A TQFT EXTENDING THE RESHETIKHIN–TURAEV TQFT TO COBORDISMS WITH CORNERS

YU TSUMURA

Abstract. We extend the Reshetikhin–Turaev TQFT to cobordisms with corners using the Kapranov–Voevodsky 2–vector spaces as a target category.

1. Introduction

One of the great breakthroughs in the understanding of physical theories was the construction of Reshetikhin–Turaev 2+1 dimensional topological field theories TFTs [Tur10, RT91]. Prior to this Atiyah axiomatized a TQFT in [Ati88]. The simpler 1+1 dimensional theory was nicely formulated by [Dij89]. The latter construction lifted to conformal field theory by Segal [Seg88]. In all these cases one has a functor from a cobordism category to an algebraic category.

Going back to Freed and Quinn [FQ93], Cardy and Lewellen [CL91], there has been an interest in including boundary conditions/information. Besides the physical challenges this poses a mathematical problem as both the geometric and the algebraic category need to be moved into higher categories. Naively a cobordism with corners is a 2–category, by viewing the corners as objects, the boundaries as cobordisms between them and the cobordism as a cobordism of cobordisms. The devil is of course in the details. There have been several approaches to the problem such as Lurie’s approach [Lur09], [FHLT10], [SP09] and the project [DSPS].

Taking a step back to the 1+1 dimensional situation, the TFTs with boundaries have been nicely characterized and given rise to new axioms such as the Cardy axiom, see e.g. [LP08] for a nice introduction or [KP06] for a model free approach. Here the objects are not quite cobordisms with corners, but more simply surfaces with boundaries and points on the boundary.

In this paper we will give a constructive solution to the problem by augmenting the setup of Reshetikhin–Turaev using ideas along the lines above. In order to do this we use an algebraic and a geometric 2–category and define a functor between them. The algebraic 2–category
is that of 2–vector spaces \[KV94\]. The geometric 2–category is constructed extending the category of cobordisms with framing of RT. For this analogous to RT we construct both a 2–category of cobordisms \(C_2\) and a category of special ribbon graphs \(Srg\).

In order to achieve this task we have to refine the standard notion of framing in order to take into account compositions on the surface level (1–morphism). This leads to a decoration essentially by separating curves, arcs, boundaries and genus which is explained in detail in Section 3.1. When computing the invariants, we have to “cap off” the surfaces to reduce to the surface without boundaries. On the ribbon side, we have to use not only the standard diagrams, but also use rainbow diagrams.

Our work is intended as a bridge between several subjects. The presentation of the categories was chosen to match with the considerations of string topology \[CS09\] in the formalism of \[Kau07, Kau08\] which will hopefully lead to some new connections between the two subjects. The inclusion of the arcs and curves connects the RT theory to the arc and curve complexes, see the overview \[MP12\] and we again expect synergies from this connection.

Once the classification of \[DSPVB\] is available, it will be interesting to see how our concrete realization of 3-2-1 TQFT fits. Their results and those of \[DSPS\] are for a different target category. We will provide a link in Section 5.1. Further studies of 3-2-1 extensions but for Turaev-Viro are in \[TV10\] and in \[BK10, Ball10, Ball11\]. It would be interesting to relate these to our construction using the relationship between RT and TV invariants \[TV10\].

The paper is organized as follows. We will start with a review of the construction of the Reshetikhin–Turaev TQFT in Section 2.1. This will serve as a quick reference of notations and definitions of \[Tur10\]. In Section 2.2 the construction of the RT TQFT will be modified. This modification is essentially the same as the original RT TQFT and not intended to extend the RT TQFT but it is this modified form of the RT TQFT that will be extended to the cobordisms with corners.

The discussion of an extension of the RT TQFT to the cobordisms with corners will starts from Section 3. In Section 3 we will introduce two 2–categories \(C_2\) and \(Srg\). The 2–category \(C_2\) is the 2–category of cobordisms with corners and \(Srg\) is the 2–category of the special ribbon graphs. These two 2–categories are closely related, in fact we will define gluings of cobordisms using the gluing of the special ribbon graphs. By definitions of these 2–categories, there will be a 2–functor \(F : C_2 \to Srg\).
In Section 4, we will introduce the Kapranov-Voevodsky 2–vector spaces 2–Vect as the target 2–category of the extended TQFT. The 2–functor (with anomaly) $G : Srg \to 2–Vect$ will be defined and several compatibility of gluing will be proved.

Finally, we will define the extended TQFT as the composition of $F : Co \to Srg$ and $G : Srg \to Co$ and state how this reduce to the RT TQFT in the case of cobordisms without corners in Section 5.

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2. The RT TQFT

2.1. The original RT TQFT. We review the construction of the Reshetikhin–Turaev topological quantum field theory (the RT TQFT) defined in [Tur10]. For the full description of the RT TQFT, see [Tur10]. In this article, a TQFT is always the Reshetikhin–Turaev type TQFT. We restrict ourselves considering only cobordisms between connected surfaces. All manifolds are assumed to be orientable. If the reader is familiar with the notation and construction of the RT TQFT in [Tur10], one may skip to Section 2.2.

2.1.1. A modular category. A modular category is an input for the RT TQFT. The objects of a modular category are used as decorations of surfaces (explained more detail later). Here we set several notations and state an important lemma.

Let $\mathcal{V}$ be a modular category with a finite set $\{V_i\}_{i \in I}$ of simple objects, where $I$ is a finite index set $I = \{1, 2, \ldots, k\}$. We assume that $V_1$ is the unit object $1$ of $\mathcal{V}$. The ring $K := \text{Hom}(1, 1)$ is called the ground ring. The ground ring $K$ is known to be commutative. We assume that $\mathcal{V}$ has an element $D$ called a rank of $\mathcal{V}$ given by the formula

$$D^2 = \sum_{i \in I} (\text{dim}(V_i))^2.$$ 

Besides a rank $D$, we need another element $\Delta$ defined as follows. The modular category has a twist $\theta_V : V \to V$ for each object of $\mathcal{V}$. Since $V_i$ is a simple object, the twist $\theta_{V_i}$ acts in $V_i$ as multiplication by a
certain \( v_i \in K \). Since the twist acts via isomorphism, \( v_i \) is invertible in \( K \). We set
\[
\Delta = \sum_{i \in I} v_i^{-1} (\dim(V_i))^2 \in K.
\]

The following lemma is very important.

**Lemma 1.** For any objects \( V, W \) of the category \( V \), there is a canonical \( K \)-linear splitting
\[
\Hom(\mathbb{1}, V \otimes W) = \bigoplus_{i \in I} (\Hom(\mathbb{1}, V \otimes V_i^*) \otimes_K \Hom(\mathbb{1}, V_i \otimes W))
\]
The isomorphism \( u \) transforming the right-hand side into the left-hand side is given by the formula
\[
(1) \quad u_i : x \otimes y \mapsto (\id_V \otimes d_{V_i} \otimes \id_W)(x \otimes y),
\]
where \( x \in \Hom(\mathbb{1}, V \otimes V_i^*) \), \( y \in \Hom(\mathbb{1}, V_i \otimes W) \). The map \( u_i \) is given graphically as in Figure 1.

For a proof, see Lemma 2.2.2 in [Tur10].

![Figure 1. The map \( u_i \)](image)

2.1.2. **Decorated surfaces and decorated cobordisms.** We define decorated surfaces and decorated cobordisms. These are the objects and morphisms of the cobordism category.

**Definition 2.** A **decorated surface** is a connected compact closed surface with ordered finite oriented arcs on it such that each such arc is decorated, namely, each arc encodes a pair \((W, \nu)\), where \( W \) is an object of \( V \) and \( \nu \) is either 1 or \(-1\). Decorated surfaces will often be called **\( d \)-surfaces** for the sake of brevity. The pair \((W, \nu)\) is called a mark, and \( W \) and \( \nu \) are called a label and a sign, respectively.

**Definition 3.** A **\( d \)-homeomorphism** of \( d \)-surfaces is a degree 1 homeomorphism of the underlying surfaces preserving the distinguished arcs together with their orientations, marks, and order.
Definition 4. For non-negative integer \( g \), an object \( W_i \) of \( \mathcal{V} \) and \( \nu_i \in \{1, -1\}, 1 \leq i \leq m \), a tuple \( (g; (W_1, \nu_1), \ldots, (W_m, \nu_m)) \) is called a the \textit{decorated type}, or in short, a \textit{type}.

Definition 5. For a \( d \)-surface \( \Sigma \) with \( m \) arcs, the \textit{decorated type}, or in short, a \textit{type} of \( \Sigma \) is defined to be a tuple \( (g; (W_1, \nu_1), \ldots, (W_m, \nu_m)) \), where \( g \) is the genus of \( \Sigma \) and \( (W_i, \nu_i) \) is a mark on the \( i \)-th arc on \( \Sigma \).

