From subdirect products to a virtual approach to extensions

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Abstract

In this article we are first investigating about the subgroups of a direct product in a fruitfully new analysis, then we use a similar approach to extensions of modules, thus also laying some of the fundamentals for the "Virtual Category", of which we are speaking elsewhere.

What is most innovative in our method, as for the first part is a suitably defined equivalence relation, which is precisely what allows for treating the confusing case of multiple factors, thus giving a deeper insight into the structure of such subgroups. Applications and new techniques arising from this approach are examined, some of them most notably concerning basic properties of homomorphisms, extending well-known elementary ones. Our innovation in the part concerning extensions stems from our "virtual" view of them, namely as constituting a quotient, which is in fact a subdirect product, of a projective cover, or dually a submodule, which is a push-out, of an injective hull. In particular we become thus able to upgrade the Yoneda correspondence to a bimodule isomorphism.

1 Introduction

Although the first part of this article concerns some very basic group theory, that is justified not only in view of the numerous applications that follow but also by the fact that there are, astonishingly, still many obscurities about the subgroups of a direct product of \( n \) groups, for \( n > 2 \), already in its general outset. There is, on the other hand, an increasing tendency to look at subdirect products in more specific contexts and instances, as one may for example see in [3].

I got my first motivation to consider this question while working on modular representation theory and trying to understand the structure of modules, then I found the present general group-theoretic question not only fascinating in itself, but merely fundamental. I then applied my results in conjunction with homomorphisms, to get by an unusual way both known and unknown facts about
them - and that was long ago, while being a young student. In this connection it should be pointed out that our results are generalizable to operator groups (/ operator subgroups) by properly extending/specializing the proofs.

Our approach is outlined here:

By use of a suitable equivalence relation defined in a subgroup $U \leq A = A_1 \times A_2 \times \ldots \times A_n$ we can finally determine a normal subgroup $I$ of $U$, whose cosets are actually the classes of our relation; this is the key to our approach, leading in particular to our theorems 6 and 22 which generalize Remak’s theorem about the structure (through some ”structural” isomorphisms, intrinsic to the subgroup inclusion of $U$ in $A$) of subgroups of the direct product of two groups to the case of any arbitrary number of factors. Namely, in theorem 6 we show an analogue of that, for any choice of a subset of the set of direct factors, while in theorem 22 we determine a necessary and sufficient condition for $U$ to have an analogue $n$-fold structure, i.e. at all places, as for the case $n = 2$.

Finally, the the optimal generalization of theorem 22 is done by theorem 31. In all these cases we proceed by means of the specifically important structure of that normal subgroup $I$, called the core of the particular inclusion of $U$ as a subgroup of $A_1 \times A_2 \times \ldots \times A_n$, always in close conjunction to its meaning with respect to our equivalence relation, elucidated here in the proofs of propositions 3 and 4. Very critical for our most general case, treated in theorem 31, is the notion of cohesive components of the core and its related (“cohesion”) decomposition as their product (propositions 18, 57).

In the following subsection 4.2 we investigate the action of a kind of ”projectivization” of $\text{Aut}_R^m$ on subdirect products over $R$, which, apart from defining some orbits in a family of subdirect products, also gives us a whole orbit once we have one of them. In subsection 4.3 we investigate conditions for a group $G$ to be expressible as a subdirect product of groups belonging to any class $\mathcal{E}$ of groups, while we also specialize in some particular classes of special interest.

Another approach proceeds in section 5, by presenting a subdirect product through ”diagonalizing” homomorphisms emanating from a single group; by that we gain an independent, quite different view of them - but then also some basic general results about homomorphisms by applying the conclusions of the previous sections on such presentations of subdirect products, results considerably extending classical/elementary ones (see proposition 47 and its corollaries). The possibilities of the techniques arising from this approach are not exhausted here, they are just opened.

In the final section 6 we turn to modules and their extensions from a new, called virtual point of view, which allows us to view and compare them as a whole family not just in an abstract way, but rather as constituting a quotient of a projective cover, which is in fact a subdirect product, or dually a submodule of an injective hull, which is indeed a push-out. That gives us also the possibility of upgrading the Yoneda correspondence to a bimodule isomorphism (see Th. 74 & its corollary). In that way we do also win a new insight into the structure of modules, an insight which in fact is a first important step to obtaining the new kind of diagram we are introducing in [5], a diagram likewise defined in virtual terms, where virtuality is crystallized in the notion of the virtual category of a
module (or even of a block of an algebra).

**Key words**: subdirect products, presentations of subdirect products, decomposition as a subdirect product, realization of an extension of simple modules in the virtual category, proportional extensions, Yoneda correspondence.

**About notation**:
We shall automatically consider subgroups of a subproduct of the original as subgroups of the latter - and vice versa, according to the context. The scythic hat \(^\hat{}\) above an index designates omitting, as usual. \(I\) may as well denote the identity element, as the trivial (sub)group. \(\pi\) with the proper indices shall designate projections from direct products. The symbol "\(\times\)" may denote not only outer, but also inner direct product, which in all cases should be clear from the context.

2 The case of two direct factors

The first theorem here is a well known one; the reason to repeat it here is, as mentioned, that we are going to get to it through a totally different, original approach, that is then also applicable to the general case of more than two factors, yielding results and insights which, to my knowledge, are new. Take notice of the fact that no use of any fundamental theorem for group homomorphisms is made in its proof.

**Theorem 1** ([11]; can be found, for example., in [16], also in [13] 8.19, p. 183 - or in [7],[12],[14]) Given the direct product \(A \times B\) of two groups and a subgroup \(U \leq A \times B\), there exists a unique isomorphism \(\sigma : \pi_A(U) \rightarrow \pi_B(U)\), which is thus "structural", as it determines the discrete "pair cosets" of which \(U\) consists; it is also natural, in the sense that, for a given subdirect product \(U\) of \(A \times B\) and a homomorphism \((f_A, f_B)\) from it to \(A \times B\), sending \(U\) to \(U'\) and inducing \(\sigma'\) from \(\sigma\), this \(\sigma'\) is precisely the "structural isomorphism" of \(U\), as above. Furthermore, the isomorphism \(\sigma\) can be (naturally) continued to \(T:=U/(U \cap A) \times (U \cap B)\). Conversely, given a normal subgroup \(K\) of a subgroup \(I\) of \(A\), respectively, a normal subgroup \(L\) of a subgroup \(J\) of \(B\) (i.e. \(K \leq I \leq A, L \leq J \leq B\)) and an isomorphism \(\sigma : I/K \rightarrow J/L\), a subgroup \(U \leq A \times B\) is uniquely determined as consisting of the \(\sigma\)-determined pair-cosets. This amounts, of course, to realizing \(U\) as a fiber product of \(\pi_A(U)\) and \(\pi_B(U)\), amalgamated over a fixed isomorphic copy \(T\) of the two sides of \(\sigma\), with respect to \(\pi_X(U)\)'s (\(X=A,B\)) epimorphisms on it, that must be so coordinated, as to induce precisely that isomorphism \(\sigma\) (as above), that determines \(U\)'s pair-bundles correctly.

**Proof.** We define a relation "\(\sim\)" on \(U \leq \pi_A(U) \times \pi_B(U)\) by stipulating first, \((a,b) \sim (a',b), (a,b) \sim (a,b')\) for any \((a,b), (a',b), (a,b') \in U\) and then taking
as "∼" the transitive hull of this first stipulation. Reflectivity and symmetricity being apparent, it is clear that "∼" is an equivalence relation on \( U \). A crucial property of this relation is that, (m) if \((a_1, b_1) \sim (a'_1, b'_1)\) and \((a_2, b_2) \sim (a'_2, b'_2)\), then \((a_1a_2, b_1b_2) \sim (a'_1a'_2, b'_1b'_2)\); the relationship between two pairs means the existence of a finite sequence of pairs starting with the first and ending with the second of those two, such that any two subsequent pairs have the same first or second coordinate (we will call such a sequence an adjacency sequence). To see our claim, consider such sequences for the two given relationships and make them of equal length by repeating the last term of the shortest one as many times as necessary; we will have to produce a sequence with subsequent terms sharing the first or the second coordinate. To achieve this, take the products of the terms of same order in those two sequences of equal length, to get first a sequence of the same length, whose first and last terms are, respectively, \((a_1a_2, b_1b_2)\) and \((a'_1a'_2, b'_1b'_2)\). Those subsequent terms in this new sequence that come from the multiplication of subsequent terms in the original sequences which in both share the same-order coordinate, do probably also share same order terms as well; there is a slight problem whenever they come from multiplication of subsequent terms sharing the first term in the one, the second in the other, hence like this:

\((\alpha, \beta), (\alpha, \delta)\) in the one (sharing the first coordinate) and \((a, b), (c, b)\) in the other (sharing the second coordinate), thus yielding the subsequent terms \((\alpha a, \beta b), (\alpha c, \delta b)\) of the new sequence (of products), which do not share any term; but then it will suffice to insert the new term \((\alpha c, \delta b) \in U\) between them.

It is, on the other hand, immediate to see that \((a, b) \sim (a_1, b_1)\) implies \((a^{-1}, b^{-1}) \sim (a_1^{-1}, b_1^{-1})\), just by taking the inverses of all terms in the finite sequence.

We shall now show, how these two properties imply that, if \((a, 1)\) (respectively, \((1, b)\)) is an element of \( U\), then the first coordinate in its equivalence class \([[(a, 1)]\) (resp., the second in \([[(1, b)]\)) runs over a normal subgroup of \(\pi_A (U)\); in order to see this, we shall show that, whenever \((\alpha, \beta) \sim (\gamma, 1)\), it turns out that \((\alpha, 1)\) is also an element of \( U\) (and, of course, in the same equivalence class). Indeed, let \(x_0 = (\gamma, 1), x_1, ..., x_{n+1} = (\alpha, \beta)\) be an adjacency sequence for the relationship \((\alpha, \beta) \sim (\gamma, 1)\); it is then immediate to see that \(x_0^{-1}x_1x_2^{-1}...x_{n+1}^{-1} = (\alpha, 1)\) or \((\alpha, 1)^{-1} = (\alpha^{-1}, 1)\), according to whether \(n\) is even or odd; in the latter case, of course also the inverse, i.e. \((\alpha, 1)\), shall belong to \( U\). Now, this means that the first coordinate of the class \([[(a, 1)]\) runs over the same set as its elements with second coordinate \(1\); on the other hand, every element of this form in \( U\) apparently belonging to the same class, they do clearly constitute a subgroup of \( U\), which must also be normal both in \( U\) and in \(\pi_A (U)\), as conjugation by elements of \( U\) yields elements of the same form, hence in the same class-subgroup and in \( A\) as well, apart from being in advance clear that the set of elements of this form in \( U\) is the group \( U \cap \pi_A (U)\). (This gives also an alternative way to see the normality of \( U \cap A = U \cap \pi_A (U)\) in \( U\) and in \(\pi_A (U)\), otherwise clear from the fact that \(\pi_A (U) \times \pi_B (U)\) normalizes \(\pi_A (U)\), hence \( U \leq \pi_A (U) \times \pi_B (U)\) must normalize \( U \cap \pi_A (U)\) - and then it
follows immediately that also \( \pi_A(U) \) has to normalize \( U \cap \pi_A(U) \); on the other hand, one sees that the kernel of the restriction to \( U \) of \( \pi_A(U) \times \pi_B(U) \) ’s projection onto its second factor is precisely \( U \cap \pi_A(U) = U \cap A \). When we transfer the same remarks to the second factor, we see immediately that the equivalence class of the elements of \( U \) of the form \((a, 1)\), also containing \((1, 1)\) and the elements \((1, b)\) in \( U \), equals the group \((U \cap A) \times (U \cap B)\).

We wish now to give another interpretation of our equivalence relation on \( U \) through its special properties which we have seen:

Assume, so, that \((a, b) \sim (c, d)\); invoking the multiplication property, we may multiply this with the trivial relationship \((a^{-1}, b^{-1}) \sim (a^{-1}, b^{-1})\) to get \((a^{-1}c, b^{-1}d) \sim (1, 1) \iff (a^{-1}c, b^{-1}d) \in [(1, 1)] = (U \cap A) \times (U \cap B) \iff c \in a(U \cap A) \& d \in b(U \cap B) \iff (c, d) \in a(U \cap A) \times b(U \cap B)\), which shows that the class of \((a, b) \in U\) is the Cartesian product of the left (and right) cosets \(a(U \cap A)\) and \(b(U \cap B)\) of \(U \cap A\), resp. of \(U \cap B\), in \(\pi_A(U)\), resp. in \(\pi_B(U)\). Now, \((a(U \cap A) \times b(U \cap B)\) being an equivalence class, in case there were also a class \((a(U \cap A) \times b(U \cap B)\) , while from the definition of our relation is \((a, b) \sim (a, d)\), this "new" class has to be identical with \((a(U \cap A) \times b(U \cap B)\), i.e. \(d(U \cap B) = b(U \cap B)\). This crucial remark establishes a bijection \(\pi_A(U) / U \cap A \to \pi_B(U) / U \cap B\), which is bound to be a group homomorphism (hence an isomorphism), considering the special property (1) of our relation. This means that \(U\) is partitioned into classes which may be described as "pair bundles" of the form \(a(U \cap A) \times (a(U \cap A))\), which at the same time determines a unique coset \((a, b)(U \cap A) \times (U \cap B)\), where \(b\) may be any element of the coset \((a(U \cap A))\) in \(\pi_B(U)\). This observation "prolongs" the bijection \(\pi_A(U) / U \cap A \leftrightarrow \pi_B(U) / U \cap B\) to \(\pi_A(U) / U \cap A \leftrightarrow \pi_B(U) / U \cap B \leftrightarrow U / (U \cap A) \times (U \cap B)\), still in a homomorphic manner, as property (m) suggests.

As for the converse assertion, we can easily prove that the given isomorphism \(\sigma\) defines a unique subgroup \(U \leq A \times B\); then \(U\), according to the previous, determines a unique such isomorphism, which hence must be \(\sigma\).

Concerning the interpretation as fiber products, that is quite clear - see, for example., \[\text{[1]}\].

Another way to realize \(U\), could be to view it as a total space, \(T\) being the base space of a bundle \((U, p, T)\) where, for \(t \in T\) and \(A_i, B_i\) its \((\sigma\cdot)\) corresponding \((U \cap A)\)-, resp. \((U \cap B)\)-, cosets in \(\pi_A(U)\), resp. \(\pi_B(U)\), we shall have \(p^{-1}(t) = A_i \times B_i\). This point of view can also be adapted to our theorems \[\text{[1]}\] and \[\text{[22]}\] below.

### 3 The generic case

Let \(U \leq A = A_1 \times A_2 \times \ldots \times A_n\); we assume further that \(\pi_i(U) \cap A_i\) is not trivial for any \(i \in \{1, \ldots, n\}\) and then introduce the following subgroups of this product:
\[ E_s = \pi_{1, \ldots, n}(U) \cap U, \] which is quite probably equal to \( \text{Ker}\pi_s \mid U = \text{Ker}\pi_s \cap U = \left( Dr \prod_{i \neq s} A_i \right) \cap U, \] consisting of the elements of \( U \), having 1 in the \( s \)-coordinate. More generally, for any set of indices \( i_1(i_2, \ldots, i_s) \) from \( \{1, \ldots, n\} \), define \( E_{i_1i_2\ldots i_s} := \pi_{i_1i_2\ldots i_s}(U) \cap U \), consisting of the elements of \( U \), having 1 in the \( i_1, i_2, \ldots, i_s \)-coordinates. It is obvious that \( E_{i_1i_2\ldots i_s} \cap E_{i_1i_2\ldots i_s} = \left( \bigcap_{i \notin \{i_1, \ldots, i_s\}} E_i \right) \)

while \( E_{i_1i_2\ldots i_s} = \bigcap_{i \in \{i_1, \ldots, i_s\}} E_i \). Define, also, for any subsequence \( \Lambda \) of indices \( i_1(i_2, \ldots, i_s) \) from \( \{1, \ldots, n\} \), \( \Lambda_\Lambda = L_{i_1i_2\ldots i_s} \) to be \((A_{i_1} \times A_{i_2} \times \ldots \times A_{i_s}) \cap \mathcal{I}\). Notice that \( E_{i_1i_2\ldots i_s} = L_s \), as they both consist of the elements of \( U \), having all but their \( s \)-coordinate equal to 1; also, \( L_{i_1i_2\ldots i_s} = L_s \).

More generally, \( L_{i_1i_2\ldots i_s} = E_{i_1i_2\ldots i_s} \). Notice that, for \( M, N \subset \{1, \ldots, n\} \), with \( M \cap N = \emptyset \), it follows that \( L_M \cap L_N = 1 \), therefore \( L_M = L_N = L \): in fact, they even commute elementwise (from the original direct product). Define also \( L_0 = 1 \).

For any \( M \subset \{1, \ldots, n\} \), \( \pi_M \) denotes the corresponding projection from \( A_1 \times A_2 \times \ldots \times A_n \) to \( Dr \prod_{i \in M} A_i \).

**Definition 2** We define, as in the case \( n = 2 \), a relation \( \sim \) on \( U \), in two steps: first, we stipulate that \( \overline{\pi} = (a_1, \ldots, a_n) \sim \overline{b} = (b_1, \ldots, b_n) \) whenever \( a_{\lambda} = b_{\lambda} \) for some \( \lambda \in \{1, \ldots, n\} \) (we will then say that \( \overline{\pi} \) and \( \overline{b} \) touch one another at the \( \lambda \)-coordinate), then take the minimal transitive extension of this first germ relation, to get an equivalence relation on \( U \). We call its equivalence classes the general connective bundles of \( U \).

This relation will lead us to a normal subgroup \( I \) of \( U \), such that the equivalence classes of \( \sim \) be the same as \( I \)'s cosets in \( U \); we may also use square brackets \([\] \) to designate equivalence classes of \( \sim \).

**Proposition 3** The equivalence class \( I = \{1, \ldots, 1\} \) of \( U \)'s identity element is a normal subgroup of \( U \), henceforth to be called the subdirect core of the subgroup \( U \) of \( A_1 \times A_2 \times \ldots \times A_n \), generated by its subgroups \( E_i, i = 1, \ldots, n \); moreover, any \( E_i \) is normal both in \( U \) and in \( \pi_{1, \ldots, n}(U) \). Furthermore, \( I = \prod_{i=1}^n E_i = \prod_{i=1}^n E_{\tau(i)} \), where \( \tau \) is any permutation in \( S_n \), a result that also holds for the subgroup of \( I \) generated by any non-empty subset of \( \{E_1, \ldots, E_n\} \).

