HIGHER ORDER SPATIAL APPROXIMATIONS FOR DEGENERATE PARABOLIC STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

ERIC JOSEPH HALL

ABSTRACT. We consider an implicit finite difference scheme on uniform grids in time and space for the Cauchy problem for a second order parabolic stochastic partial differential equation where the parabolicity condition is allowed to degenerate. Such equations arise in the nonlinear filtering theory of partially observable diffusion processes. We show that the convergence of the spatial approximation can be accelerated to an arbitrarily high order, under suitable regularity assumptions, by applying an extrapolation technique.

1. Introduction

Motivated by the nonlinear filtering theory, we consider the Cauchy problem for the stochastic partial differential equation (SPDE)

\[ du = (a^{\alpha\beta} D_\alpha D_\beta u + f) \, dt + \sum_{\rho=1}^{d_1} (b^{\alpha\rho} D_\alpha u + g^\rho) \, dw^\rho \]

with initial condition \( u(0) = u_0 \) where \((w^\rho)_{\rho=1}^{d_1}\) is a \(d_1\)-dimensional Wiener martingale for integer \(d_1 \geq 1\) and the summation convention is used with respect to the repeated indices \(\alpha, \beta \in \{0, \ldots, d\}\) for integer \(d \geq 1\). Here \(D_\alpha := \partial / \partial x_\alpha\), for \(\alpha \in \{1, \ldots, d\}\), denotes the spatial derivative with respect to the direction \(\alpha\) and we let \(D_0\) be the identity. A special case of this equation, when the free terms \(f\) and \(g\) vanish, arises as the equation for the unnormalized conditional density of a signal process with respect to an observation process in the nonlinear filtering theory and is often referred to as the Zakai equation (see [13, 17, 24]). The behavior of this equation is governed by the quadratic form

\[ \sum_{\alpha, \beta=1}^{d} A^{\alpha\beta} z_\alpha z_\beta \]

for \(A^{\alpha\beta} := 2a^{\alpha\beta} - b^{\alpha\rho} b^{\beta\rho}\) and \(z \in \mathbb{R}^d\). In [12], it is emphasized that in the setting of the nonlinear filtering theory one is only guaranteed the nonnegative definiteness of the matrices \(A\), that is, when (1.1) satisfies a degenerate stochastic parabolicity condition (cf. [11, 16] where the solvability of this equation is studied under the uniform nondegeneracy of the matrices \(A\)). In application, these problems are high dimensional in nature and the solutions are required on-line. Therefore accurate and efficient numerical methods are desired for solving the Cauchy problem for (1.1) under a degenerate parabolicity assumption.

The present manuscript concerns the accuracy of a space-time, that is, a fully discretized, finite difference scheme on uniform grids in time and space for approximating the solution to the Cauchy problem for (1.1) under the degenerate
parabolicity assumption. In general the rate of convergence of finite difference schemes is known to be unsatisfactory in high dimensional settings. We prove that the rate of convergence of the spatial approximation for our space-time scheme can be accelerated to an arbitrarily high order with respect to the computational effort by applying an extrapolation technique. That is, we show that the rate of the strong convergence of the spatial approximation to the temporal discretization can be accelerated to any order of accuracy if the initial conditions, coefficients, and free terms are sufficiently smooth in space and the matrices $A$ can be decomposed as

$$A = \sigma \sigma^T$$

for matrices $\sigma$ sufficiently smooth in space. While the requirement that the $A$ admit such a decomposition is quite restrictive, this condition is satisfied in the nonlinear filtering problem even in the general case of correlated signal and observation noises when the diffusion coefficients of the signal noise are sufficiently smooth.

The extrapolation technique that we employ to obtain higher order convergence is often referred to as Richardson’s method, after L.F. Richardson who suggested the method for accelerating the convergence of finite difference approximations for certain partial differential equations (PDEs) (see [15, 16]). The method relies on the existence of an asymptotic expansion for the error between the approximate and true solutions to a continuous problem in powers of the discretization parameter. Richardson observed that by taking appropriate weighted averages of the approximation at different mesh sizes certain lower order terms in the expansion vanish yielding a higher order rate of convergence. Therefore, it is important to give sufficient conditions under which such expansions exists. We emphasize that not only does the existence of the asymptotic expansion allow us to apply Richardson’s method to an arbitrarily high order, but also it allows us to measure the rate of convergence in the supremum norm. Richardson’s method has been thoroughly studied in the literature, see for example the book [14] which provides a study of Richardson’s method for finite difference schemes for deterministic PDEs and the survey articles [2, 10] on convergence acceleration methods. Part I of the book [22] concerns Richardson’s method and algorithms for its implementation for PDEs; our results are of a more theoretical nature and much work still needs to be done to implement them.

While finite difference schemes for PDEs and, to a lesser extent, for SPDEs are well studied in the literature (for example, see [23, 9] and the references therein) there are only a few results for degenerate parabolic equations and even fewer results concerning convergence acceleration for degenerate equations. Sharp rates of convergence are given in [3] for monotone finite difference schemes for possibly degenerate parabolic and elliptic deterministic PDEs. In [6] Richardson’s method is applied to monotone finite difference schemes for possibly degenerate parabolic deterministic PDEs to accelerate the rate of convergence. Recently, in [4] a rate of convergence is given for a class of finite difference methods, that approximate in space via finite differences while allowing time to vary continuously, for degenerate parabolic SPDEs and sufficient conditions are given for accelerating the rate of convergence for the approximation in space.

The current manuscript extends the results of [4] to a fully discretized scheme. We also mention [8], where results similar to those of the present manuscript are given under the strong parabolicity condition. A principle contribution of the present work is to provide estimates in the supremum norm in appropriate spaces for the solutions to the space-time scheme and the discretization in time under the degenerate parabolicity condition. The methods used to provide the requisite estimates in [4] are not tenable in the discrete time case. Further, we mention that we
have chosen to consider here implicit schemes as we believe that these are favored from a practical standpoint and because such schemes are unconditionally stable. We note that, to the author’s knowledge, there are no results that give the rate of convergence of the implicit time scheme for SPDEs under a degenerate parabolicity assumption and it will be the subject of a future work to give such a rate of convergence for the implicit time scheme as well as more general methods.

The paper is outlined as follows. In the next section, we begin by presenting our time scheme and our space-time scheme for approximating the solution to the Cauchy problem for (1.1) as well as some preliminaries and assumptions. We then state our main results. Theorem 2.14 gives sufficient conditions for the existence of an asymptotic expansion for the error between the space-time approximation and the temporal discretization in powers of the spatial mesh size. Theorem 2.13 gives sufficient conditions for a generalization of Theorem 2.13, namely, the existence of an expansion for differences of the solution. Then Theorems 2.15 and 2.17, using the aforementioned expansions, give an accelerated rate of convergence for the spatial approximation and for derivatives of the spatial approximation, respectively. The proof of Theorem 2.14 and hence Theorem 2.13 is given in Section 3 after some preliminary estimates are proven in Section 3.

We end this section by introducing some notation that will be used throughout this work. For an integer $d \geq 1$, let $R^d$ be the space of Euclidean points $x = (x_1, \ldots, x_d)$. We denote the $\sigma$-algebra of Borel subsets of $R^d$ by $\mathcal{B}(R^d)$. Recall that we denote by $\script{D}_\alpha := \partial/\partial x_\alpha$ for $\alpha \in \{1, \ldots, d\}$ the spatial derivative with respect to the direction $\alpha$ and let $\script{D}_0$ be the identity. For an integer $m \geq 0$, we denote by $W^m := W^m(R^d)$ the usual Hilbert-Sobolev spaces of function on $R^d$, defined as the closure of $C^\infty_0(R^d)$ functions $\phi: R^d \rightarrow R^d$ in the norm

$$\|\phi\|^2_m := \sum_{|\rho| \leq m} \int_{R^d} |D^\rho \phi(x)|^2 \, dx,$$

where $D^\rho = D^\rho_1 \cdots D^\rho_d$ for a multiindex $\rho = (\rho_1, \ldots, \rho_d)$ of length $|\rho| = \rho_1 + \cdots + \rho_d$. For an integer $s \geq 0$, we will use the notation $D^s \phi$ to denote the collection of all $s$th order spatial derivatives of $\phi$, that is, $D^s \phi := \{D^\rho \phi: |\rho| \leq s\}$ for functions $\phi = \phi(x)$ for $x \in R^d$. We note that for $L^2 := L^2(R^d) = W^0_2$ we will denote the norm by $\| \cdot \|_0$ and we will use $\langle \cdot, \cdot \rangle$ to denote the usual inner product in that space. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let $\mathcal{F}(t), t \geq 0$, be an increasing family of sub-$\sigma$-algebras of $\mathcal{F}$ such that $\mathcal{F}(0)$ is complete with respect to $(\mathcal{F}, P)$. For a fixed integer $d_1 \geq 1$ and a constant $T \in (0, \infty)$ let $(w^0_{\cdot \rho})_{\rho=1}^{d_1}$ be a given sequence of independent Wiener processes carried by the complete stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t \geq 0}, P)$. For the fundamentals of the nonlinear filtering theory, we refer the reader to the book [1] and for basic notions and results from the theory of SPDEs we refer the reader to [21].

We collect the following notation for our discretizations and differences. For fixed $\tau \in (0, 1)$, we partition $[0, T]$ into a fixed integer $n \geq 0$ with mesh size $\tau$ obtaining the time grid

$$\{i\tau; i \in \{0, 1, \ldots, n\}, \tau n = T\}.$$

We define $\phi_i := \phi(i \tau)$ for functions $\phi$ depending on $t \in [0, T]$. In particular for $i \in \{1, \ldots, n\}$, we will use the notation

$$\xi_i^\rho := w_i^\rho - w_{i-1}^\rho$$

to denote the increments of the Weiner process for each $\rho \in \{1, \ldots, d_1\}$ and $\mathcal{F}_i := \mathcal{F}(i \tau)$ to denote the filtration. For $h \in R \setminus \{0\}$ and a finite subset $\Lambda \subset R^d$ containing
the origin we define the space grids

\[ G_h := \{ \lambda_1 h + \cdots + \lambda_p h; p \in \{ 1, 2, \ldots \}, \lambda_i \in \Lambda \cup (-\Lambda) \} \]

with mesh size \(|h|\). We denote \( \Lambda_0 := \Lambda \setminus \{0\} \). For all \( h \in \mathbb{R} \setminus \{0\} \) we define first order and first order symmetric differences by

\[ \delta_{h,\lambda} := \frac{1}{h} (T_{h,\lambda} - I) \quad \text{and} \quad \delta_{\lambda} := \delta_{h,\lambda}^h := \frac{1}{2h} (\delta_{h,\lambda} + \delta_{-h,\lambda}) = \frac{1}{2h} (T_{h,\lambda} - T_{h,\lambda}^-), \]

respectively, for \( \lambda \in \mathbb{R}^d \setminus \{0\} \) where for all \( h \in \mathbb{R} \) we define the shift operator

\[ T_{h,\lambda} \phi(x) := \phi(x + h\lambda) \]

for functions \( \phi \) on \( \mathbb{R}^d \). We define \( \delta_{h,0} := I \) and \( \delta_0 := I \). We also adopt the notation \( N = N(\cdot) \) for a constant \( N \) depending only on the parameters given as arguments. For basic notions from numerical analysis we refer the reader to [15] [20].

2. Main results

To accelerate the rate of convergence of the spatial approximation for a space-time finite difference scheme, we will consider the error between the space-time approximation and the temporal discretization, the latter of which is a continuous problem in the spatial variable. Therefore we begin by considering a discretization in time for (1.1), namely the implicit Euler method. We then replace the differential operators with difference operators in this temporal discretization, yielding a space-time scheme. We then state our results, the two main results being Theorem 2.13, which gives sufficient conditions for the existence of the desired asymptotic expansion for the error between the space-time approximation and the spatial discretization in powers of the spatial mesh size, and Theorem 2.15, which uses the expansion to obtain an arbitrarily high order of convergence via Richardson’s method.

