Fusion for the one-dimensional Hubbard model

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Abstract
We discuss a formulation of the fusion procedure for integrable models which is suitable for application to non-standard $R$-matrices. It allows for construction of bound state $R$-matrices for AdS/CFT worldsheet scattering or equivalently for the one-dimensional Hubbard model. We also discuss some peculiar cases that arise in these models.

Keywords: integrable systems, Hubbard model, AdS/CFT

1. Introduction

Integrable systems constitute a special class of physical models that are exactly solvable [1]. The key ingredient that allows for the explicit construction of exact solutions is the so-called $R$-matrix. For most known integrable models, the corresponding $R$-matrices are determined by the underlying symmetry algebra of the system. This is usually an infinite-dimensional Hopf algebra of Yangian or quantum affine type.

Computing the $R$-matrix in various representations of such an algebra then describes different types of particles. For example, the Heisenberg XXX model has the Yangian of $\mathfrak{su}(2)$ as its symmetry algebra. The $R$-matrix in the fundamental representation simply describes a chain of spin-$\frac{1}{2}$ particles. Similarly, particles of higher spin can be considered by taking higher-dimensional representations of $\mathfrak{su}(2)$.

On the other hand, from basic representation theory it is well known that for example the tensor product of two spin-$\frac{1}{2}$ particles splits into a spin-1 and a spin-0 representation. In other words, one should be able to relate $R$-matrices in higher dimensional representations to the fundamental $R$-matrix. For example, the $R$-matrix of spin-1 particles should allow for some
decomposition into the $R$-matrix of spin-$\frac{1}{2}$ particles. Similarly, from the fundamental representation it should be possible to construct new $R$-matrices corresponding to other representations. This construction goes under the name of fusion [2]. For most integrable systems this procedure is well-understood, however, the established formulas do not directly apply for some of the more exotic integrable models.

Recently, there was renewed interest in the field of integrable systems due to their appearance in string and gauge theory via the AdS/CFT correspondence (see [3] and references therein). In particular, this integrable structure gave rise to an unusual $R$-matrix that displays Yangian symmetry corresponding to the centrally extended $su(2|2)$ Lie superalgebra [4, 5]. Remarkably, this $R$-matrix turned out to be directly related to Shastry’s $R$-matrix [6] describing the one-dimensional Hubbard model [7].

It soon became clear that for a full description of the string model, the $R$-matrices in higher dimensional representations corresponding to bound states were needed [8]. These bound state $R$-matrices could be computed directly by invoking the Yang–Baxter equation [9] or by using Yangian symmetry [10]. However, how to obtain these matrices directly from the fundamental $R$-matrix was unknown since the usual fusion procedure breaks down. In this note we will introduce a slight generalization of the fusion procedure for integrable models which allows us to obtain bound state $R$-matrices for the AdS/CFT S-matrix and Shastry’s $R$-matrix.

This paper is organized as follows. In section 2 we introduce our fusion procedure and study it at the level of $R$-matrices. Then in section 3 we study more advanced aspects of it. Afterwards we turn to applications: In section 4 we discuss the well-known example of the XXX spin chain before moving on to the novel case of AdS/CFT worldsheet scattering and the one-dimensional Hubbard model in section 5.

2. Fusion

Consider an integrable system whose fundamental degrees of freedom are described by an $n$-dimensional vector space $\mathcal{V}^F$. These might represent the spin degrees of freedom of an integrable spin chain or the particle flavors of an integrable scattering problem in $1 + 1$ dimensions. Their interactions are described by an $(n^2 \times n^2)$-dimensional $R$-matrix

$$ R (u_1, u_2) : \mathcal{V}^F \otimes \mathcal{V}^F \rightarrow \mathcal{V}^F \otimes \mathcal{V}^F, \quad (2.1) $$

where the parameters $u_1, u_2 \in \mathcal{M}^F$ describe the inhomogeneities of the spin sites or the particle rapidities. The space of parameters $\mathcal{M}^F$ is typically a one-dimensional complex manifold, such as the complex plane or the Riemann sphere with certain punctures.

For an integrable system, the $R$-matrix satisfies the Yang–Baxter equation and the involution property (the latter relation is understood up to an overall factor)

$$ R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}, \quad R_{12} R_{23} \sim I_{12}. \quad (2.2) $$

Here and in the following, the indices denote the spaces in a tensor product as well as the associated parameters $u_k$.

4 Additional continuous parameters or even discrete parameters are conceivable as well.
2.1. Singularities

Suppose that there are pairs of points

\[ u_{(12)} = (u_1, u_2) \in M^B \subset M^F \times M^F, \] (2.3)

where the \( R \)-matrix becomes non-invertible\(^5\). In other words, the rank of the \( R \)-matrix drops below its maximum

\[ \text{rank} R(u_1, u_2) = m < n^2. \] (2.4)

Commonly, these points form a one-dimensional sub-manifold\(^6\) \( M^B \) of \( M^F \times M^F \). The point \( u_{(12)} \in M^B \) can thus be treated as a continuous parameter, and it is on the same footing as the parameters \( u_k \in M^F \). In scattering theory, such singular behavior signals the existence of composite particles which are naturally part of the physical scattering problem. For these we will define new \( R \)-matrices to be interpreted as scattering matrices involving composite particles. We can thus extend the integrable system by adding these composite particles as additional degrees of freedom.

More precisely, we are led to the introduction of an \( m \)-dimensional vector space \( \mathcal{V}^B \) by ‘fusing’ two spaces \( \mathcal{V}^F \) such that the extended integrable system is defined on \( \mathcal{V}^{\text{ext}} := \mathcal{V}^F \otimes \mathcal{V}^B \). The corresponding \( R \)-matrix can be schematically written in block form

\[ R^{\text{ext}} : \mathcal{V}^{\text{ext}} \otimes \mathcal{V}^{\text{ext}} \rightarrow \mathcal{V}^{\text{ext}} \otimes \mathcal{V}^{\text{ext}}, \quad R^{\text{ext}} = \text{diag}(R^{FF}, R^{FB}, R^{BF}, R^{BB}). \] (2.5)

where the various blocks are maps of the types \( (A_i = F, B) \)

\[ R^{A_i A_j}(u_1, u_2) : \mathcal{V}^{A_i} \otimes \mathcal{V}^{A_j} \rightarrow \mathcal{V}^{A_i} \otimes \mathcal{V}^{A_j}, \quad (u_1, u_2) \in M^{A_i} \times M^{A_j}. \] (2.6)

In particular, \( R^{FF} \equiv R \). The fact that \( R^{\text{ext}} \) satisfies the Yang–Baxter equation is then equivalent to the statement

\[ R^{A_i A_j} R^{A_j A_k} R^{A_k A_i} = R^{A_j A_i} R^{A_i A_k} R^{A_k A_j}. \] (2.7)

Moreover, the \( R \)-matrices \( R^{A_i A_j} \) have the involution property

\[ R^{A_i A_j} R^{A_j A_i} \sim 1_{A_i A_j}. \] (2.8)

Of course the fusion procedure can be recursively applied, leading to larger and larger integrable systems.

2.2. Procedure

In the following, we shall construct the fused \( R \)-matrices and afterwards check their properties.

**Embedding and fusion matrices.** We consider the \( R \)-matrix \( R(u_{(12)}) := R(u_1, u_2) \) at a point \( u_{(12)} \) where the rank drops to \( m < n^2 \). At this point we can decompose it as a product of three matrices

\(^5\) The \( R \)-matrix can have singular points where some of its eigenvalues diverge. As we are not interested in overall factors of the \( R \)-matrix (which may well depend on \( u_1 \) and \( u_2 \)) we should rescale the \( R \)-matrix at these points to remove the singularity. In other words, what counts is that the leading contribution in a Laurent expansion is non-invertible.

