Fully frustrated Josephson junction ladders with Mobius boundary conditions as topologically protected qubits

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Abstract

We show how to realize a “protected” qubit by using a fully frustrated Josephson Junction ladder (J JL) with Mobius boundary conditions. Such a system has been recently studied within a twisted conformal field theory (CFT) approach and shown to develop the phenomenon of flux fraction-alization. The relevance of a “closed” geometry has been fully exploited in relating the topological properties of the ground state of the system to the presence of half flux quanta and the emergence of a topological order has been predicted. In this letter the stability and transformation properties of the ground states under adiabatic magnetic flux change are analyzed and the deep consequences on the realization of a solid state qubit, protected from decoherence, are presented.

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Arrays of weakly coupled Josephson junctions provide an experimental realization of the two dimensional (2D) XY model. A Josephson junction ladder (JJL) is the simplest quasi-one dimensional version of an array in a magnetic field \[5\]; recently such a system has been the subject of many investigations because of its possibility to display different transitions as a function of the magnetic field, temperature, disorder, quantum fluctuations and dissipation. In a recent paper \[3\] we analyzed the phenomenon of fractionalization of the flux quantum \[\frac{hc}{2e}\] in a fully frustrated JJL in order to investigate how the phenomenon of Cooper pair condensation could cope with properties of charge (flux) fractionalization, typical of a low dimensional system with a discrete \[Z_2\] symmetry. The role of such a symmetry was recognized to be crucial for demanding more general boundary conditions, of the Mobius type, at the end sites of the ladder \[3\]. The same feature was evidenced also in quantum Hall systems in the presence of impurities or defects \[6\] \[7\] \[8\]. Furthermore a \[Z_2\] symmetry is present in the fully frustrated XY (FFXY) model or equivalently, see Refs. \[9\] \[10\], in two dimensional Josephson junction arrays (JJA) with half flux quantum \[\frac{1}{2}\frac{hc}{2e}\] threading each square cell and accounts for the degeneracy of the ground state. We noticed how it was possible to generate non trivial topologies, i.e. the torus, in the context of a CFT approach, which allowed us to construct a ground state wave function, whose center of charge could describe a coherent superposition of localized states sharing all the non trivial global properties of the order parameter. In particular for the FFXY model they were shown to be closely related to the presence of half flux quanta, also viewed as topological defects \[8\]. The emergence of topological order in fully frustrated JJs with non trivial geometry has been predicted and fully exploited in Ref. \[4\] by means of CFT techniques. Such a concept was first introduced in order to describe the ground state of a quantum Hall fluid \[11\] but today it is of much more general interest \[12\]. Two features of topological order are very striking: fractionally charged quasiparticles and a ground state degeneracy depending on the topology of the underlying manifold, which is lifted by quasiparticles tunneling processes. In general a system is in a topological phase if its low-energy, long-distance effective field theory is a topological quantum field theory that is, if all of its physical correlation functions are topologically invariant up to corrections of the form \[e^{-\frac{\Delta}{T}}\] at temperature \[T\] for some nonzero energy gap \[\Delta\]. More recently superconductors have been proposed in which superconductivity arises from a topological mechanism rather than from a Ginzburg-Landau paradigm: the key feature is a mapping on an effective Chern-Simons gauge theory, which turns out to be exact in the case of JJA and frustrated JJA \[13\]. As we will stress in the following, topological order is crucial for the implementation of fully frustrated JJs as “protected” qubits \[13\] \[15\] in solid state quantum computation realm. The idea in all such realizations is that the systems involved (large and small size Josephson junction arrays of special geometry \[14\] \[16\]) share the property that, in the classical limit for the local superconducting variables, the ground state is highly degenerate. The residual quantum processes within such a low energy subspace lift the classical degeneracy in favor of macroscopic coherent superpositions of classical ground states \[16\]. An example of such a system has been proposed, which consists of chains of rhombi frustrated by an half flux quantum \[16\] with the property that in the classical limit each rhombus has two degenerate states. The protected degen-
eracy in all such systems emerges as a natural property of the lattice Chern-Simons gauge theories which describe them \[16\]. In general, if a physical system has topological degrees of freedom that are insensitive to local perturbations (that is noise), then information contained in those degrees of freedom would be automatically protected against errors caused by local interactions with the environment \[15\].

The aim of this letter is to show how to realize a “protected” qubit in terms of a fully frustrated Josephson Junction ladder (JJL) with Mobius boundary conditions by fully exploiting the implications of “closed” geometries on the ground state global properties of the system, already studied in Ref. \[3\]. Such a qubit would be the elementary building block of a “protected” quantum computer. The task appears to be not very simple; in general we need a quantum system with \(2^K\) quantum states \((K\text{ being the number of big openings in the Josephson system under study})\) which are degenerate in the absence of external perturbations and are robust against local random fluctuations, that is against noise. This means that any coupling to the environment doesn’t induce transitions between the \(2^K\) quantum states or change their relative phases. Summarizing, we need a system, whose Hilbert space contains a \(2^K\)-dimensional subspace characterized by the crucial property that any local operator \(\hat{O}\) has only state-independent diagonal matrix elements up to vanishingly small corrections: \(\langle n | \hat{O} | m \rangle = O_0 \delta_{mn} + o[\exp (-L)],\) \(L\) being the system size. A possible answer to such a highly non trivial requirement could be a system with a protected subspace built up by a topological degeneracy of the ground state \[15\]. An alternative approach would be to exhibit a low-energy effective field theory for the system under study which is a topological one and whose vacua are topologically degenerate and, then, robust against noise. This is the approach which we follow in the present letter; in particular we show how to get a protected subspace with \(2^K\) quantum states, \(K = 1,\) by considering a Josephson junction ladder and closing it by imposing Mobius boundary conditions, in order to get a non trivial topology. We will show that such a system is described by a low-energy effective field theory which is a twisted conformal field theory \[1, 2\] \[6\] \[7\]. Such a theory accounts very well for the topological properties of the system under study \[3\] \[4\]. In particular we analyze the stability and transformation properties of the ground state wave functions under adiabatic magnetic flux change; in this way we are able to identify the two states of a possible protected qubit and also to describe its manipulation: “flip state” processes.

