Systematic construction of F-theory models with

\[ U(1) \times \mathbb{Z}_2, \mathbb{Z}_4 \]

and transitions in discrete gauge groups

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Abstract

The systematic construction of families of F-theory models in which \( U(1) \times \mathbb{Z}_2 \) and \( U(1) \times \mathbb{Z}_4 \) forms are discussed, motivated by the proposal in the previous paper stating that a discrete \( \mathbb{Z}_n \) gauge group tends to be enhanced to, for example, \( U(1) \times \mathbb{Z}_n \), in the moduli of multisection geometry, where multisections are split into multisections of smaller degrees. We observed that a discrete \( \mathbb{Z}_2 \) gauge group is enhanced to \( U(1) \times \mathbb{Z}_2 \) along certain bisection geometries in four-section geometry. We also present an example of a family of models with \( U(1) \times \mathbb{Z}_4 \) gauge group as a demonstration of the systematic construction.
1 Introduction

F-theory [1, 2, 3] is compactified on manifolds that admit a torus fibration. Axiodilaton in type IIB superstrings and the modular parameter of elliptic curves as fibers of the torus fibration are identified in F-theory, enabling the axiodilaton to exhibit $SL_2(\mathbb{Z})$ monodromy.

In recent years, F-theory compactifications on genus-one fibrations without a global section have attracted interest, for reasons including the discrete gauge group arising in this type of compactification [19] of F-theory. A global section, geometrically, is a copy of the base space embedded in the total space of the genus-one fibration. When one chooses a point in each fiber and if one can move the chosen point throughout over the base, one obtains this copy, yielding a global section. This is possible precisely when the genus-one fibration has a global section.

There are situations in which a genus-one fibration has a global section and in which it does not have a global section; when a genus-one fibration does not have a global section, a discrete gauge group forms in F-theory on this fibration, as mentioned. Recent discussions of F-theory on genus-one fibrations without a global section can be found, for example, in [20, 19, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45].

When a genus-one fibration has a global section (in which case, the fibration is often called an elliptic fibration in the F-theory literature), the $U(1)$ gauge group forms in F-theory if the fibration has two or more independent global sections. The global sections of an elliptic fibration form a group, known as the Mordell–Weil group, which has the notion of “rank.”

\[1\] See, e.g., [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18] for recent progress of discrete gauge groups.

\[2\] [46, 47] discussed F-theory on genus-one fibrations without a global section.

\[3\] See, e.g., [28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49] for discussions of F-theory on elliptic fibrations with a global section.
The rank of the Mordell–Weil group is given by one less the number of independent global sections. The rank of the Mordell–Weil group gives the number of $U(1)$s arising in F-theory on the elliptic fibration [3].

A genus-one fibration lacking a global section still has a “multisection,” and whereas a global section (which can be seen as a “horizontal” divisor) intersects a fiber (which can be seen as a “vertical” divisor) at one point, a multisection of “degree $n$,” or more concisely, an “$n$-section,” intersects a fiber in $n$ points. A discrete $\mathbb{Z}_n$ gauge group forms in F-theory on a genus-one fibration with an $n$-section [19].

In the moduli of the $n$-section, an $n$-section deforms to split into $n$ sheets of separate global sections. It was argued in [19] that the physical viewpoint of this process reverses the geometric order and can be interpreted as a Higgsing process wherein $U(1)^{n-1}$ breaks down into a discrete $\mathbb{Z}_n$ gauge group.

However, there are various other manners in which a multisection splits into multisections of smaller degrees in the moduli of multisection geometry. When studying these, physically unnatural phenomena are identified [41, 45]. Under certain conditions, a four-section splits into a pair of bisections [34, 44]. When one considers the process [44] wherein a four-section splits into a pair of bisections, and these bisections further split into four global sections, when seen from the physical viewpoint, $U(1)^3$ breaks into a discrete $\mathbb{Z}_2$ gauge group, and this discrete $\mathbb{Z}_2$ gauge group transitions further to a discrete $\mathbb{Z}_4$ gauge group via Higgsing. It was pointed out in [44] that this process appears unnatural because a discrete $\mathbb{Z}_2$ gauge group appears “enhanced” to a discrete $\mathbb{Z}_4$ gauge group, rather than broken down into another discrete gauge group of a smaller degree. In contrast, another puzzling physical phenomenon was observed in [45], wherein an $n$-section (with $n \geq 3$) splits into a global section and an $(n-1)$-section. This process can be viewed from the physical viewpoint as a Higgsing process wherein an F-theory model without a discrete or $U(1)$ gauge group transitions to another model with a discrete $\mathbb{Z}_n$ gauge group, and this also appears puzzling [45].

