GROUP LOCALIZATION AND TWO PROBLEMS OF LEVINE

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Abstract. A. K. Bousfield’s $HZ$-localization of groups inverts homologically two-connected homomorphisms of groups. J. P. Levine’s algebraic closure of groups inverts homomorphisms between finitely generated and finitely presented groups which are homologically two-connected and for which the image normally generates. We resolve an old problem concerning Bousfield $HZ$-localization of groups, and answer two questions of Levine regarding algebraic closure of groups. In particular, we show that the kernel of the natural homomorphism from a group $G$ to its Bousfield $HZ$-localization is not always a $G$-perfect subgroup. In the case of algebraic closure of groups, we prove the analogous result that this kernel is not always an invisible subgroup.

1. Introduction

In [Bou74, Bou75, Bou77], A. K. Bousfield defines his homology localization of spaces, and initiates an investigation of a corresponding localization of groups and modules. Roughly, the $HZ$-localization of spaces is the initial functor inverting homology equivalences of spaces. If $E(G)$ is the group localization of a group $G$, and $E(X)$ the homology localization of a space $X$, then $E(\pi_1(X)) \cong \pi_1(E(X))$. Homology localization of groups inverts homomorphisms that are isomorphisms on abelianization and epimorphisms on second homology.

In analogy with Bousfield’s $HZ$-localization of spaces, P. Vogel, in unpublished work, defines a localization of spaces which inverts maps between finite complexes $X \to Y$ with a contractible cofiber. The primary significance of Vogel’s work arises from its connection to homology cobordism classes of manifolds and embedding theory. (See, for example, [Cha08, CO10, Hec12, LD88, Lev88, Lev89a, Lev89b, Lev94, Sak06].)

In [Lev89a], J. P. Levine defines his algebraic closure of groups. (An earlier version of this type of closure was defined by M. Gutierrez [Gut79].) Similarly to the connection between Bousfield’s $HZ$-localization of groups and spaces, the algebraic closure of a group $G$ describes how the Vogel localization of a space effects the fundamental group of the space. We denote both Bousfield’s $HZ$-closure of the group $G$ and Levine’s algebraic closure of $G$ by $E(G)$.

The natural homomorphism $G \to E(G)$ rarely embeds $G$. In particular, for $HZ$-localization of groups, Bousfield defines a family of subgroups which always lie in the kernel of the natural homomorphism $G \to E(G)$. A normal subgroup $N \trianglelefteq G$ is called a $G$-perfect subgroup if $N = [G, N]$. In this study of algebraically closed groups, Levine’s adds the condition that the $G$-perfect subgroup is normally generated by a finite subset of group elements, and calls these groups invisible groups.

We replace the terms $G$-perfect and invisible with the term relatively perfect subgroup. One can easily tell from context if relatively perfect means Bousfield’s $G$-perfect subgroup, or Levine’s invisible subgroup.

Bousfield, (respectively Levine) prove that $HZ$-local groups (respectively, algebraically closed groups) do not contain non-trivial relatively perfect subgroups.

Question 1.1. What is the $\ker\{G \to E(G)\}$? More precisely, we ask this question. For Bousfield’s $HZ$-localization of a group $G$, is this kernel a relatively perfect subgroup? For Levine’s group closure, is this kernel the union of the relatively perfect subgroups?

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Progress has been negligible, and constructions that might produce interesting examples rare. Bousfield does not appear to have explicitly asked the question in any paper. However, the problem has long circulated the field. J. Rodriguez and D. Scenvenls gave equivalent formulations for this problem, without resolution \cite{RS04}.

In \cite{Lev90}, Question 4, Levine explicitly asks the above question. He also asks, in Question 5 of that paper, whether the kernel of a homologically 2-connected homomorphism $G \to \Gamma$ is a union of relatively perfect subgroups when $G$ is finitely generated.

The main Theorem of this paper gives a negative answer Question\cite{Lev90} including Levine’s Question 4 and 5 from \cite{Lev90}. To answer this question we apply a new construction arising from our extension of a result by M. Bridson and A. Reid \cite{BR12}, which we call the Fiber-Product Theorem. We state our main theorem below. The Fiber-Product Theorem is stated and proven in Section \ref{sec:FP}.

**Theorem 6.1.** Let $Q$ be any finitely presented perfect group with a balanced presentation, such that $H_3(Q) \neq 0$. Suppose $Q \cong F/R$ where $F$ is free on the generators in the balanced presentation of $Q$. Consider the fiber product, $F \times_Q F \cong E/K$, where $E$ is free on the generating set of $Q$. Let $G := E/\langle [E, K] \rangle$ be the associated free central extension of $E/K$. Then

1. $G$ is a finitely generated group,
2. $\gamma_\omega(G) \neq \gamma_{\omega+1}(G) = \{1\},$
3. and if $E(G)$ represents either the Bousfield $HZ$-localization of $G$, or if $E(G)$ is Levine’s algebraic closure of $G$ and in addition in this case, $Q$ is a finite group, then

$$\gamma_\omega(G) = \ker\{G \to E(G)\}.$$ 

In particular, in both cases, the $\ker\{G \to E(G)\}$ is not a relatively perfect subgroup of $G$.