We recall that Josephson junction arrays (JJA) are a very useful tool for investigating quantum-mechanical behaviour in a wide range of parameters space, from \(E_C \gg E_J\) (where \(E_C = (2e)^2/C\) is the charging energy and \(E_J = \hbar/eI_c\) is the Josephson coupling energy; \(C\) is the capacitance of each island and \(I_c\) is the critical current of each junction) to \(E_J \gg E_C\). In fact there exists a couple of conjugate quantum variables, the charge and phase of each superconducting island, and two dual descriptions of the array can be given \[17\]: a) through the charges (Cooper pairs) hopping between the islands, b) through the vortices hopping between the plaquettes. Furthermore in the presence of an external magnetic field charges gain additional Aharonov-Bohm phases and, conversely, vortices moving around islands gain phases proportional to the average charges on the islands.
Such basic quantum interference effects found applications in recent proposals for solid state qubits for quantum computing, based on charge [19] or phase [20] degrees of freedom in JJAs. “Charge” devices operate in the regime $E_C \gg E_J$ while “phase” or “flux” devices are characterized by strongly coupled junctions with $E_J \gg E_C$.

Let us now focus on the simplest physical array one can devise in order to meet all the above requests, that is a Josephson junction ladder with $N$ plaquettes closed in a ring geometry with a half flux quantum ($\frac{1}{2}\Phi_0 = \frac{\hbar c}{2e}$) threading each plaquette [5], and describe briefly its general properties before introducing an interaction of the charges (Cooper pairs) with a magnetic impurity (defect), as drawn in Fig. 1. With each site $i$ we associate a phase $\varphi_i$ and a charge $q_i = 2e n_i$, representing a superconducting grain coupled to its neighbors by Josephson couplings; $n_i$ and $\varphi_i$ are conjugate variables satisfying the usual phase-number commutation relation. The Hamiltonian describing the system is given by the quantum phase model (QPM):

$$H = -\frac{E_C}{2} \sum_i \left( \frac{\partial}{\partial \varphi_i} \right)^2 - \sum_{\langle ij \rangle} E_{ij} \cos (\varphi_i - \varphi_j - A_{ij}),$$

(1)

where $E_C = \frac{(2e)^2}{C}$ ($C$ being the capacitance) is the charging energy at site $i$, while the second term is the Josephson coupling energy between sites $i$ and $j$ and the sum is over nearest neighbors. The most general form for the charging energy would be $\frac{1}{2} q_i C^{-1}_{ij} q_j$, where $C^{-1}_{ij}$ is the inverse capacitance matrix, but in this letter we assume for simplicity that the most important contribution arises from the self-energy of each grain [21][5]. $A_{ij} = \frac{2\pi}{\Phi_0} \int_{j}^{i} A \cdot dl$ is the line integral of the vector potential associated to an external magnetic field $B$ and $\Phi_0 = \frac{\hbar c}{2e}$ is the magnetic flux quantum. The gauge invariant sum around a plaquette is $\sum_p A_{ij} = 2\pi f$ with $f = \frac{\Phi}{\Phi_0}$, where $\Phi$ is the flux threading each plaquette of the ladder. Let us label the phase fields on the two legs with $\varphi_i^{(a)}$, $a = 1, 2$ and assume $E_{ij} = E_x$ for horizontal links and $E_{ij} = E_y$ for vertical ones. Let us also make the gauge choice $A_{ij} = +\pi f$ for the upper links, $A_{ij} = -\pi f$ for the lower ones and $A_{ij} = 0$ for the vertical ones, which corresponds to a vector potential parallel to the ladder and taking opposite values on upper and lower branches.

Thus the effective quantum Hamiltonian (11) can be written as [5]:

$$-H = \frac{E_C}{2} \sum_i \left[ \left( \frac{\partial}{\partial X_i} \right)^2 + \left( \frac{\partial}{\partial \phi_i} \right)^2 \right] + \sum_i \left[ 2E_x \cos (X_{i+1} - X_i) \cos (\phi_{i+1} - \phi_i - \pi f) + E_y \cos (2\phi_i) \right],$$

(2)

after performing the change of variables: $\varphi_i^{(1)} = X_i + \phi_i$, $\varphi_i^{(2)} = X_i - \phi_i$, where $X_i$, $\phi_i$ (i.e. $\varphi_i^{(1)}$, $\varphi_i^{(2)}$) are only phase deviations of each order parameter from the commensurate phase and should not be identified with the phases of the superconducting grains [5].
When \( f = \frac{1}{2} \) and \( E_C = 0 \) (classical limit) the ground state of the 1D frustrated quantum XY (FQXY) model displays - in addition to the continuous \( U(1) \) symmetry of the phase variables - a discrete \( Z_2 \) symmetry associated with an antiferromagnetic pattern of plaquette chiralities \( \chi_p = \pm 1 \), measuring the two opposite directions of the supercurrent circulating in each plaquette. Thus it has two symmetric, energy degenerate, ground states characterized by currents circulating in the opposite directions in alternating plaquettes in full analogy with the checkerboard ground states of the 2D system [22]. For small \( E_C \) there is a gap for creation of kinks in the antiferromagnetic pattern of \( \chi_p \) and the ground state has quasi long range chiral order [3]; furthermore the charge noise, which is the strongest noise, has less effect in such a regime [14].

The ladder under study can be closed and arranged in a Corbino disk geometry. As we will argue in the following, this is the relevant geometry for the physical implementation of an ideal quantum computer. In closing the ladder, we can distinguish two inequivalent configurations, corresponding to an even or odd number of plaquettes in the ladder. It is due to the antiferromagnetic pattern of plaquette chiralities, which characterizes the JJL ground states.

- In the even case, the plaquettes on the opposite sides of the ladder have opposite chiralities, for both the degenerate ground states. So, the closed geometry can be realized gluing the opposite sides of the ladder keeping the ground state antiferromagnetic pattern.

- In the odd case, the plaquettes on the opposite sides of the ladder have the same chiralities, for both the degenerate ground states. In this case the ladder has to be modified; a magnetic impurity has to be introduced, in the glue-point, which couples the up and down phases through its interaction with the Cooper pairs of the two legs (as represented in Fig.1).

In the odd configuration, the two degenerate ground states can be represented by \(|0\rangle\) and \(|1\rangle\) and distinguished by the value of the sum over all plaquettes \( \sum_p \chi_p \) respectively equal to \(-1\) and \(+1\).
The outlined pattern, in this closed geometry, evidences the emerging of non trivial topological properties intimately related to the twofold degeneracy of the ground states which appear to be “protected” from external perturbations [11][4][3]. Moreover, such a pattern is size independent, that is, it depends on the number of plaquettes only by its party and, in particular, persists also in the continuum limit \( N \to \infty \) and \( a \to 0 \), where \( N \) is the number of plaquettes, \( a \) is the side-size of a plaquette and \( L = aN \) is the constant length of the ladder. Our strategy is to study the continuum limit of the JJL in this closed geometry; indeed, in the continuum the powerful tools of the CFT can be used to evince the topological properties of the system which, when extended to finite ladder, allow us to propose a JJL realization of a qubit device.

