Mean-field limit and Semiclassical Expansion of a Quantum Particle System

Federica Pezzotti, Mario Pulvirenti

Abstract

We consider a quantum system constituted by \( N \) identical particles interacting by means of a mean-field Hamiltonian. It is well known that, in the limit \( N \to \infty \), the one-particle state obeys to the Hartree equation. Moreover, propagation of chaos holds. In this paper, we take care of the \( \hbar \) dependence by considering the semiclassical expansion of the \( N \)-particle system. We prove that each term of the expansion agrees, in the limit \( N \to \infty \), with the corresponding one associated with the Hartree equation. We work in the classical phase space by using the Wigner formalism, which seems to be the most appropriate for the present problem.

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1. Introduction

The Hartree equation is the following nonlinear one-particle Schrödinger equation:

\[
i\hbar \partial_t \psi = -\frac{\hbar^2}{2} \Delta \psi + (\phi \ast |\psi|^2) \psi,
\]

where the mass of the particle is chosen equal to one, \( \phi : \mathbb{R}^3 \to \mathbb{R} \) is the two-body potential and

\[
(\phi \ast |\psi|^2) (x) = \int_{\mathbb{R}^3} dy \phi(x - y) |\psi(y)|^2
\]
is the effective self-consistent interaction. Equation (1.1) arises as the reduced description of a system of $N$ identical particles interacting by means of the mean-field potential:

$$U_N(X_N) = \frac{1}{N} \sum_{1 \leq l < j \leq N} \phi(x_l - x_j),$$

(1.3)

(where $X_N = \{x_1, \ldots, x_N\}$, $x_j \in \mathbb{R}^3$, $j = 1, \ldots, N$) in the limit $N \to \infty$.

In fact, consider the $N$-particle wave function $\Psi_N = \Psi_N(X_N; t)$ solution of the Schrödinger equation:

$$i\hbar \partial_t \Psi_N = -\frac{\hbar^2}{2} \sum_{i=1}^{N} \Delta x_i \Psi_N + U_N \Psi_N,$$

(1.4)

with a completely factorized initial state given by:

$$\Psi_N(X_N; 0) = \Psi_{N,0}(X_N) = \prod_{j=1}^{N} \psi_0(x_j).$$

(1.5)

Then, it is well known that the $j$-particle reduced density matrices, defined as:

$$\rho^N_j(X_j, Y_j; t) = \int_{\mathbb{R}^{3(N-j)}} dX_{N-j} \Psi_N(X_j, X_{N-j}; t) \overline{\Psi}_N(Y_j, X_{N-j}; t),$$

(1.6)

converges, in the limit $N \to \infty$ and for any fixed $j = 1, \ldots, N$, to the factorized state:

$$\rho_j(X_j, Y_j; t) = \prod_{k=1}^{j} \psi(x_k; t) \overline{\psi}(y_k; t),$$

(1.7)

where $\psi(x; t)$ solves the one-particle Hartree equation (1.1) with initial datum $\psi_0$. This feature is usually called ”propagation of chaos”.

The previous result was originally obtained for sufficiently smooth potentials (see [1], [2], [3]); then it has been generalized to include Coulomb interactions (see [4], [5], [6]). Furthermore, some results concerning the speed of convergence of the mean-field evolution to the Hartree dynamics (for all fixed times), have been proven more recently (see [7], [8]).

The limit $N \to \infty$ for a classical system interacting by means of the same mean-field interaction (1.3), can be considered as well (see [9], [10], [11], [12], [13] for the case of smooth potential, and [14] for more singular interaction). In fact, considering as initial state of the system a probability measure $F_{N,0} = F_{N,0}(X_N, V_N)$ in the $N$-particle phase space which is completely factorized, namely:

$$F_{N,0}(X_N, V_N) = \prod_{j=1}^{N} f_0(x_j, v_j),$$

(1.8)
where $f_0$ is a given one-particle probability density, it is known that its evolution at time $t > 0$, denoted by $F_N(X_N, V_N; t)$, is obtained by solving the Liouville equation:

$$(\partial_t + V_N \cdot \nabla_{X_N}) F_N = \nabla_{X_N} U_N \cdot \nabla_{V_N} F_N. \quad (1.9)$$

Then, the $j$-particle distribution at time $t > 0$, defined as

$$f_N^j(X_j, V_j; t) = \int_{\mathbb{R}^{3(N-j)} \times \mathbb{R}^{3(N-j)}} dX_{N-j} dV_{N-j} F_N(X_j, V_j, X_{N-j}, V_{N-j}; t), \quad (1.10)$$

converges, as $N \to \infty$, to the product state:

$$f_j(X_j, V_j; t) = \prod_{k=1}^j f(x_k, v_k; t), \quad (1.11)$$

where $f(x, v; t)$ is the solution of the Vlasov equation:

$$(\partial_t + v \cdot \nabla_x) f = (\nabla \phi * f) \cdot \nabla_v f, \quad (1.12)$$

(the convolution above is with respect to both the variables $x$ and $v$) with initial datum $f_0$.

Equation (1.12) is the classical analogous of the Hartree equation (1.1).

Although the mean-field limit $N \to \infty$ is well understood for both classical and quantum systems, there is a question which seems to be still open, namely, does that limit hold for quantum systems uniformly in $\hbar$, at least for systems having a reasonable classical analogue? The proofs which are available up to now exhibit an error vanishing when $N \to \infty$ but diverging as $\hbar \to 0$, although in [16], [17], [18], [19] some efforts in the direction of a better control of the error term have been done.

If one wants to deal with the classical and quantum case simultaneously, it is natural to work in the classical phase space by using the Wigner formalism.

The one-particle Wigner function associated with the wave function $\psi(x; t)$ is given by:

$$f^h(x, v; t) = (2\pi)^{-3} \int_{\mathbb{R}^3} dy e^{i y \cdot v} \overline{\psi} \left(x + \frac{h y}{2}; t\right) \psi \left(x - \frac{h y}{2}; t\right), \quad (1.13)$$

and, similarly, the $N$-particle Wigner function associated with the wave function $\Psi_N(X_N; t)$ is defined as:

$$W^h_N(X_N, V_N; t) = (2\pi)^{-3N} \int_{\mathbb{R}^{3N}} dY_N e^{i Y_N \cdot V_N} \overline{\Psi}_N \left(X_N + \frac{h Y_N}{2}; t\right) \Psi_N \left(X_N - \frac{h Y_N}{2}; t\right). \quad (1.14)$$

Then, by using that $\psi(x; t)$ and $\Psi_N(X_N; t)$ solve equations (1.1) and (1.4) respectively, we find the equations:

$$(\partial_t + v \cdot \nabla_x) f^h = T^h f^h \quad (1.15)$$

and

$$(\partial_t + V_N \cdot \nabla_{X_N}) W^h_N = T^h_N W^h_N, \quad (1.16)$$

where $T^h$ and $T^h_N$ are suitable pseudodifferential operators.
The initial data for equations (1.15) and (1.16) are
\[
f_0^h(x, v) = (2\pi)^{-3} \int_{\mathbb{R}^3} dy \ e^{iy \cdot \psi_0 / 2} \left( x + \hbar y \right) \psi_0 \left( x - \hbar y / 2 \right),
\]
and
\[
W_{N,0}^h(X_N, V_N) = (2\pi)^{-3N} \int_{\mathbb{R}^{3N}} dY_N e^{iY_N \cdot V_N} \overline{\Psi}_{N,0} \left( X_N + \hbar Y_N / 2 \right) \Psi_{N,0} \left( X_N - \hbar Y_N / 2 \right)
= \prod_{j=1}^N f_0^h(x_j, v_j),
\]
respectively.

One can easily rephrase the result of [1] by showing that the \( j \)-particle Wigner function
\[
W_{N,j}^h(X_j, V_j; t) = \int_{\mathbb{R}^{3(N-j)} \times \mathbb{R}^{3(N-j)}} dX_{N-j} dV_{N-j} W_N^h(X_j, X_{N-j}, V_j, V_{N-j}; t)
\]
converges, in a suitable sense, to
\[
f_j^h(X_j, V_j; t) = \prod_{k=1}^j f^h(x_k, v_k; t) \quad \text{for any } t > 0.
\]
However, the convergence error is diverging when \( \hbar \to 0 \) (for example, for sufficiently small times \( t < t_0 \) it is of the form \( \frac{C}{\hbar} e^{c/\hbar} \)). The reason is that the operators \( T^h \) and \( T_N^h \) appearing in (1.15) and (1.16) are bounded as operators acting on the space in which we can prove the convergence of (1.19), but their norm diverge as \( \frac{C}{\hbar} \) when \( \hbar \to 0 \). On the other hand, the classical counterpart of this problem has been solved, so that it seems natural to look for an asymptotic expansion for the \( j \)-particle distributions \( W_{N,j}^h \), namely:
\[
W_{N,j}^h(t) = W_{N,j}^{(0)}(t) + \hbar W_{N,j}^{(1)}(t) + \hbar^2 W_{N,j}^{(2)}(t) + \ldots,
\]
and for an analogous expansion for the \( j \)-fold product of solutions of the equation (1.15), namely:
\[
f_j^h(t) = (f^h)^{\otimes j}(t) = f_j^{(0)}(t) + \hbar f_j^{(1)}(t) + \hbar^2 f_j^{(2)}(t) + \ldots
\]
The zero order term in (1.21) corresponds properly to what we previously denoted by \( f_N^j \), while the function \( f_j^{(0)} \) appearing in (1.22) is exactly the \( j \)-fold product of what we called \( f \) (see (1.10) and (1.12)). Therefore, at zero order in \( \hbar \) we obtained the classical quantities, as expected, and we know that the convergence of \( W_{N,j}^{(0)} \) to \( f_j^{(0)} \) is well established. Then, it looks natural to try to show the convergence
\[
W_{N,j}^{(k)}(t) \to f_j^{(k)}(t), \quad \text{as } N \to \infty, \text{ for any } k > 0.
\]
This is the goal of the present paper.

A complete proof of the uniformity in \( \hbar \) of the limit \( N \to \infty \) would require a control of the remainder of the expansion (1.21), but we are not able to do this. However the error term of
order $k$ in the expansion (1.22) can be proven to be $O(\hbar^{k+1})$ by adapting the proof in [15] for the linear case to the present context, under suitable smoothness assumptions.

The plan of the paper is the following. In the next two sections we present a semiclassical expansion of the Hartree equation and of the $N$-particle system. After a brief discussion of the hierarchical structures and their inadequacy as a technical tool for the present problem, we introduce the classical mean-field limit which is the basis of our analysis. After that, we explain the strategy of the proof of the convergence (1.23). The last two sections are devoted to the precise statement of our result, its proof and supplementary comments. Three Appendices contain technicalities.

2. The Hartree dynamics

The Wigner-Liouville equation associated with the Hartree equation (1.1) reads as:

\begin{align}
\begin{cases}
(\partial_t + v \cdot \nabla_x) f^\varepsilon &= T^\varepsilon f^\varepsilon,
\end{cases}
\end{align}

where, from now on, we set $\varepsilon = \hbar$. Furthermore $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $f^\varepsilon(x, v; t)$ is defined as in (1.13) and it is a real (but non necessarily positive) function on the classical phase space. We remark that in the previous section we used the notation $T^\varepsilon f^\varepsilon$ for the right hand side of (2.1) but now we use $T^\varepsilon f^\varepsilon$ to stress the nonlinearity of equation (2.1).

Here $T^\varepsilon$ acts as follows:

\begin{align}
T^\varepsilon f^\varepsilon(x, v) &= (2\pi)^{-3} i \int_{-1/2}^{1/2} \lambda d\lambda \int_{\mathbb{R}^3} dk \hat{\phi}(k) \hat{\rho}_g(k) e^{i k \cdot x (k \cdot \nabla_v)} f^\varepsilon(x, v + \varepsilon \lambda k),
\end{align}

where, as usual, $\phi = \phi(x)$ denotes the two-body interaction potential and

\begin{align}
\rho_g(x) &= \int_{\mathbb{R}^3} dv \ g(x, v).
\end{align}

Finally, we denoted by $\hat{\rho}_g$ and $\hat{\phi}$ the Fourier transforms of $\rho_g$ and $\phi$ respectively, namely:

\begin{align}
\hat{\rho}_g(k) &= \int_{\mathbb{R}^3} dx \ e^{-i k \cdot x} \rho_g(x) \quad \text{and} \quad \hat{\phi}(k) = \int_{\mathbb{R}^3} dx \ e^{-i k \cdot x} \phi(x).
\end{align}

Following [15], for a fixed $g$, $T^\varepsilon_g$ can be expanded as

\begin{align}
T^\varepsilon_g = T^{(0)}_g + \varepsilon T^{(1)}_g + \varepsilon^2 T^{(2)}_g + \ldots
\end{align}

where

\begin{align}
T^{(n)}_g = c_n (2\pi)^{-3} i \int_{\mathbb{R}^3} dk \hat{\phi}(k) \hat{\rho}_g(k) e^{i k \cdot x (k \cdot \nabla_v)^{n+1}},
\end{align}

\begin{align}
c_n = \frac{1}{2^n(n + 1)!},
\end{align}

for $n$ even and

\begin{align}
T^{(n)}_g = 0,
\end{align}

for $n$ odd.
for $n$ odd. The operator $T^{(n)}_g$, for $n$ even, can be also written as
\[ T^{(n)}_g = (-1)^{n/2} c_n \left( D^{n+1}_x \phi \ast g \right) \cdot D^{n+1}_v, \] (2.9)
where, as in (1.12), $\ast$ denotes the convolution with respect to both $x$ and $v$ and we used the notation:
\[ D^n x = \sum_{n_1, n_2, n_3 \in \mathbb{N}} \frac{\partial^n \nu}{\partial^{n_1} x^1 \partial^{n_2} x^2 \partial^{n_3} x^3} \frac{\partial^n \zeta}{\partial^{n_1} v^1 \partial^{n_2} v^2 \partial^{n_3} v^3}, \] (2.10)
with $x = (x^1, x^2, x^3) \in \mathbb{R}^3$ and $v = (v^1, v^2, v^3) \in \mathbb{R}^3$
for the one-particle functions $\nu$ and $\zeta$.

We want to determine a semiclassical expansion of $f_\varepsilon(x, v; t)$, solution of equation (2.1), namely:
\[ f_\varepsilon(t) = f_0(0) \varepsilon + \varepsilon f_1(0) \varepsilon + \varepsilon^2 f_2(0) \varepsilon + \ldots \] (2.11)
First of all, we observe that interesting physical states, such as coherent states (see e.g. the forthcoming Section 6), have a similar expansion even at time $t = 0$, namely:
\[ f_0^\varepsilon = f_0(0) + \varepsilon f_1(0) + \varepsilon^2 f_2(0) + \ldots \] (2.12)
Inserting (2.11) in (2.5) and setting:
\[ T^{(n)}_f = T^{(n)}_{f^{(k)}}, \] (2.13)
we readily arrive to the following sequence of problems for the coefficients $f^{(k)}(t)$ of the expansion (2.11):
\[ \begin{cases} (\partial_t + v \cdot \nabla_x) f^{(0)} = T^{(0)}_0 f^{(0)}, \\ f^{(0)}(x, v; t)|_{t=0} = f_0^{(0)}(x, v), \end{cases} \] (2.14)
and
\[ \begin{cases} (\partial_t + v \cdot \nabla_x) f^{(k)} = L(f^{(0)}) f^{(k)} + \Theta^{(k)}, \\ f^{(k)}(x, v; t)|_{t=0} = f^{(k)}_0(x, v), \end{cases} \] (2.15)
for $k \geq 1$, where
\[ L(h) f = T^{(0)}_h f + T^{(0)}_f h = (\nabla_x \phi * h) \cdot \nabla_v f + (\nabla_x \phi * f) \cdot \nabla_v h, \] (2.16)
and
\[ \Theta^{(k)} = \sum_{r,s,l} T^{(s)}_r f^{(l)}. \] (2.17)
Note that equation (2.14) is nothing else than the classical Vlasov equation (see (1.12)) associated with the interaction $\phi$ (which will assume to be smooth). It is well known that the Vlasov equation can be solved by means of characteristics and fixed point. Moreover, the problems (2.15) are linear and can be solved by a recursive argument (see Section 6 below). Indeed, the source terms $\Theta^{(k)}$ involve only those coefficients $f^{(n)}$ with $n < k$, so that they are known by
the previous steps. We shall give a sense to the solutions \( f^{(k)}(t) \), for \( k \geq 1 \), once we will have established precise assumptions on \( \phi \) and \( f_0^{(k)} \).

