A motivic version of Pellikaan’s two variable zeta function

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Consider the Grothendieck ring $K_0(\text{Var}_K)$ of varieties over a field $K$. It is the abelian group generated by the isomorphism classes $[X]$ of algebraic $K$-schemes subject to the relations

$$[X] = [Y] + [X \setminus Y]$$

for any closed subscheme $Y$ of $X$. The product is defined by the formula

$$[X_1] \cdot [X_2] = [X_1 \times_K X_2].$$

The class of $\mathbb{A}^1$ plays a special role. It is denoted by $\mathbb{L} = [\mathbb{A}^1]$.

For a variety $X/K$, Kapranov [K] defined the motivic zeta function of $X$ as the power series

$$Z_{X, \text{mot}}(T) = \sum_{n=0}^{\infty} [X^{(n)}]T^n \text{ in } K_0(\text{Var}_K)[[T]].$$

Here $X^{(n)} = X^n/S_n$ is the $n$-fold symmetric product, $X^{(0)} = \text{Spec}(K)$. Moreover, for a homomorphism of commutative rings with unity,

$$\mu : K_0(\text{Var}_K) \longrightarrow A,$$

a so called motivic measure, he considers the series

$$Z_{X, \mu}(T) = \sum_{n=0}^{\infty} \mu([X^{(n)}])T^n \text{ in } A[[T]].$$
The rationality of these power series is an interesting issue. See [DL] and [LL]. For $X/\mathbb{F}_q$ and $\mu : K_0(\text{Var}_{\mathbb{F}_q}) \to \mathbb{Z}$ defined by $\mu([Z]) = |Z(\mathbb{F}_q)|$, $Z_{X,\mu}(T)$ equals the usual zeta function of $X/\mathbb{F}_q$, i.e.:

$$Z_X(T) := \exp\left(\sum_{\nu \geq 1} |X(\mathbb{F}_{q^{\nu}})| \frac{T^{\nu}}{\nu}\right) = \sum_{\nu \geq 0} |X^{(\nu)}(\mathbb{F}_q)| T^{\nu}.$$ 

This is the special case of constant coefficients concentrated in degree zero of [D], lemme 4.11. It can also be proved by a direct combinatorial argument without using any cohomological expression for $Z_X(T)$.

Concerning curves, Kapranov proves the following result in [K]:

**Theorem 1 (Kapranov)** Let $X$ be a smooth projective geometrically connected curve of genus $g$ over $K$ with a degree 1 line bundle. Assume that $A$ is a field and that $\mathbb{L}_\mu := \mu(\mathbb{L}) \neq 0$ in $A$. Then we have

$$Z_{X,\mu}(T) = \frac{P_{X,\mu}(T)}{(1 - T)(1 - \mathbb{L}_\mu T)}$$

for a polynomial $P_{X,\mu}(T)$ of degree $2g$ in $A[T]$. Moreover the following functional equation holds:

$$Z_{X,\mu}(T) = \mathbb{L}_\mu^{g-1} T^{2g-2} Z_{X,\mu}(\mathbb{L}_\mu^{-1} T^{-1}).$$

For $\mu : K_0(\text{Var}_{\mathbb{F}_q}) \to \mathbb{Z}$ as above the theorem asserts the well known rationality and symmetry properties of the usual zeta function of a curve over a finite field.

We quickly recall the construction, due to R. Pellikaan, of a two-variable zeta function (see [P] for more details):

Let $X/\mathbb{F}_q$ be a curve. Its (usual) zeta function can be written as

$$Z_X(T) = \sum_{D \geq 0} T^{\deg(D)},$$

the sum being extended over all effective divisors on $X$. Summing over divisor-classes $D = [D]$ this becomes
\[ Z_X(T) = \sum_D q^{h^0(D)} - 1 \frac{T^{\deg(D)}}{q - 1}. \]

Here we write \( h^0(D) = \dim H^0(X, \mathcal{O}(D)) \) for any divisor \( D \) with \( D = [D] \).

