On Developing Piecewise Rational Mapping with Fine Regulation Capability for WENO Schemes

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Abstract
On the idea of mapped WENO scheme, properties of mapping methods are analyzed, uncertainties in mapping development are investigated, and new piecewise rational mappings are proposed. Based on our former understandings, i.e. the mapping at endpoints {0, 1} tending to identity mapping, a so-called $C_{n,m}$ condition is summarized for function development. Uncertainties, i.e., whether the pattern at endpoints of mapping would make mapped scheme behave like WENO or ENO, whether piecewise implementation of mapping would entail numerical instability, and whether WENO3 could preserve the third-order at first-order critical points by mapping, are analyzed and clarified. A new piecewise rational mapping with sufficient regulation capability is developed afterwards, where the flatness of mapping around the linear weights and the profile at endpoint tending toward identity mapping can be coordinated explicitly and simultaneously. Hence, the increase of resolution and preservation of stability can be balanced. Especially, concrete mappings are determined for {WENO3, 5, 7}. Numerical examples are tested for the new mapped WENO, which regard preservation and convergence rate of accuracy, numerical stability including that in the long-time computation, resolution and robustness. For comparison, some recent mappings such as IM by [App. Math. Comput. 232, 2014:453–468], RM by [J. Sci. Comput. 67, 2016:540–580] and AIM by [J. Comput. Phys. 381, 2019:162–188] are tested; in addition, some recent WENO-Z type schemes are chosen as well. The results manifest that new schemes can preserve optimal orders at corresponding critical points, achieve numerical stability, and indicate overall comparative advantages regarding accuracy, resolution and robustness.

Keywords WENO · Mapping method
1 Introduction

As one of the popular high-order difference methods, WENO schemes [1,2] especially WENO-JS in Ref. 2 have achieved large success. In order to improve the performance, endeavors have been made constantly, which at least lie in two aspects, namely the mapping method [3] and WENO-Z approach [4–6]. In this study, further studies regarding the former are concerned. For brevity, the “-JS” in WENO-JS will be omitted in the following.

In Ref. 3, Henrick et al. examined the accuracy relation of the fifth-order WENO5 and proposed the corresponding sufficient and necessary condition as well as the sufficient condition [3] to achieve the optimal fifth-order. They pointed out that when \( f' = 0 \) happened at \( x_j \) or nearby, the above conditions would be violated and order degradation occurred. For remedy, they pioneered the idea of mapping method by introducing \( g(\omega_k) \) for nonlinear weight \( \omega_k \in [0,1] \), and new non-normalized weights were derived accordingly. A concrete rational function, referred as \( g_M \) herein, was proposed and corresponding WENO5-M scheme was obtained. It is worthwhile to mention that Henrick et al. [3] and subsequent authors [7] did not think the third-order WENO3 could preserve its optimal order at the first-order critical point by mapping.

In Ref. 7, Feng et al. studied the errors of WENO5-M in the case of long-time computation. They found the scheme be liable to less accuracy near discontinuities, and the reason was considered as the enlargement of nonlinear weights by \( g_M \) near \( \omega_k = 0.1 \). For improvement, the authors suggested the mapped scheme to perform like ENO therein by appealing to the condition \( g'(0,1) = 0 \). A concrete piecewise polynomial mapping was proposed [7] as PM with the order \( n+2 \), where \( n \) means \( PM^{(i)}(d_k) = 0 \) for \( 1 \leq i \leq n \) and \( PM^{(n+1)}(d_k) \neq 0 \) with \( d_k \) as the linear weight. The choice of \( n=6 \) was suggested for WENO5 [7]. Still following the similar concern, Feng et al. later devised [8] a family of specific rational polynomial mapping called as IM with free parameter \( A \) and order degree \( n \) as just referred. It is noted that parameter recommendations of IM for WENO5 [8] would yield highly amplified weights near \( \omega_k = 0,1 \) than that in \( g_M \), which was opposed in Ref. 7.

In Ref. 9, motivated by enhancing the resolution of WENO through increasing flatness of mapping around \( \omega_k = d_k \), Li et al. made an investigation on polynomial mapping independently. They increased the polynomial degree to engender higher-order critical point at \( \omega_k = d_k \) and render a flatter profile there. It was found that [9] the choice of a single polynomial with high order would be prone to oscillatory distribution, whereas the piecewise polynomial method (PPM) could yield the desired result. Although the error in long-time computation [7] was not referred there, the manner of polynomial approaching endpoints was concerned for the sake of numerical stability. Consequently, the approach to identity mapping at endpoints was chosen therein and the following conditions were proposed [9]:

\[
g'(0,1) = 1 \quad \text{and} \quad g^{(i)}(0,1) = 0 \quad \text{with} \quad 2 \leq i \leq m \quad \text{for certain} \quad m.
\]

The conditions indicate the mapped scheme resembles WENO when \( \omega = \) near endpoints. As an example, a sixth-order PPM was proposed in this respect [9].

In order to explore the potential of single rational mapping functions with ENO-like feature of mapped scheme at endpoints of \( \omega \), Wang et al. [10] further studied a specific formulation called as RM with assigned orders of critical points at \( \omega_k=0 \) and \( d_k \). In order to obtain well-defined rational mapping, they claimed: (1) the condition such as \( \{ g^{(i)} = 0, \ 1 \leq i \leq m \} \) can only be imposed for one endpoint, where the point at \( \omega_k = 0 \) is chosen as the appropriate; (2) aforementioned \( m \) and \( n \) which denote the order of critical point at \( \omega_k = d_k \) should be even. One can verify that the flatness of RM at \( \omega_k = d_k \) and the rate of mapped scheme converging to ENO at \( \omega_k = 0 \) can only be adjusted through \( m \) and \( n \), and moreover, the two features
are competitive to each other. A choice of \((m, n) = (2, 6)\) is suggested. Recently, Vevek et al. [11] improved IM [8] by upgrading \(A^{-1} \omega_k (1 - \omega_k)\) in the mapping to \(A^{-1} [\omega_k (1 - \omega_k)]^{m-1}\), through which the new function can satisfy conditions at endpoints by Ref. 9 as just mentioned. Furthermore, they proposed an adaptation algorithm as that in Ref. 12 to evaluate \(A\), and the so-called AIM mapping for WENO7 was obtained. The adaptation of \(A\) is to make mapping have a flat profile around \(\omega_k = d_k\) in smooth solution, and transit to identity mapping near endpoints quickly in the case of discontinuity.

In spite of progresses, uncertainties still exist. For example, (1) Feng et al. [7,8,10] suggested the mapped scheme tending to ENO rather than WENO by mapping near endpoints, whereas Li [9] indicated the alternative pattern of WENO there. Further investigations are preferred in this regard. (2) Wang et al. [10] deemed piecewise mapping would have errors liable to numerical instability, and therefore the mapping function should be infinitely smooth or the single form was preferable. The authors even exemplified an occurrence of oscillations by WENO9 plus PM as the evidence. Whether piecewise mapping would be inferior to the single one needs further clarification. (3) Current studies except that of AIM [11] have indicated that the flatness of mapping at \(\omega = d_k\) and convergence rate to identity mapping at endpoints are competitive or unachievable simultaneously. While for AIM, on the one hand, the adaptive implementation indicates the absence of explicit fine control, on the other hand, its robustness is not fully tested and applicability for schemes such as WENO5 is unknown. It is obvious that other novel methods are expected. (4) Current studies considered the optimal order of WENO3 could not be preserved by mapping at critical points, however, which is rather needed by engineering. Targeting at above issues, clarifications are made in this study, and new piecewise rational mappings are proposed with explicit and fine control.

The paper is arranged as follows: typical mapping methods for WENO are first reviewed in Sect. 2; uncertainties mentioned above are discussed in Sect. 3; in Sect. 4, new piecewise rational mappings are proposed and intensive analysis is made; in Sect. 5, careful numerical validations are carried out; at last conclusions are drawn in Sect. 6.

2 Mapping Function Method for WENO

In order to facilitate discussion, the complete set of WENO with orders \(\{3, 5, 7, 9\}\) is revisited first. Then, typical mapping methods are reviewed.

2.1 WENO Formula [2]

Consider the following one-dimensional hyperbolic conservation law

\[
\frac{\partial u}{\partial t} + f(u)_x = 0. \tag{1}
\]

Supposing the grids are equally partitioned as \(x_j = j\Delta x\) where \(\Delta x\) denotes the interval and \(j\) is the grid index, Eq. (1) at \(x_j\) can be re-written in conservative form as:

\[
(u)_j = -(h_{j+1/2} - h_{j-1/2})/\Delta x, \quad \text{where } f(x) = \frac{1}{\Delta x} \int_{x-\Delta x/2}^{x+\Delta x/2} h(x') dx'.
\]

If \(h(x)\) is approximated by \(\hat{f}(x)\), the semi-discretized conservative scheme is written as

\[
(u)_j = -(\hat{f}_{j+1/2} - \hat{f}_{j-1/2})/\Delta x. \tag{2}
\]
To facilitate the description of WENO formula, \( f' > 0 \) is assumed tentatively. Suppose \( r \) is the number of sub-stencils and also that of grid points in each sub-stencil, WENO [2] with the order \( 2r-1 \) in smooth region can be formulated as:

\[
\hat{f}_{j+1/2} = \sum_{k=0}^{r-1} \omega_k q_k' \quad \text{with} \quad q_k' = \sum_{l=0}^{r-1} a_{kl} f(u_{j-r+k+l+1})
\] (3)

where \( q_k' \) is the candidate scheme with coefficients \( a_{kl} \), shown in Table 13 in Appendix 1 and \( \omega_k \) is the nonlinear weight. \( \omega_k \) is derived from the corresponding linear weight \( d_k \) (shown in Table 14 in Appendix 1) as

\[
\omega_k = \alpha_k \sum_{l=0}^{r-1} \alpha_l \quad \text{with} \quad \alpha_k = d_k \left( \varepsilon + IS^{(r)}_k \right)^2
\] (4)

where \( \alpha_k \) denotes the non-normalized weight, \( \varepsilon = 10^{-6} \) in WENO [2] and \( IS^{(r)}_k \) is the smoothness indicator. For WENO [2], \( IS^{(r)}_k \) can be formulated in positive semi-definite quadratic form as:

\[
IS^{(r)}_k = \sum_{m=0}^{r-2} c_m' \left( \sum_{l=0}^{r-1} b_{km} f(u_{j-r+k+l+1}) \right)^2
\] (5)

where the coefficients \( b_{km} \) and \( c_m' \) are tabulated in Table 15–16 in Appendix 1 [2,10–12].

### 2.2 Mapping Method and Typical Functions

(1) Principle and Properties of Mapping

As referred in Ref. 3, when critical points occur in computation, the necessary and sufficient condition to acquire the order \( 2r-1 \) is hard to satisfy by WENO, and accuracy loss ensues. However, the optimal order can still be achieved if the following sufficient condition could be reconstructed somehow

\[
\omega_k = d_k + O(\Delta x').
\] (6)

In order to rebuild Eq. (6), a recipe was proposed by Henrick et al., in which some prerequisite of accuracy relation between \( \omega_k \) and \( d_k \) should be established. The least requirement is:

\[
\omega_k = d_k + O(\Delta x).
\] (7)

With Eq. (7), if one function \( g(\omega_k) \) would make

\[
g(\omega_k) = d_k + O(\omega_k - d_k)'\]

then \( g(\omega_k) \) can serve as the non-normalized weight, and its normalized value can make Eq. (6) established [3]. By Taylor expansion, the requirement of \( g(\omega_k) \) can be derived as [3]:

\[
g(d_k) = d_k \quad \text{and} \quad g^{(i)}(d_k) = 0 \quad \text{for} \quad i < r.
\] (9)

In Ref. 3, Eq. (9) together with boundary conditions, namely \( g(0,1) = 0,1 \), were proposed as the requirements for mapping. However, in pursuit of other benefits as
shown in the introduction, more properties are preferred. Based on existing studies, the properties can be summarized as:

(a) Accuracy requirements such as Eq. (9) and boundary conditions [3].
(b) Monotonicity, which makes the ordinal relation of original weights preserved.
(c) Flatness of mapping profile at \( \omega_k = d_k \) and nearby. Ref. 9 indicated that the increase of flatness could enhance the resolution of WENO, which was realized by asking a higher order of critical point than that in Eq. (8). Besides of the referred one, another flatness is implied in previous investigations [8,11] at the region away from \( \omega_k = d_k \). As will be shown later (see Fig. 4), although a mapping function with a higher order of critical point at \( \omega_k = d_k \) would distribute flatter in the neighborhood of the position than the one with a lower order, it is probable that the former might appear less flat when away from the position than the latter. For convenience, we refer the flatness in the neighborhood of \( \omega_k = d_k \) as Flatness-I and the one away from the position as Flatness-II.
(d) Convergence pattern and rate of mapping towards endpoints. Not referred in Ref. 3, the issue was first concerned by Refs. 7 and 9 separately. Correspondingly, there are two patterns of endpoint convergence which is abbreviated as PEC: (i) The mapping converges to endpoints by \( g^{(i)}(0,1) = 0 \) for \( 1 \leq i \leq m \) in the purpose of making mapped scheme behave like ENO [7], which is referred as PEC with ENO correspondence for brevity; (ii) The mapping performs by \( \{ g'(0,1) = 1, g^{(i)}(0,1) = 0 \text{ for } 1 < i \leq m \} \) and corresponding scheme tends to WENO [9] when \( \omega \) is near endpoints, which is referred as PEC with WENO correspondence similarly. Further discussions on two patterns will be made in Sect. 3. It is trivial that the higher \( m \) would render the larger rate of convergence. Ref. 10 indicated it was difficult for the first pattern to acquire a well-defined mapping with higher order \( m \).

Based on above discussions, the conditions to derive a mapping function \( g(\omega) \) with given orders \( \{n, m, l\} \) regarding derivatives at \( \{d_k, 0, 1\} \), which is referred as \( C_{n,m,l} \), are summarized as:

\[
g^{(i)}(d_k) = \begin{cases} 
  d_k, & i = 0 \\
  0, & 1 \leq i \leq n, \\
  \neq 0, & i = n + 1
\end{cases}
\]

\[
g^{(i)}(0) = \begin{cases} 
  0, & i = 0 \\
  1|0, & i = 1 \text{ if } m \geq 1 \\
  0, & 2 \leq i \leq m \\
  \neq 0, & i = m + 1
\end{cases}
\]

and

\[
g^{(i)}(1) = \begin{cases} 
  1, & i = 0 \\
  1|0, & i = 1 \text{ if } i \geq 1 \\
  0, & 2 \leq i \leq l \\
  \neq 0, & i = l + 1
\end{cases}
\]  

(10)

When \( m = l \), \( C_{n,m,l} \) is simplified as \( C_{n,m} \). Particularly, \( g^{(1)}(0,1) = 1 \) in Eq. (10) corresponds to the first pattern of convergence [7] mentioned above and \( g^{(1)}(0,1) = 0 \) corresponds to the second pattern [9]. It is conceivable that the larger \( n, m \) and \( l \) are, the larger extent of two flatness and a higher rate of convergence at endpoints the mapping will have.

(2) Mapping Examples

Typical mapping functions including our recent practices will be reviewed here. To clearly demonstrate the relationship of mappings with \( C_{n,m} \) or \( C_{n,m,l} \), their names will take the form as: \( g^N_{n,m} \) or \( g^N_{n,m,l} \), where \( N \) represents the order of polynomial or that of
nominator in the case of a rational polynomial. And for piecewise mapping, the definition takes the form as: \( g(\omega_k) = \{ g^L, 0 < \omega_k < d_k; g^R, d_k \leq \omega_k < 1 \} \). In the following, the subscript “\( k \)” of \( \omega_k \) will be dropped for convenience. In addition, \( \epsilon \) in Eq. (4) will take \( 10^{-40} \) after taking mapping unless declaring explicitly.

(a) \( \text{PM}^{n+2}_{n,1}[7] \)

The mapping is a piecewise polynomial with the form

\[
\begin{align*}
\text{PM}^{n+2}_{n,1} = d_k + \frac{(-1)^p(n+1)}{d_k^{p+1}}(\omega - d_k)^{p+1} & (\omega + \frac{d_k}{n+1}) \\
\text{PM}^{n+2}_{n,1} = d_k - \frac{(-1)^p(n+1)}{(1-d_k)^{p+1}}(\omega - d_k)^{p+1} & (\omega + \frac{d_k}{n+1})
\end{align*}
\]

which satisfies \( C_{n,1} \) condition with \( g'(0,1) = 0 \). The concrete suggestion for \{WENO5, 7, 9\} is \( \text{PM}^8_{6,1} \) by Ref. 7, which was referred as PM6 therein.

(b) \( \text{PPM}^{n+m+1}_{n,m}[9] \)

As another kind of piecewise polynomial, the mapping is derived by the employment of a general polynomial \( \sum_{i=0}^{n+m+1} a_i \omega^i \) to satisfy \( C_{n,m} \) in a piecewise manner, where \( g'(0,1) = 1 \) in the case of \( m \geq 1 \). The specific cases of \( m = 0 \) at \( n = 1-4 \) and \( m = 1 \) at \( n = 4 \) were investigated in Ref. 9. One can see that \( \text{PPM}^{n+2}_{n,1} \) would be analogous to \( \text{PM}^{n+2}_{n,1}[7] \) in having the same orders in \( C_{n,m} \) while differing from the latter in PEC. Based on studies in Ref. 9, we further propose the general form of \( \text{PPM}^{n+m+1}_{n,m} \) as:

\[
\begin{align*}
\text{PPM}^{n+m+1}_{n,m} = d_k + \frac{(-1)^p(n+1)}{d_k^{p+1}}(\omega - d_k)^{p+1} & \sum_{i=0}^{m} a_i^m \omega^m - d_k^{p+1} \\
\text{PPM}^{n+m+1}_{n,m} = d_k + \frac{1}{(1-d_k)^{n+m+1}} & \sum_{i=0}^{m} a_i^m (1-\omega)^m - (1-d_k)^n
\end{align*}
\]

where \( a_i^m = \frac{\prod_{i=0}^{n} (n+i)}{(m+i)!} \) for \( i < m \) and \( a_i^m = 1 \). By means of the following theorem with its proof in Appendix 3, the satisfaction of \( C_{n,m} \) by \( \text{PPM}^{n+m+1}_{n,m} \) is indicated. Similarly, one can find that the counterpart \( \text{PPM}^{L,n+m+1}_{n,m} \) in \([0, d_k]\) satisfies \( C_{n,m} \) also.