For each fixed \( \tau \in (0,1) \), we consider

\[ v_i(x) = v_{i-1}(x) + (L_i v_i(x) + f_i(x)) \tau + \sum_{p=1}^{d_1} (M_p^{\alpha} v_{i-1}(x) + g_p^{\alpha-1}(x)) \xi_{p}^{\alpha} \]

for \( \omega \in \Omega, x \in \mathbb{R}^d \), and \( i \in \{1, \ldots, n\} \) with initial condition \( v_0(x) = v_0 \) where \( L_i \) and \( M_p^{\alpha} \) are second order and first order differential operators given by \( L_i \phi := a_i^{\alpha \beta}(x) D_{\alpha} D_{\beta} \phi \) and \( M_p^{\alpha} \phi := b_{i}^{\alpha p}(x) D_{\alpha} \phi \), for \( \rho \in \{1, \ldots, d_1\} \), where the summation convention is used with respect to the repeated indices \( \alpha, \beta \in \{0,1, \ldots, d\} \). We assume that the given \( a_i^{\alpha \beta} := a_i^{\alpha \beta}(x) \) and \( b_i^{\alpha} := (b_i^{\alpha p}(x))_{p=1}^{d_1} \) are real-valued and \( \mathbb{R}^{d_1} \)-valued, respectively, \( F_i \otimes \mathcal{B}(\mathbb{R}^d) \)-measurable functions for \( \omega \in \Omega \) and \( i \in \{0, \ldots, n\} \) for all \( \alpha, \beta \in \{0, \ldots, d\} \). The free terms \( f_i := f_i(x) \) and \( g_i^{\alpha} := g_i^{\alpha}(x) \), for \( \rho \in \{1, \ldots, d_1\} \), are \( F_i \otimes \mathcal{B}(\mathbb{R}^d) \)-measurable functions for every \( \omega \in \Omega, x \in \mathbb{R}^d \), and \( i \in \{0, \ldots, n\} \). The discretization (2.1) represents an implicit Euler method for approximating the solution to (1.1) in time. Solutions to (2.1) with appropriate initial conditions are understood as sequences of \( W_2^1 \)-valued random variables satisfying (2.1) in a weak sense in \( W_2^1 \).

As discussed in the introduction, we consider the following degenerate stochastic parabolicity condition, necessary for the well-posedness of (1.1) and hence (2.1). Note that this is a weaker condition than the strong stochastic parabolicity condition which assumes the uniform nondegeneracy of the quadratic form (cf. Assumption 2.2 in [8] for example).
Assumption 2.1. For all $\omega \in \Omega$, $i \in \{1, \ldots, n\}$, $x \in \mathbb{R}^d$, and $z = (z_1, \ldots, z_d) \in \mathbb{R}^d$,
\[
\sum_{\alpha, \beta=1}^{d} \left( 2a_i^{\alpha \beta} - b_i^{\alpha \rho}b_i^{\beta \rho} \right) z_\alpha z_\beta \geq 0,
\]
that is, the quadratic form is nonnegative definite (positive semi-definite).

To formulate existence and uniqueness results, as well as estimates, for the solution to (2.1) we also require some smoothness assumptions on the coefficients, the free terms, and the initial conditions. Let $m \geq 0$ be an integer.

Assumption 2.2. For each $\omega \in \Omega$ and $i \in \{0, \ldots, n\}$, the functions $a_i^{\alpha \beta}$ and the functions $a_i^{\alpha 0}$, $a_i^{0 \alpha}$, and $a_i^{0 0}$ are, respectively, $(m + 1) \vee 2$ times and $m + 1$ times continuously differentiable in $x$ for $\alpha, \beta \in \{1, \ldots, d\}$. For each $\omega \in \Omega$ and $i \in \{0, \ldots, n\}$, the functions $b_i^\alpha$ are $m + 2$ times continuously differentiable in $x$ for $\alpha \in \{0, \ldots, d\}$. Further, there exist constants $K_j$, for $j \in \{0, \ldots, m + 2\}$, such that
\[
\left| D^j a_i^{\alpha \beta} \right| \leq K_j \quad \text{for } j \leq (m + 1) \vee 2,
\]
\[
\left| D^j a_i^{\alpha 0} \right| + \left| D^j a_i^{0 \alpha} \right| + \left| D^j a_i^{0 0} \right| \leq K_j \quad \text{for } j \leq m + 1, \text{ and}
\]
\[
\left| D^j b_i^\alpha \right| \leq K_j \quad \text{for } j \leq m + 2
\]
for all $\alpha, \beta \in \{1, \ldots, d\}$.

For integer $l \geq 0$, we define the norm
\[
\left\| \phi \right\|_l^2 := E \sum_{i=0}^{n} \left\| \phi_i \right\|^2_l
\]
and let $W^2_2(\tau)$ be the space of $W^2_2$-valued $\mathcal{F}_\tau$-measurable processes $\phi$ such that $\left\| \phi \right\|^2_1 < \infty$. We use the shorthand notation
\[
\left\| g \right\|_l^2 := \sum_{\rho=1}^{d_1} \left\| g^\rho \right\|^2_l
\]
for functions $g = (g^\rho)_{\rho=1}^{d_1}$.

Assumption 2.3. The initial condition $v_0 \in L^2(\Omega, \mathcal{F}_0, W_{2m+2}^m)$, the space of $\mathcal{F}_0$-measurable $W_{2m+2}^m$-valued square integrable functions on $\Omega$. The free terms $f$ and $g^\rho$, for $\rho \in \{1, \ldots, d_1\}$, take values in $W_{2m+2}^m(\tau)$. Moreover,
\[
(2.2) \quad \mathcal{K}^2_m := \tau E \left\| v_0 \right\|^2_{m+2} + \left\| f \right\|^2_{m+1} + \left\| g \right\|^2_{m+1} < \infty.
\]

Remark 2.4. For $m > d/2$, we can find a continuous function of $x$ which is equal to $v_0$ almost everywhere for almost all $\omega \in \Omega$, by Sobolev’s embedding of $W_{2m}^m \subset C_b$, the space of bounded continuous functions. Similarly, for each $\omega \in \Omega$ and $i \in \{0, \ldots, n\}$ there exist continuous functions of $x$ which coincide with $f_i$ and $g_i^\rho$, for $\rho \in \{1, \ldots, d_1\}$, for almost every $x \in \mathbb{R}^d$. Thus, if Assumption 2.3 holds with $m > d/2$ we assume that $v_0$, $f_i$, and $g_i^\rho$, for $\rho \in \{1, \ldots, d_1\}$, are continuous in $x$ for all $i \in \{0, \ldots, n\}$.

For the time scheme (2.1), we give the following solvability theorem along with an estimate. The proof is provided after some preliminaries are presented in the next section.

Theorem 2.5. If Assumptions 2.1, 2.2, and 2.3 hold, then (2.1) admits a unique $W_{2m}^m$-valued $\mathcal{F}_\tau$-measurable solution $v$. Moreover,
\[
(2.3) \quad E \max_{1 \leq n} \left\| v_n \right\|^2_m \leq N \mathcal{K}^2_m,
\]
holds for a constant $N = N(d, d_1, m, T, K_0, \ldots, K_{m+2})$. 


Now we wish to approximate (2.1) in space by replacing the differential operators with difference operators. Together with (2.1) we consider, for a finite subset \( \Lambda \subset \mathbb{R}^d \) containing the origin,

\[
\begin{align*}
(2.4) \quad v_i^h(x) &= v_i^{-1}(x) + (L_i^h v_i^h(x) + f_i(x)) \tau + \sum_{\rho=1}^{d_1} \left( M_{i-1}^{h,\rho} v_{i-1}^h(x) + g_{i-1}^\rho(x) \right) \xi_i^\rho \\
&\quad \text{for } \omega \in \Omega, x \in \mathbb{R}^d, \text{ and } i \in \{1, \ldots, n\} \text{ with initial conditions } v_i^0(x) = v_0. \end{align*}
\]

For each \( i \in \{0, \ldots, n\} \), the \( L_i^h \) and \( M_i^{h,\rho} \) are given by

\[
L_i^h \phi := \sum_{\lambda,\mu \in \Lambda} a_i^{\lambda\mu}(x) \delta_{\lambda}^i \phi + \sum_{\lambda \in \Lambda_0} (p_i^\lambda(x) \delta_{h,\lambda} \phi - q_i^\lambda(x) \delta_{-h,\lambda} \phi)
\]

and

\[
M_i^{h,\rho} \phi := \sum_{\lambda \in \Lambda} b_i^{\lambda\rho}(x) \delta_{\lambda}^i \phi
\]

for \( \rho \in \{1, \ldots, d_1\} \). For all \( \lambda, \mu \in \Lambda \), we assume the given \( a_i^{\lambda\mu} := a_i^{\lambda\mu}(x) \), \( p_i^\lambda := p_i^\lambda(x) \), and \( q_i^\lambda := q_i^\lambda(x) \) are real-valued and the \( b_i^\lambda = (b_i^{\lambda\rho}(x))_{\rho=1}^{d_1} \) are \( \mathbb{R}^{d_1} \)-valued measurable functions for every \( \omega \in \Omega, x \in \mathbb{R}^d \), and \( i \in \{0, \ldots, n\} \).

In order for \( v_i^h \) to approximate the solution of (2.1) in space we require the following consistency condition, ensuring the difference operators converge to the differential operators.

**Assumption 2.6.** For all \( \alpha, \beta \in \{1, \ldots, d\} \) and \( \rho \in \{1, \ldots, d_1\} \),

\[
\sum_{\lambda \in \Lambda_0} b_i^{\lambda \alpha} = b_i^{0,\alpha}, \quad b_i^{0\rho} = b_i^{0,\rho}, \quad \sum_{\lambda,\mu \in \Lambda} a_i^{\lambda \mu} \mu^\beta = a_i^{0\beta}, \quad a_i^{00} = a_i^{0,0},
\]

and

\[
\sum_{\lambda \in \Lambda_0} a_i^{0\lambda} + \sum_{\mu \in \Lambda_0} a_i^{0\mu} + \sum_{\lambda \in \Lambda_0} p_i^\lambda - \sum_{\mu \in \Lambda_0} q_i^\mu = a_i^{00} + a_i^{0,0}
\]

for \( i \in \{0, \ldots, n\} \).

We also place the following additional assumptions on the coefficients of the difference operators.

**Assumption 2.7.** For all \( \omega \in \Omega, x \in \mathbb{R}^d \), and \( i \in \{0, \ldots, n\} \):

(i) the functions \( p_i^\lambda \geq 0 \) and \( q_i^\lambda \geq 0 \) for all \( \lambda \in \Lambda_0 \);

(ii) for integer \( d_2 \geq 1 \) and \( \lambda \in \Lambda_0 \) there exist \( F_\lambda \otimes \mathcal{B}(\mathbb{R}^d) \)-measurable real valued functions \( \sigma_\lambda^1, \ldots, \sigma_\lambda^{d_2} \) such that

\[
(2.5) \quad a_i^{\lambda\rho} := 2a_i^{\lambda\mu} - b_i^{\lambda\rho} b_i^{\mu\rho} = \sum_{r=1}^{d_2} a_i^{\lambda r} \sigma_\rho^r
\]

for all \( \lambda, \mu \in \Lambda_0 \).

**Assumption 2.8.** Let \( l \geq 1 \) be an integer. For all \( \omega \in \Omega, i \in \{0, \ldots, n\} \), \( \lambda \in \Lambda_0 \), and \( k \in \{1, \ldots, d_2\} \), the functions \( b_i^\lambda \) and \( b_i^\lambda \) are \( l+2 \) times continuously differentiable in \( x \); the functions \( \sigma_\lambda^k \) are \( l+1 \) times continuously differentiable in \( x \); and the functions \( a_i^{\lambda\lambda}, a_i^{0\lambda}, a_i^{00}, p_i^\lambda \), and \( q_i^\lambda \) are \( l \) times continuously differentiable in \( x \). Further, there exist constants \( \hat{K}_j \), for \( j \in \{0, \ldots, l+2\} \), such that

\[
|D^j b_i^\lambda| + |D^j b_i^0| \leq \hat{K}_j \quad \text{for } j \leq l + 2,
\]

\[
|D^j \sigma_\lambda^k| \leq \hat{K}_j \quad \text{for } j \leq l + 1, \text{ and}
\]

\[
|D^j a_i^{\lambda\lambda}| + |D^j a_i^0\lambda| + |D^j a_i^{00}| + |D^j p_i^\lambda| + |D^j q_i^\lambda| \leq \hat{K}_j \quad \text{for } j \leq l
\]
for all \( \omega \in \Omega, x \in \mathbb{R}^d, i \in \{0, \ldots, n\}, \lambda \in \Lambda_0, \) and \( k \in \{1, \ldots, d_2\} \).

