The exponential of the spin representation of the Lorentz algebra

jason hanson*

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Abstract

As discussed in a previous article, any (real) Lorentz algebra element possess a unique orthogonal decomposition as a sum of two mutually annihilating decomposable Lorentz algebra elements. In this article, this concept is extended to the spin representation of the Lorentz algebra. As an application, a formula for the exponential of the spin representation is obtained, as well as a formula for the spin representation of a proper orthochronous Lorentz transformation.

1 Orthogonal decomposition

Let \( g \) be a Lorentz metric on \( \mathbb{R}^4 \). That is, \( g \) is a symmetric nondegenerate inner product with determinant \(-1\). For our purposes, we need not specify a signature for \( g \). The Lorentz group \( O(g) \) is the Lie group of linear transformations \( \Lambda \) on \( \mathbb{R}^4 \) such that \( \Lambda^T g \Lambda = g \), and the Lorentz algebra \( so(g) \) is the Lie algebra of transformations \( L \) for which \( L^T g + gL = 0 \).

1.1 Decomposition of bivectors

Elements of \( so(g) \) are called Lorentz bivectors. A special type of Lorentz bivector is the simple bivector, which takes the form \( u \wedge^g v \) for four-vectors \( u, v \) in \( \mathbb{R}^4 \), where

\[
(u \wedge^g v)(w) = g(v, w)u - g(u, w)v
\]

\(^*\text{jhanson@digipen.edu}\)
when applied to the four–vector \( w \). In index notation, \((u \wedge^g v)_\beta^\alpha = u^\alpha v_\beta - v^\alpha u_\beta\). Simple bivectors are also called decomposable bivectors, and are characterized by the condition \( \det(u \wedge^g v) = 0 \).

While not every Lorentz bivector \( L \) is simple, it is the sum of simple bivectors. In fact, it can be shown that any nonsimple Lorentz bivector \( L \) admits an orthogonal decomposition: \( L = L_+ + L_- \), with \( L_\pm \) simple and \( L_+ L_- = 0 = L_- L_+ \). This decomposition is unique. Indeed, the summands of the decomposition of \( L \) are given by

\[
L_\pm = \pm \frac{L^3 - \mu_\pm L}{\mu_+ - \mu_-},
\]

where \( \mu_\pm \) are the positive and negative roots of the equation \( x^2 + (\text{tr}_2 L)x + \det L = 0 \), or equivalently, the solutions of the simultaneous equations

\[
\mu_+ + \mu_- = -\text{tr}_2 L \quad \text{and} \quad \mu_+ \mu_- = \det L.
\]

Here \( \text{tr}_2 L \) is the second order trace of \( L \), which can be computed by the formula \( \text{tr}_2 L = -\frac{1}{2} \text{tr} L^2 \) (this identity holds for any traceless matrix). In particular, \( \text{tr}_2 L_\pm = -\mu_\pm \). See [3] for details.

### 1.2 Spin representation of the Lorentz algebra

Representations of \( so(g) \) may be constructed from representations of the Clifford algebra \( Cl(g) \) on \( \mathbb{R}^4 \). Recall that \( Cl(g) \) is the quotient of the tensor algebra \( T^*(\mathbb{R}^4) \) by the subalgebra generated by the relation \( uv + vu = 2g(u, v) \) for \( u, v \in \mathbb{R}^4 \). Let \( \rho \) be a Clifford algebra representation; i.e., a (possibly complex) vector space \( V \) and a linear map \( \rho : Cl(g) \to \text{Hom}(V, V) \) that respects Clifford multiplication: \( \rho(uv) = \rho(u)\rho(v) \). We obtain the spin representation \( \sigma : so(g) \to \text{Hom}(V, V) \) by setting

\[
\sigma(u \wedge^g v) \doteq \frac{1}{2} \rho(uv - vu)
\]

for simple Lorentz bivectors, and extending linearly to all of \( so(g) \). One shows that \( \sigma \) is a Lie algebra homomorphism: \( \sigma([L_1, L_2]) = \sigma(L_1)\sigma(L_2) - \sigma(L_2)\sigma(L_1) \) for all Lorentz bivectors \( L_1, L_2 \) (see [2], for example).

