A thermoelastic theory with microtemperatures of type III

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Abstract

In this paper, we use the Green-Naghdi theory of thermomechanics of continua to derive a nonlinear theory of thermoelasticity with microtemperatures of type III. This theory permits propagation of both thermal and microtemperatures waves at finite speeds with dissipation of energy. The equations of the linear theory are also obtained. With the help of the semigroup theory of linear operators we establish that the linear anisotropic problem is well posed and we study the asymptotic behavior of the solutions. Finally, we investigate the impossibility of the localization in time of solutions.

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1. Introduction

The origin of the theories of microtemperatures in elastic solids goes back to works of [Eringen (1999), Grot (1969), Wozniak (1967a,b), Ieşan (2001, 2007) and Ieşan and Quintanilla (2006, 2009)]. The experimental observations have shown that the classical continuum theories cannot be used to describe satisfactorily some phenomena. The interest of microtemperatures is stimulated by the fact that this theory is adequate to investigate important problems related to size effects and nanotechnology. This represents an important improvement in the perspective of real applications in many engineering and geophysics applications and also in nanomanufactured activities. Analogous to the concept of micro-deformation introduced by [Eringen (1999)] in his celebrated theory of micromorphic continua, [Grot (1969)] extended the thermodynamics of a continuum with microstructure so that the points of a generic microelement are assumed to have different temperatures. In [Grot (1969)], Grot supposed that the inverse of the microelement temperature is a linear function of microcoordinates. The Clausius-Duhem inequality is modified to include microtemperatures, and the first-order moment of the energy equations are added to the usual balance laws of a continuum with microtemperatures. The concept of microtemperatures has been introduced for the first time by [Wozniak (1967a,b)]. He considered in [Wozniak (1967b)] that the continuum (macromedium) is composed of particles X with microcoordinates. Each particle is assumed at the same time as the origin of the so-called system of spatial local microcoordinates \((x_1, x_2, x_3)\). The spatial coordinates \(x'_i\) of the point \(X'\) of the microelement...
are represented in the form \( x'_i = x_i \psi_{ik} \xi_k \), where \( x_i \) are the spatial coordinates of the centroid \( X \) of the microelement; \( X'_k \) and \( X_k \) are the material coordinates of \( X' \) and \( X \), and \( \xi_k = X'_k - X_k \). The functions \( \psi_{ik} \) are called microdeformations. Wozniak (1967a,b) assumed that all the properties of the macromedium of particles \( X \) are deduced from the structural properties of the micromedium associated with each particle \( X \) and from the way in which these micromedia are interconnected. In particular, the temperatures at points belonging to micromedium are assumed to be known functions (depending on local microcoordinates) expressed by the mean temperature of the micromedium and its gradient. In (Grot, 1969), Grot established the thermodynamics of continua with microstructure when the points of a generic microelement have different temperatures. In this theory the temperature \( \theta' \) at the point \( X' \) of the microelement is a linear function of the microcoordinates \( x_k \) of the form \( \theta' = \theta + \tau_k \xi_k \), where \( \theta \) is the temperature at the centroid \( X \). The vector with the components \( T_k \) defined by \( T_k = -\tau_k / \theta \) is called the microtemperatures vector.

The usual theory of heat conduction based on the classical Fourier’s law allows the phenomena of “infinite diffusion velocity” which is not well accepted from a physical point of view. This paradox of the heat conduction, is physically unrealistic since it implies the propagation of thermal waves with infinite speed. In contrast to the classical Fourier’s law, nonclassical thermal laws came into existence during the last decades to eliminate this shortcomings. A survey article of representative these new models is due to Hetnarski and Ignaczak (1999). One of these theories may be mentioned that of Green and Naghdi who have developed in a series of articles (see (Green and Naghdi, 1991a,b, 1993)) a thermomechanical theory of deformable continua that relies on an entropy balance law rather than an entropy inequality. However, we want to mention the total compatibility of the entropy balance law with the entropy inequality. They proposed the use of the thermal displacement

\[
\alpha(x, t) = \int_{t_0}^{t} \theta(x, s) ds + \alpha_0,
\]

where \( \theta \) is the empirical temperature, and considered three theories labelled as type I, II and III, respectively. These theories were based on an entropy balance law rather than the usual entropy inequality. The type I thermoelasticity coincides with the classical one; in type II, however, the heat is allowed to propagate by means of thermal waves but without dissipating energy and, for this reason, it is also known as thermoelasticity without energy dissipation. The heat equation of type III, where the heat flux is a combination of type I and II, contains both type I and II as limiting cases. In addition, the thermoelasticity of type III allows the constitutive functions for free energy, stress tensor, entropy and heat flux to depend on the strain tensor, the time derivative of the thermal displacement, the gradient of thermal displacement and the time derivative of the gradient of thermal displacement. This theory allows the dissipation energy, but the heat flux is partially determined from the Helmholtz free energy potential. Both, type II and III, overcome the unnatural property of Fourier’s law of infinite propagation speed and imply a finite wave propagation.

It is worth citing several papers on thermoelastic theory with microtemperatures and/or in the frame of Green-Naghdi models have been published in the recent years (see, e.g., Ieşan and Quintanilla, 2000, 2009; Puri and Jordan, 2004; Quintanilla, 2009; Giorgi and Montanaro, 2016; Quintanilla and Straughan, 2004; Quintanilla, 2007; Lazzeri and Nibbi, 2008; Aouadi, 2012; Aouadi et al., 2014, 2016(a)). One of these, one can cite an interesting paper published by Ieşan and Quintanilla (2009) who derived a linear theory of thermoelastic bodies with microstructure and microtemperatures in the frame of Green-Naghdi theory of type II. This theory permits propagation of both thermal and microtemperatures waves at finite speeds; but without energy dissipation and consequently with constant energy. It is worth mentioning that few applicability studies have been developed type II model. However, mathematical and physical analysis is needed to clarify its applicability. In this paper we extend this theory to the nonlinear type III model without considering the microstructure of the material which is not our main concern here. The obtained equations for temperature and microtemperatures allow the transmission of heat with finite speeds and with energy dissipation. For this, the type III model seems physically more acceptable than type II model since it allows the propagation of the resulting deformations with decaying energy when the tensors of thermal and microthermal conductivity are both positive definite. Also a comparative test in (Giorgi and Montanaro, 2016) between type III and type I models reveals that the type III model is more preferable. On the other
hand, we need to warn the reader that our model predicts an asymptotic behavior of solution (see the fifth section) which is not possible in the frame of type II model established in [Ieșan and Quintanilla, 2009]. That is, we prove that the solution tends to zero when the time tends to infinity (see Theorem 3 in the fifth section). Moreover, it is known that in many situations the decay can be exponential. This proof is omitted in this paper for the sake of brevity. It should be noted that this paper is the first one that derives a Green-Naghdi theory of type III in the presence of microtemperatures. Hence, the theory derived in this article will be of great importance in real applications in many engineering and geophysics applications and also in nanomanufactured activities.

The organization of this paper is as follows. In Section 2 we use the theory established by Green and Naghdi (1991a, 1993) to obtain a nonlinear theory of thermoelastic with microtemperatures of type III, which admits the possibility of “second sound” with energy dissipation. The type II model is also derived as a step in the derivation of the type III which our main concern in this paper. In fact the type III model is obtained by adding to the heat flow of the type II model the dependency on the temperature gradient and on the microtemperatures gradient. The process of linearization of the obtained equations is presented in Section 3. With the help of the semigroup theory of linear operators an existence result is obtained in Section 4. In Section 5, the asymptotic behavior for the solutions of type III problem is studied. Finally, in Section 6, we investigate the impossibility of the localization in time of solutions of type III problem. It is worth noting that we focus on the analysis of the qualitative properties of solutions of type III problem. However, some particular aspects of the type II problem are also pointed out.

2. Nonlinear theory

In this section we present a nonlinear theory of thermoelasticity with microtemperatures in the context of the Green-Naghdi model of type III.

We consider a continuous body that at time \( \tau_0 \) occupies a bounded region \( \Omega \) of the Euclidean three-dimensional space with smooth boundary \( \partial \Omega \). For any sub-body we denote with \( B \) the corresponding region in the reference configuration \( \Omega \), which is bounded by a regular surface \( \partial B \), and with \( n_i \) the components of the unit outward normal to \( \partial B \). We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over the integers \((1, 2, 3)\), summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding cartesian coordinate. Moreover a superposed dot denotes the partial time derivative.

We take the configuration \( \Omega \) as reference configuration and refer the motion of the continuum to the reference configuration. Fixed system of rectangular cartesian axes, we denote with \( X_i \) the coordinates of a point in the reference configuration and with \( x_i \) the coordinates of the same point at time \( t \), where \( x_i = \hat{x}_i(X_1, X_2, X_3, t) \). We assume that the functions \( \hat{x}_i \) are continuously differentiable as it is necessary. Following Grot (1969) and Eringen (1999), we wish to extend the linear thermoelastic theory with microtemperatures of type II derived by Ieșan and Quintanilla (2009) to the nonlinear type III model. This can be done by deriving the balance laws to the case in which the temperatures of the particles are different. In the spirit of their works, let assume that \( X \) be the center of mass of a generic microelement in the reference configuration.