Remark 2.9. It is clear that (2.5) implies that

\[
\sum_{\lambda, \mu \in \Lambda_0} \hat{q}^H_{\lambda \mu} z_{\lambda} z_{\mu} \geq 0
\]

for \( \omega \in \Omega, x \in \mathbb{R}^d, i \in \{0, \ldots, d\}, \) and \( z_{\lambda} \in \mathbb{R} \) for \( \lambda \in \Lambda_0 \). This observation, together with Assumption 2.6 implies Assumption 2.1.

Solutions to (2.4) are understood as sequences of random fields taking values in \( L^2(G_h) \), the space of square summable functions on the grid points \( G_h \), satisfying (2.4) with an \( L^2(G_h) \)-valued initial condition. The following is a well known result which we include for the sake of completeness. Note that by Assumption 2.3 the \( \nu_0, f \) and \( g^\rho \), for \( \rho \in \{1, \ldots, d_1\} \), are \( L^2(G_h) \)-valued processes when restricted to the grid \( G_h \).

**Theorem 2.10.** If Assumptions 2.6 and 2.8 hold, then (2.4) admits a unique \( L^2(G_h) \)-valued solution for sufficiently small \( \tau \).

*Proof.* The proof of this solvability result relies on the invertibility of \((I - \tau L^l_1)\), for each \( i \in \{0, \ldots, n\} \), in \( L^2(G_h) \) for sufficiently small \( \tau \). Rewriting the scheme as a recursion and using this fact one can construct a unique solution to the scheme iteratively. Full details can be found, for example, in [5]. \( \square \)

We observe, however, that (2.4) is well defined not only at the points of the grid but for the whole space. Therefore, we consider (2.4) on \( \mathbb{R}^d \) and seek solutions that are sequences of \( L^2 \)-valued functions. Hence we will use the normal machinery from analysis to obtain estimates in appropriate Sobolev spaces for solutions to the space-time scheme. Then we will obtain continuous versions of these solutions, by Sobolev’s embedding, and show that these solutions agree with the “natural” solutions at the grid points.

To aid in achieving this goal one has the following lemma regarding the embedding \( W^l_2 \subset L^2(G_h) \), the proof of which can be found, for example, in [5]. Recall by Sobolev’s embedding of \( W^l_2 \) into \( C_0 \), for \( l > d/2 \) there exists a linear operator \( \mathcal{I} : W^l_2 \to C_0 \) such that \( \mathcal{I}\phi(x) = \phi(x) \) for almost every \( x \in \mathbb{R}^d \) and

\[
\sup_{x \in \mathbb{R}^d} |\mathcal{I}\phi(x)| \leq N \| \phi \|_l^2
\]

for all \( \phi \in W^l_2 \) where \( N = N(d) \).

**Lemma 2.11.** For all \( \phi \in W^l_2 \) if \( l > d/2 \) and \( h \in (0, 1) \), then

\[
\sum_{x \in G_h} |\mathcal{I}\phi(x)|^2 h^d \leq N \| \phi \|_l^2
\]

for a constant \( N = N(d) \).

With these preliminary considerations in mind, we turn to the main pursuit of this paper. To accelerate the rate of convergence of the spatial approximation to an arbitrarily high order via Richardson’s method we must first prove the existence of an asymptotic expansion in powers of the discretization parameter \( h \) for the error between the space-time approximation and the temporal discretization. Thus we prove that for an integer \( k \geq 0 \) there exists random fields \( v^{(0)}_i(x), \ldots, v^{(k)}_i(x) \) that are independent of \( h \) and satisfy certain properties for all \( i \in \{0, \ldots, n\} \) and \( x \in G_h \). Namely, that \( v^{(0)} \) is the solution to (2.1) with initial condition \( v_0 \) and for nonzero \( h \),

\[
v^H_i(x) = \sum_{j=0}^{k} \frac{h^j}{j!} v^{(j)}_i(x) + R^{\tau, h}_i(x)
\]
holds almost surely for all \( x \in G_h \) and all \( i \in \{0, \ldots, n\} \) where \( v^h \) is the solution to (2.4) with initial condition \( v_0 \) and \( R^h \) is an \( \mathcal{L}(G_h) \)-valued adapted process such that

\[
\text{E} \max_{i \leq n} \sup_{x \in G_h} \left| R_i^{\tau, h}(x) \right|^2 \leq Nh^{2(k+1)}K_m^2
\]

for a constant \( N \) independent of \( h \) and \( \tau \).

We include the following additional assumption on the coefficients of the difference operators because at certain points in the proofs to come we will require less regularity than is guaranteed by Assumption 2.8.

**Assumption 2.12.** Let \( m \geq 0 \) be a fixed integer. For \( \lambda, \mu \in \Lambda \), the spatial derivatives of \( a_\lambda^m \) and \( b^\lambda \) exist up to order \( (m-4) \land 0 \) and for \( \lambda \in \Lambda_0 \) the spatial derivatives of \( P^\lambda \) and \( q^\lambda \) exist up to order \( (m-2) \land 0 \) and the coefficients together with their spatial derivatives are bounded by constants \( C_m \) for all \( \omega \in \Omega \), \( x \in \mathbb{R}^d \), and \( i \in \{0, \ldots, n\} \).

**Theorem 2.13.** If Assumption 2.8 holds with integer \( l \geq d/2 \) and Assumptions 2.7, 2.9, 2.10, 2.11, and 2.12 hold with

\[
m = m \geq 3k + 4 + l
\]

for integer \( k \geq 0 \), then expansion (2.6) and estimate (2.7) hold for \( h > 0 \) with a constant \( N = N(d, d_1, d_2, m, l, T, K_0, \ldots, K_m, \hat{K}_0, \ldots, \hat{K}_l + 2, C_m, \Lambda) \). If, in addition, \( p^\lambda = q^\lambda = 0 \) for \( \lambda \in \Lambda_0 \), then (2.6) and (2.7) hold for all nonzero \( h \). In this case, the \( v^{(j)} \) vanish for odd \( j \leq k \) and, hence, if \( k \) is odd, then (2.8) can be replaced with \( m = m \geq 3k + 1 + l \).

This theorem follows from the next result, which will also allow us to provide higher order estimates for derivatives of the solutions. Taking differences of (2.6) yields

\[
\delta h, \lambda v^h_i(x) + \sum_{j=0}^k \frac{h_j}{j!} \delta h, \lambda v^{(j)}_i(x) + \delta h, \lambda R_i^{\tau, h}(x)
\]

for any \( \lambda := (\lambda_1, \ldots, \lambda_p) \in \Lambda^p \), for integer \( p \geq 0 \), where \( \Lambda^0 := \{0\} \) and \( \delta h, \lambda := \delta h, \lambda_1 \times \cdots \times \delta h, \lambda_p \). Although the estimate for \( \delta h, \lambda R_i^{\tau, h}(x) \) is not obvious, we have the following generalization of Theorem 2.13.

**Theorem 2.14.** Let the assumptions of Theorem 2.13 hold with

\[
m = m \geq p + 3k + 4 + l
\]

for integers \( l > d/2, p \geq 0 \), and \( k \geq 0 \) with \( \lambda \in \Lambda^p \). Then for \( h > 0 \) expansion (2.6) and

\[
\text{E} \max_{i \leq n} \sup_{x \in G_h} \left| \delta h, \lambda R_i^{\tau, h}(x) \right|^2 \leq Nh^{2(k+1)}K_m^2
\]

hold for a constant \( N = N(p, d, d_1, d_2, m, l, T, K_0, \ldots, K_m, \hat{K}_0, \ldots, \hat{K}_l + 2, C_m, \Lambda) \). If, in addition, \( p^\lambda = q^\lambda = 0 \) for \( \lambda \in \Lambda_0 \), then the terms \( v^{(j)} \) vanish for odd \( j \leq k \) and, therefore, if \( k \) is odd, then (2.8) can be replaced with \( m = m \geq p + 3k + 1 + l \).

This theorem and Theorem 2.13 follow from a more general result that is proven in Section 8 after some preliminaries are presented in Section 8. Presently we formulate our acceleration result, which says the rate of convergence of the spatial approximation can be accelerated to an arbitrarily high order by taking suitable weighted averages of the approximation at different mesh sizes.

Fix an integer \( k \geq 0 \) and let

\[
\bar{v}^h := \sum_{j=0}^k \beta_j v^{2^{-j}h} \quad \text{and} \quad \tilde{v}^h := \sum_{j=0}^k \tilde{\beta}_j v^{2^{-j}h}
\]
where $v^{2^{-j}h}$ solves, with $2^{-j}h$ in place of $h$, the space-time scheme (2.4) with initial condition $v_0$. Here $\beta$ is given by $(\beta_0, \beta_1, \ldots, \beta_k) := (1, 0, \ldots, 0)\bar{V}^{-1}$ where $\bar{V}^{-1}$ is the inverse of the Vandermonde matrix with entries $\bar{V}^{ij} = 2^{-(i-1)(j-1)}$ for $i, j \in \{1, \ldots, k + 1\}$. Similarly, $\bar{\beta}$ is given by $(\bar{\beta}_0, \bar{\beta}_1, \ldots, \bar{\beta}_k) := (1, 0, \ldots, 0)\bar{V}^{-1}$ where $\bar{V}^{-1}$ is the inverse of the Vandermonde matrix with entries $\bar{V}^{ij} = 4^{-(i-1)(j-1)}$ for $i, j \in \{1, \ldots, \tilde{k} + 1\}$ where $\tilde{k} := \lfloor \frac{k}{2} \rfloor$. Here $[c]$ denotes the integer part of $c$.

Recall that $v^{(0)}$ is the solution to (2.1) with initial condition $v_0$.

**Theorem 2.15.** Let the assumptions of Theorem 2.13 hold with
\begin{equation}
(2.11)
\end{equation}
for integers $l > d/2$ and $k \geq 0$. Then
\begin{equation}
(2.12)
\end{equation}
holds for $h > 0$ with $N = N(d, d_1, d_2, m, l, T, K_0, \ldots, K_{m+2}, \bar{K}_0, \ldots, \bar{K}_{l+2}, C_m, \Lambda)$. If, in addition, $p^\lambda = q^\lambda = 0$ for $\lambda \in \Lambda_0$, then
\begin{equation}
(2.13)
\end{equation}
holds for nonzero $h$. Moreover, if $k$ is odd, then we only require $m = m \geq 3k + 1 + l$ in place of (2.11).

**Proof.** By Theorem 2.13 we have the expansion
\begin{equation}
(2.14)
\end{equation}
for each $j \in \{0, 1, \ldots, k\}$ where $r^{2^{-j}h} := h^{-(k+1)}R^{2^{-j}h}$. Then for $\bar{r}^h := \sum_{j=0}^{\tilde{k}} r^{2^{-j}h}$,
\begin{align*}
\bar{r}^h &= \left( \sum_{j=0}^{\tilde{k}} \bar{\beta}_j \right) v^{(0)} + \sum_{j=0}^{\tilde{k}} \sum_{i=1}^{\tilde{k}} \bar{\beta}_j \frac{h^{2i}}{2i!4^i} v^{(2i)} + \bar{r}^h h^{k+1} \\
&= v^{(0)} + \sum_{i=1}^{\tilde{k}} \frac{h^{2i}}{2i!} v^{(2i)} + \bar{r}^h h^{k+1}.
\end{align*}

since $\sum_{j=0}^{\tilde{k}} \bar{\beta}_j = 1$ and $\sum_{j=0}^{\tilde{k}} \bar{\beta}_j 4^{-ij} = 0$ for each $i \in \{1, 2, \ldots, k\}$ by the definition of $\bar{\beta}$. Now using the bound on $R^{\bar{r}^h} h$ from Theorem 2.13 together with this last calculation yields (2.13). The result for (2.12) is obtained in an almost identical way and therefore we omit the proof.

**Remark 2.16.** Note that without the acceleration, that is, when $k = 0$ and $k = 1$ in (2.12) and (2.13), respectively, we have that
\begin{equation}
(2.17)
\end{equation}
and if $p^\lambda = q^\lambda = 0$ for $\lambda \in \Lambda_0$, then we have
\begin{equation}
(2.18)
\end{equation}
in the theorem above. Moreover, these estimates are sharp; see Remark 2.21 in [3] on finite difference approximations for deterministic parabolic partial differential equations.
One can also construct rapidly converging approximations for the derivatives of $v^{(0)}$ by taking suitable weighted averaged of finite differences of $\tilde{v}^h$.