The two-variable zeta function is obtained by substituting a variable \( u \) for \( q \) in this expression:

\[ Z_X(T, u) = \sum_D u^{h^0(D)} - 1 \frac{T^{\deg(D)}}{u - 1}. \]

It is a power-series in \( u \) and \( T \) with integer coefficients. Pellikaan then proves, among other things, rationality:

\[ Z_X(T, u) = \frac{P_X(T, u)}{(1 - T)(1 - uT)}. \]

for a suitable \( P_X(T, u) \in \mathbb{Z}[T, u] \). It was suggested by a question of J. Lagarias and E. Rains and proven in [N] that \( P_X(T, u) \in \mathbb{Z}[T, u] \) is absolutely irreducible.

A two-variable zeta function may be defined in the motivic case as well and as we shall see, it has similar properties as \( Z_X(T, u) \). For a curve \( X/K \) as in Kapranov’s theorem let \( \text{Pic}^n_{X/K} \) be the Picard variety of degree \( n \) line bundles on \( X \). Let \( \text{Pic}^n_{\geq \nu} \) be the closed subvariety in \( \text{Pic}^n_{X/K} \) of line bundles \( \mathcal{L} \) with \( h^0(\mathcal{L}) \geq \nu \). The algebraic \( K \)-scheme \( \text{Pic}^n_{\nu} = \text{Pic}^n_{\geq \nu} \setminus \text{Pic}^n_{\geq \nu+1} \) defines a class in \( K_0(\text{Var}_K) \)

\[ [\text{Pic}_{\nu}^n] = [\text{Pic}_{\geq \nu}^n] - [\text{Pic}_{\geq \nu+1}^n]. \]

The two-variable motivic zeta function of the curve \( X/K \) is defined as the formal power series

\[ Z_{X, \text{mot}}(T, u) = \sum_{n, \nu \geq 0} [\text{Pic}_{\nu}^n] \frac{u^n - 1}{u - 1} T^n \text{ in } K_0(\text{Var}_K)[[u, T]]. \]

For a motivic measure \( \mu \) with values in \( A \) we set:

\[ Z_{X, \mu}(T, u) = \sum_{\nu, \mu \geq 0} \mu[\text{Pic}_{\nu}^n] \frac{u^n - 1}{u - 1} T^n \text{ in } A[[u, T]]. \]
Here, again, for $X/F_q$ and $\mu : K_0(\text{Var}_{F_q}) \to \mathbb{Z}$ as above we have $Z_{X, \mu}(T, u) = Z_X(T, u)$, because for any $n, \nu \geq 0$ the set of $F_q$-rational divisor classes $D$ with $\deg(D) = n$ and $h^0(D) = \nu$ are in bijection with the $F_q$-rational points of $\text{Pic}^n$. This follows from a general result on the relative Picard functor, [BLR], 8.1, Prop. 4, and the fact that the Brauer group of a finite field is trivial.

We note that $Z_{X, \mu}(T, u)$ is constructed very much the same way as Pellikaan did: The natural morphism

$$X^{(n)} \longrightarrow \text{Pic}^n_{X/K}$$

is such that $X^{(n)} \times_{\text{Pic}^n_{X/K}} \text{Pic}^n_{\nu} \simeq \mathbb{P}^{n-1} \times \text{Pic}^n_{\nu}$ which implies that (c.f. [K], Prop. 1.2.3)

$$[X^{(n)}] = \sum_{\nu \geq 0} [\text{Pic}^n_{\nu}] \frac{\mathbb{L}^\nu - 1}{\mathbb{L} - 1} \quad \text{in } K_0(\text{Var}_K)$$

and hence

$$Z_{X, \text{mot}}(T) = \sum_{n, \nu \geq 0} [\text{Pic}^n_{\nu}] \frac{\mathbb{L}^\nu - 1}{\mathbb{L} - 1} T^n.$$

Just as Pellikaan substituted a variable for the integer $q$, we substitute a variable for the class $\mathbb{L}$ in $K_0(\text{Var}_K)$.

