The Space of Strongly Prime Gamma Subacts

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Abstract.
In this work we consider and study the structure space of gamma acts by considering strongly prime gamma subacts. Also we study compactness and connectedness properties of this space as well as the separation axioms.

Key words: gamma semigroups, gamma acts, (strongly) prime gamma subacts, Noetherian gamma acts, multiplication gamma act and uniserial gamma act.

1. Introduction
The Hausdorff property for the ring C(X) of continuous real-valued functions on X has been studied by L. Gillman in [1]. C.W. Kohls in [2] studied the space of prime ideals of an arbitrary ring while S. Chattopadhyay and S. Kar introduced and studied the structure space of gamma semigroups [3].

In this work, we introduce and study the structure space of gamma acts. For this object, let M be an SΓ-act, we consider the collection SP(M) of all strongly prime gamma subacts. By means of intersection and inclusion we define a closure operator on SP(M) and give a topology τSP(M) on SP(M). We call this topological space (SP(M), τSP(M)) the structure space of the gamma act M. We discuss separation axioms in this space, also we consider the properties of connectedness and compactness.

2. Basic Concept.
Let S and Γ be nonempty sets. Recall that S is Γ-semigroup if a, b, c ∈ S and α, β ∈ Γ. S is a Γ-semigroup with zero element if there is an element 0 ∈ S such that 0a = a0 = 0 for all a ∈ S and α ∈ Γ. A Γ-semigroup S is commutative if aαb = bαc for all a, b, c ∈ S, and α ∈ Γ [4].

Let S be a semigroup and A a nonempty set. If we have a mapping μ : S × A → A, (s, a) ↦ μ(sa) such that (st)a = s(ta) for all s, t ∈ S and a ∈ A, we call A is a left S-act and write \( \frac{s}{A} \) as [5].

The notion of gamma acts which is a generalization of acts as well as gamma semigroups has been introduced in [6].
2.1. Definition. If $S$ is a $\Gamma$-semigroup, a nonempty subset $M$ of $S$ is called a left gamma-act over $S$, denoted by $s\hookrightarrow M$, if there is a mapping $S \times \Gamma \times M \to M$, $(s, \alpha, m) \mapsto s_{\alpha}m$ (for $s \in S, \alpha \in \Gamma$ and $m \in M$) such that $s_{\alpha}(s_{\alpha}m) = (s_{\alpha}s_{\alpha})m$ for all $s, s_{1} \in S, \alpha, \alpha_{1} \in \Gamma$ and $m \in M$.

2.2. Examples (2.2).

1. Let $S = \{ 5n + 4 \mid n \in \mathbb{Z}^{+} \}$, $\Gamma = \{ 5n + 1 \mid n \in \mathbb{Z}^{+} \}$. Then $S$ is a $\Gamma$-semigroup where $s_{\alpha}s_{1} = s_{1} + \alpha + s_{1}$ (usual addition of integers). Now, let $M = \{ 5n \mid n \in \mathbb{Z}^{+} \}$. Then $M$ is a $S_{\Gamma}$-act, but $M$ is not $\Gamma$-semigroup with usual addition of integers.

2. Let $M$ be the set of all negative rational numbers. It is clear that $M$ is not $M$-act under usual multiplication of rational numbers. Let $\Gamma = \{ \frac{1}{p} \mid p \text{ is prime} \}$ and define the mapping $M \times \Gamma \times M \to M$ by $(x, \alpha, y) \mapsto x\alpha y$ (usual multiplication of rational numbers). It is an easy matter to see that $M$ is $M_{1}$-act.

A nonempty subset $N$ of $S_{\Gamma}$-act $M$ is called $S_{\Gamma}$-subact, if $S_{\Gamma}N \subseteq N$ where $S_{\Gamma}N = \{ s_{\alpha}n \mid s \in S, \alpha \in \Gamma \text{ and } n \in N \}$. An $S_{\Gamma}$-subact $N$ of an $S_{\Gamma}$-act $M$ is proper if $N \neq M$.

For $S_{\Gamma}$-acts $M$ and $N$. A mapping $f : M \to N$ is called $S_{\Gamma}$-homomorphism if $f(s_{\alpha}m) = s_{\alpha}f(m)$, for all $s \in S, \alpha \in \Gamma$ and $m \in M$. We denote $\text{Hom}(M, N)$ the set of all $S_{\Gamma}$-homomorphisms from $M$ into $N$.

2.3. Definition. Let $N$ be an $S_{\Gamma}$-subact of an $S_{\Gamma}$-act $M$. Define $(N; m) = \{ s \in S \mid s_{\Gamma}m \subseteq N \}$. In particular, for $m \in M$, $(N; m) = \{ s \in S \mid s_{\Gamma}m \subseteq N \}$. Recall that a nonempty subset $I$ of a $\Gamma$-semigroup $S$ is called ideal if $I_{\Gamma}S \subseteq I$ and $SI_{\Gamma} \subseteq I$.

We introduce the following.

2.4. Definition. Let $M$ be an $S_{\Gamma}$-act. A proper $S_{\Gamma}$-subact $P$ of $M$ called prime if for any ideal $I$ of $S$ and any $S_{\Gamma}$-subact $N$ of $M$, $I_{\Gamma}N \subseteq P$ implies that $N \subseteq P \text{ or } I \subseteq (P; m)$.

In the following, the concept of prime gamma-subacts can be reduced to elements.