A natural choice for the representation \( \rho \) would be gamma matrices; i.e., \( \rho(u) \doteq u^\alpha \gamma_\alpha \). However, a representation may be constructed directly from the Clifford algebra itself: view \( V = Cl(g) \) as a sixteen–dimensional real vector
space, and take $\rho$ to be the identity. Unlike the gamma matrix representation, this is not an irreducible Clifford algebra representation. In the following, we will not have the need to make a specific choice for $\rho$, and we simply refer to $\sigma$ as “the” spin representation of $so(g)$. We remark that all formulas, with the exception of those that appear in section 1.4, are actually valid for summands of the spin representation. In particular, they are valid for half-spin representations.

They key property of the spin representation $\sigma$ that we will make use of is the following, which makes apparent the usefulness of decomposing a bivector into a sum of simple bivectors. Here we write $I = \rho(1)$.

**Theorem 1.** Suppose $L = u \wedge^g v$ is a simple Lorentz bivector. Then $\text{tr}_2 L = g(u, u)g(v, v) - g(u, v)^2$ and $\sigma(L)^2 = -\frac{1}{4}(\text{tr}_2 L)I$.

**Proof.** From equation (1), one computes that $\text{tr}_2 L = \frac{1}{16} \rho(1)$. Now compute using equation (4) and the Clifford algebra relation:

$$
\sigma(u \wedge^g v)^2 = \frac{1}{16} \rho(uvv - uv^2u - vu^2v + vuvu)
$$

$$
= \frac{1}{16} \rho(u[-uv + 2g(v, u)]v - g(v, v)g(u, u)
$$

$$
- g(u, u)g(v, v) + v[-vu + 2g(u, v)]u)
$$

$$
= \frac{1}{16} \rho(2g(u, v)(uv + vu) - 4g(u, u)g(v, v))
$$

which implies the stated expression for $\sigma(L)^2$. \hfill \Box

### 1.3 Decomposition of a spin representation

If $L$ is a Lorentz bivector, then $L^3$ is as well. So we may apply the spin representation directly to each summand in equation (2) to obtain $\sigma(L_{\pm})$ in terms of $\sigma(L)$ and $\sigma(L^3)$. However, we would like an expression that involves only powers of $\sigma(L)$.

**Theorem 2.** If $L = L_+ + L_-$ is the orthogonal decomposition of a nonsimple Lorentz bivector, then

$$
\sigma(L_{\pm}) = \frac{\pm 2}{\mu_{\pm} - \mu_-} \left\{ \frac{1}{4}(\mu_{\mp} + 3\mu_{\pm})\sigma(L) - \sigma(L)^3 \right\}
$$

with $\mu_{\pm}$ as in equation (3).
Proof. Since \((*)\) \(\sigma(L) = \sigma(L_+) + \sigma(L_-)\), we have that \(\sigma(L)^3 = \sigma(L_+)^3 + 3\sigma(L_+)^2\sigma(L_-) + 3\sigma(L_+)\sigma(L_-) + \sigma(L_-)^3\). Using theorem 1 to reduce powers, we may rewrite this as \((**)\) \(\sigma(L)^3 = \frac{1}{4}(\mu_+ + 3\mu_-)\sigma(L_+) + \frac{1}{2}(\mu_- + 3\mu_+)\sigma(L_-)\). The determinant of the linear system \((*)\) and \((**)\) is \(\frac{1}{2}(\mu_+ - \mu_-)\), which is nonzero if \(L\) is nonsimple, and the system may be solved to yield the stated expressions for \(\sigma(L_\pm)\). \(\square\)

The summands of the orthogonal decomposition of a nonsimple Lorentz bivector are mutually annihilating. Although their images under the spin representation do not share this property, they do commute.

**Theorem 3.** If \(L = L_+ + L_-\) is the orthogonal decomposition of the nonsimple Lorentz bivector \(L\), then \(\sigma(L_+)\sigma(L_-) = \sigma(L_-)\sigma(L_+) = \frac{1}{8}(\text{tr}_2 L)I + \frac{1}{2}\sigma(L)^2\).

Proof. As \(L_+, L_-\) trivially commute, \([\sigma(L_+), \sigma(L_-)] = \sigma([L_+, L_-]) = 0\). By theorem 1 and equation (3), we then have \(\sigma(L)^2 = \sigma(L_+)^2 + 2\sigma(L_+)\sigma(L_-) + \sigma(L_-)^2 = 2\sigma(L_+)\sigma(L_-) - \frac{1}{4}(\text{tr}_2 L)I\). \(\square\)

### 1.4 A computational digression

To use the formula in theorem 2, we need to know the values of \(\mu_\pm\), which are obtained from the invariants \(\text{tr}_2 L\) and \(\det L\) of \(L\). However, we would like to deduce these values in the event we only have knowledge of \(\sigma(L)\).