We provide examples of finite, and finitely presented, perfect groups, as used in Theorem\ref{thm:6.1} within the paper. We note, for now, the particularly interesting example, the Binary Icosahedral group, sometimes called the Poincaré group, that is, the fundamental group of the Poincaré homology sphere.

2. $HZ$-localization and algebraic closure of groups

**2.1. HZ-localization of groups.** We discuss, briefly, the $HZ$-localization and Levine closure of groups, offering a little more detail than in the introduction. We recommend \cite{Bou74, Bou75} and \cite{Lev89a} for a careful treatment of these topics.

**Definition 2.1.** A homomorphism of groups $G \to \Gamma$ is called homologically 2-connected if the induced homomorphism on homology, $H_k(G) \to H_k(\Gamma)$, is an isomorphism for $k = 1$ and onto for $k = 2$.

**Definition 2.2.** A group $\Gamma$ is $HZ$-local if given any diagram of homomorphisms as follows, with $G \to H$ homologically 2-connected,

$$G \xrightarrow{G} H \xrightarrow{\Gamma} \Gamma$$

there is a unique homomorphism $H \to \Gamma$ making the diagram commute.

**Definition 2.3.** The $HZ$-localization of a group $G$ is a functor $E$ which takes groups to $HZ$-local groups, and for each group $G$ a natural, homologically 2-connected, homomorphism $G \to E(G)$ such that given any homomorphism $G \to \Gamma$ with $\Gamma$ an $HZ$-local group, there is a unique homomorphism $E(G) \to \Gamma$ making the following diagram commute.

$$G \xrightarrow{G} E(G) \xrightarrow{\Gamma} \Gamma$$
That is, the $H\mathbb{Z}$-localization is the initial local group determined by the group $G$.

Bousfield constructs and investigates the unique (up to natural equivalence of functors) $H\mathbb{Z}$-localization of groups. In particular, he shows that if $X$ is a space with $\pi_1(X) \cong G$, then $\pi_1(E(X)) \cong E(G)$, where $E(X)$ is the $H\mathbb{Z}$-localization of the space $X$.

2.2. Levine's algebraic closure of groups.

Definition 2.4. A group $\Gamma$ is algebraically closed if given any homologically 2-connected homomorphism $G \to H$ where $G$ is finitely generated and $H$ is finitely presented, and such that the image of $G \to H$ normally generates $H$, and a homomorphism $G \to \Gamma$, there is a unique extension $H \to \Gamma$ making the diagram in Definition 2.2 commute.

Levine defines closed groups using systems of equations over the group, but observes it satisfies the above properties.

As in Definition 2.4 Levine’s algebraic closure of groups is a functor, $E$, from groups to algebraically closed groups together with a natural homomorphism for each group $G$, $G \to E(G)$, which is initial among homomorphisms to algebraically closed groups.

We mention briefly that the $HR$-localization of a group has also been described as an algebraic closure of groups [DFOS89], Solving equations over groups plays no role in the theorems proven herein.

2.3. Relatively perfect subgroups.

Definition 2.5. A subgroup $N \leq G$ is relatively perfect if $N = [G, N]$.

Levine calls a group $N \leq G$ invisible if $N$ is relatively perfect and, additionally, is normally generated in $G$ by a finite set of elements. We avoid the term invisible and when working in the context of Levine’s group closure we call these groups relatively perfect as well. Whether relatively perfect includes the normal generation condition is clear from context, depending on whether we are discussing Bousfield $H\mathbb{Z}$-localization, or Levine’s group closure. That is, the notion we call relatively perfect, in the context of algebraic closure of groups will additionally require that $N$ is normally generated by a finite collection of elements.

Bousfield, and separately Levine, prove this theorem [Bon77 Proposition 1.2][Lev89a Lemma 3].

Theorem 2.6 (Bousfield, Levine). If $G$ is an $H\mathbb{Z}$-local (resp. algebraically closed group) and $N \leq G$ is a relatively perfect subgroup, then $N \cong \{1\}$.

In particular, for any group $G$, all relatively perfect subgroups of $G$ lie in $\ker\{G \to E(G)\}$.

3. Two lemmas: Group homology and the lower central series

This section contains the statement and proof of two results (of Emmanuel-Mikhailov and Mikhailov-Passi, respectively) fundamental to proving the main results of this paper. We include these short proofs to assure completeness and readability.

We first establish notation for elementary spectral sequence arguments.