**Theorem 1** Considering a function as

\[ f(\omega) = d_k + \frac{1}{(1-d_k)^{n+m+1}} \sum_{i=0}^{m} a_i^m (1-\omega)^m - (1-d_k)^n \text{ in } [d_k, 1] \text{ where } a_i^m = \frac{\prod_{i=0}^{n} (n+i)}{(m+i)!} \text{ for } i < m \text{ and } a_i^m = 1, \]

\[ f(\omega) \text{ satisfies } C_{n,m}. \]

To facilitate coding, the explicit forms of \( \text{PPM}^{n+m+1}_{n,m} \) with \( m = 0-3 \) are given as follows:

\[
\begin{align*}
\text{PPM}^{L,n+1}_{n,0} = d_k + \frac{(-1)^p(n+1)}{d_k^{p+1}}(\omega - d_k)^{p+1} & \quad \text{PPM}^{R,n+1}_{n,0} = d_k + \frac{(-1)^p(n+1)}{(1-d_k)^{p+1}}(\omega - d_k)^{p+1} \\
\text{PPM}^{L,n+2}_{n,0} = d_k + \frac{1}{(1-d_k)^{n+1}}(\omega - d_k)^{n+1} & \quad \text{PPM}^{R,n+2}_{n,0} = d_k + \frac{1}{(1-d_k)^{n+1}}(\omega - d_k)^{n+1}
\end{align*}
\]
\[
\begin{align*}
\text{(PPM}^L_{n,2})^{n+3} &= d_k + \frac{(-1)^n}{d_k^{n+3}} (\omega - d_k)^{n+1} \sum_{i=0}^{2} a_i^2 \omega_{2-i} d_k^i \\
\text{(PPM}^R_{n,2})^{n+3} &= d_k + \frac{1}{(1-d_k)^{n+2}} (\omega - d_k)^{n+1} \sum_{i=0}^{2} a_i^2 (1-\omega)^{2-i} (1-d_k)^i
\end{align*}
\]
and
\[
\begin{align*}
\text{(PPM}^L_{n,3})^{n+4} &= d_k + \frac{(-1)^n}{d_k^{n+3}} (\omega - d_k)^{n+1} \sum_{i=0}^{3} a_i^3 \omega_{3-i} d_k^i \\
\text{(PPM}^R_{n,3})^{n+4} &= d_k + \frac{1}{(1-d_k)^{n+3}} (\omega - d_k)^{n+1} \sum_{i=0}^{2} a_i^3 (1-\omega)^{2-i} (1-d_k)^i
\end{align*}
\]
where \( a_0^2 = \frac{n(n+1)}{2}, a_1^2 = n \) and \( a_2^2 = 1; a_0^3 = \frac{n(n+1)(n+2)}{3!}, a_1^3 = \frac{n(n+1)}{2}, a_2^3 = n \) and \( a_3^3 = 1 \).

(c) \text{IM}^{n+1}_{n,0;A} \text{ }^8

The mapping is a single rational polynomial defined in \([0, 1]\) as \( \text{IM}^{n+1}_{n,0;A} = d_k + \frac{a}{(\omega-d_k)^{n+1}} \) with \( A \) as the free parameter, which was called as \( \text{IM}(n, A) \) in Ref. 8. One can check \( \text{IM}^{n+1}_{n,0;A} \) satisfies \( C_{n,0} \). Feng et al. \[8\] found that \( g_M \) by Henrik et al. \[3\] can be reproduced in the form of \( \text{IM}^{2,0;A} \) by setting \( n=2 \) and \( A=1 \). \( \text{IM}^{2,0;A} \) is suitable for WENO5 in occurrence of the first-order critical points, and the suggested value of \( A \) is 0.1. As mentioned in the introduction, such choice yields a profile extremely amplifying the weights at the endpoints, which was objected in Ref. 7. However, \( \text{IM}^{n+1}_{n,0;A} \) indicates flexibility to adjust Flatness-II by choosing \( A \) other than increasing the order \( n \), which saves computational costs.

(d) \text{RM}^{n+1}_{n,m,l} \text{ }^{10}

In Ref. 10, Wang et al. proposed a specific, single rational polynomial mapping in \([0, 1]\) as \( d_k + \frac{1}{\sum_{i=0}^{m} a_i \omega (\omega - d_k)^{n+1}} \). By appealing to \( C_{n,m,l} \) with \( g'(0, 1) = 0 \), the coefficients can be obtained providing the solution exists. As the result, the meaningful formula is available only if \[^{10}\]: (i) Either \( m \) or \( l \) should be zero, and \( l = 0 \) is chosen to render better performance; then (ii) \( m \) and \( n \) should be even. The suggested mapping is: \( \text{RM}^7_{6,2,0} = d_k + \frac{1}{\sum_{i=0}^{m} a_i \omega (\omega - d_k)^{n+1}} \) with \( n=6, m=2 \) and \( l=0 \), and where \( a_0 = d_k^6 \), \( a_1 = -7d_k^5 \), \( a_2 = 21d_k^4 \), \( a_3 = (1-d_k)^6 - \sum_{i=0}^{2} a_i \). In Ref. 10 the mapping was called as \( \text{RM}(260) \), and \( \epsilon \) in Eq. (4) was told to take \( 10^{-99} \). As shown in Refs. 10–11, \( \text{RM}^7_{6,2,0} \) suits for WENO with orders up to 9, and as warned by the authors, \( \text{RM}^{n+1}_{6,1,0} \) might be singular if \( n \) or \( m \) is odd. One can check \( \text{RM}^7_{6,1,0} \) is ill-defined in the case of \( d_k = 6/10 \).

(e) \text{AIM}^{n+1}_{n,m,c;I} \text{ }^{11}

Inspired by \( \text{AIM}^{n+1}_{n,0;A} \), Vevek et al. \[11\] modified the exponent of \( (\omega(1-\omega)) \) from 1 to \( m+1 \) and obtained: \( \text{AIM}^{n+1}_{n,m} (\omega) = d_k + \frac{(\omega-d_k)^{n+1}}{(\omega-d_k)^{n+1} \omega(1-\omega))^{m+1}} \). They pointed out that \( \text{AIM}^{n+1}_{n,m} \) would satisfy \( C_{n,1} \), however one can check \( \text{AIM}^{n+1}_{n,m} \) would actually meet \( C_{n,m} \). As shown in Ref. 11, Flatness-II and endpoint convergence of the mapping are competitive to each other, and therefore the ideal mapping is hard to be obtained in the case of fixed \( s \). Concerning this, the authors further proposed an adaptive \( s \) as: \( s = c d_k^{-1} \lambda \) where \( \lambda = \frac{\min(L_s)}{\max(L_s)+\epsilon_m} \) and \( \epsilon_m = \Delta^7 \), and \( \text{AIM}^{n+1}_{n,m,c} \) is acquired accordingly with the parameter \( c \). The suggested mapping for WENO7[11] is \( \text{AIM}^5_{4,2,1;E4} \), which manifests not only good stability in the long-time computation but also high resolution in tests of Ref. 11. In Euler equations where the stencils of WENO are classified into left- and right-biased groups according to flux splitting, \( \lambda \) is further modified as \( \lambda = \min(\lambda_L, \lambda_R) \) to improve robustness according to the reference where subscripts “\( L, R \)” denote aforementioned group. The corresponding scheme is referred as \( \text{AIM}^5_{4,2,1;E4} \) in this study. As shown
later, AIM$^5_{2;1E4}$ is found not to work with WENO5 scheme, and even for WENO7 scheme, its insufficient robustness is manifested.

(f) Our recent practices on piecewise rational mapping

Rather than starting from some specific forms of mapping as in Refs. 8 and 10–11, we practice deriving piecewise rational mapping from a general formulation to satisfy $C_{n,m}$. The first step is to define a piecewise mapping in $[d_1, 1]$ as $R_{n,m}^{n+1}(\alpha) = P_{n+1}/P_{m+1}$, where $P_{n+1} = \sum_{i=1}^{n+1} a_i \alpha^i$ and $P_{m+1} = \sum_{i=0}^{m+1} b_i \alpha^i$. Without losing generality, we assume $n \geq m \geq 0$ and $a_{n+1} = 1$. It is trivial that the number of conditions in $C_{n,m}$ is $n + m + 2$, while $R_{n,m}^{n+1}$ has $n + m + 1$ unknown coefficients. Hence, there will be one free if coefficients are solved by satisfying $C_{n,m}$. For convenience, we take $b_{m+1}$ as the free one and drop its subscript for brevity. One can verify that meaningful solutions would be available only if $n \geq 1$. As examples, $R_{1,0}^{2,2}$ and $R_{2,0}^{2,3}$ can be derived as $R_{1,0}^{2,2}(\alpha) = \frac{\alpha \omega + (\omega - 2)d_1}{b \omega - d_1}$ and $R_{2,0}^{2,3}(\alpha) = \frac{\omega^3 - 3d_1 \omega^2 + (bd_1 + 3d_2^2) \omega - bd_1 - 2d_2^2 + d_3}{b \omega + (d_1 - 1)^2 - b \omega - d_4}$ which satisfy $C_{1,0}$ and $C_{2,0}$ respectively. Furthermore, they are found to be formulated as $R_{1,0}^{2,2}(\alpha) = d_k + \frac{(\omega - d_k)^2}{-b(1 - \omega) + (1 - d_k)^2}$ and $R_{2,0}^{2,3}(\alpha) = d_k + \frac{(\omega - d_k)^2}{-b(1 - \omega) + (1 - d_k)^2}$. By means of similar operations, the derived $R_{n,m}^{n+1}(\alpha)$ satisfying $C_{n,m}$ can be formulated as:

$$R_{n,m}^{n+1} = d_k + \frac{(\omega - d_k)^{n+1}}{c_{n,m,1}^{R}(\omega - d_k)^n + c_{n,m,2}^{R}(1 - \omega)^{n+1} + c_{n,m,3}^{R}(1 - d_k)^n + c_{n,m,4}^{R}(1 - \omega)^n}$$

(11)
where the exponent \( n_1 \), coefficients \( c^R_{n,m,i} \) except \( c^R_{n,m,4} \) are tabulated in Table 1 and 2 for \( m, n \leq 4 \). Almost all \( c^R_{n,m,4} \) equal to zero except \( c^R_{4,2,4} = 4(1 - d_k)^2 \).

Theoretically, \( R_{n,m}^{L,n+1} \) in \([0, d_k]\) that satisfies \( C_{n,m} \) can be derived similarly. However, on observing its symmetry with \( R_{n,m}^{R,n+1} \) about \( \omega = d_k \), \( R_{n,m}^{L,n+1} \) can be obtained through the following transformation:

(i) \( F^{(1)}(\omega) = R_{n,m}^{R,n+1}(\omega + d_k) - d_k \), where \( \omega \in [0, 1 - d_k] \) and \( F^{(1)}(\omega) \in [0, 1 - d_k] \),

(ii) \( F^{(2)}(\omega) = \frac{\partial}{\partial \omega} F^{(1)}(\frac{1 - d_k}{d_k} \omega) \), where \( \omega \in [0, d_k] \) and \( F^{(2)}(\omega) \in [0, d_k] \).

(iii) \( F^{(3)}(\omega) = -F^{(2)}(-\omega) \), where \( \omega \in [-d_k, 0] \) and \( F^{(3)}(\omega) \in [-d_k, 0] \).

(iv) Finally, \( R_{n,m}^{L,n+1} = F^{(3)}(\omega - d_k) + d_k \) where \( \omega \in [0, d_k] \) and \( R_{n,m}^{L,n+1} \subset [0, d_k] \).

The above procedure can be further assembled as: 

\[
g^L = \frac{d_k}{1 - d_k} \left[ 1 - g^R(1 - \frac{1 - d_k}{d_k} \omega) \right]
\]

where \( g^R \) and \( g^L \) represent \( R_{n,m}^{R,n+1} \) and \( R_{n,m}^{L,n+1} \). One can verify that the acquired \( R_{n,m}^{L,n+1} \) would satisfy \( C_{n,m} \) providing \( R_{n,m}^{R,n+1} \) satisfies the same condition. Finally, similar \( R_{n,m}^{L,n+1} \) can be derived as

\[
R_{n,m}^{L,n+1} = d_k + \frac{(\omega - d_k)^{n+1}}{c_{n,m,1}^{L}(\omega - d_k)^{n_1} + c_{n,m,2}^{L} \omega^{n+1} + c_{n,m,3}^{L} q_k + c_{n,m,4}^{L} \omega^n}
\]

where coefficients \( c_{n,m,i}^{L} \) are tabulated in Table 18 in Appendix 2 for \( m, n \leq 4 \) for completeness.

Regarding \( R_{n,m}^{R/L,n+1} \), the following remarks are given:

(i) Although there exists one free parameter \( b \), it cannot make either \( C_{n+1,m}^{1} \) or \( C_{n,m+1}^{R} \) further achievable. One can verify the requirement of \( C_{n+1,m}^{1} \) or \( C_{n,m+1}^{R} \) will render corresponding mapping insolvable, singular or reduced to \( R_{n,m}^{R,n+1} = d_k \) or \( \omega \).

(ii) In order to avoid zero point(s) in the denominator of Eq. (11) in \([d_k, 1]\), the range of \( b \) should be confined. Additionally, on making the mapping distribute below the identity mapping, the relation \((-1)^n q_k^{m+1} R_{n,m}^{R,n+1} (1) > 0 \) should be required, through which additional confinement of \( b \) will be derived. Combining two confinements, the range of \( b \) for \( R_{n,m}^{R,n+1} \) can be obtained and tabulated in Table 3. One can testify \( R_{n,m}^{R,n+1} \) is free of singularity and lays below the identity mapping providing \( b \) falls within the range. As expected, the same range of \( b \) exists for \( R_{n,m}^{L,n+1} \) also.

| \( m=0 \) | \( m=1 \) | \( m=2 \) | \( m=3 \) | \( m=4 \) |
|---|---|---|---|---|
| \( n=1 \) | \( b < 1 \) | \( 0 < b < 1/d_k \) | – | – | – |
| \( n=2 \) | \( b < 1 - d_k \) | \( b > 1 \) | \( b < 0 \cup b > 1/d_k \) | – | – |
| \( n=3 \) | \( b < (1 - d_k)^2 \) | \( b \geq 3(1 - d_k) \) | \( b < 1 \) | \( 0 < b < 1/d_k \) | – |
| \( n=4 \) | \( b < (1 - d_k)^3 \) | \( b \geq 6(1 - d_k)^2 \) | \( b < 3(1 - d_k) \) | \( b > 1 \) | \( b < 0 \cup b > 1/d_k \) |
Combining Table 2 and Table 3, one can get another presentation of $c_{n,m,i}^R$ as shown in Table 4 in terms of the free parameter $c_{n,m,2}^R$ and $d_k$.

The table tells that the valid range of $c_{n,m,2}^R$ is: $c_{n,m,2}^R > 0$, and it is trivial that the denominator in $R_{n,n}^{R,n+1}$, namely $(\omega - d_k)^{m} + c(1-\omega)^{m+1}$, is free of zero points at $\omega \in [d_k, 1]$ providing $c > 0$. Motivated by this observation, the following theorem is proposed:

**Theorem 2** Consider a mapping as $R_{n,n}^{R,n+1} = d_k + \frac{(\omega-d_k)^{m+1}}{(\omega-d_k)^{m} + c(1-\omega)^{m+1}}$ in $[d_k, 1]$ with $0 < d_k < 1$ and $\{n, m \geq 1\}$. $c > 0$ is the sufficient and necessary condition for $R_{n,n}^{R,n+1}$ to be out of the singularity.

For brevity, the proof of the theorem is given in Appendix 3. One can verify analogous conclusion be established, i.e. the mapping $d_k + \frac{(\omega-d_k)^{m+1}}{(\omega-d_k)^{m} + c(1-\omega)^{m+1}}$ at $\omega \in [0, d_k]$ with $0 < d_k < 1$ and $n, m \geq 1$ will be free of singularity providing $(-1)^m c > 0$.

(iii) Considering $R_{n,m}^{R,n+1}$ by Eq. (11), $\partial^{n+1}R_{n,m}^{R,n+1}(d_k)/\partial \omega^{n+1}$ and $\partial^{m+1}R_{n,m}^{R,n+1}(1)/\partial \omega^{m+1}$ can be derived and tabulated in Table 19 in Appendix 2. From the table, one can see that $\partial^{n+1}R_{n,m}^{R,n+1}(d_k)/\partial \omega^{n+1}$ will decrease with the increase of $c_{n,m,2}^R$, whereas $(-1)^m R_{n,m}^{R,n+1}(1)(m+1)$ will increase in the meanwhile; the former indicates the mapping will become flatter as away from $\omega = d_k$ and the latter indicates the mapping will diverge more from the identity mapping as away from $\omega = 1$. The reverse holds true as well. Hence, Flatness-II and endpoint convergence of $R_{n,m}^{R,n+1}$ would be competitive with each other, or they can hardly reach to a desired state at the same time.
(iv) Recalling the previous formulation of $\text{AIM}^{n+1}_{n,m,c}$, one can find that $R(L;n+1)$ resembles $\text{AIM}^{n+1}_{n,m,c}$ much if the overall term $s[\omega(1 - \omega)]^{m+1}$ in the denominator can be somehow re-viewed as $(1 - \omega)^{m+1}$ in $[d_k, 1]$ and $\omega^{m+1}$ in $[0, d_k]$. As just discussed, the fixed $s$ would constrain the performance of mapping, therefore a new formulation is proposed with sufficient controllability in Sect. 4 other than the adaptive implementation in $\text{AIM}^{n+1}_{n,m,c}$, and moreover, superior robustness is manifested. Prior to further discussions, several uncertainties of mappings will be investigated.

### 3 Investigation on Uncertainties of Mapping Method

As referred in the introduction, several uncertainties exist, and the investigation of which may help to get understanding to develop mapping methods.