**Theorem 2.17.** Let $p \geq 0$ be an integer and let $p^+ = q^+ = 0$ for $\lambda \in \Lambda_0$. If the assumptions of Theorem 2.13 hold with

$$m = m \geq p + 3k + 4 + l,$$

for integers $l > d/2$, $k \geq 0$, and $p \geq 0$, then for $\lambda \in \Lambda$ equation (2.13) holds with $\delta_{\eta,\lambda} v^h$ and $\delta_{\eta,\lambda} v^{(0)}$ in place of $\tilde{v}^h$ and $v^{(0)}$ respectively.

**Proof.** This assertion follows from Theorem 2.14 in exactly the same way that Theorem 2.15 follows from 2.13. □

We end this section with two examples of ways to choose appropriate $a$, $b$, $p$, $q$ and $\Lambda$.

**Example 1.** Let $\Lambda = \{e_0, e_1, \ldots, e_d\}$ where $e_0 = 0$ and $e_i$ is the $i$th basis vector, that is, $\Lambda$ is the basis vectors in $\mathbb{R}^d$ together with the origin. Then for $i \in \{0, 1, \ldots, n\}$, set

$$a_i^{\alpha\epsilon} := a_i^{\alpha\beta}$$

and

$$b_i^{\epsilon\gamma} := b_i^{\alpha\rho}$$

for each $\alpha, \beta \in \{0, 1, \ldots, d\}$ and $\alpha \in \{1, \ldots, d\}$. Then each spatial derivative $D_\alpha$ in (2.1) is approximated by the symmetric difference $\delta_i^{\alpha}$. 

**Example 2.** Let $\Lambda$ again be the basis vectors in $\mathbb{R}^d$ together with the origin. For $i \in \{0, \ldots, n\}$, set

$$a_i^{00} := a_0^{00}$$

for each $\alpha, \beta \in \{1, \ldots, d\}$ and

$$b_i^{\epsilon\gamma} := b_i^{\alpha\rho}$$

for $\alpha \in \{0, \ldots, d\}$. We also take $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable functions $p^{\epsilon\alpha}$ and $q^{\epsilon\alpha}$ for $\alpha \in \{1, \ldots, d\}$ such that

$$p_i^{\epsilon\alpha} - q_i^{\epsilon\alpha} := a_i^{\alpha0} + a_i^{00}$$

for $\alpha \in \{1, \ldots, d\}$.

In the next section we make observations that will be used in the proofs of Theorems 2.14 and 2.13 which are given in Section 4.

3. Auxiliary Results

We begin by delivering a proof for Theorem 2.5. For integer $m \geq 0$, recall Lemma 2.1 from [12] taking the parameter $p$ in the Lemma to be $p = 2$. This Lemma holds for all $t \in [0, T]$ so in particular we have it for each $i\tau$ for $i \in \{0, \ldots, n\}$.

**Lemma 3.1.** Let $\phi \in W_2^{m+2}$. If Assumptions 2.7 and 2.8 hold for all multiindices $\gamma$ such that $|\gamma| \leq m$, then

$$Q_i^\gamma(\phi) := \int_{\mathbb{R}^d} 2 (D^\gamma \phi) D^\gamma L_i \phi + \sum_{\rho=1}^{d_1} |D^\gamma M_i^\rho \phi|^2 \ dx \leq N \|\phi\|_{m}^2$$

for a constant $N = N(d, d_1, m, K_0, \ldots, K_m)$.

We use Lemma 3.1 to obtain estimate (2.13). The existence of a solution to (2.1) will follow from the vanishing viscosity method.
Proof of Theorem 2.2. We first assume that a sufficiently smooth solution to (2.1) exists and obtain estimate (2.3) for a constant $N$ independent of $\tau$. We begin by obtaining an expression for the square of the norm for the solution to the time scheme. Then we estimate the supremum of the expectation of the square of the norm and in particular we show that this quantity is finite. With these observations in place we are then able to estimate the expectation of the supremum of the square of the norm.

For a multiindex $|\gamma| \leq m$, considering the equality $a^2 - b^2 = 2a(a - b) - |a - b|^2$ we note that (2.4) implies

$$
\|D^\gamma v_i\|_0^2 - \|D^\gamma v_{i-1}\|_0^2 = 2(D^\gamma v_i, D^\gamma (v_i - v_{i-1})) - \|D^\gamma (v_i - v_{i-1})\|_0^2
$$

$$
= 2(D^\gamma v_i, D^\gamma (L_i v_i + f_i)) \tau - \|D^\gamma (v_i - v_{i-1})\|_0^2
$$

$$
+ \sum_{\rho=1}^{d_1} (D^\gamma (v_i - v_{i-1}), D^\gamma (M^\rho_{i-1} v_{i-1} + g^\rho_{i-1})) \xi_i^\rho
$$

$$
+ \sum_{\rho=1}^{d_1} (D^\gamma (v_i - v_{i-1}), D^\gamma (M^\rho_{i-1} v_{i-1} + g^\rho_{i-1})) \xi_i^\rho
$$

Summing up over $i$ from 1 to $j$ and over $|\gamma| \leq m$, we have

$$
\|v_j\|_m^2 \leq \|v_0\|_m^2 + H_j + I_j + J_j,
$$

where

$$
H_j := 2 \sum_{i=1}^{j} (D^m v_i, D^m (L_i v_i + f_i)) \tau,
$$

$$
I_j := 2 \sum_{i=1}^{j} \sum_{\rho=1}^{d_1} (D^m v_{i-1}, D^m (M^\rho_{i-1} v_{i-1} + g^\rho_{i-1})) \xi_i^\rho,
$$

and

$$
J_j := \sum_{i=1}^{j} \left\| \sum_{\rho=1}^{d_1} D^m (M^\rho_{i-1} v_{i-1} + g^\rho_{i-1}) \xi_i^\rho \right\|_0^2.
$$

By an application of Itô’s formula, for each $\pi, \rho \in \{1, \ldots, d_1\}$ one has that for all $i \in \{0, \ldots, n - 1\}

$$
\xi_i^\rho = (w_{i+1}^\pi - w_i^\pi)(w_{i+1}^\rho - w_i^\rho) = Y_{i+1}^{\pi,\rho} - Y_i^{\pi,\rho} + \tau \chi_{\pi,\rho}
$$

for

$$
Y_{i+1}^{\pi,\rho}(t) := \int_0^t \left( w^\rho(s) - w^\rho_{\kappa(s)} \right) dw^\pi(s) + \int_0^t \left( w^\pi(s) - w^\pi_{\kappa(s)} \right) dw^\rho(s)
$$

where $\kappa(s)$ is the piecewise defined function taking value $\kappa(s) = i$ for $s \in [i \tau, (i+1) \tau)$ and where $\chi_{\pi,\rho} = 1$ when $\pi = \rho$ and 0 otherwise. Thus we can write $J_j = J_j^{(1)} + J_j^{(2)}$ where

$$
J_j^{(1)} := \sum_{i=1}^{j} \left\| \sum_{\rho=1}^{d_1} D^m (M^\rho_{i-1} v_{i-1} + g^\rho_{i-1}) \right\|_0^2 \tau
$$

and
and

\[ J_j^{(2)} := \int_0^{\tau_j} \sum_{\pi, \nu = 1}^{d_j} \left( D^m \left( \mathcal{M}^{\pi}_{\nu}(s) v_{\nu}(s) + g^\pi_\nu(s) \right), D^m \left( \mathcal{M}^{\nu}_{\pi}(s) v_{\pi}(s) + g^\nu_\pi(s) \right) \right) dY^{\nu \pi}(s). \]

Now observe that, for each \( i \in \{1, \ldots, n\} \), by Lemma 3.1 we have

\[ H_j + J_j^{(1)} \leq N \tau \sum_{\alpha = 0}^{d_j} \|D_{\alpha} v_0\|^2_m + N \tau \sum_{i=1}^{j} \sum_{|\gamma| \leq m} \left( Q_i^\gamma(v_i) + \|D^\gamma f_i\|^2 + \|D^\gamma g_{i-1}\|^2 \right) \]

\[ \leq N \tau \|v_0\|^2_{m+1} + N \tau \sum_{i=1}^{j} \left( \|v_i\|^2_m + \|f_i\|^2_m + \|g_{i-1}\|^2_m \right) \]

(3.2)

where \( N = N(d, d_1, m, K_0, \ldots, K_{m+1}) \). Note that we only require \( b^\alpha \) to have bounded derivatives of at most order \( m+1 \); the other coefficients only need to have bounded derivatives of at most order \( m \) at this stage. Here the initial condition \( v_0 \) enters, estimated in the \( W^m_{m+1} \)-norm, due to the displacement caused by the discretization in time when we consider the quadratic form \( Q^\gamma \) from Lemma 3.1. Thus inequality (3.1) becomes

\[ \|v_j\|^2_m \leq N \tau \|v_0\|^2_{m+1} + N \tau \sum_{i=1}^{j} \left( \|v_i\|^2_m + \|f_i\|^2_m + \|g_{i-1}\|^2_m \right) + I_j + J_j^{(2)}. \]

Since \( v_i, \mathcal{M}^{\nu}_{\pi} v_i, \) and \( g^\nu_\pi \) are all \( \mathcal{F}_i \)-measurable and \( \xi^\nu_{i+1} \) is independent of \( \mathcal{F}_i \) for \( i \in \{0, \ldots, n\} \), we have that

\[ \mathbb{E} I_j = 2 \sum_{i=1}^{j} \sum_{\rho = 1}^{d_i} \mathbb{E} \left\{ \left( D^\gamma v_{i-1}, D^\gamma \left( \mathcal{M}^{\nu}_{\pi} v_{i-1} + g^\nu_\pi \right) \right) \mathbb{E} \left( \xi^\rho_{i+1} \mid \mathcal{F}_i \right) \right\} = 0. \]

Similarly, we see that \( \mathbb{E} J_j^{(2)} = 0 \) since the expectation of the stochastic integral is zero. Therefore, taking the expectation of (3.3) and the sum of \( f \) and \( g \) over \( i \in \{0, \ldots, n\} \), we have that

\[ \mathbb{E} \|v_j\|^2_m \leq N \left( \tau \mathbb{E} \|v_0\|^2_{m+1} + \|f\|^2_m + \|g\|^2_m \right) + N \tau \mathbb{E} \sum_{i=1}^{j} \|v_i\|^2_m \]

for each \( j \in \{1, \ldots, n\} \). Applying a discrete Gronwall lemma to (3.4) we have

\[ \mathbb{E} \|v_j\|^2_m \leq N \left( \tau \mathbb{E} \|v_0\|^2_{m+1} + \|f\|^2_m + \|g\|^2_m \right) (1 - N \tau)^{-j} \]

and, since

\[ (1 - N \tau)^{-j} = \left( 1 - N \frac{T \tau}{n} \right)^{-j} \leq \left( 1 - N \frac{T \tau}{n} \right)^{-n} \leq C e^{N \tau} \]

we have the following estimate for the supremum of the expectation of the square of the norm

\[ \max_{i \leq n} \mathbb{E} \|v_i\|^2_m \leq N \left( \tau \mathbb{E} \|v_0\|^2_{m+1} + \|f\|^2_m + \|g\|^2_m \right) \]

for a constant \( N = N(d, d_1, m, T, K_0, \ldots, K_{m+1}) \). In particular, we can use (3.5) to eliminate the last term on the right-hand side of (3.4) by bounding it with terms already appearing on the right-hand side (3.4).
Next we approach the estimate for the expectation of the supremum by first observing how to bound the $I$ and $J^{(2)}$ terms appearing in (3.5) using the Burkholder–Davis–Gundy inequality. For $J^{(2)}$ we have

$$\max_{i \leq n} |J_i^{(2)}| \leq C \sum_{\pi, \rho = 1}^{d_1} \mathbb{E} \left\{ \int_0^T \left\| \mathcal{M}_{\pi}^{\rho} v_{\kappa(s)} + g_{\kappa(s)}^{\rho} \right\|_m^2 \left\| \mathcal{M}_{\pi}^{\rho} v_{\kappa(s)} + g_{\kappa(s)}^{\rho} \right\|_m^2 d\langle Y^{\pi \rho} \rangle(s) \right\}^{1/2}$$

