Geometrization of Some Epidemic Models

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M.E.Kahil

Abstract

Recently, the behavior of different epidemic models and their relation both to different types of geometries and to some biological models has been revisited. Path equations representing the behavior of epidemic models and their corresponding deviation vectors are examined. A comparison between paths and their deviation vectors in Riemannian and Finslerian Geometries is presented.

1 Introduction

In this study, we are going to describe a specific susceptible-infective (SI)-model of epidemics using Antonelli’s approach of describing two species in ecology geometrically [1]. This approach has been successfully used in case of wild rabbit disease as well as coral-starfish equilibrium [2-5]. Also, allometric relationship between net production biomass and the amount of secondary compound plants produce to defend against herbivores can be geometrized as well as corresponding interaction conditions are taken to be constant. The aim of this work is to apply Antonelli’s idea of imposing geometrical paths defined in metric spaces extracted from stochastic allometry space in Riemannian geometry to obtain a better non-linear regression formula with geometrical origin [2]. The importance of allometry space, due to Antonelli’s approach, is to obtain a space having a positive definite metric with negative curvature acting as a tool to examine the behavior of growth curves. Thus, any additive terms associated with geometrical structures is useful for adjusting a proper path describing the behavior of any epidemic curve. Since 1990s Antonelli’s approach has been extending to include other types of geometries such as Finsler geometry rather than Riemannian [3]. Consequently, Antonelli and Bradbury (1997) have used Gompertz growth and allometric relationships to build a dynamical theory of ecology, evolution and development in colonial organisms using Finslerian Geometry.

Accordingly, geometrization of SI-model can be considered as an introductory step to the geometric version of SIR-model and others. Moreover, this approach has been extended to include Finslerian Geometry due to its richness of imposing

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1Mathematics Department, Modern Sciences and Arts University, Giza, EGYPT

e.mail: kahil@aucegypt.edu
many interacting factors geometrically rather than the Riemannian one [5]. The importance of geometrizing such a model is to construct the path equations and their corresponding path deviation equations to know the behavior of the growth curves and their effect after a slight perturbation on them. This can be seen by studying the stability conditions from examining the motion of path deviation equations. In the present work, we are going to utilize the nonlinear version of allometry, Riemannian and Finslerian, of Antonelli for obtaining the relevant equations of epidemics in SI-models using different types of geometries with some details on its extension to include SIR models and others by increasing the dimension of the manifold. This type of extending dimensions to examine various epidemic models will be examined in our future work.

2 Historical Background

2.1 SI-Model of Epidemics

It is well known that the simplest model of describing population growth is stemmed from the famous prey-predator model cf[7] can be expressed as follows:

\[
\frac{dS}{dt} = -\alpha_1 SI, \quad [1.a]
\]

and

\[
\frac{dI}{dt} = \alpha_1 SI, \quad [1.b]
\]

where \(S(t)\) the susceptible class \(I(t)\), the infected class and \(\alpha_1\) a parameter. Equations (1.a) and (1.b) can be related together using the following condition:

\[S(t) + I(t) = N\quad [2]\]

where \(N\) stands for the total number of the population. The above model has been modified, Lotka-Volterra model, to accommodate some other factors based on some oscillations either internally or externally. In this study, it is sufficient to display the amended version owing to this extra factor to take the following form [8]

\[
\frac{dS}{dt} = -\alpha_1 SI + \alpha_2 S \quad [3.a]
\]

and

\[
\frac{dI}{dt} = \alpha_1 SI + \alpha_2 I \quad [3.b]
\]

where \(\alpha_2\) is another a parameter with constant coefficient, used to measure this interaction. From this perspective, it is vital to construct similar sets of path equations using Riemannian and Finslerian geometries respectively.

2.2 The Concept of Allometry

In 1936, Sir Julian Huxley introduced the concept of allometry, or the experimental study of relative growth of parts of animals, via log-log plots of measurements of morphological characteristics of individuals in a Euclidean geometry. These plots
resulted straight lines via statistical method of least squares. Around the same
time Sir Joseph Needham made some experimental work to show that straight lines
allometries.