We explain a convenient way of writing $Z_{X, \mu}(T, u)$ using motivic integration: Let $K^{\text{sep}}$ denote a fixed separable closure of $K$ and let $Z$ be an algebraic scheme over $K$. A constructible $A$-valued function on $Z$ in the sense of [K] (1.2) is a function

$$f : Z(K^{\text{sep}}) \longrightarrow A$$

which can be written in the form

$$f = \sum_{i=1}^{n} a_i \chi_{W_i(K^{\text{sep}})}$$

where $a_i \in A$ and the $W_i \subset Z$ are closed subschemes. Here $\chi$ denotes the characteristic function of a set.
The integral of \( f \) with respect to \( \mu \) is defined to be

\[
\int_Z f \, d\mu = \sum_{i=1}^{n} a_i \mu(W_i) .
\]

In our case, the function

\[
\text{Pic}_X^n(K_{\text{sep}}) \to A[[u]], \quad D \mapsto \frac{u^{h^0(D)} - 1}{u - 1}
\]

defines a constructible \( A[[u]] \)-valued function on \( \text{Pic}_X^n \) and we may write \( Z_{X,\mu}(T, u) \) in the form:

\[
(1) \quad Z_{X,\mu}(T, u) = \sum_n \left( \int_{\text{Pic}_X^n} \frac{u^{h^0(D)} - 1}{u - 1} \, d\mu(D) \right) T^n .
\]

Here \( \mu \) is viewed as an \( A[[u]] \)-valued measure using the inclusion \( A \hookrightarrow A[[u]] \).

Introducing a suitable notion of convergent integrals over \( K \)-schemes which are not of finite type we could also write

\[
Z_{X,\mu}(T, u) = \int_{\text{Pic}_X/K} \frac{u^{h^0(D)} - 1}{u - 1} T^{\deg D} \, d\mu(D)
\]

where now \( \mu \) is viewed as an \( A[[u]][[T]] \)-valued measure. However we will do with (1) in the sequel. The following result is proved in the same way as Pellikaan’s original theorem for \( Z_X(T, u) \).

**Theorem 2** Let \( K \) be a field and \( X/K \) a smooth projective geometrically irreducible curve of genus \( g \) which admits a line bundle of degree one. Then we have

a) \( Z_{X,\mu}(T, u) = \frac{P_{X,\mu}(T, u)}{(1 - T)(1 - uT)} \) in \( A[T, u, (1 - T)^{-1}, (1 - uT)^{-1}] \)

where \( P_{X,\mu}(T, u) \in A[T, u] \).

b) \( P_{X,\mu}(T, u) = \sum_{i=0}^{2g} P_i(u) T^i \) with \( P_i(u) \in A[u] \), where

\( P_0(u) = 1, P_{2g}(u) = u^g, \deg P_i(u) \leq 1 + \frac{i}{2} \).

c) \( Z_{X,\mu}(T, u) = u^{g-1} T^{2g-2} Z_{X,\mu} \left( \frac{1}{Tu}, u \right) \) i.e. \( P_{2g-i}(u) = u^{g-i} P_i(u) \).

d) \( P_{X,\mu}(1, u) = \mu(\text{Pic}_X/K) \in A \subset A[u] \).
**Proof** For \( g = 0 \) we have \( Z_{X, \mu}(T, u) = (1 - T)^{-1}(1 - uT)^{-1} \), so the assertions are clear. For \( g \geq 1 \) we have:

\[
Z_{X, \mu}(T, u) = \sum_{0 \leq n \leq 2g - 2} T^n \int_{\text{Pic}^n_{X/K}} \frac{u^{h^0(D)} - 1}{u - 1} d\mu(D) \\
+ \sum_{n > 2g - 2} \frac{u^{n+1-g} - 1}{u - 1} T^n \int_{\text{Pic}^n_{X/K}} d\mu(D) \\
= \sum_{0 \leq n \leq 2g - 2} T^n \int_{\text{Pic}^n_{X/K}} \frac{u^{h^0(D)} - 1}{u - 1} d\mu(D) \\
+ \sum_{n > 2g - 2} \frac{u^{n+1-g}}{u - 1} \mu(\text{Pic}^n_{X/K}) T^n - \sum_{n \geq 0} \frac{\mu(\text{Pic}^n_{X/K})}{u - 1} T^n .
\]

