Quantum gravity models - a brief conceptual summary*

Jerzy Lukierski
University of Wrocław

Abstract

After short historical overview we describe the difficulties with application of standard QFT methods in quantum gravity (QG). The incompatibility of QG with the use of classical continuous space-time required conceptually new approach. We present briefly three proposals: loop quantum gravity (LQG), the field-theoretic framework on noncommutative space-time and QG models formulated on discretized (triangularized) space-time. We evaluate these models as realizing expected important properties of QG: background independence, consistent quantum diffeomorphisms, noncommutative or discrete structure of space-time at very short distances, finite/renormalizable QG corrections. We only briefly outline an important issue of embedding QG into larger geometric and dynamical frameworks (e.g. supergravity, (super)strings, p-branes and M-theory), with the aim to achieve full unification of all fundamental interactions.

*Invited article for the book “Mathematical Structure of the Universe”, publ. Copernicus Center Press, Copernicus Center for Interdisciplinary Studies, Cracow 2014, pp. 277-300.
Contents:

1. Introduction

2. Problems with Einstein quantum gravity (QG) as quantum field-theoretic model and dynamical noncommutative quantum space-time

3. Basic QG models
   3.1 General remarks
   3.2 Loop QG
   3.3 QG as field theory on noncommutative space-time
   3.4 Functional formulation of QG and lattice theories

4. Final remarks

1 Introduction

The first general relativity (gravity) model was formulated as nonlinear field theory of gravitational metric field $g_{\mu\nu}(x)$ and provided in 1915 the Einstein field equations following from the Einstein-Hilbert action

$$S^{EH} = -\frac{1}{2\kappa} \int d^4x \sqrt{\det g_{\mu\nu}} R,$$

(1.1)

where $R = R_{\mu\nu}^{\ \mu\nu}$ is the scalar curvature (see e.g. [Misner, Thorne & Wheeler 1973, Dirac 1975]). The gravitational field is introduced as characterizing the (pseudo)Riemannian geometry of curved space-time with the invariant length element $ds$ expressed in terms of the local metric $g_{\mu\nu}(x)$ as follows

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu.$$

The action (1.1) is invariant under the local space-time transformations (diffeomorphisms)

$$x'_\mu \equiv x'_\mu(x) = x_\mu + \chi_\mu(x).$$

(1.2)

The Einstein equations which describe the gravitational field interacting with matter take the following form

$$G_{\mu\nu}(x) \equiv R_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) R(x) = \kappa T_{\mu\nu}(x),$$

(1.3)

where $R_{\mu\nu}$ is the Ricci tensor (see e.g. [Misner, Thorne & Wheeler 1973, Dirac 1975]), $\kappa = \frac{8\pi G}{c^2} (G$ – Newton constant, $c$ – light velocity) and $T_{\mu\nu}(x)$ the local energy-momentum
tensor which describes the matter distribution in space-time. From (1.3) follows that $T_{\mu\nu}(x)$ determines the space-time curvature tensor $R_{\mu\nu}^{\rho\tau}(x)$, i.e. gravity is a dynamical theory of curved space-time providing geometric interpretation of matter distribution.

The form (1.3) of Einstein equations leads to the question of how to define its rhs, i.e. how to describe in field-theoretic framework the tensor $T_{\mu\nu}(x)$ representing gravitational matter sources. The development of field theory provided us however with the answer of how the matter is described by various fields. In particular, matter fields were described by

- in 1926 – scalar Klein-Gordon field $\phi(x)$ [Klein 1926], describing e.g. Higgs particle;
- in 1926 – spinorial Dirac field [Dirac 1926] $\psi_A(x)$, describing e.g. electron/positon and quarks;
- in 1954 – vectorial Yang-Mills (YM) field $A_i^\mu$ [Yang & Mills 1954], where the indices $i$ describe the adjoint representation of an internal symmetry group. By YM fields are described e.g. in QCD, gluons with internal symmetry $SU(3)$ or vectorial $W$-bosons in electroweak sector of Standard Model (SM) with internal symmetries $SU(2) \times U(1)$ (see e.g. [Holstein, Golowich & Donoghue 1994]).

Relativistic free fields describing noninteracting elementary particles are characterized by mass $m \geq 0$ and spin $s$ ($s = 0, \frac{1}{2}, 1, \ldots$) [Wigner 1939], with the listed above three examples of fields corresponding to $s \leq 1$. We add that the gravitational field $g_{\mu\nu}(x)$ corresponds to the choice $m = 0$ and $s = 2$.

The first effort to unify all fundamental interactions was successfully undertaken inside the local $D = 4$ field-theoretic framework in the 1970’s by coupling the gravitational sector to the collection of fields describing elementary particles in the framework of the Standard Model (SM). The action describing all elementary processes in Nature was proposed in the following form

$$S^{\text{tot}} = S^{\text{EH}} + S^{\text{SM}} + S^{\text{int}},$$

where $S^{\text{SM}}$ is the field-theoretic action of the SM and $S^{\text{int}}$ describe its gravitational interactions, with the rhs of (1.3) defined by the variation of $S^{\text{int}}$ under the change $g^{\mu\nu}(x) \rightarrow g^{\mu\nu}(x) + \delta g^{\mu\nu}(x)$, i.e.

$$T_{\mu\nu}(x) = \frac{\delta S^{\text{int}}}{\delta g^{\mu\nu}(x)}.$$

The next step is to introduce the quantization of all fields which occur in the action (1.4). It was realized that in order to calculate the scattering processes of elementary particles one should use the perturbative methods and consider perturbative expansions in quantum free fields which introduce the algebra of field oscillators. After introducing the Higgs mechanism [Higgs 1964], which generated the mass for the appropriate gauge fields, it appeared that in

---

$^1$It should be mentioned that vectorial EM field $A_\mu(x)$ with $U(1)$ internal symmetry has been known since 1861, when the Maxwell equations describing electrodynamics were introduced.
the quantized SM model, the perturbative calculations performed via Feynman diagrams techniques lead to the formulae for scattering amplitudes with infinities which one cannot remove by renormalization procedure. Unfortunately, the Einstein gravity, described by the action (1.1), after quantization leads in perturbative calculations to infinities which are not renormalizable.

2 Problems with Einstein quantum gravity (QG) as a quantum field-theoretic model and dynamical noncommutative quantum space-time

From a logical point of view it is still not excluded that gravity in Nature remains classical. Even if in the action (1.4) we quantize all the fields of SM, the requirement that the classical gravitational background has to be promoted to dynamical quantum field cannot be formally proven (see e.g. [Kiefer 2012]). This option is however usually not accepted [DeWitt 1962]. It also appeared that quantized Einstein gravity can be introduced as a suitably constrained gauge QFT of massless spin two field with local diffeomorphism (1.2) as the local gauge group (see e.g. [Ogievetsky & Polubarinov 1965]). Then, it was advocated that the aim of the theory of fundamental interactions is to incorporate the gravitational degrees of freedom in the field-theoretic quantum ‘Theory of Everything’, containing all quantized fields.

