ON THE FUNCTORIALITY OF STRATIFIED
DESINGULARIZATIONS

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To our families.

Abstract. This article is devoted to the study of smooth desingularization, which are customary employed in the definition of De Rham Intersection Co-homology with differential forms \[12\]. In this paper we work with the category of Thom-Mather simple spaces \[10\], \[14\]. We construct a functor which sends each Thom-Mather simple space into a smooth manifold called its primary unfolding. Hence we prove that the primary unfoldings are unique up Thom-Mather isomorphisms.

Introduction

Stratified spaces were initially defined by Thom \[14\]. For a recent explanation of the topic see Pflaum \[10\]. These spaces are intimately related to the study of intersection homology defined by Goresky and MacPherson \[6\], which is a loose homology theory since the homotopy axiom does not hold in this context. Nevertheless, it extends the Poincaré Duality to spaces with singular points. We define the intersection cohomology through an unfolding. This is a suitable smooth desingularization of a pseudomanifold; it consists in removing the singular part of a pseudomanifold and gluing two diffeomorphic copies of the regular part through a hypersurface, for more details see Davis \[4\], Ferrarotti \[5\] and Saralegi \[12\].

In this article we work with simple pseudomanifolds which are just stratified pseudomanifolds whose depth is \(\leq 1\). Simple pseudomanifolds are almost everywhere smooth spaces, they can be separated in two disjoint pieces: A dense open subset callet its regular part, and a closed called its singular part. Their local model has the form \(U \times c(L)\) where \(U\) and \(L\) are manifolds and \(L\) is compact; the name pseudomanifold arises from this kind of charts. This article is structured as follows: first we introduce the notion of simple pseudomanifolds and their unfoldings. Next we study the Thom-Mather simple spaces and we prove that each simple pseudomanifold has an unfolding if and only if it satisfies the Thom-Mather condition.

In the last section we construct a functor from the category of Thom-Mather simple spaces to the category of the smooth manifolds. This is done with the tool of primary unfoldings, which allows us to work in a simpler subcategory of spaces. The unfoldings in general are neither unique nor functorial objects; see Brasselet, Hector and Saralegi \[1\] and Dalmagro \[3\].

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are the main results of this article: First, every radium preserving morphism between Thom-Mather simple spaces induces a smooth map between their respective unfoldings. Second, the primary unfolding of a Thom-Mather simple space is unique.

All along this article, each time we use the word manifold we mean a smooth differentiable manifold of class $C^\infty$ without boundary.

1. The geometry

1.1. Simple Pseudomanifolds. In this section we introduce the family of simple pseudomanifolds; notice that these are stratified pseudomanifolds whose depth is $\leq 1$. Simple spaces are a previous step in regard of a clear exposition of pseudomanifolds. For more details see [6] [10].

Definition 1.1.1. A simple space is a 2nd countable metric topological space $X$ which can be written as the disjoint union of two manifolds $X = R \sqcup \Sigma$, such that $R$ is an open dense set (and therefore $\Sigma$ is closed). We refer to $R$ (resp. $\Sigma$) as the regular (resp. singular) part of $X$. A regular (resp. singular) stratum $S$ of $X$ is a connected component of $R$ (resp. $\Sigma$). The pair $(R, \Sigma)$ is a decomposition of $X$.

A simple subspace of $X$ is a subset $Y \subset X$ such that $(R \cap Y, \Sigma \cap Y)$ is a decomposition of $Y$ with the induced topology. This condition is not trivial, because according to 1.1.1 $R \cap Y$ and $\Sigma \cap Y$ must be manifolds.

Examples 1.1.2.

1. Each manifold $M$ is a simple space whose singular part $\Sigma = \emptyset$ is the empty set.
2. The canonical decomposition of any manifold with (nonempty) boundary $M$ is $(M - \partial M, \partial M)$.
3. Any open subspace of a simple space is itself a simple space.
4. If $M$ is a manifold and $X$ is a simple space, then the product space $M \times X$ is a simple space, its decomposition is $(M \times R, M \times \Sigma)$.
5. Let $L$ be a compact manifold, the open cone of $L$ is the quotient space

$$c(L) = \frac{L \times [0, \infty)}{\sim}$$

where $(l, 0) \sim (l', 0)$ for any $l, l' \in L$. The equivalence class of a point $(l, r)$ will be written $[l, r]$. The vertex is the class of any point $(l, 0)$; it will be denoted $v$. For convenience, we agree that $c(\emptyset) = \{v\}$ is a singleton. The space $c(L)$ is simple, its decomposition is $(L \times \mathbb{R}^+, \{v\})$.

