EXTREMES OF GAUSSIAN RANDOM FIELDS WITH NON-ADDITIVE DEPENDENCE STRUCTURE

LONG BAI, KRZYSZTOF DĘBICKI, AND PENG LIU

Abstract: We derive exact asymptotics of

\[ \mathbb{P}\left\{ \sup_{t \in A} X(t) > u \right\}, \quad \text{as } u \to \infty, \]

for a centered Gaussian field \( X(t), \ t \in A \subset \mathbb{R}^n, \ n > 1 \) with continuous sample paths a.s., for which \( \arg \max_{t \in A} \text{Var}(X(t)) \) is a Jordan set with finite and positive Lebesgue measure of dimension \( k \leq n \) and its dependence structure is not necessarily locally stationary. Our findings are applied to deriving the asymptotics of tail probabilities related to performance tables and chi processes where the covariance structure is not locally stationary.

Key Words: supremum of Gaussian field, exact asymptotics, GUE, performance table, chi-processes.

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1. Introduction

Let \( X(t), \ t \in \mathbb{R}^n, \ n > 1 \) be a centered Gaussian field with continuous sample paths a.s.. Due to its importance in the extreme value theory of stochastic processes, statistics and applied probability, the distributional properties of

\[ \sup_{t \in A} X(t), \]

with a bounded set \( A \subset \mathbb{R}^n \), were vastly investigated. While the exact distribution of (1) is known only for very particular processes, the asymptotics of

\[ \mathbb{P}\left\{ \sup_{t \in A} X(t) > u \right\}, \]

as \( u \to \infty \) was intensively analyzed; see the seminal monographs [1–3]. As advocated therein, the set of points that maximize the variance \( \mathcal{M}^* := \arg \max_{t \in A} \text{Var}(X(t)) \) plays an important role in the form of the exact asymptotics of (2). The best understood cases cover the situation when (i) \( v_n(\mathcal{M}^*) \in (0, \infty) \), where \( v_n \) denotes the Lebesgue measure on \( \mathbb{R}^n \) and the field \( X(t) \) is homogeneous on \( \mathcal{M}^* \) or (ii) the set \( \mathcal{M}^* \) consists of separate points. In case (i) one can argue that

\[ \mathbb{P}\left\{ \sup_{t \in A} X(t) > u \right\} \sim \mathbb{P}\left\{ \sup_{t \in \mathcal{M}^*} X(t) > u \right\} \quad \text{as } u \to \infty. \]

For the intuitive description of case (ii), suppose that \( \mathcal{M}^* = \{t^*\} \) and \( \text{Var}(X(t^*)) = 1 \). Then, the play between local behaviour of standard deviation and correlation function in the neighbourhood of \( \mathcal{M}^* \) influences the asymptotics, which takes the form

\[ \mathbb{P}\left\{ \sup_{t \in A} X(t) > u \right\} \sim f(u)\mathbb{P}\left\{ X(t^*) > u \right\} \quad \text{as } u \to \infty. \]

An applicable assumption under which one can get the exact asymptotics as given in (3), is that in the neighbourhood of \( t^* \), both the standard deviation and correlation function of \( X(t) \) factorizes according to the following
additive form
\[ 1 - \sigma(t) \sim \sum_{j=1}^{3} g_j(t^*_j - t_j), \quad 1 - \text{Corr}(s, t) \sim \sum_{j=1}^{3} h_j(\bar{x}_j - \bar{t}_j) \]
as \( s, t \to t^* \), where the coordinates of \( \mathbb{R}^n \) are split onto disjoint sets \( \Lambda_1, \Lambda_2, \Lambda_3 \) with \( \{1, \ldots, n\} = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \), \( t_j = (t_i)_{i \in \Lambda_j}, j = 1, 2, 3 \) for \( t \in \mathbb{R}^n \) and \( g_j, h_j \) are some homogeneous functions (see (13)) such that
\[ \lim_{t_1 \to \bar{t}_1} h_1(t_1) = 0, \quad \lim_{t_2 \to \bar{t}_2} h_2(t_2) \in (0, \infty), \quad \lim_{t_3 \to \bar{t}_3} h_3(t_3) = \infty. \]
Importantly, under (4)-(5)
\[ f(u) = f_1(u)f_2(u)f_3(u) \]
factorizes too and:
\( \diamond \) in the direction of coordinates \( \Lambda_1 \) the standard deviation function is relatively flat with comparison to the correlation function. Then, with respect of coordinates \( \Lambda_1 \), some substantial neighbourhood of \( M \) contributes to the asymptotics and \( f_1(u) \to \infty \) as \( u \to \infty \).
\( \diamond \) in the direction of coordinates \( \Lambda_2 \) the standard deviation function is comparable to the correlation function. Then, with respect of coordinates \( \Lambda_2 \), some relatively small neighbourhood of \( M \) is important for the asymptotics and \( f_2(u) \to \mathcal{P} \in (1, \infty) \) as \( u \to \infty \).
\( \diamond \) in the direction of coordinates \( \Lambda_3 \) the standard deviation function decreases relatively fast with comparison to the correlation. Then, with respect of coordinates \( \Lambda_3 \), only the sole optimizer \( t^* \) is responsible for the asymptotics and \( f_3(u) \to 1 \) as \( u \to \infty \). We refer to Piterbarg [1][Chapter 8] for the details.

Much less is known on the mixed cases, when set \( M^* \) is a more general subset of \( A \) and/or the local dependence structure of the analyzed process doesn’t factorize according to the additive structure as in (4)-(5). The exceptions that are available in the literature were analyzed separately and cover some particular cases as in [4–9]. We would like to point at a notable recent contribution by Piterbarg [10], which deals with analysis of high excursion probability for centered Gaussian fields on finite dimensional manifold where \( M^* \) is a smooth submanifold. In this intuitively presented work, under the assumption that the correlation function of \( X \) is locally homogeneous, three scenarios for \( M^* \subseteq A \) are worked out: (1) stationary like case, (2) transition case and (3) Talagrand case; which under the notation (4)-(5) correspond to \( \Lambda_2 = \Lambda_3 = \emptyset \) for (1), \( \Lambda_1 = \emptyset \) for (2), \( \Lambda_1 = \Lambda_2 = \emptyset \) for (3).

In view of the considered in our paper examples and transparency of the presentation of the results, we work on Euclidean space in this contribution. We derive a unified result that allows to get exact asymptotics for the class of centered Gaussian fields for which we allow that \( M^* \) is a \( k_0 \leq n \) dimensional bounded Jordan set and the dependence structure of the entire field in the neighbourhood of \( M^* \) doesn’t necessarily decompose as in (4)-(5). In comparison to [10], we allow mixed scenarios where all sets \( \Lambda_1, \Lambda_2, \Lambda_3 \) can be nonempty at the same time. Besides, we suppose that \( X \) is locally stationary only with respect to coordinates of stationary like direction (see assumption A1); this relaxation is particularly important for the examples that are worked out in Section 3.1 and Section 3.2.

One of the motivations for this contribution is the analysis of asymptotic properties of
\[ \mathbb{P} \{ D_\alpha^n > u \} := \mathbb{P} \left\{ \sup_{t \in \mathcal{S}_n} Z^\alpha(t) > u \right\}, \quad \text{as } u \to \infty, \]
where \( t = (t_1, \ldots, t_n) \), \( \mathcal{S}_n = \{ t \in \mathbb{R}^n : 0 \leq t_1 \leq \cdots \leq t_n \leq 1 \} \),
\[ Z^\alpha(t) = \sum_{i=1}^{n+1} a_i (B_i^\alpha(t_i) - B_i^\alpha(t_{i-1})) \]
with \( t_0 = 0, t_{n+1} = 1 \) and \( B_i^\alpha, i = 1, \ldots, n+1 \) are independent fractional Brownian motions with Hurst index \( \alpha/2 \in (0, 1) \). This random variable plays an important role in many areas of probability theory. In particular,
for $\alpha_i \equiv 1$ it is strongly related with the notion of the performance table and it also appears as a limit in problems describing queues in series, totally asymmetric exclusion processes or oriented percolation [11–13]. If additionally $a_i \equiv 1$, then $D_n^A$ has the same law as the largest eigenvalue of an $n$-dimensional GUE (Gaussian Unitary Ensemble) matrix [14]. However, if $\alpha = 1$ but $a_i$ are not all the same, then the size of $M^*$ depends on the number of maximal $a_i$ and the correlation structure of the entire field is not locally homogeneous.

Application of Theorem 2.1 in Section 2 allows to derive exact asymptotics of (6) as $u \to \infty$ for $\alpha \in (0, 2)$; see Proposition 3.1.

Another illustration of the applicability of Theorem 2.1 deals with the extremes of the class of chi processes $\chi(t), t \geq 0$, defined for given $X(t) = (X_1(t), \ldots, X_n(t)), t \geq 0$ where $X_i(t)$ for $i = 1, \ldots, n$ are mutually independent, as

$$\chi(t) := \sqrt{\sum_{i=1}^{n} X_i^2(t)}, \; t \geq 0.$$  

Due to their importance in statistics, asymptotic properties of high excursions of chi processes attracted substantial interest. We refer to the classical work by Lindgren [15] and more recent contributions [10, 16–20] that deal with non-stationary or noncentered cases. In Section 3.2 we apply Theorem 2.1 to the analysis of the asymptotics of tail distribution of high exceedances of $\chi(t)$ for a model, where the covariance structure of $X_i$ is not locally stationary; see Proposition 3.3.

The structure of the rest of the paper is organized as follows. The proofs of Theorem 2.1, Proposition 3.1 and Proposition 3.3 are given in Sections 4–6 respectively while the proofs of some auxiliary results are postponed to the Appendix.

2. Main result

Let $X(t), \; t \in A$ be an $n$-dimensional centered Gaussian field with continuous trajectories, variance function $\sigma^2(t)$ and correlation function $r(s, t)$, where $A$ is a bounded set in $\mathbb{R}^n$. Suppose that the maximum of variance function $\sigma^2(t)$ over $A$ is attained on a Jordan subset of $A$. Without loss of generality, we assume that $\max_{t \in A} \sigma^2(t) = 1$ and we denote $M^*: = \{t \in A : \sigma^2(t) = 1\}$.

Throughout this paper, all the operations on vectors are meant componentwise. For instance, for any given $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we write $xy = (x_1y_1, \ldots, x_ny_n), \; 1/x = (1/x_1, \ldots, 1/x_n)$ for $x_i > 0, i = 1, \ldots, n$, and $x^y = (x_1^{y_1}, \ldots, x_n^{y_n})$ for $x_i, y_i \geq 0, i = 1, \ldots, n$. Moreover, we say that $x \geq y$ for $x_i \geq y_i, i = 1, \ldots, n$.

Suppose that the coordinates of $\mathbb{R}^n$ can be exclusively split onto four disjoint sets $\Lambda_i, i = 0, 1, 2, 3$ with $k_i = \# \cup_{j=0}^{i} \Lambda_j$, $i = 0, 1, 2, 3$ (implying that $1 \leq k_0 \leq k_1 \leq k_2 \leq k_3$ with $k_3 = n$) and

$$\vec{t} := (t_i)_{i \in \Lambda_0}, \; \vec{t}_j := (t_i)_{i \in \Lambda_j}, \; j = 1, 2, 3$$

in such a way that $M^* = \{t \in A : t_i = 0, i \in \cup_{j=1,2,3} \Lambda_j\}$. Note that $M^* = A$ if $\cup_{j=1,2,3} \Lambda_j = \emptyset$. Sets $\Lambda_1, \Lambda_2, \Lambda_3$ play similar role to the described in the Introduction (see A2 below), while $\Lambda_0$ relates to $M^*$ by $M := \{\vec{t} : t \in A, t_i = 0, i \in \cup_{j=1,2,3} \Lambda_j\} \subset \mathbb{R}^{k_0}$.

Suppose that $M$ is Jordan measurable with $v_{k_0}(M) \in (0, \infty)$, where $v_{k_0}$ denotes the Lebesgue measure on $\mathbb{R}^{k_0}$, and \{$(t_1, \ldots, t_n) : \vec{t} \in M, t_i \in [0, \epsilon], i \in \cup_{j=1,2,3} \Lambda_j\} \subseteq A \subseteq \{(t_1, \ldots, t_n) : \vec{t} \in M, t_i \in [0, \infty), i \in \cup_{j=1,2,3} \Lambda_j\}$ for some $\epsilon \in (0, 1)$ small enough. Further, we shall impose the following assumptions on the standard deviation and correlation functions of $X$: 




A1: There exists a centered Gaussian random field \( W(t), \ t \in [0, \infty)^n \) with continuous sample paths and a positive continuous vector-valued function \( a(\tilde{z}) = (a_1(\tilde{z}), \ldots, a_n(\tilde{z})) \), \( \tilde{z} = (z_i)_{i \in \Lambda_0} \in \mathcal{M} \) satisfying
\[
\inf_{i=1, \ldots, n} \inf_{\tilde{z} \in \mathcal{M}} a_i(\tilde{z}) > 0
\]
such that
\[
\lim_{\delta \to 0} \sup_{\tilde{z} \in \mathcal{M}^*} \sup_{s, t \in \mathcal{A}} \left| \frac{1 - r(s, t)}{\mathbb{E}\left\{ (W(\alpha(\tilde{z})s) - W(\alpha(\tilde{z})t))^2 \right\}} - 1 \right| = 0,
\]
where the increments of \( W \) are homogeneous if we fix both \( \tilde{t}_2 \) and \( \tilde{t}_3 \), and there exists a vector \( \alpha = (\alpha_1, \ldots, \alpha_n) \) with \( \alpha_i \in (0, 2], 1 \leq i \leq n \) such that for any \( u > 0 \)
\[
\mathbb{E}\left\{ \left( W(u^{-2/\alpha_i}s) - W(u^{-2/\alpha_i}t) \right)^2 \right\} = u^{-2} \mathbb{E}\left\{ (W(s) - W(t))^2 \right\}.
\]
Moreover, there exist \( d > 0, Q_1 > 0, i = 1, 2 \) such that for any \( s, t \in \mathcal{A} \) and \( |s - t| < d \)
\[
Q_1 \sum_{i \in \cup_{j=1}^{3} \Lambda_j} |s_i - t_i|^{\alpha_i} \leq 1 - r(s, t) \leq Q_2 \sum_{i=1}^{n} |s_i - t_i|^{\alpha_i}
\]
Further, suppose that for \( s, t \in \mathcal{A} \) and \( s \neq t \)
\[
r(s, t) < 1.
\]

A2: Assume that
\[
\lim_{\delta \to 0} \sup_{\tilde{z} \in \mathcal{M}^*} \sup_{t \in \mathcal{A}} \left| \frac{1 - \sigma(t)}{\sum_{j=1}^{3} p_j(\tilde{z})g_j(\tilde{t})} - 1 \right| = 0,
\]
where \( p_j(\tilde{t}), \tilde{t} \in [0, \infty)^{k_0}, j = 1, 2, 3 \) are positive continuous functions and \( g_j(\tilde{t}), \tilde{t} \in \mathbb{R}^{k_j - k_{j-1}}, j = 1, 2, 3 \), are continuous functions satisfying \( g_i(\tilde{t}_j) > 0, \tilde{t}_j \notin \overline{\Lambda}_j, j = 1, 2, 3 \). Moreover, we shall assume the following homogeneous property on \( g_j \)'s: for any \( u > 0 \) and some \( \beta_j = (\beta_i)_{i \in \Lambda_j}, j = 1, 2, 3 \) with \( \beta_i > 0, i \in \cup_{j=1,2,3} \Lambda_j \)
\[
u g_j(\tilde{t}_j) = g_j(u^{1/\beta_j} \tilde{t}_j), \quad j = 1, 2, 3.
\]
Moreover, with \( \alpha_j = (\alpha_i)_{i \in \Lambda_j}, j = 1, 2, 3 \)
\[
\alpha_1 < \beta_1, \alpha_2 = \beta_2 \text{ and } \alpha_3 > \beta_3.
\]

We next display the main result of this paper. To the end of this paper \( \Psi(\cdot) \) denotes tail distribution of the standard normal random variable.