In the following, we will setup the CFT analysis, while some more details on our twisted model (TM) can be found in the Appendix. Performing the continuum limit of the Hamiltonian (2), one obtains:

\[
-H = \frac{E_C}{2} \int dx \left[ \left( \frac{\partial}{\partial X} \right)^2 + \left( \frac{\partial}{\partial \phi} \right)^2 \right] + \int dx \left[ E_x \left( \frac{\partial X}{\partial x} \right)^2 + E_x \left( \frac{\partial \phi}{\partial x} - \frac{\pi}{2} \right)^2 + E_y \cos(2\phi) \right]
\]  

where we see that the \( X \) and \( \phi \) fields are decoupled. In fact the \( X \) term of the above Hamiltonian is that of a free quantum field theory while the \( \phi \) one coincides with the quantum sine-Gordon model. Through an imaginary-time path-integral formulation of such a model [23] it can be shown that the 1D quantum problem maps into a 2D classical statistical mechanics system, the 2D fully frustrated XY model, where the parameter \( \alpha = \left( \frac{E_x}{E_C} \right)^{\frac{1}{2}} \) plays the role of an inverse temperature [5]. We work in the regime \( E_x \gg E_y \) where the ladder is well described by our TM with central charge \( c = 2 \).

Let us introduce in the continuum the closed geometry; in order to do so, we require the compactification of the \( \phi^{(a)} \) variables to recover the angular nature of the up and down fields. Then the even and odd configurations rising in the closed geometry of the finite ladder correspond in the continuum to two different boundary conditions for the fields, respectively, periodic (P) and Mobius (A) boundary conditions:

\[
\varphi_L^{(1)} (x = 0) = +\varphi_R^{(2)} (x = 0) \quad \text{and} \quad \varphi_L^{(1)} (x = 0) = -\varphi_R^{(2)} (x = 0),
\]

where we have indicated the compactified phases of the two legs as \( \varphi_L^{(1)} \) and \( \varphi_R^{(2)} \), \( L \) and \( R \) staying for left and right components. Indeed in the limit of strong coupling the interaction between the magnetic impurity at point \( x = 0 \) (glue-point shown in Fig. 1) and the up and down phases gives rise to these non trivial boundary conditions for the fields [6]. Such a Mobius condition is naturally satisfied by the twisted field \( \phi (z) \) of our TM (see eq. [23] in the Appendix), which describes both the left moving component \( \varphi_L^{(1)} \) and the right moving one \( \varphi_R^{(2)} \) in a folded description of a system with boundary [6] [7]. In fact the
TM results in a chiral description of the system just described in terms of the chiral fields $X$ and $\phi$ (see eqs. (22), (23)). The $m$-reduction technique [2] well accounts for these non trivial boundary conditions for the JJ ladder due to the presence of a topological defect, already built in the construction.

Our goal is the study of the stability and transformation properties of the four ground states of the JJL in the closed geometry under an adiabatic elementary flux change ($\pm \hbar c/2e$) through the central hole of the Corbino disk. Because of the energy gap, such an adiabatic transformation is believed to leave the system in a ground state which can be different from the original one, due to the occurrence of the ground state degeneracy. This analysis will be the crucial step for the identification of the two states of a possible protected qubit and for its manipulation: “flip state” processes.

We use the TM model to analyze such properties by standard conformal techniques. In the CFT description the ground state wave functions are expressed as correlation functions of the primary fields describing the elementary particles, in our case the Cooper pairs. In particular in the torus topology the characters of the theory are in one to one correspondence with the ground states. Indeed, as we are going to show, they describe the components of the “center of charge” for the corresponding ground state wave functions [24], which represent coherent states of Cooper pairs on the torus. To such an extent, let us define for a single Cooper pair on a torus $a \times b$ an effective mean-field Hamiltonian of the kind

$$H(x, y) = H_0(x, y) + V(x, y),$$

where $H_0(x, y) = \left[-i\hbar \nabla - 2eA/c\right]^2/2m$ is the Hamiltonian in the presence of an uniform magnetic field and $V(x, y)$ is a mean-field scalar potential such that $V(x, y) = V(x + a, y) = V(x, y + b)$. It is now possible to define the magnetic translations operators $\tilde{S} = e^{i\theta_x a/\hbar}$ and $\tilde{T} = e^{i\theta_y b/\hbar}$ along the two cycles $A$ and $B$ of the torus respectively, where:

$$\theta_x = \pi x - \frac{2e}{\hbar} By = -i\hbar \partial_x, \quad \theta_y = \pi y + \frac{2e}{\hbar} Bx = -i\hbar \partial_y + \frac{2e}{\hbar} Bx$$

and the gauge choice $\nabla A(x, y) = (-By, 0)$ has been made. They satisfy the relations:

$$\left[\tilde{S}, \mathcal{H}(x, y)\right] = \left[\tilde{T}, \mathcal{H}(x, y)\right] = 0, \quad \tilde{S}\tilde{T} = e^{2\pi i \Phi_{ab}/\Phi_0} \tilde{T}\tilde{S},$$

where $\Phi_{ab}$ is the magnetic flux threading the torus surface, and their action on the wave functions can be defined as:

$$\tilde{S}\varphi(x, y) = \varphi(x + a, y), \quad \tilde{T}\varphi(x, y) = e^{2\pi i B_{xy}/\Phi_0} \varphi(x, y + b).$$

Now for $\Phi_{ab} = M\Phi_0$ (i.e. when the magnetic flux $\Phi_{ab}$ is an integer number of flux quanta $\Phi_0 = \hbar c/2e$) the condition $\left[\tilde{S}, \tilde{T}\right] = 0$ holds and we can simultaneously diagonalize the operators $H(x, y), \tilde{S}, \tilde{T}$. By introducing adimensional coordinates on the torus of the kind $T = \{\omega = x + \tau y : x \in [0, 1], y \in [0, 1]\}$, eqs. (7) can be rewritten as:

$$\tilde{S}\varphi(\omega) = \varphi(\omega + 1), \quad \tilde{T}\varphi(\omega) = e^{2\pi i M\omega} \varphi(\omega + \tau).$$

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One can look for eigenfunctions of the kind \( \varphi(\omega) = e^{i\pi M y^2 \tau} f(\omega) \) and define magnetic translation operators \( S_\alpha, T_\alpha \) acting only on \( f(\omega) \):

\[
S_\alpha f(\omega) = f(\omega + \alpha), \quad T_\alpha f(\omega) = e^{i\pi M (\alpha^2 \tau + 2\omega)} f(\omega + \alpha \tau).
\]  

(9)