Since there exists a degree one line bundle on \( X \), we have \( \text{Pic}^n_{X/K} \cong \text{Pic}^0_{X/K} \). Hence we get

\[
Z_{X, \mu}(T, u) = \frac{1}{u - 1} \sum_{0 \leq n \leq 2g - 2} T^n \int_{\text{Pic}^n_{X/K}} \frac{u^{h^0(D)} - 1}{u - 1} d\mu(D) \\
+ \frac{\mu(\text{Pic}^0_{X/K})}{u - 1} \left( u^{1-g} (uT)^{2g-1} - \frac{1}{1 - T} \right) \\
= \frac{1}{u - 1} \int_{\text{Pic}^{[0,2g-2]}_{X/K}} \frac{u^{h^0(D)} T^{\deg D}}{u - 1} \left( \frac{u^g T^{2g-1}}{1 - uT} - \frac{1}{1 - T} \right) .
\]

In the last formula we have set \( \text{Pic}^{[0,2g-2]}_{X/K} = \bigsqcup_{0 \leq n \leq 2g - 2} \text{Pic}^n_{X/K} \). The rest of the proof is as in [P] but for convenience we give the details: We set

\[
G(T, u) = \int_{\text{Pic}^{[0,2g-2]}_{X/K}} \frac{u^{h^0(D)} T^{\deg D}}{u - 1} d\mu(D) \\
F(T, u) = \mu(\text{Pic}^0_{X/K}) \left( \frac{u^g T^{2g-1}}{1 - uT} - \frac{1}{1 - T} \right) .
\]

So we have \((u-1)Z_{X, \mu}(T, u) = G(T, u) + F(T, u)\). A direct computation shows that \( F(T, u) \) satisfies the functional equation. To check the functional equation for \( G(T, u) \) we observe that sending a line bundle \( \mathcal{L} \) to \( \mathcal{L}' = \mathcal{L}^{-1} \otimes \omega_{X/K} \),
where $\omega_{X/K}$ is the canonical bundle, defines an involution on the relative Picard functor of $X/K$ because formation of the canonical bundle commutes with arbitrary base-change. The theorem of Riemann-Roch together with Serre-duality on our curve implies that the corresponding involution on the $K$-scheme $\text{Pic}_{X/K} = \prod_{n \in \mathbb{Z}} \text{Pic}_{X/K}^n$ induces isomorphisms for any $n, \nu \geq 0$:

$$\text{Pic}_{\nu}^n \overset{\cong}{\longrightarrow} \text{Pic}_{\nu}^{2g-2-n}, \mathcal{D} \mapsto \mathcal{D}'$$

Using this we compute:

$$u^{g-1}T^{2g-2}G\left(\frac{1}{Tu}, u\right) = \int_{\text{Pic}_{X/K}^{[0,2g-2]}} u^{h^0(D) + g - 1 - \deg(D)}T^{2g-2 - \deg(D)} d\mu(D)$$

$$= \int_{\text{Pic}_{X/K}^{[0,2g-2]}} u^{h^0(D')} T^{\deg(D')} d\mu(D')$$

$$= G(T, u).$$

So the functional equation holds for $F(T, u)$ and $G(T, u)$, hence for $Z_{X,\mu}(T, u)$, i.e. we have proved the first assertion of part c). Consider

$$Q(T, u) := (1 - uT)(1 - T)(F(T, u) + G(T, u)) = \mu(\text{Pic}_{X/K}^0)((1 - T)u^gT^{2g-1} - (1 - uT)) + (1 - uT)(1 - T)G(T, u).$$

This is a polynomial in $u$ and $T$ with coefficients in $A$. As

$$G(T, 1) = \int_{\text{Pic}_{X/K}^{[0,2g-2]}} T^{\deg(D)} d\mu(D)$$

$$= \sum_{n=0}^{2g-2} \mu(\text{Pic}_{X/K}^n) T^n = \mu(\text{Pic}_{X/K}^0) \frac{T^{2g-1} - 1}{T - 1}$$

one computes $Q(T, 1) = 0$, hence $Q(T, u) = (u - 1)P_{X,\mu}(T, u)$ for some $P_{X,\mu}(T, u) \in A[T, u]$. Since $u - 1$ is not a zero-divisor in $A[T, u]$ we find
\[ P_{X,\mu}(T, u) = (1 - uT)(1 - T)Z_{X,\mu}(T, u), \]
i.e. we have proved part a). Clearly, \( G(T, u) \) has degree at most \( 2g - 2 \) in the variable \( T \). So from (2) the same holds true for \( P_{X,\mu}(T, u) \). We also have

