Discretely Self-Similar Singular Solutions for the Incompressible Euler Equations

Liutang Xue

Abstract. In this article we consider the discretely self-similar singular solutions of the Euler equations, and by relying on the local energy inequality of the velocity profile and using the bootstrapping method, we prove some nonexistence criteria and show some asymptotic property of the possible velocity profiles.

1. Introduction

In this paper we consider the Cauchy problem of the $N$-dimensional ($N \geq 3$) incompressible Euler equations

$$
\begin{align*}
\frac{\partial v}{\partial t} + v \cdot \nabla v + \nabla p &= 0, & (x, t) \in \mathbb{R}^N \times \mathbb{R}, \\
\text{div } v &= 0, & (x, t) \in \mathbb{R}^N \times \mathbb{R}, \\
v|_{t=0} &= v_0, & x \in \mathbb{R}^N,
\end{align*}
$$

(1.1)

where $v = (v_1, \ldots, v_N)$ is the vector-valued velocity field and $p$ is the scalar-valued pressure function. The Euler equations (1.1) describes the motion of the perfect incompressible inviscid fluids and is the fundamental system in the fluid mechanics.

For the smooth data, e.g. $v_0 \in H^k(\mathbb{R}^N)$, $k > N/2 + 1$, it is well-known that there exists a $T > 0$ such that $v \in C(-T, T; H^k(\mathbb{R}^N))$ and the pressure satisfies that $-\Delta p = \text{div} \text{div}(v \otimes v)$. Up to a function depending only on $t$, the pressure can be given by

$$
p(x, t) = -\frac{1}{N} |v(x, t)|^2 + \text{p.v.} \int_{\mathbb{R}^N} K_{ij}(x-y)v_i(y, t)v_j(y, t) \, dy,
$$

(1.2)

where

$$
K_{ij}(y) = \frac{1}{N|\mathbb{S}^{N-1}|} \frac{N y_i y_j - |y|^2 \delta_{ij}}{|y|^{N+2}}, \quad \text{for } i, j = 1, 2, \ldots, N
$$

(1.3)

is the Calderón-Zygmund kernel. It remains to be an outstanding problem if we can extend $T$ above to $\infty$ for the smooth solutions of Euler equations.

We here specially focus on the finite-time singularity of self-similar type for the Euler equations. Such singularity type is related to the basic property that the equations (1.1) are invariant under the scaling transformation

$$
v(x, t) \mapsto v_{\lambda, \alpha}(x, t) = \lambda^\alpha v(\lambda x, \lambda^{1+\alpha} t), \quad \lambda > 0,
$$

$$
p(x, t) \mapsto p_{\lambda, \alpha}(x, t) = \lambda^{2\alpha} p(\lambda x, \lambda^{1+\alpha} t).
$$

In practice, we can also combine the spacetime translation in (1.1) to show the exact formula. We call a solution $(v, p)$ of (1.1) is (backward) self-similar with respect to the origin $0$ and time $T$ on the spacetime domain $D := \mathbb{R}^N \times (-\infty, T)$ if there exist some $\alpha > -1$ and $T > 0$ such that for all $(x, t) \in D$,

$$
v(x, t) = \lambda(t)^\alpha V(\lambda(t)x), \quad p(x, t) = \lambda(t)^{2\alpha} P(\lambda(t)x),
$$

(1.4)

2010 Mathematics Subject Classification. 76B03, 35Q31, 35Q35.
where \( \lambda(t) = (T - t)^{-\frac{1}{\alpha}} > 0 \), \((V, P)\) are stationary functions. The assumption \( \alpha > -1 \) guarantees that the singular solution concentrates on the origin as \( t \to T \). Up to a spacetime translation, (1.4) corresponds to that for some \( \alpha > -1 \),

\[
v(x, t) = v_{\lambda, \alpha}(x, t), \quad p(x, t) = p_{\lambda, \alpha}(x, t), \quad \forall \lambda > 0, (x, t) \in D.
\]

(1.5)

A more general case is that the equality (1.5) holds only for one single \( \lambda > 1 \), and correspondingly we call a solution \((v, p)\) of (1.1) is \textit{discretely self-similar with a factor} \( \lambda > 1 \) with respect to the origin 0 and time \( T \) on the spacetime domain \( D := \mathbb{R}^N \times (-\infty, T) \) if there exist some \( \alpha > -1 \) and \( T > 0 \) such that for all \((x, t) \in D\),

\[
\mathcal{T}v(x, t) = v_{\lambda, \alpha}(x, t), \quad \text{for} \ \lambda > 1,
\]

(1.6)

that is,

\[
v(x, T - t) = \lambda^\alpha v(\lambda x, T - \lambda^{1+\alpha} t), \quad \text{for} \ \lambda > 1,
\]

(1.7)

where \( \mathcal{T} \) is the temporal translation \( \mathcal{T}v(x, t) = v(x, T - t) \). In terms of the similarity variables

\[
y := \frac{x}{(T - t)^{\frac{1}{1+\alpha}}}, \quad s := \log \left( \frac{T}{T - t} \right), \quad \alpha > -1,
\]

(1.8)

the discretely self-similar solution \((v, p)\) is given by that for all \((x, t) \in \mathbb{R}^N \times ]-\infty, T[\)

\[
v(x, t) = \frac{1}{(T - t)^{\frac{\alpha}{1+\alpha}}} V(y, s), \quad p(x, t) = \frac{1}{(T - t)^{\frac{\alpha}{1+\alpha}}} P(y, s),
\]

(1.9)

where \( V(y, s) \) and \( P(y, s) \) are periodic-in-\( s \) functions with period \( S_0 := (1 + \alpha) \log \lambda > 0 \). Inserting (1.9) into (1.1), we formally obtain

\[
\left\{
\begin{aligned}
    \partial_s V + \frac{\alpha}{\alpha+1} V + \frac{1}{\alpha+1} y \cdot \nabla V + V \cdot \nabla V + \nabla P &= 0, \\
    \text{div } V &= 0, \\
    V|_{s=0}(y) = T^{\frac{1}{1+\alpha}} v_0(T^{\frac{1}{1+\alpha}} y)\n\end{aligned}
\right.
\]

(1.10)

Under the mild assumption on \( V \) (e.g. \( V \in L_3^3 T_y^p(\mathbb{R}^{N+1}) \) in the sequel), from (1.2) we have

\[
P(y, s) = -\frac{1}{N} |V(y, s)|^2 + p. v. \int_{\mathbb{R}^N} K_{ij}(y - z)V_i(z, s)V_j(z, s) \, dz.
\]

(1.11)

Self-similar type singularity plays an important role in the study of singularities, and has been experimentally detected and theoretically studied in many kinds of partial differential equations (one can refer to the recent survey paper [11]). We here mainly consider the discretely self-similar singular solution for the Euler equations (1.1). Discretely self-similar singularity was firstly introduced by [9] in the context of cosmology, and has been proposed for singularities of the Euler equations (cf. [14, 15]) and other various PDEs (cf. [11]). By definition, discretely self-similar solution (1.9) is a natural generalization of the self-similar solution (1.4): if the time periodic functions \((V, P)(y, s)\) do not depend on the \( s \)-variable, i.e., \((V, P)\) are stationary, it just reduces to the usual self-similar case.

The possibility of occurring the self-similar singular solutions in the Euler equations (1.1) and their properties have been intensely considered in the mathematical literature such as [11, 2, 3, 4, 7, 12, 16, 17, 18, 19]. But the theoretic study of discretely self-similar solutions for (1.1) are relatively limited and there are only several recent works on this topic. Chae and Tsai in [8] proved some nonexistence results for the discretely self-similar solutions with time-periodic function \( V \in C^1_3 C_y^2(\mathbb{R}^{3+1}) \) based on the vorticity profile \( \Omega = \nabla \times V \): if additionally \(|V|\) and \(|\nabla V|\) has the decaying asymptotics, and \( \Omega \in L^q(\mathbb{R}^3 \times [0, S_0]) \) for some \( q \in \left[ 0, \frac{3}{3+\alpha} \right] \), then \( V \equiv 0 \) on \( \mathbb{R}^{3+1} \). They also proved the nonexistence
results for the time-periodic functions \((V, P) \in C^1_{\text{loc}}(\mathbb{R}^{3+1})\) (with \(P\) given by (1.11)) based on the velocity profile: if
\[
V \in L^2(0, S_0; L^r(\mathbb{R}^3)), \quad r \in [3, 9/2], \quad \alpha > 3/2, \quad \text{or}
\]
\[
V \in L^2(0, S_0; L^r(\mathbb{R}^3)) \cap L^3(0, S_0; L^r(\mathbb{R}^3)), \quad r \in [3, 9/2], \quad -1 < \alpha < 3/2, \quad \text{or}
\]
\[
V \in L^p(\mathbb{R}^3 \times [0, S_0]), \quad p \in [3, \infty[, \quad -1 < \alpha \leq 3/p, \quad \text{or}
\]
\[
V \in L^p(\mathbb{R}^3 \times [0, S_0]), \quad p \in [3, \infty[, \quad 3/2 < \alpha < \infty,
\]
then \(V \equiv 0\) on \(\mathbb{R}^{3+1}\). In [5], by applying the maximum principle in the far field region for the vorticity equations, Chae proved the following result for the discretely self-similar solutions with the time-periodic vector field \(V \in C^1_s C^2_{y,\text{loc}}(\mathbb{R}^{3+1})\): if additionally \(\sup_{s \in [0, S_0]} |\nabla V(y, s)| = o(1)\) as \(|y| \to \infty\), and there exists \(k > \alpha + 1\) such that the vorticity profile \(\Omega = \nabla \times V\) satisfies
\[
|\Omega(y, s)| = O(|y|^{-k}), \quad \text{as} \quad |y| \to \infty, \quad \forall s \in [0, S_0],
\]
then \(V(y, s) \equiv C(s)\) for all \(y \in \mathbb{R}^3\), where \(C : [0, S_0] \to \mathbb{R}^3\) is a closed curve satisfying \(C(s) = C(s + S_0)\) for all \(s \in [0, S_0]\). Chae in [6] also showed the unique continuation type theorem for the discretely self-similar solutions of (1.1) in \(\mathbb{R}^3\).

