Integral pinched 3-manifolds are space forms

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abstract. In this paper we prove that, under an explicit integral pinching assumption between the $L^2$-norm of the Ricci curvature and the $L^2$-norm of the scalar curvature, a closed 3-manifold with positive scalar curvature admits an Einstein metric with positive curvature. In particular this implies that the manifold is diffeomorphic to a quotient of $\mathbb{S}^3$.

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1 Introduction

One of the basic questions concerning the relation between algebraic properties of the curvature tensor and manifold topologies is under which conditions on its curvature tensor a Riemannian manifold is compact or homeomorphic to a space form (a manifold of constant sectional curvature). For example, Bonnet-Myers theorem states that a complete Riemannian manifold with positive lower bound for its Ricci curvature is compact; the theorem of Klingenberg, Berger and Rauch states that a compact, simply connected, $\frac{1}{4}$-pinched manifold with positive curvature is homeomorphic to the standard sphere.

In 1982, Hamilton \cite{13} introduced the Ricci flow and it appears to be a very useful tool to study the relationships between topology and curvature. For 2-dimensional compact manifolds, Hamilton \cite{15} and Chow \cite{5} proved that the normalized Ricci flow converges and gave by the way a new proof of the well-known uniformization theorem for compact surfaces. For 3 and 4-dimensional compact manifolds with positive curvature, Hamilton, \cite{13} and \cite{14}, proved that the initial metric can be deformed into a metric of constant positive curvature; it follows that these manifolds are diffeomorphic to the sphere $\mathbb{S}^3$ or $\mathbb{S}^4$, or a quotient space of $\mathbb{S}^3$ or $\mathbb{S}^4$ by a group of fixed point free isometries in the standard metric. In dimension 3, Hamilton’s result is the following:

\textbf{Theorem 1.1 (Hamilton)} If $(M, g)$ is a closed 3-dimensional Riemannian manifold with positive Ricci curvature, then $M$ is diffeomorphic to a spherical space form, i.e. $M$ admits a metric with constant positive sectional curvature.

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In this paper, we prove the existence of an Einstein metric of positive curvature on compact, 3-dimensional manifolds satisfying an integral pinching condition involving the second symmetric function of the Schouten tensor.

More precisely, we consider \((M, g)\), a compact, smooth, 3-dimensional Riemannian manifold without boundary. Given a section \(A\) of the bundle of symmetric two tensors, we can use the metric to raise an index and view \(A\) as a tensor of type \((1, 1)\), or equivalently as a section of \(\text{End}(TM)\). This allows us to define \(\sigma_2(g^{-1}A)\) the second elementary symmetric function of the eigenvalues of \(g^{-1}A\), namely, if we denote by \(\lambda_1\), \(\lambda_2\) and \(\lambda_3\) these eigenvalues
\[
\sigma_2(g^{-1}A) = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3.
\]

In this paper we choose the tensor \((\text{here } t \text{ is a real number})
\[
A^t_g = \text{Ric}_g - \frac{t}{4} R_g g,
\]
where \(\text{Ric}_g\) and \(R_g\) denote the Ricci and the scalar curvature of \(g\) respectively. Note that for \(t = 1\), \(A^1_g\) is the classical Schouten tensor \(A^1_g = \text{Ric}_g - \frac{1}{4} R_g g\) (see [2]). Hence, with our notations, \(\sigma_2(g^{-1}A^t_g)\) denotes the second elementary symmetric function of the eigenvalues of \(g^{-1}A^t_g\).

Our present work is motivated by a recent paper of M. Gursky and J. Viaclovsky [11]. Namely, they proved that, giving a closed 3-manifold \(M\), a metric \(g_0\) on \(M\) (with normalized volume) satisfying \(\int_M \sigma_2(g_0^{-1}A^1_0) dV_{g_0} \geq 0\) is critical (over all metrics of normalized volume) for the functional
\[
\mathcal{F} : g \rightarrow \int_M \sigma_2(g^{-1}A^1_g) dV_g
\]
if and only if \(g_0\) has constant sectional curvature.

Actually, it is not easy to exhibit a critical metric for this functional. What we prove here (this is a consequence of our main result in this paper) is that, assuming that there exists a metric \(g\) on \(M\) with positive scalar curvature and such that \(\int_M \sigma_2(g^{-1}A^1_g) dV_g \geq 0\) then the functional \(\mathcal{F}\) admits a critical point (over all metrics of normalized volume) \(g_0\) with \(\int_M \sigma_2(g_0^{-1}A^1_{g_0}) dV_{g_0} \geq 0\).

We will denote \(Y(M, [g])\) the Yamabe invariant associated to \((M, g)\) (here \([g]\) is the conformal class of the metric \(g\), that is \([g] := \{ \tilde{g} = e^{-2u} g \text{ for } u \in C^\infty(M) \}\)). We recall that
\[
Y(M, [g]) := \inf_{\tilde{g} \in [g]} \frac{\int_M R_{\tilde{g}} dV_{\tilde{g}}}{(\int_M dV_{\tilde{g}})^{\frac{3}{2}}}.
\]
An important fact that will be useful is that if \(g\) has positive scalar curvature then \(Y(M, [g]) > 0\).