Two further ways of dealing with the unsurmountable property of nonrenormalizability of perturbative Einstein QG were assumed in the standard QFT approach:

- To apply the methods of renormalization group and introduce a running energy-dependent gravitational coupling constant scenario with nonzero fixed point. Such so-called ‘asymptotic safety’ approach was initiated by Weinberg in the 1970’s [Weinberg, Hawking & Israel 1979] and further developed with some success by other authors (see e.g. [Reuter 1998, Percacci 2009]).

- To modify the Einstein-Hilbert action, in particular by the following replacement in (1.1)

\[ R \rightarrow R + aR^2 + bR_{\mu\nu}R^{\mu\nu} + cR_{\mu\nu\rho\tau}R^{\mu\nu\rho\tau} + \ldots \]  (2.1)

Various particular cases of formula (2.1) were studied, for example:

• \( c = 0 \) – Stelle family of actions [Stelle 1977], providing renormalizable theory but after quantization containing the ghost states².

²The ghost states are described by quantum states with zero or negative norm, something which is not consistent with the basic postulates of QM.
• $b = c = 0$ – Starobinsky model [Starobinsky 1979] which was successfully applied in the construction of the cosmological model describing inflation as well as nowadays the acceleration of the expansion of the Universe.

• $a = c = \frac{1}{4} b$ - Zwiebach ‘stringy’ gravity model inspired by quantum corrections in string theory [Zwiebach 1985].

In the class of modified Einstein QG models within the standard QFT approach one should include the recent modifications of Einstein theory with an asymmetric treatment of space and time, which lead to renormalizable class of Hořava-Lifschitz gravities [Hořava 2009], as well as to so-called $f(R)$ gravities (see e.g. [Capozziello & Francaviglia 2008, Sotiriou & Faraoni 2010]).

Further in this presentation we shall be interested in QG models formulated outside the standard QFT approach, with new proposals for the construction of the quantum counterparts of standard classical gravity framework. For such a purpose it is important to single out the basic reasons for the failure of standard QFT approach to QG.

There are two important difficulties with applying the standard quantum field-theoretic methods to QG:

i) **Technical problem**: The nonlinearity of Einstein-Hilbert action and the dimensionfull nature of gravitational coupling constant $\kappa \sim G$ leads to $D = 4$ perturbative nonrenormalizability. It appears that the action densities describing renormalizable $D = 4$ local field theories are low-order (up to fourth) polynomials in local field variables.

ii) **Conceptual problem**: The feature that QG describes the quantized geometrodynamics of space-time is alien to the framework of standard classical and quantum field theory. In the quantization procedure the classical space-time with numerical coordinates $x_\mu$ due to QG effects becomes dynamical, noncommutative

$$\text{classical space-time } x_\mu \xrightarrow{\text{QG}} \text{quantum space-time } \hat{x}_\mu.$$  

We recall that the standard quantum field is described on flat (static) Minkowski space $x_\mu \in \mathbb{M}^{3,1}$ as the Fourier momentum integral over the quantized field oscillators $\hat{a}(p)$

$$\hat{\phi}(x) = \int d^4 p \mu(p) \hat{a}(p) e^{ipx},$$

where the measure $\mu(p)$ for free fields incorporates the mass-shell condition. In the presence of QG effects, the classical space-time describing field arguments is replaced by the quantum space-time, i.e. the standard quantum fields due to QG are modified as follows:\footnote{The quantum space-time is usually characterized by noncommutative or discretized commutative space-time coordinates.}

$$\hat{\phi}(x) \xrightarrow{\text{QG}} \hat{\phi}(\hat{x}).$$
It has been shown [Mead 1964, Doplicher, Fredenhagen & Roberts 1995, Garay 1995, Bronstein 2012] that in the presence of a quantized gravitational field one cannot measure two different space-time coordinates with arbitrary accuracy. Assuming that:

i) gravitational field satisfies the Einstein equations (see 1.3) and

ii) quantum-mechanical Heisenberg uncertainty relations restricting the measurements of position/three-momenta and energy/time

\[ \Delta x_i \Delta p_j \geq \delta_{ij} \hbar, \]  

\[ \Delta t \Delta E \geq \hbar \]  

are valid, one can derive that

\[ \Delta x_i \Delta x_j \geq \delta_{ij} l_p^2, \quad l_p = \sqrt{\frac{\hbar G}{c^2}} \approx 10^{-33} \text{cm}. \]  

Further, due to (2.2) and relativistic covariance, the relations (2.3) can be extended to the following relativistic form of the uncertainty relations

\[ \Delta x_\mu \Delta x_\nu \geq l_p^2 (\mu \neq \nu). \]  

In the algebraic formulation that leads to (2.4) one can equivalently postulate the following noncommutativity of quantum space-time coordinates

\[ [\hat{x}_\mu, \hat{x}_\nu] = l_p^2 \theta^{(0)}_{\mu\nu} + \ldots = l_p^2 \theta_{\mu\nu} \left( \frac{\hat{x}}{l_p} \right), \]  

where \( \theta^{(0)}_{\mu\nu} = -\theta^{(0)}_{\nu\mu} \) is a constant antisymmetric tensor. In the general case the antisymmetric operator-valued function \( \theta_{\mu\nu}(\hat{x}) \) can be generalized to a function of the quantum phase space coordinates \( (\hat{x}_\mu, \hat{p}_\mu) \), namely

\[ \theta_{\mu\nu}(\hat{x}) \to \Theta_{\mu\nu}(\hat{x}, \hat{p}). \]  

Various models of quantum space-times and quantum phase spaces are characterized by different choices of \( \theta_{\mu\nu}(\hat{x}) \) or \( \Theta_{\mu\nu}(\hat{x}, \hat{p}) \). It should be added that the relations (2.5) for any \( \theta_{\mu\nu}(\hat{x}) \) necessarily break the standard Lorentz invariance, however, as it was shown firstly by Snyder [1947], if we choose particular momentum-dependent noncommutativity

\[ \Theta_{\mu\nu}(\hat{x}, \hat{p}) = M_{\mu\nu} = \hat{x}_\mu \hat{p}_\nu - \hat{x}_\nu \hat{p}_\mu, \]  

we get the Lorentz-invariant Snyder model of quantum space-time.