**Theorem 2.1.** Suppose that \( X(t), t \in \mathcal{A} \) is a \( n \)-dimensional centered Gaussian random field satisfying A1-A2. Then, as \( u \to \infty \),
\[
P\left\{ \sup_{t \in \mathcal{A}} X(t) > u \right\} \sim C u^{\sum_{i \in \Lambda_0 \cup \Lambda_1} \frac{n}{2} - \sum_{i \in \Lambda_2} \frac{n}{2} \Psi(u),
\]
where
\[
C = \int_{\mathcal{M}} \left( \mathcal{H}_{W}^{p_z(\tilde{z})g_2(a_2^{-1}(\tilde{z})\tilde{t}_2)} \left( \prod_{i \in \Lambda_0 \cup \Lambda_1} a_i(\tilde{z}) \int_{0 \leq t_1 < 1-k_0} e^{-p_1(\tilde{z})g_1(\tilde{t}_1)} d\tilde{t}_1 \right) \right) \tilde{z} \in (0, \infty),
\]
with \( a_2(\tilde{z}) = (a_i(\tilde{z}))_{i \in \Lambda_2} \) and
\[
\mathcal{H}_{W}^{p_z(\tilde{z})g_2(a_2^{-1}(\tilde{z})\tilde{t}_2)} = \lim_{\lambda \to \infty} \frac{1}{\lambda^{k_1}} \mathbb{E}\left\{ \sup_{t_1 \in [0, \lambda], \tilde{t}_1 \in [0, \Lambda_{0 \cup \Lambda_1 \cup \Lambda_2} \cup \Lambda_3]} e^{\sqrt{2} W(t) - \sigma_0^2(t) - p_z(\tilde{z})g_2(a_2^{-1}(\tilde{z})\tilde{t}_2)} \right\}.
\]

**Remark 2.2.** The result in Theorem 2.1 is also valid if some of \( \Lambda_i, i = 0, 1, 2, 3 \) are empty sets.
Next, let us consider a special case of Theorem 2.1. Suppose that

\begin{equation}
 a_i(z) \equiv a_i, z \in \mathcal{M}, \quad i = 1, \ldots, n, \quad p_j(z) \equiv 1, z \in \mathcal{M}, \quad j = 1, 2, 3,
\end{equation}

\begin{equation}
 E \left\{ (W(s) - W(t))^2 \right\} = \sum_{i=1}^{n} |s_i - t_i|^\alpha, \quad g_j(\tilde{t}_j) = \sum_{i \in \Lambda_j} b_i t_i^{\beta_j}, \quad j = 1, 2, 3.
\end{equation}

Let \( \Gamma(x) = \int_0^\infty s^{x-1} e^{-s} ds \) for \( x > 0 \) and for \( \alpha \in (0, 2) \) and \( b > 0 \), define Pickands and Piterbarg constants respectively, for \( \lambda > 0 \),

\begin{equation}
 \mathcal{H}_{B^\alpha}[0, \lambda] = E \left\{ \sup_{t \in [0, \lambda]} e^{\sqrt{2B^\alpha}(t) - t^\alpha} \right\}, \quad \mathcal{H}_{B^\alpha} = \lim_{\lambda \to \infty} \frac{\mathcal{H}_{B^\alpha}[0, \lambda]}{\lambda},
\end{equation}

\begin{equation}
 \mathcal{P}_{B^\alpha}[0, \lambda] = E \left\{ \sup_{t \in [0, \lambda]} e^{\sqrt{2B^\alpha}(t) - (1+b)t^\alpha} \right\}, \quad \mathcal{P}_{B^\alpha} = \lim_{\lambda \to \infty} \frac{\mathcal{P}_{B^\alpha}[0, \lambda]}{\lambda},
\end{equation}

where \( B^\alpha \) represents a standard fractional Brownian motion with zero mean and covariance

\[ \text{Cov}(B_\alpha(s), B_\alpha(t)) = \frac{|t|^\alpha + |s|^\alpha - |t - s|^\alpha}{2}, \quad s, t \geq 0. \]

We refer to \[1\] and the references therein for properties of Pickands and Piterbarg constants. The following proposition partially generalizes Theorems 7.1 and 8.1 of \[1\].

**Proposition 2.3.** Under the assumption of Theorem 2.1, if (15)-(16) hold, then

\[ P \left\{ \sup_{t \in A} X(t) > u \right\} \sim C u^{\sum_{i \in \Lambda_0 \cup \Lambda_1} \frac{2}{\eta_i} - \sum_{i \in \Lambda_2} \frac{1}{\eta_i}} \Psi(u), \]

where

\[ C = v_{k_0}(\mathcal{M}) \left( \prod_{i \in \Lambda_0 \cup \Lambda_1} a_i \mathcal{H}_{B^\alpha_i} \right) \left( \prod_{i \in \Lambda_1} t_i^{-1/\beta_i} \Gamma(1/\beta_i + 1) \right) \prod_{i \in \Lambda_2} \mathcal{P}_{B^\alpha_i}^{-\beta_i} \mathcal{P}_{B^\alpha_i}. \]

3. Applications

In this section, we illustrate our main results by application of Theorem 2.1 to two classes of Gaussian fields with nonstandard structures of the correlation function.

3.1. The performance table, largest eigenvalue of GUE matrix and related problems. Let

\begin{equation}
 Z^\alpha(t) := \sum_{i=1}^{n+1} a_i \left( B_i^\alpha(t_i) - B_i^\alpha(t_{i-1}) \right), \quad t = (t_1, \ldots, t_n),
\end{equation}

where \( t_0 = 0, t_{n+1} = 1 \) and \( B_i^\alpha, \quad i = 1, \ldots, n+1 \) are mutually independent fractional Brownian motions with Hurst index \( \alpha/2 \in (0, 1) \) and \( a_i > 0, \quad i = 1, \ldots, n+1 \). We are interested in the asymptotics of

\begin{equation}
 P \left\{ D_n^\alpha > u \right\} = P \left\{ \sup_{t \in S_n} Z^\alpha(t) > u \right\}
\end{equation}

for large \( u \), where \( S_n = \{ t \in \mathbb{R}^n : 0 \leq t_1 \leq \cdots \leq t_n \leq 1 \} \). Without loss of generality, we assume that \( \max_{i=1,\ldots,n+1} a_i = 1 \).

Random variable \( D_n^\alpha \) arises in many problems that are important both in theoretical and applied probability. In particular it is strongly related with the notion of performance table. More precisely, following \[11\], let \( \mathbf{w} = (w_{ij}), i, j \geq 1 \) be a family of independent random values indexed by the integer points of the first quarter of the plane. A monotonous path \( \pi \) from \((i, j)\) to \((i', j')\), \( i \leq i', j \leq j' \), is a sequence \((i, j), (i_1, j_1), \ldots, (i_k, j_k) = (i', j') \) of length \( k = i' + j' - i - j + 1 \), such that all lattice steps \((i_k, j_k) \rightarrow (i_{k+1}, j_{k+1})\) are of size one and (consequently) go to the North or to the East. The weight \( \mathbf{w}(\pi) \) of such a path is just the
sum of all entries of the array $w$ along the path. We define performance table $l(i,j), i,j \in \mathbb{N}$ as the array of largest pathweights from $(1,1)$ to $(i,j)$, that is

$$l(i,j) = \max_{\pi \text{ from } (1,1) \text{ to } (i,j)} w(\pi).$$

If $\text{Var}(w_{ij}) \equiv v > 0$ and $\mathbb{E}\{w_{ij}\} \equiv e$ for all $i,j$, then

$$D_{n,k} := \frac{l(n+1,k) - ke}{\sqrt{k^v}}$$

converges in law as $k \to \infty$ to $D_n^1$ with $a_1 \equiv 1$; see [11]. We refer to [11, 13, 21] and references therein for applications of performance tables in queueing theory and in interacting particle systems. Notably, as observed in [11], if $a_1 \equiv 1$ then $D_n^1$ has the same law as the largest eigenvalue of an $n$-dimensional Gaussian Unitary Ensemble random matrix, see [12] for details and further relations with non-colliding Brownian motions.

Denote

$$\mathcal{N} = \{i : a_i = 1, i = 1, \ldots, n + 1\}, \quad \mathcal{N}^c = \{i : a_i < 1, i = 1, \ldots, n + 1\}, \quad m = \sharp \mathcal{N}.$$

For $k^* = \max\{i \in \mathcal{N}\}$ and $x = (x_1, \ldots, x_{k^*-1}, x_{k^*+1}, \ldots, x_{n+1})$, we define

$$W(x) = 2 \sum_{i \in \mathcal{N}} B_i(s_i(x)) - B_i(s_{i-1}(x)) + \frac{\sqrt{2}}{2} \sum_{i \in \mathcal{N}^c} a_i(B_i(s_i(x)) - B_i(s_{i-1}(x))),$$

where $B_i, \tilde{B}_i$ are independent standard Brownian motions and

$$s_i(x) = \begin{cases} x_i, & \text{if } i \in \mathcal{N} \text{ and } i < k^*, \\ \sum_{j=\max\{k \in \mathcal{N} : k < i\}}^{i} x_j, & \text{if } i \in \mathcal{N}^c \text{ and } i < k^*, \\ \sum_{j=i+1}^{n+1} x_j, & \text{if } i \geq k^*, \end{cases}$$

with the convention that $\max \emptyset = 1$.

For $m$ given in (20) define

$$\mathcal{H}_W = \lim_{\lambda \to \infty} \frac{1}{\lambda^{m-1}} \mathbb{E} \left\{ \sup_{x \in [0,\lambda]^n} \sqrt{2} W(x) - \left( \sum_{i \leq k^*} x_i \right) \right\}.$$

It appears that for $\alpha = 1$ and $m < n + 1$ the field $Z^1$ satisfies A1 with $W$ as given in (21). Notably, it has stationary increments with respect to coordinates $\mathcal{N}$ while the increments of $W$ are not stationary with respect to coordinates $\mathcal{N}^c$; see (63) in the proof of the following proposition. Moreover, we have then $\Lambda_0 = \mathcal{N}$, $\Lambda_1 = \emptyset$, $\Lambda_2 = \mathcal{N}^c$, $\Lambda_3 = \emptyset$.

**Proposition 3.1.** For $Z^n$ defined in (18), we have, as $u \to \infty$,

$$\mathbb{P} \left\{ \sup_{t \in \mathbb{S}_n} Z^n(\mathcal{A}(t)) > u \right\} \sim \begin{cases} Cu^{(2 - \alpha) n} \mathcal{U} \left( \frac{u}{\sigma_u} \right), & \alpha \in (0,1), \\ \frac{1}{(m-1)!} \mathcal{H}_W u^{2(m-1)} \mathcal{U}(u), & \alpha = 1, \\ m \mathcal{U}(u), & \alpha \in (1,2), \end{cases}$$

where $\sigma_u = \left( \sum_{i=1}^{n+1} a_i^2 \right)^{1-\alpha}$ and

$$C = (\mathcal{H}_B)^n \left( \prod_{i=1}^{n} \left( a_i^2 + a_i^2 \right)^{1/2} \right)^{2(1-\frac{1}{2})^n} \left( \frac{\pi}{\alpha(1-\alpha)} \right)^{1/2} \sigma_u^{-\alpha(1-\alpha)/2} \left( \sum_{i=1}^{n+1} a_i^2 \right)^{-\alpha/2}. $$

**Remark 3.2.** i) If $1 \leq m \leq n$, then $1 \leq \mathcal{H}_W \leq m^{-1} \prod_{i \in \mathcal{N}^c} \left( 1 + \frac{2n}{1-a_i^2} \right)$.

ii) If $m = n + 1$, then $\mathcal{H}_W = 1$.

The proof of Remark 3.2 is postponed to Appendix.
3.2. Chi processes. Consider chi process $\chi$ generated by a process $X$, that is let

$$\chi(t) := \sqrt{\sum_{i=1}^{n} X_i^2(t)}, \ t \in [0, 1],$$

where $X_i, i = 1, ..., n, \text{ are iid copies of } X$. We suppose that $\{X(t), t \in [0, 1]\}$ is a centered Gaussian process with a.s. continuous trajectories, standard deviation function

$$\sigma_X(t) = \frac{1}{1 + bt^{\alpha}}, \ t \in [0, 1], \text{ for } b > 0$$

and correlation function

$$1 - r(s, t) \sim aV ar(Y(t) - Y(s)), \ s, t \to 0, \text{ for } a > 0,$$

where $\{Y(t), t \geq 0\}$ is a centered Gaussian process with a.s. continuous trajectories and satisfies:

B1: $\{Y(t), t \geq 0\}$ is self-similar with index $\alpha/2 \in (0, 1)$ (i.e. for all $r > 0$, $\{Y(rt), t \geq 0\} \overset{d}{=} \{r^{\alpha/2}Y(t), t \geq 0\}$, where $\overset{d}{=} \text{ means the equality of finite dimensional distributions}$) and $\sigma_Y(1) = 1$;

B2: there exist $c_Y > 0$ and $\gamma \in [\alpha, 2]$ such that

$$V ar(Y(1) - Y(t)) \sim c_Y|1 - t|^\gamma, \ t \uparrow 1.$$ 

Examples of Gaussian processes satisfying B1 and B2 cover such classes of Gaussian processes, as fractional Brownian motions, bi-fractional Brownian motions (see e.g. [22, 23]), sub-fractional Brownian motions (see e.g. [24, 25]), dual-fractional Brownian motions (see [26]) and time-average of fractional Brownian motions (see [26, 27]).

For a Gaussian process $Y$ satisfying B1-B2, a generalized Piterbarg constant is defined as, for $b > 0$,

$$\mathcal{P}_Y^b = \lim_{S \to \infty} \mathbb{E} \left\{ \sup_{t \in [0, S]} e^{2Y(t) - (1+b)t^\alpha} \right\} \in (0, \infty).$$

We refer to [27] for the finiteness and other properties of this constant.

The literature on the asymptotics of

$$\mathbb{P} \left\{ \sup_{t \in [0, 1]} \chi(t) > u \right\},$$

as $u \to \infty$ is focused on the case where $Y$ in (25) is a fractional Brownian motion, i.e., $1 - r(s, t) \sim a|t - s|^\alpha$ as $s, t \to$ for some $\alpha \in (0, 2]$, which means that the correlation function of $X$ is locally homogeneous at 0; see [16, 17, 19, 10]. In the following proposition, $Y$ can be a general self-similar Gaussian process satisfying B1-B2, which allows for locally nonhomogeneous structures of the correlation function of $X$, that were not investigated in the literature.