\(^6\) Discrete points with this property can also be considered along the same lines. We will, however, mostly be interested in the case of continuous values for \( u_{(12)} \).

\(^7\) In fact, they are only part of the physical problem if they are also bound. The distinction between bound and unbound composite particles makes no difference here, and we shall refer to them collectively as composite states.
with the properties
\[ \mathcal{R}(u_{(12)}) = \mathcal{E}(u_{(12)}) \mathcal{H}(u_{(12)}) \mathcal{F}(u_{(12)}), \quad \mathcal{F}(u_{(12)}) \mathcal{E}(u_{(12)}) = 1^B, \quad \mathcal{H} \text{ invertible.} \] (2.10)

Here, \( \mathcal{F} \) is a (surjective) \( m \times n^2 \) matrix which fuses the tensor product \( \mathcal{V}^F \otimes \mathcal{V}^F \) to the space \( \mathcal{V}^B \), and \( \mathcal{E} \) is an (injective) \( n^2 \times m \) matrix which embeds \( \mathcal{V}^B \) into \( \mathcal{V}^F \otimes \mathcal{V}^F \). For convenience, we assume these two matrices to be pseudo-inverses as in (2.10). Finally, \( \mathcal{H} \) is a (bijective) \( m \times m \) matrix and it represents the action of \( \mathcal{R} \) on the space \( \mathcal{V}^B \). In the following we shall describe two ways of obtaining the decomposition.

First of all, let \( \mathcal{V}^B \) be the image of \( \mathcal{R} \). Thus \( \mathcal{R} \) can be understood as a map \( \mathcal{V}^F \otimes \mathcal{V}^F \rightarrow \mathcal{V}^B \). Furthermore, define \( \mathcal{E} \) as the trivial embedding map \( \mathcal{V}^B \rightarrow \mathcal{V}^F \otimes \mathcal{V}^F \). The above decomposition can be obtained as \( \mathcal{R} = (\mathcal{R}\mathcal{E})^{-1}\mathcal{R} \) and \( \mathcal{H} = \mathcal{R}\mathcal{E} \). Note that this construction requires \( \mathcal{H} = \mathcal{R}\mathcal{E} \) to be invertible. This combination is not invertible precisely if the image of \( \mathcal{R} \) contains a vector that is in the kernel of \( \mathcal{R} \). In other words, the map \( \mathcal{R} \) contains a non-trivial nilpotent part. This case is more difficult to handle, and we shall exclude it for the time being. Later on in section 5.3, we shall discuss an explicit example.

An alternative construction of the decomposition uses eigenvectors where we assume that the Jordan decomposition is trivial (i.e. the nilpotent case is excluded). The matrix \( \mathcal{R} \) possesses \( m \) non-zero eigenvalues \( \lambda_1, \ldots, \lambda_m \). Let \( e_1, \ldots, e_m \) denote the associated right eigenvectors of \( \mathcal{R} \) and \( e_1^\dagger, \ldots, e_m^\dagger \) the left eigenvectors
\[ \mathcal{R}e_k = \lambda_k e_k, \quad e_k^\dagger \mathcal{R} = \lambda_k e_k^\dagger. \] (2.11)

All of these quantities are functions of \( u_{(12)} \). We normalize the vectors such that they form two dual bases for the space \( \mathcal{V}^B \)
\[ e_k^\dagger e_l = \delta_i^j, \quad \mathcal{R} = \sum_k \lambda_k e_k e_k^\dagger. \] (2.12)

We then define \( \mathcal{E}, \mathcal{H} \) and \( \mathcal{F} \) as the matrices of eigenvectors and eigenvalues
\[ \mathcal{E} = (e_1 \ e_2 \ \ldots \ e_m), \quad \mathcal{H} = \text{diag}(\lambda_1, \ldots, \lambda_m), \quad \mathcal{F} = (e_1^\dagger \ e_2^\dagger \ \ldots \ e_m^\dagger)^T. \] (2.13)

By their definition \( \mathcal{E} \) and \( \mathcal{F} \) satisfy the relations (2.10).
The relations (2.10) imply the following useful identities
\[ E(u_{12}) H(u_{12}) = R(u_{12}) E(u_{12}), \]
\[ H(u_{12}) F(u_{12}) = F(u_{12}) R(u_{12}), \]
\[ R(u_{12}) E(u_{12}) F(u_{12}) = R(u_{12}). \] (2.14)

These are the crucial relations that enable us to carry out the fusion procedure.

**Fused R-matrices.** We introduce R-matrices by using \( E, F \) to ‘fuse’ together two spaces \( \mathbb{V}_1^F \otimes \mathbb{V}_2^F \) into \( \mathbb{V}_1^B \) with \( u_{12} = (u_1, u_2) \). In particular, we are led to (cf. figure 2)
\[ R_{123}^{BF}(u_{12}, u_3) := F_{12}(u_{12}) R_{13}(u_1, u_3) R_{23}(u_2, u_3) E_{12}(u_{12}), \] (2.15)
\[ R_{12}^{FB}(u_1, u_{23}) := F_{23}(u_{23}) R_{13}(u_1, u_3) R_{12}(u_1, u_2) E_{23}(u_{23}). \] (2.16)

Any state from \( \mathbb{V}_1^B \otimes \mathbb{V}_3^B \) is mapped to \( \mathbb{V}_1^F \otimes \mathbb{V}_2^F \otimes \mathbb{V}_3^F \) by \( E \), acted upon with \( R \) and then mapped back by \( F \).

Notice the different ordering of the R-matrices which is needed when spaces two and three are fused rather than spaces one and two. The R-matrix \( R_{BB} \) can then be defined by applying the fusion procedure twice
\[ R_{(12)(34)}^{BB}(u_{12}, u_{34}) := F_{34}(u_{34}) R_{12}(u_{12}) R_{12}(u_{12}) R_{34}(u_{34}) E_{34}(u_{34}). \] (2.17)

In particular, from (2.14) it is readily seen that (2.17) can be cast into the symmetric form
\[ R_{(12)(34)} := F_{12} F_{34} R_{14} R_{24} R_{13} R_{23} E_{12} E_{34}, \] (2.18)
which demonstrates that it is independent of the order of fusing the underlying spaces. Here and in the following, we drop the parameters \( u \) and the labels F, B in favor of a more concise presentation. They can be recovered from the labels of the associated spaces.

**2.3. Relations**

In order to show that these R-matrices indeed describe an integrable system we have to show that they are invertible and that they satisfy the Yang–Baxter equation.

**Involution property.** We have to show the involution property (2.8) which reads more explicitly
\[ R_{123}^{BF}(u_{12}, u_3) R_{123}^{FB}(u_3, u_{12}) \sim 1, \] (2.19)
\[ R_{(12)(34)}^{BB}(u_{12}, u_{34}) R_{(34)(12)}^{BB}(u_{34}, u_{12}) \sim 1. \] (2.20)

Let us prove the first instance. For conciseness we will omit the arguments. Furthermore, we shall put brackets around the terms to be transformed in the next step.
Invertibility of $\mathcal{H}$ then gives the desired result. In general, the strategy is to remove intermediate factors of $\mathcal{E}$ and $\mathcal{F}$ by use of the Yang–Baxter equation (2.2) and the properties (2.14).

Yang–Baxter equation. Subsequently, it also follows directly from (2.14) that the above introduced $R$-matrices (2.15)–(2.17) satisfy the various versions of the Yang–Baxter equation outlined in (2.7).

