WELL-POSEDNESS AND DECAY OF SOLUTIONS TO 3D GENERALIZED NAVIER-STOKES EQUATIONS

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Abstract. The global well-posedness and large time behavior of solutions for the Cauchy problem of the three-dimensional generalized Navier-Stokes equations are studied. We first construct a local continuous solution, then by combining some a priori estimates and the continuity argument, the local continuous solution is extended to all \( t > 0 \) step by step provided that the initial data is sufficiently small. In addition, by using Strauss’s inequality, generalized interpolation type lemma and a bootstrap argument, we establish the \( L^p \) decay estimate for the solution \( u(\cdot, t) \) and all its derivatives for generalized Navier-Stokes equations with \( \max\{1, \frac{\alpha+2}{6}\} < \alpha \leq \frac{1}{2} + \min\left\{ \frac{2}{q}, \frac{3}{2p} \right\} \).

1. Introduction. In this manuscript, we consider the following Cauchy problem for generalized incompressible Navier-Stokes equations in \( \mathbb{R}^3 \):

\[
\begin{cases}
  u_t + u \cdot \nabla u + (-\Delta)^{\alpha} u + \nabla \pi = 0, \\
  \nabla \cdot u = 0, \\
  u(x, 0) = u_0(x),
\end{cases}
\]

(1)

where \( u = (u_1, u_2, u_3) \in \mathbb{R}^3 \) is the velocity field of the fluid and \( \pi \in \mathbb{R} \) is the pressure. The fractional Laplacian operator \((-\Delta)^{\alpha}\) is defined through the Fourier transform (see [27]), namely

\[
(-\Delta)^{\alpha} f(x) = \Lambda^{2\alpha} f(x) = \int_{\mathbb{R}^3} |x|^{2\alpha} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi,
\]

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and $\hat{f}$ is the Fourier transform of $f$. When $\alpha = 1$, system (1) reduces to be Navier-Stokes equations, which have drawn much attention (cf. [17, 18, 1, 12, 15] and the references therein).

The generalized Navier-Stokes equations was first studied by J. L. Lions [20]. The author proved that when $\alpha \geq \frac{5}{4}$, the 3D generalized Navier-Stokes equations has a global and unique regular solution (see also in [31] and [34], the MHD equations reduces to the Navier-Stokes equations as the magnetic field $b = 0$). However, when $\alpha < \frac{5}{4}$, the global well-posedness of 3D generalized Navier-Stokes system remains open. Accordingly, there are many literatures devoted to find regularity criteria or prove partial regularity for 3D generalized Navier-Stokes system, such as [2, 5, 32] and [34]. Another direction is to obtain its global existence of strong or smooth solutions for the generalized Navier-Stokes equations or generalized MHD equations with small initial data belonging to a variety of spaces, for example, the pseudomeasure space [22], and the space $\chi^{1-2\alpha}$ [33].

For the 3D incompressible Navier-Stokes equation, if the $L^3$-norm of the initial data is sufficiently small, we can obtain the global existence result. For the generalized Navier-Stokes equation, we also have the similar result. In this article, applying the theory of mild solution, we first establish the new global well-posedness for the solutions of system (1) with small initial data $u_0$ belongs to $L^{\frac{5}{3}}(\mathbb{R}^3)$. More precisely, the result can be stated as follows.

**Theorem 1.1.** Let $r > 0$ be any given constant and $\frac{1}{2} < \alpha < \frac{5}{4}$. Assume that $u_0(x) \in L^\infty \cap L^{\frac{5}{3}}(\mathbb{R}^3)$ with $\|u_0(x)\|_{L^\infty} \leq r$ and $\|u_0(x)\|_{L^{\frac{5}{3}}}^{\frac{5}{3}}$ is sufficiently small, then there exists a unique global solution $u(x, t) \in L^\infty(0, \infty; L^{\frac{5}{3}}(\mathbb{R}^3))$ such that

$$\|u(t, x)\|_{L^\infty} \leq 2r, \quad t \geq 0.$$ (2)

**Remark 1.** If $u(x, t)$ is a solution to system (1), then $u_\lambda(x, t)$ with any $\lambda > 0$ is also a solution, where $u_\lambda(x, t) = \lambda^{2\alpha-1}u(\lambda x, \lambda^{2\alpha}t)$. Motivated by Caffarelli-Kohn-Nirenberg [1] for the Navier-Stokes equations, we say that the norm $\|u\|_{L^{p, q}}$ is scaling dimension zero for $\frac{2\alpha}{p} + \frac{2}{q} = 2\alpha - 1$ in the sense that $\|u_\lambda\|_{L^{p, q}} = \|u\|_{L^{p, q}}$ holds for all $\lambda > 0$ if and only if $\frac{2\alpha}{p} + \frac{2}{q} = 2\alpha - 1$. A simple calculation shows that $\|u_\lambda\|_{L^{\frac{5}{3}}(\mathbb{R}^3)} = \|u\|_{L^{\frac{5}{3}}(\mathbb{R}^3)}$ holds for all $\lambda > 0$. Hence, we can consider the global well-posedness result in the critical space $L^{\frac{5}{3}}(\mathbb{R}^3)$.

For the case $\alpha \geq \frac{5}{4}$ of generalized Navier-Stokes equations, Lions [20], Wu [31] and Zhou [34] proved the existence and uniqueness of global regular solution. The authors pointed out that: If $T > 0$, $\alpha \geq \frac{5}{4}$, $u_0 \in H^s$ and $s \geq 2\alpha$, then there exists a unique classical solution for system (1), which satisfy

$$u \in L^\infty([0, T]; H^s) \bigcap L^2([0, T]; H^{s+\alpha}).$$

Here, we want to point out that if $\alpha \geq \frac{5}{4}$, by using the similar method as Theorem 1.1, we also have the global well-posedness result for system (1).

**Theorem 1.2.** Let $r > 0$ be any given constant and $\alpha \geq \frac{5}{4}$. Assume that $u_0(x) \in L^\infty \cap L^2(\mathbb{R}^3)$ with $\|u_0(x)\|_{L^\infty} \leq r$ and $\|u_0(x)\|_{L^2}$ is sufficiently small, then there exists a unique global solution $u(x, t) \in L^\infty(0, \infty; L^2)$ such that

$$\|u(t, x)\|_{L^\infty} \leq 2r, \quad t \geq 0.$$ (3)
Remark 2. As $\alpha \geq \frac{5}{4}$, a global existence can be established for arbitrary large datum. But unfortunately, the smallness of $\|u_0(x)\|_{L^2}$ is needed to get $\|u(t, x)\|_{L^\infty} \leq 2r$.

Recently, applying Fourier splitting method, Jiu and Yu [11] studied the upper bound of decay rate for $L^2$-norm of the solutions of system (1) with $0 < \alpha < \frac{5}{4}$, $u_0 \in L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ and $\max\{1, \frac{1}{3\cdot 2^{2q}}\} \leq p < 2$. It is proved that

$$\|u\|_{L^2}^2 \leq C(1 + t)^{-\frac{3}{3\cdot 2^q}(\frac{5}{4} - 1)}.$$ 

The authors also gave the result on the temporal decay estimates for high order Sobolev norms of solutions of system (1) under some assumptions. By analyzing the above result, we find that there are two interesting problems need to be investigated:

- First, if $\alpha > 1$ and $u_0 \in L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ with arbitrary $p \geq 1$, how about the decay rate of solutions for system (1)?
- Second, it seems that the previous methods only valid for $\alpha < \frac{5}{4}$. Could we find a new one to overcome the difficulty when $\alpha \geq \frac{5}{4}$?

Remark 3. One of the powerful tools to establish the decay rate of solutions is Fourier splitting method, which is introduced by Schonbek in 1980s (see [25, 26]). It is worth pointing out that Kreiss, Hagstrom, Lorenz and Zingano [16] and Zhou [34] introduced a new method to handle decay rate problems. Their method provides a very simple derivation of Wiegner’s fundamental results (see [30]). One can refer to [36, 9, 10] for details and developments.

The last goal of this paper is to extend the results on decay rate of system (1) for the parameter $\alpha \in \left(\max\{1, \frac{2+q}{n}\}, \frac{1}{2} + \min\{\frac{2}{q} - \frac{3}{p}, \frac{3}{2^q}\}\right)$ when $u_0 \in L^p \cap L^q(\mathbb{R}^3; \mathbb{R}^3)$.

To achieve this, we suppose that $\|u_0\|_{L^p}$ is sufficiently small, adopt Strauss’s inequality to obtain the $L^q$-norm estimate for the solution of system (1) and show the $L^p$-decay rate of solutions. In addition, we expand one of Kukavica and Torres’ interpolation type lemma [13, 14] to a generalized case, use parabolic interpolation lemma and a bootstrap argument, establish the $L^p$-norm temporal decay rate for higher-order derivatives. More precisely, we have the following theorems.