It was proposed in [45] that if one interprets a gauge group to be enhanced to a larger gauge symmetry at the points in the multisection geometry where a multisection splits into multisections of smaller degrees, these apparently puzzling phenomena can be naturally explained. This proposal includes a simple example in which a discrete $\mathbb{Z}_2$ gauge group is enhanced to $U(1) \times \mathbb{Z}_2$ [45]. As shown in [45], along the bisection geometries locus in a four-section geometry, one can always find a deformation of a genus-one fibration with a bisection such that it acquires the enhanced gauge group $U(1) \times \mathbb{Z}_2$ after a certain deformation.

Based on these physical motivations, points in the moduli of multisection geometry on

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4When F-theory is compactified on a Calabi–Yau genus-one fibration $Y$, the discrete gauge group arising in this compactification is identified with the discrete part of the “Tate–Shafarevich group,” $\text{III}(J(Y))$, of the Jacobian, $J(Y)$ [47, 19].

5See also, e.g., [22, 77] for discussions of F-theory models in which both $U(1)$ and a discrete gauge symmetry form.
which both $U(1)$ and discrete gauge groups form simultaneously are relevant to the physically puzzling phenomena noted in [44, 45], and studying models in which both $U(1)$ and discrete gauge groups form can be useful to providing resolutions to these puzzles. Motivated by this background, the aim of this note is to construct families of F-theory models in which $U(1) \times \mathbb{Z}_2$ and $U(1) \times \mathbb{Z}_4$ form.

$U(1)$ arises along a certain bisection geometries locus in the moduli of four-section geometry where a four-section splits into bisections, as we will see in Section 3.1 via mathematical analysis. That is, we observe an enlargement of a discrete gauge group along this bisection geometries locus. This observation can support, at least to some degree, the proposal in [45] to resolve the puzzle pointed out in [44] because the enhancement of a discrete gauge group to include $U(1)$ is observed along a locus where a four-section is split into a pair of bisections.

The reader may wonder, because a discrete gauge group arises on genus-one fibrations without a section and $U(1)$ arises when the fibration has two or more global sections, whether construction of F-theory models in which both a discrete gauge group and $U(1)$ simultaneously arise is possible; however, such construction is actually possible. Even when a genus-one fibration does not have a global section, one can consider the “Jacobian fibration” whose types of singular fibers and the discriminant locus are identical to those of the original genus-one fibration. The number of $U(1)$ arising in F-theory on a genus-one fibration lacking a global section is given by the Mordell–Weil rank of its Jacobian fibration [20]. Even when an original genus-one fibration does not admit a global section, if the Jacobian has a positive Mordell–Weil rank, specifically, if it has at least two independent global sections, $U(1)$ arises in F-theory on that genus-one fibration.

Generically, however, the Mordell–Weil rank of the Jacobian is zero, and the rank increases when the parameters, as coefficients of the Weierstrass equation of the Jacobian, assume special values. In other words, to serve our purpose, we need to clear two obstacles: first, we must analyze the moduli of genus-one fibrations with an $n$-section of fixed degree, and second, we must also find members of them of which the Jacobians have a positive Mordell–Weil rank. By making use of mathematical tricks, we resolve these issues simultaneously.

The trick we use in this note is to consider complete intersections of two quadrics in $\mathbb{P}^3$ fibered over any base to realize Calabi–Yau four-section geometry. This approach has several advantages and enables us to systematically construct families of F-theory models in which both $U(1)$ and a discrete $\mathbb{Z}_2$ gauge group, and both $U(1)$ and a discrete $\mathbb{Z}_4$ gauge group, arise. One of the advantages of this approach is that the four-section geometry realized in this fashion contains a bisection geometries locus [34, 44, 45]. Owing to this, we can analyze both four-section geometry and bisection geometry simultaneously. Another advantage is that one can construct the associated double cover of four-section geometry realized as a complete intersection. It can be determined when the Mordell–Weil rank of the Jacobian increases to

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6The relation of the moduli of genus-one fibrations and those of the Jacobian fibrations and Weierstrass models was discussed in [19] in the context of F-theory.
7Construction of the Jacobians of elliptic curves is discussed in [78].
one by studying the coefficients of this associated double cover. This will be discussed in Section 2.1.

In addition to these constructions, we also would like to point out that F-theory models of bisection geometries studied in [37] contain models with $U(1)$. Therefore, both $U(1)$ and a discrete $\mathbb{Z}_2$ gauge group arise in such models. This will be discussed in Section 3.3.

Recent model buildings of F-theory emphasized the use of local model buildings [79, 80, 81, 82]. Global aspects of models, however, need to be studied to address the issues of early universe including inflation and the issues of gravity. The compactification geometries are analyzed from the global perspective here.

This study is structured as follows: we describe the strategy to systematically construct families of F-theory models in which $U(1) \times \mathbb{Z}_2$ and $U(1) \times \mathbb{Z}_4$ form in Section 2.1. We also briefly review the construction of the Jacobian of complete intersection of two quadrics in $\mathbb{P}^3$ fibered over a base in Section 2.2. In Sections 3.1 and 3.2, we discuss some details of the systematic constructions of models with $U(1) \times \mathbb{Z}_2$ and $U(1) \times \mathbb{Z}_4$ gauge groups. We also find that the F-theory models with a discrete $\mathbb{Z}_2$ gauge group constructed in [37] include models in which $U(1)$ also forms. We discuss these models in Section 3.3. We state our concluding remarks and some open problems in Section 4.