In 1944, J. Kittredge used the same concept to estimate the crown biomass of
trees in forest stand by measuring the trunk girth, or diameter at breast height.
Again some experiments of botanist J. Harper in 1962 who found Gompertz curves
the best ones to fit for describing the growth of simple aquatic plants.

By 1965 Laird studied on vertebrate growth using Huxley’s allometric law as well.
Several applications of concept of allometry have been discussed in detail.
Recently, Karl Niklas (1994) has recorded a variety growing plants and flowers
satisfying the allometric concept as well [6].

3 Geometerization of Epidemics

3.1 Underlying Geometry of SI-model: The Riemmanian Version

It is well known that, Antonelli and Voorhees (1974) have suggested the following
metric in order to define geometrically the behavior of growth curves using allometric
space[7].

\[ g_{ij} = e^{-2a_k x^k} \delta_{ij}, \] \[ 4 \]

and its affine connection is given by

\[ \Gamma^i_{jk} = \frac{1}{2} g^{il} (g_{jk,l} + g_{ik,l} - g_{jk,l}), \] \[ 5 \]

where \( g^{ij} \) is the the matrix inverse of \( g_{ij} \). Suppose \( \alpha_i \) is a constant vector to define
the Christoffel symbol in the following way:

\[ \Gamma^i_{ii} = -\alpha_i = constant \] \[ 6.a \]

\[ \Gamma^i_{jj} = \alpha_i = constant, i \neq j \] \[ 6.b \]

\[ \Gamma^i_{ji} = 0, i \neq j \] \[ 6.c \]

\[ \Gamma^i_{jk} = \Gamma^i_{kj} = \alpha_i = constanti \neq j \neq k \] \[ 6.d \]

Thus, the metricity condition

\[ g_{ij,k} = 0 \] \[ 7 \]

becomes

\[ e^{-2a_m x^m} \delta_{ij,k} - e^{-2a_m x^m} \delta_{in} \Gamma^m_{kj} - e^{-2a_m x^m} \delta_{nj} \Gamma^m_{ki} = 0, \]

\[ -2a_k \delta_{ij} - e^{-2a_m x^m} \delta_{in} \Gamma^m_{kj} - e^{-2a_m x^m} \delta_{nj} \Gamma^m_{ki} = 0, \]

\[ \gamma_k \delta_{ij} = \delta_{mn} \Gamma^m_{kj} + \delta_{jn} \Gamma^m_{ik} \]

We can find out that the Riemann-Christoffel Curvature

\[ R^c_{abd} = \Gamma^c_{ad,b} - \Gamma^c_{ab,d} + \Gamma^m_{ad} \Gamma^c_{mb} - \Gamma^m_{ab} \Gamma^c_{md} \] \[ 8.a \]
has reduced to
\[ R^c_{abd} = \Gamma^m_{ad} \Gamma^c_{mb} - \Gamma^m_{ab} \Gamma^c_{md} \]  
[8.b]
Thus the non vanishing components of the curvature tensor become
\[ R^i_{jil} = \alpha_i \alpha_l, i \neq j \neq l \]  
[9.a]
\[ R^i_{jij} = \sum_{k \neq i,j} (\alpha_k)^2, i \neq j \]  
[8.b]
\[ R^i_{jjl} = \alpha_i \alpha_j, i \neq j \neq l. \]  
[8.c]

3.2 Equations of Epidemics in Riemannian version of SI-Model

It is well known that path equations of any growth curves can be obtained from the following applying variation principle on Lagrangian functions. This trend of imposing biological action principle has been stemmed prior to Euler by Mauphritis [2-7],

\[ L = g_{ab} U^a U^b \]  
[10]

However, in an approach to obtain path and path deviation equations from one single lagrangian using the Bazanski Lagrangian [10] which has the advantage that we obtain path and path deviation equations from the same Lagrangian:

\[ L = g_{ab} U^a \frac{D \Psi^b}{Dt} \]  
[11]
where \( a, b = 1, 2, 3, \ldots n \). By taking the variation with respect to the deviation vector \( \Psi^c \) and the tangent vector \( U^c \), and \( \frac{D \Psi^a}{Dt} \) is covariant derivative with respect to a parameter \( s \). From this perspective, it can easily be found that

\[ \frac{dU^c}{dS} + \Gamma^c_{ab} U^a U^b = 0 \]  
[12]
and

\[ \frac{D^2 \Psi^c}{Dt^2} = R^c_{abd} U^a U^b \Psi^d \]  
[13]