The idea of getting the asymptotics of (26) is based on a transformation into supremum of Gaussian random field over a sphere, see [28, 16, 10]. That is, we observe that

$$\sup_{t \in [0, 1]} \chi(t) = \sup_{t \in [0, 1], \sum_{i=1}^{n} v_i^2 = 1} X_i(t)v_i.$$ 

Next, we transform the Euclidean coordinates into spherical coordinates,

$$v_1(\theta) = \cos(\theta_1), v_2(\theta) = \sin(\theta_1) \cos(\theta_2), \ldots, v_{n-1}(\theta) = \left( \prod_{i=1}^{n-2} \sin(\theta_i) \right) \cos(\theta_{n-1}), v_n(\theta) = \prod_{i=1}^{n-1} \sin(\theta_i),$$

where $\theta = (\theta_1, \ldots, \theta_{n-1})$ and $\theta \in [0, \pi]^{n-2} \times [0, 2\pi)$. We denote

$$Z(\theta, t) = \sum_{i=1}^{n} X_i(t)v_i(\theta), \ \theta \in [0, \pi]^{n-2} \times [0, 2\pi), t \in [0, 1].$$
and we have
\[ \sup_{t \in [0,1]} \chi(t) = \sup_{(\theta, t) \in E} Z(\theta, t) \text{ with } E = [0, \pi]^{n-2} \times [0, 2\pi) \times [0, 1]. \]

Consequently,
\[ \mathbb{P} \left( \sup_{t \in [0,1]} \chi(t) > u \right) = \mathbb{P} \left( \sup_{(\theta, t) \in E} Z(\theta, t) > u \right). \]

Then, it appears that the Gaussian field \( Z \) satisfies Theorem 2.1 with \( W \) in (8) and (9) given by
\[ W(\theta, t) = \sum_{i=1}^{n-1} B_i^2(\theta_i) + \sqrt{a} Y(t), \ \theta \in \mathbb{R}^{n-1} \times \mathbb{R}^+, \]
where \( B_i^2 \) are independent fractional Brownian motions with index 2 and \( Y \) is the self-similar Gaussian process given in (25) that is independent of \( B_i^2 \). Notably, one can check that, if \( Y \) is not a fractional Brownian motion, the above defined \( W \) does not have stationary increments with respect to coordinate \( t \). Moreover, \( \Lambda_0 = \{1, \ldots, n-1\}, \Lambda_1 = \emptyset, \Lambda_2 = \{n\}, \Lambda_3 = \emptyset. \)

**Proposition 3.3.** For \( \chi \) defined in (26) with \( X \) satisfying (24) and (25), we have
\[ \mathbb{P} \left\{ \sup_{t \in [0,1]} \chi(t) > u \right\} \sim \frac{2^{3-2\gamma} \sqrt{n}}{\Gamma(n/2)} P^{\gamma-1} u^{n-1} \Psi(u), \ u \to \infty. \]

### 4. Proof of Theorem 2.1

We denote by \( Q, Q_i, i = 1, 2, 3, \ldots \) positive constants that may differ from line to line.

#### 4.1. An adapted version of Theorem 2.1 in [29].

In this subsection, we display a version of Theorem 2.1 in [29], which plays an important role in the proof of Theorem 2.1. Let \( X_{u,l}(t), t \in E \subset \mathbb{R}^n, l \in K_u \subset \mathbb{R}^m, m \geq 1 \) be a family of Gaussian random fields with variance 1, where \( E \subset \mathbb{R}^n \) is a compact set containing \( 0 \) and \( K_u \neq \emptyset \). Moreover, assume that \( g_{u,l}, l \in K_u \) is a series of functions over \( E \) and \( u_l, l \in K_u \) are positive functions of \( u \) satisfying \( \lim_{u \to \infty} \inf_{l \in K_u} u_l = \infty \). In order to get the uniform asymptotics of
\[ \mathbb{P} \left\{ \sup_{t \in E} \frac{X_{u,l}(t)}{1 + g_{u,l}(t)} > u_l \right\} \]
with respect to \( l \in K_u \), we shall impose the following assumptions:

**C1:** There exists a function \( g \) such that
\[ \lim_{u \to \infty} \sup_{l \in K_u} \sup_{t \in E} |u_l^2 g_{u,l}(t) - g(t)| = 0. \]

**C2:** There exists a centered Gaussian random field \( V(t), t \in E \) with \( V(0) = 0 \) such that
\[ \lim_{u \to \infty} \sup_{l \in K_u} \sup_{s, t \in E} |u_l^2 \text{Var}(X_{u,l}(t) - X_{u,l}(s)) - 2 \text{Var}(V(t) - V(s))| = 0. \]

**C3:** There exist \( \gamma \in (0, 2] \) and \( C > 0 \) such that for \( u \) sufficiently large
\[ \sup_{l \in K_u} \sup_{s \neq t, s, t \in E} u_l^2 \frac{\text{Var}(X_{u,l}(t) - X_{u,l}(s))}{\sum_{i=1}^{m} |s_i - t_i|^\gamma} \leq C. \]

**Lemma 4.1.** Let \( X_{u,l}(t), t \in E \subset \mathbb{R}^n, l \in K_u \) be a family of Gaussian random fields with variance 1, \( g_{u,l}, l \in K_u \) be functions defined on \( E \) and \( u_l, l \in K_u \) be positive constants. If **C1-C3** are satisfied, then
\[ \lim_{u \to \infty} \sup_{l \in K_u} \left| \mathbb{P} \left\{ \sup_{t \in E} \frac{X_{u,l}(t)}{1 + g_{u,l}(t)} > u_l \right\} - P^\gamma_{\Psi}(E) \right| = 0, \]
where
\[ P^\gamma_{\Psi}(E) = \mathbb{E} \left\{ \sup_{t \in E} e^{\sqrt{\Psi(t)} - \sigma^2(t) - g(t)} \right\}. \]
4.2. **Proof of Theorem 2.1.** In order to simplify the proof, without loss of generality, we suppose that \( \Lambda_0 = \{1, \ldots, k_0\} \) and \( \Lambda_i = \{k_{i-1} + 1, \ldots, k_i\}, \ i = 1, 2, 3, \) which in fact can be obtained by change of order of the coordinates. Thus we have \( \mathcal{M}^* = \{ \mathbf{t} \in \mathcal{A} : t_i = 0, i = k_0 + 1, \ldots, n\} \) and \( \mathcal{M} = \{ \mathbf{i} : \mathbf{t} \in \mathcal{A}, t_i = 0, i = k_0 + 1, \ldots, n\}. \) The proof is divided into three steps: In the first step, we show that the supremum of \( X(t) \) over \( \mathcal{A} \) is predominately achieved on a subset; In the second step, we split this subset into small hyperrectangles and derive the asymptotics on each hyperrectangle resorting to the so-called double-sum method in (1); Finally, step 3 is devoted to adding up the asymptotics in step 2 to obtain the asymptotics over the whole set.

4.2.1. **Step 1.** In the first step of the proof, we divide \( \mathcal{A} \) into two sets:

\[ E_2(u) = \{ \mathbf{t} \in \mathcal{A} : t_i \in [0, \delta_i(u)], k_0 + 1 \leq i \leq n\}, \quad \delta_i(u) = \left( \frac{\ln u}{u} \right)^{2/\beta_i}, \quad k_0 + 1 \leq i \leq n, \]

a neighborhood of \( \mathcal{M}^* \), which maximizes the variance of \( X(t) \) (with high probability the supremum is realized in \( E_2(u) \)) and the set \( \mathcal{A} \setminus E_2(u) \), over which the probability associated with supremum is asymptotically negligible.

For the lower bound, we only consider the process over

\[ E_1(u) = \{ \mathbf{t} \in \mathcal{A} : t_i \in [0, \delta_i(u)], k_0 + 1 \leq i \leq k_1; t_i \in [0, u^{-2/\alpha_i} \lambda], k_1 + 1 \leq i \leq k_2; t_i = 0, k_2 + 1 \leq i \leq k_3\}, \quad \lambda > 0, \]

a neighborhood of \( \mathcal{M}^* \).

To simplify notation, we denote for \( \Delta_1, \Delta_2 \subseteq \mathbb{R}^n \)

\[ P_u(\Delta_1) := P \left\{ \sup_{t \in \Delta_1} X(t) > u \right\}, \quad P_u(\Delta_1, \Delta_2) := P \left\{ \sup_{t \in \Delta_1} X(t) > u, \sup_{t \in \Delta_2} X(t) > u \right\}. \]

Then we have that for any \( u > 0 \)

\[ P_u(E_1(u)) \leq P_u(A) \leq P_u(E_2(u)) + P_u(A \setminus E_2(u)). \]

Note that in light of (1) [Theorem 8.1], by (10) and (13), for \( u \) sufficiently large,

\[ P_u(A \setminus E_2(u)) \leq C u^n(A) u^{\sum_{i=1}^n i / \alpha_i} r \psi \left( \frac{u}{1 - \Psi(\ln u)^2} \right). \]

4.2.2. **Step 2.** In the second step, we divide \( \mathcal{M} \) onto small hypercubes such that

\[ \bigcup_{\mathbf{r} \in \mathcal{V}^+} \mathcal{M}_\mathbf{r} \subset \mathcal{M} \subset \bigcup_{\mathbf{r} \in \mathcal{V}^+} \mathcal{M}_\mathbf{r}, \]

where

\[ \mathcal{M}_\mathbf{r} = \prod_{i=1}^{k_0} [r_i v, (r_i + 1) v), \quad \mathbf{r} = (r_1, \ldots, r_{k_0}), r_i \in \mathbb{Z}, 1 \leq i \leq k_0, v > 0, \]

and

\[ \mathcal{V}^+ := \{ \mathbf{r} : \mathcal{M}_\mathbf{r} \cap \mathcal{M} \neq \emptyset \}, \quad \mathcal{V}^- := \{ \mathbf{r} : \mathcal{M}_\mathbf{r} \subset \mathcal{M} \}. \]

For fixed \( \mathbf{r} \), we analyze the supremum of \( X \) over a set related to \( \mathcal{M}_\mathbf{r} \). For this, let

\[ E_{1, \mathbf{r}}(u) = \{ \mathbf{t} : \mathbf{i} \in \mathcal{M}_\mathbf{r}; t_i \in [0, \delta_i(u)], k_0 + 1 \leq i \leq k_1; t_i \in [0, u^{-2/\alpha_i} \lambda], k_1 + 1 \leq i \leq k_2; t_i = 0, k_2 + 1 \leq i \leq k_3\}, \]

\[ E_{2, \mathbf{r}}(u) = \{ \mathbf{t} : \mathbf{i} \in \mathcal{M}_\mathbf{r}; t_i \in [0, \delta_i(u)], k_0 + 1 \leq i \leq n\}. \]

Moreover, define an auxiliary set

\[ E_{3, \mathbf{r}}(u) = \{ (\mathbf{i}, \mathbf{i}_1, \mathbf{i}_2) : \mathbf{i} \in \mathcal{M}_\mathbf{r}, t_i \in [0, \delta_i(u)], k_0 + 1 \leq i \leq k_2\}. \]

We next focus on \( P_u(E_{1, \mathbf{r}}(u)) \) and \( P_u(E_{2, \mathbf{r}}(u)) \). The idea of the proof of this step is first to split \( E_{1, \mathbf{r}}(u) \) and \( E_{2, \mathbf{r}}(u) \) onto tiny hyperrectangles and uniformly derive the tail probability asymptotics on each hyperrectangle; and then to show that the asymptotics over \( E_{i, \mathbf{r}}(u), i = 1, 2 \) are the summation of the asymptotics over the corresponding hyperrectangles, respectively.
To this end, we introduce the following notation. For some $\lambda > 0$, let

$$I_{u,i}(l) = \left[ \frac{\lambda}{u^{2/\alpha_i}} (l + 1) \frac{\lambda}{u^{2/\alpha_i}}, \right], \quad l \in \mathbb{N}, \quad l = (l_1, \ldots, l_n), l_j = (l_{j-1} + 1, \ldots, l_j), \quad j = 1, 2,$$

$$D_u(l) = \left( \prod_{i=1}^{k_2} I_{u,i}(l_i) \right) \times \prod_{i=k_2+1}^{n} [0, cu^{-2/\alpha_i}], \quad C_u(l) = \left( \prod_{i=1}^{k_2} I_{u,i}(l_i) \right) \times \prod_{i=k_2+1}^{k_2} [0, \lambda u^{-2/\alpha_i}] \times \overline{u}_3,$$

with $\overline{u}_3 = (0, \ldots, 0) \in \mathbb{R}^{n-k_2}$, and

$$M_i(u) = \left[ \frac{\nu u^{2/\alpha_i}}{\lambda} \right], \quad 1 \leq i \leq k_0, \quad M_i(u) = \left[ \frac{\delta_i(u)u^{2/\alpha_i}}{\lambda} \right], \quad k_0 + 1 \leq i \leq k_2.$$

In order to derive an upper bound for $P_u(E_{2,r}(u))$ and a lower bound for $P_u(E_{1,r}(u))$, we introduce the following notation for some $\epsilon \in (0, 1),$

$$L_1(u) = \left\{ l : \prod_{i=1}^{k_2} I_{u,i}(l_i) \subset E_{3,r}(u), l_i = 0, k_1 + 1 \leq i \leq n \right\},$$

$$L_2(u) = \left\{ l : \left( \prod_{i=1}^{k_2} I_{u,i}(l_i) \right) \cap E_{3,r}(u) \neq \emptyset, l_i = 0, k_1 + 1 \leq i \leq n \right\},$$

$$L_3(u) = \left\{ l : \left( \prod_{i=1}^{k_2} I_{u,i}(l_i) \right) \cap E_{3,r}(u) \neq \emptyset, \sum_{i=k_2+1}^{k_2} l_i^2 > 0, l_i = 0, k_2 + 1 \leq i \leq n \right\},$$

$$K_1(u) = \{ (l, j) : l, j \in L_1(u), C_u(l) \cap C_u(j) \neq \emptyset \},$$

$$K_2(u) = \{ (l, j) : l, j \in L_1(u), C_u(l) \cap C_u(j) = \emptyset \},$$

$$u_{t_i}^- = u \left( 1 + (1 - \epsilon) \inf_{t_i \in [t_i, t_i+1]} p_{1,r}^{-}(u^{2/\alpha_1} \lambda t_1) \right),$$

$$u_{t_i}^+ = u \left( 1 + (1 + \epsilon) \sup_{t_i \in [t_i, t_i+1]} p_{1,r}^{+}(u^{2/\alpha_1} \lambda t_1) \right),$$

$$p_{1,r}^{-} = \sup_{z \in M_r} p_j(\tilde{z}), \quad p_{1,r}^{+} = \inf_{z \in M_r} p_j(\tilde{z}), \quad j = 1, 2, 3.$$

Bonferroni inequality gives that for $u$ sufficiently large

$$P_u(E_{1,r}(u)) \geq \sum_{l \in L_1(u)} P_u(C_u(l)) - \sum_{i=1}^{2} \Gamma_i(u),$$

$$P_u(E_{2,r}(u)) \leq \sum_{l \in L_2(u)} P_u(D_u(l)) + \sum_{l \in L_3(u)} P_u(D_u(l)),$$

where

$$\Gamma_i(u) = \sum_{(l, j) \in K_i(u)} P_u(C_u(l), C_u(j)), \quad i = 1, 2.$$

We first derive the upper bound of $P_u(E_{2,r}(u))$ as $u \to \infty$. To this end, we need to find the upper bounds of $\sum_{l \in L_2(u)} P_u(D_u(l)), j = 2, 3$, separately.