Our main result is the following:

**Theorem 1.2** Let \((M, g)\) be a closed 3-dimensional Riemannian manifold with positive scalar curvature. There exists a positive constant \(C = C(M, g)\) depending only on \((M, g)\) such that if
\[
\int_M \sigma_2(g^{-1}A^1_g) dV_g + C \left( \frac{7}{10} - t_0 \right) Y(M, [g])^2 > 0,
\]
for some \(t_0 \leq 2/3\), then there exists a conformal metric \(\tilde{g} = e^{-2u} g\) with \(R_{\tilde{g}} > 0\) and \(\sigma_2(g^{-1}A_{\tilde{g}}^{t_0}) > 0\) pointwise. Moreover we have the inequalities
\[
(1) \quad (3t_0 - 2)R_{\tilde{g}}\tilde{g} < 6 \text{Ric}_{\tilde{g}} < 3(2 - t_0)R_{\tilde{g}}\tilde{g}.
\]
As an application, when \(t_0 = 2/3\), we obtain
Theorem 1.3 Let \((M, g)\) be a closed 3-dimensional Riemannian manifold with positive scalar curvature. There exists a positive constant \(C' = C'(M, g)\) depending only on \((M, g)\) such that if

\[
\int_M \sigma_2(g^{-1}A_1^g) dV_g + C'Y(M, [g])^2 > 0,
\]

then there exists a conformal metric \(\tilde{g} = e^{-2u}g\) with positive Ricci curvature \((\text{Ric}_\tilde{g} > 0)\). In particular if \(\int_M \sigma_2(g^{-1}A_1^g) dV_g \geq 0\) then there exists a conformal metric \(\tilde{g} = e^{-2u}g\) with positive Ricci curvature \((\text{Ric}_\tilde{g} > 0)\).

Using Hamilton’s theorem \([11]\) we get:

**Corollary 1.4** Let \((M, g)\) be a closed 3-dimensional Riemannian manifold with positive scalar curvature. There exists a positive constant \(C' = C'(M, g)\) depending only on \((M, g)\) such that if

\[
\int_M \sigma_2(g^{-1}A_1^g) dV_g + C'Y(M, [g])^2 > 0,
\]

then \(M\) is diffeomorphic to a spherical space form, i.e. \(M\) admits a metric with constant positive sectional curvature. In particular, if \(\int_M \sigma_2(g^{-1}A_1^g) dV_g \geq 0\) then \(M\) is diffeomorphic to a spherical space form.

**Remark 1.5** Using the fact that \(\sigma_2(g^{-1}A_1^g) = -\frac{1}{2} |\text{Ric}_g|^2 + \frac{3}{16} R_g^2\), the assumption

\[
\int_M \sigma_2(g^{-1}A_1^g) dV_g \geq 0
\]

can be written

\[
\int_M |\text{Ric}_g|^2 dV_g \leq \frac{3}{8} \int_M R_g^2 dV_g.
\]

Actually all these results are the consequence of the following more general result:

**Theorem 1.6** Let \((M, g)\) be a closed 3-dimensional Riemannian manifold with positive scalar curvature. There exists a positive constant \(C = C(M, g)\) depending only on \((M, g)\) such that if

\[
\int_M \sigma_2(g^{-1}A_1^g) dV_g + \frac{1}{24} \left( \frac{7}{10} - t_0 \right) \inf_{g' = e^{-2u}g, |\nabla u|_{g'} \leq C} \left( \int_M R_{g'}^2 e^{-u} dV_{g'} \right) > 0,
\]

for some \(t_0 \leq 2/3\), then there exists a conformal metric \(\tilde{g} = e^{-2u}g\) with \(R_{\tilde{g}} > 0\) and \(\sigma_2(g^{-1}A_1^\tilde{g}) > 0\) pointwise. Moreover we have the inequalities

\[
(3t_0 - 2)R_{\tilde{g}} \tilde{g} < 6\text{Ric}_{\tilde{g}} < 3(2 - t_0)R_{\tilde{g}} \tilde{g}.
\]

There is a way to relate these result to the so-called \(Q\)-curvature (the curvature associated to the Paneitz operator). The Paneitz operator introduced by Paneitz in \([17]\) has demonstrated its importance in dimension 4 (see for example Chang-Gursky-Yang \([3]\) and \([4]\)). In dimension 3, the \(Q\)-curvature is defined by

\[
Q_g = -\frac{1}{4} \Delta_g R_g - 2 |\text{Ric}_g|^2 + \frac{23}{32} R_g^2,
\]

the Paneitz operator being defined (in dimension 3) by

\[
P_g = \Delta_g^2 - \text{div}_g \left( -\frac{5}{4} R_g g + 4 \text{Ric}_g \right) d - \frac{1}{2} Q_g.
\]

The Paneitz operator satisfies the conformal covariant property, that is, if \(\rho \in C^\infty(M)\), \(\rho > 0\), then for all \(\varphi \in C^\infty(M)\), \(P_{\rho^{-1}g}(\varphi) = \rho^7 P_g(\rho \varphi)\). We can now state the Corollary:
Corollary 1.7 Let \((M, g)\) be a closed 3-dimensional Riemannian manifold with non-negative Yamabe invariant. If there exists a metric \(g' \in [g]\) such that the \(Q\)-curvature of \(g'\) satisfies

\[
Q_{g'} \geq \frac{1}{48} R_{g'}^2,
\]

then \(M\) is diffeomorphic to a quotient of \(\mathbb{R}^3\) if \(Y(M, [g]) = 0\) or to a spherical space form if \(Y(M, [g]) > 0\).