In this way eqs. (8) become:

\[
\tilde{S}\varphi(\omega) = e^{i\pi M y^2 \tau} S_1 f(\omega), \quad \tilde{T}\varphi(\omega) = e^{i\pi M y^2 \tau} T_1 f(\omega).
\]  

(10)

Going back to the system of Cooper pairs, in order to describe a coherent state on a torus we look for wave functions of the kind:

\[
\psi_a(\omega_1, \ldots, \omega_M) = e^{i\pi M \tau \sum_{i=1}^M \int \frac{d\psi_a(\omega_1, \ldots, \omega_M)}{\int \frac{d\psi_a(\omega_1, \ldots, \omega_M)}}},
\]  

(11)

\[
f_a(\omega_1, \ldots, \omega_M) = \prod_{i<j=1}^M \left[ \frac{\theta_1(\omega_{ij}, \tau)}{\theta_1'(0, \tau)} \right]^4 \chi_a(\omega|\tau),
\]  

(12)

where \( \omega = \sum_{i=1}^M \omega_i \) is the “center of charge” variable and the non local functions \( \chi_a(\omega|\tau) \) are the characters of our theory (the TM). In fact it can be shown that such characters are eigenfunctions of the following generalized magnetic translations operators:

\[
S_\alpha = \prod_{i=1}^M S_{\alpha/M}^i, \quad T_\alpha = \prod_{i=1}^M T_{\alpha/M}^i,
\]  

(13)

where \( S_{\alpha/M}^i \) and \( T_{\alpha/M}^i \) are the magnetic translation operators for the single Cooper pair. In this sense our characters represent highly non local functions: all the topological properties of our system are codified in such functions.

On a pure topological base we expect for the torus a doubling of the ground state degeneracy, which can be seen at the level of the conformal blocks (characters) of our TM. Indeed we get for the periodic (even ladder) case an untwisted sector, \( P - P \) and \( P - A \), described by the four conformal blocks (35)-(38), and for the Mobius (odd ladder) case a twisted sector, \( A - P \) and \( A - A \), described by the four conformal blocks (31)-(34). Now we extract from the vacua of our theory the two states of the “protected” qubit.

Let \( A \) be the cycle of the torus which surrounds the hole of the Corbino disk; in the twisted sector it is composed by the leg 1 and the leg 2 through the gluing-point as shown in Fig. 1. As underlined in our previous publications [3], the ground state wave functions of the twisted and untwisted sectors of the TM are characterized by different monodromy properties along the \( A \)-cycle. In particular the characters of the untwisted sector are single-valued functions along the \( A \)-cycle while the characters of the twisted sector pick up a common \((-1)\) phase factor along the \( A \)-cycle. Such phase factors can be interpreted as Bohm-Aharonov phases generated while a Cooper pair is taken along the \( A \)-cycle. The above observation evidences a strong difference between the two inequivalent topological even ladder (untwisted sector) and odd ladder (twisted sector) configurations. Indeed in the odd ladder the ground state wave functions show a non trivial behavior implying the
trapping of a half flux quantum \( \left( \frac{1}{2} \left( \frac{hc}{2e} \right) \right) \) in the hole of the Corbino disk. Instead in the even ladder, due to the single-valued ground state wave functions, only integer numbers of flux quantum can be attached to the hole.

It is worth pointing out the central role played by the “isospin” (or neutral) component of the TM in producing the discussed non trivial monodromy properties. To this end let us recall that the TM is a \( c = 2 \) CFT, composed by a \( c = 1 \) charged and a \( c = 1 \) isospin CFT components, as it is well evidenced by the character decompositions in the Appendix. Now the transport of a Cooper pair along the \( A \)-cycle is implemented by a simultaneous and identical translation \( \Delta w_c = \Delta w_n = 2 \) of the charged and the isospin variables. The “charged” characters have trivial monodromy with respect to this transformation, being:

\[
K_l(w_c + 2|\tau) = K_l(w_c|\tau), \quad l = 0, \ldots, 3, \tag{14}
\]

while the “isospin” contribution is the one responsible for the non trivial monodromy of the complete ground state wave functions:

\[
\chi_{0,1}^\frac{1}{2}(2|\tau) = \chi_{0,1}^\frac{1}{2}(0|\tau), \quad \chi_{16}^\frac{1}{16}(2|\tau) = (-1)\chi_{16}^\frac{1}{16}(0|\tau) \tag{15}
\]

and the same is true for the characters \( \bar{\chi}_\beta \). Let us notice that the change in sign in the last relation of eq. 15 shows the presence in the spectrum of excitations carrying fractionalized charge quanta. More precisely the presence in the isospin component of one twist-field (with conformal dimension \( \Delta = 1/16 \)) characterizes all the conformal blocks of the twisted sector and accounts for the trapping of a half flux quantum in the hole of the closed JJL. At this point it is crucial to observe that in order to create such a fractionally charged excitation in the ground state a finite energy must be provided, so assuring the presence of a finite gap separating the ground state from the excited ones (that is in complete analogy with the presence of a gap separating the ground state from higher energy states in the Laughlin Hall fluid [25]).

We are now in the position to address the study of the stability and transformation properties of the ground state wave functions when a magnetic flux change takes place through the central hole of the closed JJL. The above analysis shows that at the level of the wave functions it has the effect to change the monodromy along the \( A \)-cycle due to the corresponding change in the Bohm-Aharonov phase. Such a modification can be implemented on the center of charge component of the wave function, i.e. the characters, with a well defined transformation. In the case of the charged component this analysis has been brought out already in [26] in the physical contest of the quantum Hall effect. Let us adapt here the results for the charged component of our TM.

On a pure physical ground the fact that we are considering a magnetic flux change, which is on one side integer in the flux quantum (one flux quantum change \( \pm \frac{hc}{2e} \)) and on the other side adiabatic, suggests both that the monodromy properties do not change and that the system remains in a degenerate ground state. Such a physical picture is in fact
confirmed for the charged component of our TM; indeed, the flux change is implemented
on the level of charged characters by the transformation $T_{1/2}^c$:

\[
T_{1/2}^c K_l (w_c|\tau) \equiv e^{i\frac{1}{2}2\pi\tau + \frac{2\pi n w_c}{2}} K_l (w_c + \frac{\tau}{2}|\tau) = K_{l+1} (w_c|\tau), \quad l = 0, \ldots, 3. \quad (16)
\]

In particular the charged component wave functions realize a flip process ($l \rightarrow l+1$) under
one magnetic flux quantum change.

