Purity of states in the theory of open quantum systems

A. Isar*

Department of Theoretical Physics, Institute of Physics and Nuclear Engineering
Bucharest-Magurele, Romania

Abstract

The condition of purity of states for a damped harmonic oscillator is considered in the framework of Lindblad theory for open quantum systems. For a special choice of the environment coefficients, the correlated coherent states are shown to be the only states which remain pure all the time during the evolution of the considered system. These states are also the most stable under evolution in the presence of the environment.

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* e-mail address: isar@theory.nipne.ro

1 Introduction

In the last two decades, more and more interest has arisen about the search for a consistent description of open quantum systems [1–5] (for a recent review see ref. [6]). Dissipation in an open system results from microscopic reversible interactions between the observable system and the environment. Because dissipative processes imply irreversibility and, therefore, a preferred direction in time, it is generally thought that quantum dynamical semigroups are the basic tools to introduce dissipation in quantum mechanics. In the Markov approximation the most general form of the generators of such semigroups was given by Lindblad [7]. This formalism has been studied for the case of damped harmonic oscillators [6, 8–12] and applied to various physical phenomena, for instance, to the damping of collective modes in deep inelastic collisions in nuclear physics [13]. A phase space representation for the open quantum systems within the Lindblad theory was given in [14, 15].

In the present study we are also concerned with the observable system of a harmonic oscillator which interacts with an environment. We discuss under what conditions the open system can be described by a quantum mechanical pure state. In Sec. 2 we present the generalized uncertainty relations and the correlated coherent states, first introduced in [16], which minimize these relations. The Lindblad theory for open quantum systems is considered in Sec. 3. Then in Sec. 4, for the one-dimensional
harmonic oscillator as an open system, we show for a special choice of the diffusion coefficients that the correlated coherent states, taken as initial states, remain pure for all time during the evolution. In some other simple models of the damped harmonic oscillator in the framework of quantum statistical theory [17, 18], it was shown that the pure Glauber coherent states remain as those during the evolution and in all other cases, the oscillator immediately evolves into mixtures. In this respect we generalize this result and also our previous result from [10] as well as the results of other authors [19], obtained by using different methods. In Sec. 5 we introduce the linear entropy and present its role for the description of the decoherence phenomenon. We claim that the correlated coherent states are the most stable under evolution in the presence of the environment and make the connection with the work done in this field by other authors [20–24]. Finally, concluding remarks are given in Sec. 6.

2 Generalized uncertainty relations

In the following we denote by $\sigma_{AA}$ the dispersion of the operator $\hat{A}$, i.e. $\sigma_{AA} = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2$, where $\langle \hat{A} \rangle \equiv \sigma_A = \text{Tr}(\hat{\rho}\hat{A})$, $\text{Tr} \hat{\rho} = 1$ and $\hat{\rho}$ is the statistical operator (density matrix). By $\sigma_{AB} = 1/2 \langle \hat{A}\hat{B} + \hat{B}\hat{A} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle$ we denote the correlation of the operators $\hat{A}$ and $\hat{B}$. Schrödinger [25] and Robertson [26] proved for any Hermitian operators $\hat{A}$ and $\hat{B}$ that the uncertainty relation (1) takes the form

$$\sigma_{AA}\sigma_{BB} - \sigma_{AB}^2 \geq \frac{1}{4} |\langle [\hat{A}, \hat{B}] \rangle|^2. \quad (1)$$

For the particular case of the operators of the coordinate $\hat{q}$ and momentum $\hat{p}$ the uncertainty relation (1) takes the form

$$\sigma_{pp}\sigma_{qq} - \sigma_{pq}^2 \geq \frac{\hbar^2}{4}. \quad (2)$$

This result was generalized for arbitrary operators (in general non-Hermitian) and for the most general case of mixed states in [16]. The inequality (2) can also be represented in the following form [16]:

$$\sigma_{pp}\sigma_{qq} \geq \frac{\hbar^2}{4(1 - r^2)}, \quad (3)$$

where

$$r = \frac{\sigma_{pq}}{\sqrt{\sigma_{pp}\sigma_{qq}}} \quad (4)$$
is the correlation coefficient. The equality in the relation (2) is realized for a special class of pure states, called correlated coherent states [16], which are represented by Gaussian wave packets in the coordinate representation. These minimizing states, which generalize the Glauber coherent states, are eigenstates of an operator of the form [16]:

\[ \hat{a}_{r,\eta} = \frac{1}{2\eta}[1 - \frac{ir}{(1 - r^2)^{1/2}}]q + i\frac{\eta}{\hbar}p \]  

(5)

with real parameters \( r \) and \( \eta, |r| < 1, \eta = \sqrt{\sigma_{qq}} \). Their normalized eigenfunctions, the correlated coherent states, have the form [16]:

\[ \Psi(x) = \frac{1}{(2\pi\eta^2)^{1/4}} \exp\left\{ -\frac{x^2}{4\eta^2} \left[ 1 - \frac{ir}{(1 - r^2)^{1/2}} \right] + \frac{\alpha x}{\eta} - \frac{1}{2}(\alpha^2 + |\alpha|^2) \right\}, \]  