For any type \( t = (g; (W_1, \nu_1), \ldots, (W_m, \nu_m)) \), there is canonical \( d \)-surface \( \Sigma_t \) of type \( t \), called a standard \( d \)-surface of type \( t \). The sketch of the construction is as follows. Let \( R_t \) denote the ribbon \((0, m)\)-graph in \( \mathbb{R}^3 \) presented by the diagram in Figure 2.

\[ \begin{array}{c}
(W_1, \nu_1) \quad \cdots \quad (W_m, \nu_m)
\end{array} \]

\[ \begin{array}{c}
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\begin{array}{c}
\text{Figure 2. } R_t
\end{array}
\end{array} \]

Fix a closed regular neighborhood \( U_t \) of \( R_t \) in the strip \( \mathbb{R}^2 \times [0, 1] \) so that the only intersections of \( \partial U_t \) and \( R_t \) are at the top of vertical bands. Then \( U_t \) is a handlebody of genus \( g \) and its boundary \( \partial U_t \) becomes a \( d \)-surface of type \( t \) as follows. The intersection of the vertical band with color \( (W_i, \nu_i) \) and \( \partial U_t \) is defined to be the \( i \)-th arc with a mark \( (W_i, \nu_i) \).

Definition 6. For any type \( t = (g; (W_1, \nu_1), \ldots, (W_m, \nu_m)) \), the \( d \)-surface constructed above is called the \textit{standard} \( d \)-surface of type \( t \) and denoted by \( \Sigma_t \).

Definition 7. A \( d \)-surface \( \Sigma \) of type \( t \) is said to be \textit{parametrized} if it is equipped with a \( d \)-homeomorphism \( \Sigma_t \to \Sigma \). This homeomorphism is called a parametrization of \( \Sigma \).
Definition 8. A decorated 3–manifold is a compact oriented 3–manifold with parametrized decorated boundary and with a $v$–colored ribbon graph sitting in this 3–manifold. A $d$–homeomorphism of decorated 3–manifolds is a degree 1 homeomorphism of 3–manifolds preserving all additional structures in question.

Let $t = (g; (W_1, \nu_1), \ldots, (W_m, \nu_m))$ be a decorated type. For $i = (i_1, \ldots, i_m) \in I^g$, we assign the simple object $V_{ij}$ to the $j$-th cap like bands in the ribbon graph $R_t$ for $j = 1, \ldots, m$. Each color $x$ of the coupon in $R_t$ determines a $v$–coloring of $R_t$. The handlebody $U_t$ with this $v$–colored ribbon graph inside is a decorated 3–manifold. It is bounded by the standard $d$–surface $\Sigma_t$ with the identity parametrization.

Definition 9. The decorated 3–manifold constructed above is called a standard handlebody and denoted by $H(U_t, R_t, i, x)$.

Definition 10. A decorated 3–cobordism is a triple $(M, \partial_–M, \partial_+M)$, where $\partial_–M$ and $\partial_+M$ are parametrized $d$–surfaces and $M$ is a decorated 3–manifold with $\partial M = (-\partial_–M) \sqcup \partial_+M$.

In this paper, we assume boundary surfaces $\partial_–M$ and $\partial_+M$ are connected.

The cobordisms category $\text{Co}_{\text{RT}}$ in the sense of Reshetikhin–Turaev consists of decorated surfaces as objects and $d$–homeomorphic classes of decorated cobordisms as morphisms. This is the symmetric monoidal category. The monoidal structure is given by disjoint unions.

2.1.3. The RT TQFT. Now that we defined the cobordism category we can construct a TQFT. A TQFT is often defined to be a symmetric monoidal functor from the cobordism category to an algebraic category. In the Reshetikhin–Turaev construction, an algebraic category is a category of projective $K$–modules of finite type. In this paper, however, we deal with the RT TQFT with anomaly, that is the RT TQFT does not preserve the composition of morphisms. Instead, we have the following lemma from [Tur10, Lemma 2.1.2].

Lemma 11. Let $\tau$ be the RT TQFT. If a decorated 3–cobordism $M = M_2M_1$ is obtained from decorated 3–cobordisms $M_1$ and $M_2$ by gluing along a $d$–homeomorphism $p: \partial_+(M_1) \rightarrow \partial_–(M_2)$ commuting with parametrizations then for some invertible $k \in K$,

$$\tau(M) = k\tau(M_2)\tau(M_1).$$

This $k$ is called the (gluing) anomaly.
Thus, the RT TQFT under consideration is not a functor in categorical sense. Instead of defining a TQFT in terms of categorical language, Turaev defined a TQFT \((\tau, T)\) in his book \([\text{Tur10}]\) as assignments of a projective space \(T(\Sigma)\) to a \(d\)-surface \(\Sigma\) and assignments of a homomorphism to a decorated cobordism satisfying several axioms. The next task is to review these assignments.

A TQFT \((\tau, T)\) is the following assignments. For each decorated (connected) surface \(\Sigma\) with a decorated type \(t = (g; (W_1, \nu_1), \ldots, (W_m, \nu_m))\), we define a projective \(K\)-module \(T(\Sigma)\) as follows. For \(i = (i_1, \ldots, i_g) \in I^g\), we set

\[
\Phi(t; i) = W_1^{i_1} \otimes W_2^{i_2} \otimes \cdots \otimes W_m^{i_m} \otimes \bigotimes_{r=1}^g (V_{i_r} \otimes V_{i_r}^*),
\]

where \(W^{+1} = W\) and \(W^{-1} = W^*\). We define

\[
T(\Sigma) = \bigoplus_{i \in I^g} \text{Hom}(1, \Phi(t; i)).
\]

This is seen to be a projective \(K\)-module. Let \((M, \partial_- M, \partial_+ M)\) be a decorated cobordism with \(\partial M = (-\partial_- M) \sqcup \partial_+ M\), where \(-\partial_- M\) is \(\partial_- M\) with the opposite orientation. We define a \(K\)-homomorphism

\[
\tau(M) = \tau(M, \partial_- M, \partial_+ M) : T(\partial_- M) \to T(\partial_+ M).
\]

The idea of the construction of \(\tau(M)\) is that we glue standard handlebodies along the boundary of \(M\) via the its parametrizations. Then \(M\) becomes a close 3-manifold with a ribbon graph inside. The new coupons attached along with the standard handlebodies are not colored. The elements of \(K\)-module \(T(\partial_- M)\) and \(T(\partial_+ M)\) can be regarded as colors of coupons in the handlebodies. Then we apply the Reshetikhin–Turaev invariant for closed 3-manifolds with \(v\)-colored ribbon inside \([\text{Tur10} \text{ p.82}]\). To give an explicit formula of \(\tau(M)\), we first need to introduce special ribbon graphs.

2.1.4. Special Ribbon Graphs. We follows the definition of a special ribbon graph in \([\text{Tur10}]\). Let \(m^-, g^-, m^+, g^+\) be non-negative integers and let \(k^- = m^- + 2g^-\) and \(k^+ = m^+ + 2g^+\). Let \(\Omega\) be a ribbon \((k^-, k^+)-\text{graph}\) in \(\mathbb{R}^3\). There are following three conditions that \(\Omega\) needs to satisfy to be a special ribbon graph.

(i) For all odd \(i\) with \(1 \leq i \leq 2g^- - 1\), the bottom boundary intervals

\([m^- + i - \epsilon, m^- + i + \epsilon] \times 0 \times 0, [m^- + i + 1 - \epsilon, m^- + i + 1 + \epsilon] \times 0 \times 0\)

are the bases of a band \(e^-_i\) of \(\Omega\) directed towards the left base.
(ii) For all odd $j$ with $1 \leq j \leq 2g^+ - 1$, the top boundary intervals 
$[m^+ + j - \epsilon, m^+ + j + \epsilon] \times 0 \times 1, [m^+ + j + 1 - \epsilon, m^+ + j + 1 + \epsilon] \times 0 \times 1$
are the bases of a band $e_j^+$ of $\Omega$ directed towards the right base.

(iii) These $g^- + g^+$ bands are not colored, all other bands and all
coupons of $\Omega$ are colored. Certain (but not necessarily all)
annuli of $\Omega$ may be colored.

**Definition 12.** With the notation above, $\Omega$ is called a (partially $v$–colored) *special ribbon graph* if it satisfies the three conditions (i)-(iii) above.

The bottom type and the top type of a special ribbon graph can be
defined from the colors and directions of vertical bands near the bottom
interval and the top interval respectively. The purpose of introducing
the notion of special ribbon graphs is the following lemma.

**Lemma 13.** Each special ribbon graph $\Omega$ with the bottom type $t^-$ and
the top type $t^+$ gives rise to a decorated 3–cobordism $(M, \Sigma_-, \Sigma_+)$,
where the parametrized $d$–surfaces $\Sigma_-, \Sigma_+$ have types $t^-, t^+$, respectively. We say that the resulting 3–cobordisms is *represented* by $\Omega$. Conversely, any decorated 3–cobordisms considered up to homeomor-
phism may be presented by a special ribbon graph in $\mathbb{R}^3$.

This extends the Lickorish–Wallace theorem that every closed 3–
manifold is obtained by surgery along a framed link in $S^3$.