2.5. Proposition. Let $P$ be a proper $S_{\Gamma}$-subact of an $S_{\Gamma}$-act $M$. Then $P$ is prime if and only if $s_{\Gamma}S_{\Gamma}m \subseteq P$ implies that $m \in P$ or $s \in (P; m)$ for all $s \in S$ and $m \in M$.

Proof. Assume that $s_{\Gamma}S_{\Gamma}m \subseteq P$ where $s \in S$ and $m \in M$. Primers of $P$ implies that $m \in P$ or $s \in (P; m)$. Conversely, assume $I_{\Gamma}V \subseteq P$ for an ideal $I$ of $S$ and $S_{\Gamma}$-subact $V$ of $M$. If $V \subseteq P$, then there is an element $x \in V$ and $x \in P$. Then for any $a \in I$ we have $a_{\Gamma}S_{\Gamma}x \subseteq I_{\Gamma}V \subseteq P$, thus $a \in (P; m)$.

Recall that a proper ideal $T$ of a $\Gamma$-semigroup $S$ is prime if for any two ideals $I$ and $J$ of $S$, $I_{\Gamma} \subseteq T$ implies that $I \subseteq T$ or $J \subseteq T$. Then we have the following corollary.

2.6. Corollary. A proper ideal $T$ of a $\Gamma$-semigroup $S$ is prime if and only if $s_{1}I_{\Gamma}s_{2} \subseteq T$ implies that $s_{1} \in T$ or $s_{2} \in T$ for all $s_{1}, s_{2} \in S$. 


2.7. **Lemma.** Let $M$ be an $S_{\Gamma}$-act. If $P$ is a prime $S_{\Gamma}$-subact of $M$, then $(P : \gamma M)$ is a prime ideal of $S$.

**Proof.** Let $s_1, s_2 \in S$ with $s_1 \Gamma s_2 \subseteq (P : \gamma M)$. Then $s_1 \Gamma s_2 \Gamma M \subseteq P$. Since $P$ is prime, then by Proposition (2.6) we have either $s_2 \Gamma M \subseteq P$ or $s_1 \subseteq (P : \gamma M)$ and hence $s_2 \in (P : \gamma M)$ or $s_1 \in (P : \gamma M)$.

For the converse we consider the following

2.8. **Definition.** An $S_{\Gamma}$-act $M$ is called multiplication if for any $S_{\Gamma}$-subact $N$ of $M$, there is an ideal $I$ of $S$ such that $N = I \Gamma M$.

It is easy matter that an $S_{\Gamma}$-subact $N$ of a multiplication $S_{\Gamma}$-act $M$ is of the form $N = (N : \gamma M) \Gamma M$.

2.9. **Theorem.** If $M$ is a multiplication $S_{\Gamma}$-act, then an $S_{\Gamma}$-subact $P$ of $M$ is prime if and only if $(P : \gamma M)$ is a prime ideal of $S$.

**Proof.** Assume that $(P : \gamma M)$ is a prime ideal of $S$, and there exist an ideal $I$ and $S_{\Gamma}$-subact $V$ of $M$ with $V \not\subseteq P$, $I \subseteq (P : \gamma M)$ and $I \Gamma V \subseteq P$.

Since $M$ is multiplication, then $V = I \Gamma M$ for some ideal $J$ of $S$. Thus $V = I \Gamma J \subseteq M$ so $J \subseteq (P : \gamma M)$, but $(P : \gamma M)$ is a prime ideal of $S$ and $I \not\subseteq (P : \gamma M)$, then $J \subseteq (P : \gamma M)$. Therefore $V = J \Gamma M \subseteq P$ which is a contradiction. Thus $P$ is prime.

It is easy matter to see that if $I$ and $J$ are two ideals of a $\Gamma$-semigroup $S$ and $P$ is a prime ideal of $S$ with $I \cap J \subseteq P$; then $I \not\subseteq P$ or $J \not\subseteq P$. This statement is no longer hold if we replace ideals of $\Gamma$-semigroup by $S_{\Gamma}$-subacts of $S_{\Gamma}$-act. However we have the following

2.10. **Theorem.** Let $N$ be a prime $S_{\Gamma}$-subact of a multiplication $S_{\Gamma}$-act $M$. If $N_1, N_2$ are $S_{\Gamma}$-subacts of $M$ with $N_1 \cap N_2 \subseteq N$, then either $N_1 \subseteq N$ or $N_2 \subseteq N$.

**Proof.** Since $(N_1 \cap N_2 : \gamma M) = (N_1 : \gamma M) \cap (N_2 : \gamma M) \subseteq (N_1 : \gamma M)$ and $(N_2 : \gamma M)$ is a prime ideal of $S$, then either $(N_1 : \gamma M) \subseteq (N_2 : \gamma M)$ or $(N_2 : \gamma M) \subseteq (N_1 : \gamma M)$. Thus either $N_1 = (N_1 : \gamma M) \Gamma M \subseteq (N_2 : \gamma M) \Gamma M = N$ or $N_2 = (N_2 : \gamma M) \Gamma M \subseteq (N_1 : \gamma M) \Gamma M = N$.

We introduce the following

2.11. **Definition.** An $S_{\Gamma}$-subact $N$ of $S_{\Gamma}$-act $M$ is called strongly prime (or finitely prime), if $S$ and $\Gamma$ contain finite subset $\bar{S}$ and $\bar{\Gamma}$ respectively such that $\bar{S} \Gamma \bar{M} \subseteq N$ implies that $\bar{m} \in N$ or $\bar{s} \in (N : \gamma M)$ for all $s \in S$ and $m \in \Gamma$.