(6)

with \( \alpha \) a complex number. If we set \( r = 0 \) and \( \eta = (\hbar/2m\omega)^{1/2} \), where \( m \) and \( \omega \) are the mass and respectively the frequency of the harmonic oscillator, the states (6) become the usual Glauber coherent states. In Wigner representation, the states (6) have the form [16]:

\[ W_{\alpha,\eta}(p, q) = \frac{1}{\pi\hbar} \exp[-\frac{2\eta^2}{\hbar^2}(p - \sigma_p)^2 - \frac{(q - \sigma_q)^2}{2\eta^2(1 - r^2)} + \frac{2r}{\hbar(1 - r^2)^{1/2}}(q - \sigma_q)(p - \sigma_p)]. \]  

(7)

This is the classical normal distribution to give dispersion

\[ \sigma_{qq} = \eta^2, \quad \sigma_{pp} = \frac{\hbar^2}{4\eta^2(1 - r^2)}, \quad \sigma_{pq} = \frac{\hbar r}{2(1 - r^2)^{1/2}} \]  

(8)

and the correlation coefficient \( r \). The Gaussian distribution (7) is the only positive Wigner distribution for a pure state [27]. All other Wigner functions that describe pure states necessarily take on negative values for some values of \( p, q \).

In the case of the relation (1) the equality is generally obtained only for pure states [16]. For any density matrix in the coordinate representation (normalized to unity) the following relation must be fulfilled:

\[ \gamma = \text{Tr}\hat{\rho}^2 \leq 1. \]  

(9)

The quantity \( \gamma \) characterizes the degree of purity of the state. For pure states \( \gamma = 1 \), for highly mixed states \( \gamma \ll 1 \) and for weakly mixed states \( 1 - \gamma \ll 1 \).

The Wigner function may be expressed as the Fourier transform of the off-diagonal matrix elements of the density operator in the coordinate representation:

\[ W(p, q, t) = \frac{1}{\pi\hbar} \int dy <q - y|\hat{\rho}|q + y>e^{2ipy/\hbar}. \]  

(10)
Then \( < x|\hat{\rho}|y > \) can be obtained from the inverse Fourier transform of the Wigner function:

\[
< x|\hat{\rho}|y > = \int dp \exp(\frac{i}{\hbar}p(x-y))W(p, \frac{x+y}{2}).
\] (11)

In the language of the Wigner function the condition (9) has the form:

\[
\gamma = 2\pi\hbar \int W^2(p, q)dpdq \leq 1.
\] (12)

Let us consider the most general mixed squeezed states described by the Wigner function of the generic Gaussian form with five real parameters:

\[
W(p, q) = \frac{1}{2\pi\sqrt{\sigma}} \exp\{-\frac{1}{2\sigma}[\sigma_{pp}(q - \sigma_q)^2 + \sigma_{qq}(p - \sigma_p)^2 - 2\sigma_{pq}(q - \sigma_q)(p - \sigma_p)]\},
\] (13)

where \( \sigma \) is the determinant of the dispersion (correlation) matrix \( M \),

\[
\sigma = \text{det} M = \sigma_{pp}\sigma_{qq} - \sigma_{pq}^2
\] (14)

and

\[
M = \begin{pmatrix}
\sigma_{pp} & \sigma_{pq} \\
\sigma_{pq} & \sigma_{qq}
\end{pmatrix}.
\] (15)

The Gaussian Wigner functions of this form correspond to the so-called quasi-free states on the \( C^* \)-algebra of the canonical commutation relations, which is the most natural framework for a unified treatment of quantum and thermal fluctuations [28].

For Gaussian states of the form (13) the coefficient of purity \( \gamma \) is given by

\[
\gamma = \frac{\hbar}{2\sqrt{\sigma}}.
\] (16)

Therefore, the dispersion matrix has to satisfy the Schrödinger-Robertson uncertainty relation (2)

\[
\sigma \geq \frac{\hbar^2}{4}.
\] (17)

This inequality must be fulfilled actually for any states, not only Gaussian. Any Gaussian pure state minimizes the relation (17). Here, \( \sigma \) is also the Wigner function area – a measure of the phase space area in which the Gaussian density matrix is localized. For \( \sigma > \hbar^2/4 \) the function (13) corresponds to mixed quantum states, while in the case of the equality \( \sigma = \hbar^2/4 \) it takes the form (7) corresponding to pure correlated coherent states.
The degree of the purity of a state can also be characterized by other quantities besides $\gamma$. The most usual one is the quantum entropy (we put the Boltzmann’s constant $k = 1$):

$$S = -\text{Tr}(\hat{\rho} \ln \hat{\rho}) = - \langle \ln \hat{\rho} \rangle. \quad (18)$$

For quantum pure states the entropy is identically equal to zero. It was shown [12, 29] that for Gaussian states with the Wigner functions (13) the entropy can be expressed through $\sigma$ only:

$$S = (\nu + 1) \ln(\nu + 1) - \nu \ln \nu, \quad \nu = \frac{1}{\hbar} \sqrt{\sigma} - \frac{1}{2}. \quad (19)$$