(1) **Patterns of Endpoint Convergence**

In Refs. 7 and 10, $\text{PEC}$ of mapping was suggested to make a mapped scheme resemble ENO by appealing to $g^{(i)}(0,1) = 0$ with $1 \leq i \leq m$ for given $m$. In the references, the long-time computations on scalar advection of square wave and combination-waves were studied, where WENO5 was tested for reference with an indication of large errors. In the meanwhile, an alternative $\text{PEC}$ was proposed to make a mapped scheme resemble WENO by Ref. 9. Hence it is necessary to clarify if the latter practice could yield results similar to that by $\text{PEC}$ to render a mapped scheme resemble ENO. Owing to the above consideration, the following numerical study is planned: first, WENO5 is employed as the underlying scheme for mapping as in Refs. 7 and 10; next, $\text{PPM}^8_{6,1}$ is chosen as representative for $\text{PEC}$ with ENO correspondence, while the commensurate $\text{PPM}^8_{6,1}$ is used as representative for $\text{PEC}$ with WENO correspondence; thirdly, the scalar advection of combination-waves at grid points $N = 800$ is checked with computation time $t = 2000$, which was regarded as the typical test to check the stability in long-time computation [7,8,10]. Corresponding results are shown in Fig. 1, where that of WENO5-M is also included for reference. The figure tells $\text{PPM}^8_{6,1}$ yields a result with little difference from that by $\text{PM}^8_{6,1}$ [7], i.e., nearly all distribution is smooth except small oscillations occur near $x = 0.6$. Hence, $\text{PEC}$ with WENO correspondence indicates the comparable performance and same applicability as $\text{PEC}$ with ENO correspondence in developing mapping function.

Besides the use of $\text{PM}^8_{6,1}$ and $\text{PPM}^8_{6,1}$ for WENO5, Sect. 4 and Table 17 in Appendix 1 will show that the minimum “$n$” in $C_{n,m}$ to preserve optimal order in the case of first-order critical points would be 2 other than 6. Hence other PPMs, e.g., $\text{PPM}^4_{2,1}$ and $\text{PPM}^5_{2,2}$, are applicable also. According to this, they are also tested with results shown in Fig. 1. The results indicate that oscillations of both mapped WENO5 decrease, especially WENO5-PPM$^5_{2,2}$ only shows a tiny undershoot that is smaller than that by WENO5-PM$^8_{6,1}$. The consequences confirm the capability and well attributes of $\text{PEC}$ with WENO correspondence.

If further comparing the results of $\text{PPM}^8_{6,1}$ and $\text{PPM}^4_{2,1}$, one can find that the former is relatively more oscillatory than the latter. The possible cause is that with the increase of “$n$” in $C_{n,m}$, both Flatness-I and -II increase, which is liable to numerical instability. Hence in order to preserve optimal order in the presence of critical point, the employment of larger “$n$” in $C_{n,m}$ than necessary is not suggested unless really needed.

(2) **Probability of Piecewise Implementation to Render Numerical Instability**
In Ref. 10, Wang et al. suggested the property of smoothness for a good mapping and expected the function to have infinite derivatives in [0, 1]. It is clear that the single continuous function would fulfill the expectation. As evidence, the computation of scalar advection of combination-waves was made by WENO9-PM86,1 and an oscillatory result was obtained. They thereupon concluded the oscillations owed to the insufficient smoothness of PM86,1, or its piecewise implementation. Because various possibilities that are not fully tested may affect numerical stability, the above conclusion deserves further inspection. In this regard, the similar commensurate PPM86,1 is first employed with WENO9 in the same problem, where 400 grid points are used and the computa-

![Fig. 1 Results of scalar combination-waves advection by WENO5-M, WENO5-PPM52,1, -PPM52,2, -PPM86,1 and -PM86,1 at t = 2000 with N = 800 a Global view b Zoomed view around x = 0.65](image-url)
tion advances to $t = 100$ as in Ref. 10. As shown in Fig. 2c–d, PPM$^8_{6,1}$ indicates an oscillatory distribution almost the same as that of PM$^8_{6,1}$, which seems to favor the conclusion of Ref. 10. However, disparate results emerge when more mappings are included in the test as follows.

Two groups that include more piecewise mappings are chosen as: the first group is PPM$^6_{4,1}$, PPM$^7_{4,2}$ and PPM$^8_{4,3}$ with $n = 4$ for $C_{n,m}$ plus PM$^8_{6,1}$, and the second group is PPM$^8_{6,1}$, PPM$^9_{6,2}$, PPM$^{10}_{6,3}$ and PM$^8_{6,1}$ with $n = 6$. Corresponding results are shown...
in Fig. 2. In the first group, on the one hand, the results of PPM$^6$, $m$ show less oscillations than that of PM$^8$, $m$, on the other hand, oscillations decrease with the increase of $m$ and almost disappear when $m=3$. In the second group, all mappings with $m$ from 1–3 yield oscillatory results. Hence the oscillations by mapped WENO9 do not owe to insufficient smoothness by piecewise function as claimed by Ref. 10, but come from the inadequate numerical stability caused by too flat profile around $\omega = d_k$ and less convergence rate of the mapping tending to identity mapping at endpoints. Based on above observations, the larger value of $m$ in $C_{n,m}$ to define mapping, e.g. $m=n$, is suggested in subsequent studies.

To further validate the above statement, the case of mapped WENO7 is investigated under $N=200$ and $t=1000$ where PPM$^3$, PM$^8$, and PPM$^8$ are employed. The results are shown in Fig. 3. One can see that derivatives of PPM$^3$ from both sides of $\omega = d_k$ only equal to each other up to the third-order, which is obviously less smooth than PM$^8$, therein. From the view of Ref. 10, PPM$^3$ would distribute discontinuously, which should entail oscillations more likely. However, Fig. 3 tells that PPM$^3$ yields a quite smooth result as that of PM$^8$, which indicates the discontinuous fourth-order derivatives would not cause numerical instability. One may observe the error in the corner of $x=-0.8$, and according to our understanding, such phenomenon owes to the dissipation of the mapping other than instability.

In short, the piecewise implementation of mapping doesn’t necessarily entail numerical stability.

(3) **Feasibility to Preserve Optimal Order by WENO3 in the Presence of First-Order Critical Points**

References 3, 8 and 10 showed the incapability of mappings to make WENO3 preserve the optimal order in the occurrence of first-order critical points. The reason is regarded as: when the critical point with the order $n_{cp}$ occurs, $\omega - d_k = O(\Delta x^{r-1-n_{cp}})$ where $r$ is the order of substencil and $n_{cp}$ indicates $f'' = ... = f^{(n_{cp})} = 0$ and $f^{(n_{cp}+1)} \neq 0$. In the case of WENO3 with $r=2$ and when $n_{cp}=1$ occurs, $\omega - d_k = O(1)$ and therefore the least condition required by mapping is violated. However, when the first-order

---

**Fig. 3** Results of WENO7-PM$^8$, -PPM$^8$, and -PPM$^3$ on scalar combination-waves advection at $t=1000$ with $N=200$
critical point occurs at \( x_j \), one can find that
\[
IS_k^{(2)} = \frac{1}{4} (f''_x \Delta x^2)^2 \left[ 1 + \frac{1 + m^2}{4} f''_x \Delta x + O(\Delta x^2) \right].
\]
So, it wonders that WENO3 could still preserve the optimal order through mapping, whereas one can verify such preservation does not happen in computations such as 1D scalar advection with the initial distribution \( sin(x) \).

In this regard, a heuristic analysis is proposed as: when the first-order critical point occurs at \( x_j \), the position will inevitably shift to the location such as \( x_c \in [x_j, x_{j+1}] \) next. For convenience, suppose \( x_j - x_c = \frac{1}{4} \Delta x \) where \( 1 < m < \infty \). One can see \( f''_x = f''_x(x_j - x_c) + O(\Delta x^2) \), and therefore
\[
IS_0^{(2)} = f''_x(x_j - x_c)^2 \Delta x^2 - (x_j - x_c) f''_x \Delta x^2 \Delta x + O(\Delta x^2) \Delta x
\]
which indicates the increase of flatness would suppress the endpoint convergence and achievement of desirable flatness and endpoint convergence at the same time is competitive, which on the one hand would make the mapping satisfy prescribed \( C_{n,m} \).

As discussed in Sect. 2.2, for existing mappings in fixed other than dynamic form, the \( IS \) of WENO3 by mapping in the occurrence of first-order critical points can be preserved through upgrading the smoothness indicator. One of the remedies for the problem is to upgrade smoothness indicators of WENO3 to that of WENO5 as suggested by Ref. 13. Specifically, \( IS_0^{(2)} \) and \( IS_1^{(2)} \) are substituted by \( IS_0^{(3)} \) and \( IS_2^{(3)} \) respectively. In the case that the first-order critical point occurs at \( x_j \), it is well-known that \( IS_k^{(3)} = \frac{13}{12} (f''_x \Delta x^2)^2 (1 + O(\Delta x)) \) and therefore the preservation of optimal order is achievable. When the critical point occurs at the aforementioned \( x_c \), one can check that \( IS_k^{(3)} = (\frac{13}{12} + \frac{13}{12} (f''_x \Delta x^2)^2 (1 + O(\Delta x)) \), therefore the least requirement of mapping is still satisfied.

In short, the order of WENO3 by mapping in the occurrence of first-order critical points can be preserved through upgrading the smoothness indicator.

### 4 New Piecewise Rational Mapping Functions

As discussed in Sect. 2.2, for existing mappings in fixed other than dynamic form, the achievement of desirable flatness and endpoint convergence at the same time is competitive, which indicates the increase of flatness would suppress the endpoint convergence and potentially impair the numerical stability. In the meanwhile, more and more concerns are paid towards the stability in long-time computation recently [7,8,10,11]. The investigations manifest that although many mappings have resolutions improved in short-time computation, they might perform oscillatory in the long run. According to the previous discussion, the mechanism is suspected to be the mismatch between flatness around \( \omega = d_k \) and endpoint convergence. In this section, we propose a new method with sufficient regulation capability which on the one hand would make the mapping satisfy prescribed \( C_{n,m} \), on the other hand could achieve desirable integration of flatness and endpoint convergence.

Based on our recent practices in point “(f)” of Sect. 2.2 as well as the observation of \( AIM_{n,m,c}^{n+1} \) [11], a new piecewise rational mapping PRM in \([d_k, 1]\) is proposed as:
\[
PRM_{n,m,c}^{R,n+1} = d_k + \frac{\omega - d_k}{\omega - d_k^n + c_2(\omega - d_k)^n_1(1 - \omega)^m_1 + c_1(1 - \omega)^m_1 + 1}
\]
(13)
where all exponents such as \( n \) and \( m \) are greater than or equal to 1, and where \( m \leq n \) is suggested. The mathematics of PRM deserves concern essentially, and as shown later, Eq. (13) will satisfy \( C_{n,m} \). Providing \( m_1 \geq m + 1 \). By aforementioned transformation, \( g^R = \frac{d_k}{\omega - d_k} \left[ 1 - g^R (1 - \frac{1 - d_k}{d_k} \omega) \right] \), corresponding PRM in \([0, d_k]\) can be obtained as:

\[ g^L = \frac{d_k}{1 - d_k} \left[ 1 - g^R (1 - \frac{1 - d_k}{d_k} \omega) \right], \]
would make mappings free of singularity. As just men-
for \{WENO3, 5, 7\} are finally acquired, which are tabulated in Table 5 cor-

\[
PRM_{L,n+1}^{m,n,m_1,m_1,c_1,c_2} = d_k + \frac{(\omega - d_k)^{n+1}}{(\omega - d_k)^n + (-1)^{m+1} \frac{1-d_k}{d_k} c_2 (\omega - d_k)^{m_1+1} + (-1)^m \left( \frac{1-d_k}{d_k} \right)^{m(n-1)} c_1 \omega^{m_1+1}}
\]

which satisfies \(C_{n,n}\) as well if \(m_1 \geq m + 1\). Especially, if \(n_1 = 1\), PRMs become:

\[
\begin{align*}
PRM_{L,n+1}^{m,n,m_1,m_1,c_1,c_2} &= d_k + \frac{(\omega - d_k)^{n+1}}{(\omega - d_k)^n + (-1)^{m+1} \frac{1-d_k}{d_k} c_2 (\omega - d_k)^{m_1} + (-1)^m c_1 \omega^{m_1+1}} \\
PRM_{R,n+1}^{m,n,m_1,m_1,c_1,c_2} &= d_k + \frac{(\omega - d_k)^{n+1}}{(\omega - d_k)^n + c_2 (\omega - d_k)^{-1} \omega^{m_1} + c_1 (1-\omega)^{m_1+1}}
\end{align*}
\]

It is conceivable that \(c_i\) can take different values in a piecewise manner. Hence it will be convenient to absorb the coefficients regarding \(d_k\) in \(PRM_{L,R}^{L,R}\) into \(c_i\) and derive the following formulations:

\[
\begin{align*}
PRM_{L,n+1}^{m,n,m_1,m_1,c_1,c_2} &= d_k + \frac{(\omega - d_k)^{n+1}}{(\omega - d_k)^n + (-1)^{n_1+1} c_2 (\omega - d_k)^{m_1} + (-1)^n c_1 \omega^{m_1+1}} \\
PRM_{R,n+1}^{m,n,m_1,m_1,c_1,c_2} &= d_k + \frac{(\omega - d_k)^{n+1}}{(\omega - d_k)^n + c_2 (\omega - d_k)^{1} (1-\omega)^{m_1} + c_1 (1-\omega)^{m_1+1}}
\end{align*}
\]

When \(m = n\), Eq. (15) further becomes

\[
\begin{align*}
PRM_{L,n+1}^{m,n,m_1,m_1,c_1,c_2} &= d_k + \frac{(\omega - d_k)^{n+1}}{(\omega - d_k)^n + (-1)^{n_1+1} c_2 (\omega - d_k)^{m_1} + (-1)^n c_1 \omega^{m_1+1}} \\
PRM_{R,n+1}^{m,n,m_1,m_1,c_1,c_2} &= d_k + \frac{(\omega - d_k)^{n+1}}{(\omega - d_k)^n + c_2 (\omega - d_k)^{1} (1-\omega)^{m_1} + c_1 (1-\omega)^{m_1+1}}
\end{align*}
\]

According to Theorem 1 and the characteristics of aforementioned transformation, it is conceivable that the positive \(c_{L,R}\) would make mappings free of singularity. As just mentioned, the relationship of \(PRM_{L,n+1}^{m,n,m_1,m_1,c_1,c_2}\) with \(C_{n,n}\) will be indicated by the following theorem with its proof shown in Appendix 3.

**Theorem 3** Consider a function as \(f(\omega) = d_k + \frac{(\omega - d_k)^{n+1}}{(\omega - d_k)^n + c_2 (\omega - d_k)^{1} (1-\omega)^{m_1} + c_1 (1-\omega)^{m_1+1}} \) in \([d_k, 1]\) where \(n \geq 1\) and \(\{c_1, c_2\}\) have the same sign. If \(m \geq 0\) and \(n_1 \geq 1\), then \(f(\omega)\) satisfies \(C_{n,\min(m,m_1+1)}\).

In the purpose of application, it is important to explore the effects of \(c_1, c_2, n_1, m_1\) in \(PRM_{L,R}^{L,R}\). In this regard, parametric studies of \(PRM_{n,n,m_1,m_1,c_1,c_2}\) is made qualitatively, and that of \(PRM_{L,R}^{L,R}\) can be conceivable accordingly. Through the investigation, the function of the above parameters and resultant regulation mechanism are achieved, and the recommendations to determine parameters are provided. For brevity, such details are shown in Appendix 4. After extensively analytical and numerical practices, the parameters of \(PRM_{n,n+1}^{L,R}\) for \{WENO3, 5, 7\} are finally acquired, which are tabulated in Table 5 corresponding to WENO(2r-1) by \(r\) and \(d_k\). Parameters for WENO9 are not investigated because of its seldom usage in applications.
For illustration, the profiles of \( \text{PRM}^{n+1}_{r,n} \) in \([0, 1]\) are shown in Figs. 4, 5, and 6 with comparisons by some other mappings. Concretely, the profiles of \( \text{PRM}^{2}_{1,1} \) regarding weights of WENO appear in Fig. 4 with comparisons by \( \text{PPM}^{2}_{1,0} \), \( \text{IM}^{3}_{2,0;0.1} \), and \( \text{PRM}^{2}_{1,1} \). Benefited from the strong regulation capability, \( \text{PRM}^{2}_{1,1} \) indicates the largest Flatness-II as well as the enhanced endpoint convergence, which favors the improvement of resolution.
smooth region and maintenance of stability near discontinuities. $\text{IM}^{2,0;0.1}$ shows a profile with the least endpoint convergence to identity mapping which might favor the increase of numerical resolution, however, such distribution is also liable to potential instability.

Moreover, although $\text{PRM}^{1,1}$ encompasses the other mappings excluding endpoints and nearby, we have checked that it appears the least flat in the neighborhood of $\omega = d_k$ when comparing with $g_M$ and $\text{IM}^{2,0;0.1}$. As previously mentioned, the latter flatness regards Flatness-I which is determined by $n$ in $C_{n,m}$. Hence the consequence indicates Flatness-II might differ from Flatness-I in some cases, and the mapping having larger Flatness-II may have smaller Flatness-I.

In Fig. 5, the profiles of combined $\text{PRM}^{3,2,2}$ regarding weights of WENO5 are shown with comparisons by $g_M$, $\text{IM}^{3,2,2}$, $\text{R}^{3,2,2}$, $\text{PM}^{6,1}$, and $\text{RM}^{7,6,2,0}$. Among comparatives $\text{R}^{3,2,2}$ is one of our practices in point “(f)” in Sect. 2 with a specific choice of parameters shown in Table 6. The parameters are so chosen that $\text{R}^{3,2,2}$ would have the similar Flatness-II as
Fig. 6 Distributions of combined PRM$_{3,3}$ with comparisons by PM$_{6,1}^8$, PPM$_{6,1}^8$ and RM$_{6,2,0}^7$ with zoomed views regarding details near endpoints a $d_0 = 1/35$ b $d_1 = 12/35$ c $d_2 = 18/35$ d $d_3 = 4/35$

Table 6 Parameters of R$_{3,2,2}^+$ to make mapping resemble Flatness-II of PRM$_{3,2,2}^+$ and that of PRM$_{3,2,2}^{*}$ to make mapping mimic PM$_{6,1}^8$ and RM$_{6,2,0}^7$ where $d_k = 6/10$ (all mappings formulated by Eq. (16))

| Mapping | $d_0$ | $c_1$ | $c_2$ | $m_1$ |
|---------|-------|-------|-------|-------|
| R$_{3,2,2}^+$ | 1/10 | 30,090 | 0 | 0 |
| | | 676,6666 | 0 | 0 |
| PRM$_{3,2,2}^*$ ⇒ PM$_{6,1}^8$ | 6/10 | 1235,6790 | 0 | 0 |
| | | 8335 | 0 | 0 |
| PRM$_{3,2,2}^*$ ⇒ RM$_{6,2,0}^7$ | 6/10 | 12,970,7047 | 0 | 0 |
| | | 929,2592 | 0 | 0 |
| PRM$_{3,2,2}^*$ ⇒ PM$_{6,1}^8$ | 6/10 | 26 | 13 | 2 |
| | | 40 | 20 | 2 |
| PRM$_{3,2,2}^*$ ⇒ RM$_{6,2,0}^7$ | 6/10 | 1 | 7500 | 5 |
| | | 1000 | 10,000 | 2 |
that of PRM$_{2,2}^3$. From the figure, PRM$_{2,2}^3$ is designed to have both the large Flatness-II and high rate of endpoint convergence toward the identity mapping. If comparing with RM$_{6,2,0}^7$, one may find that PRM$_{2,2}^3$ on the one hand has the larger Flatness-II, on the other hand it possesses the especially enforced endpoint convergence. Such regulation can only be achieved by means of Eqns. (14) - (16) with proper choices of parameters. Particularly, one can see that in the case of $d_1 = 6/10$, the profile of PRM$_{2,2}^3$ reflects a good balance between flatness and endpoint convergence, whereas RM$_{6,2,0}^7$ exhibits an exaggerated flat profile and inferior performance of endpoint convergence near $\omega = 1$. Hence PRM$_{2,2}^3$ demonstrates a superior profile from the mathematic view. Regarding R$_{2,2}^3$, although it has the similar Flatness-II as that of PRM$_{2,2}^3$, the mapping indicates less performed endpoint convergence. As will be shown in Sect. 5.3, R$_{2,2}^3$ yields oscillations in the long-time computation of combination-waves advection, which validates the importance of endpoint convergence on numerical stability.

