Quark contribution to the nucleon polarizabilities and three-body forces

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We study the response of the nucleon, as a system of three bound (constituent) non relativistic quarks, to external (quasi static) electric and magnetic fields. The approach, based on a sum rule technique, is applied to a large class of two and three-body interquark potentials. Lower and upper bounds to the electric polarizability and paramagnetic susceptibility are explicitly calculated within a large variety of constituent models and their values related to the features of the interquark interaction picture. The role of three-body forces is investigated in details as well as the effects of SU(6) breaking terms in the potential model. Our results can be used to extract the mesonic contributions to the static polarizability and susceptibility. The quark degrees of freedom give a quite sizeable contributions to both and the meson cloud accounts roughly for 30% and 60% of the electric proton and neutron polarizability respectively. The quark contribution to the paramagnetic susceptibility is even higher and the mesonic effects are rather uncertain.

I. INTRODUCTION

The electric and magnetic polarizabilities, labeled $\alpha_E$ and $\beta_M$ respectively, are fundamental observables to understand the intrinsic structure of the nucleons. They characterize the ability of the constituents to rearrange in response to (quasi) static external electric and magnetic fields and are as fundamental as other parameters like charge radii and magnetic moments.

For the proton they can be measured by means of Compton scattering experiments since the Thomson scattering amplitude $T_T = -\hat{e} \cdot \hat{e}' e^2 / M_N$ is modified, for intermediate values of the photon energies ($50 \text{ MeV} \lesssim \omega \lesssim 100 \text{ MeV}$), by first order corrections due to the nucleon polarizabilities yielding to the low-energy Compton scattering cross section $\sigma_P$:

$$
\frac{d\sigma}{d\Omega}(\omega, \theta) = \frac{d\sigma^B}{d\Omega}(\omega, \theta) - \frac{e^2}{M_N} \left( \frac{\omega'}{\omega} \right) (\omega' \omega) 
\times \left[ \frac{\tilde{\alpha}_P^E + \tilde{\beta}_M^P}{2} (1 + \cos \theta)^2 + \frac{\tilde{\alpha}_P^E - \tilde{\beta}_M^P}{2} (1 - \cos \theta)^2 \right], \tag{1}
$$

where $d\sigma^B/d\Omega$ is the Born cross section for a proton with an anomalous magnetic moment and no other structure (e.g. ref. [2]), and $e^2$ is the fine structure constant. $\tilde{\alpha}_P^E$ and $\tilde{\beta}_M^P$ are the so called dynamic (or Compton) polarizabilities containing (within a non-relativistic approach) retardation effects as well as diamagnetic contributions [3]. We will discuss such effects later on.

Eq. (1) shows that forward and backward cross sections are dominated by $\tilde{\alpha}_P^E + \tilde{\beta}_M^P$ and $\tilde{\alpha}_P^E - \tilde{\beta}_M^P$, respectively, while experiments at $90^\circ$ are sensitive to $\tilde{\alpha}_P^E$ only and the polarizabilities are obtained by measuring the deviations of the cross section from the Born values. In addition a dispersion sum rule [4] constrains the sum of $\tilde{\alpha}_P^E$ and $\tilde{\beta}_M^P$:

$$
\tilde{\alpha}_P^E + \tilde{\beta}_M^P = \frac{1}{2\pi^2} \int_{m_*}^\infty \frac{\sigma_{tot}^P(\omega) d\omega}{\omega^2} \approx 14.2 \pm 0.5 \times 10^{-4} \text{ fm}^3, \tag{2}
$$

where $\sigma_{tot}^P(\omega)$ is the total proton photoabsorption cross section and the numerical value is obtained extrapolating the available experimental data to infinite energy [5].

A global analysis of the existing experimental data has been recently performed [5] yielding to

$$
\tilde{\alpha}_P^E = (12.1 \pm 0.8 \pm 0.5) \times 10^{-4} \text{ fm}^3; \quad \tilde{\beta}_M^P = (2.1 \pm 0.8 \pm 0.5) \times 10^{-4} \text{ fm}^3; \tag{3}
$$

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The neutron has no charge and Compton interference experiments to measure $\tilde{\alpha}_E^n$ are not possible\(^1\). Recent measurements (at ORNL) have been realized by means of a precise n-Pb scattering experiment \[^8\]. The electric dipole moment, induced by the nuclear charge on the moving neutron, produces a second order effect resulting in a $1/r^4$ interaction proportional to the neutron electric polarizability which can be measured looking at the energy dependence of the neutron-$^{208}$Pb scattering cross section. The result is

$$\tilde{\alpha}_E^n = (12.3 \pm 1.5 \pm 2.0) \times 10^{-4} \text{ fm}^3; \quad \tilde{\beta}_M^n = (3.5 \pm 1.6 \pm 2.0) \times 10^{-4} \text{ fm}^3; \quad (4)$$

where $\tilde{\beta}_M^n$ is extracted by assuming $\tilde{\alpha}_E^n + \tilde{\beta}_M^n = (15.8 \pm 0.5) \times 10^{-4} \text{ fm}^3$ according to the sum rule \[^4\], and the retardation effects are included following a suggestion due to L’vov and Petrun’kin. The direct experimental result of the n-Pb scattering is $\alpha_E^n$ rather than $\tilde{\alpha}_E^n$ since the measurement has a static character, and the value is $\alpha_E^n = (12.0 \pm 1.5 \pm 2.0) \times 10^{-4} \text{ fm}^3$ which is quite close the Compton polarizability because the corrections for the neutron are small.

For our study we will use non-relativistic approaches and, in this case, the corrections assume a rather simple form. In fact the Compton polarizabilities can be written (e.g. Friar in ref. \[^1\])

$$\tilde{\alpha}_E = \frac{1}{3M} \left\langle 0 \right| \left( \sum_{i=1}^{3} e_i r_i^2 \right) \left| 0 \right> + \frac{2}{\sqrt{E_n - E_0}} \sum_{n>0} \left| \left\langle n \right| D'_{i} \left| 0 \right> \right|^2 = \Delta \alpha_E + \alpha_E \quad (5)$$

and

$$\beta_M = -\frac{1}{2M} \left\langle 0 \right| \left( \sum_{i=1}^{3} e_i z_i' \right) \left| 0 \right> - \frac{1}{6} \left\langle 0 \right| \sum_{i=1}^{3} e_i^2 \frac{r_i^2}{m_i} \left| 0 \right> + \frac{2}{\sqrt{E_n - E_0}} \sum_{n>0} \left| \left\langle n \right| \mu_z \left| 0 \right> \right|^2 = \Delta \beta_M + \beta_M \mid \text{dia} + \beta_M \mid \text{para} \quad (6)$$

In Eq. (6)

$$D'_z = \sum_{i=1}^{3} e_i z_i' \quad (7)$$

is the nucleon electric dipole operator in the $z$-direction, i.e. in the direction of the external electric field. The coordinate of the charges $z_i'$ refer to the center of mass of the nucleon (the relevance of the motion of the center of mass in dipole excitations is one of the motivation for using non-relativistic model where the spurious contributions can be separated exactly) and $\Delta \alpha_E$ is a correction due to retardation effects $\Delta \alpha_E = \left\langle r^2 \right\rangle_{\text{ch}}/3M_N$ and is related to the charge mean square radius of the system.

In Eq. (8)

$$\mu_z = \sum_{i=1}^{3} \mu_{z,i} = \sum_{i=1}^{3} \frac{e_i}{2m_i} \sigma_i \quad (8)$$

is the nucleon dipole magnetic moment operator, $\sigma_i$ the Pauli matrices. $\tilde{\beta}_M$ contains both the diamagnetic and the paramagnetic susceptibilities and, in addition, a retardation correction that can be written

$$\Delta \beta_M = -\frac{3}{2M} \left\langle 0 \right| D'_z \ D'_z \left| 0 \right> \quad (9)$$

Eqs.\((8)\) are valid both for neutrons and protons.

From a theoretical point of view the nucleon polarizabilities received much attention and, in the last few years, many approaches have been developed within complementary frameworks like constituent quark models \[^9,11\], MIT bag models \[^12\] and its chiral extensions \[^13\], soliton models \[^14,15\], chiral perturbation theory \[^16,17\] and dispersion relation method \[^18\].