### 3 Quantum Markovian master equation for damped harmonic oscillator

The simplest dynamics for an open system which describes an irreversible process is a semigroup of transformations introducing a preferred direction in time [2, 3, 7]. In Lindblad’s axiomatic formalism of introducing dissipation in quantum mechanics, the usual von Neumann-Liouville equation ruling the time evolution of closed quantum systems is replaced by the following Markovian master equation for the density operator $\hat{\rho}(t)$ in the Schrödinger picture [7]:

$$\frac{d \Phi_t(\hat{\rho})}{dt} = L(\Phi_t(\hat{\rho})). \quad (20)$$

Here, $\Phi_t$ denotes the dynamical semigroup describing the irreversible time evolution of the open system in the Schrödinger representation and $L$ is the infinitesimal generator of the dynamical semigroup $\Phi_t$. Using the Lindblad theorem [7] which gives the most general form of a bounded, completely dissipative generator $L$, we obtain the explicit form of the most general quantum mechanical master equation of Markovian type:

$$\frac{d\hat{\rho}(t)}{dt} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}(t)] + \frac{1}{2\hbar} \sum_j ([\hat{V}_j \hat{\rho}(t), \hat{V}_j^\dagger] + [\hat{V}_j, \hat{\rho}(t) \hat{V}_j^\dagger]). \quad (21)$$

Here $\hat{H}$ is the Hamiltonian operator of the system and $\hat{V}_j, \hat{V}_j^\dagger$ are bounded operators on the Hilbert space $\mathcal{H}$ of the Hamiltonian and they model the environment. We make the basic assumption that the general form (21) of the master equation with a bounded generator is also valid for an unbounded generator. In the case of an exactly solvable model for the damped harmonic oscillator, we define the possible two operators $\hat{V}_1$ and $\hat{V}_2$, which are linear in $\hat{p}$ and $\hat{q}$, as follows [6, 8, 9]:

$$\hat{V}_j = a_j \hat{p} + b_j \hat{q}, \quad j = 1, 2, \quad (22)$$
with \(a_j, b_j\) complex numbers. The harmonic oscillator Hamiltonian \(\hat{H}\) is chosen of the general form

\[
\hat{H} = \hat{H}_0 + \frac{\mu}{2} (\hat{q} \hat{p} + \hat{p} \hat{q}), \quad \hat{H}_0 = \frac{1}{2m} \hat{p}^2 + \frac{m \omega^2}{2} \hat{q}^2.
\]

(23)

With these choices and with the notations

\[
D_{qq} = \frac{\hbar}{2} \sum_{j=1,2} |a_j|^2, \quad D_{pp} = \frac{\hbar}{2} \sum_{j=1,2} |b_j|^2, \quad D_{pq} = D_{qp} = -\frac{\hbar}{2} \text{Re} \sum_{j=1,2} a_j^* b_j, \quad \lambda = -\text{Im} \sum_{j=1,2} a_j^* b_j.
\]

(24)

where \(a_j^*\) and \(b_j^*\) denote the complex conjugate of \(a_j\), and \(b_j\), respectively, the master equation (21) takes the following form [6, 9]:

\[
\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar} [\hat{H}_0, \hat{\rho}] - \frac{i}{2\hbar} (\lambda + \mu) [\hat{q}, \hat{p} \hat{\rho} + \hat{\rho} \hat{p}] + \frac{i}{2\hbar} (\lambda - \mu) [\hat{p}, \hat{q} \hat{\rho} + \hat{\rho} \hat{q}]
\]

\[
- \frac{D_{pp}}{\hbar^2} [\hat{q}, [\hat{q}, \hat{\rho}]] - \frac{D_{qq}}{\hbar^2} [\hat{p}, [\hat{p}, \hat{\rho}]] + \frac{D_{pq}}{\hbar^2} ([\hat{q}, [\hat{p}, \hat{\rho}]] + [\hat{p}, [\hat{q}, \hat{\rho}]]).
\]

(25)

Here the quantum mechanical diffusion coefficients \(D_{pp}, D_{qq}, D_{pq}\) and the friction constant \(\lambda\) satisfy the following fundamental constraints [6, 9]: \(D_{pp} > 0, D_{qq} > 0\) and

\[
D_{pp} D_{qq} - D_{pq}^2 \geq \frac{\lambda^2 \hbar^2}{4}.
\]

(26)

The relation (26) is a necessary condition for the generalized uncertainty inequality (2) to be fulfilled.

The semigroup method is valid for the weak-coupling regime, with the damping \(\lambda\) typically obeying the inequality \(\lambda \ll \omega_0\), where \(\omega_0\) is the lowest frequency typical of reversible motions.

The necessary and sufficient condition for \(L\) to be translationally invariant is \(\mu = \lambda\) [6, 8, 9]. Translation invariance means that \([p, L(\rho)] = L([p, \rho])\). In the following general values for \(\lambda\) and \(\mu\) will be considered.

By using the fact that the linear positive mapping defined by \(\hat{A} \to \text{Tr}(\hat{\rho} \hat{A})\) is completely positive, in [6, 9] the following inequality was obtained:

\[
D_{pp} \sigma_{qq}(t) + D_{qq} \sigma_{pp}(t) - 2D_{pq} \sigma_{pq}(t) \geq \frac{\hbar^2 \lambda}{2}.
\]

(27)

We have found in [10] that this inequality, which must be valid for all values of \(t\) is equivalent with the generalized uncertainty inequality (2) at any time \(t\),

\[
\sigma_{qq}(t) \sigma_{pp}(t) - \sigma_{pq}^2(t) \geq \frac{\hbar^2}{4},
\]