In this paper we consider the discretely self-similar solutions of the Euler equations (1.1) to prove some nonexistence criteria and show the asymptotic property of the possible velocity profile, which partially improves the corresponding result of [3]. The first main result reads as follows.

**Theorem 1.1.** Suppose that \(V \in C^1_s C^1_{y,\text{loc}}(\mathbb{R}^N \times \mathbb{R})\) is a periodic-in-\(s\) vector field with period \(S_0\), and \(P\) is defined from \(V\) by (1.11). We have the following statements.

1. If additionally \(V \in L^2([0, S_0]; L^p(\mathbb{R}^N))\) with some \(p \in [3, \infty[\), then for \(\alpha > \frac{N}{2}\) and \(-1 < \alpha \leq \frac{N}{p}\), we have \(V \equiv 0\), while for \(\frac{N}{p} < \alpha \leq \frac{N}{2}\), we have
\[
\sup_{s \in [0, S_0]} \int_{|y| \leq L} |V(y, s)|^2 \, dy \lesssim L^{N-2\alpha}, \quad \forall L \gg 1.
\]
In particular, for \(\frac{N}{p} < \alpha < \frac{N}{2}\), we have either \(V \equiv 0\) or
\[
\int_0^{S_0} \int_{|y| \leq L} |V(y, s)|^2 \, dy \, ds \sim L^{N-2\alpha}, \quad \forall L \gg 1.
\]

2. For \(\alpha = \frac{N}{2}\), if \(V \in L^2_{s,y}([0, S_0])\) (which is slightly weaker than (1.14)) and there exists some constant \(\kappa > 0\) such that
\[
\sup_{s \in [0, S_0]} |V(y, s)| \lesssim |y|^{1-\kappa}, \quad \forall |y| \gg 1,
\]
then we have
\[
\int_0^{S_0} \int_{L \leq |y| \leq \lambda L} |V(y, s)|^2 \, dy \, ds \lesssim \frac{1}{L^{N+2-\epsilon}}, \quad \forall L \gg 1, \quad 0 < \epsilon \ll 1.
\]

Next we consider the velocity profile with nondecreasing asymptotics, which is reasonable and interesting: indeed, from the energy inequality \(\|v(t)\|_{L^2} \leq \|v_0\|_{L^2}\) and using the scenario (1.2), we heuristically get the estimate (1.14) with \(L = (T - t)^{-1/(1+\alpha)}\) for all \(\alpha > -1\), which implies that \(V\) can have nondecreasing asymptotics for \(-1 < \alpha \leq 0\). But before stating our result, we need a refined
version of the expression formula of the pressure profile. Since the pressure profile can be defined by (1.11) up to a function depending only on $s$, we assume that for every $y \in B_L(0)$,

$$
P(y, s) = P_L(y, s) = -\frac{1}{N} |V(y, s)|^2 + \text{p.v.} \int_{\mathbb{R}^N} K_{ij}(y - z)V_i(z, s)V_j(z, s) \, dz + \tilde{P}_L(s),
$$

with

$$
\tilde{P}_L(s) := -\int_{|z| \geq 2L} K_{ij}(z)V_i(z, s)V_j(z, s) \, dz.
$$

In this way, the formula of $P$ can be expressed as that for every $y \in B_L(0)$,

$$
P(y, s) = P_L(y, s) = -\frac{1}{N} |V(y, s)|^2 + \text{p.v.} \int_{|z| \leq 2L} K_{ij}(y - z)V_i(z, s)V_j(z, s) \, dz + 
$$

$$
+ \int_{|z| \geq 2L} (K_{ij}(y - z) - K_{ij}(z))V_i(z, s)V_j(z, s) \, dz,
$$

which is reminiscent of the introducing an analogous expression formula of a singular integral operator when one proves the property that it maps $L^\infty(\mathbb{R}^N)$ to $BMO(\mathbb{R}^N)$ (cf. [10, Eq. (6.4)]). Note that for every $0 < L_1 \leq L_2 < \infty$ and for all $y \in B_{L_1}(0)$, the difference $P_{L_2}(y, s) - P_{L_1}(y, s)$ corresponds to

$$
\int_{2L_1 \leq |z| \leq 2L_2} K_{ij}(z)V_i(z, s)V_j(z, s) \, dz
$$

which is a meaningful function depending only on $s$ under the assumption (1.21), thus in such a sense $P_{L_1}(y, s)$ and $P_{L_2}(y, s)$ for all $y \in B_{L_1}(0)$ are equal.

Our second main result is as follows.

**Theorem 1.2.** Suppose that $V \in C_1^1 C_1^1_{y, \text{loc}}(\mathbb{R}^{N+1})$ is a periodic-in-$s$ vector field with period $S_0$ satisfying that for some $\delta \in ]0, \frac{1}{2}[$,

$$
1 \leq \sup_{s \in [0, S_0]} |V(y, s)| \lesssim |y|^\delta, \quad \forall |y| \gg 1,
$$

(1.21)

and $P$ is defined from $V$ through (1.18). Then for the nontrivial velocity profile $V$ associated with $-1 < \alpha < \infty$, the only possible range of $\alpha$ is $-\delta \leq \alpha \leq 0$, and the corresponding velocity profile satisfies that

$$
\sup_{s \in [0, S_0]} \int_{|y| \leq L} |V(y, s)|^2 \, dy \sim L^{N-2\alpha}, \quad \forall L \gg 1,
$$

(1.22)

and

$$
\int_0^{S_0} \int_{|y| \leq L} |V(y, s)|^2 \, dy \, ds \sim L^{N-2\alpha}, \quad \forall L \gg 1.
$$

(1.23)

**Remark 1.3.** From (1.15) and (1.22), we can expect that for every $-1 < \alpha < \frac{N}{2}$ the corresponding “typical” possible velocity profile has the following asymptotics:

$$
\sup_{s \in [0, S_0]} |V(y, s)| \sim \frac{1}{|y|^{\alpha}} + o\left(\frac{1}{|y|^{\alpha}}\right), \quad \forall |y| \gg 1,
$$

and by scaling, we can also expect that

$$
\sup_{s \in [0, S_0]} |\Omega(y, s)| \sim \frac{1}{|y|^{\alpha+1}} + o\left(\frac{1}{|y|^{\alpha+1}}\right), \quad \forall |y| \gg 1,
$$

(1.24)

with $\Omega := \nabla \times V$. Note that (1.24) is compatible with the nonexistence results of [5, 8] based on the vorticity profile (e.g. comparing (1.24) with (1.13)).
Remark 1.4. If (1.6) holds on the spacetime domain $B_r(x_0) \times (-\infty, T]$ with some $r > 0$, then the corresponding solution $(v, p)$ is called the locally discretely self-similar solution. For such singular solutions, so far it is not clear to show the analogous results as Theorem 1.1 and 1.2. Part of the reason is that the profiles $(V, P)$ are no longer genuinely time periodic functions for $(y, s) \in \mathbb{R}^{N+1}$.

The proofs of Theorem 1.1 and 1.2 are both based on the local energy inequalities of the velocity profile (2.10)-(2.11), which in turn is derived from the energy equality of the original equality (2.1). Then thanks to a careful treating of the term containing the pressure profile (cf. Lemma 5.1 and 5.2), the proofs are followed by using the bootstrapping method according to the values of $\alpha$.

The outline of this paper is as follows. In Section 2, we proved the key local energy inequality of the velocity profile. Relied on this result, we give the detailed proofs of Theorem 1.1 and 1.2 in the sections 3 and 4 respectively. At last we present two auxiliary lemmas about concerning the term including pressure profile in the section 5.

Throughout this paper, $C$ denotes a harmless constant which may be of different value from line to line. For two quantities $X, Y$, the expression $X \lesssim Y$ denotes that there is a constant $C > 0$ such that $X \leq CY$, and $X \sim Y$ means that $X \lesssim Y$ and $Y \lesssim X$. For a real number $a$, denote by $|a|$ its integer part. For $x_0 \in \mathbb{R}^N$, $r > 0$, denote by $B_r(x_0)$ the open ball of $\mathbb{R}^N$ centered at $x_0$ with radius $r$, and denote by $B_r(x_0)^c$ its complement set $\mathbb{R}^N \setminus B_r(x_0)$.