**Proof.** \( E_s \subseteq U, s = 1, 2, \ldots, n \), because \( E_s \) consists precisely of the elements of \( U \), having 1 in the \( s \)-coordinate - a property maintained through conjugation in \( U \); we shall further show that \( E_s \subseteq \pi_{1, \ldots, n}(U) \). Just for notational convenience, we will show this for \( s = 1 \), i.e. that \( E_1 \subseteq \pi_{2, \ldots, n}(U) \). Let, so, \( \overline{\pi} = (1, a_2, \ldots, a_n) \in E_1 \) and \( \overline{\pi}(b_2, \ldots, b_n) = (1, b_2, \ldots, b_n) \in \pi_{2, \ldots, n}(U) \), which means that there exists some \( b_1 \in A_1 \), such that \( \overline{b} = (b_1, \ldots, b_n) \in U \); then \( \overline{\pi} = \overline{\pi}(b_1, \ldots, b_n) \in U \), therefore \( \overline{\pi}(b_1, \ldots, b_n) \in \pi_{2, \ldots, n}(U) \cap U = E_1 \), completing the argument.
Observe that \( \mathbf{\pi} = (a_1, ..., a_n) \in I \Leftrightarrow \mathbf{\pi} \sim (1, ..., 1) \Leftrightarrow \exists a \text{ a finite sequence } a^0 = \mathbf{\pi}, a^1, ..., a^{n+1} = (1, ..., 1), \text{ set } a^\kappa = (a_\kappa', ..., a_\kappa^\kappa) \text{ for } \kappa = 0, 1, ..., \mu + 1, \text{ such that any two neighbouring terms } a^\kappa - 1, a^\kappa \text{ share, say, their } i_\kappa \text{-coordinate. (By assuming this sequence to be of minimal length, we get that } a_i^\kappa \neq 1 \forall i \in \{1, ..., n\} \text{ whenever } \kappa(\mu) \text{. Then, by taking the sequence of the inverses we get such an "adjacency sequence" yielding the relationship } \mathbf{\pi}^{-1} \sim (1, ..., 1), \text{ proving that } \mathbf{\pi}^{-1} \in I \text{ as well.}

Before proceeding to prove that, given also a \( \mathbf{\beta} = (b_1, ..., b_n) \in I \), \( \mathbf{\pi} \mathbf{\beta} \) shall belong to \( I \) too, we must make a crucial remark: in the above "adjacency sequence" for \( \mathbf{\pi} \sim (1, ..., 1) \), \( \mathbf{\pi}^{-1} a^\kappa - 1 \in U \) with 1 in the \( i_\kappa \)-coordinate, hence \( \mathbf{\pi}^{-1} a^\kappa - 1 \in E_{i_\kappa} \), allowing us to replace the condition for the existence of an "adjacency sequence" for \( \mathbf{\pi} \sim (1, ..., 1) \) with the possibility to write \( \mathbf{\pi} \) as a product of elements belonging to the several \( E_i \)'s: indeed,

\[
\mathbf{\pi} = \mathbf{\alpha} = \alpha_0 = \alpha_{0}^{0} \alpha_{0}^{1} \alpha_{0}^{2} \alpha_{0}^{3} \cdots \alpha_{0}^{i} \alpha_{0}^{i+1} \cdots \alpha_{0}^{n-1} \alpha_{0}^{n} = (1, ..., 1), \text{ where it is obvious what we have substituted the greek } \mathbf{\pi} \text{'s for; conversely, given such an expression of } \mathbf{\pi} = \alpha^0 \text{ as } \alpha_{0}^{0} \alpha_{0}^{1} \alpha_{0}^{2} \alpha_{0}^{3} \cdots \alpha_{0}^{i} \alpha_{0}^{i+1} \cdots \alpha_{0}^{n-1} \alpha_{0}^{n}, \text{ we get the adjacency sequence } \mathbf{\pi} = \alpha^{0} = \alpha_{0}^{0} \alpha_{0}^{1} \alpha_{0}^{2} \alpha_{0}^{3} \cdots \alpha_{0}^{i} \alpha_{0}^{i+1} \cdots \alpha_{0}^{n-1} \alpha_{0}^{n}, \text{ hence we may also write } \mathbf{\beta} \in I \text{ as a product of elements of the several } E_i \text{'s, say } \mathbf{\beta} = \mathbf{\beta}^{0} \mathbf{\beta}^{1} \mathbf{\beta}^{2} \mathbf{\beta}^{3} \mathbf{\beta}^{4} \mathbf{\beta}^{5}, \text{ therefore it becomes obvious through this new equivalent condition for an element of } U \text{ to belong to } I \text{ that also the product } a_0^n \mathbf{\beta} = \alpha_{0}^{0} \alpha_{0}^{1} \alpha_{0}^{2} \alpha_{0}^{3} \cdots \alpha_{0}^{i} \alpha_{0}^{i+1} \cdots \alpha_{0}^{n-1} \alpha_{0}^{n} \alpha_{0}^{0} \alpha_{0}^{1} \alpha_{0}^{2} \alpha_{0}^{3} \cdots \alpha_{0}^{i} \alpha_{0}^{i+1} \cdots \alpha_{0}^{n-1} \alpha_{0}^{n} \alpha_{0}^{0} \alpha_{0}^{1} \alpha_{0}^{2} \alpha_{0}^{3} \cdots \alpha_{0}^{i} \alpha_{0}^{i+1} \cdots \alpha_{0}^{n-1} \alpha_{0}^{n} \in I \text{ holds.}

As for the last claim, it will suffice to prove that, for any } a \in I \text{, it is possible to write it as a product } \alpha_{0}^{0} \alpha_{0}^{1} \alpha_{0}^{2} \alpha_{0}^{3} \cdots \alpha_{0}^{i} \alpha_{0}^{i+1} \cdots \alpha_{0}^{n-1} \alpha_{0}^{n}, \text{ in such a way that } i_{\mu+1} (i_{\mu+2} (i_1 i_2 (i_1 - 1) - \text{ or even in a way such that this ordering will be valid after application of the (random) permutation } \tau^{-1}; \text{ to see this, it will obviously be enough to prove that, given a product } e_i e_j \text{ with } e_i \in E_i, e_j \in E_j, \text{ it is always possible to write it in the form } e_i' e_j', \text{ where } e_i' \in E_i, e_j' \in E_j. \text{ For notational convenience we prove this for } i = 1, j = 2; \text{ so, let } e_1 \in E_1, e_2 \in E_2. \text{ By taking their commutator } [e_1, e_2], \text{ one sees directly that it belongs to } E_1 \cap E_2 = E_{12}, \text{ hence } e_1 e_2 = e_2 e_1 [e_1, e_2], \text{ which is a product of } e_2 \in E_2 \text{ and } e_1 [e_1, e_2] \in E_1. \text{ (Alternatively, it is enough to remember that the } E_i \text{'s are all normal subgroups of } U.).

We emphasize here that this does not in general mean that the elements of } E_i \text{ commute with those of } E_j \text{ (with } i \neq j), \text{ unless } n = 2, \text{ in which case the commutator above becomes the identity element of } U, \text{ as } E_{12} \text{ is then the trivial subgroup.}}
Proposition 4  For $\pi, b \in U$, it holds that $\pi \sim b$ iff $b^{-1}\pi \in I$; hence equivalence classes of $\sim$ is the same thing as $I$'s cosets in $U$.

Proof. $\pi \sim b \iff \exists$ a finite sequence $\overrightarrow{a} = \pi, a^0, ..., a^{\mu+1} = b$, such that any two neighbouring terms $a^\kappa a^{\kappa+1}, a^\kappa$ share, say, their $i_\kappa$-coordinate for $\kappa = 1, ..., \mu + 1$, meaning that $a^\kappa a^{\kappa+1} = b$. Now, $\pi = \overrightarrow{b} = a^0 = a^{\mu+1} \overrightarrow{a} \overrightarrow{a}^{\mu+1} (a^{\mu+1}a^{\mu} ... (a^{\mu+1}a^1) a^{\mu-1}a^0) = ba^{\mu+1}e_{\mu+1}e_{\mu} ... e_1 \Rightarrow b^{-1}\pi \in I$, the converse becoming apparent by expressing $b^{-1}\pi \in I$ as a product of elements of the $E_i$'s and then invoking the obtained equivalent condition for $\pi \sim b$. ■

Remark 5  Despite the "simplifying" assertion of the preceding proposition, it is however also to be able to identify the cosets of $I$ as connection bundles, while that allows us another sharp insight into the subdirect structure - which, besides being the point of view for the bulk of our general analysis here, is also later taken crucially into account in subsection.

Theorem 6  Let $U \leq A_1 \times A_2 \times ... \times A_n$ and assume also that all $\pi_i(U)$'s are non-trivial (otherwise we should have taken the maximal subproduct satisfying this condition); then, the following are true:

a. For $\overrightarrow{\pi}, b \in U$, with $\overrightarrow{\pi}$ fixed, $b$ variable, so that $\pi_s(b) = \pi_s(\pi)$, the variation space of $b$ is the coset $\pi_sE_s$ in $U$ while the variation space of $\pi_1...\pi_n(b)$ is the coset $\pi_1...\pi_n(\pi)E_s$ in $\pi_1...\pi_n(U)$; conversely, by varying only the $s$-coordinate, the variation space of $b$ is the coset $\pi_1...\pi_n E_s$ in $\pi_1...\pi_n(U)$. More generally, for any variable $\overrightarrow{\pi} = (a_1, ..., a_n) \in U$ and any sequence of indices $i_1,i_2,...,i_s$ from $\{1,...,n\}$, by fixing the $\{i_1,i_2,...,i_s\}$-coordinates $\{a_{i_1},...,a_{i_s}\}$ and varying the others, so that $\overrightarrow{\pi}$ remain in $U$, the variation space of $\overrightarrow{\pi}$ is the coset $\pi_sE_{i_1,i_2,...,i_s}$; conversely, by fixing the complementary set of coordinates, the variation space of $\overrightarrow{\pi}$ becomes $\overrightarrow{\pi}E_{i_1,i_2,...,i_s}$. Corresponding to that, the variation space of $\pi_1...\pi_s...\pi_n(\overrightarrow{\pi})E_{i_1,i_2,...,i_s}$ in $\pi_1...\pi_n(U)$; accordingly, by varying the $\{i_1,i_2,...,i_s\}$-coordinates $\{a_{i_1},...,a_{i_s}\}$ and fixing the others, so that $\overrightarrow{\pi}$ remain in $U$, the variation space of $\pi_{i_1,\pi_{i_2}},...,\pi_{i_s}$ (while lying inside $\pi_{i_1,\pi_{i_2}},...,\pi_{i_s}(U)$) is the coset $\pi_{i_1,\pi_{i_2}}...\pi_{i_s}$ $\pi_1...\pi_n(\overrightarrow{\pi})E_{i_1,i_2,...,i_s}$, for any particular ("original") value of $\overrightarrow{\pi}$.

b. For any sequence of indices $i_1,i_2,...,i_s$ from $\{1,...,n\}$, there is a unique isomorphism $\sigma : \pi_{1,\pi_{i_1}}...\pi_{i_s}$ $\pi_1...\pi_n(U) \rightarrow \pi_{i_1,i_2,...,i_s}(U) \rightarrow \pi_1...\pi_n(U)$ $(\pi_{1,\pi_{i_1}}...\pi_{i_s}(U) \rightarrow \pi_{i_1,i_2,...,i_s}(U) \rightarrow \pi_1...\pi_n(U))$, with the "structural" property that, for any $\overrightarrow{\pi} = (a_1, ..., a_n) \in U$, $\sigma$ sends the coset $\pi_{1,\pi_{i_1}}...\pi_{i_s}$ $\pi_1...\pi_n(\overrightarrow{\pi})E_{i_1,i_2,...,i_s}$ over to $\pi_{i_1,i_2,...,i_s}$ $\pi_1...\pi_n(\overrightarrow{\pi})E_{i_1,i_2,...,i_s}$. In other words, $U$ may be realized as a fiber product of $\pi_{1,\pi_{i_1}}...\pi_{i_s}$ $\pi_1...\pi_n(U)$ and $\pi_{i_1,i_2,...,i_s}(U)$, amalgamated over a fixed isomorphic copy of the two sides of $\sigma$, with respect to their apparent epimorphisms on it. The converse is also true, meaning that a subset of indices and an isomorphism $\sigma$, determine a unique subgroup $U \leq A_1 \times A_2 \times ... \times A_n$ in the prescribed way.
For \( (a) \), the first one just being a specialization of that.)

Remark 7 This theorem may be considered as a generalization of the case \( n=2 \).

Definition 8 For \( U \leq A_1 \times A_2 \times \ldots \times A_n \) and for any sequence of indices \( i_1, i_2, \ldots, i_s \) from \( \{1, \ldots, n\} \), set \( I_{i_1, i_2, \ldots, i_s} \) the subgroup of \( U \) (and, of course, of \( I \)) generated by its (normal) subgroups \( E_{i_1}, \ldots, E_{i_s} \), hence \( I_{i_1, i_2, \ldots, i_s} = E_{i_1} \cap \ldots \cap E_{i_s} \).

We shall call such an \( I_{i_1, i_2, \ldots, i_s} \) (1- or \( U \)-)connected, if there is no non-trivial partition \( \Lambda = \{i_1, i_2, \ldots, i_s\} = M \cup N \) (where \( M \cap N = \emptyset \)) of \( \Lambda \), so that \( I_{i_1, i_2, \ldots, i_s} = I_M \times I_N \).
- Obviously \( I_{\Lambda} \supseteq I_M I_N \) and \( I_{\Lambda} \supseteq I_N I_M \); that \( I_{\Lambda} = I_M \times I_N \) means that \( I_M \cap I_N = 1 \), an obvious necessary condition for that being that there is no \( \tilde{x} = (..., 1, ..., 1, ...) \in I_{\Lambda} \setminus \{1\} \), with the two aces appearing in the \( \kappa \)- and \( \lambda \)-entries, with \( \kappa \in M \) and \( \lambda \in N \) (or vice versa).

**Lemma 9** Given the non-empty subsets \( M, N \) of \( \{1, ..., n\} \), the following are true:
\[
I_M \cap I_N = I_{M \cap N}, \quad I_M I_N = I_{M \cup N} = I_N I_M.
\]
**Proof.** Quite clear. ■

**Definition 10** With the above notation, we shall speak of the total connected decomposition of the core \( I \), when we have a (possibly trivial) partition \( \{1, ..., n\} = N_1 \cup ... \cup N_m \) and an expression \( I = I_{N_1} ... I_{N_m} \), so that all of \( I_{N_1}, ..., I_{N_m} \) be maximal \( I \)-connected; in view of the former remarks, this forces that expression to be a direct product.

Conversely, if we were given a direct decomposition \( I = I_{N_1} \times ... \times I_{N_m} \) with all \( I_{N_i} \) connected, then these latter would necessarily be maximal connected.

In the last definition, these maximal \( I \)-connected subgroups \( I_{N_1}, ..., I_{N_m} \) of \( I \) shall be called the connected components of \( I \), uniquely determined by the inclusion \( U \leq A_1 \times A_2 \times ... \times A_n \).

4 Generalizations

4.1 Significant distinctions up to the general structure

The following lemma is a direct consequence of the definitions:

**Lemma 11** Let \( M, N \subseteq \{1, ..., n\} \). Then, (i) \( E_M \cap E_N = E_{M \cap N} \), (ii) \( L_{M \cup N} \supseteq L_M L_N, \quad L_N L_M \), which are both equal to \( L_M \times L_N \) in case \( M \cap N = \emptyset \).

**Proof.** About (ii): See theorem ■

In particular, for \( M \cap N = \emptyset \), we have \( L_{M \cup N} \supseteq L_M \times L_N \), where we need not have equality; this remark is crucial up to the following

**Definition 12** Let us remember that, for any set \( \Lambda \) of indices \( i_1(i_2(...)i_s \text{ from } \{1, ..., n\} \), \( L_{\Lambda} = L_{i_1} ... L_{i_s} \) denotes the subgroup \( (A_{i_1} \times A_{i_2} \times ... \times A_{i_s}) \cap I \) of \( U \), named a "subkernel" of \( U \) from now on; we shall call a non-trivial subkernel \( L_{i_1i_2...i_s} \) cohesive in \( U \) if \( A_1 \times A_2 \times ... \times A_n \) if there is no non-trivial partition \( \Lambda = \{i_1, i_2, ..., i_s\} = M \cup N \), \( M \cap N = \emptyset \), so that \( L_{\Lambda} = L_M \times L_N \). The maximal such cohesive subgroups will be called the cohesive components of \( I \). In case the cohesive components of \( I \) are precisely the \( L_{i_i} \)'s, \( i=1,2,...,n \)
, or, equivalently, \( I = L_1 \times L_2 \times \ldots \times L_n \), we shall call the subgroup \( U \leq A_1 \times A_2 \times \ldots \times A_n \) a (cohesively) smashed one. In some sense at the other extreme of such a non-trivial decomposition of the core, we want to make another distinction, that of a deltoid subkernel \( L_\Lambda \) of \( U \), meaning that for any proper subset of indices \( M \) of \( \Lambda \), \( L_M \) is trivial.

**Lemma 13** Let \( U \leq A_1 \times A_2 \times \ldots \times A_n \) be a subdirect product (i.e., \( \pi_i(U) = A_i \) for all \( i \)’s) with all \( E_i \)’s trivial; then all \( A_i \)'s are isomorphic and there is a system of ("structural") isomorphisms between any two of them, such that \( U \) consist of \( n \)-tuples of through those isomorphisms corresponding elements. The converse holds (trivially) too.

**Proof.** By assuming that we might have two elements \( \pi, b \in U \) with one, say the \( i \)'th, coordinate in common and with at least another coordinate not in common, that would give the contradiction that \( 1 \neq b^{-1} \pi \in E_i \). This shows that \( U \) entirely consists of mutually disjoint \( n \)-tuples \( \pi = (a_1, \ldots, a_n) \); this, combined with the assumption \( \pi_i(U) = A_i \) for all \( i \)'s, establishes a system of bijective maps between any two of the direct factors \( A_i \). That these are group homomorphisms, simply amounts to the group structure and the coordinatewise multiplication in \( U \).