\(^1\)Compton scattering on the neutron can be realized by means of deuteron targets, e.g. ref. \[^6\].
In the present paper we want to reconsider the non-relativistic constituent quark model which is often believed to be inadequate to describe the static response of the nucleon to external fields (cfr. e.g. the discussion in refs. [3]). As a matter of fact the constituent quark model does not incorporate meson degrees of freedom, and consequently the effects of the meson cloud cannot be included. Therefore one cannot expect to reproduce the experimental value of the electric polarizability considering quark degrees of freedom only, since the meson cloud should be responsible for a sizeable part of the electromagnetic response. However the exact amount of the quark contribution has not been calculated by using potential models which are fitted on the baryonic spectrum, but simply estimated within naive models. In the following we develop a sum rule approach to the nucleon polarizabilities which is quite general to include two-body and three-body forces between the constituent quarks. In particular three-body forces have been recently considered in the context of quark models to refine the baryon mass spectrum predictions and, in particular, to reproduce the position of the Roper resonance. Moreover three-body forces are strictly related to the gluon-gluon interaction which is one of the fundamental aspects of QCD. In addition we consider consistently the retardation effects arising from a non-relativistic approach to the electric and magnetic response. Our results should be, therefore, considered as a precise estimate of the quark contribution to the electric and magnetic response.

The paper is organized in the following way: in section II we describe the sum rule approach to the linear response of a non-relativistic system and introduce a technique to evaluate various upper and lower bounds to the polarizability sum. The relation between the interquark potential model and the electric and magnetic sum rules is discussed in detail in section III where the Isgur-Karl model, three-body hyper-radial and two-body plus three-body potential models are considered and discussed for both electric and magnetic excitations. The Hamiltonian is solved exactly by means of the Schrödinger equation and the wave functions of the nucleon used to calculate the sum rules. The role of the confining, hyperfine and three-body forces is investigated in detail and the results summarized in section IV where the comparison with experimental data is also discussed.

II. SUM RULE APPROACH TO NUCLEON POLARIZABILITY

A quantity of the type

\[ m_{-1}(\Theta) = \sum_{n>0} \frac{\langle n|\Theta|0\rangle^2}{E_n - E_0} \quad (10) \]

is involved in both the expressions \([5]\) and \([1]\) for the electric polarizability and the (para) magnetic susceptibility (\(\Theta\) corresponds to the electric dipole \([\vec{p}]\) and the magnetic dipole operator \([\vec{p}\times\vec{m}]\) respectively). A direct evaluation of \(m_{-1}(\Theta)\) from Eq. \((11)\) involves all the complications of the excitation spectrum of the system (energies and wave functions) and it cannot be easily estimated truncating the excitation space \([19]\). However one can construct upper and lower bounds to \(m_{-1}\) \([1,11,21,20]\) by using the positivity of the corresponding strength distribution

\[ S_\Theta(\omega) = \sum_{n>0} \langle n|\Theta|0\rangle^2 \delta (\omega - (E_n - E_0)) \quad , \]

and few (positive) moments of such distribution \((p \geq 0, \text{ integer})\)

\[ m_p(\Theta) = \int_0^\infty d\omega \omega^p S_\Theta(\omega) = \sum_{n>0} \langle n|\Theta|0\rangle^2 (E_n - E_0)^p = \langle 0|\Theta^\dagger (E_0 - E_0)^p \Theta|0\rangle - \delta_{p0} \langle 0|\Theta|0\rangle^2 \quad . \]

The last expression has been obtained by using the closure property of the eigenstate \(|n\rangle\) and can be expressed in a simple form for the first few moments, or sum rules ( for reviews on the sum rule techniques see e.g. refs. \([22]\) )

\[ m_0(\Theta) = \langle 0|\Theta^\dagger \Theta|0\rangle - \langle 0|\Theta|0\rangle^2 \quad , \]

\[ m_1(\Theta) = \frac{1}{2} \langle 0| [\Theta^\dagger, [H_0, \Theta]] |0\rangle \quad , \]

\[ m_2(\Theta) = \frac{1}{2} \langle 0| [[\Theta^\dagger, H_0], [H_0, \Theta]] |0\rangle \quad , \]

\[ m_3(\Theta) = \frac{1}{2} \langle 0| [[\Theta^\dagger, H_0], [H_0, [H_0, \Theta]]] |0\rangle \quad , \]

etc.
The sum rules (13 - 16) involve the ground state wave function only, and commutators (anticommutators) of the nucleon Hamiltonian ($H_0$) with the excitation operator ($\Theta = D'_{\mu z}$). In addition the dynamical aspects embodied in the Hamiltonian enter in a more and more complex way increasing the order of the sum rules and the explicit calculation are of increasing complexity. In the next sections we show how one can construct bounds to the polarizability by means of sum rules of different order.

**A. Lower bounds**

In the limiting case of a strength distribution concentrated in a very narrow region ($\delta$-like distribution), the polarizability can be easily evaluated considering sum rules of positive order only. In fact

$$m_{-1}(\Theta) = \frac{m_0^2(\Theta)}{m_1(\Theta)} = \sqrt{m_1(\Theta) m_{-3}(\Theta)} \quad \text{etc.} \quad (17)$$

However the strength is generally distributed over many eigenstates and its spread affects the lower and upper moments in a different way, and the equalities (17) become rather inequalities

$$m_{-1}(\Theta) \geq \frac{m_0^2(\Theta)}{m_1(\Theta)} : \quad m_{-1}(\Theta) \leq \sqrt{m_1(\Theta) m_{-3}(\Theta)} \quad \text{etc.} \quad (18)$$

An elegant way of taking partially into account the effects of the spreading and the width of the strength distribution has been proposed by Dalfovo and Stringari [20] minimizing, with respect to the parameters $a$ and $b$, the inequality

$$\int_0^\infty d\omega S_\Theta(\omega) (1 + a\omega + b\omega^2)^p \geq 0 \quad (p \text{ integer}), \quad (19)$$

based on the positivity of $S_\Theta(\omega)$ [4]. One obtains

$$m_{-p}(\Theta) \geq \frac{m_{-p+1}(\Theta)}{m_{-p+2}(\Theta)} 1 - \Delta_p(\Theta)/\Gamma_p(\Theta), \quad (20)$$

where

$$\Delta_p(\Theta) = \left(\frac{m_{-p+3}}{m_{-p+2}} - \frac{m_{-p+2}}{m_{-p+1}}\right)^2, \quad (21)$$

and

$$\Gamma_p(\Theta) = \left[\frac{m_{-p+4}}{m_{-p+2}} + \left(\frac{m_{-p+2}}{m_{-p+1}}\right)^2 - 2\frac{m_{-p+3}}{m_{-p+1}}\right]. \quad (22)$$

A lower bound to the polarizability sum rule is obtained for $p = 1$

$$m_{-1}(\Theta) \geq \frac{m_0^2(\Theta)}{m_1(\Theta) 1 - \Delta_1(\Theta)/\Gamma_1(\Theta)}. \quad (23)$$

In addition to the previous relations, other bounds can be obtained by using Schwartz inequalities: an example involving the first odd sum rules is

$$m_{-1}(\Theta) \geq \frac{m_0^2(\Theta)}{m_3(\Theta)} \quad (24)$$

based on the conditions

$$\frac{m_p(\Theta)}{m_{p-2}(\Theta)} \leq \frac{m_{p+2}(\Theta)}{m_p(\Theta)} \quad (25)$$

valid for the integer values of $p = 0, \pm 1, \pm 2 \ldots$ etc. Eq. (24) is obtained for $p = 1$.