**Upper bound for $\sum_{l \in L_2(u)} P_u(D_u(l))$.** By (12), we have that for $u$ sufficiently large

$$\sum_{l \in L_2(u)} P_u(D_u(l)) \leq \sum_{l \in L_2(u)} \mathbb{P} \left\{ \sup_{l \in D_u(l)} \frac{X(t)}{1 + (1 - \epsilon) p_{2,r} g_2(t_2)} > u_{t_i}^- \right\}$$

$$= \sum_{l \in L_2(u)} \mathbb{P} \left\{ \sup_{l \in E(t,u)} \frac{X_{u,t}(l)}{1 + (1 - \epsilon) p_{2,r} g_2(u^{-2/\alpha_2} (a_2(l,u))^{-1} t_2)} > u_{t_i}^- \right\},$$
where
\begin{equation}
X_{u,t}(t) = X\left(u^{-2/\alpha_1}(l_1 \lambda + (a_1(\tilde{z}(l,u)))^{-1}l_1), \ldots, u^{-2/\alpha_n}(l_n \lambda + (a_n(\tilde{z}(l,u)))^{-1}l_n)\right),
\end{equation}
with \( \tilde{z}(l,u) = (u^{-2/\alpha_1}l_1, \ldots, u^{-2/\alpha_n}l_n) \) and \( E(l,u) = \left(\prod_{i=1}^{k_2} [0, a_i(\tilde{z}(l,u))\lambda]\right) \times \prod_{i=k_2+1}^{n} [0, a_i(\tilde{z}(l,u))\epsilon]\).

Note that by (13),
\begin{equation*}
u^{-2}g^-_{2,r}(t_2) \leq g_2(u^{-2/\alpha_2}(a_2(\tilde{z}(l,u)))^{-1}t_2) = u^{-2}g_2((a_2(\tilde{z}(l,u)))^{-1}t_2) \leq u^{-2}g^+_2(t_2),
\end{equation*}
where
\begin{equation*}
g^-_{2,r}(t_2) = \inf_{\tilde{z} \in \mathcal{M}_r} g_2((a_2(\tilde{z}))^{-1}t_2), \quad g^+_2(t_2) = \sup_{\tilde{z} \in \mathcal{M}_r} g_2((a_2(\tilde{z}))^{-1}t_2).
\end{equation*}
Moreover,
\begin{equation*}
E^-_r \subset E(l,u) \subset E^+_r,
\end{equation*}
where
\begin{equation*}
E^+_r := \left(\prod_{i=1}^{k_2} [0, a^+_i \lambda]\right) \times \prod_{i=k_2+1}^{n} [0, a^+_i \epsilon], \quad E^-_r := \left(\prod_{i=1}^{k_2} [0, a^-_i \lambda]\right) \times \prod_{i=k_2+1}^{n} [0, a^-_i \epsilon]
\end{equation*}
with
\begin{equation*}
a^+_i = \sup_{\tilde{z} \in \mathcal{M}_r} a_i(\tilde{z}), \quad a^-_i = \inf_{\tilde{z} \in \mathcal{M}_r} a_i(\tilde{z}).
\end{equation*}
Hence
\begin{equation}
\sum_{t \in \mathcal{L}_2(u)} P_u(D_u(l)) \leq \sum_{t \in \mathcal{L}_2(u)} P\left\{ \sup_{t \in E^+_r} \frac{X_{u,t}(t)}{1 + (1-\epsilon)u^{-2}p_{2,r}g^-_{2,r}(t_2)} > u_1^{-\epsilon} \right\}.
\end{equation}

**Uniform asymptotics for the summands in (32).** We need to specify the notation in Lemma 4.1 for the current case. Let \( X_{u,t} \) be as was defined in (32) and let
\begin{equation*}
u_1 = u_1^{-\epsilon}, \quad g_{u,t}(t) = (1-\epsilon)u^{-2}p_{2,r}g^-_{2,r}(t_2), \quad K_u = \mathcal{L}_2(u).
\end{equation*}
We first note that \( \lim_{u \to \infty} \inf_{t \in \mathcal{L}_2(u)} u_1^{-\epsilon} = \infty \), which combined with continuity of \( g_2 \) implies
\begin{equation*}
\lim_{u \to \infty} \sup_{t \in K_u} \sup_{t \in E^+_r} \left| u_1^2 g_{u,t}(t) - (1-\epsilon)p_{2,r}g^-_{2,r}(t_2) \right| = 0.
\end{equation*}
Hence C1 holds with \( g(t) = (1-\epsilon)p_{2,r}g^-_{2,r}(t_2) \). To check C2, by (8) and (9) and using the homogeneity of \( W \) for fixed \( t_2 \) and \( \epsilon \), we have
\begin{equation*}
\lim_{u \to \infty} \sup_{t \in K_u} \sup_{s,t \in E^+_r} \left| u_1^2 Var(X_{u,t}(t) - X_{u,t}(s)) - 2Var(W(t) - W(s)) \right| = 0.
\end{equation*}
This implies that C2 is satisfied with the limiting stochastic process \( W \) defined in A1. C3 follows directly from (10). Therefore, we conclude that
\begin{equation}
\lim_{u \to \infty} \sup_{t \in K_u} \left| \mathbb{P}\left\{ \sup_{t \in E^+_r} \frac{X_{u,t}(t)}{1 + (1-\epsilon)u^{-2}p_{2,r}g^-_{2,r}(t_2)} > u_1^{-\epsilon} \right\} - \mathcal{H}(1-\epsilon)p_{2,r}g^-_{2,r}(t_2)(E^+_r) \left(\Psi(u_1^{-\epsilon})\right) \right| = 0,
\end{equation}
where
\begin{equation}
\mathcal{H}(1-\epsilon)p_{2,r}g^-_{2,r}(t_2)(E^+_r) = E\left\{ \sup_{t \in E^+_r} e^{\sqrt{2W(t)} - \sigma_{u,t}(t) - (1-\epsilon)p_{2,r}g^-_{2,r}(t_2)} \right\}.
\end{equation}
Hence we have
\begin{equation*}
\sum_{t \in \mathcal{L}_2(u)} \mathbb{P}\left\{ \sup_{t \in E} X_{u,t}(t) > u_1^{-\epsilon} \right\} \leq \sum_{t \in \mathcal{L}_2(u)} \mathcal{H}(1-\epsilon)p_{2,r}g^-_{2,r}(t_2)(E^+_r) \Psi(u_1^{-\epsilon}).
\end{equation*}
\[
\frac{1}{\lambda^{k_1}} \mathcal{H}_W^{(1-\epsilon)p_{z,r}^\gamma, \sigma_{z,r}^2}(E^+_r) \psi(u) \left( \prod_{i=1}^{k_1} \frac{v_{i,1}^{2/\alpha_1}}{\lambda} \right) \sum_{i=k_0+1}^{k_1} \sum_{i=1}^{M_i(u)} e^{-(1-\epsilon) \inf_{t_i \in [t_1, t_1+1]} p_{z,r}^\gamma g_1(u^{2/\alpha_1-2/\alpha_1} \lambda t_i)}
\]

(36)

\[
\sim \mathcal{H}_W^{(1-\epsilon)p_{z,r}^\gamma, \sigma_{z,r}^2}(E^+_r) \frac{1}{\lambda^{k_1}} \psi_0(u) u \sum_{i=1}^{k_1} \frac{\sigma_{i,r}^2}{\lambda} \sum_{i=k_0+1}^{k_1} \frac{3}{2} \int_{t_1 \in [0, \infty)^{k_1-k_0}} e^{-(1-\epsilon) p_{z,r}^\gamma g_1(t_1)} dt_1, \quad u \to \infty.
\]

Note that

\[
\lim_{\epsilon \to 0} \mathcal{H}_W^{(1-\epsilon)p_{z,r}^\gamma, \sigma_{z,r}^2}(E^+_r) = \mathcal{H}_W^{p_{z,r}^\gamma, \sigma_{z,r}^2} \left( \prod_{i=1}^{k_2} [0, a_i^+, \lambda] \right)
\]

and by dominated convergence theorem, it follows that

\[
\lim_{\epsilon \to 0} \int_{t_1 \in [0, \infty)^{k_1-k_0}} e^{-(1-\epsilon) p_{z,r}^\gamma g_1(t_1)} dt_1 = \int_{t_1 \in [0, \infty)^{k_1-k_0}} e^{-r_{z,r}^\gamma g_1(t_1)} dt_1.
\]

Hence, letting \( \epsilon \to 0 \) in (36), we have

\[
\sum_{l \in L_3(u)} P_u(D_u(l)) \leq \frac{\mathcal{H}_W^{p_{z,r}^\gamma, \sigma_{z,r}^2} \left( \prod_{i=1}^{k_2} [0, a_i^+, \lambda] \right)}{\lambda^{k_1}} \psi_0(u) \Theta^-(u), \quad u \to \infty,
\]

where

\[
\Theta^\pm(u) := \psi(u) \sum_{i=1}^{k_1} \frac{\sigma_{i,r}^2}{\lambda} \sum_{i=k_0+1}^{k_1} \frac{3}{2} \int_{t_1 \in [0, \infty)^{k_1-k_0}} e^{-r_{z,r}^\gamma g_1(t_1)} dt_1.
\]

**Upper bound for \( \sum_{l \in L_3(u)} P_u(D_u(l)) \).** Next we find a tight asymptotic upper bound for the second term displayed in the right hand side of (31). For \( u \) sufficiently large

\[
\sum_{l \in L_3(u)} P_u(D_u(l)) \leq \sum_{l \in L_3(u)} P \left( \sup_{t \in D_u(l)} \bar{X}(t) > u_{t_1,t_2}^{-\epsilon} \right) = \sum_{l \in L_3(u)} P \left( \sup_{t \in E} \bar{X}_{u,l}(t) > u_{t_1,t_2}^{-\epsilon} \right),
\]

where

\[
\bar{X}_{u,l}(t) = \bar{X} \left( u^{-2/\alpha_1}(t_1 \lambda + t_1), \ldots, u^{-2/\alpha_1}(t_n \lambda + t_n) \right), \quad E = [0, \lambda]^{k_2} \times [0, \epsilon]^{n-k_2},
\]

\[
u_{t_1,t_2}^{-\epsilon} = u \left( 1 + (1 - \epsilon) \inf_{t_1 \in [t_1, t_1+1]} g_1(u^{2/\alpha_1} \lambda t_1) + (1 - \epsilon) \inf_{t_2 \in [t_2, t_2+1]} g_2(u^{2/\alpha_2} \lambda t_2) \right).
\]

Let \( Z_u(t) \) be a homogeneous Gaussian random field with variance 1 and correlation function satisfying

\[
r_u(s,t) = e^{-u^{-2} \lambda_2 \sum_{i=1}^{n} |s_i - t_i|^\alpha_i}.
\]

By (10) and Slepian’s inequality (see, e.g., Theorem 2.2.1 in [2]) we have that for \( u \) sufficiently large

\[
P \left( \sup_{t \in E} \bar{X}_{u,l}(t) > u_{t_1,t_2}^{-\epsilon} \right) \leq P \left( \sup_{t \in E} Z_u(t) > u_{t_1,t_2}^{-\epsilon} \right), \quad l \in L_3(u).
\]

Similarly as in the proof of (34), we have

\[
\lim_{u \to \infty} \sup_{t \in L_3(u)} \left| \frac{P \left( \sup_{t \in E} Z_u(t) > u_{t_1,t_2}^{-\epsilon} \right) - \mathcal{J}(E)}{\psi(u_{t_1,t_2}^{-\epsilon})} \right| = 0,
\]

where

\[
\mathcal{J}(E) = \left( \prod_{i=1}^{k_2} \mathcal{H}_{B^{\alpha_1}} [0, (2 \lambda_2)^{1/\alpha_1}] \right) \left( \prod_{i=0}^{n} \mathcal{H}_{B^{\alpha_1}} [0, (2 \lambda_2)^{1/\alpha_1}] \right).
\]

Hence using the above asymptotics and assumption (13)

\[
\sum_{l \in L_3(u)} P_u(D_u(l)) \leq \sum_{l \in L_3(u)} \mathcal{J}(E) \psi(u_{t_1,t_2}^{-\epsilon}).
\]
\[
\begin{align*}
&\leq \mathcal{J}(E)\Psi(u) \sum_{l \in \mathcal{L}_3(u)} e^{-(1-\epsilon)\inf_{l_1} g_1(l_1, l_1+1, 1/\alpha_1) u^2 g_1(u^{2-\alpha_1/\lambda})} - (1-\epsilon) \inf_{l_2} g_2(l_2, l_2+1, 1/\alpha_2) u^2 g_2(u^{2-\alpha_2/\lambda}) \\
&\leq \mathcal{J}(E)\Psi(u) \left( \sum_{i=1}^{k_0} \frac{\prod_{l=1}^{2} \lambda^{2/\alpha_1}}{\lambda} \right) \sum_{i=k_0+1}^{k_1} \sum_{l_i=0}^{M_i(u)} e^{-(1-\epsilon)\inf_{l_1} g_1(l_1, l_1+1, 1/\alpha_1) u^2 g_1(u^{2-\alpha_1/\lambda})} \\
&\times \sum_{l_1, l_2} e^{-(1-\epsilon)\inf_{l_2} g_2(l_2, l_2+1, 1/\alpha_2) u^2 g_2(u^{2-\alpha_2/\lambda})},
\end{align*}
\]

Moreover, direct calculation shows that
\[
\sum_{i=k_0+1}^{k_1} \sum_{l_i=0}^{M_i(u)} e^{-(1-\epsilon)\inf_{l_1} g_1(l_1, l_1+1, 1/\alpha_1) u^2 g_1(u^{2-\alpha_1/\lambda})} \sim \prod_{i=k_0+1}^{k_1} \left( \frac{\lambda}{2} \right)^{l_i} \int_{t_1 \in [0, \infty)} e^{-(1-\epsilon)\inf_{l_1} g_1(t_1, l_1, l_1+1, 1/\alpha_1) u^2 g_1(u^{2-\alpha_1/\lambda})} dt_1, \quad u \to \infty.
\]

By assumption (13) and the fact that \(\alpha_2 = \beta_2\), we have for \(\lambda > 1\)
\[
\sum_{l_1, l_2} e^{-(1-\epsilon)\inf_{l_2} g_2(l_2, l_2+1, 1/\alpha_2) u^2 g_2(u^{2-\alpha_2/\lambda})} \leq \sum_{l_1, l_2} e^{-(1-\epsilon)\inf_{l_2} g_2(l_2, l_2+1, 1/\alpha_2) u^2 g_2(u^{2-\alpha_2/\lambda})} \leq Q_3 e^{-Q_2 \lambda^{\beta^*}},
\]
where \(\beta^* = \min_{l_1, l_2} (\beta_i)\). Additionally,
\[
\lim_{\epsilon \to 0} \mathcal{J}(E) = \prod_{i=1}^{k_2} \mathcal{H}, [0, (2Q_2)^{1/\alpha_1}] \lambda,
\]
and for \(\lambda > 1\)
\[
\prod_{i=1}^{k_2} \mathcal{H}, [0, (2Q_2)^{1/\alpha_1}] \lambda \leq Q_3 \lambda^{k_2}.
\]
Thus we have that for \(\lambda > 1\)
\[
\sum_{l \in \mathcal{L}_3(u)} P_u (D_u(L)) \leq Q_3 \lambda^{k_2-k_1} e^{-Q_2 \lambda^{\beta^*}} v^{k_0} \Theta^- (u), \quad u \to \infty.
\]

**Upper bound for** \(P_u (E_{2,u}(u))\). Combination of (37) and (41) yields that for \(\lambda > 1\)
\[
P_u (E_{2,u}(u)) \leq \prod_{i=0}^{k_2} \left( \mathcal{H}_W^{\alpha_1} [0, a_i] \right)^{l_i} \lambda^{l_i} + Q_3 \lambda^{k_2-k_1} e^{-Q_2 \lambda^{\beta^*}} v^{k_0} \Theta^- (u), \quad u \to \infty.
\]

We next find a lower bound of \(P_u (E_{1,u}(u))\) as \(u \to \infty\) for which we need to derive the lower bound of \(\sum_{l \in \mathcal{L}_1(u)} P_u (C_u(L))\) and upper bounds of \(\Gamma_i(u), i = 1, 2\), respectively.