For example, since $\mathcal{R}$ satisfies the Yang–Baxter equation (2.2) itself, (2.14) yields (we again suppress the explicit arguments)

$$H_{(12)} R_{(12)3} R_{3(12)} = H_{(12)} F_{(12)} R_{(12)3} E_{(12)} F_{(12)} R_{32} R_{31} E_{(12)},$$

$$= F_{(12)} [R_{12} R_{13} R_{23}] E_{(12)} F_{(12)} R_{32} R_{31} E_{(12)},$$

$$= F_{(12)} R_{23} R_{13} [R_{12} E_{(12)} F_{(12)} R_{32} R_{31} E_{(12)}],$$

$$= F_{(12)} R_{23} R_{13} R_{12} R_{32} R_{31} E_{(12)},$$

$$= [F_{(12)} R_{12} [R_{13} R_{23} R_{32} R_{31} E_{(12)}],$$

$$\sim H_{(12)} [F_{(12)} E_{(12)}],$$

$$= H_{(12)}. \quad (2.21)$$

This proves that the $R$-matrices (2.15)–(2.17) indeed describe the scattering of a new (composite) type of particle in this model.

3. Further properties

We have established the basic features of fused $R$-matrices. In the following we will discuss further properties.

3.1. Algebra

Suppose there is a Hopf algebra $H$ describing the symmetries of our integrable system. In particular, the $R$-matrix, by definition, intertwines the coproduct and opposite coproduct in the representation $\rho^F(\mu) : H \to \mathcal{V}^F$ under which our fundamental degrees of freedom transform, i.e. for any generator $J \in H$
We define a new representation $\rho^B(u_{(12)}) : \mathbb{H} \rightarrow \mathbb{B}$ for the composite degrees of freedom

$$\rho^B_{(12)}(u_{(12)})[j] := \mathcal{F}_{(12)}(u_{(12)}) \left( \rho_1^F(u_1) \otimes \rho_2^F(u_2) \right)[\Delta(j)] E_{(12)}(u_{(12)}).$$

(3.2)

We will refer to this as the fused or composite representation. This representation clearly is m-dimensional.

Let us show that (3.2) indeed defines a representation by proving that it respects the multiplicative structure. We have from (2.14) and the cocommutativity (3.1) of the coproduct, that for any $J, J' \in \mathbb{H}$

$$\mathcal{H}_{(12)} \rho_{(12)}[J] \rho_{(12)}[J'] = \mathcal{F}_{(12)} \mathcal{R}_{12} \left( \rho_1 \otimes \rho_2 \right)[\Delta(j)] E_{(12)} \mathcal{F}_{(12)} \left( \rho_1 \otimes \rho_2 \right)[\Delta(j')] E_{(12)},$$

$$= \mathcal{F}_{(12)} \mathcal{R}_{12} \left( \rho_1 \otimes \rho_2 \right)[\Delta(j)] \mathcal{R}_{12} E_{(12)} \mathcal{F}_{(12)} \left( \rho_1 \otimes \rho_2 \right)[\Delta(j')] E_{(12)},$$

$$= \mathcal{F}_{(12)} \mathcal{R}_{12} \left( \rho_1 \otimes \rho_2 \right)[\Delta(j)] \mathcal{R}_{12} \left( \rho_1 \otimes \rho_2 \right)[\Delta(j')] E_{(12)},$$

$$= \mathcal{H}_{(12)} \mathcal{F}_{(12)} \left( \rho_1 \otimes \rho_2 \right)[\Delta(j, j')] E_{(12)},$$

$$= \mathcal{H}_{(12)} \rho_{(12)}[J, J'].$$

(3.3)

Furthermore, the $R$-matrices (2.15)–(2.17) naturally intertwine the coproduct in the new representation. Explicitly,

$$\left( \rho_1^{A_1} \otimes \rho_2^{A_2} \right)[\Delta^{\mathcal{F}}(j)] R_{12}^{A_1 A_2} = R_{12}^{A_1 A_2} \left( \rho_1^{A_1} \otimes \rho_2^{A_2} \right)[\Delta(j)].$$

(3.4)

For instance, let us prove the intertwining relation for the case $A_1 = B, A_2 = F$. This is most conveniently done in the Sweedler notation

$$\sum J_{(1)(1)} \otimes J_{(1)(2)} \otimes J_{(2)} = \sum J_{(1)} \otimes J_{(2)(1)} \otimes J_{(2)(2)},$$

(3.5)

and the intertwining property (3.1) of the $R$-matrix is formulated as

$$\sum \mathcal{R}_{12} \rho_1[J_{(1)}] \rho_2[J_{(2)}] = \sum \rho_1[J_{(2)}] \rho_2[J_{(1)}] \mathcal{R}_{12}.$$

(3.6)

In this language, we have (for conciseness we suppress the arguments $u, v$ of the $R$-matrices and the sums)

$$\mathcal{R}_{12} \left( \rho_{(12)} \otimes \rho_3 \right)[\Delta(j)] = \mathcal{R}_{12} \mathcal{F}_{(12)} \mathcal{R}_{23} \mathcal{R}_{13} \rho_1[J_{(1)}] \rho_2[J_{(2)(1)}] \rho_3[J_{(2)(2)}] E_{(12)},$$

$$= \mathcal{F}_{(12)} \mathcal{R}_{23} \mathcal{R}_{13} \rho_1[J_{(1)}] \rho_2[J_{(2)(1)}] \rho_3[J_{(2)(2)}] E_{(12)},$$

$$= \mathcal{F}_{(12)} \rho_1[J_{(2)(1)}] \rho_2[J_{(2)(2)}] \rho_3[J_{(1)}] E_{(12)} \mathcal{F}_{(12)} \mathcal{R}_{13} \mathcal{R}_{23} E_{(12)},$$

$$= \left( \rho_{(12)} \otimes \rho_3 \right)[\Delta^{\mathcal{F}}(j)] \mathcal{R}_{12 \delta},$$

(3.7)

where we used (2.14) repeatedly and (3.6) in the third step. This proves that $R^{BF}$ displays the expected cocommutativity properties.
3.2. Auxiliary features

Here we will discuss some auxiliary features of the fused $R$-matrices.

Similarity transformations. We have the freedom to apply a similarity transformation $W(u_{(12)})$ to the space $\mathcal{V}^B$

\[
\begin{align*}
\mathcal{E}(u_{(12)}) &\rightarrow \mathcal{E}(u_{(12)}) W(u_{(12)})^{-1}, \\
\mathcal{F}(u_{(12)}) &\rightarrow W(u_{(12)}) \mathcal{F}(u_{(12)}), \\
\mathcal{H}(u_{(12)}) &\rightarrow W(u_{(12)}) \mathcal{H}(u_{(12)}) W(u_{(12)})^{-1}.
\end{align*}
\] (3.8)

This transformation affects none of the relations (2.10), and therefore all the above results apply to the transformed system without further ado. In the construction of $\mathcal{E}$, $\mathcal{H}$, $\mathcal{F}$ via eigenvectors (2.13), the similarity transformation does not preserve the diagonal nature of $\mathcal{H}$ (unless $W$ is diagonal as well). However, we have not made explicit use of this property in the constructions.

