A Murnaghan–Nakayama Rule for Grothendieck Polynomials of Grassmannian Type

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Abstract

We consider the Grothendieck polynomials appearing in the $K$-theory of Grassmannians, which are analogs of Schur polynomials. This paper aims to establish a version of the Murnaghan–Nakayama rule for Grothendieck polynomials of the Grassmannian type. This rule allows us to express the product of a Grothendieck polynomial with a power-sum symmetric polynomial into a linear combination of other Grothendieck polynomials.

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1 Introduction

The $K$-theory of flag varieties was studied by Kostant and Kumar [KK87], and by Demazure [Dem74]. Lascoux and Schutzenberger introduced the Grothendieck polynomials as representatives for the structure sheaves of the Schubert varieties of a flag variety [Ghe06, LS82, Las07]. For any permutation $w \in \bigcup_{m \geq 1} S_m$, the Grothendieck polynomial $G_w := G_w(x_1, x_2, \ldots)$ is defined by isobaric divided difference operators. Fomin and Kirilov studied combinatorics of these polynomials in [FK96, FK94].

Let $s_\lambda$ be the Schur function associated with a partition $\lambda$, and $p_k$ be the powersum symmetric functions of degree $k$ [Mac91]. The classical Murnaghan–Nakayama rule describes the decomposition of the product $s_\lambda p_k$ into a sum of Schur functions [Mac91] as follows. We have

$$s_\lambda p_k = \sum_{\mu} (-1)^{r(\mu/\lambda)+1} s_\mu,$$

where the sum runs over all partitions $\mu$ such that $\mu/\lambda$ is a ribbon of size $k$ and $r(\mu/\lambda)$ is the number of rows of skew shape $\mu/\lambda$.

The classical Murnaghan–Nakayama rule plays an important role in the representation theory of symmetric groups. It gave a formula for the character table [Nak40b, Nak40a, Mur37]. Because of the classical story, many extensions and generalizations of the classical Murnaghan–Nakayama rule were studied. Indeed, a version for non-commutative symmetric functions is given by Fomin and Green in [FG98] (it led to formulas for characters of representations associated with stable Schubert and Grothendieck polynomials). A skew version and its generalization of multiplication with quantum power-sum function are given by [Kon12, AM11]. A version for non-commutative Schur functions can be found in [Tew16]. A plethystic version is given by [Wil16]. A version for loop Schur
functions is given by [Ros14] (it provides a fundamental step in the orbifold Gromov–Witten/Donaldson–Thomas correspondence in [RZ13]). A version in the cohomology of an affine Grassmannian can be found in [BSZ11]. An extended version of Schubert polynomials and the quantum cohomology of Grassmannians can be found in [MS18].

In this paper, we restrict our attention to the simplest complex flag variety: the Grassmann variety of \( n \)-dimensional subspaces of \( \mathbb{C}^{n+m} \). The operator-definition of \( \mathcal{G} \)-reduces in the special case of Grassmannian permutations of descent \( n \) to symmetric polynomials in \( n \) variables indexed by partitions \( \lambda \) with at most \( n \) parts, commonly relabeled as \( G_\lambda \) and called Grassmannian Grothendieck polynomials. These \( G_\lambda \) are \( K \)-theoretic analogs of Schur functions, representing the \( K \)-theory of the Grassmannian variety of \( n \)-dimensional subspaces of \( \mathbb{C}^{n+m} \). Like the Schur functions, they are given by a number of formulas (see Sect. 2.3). For example

\[
G_\lambda(x_1, \ldots, x_n) = \frac{\det(x_i^{\lambda_i+n-j}(1+\beta x_i)^{j-1})_{n \times n}}{\prod_{1 \leq i < j \leq n}(x_i - x_j)},
\]

where \( \beta \) is a formal parameter. Recall that, if \( \beta = 0 \), then \( G_\lambda \) is identified with the Schur function \( s_\lambda(x_1, \ldots, x_n) \). The products of \( G_\lambda \) with other special symmetric polynomials \( e_k, h_k \) are mentioned in [Len00], in which Lenart studied the Pieri rules of the Grassmannian Grothendieck polynomials. Our work on the product \( G_\lambda p_k \) can be considered as a \( K \)-theoretic version of the classical Murnaghan–Nakayama rule.

