IRREDUCIBLE CONTACT CURVES VIA GRAPH STRATIFICATION

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Abstract. We prove that the moduli space of contact stable maps to $\mathbb{P}^{2n+1}$ of degree $d$ admits a stratification parameterized by graphs. We use it to determine the number of irreducible rational contact curves in $\mathbb{P}^{2n+1}$ with any Schubert condition. We give explicitly some of these invariants for $\mathbb{P}^3$ and $\mathbb{P}^5$. We give another proof of the formula for the number of plane contact curves in $\mathbb{P}^3$ meeting the appropriate number of lines.

1. Introduction

A fast-growing area of mathematics is Contact Geometry, that is the study of manifold with a contact structure. First defined by [Kob59], a contact structure on a smooth complex manifold $X$ of odd dimension is a corank 1 non-integrable holomorphic distribution. All projective spaces of odd dimension admit a (unique) contact structure.

The interest for these manifolds comes from real geometry. Calabi [Cal67] proved that the study of harmonic maps of spheres $f: S^2 \to S^4$ is equivalent, modulo an involution, to the study of holomorphic horizontal curves $\tilde{f}: \mathbb{C}P^1 \to \mathbb{C}P^3$ of fixed degree. Roughly speaking, $f$ minimizes the energy functional if $f$ or $-f$ has a lift $\tilde{f}$, such that $\tilde{f}(\mathbb{C}P^1)$ is integral with respect to the contact structure of $\mathbb{P}^3$.

The enumeration of contact curves (i.e., rationally connected curves integral with respect to the contact structure) with geometric conditions started in [LV11]. Levcovitz and Vainsencher found, among other results, the number of contact plane curves of degree $d$ meeting $3 + d$ general lines in $\mathbb{P}^3$. On the other hand, the first systematic study of enumerative invariants of those curves started in [Mur22] (strengthening results from [Amo14]). The strategy was to consider the closed subscheme $S_m(\mathbb{P}^n, d) \subset \overline{M}_{0,m}(\mathbb{P}^n, d)$ of stable maps whose image is a contact curve. Certain Gromov-Witten invariants related to $S_m(\mathbb{P}^n, d)$ are enumerative. Using localization, the author was able to give enumerative invariants of contact curves with any Schubert condition.

The motivation to this paper is that very few enumerative invariants are known for irreducible rational contact curves. The reason is that $S_m(\mathbb{P}^n, d)$ has many irreducible components. As already pointed out in [Mur22, Remark 3.10], the subscheme of $S_m(\mathbb{P}^n, d)$ parameterizing reducible curves has codimension 0. That is, the “boundary” of $S_m(\mathbb{P}^n, d)$ is not a divisor, but a component purely of the same
dimension. So that, the enumerative invariants get contribution from all irreducible components.

In this paper we give a decomposition of $\mathcal{S}_m(\mathbb{P}^n,d)$ in components parameterized by graphs, with a complete description of the dimension of all components (Theorem 4.4). This stratification is induced by a similar one of $\overline{M}_{0,m}(\mathbb{P}^n,d)$. We use this stratification to give an explicit formula to compute the number of reducible contact curves with any dual graph (Corollary 5.2). Moreover, we are able to give another proof of Levcovitz and Vainsencher’s result (Proposition 5.7).

In Tables 5.1 and 5.2 we give enumerative invariants of irreducible rational contact curves in $\mathbb{P}^3$ and $\mathbb{P}^5$ of low degree. The strategy is to subtract, from the number of contact curves, the number of reducibles contact curves with the same Schubert conditions. Note that the enumerative numbers of contact curves in $\mathbb{P}^{2n+1}$ are all known thanks to [MS22].

An alternative way to get enumerative invariants of irreducible contact curves may be to use another moduli space, like the Hilbert scheme. We plan to pursue this research path in the future.

I would like to thank Carlos Florentino for his many suggestions. This problem came out from one of the many conversations I had with Israel Vainsencher during my work at UFMG. I would like to thank him for his time. I also thank Giordano Cotti, Angelo Lopez and Filippo Viviani for useful conversations. Finally, I thank Csaba Schneider for his computational support.

The author is supported by FCT - Fundação para a Ciência e a Tecnologia, under the project: UIDP/04561/2020. The author is a member of GNSAGA (INdAM).

2. Contact Structures

All varieties in this article are defined over $\mathbb{C}$. In this section $n$ is any non-negative integer.

**Definition 2.1.** Let $X$ be a $(2n+1)$-dimensional complex manifold. A contact structure on $X$ is an open cover $\{U_i\}_i$ of $X$ together with 1-forms $\alpha_i$ on each $U_i$ such that:

1. at every point of $U_i$, the form $\alpha_i \wedge (d\alpha_i)^n$ is non-zero, and
2. if the intersection $U_i \cap U_j$ is non empty, then there exists a non-vanishing holomorphic function $f_{ij}$ on $U_i \cap U_j$ such that $\alpha_i = f_{ij}\alpha_j$ on $U_i \cap U_j$.

A contact curve is a rationally connected curve $C \subset X$ such that any local 1-form $\alpha_i$ vanishes at every smooth point of $C \cap U_i$.

**Remark 2.2.** An equivalent definition is the following: A contact structure on $X$ is a pair $(X, L)$ where $L$ is a line subbundle of $\Omega^1_X$ such that if $s$ is a non trivial local section of $L$, then $s \wedge (ds)^n$ is everywhere non-zero.

Any symplectic 1-form on $\mathbb{C}^{2n+2}$ defines a contact structure on $\mathbb{P}^{2n+1}$ by the projection $\mathbb{C}^{2n+2} \setminus \{0\} \to \mathbb{P}^{2n+1}$. Vice versa any contact structure on $\mathbb{P}^{2n+1}$ is induced by a symplectic form on $\mathbb{C}^{2n+2}$. Let $(x_0, \ldots, x_n, y_0, \ldots, y_n)$ be local coordinates of $\mathbb{C}^{2n+2}$. All symplectic 1-forms on $\mathbb{C}^{2n+2}$ are conjugated to

$$\alpha := \sum_{i=0}^{n} x_i dy_i - y_i dx_i.$$
In particular, there exists a unique contact structure on $\mathbb{P}^{2n+1}$ modulo conjugation. See [OSS11, 1.4.2].

**Example 2.3.** On $\mathbb{P}^3$, the distribution induced by $\alpha$ assigns to $p = [a : b : c : d]$ the plane with equation

$$H_p := \{ [X : Y : Z : W] \in \mathbb{P}^3 / bX - aY - dZ + cW = 0 \}.$$

A curve is said to be a contact curve if the tangent line at each smooth point $p$ is contained in the plane $H_p$. The contact curves of degree 1 in $\mathbb{P}^{2n+1}$ are parameterized by the symplectic Grassmannian of isotropic 2-spaces of $\mathbb{C}^{2n+2}$. See [LVT11] for other examples.

**Remark 2.4.** We do not require that a contact curve $C$ is irreducible. Since the property of being contact is local, every component of $C$ is contact in its own.

### 2.1. Symplectic group.

Let us denote by $\text{Sp}(2n+2, \mathbb{C})$ the set of square matrices $M$ of order $2n+2$ with complex coefficients such that $M^T\Omega M = \Omega$, where $\Omega$ is the block matrix

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

and $I$ is the unit matrix of order $n+1$. The following facts are well-known:

- The set $\text{Sp}(2n+2, \mathbb{C})$ is a subgroup of the group of matrices with determinant 1. The two groups coincide if and only if $n = 0$.
- The standard action of $\text{Sp}(2n+2, \mathbb{C})$ on $\mathbb{C}^{2n+2}$ is transitive, and preserves the 1-form $\alpha$.