We assume that deriving the particle temperature \( T' \) have the form (Grot, 1969)

\[
T'(X', t) = T(X, t) + T_i(X, t)(X'_i - X_i)
\]

where the function \( T_i \) is called microtemperatures. The variables in (1) form a vector and represent the variation of the temperature within a microvolume. They will be considered as independent thermodynamic variables to be determined by the balance laws.

The balance of linear momentum can be written in the form

\[
\dot{\rho} \hat{x}_i = t_{ki,k} + \rho f_i
\]

where \( t_{ki} \) is the first Piola Kirchhoff stress, \( \rho \) is the reference mass density and \( f_i \) is the body force. The surface traction \( t_i \) at regular points of \( \partial B \) are given by

\[
t_i = t_{ik} n_k.
\]
Following Green and Naghdali (1991b,a, 1993) we postulate, for each microelement, the local balance of entropy

\[ \rho' S' = \Phi_k,k + \rho'(s' + \xi') \]

where \( \rho' \) is the mass density of the microelement, \( S' \) the entropy density per unit mass of the microelement, \( \Phi'_k \) the microentropy flux vector, \( s' \) the external rate of supply of entropy per unit mass of the microelement and \( \xi' \) the internal rate of production of entropy per unit mass of the microelement. The heat flux vector associated with the microelement is given by \( q'_k = T'\Phi'_k \) and the external rate of supply of heat per unit mass is defined by \( r' = T's' \).

According to Green and Naghdali (1991b,a, 1993), we postulate the local balance of entropy

\[ \rho' \dot{S} = \Phi_{k,k} + \rho(s + \xi) \]

and the balance of first moment of entropy

\[ \rho' \dot{\varepsilon}_i = \Lambda_{ji,j} + \Phi_i - H_i + \rho(Q_i + \xi_i) \]

where \( S \) is the entropy per unit mass of the body, \( \Phi_i \) is the entropy flux vector, \( s \) is the external rate of supply of entropy per unit mass, \( \xi \) is the internal rate of production of entropy per unit mass, \( \varepsilon_i \) is the first entropy moment vector, \( \Lambda_{ji} \) is the first entropy flux moment tensor, \( H_i \) is the mean entropy flux vector, \( Q_i \) is the first moment of the external rate of supply of entropy and \( \xi_i \) is the first moment of the internal rate of production of entropy.

Moreover, following the arguments of Green and Naghdali (1991a) and Iesan and Quintanilla (2009), we postulate for every subregion \( B \) of \( \Omega \) and every time \( t \), the following balance equations for the energy

\[ \int_B \rho(\dot{x}_i \dot{x}_i + \varepsilon)dv = \int_B \rho(f_i \dot{x}_i + sT + Q_i T_i)dv + \int_{\partial B} (t_i \dot{x}_i + \Phi T + \sigma_i T_i)da, \]

where \( \varepsilon \) is the internal energy per unit mass. The entropy flux \( \Phi \) and the first entropy moment flux vector \( \sigma_i \) at regular points of \( \partial B \) are given by

\[ \Phi = \Phi_k n_k, \quad \sigma_k = \Lambda_{jk} n_j. \]

Therefore, thanks to the arbitrariness of \( B \), equations (2)-(6), we obtain the following local form for the balance equations

\[ \rho \dot{c} = t_{ki} \dot{x}_{i,k} + \rho sT + \rho Q_i T_i + \Phi_k T_k + \Phi_{k,k} T_{k,k} + \Lambda_{kj,k} T_{j,k} + \Lambda_{kj} T_{j,k}. \]

By eliminating \( \rho sT \) and \( \Lambda_{kj,k} T_{j,k} \) from (8) through the use of (4) and (5), we obtain

\[ \rho \dot{c} = t_{ki} \dot{x}_{i,k} + \rho \dot{S}T - \rho \dot{\xi}T + \Phi_k T_{k,k} + \Lambda_{kj} T_{j,k} + \rho \dot{\varepsilon}T_k + (H_k - \Phi_k)T_k - \rho \xi_k T_k. \]

If we introduce the specific Helmholtz free energy per unit mass

\[ \Psi = c - TS - \varepsilon_k T_k, \]

the energy equation (9) becomes

\[ \rho(\dot{\Psi} + \dot{T}S + \dot{T}_k \dot{\varepsilon}_k) = t_{ki} \dot{x}_{i,k} - \rho \dot{\xi}T + \Phi_k T_{k,k} + \Lambda_{kj} T_{j,k} + (H_k - \Phi_k)T_k - \rho \xi_k T_k. \]

We introduce the notations

\[ q_j = \Phi_j T, \quad q_{ji} = \Lambda_{ji} T. \]

The heat flux \( q \) and the heat flux moment vector \( \Lambda_i \) at regular points of \( \partial B \) are given by

\[ q = q_j n_j, \quad \Lambda_i = q_{ji} n_j \]

respectively. We shall now introduce the nonlinear model of type II to deduce the type III model.
2.1. Type II - dissipationless theory

According to the Green-Naghdi theory (Green and Naghdi, 1991b), we introduce the thermal displacement \( \alpha \) whose derivative coincides with the absolute temperature, i.e., \( \dot{\alpha} = T \). This scalar, on the macroscopic scale, is regarded as representing some “mean” thermal displacement magnitude on the molecular scale. In a similar way, we introduce a scalar function \( \beta_i \) related to the microtemperatures by the equation \( \dot{\beta}_i = T_i \) (see (Green and Naghdi, 1991b; Ieşan and Quintanilla, 2009)).

We assume that the response functions

\[
\Psi, \ t_{kj}, \ S, \ \varepsilon_i, \ \Phi_k, \ \Lambda_{jk}, \ H_k, \ \xi, \ \xi_k
\]

depend on the set of the independent variables

\[
A_1 = (x_{i,k}, \ T, \ T_k, \ \alpha_{k}, \ \beta_{i,k}).
\]

Thus, we consider constitutive equations of the form

\[
\mathcal{F} = \hat{\mathcal{F}}(A_1)
\]

and we assume that the response functions are of \( C^1 \)-class. Using the chain rule

\[
\dot{\Psi} = \frac{\partial \hat{\Psi}}{\partial x_{j,k}} \dot{x}_{j,k} + \frac{\partial \hat{\Psi}}{\partial T} \dot{T} + \frac{\partial \hat{\Psi}}{\partial T_k} \dot{T}_k + \frac{\partial \hat{\Psi}}{\partial \alpha_{k}} \dot{\alpha}_{k} + \frac{\partial \hat{\Psi}}{\partial \beta_{j,k}} \dot{\beta}_{j,k},
\]

the comparison of Eqs. (11) and (16) yields

\[
\left( \rho \frac{\partial \hat{\Psi}}{\partial T} + \rho \dot{S} \right) \dot{T} + \left( \rho \frac{\partial \hat{\Psi}}{\partial T_i} + \rho \dot{\varepsilon}_i \right) \dot{T}_i + \left( \rho \frac{\partial \hat{\Psi}}{\partial x_{j,k}} - \dot{\hat{T}}_{kj} \right) \dot{x}_{j,k} + \left( \rho \frac{\partial \hat{\Psi}}{\partial \alpha_{k}} - \dot{\hat{\beta}}_{i,k} \right) \dot{\alpha}_{k}
\]

\[
+ \left( \rho \frac{\partial \hat{\Psi}}{\partial \beta_{i,k}} - \dot{\hat{\Lambda}}_{i,k} \right) \dot{\beta}_{i,k} + \rho T_{\xi} + (\Phi_k - H_k)T_k + \rho \hat{\xi}_k T_k = 0
\]

which must hold for all choice of \( \dot{T}, \ \dot{T}_i, \ \dot{x}_{j,k}, \ \dot{\alpha}_{k} \) and \( \dot{\beta}_{i,k} \). From this equality we see that the constitutive equations are compatible with the energy equation if satisfy the following relations

\[
\Psi = \hat{\Psi}(A_1), \quad S = -\frac{\partial \hat{\Psi}(A_1)}{\partial T}, \quad \varepsilon_i = -\frac{\partial \hat{\Psi}(A_1)}{\partial T_i}, \quad t_{kj} = \rho \frac{\partial \hat{\Psi}(A_1)}{\partial x_{j,k}}.
\]

\[
\Phi_k = \rho \frac{\partial \hat{\Psi}(A_1)}{\partial \alpha_{k}}, \quad \Lambda_{ki} = \rho \frac{\partial \hat{\Psi}(A_1)}{\partial \beta_{i,k}}, \quad \rho T_{\xi} + (\Phi_k - H_k)T_k + \rho \hat{\xi}_k T_k = 0.
\]

**Remark 1.** Since the last equation of (18) must be satisfied for all process and the temperature and the microtemperatures can not vanish for all process, we conclude that

\[
\xi = 0 \quad \text{whenever} \quad \Phi_k - H_k + \rho \hat{\xi}_k = 0.
\]

The thermal displacement \( \alpha \) and the microtemperatures displacement \( \beta_k \) are defined analogously to the well-known mechanical displacement. For this, the entropy flux vector \( \Phi_k \) and the first entropy flux moment tensor \( \Lambda_{ij} \) are deduced from a potential in the same way as the stress tensor is derived in mechanics.
2.2. Type III - dissipation theory

