Quantized phase shifts and a dispersive universal quantum gate

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A single $^{40}$Ca$^+$-ion is trapped and laser cooled to its motional ground state. Laser radiation which couples off-resonantly to a motional sideband of the ion’s $S_{1/2}$ to $D_{5/2}$ transition causes a phase shift proportional to the ion’s motional quantum state $|n\rangle$. As the phase shift is conditional upon the ion’s motion, we are able to demonstrate a universal 2-qubit quantum gate operation where the electronic target state $\{S, D\}$ is flipped depending on the motional qubit state $|n\rangle = \{|0\rangle, |1\rangle\}$. Finally, we discuss scaling properties of this universal quantum gate for linear ion crystals and present numerical simulations for the generation of a maximally entangled state of five ions.

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Trapped ions are successfully used for quantum information processing [1, 2, 3] as experiments with a single ion [4, 5, 6] and two ion crystals [7, 8] have proven. Even four-ion crystals [9] have been transferred into an entangled state. For the implementation of quantum algorithms, single- and two-qubit operations are required [2]. For most implementations of two-qubit gate operations, a well defined phase shift is induced, depending on the control qubit’s quantum state. Its origin might be either a geometric phase [10, 11] as the quantum state follows a closed loop in phase space [7], or the sign of the wave function is changed as a resonant $2\pi$-Rabi rotation is performed [4, 8, 12]. Alternatively, the phase shift which is required for the gate operation may be derived from a non-resonant laser-atom interaction. The resulting quantum phase gate $\Phi$ transforms the state of two qubits $|a, b\rangle \rightarrow \exp(i\phi\delta_{1,a}\delta_{1,b})|a, b\rangle$, where $\delta_{1,a}$ and $\delta_{1,b}$ denote Kronecker symbols with $\delta_{1,a} = 1$ if qubit $a$ is in the state $|1\rangle$ and $\delta_{1,a} = 0$ otherwise. Only for both qubits in $|1\rangle$, the quantum state’s phase is shifted by $\phi$. This phase gate may be converted into a controlled-NOT operation (CNOT) by single qubit rotations before and after performing $\Phi$ to yield $|a, b\rangle \rightarrow |a, a \oplus b\rangle$, where $\oplus$ represents an addition modulo 2.

A quantum phase gate [13] also has been demonstrated using long lived superpositions of Rydberg atoms which traverse a high-Q microwave cavity. The phase $\phi$ of a superposition $(|0\rangle + e^{i\phi}|1\rangle) / \sqrt{2}$, where $|0\rangle$ and $|1\rangle$ denote the qubit states, is changed conditioned on the quantum state of the cavity mode. If the cavity resonance is detuned slightly from that of the atomic qubit transition, superpositions acquire a significant light shift already by a single photon stored in the microwave cavity [14].

In this paper we study experimentally and theoretically the dispersive (off-resonant) coupling of laser radiation for quantum information processing with trapped ions. The quantum information is encoded in the ion’s electronic ground state $S_{1/2}, (m = -1/2) \equiv |S\rangle$ and a long lived metastable state $D_{5/2}, (m = -1/2) \equiv |D\rangle$ (see Fig. 1). The qubit can be manipulated by laser radiation near 729 nm on the corresponding quadrupole transition. The motional state of the ion is prepared in a Fock state $|n\rangle$ for a population of the state $|S\rangle$ to $|D\rangle$.