2. Local energy inequality of the velocity profile

We start with the local energy equality of the original velocity
\[
\int_{\mathbb{R}^N} |v(x, t_2)|^2 \chi(x, t_2) \, dx - \int_{\mathbb{R}^N} |v(x, t_1)|^2 \chi(x, t_1) \, dx
= \int_{t_1}^{t_2} \int_{\mathbb{R}^N} |v(x, t)|^2 \partial_t \chi(x, t) \, dx \, dt + \int_{t_1}^{t_2} \int_{\mathbb{R}^N} (|v|^2 v + 2pv) \cdot \nabla \chi(x, t) \, dx \, dt,
\]
with $-\infty < t_1 < t_2 < T$ and $\chi \in \mathcal{D}(\mathbb{R}^N \times [-\infty, T])$. The equality can hold if the velocity field is regular enough, e.g. $v \in C^1_{\text{loc}}(\mathbb{R}^N \times [-\infty, T]) \cap L^\infty([a, b]; L^2(\mathbb{R}^N))$.

Let $\phi \in \mathcal{D}(\mathbb{R}^N)$ be a cutoff function supported on $B_1(0)$ such that $\phi \equiv 1$ on $B_1(0)$ and $0 \leq \phi \leq 1$ ($\lambda > 1$ is just the DSS factor in (1.6)). Set $\chi(x, t) = \phi(x)$, then for any $t_1 < t_2 < T$, (2.1) reduces to
\[
\int_{\mathbb{R}^N} |v(x, t_2)|^2 \phi(x) \, dx - \int_{\mathbb{R}^N} |v(x, t_1)|^2 \phi(x) \, dx
= \int_{t_1}^{t_2} \int_{\mathbb{R}^N} (|v|^2 v + 2pv)(x, t) \cdot \nabla \phi(x) \, dx \, dt.
\]
Inserting the ansatz (1.9) into (2.2), and denoting $s_2 := \log \frac{1}{T-t_2}$, $s_1 := \log \frac{1}{T-t_1}$, we obtain that for any $-\infty < s_1 < s_2 < \infty$,
\[
e^{s_2 \frac{2\alpha}{1+\alpha}} \int_{\mathbb{R}^N} |V(y, s_2)|^2 \phi(y e^{-\frac{1}{1+\alpha} s_2}) \, dy - e^{s_1 \frac{2\alpha}{1+\alpha}} \int_{\mathbb{R}^N} |V(y, s_1)|^2 \phi(y e^{-\frac{1}{1+\alpha} s_1}) \, dy
= \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \frac{1}{(T-t)^{\frac{2\alpha}{1+\alpha}} (1+\alpha)} (|V|^2 V + 2PV(y, s)) \cdot \nabla \phi(y(T-t)^\frac{1}{1+\alpha}) \, dy \, dt
= \int_{s_1}^{s_2} \int_{\mathbb{R}^N} e^{s_2 \frac{2\alpha}{1+\alpha} - s_1 \frac{2\alpha}{1+\alpha}} (|V|^2 V + 2PV(y, s)) \cdot \nabla \phi(y e^{-s_1 \frac{1}{1+\alpha}}) \, dy \, ds.
\]
With no loss of generality, we assume that $s_1 + S_0 < s_2$ with $S_0$ the period. Let $t_1, t_2 \in [0, S_0]$ be arbitrary, and by replacing $s_1$ with $s_1 + \tau_1$ in (2.3) (if $s_1 + S_0 \geq s_2$, we may use $s_1 + \tau_1$ and $s_1 + \tau_1 + \tau_2$,
to replace $s_1$ and $s_2$ respectively, we get
\[
e^{(s_2 + \tau_2)\frac{2\alpha - N}{1 + \alpha}} \int_{\mathbb{R}^N} |V(y, s_2 + \tau_2)|^2 \phi\left(y e^{-\frac{s_2 + \tau_2}{1 + \alpha}}\right) dy - e^{(s_1 + \tau_1)\frac{2\alpha - N}{1 + \alpha}} \int_{\mathbb{R}^N} |V(y, s_1 + \tau_1)|^2 \phi\left(y e^{-\frac{s_1 + \tau_1}{1 + \alpha}}\right) dy
\]
\[
= \int_{s_1 + \tau_1}^{s_2 + \tau_2} \int_{\mathbb{R}^N} e^{\frac{2\alpha - N - 1}{1 + \alpha}} \left(|V|^2 V + 2PV(y, s)\right) \cdot \nabla \phi\left(y e^{-\frac{s}{1 + \alpha}}\right) dyds.
\] (2.4)

For $i = 1, 2$, we set
\[
I_i := \int_{0}^{S_0} \int_{\mathbb{R}^N} e^{(s_i + \tau_i)\frac{2\alpha - N}{1 + \alpha}} |V(y, s_i + \tau_i)|^2 \phi\left(y e^{-\frac{s_i + \tau_i}{1 + \alpha}}\right) dyd\tau_i,
\]
\[
J_i := \sup_{\tau_i \in [0, S_0]} \int_{\mathbb{R}^N} e^{(s_i + \tau_i)\frac{2\alpha - N}{1 + \alpha}} |V(y, s_i + \tau_i)|^2 \phi\left(y e^{-\frac{s_i + \tau_i}{1 + \alpha}}\right) dy.
\] (2.5)

By the periodicity property of $V$, and denoting
\[
l_i := e^{\frac{s_i}{1 + \alpha}} \in ]0, \infty[, \quad \mu := e^{\frac{S_0}{1 + \alpha}},
\] (2.6)
we directly see that for $i = 1, 2$,
\[
c_{\alpha i}^{2\alpha - N} \int_{0}^{S_0} \int_{|y| \leq \frac{1}{\mu}} |V(y, s)|^2 dyds \leq I_i \leq C_{\alpha i}^{2\alpha - N} \int_{0}^{S_0} \int_{|y| \leq \mu l_i} |V(y, s)|^2 dyds,
\]
\[
c_{\alpha i}^{2\alpha - N} \sup_{s \in [0, S_0]} \int_{|y| \leq \frac{1}{\mu}} |V(y, s)|^2 dy \leq J_i \leq C_{\alpha i}^{2\alpha - N} \sup_{s \in [0, S_0]} \int_{|y| \leq \mu l_i} |V(y, s)|^2 dy,
\] (2.7)
where $c_{\alpha} := \min\{1, e^{S_0^{2\alpha - N}}\}$, $C_{\alpha} := \max\{1, e^{S_0^{2\alpha - N}}\}$. Taking the supremum over the $\tau_2$-variable and integrating on the $\tau_1$-variable in (2.4), we obtain
\[
|S_0 J_2 - I_1| \leq K_1
\]
with
\[
K_1 := \sup_{\tau_2 \in [0, S_0]} \int_{0}^{S_0} \int_{s_1 + \tau_1}^{s_2 + \tau_2} \int_{\mathbb{R}^N} e^{\frac{2\alpha - N - 1}{1 + \alpha}} \left(|V|^3 + 2|PV|(y, s)\right) |\nabla \phi\left(y e^{-\frac{s}{1 + \alpha}}\right)| dydsd\tau_1.
\]

Denoting by
\[
k_1 := \lfloor \log_\lambda (\mu l_2/l_1) \rfloor, \quad B_k := \{s : l_1 \lambda^k \leq e^{\frac{s}{1 + \alpha}} \leq l_1 \lambda^{k+1}\},
\] (2.8)
and from the support property of $\phi$ and the periodicity property of $(V, P)$, we infer that
\[
K_1 \leq S_0 \int_{s_1}^{s_2 + S_0} \int_{\mathbb{R}^N} e^{\frac{2\alpha - N - 1}{1 + \alpha}} \left(|V|^3 + 2|PV|(y, s)\right) |\nabla \phi\left(y e^{-\frac{s}{1 + \alpha}}\right)| dyds
\]
\[
\leq S_0 \sum_{k=0}^{k_1} \int_{s_1}^{s_2} \int_{\frac{k}{1 + \alpha} \leq |y| \leq \frac{1}{1 + \alpha}} 1_B_k(s) e^{\frac{2\alpha - N - 1}{1 + \alpha}} \left(|V|^3 + 2|PV|(y, s)\right) |\nabla \phi\left(y e^{-\frac{s}{1 + \alpha}}\right)| dyds
\]
\[
\leq S_0 \sum_{k=0}^{k_1} \frac{1}{(l_1 \lambda^k)^{N+1-2\alpha}} \int_{(1+\alpha) log(l_1 \lambda^{k+1})} \int_{l_1 \lambda^{k-1} \leq |y| \leq l_1 \lambda^{k+1}} \frac{|V|^3 + 2|P||V|}{|y|^{N+1-2\alpha}} |\nabla \phi\left(y e^{-\frac{s}{1 + \alpha}}\right)| dyds
\]
\[
\leq \frac{C S_0}{\lambda} \sum_{k=0}^{k_1} \frac{1}{(l_1 \lambda^k)^{N+1-2\alpha}} \int_{0}^{S_0} \int_{l_1 \lambda^{k-1} \leq |y| \leq l_1 \lambda^{k+1}} (|V|^3 + |P||V|)(y, s) dyds =: K_2,
\] (2.9)
where in the last line we have used the fact that \(|B_k| = (1 + \alpha) \log \lambda = S_0\). On the other hand, we also get