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2The authors of ref. [20] considered the case $p = 1$ only.
B. Upper bounds

Upper bounds to the polarizability are easily found considering that the energy transfer $\omega$ cannot be smaller than the energy difference between the first excited state of the system and the ground state, i.e. $\omega \geq E_1 - E_0 (= E_{10})$. The simplest way is just using the definition (10) and applying closure

$$m_{-1}(\Theta) = \sum_{n>0} |\langle n|\Theta|0\rangle|^2 / E_n - E_0 \leq \sum_{n>0} |\langle n|\Theta|0\rangle|^2 / E_{10} = m_0(\Theta) / E_{10}. \quad (26)$$

Corrections to the previous bound can be considered following again ref. [20]:

$$\int_0^\infty d\omega \frac{S_{\Theta}(\omega)}{\omega} (1 + \gamma \omega)^2 \leq \int_0^\infty d\omega \frac{S_{\Theta}(\omega)}{E_{10}} (1 + \gamma \omega)^2, \quad (27)$$

which leads to

$$m_{-1}(\Theta) \leq \frac{m_0(\Theta)}{E_{10}} \left[ 1 - \frac{m_0}{m_1} \left( \frac{m_1}{m_0} - E_{10} \right)^2 \left( \frac{m_2}{m_1} - E_{10} \right)^{-1} \right] = \frac{m_0(\Theta)}{E_{10}} \Lambda(\Theta). \quad (28)$$

An even more stringent upper bound can be obtained again from Eq.(19): in fact, for $p = 2$ one obtains

$$m_{-1}(\Theta) \leq \frac{m_1(\Theta) m_0(\Theta)}{m_2(\Theta)} \left[ 1 + \sqrt{\left( \frac{m_0 m_2}{m_1^2} - 1 \right) \left( \frac{m_2 m_{-2}}{m_0^2} - 1 \right)} \right] = \frac{m_1 m_0}{m_2} \Sigma(\Theta). \quad (29)$$

Note that the root argument is always positive (due to Schwartz inequalities and the relation (19) for $p = 1$). Eq. (29) involves inverse quadratic energy weighted sum rules.

Finally, from the Schwartz inequalities (28), one can obtain also an upper limit for $m_{-1}$ from

$$m_{-1}(\Theta) \leq \sqrt{m_1(\Theta) m_{-3}(\Theta)}, \quad (30)$$

which involves the inverse cubic moment.

$m_{-2}(\Theta)$ and $m_{-3}(\Theta)$ have no closed form in terms of commutators and/or anticommutators; due to the quadratic and cubic energy power at the denominator they can be easily estimated including the first excited states only (in the following we will make use of the first two states).

III. SUM RULES AND INTERQUARK POTENTIAL MODELS

The main ingredients entering the sum rules are: i) the nucleon ground state, and ii) the corresponding Hamiltonian

$$H_0 = T + V_{q-q}^{(2)} + V_{q-q-q}^{(3)}, \quad (31)$$

which includes, in principle, two and three-body interaction terms.

A. Harmonic oscillator potential model

The most simple choice is the harmonic oscillator two-body confining potential. It has analytic solutions and supplies a convenient classification scheme of the baryon resonances. If we assume a (spin-independent) harmonic form ($V_{q-q-q}^{(3)} = 0$), $V_{q-q}^{(2)} = 1/2 K \sum_{i<j} (r_i - r_j)^2 = 1/6 m \omega_{h.o.}^2 \sum_{i<j} (r_i - r_j)^2$, it is rather easy to derive the sum rules for the electric transitions by noting that all the excited electric dipole states are degenerates at $E^D = E_{10}^D = 1 \hbar \omega_{h.o.}$
and that $D_z', V_{q-q}^{(2)} + V_{q-q}^{(3)} = 0$. As a consequence the linear energy-weighted sum rule assume a model independent form \[23\]

$$m_1(D_z') = \frac{1}{2} \langle 0 \left| D_z', [H_0, D_z'] \right| 0 \rangle = \frac{1}{2} \langle 0 \left| D_z', [T, D_z'] \right| 0 \rangle = \frac{e^2}{3m}$$  \hspace{1cm} (32)

and the sum rules can be written

$$m_p(D_z') = m_1(D_z') (E^D)^{p-1} = \frac{e^2}{3m} \omega_{h.o.}^{p-1} = \frac{e^2}{3} \frac{\alpha_{h.o.}}{m_p}; \hspace{0.5cm} p = 0, \pm 1, \pm 2, \text{ etc.},$$  \hspace{1cm} (33)

with $\alpha_{h.o.}^2 = m \omega_{h.o.}$. When $p = -1$ one derives the prediction of the harmonic oscillator model for the electric polarizability of neutron and proton \[10\]

$$\alpha_{E}^{n,p} = 2 m_{-1}(D_z') = \frac{2}{9} e^2 \frac{M_N}{\alpha_{h.o.}}.$$  \hspace{1cm} (34)

The previous equation can be expressed in a more general form as function of the proton charge mean square radius $\langle r_p^2 \rangle_{ch} = 1/\alpha_{h.o.}$,

$$\alpha_{E}^{n,p} = \frac{2}{9} e^2 M_N (\langle r_p^2 \rangle_{ch})^2,$$  \hspace{1cm} (35)

which can be derived within a variational approach to the problem as it will be extensively discussed in section IV A.

The paramagnetic susceptibility vanishes in the h.o. model since the energy variation to flip a quark spin is zero (the interquark potential does not depend on the spin degrees of freedom and therefore the $\Delta$'s mass does not differ from the nucleon mass). From a sum rule point of view we note that, since $[H_0, \mu_z] = 0$ in the h.o. potential model, $m_1(\mu_z) = m_2(\mu_z) = m_3(\mu_z) = 0$, while $m_0(\mu_z) = \frac{e^2}{2m} (\langle r_p^2 \rangle_{ch})^2$ (cfr. section II C 2) and the bounds previously introduced are not well defined.

**B. Potential models and variational approaches**

In more sophisticated versions of the interquark interaction, the confining potential consists of the h.o. part plus terms which removes its degeneracy. This is the case of the Isgur-Karl (IK) model \[24\] which contains an unknown $U$-term added to shift the energies of some states. The model includes a delta-like hyperfine interaction derived from the one-gluon-exchange $qq$-potential. The baryonic states (obtained diagonalizing the Hamiltonian within a h.o. basis) result in a superposition of different $SU(6)$ configurations and lead to a rather good description of the baryonic spectrum.

A sum rule approach to the nucleon polarizabilities can be developed also for that kind of potential and we discussed results on $\alpha_E$ in a recent work \[11\]. However the use of sum rule technique is limited to potentials and wave functions which are well defined and self-consistent. The IK model, making use of an unknown potential term, prevents explicit calculations of the sum rules. As a consequence the approach proposed in ref. \[11\] was limited to few sums (namely $m_1, m_3$ and $m_{-3}$) and to the bounds

$$2 \frac{m_1^2(D_z')} {m_3^2(D_z')} \leq \alpha_E \leq 2 \sqrt{m_1(D_z') m_{-3}(D_z')},$$  \hspace{1cm} (36)

because they are the only ones which still contain some informations. In fact $m_1$ is model independent and the unknown (central) $U$-potential is irrelevant. $m_{-3}$ can be evaluated explicitly making use of the first two excited electric dipole states of the nucleon spectrum ($S_{11}$ and $D_{13}$) and therefore the upper limit is well defined. The lower bound, even if it is written as a combination of sum rules, defines a variational approach to $\alpha_E$ which assumes that the global effect of the external electric field ($E$) is a rigid translation of the $u$-quark wave function against the $d$-quarks without any additional deformation \[23\]. As a matter of fact variational and sum rule approaches are often related and we discuss this point in some detail.
Let us assume that the wave function of the polarized nucleon can be obtained from the unperturbed ground state by means of a unitary transformation shifting charges in opposite directions in a rigid way, one can write a set of trial states of the form

$$|\phi_E\rangle = e^{\eta(E) A} |0\rangle ,$$

(37)

$A$ is an anti-hermitian operator and $\eta(E)$ the variational parameter. Assuming $A = [T, D'_z]$ one has

$$|\phi_E\rangle = e^{\eta(E) [T, D'_z]} |0\rangle = e^{-i \eta(E) 2m \sum_i \epsilon_i' \varphi_{i',i} |0\rangle .}$$

(38)

The states $|\phi_E\rangle$, driven by the translation operator $p'_z$ are shifted, in the z-direction, in a rigid way. For weak external fields $\eta(E) \to 0$ and one can expand $|\phi_E\rangle$ to evaluate the variation of the total energy. One gets

$$E_{\text{tot}} = \langle \phi_E | H_0 - D'_z E | \phi_E \rangle$$

$$\to \langle 0 | H_0 | 0 \rangle - \frac{\eta^2}{2} \langle 0 | A | H_0, A \rangle | 0 \rangle - E \eta \langle 0 | D'_z, A | 0 \rangle$$

