Finite width of the sonic event horizon and grey body Hawking radiation.

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Abstract

Finite width of the analog event horizon is determined by the nonlinearity length in the Kerr nonlinear optical system, which is discussed here, or by the healing length in Bose-Einstein condensates. The various eigen modes of fluctuations are found in the immediate vicinity of the event horizon and the scattering matrix due to the finite width horizon is calculated to within the leading order corrections in the nonlinearity length. The Hawking radiation is found to be that of a grey body with the emissivity larger than one. A procedure of paraxial quantization of the fluctuation field is discussed and its connection to the conventional quantization of the electromagnetic field is demonstrated.

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I. INTRODUCTION

Analogue gravity modelled in various laboratory systems is a rapidly developing field both from theoretical and experimental points of view. The seminal paper by Unruh\(^1\) proposed a transonic flow of a barotropic isentropic fluid as a simulator of a black hole event horizon. The prediction was that a radiation analogous to the celebrated Hawking radiation\(^2,3\) could be observed in such a system. Since then a number of various systems were proposed as playgrounds for simulating the event horizon of black hole\(^4–12\). There is also an important progress achieved in experiment, e.g. a white-hole horizon, created by a moving soliton, was observed in optical fibers\(^13\). An observation of analog Hawking radiation in optical fibers was reported\(^14\). A black-hole horizon was observed in a Bose-Einstein condensate (BEC) system\(^15\), and quite recently an observation of laser type amplification of Hawking radiation\(^16\) has been reported. A ”horizon physics” is studied also in the surface water waves\(^17–19\). A possibility of creating an event horizon in the coherent light propagating in the Kerr nonlinear defocusing medium was discussed in Refs.\(^11,20,21\). Reviews on the progress in the field and relevant topics can be found in recent papers\(^22–24\).

One of the intriguing questions, which is common for general relativity (GR) black holes and analogue gravity models is the behavior of the radiation in the immediate vicinity of the event horizon. The wave length becomes of the order of Planck length in GR and healing length in BEC or nonlinearity length in the optical analogues of the horizon. The issue was addressed in Refs.\(^25–27\). Quanti-
tative results in GR were obtained by intro-
ducing a sub- or superluminal deviation of the
otherwise linear spectrum of massless particles,
which may happen on the Planckian (healing
length) scale. A review can be found in
Ref.\textsuperscript{40}

Analysis of the role of the quantum poten-
tial in Ref.\textsuperscript{41} shows that the behavior of fluc-
tuations becomes regular near the event hori-
zon on the scale $l_r$, which is somewhat larger
than the healing length. A similar regulariza-
tion length appears in the numerical study\textsuperscript{39}
(see also discussion of the various length scales
appearing in the problem in the GR context,
Ref.\textsuperscript{42}).

The approach outlined in Ref.\textsuperscript{41} will allow
us to find explicit formulas for all eigenmodes
for fluctuations near the event horizon. Two
of these eigenmodes are evanescent in the sub-
sonic flow but become real after a certain criti-
cal distance from the horizon in the supersonic
region of the flow.

Considering Hawking radiation in an all-
optical configuration\textsuperscript{11,20,21} we have to address
several issues, some of which are also relevant
to other experimental setups. The most impor-
tant one is the finite width of the event horizon
(see, e.g. Ref.\textsuperscript{42}), due to the processes taking
place on the scale of the nonlinearity length.
They determine formation of the six types of fluctuation — two positive frequency modes in
the subsonic region and four positive and nega-
tive frequency modes in the supersonic region.

Calculation of the scattering matrix and spec-
trum of the Hawking radiation for finite val-
ues of the nonlinearity length becomes then
straightforward. (Scattering problem in BEC
was considered for several specific density and
interaction coefficient shapes.\textsuperscript{9,43,44})