Let \( \lambda \) and \( \mu \) be partitions of length at most \( n \) and \( \lambda \leq \mu \). Let \( |\mu/\lambda|, c(\mu/\lambda), r(\mu/\lambda) \) be the size, number of columns, number of rows of the skew shape \( \mu/\lambda \). We say two boxes in a skew shape are adjacent whenever they share an edge, and we say a skew shape \( \mu/\lambda \) is connected whenever every pair of its boxes is connected by a sequence of adjacent boxes. A ribbon is a connected skew shape with no \( 2 \times 2 \) square. When \( \mu/\lambda \) is connected, the maximal ribbon along the northwest border of \( \mu/\lambda \) is the ribbon \( \nu/\lambda \) of size as max as possible, contained in \( \mu/\lambda \).

The main result of this paper is stated as follows.

**Theorem 1.1.** For any partition \( \lambda \) of length at most \( n \) and \( k \in \mathbb{Z}_{>0} \), we have

\[
G_{\lambda p_k} = \sum_\mu (-\beta)^{|\mu/\lambda|-k}(-1)^{k-c(\mu/\lambda)} \left( \frac{r(\mu/\lambda) - 1}{k - c(\mu/\lambda)} \right) G_\mu,
\]

where the sum runs over all partitions \( \mu \) of length at most \( n \), \( \mu \geq \lambda \) such that \( c(\mu/\lambda) \leq k \), \( \mu/\lambda \) is connected and the maximal ribbon along its northwest border has size at least \( k \).

This paper is organized as follows. In Sect. 2 we recall the basic knowledge related to symmetric polynomials, partitions, diagrams, binary tableaux, Grothendieck polynomials of Grassmannian type. In Sect. 3 we prove our main result.

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2 Preliminaries

2.1 Symmetric Polynomials

A polynomial $f(x_1, \ldots, x_n)$ in $n$ variables is said to be symmetric if for all permutations $\sigma \in S_n$, then we have

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n).$$

There are fundamental symmetric polynomials: The $k$-th elementary symmetric polynomial

$$e_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k},$$

the $k$-th complete homogeneous symmetric polynomial

$$h_k = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} x_{i_1} \cdots x_{i_k},$$

and the $k$-th power-sum symmetric polynomial

$$p_k = \sum_{i=1}^{n} x_i^k.$$

Let $k$ be a positive integer. The following formula is key to the proof of Theorem 1.1.

**Lemma 2.1.**

$$p_k = \sum_{i=0}^{k-1} (-1)^i (k - i) e_i h_{k-i}. \tag{1}$$

**Proof.** The proof of the equality (1) is as follows. We consider the following generating functions

$$H(t) = \sum_{k \geq 0} h_k t^k = \prod_{i=1}^{n} \frac{1}{1 - x_i t},$$

$$E(t) = \sum_{k \geq 0} e_k t^k = \prod_{i=1}^{n} (1 + x_i t),$$

$$P(t) = \sum_{k \geq 1} p_k t^{k-1} = \sum_{i=1}^{n} \frac{x_i}{1 - x_i t}.$$  

By (2.6), (2.10) in [Mac98], we have

$$P(t) = H'(t)/H(t) = H'(t)E(-t).$$

By comparing the coefficients of $t^{k-1}$ in both sides of the identity, we get the conclusion. \qed
2.2 Partitions, Diagrams and Binary Tableaux

A non-negative integer sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ is called a partition if $\lambda_1 \geq \lambda_2 \geq \ldots$. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ with $\lambda_1 > 0$ and $\sum_{i=1}^{l} \lambda_i = m$, we write $l(\lambda) = l$, $|\lambda| = m$. We call $l(\lambda)$ the length, and $|\lambda|$ the size of the partition $\lambda$. Each partition $\lambda$ is presented by a Young diagram that is a collection of boxes such that: The leftmost boxes of each row are in a column, and the numbers of boxes from top row to bottom row are $\lambda_1, \lambda_2, \ldots$, respectively.