However the analysis for the complete TM, with charged and isospin components, is
more involved. In particular the problem of the stability of the ground state wave func-
tions under the change of one flux quantum in the central hole has to be clearly brought
out. This is mainly due to the non trivial interplay between charged and isospin components summarized in the so-called $m$-ality parity rule, which characterizes the gluing condition for the charged and isospin excitations (see Appendix). The main point being the compatibility between such parity rule and the transformation of the complete characters of TM under the insertion of a flux quantum through the hole of the closed ladder, which reads as:

\[
T_{1/2} f(w_n|w_c|\tau) = e^{2i\pi(\alpha^2\tau + \alpha(w_n+w_c))} f(w_n + \alpha\tau|w_c + \alpha\tau|\tau)|_{\alpha=1/2}, \quad (17)
\]

where $f(w_n = 0|w_c|\tau)$ stays for any character of TM. The full list of such transformations
is presented in Appendix, here we only comment on the very simple and clear picture
which emerges for the stability and transformations of the ground states of the closed JJL.

The even configuration of the closed JJL (periodic case) is proven to be unstable under
this transformation. Indeed eq. (10) and (11) show the decoupling of the untwisted $P-A$
sector and of the state $\tilde{\chi}_{\alpha}^+$ of the $P-P$ sector while eq. (12) shows that the state $\tilde{\chi}_{\beta}^+$ of
$P-P$ sector gets excited by this transformation.

For the odd configuration of the closed JJL (Mobius case), eq. (13) shows that the
twisted $A-A$ sector decouples. So we are left only with the $A-P$ sector, with the two
ground states flipping one into the other under an adiabatic flux change of $\pm \frac{hc}{2e}$ through
the central hole, as it can be seen from eq. (14).

Summarizing, between the two inequivalent configurations for the closed JJL, corre-
sponding to even and odd ladder, just the odd one is proven to be stable under an adiabatic
flux change of $\pm \frac{hc}{2e}$ through the central hole. Then in terms of the ground states center
of charge wave functions (characters), we can make the following identifications:

\[
|0\rangle \sim \chi_{(0)}^+(0|w_c|\tau), \quad |1\rangle \sim \chi_{(1)}^+(0|w_c|\tau). \quad (18)
\]

Then $|0\rangle$ and $|1\rangle$ are the two ground states of the odd closed JJL characterized by the
size invariant sum over all plaquettes $\sum_p \chi_p$ respectively equal to $-1$ and $+1$. 
Figure 2: The two logical states of the JJL qubit: a) for the state $|1\rangle$, b) for the state $|0\rangle$.

Based on the above consideration we are ready to propose the odd closed JJL as our protected qubit. In fact the two ground states $|0\rangle$ and $|1\rangle$ work as the two logical states of the qubit and the required one qubit operations:

$$|0\rangle \rightarrow |1\rangle, \quad |1\rangle \rightarrow |0\rangle,$$

are simply implemented by insertion of a flux quantum $(\pm \frac{hc}{2e})$ through the central hole.

Let us now make a comment on the stability of such qubit device under local perturbations. Local perturbations can be viewed in such a context as creating a finite energy excitation above the ground states in the form of double kinks. A double kink can be produced from the ground state by exchanging the chirality of two nearest neighbor plaquettes in the ladder and, as such, it is local and it leaves invariant the chirality sum $\sum_p \chi_p$ over all plaquettes and so the characterization of the two logical states. Furthermore, since a double kink can be described by the presence of two elementary half flux quanta of opposite sign $(\pm \frac{hc}{2e})$ localized in between the pairs of plaquettes with the same chirality, it doesn’t produce any flux change in the central hole. In this way the excited logical state wave function shows the same monodromy properties along the $A$-cycle as the corresponding ground state one and, in particular, satisfies the same transformation rules under an adiabatic flux change of $\pm \frac{hc}{2e}$ through the central hole. Summarizing, the characterization of the two logical states and their flipping processes are left unchanged under local perturbations, which produce a finite energy excitation above the ground state.

The minimal configuration for such a protected qubit is represented in Fig. 2 by a closed fully frustrated JJL with $N = 3$ plaquettes, 3 being the minimum odd number of plaquettes needed in order to fulfill all the above requests.

Now it should be possible to construct symmetric ($s$) and antisymmetric ($a$) linear combinations of such degenerate ground states and then to control their amplitude and
relative phase: such operations are needed in order to prepare the qubit in a definite state and to manipulate it \[27\]. In order to realize the logical \textit{NOT} we must perform an adiabatic change of local magnetic fields that drags one vortex across the system, i.e. a flux quantum through the $A$-cycle of the torus, and flips the state of the system, so lifting the degeneracy \[14\].

Josephson junction ladders with annular geometry have been fabricated within the trilayer $Nb/Al - AlO_x/Nb$ technology and experimentally investigated \[28\], but in such a case the application of an external transverse magnetic field is needed in order to fulfill the requirement of full frustration and that could be another source of decoherence. It is now possible to avoid such a problem by realizing arrays with a built-in frustration. In fact high-$T_c$ Josephson junction arrays have been recently proposed \[29\], which support degenerate spontaneous current states in zero magnetic field due to the presence of plaquettes containing an odd number of $\pi$-junctions \[30\]. Such unconventional junctions can be realized because of the $d$-wave symmetry of high-$T_c$ superconductors \[31\], which produces a $\pi$-shift in the phase of the wave function on one side of the junction. Furthermore $\pi$-junctions can be obtained also with superconducting-ferromagnetic-superconducting junctions (SFS) \[32\]. In this way it is possible to avoid the external frustration bias but, in any case, external magnetic fields are needed for control and read-out operations: in fact our JJL qubit is a flux device \[20\].

So in principle an experimental setup for the realization of our protected qubit can be conceived: the JJL, arranged in a ring geometry (Corbino disk) with an odd number of plaquettes along the inner hole, should be equipped with a coil $P_{\text{bias}}$ which can be used to set the system in one of the two ground states $|0\rangle$, $|1\rangle$; another coil $P_{\text{hole}}$ is needed in order to control the mixing of the two ground states and to carry out the flipping. Finally $N$ read-out coils, coupled to external conventional SQUIDs, are needed in order to read out the state of the system after the quantum evolution. The whole device has to be embedded in a superconducting cavity in order to guarantee the stability of the boundary conditions. In this way a reliable qubit is built up, whose quantum evolution can be controlled in order to perform all possible single qubit logical operations. Then such qubits can be antiferromagnetically coupled by means of suitable superconducting flux transformers ("eight" coils) which provide an inductive coupling and whose strength can be controlled within a wide range of useful values: in this way a quantum register can be realized and all multi-qubits logical operations can be performed.