Another important issue is the quantization
of fluctuations. Analysis of the analogue Hawk-
ing radiation in an all-optical setup is based on
the nonlinear Schrödinger equation (NLS)

$$i \partial_z A = -\frac{1}{2\beta_0} \nabla^2 A + g|A|^2 A$$ \hspace{1cm} (1)

deduced from the classical Maxwell equations
in the paraxial approximation. Here $A$ is the
amplitude of the electric field, the wave vec-
tor $\beta_0$ of the light plays the role analogous to
the mass of a "quantum particle". The prop-
gagation distance $z$ is now "time". The Lapla-
cian now contains derivatives with respect to
the coordinates $x$, $y$ and real time $t$. Since the
part of time is played now by the propagation
distance $z$ the quantization in this approxima-
tion, i.e. introduction of "paraxial photons"
becomes rather tricky. A general approach to
the paraxial quantization was proposed in Refs.
\textsuperscript{45,47}, which allows one to connect the stan-
dard quantization of the electromagnetic field
to the paraxial photons. Paraxial quantization
of the fluctuations near the all-optical event
horizon will be carried out below.
II. FLUCTUATIONS NEAR THE EVENT HORIZON IN A LUMINOUS FLUID

The propagation of coherent light in a Kerr nonlinear medium in the paraxial approximation can be mapped on a flow of an equivalent luminous fluid. Madelung transformation $A = f e^{-i\varphi}$ allows one to represent the NLS equation (1) in the form of two hydrodynamic equations for the density $\rho(r,z) = \beta_0 f_0^2(r,z)$, which is in fact the light intensity, and velocity $v(r,z) = -\frac{1}{\beta_0} \nabla \varphi(r,z)$.

We will consider here small fluctuation of the amplitude $\delta A = A - A_0$ with respect to a stationary solution $A_0 = f_0 e^{-i\varphi_0}$. Their dynamics is described by the equations

\begin{align}
\hat{D} \chi - \frac{1}{\beta_0} \frac{1}{f_0^2} \nabla (f_0^2 \nabla \xi) &= 0 \quad (2) \\
\hat{D} \xi + \frac{1}{4\beta_0} \frac{1}{f_0^2} \nabla (f_0^2 \nabla \chi) - g f_0^2 \chi &= 0 \quad (3)
\end{align}

which are obtained by linearizing Eq. (1) around the stationary solution (see, e.g. Refs. 41,52). Here $\hat{D} = \partial_z + v_0 \cdot \nabla$ and

\begin{align}
\chi &= \frac{1}{f_0} [e^{-i\varphi_0} \delta A^* + e^{i\varphi_0} \delta A] \\
\xi &= \frac{1}{2i f_0} [e^{-i\varphi_0} \delta A^* - e^{i\varphi_0} \delta A],
\end{align}

are classical scalar fields describing fluctuations of the amplitude and phase, respectively. Although the functions $\chi$ and $\xi$ are explicitly real, we will consider below the general properties of the complex solutions of the linear equations (2) and (3). However when calculating the physically measurable quantities only the real part of the functions should be considered.

We now assume that the stationary solution behaves as $\rho_0 = \beta_0 f_0^2(x) = \frac{\beta_0 \pi}{g} (1 - \alpha x)$ and $v_0(x) = \bar{s}(1 + \alpha x)$ with a parameter $\alpha$. Here $\bar{s}$ is the sound velocity of the luminous fluid at $x = 0$, and $x$ is the distance from the horizon surface along the streamline normal to it. This approximation holds at $\alpha x \ll 1$.

It is sufficient to limit the discussion to $1 + 1$ dimensions. Then following the derivation outlined in Refs. 21,41 we get the solutions of the equations (2) and (3) as integrals with the properly chosen integration contours:

\begin{align}
\chi(x,z) &= \int dv e^{-i\nu z} \int_C dk k^{\gamma_1} \left( k - \frac{2}{3} \nu - \frac{i}{3} \alpha \right)^{\gamma_2} \exp \{A(k,\nu) + ikx\}. \quad (5)
\end{align}

where

$$\gamma_1 = \frac{1}{4} - \frac{i\nu}{2\alpha},$$

and the $l_n$ dependent part is given by
\[ \Lambda(k, \nu) = \frac{l_n^2}{\alpha} \left\{ -\frac{i}{18} k^3 + \frac{5}{36} \alpha k^2 - \frac{i}{18} \nu k^2 - \frac{2i}{27} \nu^2 k + \frac{4}{27} \nu \alpha k \right\}. \]  \tag{6}

\[ l_n^2 = \frac{1}{2 \beta_0 s^2} \] is the nonlinearity length.