The profiles of combined PRM$_{3,3}^4$ regarding weights of WENO7 are shown in Fig. 6 with comparisons by PM$_{6,1}^8$, PPM$_{6,1}^8$, and RM$_{6,2,0}^7$. With the help of sufficient regulation capability, PRM$_{3,3}^4$ is designed to have a profile with the balanced large Flatness-II and well-performed endpoint convergence and outweighs the comparatives.

Finally, in order to demonstrate the sufficient regulation capability of PRM, two particular PRM$_{2,2}^3$ with specific choice of parameters, denoted as PRM$_{2,2}^{3*}$, are provided as examples to mimic profiles of PM$_{6,1}^8$ and RM$_{6,2,0}^7$ in the case of $d_k = 6/10$. It is a reminder that PRM$_{2,2}^{3*}$ will differ from RM$_{6,2,0}^7$ and PM$_{6,1}^8$ near $\{\omega = 0, 1\}$ anyway because of the different endpoint conditions in satisfaction of $C_{n,m,l}$. Corresponding results are shown in Fig. 7, and the parameters regarding PRM$_{2,2}^{3*}$ are also given in Table 6.

Once PRM is ascertained, the corresponding WENO-PRM will be defined. Unless otherwise specified, $\varepsilon$ in Eq. (4) will take $10^{-40}$, $10^{-45}$ and $10^{-70}$ respectively for WENO3-PRM$_{1,1}^2$, WENO5-PRM$_{2,2}^3$, and WENO7-PRM$_{3,3}^4$. 

Fig. 7 Imitation of PM$_{6,1}^8$ and RM$_{6,2,0}^7$ by PRM$_{2,2}^{3*}$ in the case of $d_k = 6/10$ through specific choices of parameters.
5 Numerical Tests

5.1 Case Descriptions

The following 1-D problems of scalar advection and that by Euler equations are chosen to explore the performance of PRMs.

(1) 1-D Scalar Advection Problems

The governing equation is: \( \partial u / \partial t + \partial u / \partial x = 0 \) with various initial conditions corresponding to specific problems. Concretely, the initial conditions include:

(a) Sinusoidal-Like Wave Advection I (SWA-I): \( u(x, 0) = \sin \left( \pi x - \frac{\sin(\pi x)}{a} \right) \) with \( x \in [-1, 1] \)

The case describes advection of sinusoidal-like wave with only two first-order critical points when \( a > 1/\pi \), and the choice of \( a = 1 \) has been used in Ref. 3. Other than canonical one as \( u(x, 0) = \sin(\pi x) \), we find SWA-I would make numerical schemes behave disparately in convergence rate of accuracy. Empirically, the scheme having a fast convergence rate is expected to achieve better accuracy in less grids, which is favored by applications. Specifically, \( a = 1 \) and 1.005/\( \pi \) are chosen for tests of the third and fifth-order schemes respectively in Sect. 5.3. The fourth-order Runge–Kutta method is employed for temporal discretization with \( \Delta t < \Delta x^{4/2} \). A series of grids with the numbers \{20, 40, 80 … \} are employed and sequentially referred as the first, second … grids for convenience. The evolution of \( L_\infty \)-norm errors with \( \Delta t \) after one period of advection is acquired to indicate the accuracy order.

(b) Sinusoidal-Like Wave Advection II (SWA-II): \( u(x, 0) = \sin^3(\pi x - \frac{\sin(\pi x)}{a}) \) with \( x \in [-1, 1] \)

This case is also designed to test the accuracy order of schemes. Besides two first-order critical points, there are two second-order critical points at \( x = 0, 1 \) (or -1). \( a = 0.32 \) is chosen herein with two first-order critical points occurring at \( x \approx \pm 0.7345 \). Numerical schemes will usually show more disparate rates of order-convergence than in the case of \( u(x, 0) = \sin^3(\pi x) \). The temporal scheme and computation setup such as grids, \( \Delta t \) and run period are the same as that in SWA-I.

(c) Combination-Waves Advection 4

\[
\begin{align*}
\frac{1}{6} (G(x, \beta, z - \delta) + G(x, \beta, z + \delta) + 4G(x, \beta, z)), & \quad -0.8 \leq x \leq -0.6 \\
1, & \quad -0.4 \leq x \leq 0.2 \\
1 - |10(x - 0.1)|, & \quad 0 \leq x \leq 0.2 \\
\frac{1}{6} (F(x, \alpha, a - \delta) + F(x, \alpha, a + \delta) + 4F(x, \alpha, a)), & \quad 0.4 \leq x \leq 0.6 \\
0, & \quad \text{otherwise}
\end{align*}
\]

where \( x \in [-1, 1] \), \( G(x, \beta, z) = e^{-\beta(x-z)^2} \), \( F(x, \alpha, a) = \sqrt{\max(1 - a^2(x-a)^2, 0)} \), \( a = 0.5 \), \( z = -0.7 \), \( \delta = 0.005 \), \( \alpha = 10 \) and \( \beta = \log 2/36\delta^2 \). Typical grid numbers include: 200, 400 and 800 as in Refs. 7–8, 10–11. Computation time varies from the short 2 to the long 4000. In computations, the third-order TVD Runge–Kutta method \(^{21}\) is employed with the CFL number as 0.1.

Although this example has already served as a canonical test to check numerical stability in a short time period, the stability along with errors in the long-time computation was at least concerned by Ref. 7 initially regarding WENO5 schemes. Later,
computations in short and long periods were intensively investigated by Refs. 7–8 and 10–11 regarding mapped WENO5 and WENO7. Because of the potential importance of referred issues in applications, they are studied in detail herein.

(2) **1-D Problems by Euler Equations**

Because all schemes discussed next can pass standard tests such as the Sod problem, they will not be revisited here for brevity. The chosen problems are as follows:

(a) **Strong Shock Wave**

This case puts forward a trial regarding the computational robustness. The initial condition is: \((\rho, u, p) = \begin{cases} (1, 0, 0.1PR) & -5 \leq x < 0 \\ (1, 0, 0.1) & 0 < x \leq 5 \end{cases}\) In this study, \(PR\) takes the value of \(10^3\) and \(10^6\) respectively, and corresponding computations advance to \(t = 0.3\) and 0.01. The typical grid number is \(N = 200\). Because the schemes that fulfill the computation with \(PR = 10^3\) are found to usually have quite similar distributions, only results of \(PR = 10^6\) will be shown in the following.

(b) **Blast Wave**

This canonical example serves as another trial regarding the computational robustness. The initial condition is:

\[
\begin{align*}
\rho, u, p &= \begin{cases} 
(1, 0, 1000) & 0 \leq x < 0.1 \\
(1, 0, 0.1) & 0.1 \leq x \leq 0.9 \\
(1, 0, 100) & 0.9 \leq x \leq 1
\end{cases}
\end{align*}
\]

and solid wall condition is imposed at boundaries. The grid number is \(N = 200\) and the computation advances to \(t = 0.038\). By convention, a result on 15,001 grids by WENO5 is regarded as the “Exact” solution for reference.

(c) **Shu-Osher Problem**

This problem is a benchmark test on the numerical resolution. The initial condition is:

\[
\begin{align*}
\rho, u, p &= \begin{cases} 
(3.857143, 2.629369, 10.3333) & -5 \leq x < -4 \\
(1 + 0.2 \sin(5x), 0, 1) & -4 \leq x \leq 5
\end{cases}
\end{align*}
\]

The computation advances to \(t = 1.8\). By convention, the result of WENO5 at 10,001 grids is regarded as the “Exact” solution for reference.

(d) **Titarev-Toro Problem**

This example describes a Mach 1.1 shock interacting with the density fluctuations with high frequency, which serves as another trial to test the numerical resolution. The initial condition is:

\[
\begin{align*}
\rho, u, p &= \begin{cases} 
(1.515695, 0.523346, 1.805) & -5 \leq x < -4.5 \\
(1 + 0.1 \sin(20\pi x), 0, 1) & -4.5 \leq x < 5
\end{cases}
\end{align*}
\]

The typical grid number is \(N = 1000\) and the computation advances to \(t = 5\). By convention, the result of WENO5 at 10,001 grids is referenced as the “Exact” solution.

### 5.2 Additional WENO-Z Type Schemes for Comparison

In order to further evaluate the performance of PRMs, not only mappings in Sect. 2.2 but also some typical/updated WENO-Z type schemes are chosen for comparison. Under the framework of Eqns. (3)–(5), their formulations are described next for completeness.

1. **Third-Order Schemes**: WENO3-P+3 [14] and WENO3-F3 [15]

   WENO3-P+3 is obtained by applying new \(\alpha_k\) and \(\tau_p\) as

   \[
   \alpha_k = d_k \left[ 1 + \frac{\tau_p}{\kappa_1 + \epsilon} + \lambda \left( \frac{I_0^{(2)} + I_1^{(2)}}{\kappa_1 + \epsilon} \right) \right]
   \]

   with

   \[
   \tau_p = \left( \frac{1}{2} (I_0^{(2)} + I_1^{(2)}) - \frac{1}{4} (f_i - f_{i+1})^2 \right),
   \]

   \[
   \lambda = (\Delta x)^{1/6} \text{ and } \epsilon = 10^{-40}.
   \]
WENO3-F3 is acquired by applying new $\alpha_k$, $IS_3$, and $\tau_{F3}$ as $\alpha_k = d_k \left( 1 + \frac{\tau_{F3}}{IS_3^{0,+}} \right)$ with $\tau_{F3} = \left| \frac{1}{2} (IS_0^{(2)} + IS_1^{(2)}) - IS_3^{0} \right|^p$, $IS_3 = \frac{1}{12} (f_{i-1} - 2f_i + f_{i+1})^2 + \frac{1}{4} (f_{i-1} - f_{i+1})^2$, $p = 3/2$ and $\varepsilon = 10^{-10}$.

(2) Fifth-Order Schemes: canonical WENO5-Z scheme [4] and WENO5-NIS [16]

Canonical WENO5-Z scheme is derived by applying new $\alpha_k$ and $\tau_z$ as $\alpha_k = d_k \left[ 1 + \left( \frac{\tau_z}{IS_3^{0,+}} \right)^q \right]$ with $\tau_z = \left| IS_0^{(3)} - IS_2^{(3)} \right|$. When the first-order critical points occur, WENO5-Z would have the fourth-order accuracy with $q = 1$ but preserve the fifth-order with $q = 2$, however, the choice of $q = 1$ is usually regarded to yield higher resolution [4].

WENO5-NIS is formulated by proposing the new smoothness indicators $NIS_k^{(3)}$ from the original $IS_k^{(3)}$ as

\[
\begin{pmatrix}
NIS_0^{(3)} \\
NIS_1^{(3)} \\
NIS_2^{(3)}
\end{pmatrix} = \begin{pmatrix}
IS_0^{(3)} - \left( f_i - 2f_{i+1} + f_{i+2} \right) \left( 3f_i - 4f_{i+1} + f_{i+2} \right) \\
IS_1^{(3)} - \left( f_{i-1} - 2f_i + f_{i+1} \right) \left( f_{i-1} - f_{i+1} \right) \\
IS_2^{(3)} - \left( f_{i-2} - 2f_{i-1} + f_i \right) \left( f_{i-2} - 4f_{i-1} + 3f_i \right)
\end{pmatrix}.
\]

5.3 1-D Scalar Problems

The main purpose here is to explore the performance of proposed schemes on rate of order convergence in the presence of critical point and stability in long time computation in a comparative manner. The discussions are organized according to WENO-PRM series with third to seventh-orders.

(1) Results of WENO3-PRM$^2_{1,1}$ and Corresponding Third-Order Comparatives

(a) Accuracy Preservation and Convergence Rate by Swa-I with Occurrences of Critical Points

In this situation, SWA-I is tested by WENO3-PRM$^2_{1,1}$ at the choice of $a = 1$ where first-order critical points occur. For comparison, canonical WENO3 and other mapped WENOs are tested as well. It is worth mentioning that the smoothness indicators in the mappings, including that in PRM$^2_{1,1}$, should employ the ones in WENO5 as described.
in Sect. 3. Comparative mappings include: $\text{PPM}^2_{1,0}, g_M$, $\text{PPM}^3_{2,0}$ and $\text{IM}^3_{2,0,0,1}$. $\text{PPM}^2_{1,0}$ has the minimum requirement of $n=1$ in $C_{n,m}$ as that of $\text{PRM}^2_{1,1}$, whereas the rest have one order higher as $n=2$ in $C_{n,m}$.

Figure 8 illustrates that all mapped schemes can recover the third-order accuracy, however, different schemes demonstrate disparate rates of order convergence. Among them, WENO3-PRM$^2_{1,1}$ shows the quickest rate which achieves the 2.8th-order on the second grids and the near third-order on the third grids, in the meanwhile WENO3-IM$^3_{2,0,0,1}$ performs similarly on the first two grids but indicates the order slightly less than three on the third grids. The results of other mappings show a definite slower rate of convergence even if $g_M$ and $\text{PPM}^3_{2,0}$ have one order larger of $n$ in $C_{n,m}$ than that of $\text{PRM}^2_{1,1}$. As expected, WENO3 shows a degraded order of 2 because of the critical points.

Table 7  \(L_\infty\)-error and order convergence of schemes with grid number $N$ in sinusoidal-like wave advection I

| $N$ | WENO3 | $L_\infty$-error | Order | WENO3-PPM$^2_{1,0}$ | $L_\infty$-error | Order |
|-----|-------|------------------|------|------------------|------------------|------|
| 10  | 0.564973486836035 | – | – | 0.369561268000000 | – | – |
| 20  | 0.246534549054767 | 1.196 | 0.098486793000000 | 1.908 | 0.098486793000000 | 1.908 |
| 40  | 0.10540985596549 | 1.226 | 0.048272400000000 | 1.029 | 0.048272400000000 | 1.029 |
| 80  | 0.04257691614799 | 1.308 | 0.021146640000000 | 1.191 | 0.021146640000000 | 1.191 |
| 160 | 0.016510698827055 | 1.367 | 0.005949088000000 | 1.830 | 0.005949088000000 | 1.830 |
| 320 | 0.006240184505551 | 1.404 | 0.001711034000000 | 1.798 | 0.001711034000000 | 1.798 |
| 640 | 0.0023544237958584 | 1.406 | 0.00002171855024 | 9.622 | 0.00002171855024 | 9.622 |
| 1280| 0.000873849276864 | 1.430 | 0.000000250598385 | 3.115 | 0.000000250598385 | 3.115 |
| 2560| 0.000321603836272 | 1.442 | 0.000000031325051 | 3.000 | 0.000000031325051 | 3.000 |

| $N$ | WENO3-M | $L_\infty$-error | Order | WENO3-PPM$^3_{2,0}$ | $L_\infty$-error | Order |
|-----|-------|------------------|------|------------------|------------------|------|
| 10  | 0.344508199636262 | – | – | 0.341486487957917 | – | – |
| 20  | 0.086458257980084 | 1.994 | 0.091550810389959 | 1.899 | 0.091550810389959 | 1.899 |
| 40  | 0.036297539306736 | 1.252 | 0.043636486031413 | 1.069 | 0.043636486031413 | 1.069 |
| 80  | 0.002955482623476 | 3.618 | 0.013783095139290 | 1.663 | 0.013783095139290 | 1.663 |
| 160 | 0.000128049396529 | 4.529 | 0.000128058937886 | 6.750 | 0.000128058937886 | 6.750 |
| 320 | 0.000016033305547 | 2.998 | 0.00001603286540 | 2.998 | 0.00001603286540 | 2.998 |
| 640 | 0.000002004688932 | 3.000 | 0.000002004688931 | 3.000 | 0.000002004688931 | 3.000 |
| 1280| 0.000000250598381 | 3.000 | 0.000000250598376 | 3.000 | 0.000000250598376 | 3.000 |

| $N$ | WENO3-IM$^3_{2,0,0,1}$ | $L_\infty$-error | Order | WENO3-PRM$^2_{1,1}$ | $L_\infty$-error | Order |
|-----|-------|------------------|------|------------------|------------------|------|
| 10  | 0.209640727631624 | – | – | 0.236350335063057 | – | – |
| 20  | 0.045150235304096 | 2.215 | 0.053749378560005 | 2.137 | 0.053749378560005 | 2.137 |
| 40  | 0.007745532657626 | 2.543 | 0.007908840385026 | 2.765 | 0.007908840385026 | 2.765 |
| 80  | 0.001021083507328 | 2.923 | 0.001019730459840 | 2.955 | 0.001019730459840 | 2.955 |
| 160 | 0.000128083305016 | 2.995 | 0.000128091167729 | 2.993 | 0.000128091167729 | 2.993 |
| 320 | 0.000016033324597 | 2.998 | 0.000016033295950 | 2.998 | 0.000016033295950 | 2.998 |
| 640 | 0.000002004689632 | 3.000 | 0.000002004689662 | 3.000 | 0.000002004689662 | 3.000 |
For clarity, details of $L_{\infty}$-error and corresponding numerical orders of schemes are provided in Table 7.