Summarizing, a single non-interacting qubit is described by a double well potential and the external magnetic flux controls the energy difference between the minima, the symmetric situation being for $\Phi^e = 0$. Each logical state, $|0\rangle$ or $|1\rangle$, is represented by a wave function localized in a distinct potential well and corresponds to distinguishable flux states trapped in the plaquettes of the ladder with current flowing in opposite directions in alternating plaquettes. When the energy difference $\varepsilon$ of the minima of the two different wells is small with respect to the oscillation frequency $\omega$ around the minima, $\varepsilon \ll \omega$, these states become coupled and the wave functions spread over both the wells, the coupling being maximum in resonance conditions ($\varepsilon = 0$), while the energy eigenstates tend to be
localized in one of the well away from resonance as $\varepsilon$ is increasing. The coupling of the states can be described by a tunneling amplitude $\Delta(\varepsilon)$ and the effective Hamiltonian of any qubit reduces to the regular two-state form in the basis of these logical states:

$$H_{\text{eff}} = \frac{1}{2} \left[ \varepsilon (|0\rangle \langle 0| - |1\rangle \langle 1|) - \Delta (|0\rangle \langle 1| + |1\rangle \langle 0|) \right] = \frac{1}{2} (\varepsilon \sigma^z - \Delta \sigma^x) ,$$

where $\sigma^x$, $\sigma^z$ are the Pauli spin matrices.

The diagonal elements of $H_{\text{eff}}$ can be easily controlled by an external magnetic field in the $z$-direction producing an external flux $\Phi^e$ while the off-diagonal elements are related to the tunneling amplitude and thus are controlled by the adiabatic change of the magnetic flux in the central hole. The general state vector of such a qubit is the linear combination of the basis states:

$$|\Psi\rangle = \alpha |0\rangle + \beta |1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} ,$$

so it is described by two complex numbers $\alpha$ and $\beta$.

When the inductive coupling among qubits is turned on, there could be a bias in, say, the $j$-th qubit even though $\Phi^e_j = 0$ and, as a consequence, its logical states may be asymmetric. In the approximation in which every JJL can be considered as a two level system, the system of flux linked qubits can be described by an effective Hamiltonian of the kind:

$$\mathcal{H}_{\text{eff}} = \sum_j \varepsilon_j \sigma^z_j + \sum_j \Delta_j \sigma^x_j + \sum_{jk} \Lambda_{jk} \sigma^z_k \sigma^z_j .$$

In order to control such Hamiltonian, one should be able to modulate the tunneling amplitude of each qubit as well as to switch on and off the magnetic coupling between neighbors qubits. The analysis of multi-qubit logical operations will be the subject of a future publication.

In conclusion in this letter we have presented a simple collective description of a fully frustrated ladder of Josephson junctions arranged in a non trivial geometry, with a macroscopic half flux quantum trapped in the hole. The powerful tools of the CFT have been used to evince the topology, the stability and the transformation properties of the system. In particular it has been shown how such features can be exploited for the realization of a “protected” qubit: a simple device has been proposed and its operation mode has been briefly sketched.

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Appendix: reminder of TM for the JJL

Here we briefly summarize the main results of our theory, the TM, for the fully frustrated JJL \[4\][3]. We first construct the bosonic theory and show that its energy momentum tensor fully reproduces the Hamiltonian of eq. (3) for the JJL. That allows us to describe the JJL excitations in terms of the primary fields \(V_\alpha(z)\). Then we show that it is possible to construct the \(N\)-vertices correlator for the torus topology in 2\(D\) (basically by letting the edge to evolve in “time” and to interact with external vertex operators placed at different points). We assume that a suitable correlator is apt to describe the ground state wave function of the JJL at \(T = 0\) temperature and then perform an analysis of the symmetry properties of its center of charge wave function (conformal blocks), which emerge in the presence of vortices carrying half quantum of flux \((\frac{1}{2} (\frac{hc}{2e}))\).

Let us focus on the \(m\)-reduction procedure \[2\] for the special \(m = 2\) case (see Ref. \[1\] for the general case), since we are interested in a system with a \(Z_2\) symmetry and choose the “bosonic” theory \[4\][3], which well adapts to the description of a system with Cooper pairs of electric charge \(2e\) in the presence of a topological defect \[6\], i.e. a fully frustrated JJL. To each of the two legs (edges) of the ladder we assign a chirality, so making a correspondence between up-down leg and left-right chirality states.

Let us now write each phase field as the sum \(\varphi(x) = \varphi_L(x) + \varphi_R(x)\) of left and right moving fields defined on the half-line because of the topological defect located in \(x = 0\). Then let us define for each leg the two chiral fields \(\varphi_{e,o}(x) = \varphi_L(x) \pm \varphi_R(-x)\), each defined on the whole \(x\)-axis \[35\]. In such a framework the dual fields \(\varphi_o(x)\) are fully decoupled because the corresponding boundary interaction term in the Hamiltonian does not involve them \[36\]; they are involved in the definition of the conjugate momenta \(\Pi_o = (\partial_x \varphi_o) = (\frac{\partial}{\partial \varphi_o})\) present in the quantum Hamiltonian. Performing the change of variables \(\varphi^{(1)} = X + \phi\), \(\varphi^{(2)} = X - \phi\) (\(\varphi_X = X + \phi\), \(\varphi_\phi = X - \phi\) for the dual ones) we get the quantum Hamiltonian \[3\] but now all the fields are chiral ones. Finally let us identify in the continuum such chiral phase fields \(\varphi_e, a = 1, 2\), each defined on the corresponding leg, with the two chiral fields \(Q^{(a)}, a = 1, 2\) of the TM with central charge \(c = 2\).

As a result of the 2-reduction procedure \[1\][2] we get a \(c = 2\) orbifold CFT, the TM, whose fields have well defined transformation properties under the discrete \(Z_2\) (twist) group, which is a symmetry of the TM. Its primary fields content can be expressed in terms of a \(Z_2\)-invariant scalar field \(X(z)\), given by

\[
X(z) = \frac{1}{2} (Q^{(1)}(z) + Q^{(2)}(z)),
\]

(22)

describing the continuous phase sector of the theory, and a twisted field

\[
\phi(z) = \frac{1}{2} (Q^{(1)}(z) - Q^{(2)}(z)),
\]