6. A pseudo-Euclidean model (or pem for short), is a product $U \times c(L)$ with the decomposition given by the above examples, i.e., $(V \times L \times \mathbb{R}^+, U \times \{v\})$; where $U, L$ are manifolds and $L$ is compact; this $L$ is said to be the link of $U$ on $U \times c(L)$. Our convention for $L = \emptyset$ implies that any euclidean nbhd is a pem.
Since any pem is the quotient of a product manifold; each morphism 
\[ U \times c(L) \xrightarrow{f} V \times c(N) \] can be written as 
\[ f(u, [l, r]) = (a_1(u, l, r), [a_2(u, l, r), a_3(u, l, r)]) \]
where \( a_1, a_2, a_3 \) are maps defined on \( U \times L \times [0, \infty) \), and they are "smooth by pieces", i.e., on \( U \times L \times \{0\} \) and \( U \times L \times \mathbb{R}^+ \). Notice that \( a_3(u, l, 0) = 0 \) for any \( u, l \).

**Definition 1.1.3.** A simple pseudomanifold or spm for short, is a simple space \( X \) such that each singular point \( x \in \Sigma \) has an open nbhd \( x \in V \subset X \) which is the image of an embedding 
\[ \alpha : U \times c(L) \longrightarrow X \]
We call \( V = \text{Im}(\alpha) \) a pem-nbhd of \( x \), while the pair \((U, \alpha)\) is a chart. Since any euclidean nbhd is a pem-nbhd (see example §1.1.2-(6)), the above condition is non-trivial just for singular points.

Up to some minor details, we assume that \( \alpha(u, v) = u \) for all \( u \in U \), so \( U = V \cap \Sigma \) is an open nbhd of \( x \) on the corresponding stratum \( S \subset \Sigma \) containing \( x \). We usually ask the points of \( S \) to have the same link \( L \), so it does not depend on the choice of \( x \). We call \( L \) the link of \( S \).

**Examples 1.1.4.**
(a) The examples §1.1.2-(1),(2),(5) and (6) are spm’s.
(b) If \( M \) is a manifold and \( X \) is a spm then \( M \times X \) is a spm.
(c) Any open set of a spm is also a spm.

### 2. Desingulatizations

It is well known how some usual (co)homological properties of smooth manifolds are lost when we add singularities. This is the case of the Poincaré Duality, for instance. In order to recover these properties on a larger family of spaces, the original works of Goresky and MacPherson defined the Intersection Homology with singular chains \cite{6}. Later on \cite{7}, \cite{12}, Hector and Saralegi provided a smooth approach to the Intersection Cohomology. For more details see also \cite{1}, \cite{5}. Their viewpoint strongly depends on two geometric objects associated to any spm \( X \), which are built in order to study the way we reach the singular part and how we can recover the usual cohomological data; these are the Thom-Mather tubular nbhds \cite{14} and the smooth unfoldings \cite{4}. In the sequel, we will study both of them.

#### 2.1. Unfoldings
Recall the definition of smooth unfoldings \cite{12}. Such an object is obtained from an spm \( X \) with decomposition \((R, \Sigma)\) by gluing a finite number of copies of \( R \) and replacing \( \Sigma \) with a suitable smooth hypersurface.

**Definition 2.1.1.** An unfolding of a spm \( X \) is a manifold \( \widetilde{X} \) together a continuous proper map \( \widetilde{X} \xrightarrow{\mathcal{L}} X \) such that:

1. The restriction \( \mathcal{L}^{-1}(R) \xrightarrow{\mathcal{L}} R \) is a smooth trivial covering.
2. For each \( z \in \mathcal{L}^{-1}(\Sigma) \) there is a commutative square diagram:

\[
\begin{array}{ccc}
U \times L \times \mathbb{R} & \xrightarrow{\tilde{\alpha}} & \widetilde{X} \\
\downarrow{\varepsilon} & & \downarrow{\varepsilon} \\
U \times c(L) & \xrightarrow{\alpha} & X
\end{array}
\]
such that
(a) \((U, \alpha)\) is a chart (see §1.1.3).
(b) \(c(u, l, t) = (u, |l|, |t|)\).
(c) \(\tilde{\alpha}\) is a diffeomorphism on \(L^{-1}(\text{Im}(\alpha))\).