Example 2.2. The Young diagram associated to the partition $\lambda = (3, 3, 2, 1, 0, 0)$ is

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\nu = (\nu_1, \nu_2, \ldots)$ be partitions. We define the sum of two partitions by $\lambda + \nu = (\lambda_1 + \nu_1, \lambda_2 + \nu_2, \ldots)$. For a non-negative integer $n$, we denote $P_n$ the set of all partition of length at most $n$. Let $(1^n)$ be the $n$-tuple partition $(1, \ldots, 1)$, and $\lambda = (\lambda_1, \ldots, \lambda_n) \in P_n$, then we have $\lambda + (1^n) = (\lambda_1 + 1, \ldots, \lambda_n + 1)$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ be two partitions. We say that $\lambda$ is smaller than $\mu$ if and only if $\lambda_i \leq \mu_i$ for all $i$, and we write $\lambda \leq \mu$. In this case, we define the skew Young diagram $\mu/\lambda$ as the result of removing boxes in the Young diagram $\lambda$ from the Young diagram $\mu$. We write $|\mu/\lambda| = |\mu| - |\lambda|$ for the size, and $r(\mu/\lambda), c(\mu/\lambda)$ for the number of rows, columns of the skew Young diagram $\mu/\lambda$, respectively. We say two boxes in a skew shape are adjacent whenever they share an edge, and we say a skew shape $\mu/\lambda$ is connected whenever every pair of its boxes is connected by a sequence of adjacent boxes. A ribbon is a connected skew shape with no $2 \times 2$ square. The maximal ribbon along the northwest border of a connected skew Young diagram $\mu/\lambda$ is the ribbon $\nu/\lambda$ of size as max as possible, contained in $\mu/\lambda$. A binary tableau $T$ of skew shape $\mu/\lambda$ is a result of filling the skew Young diagram $\mu/\lambda$ by the alphabet $\{0, 1\}$ such that the entry in the bottom of each column is $1$. A binary tableau $T$ is said to have content $\alpha(T) = (\alpha_0, \alpha_1)$ if $\alpha_i = \alpha_i(T)$ is the number of entries $i$ in $T$. We write $sh(T)$ for the shape of the tableau $T$.

Example 2.3. We consider partitions in $P_6$: $\lambda = (3, 3, 2, 1, 0, 0)$ and $\mu = (4, 3, 3, 3, 1, 1)$. Then $\mu \geq \lambda$ and the skew diagram $\mu/\lambda$ has $r(\mu/\lambda) = 5$ rows, $c(\mu/\lambda) = 4$ columns. In this case $\mu/\lambda$ is not connected and is not a ribbon. However, it contains ribbons of size $1, 2, 3$, for example

The following tableau $T$ is a binary tableau of skew shape $sh(T) = \mu/\lambda$. 

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Here the diagram in gray means the Young diagram $\lambda$ removed from the Young diagram $\mu$. We bold-outline the skew shape $\mu/\lambda$. The content of the binary tableau $T$ is $\alpha(T) = (1, 5)$. Other example, let $\nu = (2, 1, 0, 0, 0, 0)$, then $\mu/\nu$ is connected and is not a union of ribbons. The following is a binary tableau of skew shape $\mu/\nu$, with content $(5, 7)$.