There exists an explicit description of $\text{Sp}(2n+2, \mathbb{C})$. Let $M$ be a block matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A, B, C, D$ are matrices of order $n+1$. Then $M$ is in $\text{Sp}(2n+2, \mathbb{C})$ if and only if

- $C^T A = A^T C$ (2.1)
- $A^T D - C^T B = I$ (2.2)
- $D^T B = B^T D$ (2.3)

Main references are [FH91, Lecture 16] and [O'M78]. It follows that the standard action of $\text{Sp}(2n+2, \mathbb{C})$ on $\mathbb{P}^{2n+1}$ preserves the contact structure given by $\alpha$. More generally, for any contact structure on $\mathbb{P}^{2n+1}$ there is an action of $\text{Sp}(2n+2, \mathbb{C})$ preserving it. This follows from the fact that all contact structures are equivalent by a change of coordinates.

Let $\{ U_i \}_{i=0}^n$ be the standard open cover of $\mathbb{P}^n$. It is well known that there exist subgroups $\{ H_i \}_{i=0}^n$ of $\text{GL}(n+1, \mathbb{C})$ isomorphic to $\mathbb{C}^n$, such that the action $H_i \curvearrowright U_i$ is the translation (see [GH78, 1.5]). Unfortunately, such subgroups do not preserve the contact form. The following proposition may be known to experts, but I could not find any reference.

**Proposition 2.5.** Consider the standard action of $\text{Sp}(2n+2, \mathbb{C})$ on $\mathbb{P}^{2n+1}$. For every $i = 0, \ldots, 2n+1$ there exist

- a subgroup $H_i \subset \text{Sp}(2n+2, \mathbb{C})$,
- an open subset $U_i \subset \mathbb{P}^{2n+1}$, and
two isomorphisms \( \varphi_i: \mathbb{C}^{2n+1} \to H_i \) and \( \phi_i: \mathbb{C}^{2n+1} \to U_i \), such that

1. \( U_i \) is \( H_i \)-invariant,
2. the action of \( H_i \) on \( U_i \) is free and transitive, and
3. the open subsets \( \{ U_1 \}_{i=0}^{2n+1} \) form an open cover of \( \mathbb{P}^{2n+1} \).

**Proof.** Let \( \{ x_0, \ldots, x_n, y_0, \ldots, y_n \} \) be a basis of \( \mathbb{C}^{2n+2} \), and \( i \in \{0, \ldots, n\} \). Let \( U_i \) be the open subset of \( \mathbb{P}^{2n+1} \) of vectors with coefficient of \( x_i \) non-zero, and let \( \phi_i: \mathbb{C}^{2n+1} \to U_i \) be the standard isomorphism.

Let us define the map \( \varphi_i: \mathbb{C}^{2n+1} \to \text{Sp}(2n + 2, \mathbb{C}) \) in the following way. For every \( \vec{a}, \vec{c} \) = \( (a_0, a_1, \ldots, a_n, c_0, c_1, \ldots, c_n) \in \mathbb{C}^{2n+1} \), \( \varphi_i(\vec{a}, \vec{c}) \) is the linear map sending

\[
\begin{align*}
x_j &\mapsto x_j + c_j y_i \\
x_i &\mapsto x_i + c_i y_i + \sum_{k \neq i} a_k x_k + c_k y_k \\
y_j &\mapsto y_j - a_j y_i \\
y_i &\mapsto y_i
\end{align*}
\]

where \( j = 0, \ldots, i, \ldots, n \). It is an easy exercise to prove that \( \varphi_i(\vec{a}, \vec{c}) \) defines a linear map, whose inverse is \( \varphi_i(-\vec{a}, -\vec{c}) \) and \( \varphi_i(\vec{a}, \vec{c}) \phi_i(\vec{b}, \vec{d}) = \varphi_i(\vec{q}, \vec{w}) \) where

\[
\begin{align*}
q_j &= a_j + b_j \\
w_j &= c_j + d_j \\
\end{align*}
\]

By direct computation, \( \varphi_i(\vec{a}, \vec{c}) \) fixes the 1-form \( \sum_{k=0}^{n} x_k dy_k - y_k dx_k \) (see also Remark 2.6). Finally the coefficient of \( x_i \) is preserved, hence \( U_i \) is \( \varphi_i(\vec{a}, \vec{c}) \)-invariant. We proved that the map \( \varphi_i: \mathbb{C}^{2n+1} \to \text{Sp}(2n + 2, \mathbb{C}) \) is an injective morphism and \( H_i = \varphi_i(\mathbb{C}^{2n+1}) \) is a subgroup acting on \( U_i \).

Note that \( \varphi_i(\vec{a}, \vec{c}) \cdot \phi_i(\vec{b}, \vec{d}) = \phi_i(\vec{q}, \vec{w}) \) where \( \vec{q} \) and \( \vec{w} \) are the vectors whose coordinates are in Eq. (2.4). It follows that the isomorphism \( \phi_i \circ \varphi_i^{-1}: H_i \to U_i \) is \( H_i \)-linear. Hence the action of \( H_i \) on \( U_i \) is free and transitive.

Now, we prove that such a construction works for every \( i \). Suppose \( n < i \leq 2n+1 \) and let \( f \) be the involutive linear automorphism of \( \mathbb{C}^{2n+2} \) such that \( f(x_k) = y_k \) for \( k = 0, \ldots, n \). Let \( U_i \) be the open subset of \( \mathbb{P}^{2n+1} \) of vectors with coefficient of \( y_{i-n-1} \) non-zero, and let \( \phi_i: \mathbb{C}^{2n+1} \to U_i \) be the standard isomorphism. Let us define \( \varphi_i := f \circ \varphi_{i-n-1} \circ f \). It is clear that such a morphism satisfies the conditions, and \( \{ U_1 \}_{i=0}^{2n+1} \) is the standard open cover.

**Remark 2.6.** We can see that \( \varphi_i(\vec{a}, \vec{c}) \) is a symplectic linear isomorphism by looking at its matrix description. With respect to the basis \( \{ x_0, \ldots, x_n, y_0, \ldots, y_n \} \), let us consider the representation

\[
\varphi_i(\vec{a}, \vec{c}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

where \( A, B, C, D \) are square matrices of order \( (n+1) \). For example, when \( i = 0 \),

\[
A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_1 & a_1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & a_n & \cdots & 1 \end{pmatrix}
\]

\[
C = \begin{pmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_1 & \cdots & c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_n & \cdots & 1 \end{pmatrix}
\]

\[
D = \begin{pmatrix} 1 & -a_1 & \cdots & -a_n \\ 1 & 1 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \cdots & 1 \end{pmatrix}
\]
3. Contact Stable Maps

3.1. Stable trees. We give an introduction to stable trees as in [BM96] Part I. A graph $\tau$ is a quadruple $(F_\tau, V_\tau, \partial_\tau, j_\tau)$, where $F_\tau$ is a finite set of flags, $V_\tau$ is a finite set of vertices, $\partial_\tau: F_\tau \to V_\tau$ is a map, and $j_\tau: F_\tau \to F_\tau$ is an involution. The elements of $E_\tau := \{\{f, f\} \subseteq F_\tau / f' = j_x(f) \neq f\}$ are called the edges of $\tau$, and the elements of $L_\tau := \{f \in F_\tau / j_x(f) = f\}$ are called the leaves (or tails) of $\tau$. For each vertex $v \in V_\tau$, its valence $n(v)$ is the cardinality of $F_\tau(v) := \partial_\tau^{-1} (\{v\})$.