**Lower bound for** \(\sum_{l \in \mathcal{L}_1(u)} P_u (C_u(L))\). Analogously to (37), we derive
\[
\sum_{l \in \mathcal{L}_1(u)} P_u (C_u(L)) \geq \prod_{i=1}^{k_2} \left( \mathcal{H}_W^{\alpha_1} [0, a_i] \right)^{l_i} \lambda^{l_i} v^{k_0} \Theta^+ (u), \quad u \to \infty, \epsilon \to 0.
\]

Upper bound for \(\Gamma_i(u), i = 1, 2\). Applying an approach analogous to that of the proof of Theorem 8.2 in [1], we have, for \(\lambda > 1\) and as \(u \to \infty\),
\[
\Gamma_1(u) \leq Q_4 \lambda^{-1/2} \lambda^{2k_2-k_1} v^{k_0} \Theta^- (u),
\]
\[
\Gamma_2(u) \leq Q_5 \lambda^{2k_2-k_1} e^{-Q_6 \lambda^{\alpha^*}} v^{k_0} \Theta^- (u),
\]
where \(\alpha^* = \max(\alpha_1, \ldots, \alpha_{k_1})\) and \(Q_i, i = 4, 5, 6\) are some positive constants.
Lower bound for $P_u(E_{1,r}(u))$. Inserting (43), (44) and (45) into (30), we obtain for $\lambda > 1$, as $u \to \infty$,

\[
(46) \quad P_u(E_{1,r}(u)) \geq \left( \frac{\mathcal{H}^{g_{r}, g_{r}^{+}}(\tilde{t}_2)}{\lambda^{k_1}} \prod_{i=1}^{k_1} a_{i,r} \int_{t_1 \in [0,\infty)} e^{-p_{i,r}^{+}(t)} \eta dt \right)^{-1} Q_{4} \lambda^{-1/2} - Q_{5} \lambda^{2k_2-k_1} e^{-Q_{6} \lambda^{\alpha^{*}}} \right) v^{k_0} \Theta^{+}(u).
\]

Existence of $\mathcal{H}_W^{g_{r}}(\tilde{t}_2)$ The idea used here is similar to that of Lemmas 7.1 and 8.3 in [1]. Thus we present only main steps of argumentation. We assume

\[
a(\tilde{z}) = 1, p_j(\tilde{z}) = 1, j = 1, 2, 3, \tilde{z} \in \mathcal{M}_r.
\]

Dividing (42) and (46) by $v^{k_0}\Theta^{-}(u)$ and letting $u \to \infty$, we derive that, for some $\lambda, \lambda_1 > 1$,

\[
\lim_{\lambda \to \infty} \frac{\mathcal{H}_W^{g_{r}}(\tilde{t}_2)(0, \lambda^{k_2})}{\lambda^{k_1}} \leq \lim_{\lambda \to \infty} \inf_{\lambda \to \infty} \frac{\mathcal{H}_W^{g_{r}}(\tilde{t}_2)(0, \lambda^{k_2})}{\lambda^{k_1}} < \infty.
\]

The positivity of the above limit follows from the same arguments as in [1]. Therefore,

\[
(47) \quad \mathcal{H}_W^{g_{r}}(\tilde{t}_2) := \lim_{\lambda \to \infty} \frac{\mathcal{H}_W^{g_{r}}(\tilde{t}_2)(0, \lambda^{k_2})}{\lambda^{k_1}} \in (0, \infty).
\]

Moreover, using (42) and (46), we have, for $\lambda > 1$,

\[
(48) \quad \left| \frac{\mathcal{H}_W^{g_{r}}(\tilde{t}_2)(0, \lambda^{k_2})}{\lambda^{k_1}} - \mathcal{H}_W^{g_{r}}(\tilde{t}_2) \right| \leq Q_{7} \left( \lambda^{-1/2} + \lambda^{2k_2-k_1} e^{-Q_{6} \lambda^{\alpha^{*}}} + \lambda^{k_2-k_1} e^{-Q_{2} \lambda^{\alpha^{*}}} \right).
\]

Let $G := \{g_2 : g_2$ is continuous, $u g_2(\tilde{t}_2) = g_2(\tilde{t}_2)^{1/\beta_2} \tilde{t}_2, u > 0, \inf_{\tilde{t}_2 \in \mathcal{G}, t \in \mathcal{M}} g_2(\tilde{t}_2) > c \geq 0\}$, where $c$ and $\beta_2$ are fixed. For any $g_2 \in G$, (37) and (41)-(46) are still valid. Hence, (48) also holds. This implies that for any $\lambda > 1$,

\[
(49) \quad \sup_{g_2 \in G} \left| \frac{\mathcal{H}_W^{g_{r}}(\tilde{t}_2)(0, \lambda^{k_2})}{\lambda^{k_1}} - \mathcal{H}_W^{g_{r}}(\tilde{t}_2) \right| \leq Q_{7} \left( \lambda^{-1/2} + \lambda^{2k_2-k_1} e^{-Q_{6} \lambda^{\alpha^{*}}} + \lambda^{k_2-k_1} e^{-Q_{2} \lambda^{\alpha^{*}}} \right).
\]

4.2.3. Step 3 In this step of the proof, we sum up the asymptotics derived in step 2. Set

\[
\Theta_1(u) = u^{\sum_{i=1}^{k_1} \tilde{z}_i - \sum_{i=k+1}^{k_1} \tilde{z}_i} \Psi(u).
\]

Letting $\lambda \to \infty$ in (42) and (46), it follows that

\[
P_u(E_{1,r}(u)) \geq \mathcal{H}_W^{g_{r}, g_{r}^{+}}(\tilde{t}_2) \prod_{i=1}^{k_1} a_{i,r}^{+} \int_{t_1 \in [0,\infty)} e^{-p_{i,r}^{+}(t)} \eta dt \right)^{-1} Q_{4} \lambda^{-1/2} - Q_{5} \lambda^{2k_2-k_1} e^{-Q_{6} \lambda^{\alpha^{*}}} \right) v^{k_0} \Theta^{+}(u),
\]

\[
P_u(E_{2,r}(u)) \leq \mathcal{H}_W^{g_{r}, g_{r}^{+}}(\tilde{t}_2) \prod_{i=1}^{k_1} a_{i,r}^{+} \int_{t_1 \in [0,\infty)} e^{-p_{i,r}^{+}(t)} \eta dt \right)^{-1} Q_{4} \lambda^{-1/2} - Q_{5} \lambda^{2k_2-k_1} e^{-Q_{6} \lambda^{\alpha^{*}}} \right) v^{k_0} \Theta^{+}(u).
\]

We add up $P_u(E_{1,r}(u))$ and $P_u(E_{2,r}(u))$ with respect to $\mathcal{V}$ respectively to get a lower bound of $P_u(E_{1}(u))$ and an upper bound of $P_u(E_{2}(u))$. Observe that

\[
P_u(E_{1}(u)) \geq \sum_{r \in \mathcal{V}^{-}} P_u(E_{1,r}(u)) - \sum_{r, r' \in \mathcal{V}^{-}, r \neq r'} P_u(E_{1,r}(u), E_{1,r'}(u)),
\]

\[
P_u(E_{2}(u)) \leq \sum_{r \in \mathcal{V}^{+}} P_u(E_{2,r}(u)).
\]

Note that $g_{2,r}(\tilde{t}_2) \in \mathcal{G}, r \in \mathcal{V}^{+}$ and $p_2(\tilde{z}) g_2(\tilde{z}) \tilde{t}_2) \in \mathcal{G}, \tilde{z} \in \mathcal{M}$ with fixed $c$ and $\beta_2$. Thus (49) implies that for any $\epsilon > 0$ there exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0 > 0$ and $r \in \mathcal{V}^{+}$ and $\tilde{z} \in \mathcal{M}$

\[
|\mathcal{H}_W^{g_{r}, g_{r}^{+}}(\tilde{t}_2) - \mathcal{H}_W^{g_{r}, g_{r}^{+}}(\tilde{t}_2)(0, \lambda^{k_2})\lambda^{-k_1}| < \epsilon, \left| \mathcal{H}_W^{g_{r}, g_{r}^{+}}(\tilde{t}_2)(0, \lambda^{k_2})\lambda^{-k_1} \right| < \epsilon.
\]

Hence it follows that, as $u \to \infty$ and $\lambda > \lambda_0$,

\[
\sum_{r \in \mathcal{V}^{-}} \frac{P_u(E_{1,r}(u))}{\Theta_1(u)} \geq \sum_{r \in \mathcal{V}^{-}} \mathcal{H}_W^{g_{r}, g_{r}^{+}}(\tilde{t}_2) \prod_{i=1}^{k_1} a_{i,r}^{+} \int_{t_1 \in [0,\infty)} e^{-p_{i,r}^{+}(t)} \eta dt v^{k_0}.
\]
\[ \geq \int_{\mathcal{M}} \sum_{\tau \in V'} \left( (\mathcal{H}_{W}^{\mathcal{P}_{2}^{+}} g_{2}^{+,(t_{2})}(\tau_{2})) ([0, \lambda]_{k_{1}}) \lambda^{-k_{1}} - \epsilon \sum_{i=1}^{k_{1}} a_{i, \tau} \int_{t_{1} \in [0, \infty)^{k_{1} - k}} e^{-p_{i, \tau', g_{1}(t_{1})}} d\tau_{1} \right) \mathbb{I}_{\mathcal{M}_{\tau}}(z) \, dz. \]

Note that for any fixed \( \bar{z} \in \mathcal{M}^{o} \), where \( \mathcal{M}^{o} \subset \mathcal{M} \) is the interior of \( \mathcal{M} \),

\[ \lim_{\epsilon \to 0} \sum_{\tau \in V'} \left( (\mathcal{H}_{W}^{\mathcal{P}_{2}^{+}} g_{2}^{+,(t_{2})}(\tau_{2})) ([0, \lambda]_{k_{1}}) \lambda^{-k_{1}} - \epsilon \sum_{i=1}^{k_{1}} a_{i, \tau} \int_{t_{1} \in [0, \infty)^{k_{1} - k}} e^{-p_{i, \tau', g_{1}(t_{1})}} d\tau_{1} \right) \mathbb{I}_{\mathcal{M}_{\tau}}(z) \]

\[ = \left( \mathcal{H}_{W}^{\mathcal{P}_{2}^{+}} g_{2}^{+,(t_{2})}(\tau_{2}) \right) ([0, \lambda]_{k_{1}}) \lambda^{-k_{1}} - \epsilon \sum_{i=1}^{k_{1}} a_{i, \tau} \int_{t_{1} \in [0, \infty)^{k_{1} - k}} e^{-p_{i, \tau', g_{1}(t_{1})}} d\tau_{1} \]

\[ \geq \left( \mathcal{H}_{W}^{\mathcal{P}_{2}^{+}} g_{2}^{+,(t_{2})}(\tau_{2}) \right) \sum_{i=1}^{k_{1}} a_{i, \tau} \int_{t_{1} \in [0, \infty)^{k_{1} - k}} e^{-p_{i, \tau', g_{1}(t_{1})}} d\tau_{1} \]

Moreover, it is clear that there exits \( \mathcal{Q} < \infty \) such that for any \( \lambda > 1 \) and \( v > 0 \)

\[ \left( \mathcal{H}_{W}^{\mathcal{P}_{2}^{+}} g_{2}^{+,(t_{2})}(\tau_{2}) \right) \sum_{i=1}^{k_{1}} a_{i, \tau} \int_{t_{1} \in [0, \infty)^{k_{1} - k}} e^{-p_{i, \tau', g_{1}(t_{1})}} d\tau_{1} \mathbb{I}_{\mathcal{M}_{\tau}} < \mathcal{Q}_{\mathcal{Q}}. \]

Consequently, dominated convergence theorem gives

\[ \lim_{u \to \infty} \sup_{\tau \in V} \frac{\mathbb{P}_{u}(E_{1,\tau}(u))}{\Theta_{1}(u)} \geq \int_{\mathcal{M}} \left( \mathcal{H}_{W}^{\mathcal{P}_{2}^{+}} g_{2}^{+,(t_{2})}(\tau_{2}) \right) \sum_{i=1}^{k_{1}} a_{i, \tau} \int_{t_{1} \in [0, \infty)^{k_{1} - k}} e^{-p_{i, \tau', g_{1}(t_{1})}} d\tau_{1} \, dz. \]

(52)

Next we focus on the double-sum term in (50). For \( \tau \in V^{r}, \tau' \in V^{r}, M_{\tau} \cap M_{\tau'} = \emptyset \), we have

\[ \mathbb{P}_{u}(E_{1,\tau}(u)) \leq \mathbb{P} \left( \sup_{s \in E_{1,\tau}, t \in E_{1,\tau'}} X(s) + X(t) > 2u \right). \]

By A1 and (11), there exists \( 0 < \delta < 1 \) such that for all \( \tau \in V^{r}, \tau' \in V^{r}, M_{\tau} \cap M_{\tau'} = \emptyset, \)

\[ \sup_{s \in E_{1,\tau}, t \in E_{1,\tau'}} \text{Var}(X(s) + X(t)) < 4 - \delta. \]

By Borell-TIS inequality (see, e.g., Theorem 2.1.1 in [2]), we have for \( u > a \)

\[ \mathbb{P} \left( \sup_{s \in E_{1,\tau}, t \in E_{1,\tau'}} X(s) + X(t) > 2u \right) \leq e^{- \frac{4(u-a)^{2}}{\delta}}. \]

where \( a = \mathbb{E}(\sup_{s \in E_{1,\tau}, t \in E_{1,\tau'}} X(s) + X(t)) = \mathbb{E}(\sup_{t \in A} X(t)) \). Consequently,

\[ \sum_{\tau, \tau' \in V^{r}, M_{\tau} \cap M_{\tau'} = \emptyset} \mathbb{P}_{u}(E_{1,\tau}(u), E_{1,\tau'}(u)) \leq \mathcal{Q} e^{- \frac{4(u-a)^{2}}{2(4-\delta)}} = o(\Theta_{1}(u)), \ u \to \infty. \]

For \( \tau, \tau' \in V^{r}, \tau \neq \tau', M_{\tau} \cap M_{\tau'} \neq \emptyset, \)

\[ \mathbb{P}_{u}(E_{1,\tau}(u), E_{1,\tau'}(u)) = \mathbb{P}_{u}(E_{1,\tau}(u)) + \mathbb{P}_{u}(E_{1,\tau'}(u)) - \mathbb{P}_{u}(E_{1,\tau}(u), E_{1,\tau'}(u)). \]

Hence in light of arguments of (51) and (52), we have

\[ \sum_{\tau, \tau' \in V^{r}, \tau \neq \tau', M_{\tau} \cap M_{\tau'} \neq \emptyset} \mathbb{P}_{u}(E_{1,\tau}(u), E_{1,\tau'}(u)) = o(\Theta_{1}(u)), \ u \to \infty, v \to 0. \]

Therefore we have

\[ \sum_{\tau, \tau' \in V^{r}, \tau \neq \tau'} \mathbb{P}_{u}(E_{1,\tau}(u), E_{1,\tau'}(u)) = o(\Theta_{1}(u)), \ u \to \infty, v \to 0, \]
We skip its proof as it only needs some standard but tedious calculations.