Integral (5) can be also calculated by means of the steepest descent technique. For this we first have to find the saddle points determined by the equation

\[ \nu - kv(x) = \pm \Omega(k, \nu, x) \]  \tag{7}

where

\[ \Omega^2(k, \nu, x) = \frac{l_n^2}{2} (i \alpha k + k^2)^2 s^2 + k^2 s^2(x) \]  \tag{8}

obtained in Ref. [41].

Neglecting the small \( \alpha/k \ll 1 \) corrections in the quartic term, we have an equation that looks exactly as the Bogolubov dispersion relation for the above condensate excitations in the moving frame. The important difference, however, is that both the sound velocity and the flow velocity depend on the coordinate and the solutions may change drastically when crossing the event horizon (at \( x = 0 \)). The conventional Bogolubov spectrum of excitations in NLS equation is obtained under the condition that the amplitude \( f_0 \) and velocity \( v_0 \) are constants. In the context of our problem these conditions may be fulfilled at large distances from the sonic horizon, whereas Eq. (7) holds in its immediate vicinity. It allows one to follow the evolution and interconnection of the eigen functions, when passing from the subsonic region \( (x < l_r) \) via regularization region \( |x| < l_r \) to the supersonic region \( x > l_r \). Here \( l_r = l_n/(\alpha l_n)^{1/3} \) is the regularization length.

Graphical solutions of Eq. (7) are shown in Figure 1. In principle, \( \Omega(k, \nu, x) \) also varies with \( x \) but we ignore it in the graph in order not to overload it. In the analytical calculations this dependence is certainly taken into account. Three blue circles show the solution with the saddle point at \( k_r \approx \nu/2 \pi \), which we call regular. It is obtained under the assumption that \( k_r l_n \ll 1 \) and corresponds to the

![Graphical analysis of equation (7). Two branches of the function \( \Omega(k, \nu, x) \) are plotted. They are crossed by the straight dashed lines \( \nu - kv(x) \) for three positions of an "observer": \( x < 0 \) — subsonic regime, when \( v < s \), \( x = 0 \), at the horizon, when \( v = s \); and \( x > 0 \) — supersonic regime, when \( v > s \).]
fluctuation \( \chi_r = e^{-i\nu(z-x)} \) propagating downstream with the double sound velocity. This solution changes only slightly when moving from subsonic to supersonic region.

The red squares show the evolution of the solution corresponding to the saddle point \( k_s = 2\nu/3\pi\alpha x \) in the subsonic region \( (x < -l_r) \) also obtained under the condition \( k_r l_n \ll 1 \), i.e. \( |x|/l_n \gg 2\nu/3\pi\alpha \). When moving closer to the horizon and crossing it (see figure [1]) this solution moves towards large \( k \) values where the limit \( kl_n \gg 1 \) should be taken. Then equation (7) becomes

\[
\frac{l_r^2}{2}k^3 - 3k\alpha x + 2\nu = 0 \tag{9}
\]

This equation has three solutions one of which is real and the other two become real only at

\[
x \geq x_c = \left( \frac{2\nu^2}{3\alpha^2} \right)^{1/3} \frac{l_r}{2^{1/3}} \left( \frac{\nu}{\alpha} \right),
\]

with the corresponding wave vector \( k_c = \left( \frac{2\nu}{3\alpha} \right)^{1/3} \). These two modes appear due to bifurcation in the lower half plane in Fig. 1 when the straight line representing the l.h.s. of Eq. (7) touches the curve \( -\Omega(k, \nu, x) \). This bifurcation point coincides to within a numerical factor with the turning point found in Ref. 48.

The real solution of (9) is \( k_{1h} = (4\nu/l_n^2)^{1/3} \) within the width of the horizon at \( |x| \ll x_c \) and \( k_{e1} = (6\alpha x)^{1/2}/l_n \) outside at \( x \gg x_c \). The corresponding eigenfunction describes a mode, which propagates upstream. That is why its character changes drastically when moving from the subsonic to supersonic region, from the singular function \( \chi_{s1} = e^{-i\nu x}x^{\gamma-1} \) to \( \chi_{1h} = e^{-i\nu x-i(4\nu/l_n^2)^{1/3}x} \) at \( |x| \ll x_c \) to \( \chi_{e1} = e^{-i\nu x-i(2\nu/3\alpha x^{3/2})} \). Here \( \gamma = -\gamma_1 - \gamma_2 \).