$$\leq C \sum_{\pi, \rho = 1}^{d_1} \mathbb{E} \left\{ \int_0^T \left\| \mathcal{M}_{\pi}^{\rho} v_{\kappa(s)} + g_{\kappa(s)}^{\rho} \right\|_m^4 \left| w^\pi(s) - w^{\pi \kappa(s)} \right|^2 ds \right\}^{1/2}$$

$$\leq C \sum_{\pi, \rho = 1}^{d_1} \mathbb{E} \max_{i \leq n} \sqrt{T} \left\| \mathcal{M}_{\pi}^{\rho} v_i + g_i^{\rho} \right\|_m$$

$$\times \left\{ \frac{1}{T} \int_0^T \left\| \mathcal{M}_{\pi}^{\rho} v_{\kappa(s)} + g_{\kappa(s)}^{\rho} \right\|_m^2 \left| w^\pi(s) - w^{\pi \kappa(s)} \right|^2 ds \right\}^{1/2}$$

where $C$ is a constant independent of the parameters and functions under consideration and is allowed to change from one instance to the next. Therefore,

$$\max_{i \leq n} |J_i^{(2)}| \leq d_1 C \sum_{\pi, \rho = 1}^{d_1} \mathbb{E} \max_{i \leq n} \left\| \mathcal{M}_{\pi}^{\rho} v_i + g_i^{\rho} \right\|_m^2$$

$$+ C \frac{\tau}{T} \sum_{\pi, \rho = 1}^{d_1} \mathbb{E} \int_0^T \left\| \mathcal{M}_{\pi}^{\rho} v_{\kappa(s)} + g_{\kappa(s)}^{\rho} \right\|_m^2 \left| w^\pi(s) - w^{\pi \kappa(s)} \right|^2 ds$$

by Young’s inequality. We observe that the second term on the right-hand side of (3.6) can be estimated by

$$\frac{1}{T} \sum_{\pi, \rho = 1}^{d_1} \mathbb{E} \int_0^T \left\| \mathcal{M}_{\pi}^{\rho} v_{\kappa(s)} + g_{\kappa(s)}^{\rho} \right\|_m^2 \left| w^\pi(s) - w^{\pi \kappa(s)} \right|^2 ds$$

$$\leq \frac{1}{T} \sum_{\pi, \rho = 1}^{d_1} \mathbb{E} \left\{ \int_0^T \left\| \mathcal{M}_{\pi}^{\rho} v_{\kappa(s)} + g_{\kappa(s)}^{\rho} \right\|_m^2 \mathbb{E} \left( \left| w^\pi(s) - w^{\pi \kappa(s)} \right|^2 \left| F_{\pi(s)} \right| \right) ds \right\}$$

$$\leq N \tau \mathbb{E} \sum_{i = 0}^n \left\| v_i \right\|_{m+1}^2 + N \left\| g_i \right\|_{m+1}^2$$

using the tower property for conditional expectations. Further, the first term on the right-hand side of (3.6) is bounded from above by the sum over all $i \in \{1, \ldots, n\}$ and can be estimated by the same quantity. Combining these estimates and using (3.5) with $m + 1$ in place of $m$ we see that $\max_{i \leq n} |J_i^{(2)}|$ is estimated by

$$\max_{i \leq n} |J_i^{(2)}| \leq N \left( \tau \mathbb{E} \left\| v_0 \right\|_{m+2}^2 + \left\| I \right\|_{m+1}^2 + \left\| \mathcal{F}_{i(s)} \right\|_{m+1}^2 \right)$$

for a constant $N = N(d, d_1, m, T, K_0, \ldots, K_{m+2})$. Moving on to $I$, we note that

$$I_j = 2 \sum_{i = 1}^{d_1} \sum_{\rho = 1}^{d_1} \left( D^m v_i, D^m \left( \mathcal{M}_{\pi}^{\rho} v_{i-1} + g_{i-1}^{\rho} \right) \right) \left( w_i^{\rho} - w_i^{\rho \pi} \right)$$

$$= 2 \sum_{\rho = 1}^{d_1} \int_0^T \left( D^m v_{\kappa(s)}, D^m \left( \mathcal{M}_{\pi}^{\rho} v_{\kappa(s)} + g_{\kappa(s)}^{\rho} \right) \right) dw^\rho(s).$$
Applying the Burkholder–Davis–Gundy inequality once again, we obtain
\[
E \max_{t \leq n} |I_t| \leq C \sum_{p=1}^{d_1} E \left\{ \int_0^T \| v_{\kappa(s)} \|^2_m \left\| \mathcal{M}_{\kappa(s)}^p v_{\kappa(s)} + g_{\kappa(s)}^p \right\|^2_m \, ds \right\}^{1/2}
\leq C \sum_{p=1}^{d_1} E \left\{ \max_{t \leq n} \| v_t \|_m \left( \int_0^T \left\| \mathcal{M}_{\kappa(s)}^p v_{\kappa(s)} + g_{\kappa(s)}^p \right\|^2_m \, ds \right) \right\}^{1/2}
\]
and then using Young’s inequality followed by (3.5) with \( m + 1 \) in place of \( m \) we see that \( E \max |I| \) is estimated by the same quantity appearing on the right and side of (3.7).

Returning to (3.8), taking the maximum directly followed by the expectation, and using the estimates for the expectation of the supremum of \( |J^{(2)}| \) and \( |I| \), we see that
\[
E \max_{t \leq n} \| v_t \|^2_m \leq N \left( \tau E \| v_0 \|^2_{m+2} + \| f \|^2_{m+1} + \| g \|^2_{m+1} \right) = NK^2_{m+1},
\]
holds with a constant \( N = N(d, d_1, m, T, K_0, \ldots, K_{m+2}) \), thus establishing (2.20).

Next, we use the vanishing viscosity method to show that (2.1) admit a solution. Notice the leading coefficient of the operator \( \mathcal{L}_i^\tau \) is given by \( a_{i\beta}^\alpha = d_{i\beta}^\alpha + \varepsilon \chi_{\alpha \beta} \), where \( \chi_{\alpha \beta} = 1 \) for \( \alpha = \beta \) and zero otherwise. We then consider the equation
\[
v_i^\tau = v_{i-1}^\tau + (\mathcal{L}_i^\tau v_i^\tau + f_i) \tau + \sum_{p=1}^{d_1} (\mathcal{M}_{i-1}^p v_{i-1}^\tau + g_{i-1}^p) \xi_i^p
\]
for each \( i \in \{1, \ldots, n\} \) with initial condition \( v_0^\tau = v_0 \). Proving the solvability of (3.9) reduces to solving, for each \( \omega \in \Omega \), the elliptic problem
\[
A_i v_i^\tau = F_i
\]
for each \( i \in \{1, \ldots, n\} \) with free term
\[
F_i := v_{i-1}^\tau + \tau f_i + \sum_{p=1}^{d_1} \xi_i^p (\mathcal{M}_{i-1}^p v_{i-1}^\tau + g_{i-1}^p)
\]
and operator
\[
A_i := (I - \tau \mathcal{L}_i^\tau)
\]
where \( I \) is the identity. That is, we claim that \( A_i \) is
(i) bounded, i.e. \( \| A_i \phi \|^2_m \leq K \| \phi \|^2_{m+2} \) for a constant \( K \),
(ii) and coercive for sufficiently small \( \tau \), i.e. \( \langle A_i \phi, \phi \rangle \geq \frac{\tau}{2} \| \phi \|^2_{m+2} \),
for all \( \phi \in W^{m+2}_2 \) for every \( i \in \{1, \ldots, n\} \), where \( \langle \cdot, \cdot \rangle \) denotes the duality pairing between \( W^{m+2}_2 \) and \( W^m_2 \) based on the inner product in \( W^{m+1}_2 \). We will obtain the existence of a solution \( v_t^\tau \) to (3.10) for each \( i \in \{1, \ldots, n\} \) via Galerkin approximations (of course, in this instance, one could also use the Lax–Milgram Theorem).

For integer \( p \geq 0 \), let \( E_p \) be the \( p \)-dimensional subspace of \( W^{m+2}_2 \) spanned by the first \( p \) elements of \( \{e_j; j \in N\} \), a collection of vectors from \( W^{m+2}_2 \) forming an orthonormal basis for \( W^{m+1}_2 \). We seek an approximate solution \( \phi_i^p \in E_p \) to
\[
\langle A_i \phi^p_i, e_k \rangle = \langle F_i, e_k \rangle
\]
for each \( k \in \{1, \ldots, p\} \). Rewriting \( \phi_i^p = c_i^p e_j \) for coefficients \( c_i^p \), where the summation convention is used with respect to the repeated index \( j \in \{1, \ldots, p\} \), we see that \( \phi_i^p \) is an approximate solution if and only if \( c_i^p \) is \( W^{m+1}_2 \)-valued and satisfies the system of ordinary differential equations
\[
c_i^p \langle A_i e_j, e_k \rangle = \langle F_i, e_k \rangle
\]
for each $p$. We derive the estimate

$$
\frac{c}{2} \| \phi_i^p \|^2_{m+2} \leq \langle A_i \phi_i^0, \phi_i^p \rangle = \langle F_i, \phi_i^p \rangle \leq \| F_i \|_m \| \phi_i^p \|_{m+2}.
$$

From this we see that $(A_i \varepsilon_j, \varepsilon_k)$ is invertible and thus a solution $\phi_i^p$, and hence an approximate solution $\phi_i^p$, exists and moreover we have the estimate

$$
E \| \phi_i^p \|_{m+1} \leq 2e^{-1} E \| F_i \|_m
$$

uniformly in $p$. Thus, there exists a $v_i^p \in W_2^{m+2}$ and a subsequence $\rho_k$ such that

$$
\phi_i^p \rightarrow v_i^p \text{ weakly in } W_2^{m+2}.
$$

Therefore, for each $i \in \{1, \ldots, n\}$ there exists a $v_i^p$ satisfying (3.10) and, moreover, this solution is easily seen to be unique. Hence, we construct a unique solution to (3.9) iteratively.

Using the existence and uniqueness to the elliptic problem in each interval, we note that

$$
v_0 + \tau f_1 + \sum_{\rho=1}^{d_1} (M_0^\rho v_0 + g_0^\rho) \xi_1^\rho \in W_2^m,
$$

by the assumptions on $v_0$, $f$, and $g^\rho$, for $\rho \in \{1, \ldots, d_1\}$, and therefore there exists a $v_1^p \in W_2^{m+2}$ satisfying

$$
A_i v_1^p = v_0 + \tau f_1 + \sum_{\rho=1}^{d_1} (M_0^\rho v_0 + g_0^\rho) \xi_1^\rho.
$$

Further, assuming that there exists a $v_i^p \in W_2^{m+2}$ satisfying (3.9), we have that

$$
v_i^p + \tau f_{i+1} + \sum_{\rho=0}^{d_1} (M_0^\rho v_i^p + g_0^\rho) \xi_{i+1}^\rho \in W_2^m
$$

by the induction hypothesis and Assumption 2.3, and therefore there exists a $v_i^{p+1} \in W_2^{m+2}$ satisfying (3.9). Hence, we obtain $v^p = (v_i^p)_{i=1}^n$ such that each $v_i^p \in W_2^{m+2}$ satisfying (3.9).

Finally, we observe that the estimate (3.8) can be obtained for the solution $v^p$ to (3.9) in a similar manner. In particular, this gives a uniform estimate in $\varepsilon$ for the solution to (3.9). Therefore, there exists a subsequence $\varepsilon_k \rightarrow 0$ and a $W_2^m$-valued $F_i$-measurable $v_i$ such that $v_i^\varepsilon_k$ converges weakly to $v_i$ as $k \rightarrow \infty$ in $W_2^m$ for each $i \in \{1, \ldots, n\}$. This $v = (v_i)_{i=1}^n$ is the solution to (2.1) and is easily seen to be unique.

\begin{proof}

\end{proof}

\begin{remark}

We consider an implicit scheme where the operators $\mathcal{L}$ and $\mathcal{M}^\rho$ take values at the points of the time grid. The displacement observed in (3.2) caused by the discretization in time can be avoided if we consider a modified implicit scheme. Namely, if we consider operators defined to be the average over the intervals defined by consecutive points of the time grid, as in [7], we could then take $\mathcal{M}_0^\rho v_0 := 0$. However, we believe such a scheme would be less practical from a computational standpoint.