For a group extension

$$1 \to R \to G \to Q \to 1$$

and any $\mathbb{Z}Q$-module $M$, the Hochschild-Serre spectral sequence gives a filtration

$$\{0\} \subset D_{0,n} \subset D_{1,n-1} \subset \cdots \subset D_{n,0} = H_n(G; M)$$

with associated graded

$$E^\infty_{p,q} = D_{p,n-p}/D_{p-1,n-p+1}$$

and

$$E^2_{p,q} = H_p(Q, H_q(R; M)) \Rightarrow H_{p+q}(G; M).$$

Definition 3.1. A group $G$ is super-perfect if $H_1(G) \cong H_2(G) \cong 0$.

Example 3.2. The Binary Icosahedral group

$$Q := \langle a, b \mid a^5 = b^3, b^3 = (ab)^2 \rangle$$

is an order 120 super-perfect group that arises as the fundamental group of the Poincaré sphere, a compact, orientable 3 manifold with perfect fundamental group.
Definition 3.5. A group is residually nilpotent if

\[ \text{for a discrete ordinal } \theta \]

Definition 3.4. Here the second isomorphism follows since

\[ R \]

Recall as well the Dwyer filtration on the second homology of a group [Dwy75].

Lemma 3.3. Suppose we have a group presentation

\[ 1 \to R \to F \to Q \to 1 \]

Lemma 3.2. Here the second isomorphism follows since.

\[ \text{Then there is an exact sequence} \]

\[ H_4(Q) \to H_0(Q; R_{ab} \otimes R_{ab}) \to H_2(G) \to H_3(Q) \to 0. \]

Proof. First note that \( H_2(G) \cong H_1(F; R_{ab}) \). Here \( R_{ab} \) is the abelianization of the group of relations \( R \), which can be viewed as a \( \mathbb{Z}Q \)-module, where \( Q \) acts via conjugation in \( F \). One easily sees the above isomorphism using the spectral sequence for the extension \( 1 \to R \to G \to F \to 1 \). More precisely since \( F \) and \( R \) are free groups,

\[ H_2(G) \cong H_2(R \rtimes F) = D_{2,0} = E_{1,1}^2 = H_1(F; H_1(R)) = H_1(F; R_{ab}). \]

Similarly, using the spectral sequence of the extension \( F \to Q \), and that \( R \) is free, we observe that for all \( p \geq 1 \),

\[ H_{p+2}(Q) \cong H_p(Q; H_1(R)). \]

Now consider the homology spectral sequence with \( R_{ab} \) coefficients associated to

\[ 1 \to R \to F \to Q \to 1. \]

Then

\[ E_{p,0}^2 = H_p(Q; H_0(R; R_{ab})) \cong H_p(Q; R_{ab}) \cong H_{p+2}(Q), \quad p \geq 1. \]

Here the second isomorphism follows since \( R \) acts on \( R \) by conjugation, and so acts trivially on \( R_{ab} \). The last isomorphism is the observation made in the previous paragraph.

\[ E_{p,q}^2 = H_p(Q; H_q(R; R_{ab})) \cong 0, \quad q \geq 2, \]

since \( R \) is free. Note as well that \( E_{1,0}^\infty = E_{1,0}^2 \cong H_3(Q) \).

For \( q = 1 \),

\[ E_{p,1}^2 = H_p(Q; H_1(R; R_{ab})) \cong H_p(Q; R_{ab} \otimes R_{ab}). \]

Hence we have the usual exact sequence of low order terms

\[ E_{2,0}^2 \xrightarrow{d_2^0} E_{0,1}^2 \to D_{1,0} \to E_{1,0}^\infty \to 0 \]

and using the above exact sequence, this completes the proof.

Recall the transfinite lower central series of a group \( G \).

Definition 3.4. For a discrete ordinal \( k \), the \( k \)-th lower central series subgroup of \( G \) is defined by \( \gamma_1(G) = G \), and \( \gamma_{k+1}(G) = [G, \gamma_k(G)] \), where for normal subgroups \( H, K \triangleleft G \), the symbol \( [H, K] \) denotes the subgroup generated by commutators \( [h, k] := h^{-1}k^{-1}hk \in G \). For \( \lambda \) a limit ordinal,

\[ \gamma_\lambda(G) = \bigcap_{\delta < \lambda} \gamma_\delta(G). \]

Definition 3.5. A group is residually nilpotent if \( \gamma_\omega(G) = \cap_k \gamma_k(G) = 1 \).

Recall as well the Dwyer filtration on the second homology of a group \( G \) [Dwy75].
Definition 3.6. For $K$ finite, the $k^{th}$ term in the Dwyer filtration of the second homology of a group $G$ is defined as
\[
\phi_k(G) := \ker \{ H_2(G) \to H_2(G/\gamma_{k-1}(G)) \}, \quad k \geq 2
\]
and the first transfinite term by
\[
\phi_\omega(G) = \cap_k \phi_k(G).
\]

The next Lemma appears in [MP06].