**Remark 14** As \( L_i \leq E_j \) for any \( j \neq i \), the hypothesis of the proposition yields that also all \( L_i \)’s are trivial.

It is immediately clear from the definitions (and also a consequence of the previous lemma, by taking the subkernel itself as \( U \)) that in a deltoid subkernel any two non-equal elements have at no \( \Lambda \)-entry \( i \) the same element of \( A_i \). This fact makes things very clear here, enabling us to continue modulo this case, by simply substituting the simple direct factor \( \pi_\Lambda(U) \) for \( Dr \prod_{i \in \Lambda} A_i \) in \( Dr \prod_{i=1}^n A_i \), with corresponding \( U \cap \pi_\Lambda(U) = L_\Lambda \), for any maximal deltoid subkernel \( L_\Lambda \) of \( U \), which legitimizes "at no cost" the following

**Condition 15** From now on we may assume that there are no deltoid subkernels for our subgroup \( U \) of \( Dr \prod_{i=1}^n A_i \).

As \( L_\Lambda = \left( Dr \prod_{i \in \Lambda} A_i \right) \cap U \), our theorem [6] is applicable to the subgroup \( L_\Lambda \) of \( Dr \prod_{i \in \Lambda} A_i \), while the definition of \( L_M, L_N \) as subgroups of \( L_\Lambda \leq Dr \prod_{i \in \Lambda} A_i \) still remains unchanged inside \( U \leq A_1 \times A_2 \times \ldots \times A_n \) (since they were already subgroups of \( Dr \prod_{i \in \Lambda} A_i \) inside \( Dr \prod_{i=1}^n A_i \)); hence, by that theorem, at any rate is \( L_M \times L_N \leq L_\Lambda \), while equality here would
mean \( \pi_M (L_\Lambda) = L_M \) and, equivalently, \( \pi_N (L_\Lambda) = L_N \), since in this case the groups of the isomorphism \( \sigma \) in theorem 6(b) are trivial. In this connection it is important to notice that, considering a subgroup \( U \) of \( \prod_{i=1}^{n} A_i \) in the case that \( U \)'s projection on some of the direct factors \( A_i \) is trivial makes our analysis too unfruitful, by short-circuiting it in effect at a trivial level; consequently one should have to exclude at least some of the direct factors \( A_i \), on which \( U \)'s projection is trivial, by taking the subkernel that corresponds to the direct factors, on which the projection of \( U \) is non-trivial, and which in this case is \( U \) itself. Further, analyzing all subkernels of \( U \), could take us closer to a diagrammatic representation of \( U \)'s structure - which at any rate is limited by the (complexity of the) structure of the direct factors \( A_i \) themselves.

- In elementary terms, the condition \( L_M \times L_N = L_\Lambda \) means (by theorem 6 applied on \( L_\Lambda \)) that, for any \( x \in L_\Lambda \), the element \( \pi_M (x) \) of \( \prod_{i \in M} A_i \) also belongs to \( L_M \) - or, equivalently, \( \pi_N (x) \in L_N \). By now viewing \( L_\Lambda \) as a subgroup of \( \prod_{i \in \Lambda} A_i \), while forgetting for a moment about the original \( U \subseteq A_1 \times A_2 \times ... \times A_n \), we get the following

**Lemma 16** For \( U \subseteq A_1 \times A_2 \times ... \times A_n \), \( \emptyset \neq M \subseteq \{1, ..., n\} \), \( M \neq \{1, ..., n\} \), the condition \( \pi_M (U) \leq U \) implies the following (by theorem 6 equivalent) facts:

\[ \pi_M (U) = L_M, \quad \pi_M (U) = L_M, \quad U = L_M \times L_M \]

**Proof.** Enhance the preceding discussion with the remark, following from the definition of the core \( I \) of \( U \), that \( M \) being a proper subset of \( \{1, ..., n\} \) immediately means that the relation \( \pi_M (U) \leq U \) implies \( \pi_M (U) \leq I(U) \), which then forces \( \pi_M (U) = L_M \).

- We are now pointing out a relevant implication of part (a) of theorem 6 in order to prove, on the contrary, that \( L_M \times L_N \neq L_\Lambda \), it is enough just to find one \( x \in L_\Lambda \), with the property that \( \pi_M (x) \notin L_\Lambda \) (or, equivalently, \( x \notin L_M \))!

**Lemma 17** The cohesive components of \( I \) intersect each other trivially.

**Proof.** At first, notice that \( M \cap N = \emptyset \Rightarrow L_M \cap L_N = 1 \) and, therefore,

\[ L_M \cap L_N \neq 1 \Rightarrow M \cap N \neq \emptyset \] . In view of this, combined with the maximality of the cohesive components from their definition, it will suffice to prove the following:

"If \( M, N, P \) are mutually disjoint non empty index sets, such that \( L_{M \cup P}, L_{N \cup P} \) be cohesive, then \( L_{M \cup N \cup P} \) is cohesive too."

Assume to the contrary, that there is a non-trivial decomposition \( (1) L_{M \cup N \cup P} = L_R \times L_S \) (\( R, S \) non-empty, \( R \cap S = \emptyset, R \cap S = M \cup N \cup P \.

On account of lemma 16 cohesiveness of \( L_{M \cup P}, L_{N \cup P} \) implies that neither \( R \) nor \( S \) may be contained in either \( M \cup P \) or \( N \cup P \), which in turn means that \( R \) as well as \( S \) have non-trivial intersections with \( M \) and \( N \). So, by means of of lemma 16 we get through (1) a non-trivial decomposition of the subgroups \( L_{M \cup P}, L_{N \cup P} \) of \( L_{M \cup N \cup P} \), contrary to their cohesiveness. \( \blacksquare \)
Proposition 18 There is always a unique (up to ordering of factors) decomposition \( I = L_{N_1} \times ... \times L_{N_n} \) of the core \( I \) as the product of its cohesive components; this will be referred to as the (total) cohesiveness decomposition of the core \( I \).

Proof. If \( I = L_{1...n} \) is cohesive, then we are done with \( m=1 \); otherwise, we continue examining its factors, until they cannot be any further decomposed, meaning that they are cohesive. Uniqueness is a consequence of the previous lemma.

Remark 19 Given that one might have the situation \( N \subset M \) \( (N \neq M) \) and still \( L_N = L_M \), to a cohesion decomposition is, to begin with, not necessarily attached a unique partition of \( \{1,...,n\} \). To remedy that, we agree from now on (unless otherwise specified) to take the maximal such subsets of \( \{1,...,n\} \).

Definition 20 We shall call a subgroup \( U \leq A_1 \times A_2 \times ... \times A_n \), for \( n > s + 2 \), an \( r \)-weakly smashed one if there exists a partition of \( \{1,...,n\} \) into subsets of cardinality at least \( r \), such that for every subset \( N = \{i_1,i_2,...,i_s\} \) in the partition \( (r \leq s) \), \( E_{i_1i_2...i_s} \) is contained in (the direct product) \( \prod_{k \notin N} L_k \).

(Trivially, for \( t < s \), \( t \)-weak smashedness also implies \( s \)-weak smashedness.)

For \( n > 2 \) the condition that all \( E_{ij}, i \neq j \), be trivial, of course also implies that all \( L_i \)'s are trivial.

Lemma 21 "1-weakly smashed" means for \( U \) the same as "(cohesively) smashed".

Proof. \( \Rightarrow \): As the core \( I \) is generated by the \( E_i \)'s, \( E_i \subset L_1 \times ... \times L_n \Rightarrow I \subset L_1 \times ... \times L_n \), while the converse inclusion is trivial.

\( \Leftarrow \): Trivial.

Theorem 22 Let \( U \leq A_1 \times A_2 \times ... \times A_n \) \( (n\geq 2) \) be a subdirect product (i.e., \( \pi_i(U) = A_i \) for all \( i \)'s) which is smashed. Then there is a (uniquely determined) "structural" system of isomorphisms of the \( A_i/L_i \)'s, all those being isomorphic to \( U/I = U/(L_1 \times L_2 \times ... \times L_n) \), in a perfect generalization of the case \( n=2 \). This, again, amounts to realizing \( U \) as a fiber product of the \( A_i \)'s, amalgamated over \( R := A_1/L_1 \), with respect to each \( A_i \)'s epimorphism on it, given by composing the canonical \( A_i \rightarrow A_i/L_i \), with \( A_i/L_i \rightarrow A_1/L_1 \) from the mentioned "structural" system of isomorphisms. The converse is again true. Also, for any subset of indices \( i_1(i_2(...i_s \text{ from } \{1,...,n\}) \), we have uniquely determined structural isomorphisms \( R \approx \pi_{i_1i_2...i_s}(U)/(L_{i_1} \times ... \times L_{i_s}) \).

Proof. Set, now, \( A' \equiv A_i/L_i \) and use the previous proposition for the projection \( U' \) of \( U \), as a subgroup of
\[ A'_1 \times A'_2 \times ... \times A'_n \]
terminology that we have established above; it is immediate to see the following facts:

\[ \pi_i(A) \cong A_i \text{ for all } i, \]

where \( \pi_i \) is the natural projection onto the \( i \)-th coordinate. Consequently, \( U = \pi_1 \times \pi_2 \times \cdots \times \pi_n \) is isomorphic to the direct product of its projections, \( A = A_1 \times A_2 \times \cdots \times A_n \), and we set \( U = \pi_1(A_1) \times \pi_2(A_2) \times \cdots \times \pi_n(A_n) \).

As for the converse, we set \( I = L_1 \times L_2 \times \cdots \times L_n, \) and consider \( (A_1 \times A_2 \times \cdots \times A_n) / I \cong A'_1 \times A'_2 \times \cdots \times A'_n \), where \( A'_i = A_i / L_i \), for all \( i \). Then the preceding theorem and lift back.

Alternatively, we could again use the method of determining the "bundles of \( n \)-tuples" (turning out to be cosets of \( I \) in \( U \)) as equivalence classes in \( U \), as we did in the case \( n = 2 \).

As for the last assertion, it suffices to apply theorem\(^6\) since \( E_{i_1 \cdots i_s} = \pi_{i_1 \cdots i_s}(U) \cap U \) which, as it lies inside the "equivalence class" \( I \subset U \) (for \( s < n \)), is the same as \( \pi_{i_1 \cdots i_s}(U) \cap (L_1 \times \cdots \times L_n) = L_{i_1} \times \cdots \times L_{i_s} \).

\[ \boxed{\text{Remark 23}} \]

This theorem may also be viewed as a generalization, in another direction, of the case \( n = 2 \).

\[ \boxed{\text{Corollary 24}} \]

A smashed subdirect product of \( A_1 \times A_2 \times \cdots \times A_n \) may always be taken as a pull-back of \( n \) epimorphisms.

\[ \boxed{\text{Corollary 25}} \]

If \( U \leq A_1 \times A_2 \times \cdots \times A_n \) such that, for all \( i \in N = \{i_1, i_2, \ldots, i_s\} \), \( E_i \) is contained in \( \left( L_{i_1} \times \cdots L_{i_s} \times \cdots \times L_{i_s} \right) \prod_{\kappa \notin N} E_\kappa \) (equivalently, just in \( \left( L_{i_1} \times \cdots L_{i_s} \right) \prod_{\kappa \notin N} E_\kappa \)), then the preceding theorem is applicable for \( \pi_{i_1 \cdots i_s}(U) \) as a subgroup of the direct product of its projections on \( A_{i_1}, \ldots, A_{i_s} \).

\[ \boxed{\text{Example 26}} \]

Assume that we have a normal subgroup \( B = B_1 \times \cdots \times B_n \) of a group \( G \), \( n \geq 2 \), hence every single direct factor \( B_i \) is normal in \( G \). Let \( G / B \approx R \).

Set \( K_i = Dr \prod_{j \neq i} B_j \), \( \sigma_i: G \rightarrow G / K_i, i = 1, \ldots, n \), the natural epimorphisms and, finally, \( A_i = G / K_i \), \( A = Dr \prod_{i=1}^n A_i \).

Then we get a faithful representation \( \sigma \) of \( G \) as a subdirect product of the direct product \( A \), as follows:

\[ \sigma: G \ni g \mapsto (\sigma_1(g), \ldots, \sigma_n(g)) \in A \]

It is obvious that this is a monomorphism (we are in the just following showing its injectivity) - and we set \( U = \sigma(G) \leq A = Dr \prod_{i=1}^n A_i \). We will be using the terminology that we have established above; it is immediate to see the following facts: \( \pi_i(U) = \sigma_i(G) = A_i \), \( L_i = \sigma(B_i) \cong (\text{realizable as } "\cong" \text{ inside } A) \sigma_i(B_i) = (\text{inside } A_i) B_i K_i / K_i = B_i / K_i \cong B_i \text{, therefore also } \sigma^{-1}(1) = \sigma^{-1}(L_1 \cap L_2) = \sigma^{-1}(L_1) \cap \sigma^{-1}(L_2) = B_1 \cap B_2 = 1 \) (alternatively, \( \sigma^{-1}(1) = \sigma^{-1} \left( \bigcap_{i=1}^n E_i \right) = \bigcap_{i=1}^n \sigma^{-1}(E_i) = \bigcap_{i=1}^n K_i = 1 \).
\(\sigma\) of \(G\) as a smashed subdirect product of groups of a group \(G\) and set \(B\) where we may for example use as \(\{\}
\)
\[
\text{Example 28}
\]
\[
\text{Theorem 29 (R. Remak) [10]} \text{ Let } G, G_i \text{ be minimal normal subgroups of a group } G \text{ and set } B = \prod_{i=1}^m B_i. \text{ Then there exists a subset } \{i_1, i_2, \ldots, i_n\} \subset \{1, \ldots, m\}, \text{ such that } B = B_{i_1} \times \cdots \times B_{i_n}.
\]

\[
\text{Problem 30 Conclusion[27] may also prompt us to the more general "inverse" problem, of investigating the ways to (faithfully) represent a given group as a}
\]

\[
1\), proving the injectivity of \(\sigma\); on the other hand, \(\sigma(B) = Dr \prod_{i=1}^n \sigma_i(B_i) = Dr \prod_{i \neq i}^n \sigma_j(B_j) = Dr \prod_{j \neq i}^n L_j\); in particular, \(E_i = \sigma(K_i) = Dr \prod \sigma(B_j) = Dr \prod \sigma_j(B_j) = Dr \prod L_j\), showing that \(U\) is a smashed subgroup of the direct product \(A\), hence our last theorem 19 applies; therefore, its core is just being \(I = L_1 \times L_2 \times \cdots \times L_n\), we have \(U/I = U/Dr \prod_{i=1}^n L_i \simeq \)
\(R\) and then, according to theorem 19, also \(\pi_i(U)/L_i \simeq R\) - a fact at which we also can arrive directly, as \(\pi_i(U)/L_i = (G/K_i)/(B_iK_i/K_i) \simeq G/B \simeq R\).

Also from the same theorem, more generally \(R \simeq \pi_{i_1i_2i_3}(U)/L_{i_1} \times L_{i_2} \times L_{i_3}\), for any proper subset \(\{i_1, i_2, \ldots, i_n\} \subset \{1, \ldots, n\}\); if we assumed that \(\pi_{i_1i_2i_3}(U) \leq U\), which would immediately also imply, due to the core \(I\)'s definition, that \(\pi_{i_1i_2i_3}(U) = L_{i_1i_2i_3}(U)\), then, according to theorem 5, all three isomorphic factor groups given by it should be trivial, hence \(U = L_{i_1i_2i_3} \times L_{1_i} \times \cdots \times L_{n_i}\), which in our case equals \(L_1 \times \cdots \times L_n\) - which, through the isomorphism \(\sigma\), would then yield that \(R \simeq G/B = 1\) and \(G = B = B_1 \times \cdots \times B_n\) (compare as well with lemma 12).

\[
\text{Conclusion 27} \text{ Given a subgroup } B = B_1 \times \cdots \times B_n \text{ of a group } G, n \geq 2, \text{ so that every single direct factor } B_i \text{ is normal in } G, \text{ we get a faithful representation } \sigma \text{ of } G \text{ as a smashed subdirect product } U \text{ of the direct product } A = Dr \prod_{i=1}^n G/K_i,
\]

\[
\text{where } K_i = Dr \prod_{j \neq i} B_j.
\]

\[
\text{Example 28 As a case of particular interest for our (quite general) example, we may for example, use as } B \text{ one of the normal subgroups (for example, the maximal of them, by starting off with all minimal normal subgroups, at least for } G \text{ finite) of a group } G \text{ given by the following theorem of R. Remak (as well)}:
\]

\[
\text{Theorem 29 (R. Remak) [10]} \text{ Let } B_1, \ldots, B_m (m > 0) \text{ be minimal normal subgroups of a group } G \text{ and set } B = \prod_{i=1}^m B_i. \text{ Then there exists a subset } \{i_1, i_2, \ldots, i_n\} \subset \{1, \ldots, m\}, \text{ such that } B = B_{i_1} \times \cdots \times B_{i_n}.
\]

\[
\text{Problem 30 Conclusion[27] may also prompt us to the more general "inverse" problem, of investigating the ways to (faithfully) represent a given group as a}
\]
subdirect product; of particular interest would be to get to non-smashed representations. We are dealing with this problem in our last section here.

Some orientation on previous investigation and application of this kind of problems in general Group Theory can be found in [6]; but it was addressed to, at least already in [11].

**Theorem 31** Given \( U \leq A = A_1 \times A_2 \times \ldots \times A_n \), let \( I = L_{N_1} \times \ldots \times L_{N_m} \) be the total cohesive decomposition of its core \( I \); denote, also, by \( \pi^i, i = 1, \ldots, m \), the projection from the product \( A \) to its subproduct attributed to the subset \( N_i \) of \( \{1, \ldots, n\} \). Set \( R = U/I \); then all quotients \( \pi^i(U)/L_{N_i}, \) for \( i = 1, \ldots, m \), are isomorphic to each other and to \( R \) in a "structural" way, as in our previous theorems (see theorem 22). \( U \) may be realized as a fiber product (pull-back) of the \( \pi^i(U) \)'s ("structurally coordinated") epimorphisms onto \( R \); in other words, \( U \) may be realized as a smashed subdirect product.