The lemma below is given in [4] for all $t \in [0, T]$ so, in particular, we have the following for each $i \in \{0, \ldots, n\}$. This lemma plays the role of Lemma 5.1 for obtaining the estimate for the space-time scheme.

\begin{lemma}

Let $\phi \in W_2^{m+2}$. If Assumptions 2.7 and 2.8 hold, then for all multi-indices $\gamma$ such that $|\gamma| \leq l$, then

$$
Q^\gamma_i(\phi) = \int_{\mathbb{R}^d} 2 (D^{\gamma} \phi(x)) D^{\gamma} L_i^\rho \phi(x) + \sum_{\rho=1}^{d_1} \left| D^{\gamma} M_i^\rho \phi(x) \right|^2 dx \leq N \| \phi \|^2_l
$$

for all $i \in \{1, \ldots, n\}$ for a constant $N = N(d, d_1, d_2, l, \hat{K}_0, \ldots, \hat{K}_{l+1}, \Lambda)$.

\end{lemma}
We are now able to give an estimate for solutions to the space-time scheme that is independent of \( h \). Recall that for integer \( l \geq 0 \), we define the norm
\[
\| \phi \|_{l}^{2} := E \sum_{i=0}^{n} \tau \| \phi_{i} \|_{l}^{2}
\]
and let \( W_{2}(\tau) \) be the space of \( W_{2}^{1} \)-valued \( F_{\tau} \)-measurable processes \( \phi \) such that \( \| \phi \|_{l}^{2} < \infty \).

**Theorem 3.4.** If Assumptions 2.3, 2.7, and 2.8 hold, then for each nonzero \( h \) equation (2.3) admits a unique \( W_{2}^{1} \)-valued \( F_{\tau} \)-measurable solution \( v^{h} \). Moreover, \( v^{h} \) satisfies
\[
(3.11) \quad E \max_{1 \leq n} \| v^{h} \|_{l}^{2} \leq N K_{2}^{2}
\]
for a constant \( N = N(d, d_{1}, d_{2}, l, T, \bar{K}_{0}, \ldots, \bar{K}_{i+2}, \Lambda) \). If, in addition, \( p^{\lambda} = q^{\lambda} = 0 \) for \( \lambda \in \Lambda_{0} \), then (3.11) holds for all nonzero \( h \).

**Proof.** That (2.3) admits a unique \( L^{2} \)-valued solution follows immediately from the considerations in the proof of Theorem 2.10. In particular, (3.11) can be achieved easily for a constant \( N \) depending on \( h \), so we see that the solution \( v^{h} \) is \( W_{2}^{1} \)-valued and \( F_{\tau} \)-measurable. To achieve (3.11) for a constant \( N \) independent of \( h \) (and \( \tau \)) follows almost immediately from the derivation of the estimate (2.3) in the proof of Theorem 2.5 using Lemma 3.3 in place of Lemma 3.1.

We obtain
\[
(3.12) \quad \| v_{j}^{h} \|_{l}^{2} \leq \| v_{0} \|_{l}^{2} + H_{j} + I_{j} + J_{j},
\]
in the same manner as (3.11), with
\[
H_{j} = 2 \sum_{i=1}^{d_{j}} \left( D^{\gamma} v_{i}^{h}, D^{\gamma} \left( L_{i-1}^{h} v_{i-1}^{h} + f_{i} \right) \right) \tau,
\]
\[
I_{j} = 2 \sum_{i=1}^{d_{j}} \sum_{\rho=1}^{d_{i}} \left( D^{\gamma} v_{i-1}^{h}, D^{\gamma} \left( M_{i-1}^{h, \rho} v_{i-1}^{h} + g_{i-1}^{\rho} \right) \right) \xi_{i}^{\rho},
\]
and
\[
J_{j} = \left( \sum_{i=1}^{d_{j}} \sum_{\rho=1}^{d_{i}} D^{\gamma} \left( M_{i-1}^{h, \rho} v_{i-1}^{h} + g_{i-1}^{\rho} \right) \xi_{i}^{\rho} \right)^{2}.
\]

Then by an application of Itô’s formula, we rewrite \( J_{j} = J_{j}^{(1)} + J_{j}^{(2)} \) using
\[
J_{j}^{(1)} = \left( \sum_{i=1}^{d_{j}} \sum_{\rho=1}^{d_{i}} D^{\gamma} \left( M_{i-1}^{h, \rho} v_{i-1}^{h} + g_{i-1}^{\rho} \right) \right)^{2} \tau
\]
and
\[
J_{j}^{(2)} = \int_{0}^{\tau} \sum_{\rho, \pi=1}^{d_{i}} \left( D^{\gamma} \left( M_{\pi}^{h, \rho} v_{\pi}^{h} + g_{\pi}^{\rho} \right), D^{\gamma} \left( M_{\pi}^{h, \rho} v_{\pi}^{h} + g_{\pi}^{\rho} \right) \right) dY^{\pi, \rho}(s)
\]
where \( Y^{\pi, \rho}(t), t \in [0, T], \) is defined in the proof of Theorem 2.5.

Now observe that, by Lemma 3.3 for each \( i \in \{0, \ldots, n\} \), we have
\[
H_{j} + J_{j}^{(1)} \leq \tau \sum_{\lambda \in \Lambda} \| \delta_{h, \lambda} v_{0} \|_{l_{i}}^{2} + N \tau \sum_{i=1}^{j} \left( Q_{i}^{h} (v_{i}^{h}) + (D^{\gamma} v_{i}^{h}, D^{\gamma} f_{i}) + \| D^{\gamma} g_{i-1} \|_{0}^{2} \right)
\]
\[
\leq N \tau \| v_{0} \|_{l_{i+1}}^{2} + N \tau \sum_{i=1}^{j} \left( \| v_{i}^{h} \|_{l_{i}}^{2} + \| f_{i} \|_{l_{i}}^{2} + \| g_{i-1} \|_{l_{i}}^{2} \right)
\]
for $h > 0$ where $N = N(l, d, d_2, \bar{K}_0, \ldots, \bar{K}_{l+1}, \Lambda)$. Again, here the initial condition $v_0$ enters, estimated in the $W^{l+1}_2$-norm, due to the displacement caused by the discretization in time when we consider the quadratic form $Q^\gamma$ from Lemma 3.3. If, in addition, $p^\lambda = q^\lambda = 0$ for $\lambda \in \Lambda_0$, then this last calculation holds for all nonzero $h$. Thus, inequality (3.12) becomes

\begin{equation}
(3.13) \quad \|v_i^h\|^2 \leq N\tau \|v_0\|^2_{l+1} + N\tau \sum_{j=1}^{\tau} \left( \|v_i^h\|^2 + \|f_i\|^2 + \|g_{i-1}\|^2 \right) + I_j + J_j^{(2)}.
\end{equation}

Since $EI_j = 0$ and $EJ_j^{(2)} = 0$, taking the expectation of (3.13) and taking the sum of $f$ and $g$ over $i \in \{0, \ldots, n\}$, we have that

\begin{equation}
(3.14) \quad E \|v_i^h\|^2 \leq N \left( \tau E \|v_0\|^2_{l+1} + \|f_i\|^2 + \|g_i\|^2 \right) + N\tau E \sum_{i=1}^{\tau} \|v_i^h\|^2
\end{equation}

for each $j \in \{1, \ldots, n\}$. Applying a discrete Gronwall lemma to (3.14) we have

\begin{equation}
E \|v_i^h\|^2 \leq N \left( \tau E \|v_0\|^2_{l+1} + \|f_i\|^2 + \|g_i\|^2 \right) (1 - N\tau)^{-j}
\end{equation}

and thus

\begin{equation}
(3.15) \quad \max_{i \leq n} E \|v_i^h\|^2 \leq N \left( \tau E \|v_0\|^2_{l+1} + \|f_i\|^2 + \|g_i\|^2 \right)
\end{equation}

for a constant $N = N(d, d_1, d_2, l, T, \bar{K}_0, \ldots, \bar{K}_{l+1}, \Lambda)$. In particular, we can use (3.15) to eliminate the last term on the right-hand side of (3.14) by bounding it with terms already appearing on the right-hand side (3.14).

The terms $I$ and $J_j^{(2)}$ are estimated as in the proof of Theorem 2.5 with $M^{h, p}$ in place of $M_e$, using the Burkholder–Davis–Gundy inequality. In particular, we have

\begin{equation}
E \max_{i \leq n} I_i + E \max_{i \leq n} J_i^{(2)} \leq N \left( \tau E \|v_0\|^2_{l+2} + \|f_i\|^2_{l+1} + \|g_i\|^2_{l+1} \right)
\end{equation}

for a constant $N = N(d, d_1, d_2, l, T, \bar{K}_0, \ldots, \bar{K}_{l+2}, \Lambda)$. Thus, returning to (3.13) and taking the maximum followed by the expectation we have that

\begin{equation}
E \max_{i \leq n} \|v_i^h\|^2 \leq N \left( \tau E \|v_0\|^2_{l+2} + \|f_i\|^2_{l+1} + \|g_i\|^2_{l+1} \right) = NK^2 < \infty,
\end{equation}

for a constant $N = N(d, d_1, d_2, l, T, \bar{K}_0, \ldots, \bar{K}_{l+2}, \Lambda)$.

For the convenience of the reader we record the following lemma, found in [4].

**Lemma 3.5.** Let $\phi \in W^{p+1}_2$ and $\psi \in W^{p+2}_2$ for an integer $p \geq 0$ and let $\lambda, \mu \in \Lambda_0$. Set

$$\partial_{\lambda} \phi := \lambda^l D_j \phi \quad \text{and} \quad \partial_{\mu} \phi := \partial_{\lambda} \partial_{\mu} \phi.$$ 

Then we have

\begin{equation}
(3.16) \quad \frac{\partial^p}{(\partial h)^p} \delta_{h, \lambda} \phi(x) = \int_0^1 \theta^p \partial^p_{\lambda} \phi(x + h\theta) \, d\theta,
\end{equation}

\begin{equation}
(3.17) \quad \frac{\partial^p}{(\partial h)^p} \delta_{\lambda} \phi(x) = \frac{1}{2} \int_{-1}^1 \theta^p \partial^p_{\lambda} \phi(x + h\theta) \, d\theta
\end{equation}

and

\begin{equation}
(3.18) \quad \frac{\partial^p}{(\partial h)^p} \delta_{\lambda} \partial_{\mu} \phi(x) = \frac{1}{4} \int_{-1}^1 \int_{-1}^1 (\theta_1 \partial_{\lambda} - \theta_2 \partial_{\mu}) \partial_{\lambda} \partial_{\mu} \psi(x + h(\theta_1 \lambda - \theta_2 \mu)) \, d\theta_1 \, d\theta_2
\end{equation}
Furthermore, for integer \( l \in \mathbb{R}^d \) for each \( h \in \mathbb{R} \). Thus

\[
(3.19) \quad \frac{\partial^p}{(\partial h)^p} \delta_{h,\lambda}\phi \bigg|_{h=0} = \frac{1}{p+1} \partial^{p+1}_\lambda \phi, \quad \frac{\partial^p}{(\partial h)^p} \delta_\lambda \phi \bigg|_{h=0} = \frac{B_p}{p+1} \partial^{p+1}_\lambda \phi,
\]

and

\[
(3.20) \quad \frac{\partial^p}{(\partial h)^p} \delta_\lambda \psi \bigg|_{h=0} = \sum_{r=0}^p A_{p,r} \partial^{r+1}_\lambda \partial^{p-r+1}_\mu \psi,
\]

where

\[
(3.21) \quad B_p := \begin{cases} 0 & \text{if } p \text{ is odd} \\ 1 & \text{if } p \text{ is even} \end{cases}, \quad A_{p,r} := \begin{cases} 0 & \text{if } p \text{ or } r \text{ is odd} \\ \frac{p^r}{(r+1)!} \frac{1}{(p-r+1)!} & \text{if } p \text{ and } r \text{ are even} \end{cases}.
\]

Furthermore, for integer \( l \geq 0 \), \( \phi \in W_{2}^{p+2+l} \), and \( \psi \in W_{2}^{p+3+l} \) one has

\[
(3.22) \quad \left\| \delta_{h,\lambda} \phi - \sum_{j=0}^p \frac{|h|^j}{(j+1)!} \partial^{j+1}_\lambda \phi \right\|_l \leq \frac{|h|^{p+1}}{(p+2)!} \left\| \partial^{p+2}_\lambda \phi \right\|_l,
\]

\[
(3.23) \quad \left\| \delta_\lambda \phi - \sum_{j=0}^p \frac{|h|^j}{(j+1)!} B_j \partial^{j+1}_\lambda \phi \right\|_l \leq \frac{|h|^{p+1}}{(p+2)!} \left\| \partial^{p+2}_\lambda \phi \right\|_l,
\]

and

\[
(3.24) \quad \left\| \delta_\lambda \psi - \sum_{j=0}^p \frac{|h|^j}{(j+1)!} \sum_{r=0}^j A_{j,r} \partial^{r+1}_\lambda \partial^{p-r+1}_\mu \psi \right\|_l \leq N|h|^{p+1} \left\| \psi \right\|_{l+p+3},
\]

where \( N = N(|\lambda|, |\mu|, d, p) \).