Let us emphasize the fact that, in our results, we don’t make any assumption on the positivity of the Ricci tensor, we only assume that its trace is positive and a pinching on its \(L^2\)-norm.

During the preparation of the manuscript of this paper, we learned that Y. Ge, C.S. Lin and G. Wang \cite{ge-lin-wang} proved a weaker version of Corollary 1.4, namely they prove that if \((M, g)\) is a closed 3-dimensional Riemannian manifold with positive scalar curvature and if \(\int_M \sigma_2(g^{-1}A_{g}^1) dV_g > 0\), then \(M\) is diffeomorphic to a spherical space form. Their proof is completely different from ours since they use a very specific conformal flow.

For the proof of Theorem 1.2 and Theorem 1.3 we will be concerned with the following equation for a conformal metric \(\tilde{g} = e^{-2u}g\):

\[
(\sigma_2(g^{-1}A_{\tilde{g}}^1))^{1/2} = fe^{2u},
\]

where \(f\) is a positive function on \(M\). Let \(\sigma_1(g^{-1}A_{g}^1)\) be the trace of \(A_{g}^1\) with respect to the metric \(g\). We have the following formula for the transformation of \(A_{g}^t\) under this conformal change of metric:

\[
A_{\tilde{g}}^t = A_{g}^t + \nabla_g^2 u + (1 - t)(\Delta_g u)g + du \otimes du - \frac{2 - t}{2} |\nabla_g u|_{g}^2 g.
\]

Since

\[
A_{g}^t = A_{g}^1 + (1 - t)\sigma_1(g^{-1}A_{g}^1)g,
\]

this formula follows easily from the standard formula for the transformation of the Schouten tensor (see \cite{schouten}):

\[
A_{g}^1 = A_{g}^1 + \nabla_g^2 u + du \otimes du - \frac{1}{2} |\nabla_g u|_{g}^2 g.
\]

Using this formula we may write \cite{schouten} with respect to the background metric \(g\)

\[
\sigma_2 \left( g^{-1} \left( A_{g}^t + \nabla_g^2 u + (1 - t)(\Delta_g u)g + du \otimes du - \frac{2 - t}{2} |\nabla_g u|_{g}^2 g \right) \right)^{1/2} = f(x)e^{2u}.
\]

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2 Ellipticity

Following \cite{schouten}, we will discuss the ellipticity properties of equation \cite{schouten}.

Definition 2.1 Let \((\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3\). We view the second elementary symmetric function as a function on \(\mathbb{R}^3\):

\[
\sigma_2(\lambda_1, \lambda_2, \lambda_3) = \sum_{1 \leq i < j \leq 3} \lambda_i \lambda_j,
\]

and we define

\[
\Gamma_2^+ = \{ \sigma_2(\lambda_1, \lambda_2, \lambda_3) > 0 \} \cap \{ \sigma_1(\lambda_1, \lambda_2, \lambda_3) > 0 \} \subset \mathbb{R}^3,
\]

where \(\sigma_1(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 + \lambda_2 + \lambda_3\) denotes the trace.
For a symmetric linear transformation $A : V \to V$, where $V$ is an $n$-dimensional inner product space, the notation $A \in \Gamma^n_2$ will mean that the eigenvalues of $A$ lie in the corresponding set. We note that this notation also makes sense for a symmetric 2-tensor on a Riemannian manifold. If $A \in \Gamma^n_2$, let $\sigma_2^{1/2}(A) = \{\sigma_2(A)\}^{1/2}$.

**Definition 2.2** Let $A : V \to V$, where $V$ is an $n$-dimensional inner product space. The first Newton transformation associated with $A$ is (here $I$ is the identity map on $V$)

$$ T_1(A) := \sigma_1(A) \cdot I - A. $$

Also, for $t \in \mathbb{R}$ we define the linear transformation

$$ L_t(A) := T_1(A) + (1 - t)\sigma_1(T_1(A)) \cdot I. $$

We have the following:

**Lemma 2.3** If $A : \mathbb{R} \to \text{Hom}(V, V)$, then

$$ \frac{d}{ds} \sigma_2(A)(s) = \sum_{i,j} T_1(A)_{ij}(s) \frac{d}{ds} (A)_{ij}(s), $$

i.e., the first Newton transformation is what arises from differentiation of $\sigma_2$.

**Proof** The proof of this lemma is a consequence of an easy computation. See Gursky-Viaclovsky [11]

**Proposition 2.4** (Ellipticity property) Let $u \in C^2(M)$ be a solution of equation (3) for some $t \leq 2/3$ and let $\bar{g} = e^{-2u}g$. Assume that $A^t_g \in \Gamma^n_{2+}$. Then the linearized operator at $u$, $L^t : C^{2,\alpha}(M) \to C^{\alpha}(M)$, is invertible ($0 < \alpha < 1$).