An isomorphism of graphs $\varphi: \tau \to \sigma$ is a pair $(\varphi_F, \varphi_V)$, where $\varphi_F: F_\tau \to F_\sigma$ and $\varphi_V: V_\tau \to V_\sigma$ are bijective maps such that $\varphi_V \circ \partial_\tau = \partial_\sigma \circ \varphi_F$ and $\varphi_F \circ j_\tau = j_\sigma \circ \varphi_F$. A graph is connected if for every two vertices $v, v' \in V_\tau$, there exists a sequence of vertices $\{v_i\}_{i=0}^r$ such that

- for each $i = 1, \ldots, r$, there exists an edge $e = \{f, f'\}$ such that $\partial_\tau(f') = v_{i-1}$, $\partial_\tau(f) = v_i$, and
- $v_0 = v$, $v_r = v'$.

Definition 3.1. A tree is a connected graph such that $|E_\tau| + |L_\tau| + |V_\tau| = 1 + |F_\tau|$.

For any pair $(m, d)$ of non negative integers, a $(m, d)$-tree is the datum $(\tau, l_\tau, d_\tau)$ where $\tau$ is a tree, $A_\tau$ is a set, $|A_\tau| = m$, $l_\tau: L_\tau \to A_\tau$ is a bijective map, and $d_\tau: V_\tau \to \mathbb{Z}$ is a map such that $\sum_{v \in V_\tau} d_\tau(v) = d$ and $d_\tau(v) \geq 0$ for every $v \in V_\tau$.

Definition 3.2. A $(m, d)$-tree $(\tau, l_\tau, d_\tau)$ is stable if for each $v \in V_\tau$, either $d_\tau(v) \neq 0$ or $n(v) \geq 3$.

Two stable $(m, d)$-trees $(\tau, l_\tau, d_\tau)$ and $(\sigma, l_\sigma, d_\sigma)$ are isomorphic if there exist isomorphisms $\varphi: \tau \to \sigma$, $\theta: A_\tau \to A_\sigma$ such that $\theta \circ l_\tau = l_\sigma \circ \varphi$ and $d_\sigma = d_\sigma \circ \varphi_V$. By abuse of notation, we may denote a stable $(m, d)$-tree $(\tau, l_\tau, d_\tau)$ simply by $\tau$.

Definition 3.3. The group of automorphisms of the stable $(m, d)$-tree $\tau$ is denoted by $Aut(\tau)$.

The set of all stable $(m, d)$-tree, modulo isomorphism, is denoted by $\Gamma(m, d)$. It is finite by [GP06] Proposition 4.3. The subset of those trees such that $d_\tau(v) > 0$ for every $v \in V_\tau$ is denoted by $\Gamma(m, d)^+$.

Definition 3.4. The unique stable $(m, d)$-tree with only one vertex $v$ is denoted by $\tau_v$.

Since each edge joins two vertices, in a stable $(m, d)$-tree $\tau$ we have

\[ |F_\tau| - 2|V_\tau| = m - 2. \]

3.2. Stable maps. In this section by $n$ we will denote any odd number.

A stable map is the datum $(C, f, p_1, \ldots, p_m)$ where $C$ is a projective, connected, nodal curve of arithmetic genus 0, the markings $p_1, \ldots, p_m$ are distinct nonsingular points of $C$, and $f: C \to \mathbb{P}^m$ is a morphism such that $f_*([C])$ is a 1-cycle of degree $d$. Moreover, for every rational component $E \subseteq C$ mapped to a point, $E$ contains at least three points among marked points and nodal points.

The dual graph $\tau$ of the stable map $(C, f, p_1, \ldots, p_m)$ is defined as follows.

- For each vertex $v \in V_\tau$, there is an irreducible component $C_v$ of $C$.
• For each nodal point of $C$, there is an edge in $\tau$ connecting the vertices corresponding to those components.
• For each $p_i$, there is a leaf of $\tau$ attached to the vertex corresponding to the irreducible component containing $p_i$.
• There is a map $d_\tau : V_\tau \to \mathbb{Z}$ such that $d_\tau(v)$ is the degree of the restriction of $f$ to $C_v$.

A map $f : C \to \mathbb{P}^n$ of genus 0 with $m$ marked points and degree $d$ is stable if and only if its dual graph is a stable $(m, d)$-tree \cite[Proposition 4.4]{GP06}.

Let $M_{0,m}(\mathbb{P}^n, d)$ be the coarse moduli space of irreducible stable maps of genus 0 to $\mathbb{P}^n$, and let $\overline{M}_{0,m}(\mathbb{P}^n, d)$ be its compactification. Let us denote by $ev_i$ the evaluation map $ev_i : M_{0,m}(\mathbb{P}^n, d) \to \mathbb{P}^n$ at the marked point $p_i$. Given an isomorphism class $\tau$ of stable $(m, d)$-trees, the locus $M(\tau) \subset \overline{M}_{0,m}(\mathbb{P}^n, d)$ parameterizes maps whose dual graph is isomorphic to $\tau$. It is locally closed and of codimension $|E_\tau|$. For example, the unique stable $(m, d)$-tree with only one vertex corresponds to $M_{0,m}(\mathbb{P}^n, d)$. There is a stratification of $\overline{M}_{0,m}(\mathbb{P}^n, d)$ given by $M(\tau)$ for all $\tau \in \Gamma(m, d)$. Finally, there is a canonical isomorphism $M(\tau) \cong M(\square)(\tau)/\text{Aut}(\tau)$ where $M(\square)(\tau)$ is the fibred product

\begin{equation}
M(\square)(\tau) \longrightarrow \prod_{v \in V_\tau} M_{0,F_r(v)}(\mathbb{P}^n, d_\tau(v)) \\
(\mathbb{P}^n)_{E_\tau \cup L_\tau} \longrightarrow (\mathbb{P}^n)^{F_\tau}.
\end{equation}

We denote by $\overline{M}(\tau)$ the topological closure of $M(\tau)$ in $\overline{M}_{0,m}(\mathbb{P}^n, d)$.

Remark 3.5. We may define a scheme $\overline{M}(\square)(\tau)$ using the same square of Eq. (3.2), but using $M_{0,F_r(v)}(\mathbb{P}^n, d_\tau(v))$ instead of $M_{0,F_r(v)}(\mathbb{P}^n, d_\tau(v))$. Such a scheme has a natural ramified map to $\overline{M}_{0,m}(\mathbb{P}^n, d)$ of degree $\text{Aut}(\tau)$. The image is the closure of $M(\tau)$. See \cite{BM96} or \cite{MM07}.

Let us now focus on contact curves.

Definition 3.6. Let $(X, L)$ be a contact structure. A contact stable map is a stable map $(C, f, p_1, \ldots, p_m)$ such that for any local section $s$ of $L$, $f^* s = 0$.