(b) Long-Time Computation of Combination-Waves Advection

A long period up to $t = 4000$ is chosen in computation. The following comparative mappings are selected: $g_M$, PPM$^3_{2,0}$ and IM$^3_{2,0,0.1}$, where the order $n$ in $C_{n,m}$ is 2 and one larger than that of PRM$^2_{1,1}$; likewise, WENO3 is also included. In Fig. 9, the distribution of WENO3-PRM$^2_{2,1,1}$ on 800 grids is drawn with comparisons by other schemes. One can see that after long-time computation, WENO3-PRM$^2_{1,1}$ and -IM$^3_{2,0,0.1}$ show quite similar results, which are relatively more accurate than that by WENO3-M and -PPM$^3_{2,0}$, whereas WENO3 (with $\varepsilon = 10^{-6}$ in Eq. (4)) yields a rather dissipative distribution. From the figure, no instability occurs and all schemes perform stably.

In order to reveal numerical errors of schemes, additional 200, 400 grids are adopted and the corresponding $L_1$-error is evaluated. As shown in Fig. 10, WENO3-PRM$^2_{2,1,1}$ indicates the least error among four mapped WENOs, whereas WENO3 exhibits an
obvious large error. Moreover, carefully checking shows WENO3-PRM2,2 achieves the approximate first-order on the third grids.

Similarly, details of $L_1$-error and corresponding numerical orders of schemes are shown in Table 8.

(2) **Results of WENO5-PRM$^3_{2,2}$ and Corresponding Fifth-Order Comparatives**

(a) **Accuracy Preservation and Convergence Rate by SWA-I with Occurrences of Critical Points**

| $N$ | WENO3 $L_1$-error | Order | WENO3-M $L_1$-error | Order |
|-----|-------------------|-------|---------------------|-------|
| 200 | 0.327287380772524 | –     | 0.308615073500625   | –     |
| 400 | 0.30493696544888  | 0.102 | 0.275658010531971   | 0.163 |
| 800 | 0.268253462970276 | 0.185 | 0.14356027114544    | 0.941 |

| $N$ | WENO3-PPM$^3_{2,0}$ $L_1$-error | Order | WENO3-IM$^3_{2,0,0,1}$ $L_1$-error | Order |
|-----|----------------------------------|-------|----------------------------------|-------|
| 200 | 0.30911342139155                | –     | 0.30654754938900               | –     |
| 400 | 0.279885198449823              | 0.143 | 0.27076795241797              | 0.179 |
| 800 | 0.143693467797502              | 0.962 | 0.140915069845183          | 0.942 |

| $N$ | WENO3-PRM$^2_{1,1}$ $L_1$-error | Order |
|-----|---------------------------------|-------|
| 200 | 0.304502396829442               | –     |
| 400 | 0.26909806670935               | 0.178 |
| 800 | 0.139913229301360              | 0.944 |

Fig. 11 Evolution of accuracy convergence by scalar wave advection with the initial condition

$$u(x, 0) = \sin \left( \pi x - \frac{\sin(\pi x)}{1.005} \right)$$

WENO5, WENO5-M, -PPM$^3_{2,0}$, -PPM$^8_{6,1}$, -PM$^8_{6,1}$, -RM$^7_{6,2,0}$ and -PRM$^2_{2,2}$
As shown in Sect. 3 and Table 17 in Appendix 1, the minimum requirement of \( n \) in \( C_{n,m} \) to preserve the optimal order is 2 and the largest \( n_{cp} \) where the 5th-order could be preserved is 1 for mapped WENO5. Therefore, WENO5-PRM\(^{3,2,2}\) is still tested by SWA-1 to check accuracy order and convergence rate, where \( a \) is taken as 1.005/\( \pi \) in the initial condition. For comparison, the following mappings satisfying the same \( C_{2,0} \) are tested: \( g_M \), PPM\(^{3,2,0}\), and IM\(^{2,0,0,1}\); besides, mappings with larger \( n \) in \( C_{n,m} \) such as PPM\(^{6,1}\) and that proposed in Refs. 8 and 10 for WENO5 are also chosen, namely PM\(^{6,1}\) and RM\(^{6,2,0}\). It is worthwhile to mention that AIM\(^{5,2,1E4}\) fails to work in the case of WENO5 probably because of its parameters, namely \( c = 10^4 \), only applicable to WENO7. According to the analysis above and results in Fig. 11, all mapped schemes achieve the fifth-order theoretically and numerically. Furthermore, WENO5-PRM\(^{3,2,2}\) demonstrates the quickest convergence rate with the achievement of the fifth-order on the fourth grids, whereas the others achieve the order on the sixth or seventh grids (WENO5-M). As expected, WENO5 can only obtain a degraded third-order.

| \( N \) | WENO5 \( L_{\infty}-error \) | Order | WENO5-M \( L_{\infty}-error \) | Order |
|---|---|---|---|---|
| 10 | 0.587459230146862 | – | 0.467397869549396 | – |
| 20 | 0.100645097327154 | 2.545 | 0.035573591263660 | 3.716 |
| 40 | 0.009996142909834 | 3.332 | 0.002093813288409 | 4.087 |
| 80 | 0.000687461442217 | 3.862 | 0.000276703544324 | 2.920 |
| 160 | 0.000060104415060 | 3.516 | 0.000004252833032 | 6.024 |
| 320 | 0.000000518939685 | 3.582 | 0.00000067081118 | 5.986 |
| 640 | 0.000000513840014 | 3.288 | 0.00000002098880 | 4.998 |

| \( N \) | WENO5–PPM\(^{3,2,0}\) \( L_{\infty}-error \) | Order | WENO5–IM\(^{3,2,0,0,1}\) \( L_{\infty}-error \) | Order |
|---|---|---|---|---|
| 10 | 0.514871540786020 | – | 0.36061027738361 | – |
| 20 | 0.045062099498991 | 3.514 | 0.039335408137773 | 3.197 |
| 40 | 0.002102061012292 | 4.422 | 0.002018347705772 | 4.285 |
| 80 | 0.000137403397668 | 3.935 | 0.00009259671312 | 4.446 |
| 160 | 0.000002201347390 | 5.964 | 0.00000213724887 | 5.437 |
| 320 | 0.000000670932725 | 5.036 | 0.00000067047951 | 4.994 |
| 640 | 0.000000020988789 | 4.998 | 0.00000002098876 | 4.998 |

| \( N \) | WENO5–PPM\(^{8,6,1}\) \( L_{\infty}-error \) | Order | WENO5–PM\(^{8,6,1}\) \( L_{\infty}-error \) | Order |
|---|---|---|---|---|
| 10 | 0.519121673737166 | – | 0.531715079636646 | – |
| 20 | 0.044882636747115 | 3.532 | 0.045419487273297 | 3.549 |
| 40 | 0.002058261174730 | 4.447 | 0.002149230451618 | 4.401 |
| 80 | 0.000076152268613 | 4.756 | 0.000083827920891 | 4.680 |
| 160 | 0.000002137079114 | 5.155 | 0.000002137042801 | 5.294 |
| 320 | 0.00000067047951 | 4.994 | 0.00000067047275 | 4.994 |
| 640 | 0.000000020988789 | 4.998 | 0.00000002098866 | 4.998 |
Similarly, details of $L_{\infty}$-error and corresponding numerical orders of schemes are provided in Table 9.

(b) **Long-Time Computation of Combination-Waves Advection**

WENO5-PRM$^{2,2}$ is tested on 200, 400 and 800 grids with $t = 2000$. In Refs. 7, 8 and 10, the stability and numerical errors of this case were intensively studied, where WENO5-RM$^{7,6,2,0}$ was considered to have outstanding performance in both short time and long-time computations. For comparison, the following schemes are checked: WENO5, WENO5-M, -IM$^{3,2,0,0.1}$, -PM$^{8,6,1}$, and -RM$^{7,6,2,0}$. WENO5-IM$^{3,2,0,0.1}$ is absent again because of its failure in computation. The results on 800 grids are shown in Fig. 12, and the results of schemes for comparison have been checked to coincide with that in Ref. 7. The figure tells that WENO5 and WENO5-M show smeared distributions after long-time computation, WENO5-IM$^{3,2,0,0.1}$ yields relatively dissipative result (see the distribution at $x = -0.7$ and -0.8) and the rest schemes perform similarly except at the corner $x = 0.6$. At the corner, WENO5-IM$^{3,2,0,0.1}$ and -RM$^{7,6,2,0}$ yield results without oscillations as in Refs. 8 and 10, whereas WENO5-PM$^{8,6,1}$ and -PRM$^{3,2,2}$ show results with smaller ones. Refs. 3 and 10 implied that $\varepsilon$ in Eq. (4) may affect the stability of computations especially in the long-time run. As shown in Fig. 12(c), WENO5-RM$^{7,6,2,0}$ will yield similar oscillations when $\varepsilon$ takes $10^{-45}$ and that of WENO5-PRM$^{3,2,2}$ can be totally free of fluctuations when $\varepsilon = 10^{-101}$.

Next, $L_1$-error of WENO5-PRM$^{3,2,2}$ is investigated with the comparisons by WENO5, WENO5-M, -IM$^{3,2,0,0.1}$, -PM$^{8,6,1}$, and -RM$^{7,6,2,0}$. As shown in Fig. 13, WENO5-PRM$^{3,2,2}$ indicates a low error level as that of WENO5-PM$^{8,6,1}$ and -RM$^{7,6,2,0}$, and they nearly achieve the first-order on the second grids. As the comparison, WENO5-IM$^{3,2,0,0.1}$ shows a relatively larger error whereas that of WENO5-M and WENO5 falls into the group with the largest errors.

Similarly, details of $L_1$-error and corresponding numerical orders of schemes are shown in Table 10.

In Sect. 4, a comparative $R^{3,2,2}$ is devised which has the similar Flatness-II as that of PRM$^{3,2,2}$. As shown in Fig. 5, $R^{3,2,2}$ indicates relatively inferior performance on endpoint convergence due to the lack of sufficient regulation. Two mappings are also compared in current computation at $t = 2000$ on 800 grids and under $\varepsilon = 10^{-45}$ in Eq. (4) with results shown in Fig. 14. The figure tells that $R^{3,2,2}$ yields distinct oscillations than PRM$^{3,2,2}$ does, which verifies the importance of $PEC$ on enhancing numerical stability and validates the effectiveness of new implementation in $PRM^{3,2,2}$.

(3) **Results of WENO7-PRM$^{3,3}$ and Corresponding Seventh-Order Comparatives**

(a) **Accuracy Preservation and Convergence Rate By SWA-II with Occurrences of Critical Points**

Recalling the analysis in Sect. 3 and Table 17 in Appendix 1, SWA-II is chosen with the existence of two groups of first- and two second-order critical points. The case represents an utmost situation that canonical WENO7 could preserve its optimal order by mappings, and the minimum requirement of $n$ in $C_{n,m}$ for order preservation is 3. To compare with WENO7-PRM$^{3,3}$, canonical WENO7 and some recent mappings are chosen: PM$^{8,6,1}$, RM$^{7,6,2,0}$, and AIM$^{5,4,2,1E4}$. The reason to choose the three latter is that they are specially designed and/or tested for WENO7$^{[10,11]}$, whereas aforementioned IM$^{3,2,0,0.1}$ is not chosen because it does not have the required $n = 3$ in $C_{n,m}$. The evolution of $L_1$-errors with $\Delta x$ is shown in Fig. 15, which indicates all schemes have achieved the optimal orders in the presence of critical points. Carefully checking shows WENO7-PM$^{8,6,1}$ acquires the seventh-order on the fourth grids, while the rest do on the fifth grids.
Fig. 12 Results of combination-waves advection by WENO5-PRM$^3_{2,2}$ with the comparisons by WENO5, WENO5-M, IM$^{3,0,0,1}$, PM$^8_{6,1}$ and RM$^7_{6,2,0}$ at $t=2000$ on 800 grids a Global view b Zoomed view c Results of PRM$^3_{2,2}$ with Different $\varepsilon$ in Eq. (4) and RM$^7_{6,2,0}$ with $\varepsilon=10^{-45}$
Similarly, details of $L_{\infty}$-error and corresponding numerical orders of schemes are provided in Table 11.

(b) Long-Time Computation of Combination-Waves Advection

This case is especially concerned and intensively tested among the seventh-order mapped WENO5s in Refs. 10–11. The longest computation time in the references is chosen, namely $t = 2000$, and the full series of grids therein are used with numbers $\{200, 400, 800\}$. Besides WENO7-PRM$_{3,2}$, similar schemes are chosen as those in the previous case. Prior to further discussion, it is worthy of mention that we have repeated the result of WENO7-PM$_{6,1}$ on 200 grids at $t = 1000$ by Ref. 10. The results of 200 grids are first shown in Fig. 16. In Fig. 16, all schemes have shown discrepancies with respect to the exact solution, where WENO7-AIM$_{4,2;1E4}$ yields an obvious under-estimation
of the second square and overshoots about the fourth oval, and the other schemes produce small overshoots and obvious under-estimations also. Because the issue of numerical stability is the focal in Ref. 10, 11, the distributions at the corner $x \approx -0.78$ are zoomed in Fig. 16b. The figure tells that both WENO7-PRM$^{3,3}$ and -RM7$^{6,2,0}$ yield oscillations while WENO7-AIM$^{4,2;1E4}$ does not; besides, WENO7-PRM$^{3,3}$ indicates large deviation from the exact solution but not oscillation there. Such deviation would definitely increase computation error, however they should be attributed to numerical dissipation other than instability according to our experience and discussions in Ref. 11. In short, in the case of 200 grids, WENO7-PRM$^{3,3}$ and AIM$^{4,2;1E4}$ yield results without oscillations at $t = 2000$.

Next, the computations are carried out on 400 grids with $t = 2000$, and the results are shown in Fig. 17. All mapped schemes perform normally except at the foot of the first peak and the region between the first two structures with the occurrence of
oscillations. Besides, WENO7-PRM\textsuperscript{4,3,3} yields a relatively smeared description on the first peak. In order to clearly visualize the oscillations, the zoomed view is shown in Fig. 17b. The figure tells that at the left foot of the first peak, WENO7-PM\textsuperscript{8,6,1}, -RM\textsuperscript{7,6,2,0} and -AIM\textsuperscript{5,4,2;1E4} generate oscillations whereas WENO7-PRM\textsuperscript{4,3,3} produces smooth distribution; at the right foot of the peak, all mapped schemes yield oscillations except WENO7-AIM\textsuperscript{5,4,2;1E4}; additionally, WENO7-PRM\textsuperscript{4,3,3} produces more perturbations between the first two structures. Overall, all schemes yield oscillations in this case, whereas WENO7-PRM\textsuperscript{4,3,3} indicates more error because of its deviation from the exact solution at the first peak.

Thirdly, the schemes are tested on 800 grids with $t=2000$. As shown in Fig. 18, all mapped schemes perform nicely except that small perturbations by some schemes appear at the right foot of the fourth oval. The zoomed view in Fig. 18b indicates that WENO7-PRM\textsuperscript{4,3,3}, -PM\textsuperscript{8,6,1} and -RM\textsuperscript{7,6,2,0} claim this behavior, whereas WENO7-AIM\textsuperscript{5,4,2;1E4} yields a distribution free of oscillations which coincides with reports in Ref. 11. It is worth

| \( N \) | WENO7 & WENO7-\textsuperscript{PM}\textsuperscript{8,6,1} & WENO7-\textsuperscript{RM}\textsuperscript{7,6,2,0} & WENO7-\textsuperscript{AIM}\textsuperscript{5,4,2;1E4} |
|---|---|---|---|
| \( L_\infty \)-error & Order & \( L_\infty \)-error & Order & \( L_\infty \)-error & Order |
| 10 | 0.520557829849167 – | 0.414194127655134 – | 0.246785762278284 – |
| 20 | 0.268330152424100.953 | 0.233568424563760.826 | 0.2216767377552820.155 |
| 40 | 0.0408385104063027.19 | 0.0334698540229142.803 | 0.000022634456116.915 |
| 80 | 0.0010748266127545.248 | 0.0002731509093786.937 | 0.0000000181108636.966 |
| 160 | 0.0000262610925055.355 | 0.0000022634456116.915 | 0.0000000181108636.966 |
| 320 | 0.0000000448944035.870 | 0.0000000181108636.966 | 0.0000000181108636.966 |
| 640 | 0.0000000084070775.739 | 0.0000000181108636.966 | 0.0000000181108636.966 |

Table 11: \( L_\infty \)-error and order convergence of schemes with grid number \( N \) in sinusoidal-like wave advection II
mentioning that in our computation, WENO7-RM7,6,2,0 performs well in the rest regions, which is contrary to oscillations at the foot of the second and fourth structures reported in Ref. 11.

Based on above results, the evolution of $L_1$-error of schemes with $\Delta x$ is derived and shown in Fig. 19. The figure shows that WENO7-PRM4,3,3 indicates relatively larger errors on the first two grids but shows almost the same error on the third grids. As just discussed, the larger error of WENO7-PRM4,3,3 on the first grids owes to the deviation at the left foot of the first peak (see Fig. 16b), while the error on the second grids owes to the smeared description of left side of the first peak and additional oscillations between the first two structures (see Fig. 17b). It seems that WENO7-AIM5,4,2;1E4 indicates the performance with overall fewer oscillations, however, such performance does not guarantee its robustness in subsequent tests by Euler equations.

Similarly, details of $L_1$-error and corresponding numerical orders of schemes are shown in Table 12.
5.4 1-D Problems by Euler Equations

In this situation, the third-order TVD Runge–Kutta [2] is used for temporal discretization and the Steger-Warming scheme is employed for flux splitting. The main purpose is to explore the performance of proposed schemes on computational robustness and numerical resolution in a comparative manner.