2.12. **Proposition.** Every strongly prime $S_{\Gamma}$-subact of $S_{\Gamma}$-act $M$ is prime.

**Proof.** Let $N$ be a strongly $S_{\Gamma}$-subact of $S_{\Gamma}$-act $M$. For $s \in S$ and $m \in \Gamma$, if $s \bar{S} \Gamma \bar{m} \subseteq N$, then there are finite subsets $\bar{S}$ and $\bar{\Gamma}$ respectively and $s \bar{S} \Gamma \bar{m} \subseteq s \bar{S} \Gamma \bar{m} \subseteq N$. This implies that $\bar{m} \in N$ or $\bar{s} \in (N : \gamma M)$.

In the following consider intersection of (strongly) prime gamma subacts.

2.13. **Proposition.** Let $\{N_\alpha : \alpha \in \Lambda\}$ be a collection of prime $S_{\Gamma}$-subacts of an $S_{\Gamma}$-act $M$ such that $\{N_\alpha : \alpha \in \Lambda\}$ forms a chain. Then $\bigcap_{\alpha \in \Lambda} N_\alpha$ is a prime $S_{\Gamma}$-subact of $M$.  

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Proof: For any ideal I of S and sΓ-subact V of M, if I \( \Gamma \) \( V \subseteq \bigcap_{\alpha \in A} N_{\alpha} \) with I \( \not\subseteq \bigcap N_{\alpha} \) and \( V \not\subseteq N_{\alpha} \), then there are \( \alpha, \beta \in A \) such that \( I \Gamma \) \( M \not\subseteq N_{\alpha} \) and \( V \not\subseteq N_{\alpha} \). No loss of generality if we assume \( N_{\alpha} \subseteq N_{\beta} \). This implies that \( V \not\subseteq N_{\beta} \), a contradiction. Thus \( \bigcap_{\alpha \in A} N_{\alpha} \) is a prime SΓ-subact of M.

A sΓ-act M is called uniserial, if for any two SΓ-subacts N and K of M, either \( N \subseteq K \) or \( K \subseteq N \).

2.14. Corollary. Let M be a uniserial SΓ-act. If \( \{ N_{\alpha} \mid \alpha \in \Lambda \} \) is a family of (strongly) prime SΓ-subacts of M, then \( \bigcap N_{\alpha} \subseteq N_{\alpha} \) is a prime SΓ-subact of M.

3. Structure space of SΓ-acts.

Let M be an SΓ-act. Denote by SP(M) the collection of all strongly prime SΓ-subacts of M. For any \( N \subseteq SP(M) \), we define \( \overline{N} = \{ K \in SP(M) \mid K \subseteq N \} \). It is clear that \( \overline{\emptyset} = \emptyset \) and \( N \subseteq \overline{N} \) for any subset N of SP(M).

3.1. Theorem. For any two subsets \( N \) and \( L \) of \( SP(M) \), the following hold:

1. \( \overline{\overline{N}} = \overline{N} \)
2. \( N \subseteq L \) implies \( \overline{N} \subseteq \overline{L} \)
3. If \( M \) is a multiplication SΓ-act, then \( \overline{N \cup L} = \overline{\overline{N \cup L}} \).

Proof. (1). It is clear that \( \overline{\overline{N}} = \overline{N} \). For other inclusion, let \( k_{\beta} \in \overline{N} \). Then \( \bigcap_{K_{\alpha} \subseteq \overline{N}} K_{\alpha} \subseteq K_{\beta} \) and \( K_{\alpha} \subseteq \overline{N} \) implies that \( \bigcup_{\alpha \in S} K_{\alpha} \subseteq K_{\beta} \) for all \( \alpha \in \Lambda \). Thus \( \bigcap_{K_{\alpha} \subseteq \overline{N}} K_{\alpha} \subseteq K_{\beta} \) that is \( \bigcap_{K_{\alpha} \subseteq \overline{N}} K_{\alpha} \subseteq K_{\beta} \) and so \( K_{\beta} \in \overline{N} \) hence \( \overline{\overline{N}} = \overline{N} \).

2. Suppose \( N \subseteq L \) and \( K_{\alpha} \in \overline{N} \). Then \( \bigcap_{K_{\beta} \subseteq L} K_{\beta} \subseteq K_{\alpha} \). Since \( N \subseteq L \), then \( \bigcap_{K_{\beta} \subseteq L} K_{\beta} \subseteq \bigcap_{K_{\beta} \subseteq N} K_{\beta} \subseteq K_{\alpha} \) and this implies that \( K_{\alpha} \subseteq L \) and hence \( \overline{N} \subseteq \overline{L} \).