From the mathematical point of view, quantum space-time is a noncommutative algebraic manifold, with a basic relation (2.5) (see also (2.6) and footnote3). Physically, the

\footnote{In such a case we should supplement (2.5) with the formulae for the commutators \([\hat{x}_\mu, \hat{p}_\nu]\) and \([\hat{p}_\mu, \hat{p}_\nu]\) in a way consistent with Jacobi identities.}
appearance of a noncommutative structure can be explained by the gravitational effect of creating microscopic black holes due to the appearance of very high energy densities in the region where the space-time localization measurement is performed (such an effect was predicted firstly by M.P. Bronstein [2012]). In consequence the QG effects lead to the atomization of physical space-time at very short distances.

\[ \text{reaction of dynamical space-time to quantum measurement process } \iff \text{below } 10^{-33} \text{cm the notion of classical space-time looses operational meaning} \]

In the physical models formulated in noncommutative space-time, the uncertainty relations (2.4) are incorporated on a basic algebraic level i.e. the limitations of local space-time measurements are becoming an ‘ontological’ property of such an approach.

In the framework with quantum space-time, the Planck length or Newton constant becomes, besides \( c \) and \( \hbar \), the third fundamental geometric parameter. In particular, one can introduce the following three frameworks describing three basic domains of physical dynamical theories:

| Classical mechanics and classical field theory – | Quantum mechanics and standard QFT – | QM and QFT in presence of QG effects – |
|-----------------------------------------------|----------------------------------------|---------------------------------------|
| \( h = 0, \ G = 0; (x_\mu, p_\mu) \) commutative | \( h \neq 0, \ G = 0; \) standard quantum phase space \( (\hat{x}_\mu, \hat{p}_\mu) \) with commuting space-time | \( h \neq 0, \ G \neq 0; \) deformed quantum phase space \( (\hat{x}_\mu, \hat{p}_\mu) \) with noncommutative space-time |

\[ \Downarrow h \neq 0 \]

\[ \Downarrow G \neq 0 \]

The third sector \( (h \neq 0, G \neq 0) \) of quantum theories containing the QG effects can be described by new mathematical tools based on the notions of algebraic geometry: quantum groups, noncommutative geometries and quantum spaces (see e.g. [Drinfeld 1987, Reshetikhin, Takhtadzhyan & Faddeev 1989, Connes 1994, Majid 1995]).

Similarly as the existence of energy quanta follows from basic postulates of quantum mechanics, the quantum space-time geometry implies that at very short distances we are dealing with quanta of space-time heuristically illustrated in \( D = 4 \) by cells in space-time with volume \( \sim l_p^4 \). This picture leads to a popular conclusion that due to QG effects the space-time at very short Planck lengths has a discrete structure.
3 Basic QG models

3.1 General remarks

The modification at short distances of classical space-time structure distinguishes new QG models from more conventional treatments of QG which propose embedding of gravity into a larger geometric structure, with the subsector of standard classical space-time and standard canonical phase space coordinates.

In the conventional approaches to QG the construction of a finite (renormalizable) unified theory is achieved by introducing additional degrees of freedom coupled to gravity sector in order to ‘smooth’ the infinities occurring as nonrenormalizable results of quantum calculations. The first efforts towards such extended space-time framework were based on a Kaluza-Klein scenario, with space-time enlarged by new $n$ dimensions ($D = 4 \rightarrow D = 4 + n$; usually $1 \leq n \leq 7$) (see e.g. [Duff 1994]). Due to the appearance of bosons and fermions in the observable spectrum of elementary particles, the geometric programme of the unification of elementary interactions naturally leads to supersymmetric theories, with $D = 11$ supergravity as the first Theory of Everything unifying all interactions [Freedman & Van Proeyen 2012]. The description which requires $D > 4$ as well as supersymmetry was further conceptually widened by the introduction of extended elementary objects with mostly studied (super)string theory. Chronologically, the next Theory of Everything containing QG was described by $D = 10$ superstrings [Green, Schwarz & Witten 1987] providing after second quantization (i.e. in the framework of quantum string theory) the finite renormalization corrections in its gravity sector.

At present, QG is studied in the literature in basically two ways:

- As embedded in covariant theory of extended $p$-dimensional elementary objects, called $p$-branes ($p = 1, 2, \ldots, D - 1$). In such approach the (super)string formalism is generalized to so-called $M$-theory formulated in $D = 11$ [Becker, Becker & Schwarz 2007], which is considered also as the actual Theory of Everything.

- By introducing nonstandard new models of quantized Einstein gravity. These models are supposed to provide finite (or renormalizable) QG by modifying the canonical field-theoretic approach and lead to the various ways of dealing with quantum space-time.

In this article we shall restrict our comments to the following three basic nonstandard QG approaches:

- Loop quantum gravity (LQG) (see e.g. [Thiemann 2007]);

- QG formulated as QFT on noncommutative space, which may be equivalently described as a particular class of nonlocal field theories on standard space-time (see e.g. [Rovelli 2004, Majid 2009]);

8
Lattice models of QG which are linked with the functional formulation of QG by the use of discretized space-time and its triangulations (see e.g. [Ambjørn, Jurkiewicz & Loll 2010]).

We shall discuss the QG models by taking into consideration the following desirable features:

A. The description should not be attached to any particular choice of space-time, i.e. the formulation should be background-independent.

B. The model should incorporate a quantum counterpart of the diffeomorphism invariance.

C. At very small distances the quantum space-time structure should appear to be described by noncommutative or commutative but discrete manifold.

D. The model should provide renormalizable, or even better finite, formulae which describe the quantum fundamental processes involving QG.

Various new QG models address these requirements in different ways. We mention that we did not include here as a fifth desired property, namely the requirement of the unification of all fundamental interactions, however it is quite plausible that the formulation of a finite unified theory will be related with the embedding of QG in a new, as yet unknown, quantum Theory of Everything.

### 3.2 Loop quantum gravity (LQG)

Einstein gravity based on Einstein-Hilbert action (1.1) is a nonlinear local field theory of a fundamental gravitational field, with local diffeomorphism invariance. As early as the 1960’s (see e.g. [Mandelstam 1962]), by exploiting the analogy between local diffeomorphisms and local non-Abelian internal symmetries in Yang-Mills theories a new kinematic description of gravity was conjectured by introducing the nonlocal loop (or holonomy) variables ($\mathcal{L}$ - closed loop; $i = 1, 2, 3$)

$$A^i[\mathcal{L}; t] = \exp \oint_{\mathcal{L}} A_i^r(\bar{x}, t)dx^r \quad (r = 1, 2, 3), \quad (3.1)$$

which are invariant under local three-dimensional space diffeomorphism transformations. The generalized field coordinates (3.1) were further supplemented by generalized field momenta described by nonlocal flux variables ($S$ - closed surface; $i, j, k = 1, 2, 3$)

$$E^i[S; t] = \varepsilon_{rst} \oint_S E_{i}^r(\bar{x}, t)dx_s \wedge dx_t. \quad (3.2)$$
The fields $A_i^r(x)$, $E_i^r(x)$ are the conjugate Ashtekar SU(2) gauge variables [Ashtekar 1987] which express locally the curved space metric tensor field $g_{rs}(x)$ ($r, s = 1, 2, 3$) by means of the triad $E_i^r$ ('Ashtekar electric field') as follows

$$g_{rs} = \sum_i E_i^r E_i^s \cdot (\det E)^{-1}.$$  

The variables (3.1–3.2) define new phase space with holonomy-flux Poisson algebra which provides the quantization rules

$$\{A_i^r[L; t], E_i^s[S; t]\} \xrightarrow{\hbar \neq 0} \left[\hat{A}_i^r[L; t], \hat{E}_i^s[S; t]\right].$$  

(3.3)

The description of LQG by basic variables (3.1–3.2) is linked with the canonical approach – i.e. the nonlocal phase-space coordinates are defined nonlocally in space but for fixed ‘local’ value of global time $t$. The quantization rules (3.3) are in principle derivable from the canonical formulation of Einstein gravity. In such new phase-space formulation of gravity the dynamics is provided by the following kinematic and dynamical constraints [Thiemann 2007]

→ three SU(2) (Gauss) kinematic constraints,
→ three kinematic constraints generated by space diffeomorphisms,
→ dynamical Hamiltonian constraint following from the covariance under the time translations (time reparametrisations).

