Around some extensions of Casas-Alvero conjecture for non-polynomial functions

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Abstract: We show that two natural extensions of the real Casas-Alvero conjecture in the non-polynomial setting do not hold.

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1. Introduction

The Casas-Alvero conjecture affirms that if a complex polynomial $P$ of degree $n > 1$ shares roots with all its derivatives, $P^{(k)}$, $k = 1, 2, \ldots, n - 1$, then there exist two complex numbers, $a$ and $b \neq 0$, such that $P(z) = b(z - a)^n$. Notice that, in principle, the common root between $P$ and each $P^{(k)}$ might depend on $k$. Casas-Alvero arrived to this problem at the turn of this century, when he was working in his paper [1] trying to obtain an irreducibility criterion for two variable power series with complex coefficients. See [2] for an explanation of the problem in his own words.

Although several authors have got partial answers, to the best of our knowledge the conjecture remains open. For $n \leq 4$ the conjecture is a simple consequence of the wonderful Gauss-Lucas Theorem ([6]). In 2006 it was proved in [3], by using Maple, that it is true for $n \leq 8$. Afterwards in [6, 7] it was proved that it holds when $n$ is $p^m$, $2p^m$, $3p^m$ or $4p^m$, for some prime number $p$ and $m \in \mathbb{N}$. The first cases left open are those where $n = 24, 28$ or $30$. See again [6] for a very interesting survey or [3, 8] for some recent contributions on this question.

Adding the hypotheses that $P$ is a real polynomial and all its $n$ roots, taking into account their multiplicities, are real, the conjecture has a real
counterpart, that also remains open. It says that \( P(x) = b(x - a)^n \) for some real numbers \( a \) and \( b \neq 0 \). For this real case, the conjecture can be proved easily for \( n \leq 4 \), simply by using Rolle’s Theorem. This tool does not suffice for \( n \geq 5 \), see for instance [4] for more details, or next section.

Also in the real case, in [6] it is proved that if the condition for one of the derivatives of \( P \) is removed, then there exist polynomials satisfying the remaining \( n - 2 \) conditions, different from \( b(x - a)^n \). The construction of some of these polynomials presented in that paper is very nice and is a consequence of the Brouwer’s fixed point Theorem in a suitable context.

Finally, it is known that if the conjecture holds in \( \mathbb{C} \), then it is true over all fields of characteristic 0. On the other hand, it is not true over all fields of characteristic \( p \), see again [7]. For instance, consider \( P(x) = x^2(x^2 + 1) \) in characteristic 5 with roots 0, 0, 2 and 3. Then \( P'(x) = 2x(2x^2 + 1) \), \( P''(x) = 12x^2 + 2 = 2(x^2 + 1) \) and \( P'''(x) = 4x \) and all them share roots with \( P \).

The aim of this note is to present two natural extensions of the real Casas-Alvero conjecture to smooth functions and show that none of them holds.

**Question 1.** Fix \( 1 < n \in \mathbb{N} \). Let \( F \) be a class \( C^n \) real function such that \( F^{(n)}(x) \neq 0 \) for all \( x \in \mathbb{R} \), and has \( n \) real zeroes, taking into account their multiplicities. Assume that \( F \) shares zeroes with all its derivatives, \( F^{(k)} \), \( k = 1, 2, \ldots, n - 1 \). Is it true that \( F(x) = b(f(x))^n \) for some \( 0 \neq b \in \mathbb{R} \) and some \( f \), a class \( C^n \) real function, that has exactly one simple zero?

Notice that one of the hypotheses of the real Casas-Alvero conjecture can be reformulated as follows: The polynomial \( F \) shares roots with all its derivatives but one, precisely the one corresponding to its degree. Trivially, this is so, because all the derivatives of order higher than \( n \) are identically zero. The second question that we consider is:

**Question 2.** Fix \( 1 < n \in \mathbb{N} \). Let \( F \) be a real analytic function that shares zeroes with all its derivatives but one, say \( F^{(n)} \). Is it true that \( F(x) = b(f(x))^n \) for some \( 0 \neq b \in \mathbb{R} \) and some real analytic function \( f \), that has exactly one simple zero?