For integers \( l \geq 0 \) and \( r \geq 1 \), denote by \( W_{h,2}^{l,r} \) the Hilbert space of functions \( \phi \) on \( \mathbb{R}^d \) such that

\[
(3.25) \quad \left\| \phi \right\|_{l,r,h}^2 := \sum_{\lambda_1,\ldots,\lambda_r \in \Lambda} \left\| \delta_{h,\lambda_1} \cdots \delta_{h,\lambda_r} \phi \right\|_l^2 < \infty
\]

and set \( W_{h,2}^{l,0} := W^{l}_2 \).

**Remark 3.6.** Formula (3.10) with \( p = 0 \) and Minkowski’s integral inequality imply that

\[
\left\| \delta_{h,\lambda} \phi \right\|_0 \leq \left\| \partial_\lambda \phi \right\|_0.
\]

By applying this inequality to finite differences of \( \phi \) and using induction we can conclude that \( W_{2}^{l+r} \subset W_{h,2}^{l,r} \). Further, for any \( \phi \in W_{h,2}^{l+r} \) we have

\[
\left\| \phi \right\|_{l+r,h} \leq N \left\| \phi \right\|_{l+r},
\]

where \( N \) depends only on \( |A_0| = \sum_{\lambda \in A_0} |\lambda|^2 \) and \( r \).

We now use the preceding observations to obtain estimates in appropriate Sobolev spaces for a system of time discretized equations. Here we use the summation convention with respect to the repeated indices \( \lambda, \mu \in \Lambda_0 \). For \( i \in \{0, \ldots, n\} \), let

\[
L_i^{(0)} := a_i^{\lambda\mu} \partial_\lambda \partial_\mu + (p_i^{\lambda} - q_i^{\lambda}) \partial_\lambda,
\]

\[
M_i^{(0)} := b_i^{\lambda\mu} \partial_\lambda,
\]
for each $\rho \in \{1, \ldots, d_1\}$, and for an integer $p \geq 1$ let

$$L_i^{(p)} := \sum_{j=0}^{p} A_{p,j} a_i^{\lambda_j} \partial_{\lambda_j}^{p+j+1} \partial_{\lambda_j}^{p-j} + \frac{B_p}{p+1} (a_i^{\lambda_0} + a_i^{\lambda_3}) \partial_{\lambda_3}^{p+1}$$

$$\nu_{i-1}^{(p)} + \left( L_i \nu_{i-1}^{(p)} + \sum_{j=1}^{p} C_{p,j} L_i^{(j)} \nu_{i-1}^{(p-j)} \right) \tau$$

for $p \in \{1, \ldots, k\}$ where $C_{p,j}$ is the binomial coefficient and $\nu_{i-1}^{(p)}$ is the solution to (2.1) with initial condition $v_0$.

**Theorem 3.8.** If Assumptions 2.1, 2.2, 2.3, and 2.12 hold with $m = m \geq 3k$, then the system (3.28) admits a unique solution $\nu^{(1)}, \ldots, \nu^{(k)}$ for initial condition $\nu_0^{(1)} = \cdots = \nu_0^{(k)} = 0$ such that $\nu^{(p)}$ is $W_2^{m-3p}$-valued $F_t$-measurable. Moreover, for each $p \in \{1, \ldots, k\}$ if $v^{(p)}$ is a solution, then

$$E \max_{t \leq n} \left\| v_t^{(p)} \right\|_{m-3p}^2 \leq NK_m^2$$

holds for $h > 0$ with a constant $N = N(d, d_1, m, k, T, K_0, \ldots, K_{m+2}, C_m)$. If, in addition, $p^\lambda = q^\lambda = 0$ for $\lambda \in \Lambda_0$, then (3.29) holds for all nonzero $h$ and $\nu^{(p)} = 0$ for odd $p \leq k$. 
Proof. For convenience let
\[ F^{(p)}_i = \sum_{j=1}^{P} C^{(j)}_p L^{(j)}_i \nu^{(p-j)}_i, \]
and
\[ G^{(p)}_i = \sum_{j=1}^{P} C^{(j)}_p M^{(j)}_i \nu^{(p-j)}_i, \]
where we will write \( G^{(p)} = \sum_{\rho=1}^{d_1} G^{(p)}_\rho \). Observe that for each \( p \in \{1, \ldots, k\} \) the equation for \( \nu^{(p)} \) in (3.28) depends only on \( \nu^{(j)} \) for \( j < p \) and does not involve any of the unknown processes \( \nu^{(j)} \) with indices \( j \geq p \). Therefore, we shall prove the solvability of the system and the desired properties on \( \nu^{(p)} \) recursively using Theorem 2.5.

By Theorem 2.5, \( \nu^{(0)} \) is \( W^m \)-valued, \( F_1 \)-measurable, and satisfies (3.29) with \( p = 0 \). Observe that
\[ \left\| L^{(1)}_i \nu^{(0)}_i \right\|_{m-2} \leq N \left\| \nu^{(0)}_i \right\|_m \]
for a constant \( N = N(m, C_m) \), recalling the constant \( C_m \) from Assumption 2.12 in this instance since we only require that \( (m-2) \) derivatives of the coefficients exist and are bounded, and further that \( M^{(1)}_i \nu^{(0)}_i = 0 \). Therefore, by Theorem 2.5 there exists a unique \( W^{m-3p}_2 \)-valued \( F_1 \)-measurable \( \nu^{(1)} \) satisfying (3.28) with zero initial condition. Moreover, the estimate (3.29) is clearly satisfied in for \( p = 1 \).

Now we induct on \( p \), assuming that for \( m \geq 3k \geq 2 \) and \( p \in \{2, \ldots, k\} \) we have unique solutions \( \nu^{(1)}, \ldots, \nu^{(p-1)} \) satisfying the desired properties. Observe that
\[ \left\| L^{(1)}_i \nu^{(p-1)}_i \right\|_{m-2} \leq N \left\| \nu^{(p-1)}_i \right\|_{m-2} \]
for a constant \( N = N(m, C_m) \), by Assumption 2.12 and that \( M^{(1)}_i \nu^{(p-1)}_i = 0 \). Further for \( j \geq 2 \), if \( j \) is even, then
\[ \left\| L^{(j)}_i \nu^{(p-j)}_i \right\|_{m-2} \leq N \left\| L^{(j)}_i \nu^{(p-j)}_i \right\|_{m-2} \]
and
\[ \sum_{\rho=1}^{d_1} \left\| M^{(j)}_i \nu^{(p-j)}_i \right\|_{m-2}^{2} \leq N \sum_{\rho=1}^{d_1} \left\| M^{(j)}_i \nu^{(p-j)}_i \right\|_{m-2}^{2} \]
or if \( j \) is odd, then
\[ \left\| L^{(j)}_i \nu^{(p-j)}_i \right\|_{m-2} \leq N \left\| L^{(j)}_i \nu^{(p-j)}_i \right\|_{m-2} \]
and \( M^{(j)}_i \nu^{(p-j)}_i = 0 \), for constants \( N = N(m, C_m) \), by Assumption 2.12. Therefore, by the induction hypothesis, \( F^{(p)} \) is \( W^{m-3p+1}_2 \)-valued and \( F_1 \)-measurable, \( G^{(p)} \) is \( W^{m-3p+1}_2 \)-valued and \( F_1 \)-measurable, and
\[ \left( F^{(p)} \right)^2_{m-3p+1} + \left( G^{(p)} \right)^2_{m-3p+1} \leq N K^2_{m}. \]
That is, \( F^{(p)} \in W^{m-3p+1}_2(\tau) \) and \( G^{(p)} \in W^{m-3p+1}_2(\tau) \). Thus by Theorem 2.5 there exists a \( W^{m-3p}_2 \)-valued \( F_1 \)-measurable \( \nu^{(p)} \) satisfying (3.28) with zero initial condition. Moreover, Theorem 2.5 yields the estimate
\[ \mathbb{E} \max_{i \leq n} \left\| \nu^{(p)}_i \right\|_{m-3p}^{2} \leq N \left( \left\| F^{(p)} \right\|_{m-3p+1} + \left\| G^{(p)} \right\|_{m-3p+1} \right)^2 \]
for a constant $N = N(d, d_1, m, k, T, K_0, \ldots, K_{n+1}, C_m)$. Combining this with (3.30) yields (3.29). We remark further that the uniqueness of each $\nu^{(p)}$ follows from the uniqueness of the solutions obtained from Theorem 2.4.

Note that $\mathcal{M}^{(p)} = 0$ for odd $p \leq k$ by (3.19). Assume, in addition, that $p^* = q^* = 0$ for $\lambda \in \Lambda_0$. Then also $\mathcal{L}^{(p)} = 0$ for odd $p \leq k$ by (3.19) and (3.20). Therefore, $F^{(1)} = 0$ and $G^{(1)} = 0$, which implies $\nu^{(1)} = 0$. Assume that $k \geq 2$ and that for an odd $p \leq k$ we have $\nu^{(j)} = 0$ for all odd $j \leq p$. Then $\mathcal{L}^{(p-j)}\nu^{(j)} = 0$ and $\mathcal{M}^{(p-j)}\nu^{(j)} = 0$ for all $j \in \{1, \ldots, p\}$ since either $j$ or $p-j$ is odd. Thus $F^{(p)} = 0$ and $G^{(p)} = 0$. Hence $\nu^{(p)} = 0$ for all odd $p \leq k$.

For integers $k \geq 0$ let $\nu^{(1)}, \ldots, \nu^{(k)}$ be the solutions to the system (3.28) with zero initial condition coming from Theorem 3.3. Let

$$
(3.31) \quad \tau_{i, h}^{\nu, h} := \nu_{i, h}^{(0)} - \sum_{j=1}^{k} \frac{h_j}{j!} \nu_{i, h}^{(j)}
$$

for $i \in \{1, \ldots, n\}$, where $\nu^{(h)}$ and $\nu^{(0)}$ are the unique $W^m_2$-valued solutions to (2.4) and (2.1), respectively, with initial condition $v_0$.

**Lemma 3.9.** Let $\tau_{i, h}^{\nu, h}$ be defined as in (3.31). If Assumptions 2.7, 2.8, and 2.12 hold with $m = m \geq 3k + 4 + l$ for integers $k \geq 0$ and $l \geq 0$, then $\tau_{0, h}^{0, h} = 0$ and $\tau_{i, h}^{\nu, h}$ is $W^m_2$-valued $\mathcal{F}_i$-measurable such that

$$
\sup_{i \leq n} \left\| \tau_{i, h}^{\nu, h} \right\|^2 < \infty
$$

and $\tau_{i, h}^{\nu, h}$ satisfies

$$
(3.32) \quad \tau_{i, h}^{\nu, h} = \tau_{i-1, h}^{\nu, h} + \left( L_i\tau_{i, h}^{\nu, h} + \psi_{i, h}^{0} \right) \tau + \sum_{j=1}^d \left( M_i^{h, \nu} \tau_{i-1, h}^{\nu, h} + \phi_{i-1, h}^{0} \right) \tau
$$

for $i \in \{1, \ldots, n\}$, where

$$
\psi_{i, h}^{0} = \sum_{j=0}^{k} \frac{h_j}{j!} \mathcal{O}_i^{h(k-j)} \nu_{i-1, h}^{(j)}
$$

and

$$
\phi_{i-1, h}^{0} = \sum_{j=0}^{k} \frac{h_j}{j!} \mathcal{R}_i^{h(k-j)} \nu_{i-1, h}^{(j)}
$$

Moreover $\psi_{i, h}^{0} \in W^{l+1}_2(\tau)$ and $\phi_{i-1, h}^{0} \in W^{l+1}_2(\tau)$.