2 Strategies to systematically construct models with $U(1) \times \mathbb{Z}_2$ and $U(1) \times \mathbb{Z}_4$ gauge groups

2.1 Strategies of systematic construction

As pointed out in [44, 45], physically puzzling phenomena can be observed when some splitting processes in the multisection geometry are analyzed. It was proposed in [45] that an interpretation that a (discrete) gauge group tends to enlarge at such points in the moduli where multisections split into multisections of smaller degrees, which can naturally explain the puzzling phenomena. Given these backgrounds, as noted in the introduction, the points in the moduli of multisection geometry where a discrete $\mathbb{Z}_n$ gauge group is enhanced to $U(1) \times \mathbb{Z}_n$ relates to a possible solution of the puzzles proposed in [45], and analyzing these points can be physically interesting in this context.

We introduce in this study systematic ways to construct F-theory models with $U(1) \times \mathbb{Z}_2$ and $U(1) \times \mathbb{Z}_4$ gauge groups. We describe the strategies of systematic constructions of these models here. Some details of these constructions will be discussed in Sections 3.1 and 3.2.

Models with $U(1) \times \mathbb{Z}_2$ gauge group relate to the puzzle pointed out in [44] and to a possible solution of this as proposed in [45]. We actually find in Section 3.1 that along

\footnote{[23] also discussed a method to construct the Jacobian fibrations of genus-one fibrations built as the complete intersections. We take a different approach here.}
a certain bisection geometries locus in the four-section geometry, $U(1) \times \mathbb{Z}_2$ forms and this observation can, to some degree, support the proposal [45] of the puzzle raised in [44]. Models with $U(1) \times \mathbb{Z}_4$ can be relevant to the situation where a multisection splits into multisections including a four-section.

The constructions we describe here and in Sections 3.1 and 3.2 do not depend on the dimension of the space. Our argument particularly applies both to six-dimensional (6D) and four-dimensional (4D) F-theory models, at least at the geometrical level. However, when one considers four-dimensional F-theory models, the issues of flux [83, 84, 85, 86, 87], including the effect of the superpotential that it generates, also need to be considered [19]. We do not discuss the effects of flux in this study. To this end, we mainly focus on 6D F-theory models in most of this note. (However, the models we consider in Section 3.3 are 4D and eight-dimensional (8D).)

In this study, particularly in this section and in Sections 3.1 and 3.2, we consider, as four-section geometry [26, 34, 44], the complete intersection of two quadric hypersurfaces in $\mathbb{P}^3$ fibered over any base space. The general form of this type of complete intersection is given by the following equation:

\[
\begin{align*}
\alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 + \alpha_4 x_4^2 \\
+ 2\alpha_5 x_1 x_2 + 2\alpha_6 x_1 x_3 + 2\alpha_7 x_1 x_4 + 2\alpha_8 x_2 x_3 + 2\alpha_9 x_2 x_4 + 2\alpha_{10} x_3 x_4 &= 0 \\
\beta_1 x_1^2 + \beta_2 x_2^2 + \beta_3 x_3^2 + \beta_4 x_4^2 \\
+ 2\beta_5 x_1 x_2 + 2\beta_6 x_1 x_3 + 2\beta_7 x_1 x_4 + 2\beta_8 x_2 x_3 + 2\beta_9 x_2 x_4 + 2\beta_{10} x_3 x_4 &= 0.
\end{align*}
\]

$[x_1 : x_2 : x_3 : x_4]$ gives the coordinates of $\mathbb{P}^3$, and $\alpha_i$ and $\beta_j$, $i, j = 1, \ldots, 10$, are sections of line bundles over the base. For these complete intersections, a method to construct the Jacobian fibration is known [20, 44]. This method is described in [20, 44], and we briefly review this in Section 2.2. The construction of the Jacobian consists of two steps: one can build the associated double cover of a quartic polynomial from the complete intersection, and this double cover is generally a bisection geometry [20, 19]. The associated double cover can be expressed as follows:

\[
\tau^2 = e_0 \lambda^4 + e_1 \lambda^3 + e_2 \lambda^2 + e_3 \lambda + e_4.
\]

The Jacobian fibration of the double cover is known [20, 19], and the resulting Jacobian of the double cover yields the Jacobian fibration of the original complete intersection.

The following fact is crucial to the systematic constructions of models in which both $U(1)$ and a discrete $\mathbb{Z}_2$ or $\mathbb{Z}_4$ gauge group form: when either the constant term $e_4$ or the coefficient

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9Recent progress of F-theory compactifications with four-form flux can be found, for example, in [88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 28, 100].