2.1.5. *The explicit formula for $\tau(M)$*. Let $M$ be a decorated cobor-
dism represented by a special ribbon graph $\Omega$ whose bottom type is
$t^-$ and top type is $t^+$. We now give an explicit formula for computing
the homomorphism $\tau(M) : \mathcal{T}(\partial_- M) \to \mathcal{T}(\partial_+ M)$ from the operator
invariants of $\Omega$. The operator invariant $\mathcal{F}$ is a covariant functor from
the category of ribbon graphs to $\mathcal{V}$. (See [Tur10, Theorem 2.5] for
detail.) With respect to the splittings $\mathcal{B}$ of $\mathcal{T}(\partial_- M)$ and $\mathcal{T}(\partial_+ M)$
the homomorphism $\tau(M)$ is presented by a block matrix $\tau^i_j$, where $i$
runs over sequences $i = (i_1, \ldots, i_{g^-}) \in I_g^-$ and $j$ runs over sequences
$j = (j_1, \ldots, j_{g^+}) \in I_g^+$. Each such sequence $i \in I_g^-$ determines a
coloring $e_n^- \mapsto i_n$ of the uncolored cap-like bands of $\Omega$ incident to its
top boundary. Similarly, each sequence $j \in I_g^+$ determines a coloring
$e_n^+ \mapsto j_n$ of the uncolored rainbow cup-like bands of $\Omega$ incident to its
top boundary. Therefore a pair $(i \in I_g^-, j \in I_g^+)$ determines a coloring
of uncolored bands of $\Omega$. Let $L = L_1 \cup \cdots \cup L_m$ be the framed link
formed by the uncolored annuli of $\Omega$. Every $\lambda \in \text{col}(L)$ determines (to-
gether with $i$ and $j$) a $v$–coloring of $\Omega$. Denote the resulting $v$–colored
ribbon graph in $\mathbb{R}^3$ by $(\Omega, i, j, \lambda)$. The operator invariant $\mathcal{F}(\Omega, i, j, \lambda)$ is a morphism of $\mathcal{V}$ from $\Phi(t^-; i)$ to $\Phi(t^+; j)$. The composition of a morphisms $1 \rightarrow \Phi(t^-; i)$ with $\mathcal{F}(\Omega, i, j, \lambda)$ defines a $K$-linear homomorphism $\text{Hom}(1, \Phi(t^-; i)) \rightarrow \text{Hom}(1, \Phi(t^+; j))$ denoted by $\mathcal{F}_0(\Omega, i, j, \lambda)$. Finally we can state the formula calculating the $K$-homomorphism $\tau(M)$.

**Theorem 14.** The $(i, j)$ block-matrix $\tau^j_i$ of $\tau(M)$ is calculated by the formula:

$$
(5) \quad \tau^j_i = \Delta^{\sigma(L)} D^{g^+-\sigma(L)-m} \dim(j) \sum_{\lambda \in \text{col}(L)} \dim(\lambda) \mathcal{F}_0(\Omega, i, j, \lambda),
$$

where $\sigma$ is the signature of $L$ and

$$
\dim(j) = \prod_{l=1}^{g^+} \dim(V_{j_l}),
$$

$$
\dim(\lambda) = \prod_{n=1}^m \dim(\lambda(L_n)).
$$

### 2.2. A modification of the RT TQFT.

Here we slightly modify the construction of the RT TQFT described above. The modified version of the TQFT is essentially the same as the original RT TQFT and it is still a non-extended TQFT. The modification is made so that the setting become more appropriate to extend the RT TQFT to the category of cobordisms with corners. We may use the same terminologies as before but we use newer definitions in the rest of the paper.

Let us first modify the concept of decorated type. The decorated type defined in Definition 4 is of the form $t = (g; (W_1, \nu_1), \ldots, (W_m, \nu_m))$. Here $g$ represents the genus of a surface $\Sigma$ and its effect in $\mathcal{T}(\Sigma)$ is the object $\bigotimes_{i=1}^g (V_i \otimes V_i^*)$ of $\mathcal{V}$ (see (2)). This object is used to color cap like bands (also cup like bands) like in Figure 3 to calculate $\tau(M)$.

When we consider an extension of the RT TQFT to cobordisms with corners, it is useful to consider “rainbow” like bands like in Figure 4. The reason that rainbow like bands appear is as follows. When we consider the horizontal gluing of surfaces, it will corresponds to the horizontal gluing of special ribbon graphs. If we use non-rainbow like bands, we need to rearrange the order of bands after two special ribbon graphs are glued horizontally. To avoid this rearrangements, we choose to use the rainbow like bands.

Using the rainbow like bands is the same as the idea of partitioning the genus $g$ into several positive integers. This leads to the following new definitions of decorated types.
Definition 15. A decorated type is the tuple \( t = (a_1, a_2, \ldots, a_p) \), where \( a_i \) is either a positive integer or a tuple \((W, \nu)\), where \( W \) is an object of \( V \) and \( \nu \) is a sign \( \pm 1 \).

Turaev's decorated type is canonically considered as a new decorated type by regarding Turaev's decorated type \( (g; (W_1, \nu_1), \ldots, (W_m, \nu_m)) \) as the decorated type \( ((W_1, \nu_1), \ldots, (W_m, \nu_m), 1, 1, \ldots, 1) \), where the number of 1 is \( g \). The new definition allows the permutations of the elements in the tuple.

Next, we modify the definition of decorated surfaces. Let \( \Sigma \) be a closed compact 3-manifolds with disjoint separating curves \( c_1, \ldots, c_{p+1} \) for some integer \( p \) on \( \Sigma \). Each \( c_i \) is a simple curve homeomorphic to a circle. Let \( \Sigma_i, i = 0, \ldots, p \) be connected components of the complement
of $c_i$’s, namely
\[
\bigcup_{i=0}^{p+1} \Sigma_i = \Sigma \setminus \bigcup_{i=1}^{p+1} c_i.
\]
We index these curves $c_i$ so that the boundaries of $\Sigma_i$’s are $\partial \Sigma_0 = c_1$, $\partial \Sigma_i = c_i \sqcup c_{i+1}$ for $i = 1, \ldots, p$ and $\partial \Sigma_{p+1} = c_{p+1}$.

**Definition 16.** With the convention above, $\Sigma$ is called a *decorated surface* or in brevity a *d–surface* if $\Sigma_0$ and $\Sigma_{p+1}$ have no marks and genera zero and for $i = 1, \ldots, p$ each $\Sigma_i$ is either genus zero and has a marked arc on it or genus greater than 0 and has no marked arcs on it.

The definitions of decorated 3–manifolds and decorated cobordisms are modified accordingly.

It might seem that the components $\Sigma_0$ and $\Sigma_{p+1}$ are unnecessary. This is so defined to take an extension to surfaces with boundary in account later.

**Definition 17.** A *decorated type*, or in short *type* of a decorated surface $\Sigma$ is the tuple $t = (a_1, a_2, \ldots, a_p)$, where $a_i$ is the mark of the arc of $\Sigma_i$ if it exists or $a_i$ is the genus of $\Sigma_i$ if there is no marked arc on $\Sigma_i$.

Finally we modify the definition of $\Phi$ in (2) as follows. For a positive integer $a$ and $i = (i_1, \ldots, i_a) \in I^a$, we set
\[
H^a_i = V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_a} \otimes V_{i_a}^* \otimes \cdots \otimes V_{i_2}^* \otimes V_{i_1}^*.
\]
If $a = (W, \nu)$ is a signed object of $\mathcal{V}$, we set $I^a$ to be a set of only one element and set $H^a_i = W^\nu$ for $i \in I^a$. Note that the tensor product in (6) can be used as a color for rainbow like bands. Let $\Sigma$ be a decorated (connected) surface with a decorated type $t = (a_1, a_2, \ldots, a_p)$. We write
\[
I^t := I^{a_1} \times \cdots \times I^{a_p}.
\]
For $\zeta = (\zeta_1, \ldots, \zeta_p) \in I^t$ with $\zeta_1 = (\zeta_1^1, \ldots, \zeta_1^{a_1}), \ldots, \zeta_p = (\zeta_p^1, \ldots, \zeta_p^{a_p})$, we set
\[
\Phi(t; \zeta) = H_{\zeta_1}^{a_1} \otimes H_{\zeta_2}^{a_2} \otimes \cdots \otimes H_{\zeta_p}^{a_p}.
\]
We define
\[
\mathcal{T}(\Sigma) = \bigoplus_{\zeta \in I^t} \text{Hom}(\mathbb{1}, \Phi(t; \zeta)).
\]

In addition to the standard surfaces in \cite{Turaev10}, we add the standard surfaces corresponding to rainbow like bands. The special ribbon graphs is also extended to include rainbow like bands. All arguments of the construction of the RT TQFT in \cite{Turaev10} can be modified with newly defined decorated surfaces. For instance, Theorem 14 is modified as follows.
Theorem 18. Let $M$ be a decorated cobordism represented by a special ribbon graph $\Omega$ whose bottom type is $t^- = (a_1, \ldots, a_p)$ and top type is $t^+ = (b_1, \ldots, b_q)$. Suppose the number of the uncolored annuli of $\Omega$ is $m$. For $\zeta \in \Gamma t^-$ and $\eta \in \Gamma t^+$, the block matrix $\tau_{\zeta}^{\eta}$ is given by the formula

$$\tau_{\zeta}^{\eta} = \Delta^\sigma(L) D^{g^+ - \sigma(L) - m} \sum_{\lambda \in \text{col}(L)} \dim(\lambda) F_0(\Omega, \zeta, \eta, \lambda),$$

where $g^+$ is the sum of the integer entries of $t^+$ and

$$\dim(\eta) := \prod_{i=1}^p \dim(\eta_i)$$

with

$$\dim(\eta_i) := \begin{cases} \prod_{i=1}^{a_i} \dim(\zeta_i^i) & \text{if } a_i \in \mathbb{Z} \\ 1 & \text{if } a_i \text{ is a mark.} \end{cases}$$

3. An Extension to Cobordisms with Corners

3.1. Decorated surfaces. We extend the concept of decorated surface to a decorated surface with boundary. We first extend the definition of decorated types of Definition 15.