**Lemma 1.** For any four-vectors \(a, b, u, v\)

\[
\text{tr}_\rho(abuv) = \text{tr}I \{g(a, b)g(u, v) - g(a, u)g(b, v) + g(a, v)g(b, u)\}. \square
\]

This generalizes the well-known identity for gamma matrices, so we need not repeat the computation here. We do note however, that \(\text{tr}I = \text{tr}_\rho(1)\) is the dimension of the representation: \(\text{tr}I = 16\) for the Clifford algebra \(\mathcal{C}l(g)\) itself, and \(\text{tr}I = 4\) for the gamma matrix representation.

**Lemma 2.** If \(L = L_+ + L_-\) is the orthogonal decomposition of a Lorentz bivector with \(L_+ = a \wedge^g b\) and \(L_- = u \wedge^g v\), then

\[
g(a, v)g(b, u) - g(a, u)g(b, v) = 0.
\]

Proof. From equation (1), one computes \((a \wedge^g b)(u \wedge^g v) = g(b, u)av^T g - g(b, v)au^T g - g(a, u)bv^T g + g(a, v)bu^T g\). Taking the trace of both sides, we get \(\text{tr}\{(a \wedge^g b)(u \wedge^g v)\} = 2g(b, u)g(v, a) - 2g(a, u)g(v, b)\), which is necessarily zero, since \((a \wedge^g b)(u \wedge^g v) = L_+ L_- = 0\). \(\square\)
Theorem 4. If \( L = L_+ + L_- \) is the orthogonal decomposition of a Lorentz bivector, then \( \text{tr}\{\sigma(L_+)\sigma(L_-)\} = 0 \).

Proof. Write \( L_+ = a \wedge^g b \) and \( L_- = u \wedge v. \) Then \( \sigma(L_+)\sigma(L_-) = \frac{1}{16}\rho(ab - ba)\rho(uv - vu) = \frac{1}{16}(\rho(abuv) - \rho(abvu) - \rho(bauv) + \rho(bavu)). \) Take the trace and apply the previous two lemmas. \( \square \)

Theorem 5. For any Lorentz bivector \( L, \text{tr}_2L = -4\text{tr}\sigma(L)^2/\text{tr}I \) and \( \det L = 4\text{tr}\sigma(L)^4/\text{tr}I - 4\text{tr}_2\sigma(L)^2/\text{tr}^2I. \)

Proof. For the first formula, take the trace of the formula in theorem 3. For the second, use the orthogonal decomposition \( L = L_+ + L_- \) and theorem 4 to compute \( \sigma(L)^4 = \{\sigma(L_+) + \sigma(L_-)\}^4 = \frac{1}{16}\mu^2 I - \mu_+ \sigma(L_+)\sigma(L_-) + \frac{3}{8}\mu_+ \mu_- I - \mu_- \sigma(L_+)\sigma(L_-) + \frac{1}{16}\mu^2 I. \) Taking traces, we get \( \text{tr}\sigma(L)^4 = \frac{1}{16}(\text{tr}I)(\mu^2_+ + 6\mu_+ \mu_- + \mu_-^2) = \frac{1}{16}(\text{tr}I)((\mu_+ + \mu_-)^2 + 4\mu_+ \mu_-) = \frac{1}{16}(\text{tr}I)(\text{tr}_2^2L + 4\det L), \) courtesy of equation 3. Solving for \( \det L \) and using the first formula yields the desired expression. \( \square \)

2 Exponential of the spin representation

By general principles, the exponential operation \( \exp(L) = \sum_{n \geq 0} L^n/n! \) is a map \( \exp : so(g) \to O(g), \) whose image is the connected component \( SO^+(g) \) of \( O(g) \) containing the identity transformation; i.e., the set of all proper orthochronous Lorentz transformations. A closed formula for \( \exp(L) \) was obtained in [1]. Here we give a closed expression for \( \exp(\sigma(L)) \) for both simple and nonsimple Lorentz bivectors.