(28)
if the initial values $\sigma_{qq}(0), \sigma_{pp}(0)$ and $\sigma_{pq}(0)$ for $t = 0$ satisfy this inequality. If the initial state is the ground state of the harmonic oscillator, then

$$\sigma_{qq}(0) = \frac{\hbar}{2m\omega}, \quad \sigma_{pp}(0) = \frac{m\hbar\omega}{2}, \quad \sigma_{pq}(0) = 0.$$  

(29)

By using the complete positivity property of the dynamical semigroup $\Phi_t$, it was shown in [9] that the relation

$$\text{Tr}(\Phi_t(\hat{\rho}) \sum_j \hat{V}_j^\dagger \hat{V}_j) = \sum_j \text{Tr}(\Phi_t(\hat{\rho}) \hat{V}_j^\dagger) \text{Tr}(\Phi_t(\hat{\rho}) \hat{V}_j)$$  

(30)

represents the necessary and sufficient condition for $\hat{\rho}(t) = \Phi_t(\hat{\rho})$ to be a pure state for all times $t \geq 0$. This equality is a generalization of the pure state condition [30–32] to all Markovian master equations (21). If $\hat{\rho}^2(t) = \hat{\rho}(t)$ for all $t \geq 0$, there exists a wave function $\psi \in \mathcal{H}$ which satisfies the nonlinear Schrödinger-type equation

$$i\hbar \frac{\partial \psi(t)}{\partial t} = \hat{H}' \psi(t),$$  

(31)

with the non-Hermitian Hamiltonian

$$\hat{H}' = \hat{H} + i \sum_j <\psi(t), \hat{V}_j^\dagger \psi(t) > \hat{V}_j - \frac{i}{2} <\psi(t), \sum_j \hat{V}_j^\dagger \hat{V}_j \psi(t) > - \frac{i}{2} \sum_j \hat{V}_j^\dagger \hat{V}_j.$$

(32)

For environment operators $\hat{V}_j$ of the form (22), the pure state condition (30) takes the following form [9], corresponding to equality in the relation (27):

$$D_{pp}\sigma_{qq}(t) + D_{qq}\sigma_{pp}(t) - 2D_{pq}\sigma_{pq}(t) = \frac{\hbar^2 \lambda}{2}$$  

(33)

and the Hamiltonian (32) becomes

$$\hat{H}' = \hat{H} + \lambda(\sigma_p(t)\dot{q} - \sigma_q(t)\dot{p}) + \frac{i}{\hbar}[\lambda \hbar^2 - D_{qq}((\dot{q} - \sigma_q(t))^2 + \sigma_{pp}(t)) - D_{pp}((\dot{q} - \sigma_q(t))^2 + \sigma_{qq}(t)) + D_{pq}((\dot{q} - \sigma_q(t))(\dot{q} - \sigma_q(t)) + (\dot{q} - \sigma_q(t))(\dot{q} - \sigma_q(t))) + 2\sigma_{pq}(t))].$$  

(34)

From the master equation (25) we obtain the following equations of motion for the expectation values and variances of the coordinate and momentum:

$$\frac{d\sigma_q(t)}{dt} = - (\lambda - \mu)\sigma_q(t) + \frac{1}{m}\sigma_p(t),$$  

(35)

$$\frac{d\sigma_p(t)}{dt} = - m\omega^2\sigma_q(t) - (\lambda + \mu)\sigma_p(t)$$  

(36)
and
\[
\frac{d\sigma_{qq}(t)}{dt} = -2(\lambda - \mu)\sigma_{qq}(t) + \frac{2}{m}\sigma_{pq}(t) + 2D_{qq}, \tag{37}
\]
\[
\frac{d\sigma_{pp}(t)}{dt} = -2(\lambda + \mu)\sigma_{pp}(t) - 2m\omega^2\sigma_{pq}(t) + 2D_{pp}, \tag{38}
\]
\[
\frac{d\sigma_{pq}(t)}{dt} = -m\omega^2\sigma_{qq}(t) + \frac{1}{m}\sigma_{pp}(t) - 2\lambda\sigma_{pq}(t) + 2D_{pq}. \tag{39}
\]

In the underdamped case \((\omega > \mu)\) considered in this paper, with the notation \(\Omega^2 = \omega^2 - \mu^2\), we obtain [6, 9]:
\[
\sigma_q(t) = e^{-\lambda t}((\cos \Omega t + \frac{\mu}{\Omega} \sin \Omega t)\sigma_q(0) + \frac{1}{m\Omega} \sin \Omega t\sigma_p(0)), \tag{40}
\]
\[
\sigma_p(t) = e^{-\lambda t}(-\frac{m\omega^2}{\Omega^2} \sin \Omega t\sigma_q(0) + (\cos \Omega t - \frac{\mu}{\Omega} \sin \Omega t)\sigma_p(0)) \tag{41}
\]

and \(\sigma_q(\infty) = \sigma_p(\infty) = 0\). It is convenient to consider the vectors
\[
X(t) = \begin{pmatrix} m\omega\sigma_{qq}(t) \\ \sigma_{pp}(t)/m\omega \\ \sigma_{pq}(t) \end{pmatrix} \tag{42}
\]
and
\[
D = \begin{pmatrix} 2m\omega D_{qq} \\ 2D_{pp}/m\omega \\ 2D_{pq} \end{pmatrix}. \tag{43}
\]