In LQG, the local field operators describing standard formulation of Einstein gravity are replaced by the functionals depending on the loop variable (3.1)

$$g_{\mu \nu} \longrightarrow \Phi \left[ A_i^r[L]; t \right].$$

The standard quantization with excitations described by the field-theoretical creation/annihilation operators (e.g. gravitons realized in Fock space) is replaced in LQG by the quantized functionals $\Phi$ acting on nonseparable ‘polymer’ Hilbert spaces which are spanned by the superposition of SU(2) spin network states. These states form a quantum spin lattice and in particular permit to calculate (see e.g. [Thiemann 2007])

- the spectrum of areas describing elementary quantum surface elements, proportional to $\gamma l_p^2$ ($\gamma$ - Barbero-Immirzi parameter);
- the spectrum of elementary quantum volume elements ($\sim l_p^3$).

Both area and volume spectra are bounded from below and introduce dynamically determined granular structure of space, with minimal surfaces and minimal volumes defining the sizes of ‘space quanta’.
Subsequently it was proposed a spacetime covariant extension of canonical LQG – the
covariant LQG framework, with nonrelativistic SU(2) spin network states replaced by space-
time lattices and SU(2) algebra indices replaced by SL(2;C) labeling. The covariant LQG
approach is closely linked with so-called spin foam models (see e.g. [Bahr, Hellmann,
Kamiński, Kisielowski & Lewandowski 2011]), with the space of states defined by the spin
foam graphs. It can also be argued that the covariant LQG model is closely linked to the
four-dimensional lattice approach to QG which will be considered in Sect. 3.3.

One can summarize the properties of the LQG approach by providing the following
answers to the list of desirable properties of QG models (see A-D in Sect. 3.1).

A. The LQG formulation is background-independent by construction; the notion of quant-
   um space-time emerges in a dynamical way.

B. The diffeomorphism-invariant spaces are obtained by introducing the equivalence classes
   of graphs defining the spin network states [Thiemann 2007].

C. An advantage of LQG approach is the dynamical derivation of discrete noncommutative
   structure of quantum space-time, described in particular by minimal surfaces and
   minimal volumes.

D. The LQG calculations, due to the appearance of dynamical discrete structure which
   is the physical regularization, should lead to finite results, with lattice lengths de-
   termined by gravity dynamics. At present, however, the explicit LQG calculations
   provide rather modest results.

One can add that there are still problems with the description in LQG framework of the
classical general relativity results, in particular the derivation of known forms of Einstein
equations and gravitational classical solutions. Also, in LQG formalism the description
of gravitons as quantum excitations requires nontrivial dynamical considerations. On the
other hand, the simplified one-dimensional FRW models have been successfully quantized
and applied in cosmology. Using such ‘toy’ cosmological LQG models it has been deduced
that the evolution of the Universe need not begin with a primary singularity, i.e. the Big
Bang postulate appears not to be supported by the LQG approach.

3.3 QG as QFT on noncommutative space-time

In such an approach one inserts in QG geometric framework explicitly the noncommutativity
of space-time. In place of standard commuting space-time coordinates $x_\mu$ one uses the
following noncommutativity relations for generators $\hat{x}_\mu$ of quantum space-time (see also
(2.5))

$$[x_\mu, x_\nu] = 0 \quad \Rightarrow \quad [\hat{x}_\mu, \hat{x}_\nu] = i d_p^2 \theta_{\mu\nu} \left( \frac{\hat{x}}{l_p} \right), \quad (3.4)$$

11
where the operator-valued tensorial field $\theta_{\mu\nu}$ can be expanded in the gravitational coupling constant as the power expansion 

$$
\left( \frac{y}{l_p} \right) = \frac{y_1}{l_p} + \frac{y_2}{l_p^2} + \cdots,
$$

$$
\theta_{\mu\nu}(y) = \theta_{\mu\nu}(1) + \theta_{\mu\nu}(2) y_2 + \cdots,
$$

where $\theta_{\mu\nu}^{(n)}$ are constant tensors.

The first constant term defines canonical (called also (DFR) Doplicher-Fredenhagen-Roberts) deformation [Doplicher, Fredenhagen & Roberts 1995]; the second one describes Lie-algebraic deformation of space-time algebra, which in particular includes quite popular $\kappa$-deformation [Lukierski, Ruegg, Nowicki & Tolstoy 1991, Majid & Ruegg 1994, Lukierski, Ruegg & Zakrzewski 1995]. The relations (3.4) in quantum phase space formulation can be generalized by assuming that the commutator $[\hat{x}_\mu, \hat{x}_\nu]$ depends also on momenta or even other dynamical primary variables (see e.g. [Lukierski 1999]).

It should be added that at the turn of the millennium there [Amelino-Camelia 2001, Bruno, Amelino-Camelia & Kowalski-Glikman 2001] the modification of special relativity called ‘Doubly Special Relativity’ (DSR) with Planck mass as (in addition to light velocity $c$) the second kinematic invariant was introduced. The postulates of DSR, linked with the earlier description of quantum $\kappa$-deformed Poincaré symmetries [Kowalski-Glikman & Nowak 2002, Lukierski & Nowicki 2003], became in recent years a fertile incentive for proposing the calculations aiming at the description of possible QG effects (see e.g. [Borowiec & Pachol 2010, Freidel & Smolin 2011]).

In the approach using noncommutative (quantum) space-time the QG corrections are introduced into field theory in an algebraic way, by the substitution of the classical space-time coordinates $x_\mu$ in standard quantum fields (e.g. describing Standard Model) by the noncommutative $\hat{x}_\mu$. In such a way one can introduce the noncommutative classical fields $\phi_A(\hat{x})$ and the noncommutative quantum fields $\hat{\phi}_A(\hat{x})$. If we take into consideration the QG corrections, the quantization of classical fields is obtained by the substitution

$$
\phi_A(x) \sim \sum_p a(\vec{p}) e^{ipx} \rightarrow \hat{\phi}_A(\hat{x}) \sim \sum_p \hat{a}(\vec{p}) e^{ip\hat{x}},
$$

with the noncommutativity of quantum fields $\hat{\phi}_A(\hat{x})$ introduced by quantized field oscillators as well as by the noncommutative space-time coordinates.