We will refer to the above diagram as an **unfolded chart** at \(x = \mathcal{L}(z)\).

A spm \(X\) is **unfoldable** if there is (at least) a (smooth) unfolding as above. An **unfoldable morphism** is a commutative square diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X}' \\
\downarrow c & & \downarrow c' \\
X & \xrightarrow{f} & X'
\end{array}
\]

such that \(f\) is a morphism, \(\tilde{f}\) is smooth and the vertical arrows are unfoldings.

**Examples 2.1.2.**

(1) For any manifold \(M\) the identity map \(M \xrightarrow{id} M\) is an unfolding.

(2) The map \(c\) given in §2.1.1-(2).(b) is an unfolding of the pem \(U \times c(L)\).

(3) For any manifold \(M\) with non-empty border \(\partial M \neq \emptyset\); the link of \(\partial M\) is a point \(L = \{l\}\). Define \(\tilde{M} = M \times \{\pm 1\}/\sim\) as the quotient set obtained by gluing two copies of \(M\) along \(\partial M\), i.e. \((m, 1) \sim (m, -1)\forall m \in \partial M\). Write \([m, j]\) for the class of \((m, j)\). Consider the map

\[
\tilde{M} \xrightarrow{\mathcal{L}} M \xrightarrow{[m, j]} m
\]

Since the border submanifold \(\partial M\) has always a smooth collar then, locally, \(\mathcal{L}\) behaves like \(c\) at §2.1.1-(2).(b). So \(\tilde{M}\) has a unique smooth structure such that \(\mathcal{L}\) is an unfolding.

(4) If \(\tilde{X} \xrightarrow{\mathcal{L}} X\) is an unfolding, then for any open subset \(A \subset X\) the restriction \(\mathcal{L}^{-1}(A) \xrightarrow{\mathcal{L}} A\) is an unfolding.

(5) If \(\tilde{X} \xrightarrow{\mathcal{L}} X\) is an unfolding, then for any singular stratum \(S \subset \Sigma\) the restriction \(\mathcal{L}^{-1}(S) \xrightarrow{\mathcal{L}} S\) is a smooth bundle with typical fiber \(F = L\) and \(\mathcal{L}^{-1}(S)\) is a hypersurface of \(\tilde{X}\).

(6) The 2-torus \(T^2\) provides an unfolding for the real projective plane \(T^2 \xrightarrow{\mathcal{L}} \mathbb{R}P^2\). We replace the singular point \(\Sigma = \{\infty\}\) with two disjoint circles \(\mathbb{S}^1 \times \{\pm 1\}\) and glue them with two copies of the cylinder \(\mathbb{S}^1 \times \mathbb{R}\), obtaining so \(T^2\). The map \(\mathcal{L}\) is given by \(\mathcal{L}(e^{i\theta}, e^{i\varphi}) = [e^{i2\theta}, e^{i\varphi}]\).

(7) The unfoldings are related to some other geometric objects, such as tubular nbhds and normalizations. Although we we do not explicitly mention it, we always can assume that the links are connected [9].

2.2. **Tubular neighborhoods.** The construction of tubular nbhds is due, among others, to Gleason and Palais. It was developed in the context of compact transformation groups. In the smooth context, a tubular can be constructed through an invariant riemannian metric and they are related to the existence of equivariant slices [2]. All these results have been extended to the singular context [7, 10].
When we move on a spm $X$, as we approach to the singular part we must preserve the geometric notion of conical radius.

**Definition 2.2.1.** Given an spm $X$ let us consider a singular stratum $S \subset \Sigma$. A **tubular nbhd** of $S$ is a fiber bundle $\xi = (T, \tau, S, c(L))$ satisfying

1. $T$ is an open nbhd of $S$ in $X$.
2. The abstract fiber of $\xi$ is $F = c(L)$, the open cone of the link $L$ of $S$.
3. $\tau(x) = x$ for any $x \in S$. In other words, the inclusion $S \subset T$ is a section of the fiber bundle.
4. The structure group of $\xi$ is a subgroup of $\text{Difeo}(L)$; i.e., if $(U, \alpha), (V, \beta)$ are two bundle charts and $U \cap V \neq \emptyset$ then the change of charts can be written as
   \[ \beta^{-1}\alpha : U \cap V \times c(L) \to U \cap V \times c(L) \quad \beta^{-1}(\alpha(u, [l, r])) = (u, [g_{\alpha\beta}(u)(l), r]) \]
   where $g_{\alpha\beta}(u)$ is a diffeomorphism of $L$ for all $u \in U \cap V$.