Definition 19. A decorated type, or for brevity type, is a tuple $t = (m, n; a_1, a_2, \ldots, a_p)$ such that $m, n$ are non negative integers and $a_i$ is either a positive integer or a signed object $(W, \nu)$ of $V$.

A decorated type $t = (a_1, a_2, \ldots, a_p)$ in the sense of Definition 15 is canonically regarded as the newly defined decorated type by identifying $t$ with the type $(0, 0; a_1, \ldots, a_p)$. Thus this definition extends the Definition 15.

Let $\Sigma$ be a compact 3–manifolds with boundaries and disjoint separating curves $c_1, \ldots, c_{p+1}$ for some integer $p$ on it. Each $c_i$ is a simple curve homeomorphic to a circle. For $i = 0, \ldots, p+1$, let $\Sigma_i$ be connected components of the complement of $c_i$’s, namely

$$\bigcup_{i=0}^{p+1} \Sigma_i = \Sigma \setminus \bigcup_{i=1}^{p+1} c_i.$$

We index these curves $c_i$ so that the boundaries of $\Sigma_i$’s are $\partial \Sigma_0 = \partial_m \Sigma \sqcup c_1$, $\partial \Sigma_i = c_i \sqcup c_{i+1}$ for $i = 1, \ldots, p$ and $\partial \Sigma_{p+1} = c_{p+1} \sqcup \partial_{\text{out}} \Sigma$, where $\partial_m \Sigma$ and $\partial_{\text{out}} \Sigma$ are sets of boundary circle such that $\partial \Sigma = \partial_m \Sigma \sqcup \partial_{\text{out}} \Sigma$.

Definition 20. With the convention above, $\Sigma$ is called a decorated surface or in brevity a $d$–surface if $\Sigma_0$ and $\Sigma_{p+1}$ have no marks and genera zero and for $i = 1, \ldots, p$ each $\Sigma_i$ is either genus zero and has a marked arc on it or genus greater than 0 and has no marked arcs on it.
Definition 21. A *decorated type*, or in short a *type*, of a decorated surface $\Sigma$ is the tuple $t = (m, n; a_1, a_2, \ldots, a_p)$ such that $m, n$ are the numbers of connected components of $\partial_\text{in} \Sigma, \partial_\text{out} \Sigma$ respectively, and $a_i$ is the mark of the arc of $\Sigma_i$ if it exists or $a_i$ is the genus of $\Sigma_i$ if there is no marked arc on $\Sigma_i$.

See Figure 5 for an example with type $(2, 3; (W_1, \nu_1), 1, (W_2, \nu_2), 3, (W_3, \nu_3), 2)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{A decorated surface with boundary}
\end{figure}

Let $J$ be an index set so that $a_j$ is a positive number for $j \in J$. The genus of a decorated surface is equal to $\sum_{i \in J} a_i$. We can think of the integer $m$ in $t$ is the number of left/inboundary circles and $n$ is the number of right/outboundary circles.

Next, for each type $t = (m, n; a_1, \ldots, a_p)$ we define a standard surface with boundary. First we define a “block” of a ribbon graph inside a standard surface with boundary. For a mark $a = (W, \nu)$, the block for $(W, \nu)$ is defined to be a $1 \times 1$ square coupon in $\mathbb{R}^2$ and and a short band attached to the top of it and the band is colored by $W$ if $\nu = 1$ and $W^*$ if $\nu = -1$. See Figure 6.

The block for a positive integer $a$ consists of a $1 \times 1$ square coupon in $\mathbb{R}^2$ and rainbow like bands with $a$ bands on the top of the square. These bands are not colored and their cores are oriented from right to left. See Figure 7.

For the integer $x \in \{m, n\}$ corresponding to the either of the first two entries in a type, the block for $x$ is defined to be a $1 \times 1$ square coupon and $x$ vertical uncolored bands. For a type $t = (m, n; a_1, \ldots, a_p)$, Let $R_t$ be a ribbon graph in $\mathbb{R}^3$ constructed by arranging, in the strip
Consider a closed regular neighborhood $U_t$ of $R_t$ so that it intersects with $\Omega$ only at the top of all bands. The intersections of $\partial U_t$ and the bands of $\Omega$ induce the arcs on $\Sigma_t$ and it is colored if the bands have marks as in the Turaev’s case. Let $c_i = \partial U_t \cap P_i$, where $P_i$ is the plane in $\mathbb{R}^3$ with the fixed $x$-coordinates $i$ for $i = 1, \ldots, p + 1$. Then $c_i$ are separating curves on $\partial U_t$ such that the boundaries of complements $\Sigma_i$, $i = 0, \ldots, p + 1$.
Definition 22. The resulting surface with marks and separating curve is called the standard surface of type \((m,n; a_1,\ldots, a_p)\).

Definition 23. For decorated types

\[ t_1 = (m_1, n_1; a_1, a_2, \ldots, a_p) \] and \[ t_2 = (m_2, n_2; b_1, b_2, \ldots, b_q), \]

if \(n_1 = m_2\) then we define a composition \(t_1 \circ t_2\) to be the type

\[ (11) \quad t_1 \circ t_2 := (m_1, n_2; a_1, a_2, \ldots, a_p, n_1 - 1, b_1, b_2, \ldots, b_q). \]

Here if we have \(n_1 = 1\) then we delete the term \(n_1 - 1\) from the sequence.

3.2. 2-categories.

Definition 24. A 2-category is a category enriched over the category of categories and functor. We call the usual morphism 1-morphism and call the morphisms in a category of 1-morphisms 2-morphisms.

This, in particular, means that beside the composition of 1-morphisms there are horizontal compositions and vertical compositions of 2-morphisms and these compositions satisfy the interchange law. The 2-category is strict in a sense that the compositions are all associative.
3.3. Cobordisms with corners. We define a 2–category $\text{Co}$ of cobordisms with corners. An object of $\text{Co}$ is ordered oriented circles with base points for each circle. A circle with a base point is a circle with a specific point on it.

Let $X$ and $Y$ be objects of $\text{Co}$ and suppose $X$ has $m = |X|$ components and $Y$ has $n = |Y|$ components. A 1–morphism $S$ from $X$ to $Y$ is a connected oriented compact $d$–surface of decorated type $(m, n; a_1, \ldots, a_2)$ whose boundary is $-X \sqcup Y$, where $-X$ is $X$ with orientation reversed. We denote this by $S : X \to Y$.

Let $S$ and $T$ be 1–morphisms from $X$ to $Y$ with types $s$ and $t$. We define a 2–morphism from $S$ to $T$. Let $S_1$ and $T_1$ be parametrized $d$–surfaces with parametrizations $f_1$ and $g_1$, respectively. Let $M_1$ be an oriented 3–manifold with a ribbon graph inside of it such that $\partial M_1 = (-S_1 \sqcup T_1) \cup (\partial S_1 \times [0, 1])$. Here we identify $\partial (-S_1) = \partial S_1 \times \{0\}$ and $\partial T_1 = \partial S_1 \times \{1\}$. We call $S_1$ and $T_2$ vertical boundaries and call $\partial S_1 \times [0, 1]$ horizontal boundaries of $M$. A ribbon graph in $M$ intersects with $\partial M$ only at the marked arcs on $S$ and $T$. Two such data $(M_1, f_1, g_1)$ and $(M_2, f_2, g_2)$ are equivalent if there is an orientation preserving homeomorphism $h$ such that it commutes with parametrizations, i.e., $h \circ f_1 = f_2$ and $h \circ g_1 = g_2$ and also $h$ preserve the ribbon graphs. This is clearly an equivalence relation. A 2–morphism from $S$ to $T$ is defined to be an equivalence class of $(M, f, g)$. We write this as $(M, f, g) : S \Rightarrow T$, or when there is no confusion, we simply write $M : S \Rightarrow T$. Note that 1–morphisms are, by definition, not assumed to have parametrizations and parametrizations are given on the level of 2–morphisms.