3. Clearly by (2) \( \overline{N \cup L} \subseteq \overline{N \cup L} \). Let \( K_{\alpha} \in \overline{N \cup L} \). Then \( \bigcap_{K_{\beta} \subseteq \overline{N \cup L}} K_{\beta} \subseteq K_{\alpha} \). It is easy to see that \( \bigcap_{K_{\beta} \subseteq \overline{N \cup L}} K_{\beta} = ( \bigcap_{K_{\beta} \subseteq N} K_{\beta} ) \cap ( \bigcap_{K_{\beta} \subseteq L} K_{\beta} ) \subseteq K_{\alpha} \). Since \( K_{\alpha} \) is strongly prime for each \( \alpha \), then \( K_{\alpha} \) is prime, Proposition (2.14). By multiplication property of M and Proposition (1.12), we have \( \bigcap_{K_{\beta} \subseteq N} K_{\beta} \subseteq K_{\alpha} \) or \( \bigcap_{K_{\beta} \subseteq L} K_{\beta} \subseteq K_{\alpha} \), this is \( K_{\alpha} \in \overline{N} \) or \( K_{\alpha} \in \overline{L} \) and hence \( \overline{N \cup L} = \overline{N \cup L} \).

3.2. Definition. Let M be a multiplication SΓ-act. The closure operator \( N \rightarrow \overline{N} \) gives a topology \( \tau_{SP(M)} \) on SP(M). This topology is called the strongly prime topology and the topology space \( ( \tau_{SP(M)} , SP(M) ) \) is called the structure space of the SΓ-act M.
For $S \Gamma$-subact $N$ of an $S \Gamma$-act $M$. We define $\Delta(N) = \{ N' \in SP(M) \mid N \subseteq N' \}$ and $C\Delta(N) = SP(M) \setminus \Delta(N)$. In the following we describe the closed set in $SP(M)$.

### 3.3. Proposition
Let $M$ be a multiplication $S \Gamma$-act. Then for any closed set $W \subseteq SP(M)$, there is an $S \Gamma$-subact $N$ of $M$ such that $W = \Delta(N)$.

**Proof.** Let $W$ be a closed subset in $SP(M)$ where $W \subseteq SP(M)$. Then $W = \{ N_a \in SP(M) \mid a \in \Lambda \}$. Let $N = \bigcap_{N_a \in W} N_a$. Then $N$ is an $S \Gamma$-subact of $M$ if $N' \in W$, then $\bigcap_{N_a \in W} N_a \subseteq N'$. This implies that $N \subseteq N'$ and hence $N' \in \Delta(N)$ so $W \subseteq \Delta(N)$. Conversely, let $N' \in \Delta(N)$. Then $N \subseteq N'$, that is $\bigcap_{N_a \in W} N_a \subseteq N'$, this implies that $N' \subseteq W$ and hence $\Delta(N) \subseteq W$.

### 3.4. Corollary (3.4)
Any open set in $SP(M)$ is of the form $C\Delta(N)$ for some $S \Gamma$-subact $N$ of multiplication $S \Gamma$-act $M$.

Let $M$ be an $S \Gamma$-subact and $m \in M$. We define $\Delta(m) = \{ N \in SP(M) \mid m \in N \}$ and $C\Delta(m) = SP(M) \setminus \Delta(m)$.

### 3.5. Proposition
If $M$ is a multiplication $S \Gamma$-act. Then $\{ C\Delta(m) \mid m \in M \}$ forms an open base for the topology $\tau_{SP(M)}$ on $SP(M)$.

**Proof.** Let $U \in \tau_{SP(M)}$. Then by Corollary (3.4), there is an $S \Gamma$-subact $N$ of $M$ such that $U = \Delta(N)$. Let $K \in U$ then $N \not\subseteq K$ and there is $x \in N$ with $x \not\in K$. Thus $K \in \Delta(X)$. To see $\Delta(X) \subseteq U$. Let $K \in \Delta(m)$. Then $m \not\in K$. It follows that $N \not\subseteq K$ and hence $K \in U$ and so $\Delta(m) \subseteq U$. Thus $\{ C\Delta(m) \mid m \in M \}$ is an open base for $\tau_{SP(M)}$. □

### 3.6. Theorem
The space $(SP(M), \tau_{SP(M)})$ is $T_1$-space for any multiplication $S \Gamma$-act $M$.

**Proof.** Suppose $N_1$ and $N_2$ are two distinct elements in $SP(M)$. Without loss of generality, we assume that there is an element $x \in N_1$, and $x \not\in N_2$. Then $\Delta(x)$ is a neighborhood of $N_2$ not containing $N_1$.

### 3.7. Theorem
The following statements are equivalent for a multiplication $S \Gamma$-act $M$

1. $(SP(M), \tau_{SP(M)})$ is $T_1$-space

2. No element of $SP(M)$ is contained in any other element of $SP(M)$.

**Proof.** (1) $\rightarrow$ (2). Suppose $(SP(M), \tau_{SP(M)})$ is a $T_1$-space and $N_1, N_2$ be distance elements of $SP(M)$. Then each of $N_1$ and $N_2$ has a neighborhood not containing the other. Since $N_1$ and $N_2$ are any elements. This implies that no element of $SP(M)$ is containing in any other element of $SP(M)$.