**Theorem 1.3** ($L^p$-decay estimate). Assume that $p, q$ are two positive constants, which satisfy $2 \leq p \leq \infty$, $1 \leq q < p$, $\min\{\frac{2}{q} - \frac{3}{p}, \frac{3}{2^q}\} > \frac{1}{2}$, $\max\{1, \frac{3\cdot q + 9}{2^q}\} < \alpha \leq \frac{1}{2} + \min\{\frac{2}{q} - \frac{3}{p}, \frac{3}{2^q}\}$, $\nabla \cdot u_0 = 0$, $u_0 \in L^q \cap L^p(\mathbb{R}^3)$ and $\|u_0\|_{L^p}$ sufficiently small. Then there exists a positive constant $C$ such that the small global-in-time solution satisfies

$$\|u(x, t)\|_{L^p(\mathbb{R}^3)} \leq C(1 + t)^{-\frac{3\cdot q}{2^q}(\frac{5}{4} - 1)}, \text{ for large } t. \quad (4)$$

**Theorem 1.4** ($\dot{W}^{s,p}$-decay estimate). Assume that $p, q$ are two positive constants, which satisfy $2 \leq p \leq \infty$, $1 \leq q < p$, $\min\{\frac{2}{q} - \frac{3}{p}, \frac{3}{2^q}\} > \frac{1}{2}$, and $pq \leq \min\{9, 18 - \frac{2}{p}\}$. Suppose that $\max\{1, \frac{3\cdot q + 9}{2^q}\} < \alpha \leq \frac{1}{2} + \min\{\frac{2}{q} - \frac{3}{p}, \frac{3}{2^q}\}$, $\nabla \cdot u_0 = 0$, $u_0 \in L^q \cap \dot{W}^{s,p}(\mathbb{R}^3)$ and $\|u_0\|_{L^p}$ sufficiently small. Then, for all multi-index $\kappa \in \mathbb{N}_0^3$, there exists a positive constant $C$ such that the small global-in-time solution satisfies

$$\|\partial_\kappa u(x, t)\|_{L^p(\mathbb{R}^3)} \leq (1 + t)^{-\frac{n}{2}(\frac{5}{4} - 1)} \cdot \frac{1}{\kappa!} \quad \text{for large } t. \quad (5)$$
Remark 4. If $0 < \alpha \leq 1$, $m \in \mathbb{N}_0$ and $u_0 \in L^2(\mathbb{R}^3) \cap L^9(\mathbb{R}^3)$, the $L^2$-decay rate of solutions for system (1) is

$$\|u\|_{L^2} \leq C(1 + t)^{-\frac{3}{2(n-1)}},$$

and the $H^m$-decay rate of small global-in time solutions is

$$\|D^m u\|_{L^2} \leq C(1 + t)^{-\frac{3}{2(n-1)} - \frac{m}{n}},$$

which can be found in Jiu and Yu [11]. Then by Gagliardo-Nirenberg inequality, for $2 \leq p \leq \infty$, we have

$$\|D^m u(x, t)\|_{L^p(\mathbb{R}^3)} \leq C(1 + t)^{-\frac{3}{2(n-1)} - \frac{m}{n}}, \quad \text{for large } t.$$

A typical case for Theorems 1.3 and 1.4 is $q = 1$, $p = 2$, $1 \leq \alpha \leq 2$ and $\|u_0\|_{L^3}$ sufficiently small. At this time, we have the following results:

$$\|u(x, t)\|_{L^2(\mathbb{R}^3)} \leq C(1 + t)^{-\frac{1}{2}}, \quad \text{for large } t,$n

$$\|\partial_t u(x, t)\|_{L^2(\mathbb{R}^3)} \leq (1 + t)^{-\frac{3}{4} - \frac{n}{2(n-1)}}, \quad \text{for large } t.$$

Another interesting circumstance is $q = \frac{3}{2(n-1)} < 2 = p$, $1 \leq \alpha \leq \frac{5}{4}$ and $\|u_0\|_{L^{\frac{3}{2(n-1)}}}$ sufficiently small. Let us remind $L^{\frac{3}{2(n-1)}}$ is the critical space, and the global well-posedness of solutions for generalized Navier-Stokes equations in this space has been established in Theorem 1.2. Setting $p = 2$ and $q = \frac{3}{2(n-1)}$ in (4) and (5), we derive that

$$\|u(x, t)\|_{L^2(\mathbb{R}^3)} \leq C(1 + t)^{-\frac{1}{2}}, \quad \text{for large } t,$$

$$\|\partial_t u(x, t)\|_{L^2(\mathbb{R}^3)} \leq (1 + t)^{-\frac{3}{4} - \frac{n}{2(n-1)}}, \quad \text{for large } t.$$

We find out that $\alpha = \frac{5}{4}$ is a special critical value, which is coincide with Lions [20], Wu [31] and Zhou [34].

Theorems 1.3 and 1.4 are the first $L^p$ and $\dot{W}^{s,p}$ decay estimates for the hydrodynamic equations with fractional Laplacians.

The rest of the paper is organized as follows. In the next section, we introduce some preliminary results. Section 3 is devoted to the proof of Theorem 1.1 whereas Section 4 deals with the proof of Theorem 1.2. In the last section, we give the proof of Theorems 1.3 and 1.4.

2. Preliminaries. In the proof of lemmas and theorems, we frequently employ the Gagliardo-Nirenberg inequality:

Lemma 2.1 (Gagliardo-Nirenberg inequality [6]). Let $u \in L^q(\mathbb{R}^n)$, $\nabla^m u \in L^r(\mathbb{R}^n)$, $1 \leq q, r \leq \infty$. Then, there exists a positive constant $C = C(n, m, j, a, q, r)$, such that

$$\|\nabla^j u\|_{L^p} \leq C\|\nabla^m u\|_{L^r}^{\frac{p}{r}}\|u\|_{L^q}^{\frac{1}{r} - \frac{a}{p}},$$

where

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1 - a)\frac{1}{q}, \quad 1 \leq p \leq \infty, \quad 0 \leq j \leq m, \quad \frac{j}{m} \leq a \leq 1.$$

The following two lemmas will be used in this section.

Lemma 2.2 (Strauss’s inequality [28]). Suppose that $M(t)$ is a nonnegative continuous function of $t$. $M(t)$ satisfy

$$M(t) \leq d_1 + d_2 M(t)^r$$

in some interval containing 0, where $d_1$ and $d_2$ are positive constants and $r > 1$. If $M(0) \leq d_1$ and

$$d_1 d_2 < (1 - r^{-1}) r^{-(r-1)^{-1}},$$
then in the same interval

\[ M(t) \leq \frac{d_1}{1 - r^{-1}}. \]

**Lemma 2.3** (Singular Gronwall’s inequality [23]). Assume that \( g(t) \) is a nonnegative continuous function defined on \([s, T]\) and satisfies

\[ g(t) \leq N_1(t - b)(t - a)^{-\alpha} + N_2(t - b) \int_a^t (t - s)^{-\alpha} g(s)ds, \]

where \( s, \alpha, a \) and \( b \) are positive constants satisfying

\[ 0 < \alpha < 1, \ s > \max\{a, b\}, \]

and \( N_i(t - b) \) (\( i = 1, 2 \)) are continuous increasing functions of \( t \). Then,

\[ g(t) \leq (t - a)^{-\alpha}N(t - a, t - b) < \infty, \ s \leq t \leq T, \]

with

\[ N(t - a, t - b) = N_1(t - b) \left\{ 1 + \sum_{j=1}^{\infty} \frac{\Gamma(1 - \alpha)}{\Gamma((j + 1)(1 - \alpha))} \times [\Gamma(1 - \alpha)N_2(t - b)(t - a)^{1-\alpha}]^j \right\}, \]

It is easy to see that \( N(t - a, t - b) \) is a continuous increasing function of \( t \).

In [24], Miao et. al. establish the following space-time estimates for the generalized heat equation

\[ \begin{cases} \partial_t \tilde{u} + (-\Delta)^{\alpha} \tilde{u} = 0, & x \in \mathbb{R}^3, \\ \tilde{u}(x, 0) = u_0(x), & x \in \mathbb{R}^3. \end{cases} \]  

**Lemma 2.4** ([24]). Let \( 1 \leq r \leq q \leq \infty \) and \( v_0(x) \in L^r(\mathbb{R}^3) \). Then, for the solution \( v(x, t) \) of problem (6), we have

\[ \|\tilde{u}(x, t)\|_{L^p} \leq C t^{-\frac{\mu}{\alpha} - \frac{\mu}{1-\frac{\mu}{q}} + \frac{1}{q}} \|u_0\|_{L^r}, \]  

and

\[ \|(-\Delta)^{\frac{\mu}{2}}\tilde{u}(x, t)\|_{L^p} \leq C t^{-\frac{\mu}{2\alpha} - \frac{\mu}{1-\frac{\mu}{q}} + \frac{1}{q}} \|u_0\|_{L^r}, \]

for \( \alpha > 0 \) and \( \mu > 0 \).