The eigenfunction \( \chi_{e1} \) appears only due to the quartic term in the fluctuation spectrum in Eq. (7) and is related to the evanescent solution in the subsonic region. The other two eigenfunctions appear due to bifurcation in the lower half plane in Fig. 1 which takes place at \( x = x_c \). The two emerging saddle points, one moving towards smaller \( k \) (\( k^3 \) can be neglected in (9)) and the other one towards large \( k \) (free term can be neglected) produce two eigenfunctions: the singular \( \chi_{s2} = e^{-i\nu x}x^{\gamma-1} \) and the one related to the second evanescent function \( \chi_{e2} = e^{-i\nu x+i(2\nu/3\alpha x^{3/2})} \). Both functions \( \chi_{e1} \) and \( \chi_{e2} \) exist as propagating waves only in the supersonic region, otherwise they become evanescent (see also discussion in Ref. 44).
III. SCATTERING MATRIX.

Now we are in a position to calculate the scattering matrix for an event horizon of a small but finite width. First we have to make use of the coordinates

$$x \rightarrow \tilde{x} = x$$

$$z \rightarrow z + \int \frac{v_0(x)dx}{s^2(x) - v_0^2(x)} \approx z - \frac{\ln(x)}{3\alpha s}$$

This coordinate transformation was used in Ref. [1] in order to diagonalize the metric and represent it in the conventional Schwarzschild form. Then the density and current flow corresponding to the canonical pair of fields $\chi$ and $\xi$ take the form [11]

$$\varrho = -i \frac{s^2(\tilde{x})}{s^2(\tilde{x}) - v_0^2(\tilde{x})}[\partial_\tilde{x}^*\xi - \xi^*(\partial_\tilde{x}\xi)]$$

$$j = -i[v_0^2(\tilde{x}) - s^2(\tilde{x})][\partial_\tilde{x}^*\xi - \xi^*(\partial_\tilde{x}\xi)]$$

where the relation $\chi = \frac{1}{\alpha} \partial_\tilde{x} \xi$, holding outside the regularization region ($|x| \gg l_r$), has been used.

The scattering matrix $S$ transforms three incoming waves $(r1, r2, e2)$, of which the two last waves are negative frequency waves, into three outgoing waves $(s1, s2, e1)$. Hence the unitarity condition $S^† US = U$ is defined with $U = \text{diag}(1, -1, -1)$. The balance of the incoming and outgoing currents reads

$$j_{s1} = |S_{11}|^2 j_{r1} - |S_{12}|^2 j_{r2} - |S_{13}|^2 j_{e2}$$

$$j_{s2} = |S_{21}|^2 j_{r1} - |S_{22}|^2 j_{r2} - |S_{23}|^2 j_{e2}$$

$$i_{e1} = |S_{31}|^2 j_{r1} - |S_{32}|^2 j_{r2} - |S_{33}|^2 j_{e2}$$

which holds under the condition that the incoming functions

$$\xi_{r1}(x) = |\tilde{x}| \frac{\alpha}{6} e^{-i\nu \tilde{x}}$$

$$\xi_{r2}(x) = \tilde{x} \frac{\alpha}{6} e^{-i\nu \tilde{x}}$$

$$\xi_{e2}(x) = \sqrt{\frac{2\alpha \nu}{(6\alpha x)^{3/2}}} \tilde{x}^{-\frac{\alpha}{6}} e^{-i\nu \tilde{x} + i\frac{\alpha}{3\alpha} x^{3/2}}$$

and outgoing functions

$$\xi_{s1}(x) = |\tilde{x}| \frac{\alpha}{6} e^{-i\nu \tilde{x}}$$

$$\xi_{s2}(x) = \tilde{x} \frac{\alpha}{6} e^{-i\nu \tilde{x}}$$

$$\xi_{e1}(x) = \sqrt{\frac{2\alpha \nu}{(6\alpha x)^{3/2}}} \tilde{x}^{-\frac{\alpha}{6}} e^{-i\nu \tilde{x} - i\frac{\alpha}{3\alpha} x^{3/2}}$$