\[
K_1 \leq CS_0 \int_{s_1}^{s_2 + S_0} \int_{\frac{1}{4} \mu t_1 \leq |y| \leq e^{\frac{t}{1+\alpha}}} \frac{|V|^3 + |P||V(y,s)|}{|y|} |\nabla \phi(y e^{-\frac{t}{1+\alpha}})|\,dy\,ds
\]

\[
\leq CS_0 \int_{s_1}^{s_2 + S_0} \int_{\frac{1}{4} \mu t_1 \leq |y| \leq e^{\frac{t}{1+\alpha}}} \frac{|V|^3 + |P||V(y,s)|}{|y|} |\nabla \phi(y e^{-\frac{t}{1+\alpha}})|\,dy\,ds
\]

\[
\leq \frac{CS_0}{\lambda} \int_0^{S_0} \int_{\frac{1}{4} \mu t_1 \leq |y| \leq e^{\frac{t}{1+\alpha}}} \frac{|V|^3 + |P||V(y,s)|}{|y|} \,dy\,ds =: K_3,
\]

where the interval \(\{s : |y/\lambda \leq e^{\frac{t}{1+\alpha}} \leq |y|\}\) has the length \((1 + \alpha) \log \lambda = S_0\) and is just of a period. Hence we find

\[
|S_0 J_2 - I_1| \leq CK_2, \quad \text{and} \quad |S_0 J_2 - I_1| \leq CK_3.
\]

Similarly, by using the different treating of \(\tau_1, \tau_2\) in (2.2), i.e. taking the supremum norm and \(L^1\)-norm on the \(\tau_1, \tau_2\) variables in different order, we also have

\[
|I_2 - I_1| + |I_2 - S_0 J_1| + |J_2 - J_1| \leq CK_2,
\]

and

\[
|I_2 - I_1| + |I_2 - S_0 J_1| + |J_2 - J_1| \leq CK_3.
\]

3. Proof of Theorem 1.1

3.1. Proof of Theorem 1.1 (1). We here mainly focus on the case of \(\alpha > \frac{N}{p}\), especially \(\frac{N}{p} < \alpha \leq \frac{N}{2}\).

We start with the inequality (2.10): \(|S_0 J_2 - I_1| \leq CK_2\) (or the inequality \(|J_2 - J_1| \leq CK_2\), and by setting \(t_1 = \lambda\) and \(t_2 = \lambda \Lambda \gg 1\), we get

\[
L^{2\alpha - N} \sup_{s \in [0, S_0]} \int_{|y| \leq L} |V(y,s)|^2 \,dy
\]

\[
\lesssim \int_0^{S_0} \int_{|y| \leq \lambda \mu} |V(y,s)|^2 \,dy\,ds + \sum_{k=0}^{[\log_\lambda (\mu L)]} \frac{1}{\lambda^k (N + 1 - 2\alpha)} \int_0^{S_0} \int_{|y| \leq \lambda^k} (|V|^3 + |P||V|)(y,s) \,dy\,ds.
\]

It directly leads to

\[
\sup_{s \in [0, S_0]} \int_{|y| \leq L} |V(y,s)|^2 \,dy \leq CL^{N - 2\alpha} + CE(L),
\]

with

\[
E(L) := L^{N - 2\alpha} \sum_{k=0}^{[\log_\lambda (\mu L)]} \frac{1}{\lambda^k (N + 1 - 2\alpha)} \int_0^{S_0} \int_{|y| \leq \lambda^k} (|V|^3 + |P||V|)(y,s) \,dy\,ds.
\]

Thanks to Hölder’s inequality and Lemma 5.1 below, we see that

\[
E(L) \lesssim L^{N - 2\alpha} \sum_{k=0}^{[\log_\lambda (\mu L)]} \frac{1}{\lambda^k (N + 1 - 2\alpha)} \int_0^{S_0} \left( \int_{|y| \leq \lambda^k} |V|^p \,dy \right)^{3/p} + \left( \int_{|y| \leq \lambda^k} |P|^2 \,dy \right)^{3/p} \,ds
\]

\[
\lesssim L^{N - 2\alpha} \sum_{k=0}^{[\log_\lambda (\mu L)]} \frac{1}{\lambda^k (N + 1 - 2\alpha)} \int_0^{S_0} \left( \int_{|y| \leq \lambda^k} |V(y,s)|^p \,dy \right)^{3/p} \,ds
\]

\[
\lesssim L^{N - 2\alpha} \sum_{k=0}^{[\log_\lambda (\mu L)]} \lambda^{k(1 - 2\alpha + \frac{N}{p})}.
\]
If $1 - 2\alpha + \frac{3N}{p} \geq 0$, i.e. $2\alpha \leq 1 + \frac{3N}{p}$, from the above estimate of $E(L)$ we get
\[
\sup_{s \in [0, S_0]} \int_{|y| \leq L} |V(y, s)|^2 \, dy \leq CL^{N-2\alpha} [\log \lambda L].
\]
Otherwise, if $1 - 2\alpha + \frac{3N}{p} < 0$, we obtain
\[
\sup_{s \in [0, S_0]} \int_{|y| \leq L} |V(y, s)|^2 \, dy \leq CL^{\beta p}, \quad \text{with } \beta_p := N - 1 - \frac{3N}{p} > 0. \quad (3.3)
\]
Next we intend to improve the estimate (3.3) in this case. By interpolation and Hölder’s inequality, we infer that
\[
\int_0^{S_0} \int_{|y| \leq L} |V(y, s)|^3 \, dy \, ds \lesssim \left( \int_0^{S_0} \left( \int_{|y| \leq L} |V(y, s)|^2 \, dy \right)^{\theta_p} \left( \int_{|y| \leq L} |V(y, s)|^p \, dy \right)^{1-\theta_p} \, ds \right)^{\frac{p}{\theta_p (p-2)}}
\]
\[
\lesssim L^{\beta \theta_p}, \quad (3.4)
\]
with $\theta_p := \frac{p-3}{p-2}$. We use this estimate and Lemma 5.1 to improve the bound of $E(L)$ as follows
\[
E(L) \lesssim L^{N-2\alpha} \sum_{k=0}^{\lfloor \log_2 (\mu L) \rfloor} \frac{1}{\lambda^{N+1-2\alpha}} \int_0^{S_0} \left( \int_{|y| \sim \lambda^k} |V|^3 \, dy + \int_{|y| \sim \lambda^k} |P|^3 \, dy \right) \, ds
\]
\[
\lesssim L^{N-2\alpha} \sum_{k=0}^{\lfloor \log_2 (\mu L) \rfloor} \frac{1}{\lambda^{N+1-\alpha}} \int_0^{S_0} \int_{|y| \leq \lambda^k} |V(y, s)|^3 \, dy \, ds
\]
\[
\lesssim L^{N-2\alpha} \sum_{k=0}^{\lfloor \log_2 (\mu L) \rfloor} \lambda^{-k(N+1-2\alpha-\beta_p \theta_p)}. \quad (3.5)
\]
If $N + 1 - 2\alpha - \beta_p \theta_p \geq 0$, i.e. $N - 2\alpha \geq \beta_p \theta_p - 1$, then we directly get
\[
\sup_{s \in [0, S_0]} \int_{|y| \leq L} |V(y, s)|^2 \, dy \lesssim L^{N-2\alpha} [\log \lambda L].
\]
Otherwise if $N + 1 - 2\alpha - \beta_p < 0$, then we obtain that
\[
\sup_{s \in [0, S_0]} \int_{|y| \leq L} |V(y, s)|^2 \, dy \lesssim L^{\beta_p \theta_p - 1},
\]
and thus by interpolation,
\[
\int_0^{S_0} \int_{|y| \leq L} |V(y, s)|^3 \, dy \, ds \lesssim L^{\beta_p \theta_p^2 - \theta_p}.
\]
The above process can be iteratively repeated in finite time, and for every $\alpha > \frac{N}{p}$, there exists $n \in \mathbb{N}$ so that $N - 2\alpha - (\beta_p \theta_p^n - \theta_p^{n-1} - \cdots - 1) \geq 0$, and we have
\[
\sup_{s \in [0, S_0]} \int_{|y| \leq L} |V(y, s)|^2 \, dy \lesssim L^{N-2\alpha} [\log \lambda L]. \quad (3.6)
\]
By passing $L$ to $\infty$, this already implies that $V \equiv 0$ for all $(y, s) \in \mathbb{R}^{N+1}$ at the case $\alpha > \frac{N}{2}$.
Now we intend to remove the additional term $[\log \lambda L]$ appearing in (3.6). Let $\epsilon \in [0,1]$, then from (3.6) we deduce that
\[
\sup_{s \in [0,S_0]} \int_{|y| \leq L} |V(y,s)|^2 \, dy \lesssim e^{L^{N-2\alpha+\epsilon}},
\]
and by interpolation,
\[
\int_0^{S_0} \int_{|y| \leq L} |V(y,s)|^3 \, dyds \lesssim L^{(N-2\alpha+\epsilon)\theta_p}.
\]
Similarly as obtaining (3.5) we have
\[
E(L) \lesssim L^{(N-2\alpha)} \sum_{k=0}^{[\log(\alpha L)]} \lambda^{-k \left( N+1-2\alpha -(N-2\alpha+\epsilon)\theta_p \right)}
\lesssim L^{N-2\alpha} \sum_{k=0}^{[\log(\alpha L)]} \lambda^{-k(1-\epsilon)} \lesssim L^{N-2\alpha}.
\]
Hence the desired estimate (1.14) is derived for any $\alpha \in \left[ \frac{N}{p}, \frac{N}{2} \right]$.