### 2.3 Grothendieck Polynomials of Grassmannian Type

In this paper, we restrict our attention to the simplest complex flag variety: the Grassmann variety of $n$ dimensional subspaces of $\mathbb{C}^{n+m}$. The Grothendieck polynomials in this case are indexed by Grassmannian permutations [Buc02]. Namely, let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition of length at most $n$. The Grassmannian permutation $w_\lambda$ of descent $n$ is defined by $w_\lambda(i) = i + \lambda_{n+1-i}$ for $i \in [1, n]$ and $w_\lambda(i) < w_\lambda(i+1)$ for all $i \neq n$. Set $G_\lambda = G_{w_\lambda}$. There are several new formulas for $G_\lambda$, for example, in the terms of set-valued tableaux [Buc02] or Jacobi–Trudy identity [Kir16]. We here recall the Weyl identity given by Ikeda and Naruse [IS14, IN13]:

$$G_\lambda(x_1, \ldots, x_n) = \frac{\det(x_i^{\lambda_i+n-j}(1 + \beta x_i)^{j-1})_{n \times n}}{\prod_{1 \leq i < j \leq n}(x_i - x_j)}.$$ 

They are polynomial representatives of Schubert classes of the $K$-theory of Grassmann varieties of $n$ dimensional subspaces of $\mathbb{C}^{n+m}$.

**Example 2.4.** For $n = 3$ and $\lambda = (2, 1, 0)$, we have

$$G_{(2,1,0)}(x_1, x_2, 0) = \begin{vmatrix} x_1^4 & x_1^2(1 + \beta x_1) & (1 + \beta x_1)^2 \\ x_2^4 & x_2^2(1 + \beta x_2) & (1 + \beta x_2)^2 \\ 0 & 0 & 1 \end{vmatrix} = (x_1 - x_2)x_1x_2$$

$$= x_1^2x_2 + x_1x_2^2 + \beta x_1^2x_2^2.$$ 

We can check that $G_{(2,1,0)}(x_1, x_2, 0) = G_{(2,1)}(x_1, x_2)$.

**Remark 2.5.** For $\beta = 0$, the Grothendieck polynomial $G_\lambda(x_1, \ldots, x_n)$ coincides with the Schur polynomial $s_\lambda(x_1, \ldots, x_n)$.
3 Proof of the Main Theorem

We only need to focus on the case $\beta \neq 0$. Indeed, if $\beta = 0$, then $G_\lambda$ is the Schur function associated with partition $\lambda$. The Murnaghan–Nakayama rule for Schur polynomials is very well known [Mac91]. When $\beta \neq 0$, set

$$\tilde{G}_\lambda(x_1, \ldots, x_n) = \beta^{\lambda\lambda} G_\lambda\left(\frac{x_1}{\beta}, \ldots, \frac{x_n}{\beta}\right).$$

Theorem 1.1 can be reduced to the following theorem.

**Theorem 3.1.** For any partition $\lambda \in \mathcal{P}_n$ and $k \in \mathbb{Z}_{\geq 0}$, we have

$$\tilde{G}_\lambda p_k = \sum_{\mu} (-1)^{\mu\lambda - c(\mu/\lambda)} \left(\frac{r(\mu/\lambda) - 1}{k - c(\mu/\lambda)}\right) \tilde{G}_\mu,$$

where the sum runs over all partitions $\mu \in \mathcal{P}_n$, $\mu \geq \lambda$ such that $c(\mu/\lambda) \leq k$, $\mu/\lambda$ is connected and the maximal ribbon along its northwest border has size at least $k$.

Before going to the proof, we need to restate the following lemma. It was key to obtain the Pieri rules for Grothendieck polynomials of Grassmannian type [Len00].