are properly normalized. Here $\tilde{x}$ gives us a scale where the linear $x$ dependence of the flow velocity holds, $\tilde{x} \approx 1/\alpha$. It corresponds to one of the length scales discussed in Ref. [42]. The normalization cannot be carried out directly, since we know the eigen functions only in a limited part of the space and cannot integrate the density in the whole space. However, we can find a relative normalization. So that all the currents in (14) were equal. Then Eqs. (14) become compatible with the unitarity of the scattering matrix. This has allowed us to choose the coefficients in Eqs. (15) and (16), so that this functions are now defined to within a common factor. Similar eigen modes are obtained in Refs. [48, 49] in the GR context. The mode equation in this case differs from our equations (2) and (3) and produce eigen modes which differ from those obtained in the previous section. They become similar only after the transformation (10) and (11).
The scattering matrix has the simple form

\[ S^{(0)} = \begin{pmatrix} \alpha_B & \beta_B & 0 \\ \beta_B & \alpha_B & 0 \\ 0 & 0 & 1 \end{pmatrix} \]  \tag{17}

in the limit \( l_n \to 0 \). Here the condition for the Bogolubov coefficients \( \alpha_B^2 - \beta_B^2 = 1 \) follows from the unitarity of \( S^{(0)} \) and \( \alpha_B / \beta_B = e^{\pi i \gamma} \) results from the branch point, characteristic of the functions \( \xi_{s1} \) and \( \xi_{s2} \). This S-matrix approach is just another version of the analysis of Hawking radiation as presented in Refs. 50, 51.

Now we will calculate a matrix \( S_{ij} = S_{ij}^{(0)} + S_{ij}^{(1)} + S_{ij}^{(2)} \) with the first and second order corrections due to small but nonzero \( l_n \). The unitarity of this matrix results in six equations

\[
\begin{align*}
|S_{11}|^2 - |S_{12}|^2 - 1 &= |S_{13}|^2 \\
|S_{22}|^2 - |S_{12}|^2 - 1 &= -|S_{23}|^2 \\
|S_{33}|^2 - 1 &= |S_{13}|^2 - |S_{23}|^2 \\
S_{11}S_{12} - S_{21}S_{22} &= S_{31}S_{32} \\
S_{11}S_{13} - S_{21}S_{23} &= S_{31}S_{33} \\
S_{12}S_{13} - S_{22}S_{23} &= S_{32}S_{33}
\end{align*}
\]

(18)

The matrix elements \( S_{13} \) and \( S_{23} \) will be treated as small parameters. The matrix \( S_{ij} \) will be assumed to be real in what follows. The last two equations in (18) in the leading order become

\[
\begin{align*}
(\alpha_B - 1)S_{13} &= \beta_BS_{23} \\
\beta_BS_{13} &= (\alpha_B + 1)S_{23}
\end{align*}
\]

and we get that

\[
\frac{S_{13}}{S_{23}} = e^{\pi i \gamma} + \sqrt{e^{2\pi i \gamma} - 1} = h. \tag{19}
\]

The first order corrections in the fourth equation in (18) must compensate each other, which yields the ratio

\[
\frac{S_{11}^{(1)}}{S_{12}^{(1)}} = \frac{S_{22}^{(1)}}{S_{12}^{(1)}} = \frac{\beta_B}{\alpha_B} = e^{-\pi i \gamma}. \tag{20}
\]

It is also consistent with the first two equations.

The third equation in (18) is solved straightforwardly, \( S_{33}^{(1)} = 0 \), \( S_{33}^{(2)} = S_{23}^2/g \). The remaining first, second and fourth equations, containing only second order corrections are linearly dependent and it is sufficient to consider only two of them,

\[
\begin{align*}
2\alpha S_{11}^{(2)} - 2\beta S_{12}^{(2)} &= |S_{23}|^2 h^2 - S_{11}^{(1)} + S_{12}^{(1)}^2 \\
2\alpha S_{22}^{(2)} - 2\beta S_{12}^{(2)} &= -|S_{23}|^2 - S_{22}^{(1)} + S_{12}^{(1)}^2
\end{align*}
\]

(21)