Lemma 3.7. Let $G \cong F/R$ where $F$ is a free group. If $G$ is residually nilpotent, then
\[
\gamma_\omega \left( \frac{F}{[F,R]} \right) \cong \phi_\omega(G)
\]

Proof. We first compute the $k^{th}$ term in the Dwyer filtration of $H_2(G), k$ finite. Note that
\[
H_2(G) \cong \frac{R \cap [F,F]}{[F,R]}, \quad \text{and} \quad (1)
\]
and
\[
H_2 \left( \frac{G}{\gamma_k(G)} \right) \cong H_2 \left( \frac{F}{\gamma_k(F) \cdot R} \right) \cong \frac{\gamma_k(F) \cdot R \cap [F,F]}{[F,\gamma_k(F) \cdot R]} = \frac{\gamma_k(F) \cdot R \cap [F,F]}{\gamma_{k+1}(F) \cdot [F,R]}, \quad (2)
\]
where the last isomorphism follows since $\gamma_k(F)$ and $R$ are normal subgroups of $F$. Combining the equations (1) and (2) above,
\[
\phi_{k+1}(G) \cong \ker \left\{ \frac{R \cap [F,F]}{[F,R]} \to \frac{\gamma_k(F) \cdot R \cap [F,F]}{\gamma_{k+1}(F) \cdot [F,R]} \right\} = \frac{R \cap \gamma_{k+1}(F) \cdot [F,R]}{[F,R]} \subset \frac{\gamma_{k+1}(F) \cdot [F,R]}{[F,R]} = \gamma_{k+1} \left( \frac{F}{[F,R]} \right).
\]
Therefore
\[
\phi_\omega(G) = \bigcap_k \phi_k(G) \subset \gamma_\omega(G).
\]

To prove the other inclusion, we use the hypothesis that $G$ is residually nilpotent.

There is a quotient homomorphism $\frac{R}{[F,R]} \to \frac{F}{[F,R]}$. Given $g \cdot [F,R] \in \gamma_\omega \left( \frac{F}{[F,R]} \right)$ it follows that $g \cdot R \in \gamma_\omega \left( \frac{R}{[F,R]} \right)$, and since $F/R$ is residually nilpotent, $g \in R$. Thus we have the following two relations.
\[
\gamma_\omega \left( \frac{F}{[F,R]} \right) \subset \frac{R}{[F,R]} \quad \text{and} \quad \gamma_\omega \left( \frac{F}{[F,R]} \right) = \bigcap_k \frac{\gamma_k(F) \cdot [F,R]}{[F,R]}
\]
So for any $k$,
\[
\gamma_\omega \left( \frac{F}{[F,R]} \right) \subset \frac{R}{[F,R]} \cap \frac{\gamma_{k+1}(F) \cdot [F,R]}{[F,R]} = \frac{R \cap \gamma_{k+1}(F) \cdot [F,R]}{[F,R]} = \phi_{k+1}(G),
\]
where the middle equality follows since $[F,R]$ is normal in $R$ and in $\gamma_{k+1}(F) \cdot [F,R]$.

Thus, $\gamma_\omega \left( \frac{F}{[F,R]} \right) \subset \phi_\omega(G)$, proving the Lemma. \qed

4. The Fiber-Product Theorem

We now prove a theorem which we view as an extension of Propositions 3.1 and 3.2 of Bridson and Reid [BR12]. Most notably, we make no assumptions about finite index subgroups.

We offer a further extension in a future paper with G. Baumslag, whose proof uses simplicial methods. In spite of the elementary proof given below, this result underpins the remainder of the paper.

Fiber-Product Theorem 4.1. Suppose $Q$ is a super-perfect group with the following presentation, where $F$ is a free group.
\[
1 \to R \to F \to Q \to 1
\]
Consider the following pull-back diagram

\[
\begin{array}{ccc}
P & \xrightarrow{p} & F \\
\downarrow & & \downarrow \\
F & \xrightarrow{p} & Q
\end{array}
\]

and denote \( P := F \times_Q F \), that is, \( P = \{(x, y) \mid p(x) = p(y)\} \) is the fiber product over \( p: F \to Q \).

Then,

1. The inclusion homomorphism \( F \times_Q F \to F \times F \) is homologically 2-connected.
2. If \( \ker \{ H_2(F \times_Q F) \to H_2(F \times F) \} \cong 0 \) then \( H_3(Q) \cong 0 \).

Proof. The homomorphism \( F \times_Q F \to F \) splits since \( F \) is free, and has kernel \( R = \ker \{ F \to Q \} \) since \( P = F \times_Q F \) is a pullback with fiber \( R \). Thus, \( F \times_Q F \cong R \times F \). Hence,

\[
H_1(F \times_Q F) \cong H_1(R \times F) \cong R/[F, R] \oplus F_{ab},
\]

where \( F_{ab} \) is the abelianization of the free group \( F \). We have a commutative diagram where the bottom horizontal homomorphism is a sum of a homomorphism induced by the inclusion \( \text{inc}: R \to F \) and the identity.

\[
\begin{array}{ccc}
H_1(F \times_Q F) & \xrightarrow{\cong} & H_1(F \times F) \\
\downarrow & & \downarrow \\
R/[F, R] \oplus F_{ab} & \xrightarrow{(\text{inc}) \oplus \text{id}} & F_{ab} \oplus F_{ab}
\end{array}
\]