**Proof** The proof of this proposition, adapted in dimension 3, may be found in [12].

### 3 Upper bound and gradient estimate

Throughout the sequel, $(M, g)$ will be a closed 3-dimensional Riemannian manifold with positive scalar curvature. Since $R_g > 0$, there exists $\delta > -\infty$ such that $A^t_{g, \delta}$ is positive definite (i.e. $\text{Ric}_g - \frac{\delta}{2} R_g g > 0$ on $M$). Note that $\delta$ only depends on $(M, g)$. For $t \in [\delta, 2/3]$, consider the path of equations (in the sequel we use the notation $A^{t, u_t} := A^{t, g_t}$ for $g_t$ given by $g_t = e^{-2u_t}g$)

$$ \sigma_2^{1/2}(g^{-1}A^{t, u_t}) = fe^{2u_t}, $$

where $f = \sigma_2^{1/2}(g^{-1}A^\delta_g) > 0$. Note that $u \equiv 0$ is a solution of (6) for $t = \delta$.

**Proposition 3.1** (Upper bound) Let $u_t \in C^2(M)$ be a solution of (6) for some $t \in [\delta, 2/3]$. Then $u_t \leq \bar{\delta}$, where $\delta$ depends only on $(M, g)$.

**Proof** From Newton’s inequality $\sqrt{3} \sigma_2^{1/2} \leq \sigma_1$, so for all $x \in M$

$$ \sqrt{3}fe^{2u_t} \leq \sigma_1(g^{-1}A^{t, u_t}). $$

Let $p \in M$ be a maximum of $u_t$, then using (6), since the gradient terms vanish at $p$ and $(\Delta u_t)(p) \leq 0$,

$$ \sqrt{3}f(p)e^{2u_t(p)} \leq \sigma_1(g^{-1}A^{t, u_t})(p) = \sigma_1(g^{-1}A^\delta_g)(p) + (4 - 3t)(\Delta u_t)(p) \leq \sigma_1(g^{-1}A^\delta_g)(p). $$

Since $t \geq \delta$, this implies $u_t \leq \bar{\delta}$, for some $\bar{\delta}$ depending only on $(M, g)$. 

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Proposition 3.2 (Gradient estimate) Let \( u_t \in C^3(M) \) be a solution of (6) for some \( \delta \leq t \leq 2/3 \). Assume that \( u_t \leq \delta \). Then \( \| \nabla g u \|_{g, \infty} < C_1 \), where \( C_1 \) depends only on \((M, g)\) and \( \delta \).

The proof of this lemma can be found in the paper Gursky-Viaclovsky [12].

Remark 3.3 Note that we will use this proposition with \( \delta \) given by Proposition 3.1 and then, since \( \delta \) depends only on \((M, g)\), we infer that \( C_1 \) only depends on \((M, g)\).

4 A technical lemma

As we proved in the previous section, there exists two constants \( \bar{\delta} \) and \( C_1 \) depending only on \((M, g)\) such that all solutions of (6) for some \( \delta \leq t \leq 2/3 \), satisfying \( u_t \leq \bar{\delta} \) satisfies \( \| \nabla g u \|_{g, \infty} < C_1 \).

We consider the following quantity:

\[
I(M, g) := \inf_{g' = e^{-2\varphi} g, |\nabla g \varphi| \leq C_1} \left( \int_M R_{g'}^2 e^{-\varphi} dV_{g'} \right).
\]

We let, for \( g' = e^{-2\varphi} g \)

\[
i(g') := \int_M R_{g'}^2 e^{-\varphi} dV_{g'}.
\]

As one can easily check, if two metrics \( g_1 \) and \( g_2 \) are homothetic, then \( i(g_1) = i(g_2) \). So, we have

\[
I(M, g) = \inf_{g' = e^{-2\varphi} g, Vol(M, g') = 1 \text{ and } |\nabla g \varphi| \leq C_1} \left( \int_M R_{g'}^2 e^{-\varphi} dV_{g'} \right).
\]

We have the following

Lemma 4.1 There exists a positive constant \( C = C(M, g) \) depending only on \((M, g)\) such that

\[
I(M, g) \geq C (Y(M, [g]))^2.
\]

Proof As we have seen

\[
I(M, g) = \inf_{g' = e^{-2\varphi} g, Vol(M, g') = 1 \text{ and } |\nabla g \varphi| \leq C_1} \left( \int_M R_{g'}^2 e^{-\varphi} dV_{g'} \right).
\]

Take \( \varphi \in C^\infty(M) \) such that, for \( g' = e^{-2\varphi} g, Vol(M, g') = 1 \) and such that \( |\nabla g \varphi|_g \leq C_1 \) where \( C_1 \) is given by Proposition 3.2 Since \( Vol(M, g') = 1 \), if \( p \) is a point where \( \varphi \) attains its minimum we have

\[
e^{-3\varphi(p)} Vol(M, g) \geq 1,
\]

and then, there exists \( C_0 \) depending only on \((M, g)\) such that \( \varphi(p) \leq C_0 \). Now, using the mean value theorem, it follows since \( |\nabla g \varphi|_g \) is controlled by a constant depending only on \((M, g)\), that \( \max \varphi \leq C_0' \) where \( C_0' \) depends only on \((M, g)\).