Also, for any sequence of indices \( i_1\langle i_2\langle \ldots\langle i_s \) from \( \{1, \ldots, m\} \), we have structural isomorphisms \( R \cong \pi^{i_1i_2\ldots i_s}(U)/(L_{N_{i_1}} \times \ldots \times L_{N_{i_s}}) \), where \( \pi^{i_1i_2\ldots i_s} \) denotes the projection from the product \( A \) to its subproduct attributed to the subset \( N_{i_1} \cup \ldots \cup N_{i_s} \) (a disjoint union) of \( \{1, \ldots, n\} \).

**Proof.** Thanks to the commutativity amongst the factors \( A_1, A_2, \ldots, A_n \) of the direct product \( A \), we may rearrange them in an order that fits into the sequence \( N_1, \ldots, N_m \) of our partition of \( \{1, \ldots, n\} \) and renumber; then, by setting \( B_\kappa = D_{\kappa} \prod_{j \in N_\kappa} A_j, \kappa = 1, \ldots, m \), we get \( A = B_1 \times B_2 \times \ldots \times B_m \), indeed a smashed product, whereupon we now may apply our theorem 11 and get exactly what we are looking for. ■

This last theorem 31 is a generalization of theorem 22; this may also be fruitfully combined to theorem 6.

### 4.2 Subdirect \( AutR \)-classes

In what follows in this subsection we start off at our last theorem 31; however we could equally well apply this theory in the situation of any of the previous theorems of similar nature.

Let us so use the notation of theorem 31

We shall denote by \( \Psi^{m-1}(AutR) \) the set of equivalence classes in \( (AutR)^m \) under the following relation: \( (\sigma_1, \ldots, \sigma_m) \sim (\tau_1, \ldots, \tau_m) \) iff there exists \( \rho \in AutR: (\sigma_1\rho, \ldots, \sigma_m\rho) = (\tau_1, \ldots, \tau_m) \). We may also define multiplication in \( (AutR)^m \)
in the obvious way, and it is then equally apparent that left multiplication by another element preserves equivalence of two elements.

**We need also to identify all** \( m \) **factor groups** \( \pi^i(U)/L_{N_R} \) **with** \( R \): we allow ourselves further a notational convention, that we may w.r.t. the canonical epimorphisms \( \xi_i: \pi^i(U) \to \pi^i(U)/L_{N_R} \) identify the preimage \( \xi_i^{-1}(r) \) as \( rL_{N_R} \), which may not cause any confusion, insasmuch as these coset representatives \( \pi^i(U) \) shall never interact with one another. We shall then remember that, whenever considering the fixed representatives (transversals) \( \pi^i(U) \) in different \( \pi^i(U) \), they do only make a group when considered modulo \( L_{N_R} \). This identification may be viewed as a step toward the idea of "virtuality"/"virtual category".

It is immediate to check through the universal property of a fiber product that the fiber products \( \{ (\xi_i, i = 1, \ldots, m); R \} \) and \( \{ (\rho \circ \xi_i, i = 1, \ldots, m); R \} \) for any \( \rho \in AutR \) are identical; on the other hand we may also look directly into these fiber products as subsets (and subgroups) of \( \prod_{i=1}^m \pi^i(U) \leq A = A_1 \times A_2 \times \ldots \times A_n \), by looking at their connective bundles:

We can immediately see that composing all the \( m \) canonical epimorphisms \( \xi_i: \pi^i(U) \to \pi^i(U)/L_{N_R} \) (and similarly any other set of epimorphisms \( \sigma_i \circ \xi_i: \pi^i(U) \to \pi^i(U)/L_{N_R} \) for some isomorphisms \( \sigma_i \)) with a \( \rho \in AutR \) from the left and then taking the corresponding fiber-product for \( \{ (\rho \circ \xi_i, i = 1, \ldots, m); R \} \) results in precisely the same subgroup, as the one gotten as the original pullback \( U \) of \( \{ (\xi_i, i = 1, \ldots, m); R \} \) over \( R \), insasmuch as it gives precisely the same connective bundles: The connective bundles that constitute the original fiber product \( U \) are precisely \( \{ (\xi_1^{-1}(r), \ldots, \xi_m^{-1}(r)), r \in R \} \) which in somewhat loose notation shall also be describing as \( \{ (rL_{N_1}, \ldots, rL_{N_m}), r \in R \} \), wherein we are also recalling our standard notation around \( U \) (see theorem 31). By letting \( \sigma = (\rho, \ldots, \rho) \in (AutR)^m \) act on the \( m \) canonical epimorphisms \( \xi_i: \pi^i(U) \to \pi^i(U)/L_{N_R} \) through composition as above, we now get the new fiber product as a subgroup of \( \prod_{i=1}^m \pi^i(U) \), consisting of the connective bundles

\[
\begin{align*}
\{ (\rho \circ \xi_1)^{-1} \cdots (\rho \circ \xi_m)^{-1} (r), r \in R \} &= \\
= \{ (\xi_1^{-1} \rho^{-1} \cdots \xi_m^{-1} \rho^{-1} (r)), r \in R \} &= \\
= \{ (\xi_1^{-1} r \cdot i, \ldots, \xi_m^{-1} r \cdot i), r \in R \} \text{ which, by substituting } r \to r^\sigma \text{ is rewritten as } \\
\{ (\xi_1^{-1}(r), \ldots, \xi_m^{-1}(r)), r \in R \} &= \{ (rL_{N_1}, \ldots, rL_{N_m}), r \in R \} \text{ precisely as for } U, \\
\text{and with the same multiplication.}
\end{align*}
\]

We may also more generally let any \( \sigma = (\sigma_i, i = 1, \ldots, m) \in (AutR)^m \) act on \( \prod_{i=1}^m \pi^i(U)/L_{N_R} \), thus yielding a new subdirect product in \( \prod_{i=1}^m \pi^i(U) \), out of the original one \( U \), obtained as the fiber product \( U^\sigma: \{ (\sigma_i \circ \xi_i, i = 1, \ldots, m); R \} \) over \( R \): The resulting \( U^\sigma \) consists of the connective bundles

\[
\begin{align*}
\{ (\sigma_1 \circ \xi_1)^{-1} \cdots (\sigma_m \circ \xi_m)^{-1} (r), r \in R \} &= \\
= \{ (\xi_1^{-1} \sigma_1^{-1} \cdots \xi_m^{-1} \sigma_m^{-1} (r)), r \in R \} &= \\
= \{ (r^{\sigma_1 \cdot i} L_{N_1}, \ldots, r^{\sigma_m \cdot i} L_{N_m}), r \in R \} \text{.}
\end{align*}
\]
had we acted with its "\sim"-equivalent m-tuple $\varphi=(\tau_1, ..., \tau_m)=(\sigma_1 \rho, ..., \sigma_m \rho)$ of $R$-automorphisms above, we would again get the exactly same set of connective bundles, now described as $$\left\{ \left( r^{\tau_1^{-1}} L_{N_1}, ..., r^{\tau_m^{-1}} L_{N_m} \right), r \in R \right\} = \left\{ \left( (r^{n_{\rho^{-1}}} \sigma_1^{-1} L_{N_1}, ..., (r^{n_{\rho^{-1}}} \sigma_m^{-1} L_{N_m}) \right), r \in R \right\}$$ which, by substituting $r \rightarrow r^\rho$ is rewritable as $$\left\{ \left( r^{\sigma_1^{-1}} L_{N_1}, ..., r^{\sigma_m^{-1}} L_{N_m} \right), r \in R \right\};$$ thus we have obtained an equality between the subgroups $U^\varphi$ and $U^\sigma$. Conversely, if we had $U^\varphi=U^\sigma$ for some $\varphi= (\sigma_i, i=1, ..., m), \varphi= (\tau_1, ..., \tau_m) \in (\text{Aut} R)^m$, then their corresponding sets of connective bundles must be identical, i.e.

$$\left\{ \left( r^{\sigma_1^{-1}} L_{N_1}, ..., r^{\sigma_m^{-1}} L_{N_m} \right), r \in R \right\} = \left\{ \left( (r^{n_{\tau_1^{-1}}} \sigma_1^{-1} L_{N_1}, ..., (r^{n_{\tau_m^{-1}}} \sigma_m^{-1} L_{N_m}) \right), r \in R \right\}$$

from which we conclude the existence of a unique bijection $\rho : R \rightarrow R$ such that

$$\left\{ \left( r^{\tau_1} L_{N_1}, ..., r^{\tau_m} L_{N_m} \right), r \in R \right\} = \left\{ \left( (r^{n_{\rho^{-1}}} \sigma_1^{-1} L_{N_1}, ..., (r^{n_{\rho^{-1}}} \sigma_m^{-1} L_{N_m}) \right), r \in R \right\},$$

where this last expression is the fiber product of $\{(\sigma_i \circ \xi_i, i=1, ..., m) : R\}$ over $R$, while the first is the one of $\{ (\tau_i \circ \xi_i, i=1, ..., m) : R \}$ over $R$, forcing $\sigma_i \circ \rho=\tau_i$. That $\rho$ is homomorphic modulo $I$ follows from the multiplication rules of the connective bundles.

These findings amount to the following

**Proposition 32** In the above described way, two m-tuples $\varphi, \sigma \in (\text{Aut} R)^m$ give rise to the same subgroup if and only if the m-tuples $\varphi, \sigma$ of automorphisms of $R$ are equivalent under the introduced "right projective" equivalence relation "\sim". The so determined action of such a $\sigma$ on $\prod_{i=1}^m \pi_i^m (U)/L_{N_i}$ may be realized by changing the coordinated structural isomorphisms of theorem 27 all the way through, a change effectuated by "twisting" every $\pi_i^m (U)/L_{N_i}$ by $\sigma_i^{-1}$. By denoting the $(\text{Aut} R)^m$-orbit of $U$ as $\mathfrak{P}(U)$, we do so finally get an induced faithful and transitive action of $\mathfrak{P}^{m-1}(\text{Aut} R)$ on $\mathfrak{P}(U)$, where by $\mathfrak{P}^{m-1}(\text{Aut} R)$ we mean $(\text{Aut} R)^m/\sim$.

We point out that the twistings above establish the new "alignments", that determine the new connective bundles that constitute $U^\varphi$ out of those of the original $U$.

**Proposition 33** Let $\varphi= (\sigma_1, ..., \sigma_m) \in (\text{Aut} R)$, where $\sigma_1 = id_R$, and let $\Sigma = \langle \sigma_2, ..., \sigma_m \rangle$; then $U^\varphi \cap U$ consists of the $R^\Sigma$-connective bundles in $U$, where $R^\Sigma$ is the subgroup of $R$, consisting of the $\Sigma$-fixed points on it - that is $U^\varphi \cap U = \{ rL_{N_1}, ..., rL_{N_m}, r \in R^\Sigma \}$, by using our notation explained above. In particular, all groups in the $\mathfrak{P}^{m-1}(\text{Aut} R)$-orbit $\mathfrak{P}(U)$ of $U$ contain the core $I = L_{N_1} \times ... \times L_{N_m}$ of the original $U$.

**Proof.** It follows from our discussion above, by comparing the connective bundles of $U^\varphi$ and $U$. ■
It is clear that every equivalence class in \((\text{Aut}R)^m\) contains a representative of the form of \(\sigma\) in the proposition above, i.e. with the first component equal to \(id_R\); We shall call it its \((1-)\text{canonical representative}\); further we shall call the subgroup \(\Sigma\) of \(\text{Aut}R\) generated by all components of the canonical representative its breadth group. We could have defined corresponding breadth groups by demanding the \(i\)-th automorphism to be trivial instead; it is nonetheless immediate to check that \(any\ of\ these\ choices\ results\ in\ the\ same\ breadth\ group.\)

We want next to examine, whether we can determine \(\text{conditions to ensure the existence of a homomorphism } \alpha \text{ of } U, \text{ coinduced by } \sigma = (\sigma_1, \ldots, \sigma_m);\) i.e. which acts trivially on the core \(I\) of \(U\) and induces \(\sigma_i\) on \(\pi^i(U) / L_{N_i};\) such one would probably establish a very convenient isomorphism between \(U\) and \(U^\sigma.\)

Due to the original direct product, this issue boils down to the corresponding question for every \(\pi^i(U)\) (except that we are now looking for automorphisms \(\alpha : \pi^i(U) \to \pi^i(U)\)).

This would in general seem too good to be true: In order to come closer to some sufficient conditions for such a cute set-up, let us further \(\text{assume that } U \text{ splits over } I, \text{ i.e. that } U \cong R \times I, \text{ which again is equivalent to } \pi^i(U) = R_i \times L_{N_i}, \text{ for } i = 1, \ldots, m, \text{ where } R_i \cong R.\)

Let us therefore define such a map \(\alpha\) on \(U\), by determining its \(N_i\)-coordinates through \(\pi^i(\alpha(r)) = \sigma_i(r)\) or, with exponential notation, \(r^{\sigma_i}\), for \(r \in R_i\), \(\pi^i(\alpha(l)) = l\) for \(l \in L_{N_i}\). This is clearly a bijective map; we are now going to find conditions for it to be homomorphic:

Let \(x, x' \in U\), with \(\pi^i(x) = r_i l_i, \pi^i(x') = r_i' l_i'\), where \(l_i, l_i' \in L_{N_i}, \quad r_i, \quad r_i' \in R_i\). On the one side we have that \(\pi^i(\alpha(xx')) = \alpha(\pi^i(xx')) = \alpha(r_i l_i r_i' l_i') = \alpha(\sigma_i(r_i l_i r_i' l_i')) = \alpha(r_i r_i' l_i l_i') = r_i^\sigma_i r_i'^\sigma_i l_i l_i' (1)\), on the other is \(\pi^i(\alpha(x) \alpha(x')) = \pi^i(\alpha(x)) \pi^i(\alpha(x')) = \alpha(\pi^i(x)) \alpha(\pi^i(x')) = \alpha(r_i l_i) \alpha(r_i' l_i') = r_i^\sigma_i r_i'^\sigma_i l_i l_i' (2).\)

Then for \(\alpha\) to be a homomorphism one should have \(\alpha(xx') = \alpha(x) \alpha(x'),\) or equivalently that \(\pi^i(\alpha(xx')) = \pi^i(\alpha(x)) \pi^i(\alpha(x'))\), which by (1) & (2) means \(l_i l_i' = l_i'^{\sigma_i}\), that is, \(r_i^{-1} r_i'^{\sigma_i}\) centralizes \(l_i \forall l_i \in L_{N_i}, \forall r_i \in R_i\) - i.e. that every element of the form \(r_i^{-1} r_i'^{\sigma_i}\) in each \(R_i\) centralizes \(L_{N_i}, i = 1, \ldots, m.\) Assume now further that \(\text{the map } \sigma \in (\text{Aut}R)^m\) is the 1-canonical representative of its class in \(\mathfrak{P}^{m-1}(\text{Aut}R)\). Notice that, with such an 1-canonical \(\sigma \in (\text{Aut}R)^m\), the condition we have found is automatically trivially satisfied by \(R_1.\)

**Proposition 34**  
\(\text{a. If } U \text{ splits over its core } I, \text{ then the necessary and sufficient condition for the existence of an isomorphism } \alpha \text{ from } U \text{ to } U^\sigma \text{ (with } \sigma \text{ 1-canonical) acting trivially on the core } I \text{ of } U \text{ and inducing } \sigma_i \text{ on } R_i, i = 1, \ldots, m \text{ is that every element of the form } r_i^{-1} r_i'^{\sigma_i} \text{ in each } R_i \text{ centralizes } L_{N_i}, i = 1, \ldots, m.\)

\(\text{b. Assuming further that, for every } i = 2, \ldots, m, \text{ } \sigma_i \text{ is a fixed-point free automorphism of } R_i \text{ and that every } R_i \text{ is either finite or abelian and Artinian (as a } \mathbb{Z}\text{-module), the condition above is equivalent to the statement that } R_i \text{ is contained in the center } \mathfrak{Z}(\pi^i(U)) \text{ of } \pi^i(U), \text{ hence that } \pi^i(U) = L_{N_i} \times R_i, \text{ for } i = 2, \ldots, m.\)

In particular, that latter is the case if already \(R\) itself centralizes \(I, \text{ i.e. if } U \cong R \times I, \text{ which is equivalent to } \pi^i(U) = L_{N_i} \times R_i, \text{ for } i = 1, \ldots, m.\)
Proof. It is now sufficient to prove part (b).

Assuming that the automorphism \( \sigma_i \) is fixed-point free, the map (not a homomorphism, in general) \( \phi_i \) sending \( x \in R_i \) to \( x^{-1}x^\sigma_i \in R_i \) is injective: for \( x^{-1}x^\sigma_i = y^{-1}y^\sigma_i \Leftrightarrow yx^{-1} = (yx^{-1})^\sigma_i \), whence the assumption on \( \sigma_i \) gives \( y = x \).

If \( R_i \) is finite, then \( \phi_i \) is clearly bijective.

Let us now suppose that \( R_i \) is abelian and Artinian (as a \( \mathbb{Z} \)-module).

Then \( \phi_i \) is suddenly homomorphic, actually a monomorphism. Then the Artinian property (DCC) forces the monomorphism \( \phi_i \) to be surjective, hence an automorphism: Suppose that \( \phi_i \) is not surjective. So, there exists some \( 0 \neq y \in R_i \) that does not belong to \( \text{Im} \phi_i \), therefore \( \phi_i(y) \notin \text{Im} \phi_i^2 \). On the other hand \( \phi_i(y) \), obviously belonging to \( \text{Im} \phi_i \), cannot be 0, due to \( \phi_i \)'s injectivity; that proves that the obvious inclusion \( \text{Im} \phi_i \supset \text{Im} \phi_i^2 \) is strict; by a similar argument the strictness of inclusion continues inductively in the infinite tower \( \text{Im} \phi_i \supset \text{Im} \phi_i^2 \supset \text{Im} \phi_i^3 \supset \text{Im} \phi_i^4 \supset \ldots \) which contradicts the Artinian DCC. Therefore is \( \phi_i \) an automorphism.

That means that, in both cases of (b) every element of \( R_i \) may be written in the form \( x^{-1}x^\sigma_i \), for some \( x \in R_i \); therefore, the condition existence of a \( \sigma_i \)-coinduced homomorphism \( \alpha \) of \( U \), becomes that every element of \( R_i \) centralize \( L_{N_i} \), according to (a), i.e., that \( R_i \) centralizes \( L_{N_i} \).