(1) Results of WENO3-PRM2,1,1 and Corresponding Third-Order Comparatives

(a) Strong Shock Wave

Besides WENO3-PRM2,1,1, the following mapped WENO3 schemes are chosen for comparison, namely WENO3-M, -PPM3,2,0 and -IM3,2,0,0,1, where incorporated mappings satisfy $C_{2,0}$; in addition, two recently proposed WENO-Z-type schemes are selected as well: WENO3-P + 3[14] and WENO3-F3[15]. In two computations with $PR = 10^3$ and $10^6$, all schemes except WENO3-P + 3 fulfill the computation. Because
Fig. 18 Results of combination-waves advection at $t = 2000$ on 800 grids by WENO7-PRM$^3_{3,3}$ with the comparisons by WENO7, WENO7-PM$^8_{6,1}$, -RM$^7_{6,2,0}$ and -AIM$^5_{4,2;1E4}$. a Global view b Zoomed view where $x \in [0.59,0.71]$

Fig. 19 $L_1$-errors with $\Delta x$ from computations of combination-waves advection at $t = 2000$ by WENO7-PRM$^3_{3,3}$ with the comparisons by WENO7, WENO7-PM$^8_{6,1}$, -RM$^7_{6,2,0}$ and -AIM$^5_{4,2;1E4}$
Table 12 $L_1$-error and order convergence of schemes with grid number $N$ in combination-waves advection

| $N$ | WENO7 $L_1$-error | Order | WENO7-PM$^6_{6,1}$ $L_1$-error | Order |
|-----|-------------------|-------|---------------------------------|-------|
| 200 | 0.2171214177075816 | –     | 0.05703843722886646 | –     |
| 400 | 0.142013579585604  | 0.612 | 0.0265336205060503  | 1.104 |
| 800 | 0.0436110283020527 | 1.703 | 0.0112982613966643  | 1.232 |

| $N$ | WENO7-RM$^7_{6,2,0}$ $L_1$-error | Order | WENO7-AIM$^5_{4,2,1E4}$ $L_1$-error | Order |
|-----|---------------------------------|-------|---------------------------------|-------|
| 200 | 0.0573294334786627 | –     | 0.0577603286031427 | –     |
| 400 | 0.0261871374783609 | 1.130 | 0.0254553278492772 | 1.182 |
| 800 | 0.011287389625161 | 1.214 | 0.011090751924606 | 1.199 |

| $N$ | WENO7-PRM$^4_{3,3}$ $L_1$-error | Order |
|-----|---------------------------------|-------|
| 200 | 0.0640933779863998 | –     |
| 400 | 0.0297198912321625 | 1.109 |
| 800 | 0.011290902147484 | 1.396 |

Fig. 20 Density distributions of strong shock wave at $t=0.01$ on 200 grids with initial pressure ratio $PR = 10^6$ by WENO3-PRM$^2_{1,1}$ with the comparisons by WENO3, WENO3-F3, -M, -PPM$^3_{2,0}$, -IM$^2_{2,0;0.1}$

all results are similar in the first $PR$ case, only that of the second case are illustrated in Fig. 20. The figure tells that all schemes do not well simulate the density platform after the shock, where WENO3-F3 and -PPM$^3_{2,0}$ seem to yield a result with a relatively lower peak. Because WENO3-M shows a peak with a higher height than that of WENO3-IM$^2_{2,0;0.1}$, we would rather regard this case as a test of numerical robustness other than an indication of resolution.

(b) Blast Wave

Besides WENO3-PRM$^2_{1,1}$, similar third-order comparatives are chosen as above. Corresponding density distributions are shown in Fig. 21. It is found that all schemes can fulfill the computation. Although WENO3-P+3 shows relatively more resolu-
tion in zoomed view, its failure in the previous test indicates current performance might attribute to its inadequate numerical dissipation. Considering the performance of WENO3-PRM$^{2,1,1}$ later on Shu-Osher problem, the scheme indicates a good balance between the robustness and resolution.

(c) Shu-Osher Problem

Besides WENO3-PRM$^{2,1,1}$, similar comparatives are adopted as in the previous problem. Other than grids with numbers usually from 400–600, 230 grids are employed here, which poses a tough test on numerical resolution. Corresponding results are shown in Fig. 22. One can see that only the first three waves after the shock wave are distinguishable by WENO3-PRM$^{2,1,1}$, -IM$^{3,2,0;0.1}$ and -P$+3$ among tested schemes. Regarding the three schemes, WENO3-PRM$^{2,1,1}$ and -IM$^{3,2,0;0.1}$ yield results with the best resolution, which tells the density fluctuation of the fourth wave structure.
and therefore manifests their superior resolution. A closer look shows that WENO3-PRM$^{1,1}$ indicates slightly more resolution on density fluctuation.

(2) **Results of WENO5-PRM$^{2,2}$ and Corresponding Fifth-Order Comparatives**

(a) **Strong Shock Wave**

To compare with PRM$^{3,2,2}$, the following representative mappings in fifth-order scenario are chosen: $g_{5M}$, IM$^{3,2,0;0.1}$, PM$^{8,1}_{6,1}$ and RM$^{7,2,0}$, where the latter three are intensively studied in Refs. 7, 8 and 10 as improvements for WENO5-M. Besides comparative mappings, some WENO-Z type schemes are chosen as well, namely WENO5-Z with $q=1$ and 2 [4] and WENO5-NIS [16]. All schemes can accomplish the computations with density distributions shown in Fig. 23. One can see that the density platform after the shock is still not well resolved. Specifically, WENO5-Z at $q=1$ and WENO5-IM$^{3,2,0;0.1}$ yield a relatively larger overshoot over the exact solution while WENO5-PRM$^{3,2,2}$ indicates a moderate distribution.

Fig. 23  Density distributions of strong shock wave with initial pressure ratio $PR=10^6$ at $t=0.01$ on 200 grids by WENO5-PRM$^{3,2,2}$ with the comparisons by WENO5, WENO5-Z($q=1$ and 2), -NIS, -M, -PM$^{8,1}_{6,1}$, -IM$^{3,2,0;0.1}$ and -RM$^{7,2,0}$

Fig. 24  Density distributions of blast waves at $t=0.038$ on 200 grids by WENO5-PRM$^{3,2,2}$ with the comparisons by WENO5, WENO5-NIS, -Z($q=2$), -M, -PM$^{8,1}_{6,1}$ and -RM$^{7,2,0}$
(b) **Blast Wave**

To compare with WENO5-PRM\(^3\)\(_{2,2}\), similar fifth-order comparatives are chosen as above. The density distributions are shown in Fig. 24. All schemes have accomplished the test. Differences in density distributions are locally enlarged in the zoomed view for clarity.

(c) **Shu-Osher Problem**

To compare with WENO5-PRM\(^3\)\(_{2,2}\), similar comparatives are chosen for testing as just now. For clarity, the results of comparatives are categorized in two groups, i.e. results of mapped WENO5 in Fig. 25a and that of WENO5-Z type schemes in Fig. 25b. From Fig. 25a, one can see that WENO5-PRM\(^3\)\(_{2,2}\) and -IM\(^3\)\(_{2,0,0.1}\) show the best resolutions and indicate quite similar performance; WENO5-RM\(^7\)\(_{6,2,0}\) shows a resolution of the second class, while the rest schemes exhibit relatively inferior resolutions.
Fig. 26  Local and zoomed views of density distributions of Titarev-Toro problem at $t=5$ on 1000 grids by WENO5-PRM$^{3}_{2,2}$ with the comparisons by WENO5, WENO5-M, -IM$^{3}_{2,0,0,1}$, -PM$^{6}_{6,1}$, -RM$^{6,2}_{6,0}$, and WENO5-Z ($q=1$ and 2) a Global view where $x \in [1.33, 3.5]$ b Zoomed view 1 where $x \in [2.1, 3.2]$ c Zoomed view 2 where $x \in [1.36, 2.12]$
Next, the result of WENO5-PRM$^{3,2,2}$ is compared with that of WENO5-Z type schemes as shown in Fig. 25b. The figure tells that WENO5-PRM$^{3,2,2}$ and WENO5-Z at $q = 1$ achieve the best resolution and outperform the rest WENO5-Z schemes.

(d) Titarev-Toro Problem

To compare with WENO5-PRM$^{3,2,2}$, similar mapped schemes and WENO-Z type methods are employed. Because of the high frequency of the density fluctuation, only part of the distributions are displayed such that possible differences can be told, i.e. the region at $x \in [1.33, 3.5]$ as shown in Fig. 26a, whilst zoomed views are further displayed subsequently. The area just after the shock is first zoomed in Fig. 26b, where the fluctuations experience numerical dissipation shortly; afterwards, a further suffering of dissipation at $x \in [1.36, 2.12]$ is zoomed in Fig. 26c. The figures tell that WENO5-PRM$^{3,2,2}$, -IM$^{3,2,0;0.1}$ and WENO5-Z with $q = 2$ achieve the best resolution; WENO5-RM$^{7,6,2,0}$ and WENO5-Z with $q = 1$ take the second place; WENO5-M and -PM$^{8,6,1}$ take the third position where the former indicates a relatively better resolution. From Fig. 26c, WENO5-PRM$^{3,2,2}$ even outperforms WENO5-IM$^{3,2,0;0.1}$ at $x \in [1.36, 2.12]$ and indicates a better resolution. Note that $n$ in $C_{n,m}$ satisfied by PM$^{8,6,1}$, is 6, much larger than 2 in PRM$^{3,2,2}$ and IM$^{3,2,0;0.1}$, therefore the mappings with smaller $n$ but larger Flatness-II can outperform the one with larger $n$ in $C_{n,m}$ but smaller flatness. WENO5-NIS fails in the computation, which reminds again the importance of robustness in developing high-order schemes.

Another point worthy of attention is that WENO5-Z with $q = 2$ slightly outperforms the scheme with $q = 1$, which is opposite to the resolution relationship shown in Shu-Osher problem (see Fig. 25b). In Ref. 4, WENO5-Z at $q = 1$ was thought to be one order lower than the scheme at $q = 2$ at first-order critical points; however, the former would allocate more weights to unsmooth stencils and higher resolution would be usually achieved. The distributions in Fig. 26 indicate the advantage of accuracy order would sometimes exceed the effect of nonlinear techniques. Unsurprisingly, WENO5 indicates a poor resolution and almost fails to tell downstream fluctuations, which is the same as that in Refs. 10–11.

(3) Results of WENO7-PRM$^{4,3,3}$ and Corresponding Seventh-Order Comparatives

(a) Strong Shock Wave

Fig. 27 Density distributions of strong shock wave with initial pressure ratio $PR = 10^6$ at $t = 0.01$ on 200 grids by WENO7-PRM$^{4,3,3}$ with the comparisons by WENO7, WENO7-PM$^{8,6,1}$ and -RM$^{7,6,2,0}$.
As in previous scalar problems, similar mappings are chosen to compare with PRM$^{4,3,3}$, i.e., PM$^{8,6,1}$, RM$^{7,6,2,0}$ and AIM$^{5,4,2,1E4-M}$. The use of AIM$^{5,4,2,1E4-M}$ other than AIM$^{5,4,2,1E4}$ is to follow the suggestion of Ref. 11 that the former will behave more robustly. And as usual, canonical WENO7 is included. It is surprising to note that WENO7-AIM$^{5,4,2,1E4-M}$ fails to accomplish the computation at both pressure ratios because of the blow up, which indicates less robustness despite of its good performance in previous scalar computations. The other schemes fulfill the test, and the results with $PR = 10^6$ are shown in Fig. 27. The figure shows that the schemes yield similar results in which the density platform is still not well resolved.

Because IM$^{3,2,0,0,1}$ had been integrated with WENO7 for computations in Refs. [10, 11], we also have WENO7-IM$^{3,2,0,0,1}$ tested in this case. It is found the scheme cannot fulfill the computation because of the blow-up, which indicates its risk of insufficient robustness in computations.
(b) **Blast Wave**

Besides WENO7-PRM$^4_{3,3}$, similar comparatives are chosen as above. WENO7-AIM$^3_{4,2;1E4}$-M fails in the computation again due to the blow up, which further confirms its relatively inferior robustness. The other schemes yield similar results as shown in Fig. 28 with details in zoomed view.

Considering the failure of WENO7-IM$^3_{2,0;0,1}$ in the previous case, we test the scheme in this problem also. It turns out that the scheme can not accomplish the computation because of the blow-up, which reminds again the potential of insufficient robustness.

(c) **Shu-Osher Problem**

To compare with WENO7-PRM$^4_{3,3}$, similar comparatives are adopted for testing as above, and corresponding results are shown Fig. 29. The figure tells that WENO7-PRM$^4_{3,3}$, RM$^7_{6,2,0}$ and AIM$^5_{4,2;1E4}$-M yield results with similar resolution, whereas the result of WENO7-PM$^8_{6,1}$ appears slightly dissipated.

Likewise, we have WENO7-PRM$^4_{3,3}$ and comparatives tested in Titarev-Toro problem. In the scenario of seventh-order schemes, the performance of WENO7-PRM$^4_{3,3}$, RM$^7_{6,2,0}$ and AIM$^5_{4,2;1E4}$-M are similar, whereas WENO7-PM$^8_{6,1}$ behaves relatively dissipative. To save space, this part is shown in Appendix 5.

### 6 Conclusions

Comprehensive and intensive investigations are carried out regarding piecewise rational mapping, and the new PRM method is proposed with sufficient regulation capability. Through theoretical analysis and numerical tests, the following conclusions are drawn:

1. The so-called $C_{n,m}/C_{n,m,l}$ condition is summarized to develop rational mapping, which incarnates the favorable properties of an ideal mapping. In $C_{n,m}$, the pattern of endpoint convergence to make mapped schemes behave as WENO, which was first proposed by Ref. 9, is utilized in this paper. On the one hand, numerical example indicates the pattern with WENO correspondence will yield performance similar to that by patterns with ENO correspondence [7,10], on the other hand, the former has no restrictions on the choice of $n$ and $m$ of $C_{n,m}$ as the latter does in the case of RM$^{n+1}_{n,m,l}$.

2. Numerical examples indicate that the piecewise implementation of mapping does not necessarily entail numerical instability in high-order mapped WENO (e.g., mapped WENO9), which is contrary to the conclusion in Ref. 10.

3. By upgrading the smoothness indicators of WENO3 to that of WENO5 as in Ref. 13, WENO3 can preserve its optimal order through mapping in the occurrence of first-order critical points, which is thought to be unavailable in Refs. 3, 7–8 and 10–11.

4. New piecewise rational mapping, PRM$^{n+1}_{n,m}$, is proposed. The mapping can explicitly and simultaneously achieve the desired flatness and performance of endpoint convergence by means of sufficient regulation capability. As an example of the capability, the particular PRM$^3_{2,2}$ can resemble two distinct mappings, namely PM$^8_{6,1}$ [7] and RM$^7_{6,2,0}$ [10], by specific choices of parameters. Two theorems are provided with proofs to guarantee PRM$^{n+1}_{n,m}$ satisfy $C_{n,m}$ and to define the valid range of parameters of the mapping. As a byproduct, the general form of PPM$^{n+m+1}_{n,m}$ is proposed and one theorem is provided with proof to guarantee $C_{n,m}$ satisfied. Three concrete ones, i.e., PRM$^3_{1,1}$, PRM$^3_{2,2}$ and PRM$^3_{3,3}$, are determined for {WENO3, 5, 7} with parameters.
specified, and corresponding WENO3-PRM$^2_{1,1}$, WENO5-PRM$^3_{2,2}$, WENO7-PRM$^4_{3,3}$ are obtained.

(5) Regarding mappings for WENO3, although PRM$^2_{1,1}$ does not have a larger Flatness-I than $g_M$ and IM$^3_{2,0,0,1}$ in the neighborhoods of linear weights, it has the largest Flatness-II among comparatives as well as the best performance of endpoint convergence. Numerically, WENO3-PRM$^2_{1,1}$ not only can preserve the third-order accuracy in the occurrence of first-order critical points but also shows the quickest rate of order convergence among comparatives. In computations regarding robustness and stability, WENO3-PRM$^2_{1,1}$ shows one of the best performance whereas WENO3-P + 3 fails in tests of strong shock; besides, WENO3-PRM$^2_{1,1}$ shows a fine resolution in Shu-Osher problem on 230 grids, which slightly surpasses that of WENO3-IM$^3_{2,0,0,1}$. Hence WENO3-PRM$^2_{1,1}$ indicates the overall advantages on accuracy, robustness and resolution.

(6) Regarding mappings for WENO5, PRM$^3_{2,2}$ shows excellent Flatness-II and satisfactory performance of endpoint convergence concurrently among comparatives such as $g_M$, IM$^3_{2,0,0,1}$, PM$^8_{6,1}$ and RM$^7_{6,2,0}$, moreover, the drawbacks such as excessive Flatness-II and poor performance of endpoint convergence in the latter three are overcome. Numerically, WENO5-PRM$^3_{2,2}$ shows one of the quickest rates of order convergence with the preservation of the fifth-order accuracy at first-order critical points. Regarding the stability of the long-time computation, WENO5-PRM$^3_{2,2}$ indicates a similar performance as that of PM$^8_{6,1}$ and RM$^7_{6,2,0}$. A testing R$^3_{2,2}$ which has similar Flatness-II as that of PRM$^3_{2,2}$ produces oscillations in the long-time computation, which validates the necessity of extra implementation in PRM$^3_{2,2}$ to enforce numerical stability. In computations regarding robustness and stability, WENO5-PRM$^3_{2,2}$ shows satisfactory performance, in the meanwhile the scheme indicates one of the best resolutions in Shu-Osher and Titarev-Toro problems, whereas WENO5-NIS fails in the computation. Hence WENO5-PRM$^3_{2,2}$ indicates the overall advantages on accuracy, robustness and resolution. It is worth mentioning that AIM$^5_{4,2;1E4}$ is found to fail to work with WENO5.

(7) Regarding mappings for WENO7, PRM$^4_{3,3}$ shows the profile with well-performed Flatness-II and satisfactory endpoint convergence in balance among comparatives such as PM$^8_{6,1}$ and RM$^7_{6,2,0}$. Numerically, WENO7-PRM$^4_{3,3}$ shows a similar quick rate of order convergence with the preservation of seventh-order accuracy at first- and second-order critical points as WENO7-PM$^8_{6,1}$, -RM$^7_{6,2,0}$ and -AIM$^5_{4,2;1E4}$. In the long-time computation, WENO7-PRM$^4_{3,3}$ indicates similar performance as that of WENO7-PM$^8_{6,1}$ and -RM$^7_{6,2,0}$, while WENO7-AIM$^5_{4,2;1E4}(-M)$ shows tiny improvements. Similar situations occur in computations of Shu-Osher and Titarev-Toro problems. However, WENO7-AIM$^5_{4,2;1E4}(-M)$ indicates obviously less robustness because of its failure in strong shock and blast wave problems. In summary, according to our tests, WENO7-PRM$^4_{3,3}$ shows the performance similar to that of WENO7-RM$^7_{6,2,0}$, while the profile of PRM$^4_{3,3}$ indicates more advantage on stability potentially due to its improved performance of endpoint convergence.