The path and path deviation equations of the SI-model can be obtained from the following Bazanski Lagrangian

\[ L = g_{\mu \nu} U^\mu \frac{D \Psi^\nu}{Dt} \]  
[14]

By taking the variation with respect to the deviation vector \( \Psi^\sigma \) to get the following path equation

\[ \frac{dS}{dt} + \Gamma^1_{11} S^2 + \Gamma^1_{22} I^2 + 2 \Gamma^1_{12} SI = 0, \]  
[15.a]
and

\[ \frac{dI}{dt} + \Gamma^2_{11} S^2 + \Gamma^2_{22} I^2 + 2 \Gamma^2_{12} SI = 0, \]  
[15.b]
While taking the variation with respect to velocity vector $U^{\alpha}$ to get their corresponding path deviation equations:

$$\frac{D^{2}\Psi_{S}}{Dt^{2}} = R_{112}^{1} S I \Psi_{I} + R_{121}^{1} S I \Psi_{S} + R_{212}^{1} S I \Psi_{I} + R_{221}^{1} S I \Psi_{S}, \quad [16.a]$$

and

$$\frac{D^{2}\Psi_{I}}{Dt^{2}} = R_{112}^{2} S I \Psi_{I} + R_{121}^{2} S I \Psi_{S} + R_{212}^{2} S I \Psi_{I} + R_{221}^{2} S I \Psi_{S}, \quad [16.b]$$

i.e.

$$\frac{D^{2}\Psi_{S}}{Ds^{2}} = (\alpha_{1}^{2} + 2\alpha_{1}\alpha_{2} + \alpha_{2}^{2}) S I (\Psi_{S} - \Psi_{I}) \quad [16.c],$$

and

$$\frac{D^{2}\Psi_{I}}{Ds^{2}} = (\alpha_{1}^{2} + 2\alpha_{1}\alpha_{2} + \alpha_{2}^{2}) S I (\Psi_{S} - \Psi_{I}) \quad [16.d].$$

Consequently, we can suggest the path path and path deviation equations of a modified SI model can be following Lagrangian:

$$L = g_{\mu\nu} U^{\mu} \frac{D\Psi^{\nu}}{Dt} + \lambda \Psi^{\mu} U^{\mu} \quad [17]$$

to give,

$$\frac{dU^{a}}{ds} + \Gamma^{a} b c U^{b} U^{c} = \lambda_{a} U^{a} \quad [18.a]$$

i.e

$$\frac{dS}{dt} + \Gamma_{11}^{1} S^{2} + 2\Gamma_{12}^{1} S I + \Gamma_{22}^{1} I^{2} = \lambda S \quad [18.b]$$

and

$$\frac{dI}{dt} + \Gamma_{11}^{2} S^{2} + 2\Gamma_{12}^{2} S I + \Gamma_{22}^{2} I^{2} = \lambda I \quad [18.c]$$

in which can be expressed as follows

$$\frac{dS}{dt} + 2\beta S I - \alpha I^{2} + \alpha S^{2} = \lambda S \quad [18.d]$$

and

$$\frac{dI}{dt} + 2(\alpha - \beta L) S I - (\beta + \alpha L) S + (\alpha - \beta L) I^{2} = \lambda I \quad [18.e]$$

And their corresponding path deviation equations become as follows:

$$\frac{D^{2}\Psi^{a}}{Dt^{2}} = R^{a}_{bd} U^{b} U^{c} \Psi^{d} + \lambda \frac{D\Psi^{a}}{Dt} \quad [19.a]$$

$$\frac{D^{2}\Psi_{S}}{Ds^{2}} + \lambda \frac{D\Psi_{S}}{Dt} = R_{112}^{1} S I \Psi_{I} + R_{121}^{1} S I \Psi_{S} + R_{212}^{1} S I \Psi_{I} + R_{221}^{1} S I \Psi_{S}, \quad [19.b]$$

and

$$\frac{D^{2}\Psi_{I}}{Ds^{2}} + \lambda \frac{D\Psi_{I}}{Dt} = R_{112}^{2} S I \Psi_{I} + R_{121}^{2} S I \Psi_{S} + R_{212}^{2} S I \Psi_{I} + R_{221}^{2} S I \Psi_{S}, \quad [19.c]$$
which can be expressed as follows:

\[
\frac{D^2 \Psi_S}{D s^2} + \lambda \frac{D \Psi_S}{D t} = (\alpha_1^2 + 2\alpha_1\alpha_2 + \alpha_2^2) S I (\Psi_S - \Psi_I), \tag{19.d}
\]

and

\[
\frac{D^2 \Psi_I}{D s^2} + \lambda \frac{D \Psi_I}{D t} = (\alpha_1^2 + 2\alpha_1\alpha_2 + \alpha_2^2) S I (\Psi_S - \Psi_I) \tag{19.e}
\]

### 3.3 Berwald type

The metric tensor of Berwald type in Finslerian geometry can be described as follows

\[
g_y = \dot{\partial}_i \dot{\partial}_j \left( \frac{1}{2} F^2 \right)
\]

such a metric tensor is called a Minkowoski space with Finslerian norm \(ds = f(dx^1, dx^2)\). Thus, its corresponding geodesic can be described as

\[
\frac{d^2 x^i}{ds^2} + \hat{\Gamma}^i_{jk}(x, y) = 0 \tag{20}
\]

such that

\[
\hat{\Gamma}^i_{jh} = \frac{1}{2} g^{im} (\delta_h g_{jm} + \delta_j g_{mh} - \delta_m g_{jh}), \tag{21}
\]

where \(\delta_h\) is a partial derivative with respect to non-linear connection.

\[
\delta_h = \partial_h - \Gamma^i_{jh} (x) \frac{\partial}{\partial y^h}, \tag{22}
\]

\[
\hat{C}^i_{jk} = \frac{1}{2} g^{il} \left( \frac{\partial g_{lk}}{\partial y^j} + \frac{\partial g_{il}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^l} \right) \tag{23}
\]

The Berwald type has some associated curvatures are defined in the following way [7]:

\[
B^i_{kjh} = R^i_{kjh} + G^i_j D^i_{rhk} - G^r_k D^i_{rhj} \tag{24}
\]

where

\[
G^i_j = \hat{\Gamma}^i_{jk} y^j - \frac{1}{2} g^{im} \hat{\Gamma}^i_{jk} \frac{\partial g_{ml}}{\partial y^i} y^j y^k \tag{25.a}
\]

and

\[
D^i_{kjh} = \frac{\partial^3 H^i_2}{\partial y^i \partial y^k \partial y^l}, \tag{25.b}
\]

such that

\[
\hat{\Gamma}^i_{\alpha_1 \alpha_2 \alpha_3 \ldots \alpha_m} = \frac{1}{m!} \frac{\partial^m H_{(m)}}{\partial y^\alpha_1 \partial y^\alpha_2 \partial y^\alpha_3 \ldots \partial y^\alpha_m} \tag{25.c}
\]
3.4 SI-Model in Finsler Geometry

In 1991 Antonelli developed such a metric for 2-dimensional Berwald space with locally constant coefficients to become [3]

\[ F^2 = e^{2\alpha_i X^i (L^2 + 1)} ((\dot{X}^1)^2 + (\dot{X}^2)^2), i = 1, 2 \]  

where \( X^1 \) and \( X^2 \) are Cartesian coordinates on \( R^2 \) and \( L \) acts as a perturbation parameter. The above relation can be related to Riemannian geometry by relaxing the term \( L \)

\[ \tilde{g}_{ij}(X, \dot{X}) = e^{2\alpha_i X^i (L^2 + 1)} g_{ij} \]  

The above system of equations together with their corresponding deviation vector equations can be obtained from taking the action principle to the following Lagrangian.

\[ L_{BF} = g_{ij}(x, y) X^i \frac{D\hat{\psi}^j}{Dt} \]  

Such that

\[ \frac{D\hat{\psi}^i}{Dt} = \frac{d\hat{\psi}^i}{dt} + \hat{\Gamma}^i_{jh} \psi^j \dot{U}^h + C^i_{jh} \dot{\psi}^j \dot{V}^h, \]  

where

\[ V^h = \frac{\delta y}{\delta t}. \]  