The local multiplication of classical and quantum fields modified by QG corrections can be realized by the introduction of a nonlocal star multiplication of corresponding standard local fields (see e.g. [Blohmann 2003, Kulish 2006]).

$$
\begin{align*}
\phi(\hat{x}) \cdot \chi(\hat{x}) \quad &\xrightarrow{\text{Weyl map}} \quad \phi(x) \star \chi(x) \quad \text{(classical)} \\
\hat{\phi}(\hat{x}) \cdot \hat{\chi}(\hat{x}) \quad &\xrightarrow{\text{(quantum) Weyl map}} \quad \hat{\phi}(x) \star \hat{\chi}(x) \quad \text{(quantum)}
\end{align*}
$$
The Weyl maps (3.6) provide homomorphisms of noncommutative field algebras, and replace the local noncommutative field theory with some specific nonlocal field theory on standard Minkowski space. Effectively one can obtain such noncommutative QG-modified field theory by introducing in classical action the $\star$-product multiplication of fields. For example, in such an approach the QG corrections in QED will be described by the following replacement of standard Maxwell action density

$$F_{\mu\nu}(x) F^{\mu\nu}(x) + e J_\mu(x) A^\mu(x) \xrightarrow{\text{QG corrections}} F_{\mu\nu} \star F^{\mu\nu} + e J_\mu(x) \star A^\mu(x)$$

where $J_\mu(x)$ denotes the external electric current.

The $\star$-multiplication for simplest canonical (DFR) noncommutative deformation is represented by the Moyal $\star$-product [Moyal 1949]. In such a case it was shown [Filk 1996] that for QED only the vertices are modified in perturbative Feynmann diagrams.

The description of QG corrections in particle physics can be described therefore in two equivalent ways

| local field-theoretic | nonlocal SM on |
|-----------------------|---------------|
| SM on noncommutative  | classical space-time |
| space-time             | with $\star$-product of fields |

The noncommutativity (e.g. the tensors $\theta^{(n)}_{\mu(\rho,...,\rho)}$ in (3.5)) should be determined dynamically by QG, e.g. in standard canonical approach it should depend on three-dimensional 2-tensors fields $h_{ij}(\vec{x},t)$, $p_{ij}(\vec{x},t)$ (see e.g. [Misner, Thorne & Wheeler 1973, Dirac 1975]). In particular in formula (3.4) the rhs should appear as some dynamically determined functional of quantized gravitational canonical fields

$$\theta_{\mu\nu}(\hat{x}) \rightarrow \theta_{\mu\nu}[\hat{h}_{ij}, \hat{p}_{ij}; \hat{x}] . \quad (3.7)$$

Unfortunately the dynamical equations specifying the functional noncommutativity factors (3.7) are not known. As another option (see e.g. [Amorim 2008]) it has also been postulated that the fields $\theta_{\mu\nu}$ introduce new degrees of freedom in QG.

From a geometric point of view one can consider the noncommutative approach to QG as described by the formalism of noncommutative Riemannian geometry. In such a framework QG is described by the noncommutative deformation of standard Riemannian geometry, with the use of noncommutative counterparts of basic notions of differential geometry (connections, curvatures, diffeomorphism maps) (see e.g. [Aschieri 2006, Majid 2013]). It can be added that the noncommutative curved space-times has been also linked with the quantum group approach to local diffeomorphism generated by $\infty$-dimensional quantum group, with twist deformation of standard Riemannian geometry described by Drinfeld twist [Aschieri 2012].

There have already been numerous papers with the calculation of noncommutative corrections to Einstein action obtained for some particular classes of deformations (some
classes of $\star$-products) (see e.g. [Aschieri, Dimitrijević, Meyer & Wess 2006, Calmet & Kobakhidze 2006, Banerjee, Mukherjee & Samanta 2007]). It is interesting to mention that for two basic deformations – canonical (DSR) and $\kappa$-deformation – the leading corrections are of second order, i.e. proportional to $L_p^2$, far below the present observability threshold.

The QG approach based on noncommutative space-time geometry provides the following responses to the list of desirable properties listed as A–D in Sect. 3.1.

A. At present, the noncommutative framework does not provide a background–independent approach, because the noncommutative space-time structure is postulated. The link of noncommutativity with QG dynamics and possible noncommutative background-independent formulation is according to our opinion (see also [Doplicher 2006]) one of the major challenges in QG. It requires the discovery of new fundamental QG equations that determine dynamically the noncommutativity of QG-deformed coordinates and fields.

B. In noncommutative Riemannian geometry the standard diffeomorphisms are becoming quantum operator-valued diffeomorphisms. If the deformation of Riemannian geometry is generated by a twist, one can introduce explicit formulae describing the noncommutative Riemannian geometry (see e.g. [Aschieri 2012]). If one uses $\star$-product formalism, one gets the calculable modifications of standard formalism of general relativity (action, field equations, quantum corrections).

C. The noncommutativity of quantum space-time is present in this approach as the consequence of a primary postulate. This postulate should be supplemented with the above-mentioned derivation of noncommutativity from QG dynamics. In this respect one can mention the functional method of integration of gravitational degrees of freedom, which are coupled to quantum matter, as the way of generating the QG-generated noncommutativity of matter degrees of freedom (see e.g. [Freidel & Livine 2006]).

D. In the noncommutative approach to the models of QFT only in particular cases some infinities were removed, but a general scheme for the cancellation of nonrenormalizable divergencies due to the presence of space-time noncommutativity is not known. The question whether noncommutativity improves the finiteness of the corresponding quantum field theory remains open.

3.4 Functional formulation of QG and lattice approach

The Feynman quantization method in canonical field theory leads to the functional integral over the configurations of classical field $\phi$

$$Z[\phi] = \int \mathcal{D}\phi \, e^{iS[\phi]},$$

(3.8)
where $S[\phi]$ describes the action functional determining the field dynamics. If $S \equiv S^{EH}$ (see (1.1)) one gets QG functional integral [Misner 1957]

$$Z^{EH}[g_{\mu\nu}] = \int D[g_{\mu\nu}] e^{iS^{EH}[g_{\mu\nu}]}$$

(3.9)

describing Einstein QG, with functional measure $D[g_{\mu\nu}]$ providing the functional integral over all metrics divided by the diffeomorphisms. One can say alternatively that the formula (3.9) describes the functional over all possible curved space-time geometries represented by diffeo-invariant configurations of the gravitational field.

