The Inhomogeneous Invariance Quantum Supergroup of Supersymmetry Algebra

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Abstract

We consider an inhomogeneous quantum supergroup which leaves invariant a supersymmetric particle algebra. The quantum sub-supergroups of this inhomogeneous quantum supergroup are investigated.

1 Introduction

Symmetry transformations based on Lie groups and Lie algebras are most known in all areas of physics. Symmetry transformations are algebraic objects and they contain Lie groups and Lie algebras as special cases. Theory of quantum integrable systems has initiated a new type of symmetry and mathematical

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objects called quantum groups. The quantum groups are related to usual Lie
groups as quantum mechanics is related to its classical limit[1].

Since the method of derivation of a quantum group from a group is to make
deformations on it, almost all examples of quantum groups considered in physics
have been deformations of ordinary groups depending on one or more param-
ters. However, one can build up a matrix whose elements satisfy Hopf algebra
axioms just as the quantum groups which are derived from ordinary groups.[2]
In this paper we use the term “invariance quantum supergroup” to describe a
Hopf superalgebra such that a supersymmetry algebra forms a right module of
the Hopf superalgebra.

An ordinary Lie algebra is defined over the field of complex numbers $\mathbb{C}$, how-
ever if an algebra is defined over a vector space which has bosonic and fermionic
elements, the algebra is called a superalgebra[3]. An algebraic object which
generalizes Lie superalgebras and their supergroups, is called a quantum super-
group, more commonly Hopf superalgebra. Two algebras in a braided category
have a natural braided tensor product structure[4]. Thus, in a superalgebra
coproduct is defined via a braided tensor product.

The Standard Model does not explain some aspects of cosmology, concerning
the large scale universe. For example, the Standard Model can not explain the
dark matter. However, supersymmetry suggests explanations. The basic idea
of supersymmetry is that the equations which represent basic laws of nature
do not change if certain particles in the equations are interchanged with one
another[5]. These equations have a supersymmetry because they contain both
fermionic and bosonic elements. These are called SUSY algebras. The simplest
SUSY algebra contains $n$ bosons and $n$ fermions and these commute among each
other. This algebra is called the associative unital superalgebra.

Since the associative unital superalgebra contains bosonic and fermionic creation and annihilation operators one can build up a noncommutative (NC) QFT. It is well known that QFT on 4-dimensional NC space-time is invariant under the $SO(1,1) \times SO(2)$ subgroup of the Lorentz group. However representation of this group is different from the representation of Lorentz group. Using the notion of twisted Poincaré symmetry one can show that the interpretation of NC coordinates is extended from Lie algebra framework to Hopf algebras. A consistent frame for NC QFT can be realized in terms of twisted Poincaré symmetry.

For an ordinary particle algebra, its quantum inhomogeneous invariance quantum group has been discovered. For the fermion algebra it is $FIO(2d)$ and for the boson algebra it is $BISp(2d)$ where $d$ is number of fermions(bosons). In this paper we look for an inhomogeneous quantum supergroup which leaves the associative unital superalgebra invariant.

2 FBIOSp(2n|2m)

The associative unital superalgebra contains fermions and bosons. Since the algebra does not change if the particles interchanged with one to another, the associative unital superalgebra is an example of SUSY algebra. Using the definition of graded commutator the commutation relations of the particle algebra
can be written explicitly as
\[
\{f_i, f_j\} = 0 \quad \{f_i, f_j^*\} = \delta_{ij},
\]
\[
[b_k, b_l] = 0 \quad [b_k, b_l^*] = \delta_{kl}
\]
\[
[f_i, b_k] = 0 \quad [f_i, b_k^*] = 0,
\]

(1)