(39)

and the expectation value of the induced dipole defines the polarizability

$$\langle \phi_E | D'_z | \phi_E \rangle - \langle 0 | D'_z | 0 \rangle = \eta \langle 0 | D'_z, A | 0 \rangle \equiv \alpha_E E .$$

(40)

Minimizing the energy variation one can extract the equilibrium value of the variational parameter

$$\eta = -E \frac{-\langle 0 | D'_z, A | 0 \rangle}{\langle 0 | A, [H_0, A] | 0 \rangle}$$

(41)

and substituting this expression in Eq. (40) one finally obtains

$$\alpha_E = \frac{\langle 0 | D'_z, A | 0 \rangle^2}{\langle 0 | A, [H_0, A] | 0 \rangle}$$

$$= \frac{\langle 0 | [D'_z, [T, D'_z]] | 0 \rangle^2}{\langle 0 | [D'_z, T], [H_0, [T, D'_z]] | 0 \rangle} .$$

(42)

Eq. (42) represents a lower bound to the polarizability since the solutions of the variational equation for $|\phi_E\rangle$ are restricted to the wave functions obtained just shifting the ground state charge densities of the $u$ and $d$ quarks in opposite directions as described by Eqs. (37) and (38). However when the Hamiltonian $H_0$ of the system in study has the property $[H_0, D'_z] = [T, D'_z]$ because the potential commutes with the electric dipole operator, Eq. (42) can be written as combination of sum rules

$$\alpha_E = \frac{\langle 0 | [D'_z, [H_0, D'_z]] | 0 \rangle^2}{\langle 0 | [D'_z, H_0], [H_0, [H_0, D'_z]] | 0 \rangle} \equiv 2 \frac{m_i^2 (D'_z)}{m_3 (d'_z)} .$$

(43)

which is valid for all the models without velocity dependent (or non-local) potentials, and is equivalent to the lower bound disussed in section [11A] and used in ref. [11].

Other possible variational approaches have a close relation with sum rules and we discuss a second example which assumes that the wave function of the system, under the influence of the electric field, can be approximated by means of a simple deformation driven by an operator $F$ such that

$$|\psi_E\rangle = \frac{1}{\sqrt{N_E}} [1 + a F] |0\rangle .$$

(44)

$a$ is a variational parameter, $|0\rangle$ the unperturbed ground state of the nucleon, and $F = \sum_i f(r'_i)$ a single particle (local) operator depending on the (intrinsic) quark coordinates only. $\sqrt{N_E}$ is a normalization factor which ensures the condition $\langle \psi_E | \psi_E \rangle = 1$ and $f(r'_i)$ a function of $r'_i$ to be guessed in order to introduce physical "dipole" deformations on the wave function. We note that $F$ commutes with the dipole operator so that $[F, D'_z] = 0$, $\langle 0 | F | 0 \rangle = 0$ because of parity and the "dipole" character of the function $F$, and $N_E = 1 + a^2 \langle 0 | F^2 | 0 \rangle$. Calculating the total energy variation one gets
Minimizing (45) and substituting in (46) the obtained equilibrium value of \( a \), one has

\[
\alpha_E = 4 \frac{(\langle 0 | D'_z D'_z | 0 \rangle)^2}{\langle 0 | D'_z [H_0, D'_z] | 0 \rangle} = 2 \frac{m_0^2(D'_z)}{m_1(D'_z)},
\]

and a simplified lower bound is found. It is rather similar to the bound (23) and can be obtained from the constraint (43) in the limit \( p = 1 \) and \( b = 0 \).

C. Hyperradial potentials and three-body forces

Just at the opposite side of the simple two-body harmonic potential, one can locate the three-body force model (TBM) recently proposed by Ferraris, Giannini, Pizzo, Santopinto and Tiator [27]. The model assumes that the interquark potential can be written as an hypercentral potential, i.e. a potential depending on the hyperradius only. The idea of hyperradius (\( \xi \)) is introduced in the so called hyperspherical formalism [28] together with the hyperangle (\( \phi \)), to define the hyperspherical set of coordinates in a six-dimensional space: \( \Omega_\rho, \Omega_\lambda, \xi \) and \( \phi \), with \( \xi = \sqrt{\rho^2 + \lambda^2} \), \( \phi = \arctan(\xi) \) and \( \rho \) and \( \lambda \) are the absolute values of the Jacobi coordinates \( \hat{\rho} = (r_1 - r_2)/\sqrt{2} \), and \( \hat{\lambda} = (r_1 + r_2 - 2 r_3)/\sqrt{6} \).

Under the assumption of an hypercentral potential the intrinsic Hamiltonian \( H_0 \) can be written

\[
H_0 = -\frac{1}{2 m} \left( \nabla^2_\rho + \nabla^2_\lambda \right) + V^{(3)}(\xi) = \frac{1}{2 m} \left( \frac{d^2}{d\xi^2} + \frac{5}{\xi} \frac{d}{d\xi} + \frac{L^2(\Omega)}{\xi^2} \right) + V^{(3)}(\xi),
\]

where \( L^2(\Omega)/\xi^2 \) is the analogous, in six dimensions, of the three-dimensional centrifugal barrier and \( \Omega \) embodies \( \Omega_\rho, \Omega_\lambda, \phi \). The eigenfunctions of the grand-angular operator \( L^2 \) are called hyperspherical harmonics and denoted by \( Y_{\gamma}(\Omega) \). They are written as products of spherical harmonics in \( \Omega_\rho, \Omega_\lambda \) with angular momentum \( l_\rho \) and \( l_\lambda \) (corresponding to the coordinates \( \hat{\rho} \) and \( \hat{\lambda} \)) and Jacobi polynomials in the hyperangle \( \phi \) [28]. They form a complete orthogonal basis in the space of the five-dimensional functions of \( \Omega_\rho, \Omega_\lambda \) and \( \phi \). The eigenvalues of \( L^2 \) are \( -\gamma (\gamma + 4) \), \( \gamma \) are called the grand-angular quantum numbers and are given by \( \gamma = 2 k + l_\rho + l_\lambda \) with \( k \) integer and non-negative. As long as the potential is hypercentral the complete wave function \( \Psi(\xi, \Omega) \) can separated as \( \Psi(\xi, \Omega) = \psi_{\nu,\gamma}(\xi) Y_{\gamma}(\Omega) \) and the hyperradial wave function \( \psi_{\nu,\gamma}(\xi) \) is solution of a Schrödinger equation. For fixed value of \( \gamma \), the label \( \nu \) indicates the number \( (\nu + 1) \) of nodes in the wave function.

The hypercentral character of the potential means (in general) that the interquark interaction has a genuine three-body character, in the sense that the coordinates of a specific pair cannot be disentangled from the third one [29]. The idea of a three-quark force is related to the non-abelian character of QCD, in particular to the existence of a direct gluon-gluon interaction which represents a fundamental justification for the introduction of three-body forces.

3 The two example given in this section could be unified within the unitary transformation formalism of Eq. (27), considering the operator \( A \) as a many-body operator satisfying the relations \( \langle 0 | [A, D'_z] | 0 \rangle = \langle 0 | F D'_z | 0 \rangle \) and \( \langle 0 | [A, [H_0, A]] | 0 \rangle = \langle 0 | [F, [H_0, F]] | 0 \rangle \).
From a more phenomenological point of view it has been argued [31,32] that the contribution of a three-body force is of some help in solving the "Roper puzzle" [33] giving a consistent explanation to its relative position in the spectrum.

For a given hypercentral potential the eigenvalue equation has to be solved numerically obtaining energies and hypercentral wave functions which, depending on the hyperradius only, result to be completely symmetric. The correct global symmetry of the state is obtained combining $\psi_{\nu,\gamma}(\xi)$ with the appropriate hyperangular, angular, spin and isospin wave functions [34]. In particular the nucleon wave function is written as $|N, J^P = 1/2^+\rangle = \psi_{00}(\xi) Y^{(0,0)}_{[0,0]}(\Omega) \chi_S(1/2; 1/2)$, (50)

where $\chi_S(1/2; 1/2)$ is the symmetric $SU(6)$ combination of spin and isospin wave functions of the three quarks, and we have explicitly indicated the symmetry required for the hyperspherical wave function and the total angular momentum $\tilde{L} = \tilde{L}_\rho + \tilde{L}_\lambda$: $Y^{(L,M)}_{[\gamma,\text{symmetry}]}$. For any given $SU(6)$-configuration only $\psi_{\nu,\gamma}(\xi)$ will depend on the potential and we can write general results for all the sum rules when $H_0 = T + V^{(3)}(\xi)$ [35].