To check $\text{Co}$ is actually a 2–category, we need to specify several compositions. If $S : X \to Y$ and $T : Y \to Z$ are 1–morphisms then the composition $S \circ T$ is defined to be a decorated surface defined as follows. If $Y$ is the empty set, then we define the composition $S \circ T$ to be the empty set. If $Y$ is not empty, then as a surface $S \circ T$ is obtained by gluing $S$ and $T$ along the common boundary $Y$. Let

$$s = (m, n; a_1, a_2, \ldots, a_p) \quad \text{and} \quad t = (n, l; b_1, b_2, \ldots, b_q)$$

be the types of $S$ and $T$, respectively and let $c_1, \ldots, c_p$ and $c'_1, \ldots, c'_q$ be the separating curves of $S$ and $T$, respectively. The curves $c_1, \ldots, c_p$ and $c'_1, \ldots, c'_q$ and the marked arcs on $S$ and $T$ naturally induce separating curves and the marked arcs on $S \circ T$. If $n = 1$, we delete $c'_1$ and use the rest for the separating curves for $S \circ T$. The decorated type of
$S \circ T$ is the composition of the types $s$ and $t$, namely

$$s \circ t = \begin{cases} (m, l; a_1, \ldots, a_p, n - 1, b_1, \ldots, b_q) & \text{if } n > 1 \\ (m, l; a_1, \ldots, a_p, b_1, \ldots, b_q) & \text{if } n = 1. \end{cases}$$

The gluing of 2–morphisms will be defined later after introducing the special ribbon graph for cobordisms with corners.

A decorated surface with boundary is essentially equivalent to a decorated surface without boundary but with non-marked arcs on it. This can be seen as follows. Let $B^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ be a disk with standard orientation and let $S^1$ be its boundary with induced orientation. The point $(0, -1)$ is the base point of $B^2$. The segment $\{(x, 0) \mid -1 \leq x \leq 1\} \subset B^2$ is a (non-marked) arc of $B^2$. For each boundary component of a 1–morphism $S : X \to Y$ of Co, we glue the disk $B^2$ via an orientation and base points preserving homeomorphism of $S^1$ to the boundary component. (The choice of this homeomorphism does not matter. See Appendix III in [Tur10].) The resulting surface $\tilde{S}$ is closed and $\tilde{S}$ has marked arcs from $S$ and non-marked arcs from $B^2$'s. We call this process capping $S$.

Let $M$ be a 3–manifold with a ribbon graph such that

$$\partial M = (-S \sqcup T) \cup (\partial S \times [0, 1]),$$

where $S$ and $T$ are parametrized $d$–surfaces. Let $C = B^2 \times [0, 1]$ be a cylinder. The product of the arc in $B^2$ and $[0, 1]$ is a band in the cylinder $C$. The front side of the band is the one facing $(0, -1) \times [0, 1] \subset B^2 \times [0, 1] \subset \mathbb{R}^3$. For each circle $s$ in $X$ or $Y$, we glue the cylinder $C$ to $M$ along an orientation preserving homeomorphism from the boundary of $C$ to the component $s \times [0, 1]$ of horizontal boundaries. This homeomorphism should send the segment $(0, -1) \times [0, 1] \subset C$ to the segment $x \times [0, 1]$, where $x$ is the base point of the circle $s$. When the gluing is restricted to $B^2 \times \{0\}$ and $S$ in $M$, then it induces the capping $\tilde{S}$ of $S$. Similarly, when it is restricted to $B^2 \times \{1\}$ and $T$ in $M$, it induces the capping $\tilde{T}$ of $T$. Thus the resulting manifold $\tilde{M}$ is a cobordism from $\tilde{S}$ to $\tilde{T}$. The cobordism $\tilde{M}$ has a ribbon graph inside of it from $M$ and from $C$’s. Those newly added bands are not colored.

In the capped setting, the standard surfaces are also capped. Actually, when we defined the standard surface with corners, we cut out the neighborhoods of uncolored arcs. The upped version of the standard surface is the one that the neighborhoods are not removed. Thus the parametrization of boundary surfaces $S$ and $T$ of a cobordism with corners $M : S \Rightarrow T$ can be extended to parametrizations of $\tilde{S}$ and $\tilde{T}$. 
3.4. **Special ribbon graphs.** We extend the definition of a special ribbon graph. A special ribbon graph in [Tur10] has a bottom type \( t^- = (0,0;a_1,\ldots,a_p) \) and a top type \( t^+ = (0,0;b_1,\ldots,b_q) \) with notation defined in this article. We define a special ribbon graph whose bottom type is \( t^- = (m,n;a_1,\ldots,a_p) \) and a top type \( t^+ = (m,n;b_1,\ldots,b_q) \). This special ribbon graph is obtained from a special ribbon graph \( \Omega \) whose bottom type \( a(t^-) = (0,0;a_1,\ldots,a_p) \) and whose top type \( a(t^+) = (0,0;b_1,\ldots,b_q) \) by adding a \( m \) vertical bands directed upwards to the left of \( \Omega \) and \( n \) vertical bands directed downwards on the right of \( \Omega \). These newly added bands are not colored. See Figure 9.

As an old special ribbon graph represents a Turaev's decorated cobordism, a new special ribbon graph represents a decorated cobordisms with partially \( v \)-colored ribbon graph. (The newly added bands are not colored.) Cutting out the neighborhood of these uncolored bands, we have a decorated cobordism with corners. Two special ribbon graphs are equivalent if they represent \( d \)-homeomorphic decorated cobordisms.

We define a 2–category \( \text{Srg} \) of special links as follows. A object of \( \text{Srg} \) is a nonnegative integer. Let \( n \) and \( m \) be objects of \( \text{Srg} \), namely \( n, m \in \mathbb{Z}_{\geq 0} \). A 1–morphism from \( n \) to \( m \) is a decorated type \( t = (n,m;a_1,\cdots,a_p) \). Let \( t_1 \) and \( t_2 \) be decorated types from \( n \) to \( m \). A 2–morphism from \( t_1 \) to \( t_2 \) is an equivalent class of special ribbon graph \( \Omega \) whose bottom type is \( t_1 \) and whose top type is \( t_2 \). We abuse the
notation and denote by $\Omega$ the equivalence class of the special ribbon graph $\Omega$.

**Definition 25.** The vertical composition $\Omega_1 \cdot \Omega_2$ of two special ribbon graphs $\Omega_1 : t_1 \Rightarrow t_2$ and $\Omega_2 : t_2 \Rightarrow t_3$ is a concatenation of $\Omega_1$ and $\Omega_2$ along the top boundary of $\Omega_1$ and the bottom boundary of $\Omega_2$.

The vertical composition $\Omega_1 \cdot \Omega_2$ has the bottom type $t_1$ and the top type $t_2$, hence $\Omega_1 \cdot \Omega_2 : t_1 \Rightarrow t_3$.

**Definition 26.** The horizontal composition $\Omega_1 \circ \Omega_2$ of special ribbon graphs $\Omega_1 : t_1 \Rightarrow t_2 : l \rightarrow m$ and $\Omega_2 : t'_1 \Rightarrow t'_2 : m \rightarrow n$ is defined as follows. First, we arrange $\Omega_1$ on the left side of $\Omega_2$. There are $m$ vertical bands on the rightmost of $\Omega_1$ and $m$ vertical bands on the leftmost of $\Omega_2$. We replace these $2m$ ribbons by the ribbon graph defined in Figure 10. The resulting ribbon graph is a special ribbon graph and we call it the horizontal gluing of $\Omega_1$ and $\Omega_2$.

By definitions the bottom type and the top type of $\Omega_1 \circ \Omega_2$ are compositions of types $t_1 \circ t'_1$ and $t_2 \circ t'_2$, respectively. Thus

$$\Omega_1 \circ \Omega_2 : t_1 \circ t'_1 \Rightarrow t_2 \circ t'_2 : l \rightarrow n.$$ 

It is not hard to see that the vertical and horizontal compositions satisfy the interchange law and $\text{Srg}$ is a 2–category. This can be seen by noting that the difference of the two ways to glue four special ribbon graphs results in the identity cobordism.

3.5. **Cobordisms with corners represented by special ribbon graphs.** For a decorated cobordism, there is a special ribbon graph representing it (Lemma 13). We extend this to a decorated cobordism
with corners. Let \( M : S \Rightarrow T : X \rightarrow Y \) be a cobordism with corners (2–morphism of \( \text{Co} \)). Suppose that \( X \) has \( m = |X| \) circles and \( Y \) has \( n = |Y| \) circles. Let \( t_1, t_2 \) be the decorated types of \( S, T \) respectively. Capping horizontal boundaries, we have a decorated cobordism \( \tilde{M} \) (without corners). Then there exists a special ribbon graph \( \Omega \) representing \( \tilde{M} \). This special ribbon graph \( \Omega \) has bottom type \( t_1 \) and top type \( t_2 \). Thus there are \( m \) bands on the right part and \( n \) bands on the left part of \( \Omega \). We call \( \Omega \) a special ribbon graph representing \( M \). To recover \( M \) from \( \tilde{M} \), we cut out the neighborhoods of uncolored \( m + n \) bands of \( \Omega \). As in Turaev’s case, if \( M' \) is equivalent to \( M \), then their special ribbon are equivalent.