(2) $\rightarrow$ (1), assume that no element of $SP(M)$ is contained in any other element of $SP(M)$. Let $N_1$ and $N_2$ be two different elements of $SP(M)$. Then by hypothesis, there exist $x, y \in M$ with $x \in N_1 \setminus N_2$ and $y \in N_2 \setminus N_1$. Then $x \not\in N_2$ and $y \not\in N_1$. Therefore, $\Delta(x) \setminus \Delta(y) = N_1 \setminus N_2$ is a neighborhood of $N_1$ not containing $N_2$. Since $\Delta(x) \setminus \Delta(y) = N_1 \setminus N_2$, this implies that $\Delta(x) \not\subseteq N_2$ and hence $x \not\in N_2$. Similarly, $\Delta(y) \not\subseteq N_1$ and hence $y \not\in N_1$. Therefore, $(SP(M), \tau_{SP(M)})$ is a $T_1$-space.
Thus, we have $N_1 \subseteq C \Delta(y)$ but $N_1 \notin C \Delta(x)$ and $N_2 \in C \Delta(x)$, but $N_1 \notin C \Delta(y)$. Thus each of $N_1$ and $N_2$ has a neighborhood no containing the other. Hence $(SP(M), \tau_{SP(M)})$ is a $T_1$-space. □

3.8. Corollary. Let $S$ be a commutative $\Gamma$-semigroup and $M$ a multiplication $S_l$-act. If $Max(M)$ is the class of maximal $S_l$-subacts of $M$, then $(Max(M), \tau_{Max(M)})$ is a $T_1$-space where $\tau_{Max(M)}$ is the induced topology on $Max(M)$ from $(SP(M), \tau_{SP(M)})$.

3.9. Theorem. If $M$ is a multiplication $S_l$-act. Then the following conditions are equivalent

(1) $(SP(M), \tau_{SP(M)})$ is a Hausdorff space.

(2) Any two distinct elements $N$ and $K$ of $SP(M)$, there exist $x, y \in M$ such that $x \notin N, y \notin K$ and does not exist any $W \in SP(M)$ such that $x, y \notin W$.

Proof. (1) $\rightarrow$ (2). Assume that $(SP(M), \tau_{SP(M)})$ is a Hausdorff space. Then for any two distinct $N_1$ and $N_2$ of $SP(M)$, there is an open set $C \Delta(x)$ and $C \Delta(y)$ such that $N_1 \subseteq C \Delta(x), N_2 \subseteq C \Delta(y)$ and $C \Delta(x) \cap C \Delta(y) = \emptyset$. This implies that $x \notin N_1$ and $y \notin N_2$. If there is $K \in SP(M)$ such that $x \notin K, y \notin K$. Then $K = C \Delta(x) \cap C \Delta(y) = \emptyset$ a contradiction. Thus there does not exist any $K \in SP(M)$ with $x \notin K$ and $y \notin K$. (2) $\rightarrow$ (1). Assume the given condition holds and $N_1, N_2 \in SP(M)$ with $N_1 \neq N_2$. Let $a, b \in M$ with $a \notin N_1$ with $b \notin N_2$ and there does not exist any $K \in SP(M)$ such that $a \notin K, b \notin K$. This exactly implies $N_1 \subseteq C \Delta(x), N_2 \subseteq C \Delta(y)$ and $C \Delta(x) \cap C \Delta(y) = \emptyset$ and hence $(SP(M), \tau_{SP(M)})$ is a Hausdorff space.

3.10. Proposition Let $M$ be a multiplication $S_l$-act and $(SP(M), \tau_{SP(M)})$ is a Hausdorff space. Then

(1) No proper strongly prime $S_l$-subact of $M$ contains any other proper strongly prime $S_l$-subact

(2) If $(SP(M), \tau_{SP(M)})$ contains more than one element, then there exist $x, y \in M$ where $SP(M) = C \Delta(x) \cup C \Delta(y) \cup \Delta(W)$, where $W$ is the $S_l$-subact of $M$ generating by $x$ and $y$.

Proof. (1). It's clear by Theorem (3.7) and the fact that every Hausdorff space is a $T_1$-space.

(2). Let $N$ and $K$ be two distinct strongly prime $S_l$-subacts of $M$. Then there is an open set $C \Delta(x)$ and $C \Delta(y)$ such that $N \subseteq C \Delta(x), K \subseteq C \Delta(y)$ and $C \Delta(x) \cap C \Delta(y) = \emptyset$. Suppose $W$ is the $S_l$-subact of $M$ generating by $x$ and $y$, namely $W$ is the smallest $S_l$-subact of $M$ containing $x$ and $y$ and $W = SF_x \cup SF_y$. Let $L \in SP(M)$.

Then we have the following cases. (1) $x, y \in L$, (2) $x \in L, y \notin L$, (3) $x \notin L, y \in L$ and (4) $x \notin L, y \notin L$. Case (4) is not possible since $C \Delta(x) \cap C \Delta(y) = \emptyset$, case (2) implies that $L \in C \Delta(y)$, similarity case (3) implies that $L \in C \Delta(x)$ and finally case (1) implies that $L \in \Delta(W)$ and thus $PS(M) \subseteq C \Delta(x) \cup C \Delta(y) \cup \Delta(W)$. □

3.11. Theorem. The following conditions are equivalent for a multiplication $S_l$-act $M$.

(1) $(SP(M), \tau_{SP(M)})$ is a regular space

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This page seems to be discussing topological concepts in the context of semigroups and acts. The text outlines various theorems and propositions, including conditions for Hausdorff spaces, properties of strongly prime subacts, and equivalences in regular spaces. The proofs provided rely on the definitions and properties given earlier in the document.
(2) For \( N \in \text{SP}(M) \) and \( x \in M \setminus N \), there exist an \( S_f \)-subact \( K \) of \( M \) and \( y \in M \) such that \( N \in C \Delta(y) \subseteq \Delta(K) \). 