(8) Although both WENO5-IM$^3_{2,0,0,1}$ and WENO7-AIM$^5_{4,2;1E4}$ perform quite well in the long-time computation of scalar combination-waves advection, the latter fails in both strong shock and blast wave problems, while WENO7-IM$^3_{2,0,0,1}$ fails in two problems either. Such consequences indicate that the stability in the long-time computation does not necessarily entail the conventional stability/robustness which is essential for applications.
In spite of parameter recommendations of PRM for \{WENO3, 5, 7\}, they might not be the most appropriate. The potential users can still make adjustments according to needs of applications.

Appendix 1

For reference, the coefficients of candidate schemes, linear weights, and coefficients of smoothness indicators of WENO which correspond to Eqns. (3) - (5) are tabulated in Tables 13, 14, 15, 16 and 17.

Considering the occurrence of critical point with the order $n_{cp}$, suppose $r_c$-WENO is corresponding order of WENO and $r_c$-WENO-M is that of WENO-M. According to Ref. 3, the above orders together with $n$ in $C_{n,m,l}$ required for order preservation can be tabulated in Table 17 for WENO with order $(2r-1)$ from $r = 2$ to 5.

It is worthwhile to mention that when the upgradation of smoothness indicator described in Sect. 3.1 is employed for WENO3, its optimal order could be recovered by asking $n$ in $C_{n,m,l}$ being at least 1.

### Table 13
Coefficients $a'_{kl}$ of candidate schemes $q'_{k}$ of WENO with $r = 2–5$ [2]

| $r$ | $k$ | $d'_{k0}$ | $d'_{k1}$ | $d'_{k2}$ | $d'_{k3}$ | $d'_{k4}$ |
|-----|-----|-----------|-----------|-----------|-----------|-----------|
| 2   | 0   | −1/2      | 3/2       | −         | −         | −         |
|     | 1   | 1/2       | 1/2       | −         | −         | −         |
| 3   | 0   | 2/6       | −7/6      | 11/6      | −         | −         |
|     | 1   | −1/6      | 5/6       | 2/6       | −         | −         |
|     | 2   | 2/6       | 5/6       | −1/6      | −         | −         |
| 4   | 0   | −3/12     | 13/12     | −23/12    | 25/12     | −         |
|     | 1   | 1/12      | −5/12     | 13/12     | 3/12      | −         |
|     | 2   | −1/12     | 7/12      | 7/12      | −1/12     | −         |
|     | 3   | 3/12      | 13/12     | −5/12     | 1/12      | −         |
| 5   | 0   | 12/60     | −63/60    | 137/60    | −163/60   | 137/60    |
|     | 1   | −3/60     | 17/60     | −43/60    | 77/60     | 12/60     |
|     | 2   | 2/60      | −13/60    | 47/60     | 27/60     | −3/60     |
|     | 3   | −3/60     | 27/60     | 47/60     | −13/60    | 2/60      |
|     | 4   | 12/60     | 77/60     | −43/60    | 17/60     | −3/60     |

### Table 14
Linear weights $d_k$ of WENO with $r = 2–5$ [2]

| $r$ | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ |
|-----|-------|-------|-------|-------|-------|
| 2   | 1/3   | 2/3   | −     | −     | −     |
| 3   | 1/10  | 6/10  | 3/10  | −     | −     |
| 4   | 1/35  | 12/35 | 18/35 | 4/35  | −     |
| 5   | 1/126 | 20/126| 60/126| 40/126| 5/126 |
Table 15 Coefficients $b'_{km}$ of smoothness indicators of WENO in Eq. (5) with $r=2–5$ [2,10–12]

| $r$ | $k$ | $m$ | $b'_{km0}$ | $b'_{km1}$ | $b'_{km2}$ | $b'_{km3}$ | $k$ | $m$ | $b'_{km0}$ | $b'_{km1}$ | $b'_{km2}$ |
|-----|-----|-----|------------|------------|------------|------------|-----|-----|------------|------------|------------|
| 2   | 0   | 0   | −1         | 1          | −          | −          |     |     |            |            |            |
| 1   | 0   | −1  | 1          | −          | −          |            |     |     |            |            |            |
| 3   | 0   | 0   | 1          | −4         | 3          | −          | 2   | 0   | 3          | −4         | 1          | −          |
|     | 1   | −2  | 1          | −          | −          |            | 1   | 1   | −2         | 1          | −          |
| 1   | 0   | −1  | 0          | 1          | −          | −          |     |     |            |            |            |
| 1   | 1   | −2  | 1          | −          | −          |            |     |     |            |            |            |
| 4   | 0   | 0   | −2         | 9          | −18        | 11         | 2   | 0   | −2         | −3         | 6          | −1         |
|     | 1   | −1  | 4          | −5         | 2          | −          | 1   | 1   | −2         | 1          | 0          | −          |
|     | 2   | −1  | 3          | −3         | 1          | −          | 2   | −1  | 3          | −3         | 1          | −          |
| 1   | 0   | −6  | 3          | 2          | −          | 3          | 0   | −11 | 18         | −9         | 2          | −          |
|     | 1   | 0   | 1          | −2         | 1          | −          | 1   | 2   | −5         | 4          | −1         | −          |
|     | 2   | −1  | 3          | −3         | 1          | −          | 2   | −1  | 3          | −3         | 1          | −          |
| 5   | 0   | 0   | −16        | 36         | −48        | 25         | 3   | 0   | 18         | 10         |
|     | 1   | 119 | −606       | 1234       | −112       | 379        | 1   | 119 | −216      | 64         |
|     | 2   | 3   | −14        | 24         | −18        | 5          | 2   | 3   | −10       | 12         |
|     | 3   | −4  | 6          | −4         | 1          | 3          | 1   | −4  | 6          |
| 1   | 0   | −6  | 18         | −10        | −3         | 4          | 0   | 25  | −48       |
|     | 1   | 119 | −64        | 216        | −119       | 1          | 379 | −112 | 1234       |
|     | 2   | −6  | 12         | −10        | 3          | 2          | 5   | −18 | 24         |
|     | 3   | −4  | 6          | −4         | 1          | 3          | 1   | −4  | 6          |
|     | 2   | 0   | −8         | 0          | 8          | −1         | 1   | 119 | −174      |
|     | 2   | −2  | 0          | 2          | −1         |            | 1   | −4  | 6          |

Table 16 Coefficients $c'_n$ of smoothness indicators of WENO in Eq. (5) with $r=2–5$ [2,10–12]

| $r$ | $c'_0$ | $c'_1$ | $c'_2$ | $c'_3$ |
|-----|--------|--------|--------|--------|
| 2   | 1      | −      | −      | −      |
| 3   | 1/4    | 13/12  | −      | −      |
| 4   | 1/36   | 13/12  | 781/720| −      |
| 5   | 1/144  | 13/202800| 781/2880| 1,421,461/1310400|
Appendix 2

Given the mapping \( R_{L,n}^{L,n+1} \) in \([d_k, 1]\) in Eq. (11), the symmetric \( R_{L,n}^{L,n+1} \) can be derived by

\[
R_{L,n}^{L,n+1} = \frac{d_k}{1-d_k} \left[ 1 - R_{n,m+1}^{R,n+1}(1 - \frac{1-d_k}{d_k} \omega) \right]
\]

in the form as Eq. (12). Corresponding coefficients \( c_{L,n,m,i} \) except \( c_{L,n,m,4} \) are tabulated in Table 18 for \( m, n \leq 5 \). Almost all \( c_{L,n,m,4} \) equals zero except \( c_{L,4,2,4} = -d_k \left( \frac{b}{1-d_k} - 4 \right) \).

Considering \( R_{n,m}^{R,n+1} \) by Eq. (11), \( \partial \omega^{n+1} R_{n,m}^{R,n+1}(d_k) / \partial \omega^{n+1} \) and \( \partial \omega^{n+1} R_{n,m}^{R,n+1}(1) / \partial \omega^{n+1} \) can be derived and tabulated in Table 19.

### Table 17 Relationship of \( r_c \)-WENO, \( r_c \)-WENO-M and \( n \) in \( C_{n,m,l} \) in terms of \( r \) and \( n_{ip} \)

| \( r \) | \( n_{ip} \) | \( r_c \)-WENO | \( r_c \)-WENO-M | \( n \) in \( C_{n,m,l} \) |
|---|---|---|---|---|
| 2 | 0 | 3 | - | - |
| | 1 | 1 | N/A | N/A |
| 3 | 0 | 5 | - | - |
| | 1 | 3 | 5 \( \geq 2 \) | - |
| | 2 | 2 | 2 | N/A |
| 4 | 0 | 7 | - | - |
| | 1 | 5 | 7 \( \geq 2 \) | - |
| | 2 | 4 | 6 \( \geq 3 \) | - |
| | 3 | 3 | 3 | N/A |
| 5 | 0 | 9 | 9 | - |
| | 1 | 7 | 9 \( \geq 2 \) | - |
| | 2 | 6 | 9 \( \geq 2 \) | - |
| | 3 | 5 | 7 \( \geq 4 \) | - |
| | 4 | 4 | 4 | N/A |

### Table 18 Coefficients \( c_{L,n,m,i} \) except \( c_{L,n,m,4} \) of mapping \( R_{n,m}^{L,n+1} \)

| \( n \) | \( m = 0 \) | \( m = 1 \) | \( m = 2 \) | \( m = 3 \) | \( m = 4 \) |
|---|---|---|---|---|---|
| 1 | \( i = 1 \) | 0 | 1 | - | - |
| | \( i = 2 \) | \( b \) | \( \frac{-b}{1-b-d_k} \) | - | - |
| | \( i = 3 \) | -1 | 0 | - | - |
| 2 | \( i = 1 \) | 0 | -2d_k | 1 | - |
| | \( i = 2 \) | \( \frac{-b_d}{1-d_k} \) | \( b \) | \( b \frac{b_d-b}{d_k(1-b-d_k)} \) | - |
| | \( i = 3 \) | 1 | -1 | 0 | - |
| 3 | \( i = 1 \) | 0 | 3d^2_k | 1 | 1 | - |
| | \( i = 2 \) | \( \frac{b_d^2}{(1-d_k)^3} \) | \( \frac{-b_d}{1-d_k} \) | \( b-1 \) | \( \frac{b}{1-b-d_k} \) | - |
| | \( i = 3 \) | -1 | 2 | 0 | 0 | - |
| 4 | \( i = 1 \) | 0 | -4d^3_k | 1 | 1 | 1 |
| | \( i = 2 \) | \( \frac{-b_d^3}{(1-d_k)^3} \) | \( \frac{b_d}{(1-d_k)^3} \) | \( -1 \) | \( \frac{-b(1-d_k)}{d_k(1-b-d_k)} \) | - |
| | \( i = 3 \) | 1 | -3 | 0 | 0 | 0 |
Table 19 $R^{R,n+1(α+1)}_{n,m}(d_k)$ and $R^{R,n+1(α+1)}_{n,m}(1)$ with $d_{1k}$ denoting $(1-d_k)$

| $m$  | $m=0$ | $m=1$ | $m=2$ | $m=3$ | $m=4$ |
|------|-------|-------|-------|-------|-------|
| $R^{2,2}_{2,1}(d_k)$ | $\frac{2}{c_{1,2}^{n+1}d_{1k}}$ | $\frac{2}{c_{1,2}^{n+2}}$ | $-2\frac{d_{1k}}{c_{1,2}^{n+2}}$ | $-\frac{d_{1k}}{c_{1,2}^{n+2}}$ | $-\frac{d_{1k}}{c_{1,2}^{n+2}}$ |
| $R^{2,2}_{1,1}(1)$ | $\frac{6}{c_{2,2}^{n+1}d_{1k}+d_{1k}}$ | $\frac{6}{c_{2,2}^{n+2}}$ | $\frac{6}{c_{2,2}^{n+2}}$ | $\frac{6}{c_{2,2}^{n+2}}$ | $\frac{6}{c_{2,2}^{n+2}}$ |
| $R^{3,3}_{2,1}(1)$ | $\frac{\frac{n}{c_{2,2}^{n+1}+3d_{1k}}}{d_{1k}}$ | $\frac{n}{c_{2,2}^{n+2}}$ | $\frac{n}{c_{2,2}^{n+2}}$ | $\frac{n}{c_{2,2}^{n+2}}$ | $\frac{n}{c_{2,2}^{n+2}}$ |
| $R^{4,4}_{3,1}(1)$ | $\frac{\frac{120}{c_{2,2}^{n+1}d_{1k}+d_{1k}}}{d_{1k}}$ | $\frac{120}{c_{2,2}^{n+2}}$ | $\frac{120}{c_{2,2}^{n+2}}$ | $\frac{120}{c_{2,2}^{n+2}}$ | $\frac{120}{c_{2,2}^{n+2}}$ |
| $R^{3,3}_{1,1}(1)$ | $\frac{\frac{n}{c_{2,2}^{n-1}+4d_{1k}}}{d_{1k}}$ | $\frac{n}{c_{2,2}^{n+2}}$ | $\frac{n}{c_{2,2}^{n+2}}$ | $\frac{n}{c_{2,2}^{n+2}}$ | $\frac{n}{c_{2,2}^{n+2}}$ |
| $R^{4,4}_{4,1}(1)$ | $\frac{\frac{120}{c_{2,2}^{n+1}d_{1k}+d_{1k}}}{d_{1k}}$ | $\frac{120}{c_{2,2}^{n+2}}$ | $\frac{120}{c_{2,2}^{n+2}}$ | $\frac{120}{c_{2,2}^{n+2}}$ | $\frac{120}{c_{2,2}^{n+2}}$ |

Appendix 3

The following proofs are provided for proposed theorems.

**Theorem 1** Considering a function as $f(ω) = d_k + \frac{1}{(1-d_k)^{n+1}}(ω - d_k)^{n+1} \sum_{j=0}^{m} a_i^{n}(1-ω)^{m-j}(1-d_k)^j$ for $[d_k, 1]$ where $a_i^{m} = \prod_{j=0}^{n-1-i} (n+j)$ for $i < m$ and $a_i^{m} = 1$, $f(ω)$ satisfies $C_{n,m}$.

**Proof** It is trivial that $f(d_k) = d_k$ and $f(1) = 1$. Let $A = \frac{1}{(1-d_k)^{n+1}}$, $u = (ω - d_k)^{n+1}$ and $v = \sum_{j=0}^{m} a_i^{n} (1-ω)^{m-j}(1-d_k)^j$, then $f(ω) = d_k + Auv$. Using Leibnitz law, $f(ω)^{(k)} = A \sum_{j=0}^{m} C_k^{(k)} u^{(j)} v^{(k-j)}$. Subsequently, $f(ω)^{(k)}$ at two endpoints will be discussed respectively.

1) $f(d_k)^{(k)}$. Because $u^{(k)}(ω) = \left[ \prod_{i=0}^{k-1} (n+1-i) \right] (ω - d_k)^{n-k+1}$ when $1 ≤ k ≤ n$, then $f(d_k)^{(k)} = 0$ for the prescribed $k$; moreover, considering $u^{(n+1)}(d_k) = \prod_{i=0}^{k-1} (n+1-i) ≠ 0$ and $v(d_k) = \sum_{j=0}^{m} a_i^{n} (1-d_k)^{m-j}(1-d_k)^j ≠ 0$, $f(d_k)^{(n+1)} = (A(u^{(n+1)}v)^{(n+1)})_{ω=d_k} ≠ 0$ for $1 ≤ m$.

2) $f(1)^{(k)}$. Because $v(k)(ω) = (-1)^k \sum_{i=0}^{m-k} \prod_{j=0}^{k-1} (n+j)(1-ω)^{m-k-i}(1-d_k)^j$ when $1 ≤ k ≤ m$, then $v^{(k)}(1) = (-1)^k \sum_{i=0}^{m-k} (n+j)(1-d_k)^{m-k-i}$ for the prescribed $k$. Consequently, one can find that $f'(1) = A(u'v + uv') |_{ω=1} = 1$. When $1 < k ≤ m$,
Through symbolic operation, the term within the bracket turns out to be zero for given \( k \geq 2 \), which stands even at \( k = m + 1 \), and therefore \( f(1)^{(k)} = 0 \). When \( k = m + 1 \), it is trivial \( \psi^{(m+1)}(\omega) = 0 \), and

\[
 f(1)^{(m+1)} = A \sum_{p=0}^{m+1} C_k^p u^{(p)}(\omega^{(m+1-p)}) \bigg|_{s=1} = A \left( u^{(k)} + \sum_{p=1}^{k-1} C_k^p u^{(p)}(\omega^{k-p}) + u^{(k)} \right) \bigg|_{s=1}
\]

\[
= \frac{1}{(1-d_k)^{m+m}} \left[ (1-d_k)^{m+1}(-1)^m \prod_{i=0}^{m-1} (n+1-i)(1-d_k)^{n-p+1}(-1)^{m-p}(n+j)(1-d_k)^{j-1} \right. \\
+ \sum_{p=1}^{k-1} C_k^p \left. \prod_{i=0}^{p-1} (n+1-i)(1-d_k)^{n-p+1}(1-d_k)^m \right] \\
= \frac{1}{(1-d_k)^m} \left[ (1-d_k)^{m+1}(-1)^m \prod_{i=0}^{m-1} (n+1-i)(1-d_k)^{n-p+1}(1-d_k)^m \right.
\]

Comparing the last term in above \( f(1)^{(m+1)} \) with the last term in \( f(1)^{(k)} \) at \( k = m + 1 \), one can find that \( f(1)^{(m+1)} = \frac{1}{(1-d_k)^m}(-1)^m \prod_{j=0}^{m} (n+j) \neq 0 \).

In short, \( f(\omega) \) satisfies \( C_{n,m} \) condition.

**Theorem 2** Consider a mapping as \( R_{n,m}^{n+1} = d_k + \frac{(\omega-d_k)^{n+1}}{(\omega-d_k)^{n+1} + c(1-\omega)^{m+1}} \) in \([d_k, 1]\) with \( 0 < d_k < 1 \) and \( n, m \geq 1 \). \( c > 0 \) is the sufficient and necessary condition for \( R_{n,m}^{n+1} \) to be out of singularity.