### 3.6. Horizontal and vertical gluings of cobordisms with corners

Now we define horizontal and vertical gluings of decorated cobordisms with corners that we postponed in Section 3.3.

**Definition 27.** Let \( M_1 : S \Rightarrow T : l \rightarrow m \) and \( M_2 : S' \Rightarrow T' : m \rightarrow n \) be 2–morphisms in \( \text{Co} \). Let \( \Omega_1 \) and \( \Omega_2 \) be special ribbon graphs representing \( M_1 \) and \( M_2 \), respectively. We define a horizontal gluing \( M_1 \circ M_2 \) of \( M_1 \) and \( M_2 \) to be a decorated cobordism represented by the horizontal gluing of \( \Omega_1 \) and \( \Omega_2 \).

By definitions, we have

\[
M_1 \circ M_2 : S \circ S' \Rightarrow T \circ T' : l \rightarrow n.
\]

**Definition 28.** Let \( M_1 : S \Rightarrow T : m \rightarrow n \) and \( M_2 : T \Rightarrow U : m \rightarrow n \) be 2–morphisms in \( \text{Co} \). Let \( \Omega_1 \) and \( \Omega_2 \) be special ribbon graphs representing \( M_1 \) and \( M_2 \), respectively. We define a vertical gluing \( M_1 \cdot M_2 \) of \( M_1 \) and \( M_2 \) to be a decorated cobordism represented by the vertical gluing of \( \Omega_1 \) and \( \Omega_2 \).

By definitions, we have \( M_1 \cdot M_2 : S \Rightarrow U : m \rightarrow n \).

With these compositions, \( \text{Co} \) is a 2–category.

### 3.7. A 2–functor \( F \) from \( \text{Co} \) to \( \text{Srg} \)

The construction of a 2–category \( \text{Co} \) uses essentially the structure of \( \text{Srg} \). In a precise statement, there is a 2–functor \( F \) from \( \text{Co} \) to \( \text{Srg} \). To an object \( X \) of \( \text{Co} \) consisting \( n \) circles, \( F \) assigns \( n \) as an object of \( \text{Srg} \). To a 1–morphism surface \( S \), \( F \) assigns the decorated type of \( S \). To a decorated cobordism with corners \( M \), \( F \) assigns the equivalent class of a special ribbon graph representing \( M \).
4. The Kapranov–Voevodsky 2–vector spaces

4.1. The definition of the KV 2–vector spaces. The target 2–category of the extended TQFT will be the Kapranov-Voevodsky (KV) 2–category 2–Vect. The objects of 2–Vect are non-negative integers.

The definition of the KV 2–vector spaces. Let $V$ be a projective module. Two 2–matrices $V$ and $W$ are isomorphic if each $(i,j)$–components is isomorphic for $1 \leq i \leq m$ and $1 \leq j \leq n$. The isomorphism class of $V$ is denoted by $[V]$. We denote a 1–morphism from $m$ to $n$ by $[V] : m \to n$ or simply $V : m \to n$.

Let $V,W,V',W'$ be $(m \times n)$ 2–matrices and suppose we have two matrices of homomorphisms $\{T_{i,j}\}$ and $\{T'_{i,j}\}$ of homomorphisms $T_{i,j} : V_{i,j} \to W_{i,j}$ and $T'_{i,j} : V'_{i,j} \to W'_{i,j}$. We say that $\{T_{i,j}\}$ and $\{T'_{i,j}\}$ are isomorphic if $V$ is isomorphic to $V'$ via an isomorphism $f$ and $W$ is isomorphic to $W'$ via an isomorphism $g$ and we have $g_{i,j} \circ T_{i,j} = T'_{i,j} \circ f_{i,j}$. Now we define a 2–morphism from $[V] : m \to n$ to $[W] : m \to n$. Let $V' \in [V]$ and $W' \in [V]$ and let $T = \{T_{i,j}\}$ be a matrix of homomorphisms from $V'$ to $W'$. Then the isomorphism class $[T]$ of $T$ is a 2–morphism from $[V] : m \to n$ to $[W] : m \to n$.

Usual matrix calculations extend to this setting if we replace a multiplication by $\otimes$ and an addition by $\oplus$. The horizontal composition is given by matrix multiplication and the vertical composition is given by the composition of each entries. (For details, see [KV94] p. 226.)

4.2. A 2–functor $G$ from $Srg$ to 2–Vect. We define a 2–functor $G$ from $Srg$ to $2$–Vect. In fact, $G$ will be a 2–functor “with anomaly”.

The meaning of this will be clear in Proposition 31. Composing with a 2–functor $F$ from Section 3.7, we obtain a 2–functor from $Co$ to $2$–Vect. This 2–functor is the one that extends the RT TQFT to the cobordisms with corners.

The construction of $G$ is as follows. Recall that the modular category $\mathcal{Y}$ has $k$ simple objects. On the objects, to $n \in \mathbb{Z}_{\geq 0}$ as an object of $Srg$, $G$ assigns an integer $k^n$. For a 1–morphism $t : m \to n$ of $Srg$, which is a decorated type $t = (m,n : a_1,a_2,\ldots,a_p)$, we construct a $(k^m \times k^n)$ 2–matrix $g(t)$ and its class is defined to be $G(t)$. We consider sets $I_m = \{(i_1,i_2,\ldots,i_m) | 1 \leq i_1,\ldots,i_m \leq k\}$ and $J_n = \{(j_1,j_2,\ldots,j_n) | 1 \leq j_1,\ldots,j_n \leq k\}$. The set $I_m$ has $k^m$ elements and the set $J_n$ has $k^n$ elements. We order these sets. The ways to order the set $I_m$ and $J_n$ are different (dual). We define $(i_1,i_2,\ldots,i_m) \leq (i'_1,i'_2,\ldots,i'_m)$ if

1. $i_1 \leq i'_1$, or
(2) \( i_1 = i_1', i_2 = i_2', \ldots, i_{l-1} = i_{l-1}' \) and \( i_l \leq i_l' \) for some \( l \in \{2, \ldots, m\} \).

Similarly but slightly differently we define \((j_1, j_2, \ldots, j_n) \leq (j_1', j_2', \ldots, j_n')\) if

1. \( j_n \leq j_n' \), or
2. \( j_n = j_n', j_{n-1} = j_{n-1}', \ldots, j_{i+1} = j_{i+1}' \) and \( j_i \leq j_i' \) for some \( l \in \{1, \ldots, n-1\} \).

Now we define \((i, j)\) components of a 2–matrix \(g(t)\). Let \((i_1, i_2, \ldots, i_m)\) be the \(i\)-th order element of \(I_m\) and let \((j_1, j_2, \ldots, j_n)\) be the \(j\)-th order element of \(J_n\). The \((i, j)\) component module \(g(t)_{i,j}\) is defined as follows. For a decorated type \(t\) put \(a(t) = (a_1, \ldots, a_p) = (0, 0 : a_1, \ldots, a_p)\). We set \(I^i := I^{a(t)}\), see (7). For \(\zeta \in I^i\) we set

\[
\Phi(t; \zeta; i, j) = V_{i_1}^* \otimes V_{i_2}^* \otimes \cdots \otimes V_{i_m}^* \otimes \Phi(a(t), \zeta) \otimes V_{j_1} \otimes V_{j_2} \otimes \cdots \otimes V_{j_n},
\]

where the object \(\Phi(a(t), \zeta)\) is defined in (8). The module \(g(t)_{i,j}\) is defined to be

\[
g(t)_{i,j} := \bigoplus_{\zeta \in I^{a(t)}} \text{Hom}(1, \Phi(t; \zeta; i, j)).
\]

Then the isomorphism class of the 2–matrix where \((i, j)\)–component is \(g(t)_{i,j}\) is set to be \(G(t)\).

We now discuss the assignment of \(G\) on the 2–morphisms level. Let \(\Omega : t_1 \Rightarrow t_2 : m \rightarrow n\) be a special ribbon graph. For each \((i, j)\) components of \(g(t_1)\) and \(g(t_2)\), we define a homomorphism \(g(\Omega)_{i,j}\) from \(g(t_1)_{i,j}\) to \(g(t_2)_{i,j}\). Let \((i_1, i_2, \ldots, i_m)\) be the \(i\)-th order element of \(I_m\) and let \((j_1, j_2, \ldots, j_n)\) be the \(j\)-th order element of \(J_n\). The special ribbon graph \(\Omega\) has \(m\) uncolored bands on left and \(n\) uncolored bands on right.

We color the left \(m\) bands with objects \(V_{i_1}^*, V_{i_2}^*, \ldots, V_{i_m}^*\) from the left and color the right \(n\) bands with objects \(V_{j_1}, V_{j_2}, \ldots, V_{j_n}\) from the left. Then we have a special ribbon graph in the sense Turaev, which we denote by \(i_1 \Omega j\). Apply the RT TQFT to \(i_1 \Omega j\), we get a homomorphism from \(g(t_1)_{i,j}\) to \(g(t_2)_{i,j}\) and we define the isomorphism class of the homomorphism to be \(G(\Omega)_{i,j}\).

We check this assignments \(G\) is compatible with compositions.