1. sum rules for the electric dipole excitations

i) $m_0(D'_z)$:

$$m_0(D'_z) = \left\langle \sum_i e_i^2 z_i^2 \right\rangle + \left\langle \sum_{i \neq j} e_i e_j z_i^2 z_j^2 \right\rangle = \frac{1}{9} e^2 \int d\xi \xi^7 |\psi_{00}(\xi)|^2 = \frac{1}{3} e^2 \langle r_p^2 \rangle_{\text{ch}}$$ (51)

both for protons and neutrons. The non-energy-weighted sum rule is therefore proportional to the mean square mass radius of the nucleon, which, at least as long as the nucleon wave function belongs to a $SU(6)$ multiplet, is also the proton charge radius.

ii) $m_1(D') = e^2/3m$ remains unmodified for all the velocity independent potentials as already discussed (cfr. Eq. (32)) and does not distinguish neutron and proton. Its model independence is based on the property $[V^{(3)}(\xi), D'_z] = 0$

iii) $m_2(D'_z)$

$$m_2(D'_z) = -\frac{1}{2m^2} \left( \frac{1}{3} (e_1 + e_2 - 2e_3)^2 \nabla_\rho^2 \nabla_\lambda^2 + (e_1 - e_2)^2 \nabla_\rho^2 \nabla_\lambda^2 + \right.$$  
$$+ \frac{2}{\sqrt{3}} (e_1 + e_2 - 2e_3) (e_1 - e_2) \nabla_\rho^2 \nabla_\lambda^2 \right) =$$  
$$= \frac{e^2}{9m} \langle T \rangle = -\frac{e^2}{9m^2} \int d\xi \xi^5 |\psi_{00}(\xi)|^2 \left( \frac{d^2}{d\xi^2} + \frac{5}{\xi} \frac{d}{d\xi} \right) \psi_{00}(\xi) .$$ (52)

$m_2$ does not depend on the interquark interaction explicitly, however is proportional to the mean kinetic energy of the system and measures the presence of high momentum components in the nucleon wave functions. If the potential is highly confining the high momentum components are larger and the sum increases.

iv) $m_3(D'_z)$

$$m_3(D'_z) = \frac{1}{2m^2} \left( \frac{1}{6} (e_1 + e_2 - 2e_3)^2 (\nabla_\lambda^2 \nabla_\lambda^2 V^{(3)}(\xi)) + \right.$$  
$$+ \frac{1}{2} (e_1 - e_2)^2 (\nabla_\rho^2 \nabla_\lambda^2 V^{(3)}(\xi))) +$$  
$$+ \frac{1}{\sqrt{3}} (e_1 + e_2 - 2e_3) (e_1 - e_2) (\nabla_\lambda^2 \nabla_\rho^2 V^{(3)}(\xi)) \right) =$$  
$$= \frac{e^2}{18m^2} \int d\xi \xi^5 |\psi_{00}(\xi)|^2 \left( \frac{d^2 V^{(3)}(\xi)}{d\xi^2} + \frac{5}{\xi} \frac{d V^{(3)}(\xi)}{d\xi} \right) .$$ (53)

The three-times energy-weighted sum is particularly interesting because it depends crucially on the potential model and on its derivatives.
2. sum rules for magnetic dipole excitations

Magnetic excitations induced by the dipole operator \(\mathcal{V}_{\text{Hyp}}(2)\) are easily understood in the case of spin-independent interactions. In fact the magnetic operator commutes with the nucleon Hamiltonian \([T + V^{(3)}\xi],\mu_z] = 0\) and the total strength is concentrated at a value of the excitation energy equal to the ground state and one gets:

i) \(m_0(\mu_z)\)

\[
m_0(\mu_z) = \frac{1}{(2m)^2} \left[ \sum_{i \neq j} e_i e_j \sigma_i^z \sigma_j^z + \sum_i e_i^2 \right] - \left| \left( \sum_i e_i \sigma_i^z \right) \right|^2
\]

\[
= \left( \frac{e}{2m} \right)^2 \left[ \frac{8}{9} + 1 - 1 \right] = \frac{8}{9} \mu_0^2.
\]  

The result (54) is valid for both protons and neutrons (in the neutron case the individual contributions in parenthesis become \(2/3 + 2/3 - 4/9\)), and we have defined \(\mu_0 = e/2m\).

ii) higher sum rules:

\[
m_1(\mu_z) = m_2(\mu_z) = m_3(\mu_z) = 0
\]  

vanish because we did not consider the effects due to hyperfine interaction. In the following we discuss a variety of hypercentral potential investigated in ref. [27], and to this hand we discuss first the role of the spin-spin interaction terms.

D. The role of the hyperfine interaction

The authors of the ref. [27] include a perturbative Hyperfine interaction in the three-body Hamiltonian in order to reproduce the correct \(N-\Delta\) mass splitting. Such contribution is seen as the non-relativistic reduction of the one-gluon-exchange interaction [37]. We restrict the Hyperfine contribution to the dominant spin-spin zero-range term

\[
V^{(2)}_{\text{Hyp}}(r_{12}) = \frac{2\alpha_S}{3m^2} \frac{8\pi}{3} \mathbf{S}_1 \cdot \mathbf{S}_2 \delta(\mathbf{r}_{12})
\]  

(56)

neglecting the (small) tensor contribution. The effective coupling constant \(\alpha_S\) is fixed by the \(N-\Delta\) mass splitting

\[
M_\Delta - M_N = \frac{\sqrt{2}}{3} \frac{\alpha_S}{m^2} \frac{8}{\pi} \int d\xi \xi^2 |\psi_{00}(\xi)|^2.
\]  

(57)

The additional contributions due to the Hyperfine potential (56) prevents the magnetic dipole sums to vanish, and modifies the energy-weighted electric dipole sums.

I. sum rules for the electric dipole excitations

i) \(m_1(D'_z)\) and \(m_2(D'_z)\) remain unmodified since \([D'_z, V^{(2)}_{\text{Hyp}}] = 0\)

ii) \(m_3(D'_z)\) gains an additional contribution:

\[
m_3(D'_z)_{\text{Hyp}} = \frac{\sqrt{2}\pi}{9} \frac{\alpha_S}{m^2} \langle (e_1 - e_2)^2 \mathbf{S}_1 \cdot \mathbf{S}_2 [\nabla_\rho^2 \delta(\hat{\rho})] \rangle
\]

\[
= - \frac{1}{3} \frac{e^2}{m^2} (M_\Delta - M_N) \int d\xi \psi_{00}(\xi) \psi_{00}'(\xi) \int d\xi \xi^2 |\psi_{00}(\xi)|^2,
\]  

(58)

which has to be added incoherently to the Eq. (53). In Eq. (58) \(\psi_{00}(\xi)\) indicates the total derivative of the wave function and the relation (57) has been used [36].
The crucial role of the Hyperfine interaction for the magnetic energy-weighted sum rules has been already pointed out in the previous sections by stressing the fact that these sum rules simply vanish if the spin-spin term \( (56) \) is neglected. Hyperfine potential is, however, a two-body operator and the commutators with the magnetic dipole moment \( (6) \) are not as simple as the electric case. For simplicity we will restrict to the linear energy-weighted sum rule which, differently from \( m_1(D^0) \), is not model independent and embodies the interesting aspects of the interaction-dependence we recognize in \( m_3(D^0) \).