The spm $X$ is a **Thom-Mather** iff every singular stratum has a tubular nbhd. If $X, X'$ are Thom-Mather spms a **Thom-Mather morphism** is a morphism $f$ in the sense of §1.1.1 which preserves the tubular nbhd.

**Remark 2.2.2.** Given a tubular nbhd $T \xrightarrow{\tau} S$, by §2.2.1-(4), there is global sense of radium $T \xrightarrow{\rho} [0, \infty)$ in the whole tube $T$ given by $\rho(u, [l, r]) = r$, this function is the **radium** of $T$. Notice that $\rho^{-1}(\{0\}) = S$ and $\rho^{-1}(\mathbb{R}^+) = (T - S)$. There is also an action $\mathbb{R}^+ \times T \to T$ by **radium stretching**, given by $\lambda \cdot \alpha(u, [l, r]) = \alpha(u, [l, \lambda r])$.

**Remark 2.2.3.** Given a spm $X$, the singular strata are disjoint connected manifolds. Since $X$ is normal, they can be separated through a disjoint family of open subsets. Therefore, and up to some minor details, if $X$ is Thom-Mather then we can find a disjoint family of tubular nbhds. This allows us to simplify some things. In the rest of this section we fix a Thom-Mather spm $X$ and we will assume, without loss of generality, that $X$ has a unique singular stratum $S = \Sigma$, with a tubular nbhd as above.

The main goal of this section is to study how the unfoldings and the tubes relate to each other. We state this relationship in the following

**Theorem 2.2.4.** A spm is Thom-Mather if and only if it is unfoldable.

We will prove separately each implication, so next we show how to obtain an unfolding from a tubular nbhd. For an equivalent way of constructing unfoldings, the reader can see [4] [7].

**Proposition 2.2.5.** Every Thom-Mather spm is unfoldable.

**Proof** For the singular stratum $S = \Sigma$ and a the tubular nbhd $T \xrightarrow{\tau} S$, let’s fix a bundle atlas $U = \{(U_\alpha, \alpha)\}_{\alpha \in \mathcal{A}}$. We proceed in three steps.

- **Unfolding a chart:** For any chart $(U, \alpha) \in U$, the unfolding of $\tau^{-1}(U)$ is the composition
  \[ U \times L \times \mathbb{R} \xrightarrow{\epsilon} U \times c(L) \xrightarrow{\alpha} \tau^{-1}(U) \]
  where $c$ is the map given at §2.1.1(2).(b).
• **Unfolding the tube:** Define

\[
\widetilde{T} = \bigcup_{\alpha} U_\alpha \times L \times \mathbb{R} \quad \sim \quad (u, l, t) \sim (u, g_{\alpha\beta}(u)(l), t) \forall \alpha, \beta \forall u \in U_\alpha \cap U_\beta
\]

as the quotient of the disjoint union, with the above equivalence relation. Write \([u, l, t]\) for the equivalence class of a triple \((u, l, t)\). According to [13, p. 14], the above operation defines a fiber bundle

\[
\tilde{T} \xrightarrow{\pi} S \quad \tilde{T}([u, l, t]) = u
\]

with abstract fiber \(F = L \times \mathbb{R}\) and the same structure group of \(T\). Since the cocycles are smooth, notice that \(\tilde{T}\) is a manifold. Let’s now define

\[
\tilde{T} \xrightarrow{\mathcal{L}} T \quad \mathcal{L}([u, l, r]) = \alpha(u, [l, |t|]) \forall x \in U_\alpha \forall \alpha
\]

In order to show that the above arrow is an unfolding of \(T\), the reader only needs to check, for any chart \((U_\alpha, \alpha) \in \mathcal{U}\), the smoothness of \(\tilde{\alpha}\) in the next commutative square diagram:

\[
\begin{array}{ccc}
U_\alpha \times L \times \mathbb{R} & \xrightarrow{\tilde{\alpha}} & \tilde{T} \\
\downarrow & & \downarrow \mathcal{L} \\
U_\alpha \times \mathcal{L}(L) & \xrightarrow{\alpha} & T
\end{array}
\]

where \(\tilde{\alpha}(u, l, r) = [u, l, r]\) is just to pick the respective equivalence class.