Theorem 6. If \( L \) is a simple Lorentz bivector, then \( \exp(\sigma(L)) = \tilde{c} + \tilde{s}\sigma(L), \) where

\[
\begin{align*}
\text{if } \text{tr}_2L > 0, \quad & \tilde{c} \doteq \cos \frac{1}{2} \sqrt{\text{tr}_2L} \quad \text{and} \quad \tilde{s} \doteq \frac{2}{\sqrt{\text{tr}_2L}} \sin \frac{1}{2} \sqrt{\text{tr}_2L} \\
\text{if } \text{tr}_2L < 0, \quad & \tilde{c} \doteq \cosh \frac{1}{2} \sqrt{-\text{tr}_2L} \quad \text{and} \quad \tilde{s} \doteq \frac{2}{\sqrt{-\text{tr}_2L}} \sinh \frac{1}{2} \sqrt{-\text{tr}_2L} \\
\text{and if } \text{tr}_2L = 0, \quad & \tilde{c} = 1 = \tilde{s}.
\end{align*}
\]

Proof. In the case \( \text{tr}_2L > 0, \) theorem 4 implies \( \sigma(L)^{2p} = (-\theta^2)^p = (-1)^p\theta^{2p}, \) where \( \theta = \frac{1}{2}\sqrt{\text{tr}_2L}. \) Consequently, we may write \( \sigma(L)^{2p+1} = \sigma(L)^{2p}\sigma(L) = (-1)^p\theta^{2p+1}\sigma(L)/\theta. \) Thus the series \( \sum_{n \geq 0} \sigma(L)^n/n! = \sum_{p \geq 0} \sigma(L)^{2p}/(2p)! + \)
\[ \sum_{p \geq 0} \sigma(L)^{2p+1}/(2p + 1)! \text{ is summed using the usual Taylor series expansion for sine and cosine. The case when } \text{tr}_2 L < 0 \text{ is similar, and the case } \text{tr}_2 L = 0 \text{ is trivial.} \]

Recall that \( \exp(A + B) = \exp(A) \exp(B) \) whenever the matrices \( A, B \) commute. Thus, \( \exp(\sigma(L)) = \exp(\sigma(L_+)) \exp(\sigma(L_-)) \). Theorems 3 and 6 then lead to the following.

**Theorem 7.** Suppose \( L = L_+ + L_- \) is the orthogonal decomposition of a nonsimple Lorentz bivector. Define \( \theta_\pm \equiv \frac{1}{2} \sqrt{\pm \text{tr}_2 L_\pm}, \ c_\pm \equiv \cosh \theta_\pm, \ c_\pm \equiv \cos \theta_\pm, \ s_\pm \equiv \sinh \theta_\pm/\theta_\pm, \) and \( \bar{s}_\pm \equiv \sin \theta_\pm/\theta_\pm \). Then

\[
\exp(\sigma(L)) = \bar{c}_+ \bar{c}_- + \bar{s}_+ \bar{s}_- \sigma(L_+) + \bar{c}_+ \bar{s}_- \sigma(L_-) + \bar{s}_+ \bar{s}_- \sigma(L_+) \sigma(L_-)
\]

We derive an alternative formula for \( \exp(\sigma(L)) \) as a polynomial in \( \sigma(L) \).

**Theorem 8.** If \( L \) is a nonsimple Lorentz bivector, then

\[
\exp(\sigma(L)) = \alpha_0 + \alpha_1 \sigma(L) + \alpha_2 \sigma(L)^2 + \alpha_3 \sigma(L)^3
\]

\[
\begin{align*}
\alpha_0 & \equiv \bar{c}_+ \bar{c}_- - \frac{1}{8}(\mu_+ + \mu_-) \bar{s}_+ \bar{s}_- \\
\alpha_1 & \equiv \frac{1}{4} N \{ (\mu_- + 3 \mu_+) \bar{s}_+ \bar{c}_- - (\mu_+ + 3 \mu_-) \bar{c}_+ \bar{s}_- \} \\
\alpha_2 & \equiv \frac{1}{2} \bar{s}_+ \bar{s}_- \\
\alpha_3 & \equiv N (\bar{c}_+ \bar{s}_- - \bar{s}_+ \bar{c}_-)
\end{align*}
\]

with \( \mu_\pm \) as in equation (3), \( \bar{c}_\pm, \bar{s}_\pm \) as in theorem 7, and \( N \equiv 2/(\mu_+ - \mu_-). \)

### 3 Spin representation of a Lorentz transformation

The spin representation \( \sigma : \mathfrak{so}(g) \rightarrow \text{Hom}(V, V) \) on the Lie algebra level induces a projective representation \( \Sigma : \text{SO}^+(g) \rightarrow \text{SL}(V)/\pm \) on the Lie group level ([1]). We would like to deduce an explicit formula for \( \Sigma \).