Next we consider the statement (1.15) for the case $\frac{N}{p} < \alpha < \frac{N}{2}$. In order to show (1.15) for the nontrivial velocity profile, it suffices to prove the following inequality
\[
\frac{1}{L^{N-2\alpha}} \int_0^{S_0} \int_{|y| \leq L} |V(y,s)|^2 \, dyds \gtrsim 1, \quad \forall L \gg 1.
\]
(3.7)
The method is by contradiction. Suppose (3.7) does not hold, then there is a sequence of numbers $L_n \gg 1$ such that as $L_n \to \infty$, one has
\[
\frac{1}{L_n^{N-2\alpha}} \int_0^{S_0} \int_{|y| \leq L} |V(y,s)|^2 \, dyds \to 0.
\]
We shall use the local energy inequality $|I_2 - S_0 J_1| \leq CK_2$, and by setting $l_2 = L_n \to \infty$ and $l_1 = \lambda L$, we get
\[
\sup_{s \in [0,S_0]} \int_{|y| \leq L} |V(y,s)|^2 \, dy \lesssim L^{N-2\alpha} \sum_{k=0}^{\infty} \frac{1}{(\lambda^k L)^{N+1-2\alpha}} \int_0^{S_0} \int_{|y| \sim \lambda^k L} (|V|^3 + |P||V|)(y,s) \, dyds.
\]
(3.8)
Since we already have (1.14), thanks to the inequality (3.4), we deduce
\[
\int_0^{S_0} \int_{|y| \leq L} |V|^3 \, dyds \lesssim L^{(N-2\alpha)\theta_p}, \quad \forall L \gg 1.
\]
By virtue of Lemma 5.1 again (similar to obtaining (3.5)), we find that
\[
\sup_{s \in [0,S_0]} \int_{|y| \leq L} |V(y,s)|^2 \, dy \lesssim \frac{1}{L} \sum_{k=0}^{\infty} 2^{-k(2N-2\alpha+1)} (2^k L)^{(N-2\alpha)\theta_p} \lesssim L^{(N-2\alpha)\theta_p-1}.
\]
(3.9)
By interpolation we further get
\[
\int_0^{S_0} \int_{|y| \leq L} |V(y,s)|^3 \, dyds \lesssim L^{(N-2\alpha)\theta_p^2-\theta_p}.
\]
Using this improved estimate in (3.8) we further obtain a more refined estimate than (3.9). By repeating such iterative process, after a finite $n$-times, we obtain
\[
\sup_{s \in [0,S_0]} \int_{|y| \leq L} |V(y,s)|^2 \, dy \lesssim L^{(N-2\alpha)\theta_p^n-\theta_p^{n-1}-\cdots-1}.
\]
For $n$ large enough, the power of $L$ becomes negative, which guarantees $\sup_{s \in [0, S_0]} \int_{\mathbb{R}^N} |V(y, s)|^2 \, dy \equiv 0$, and thus $V \equiv 0$ for all $(y, s) \in \mathbb{R}^{N+1}$.

For the case $-1 < \alpha \leq N/p$, since the treating is similar to the obtaining of (1.15) or that of the case $-1 < \alpha < -\delta$ in the next section, we omit the details and we only note that for all $-1 < \alpha \leq N/p$,

$$
\frac{1}{l^2} \int_0^{S_0} \int_{|y| \leq \mu_2} |V|^2 \, dy \, ds \leq \frac{1}{l^{2-N-2\alpha}} \int_0^{S_0} \int_{|y| \leq M} |V|^2 \, dy \, ds + \frac{1}{l^{2-N-2\alpha}} \int_0^{S_0} \int_{M \leq |y| \leq \mu_2} |V|^2 \, dy \, ds \\
\lesssim \frac{M(N-\frac{3}{p})}{l^{2-N-2\alpha}} \left( \int_0^{S_0} \left( \int_{|y| \leq M} |V|^p \, dy \right)^{\frac{2}{p}} \, ds \right)^{\frac{2}{3}} + l^2 \left( \frac{1}{p} \right) \left( \int_0^{S_0} \left( \int_{|y| \geq M} |V|^p \, dy \right)^{\frac{2}{p}} \, ds \right)^{\frac{2}{3}} \\
\to 0, \quad \text{as} \quad l_2 \to \infty, \quad \text{and then} \quad M \to \infty,
$$

and at the first step of iteration

$$
\int_0^{S_0} \int_{|y| \sim \lambda^k L} \left(|V|^3 + |P||V|\right) \, dy \, ds \lesssim (\lambda^k L)^{(N-\frac{3}{p})} \left( \int_0^{S_0} \left( \int_{|y| \sim \lambda^k L} |V|^p \, dy \right)^{\frac{3}{p}} \left( \int_0^{S_0} \left( \int_{|y| \sim \lambda^k L} |P|^p \, dy \right)^{\frac{3}{p}} \right) \right) \\
\lesssim (\lambda^k L)^{(N-\frac{3}{p})} \int_0^{S_0} \left( \int_{|y| \sim \lambda^k L} |V|^p \, dy \right)^{\frac{3}{p}} \, ds \lesssim (\lambda^k L)^{(N-3/p)}.
$$

3.2. Proof of Theorem 1.1.1(2). We begin with the local energy inequality (2.12): $|I_2 - I_1| \leq CK_3$ at the $\alpha = N/2$ case, and from (2.5) and $c_\alpha = C_\alpha = 1$ for $\alpha = N/2$, it also leads to

$$
\int_0^{S_0} \int_{|y| \leq l_2^\lambda} |V|^2 \, dy \, ds \leq \int_0^{S_0} \int_{|y| \leq \mu_1} |V|^2 \, dy \, ds + \int_0^{S_0} \int_{|y| \leq \mu_2} \frac{|V|^3 + |P||V(y,s)|}{|y|} \, dy \, ds.
$$

By letting $\mu_1 = L$ and $l_2 = \lambda^2 L$, we get

$$
\int_0^{S_0} \int_{L \leq |y| \leq \lambda L} |V(y,s)|^2 \, dy \, ds \leq \frac{C}{L^\nu} \int_0^{S_0} \int_{\lambda \leq |y| \leq \mu \lambda^2 L} \left(|V|^3 + |P||V|\right)(y,s) \, dy \, ds \\
\leq \frac{C}{L^\nu} \int_0^{S_0} \int_{\lambda \leq |y| \leq \mu \lambda^2 L} \left(|V|^3 + |P||V(y,s)|\right) \, dy \, ds
$$

with $\nu := \lfloor \log \lambda \mu \rfloor + 1$. From (1.16), we see that

$$
\int_0^{S_0} \int_{L \leq |y| \leq \lambda L} |V|^2 \, dy \, ds \leq \frac{C}{L^\nu} \int_0^{S_0} \int_{\lambda \leq |y| \leq \mu \lambda^2 L} |V|^2 \, dy \, ds + \frac{C}{L^\nu} \int_0^{S_0} \int_{\lambda \leq |y| \leq \mu \lambda^2 L} |P||V| \, dy \, ds.
$$

In order to treat the term involving $P$, we make the following decomposition

$$
P(y, s) = -\frac{|V(y, s)|^2}{N} + \text{p.v.} \int_{|z| \leq \frac{L}{2}} K_{ij}(y - z) V_i(z, s) V_j(z, s) \, dz \\
+ \int_{\lambda \leq |z| \leq 2\lambda^2 L} K_{ij}(y - z) V_i(z, s) V_j(z, s) \, dz + \int_{|z| \geq \lambda^3 L} K_{ij}(y - z) V_i(z, s) V_j(z, s) \, dz \\
:= P_{1,L}(y, s) + P_{2,L}(y, s) + P_{3,L}(y, s) + P_{4,L}(y, s).
$$

The treating of the term containing $P_{1,L}$ is obvious:

$$
\frac{1}{L} \int_0^{S_0} \int_{\lambda \leq |y| \leq \lambda^2 L} |P_{1,L}| |V| \, dy \, ds \leq \frac{C}{L^\nu} \int_0^{S_0} \int_{\lambda \leq |y| \leq \lambda^2 L} |V(y, s)|^2 \, dy \, ds.
$$
For $P_{2,L}$, from the support property, we infer that for every $|y| \geq \frac{L}{2}$,

\[ |P_{2,L}(y,s)| \leq \frac{C}{L^N} \int_{|z| \leq \frac{L}{L^{N/2}}} |V(z,s)|^2 \, dz \, ds \leq \frac{C \|V\|_{L^2}^2}{L^N}, \]

and thus by the H"older inequality we obtain

\[
\frac{1}{L} \int_0^{S_0} \int_{\frac{L}{L^{N/2}}} \frac{1}{|y| \leq \lambda^{\nu+2}L} |P_{2,L}||V| \, dy \, ds \leq \frac{C}{L^{N+1}} \int_{|y| \leq \lambda^{\nu+2}L} |V| \, dy \, ds \leq \frac{C}{L^{N/2+1}} \left( \int_{|y| \leq \lambda^{\nu+2}L} |V|^2 \, dy \, ds \right)^{\frac{1}{2}}.
\]