Whereas in case of heat flow of type II the response functions (14) are assumed to depend on the material deformation gradient \( x_{i,k} \), the temperature \( T \), the microtemperatures \( T_i \), the thermal displacement gradient \( \alpha_{i,k} \), and the microtemperatures displacement gradient \( \beta_{i,k} \), for type III we now add the dependency on the temperature gradient \( \dot{\alpha}_{i,k} \) and on the microtemperatures gradient \( \dot{\beta}_{i,k} \). Hence, we assume that the response functions (14) depend on the set of the independent variables

\[ \mathcal{A}_2 = (x_{i,k}, T, T_i, \alpha_{i,k}, \beta_{i,k}, \dot{\alpha}_{i,k}, \dot{\beta}_{i,k}) = (x_{i,k}, T, T_i, \alpha_{i,k}, \beta_{i,k}, T_k, T_{i,k}). \]

In this case, using the chain rule

\[
\dot{\Psi} = \frac{\partial \hat{\Psi}}{\partial x_{j,k}} \dot{x}_{j,k} + \frac{\partial \hat{\Psi}}{\partial T} \dot{T} + \frac{\partial \hat{\Psi}}{\partial T_k} \dot{T}_k + \frac{\partial \hat{\Psi}}{\partial \alpha_{i,k}} \dot{T}_{i,k} + \frac{\partial \hat{\Psi}}{\partial \beta_{i,k}} \dot{T}_{i,k} + \frac{\partial \hat{\Psi}}{\partial T_{i,k}} \dot{T}_{i,k},
\]

the comparison of Eqs. (11) and (19) yields

\[
\frac{\partial \hat{\Psi}}{\partial T_{i,k}} = 0, \quad \frac{\partial \hat{\Psi}}{\partial T_{i,k}} = 0,
\]

that is \( \Psi = \hat{\Psi}(x_{i,k}, T, T_k, \alpha_{i,k}, \beta_{i,k}) = \hat{\Psi}(A_1) \), and

\[
S = -\frac{\partial \hat{\Psi}(A_1)}{\partial T}, \quad \varepsilon_i = -\frac{\partial \hat{\Psi}(A_1)}{\partial T_i}, \quad t_{kj} = \rho \frac{\partial \hat{\Psi}(A_1)}{\partial x_{j,k}},
\]

\[
\left( \rho \frac{\partial \hat{\Psi}}{\partial \alpha_{i,k}} - \hat{\Phi}_k \right) T_k + \left( \rho \frac{\partial \hat{\Psi}}{\partial \beta_{i,k}} - \hat{\Lambda}_{ki} \right) T_{i,k} + \rho T\xi + (\Phi_k - H_k)T_k + \rho \xi_k T_k = 0.
\]

3. Linear theory

We consider a reference configuration which is in thermal equilibrium and free from stresses, with \( \alpha \) and \( \beta_k \) constant. We assume that the deformations and the changes of temperature and microtemperatures are very small with respect to the reference configuration in such way that, if \( T_0 \) and \( T^0_0 \) are respectively the (constant) absolute temperature and the (constant) absolute microtemperatures of the body in the reference configuration, we can write

\[
x_i - X_i = u_i = \varepsilon u'_i, \quad T - T_0 = \theta = \varepsilon \theta', \quad T_i - T^0_i = M_i = \varepsilon M'_i
\]

where \( \varepsilon \) is a constant small enough for squares and higher powers to be neglected, and \( u'_i, \theta' \) and \( M'_i \) are independent on \( \varepsilon \). Under these hypotheses, the strain tensor is approximated with

\[
\varepsilon_{ik} = \frac{1}{2}(u_{i,k} + u_{k,i}).
\]

3.1. Green-Naghdi theory of type II

The set of the independent variables for the Green-Naghdi model of type II (without energy dissipation) becomes

\[
\hat{\mathcal{A}}_1 = (\varepsilon_{ik}, \theta, M_i, \tau_i, R_{i,k})
\]

where

\[
\tau = \int_{t_0}^{t} \theta ds, \quad R_i = \int_{t_0}^{t} M_i ds.
\]
To obtain a linear theory, we consider the free energy $\Psi$ function in the quadratic approximation
\[
\rho\Psi = \frac{1}{2}A_{ijkl}e_{ij}e_{kl} - a_{ij}e_{ij}\theta + B_{ijkl}e_{ij}R_{k,l} - b_{ij}R_{i,j}\theta - \frac{p_cE}{2T_0}\theta^2 + \frac{1}{2}K_{ij}\tau_{i,j}\tau_{j,i} + \frac{1}{2}C_{ijkl}R_{i,j}R_{k,l} - d_{ij}M_i\tau_{j,i} - \frac{1}{2}c_{ij}M_iM_j
\]
where $c_c$ is the specific heat at constant strain, $A_{ijkl}$ is the tensor of elastic constants, $a_{ij}$ is the tensor of thermal expansion, $B_{ijkl}$ is the tensor of microtemperatures expansion, $K_{ij}$ is the tensor of thermal conductivity and $c_{ij}$ is the tensor of microthermal conductivity. The tensors $d_{ij}$ and $b_{ij}$ are, respectively, measure of thermal and microthermal gradient displacement. The constitutive coefficients have the following symmetries
\[
A_{ijkl} = A_{kllj}, \quad a_{ij} = a_{ji}, \quad B_{ijkl} = B_{jikl}, \quad K_{ij} = K_{ji}
\]
\[
C_{ijkl} = C_{klij}, \quad c_{ij} = c_{ji}, \quad b_{ij} = b_{ji}, \quad d_{ij} = d_{ji}.
\]
For centrosymmetric materials, the compatibility conditions \[\text{(18)}\] give the following linear constitutive equations
\[
t_{ij} = A_{ijkl}e^{kl} - a_{ij}\theta + B_{ijkl}R_{k,l}, \\
\rho S = a_{ij}e_{ij} + \frac{p_cE}{T_0}\theta + b_{ij}R_{i,j}, \\
\rho\varepsilon_i = c_{ij}M_j + d_{ij}\tau_{j,i}, \\
\Phi_i = -d_{ij}M_j + K_{ij}\tau_{j,i}, \\
\Lambda_{ij} = B_{ijkl}e^{kl} - b_{ij}\theta + C_{ijkl}R_{k,l}.
\]
Moreover, by using \[\text{(25)}\], the linear approximation of \[\text{(12)}\] is given by
\[
q_i = T_0\Phi_i = T_0(-d_{ij}M_j + K_{ij}\tau_{j,i}), \\
q_{ij} = T_0\Lambda_{ij} = T_0(B_{ijkl}e^{kl} - b_{ij}\theta + C_{ijkl}R_{k,l}).
\]
By Remark \[\text{(1)}\], we find that, in the linear theory of centrosymmetric materials, we have $\xi = 0$ and consequently $\Phi_k - H_k + \rho\xi_k = 0$. Thus, \[\text{(4)}\] and \[\text{(5)}\] become
\[
\rho\ddot{S} = \Phi_{k,k} + \rho s, \quad \rho\ddot{\varepsilon}_i = \Lambda_{jj,i} + \rho Q_i.
\]
The basic equations of the linear theory of type II consist of the equations of motion \[\text{(2)}\] and \[\text{(3)}\], the energy equations \[\text{(27)}\], the constitutive equations \[\text{(25)}\] and the geometrical equation \[\text{(22)}\]. These equations furnish the following system of partial differential equations for the unknown functions $u_i, \tau, R_i$
\[
\rho\ddot{u}_i = (A_{ijkl}e^{kl} - a_{ij}\theta + B_{ijkl}R_{k,l}) + \rho f_i, \\
c\ddot{\tau} = -a_{ij}\dot{e}_{ij} + (d_{ij}\dot{R}_i + K_{ij}\dot{\tau}_{j,i}, -b_{ij}\dot{R}_{i,j} + \rho s, \\
c_{ij}\dot{R}_i = (B_{ijkl}e^{kl} - b_{ij}\theta + C_{ijkl}R_{k,l}) + d_{ij}\dot{\tau}_{j,i} + \rho Q_i,
\]
where $c = \frac{p_cE}{T_0}$. If the material is isotropic, then the constitutive equations \[\text{(25)}\] become
\[
t_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \beta\theta\delta_{ij} + \gamma_1R_{k,k}\delta_{ij} + 2\gamma_2(R_{i,j} + R_{j,i}), \\
\rho S = \beta e_{kk} + \frac{p_cE}{T_0}\theta + \nu R_{k,k}, \\
\rho\varepsilon_i = \alpha M_i + h\tau_{i,i}, \\
\Phi_i = -hM_i + K\tau_{i,i}, \\
\Lambda_{ij} = \gamma_1e_{kk}\delta_{ij} + 2\gamma_2e_{ij} - \nu\theta\delta_{ij} + \gamma_1R_{k,k}\delta_{ij} + \gamma_2R_{i,j} + \gamma_3R_{j,i},
\]
where $\delta_{ij}$ is the Kronecker’s delta, $\lambda$ and $\mu$ are Lamé’s constants, $\beta = (3\lambda + 2\mu)\alpha_t$ and $\alpha_t$ is the coefficient of linear thermal expansion.