10Genus-one fibration structures of 3-folds are analyzed in [101, 102, 103].

11These are subject to the conditions, so when the Jacobian fibration is taken, the Weierstrass form of which is $y^2 = x^3 + f x + g$, then $[f] = -4K$, $[g] = -6K$, to ensure that the total space yields the Calabi–Yau genus-one fibration, as described in [44]. ($K$ denotes the canonical divisor of the base space.)
$e_0$ of $\lambda^4$ of the double cover is a perfect square, the double cover admits two global sections \[19\]. In this case, the double cover yields the Jacobian of the complete intersection and one does not need to compute the Jacobian of the double cover. When the Jacobian has two global sections, its Mordell–Weil rank is one. Making use of this fact of the associated double cover leads us to the following strategy: when we choose the specific coefficients of the complete intersection \[1\] such that the associated double cover has either $e_0$ or $e_4$ as a perfect square, then we obtain four-section geometries whose Jacobian fibrations have Mordell–Weil rank one. $U(1) \times \mathbb{Z}_4$ forms in F-theory on this family.

Along a certain locus in the complete intersection \[1\], a four-section splits into a pair of bisections \[34, 44\]. The bisection geometries locus is given by the following conditions \[44\]:

$$
\begin{align*}
a_3 &= 1 \\
a_8 = a_{10} &= 0 \\
b_4 &= b_2 \\
b_3 = b_8 = b_9 = b_{10} &= 0.
\end{align*}
$$

Further, by inspection it can be confirmed that the bisection geometries locus \[3\] is in fact contained in a larger bisection geometries locus, given by the following condition:

$$
\begin{align*}
b_4 &= b_2 \\
b_3 = b_8 = b_9 = b_{10} &= 0.
\end{align*}
$$

A four-section still splits into a pair of bisections along the locus \[4\]. (Some detailed explanation of this will be provided in section \[3.1\].) Owing to this fact, we rather use here and in Section \[3.1\] the bisection geometries locus \[4\], which contains the bisection geometries locus \[3\].

It turns out that the associated double covers along this bisection geometries locus have the coefficients $e_0$ of $\lambda^4$ perfect squares. Therefore, $U(1) \times \mathbb{Z}_2$ forms along this locus. This will be discussed in Section \[3.1\]. As noted in the introduction, this result can support to some degree the interpretation proposed in \[45\] to possibly resolve the puzzle in \[44\], as an enlargement of a discrete $\mathbb{Z}_2$ gauge group can be seen along this locus. (Whether similar enlargements occur along other bisection geometries is open, as will be mentioned at the end of this note.)

We provide in Section \[3.2\] an example of a family on which $U(1) \times \mathbb{Z}_4$ forms in F-theory, as an explicit demonstration of our strategy that we described to systematically construct families on which $U(1) \times \mathbb{Z}_4$ gauge group forms. We give a brief summary of the results in Section \[3.2\] here. The details can be found in Section \[3.2\]. If we consider the following complete intersections:

$$
\begin{align*}
a_1 x_1^2 + a_2 x_2^2 + a_1 x_3^2 + a_2 x_4^2 + 2a_2 x_1 x_3 + 2a_1 x_2 x_4 &= 0 \\
b_1 x_1^2 + b_2 x_2^2 + b_3 x_3^2 + b_4 x_4^2 + 2b_6 x_1 x_3 + 2b_9 x_2 x_4 &= 0,
\end{align*}
$$

\[5\]
they yield four-section geometry, and the associated double covers have constant terms $e_4$ that are perfect squares. Thus, $U(1) \times \mathbb{Z}_4$ forms on the complete intersections (5).

In Sections 3.1 and 3.2, the conditions when $U(1)$ is further enhanced to $SU(2)$ are also mentioned, along the lines of the arguments as in [19, 44].

2.2 Review of the construction of the Jacobian

We briefly review the construction of the Jacobian [20, 44] of the complete intersection of two quadrics in $\mathbb{P}^3$ fibered over a base. We set this section as preliminary for the computations in Sections 3.1 and 3.2. This section can be skipped if the reader is familiar with the construction of the Jacobian.

As we stated previously in Section 2.1, complete intersection of two quadrics fibered over a base yields a four-section geometry, and the base can be any space. At the geometrical level, the base can have any dimension, but the issue of flux arises when we discuss 4D F-theory models, so the central focus here is on 6D F-theory models. Thus, we mainly consider the cases where the base is a complex surface.