2.1 The expansions of \( z(55) \) where \( \alpha \) at finite number of points. However, the case \( \alpha = 1 \) is essentially different from the aforementioned two cases in the sense that depending on \( a_i \) the maximum of the variance can be achieved at a set of positive Lebesgue measure of dimension \( m \), 1, where \( m \) is defined in (20). We apply Theorem 2.1 to this case.

For \( Z^\alpha(t) \) introduced in (18) with \( \alpha = (0, 2) \), we write \( \sigma_Z^2 \) for the variance of \( Z^\alpha \) and \( r_Z \) for its correlation function. Moreover, we denote \( \sigma_\alpha = \max_{t \in S_n} \sigma_Z(t) \) and recall that \( S_n = \{0 = t_0 \leq t_1 \leq \cdots \leq t_n \leq t_{n+1} = 1\} \). The expansions of \( \sigma_Z \) and \( r_Z \) are displayed in the following lemma which is crucial for the proof of Proposition 3.1. We skip its proof as it only needs some standard but tedious calculations.

Lemma 5.1. i) For \( \alpha \in (0, 1) \), the standard deviation \( \sigma_Z \) attains its maximum on \( S_n \) at only one point \( z_0 = (z_1, \ldots, z_n) \in S_n \) with \( z_i = \frac{\sum_{j=1}^i a_j^2}{\sum_{j=1}^{n+1} a_j^2} \), \( i = 1, \ldots, n \), and its maximum value is \( \sigma_* = \left( \sum_{i=1}^{n+1} a_i^{2\alpha} \right)^{\frac{1-\alpha}{2\alpha}} \).

Moreover,

\[
\lim_{\delta \to 0} \sup_{t \in S_n \atop t \sim z_0} \left| \frac{1 - \sigma_Z(t)}{\sigma_*} \right| = 0
\]

with \( z_0 := 0, z_{n+1} := 1 \), and

\[
\lim_{\delta \to 0} \sup_{\substack{s \neq t, s, t \in S_n \atop |s - z_0|, |t - z_0| < \delta}} \left| \frac{1 - r_Z(s, t)}{\sigma_*} \right| = 0.
\]

ii) For \( \alpha = 1 \) and \( m \) defined in (20), if \( m = n + 1 \), \( \sigma_\alpha(t) \equiv 1 \), \( t \in S_n \), and if \( m < n + 1 \), function \( \sigma_\alpha \) attains its maximum equal to 1 on \( S_n \) at \( M = \{t \in S_n : \sum_{j \in N} |t_j - t_{j-1}| = 1\} \) and satisfies

\[
\lim_{\delta \to 0} \sup_{t \in S_n \atop |t - z| \leq \delta} \left| \frac{1 - \sigma_\alpha(t)}{1/2} \right| = 0.
\]

In addition, for \( 1 \leq m \leq n - 1 \), we have

\[
\lim_{\delta \to 0} \sup_{\substack{z \in M \atop |z - s|, |z - t| < \delta}} \left| \frac{1 - r_Z(s, t)}{\sigma_*} \right| = 0.
\]

iii) For \( \alpha \in (1, 2) \), function \( \sigma_Z \) attains it maximum on \( S_n \) at \( m \) points \( z^{(j)} \), \( j \in N = \{i : a_i = 1, i = 1, \ldots, n+1\} \), where \( z^{(j)} = (0, \ldots, 0, 1, \ldots, 1) \) (the first 1 stands at the \( j \)-th coordinate) if \( j \in N \) and \( j < n + 1 \), and \( z^{(n+1)} = (0, \ldots, 0) \) if \( n + 1 \in N \). We further have that \( \sigma_* = 1 \) and as \( t \to z^{(j)} \)

\[
\lim_{\delta \to 0} \sup_{t \in S_n \atop |t - z^{(j)}| \leq \delta} \left| \frac{1 - \sigma_Z(t)}{\sigma_*} \right| = 0.
\]
Case 1. $\alpha \in (0,1)$: From Lemma 5.1 i), we have that $\sigma_Z$ on $\mathcal{S}_n$ attains its maximum $\sigma_*$ at the unique point $z_0 = (z_1, \ldots, z_n)$ with

$$z_i = \frac{\sum_{j=1}^{n} a_j^{-\frac{1}{\alpha}}}{\sum_{j=1}^{n+1} a_j^{-\frac{1}{\alpha}}}, \quad i = 1, \ldots, n.$$ 

Moreover, from (53) we have for $t \in \mathcal{S}_n$

$$1 - \frac{\sigma_Z(t)}{\sigma_*} \sim \frac{\alpha(1-\alpha)}{4} \left( \sum_{i=1}^{n+1} a_i^{-\frac{2}{\alpha}} \right)^{-1} \left( a_1^{-2} (t_1 - z_1)^2 + a_n^{-2} (tn - z_n)^2 + \sum_{i=2}^{n} a_i^{-2} ((t_i - z_i) - (t_{i-1} - z_{i-1}))^2 \right),$$

as $|t - z_0| \to 0$ and from (54) for $t, s \in \mathcal{S}_n$

$$1 - r_Z(s, t) \sim \frac{1}{2\sigma_*^2} \left( \sum_{i=1}^{n} (a_i^{-2} + a_i^{-2}) |s_i - t_i|^\alpha \right),$$

as $|s - z_0|, |t - z_0| \to 0$. Further, we have

$$\mathbb{E} \{(Z^\alpha(s) - Z^\alpha(t))^2\} \leq 4 \sum_{i=1}^{n} |t_i - s_i|^\alpha.$$ 

Thus by [1] [Theorem 8.2] we obtain that, as $u \to \infty$,

$$\mathbb{P} \left\{ \sup_{t \in \mathcal{S}_n} Z^\alpha(t) > u \right\} \sim (\mathcal{H}_{\mathbb{R}^n})^n \prod_{i=1}^{n} \left( \frac{a_i^2 + a_i^{-2}}{2\sigma_*^2} \right)^{1/\alpha} \left( \frac{u}{\sigma_*} \right)^{(2\alpha-1)n} \int_{\mathbb{R}^n} e^{-f(x)} dx \Psi \left( \frac{u}{\sigma_*} \right),$$

where

$$f(x) = \frac{\alpha(1-\alpha)}{4} \left( \sum_{i=1}^{n+1} a_i^{-\frac{2}{\alpha}} \right)^{-1} \left( a_1^{-2} x_1^2 + a_n^{-2} x_n^2 + \sum_{i=2}^{n} a_i^{-2} (x_i - x_{i-1})^2 \right), \quad x \in \mathbb{R}^n.$$ 

Direct calculation shows

$$\int_{\mathbb{R}^n} e^{-f(x)} dx = \left( \frac{4\pi}{\alpha(1-\alpha)} \right)^{\frac{n}{2}} \sigma_*^{-\frac{n-1}{2}} \left( \sum_{j=1}^{n+1} a_j^{-\frac{2}{\alpha}} \right)^{-\frac{1}{2}}.$$ 

This completes the proof of this case.

Case 2. $\alpha = 1$: First we consider the case $m < n + 1$. Let $k^* = \max\{i \in \mathcal{N}\}$ and denote

$$\mathcal{N}_0 = \{i \in \mathcal{N}; i < k^*\}, \quad \mathcal{N}_0^c = \{i \in \mathcal{N}^c; i < k^*\}.$$ 

In order to facilitate our analysis, we make the following transformation:

$$x_i = t_i, \quad i \in \mathcal{N}_0, \quad x_i = t_i - t_{i-1}, \quad i \in \mathcal{N}_0^c,$$

implying that $x = (x_1, \ldots, x_{k^*-1}, x_{k^*+1}, \ldots, x_{n+1}) \in [0,1]^n$ and

$$t_i = t_i(x) = \begin{cases} x_i, & \text{if } i \in \mathcal{N}_0, \\ 1 - \sum_{j=i+1}^{n+1} x_j, & \text{if } i \geq k^*, \\ \sum_{j=\max\{k \in \mathcal{N}; k < i\}}^{i} x_j, & \text{if } i \in \mathcal{N}_0^c, \end{cases}$$

with the convention that $\max \emptyset = 0$. Define $Y(x) = Z(t(x))$ and $\mathcal{S}_n = \{x : t(x) \in \mathcal{S}_n\}$ with $t(x)$ given in (59).

By Lemma 5.1 ii) it follows that $\sigma_Y(x)$, the standard deviation of $Y(x)$, attains its maximum equal to 1 at

$$\{x \in \mathcal{S}_n : x_i = 0, \text{ if } i \in \mathcal{N}_0^c\}.$$ 

Moreover, let $\bar{x} = (x_i)_{i \in \mathcal{N}_0}, \ x = (x_i)_{i \in \mathcal{N}^c}$ and denote for any $\delta \in (0, \frac{1}{(n+1)^2})$

$$\mathcal{S}_n^\delta(\delta) = \left\{ x \in \mathcal{S}_n : 0 \leq x_i \leq \frac{\delta}{(n+1)^2}, \text{if } i \in \mathcal{N}_0^c \right\},$$
\[ \bar{M} = \{ \bar{x} \in [0,1]^{m-1} : x_i \leq x_j, \text{ if } i, j \in \mathcal{N}_0 \text{ and } i < j \}, \]
\[ \bar{M}(\delta) = \{ \bar{x} \in [\delta, 1-\delta]^{m-1} : x_j - x_i \geq \delta, \text{ if } i, j \in \mathcal{N}_0 \text{ and } i < j \} \subseteq \bar{M}, \]
\[ \bar{S}_n(\delta) = \{ x \in \bar{S}_n^*(\delta) : \bar{x} \in \bar{M}(\delta) \}. \]

We notice that
\[ P \left\{ \sup_{x \in \bar{S}_n} Y(x) > u \right\} \geq P \left\{ \sup_{x \in \bar{S}_n} Y(x) > u \right\}, \]
and
\[ P \left\{ \sup_{x \in \bar{S}_n} Y(x) > u \right\} \leq P \left\{ \sup_{x \in \bar{S}_n} Y(x) > u \right\} + P \left\{ \sup_{x \in \bar{S}_n} Y(x) > u \right\}. \]

The idea of the proof is first to apply Theorem 2.1 to obtain the asymptotics of \( P \left\{ \sup_{x \in \bar{S}_n} Y(x) > u \right\} \) as \( u \to \infty \) and then to show that the other two terms in (61) are asymptotically negligible. Let us begin with finding the asymptotics of \( P \left\{ \sup_{x \in \bar{S}_n(\delta)} Y(x) > u \right\} \). First observe
\[ \bar{S}_n(\delta) = \{ x : \bar{x} \in \bar{M}(\delta), 0 \leq x_i \leq \frac{\delta}{(n+1)^2} \text{ if } i \in \mathcal{N}_c \}, \]
which is a set satisfying the assumption in Theorem 2.1. Moreover, it follows from (55) that
\[ \lim_{\delta \to 0} \sup_{x \in \bar{S}_n(\delta)} \left| \frac{1 - a_y(x)}{\sum_{i \in \mathcal{N}_c} (1 - a_i^2) x_i} - 1 \right| = 0. \]

Taking \( \bar{t} = \bar{x} \) and \( \bar{t}_2 = \bar{x} \) in Theorem 2.1, (62) implies that \( A2 \) holds with \( g_2(\bar{x}) = \frac{1}{2} \sum_{i \in \mathcal{N}_c} (1 - a_i^2) x_i \) and \( p_2(\bar{x}) = 1 \) for \( \bar{x} \in \bar{S}_n^*(\delta) \). We note that \( \Lambda_1 = \Lambda_3 = \emptyset \) in this case.

We next check Assumption A1. To compute the correlation, we need the following inequalities. Note that for \( x, y \in \bar{S}_n(\delta) \) and \( |x - y| < \frac{\delta}{(n+1)^2} \), if \( i \in \mathcal{N}_0 \),
\[ |x_i - y_i| + |t_{i-1}(x) - t_{i-1}(y)| < \frac{\delta}{(n+1)^2} + \frac{n\delta}{(n+1)^2} = \frac{\delta}{n+1} \leq \frac{\delta}{2}, \]
and
\[ |t_i(x) - t_{i-1}(x)| = \begin{cases} |x_i - x_{i-1}| \geq \delta & \text{if } i - 1 \in \mathcal{N}_0, \\ |x_i - \sum_{j=i}^{i-1} x_j| \geq \delta - \frac{n\delta}{(n+1)^2} > \frac{\delta}{2} & \text{if } i - 1 \in \mathcal{N}_c, \end{cases} \]
if \( i = k^* \),
\[ |t_{k^*-1}(y) - t_{k^*-1}(x)| + |t_{k^*}(y) - t_{k^*}(x)| < \frac{n\delta}{(n+1)^2} < \frac{\delta}{2}, \]
and
\[ |t_{k^*}(x) - t_{k^*-1}(x)| = \begin{cases} 1 - \sum_{j=k^*+1}^{n+1} x_j - x_{k^*+1} \geq 1 - (1 - \delta) - \frac{n\delta}{(n+1)^2} > \frac{\delta}{2} & \text{if } k^* - 1 \in \mathcal{N}_0, \\ 1 - \sum_{j=k^*+1}^{n+1} x_j - \sum_{j=k^*+1}^{n+1} x_j \geq 1 - (1 - \delta) - \frac{n\delta}{(n+1)^2} > \frac{\delta}{2} & \text{if } k^* - 1 \in \mathcal{N}_c. \end{cases} \]

Hence for \( r_y(x, y) \), the correlation function of \( Y(x) \), we derive from ii) of Lemma 5.1 that for \( x, y \in \bar{S}_n(\delta) \) and \( |x - y| < \frac{\delta}{(n+1)^2} \), as \( \delta \to 0 \)
\[ 1 - r_y(x, y) = 1 - r_2(t(x), t(y)) \]
\[ = \frac{1}{2} \sum_{i=1}^{n+1} a_i^2 \min \{ |t_{i-1}(y) - t_{i-1}(x)|, |t_i(y) - t_i(x)|, |t_i(y) - t_{i+1}(y)| + |t_{i}(x) - t_{i-1}(x)| \} \]
\[ - \frac{1}{2} \sum_{i \in \mathcal{N}} (|t_{i-1}(y) - t_{i-1}(x)| + |t_i(y) - t_i(x)|) \]
+ \frac{1}{2} \sum_{i \in \mathcal{N}^c} a_i^2 \min \left( |t_{i-1}(y) - t_{i-1}(x)|, |t_i(y) - t_i(x)|, |t_i(x) - t_{i-1}(y)|, |t_i(x) - t_{i-1}(x)| \right) \\
= \frac{1}{2} \sum_{i \in \mathcal{N}_0} (|x_i - y_i| + |t_{i-1}(x) - t_{i-1}(y)|) \\
+ \frac{1}{2} |t_{k^* - 1}(x) - t_{k^* - 1}(y)| + \frac{1}{2} \sum_{j = k^* + 1}^{n+1} (x_j - y_j) \\
+ \frac{1}{2} \sum_{i \in \mathcal{N}_0} a_i^2 \min \left( |t_{i-1}(x) - t_{i-1}(y)|, |t_i(x) - t_i(y)|, x_i + y_i \right) \\
+ \frac{1}{2} \sum_{i = k^* + 1}^{n+1} a_i^2 \min \left( \sum_{j = i}^{n+1} (x_j - y_j), \sum_{j = i-1}^{n+1} (x_j - y_j), x_i + y_i \right).