As long as the eigenstates of the hypercentral potential are assumed to belong to a specific \( SU(6) \) multiplet, the magnetic operator can excite one state only: the \( \Delta_{33} \) resonance. Within such simple \( N-\Delta \) excitation model the magnetic susceptibility is given by

\[
\beta_M = \frac{|\langle \Delta | \mu_z | N \rangle|^2}{M_\Delta - M_N} = \frac{16}{9} \mu_0^2 \frac{1}{M_\Delta - M_N} \approx 8.7 \cdot 10^{-4} \text{fm}^3 . \tag{59}
\]

The sum rule approach cannot add any information to this simple scheme as it can be demonstrated calculating the first moments of the magnetic strength distribution:

i) \( m_0(\mu_z) \) has the form \((54)\).

ii) \( m_1(\mu_z) \)

\[
m_1(\mu_z) = -\frac{12}{m^2} \left[e_1 - e_2 \right] \left(S_1 \cdot S_2 - S_1^z S_2^z \right) \frac{2\pi}{9} \frac{\alpha_S}{m^2} \delta(12) \]
\[
= \frac{\alpha_S}{27\sqrt{2}} \frac{e^2}{m^2} \frac{16}{\pi} \int d\xi \xi^2 |\psi_0(\xi)|^2 \]
\[
= \frac{8}{9} \mu_0^2 \left( M_\Delta - M_N \right) , \tag{60}
\]

where Eq. \((57)\) has been used.

By using the bounds involving \( m_1, m_0 \) and \( E_{10}^M = M_\Delta - M_N \) one obtains

\[
2 \frac{m_0(\mu_z)}{m_1(\mu_z)} = \frac{16}{9} \frac{\mu_0^2}{M_\Delta - M_N} \leq \beta_M \leq 2 \frac{m_0(\mu_z)}{E_{10}^M} = \frac{16}{9} \frac{\mu_0^2}{M_\Delta - M_N} , \tag{61}
\]

and the two limits coincides with the result \((59)\) as expected.

An estimate of \( \beta_M \) different from the previous result comes from \( SU(6) \) breaking components in the nucleon wave function. In the next section we discuss a potential model whose nucleon wave function results in a superposition of different \( SU(6) \) configurations.

### E. Two-body + three-body interactions

The pathologic aspect of non-relativistic quark models involving confining potentials plus (at least part of) the one-gluon-exchange interaction remains the impossibility to predict the masses of the first (negative parity) excitations in the baryonic octet and decuplet correctly. The authors of ref. \[32\] proposed to add a phenomenological three-quark one-gluon-exchange interaction remains the impossibility to predict the masses of the first (negative parity) excitations \( (6) \) multiplet, the magnetic operator is confined, Coulomb-like and spin-spin \( qq \) interaction of the form \[38\]

\[
V_{q\bar{q}}^{(2)}(\xi) = \frac{1}{2} \left( -\frac{\kappa}{r_{12}} + \frac{r_{12}}{a^2} + \frac{4}{m^2} \frac{\exp(-r_{12}/r_0)}{r_0^2 r_{12}} S_1 \cdot S_2 - D \right) , \tag{62}
\]

where the Yukawa form of the spin-spin term replaces the delta contact interaction of the OGE potential \[36\] to avoid an unbounded spectrum when the Schrödinger equation is solved.

The three-body term suggested by Cano et al. in ref. \[32\]

\[
V_{q\bar{q}q}^{(3)} = V_0 \exp \left( -\sum_{i<j} \frac{r_{ij}^2}{\lambda_0^2} \right) = V_0 \exp \left( -3\xi^2/\lambda_0^2 \right) \tag{63}
\]
is identified as $V_{III}^{(3)}$ by the same authors and assumes a simple form within the hyperspherical formalism. The parameters of the interaction can be found in table 1 of their paper and in the following we discuss the results of such quite recent version of the QCD inspired potential including two- and three-body contributions.

The solutions of the Schrödinger equation can be expanded on the hyperspherical basis and this is the actual procedure followed in ref. [32]. In practical calculations only two terms have been retained and the nucleon wave function can be written:

$$|N, J^P = 1/2^+\rangle = \Psi_1(\xi) Y_{[\mu,S]}^{(0,0)}(\chi_{[S]})(1/2; 1/2) +$$

$$+ \Psi_3(\xi) \frac{1}{\sqrt{2}} \left[ Y_{[\mu,MS]}^{(0,0)}(\chi_{[MS]})(1/2; 1/2) + Y_{[\mu,MA]}^{(0,0)}(\chi_{[MA]})(1/2; 1/2) \right],$$

(64)

where the second term of the expansion is clearly an $SU(6)$-breaking contribution, and $\chi_{[MS]}(1/2; 1/2)$, $\chi_{[MA]}(1/2; 1/2)$ are the mixed-symmetric (antisymmetric) $SU(6)$ combinations of spin and isospin wave functions of the three quarks. Owing to the $SU(6)$-breaking term in the nucleon wave function, some results of the pure hyper-radial potentials are modified. For instance the charge root mean square radius of the neutron is not vanishing and the values depend on the contributions coming from the three-body part of the potential.

1. sum rules for electric dipole excitations

i) $m_0(D'_z)$:

$$m_0(D'_z) = \frac{1}{9} e^2 \left\{ \int d\xi \xi^7 \left[ |\Psi_1(\xi)|^2 + |\Psi_3(\xi)|^2 \right] - \frac{1}{\sqrt{2}} \int d\xi \xi^7 \Psi_1(\xi) \Psi_3(\xi) \right\}$$

$$= \frac{1}{3} e^2 \langle r_0^2 \rangle_{ch} + \frac{2}{3} e^2 \langle r_0^2 \rangle_{ch},$$

(65)

both for protons and neutrons. The non-energy-weighted sum rule, therefore, is no longer proportional to the charge mean square radius of the proton when $SU(6)$ breaking component in the nucleon wave function are taken into account.

ii) $m_1(D'_z) = e^2/3m$ again remains unmodified because $[V_{q-q}^{(2)}(\xi) + V_{q-q}^{(3)}(\xi), D'_z] = 0$ as already discussed and does not distinguish neutron and proton.

iii) $m_2(D'_z)$ is no longer proportional to the mean kinetic energy and contains an additional term:

$$m_2(D'_z) = \frac{2e^2}{9m} \langle T \rangle + \frac{e^2}{9m^2} \frac{1}{\sqrt{2}} \int d\xi \xi^5 \Psi_1^*(\xi) \left( \frac{d^2}{d\xi^2} - \frac{1}{\xi} \frac{d}{d\xi} \right) \Psi_1(\xi);$$

(66)

iv) $m_3(D'_z)$

Both $V_{q-q}^{(2)}$ and $V_{q-q}^{(3)}$ contribute to the sum rule which can be written $m_3(D'_z) = m_3(D'_z)|_{q-q} + m_3(D'_z)|_{q-q-q}$, where

$$m_3(D'_z)|_{q-q} = \frac{1}{4m^2} \langle (e_1 - e_2)^2 \rangle_{r_{12}} \left[ \frac{\kappa_C}{r_{12}} + \frac{r_{12}}{a^2} + 4 \frac{\kappa_{\sigma}}{m^2} \exp(-r_{12}/r_0) \frac{S_1 \cdot S_2}{r_{12}^3} \right],$$

(67)

and the three-body $m_3(D'_z)|_{q-q-q}$ is again given by Eq. (68) with the obvious replacement $V_{q-q}^{(3)}(\xi) \rightarrow V_{q-q-q}^{(3)}(\xi)$.

2. sum rules for magnetic excitations

The role of $SU(6)$ breaking components in the nucleon wave function is particularly relevant for the magnetic excitations because the operator $\mu_z$ can now mix different $SU(6)$ configurations. We discuss the simple bounds which involve $m_0$, $m_1$ and $E_{[6]}^H$ only. One gets

i) $m_0(\mu_z)$

$$m_0(\mu_z) = \mu_0^2 \left[ 1 + \frac{8}{9} P_1 - \left( P_1 + \frac{1}{3} P_3 \right)^2 \right],$$

(68)
where $P_\alpha = \int d\xi \xi^5 |\psi_\alpha(\xi)|^2$ and one recovers the result (54) in the limit $P_1 = 1$ and $P_3 = 0$. In the neutron case the sum rule reads

$$m_0(\mu_z) = \mu_0^2 \frac{2}{3} \left[ 1 + \left( P_1 + \frac{1}{3} P_3 \right) - \frac{2}{3} P_1^2 \right]. \quad (69)$$

However, despite the different form of the sum rules the numerical values are quite close each other and do not produce differences in the estimates of the $\beta_M$ for protons and neutrons.

ii) $m_1(\mu_z)$

$$m_1(\mu_z) = -\frac{12}{m^2} \left( (e_1 - e_2)^2 (S_1 \cdot S_2 - S^z_1 S^z_2) \frac{1}{4} \kappa_\sigma \exp(-r_{12}/r_0) \right). \quad (70)$$