Opposite form. A similarity transformation by $H_{(12)}$ has a curious effect on the fused $R$-matrices

\[
\begin{align*}
H_{(12)} R_{(12)3}^\dagger H_{(12)}^{-1} &= H_{(12)} F_{(12)} R_{13} R_{23} E_{(12)} H_{(12)}^{-1}, \\
&= F_{(12)} R_{13} R_{23} E_{(12)} H_{(12)}^{-1}, \\
&= F_{(12)} R_{23} R_{13} E_{(12)} H_{(12)}^{-1}, \\
&= F_{(12)} R_{23} E_{(12)}. \tag{3.9}
\end{align*}
\]

Compared to the original definition $R_{(12)3} = F_{(12)} R_{13} R_{23} E_{(12)}$, we observe that conjugation by $H_{(12)}$ flips the order of the $R$-matrix factors within the fused $R$-matrix.

This observation goes hand in hand with the definition (3.2) of the fused representation via the coproduct. If instead of the coproduct we use the opposite coproduct, the resulting representation is related to the original one by a simple similarity transformation

\[
F_{(12)} \left( \rho_1 \otimes \rho_2 \right) [\Delta(J)] E_{(12)} = H_{(12)}^{-1} F_{(12)} \left( \rho_1 \otimes \rho_2 \right) [\Delta^o(J)] E_{(12)} H_{(12)}^{-1}. \tag{3.10}
\]

As usual we used (2.14) and the intertwining property of the $R$-matrix.

Symmetric $R$-matrices. In many practical applications $R(u_{(12)})$, acting as an operator on the space $\mathcal{V}^F \otimes \mathcal{V}^F$, is symmetric w.r.t. some inner product, e.g. the standard inner product $\langle a, b \rangle := a^\dagger b$ defined on $\mathcal{V}^F$. In most cases, $R(u_{(12)}) = R(u_{(12)})^\dagger$ admits an orthonormal basis of eigenvectors.\(^8\) In addition, the fusion and embedding matrices are conjugate to each other if $F = E^\dagger$.

One minor problem is that the resulting $R$-matrices (2.15) are not symmetric. Transposition reverses the order of the constituent $R$-matrices which corresponds to a similarity transformation according to (3.9)

\[
R_{(12)3}^\dagger = F_{(12)} R_{23} R_{13} E_{(12)} = H_{(12)} R_{(12)3} H_{(12)}^{-1}. \tag{3.11}
\]

By applying a similarity transformation (3.8) defined by a $W$ such that $H = W^TW$, the resulting $R$-matrices become symmetric

\(^8\) This is evident if the inner product is positive definite. For indefinite inner products (including complex symmetric matrices), eigenvectors can be null. In this case the eigenvectors cannot be normalized, and even a non-trivial Jordan decomposition may arise.
3.3. Complementary fusion

A fused \( R \)-matrix can also be defined for the complement \( \mathcal{V}_{(12)}^B \) of the space \( \mathcal{V}_{(12)} \)

\[
\mathcal{V}_{(12)}^F \otimes \mathcal{V}_{(12)}^F = \mathcal{V}_{(12)} \oplus \mathcal{V}_{(12)}^B, \quad \mathcal{R}_{(12)3}^B : \mathcal{V}_{(12)}^F \otimes \mathcal{V}_{(12)}^F \rightarrow \mathcal{V}_{(12)}^B \otimes \mathcal{V}_{(12)}^F. \tag{3.13}
\]

As we shall see, this space is even better suited for fusion.

Complementary space. To define an \( R \)-matrix for the complement, we need to define fusion and embedding matrices for the complementary space

\[
\mathcal{E}(u_{(12)}) : \mathcal{V}_F \rightarrow \mathcal{V}_F \otimes \mathcal{V}_F, \quad \mathcal{F}(u_{(12)}) : \mathcal{V}_F \otimes \mathcal{V}_F \rightarrow \mathcal{V}_B.	ag{3.14}
\]

They are defined to obey the following orthogonality properties with the original embedding and fusion matrices in (3.15):

\[
\mathcal{F}(u_{(12)}) \mathcal{E}(u_{(12)}) = 1_B, \quad \mathcal{F}(u_{(12)}) \mathcal{E}(u_{(12)}) = 0, \quad \mathcal{F}(u_{(12)}) \mathcal{E}(u_{(12)}) = 0. \tag{3.15}
\]

They directly imply the completeness relations

\[
\mathcal{E}(u_{(12)}) \mathcal{F}(u_{(12)}) + \mathcal{E}(u_{(12)}) \mathcal{F}(u_{(12)}) = 1_{12} \tag{3.16}
\]

as well as orthogonality relations with the \( R \)-matrix

\[
\mathcal{R}(u_{(12)}) \mathcal{E}(u_{(12)}) = 0, \quad \mathcal{F}(u_{(12)}) \mathcal{R}(u_{(12)}) = 0. \tag{3.17}
\]

Complementary \( R \)-matrix. One can define a complementary \( R \)-matrix in analogy to (2.15)

\[
\mathcal{R}_{(12)3}^B(u_{(12)}, u_3) := \mathcal{F}_{(12)}\mathcal{E}_{(12)} \mathcal{R}_{13}(u_1, u_3) \mathcal{R}_{23}(u_2, u_3) \mathcal{E}_{(12)} \mathcal{F}_{(12)}(u_{(12)}). \tag{3.18}
\]

The other related \( R \)-matrices follow in a similar fashion. The various integrability relationships can be derived in a similar fashion to the above. The general strategy is to remove the factors of \( \mathcal{E}_{(12)} \mathcal{F}_{(12)} \), which typically appear between the various \( R \)-matrices. The starting point is the relationship \( \mathcal{F}_{(12)} \mathcal{R}_{13} \mathcal{R}_{23} \mathcal{E}_{(12)} = 0 \) which follows from the above definitions

\[
\mathcal{H}_{(12)} \mathcal{F}_{(12)} \mathcal{R}_{13} \mathcal{R}_{23} \mathcal{E}_{(12)} = \mathcal{F}_{(12)} \mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} \mathcal{E}_{(12)} = \mathcal{F}_{(12)} \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12} \mathcal{E}_{(12)} = 0. \tag{3.19}
\]

Together with the completeness relationship (3.16) one can show

\[
\mathcal{E}_{(12)} \mathcal{F}_{(12)} \mathcal{R}_{13} \mathcal{R}_{23} \mathcal{E}_{(12)} = \mathcal{R}_{13} \mathcal{R}_{23} \mathcal{E}_{(12)} - \mathcal{E}_{(12)} \mathcal{F}_{(12)} \mathcal{R}_{13} \mathcal{R}_{23} \mathcal{E}_{(12)} = \mathcal{R}_{13} \mathcal{R}_{23} \mathcal{E}_{(12)}. \tag{3.20}
\]

Therefore, all intermediate factors of \( \mathcal{E}_{(12)} \mathcal{F}_{(12)} \) can be removed iteratively from products of \( R \)-matrices from the left to the right.

Similarly, one can prove that

\[
\mathcal{F}_{(12)} \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{E}_{(12)} = 0, \quad \mathcal{E}_{(12)} \mathcal{F}_{(12)} \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{E}_{(12)} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{E}_{(12)}. \tag{3.21}
\]

On the level of the \( R \)-matrix this simply corresponds to a similarity transformation on the composite particle space applied to the original fused \( R \)-matrix (3.9).

Put differently, the combinations \( \mathcal{R}_{13} \mathcal{R}_{23} \) and \( \mathcal{R}_{23} \mathcal{R}_{13} \), when viewed as a block diagonal matrix, effectively possess a triangular shape:
Composite states from the complementary subspace $\mathcal{V}^B_{(12)}$ are mapped to the same subspace by $R_{13}R_{23}$. Conversely, composite states from the original subspace $\mathcal{V}^B_{(12)}$ can map to both subspaces. Fusion as defined in (2.15) works only due to the presence of the projector $P_{(12)}$. However, for the opposite $R$-matrix (3.9) this reverses and states from $\mathcal{V}^B_{(12)}$ are mapped to the same subspace. This is at the cost of a similarity transformation on $\mathcal{V}^B_{(12)}$.