In \cite{Mur22} we proved the following properties:
• The moduli space of contact stable maps $S_m(\mathbb{P}^n, d) \subset \overline{M}_{0,m}(\mathbb{P}^n, d)$ is the zero locus of a vector bundle $E$ of rank $2d - 1$.
• If $d > 0$, the irreducible components of $S_m(\mathbb{P}^n, d)$ are pure of dimension $d(n - 1) + n + m - 2$.
• The Gromov-Witten invariant

\begin{equation}
\int_{S_m(\mathbb{P}^n, d)} ev_1^*(\Gamma_1) \cdots ev_n^*(\Gamma_m) \cdot c_{2d-1}(E) = \int_{S_m(\mathbb{P}^n, d)} ev_1^*(\Gamma_1) \cdots ev_n^*(\Gamma_m)
\end{equation}

is enumerative.

Enumerative means that if $\{\Gamma_i\}_{i=1}^m$ are subvarieties of $\mathbb{P}^n$ in general position such that $\sum_{i=1}^m \text{codim}(\Gamma_i) = \dim S_m(\mathbb{P}^n, d)$, then \((3.3)\) equals the number of contact curves passing through all $\Gamma_i$. 
If the stable map \((C, f, p_1, \ldots, p_m)\) contracts \(C\), then it is contact. It follows that \(S_m(\mathbb{P}^n, 0) = \mathcal{M}_{0,m}(\mathbb{P}^n, 0) \cong \mathcal{M}_{0,m} \times \mathbb{P}^n\). Hence

\[
\int_{S_m(\mathbb{P}^n, 0)} \ev^*_i(\Gamma_1) \cdots \ev^*_m(\Gamma_m) = \begin{cases} \Gamma_1 \cdots \Gamma_m & m = 3 \\ 0 & m \neq 3 \end{cases}
\]

The WDVV differential equations is the system of PDE:

\[
I_d(m_0, \ldots, m_n) = \int_{S_{m_0+\ldots+m_n}(\mathbb{P}^n, d)} \prod_{i=0}^n \ev^*(H^i)^{m_i}
\]

\[
\Phi = \sum_{m_0+\ldots+m_n \geq 3} I_d(m_0, \ldots, m_n) \frac{h_0^{m_0}}{m_0!} \cdots \frac{h_n^{m_n}}{m_n!}.
\]

For example, in the case \(n = 3\),

\[
\Phi = h_0 h_1 h_2 + \frac{h_1^3}{3!} + \frac{h_2^3}{2!} + 2 \frac{h_3^3}{3!} + h_1 h_2 h_3 + 2 \frac{h_2 h_3^2}{2} + \cdots.
\]

The WDVV differential equations is the system of PDE:

\[
\sum_{e=0}^n \Phi_{ij} \Phi_{kl(n-e)} = \sum_{e=0}^n \Phi_{jkek}(n-e),
\]

where \(\Phi_{ijk} := \frac{\partial^3}{\partial h_i \partial h_j \partial h_k} \Phi\). The usual GW-potential satisfies the WDVV equations [KM94]. The contact potential does not. In order to see this, it is enough to compute all derivatives \(\Phi_{ijk}\), and evaluate them at 0. There are not known differential equations satisfied by \(\Phi\).

4. Stratification of the Moduli of Contact Stable Maps

**Definition 4.1.** For every \(\tau \in \Gamma(m, d)\), we denote by \(S(\tau)\) the space given by the intersection \(S_m(\mathbb{P}^n, d) \cap \mathcal{M}(\tau)\).

We will denote by \(\overline{S(\tau)}\) the topological closure of \(S(\tau)\) in \(S_m(\mathbb{P}^n, d)\). The restriction of the evaluation maps \(\ev_i: \mathcal{M}_{0,m}(\mathbb{P}^n, d) \rightarrow \mathbb{P}^n\) to \(S(\tau)\) or \(\overline{S(\tau)}\) will still be denoted by \(\ev_i\). We denote by \(S_m(\mathbb{P}^n, d)^c\) the open subset \(S_m(\mathbb{P}^n, d) \cap \mathcal{M}_{0,m}(\mathbb{P}^n, d)\). The following result follows from Proposition 2.5. See also [Bag22, Proposition 3.8].

**Proposition 4.2.** Let \(\tau \in \Gamma(m, d)\). The map \(\ev_i: S(\tau) \rightarrow \mathbb{P}^n\) is Zariski locally trivial.

**Proof.** We start by proving that \(\ev_i: S_m(\mathbb{P}^n, d) \rightarrow \mathbb{P}^n\) is Zariski locally trivial. As we saw in Proposition 2.5, there exists an open cover of \(\mathbb{P}^n\) such that each open set \(U\) is isomorphic to a subgroup \(H\) of \(\text{Sp}(n+1, \mathbb{C})\), in such a way that \(U\) is \(H\)-invariant and the action of \(H\) on \(U\) is free and transitive. This induces an action \(\beta: H \times \ev^{-1}_i(U) \rightarrow \ev^{-1}_i(U)\) sending \((h, (C, f, p_1, \ldots, p_m))\) to \((C, h \cdot f, p_1, \ldots, p_m)\). Let \(o \in U\) be a point. There exists a map \(\gamma: H \times \ev^{-1}_i(\{o\}) \rightarrow \ev^{-1}_i(U)\) given by

\[
H \times \ev^{-1}_i(\{o\}) \rightarrow H \times \ev^{-1}_i(U) \rightarrow \ev^{-1}_i(U),
\]

where the first map is the embedding and the second is \(\beta\). Let us prove that \(\gamma\) is invertible. Since the action of \(H\) on \(U\) is free and transitive, there exists a unique isomorphism of varieties \(g: U \rightarrow H\) such that \(g(x) \cdot x = o\) for all \(x \in U\). Let us
consider the map $((g \circ ev), id) : ev^{-1}_i(U) \to H \times ev^{-1}_j(U)$. The composition of that map with $\beta$ has image contained in $H \times ev^{-1}_j(\{g\})$ and it is the inverse of $f$.

Since $H$ preserves the dual graph of any contact stable map, it induces an action on any $S(\tau)$, in such a way that $ev_1 : S(\tau) \to \mathbb{P}^n$ is Zariski locally trivial by restriction.

**Proposition 4.3.** Let $\tau \in \Gamma(m, d)$. Then $S(\tau) \cong S_{\Box}(\tau)/\Aut(\tau)$ where $S_{\Box}(\tau)$ is the fibred product

\[
S_{\Box}(\tau) \longrightarrow \prod_{v \in V_r} S_{F_r(v)}(\mathbb{P}^n, d_r(v))^o
\]

\[
\bigl(\mathbb{P}^n\bigr)^{E_r, L_r} \longrightarrow \bigl(\mathbb{P}^n\bigr)^{F_r}.
\]

Moreover, $S(\tau)$ and $S_{\Box}(\tau)$ are purely of the same dimension.

**Proof.** By standard properties of fibred product (e.g., [GW20, Proposition 4.16]), there is a double fibred product diagram involving $S_{\Box}(\tau)$

\[
S_{\Box}(\tau) \longrightarrow M_{\Box}(\tau) \longrightarrow \bigl(\mathbb{P}^n\bigr)^{E_r, L_r}
\]

\[
\prod_{v \in V_r} S_{F_r(v)}(\mathbb{P}^n, d_r(v))^o \longrightarrow \prod_{v \in V_r} M_{0, F_r(v)}(\mathbb{P}^n, d_r(v)) \longrightarrow \bigl(\mathbb{P}^n\bigr)^{F_r}.
\]

That is, $S_{\Box}(\tau)$ is embedded in $M_{\Box}(\tau)$ and it is $\Aut(\tau)$-invariant. This induces an action of $\Aut(\tau)$ on $S_{\Box}(\tau)$, in particular $S_{\Box}(\tau)/\Aut(\tau) \subseteq M(\tau)$. As we said in Remark 2.4, a reducible curve is contact if and only if each one of its components is contact in its own. That is, if $(C, f, p_1, \ldots, p_m) \in M(\tau)$ is contact then for every irreducible component $C_i \subseteq C$, the restriction $f_{|C_i} : C_i \to \mathbb{P}^n$ is contact too. Thus both $S_{\Box}(\tau)/\Aut(\tau) \subseteq S(\tau)$ and $S(\tau) \subseteq S_{\Box}(\tau)/\Aut(\tau)$ are true.