With these notations the solutions for the variances can be written in the form [6, 9]:
\[
X(t) = (Te^{KT}T)(X(0) - X(\infty)) + X(\infty), \tag{44}
\]
where the matrices \(T\) and \(K\) are given by
\[
T = \frac{1}{2i\Omega} \begin{pmatrix} \mu + i\Omega & \mu - i\Omega & 2\omega \\ \mu - i\Omega & \mu + i\Omega & 2\omega \\ -\omega & -\omega & -2\mu \end{pmatrix}, \tag{45}
\]
\[
K = \begin{pmatrix} -2(\lambda - i\Omega) & 0 & 0 \\ 0 & -2(\lambda + i\Omega) & 0 \\ 0 & 0 & -2\lambda \end{pmatrix}. \tag{46}
\]
and

\[ X(\infty) = -(TK^{-1}T)D. \]  

(47)

The formula (47) is remarkable because it gives a very simple connection between
the asymptotic values \((t \to \infty)\) of \(\sigma_{qq}(t), \sigma_{pp}(t), \sigma_{pq}(t)\) and the diffusion coefficients
\(D_{qq}, D_{pp}, D_{pq}:\)

\[\sigma_{qq}(\infty) = \frac{1}{2(m\omega)^2\lambda(\lambda^2 + \omega^2 - \mu^2)}((m\omega)^2(2\lambda(\lambda + \mu) + \omega^2)D_{qq} + \omega^2D_{pp} + 2m\omega(\lambda + \mu)D_{pq}),\]

(48)

\[\sigma_{pp}(\infty) = \frac{1}{2\lambda(\lambda^2 + \omega^2 - \mu^2)}((m\omega)^2D_{qq} + (2\lambda(\lambda - \mu) + \omega^2)D_{pp} - 2m\omega(\lambda - \mu)D_{pq}),\]

(49)

\[\sigma_{pq}(\infty) = \frac{1}{2m\lambda(\lambda^2 + \omega^2 - \mu^2)}(-\lambda + \mu)(m\omega)^2D_{qq} + (\lambda - \mu)D_{pp} + 2m(\lambda^2 - \mu^2)D_{pq}).\]

(50)

These relations show that the asymptotic values \(\sigma_{qq}(\infty), \sigma_{pp}(\infty), \sigma_{pq}(\infty)\) do not depend
on the initial values \(\sigma_{qq}(0), \sigma_{pp}(0), \sigma_{pq}(0)\). In the considered underdamped case we have

\[Te^{Kt}T = -\frac{e^{-2\lambda t}}{2\Omega^2} \left( \begin{array}{ccc} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{array} \right),\]

(51)

where \(b_{ij}, i, j = 1, 2, 3\) are time-dependent oscillating functions given by (3.78) in [9].

4 Purity of states

We will be interested to find the Gaussian states which remain pure during the evo-
lution of the system for all times \(t\). We start by considering the pure state condition
(33) and the generalized uncertainty relation (2) which transforms into the following
minimum uncertainty equality for pure states:

\[\sigma_{pp}(t)\sigma_{qq}(t) - \sigma_{pq}^2(t) = \frac{\hbar^2}{4}.\]

(52)

By eliminating \(\sigma_{pp}\) between the equalities (33) and (52), like in [33], we obtain:

\[\left(\sigma_{qq}(t) - \frac{D_{pq}\sigma_{pq}(t) + \frac{1}{4}\hbar^2\lambda}{D_{pp}}\right)^2 + \frac{D_{pp}D_{pp} - D_{pq}^2}{D_{pp}^2}\left[\sigma_{pq}(t) - \left\{\sigma_{pq}(t) - \frac{1}{4}\hbar^2\lambda D_{pq} \right\}^2\right] \]

\[+ \frac{1}{4}\hbar^2 \frac{D_{pq}D_{pp} - D_{pq}^2 - \frac{1}{4}\hbar^2\lambda^2}{(D_{pp}D_{pp} - D_{pq}^2)^2}D_{pp}D_{qq} = 0.\]

(53)
Since the opening coefficients satisfy the inequality (26), from the relation (53) we obtain the following relations which have to be fulfilled at any moment of time:

\[ D_{pp}D_{qq} - D_{pq}^2 = \frac{\hbar^2\lambda^2}{4}, \]  

\[ D_{pp}\sigma_{qq}(t) - D_{pq}\sigma_{pq}(t) - \frac{\hbar^2\lambda}{4} = 0, \]  

\[ \sigma_{pq}(t)(D_{pp}D_{qq} - D_{pq}^2) - \frac{\hbar^2\lambda}{4}D_{pq} = 0. \]  

From the relations (52) and (54)–(56) it follows that the pure states remain pure for all times only if their variances are constant in time and have the form:

\[ \sigma_{pp}(t) = \frac{D_{pp}}{\lambda}, \quad \sigma_{qq}(t) = \frac{D_{qq}}{\lambda}, \quad \sigma_{pq}(t) = \frac{D_{pq}}{\lambda}. \]  

If these relations are fulfilled, then the inequalities (27) and (28) are both equivalent to (26), including also the corresponding equalities (33), (52) and (54). From Eq. (44) it follows that the variances remain constant and do not depend on time only if \( X(0) = X(\infty) \), which means \( \sigma_{pp}(0) = \sigma_{pp}(\infty) \), \( \sigma_{qq}(0) = \sigma_{qq}(\infty) \), \( \sigma_{pq}(0) = \sigma_{pq}(\infty) \).