**Proof.** First recall that by Theorem 3.4 the solution to the space-time scheme $\nu^{(h)}$ is $W^m_2$-valued $\mathcal{F}_t$-measurable and satisfies estimate (3.11) owing to Assumptions 2.3, 2.7, and 2.8 with $m = l$. By Theorem 3.3 $\nu^{(0)}$ is $W^m_2$-valued $\mathcal{F}_t$-measurable and satisfies estimate (2.3) by Assumptions 2.1, 2.2, and 2.3 with $m = l$. By Theorem 3.8 the $\nu^{(j)}$ are $W^m_2$-valued $\mathcal{F}_t$-measurable processes satisfying estimate (3.29) for all $j \in \{1, \ldots, k\}$ owing to Assumptions 2.1, 2.2, 2.3, and 2.12 with $m = m = l$. Thus $\tau_{i, h}^{\nu, h}$ is $W^m_2$-valued $\mathcal{F}_i$-measurable and satisfies

$$
\sup_{i \leq n} \left\| \tau_{i, h}^{\nu, h} \right\|^2 < \infty
$$

for $m = m = l \geq 3k$.

One can easily show that $\tau_{i, h}^{\nu, h}$ satisfies (3.32) using (2.1), (2.4), and (3.28) by noting that we can rewrite $\psi_{i, h}^{0}$ and $\phi_{i-1, h}^{0}$ as

$$
\psi_{i, h}^{0} = L_i\nu^{(0)} - \mathcal{L}\nu^{(0)} + \sum_{j=1}^{k} \frac{h_j}{j!} L_i^{h}\nu^{(j)} - \sum_{j=1}^{k} \frac{h_j}{j!} \mathcal{L}\nu^{(j)} - I
$$

and

$$
\phi_{i-1, h}^{0} = \sum_{j=0}^{k} \frac{h_j}{j!} \mathcal{R}_i^{h(k-j)} \nu_{i-1, h}^{(j)}
$$
Theorem 4.1. Let $l, m, \tau, h$ be defined as in (3.30). If Assumption 2.8 holds with integer $l \geq 0$ and Assumptions 2.1, 2.4, 2.8, 2.9, 2.10, 2.12 hold with
\begin{equation}
\left(4.1\right)
m = 3k + 4 + l
\end{equation}
for integer $k \geq 0$, then, for $h > 0$,
\begin{equation}
\left(4.2\right)
\mathbb{E} \max_{i \leq n} \left| \tau^{\tau,h}_{i} \right|^2 \leq N |h|^{2(k+1)} L^{-2}
\end{equation}
holds with a constant $N = N(d, d_1, d_2, m, l, T, K_0, \ldots, K_{m+2}, \Lambda)$. If, in addition, $p^\lambda = q^\lambda = 0$ for $\lambda \in \Lambda_0$, then (4.2) holds for nonzero $h$ and it suffices to assume $m \geq 3k + 1 + l$ in place of (4.1).
Proof. By Lemma 3.3 we have that \( f^h \in W_2^{l+1}(\tau) \) and \( g^{h,\rho} \in W_2^{l+1}(\tau) \). Thus, by Lemma 3.3 and Theorem 3.3

\[
\begin{align*}
E \max_{t \leq \tau} \|\tau_{t}^h\|_{l}^{2} & \leq N \left( \|f^h\|_{l+1}^{2} + \|g^h\|_{l+1}^{2} \right)
\end{align*}
\]

for a constant \( N = N(d, d_1, d_2, l, T, K_0, \ldots, K_{m+2}) \). Then by (4.1), for \( j \in \{0, \ldots, k\} \) and by Remark 3.26, we have

\[
\begin{align*}
\left\| \sigma^{h(k-j)}_{t} \nu^{(j)}_{t} \right\|_{l+1} & \leq N |h|^{k-j+1} \left\| \nu^{(j)}_{t} \right\|_{l+k-j+4} \leq N |h|^{k-j+1} \left\| \nu^{(j)}_{t} \right\|_{m-3j},
\end{align*}
\]

and

\[
\begin{align*}
\left\| \mathcal{R}^{h(k-j)p}_{t} \nu^{(j)}_{t} \right\|_{l+1} & \leq N |h|^{k-j+1} \left\| \nu^{(j)}_{t} \right\|_{l+k-j+3} \leq N |h|^{k-j+1} \left\| \nu^{(j)}_{t} \right\|_{m-3j-1}.
\end{align*}
\]

Now using Theorem 3.5 we see that

\[
\begin{align*}
\|f^h\|_{l+1}^{2} + \|g^h\|_{l+1}^{2} & \leq N |h|^{2(k+1)} K_m
\end{align*}
\]

which, when taken together with (4.3), implies (4.2).

If, in addition, \( p^h = q^h = 0 \) for \( \lambda \in \Lambda_0 \), then as in Theorem 3.8 it follows that \( \nu^{(j)} = 0 \) for all odd \( j < k \). If \( k \) is odd, then clearly \( \nu^{(k)} = 0 \) and (4.1) holds for \( j = k \) and also for \( j \geq k-1 \) and we need only \( m = 3k+1+l \). We mention further that \( \nu^{(j)} = 0 \) for all odd \( j < k \) in the case \( p^h = q^h = 0 \) for \( \lambda \in \Lambda_0 \) also follows from (4.1), now valid for all nonzero \( h \), since \( \nu^h \) and \( \nu^{-h} \) are the \( L^2 \)-valued solutions to (2.4) with initial condition \( v_0 \) and we must have \( \nu^h = \nu^{-h} \) due to the uniqueness of solutions.

We have the following corollary to Theorem 4.1 which implies Theorem 2.14 and hence Theorem 2.13. Let \( R^t = \mathcal{I}^t \hat{\varphi}^h \), where \( \mathcal{I} \) is the embedding operator from \( 2 \) and notice that \( R^t \in L^2(G_h) \) for all \( i \in \{0, \ldots, n\} \).

**Corollary 4.2.** If in Theorem 4.1 we have \( l > p + d/2 \) for integer \( p \geq 0 \), then, for \( \lambda \in \Lambda^p \) and \( h > 0 \),

\[
\begin{align*}
E \max_{t \leq \tau} \sup_{x \in \mathbb{R}^d} \left| \delta_{t, \lambda} R^t \right|^{2} \leq N h^{2(k+1)} K_m
\end{align*}
\]

holds with a constant \( N = N(d, d_1, d_2, m, l, T, K_0, \ldots, K_{m+2}, \hat{K}_0, \ldots, \hat{K}_{l+2}, C_m, \Lambda) \).

**Proof.** By Sobolev’s embedding of \( W_2^{l-p} \) into \( C_b \) and Remark 3.6 we have that

\[
\begin{align*}
E \max_{t \leq \tau} \sup_{x \in \mathbb{R}^d} \left| \delta_{t, \lambda} R^t \right|^{2} \leq NE \max_{t \leq \tau} \left| \tau_{t}^h \right|_{l-p, p, h}^{2} \leq NE \max_{t \leq \tau} \left| \tau_{t}^h \right|_{l}^{2}
\end{align*}
\]

which, together with Theorem 4.1 yields the desired result.

It only remains to explain how the corollary implies Theorem 2.14 and hence Theorem 2.13. Using Sobolev’s embedding, we find continuous versions of the \( L^2 \)-valued solutions to the space-time scheme, the time scheme, and the system of time discretized equations. Then we notice that the restriction of the \( L^2 \)-valued solution to (2.4) onto \( G_h \) agrees with the unique \( L^2(G_h) \) valued solution. This argument can be found in [8], nevertheless we reproduce it here for the convenience of the reader.

For \( \mathcal{I} : W_2^{l} \to C_b \) from Lemma 2.11 Theorem 2.14 follows by considering the embeddings \( \hat{\varphi}^h = \mathcal{I} \hat{\varphi}^h \), where \( \hat{\varphi}^h \) is the unique \( L^2 \)-valued solution to (2.4) with initial condition \( v_0 \), and \( \nu^{(j)} = \mathcal{I} \nu^{(j)} \), for \( j \in \{0, \ldots, k\} \), where \( \nu^{(0)} \) is the unique \( L^2 \)-valued solution to (2.1) with initial condition \( v_0 \) and the processes \( \nu^{(i)} \) are the solutions to the system of time discretized SPDEs (3.28) as given in Theorem 3.5. By Theorem 3.1 \( \hat{\varphi}^h \) is \( W_2^{l} \)-valued and \( \mathcal{F}_t \)-measurable for all \( i \in \{1, \ldots, n\} \). For
each \( j \in \{1, \ldots, k\} \), the \( \nu^{(j)} \) are \( W^{m - 3k} \)-valued processes by Theorem 2.5. Since \( l > d/2 \) and \( m - 3k > d/2 \), the processes \( \tilde{v}^h \) and \( v^{(i)} \) are well defined and clearly (3.31) implies (2.6) with \( \tilde{v}^h \) in place of \( v^h \). That is, we have the expansion for a continuous version of the \( L^2 \)-valued solution.

Next we show that the restriction of the \( L^2 \)-valued solution to the grid \( \mathcal{G}_h \), a set of Lebesgue measure zero, is indeed equal almost surely to the unique \( \ell^2(\mathcal{G}_h) \)-valued solution that one would naturally obtain from (2.4). That is, we show that

\[
\hat{v}^h_i(x) = v^h_i(x)
\]

almost surely for all \( i \in \{1, \ldots, n\} \) and for all \( x \in \mathcal{G}_h \) where \( v^h_i \) is the unique \( \mathcal{F}_t \)-measurable \( \ell^2(\mathcal{G}_h) \)-valued solution of (2.4) from Theorem 3.3. Therefore, for a compactly supported nonnegative smooth function \( \phi \) on \( \mathbb{R}^d \) with unit integral and for a fixed \( x \in \mathcal{G}_h \) we define

\[
\phi_\varepsilon(y) := \phi \left( \frac{y - x}{\varepsilon} \right)
\]

for \( y \in \mathbb{R}^d \) and \( \varepsilon > 0 \). Recall, by Remark 2.4, that we can obtain versions of \( v_0, f_i \), and \( g_\rho^\rho \), for \( \rho \in \{1, \ldots, d_i\} \), that are continuous in \( x \). Since \( \tilde{v}^h \) is a \( L^2 \)-valued solution of (2.4) for each \( \varepsilon \), almost surely

\[
\int_{\mathbb{R}^d} \hat{v}^h_i(y)\phi_\varepsilon(y) \, dy = \int_{\mathbb{R}^d} v^h_{i-1}(y)\phi_\varepsilon(y) \, dy + \tau \int_{\mathbb{R}^d} (L^h_i \hat{v}^h_i + f_i)(y)\phi_\varepsilon(y) \, dy + \sum_{\rho=1}^{d_i} \xi^\rho_{i-1} \int_{\mathbb{R}^d} (M^\rho_{i-1} \hat{v}^h_{i-1} + g^\rho_{i-1})(y)\phi_\varepsilon(y) \, dy
\]

for each \( i \in \{1, \ldots, n\} \). Letting \( \varepsilon \to 0 \), we see that both sides converge for all \( i \in \{1, \ldots, n\} \) and \( \omega \in \Omega \). Therefore, almost surely

\[
\hat{v}^h_i(x) = v^h_{i-1}(x) + (L^h_i \hat{v}^h_i + f_i)(x)\tau + \sum_{\rho=1}^{d_i} \left( M^\rho_{i-1} \hat{v}^h_{i-1}(x) + g^\rho_{i-1}(x) \right) \xi^\rho_{i-1}
\]

for all \( i \in \{1, \ldots, n\} \). Moreover, by Lemma 2.11, the restriction of \( \hat{v}^h \), the continuous version of \( v^h \), onto \( \mathcal{G}_h \) is an \( \ell^2(\mathcal{G}_h) \)-valued process. Hence, (4.6) holds, due to the uniqueness of the \( \mathcal{F}_t \)-measurable \( \ell^2(\mathcal{G}_h) \)-valued solution of (2.4) for any \( \mathcal{F}_0 \)-measurable \( \ell^2(\mathcal{G}_h) \)-valued initial data. This finishes the proof of Theorems 2.14 and 2.13.

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School of Mathematics, University of Edinburgh, King’s Buildings, Edinburgh, EH9 3JZ, UK
E-mail address: e.hall@ed.ac.uk