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FIG. 1: a) $^{40}$Ca$^+$ level scheme. A qubit is encoded in the $S_{1/2}, (m = -1/2)$ ground and $D_{5/2}, (m = -1/2)$ metastable state. b) The lowest two number states $n$ of the axial vibrational motion in the trap form the other qubit. c) Non-resonant interaction with the laser tuned close to the upper motional sideband (indicated by arrows). d) Laser pulse sequence for the quantum gate, consisting of the preparation of qubit states, single bit operations R1 and R2 and controlled phase shift operation $\Phi$ of duration $t$. Finally the qubits are analyzed for their electronic and motional quantum state. (see Fig. 1b) and serves as control qubit. The interaction with off-resonant laser light (see Fig. 1b) induces a phase shift of the target qubit (electronic states) conditioned upon the ion’s motional state. In our experiment, a single $^{40}$Ca$^+$ ion is stored in a linear Paul trap with a radial and axial frequency of $\omega_{rad}/2\pi = 4.9$ MHz and $\omega_{ax}/2\pi = 1.712$ MHz, respectively. Doppler cooling, followed by sideband ground state cooling and optical pumping leads to a population of the state $S_{1/2}, (m = -1/2) \equiv |S\rangle$ and its lowest motional quantum state $|n = 0\rangle$ with more than 98% probability. The internal state is detected by electron shelving [5] with an efficiency $> 99%$. We de-
termine the occupation probability of the \( |D\rangle \) state, \( P_D \), by averaging over 100 experimental sequences. Details of the cooling method and the experimental apparatus are described elsewhere \[13\]. Starting from the state \( |S, n = 0\rangle \), laser pulses resonant to the blue axial sideband \( (|S, n\rangle \leftrightarrow |D, n + 1\rangle) \) and resonant to the carrier transition \( (|S, n\rangle \leftrightarrow |D, n\rangle) \) of the qubit transition are used to transfer the motional state to Fock states \[13\]. The transfer quality of blue sideband \( \pi \)-pulse and carrier \( \pi \)-pulse exceed 98% and 99%, respectively. Thus, we obtain the desired motional Fock state with fidelities better than 96%, 94% and 92% for \( n = 1 \), 2 and 3, respectively.

The phase evolution of the atomic wavefunction due to interaction with non-resonant laser radiation is probed in a Ramsey experiment: A resonant \( \pi/2 \) pulse \( (R_1 \text{ in Fig. } 1b) \) on the quadrupole transition prepares the electronic qubit in the superposition state \( (|S\rangle + |D\rangle)/\sqrt{2} \). A second resonant \( \pi/2 \) laser pulse, \( R_2 \) is applied with opposite laser phase and a delay of \( t = 260 \mu s \). If the atomic wavefunction does not acquire any additional phase during the delay time, \( R_2 \) would undo the operation of \( R_1 \) and transfer the electronic state back to \( |S\rangle \). For generation of a deterministic phase shift during the delay time, the ion is exposed to laser light which is blue-detuned by \( \Delta \) from the blue axial sideband of the qubit transition. The resulting phase shift is due to \( i \)-off-resonant coupling to dipole transitions (see Fig. 1b), \( ii \)-off-resonant coupling to carrier transitions of the \( S_{1/2} \) to \( D_{3/2} \) Zeeman manifold and \( iii \)-off-resonant coupling to the motional sidebands of the qubit transition. An extensive study of the first two contributions is given in \[13\].

The relative strengths of the three contributions scale with their respective Rabi frequencies. Thus, the light (AC Stark) shift \( \delta_{\text{res}} = \Omega_{n,n+1}^2/4\Delta \) due to the third contribution is weaker by almost two orders of magnitude compared with \( i \) and \( ii \), as the resonant Rabi frequency of the blue sideband between \( |S, n\rangle \) and \( |D, n+1\rangle \) is given by

\[
\Omega_{n,n+1} = \eta \Omega_0 \sqrt{n + 1}
\]

with the Lamb-Dicke factor \( \eta = 0.068 \ll 1 \). As \( \delta_{\text{res}} \) strongly depends on the motional quantum number it can be utilized to generate conditional phase shifts. However, the stronger contributions \( i \) and \( ii \) mask this effect. Therefore, we cancel \( i \) and \( ii \) using an additional off-resonant light field with equal, but opposite phase shift (“compensating” light field) which illuminates the ions during the phase shift operation \[15\]. This light field is derived from the same laser source near 729 nm which we use for the qubit transition using an acousto-optical modulator. By optimizing the compensation light field parameters, i.e., laser intensity and detuning, we also cancel the light shift for the motional Fock state \( |n = 0\rangle \). For this state the compensation leads to a residual light shift of \( \delta_{\text{res}}/2\pi \leq 300 \text{ Hz} \) (see Fig. 2). Thus, after compensation, the phase shift \( \Delta \phi = \delta_{\text{res}} t \) is directly proportional to the motional quantum number \( n \) of the ion,

\[
\Delta \phi = \frac{\eta^2 \Omega_0^2 t}{4\Delta} n
\]

where \( t \) denotes the duration of the phase shift operation in the sequence. For the Fock states \( |n = 1\rangle \) to \( |n = 3\rangle \), we observe a linear slope of \( \delta_{\text{res}}/2\pi = 2.71(6) \text{ kHz} \) per phonon (see Fig. 2).