**Theorem A.**
(i) The answer to the Question 1 is “yes” for \( n \leq 4 \) and “no” for \( n = 5 \).

(ii) The answer to the Question 2 is already “no” for \( n = 2 \).

Our result reinforces the intuitive idea that Casas-Alvero conjecture is mainly a question related with the rigid structure of the polynomials.
2. Proof of Theorem A

(i) The answer to Question 1 is “yes” for \(n = 2, 3, 4\) because the proof of the real Casas-Alvero conjecture for the same values of \(n\), based on the Rolle’s Theorem and given in [4], does not use at all that \(P\) is a polynomial. Let us adapt it to our setting. Since \(F^{(n)}\) does not vanish we know that \(F\) has exactly \(n\) real zeroes, taking into account their multiplicities. Moreover we know that \(F\) has to have at least a double zero, that without loss of generality can be taken as 0. Next we can do a case by case study to discard all situations except that \(F\) has only a zero and it is of multiplicity \(n\). For the sake of brevity, we give all the details only in the most difficult case, \(n = 4\).

Assume, to arrive to a contradiction, that \(n = 4\), \(F\) is under the hypotheses of Question 1 and \(x = 0\) is not a zero of multiplicity four. Notice that by Rolle’s theorem, for \(k = 1, 2, 3\), each \(F^{(k)}\) has exactly \(4 - k\) zeroes, taking into account their multiplicities. Moreover, the only zero of \(F''\) must be one of the zeroes of \(F'\). Moreover, the only zero of \(F''\) must be one of the zeroes of \(F\).

If \(F''(0) = 0\) and \(F'''(0) \neq 0\) then \(F\) has only another zero at \(x = a\) and, without loss of generality, we can assume that \(a > 0\). Applying three times Rolle’s theorem we get that \(F'''(b) = 0\) for some \(b \in (0, a)\) which is a contradiction with the hypotheses. If \(F''(0) \neq 0\) then \(F\) has two more zeroes counting multiplicities. There are three possibilities. The first one is that there is \(a > 0\) such that \(F(a) = F'(a) = 0\). In this case, applying two times Rolle’s theorem we obtain that there exist \(b, c \in (0, a)\) with \(F''(b) = F''(c) = 0\) and they are the only zeroes of \(F''\). This fact gives again a contradiction because none of them is a zero of \(F\). The second one is that there exist \(a_1, a_2 \in \mathbb{R}\) with \(0 \in (a_1, a_2)\) such that \(F(a_1) = F(a_2) = 0\). Also in this case, by applying two times Rolle’s theorem we obtain that there exist \(b, c \in (a_1, a_2)\) such that \(0 \in (b, c)\) and \(F''(b) = F''(c) = 0\) giving us the desired contradiction. Lastly, assume that the other two zeroes of \(F\) are \(a_1\) and \(a_2\), with \(0 < a_1 < a_2\). By Rolle’s Theorem the zeroes of \(F'\) are \(0, b_1\) and \(b_2\) and satisfy \(0 < b_1 < a_1 < b_2 < a_2\). Then, since \(F''\) has to have two zeroes, say \(c_1, c_2\), and they satisfy \(0 < c_1 < b_1 < c_2 < b_2\), the hypotheses force that \(c_2 = a_1\). Hence the zero of \(F''\) has to be between \(c_1\) and \(c_2 = a_1\), that is in particular in \((0, a_1)\), interval that contains no zero of \(F\), arriving once more to the desired contradiction.

In short, we have proved for \(n \leq 4\), that \(F(x) = x^n G(x)\), for some class \(C^n\) function \(G\), that does not vanish. Hence

\[
F(x) = \text{sign}(G(0)) \left( x + \sqrt[4]{\frac{G(x)}{\text{sign}(G(0))}} \right)^n = b(f(x))^n,
\]
where \( f \) has only one zero, \( x = 0 \), that is simple, as we wanted to prove.