Notice that, in case the condition of (b) on \( R_i \), \( i = 1, \ldots, m \), is satisfied for all but for \( i = \kappa \), then one should prefer the \( \kappa \)-canonical representative of the class of \( \mathfrak{F} \) in \( \mathfrak{F}^{m-1}(\text{Aut}R) \), afterwards examine if all \( \sigma_i \) on \( R_i \) for \( i \neq \kappa \) are fixed-point free.

Remark 35 Here is a situation, where the above proposition is applicable, possibly (depending on \( \mathfrak{F} \)) its part (b) too: Let \( U \) above be of finite order \( \kappa \lambda \), \( (\kappa, \lambda) = 1, |R| = \kappa, |I| = \lambda \), with \( I \) abelian; in that case it is known (for example. [4, IV 3.13, Remark]) that the sequence \( I \rightarrow U \rightarrow R \) splits - and, consequently (or just by the same arguments, as \( |L_{N_i}| \) is a divisor of \( |I| \)), \( \pi^i(U) \) splits over \( L_{N_i} \) (i.e., \( L_{N_i} \rightarrow \pi^i(U) \rightarrow R_i \) splits).

Lemma 36 Define \( \phi_i : R_i \rightarrow R_i \) as the map sending \( x \) to \( x^{-1}x^\sigma_i \); by assuming \( R_i \) to be abelian, \( \phi_i \) becomes a group homomorphism. Then the restriction of \( \sigma_i \) to the image \( \phi_i(R_i) = \text{Im} \phi_i \) is fixed-point free.

Proof. Observe that \( \ker \phi_i = R_i^{(\sigma_i)} \), where \( R_i^{(\sigma_i)} \) is the subgroup of \( \sigma_i \)-fixed points, hence we get the natural isomorphism \( \text{Im} \phi_i \cong R_i/R_i^{(\sigma_i)} \) so that the \( \sigma_i \)-action on \( \text{Im} \phi_i \) be equivalent to the one induced on \( R_i/R_i^{(\sigma_i)} \).
Example 37 Let us just take a simple example, just to assist visualization:

Consider the epimorphisms \( \xi_1 : \mathbb{Z}_{15} \ni x \mapsto x \mod 3 \in \mathbb{Z}_3 \) and \( \xi_2 : \mathbb{Z}_{21} \ni y \mapsto y \mod 3 \in \mathbb{Z}_3 \); their fiber product over \( \mathbb{Z}_3 \) is then the subgroup \( U = \{(x, y) \in \mathbb{Z}_{15} \times \mathbb{Z}_{21} : x \mod 3 = y \mod 3 \} \), clearly a subdirect product of \( \mathbb{Z}_{15} \times \mathbb{Z}_{21} \). Let \( \text{Aut} \mathbb{Z}_3 = \langle \sigma \rangle, \sigma^2 = 1 \); let us determine the subgroup \( U^\sigma \), with \( \sigma = (1, \sigma) \sim (\sigma^{-1}, 1) \) which, according to proposition 32, is effectuated through "twisting" by \( (\sigma, 1) \): That means, \( U^\sigma = \{(x, y) \in \mathbb{Z}_{15} \times \mathbb{Z}_{21} : (x \mod 3)^\sigma = y \mod 3 \} \).

Notice that both \( U \) and \( U^\sigma \) convey similar diagrammatic depictions as \( \mathbb{Z}_5 \upharpoonright \mathbb{Z}_3 \downharpoonleft \mathbb{Z}_7 \).

4.3 Subdirect \( \mathcal{E} \)-(in)decomposability

One might wonder about the decomposability of any arbitrary group as a subdirect product and what does a particular kind of decomposition mean in terms of the structure of the group. To meet this kind of questions, also by getting inspiration from our example 26 above, we come to the propositions below.

Before proceeding to see them, we wish to generalize the notion of a subdirect group product, by allowing its definition up to isomorphism and without the restriction about the finite number of direct factors:

**Definition 38** \( U \) shall be called a (generalized) subdirect product of the family \( \{A_i, i \in I\} \) if there exists a monomorphism \( \mu : U \to \text{Dr} \prod_{i \in I} A_i \), such that for any canonical projection \( \pi_i : \text{Dr} \prod_{i \in I} A_i \to A_i \) the composite \( \pi_i \circ \mu \) is an epimorphism.

**Proposition 39** \( U \) is a subdirect product of the family \( \{A_i, i \in I\} \) if and only if there exists a family \( \{E_i, i \in I\} \) of normal subgroups of \( U \), so that \( U \uparrow E_i \cong A_i \), and \( \bigcap_{i \in I} E_i = 1 \).

**Proof.** If the family of normal subgroups is given, in order to define \( \mu : U \to \text{Dr} \prod_{i \in I} U \uparrow E_i \) it suffices to define all \( \pi_i \circ \mu : U \to U \uparrow E_i \); we simply define them as the canonical maps.

Conversely, given a subdirect product \( U \) as in the definition, let \( E_i \) be defined the way we have done it earlier, i.e. \( E_i = \ker (\pi_i \circ \mu) \); but since \( \pi_i \circ \mu : U \to A_i \) has been assumed to be epi-, we get readily \( U \uparrow E_i \cong A_i \), as wished. On the other hand the kernel of the monomorphism \( \mu \), being trivial, is also equal to \( \bigcap_{i \in I} \ker (\pi_i \circ \mu) = \bigcap_{i \in I} E_i \), and we are done. ■
**Corollary 40** A group cannot be (non-trivially) written as a subdirect product if and only if the intersection of all its non-trivial normal subgroups is non-trivial.

Such groups may be called *subdirectly indecomposable*.

There is much more that can in a similar manner be derived from the last proposition; to state them in generality, let $\mathcal{E}$ be a property referring to factor groups by normal subgroups of a given group $G$. We might also refer to $\mathcal{E}$ as a class of groups, and consider then the normal subgroups of $G$, such that the corresponding factor group belong to the class $\mathcal{E}$. We shall call the intersection of all such normal subgroups the *$\mathcal{E}$-residual of $G$*. It becomes then immediate to see the following

**Proposition 41** The necessary and sufficient condition for a group $G$ to be expressible as a subdirect product of groups belonging to the class $\mathcal{E}$ is that the $\mathcal{E}$-residual of $G$ is trivial - and there are at least two non-trivial, proper normal subgroups of $G$, such that the corresponding factor group belongs to $\mathcal{E}$.

We mention some examples in the next

**Corollary 42** If $G$ is a torsion group, $\omega_1$, $\omega_2$, $\ldots$, $\omega_s$ sets of primes, then $G$ can be expressed as a subdirect product of respectively $\omega_1$, $\omega_2$, $\ldots$, $\omega_s$ -groups if and only if $\bigcap_{i=1}^{s} \mathcal{O}^{\omega_i} = 1$, where $\mathcal{O}^{\omega}$ is the $\omega$-residual of $G$ (for def. see for example. [13, 3.44]).

We remind that a group $G$ is called quasisimple if it is perfect (″$[G, G] = G$″) and $G/Z(G)$ is simple. The quasisimple residual $QCR(G)$ of a group $G$ is defined as the intersection of all kernels of epimorphisms of $G$ onto quasisimple groups (see also [6] intr., where it is namely called "q.s. radical" - but I think "residual" is the proper name).

**Corollary 43** A group $G$ is expressible as a subdirect product of quasisimple groups if and only if its quasisimple residual $QCR(G)$ is trivial. In that case it follows that it actually becomes a direct or central product.

**Corollary 44** The quasisimple residual $QCR(G)$ of a group $G$ is its unique smallest normal subgroup such that the corresponding factor group is a direct or central product of quasisimple groups.

**Proposition 45** Every group $G$ may be written as a subdirect product of groups that are either simple or subdirectly indecomposable.
Proof. Let us assign to each $x \neq 1$ in $G$ a normal subgroup $K_x$, maximal among the normal subgroups not containing $x$; obviously $\bigcap_{x \neq 1} K_x = 1$. By invoking to the elementary fact of the $1 - 1$ correspondence between the lattices of normal subgroups of $G$ containing $K_x$, and of normal subgroups of $G/K_x$ (see for example, [13, 3.29]), we see immediately that any normal subgroup of $G/K_x$ has to contain the (non-trivial) canonical image of $x$ in $G/K_x$, otherwise the $1 - 1$ correspondence would yield a normal subgroup of $G$, properly containing $K_x$, contradicting the maximality of $K_x$. In case there is no normal subgroup of $G/K_x$ either, containing the canonical image of $x$ in $G/K_x$, this group is obviously simple, while otherwise is $G/K_x$ subdirectly indecomposable (corollary 40). Proposition 39 now gives the result.

Remark 46 Of course this proposition does not tell us anything about how interesting the guaranteed subdirect decomposition might be. For example in the case of a simple $G$ the proof of the proposition actually gives us a subdirect representation of $G$ as the diagonal in the product $\prod_{x \in G} G_x$, where each $G_x = G$.

That is, in the case of a simple $G$ (and only in this!), the procedure in the proof of proposition 45 yields a "deltoid subdirect product" (compare our definition 12), in the sense that its whole core $I$ is trivial. That is of course uninteresting, it makes therefore sense to substitute $H$ for any diagonal $\Delta H$ like in the case of a simple $G$ in the outcome of the procedure, in the spirit of condition 13 still to get a subdirect decomposition of the form guaranteed by proposition 45 (except only for the case that the given group was simple) but a more interesting one. We notice also that such a decomposition is not in general uniquely determined, as the choice of a maximal $K_x$ is not so either.

I believe that this new view/realization of our subject may provide a key to progress on other questions, even in better understanding some already known results and thereby also enhancing their deepening or implementation; as a possible such reviewing might be thought the issue of the lattice of such subgroups, also of the normal subgroups; on this subject it should anyway be expedient to revisit, among many others of course, [2], [13], [16].

5 Subdirect presentations & applications to homomorphisms

It is in itself interesting to look at subdirect products from another point of view (and compare), but we may furthermore gain important new insight and basic results about homomorphisms/endomorphisms on the way, considerably extending classical/elementary ones; results which hitherto have (amazingly)
remained hidden, while we come across them very naturally and unrestrained by the present approach. Also our approach here, for which we already have been predisposed by example 24, remains general - but it would be interesting and very fruitful, I believe, to apply our results and techniques in more specific contexts or in concrete situations.

5.1 The case of two factors

Let \( f_i : A \to G_i, \ i = 1, 2 \), be non-trivial group epimorphisms (we may always get to epimorphisms, by substituting the target group with the image of the homomorphism). Let also \( \Delta : A \ni a \mapsto (a, a) \in A \times A \) be the diagonal monomorphism, \( F : A \times A \ni (a_1, a_2) \mapsto (f_1(a_1), f_2(a_2)) \in G_1 \times G_2 \) and define \( u := F \circ \Delta, U := u(A) \leq G_1 \times G_2 \).

We shall subsequently be using all our previous terminology and symbols about \( U \leq G_1 \times G_2 \). Apparently, \( \ker (F) = \ker (f_1) \times \ker (f_2) \leq A \times A \), therefore is \( \ker (u) = \ker (f_1) \cap \ker (f_2) \) (1). Of course, the assumed surjectivity of the \( f_i \)'s means that the chosen \( U \) indeed is a subdirect product. Due to (1), the condition \( \bigcap_{i=1}^{2} \ker (f_i) = 1 \) is equivalent to \( u \) being injective; in this case, we may describe \( U \) more explicitly (set-theoretically), as \( U = \{(f_1(a), f_2(a)) \mid a \in A\} \).

As \( L_1 = E_2 = U \cap G_1 = f_1(\ker (f_2)) \) and \( L_2 = E_1 = U \cap G_2 = f_2(\ker (f_1)) \), theorem 47 in this case becomes:

**Proposition 47** \( G_1/f_1(\ker (f_2)) \cong G_2/f_2(\ker (f_1)) \cong U/f_1(\ker (f_2)) \times f_2(\ker (f_1)) \).

By taking \( G_2 = A \) and \( f_2 = id_A \), we get

**Corollary 48** Given an epimorphism \( f_1 : A \to G_1 \) of groups, we have \( G_1 \cong A/\ker (f_1) \)

which, of course, is the elementary first homomorphism theorem: at this point, it is essential to notice that our proof of theorem 47 on which proposition 48 depends, does not apply this homomorphism theorem!

In this view, the first isomorphism in Prop. 47 above is seen to be a generalization of the first homomorphism theorem; as such one, we reformulate it here:

**Corollary 49** Given group homomorphisms \( f_i : A \to G_i, \ i = 1, 2 \), we have \( Im (f_1)/f_1(\ker (f_2)) \cong Im (f_2)/f_2(\ker (f_1)) \).

Of particular interest is to specialize to the case, in which we have endomorphisms instead of homomorphisms, when we shall drop surjectivity, so as to have the same target group \( A \) in all cases; it is often convenient to take isomorphic copies \( A_i \) of \( A \), through fixed isomorphisms: we may subsequently do it at convenience, even without special notification.
Corollary 50  Given \( \rho_i \in \text{End}(A) \), \( i = 1,2 \), it holds that \( \text{Im}(\rho_1) / \rho_1(\ker(\rho_2)) \simeq \text{Im}(\rho_2) / \rho_2(\ker(\rho_1)) \simeq U / \rho_1(\ker(\rho_2)) \times \rho_2(\ker(\rho_1)) \), where \( U = u(A) \), as above.

5.2  The case of \( n (>2) \) factors

The general outset is just a generalization of the case \( n = 2 \); thus, let \( f_i : A \rightarrow G_i \), \( i = 1, \ldots, n \), be non-trivial group epimorphisms (we can always reduce to the case of epimorphisms, by taking the images as domains of the \( f_i \)'s). Let also \( \Delta : A \ni a \mapsto (a, \ldots, a) \in A^n \) be the diagonal monomorphism, \( F : A^n \ni (a_1, \ldots, a_n) \mapsto (f_1(a_1), \ldots, f_n(a_n)) \in Dr \prod_{i=1}^n G_i \) and define \( u := F \circ \Delta \), \( U := u(A) \leq Dr \prod_{i=1}^n G_i \). As before, \( \ker(F) = Dr \prod_{i=1}^n \ker(f_i) \leq A^n \), whence \( \ker(u) = \bigcap_{i=1}^n \ker(f_i) \), so that the assumption of its triviality, i.e. that \( \bigcap_{i=1}^n \ker(f_i) = 1 \), amount to \( u \)’s injectivity, in which case we may describe \( U \) set-theoretically as, \( U = \{(f_1(a), \ldots, f_n(a)) / a \in A\} \).

Our previous terminology shall apply to our \( U \) here too.

\( u \) may of course be viewed as a representation of the group \( A \) as subdirect product; by taking our outview from a subdirect product, however, one might be looking for a suitable \( u \), i.e. an \( A \) with the right homomorphisms, to get to such a "presentation" of a given \( U \):

Definition 51  For a subdirect product \( U \) of \( Dr \prod_{i=1}^n G_i \), we shall be calling a homomorphism \( u = F \circ \Delta : A \rightarrow Dr \prod_{i=1}^n G_i \) as above, such that \( u(A) = U \), a "presentation of \( U \) by homomorphisms"; we shall also denote this subdirect product \( U \) by \( [A; (f_1, \ldots, f_n)] \). It shall be called "terse", if it is injective, i.e. if \( \bigcap_{i=1}^n \ker(f_i) = 1 \).

Let \( i_1 < i_2 < \ldots < i_s \) sequence of indices from \( \{1, \ldots, n\} \); set \( \Lambda = \{i_1, i_2, \ldots, i_s\} \) and write \( \{1, \ldots, n\} = \Lambda \cup \hat{\Lambda} \), a disjoint union.

Set \( K_i = \ker(f_i) \) and, for any subset \( \Lambda \) of indices as above, \( K_\Lambda = K_{i_1i_2\ldots i_s} = K_{i_1} \cap K_{i_2} \cap \ldots \cap K_{i_s} \). Set, furthermore, \( \xi_\Lambda = \pi_\Lambda \circ u \). Clearly, \( \xi_\Lambda(A) = \pi_\Lambda(U) \).

We see immediately that

Lemma 52  (a) \( \ker(\xi_\Lambda) = K_\Lambda \), (b) \( L_\Lambda(U) = u(K_\Lambda) = E_\Lambda(U) \) and (c) The core \( I \) of \( U \) is, \( I = \langle E_i / i = 1, \ldots, n \rangle = \langle u(K_i) / i = 1, \ldots, n \rangle = u(\langle K_i / i = 1, \ldots, n \rangle) \) = \( u \left( \prod_{i=1}^n K_i \right) \).

25
Lemma 53 Given a presentation by homomorphisms u of the subdirect product $U$ as above, we can always get to a terse presentation of $U$ as a subdirect product of the same direct product.

Proof. Since $K_{12...n} = \bigcap_{i=1}^{n} \ker(f_i)$ is contained in the kernel of any $f_i$, all $f_i$’s factor through $\overline{A} = A/K_{12...n}$, giving rise to $\overline{f_i}: \overline{A} \rightarrow G_i, i = 1, ..., n$ and, thus, a terse presentation. □

It is immediate to verify the following remarkable lemma:

Lemma 54 Given a "tersely presented by homomorphisms" smashed subdirect product $U = [A; (f_1, ..., f_n)]$, we can readily get a usual definition of $U$ as a pull-back out of it. Conversely, given a definition of a subdirect product $U$ as a pull-back, we get to a terse presentation of it as $U = [U; (p_1, ..., p_n)]$, where $p_i$ is the $i$'th projection from the direct product, which may be considered as trivial in the sense that the homomorphism $u = (p_1, ..., p_n) \circ \Delta$ is the identity map on $U$.