Opposite fusion. The involution property (2.2) also has an interesting implication on the complementary space $\mathcal{V}^B$ which we shall discuss in the following. According to our assumptions, the first $m$ eigenvalues of $R(u_{(12)})$ are non-zero while the other $n^2 - m$ vanish at the point $u_{(12)} = (u_1, u_2)$. We assume that the latter fall off linearly in $\epsilon$ when approaching the singular point as in $(u_1, u_2 + \epsilon) \rightarrow u_{(12)}$. Furthermore, also the product $R_{12}R_{21}$ is assumed to be proportional to $\epsilon$.

By the involution property (2.2) we have that away from the singular points $R_{12}$ is invertible with inverse proportional to $R_{21}$. Therefore $R_{12}$ and $R_{21}$ share the same eigenvectors, but with inverse eigenvalues. Now, by our assumptions on the behavior near the singular point, we are led to the conclusion that the first $m$ eigenvalues of $R_{21}$ fall off linearly with $m$ while the remaining $n^2 - m$ eigenvalues approach a constant$^9$.

Therefore, the null space of $R_{21}(u_{(21)})$ at $u_{(21)} := (u_2, u_1)$ is given by the image of $E_{(12)}(u_{(12)})$ whereas the non-trivial eigenspace is the image of $E_{(21)}(u_{(21)})$. Consequently, the space $\mathcal{V}^B_{(21)}$ is the complement of $\mathcal{V}^B_{(12)}$ within $\mathcal{V}^F_1 \otimes \mathcal{V}^F_2$, and it has dimension $\tilde{m} := n^2 - m$.

We therefore reproduce the elements of complementary fusion

$$\mathcal{V}^B_{(21)} = \mathcal{V}^B_{(12)}, \quad R_{(21)} \sim T_{(12)}, \quad E_{(21)} \sim E_{(12)}.$$  

(3.23)

All the constructions for the complementary fused states and operators proceed as before with the roles of spaces 1 and 2 interchanged.

### 3.4. Algebraic Bethe ansatz

Let us briefly touch upon the effect of fusion on monodromy and transfer matrices that play a central role in the algebraic Bethe Ansatz.

RTT-relation. The key relation in the algebraic Bethe ansatz is the so-called RTT-relation. The RTT-relation describes the commutation relations between the elements of an $n \times n$ dimensional operator valued matrix $T^F(u) : \mathcal{V}^F \rightarrow \mathcal{V}^F \otimes \mathcal{O}$, called the monodromy matrix. We have

$$R_{12}^{FF}(u_1, u_2) T_1^F(u_1) T_2^F(u_2) = T_2^F(u_2) T_1^F(u_1) R_{12}^{FF}(u_1, u_2).$$  

(3.24)

However, rather than taking $R_{12}^{FF}(u_1, u_2)$ one can also consider taking fused $R$-matrices and consider the RTT relation this would generate. To this end, we introduce a fused monodromy

$^9$ If some of the eigenvalues of $R_{12}$ scale with a higher power of $\epsilon$ (consequently also $R_{13}R_{23}$), only fewer than $n^2 - m$ eigenvalues will be finite. The following considerations would have to be adapted accordingly. See section 5.3 for a concrete example of this case.
It is straightforward to show that $\mathcal{T}_{(12)}^B(u_{(12)})$ satisfies the RTT-relation for fused $R$-matrices.

An object of special interest is the transfer matrix, which is defined as the trace of the monodromy matrix

$$t^F(u) = trT^F(u), \quad t^B(u_{(12)}) = trT^B(u_{(12)}).$$

This object generates the mutually commuting set of operators that is the defining property of integrable systems.

**Fusion for transfer matrices.** We can formulate the relation between the transfer matrices in different representations. Consider the product of two transfer matrices and use the completeness relation (3.16)

$$t^F(u_1) t^F(u_2) = trT^F(u_1) T^F(u_2),$$

$$= trT^F(u_1) T^F(u_2) E_{(12)} + trT^F(u_1) T^F(u_2) E_{(21)} + t^B(u_{(12)}) + t^B(u_{(21)}).$$

(3.27)

For the latter term, we note that a similarity transformation by $H_{(21)}$ is required to interchange the order of monodromy matrices in analogy to (3.9).

### 4. The Heisenberg XXX spin chain

To illustrate our fusion procedure, let us first consider the Heisenberg XXX spin chain. The fundamental particles transform in the spin-$\frac{1}{2}$ representation of $su(2)$. The corresponding $R$-matrix is of difference form $R_{XXX}(u_1, u_2) \equiv R_{XXX}(u_1 - u_2)$ and is given by

$$R_{XXX}(u) = \begin{pmatrix}
  u + 1 & 0 & 0 & 0 \\
  0 & u & 0 & 0 \\
  0 & 1 & u & 0 \\
  0 & 0 & 0 & u + 1
\end{pmatrix}. \quad (4.1)

There are two points at which the rank of $R_{XXX}(u)$ is not maximal, namely at $u = \pm 1$. At $u = 1$ the rank is three, while at $u = -1$ the rank is one. These points correspond to the singlet and triplet that arise in the decomposition of the tensor product of two spin-$\frac{1}{2}$ representations.

**Singlet.** At the point $u_1 - u_2 = -1$, there is only one eigenvector with non-zero eigenvalue and our projector is

$$E = F^\top = \frac{1}{\sqrt{2}} \begin{pmatrix}
  0 \\
  1 \\
  -1 \\
  0
\end{pmatrix}, \quad H = -2. \quad (4.2)

It is easy to check that the identities from (2.14) hold. The representation that is generated according to (3.2) is the trivial representation. Consequently, the fused $R$-matrix describing the scattering of a singlet state with a fundamental particle is given by
In other words, we see that the singlet has trivial scattering with a doublet (up to an overall factor which depends on the definition of the overall factor of $R_{XYY}$).

**Triplet.** The other point $u_1 - u_2 = 1$ is the opposite of $u_1 - u_2 = -1$. The resulting space therefore is the complement of the above singlet. More concretely, there is only one null vector and from the three remaining eigenvectors we find

$$\mathcal{E} = F^T = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}, \quad \mathcal{H} = \text{diag}(2, 2, 2).$$

It is easily seen that this will give rise to the usual spin-1 representation of $\mathfrak{su}(2)$. Indeed, (3.2) yields for the simple roots

$$\rho_{(12)}[S^+] = \begin{pmatrix}
\sqrt{2} & 0 \\
0 & 0 \\
0 & 0 \\
\end{pmatrix}, \quad \rho_{(12)}[S^-] = \begin{pmatrix}
0 & 0 & 0 \\
\sqrt{2} & 0 & 0 \\
0 & \sqrt{2} & 0 \\
\end{pmatrix}.$$ (4.5)

Consequently, we recover the standard composite state $R$-matrix $R_{(12)3}(u_{(12)}, u_3)$ for the triplet-doublet case from (2.15)

$$\begin{pmatrix}
u_2 - u_3 + 2 & 0 & 0 & 0 & 0 & 0 \\
0 & u_2 - u_3 & \sqrt{2} & 0 & 0 & 0 \\
0 & \sqrt{2} & u_2 - u_3 + 1 & 0 & 0 & 0 \\
0 & 0 & 0 & u_2 - u_3 + 1 & \sqrt{2} & 0 \\
0 & 0 & 0 & \sqrt{2} & u_2 - u_3 & 0 \\
0 & 0 & 0 & 0 & 0 & u_2 - u_3 + 2 \\
\end{pmatrix}.$$ (4.6)

One can now easily check that the Yang–Baxter equation holds.