(23)
which satisfies the twisted boundary conditions $\phi(e^{i\pi}z) = -\phi(z)$ \[1\]. More explicitly such a field can be written in terms of the left and right moving components $\varphi^{(1)}_L$, $\varphi^{(2)}_R$ as we stated above; then the Mobius boundary conditions given in eq. (4) are described by the boundary conditions for $\phi$. This will be more evident for closed geometries, i.e. for the torus case, where the magnetic impurity gives rise to a line defect in the bulk, so allowing us to resort to the folding procedure and introduce boundary states \[6\][7]. Such a procedure is used in the literature to map a problem with a defect line (as a bulk property) into a boundary one, where the defect line appears as a boundary state of a theory which is not anymore chiral and its fields are defined in a reduced region which is one half of the original one. Our approach, the TM, is a chiral description of that, where the chiral $\phi$ field defined in $(-L/2, L/2)$ describes both the left moving component and the right moving one defined in $(-L/2, 0), (0, L/2)$ respectively, in the folded description \[6\][7]. Furthermore to make a connection with the TM we consider more general gluing conditions:

$$\phi_L(x = 0) = \mp \phi_R(x = 0),$$

the $-(+)$ sign staying for the twisted (untwisted) sector. We are then allowed to use the boundary states given in \[37\] for the $c = 1$ orbifold at the Ising$^2$ radius. The $X$ field, which is even under the folding procedure, does not suffer any change in boundary conditions \[6\] while condition (4) is naturally satisfied by the twisted field $\phi(z)$. So topological order can be discussed referring to the characters with the implicit relation to the different boundary states (BS) present in the system \[6\]. These BS should be associated to different kinds of linear defects compatible with conformal invariance.

The fields in eqs. (22)-(23) coincide with the ones introduced in eq. (3). In fact the energy momentum tensor for such fields fully reproduces the second quantized Hamiltonian of eq. (3). The whole TM theory decomposes into a tensor product of two CFTs, a twisted invariant one with $c = \frac{3}{2}$ and the remaining $c = \frac{1}{2}$ one realized by a Majorana fermion in the twisted sector. In the $c = \frac{3}{2}$ sub-theory the primary fields are composite vertex operators $V(z) = U^\alpha_U (z) \psi(z)$ or $V_{qh} (z) = U^\alpha_X (z) \sigma(z)$, where

$$U^\alpha_X (z) = \frac{1}{\sqrt{z}} : e^{i\alpha_l X(z)} :$$

(24)

is the vertex of the continuous sector with $\alpha_l = \frac{l}{2}$, $l = 1, ..., 4$ for the $SU(2)$ Cooper pairing symmetry used here. Regarding the other component, the highest weight state in the isospin sector, it can be classified by the two chiral operators:

$$\psi(z) = \frac{1}{2\sqrt{z}} \left( : e^{i\sqrt{2} \phi(z)} : + : e^{i\sqrt{2} \phi(-z)} : \right), \quad \overline{\psi}(z) = \frac{1}{2\sqrt{z}} \left( : e^{i\sqrt{2} \phi(z)} : - : e^{i\sqrt{2} \phi(-z)} : \right),$$

(25)

which correspond to two $c = \frac{1}{2}$ Majorana fermions with Ramond (invariant under the $Z_2$ twist) or Neveu-Schwartz ($Z_2$ twisted) boundary conditions \[1\][2] in a fermionized version of the theory. The Ramond fields are the degrees of freedom which survive after the tunnelling and the parity symmetry, which exchanges the two Ising fermions, is broken.
Besides the fields appearing in eq. (25), there are the $\sigma(z)$ fields, also called the twist fields, which appear in the quasi-hole primary fields $V_{gh}(z)$. The twist fields have non local properties and decide also for the non trivial properties of the vacuum state, which in fact can be twisted or not in our formalism.

Starting from the primary fields $V_\alpha(z)$ we can now construct the non perturbative ground state wave function of the JJL system for the torus topology. It turns out that by construction it results as a coherent superposition of gaussian states with all the non trivial global properties of the order parameter. In fact by using standard conformal field theory techniques it is now possible to generate the torus topology, starting from the edge theory, just defined above. That is realized by evaluating the $N$-vertices correlator

$$\langle n| V_\alpha(z_1) \ldots V_\alpha(z_N) e^{2\pi i \tau L_0} |n\rangle,$$

where $V_\alpha(z_i)$ is the generic primary field representing the excitation at $z_i$, $L_0$ is the Virasoro generator for dilatations and $\tau$ the proper time. The neutrality condition $\sum \alpha = 0$ must be satisfied and the sum over the complete set of states $|n\rangle$ is indicating that a trace must be taken. It is very illuminating for the non expert reader to pictorially represent the above operation in terms of an edge state (that is a primary state defined at a given $\tau$) which propagates interacting with external fields at $z_1 \ldots z_N$ and finally getting back to itself. In such a way a 2D surface is generated with the torus topology. It is interesting to observe that such a procedure is equivalent to the coherent insertion of correlated relevant vortices (as provided by the CFT description) at positions $z_1 \ldots z_N$, as they appear in the non perturbative ground state of the physical JJL system. From such a picture it is evident then how the degeneracy of the non perturbative ground state is closely related to the number of primary states. Furthermore, since in this letter we are interested in the understanding of the topological properties of the system, we can consider only the center of charge contribution in the above correlator, so neglecting its short distances properties.

To such an extent the one-point functions are extensively reported in the following.

On the torus [2] the TM primary fields are described in terms of the conformal blocks of the $Z_2$-invariant $c = \frac{3}{2}$ sub-theory and of the non invariant $c = \frac{1}{2}$ Ising model, so reflecting the decomposition on the plane above outlined. The characters $\chi_0(0|\tau), \chi_\pm(0|\tau), \chi_+(0|\tau)$ express the primary fields content of the Ising model with Neveu-Schwartz ($Z_2$ twisted) boundary conditions [2], while

$$\chi_{(0)}^{c=3/2}(0|w_c|\tau) = \chi_0(0|\tau) K_0(w_c|\tau) + \chi_\pm(0|\tau) K_2(w_c|\tau),$$

$$\chi_{(1)}^{c=3/2}(0|w_c|\tau) = \chi_\pm(0|\tau) (K_1(w_c|\tau) + K_3(w_c|\tau)), $$

$$\chi_{(2)}^{c=3/2}(0|w_c|\tau) = \chi_+(0|\tau) K_0(w_c|\tau) + \chi_0(0|\tau) K_2(w_c|\tau)$$

represent those of the $Z_2$-invariant $c = \frac{3}{2}$ CFT. They are given in terms of a “charged” $K_\alpha(w_c|\tau)$ contribution:

$$K_{2l+i}(w|\tau) = \frac{1}{\eta(\tau)} \Theta \left[ \begin{array}{c} \frac{2l+i}{4} \\ 0 \end{array} \right] (2w|4\tau), \quad \text{with } l = 0, 1 \text{ and } i = 0, 1, \quad (30)$$
and a “isospin” one $\chi_\beta(0|\tau)$, (the conformal blocks of the Ising Model), where $w_c = \frac{1}{2\pi i} \ln z_c$ is the torus variable of the “charged” component while the corresponding argument of the isospin block is $w_n = 0$ everywhere.