IV. NUMERICAL RESULTS AND DISCUSSION

We discuss, first, numerical results for a variety of hyperradial potentials introduced by Ferraris et al. [27], namely

1. $V_1^{(3)}(\xi) = -\frac{\tau}{\xi} + k_1 \xi$
2. $V_2^{(3)}(\xi) = -\frac{\tau}{\xi} + k_1 \xi + \frac{b}{\xi^2}$
3. $V_3^{(3)}(\xi) = -\frac{\tau}{\xi} + k_1 \xi + \frac{b}{\xi^2} + c \log \xi$
4. $V_4^{(3)}(\xi) = -\frac{\tau}{\xi}$
5. $V_5^{(3)}(\xi) = -\frac{\tau}{\xi} + \frac{b}{\xi^2}$

whose parameters are summarized, for convenience, in table I. The potentials $V_1 - V_3$ have been fitted on the baryon mass spectrum, the hypercoulomb parametrization (74) has been considered as check of numerical calculations, while the version (75) has been introduced in ref. [27] because it reproduces the electric dipole form factor and root mean square radius of the proton. In the same table the predictions of the mass radius are also presented.

| potentials | $\tau$ [u] | $k_1$ [fm$^{-2}$] | $b$ [fm] | $c$ [fm$^{-1}$] | $\langle r^2 \rangle$ [fm$^2$] |
|------------|-----------|------------------|--------|-------------|-----------------|
| $V_1^{(3)}$ | 4.59     | 1.61             | -      | -           | (0.516)$^2$     |
| $V_2^{(3)}$ | 2.50     | 1.14             | -0.80 | -           | (0.483)$^2$     |
| $V_3^{(3)}$ | 2.50     | 1.14             | -0.80 | 0.1         | (0.475)$^2$     |
| $V_4^{(3)}$ | 6.39     | -                | -      | -           | (0.462)$^2$     |
| $V_5^{(3)}$ | 1.78     | -                | -0.78 | -           | (0.88)$^2$      |
A. Electric dipole polarizability

1. Results of three-body force models (TBM)

The electric dipole sum rules are shown in Table II for the hyperradial interactions (71) - (75), and the corresponding lower and upper limits in Tables III and IV. The smaller is the predicted radius of the system, the smaller is the non-energy-weighted sum and the larger the $m_2$ moment. In fact $m_0(D'_z)$ is proportional to the proton charge radius and $m_2(D'_z)$ to the mean kinetic energy which is larger in smaller systems because of the indetermination principle. The role of the hyperfine interaction is clearly seen in $m_3(D'_z)$ (cfr. Table III). A large part of the sum comes from the spin-spin interaction and the lower limit $2m_1^2/m_3$ can become significantly small (cfr. Table III) loosing a direct connection with reasonable values of the polarizability. One can conclude that the charge deformation induced by the external electric field on the quark distribution is far from being approximated by a rigid oscillation of $u$ and $d$ charge densities (opposite in phase) as assumed in the relations (42), and (43). The inclusion of hyperfine contributions to the $m_3$ sum lowers the polarizability by few percent enlarging the difference between the upper and lower limits.

TABLE II. Electric dipole sum rule values for the potential models (71)-(75) (cfr. also Table I). For the $m_3$ sum the contributions coming from the hyperradial part of the potential is shown, the total result contains, in addition, the Hyperfine contribution coming from the interaction term (56).

| potentials | $m_0(D'_z)$ [fm$^3$] | $m_1(D'_z)$ [fm] | $m_2(D'_z)$ [u] | $m_3(D'_z)|_{V^{(3)}}$ [fm$^{-1}$] | $m_3(D'_z)|_{tot}$ [fm$^{-1}$] |
|------------|---------------------|-----------------|-----------------|---------------------------------|-------------------------------|
| $V_1^{(3)}$ | 0.0006              | 0.0015          | 0.0040          | 0.0137                          | 0.0278                        |
| $V_2^{(3)}$ | 0.0006              | 0.0015          | 0.0056          | 0.0760                          | 0.2085                        |
| $V_3^{(3)}$ | 0.0005              | 0.0015          | 0.0058          | 0.0799                          | 0.2149                        |
| $V_4^{(3)}$ | 0.0005              | 0.0015          | 0.0053          | 0.0274                          | 0.0498                        |
| $V_5^{(3)}$ | 0.0019              | 0.0015          | 0.0018          | 0.0098                          | 0.0602                        |

TABLE III. Lower bounds to the electric polarizability as predicted by the potential models (71)-(75). In parenthesis the numerical results obtained neglecting the Hyperfine contribution in $m_3$ (cfr. Table III). The polarizability is expressed in $10^{-4}$ fm$^3$.

| potentials | $2 \frac{m_1^2(D'_z)}{m_3(D'_z)}$ | $2 \frac{m_2^2(D'_z)}{m_3(D'_z)}$ | $2 \frac{m_3^2(D'_z)}{m_3(D'_z)|_{tot}}$ |
|------------|---------------------------------|---------------------------------|---------------------------------|
| $V_1^{(3)}$ | 1.69 (3.44)                     | 5.48                            | 5.50 (5.60)                     |
| $V_2^{(3)}$ | 0.23 (0.62)                     | 4.19                            | 4.22 (4.29)                     |
| $V_3^{(3)}$ | 0.22 (0.59)                     | 3.94                            | 3.97 (4.04)                     |
| $V_4^{(3)}$ | 0.94 (1.72)                     | 3.51                            | 3.55 (3.65)                     |
| $V_5^{(3)}$ | 0.78 (4.79)                     | 45.7                            | 45.9 (47.1)                     |
Looking at the comparison between the simple lower bound

\[ \alpha_E^{p,n} = 2 \frac{m_0^2(D'_1)}{m_1(D'_2)} = \frac{2}{9} e^2 (3 m) \langle r_p^2 \rangle_{ch} \]  

\[ (76) \]

shown in table III and the results of table IV, one can see that the approximation (76) is rather good for all the potential models suggesting that Eq. (44) with \( F = D'_1 \) represents a more reliable nucleon charge deformation induced by the electric field.

**TABLE IV.** Upper bounds to the electric polarizability as predicted by the potential models (71)-(75). The polarizability is expressed in \( 10^{-4} \) fm\(^3\).

| potentials | \( m_1(D'_1) m_0(D'_0) \) \( \Sigma \) | \( \frac{2 m_0(D'_1)}{E_{ch}} \) | \( \frac{2 m_0(D'_1)}{E_{ch}} \) \( \Lambda \) | \( 2 \sqrt{m_1(D'_1) m_{-3}(D'_2)} \) |
|------------|---------------------------------|------------------|-----------------|------------------|
| \( V_1^{(3)} \) | 5.65 | 5.86 | 5.71 | 5.88 |
| \( V_2^{(3)} \) | 4.53 | 4.09 | 3.82 | 3.95 |
| \( V_3^{(3)} \) | 4.27 | 4.79 | 4.51 | 4.75 |
| \( V_4^{(3)} \) | 3.64 | 66.2 | 58.1 | 56.0 |
| \( V_5^{(3)} \) | 46.5 | | | |

**TABLE V.** Bounds to the electric polarizability as predicted by the potential models (71)-(75). The results obtained neglecting Hyperfine contributions are also shown. The polarizability is expressed in \( 10^{-4} \) fm\(^3\).

| potentials | \( V^{(3)}(\xi) \) [only] | \( V^{(3)}(\xi) + V^{(2)}_{\text{Hyf}} \) |
|------------|------------------|------------------|
| \( V_1^{(3)} \) | \( \alpha_{E}^{p,n} = 5.65 \pm 0.05 \) | \( \alpha_{E}^{p,n} = 5.60 \pm 0.10 \) |
| \( V_2^{(3)} \) | \( \alpha_{E}^{p,n} = 4.54 \pm 0.25 \) | \( \alpha_{E}^{p,n} = 4.50 \pm 0.28 \) |
| \( V_3^{(3)} \) | \( \alpha_{E}^{p,n} = 4.27 \pm 0.23 \) | \( \alpha_{E}^{p,n} = 4.24 \pm 0.27 \) |
| \( V_4^{(3)} \) | \( \alpha_{E}^{p,n} = 3.74 \pm 0.09 \) | \( \alpha_{E}^{p,n} = 3.69 \pm 0.14 \) |
| \( V_5^{(3)} \) | \( \alpha_{E}^{p,n} = 52.6 \pm 5.50 \) | \( \alpha_{E}^{p,n} = 52.9 \pm 6.10 \) |

**2. results of Two-body + three-body force model**

In the following we present results of two sets of potentials (cfr. section III E): i) \( V_I \): includes two-body interaction only (see Eq. (62)); \( V_{III} \): includes a three-body term also as in Eq. (63). The numerics is summarized in tables VI - IX, where also the effects of the hyperfine interaction and three-body terms are emphasized. In particular the role of three-body interaction is quite important. Neglecting that contribution to \( m_3(D'_1) \) yields to a negative value of the lower bound (cfr. table VII), while the hyperfine interaction plays a role similar to that described in relation with the hyperradial models. The overall impression is that, differently from the hyperradial models, the two body models have a quite small radius and therefore a smaller electric polarizability. Adding the three-body term does not improve the situation and the resulting polarizability is even smaller.