As we have seen, the general complete intersection of the two quadrics is given by the following equation:

$$a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2 + 2a_5 x_1 x_2 + 2a_6 x_1 x_3 + 2a_7 x_1 x_4 + 2a_8 x_2 x_3 + 2a_9 x_2 x_4 + 2a_{10} x_3 x_4 = 0$$

$$b_1 x_1^2 + b_2 x_2^2 + b_3 x_3^2 + b_4 x_4^2 + 2b_5 x_1 x_2 + 2b_6 x_1 x_3 + 2b_7 x_1 x_4 + 2b_8 x_2 x_3 + 2b_9 x_2 x_4 + 2b_{10} x_3 x_4 = 0.$$  

One can construct the associated double cover from the complete intersection (6). As discussed in [44], one subtracts $\lambda$ times the second equation from the first equation, and one arranges the coefficients of the resulting equation into a $4 \times 4$ symmetric matrix. ($\lambda$ serves as a variable.) Taking the double cover of the determinant of this symmetric matrix, one arrives at the equation of the associated double cover as follows:

$$\tau^2 = \begin{vmatrix} a_1 - \lambda b_1 & a_5 - \lambda b_5 & a_6 - \lambda b_6 & a_7 - \lambda b_7 \\ a_5 - \lambda b_5 & a_2 - \lambda b_2 & a_8 - \lambda b_8 & a_9 - \lambda b_9 \\ a_6 - \lambda b_6 & a_8 - \lambda b_8 & a_3 - \lambda b_3 & a_{10} - \lambda b_{10} \\ a_7 - \lambda b_7 & a_9 - \lambda b_9 & a_{10} - \lambda b_{10} & a_4 - \lambda b_4 \end{vmatrix}.$$  

$\vert \cdot \vert$ on the right-hand side means to take the determinant of the matrix. When the determinant on the right-hand side is expanded, the resulting associated double cover (7) takes the following form:

$$\tau^2 = e_0 \lambda^4 + e_1 \lambda^3 + e_2 \lambda^2 + e_3 \lambda + e_4,$$  

and the double cover, generically, is a bisection geometry lacking a global section. The construction of the Jacobian of the double cover (8) was discussed in [20, 19]. The Jacobian of the double cover (7) yields the Jacobian fibration of the original complete intersection (6).
When either coefficient $e_4$ or $e_0$ is a perfect square, the double cover [8] has two global sections [19], and therefore we do not need to take the Jacobian of the double cover to obtain the Jacobian of the complete intersection, as noted in Section 2.1. For these particular situations, that is, when either $e_0$ or $e_4$ is a perfect square, the associated double cover yields the Jacobian of the complete intersection [6].

3 Some details of systematic constructions of F-theory models with $U(1) \times \mathbb{Z}_2$ and $U(1) \times \mathbb{Z}_4$

3.1 Construction of models with $U(1) \times \mathbb{Z}_2$

Complete intersection of two quadrics in $\mathbb{P}^3$ fibered over a base space:

$$a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2$$

$$+ 2a_5 x_1 x_2 + 2a_6 x_1 x_3 + 2a_7 x_1 x_4 + 2a_8 x_2 x_3 + 2a_9 x_2 x_4 + 2a_{10} x_3 x_4 = 0$$

$$b_1 x_1^2 + b_2 x_2^2 + b_3 x_3^2 + b_4 x_4^2$$

$$+ 2b_5 x_1 x_2 + 2b_6 x_1 x_3 + 2b_7 x_1 x_4 + 2b_8 x_2 x_3 + 2b_9 x_2 x_4 + 2b_{10} x_3 x_4 = 0$$

yields a four-section geometry [26, 44]. $a_i, b_j$ are sections of line bundles, subject to certain conditions, so the genus-one fibration yields a Calabi–Yau manifold [44] as previously noted in Section 2.1. $[x_1 : x_2 : x_3 : x_4]$ are the coordinates of $\mathbb{P}^3$.

When the complete intersection (9) satisfies the condition

$$b_1 = b_2$$

$$b_3 = b_8 = b_9 = b_{10} = 0,$$

a four-section splits into bisections as we stated in section 2.1. When explicitly written, the complete intersection (9) with the condition (10) imposed is given by the following equation:

$$a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2$$

$$+ 2a_5 x_1 x_2 + 2a_6 x_1 x_3 + 2a_7 x_1 x_4 + 2a_8 x_2 x_3 + 2a_9 x_2 x_4 + 2a_{10} x_3 x_4 = 0$$

$$b_1 x_1^2 + b_2 (x_2^2 + x_4^2)$$

$$+ 2b_5 x_1 x_2 + 2b_6 x_1 x_3 + 2b_7 x_1 x_4 = 0.$$

A pair of bisections into which a four-section splits along this locus is given by: $\{x_1 = 0, x_2 = ix_4\}$ and $\{x_1 = 0, x_2 = -ix_4\}$ [27]. We would like to provide a demonstration that these solutions actually yield bisections: When restricted to a genus-one fiber, the intersection of the vanishing of the first equation in (11) with each of these solutions yields intersection of

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[8] The bisections were obtained for smaller bisection geometries loci contained in the bisection geometries locus [18] in [19, 43].
two hypersurfaces of degree 1 and a quadric in \( \mathbb{P}^3 \), which gives two points \(^{13}\). Therefore, each of the solutions \( \{ x_1 = 0, \ x_2 = ix_4 \} \) and \( \{ x_1 = 0, \ x_2 = -ix_4 \} \) (along the vanishing of the first equation in (11)) globally yields a fibration of two points over the base, that is, a bisection. We learn from this argument that the solutions yield bisections.