3. The \( N \)-particle dynamics

Consider a quantum system constituted by \( N \) identical particles. Its time evolution in the classical phase space is given by the Wigner-Liouville equation:

\[
(\partial_t + V_N \cdot \nabla X_N) W_N^\varepsilon = T_N^\varepsilon W_N^\varepsilon,
\]

where \( W_N^\varepsilon(X_N, V_N; t) \) is the Wigner function describing the state of the system,

\[
X_N = (x_1, \ldots, x_N) \in \mathbb{R}^{3N}, \quad V_N = (v_1, \ldots, v_N) \in \mathbb{R}^{3N},
\]

and the pair \( Z_N := (X_N, V_N) \) denotes the generic point in the classical \( N \)-particle phase space. Moreover,

\[
(T_N^\varepsilon W_N^\varepsilon)(Z_N) = \frac{i}{(2\pi)^{3N}} \int_{-1/2}^{1/2} d\lambda \int dK_N \hat{U}(K_N)e^{iK_N \cdot V_N} (K_N \cdot \nabla V_N) W_N^\varepsilon(X_N, V_N + \lambda \varepsilon K_N),
\]

where \( K_N = (k_1, \ldots, k_N) \in \mathbb{R}^{3N} \) and \( U_N \) is the (mean-field) interaction potential given by:

\[
U_N(X_N) = \frac{1}{N} \sum_{1 \leq l < j \leq N} \phi(x_l - x_j).
\]

We choose, as initial datum, the factorized state:

\[
W_{N,0}^\varepsilon(X_N, V_N) = \prod_{j=1}^N f_0^\varepsilon(x_j, v_j),
\]

where \( f_0^\varepsilon \) is the initial datum of the equation \( (2.1) \). Following [15], we expand

\[
T_N^\varepsilon = T_N^{(0)} + \varepsilon T_N^{(1)} + \varepsilon^2 T_N^{(2)} + \ldots
\]

where, for \( n \) even we have

\[
T_N^{(n)} = i(2\pi)^{-3N}C_n \int_{\mathbb{R}^{3N}} dK_N \hat{U}(K_N)e^{iK_N \cdot X_N} (K_N \cdot \nabla V_N)^{n+1},
\]

\( C_n \) being constants depending on \( n \), and, for \( n \) odd, we find

\[
T_N^{(n)} = 0.
\]

Looking for a semiclassical expansion

\[
W_N^\varepsilon(t) = W_N^{(0)}(t) + \varepsilon W_N^{(1)}(t) + \varepsilon^2 W_N^{(2)}(t) + \ldots,
\]

we first expand the initial datum

\[
W_{N,0}^\varepsilon = W_{N,0}^{(0)} + \varepsilon W_{N,0}^{(1)} + \varepsilon^2 W_{N,0}^{(2)} + \ldots.
\]
The coefficients $W_{N,0}^{(k)}$ are determined by (3.4) and (2.12) as

$$W_{N,0}^{(0)} = \prod_{j=1}^{N} f_0^{(0)}(x_j, v_j),$$

(3.10)

...$

$$W_{N,0}^{(k)} = \sum_{s_1,\ldots,s_N} \prod_{j=1}^{N} f_0^{(s_j)}(x_j, v_j).$$

(3.11)

Note that $W_{N,0}^{(k)}$ is factorized only for $k = 0$.

By (3.8) and (3.5), we arrive to the sequence of problems:

$$\begin{cases}
(\partial_t + V \cdot \nabla X) W_N^{(0)} = T_N^{(0)} W_N^{(0)}, \\
W_N^{(0)}(X_N, V_N; t) \bigg|_{t=0} = W_{N,0}^{(0)}(X_N, V_N),
\end{cases}$$

(3.12)

and

$$\begin{cases}
(\partial_t + V \cdot \nabla X) W_N^{(k)} = T_N^{(0)} W_N^{(k)} + \Theta_N^{(k)}, \\
W_N^{(k)}(X_N, V_N; t) \bigg|_{t=0} = W_{N,0}^{(k)}(X_N, V_N),
\end{cases}$$

(3.13)

for $k \geq 1$, where

$$\Theta_N^{(k)} = \sum_{0 \leq l < k} T_N^{(k-l)} W_N^{(l)}.$$ 

(3.14)

Note that $T_N^{(0)} = \nabla X \cdot U_N \cdot \nabla V_N = \frac{1}{N} \sum_{1 \leq i < j \leq N} \nabla_x \phi(x_i - x_j) \cdot \nabla v_i$ is the classical Liouville operator, while the source terms $\Theta_N^{(k)}$, at each order $k$, are known by the previous steps. Note also that, under reasonable assumptions on the interaction potential $\phi$, equation (3.12) can be solved by considering the Hamiltonian flow $\Phi_t(X_N, V_N)$, solution of the problem

$$\begin{cases}
\dot{x}_i = v_i \\
\dot{v}_i = -\frac{1}{N} \sum_{j \neq i} \nabla_x \phi(x_i - x_j).
\end{cases}$$

(3.15)

Indeed

$$W_N^{(0)}(X_N, V_N; t) = S_N(t) W_{N,0}^{(0)}(X_N, V_N) = W_{N,0}^{(0)}(\Phi^{-t}(X_N, V_N)).$$

(3.16)

where, from now on, we denote by $S_N$ the flow generated by the Liouville operator $T_N^{(0)}$, while equations (3.13) can be solved by recurrence thanks to the Duhamel formula:

$$W_N^{(k)}(t) = S_N(t) W_{N,0}^{(k)} + \int_{0}^{t} dt_1 S_N(t - t_1) \Theta_N^{(k)}(t_1).$$

(3.17)
We conclude this section by expressing the operators $T_N^{(n)}$ ($n$ even) in terms of the variables $X_N, V_N$. From (3.6), we find that:

$$T_N^{(n)} = \hat{T}_N^{(n)} + R_N^{(n)},$$

where

$$\hat{T}_N^{(n)} = c_n \left( -1 \right)^{n/2} \frac{1}{N} \sum_{1 \leq l < j \leq N} D_{x_l}^{n+1} \phi(x_l - x_j) \cdot D_{v_l}^{n+1},$$

where $c_n$ is the same of (2.7), and

$$R_N^{(n)} = \frac{1}{N} \sum_{1 \leq l < j \leq N} \sum_{k_1, k_2 \in \mathbb{N}} C_{k_1, k_2} \frac{\partial^{n+1}}{\partial x_l^{k_1} \partial x_j^{k_2}} \phi(x_l - x_j) \cdot \frac{\partial^{n+1}}{\partial v_l^{k_1} \partial v_j^{k_2}},$$

where, for $i = 1, 2, k_i = (k_{i,1}, k_{i,2}, k_{i,3})$, $|k_i| = k_{i,1} + k_{i,2} + k_{i,3}$, and

$$\frac{\partial^{n+1}}{\partial x_l^{k_1} \partial x_j^{k_2}} = \frac{\partial^{k_1}}{\partial x_l^{k_1}} \frac{\partial^{k_2}}{\partial x_j^{k_2}} = \frac{\partial^{k_1}}{\partial x_l^{k_1}} \frac{\partial^{k_2}}{\partial v_l^{k_1}} \frac{\partial^{k_2}}{\partial v_j^{k_2}} \phi(x_l - x_j) \cdot \frac{\partial^{n+1}}{\partial v_l^{k_1} \partial v_j^{k_2}},$$

with $x_j = (x_{j,1}, x_{j,2}, x_{j,3})$, $x_l = (x_{l,1}, x_{l,2}, x_{l,3})$,

while $C_{k_1, k_2}$ are suitable coefficients. The same holds for the derivatives with respect to the velocities.

We observe that, by the expression (3.20), we mean that the derivative of order $|k_1|$ is distributed over the three components of $x_l$ in the same way in which it is distributed over the three components of $v_l$, and the same holds for the derivative of order $|k_2|$.

4. **Hierarchies**

One way to investigate the behavior of the $N$-particle system in the limit $N \to \infty$, is to consider the hierarchy associated with equation (3.1). More precisely, introducing the $j$-particle functions:

$$W_N^{\varepsilon_j}(X_j, V_j; t) = \int_{\mathbb{R}^{3(N-j)} \times \mathbb{R}^{3(N-j)}} dX_{N-j} dV_{N-j} W_N^{\varepsilon}(X_j, X_{N-j}, V_j, V_{N-j}; t),$$

a straightforward computation yields the following sequence of equations:

$$(\partial_t + V_j \cdot \nabla X_j) W_N^{\varepsilon_{j+1}} = T_j^{\varepsilon_j} W_N^{\varepsilon_j}(X_j, X_{N-j}, V_j, V_{N-j}; t),$$

$$+(\frac{N-j}{N}) C_{j+1}^{\varepsilon_j} W_N^{\varepsilon_{j+1}}, \quad j = 1, 2, \ldots, N,$$

(4.2)

with $W_N^{\varepsilon_{N+1}} = W_N^{\varepsilon}$ and $C_N^{\varepsilon_{N+1}} \equiv 0$.

which, as usual, is called "hierarchy" because each equation is linked to the subsequent one. The operator $T_j^{\varepsilon_j}$ (for a fixed $j$) describes the interaction of the first $j$ particles (we recall that we are dealing with identical particles, thus, in order to refer to any group of $j$ particles we can say "the first $j$ particles" because we can rearrange them as we want) among themselves, while
the operator $C_{j+1}^\varepsilon$ describes the interaction of the first $j$ particles with the remaining $N - j$. The explicit form of such operators is:

$$
\left( T_j^\varepsilon W_{N,j}^\varepsilon \right) (X_j, V_j) = \frac{i(2\pi)^{3N}}{N} \sum_{1 \leq l < r \leq j} \int_{-1/2}^{1/2} d\lambda \int_{\mathbb{R}^3} dk \hat{\phi}(k) e^{ik(x_l - x_r)} (k \cdot \nabla v_l) W_{N,j}^\varepsilon (X_j, V_{l-1}, v_l + \lambda \varepsilon k, V_{j-l}),
$$

(4.3)

and

$$
\left( C_{j+1}^\varepsilon W_{N,j+1}^\varepsilon \right) (X_j, V_j) = \frac{i(2\pi)^{3N}}{N} \sum_{l=1}^{j} \int_{-1/2}^{1/2} d\lambda \int_{\mathbb{R}^3} dk \hat{\phi}(k) \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_{j+1} dv_{j+1} e^{ik(x_l - x_{j+1})} (k \cdot \nabla v_l) W_{N,j+1}^\varepsilon (X_j, x_{j+1}, V_{l-1}, v_l + \lambda \varepsilon k, V_{j-l}, v_{j+1}).
$$

(4.4)

Note now that $T_j^\varepsilon = O \left( \frac{j^2}{N} \right)$, thus we expect its action to be negligible in the limit. On the other hand, if we consider the sequence $\{f_j^\varepsilon(t)\}_j$, where $f_j^\varepsilon(t) = f_j^\varepsilon(X_j, V_j; t)$ is given by:

$$
f_j^\varepsilon(X_j, V_j; t) = \prod_{k=1}^{j} f^\varepsilon(x_k, v_k; t)
$$

(4.5)

and $f^\varepsilon(t)$ is the solution of the Wigner-Liouville equation associated with the Hartree dynamics (see (2.1)), we easily deduce the following hierarchy:

$$
(\partial_t + V_j \cdot \nabla x_j) f_j^\varepsilon = C_{j+1}^\varepsilon f_{j+1}^\varepsilon.
$$

(4.6)

Therefore, we expect that

$$
W_{N,j}^\varepsilon(t) \to f_j^\varepsilon(t), \quad \text{as } N \to \infty,
$$

(4.7)

for any $t > 0$, provided that the same convergence holds at time $t = 0$. As we have already recalled in Section 1, such a result can indeed be proven under various assumptions on the interaction potential $\phi$, both in the reduced density matrix formalism and in the Wigner one. This constitutes a validation of the Hartree equation in the mean-field limit. Nevertheless, as we have already remarked, a common feature of these results is that the limit is singularly behaving when $\varepsilon \to 0$. In fact, the operators involved in the above hierarchies are bounded in the norm appropriate to study the convergence, both in the reduced density matrix formalism and in the Wigner one, but their norm is diverging when $\varepsilon$ goes to zero. This suggests to consider the semiclassical equations described in Sections 1 and 2. In this way, considering equations at each order in $\varepsilon$ and analyzing the hierarchies associated with each of those equation, we have to deal with operators which are clearly independent of $\varepsilon$ ($T_N^{(n)}$, for the $N$-particle case, and $T_k^{(n)}$ for the expansion associated with the Hartree dynamics), and we have to investigate only the limit $N \to \infty$ without any dependence on $\varepsilon$. The price we have to pay is that now
those operators are unbounded, as it comes out for the classical mean-field limit we are going
to discuss in the next section. Therefore, if we want to prove that the coefficient of order
\( \varepsilon^k \) of the expansion of \( W_{N,j}^{(k)} \), namely:

\[
W_{N,j}^{(k)}(X_j, V_j; t) = \int_{\mathbb{R}^{3(N-j)} \times \mathbb{R}^{3(N-j)}} dX_{N-j}dV_{N-j} W_{N}^{(k)}(X_j, X_{N-j}, V_j, V_{N-j}; t),
\]

(4.8)

converges to the corresponding object for the Hartree flow, which is:

\[
f_{j}^{(k)}(X_j, V_j; t) = \sum_{s_1 \ldots s_j} \prod_{r=1}^{j} f^{(s_r)}(x_r, v_r; t),
\]

(4.9)

(where the one-particle functions \( f^{(s_r)}(x_r, v_r; t) \) solve equation (2.14), if \( s_r = 0 \), and (2.15), for \( s_r > 0 \)), the use of the hierarchy solved by \( W_{N,j}^{(k)}(t) \) does not seem a good idea. In fact, even
at level zero, when we have to deal with the classical mean-field limit, the hierarchy is very
difficult to handle with. Such a hierarchy is given by:

\[
(\partial_t + V_j \cdot \nabla X_j) W_{N,j}^{(0)} = T_{j}^{(0)} W_{N,j}^{(0)} + \frac{N - j}{N} C_{j+1}^{(0)} W_{N,j+1}^{(0)},
\]

(4.10)

with

\[
\left( T_{j}^{(0)} W_{N,j}^{(0)} \right)(X_j, V_j) = \frac{1}{N} \sum_{1 \leq l \leq r \leq j} \nabla x_l \phi(x_l - x_r) \cdot \nabla v_l W_{N,j}^{(0)}(X_j, V_j),
\]

(4.11)

and

\[
\left( C_{j+1}^{(0)} W_{N,j+1}^{(0)} \right)(X_j, V_j) =
\]

\[
= \sum_{l=1}^{j} \int_{\mathbb{R}^{3} \times \mathbb{R}^{3}} dx_{j+1} dv_{j+1} \nabla x_l \phi(x_l - x_{j+1}) \cdot \nabla v_l W_{N,j+1}^{(0)}(X_j, x_{j+1}, V_j, v_{j+1}).
\]

(4.12)

Clearly these operators are unbounded (unless to make them act on analytical functions) be-
cause they involve derivatives with respect to the velocity variables. For this reason, to deal
with the hierarchy is quite difficult and the obstacle which occurs in facing the higher order
terms is precisely the same. However, in the classical case we can treat the convergence in a
more natural way, avoiding to use the hierarchy. The idea is to control the \( j \)-particle marginals
\( W_{N,j}^{(0)} \) in terms of the expectation of the \( j \)-fold product of empirical measures with respect to the
initial \( N \)-particle probability distribution (see Section 5 below). In the present paper we follow
a similar strategy in dealing with the convergence of the higher order terms of the expansion.
More precisely, we will express \( W_{N,j}^{(k)} \) in terms of the expectation, with respect to the initial
\( N \)-particle zero order coefficient (which is known to be a probability distribution), of suitable
operators acting on empirical measures. The control of these objects will be obtained thanks
to some estimates of the derivatives of the classical flow with respect to the initial data (see Proposition 5.1).

5. The classical mean-field limit

The semiclassical expansion of the \( N \)-particle system leads us to consider the sequence of problems (3.12)-(3.13). The zero order equation (3.12) is purely classical and well understood. At this regard, let us remind some basic facts concerning the case of smooth potentials, which will be crucial in what follows.