We define a Lorentz transformation \( \Lambda \in \text{SO}^+(g) \) to be simple if it is the image of a simple Lorentz bivector under the exponential map. A criterion for simplicity is that \( \text{tr}_2 \Lambda = 2(\text{tr}\Lambda - 1) \). Here, the second order trace may be computed from the general formula \( \text{tr}_2 \Lambda = \frac{1}{2}(\text{tr}\Lambda^2 - 2\text{tr}\Lambda + 3) \). Moreover, it should be noted that for any proper orthochronous Lorentz transformation (simple or not), \( \text{tr}\Lambda \geq 0 \). See [3] for more details.
We will need the following fact for computing the logarithm of a simple Lorentz transformation, as given in [3]. The special case when \( \text{tr}\Lambda = 0 \) will be handled later.

**Proposition 1.** If \( \Lambda \in SO^+(g) \) is a simple Lorentz transformation with \( \text{tr}\Lambda > 0 \), then \( \Lambda = \exp(L) \) and \( \text{tr}_2L = -\mu \), where \( L = \frac{1}{2}k(\Lambda - \Lambda^{-1}) \) and

1. if \( 0 < \text{tr}\Lambda < 4 \), then \( k = \frac{\sqrt{-\mu}}{\sin \sqrt{-\mu}} \) and \( \sqrt{-\mu} = \cos^{-1}(\frac{1}{2}\sqrt{-\mu} - 1) \),
2. if \( \text{tr}\Lambda > 4 \), then \( k = \frac{\sqrt{\mu}}{\sinh \sqrt{\mu}} \) and \( \sqrt{\mu} = \cosh^{-1}(\frac{1}{2}\sqrt{\mu} - 1) \),
3. if \( \text{tr}\Lambda = 4 \), then \( k = 0 \) and \( \mu = 0 \).

**Theorem 9.** Suppose \( \Lambda \in SO^+(g) \) is simple. If \( \text{tr}\Lambda > 0 \), then up to an overall sign,

\[
\Sigma(\Lambda) = \frac{1}{2\sqrt{\text{tr}\Lambda}} \left\{ \text{tr}\Lambda + 2\sigma(\Lambda - \Lambda^{-1}) \right\}.
\]

**Proof.** Let \( L \) be a simple Lorentz bivector such that \( \Lambda = \exp(L) \). By the general properties of the exponential map on a Lie algebra, \( \Sigma(\Lambda) = \exp(\sigma(L)) \). Writing \( L = \frac{1}{2}k(\Lambda - \Lambda^{-1}) \) as in proposition 1, we have \( \exp(\sigma(L)) = \bar{c} + \frac{1}{2}k\bar{s}\sigma(\Lambda - \Lambda^{-1}) \), according to theorem 6. The values of \( \bar{c}, \bar{s} \) depend on the value of \( \text{tr}_2L \). We consider the case \( \text{tr}_2L > 0 \), so that \( \mu < 0 \) in the notation of proposition 1 which occurs when \( 0 < \text{tr}\Lambda < 4 \). The other two cases are similar. In this case, we have \( \cos \sqrt{-\mu} = \frac{1}{2}\sqrt{\text{tr}\Lambda} - 1 \). Now, \( \bar{c} = \cos \frac{1}{2}\sqrt{\text{tr}_2L} = \cos \frac{1}{2}\sqrt{-\mu} \).

Similarly,

\[
\frac{1}{2}k\bar{s} = \frac{1}{2} \frac{\sqrt{-\mu}}{\sin \sqrt{-\mu}} \frac{2}{\sqrt{-\mu}} \sin \frac{1}{2}\sqrt{-\mu} = \frac{\sin \frac{1}{2}\sqrt{-\mu}}{\sin \sqrt{-\mu}} = \frac{1}{2\cos \frac{1}{2}\sqrt{-\mu}}
\]

On the other hand, we have \( \cos^2 \frac{1}{2}\sqrt{-\mu} = \frac{1}{2}(1 - \cos \sqrt{-\mu}) = \frac{1}{4}\text{tr}\Lambda \), so that \( \cos \frac{1}{2}\sqrt{-\mu} = \pm \frac{1}{2}\sqrt{\text{tr}\Lambda} \). Although the choice of \( L \) determines the sign here, the choice of \( L \) such that \( \exp(L) = \Lambda \) is not unique. Indeed, if we take \( L' = \alpha L \), with \( \alpha \) chosen such that \( \sqrt{\text{tr}_2L'} = \sqrt{\text{tr}_2L} + 2\pi \), then \( \exp(L') = \Lambda \). However, \( \frac{1}{2}\sqrt{-\mu'} = \frac{1}{2}\sqrt{-\mu} + \pi \), so that \( \exp(\sigma(L')) = -\exp(\sigma(L)) \).