Thus, complete intersection (11) is a bisection geometry, and a discrete \( \mathbb{Z}_2 \) gauge group forms in F-theory on this space. We find that \( U(1) \) also forms along this bisection geometries locus, as we shortly demonstrate using a mathematical technique.

Now, we construct the associated double cover of the bisection geometries locus (11) using the method we just reviewed in Section 2.2. The resulting associated double cover is given by

\[
\tau^2 = \begin{vmatrix}
  a_1 - \lambda b_1 & a_5 - \lambda b_5 & a_6 - \lambda b_6 & a_7 - \lambda b_7 \\
  a_5 - \lambda b_5 & a_2 - \lambda b_2 & e_8 & e_9 \\
  a_6 - \lambda b_6 & a_8 & a_3 & a_{10} \\
  a_7 - \lambda b_7 & a_9 & a_{10} & a_4 - \lambda b_2 \\
\end{vmatrix}.
\] (12)

Expanding the determinant on the right-hand side, we find that the term \( e_0 \) of \( \tau^2 = e_0 \lambda^4 + e_1 \lambda^3 + e_2 \lambda^2 + e_3 \lambda + e_4 \) is a perfect square:

\[
e_0 = -b_2^2 b_6^2.
\] (13)

Thus, the double cover (12) actually has two global sections \(^{19}\), and therefore yields the Jacobian of the complete intersection (11) as noted in Section 2.1. Because the Jacobian (12) has two global sections, the Mordell–Weil rank is one, and we deduce that \( U(1) \times \mathbb{Z}_2 \) forms in F-theory on the bisection geometries locus (11).

We have seen that the bisection geometries locus (11) yields a family on which \( U(1) \times \mathbb{Z}_2 \) forms in F-theory. \( a_i, i = 1, \ldots, 10, \ b_1, b_2, b_5, b_6, b_7 \) are the parameters of this family.

As discussed in \(^{19}\), in the limit at which \( e_0 \to 0 \), namely, when \( b_6 \to 0 \) \(^{14}\) \( U(1) \) is further un-Higgsed to \( SU(2) \). Additionally, un-Higgsing of \( U(1) \) also occurs when \( e_1 \) vanishes \(^{15}\), and \( U(1) \) is enhanced to \( SU(2) \) for this situation. \( e_1 \) of the associated double cover (12) is given by

\[
e_1 = -a_3 b_1 b_2^2 + a_3 b_2 b_3^2 + a_2 b_2 b_5^2 + a_4 b_2 b_6^2 + a_3 b_2 b_7^2 + 2a_6 b_2 b_6 - 2a_8 b_2 b_5 b_6 - 2a_{10} b_2 b_6 b_7.
\] (14)

In Section 3.2 we will discuss a systematic construction of models in which \( U(1) \times \mathbb{Z}_4 \) forms.

Before we move on to the next section, we would like to make a remark: we have just seen that a discrete \( \mathbb{Z}_2 \) gauge group is enhanced and \( U(1) \) arises in a certain bisection geometries

\(^{13}\)Intersection of two hypersurfaces of degree 1 and a quadric in \( \mathbb{P}^3 \) is equivalent to an intersection of a degree 1 hypersurface and a quadric in \( \mathbb{P}^2 \), because a degree 1 hypersurface in \( \mathbb{P}^3 \) is isomorphic to \( \mathbb{P}^2 \). Now an intersection of a degree 1 hypersurface and a quadric in \( \mathbb{P}^2 \) yields two points as a consequence of an elementary algebrogeometric argument.

\(^{14}\)We do not discuss the case \( b_2 \to 0 \) in this note.

\(^{15}\)Discussions of some F-theory models belonging to this locus can be found in \(^{30, 34, 44, 45}\).
locus in the four-section geometry. This fact can support the proposed interpretation in [45] that can resolve the puzzle raised in [44]. Can similar enhancements in a discrete gauge group be observed along other bisection geometries loci? We also make a brief comment on this at the end of this work.

3.2 Construction of models with $U(1) \times \mathbb{Z}_4$

As noted in Section 2.1, if we can construct complete intersections of two quadrics in $\mathbb{P}^3$ fibered over a base whose Jacobians have the Mordell–Weil rank one, this yields a family of Calabi–Yau genus-one fibrations on which $U(1) \times \mathbb{Z}_4$ forms in F-theory, and it suffices to check either the coefficient $e_0$ or $e_4$ of the associated double cover is a perfect square to achieve this construction. We provide a demonstration of the strategy, by presenting an explicit example of such a family here.