The following parabolic interpolation lemma was proved in [29].

**Lemma 2.5** ([29]). Let \( p \in [1, \infty] \) and \( T > 0 \). Suppose that \( u \in L^\infty(0, T; L^p(\mathbb{R}^n)) \)

and \( t(u_t + (-\Delta)^{\alpha}u) \in L^\infty(0, T; L^p(\mathbb{R}^n)) \). Then, for \( \alpha > \frac{1}{2} \), there exists a positive constant \( C = C(\alpha) \) such that

\[ \sup_{\frac{1}{4} \leq \tau \leq t} \|\nabla u(\cdot, \tau)\|_{L^p} \leq C \left( \sup_{\frac{1}{4} \leq \tau \leq t} \|u(\cdot, \tau)\|_{L^p} \right)^{\frac{1}{p}} \left( \sup_{\frac{1}{4} \leq \tau \leq t} \|(-\Delta)^{\frac{\alpha}{2}}u(\cdot, \tau)\|_{L^p} \right)^{1 - \frac{1}{p}} \]

\[ + Ct^{-\frac{1}{2}} \sup_{\frac{1}{4} \leq \tau \leq t} \|u(\cdot, \tau)\|_{L^p}. \]

In order to prove the main results, we first give the \( L^p(\mathbb{R}^3, \mathbb{R}) \)-estimate on the fundamental solution to the generalized Navier-Stokes equations.
Lemma 2.6. Suppose that \( k(t, x) = F^{-1}(e^{-|\xi|^2 t}) \), where \( \xi, x \in \mathbb{R}^N \) and \( t > 0 \). We have
\[
\|k(t)\|_{L^p(\mathbb{R}^N)} \leq \frac{c_p}{c_q} t^{-\frac{N}{2p}(\frac{1}{p} - \frac{1}{q})} \|k(t)\|_{L^q(\mathbb{R}^N)}, \tag{9}
\]
\[
\|D^j k(t)\|_{L^p(\mathbb{R}^N)} \leq \frac{c_{p,j}}{c_q} t^{-\frac{N}{2p}(\frac{1}{p} - \frac{1}{q})} \|k(t)\|_{L^q(\mathbb{R}^N)}, \quad j = 1, 2, \ldots, \tag{10}
\]
where \( c_p, c_q \) and \( c_{p,j} \) are positive constants with \( c_1 = 1 \) and \( F^{-1} \) denotes the inverse Fourier transformation with respect to \( \xi \).

Proof. Let \( \xi = \eta t^{-\frac{1}{2p}} \). Hence
\[
k(t, x) = \int_{\mathbb{R}^N} e^{-|\xi|^2 t} e^{ix \cdot \xi} d\xi = t^{-\frac{N}{2p}} \int_{\mathbb{R}^N} e^{-|\eta|^2 t} e^{ix \cdot \eta} d\eta.
\]
Let \( G(y) = \int_{\mathbb{R}^N} e^{-|\eta|^2 t} e^{iy \cdot \eta} d\eta \). Clearly, \( G(y) \) is a rapidly decreasing function. Then
\[
\left( \int_{\mathbb{R}^N} |k(t, x)|^p dx \right)^{\frac{1}{p}} = t^{-\frac{N}{2p}} \left( \int_{\mathbb{R}^N} |G(xt^{-\frac{1}{2p}})|^p dx \right)^{\frac{1}{p}}
\]
\[
= t^{-\frac{N}{2p}} t^{\frac{N}{2p}} \left( \int_{\mathbb{R}^N} |G(z)|^p dz \right)^{\frac{1}{p}} = c_p t^{-\frac{N}{2p}(\frac{1}{p} - \frac{1}{q})}.
\]
Similarly, we have
\[
\left( \int_{\mathbb{R}^N} |k(t, x)|^q dx \right)^{\frac{1}{q}} = c_q t^{-\frac{N}{2p}(\frac{1}{q} - \frac{1}{p})}.
\]
Combining the above two equalities together gives (9). In addition,
\[
D^k k(t, x) = t^{-\frac{N+k}{2p}} D^k G(xt^{-\frac{1}{2}}).
\]
By simple calculations, we obtain (10). The proof is completed. \( \Box \)

To consider the decay rate of solutions for the generalized Navier-Stokes equations, we establish the following interpolation type lemma.

Lemma 2.7. Suppose \( \tau_0 > 0 \) and \( F : [\tau_0, \infty) \to [0, \infty) \) satisfies \( \sup_{\tau_0 \leq \tau \leq A} F(\tau) < \infty \) for all \( A > 0 \). Assume that there exists two constants \( C_0 > 0 \) and \( \gamma \in \mathbb{R} \) such that
\[
\sup_{\frac{t}{2} \leq \tau \leq t} F(\tau)^p \leq C_0 t^{-\gamma p} + C_0 t^{-\frac{p}{2N}} \left( \sup_{\frac{t}{4} \leq \tau \leq t} F(\tau) \right)^{\frac{2a-1}{2a} p}, \quad t \geq 4\tau_0, \tag{11}
\]
then \( F(t) = O(t^{-\gamma}) \) as \( t \to \infty \).

Proof. Based on (11), we derive that
\[
\sup_{\frac{t}{2} \leq \tau \leq t} F(\tau)^p \leq C t^{-\gamma p} + C t^{-\frac{p}{2N}} \left( \sup_{\frac{t}{4} \leq \tau \leq t} F(\tau) \right)^{\frac{2a-1}{2a} p} + C t^{-\frac{p}{2N}} \left( \sup_{\frac{t}{4} \leq \tau \leq t} F(\tau) \right)^{\frac{2a-1}{2a} p}
\]
\[
\leq C t^{-\gamma p} + C t^{-\frac{p}{2N}} \left( \sup_{\frac{t}{4} \leq \tau \leq t} F(\tau) \right)^{\frac{2a-1}{2a} p} + C t^{-\frac{p}{2N}} \left( \sup_{\frac{t}{4} \leq \tau \leq t} F(\tau) \right)^{\frac{2a-1}{2a} p} + C t^{-\gamma p} + \frac{1}{2} t^{-\gamma} \sup_{\frac{t}{4} \leq \tau \leq t} F(\tau)^p, \forall t \geq 4\tau_0.
\]
Hence
\[ \sup_{\frac{1}{2} \leq \tau \leq t} F(\tau)^p \leq Ct^{-\gamma} + Ct^{-\frac{12a-1}{3\alpha}} \left( \sup_{\frac{1}{4} \leq \tau \leq \frac{1}{2}} F(\tau) \right)^{\frac{2a-1}{2\alpha}}, \quad \forall t \geq 4\tau_0. \]

Multiplying both sides by \( t^{p\gamma} \) gives
\[ t^{p\gamma} \sup_{\frac{1}{2} \leq \tau \leq t} F(\tau)^p \leq C + Ct^{-\frac{12a-1}{3\alpha}} \left( \sup_{\frac{1}{4} \leq \tau \leq \frac{1}{2}} F(\tau) \right)^{\frac{2a-1}{2\alpha}}, \quad \forall t \geq 4\tau_0, \quad (12) \]
that is
\[ \left( \sup_{\frac{1}{2} \leq \tau \leq t} \tau^\gamma F(\tau) \right)^p \leq C + 4^{\frac{2a-1}{2\alpha}} r^\gamma C \left( \sup_{\frac{1}{4} \leq \tau \leq \frac{1}{2}} \tau^\gamma F(\tau) \right)^{\frac{2a-1}{2\alpha}}. \quad (13) \]

Suppose that \( G(\tau) =\tau^\gamma F(\tau) \). By using the Cauchy-Schwartz inequality, we derive that
\[ \sup_{\frac{1}{4} \leq \tau \leq \frac{1}{2}} G(\tau)^p \leq C_1 + \frac{1}{2} \sup_{\frac{1}{4} \leq \tau \leq \frac{1}{2}} G(\tau)^p, \quad \forall t \geq 4\tau_0, \quad (14) \]
where the positive constant \( C_1 \) depends on \( \gamma \). Using (14) with \( \frac{1}{4} \) instead of \( t \) leads to
\[ \sup_{\frac{1}{4} \leq \tau \leq \frac{1}{2}} G(\tau)^p \leq C_1 + \frac{1}{4} \sup_{\frac{1}{4} \leq \tau \leq \frac{1}{2}} G(\tau)^p, \quad \forall t \geq 8\tau_0. \quad (15) \]