Using the asymptotic values (48)–(50) of the variances and the relations (57), we obtain the following expressions of the diffusion coefficients which assure that the initial pure states remain pure for any \( t \):

\[ D_{qq} = \frac{\hbar\lambda}{2m\Omega}, \quad D_{pp} = \frac{\hbar\lambda m\omega^2}{2\Omega}, \quad D_{pq} = -\frac{\hbar\lambda\mu}{2\Omega}. \]  

Formulas (58) are generalized Einstein relations and represent examples of quantum fluctuation-dissipation relations, connecting the diffusion with both Planck’s constant and damping constant. With the coefficients (58), the variances (48)–(50) become:

\[ \sigma_{qq} = \frac{\hbar}{2m\Omega}, \quad \sigma_{pp} = \frac{\hbar m\omega^2}{2\Omega}, \quad \sigma_{pq} = -\frac{\hbar\mu}{2\Omega}. \]  

Therefore, the quantity \( \sigma \) (see Eq. (14)) is equal to its minimum possible value \( \hbar^2/4 \), according to the generalized uncertainty relation (17). Then the corresponding state described by a Gaussian Wigner function is a pure quantum state, namely a correlated coherent state [16] with the correlation coefficient (4) \( r = -\mu/\omega \). Given \( \sigma_{pp} \), \( \sigma_{qq} \) and \( \sigma_{pq} \), there exists one and only one such a state minimizing the uncertainty \( \sigma \) [34]. A particular case of our result (corresponding to \( \lambda = \mu \) and \( D_{pq} = 0 \)) was obtained by
Halliwell and Zoupas by using the quantum state diffusion method [19]. We remark that the minimization of the quantity (14) is equivalent, by virtue of the relation (19), to the minimization of the entropy $S$. As we have mentioned before, we have considered general coefficients $\mu$ and $\lambda$ and in this respect our expressions for the diffusion coefficients and variances generalize also the ones obtained by Dekker and Valsakumar [33] and Dodonov and Man’ko [35], who used models where $\mu = \lambda$ was chosen. If $\mu = 0$, we get from (58) $D_{pq} = 0$. This case, which was considered in [10], where we have obtained a density operator describing a pure state for any $t$, is also a particular case of the present results which give the most general Gaussian pure state which remains pure for any $t$. For $\mu = 0$, the expressions (59) become

$$\sigma_{qq} = \frac{\hbar}{2m\omega}, \quad \sigma_{pp} = \frac{\hbar m \omega}{2}, \quad \sigma_{pq} = 0,$$

which are the values of variances (29) for the ground state of the harmonic oscillator and the correlation coefficient (4) takes the value $r = 0$, corresponding to the case of usual coherent states.

The Lindblad equation or its equivalent Fokker-Planck equation for the Wigner function with the diffusion coefficients (58) can be used only in the underdamped case, when $\omega > \mu$. Indeed, for the coefficients (58) the fundamental constraint (26) implies that $m^2(\omega^2 - \mu^2)D_{qq}^2 \geq \hbar^2 \lambda^2/4$, which is satisfied only if $\omega > \mu$. It can be shown that there exist diffusion coefficients which satisfy the condition (54) and make sense for $\omega < \mu$, but in this overdamped case we have always $\sigma > \hbar^2/4$ and the state of the oscillator cannot be pure for any diffusion coefficients [35].

The fluctuation energy of the open harmonic oscillator is

$$E(t) = \frac{1}{2m} \sigma_{pp}(t) + \frac{1}{2} m \omega^2 \sigma_{qq}(t) + \mu \sigma_{pq}(t).$$

If the state remains pure in time, then the variances are given by (57) and the fluctuation energy is also constant in time and is given by

$$E = \frac{1}{\lambda} (\frac{1}{2m} D_{pp} + \frac{1}{2} m \omega^2 D_{qq} + \mu D_{pq}).$$

Minimizing this expression with the condition (54), we obtain just the diffusion coefficients (58) and $E_{\text{min}} = \hbar \Omega/2$. Therefore, the conservation of the purity of state implies that the fluctuation energy of the system has all the time the minimum possible value $E_{\text{min}}$. Moreover, we can consider that the case when the diffusion coefficients satisfy the equality (54) corresponds to a zero temperature of the environment (bath). Then the influence on the oscillator is minimal.
If we choose the coefficients of the form (58), then the equation for the density operator can be represented in the form (21) with only one operator \( \hat{V} \), which up to a phase factor can be written in the form:

\[
\hat{V} = \sqrt{\frac{2}{\hbar D_{qq}}} \left[ \left( \frac{\lambda}{2} - iD_{pp}\hat{q} + iD_{qq}\hat{p} \right) \right],
\]

with \([\hat{V}, \hat{V}^\dagger] = 2\hbar \lambda\).