Using this, we clearly have that

\[
\int_M R_{g'}^2 e^{-\varphi} dV_{g'} \geq e^{-C_0'} \int_M R_{g'}^2 dV_{g'}.
\]

Using Hölder inequality and the definition of the Yamabe invariant, we get (recall that \( Vol(M, g') = 1 \))

\[
\int_M R_{g'}^2 e^{-\varphi} dV_{g'} \geq e^{-C_0'} (Y(M, [g]))^2,
\]

and then \( I(M, g) \geq e^{-C_0'} (Y(M, [g]))^2 \). This ends the proof.
5 Lower bound

For the lower bound, we need the following lemmas:

**Lemma 5.1** For a conformal metric \( \tilde{g} = e^{-2u}g \), we have the following integral transformation

\[
\int_M \sigma_2 (\tilde{g}^{-1} A_{\tilde{g}}^1) e^{-4u} dV_{\tilde{g}} = \int_M \sigma_2 (g^{-1} A_g^1) dV_g + \frac{1}{8} \int_M R_g |\nabla_g u|_{g}^2 dV_g - \frac{1}{4} \int_M |\nabla_g u|_{g}^4 dV_g + \frac{1}{2} \int_M \Delta_g u |\nabla_g u|_{g}^2 dV_g - \frac{1}{2} \int_M A_g^1 (\nabla_g u, \nabla_g u) dV_g.
\]

**Proof** Denote \( \tilde{\sigma}_1 = \sigma_1 (\tilde{g}^{-1} A_{\tilde{g}}^1), \) \( \sigma_1 = \sigma_1 (g^{-1} A_g^1), \) \( \tilde{\sigma}_2 = \sigma_2 (\tilde{g}^{-1} A_{\tilde{g}}^1), \) \( \sigma_2 = \sigma_2 (g^{-1} A_g^1). \) We have

\[
2\tilde{\sigma}_2 = \tilde{\sigma}_1^2 - |A_{\tilde{g}}^1|_{\tilde{g}}^2.
\]

By equation (5), we have

\[
\sigma_1 e^{-2u} = \sigma_1 + \Delta_g u - \frac{1}{2} |\nabla_g u|_{g}^2,
\]

so

\[
\sigma_1^2 e^{-4u} = \sigma_1^2 + (\Delta_g u)^2 + \frac{1}{4} |\nabla_g u|_{g}^2 + 2\sigma_1 \Delta_g u - \Delta_g u |\nabla_g u|_{g}^2 - \sigma_1 |\nabla_g u|_{g}^2.
\]

After an easy computation, we get

\[
|A_{\tilde{g}}^1|_{\tilde{g}}^2 e^{-4u} = |A_g^1|_{g}^2 + |\nabla_g u|_{g}^2 + \frac{3}{4} |\nabla_g u|_{g}^4 - \sigma_1 |\nabla_g u|_{g}^2 - \Delta_g u |\nabla_g u|_{g}^2 + 2 (A_g^1)_{ij} \nabla_{g}^2 u \nabla_{g}^i u + \nabla_{g}^i u \nabla_{g}^j u + \nabla_{g}^i u \nabla_{g}^j u - 2\sigma_1 \Delta_g u
\]

Putting all together, we obtain

\[
2\tilde{\sigma}_2 e^{-4u} = 2 \sigma_2 + (\Delta_g u)^2 - |\nabla_g u|_{g}^2 - \frac{1}{2} |\nabla_g u|_{g}^4 + 2\sigma_1 \Delta_g u
\]

Now, by simple computation, we have the following identities

\[
-2 \int_M (A_g^1)_{ij} \nabla_{g}^2 u dV_g = -2 \int_M \sigma_1 \Delta_g u dV_g,
\]

\[
-2 \int_M \nabla_{g}^2 u \nabla_{g}^i u \nabla_{g}^j u dV_g = \int_M \Delta_g u |\nabla_g u|_{g}^2 dV_g,
\]

where we integrated by parts and we used the Schur’s Lemma for the first identity. Finally we get

\[
2 \int_M \tilde{\sigma}_2 e^{-4u} dV_g = 2 \int_M \sigma_2 dV_g + \int_M [(\Delta_g u)^2 - |\nabla_g u|_{g}^2 + \frac{1}{2} |\nabla_g u|_{g}^4 + \Delta_g u |\nabla_g u|_{g}^2 - 2A_g^1 (\nabla_g u, \nabla_g u)] dV_g.
\]

Now using the integral Bochner formula

\[
\int_M |\nabla_g u|_{g}^2 dV_g + \int_M Ric_g (\nabla_g u, \nabla_g u) dV_g - \int_M (\Delta_g u)^2 dV_g = 0,
\]

we get the final result.

In the sequel of the proof, we will need the following proposition (see [12] for the proof)

**Proposition 5.2** If for some metric \( g_1 \) on \( M \) we have \( A_{g_1}^t \in \Gamma^+_2 \), then

\[
-A_{g_1}^t + \sigma_1 (g_1^{-1} A_{g_1}^t) g_1 > 0,
\]

\[
A_{g_1}^t + \frac{1}{3} \sigma_1 (g_1^{-1} A_{g_1}^t) g_1 > 0.
\]
Going on with the proof for the lower bound, we have the Lemma:

**Lemma 5.3** If \( A^t_y \in \Gamma_2^+ \), then we have the following estimate

\[
\frac{1}{2} \int_M A_y^t(\nabla_g u, \nabla_g u) dV_g < \frac{3 - 2t}{8} \int_M R_g |\nabla_g u|^2 e^{-2u} dV_g + \frac{1}{4} \int_M \Delta_g u |\nabla_g u|^2 dV_g - \frac{1}{4} \int_M |\nabla_g u|^2 dV_g.
\]