**Proposition 29.** Let \(t_1 : l \rightarrow m\) and \(t_2 : m \rightarrow n\) be decorated types. Then \(G(t_1 \circ t_2) = G(t_1)G(t_2)\) as an isomorphism class.

**Proof.** The \((h, j)\)–component of the product of 2–matrices \(g(t_1)\) and \(g(t_2)\) is

\[
\bigoplus_{1 \leq i \leq 2^m} g(t_1)_{h,i} \otimes g(t_2)_{i,j}.
\]
Using Lemma 1, the module (12) is isomorphic to
\begin{equation}
\bigoplus_{i=(i_1, \ldots, i_{m-1}) \in I^{m-1}} \bigoplus_{\zeta \in I^1, \eta \in I^2} \text{Hom } (\mathbb{1}, U(i, \zeta, \eta)) ,
\end{equation}
where $U(i, \zeta, \eta)$ is the following module.

\[
V_{h_1}^* \otimes \cdots \otimes V_{h_p}^* \otimes \Phi(a(t_1), \zeta) \\
\otimes V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_{m-1}} \otimes V_{i_{m-2}}^* \otimes \cdots \otimes V_{i_1}^* \\
\otimes \Phi(a(t_2), \eta) \otimes V_{j_1} \otimes \cdots \otimes V_{j_n}
\]

Note that we have the equality
\[
\bigoplus_{i=(i_1, \ldots, i_{m-1}) \in I^{m-1}} \bigoplus_{\zeta \in I^1, \eta \in I^2} \bigoplus_{\xi \in I^1 \circ t_2} = 
\bigoplus_{\xi \in I^1 \circ t_2}
\]
and for $\xi = (\zeta, i, \eta) \in I^1 \circ t_2$ with $\zeta \in I^1, i \in I^{m-1}, \eta \in I^2$, the object $\Phi(t_1 \circ t_2, \xi)$ is equal to
\[
\Phi(t_1', \zeta) \otimes V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_{m-1}} \otimes V_{i_{m-2}}^* \otimes \cdots \otimes V_{i_1}^* \otimes \Phi(t_2', \eta)
\]
Thus the module (13) is $g(t_1 \circ t_2)_{h,j}$. Note that the isomorphism from $g(t_1) \circ g(t_2)$ to $g(t_1 \circ t_2)$ is given by the isomorphism $u$ of Lemma 1.

This fact will be used in the proof of Proposition 30 below.

**Proposition 30** (Horizontal composition). Let $\Omega_1 : s_1 \Rightarrow s_2 : l \to m$ and $\Omega_2 : t_1 \Rightarrow t_2 : m \to n$ be special ribbon graphs. Then $G(\Omega_1 \circ \Omega_2) = G(\Omega_1) \circ G(\Omega_2)$.

**Proof.** We may assume that $\Omega_1$ is a disjoint union of a ribbon graph $\Omega_1'$ and $m$ vertical lines directed downwards which is placed on the right of $\Omega_1'$. Similarly, $\Omega_2$ is a disjoint union of a ribbon graph $\Omega_2'$ and $m$ vertical lines directed upwards which is placed on the left of $\Omega_2'$. See Figure 11. For $i = 1, 2$, we define several notations. Let $g_i^+$ be the number of

\[
\begin{array}{c}
\Omega_1 = \Omega_1' \downarrow \cdots \downarrow \\
m = |Y|
\end{array}
\quad
\begin{array}{c}
\Omega_2 = \cdots \uparrow \\
m = |Y|
\end{array}
\quad
\begin{array}{c}
\Omega_2' \uparrow \cdots \uparrow \\
m = |Y|
\end{array}
\]

**Figure 11.** Special links $\Omega_1$ and $\Omega_2$
cup like bands in $\Omega'$. Let $L_i$ be surgery links in $\Omega_i$ and let $\nu_i$ be the number of connected components of $L_i$. Denote by $\Omega$ the horizontal composition of $\Omega_1$ and $\Omega_2$, hence we have $\Omega = \Omega'_1 \sqcup \omega_{m-1} \sqcup \Omega'_2$. Let $g^+$ be the number of cup like band in $\Omega$. Clearly $g^+ = g^+_1 + g^+_2 + (m-1)$. Denote by $L$ the surgery links of $\Sigma$. Then $L$ is a disjoint union of $L_1$ and $L_2$ and the $(m-1)$ circles in $\omega_{m-1}$, which we denote by $L_3$. Let $\mu$ be the number of components of $L$. We have $\mu = \mu_1 + \mu_2 + m - 1$.

The homomorphism $g(\Omega_1 \circ \Omega_2)_{h,j} : g(s_1 \circ t_1)_{h,j} \to g(s_2 \circ t_2)_{h,j}$ can be calculated by the formula (5). Let $\zeta$ and $\eta$ be a color for cap like bands and cup like bands of $\Omega$, respectively. By the formula (10), we calculate

$$\tau_\zeta^\eta := (g(\Omega_1 \circ \Omega_2)_{h,j})^\eta_{\zeta}.$$ 

We have

$$\tau_\zeta^\eta = \Delta^{\sigma(L)} F^{g^+ - \sigma(L) - \nu} \dim(\eta) \sum_{\lambda \in \text{col}(L)} \dim(\lambda) F_0(h, \Omega_j, \zeta, \eta, \lambda).$$

Note that we have $\sigma(L) = \sigma(L_1) + \sigma(L_2) + \sigma(L_3) = \sigma(L_1) + \sigma(L_2)$, since $\sigma(L_3) = 0$. We write $\eta = \eta_1 + \eta_2 + \eta_3$, where $\eta_i$ is a color of cup like bands of $\Omega_i$, for $i = 1, 2$ and $\eta_3$ is a color of $\omega_{m-1}$. Then we have $\dim(\eta) = \dim(\eta_1) \dim(\eta_2) \dim(\eta_3)$. Write analogously $\zeta = \zeta_1 + \zeta_2 + \zeta_3$ for cap like bands. Similarly, we decompose a color $\lambda = \lambda_1 + \lambda_2 + \lambda_3$, where $\lambda_i$ is a color of $L_i$ for $i = 1, 2, 3$. Then we have $\dim(\lambda) = \dim(\lambda_1) \dim(\lambda_2) \dim(\lambda_3)$. Since $\Omega$ is a disjoint union of $\Omega'_1$ and $\omega_{m-1}$ and $\Omega'_2$ and $F$ is a monoidal functor, we have $F(h, \Omega_j, \zeta, \eta, \lambda)$

$$= F(h(\Omega'_1), \zeta_1, \eta_1, \lambda_1) \otimes F(\omega_{m-1}, \zeta_3, \eta_3, \lambda_3) \otimes F((\Omega'_2)_j, \zeta_2, \eta_2, \lambda_2).$$

Then $\tau_\zeta^\eta$ is the composition of a morphism $1 \to \Phi(t; \zeta, h, j)$ with

$$\Delta^{\sigma(L_1)} F^{g^+_1 - \sigma(L_1) - \nu_1} \dim(\eta_1) \sum_{\lambda_1 \in \text{col}(L_1)} \dim(\lambda) F(h(\Omega'_1), \zeta_1, \eta_1, \lambda_1)$$

$$\otimes \Delta^{\sigma(L_3)} F^{-(m-1) - \sigma(L_3) - \nu_3} \dim(\eta_3) \sum_{\lambda_3 \in \text{col}(L_3)} \dim(\lambda_3) F(\omega_{m-1}, \zeta_3, \eta_3, \lambda_3)$$

$$\otimes \Delta^{\sigma(L_2)} F^{g^+_2 - \sigma(L_2) - \nu_2} \dim(\eta_2) \sum_{\lambda_2 \in \text{col}(L_2)} \dim(\lambda_2) F((\Omega'_2)_j, \zeta_2, \eta_2, \lambda_2).$$

We compute the second (middle) term. First, since $\sigma(L_3) = 0$ and $\nu_3 = m - 1$, the second term reduces to

$$\Delta^{2(m-1)} \dim(\eta_3) \sum_{\lambda_3 \in \text{col}(L_3)} \dim(\lambda_3) F(\omega_{m-1}, \zeta_3, \eta_3, \lambda_3).$$

By graphical calculations, this is zero unless $\zeta_3 = \eta_3$ and if $\zeta_3 = \eta_3$, then this is equal to the operator invariant of $2(m - 1)$ vertical bands.
A TQFT extending the RT TQFT

whose left $m-1$ bands are directed downwards and right $m-1$ bands are directed upwards. (See the proof of Lemma 2.1.1 of [Tur10], p.174 for a similar graphical calculation.)