The fact that $S_{\Box}(\tau)$ is purely dimensional follows from Proposition 4.2. Indeed, each $S_{F_r(v)}(\mathbb{P}^n, d_r(v))^o$ is purely dimensional, so each fiber $F$ of the locally trivial map $S_{F_r(v)}(\mathbb{P}^n, d_r(v))^o \to \mathbb{P}^n$ is also. It follows that $S_{\Box}$ is locally isomorphic to $F \times U$ where $U$ is an open subset of $(\mathbb{P}^n)^{E_r, L_r}$. Since $\Aut(\tau)$ is finite, $S(\tau)$ is also purely dimensional and $\dim S(\tau) = \dim S_{\Box}(\tau)$.

Even if we use the fact that $S_m(\mathbb{P}^n, d)^o$ is purely dimensional, we do not say that $S_{F_r(v)}(\mathbb{P}^n, d_r(v))^o$ is irreducible. As already noted in the proof of [Mur22, Lemma 3.7], $S_m(\mathbb{P}^n, d)^o$ is a quotient $M_d/\Aut(\mathbb{P}^1)$ where $M_d$ is the moduli space of harmonic maps $\mathbb{C}P^1 \to \mathbb{C}P^3$ we cited in Introduction. The manifold $M_d$ need to be irreducible [Loo99, KL98]. In the same Lemma, it is claimed that

\[
\dim(S_1(\mathbb{P}^n, d_1) \times_{\mathbb{P}^n} S_1(\mathbb{P}^n, d_2)) = \dim S_1(\mathbb{P}^n, d_1) + \dim S_1(\mathbb{P}^n, d_2) - n.
\]

This follows from the fact the action of $Sp(n+1, \mathbb{C})$ on $\mathbb{P}^n$ is transitive and preserves the fibers of $ev_1 : S_1(\mathbb{P}^n, d) \to \mathbb{P}^n$. So $ev_1$ is flat.

We are now ready to state the first important result of the paper.

**Theorem 4.4.** Let $n$ be an odd number and $S_m(\mathbb{P}^n, d)$ be the moduli space of contact stable maps.

1. Every subspace $S(\tau)$ is locally closed in $S_m(\mathbb{P}^n, d)$. 


(2) There exists a stratification of $S_m(\mathbb{P}^n, d)$ given by the union of all $S(\tau)$.

(3) For every $\tau \in \Gamma(m, d)$, the spaces $S(\tau)$ are of pure codimension equal to $|V^0_\tau|$, where $V^0_\tau = \{v \in V_\tau/d_\tau(v) = 0\}$.

In particular there exists a decomposition of $S_m(\mathbb{P}^n, d)$ as union of components $\overline{S(\tau)}$ for $\tau \in \Gamma(m, d)^+$, so that

$$\int_{S_m(\mathbb{P}^n, d)} e^*_\tau(\Gamma_1) \cdots e^*_m(\Gamma_m) = \sum_{\tau \in \Gamma(m, d)^+} \int_{\overline{S(\tau)}} e^*_\tau(\Gamma_1) \cdots e^*_m(\Gamma_m).$$

Proof. The first two points follow from analogous properties of $\overline{M}_{0,m}(\mathbb{P}^n, d)$, so let us prove Point (3). By Proposition 4.3 we have

$$\dim S(\tau) = n(|E_\tau| + |L_\tau| - |F_\tau|) + \sum_{v \in V_\tau} \dim S_{F_\tau(v)}(\mathbb{P}^n, d_\tau(v))^0.$$

Recall that if $d > 0$ (resp., $d = 0$) then $\dim S_m(\mathbb{P}^n, d)^0 = d(n - 1) + n + m - 2$ (resp., $n + m - 3$). Thus the dimension of $S(\tau)$ is, using Eq. (3.1),

$$n(1 - |V_\tau|) + \sum_{v \in V^0_\tau} S_{F_\tau(v)}(\mathbb{P}^n, d_\tau(v))^0 + \sum_{v \in V_\tau \setminus V^0_\tau} S_{F_\tau(v)}(\mathbb{P}^n, d_\tau(v))^0$$

$$= n(1 - |V_\tau|) + n|V_\tau| + |F_\tau| - 3|V^0_\tau| + d(n - 1) - 2(|V_\tau| - |V^0_\tau|)$$

$$= n + |F_\tau| - 2|V_\tau| - |V^0_\tau| + d(n - 1)$$

$$= d(n - 1) + n + m - 2 - |V^0_\tau|$$

$$= \dim S_m(\mathbb{P}^n, d) - |V^0_\tau|.$$

In particular for those $\tau \in \Gamma(m, d)^+$ we have $\dim S(\tau) = \dim S_m(\mathbb{P}^n, d)$. Since $S_m(\mathbb{P}^n, d)$ is purely of the expected dimension, it can be decomposed as union of $\overline{S(\tau)}$ for $\tau \in \Gamma(m, d)^+$. Equation (4.2) follows from linearity of the intersection product [Ful98, Example 1.8.1].

In the next section, we will see how to use the theorem to compute all enumerative invariants of contact curves.

5. Applications

Let us briefly introduce the decomposition of stable trees. Given a stable $(m, d)$-tree $(\tau, l_\tau, d_\tau)$, for every edge $\{f, f'\} = e \in E_\tau$ there exist two unique stable trees $(\sigma, l_\sigma, d_\sigma)$ and $(\sigma', l_{\sigma'}, d_{\sigma'})$ with the following properties:

- there is a partition $F_\tau = F_\sigma \sqcup F_{\sigma'}$ such that $f \in F_\sigma$ and $f' \in F_{\sigma'}$,
- for every $x \in F_\tau \setminus \{f, f'\}$ such that $x \in F_\sigma$ (resp., $F_{\sigma'}$), $j_\sigma(x)$ (resp., $j_{\sigma'}(x)$) is equal to $j_\tau(x)$.
- $f$ and $f'$ are fixed points of, respectively, $j_\sigma$ and $j_{\sigma'}$,
- $V_\sigma = \partial_\tau(F_\sigma)$ and $V_{\sigma'} = \partial_\tau(F_{\sigma'})$, and,
- $d_\sigma$ (resp., $d_{\sigma'}$) is the restriction of $d_\tau$ to $V_\sigma$ (resp., $V_{\sigma'}$).

It is immediate to deduce that $E_\tau \setminus \{e\} = E_\sigma \sqcup E_{\sigma'}$ and $L_\tau \setminus \{f, f'\} = L_\sigma \sqcup L_{\sigma'}$. We will use the following notation: $E^*_\tau = E_\tau \setminus \{e\}$, $L^*_\tau = L_\tau \setminus \{f\}$ and $L^*_{\sigma'} = L_{\sigma'} \setminus \{f'\}$. The maps $l_\sigma$ and $l_{\sigma'}$ are defined in the obvious way.