Combining (14) and (15) together gives
\[ \sup_{\frac{1}{2} \leq \tau \leq t} G(\tau)^p \leq \frac{3}{2} C_1 + \frac{1}{4} \sup_{\frac{1}{4} \leq \tau \leq \frac{1}{2}} G(\tau)^p, \quad \forall t \geq 8\tau_0. \]

Repeating this process, we deduce that
\[ \sup_{\frac{1}{2} \leq \tau \leq t} G(\tau)^p \leq C_1 \sum_{k=0}^{t} \left( \frac{1}{2^{k+1}} \right) \sup_{\frac{1}{2^{k+1}} \leq \tau \leq \frac{1}{2^k}} G(\tau)^p, \quad \forall k \in \mathbb{N}_0 \text{ and } t \geq 2^{k+1}\tau_0. \]

Then, for a given \( t \geq \tau_0 \), there exists a \( k \in \mathbb{N}_0 \) such that \( 2\tau_0 \leq \frac{t}{2^k} \leq 4\tau_0 \). For a positive constant \( C_2 \), since
\[ \sup_{\frac{t}{2^{k+1}} \leq \tau \leq \frac{t}{2^k}} G(\tau)^p \leq \sup_{\tau_0 \leq \tau \leq 4\tau_0} G(\tau)^p \leq C_2, \]
we have \( G(\tau) \leq C \) for all \( \tau \in [\frac{1}{2}, t] \). Then, the proof is completed. \( \square \)

3. Proof of Theorem 1.1. If \( u(x, t) \) is the solution of system (1), it should satisfy the following integro-differential equation
\[ u(x, t) = e^{-t(-\Delta)^s} u_0 - \int_0^t e^{-(t-s)(-\Delta)^s} \mathbb{P}(u \cdot \nabla u)(s) ds, \quad (16) \]
where \( \mathbb{P} \) is the Leray projection operator. The basic estimates we use are
\[ \|e^{-t(-\Delta)^s} u\|_{L^p} \leq Ct^{-\frac{s}{2}(\frac{1}{p} - \frac{1}{2})}\|u\|_{L^p}, \quad 1 < p \leq q < \infty, \quad (17) \]
\[ \|De^{-t(-\Delta)^s} u\|_{L^p} \leq Ct^{-\frac{s}{2}(1+\frac{1}{p} - \frac{1}{2})}\|u\|_{L^p}, \quad 1 < p \leq q < \infty, \quad (18) \]
and
\[ \|\mathbb{P}(u \cdot \nabla u)\|_{L^p} \leq C\|u\|_{L^r}\|Du\|_{L^s}, \quad \frac{1}{p} = \frac{1}{r} + \frac{1}{s}. \quad (19) \]
Note that \( u \cdot \nabla u = \nabla \cdot (u \otimes u) \). Combining (17)-(19) together, we obtain, with a slight change of notation,

\[
\left\| \int_0^t e^{(t-s)\partial_\alpha} P(u \cdot \nabla u) ds \right\|_{L^\frac{d}{2}} \leq C \int_0^t (t-s)^{-\frac{1}{2} \frac{(a+b+1-\gamma)}{2}} \|u(s)\|_{L^\frac{d}{2}} \|\nabla u(s)\|_{L^\frac{d}{2}} ds.
\]

(20)

where \( a, b, \gamma > 0 \) and \( \gamma \leq a + b < 3 \).

In \cite{3}, The authors established the following weak-strong uniqueness for the generalized Navier-Stokes equations.

**Lemma 3.1** (\cite{3}). Let \( u_0 \in L^2, \alpha > \frac{1}{2} \). Assume that

\[
u \in L^r(0,T;L^p) \text{ with } \frac{2\alpha}{r} + \frac{d}{p} = 2\alpha - 1, \ p > \frac{d}{2\alpha - 1}.
\]

Then, the solution \( u(x,t) \) to the generalized Navier-Stokes equation (1) is uniqueness.

We state our local existence and uniqueness result.

**Lemma 3.2.** [Local existence and uniqueness] Suppose that the conditions listed in Theorem 1.1 are satisfied, \( \max_{u \in B_{(u,v)}}, 0 < k \leq 3 \) and \( t_1 = \min \{1, \left( \frac{2\alpha - 1}{4\alpha \eta_{1,2}} \right)^{\frac{2\alpha}{2\alpha - 1}} \} \).

Then, there exists a unique local solution \( u(x,t) \) on the strip \( \Pi_{t_1} := \{(x,t): x \in \mathbb{R}^3, 0 < t \leq t_1 \} \) such that

\[
\|u(x,t)\|_{L^\infty} \leq 2r, \ 0 \leq t \leq t_1.
\]

(21)

**Proof.** Note that (16) is equivalent to

\[
u(x,t) = \int_{\mathbb{R}^3} k(x - y, t) u_0 dy - \int_0^t \int_{\mathbb{R}^3} \mathbb{P} Dk(x - y, t - s)(u \otimes u)(s, y) dy.
\]

(22)

We first prove that there is a sufficiently small \( t_1 > 0 \) such that there exists a unique continuous solution \( u(x,t) \) on the strip \( \Pi_{t_1} \) for equation (22).

Suppose that \( T(t) = k(x,t) * u(x,t) \), (22) is equivalent to

\[
u(x,t) = T(t)u_0 - \int_0^t \mathbb{P} D\mathbb{T}(t - s)(u \otimes u)(s) ds.
\]

(23)

It is easy to see that \( T(t)1 = 1 \). In order to prove (22) has a local continuous solution, we just need to prove that there exists a local continuous solution for the integro-differential equation

\[
u(x,t) = T(t)u_0(x) - \int_0^t \mathbb{P} D\mathbb{T}(t - s)(u \otimes u)(s) ds.
\]

(24)

In the following, we use the standard method of successive approximations (see\cite{4, 7, 8, 23}): Let \( u^{(0)}(x,t) = u_0(x) \). For \( n \geq 1 \), define

\[
u^{(n)}(x,t) = T(t)u^{(n-1)} - \int_0^t \mathbb{P} D\mathbb{T}(t - s)(u \otimes u)(s) ds.
\]
Clearly, \( u^{(n)}(x,t) \) is well defined on \( \mathbb{R}^3 \times [0, \infty) \) for each \( n \geq 0 \). Set \( \|f\| = \sup_{(x,t) \in B_1} |f(x,t)|. \) We prove by induction that if \( t_1 = \min \left\{ 1, \left( \frac{2\alpha-1}{4\alpha bc_{1,1}} \right)^{\frac{2\alpha-1}{2\alpha}} \right\} \),

\[
\|\|u^{(n)}\|\| \leq 2r. \tag{26}
\]

In fact, on the basis of the assumption we imposed on the initial data, the case \( n = 0 \) of (26) obtained immediately. As to \( n = 1 \), follows from the fact \( u_0(x) \in B(0,r) \) that

\[
\left| \int_{\mathbb{R}^3} k(x-y,t)u_0(y)dy \right| \leq \|u_0\|_{L^\infty} \leq r.
\]

We have from (25) that

\[
|u^{(1)}(x,t)| \leq r + bc_{1,1} \int_0^t (t - \tau)^{-\frac{s}{2}} \left\| u^{(0)} \right\| d\tau \leq r + \frac{4\alpha rbc_{1,1}}{2\alpha - 1}(t_1)^{\frac{2\alpha-1}{2\alpha}} \leq 2r.
\]

Hence, (26) holds for \( n = 1 \). Assume that (26) holds for \( n \leq m - 1 \) with \( m \in \mathbb{N}_+ \), we now prove that (26) is also true for \( n = m \). By (25) and Hausdorff-Young’s inequality, we deduce that

\[
\left| u^{(m)} \right| \leq r + b \int_0^t \|DK(t-s)\|_{L^1(\mathbb{R}^3)} \left\| u^{(m-1)} \right\| ds
\]

\[
\leq r + bc_{1,1} \left\| u^{(m-1)} \right\| \int_0^t (t - \tau)^{-\frac{s}{2}} d\tau \leq r + \frac{4\alpha rbc_{1,1}}{2\alpha - 1}(t_1)^{\frac{2\alpha-1}{2\alpha}} \leq 2r,
\]

which implies that (26) holds for \( n = m \). By induction, it is easy to see that (26) holds for any \( n \geq 0 \).