Accordingly, geodesic equation may be as follows

\[ \frac{d^2 x^i}{dt^2} + \hat{\Gamma}^i_{jh}(x, y) y^j y^h = 0, \]  

i.e.

\[ \frac{dS}{dt} + 2(\beta + \alpha L) SI + (\beta L - \alpha) I^2 + (\alpha - \beta L) S^2 = 0 \]  

and

\[ \frac{dI}{dt} + 2(\alpha - \beta L) \dot{S} I - (\beta + \alpha L) \dot{S} + (\alpha - \beta L) I^2 = 0 \]  

and its deviation equation becomes

\[ \frac{D^2 \hat{\psi}^a}{Dt^2} = B^a_{bcd} U^b U^c \hat{\psi}^d, \]  

to become

\[ \frac{D^2 \hat{\psi}_S}{Ds^2} = B^1_{112} SI \hat{\psi}_I + B^1_{121} SI \hat{\psi}_S + B^1_{212} SI \hat{\psi}_I + B^1_{222} SI \hat{\psi}_S, \]  

and

\[ \frac{D^2 \hat{\psi}_I}{Ds^2} = B^2_{112} SI \hat{\psi}_I + B^2_{121} t SI \hat{\psi}_S + B^2_{212} SI \hat{\psi}_I + B^2_{222} SI \hat{\psi}_S, \]  

i.e.

\[ \frac{D^2 \hat{\psi}_S}{Ds^2} = 0, \]  

[32.d]
and
\[ \frac{\hat{D}^2 \Psi_I}{\hat{D}s^2} = 0. \] \[32.e\]

Similarly, applying Antonelli’s method to define Volterra’s equation in ecology \([2-7]\), we obtain its corresponding part of the SI-model in Berwald space,
\[ \frac{d^2 S}{dt^2} + \hat{\Gamma}_{11}^1 (\frac{dS}{dt})^2 + 2 \hat{\Gamma}_{12}^1 (\frac{dS}{dt})(\frac{dI}{dt}) + \hat{\Gamma}_{22}^1 (\frac{dI}{dt})^2 = \lambda (\frac{dS}{dt}), \] \[33.a\]
and
\[ \frac{d^2 I}{dt^2} + \hat{\Gamma}_{11}^2 (\frac{dS}{dt})^2 + 2 \hat{\Gamma}_{12}^2 (\frac{dS}{dt})(\frac{dI}{dt}) + \hat{\Gamma}_{22}^2 (\frac{dI}{dt})^2 = \lambda (\frac{dI}{dt}), \] \[33.b\]
Thus, the coefficients of its affine connection become:
\[ \hat{\Gamma}_{11}^1 = \alpha_1 - \alpha_2 L, \] \[34.a\]
\[ \hat{\Gamma}_{22}^1 = - (\alpha_1 - \alpha_2 L), \] \[34.b\]
\[ \hat{\Gamma}_{12}^1 = \hat{\Gamma}_{21}^1 = - (\alpha_2 + \alpha_1 L) \] \[34.c\]
\[ \hat{\Gamma}_{12}^2 = \hat{\Gamma}_{21}^2 = - (\alpha_1 - \alpha_2 L) \] \[34.d\]
\[ \hat{\Gamma}_{11}^2 = - (\alpha_2 + \alpha_1 L) \] \[34.e\]
\[ \hat{\Gamma}_{22}^2 = - (\alpha_2 + \alpha_1 L). \] \[34.f\]
Thus, the components of path equations are expressed as follows:
\[ \frac{dS}{dt} + 2(\beta + \alpha L)SI + (\beta L - \alpha)(I)^2 + (\alpha - \beta L)(S)^2 = \lambda S, \] \[35.a\]
and
\[ \frac{dI}{dt} + 2(\alpha - \beta L)SI - (\beta + \alpha L)S^2 + (\alpha - \beta L)I^2 = \lambda I. \] \[35.b\]
Equations (35.a) and (35.b) are obtained by taking the variation with respect the corresponding deviation vector of the following modified Bazanski Lagrangian
\[ L = \bar{g}_{ab} \dot{X}^a \dot{\Psi}^b + \lambda \Psi_a \dot{X}^a \] \[36\]
Also, after some multiplications, we can find Their corresponding deviation equations by taking the variation with respect to \( \dot{X}^c \) to become:
\[ \frac{\hat{D}^2 \Psi_S}{\hat{D}s^2} + \lambda \frac{\hat{D} \Psi_S}{\hat{D}t} = B_{112}^1 SI \dot{\Psi}_I + B_{121}^1 SI \dot{\Psi}_S + B_{212}^1 SI \dot{\Psi}_I + B_{221}^1 SI \dot{\Psi}_S, \] \[37.a\]
and
\[ \frac{\hat{D}^2 \Psi_I}{\hat{D}s^2} + \lambda \frac{\hat{D} \Psi_I}{\hat{D}t} = B_{112}^2 SI \dot{\Psi}_I + B_{121}^2 SI \dot{\Psi}_S + B_{212}^2 SI \dot{\Psi}_I + B_{221}^2 SI \dot{\Psi}_S \] \[37.b\]
\( i.e. \)
\[ \frac{\hat{D}^2 \Psi_S}{\hat{D}s^2} + \lambda \frac{\hat{D} \Psi_S}{\hat{D}t} = 0 \] \[37.c\]
and
\[ \frac{\hat{D}^2 \Psi_I}{\hat{D}s^2} + \lambda \frac{\hat{D} \Psi_I}{\hat{D}t} = 0 \] \[37.d\]
3.5 Antonelli-Finsler Metric Function