To find a map \( F \) for which the answer to Question 1 is “no” we consider \( n = 5 \) and a configuration of zeroes of \( F \) and its derivatives proposed in \([4]\) as the simplest one, compatible with the hypotheses of the Casas-Alvero conjecture and Rolle’s Theorem. Specifically, we will search for a function \( F \), of class at least \( C^5 \), with the five zeroes 0, 0, 1, \( c \), \( d \), to be fixed, satisfying

\[
0 < 1 < c < d,
\]

and moreover

\[
F'(0) = 0, \quad F''(1) = 0, \quad F'''(c) = 0, \quad F^{(4)}(1) = 0, \tag{2.1}
\]

and such that \( F^{(5)} \) does not vanish. Notice that \( F'(0) = 0 \) is not a new restriction.

We start assuming that \( F^{(5)}(x) = r - \sin(x) \), for some \( r > 1 \) to be determined. By imposing that conditions (2.1) hold, together with \( F(0) = 0 \), we get that

\[
F(x) = \int_0^x \int_0^u \int_0^w \int_0^z \int_0^y (r - \sin(t)) \, dt \, dy \, dz \, dw \, du.
\]

Some straightforward computations give that

\[
F(x) = \frac{r}{120} x^5 - \frac{r + \cos(1)}{12} x^4 + \frac{2rc - 2\sin(c) + 2\cos(1)c - rc^2}{12} x^3
\]

\[
+ \frac{6\sin(c) + 2r + 9\cos(1) - 6rc + 3rc^2 - 6\cos(1)c}{12} x^2 - 1 + \cos(x).
\]

Imposing now that \( F(1) = 0 \) we obtain that

\[
r = \frac{5(8\cos(1)c - 41\cos(1) - 8\sin(c) + 24)}{4(5c^2 - 10c + 4)} = R(c).
\]

Next we have to impose that \( F(c) = 0 \). By replacing the above expression of \( r \) in \( F \) we obtain that

\[
F(c) = \frac{G(c)}{96(5c^2 - 10c + 4)},
\]

where

\[
G(c) = -c^2 (12 c^4 - 369 c^3 + 1437 c^2 - 1708 c + 532) \cos(1)
\]

\[
- 8 c^2 (c - 1) (c - 2)^2 \sin(c) + (480 c^2 - 960 c + 384) \cos(c)
\]

\[
- 24 (c - 1) (9 c^4 - 36 c^3 + 24 c^2 + 24 c - 16).
\]
A carefully study shows that $G$ has exactly one real zero $c_1 \in (17/10, 19/10) = I$, with $c_1 \approx 1.79343096$. To prove its existence it suffices to show that

$$G\left(\frac{17}{10}\right) = -\frac{99211099}{500000} \cos\left(\frac{17}{10}\right) - \frac{18207}{12500} \sin\left(\frac{17}{10}\right) + \frac{696}{5} \cos\left(\frac{17}{10}\right) + \frac{1583211}{12500} > 0,$$

$$G\left(\frac{19}{10}\right) = -\frac{180110481}{500000} \cos\left(\frac{19}{10}\right) - \frac{3249}{12500} \sin\left(\frac{19}{10}\right) + \frac{1464}{5} \cos\left(\frac{19}{10}\right) + \frac{3616677}{12500} < 0.$$

By using Taylor’s formula we know that for any $c > 0$, $S^-(c) < \sin(c) < S^+(c)$ and $C^-(c) < \cos(c) < C^+(c)$ where

$$S^\pm(c) = c - \frac{c^3}{3!} + \frac{c^5}{5!} - \frac{c^7}{7!} + \frac{c^9}{9!} \pm \frac{c^{11}}{11!}$$

and

$$C^\pm(c) = 1 - \frac{c^2}{2!} + \frac{c^4}{4!} - \frac{c^6}{6!} + \frac{c^8}{8!} \pm \frac{c^{10}}{10!}.$$

Hence we can replace the values of the trigonometric functions in $G$ by rational numbers to have upper or lower bounds of this function evaluated at $1, 17/10$ or $19/10$. For instance,

$$0.5403023 \approx \frac{1960649}{3628800} = C^-(1) < \cos(1) < C^+(1) = \frac{280093}{518400} \approx 0.5403028.$$