By following the previous procedure, it follows from \[\text{(29)}\] that the field equations of the theory of homogeneous and isotropic bodies for the functions $u_i, \tau, R_i$ can be expressed as
\[
\rho\ddot{u}_i = \mu\Delta u_i + (\mu + \lambda)u_{j,j,i} - \beta\dot{\tau}_{i,i} + \gamma_2\Delta R_i + (\gamma_1 + \gamma_2)R_{j,j,i} + \rho f_i, \\
c\ddot{\tau} = -\beta u_{r,r} + K\Delta\tau - (\nu + h)\dot{R}_{i,i} + \rho s, \\
\alpha R_i = \gamma_2\Delta u_i + (\gamma_1 + \gamma_2)u_{j,j,i} + \gamma_2\Delta R_i + (\gamma_1 + \gamma_3)R_{j,j,i} - h\dot{\tau}_{i,i} + \rho Q_i,
\]
where $\Delta$ is the Laplacian.
Remark 2. System \((28)\) can be deduced from system \((3.18)\) of [Ieşan and Quintanilla, 2004] by eliminating the porosity. The model of the Green-Naghdi theory of type III (with energy dissipation) becomes

\[ \tilde{A}_2 = (e_{ik}, \theta, M_i, \tau_k, R_{i,k}, \theta_{k,i}, M_{i,k}) \]

and, as a consequence of \((20)\), the quadratic approximation of the free energy \(\Psi\) is given by \((28)\). Moreover, the constitutive equations for \(t_{k,j}, S\) and \(\rho e_i\) are equal to \((25)_{1,2,3}\), respectively. The condition \((20)\) leads, in the linear context, to

\[
\left( \rho \frac{\partial \Psi}{\partial \tau_i} - \tilde{F}_i \right) \theta_i + \left( \rho \frac{\partial \Psi}{\partial R_{i,j}} - \tilde{A}_{ij} \right) M_{i,j} + \rho T_0 \xi_i + \left( \Phi_i - H_i \right) T_0^0 + \rho \xi_i T_0^0 = 0. \tag{30}
\]

The condition \((30)\) is satisfied if we choose for a centrosymmetric material

\[
\begin{align*}
\Phi_i &= -d_{ij} M_j + K_{ij} \tau_j + \tilde{K}_{ij} \theta_j, \\
A_{ij} &= B_{kl} e_{kl} - b_{ij} \theta + C_{jikl} R_{k,l} + \tilde{C}_{jikl} M_{k,l}, \\
q_i &= T_0 \Phi_i = T_0 \left[ -d_{ij} M_j + K_{ij} \tau_j + \tilde{K}_{ij} \theta_j \right], \\
q_{ij} &= T_0 A_{ij} = T_0 \left[ B_{kl} e_{kl} - b_{ij} \theta + C_{jikl} R_{k,l} + \tilde{C}_{jikl} M_{k,l} \right], \\
\rho T_0 \xi_i &= \tilde{K}_{ij} \theta_j + \tilde{C}_{jikl} M_{i,k,l}, \\
\rho \xi_i &= H_i - \Phi_i,
\end{align*}
\tag{31}
\]

where \(\tilde{K}_{ij}\) and \(\tilde{C}_{ijkl}\) are tensors characteristic of the type III model. In view of \((31)_{5,6}\), the local balance of entropy and the balance of first moment of entropy, in the linear context, are given by \((27)\). Using \((31)\) and \((27)\), we obtain the following evolutive equations of the theory of thermoelastic centrosymmetric materials with microtemperatures of type III (with energy dissipation)

\[
\begin{align*}
\rho \tilde{u}_i &= (A_{ijkl} e_{kl} - a_{ij} \tilde{r} + B_{ijkl} R_{k,l})_j + \rho f_i, \\
\tilde{c} \tau &= -a_{ij} \tilde{e}_{ij} + (-d_{ij} \tilde{R}_j + K_{ij} \tau_j + \tilde{K}_{ij} \tilde{r}_j)_i - b_{ij} \tilde{R}_{i,j} + \rho s, \\
\tilde{c}_i \tilde{R}_{ij} &= (B_{kl} e_{kl} - b_{ij} \tilde{r} + C_{jikl} R_{k,l} + \tilde{C}_{jikl} R_{k,l})_j - d_{ij} \tilde{r}_j + \rho Q_i,
\end{align*}
\tag{32}
\]

where the above constitutive coefficients satisfy the following symmetry relations

\[
\begin{align*}
A_{ijkl} &= A_{klji} = A_{ij}, \quad a_{ij} = a_{ji}, \quad B_{ijkl} = B_{klji} = B_{klij}, \quad K_{ij} = K_{ji}, \\
C_{ijkl} &= C_{klji} = C_{ij}, \quad c_{ij} = c_{ji}, \quad b_{ij} = b_{ji}, \quad d_{ij} = d_{ji}, \quad K_{ij} = K_{ji}, \quad \tilde{C}_{ijkl} = \tilde{C}_{klij}.
\end{align*}
\tag{33}
\]

Remark that the evolutive equations \((28)\) of the thermoelastic diffusion theory of type II (without energy dissipation) can be deduced from \((32)\) by taking \(\tilde{K}_{ij} = \tilde{C}_{ijkl} = 0\).

If the material is isotropic, then the constitutive equations become

\[
\begin{align*}
t_{ij} &= \lambda e_{kk} \delta_{ij} + 2 \mu e_{ij} - \beta \theta \delta_{ij} + \gamma_1 R_{k,k} \delta_{ij} + 2 \gamma_2 (R_{i,j} + R_{j,i}), \\
\rho S &= \beta c_{kk} + \frac{\rho c_p}{T_0} \theta + \pi R_{k,k}, \\
\rho e_i &= \alpha M_i + h \tau_i, \\
\Phi_i &= -h M_i + K \tau_i + H \theta_j, \\
A_{ij} &= \gamma_1 e_{kk} \delta_{ij} + 2 \gamma_2 e_{ij} - \pi \theta \delta_{ij} + \eta_1 R_{k,k} \delta_{ij} + \eta_2 R_{i,j} + \eta_3 R_{j,i} + \eta_4 M_{k,k} \delta_{ij} + \eta_5 M_{i,j} + \eta_6 M_{j,i}.
\end{align*}
\]
By following the previous procedure, it follows that the field equations of the theory of type III for homogeneous and isotropic bodies for the functions $u, \tau, R$ are

$$\rho \ddot{u} = \mu \Delta u + (\mu + \lambda)u_{j;i} - \beta \ddot{\tau}, + \gamma_2 \Delta R_i + (\gamma_1 + \gamma_2)R_{j;i} + \rho \ddot{f},$$

$$v = -\dot{\theta}u, + K \Delta \tau + H \Delta \tau - (\omega + h)\dot{R}_i + \rho s,$$

$$\alpha \ddot{R}_i = \gamma_2 \Delta u_i + (\gamma_1 + \gamma_2)u_{j;i} + \eta_2 \Delta R_i + (\eta_1 + \eta_3)R_{j;i} + \varphi_2 \Delta \dot{R}_i + (\varphi_1 + \varphi_3)\dot{R}_{j;i} - h \dot{\tau}, + \rho Q_i.$$

(34)

To the field equations (32) or (28) we add initial and boundary conditions. Summarizing, the following initial boundary value problems are to be solved for type III model (or Type II model):

Find $(u_i, v_i, \tau, \theta, M_i, R_i)$ solution to (32) or (28) subject to the initial conditions

$$u_i(\cdot, 0) = u_i^0, v_i(\cdot, 0) = v_i^0, \tau(\cdot, 0) = \tau_0,$$

$$\theta(\cdot, 0) = \theta_0, M_i(\cdot, 0) = M_i^0, R_i(\cdot, 0) = R_i^0$$

and the boundary conditions

$$u_i = \tilde{u}_i \text{ on } \partial \Omega_u \times (0, \infty), \quad \tau = \tilde{\tau} \text{ on } \partial \Omega_\tau \times (0, \infty), \quad M_i = \tilde{M}_i \text{ on } \partial \Omega_{M_i} \times (0, \infty),$$

$$t_{ji}, n_j = \tilde{t}_i \text{ on } \partial \Omega_t \times (0, \infty), \quad \Phi_i n_i = \tilde{\Phi} \text{ on } \partial \Omega_{\Phi} \times (0, \infty), \quad \Lambda_{ji}, n_j = \tilde{\sigma}_i \text{ on } \partial \Omega_{\sigma_i} \times (0, \infty),$$

(36)

where $\tilde{u}_i, \tilde{\tau}, \tilde{M}_i, \tilde{t}_i, \tilde{\Phi}$ and $\tilde{\sigma}_i$ are prescribed functions, $u_i^0, v_i^0, \tau_0, \theta_0, M_i^0$ and $R_i^0$ are given and

$$\partial \Omega = \partial \Omega_u \cup \partial \Omega_\tau \cup \partial \Omega_\Phi = \partial \Omega_{M_i} \cup \partial \Omega_{\sigma_i} \quad \text{and} \quad \partial \Omega_u \cap \partial \Omega_\tau = \partial \Omega_\Phi \cap \partial \Omega_\sigma = \partial \Omega_{M_i} \cap \partial \Omega_{\sigma_i} = \emptyset.$$

In the following, some qualitative properties of the solutions to type III problem are studied. However, some particular aspects of the type II problem are also pointed out.