**Proof.** (1) \( \rightarrow \) (2). Let \( N \in \text{SP}(M) \) and \( x \in M \setminus N \). Then \( N \in C \Delta(x) \) and \( \text{SP}(M) \setminus C \Delta(x) \) is closed set not containing \( N \). By (1) there is disjoint open sets \( U \) and \( V \) such that \( N \in U \) and \( \text{SP}(M) \setminus C \Delta(x) \subseteq V \). This implies that \( \text{SP}(M) \setminus V \subseteq C \Delta(x) \). Since \( \text{SP}(M) \setminus V \) is closed, then by Proposition (3.3), there is an \( S_f \)-subact \( K \) of \( M \) such that \( \text{SP}(M) \setminus V = C \Delta(K) \) and hence we get \( \Delta(K) \subseteq C \Delta(x) \). Since \( U \cap V = \emptyset \), then \( V \subseteq \text{SP}(M) \setminus U \). Again since \( \text{SP}(M) \setminus U \) is closed, then there exists an \( S_f \)-subact \( W \) of \( M \) such that \( \text{SP}(M) \setminus \Delta(W) = \Delta(K) \), this is \( V \subseteq \Delta(W) \). Since \( N \in U \), then \( N \notin \text{SP}(M) \setminus U = \Delta(W) \). It follows that \( W \nsubseteq N \), and hence there is \( y \in W \setminus N \) so \( N \in C \Delta(y) \).

Now we show that \( V \subseteq \Delta(y) \). Let \( L \in V \subseteq \Delta(W) \). Then \( W \subseteq L \). Since \( y \in W \), then \( y \in L \) and hence \( L \subseteq \Delta(y) \), so \( V \subseteq \Delta(y) \) this implies that \( \text{SP}(M) \setminus \Delta(y) \subseteq \text{SP}(M) \setminus \Delta(K) \) and hence \( C \Delta(y) \subseteq C \Delta(K) \). This shows that \( N \in C \Delta(y) \) and \( \Delta(K) \subseteq C \Delta(x) \). 

(2) \( \rightarrow \) (1). Let \( I \in \text{SP}(M) \) and \( \Delta(K) \) be any closed set not containing \( I \). Since \( I \notin \Delta(K) \), we have \( K \nsubseteq I \). Then there is an element \( a \in K \) of \( M \) and \( b \in M \) such that \( I \in C \Delta(b) \subseteq \Delta(J) \subseteq C \Delta(a) \). Since \( a \in K \) and \( C \Delta(a) \cap \Delta(K) = \emptyset \), it follows that \( \Delta(K) \subseteq \text{SP}(M) \setminus C \Delta(a) \subseteq \text{SP}(M) \setminus \Delta(J) \). Since \( \Delta(J) \subseteq \text{SP}(M) \) is closed, then \( \text{SP}(M) \setminus \Delta(J) \) is an open set containing the closed \( \Delta(K) \). Clearly \( C \Delta(b) \cap (\text{SP}(M) \setminus \Delta(J)) = \emptyset \), so we find that \( C \Delta(b) \) and \( \text{SP}(M) \setminus \Delta(J) \) are two disjoint open sets containing \( I \) and \( \Delta(K) \) respectively. This shows that \( \text{SP}(M), \tau_{\text{SP}(M)} \) is a regular space. \( \square \)

### 3.12. Theorem

Let \( M \) be a multiplication \( S_f \)-act. Then the following are equivalent

1. \( (\text{SP}(M), \tau_{\text{SP}(M)}) \) is a compact space.

2. For any set \( \{ x_{\alpha} \in M \mid \alpha \in \Lambda \} \) there is a finite subset \( \{ x_i \mid i = 1, 2, \ldots, n \} \) such that for any \( N \in \text{SP}(M) \) there exists \( x_i \) such that \( x_i \notin N \).

**Proof.** (1) \( \rightarrow \) (2). Let \( \{ x_{\alpha} \in M \mid \alpha \in \Lambda \} \) and \( N \) be any element in \( \text{SP}(M) \). Then \( \{ C \Delta(x_{\alpha}) \mid x_{\alpha} \in M, \alpha \in \Lambda \} \) is an open cover of \( \text{SP}(M), \tau_{\text{SP}(M)} \) ) but (1) \( \text{SP}(M), \tau_{\text{SP}(M)} \) ) has a finite subcover \( \{ C \Delta(x_i) \mid i = 1, 2, \ldots, n \} \) and hence \( N \in C \Delta(x_i) \) for some \( x_i \in M \). This implies that \( x_i \notin N \).

(2) \( \rightarrow \) (1). Assume that \( \{ C \Delta(x_{\alpha}) \mid x_{\alpha} \in M, \alpha \in \Lambda \} \) is an open cover of \( \text{SP}(M) \) which has no finite sub cover \( \{ C \Delta(x_i) \mid i = 1, 2, \ldots, n \} \) of \( \text{SP}(M) \). This means that for any finite subset \( \{ x_1, x_2, \ldots, x_n \} \) of \( M \), \( C \Delta(x_1) \cup C \Delta(x_2) \cup \ldots \cup C \Delta(x_n) \neq \text{SP}(M) \) and have \( \Delta(x_i) \cap \Delta(x_j) \cap \ldots \cap \Delta(x_n) \neq \emptyset \). Then there is \( N \in \text{SP}(M) \) such that \( N \in \Delta(x_i) \cap \Delta(x_j) \cap \ldots \cap \Delta(x_n) \). Thus, \( x_1, x_2, \ldots, x_n \in N \) which is contradicts (2). This shows that \( \text{SP}(M), \tau_{\text{SP}(M)} \) ) is a compact space. \( \square \)

An \( S_f \)-act \( M \) is called finitely generated if there exists a finite subset \( X \) of \( M \) such that \( M = \langle X \rangle = \bigcup_{u \in \Sigma} S \Gamma u \) where \( S \Gamma u = \{ s \alpha u \mid s \in S \text{ and } \alpha \in \Gamma \} \).
3.13. **Corollary.** If $M$ is a finitely generating multiplication $S_I$-act. Then $(SP(M), \tau_{SP(M)})$ is a compact space.