In order to realize a phase gate operation, we chose \( t = t_0 \) such that the residual phase shift for states with \( n = 1 \) yields precisely \( \pi/2 \). This accomplishes a quantum gate in the two-qubit computational space composed by the states \( |S, 0\rangle, |D, 0\rangle, |S, 1\rangle \) and \( |D, 1\rangle \): The states \( |S, 1\rangle \) and \( |D, 1\rangle \) acquire phases of \( e^{\pm i\pi/2} = \pm i \), respectively, while \( |S, 0\rangle \) and \( |D, 0\rangle \) do not acquire any phase. Thus the phase shift operation \( \Phi \) reads:

\[
\Phi(t_0) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & i & 0 \\
0 & 0 & 0 & -i
\end{pmatrix}
\]

The Ramsey pulses \( R_1 \) and \( R_2 \) are represented by

\[
\frac{1}{\sqrt{2}} \begin{pmatrix}
1 & \pm i & 0 & 0 \\
\pm i & 1 & 0 & 0 \\
0 & 0 & 1 & \pm i \\
0 & 0 & \pm i & 1
\end{pmatrix},
\]

where the \( + \) (-) sign refers to \( R_1(R_2) \). The full sequence \( C \) represents a universal two-qubit gate:

\[
C = R_2 \Phi(t_0) R_1 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]

In the experiment, the truth-table of this gate is probed.
we determine $\Omega$ blue axial sideband. These Rabi oscillations are plotted i.e. the control qubit, we drive Rabi oscillations on the shelving. For detection of the motional state the ion interacts on the blue sideband (FIG. 3: State analysis after the gate operation for input states $|S, 0\rangle$ (a), $|D, 0\rangle$ (b), $|S, 1\rangle$ (c) and $|D, 1\rangle$ (d). The ion interacts on the blue sideband ($|S, n\rangle \leftrightarrow |D, n + 1\rangle$) resonant for a time $\tau$. A fit of pure sine-functions yields Rabi frequencies of $\Omega_{1/2\pi}=11.9(1) \text{ kHz}$ (a), $12.0(2) \text{ kHz}$ (c) and $\Omega_{1/2\pi}=16.95(5) \text{ kHz}$ (d). The frequency ratios of (d) and (a): $\sqrt{2}(0.024)$, and of (d) and (c): $\sqrt{2}(1.99(7))$ agree well with the expected factor $\sqrt{2}$ from eq. 3. The solid lines correspond to a fit of the model function eq. 3.

by preparing the four computational basis input states. After the gate operation $C$ the internal state $\{S, D\}$ which served as the target qubit is detected by electron shelving. For detection of the motional state $n = \{0, 1\}$, i.e. the control qubit, we drive Rabi oscillations on the blue axial sideband. These Rabi oscillations are plotted in Fig. 3. In a first step we determine the Rabi frequencies by fitting the data with pure sine-functions. From this fit we determine $\Omega_{0,1}/2\pi=11.9(1) \text{ kHz}$ and $\Omega_{1,2}/2\pi=\sqrt{2}\Omega_{0,1}$ (c.f. Fig. 3). In a next step we fit the respective experimental data in Fig. 3a-d with a model function:

$$P_D(\tau) = a_{S0}\sin^2(\Omega_{0,1}\tau) + a_{D0} + a_{S1}\sin^2(\Omega_{1,2}\tau) + a_{D1}\cos^2(\Omega_{0,1}\tau)$$

The coefficients $a_{S0}, a_{D0}, a_{S1}, a_{D1}$ account for the contributions of the four computational basis states $|S, 0\rangle, |D, 0\rangle, |S, 1\rangle, |D, 1\rangle$ and obey $a_{S0} + a_{D0} + a_{S1} + a_{D1} = 1$. Table I lists these coefficients for the four different input states. We find transfer efficiencies between the ground state $|S, 0\rangle$ and the desired output states of $\geq 81\%$. The measured efficiencies are attributed to the following experimentally determined imperfections: (i) state preparation as mentioned above (ground state cooling and preparation pulses) (ii) the chosen gate interaction time $t_0 = 200 \mu s$ which has been off by $+8\%$ from its ideal value, (iii) the independently measured loss of coherence ($\approx 6\%$) of the qubit within the delay time between $R1$ and $R2$ and (iv) off-resonant sideband excitations during $\Phi$, measured here to account for an error of $4\%$.

The demonstrated dispersive two-qubit gate can be extended to a larger number of $N$ qubits. An important advantage of dispersive gates, like e.g. the gate presented in Ref. 13, is that the computational subspace is conserved automatically as the off-resonant interaction $\Phi$ only modifies the phase. Other gate schemes need auxiliary levels or composite pulse techniques thus increasing the technical complexity of the experimental sequence.