The coefficients $e_0$ and $e_4$ of the associated double cover of complete intersection of the general form (9) are not perfect squares, so some specific coefficients of the complete intersection need to be chosen.

To this end, we choose the following specific complete intersection:

\[
\begin{align*}
    a_1 x_1^2 + a_2 x_2^2 + a_1 x_3^2 + a_2 x_2^2 + 2a_2 x_1 x_3 + 2a_1 x_2 x_4 &= 0 \\
    b_1 x_1^2 + b_2 x_2^2 + b_3 x_3^2 + b_4 x_4^2 + 2b_6 x_1 x_3 + 2b_9 x_2 x_4 &= 0
\end{align*}
\]

This is a four-section geometry, and as we demonstrate shortly, the associated double cover of this complete intersection has two global sections.

The associated double cover is given using the method we reviewed in Section 2.2, by the following equation:

\[
\tau^2 = \begin{vmatrix} a_1 - \lambda b_1 & 0 & a_2 - \lambda b_6 & 0 \\
0 & a_2 - \lambda b_2 & 0 & a_1 - \lambda b_9 \\
0 & a_2 - \lambda b_6 & 0 & a_1 - \lambda b_3 \\
0 & 0 & a_1 - \lambda b_9 & a_2 - \lambda b_4 \end{vmatrix}.
\]

Expanding the determinant on the right-hand side, we find that the term $e_4$ of $\tau^2 = e_0 \lambda^4 + e_1 \lambda^3 + e_2 \lambda^2 + e_3 \lambda + e_4$ is a perfect square,

\[
e_4 = -(a_1^2 - a_2^2)^2;
\]

Therefore, the associated double cover \[16\] has two global sections. Thus, the associated double cover \[16\] yields the Jacobian fibration of the complete intersection \[15\], and the Mordell–Weil rank of the Jacobian is one. $U(1) \times \mathbb{Z}_4$ forms in F-theory on the complete intersections \[15\]. $a_1, a_2, b_1, b_2, b_3, b_4, b_6, b_9$ are the parameters of this family.

For the case $e_3$ vanishes, $e_3 = 0$, $U(1)$ is further un-Higgsed to $SU(2)$ as discussed in [45]. $e_3$ of the associated double cover \[16\] is given by

\[
\begin{align*}
    e_3 &= a_1^3 b_1 - a_1 a_2^2 b_1 + a_2^3 b_2 - a_1^2 a_2 b_2 + a_3 b_3 - a_1 a_2 b_3 \\
    &+ a_2^3 b_4 - a_1^2 a_2 b_4 + 2a_2^3 b_6 - 2a_1^2 a_2 b_6 + 2a_1^3 b_9 - 2a_1 a_2^2 b_9.
\end{align*}
\]
3.3 Models with $U(1) \times \mathbb{Z}_2$ using Halphen surfaces of index 2

Halphen surfaces of index 2 are genus-one fibered rational surfaces and yield examples of rational elliptic surfaces without a global section. General discussions of Halphen surfaces can be found in [104, 105]. Examples of Halphen surfaces of index 2 possessing various type $I_n$ fibers were constructed in the context of F-theory in [35, 37].

The fiber of a Halphen surface of index 2 is twice a certain divisor, as a divisor class, therefore intersection number of a fiber of a Halphen surface of index 2 with any divisor is a multiple of two. Owing to this, Halphen surfaces of index 2 do not have a global section, because the intersection number of a global section, if a surface possesses, with a fiber is one. (Details of this can be found in [35, 37].) Halphen surfaces of index 2 are bisection geometries.

Double covers of Halphen surfaces of index 2 (branched over an appropriate divisor) yield genus-one fibered K3 surface. The resulting K3 surfaces are also bisection geometries [35, 37]. For generic parameters of taking a double cover, the singular fibers of the resulting K3 surface is precisely twice those of the original Halphen surface [35, 37].

Among the Halphen surfaces with type $I_n$ fibers constructed in [37], the ones with a type $I_7$ fiber is particularly relevant to the contents of this study. As we have just stated, double cover of a Halphen surface yields a K3 surface the singular fibers of which are exactly twice those of the Halphen surface. Therefore, the double covers of the Halphen surfaces with a type $I_7$ fiber yield K3 surfaces, the singular fibers of which include two type $I_7$ fibers [37]. By applying techniques in algebraic geometry, it can be shown that the Jacobians of these K3 surfaces have Mordell–Weil rank either one or two.

We would like to give a sketch of a proof of this: The Shioda–Tate formula [107, 108, 109] can be used as a key tool to show this. The formula, when applied to rational elliptic surfaces $S$ with a section (such as the Jacobians of the Halphen surfaces), states that:

$$\text{rk MW}(S) + \text{rk } ADE(S) = 8.\quad (19)$$

rk $ADE(S)$ in the equation denotes the rank of the singularity type of a rational elliptic surface $S$. This means that the Mordell–Weil rank and the singularity rank of a rational elliptic surface with a section always add to 8. Making use of the classification result in [110], it can be deduced that the singularity type of a Halphen surface of index 2 with a type $I_7$ fiber is always either $A_6$ or $A_6A_1$, and the Jacobian has the singularity rank either 6 or 7 [17].