To obtain an analogous formula for a nonsimple Lorentz transformation, we will make use of the fact that such a transformation is a product of
commuting simple transformations. Indeed, since $SO^+(g)$ is exponential, we may write $\Lambda = \exp(L)$ for some Lorentz bivector $L$. Using the orthogonal decomposition $L = L_+ + L_-$, we have $\exp(L) = \exp(L_+) \exp(L_-)$. Taking $\Lambda_\pm = \exp(L_\pm)$, we may write $\Lambda = \Lambda_+ \Lambda_-$, with $\Lambda_+ , \Lambda_-$ commuting simple Lorentz transformations. In [3], the following explicit formula is obtained.

**Proposition 2.** If $\Lambda \in SO^+(g)$ is nonsimple, then $\Lambda = \Lambda_+ \Lambda_-$, where $\Lambda_\pm$ are the commuting simple Lorentz transformations

$$\Lambda_\pm = \pm \frac{1}{2(c_+ - c_-)} \left\{ (1 + 2c_\pm)I - \Lambda^{-1} - (1 + 2c_\mp)\Lambda + \Lambda^2 \right\}$$

with $c_\pm = \frac{1}{4}(\text{tr} \Lambda \pm \sqrt{\Delta})$ and $\Delta = \text{tr}^2 \Lambda - 4\text{tr}_2 \Lambda + 8$, $c_+ > 1$, and $1 \leq c_- < 1$.

Using this decomposition, we use the fact that $\Sigma$ is a group homomorphism to write $\Sigma(\Lambda) = \Sigma(\Lambda_+) \Sigma(\Lambda_-)$, where each factor on the right hand side may be computed using theorem 9. Note that $\Sigma(\Lambda_\pm)$ necessarily commute.

**Theorem 10.** If $\Lambda$ is a nonsimple proper orthochronous Lorentz transformation with $2 + 2\text{tr} \Lambda + \text{tr}^2 \Lambda \neq 0$, then up to sign

$$\Sigma(\Lambda) = \frac{1}{2\sqrt{2 + 2\text{tr} \Lambda + \text{tr}^2 \Lambda}} \left\{ (2 + \text{tr} \Lambda + \text{tr}^2 \Lambda - \frac{1}{4}\text{tr}^2 \Lambda) + (\text{tr} \Lambda + 2)\sigma(\Lambda - \Lambda^{-1}) - \sigma(\Lambda^2 - \Lambda^{-2}) + \sigma(\Lambda - \Lambda^{-1})^2 \right\}$$

**Proof.** For brevity, we set $\tau_k^\pm = \text{tr}_k \Lambda_\pm$. From theorem[9] we compute $\Sigma(\Lambda) = \Sigma(\Lambda_+) \Sigma(\Lambda_-)$ to be:

$$\Sigma(\Lambda) = \frac{1}{4\sqrt{\tau_1^+ \tau_1^-}} \left\{ \tau_1^+ \tau_1^- + 2\tau_1^+ \sigma(\Lambda_+ - \Lambda_-^1) + 2\tau_1^- \sigma(\Lambda_+ - \Lambda_+^1) + 4\sigma(\Lambda_+ - \Lambda_-^1)\sigma(\Lambda_- - \Lambda_-^1) \right\}$$

Now from proposition[2] one shows that $(\ast)$ $\tau_1 \doteq \text{tr} \Lambda = 2(c_+ + c_-)$ and $\tau_2 \doteq \text{tr}_2 \Lambda = 4c_+ c_- + 2$, and that $(\ast\ast)$ $\tau_1^\pm = 2(1 + c_\pm)$. Moreover, since for any Lorentz transformation $\Lambda^{-1} = g^{-1} \Lambda^T g$, we find that

$$\Lambda_\pm - \Lambda_\pm^{-1} = \mp \frac{1}{2(c_+ - c_-)} \left\{ 2c_\mp (\Lambda - \Lambda^{-1}) - (\Lambda^2 - \Lambda^{-2}) \right\}$$