The Vlasov equation (1.12) makes sense even for a generic probability measure \( \nu \) because \( \nabla \phi \ast \nu \in C^\infty(\mathbb{R}^3) \) (thanks to the smoothness of \( \phi \)) and, by using the characteristic flow associated with the equation and a fixed point argument, it is possible to prove the existence and uniqueness of the solution. Furthermore, introducing the Wasserstein distance \( W \) (e.g. [13]) based on a bounded metric in \( \mathbb{R}^6 \) to avoid unnecessary boundedness assumptions on the moments of the measures we deal with, it is possible to prove the following continuity property:

\[
W(\nu^1_t, \nu^2_t) \leq e^{Ct} W(\nu^1_0, \nu^2_0)
\]

where \( \nu^1_0 \) and \( \nu^2_0 \) are two probability measures and \( \nu^1_t \) and \( \nu^2_t \) are the weak solutions of the Vlasov equation with initial data given by \( \nu^1_0 \) and \( \nu^2_0 \) respectively (see again [13]). Moreover, for a configuration \( Z_N = \{z_1, \ldots, z_N\} \), where \( z_j = (x_j, v_j) \in \mathbb{R}^6 \), consider the Hamiltonian flow

\[
\Phi^t(X_N, V_N) = Z_N(t) = Z_N(t; Z_N),
\]

with initial datum \( Z_N \) (see (3.15)), and construct the empirical measure \( \mu_N(t) \) as follows:

\[
\mu_N(t) := \mu_N(z|Z_N(t)) = \frac{1}{N} \sum_{j=1}^{N} \delta(z - z_j(t)).
\]

The basic remark is that \( \mu_N(t) \) is a weak solution of the Vlasov equation, so that, by (5.1), we have:

\[
\mu_N(t) \to f^{(0)}(t), \quad \text{as } N \to \infty,
\]

provided that

\[
\mu_N \to f^{(0)}(0) = f^{(0)}_0, \quad \text{as } N \to \infty,
\]

where

\[
\mu_N := \mu_N(z|Z_N) = \frac{1}{N} \sum_{j=1}^{N} \delta(z - z_j)
\]

is the empirical distribution at time \( t = 0 \). Moreover, \( f^{(0)}_0 \) is a (possibly smooth) probability distribution and \( f^{(0)}(0) \) is the solution of the Vlasov equation with initial datum \( f^{(0)}_0 \). Clearly, the convergences (5.4) and (5.5) hold with respect to the metric induced by \( W \) on the space of probability measures on \( \mathbb{R}^6 \) and this is equivalent to the weak topology of probability measures.
Next, let us consider the (factorized) \( N \)-particle probability distribution \( W^{(0)}_{N,0}(Z_N) = \prod_{k=1}^{N} f_0^{(0)}(x_k, v_k) \) and let \( W^{(0)}_N(t) \) be its time evolution according to the Liouville equation (3.16). We want to investigate the behavior of the \( j \)-particle marginals \( W^{(0)}_{N,j}(t) \). Denoting by \( E_N \) the expectation with respect to \( W^{(0)}_{N,0}(Z_N) \), after straightforward computations, we obtain:

\[
E_N \left[ \mu_N (z'_1|Z_N(t)) \ldots \mu_N (z'_j|Z_N(t)) \right] = \frac{N(N-1)\ldots(N-j+1)}{N^j} W^{(0)}_{N,j}(Z'_j; t) + O \left( \frac{1}{N} \right),
\]

(5.7)

where \( Z'_j = (z'_1 \ldots z'_j) \) and \( W^{(0)}_{N,j}(Z'_j; t) = W^{(0)}_{N,j}(Z'_j(t)) \) (see (5.2)).

Consider now a typical sequence \( Z_N \) with respect to \( f_0^{(0)} \), namely such that (5.5) holds. By the strong law of large numbers this happens a.e. with respect to \( (f_0^{(0)})^\otimes \infty \) and by (5.4) and (5.7) we have:

\[
\lim_{N \to \infty} E_N \left[ \mu_N (z'_1|Z_N(t)) \ldots \mu_N (z'_j|Z_N(t)) \right] = \lim_{N \to \infty} W^{(0)}_{N,j}(Z'_j; t) = (f^{(0)})^\otimes j (Z'_j; t),
\]

(5.8)

in the weak topology of probability measures. Thus propagation of chaos is proven, and, this is the remarkable fact, it has been done without using the hierarchy.

For fixed \( \varepsilon > 0 \), the quantum hierarchy is, in a certain sense, easier. In fact, in that situation the operators involved are bounded (as operators acting on the spaces appropriate for that context) and it is possible to realize the limit by using the hierarchy. On the contrary, in the quantum context we cannot use any characteristic flow and there is not any object analogous to the empirical measure. Nevertheless, if we consider the semiclassical expansion of the time evolved Wigner function, the higher order terms can be viewed as quantum corrections to the classical dynamics.

We now explain heuristically our approach fully exploited in Section 7.

The first correction to the Vlasov equation in the Hartree dynamics satisfies (see (2.11) and (2.12)):

\[
\begin{align*}
(\partial_t + v \cdot \nabla_x) f^{(1)} &= L(f^{(0)}) f^{(1)}, \\
|f^{(1)}(x, v; t)|_{t=0} &= f_0^{(1)}(x, v),
\end{align*}
\]

(5.9)

(looking at the expression (2.17) for the source terms \( \Theta^{(k)} \), we verify that \( \Theta^{(1)} \equiv 0 \)). As we shall see in detail in the following section, our choice for the initial one-particle datum is a mixture of coherent states and each coefficient of the expansion it is given by suitable derivatives of the zero order distribution. In particular, the explicit form for \( f_0^{(1)} \) is:

\[
f_0^{(1)}(x, v) = D_G f_0^{(0)}(x, v),
\]

(5.10)
where $D_G^2$ is a suitable second order derivation operator (see formula (6.8) below in the case $k = 2$) involving derivatives with respect to the initial variables $z_1, \ldots, z_N$.

As regard to the $N$-particle dynamics, looking at (3.11) in the case $k = 1$, we know that the initial datum for the coefficient of order one in $\varepsilon$ is:

$$W_{N,0}^{(1)}(Z_N) = \sum_{j=1}^{N} f_0^{(1)}(z_j) \prod_{l \neq j} f_0^{(0)}(z_l) = D^2W_{N,0}^{(0)}(Z_N), \quad (5.11)$$

where

$$D^2 = \sum_{j=1}^{N} D_{G,j}^2, \quad (5.12)$$

and $D_{G,j}^2$ is the operator $D_G^2$ relative to the variable $z_j \in \mathbb{R}^6$. Let us now define $D^2\mu_N(t)$ as the distribution acting on a test function $u$ in the following way:

$$(u, D^2\mu_N(t)) = D^2 \left( \frac{1}{N} \sum_{i=1}^{N} u(z_i(t)) \right) = \frac{1}{N} \sum_{l,j=1}^{N} D_{G,j}^2 u(z_l(t)). \quad (5.13)$$

We remind that the operator $D_{G,j}^2$ involves derivatives with respect to the initial variables $z_1, \ldots, z_N$, thus, if at time $t = 0$ we have $\mu_N \rightarrow f_0^{(0)}$ when $N \rightarrow \infty$ (in the weak sense of probability measures), it follows that:

$$(u, D^2\mu_N) = D^2 \frac{1}{N} \sum_{i=1}^{N} u(z_i) = \frac{1}{N} \sum_{l,j=1}^{N} D_{G,j}^2 u(z_l) = \frac{1}{N} \sum_{j=1}^{N} D_{G,j}^2 u(z_j) = (D_G^2 u, \mu_N) \rightarrow \left( D_G^2 u, f_0^{(0)} \right) = \left( u, D_G^2 f_0^{(0)} \right) = \left( u, f_0^{(1)} \right) \quad (5.14)$$

as $N \rightarrow \infty$. Moreover, by (5.11) and (5.14), we can conclude that:

$$(u, W_{N,1}^{(1)}(t)|_{t=0}) = (u, E_N \left[ D^2\mu_N \right]) \rightarrow \left( u, f_0^{(1)} \right) \quad \text{as} \quad N \rightarrow \infty. \quad (5.15)$$

By equation (3.13) for $k = 1$, we have:

$$(\partial_t + V_N \cdot \nabla_{X_N}) W_N^{(1)} = \nabla_{X_N} U_N \cdot \nabla_{V_N} W_N^{(1)},$$

$$W_N^{(1)}(Z_N; t)|_{t=0} = W_{N,0}^{(1)}(Z_N), \quad (5.16)$$

namely, the classical Liouville equation. Therefore:

$$W_N^{(1)}(Z_N; t) = S_N(t)W_{N,0}^{(1)}(Z_N). \quad (5.17)$$
Finally, by virtue of (5.17) and (5.11), we obtain
\[
\left(u, W_{N,1}^{(1)}(t)\right) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} dZ_N S_N(t) W_{N,0}^{(1)}(Z_N)(u, \mu_N) = \\
= \int_{\mathbb{R}^3 \times \mathbb{R}^3} dZ_N W_{N,0}^{(1)}(Z_N)(u, \mu_N(t)) = \\
= \int_{\mathbb{R}^3 \times \mathbb{R}^3} dZ_N D^2 W_{N,0}^{(0)}(Z_N)(u, \mu_N(t)) = \\
= \int_{\mathbb{R}^3 \times \mathbb{R}^3} dZ_N W_{N,0}^{(0)}(Z_N)(u, D^2 \mu_N(t)) = \\
= (u, E_N [D^2 \mu_N(t)]). \tag{5.18}
\]

Therefore, the behavior of $W_{N,1}^{(1)}(t)$ is determined by that of $D^2 \mu_N(t)$ for any initial configuration $Z_N$ which is typical with respect to $f^{(0)}$. Finally, since $\mu_N(t)$ solves:
\[
\begin{cases}
(\partial_t + v \cdot \nabla_x) \mu_N(t) = (\nabla \phi \ast \mu_N(t)) \cdot \nabla v \mu_N(t) \\
\mu_N(t)|_{t=0} = \mu_N,
\end{cases} \tag{5.19}
\]
applying $D^2$, we get:
\[
\begin{cases}
(\partial_t + v \cdot \nabla_x) D^2 \mu_N(t) = L(\mu_N(t)) D^2 \mu_N(t) + R_N, \\
D^2 \mu_N(t)|_{t=0} = D^2 \mu_N, \tag{5.20}
\end{cases}
\]
where $R_N$ is a term involving objects of the form $\sum_j (D G,j \mu_N(t)) (D G,j \mu_N(t))$ which, as we shall see later, are of order $1/N$ when tested versus smooth functions. The equation (5.20) is similar to (5.9), except for the presence of the term $R_N$ and for the fact that we have $L(\mu_N(t))$ instead of $L(f^{(0)})$. Therefore, the proof of the convergence of $W_{N,1}^{(1)}(t)$ to $f^{(1)}(t)$ reduces to that of a stability property for the solution of (5.9) with respect to suitable weak topologies. Propositions 6.1 and 6.2 below will provide us such property.

The general case $k > 1$ is only technically more complicated because of the presence of source terms, but the main ideas are those presented here.

We conclude by establishing a Proposition controlling the size of the derivatives of the Hamilton flow (3.15) with respect to the initial data.

From now on we shall denote by $C$ a positive constant, independent of $N$, possibly changing from line to line.

**Proposition 5.1**

Let $z_i(t) = (x_i(t), v_i(t))$, $i = 1, \ldots, N$ be the solution of equations (3.15) with initial datum $z_i = (x_i, v_i)$, $i = 1, \ldots, N$. Let $z_i^\beta$, $\forall \beta = 1, \ldots, 6$ be the $\beta$-th component of $z_i \in \mathbb{R}^6$. If the pair interaction potential $\phi$ is $C^\infty(\mathbb{R}^3)$ and the derivatives of any order of $\phi$ are uniformly bounded,
then, for each $k \in \mathbb{N}$:

$$\left| \frac{\partial^k z_i^\beta(t)}{\partial z_{j_1}^{\alpha_1} \ldots \partial z_{j_k}^{\alpha_k}} \right| \leq \frac{C}{N^{d_k(i)}},$$  \hspace{1cm} (5.21)

where $I := (j_1, \ldots, j_k)$ is any sequence of possibly repeated indices and $d_k(i)$ is the number of different indices in $I$ which are also different from $i$.

The physical significance of (5.21) is obvious. In the mean-field context, the quantity $z_i(t)$ depends weakly on $z_j$ if $j \neq i$ for each $t > 0$. Actually $\frac{\partial z_i^\beta(t)}{\partial z_j^\beta} = O \left( \frac{1}{N} \right)$ while $\frac{\partial z_i^\beta(t)}{\partial z_i^\alpha} = O(1)$ and these two estimates give rise to (5.21) in the case $k = 1$. Estimate (5.21) says that for each derivative of any order with respect to some $z_j$ of $z_i(t)$, we gain a factor $1/N$. We have also the following corollary whose straightforward proof will be omitted.

**Proposition 5.2**

Let $U = U(Z_N(t))$ be a function of the time evolved configuration $Z_N(t)$ of the form:

$$U(Z_N(t)) = \frac{1}{N} \sum_{i=1}^{N} u(z_i(t)),$$

where $u \in C^\infty_b(\mathbb{R}^3 \times \mathbb{R}^3)$. Then, if the pair interaction potential $\phi$ satisfies the assumptions of Proposition 5.1, the following estimate holds:

$$\left| \frac{\partial^k U(Z_N(t))}{\partial z_{j_1}^{\alpha_1} \ldots \partial z_{j_k}^{\alpha_k}} \right| \leq \frac{C}{N^{d_k}},$$  \hspace{1cm} (5.22)

where $d_k$ is the number of different indices in the sequence $I = (j_1, \ldots, j_k)$.

The proof of Proposition 5.1 will be given in Appendix A.

### 6. Results and technical preliminaries

We choose, as initial condition for the one-particle Wigner function, a mixture of coherent states. The Wigner function associated with a pure coherent state centered at the point $(x_0, v_0)$ is given by:

$$w(x, v|x_0, v_0) = \frac{1}{(\pi \varepsilon)^{3/2}} e^{-\frac{(x-x_0)^2}{\varepsilon}} e^{-\frac{(v-v_0)^2}{\varepsilon}}.$$ \hspace{1cm} (6.1)

Let now $g = g(x, v)$ be a smooth probability density on the one-particle phase space independent of $\varepsilon$ (see Hypotheses H below). Then we define:

$$f_0(x, v) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_0 dv_0 \ w(x, v|x_0, v_0) g(x_0, v_0).$$ \hspace{1cm} (6.2)
Using the standard notation \( z = (x, v) \) and \( z_0 = (x_0, v_0) \), (6.2) is equivalent to:

\[
f_0(z) = \frac{1}{(\pi \varepsilon)^3} \int_{\mathbb{R}^6} d z_0 \, e^{-\frac{(z-z_0)^2}{8 \varepsilon}} g(z_0) =
\]

\[
= \frac{1}{(\pi \varepsilon)^3} \int_{\mathbb{R}^6} d \zeta \, e^{-\zeta^2} g(z - \sqrt{\varepsilon} \zeta).
\]

(6.3)

Expanding

\[
g(z - \sqrt{\varepsilon} \zeta) = g(z) - (\zeta \cdot \nabla_z) g(z) \sqrt{\varepsilon} + (\zeta \cdot \nabla_z)^2 g(z) \frac{(\sqrt{\varepsilon})^2}{2} + \ldots
\]

\[
\ldots - (\zeta \cdot \nabla_z)^{2n-1} g(z) \frac{(\sqrt{\varepsilon})^{2n-1}}{(2n-1)!} + (\zeta \cdot \nabla_z)^{2n} g(z) \frac{(\sqrt{\varepsilon})^{2n}}{(2n)!} + \ldots,
\]

(6.4)

and performing the gaussian integrations (which cancels the terms with the odd powers of \( \sqrt{\varepsilon} \)), we readily arrive to the following expansion for the Wigner function \( f_0 \):

\[
f_0 = f_0^{(0)} + \varepsilon f_0^{(1)} + \ldots + \varepsilon^n f_0^{(n)} + \ldots,
\]

(6.5)

where

\[
f_0^{(0)} = g,
\]

(6.6)

\[
f_0^{(n)} = D_G^{2n} f_0^{(0)} \text{ for } n \geq 1,
\]

(6.7)

and \( D^k_G \) (\( G \) stands for "Gaussian"), for each \( k > 0 \), is the following derivation operator with respect to the variable \( z = (x, v) \):

\[
D^k_G = \sum_{\alpha_1, \ldots, \alpha_k} C_G(\alpha_1 \ldots \alpha_k) \frac{\partial^k}{\partial z^{\alpha_1} \ldots \partial z^{\alpha_k}},
\]

(6.8)

where

\[
C_G(\alpha_1 \ldots \alpha_k) = \frac{1}{k!} \int_{\mathbb{R}^6} d \zeta \, e^{-\zeta^2} \prod_{j=1}^{k} \zeta^{\alpha_j}.
\]

(6.9)

Therefore, \( C_G(\alpha_1 \ldots \alpha_k) \) is equal to zero for each sequence \( \alpha_1 \ldots \alpha_k \) in which at least one index appears an odd number of times.