For $P_{3,L}$, taking advantage of the Calderón-Zygmund theorem and (1.16) again, we find that

\[
\frac{1}{L} \int_0^{S_0} \int_{\frac{L}{L^{N/2}}} \frac{1}{|y| \leq \lambda^{\nu+2}L} |P_{3,L}| |V| \, dy \, ds \leq \frac{1}{L} \left( \int_0^{S_0} \int_{\frac{L}{L^{N/2}}} \frac{1}{|y| \leq \lambda^{\nu+2}L} |V|^2 \, dy \, ds \right)^{\frac{1}{2}} \left( \int_0^{S_0} \int_{\frac{L}{L^{N/2}}} \frac{1}{|y| \leq \lambda^{\nu+2}L} |P_{3,L}|^2 \, dy \, ds \right)^{\frac{1}{2}} \leq \frac{C}{L} \left( \int_0^{S_0} \int_{\frac{L}{L^{N/2}}} \frac{1}{|y| \leq \lambda^{\nu+2}L} |V|^2 \, dy \, ds \right)^{\frac{1}{2}} \left( \int_0^{S_0} \int_{\frac{L}{L^{N/2}}} \frac{1}{|y| \leq \lambda^{\nu+2}L} |V|^4 \, dy \, ds \right)^{\frac{1}{2}} \leq \frac{C}{L^\kappa} \int_0^{S_0} \int_{\frac{L}{L^{N/2}}} \frac{1}{|y| \leq \lambda^{\nu+3}L} |V|^2 \, dy \, ds.
\]

By virtue of the dyadic decomposition and (1.16), we estimate the term containing $P_{4,L}$ as follows

\[
\frac{1}{L} \int_0^{S_0} \int_{\frac{L}{L^{N/2}}} \frac{1}{|y| \leq \lambda^{\nu+2}L} |P_{4,L}| |V| \, dy \, ds \leq L^{N-\kappa} \int_0^{S_0} \sup_{|y| \leq \lambda^{\nu+2}L} |P_{4,L}(y,s)| \, ds \leq CL^{N-\kappa} \int_0^{S_0} \sup_{|y| \leq \lambda^{\nu+2}L} \left( \sum_{k=\nu+3}^{\infty} \int_{\lambda^k L \leq |z| \leq \lambda^k L} \frac{1}{|y-z|^N} |V(z,s)|^2 \, dz \right) \, ds \leq \frac{C}{L^\kappa} \sum_{k=\nu+3}^{\infty} \frac{1}{\lambda^{N\kappa}} \int_0^{S_0} \int_{\lambda^k L \leq |z| \leq \lambda^{k+1}L} |V(z,s)|^2 \, dz \, ds.
\]

Gathering the above estimates leads to

\[
\int_0^{S_0} \int_{L \leq |y| \leq 2L} |V|^2 \, dy \, ds \leq \frac{C}{L^{N/2+1}} \sum_{j=-\nu-1}^{\nu+2} \left( \int_0^{S_0} \int_{\lambda^j L \leq |y| \leq \lambda^{j+1}L} |V|^2 \, dy \, ds \right)^{\frac{1}{2}} + \frac{C}{L^\kappa} \sum_{k=-\nu-2}^{\nu+2} \frac{1}{\lambda^{N\kappa}} \int_0^{S_0} \int_{\lambda^k L \leq |y| \leq \lambda^{k+1}L} |V|^2 \, dy \, ds. \tag{3.11}
\]

By denoting $A_k = A_k(L) := \int_0^{S_0} \int_{\lambda^k L \leq |y| \leq \lambda^{k+1}L} |V|^2 \, dy \, ds$ for every $k \in \mathbb{Z}$, we rewrite (3.11) as follows

\[
A_0 \leq \frac{C}{L^{N/2+1}} \sum_{j=-\nu-1}^{\nu+2} A_j^{1/2} + \frac{C}{L^\kappa} \sum_{k=-\nu-2}^{\nu+2} \frac{1}{\lambda^{N\kappa}} A_k, \tag{3.12}
\]
which also ensures that for every \( i \in \mathbb{Z} \),

\[
A_i \leq \frac{C}{(\lambda L)^{N/2+1}} \sum_{j=-\nu-1}^{\nu+2} A_{i+j}^{1/2} + \frac{C}{L^\kappa} \sum_{k=-\nu-2}^{\infty} \frac{1}{\lambda^{N_k}} A_{i+k}.
\]  

(3.13)

Using (3.13) in estimating the righthand side of (3.12), we get

\[
A_0 \leq \frac{C}{L^{N/2+1}} \sum_{j_1=-\nu-1}^{\nu+2} \left( \frac{C}{(\lambda L)^{(N/2+1)/2}} \sum_{j_2=-\nu-1}^{\nu+2} A_{j_1+j_2}^{1/4} + \frac{C}{L^{\kappa/2}} \sum_{k_1=-\nu-2}^{\infty} \frac{1}{\lambda^{N_{k_1}} A_{k_1}} \right)
\]

\[
+ \frac{C}{L^{N/2+1}(1+\kappa)} \sum_{j_1,j_2=-\nu-1}^{\nu+2} \frac{1}{\lambda^{N_{k_1}}} A_{j_1+j_2}^{1/2} + \frac{C}{L^{2\kappa}} \sum_{k_1,k_2=-\nu-2}^{\infty} \frac{1}{\lambda^{N_{k_1+k_2}}} A_{k_1+k_2}.
\]

By repeating this process \( n \)-times, we obtain

\[
A_0 \leq \frac{C}{L^{(N/2+1)(1+\cdots+1/2^n)}} \sum_{j_1,\ldots,j_n=-\nu-1}^{\nu+2} A_{j_1+\cdots+j_n}^{1/2n+1}
\]

\[
+ \frac{C}{L^{(N/2+1)(1+\cdots+1/2^n)+\kappa/2}} \sum_{j_1,\ldots,j_n,-\nu-1}^{\nu+2} \sum_{k_n=-\nu-2}^{\infty} \frac{1}{\lambda^{N_{k_1}+\cdots+k_n}} A_{j_1+\cdots+j_n+k_n}^{1/2n+1}
\]

\[
+ \cdots + \frac{C}{L^{(n+1)\kappa}} \sum_{k_1,\ldots,k_n,-\nu-2}^{\nu+2} \sum_{j_n=-\nu-1}^{\nu+2} \frac{1}{\lambda^{(k_1+\cdots+k_n)N}} A_{k_1+\cdots+k_n}^{1/2n+1}
\]

For every small \( \epsilon \) > 0, due to \( A_k \leq C \) for all \( k \in \mathbb{Z} \), we can let \( n \) large enough so that

\[
A_0(L) \leq \frac{C}{L^{(N/2+1)(1+\cdots+1/2^n)}} + \cdots + \frac{C}{L^{(N/2+1)(1+\cdots+1/2^n)+(n+1-m)\kappa/2}} + \cdots + \frac{C}{L^{(n+1)\kappa}} \leq \frac{C}{L^{N+2-\epsilon}}.
\]

This finishes the proof of (3.17).

4. PROOF OF THEOREM 1.2

We first consider the case \(-1 < \alpha < -\delta\), and we begin with the local energy inequality \( |J_2 - J_1| \leq CK_2 \) (or (2.10)). Note that thanks to (1.21), we see that for all \( \alpha \in (-1,-\delta) \),

\[
\frac{1}{l_2^{-2\alpha}} \int_0^{S_0} \int_{|y| \leq \mu_{l_2}} |V(y,s)|^2 \, dy \, ds \lesssim l_2^{-N+2\alpha} \int_0^{S_0} \int_{|y| \leq \mu_{l_2}} |y|^{2\delta} \, dy \, ds
\]

\[
\lesssim l_2^{2\delta+2\alpha} \to 0, \quad \text{as} \quad l_2 \to \infty,
\]
thus by letting \( l_1 = \lambda L \gg 1 \) and \( l_2 \to \infty \) and using (2.7), we have

\[
\sup_{s \in [0,S_0]} \int_{|y| \leq L} |V(y,s)|^2 \, dy \leq C L^{N-2\alpha} \sum_{k=0}^{\infty} \frac{1}{(\lambda^k L)^{N+1-2\alpha}} \int_{0}^{S_0} \int_{\lambda^k L \leq |y| \leq \lambda^{k+2} L} (|V|^3 + |P||V|) \, dy \, ds,
\]

where \( P = -\frac{1}{N} |V|^2 + P'_{1,L} + P'_{2,L} \) is defined by (1.18) or (1.20) for every \( y \in B_L(0) \) with

\[
P'_{1,L}(y,s) := \text{p.v.} \int_{|y| \leq 2L} K_{ij}(y-z)V_i(z,s)V_j(z,s) \, dz,
\]

\[
P'_{2,L}(y,s) := \int_{|y| \geq 2L} \left( K_{ij}(y-z) - K_{ij}(z) \right) V_i(z,s)V_j(z,s) \, dz.
\]