If we now turn to the whole $c = 2$ theory, the characters of the twisted sector are given by:

$$\chi^+_0(0|w_c|\tau) = \bar{\chi}^+_0(0|\tau) \left( \chi_0 + \chi_+ \right) (0|\tau)(K_0 + K_2)(w_c|\tau),$$

$$\chi^+_1(0|w_c|\tau) = \chi^+_1(0|\tau) \left( \chi_0 + \bar{\chi}_+ \right) (0|\tau)(K_1 + K_3)(w_c|\tau),$$

for the $A - P$ sector and by:

$$\chi^-_0(0|w_c|\tau) = \bar{\chi}^-_0(0|\tau) \left( \chi_0 - \chi_+ \right) (0|\tau)(K_0 - K_2)(w_c|\tau),$$

$$\chi^-_1(0|w_c|\tau) = \chi^-_1(0|\tau) \left( \chi_0 - \bar{\chi}_+ \right) (0|\tau)(K_1 + K_3)(w_c|\tau),$$

for the $A - A$ one. Furthermore the characters of the untwisted sector are [2]:

$$\bar{\chi}^-_0(0|w_c|\tau) = \left( \bar{\chi}_0 \chi_0 - \bar{\chi}_+ \chi_+ \right) (0|\tau)K_0(w_c|\tau) + \left( \bar{\chi}_0 \chi_+ - \bar{\chi}_+ \chi_0 \right) (0|\tau)K_2(w_c|\tau),$$

$$\bar{\chi}^-_1(0|w_c|\tau) = \left( \bar{\chi}_0 \chi_+ - \bar{\chi}_+ \chi_0 \right) (0|\tau)K_0(w_c|\tau) + \left( \bar{\chi}_0 \chi_0 - \bar{\chi}_+ \chi_+ \right) (0|\tau)K_2(w_c|\tau),$$

for the $P - A$ sector while for the $P - P$ sector we have:

$$\bar{\chi}^+_0(0|w_c|\tau) = \frac{1}{2} \left( \bar{\chi}_0 - \bar{\chi}_+ \right) (0|\tau) \left( \chi_0 - \chi_+ \right) (0|\tau)(K_0 - K_2)(w_c|\tau),$$

$$\bar{\chi}^+_1(0|w_c|\tau) = \frac{1}{2} \left( \bar{\chi}_0 + \bar{\chi}_+ \right) (0|\tau) \left( \chi_0 + \chi_+ \right) (0|\tau)(K_0 + K_2)(w_c|\tau),$$

and

$$\bar{\chi}^+_\gamma(0|w_c|\tau) = \bar{\chi}^+_\gamma(0|\tau) \chi^+_\gamma(0|\tau) (K_1 + K_3)(w_c|\tau).$$

Let us comment that the above factorization expresses the parity selection rule ($m$-ality), which gives a gluing condition for the “charged” and “isospin” excitations.

It is worth underlining that in the $P - P$ sector, unlike for the other sectors, modular invariance constraint requires the presence of three different characters. The isospin operator content of the character $\bar{\chi}^+_\gamma(0|w_c|\tau)$ clearly evidences its peculiarity with respect to the other states of the periodic (even ladder) case. Indeed it is characterized by two twist fields ($\Delta = 1/16$) in the isospin components. The occurrence of the double twist in the state described by $\bar{\chi}^+_\gamma(0|w_c|\tau)$ is simply the reason why such a state is a periodic state. Indeed, being an isospin twist field the representation in the continuum limit of a magnetic impurity (a half flux quantum trapping or equivalently a kink), the double twist corresponds to a double half flux quantum trapping, i.e. one flux quantum, typical of the periodic configuration.
The above analysis would suggest that the $P - P$ state described by $\tilde{\chi}^+(0|w_c|\tau)$ embeds in the continuum limit a kink-antikink excitation, i.e. it represents an excited state in the $P - P$ sector. In this way, as it happens for all the other sectors, the $P - P$ sector is left with just two degenerate ground states ($\tilde{\chi}_0^+(0|w_c|\tau)$ and $\tilde{\chi}_0^+(0|w_c|\tau)$) and, as expected on a pure topological base, the ground state degeneracy in the torus topology is the double of that of the disk.

Let us now present the full list of character transformations under the insertion of a magnetic flux quantum through the hole of the closed ladder.

In the even closed JJ ladder configuration, we have that the two ground state wave functions of the $P - A$ sector decouple, being

$$ T_{1/2}\tilde{\chi}_{(0)}(0|w_c|\tau) = 0, \quad T_{1/2}\tilde{\chi}_{(1)}(0|w_c|\tau) = 0. \quad (40) $$

Concerning the $P - P$ sector, we have:

$$ T_{1/2}\tilde{\chi}_0^+(0|w_c|\tau) = 0 \quad (41) $$

and

$$ T_{1/2}\tilde{\chi}_0^+(0|w_c|\tau) = \tilde{\chi}_0^+(0|w_c|\tau) \quad ( T_{1/2}\tilde{\chi}_0^+(0|w_c|\tau) = \tilde{\chi}_0^+(0|w_c|\tau) ). \quad (42) $$

Such transformations show the instability of the $P - P$ sector under the insertion of a flux quantum through the hole of the closed ladder. More precisely the state $\tilde{\chi}_0^+(0|w_c|\tau)$ decouples while the state $\tilde{\chi}_0^+(0|w_c|\tau)$ gets excited to the state with a kink-antikink configuration $\tilde{\chi}_0^+(0|w_c|\tau)$.

Furthermore in the odd closed JJ ladder configuration, we have that the two ground state wave functions of the $A - A$ sector decouple, being

$$ T_{1/2}\tilde{\chi}_{(0)}(0|w_c|\tau) = 0, \quad T_{1/2}\tilde{\chi}_{(1)}(0|w_c|\tau) = 0. \quad (43) $$

Concerning the $A - P$ sector, we have that the two ground state wave functions transform as:

$$ T_{1/2}\tilde{\chi}_{(0)}(0|w_c|\tau) = \chi_{(0)}(0|w_c|\tau), \quad T_{1/2}\tilde{\chi}_{(1)}(0|w_c|\tau) = \chi_{(1)}(0|w_c|\tau). \quad (44) $$

Concluding, the full set of transformations, here presented, allows to claim the following simple and clear picture: the odd closed JJJ configuration is the only one which is stable under the insertion of a magnetic flux quantum through the central hole; moreover, in such odd JJJ configuration such a magnetic flux insertion simply implements the flipping process between the two degenerate ground states $|0\rangle$ and $|1\rangle$.

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