• **Unfolding the whole spm \(X\):** Remark that

\[
\mathcal{L}^{-1}(T - S) = T^+ \sqcup T^-
\]

has two connected components, each of them being a smooth bundle over \(S\) with abstract fibers \(F^+ = L \times \mathbb{R}^+\) and \(F^- = L \times \mathbb{R}^-\) respectively. These two components are disjoint because the cocycles are radium-independent. The unfolding of the whole space \(X\) can be made by taking two copies of the regular part \(R = X - \Sigma\); say \(R^+, R^-\), and gluing them together and suitably with \(\tilde{T}\) along \(\mathcal{L}^{-1}(T - S)\). \(\square\)

In order to prove the converse statement, let’s recall that any manifold \(M\) with nonempty boundary \(\partial M \neq \emptyset\) has a **collar**, i.e. a transverse nbhd given by a smooth embedding \(\Gamma : \partial M \times [0, \infty) \longrightarrow M\) such that \(\text{Im}(\Gamma)\) is open in \(M\) and \(\Gamma(m, 0) = m\) for all \(m \in \partial M\).

**Proposition 2.2.6.** Every unfoldable pseudomanifold \(X\) is Thom-Mather.

**[Proof]** Let \(\bar{X} \xrightarrow{\mathcal{L}} X\) be an unfolding of \(X\). Then \(\mathcal{L}^{-1}(X - \Sigma)\) is a finite trivial smooth covering of \(R = (X - \Sigma)\), i.e. a disjoint union of finitely many diffeomorphic copies of \(R\). Pick one, say \(R_0 \cong R\), such that \(\overline{R_0}\) is a manifold with boundary \(\partial(\overline{R_0}) = \mathcal{L}^{-1}(S)\), and take a collar \(\mathcal{L}^{-1}(S) \times \mathbb{R} \xrightarrow{\Gamma} \overline{R_0}\) of \(\mathcal{L}^{-1}(S)\) in \(\overline{R_0}\). Define \(T = \mathcal{L}(\text{Im}(\Gamma))\) and

\[
T \xrightarrow{\tau} S \quad \tau(\mathcal{L}(\Gamma(z, r))) = \mathcal{L}(z)
\]
Following [7], the above map provides a tubular nbhd of $S$ and each unfolded chart as in §2.1.1(2) induces a bundle chart

$$U \times c(L) \xrightarrow{\tilde{\alpha}} \tau^{-1}(U) \quad \tilde{\alpha}(u, [l, r]) = L(\Gamma(\tilde{\alpha}(u, l, 0), r))$$

We leave the details to the reader. □

3. Functorial constructions

In the previous sections we dealt with the existence of unfoldings in terms of the tubular nbhds. Now we will see in more detail their categorical properties. It can be easily deduced from §2.1.1 that the unfoldings are neither unique nor functorial objects. We will restrict ourselves to a narrower family of spaces in order to develop some ideas concerning the smooth desingularization of pseudomanifolds and their functoriality, when considered as a topological process.

3.1. Primary unfoldings. Primary unfoldings are the smallest unfoldings one can find for a given pseudomanifold. They were originally presented by Brasselet, Hector and Saralegi [1] and later redefined by Dalmagro [3], whose points of view constitute the aim of this section.

Definition 3.1.1. A primary unfolding is an unfolding $\tilde{X} \xrightarrow{\mathcal{L}} X$ in our previous sense, such that the preimage of the regular part $\mathcal{L}^{-1}(R) = R_0 \sqcup R_1$ is a double (smooth, trivial) covering, i.e., the union of exactly two diffeomorphic copies of $R$.

Remark 3.1.2. The family of primary unfoldings is representative in the category of unfoldable spms. Starting from any unfolding $\tilde{X} \xrightarrow{\mathcal{L}} X$, we can construct a primary unfolding $\tilde{X}' \xrightarrow{\mathcal{L}'} X$ by taking a manifold with border $M = R_0$ as in the proof of §2.1.1 and then proceeding as in example §2.1.2-(3) in order to get $\tilde{M} \xrightarrow{\mathcal{L}''} M$. Then $\tilde{X}' = \tilde{M}$ and $\mathcal{L}' = \mathcal{L}\mathcal{L}''$ is the composition.