**Hypotheses H:**

In the present paper we assume that the probability density \( g = f_0^{(0)} \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R}^3) \), thus (6.7) make sense for any \( n \geq 1 \). As regard to the pair interaction potential \( \phi \), we assume that \( \phi \in C^\infty(\mathbb{R}^3) \), that any derivative of \( \phi \) is uniformly bounded (in order to be able to apply Proposition 5.1) and that \( \phi \) is spherically symmetric.
Remark 6.1:
In this paper we consider a completely factorized $N$-particle initial state. Furthermore the one-particle state is a mixture and this automatically excludes the Bose statistics.

Remark 6.2:
We made the choice to expand fully the initial state $f_0$ according to equation (6.5). Another possibility is to assume the ($\varepsilon$ dependent) state $f_0$ (which is a probability measure in the present case) as initial condition for the Vlasov problem and, consequently, $f_0^{(k)} = 0$ for the problems (2.15). Now the coefficients $f_0^{(k)}(t)$ are $\varepsilon$ dependent but this does not change deeply our analysis because $f_0$ is smooth, uniformly in $\varepsilon$.

Under hypotheses H, we can give a sense to the linear problem (2.15) for any $k \geq 1$, by virtue of the following proposition, whose (straightforward) proof will be given in Appendix B.

**Proposition 6.1**

Consider the following initial value problem:

$$
\begin{align*}
(\partial_t + v \cdot \nabla_x) \gamma &= L(h)\gamma + \Theta, \\
\gamma(x,v,t)|_{t=0} &= \gamma_0(x,v),
\end{align*}
(6.10)
$$

with $\gamma_0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$, $h = h(x,v,t)$ is such that $|\nabla_v h| \in C^0(L^1(\mathbb{R}^3 \times \mathbb{R}^3), \mathbb{R}^+)$, $\Theta = \Theta(x,v,t)$ is such that $\Theta \in C^0(L^1(\mathbb{R}^3 \times \mathbb{R}^3), \mathbb{R}^+)$. Then, there exists a unique solution $\gamma = \gamma(x,v,t)$ of (6.10), such that $\gamma \in C^0(L^1(\mathbb{R}^3 \times \mathbb{R}^3), \mathbb{R}^+)$, given by an explicit series expansion.

Furthermore, denoting by $\Sigma_h$ the flow generated by $L(h)$, we have that $\Sigma_h(t,0)\gamma_0 \in C^k(\mathbb{R}^3 \times \mathbb{R}^3)$ provided that $\nabla_v h \in C^k(\mathbb{R}^3 \times \mathbb{R}^3)$ and $\gamma_0 \in C^k(\mathbb{R}^3 \times \mathbb{R}^3)$.

The main goal of the present paper is to compare the $j$-particle semiclassical expansion associated with the $N$-particle flow, namely $W_{N,j}^{(k)}(t)$, $k = 0, 1, 2, \ldots$, with the corresponding coefficients $f_0^{(k)}(t)$ of the expansion:

$$
f_j^{(k)}(t) = f_j^{(0)}(t) + \varepsilon f_j^{(1)}(t) + \cdots + \varepsilon^k f_j^{(k)}(t) + \ldots,
(6.11)
$$

where $f_j^{(k)}(t)$ is given by (4.9). The main result is the following.

**Theorem 6.1**

Under the Hypotheses H, for all $t > 0$, for any integers $k$ and $j$, the following limit holds in $S'(\mathbb{R}^{3j} \times \mathbb{R}^{3j})$:

$$
W_{N,j}^{(k)}(t) \to f_j^{(k)}(t).
(6.12)
$$
as $N \to \infty$.

**Remark 6.3:**

As we shall see in the sequel, the convergence (6.12) is slightly stronger than the convergence in $S'(\mathbb{R}^3 \times \mathbb{R}^3)$. Indeed, the sequence $W_{N,j}^{(k)}(t)$ converges also when it is tested on functions in $C_0^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$, namely, the space of functions which are uniformly bounded and infinitely differentiable. Such kind of convergence, which is natural in the present context, will be called $C_0^\infty$-weak convergence.

**Proposition 6.2**

Let $\gamma_N(x,v; t)$ be a sequence in $S'(\mathbb{R}^3 \times \mathbb{R}^3)$ (for each $t$) satisfying:

\[
\begin{aligned}
\left\{ \begin{array}{l}
(\partial_t + v \cdot \nabla_x) \gamma_N = L(h_N)\gamma_N + \Theta_N, \\
\gamma_N(x,v; t)|_{t=0} = \gamma_{N,0}(x,v),
\end{array} \right.
\]

(6.13)

where $\gamma_{N,0}, \Theta_N$ are sequences in $S'(\mathbb{R}^3 \times \mathbb{R}^3)$. We assume that:

i) $h_N(x,v; t)$ is a sequence of probability measures converging, as $N \to \infty$, to a measure $h(t)dx dv$ with a density $h(t) \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ and such that $|\nabla_v h| \in C^0(L^1(\mathbb{R}^3 \times \mathbb{R}^3), \mathbb{R}^+)$.

ii) for all $u_1, u_2$ in $C_0^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$, there exists a constant $C = C(u_1, u_2) > 0$, not depending on $N$, such that:

\[
\| u_1 * (u_2 \gamma_N) \|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} < C < +\infty \quad \text{for any } t.
\]

(6.14)

iii) $\gamma_{N,0} \to \gamma_0$, as $N \to \infty$, $C_0^\infty$-weakly, $\gamma_0 = \gamma_0(x,v)$ is a function in $L^1(\mathbb{R}^3 \times \mathbb{R}^3)$.

iv) $\Theta_N \to \Theta$, as $N \to \infty$, $C_0^\infty$-weakly, $\Theta = \Theta(x,v; t)$ is a function in $C^0(L^1(\mathbb{R}^3 \times \mathbb{R}^3), \mathbb{R}^+)$.

Then:

\[
\gamma_N \to \gamma, \quad \text{as } N \to \infty \quad C_0^\infty \text{-weakly,}
\]

(6.15)

where $\gamma$ is the unique solution of the problem (6.10) in $C^0(L^1(\mathbb{R}^3 \times \mathbb{R}^3), \mathbb{R}^+)$. For the proof, see Appendix B.

### 7. Convergence

This section is devoted to the proof of Theorem 6.1.

By (3.17) and (3.14), for $k \geq 0$ we have:

\[
W_{N}^{(k)}(Z_N; t) = \sum_{n \geq 0} \sum_{r_0 = 0}^{k} \sum_{r_1 \ldots r_n: \sum r_j > k-r} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \sum_{r_j > 0} S_N(t - t_1)T_{N}^{(r_1)} S_N(t_1 - t_2) \ldots T_{N}^{(r_n)} S_N(t_n)W_{N,0}(Z_N).
\]

(7.1)
It is useful to remind that the only non-vanishing terms in (7.1) are those for which all \( r_1, \ldots, r_n \) are even.

According to (3.11) and (6.7),
\[
W^{(r)}_{N,0}(Z_N) = \sum_{s_1, \ldots, s_N \geq 0} \prod_{j=1}^N \left( D_{G,j}^{2s_j} f_0^{(0)}(z_j) \right),
\]
(7.2)

where \( D_{G,j}^{k} \) is defined in (6.8) and the extra symbol \( j \) means that this operator acts on the variable \( z_j \). Defining the operator \( D^{2r} \) as:
\[
D^{0} = 1,
D^{2r} = \sum_{s_1, \ldots, s_N \geq 0} \prod_{j=1}^N D_{G,j}^{2s_j}, \quad r \geq 1,
\]
(7.3)

we have:
\[
W^{(r)}_{N,0}(Z_N) = D^{2r} W^{(0)}_{N,0}(Z_N) \; \forall \; r \geq 0.
\]
(7.4)

In order to investigate the behavior of the \( j \)-particle functions \( W^{(k)}_{N,j}(Z_j; t) \) when \( N \to \infty \),
we consider the following object, for a given configuration \( Z_j' = (z_1' \ldots z_j') \):
\[
\omega^{(k)}_{N,j}(Z_j'; t) = \int_{\mathbb{R}^{6N}} dZ_N \ W^{(k)}_{N}(Z_N; t) \mu_N(z_1'|Z_N) \ldots \mu_N(z_j'|Z_N).
\]
(7.5)

In the end of the section, we will show that (7.5) is asymptotically equivalent to \( W^{(k)}_{N,j}(Z_j'; t) \).

From (7.1), (7.4) and (7.5), it follows that:
\[
\omega^{(k)}_{N,j}(Z_j'; t) = \sum_{n \geq 0} \sum_{r=0}^k \sum_{r_1, \ldots, r_n \geq 0} \prod_{j=k-r} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \int_{\mathbb{R}^{6N}} dZ_N \mu_{N,j}(Z_j'|Z_N)
\]
(7.6)

where
\[
\mu_{N,j}(Z_j'|Z_N) = \mu_N(z_1'|Z_N) \ldots \mu_N(z_j'|Z_N).
\]
(7.7)
Integrating by parts, reminding that each \( r_j \) is even and that each \( T^{(r_j)}_N \) involves derivatives of order \( r_j + 1 \), we have:

\[
\omega_{N,j}^{(k)}(Z'_j; t) = \sum_{n \geq 0} (-1)^n \sum_{r=0}^{k} \sum_{\eta_n: r_j > 0 \atop |\eta_n| = k-r} \int^t \mathcal{d}t_n \mathbb{E}_N \left[ D^{2r} T^{(r_n)}_N (t_n) T^{(r_{n-1})}_N (t_{n-1}) \ldots T^{(r_1)}_N (t_1) \mu_{N,j} (Z'_j | Z_N(t)) \right],
\]

(7.8)

where \( \eta_n \) is the sequence of positive integers \( r_1, \ldots, r_n \), \( |\eta_n| = \sum_{j=1}^n r_j \) and \( Z_N(t) \) is the Hamiltonian flow defined in \([5,2]\). Moreover \( t_n = t_1 \ldots t_n \) and \( \int^t \mathcal{d}t_n \) denotes the integral over the simplex \( 0 < t_n < t_{n-1} < \ldots < t_1 < t \). Finally, \( \mathbb{E}_N \) stands for the expectation with respect to the \( N \)-particle density \( W^{(0)}_{N,0} \) and

\[
T^{(r)}_N (t) = S_N (-t) T^{(r)}_N S_N (t).
\]

(7.9)

Therefore, the objects we have to investigate in the limit \( N \to \infty \) are:

\[
\nu_j^{(k)}(Z'_j; t) = \sum_{n \geq 0} (-1)^n \sum_{r=0}^{k} \sum_{\eta_n: r_j > 0 \atop |\eta_n| = k-r} \int^t \mathcal{d}t_n \eta_j (Z'_j; t, r, \eta_n, t_n, Z_N),
\]

(7.10)

(for any configuration \( Z_N \), typical with respect to \( f^{(0)}_0 \)), where \( \eta_j \) is given by:

\[
\eta_j (Z'_j; t, r, \eta_n, t_n, Z_N) = D^{2r} T^{(r_n)}_N (t_n) T^{(r_{n-1})}_N (t_{n-1}) \ldots T^{(r_1)}_N (t_1) \mu_{N,j} (Z'_j | Z_N(t)).
\]

(7.11)

Note that:

\[
\nu_j^{(0)}(Z'_j; t) = \mu_{N,j} (Z'_j | Z_N(t)).
\]

(7.12)

We start by analyzing the behavior of \( \nu_j^{(k)} \) in the cases \( j = 1, 2 \), thus we are lead to consider:

\[
\eta_1 (z'_1; t, r, \eta_n, t_n, Z_N) = D^{2r} T^{(r_n)}_N (t_n) T^{(r_{n-1})}_N (t_{n-1}) \ldots T^{(r_1)}_N (t_1) \mu_N (z'_1 | Z_N(t)),
\]

(7.13)

and

\[
\eta_2 (z'_1, z'_2; t, r, \eta_n, t_n, Z_N) = D^{2r} T^{(r_n)}_N (t_n) T^{(r_{n-1})}_N (t_{n-1}) \ldots T^{(r_1)}_N (t_1) \mu_{N,2} (Z'_2 | Z_N(t)).
\]

(7.14)

It is useful to stress that the operators \( T^{(r_j)}_N (t_j) (j = 1, \ldots, n) \) and \( D^{2r} \) act as suitable distributional derivatives with respect to the variables \( Z_N \). To evaluate \( \eta_1 \), let us first analyze...
the action of \( T_N^{(r)}(\tau) \). By (7.9) and (3.18), for any function \( G = G(Z_N) \), we have:

\[
\left( T_N^{(r)}(\tau) G \right)(Z_N) = S_N(\tau) \left( \hat{T}_N^{(r)} + \hat{R}_N^{(r)} \right)(S_N(\tau) G)(Z_N) = \\
= (-1)^{r/2} \frac{C_r}{N} \sum_{j,l} S_N(\tau) D_{x_j}^{r+1} \phi(x_j - x_l) \cdot D_{v_j}^{r+1} (S_N(\tau) G)(Z_N) + \\
+ \frac{1}{N} \sum_{l,j=1}^{N} \sum_{k_1,k_2 \in \mathbb{N}^3 \atop |k_1| + |k_2| = r+1} C_{k_1,k_2} S_N(\tau) \frac{\partial^{r+1}}{\partial x_l \partial x_j} \phi(x_l - x_j) \cdot \frac{\partial^{r+1}}{\partial v_l \partial v_j} (S_N(\tau) G)(Z_N).
\]

(7.15)

Note that the derivatives involved here are done with respect to the variables at time \( t = 0 \).

Denoting by \( D_{z_j}^r \) any derivative of order \( r \) with respect to a variable \( z_j \) at time \( t = 0 \), we observe that:

\[
S_N(-t) D_{z_j}^r G(Z_N) = \left( D_{z_j}^r G \right)(Z_N(t)) = D_{z_j}^r (S_N(-t)G)(Z_N),
\]

where, by (7.16), we denote the same derivative of order \( r \) with respect to the variable \( z_j(t) \). Then, by (7.16) and (7.15):

\[
\left( T_N^{(r)}(\tau) G \right)(Z_N) = S_N(\tau) \left( \hat{T}_N^{(r)} + \hat{R}_N^{(r)} \right) S_N(\tau) G(Z_N) = \\
= (-1)^{r/2} \frac{C_r}{N} \sum_{j,l} \left( D_{x_j}^{r+1} \phi \right)(x_j(\tau) - x_l(\tau)) \cdot D_{v_j}^{r+1} (\tau) G(Z_N) + \\
+ \frac{1}{N} \sum_{l,j=1}^{N} \sum_{k_1,k_2 \in \mathbb{N}^3 \atop |k_1| + |k_2| = r+1} C_{k_1,k_2} \left( \frac{\partial^{r+1}}{\partial x_l \partial x_j} \phi \right)(x_l(\tau) - x_j(\tau)) \cdot \frac{\partial^{r+1}}{\partial v_l \partial v_j} (\tau) G(Z_N).
\]

(7.17)

Therefore, in computing the action of \( T_N^{(r)}(\tau) \), we have to consider derivatives with respect to the variables at time \( \tau \). As a consequence, we have to deal with a complicated function of the configuration \( Z_N \) which, however, we do not need to make explicit, as we shall see in a moment.