It is not known how to calculate explicitly the functional integral (3.8) unless the action $S[\phi]$ is free, i.e. linear or quadratic in field variable $\phi$, with linear term describing an interaction with external classical current. For more general choices of field actions, describing nonlinear interactions, one can however perform the approximate calculations if we discretize the functional integral, i.e. introduce a lattice counterpart of the functional integration. In such a way from (3.9) one can derive the lattice formulation of QG. Such discretized formulation permits the introduction of new nonperturbative methods for performing QG calculations. It should also be added that the lattice discretization is required if we wish to use advanced computer techniques for the approximate calculations of functional integrals.

The history of lattice approaches to QG began more than half a century ago [Regge 1961]. Further substantial progress in lattice methods was achieved in the framework of QCD calculations [Creutz, Jacobs & Rebbi 1983, Creutz 1985], with the aim of deriving the properties of hadronic matter from basic theory of its elementary constituents: quarks and gluons. The discretization of functional integral in QG is however more complicated than in QCD, because in the theories with local internal symmetries (YM field theory) one can introduce fixed, static lattices.

In discretized QG functional integral however, the lattices cannot be fixed because after dynamical triangulation procedure they represent variable geometries which are ‘latticized’. In particular the QG triangulation should provide the lattice version of local coordinate invariance and describe consistently the geometric gravity data, such as e.g. space-time curvature.

There are two versions of dynamical triangulations. The older one, known as dynamical triangulation (DT), was used to discretize the functional integral for Euclidean field theory, which on the lattice can be interpreted as describing a statistical infinite-dimensional system. A more physical and recent approach is provided by the Minkowskian dynamical triangulation with triangularized space and discretized global time, called causal dynamical triangulation (CDT) [Ambjørn 2002, Ambjørn, Jurkiewicz & Loll 2004, Ambjørn, Görlich, Jurkiewicz & Loll 2014]. In the CDT approach one assumes the triangular discretization of curved three-space and the time evolution is discretized in a way consistent with the Hamiltonian lattice framework. The triangularization in the CDT method is fitted therefore into a space-time layer between two discrete global time values $t$ and $t + \Delta t$, i.e. similarly
as in nonrelativistic approach we strongly distinguish space and time discretizations. It should be added however that in a parallel way to the development of the covariant formulation of LQG, recently also a covariant CDT approach without singling out the global time coordinate has been investigated, mostly studied in $D = 2 + 1$ QG [Jordan & Loll 2013].

The dynamical lattice approach to QG in the following way addresses the list of desirable properties of QG models:

A. CDT technique, with variable dynamical triangulation, does not require fixed gravitational three-dimensional background, however the modeling of discretization in lattice approach does not follow from physical properties of field-theoretic system under consideration. Lattice construction is a formal regularization method required also by the efficiency of calculational method. In CDT method the requirement of background independence should be replaced by the independence of obtained results from the choice of imposed discretization scheme.

B. The diffeomorphism invariance with the choice of finite lattice size $a$ is approximate. Further, it is an important and difficult step to derive the theory with consistent local diffeomorphism invariance in the limit $a \rightarrow 0$.

C. Because the discrete eigenvalues imitate the ‘quanta’ of quantum space-time, sometimes the discretization in lattice approach to QG is treated as a simulation of physical noncommutativity. This argument is however rather misleading, because in lattice approach the choice of discretization is not derived dynamically.

D. The strong side of the CDT approach is its good adjustment to computer calculations, i.e. the method did provide remarkable calculational results. In particular, following QCD lattice calculations which detected phase transitions in quark-gluon plasma, phase transitions were detected in QG as well. Further, in Euclidean lattice calculations (DT approach) one can represent the evolution of dynamical system as a discrete diffusion process, and calculate so-called spectral dimensions at short distances. For $D = 4$ QG this spectral dimension appears to be equal to two, so subsequently one can argue that one gets the two-dimensional field-theoretic picture of QG at very short distances, what suggests renormalizability.

4 Final remarks

We recall again that we did not consider in this note the quite popular description of QG as a low energy effective sector of quantum (super)string field theory, or more recently as the sector of quantum $M$-theory. The idea of embedding QG in quantum (super)string field theory (see e.g. [Thorn 1989, Witten 1992]) aims at solving at once two basic problems in the theory of fundamental interactions: to unify all interactions and providing finite quantum theory. It should be stressed, however, that the basic unsolved questions in the
The renormalizability or finiteness of a string or \( M \)-theory extension of Einstein QG requires the use of several post-Einsteinian geometric concepts, namely

i) additional space-time dimensions (\( D = 10 \) or \( D = 11 \)),

ii) local supersymmetry,

iii) infinite collection of stringy particles (string excitations) which in a somewhat miraculous way leads to the finite interactions of infinite spin multiplets (see e.g. \[Vasilev 2013\]).

Also strongly linked to the string field theory formulated in AdS space-time is the very successful idea of AdS/CDT correspondence [Maldacena 1997, Gubser, Klebanov & Polyakov 1998, Aharony, Gubser, Maldacena, Ooguri & Oz 2000], which provides in a field-theoretic framework the dual description of gravity and YM gauge theories. Such a surprising dynamical picture indicates that at the nonperturbative level QG is holographic, i.e. described in space-time by the degrees of freedom defined on its boundary [\'t Hooft 1993, Susskind 1995]. We recall that a spectacular application of the AdS/CFT correspondence is the relation of a weak coupling limit of gravity with a strong coupling sector in YM theory, what was used to deduce new properties of quark-gluon plasma from known perturbative gravity solutions (see e.g. \[Janik & Peschanski 2006\]).

In Sect. 3 we described three approaches to QG based on different quantization schemes. All these approaches are in continuous progress, but they still do not indicate clearly how a fully satisfactory QG model will appear. There is still room for entirely new ideas, for example

- Following Penrose one can consider as fundamental the geometry of conformal spinors (twistors), with space-time emerging as a nonprimary ‘composite’ geometry. Twistor ideas have been studied already for fifty years (see e.g. \[Penrose 1967, Penrose & MacCallum 1973, Hughston 1979\]), however recently an increasing number of physically important models were formulated with the tools of twistor geometry (see e.g. \[Witten 2004, Mason 2005, Adamo & Mason 2014\]).

- There have been formulated approaches with the pregeometry and emergent gravity concepts. As examples of pregeometry can be listed e.g. spin models of Ising type and the emergence of space-time and gravity due to collective phenomena (see e.g. \[Dreyer 2007\]).

Finally, we stress that what matters for physicists is the experimental confirmation of QG effects which we expect to be present at very short (Planckian) distances or at a very early stage in the evolution of the Universe. The prospects of a detection of such effects favours the astrophysical measurements. Unfortunately, till present time such effects have not been observed, but there were proposed many more or less justified theoretical predictions of phenomena indirectly related with QG [Amelino-Camelia & Kowalski-Glikman 2005, Amelino-Camelia 2013] (violation of Lorentz invariance, modification of light
velocity, particular anisotropies of CMB, primordial black holes etc.). All these proposals are under numerous studies with a common hope that the experimental ‘Planck window’ for QG effects will open in the future. The present models of QG are therefore continuously developing, but it is not excluded, that the first confirmation of QG and the construction of a ‘correct’ QG model will come from an unexpected direction (linked with dark matter?).