Lemma 3.1.3. Let $X, X'$ be two unfoldable pseudomanifolds. Then, for any pair of unfoldings $\tilde{X} \xrightarrow{\mathcal{L}} X$, $\tilde{X}' \xrightarrow{\mathcal{L}'} X'$ and any morphism $X \xrightarrow{f} X'$; there is a unique continuous and almost everywhere smooth map $\tilde{X} \xrightarrow{\tilde{f}} \tilde{X}'$ such that the square diagram

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X}' \\
\mathcal{L} \downarrow & & \downarrow \mathcal{L}' \\
X & \xrightarrow{f} & X'
\end{array}$$

is commutative; we call $\tilde{f}$ the lifting of $f$.

[Proof] In order to simplify the exposition, by the above remark §3.1.2 we assume that

$$\tilde{X} \xrightarrow{\mathcal{L}} X, \quad \tilde{X}' \xrightarrow{\mathcal{L}'} X'$$

are primary unfoldings.
(a) **Lifting \( f \) on the regular part:** If \( \Sigma = \emptyset \) then there is no singular part, and \( X = \mathbb{R} \) is a manifold. It follows that \( \tilde{X} = \mathbb{R}_0 \cup \mathbb{R}_1 \). Take \( \tilde{f} = f \times \text{id} \), then [3.1.3] trivially holds. So let’s suppose that \( \Sigma \neq \emptyset \) and moreover, by [2.2.3] we will assume that \( \Sigma = S \) is a single stratum. By these arguments, we have already defined a continuous function \( \tilde{f} \) satisfying \([3.1.3]\) on \( \mathcal{L}^{-1}(R) \). Therefore, we only must find a continuous extension of \( f \) to the entire \( \tilde{X} \), i.e., to \( \mathcal{L}^{-1}(S) \).

(b) **Extension of the lifting:** Pick some \( \tilde{z} \in \mathcal{L}^{-1}(S) \). We must show a way to choose \( \tilde{f}(\tilde{z}) \). For this sake, let \( \{ \tilde{z}_n \}_n \subset \mathcal{L}^{-1}(R) \) be a sequence converging to \( \tilde{z} \). Since \( \mathcal{L}, f \) are continuous maps and \( \mathcal{L}' \) is a continuous and proper map; by an argument of compactness, and up to some little adjusts, we may assume that the sequence \( \{ \tilde{f}(\tilde{z}_n) \}_n \) converges in \( \mathcal{X}' \). We define

\[
\tilde{f}(\tilde{z}) = \lim_{n \to \infty} \tilde{f}(\tilde{z}_n)
\]

If our limit-definition makes sense then it is also continuous; so next we will show the non ambiguity of \( \tilde{f} \). Since the former is a local definition, we first study the

(c) **Local writing of the lifting:** By [2.1.1](2) we can restrict to unfoldables charts; so we will assume that \( X = U \times c(L) \) and \( Y = V \times c(N) \) are trivial pem nbhds and their respective unfoldings are the canonical ones - see [2.1.2](2). Then \( f \) can be written as in [1.1.2](7). The point

\[
\tilde{z} = (u, l, 0) \in U \times L \times \{0\}
\]

is the limit of a sequence

\[
\{ \tilde{z}_n = (u_n, l_n, t_n) \}_n \subset U \times L \times (\mathbb{R} - \{0\})
\]

So the sequences \( \{u\}_n \), \( \{l\}_n \) and \( \{t\}_n \) respectively converge to \( u, l \) and \( 0 \). Since \( \mathcal{L}, f \) are continuous maps, the sequence

\[
w_n = f(\mathcal{L}(\tilde{z}_n)) = (a_1(u_n, l_n, |t_n|), a_2(u_n, l_n, |t_n|), a_3(u_n, l_n, |t_n|))
\]

converges to \( w = f(\mathcal{L}(\tilde{z})) = (a_1(u, l, 0), v) \). By the continuity of the functions \( a_j \) for \( j = 1, 2, 3 \) and up to some little adjust on \( a_2 \) concerning the compactness arguments; we get that

\[
\tilde{w}_n = (a_1(u_n, l_n, |t_n|), a_2(u_n, l_n, |t_n|), \pm a_3(u_n, l_n, |t_n|))
\]

converges to \( \tilde{w} = (a_1(u, l, 0), a_2(u, l, 0), 0) \).