The correlated coherent states (6) with nonvanishing momentum average, can also be written in the form:

\[
\Psi(x) = \left( \frac{1}{2\pi \sigma_{qq}} \right)^{\frac{1}{4}} \exp\left\{ -\frac{1}{2\sigma_{qq}}(x - \sigma_q)^2 - \frac{i}{\hbar} \sigma_p x \right\}
\]

and the most general form of Gaussian density matrices compatible with the generalized uncertainty relation (2) is the following:

\[
<x|\hat{\rho}|y> = \left( \frac{1}{2\pi \sigma_{qq}} \right)^{\frac{1}{2}} \exp\left\{ -\frac{1}{2\sigma_{qq}}(x + y)^2 - \sigma_q^2 \right\}
- \frac{1}{2\hbar} \left( \sigma_{pp} - \frac{\sigma_{pq}^2}{\sigma_{qq}} \right) (x - y)^2 + \frac{i\sigma_{pq}}{\hbar\sigma_{qq}} \left( \frac{x + y}{2} - \sigma_q \right)(x - y) + \frac{i}{\hbar} \sigma_p (x - y).
\]

For these matrices we can verify that \( \text{Tr}\rho^2 = \hbar / 2\sqrt{\sigma} \) and they correspond to the correlated coherent states (64) if \( \sigma_{pp}, \sigma_{qq} \) and \( \sigma_{pq} \) in (65) satisfy the equality in (2).

In [11, 14, 15] we have transformed the master equation for the density operator into the following Fokker-Planck equation satisfied by the Wigner distribution function \( W(q,p,t) \):

\[
\frac{\partial W}{\partial t} = -\frac{p}{m} \frac{\partial W}{\partial q} + m\omega^2 q \frac{\partial W}{\partial p} + (\lambda - \mu) \frac{\partial}{\partial q} (qW) + (\lambda + \mu) \frac{\partial}{\partial p} (pW) + D_{qq} \frac{\partial^2 W}{\partial q^2} + D_{pp} \frac{\partial^2 W}{\partial p^2} + D_{pq} \frac{\partial^2 W}{\partial p \partial q}.
\]

For an initial Gaussian Wigner function the solution of Eq. (66) is

\[
W(p,q,t) = \frac{1}{2\pi \sqrt{\sigma}} \times \exp\left\{ -\frac{1}{2\sigma} \left[ \sigma_{pp}(q - \sigma_q(t))^2 + \sigma_{qq}(p - \sigma_p(t))^2 - 2\sigma_p q (q - \sigma_q(t))(p - \sigma_p(t)) \right] \right\}.
\]

We see that the initial Wigner function remains Gaussian and therefore the property of positivity is preserved in time. Consider the harmonic oscillator initially in a correlated coherent state of the form (64), with the corresponding Wigner function (7). For our environment described by the diffusion coefficients (58), the solution for the Wigner
function at time \( t \) is given by (67), where \( \sigma_q(t) \) and \( \sigma_p(t) \) are given respectively by (40) and (41) and the variances by (59). Using either (11) and (67) or (65), we get for the density matrix the following time evolution:

\[
\langle x | \hat{\rho}(t) | y \rangle = \left( \frac{m \Omega}{\pi \hbar} \right)^{\frac{1}{2}} \exp \left[ -\frac{m \Omega}{\hbar} \left( \frac{x + y}{2} - \sigma_q(t) \right)^2 \right. \\
\left. - \frac{m \Omega}{4 \hbar} (x - y)^2 - \frac{im \mu}{\hbar} \left( \frac{x + y}{2} - \sigma_q(t) \right)(x - y) + \frac{i}{\hbar} \sigma_p(t)(x - y) \right].
\]

(68)

In the long time limit \( \sigma_q(t) = 0, \sigma_p(t) = 0 \) and we have

\[
\langle x | \hat{\rho}(\infty) | y \rangle = \left( \frac{m \Omega}{\pi \hbar} \right)^{\frac{1}{2}} \exp \left\{ -\frac{m}{2 \hbar} \Omega \left( x^2 + y^2 \right) + i \mu (x^2 - y^2) \right\}.
\]

(69)

The corresponding Wigner function has the form

\[
W_{\infty}(p, q) = \frac{1}{\pi \hbar} \exp \left[ -\frac{1}{\hbar \Omega} \left( \frac{p^2}{m} + m \omega^2 q^2 + 2 \mu pq \right) \right].
\]

(70)

We see that the time evolution of the initial correlated coherent state of the damped harmonic oscillator is given by a Gaussian density matrix with variances constant in time. According to known general results [22, 23], the initial Gaussian density matrix remains Gaussian centered around the classical path. So, the correlated coherent state remains a correlated coherent state and \( \sigma_q(t) \) and \( \sigma_p(t) \) give the average time dependent location of the system along its trajectory in phase space.

## 5 Entropy and decoherence

Besides the von Neumann entropy \( S(18) \), there is another quantity which can measure the degree of mixing or purity of quantum states. It is the linear entropy \( S_l \) defined as

\[
S_l = \text{Tr}(\hat{\rho} - \hat{\rho}^2) = 1 - \text{Tr}\hat{\rho}^2.
\]

(71)

For pure states \( S_l = 0 \) and for a statistical mixture \( S_l > 0 \).

As it is well-known, the increasing of the linear entropy \( S_l \) (as well as of the von Neumann entropy \( S \)) due to the interaction with the environment is associated with the decoherence phenomenon (loss of quantum coherence), given by the diffusion process [20, 21]. Dissipation increases the entropy and the pure states are converted into mixed states. The rate of entropy production is given by

\[
\dot{S}_l(\hat{\rho}) = -2\text{Tr}(\dot{\rho} \hat{\rho}) = -2\text{Tr}(\dot{\rho} L(\hat{\rho})),
\]

(72)

where \( L \) is the evolution operator. According to Zurek’s theory, the maximally predictive states are the pure states which minimize the entropy production in time. These
states remain least affected by the openness of the system and form a "preferred set of states" in the Hilbert space of the system, known as the "pointer basis". Their evolution is predictable with the principle of least possible entropy production.