The extension \( R \to F \to Q \) determines the usual 5-term exact sequence on homology groups,

\[
0 \to H_2(Q) \to R/[F, R] \xrightarrow{\text{inc}} F_{ab} \to H_1(Q) \to 0.
\]

Combining this with the above commutative square of homomorphisms, we get the following elegant exact sequence:

\[
0 \to H_2(Q) \to H_1(F \times_Q F) \xrightarrow{\text{inc}} H_1(F \times F) \to H_1(Q) \to 0. \tag{3}
\]

Thus, \( H_1(F \times_Q F) \to H_1(F \times F) \) is an isomorphism if and only if \( Q \) is super-perfect, that is, \( H_1(Q) \cong H_2(Q) \cong 0 \).

To compute the image of the second homology, the homomorphism of presentations

\[
\begin{array}{c}
1 \to R \to F \to Q \to 1 \\
\downarrow & & \downarrow & & \downarrow \\
1 \to F \to F \to 1
\end{array}
\]

yields a homomorphism of exact sequences, as given in Lemma 3.3 as follows.

\[
\begin{array}{cccc}
H_4(Q) & \xrightarrow{H_0(Q; R_{ab} \otimes R_{ab})} & H_2(F \times_Q F) & \xrightarrow{H_3(Q)} \\
\downarrow & & \downarrow & & \downarrow \\
H_0(1; F_{ab} \otimes F_{ab}) & \xrightarrow{=} & H_2(F \times F)
\end{array}
\]

Since \( R_{ab} \to F_{ab} \) is onto (that is, since \( Q \) is perfect and using exact sequence 3 given above,) the left vertical homomorphism is onto as shown below.

\[
H_0(Q; R_{ab} \otimes R_{ab}) \cong (R_{ab} \otimes R_{ab}) \otimes \mathbb{Z}_Q \mathbb{Z} \to F_{ab} \otimes F_{ab} \cong H_0(1; F_{ab} \otimes F_{ab})
\]

Thus, \( H_3(F \times_Q F) \to H_2(F \times F) \) is onto as claimed.

To prove statement (2), if \( H_2(F \times_Q F) \to H_2(F \times F) \) is 1-1, then it is an isomorphism, and hence, \( H_0(Q, R_{ab} \otimes R_{ab}) \to H_2(F \times_Q F) \) is onto, implying that \( H_3(Q) \cong 0 \) as claimed. \( \square \)
Remark 4.2. Fibre Product Theorem \[\text{[4,1]}\] can be generalized to the case of fibre products with many summands. For \( n \geq 2 \) and an epimorphism \( p : F \to Q \), consider the higher fibre product \( F_n(p) := F \times_Q \cdots \times_Q F = \{ (x_1, \ldots, x_n) \in F^n \mid p(x_1) = \cdots = p(x_n) \} \).

Then the natural inclusion \( F_n(p) \hookrightarrow F^n \) is homologically 2-connected provided \( Q \) is super-perfect.

The proof follows by induction on \( n \), using the natural split exact sequence, \( (n \geq 3) \):

\[
1 \to R \to F_n(p) \to F_{n-1}(p) \to 1.
\]

5. Applications to \( H\mathbb{Z}\)-localization and closure of groups

We begin by describing a new construction using the Fiber-Product Theorem \[\text{[4,1]}\].

Let \( Q \) be a finitely presented perfect group with a balanced presentation, that is \( Q \) has a presentation with an equal number of generators and relations. As observed following Definition \[\text{[8,4]}\] this implies \( Q \) is super-perfect.

Let \( F \) be the free group on the generating set for \( Q \), with \( F \twoheadrightarrow Q \) the associated epimorphism. Let \( R = \ker\{ F \twoheadrightarrow Q \} \). As noted previously, \( F \times_Q F \cong R \times F \) where \( F \) acts on \( R \) by conjugation.

Note that if \( Q \) has \( k \) generators, and therefore \( k \) relations, then \( R \) is normally generated by \( k \) elements, and the group \( F \times_Q F \cong R \times F \) is generated by \( 2k \)-elements. Since \( Q \) is super-perfect, \( H_1(F \times F) \cong H_1(F \times F) \cong \mathbb{Z}^{2k} \) by statement (2) of the Fiber Product Theorem \[\text{[4,1]}\] and so \( F \times_Q F \) is generated by at least \( 2k \) generators, and thus by exactly \( 2k \) generators.