4. Well-posedness

We will prove the existence, uniqueness and continuous dependence from the initial values and the external loads of the solution for system (32) using the semigroups theory. Seeking for simplicity, we will restrict ourselves to homogeneous boundary conditions

$$u = 0, \quad \tau = 0, \quad R = 0, \quad \text{on } \partial \Omega \times (0, \infty)$$

(37)

where $u$ and $R$ denote the vectors of components $u_i$ and $R_i$, respectively.

In the rest of the paper we assume:

(i) relations (33) are satisfied;

(ii) $\rho > 0$ and

$$\tilde{K}_{i,j} \theta_1 \theta_2 + \tilde{C}_{ijkl} M_{i,k} M_{j,l} \geq 0;$$

(iii) there exists a positive constant $c_0$ such that

$$\int_\Omega \left( A_{ijkl} e_{kl} e_{ij} + 2 B_{ijkl} e_{ij} R_{k,l} + K_{i,j} \tau_1 \tau_2 + C_{ijkl} R_{i,j} R_{k,l} \right) dv \geq c_0 \int_\Omega (e_{ij} e_{ij} + \tau_1 \tau_2 + R_{i,j} R_{i,j}) dv.$$

(39)

We now wish to transform the initial boundary value problem defined by the equations (32), the initial conditions (35) and the boundary conditions (37) to an abstract problem on a suitable Hilbert space. In what follows we use the notations $v_i = u_i, \quad \theta = \tau, \quad M_i = R_i$. Let define the following Hilbert space:

$$\mathcal{H} = \left\{ (u_i, v_i, \tau, \theta, M_i); \quad u_i, \quad R_i \in W^{1,2}_0(\Omega); \quad v_i, \quad M_i \in L^2(\Omega); \quad \tau \in W^{1,2}_0(\Omega); \quad \theta \in L^2(\Omega) \right\}.$$

1The inequality sign is a consequence of the Second Law of Thermodynamics, which requires the non-negativeness of the functional $\xi$ (see [Green and Naghdi, 1991]) and of our choice (41)5.
where $W_0^{1,2}(\Omega)$ and $L^2(\Omega)$ are the familiar Sobolev spaces and

$$W_0^{1,2}(\Omega) = |W_0^{1,2}(\Omega)|^3, \quad L^2(\Omega) = |L^2(\Omega)|^3.$$ 

We introduce the inner product in $\mathcal{H}$ defined by

$$\langle U, U^* \rangle = \frac{1}{2} \int_{\Omega} \left( \rho v_i v_i^* + c \theta \tau^* + c_{ij} M_i M_j^* + 2W((u_i, \tau, R_i), (u_i^*, \tau^*, R_i^*)) \right) dv,$$ 

where $U = (u_i, v_i, \tau, R_i, M_i)$, $U^* = (u_i^*, v_i^*, \tau^*, \theta, R_i^*, M_i^*)$ and

$$2W((u_i, \tau, R_i), (u_i^*, \tau^*, R_i^*)) = A_{ijkl} e_{ij} e_{kl}^* + B_{ijkl} (e_{ij} R_{k,l}^* + e_{kl} R_{i,j}^*) + K_{ij} \tau_j \tau_i^* + C_{ijkl} R_{i,j} R_{k,l}^*.$$ 

If we recall the assumption (39), the first Korn inequality and the Poincaré inequality, we conclude that

$$\int_{\Omega} W((u, \tau, R), (u, \tau, R)) dv$$

defines a norm that is equivalent to the usual norm in $W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$. Hence, the bilinear form (40) defines an inner product equivalent to the usual one in $\mathcal{H}$.

We introduce the following operators

$$A_{i} u = \rho^{-1} (A_{ijkl} u_{k,l})_{,j}, \quad B_{i} \theta = -\rho^{-1} (B_{ijkl} R_{k,l})_{,j},$$

$$C_{i} R = \rho^{-1} (C_{ijkl} R_{k,l})_{,j}, \quad D_{i} v = -c^{-1} (a_{ij} v_{i,j}),$$

$$E_{i} \tau = c^{-1} (K_{ij} \tau_{j})_{,i}, \quad G_{i} \theta = c^{-1} (K_{ij} \theta_{j})_{,i},$$

$$J_{i} M = -c^{-1} ((d_{ij} M_{j,i})_{,i} + b_{ij} M_{i,j}),$$

$$L_{i} u = \ell_{si} (B_{ijkl} u_{k,l})_{,i}, \quad Z_{i} \theta = -\ell_{si} (b_{ij} \theta_{j} + d_{ij} \theta_{j}),$$

$$N_{i} R = \ell_{si} (C_{ijkl} R_{k,l})_{,j}, \quad P_{i} M = \ell_{si} (C_{ijkl} \theta_{k,l})_{,j},$$

where $\ell_{si}$ is defined by $\ell_{si} e_{ij} = \delta_{sj}$. We consider the matrix operator $\mathcal{A}$ on $\mathcal{H}$ by

$$\mathcal{A} = \begin{bmatrix} 0 & \Id & 0 & 0 & 0 \\ A & 0 & 0 & B & C \\ 0 & 0 & 0 & \Id & 0 \\ 0 & D & E & G & 0 \\ 0 & 0 & 0 & 0 & \Id \\ L & 0 & 0 & Z & N & P \end{bmatrix}$$

where $\Id$ and $\Id$ are the identity operators in the respective spaces, $A = (A_i)$, $B = (B_i)$, $C = (C_i)$, $L = L_i$, $Z = Z_i$, $N = N_i$ and $P = P_i$. The domain of $\mathcal{A}$ is

$$\mathcal{D} = \mathcal{D}(\mathcal{A}) = \{(u_i, v_i, \tau, R_i, M_i) \in \mathcal{H}; \quad \mathcal{A}(u_i, v_i, \tau, R_i, M_i) \in \mathcal{H}\}.$$ 

It is clear that

$$(W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)) \times W_0^{1,2}(\Omega) \times (W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)) \times W_0^{1,2}(\Omega) \times (W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)) \times W_0^{1,2}(\Omega)$$

is a subset of $\mathcal{D}$ which is dense in $\mathcal{H}$.

In the frame of type II theory, we have that $\tilde{K}_{ij} = 0$ and $\tilde{C}_{ijkl} = 0$, so that $G = 0$ and $P = 0$ in Eqs. (11) and (12).
The initial boundary value problem (32), (35) and (37) can be transformed into the following Cauchy problem in the Hilbert space $H$,

$$\frac{d\mathcal{U}(t)}{dt} = \mathcal{A}\mathcal{U}(t) + \mathcal{F}(t), \quad \mathcal{U}(0) = \mathcal{U}_0,$$

where $\mathcal{U} = (u_i, v_i, \tau, R_i, M_i)$, $\mathcal{F} = (0, \rho f_i, 0, \rho s, 0, \rho Q_i)$, $\mathcal{U}_0 = (u_i^0, v_i^0, \tau^0, R_i^0, M_i^0)$.

Now, we use the theory of semigroups of linear operators to obtain the existence of solutions to the Cauchy problem (43).

**Lemma 1.** The operator $\mathcal{A}$ satisfies the inequality

$$<\mathcal{A}\mathcal{U}, \mathcal{U}> \leq 0$$

for every $\mathcal{U} \in \mathcal{D}(\mathcal{A})$, solution to (43).

**Proof:** Let $\mathcal{U} = (u_i, v_i, \tau, R_i, M_i) \in \mathcal{D}(\mathcal{A})$. Using the divergence theorem and the boundary conditions, we have

$$<\mathcal{A}\mathcal{U}, \mathcal{U}> = \int_{\Omega} \left[ W((u_i, \tau, R_i), (v_i, \theta, M_i)) - v_{i,j}^t j_i - \Phi_i \theta_j, i - \Lambda_j M_{i,j} \right] dv = - \int_{\Omega} (\tilde{K}_{ij} \theta_j, i + \tilde{C}_{ijkl} M_{i,k} M_{i,j}) dv.$$

The thesis follows from our hypothesis (38). \hfill \Box

In the context of type II theory ($\tilde{K}_{ij} = \tilde{C}_{ijkl} = 0$), this lemma implies $<\mathcal{A}\mathcal{U}, \mathcal{U}> = 0$, which means conservation of the energy

$$E(t) = \frac{1}{2} \int_{\Omega} \left( \rho v_i v_i + c \theta^2 + c_{ij} M_i M_j + A_{ijkl} e_{kl} e_{ij} + 2 B_{ijkl} e_{kl} R_{k,l} + K_{ij} \tau_i \tau_j + C_{ijkl} R_{i,j} R_{k,l} \right) dv. \quad (44)$$

This confirms the result obtained in [Ieşan and Quintanilla, 2009].

It is worth remarking that this quantity is also conserved even if we do not impose conditions (i) – (iii).