In order to generalize the previous metric in (26) Antonelli has modified Finsler metric to become [7]:

$$F = e^{\phi} \left( \sum_{i=1}^{n} (\dot{x}^i)^m \right)^{\frac{1}{m}},$$  \[38\]

where $m \geq 2$ and $n = 2$. Thus their corresponding components of its affine connection are given as:

$$\hat{\Gamma}_{11}^1 = \sigma_1 - \frac{1}{9} \beta_1 \left( \frac{\dot{I}}{S} \right)^{\frac{4}{3}};$$  \[39.a\]

$$\hat{\Gamma}_{21}^1 = \frac{4}{9} \beta_1 \left( \frac{\dot{I}}{S} \right)^{\frac{4}{3}} + \frac{1}{2} \gamma_{12};$$  \[39.b\]

$$\hat{\Gamma}_{12}^1 = \frac{2}{9} \beta_1 \left( \frac{\dot{S}}{I} \right)^{\frac{4}{3}};$$  \[39.c\]

$$\hat{\Gamma}_{22}^1 = \sigma_2 - \frac{1}{9} \beta_2 \left( \frac{\dot{S}}{I} \right)^{\frac{4}{3}}.$$  \[39.d\]

Consequently, the Bazanski method to obtain path and path deviation equations will become in the following way:

[i] The components of Path Equations:

$$\frac{d\dot{S}}{dt} + \sigma_1 S^2 + \sigma_2 \left( \frac{m}{m-1} \right) SI + \frac{\sigma_1}{m-1} \left( \frac{I}{S} \right)^{m-2} I^2 = \lambda_1 S$$  \[40.a\]

and

$$\frac{dI}{dt} + \sigma_2 I^2 + \sigma_1 \left( \frac{m}{m-1} \right) SI + \frac{\sigma_2}{m-1} \left( \frac{\dot{I}}{S} \right)^{m-2} I^2 = \lambda_2 I.$$  \[40.b\]

[ii] The components of Path deviation Equations:

$$\frac{D^2 \Psi_S}{DS^2} + \lambda \frac{D\Psi_S}{Dt} = 0;$$  \[41.a\]

and

$$\frac{D^2 \Psi_I}{DS^2} + \lambda \frac{D\Psi_I}{Dt} = 0;$$  \[41.b\]

It is well known that the interaction between $S$I can be vitally important when $m > 2$ and the cases of increasing dimensions such as the following section to examine the possibility to geometrize SIR models.

4 Geomertization of SIR Model

In a similar way, we can extend our study to examine SIR model using the Bazanski method in each Riemannian and Finslerian to become:

(i) For Riemannian Geometry

$$\frac{DS}{Dt} = 0;$$  \[42.a\]

$$\frac{DI}{Dt} = \alpha I;$$  \[42.b\]
and
\[ \frac{DR}{Dt} = 0, \]  \[42.c\]
promised that
\[ S + R + I = N \]  \[43\]
where \( R \) is the number of recovered or dead group.