We now rewrite the summands of equation (5) in terms of $\Lambda$ and its first and second order traces. For the first summand, using $(\ast\ast)$ and $(\ast)$ we obtain

$$\tau_1^+ \tau_1^- = 4(1 + c_+)(1 + c_-) = 2 + 2\tau_1 + \tau_2$$

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For the second and third summands in (5), using (***) and (6) one computes
\[ \tau_1^+ \sigma(\Lambda_+ - \Lambda_-^{-1}) + \tau_1^- \sigma(\Lambda_+ - \Lambda_-^{-1}) = 2(1 + c_+ + c_-)\sigma(\Lambda - \Lambda^{-1}) - \sigma(\Lambda^2 - \Lambda^{-2}). \]
Thus by (\*),
\[ \tau_1^+ \sigma(\Lambda_- - \Lambda_-^{-1}) + \tau_1^- \sigma(\Lambda_+ - \Lambda_-^{-1}) = (\tau_1 + 2)\sigma(\Lambda - \Lambda^{-1}) - \sigma(\Lambda^2 - \Lambda^{-2}) \quad (8) \]
For the fourth summand in (5), we similarly compute (although we also need to use the fact that \( \sigma(\Lambda - \Lambda^{-1}) \) and \( \sigma(\Lambda^2 - \Lambda^{-2}) \) commute, as \( \sigma \) is a Lie algebra homomorphism)
\[ 4\sigma(\Lambda_+ - \Lambda_-^{-1})\sigma(\Lambda_- - \Lambda_-^{-1}) = \frac{1}{(c_+ - c_-)^2} \{(2 - \tau_2)\sigma(\Lambda - \Lambda^{-1}) \]
\[ + \tau_1 \sigma(\Lambda - \Lambda^{-1})\sigma(\Lambda^2 - \Lambda^{-2}) - \sigma(\Lambda^2 - \Lambda^{-2})^2 \} \quad (9) \]
However, the terms in this expression are not algebraically independent. To find a relation, we make use of the fact that \( \Lambda_\pm \) are simple: from theorem \( \bullet \) \( \sigma(L_\pm)^2 = -\frac{1}{4}\text{tr}_2 L_\pm \), where \( L_\pm \) are such that \( \exp(L_\pm) = \Lambda_\pm \), and may be computed using proposition \( \bullet \). Indeed, \( L_+ = \frac{1}{2}k_\pm(\Lambda_+ - \Lambda_+^{-1}) \), with \( \text{tr}_2 L_+ = -\mu_+ \), \( c_+ = \cosh \sqrt{\mu_+} \), and \( k_+ = \sqrt{\mu_+}/\sinh \sqrt{\mu_+} \). The equation \( \sigma(L_+)^2 = -\frac{1}{4}\text{tr}_2 L_+ \) and (6) then lead to
\[ 4c_+^2\sigma(\Lambda - \Lambda^{-1})^2 - 4c_-\sigma(\Lambda - \Lambda^{-1})\sigma(\Lambda^2 - \Lambda^{-2}) + \sigma(\Lambda^2 - \Lambda^{-2})^2 \]
\[ = 4(c_+ - c_-)^2(c_+^2 - 1) \]
(note that \( \sinh^2 \sqrt{\mu_+} = c_+^2 - 1 \)). Similarly, \( \sigma(L_-)^2 = -\frac{1}{4}\text{tr}_2 L_- \), proposition \( \bullet \) and (6) imply
\[ 4c_-^2\sigma(\Lambda - \Lambda^{-1})^2 - 4c_+\sigma(\Lambda - \Lambda^{-1})\sigma(\Lambda^2 - \Lambda^{-2}) + \sigma(\Lambda^2 - \Lambda^{-2})^2 \]
\[ = 4(c_+ - c_-)^2(c_-^2 - 1) \]
Adding these two equations together and using (\*) then yields the relation
\[ (\tau_1^2 - 2\tau_2 + 4)\sigma(\Lambda - \Lambda^{-1})^2 - 2\tau_1 \sigma(\Lambda - \Lambda^{-1})\sigma(\Lambda^2 - \Lambda^{-2}) \]
\[ + 2\sigma(\Lambda^2 - \Lambda^{-2})^2 = (c_+ - c_-)^2(\tau_1^2 - 2\tau_2 - 4) \quad (10) \]
Combining (5) and (10) together, we see that we may write the fourth summand in (5) as
\[ 4\sigma(\Lambda_+ - \Lambda_-^{-1})\sigma(\Lambda_- - \Lambda_-^{-1}) = 2\sigma(\Lambda - \Lambda^{-1})^2 - \frac{1}{2}(\tau_1^2 - 2\tau_2 - 4) \quad (11) \]
Combining (5), (7), (8), and (11) give the desired formula for \( \Sigma(\Lambda) \). \( \square \)
3.1 Special case