By using the expression of \( P \) as \( P = -\frac{1}{N} |V|^2 + P'_{1,\lambda^{k+2} L} + P'_{2,\lambda^{k+2} L} \), and noting that roughly

\[
\sup_{s \in [0,S_0]} \int_{|y| \leq \lambda^{k+2} L} |V|^2 \, dy \lesssim (\lambda^k L)^{N+2\delta}, \quad \text{with} \ 0 \leq \delta < \frac{1}{2},
\]

we have the following estimate thanks to Lemma 5.2 below,

\[
\int_{0}^{S_0} \int_{\lambda^k L \leq |y| \leq \lambda^{k+2} L} |P||V| \, dy \, ds \lesssim (2^k L)^{N+3\delta}.
\]

Hence we find

\[
\sup_{s \in [0,S_0]} \int_{|y| \leq L} |V(y,s)|^2 \, dy \leq \frac{C}{L} \sum_{k=1}^{\infty} 2^{-k(N-2\alpha+1)} (2^k L)^{N+3\delta} \leq CL^{N+3\delta-1}.
\]

Note that (4.5) is better than (4.3), and we can use this refined estimate to get a better estimate than (4.4), that is,

\[
\int_{0}^{S_0} \int_{|y| \sim 2^k L} |P||V| \, dy \, ds \lesssim (2^k L)^{N+4\delta-1}.
\]

Thus inserting the above estimate into (4.1) we further obtain

\[
\sup_{s \in [0,S_0]} \int_{|y| \leq L} |V(y,s)|^2 \, dy \leq \frac{C}{L} \sum_{k=1}^{\infty} 2^{-k(N-2\alpha+1)} (2^k L)^{N+4\delta-1} \lesssim L^{N+4\delta-2}.
\]

By repeating the above process, after \( n \)-times (\( n \in \mathbb{N} \)) we deduce that

\[
\sup_{s \in [0,S_0]} \int_{|y| \leq L} |V(y,s)|^2 \, dy \leq C L^{N+2\delta+n(\delta-1)}, \quad \forall L \gg 1.
\]

For \( n \) large enough, the power of \( L \) is negative, which implies that \( |V(y,s)| \equiv 0 \) for all \( (y,s) \in \mathbb{R}^{N+1} \).

Next we consider the case \( \alpha \geq -\delta \), and we still begin with the inequality \( |J_2 - J_1| \leq CK_2 \). From (2.7), and by setting \( l_1 = \lambda \) and \( l_2 = \lambda L \gg 1 \), we have

\[
\sup_{s \in [0,S_0]} \int_{|y| \leq L} |V(y,s)|^2 \, dy \lesssim L^{N-2\alpha} + L^{N-2\alpha} \sum_{k=0}^{[\log_\lambda(\mu L)]} \frac{1}{\lambda^k (N+1-2\alpha)} \int_{0}^{S_0} \int_{\lambda^k \leq |y| \leq \lambda^{k+2}} (|V|^3 + |P||V|) \, dy \, ds.
\]

By using Lemma 5.2 and the expression \( P = -\frac{1}{N} |V|^2 + P'_{1,\lambda^{k+2} L} + P'_{2,\lambda^{k+2} L} \) for \( y \in B_{\lambda^{k+2}}(0) \), we have

\[
\int_{0}^{S_0} \int_{|y| \leq \lambda^{k+2}} |P||V| \, dy \, ds \lesssim \lambda^{k(N+3\delta)}.
\]
Thus we infer that

\[
\sup_{s \in [0, S_0]} \int_{|y| \leq L} |V(y, s)|^2 \, dy \leq CL^{N-2\alpha} + CL^{N-2\alpha} \sum_{k=0}^{\lfloor \log_\lambda (\mu L) \rfloor} 2^{-k(1-3\delta-2\alpha)} \leq \begin{cases} CL^{N-2\alpha} \lfloor \log_\lambda L \rfloor, & \text{if } -2\alpha \geq 3\delta - 1, \\ CL^{N+3\delta-1}, & \text{if } -2\alpha < 3\delta - 1. \end{cases}
\]

For the case \(-2\alpha < 3\delta - 1\), i.e. \(2\alpha > 1 - 3\delta\), we apply Lemma 5.2 to find a better estimate of (4.8):

\[
\int_0^{S_0} \int_{|y| \leq \lambda^{k+2}} |P||V| \, dy \, ds \lesssim \lambda^{k(N+4\delta-1)}. \tag{4.9}
\]

Furthermore, we obtain

\[
\sup_{s \in [0, S_0]} \int_{|y| \leq L} |V(y, s)|^2 \, dy \leq CL^{N-2\alpha} + CL^{N-2\alpha} \sum_{k=0}^{\lfloor \log_\lambda (\mu L) \rfloor} 2^{-k(2-4\delta-2\alpha)} \leq \begin{cases} CL^{N-2\alpha} \lfloor \log_\lambda L \rfloor, & \text{if } -2\alpha \geq 4\delta - 2, \\ CL^{N+4\delta-2}, & \text{if } -2\alpha < 4\delta - 2. \end{cases}
\]

By repeating the process as above, if there exists some \(n \in \mathbb{N}^+\) such that \(-2\alpha < 2\delta - n(1 - \delta)\), we can show that

\[
\int_0^{S_0} \int_{|y| \leq L} |V(y, s)|^2 \, dy \, ds \leq \begin{cases} CL^{N-2\alpha} \lfloor \log_\lambda L \rfloor, & \text{if } -2\alpha \geq 2\delta - (n+1)(1 - \delta), \\ CL^{N+2\delta - (n+1)(1 - \delta)}, & \text{if } -2\alpha < 2\delta - (n+1)(1 - \delta). \end{cases} \tag{4.10}
\]

Since for all \(\alpha \geq -\delta\), there is some \(n_0 \in \mathbb{N}^+\) large enough such that \(-2\alpha \geq 2\delta - n_0(1 - \delta)\), and thus we get

\[
\sup_{s \in [0, S_0]} \int_{|y| \leq L} |V(y, s)|^2 \, dy \leq CL^{N-2\alpha} \lfloor \log_\lambda L \rfloor, \quad \forall L \gg 1. \tag{4.11}
\]

From the lower bound in (1.21), there is a constant \(c > 0\) independent of \(L\) so that

\[
\sup_{s \in [0, S_0]} \int_{|y| \leq L} |V(y, s)|^2 \, dy \geq cL^N, \quad \forall L \gg 1,
\]

thus for any \(\alpha > 0\) this leads to a contradiction with (4.11). We can also improve (4.11) by removing the additional \(\lfloor \log_\lambda L \rfloor\) term. For any \(\epsilon \in [0, 1 - \delta]\), we have a rougher estimate

\[
\sup_{s \in [0, S_0]} \int_{|y| \leq L} |V(y, s)|^2 \, dy \leq C_\epsilon L^{N-2\alpha + \epsilon}.
\]

Using this estimate in the treating of (4.7), and in a similar way as above, we further get

\[
\sup_{s \in [0, S_0]} \int_{|y| \leq L} |V(y, s)|^2 \, dy \leq CL^{N-2\alpha} + CL^{N-2\alpha} \sum_{k=0}^{\lfloor \log_\lambda (\mu L) \rfloor} 2^{-k(N-2\alpha+1)\lambda^{k(N-2\alpha+\epsilon+\delta)}} \leq CL^{N-2\alpha} + C_\epsilon L^{N-2\alpha} \sum_{k=0}^{\lfloor \log_\lambda (\lambda L) \rfloor} 2^{-k(1-\delta-\epsilon)} \leq C_\epsilon L^{N-2\alpha}.
\]

Now in order to prove (1.22) for \(-\delta \leq \alpha \leq 0\), and it suffices to prove that