On the basis of the previous considerations, we compute the time derivative of \( \eta_1 \) by applying the operators \( D^{2r} T_N^{(r_n)}(t_n) T_N^{(r_{n-1})}(t_{n-1}) \ldots T_N^{(r_1)}(t_1) \) to the Vlasov equation:

\[
(\partial_t + v'_1 \cdot \nabla x'_1) \mu_N(t) = (\nabla x'_1 \phi \ast \mu_N(t)) \cdot \nabla v'_1 \mu_N(t).
\]

(7.18)

In doing this we have to compute

\[
D^{2r} T_N^{(r_n)}(t_n) T_N^{(r_{n-1})}(t_{n-1}) \ldots T_N^{(r_1)}(t_1) \mu_N(z'_1|Z_N(t)) \mu_N(z'_2|Z_N(t)).
\]

(7.19)

Now we select the contribution in which each \( T_N^{(r)}(t_\ell) \) and \( D^{2r} \) apply either on \( \mu_N(z'_1|Z_N(t)) \) or to \( \mu_N(z'_2|Z_N(t)) \). The other contribution involves terms in which are present products of derivatives with respect to the same variable. By Proposition 5.1 and Proposition 5.2 we expect
those terms to be negligible (in the $C^\infty$-weak sense) in the limit $N \to \infty$. Therefore we obtain the following equation:

\[
(\partial_t + v_1' \cdot \nabla x_1') \eta_1(z_1', t, r, \underline{x}, \underline{t}, Z_N) = L(\mu_N(t))\eta_1(z_1', t, r, \underline{x}, \underline{t}, Z_N) + \\
+ \sum_{0 \leq \ell \leq r} \sum_{0 \leq m \leq n} \sum_{I \subset I_n: |I| = m, \ell = m, 0 < |\underline{x}_I| + \ell < k} \left( \nabla_{x_I'} \phi * \eta_1(\cdot, t, \ell, \underline{x}_I, \underline{t}_I, Z_N) \right) \cdot \nabla_{v_1'} \eta_1(z_1', t, r - \ell, \underline{x}_{I_n \setminus I}, \underline{t}_{I_n \setminus I}, Z_N) + \\
+ E^1_N,
\]

(7.20)

where $E^1_N$ is an error term which will be proven to be negligible in the limit $N \to \infty$ in Appendix C. In (7.20) we used the notations:

\[
I_n = \{1, 2, \ldots, n\}, \quad I \text{ is any subset of } I_n, \quad \underline{x}_I = \{r_j\}_{j \in I}, \quad \underline{t}_I = \{t_j\}_{j \in I}.
\]

(7.21)

Next, we compute the time derivative of $\nu^{(k)}_1(t)$. We have:

\[
(\partial_t + v_1' \cdot \nabla x_1') \nu^{(k)}_1 = \\
\sum_{n \geq 0} (-1)^n \sum_{r=0}^k \int_0^t dt_2 \int_0^{t_2} dt_3 \ldots \int_0^{t_{n-1}} dt_n \eta_1(z_1'; t, r, \underline{x}_n, \underline{t}_n, Z_N) \bigg|_{t_1 = t} + \\
+ \sum_{n \geq 0} (-1)^n \sum_{r=0}^k \int_0^t dt_n (\partial_t + v_1' \cdot \nabla x_1') \eta_1(z_1'; t, r, \underline{x}_n, \underline{t}_n, Z_N).
\]

(7.22)

In evaluating the first term on the right hand side of (7.22), we are lead to consider $\eta_1$ evaluated in $t = t_1$. Thus, according to the expression of $\eta_1$ (see (7.13)), we have to deal with:

\[
T^{(r_1)}_N(t) \mu_N(z_1'|Z_N(t)) = S_N(-t)T^{(r_1)}_N \mu_N(z_1'|Z_N).
\]

(7.23)

Therefore:

\[
T^{(r_1)}_N(t) \mu_N(z_1'|Z_N(t)) = (-1)^{r_1/2} c_{r_1} \left(D^{r_1+1}_{x_1'} \phi * \mu_N(t)\right)(x_1') \cdot D^{r_1+1}_{v_1'} \mu_N(z_1'|Z_N(t)) = \\
(-1)^{r_1/2} c_{r_1} \int dx_2' dv_2' D^{r_1+1}_{x_1'} \phi(x_1' - x_2') \cdot D^{r_1+1}_{v_1'} \mu_N(x_1', v_1'|Z_N(t)) \mu_N(x_2', v_2'|Z_N(t)),
\]

(7.24)

where the term involving off-diagonal derivatives, namely $R^{(r_1)}_N$ (see (3.20)), disappears because both the derivatives and the empirical distribution are evaluated at time $t$. Hence we compute
Then, putting together (7.22), (7.25), (7.20) and (7.26), we obtain the following equation for \( E \):

\[
\sum_{n \geq 0} (-1)^n \sum_{r = 0}^k \sum_{r_1 > r} \int_0^t \int_0^{t_2} \int_0^{t_3} \cdots \int_0^{t_{n-1}} dt_n \eta_1 (z'_1; t, r, \xi, \nu, t_n, Z_N) \bigg|_{t = t} = \sum_{0 < r_1 \leq k} (-1)^{r_1/2} c_{r_1} \int dx'_2 dv'_2 D^{r_1+1} \phi (x'_1 - x'_2) \cdot D^{r_1+1} \nu_2^{(k-r_1)} (x'_1, v'_1, x'_2, v'_2; t). \]  

(7.25)

Let us come back now to equation (7.22). It is useful to observe that:

\[
\int_{ord}^t dt, \sum_{I \subseteq I_1; |I| = m} = \int_{ord}^t dt \int_{ord}^t dt_{|n|}. \]  

(7.26)

Then, putting together (7.22), (7.25), (7.20) and (7.26), we obtain the following equation for \( \nu_1^{(k)} \):

\[
\begin{align*}
(\partial_t + v'_1 \cdot \nabla_{x'_1}) \nu_1^{(k)} (x'_1, v'_1; t) &= L (\mu_N (t)) \nu_1^{(k)} (x'_1, v'_1; t) + \\
&+ \sum_{0 < r_1 \leq k} (-1)^{r_1/2} c_{r_1} \int dx'_2 dv'_2 D^{r_1+1} \phi (x'_1 - x'_2) \cdot D^{r_1+1} \nu_2^{(k-r_1)} (x'_1, v'_1, x'_2, v'_2; t) + \\
&+ \sum_{0 < \ell < k} \left( \nabla_{x'_1} \phi \ast \nu_1^{(\ell)} (t) \right) \cdot \nabla_{v'_1} \nu_1^{(k-\ell)} (t) + E_N^2,
\end{align*}
\]  

(7.27)

with initial datum given by:

\[
\nu_1^{(k)} (x'_1, v'_1; t)|_{t = 0} = \eta_1 ((z'_1, 0, k, \xi, \nu, t_0, Z_N) = D^{2k} \mu_N (z'_1 | Z_N). \]  

(7.28)

Here \( E_N^2 \) arises from \( E_N^1 \) (see (7.20)). Now, we want to prove that:

\[
\nu_1^{(k)} (t) \to f^{(k)} (t), \quad \text{as} \quad N \to \infty, \quad C_b^\infty \quad \text{weakly},
\]  

(7.29)

and

\[
\nu_2^{(k)} (t) \to f^{(k)}_2 (t), \quad \text{as} \quad N \to \infty, \quad C_b^\infty \quad \text{weakly},
\]  

(7.30)

for any configuration \( Z_N \) such that \( \mu_N \to f^{(0)}_0 \) in the weak sense of probability measure (namely, for any \( Z_N \) typical with respect to \( f^{(0)}_0 \)). As a consequence, reminding that \( \nu_1^{(k)} (t) \) and \( \nu_2^{(k)} (t) \) are equal to \( \omega^{(k)}_{N^1} (t) \) and \( \omega^{(k)}_{N^2} (t) \) respectively, a.e. with respect to \( W^{(0)}_{N,0} \), (7.29) and (7.30) are equivalent to:

\[
\omega^{(k)}_{N,1} (t) \to f^{(k)} (t), \quad \text{as} \quad N \to \infty, \quad C_b^\infty \quad \text{weakly},
\]  

(7.31)

and

\[
\omega^{(k)}_{N,2} (t) \to f^{(k)}_2 (t), \quad \text{as} \quad N \to \infty, \quad C_b^\infty \quad \text{weakly}.
\]  

(7.32)
As we already remarked, the $C^\infty$-weak convergence implies the convergence in $S'$, therefore, (7.31) and (7.32) imply the convergence of $\omega^{(k)}_{N,1}(t)$ to $f^{(k)}(t)$ in $S'(\mathbb{R}^3 \times \mathbb{R}^3)$ and of $\omega^{(k)}_{N,2}(t)$ to $f^{(k)}_2(t)$ in $S'(\mathbb{R}^6 \times \mathbb{R}^6)$.

7.1. One and two-particle convergence. In evaluating the behavior of $\nu^k(t)$ when $N \to \infty$, we note that it solves the initial value problem (7.27)-(7.28) for which we want to use Proposition 6.2. First, however, we have to verify the assumptions. The first one, namely i), is verified as follows by the considerations developed in Section 5.

Now, we have to check that assumption ii) is satisfied, namely, we have to prove that

$$\forall u_1, u_2 \in C^\infty_c(\mathbb{R}^3 \times \mathbb{R}^3),$$

there exists a constant $C = C(u_1, u_2) > 0$, independent of $N$, such that:

$$\left\| u_1 \ast \left( u_2 \nu^k_1(t) \right) \right\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} < C \quad \text{for any } t. \quad (7.33)$$

We have:

$$\left\| u_1 \ast \left( u_2 \nu^k_1(t) \right) \right\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} = \sup_{x', \nu'_1} \left| \int dydw \ u_1(x' - y, \nu'_1 - w)u_2(y, w)\nu^k_1(y, w; t) \right| \leq$$

$$\leq \sum_{n \geq 0} \sum_{r=0}^{k} \sum_{\mathbb{E}_n ; r_j > 0} \int_{\text{ord}}^t dt_n \sup_{x', \nu'_1} \left| \int dydw \ u_1(x' - y, \nu'_1 - w)u_2(y, w)\eta_1(y, w; t; r, \mathbb{E}_n, t_n, Z_N) \right| =$$

$$= \sum_{n \geq 0} \sum_{r=0}^{k} \sum_{\mathbb{E}_n ; r_j > 0} \int_{\text{ord}}^t dt_n \sup_{x', \nu'_1} \left| \int dydw \ (u_1(x' - y, \nu'_1 - w)u_2(y, w))D^{2r}T^{(r_n)}_N(t_n) \cdots T^{(r_1)}_N(t_1)\mu_N(y, w|Z_N(t)) \right| =$$

$$= \sum_{n \geq 0} \sum_{r=0}^{k} \sum_{\mathbb{E}_n ; r_j > 0} \int_{\text{ord}}^t dt_n \sup_{x', \nu'_1} \left| \int dydw \ g(x', \nu'_1, y, w)D^{2r}T^{(r_n)}_N(t_n) \cdots T^{(r_1)}_N(t_1)\mu_N(y, w|Z_N(t)) \right|, \quad (7.34)$$

where $\nu^k_1(t)$ is verified, therefore, (7.31) and (7.32) imply the convergence of $\omega^{(k)}_{N,1}(t)$ to $f^{(k)}(t)$ in $S'(\mathbb{R}^3 \times \mathbb{R}^3)$ and of $\omega^{(k)}_{N,2}(t)$ to $f^{(k)}_2(t)$ in $S'(\mathbb{R}^6 \times \mathbb{R}^6)$.
versus a function in $C^\infty_b(\mathbb{R}^3 \times \mathbb{R}^3)$ we obtain a quantity uniformly bounded in $N$. This feature, by virtue of the good properties of the function $g$ ensures that (7.34) is finite.

Let us now look at the initial datum for $\nu_{1}^{(k)}(t)$, in order to verify assumption iii).

From (7.28) we know that $\nu_{1}^{(k)}(0) = D^{2k} \mu_N \in \mathcal{S}'(\mathbb{R}^3 \times \mathbb{R}^3)$. As regard to its limiting behavior, we find that:

$$
\nu_{1}^{(k)}(t)\Big|_{t=0} = D^{2k} \mu_N = \sum_{n=1}^{N} \sum_{I \subseteq I_N} \sum_{|I|=n} \prod_{j \in I} D^{s_j}_{G,j} \mu_N,
$$

(7.35)

where $I_N = \{1, \ldots, N\}$. For our convenience, we have written the action of the operator $D^{2k}$ in an equivalent and slightly different way from that we used in (7.3).

We realize that the only surviving term in the sum (7.35) is that with $n = 1$. Hence:

$$
\nu_{1}^{(k)}(t)\Big|_{t=0} = \sum_{j=1}^{N} D^{2k}_{G,j} \delta(z'_j - z_j) = D^{2k}_G \mu_N.
$$

(7.36)

Therefore we can conclude, by using the mean-field limit:

$$
\left( u, \nu_{1}^{(k)}(t)\Big|_{t=0} \right) = (u, D^{2k}_G \mu_N) =
$$

$$
= \left( D^{2k}_G u, \mu_N \right) \to \left( D^{2k}_G u, f_{0}^{(0)} \right) = \left( u, D^{2k}_G f_{0}^{(0)} \right) = \left( u, f_{0}^{(k)} \right), \text{ as } N \to \infty,
$$

$$
\forall u \text{ in } C^\infty_b(\mathbb{R}^3 \times \mathbb{R}^3).
$$

(7.37)

Thus, $f_{0}^{(k)}$ plays the role of $\gamma_0$ in Proposition 6.2 and it is in $L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ because $f_{0}^{(0)} \in \mathcal{S}(\mathbb{R}^3 \times \mathbb{R}^3)$.

We conclude the convergence proof (for the one and two-particle functions) by induction.

For $k = 0$ we know that, for any configuration $Z_N$ which is typical with respect to $f_{0}^{(0)}$, we have:

$$
\nu_{1}^{(0)}(t) = \mu_N(t) \to f^{(0)}(t), \text{ as } N \to \infty,
$$

(7.38)

in the weak sense of probability measures, and, as a consequence, the convergence holds $C^\infty_b$—weakly. Moreover

$$
\nu_{2}^{(0)}(t) = \mu_N(t) \otimes \mu_N(t) \to f_{2}^{(0)}(t) = f^{(0)}(t) \otimes f^{(0)}(t), \text{ as } N \to \infty,
$$

(7.39)

in the weak sense of probability measures, and, as a consequence, the convergence holds $C^\infty_b$—weakly.

We make the following inductive assumptions for all $h < k$:

$$
\nu_{1}^{(h)}(t) \to f^{(h)}(t), \text{ as } N \to \infty, \text{ } C^\infty_b \text{—weakly},
$$

(7.40)
for any configuration $Z_N$ which is typical with respect to $f_0^{(0)}$, and
\[
\nu_2^{(h)}(t) \to f_2^{(h)}(t) = \sum_{0 \leq q \leq h} f^{(q)}(t) f^{(h-q)}(t), \text{ as } N \to \infty, \quad C^\infty_b \text{ weakly,} \quad (7.41)
\]
for any configuration $Z_N$ which is typical with respect to $f_0^{(0)}$.

Now we want to prove that (7.40) and (7.41) hold also for $h = k$.

Thanks to (7.40), we can affirm that:
\[
\sum_{0 \leq \ell < k} \left( \nabla_{x'_1} \phi \ast \nu_1^{(\ell)} \right) \cdot \nabla_{v'_1} \nu_1^{(k-\ell)} \to \sum_{0 \leq \ell < k} \left( \nabla_{x'_1} \phi \ast f^{(\ell)} \right) \cdot \nabla_{v'_1} f^{(k-\ell)} = \sum_{0 \leq \ell < k} T_\ell^{(0)} f^{(k-\ell)},
\]
which plays the role of \(\Theta\) in Proposition 6.2 and it is easy to check that it is in $C^0(L^1(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{R}^+))$. Therefore, we can apply Proposition 6.2 claiming that, for any typical configuration $Z_N$ with respect to $f_0^{(0)}$, $\nu_1^{(k)}(t)$ converges $C^\infty_b$-weakly to the solution of the problem (6.10). Looking at (2.15) and (2.17), we realize that we obtained the equation satisfied by $f^{(k)}(t)$.