here \(f_i\)'s are fermion annihilation operators and \(b_k\)'s are boson annihilation operators. In the algebra \(i = 1, \cdots, n \) and \(k = 1, \cdots, m\) where \(n\) is number of fermions, \(m\) is number of bosons. Also hermitian conjugates of the commutation relations are valid. With respect to the commutation relations the associative unital superalgebra is also a Lie superalgebra[11]. It is well known that the supergroup \(OSP(2n|2m)\) acting homogeneously on the associative unital superalgebra leaves the commutation relations (1) invariant[12]. Here we would like to consider an inhomogeneous transformation on bosons and fermions and want the algebra to remain invariant under the transformation. We write transformed fermionic and bosonic creation and annihilation operators
\[
\begin{pmatrix}
    f_i' \\
    f_i'^* \\
    b_k' \\
    b_k'^*
\end{pmatrix}
= \begin{pmatrix}
    \alpha_{ij} & \beta_{ij} & a_{il} & c_{il} & \gamma_i \\
    \beta_{ij}^* & \alpha_{ij}^* & a_{il}^* & c_{il}^* & \gamma_i^* \\
    d_{kj} & e_{kj} & \varepsilon_{kl} & \theta_{kl} & \phi_k \\
    \varepsilon_{kj}^* & \theta_{kj}^* & d_{kl}^* & e_{kl}^* & \phi_k^*
\end{pmatrix}
\otimes
\begin{pmatrix}
    f_j \\
    f_j^* \\
    b_l \\
    b_l^*
\end{pmatrix}.
\]

(2)

The transformation matrix \(T\) is given in terms of sub-matrices by
\[
T = \begin{pmatrix}
    \alpha_{ij} & \beta_{ij} & a_{il} & c_{il} & \gamma_i \\
    \beta_{ij}^* & \alpha_{ij}^* & a_{il}^* & c_{il}^* & \gamma_i^* \\
    d_{kj} & e_{kj} & \varepsilon_{kl} & \theta_{kl} & \phi_k \\
    \varepsilon_{kj}^* & \theta_{kj}^* & d_{kl}^* & e_{kl}^* & \phi_k^*
\end{pmatrix}
= \begin{pmatrix}
    \alpha & A & \Gamma \\
    D & \varepsilon & \Phi \\
    0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
    H & K \\
    0 & 1
\end{pmatrix},
\]

(3)
\[ \alpha = \begin{pmatrix} \alpha_{ij} & \beta_{ij} \\ \beta_{ij}^* & \alpha_{ij}^* \end{pmatrix}, \quad A = \begin{pmatrix} a_{il} & c_{il} \\ c_{il}^* & a_{il}^* \end{pmatrix}, \quad D = \begin{pmatrix} d_{kj} & e_{kj} \\ e_{kj}^* & d_{kj}^* \end{pmatrix}, \quad \epsilon = \begin{pmatrix} \varepsilon_{kl} & \theta_{kl} \\ \theta_{kl}^* & \varepsilon_{kl}^* \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \gamma_i \\ \gamma_i^* \end{pmatrix}, \quad \Phi = \begin{pmatrix} \phi_k \\ \phi_k^* \end{pmatrix}. \]

Where \( H \) is the homogeneous and \( K \) is the inhomogeneous part of the matrix \( T \). Here dimensions of the sub-matrices are: \( \text{dim}(\alpha) = (2n) \times (2n) \), \( \text{dim}(A) = (n+m) \times (n+m) \), \( \text{dim}(D) = (m+n) \times (m+n) \) and \( \text{dim}(\epsilon) = (2m) \times (2m) \). \( \Gamma \) is a \( 2n \times 1 \) and \( \Phi \) is a \( 2m \times 1 \) dimensional matrix. First of all, we should mention that the submatrices \( A \), \( D \) and \( \Gamma \) have fermionic elements and the others have bosonic elements. Secondly, we define a braided tensor product. This is given by:

\[
(P \otimes Q)(R \otimes S) = (-1)^{\text{deg}(Q)\text{deg}(R)} PR \otimes QS.
\]

\[
\text{deg}(X) = \begin{cases} 
1 & \text{fermionic}, \\
0 & \text{bosonic}.
\end{cases}
\]

We should note that the above definition of braided tensor product is true for only the \( Z_2 \) graded case namely, the algebra has only bosonic and fermionic elements. We use transformed operators in equation (11) and want the algebra to remain unchanged. Therefore, we get some relations involving the elements of the transformation matrix \( T \). We should mention that braided tensor product...
was used to find the relations.