**Proof.** Let $\{ u_i \mid i = 1, 2, \ldots, n \}$ be a generated set of $M$, and $N$ a strongly prime $S_I$-subact of $M$. Then there exists some $u_i$ such that $u_i \not\in N$. Hence by Theorem (2.13), $(SP(M), \tau_{SP(M)})$ is a compact space.

\[ \square \]

An $S_I$-act $M$ is called Noetherian if any ascending chain $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n \subseteq \ldots$ of $S_I$-subacts of $M$, there is a positive integer $n$ such that $N_m = N_n$ for $m \geq n$.

3.14. **Theorem.** If $M$ is a Noetherian $S_I$-act. Then $(SP(M), \tau_{SP(M)})$ is countably compact.

**Proof.** Let $\{ \Delta(N_i) \mid i = 1, 2, \ldots, \infty \}$ be a countable collection of closed sets in $SP(M)$ with finite intersection property where $N_i$ is an $S_i$-subact of $M$ for each $i$. Consider the following ascending chain $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n \subseteq \ldots$ of $S_I$-subacts of $M$. Then there is a positive integer $n$ such that $N_i \cup N_2 \cup \ldots \cup N_n = N_1 \cup N_2 \cup \ldots \cup N_{n-1}$. Thus it follows that $N_i \cup N_2 \cup \ldots \cup N_n \in \bigcap_{i=1}^{n} \Delta(N_i)$. Consequently $\bigcap_{i=1}^{n} \Delta(N_i) \neq \emptyset$ and hence $(SP(M), \tau_{SP(M)})$ is countably compact.

\[ \square \]

The following follows from Theorem (3.14) and the fact that a second countable space is compact if it is countably compact.

3.15. **Corollary.** If $M$ is a Noetherian $S_I$-act and $(SP(M), \tau_{SP(M)})$ is second countable, then it is compact.

3.16. **Definition.** The structure space $(SP(M), \tau_{SP(M)})$ is called irreducible if for any decomposition $SP(M) = A_1 \cup A_2$ where $A_1$ and $A_2$ are closed subsets of $SP(M)$ we have $SP(M) = A_1$ or $SP(M) = A_2$.

3.17. **Theorem.** Let $M$ be a multiplication $S_I$-act. Then the following statements are equivalent for any closed subset $A$ of $SP(M)$.

(1) $A$ is irreducible

(2) $\Lambda_{g_{A\alpha} A} N_{\alpha}$ is a prime $S_I$-subact of $M$.

**Proof.** (1) $\rightarrow$ (2). Let $I$ be an ideal of $S$ and $\Lambda$ a $S_I$-subact of $M$ with $I \in \Lambda \subseteq \bigcap_{\alpha \in \Lambda} N_{\alpha}$. Then $I \in \Lambda \subseteq \bigcap_{\alpha \subseteq \Lambda} N_{\alpha}$ for each $\alpha$. Since $N_{\alpha}$ is prime, then either $V \subseteq N_{\alpha}$ or $\Pi M \subseteq N_{\alpha}$ which implies that for $N_{\alpha} \subseteq A$, either $N_{\alpha} \subseteq \{ \emptyset \}$ or $N_{\alpha} \subseteq \{ \Pi M \}$. Hence $A = (A \cap V) \cup (A \cap \Pi M)$, since $A$ is irreducible and both $A \cap V$ and $A \cap \Pi M$ are closed. Then it follows that either $V \subseteq \bigcap_{\alpha \subseteq A} N_{\alpha}$ or $I \subseteq \bigcap_{\alpha \subseteq \Pi M}$ and hence $A \subseteq V$ or $A \subseteq \Pi M$. This implies that $V \subseteq \bigcap_{\alpha \subseteq A} N_{\alpha}$ or $I \subseteq \bigcap_{\alpha \subseteq \Pi M}$ and so $\bigcap_{\alpha \subseteq A} N_{\alpha}$ is a prime in $M$.