\[
\frac{1}{L^{N-2\alpha}} \sup_{s \in [0, S_0]} \int_{|y| \leq L} |V(y, s)|^2 \, dy \gtrsim 1, \quad \forall L \gg 1. \tag{4.12}
\]
Suppose \((1.12)\) is not correct, then necessarily there exists a sequence of numbers \(L_n\) such that
\[
\frac{1}{L_n^{N-2\alpha}} \sup_{s \in [0, S_0]} \int_{|y| \leq L_n} |V(y, s)|^2 \, dy \to 0, \quad \text{as} \quad L_n \to \infty \quad \text{or} \quad n \to \infty. \tag{4.13}
\]
We also start from the local energy inequality \(|J_2 - J_1| \leq CK_2\), and by letting \(l_2 \to \infty\) and \(l_1 = \lambda L\), we have
\[
\sup_{s \in [0, S_0]} \int_{|y| \leq L} |V(y, s)|^2 \, dy \leq CL^{N-2\alpha} \sum_{k=0}^{\infty} \left(\frac{\lambda^k}{L}\right)^{N+1-2\alpha} \int_0^{S_0} \int_{\lambda^k L \leq |y| \leq \lambda^{k+2} L} (|V|^3 + |P||V|) \, dy \, ds
\tag{4.14}
\]
Inserting this estimate into (4.1) leads to
\[
P \equiv \frac{1}{2} |V|^2 + P_{1, \lambda^k L} + P_{2, \lambda^{k+2} L}
\]
for every \(y \in B_{\lambda^{k+2} L}(0)\). Since we already have the estimate
\[
\sup_{s \in [0, S_0]} \int_{|y| \leq L} |V(y, s)|^2 \, dy \leq CL^{N-2\alpha}, \quad \forall L \gg 1,
\]
and thanks to Lemma \(\ref{lem:5.1}\) we get
\[
\int_0^{S_0} \int_{|y| \leq 2^{k+1} L} |P||V| \, dy \, ds \leq CL^{N-2\alpha + \delta}.
\]
Inserting this estimate into (4.14) leads to
\[
\sup_{s \in [0, S_0]} \int_{|y| \leq L} |V(y, s)|^2 \, dy \leq CL^{N-2\alpha + \delta - 1} \sum_{k=0}^{\infty} 2^{-k(1-\delta)} \leq CL^{N-2\alpha + \delta - 1}.
\]
Similarly as obtaining (1.10), by iteration we infer that for \(n\) large enough,
\[
\sup_{s \in [0, S_0]} \int_{|y| \leq L} |V(y, s)|^2 \, dy \leq CL^{N-2\alpha - n(1-\delta)}, \quad \forall L \gg 1, \tag{4.15}
\]
which guarantees \(V \equiv 0\) for all \((y, s) \in \mathbb{R}^{N+1}\). Thus it contradicts with the assumption that \(V\) is nontrivial.

The proof of (1.23) is quite similar to that of (1.22) by starting from the local energy inequality \(|S_0 J_2 - J_1| \leq CK_2\), and we omit the details.

5. Auxiliary lemmas: estimation of the pressure profile

**Lemma 5.1.** Suppose that \(V \in C^1_s C^1_{y, \text{loc}}(\mathbb{R}^{N+1})\) is a locally periodic-in-\(s\) vector field with period \(S_0 > 0\), which additionally satisfies that for every \(L \gg 1\), \(2 \leq p < \infty\) and \(2 \leq r \leq \infty\),
\[
\left\| \left( \int_{|y| \leq L} |V(y, s)|^p \, dy \right)^{\frac{1}{p}} \right\|_{L^r([0, S_0])} \lesssim L^{\frac{2a}{p}}, \quad \text{with} \quad 0 \leq a < N.
\]
Let \(P(y, s)\) be a scalar-valued function defined from \(V\) by
\[
P(y, s) := c_0 |V(y, s)|^2 + \text{p.v.} \int_{\mathbb{R}^N} K_{ij}(y-z)V_i(z, s)V_j(z, s) \, dz \tag{5.1}
\]
with \(c_0 \in \mathbb{R}\) and \(K_{ij}(z)\) \((i, j = 1, \cdots, N)\) some Calderón-Zygmund kernel, then we have
\[
\left\| \left( \int_{|y| \leq L} |P(y, s)|^p \, dy \right)^{\frac{1}{p}} \right\|_{L^r([0, S_0])} \lesssim L^{\frac{2a}{p}}. \tag{5.2}
\]
Proof of Lemma 5.1. We only suffice to treat the integral term in the expression formula (5.1), denoting by \( \tilde{P}(y, s) \), and we use the following decomposition

\[
\tilde{P}(y, s) = \text{p.v.} \int_{|z| \leq 2L} K_{ij}(y - z)V_i(z, s)V_j(z, s) \, dz + \int_{|z| \geq 2L} K_{ij}(y - z)V_i(z, s)V_j(z, s) \, dz
\]

\[
:= \tilde{P}_{1,L}(y, s) + \tilde{P}_{2,L}(y, s).
\]

By the Calderón-Zygmund theorem, we first see that

\[
\| \left( \int_{|y| \leq L} |\tilde{P}_{1,L}(y, s)|^{\frac{p}{2}} \, dy \right)^{\frac{2}{p}} \|_{L^{s/2}_s} \lesssim \left\| \left( \int_{|y| \leq 2L} |V(y, s)|^p \, dy \right)^{1/p} \right\|_{L^{s}_s}^2 \lesssim L^{\frac{2n}{p}}.
\]

For \( \tilde{P}_{2,L} \), by the dyadic decomposition, Minkowski’s inequality and Hölder’s inequality we have

\[
\| \left( \int_{|y| \leq L} |\tilde{P}_{2,L}(y, s)|^{\frac{p}{2}} \, dy \right)^{\frac{2}{p}} \|_{L^{s/2}_s} \lesssim \left( \sum_{k=1}^{\infty} \left\| \left( \int_{|y| \leq L} \left( \int_{|z| \sim 2^k L} \frac{1}{|y - z|^N} |V(z, s)|^2 \, dz \right)^{\frac{p}{2}} \, dy \right)^{\frac{2}{p}} \right\|_{L^{s/2}_s} \right)^{\frac{2}{p}}
\]

\[
\lesssim L^{\frac{2N}{p}} \sum_{k=1}^{\infty} (2^k L)^{-N} \left\| \int_{|z| \sim 2^k L} |V(z, s)|^2 \, dz \right\|_{L^{s/2}_s} \lesssim L^{\frac{2N}{p}} \sum_{k=1}^{\infty} (2^k L)^{-2N/p} \left\| \left( \int_{|z| \sim 2^k L} |V(z, s)|^p \, dz \right)^{\frac{2}{p}} \right\|_{L^{s/2}_s} \lesssim L^{\frac{2N}{p}} \sum_{k=1}^{\infty} (2^k L)^{-2(N-a)/p} \lesssim L^{\frac{2n}{p}}.
\]

Hence gathering the above estimates yields (5.2).

Lemma 5.2. Suppose \( V \in C^1_c C^1_{\text{v,loc}}(\mathbb{R}^{N+1}) \) is a periodic-in-\( s \) vector field with period \( S_0 \) which additionally satisfies that for every \( 0 \leq b < N + 1, \delta \in [0, 1] \),

\[
|V(y, s)| \lesssim |y|^\delta, \quad \forall |y| \gg 1, s \in [0, S_0], \quad (5.3)
\]

and

\[
\sup_{s \in [0, S_0]} \int_{|y| \leq L} |V(y, s)|^2 \, dy \, ds \lesssim L^b, \quad \forall L \gg 1. \quad (5.4)
\]

Let \( Q(y, s) \) be a scalar-valued function defined from \( V \) by that for every \( y \in B_L(0) \),

\[
Q(y, s) = c_0 |V(y, s)|^2 + \text{p.v.} \int_{|z| \leq 2L} K_{ij}(y - z)V_i(z, s)V_j(z, s) \, dz + \int_{|z| \geq 2L} (K_{ij}(y - z) - K_{ij}(z))V_i(z, s)V_j(z, s) \, dz
\]

with \( c_0 \in \mathbb{R} \) and \( K_{ij}(z) \) some Calderón-Zygmund kernel, then we have

\[
\int_0^{S_0} \int_{|y| \leq L} |Q(y, s)||V(y, s)| \, dy \, ds \lesssim L^{b+\delta}. \quad (5.6)
\]
Proof of Lemma 5.2. By denoting the right-hand side of (5.5) is \( Q \) respectively, we rewrite (5.5) as \( Q \) is direct. For the term involving \( Q_{2,L} \), by the Hölder inequality and Calderón-Zygmund theorem, we directly get
\[
\int_0^{S_0} \int_{|y| \leq L} |Q_{2,L}| |V| \, dy \, ds \leq \left( \int_0^{S_0} \int_{|y| \leq L} |Q_{2,L}|^2 \, dy \, ds \right)^{\frac{1}{2}} \left( \int_0^{S_0} \int_{|y| \leq L} |V|^2 \, dy \, ds \right)^{\frac{1}{2}} \lesssim \int_0^{S_0} \int_{|y| \leq 2L} |V|^2 \, dy \, ds \lesssim L^{b+\delta}.
\]
For the term containing \( Q_{3,L} \), thanks to the support property and the dyadic decomposition, we find
\[
\int_0^{S_0} \int_{|y| \leq L} |Q_{3,L}| |V| \, dy \, ds \lesssim L^{N+\delta} \sup_{s \in [0,S_0]} \sup_{|y| \leq L} |Q_{3,L}(y,s)| \lesssim L^{N+\delta} \sup_{s \in [0,S_0]} \sup_{|z| \geq 2L} |y|^{N+1} |V(z,s)|^2 \, dz \lesssim L^{N+\delta+1} \sum_{k=1}^{\infty} (2^k L)^{-N-1} \left( \sup_{s \in [0,S_0]} \int_{2^k L \leq |z| \leq 2^{k+1} L} |V(y,s)|^2 \, dz \right) \lesssim L^{N+\delta+1} \sum_{k=1}^{\infty} (2^k L)^{-N-1+b} \lesssim L^{b+\delta}.
\]
Combining these estimates leads to the desired inequality (5.6). \( \square \)

Acknowledgements. The research was partially supported by a special fund from the Laboratory of Mathematics and Complex Systems, Ministry of Education.

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School of Mathematical Sciences, Beijing Normal University and Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, P.R. China

E-mail address: xuelt@bnu.edu.cn