\[ [\alpha_{ij}, \alpha_{kl}] = 0 \quad [\alpha_{ij}, A_{kl}] = 0, \]
\[ [\alpha_{ij}, D_{kl}] = 0 \quad [\alpha_{ij}, \epsilon_{kl}] = 0, \]
\[ [\alpha_{ij}, \Gamma_k] = 0 \quad [\alpha_{ij}, \Phi_k] = 0, \]
\[ \{ A_{ij}, D_{kl} \} = 0 \quad [A_{ij}, \epsilon_{kl}] = 0, \]
\[ \{ A_{ij}, \Gamma_k \} = 0 \quad [A_{ij}, \Phi_k] = 0, \]
\[ [D_{ij}, \epsilon_{kl}] = 0 \quad [D_{ij}, \Gamma_k] = 0, \]
\[ [D_{ij}, \Phi_k] = 0 \quad \{ \epsilon_{ij}, \Gamma_k \} = 0, \]
\[ \{ D_{ij}, D_{kl} \} = 0 \]
\[ \{ D_{ij}, A_{kl} \} = 0, \]
\[ \{ D_{ij}, \Phi_k \} = 0 \]
\[ \{ D_{ij}, \Phi_k \} = 0 \]
\[ \{ \epsilon_{ij}, \Phi_k \} = 0 \]
\[ \{ \epsilon_{ij}, \Phi_k \} = 0 \]
\[ \{ D_{ij}, D_{kl} \} = 0 \]

(4)

also there are relations between the elements of matrices $\Gamma$ and $\Phi$. These can be written as:

\[ \{ \gamma_i, \gamma_j \} = -\alpha_{ik} \beta_{jk} - \beta_{ik} \alpha_{jk} - c_{il} a_{jl} + a_{il} c_{jl}, \]
\[ \{ \gamma^*_i, \gamma^*_j \} = \delta_{ij} - \alpha_{ik} \alpha^*_j - \beta_{ik} \beta^*_j - c_{il} c^*_j + a_{il} a^*_j, \]
\[ [\gamma_i, \phi_j] = \alpha_{ik} e_{jk} + \beta_{jk} e_k + c_{il} \epsilon_{jl} + a_{il} \theta_{jl}, \]
\[ [\gamma^*_i, \phi^*_j] = \alpha_{ik} e^*_j + \beta_{jk} e^*_k - c_{jl} \epsilon_{jl} + a_{jl} \theta^*_{jl}, \]
\[ [\phi_i, \phi_j] = d_{ik} e_{jk} + e_{ik} d_{jl} - \theta_{il} \epsilon_{jl} + \theta_{jl} \epsilon_{jl}, \]
\[ [\phi_i, \phi^*_j] = \delta_{ij} + d_{ik} d^*_j + e_{ik} e^*_j - \theta_{il} \theta^*_{jl} + \epsilon_{jl} \epsilon^*_{jl}. \]

(5)

We look for a Hopf superalgebra such that under this transformation equation (1) remains invariant. In order to do this, one should first check the coproduct. We want the elements of the matrix $T$ to belong to a Hopf superalgebra $H$ where the coproduct is given by a matrix multiplication

\[ \Delta(T) = T \otimes T. \]

(6)

One can see that coproduct of commutation relations are satisfied by defining braided tensor product. The counit is

\[ \Delta(T) = T \otimes T. \]
\[\varepsilon(T) = I.\]  

Finally, the antipode should be found.

\[S(T) = T^{-1}\]  

\[T^{-1} = \begin{pmatrix} H^{-1} & -H^{-1}K \\ 0 & 1 \end{pmatrix}.\]  

Now we should find the inverse of homogeneous part \(H\) of the transformation matrix. Since the matrix \(H\) is a supermatrix, to find its inverse the sub-matrices \(\alpha\) and \(\epsilon\) should be invertible. We use the same technique which is used to find the inverse of a supermatrix. The inverse matrix can be written as;

\[H^{-1} = \begin{pmatrix} \alpha' & A' \\ D' & \epsilon' \end{pmatrix},\]  

and the elements of \(H^{-1}\) are:

\[\alpha' = (\alpha - \alpha^{-1}A)\]  

\[\epsilon' = (\epsilon - D\epsilon^{-1}A)^{-1},\]  

\[A' = -\alpha^{-1}A\epsilon',\]  

\[D' = -\epsilon^{-1}D\alpha'.\]  

Since the elements of \(\alpha\) and \(\epsilon\) are commutative, \(\alpha^{-1}\) and \(\epsilon^{-1}\) can be found using the ordinary matrix inverse rule.