Proof. We restrict ourselves to show it here for $n = 2$ (in which case $U$ is always smashed), as the technic is the same for any $n$; the generalization for an arbitrary $n > 2$ is obtained by use of theorem[22]

For the first part, we set $G_1/f_1(\ker(f_2)) \cong G_2/f_2(\ker(f_1)) \cong R$; for the pull-back, we set off from the epimorphisms $\tau_i: G_i \rightarrow R$, that have to be chosen so that together they induce the structural $\sigma: G_1/f_1(\ker(f_2)) \rightarrow G_2/f_2(\ker(f_1))$ (: the structural correspondence of pair-bundles), for example, by letting $G_2/f_2(\ker(f_1)) := R$ and, by denoting as $\pi_i: G_i \rightarrow G_1/f_1(\ker(f_2))$, $\{i,j\} = \{1,2\}$, the natural projections, take $\tau_1 = \sigma \circ \pi_1$ and $\tau_2 = \pi_2$; i.e.

\[
\begin{array}{ccc}
U & \rightarrow & G_1 \\
\downarrow & \downarrow & \downarrow \\
G_2 & \tau_2 & R
\end{array}
\]

For the other direction, let $U$ be given as the pull-back of the epimorphisms $\tau_i: G_i \rightarrow R$, i.e. $U = \{(g_1, g_2) \in G_1 \times G_2 / \tau_1(g_1) = \tau_2(g_2)\}$; take $[\hat{U}; (p_1, p_2)]$, set $K_i = \ker(f_i)$ (i=1,2) and observe that $\ker(p_1) = 1 \times L_2$, $\ker(p_2) = L_1 \times 1$, $p_i(\ker(p_j)) = L_i$ (i $\neq j$). □

So, we see that homomorphic presentation of a subgroup of a direct product is not necessarily bound to its smashedness, as its definition as a pull-back does.

We may apply theorem[23] to our "homomorphically" presented" subdirect product $U$, even without demanding our $u$ to be injective ("terse"). Taking into account lemma[52](b), we get:

Proposition 55 With the notation above, $\xi_A(A) / u(K_A) \cong u(\xi_A(A) / u(K_A) \cong u(A) / (u(K_A) \times u(K_A))$.

For $n=2$, this amounts to proposition[17] let us see what we get for $n=3$: 26
Corollary 56 Given the non-trivial group epimorphisms \( f_i : A \to G_i, i = 1, 2, 3 \), we have the double isomorphism:
\[
G_1 / f_1(K_{23}) \cong \xi_{23}(A) / \xi_{23}(K_1) \cong u(A) / (f_1(K_{23}) \times \xi_{23}(K_1)) - \text{and two more, symmetrically.}
\]

We recall now a previous note, from just above lemma 16:

- In elementary terms, this condition means that \( L_M \times L_N = L_A \) iff, for any \( x \in L_A \), the element \( \pi_M(x) \) of \( Dr \prod_{i \in M} A_i \) also belongs to \( L_M \) - or, equivalently, for any \( x \in L_A \), \( \pi_N(x) \in L_N \).

By using this criterion, we get readily to the following proposition.

Proposition 57 Assume that \( K_{12\ldots n} = 1 \), i.e. that \( u \) is injective (: a terse presentation of \( U \)) and let \( \Lambda = \{i_1, i_2, \ldots, i_n\} = M \cup N \) (\( M \cap N = \emptyset \)) be a partition of the subset \( \Lambda \) of \( \{1, \ldots, n\} \). The condition \( L_A = L_M \times L_N \) is in our case of such a "homomorphically" presented" subdirect product \( U \) equivalent to \( K_\Lambda = K_M \times K_N \) (an "internal" direct product), where \( M \), \( N \) are the complements of \( M \), \( N \) (respectively) inside the index set \( \{1, \ldots, n\} \).

Proof. By implementing the above mentioned criterion, if \( L_A = L_M \times L_N \) then, for any \( x \in K_\Lambda \) (\( \iff u(x) \in L_A \)), \( \pi_M(u(x)) = \xi_M(x) \in L_M \), meaning that there is some \( x_M \in A \), with \( u(x_M) \in L_M \) (\( \iff f_i(x_M) = 1 \) for any \( i \in \hat{M} \iff x_M \in K_M \)), such that \( \pi_M(u(x_M)) = \pi_M(u(x)) \); correspondingly, we have that \( \pi_N(u(x)) = \xi_N(x) \in L_N \), meaning that there is some \( x_N \in K_N \), such that \( \pi_N(u(x_N)) = \pi_M(u(x)) \).

We claim now that \( u(x_Mx_N) = u(x) \); since both parts do apparently belong to \( L_A \) and \( \Lambda = M \cup N \), it suffices to prove that both projections \( \pi_M \) and \( \pi_N \), when applied to them, give the same result.

However, \( \pi_M(u(x_Mx_N)) = \pi_M(u(x_M)) \pi_M(u(x)) = \pi_M(u(x_N)) = \pi_M(u(x)) \), as \( x_N \in K_N \subset K_M = \ker(\pi_M \circ u) \) and, similarly, \( \pi_N(u(x_Mx_N)) = \pi_N(u(x)) \); hence, as noticed, \( u(x_Mx_N) = u(x) \), which, by invoking to the injectivity of \( u \), implies that \( x = x_Mx_N \).

We have thus proven that \( K_\Lambda \subset K_M \times K_N \); the other inclusion being apparent, this implies \( K_\Lambda = K_M \times K_N \). But, as \( M \cap N = \emptyset \) implies \( \hat{M} \cup \hat{N} = \{1, \ldots, n\} \), we get that \( K_{\hat{M}} \cap K_{\hat{N}} = K_{1\ldots n} = 1 \), as \( U \)'s presentation is "terse", implying that the product of those two normal subgroups of \( K_\Lambda \) is, indeed, direct.

The converse implication becoming now quite apparent, our proposition has been proven.

Corollary 58 For the tersely presented \( U \) above, the condition for it to be smashed (see def. 11) is, that the subgroup of \( A \), generated by the normal subgroups \( K_i \), \( i = 1, \ldots, n \), is the (necessarily direct, due to terseness) product of all \( K_i \)'s - i.e., \( \prod_{i=1}^{n} K_i = \prod_{i=1}^{n} K_i \).
We may also specialize to endomorphisms instead of the above homomorphisms \( f_i \) - in which case, of course, we shall have to abandon the choice of surjectivity for the defining homomorphisms.

**Example 59** Let \( U = [A; (f_1, \ldots, f_n)], f_i : A \twoheadrightarrow G_i, i = 1, \ldots, n \) (\( n \geq 2 \)), where \( A \) has a normal subgroup \( B = B_1 \times \cdots \times B_n \), hence every single direct factor \( B_i \) is normal in \( A \). Let \( G/B \simeq R \).

Set \( K_i = Dr \prod_{j \neq i} B_j, \text{ker}(f_i) = K_i, G = Dr \prod_{i=1}^n G_i \).

In the notation that we have introduced in this last section, \( K_i = \bigcap_j K_j = B_i \) and, by the corollary above, \( U \) is smashed.

Of course, this here is a re-visiting (and notational updating) of example 26.

**Example 60** We construct an example of a non-smashed subdirect product:

Let \( U = [A; (f_1, \ldots, f_5)], f_i : A \twoheadrightarrow G_i, i = 1, \ldots, 5 \), where \( A \) has a normal subgroup \( B = B_{12} \times B_5 \times B_3 \times B_2 \), and let \( C_1, C_2 \) be normal subgroups of \( A \) contained in \( B_{12} \), with trivial intersection (or otherwise the presentation wouldn’t be terse) so that \( C_1C_2 \neq B_{12} \), with \( K_1 = \ker(f_1) = C_2B_3B_4B_5, K_2 = \ker(f_2) = C_1B_3B_4B_5, K_3 = \ker(f_3) = B_{12}B_3B_4B_5, K_4 = \ker(f_4) = B_{12}B_3B_5, K_5 = \ker(f_5) = B_{12}B_3B_4 \). The presentation is terse, as \( K_{12345} = 1 \).

By lemma 35(b), we have: \( L_1 = u(K_{2345}) = u(C_1), L_2 = u(K_{1345}) = u(C_2), L_12 = u(K_{1345}) = u(B_{12}); \) by applying proposition 40, we see that \( L_{12} \) is cohesive, which means that \( U \) here is not smashed. The cohesive components are readily seen to be \( L_{12}, L_3 = u(K_{1235}) = u(B_3), L_4 = u(K_{1325}) = u(B_4), L_5 = u(K_{1324}) = u(B_5) \). We may now apply theorem 29, to deduce that:

\[ [A; (f_1, f_2)]/u(B_{12}) \simeq G_3/\ker(f_3) \simeq G_4/\ker(f_4) \simeq G_5/\ker(f_5) \simeq \cdots \]

It is not difficult to make a generalization of this example.

**Remark 61** If we let \( f_i : A \twoheadrightarrow G_i, i = 1, \ldots, n \), be just group homomorphisms, not necessarily surjective, we get obvious actions of \( Aut(A) \) and \( Dr \prod_i Aut(G_i) \), further also of \( End(A) \) and \( Dr \prod_i End(G_i) \), on the set (/semigroup) of subgroups of \( Dr \prod_i G_i \). Of particular interest is the case, when all \( G_i \)'s are isomorphic - in particular, to \( A \).

6 From pull-backs & push-outs to diagrams

6.1 General approach

As we have seen, subgroups of direct products may be viewed as pull-backs, i.e., fiber products. Their structure conveys "naturally" diagrammatic depictions of
the form \( \rightarrow/\) - with \( m \) edges in the general case of theorem 31. Although we have not given a proper general definition for diagrams, its suggested use in this case just corresponds to the structure theorem 31 and certainly has the special restriction that it refers to a particular representation of a group \( U \) as a subdirect product. It has, nevertheless, the basic characteristic that we might expect of any diagram: Consisting of just two layers (levels), the vertices of the lower correspond to (a direct product of) subobjects (subgroups), the vertex on top is a factor group, namely one that is a factor group in many different ways according to our general theorem 31. Let us, for our ease, allow ourselves to call the vertex on top the head of our depiction of \( U \), the direct product of the lower level its socle; notice that this refers only to the particular embedding of \( U \) as a subdirect product.

If we now, conversely, use the investigated structure of such subgroups in order to deduce properties for such a basic diagram, our theorem 31 and our whole analysis of the structure makes it clear that:

**Lemma 62** a. Any subdiagram of the suggested subdirect-representation diagram of such a subgroup \( U \) comprising a single edge (or any proper subset of the set of edges), corresponds to a certain factor group - but never to a subgroup of \( U \). Consequently, there is no proper subgroup of \( U \) that corresponds to any subdiagram containing the top vertex.

b. The diagrammatic properties of any subdiagram as in (a), comprising any number of edges, corresponding to a factor group of \( U \), are the same as of the whole diagram of \( U \) - i.e., property (a) "repeats itself".

Notice that, whenever we speak of subdiagrams, we mean that they are connected and that they include the ends of their edges.

But there is another major feature to justify taking this kind of simple diagrams as a major cornerstone for a diagrammatic theory: Namely, what makes such a diagrammatic depiction especially interesting and worth studying for us is its **virtuality**, in the sense that the multiple direct factors of the "socle" are also determined set-theoretically (elementwise in set-theoretically well defined sections of \( U \)) and not just up to isomorphism - meaning that: Their vertices correspond to well-defined subsets of well-defined subsections. Notice that this virtuality could not possibly be claimed just by reference to pull-backs, as these are only defined up to isomorphism; this is why we our first approach has been through "subgroups of direct products".

Our main focus with diagrammatic methods shall from now on however shift from groups to modules and representation theory. We want to move toward a new kind of diagrams there, one having "virtual properties" in a sense that generalizes the basic "virtuality" described above. The original motivation toward the main subject of this article has actually been to begin understanding and substantializing "virtuality" as well as possible: then I chose to generalize by considering the more difficult category of groups instead of that of abelian groups or modules, while viewing it as very interesting also for its own sake.
The next natural step would now be to consider the dual case, i.e. push-outs. Things do however get somewhat complicated, if we attempt to dualize these ideas in the category of groups: In that category coproducts are namely given by free products - and push-outs by amalgamated free products. That deviates quite from our original motive - and we shall therefore not look at them in this article. Of particular interest is, however, the case of extensions of an arbitrary group \( G \) by an abelian group \( A \): then the push-out gives the extension that is (functorially) induced by homomorphisms \( A \to A' \) of \( G \)-modules, while the ones induced by homomorphisms \( G' \to G \) are as usually (see below) obtained as pull-backs. The push-out mentioned here is in this case a quotient of the semidirect, rather than the direct, product (see [3] IV 3, exercise 1(b); see also ex. 2).

On the contrary, things become much more (dually) analogue to what we have seen about pull-backs also in the case of push-outs, if we confine ourselves to the category \( \text{Ab} \) of abelian groups: then the push-out of a family \( S \to A_i, \ i \in I, \) of morphisms is obtained as a certain factor group of their direct sum (coproduct); one may compare this with example [7].

So, in \( \text{Ab} \) we may draw diagrammatic fan-like depictions for push-outs as well like the above on pull-backs but "dual" to them, i.e. the common vertex of all edges shall now be at the bottom, in a form like \( \backslash/ \).

Speaking of pull-backs and push-outs with corresponding fan-like diagrams, we must point out that such a "fan"-diagram may in both cases even degenerate to a single edge; we are also going to investigate this case - but not even in the category \( \text{Ab} \): From now on we shall be considering modules instead, by letting a ring (/an algebra, f.ex. a group ring) act on abelian groups - to begin with, generally in a category of (left) Artinian and Noetherian \( R \)-modules, \( R \) being a ring with 1: For example, in a category of finitely generated modules over an Artinian ring.

In the following we shall often be referring to [5] Chapter III, especially sections 1, 3, 5]. We shall look at (1-) extensions of \( R \)-modules, and first of all we want to discuss the case of "proportional" extensions, as we have called them in [5], corresponding to "composites of a small exact sequence with a homomorphism (in our case: an automorphism)" according to their exposition in [5] Chapter III, section 1:

For \( R \)-modules \( A, C \) we consider the set of extensions \( Y \text{Ext}^1_R (C, A) \) as consisting of the equivalence classes in the class of short exact sequences \( \mathcal{E} \) (by which we may also denote its extension class) of the form \( \mathcal{E} : 0 \to A \to E \to C \to 0 \) in the usual manner (the corresponding notation in [5] is \( \text{Ext}_R (C, A) \)).

\( Y \text{Ext}^1_R (\_ , \_ ) \) is a bifunctor, contravariant in the first, covariant in the second argument: Let, so, \( \alpha : A \to A'' \), \( \gamma : C' \to C \) be non-zero \( R \)-module homomorphisms; our bifunctor transforms them to \( \alpha_* : \mathcal{E} \to \mathcal{E}'' \), \( \gamma^* : \mathcal{E}' \to \mathcal{E} \) by using identity map on the other extreme term and by getting the middle term of \( \mathcal{E}' , \mathcal{E} \) as push-out and pull-back, respectively. We are also adopting MacLane’s designation of the resulting extensions \( \mathcal{E}'' = \alpha_* \mathcal{E} \), \( \mathcal{E}' = \gamma^* \mathcal{E} \) as, respectively, \( \alpha \mathcal{E} \).
and $E\gamma$; we shall, correspondingly, denote by $\alpha E$ and $E\gamma$ the middle terms resulting from the original $E$. It is also important to understand $\alpha E$, resp. $E\gamma$, as the obstruction for extending $\alpha$ to $E$, resp. for lifting $\gamma$ to $E$, and the obstruction homomorphisms $\tau^* = \tau^*_E : \text{Hom}_R\left(C', C\right) \ni \gamma \mapsto E\gamma \in Y\text{Ext}_R(C', A)$, $\tau_* = \tau_*E : \text{Hom}_R(A, A'') \ni \alpha \mapsto \alpha E = \alpha E \in Y\text{Ext}_R(C, A'')$ as the connecting homomorphisms for the $E$-derived covariant, resp. contravariant, long exact sequences, see [8], Chapter III, section 3 - where also notably its involved higher steps (or at least the first one, involving 1-extensions) are realized in a somehow self-contained manner, i.e. just by using extension classes of sequences, with no reference to the proper concept of the derived functors: We have of course also the Yoneda identification of the above mentioned functors $Y\text{Ext}$ with the derived functors $\text{Ext}$, see for example [8], III 2.4, IV 9.1: "There is natural equivalence between the set-valued bifunctors $Y\text{Ext}^n_R(C, A)$ (of equivalence classes of exact $n$-sequences from $C$ to $A$) and $\text{Ext}^n_R(M, A)$, $n = 1, 2, ..."$

We are now going to specialize even more: Let $A' = A, C' = C$ be simple $R$-modules. Assume also that $E$ is non-split. Then $\alpha, \gamma$ are automorphisms (unless they are 0) and the middle terms $E'' = \alpha E, E' = E\gamma$ of the (non-congruent to $E$, unless those automorphisms are the identity maps!) short exact sequences $E'' = \alpha E, E' = E\gamma$ are isomorphic to $E$ (5-lemma), where $E$ is uniserial of length 2, with series $C \to A$. The middle term of $E\gamma$ is the pull-back of $\gamma$ and $\sigma : E \to C$ in $E$, while the middle term of $\alpha E$ is the push-out of $\alpha$ and the map $A \to E$ in $E$.

We have first to define "upper and lower proportionals", respectively of type $E\gamma$ and $\alpha E$, as two distinct concepts; it is immediate to see that each of them defines an equivalence relation: we shall accordingly speak of "upper" or "lower" proportionality.

From the short exact sequence $E$ above we get by applying the functor $\text{Hom}(\_, \_)$ the long exact sequence ([8], III 3.4])

\[ 0 \to \text{Hom}_R(C, A) \to \text{Hom}_R(C, E) \to \text{Hom}_R(C, C) \to Y\text{Ext}^1_R(C, A) \to Y\text{Ext}^1_R(C, E) \to ... \]

giving here the monomorphism $\tau^* = \tau^*_E : \text{Hom}_R(C, C) \to Y\text{Ext}^1_R(C, A)$, which is implemented by the assignment $\gamma \mapsto \gamma^*E = E\gamma$ in the way mentioned in the previous paragraph. By (a variant of) Schur’s lemma is $\text{Hom}_R(C, C) \cong \mathcal{O}_c$, a division ring. It is easily verified that $(\gamma \circ \gamma')^* = \gamma'^* \circ \gamma^*$, i.e. $E(\gamma'\gamma) = (E\gamma')(\gamma)$, and that $E(id_{\gamma}) = E$, for any extension class $E$ in $Y\text{Ext}^1_R(C, A)$, meaning that:

**Lemma 63** $\mathcal{O}_c$ acts on $Y\text{Ext}^1_R(C, A)$ from the right. In particular we have that $(E\gamma)\gamma^{-1} = E$ (1).