Now by applying the Shioda–Tate formula, one finds that the Mordell–Weil rank of the

---

16 When the parameters of taking a double cover assume special values, type $I_7$ fibers of the resulting K3 surface collide, and they are enhanced to a type $I_{14}$ fiber [37], consistent with the arguments in [109].

17 This follows from the following observation: the types of the singular fibers of the rational elliptic surfaces with a section with singularity rank 8, or “extremal rational elliptic surfaces,” were classified [110]. It can be seen from the classification result of extremal rational elliptic surfaces in [110] that these surfaces do not have a type $I_7$ fiber. (The classification result of the singular fibers of the extremal rational elliptic surfaces can be found in the table in Theorem 4.1 in [110]. Type $I_7$ fiber does not appear in this classification result.) This means that the Jacobian of Halphen surface of index 2 with a type $I_7$ fiber is not extremal, namely its singularity type has rank strictly less than 8. Because the singularity corresponding to type $I_7$ fiber is $A_6$, the singularity rank the Jacobian has must be at least 6. Therefore, the Jacobian has the singularity rank either 6 or 7.
Jacobian of a Halphen surface of index 2 with a type $I_7$ fiber should be either one or two. It only remains to confirm that the Jacobian of the K3 surface obtained as double cover of this Halphen surface also has Mordell–Weil rank one or two. By applying an argument similar to those given in [69, 74], it can be seen that (for generic parameters) the Mordell–Weil ranks are invariant under the operation of taking the double cover. In other words, the Mordell–Weil rank of the Jacobian of the resulting K3 surface is identical to the Mordell–Weil rank of the Jacobian of the Halphen surface of index 2 with a type $I_7$ fiber, that is, it has the rank one or two.

The resulting K3 surfaces are bisection geometries, as stated previously, and as we have just seen, the Jacobians have a Mordell–Weil rank of either one or two. Thus, $U(1) \times \mathbb{Z}_2$ forms in F-theory on these K3 surfaces \(^{18}\).

F-theory compactifications of these surfaces yield 8D examples in which $U(1) \times \mathbb{Z}_2$ forms. If we consider direct products of these K3 surfaces with a K3 surface, F-theory compactifications on the direct products yield 4D examples with $N = 2$ supersymmetry with $U(1) \times \mathbb{Z}_2$ gauge group. By including flux, half the supersymmetry is broken, left with $N = 1$, as discussed in [111, 29, 30]; however, the complex structure of the K3 surface needs to be analyzed to determine whether a consistent flux exists [111, 112, 29, 30].

4 Concluding remarks

We discussed systematic constructions of F-theory models with $U(1) \times \mathbb{Z}_2$ and $U(1) \times \mathbb{Z}_4$ gauge groups. For realizing these constructions, using of four-section geometry as complete intersections of two quadric hypersurfaces in $\mathbb{P}^3$ fibered over a base was useful.

We observed in Section 3.1 that along a bisection geometries locus (11), a discrete $\mathbb{Z}_2$ gauge group becomes enlarged and $U(1)$ also forms. This observation can support, at least to some level, the possible interpretation proposed in [45] that can resolve the puzzle raised in [44].

Can similar enlargements of a discrete gauge $\mathbb{Z}_2$ group be observed along other bisection geometries loci? Does this enlargement occur whenever a four-section splits into bisections in the four-section geometries? Studying these may be interesting, and these are likely directions of future study.

It may be also interesting to study whether a $U(1)$ arises when a multisection splits for multisectors of other degrees.

The spaces we considered in Section 3.3 were K3 surfaces and/or products of K3 surfaces (although general genus-one fibered Calabi–Yau spaces were considered as compactification spaces in the rest of this study. As noted in Section 2.1 the discussions in the sections other than Section 3.3 do not depend on the dimension of the space at the geometrical level; however, if one considers 4D F-theory, the issue of flux arises. Owing to this, we mainly focused on 6D F-theory in the rest sections, and thus genus-one fibered Calabi–Yau 3-folds

\(^{18}\)Gauge groups arising on F-theory on K3 surfaces obtained as double covers of Halphen surfaces of index 2 with an $I_7$ fiber were analyzed in [37]; however, the appearance of $U(1)$ was not discussed in [37].
were mainly considered therein). It can be interesting to generalize the discussion in Section 3.3 to F-theory on Calabi–Yau genus-one fibrations of higher dimensions. This is also a likely target of future study. Because K3 surfaces and Halphen surfaces discussed in Section 3.3 are bisection geometries, and the Mordell–Weil rank of the Jacobians of them is one, it might be interesting to study if these surfaces belong to some bisection geometries loci of four-section geometry, or multisection geometry of higher degree.

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