We consider the case when a Lorentz transformation $\Lambda \in SO^+(g)$ satisfies the identity

$$2 + 2\text{tr}\Lambda + \text{tr}\Lambda = 0 \quad (12)$$

In the case when $\Lambda$ is simple, so that $\text{tr}\Lambda^2 = 2(\text{tr}\Lambda - 1)$, this condition reduces to $\text{tr}\Lambda = 0$. It should be noted that a simple Lorentz transformation is traceless only if $\Lambda = \exp(L)$ for some simple Lorentz bivector with $\text{tr}\Lambda^2 = \pi^2$. A nonsimple transformation satisfies (12) only if its decomposition in proposition 2, $\Lambda = \Lambda_+ \Lambda_-$, is such that the simple factor $\Lambda_-$ is traceless.

We will not be able to obtain an explicit formula for the spin representation $\Sigma(\Lambda)$ of such a Lorentz transformation. However, we can give an algorithm. The key lies in the following two facts from [3].

Proposition 3. Let $u, v$ be four–vectors, and set $\mathcal{L} = u \wedge g v$. The two–plane $P$ spanned by $u, v$ is nondegenerate (that is, $g$ is nondegenerate when restricted to $P$) if and only if $\text{tr}\Lambda^2 \neq 0$, in which case $P_\mathcal{L} = -\Lambda^2/\text{tr}\Lambda^2$ is $g$–orthogonal projection onto $\mathcal{P}$. Conversely, if $P$ is $g$–orthogonal projection onto a two–plane, then the two–plane is nondegenerate and $P = P_\mathcal{L}$, where $\mathcal{L} = u \wedge g v$ and $u, v$ are any linearly independent four–vectors in the image of $P$.

Proposition 4. If $\Lambda$ is a simple Lorentz transformation with $\text{tr}\Lambda = 0$, then $\Lambda^2 = I$, $P_\Lambda = \frac{1}{2}(I - \Lambda)$ is $g$–orthogonal projection onto a nondegenerate two–plane, and $-\pi^2 P_\Lambda$ is the square of a simple Lorentz bivector $L_\Lambda$ with $\exp(L_\Lambda) = \Lambda$ and $\text{tr}\Lambda^2 = \pi^2$.

Theorem 11. Suppose $\Lambda \in SO^+(g)$ is simple with $\text{tr}\Lambda = 0$. Let $u, v$ be any two linearly independent vectors in the image of $P_\Lambda = \frac{1}{2}(I - \Lambda)$. Then up to sign, $\Sigma(\Lambda) = (2/\sqrt{\text{tr}(u \wedge g v)})\sigma(u \wedge g v)$.

Proof. Set $\mathcal{L} = u \wedge g v$. Then $P_\mathcal{L}$ is $g$–orthogonal projection onto the (necessarily nondegenerate) two–plane $\mathcal{P}$ spanned by $u, v$. By uniqueness of $g$–orthogonal projection, we must have $P_\mathcal{L} = P_\Lambda$. Now $L_\Lambda = u' \wedge g v'$, where $u', v'$ lie in $\mathcal{P}$. Writing $u' = au + bv$ and $v' = cu + dv$, we compute that $u' \wedge g v' = (ad - bc)u \wedge g v$; i.e., $L_\Lambda = \alpha L$ for some scalar $\alpha$. Taking 2–traces, we get $\pi^2 = \text{tr}_2 L_\Lambda = \alpha^2 \text{tr}_2 L$, so that $\alpha = \pm \pi/\sqrt{\text{tr}_2 L}$. Since $\exp(L_\Lambda) = \Lambda$, $\Sigma(\Lambda) = \exp\sigma(L_\Lambda)$. On the other hand, by theorem 6, $\exp\sigma(L_\Lambda) = (2/\pi)\sigma(L_\Lambda) = (2\alpha/\pi)\sigma(L)$. $\square$
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