Let \( E \) and \( F \) be free on \( 2k \) letters such that \( F \times_Q F \cong E/K \), and \( F \times F \cong F/L \). Since \( E \) is free, one easily constructs a homomorphism \( E \to F \) inducing the following commutative diagram of groups.

\[
\begin{array}{c}
E \cong (F/K) \quad \xrightarrow{\phi_{(Q,P)}} \quad (F/L) \quad \xrightarrow{\gamma} \quad F \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
(F/Q) \quad \cong \quad F \times_Q F \\
\end{array}
\]

Notation. Let \( Q \) be a perfect group with balanced presentation \( P \). We denote the homomorphism above,

\[
\phi_{(Q,P)} : \frac{E}{[E,K]} \to \frac{F}{[F,L]}.
\]

Proposition 5.1. Let \( Q \) be a perfect group with a balanced presentation \( P \), such that \( H_3(Q) \neq 0 \). Then the following holds:

1. \( \frac{E}{[E,K]} \) is a finitely generated group.
2. \( \phi_{(Q,P)} : \frac{E}{[E,K]} \to \frac{F}{[F,L]} \) is homologically 2-connected.
3. \( \gamma_\omega \left( \frac{E}{[E,K]} \right) \neq \{1\} \) and \( \gamma_{\omega+1} \left( \frac{E}{[E,K]} \right) = \{1\} \)
4. \( \frac{F}{[F,K]} \) is residually nilpotent.
5. If \( Q \) is finite, then \( \frac{E}{[E,K]} \) is a finitely presented group.

Proof. To see (1), \( E \) is free on twice the number of generators of \( Q \) and thus \( E/[E,L] \) is finitely generated.

To prove statement (2), first note that since \( Q \) is super-perfect, the homomorphism \( H_1(F \times Q) \to H_1(F \times F) \) is an isomorphism by the Fiber Product Theorem \[\text{[4,1]}\] and these first homology groups have rank \( 2k \) since \( F \times F \) has rank \( 2k \). Since \( E \) and \( F \) are free of rank \( 2k \) as well, it follows that \( H_1(E/[E,K]) \to H_1(F/[F,L]) \) is an isomorphism of rank \( 2k \) groups.
To prove $\Phi_{(Q,P)}$ induces an epimorphism on second homology, we compare the spectral sequences associated to the extensions
\[
\begin{array}{ccc}
K & \overset{\alpha}{\rightarrow} & \mathcal{E}/[\mathcal{E},K] \\
[\mathcal{E},K] & \overset{\beta}{\rightarrow} & \mathcal{E}/K \\
[\mathcal{E},K] & \overset{\phi_{(Q,P)}}{\rightarrow} & \mathcal{E} \\
L & \overset{\gamma}{\rightarrow} & \mathcal{F}/[\mathcal{F},L] \\
[\mathcal{F},L] & \overset{\delta}{\rightarrow} & \mathcal{F}/L \\
[\mathcal{F},L] & \overset{\phi_{(Q,P)}}{\rightarrow} & \mathcal{F} \\
\end{array}
\] (4)

For each of these spectral sequences the differential $d_{2,0} : E^2_{2,0} \rightarrow E^2_{0,1}$ are onto since we have shown that the quotient homomorphisms in the extensions above are isomorphisms on abelianization. So we have an exact sequence
\[
E^\infty_{0,2} \rightarrow H_2 \rightarrow E^\infty_{1,1} \rightarrow \{0\}.
\]

So we have a homomorphism of short exact sequences, where $\Lambda^2(A)$ is the group of second exterior powers of $A$, that is $E^2_{2,0}$, modulo the image of differentials, and the right-most terms are quotients of $E^2_{1,1}$ by the image of $d^2_{3,0}$.

By statement (1) of the Fiber Product Theorem and since $H_2(\mathcal{E}/K) \cong K/\mathcal{E},K$ and $H_2(\mathcal{F}/L) \cong L/[\mathcal{F},L]$, it follows that the homomorphism $K/[\mathcal{E},K] \rightarrow L/[\mathcal{F},L]$ is onto. Hence $\alpha$ is onto in the above diagram.

Since the displayed extensions above, numbered \([\mathbf{4}]\), are central extensions, $H_1(\mathcal{E}/K;K/[\mathcal{E},K]) \cong H_1(\mathcal{E}/K) \otimes K/[\mathcal{E},K]$, and $H_1(\mathcal{F}/L;L/[\mathcal{F},L]) \cong H_1(\mathcal{F}/L) \otimes L/[\mathcal{F},L]$. Since $H_1(\mathcal{E}/K) \cong H_1(\mathcal{F}/L)$, the homomorphism $\beta$ from the above diagram is onto as well.

Thus,
\[
\Phi_{(Q,P)} : H_2(\mathcal{E}/[\mathcal{E},K]) \rightarrow H_2(\mathcal{F}/[\mathcal{F},L])
\]
is onto as claimed, proving statement (2) of Proposition 5.1.

To prove statement (3), first observe that $F \times Q F \subset F \times F$. The latter is residually nilpotent since the free group $F$ is residually nilpotent. Hence, $\mathcal{E}/K \cong F \times Q F$ is residually nilpotent. By Lemma 3.7, $\gamma_{\omega}(\mathcal{E}/K) \cong \phi_{\omega}(\mathcal{E}/K)$.

We now show that $\phi_{\omega}(\mathcal{E}/K) \not\cong 0$.