In conclusion, one can say that the present QG is described by several very promising but still hypothetical models, which are definitely much more than speculations but less than the long awaited theory of quantum gravitational phenomena.

Acknowledgments

The article is based on talks presented at the 42nd Meeting of Polish Physicists in Poznań (September 2013) and at the CERN TH Seminar (November 2013). This research was supported by Polish National Centre of Science (NCN)-Research Project No 2011/01/ST2/03354.
References

Adamo, T. & Mason, L. 2014. “Conformal and Einstein gravity from twistor actions.” *Classical and Quantum Gravity* 31(4):045014.

Aharony, O., Gubser, S.S., Maldacena, J., Ooguri, H. & Oz, Y. 2000. “Large N field theories, string theory and gravity.” *Physics Reports* 323(3):183–386.

Ambjørn, J. 2002. Simplicial Euclidean and Lorentzian quantum gravity. In *General Relativity and Gravitation, Proceedings of the 16th International Conference of the International Society on General Relativity and Gravitation (GR 16)*. World Scientific p. 3.

Ambjørn, J., Görlich, A., Jurkiewicz, J. & Loll, R. 2014. Quantum gravity via causal dynamical triangulations. In *Handbook of Spacetime*, ed. A. Ashtekar & V. Petkov. Springer.

Ambjørn, J., Jurkiewicz, J. & Loll, R. 2004. “Emergence of a 4D world from causal quantum gravity.” *Physical Review Letters* 93:131301.

Ambjørn, J., Jurkiewicz, J. & Loll, R. 2010. Quantum gravity as sum over spacetimes. In *New Paths Towards Quantum Gravity*, ed. B. Booss-Bavnbek, G. Esposito & M. Lesch. Vol. 807 of *Lecture Notes in Physics*. Springer pp. 59–124.

Amelino-Camelia, G. 2001. “Testable scenario for relativity with minimum length.” *Physics Letters B* 510(1):255–263.

Amelino-Camelia, G. 2013. “Quantum-spacetime phenomenology.” *Living Reviews in Relativity* 16(5).

Amelino-Camelia, G. & Kowalski-Glikman, J. 2005. *Planck Scale Effects in Astrophysics and Cosmology*. Springer.

Amorim, R. 2008. “Dynamical symmetries in noncommutative theories.” *Physical Review D* 78:105003.

Aschieri, P. 2006. “Noncommutative Gravity and the *-Lie algebra of diffeomorphisms.” *Fortschritte der Physik* 55:649–654.

Aschieri, P. 2012. “Twisting all the way: from algebras to morphisms and connections.” In *Proceedings of J. Wess Memorial Workshop, August 2011, Serbia, International Journal of Modern Physics: Conference Series* 13:1–19.

Aschieri, P., Dimitrijević, M., Meyer, F. & Wess, J. 2006. “Noncommutative geometry and gravity.” *Classical and Quantum Gravity* 23(6):1883.
Ashtekar, A. 1987. “New Hamiltonian formulation of general relativity.” Physical Review D 36:1587–1602.

Bahr, B., Hellmann, F., Kamiński, W., Kisielowski, M. & Lewandowski, J. 2011. “Operator spin foam models.” Classical and Quantum Gravity 28(10):105003.

Banerjee, R., Mukherjee, P. & Samanta, S. 2007. “Lie algebraic noncommutative gravity.” Physical Review D 75:125020.

Becker, K., Becker, M. & Schwarz, J.H. 2007. String Theory and M-Theory: a Modern Introduction. Cambridge University Press.

Blohmann, C. 2003. “Covariant realization of quantum spaces as star products by Drinfeld twists.” Journal of Mathematical Physics 44(10):4736–4755.

Borowiec, A. & Pachol, A. 2010. “κ-Minkowski spacetimes and DSR algebras: fresh look and old problems.” Symmetry, Integrability and Geometry: Methods and Applications 6:086.

Bronstein, M. 2012. “Republication of: Quantum theory of weak gravitational fields.” General Relativity and Gravitation 44(1):267–283.

Bruno, N.R., Amelino-Camelia, G. & Kowalski-Glikman, J. 2001. “Deformed boost transformations that saturate at the Planck scale.” Physics Letters B 522(1):133–138.

Calmet, X. & Kobakhidze, A. 2006. “Second order noncommutative corrections to gravity.” Physical Review D 74:047702.

Capozziello, S. & Francaviglia, M. 2008. “Extended theories of gravity and their cosmological and astrophysical applications.” General Relativity and Gravitation 40(2-3):357–420.

Connes, A. 1994. Noncommutative Geometry. Academic Press.

Creutz, M. 1985. Quarks, Gluons and Lattices. Cambridge Monographs on Mathematical Physics. Cambridge University Press.

Creutz, M., Jacobs, L. & Rebbi, C. 1983. “Monte Carlo computations in lattice gauge theories.” Physics Reports 95(4):201–282.

DeWitt, B.S. 1962. Gravitation: an Introduction to Current Research. Wiley.

Dirac, P.A.M. 1926. “On the theory of quantum mechanics.” Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science 112(762):661–677.

Dirac, P.A.M. 1975. General Theory of Relativity. Wiley.
Doplicher, S. 2006. “Quantum field theory on quantum spacetime.” *Journal of Physics: Conference Series* 53:793–798.

Doplicher, S., Fredenhagen, K. & Roberts, J.E. 1995. “The quantum structure of spacetime at the Planck scale and quantum fields.” *Communications in Mathematical Physics* 172(1):187–220.

Dreyer, O. 2007. Emergent relativity. In *Approaches to Quantum Gravity: Towards a New Understanding of Space, Time and Matter*, ed. D. Oriti. Cambridge University Press pp. 99–100.

Drinfeld, V.G. 1987. “Proceedings of the International Congress of Mathematicians.” *Quantum Groups* pp. 798–819.

Duff, M.J. 1994. “Kaluza-Klein theory in perspective.” *Preprint* arXiv:hep-th/9410046.

Filk, T. 1996. “Divergencies in a field theory on quantum space.” *Physics Letters B* 376(1):53–58.

Freedman, D.Z. & Van Proeyen, A. 2012. *Supergravity*. Cambridge University Press.

Freidel, L. & Livine, E.R. 2006. “3D quantum gravity and effective noncommutative quantum field theory.” *Physical Review Letters* 96:221301.

Freidel, L. & Smolin, L. 2011. “Gamma ray burst delay times probe the geometry of momentum space.” *Preprint* arXiv:1103.5626.

Garay, L.J. 1995. “Quantum gravity and minimum length.” *International Journal of Modern Physics A* 10(2):145–165.

Green, M.B., Schwarz, J.H. & Witten, E. 1987. *Superstring theory, Vol. 1, 2*.