### 5. The Hubbard model

The $R$-matrix for the Hubbard model is best described in terms of its symmetry algebra; centrally extended $\mathfrak{su}(2|2)$. This algebra is obtained from regular $\mathfrak{su}(2|2)$ by adjoining two additional central charges to it. The $R$-matrix in the fundamental representation is completely fixed by the intertwining property (3.1) [4, 11]. The $R$-matrices involving composite state representations are fixed by Yangian invariance [10].

It turns out that there are two cases where the $R$-matrix becomes of lower rank. There are two cases where it becomes of rank 8 and the fused representation is a (a)symmetric short representation. Moreover, there is a point where the $R$-matrix reduces to rank 1 corresponding to a singlet representation.

We will show that the bound state $R$-matrices found in the literature [9] follow from our fusion procedure.
5.1. The Hubbard model R-Matrix

The \( R \)-matrix for the Hubbard model is a \( 4^2 \times 4^2 \) matrix. It acts on the tensor product of two 2|2-dimensional spaces with bosonic basis vectors \(|\phi^a\rangle, a = 1, 2\) and their fermionic counterparts \(|\psi^a\rangle\), \(a = 3, 4\). For convenience let us introduce the 2|2-dimensional basis vector \( E_{A} = (|\phi^1\rangle, |\phi^2\rangle, |\psi^3\rangle, |\psi^4\rangle) \) and let \( E_{A}^{\dagger} \) be the matrix unities with a \((-1)^{|B|}\) in row \(A\) and column \(B\). The fundamental \( R \)-matrix is then of the form

\[
\mathcal{R}(u_1, u_2) = (-1)^{|B|+|C|} E_{A}^{\dagger} \otimes E_{B}^{\dagger} \mathcal{R}_{A}^{B} D_{C}(u_1, u_2) \quad (5.1)
\]

with the only non-zero entries given by

\[
\begin{align*}
\mathcal{R}_{\alpha\beta}^{a\beta} &= \delta_{\alpha}^{a} \delta_{\beta}^{\beta} + \left( \delta_{\alpha}^{a} \delta_{\beta}^{\beta} - \delta_{\alpha}^{a} \delta_{\beta}^{\beta} \right) \frac{x_{1}^{\alpha} - x_{2}^{\beta}}{x_{1}^{\alpha} - x_{2}^{\beta}} \frac{x_{1}^{\alpha} - x_{2}^{\beta}}{x_{1}^{\alpha} - x_{2}^{\beta}} - \frac{1}{\mbox{1}}. \\
\mathcal{R}_{\alpha\beta}^{a\beta} &= \frac{U_{2}}{U_{1}} \frac{x_{1}^{a} - x_{2}^{\beta}}{x_{1}^{a} - x_{2}^{\beta}} \left[ \delta_{\alpha}^{a} \delta_{\beta}^{\beta} + \left( \delta_{\alpha}^{a} \delta_{\beta}^{\beta} - \delta_{\alpha}^{a} \delta_{\beta}^{\beta} \right) \frac{x_{1}^{a} - x_{2}^{\beta}}{x_{1}^{a} - x_{2}^{\beta}} \frac{x_{1}^{a} - x_{2}^{\beta}}{x_{1}^{a} - x_{2}^{\beta}} - \frac{1}{\mbox{1}} \right]. \\
\mathcal{R}_{\alpha\beta}^{a\beta} &= -e_{ab} e_{cd} \frac{\gamma_{2} U_{2} x_{1}^{a} x_{2}^{\beta}}{x_{1}^{a} x_{2}^{\beta}} - \frac{1}{\mbox{1}} (x_{1}^{a} x_{2}^{\beta} - 1), \\
\mathcal{R}_{\alpha\beta}^{a\beta} &= e_{ab} e_{cd} \frac{\gamma_{2} U_{1} (x_{1}^{a} x_{2}^{\beta} - 1)}{x_{1}^{a} x_{2}^{\beta}} (x_{1}^{a} x_{2}^{\beta} - 1) \frac{1}{\mbox{1}} .
\end{align*}
\] (5.2)

as well as

\[
\begin{align*}
\mathcal{R}_{\alpha\beta}^{a\beta} &= \delta_{\alpha}^{a} \delta_{\beta}^{\beta} \frac{1}{U_{1}} \frac{x_{1}^{a} - x_{2}^{\beta}}{x_{1}^{a} - x_{2}^{\beta}}, & \mathcal{R}_{\alpha\beta}^{a\beta} &= \delta_{\alpha}^{a} \delta_{\beta}^{\beta} \frac{1}{U_{2}} \frac{x_{1}^{a} - x_{2}^{\beta}}{x_{1}^{a} - x_{2}^{\beta}} \frac{1}{\gamma_{1}}, \\
\mathcal{R}_{\alpha\beta}^{a\beta} &= \delta_{\alpha}^{a} \delta_{\beta}^{\beta} \frac{1}{U_{1}} \frac{x_{1}^{a} - x_{2}^{\beta}}{x_{1}^{a} - x_{2}^{\beta}} \frac{1}{\gamma_{1}}, & \mathcal{R}_{\alpha\beta}^{a\beta} &= \delta_{\alpha}^{a} \delta_{\beta}^{\beta} \frac{1}{U_{2}} \frac{x_{1}^{a} - x_{2}^{\beta}}{x_{1}^{a} - x_{2}^{\beta}} \frac{1}{\gamma_{2}}.
\end{align*}
\] (5.3)

The parameters \( x_{1,2} \) are related to the spectral parameters \( u_{1,2} \), respectively in the following way

\[
u = x^{+} + \frac{1}{x^{+}} - \frac{h}{2} = x^{-} + \frac{1}{x^{-}} + \frac{h}{2}, \quad (5.4)
\]

The parameter \( h \) corresponds to the coupling constant and the parameters \( U \) are related to the above by \( U^{2} = x^{+} x^{-} \). Finally, the additional parameter \( \gamma \) defines the relative normalization of bosons \( (\phi) \) and fermions \( (\psi) \). The \( R \)-matrix is the intertwiner of the centrally extended \( su(2|2) \) superalgebra. This algebra contains two \( su(2) \) subalgebras that act on the bosons and fermions respectively. The \( R \)-matrix is a symmetric matrix with respect to an appropriately chosen inner product for the states

\[
\langle \phi^{a} | \phi^{b} \rangle = \delta^{ab}, \quad \langle \psi^{a} | \psi^{b} \rangle = \delta^{ab} \frac{U}{\gamma^{2}} (x^{+} - x^{-}) = \delta^{ab} \frac{x^{+}}{\gamma^{2}} (U - U^{-1}).
\] (5.5)