(2) $\rightarrow$ (1). Assume $A = A_1 \cup A_2$ where $A_1$ and $A_2$ are closed of $A$. Then $\bigcap_{\alpha \subseteq A} N_{\alpha} \subseteq \bigcap_{\alpha \subseteq A_1} N_{\alpha}$ and $\bigcap_{\alpha \subseteq A} N_{\alpha} \subseteq \bigcap_{\alpha \subseteq A_2} N_{\alpha}$. Also $\bigcap_{\alpha \subseteq A} N_{\alpha} = \bigcap_{\alpha \subseteq A_1} N_{\alpha} \cup \bigcap_{\alpha \subseteq A_2} N_{\alpha}$. For each ideal $I$
of S, \( \Gamma(\cap N_\alpha A_1 N_\alpha) \subseteq \cap N_\alpha A_1 N_\alpha \) and \( \Gamma(\cap N_\alpha A_2 N_\alpha) \subseteq \cap N_\alpha A_2 N_\alpha \) so \( \Gamma(\cap N_\alpha A_1 N_\alpha) \cap (\cap N_\alpha A_2 N_\alpha) = \cap N_\alpha A N_\alpha \). Since \( \cap N_\alpha A N_\alpha \) is prime it follows that \( \cap N_\alpha A_1 N_\alpha \subseteq \cap N_\alpha A N_\alpha \) or \( \Gamma M \subseteq \cap N_\alpha A N_\alpha \) and hence \( \cap N_\alpha A N_\alpha = \cap N_\alpha A_1 N_\alpha \) and \( \Gamma M \subseteq \cap N_\alpha A_2 N_\alpha \) similarly \( \cap N_\alpha A_2 N_\alpha = \cap N_\alpha A N_\alpha \) and \( \Gamma M \subseteq \cap N_\alpha A N_\alpha \). It follows that \( \cap N_\alpha A N_\alpha = \cap N_\alpha A_1 N_\alpha \) and \( \cap N_\alpha A_2 N_\alpha = \cap N_\alpha A N_\alpha \). Let \( N_\beta \in A \). Then we have \( \cap N_\alpha A_1 N_\alpha \subseteq N_\beta \) or \( \cap N_\alpha A_2 N_\alpha \subseteq N_\beta \). Since \( A_1, A_2 \subseteq A \), so either \( N_\alpha \subseteq N_\beta \) for all \( N_\alpha \in A_1 \) or \( N_\alpha \subseteq N_\beta \) for all \( N_\alpha \in A_2 \). Thus \( N_\alpha \subseteq A_1 = A_2 \), since \( A_1 \) and \( A_2 \) are closed i.e \( A = A_1 \) or \( A = A_2 \). This proves (1). □

3.18. **Corollary.** Let \( M \) be a uniserial multiplication \( S_t \)-act. Then any closed subset of \( \text{SP}(M) \) is irreducible.

**Proof.** Let \( A \) be a closed subset of \( \text{SP}(M) \). Then by Corollary (2.1.4) we have \( \cap N_\alpha A N_\alpha \) is a prime \( S_t \)-subact of \( M \). Hence by Theorem (3.16) we get \( A \) is irreducible. □

Let \( M \) be a uniserial multiplication \( S_t \)-act and \( N, K \) two \( S_t \)-subacts of \( M \). We define \( NK = \text{Hom}(M, K)N = \cup \{ \alpha(N) \mid \alpha : M \to K \} \).

An \( S_t \)-subact \( N \) of \( M \) is called idempotent if \( N = NN = U(N) \) where the union runs among all \( S_t \)-homomorphism \( \varphi : M \to N \). This is equivalent to saying that for each \( n \in N \), there exist an \( S_t \)-homomorphism \( \varphi : M \to N \) and an element \( n' \in N \) such that \( n = \varphi(n') \). An element \( m \in M \) is called idempotent if it generates an idempotent \( S_t \)-subact of \( M \), namely \( SFm \) is idempotent \( S_t \)-subact of \( M \). We denote \( e(M) \) for the set of all idempotent elements of \( M \).

3.19. **Definition.** An \( S_t \)-subact \( N \) of an \( S_t \)-act \( M \) is called id-full if \( e(M) \subseteq N \).

Let \( W \) be the collection of all strongly prime id-full \( S_t \)-subacts of an \( S_t \)-act \( M \). Then clearly \( W \subseteq \text{SP}(M) \) and hence \( (W, \tau_W) \) is a topological space where \( \tau_W \) is the subspace topology generally \( (\text{SP}(M), \tau_{\text{SP}(M)}) \) is neither compact nor connected. But in particular we have the following results.

3.20. **Proposition.** Let \( M \) be a uniserial multiplication \( S_t \)-act. Then every closed subset of \( \text{SP}(M) \) is connected.

**Proof.** Let \( A \) be a closed subset of \( \text{SP}(M) \). By Theorem (3.17), \( A \) is irreducible. Hence \( A \) is connected.

3.21. **Theorem.** Let \( M \) be a multiplication \( S_t \)-act. Then \( (W, \tau_W) \) is a connected space.

**Proof.** Let \( N \) be the strongly prime \( S_t \)-subact of \( M \) generated by \( e(M) \). Since every strongly prime id-full \( K \) of \( M \) contains \( e(M) \), contains \( N \). Thus \( N \) belongs to any closed subset \( \Delta(N') \) of \( W \). This implies that any two closed subsets of \( W \) are not disjoint. Hence \( (W, \tau_W) \) is a connected space. □

3.22. **Theorem.** Let \( M \) be a multiplication \( S_t \)-act. Then \( (W, \tau_W) \) is a compact space.

**Proof.** Let \( \{ \Delta(N_\alpha) \mid \alpha \in \Lambda \} \) be any collection of closed subsets on \( W \) with finite intersection property, and \( N \) be the strongly prime \( S_t \)-subact generated by \( e(M) \). Since any strongly prime id-full \( S_t \)-subact \( K \) contains \( e(M) \),
contains $N$. Hence $N \in \bigcap_{a \in \lambda} \Delta(N_a)$ and so $\bigcap_{a \in \lambda} \Delta(N_a)$ is nonempty. This implies that $(W, \tau_W)$ is a compact space. □

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