(d) **The lifting is well defined:** From the continuity of the functions \( a_i \), the element \( w \) does not depend on the choice of a particular sequence \( \{ \tilde{z}_n \}_n \).

Notice that, the lifting \( \tilde{f} \) is always smooth on \( \mathcal{L}^{-1}(X - \Sigma) \), the preimage of the regular part.

\( \square \)

**Remark 3.1.4.** Given an unfolding \( \tilde{X} \xrightarrow{\mathcal{L}} X \); a **bubble** of \( \tilde{X} \) is a connected component of \( \mathcal{L}^{-1}(X - \Sigma) \). The above proof is still valid if we take any other permutation of the bubbles. Along the rest of this paper, we assume that we are working with the identity permutation, unless we state the opposite.
Lemma 3.1.5. A morphism \( f \) between pms nhbds is \textbf{lifting} if its lifting \( \tilde{f} \) is globally smooth on every \( X \). This is equivalent to ask \( \tilde{f} \) to be smooth on a nhbd of \( L^{-1}(\Sigma) \).

Proposition 3.1.6. A morphism between pm nhbds

\[
U \times c(L) \xrightarrow{f} U' \times c(L') \quad f(u, [l, r]) = (a_1(u, l, r), a_2(u, l, r), a_3(u, l, r))
\]

is liftable into

\[
U \times L \times \mathbb{R} \xrightarrow{\tilde{f}} U' \times L' \times \mathbb{R} \quad \tilde{f}(u, l, t) = (\tilde{a}_1(u, l, t), \tilde{a}_2(u, l, t), \tilde{a}_3(u, l, t))
\]

if and only if

(a) \( \tilde{a}_1, \tilde{a}_2 \) are smooth even extensions of, respectively, \( a_1, a_2 \).

(b) \( \tilde{a}_3 \) is either an odd (and therefore smooth) extension of \( a_3 \) or it is a smooth even extension and \( \tilde{a}_3(u, l, 0) = a_3(u, l, 0) = 0 \) for all \( u, l \).

[Proof] If \( \tilde{f} \) is a lifting of \( f \), then \( fc = c'\tilde{f} \) where \( c \) and \( c' \) are canonical unfoldings as in §2.1.1(2). Checking both sides of this equality we get

\[
f(c(u, l, t)) = f(u, [l, |t|]) = (a_1(u, l, |t|), a_2(u, l, |t|), a_3(u, l, |t|))
\]

and

\[
c'(\tilde{f}(u, l, t)) = c'((\tilde{a}_1(u, l, t), \tilde{a}_2(u, l, t), \tilde{a}_3(u, l, t))) = (\tilde{a}_1(u, l, t), [\tilde{a}_2(u, l, t), [\tilde{a}_3(u, l, t)]])
\]

we conclude that

\[
(a_1(u, l, |t|), a_2(u, l, |t|), a_3(u, l, |t|)) = (\tilde{a}_1(u, l, t), [\tilde{a}_2(u, l, t), [\tilde{a}_3(u, l, t)]])
\]

There are two cases; \( t = 0 \) and \( t \neq 0 \), from which we get 3.1.6. \( \Box \)

Lemma 3.1.7. Let \( X \) be a Thom-Mather spm, then the cocycles of a tubular neighborhood are liftable.

[Proof] It is enough to take \( \varphi = \beta^{-1}\alpha \) as in §2.2.1(4) and check that \( a_1(u, l, r) = u, a_2(u, l, r) = g(u)(l), a_3(u, l, r) = r \) satisfy the hypothesis of 3.1.6. \( \Box \)

Lemma 3.1.8. Let \( f, f' : U \times c(L) \longrightarrow U' \times c(L') \) be morphisms, \( \varphi : U \times c(L) \longrightarrow U \times c(L), \varphi' : U' \times c(L') \longrightarrow U' \times c(L') \) be isomorphisms as in §2.2.1(4). Then \( f'\varphi = \varphi'f \) if and only if \( a_1, a_3 \) are invariant respect to the group action on the coordinate \( l \) and \( a_2 \) commutes with the cocycles i.e, \( a_1, a_2 \) and \( a_3 \) satisfy:

\[
a_1(u, l, r) = a'_1(u, g(u)(l), r) \quad (5) \quad a_2(u, l, r) = a'_2(u, g(u)(l), r) \quad a_3(u, l, r) = a'_3(u, g(u)(l), r)
\]