**Proof** Since \( A^t_y \in \Gamma_2^+ \), by Proposition 5.2 we get

\[-A^t_y > -\sigma_1(\tilde{g}^{-1} A^t_y) \tilde{g} = -(4 - 3t) \sigma_1(\tilde{g}^{-1} A^t_y) e^{-2u} g.\]

Hence we get

\[-A^t_y - (1 - t) \sigma_1(\tilde{g}^{-1} A^t_y) e^{-2u} g > -(4 - 3t) \sigma_1(\tilde{g}^{-1} A^t_y) e^{-2u} g,\]

which implies that

\[A^t_y < (3 - 2t) \sigma_1(\tilde{g}^{-1} A^t_y) e^{-2u} g.\]

Applying this to \( \nabla_g u \) we obtain

\[\frac{1}{2} A^t_y(\nabla_g u, \nabla_g u) < \frac{3 - 2t}{8} R_g |\nabla_g u|^2 e^{-2u}.\]

Using the conformal transformation law of the tensor \( A^t_y \), integrating over \( M \), we have the result.

Now we are able to prove the following lower bound (recall that \( C_1 \) is given by Lemma 3.2)

**Proposition 5.4 (Lower Bound)** Assume that for some \( t \in [\delta, 2/3] \) the following estimate holds

\[
(7) \quad \int_M \sigma_2(g^{-1} A^t_y) dV_g + \frac{1}{24} \left( \frac{7}{10} - t \right) \inf_{g' \in \mathcal{E}_{e^{-2\varphi}}} \left( \int_M R_{g'}^2 e^{-\varphi} dV_{g'} \right) := \mu_t > 0.
\]

Then there exists \( \delta \) depending only on \((M, g)\) such that if \( u_t \in C^2(M) \) is a solution of (6) and if \( A^t_{u_t} \in \Gamma_2^+ \) then \( u_t \geq \delta \).

**Proof** Since \( A^t_y = A^t_{u_t} + (1 - t) \sigma_1(g^{-1} A^t_y) g \), we easily have that

\[\sigma_2(A^t_y) = \sigma_2(A^t_{u_t}) + (1 - t)(5 - 3t) \sigma_1(g^{-1} A^t_y)^2.\]

Letting \( \tilde{g} = e^{-2u_t} g \),

\[e^{4u_t} f^2 = \sigma_2(g^{-1} A^t_{u_t}) = \sigma_2(g^{-1} A^t_{u_t}) + (1 - t)(5 - 3t) \left( \sigma_1(g^{-1} A^t_{u_t}) \right)^2 = e^{-4u_t} \left( \sigma_2(g^{-1} A^t_{u_t}) + \frac{1}{16}(1 - t)(5 - 3t) R_{u_t}^2 \right).\]

Integrating this with respect to \( dV_g \), we obtain

\[C \int_M e^{4u_t} dV_g \geq \int_M f^2 e^{4u_t} dV_g\]

\[= \int_M \sigma_2(\tilde{g}^{-1} A^t_{u_t}) e^{-4u_t} dV_g + \frac{1}{16}(1 - t)(5 - 3t) \int_M R_{u_t}^2 e^{-4u_t} dV_g\]

\[= \int_M \sigma_2(\tilde{g}^{-1} A^t_{u_t}) e^{-4u_t} dV_g + \frac{1}{16}(1 - t)(5 - 3t) \int_M R_{\tilde{g}}^2 e^{-u_t} dV_{\tilde{g}},\]

where \( C > 0 \) is chosen so that \( f^2 \leq C \) (recall that, since \( f = \sigma_2(g^{-1} A^t_y) \), \( C \) depends only on \((M, g)\)).

Using the fact that

\[R_{\tilde{g}} e^{-u_t} = R_{\tilde{g}} + 4 \Delta_{\tilde{g}} u_t - 2 |\nabla_{\tilde{g}} u_t|^2,\]

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from Lemma\textsuperscript{5.1} we get
\[
\int_M \sigma_2(\tilde{g}^{-1}A_{ui}^1)e^{-4ut}dV_g = \int_M \sigma_2(g^{-1}A_{y_i}^1)dV_g + \frac{1}{8} \int_M R_g |\nabla_g u|^2 e^{-2ut}dV_g \\
- \frac{1}{2} \int_M A_{y_i}^1(\nabla_g u, \nabla_g u) dV_g.
\]

Notice that, since $A_{ui}^1 \in \Gamma_2^+$, we have
\[
0 < \sigma_1(\tilde{g}^{-1}A_{ui}^1) = (4 - 3t)\sigma_1(g^{-1}A_{ui}^1),
\]
and so $R_{\tilde{g}} > 0$. By Lemma\textsuperscript{5.3} we obtain
\[
\int_M \sigma_2(\tilde{g}^{-1}A_{ui}^1)e^{-4ut}dV_g \geq \int_M \sigma_2(g^{-1}A_{y_i}^1)dV_g - \frac{1 - t}{4} \int_M R_g |\nabla_g u|^2 e^{-2ut}dV_g \\
- \frac{1}{4} \int_M \Delta_g u |\nabla_g u|^2 dV_g + \frac{1}{4} \int_M |\nabla_g u|^4 dV_g.
\]