Gubser, S.S., Klebanov, I.R. & Polyakov, A.M. 1998. “Gauge theory correlators from non-critical string theory.” *Physics Letters B* 428(1):105–114.

Higgs, P.W. 1964. “Broken symmetries and the masses of gauge bosons.” *Physical Review Letters* 13:508–509.

Hořava, P. 2009. “Quantum gravity at a Lifshitz point.” *Physical Review D* 79:084008.

Holstein, B.R., Golowich, E. & Donoghue, J.F. 1994. *Dynamics of the Standard Model*. Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology. Cambridge University Press.

Hughston, L.P. 1979. *Twistors and Particles*. Springer-Verlag.
Janik, R.A. & Peschanski, R. 2006. “Asymptotic perfect fluid dynamics as a consequence of AdS/CFT correspondence.” Physical Review D 73:045013.

Jordan, S. & Loll, R. 2013. “De Sitter Universe from causal dynamical triangulations without preferred foliation.” Physical Review D 88:044055.

Kiefer, C. 2012. Quantum Gravity. Vol. 155 of International Series of Monographs on Physics. 3 ed. Oxford University Press.

Klein, O. 1926. “Quantentheorie und fünfdimensionale Relativitätstheorie.” Zeitschrift für Physik 37(12):895–906.

Kowalski-Glikman, J. & Nowak, S. 2002. “Doubly special relativity theories as different bases of $\kappa$-Poincaré algebra.” Physics Letters B 539(1):126–132.

Kulish, P.P. 2006. “Twists of quantum groups and noncommutative field theory.” Preprint arXiv:0606056.

Lukierski, J. 1999. Deformed Quantum Relativistic Phase Spaces. In 3rd International Workshop “Classical and Quantum Integrable Systems”, Yerevan, July 1998, ed. JINR Dubna.

Lukierski, J. & Nowicki, A. 2003. “Doubly special relativity versus $\kappa$-deformation of relativistic kinematics.” International Journal of Modern Physics A 18(1):7–18.

Lukierski, J., Ruegg, H., Nowicki, A. & Tolstoy, V.N. 1991. “$q$-deformation of Poincaré algebra.” Physics Letters B 264(3):331–338.

Lukierski, J., Ruegg, H. & Zakrzewski, W.J. 1995. “Classical and quantum mechanics of free $\kappa$-relativistic systems.” Annals of Physics 243(1):90–116.

Majid, S. 1995. Foundations of Quantum Group Theory. Cambridge University Press.

Majid, S. 2009. Algebraic approach to quantum gravity II: noncommutative spacetime. In Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter, ed. D. Oriti. Cambridge University Press pp. 466–492.

Majid, S. 2013. “Reconstruction and quantization of Riemannian structures.” Preprint arXiv:1307.2778.

Majid, S. & Ruegg, H. 1994. “Bicrossproduct structure of $\kappa$-Poincaré group and noncommutative geometry.” Physics Letters B 334(3):348–354.

Maldacena, J.M. 1997. “The large N limit of superconformal field theories and supergravity.” Advances in Theoretical and Mathematical Physics 2:231–252.
Mandelstam, S. 1962. “An extension of the Regge formula.” *Annals of Physics* 19(2):254–261.

Mason, L.J. 2005. “Twistor actions for non-self-dual fields; a new foundation for twistor-string theory.” *Journal of High Energy Physics* 10(2005):009.

Mead, C.A. 1964. “Possible connection between gravitation and fundamental length.” *Physical Review* 135:B849–B862.

Misner, C.W. 1957. “Feynman quantization of general relativity.” *Reviews of Modern Physics* 29:497–509.

Misner, C.W., Thorne, K.S. & Wheeler, J.A. 1973. *Gravitation*. Freeman.

Moyal, J.E. 1949. “Quantum mechanics as a statistical theory.” *Mathematical Proceedings of the Cambridge Philosophical Society* 45:99–124.

Ogievetsky, V.I. & Polubarinov, I.V. 1965. “Interacting field of spin 2 and the Einstein equations.” *Annals of Physics* 35(2):167–208.

Penrose, R. 1967. “Twistor algebra.” *Journal of Mathematical Physics* 8(2):345–366.

Penrose, R. & MacCallum, M.A.H. 1973. “Twistor theory: an approach to the quantisation of fields and space-time.” *Physics Reports* 6(4):241–315.

Percacci, R. 2009. "Asymptotic safety". In *Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter*, ed. D. Oriti. Cambridge University Press pp. 111–128.

Regge, T. 1961. “General relativity without coordinates.” *Il Nuovo Cimento* 19(3):558–571.

Reshetikhin, N.Y., Takhtadzhyan, L.A. & Faddeev, L.D. 1989. “Quantization of Lie groups and Lie algebras.” *Algebra i Analiz* 1(1):178–206.

Reuter, M. 1998. “Nonperturbative evolution equation for quantum gravity.” *Physical Review D* 57:971–985.

Rovelli, C. 2004. *Quantum Gravity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press.

Snyder, H.S. 1947. “Quantized space-time.” *Physical Review* 71:38–41.

Sotiriou, T.P. & Faraoni, V. 2010. “$f(R)$ theories of gravity.” *Reviews of Modern Physics* 82:451–497.

Starobinsky, A.A. 1979. “Relict gravitation radiation spectrum and initial state of the Universe.” *JETP Letters* 30:682.
Stelle, K.S. 1977. “Renormalization of higher-derivative quantum gravity.” *Physical Review D* 16:953–969.

Susskind, L. 1995. “The world as a hologram.” *Journal of Mathematical Physics* 36(11):6377–6396.

’t Hooft, G. 1993. “Dimensional reduction in quantum gravity.” Preprint arXiv:gr-qc/9310026.

Thiemann, T. 2007. *Modern Canonical Quantum General Relativity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press.

Thorn, C.B. 1989. “String field theory.” *Physics Reports* 175(1):1–101.

Vasiliev, M.A. 2013. “Multiparticle extension of the higher spin algebra.” *Classical and Quantum Gravity* 30(10):104006.

Weinberg, S., Hawking, S.W. & Israel, W. 1979. *General Relativity: an Einstein Centenary Survey*. Cambridge University Press.

Wigner, E. 1939. “On unitary representations of the inhomogeneous Lorentz group.” *Annals of Mathematics* pp. 149–204.

Witten, E. 1992. “On background-independent open-string field theory.” *Physical Review D* 46:5467–5473.

Witten, E. 2004. “Perturbative gauge theory as a string theory in twistor space.” *Communications in Mathematical Physics* 252(1–3):189–258.

Yang, C.N. & Mills, R.L. 1954. “Conservation of isotopic spin and isotopic gauge invariance.” *Physical Review Letters* 96:191–195.

Zwiebach, B. 1985. “Curvature squared terms and string theories.” *Physics Letters B* 156(5):315–317.