[Proof] Let \( f \) as above, and \( \varphi, \varphi' \) be cocycles as in §2.2.1(4), then

\[
f'(\varphi(u, [l, r])) = f'(u, [g(u)(l), r]) = (a'_1(u, g(u)(l), r), a'_2(u, g(u)(l), r), a'_3(u, g(u)(l), r))
\]
On the other hand
\[ \varphi'(f(u, [l, r])) = \varphi'(a_1(u, l, r), [a_2(u, l, r), a_3(u, l, r)]) = (a_1(u, l, r), [g'(a_1(u, l, r)(a_2(u, l, r)), a_3(u, l, r))]) \]

If \( f' \varphi = \varphi' f \) then, after checking the cases \( r = 0 \) and \( r \neq 0 \), we get \( 3.1.8 \) \( \Box \)

**Lemma 3.1.9.** Let \( X \xrightarrow{\Psi} X' \) be a Thom-Mather morphism (see \( 2.2.1 \)) then the restriction of \( \Psi \) on each respective local trivialization satisfies \( 3.1.8 \).

**Proof** Without any loss of generality, we can assume that \( T = X \) and \( T' = X' \) (see \( 2.2.3 \)). Let \( \alpha, \beta \) and \( \alpha', \beta' \) be two bundle charts of \( X \) and \( X' \) respectively defined on \( U \times c(L) \) and \( U' \times c(L') \). If \( \varphi = \beta^{-1} \alpha \) and \( \varphi' = (\beta')^{-1} \alpha' \) are their respective cocycles (c.f., \( 2.2.1 (4) \)). It follows that \( f = (\alpha')^{-1} \Psi \alpha \) and \( f' = (\beta')^{-1} \Psi \beta \) satisfy \( 3.1.8 \) \( \Box \)

These results imply that there is a functor from the category of Thom-Mather spaces to the category of the smooth manifolds. Formally we state the next theorem.

**Theorem 3.1.10.** Every Thom-Mather morphism has a smooth Thom-Mather lifting.

**Proof** Let \( X, X' \) be two Thom-Mather spms, if \( f \) has a smooth lifting \( \tilde{f} \) as in \( 3.1.6 \) then, according to \( 3.1.3 \) there is a unique almost everywhere smooth map \( \tilde{X} \xrightarrow{\tilde{f}} \tilde{X}' \). by \( 3.1 \) \( \tilde{f} \) is locally smooth on a nbhd of \( L^{-1}(\Sigma) \) then \( \tilde{f} \) is smooth. \( \Box \)

### 3.2. Uniqueness of the primary unfoldings

**Theorem 3.2.1.** Let \( X, X' \) be two Thom-Mather spaces and let \( X \xrightarrow{f} X' \) be a Thom-Mather isomorphism then the smooth lifting of \( f, \tilde{f} \) is a diffeomorphism between the manifolds \( \tilde{X} \) and \( \tilde{X}' \).

**Proof** We know that by \( 3.1.3 \) the lifting of \( f, \tilde{f} \) is unique up to permutation of the bubbles. Let us fix a bubble permutation \( \sigma \) as in \( 3.1.4 \) on \( L^{-1}(X - \Sigma) \). As \( f \) is a Thom-Mather morphism its lifting \( \tilde{f} \) is a smooth Thom-Mather map between \( \tilde{X} \) and \( \tilde{X}' \) (see, \( 3.1.10 \)). On the other hand, the inverse morphism of \( f, g \) is also a Thom-Mather map which lifts into a smooth manifold map \( \tilde{g} \) between \( \tilde{X}' \) and \( \tilde{X} \). In order to satisfy the equations \( \tilde{g} \tilde{f} = id_{\tilde{X}} \) and \( \tilde{f} \tilde{g} = id_{\tilde{X}'} \) we take the lifting \( \tilde{g} \) of \( g \) induced with the inverse permutation \( \sigma^{-1} \) of the bubbles. So, therefore \( \tilde{f} \) is a diffeomorphism. \( \Box \)

**Corollary 3.2.2.** The primary unfolding of a Thom-Mather space is unique.

**Proof** Take \( X = X' \) as in \( 3.2.1 \) and apply the same argumentation of this lemma to the identity map on \( X \). According to \( 3.1.4 \) this is true up to bubble labeling. \( \Box \)
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