We also need to prove that \( u^{(n)}(x,t) \) satisfies the following estimate

\[
\left\| u^{(n)} - u^{(n-1)} \right\| \leq \frac{C_0(t_1)^{\frac{(2\alpha-1)(n-1)}{2\alpha - 1}}}{\Gamma\left(\frac{m-1}{2} + \frac{1}{2}\right)} M_0 \leq \frac{C_0^{n-1}}{\Gamma\left(\frac{m-1}{2} + \frac{1}{2}\right)} M_0, \quad n \geq 1, \tag{27}
\]

where \( M_0 = \frac{2\alpha-1}{2\alpha - 1}bc_{1,1}\sqrt{r} \) and \( C_0 = bc_{1,1}\sqrt{r} \).

As to \( n = 1 \), we can obtain from (25) that

\[
|u^{(1)}(x,t) - u^{(0)}(x,t)| \leq \int_0^t \|DK(t-s)\|_{L^1} \left\| u^{(0)} \right\| ds
\]

\[
\leq bc_{1,1} \int_0^t (t-s)^{-\frac{s}{2}} ds \leq \frac{2\alpha rbc_{1,1}}{2\alpha - 1}(t_1)^{\frac{2\alpha-1}{2\alpha}} \leq \frac{M_0}{\sqrt{r}}. \tag{28}
\]

Therefore, (27) is true for \( n = 1 \). Suppose that (27) is true for \( n \leq m - 1 \) for some positive integer \( m \geq 2 \), by (26), we derive that

\[
\left| u^{(m)}(x,t) - u^{(m-1)}(x,t) \right|
\]

\[
\leq \int_0^t \|DK(t-s)\|_{L^1} \left\| u^{(m-1)} \otimes u^{(m-1)} - u^{(m-2)} \otimes u^{(m-2)} \right\| ds
\]

\[
\leq bc_{1,1} \int_0^t (t-s)^{-\frac{s}{2}} \left( \frac{C_0^m}{\sqrt{r}} \right)^{m-2} M_0 ds
\]

\[
\leq bc_{1,1} \frac{C_0^{m-2} M_0 \sqrt{r} \Gamma\left(\frac{m-2}{2} + 1\right)}{\Gamma\left(\frac{m-2}{2} + \frac{1}{2}\right)} (t_1)^{\frac{(2\alpha-1)(m-1)}{2\alpha}}
\]

\[
\leq \frac{(t_1)^{\frac{(2\alpha-1)(m-1)}{2\alpha}} C_0^{m-2} bc_{1,1} \sqrt{r} M_0}{\Gamma\left(\frac{m-2}{2} + \frac{1}{2}\right)} \leq \frac{C_0^{m-1} M_0}{\Gamma\left(\frac{m-1}{2} + \frac{1}{2}\right)}. \tag{29}
\]
which implies (27) holds for \( n = m \). Then, by induction again, we can deduce that (27) is true for any \( n \geq 1 \).

Since \( \sum_{n=0}^{\infty} \frac{C_0^{n-1}M_0}{t^{(n-1)}} \) is convergent, by (27), we know that \( v^{(n)}(x,t) \) converges uniformly on the strip \( \Pi_t \), whose limit is denoted by \( v(x,t) = u(x,t) - \bar{u} \). It is easy to see that the unique limit \( v(x,t) \) is a continuous solution of integro-differential equation (22) on the strip \( \Pi_t \). In addition, by Lemma 3.1, we can obtain the uniqueness of the solutions. The proof is completed.

**Lemma 3.3.** If the local solution \( u(x,t) \) obtained in Lemma 3.2 has been extended up to time \( T \) (\( T \geq t_1 > 0 \)) while the smooth properties and the a priori estimate (21) is kept unchanged, then for any \( 0 < s_1' < s_2' < s_3' < s_4' < t \leq T \), we have

\[
\| D^{2\alpha-1}u(x,t) \|_{L^{\frac{2}{1+2\alpha}}} 
\leq (t - s_k')^{-\frac{2\alpha-1}{2\alpha}} \sup_{[0,T]}\| u(x,t) \|_{L^{\frac{2}{2\alpha}}} M_k(r, s_k' - s_1', s_3' - s_1', t - s_k') < \infty, \tag{30}
\]

where \( k = 1, 2, 3 \) and \( M_k \) is a continuous increasing function of \( t - s_k' \).

**Proof.** Note that

\[
u(x,t) = T(t - s_k')u(x,s_k') - \int_{s_k'}^{t} \mathbb{P}DT(t-s)(u \otimes u)(s)ds,
\]

we have

\[
D^{2\alpha-1}u(x,t) = T(t-s_k')D^{2\alpha-1}(u(x,s_k') - \bar{u})
- \int_{s_k'}^{t} \mathbb{P}DT(t-s)D^{2\alpha-1}(u \otimes u)(s)ds,
\]

Applying (17), (18) and (20), we derive that

\[
\| D^{2\alpha-1}u(x,t) \|_{L^{\frac{2}{1+2\alpha}}}
\leq C_1(t - s_k')^{-\frac{2\alpha-1}{2\alpha}} \| u(s_k') - \bar{u} \|_{L^{\frac{2}{2\alpha}}}
+ C_2 \int_{s_k'}^{t} (t-s)^{-\frac{2\alpha-1}{2\alpha}} \| D^{2\alpha-1}u(s) \|_{L^{\frac{2}{2\alpha}}} \| u \|_{L^{\infty}} ds
\leq C_1(t - s_k')^{-\frac{2\alpha-1}{2\alpha}} \| u(s_1') - \bar{u} \|_{L^{\frac{2}{2\alpha}}}
+ 2rC_2 \int_{s_k'}^{t} (t-s)^{-\frac{2\alpha-1}{2\alpha}} \| D^{2\alpha-1}u(s) \|_{L^{\frac{2}{2\alpha}}} ds. \tag{32}
\]

On the basis of the singular Gronwall’s inequality and (32), we can easy deduce that (30) holds.

We also have the following lemma, which is concerned with the time-independent \( L^{\frac{1}{2\alpha}}(\mathbb{R}^d) \)-estimate on the solution \( u(t,x) \). This estimate is very important in extending the local solution step by step to a global one.

**Lemma 3.4.** Suppose that the assumption listed in Lemma 3.3 are satisfied, then \( u(x,t) \) satisfies

\[
\| u(x,t) \|_{L^{\frac{1}{2\alpha}}} + t^{\frac{2\alpha-1}{2\alpha}} \| u(x,t) \|_{L^{\frac{2}{2\alpha}}} \leq C_1(r) \| u_0(x) \|_{L^{\frac{1}{2\alpha}}}, \tag{33}
\]

where \( C_1(r) \) is a positive constant depending only on \( r \).
Suppose that $C$ can get (33) immediately.

Proof of Theorem 1.1. Let

$$X = \{ u(t, x) : u(x, t) \in C(\mathbb{R}^+, L^\infty(\mathbb{R}^3)), t^{\frac{2n-1}{2n}} u(x, t) \in C(\mathbb{R}^+, L^\infty(\mathbb{R}^3, \mathbb{R})) \},$$

with its norm defined by

$$\|u(t)\|_X = \sup_{\mathbb{R}^+} \{ \|u(t)\|_{L^{\infty}(\mathbb{R}^3)} + t^{\frac{2n-1}{2n}} \|u(t)\|_{L^{\infty}(\mathbb{R}^3, \mathbb{R})} \}.$$

On the basis of the integro-differential representation, we have

$$A \text{ simple calculation shows that } \beta > 0 \text{ where } \beta = \sup \{ \|T(t)u_0\|_{L^\frac{3}{2n-1}(\mathbb{R}^3)} + t^{\frac{2n-1}{2n}} \|T(t)u_0\|_{L^\infty(\mathbb{R}^3, \mathbb{R})} \} \leq C \|u_0\|_{L^\frac{3}{2n-1}}.$$

For $I_2$, by using (20), we have

$$I_2 \leq \sup_{\mathbb{R}^+} \left\{ \int_0^t \|DT(t - s)|u(s)|^2\|_{L^\frac{3}{2n-1}(\mathbb{R}^3)} ds + t^{\frac{2n-1}{2n}} \int_0^t \|DT(t - s)|u(s)|^2\|_{L^\infty(\mathbb{R}^3)} ds \right\} \leq C \sup_{\mathbb{R}^+} \left\{ \int_0^t (t - s)^{-\frac{1}{2n}} \|u(t) - \bar{u}\|^2_{L^\frac{6}{n-1}} ds + t^{\frac{2n-1}{2n}} \int_0^t \int_0^s (t - s)^{-\frac{2n-1}{4n}} s^{-\frac{2n-1}{2n}} ds \right\} \|u(t) - \bar{u}\|_X \leq C \|u(t)\|^2_X.$$

Summing up, we obtain

$$\|u(t)\|_X \leq C \|u_0\|_{L^\frac{3}{2n-1}} + C \|u(t)\|^2_X.$$

If we suppose that $\|u_0 - \bar{u}\|_{L^\frac{3}{2n-1}}$ is sufficiently small, by Strauss’s inequality, we can get (33) immediately.