**Lemma 2.** The operator $\mathcal{A}$ has the property that

$$\text{Range}(I - \mathcal{A}) = \mathcal{H}.$$ 

**Proof:** Let $\mathcal{U}^* = (u_i^*, v_i^*, \tau^*, \theta^*, R_i^*, M_i^*) \in \mathcal{H}$. We must prove that the equation

$$\mathcal{U} - \mathcal{A}\mathcal{U} = \mathcal{U}^*$$

has a solution $\mathcal{U} = (u_i, v_i, \tau, R_i, M_i) \in \mathcal{D}$. This equation leads to the system

$$u^* = u - v,$$

$$\tau^* = \tau - \theta,$$

$$R^* = R - M,$$

$$v^* = v - (A u + B \theta + C R),$$

$$\theta^* = \theta - (D v + E \tau + G \theta + J M),$$

$$M^* = M - (L u + Z \theta + N R + P M).$$

(45)

Substituting the first three equations in the others, we obtain

$$\mathcal{S}(u, \tau, R) = (u^* + v^* - B \tau^* - \theta^* + \tau^* - D u^* - G \tau^* - J R^*, R^* + M^* - Z \tau^* - P R^*)$$

(46)
where
\[
\mathcal{B} = \begin{pmatrix}
\text{Id} - A & -B & -C \\
-D & \text{Id} - (E + G) & -J \\
-L & -Z & \text{Id} - (N + P)
\end{pmatrix}.
\]

To solve the system (46), we introduce the following bilinear form on \(W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)\),
\[
\mathcal{B}[(u, \tau, R), (u', \tau', R')] = < \mathcal{B}(u, \tau, R), (u', \tau', R') >_{L^2 \times L^2 \times L^2}.
\]

A direct calculation shows that \(\mathcal{B}\) is bounded. Using the divergence theorem, we have
\[
\mathcal{B}[(u, \tau, R), (u, \tau, R)] = \int_\Omega \left[ \rho \mu\dot{u} + A_{ijkl}e_{ij}e_{kl} + c\tau^2 + c_{ij}R_{i}R_{j} + \tilde{K}_{ij}\tau_{i}\tau_{j} + \tilde{C}_{ijkl}R_{i,k}R_{j,l} + 2W([u, \tau, R], (u, \tau, R)) \right] dv.
\]

In view of our assumptions on the constitutive coefficients, we see that \(\mathcal{B}\) is coercive on \(W^{-1,2}(\Omega) \times W^{1,2}(\Omega) \times W^{1,2}(\Omega)\). On the other hand, it is easy to see that the vector
\[
(u^* + v^* - B\tau^*, \theta^* + \tau^* - D\mu^* - G\tau^* - J\mathcal{R}^*, \mathcal{R}^* + \mathcal{M}^* - \mathcal{Z}\tau^* - \mathcal{P}\mathcal{R}^*)
\]
lies in \(W^{1,2}(\Omega) \times W^{1,2}(\Omega) \times W^{1,2}(\Omega)\). Hence the Lax-Milgram theorem (Gilbarg and Trudinger, 1983) implies the existence of \((u, \tau, R) \in W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)\) which solves equation (46). Now, we may also conclude the existence of \(v \in W_0^{1,2}(\Omega)\), \(\theta \in W_0^{1,2}(\Omega)\) and \(\mathcal{M} \in W_0^{1,2}(\Omega)\) solving system (45).

The previous lemmas lead to next theorem.

**Theorem 1.** The operator \(\mathcal{A}\) generates a semigroup of contraction in \(\mathcal{H}\).

**Proof.** The proof follows from Lumer-Phillips corollary to the Hille-Yosida theorem (Pazy, 1983).

It is worth remarking that this theorem implies that the dynamical system generated by the equations of thermoelasticity with diffusion of type III (or type II) is stable in the sense of Lyapunov.

**Theorem 2.** Assume that the conditions (i) - (iii) hold, \(f_i, s, Q_i \in C^1([0, \infty), L^2) \cap C^0([0, \infty), W_0^{1,2})\) and \(\mathcal{U}_0\) is in the domain of the operator \(\mathcal{A}\). Then, there exists a unique solution \(\mathcal{U}(t) \in C^1([0, \infty), \mathcal{H}) \cap C^0([0, \infty), \mathcal{D}(\mathcal{A}))\) to the problem (13).

Since the solutions are defined by means of a semigroup of contraction, we have the estimate
\[
\|\mathcal{U}(t)\|_\mathcal{H} \leq \|\mathcal{U}_0\|_\mathcal{H} + \int_0^t \left( \|f_i(\xi)\|_{L^2} + \|s(\xi)\|_{L^2} + \|Q_i(\xi)\|_{L^2} \right) d\xi
\]
which proves the continuous dependence of the solutions upon initial data and body loads. Thus, under assumptions (i) - (iii) the problem of linear thermoelasticity with microtemperatures of type III (or type II) is well posed.

5. **Asymptotic behavior of solutions**

In this section we study the asymptotic behavior of solutions, whose existence has been proved previously. We consider in the following sections the homogenous case \((f_i=0, s=0, Q_i = 0)\). In particular we are interested in the relation between dissipation effects and time decay of solutions. Therefore, we will continue to assume that the assumptions (i) - (iii) considered in the previous section hold. However it is worth noting that the results for this section only hold for type III theory.

To this end, we recall that for a semigroup of contraction, the precompact orbits tend to the \(\omega\)-limit sets if its generator \(\mathcal{A}\) has only the fixed point \(0\) (see Dafermos, 1976) and the structure of the \(\omega\)-limit sets is determined by the eigenvectors of eigenvalue \(i\lambda\) (where \(\lambda\) is a real number) in the closed subspace
\[
\mathcal{L} = \mathcal{C} \setminus \{U \in \mathcal{H} : \langle \mathcal{A}U, U \rangle = 0 \},
\]
where \(\mathcal{C} \setminus \{U \in \mathcal{H} : \langle \mathcal{A}U, U \rangle = 0 \}\) denotes the closed vectorial subspace generated by the set \(\mathcal{C}\).

From the assumptions (i)-(iii) it is easy to check that \(\mathcal{A}^{-1}(0) = \{0\}\), while the precompactness of the orbits starting in \(\mathcal{D}\) is a consequence of the following Lemma (Pazy, 1983).
Lemma 3. The operator \((I - \mathcal{A})^{-1}\) is compact.

Proof. Let \((u_n, v_n, \tau_n, \theta_n, R_n, M_n)\) be a bounded sequence in \(\mathcal{H}\) and let \(U_n = (u_n, v_n, \tau_n, \theta_n, R_n, M_n)\) be the sequence of the respective solutions to the system (43). We have

\[
\mathcal{P}(U_n, U_n) = \mathcal{P}(u_n, \tau_n, R_n, v_n, \tau_n, R_n) \leq \text{Constant} \times \mathcal{P}(U_n, U_n)^{\frac{1}{2}}.
\]  

(47)

Inequality (47) implies that \((u_n, \tau_n, R_n)\) is a bounded sequence in \(W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)\). The theorem of Rellich-Kondrasov (Ciarlet, 1988) implies that there exists a subsequence converging in \(L^2(\Omega) \times L^2(\Omega)\). In a similar way

\[
v_{n_j} = u_{n_j} - \hat{u}_{n_j}, \quad \theta_{n_j} = \tau_{n_j} - \hat{\tau}_{n_j}, \quad M_{n_j} = R_{n_j} - \hat{R}_{n_j}
\]

has a sub-sequence converging in \(L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)\). Thus we conclude the existence of a sub-sequence

\[
(u_{n_{jk}}, v_{n_{jk}}, \tau_{n_{jk}}, \theta_{n_{jk}}, R_{n_{jk}}, M_{n_{jk}})
\]

which converges in \(\mathcal{H}\).

Now, we can state a theorem on the asymptotic behavior of solutions.

Theorem 3. Let \(U_0 = (u^0, v^0, \tau^0, \theta^0, R^0, M^0) \in \mathcal{P}(\mathcal{A})\) and \(U(t)\) be the solution to the problem (43) with \(\mathcal{F} = 0\). Then

\[
\tau(t) \to 0 \text{ as } t \to \infty \text{ in } W_0^{1,2}(\Omega) \quad \text{and} \quad \theta(t) \to 0 \text{ as } t \to \infty \text{ in } L^2(\Omega).
\]

Moreover

\[
u(t) \to 0 \text{ as } t \to \infty \text{ in } W_0^{1,2}(\Omega) \quad \text{and} \quad v(t) \to 0 \text{ as } t \to \infty \text{ in } L^2(\Omega)
\]

\[
R(t) \to 0 \text{ as } t \to \infty \text{ in } W_0^{1,2}(\Omega) \quad \text{and} \quad M(t) \to 0 \text{ as } t \to \infty \text{ in } L^2(\Omega)
\]

whenever the system

\[
\begin{align*}
Au + \lambda^2 u &= 0 \text{ in } \Omega \\
Dv &= 0 \text{ in } \Omega \\
Lu &= 0 \text{ in } \Omega \\
u &= 0 \text{ on } \partial \Omega
\end{align*}
\]

(48)

has only the null solution.