By Young’s inequality, one has
\[
\int_M R_{\tilde{g}}^2 e^{-ut} dV_{\tilde{g}} \geq \frac{2}{\varepsilon} \int_M R_{\tilde{g}} |\nabla_g u|^2 e^{-2ut} dV_g - \frac{1}{\varepsilon^2} \int_M |\nabla_g u|^4 dV_g,
\]
for all $\varepsilon > 0$. By an easy computation, we have
\[
\frac{1}{16}(1-t)(5-3t) = \frac{1}{24}(\frac{7}{10} - t) + P_2(t),
\]
where $P_2(t)$ is a positive, second order, polynomial in $t$. Putting all together, we obtain (for $C > 0$ depending only on $(M, g)$)
\[
C \int_M e^{4ut} dV_g \geq \int_M \sigma_2(g^{-1}A_{ui}^1)e^{-4ut}dV_g + \frac{1}{16}(1-t)(5-3t) \int_M R_{\tilde{g}}^2 e^{-ut} dV_{\tilde{g}} \\
= \int_M \sigma_2(\tilde{g}^{-1}A_{ui}^1)e^{-4ut}dV_g + \left( \frac{1}{24}(\frac{7}{10} - t) + P_2(t) \right) \int_M R_{\tilde{g}}^2 e^{-ut} dV_{\tilde{g}} \\
\geq \int_M \sigma_2(g^{-1}A_{y_i}^1)dV_g + \frac{1}{24}(\frac{7}{10} - t) \int_M R_{\tilde{g}}^2 e^{-ut} dV_{\tilde{g}} \\
+ P_2(t) \int_M R_{\tilde{g}}^2 e^{-ut} dV_{\tilde{g}} - \frac{1 - t}{4} \int_M R_g |\nabla_g u|^2 e^{-2ut} dV_g \\
- \frac{1}{4} \int_M \Delta_g u |\nabla_g u|^2 dV_g + \frac{1}{4} \int_M |\nabla_g u|^4 dV_g.
\]

Now using Young’s inequality and the conformal change equation of the scalar curvature, we get (for a certain $C > 0$ depending only on $(M, g)$)
\[
C \int_M e^{4ut} dV_g \geq \int_M \sigma_2(g^{-1}A_{y_i}^1)dV_g + \frac{7}{24} \left( \frac{7}{10} - t \right) \int_M R_{\tilde{g}}^2 e^{-ut} dV_{\tilde{g}} \\
+ \frac{2P_2(t)}{\varepsilon} - \frac{1-t}{4} \int_M R_g |\nabla_g u|^2 dV_g \\
+ \left( \frac{8P_2(t)}{\varepsilon} - (1-t) - \frac{1}{4} \right) \int_M \Delta_g u |\nabla_g u|^2 dV_g \\
+ \left( \frac{3 - 2t}{4} - \frac{P_2(t)}{\varepsilon^2} - \frac{4P_2(t)}{\varepsilon} \right) \int_M |\nabla_g u|^4 dV_g.
\]
We choose \( \varepsilon = \varepsilon(t) > 0 \), such that \( \frac{8P_2(t)}{\varepsilon} - (1 - t) - \frac{1}{t} = 0 \). One can easily check that, with this choice,
\[
\frac{2P_2(t)}{\varepsilon} - \frac{1 - t}{4} \geq 0 \quad \text{and} \quad \frac{3 - 2t}{4} - \frac{P_2(t)}{\varepsilon} - \frac{4P_2(t)}{\varepsilon} \geq 0.
\]

Finally, recalling that according to lemma 3.2 \( \| \nabla u_t \|_{g, \infty} \leq C_1 \) with \( C_1 \) depending only on \((M, g)\), we obtain the following estimate (for a certain \( C > 0 \) depending only on \((M, g)\))
\[
C \int_M e^{4u_t} dV_g \geq \int_M \sigma_2(g^{-1}A_g^1) dV_g + \frac{1}{24} \int_M R_g^2 e^{-u_t} dV_g \\
\geq \int_M \sigma_2(g^{-1}A_g^1) dV_g + \frac{1}{24} \int_M R_g^2 e^{-u_t} dV_g \\
\geq \int_M \sigma_2(g^{-1}A_g^1) dV_g + \frac{1}{24} \left( \frac{7}{10} - t \right) \int_M R_g^2 e^{-u_t} dV_g \\
\geq \int_M \sigma_2(g^{-1}A_g^1) dV_g + \frac{1}{24} \left( \frac{7}{10} - t \right) \inf_{g' = e^{-2\varphi}g, |\nabla \varphi|_g \leq C_1} \left( \int_M R_g^2 e^{-\varphi} dV_g' \right) = \mu_t > 0.
\]

This gives
\[
\max_M u_t \geq \log \mu_t - C(g).
\]
Since \( \| \nabla u_t \|_{g, \infty} < C_1 \) this implies the Harnack inequality
\[
\max_M u_t \leq \min_M u_t + C(M, g),
\]
by simply integrating along a geodesic connecting points at which \( u_t \) attains its maximum and minimum. Combining this two inequalities, we obtain
\[
\min_M u_t \geq \log \mu_t - C,
\]
where \( C \) only depends on \((M, g)\). This ends the proof of the Lemma.