To show $G(\Omega_1 \circ \Omega_2) = G(\Omega_1) \circ G(\Omega_2)$, it suffices to show

\begin{equation}
 u(g(\Omega_1)_{h,i} \otimes g(\Omega_2)_{i,j}) = g(\Omega_1 \circ \Omega_2)_{h,j}u,
\end{equation}

where $u$ is an isomorphism in Lemma 1. The right hand side of (14) is equal to

\begin{equation}
 \bigoplus_{\zeta, \eta} \Delta^{\sigma(L)} D^{-g_1^+ - \sigma(L) - \nu_1} \dim(\eta) \sum_{\lambda_1} \dim(\lambda) F_0(h(\Omega_1)_{i}, \zeta_1, \eta_1, \lambda_1) \otimes \Delta^{\sigma(L)} D^{-g_2^+ - \sigma(L) - \nu_2} \dim(\eta_2) \sum_{\lambda_2} \dim(\lambda_2) F_0((\Omega_1)_{j}, \zeta_2, \eta_2, \lambda_2) u,
\end{equation}

where $\zeta \in I^{s_1}$ and $\eta \in I^{t_1}$ decompose, by the argument above, so that $\zeta = \zeta_1 + \zeta_2 + \zeta_3$ and $\eta = \eta_1 + \eta_2 + \zeta_3$ with notations as above. Hence we can write the equation (15) as follows.

\begin{equation}
 \bigoplus_{i \in I^{m-1}} \bigoplus_{\zeta_1, \zeta_2} \bigoplus_{\eta_1, \eta_2} \left[ \Delta^{\sigma(L_1)} D^{-g_1^+ - \sigma(L_1) - \nu_1} \dim(\eta_1) \sum_{\lambda_1} \dim(\lambda) F_0(h(\Omega_1)_{i}, \zeta_1, \eta_1, \lambda_1) \otimes \Delta^{\sigma(L_2)} D^{-g_2^+ - \sigma(L_2) - \nu_2} \dim(\eta_2) \sum_{\lambda_2} \dim(\lambda_2) F_0((\Omega_2)_{j}, \zeta_2, \eta_2, \lambda_2) \right] u,
\end{equation}

where $\Omega_1^-$ is obtained from $\Omega_1$ by deleting the rightmost vertical band and $\Omega_2^-$ is obtained from $\Omega_2$ by deleting the leftmost vertical band. Furthermore the equation (16) is equal to

\begin{equation}
 u \bigoplus_{i \in I^{m}} \bigoplus_{\zeta_1, \zeta_2} \bigoplus_{\eta_1, \eta_2} \left[ \Delta^{\sigma(L_1)} D^{-g_1^+ - \sigma(L_1) - \nu_1} \dim(\eta_1) \sum_{\lambda_1} \dim(\lambda) F_0(h(\Omega_1)_{i}, \zeta_1, \eta_1, \lambda_1) \otimes \Delta^{\sigma(L_2)} D^{-g_2^+ - \sigma(L_2) - \nu_2} \dim(\eta_2) \sum_{\lambda_2} \dim(\lambda_2) F_0((\Omega_2)_{j}, \zeta_2, \eta_2, \lambda_2) \right],
\end{equation}

This follows from the graphical calculation in Figure 12. Finally by the formula (10), this is equal to

\begin{equation}
 u(g(\Omega_1) \circ g(\Omega_2))_{h,j}.
\end{equation}
Proposition 31 (Vertical composition). Let $\Omega_1 : t_1 \Rightarrow t_2 : m \rightarrow n$ and $\Omega_2 : t_2 \Rightarrow t_3 : m \rightarrow n$ be special ribbon graphs. Then we have $G(\Omega_1 \cdot \Omega_2) = kG(\Omega_1)G(\Omega_2)$, where $k \in K$ is a gluing anomaly of the pair $(\Omega_1, \Omega_2)$ given as follows. Let $L_1, L_2$ and $L$ be surgery links of $\Omega_1$, $\Omega_2$ and $\Omega_1 \cdot \Omega_2$, respectively. Then $k = (D\Delta)^{\sigma(L_1) + \sigma(L_2) - \sigma(L)}$.

Proof. This follows easily from Lemma 11. For the value of $k$, see the page 176 of [Tur10].

Proposition 32 (Interchange law). Let $\Omega_1 : s_1 \Rightarrow s_2 : l \rightarrow m$, $\Omega_2 : s_2 \Rightarrow s_3 : l \rightarrow m$, $\Omega'_1 : t_1 \Rightarrow t_2 : m \rightarrow n$, $\Omega'_2 : t_2 \Rightarrow t_3 : m \rightarrow n$ be special ribbon graphs. Then we have

\[ G(\Omega_1 \circ \Omega'_1) \cdot G(\Omega_2 \circ \Omega'_2) = G(\Omega_1 \cdot \Omega_2) \circ G(\Omega'_1 \cdot \Omega'_2) \]
Proof. By the interchange law of Srg, we have
\[(\Omega_1 \circ \Omega'_1) \cdot (\Omega_2 \circ \Omega'_2) = (\Omega_1 \cdot \Omega_2) \circ (\Omega'_1 \cdot \Omega'_2).\]
Applying \(G\) to this and using Proposition 30 and Proposition 31, we have
\[(18) \quad k_1 G(\Omega_1 \circ \Omega'_1) \cdot G(\Omega_2 \circ \Omega'_2) = k_2 G(\Omega_1 \cdot \Omega_2) \circ k_3 G(\Omega'_1 \cdot \Omega'_2),\]
where \(k_1, k_2, k_3\) are gluing anomalies of pairs \((\Omega_1 \circ \Omega'_1, \Omega_2 \circ \Omega'_2), (\Omega_1, \Omega_2), (\Omega'_1, \Omega'_2)\) respectively. Note that the signature of surgery links in \(\omega_m\) is 0 and so is the signature of the concatenation of two \(\omega_m\). By the explicit formula for a gluing anomaly in Proposition 31, it follows that \(k_1 = k_2 k_3\). Hence the equality (17) follows. □

If the value of a gluing anomaly \(k\) is always 1, then the assignment \(G\) is the usual 2–functor. However, in general \(k\) might be not 1. We call this type of assignment 2–functor with anomaly.

5. The extended TQFT

Definition 33. The composition of 2–functors \(F : Co \to Srg\) and \(G : Srg \to 2–Vect\) is said to be the extended TQFT \(Z = GF : Co \to 2–Vect\).

Since \(G\) is the 2–functor with anomaly, the extended TQFT \(Z\) also has anomalies.

Now let us see how the above construction of the extended TQFT \(Z\) for cobordisms with corners extends the original the RT TQFT for cobordisms without corners. Suppose \((M, \partial_- M, \partial_+ M)\) is a cobordism without corners. Then \(M\) is represented by a special ribbon graph with a bottom type \(t^- = (0, 0; a_1, \ldots, a_p)\) and a top type \(t^+ = (0, 0; b_1, \ldots, b_q)\). Since there are no left and right circles on boundary surfaces of \(M\), \(Z(\partial_- M)\) is a \(1 \times 1\) 2–matrix and we canonically identify it with its element \(\bigoplus_{i \in H^-} \Hom(1, \Phi(t^-; i))\). This module is what the RT TQFT assigns to \(\partial_- M\). Similarly for \(\partial_+ M\). Then we can identify \(Z(M)\) as a homomorphism from the module \(\partial_- M\) to the module \(\partial_+ M\), which is the same as the RT TQFT by definition. In summary, the extended TQFT is reduced to the RT TQFT for cobordisms without corners.

5.1. Comments. To make a connection with the work of [DSPS], we consider 2–functors \(2–Vect \to \text{Bimod} \to \text{Cat}\). Here \(\text{Bimod}\) is a 2-category whose objects are \(K\)–algebra, where \(K\) is the ground ring \(K\), and whose 1–morphisms are equivalence classes of bimodules and 2–morphisms are equivalence classes of \(K\)-homomorphisms.
The 2–category Cat consists of categories for objects, functors for 1–morphisms and natural transformations for 2–morphisms.

First let us define the 2-functor $2–\text{Vect} \to \text{Bimod}$. On object level, to each object $n \in \mathbb{Z}$ of $2–\text{Vect}$ we assign $K^n$. On 1–morphism level, to each $m \times n$ 2-matrix $(V_{ij})$ we assign a $(K^m, K^n)$–bimodule $\bigoplus_{i,j} V_{ij}$, where the bimodule structure is induced by the multiplication of $1 \times m$ matrix from the left of $(V_{ij})$ and the multiplication of $n \times 1$ matrix from the right of $(V_{ij})$. Clearly the equivalent 2-matrices are mapped to the equivalent bimodules. On 2–morphism level, if $[(T_{ij})]$ is a 2–morphism from $[(V_{ij})]$ to $[(W_{ij})]$, we assign $[\bigoplus_{i,j} T_{i,j}] : [\bigoplus_{i,j} V_{ij}] \to [\bigoplus_{i,j} W_{ij}]$. These assignments are easily seen to be a 2–functor.

The 2–functor from Bimod to Cat assigns on object level, to each $K$–algebra $A$, the category of right $A$–modules whose objects are isomorphism classes of $A$–modules and morphisms are equivalence classes of homomorphisms. On 1–morphism level, the assignment is induced by tensoring a bimodule from the right. On 2–morphism level, natural transformations are induced by the homomorphisms of bimodules.

Composing this 2–functor with the extended TQFT $Z : \text{Co} \to 2–\text{Vect}$, we have a 2-functor from the 2–category of cobordisms with corners to the 2–category Cat.

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E-mail address: ytsumura@math.purdue.edu

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE IN 47907, USA