Dually, through application of the functor $\text{Hom}(\_, A)$ on the above short exact sequence $E$ we get a long exact sequence ([8], III 3.2], which yields the monomorphism $\tau_* = \tau_*E : \text{Hom}_R(A, A) \to Y\text{Ext}^1_R(C, A)$, amounting in terms
of $Y\text{Ext}_R^1(C,A)$ to what has explicitly been explained above by the annotation $\alpha \mapsto \alpha_\ast$, where again $\text{Hom}_R(A,A)$ is a division ring $\mathcal{D}_A$.

In that way we get two different actions on $Y\text{Ext}_R^1(C,A)$, one by taking automorphisms on $C$, the other one through automorphisms of $A$. By looking at the second action, giving rise to the notion of lower proportional extensions of $C$ ([5]), let us consider the projective cover $P$ of $C$; let us further assume now that $R$ is Artinian. Then it is well known (see for example [2, Cor. 2.5.4]) that we have the following isomorphisms of $R$-modules:

"Given an Artinian ring $R$, the $R$-modules $A, M$ where $A$ is simple, for $n > 0$ we have the following isomorphisms of $R$-modules: $\text{Ext}_R^n(M,A) \cong \text{Hom}_R(\Omega^nM, A)$, $\text{Ext}_R^n(A,M) \cong \text{Hom}_R(A, \Omega^{-n}M)$.""

In particular, we have an 1-1 correspondence (in fact, a natural equivalence of set-valued bifunctors) between $Y\text{Ext}_R^1(C,A)$, $\text{Hom}_R(\Omega C, A)$ and $\text{Hom}_R(C, \Omega^{-1}A)$, where $\Omega C = \text{rad}P$ and $\Omega^{-1}A = I/\text{soc}I$, with $P$ the projective cover of $C$, $I$ the injective hull of $A$. That means that, by considering a virtual radical series of $P/\text{rad}^2P$, we get an identification of $Y\text{Ext}_R^1(C,A)$ with $\text{Hom}_R(\Sigma A, A)$ where $\Sigma A$ is the $A$-part of $P$'s second radical layer. We are going to investigate how this identification becomes a natural $\mathcal{D}$-module isomorphism (where $\mathcal{D} = \mathcal{D}_A$), by viewing $Y\text{Ext}_R^1(C,A)$ as a (left) $\mathcal{D}$-module, by virtue of the above mentioned "$\mathcal{D}$-proportionality action" on $A$. Let $\Sigma A \cong A^m$, i.e. a sum of $m$ copies of the irreducible $A$; from now on we shall be looking at those copies as virtually identifiable ([5]).

On the other hand it is clear that $\text{Hom}_R(\Omega C, A) = \text{Hom}_R(\text{rad}P, A) \cong \text{Hom}_R(\Sigma A, A) \cong \mathcal{D}^m$.

We shall at first see, why we may view the proportionals of any factor module $E$ of $P$ of the type $\begin{array}{c} \ldots \\ \rightarrow C \rightarrow E \rightarrow A \rightarrow 0 \\ \rightarrow \end{array}$ as factor modules of $P$ again. So, we are looking at the short exact sequence $0 \rightarrow A \rightarrow E \rightarrow C \rightarrow 0$, while viewing $A, E$ and, of course, $C$ as virtually determined w.r.t. $P$. Denote by $E$ the extension class of the above sequence, then we will denote the extension module of $\gamma E = E\gamma$ by $E\gamma$, that of $\alpha E = \alpha E$ by $\alpha E$.

Our virtual approach leads us to the following concern, which is also fundamental for the whole setup:

**Problem 64** The projective property of the projective cover $P$ of $C$ guarantees that any extension of $C$ by a simple $A$ may be realized as a quotient module of $P$. How can we get a "virtual" overview of those quotients, compared to the respective extension classes?

Dually, the injective property of the injective hull $I$ of $A$ ensures that any extension of any simple module $C$ by $A$ may be viewed as a submodule of $I$. How can we get a "virtual" overview of those submodules, compared to the respective extension classes?

It is crucial to realize that in any such 1-extension the above suggested epimorphism must be understood as the composition of an automorphism of $C$
with the canonical one; we may, dually, look at the suggested monomorphism as the composition of the inclusion with an automorphism of $A$.

This suggests that one and the same (virtually!) module, let us say here, one with series $\frac{C}{A}$, is not just assignable to one but to a whole family of ("proportional", as we are going to see) extensions! We may sometimes occasionally allow ourselves to use different letters for the same module, inasmuch as it is drawn in different extensions. In the following we want to show a kind of converse to this assertion, namely that proportional extensions are realizable by the same (not just an isomorphic!) module: But such a statement may only sense make in the frame of a virtual category! This shows that the (proper to the case) virtual category is actually the proper frame to consider extensions in.

Let so $P=P_C$ be the projective cover of $C$ as above, and apply $\text{Hom}_R(P,\_)$ on $E$: since $\text{Hom}_R(P,A)=0=Y\text{Ext}^1_R(P,E)$, that yields an $R$-module isomorphism (2) $\text{Hom}_R(P,E)\cong \text{Hom}_R(P,C)$, induced by the canonical $\sigma : E \to C$ in $E$. It is also clear that $\text{Hom}_R(P,C) \cong \text{End}_R C$, naturally given by the assignment $\text{End}_R C \ni \gamma \mapsto \gamma \circ \xi \in \text{Hom}_R(P,C)$, where $\xi$ is the canonical epimorphism $P \to C$, inducing $id_C : \text{End}_A C \to C$, in virtual terms. To work in the virtual category, let us consider the middle term $E$ of $E$ as a quotient of $P$ by a submodule $L$. Then we fix the canonical epimorphism $P \to C$, as the one corresponding to $E$, while its distortions by automorphisms of $C$ shall give its "upper proportional" extensions; we illustrate this discussion by the following diagram, in which we are also using (1) from lemma 63:

\[
\begin{array}{ccc}
P & \xrightarrow{\xi} & E \\
\downarrow{\text{id}_A} & & \downarrow{\gamma^{-1}} \\
E : A & \xrightarrow{\gamma} & C.
\end{array}
\]

We are pointing out that the (uniquely through $E$ and $P \to C$ determined) homomorphism $P \to E$ has to be surjective, inasmuch as it cannot split, since $E$ is "of type" $\frac{C}{A}$ and not semisimple. The crucial observation to make here is that, not only is $\ker(P \to E\gamma) = \ker(P \to E)$ but, furthermore, $P \to E\gamma$ is "almost" the canonical epimorphism $P \to E$, since it twists the canonical one by an automorphism $\gamma^{-1}$, inducing the identity on the socle $A$ of $E$, thus just meaning a "reordering" of the $A$-cosets in $E$: it is still the same set! If we conversely start with an isomorphism from an object $E'$ to another $E$ in $Y\text{Ext}^1_R(C,A)$, with $C$, $A$ simple modules, with the identity map on $A$, then it induces some automorphism $\gamma'$ on $C$, so that $E' = E\gamma'$.

That means that:

Lemma 65 An upper proportionality class in $Y\text{Ext}^1_R(C,A)$ is "virtually determined" by a certain quotient of the projective cover of $C$, which in turn is of course determined by a certain submodule, to be called the "proportionality kernel".

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In the frame of the virtual category $\mathfrak{V}(P)$ of $P$ we need a special ordering "$\sqsubseteq$", defined as follows:

**Definition 66** In the frame of the virtual category $\mathfrak{V}(P)$ of $P$ define on the family of submodules of subquotients of $P$ an ordering "$\sqsubseteq$" generated by the rule, that "less than" mean to be a submodule or a (sub)quotient of; we shall accordingly speak of "slimmer" or, on the contrary, of "broader" subquotients.

It turns out to be futile, if we attempt to virtually determine "lower proportionality" classes in $Y\text{Ext}^1_R(C, A)$, as quotients of the projective cover of $C$, again by just looking at obstructions. Instead of that, we now turn toward the proof for the Yoneda equivalence $\text{Yon} : \text{Ext} \rightarrow Y\text{Ext}$:

Let us denote by $\Sigma A$ the $A$-part of the second radical layer $\text{rad}P/\text{rad}^2P$ of $P$. The Yoneda correspondence establishes a bijection between $Y\text{Ext}^1_R(C, A)$ and $\text{Hom}_R(\Omega C, A) = \text{Hom}_R(\text{rad}P, A) \cong \text{Hom}_R(\text{rad}P/\text{rad}^2P, A) \cong \text{Hom}_R(\Sigma A, A)$. Let us see closer to how extension classes in $Y\text{Ext}^1_R(C, A)$ may be identified by homomorphisms $\overline{\alpha} : \Sigma A \rightarrow A$: Such a homomorphism has to split, due to semisimplicity of $\Sigma A$. There is therefore a well defined submodule $\overline{A}$ ($\cong A$) of $A$, henceforth to be called the support of $\overline{\alpha}$, such that $\Sigma A = \ker\overline{\alpha} \oplus \overline{A}$ - and so that we may identify $\overline{\alpha}$ by the submodule $\overline{A}$ of $\Sigma A$ (in fact, also identifiable as $\text{Coim}(\overline{\alpha})$) and its (isomorphic, since it is onto and we are in an exact category) restriction $a : \overline{A} \rightarrow A$ to it.

Let us again return to an epimorphism $\zeta : \Omega C = \text{rad}P \rightarrow A$ of kernel, say, $L_\zeta$: Then $\zeta$ factors through an induced isomorphism $\text{Coim}(\zeta) \cong A$, where $\text{Coim}(\zeta) = \Omega C/\ker\zeta = \Omega C/L_\zeta$, by means of the canonical $\xi : \Omega C \rightarrow \Omega C/L_\zeta$. Write also $\text{rad}P/\text{rad}^2P = \Sigma A \oplus N$, $\rho : \text{rad}P \rightarrow \text{rad}P/\text{rad}^2P$ for the canonical epimorphism there and set $R' := \rho^{-1}(N)$, then take as $\overline{\alpha} : \Sigma A \rightarrow A$ that $\zeta$ considered modulo $R'$: inasmuch as $\text{Coim}(\zeta)$ considered modulo $R'$ is virtually nothing but the $\text{Coim}(\overline{\alpha}) = A$ we have seen above, and this $\zeta$-induced isomorphism $\text{Coim}(\zeta) \cong A$, which $\zeta$ factors through and which also (together with $\text{Coim}(\zeta)$, of course) determines $\zeta$ completely. Therefore $\zeta$ may naturally and unambiguously be identified by this $\overline{\alpha} : \Sigma A \rightarrow A$, which precisely induces a on its "support" $\overline{A}$: if we call $p$ the projection of $\Sigma A$ onto $A$ along the decomposition (3), then we have $\overline{\alpha} = a \circ p$ and $a = \overline{\alpha}|_{\overline{A}}$.

Also, we may write $\xi = p \circ r$, where $r : \Omega C \rightarrow \Omega C/R = \Sigma A$ is the canonical epimorphism. Hence is $\xi = a \circ \xi = a \circ p \circ r$.

So we have a well defined epimorphism $\zeta : \Omega C = \text{rad}P \rightarrow A$ of kernel $L_\zeta$, which precisely induces a on $\overline{A}$, and which may be viewed as gotten through twisting the canonical epimorphism $\xi : \text{rad}P \rightarrow \overline{A}$, virtually equal to $\text{rad}P/L_\zeta$, by a (i.e. $\zeta = a \circ \xi$), similarly to what we have seen before.

We identify for a moment $A$ with $\overline{A}$, so that we may virtually look at arbitrary lower proportionals. Set $E = P/L_\zeta$, and call $E_\alpha$ its lower $a$-proportional (to $E$ denoted as $aE$, like above), i.e. $E_\alpha$ is a push-out, but a virtual one, as in the diagram
in which the extension \( aE \) is realizable is precisely \( P/L_\zeta \), i.e. the same as with \( E \): In the virtual category \( \mathcal{V}(P) \) of \( P \) we may look at them as quotients of that quotient of \( P \), which results from factoring out the largest possible submodule, so that both \( E \) and \( E_\alpha \) still be quotients of that quotient. That quotient is easily seen to be the pull-back of \( E \) and \( E_\alpha \) over the top \( C \) in \( \mathcal{V}(P) \).

If \( E \) and \( E_\alpha \) were different quotients of \( P \), then that pull-back in the virtual category of \( P \) would have the diagrammatic form \( \text{\shortdownarrow} \), with \( C \) on top; however this is contradicted by the fact that the pull-back of \( E \) and \( E_\alpha \) is just \( E \), as it is immediately verified by the universal property from the diagram:

\[
\begin{array}{ccc}
E & \xrightarrow{id_E} & E \\
\downarrow & & \downarrow \\
E_\alpha & \rightarrow & C
\end{array}
\]

Therefore \( E_\alpha \) in \( aE \) is realized as the same quotient module of \( P \), as \( E = P/L_\zeta \) in \( E \).

Seen in a more set-theoretic way, \( E \) is equal to \( \bigcup_{x \in C} \sigma^{-1}(x) \), each \( \sigma^{-1}(x) \) being an \( \tilde{A} \)-coset, while the meaning of the identity being induced in the previous diagram, that shows the isomorphism \( E \rightarrow aE \), is that \( E_\alpha \) still consists of precisely the same (set-theoretically!) cosets, where only they are "rearranged", by application of the automorphism \( a^{-1} \) on \( \tilde{A} \).

We now illustrate the situation with the following diagram:

\[
\begin{array}{ccccccccc}
0 & 0 & \rightarrow & \Omega C & \rightarrow & P & \rightarrow & C & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
L_\zeta & = & L_\zeta & & L_\zeta & & L_\zeta & & L_\zeta \\
\zeta = a \circ \xi, & (D_1) & & (\downarrow) & & (\downarrow) & & (\downarrow) & (\downarrow) \\
\end{array}
\]

Notice that unless \( A \) be identified with \( \tilde{A} \), \( aE \) is somehow "abstract": we shall however use the above discussion and diagram \((D_1)\) as a kind of springboard, to enhance its scope in a virtual direction:

We may identify the automorphisms of \( \tilde{A} \) with those of \( A \), by fixing an (arbitrary!) isomorphism; such an identification is therefore only conceivable
up to an automorphism of $A$. We shall call such an automorphism-identifying isomorphism an identomorphism. However this relativity is a consideration that might only be taken when comparing identomorphisms to the same $A$ of supports of different such epimorphisms $\Sigma A \to A$; this issue remains anyway insignificant when talking of "lower proportional" of the same extensions, as the approach is then done by fixing just one such identomorphism.

That is, having established such an identification $j : A \to A$ for our $\tilde{A}$ here, we may then view $a$ in the above diagram $(D_1)$ as equal to $\alpha \circ j$, for an element $\alpha$ of $\text{Aut}A$, thus also identifying $jE$ with $E$, so that we now may get all the lower proportionals of $E$ through automorphisms of $A$.

**Remark 67** It must be here observed that, in identifying an extension class in $Y \text{Ext}^1_R(C, A)$, by means of an arbitrary epimorphism $\Sigma A \to A$, where $A$ is, say, a "prototypic isomorphic copy", this may at first glance seem only to be possible up to lower proportionality; however we shall show in the following subsection that this seemingly innate relativity may easily be overcome.

We may now dualize everything above, by working on $E$ as embedded in the injective hull $I$ of $A$; in particular, if $E\gamma$ were realized by a different submodule of $I$, then the push-out of these two over $A$ would contain them both properly: however it is easy to verify again (by the universal property) that this push-out is actually $E$ itself. It may on the other hand again be precisely seen, how any upper proportional $E\gamma$ of an extension $E$ of $C$ by $A$ is also a lower upper proportional $\alpha E$ of $E$, for some automorphism $\alpha$ of $A$.

We sum our results up (and the similarly deducible dual analogues) in the following

**Proposition 68** A lower proportionality class in $Y \text{Ext}^1_R(C, A)$ is "virtually determined" by a certain quotient of the projective cover $P$ of $C$, as its "virtual middle term", identifiable by a direct summand of $\text{rad}P/\text{rad}^2P$ isomorphic to $A$; dually, an upper proportionality class in $Y \text{Ext}^1_R(C, A)$ is "virtually determined" by a certain submodule of the injective hull of $A$ as its "virtual middle term".
term”, identifiable by a direct summand of $\text{soc}^2 I / \text{soc} I$ isomorphic to $C$, where $I$ is the injective hull of $A$.

There exists, further, a bijection between $\text{YExt}^1_R(C, A)$, $\text{Hom}_R(\Omega C, A)$ and $\text{Hom}_R(\Sigma A, A)$ - and, similarly, a bijection between $\text{YExt}^1_R(C, A)$, $\text{Hom}_R(C, \Omega^{-1} A)$ and $\text{Hom}_R(C, \Sigma C)$, where $\Omega C = \text{rad} P$ and $\Omega^{-1} A = I / \text{soc} I$, with $P$ the projective cover of $C$, $\Sigma A$ is the $A$-part of $\text{rad} P / \text{rad}^2 P$, $\Sigma C$ the $C$-part of $\text{soc}^2 I / \text{soc} I$.

We might finally as well attempt a more set-theoretic view of the identical realizations of the extension modules $E$ and $E\gamma$:

Lemma 69 (i) The extension module $E \circ \text{id}_C$ is identifiable with $E$. (ii) The extension module $E\gamma$, for any $\gamma \in \text{Aut} C$, viewed as a submodule of $E \times C$ is in fact $(E \circ \text{id}_C)^{(1, \gamma)}$ - and it is virtually identifiable with $E$ in $P$, as the same quotient module.

Proof. (i) We have to remind that the extension module $E \circ \text{id}_C$ is only defined up to isomorphism, not set-theoretically; it is also clear that it is isomorphic to $E$, it is only its extension class that varies. Its standard concrete construction is given as a submodule of $E \times C$, in this case $= \bigcup_{c \in C} \{(\xi^{-1}(c), c)\}$, where $\xi : E \to C$ in the exact sequence, whose elements may be viewed as being the same as those of $E$, just written with the superfluous suffixes $c$. It is in fact easy to check that they are naturally isomorphic.

On the other hand lemma guarantees that $(\text{id}_C)^* \mathcal{E} = \mathcal{E} (\text{id}_C) = \mathcal{E}$, therefore in the frame of a virtual category the fact that $\mathcal{E}$ still remains in the same equivalence after appliance of $(\text{id}_C)^*$, notably also while inducing identity on $C$, it is clear that $(\text{id}_C)^*$ also induces the identity map on $A$, meaning that, in virtual terms, $(\text{id}_C)^* E = E (\text{id}_C)$ is identical with $E$.