**Lemma 3.2 (Theorem 3.2, [Len00]).** For any partition $\lambda \in \mathcal{P}_n$ and $k \in \mathbb{N}$, we have

$$\tilde{G}_\lambda e_k = \sum_T (-1)^{\alpha_0(T)} \tilde{G}_\mu,$$

$$\tilde{G}_\lambda h_k = \sum_T (-1)^{\alpha_0(T)} \tilde{G}_\mu.$$  \hspace{1cm} (3)  \hspace{1cm} (4)

The first sum runs over all binary tableaux $T$ of shape $\mu/\lambda$ with $\mu \in \mathcal{P}_n$, $\lambda \leq \mu \leq \lambda + (1^n)$, $\alpha_1(T) = k$. The second sum runs over all binary tableaux $T$ of shape $\mu/\lambda$ with $\mu \in \mathcal{P}_n$, $\lambda \leq \mu$, $\alpha_1(T) = k$, no two 1’s in the same column.

**Proof of Theorem 3.1.** By equalities (1), (3), (4), we have

$$\tilde{G}_\lambda p_k = \sum_{i=0}^{k-1} (-1)^i (k-i) \tilde{G}_\lambda e_i h_{k-i}$$

$$= \sum_{i=0}^{k-1} (-1)^i (k-i) \sum_T (-1)^{\alpha_0(T)} \tilde{G}_\mu h_{k-i}$$

where $T_1$ has shape $\nu/\lambda$ with $\nu \in \mathcal{P}_n$, $\lambda \leq \nu \leq \lambda + (1^n)$, $\alpha_1(T_1) = i$,

$$= \sum_{i=0}^{k-1} (-1)^i (k-i) \sum_{T_1} (-1)^{\alpha_0(T_1)} \tilde{G}_\mu h_{k-i}$$

where $T_2$ has shape $\mu/\nu$ with $\mu \in \mathcal{P}_n$, $\nu \leq \mu$, $\alpha_1(T_2) = k-i$, no two 1’s in the same column,

$$= \sum_{\mu} \sum_{T=T_1\cup T_2} (-1)^{|\mu/\lambda|-\alpha_1(T_2)} \alpha_1(T_2) \tilde{G}_\mu,$$

because

$$i = \alpha_1(T_1), k-i = \alpha_1(T_2),$$
and
\[ \alpha_0(T_1) + \alpha_1(T_1) + \alpha_0(T_2) + \alpha_1(T_2) = |\mu/\lambda|. \]

In (8), the sum runs over binary tableaux \( T = T_1 \sqcup T_2 \) of shape \( \mu/\lambda \), with \( \mu \in P_n, \mu \geq \lambda \), \( \alpha_1(T) = k \), where \( T_1 \) and \( T_2 \) are filled according to (6) and (7). Fix such a shape \( \mu \) containing \( \lambda \), we are going to determine the form of \( \mu \) and the coefficient of \( \tilde{G}_\mu \) appearing in the decomposition of \( \tilde{G}_\lambda \).

First step: Construct all tableaux \( T \) mentioned in (8) of given skew shape \( \mu/\lambda \) such that the numbers of entries with value 1 in \( T_2 \) is a fixed number \( j \). We proceed as follows.

- First, we label all boxes in the bottom of each column of the skew diagram \( \mu/\lambda \) by 1. Let \( B \) be the set of boxes we have created.
- Now, we will choose a subset of boxes in \( B \) and set it as the set of boxes in the bottom of \( T_2 \), say \( B_2 \). In fact, we cannot choose such a subset arbitrarily because its complement in \( B \), say \( B_1 \), will be a subset of boxes in the bottom of \( T_1 \). Hence, it must satisfy a strict condition that the boxes in \( B_1 \) are located in the skew diagram \( (\lambda + (1^n))/\lambda \). Therefore, to choose a subset \( B_2 \), we should start by choosing \( B_1 \).

Fixing a number of entries with value 1 in \( T_2 \), say \( \alpha_1(T_2) = j \), we have

- The cardinality of \( B_1 \) is \( c(\mu/\lambda) - j \).
- The elements in \( B_1 \) are chosen arbitrarily from \( \gamma := B \cap (\lambda + (1^n))/\lambda \).