(63)

By (59), we have for any \( i = 1, \ldots, n+1 \)

\[ |t_i(y) - t_i(x)| \leq \sum_{i \neq k^*}^{n+1} |x_i - y_i|. \]

Then for \( x, y \in \tilde{S}_N(\delta) \) and \( |x - y| < \frac{\delta}{(n+1)^2} \) with \( \delta > 0 \) sufficiently small

\[ \frac{1}{2} \sum_{i \in \mathcal{N}_0} |x_i - y_i| \leq 1 - r_N(x, y) \leq Q \sum_{i = 1}^{n+1} |x_i - y_i|, \]

implying that (10) holds.

Recall

\[ W(x) = \frac{\sqrt{2}}{2} \sum_{i \in \mathcal{N}} \left[ B_i(s_i(x)) - \tilde{B}_i(s_{i-1}(x)) \right] + \frac{\sqrt{2}}{2} \sum_{i \in \mathcal{N}^c} a_i \left( B_i(s_i(x)) - B_i(s_{i-1}(x)) \right), \]

where \( B_i, \tilde{B}_i \) are iid standard Brownian motions and

\[ s_i(x) = \begin{cases} 
 x_i, & \text{if } i \in \mathcal{N}_0, \\
 \sum_{j = \max\{k \in \mathcal{N}, k < i\}}^{i} x_j, & \text{if } i \in \mathcal{N}_0^c, \\
 \sum_{j = i+1}^{n+1} x_j, & \text{if } i \geq k^*, 
\end{cases} \]

with the convention that \( \max \emptyset = 0 \). Direct calculation gives that \( \mathbb{E} \left\{ (W(x) - W(y))^2 \right\} \) coincides with (63) for any \( x, y \in [0, \infty)^n \).

This implies that (8) holds with \( W \) given in (64) and \( a(\tilde{x}) \equiv 1 \) for \( \tilde{x} \in \tilde{M}(\delta) \).

Using (63) and the fact that for any \( i = 1, \ldots, n, s_i(x) - s_i(y) \) is the absolute value of the combination of \( x_j - y_j, j \in \{1, \ldots, k^* - 1, k^* + 1, \ldots, n + 1\} \), we derive that for a fixed \( \bar{x} \) the increments of \( W(x) = W(\bar{x}, x) \) are homogeneous with respect to \( \bar{x} \). In addition, it is easy to check that (11) also holds. Hence \( A_1 \) is satisfied.

Consequently, by Theorem 2.1, as \( u \to \infty \), we have

\[ \mathbb{P} \left\{ \sup_{x \in \tilde{S}_N(\delta)} Y(x) > u \right\} \sim v_{m-1} \left( \tilde{M}(\delta) \right) \mathcal{H}_W u^{2(m-1)} \psi(u), \]

where

\[ \mathcal{H}_W = \lim_{\lambda \to \infty} \frac{1}{\lambda^{m-1}} \mathbb{E} \left\{ \sup_{x \in [0, \lambda]^n} e^{\sqrt{2} W(x) - \sigma_W(x) - \frac{1}{2} \sum_{j \in \mathcal{N}^c} (1 - a_j^2) x_j} \right\} \\
= \lim_{\lambda \to \infty} \frac{1}{\lambda^{m-1}} \mathbb{E} \left\{ \sup_{x \in [0, \lambda]^n} e^{\sqrt{2} W(x) - \left( \sum_{j = k^*}^{n+1} x_j \right)} \right\}. \]
We now proceed to the negligibility of the other two terms in (61). In light of Borell-TIS inequality, we have

\[
\mathbb{P}\left\{ \sup_{x \in \tilde{S}_n \setminus \tilde{S}_n^{\delta}} Y(x) > u \right\} \leq \exp\left( \frac{(u - \mathbb{E}(\sup_{x \in \tilde{S}_n \setminus \tilde{S}_n^{\delta}} Y(x)))^2}{2(1 - \varepsilon)^2} \right) = o(\Psi(u)), \ u \to \infty,
\]

where \(\varepsilon = 1 - \sup_{x \in \tilde{S}_n \setminus \tilde{S}_n^{\delta}} \sigma_Y(x).\) By Slepian’s inequality and Theorem 2.1, we have

\[
\mathbb{P}\left\{ \sup_{x \in \tilde{S}_n^{\delta} \setminus \tilde{S}_n^{\delta}} Y(x) > u \right\} \leq v_{m-1} (\tilde{M} \setminus \tilde{M}(\delta)) \tilde{H}_W u^{2(m-1)} \Psi(u)
\]

(67)

Combination of the fact that

\[
\lim_{\delta \to 0} v_{m-1} (\tilde{M}(\delta)) = v_{m-1} (\tilde{M}) = \frac{1}{(m-1)!}
\]

with (60), (61), and (65)-(67) leads to

\[
\mathbb{P}\left\{ \sup_{t \in S_n} Z(t) > u \right\} = \mathbb{P}\left\{ \sup_{x \in \tilde{S}_n} Y(x) > u \right\} \sim \frac{1}{(m-1)!} \mathcal{H}_W u^{2(m-1)} \Psi(u), \ u \to \infty.
\]

Case \(m = n + 1:\) For some small \(\varepsilon \in (0,1),\) define \(E(\varepsilon) = \{t \in S_n : t_i - t_{i-1} \geq \varepsilon, i = 1, \ldots, n + 1\}.\) Then we have

\[
\mathbb{P}\left\{ \sup_{t \in S_n \setminus E(\varepsilon)} Z(t) > u \right\} \leq \mathbb{P}\left\{ \sup_{t \in S_n \setminus E(\varepsilon)} Z(t) > u \right\} \leq \mathbb{P}\left\{ \sup_{t \in S_n \setminus E(\varepsilon)} Z(t) > u \right\} + \mathbb{P}\left\{ \sup_{t \in E(\varepsilon)} Z(t) > u \right\}.
\]

Let us first derive the asymptotics of \(Z\) over \(E(\varepsilon).\) For \(s, t \in E(\varepsilon),\) by (56) we have

\[
1 - r(s, t) \sim \sum_{i=1}^{n} |s_i - t_i|, |t - s| \to 0.
\]

Moreover, it follows straightforwardly that \(\text{Var}(Z(t)) = 1\) for \(t \in E(\varepsilon)\) and \(\text{Corr}(Z(t), Z(s)) < 1\) for any \(s \neq t\) and \(s, t \in E(\varepsilon).\) Hence by [1] [Lemma 7.1], we have

\[
\mathbb{P}\left\{ \sup_{t \in E(\varepsilon)} Z(t) > u \right\} \sim v_n(E(\varepsilon)) u^{2n} \Psi(u) \sim v_n(S_n) u^{2n} \Psi(u), \ u \to \infty, \varepsilon \to 0.
\]

Moreover, by Slepian’s inequality and [1] [Lemma 7.1],

\[
\mathbb{P}\left\{ \sup_{t \in S_n \setminus E(\varepsilon)} Z(t) > u \right\} \leq v_n(S_n \setminus E(\varepsilon))(2\mathcal{H}_W \mathbb{Q}_4)^n u^{2n} \Psi(u) = o(u^{2n} \Psi(u)), \ u \to \infty, \varepsilon \to 0.
\]

Inserting (69) and (70) into (68), we obtain

\[
\mathbb{P}\left\{ \sup_{t \in S_n} Z(t) > u \right\} \sim \frac{1}{n!} u^{2n} \Psi(u), \ u \to \infty.
\]

The claim is established by ii) of Remark 3.2.

Case 3. \(\alpha \in (1, 2):\) For \(s, t \in S_n,\) one can easily check that

\[
r_Z(s, t) = \frac{\mathbb{E}\{ Z^\alpha(t) Z^\alpha(s) \}}{\sigma_Z(s) \sigma_Z(s)} = \frac{\sum_{i=1}^{n+1} \alpha_i^2 \mathbb{E}\{(B^\alpha_i(t_i) - B^\alpha_i(t_{i-1}))(B^\alpha_i(s_i) - B^\alpha_i(s_{i-1}))\}}{\sigma_Z(s) \sigma_Z(s)} < 1
\]

if \(s \neq t.\) In light of Lemma 5.1 iii), \(\sigma_Z\) attains its maximum at \(m\) distinct points \(z^{(j)}, j \in \mathcal{N}.\) Consequently, by [1] [Corollary 8.2], we have

\[
\mathbb{P}\left\{ \sup_{t \in S_n} Z^\alpha(t) > u \right\} \sim \sum_{j \in \mathcal{N}} \mathbb{P}\left\{ \sup_{t \in \Pi_{x,j}} Z^\alpha(t) > u \right\}, \ u \to \infty,
\]
where $\Pi_{\delta,j} = \{ t \in S_n : |t - z^{(j)}| \leq \frac{1}{\delta} \}$.

Define $E_j(u) := \{ t \in \Pi_{\delta,j} : 1 - (\frac{\ln u}{\alpha})^2 \leq t_j - t_{j-1} \leq 1 \} \ni z^j$. Observe that

$$
\mathbb{P}\left\{ \sup_{t \in E_j(u)} Z^\alpha(t) > u \right\} \leq \mathbb{P}\left\{ \sup_{t \in \Pi_{\delta,j}} Z^\alpha(t) > u \right\} \leq \mathbb{P}\left\{ \sup_{t \in E_j(u)} Z^\alpha(t) > u \right\} + \mathbb{P}\left\{ \sup_{t \in \Pi_{\delta,j} \setminus E_j(u)} Z^\alpha(t) > u \right\}.
$$

We first find the exact asymptotics of $\mathbb{P}\left\{ \sup_{t \in E_j(u)} Z^\alpha(t) > u \right\}$ as $u \to \infty$. Clearly, for any $u \in \mathbb{R}$,

$$
\mathbb{P}\left\{ \sup_{t \in E_j(u)} Z^\alpha(t) > u \right\} \geq \mathbb{P}\left\{ Z^\alpha(z^j) > u \right\} = \Psi(u).
$$

Moreover, for $s, t \in S_n$, there exists a constant $c > 0$ such that $\inf_{t \in S_n} \sigma_Z(t) \geq \frac{1}{\sqrt{2\pi}}$. Hence in light of (58) we have

$$(71) \quad 1 - r_Z(s, t) \leq 4e \sum_{i=1}^{n+2} |t_i - s_i|^\alpha.
$$

Let $U_2(t), t \in \mathbb{R}^n$ be a centered homogeneous Gaussian field with continuous trajectories, unit variance and correlation function $r_{U_2}(s, t)$ satisfying

$$
r_{U_2}(s, t) = 1 - \exp\left(8c \sum_{i=1}^{n+2} |t_i - s_i|^\alpha\right).
$$

Set $E_j(u) = [0, \varepsilon_1 u^{-2/\alpha}]^{j-1} \times [1 - \varepsilon_1 u^{-2/\alpha}, 1]^{n-j+1}$ for some constant $\varepsilon_1 \in (0, 1)$. Then it follows that $E_j(u) \subset \tilde{E}_j(u)$ for $u$ sufficiently large. By Slepian’s inequality and [1] [Lemma 6.1]

$$
\mathbb{P}\left\{ \sup_{t \in \tilde{E}_j(u)} Z^\alpha(t) > u \right\} \leq \mathbb{P}\left\{ \sup_{t \in E_j(u)} U_2(t) > u \right\} \sim \left(\mathcal{H}_{B^n}(0, (8c)^{1/\alpha} \varepsilon_1^n)\right)^n \Psi(u) \sim \Psi(u),
$$

as $u \to \infty, \varepsilon_1 \to 0$, where

$$
\lim_{\lambda \to 0} \mathcal{H}_{B^n}(0, \lambda) = \lim_{\lambda \to 0} \mathbb{E}\left\{ \sup_{t \in [0, \lambda]} e^{\sqrt{2}B^n(t) - \lambda^\alpha} \right\} = 1.
$$

Consequently,

$$(72) \quad \mathbb{P}\left\{ \sup_{t \in E_j(u)} Z^\alpha(t) > u \right\} \sim \Psi(u), \ u \to \infty.
$$

Note that for $t \in S_n$

$$
\sum_{i=1}^{n+2} a_i^2 |t_i - t_{i-1}|^\alpha \leq |t_j - t_{j-1} - 1|.
$$

Hence, by (57), for $u$ sufficiently large,

$$
\sup_{t \in \Pi_{\delta,j} \setminus E_j(u)} \sigma_Z(t) \leq \sup_{t \in \Pi_{\delta,j} \setminus E_j(u)} \left(1 - \frac{(1 - \varepsilon)(\alpha - 1)}{2} \frac{\ln u}{u} \right)^2 \leq 1 - \frac{(1 - \varepsilon)(\alpha - 1)}{2} \left(\frac{\ln u}{u}\right)^2,
$$

where $\varepsilon \in (0, 1)$ is a constant. In light of (71) and (73), by [1] [Theorem 8.1] we have, for $u$ sufficiently large,

$$
\mathbb{P}\left\{ \sup_{t \in \Pi_{\delta,j} \setminus E_j(u)} Z^\alpha(t) > u \right\} \leq \mathcal{Q}_0 u^{2n/\alpha} \Psi\left(\frac{u}{1 - \frac{(1 - \varepsilon)(\alpha - 1)}{2} \left(\frac{\ln u}{u}\right)^2}\right) = o(\Psi(u)), \ u \to \infty,
$$

which combined with (72) leads to

$$
\mathbb{P}\left\{ \sup_{t \in \Pi_{\delta,j}} Z^\alpha(t) > u \right\} \sim \mathbb{P}\left\{ \sup_{t \in E_j(u)} Z^\alpha(t) > u \right\} \sim \Psi(u), \ u \to \infty.
$$
Consequently, with \( m = \mathcal{N} \), we obtain
\[
\mathbb{P} \left( \sup_{t \in \mathcal{S}_n} Z^a(t) > u \right) \sim \sum_{j \in \mathcal{N}} \mathbb{P} \left( \sup_{t \in \Pi_{i,j}} Z^a(t) > u \right) \sim m \Psi(u), \ u \to \infty.
\]
This completes the proof. \( \square \)

6. Proof of Proposition 3.3

Observe that for \( 0 < \epsilon < \pi/4 \)
\[
\mathbb{P} \left( \sup_{(\theta, t) \in E_{1, \epsilon}} Z(\theta, t) > u \right) \leq \mathbb{P} \left( \sup_{(\theta, t) \in E} Z(\theta, t) > u \right) \leq \sum_{i=1}^{3} \mathbb{P} \left( \sup_{(\theta, t) \in E_{1, \epsilon}} Z(\theta, t) > u \right),
\]
where
\( E_{1, \epsilon} = [\epsilon, \pi - \epsilon]^{n-2} \times [0, 2\pi - \epsilon] \times [0, \epsilon], \ E_{2, \epsilon} = [0, \pi]^{n-2} \times [0, 2\pi) \times [\epsilon, 1], \ E_{3, \epsilon} = E/(E_{1, \epsilon} \cup E_{2, \epsilon}). \)

We will first apply Theorem 2.1 to obtain the asymptotics over \( E_{1, \epsilon} \) and then show that the asymptotics over \( E_{2, \epsilon} \) and \( E_{3, \epsilon} \) are negligible as \( u \to \infty \) and \( \epsilon \to 0 \).