(ii) Compared to this writing of $E = E \circ \text{id}_C$, $E \circ \gamma$ is written as 

\[
\bigcup_{c \in C} \{(\xi^{-1}(c), \gamma^{-1} c)\} = (E \circ \text{id}_C)^{(1, \gamma)}
\]

which is clearly not set-theoretically identical to $E$. We can, nevertheless, similarly to (i) above see that $(E \circ \gamma_1) \circ \gamma_2 = E \circ (\gamma_1 \circ \gamma_2)$ as sets, implying that the functor $Y \text{Ext} (\gamma, A)$ may also give rise to a functor from modules to sets, the sets of the extension modules. On the other hand it is immediate to check that $(E \gamma_1) \gamma_2 = E (\gamma_1 \gamma_2)$ and, similarly, that $\alpha_1 (\alpha_2 \mathcal{E}) = (\alpha_1 \alpha_2) \mathcal{E}$ for $\alpha_1, \alpha_2 \in \text{End} A$.

We have clearly a bijection between $\text{Aut} C$ and $\text{Hom} (P, C)$, $\gamma \mapsto \tilde{\gamma} := \gamma \circ \xi$, in which we may to $\text{id}_C$ attach the canonical projection $\xi$, that virtually corresponds to the extension module $E$. But then $\tilde{\gamma}^{-1} = \gamma^{-1} \circ \xi$ factors through the extension module $E \circ \gamma$ in virtual terms, hence we get an epimorphism $P \to E \circ \gamma$ which has the same kernel as $\xi$. This proves that $E$ and $E \circ \gamma$ are virtually the same factor modules in $P$. ■
6.2 \( \Omega \)- or \( \Phi \)-space of virtual extensions of irreducibles

We wish now to enhance our analysis above, prior to proposition 68.

Let us so again consider \( \Sigma A \) as there and let us choose some decomposition \( \Sigma A = \bigoplus_{i=1}^{m} A_i \), with \( A_i \) fixed; let also \( s_i : A_i \rightarrow A \) be some fixed isomorphisms ("identomorphisms"), with \( A \) a "prototypic isomorphic copy" for them all: Notice that such a "prototypic isomorphic copy" of the \( A_i \)'s also allows us now by virtue of the "identomorphisms" \( s_i \) to "coordinate" (or identify, if you prefer) their automorphisms. Denote by \( p_i, i = 1,...,m \), the standard projections of \( \bigoplus_{i=1}^{m} A_i \). We have a natural isomorphism between \( \text{Hom}_R(\Omega C, A) \) and \( \text{Hom}_R(\Sigma A, A) \), therefore we may identify any element of the first by an element of the second - and vice versa. Finally we get (4) \( \text{Hom}_R(\Omega C, A) \cong \bigoplus_{i=1}^{m} \text{Hom}_R(A_i, A) \).

Let us also denote the family of lower proportionality classes in \( \text{Y Ext}_R^{1}(C, A) \) by \( \text{Y Ext}_R^{1}(C, A) \).

We are now ready to define addition inside \( \text{Y Ext}_R^{1}(C, A) \); let us first introduce some relevant notation:

We shall represent the homomorphism \( s_i \circ p_i : \Sigma A \rightarrow A \) by a column of length \( m \), having 1 as the \( i \)'th and 0 at all other entries. It is clear that such a homomorphism \( s_i \circ p_i \) corresponds then to the "default" extension \( E_i \), that is realized as a quotient \( E_i \) of \( P \), "virtually containing" \( A_i \) and in which the epimorphism is just the canonical one.

This quotient \( E_i \), virtually corresponding to an edge \( \frac{C}{A_i} \), is the neuron (or factor span) of \( A_i \) ([5]), i.e. the slimmest factor module of \( P \), that has the virtual \( A_i \) as its socle. Notice that the uniqueness of such a neuron is quite apparent in this case.

We shall consider all homomorphisms \( \Sigma A \rightarrow A \) written up as

\[
\sum_{i=1}^{m} \alpha_i \circ s_i \circ p_i = \begin{bmatrix}
\alpha_1 & \ldots & \alpha_m
\end{bmatrix} \circ \begin{bmatrix}
s_1 \circ p_1 \\
\vdots \\
s_m \circ p_m
\end{bmatrix} = \begin{bmatrix}
\alpha_1 & \ldots & \alpha_m
\end{bmatrix} \circ \begin{bmatrix}
1 \\
\vdots \\
1
\end{bmatrix},
\]

which we shall then briefly denote by \( [\alpha_1 \ldots \alpha_m] \) or, even simpler, as \( (\alpha_1, \ldots, \alpha_m) \), where \( \alpha_i \in \text{End}(A) \).

For \( E \in \text{Y Ext}_R^{1}(C, A) \) denote so by \( \phi(E) \) its attached homomorphism \( \Sigma A \rightarrow A \), and let \( \overline{\alpha}(E) \) be the element of \( (\text{End}(A))^m \), corresponding to \( \phi(E) \) in the way just described, and thus attached to \( E \); for \( \kappa \) such a homomorphism \( \Sigma A \rightarrow A \), let \( \sup(\kappa) \subset \Sigma A \) denote its support, \( \overline{\alpha}_\kappa \in (\text{End}(A))^m \) its attached \( m \)-tuple of \( A \)-endomorphisms and, by a slight slackness of notation, \( ex(\kappa) = ex(\overline{\alpha}_\kappa) \) the corresponding extension.
Definition 70 For $E_1, E_2 \in Y\text{Ext}^1_R(C, A)$, define their sum $E_1 + E_2$ as the extension class $\text{ex}(\phi(E_1) + \phi(E_2))$.

Notice that our definition may be viewed as a generalized analogue to, but it may not be compared with the well known Baer sum, as these are two different things: That is, the one is a formal and abstract construction, the other is virtual.

Example 71 Let us look at some easy examples: Let $m = 2$, $\alpha_1 = \alpha_2 = \text{id}_A$ and set $E_1 = \text{ex}(\alpha_1, 0)$, $E_2 = \text{ex}(0, \alpha_2)$; then the middle term of the sum $\text{ex}(\phi(E_1)) + \text{ex}(\phi(E_2))$ corresponds to the support gotten by the (split) factoring out of the kernel of $(\alpha_1, \alpha_2) = (\text{id}_{A_1}, \text{id}_{A_2})$ on $\Sigma A = A_1 \oplus A_2$, reminding of taking the codiagonal $\bigtriangleup_A$ in the Baer definition. If we generalize to the similar example for an arbitrary $m$, we shall then have to (split-)outquotient on the $D_A$-space $\Sigma A$ (where $D_A = \text{Hom}_R(A, A)$) the ”hyperplane” defined by the equation $a_1 + \ldots + a_m = 0$, where the choice of coordinate system corresponds to choosing the summands $A_i$ of $\Sigma A$ together with the choice of the $m$ identomorphisms $s_i : A_i \to A$.

We point out that this analysis actually takes place in a subdirect product: Indeed, by recalling the ordering 66 on the family of submodules of subquotients of $P$, let $U$ be defined as the slimmest quotient of $P$, so that $\Sigma A$ be contained in it as a submodule. It is quite clear that $U$ is the virtual pull-back of $E_i$’s pull-back over (their canonical epimorphisms onto) $C$. In that way we somehow turn back to the beginning of this article.

Remark 72 We might as well have approached the issue dually, by considering the extension classes $Y\text{Ext}^1_R(C, A)$ embedded in the injective hull of $A$, identifiable then so by homomorphisms in $\text{Hom}_R(C, \Omega^{-1}A)$, as we have seen.

It is clear that, for any given extension $E \in Y\text{Ext}^1_R(C, A)$, $\alpha E$ is realized by the same quotient $E$ of $P$ as $E$ does. Now we shall prove the statement of lemma 65 above anew, by involving the machinery that we have developed:

Lemma 73 For any $E \in Y\text{Ext}^1_R(C, A)$ and any $\gamma \in \text{Aut}C$, $E\gamma$ is also realizable by the same factor module $E$ in $P$.

Proof. In view of the above proposition, as well as definition 70 and relation (4) it suffices to prove the claim for an extension having some of the fixed $A_i$’s as its support, therefore with one of the $E_i$’s as its realizing factor module; without loss of generality we may assume that its support is $A_1$, i.e. that our extension is $\alpha E_1$, for some $\alpha \in \text{Aut}A$. Now we may take the injective hull $I$ of $A$.
in such a way, that $E_1$ is virtually embedded in it as a submodule. If now $\alpha E_1 \gamma$ had another support than $A_1$ in $P$, then that should also be the case inside $I$, as multiplication $(\alpha E_1) \gamma$ is similarly defined in both cases. But that is a contradiction.

Thus we may in fact simply speak of proportionality classes in $Y\text{Ext}^1_R(C,A)$ and of their virtual realizations in the virtual category $\mathfrak{V}(P)$; however we have still two dual descriptions of those classes, in a way that becomes somewhat more clarified in the following theorem.

Set again $\text{Hom}_R(C,C) \cong \mathcal{D}_e$. Then we shall prove that the mentioned bijection between certain homomorphisms and the extension classes $Y\text{Ext}^1_R(C,A)$, is in reality a $\mathcal{D}_A$-$\mathcal{D}_C$-bimodule isomorphism, in a way that gives us a very concrete virtual insight into the module structure of $Y\text{Ext}^1_R(C,A)$:

**Theorem 74**

a. $Y\text{Ext}^1_R(C,A)$ has a $\mathcal{D}_A$-$\mathcal{D}_C$-bimodule structure.

b. There exists a (left) $\mathcal{D}_A$-module isomorphism between $Y\text{Ext}^1_R(C,A)$ and $\text{Hom}_R(\Omega C,A)$ and, similarly, a (right) $\mathcal{D}_C$-module isomorphism between $Y\text{Ext}^1_R(C,A)$, $\text{Hom}_R(C,\Omega^{-1}A)$, where $\Omega C = \text{rad} P$ and $\Omega^{-1}A = I/\text{soc} I$, with $P$ the projective cover of $C$, $I$ the injective hull of $A$.

c. $Y\text{Ext}^1_R(C,A)$ has the the structure of a $\mathcal{D}_A$-projective space and it is isomorphic to $P^{m-1}(\mathcal{D}_A)$. Dually, the family $Y\text{Ext}^1_R(C,A)$ of upper proportionality classes in $Y\text{Ext}^1_R(C,A)$ has the structure of a $\mathcal{D}_C$-projective space and it is isomorphic to $P^{m-1}(\mathcal{D}_C)$.

**Proof.** In view of definition 70 and relation (4), it suffices to prove the $\mathcal{D}_A$-module pseudo-distributivity properties for extensions having the $A_i$’s as their support.

The properties $id_A \mathcal{E} = \mathcal{E}$, $\sigma (E_1 + E_2) = \sigma E_1 + \sigma E_2$, $\sigma \mathcal{E} + \tau \mathcal{E} = (\sigma + \tau) \mathcal{E}$ are then a direct consequence of definition 70 where in the last we have also to notice that the extension classes on both sides of the equality indeed have the same support.

We use the dual arguments for the right $\mathcal{D}_C$-module structure.

As for the left & right pseudoassociativity and the blended one, $(\alpha \mathcal{E}) \gamma = \alpha (\mathcal{E} \gamma)$, we refer to [8. Ch. III, lemmata 1.2, 1.4, 1.6].

Regarding (c), it is clear that "multiplying" from left with an $\alpha \in \text{Aut} A$, correspondingly from the right with a $\gamma \in \text{Aut} C$, doesn’t change the proportionality class. The rest is easily checked.

We wish to close here by specializing the last theorem to a very usual case, in which $R$ shall be a finite dimensional $\mathfrak{A}$-algebra, where $\mathfrak{A}$ is an algebraically closed field. Before doing that, we go through a quick review of some basic relevant facts.
Let us begin by looking at the $R$-endomorphisms of $\Sigma A \cong A^m$, $A$ a simple $R$-module, beginning in a more general context: $\text{End}_R(A^m) \cong M_m(\mathfrak{D})$, the ring of $m \times m$ matrices with entries from $\mathfrak{D} = \text{End}_R(A)$.

It is an immediate consequence of the Density Theorem that, if $M$ is a semisimple $R$-module, which in case $R$ the canonical homomorphism $\text{End}_R(M) \to \text{End}_{\text{End}_R(M)}(M)$ is surjective. If $A$ is a simple $R$-module, then as a (finitely generated) module over the division ring $\mathfrak{D} = \text{End}_R(A)$ it must be free, say $\cong \mathfrak{D}^k$, therefore is $\text{End}_{\mathfrak{D}}(A) \cong M_k(\mathfrak{D}^\text{op})$ and we get the canonical surjective homomorphism $R \to \text{End}_{\mathfrak{D}}(A) \cong M_k(\mathfrak{D}^\text{op})$, which in case $A$ is also a faithful $R$-module (i.e. by substituting the appropriate $R$-block, in this case just meaning the appropriate simple summand of $R$, for $R$) becomes an isomorphism. Notice that here, although we haven’t assumed semisimplicity of the ring $R$, it is its faithful action on a simple module that implies its simplicity.

If we now take $R$ to be a $\mathfrak{A}$-algebra, $\mathfrak{A}$ a field, then is $\text{End}_R(M)$ a $\mathfrak{A}$-algebra too, therefore is $M$ a $\mathfrak{A}$-vector space. In case $\dim \mathfrak{A} M < \infty$, then is $\dim_\mathfrak{A} \text{End}_R(M) < (\dim_\mathfrak{A} M)^2 < \infty$ too, as $\text{End}_R(M) \subset \text{End}_R(M)$.

By assuming further $R$ to be a finite dimensional $\mathfrak{A}$-algebra (notably also implying that it is Artinian), every simple $R$-module $A$, being an $R$-epimorphic image of $R$, shall necessarily be finite $\mathfrak{A}$-dimensional. Since $\mathfrak{A} \cong \mathfrak{A} \cdot \text{id}_A \subset \text{End}_R(A) = \mathfrak{D}$, the assumed finite $\mathfrak{A}$-dimensionality of $A$ implies certainly that $A$ is finitely generated as a $\mathfrak{D}$-module; but then, as we have seen, the canonical homomorphism $R \to \text{End}_{\mathfrak{D}}(A)$ is surjective.

If we also assume that $\mathfrak{A}$ is algebraically closed then, by the well known Schur lemma, $\mathfrak{D} = \text{End}_R(A) = \mathfrak{A} \cdot \text{id}_A \cong \mathfrak{A}$.

We have not started from this assumption, while it is very essential from our point of view to look at the action of $\text{End}_R(A) (\cong \mathfrak{A}$ in this last case) on $Y \text{Ext}^1_R(C, A)$. However in this case the last theorem becomes:

**Corollary 75** Assume $R$ to be a finite dimensional $\mathfrak{A}$-algebra, with $\mathfrak{A}$ an algebraically closed field. With the same notation as above, we then have:

a. $Y \text{Ext}^1_R(C, A)$ has a $\mathfrak{A}$-bimodule structure.

b. There exists a (left) $\mathfrak{A}$-isomorphism between $Y \text{Ext}^1_R(C, A)$ and $\text{Hom}_R(\Omega C, A)$ and, similarly, a (right) $\mathfrak{A}$-module isomorphism between $Y \text{Ext}^1_R(C, A)$, $\text{Hom}_R(C, \Omega^{-1} A)$, where $\Omega C = \text{rad} P$ and $\Omega^{-1} A = I/\text{soc} I$, with $P$ the projective cover of $C$, $I$ the injective hull of $A$.

c. $Y \text{Ext}^1_R(C, A)$ has the structure of a $\mathfrak{A}$-projective space and it is isomorphic to $P^{m-1}(\mathfrak{A})$. Dually, the family $Y \text{Ext}^1_R(C, A)$ of upper proportionality classes in $Y \text{Ext}^1_R(C, A)$ has the structure of a $\mathfrak{A}$-projective space and it is isomorphic to $P^{m-1}(\mathfrak{A})$.

Notice that, as long as we have not "coordinated" the left and right actions of $\mathfrak{A}$ on $Y \text{Ext}^1_R(C, A)$, we have to be careful with the non-commutable $\mathfrak{A}$-bimodule structure of the latter.
References

[1] M. Barakat, Barbara Bremer, Higher Extension Modules and the Yoneda Product, arXiv:0802.3179 [math.KT], 2008

[2] David J. Benson, Representations and Cohomology I, Cambridge Studies in Advanced Mathematics, 1991 (No. 30)

[3] Bridson Martin R., Howie James, Miller III Charles F., Short H., On the finite presentation of subdirect products and the nature of residually free groups, American Journal of Mathematics, vol. 135 (2013) p.891-933.

[4] Kenneth S. Brown, Cohomology of Groups, Springer Verlag, Graduate Texts in Mathematics, Vol. 87, [DOI 10.1007/978-1-4684-9327-6] 1982.

[5] Stephanos Gekas, A new type of diagrams for modules, arXiv:1603.06506 [math.RT], [DOI:10.13140/RG.2.1.1168.2964] 2016

[6] Derek F. Holt, W. Plesken, Perfect Groups, Oxford Mathematical Monographs, 1989.

[7] B. Huppert, “Endliche Gruppen I”, Springer-Verlag, 1967.

[8] Saunders MacLane, Homology. Die Grundlehren der mathematischen Wissenschaften, Bd. 114, Springer - Berlin 1963 x+422 pp.

[9] M.D. Miller, On the lattice of normal subgroups of a direct product, Pacific J. Math. 60 no.2 (1975), 153–158.

[10] R. Remak, Über minimale invariante Untergruppen in der Theorie der endlichen Gruppen, J. reine angew. Math. 162 (1930), 1-16.

[11] R. Remak, Über die Darstellung der endlichen Gruppen als Untergruppen direkter Produkte, J. reine angew. Math. 163 (1930), 1-44.

[12] Derek J.S. Robinson, A Course in the Theory of Groups, Springer Verlag, Graduate Texts in Mathematics, Vol. 80, 1995.

[13] John S. Rose, A Course on Group Theory, Cambridge University Press, 1978

[14] W.R. Scott, Group Theory, Prentice-Hall, 1964

[15] M. Suzuki, Structure of a Group and the Structure of its Lattice of Subgroups, Springer-Verlag, 1967.

[16] J. Thévenaz, Maximal subgroups of direct products, Journal of Algebra, vol. 198, num. 2, p. 352-361, 1997.