The determinant of the scattering matrix

\[
\det S = 1 + (S_{11}^{(1)}S_{22}^{(1)} - S_{21}^{(1)}S_{12}^{(1)}) + S_{31}^2(h^2 - 1) + S_{13}^2[2\beta_B h - \alpha_B(h^2 + 1)]
\]

is calculated using the above relations between the matrix elements. Requiring that the determinant equals to one and applying (20) we get

\[
S_{11}^{(1)} = S_{22}^{(1)} = S_{12}^{(1)} = 0.
\]

Now we have to go back to the fifth and sixth equations and consider the higher order terms

\[
\begin{align*}
S_{11}^{(2)}S_{13} - S_{21}^{(2)}S_{23} &= S_{31}S_{33} \\
S_{12}^{(2)}S_{13} - S_{22}^{(2)}S_{23} &= S_{32}S_{33}
\end{align*}
\]

Then using the above relations between the matrix elements and Eqs. (21) we get

\[
\begin{align*}
S_{11}^{(2)} &= \frac{1}{2} h^2 S_{32}^2 \\
S_{22}^{(2)} &= \frac{1}{2} S_{32}^2 \\
S_{12}^{(2)} &= \frac{h}{2} S_{32}^2
\end{align*}
\]

(22)
As a result we get the scattering matrix 

\[ S = \begin{pmatrix} \alpha_B + \frac{1}{2} h S_{32}^2 & \beta_B + \frac{1}{2} h S_{32}^2 & h S_{32} \\ \beta_B + \frac{1}{2} h S_{32}^2 & \alpha_B + \frac{1}{2} S_{32}^2 & S_{32} \\ h S_{32} & S_{32}^2 & 1 + \frac{h}{2} S_{32}^2 \end{pmatrix} \]  

(23)

The fact that the functions \( \xi_{s2} \) and \( \xi_{e2} \) stem from the same bifurcation point where they must coincide, allows us to assume that \( S_{32} \approx \sqrt{2l_n \nu} \propto \sqrt{\ln \nu} \), which is a rather crude estimate, holding to within a numerical factor. However, it may correctly reflect the dependence on \( \ln \nu \) which is of primary importance for us here.

We can now find the spectrum of the Hawking radiation (in the subsonic region)

\[ N_{H<}(\omega) = S_{12}^2 = \frac{g_< e^{\nu/T_H(\nu)}}{e^{\nu/T_H(\nu)} - 1}. \]

This is radiation of a grey body with the emissivity

\[ g_< = 1 + l_n \nu h(\nu) \beta_B(\nu) \]  

(24)

with

\[ \beta_B(\nu) = \frac{1}{\sqrt{e^{\nu/T_H(\nu)} - 1}}, \]
\[ h(\nu) = e^{\nu/2T_H(\nu)} + \sqrt{e^{\nu/T_H(\nu)} - 1} \]

and Hawking temperature

\[ T_H(\nu) = \frac{3h_\alpha}{4\pi k_B} \left[ 1 + \frac{2l_n^2 \nu^2}{27 \nu^2} \right]^{-1} \].

It is important to emphasize a surprising result that the emissivity \( g_< \) is larger than one. It is well known that emissivity of any grey body at equilibrium with the photon gas is always smaller than one. Here, however we deal with an essentially nonequilibrium system. That is why the emissivity may be larger than one. It depends on the wave number \( \nu \) of the emitted photon and increases with it.

The radiation in the supersonic region ("inside the black hole") occurs in two modes: (1) The negative frequency singular mode \( s2 \) radiates with the spectrum

\[ N_{H>}(\omega) = S_{22}^2 = \frac{g_> e^{\nu/T_H(\nu)}}{e^{\nu/T_H(\nu)} - 1} \]

where \( g_> = 1 + l_n \nu \sqrt{1 - e^{-\nu/T_H(\nu)}} \); (2) There is also a weak radiation \( O(l_n \nu) \) due to the \( e1 \) mode.