With the above preparations in hand, we now prove Theorem 1.1.

Proof of Theorem 1.1. Choose $0 < s_1 < s_2 < s_3 < s_4 \leq T$ sufficiently small such that $s_3 \leq t_1$ and

$$t_1 - s_3 = s_3 - s_2 = s_2 - s_1 = s_1 - s_3 = \beta,$$

where $\beta > 0$ is a sufficiently small positive constant. By (30) and (32), we obtain

$$\begin{cases}
\|u(x, t)\|_{L^\frac{3}{2n-1}} \leq C_1(r) \|u_0\|_{L^\frac{3}{2n-1}}, & 0 \leq t \leq t_1, \\
\|u(t_1)\|_{W^{2n-1, \frac{3}{2n-1}}} \leq C_2(\beta, r, t_1) \sup_{[0, t_1]} \|u(x, t)\|_{L^\frac{3}{2n-1}}.
\end{cases}$$

(34)

Suppose that $C$ is a constant in Sobolev’s inequality $\|u\|_{L^{\infty}} \leq C \|u\|_{W^{2n-1, \frac{3}{2n-1}}}$. If $\|u_0\|_{L^\frac{3}{2n-1}}$ is sufficiently small such that

$$CC_1(r)C_2(\beta, r, t_1) \|u_0\|_{L^\frac{3}{2n-1}} \leq \|u_0\|_{L^{\infty}},$$

(35)
then
\[ \|u(t_1)\|_{L^\infty} \leq C\|u(t_1)\|_{W^{2a-1, \frac{3}{2a-1}}} \leq CC_2(\beta, r, t_1) \sup_{[0,t_1]} \|u\|_{L^{\frac{3}{\alpha}}} \]
\[ \leq CC_1(r)C_2(\beta, r, t_1)\|u_0\|_{L^{\frac{3}{\alpha}}} \leq \|u_0\|_{L^\infty} \leq r. \]

By Lemmas 3.2 and 3.4, \( u(x, t) \) can be extended up to \( 2t_1 \) and satisfies
\[
\begin{cases}
\|u\|_{L^\infty} \leq 2r, & 0 \leq t \leq 2t_1,
\|u(t)\|_{L^{\frac{3}{\alpha}}} \leq C_1(\beta)\|u_0\|_{L^{\frac{3}{\alpha}}}, & 0 \leq t \leq 2t_1.
\end{cases} 
\] (36)

Setting \( t = 2t_1, s'_i = s_i + t_1, \bar{s}'_i = \bar{s}_i + t_1 \) (\( i = 1, 2, 3 \)) in (30) and (32), we easily obtain
\[
\|u(2t_1)\|_{W^{2a-1, \frac{3}{2a-1}}} \leq C_2(\beta, r, t_1) \sup_{[0,2t_1]} \|u\|_{L^{\frac{3}{\alpha}}}. 
\] (37)

Now, assume that \( u(x, t) \) has been defined up to time \( kt_1 \) for some \( k \in \mathbb{N}_+ \) such that
\[
\begin{cases}
\|u\|_{L^\infty} \leq 2r, & 0 \leq t \leq kt_1,
\|u(t)\|_{L^{\frac{3}{\alpha}}} \leq C_1(\beta)\|u_0\|_{L^{\frac{3}{\alpha}}}, & 0 \leq t \leq kt_1.
\end{cases} 
\] (38)

Setting \( t = kt_1, s'_i = s_i + (k-1)t_1, \bar{s}'_i = \bar{s}_i + (k-1)t_1 \) (\( i = 1, 2, 3 \)) in (30) and (32), we have
\[
\|u(kt_1)\|_{W^{2a-1, \frac{3}{2a-1}}} \leq C_2(\beta, r, t_1) \sup_{[0,kt_1]} \|u\|_{L^{\frac{3}{\alpha}}}. 
\] (39)

It follows from (35), (38) and (39) that
\[
\|u(kt_1)\|_{L^\infty} \leq C\|u(kt_1)\|_{W^{2a-1, \frac{3}{2a-1}}} \leq CC_2(\beta, r, t_1) \sup_{[0,kt_1]} \|u\|_{L^{\frac{3}{\alpha}}} 
\[ \leq CC_1(r)C_2(\beta, r, t_1)\|u_0\|_{L^{\frac{3}{\alpha}}} \leq \|u_0\|_{L^\infty} \leq r. \]

Applying Lemmas 3.2 and 3.4 again, \( u(x, t) \) can be extended up to time \( (k+1)t_1 \) and \( u(x, t) \) satisfies
\[
\begin{cases}
\|u\|_{L^\infty} \leq 2r, & 0 \leq t \leq (k+1)t_1,
\|u(t)\|_{L^{\frac{3}{\alpha}}} \leq C_1(\beta)\|u_0\|_{L^{\frac{3}{\alpha}}}, & 0 \leq t \leq (k+1)t_1.
\end{cases} 
\] (40)

Proceeding inductively, we thus establish the existence of solution
\[ u(x, t) \in L^\infty(0, \infty; L^{\frac{3}{\alpha}}(\mathbb{R}^3)). \]
The proof is completed. \( \square \)

4. **Proof of Theorem 1.2.** Note that \( u(x, t) \in L^\infty(0, \infty; L^2) \cap L^2(0, \infty; H^\alpha) \) if \( \alpha \geq \frac{3}{4} \). We will prove (3) under the assumption \( \|u_0\|_{L^\infty} \leq r \) and \( \|u_0\|_{L^2} \) sufficiently small.

We give the natural energy relation. Multiplying (1) by \( u(t, x) \), integrating over \( \mathbb{R}^3 \), we derive that
\[
\frac{d}{dt}\|u(t)\|_{L^2(\mathbb{R}^3)}^2 + 2\|\Lambda^\alpha u(t)\|_{L^2(\mathbb{R}^3)}^2 = 0,
\]
which means
\[
\|u(t)\|_{L^2(\mathbb{R}^3)}^2 + 2\int_0^t \|\Lambda^\alpha u(t)\|_{L^2(\mathbb{R}^3)}^2 dt = \|u_0\|_{L^2(\mathbb{R}^3)}^2. \] (41)

It is worth pointing that Lemma 3.2 holds for all \( \alpha \geq \frac{1}{2} \). We first prove the following lemma.
Lemma 4.1. Let $\alpha \geq \frac{5}{4}$. If the local solution $u(x,t)$ obtained in Lemma 3.2 has been extended up to time $T$ ($T \geq t_1 > 0$) while the smooth properties and the a priori estimate (21) is kept unchanged, then for any $0 < s'_1 < s_1 < s'_2 < s_2 < s'_3 < s_3 < T$, we have
\[
\|D^m u(x,t)\|_{L^2} \\
\leq (t - s'_k)^{-\frac{m}{2n}} \sup_{[0,t_1]} \|u(x,t)\|_{L^2} M_k(r, s'_k - s'_1, s'_k - s'_1, t - s'_k) < \infty, \quad (42)
\]
where $k = 1, 2, 3$, $m = 1, 2$ and $M_k$ is a continuous increasing function of $t - s'_k$.

Proof. We have
\[
D^m u(x,t) = T(t - s'_k) D^m u(x,s'_k) - \int_{s'_k}^t \mathbb{P} DT(t-s) D^m(u \otimes u)(s)ds, \quad (43)
\]
Applying (17), (18) and (20), we derive that
\[
\|D^m u(x,t)\|_{L^2} \\
\leq C_1(t - s'_k)^{-\frac{m}{2n}} \|u(s'_k)\|_{L^2} + C_2 \int_{s'_k}^t (t-s)^{-\frac{m}{2n}} \|D^m u(s)\|_{L^2} \|u\|_{L^\infty} ds \quad (44)
\]
\[
\leq C_1(t - s'_1)^{-\frac{m}{2n}} \|u(s'_1)\|_{L^2} + 2r \int_{s'_1}^t (t-s)^{-\frac{m}{2n}} \|D^m u(s)\|_{L^2} ds.
\]
By singular Gronwall’s inequality and (44), we can easy deduce that (42) holds. \qed

