Design Method for N-Lobed Noncircular Bevel Gears

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Abstract
As a type of spatial transmission mechanism, noncircular bevel gears (NBGs) can transfer power and motion between two intersecting axes with variable transmission following a suitable program of motion. Utilizing the spherical triangle theorem and meshing principle, parametric equations are established in the spherical polar coordinate system for the driving and driven gears, for the pitch curves, and for the addendum and dedendum curves of a NBG for a given transmission ratio and axis angle. A formulation of the tooth profile of a NBG is deduced using an analytic method. Three-dimensional models of the 3- and 4-lobed NBGs are derived in verifying this method.

Keywords
intersection axis, variable transmission, noncircular bevel gears, tooth profile, pitch curve, addendum and dedendum curves

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Introduction
With noncircular bevel gears (NBGs), power and motion can be transferred between two intersecting axes with a variable transmission executed by a suitable program of motion. NBGs have many advantages such as smooth motion, compact structure, and accurate transmission. Because of their performance, they have also been used in highly specialized applications such as limited-slip differentials. Compared with cylindrical gears, the pitch curves of NBGs are spatial curves. Therefore, designing them is more difficult than either noncircular or bevel gears. Their theory of transmission is incomplete, the research into applications is not mature, and analyses and calculations need to be improved if NBGs are to be developed further.

On the basis of the research method for planar noncircular gears, Ollson4 proposed a design and manufacture method for NBGs employing the spherical polar coordinate system. The pitch surface of a N-lobed elliptical bevel gear was analyzed by Figliolini and Angeles5. Many scholars have studied the tooth shape, pitch curve, and machining method of NBGs. Because of its variable transmission ratio, Wang and collaborators4,5 applied noncircular gears to a limited-slip differential and were granted patents for the device. Jia and collaborators6,7 applied NBGs to a limited-slip differential with a variable transmission ratio and studied methods for meshing NBGs. Jia and collaborators also put forward a method of machining a tooth surface by wire cutting on a NC machine tool and thereby solved the problem of requiring a small taper for the domestic machine tool8. Zhao and collaborators9 also applied this machining method for their NBGs. Lin and collaborators10 studied the method of calculation and transmission performance of ellipse bevel gears. Using screw theory, Lin and collaborators11 studied the compound
transmission mechanism of a curved-face gear. Xia and collaborators\textsuperscript{12} studied the geometric parametric equation and CAD modeling of NBGs. Lv and collaborators\textsuperscript{13} proposed a new kind of shaping method for the pitch surface that solves the problems caused by convex and concave tips. Zheng and collaborators\textsuperscript{14} proposed a universal method that is applicable to tooth profiles. They also analyzed the generation concept of a crown tooth to generate a tooth surface\textsuperscript{15}. Shi and collaborators\textsuperscript{16} analyzed the minimum teeth number to avoid undercutting. Additionally, the varying-coefficient-profile-shift-modification method is used to avoid undercutting, thus ensuring the root part of the tooth face does not participate during meshing. They further presented a design method for NBGs having a concave tip; therefore, the transmission ratio is given by

\begin{equation}
\frac{\tan \varphi_1}{\cos \varphi_0 + f(\theta_1)} = \frac{\sin \varphi_0}{\cos \varphi_0 + f(\theta_1)}.
\end{equation}

and from equation (1) we have

\begin{equation}
\begin{aligned}
\theta_2 &= \int_{\theta_1}^{\theta_2} \frac{\cos \phi - \cos \lambda}{\sin \lambda - \sin \phi \cdot \cos \theta_1} d\theta_1, \\
\varphi_2 &= \varphi_0 - \varphi_1.
\end{aligned}
\end{equation}

Using Figure 2, the sum of the angles from P to C and D is a constant \(\lambda\), and the number of cycles of \(\theta_2\) is \(n_2\). Therefore, the equation for the pitch curve of an elliptical bevel gear is then

\begin{equation}
\tan \varphi_1 = \frac{\cos \phi - \cos \lambda}{\sin \lambda - \sin \phi \cdot \cos \theta_1},
\end{equation}