The braided coproduct, the counit and the antipode of the transformation matrix \(T\) as given by Eqs (6-8) have been constructed. Thus, the transformation matrix \(T\) is an element of a quantum supergroup. We may call this quantum supergroup \(FBIOSp(2n|2m)\), the Fermionic-Bosonic Inhomogeneous Orthosymplectic quantum supergroup.
To find the quantum subgroups of $FBIOSp(2n|2m)$, we should impose additional constraints such as:

1. $-\alpha_{ik}\beta_{jk} - \beta_{ik}\alpha_{jk} - c_{il}a_{jl} + a_{il}c_{jl} = 0,$
   \[ \delta_{ij} - \alpha_{ik}\alpha^*_{jk} - \beta_{ik}\beta^*_{jk} - c_{il}c^*_{jl} + a_{il}a^*_{jl} = 0, \]
   \[ \alpha_{ik}e_{jk} + \beta_{ik}d_{jk} - c_{il}\varepsilon_{jl} + a_{il}\theta_{jl} = 0, \]
   \[ d_{ik}\varepsilon_{jk} + e_{ik}d_{jk} - \theta_{il}\varepsilon_{jl} + \theta_{il}\varepsilon_{jl} = 0, \]
   \[ \delta_{ij} + d_{ik}d^*_{jk} + e_{ik}e^*_{jk} - \theta_{il}\theta^*_{jl} + \varepsilon_{il}\varepsilon^*_{jl} = 0. \]

2. $\Gamma = \Phi = 0.$

3. $A = D = 0.$

4. $\alpha = \epsilon = 0.$

Applying the above relations on $FBIOSp(2n|2m)$ one can get the quantum subgroups and these are:

$$
\begin{align*}
FBIOSp(2n|2m) & \xrightarrow{(i)} IOSp(2n|2m) \xrightarrow{(ii)} OSp(2n|2m) \\
\downarrow{(iii)} & \downarrow{(iii)} & \downarrow{(iii)} \\
FIO(2n) \times BISp(2m) & \xrightarrow{(i)} GrIO(2n) \times ISp(2m) \xrightarrow{(ii)} O(2n) \times Sp(2m) \\
\downarrow{(iv)} & & \\
SA(n|m) & 
\end{align*}
$$

Here $IOSp(2n|2m)$ is that the Inhomogeneous Orthosymplectic supergroup where the elements of inhomogeneous part have graded commutation relations among themselves. $OSp(2n|2m)$ is the Orthosymplectic supergroup \cite{13}. $FIO(2d)$ is the inhomogeneous invariance quantum group of the fermion algebra \cite{8, 9}. $BISp(2d)$ is the inhomogeneous invariance quantum group of the boson algebra \cite{10}. $GrIO(2d)$ is the grassmanian inhomogeneous orthogonal group. $ISp(2d)$
is the Inhomogeneous Symplectic group \[ \text{14} \]. \( SA(n|m) \) is the associative unital superalgebra whose elements satisfy equation \[ \text{11} \]. We should mention that \( FIO(2n) \times BISp(2m) \) has quantum group structure. The others have group structure.

## 3 Conclusion

Supersymmetry claims that the equations of the laws of nature should be invariant under transformations which change all basic particles into each other. To describe laws of nature, supersymmetric models are more convenient although there is yet no experimental evidence. According to supersymmetry, every particle has a partner particle which is a fermion for a boson and a boson for a fermion. Supersymmetry algebra describes the symmetry between bosons and fermions.

Graded Lie algebras and their graded Lie groups are used in various branches of theoretical physics such as supersymmetric field theory \[ \text{15} \]. Both in graded Lie algebras and in supersymmetry a grading factor is defined. For quantum groups, the same grading can be defined, in which case they are called quantum supergroups or braided quantum groups.

In this paper, we have shown that for the associative unital superalgebra, an inhomogeneous quantum invariance supergroup can be constructed and we have investigated its quantum sub-supergroups.

The important point is that \( FBIOSp(2n|2m) \) does not have a deformation parameter. Other examples of quantum groups which do not contain a deformation parameter are \( FIO(2d,R) \) and \( U_G(d+1) \) \[ \text{16} \].
The well known supergroups $OSp(2n|2m)$ and $O(2n) \times Sp(2m)$ are sub-supergroups of homogeneous part of $FBIOSp(2n|2m)$. In this paper, we have shown that there is a general structure which contains all the well known supergroups $IOSp(2n|2m)$, $OSp(2n|2m)$ and $O(2n) \times Sp(2m)$. This general structure is called as $FBIOSp(2n|2m)$ and it is a quantum supergroup.

As a further work, one can search for the inhomogeneous invariance quantum group of a NC QFT and look at whether any relation between this inhomogeneous quantum group and twisted Poincaré symmetry.

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