The asymptotics over \( E_{1, \epsilon} \). To this end, we next analyze the variance and correlation of \( Z \). By (24), we have
\[
\sigma_Z(\theta, t) = \frac{1}{1 + bt^a}, \ t \in [0, 1].
\]
Hence \( \sigma_Z(\theta, t) \) attains its maximum equal to 1 at \( [0, \pi]^{n-2} \times [0, 2\pi) \times \{0\} \) and
\[
\lim_{\delta \to 0} \sup_{\theta \in [0, \pi]^{n-2} \times [0, 2\pi), 0 < t < \delta} \left| \frac{1 - \sigma_Z(\theta, t)}{bt^a} - 1 \right| = 1.
\]
This implies that \( A2 \) is satisfied. For \( A1 \), by (25), we have
\[
1 - Corr(Z(\theta, t), Z(\theta', t')) \sim aVar(Y(t) - Y(t')) + \frac{1}{2} \sum_{i=1}^{n} (v_i(\theta) - v_i(\theta'))^2
\]
\[
\sim aVar(Y(t) - Y(t')) + \frac{1}{2} \frac{(\theta_1 - \theta_1')^2}{2} + \frac{1}{2} \sum_{i=2}^{n-1} \left( \prod_{j=1}^{i-1} \sin(\theta_j) \right)^2 (\theta_i - \theta_i')^2,
\]
as \( (\theta, t), (\theta', t') \in E \) and \( |t - t'|, |\theta - \theta'| \to 0 \). Let
\[
W(\theta, t) = \sum_{i=1}^{n-1} B_i^2(\theta_i) + \sqrt{n}Y(t), \ \theta \in \mathbb{R}^{n-1} \times \mathbb{R}^+,
\]
where \( B_i^2 \) are independent fractional Brownian motions with index 2 and \( Y \) is the self-similar Gaussian process given in (25) that is independent of \( B_i^2 \). Moreover, we denote \( a(\varphi) = (a_1(\varphi), \ldots, a_{n-1}(\varphi)) \), \( \varphi \in [0, \pi]^{n-2} \times [0, 2\pi) \) with
\[
a_1(\varphi) = \frac{1}{\sqrt{2}} \text{ and } a_i(\varphi) = \frac{1}{\sqrt{2}} \prod_{j=1}^{i-1} \sin(\varphi_j), \ i = 2, \ldots, n - 1.
\]
It follows that for \( 0 < \epsilon < \pi/4 \)
\[
\lim_{\delta \to 0} \sup_{\varphi \in [\epsilon, \pi - \epsilon]^{n-2} \times [0, 2\pi)} \sup_{(\theta, t), (\theta', t') \in E, |(\theta, t) - (\varphi, 0)|, |(\theta', t') - (\varphi, 0)| < \delta} \left| \frac{1 - Corr(Z(\theta, t), Z(\theta', t'))}{(W(a(\varphi)\theta, t) - W(a(\varphi)\theta', t'))^2} \right| \mathbb{E} \left\{ \left( W(a(\varphi)\theta, t) - W(a(\varphi)\theta', t') \right)^2 \right\} = 0.
\]
By the fact that
\[
Var(W(\theta, t) - W(\theta', t')) = aVar(Y(t) - Y(t')) + \sum_{i=1}^{n-1} (\theta_i - \theta_i')^2,
\]
we know that \( W(\theta, t) \) is homogeneous with respect to \( \theta \) if \( t \) is fixed. This implies that (8) holds with \( W \) defined in (76).
Moreover, by self-similarity of \( Y \) and (77) we have
\[
\text{Var}(W(u^{-1}\theta, u^{-2/\alpha}t) - W(u^{-1}\theta', u^{-2/\alpha}t')) = u^{-2}\text{Var}(W(\theta, t) - W(\theta', t')),
\]
showing that (9) holds with \( \alpha_i = 2, \ i = 1, \ldots, n-1 \) and \( \alpha_n = \alpha \). In addition, by B1-B2, there exists \( d > 0 \) such that for \(|\theta, t) - (\theta', t')| < \delta \) with \((\theta, t), (\theta', t') \in E_{1,\varepsilon},
\[
\mathbb{Q}_1 \sum_{i=1}^{n-1} (\theta_i - \theta_i')^2 \leq 1 - \text{Corr}(Z(\theta, t) \leq \mathbb{Q}_2 \left(|t - t'|^\alpha + \sum_{i=1}^{n-1} (\theta_i - \theta_i')^2 \right).
\]
Hence (10) is confirmed. Moreover, (11) is clearly satisfied over \( E_{1,\varepsilon} \). Therefore, A1 is verified for \( Z \) over \( E_{1,\varepsilon} \).
Consequently, it follows from Theorem 2.1 that, as \( u \to \infty, \)
\[
\mathbb{P} \left( \sup_{(\theta, t) \in E_{1,\varepsilon}} Z(\theta, t) > u \right) \sim \mathcal{H}_W^{bt\alpha} \int_{\theta \in [0,\pi]^{n-2} \times [0,2\pi]} 2^{-(n-1)/2} \prod_{i=1}^{n-1} |\sin(\theta_i)|^{n-1-i}d\theta_1 \ldots d\theta_{n-1} u^{n-1} \Psi(u),
\]
where \( W \) is given in (76).
Upper bound for the asymptotics over \( E_{2,\varepsilon} \). By (75), there exists \( 0 < \delta < 1 \) such that
\[
\sup_{(\theta, t) \in E_{2,\varepsilon}} \text{Var}(Z(\theta, t)) \leq 1 - \delta.
\]
It follows from Borell-TIS inequality that
\[
\mathbb{P} \left( \sup_{(\theta, t) \in E_{2,\varepsilon}} Z(\theta, t) > u \right) \leq \exp \left( -\frac{(u - \mathbb{E} \left\{ \sup_{(\theta, t) \in E_{2,\varepsilon}} Z(\theta, t) \right\})^2}{2(1 - \delta)} \right) = o(u^{n-1}\Psi(u)), \ u \to \infty.
\]
Upper bound for the asymptotics over \( E_{3,\varepsilon} \). Direct calculation shows that
\[
1 - \text{Corr}(Z(\theta, t) \leq \mathbb{Q}_2 \left(|t - t'|^\alpha + \sum_{i=1}^{n-1} (\theta_i - \theta_i')^2 \right)
\]
holds for \((\theta, t), (\theta', t') \in E_{3,\varepsilon} \). Define \( U_3(\theta, t), (\theta, t) \in \mathbb{R}^n \) to be a centered homogeneous Gaussian field with continuous trajectories, unit variance and correlation function \( r_{U_3}(\theta, t, \theta', t') \) satisfying
\[
r_{U_3}(\theta, t, \theta', t') = 1 - \exp \left( -2\mathbb{Q}_2 \left(|t - t'|^\alpha + \sum_{i=1}^{n-1} (\theta_i - \theta_i')^2 \right) \right).
\]
By Slepian’s inequality and Theorem 2.1, we have
\[
\mathbb{P} \left( \sup_{(\theta, t) \in E_{3,\varepsilon}} Z(\theta, t) > u \right) \leq \mathbb{P} \left( \sup_{(\theta, t) \in E_{3,\varepsilon}} \frac{U_3(\theta, t)}{1 + bt\alpha} > u \right) \leq \mathbb{Q} v_n(E_{3,\varepsilon}) u^{n-1} \Psi(u), \ u \to \infty.
\]
Noting that \( \lim_{\varepsilon \to 0} v_n(E_{3,\varepsilon}) = 0 \), combination of the above asymptotics and upper bounds leads to
\[
\mathbb{P} \left( \sup_{(\theta, t) \in E} Z(\theta, t) > u \right) \sim \mathcal{H}_W^{bt\alpha} \int_{\theta \in [0,\pi]^{n-2} \times [0,2\pi]} 2^{-(n-1)/2} \prod_{i=1}^{n-1} |\sin(\theta_i)|^{n-1-i}d\theta_1 \ldots d\theta_{n-1} u^{n-1} \Psi(u), \ u \to \infty.
\]
By the fact that
\[
\int_{\theta \in [0,\pi]^{n-2} \times [0,2\pi]} \prod_{i=1}^{n-1} |\sin(\theta_i)|^{n-1-i}d\theta_1 \ldots d\theta_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)},
\]
and \( \mathcal{H}_W = \mathcal{P}_{\pi^b}(\mathcal{H}_{B2})^{n-1} = \mathcal{P}_{\pi^{-1/2}}^{n-1}, \) where we used that \( \mathcal{H}_{B2} = \pi^{-1/2}, \) we have
\[
\mathbb{P} \left( \sup_{(\theta, t) \in E} Z(\theta, t) > u \right) \sim \frac{2\pi^{n/2}}{\Gamma(n/2)} \mathcal{P}_{\pi^{-1/2}}^{n-1} u^{n-1} \Psi(u), \ u \to \infty.
\]
□
i) For the case $1 \leq m \leq n$, we first show that $\mathcal{H}_W \geq 1$. Recall that $\mathcal{N}_0 = \{ i \in \mathcal{N}, i < k^* \}$, $\mathcal{N}^c = \{ i : a_i < 1, i = 1, \ldots, n + 1 \}$ and $\mathcal{K} = (x_i)_{i \in \mathcal{N}_0}$.

For $x_i = 0, \ i \in \mathcal{N}^c$, by the definition of $W$ in (21), we have

$$\left\{ \sqrt{2}W(x) - \sum_{i=1, i \neq k^*}^{n+1} x_i, \ \tilde{x} \in [0, \lambda]^{m-1} \right\} = \left\{ \sum_{i \in \mathcal{N}_0} \sqrt{2}B_i(x_i) - \sum_{i \in \mathcal{N}_0} x_i, \ \tilde{x} \in [0, \lambda]^{m-1} \right\}.$$ 

Hence

$$\mathcal{H}_W \geq \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda^{m-1}} \mathbb{E} \left\{ \sup_{\tilde{x} \in [0, \lambda]^{m-1}} e^{\sum_{i \in \mathcal{N}_0} \sqrt{2}B_i(x_i) - \sum_{i \in \mathcal{N}_0} x_i \tilde{x} \in [0, \lambda]^{m-1}} \right\} = \prod_{i \in \mathcal{N}_0} \mathcal{H}_B_i,$$

where $H_B_i$ is defined in (17). Note that $H_B_1 = 1$, see e.g., [1] (or [30]). Therefore, $\mathcal{H}_W \geq 1$. We next derive the upper bound of $\mathcal{H}_W$ for $1 \leq m \leq n$, for which we need to use the notation in the proof of ii) of Proposition 3.1, e.g., $Y$ and $\tilde{S}_n(\delta)$. For $\delta \in (0, \frac{1}{(n+1)^2})$, let

$$A(\delta) = \{ x : \tilde{x} \in B(\delta), \ 0 \leq x_i \leq \frac{\delta}{(n+1)^2}, \text{if } i \in \mathcal{N}^c \},$$

where $B(\delta) = \prod_{i=1}^{m-1} [2i, (2i+1)\delta]$. Clearly, $A(\delta) \subseteq \tilde{S}_n(\delta)$. Moreover, by (63) we have that for any $\epsilon > 0$, there exists $\delta \in (0, \frac{1}{(n+1)^2})$ such that for any $x, y \in A(\delta),$

$$1 - \rho_Y(x, y) \leq (n + \epsilon) \sum_{i=1, i \neq k^*}^{n+1} |x_i - y_i|.$$

Let us introduce a centered homogeneous Gaussian fields $U_4(x), \ x \in [0, \infty)^n$ with continuous trajectories, unit variance and correlation functions

$$\rho_{U_4}(x, y) = \exp \left\{ -\mathbb{E} \left\{ (W_4(x) - W_4(y))^2 \right\} \right\}, \ \text{with } W_4(x) = \sqrt{n + \epsilon} \sum_{i=1, i \neq k^*}^{n+1} B_i(x_i),$$

where $B_i, \ i = 1, \ldots, k^* - 1, k^* + 1, n + 1$ are iid standard Brownian motions. By (62) and Slepian’s inequality, we have, for $0 < \epsilon < 1,$

$$\mathbb{P} \left\{ \sup_{x \in A(\delta)} \frac{U_4(x)}{1 + \sum_{i \in \mathcal{N}^c} \frac{1}{2+\epsilon} x_i} > u \right\} \geq \mathbb{P} \left\{ \sup_{x \in A(\delta)} Y(x) > u \right\}.$$

Analogously to (65), we have

$$\mathbb{P} \left\{ \sup_{x \in A(\delta)} Y(x) > u \right\} \sim u_{m-1} \mathcal{H}_W u^{2(m-1)} \Psi(u),$$

and

$$\mathbb{P} \left\{ \sup_{x \in A(\delta)} \frac{U_4(x)}{1 + \sum_{i \in \mathcal{N}^c} \frac{1}{2+\epsilon} x_i} > u \right\} \sim u_{m-1} \mathcal{H}_W u^{2(m-1)} \Psi(u),$$

where

$$\mathcal{H}_W = \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda^{m-1}} \mathbb{E} \left\{ e^{\sum_{i=1, i \neq k^*}^{n+1} B_i(x_i) - (n+\epsilon) \sum_{i=1, i \neq k^*}^{n+1} x_i - \sum_{i \in \mathcal{N}^c} \frac{1}{2+\epsilon} x_i} \right\} = (n + \epsilon)^{-\frac{1}{2}} \prod_{i \in \mathcal{N}_0} \mathcal{H}_B_i \prod_{i \in \mathcal{N}^c} \mathcal{H}_{B_i}^{\frac{1}{2}}.$$
with $\mathcal{P}_{B_i}$ for $c > 0$ being defined in (17). Using the fact that $\mathcal{H}_{B_i} = 1$ and for $c > 0$ (see, e.g., [30]), $\mathcal{P}_{B_i} = 1 + \frac{1}{c}$, we have

$$\mathcal{H}_{W_4} = (n + \epsilon)^{m-1} \prod_{i \in \mathcal{N}_c} \left( 1 + \frac{(2 + \epsilon)(n + \epsilon)}{1 - a_i^2} \right).$$

Hence

$$\mathcal{H}_{W} \leq \mathcal{H}_{W_4} = (n + \epsilon)^{m-1} \prod_{i \in \mathcal{N}_c} \left( 1 + \frac{(2 + \epsilon)(n + \epsilon)}{1 - a_i^2} \right).$$

We establish the claim by letting $\epsilon \to 0$.

ii) If $m = n + 1$, we have $\mathcal{N}_0 = \{1, \ldots, n\}$ and

$$\mathcal{H}_{W} = \lim_{\lambda \to \infty} \frac{1}{\lambda^n} \mathbb{E} \left\{ \sup_{\tilde{x} \in [0,\lambda]^n} e^{\sum_{i \in \mathcal{N}_0} \sqrt{\mathcal{P}_{B_i}(x_i)} - \sum_{i \in \mathcal{N}_0} x_i} \right\} = \prod_{i \in \mathcal{N}_0} \mathcal{H}_{B_i} = 1.$$ 

\[\square\]

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