IV. PARAXIAL QUANTIZATION

The analysis of Hawking radiation carried out above is based on the equations (2) and (3) deduced from the classical Maxwell equations in the paraxial approximation. However, the phenomenon of Hawking radiation is a quantum effect. In this context quantization of the fluctuations becomes an important issue. As was shown in Ref. 21, equations for fluctuations are Euler- Lagrange equations produced by the Lagrangian.
\[
\mathcal{L} = \frac{1}{2} f_0^2 (\chi \partial_z \xi - \xi \partial_z \chi) + \frac{1}{2} f_0^2 \mathbf{v}_0 \cdot (\chi \partial_z \xi - \xi \partial_z \chi) - \frac{1}{2} g f_0^4 \chi^2 - \frac{1}{2} \beta_0 f_0^2 (\partial_z \xi)^2 - \frac{1}{8} \beta_0 f_0^2 (\partial_x \chi)^2 \tag{25}
\]

Choosing \( \xi \) as a canonical coordinate we get that the corresponding canonical momentum becomes

\[
p_\xi = \frac{\beta}{\partial_t} \left[ \mathcal{L} + \frac{1}{2} \partial_t (f_0 \chi) \right] = f_0^2 \chi \tag{26}
\]

This makes the quantization procedure seemingly trivial. It would be sufficient to request the commutation relation

\[
[\xi(x', z), p_\xi(x, z)] = [\xi(x', z), f_0^2 \chi(x, z)] = i\hbar \delta(x - x') \tag{27}
\]

for the corresponding operators of the canonical momentum and coordinate. However, the paraxial quantization condition (27) is imposed on the operators acting at the same propagation distance \( z \) rather than at the same time as is usually done.

The question of how the “paraxially” quantized operators relate to the usual photons is in order. The issue of paraxial quantization was addressed in Ref. 47. It was shown that the positive frequency part of the electric field operator (Coulomb gauge) in the paraxial approximation has the form

\[
\hat{E} = i \int_0^\infty d\omega \sqrt{\frac{\hbar \omega}{4\pi \epsilon_0 c}} e^{i\omega(t-z)} \sum_{\mu, m, n} \hat{a}_{\mu, m, n}(\omega) \left( \hat{x}_\mu + i \frac{k_\perp^2}{\omega} \hat{x}_\mu \nabla_\perp \right) \psi_{\mu, m, n}(x, z; \omega) \tag{28}
\]

where \( \hat{x}_\mu \) are the unit polarization vectors, \( \mu = 1, 2 \), \( x = \hat{x} + y\hat{y} \) is normal to the propagating direction, \( k_\perp = k_x \hat{x} + k_y \hat{y} \) is the transverse part of the wave vector, \( \nabla_\perp = \hat{x} \partial_x + \hat{y} \partial_y \), \( \psi_{\mu, m, n}(x, z; \omega) \) make a set of orthogonal polynomials e.g. Hermite or Laguerre polynomials. \( \hat{a}_{\mu, m, n}(\omega) \) and \( \hat{a}_{\mu, m, n}^\dagger(\omega) \) are annihilation and creation operators of a photon with the spatial mode \( m, n \) and polarization \( \mu \) satisfying the standard boson commutation relations.

In the current paper we keep only one polarization, neglect the terms \( O(k_\perp^2) \). Then Eq. (28) becomes

\[
\sum_{m,n} \psi_{m,n}(x, z; \omega) \psi_{m,n}^*(x', z; \omega) = \delta(x - x') \tag{29}
\]
\[ E = i \int_0^\infty d\omega \sqrt{\frac{\hbar \omega}{4\pi\epsilon_0 c}} e^{i\omega(t-\frac{z}{c})} \sum_{m,n} \hat{a}_{m,n}(\omega) \psi_{m,n}(x, z; \omega) = \int \frac{d\omega}{2\pi} \sqrt{\frac{i\omega}{4\pi\epsilon_0 c}} \bar{A}(x, y, z, \omega)e^{i(\beta_{t0}z-\omega t)} \]  

(30)

Here the operator \( \hat{A} \) represents the classical amplitude \( A \) satisfying the NLS equation (11) (up to the square root factor). Applying the procedure similar to that used in BEC (see, e.g. Ref. 53) we write \( \hat{A} = A_0 + \delta \hat{A} \) where the classical amplitude \( A_0 = \langle \hat{A} \rangle \) is obtained by averaging the operator \( \hat{A} \) over the coherent state describing the field in the stationary laser beam.