Using (25), in our model the rate of entropy production (72) is given by:

\[
\dot{S}_l(t) = \frac{4}{\hbar^2}[D_{pp}\text{Tr}(\hat{\rho}^2\hat{q}^2 - \hat{p}\hat{q}\hat{p}\hat{q}) \\
+ D_{qq}\text{Tr}(\hat{\rho}^2\hat{p}^2 - \hat{p}\hat{q}\hat{p}\hat{q}) - D_{pq}\text{Tr}(\hat{\rho}^2(\hat{q}\hat{p} + \hat{p}\hat{q}) - 2\hat{p}\hat{q}\hat{p}\hat{q}) - \frac{\hbar^2}{2}\lambda\text{Tr}(\hat{\rho}^2)].
\] (73)

When the state remains approximately pure (\(\hat{\rho}^2 \approx \hat{\rho}\)), we obtain:

\[
\dot{S}_l(t) = \frac{4}{\hbar^2}(D_{pp}\sigma_{qq}(t) + D_{qq}\sigma_{pp}(t) - 2D_{pq}\sigma_{pq}(t) - \frac{\hbar^2}{2}\lambda) \geq 0,
\] (74)

according to (27). We see that \(\dot{S}_l(t) = 0\) when the condition (33) of purity for any time \(t\) is fulfilled. The entropy production \(S_l\) is also equal to 0 at \(t = 0\) if the initial state is a pure state. But we have shown before that the only initial states which remain pure for any \(t\) are the correlated coherent states. Therefore, we can state that in the Lindblad theory for the open quantum harmonic oscillator the correlated coherent states, which are generalized coherent states, are the maximally predictive states. Our result generalizes the previous results which assert that for many models of quantum Brownian motion in the high temperature limit the usual coherent states correspond to minimal entropy production and, therefore, they are the maximally predictive states. In our model the coherent states can be obtained as a particular case of the correlated coherent states by taking \(\mu = 0\), so that the correlation coefficient \(r = 0\) (see Eq. (4)).

Paz, Habib and Zurek [20, 21] considered the harmonic oscillator undergoing quantum Brownian motion in the Caldeira-Leggett model and concluded that the minimizing states which are the initial states generating the least amount of von Neumann or linear entropy and, therefore, the most predictable or stable ones under evolution in the presence of an environment are the ordinary coherent states. Using an information-theoretic measure of uncertainty for quantum systems, Anders and Halliwell showed in [22] that the minimizing states are more general Gaussian states. Anastopoulos and Halliwell [23] offered an alternative characterization of these states by noting that these states minimize the generalized uncertainty relation. According to this assertion, we can say that in our model the correlated coherent states are the most stable states which minimize the generalized uncertainty relation (2). Our result confirms the one of [23], but the model used in [23] is different, namely the open quantum system consists of a particle moving in a harmonic oscillator potential and is linearly coupled to
an environment consisting of a bath of harmonic oscillators in a thermal state. At the same time we remind that the Caldeira-Leggett model considered in [20, 21] violates the positivity of the density operator at short time scales [36, 37], whereas in the Lindblad model considered here the property of positivity is automatically fulfilled.

The rate of predictibility loss, measured by the rate of linear entropy increase, for a damped harmonic oscillator is also calculated in the framework of Lindblad theory in Ref. [38]. The initial states which minimize the predictability loss are identified as quasi-free states with a symmetry dictated by the environment diffusion coefficients. For an isotropic diffusion in phase space, the coherent states or mixtures of coherent states are selected as the most stable ones.

In order to generalize the results of Zurek and collaborators, the entropy production was considered by Gallis [24] within the Lindblad theory of open quantum systems, treating environment effects perturbatively. Gallis considered the particular case with $D_{pq} = 0$ and found out that the squeezed states emerge as the most stable states for intermediate times compared to the dynamical time scales. The amount of squeezing decreases with time, so that the coherent states are most stable for large time scales. For $D_{pq} \neq 0$ we have generalized the result of Gallis and established that the correlated coherent states are the most stable under the evolution in the presence of an environment.

6 Concluding remarks

Recently there is a revival of interest in quantum Brownian motion as a paradigm of quantum open systems. The possibility of preparing systems in macroscopic quantum states led to the problems of dissipation in tunneling and of loss of quantum coherence (decoherence). These problems are intimately related to the issue of quantum to classical transition and all of them point the necessity of a better understanding of open quantum systems. The Lindblad theory provides a selfconsistent treatment of damping as a general extension of quantum mechanics to open systems and gives the possibility to extend the model of quantum Brownian motion. In the present paper we have studied the one-dimensional harmonic oscillator with dissipation within the framework of this theory. We have shown that the only states which stay pure during the evolution in time of the system are the correlated coherent states under the condition of a special choice of the environment coefficients. These states are also connected with the decoherence phenomenon and they are the most stable under the evolution in the presence of the environment. In a future work in the framework of the
Lindblad theory we plan to discuss the connection between uncertainty, decoherence and correlations of open quantum systems with their environment in more details.

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