**Proof** Other than targeting at the direct problem, we first study the solution of \( c \) through which the denominator in \( R_{n,m}^{n+1} \) has zero value. Consider \((\omega-d_k)^n + c(1-\omega)^{m+1} = 0\) at \( \omega \in [d_k, 1] \), which actually defines a function \( c(\omega) \) as \( c = -\frac{(\omega-d_k)^n}{(1-\omega)^{m+1}} \). Because \( \frac{dc(\omega)}{d\omega} = -\frac{(\omega-d_k)^{n-1}}{(1-\omega)^{m+2}} \left[ n(1-\omega) + (m+1)(\omega-d_k) \right] \), then \( \frac{dc(\omega)}{d\omega} > 0 \) at \( \omega \in [d_k, 1] \) for \( 0 < d_k < 1 \) and \( n, m \geq 1 \). Hence the function implied in the solution is monotone and there is one-to-one correspondence between \( c \) and \( \omega \) at \( \omega \in [d_k, 1] \). Considering \( c(d_k) = 0 \) and \( c(1) = -\infty \), the range of \( c \) turns out to be \((-\infty, 0]\). In other words, for any \( c \in (-\infty, 0] \), there is a \( \omega \in [d_k, 1] \) such that \((\omega-d_k)^n + c(1-\omega)^{m+1} = 0\). Hence the range of \( c \) such that \((\omega-d_k)^n + c(1-\omega)^{m+1} \neq 0 \) at \( \omega \in [d_k, 1] \) is \( c > 0 \).
Theorem 3 Consider a function as \( f(\omega) = d_k + \frac{(\omega-d_k)^{n+1}}{(\omega-d_k)^n+c_2(\omega-d_k)^{n+1}(1-\omega)^{m_1}+c_1(1-\omega)^{m+1}} \) in \([d_k, 1]\) where \( n \geq 1 \), and \( \{c_1, c_2\} \) have the same sign. If \( m \geq 0 \) and \( m_1 \geq 1 \), then \( f(\omega) \) satisfies \( C_{n,\min(m,m_1-1)} \).

Prior to the proof of the theorem, the following lemma is proposed and proved.

Lemma
Supposing a function \( f(\omega) \) which is defined as \( f(\omega) = (\omega - a)^N \varphi(\omega) \) with \( \varphi(a) \neq 0 \) and \( N \geq 1 \), then \( f(\omega) \) satisfies \( f^{(i)}(a) = \left\{ \begin{array}{ll} 0, & 0 \leq i < N \ \text{N!}\varphi(a), & i = N \end{array} \right. \)

Proof
It trivial that \( f^{(0)}(a) = 0 \). Using Leibnitz law, \( f^{(i)}(\omega) = \sum_{k=0}^{i} C_k^i \left[ (\omega - a)^N \right]^{(k)} \varphi^{(i-k)}(\omega) \) with \( 0 < i \leq N \). Because \( \left[ (\omega - a)^N \right]^{(k)} \varphi^{(i-k)}(\omega) \) then \( \left[ (\omega - a)^N \right]^{(k)} \varphi^{(i-k)}(\omega) \) of \( a = 0, 0 < k < N \). \( N! \). \( k = N \).

Furthermore, when \( 1 \leq i < N \), \( f^{(i)}(a) = \left( \sum_{k=0}^{i} C_k^i \left[ (\omega - a)^N \right]^{(k)} \varphi^{(i-k)}(\omega) \right) \) of \( a = 0, 0 \leq i < N \).

When \( i = N \), \( f^{(N)}(a) = \left( C_N^N \left[ (\omega - a)^N \right] \varphi^{(N-N)}(\omega) \right) \) of \( a = 0, N! \varphi(a) \). Hence,
\[
\begin{align*}
f^{(i)}(a) = \left\{ \begin{array}{ll}
0, & 0 \leq i < N \\
N! \varphi(a), & i = N \end{array} \right.
\end{align*}
\]

Based on the above Lemma, the proof of Theorem 3 is given as follows.

Proof of Theorem 3
The following steps are taken:

1. It is trivial that \( f(d_k) = d_k \) and \( f(1) = 1 \) under \( n \geq 1 \), \( m \geq 0 \) and \( m_1 \geq 1 \). It is worth noting that \( f(\omega) \) satisfies the zero-order condition at \( \omega = 1 \).

2. Suppose \( \varphi(\omega) = (\omega - d_k)^m + c_2(\omega - d_k)(1 - \omega)^{m_1} + c_1(1 - \omega)^{m+1} \). Because \( \varphi(d_k) = c_1(1 - d_k)^{m+1} \neq 0 \), then \( \left( f(\omega) - d_k \right)^{(i)} \) of \( a = d_k \) then \( \left( \frac{(n+1)!}{c_1(1-d_k)^{m+1}}(\neq 0), i = n + 1 \right) \) according to above Lemma.

3. Through symbolic operation, the first-order differentiation of \( f(\omega) \) can be derived as:
\[
f'(\omega) = (f(\omega) - d_k) \left[ \frac{n + 1}{\omega - d_k} - \frac{(n(\omega - d_k)^n - c_2(\omega - d_k)^{n+1}(1 - \omega)^{m_1} \times (n_1(1 - \omega) + m_1(\omega - d_k)) - c_1(m + 1)(1 - \omega)^m)}{(\omega - d_k)^n + c_2(\omega - d_k)^{n+1}(1 - \omega)^{m_1} + c_1(1 - \omega)^{m+1}} \right].
\]

Consider the situation where \( m \geq 1 \). Then if \( m_1 = 1 \) or/and \( m = 0 \),
\[
f'(1) = \left\{ \begin{array}{ll}
1 + c_2(1 - d_k)^{1+n_1-n}, & m_1 = 1, m \geq 1 \\
1 + c_1(1 - d_k)^{1-n}, & m = 0, m_1 \geq 2; \text{if } m_1 \geq 2 \text{ and } m \geq 1,
\end{array} \right.
\]
\[
h_0 = \left\{ \begin{array}{ll}
1 + c_2(1 - d_k)^{1+n_1-n} + c_1(1 - d_k)^{1-n}, & m_1 = 0, m = 1 \\
1 + c_2(1 - d_k)^{1+n_1-n} + c_1(1 - d_k)^{1-n}, & m_1 = 1, m \geq 1
\end{array} \right.
\]
\[
f'(1) = n + 1 - n = 1. \]

We further derive the second-order derivatives of \( f(\omega) \), which can be formulated as:
\[
f''(\omega) = \frac{h(\omega)}{\left[(\omega - d_k)^n + c_2(\omega - d_k)^{n+1}(1 - \omega)^{m_1} + c_1(1 - \omega)^{m+1}\right]^3}.
\]
where \( h(\omega) = \sum (1 - \omega)^{n_h} R_i \) and \( R_i \) denotes (rational) polynomial. We have testified that when \( R_i \) is rational polynomial, its denominator has the form as \((\omega - d_k)^{n_h}\) with \( n_R \) as a certain integer. The exponent \( n_h \) takes values within \{\( m-1, m, 2m-1, 2m; 2m_1-2, 2m_1-1, 2m_1; m_1, m_1+m-2, m_1+m-1; m_1-2, m_1-1, m_1 \}\). It can be seen the minimum of \( n_h \) will be either \( m-1 \) or \( m_1-2 \), and therefore:

(i) If \( m_1-2 \geq m-1 \) or \( m_1 \geq m-1 \), the exponent \( n_h \geq m-1 \), and \( f''(\omega) \) can be re-formulated as \( f''(\omega) = (1 - \omega)^{m-1} \varphi(\omega) \) where \( \varphi(\omega) \neq 0 \). So \( f''(1) = \begin{cases} \varphi(1), & m = 1 \\ 0, & m \geq 2 \end{cases} \) Let

\[
F(\omega) = (-1)^{m-1} f''(\omega) = (\omega - 1)^{m-1} \varphi(\omega),
\]

then

\[
F^{(i)}(1) = \begin{cases} i = 1 & 0 \leq i \leq m-2 \\ i = m - 1 & 0 \leq i \leq m \end{cases}
\]

if \( m \geq 2 \) according to the previous Lemma and \( F(1) = \varphi(1) \) if \( m = 1 \). One can find by derivation that if \( m_1 = m+1 \), \( \varphi(1) = (-1-d_k)^{2m+1} c_1(m+m^2) + (1-d_k)^{2m+1+n_1} c_2(m_1 - m^2) \), and if \( m_1 > m+1 \), \( \varphi(1) = (-1-d_k)^{2m+1} c_1(m+m^2) \). Therefore if \( m \geq 1 \) and \( \{c_1, c_2\} \) have the same sign, \( \varphi(1) \neq 0 \), and so \( F^{(i)}(1) = \begin{cases} (-1)^{m-1} F^{(i-2)}(1) = 0, & 2 \leq i \leq m \\ (-1)^{m-1} F^{(m-1)}(1) \neq 0, & i = m \end{cases} \) if \( m \geq 2 \) and \( f''(1) = (-1)^{m-1} F(1) = \varphi(1) \) if \( m = 1 \). In summary, if \( c_1 \) and \( c_2 \) have the same sign,

\[
f^{(i)}(1) = \begin{cases} 1 & i = 1 \\ 0 & 2 \leq i \leq m \text{ if } m \geq 2 \text{ and } f''(1) = (-1)^{m-1} F(1) = \varphi(1) \text{ if } m = 1. \\ \neq 0 & i = m + 1 \end{cases}
\]

(ii) If \( m_1-2 < m-1 \) under \( m_1-3 \geq 0 \), or \( m+1 > m_1 \geq 3 \), the minimum of \( n_h \) will be \( m_1-2 \). Following similar procedures as above, one can find:

\[
\varphi(1) = (1-d_k)^{2m+1+n_1} c_2(m_1 - m^2).
\]

Therefore if \( m \),

\[
m_1 \geq 3, f^{(i)}(1) = \begin{cases} 1 & i = 1 \\ 0 & 2 \leq i \leq m_1 - 1. \end{cases}
\]

When \( m_1 = 2, n_h = 0, \) and then

\[
f^{(i)}(1) = \begin{cases} 1 & i = 1 \\ \neq 0 & i = 2 \end{cases}
\]

In short, the given \( f(\omega) \) will satisfy \( C_{n,\min(m, m_1-1)} \) if \( m \geq 0 \) and \( m_1 \geq 1 \).

### Appendix 4

In order to explore the effects of \( c_1, c_2, n_1, m_1 \) in \( PRM^{L/R} \), parametric studies of \( PRM^{R,n+1}_{n,m,n_1,m_1,c_1,c_2} \) is made qualitatively, and that of \( PRM^{L,n+1}_{n,m,n_1,m_1,c_1,c_2} \) can be understood accordingly. Specifically, \( PRM^{R,3}_{2,2,n_1,m_1,c_1,c_2} \) and its profile regarding \( d_k = 3/10 \) are chosen for illustration. For convenience, the whole or part of super—and/or sub-scripts of \( PRM \) and \( c_1/c_2 \) are omitted sometimes.

First, the effect of \( c_1 \) is discussed. As just mentioned, \( c_1 \) should be positive in order to make mappings free of singularity. A series of \( c_1 \) is taken as \{1, 10, 100\} under \( c_2 = 10, n_1 = 1 \) and \( m_1 = 5 \) and corresponding profiles are shown in Fig. 30a. One can see the decrease of \( c_1 \) would yield less flatness of mapping around \( \omega = d_k \) and the increased convergence to identity mapping around \( \omega = 1 \), and vice versa. Next, the effect of \( c_2 \) is studied by taking values \{1, 10^2, 10^6\} under the choice \( c_1 = 1, n_1 = 1 \) and \( m_1 = 5 \) with results shown in Fig. 4b. The figure tells the increase of \( c_2 \) will extend the flatness and accelerate
endpoint convergence by pushing the transition position of PRM to identity mapping toward endpoints, and vice versa. Thirdly, the effect of $n_1$ is explored by taking $\{1, 4, 10\}$ under $c_1 = 1$, $c_2 = 100$ and $m_1 = 2$, and corresponding results are shown in Fig. 30c. One can see that the enlargement of $n_1$ will decrease the flatness of mapping, however desirable improvement on endpoint convergence is not achieved. Based on this observation and in save of cost, $n_1$ is suggested to take 1 next. At last, the effect of $m_1$ is studied by taking $\{2, 3, 5\}$ under $c_1 = 1$, $c_2 = 100$ and $n_1 = 1$. Theoretically $m_1$ controls how fast the influence of $c_2(\omega - d_k)^{n_1}(1 - \omega)^{m_1}$ will vanish when approaching $\omega = 1$. One can observe that on the one hand $m_1$ can adjust the flatness (i.e., the smaller the flatter) and rate of endpoint convergence, on the other hand it controls the transition slope of a profile from $\omega = d_k$ to $\omega = 1$. The larger $m_1$ is, the more abruptly PRM will transit.

![Effects of $c_1$, $c_2$, $n_1$, $m_1$ on the profile of PRM](image)

**(a)** Effects of $c_1$ by taking values $\{1, 10, 100\}$ under $c_2 = 10$, $n_1 = 1$ and $m_1 = 5$

**(b)** Effects of $c_2$ by taking values $\{1, 10^3, 10^6\}$ under $c_1 = 1$, $n_1 = 1$ and $m_1 = 5$

**(c)** Effects of $n_1$ by taking values $\{1, 4, 10\}$ under $c_1 = 1$, $c_2 = 100$ and $m_1 = 2$

**(d)** Effects of $m_1$ by taking values $\{2, 3, 5\}$ under $c_1 = 1$, $c_2 = 100$ and $n_1 = 1$

**Fig. 30** Effects of $c_1$, $c_2$, $n_1$, $m_1$ on the profile of PRM $R^{33}_{13}$ where $\omega \in [d_k, 1]$ regarding $d_k = 3/10$ (solid, dash and dash-dot lines correspond to parameter values from small to big): a) Effects of $c_1$ by taking values $\{1, 10, 100\}$ under $c_2 = 10$, $n_1 = 1$ and $m_1 = 5$ b) Effects of $c_2$ by taking values $\{1, 10^3, 10^6\}$ under $c_1 = 1$, $n_1 = 1$ and $m_1 = 5$ c) Effects of $n_1$ by taking values $\{1, 4, 10\}$ under $c_1 = 1$, $c_2 = 100$ and $m_1 = 2$ d) Effects of $m_1$ by taking values $\{2, 3, 5\}$ under $c_1 = 1$, $c_2 = 100$ and $n_1 = 1$
Based on the above studies, extensive numerical practices are made afterwards, and the following recommendations are acquired to determine parameters including $n$ and $m$:

1. **The Recommendation of $n$** According to understandings in Refs. 3, 8 and 10, given WENO$k$, the minimum required order $n$ of $g(\omega_k)$ such that $g^{(i)}(\omega_k) = 0$ for $0 < i \leq n$ to recover the optimal order at critical point is tabulated in Table 17 in Appendix 1. As previously mentioned, $n$ defines the extent of Flatness-I, and obviously, is the same one in $C_{n,m}$. Hence on considering situations of critical points, $n$ in Eqns. (13)-(14) can take 1 or larger for WENO3; for WENO5, $n \geq 2$; for WENO7, $n \geq 3$.

2. **The Recommendation of $m$** As $m$ corresponds to the maximum order of $g^{(i)}(0,1) = 0$ in $C_{n,m}$ (also the degree of PRM at the endpoint to make mapped scheme approach WENO) when $m \geq 2$, the larger $m$ will favor numerical stability as indicated in Sect. 3. The choice of $m = n$ is suggested in this study.

3. **The Recommendation of $c_1$** As $c_1$ affects the extent of Flatness-II and also that of endpoint convergence, a relatively small value is suggested to ensure sufficient stability and also space for subsequent regulations. The recommendation is $c_1 \leq 1$ or likewise.

4. **The Recommendation of $c_2$** As $c_2$ affects the extent of Flatness-II directly, its value will affect the resolution of mapped WENO scheme. Usually, $c_2$ can take the value larger than $10^3 \sim 10^4$ or likewise; besides, the value might differ from a case by case of schemes and should take values in a piecewise manner.

5. **The Determination of $n_1$** $n_1 = 1$ is suggested in terms of computation efficiency, which implies the relatively large flatness.

6. **The Determination of $m_1$** As $m_1$ is critical to control the transition of PRM from the flat profile around $\omega = d_{\pm}$ to identity mapping at endpoints, its proper choice helps to achieve desired flatness and endpoint convergence concurrently. Additionally, our experience indicates too large $m_1$ will on the one hand increase the computation cost, on the other hand engender abrupt transition prone to numerical instability. Similarly, the value of $m_1$ should be taken according to specific WENO.

From above discussions, the function interaction of $c_1$, $c_2$, and $m_1$ is demonstrated, nevertheless, through which the desired flatness and endpoint convergence can be achieved.

### Appendix 5

The results of Titarev-Toro problem by WENO7-PRM$^{4,3,3}$ are presented here with comparisons of that by WENO7-AIM$^{5,4,2;1E4}$-M, -RM$^{7,6,2,0}$, -PM$^{8,6,1}$ and -JS. Corresponding results in local region are shown in Fig. 31. First, the overview of density fluctuation after the shock is shown in Fig. 31a. To illustrate the details, the region in Fig. 31a is separately zoomed in three sub-views by Figs. 31b–d. One can see that WENO7-PRM$^{4,3,3}$ performs similarly as WENO7-AIM$^{5,4,2;1E4}$-M and WENO7-RM$^{7,6,2,0}$, while WENO7-AIM$^{5,4,2;1E4}$-M indicates slightly higher resolution especially in Fig. 31c. By contrast, WENO7-PM$^{8,6,1}$ yields relatively less-resolved density fluctuations although the approximate differentiation on the structures. As comparison, WENO7 indicates a result with poor resolution and even fails to resolve fluctuations in the downstream region as shown in Fig. 31c–d.

Although WENO7-AIM$^{5,4,2;1E4}$-M indicates slight superiority comparing with WENO7-PRM$^{4,3,3}$ and -RM$^{7,6,2,0}$ in this case, its failure in strong shock wave and blast wave problems...
Fig. 31 Local and zoomed views of density distributions of Titarev-Toro problem at \( t = 5 \) on 1000 grids by WENO7-PRM\(^4_{3.3}\) with the comparisons by WENO7, WENO7-PM\(^8_{6.1}\), -RM\(^7_{6.2,0}\) and AIM\(^5_{4.2,444} - M\). 

a Local view where \( x \in [-2, 3.5] \)

b Zoomed view 1 where \( x \in [2, 3.3] \)

c Zoomed view 2 where \( x \in [0, 2] \)

d Zoomed view 3 where \( x \in [-2, 0] \)
reminds numerical stability/robustness should be especially noticed whilst improvement of resolution.

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