We obtain that

$$G\left(\frac{17}{10}\right) > -\frac{99211099}{500000} C^+(1) - \frac{18207}{12500} S^+\left(\frac{17}{10}\right) + \frac{696}{5} C^-\left(\frac{17}{10}\right) = \frac{3444600099561969856969}{49896000000000000000000} > 0$$

and

$$G\left(\frac{19}{10}\right) < -\frac{180110481}{500000} C^-(1) - \frac{3249}{12500} S^-\left(\frac{19}{10}\right) + \frac{1464}{5} C^+\left(\frac{19}{10}\right) = -\frac{1689627895469649855823}{16632000000000000000000} < 0.$$
To show the uniqueness of the zero in $I$, we will prove that $G$ is strictly decreasing in this interval. It holds that

$$G'(c) = T(c) \cos(c) + U(c) \sin(c) + V(c \cos(c)) + W(c),$$

with

$$T(c) = -c \left(72c^4 - 1845c^3 + 5748c^2 - 5124c + 1064\right),$$
$$U(c) = -8 (5c^2 - 10c + 4) (c^2 - 2c + 12),$$
$$V(c) = -8 (c - 1) (c^4 - 4c^3 + 4c^2 - 120),$$
$$W(c) = -120 (9c^4 - 36c^3 + 36c^2 - 8).$$

By computing the Sturm sequences of $T$, $U$ and $V$ we can prove that $T(c) < 0$, $U(c) < 0$ and $V(c) > 0$ for all $c \in I$. Hence, for $c \in I$,

$$G'(c) < T(c)C^-(c) + U(c)S^-(c) + V(c)C^+(c) + W(c) = Q(c),$$

where

$$Q(c) = \frac{72469}{64800} c - \frac{699211}{43200} c^2 + \frac{18852329}{302400} c^3 - \frac{8854991}{80640} c^4$$
$$+ \frac{4732471}{50400} c^5 - \frac{532}{15} c^6 + \frac{8}{7} c^7 + \frac{191}{70} c^8$$
$$- \frac{797}{1890} c^9 - \frac{34}{405} c^{10} + \frac{1651}{103950} c^{11} + \frac{3533}{2494800} c^{12}$$
$$- \frac{193}{623700} c^{13} + \frac{1}{142560} c^{14} - \frac{1}{831600} c^{15}. $$

The Sturm sequence of $Q$ shows that it has no zeroes in $I$. Moreover, it is negative in this interval, and as a consequence, $G'$ is also negative, as we wanted to prove.

We fix $c = c_1$. Then, $r = R(c_1)$ and $F$ is also totally fixed. Moreover, by using the same techniques we get that $r = R(c_1) > R(19/10) > 1$ and as a consequence $F^{(5)}$ does not vanish. In fact, $r = R(c_1) \approx 1.04591089$. Finally, $F$ has one more real zero $d \in (33/10, 34/10)$. In fact, $d \approx 3.32178369$. This $F$ gives our desired example, see Figure 1.

(ii) Consider $F(x) = 4x^2 + \pi^2 (\cos(x) - 1)$ that has a double zero at 0 and also vanishes at $\pm \pi/2$. Moreover, $F'(x) = 8x - \pi^2 \sin(x)$ vanishes at $x = 0$, $F''(x) = 8 - \pi^2 \cos(x)$ has no common zeroes with $F$ and, for any $k > 1$,
$|F^{(2k)}(x)| = \pi^2|\cos(x)|$ vanishes at $x = \pi/2$ and $|F^{(2k-1)}(x)| = \pi^2|\sin(x)|$ vanishes at $x = 0$.

A similar example for $n = 3$ is $F(x) = 4x^3 - 6\pi x^2 + \pi^3(1 - \cos(x))$, that vanishes at $0, \pi$ (double zeroes) and $\pi/2$.

Figure 1: Plot of a map $F$ for which the answer to Question 1 for $n = 5$ is “no”.

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