In order to "close" the recurrence procedure, it remains to show the two-particle convergence at order $k$. It follows from the one-particle analysis and from the following computation

\[
\sum_{0 \leq \ell < k} (\nabla_{x'_1} \phi \ast \nu_1^{(\ell)}) \cdot \nabla_{v'_1} \nu_1^{(k-\ell)} = \sum_{0 \leq \ell < k} T_\ell^{(0)} f^{(k-\ell)},
\]
and, thanks to (7.41), have:
\[
\sum_{0 \leq r_1 \leq k} \frac{(-1)^{r_1/2}}{c_{r_1}} \int dx'_2 dv'_2 \nu_2^{r_1}(x'_1 - x'_2) \cdot \nabla_{v'_1} \nu_1^{(k-r_1)}(x'_1, v'_1, x'_2, v'_2, t) \Delta_{C^\infty_b} \text{ weakly,}
\]

\[
\sum_{0 \leq r_1 \leq k} \frac{(-1)^{r_1/2}}{c_{r_1}} \int dx'_2 dv'_2 \nu_2^{r_1}(x'_1 - x'_2) \cdot \nabla_{v'_1} \nu_1^{(k-r_1)}(x'_1, v'_1, x'_2, v'_2, t) = \sum_{0 \leq r_1 \leq k} \sum_{0 \leq q \leq k-r_1} (-1)^{r_1/2} \int dx'_2 dv'_2 \nu_2^{r_1+1}(x'_1 - x'_2) f^{(k-r_1)}(x'_2, v'_2, t) \cdot \nabla_{v'_1} f^{(q)}(x'_1, v'_1, t) = \sum_{0 \leq r_1 \leq k} \sum_{0 \leq q \leq k-r_1} T_q^{(r_1)} f^{(k-r_1-q)}(t).
\]

At the end, putting together (7.42) and (7.43), we find that the sum of the source terms in equation (7.27) converges $C^\infty_b$-weakly to:
\[
\sum_{0 \leq \ell < k} T_\ell^{(0)} f^{(k-\ell)} + \sum_{0 \leq r_1 \leq k} \sum_{0 \leq q \leq k-r_1} T_q^{(r_1)} f^{(k-r_1-q)},
\]
which plays the role of \(\Theta\) in Proposition 6.2 and it is easy to check that it is in $C^0(L^1(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{R}^+))$. Therefore, we can apply Proposition 6.2 claiming that, for any typical configuration $Z_N$ with respect to $f_0^{(0)}$, $\nu_1^{(k)}(t)$ converges $C^\infty_b$-weakly to the solution of the problem (6.10).
\[ \eta_2(z_1', z_2', t, r, r_n, t_n, Z_N) = \sum_{0 \leq \ell \leq k} \sum_{0 \leq m \leq n} \sum_{I:I \subseteq I_n, |I|=m} \eta_1(z_1', t, \ell, r_I, t_I, Z_N) \eta_1(z_2', t, k - \ell, r_{I_n \setminus I}, t_{I_n \setminus I}, Z_N) + R^2_N, \quad (7.45) \]

where \( R^2_N \) is a remainder arising from the action of the operator \( D^2_{r_1} T_{N_n}^{(r_n)}(t_n) \ldots T_{N_1}^{(r_1)}(t_1) \) on a product of two empirical measures \( \mu_N(t) \). In Appendix C we will see that it is vanishing in the limit. As a consequence, \( \nu_2^{(k)} \) (see (7.10) for \( j = 2 \)) is such that:

\[ \nu_2^{(k)}(t) = \sum_{0 \leq q \leq k} \nu_1^{(q)}(t) \nu_1^{(k-q)}(t) + o(1), \quad (7.46) \]

in the limit \( N \to \infty \). Therefore, from the inductive assumption (7.40) and from the one-particle convergence at order \( k \), we conclude that:

\[ \nu_2^{(k)}(t) \to \sum_{0 \leq q \leq k} f^{(q)}(t) f^{(k-q)}(t) = f_2^{(k)}(t), \text{ as } N \to \infty, C^\infty - \text{weakly}, \quad (7.47) \]

for any configuration \( Z_N \) which is typical with respect to \( f_0^{(0)} \). Thus, we have just proven the convergence of \( \omega_{N,j}^{(k)} \) in the cases \( j = 1, j = 2 \).

### 7.2. \( j \)-particle convergence.

As for \( j = 2 \), the \( j \)-particle convergence can be reduced by the one-particle control. Indeed by (7.10) and (7.11) we have:

\[ \nu_j^{(k)}(t) = \sum_{0 \leq s \leq k} \prod_{m=1}^j \nu_1^{(s_m)}(t) + R^j_N, \quad (7.48) \]

with \( R^j_N \to 0 \) when \( N \to \infty \).

Again the error term \( R^j_N \) arises from the presence of products of derivatives with respect to the same variable. In conclusion, the result we proved for \( \nu_1^{(k)}(t) \), together with the estimates proven in Appendix C, is sufficient to guarantee the \( C^\infty \)-weak convergence of \( \nu_j^{(k)}(t) \) to \( f_j^{(k)}(t) \) for any \( j \) (for any typical configuration \( Z_N \) with respect to \( f_0^{(0)} \)), and, as a consequence, the \( C^\infty \)-weak convergence of \( \omega_{N,j}^{(k)}(t) \) is \( f_j^{(k)}(t) \), for any \( j \).

The final step is to realize that this convergence does imply that for the coefficients \( W_{N,j}^{(k)}(t) \), namely what is established by Theorem 6.1.
First of all, we observe that, for any test function $u$ we have:

\[
\left( u, W_{N,1}^{(k)}(t) \right) = \int_{\mathbb{R}^6} d z_1 W_{N,1}^{(k)}(z_1; t) u(z_1) = \\
= \int_{\mathbb{R}^{3N} \times \mathbb{R}^{3N}} d Z_N W_{N}^{(k)}(Z_N; t) u(z_1) = \\
= \int_{\mathbb{R}^{3N} \times \mathbb{R}^{3N}} d Z_N W_{N}^{(k)}(Z_N; t) \frac{1}{N} \sum_{l=1}^{N} u(z_l) = \\
= \int_{\mathbb{R}^{3N} \times \mathbb{R}^{3N}} d Z_N W_{N}^{(k)}(Z_N; t) (u, \mu_N) = \left( u, \omega_{N,1}^{(k)}(t) \right),
\]

(7.49)

where we made use of the symmetry of the coefficient $W_{N}^{(k)}(Z_N; t)$ with respect to any permutation of the variables (the computation is the same we did in Section 5 for $W_{N,1}^{(1)}(t)$). From (7.49), we can see that $W_{N,1}^{(k)}(t)$ and $\omega_{N,1}^{(k)}(t)$ are equal as distributions in $\mathcal{S}'(\mathbb{R}^3 \times \mathbb{R}^3)$ (in particular, we can choose test functions belonging to $C^\infty_b(\mathbb{R}^3 \times \mathbb{R}^3)$), then the convergence of $W_{N,1}^{(k)}(t)$ is proven. Moreover, for $j \geq 2$, a straightforward computation shows that, by fixing an index $j$, we have

\[
\left( u_j, \omega_{N,j}^{(k)}(t) \right) = \frac{N(N-1) \ldots (N-j+1)}{N^j} \left( u_j, W_{N,j}^{(k)}(t) \right) + \frac{C_{j < j}}{N},
\]

(7.50)

where $C_{j < j} < \infty$ provided that $\left( u_j, W_{N,j}^{(k)}(t) \right)$ is uniformly bounded for each $j < \overline{j}$. Then, to conclude the proof of Theorem 6.1, it is enough to use a recurrence argument.
Appendix A

Proof of Proposition 5.1

To avoid inessential notational complications, we deal with the one-dimensional case. By the Newton equations, we have:

\[
\frac{\partial x_i(t)}{\partial v_r} = \delta_{ir}t + \int_0^t ds(t - s)\frac{1}{N} \sum_{j \neq i} \partial_x F(x_i(s) - x_j(s)) \left( \frac{\partial x_i(s)}{\partial v_r} - \frac{\partial x_j(s)}{\partial v_r} \right),
\]

(A.1)

\[
\frac{\partial v_i(t)}{\partial v_r} = \delta_{ir} + \int_0^t ds \frac{1}{N} \sum_{j \neq i} \partial_x F(x_i(s) - x_j(s)) \left( \frac{\partial x_i(s)}{\partial v_r} - \frac{\partial x_j(s)}{\partial v_r} \right),
\]

(A.2)

where:

\[
F = -\nabla x \phi,
\]

(A.3)

is the force associated with the potential \( \phi \).

Let us analyze in detail the derivative of \( x_i(t) \). From (A.1), we get:

\[
\max_{i,r,t \leq T} \left| \frac{\partial x_i(t)}{\partial v_r} \right| \leq C.
\]

(A.4)

Inserting this estimate again in (A.1), we realize that we can obtain a better bound for \( \frac{\partial v_i(t)}{\partial v_r} \) in the case \( r \neq i \) (see [17]), namely:

\[
\left| \frac{\partial x_i(t)}{\partial v_r} \right| \leq C \int_0^t ds(t - s) \left| \frac{\partial x_i(s)}{\partial v_r} \right| + C \int_0^t ds(t - s) \frac{1}{N} \left| \frac{\partial x_i(s)}{\partial v_r} \right| + C \int_0^t ds(t - s) \frac{1}{N} \sum_{j \neq i} \partial_x F(x_i(s) - x_j(s)) \left| \frac{\partial x_j(s)}{\partial v_r} \right|.
\]

(A.5)

Hence, by virtue of the Gronwall lemma, we find:

\[
\max_{i \neq r,t \leq T} \left| \frac{\partial x_i(t)}{\partial v_r} \right| \leq \frac{C}{N}.
\]

(A.7)

By (A.2), we find that the same estimate holds for the derivative of \( v_i(t) \) with respect to \( v_r \). Analogous estimates hold for the derivatives with respect to the initial positions (see also [17]). Therefore the claim of Proposition 5.1 is proven for derivatives of order one.
Now, let us consider a sequence $I := (j_1, \ldots, j_k)$ of possibly repeated indices. We show that:

$$\frac{1}{N} \sum_{i=1}^{N} \left| \frac{\partial^k x_i(t)}{\partial v_{j_1} \cdots \partial v_{j_k}} \right| \leq \frac{C}{N^{d_k}}, \quad (A.8)$$

where $d_k$ is the number of different indices in the sequence $j_1, \ldots, j_k$. We know that (A.8) is verified for $k = 1$ (it follows directly by (A.4) and (A.7)), thus we prove (A.8) by induction on $k$. Denoting by:

$$D(I) := \frac{\partial^k}{\partial v_{j_1} \cdots \partial v_{j_k}}, \quad (A.9)$$

estimate (A.8) can be rewritten as:

$$\frac{1}{N} \sum_{i=1}^{N} |D(I)x_i(t)| \leq \frac{C}{N^{d_k}}. \quad (A.10)$$

By (A.1) we derive the following estimate for $D(I)x_i(t)$:

$$|D(I)x_i(t)| \leq \int_0^t ds(t-s) \sum_{n=2}^{k} \sum_{\mathcal{P}_n} C(\mathcal{P}_n) \sum_{j=1}^{N} \prod_{H \in \mathcal{P}_n} \left| D(H)(x_i(s) - x_j(s)) \right| + M_i(t), \quad (A.11)$$

where the term $M_i(t)$ can be computed from (A.1) according to the Leibniz rule. Let $\mathcal{P}_n := \{I_1, \ldots, I_n\}$ be a partition of the set $I$ of cardinality $n$, with $2 \leq n \leq k$, then we have:

$$M_i(t) \leq \int_0^t ds(t-s) \frac{1}{N} \sum_{n=2}^{k} \sum_{\mathcal{P}_n} C(\mathcal{P}_n) \sum_{j=1}^{N} \prod_{H \in \mathcal{P}_n} \left| D(H)(x_i(s) - x_j(s)) \right| \leq$$

$$\leq \int_0^t ds(t-s) \sum_{n=2}^{k} \sum_{\mathcal{P}_n} C(\mathcal{P}_n) \frac{1}{N} \sum_{j=1}^{N} \prod_{H \in \mathcal{P}_n} \left| D(H)(x_i(s) - x_j(s)) \right|, \quad (A.12)$$

where $D(H) := \prod_{h \in H} \frac{\partial}{\partial v_h}$ and $C(\mathcal{P}_n)$ are coefficients depending on the partition $\mathcal{P}_n$ and on suitable derivatives of $F$. By (A.11), it follows that:

$$\frac{1}{N} \sum_{i=1}^{N} |D(I)x_i(t)| \leq \int_0^t ds(t-s) \frac{C}{N} \sum_{i=1}^{N} |D(I)x_i(s)| + M(t), \quad (A.13)$$

where $M(t) = \frac{1}{N} \sum_{i=1}^{N} M_i(t)$ and, by (A.12), we have:

$$M(t) \leq \int_0^t ds(t-s) \sum_{n=2}^{k} \sum_{\mathcal{P}_n} C(\mathcal{P}_n) \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \prod_{H \in \mathcal{P}_n} \left| D(H)(x_i(s) - x_j(s)) \right|, \quad (A.14)$$
We observe that:

\[
\frac{1}{N^2} \sum_{i,j=1}^{N} \left| \prod_{H \in P_n} [D(H)(x_i(s) - x_j(s))] \right| \leq \frac{1}{N} \sum_{i=1}^{N} \prod_{H \in P_n} |D(H)x_i(s)| + \\
+ \frac{1}{N} \sum_{j=1}^{N} \prod_{H \in P_n} |D(H)x_j(s)| + \\
+ \sum_{Q \subset P_n} C(Q) \left( \frac{1}{N} \sum_{i=1}^{N} \prod_{Q \in Q} |D(Q)x_i(s)| \right) \left( \frac{1}{N} \sum_{j=1}^{N} \prod_{J \in P_n \setminus Q} |D(J)x_j(s)| \right),
\]

where \( Q \) is any subpartition of \( P_n \) and \( C(Q) \) are coefficients depending on \( Q \).

We assume that the estimate (A.10) holds for any \( m \leq k - 1 \), namely:

\[
\frac{1}{N} \sum_{i=1}^{N} |D(M)x_i(t)| \leq \frac{C}{N^d_m}, \text{ for any } M \subset I \text{ s.t. } |M| = m \leq k - 1,
\]

where \( d_m \) is the number of different indices in the sequence \( M \).

Indeed, if we consider a partition \( P_n \) of cardinality \( n \geq 2 \), we are guaranteed that \( |M| \leq k - 1 \) for each \( M \in P_n \). Then, by noting that:

\[
\frac{1}{N} \sum_{i=1}^{N} \prod_{H \in \mathcal{H}} |D(H)x_i(t)| \leq \prod_{H \in \mathcal{H}} \frac{1}{N} \sum_{i=1}^{N} |D(H)x_i(t)|, \ \forall \text{ subpartition } \mathcal{H} \subseteq P_n,
\]

we can apply the inductive hypotheses (A.16) to estimate the derivatives of \( x_i(s) \) and \( x_j(s) \) appearing in (A.15). Thus, we obtain:

\[
\frac{1}{N} \sum_{i=1}^{N} \prod_{H \in P_n} |D(H)x_i(s)| \leq \prod_{H \in P_n} \frac{1}{N} \sum_{i=1}^{N} |D(H)x_i(s)| \leq \\
\leq \prod_{H \in P_n} \frac{C}{N^{d_h}} = \frac{C}{N^{\sum d_h}} \leq \frac{C}{N^{d_h}},
\]

where \( d_h \) is the number of different indices in the sequence \( H \) and we used that \( \sum_{H \in P_n} d_h \geq d_k \).