Now, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. Choose $0 < s_1 < s_2 < s_2 < s_3 < s_3 \leq T$ sufficiently small such that $s_3 \leq t_1$ and
\[
t_1 - s_3 = s_3 - s_3 = s_3 - s_2 = \bar{s}_2 - s_2 = s_2 - s_1 = s_1 - s_1 = \beta,
\]
where $\beta > 0$ is a sufficiently small positive constant. By (34) and (44), we obtain
\[
\begin{aligned}
\|u(x,t)\|_{L^2} &= \|u_0\|_{L^2}, \quad 0 \leq t \leq t_1, \\
\|u(t_1)\|_{H^{\frac{3}{2}}} &\leq \|u(t_1)\|_{H^{\frac{3}{2}}} \leq C_3(\beta, r, t_1) \sup_{[0,t_1]} \|u(x,t)\|_{L^2}.
\end{aligned}
\]
(45)
Suppose that $C$ is a constant in Sobolev’s inequality
\[
\|u\|_{L^\infty} \leq C \|u(t_1)\|_{H^{\frac{3}{2}}}, \quad 0 \leq t \leq t_1.
\]
If $\|u_0\|_{L^2}$ is sufficiently small such that
\[
CC_3(\beta, r, t_1) \|u_0\|_{L^2} \leq \|u_0\|_{L^\infty}, \quad (46)
\]
then
\[
\|u(t_1)\|_{L^\infty} \leq C \|u(t_1)\|_{H^{\frac{3}{2}}} \|u(t_1)\|_{H^{\frac{3}{2}}} \leq CC_3(\beta, r, t_1) \sup_{[0,t_1]} \|u\|_{L^2}
\]
\[
= CC_3(\beta, r, t_1) \|u_0\|_{L^2} \leq \|u_0\|_{L^\infty} \leq r.
\]
By Lemma 3.2 and (34), $u(x,t)$ can be extended up to $2t_1$ and satisfies
\[
\begin{aligned}
\|u\|_{L^\infty} &\leq 2r, \quad 0 \leq t \leq 2t_1, \\
\|u(t)\|_{L^2} &= \|u_0\|_{L^2}, \quad 0 \leq t \leq 2t_1.
\end{aligned}
\]
(47)
Similar to Section 2, we set Proof of Theorem 1.3. Strauss’s inequality (Lemma 2.2).

In the following, we suppose that \( u(x, t) \) has been defined up to time \( kt_1 \) for some \( k \in \mathbb{N}_+ \) such that
\[
\begin{cases}
\|u\|_{L^\infty} \leq 2r, & 0 \leq t \leq kt_1, \\
\|u(t)\|_{L^2} = \|u_0\|_{L^2}, & 0 \leq t \leq kt_1.
\end{cases}
\]
Setting \( t = kt_1, s_i^t = s_i + t_1, \) \( s_i^t = s_i + (k-1)t_1 \) \( (i = 1, 2, 3) \) in (42) and (44), we have
\[
\|u(kt_1)\|_{H^1}^2 \leq C_3(\beta, r, t_1) \sup_{[0, kt_1]} \|u\|_{L^2}.
\]
It follows from (46), (49) and (50) that
\[
\|u(kt_1)\|_{L^\infty} \leq C \|u(kt_1)\|_{H^1}^2, \quad \|u(kt_1)\|_{H^2}^2 \leq C C_2(\beta, r, t_1) \sup_{[0, kt_1]} \|u\|_{L^2}
\]
\[
\leq C (r) C_1(\beta, r, t_1) \|u_0\|_{L^2} \leq \|u_0\|_{L^\infty} \leq r.
\]
Applying Lemma 3.2 and (34) again, \( u(x, t) \) can be extended up to time \( (k+1)t_1 \) and \( u(x, t) \) satisfies
\[
\begin{cases}
\|u\|_{L^\infty} \leq 2r, & 0 \leq t \leq (k+1)t_1, \\
\|u(t)\|_{L^2} = \|u_0\|_{L^2}, & 0 \leq t \leq (k+1)t_1.
\end{cases}
\]
Proceeding inductively, we establish the existence of solution \( u(x, t) \) for system (1) with \( \alpha \geq \frac{5}{4} \) in all \( t > 0 \). The proof is completed. \( \square \)

5. Proof of Theorems 1.3 and 1.4. First of all, we consider the \( L^p \)-decay estimate of solutions for the generalized Navier-Stokes equations.

It is worth pointing out that the weak solution \( u(x, t) \) of system (1) satisfies the following integro-differential equation
\[
u(t, x) = \int_{\mathbb{R}^3} k(t, x - y)u_0dy - \int_0^t \int_{\mathbb{R}^3} \mathbb{P}Dk(t - s, x - y)(u \otimes u)(s, y)dyds,
\]
Let \( T(t)u = k(t, x) * u(t, x) \). Then, (52) is equivalent to
\[
u(t, x) = T(t)u_0 - \int_0^t \mathbb{P}DT(t - s)(u \otimes u)(s, y)dyds.
\]
In the following, we give the proof of Theorem 1.3. The main tool we use is also Strauss’s inequality (Lemma 2.2).

Proof of Theorem 1.3. Similar to Section 2, we set
\[
X = \left\{ u(x, t) : u(x, t) \in C(\mathbb{R}^+, L^9(\mathbb{R}^3, \mathbb{R}^3)), t^{\frac{3}{2} - \frac{1}{p}} u(t, x) \in C(\mathbb{R}^+, L^p(\mathbb{R}^3, \mathbb{R}^3)) \right\},
\]
with its norm defined by
\[
\|u(t)\|_X = \sup_{\mathbb{R}^+} \left\{ \|u(t)\|_{L^9(\mathbb{R}^3)} + t^{\frac{3}{2} - \frac{1}{p}} \|u(t)\|_{L^p(\mathbb{R}^3)} \right\}.
\]
It follows from (53) that
\[
\|u\|_X \leq \|T(t)u_0\|_X + \left\| \int_0^t DT(t - s)|u(s)|^2ds \right\|_X := J_1 + J_2.
\]
Using Hausdorff-Young’s inequality, we deduce that
\[ J_1 = \sup_{\mathbb{R}^+} \left\{ \| T(t)u_0 \|_{L^q(\mathbb{R}^3)} + t^{\frac{3}{2p}} \left( 1 - \frac{3}{p} \right) \| T(t)u_0 \|_{L^p(\mathbb{R}^3)} \right\} \]
\[ \leq \sup_{\mathbb{R}^+} \left\{ (1 + C t^{\frac{3}{2p}} (\frac{1}{q} - \frac{1}{p})) \| u_0 \|_{L^q(\mathbb{R}^3)} \right\} \]
\[ \leq (1 + C) \| u_0 \|_{L^q(\mathbb{R}^3)}, \quad \forall t \geq 0. \]

Since \( \max \{1, \frac{3+4}{6} \} < \alpha \leq \frac{1}{2} + \min \{ \frac{3}{q} - \frac{3}{p}, \frac{3}{2p} \} \), using Hölder’s inequality and Hausdorff-Young’s inequality, we have
\[ J_2 \leq \sup_{\mathbb{R}^+} \left\{ \int_0^t \| DT(t-s)\|_{L^q(\mathbb{R}^3)} \| u(t) \|^2_{L^p(\mathbb{R}^3)} \right\} \]
\[ + t^{\frac{3}{2p}} \left( \frac{1}{q} - \frac{1}{p} \right) \int_0^t \| DT(t-s)\|_{L^q(\mathbb{R}^3)} \| u(t) \|^2_{L^p(\mathbb{R}^3)} \right\} \]
\[ \leq C \sup_{\mathbb{R}^+} \left\{ \int_0^t (s-t)^{\frac{1}{p} - \frac{1}{q}} \| u(t) \|^2_{L^p(\mathbb{R}^3)} \right\} \]
\[ + t^{\frac{3}{2p}} \left( \frac{1}{q} - \frac{1}{p} \right) \int_0^t (s-t)^{\frac{1}{p} - \frac{1}{q}} \| u(t) \|^2_{L^p(\mathbb{R}^3)} \right\} \]
\[ \leq C \sup_{\mathbb{R}^+} \left\{ \int_0^t (s-t)^{\frac{1}{p} - \frac{1}{q}} \| u(t) \|_{L^q(\mathbb{R}^3)} \right\} \]
\[ + t^{\frac{3}{2p}} \left( \frac{1}{q} - \frac{1}{p} \right) \int_0^t (s-t)^{\frac{1}{p} - \frac{1}{q}} \| u(t) \|_{L^p(\mathbb{R}^3)} \right\} \]
\[ \leq C \sup_{\mathbb{R}^+} \left\{ \int_0^t (s-t)^{\frac{1}{p} - \frac{1}{q}} \| u(t) \|_{L^q(\mathbb{R}^3)} \right\} \]
\[ + t^{\frac{3}{2p}} \left( \frac{1}{q} - \frac{1}{p} \right) \int_0^t (s-t)^{\frac{1}{p} - \frac{1}{q}} \| u(t) \|_{L^p(\mathbb{R}^3)} \right\} \]
\[ \leq C \sup_{\mathbb{R}^+} \left\{ \int_0^t (s-t)^{\frac{1}{p} - \frac{1}{q}} \| u(t) \|_{L^q(\mathbb{R}^3)} \right\} \]
\[ + t^{\frac{3}{2p}} \left( \frac{1}{q} - \frac{1}{p} \right) \int_0^t (s-t)^{\frac{1}{p} - \frac{1}{q}} \| u(t) \|_{L^p(\mathbb{R}^3)} \right\} \]
\[ \leq C \| u(t) \|_{L^q(\mathbb{R}^3)}, \quad \forall t \geq 1. \]