5.2. Symmetric states

Let us first consider the points where the \( R \)-matrix becomes of rank 8. They correspond to the two special points \( x_{1}^{+} = x_{-}^{-} \) (corresponding to \( u_{1} = u_{2} + h \)) as well as \( x_{1}^{a} = x_{2}^{b} \) (corresponding to \( u_{1} = u_{2} - h \)). In the following we will consider the point \( x_{1}^{+} = x_{2}^{+} \). The considerations for the other point are completely analogous.
This first ingredient we need is the matrix of normalized eigenvectors $E$ of the fundamental $R$-matrix (5.1). There are four bosonic vectors $|S^{(ab)}\rangle$ and $|S^{[34]}\rangle$. Three of them have unit eigenvalue $\lambda_{ab} = 1$ and form a standard triplet under the bosonic $su(2)$

$$|S^{(ab)}\rangle = |\phi_a\rangle \otimes |\phi_b\rangle, \quad |S^{[12]}\rangle = \frac{1}{\sqrt{2}} \left( |\phi_1^1\rangle \otimes |\phi_2^2\rangle + |\phi_2^1\rangle \otimes |\phi_1^2\rangle \right),$$

and the fourth contains two fermionic states and is a singlet under both $su(2)$'s

$$|S^{[34]}\rangle = \frac{\gamma(U_2 U_2^{-1}) \phi^{ab} |\psi_a\rangle \otimes |\psi_b\rangle}{\sqrt{U_1 U_2^{-1} (x_2^+ - x_2^-)(x_1^+ - x_1^-) + \left(1 - \frac{1}{2} \left(U_2^{-2} + U_2^2\right)\right)^2}} + \frac{2}{U_1 U_2} \frac{1 - x_1^- x_2^+}{x_2^+ - x_1^-}. \quad \lambda_{[34]} = \frac{x_2^+ - x_2^-}{1 - x_1^+ x_1^-} + \frac{2}{U_1 U_2} \frac{1 - x_1^- x_2^+}{x_2^+ - x_1^-}. \quad \lambda_{[34]} = \frac{x_2^+ - x_2^-}{1 - x_1^+ x_1^-} + \frac{2}{U_1 U_2} \frac{1 - x_1^- x_2^+}{x_2^+ - x_1^-}.$$

There are four fermionic eigenstates $|S^{(a)}\rangle$

$$|S^{(a)}\rangle = \frac{\gamma(U_2 U_2^{-1}) \phi^{a} |\psi\rangle}{\sqrt{U_2 (x_2^+ - x_2^-) + U_1 U_2^2 (x_1^+ - x_1^-)}} + \frac{1}{U_2} \frac{x_2^+ - x_2^-}{x_2^+ - x_1^-} + \frac{U_1}{U_2} \frac{x_1^- - x_1^+}{x_1^- - x_2^+}.$$

After packing these vectors in the matrix $E$ and taking $F = E^T$, it is readily checked that (2.10) and (2.14) hold.

We are now in a position to apply our fusion procedure. Let us consider the fused matrix (2.15) and make it symmetric by applying the transformation $W$ as in (3.12). It is then straightforward to check that this $R$-matrix coincides with $S^{BA}$ from [9].

As a non-trivial example let us compute two scattering processes

$$R^{BA}(u_{(12)}, u_3)|S^{(11)}\rangle \otimes |\phi^1\rangle, \quad R^{BA}(u_{(12)}, u_3)|S^{[13]}\rangle \otimes |\psi^3\rangle.$$

It is easy to see that

$$R^{BA}(u_{(12)}, u_3)|S^{(11)}\rangle \otimes |\phi^1\rangle = |S^{(11)}\rangle \otimes |\phi^1\rangle,$$

which agrees with $a_1 = 1$ from section 6.1.2 of [9]. This shows that we have the same normalization for the bound state S-matrix. Subsequently, we have by definition

$$R^{BA} |S^{[13]}\rangle \otimes |\psi^3\rangle = W U_{12} R_{13} U_{32} W^{-1} |S^{[13]}\rangle \otimes |\psi^3\rangle$$

$$= \frac{U_2 (x_2^+ - x_2^-) (x_1^+ - x_1^-) (x_1^+ - x_1^-) (x_2^+ - x_2^-)}{(x_1^- - x_1^+)(x_2^- - x_2^+)} + \frac{(x_2^+ - x_2^-) (x_1^+ - x_1^-) (x_2^+ - x_2^-)}{U_1 (x_1^- - x_1^+)(x_2^- - x_2^+)}$$

$$+ \frac{U_2 (x_2^+ - x_2^-) (x_1^- - x_1^+)(x_3^- - x_3^+)}{(x_1^- - x_1^+)(x_3^- - x_3^+)}$$

$$\cdot \frac{U_5}{U_1 U_2} \frac{U_2 (x_2^+ - x_2^-) + U_1 U_2^2 (x_1^- - x_1^+)}{|S^{[13]}\rangle \otimes |\psi^3\rangle} = \frac{U_5}{U_1 U_2} \frac{U_2 (x_2^+ - x_2^-) + U_1 U_2^2 (x_1^- - x_1^+)}{|S^{[13]}\rangle \otimes |\psi^3\rangle}, \quad \lambda_{[13]} = \frac{x_2^+ - x_2^-}{1 - x_1^+ x_1^-} + \frac{2}{U_1 U_2} \frac{1 - x_1^- x_2^+}{x_2^+ - x_1^-}.$$
where we used the inner product (5.5) and $x_1^+ = x_2^-$. This result exactly coincides with the literature, in particular it is the coefficient $a_7$ from section 6.1.2 of [9]. In fact it is not hard to check that the representation (3.2) is exactly the two particle bound state representation from [9].

Let us conclude this section by considering the complementary fusion procedure. The complement is spanned by antisymmetric states $|\Lambda\rangle = (|A\rangle^{a\beta} , |A^{12}\rangle)$. In particular, it is the coefficient $a_7$ from section 6.1.2 of [9]. In fact it is not hard to check that the representation (3.2) is exactly the two particle bound state representation from [9].

For these states, one can then easily compute the complementary $R$-matrix (3.18). In particular, due to the upper triangular structure (3.22), the computation simplifies somewhat.

For instance, let us consider the analogue of the states (5.9)

$$|\Lambda^{(33)}\rangle \otimes |\psi^3\rangle, \quad |\Lambda^{(1)}\rangle \otimes |\phi^1\rangle.$$

The action of $R_{13} R_{23}$ on both states is simply multiplicative and yields

$$R_{13} R_{23} |\Lambda^{(33)}\rangle \otimes |\psi^3\rangle = \frac{x_1^+ - x_2^-}{x_1^+ - x_3^+} U_1, \quad \frac{x_2^+ - x_3^-}{x_2^+ - x_3^+} U_2 |\Lambda^{(33)}\rangle \otimes |\psi^3\rangle,$$

$$R_{13} R_{23} |\Lambda^{(1)}\rangle \otimes |\phi^1\rangle = \frac{x_1^+ - x_2^-}{x_1^+ - x_3^+} U_3, \quad \frac{x_2^+ - x_3^-}{x_2^+ - x_3^+} U_2, \quad \frac{U_1 U_2}{U_3} \frac{x_3^- - x_3^+}{x_3^- - x_3^+} |\Lambda^{(1)}\rangle \otimes |\phi^1\rangle.$$

This then corresponds to the fused $R$-matrix on antisymmetric states corresponding to the other point of lower rank $x_1^- = x_2^+$. Notice that the matrix has a different normalization and the inverse of the coefficient (5.11) appears, which is in agreement with [12] where the relation between the symmetric and antisymmetric $R$-matrices is discussed.