Since $F \times Q F \rightarrow F \times F$ is homologically 2-connected by part (1) of the Fiber Product Theorem, it follows from Stallings’ Theorem that $F \times Q F \rightarrow F \times F$ induces an isomorphism of lower central series quotients \([\text{Stal65}]\). Thus, for all $k \geq 1$,
\[
\ker\left(\frac{H_2(F \times Q F)}{\gamma_k(F \times F)} \right) = \phi_{k+1}(\mathcal{E}/K).
\]
Since $H_3(Q) \not\cong 0$, statement (2) of the Fiber Product Theorem asserts that
\[
\ker\left(\frac{H_2(F \times Q F)}{\gamma_k(F \times F)} \right) \not\cong 0.
\]

Hence, for all $k$
\[
\ker\left(\frac{H_2(F \times Q F)}{\gamma_k(F \times F)} \right) < \phi_{k}(\mathcal{E}/K).
\]
Hence $\phi_{\omega}(F \times Q F) \not\cong 0$.

However, $\mathcal{E}/[\mathcal{E},K] \rightarrow \mathcal{E}/K$ is a central extension by construction, and since $F \times Q F \subset F \times F$ is residually nilpotent it follows that $\gamma_{\omega+1}(\mathcal{E}/[\mathcal{E},K]) = \{1\}$. This proves part (3) of Proposition 5.1.

Statement (4) of Proposition 5.1 follows from Lemma 3.7 as well, since $F \times F$ is residually nilpotent and one easily computes the second equality below.
\[
\gamma_{\omega}(\mathcal{F}/[\mathcal{F},L]) = \phi_{\omega}(F \times F) = 0
\]
Finally, assume $Q$ is finite. By the 1-2-3 Lemma of G. Baumslag, M. Bridson, C. Miller III and H. Short [BBMIS00], it follows that $E/K \cong F \times_Q F$ is finitely presented. So $H_3(F \times_Q F) \cong K/\langle E, K \rangle$ is finitely presented, and therefore the group $E/\langle E, K \rangle$ in the following extension is finitely presented as well.

$$1 \rightarrow K \rightarrow E \rightarrow \frac{E}{\langle E, K \rangle} \rightarrow \frac{E}{K} \rightarrow 1$$

\[\square\]

6. LOCALIZATION, CLOSURE, AND RELATIVELY PERFECT SUBGROUPS

We prove our main theorem.

**Theorem 6.1.** Let $Q$ be any finitely presented perfect group with a balanced presentation, such that $H_3(Q) \neq 0$. (For instance, let $Q$ be the binary icosahedral group.) Let $F \times_Q F \cong E/K$ where $E$ is free on the generating set of $Q$. Let $G := E/\langle E, K \rangle$ be the associated free central extension of $E/K$. Then

1. $G$ is a finitely generated group,
2. $\gamma_\omega(G) \neq \gamma_{\omega+1}(G) = \{1\}$,
3. and if $E(G)$ represents either the Bousfield $H\mathbb{Z}$-localization of $G$, or if $E(G)$ is Levine’s algebraic closure of $G$, and in addition, $Q$ is a finite group, then $\gamma_\omega(G) = \ker\{G \rightarrow E(G)\}$.

In particular, in both cases, the $\ker\{G \rightarrow E(G)\}$ is not a relatively perfect subgroup of $G$.

**Proof.** All of the steps in this theorem follow from Proposition 5.1.

Observe that $E/\langle E, K \rangle$ is finitely generated by Proposition 5.1 statement (1). Also, by statement (2) of Proposition 5.1, the homomorphism

$$\frac{E}{\langle E, K \rangle} \rightarrow \frac{F}{[F, L]}$$

is homologically 2-connected. $\gamma_\omega(G) \neq \gamma_{\omega+1}(G) = \{1\}$ by statement (3) of Proposition 5.1.

We first prove the result for Bousfield $H\mathbb{Z}$-localization of groups.

Since $F \times_Q F \rightarrow F \times F$ is homologically 2-connected, this homomorphism induces an isomorphism of $H\mathbb{Z}$-localizations, $E(F \times_Q F) \rightarrow E(F \times F)$. By statements (3) and (4) of Proposition 5.1, $\gamma_\omega(G) = \gamma_\omega(E/\langle E, K \rangle) \neq \{1\}$, and has trivial image in $F/[F, L]$ by statement (4). But $\gamma_\omega(E/\langle E, K \rangle)$ is not relatively perfect, by statement (3).

The proof for Levine’s closure of groups needs only a small modification.