Hence, for a fixed \( \alpha_1(T_2) = j \), the number of choices of \( B_2 \) is equal to the number of choices of \( B_1 \) and it is

\[
\binom{|\gamma|}{c(\mu/\lambda) - j}. \tag{9}
\]

Since \( B_2 = B \setminus B_1 \), we have

\[
j = |B_2| \in [|B \setminus B_1|, |B|] = [c(\mu/\lambda) - |\gamma|, c(\mu/\lambda)]. \tag{10}
\]

- Now, the last step to construct tableau \( T \) is locating remaining entries with value 1 of \( T \) which are not in the bottom \( B \) in the skew diagram \( (\lambda + (1^n))/\lambda \cap (\mu/\lambda) \). We have

- The number of remaining entries with value 1 is \( k - c(\mu/\lambda) \).
- Such entries with value 1 are chosen arbitrarily from \( \eta := (\lambda + (1^n))/\lambda \cap (\mu/\lambda) \setminus \gamma \).

Hence, the number of choices of this step is

\[
\binom{|\eta|}{k - c(\mu/\lambda)}. \tag{11}
\]

Therefore, we have described a way to construct tableaux \( T \) of given skew shape \( \mu/\lambda \) such that the number of entries with value 1 in \( T_2 \) is a fixed number \( j \).

Second step: Substitute (9), (10), (11) to (8) and simplify it. We have:

\[
\tilde{G}_{\lambda \mu} = \sum_{\mu} \sum_{j=c(\mu/\lambda) - |\gamma|}^{c(\mu/\lambda)} (-1)^{|\mu/\lambda| - j} j \binom{|\gamma|}{c(\mu/\lambda) - j} \binom{|\eta|}{k - c(\mu/\lambda)} \tilde{G}_\mu. \tag{12}
\]
We note that the binomial coefficient
\[
\binom{|\eta|}{k - c(\mu/\lambda)}
\]
depends only on \( \lambda, \mu \) and \( k \). Thus, in order to simplify the coefficient of \( \tilde{G}_\mu \), we only need to determine the sum
\[
\sum_{j=c(\mu/\lambda)-|\gamma|}^{c(\mu/\lambda)} (-1)^{c(\mu/\lambda) - j} j \binom{|\gamma|}{c(\mu/\lambda) - j}.
\]

Since \( k > 0 \), then \( |\gamma| \geq 1 \). We prove the following lemma.

**Lemma 3.3.** The sum
\[
\sum_{j=c(\mu/\lambda)-|\gamma|}^{c(\mu/\lambda)} (-1)^{c(\mu/\lambda) - j} j \binom{|\gamma|}{c(\mu/\lambda) - j}
\]
is equal to 0 if \( |\gamma| > 1 \) and 1 if \( |\gamma| = 1 \).

**Proof.** First, we consider the following identity:
\[
(1 - x)^m = \sum_{i=0}^{m} (-x)^i \binom{m}{i}.
\]
When \( m \geq 1 \), \( x = 1 \), from (14), we get
\[
0 = \sum_{i=0}^{m} (-1)^i \binom{m}{i}.
\]
Differentiating both sides of (14), we get
\[
m(1 - x)^{m-1} = \sum_{i=0}^{m} i(-x)^{i-1} \binom{m}{i}.
\]
When \( m > 1 \), \( x = 1 \), from (16), we get
\[
0 = \sum_{i=0}^{m} (-1)^i i \binom{m}{i}.
\]
Now, we use the equalities above to prove the lemma. Set \( i = c(\mu/\lambda) - j \) and \( c = c(\mu/\lambda) \). Then, (13) can be rewritten as
\[
\sum_{i=0}^{|\gamma|} (-1)^i (c - i) \binom{|\gamma|}{i} = c \sum_{i=0}^{|\gamma|} (-1)^i \binom{|\gamma|}{i} - \sum_{i=0}^{|\gamma|} (-1)^i i \binom{|\gamma|}{i}.
\]
Since \( |\gamma| \geq 1 \), then by (15), we get
\[
\sum_{i=0}^{|\gamma|} (-1)^i \binom{|\gamma|}{i} = 0.
\]
If $|\gamma| > 1$, then by (17), we get
\[
\sum_{i=0}^{|\gamma|} (-1)^i \binom{|\gamma|}{i} = 0.
\]