5.3. Singlet state

The $R$-matrix coefficients in (5.2) have a common factor of $(x_1^+ - x_2^-)/(x_1^- x_2^- - 1)$, which has a potential singularity at $x_2^- = 1/x_1^-$. Furthermore, $x_2^- \neq x_1^+$ in order for the numerator to be non-zero, i.e. $x_2^- = 1/x_1^+$. The singularity affects only the $su(2) \times su(2)$ singlet sector spanned by the two states

$$|BB\rangle = \epsilon^{ab} \phi^a \otimes \phi^b, \quad |FF\rangle = \epsilon^{ab} \psi^a \otimes \psi^b.$$

The action on the remaining 14 states is finite. Acting on the singlet, the $R$-matrix reduces to a $2 \times 2$ matrix $M$. In the limit $x_2^+ \rightarrow 1/x_1^-$, it has the following singularity structure

$$M = \frac{1}{\epsilon} M^{(\epsilon)} + M^{(0)} + \cdots.$$ 

Up to an overall factor we find for the residue

$$M^{(\epsilon)} \sim \begin{pmatrix} 1 & -(U_1 - U_1^{-1})/x_1^- \\ U_1 U_2 / (U_1 - U_1^{-1}) & -U_1 U_2 \end{pmatrix}.$$

We know that $U_2^2 = x^+ / x^-$ and hence $U_1 U_2 = \pm 1$. Both values of the latter signs are permitted, and we have to discuss the two cases separately, as they lead to rather distinct behavior. 

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10 Here, we do not explicitly multiply the $R$-matrix by an overall factor to compensate the singularity. Hence we will consider the most singular contributions to the matrix.
We start with $U_1 U_2 = -1$ which is analogous to the cases discussed above. There are two eigenvectors

$$
\left( U_1 - U_1^{-1} \right) |BB\rangle + \gamma_1 \gamma_2 |FF\rangle \quad \text{and} \quad \left( U_1 - U_1^{-1} \right) |BB\rangle - \gamma_1 \gamma_2 |FF\rangle.
$$

(5.18)

The eigenvalues are 2 and 0, respectively, therefore only the former state is singular. It is a singlet of the Yangian algebra, however it has a non-trivial charge $U_{12} = U_1 U_2 = -1$ which plays a role for the coproduct. The other state belongs to an adjoint representation of $\text{psu}(2|2)$.

The singularity structure of this case is peculiar with respect to complementary fusion: In the limit $x_2^+ \to 1/x_1^+$, the eigenvalue of the above non-singular singlet state approaches zero even faster than for the 14 non-singlet states. We find

$$
\lambda_{1a} \sim \frac{1}{e}, \quad \lambda_{14} \sim 1, \quad \lambda_{1b} \sim e.
$$

(5.19)

This means that complementary fusion based on $R_{21}$ produces merely one composite state rather than 15. This state is just the other singlet, and in fact one can see that exchanging the two sites in (5.18) interchanges the two states.

The other case $U_{12} = U_1 U_2 = +1$ has only one eigenvector

$$
\left( U_1 - U_1^{-1} \right) |BB\rangle + \eta \gamma_2 |FF\rangle,
$$

(5.20)

and its eigenvalue is 0. The matrix $M^{(1)}$ does not admit a second eigenvector because it has a non-trivial Jordan decomposition. This makes the case very special, and the fusion procedure described in this paper does not immediately apply. Let us therefore try to understand what is going on: The eigenstate is the singlet state discussed in [4]. There it was shown that the state behaves just like a fused state under scattering with other states, i.e. it preserves its form. To understand the role of the other singlet state, it makes sense to take a closer look at the limit $x_2^+ \to 1/x_1^+$. Here, both singlet eigenvalues remain finite, whereas the eigenvectors become collinear. Therefore, at $x_2^+ = 1/x_1^+$ there is only one meaningful eigenvector, the difference of the eigenvectors plays no significant role. Even though all eigenvalues remain finite at this point, fusion does take place due to the coincidence of eigenvectors.

The fused $R$-matrix $R_{(12)3}$ of the singlet state with $U_{12} = +1$ was shown to be trivial in [4] up to an overall phase factor related to crossing symmetry [13]. Based on this result one can easily derive the fused $R$-matrix of the singlet state with $U_{12} = -1$. The point is that the factor $U_1$ appears in odd powers in (5.2) and (5.3) only when the third index is fermionic. Flipping the sign $U_1$ flips the sign of $R_{(12)3}$ precisely if state 3 is fermionic (5.1). This means that the state with $U_{12} = -1$ behaves like a fermionic singlet, while $U_{12} = +1$ corresponds to a bosonic singlet. This observation is in line with the coproduct rule of odd generators.

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11 The adjoint is reducible but indecomposable, and the former singlet also acts as the top components of this representation.

12 One has to take into account the sign from exchanging fermions and that $U_1 \to -U_1^{-1}$.
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References

[1] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)

[2] Karowski M 1979 On the bound state problem in (1+1)-dimensional field theories *Nucl. Phys.* B 153 244

Kulish P P, Reshetikhin N Yu and Sklyanin E K 1981 Yang–Baxter equation and representation theory: I *Lett. Math. Phys.* 5 393

Jimbo M 1985 A q difference analog of U(g) and the Yang–Baxter equation *Lett. Math. Phys.* 10 63

Jimbo M 1986 A q Analog of U(gl(n + 1)), Hecke algebra and the Yang–Baxter equation *Lett. Math. Phys.* 11 247

Kirillov A N and Reshetikhin N Yu 1987 Exact solution of the integrable XXZ Heisenberg model with arbitrary spin: I. The ground state and the excitation spectrum *J. Phys. A: Math. Gen.* 20 1565

Mezincescu L and Nepomechie R I 1992 Fusion procedure for open chains *J. Phys. A: Math. Gen.* 25 2533

[3] Beisert N et al 2012 Review of AdS/CFT integrability: an overview *Lett. Math. Phys.* 99 3

[4] Beisert N 2008 The SU(2/2) dynamic s-matrix *Adv. Theor. Math. Phys.* 12 945

[5] Arutyunov G, Frolov S, Plefka J and Zamaklar M 2007 The off-shell symmetry algebra of the light-cone AdS$_5 \times S^5$ *J. Phys. A: Math. Theor.* 40 3583

Beisert N 2006 The S-matrix of AdS / CFT and Yangian symmetry *PoS SOLVAY 002*

[6] Shastry B S 1988 Decorated star-triangle relations and exact integrability of the one-dimensional Hubbard model *J. Stat. Phys.* 50 57

Beisert N 2007 The analytic bethe ansatz for a chain with centrally extended su(2/2) symmetry *J. Stat. Mech.* P01017

Martins M J and Melo C S 2007 The Bethe ansatz approach for factorizable centrally extended s-matrices *Nucl. Phys.* B 785 246

[8] Dorey N 2006 Magmon bound states and the AdS/CFT correspondence *J. Phys. A: Math. Gen.* 39 13119

Chen H-Y, Dorey N and Okamura K 2006 Dyonic giant magnons *J. High Energy Phys.* JHEP09 (2006)024

Arutyunov G and Frolov S 2007 On string s-matrix, bound states and TBA *J. High Energy Phys.* JHEP12(2007)024

[9] Arutyunov G and Frolov S 2008 The s-matrix of string bound states *Nucl. Phys.* B 804 90

[10] de Leeuw M 2008 Bound states, yangian symmetry and classical r-matrix for the AdS$_3 \times S^3$ superstring *J. High Energy Phys.* JHEP06(2008)085

Arutyunov G, de Leeuw M and Torrielli A 2009 The bound state s-Matrix for AdS$_3 \times S^3$ superstring *Nucl. Phys.* B 819 319

[11] Arutyunov G, Frolov S and Zamaklar M 2007 The Zamolodchikov–Faddeev algebra for AdS$_3 \times S^3$ superstring *J. High Energy Phys.* JHEP04(2007)002

[12] Bajnok Z and Janik R A 2009 Four-loop perturbative Konishi from strings and finite size effects for multiparticle states *Nucl. Phys.* B 807 625

[13] Janik R A 2006 The AdS$_3 \times S^3$ superstring worldsheet S-matrix and crossing symmetry *Phys. Rev.* D 73 086006

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