Hence
\[ \| u(t) \|_{L^q(\mathbb{R}^3)} \leq (1 + C) \| u_0 \|_{L^q(\mathbb{R}^3)} + C \| u(t) \|_{L^q(\mathbb{R}^3)}. \]

If we suppose that \( \| u_0 \|_{L^q(\mathbb{R}^3)} \) is sufficiently small, by Lemma 2.2, we can get
\[ \| u(t) \|_{L^q(\mathbb{R}^3)} \leq C, \]
which implies \( t^{\frac{3}{2p}} (\frac{1}{q} - \frac{1}{p}) \| u(t) \|_{L^p(\mathbb{R}^3)} \leq C. \) The proof is completed. □

**Remark 5.** It follows from the above result, we can make sure that if \( u_0 \in L^q \cap L^\infty(\mathbb{R}^3; \mathbb{R}^3), \)
\[ \| u \|_{L^\infty} \leq C(1 + t)^{-\frac{1}{2p}}, \quad \text{for large } t. \]

The proof of Theorem 1.4 is in the following.

Our proof is based on parabolic interpolation Lemma 2.5, interpolation type Lemma 2.7 and a boot strap argument, which is different from [11].
Proof of Theorem 1.4. Since the solution $u(\cdot,t)$ satisfies (1), we can derive

$$-\Delta \pi = \sum_{i,j=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j} (u^i u^j).$$

Then,

$$\|\nabla \pi\|_{L^p} \leq \|u \cdot \nabla u\|_{L^p}.$$ 

If $\alpha \geq \max\{1, \frac{3+q}{6}\}$, a simple calculation shows that

$$\frac{1}{2\alpha} \left( \frac{p(3+q)}{2\alpha q} - \frac{3}{2\alpha} \right) \leq \frac{p}{\alpha^2} - \frac{3}{4\alpha^2}.$$ 

Note that we consider the decay rate for system (1) in the case $\max\{1, \frac{3+q}{6}\} < \alpha \leq \frac{1}{2} + \min\{\frac{3}{q} - \frac{3}{p}, \frac{3}{2p}\}$. Then, applying Lemma 2.5, we obtain

$$\sup_{\frac{1}{2} \leq \tau \leq t} \|\nabla u(\tau)\|_{L^p}^p \leq C \left( \sup_{\frac{1}{2} \leq \tau \leq t} \|u(\tau)\|_{L^p}^{\frac{p}{2}} \sup_{\frac{1}{4} \leq \tau \leq t} \|u \cdot \nabla u\|_{L^p} + \|\nabla \pi\|_{L^p} \right)^{\frac{2q-1}{2\alpha q}} p + C t^{-\frac{p}{\alpha q}} \sup_{\frac{1}{4} \leq \tau \leq t} \|u\|_{L^p}^p$$

$$\leq C \sup_{\frac{1}{4} \leq \tau \leq t} \|u\|_{L^p}^p \sup_{\frac{1}{4} \leq \tau \leq t} \left( \|u\|_{L^\infty}^{\frac{2q-1}{2\alpha q}} \|\nabla u\|_{L^p}^{\frac{2q-1}{2\alpha q}} p \right) + C t^{-\frac{p}{\alpha q}} \sup_{\frac{1}{4} \leq \tau \leq t} \|u\|_{L^p}^p$$

$$\leq C \left( (1 + t)^{-\frac{3}{2\alpha} \left( \frac{1}{q} - \frac{1}{2} \right)} \right)^{\frac{1}{2\alpha q}} \left( (1 + t)^{-\frac{3}{2\alpha} \left( \frac{1}{q} - \frac{1}{2} \right)} \right)^{\frac{2q-1}{2\alpha q}} \|u\|_{L^p}^{\frac{2q-1}{2\alpha q}} p + C t^{-\frac{p}{\alpha q}} (1 + t)^{-\frac{3}{2\alpha} \left( \frac{1}{q} - \frac{1}{2} \right)}$$

$$\leq C(1 + t)^{-\left(\frac{3}{2\alpha} \left( \frac{1}{q} - \frac{1}{2} \right) + \frac{3}{2\alpha q} \right)} \|\nabla u\|_{L^p}^{\frac{2q-1}{2\alpha q}} p + C(1 + t)^{-\left(\frac{p(3+q)}{2\alpha q} - \frac{3}{2\alpha q} \right)}$$

$$\leq C(1 + t)^{-\left(\frac{p(3+q)}{2\alpha q} - \frac{3}{2\alpha q} \right)} \|\nabla u\|_{L^p}^{\frac{2q-1}{2\alpha q}} p + C(1 + t)^{-\left(\frac{p(3+q)}{2\alpha q} - \frac{3}{2\alpha q} \right)} \|\nabla u\|_{L^p}^{\frac{2q-1}{2\alpha q}} p$$

for large $t$.

It follows from Lemma 2.7 that

$$\|\nabla u\|_{L^p}^p \leq C(1 + t)^{-\left(\frac{p(3+q)}{2\alpha q} - \frac{3}{2\alpha q} \right)} = C(1 + t)^{-\left[p \left(\frac{1}{2} - \frac{1}{q} \right) - \frac{3}{2\alpha q} \right]} \quad \text{for large } t. \quad (54)$$

The case $m = 1$ for (5) has been proved in the above. Suppose that (5) holds for any $|\kappa| \leq \hat{k}$, where $k \in \mathbb{N}$, we will prove that (5) is also true for the case $|\kappa| = k + 1$. It is worth pointing out that $\partial_\kappa u$ satisfies

$$(\partial_t + (-\Delta)^\kappa) \partial_\kappa u = \nabla \partial_\kappa \pi - \partial_\kappa (u \cdot \nabla u).$$
Lemma 2.7 implies

\[
\|
\n\|
\|\n\|\n\]

we get for large \( t \),

\[
\sup_{\frac{1}{2} \leq \tau \leq t} \left\| \nabla \partial_k u(\tau) \right\|_{L^p} \leq C \left( \begin{array}{c}
\sup_{\frac{1}{2} \leq \tau \leq t} \left\| \partial_k u(\tau) \right\|_{L^p} \\
+ C t^{-\frac{3}{2p}} \sup_{\frac{1}{4} \leq \tau \leq t} \left\| \partial_k u(\tau) \right\|_{L^p}
\end{array} \right) \sup_{\frac{1}{4} \leq \tau \leq t} \left( \left\| \nabla \partial_k \pi \right\|_{L^p} + \left\| \partial_k (u \cdot \nabla u)(\tau) \right\|_{L^p} \right)^{\frac{1}{2} - \frac{1}{n}}
\]

\[
\leq C (1 + t)^{-\frac{3}{4n^2} \left( \frac{1}{q} - \frac{1}{p} \right) - \frac{|\alpha|}{4n^2}} \sup_{\frac{3}{4} \leq \tau \leq t} \left\| \partial_k u(\tau) \right\|_{L^p} \left( \begin{array}{c}
\sup_{\frac{3}{4} \leq \tau \leq t} \left\| \partial_k u(\tau) \right\|_{L^p} + \sum_{0 < \beta \leq \kappa} \left\| \partial_\beta u(\tau) \right\|_{L^p} \left\| \nabla \partial_k - \beta u(\tau) \right\|_{L^{2p}}
\end{array} \right)^{\frac{2q-1}{2n-1}}
\]

\[
\leq C (1 + t)^{-\frac{3}{4n^2} \left( \frac{1}{q} - \frac{1}{p} \right) - \frac{|\alpha|}{4n^2} + \left( \frac{1}{q} - \frac{1}{p} \right) + \left( \frac{2q-1}{2n-1} \right) - \frac{|\alpha|}{4n^2}} \left( \begin{array}{c}
\sup_{\frac{3}{4} \leq \tau \leq t} \left\| \partial_k u(\tau) \right\|_{L^p} + \left( \frac{1}{q} - \frac{1}{p} \right) - \frac{|\alpha|}{4n^2},
\end{array} \right)
\]

\[
\leq C (1 + t)^{-\frac{3}{4n^2} \left( \frac{1}{q} - \frac{1}{p} \right) - \frac{|\alpha|}{4n^2}}, \quad \text{for large } t.
\] (55)

Therefore, we complete the proof.

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