In a similar way, we find

\[
\frac{1}{N} \sum_{i=1}^{N} \prod_{Q \in Q} |D(Q)x_i(s)| \leq \prod_{Q \in Q} \frac{C}{N^{d_q}},
\]

(A.19)
where \(d_q\) is the number of different indices in the sequence \(Q\). Moreover, we have:

\[
\frac{1}{N} \sum_{j=1}^{N} \prod_{H \in P_n} |D(H)x_j(s)| \leq \prod_{H \in P_n} \frac{C}{N^{d_h}} = \frac{C}{N^{d_h}} \leq \frac{C}{N^{d_k}}, \tag{A.20}
\]

and

\[
\frac{1}{N} \sum_{j=1}^{N} \prod_{J \in \mathcal{P} \setminus Q} |D(J)x_j(s)| \leq \prod_{J \in \mathcal{P} \setminus Q} \frac{C}{N^{d_j}}, \tag{A.21}
\]

where \(d_j\) is the number of different indices in the sequence \(J\). Then, putting together (A.19) and (A.21), we find:

\[
\sum_{Q \subset \mathcal{P}_n} C(Q) \left( \frac{1}{N} \sum_{i=1}^{N} \prod_{Q \in \mathcal{Q}} |D(Q)x_i(s)| \right) \left( \frac{1}{N} \sum_{j=1}^{N} \prod_{J \in \mathcal{P}_n \setminus Q} |D(J)x_j(s)| \right) \leq \sum_{Q \subset \mathcal{P}_n} C(Q) \prod_{Q \in \mathcal{Q}} \prod_{J \in \mathcal{P}_n \setminus Q} \frac{C}{N^{d_q + d_j}} \leq \sum_{Q \subset \mathcal{P}_n} C(Q) \prod_{Q \in \mathcal{Q}} \prod_{J \in \mathcal{P}_n \setminus Q} \frac{C}{N^{d_k}} \leq \frac{C}{N^{d_k}}. \tag{A.22}
\]

In the end, we have just proven that each term in (A.19) is bounded by \(\frac{C}{N^{d_k}}\). Therefore, by using this estimate in (A.14), we find:

\[
M(t) \leq \frac{C}{N^{d_k}}. \tag{A.23}
\]

By (A.23) and (A.13), it follows that:

\[
\frac{1}{N} \sum_{i=1}^{N} |D(I)x_i(t)| \leq \int_0^t ds (t - s) \frac{C}{N} \sum_{i=1}^{N} |D(I)x_i(s)| + \frac{C}{N^{d_k}}. \tag{A.24}
\]

Therefore, by using the Gronwall lemma, we find:

\[
\frac{1}{N} \sum_{i=1}^{N} |D(I)x_i(t)| \leq \frac{C}{N^{d_k}}. \tag{A.25}
\]

As regard to the derivatives of \(v_i(t)\) with respect to some initial velocities \(v_{j_1}, \ldots, v_{j_k}\), an analogous estimate holds and the proof works in the same way. Furthermore, this strategy leads to the same estimate for the derivatives of the function \(\frac{1}{N} \sum_{i=1}^{N} z_i(t)\) with respect to some initial positions \(x_{j_1}, \ldots, x_{j_k}\).
Now, thanks to the estimate we have just proven for the derivatives of the function $\frac{1}{N} \sum_{i=1}^{N} z_i(t)$, we are able to prove the claim of Proposition 5.1. In fact, we have:

$$\frac{1}{N} \sum_{i=1}^{N} |D(I)z_i(t)| = \frac{1}{N} \sum_{i \in D}^{N} |D(I)z_i(t)| + \frac{1}{N} \sum_{i \notin D}^{N} |D(I)z_i(t)| \leq \frac{C}{Nd_k}, \quad (A.26)$$

where $D \subset I$ contains the different indices appearing in the sequence $I$. Thus, according to our previous notation, $|D| = d_k$ and we denote the elements of $D$ by $\tilde{j}_1, \ldots, \tilde{j}_d_k$. Then by (A.26) we find:

$$\frac{1}{N} \sum_{i=1}^{N} |D(I)z_i(t)| = \frac{1}{N} \left| D(I)z_{\tilde{j}_1}(t) \right| + \cdots + \frac{1}{N} \left| D(I)z_{\tilde{j}_{d_k}}(t) \right| +$$

$$+ \frac{1}{N} \sum_{i \notin D}^{N} |D(I)z_i(t)| \leq \frac{C}{Nd_k}, \quad (A.27)$$

which implies

$$|D(I)z_i(t)| \leq C \left( \sum_{\ell=1}^{d_k} \delta_{i\ell} + \frac{1}{Nd_k} \right), \quad (A.28)$$

or

$$|D(I)z_i(t)| \leq \frac{C}{Nd_k^{(d)}}, \quad (A.29)$$

where $d_k^{(i)}$ is the number of different indices in the sequence $I$ which are also different from $i$. 

□

APPENDIX B

Proof of Proposition 6.1

Let $U_h(t, s)$ be the two parameters semigroup solution of the linear problem:

$$\begin{cases}
(\partial_t + v \cdot \nabla_x) U_h(t, s) \gamma_0 = (\nabla \phi * h) * \nabla_v U_h(t, s) \gamma_0, \\
U_h(s, s) \gamma_0 = \gamma_0.
\end{cases} \quad (B.1)$$

The solution of (B.1) is obtained by carrying the initial datum $\gamma_0$ along the characteristic flow

$$\begin{cases}
\dot{x} = v, \\
\dot{v} = -\nabla \phi * h.
\end{cases} \quad (B.2)$$
Next, we consider the problem
\[
\begin{cases}
(\partial_t + v \cdot \nabla_x) \tilde{\gamma} = L(h)\tilde{\gamma}, \\
\tilde{\gamma}|_{t=0} = \gamma_0.
\end{cases}
\] (B.3)

which can be reformulated in integral form:
\[
\tilde{\gamma}(t) = U_h(t, 0)\gamma_0 + \int_0^t ds \, U_h(t, s) [\nabla \phi * \tilde{\gamma}(s)] \cdot \nabla_v h(s).
\] (B.4)

The above formula can be iterated to yield the formal solution
\[
\tilde{\gamma}(x, v; t) = U_h(t, 0)\gamma_0(x, v) + \sum_{n \geq 1} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \int dx_1 \int dv_1 \ldots \int dx_n \int dv_n
\]
\[
U_h(t_1, t_2) [\nabla_v h(x_1, v_1; t_1) \cdot \nabla_x \phi(x_1 - x_2)]
\]
\[\ldots\]
\[
U_h(t_{n-1}, t_n) [\nabla_{v_{n-1}} h(x_{n-1}, v_{n-1}; t_n) \cdot \nabla_{x_{n-1}} \phi(x_{n-1} - x_n)]
\]
\[
U_h(t_n, 0)\gamma_0(x_n, v_n).
\] (B.5)

We remark that $U_h(t_k, t_{k+1})$ acts on the variables $x_k, v_k$ with the convention that $(x_0, v_0) = (x, v)$ and, furthermore, $U_h$ is multiplicative and preserves the $L^p(\mathbb{R}^3 \times \mathbb{R}^3)$ norms ($p = 1, 2, \ldots, \infty$).

Under the assumptions of Proposition 6.1, the above series is bounded in $L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ by:
\[
\sum_{n \geq 0} \frac{t^n}{n!} \left( \sup_{\tau \in [0, t]} \|\nabla_v h(\tau)\|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)} \right)^n \|\nabla_x \phi\|^n_{L^\infty(\mathbb{R}^3)} \|\gamma_0\|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)},
\] (B.6)

which is converging for each $t$. Now, we denote by $\Sigma_h(t, s) : L^1(\mathbb{R}^3 \times \mathbb{R}^3) \to L^1(\mathbb{R}^3 \times \mathbb{R}^3)$, the two parameters semigroup given by the series $U_h$. Then, the solution $\gamma$ to the problem (6.10) is given by:
\[
\gamma(t) = \Sigma_h(t, 0)\gamma_0 + \int_0^t ds \, \Sigma_h(t, s)\Theta(s),
\] (B.7)

and, thanks to the assumption we made on $\Theta$ and to the fact that the above series is converging for any $t$, we are guaranteed that $\gamma \in C^0(L^1(\mathbb{R}^3 \times \mathbb{R}^3), \mathbb{R}^+)$. The $C^k$ regularity of $\tilde{\gamma}(t) = \Sigma_h(t, 0)\gamma_0$ follows by $U_h$ and the fact that $U_h(t_1)$ propagates the $C^k$ regularity.

\[\square\]

\textbf{Proof of Proposition 6.2}
The proof consists of two steps.

**Step 1):**
Let $\gamma_N$ be as in Proposition 6.2. Then, we show that $\gamma_N$ solves the problem:

\[
\begin{aligned}
(\partial_t + v \cdot \nabla_x) \gamma_N &= L(h)\gamma_N + \Theta'_N, \\
\gamma_N|_{t=0} &= \gamma_{N,0},
\end{aligned}
\]  

(B.8)

with

\[ \Theta'_N = \Theta_N + R_N, \]

(B.9)

and $R_N$ is such that:

\[ R_N \to 0, \quad C^\infty_b - \text{weakly.} \]

In proving (B.10), the assumption ii) on $\gamma_N$ is crucial.

**Step 2):**
By virtue of Step 1), the hypotheses we made on $\nabla_v h$ and Proposition 6.1, we find that:

\[ \gamma_N(t) = \Sigma_h(t,0)\gamma_{N,0} + \int_0^t \frac{\partial}{\partial \gamma_N} \Theta'_N(s). \]

(B.11)

Then, reminding that:
\[\circ h(t) \in C^\infty_b(\mathbb{R}^3 \times \mathbb{R}^3) \text{ for any } t,\]
\[\circ \text{ the flow } \Sigma_h \text{ propagates the } C^k \text{ regularity,}\]
\[\circ R_N \to 0, \quad C^\infty_b - \text{weakly},\]

and by virtue of the assumptions on $\gamma_{N,0}$ and $\Theta_N$, we can easily show that:

\[ \gamma_N \to \gamma, \quad \text{as } N \to \infty, \quad C^\infty_b - \text{weakly,} \]

(B.12)

where

\[ \gamma(t) = \Sigma_h(t,0)\gamma_0 + \int_0^t \frac{\partial}{\partial \gamma_N} \Theta(s). \]

(B.13)

Therefore, we recognize that $\gamma$ solves the problem (6.10) and, by virtue of Proposition 6.1, it is uniquely determined by (B.13) and hence it is in $C^0(L^1(\mathbb{R}^3 \times \mathbb{R}^3), \mathbb{R}^+)$. 

**Proof of Step 1):**
We have:

\[
\begin{aligned}
(\partial_t + v \cdot \nabla_x) \gamma_N &= L(h)\gamma_N + \Theta_N + L(h_N - h)\gamma_N \\
\gamma_N(x,v;t)|_{t=0} &= \gamma_{N,0}(x,v),
\end{aligned}
\]

(B.14)

where

\[ R_N = R_N(x,v;t) := L(h_N - h)\gamma_N. \]

(B.15)

We want to show that $R_N \to 0, \quad C^\infty_b - \text{weakly.}$ According to the definition of the operator $L$, we have:

\[ R_N = (\nabla_x \phi \ast (h_N - h)) \nabla_v \gamma_N + (\nabla_x \phi \ast \gamma_N) \nabla_v (h_N - h), \]

(B.16)
thus, we have to show that

\[ (u, (\nabla_x \phi \ast (h_N - h)) \nabla_v \gamma_N) \to 0, \quad \text{as} \quad N \to \infty, \quad \forall \ u \in \mathcal{C}^\infty_b(\mathbb{R}^3 \times \mathbb{R}^3), \quad (B.17) \]

and

\[ (u, (\nabla_x \phi \ast \gamma_N) \nabla_v (h_N - h)) \to 0, \quad \text{as} \quad N \to \infty, \quad \forall \ u \in \mathcal{C}^\infty_b(\mathbb{R}^3 \times \mathbb{R}^3). \quad (B.18) \]

We show only (B.18) in detail because (B.17) will follow the same line. We have:

\[ (u, (\nabla_x \phi \ast \gamma_N) \nabla_v (h_N - h)) = \int dx dv \int dy dw \ u(x,v) \nabla_x \phi(x - y) \gamma_N(y,w,t) \cdot \nabla_v (h_N(x,v; t) - h(x,v;t)) = \]

\[ = - \int dx dv \int dy dw \ \nabla_v u(x,v) \nabla_x \phi(x - y) \gamma_N(y,w; t) \cdot (h_N(x,v; t) - h(x,v; t)) = \]

\[ = \int dx dv \int dy dw \ \nabla_v u(x,v) (\nabla_x \phi \ast \gamma_N) (x,v; t)(h - h_N)(x,v; t). \quad (B.19) \]

Setting

\[ \zeta_N(x,v) := \nabla_v u(x,v) \int dy dw \nabla_x \phi(x - y) \gamma_N(y,w; t), \quad (B.20) \]

we can write (B.19) as:

\[ (u, (\nabla_x \phi \ast \gamma_N) \nabla_v (h_N - h)) = \int dx dv \zeta_N(x,v)(h(x,v; t) - h_N(x,v; t)) = \]

\[ = \int dx dv \int dx'dv' (\zeta_N(x,v) - \zeta_N(x',v')) P_N(x,v;x',v';t), \quad (B.21) \]

where \( P_N \) is a coupling of \( h \) and \( h_N \), namely a probability density in \( \mathbb{R}^6 \times \mathbb{R}^6 \) with marginals given by \( h \) and \( h_N \). Now we observe that:

\[ \nabla_{x,v} \zeta_N(x,v) := \int dy dw \nabla_{x,v} [\nabla_v u(x,v) \nabla_x \phi(x - y)] \gamma_N(y,w; t), \quad (B.22) \]

and, thanks to the assumption ii) we made on \( \gamma_N \), we know that there exists a constant \( C = C(u, \phi) > 0 \) such that:

\[ \sup_{x,v} |\nabla_{x,v} \zeta_N(x,v)| = \| \nabla \zeta_N \|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} < C < +\infty. \quad (B.23) \]
Therefore, coming back to (B.21), we find:
\[
|(u, (\nabla_x \phi \ast \gamma_N) \nabla_v (h_N - h))| \leq \int dz \int dz' |\zeta_N(z) - \zeta_N(z')| P_N(z; z'; t)
\leq \int dz \int dz' C |z - z'| P_N(z; z'; t).
\]
(B.24)
where we used the standard notation \(z = (x, v)\) and \(z' = (x', v')\). Then, taking in (B.24) the infimum over all couplings between \(h\) and \(h_N\), we obtain that:
\[
|(u, (\nabla_x \phi \ast \gamma_N) \nabla_v (h_N - h))| \leq C W(h_N, h),
\]
(B.25)
where, as in Section 5, \(W\) denotes the Wasserstein distance. But we know that the right hand side of (B.25) goes to zero because of the assumption i), then we have just proven that:
\[
|(u, (\nabla_x \phi \ast \gamma_N) \nabla_v (h_N - h))| \to 0, \quad \forall u \in C^\infty_b(\mathbb{R}^3 \times \mathbb{R}^3).
\]
(B.26)
Analogously, we can prove that
\[
|(u, (\nabla_x \phi \ast (h_N - h)) \nabla_v \gamma_N)| \to 0, \quad \forall u \in C^\infty_b(\mathbb{R}^3 \times \mathbb{R}^3).
\]
(B.27)
Therefore we have just proven that \(R_N\) goes to zero in the \(C^\infty_b\)-weak sense and the proof of Step 1) is done.

Proof of Step 2):
Thanks to Step 1) and to the assumption on \(\nabla_v h\), we know that \(\gamma_N(t)\) can be written as in (B.11). Then, for any function \(u\) in \(C^\infty_b(\mathbb{R}^3 \times \mathbb{R}^3)\), we have that:
\[
(u, \gamma_N(t)) = (u, \Sigma_h(t, 0) \gamma_{N,0}) + \int_0^t ds \ (u, \Sigma_h(t, s) \Theta_N'(s)),
\]
(B.28)
namely
\[
(u, \gamma_N(t)) = ((\Sigma_h(t, 0))^* u, \gamma_{N,0}) + \int_0^t ds \ ((\Sigma_h(t, s))^* u, \Theta_N'(s)),
\]
(B.29)
where \(\Sigma_h^*\) is the adjoint of \(\Sigma_h\). We remind that the two-parameters semigroup \(\Sigma_h(t, s)\) propagates the \(C^k\) regularity, provided that \(\nabla_v h \in C^k(\mathbb{R}^3 \times \mathbb{R}^3)\). In particular, if \(\Sigma_h\) acts on a function \(u\) which is in \(C^\infty_b(\mathbb{R}^3 \times \mathbb{R}^3)\) and the function \(h(t)\) is supposed to be in \(C^\infty_b(\mathbb{R}^3 \times \mathbb{R}^3)\) for any \(t\), as it is in the assumptions of Proposition 6.2, we are clearly guaranteed that \(\nabla_v h(t)\) is in \(C^\infty_b(\mathbb{R}^3 \times \mathbb{R}^3)\) for any \(t\), and then, \(u(t) := \Sigma_h(t, 0) u(x, v)\) is also in \(C^\infty_b(\mathbb{R}^3 \times \mathbb{R}^3)\) for any \(t\). Obviously, the same holds for \(\Sigma_h^*\). Thus, the functions \((\Sigma_h(t, 0))^* u\) and \((\Sigma_h(t, s))^* u\) appearing in (B.29) are in \(C^\infty_b(\mathbb{R}^3 \times \mathbb{R}^3)\) for any \(t\). Therefore, thanks to the assumptions we made on \(\gamma_{N,0}\)
and \( \Theta_N \), and of what we know about \( R_N \), we find that:

\[
((\Sigma_h(t,0))^* u, \gamma_{N,0}) + \int_0^t ds \ ((\Sigma_h(t,s))^* u, \Theta'_N(s)) \\
\downarrow \quad N \to \infty
\]

\[
((\Sigma_h(t,0))^* u, \gamma_0) + \int_0^t ds \ ((\Sigma_h(t,s))^* u, \Theta(s)) = \\
= (u, \Sigma_h(t,0)\gamma_0) + \int_0^t ds \ (u, \Sigma_h(t,s)\Theta(s)).
\]

(B.30)

Finally, by Proposition 6.1, we know that the expression (B.30) identifies properly the unique solution of the problem (6.10) in \( C_0^\infty(L^1(\mathbb{R}^3 \times \mathbb{R}^3), \mathbb{R}^+ \) and Proposition 6.2 is proven.

\[\square\]

Appendix C

Lemma C.1: For each time \( \tau > 0 \), let us define the operator \( \hat{T}_N^{(n)}(\tau) \) as follows:

\[ \hat{T}_N^{(n)}(\tau) := S_N(-\tau)\hat{r}_N^{(n)}S_N(\tau). \]

Then, for each \( m \geq 0 \) and for each \( u \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \), there exists a constant \( C > 0 \), not depending on \( N \), such that:

i) \[ \left\| \left(u, \hat{T}_N^{(r_m)}(t_m) \cdots \hat{T}_N^{(r_1)}(t_1)\mu_N(t) \right) \right\| < C. \] (C.1)

Moreover, we have:

ii) \[ \left\| \left(u, \hat{T}_N^{(r_m)}(t_m) \cdots \hat{T}_N^{(r_1)}(t_1)\mu_N(t) \right) \right\| \leq \left\| \left(u, \hat{T}_N^{(r_m)}(t_m) \cdots \hat{T}_N^{(r_1)}(t_1)\mu_N(t) \right) \right\| + O \left( \frac{1}{N} \right). \] (C.2)

Proof:

We observe that:

\[
\left(u, \hat{T}_N^{(r_m)}(t_m) \cdots \hat{T}_N^{(r_1)}(t_1)\mu_N(t) \right) = \hat{T}_N^{(r_m)}(t_m)\hat{T}_N^{(r_{m-1})}(t_{m-1}) \cdots \hat{T}_N^{(r_1)}(t_1)U(Z_N(t)),
\]

(C.3)

where:

\[
U(Z_N(t)) := (u, \mu_N(t)) = \frac{1}{N} \sum_{\ell=1}^N u(z_\ell(t)).
\] (C.4)
We assume $m > 0$ being the case $m = 0$ obvious.