\[ \delta \hat{A} = \sqrt{i\hbar} \sum_{m,n} \hat{b}_{m,n}(\omega) \psi_{m,n}(x, z; \omega) \]  

(31)

is the fluctuation operator. We have introduced here the new photon operators \( \hat{b}_{m,n}(\omega) = \hat{a}_{m,n}(\omega) - \langle \hat{a}_{m,n}(\omega) \rangle \) and \( \hat{b}_{m,n}^\dagger(\omega) = \hat{a}_{m,n}^\dagger(\omega) - \langle \hat{a}_{m,n}^\dagger(\omega) \rangle \). These new operators correspond to the fluctuations of the electric field. They obviously satisfy the boson commutation relations and \( \langle \hat{b}_{m,n}(\omega) \rangle = \langle \hat{b}_{m,n}^\dagger(\omega) \rangle = 0 \).

As the last step we assume that \( A_0 = \bar{f}_0 e^{-i\phi_0} \) and write

\[ \delta \hat{A} = f_0 e^{-i\phi_0} \left[ \frac{1}{2} \hat{\chi}(x, z; \omega) + i \hat{\xi}(x, z; \omega) \right]. \]  

(32)

The two operators in Eq. (32) represent fluctuations of the amplitude and the phase, respectively. Since these quantities are real the operators

\[ \hat{\chi}(x, z; \omega) = \frac{\sqrt{i\hbar}}{f_0} \sum_{m,n} [\hat{b}_{m,n}(\omega) \psi_{m,n}^*(x, z; \omega)e^{-i\phi_0} + \hat{b}_{m,n}(\omega) \psi_{m,n}(x, z; \omega)e^{i\phi_0}], \]

\[ \hat{\xi}(x, z; \omega) = \frac{\sqrt{i\hbar}}{2i f_0} \sum_{m,n} [\hat{b}_{m,n}(\omega) \psi_{m,n}^*(x, z; \omega)e^{-i\phi_0} - \hat{b}_{m,n}(\omega) \psi_{m,n}(x, z; \omega)e^{i\phi_0}] \]  

(33)

are Hermitian. Using the orthogonality condition (29) and bosonic commutation relation for the operators \( \hat{b}_{m,n}(\omega) \) and \( \hat{b}_{m,n}^\dagger(\omega) \) we may readily verify that the commutation relation (27) holds. Equations (33) connect paraxially quantized quantities (27) with the photon operators \( \hat{b}_{m,n}(\omega) \) and \( \hat{b}_{m,n}^\dagger(\omega) \).

V. CONCLUDING REMARKS

We discuss here the role that the finite width of the analog event horizon plays in the dynamics of fluctuations and formation of the
spectrum of the Hawking radiation. Fluctuations near the GR event horizon, discussed in the recent papers\textsuperscript{48,49}, are described by an equation for one field, which differs from equations (2) and (3) for two fields that follow from the NLS equation for the optical analog event horizon. As a result, the fluctuation modes obtained in Section II differ from those obtained in Ref \textsuperscript{48,49}. Actually it means that the laboratory frame of the analog systems is not fully compatible with the Schwarzschild frame in GR. The transformation (10) and (11) needed in order to reach better compatibility. Then the fluctuation modes (15) and (16) become really analogous to those obtained in Ref \textsuperscript{48,49}. This transformation is singular at \(|x| \to 0\), therefore the relevant results hold only outside the width of the horizon \(|x| > l_r\). Nevertheless, it is sufficient for our analysis of the scattering matrix in Section III.

Calculating the scattering matrix to within the leading order corrections in the nonlinearity length (which determines the width of the horizon) we come to the conclusion that the Hawking radiation is that of a grey body with the emissivity larger than one. This result is quite understandable since the system is stationary but out of equilibrium and there is a permanent source of energy. This conclusion is certainly not specific for the Kerr nonlinear optical systems, discussed here, and can be readily extended to other systems such as, say, Bose-Einstein condensates.

We also show here how the paraxial quantization (commutation relations at the same propagation distance rather than at the same time) is connected with the conventional quantization of the electromagnetic field. As a result, the paraxial operators $\hat{\chi}$ and $\hat{\xi}$ describing the Hawking radiation can be now converted into regular photon operators.

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