We will denote by $\tau = \sigma \times \sigma'$ the decomposition of $\tau$ with respect to the edge $e$. Note that every vertex $v$ of $\sigma$ or $\sigma'$ has the same valence as a vertex of $\tau$, so $\sigma$ and $\sigma'$ are always stable.
Corollary 5.1. Let $\tau \in \Gamma(m, d)$ such that $\tau = \sigma \times_{e} \sigma'$ for an edge $e = \{f, f'\}$. Then

1. There exists a projective variety $\overline{S}(\tau)$ and a map $\overline{S}(\tau) \rightarrow S(\tau)$ extending $\overline{S}(\tau) \rightarrow S(\tau)$.

2. There exist isomorphisms

$$\overline{S}(\tau) \cong S(\sigma) \times_{\mathbb{P}^{n}} S(\sigma'), \quad \overline{S}(\tau) \cong S(\sigma) \times_{\mathbb{P}^{n}} S(\sigma').$$

Proof. Let us define $\overline{S}(\tau)$ as the fiber product

$$\overline{S}(\tau) \rightarrow \prod_{v \in V_{e}} S_{F_{v}(e)}(\mathbb{P}^{n}, d_{e}(v))^\circ \rightarrow (\mathbb{P}^{n})_{E_{e} \cup L_{e}} \rightarrow (\mathbb{P}^{n})_{F_{e}}.$$ 

Point (1) follows from Remark 3.3. The image of the natural map $S_{\sigma}(\tau) \hookrightarrow \overline{S}(\tau)$ is dense in every irreducible component of $\overline{S}(\tau)$, and it factorizes the inclusion $S_{\sigma}(\tau) \hookrightarrow \mathcal{M}(\tau)$. It follows that the restriction of $\mathcal{M}(\tau) \rightarrow \mathcal{M}_{0,m}(\mathbb{P}^{n}, d)$ to $S_{\sigma}(\tau)$ maps to $S(\tau)$.

Let us prove Point (2). The isomorphism $V_{e} \cong V_{e} \cup V_{e}'$ preserves the valence of each vertex $v \in V_{e}$. So there exists an isomorphism

$$X := \prod_{v \in V_{e}} S_{F_{v}(e)}(\mathbb{P}^{n}, d_{e}(v))^\circ \cong \prod_{v \in V_{e}} S_{F_{v}(e)}(\mathbb{P}^{n}, d_{e}(v))^\circ \times \prod_{v \in V_{e}} S_{F_{v}(e)}(\mathbb{P}^{n}, d_{e}(v))^\circ.$$ 

Using the equality $F_{e} = F_{e}' \cup F_{e}'$, it follows that we have a diagram

$$S(\tau) \rightarrow S(\sigma) \times_{\mathbb{P}^{n}} S(\sigma') \rightarrow X.$$ 

The right-hand square is a fibred product diagram, it is obtained by (2) applied to $\sigma$ and $\sigma'$. By applying again [GW20] Proposition 4.16, we deduce that also the left-hand square is a fibred product diagram. The immersion $(\mathbb{P}^{n})_{e} \rightarrow (\mathbb{P}^{n})_{e}' \times (\mathbb{P}^{n})_{e}'$ is the diagonal, hence $S_{\sigma}(\tau) \cong S_{\sigma}(\sigma) \times_{\mathbb{P}^{n}} S_{\sigma}(\sigma')$. The same proof applies equally well in $\overline{S}(\tau) \cong S(\sigma) \times_{\mathbb{P}^{n}} S(\sigma')$. \hfill \square

The next corollary allows us to compute intersections in $\overline{S}(\tau)$ recursively.

Corollary 5.2. Let $\tau \in \Gamma(m, d)$ such that $\tau = \sigma \times_{e} \sigma'$ for an edge $e = \{f, f'\}$. Then

$$\int_{\overline{S}(\tau)} \text{ev}_{1}^{t}(\Gamma_{1}) \cdots \text{ev}_{m}^{t}(\Gamma_{m})$$

is equal to

$$\frac{|\text{Aut}(\sigma)||\text{Aut}(\sigma')|}{|\text{Aut}(\tau)|} \sum_{j=0}^{n} \int_{S(\sigma)} \prod_{i \in L_{\sigma}} \text{ev}_{i}^{t}(\Gamma_{i}) \cdot \text{ev}_{j}^{t}(H^{n-j}) \int_{S(\sigma') \prod_{i \in L_{\sigma'}}} \text{ev}_{i}^{t}(\Gamma_{i}) \cdot \text{ev}_{j}^{t}(H^{n-j}).$$
In particular, if $\sum_{\Gamma_1}^{\Gamma_m}(\Gamma_1) \cdot \cdots \cdot \Gamma_m(\Gamma_m) = \frac{1}{|\text{Aut}(\tau)|} \int_{\overline{\mathcal{S}(\tau)}} \text{ev}_1^* \cdot \cdots \cdot \text{ev}_m^*(\Gamma_m)$.

Let us denote again by $H$ the generator of $H^2(P^n, \mathbb{Z})$. The class of the diagonal of $P^n \times P^n$ is $\sum_{j=0}^n H^{n-j} \boxtimes H^j$, so using again the corollary:

$$\sum_{j=0}^n \int_{S(\sigma)} \prod_{i \in L_{\sigma}} \text{ev}^*_i(\Gamma_1) \cdot \text{ev}^*_j(H^{n-j}) \int_{S(\sigma)} \prod_{i \in L_{\sigma}} \text{ev}^*_i(\Gamma_1) \cdot \text{ev}^*_j(H^{n-j})$$

$$= \frac{|\text{Aut}(\tau)|}{|\text{Aut}(\sigma)|-1} \int_{\Sigma(\tau)} \prod_{i \in L_{\sigma}} \text{ev}^*_i(\Gamma_1) \cdot \text{ev}^*_j(H^{n-j}) \int_{\Sigma(\sigma)} \prod_{i \in L_{\sigma}} \text{ev}^*_i(\Gamma_1) \cdot \text{ev}^*_j(H^{n-j}).$$

The last equation implies (5.2).

**Corollary 5.3.** Let $\tau \in \Gamma(m, d)$ and let $\Gamma_1, \ldots, \Gamma_m$ be subvarieties of $P^n$ in general position such that

$$(5.3) \sum_{i=1}^m \text{codim}(\Gamma_i) = \dim \mathcal{S}(\tau).$$

Then the number of contact stable maps in $P^n$ of degree $d$ whose images meet all $\Gamma_1, \ldots, \Gamma_m$, and with dual graph $\tau$, is

$$\int_{\mathcal{S}(\tau)} \text{ev}_1^* \cdot \cdots \cdot \text{ev}_m^*(\Gamma_m).$$

In particular, if $\sum_{i=1}^m \text{codim}(\Gamma_i) = d(n-1) + n + m - 2$, the number of irreducible contact curves in $P^n$ of degree $d$ meeting all $\Gamma_1, \ldots, \Gamma_m$ is

$$\int_{\mathcal{S}(\tau)} \text{ev}_1^* \cdot \cdots \cdot \text{ev}_m^*(\Gamma_m) - \sum_{\tau \in \Gamma(m, d) \setminus \{\tau_0\}} \int_{\mathcal{S}(\tau)} \text{ev}_1^* \cdot \cdots \cdot \text{ev}_m^*(\Gamma_m).$$