By using the notations:

$$S(\mathbf{r}_m, \mathbf{t}_m) := T_N^{(r_m)}(t_m) \cdots T_N^{(r_1)}(t_1)$$  \hfill (C.5)

and

$$\dot{S}(\mathbf{r}_m, \mathbf{t}_m) := \dot{T}_N^{(r_m)}(t_m) \cdots \dot{T}_N^{(r_1)}(t_1),$$  \hfill (C.6)

we have (see the first term in the right hand side of (C.15)):

$$\dot{S}(\mathbf{r}_m, \mathbf{t}_m) U(Z_N(t)) = \frac{C}{N^m} \sum_{j_1 \cdots j_m} \sum_{l_1 \cdots l_m} D_x^{r_m+1} \phi(x_{j_m}(t_m) - x_{l_m}(t_m)) \cdot D_{v_{j_m}}^{r_m+1}(t_m)$$

$$D_x^{r_m-1+1} \phi(x_{j_m-1}(t_{m-1}) - x_{l_m-1}(t_{m-1})) \cdot D_{v_{j_m-1}}^{r_m-1+1}(t_{m-1})$$

$$\cdots$$

$$D_x^{r_1+1} \phi(x_{j_1}(t_1) - x_{l_1}(t_1)) \cdot D_{v_{j_1}}^{r_1+1}(t_1) U(Z_N(t)), \quad (C.7)$$

$C$ depending on $\mathbf{r}_m$. By setting:

$$\Phi_{j_n}(Z_N(t_n)) := \frac{1}{N} \sum_{l_n=1}^{N} D_x^{r_n+1} \phi(x_{j_n}(t_n) - x_{l_n}(t_n)) \quad (C.8)$$

$\forall ~ n = 1, 2, \ldots, m$

(C.7) can be rewritten as

$$\dot{S}(\mathbf{r}_m, \mathbf{t}_m) U(Z_N(t)) = C \sum_{j_1 \cdots j_m} \Phi_{j_m}(Z_N(t_m)) \cdot D_{v_{j_m}}^{r_m+1}(t_m)$$

$$\Phi_{j_{m-1}}(Z_N(t_{m-1})) \cdot D_{v_{j_{m-1}}}^{r_{m-1}+1}(t_{m-1})$$

$$\cdots$$

$$\Phi_{j_1}(Z_N(t_1)) \cdot D_{v_{j_1}}^{r_1+1}(t_1) U(Z_N(t)). \quad (C.9)$$

We observe that, thanks to the smoothness of the potential $\phi$, $\Phi_{j_n}$ (for each $n$) is a uniformly bounded function of the configuration $Z_N$, together with its derivatives.

Performing the derivatives in (C.9), we realize that $\dot{S}(\mathbf{r}_m, \mathbf{t}_m) U(Z_N(t))$ is a linear combination of terms of the following type:

$$\sum_{j_1 \cdots j_m} \Phi_{j_m}(Z_N(t_m)) \cdot D_{v_{j_m}}^{a_{m,1}}(t_m) \cdots D_{v_{j_2}}^{a_{2,1}}(t_2) D_{v_{j_1}}^{a_{1,1}}(t_1) U(Z_N(t))$$

$$D_{v_{j_m}}^{a_{m,2}}(t_m) \cdots D_{v_{j_2}}^{a_{2,2}}(t_2) \Phi_{j_1}(Z_N(t_1))$$

$$\cdots$$

$$D_{v_{j_m}}^{a_{m,m-1}}(t_m) D_{v_{j_{m-1}}}^{a_{m-1,m-1}}(t_{m-1}) \Phi_{j_{m-2}}(Z_N(t_{m-2}))$$

$$D_{v_{j_m}}^{a_{m,m}}(t_m) \Phi_{j_{m-1}}(Z_N(t_{m-1})), \quad (C.10)$$
with the constraint

\[
\begin{align*}
  a_{1,1} &= r_1 + 1 \\
  a_{2,1} + a_{2,2} &= r_2 + 1 \\
  \vdots \\
  a_{m,1} + a_{m,2} + \cdots + a_{m,m} &= r_m + 1.
\end{align*}
\] (C.11)

For a fixed sequence \(a_{\ell,s}\), we have to compensate the divergence arising from the sum \(\sum_{j_1 \ldots j_m}\), which is \(O(N^m)\), by the decay of the derivatives as given by Proposition 5.1 and Proposition 5.2. Indeed we have:

\[
\left| D^{a_{m,1}}(t_m) D^{a_{2,1}}(t_2) D^{a_{1,1}}(t_1) U(Z_N(t)) \right| \leq \frac{C}{N^d},
\] (C.12)

where \(d\) is the number of different indices in the sequence \(j_1, j_2, \ldots, j_m\) for which \(a_{m,1}, \ldots, a_{2,1}, a_{1,1}\) are strictly positive. Note that the fact that the derivatives are not computed at time \(t = 0\) but at different times \(t_1, t_2, \ldots, t_m\), does not change the estimate in an essential way.

An analogous estimate holds when we replace \(U\) by some \(\Phi_{j_s}\), namely

\[
\left| D^{a_{m,k}}(t_m) D^{a_{m-1,k}}(t_{m-1}) \cdots D^{a_{k,k}}(t_k) \Phi_{j_k}(Z_N(t_k)) \right| \leq \frac{C}{N^{d_{k-1}}},
\] (C.13)

where \(d_{k-1}\) is the number of different indices in the sequence \(j_k, \ldots, j_m\) which are also different from \(j_{k-1}\) and from which \(a_{m,k}, \ldots, a_{k,k}\) are strictly positive.

As regard to the term in the sum \(\sum_{j_1 \ldots j_m}\) in which all the indices are different (which is the only one of size \(O(N^m)\)), the constraints (C.11) together with estimates (C.12) and (C.13) ensure that the product of derivatives on the right hand side of (C.10) is bounded by \(1/N^m\). Thus this term is of order one. Now for each \(s = 1, \ldots, m - 1\) consider the \(\frac{m!}{s!(m-s)!}\) terms in the sum \(\sum_{j_1 \ldots j_m}\) in which \(s\) indices are equal. The sum is bounded by \(N^{m-s}\). On the other hand, the constraints (C.11) together with (C.12) and (C.13) ensure that the product of derivatives on the right hand side of (C.10) is bounded by \(1/N^{m-s}\). Thus even these terms are of size one and i) is proven.

To prove ii) we observe that:

\[
S(\mathbf{t}_m, \mathbf{\xi}_m) U(Z_N(t)) - \hat{S}(\mathbf{t}_m, \mathbf{\xi}_m) U(Z_N(t))
\] (C.14)

can be expanded as in (C.7) and (C.10). However now we have an extra derivative, arising from the definition of \(R_N^{(n)}\) (see (3.20)), which yields an additional \(1/N\). We omit the details of the proof which follows the same line of i).

\[\square\]

In the same way we can also prove the following

**Lemma C.2:** For each \(m \geq 0\), \(k > 0\) and \(u \in C_b^\infty(\mathbb{R}^3 \times \mathbb{R}^3)\), there exists a constant \(C > 0\), not depending on \(N\), such that:

\[
| D^{2k} S(\mathbf{t}_m, \mathbf{\xi}_m) U(Z_N(t)) | < C.
\] (C.15)
where $U(Z_N(t))$ is defined as in (C.4).

**Proof:**

First we look at the case $m > 0$. Reminding the structure of the operator $\mathcal{D}^{2k}$ (see (7.3)), we are led to consider the term $D_{G,j}^{2s}S(\mathbf{r}_m, \mathbf{t}_m)U(Z_N(t))$. We remind that $D_{G,j}^{2s}$ is a derivation operator with respect to the variable $z_j$ that acts as specified by (6.8). By the expansion (C.10) we readily arrive to the bound:

$$
|D_{G,j}^{2s}S(\mathbf{r}_m, \mathbf{t}_m)U(Z_N(t))| \leq \frac{C}{N}. \tag{C.16}
$$

Indeed by applying $D_{G,j}^{2s}$ to (C.10) either $j \notin (j_1, \ldots, j_m)$ so that we gain $1/N$ by the extra derivative, or $j \in (j_1, \ldots, j_m)$ so that we reduce the sum $\sum_{j_1, \ldots, j_m}$ by a factor $1/N$. More generally, by the same argument we find:

$$
\left| \prod_{j \in I} D_{G,j}^{2s}S(\mathbf{r}_m, \mathbf{t}_m)U(Z_N(t)) \right| \leq \frac{C}{N^n}, \tag{C.17}
$$

where $n = |I|$.

Finally by writing the action of the operator $\mathcal{D}^{2k}$ as in (7.35), we obtain

$$
\left| \mathcal{D}^{2k}S(\mathbf{r}_m, \mathbf{t}_m)U(Z_N(t)) \right| \leq \sum_{n=1}^{N} \frac{N!}{n!(N-n)!} \sum_{1 \leq s_j \leq k} \left( \sum_{\sum s_j=k} \right) \frac{C}{N^n} \leq B^k \sum_{n=1}^{N} \frac{N!}{n!(N-n)!} \left( \sum_{s_j} \right) \leq C,
$$

where $B, C$ being positive constants not depending on $N$. Again $\mathcal{D}^{2k}S(\mathbf{r}_m, \mathbf{t}_m)U(Z_N(t))$ is the leading term of $\mathcal{D}^{2k}S(\mathbf{r}_m, \mathbf{t}_m)U(Z_N(t))$ for the same reasons we discussed in Lemma C.1.

If $m = 0$, the estimates (C.16) and (C.17) follow directly by Proposition 5.2. Thus, even in this case, the proof is concluded by (C.18).

The fact that the error term $E_1^N$ (see (7.20)) and hence $E_2^N$ (see (7.27)) are $C^\infty$-weakly vanishing when $N \to \infty$ is an immediate consequence of the following

**Lemma C.3:** Let $\mathbf{r}_I$ and $\mathbf{t}_I$ be defined as in Section 7, for any $J \subset I_n$ with $I_n = \{1, 2, \ldots, n\}$. 

For any \( r \geq 0 \) we have:

\[
\mathcal{D}^{2r} S(\mathbf{x}_n, \mathbf{t}_n) \mu_N(z_1^T Z_N(t)) \mu_N(z_2^T Z_N(t)) = \\
\sum_{0 \leq t \leq r} \sum_{0 \leq m \leq n} \sum_{I \subseteq I_n \mid |I| = m} (\mathcal{D}^{2r} S(\mathbf{x}_I, \mathbf{t}_I) \mu_N(z_1^T Z_N(t))) (\mathcal{D}^{2(r-t)} S(\mathbf{x}_{I_n \setminus I}, \mathbf{t}_{I_n \setminus I}) \mu_N(z_2^T Z_N(t))) + \epsilon_{r,N}
\]

(C.19)

where

\[
\epsilon_{r,N} \to 0 \quad \text{as} \quad N \to \infty \quad \mathcal{O}_b^\infty \quad \text{weakly.}
\]

(C.20)

Proof:

It is enough to prove (C.19) and (C.20) replacing each streak \( S \) with the corresponding \( \hat{S} \), being the difference \( S - \hat{S} \) negligible in the limit.

We start by assuming \( r = 0 \). In that case, testing the left hand side of (C.19) against a product of two test functions \( u_1, u_2 \), we are led to consider:

\[
\hat{S}(\mathbf{x}_n, \mathbf{t}_n) U_1(Z_N(t)) U_2(Z_N(t))
\]

(C.21)

for which we can apply the expansion (C.7).

Proceeding as in the proof of Lemma C.1 (see (C.10)), we have to consider:

\[
D_{v_{j_k}}^{a_{k,1}}(t_m) \ldots D_{v_{j_2}}^{a_{2,1}}(t_2) D_{v_{j_1}}^{a_{1,1}}(t_1) U_1(Z_N(t)) U_2(Z_N(t)),
\]

(C.22)

where \( a_{1,1} = r_1 + 1 > 0 \). Now any contribution of the form

\[
D_{v_{j_1}}^{\alpha}(t_1) U_1(Z_N(t)) D_{v_{j_1}}^{\beta}(t_1) U_2(Z_N(t)),
\]

(C.23)

with \( \alpha > 0, \beta > 0, \alpha + \beta = a_{1,1} \) is \( O \left( \frac{1}{N^2} \right) \), therefore it is negligible in the limit. The same argument applies to \( D_{v_{j_k}}^{a_{k,1}}(t_k) \) whenever \( a_{k,1} > 0 \). This means that each derivative appearing in \( \hat{S} \) either applies to \( \mu_N(z_1^T Z_N(t)) \) or to \( \mu_N(z_2^T Z_N(t)) \) up to an error \( \epsilon_{0,N} \) vanishing in the limit. This is exactly what (C.19) and (C.20) say for \( r = 0 \).

For \( r > 0 \) we have to apply \( \mathcal{D}^{2r} \) to (C.19) (replacing \( S \) by \( \hat{S} \)) with \( r = 0 \). Clearly \( \mathcal{D}^{2r} \epsilon_{0,N} \) vanishes in the limit. Moreover:

\[
\mathcal{D}_{G,j}^{2s_j} \left[ \hat{S}(\mathbf{x}_I, \mathbf{t}_I) U_1(Z_N(t)) \hat{S}(\mathbf{x}_{I_n \setminus I}, \mathbf{t}_{I_n \setminus I}) U_2(Z_N(t)) \right] = \\
\left( \mathcal{D}_{G,j}^{2s_j} \hat{S}(\mathbf{x}_I, \mathbf{t}_I) U_1(Z_N(t)) \right) \hat{S}(\mathbf{x}_{I_n \setminus I}, \mathbf{t}_{I_n \setminus I}) U_2(Z_N(t)) + \\
+ \hat{S}(\mathbf{x}_I, \mathbf{t}_I) U_1(Z_N(t)) \left( \mathcal{D}_{G,j}^{2s_j} \hat{S}(\mathbf{x}_{I_n \setminus I}, \mathbf{t}_{I_n \setminus I}) U_2(Z_N(t)) \right) + O \left( \frac{1}{N^2} \right)
\]

(C.24)

By simple algebraic manipulation we finally arrive to (C.19) and (C.20).
References

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