $D = G_x \setminus (G_x \cap Q_x)$. Let us consider first in the next section three examples of grids for complicated deformed bodies of revolution (a deforming surface composed of more than one surface).

4.2. Examples for complicated deformed bodies of revolutions

In figure 6 (a), (b), the grids are shown for the volume in which the deformation is carried out by the volume of revolution shown in figure 5 (c) where the deforming surface is formed by conical and spherical surfaces, and, in figure 6 (c) by the body shown in figure 6 (d) where the deforming surface is formed by conical, cylindrical and spherical surfaces. In grid generation, the criterion of adaptation is not applied. All stages of the non-stationary algorithm for generating above
grids are presented in [11]. In the next section, we describe the new changes concerning a stage of optimization.

4.3. A stage of optimization

New changes are made on the basis of testing for the following monitoring functions (see figure 7)

$$
\Phi(x_1, x_2, x_3) = \exp\left(-\frac{(x_3 - 30)^2}{\varepsilon}\right), \varepsilon = 1, \partial \Phi / \partial x_3 = -2(x_3 - 30)/\varepsilon \exp\left(-\frac{(x_3 - 30)^2}{\varepsilon}\right).
$$
In figure 7 (a), an adaptive grid is shown on the left and the values of the monitoring function are given on the right for a deformed volume. For the construction shown in figure 7, variables $x_1, x_2, x_3$ vary on segments $[-20, 20], [-20, 0], [-40, 40]$, correspondingly. The monitoring function has high derivatives along a plane $x_3 = 30$ from its both sides (see figure 7 (b) on the right), so the condensing of nodes is viewed near this plane. On the plane $x_3 = 30$, the derivative $\partial \Phi / \partial x_3$ is equal to zero and the increase of the grid interval is visualized there for adaptive grids (see figure 7 (a)). When this property is not desirable, an adaptation under the first derivative (one-dimensional and two-dimensional cases see in [13, 14]) allows to get rid of the increase of grid intervals on the plane $x_3 = 30$ (see figure 7 (b)). For this purpose, the optimization procedure for adaptation [11, 17] is modified: now, instead of squared of variations

$$[\Phi(H_{i+1,j,k}) - \Phi(H_{ij,k})]^2, [\Phi(H_{i,j+1,k}) - \Phi(H_{ij,k})]^2, [\Phi(H_{i,j,k+1}) - \Phi(H_{ij,k})]^2,$$

the following weight functions $w_1 = 1 + A_A [\Phi(H_{i+1,j,k}) - \Phi(H_{ij,k})]^2, w_2 = 1 + A_A [\Phi(H_{i,j+1,k}) - \Phi(H_{ij,k})]^2, w_3 = 1 + A_A [\Phi(H_{i,j,k+1}) - \Phi(H_{ij,k})]^2$, are used in (3) for adaptation under the given monitoring function $\Phi$, and

$$w_1 = 1 + A_A [\Phi(H_{i+1,j,k}) - \Phi(H_{ij,k})]^2 + A_{x_3}^2 \left[ \partial \Phi / \partial x_3(H_{i+1,j,k}) - \partial \Phi / \partial x_3(H_{ij,k}) \right]^2,$$

$$w_2 = 1 + A_A [\Phi(H_{i,j+1,k}) - \Phi(H_{ij,k})]^2 + A_{x_3}^2 \left[ \partial \Phi / \partial x_3(H_{i,j+1,k}) - \partial \Phi / \partial x_3(H_{ij,k}) \right]^2,$$

$$w_3 = 1 + A_A [\Phi(H_{i,j,k+1}) - \Phi(H_{ij,k})]^2 + A_{x_3}^2 \left[ \partial \Phi / \partial x_3(H_{i,j,k+1}) - \partial \Phi / \partial x_3(H_{ij,k}) \right]^2,$$

for the adaptation under the given monitoring function $\Phi$ and its derivative. In the case with the weight functions, the weight coefficient $A_A$ is not used before the functional $D_A$ in (2). It is inserted in the weight function along with the additional weight coefficient $A_{x_3}^2$ for the adaptation under the partial derivative $\partial \Phi / \partial x_3$. For figure 7 (a) $A_A = 100, A_{x_3}^2 = 0$, for figure 7 (b) $A_A = 100, A_{x_3}^2 = 30$.

5. Grid generation for generalizations of the volumes of revolutions

![Figure 8](image-url)  

**Figure 8.** An illustration of a face forming for a generalization of a volume of revolution (a)–(d), an optimal grid (e), surfaces of revolution numbered by colors 1–6 (f).

For determining these constructions [5], two simply-connected domains $U_L, U_R$ (see figure 8 (a)) with the boundaries $\partial U_L = \bigcup e_i^L, \partial U_R = \bigcup e_i^R, e_i^L = e_i^R$ and the set of axes of rotation $\alpha_i$ for
elements $e_i$, which can be straight line segments, arc of circles or ellipses are given in the plane $x_1x_3$. The rotation of the element $e_i$ around the axis $a_i$ forms the surfaces $S_i$ (figure 8 (c)). Surfaces $S_i$ define one of the faces $S$ of the curvilinear hexahedron $G$ (see figure 8 (d)). The lines $L_i^0$ are orthogonal projections onto the plane $x_1x_3$ of intersection lines $L_i$ of the surfaces $S_i$, $S_{i+1}$ (figure 8 (b), (c), (d)). The plane domain $S^0$ bounded by the elements $e_i^L$, $e_i^R$ is divided by lines $L_i^0$ into subdomains $S_i^0 : S^0 = \bigcup S_i^0$. The surface $S$ is formed by the parts $S_i$ orthogonal projection of which onto the plane $x_1x_3$ coincides with $S_i^0$. The generation of a grid (see examples in figure 8 (e)–(f)) in such constructions is carried out by the method described in section 2 through the optimization ([9], [12]) of the initial grid [5] constructed by the geometrical approach.

Conclusion

New developments of the technology are realized in computer codes written in C++. The developing technology allowed to improve essentially the effectiveness of modeling in comparison with traditionally used rotational grids. The further development of the technology will be performed for the cases of deformation of the volume of revolution by the generalization of the volume of revolution.

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