**Proof.** This proof follows the same line of [Mur22, Theorem 3.8]. Let us give a sketch. Let $G$ be the group of invertible complex matrices of order $n + 1$, and let $X = P^n$. Consider the diagram

$$\begin{align*}
\overline{\mathcal{S}(\tau)} \\
\Gamma_1 \times \cdots \times \Gamma_m =: \Gamma \arrow{e} X^m
\end{align*}$$

By applying the Kleiman-Bertini Theorem [Kle74], we deduce that for a general $\sigma \in G^m, \Gamma_1 \times X \in \overline{\mathcal{S}(\tau)}$ is either empty, or of dimension 0. We assume, without
loss of generality, that $\Gamma^s$ is smooth by applying again Kleiman-Bertini. In the same way we may prove\textsuperscript{1} that the intersection
\[
\overline{S(\tau)} \cap \ev_1^{-1}(g_1 \Gamma_1) \cap \cdots \cap \ev_m^{-1}(g_m \Gamma_m)
\]
is supported in the intersection between the automorphisms-free part of $M_{0,m}(X,d)$ and the smooth part of $S(\tau)$. So, when $\Gamma^s \times_{X \times \cdots} \overline{S(\tau)}$ is not empty, it is reduced of dimension 0. Points of that intersection represent maps whose images meet all $\Gamma_1, \ldots, \Gamma_m$, and it is equal to (5.4).

Finally, $\dim S(\tau) = d(n - 1) + n + m - 2$ implies that $\tau$ has no contracted components by Theorem 4.4. So if we are interested in a unique non-contracted component, it must be $\tau = \tau_v$. Equation (5.5) follows from (4.2). □

Remark 5.4. The last results imply that for every $\tau \in \Gamma(m,d)$, we can compute every integral like in (5.4). Indeed, if $d = 1$ then $\overline{S(\tau)} = S_m(\mathbb{P}^n, 1)$ and we can use the package AtiyahBott of [MS22]. If $d > 1$, then Eqs. (5.2) and (5.5) allow us to reduce the computation to intersections in $S_m(\mathbb{P}^n, d)$ and $S(\tau')$, for some $\tau' \in \Gamma(m', d')$ where $m' \in \mathbb{Z}$ and $d' < d$. The result follows by using again the package and recursion on $d$.

Let us see an example of these applications.

Example 5.5. Suppose we want to compute the number of irreducible contact curves of degree 2 in $\mathbb{P}^3$ passing through two points and a line in general position. Let $H$ be the class of a hyperplane in $\mathbb{P}^3$. The number of contact curves is given by
\[
\int_{S_3(\mathbb{P}^3, 2)} \ev_1^*(H^2) \cdot \ev_2^*(H^3) \cdot \ev_3^*(H^3) = 2.
\]
The set $\Gamma(3,2)^+$ consists of the following stable trees:
\[
\begin{array}{cccccccc}
\tau_1 & \tau_2 & \tau_3 & \tau_4 & \tau_5 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{1,2,3\} & \{1,2\} & \{3\} & \{1,3\} & \{2\} & \{2,3\} & \{1\} & \{1,2,3\}
\end{array}
\]

The vertices are labeled by their degree, under any vertex $v$ we put its leaves. We want to compute $\int_{S(\tau_2)} \ev_1^*(H^2) \cdot \ev_2^*(H^3) \cdot \ev_3^*(H^3)$. Note that $\tau_2 = \sigma \times_e \sigma'$ where
\[
\begin{array}{c}
\sigma \\
1 \\
\{1,2,f\}
\end{array}
\quad
\begin{array}{c}
\sigma' \\
1 \\
\{3,f'\}
\end{array}
\]
Using Corollary 5.2 and Remark 5.4 we get
\[
\begin{align*}
\int_{S(\tau_2)} & \ev_1^*(H^2) \cdot \ev_2^*(H^3) \cdot \ev_3^*(H^3) \\
= & \sum_{j=0}^3 \int_{S(\sigma)} \ev_1^*(H^2) \cdot \ev_2^*(H^3) \cdot \ev_3^*(H^{3-j}) \int_{S(\sigma')} \ev_3^*(H^j) \cdot \ev_{j'}(H^1) \\
= & \int_{S(\sigma)} \ev_1^*(H^2) \cdot \ev_2^*(H^3) \cdot \ev_f^*(H^1) \int_{S(\sigma')} \ev_3^*(H^3) \cdot \ev_{j'}(H^2) \\
= & 1 \cdot 1 = 1.
\end{align*}
\]
\textsuperscript{1}Details can be found in [Alu97, Lemma p. 5] or [FP97, Lemma 14].
The computation of \( \int_{S(\tau)} \ev^*_\tau(H^2) \cdot \ev^*_\tau(H^3) \cdot \ev^*_\tau(H^3) = 1 \) is similar. Finally, one can easily figure out that \( \int_{S(\tau)} \ev^*_\tau(H^2) \cdot \ev^*_\tau(H^3) \cdot \ev^*_\tau(H^3) = 0 \) for \( \tau \in \{\tau_4, \tau_5\} \). It follows that

\[
\int_{S(\tau)} \ev^*_\tau(H^3) \cdot \ev^*_\tau(H^3) \cdot \ev^*_\tau(H^2) = 2 - 1 - 1 - 0 - 0 = 0.
\]

So, there are no irreducible contact conics through two points and a line in general position. This result is completely expected since there are no irreducible contact conics in \( \mathbb{P}^3 \) \( \left( \text{LM07, Proposition 17.1} \right) \). Anyway, this shows that our method works.

Remark 5.6. If the value of (5.2) is non-zero, then \( |\text{Aut}(\tau)| = 1 \). Indeed, if \( \sigma' = \tau_v \) where \( v \) is an extremal vertex with no leaves, then Eq. (5.2) becomes

\[
\frac{|\text{Aut}(\tau)|}{|\text{Aut}(\tau)|} \sum_{j=0}^{n} \int_{S(\tau)} \prod_{i \in L_v} \ev^*_\tau(\Gamma_i) \cdot \ev^*_\tau(H^{n-j}) \int_{S(\tau)} \ev^*_\tau(H^j).
\]

The value of \( \int_{S(\tau)} \ev^*_\tau(H^j) \) is zero for dimensional reasons, meaning that also (5.2) is zero. This implies that every vertex of \( \tau \) with only one edge must have leaves. Since any automorphism preserves the leaves, all vertices with only one edge must be fixed by any automorphism, hence the identity is the only possible automorphism. Moreover, the sum (5.2) is non zero for at most one value of \( j \), which is \( j_0 = \dim S(\tau_v) - \sum_{i \in L_v} \text{codim}(\Gamma_i) \), if \( 0 \leq j_0 \leq n \).

The following result is \( \left( \text{LV11, Proposition 2.3(i)} \right) \). We provide another proof.

**Proposition 5.7.** The number of contact plane curves of degree \( d \) meeting \( d + 3 \) general lines in \( \mathbb{P}^3 \) is

\[
\frac{d^2}{6} (d + 3)(d + 2)(d + 1)(d - 1) = 20d \binom{d + 3}{5}.
\]

**Proof.** We give a sketch of the proof, showing that it can be proved using graphs. A contact plane curve is linearly degenerate in \( \mathbb{P}^3 \), then it is cone \( \left( \text{LM07, Proposition 17.1} \right) \). On the other hand, cones in \( \mathbb{P}^3 \) are contact if they are linearly degenerate. The cases \( d = 1, 2 \) are trivial, so let us focus on the case \( d \geq 3 \). We are looking for curves given by theunion of \( d \) contact lines, all of them passing through a fixed point. Let \( X \) be one of those contact curves. Any stable map \( (C, f, p_1, \ldots, p_m) \) with dual graph \( \tau \) whose image is \( X \) must have \( m = d + 3 \) marked points (since we are imposing \( d + 3 \) Schubert conditions). By condition (5.3), the number of contracted vertices of degree 0 of \( \tau \) is \( d - 2 \). The inverse image under \( f \) of the singular point of \( X \) must be connected. Hence, the degree 0 vertices form a connected subtree. Since we are not imposing any condition on the singular point, no leaf is attached to a degree 0 vertex. So all leaves lie on \( d \) non-contracted components of degree 1.