\[ Q(1, u) = \mu(Pic_0^0 X/K)(u - 1) = (u - 1)P_{X,\mu}(1, u), \]
which proves part d). It is left to the reader to check that the functional equation is indeed equivalent to \( P_{2g-i}(u) = u^{g-i}P_i(u) \) for all \( i \geq 0 \). To prove part b) we have

\[ Q(0, u) = -\mu(Pic_0^0 X/K) + G(0, u). \]
Observe that \( Pic_{\geq 1}^0 = Pic_1^0 \) is the zero section of the abelian variety \( Pic_0^0 X/K \) and hence \( \mu(Pic_1^0) = 1 = \mu(Pic_0^0 X/K) - \mu(Pic_0^0) \) from which we get

\[ Q(0, u) = -\mu(Pic_0^0 X/K) + \int_{Pic_0^0 X/K} u^{h^0(D)} d\mu(D) \]
\[ = -\mu(Pic_0^0 X/K) + \mu(Pic_0^0) + u = u - 1, \]
i.e. \( P_0(u) = 1 \) and from the functional equation \( P_{2g}(u) = u^g \) also. The last assertion, \( \deg P_i(u) \leq 1 + i/2 \), follows from Clifford’s theorem (c.f. [11], IV, thm. 5.4), which can be formulated as asserting that if a divisor class \( D \) has \( h^0(D) \geq \max\{1, \deg(D) - g + 2\} \) then \( h^0(D) \leq \frac{1}{2} \deg(D) + 1 \). This implies that for any \( n, \nu \geq 0 \), if \( \nu \geq \max\{1, n - g + 2\} \) and \( \nu > \frac{1}{2}n + 1 \), then \( Pic_\nu^0 = \emptyset \). Indeed, Clifford’s theorem gives that for such couples \( (n, \nu) \) we have \( Pic_\nu^0(K) = \emptyset \) and as \( Pic_\nu^0 \) is of finite type over \( K \) this implies \( Pic_\nu^0 = \emptyset \). The rest is left to the reader. \( \square \)

There is the following irreducibility result for the two-variable motivic zeta function which contains the analogous result for Pellikaans two-variable zeta function as a special case:
Theorem 3. Let $K$ be a field and $X/K$ a smooth projective geometrically irreducible curve of genus at least one. Assume $\mu : K_0(\text{Var}_K) \to A$ is a motivic measure with values in a field $A$. Then $P_{X,\mu}(T, u)$ is (absolutely) irreducible if and only if $\mu(\text{Pic}^0_{X/K}) \neq 0$.

Proof

If $\mu := \mu(\text{Pic}^0_{X/K}) = 0$ then $P_{X,\mu}(1, u) = 0$ by theorem 2 d) so $P_{X,\mu}(T, u)$ is divisible by $T - 1$.

The proof of absolute irreducibility of $P_{X,\mu}(T, u)$ in case $\mu \neq 0$ is virtually the same as for Pellikaan’s function, which can be found in [N]. So we only give a short sketch here: We prove absolute irreducibility for $F(T, u) := T^{2g} P_{X,\mu}(T^{-1}, u)$ which implies the result because $P_{X,\mu}(0, u) = 1 \neq 0$. The polynomial $F$ is monic in $T$ and satisfies $F(1, u) = \mu$ a non-zero constant.

An easy exercise in commutative algebra shows that to finish the proof it is sufficient to show that the leading coefficient of $F$ as a polynomial in $u$ is irreducible in $A[T]$. Indeed, we have more precisely:

$$F(T, u) = (1 - T)u^g + O(u^{g-1}).$$

Using $\deg P_i(u) \leq \frac{1}{2} + i$ this boils down to an assertion about $P_1(u)$ and $P_2(u)$ for which we refer to loc.cit. where one has to replace $b_{nk}$ by $\mu(\text{Pic}^n_k)$. A final point is that the vanishing of certain $b_{nk}$ deduced from Clifford’s theorem in loc.cit. carries over to the $\mu(\text{Pic}^n_k)$ as explained at the end of the proof of theorem 2.

\[\square\]

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