Proof. To prove the theorem we have to study the structure of the \(\omega\)-limit set. Thus, we must study the equation

\[
\mathcal{A}U = i\lambda U
\]

(49)

for some real number \(\lambda\), where \(U \in \mathcal{P}(\mathcal{A})\) and \(\mathcal{A} = \mathcal{A}\mathcal{L}\) is the generator of a group on \(\mathcal{L}\). If \(U \in \mathcal{L}\) then \(\mathcal{A}U, U > 0\). Under the assumption that the tensors \(\tilde{K}_{ij}\) and \(\tilde{C}_{ijkl}\) are definite positive, it follows that \(\theta = M_i = 0\) and then \(\tau = R_i = 0\). Thus, the asymptotic behavior of the temperature and the microtemperatures is proved. Now, Eq. (49) can be rewritten as

\[
\begin{pmatrix}
0 & I & 0 & 0 & 0 & 0 \\
A & 0 & 0 & B & C & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
0 & D & E & G & 0 & J \\
0 & 0 & 0 & 0 & I & 0 \\
L & 0 & 0 & Z & N & P
\end{pmatrix}
\begin{pmatrix}
u \\
\Theta
\end{pmatrix}
= i\lambda
\begin{pmatrix}
u \\
\Theta
\end{pmatrix}.
\]

Introducing the first equation into the others we obtain the system (43).

If the system (43) has only the trivial solution, we obtain \(\omega\)-limit\((U_0) = 0\) and \(u(t) \to 0 \text{ in } W_0^{1,2}(\Omega)\)

\[
\text{and } v(t) \to 0 \text{ in } L^2(\Omega) \text{ when } t \to \infty.
\]
6. Impossibility of localization in time

In previous sections we have proved that the solutions of type III theory are stable asymptotically and also in the sense of Lyapunov. A natural question is to ask if the decay is fast enough to guarantee that the solutions vanishing in a finite time. In fact, when the dissipation mechanism in a system is sufficiently strong, the localization of solutions in the time variable can hold. This means that the decay of the solutions is sufficiently fast to guarantee that they vanish after a finite time.

In the context of Green-Naghdi thermoelasticity of type III, Quintanilla (2007) has shown that the thermal dissipation is not strong enough to obtain the localization in time of the solutions. In this section we assume the quadratic form \((38)\) positive definite and prove that the further dissipation effects due to the microtemperatures of type III are not sufficiently strong to guarantee that the thermomechanical deformations vanish after a finite interval of time. This means that, in absence of sources, the only solution for the evolutive problem that vanishes after a finite time is the null solution, that is the following theorem holds.

**Theorem 4.** Let \((u_i, \tau, R_i)\) be a solution of the system \((32), (35)\) and \((37)\) which vanishes after a finite time \(t\). Then \((u_i, \tau, R_i) \equiv (0, 0, 0)\) for every \(t \geq 0\).

In order to prove this theorem, generalizing the technique used in Quintanilla (2007), we show the uniqueness of solutions for the related backward in time problem. Backward in time problems are relevant from the mechanical point of view when we want to have some information about what happened in the past by means of the information that we have at this moment.

For our model, the system of equations which govern the backward in time problem is given by

\[
\begin{aligned}
\rho \ddot{u}_i &= (A_{ijkl}e_{kl} + a_{ij}\dot{\tau} + B_{ijkl}R_{i,k,j})_j + \rho f_i, \\
c_i \dot{R}_j &= (B_{ijkl}e_{kl} + b_{ij}\dot{\tau} + C_{ijkl}R_{i,k,l} - \tilde{C}_{ijkl}\dot{R}_{k,l})_j + d_{ij} \dot{\tau}_j + \rho Q_i, \\
b_{ij} \dot{\tau}_j &= (B_{ijkl}e_{kl} - C_{ijkl}R_{i,k,l} + \tilde{C}_{ijkl}\dot{R}_{k,l})_j + d_{ij} \dot{\tau}_j + \rho Q_i.
\end{aligned}
\]

**Proposition 1 (Uniqueness).** Let \((u_i, \tau, R_i)\) be a solution to the problem \((51), (57)\) with null initial data and sources. Then \((u_i, \tau, R_i) = (0, 0, 0)\) for every \(t \geq 0\).

**Proof.** Let us introduce the following functionals

\[
\begin{aligned}
E_1(t) &= \frac{1}{2} \int_{\Omega} \left( \rho \dot{u}_i \ddot{u}_i + c_i \dot{R}_j \ddot{R}_j + A_{ijkl} \dot{u}_{i,j} \dot{u}_{k,l} + 2B_{ijkl} \dot{u}_{i,j} R_{k,l} + K_{ij} \dot{\tau}_i \dot{\tau}_j + C_{ijkl} R_{i,j,k,l} \right) dv, \\
E_2(t) &= \frac{1}{2} \int_{\Omega} \left( \rho \dot{u}_i \ddot{u}_i - c_i \dot{R}_j \ddot{R}_j + A_{ijkl} \dot{u}_{i,j} \dot{u}_{k,l} - K_{ij} \dot{\tau}_i \dot{\tau}_j - C_{ijkl} R_{i,j,k,l} \right) dv, \\
E_3(t) &= \int_{\Omega} \left( \rho \dot{u}_i \ddot{u}_i + c_i \dot{R}_j \ddot{R}_j + \frac{1}{2} K_{ij} \dot{\tau}_i \dot{\tau}_j + \frac{1}{2} \tilde{C}_{ijkl} R_{i,j,k,l} + a_{ij} \tau_{i,j} \right) dv,
\end{aligned}
\]

and compute their time derivatives. By multiplying the first equation of \((50)\) by \(\dot{u}_i\), the second one by \(\dot{\tau}\) and the third one by \(\dot{R}_i\), we get

\[
\dot{E}_1(t) = \int_{\Omega} \left( K_{ij} \dot{\tau}_i \dot{\tau}_j + \tilde{C}_{ijkl} \dot{R}_{i,j,k,l} \right) dv.
\]

If we multiply the first equation of \((50)\) by \(\ddot{u}_i\), the second one by \(\ddot{\tau}\) and the third one by \(\ddot{R}_i\), we have

\[
\dot{E}_2(t) = - \int_{\Omega} \left( - 2a_{ij} \dot{u}_i \ddot{\tau}_j + B_{ijkl}(\dot{u}_{i,j} R_{k,l} - u_{k,l} \dot{R}_{i,j}) + \tilde{K}_{ij} \dot{\tau}_i \dot{\tau}_j + \tilde{C}_{ijkl} \dot{R}_{k,l} \dot{R}_{i,j} \right) dv
\]

and, finally, if we multiply the first equation of \((50)\) by \(\dddot{u}_i\), the second one by \(\dddot{\tau}\) and the third one by \(\dddot{R}_i\), we obtain

\[
\dot{E}_3(t) = - \int_{\Omega} \left( A_{ijkl} u_{k,l} u_{i,j} + b_{ij}(\dot{R}_{i,j} - \dot{R}_{i,j}) + c_i \dot{R}_j \dddot{R}_j - K_{ij} \dot{\tau}_i \dot{\tau}_j - C_{ijkl} R_{i,j,k,l} - \rho \dddot{u}_i \right) dv.
\]
Moreover, a well-known identity for type III thermoelasticity (see equation (3.9) in [Quintanilla and Straughan 2000]) for our model becomes

\[
\int_\Omega \left( A_{ijkl} u_{i,j} u_{k,l} + c \tau^2 + c_{ij} \dot{R}_i \dot{R}_j \right) dv = \int_\Omega \left( \rho \ddot{u}_i + K_{ij} \tau_i \tau_j + C_{ijkl} R_{i,j} R_{k,l} \right) dv. \quad (51)
\]

Then we have

\[
E_2(t) = \int_\Omega \left( A_{ijkl} u_{i,j} u_{k,l} - K_{ij} \tau_i \tau_j - C_{ijkl} R_{i,j} R_{k,l} \right) dv
\]

and

\[
\dot{E}_2(t) = - \int_\Omega \left( b_{ij} (\tau \dot{R}_{i,j} - \tau R_{i,j}) + d_{ij} (\tau \dot{R}_i - \tau R_i) \right) dv.
\]

We consider the function

\[
\mathcal{E}(t) = \int_0^t [\epsilon E_1(s) + E_2(s) + \lambda E_3(s)] ds
\]

where \( \epsilon \) and \( \lambda \) are positive suitable constants such that the quadratic form

\[
\int_\Omega \left\{ [\lambda \dot{K}_{ij} + (\epsilon - 2) K_{ij}] \tau_i \tau_j + [\lambda \dot{C}_{ijkl} + (\epsilon - 2) C_{ijkl}] R_{i,j} R_{k,l} \right\} dv
\]

is positive definite. By using the null initial data hypothesis and the Poincaré inequality we have

\[
\lambda \int_0^t \int_\Omega \left[ \rho u_{i,j} - c \tau_i \tau_j - c_{ij} R_i \dot{R}_j \right] dv \leq \frac{\epsilon}{4} \int_0^t \int_\Omega [\rho \ddot{u}_i + c \tau^2 + c_{ij} \dot{R}_i \dot{R}_j] dv ds
\]

for any \( t \leq t_0 \), where \( t_0 \) is a positive time which depends on \( \lambda, \epsilon \) and the constitutive coefficients. Therefore \( \mathcal{E}(t) \) is a positive definite quadratic form for \( 0 \leq t \leq t_0 \), in particular

\[
\mathcal{E}(t) \geq \frac{1}{4} \int_0^t \int_\Omega \left[ \rho \ddot{u}_i + c \tau^2 + c_{ij} \dot{R}_i \dot{R}_j + (\epsilon + 2) A_{ijkl} u_{i,j} u_{k,l} + 2 \epsilon B_{ijkl} u_{i,j} R_{k,l} + 2 \lambda a_{ij} \tau u_{i,j} \right] dv ds
\]

\[
+ \frac{1}{4} \int_0^t \int_\Omega \left\{ [\lambda \dot{K}_{ij} + (\epsilon - 2) K_{ij}] \tau_i \tau_j + [\lambda \dot{C}_{ijkl} + (\epsilon - 2) C_{ijkl}] R_{i,j} R_{k,l} \right\} dv ds. \quad (52)
\]