\textbf{Pitch Curve}

The two pitch surfaces in the mesh (Figure 1) can be formally expressed as \(\varphi_1 = \varphi_1(\theta_1)\) and \(\varphi_2 = \varphi_2(\theta_2)\). Here, \(\varphi\) denotes the pitch curve polar angle, \(\theta\) the perigon, and \(\omega\) the instantaneous angular velocity. The transmission ratio of the NBGs is\textsuperscript{16}

\begin{equation}
i_{12} = \frac{\omega_1}{\omega_2} = \frac{d\theta_1/dt}{d\theta_2/dt} = \frac{1}{F''(\theta_1)} = f(\theta_1).
\end{equation}

At point \(P\), \(|\vec{v}_p| = |\vec{v}_{p_1}|, \vec{v}_p\) represents the velocity in the common normal direction, and \(\varphi_0\) the original angle; therefore, the transmission ratio is given by

\begin{equation}
i_{12} = f(\theta_1) = \frac{\sin(\varphi_0 - \varphi_1)}{\sin \varphi_1} = \frac{\sin \varphi_0}{\tan \varphi} - \cos \varphi_0.
\end{equation}

The last identity is obtained using a basic trigonometric identity. The pitch curve equation is

\begin{equation}
\tan \varphi_1 = \frac{\sin \varphi_0}{\cos \varphi_0 + f(\theta_1)},
\end{equation}

\begin{equation}
\left\{
\begin{aligned}
\theta_2 &= \int_{\theta_1}^{\theta_2} \frac{\cos \phi - \cos \lambda}{\sin \lambda - \sin \phi \cdot \cos \theta_1} d\theta_1, \\
\varphi_2 &= \varphi_0 - \varphi_1.
\end{aligned}
\right.
\end{equation}

\begin{equation}
\cos \theta_1 = \frac{A_1 \cdot \cos^2(\frac{n_2 \varphi_2}{2}) - B_1 \cdot \sin^2(\frac{n_2 \varphi_2}{2})}{A_1 \cdot \cos^2(\frac{n_2 \varphi_2}{2}) + B_1 \cdot \sin^2(\frac{n_2 \varphi_2}{2})},
\end{equation}

with

\begin{equation}
\left\{
\begin{aligned}
A_1 &= \sin \varphi_0 \cdot \sin \lambda + \cos \varphi_0 \cdot \cos \lambda_1 \\
&- \cos \varphi_0 \cdot \cos \phi + \sin \varphi_0 \cdot \sin \phi, \\
B_1 &= \sin \varphi_0 \cdot \sin \lambda + \cos \varphi_0 \cdot \cos \lambda_1 \\
&- \cos \varphi_0 \cdot \cos \phi - \sin \varphi_0 \cdot \sin \phi.
\end{aligned}
\right.
\end{equation}
Dedendum and Addendum Curves

The equations of the dedendum and addendum curves (Figure 3) are\footnote{Shi et al.}
\[
\begin{align*}
\cos \gamma &= \cos \varphi \cdot \cos \alpha_a + \sin \varphi \cdot \sin \alpha_a \cdot \cos \xi \\
\cos \Delta \theta_\gamma &= \frac{\cos \alpha_a - \cos \varphi \cdot \cos \gamma}{\sin \varphi \cdot \sin \gamma}
\end{align*}
\]

where
\[
\begin{align*}
\theta_\gamma &= \begin{cases} 
\theta - \Delta \theta_\gamma, & 0 \leq \theta \leq \pi \\
\theta + \Delta \theta_\gamma, & \pi \leq \theta \leq 2\pi
\end{cases} \\
\theta_\beta &= \begin{cases} 
\theta + \Delta \theta_\beta, & 0 \leq \theta \leq \pi \\
\theta - \Delta \theta_\beta, & \pi \leq \theta \leq 2\pi
\end{cases}
\end{align*}
\]

Tooth Profile

The included angle, denoted by \(\lambda\), is defined as the angle between the normal arc of a tooth profile and the spherical orthodrome and is the polar angle between the normal arc of the left and right tooth profiles of a NBG. With \(\delta\) denoting the azimuth angle of the tangent of the intersection point of the pitch curves, the difference \(\lambda - \delta\) is then a constant value. The tooth profiles can be derived from the Willis theorem.\footnote{Shi et al.} The concave pitch curve of NBGs can be processed by the bevel gear milling cutter, so the deduced tooth profile equation is universal. The tooth profile of the bevel gear milling cutter is shown in Figure 4.