For the entangling operation, we select a symmetric vibrational mode (bus mode) where the absolute values of all the ions’ Lamb-Dicke parameters are equal, e.g. the center of mass mode. Similar to the one-ion gate operation, the $N$-ion string is assumed to be cooled close to the motional ground state (e.g. by EIT-cooling). To create an entangled state, we use a pulse sequence $R1\Phi(t_0)R2'$ acting in the following way: $R1\Phi(t_0)R2'|SS \ldots S| = |SS \ldots S\rangle + e^{i\phi}|DD \ldots D\rangle$. First we consider the two ion case. The two ions are excited by a $R1$ pulse as defined above yielding the superposition state $1/2(|SS\rangle + |SD\rangle + |DS\rangle + |DD\rangle)$. The phase operation $\Phi(t_0)$ then couples the ions via the bus mode. The phase of the final $\pi/2$ Ramsey pulse $R2'$ has to be chosen $\pi(\pi/2)$ relative to $R1$ for an odd (even) number of ions. Note that this scheme only requires the same Rabi frequency for all ions, but no individual addressing of the ions. This can be conveniently achieved by illuminating the ions along the trap axis. The crucial part of the operation is contained in $\Phi(t_0)$: The fraction of the superposition being in the $|DD\rangle$-state is dark and thus remains completely unchanged. Similarly to the one-ion case, the $|SS\rangle$-part, however, acquires a phase factor without a change in population. Choosing the pulse length $t_0$, Rabi frequency $\Omega$ and detuning $\Delta$ of the sideband interaction such that $t_0 = \Delta/(\hbar^2\Omega^2)$ the acquired phase factor of the $|SS\rangle$-state is $-1$. For ions where both qubit levels are populated, Raman processes

TABLE I: Experimentally obtained truth table of the gate including the imperfect input state preparation.

| $|\text{input}\rangle$ | $|S, 0\rangle$ | $|D, 0\rangle$ | $|S, 1\rangle$ | $|D, 1\rangle$ |
|---|---|---|---|---|
| $|S, 0\rangle$ | $0.90(1)$ | $0.06(1)$ | $0.01(2)$ | $0.03(1)$ |
| $|D, 0\rangle$ | $0.09(1)$ | $0.89(1)$ | $0.00(1)$ | $0.02(1)$ |
| $|S, 1\rangle$ | $0.00(1)$ | $0.03(1)$ | $0.16(2)$ | $0.81(2)$ |
| $|D, 1\rangle$ | $0.07(1)$ | $0.00(1)$ | $0.84(2)$ | $0.09(2)$ |
take place (see Fig. 4) where the populations of the |SD⟩ and |DS⟩ are exchanged and a phase factor of −1 is acquired. The entangling mechanism is reminiscent of the Mølmer-Sørensen scheme which, however, uses a non-resonant bichromatic light field.

A similar reasoning holds for N ions as verified by numerical simulations. Taking N = 5 as example, we chose the parameters Ω = 2π × 230 kHz, Δ = 2π × 60 kHz, and η = 0.068/√5. In addition to the bus mode (center of mass) the two closest spectator modes were taken into account. After an interaction time of 900 µs a state

\[ |\Psi⟩ = \sqrt{0.48} |SSSS,000⟩ + e^{i\phi} \sqrt{0.45} |DDDDD,000⟩ + \sqrt{0.07}|ε⟩ \]

is achieved, where |n_b,n_{s1},n_{s2}⟩ refers to the phonon numbers of the bus-mode and two of the spectator modes, respectively. |ε⟩ is a superposition of all undesired states with mostly zero phonon excitation. The population in |ε⟩ could be reduced further by reducing the gate speed. The simulations take into account most of our current experimental parameters and imperfections.

A spin-echo technique reduces the influence of frequency fluctuations which are slow compared to the entangling time. Applying this technique, we observe an interference contrast of more than 0.9 over 2 ms with a single ion. Thus, we find that – for our current experimental imperfections – it is reasonable to entangle 5 ions with fidelities of about 80%. Unlike most other proposals the entangling time is \( \sim \sqrt{N} \) and works for any number of ions.

In conclusion, we have demonstrated a dispersive quantum gate operation employing light shifts conditional on the vibrational quantum number in a single ion. With numerical simulations, we explore the scheme for larger ion numbers and propose the generation of maximally entangled states for ion crystals (Eq. 4) with high fidelities.

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