Since $Q$ is finite in this case, $E/\langle E, K \rangle$ is finitely presented (by statement (1) of Proposition 5.1). The group $F/[F, L]$ is finitely presented as well. Since, in addition, $E/\langle E, K \rangle \rightarrow F/[F, L]$ is homologically 2-connected, we get an isomorphism of closures, $E(E/\langle E, K \rangle) \rightarrow E(F/[F, L])$. So $\gamma_\omega(K/\langle E, K \rangle) \neq \ker\{E/\langle E, K \rightarrow F/[F, L]\}$. But $K/[F, K]$ is not relatively perfect in the sense of Levine, proving the theorem. \[\square\]

**References**

[BBMIS00] Gilbert Baumslag, Martin R. Bridson, Charles F. Miller III, and Hamish Short, *Fibre products, non-positive curvature, and decision problems*, Comment. Math. Helv. **75** (2000), no. 3, 457–477. MR 1793798 (2001k:20091)

[Bou74] A. K. Bousfield, *Homological localizations of spaces, groups, and $\pi$-modules*, Localization in group theory and homotopy theory, and related topics (Sympos., Battelle Seattle Res. Center, Seattle, Wash., 1974), Springer, Berlin, 1974, pp. 22–30. Lecture Notes in Math., Vol. 418. MR MR0380779 (52 #5124)

[Bou75] , *The localization of spaces with respect to homology*, Topology **14** (1975), 133–150. MR MR0380779 (52 #1676)

[Bou77] , *Homological localization towers for groups and $\Pi$-modules*, Mem. Amer. Math. Soc. **10** (1977), no. 186, vii+68. MR 0447376 (56 #5688)

[BR12] Martin. R. Bridson and Alan. W. Reid, *Nilpotent completions of groups, Grothendieck pairs, and four problems of Baumslag*, arXiv:1211.0493, November 2012.

[Cha08] Jae Choon Cha, *Injectivity theorems and algebraic closures of groups with coefficients*, Proc. London Math. Soc. **96** (2008), no. 1, 227–250.

[CO10] Jae Choon Cha and Kent E. Orr, *Hidden torsion and homology cobordism of $3$-manifolds*, To appear in Journal of Topology, 2010.

\[1\] A partial list of finite superperfect groups with non-trivial third homology group can be found at [http://hamilton.nuigalway.ie/Hap/www/SideLinks/About/aboutSuperperfect.html](http://hamilton.nuigalway.ie/Hap/www/SideLinks/About/aboutSuperperfect.html)
[DFO89] E. Dror Farjoun, K. Orr, and S. Shelah, *Bousfield localization as an algebraic closure of groups*, Israel J. Math. **66** (1989), no. 1-3, 143–153. MR MR1017158 (90j:55016)

[Dwy75] William G. Dwyer, *Homology, Massey products and maps between groups*, J. Pure Appl. Algebra **6** (1975), no. 2, 177–190. MR 0385851 (52 #6710)

[EM07] Ioannis Emmanouil and Roman Mikhailov, *A limit approach to group homology*, J. Algebra **319** (2008), no. 4, 1450–1461. MR 2383055 (2008m:20086)

[GM79] M. A. Gutierrez, *Concordance and homology. I. Fundamental group*, Pacific J. Math. **82** (1979), no. 1, 75–91. MR 549834 (82a:57020)

[Hec12] Prudence Heck, *Twisted homology cobordism invariants of knots in aspherical manifolds*, Int. Math. Res. Not. IMRN **2012** (2012), no. 15, 3434–3482.

[LD88] Jean-Yves Le Dimet, *Cobordisme d’enlacements de disques*, Mém. Soc. Math. France (N.S.) (1988), no. 32, ii+92. MR 90e:57046

[Lev88] Jerome P. Levine, *Link concordance*, Algebra and Topology 1988 (Taejŏn, 1988), Korea Inst. Tech., Taejŏn, 1988, pp. 57–76. MR 90k:57030

[Lev89a] Jerome P. Levine, *Link concordance and algebraic closure. II*, Invent. Math. **96** (1989), no. 3, 571–592. MR 91g:57007

[Lev90] J. P. Levine, *Algebraic closure of groups*, Combinatorial group theory (College Park, MD, 1988), Contemp. Math., vol. 109, Amer. Math. Soc., Providence, RI, 1990, pp. 99–105. MR 1076380

[Lev94] Jerome P. Levine, *Link invariants via the eta invariant*, Comment. Math. Helv. **69** (1994), no. 1, 82–119. MR 95a:57009

[MP06] Roman Mikhailov and Inder Bir S. Passi, *Faithfulness of certain modules and residual nilpotence of groups*, Internat. J. Algebra Comput. **16** (2006), no. 3, 525–539. MR 2241621 (2008k:20076)

[RS04] José L. Rodríguez and Dirk Scevenels, *Homology equivalences inducing an epimorphism on the fundamental group and Quillen’s plus construction*, Proc. Amer. Math. Soc. **132** (2004), no. 3, 891–898. MR 2019970 (2005a:55006)

[Sak06] Takuya Sakasai, *Homology cylinders and the acyclic closure of a free group*, Algebr. Geom. Topol. **6** (2006), 603–631 (electronic). MR 2206961 (2007m:57002)

[Sta65] John Stallings, *Homology and central series of groups*, J. Algebra **2** (1965), 170–181. MR 31 #232

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