The initial meshing point \(A_0\) is on the pitch circle of the cutter. When the cutter rotates to \(A_1\), they are meshed in \(N_1\), and

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**Figure 3.** Angles and curves required for the analysis of gears.
(a) \(\delta<\pi/2\).
(b) \(\delta>\pi/2\).
1– Dedendum curve 2– Pitch curve 3– Addendum curve.
\[
\begin{align*}
CE &= A_1N_1 \\
A_0A_1 \cdot \sin \phi_B &= CE \cdot \sin \phi \\
\sin \phi \cdot \cos \alpha_n &= \sin \phi_B
\end{align*}
\]

(9)

where \( \phi_B \) denotes the polar angle of the base circle, \( \phi \) the polar angle of the pitch circle, and \( \alpha_n \) the profile angle of the bevel gear cutter. With \( A_nN_n \) denoting the length between \( A_n \) on the pitch circle and \( N_n \) in the tooth profile, we deduce from Eq. (9)

\[
A_nN_n = S_n \cos \alpha_n,
\]

(10)

where \( S_n \) denotes the pitch circle length between the two intersection points, and \( n = 1, 2, \ldots \).

By the principle of gear engagement, the arc length of the NBGs and the helical curve arc of the cutter are the same. The tooth profile of any point \( A_0 \) on the pitch curve of the NBGs is as follows: make a right spherical triangle \( \triangle A_1T_1 \) at any point \( A_1 \) on the pitch circle; find the tangent \( \hat{A}_1T_1 \) to the pitch curve at point \( A_1 \), with

\[
\hat{A}_1T_1 = a \tan \left( \frac{\tan(\alpha_n \cos \alpha_n)}{\cos \omega} \right),
\]

where \( \angle T_1A_1N_1 = \alpha_n \) and \( \hat{A}_0A_1 \) denotes the arc from the intersection point of the pitch curve of the NBGs and the tooth profile to the other intersection point; hence, point \( N_1 \) is the trail of the tooth profile, which is above the pitch curve (see Figure 5).

Similarly, make a right spherical triangle \( \triangle A_2N_2T_2 \) at any point \( A_2 \) on the pitch curve, a tangent \( \hat{A}_2T_2 \) to the pitch curve at point \( A_2 \), where

\[
\hat{A}_2T_2 = a \tan \left( \frac{\tan(\alpha_n \cos \alpha_n)}{\cos \omega} \right) \text{ with } \angle T_2A_2N_2 = \alpha_n.
\]

The point \( N_2 \) is the trail of the tooth profile, which is below the pitch curve.

Design Examples

The driving gear is a 3-order NBG whereas the driven gear is of 4-order for the given parameter settings \( \lambda = 74^\circ \) and \( \varphi_0 = 90^\circ \). The equations for the pitch curve, and the addendum and dedendum curves of the driving gear were obtained along with those of the driven gear. The curvature of the pitch curve was deduced, with the minimum number of teeth of the NBGs being 38.

To ensure a completed tooth shape, the number of teeth should satisfy condition

\[
\frac{S}{S_i} = Z \frac{Z}{Z_0},
\]

(11)

where \( S \) denotes the length of the pitch curve of the shaper cutter and \( S = 2\pi \cdot \sin \omega \), \( S_i \) the length of the pitch curve of the NBGs, obtained from \( S_i = \int_0^{\varphi_0} \sqrt{\sin^2 \varphi + \varphi^2(\theta)} d\theta \), \( Z \) the number of teeth for the shaper cutter, \( Z_i \) the number of teeth for driving gear and driven gear, and \( \omega \) the cone angle of the shaper cutter. Three-dimensional models of the 3-order and 4-order NBGs are illustrated in Figure 6.

Conclusions

By applying the spherical triangle theorem and adopting the meshing principle, a general design method for NBGs has been proposed. The following summarizes the results obtained:

1. The equations that determine the pitch curve of the NBGs were obtained for any order and in any configuration during their pure rolling motion for a given transmission ratio and axis.
angle; the equations are expressed in the spherical polar coordinate system.

(2) The equations of the addendum and dedendum curves for the driving and driven gear were derived.

(3) Using an analytic method, a formulation of the tooth profile for the NBGs was deduced and three-dimensional models of a pair of conjugate NBGs were developed in a verification of the correctness and reliability of this modification method.

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Figure 6. 3D models of the 3- and 4-lobed NBGs. 
(a) $n_1 = 3$, $Z_1 = 39$.
(b) $n_2 = 4$, $Z_2 = 52$.
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