6 \( C^{2, \alpha} \) estimate

We have the following \( C^{2, \alpha} \) estimate for solutions of the equation \( (8) \). For the proof, see [12] and [10].

**Proposition 6.1 (\( C^{2, \alpha} \) estimate)** Let \( u_t \in C^4(M) \) be a solution of \( (8) \) for some \( \delta \leq t < 2/3 \), satisfying \( \delta < u_t < \delta \), and \( \| \nabla u_t \|_{g, \infty} < C_1 \). Then for \( 0 < \alpha < 1 \), \( \| u_t \|_{g, C^{2, \alpha}} \leq C_2 \), where \( C_2 \) depends only on \((M, g)\).

7 Proof of Theorem 1.6

We use the continuity method. Our 1-parameter family of equations, for \( t \in [\delta, t_0] \), is
\[
(8) \quad \sigma_2^{1/2}(g^{-1}A_{u_t}) = f(x)e^{2u_t},
\]
with \( f(x) = \sigma_2^{1/2}(g^{-1}A_g^0) > 0 \), and \( \delta \) was chosen so that \( A_g^0 \) is positive definite. Define
\[
S = \{ t \in [\delta, t_0] | \exists \text{a solution } u_t \in C^{2, \alpha}(M) \text{ of } (8) \text{ with } A_{u_t} \in \Gamma_2^+ \}.
\]
Clearly, with our choice of \( f, u \equiv 0 \) is a solution for \( t = \delta \). Since \( A_g^0 \) is positive definite, \( \delta \in S \), and \( S \neq \emptyset \). Let \( t \in S \), and \( u_t \) be a solution. By Proposition 2.3 the linearized operator at \( u_t \), \( L^t : C^{2, \alpha}(M) \to C^0(M) \), is invertible. The implicit function theorem tells us that \( S \) is open. From classical elliptic theory, it follows that \( u_t \in C^\infty(M) \), since \( f \in C^\infty(M) \). By Proposition 3.1 we get an uniform upper bound on the solutions \( u_t \), independent of \( t \). We may then apply Proposition 3.2 to obtain a uniform gradient bound on \( u_t \), and by Proposition 5.1 we get a uniform lower bound. Finally using Proposition 6.1 and the classical Ascoli-Arzelà’s Theorem, then implies that \( S \) must be closed, therefore \( S = [\delta, t_0] \). The metric \( \tilde{g} = e^{-2u_0}g \) then satisfies \( \sigma_2(\tilde{A}_g^0) > 0 \) and \( R_g > 0 \). The inequalities [2] follow from proposition 5.2.
8 Proof of Theorem 1.2

Theorem 1.2 is a direct consequence of Theorem 1.6 and of Lemma 4.1.

9 Proof of Corollary 1.7

Assume that $M$ admits a metric $g'$ such that $Q_{g'} \geq \frac{1}{48} R_{g'}^2$ and $Y(M,[g']) \geq 0$. Recall that

$$Q_{g'} = -\frac{1}{4} \Delta_{g'} R_{g'} - 2 |\text{Ric}_{g'}|^2 + \frac{23}{32} R_{g'}^2.$$ 

Integrating $Q_{g'}$ on $M$ with respect to $dV_{g'}$ we obtain (since $Q_{g'} \geq 0$)

$$\int_M |\text{Ric}_{g'}|^2 dV_{g'} \leq \frac{23}{64} \int_M R_{g'}^2 dV_{g'}.$$ 

Now if we compute $\int_M \sigma_2(g'^{-1}A_{g'}^1)$ using (9), we have (recall that $\sigma_2(g'^{-1}A_{g'}^1) = -\frac{1}{2} |\text{Ric}_{g'}|^2 + \frac{2}{16} R_{g'}^2$):

$$\int_M \sigma_2(g'^{-1}A_{g'}^1) \geq \frac{1}{128} \int_M R_{g'}^2 dV_{g'} \geq 0.$$ 

Now, consider the conformal laplacian operator $L_{g'} := \Delta_{g'} - \frac{1}{8} R_{g'}$. We have using the assumption $Q_{g'} \geq \frac{1}{48} R_{g'}^2$

$$L_{g'} R_{g'} = \Delta_{g'} R_{g'} - \frac{1}{8} R_{g'}^2 \leq -8 |\text{Ric}_{g'}|^2 + \frac{22}{8} R_{g'}^2 - \frac{1}{12} R_{g'}^2 \leq \left( -\frac{8}{3} + \frac{22}{8} - \frac{1}{12} \right) R_{g'}^2 = 0.$$ 

Applying a Lemma due to Gursky [9], since $Y(M,[g']) \geq 0$ we have either $R_{g'} > 0$ (if $Y(M,[g']) > 0$) or $R_{g'} \equiv 0$ (if $Y(M,[g']) = 0$). If $Y(M,[g']) > 0$ we can apply Theorem 1.3 to conclude that $m$ is diffeomorphic to a spherical space form. Otherwise, if $Y(M,[g']) = 0$, since $Q_{g'} \geq \frac{1}{18} R_{g'}^2$ and $R_{g'} \equiv 0$, we deduce, using the expression giving $Q_{g'}$, that $\text{Ric}_{g'} \equiv 0$ and then $M$ is diffeomorphic to a quotient of $\mathbb{R}^3$.

This ends the proof of the Corollary.

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