The dual graph \( \tau \) with these properties is not univoquely determinated. An example is \( \tau_1 \) in Figure 5.1 (we use the same notation of Example 5.5).

Equation (5.2) applied to any of these stable trees is equal to 4 (e.g., by induction on \( d \)), so there are exactly 4 maps having \( \tau_1 \) as dual graph \( \left( \text{Corollary 5.3} \right) \). The image of \( \tau_1 \) only depends on the vertices with 2 and 3 leaves. There are exactly \( \binom{d + 3}{3} \binom{d}{2} \) different dual graphs with the same support as \( \tau_1 \), but with different leaves in those vertices.

\(^2\text{Otherwise, the symplectic form defining the contact structure in } \mathbb{C}^4 \text{ would be degenerate.}\)
If we consider another stable tree $\tau'_1$ in Figure 5.2, this gives different stable maps but with the same images as $\tau_1$.

Figure 5.1. Stable tree $\tau_1$.

Since we are interested in the contact stable curves rather than in the stable maps, we can ignore all other stable trees if they give the same image. There is another family of graphs interesting to us, it is $\tau_2$ in Figure 5.3.

Figure 5.2. Stable tree $\tau'_1$.

Figure 5.3. Stable tree $\tau_2$.

Arguing like before, we get $\frac{1}{3!} \binom{d + 3}{2} \binom{d + 1}{2} \binom{d - 1}{2}$ different dual graphs with different images, each one with a contribution equal to 8. It is easy to see that there are no more interesting stable trees, for several reasons. Each non-contracted component must have at least one leaf by Remark 5.6, and no more than three leaves (there are no contact lines through 4 general lines). Each degree zero vertex $v$ must have valence $n(v) = 3$ by Eq. (3.4). Moreover, it is not important the stable tree itself but how the $d + 3$ leaves are distributed among the $d$ vertices of degree 1. Hence $\tau_1$ and $\tau_2$ are the only interesting configurations. It follows that the number $N$ of curves we are looking for is

$$\binom{d + 3}{3} \binom{d}{2} \int_{S(\tau_1)} \prod_{i=1}^{d+3} \text{ev}^*_i(H^2) + \frac{1}{3!} \binom{d + 3}{2} \binom{d + 1}{2} \binom{d - 1}{2} \int_{S(\tau_2)} \prod_{i=1}^{d+3} \text{ev}^*_i(H^2).$$

Using

$$\int_{S(\tau_1)} \prod_{i=1}^{d+3} \text{ev}^*_i(H^2) = 4, \quad \int_{S(\tau_2)} \prod_{i=1}^{d+3} \text{ev}^*_i(H^2) = 8,$$

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we get

\[
N = \binom{d+3}{3} \binom{d}{2} \frac{1}{3!} \binom{d+3}{2} \binom{d+1}{2} \binom{d-1}{2} 8
= 20d \binom{d+3}{5},
\]

as expected. \qed

5.1. **Number of irreducible contact curves.** We denote by \(N^{irr}_d(a_2, \ldots, a_n)\) the number of irreducible contact curves of degree \(d\) meeting \(a_i\) general linear subspaces of codimension \(i\) in \(\mathbb{P}^n\). As application of Corollary 5.3, we give tables containing such numbers for \(n = 3, 5\) and low values of \(d\). We subtract from the number of contact curves given in [Mur22, MS22], the number of reducible ones. Using Julia ([BEKS17]), we implemented Eq. (5.2) in [Mur]. Csaba Schneider wrote the part based on the algorithm for generation of trees of [WROM86].

**Remark 5.8.** The number \(N^{irr}_3(1,3) = 3\) was first computed by [Kal18]. The number \(N^{irr}_3(7,0) = 1080\) was first computed by [Amo14] using hand-done computations of reducibles contact curves of degree 3. He also claimed, using the same strategy, that \(N^{irr}_4(9,0) = 378944\). Unfortunately, his computation is not complete.

| \(d\) | \(a = (a_2, a_3)\) | \(N^{irr}_3(a)\) | \(d\) | \(a = (a_2, a_3)\) | \(N^{irr}_3(a)\) | \(d\) | \(a = (a_2, a_3)\) | \(N^{irr}_3(a)\) |
|------|----------------|----------------|------|----------------|----------------|------|----------------|----------------|
| 3    | (7, 0)        | 1080           | 4    | (9, 0)        | 145664         | 5    | (11, 0)       | 65619360       |
|      | (5, 1)        | 132            |      | (7, 1)        | 12800          |      | (9, 1)        | 4501008        |
|      | (3, 2)        | 18             |      | (5, 2)        | 1216           |      | (7, 2)        | 328824         |
|      | (1, 3)        | 3              |      | (3, 3)        | 128            |      | (5, 3)        | 25884          |
|      | (1, 4)        | 16             |      | (1, 4)        | 16             |      | (3, 4)        | 2250           |
|      | (1, 5)        | 225            |      | (1, 5)        | 225            |      |                |                |

**Table 5.1.** Irreducible contact curves in \(\mathbb{P}^3\) of degree \(d \leq 5\).

| \(a = (a_2, a_3, a_4, a_5)\) | \(N^{irr}_2(a)\) | \(a = (a_2, a_3, a_4, a_5)\) | \(N^{irr}_2(a)\) | \(a = (a_2, a_3, a_4, a_5)\) | \(N^{irr}_2(a)\) |
|-------------------------------|----------------|-------------------------------|----------------|-------------------------------|----------------|
| (11, 0, 0, 0)                | 27184          | (4, 2, 1, 0)                 | 100            | (1, 5, 0, 0)                 | 48             |
| (9, 1, 0, 0)                 | 7554           | (4, 0, 1, 1)                 | 30             | (1, 3, 0, 1)                 | 2              |
| (8, 0, 1, 0)                 | 1262           | (3, 4, 0, 0)                 | 168            | (1, 2, 2, 0)                 | 4              |
| (7, 2, 0, 0)                 | 2112           | (3, 2, 0, 1)                 | 22             | (1, 1, 0, 2)                 | 0              |
| (7, 0, 1, 0)                 | 432            | (3, 1, 2, 0)                 | 16             | (1, 0, 2, 1)                 | 0              |
| (6, 1, 1, 0)                 | 355            | (3, 0, 0, 2)                 | 8              | (0, 4, 1, 0)                 | 8              |
| (5, 3, 0, 0)                 | 594            | (2, 3, 1, 0)                 | 28             | (0, 2, 1, 1)                 | 0              |
| (5, 1, 0, 1)                 | 119            | (2, 1, 1, 1)                 | 3              | (0, 1, 3, 0)                 | 0              |
| (5, 0, 2, 0)                 | 58             | (2, 0, 3, 0)                 | 2              | (0, 0, 1, 2)                 | 0              |

**Table 5.2.** Irreducible contact conics in \(\mathbb{P}^5\).
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