If $|\gamma| = 1$, then
\[
\sum_{i=0}^{|\gamma|} (-1)^i \binom{|\gamma|}{i} = \sum_{i=0}^1 (-1)^i \binom{1}{i} = -1.
\]
We obtain the result as desired.

Now, we consider two cases.

- If $|\gamma| = 1$, then $\mu/\lambda$ is connected (by definition of $\gamma$, the cardinality of $\gamma$ counts the number of connected components of $\mu/\lambda$). The reader may check that connectedness implies $|(\lambda + \binom{n}{1})/\lambda \cap (\mu/\lambda)| = r(\mu/\lambda)$. Therefore, $|\eta| = r(\mu/\lambda) - 1$. The coefficient of $\tilde{G}_\mu$ in (12) is
\[
(-1)^{|\mu/\lambda| - c(\mu/\lambda)} \binom{r(\mu/\lambda) - 1}{k - c(\mu/\lambda)}.
\]

- If $|\gamma| > 1$, the coefficient of $\tilde{G}_\mu$ in (12) is 0.

When $|\gamma| = 1$, the conditions that $\mu/\lambda$ is the shape of a tableau $T = T_1 \sqcup T_2$, where $T_1, T_2$ are of the form in (3), (4) and that $a_1(T) = k$ are equivalent to the conditions $c(\mu/\lambda) \leq k$ (entries with value 1 in bottom $\mathcal{B}$ is a part of all entries with value 1 of $T$) and $k - c(\mu/\lambda) \leq r(\mu/\lambda) - 1$ (entries with value 1 not in bottom $\mathcal{B}$ can be filled into $\eta$). The last inequality condition can be rewritten as
\[
k \leq c(\mu/\lambda) + r(\mu/\lambda) - 1.
\]

Since $\mu/\lambda$ is connected, the right-hand side of (18) counts the size of the maximal ribbon contained in the skew shape $\mu/\lambda$ along its northwest border. Hence, the conditions of $\mu$ such that $\tilde{G}_\mu$ appears in the decomposition of $\tilde{G}_{\lambda \mu p_k}$ are: $\mu \in \mathcal{P}_n, \mu \geq \lambda$ such that $c(\mu/\lambda) \leq k$, $\mu/\lambda$ is connected and the maximal ribbon along its northwest border has size at least $k$.

The example below visualizes the first step, constructing tableaux $T$, in the proof of Theorem 3.1.

**Example 3.4.** We continue Example 2.3. In the picture below, $\mathcal{B}$ is the set of four boxes $\begin{array}{c} 1 \end{array}$, and the skew diagram $(\lambda + \binom{n}{1})/\lambda$ is colored in yellow. Therefore, $\gamma$ is the set of three yellow boxes with blue entries with value 1 inside.
The range of the number of entries with value 1 in $T_2$ is $j \in [1, 4]$. If we fix $j = 2$, then there are three choices of $B_1$ (also $B_2$) as in the picture below (the entries with value 1 in $B_1$ are circled).

![Diagram of entries with value 1 circled]

The skew shape $\eta$ contains two boxes where we put $*$ inside in the picture as follows.

![Diagram of skew shape with marked box]

If $k = 5$, then we just need to put only one remaining entry 1 arbitrarily to the boxes marked by $*$. The remaining boxes of $\mu/\lambda$ are labeled by 0. For example, if we fix the first choice of $B_1$ in the picture above ($j = 2$), we have two tableaux below (empty yellow boxes are not counted in tableaux).

![Diagram of tableaux]

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