The above equation can easily be obtained by assuming the following Lagrangian:
\[ L = g_{\mu\nu} U^\mu \frac{D\Psi^\nu}{Dt} + \alpha_{(\mu)} \Psi^\nu U^\nu, \quad a, b = 1, 2, 3 \]  \[44\]
where \( \alpha_{(\mu)} \) is an arbitrary constant.

From this perspective, we can develop the SIR model in its geometric version as follows
\[ \frac{DU^\mu}{Ds} = \alpha_{(\mu)} U^\mu \]  \[45\]
and their corresponding deviation equations become:
\[ \frac{D^2\hat{\Psi}^\mu}{Dt^2} + \alpha_{(\mu)} \frac{D\hat{\Psi}^\mu}{Dt} = R_{\nu\rho\sigma} U^\nu U^\rho \hat{\Psi}^\sigma. \]  \[46\]

(ii) For Finslerian Geometry (Berwald Type):

In a similar way to equations [42a-45], we can easily construct the following equations:
\[ \frac{D\hat{S}}{Dt} = 0, \]  \[47.a\]
\[ \frac{D\hat{I}}{Dt} = \alpha I, \]  \[47.b\]
and
\[ \frac{D\hat{R}}{Dt} = 0. \]  \[47.c\]

Equations [47 a-c] are obtained by taking the action of the following Largangian:
\[ L = g_{\mu\nu} \hat{U}^\mu \frac{D\hat{\Psi}^\nu}{Dt} + \alpha_{(\mu)} \hat{\Psi}^\nu U^\nu, \quad \mu, \nu = 1, 2, 3. \]  \[48\]

We also can find their corresponding deviation equations to become:
\[ \frac{D^2\hat{\Psi}^\mu}{Dt^2} + \alpha_{(\mu)} \frac{D\hat{\Psi}^\mu}{Dt} = B_{\nu\rho\sigma} U^\nu U^\rho \hat{\Psi}^\sigma \]  \[49\]

For a complete description of this model will be examined in our future work.
5 Discussion and Concluding Remarks

The paper deals with geometrization some epidemic models using their corresponding path equations. Also, we have obtained their corresponding path deviation equations from one single Lagrangian for different types of geometries based under a positive definite metric with constant affine connections and negative curvature. This process is based on a geometrized vision of concept of allometry to examine epidemic curves as by finding their path equations. The paper has also opened the window to impose non conventional types of geometries in future work to describe more complex models of SI or briefly in SIR model. One of good results is increasing the dimensions with preserving the concept of a positive metric with constant affine connections and negative curvature will include many other epidemic models such as susceptible-exposed-infective-recovered models (SEIR).

Finally, this study can be extended to obtain such an appropriate prospective vision to visualize how an epidemic behaves on a longer times without many parameters as needed in case of the traditional regression analysis. This is not the optimal case to get an exact form of epidemic curve, but it is a tend to apply such a concept of geometrization. In addition in our future work we may in need to develop these geometries to include some new parameters affecting the SI-model such as transmission rate \[12\] to become:

\[
\frac{dS}{dt} = -\alpha S^p I^p \tag{50}
\]

can be studied in future to be geometrized.

Also, for the SIR model can also be extended to include some interactions like population fertility or immigration to be considered in demographic using partial differential equations of McKendrick- von Forester with boundary conditions \[13\].

\[
(\frac{\partial}{\partial t} + \frac{\partial}{\partial a})n(a,t) = -\mu n(a,t) + I(a,t), \quad 0 < a < \omega, t > 0 \tag{5}
\]
such that

\[
n(0,t) = B(t) = \int_0^\omega m(a)n(a,t)da, \quad t \geq 0
\]

\[
n(a,t) = n_0(a), 0 < a < \omega
\]

where \(\omega\) represents the maximum life span of individuals, \(B(t)\) is the number of newborn per unit time \(t\), \(m(a)\) is a mortality rates and \(\mu(a)\) is a constant. The evolution of density \(n(a,t)\) of individuals aged \(a\) and time \(t\), will open the window to be included geometrically as non linear connections in Finslerian approach.

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