Moreover, recalling the null initial data assumption, we have

\[
\dot{\mathcal{E}}(t) = (\epsilon - 1) \int_0^t \int_\Omega \left( K_{ij} \dot{\tau}_i \dot{\tau}_j + \dot{C}_{ijkl} \dot{R}_{i,j} \dot{R}_{k,l} \right) dv ds - \int_0^t \int_\Omega \left( -2 a_{ij} \dot{u}_i \tau_j + B_{ijkl} \dot{u}_{i,j} R_{k,l} - u_{k,l} \dot{R}_{i,j} \right) dv ds
\]

\[
+ \lambda b_{ij} (\tau \dot{R}_{i,j} - \tau R_{i,j}) + \lambda d_{ij} (\tau \dot{R}_i - \tau R_i) \right\} dv ds.
\]

Choosing \( 0 < \epsilon < 1 \) and using the inequality of arithmetic and geometric means, we have

\[
\left| \int_0^t \int_\Omega \left( -2 a_{ij} \dot{u}_i \tau_j + B_{ijkl} \dot{u}_{i,j} R_{k,l} - u_{k,l} \dot{R}_{i,j} \right) dv ds + \lambda b_{ij} (\tau \dot{R}_{i,j} - \tau R_{i,j}) + \lambda d_{ij} (\tau \dot{R}_i - \tau R_i) \right\} dv ds \right| \leq
\]

\[
(1 - \epsilon) \int_0^t \int_\Omega \left( K_{ij} \dot{\tau}_i \dot{\tau}_j + \dot{C}_{ijkl} \dot{R}_{i,j} \dot{R}_{k,l} \right) dv ds + K_1 \int_0^t \int_\Omega \rho \ddot{u}_i dv ds + K_2 \int_0^t \int_\Omega c \tau^2 + c_{ij} \dot{R}_i \dot{R}_j dv ds + K_3 \int_0^t \int_\Omega A_{ijkl} u_{i,j} u_{k,l} dv ds + K_4 \int_0^t \int_\Omega B_{ijkl} u_{i,j} R_{k,l} dv ds + K_5 \int_0^t \int_\Omega a_{ij} u_{i,j} \tau dv ds
\]

\[
+ \frac{1}{2} \int_0^t \int_\Omega \left\{ [\lambda \dot{K}_{ij} + (\epsilon - 2) K_{ij}] \tau_i \tau_j + [\lambda \dot{C}_{ijkl} + (\epsilon - 2) C_{ijkl}] R_{i,j} R_{k,l} \right\} dv ds
\]

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where the positive constants $K_i$ can be calculated by standard methods, so that
\[
\dot{E}(t) \leq K \int_0^t \int_\Omega \left\{ \epsilon \rho \dot{u}_i \dot{u}_i + \epsilon \sigma^2 + \epsilon \gamma \tau_i \tau_i + (\epsilon + 2) A_{ijkl} R_{ik} R_{jl} + 2 \lambda a_{ijkl} \tau_i \tau_j \right\} \, dv \, ds
\]
\[
+ K \int_0^t \int_\Omega \left\{ |\lambda \hat{K}_{ij} + (\epsilon - 2) K_{ij} | \tau_i \tau_j + |\lambda \hat{C}_{ijkl} + (\epsilon - 2) C_{ijkl} | R_{ik} R_{jl} \right\} \, dv \, ds
\]
with $K = \max \{ \frac{1}{2}, K_1, K_2, K_3, K_4, K_5 \}$. Inequalities (52) and (53) yield
\[
\dot{E}(t) \leq 4K \mathcal{E}(t), \quad 0 \leq t \leq t_0.
\]
This inequality and the null initial data imply $\dot{E}(t) \equiv 0$ if $0 \leq t \leq t_0$. Reiterating this argument on each subinterval $[(n-1)t_0, nt_0]$ we obtain $\dot{E}(t) \equiv 0$ for $t \geq 0$.
If we take into account the definition of $\dot{E}(t)$, the uniqueness result is proved. \hfill \Box

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References

Aouadi, M., 2012. Uniqueness and Existence Theorems in Thermoelasticity with Voids without Energy Dissipation. J. Franklin Inst. 349, 128–139.
Aouadi, M., Ciarietlatta, M., Iovane, G., 2016a. A Porous Thermoelastic Diffusion Theory of Types II and III. Acta Mech., 1–19.
Aouadi, M., Ciarietlatta, M., Tibullo, V., 2016b. A Thermoelastic Diffusion Theory with Microtemperatures and Microconcentrations. J. Therm. Stresses, 1–16.
Aouadi, M., Lazzari, B., Nibbi, R., 2014. A Theory of Thermoelasticity with Diffusion Under Green-Naghdi Models. Z. Angew. Math. Mech. 94, 837–852.
Ciarietlatta, P.G., 1988. Mathematical Elasticity, Volume I: Three-Dimensional Elasticity. Nort-Holland, Amsterdam.
Dafermos, C.M., 1976. Contraction Semigroups and Trend to Equilibrium in Continuum Mechanics, in: German, P., Nayroles, B. (Eds.), Applications of Methods of Functional Analysis to Problems in Mechanics. Springer-V erlag, Berlin, pp. 295–306.
Eringen, A.C., 1999. Microcontinuum Field Theories. I. Foundations and Solids. Springer, New York.
Gilbarg, D., Trudinger, N.S., 1983. Elliptic Partial Differential Equations of Second Order. Springer-Verlag, Berlin.
Giorgi, C., Montanaro, A., 2016. Constitutive Equations and Wave Propagation in Green-Naghdi Type II and III Thermoelectroelasticity. J. Thermal Stresses 39, 1051–1073.
Green, A.E., Naghdi, P.M., 1991a. A Demonstration of Consistency of an Entropy Balance with Balance Energy. Z. Angew. Math. Phys. 42, 159–168.
Green, A.E., Naghdi, P.M., 1991b. A Re-Examination of the Basic Postulates of Thermomechanics. P. Roy. Soc. A/Math. Phy. 432, 171–194.
Green, A.E., Naghdi, P.M., 1993. Thermoelasticity without Energy Dissipation. J. Elast. 31, 189–208.
Grot, R.A., 1969. Thermodynamics of a Continuum with Microstructure. Int. J. Eng. Sci. 7, 801–814.
Hetaarsaki, R., Ignaczak, J., 1999. Generalized Thermoelasticity. J. Therm. Stresses 22, 451–476.
Ieșan, D., 2001. On a Theory of Micromorphic Elastic Solids with Microtemperatures. J. Therm. Stresses 24, 737–752.
Ieșan, D., 2007. Thermoelasticity of Bodies with Microstructure and Microtemperatures. Int. J. Solids Struct. 44, 8648–8662.
Ieșan, D., Quintanilla, R., 2000. On a Theory of Thermoelasticity with Microtemperatures. J. Therm. Stresses 23, 199–215.
Ieșan, D., Quintanilla, R., 2009. On Thermoelastic Bodies with Inner Structure and Microtemperatures. J. Math. Anal. Appl. 354, 12–23.
Lazzari, B., Nibbi, R., 2008. On The Exponential Decay in Thermoelasticity without Energy Dissipation and of Type III in Presence of an Absorbing Boundary. J. Math. Anal. Appl. 358, 317–329. doi:10.1016/j.jmaa.2007.05.017
Pazy, A., 1983. Semigroups of Linear Operators and Applications to Partial Differential Equations, volume 44 of Applied Mathematical Sciences. Springer-Verlag, New York.
Puri, P., Jordan, P.M., 2004. On the Propagation of Plane Waves in Type-III Thermoelastic Media. P. Roy. Soc. A/Math. Phy. 460, 3203–3221.
Quintanilla, R., 2007. On the Impossibility of Localization in Linear Thermoelasticity. P. Roy. Soc. A/Math. Phy. 463, 3311–3322.
Quintanilla, R., 2009. Uniqueness in Thermoelectricity of Porous Media with Microtemperatures. Arch. Mech. 61, 371–382.
Quintanilla, R., Straughan, B., 2000. Growth and Uniqueness in Thermoelectricity. P. Roy. Soc. A/Math. Phy. 456, 1419–1429.
Wozniak, C., 1967a. Thermoelectricity of Bodies with Micro-structure. Arch. Mech. Stos. 19, 335.
Wozniak, C., 1967b. Thermoelectricity of Non-Simple Oriented Materials. Int. J. Eng. Sci. 5, 605–612.