**TABLE VI.** Electric dipole sum rule values for the potential models \( V_I, V_{III} \) (62)-(63). For the \( m_3 \) sum the contributions arising from the various potential terms of Eq. (57) are shown separately; the last value refers to the three-body term and it vanishes for the \( V_I \) version.

| potentials | \( m_0(D'_1) \) | \( m_1(D'_1) \) | \( m_2(D'_1) \) | \( m_3(D'_1) \) \( \text{tot} \) |
|------------|----------------|----------------|----------------|----------------|
| \( V_I \) | 0.0005 | 0.0014 | 0.0044 | 0.0168 = 0.0022 + 0.0073 + 0.0072 + \text{zero} |
| \( V_{III} \) | 0.0003 | 0.0015 | 0.0197 | 0.7441 = 0.1214 + 0.0040 + 0.0524 + 0.5662 |
TABLE VII. Lower bounds to the electric polarizability as predicted by the potential models $V_I, V_{III}$ (22)-(23). The results in brackets () for the version $I$ are obtained neglecting the Hyperfine contributions; those ones in [ ] for the version $III$ are obtained excluding the three-body contributions. The polarizability is expressed in $10^{-4}$ fm$^3$.

| potentials | $2\frac{m_1^2(D'_0)}{m_1(D'_0)}$ | $2\frac{m_1^2(D'_0)}{m_1(D'_0)}$ | $2\frac{m_1^2(D'_0)}{m_1(D'_0)}$ |
|------------|-----------------|-----------------|-----------------|
| $V_I$      | 2.42 (4.25)     | 3.27            | 3.30 (3.24)     |
| $V_{III}$ | 0.06 [0.25]     | 0.97            | 1.15 [−0.061]   |

Another interesting conclusion can be drawn for the approximated expression (48). Its values are still not far from the exact results, but the inclusion of three-boy forces worsens the situation. One has to emphasize, however, that the bound $2 \frac{m_3^2}{m_1}$ cannot be expressed in terms of the mean charge radius of the proton because of the $SU(6)$ breaking components present in the wave function (64) and the second identity (76) is no longer valid. If one assumes $\alpha_E^p = \frac{2}{\pi} e^2 (3m) \langle r_p^2 \rangle_{ch}$ one should have $\alpha_E = 4.37 \times 10^{-4}$ fm$^3$ for $V_I$ and $\alpha_E = 1.12 \times 10^{-4}$ fm$^3$ for $V_{III}$, values rather far from the exact result of table XII: the $SU(6)$ breaking components produce relevant effects on the polarizability which are not included in the most simple estimates. In particular the bounds which involves higher sum rules ($m_3$ and $m_2$) are quite sensitive to the tuning effects due to the $\Psi_3$ components and if one puts $\Psi_3 = 0$ in the calculations one cannot satisfy the correct inequalities obtaining, for the $V_I$ potential model, a lower bound ($\alpha_E = 3.82 \times 10^{-3}$ fm$^3$) larger than the upper bound ($\alpha_E = 3.73 \times 10^{-3}$ fm$^3$).
TABLE VIII. Upper bounds to the electric polarizability as predicted by the potential models $V_I$, $V_{III}$ \((62)-(63)\). The polarizability is expressed in \(10^{-4}\) fm\(^3\).

| potentials | $2 \frac{m_0(D'_z)}{E_{0z}^L}$ | $2 \frac{m_0(D'_z)}{E_{0z}^L} \Lambda$ |
|------------|---------------------------------|---------------------------------|
| $V_I$      | 3.58                            | 3.38                            |
| $V_{III}$  | 1.87                            | 1.64                            |

TABLE IX. Upper and lower bounds to the nucleon polarizability as predicted by the potential models $V_I$, $V_{III}$ \((62)-(63)\). Notations for the brackets as in table VII. The polarizability is expressed in \(10^{-4}\) fm\(^3\).

| potentials | $\alpha_E^{''}$ | $\alpha_E^{''}$                     |
|------------|----------------|--------------------------------------|
| $V_I$      | 3.34 ± 0.04    | $\alpha_E^{''} = 3.31 ± 0.07$      |
| $V_{III}$  | 1.39 ± 0.24    | $\alpha_E^{''} = 1.42 ± 0.45$      |

B. Magnetic susceptibility

Paramagnetic susceptibility of hyperradial models does not differ from the results of Eqs. \((59)\) and \((61)\) because the Hamiltonian \((49)\) commutes with $\mu_z$. On the contrary the $SU(6)$ breaking model with three-body forces has a much more complex spin structure which is evident in Eqs. \((68)\), and \((70)\). The results are shown in tables X - XIII.

TABLE X. Values of the magnetic sum rules for the potential models $V_I$, $V_{III}$ \((62)-(63)\).

| potentials | $m_0(\mu_z)$ [fm\(^2\)] | $m_1(\mu_z)$ [fm] |
|------------|---------------------------|-------------------|
| $V_I$      | 5.60 \text{E}^{-4}       | 0.0011            |
| $V_{III}$  | 6.17 \text{E}^{-4}       | 0.0015            |

TABLE XI. Lower bounds to the paramagnetic susceptibility as predicted by the potential models $V_I$, $V_{III}$ \((62)-(63)\). The susceptibility is expressed in \(10^{-4}\) fm\(^3\).

| potentials | $2 \frac{m_0(\mu_z)}{E_{0z}^L}$ |
|------------|---------------------------------|
| $V_I$      | 5.89                            |
| $V_{III}$  | 5.23                            |

Since we are calculating the simplest bounds only, the uncertainties are larger than those ones for the polarizability (13% and 23% for $V_I$ and $V_{III}$ respectively), however the reduction with respect the $N-\Delta$ approximation is quite relevant (roughly $-30\%$) and is due to inclusion of the $\gamma = 2$ component in the hyperspherical expansion \((64)\). Again the results are identical for protons and neutrons.

TABLE XII. Upper bounds to the paramagnetic susceptibility as predicted by the potential models $V_I$, $V_{III}$ \((62)-(63)\). The susceptibility is expressed in \(10^{-4}\) fm\(^3\).

| potentials | $2 \frac{m_0(\mu_z)}{E_{0z}^L}$ |
|------------|---------------------------------|
| $V_I$      | 7.55                            |
| $V_{III}$  | 8.31                            |
TABLE XIII. Upper and lower bounds to the paramagnetic susceptibility as predicted by the potential models $V_I$, $V_{III}$ (72)-(73). The susceptibility is expressed in $10^{-4}$ fm$^3$.

| potentials | $\Delta \alpha_p^{V_I}$ | $\alpha_p^{V_I}$ | $\Delta \alpha_p^{V_{III}}$ | $\alpha_p^{V_{III}}$ | $\Delta \beta_p$ | $\beta_p^{\text{dia}}$ | $\beta_p^0$ | $\beta_p^{\text{dia}}$ | $\beta_p^0$ |
|------------|-----------------|--------------|-----------------|--------------|----------------|----------------|---------------|----------------|---------------|
| $V_I^{(3)}$ | 1.36            | 6.96         | -2.04           | -2.04        | 4.61           | 5.13           | 5.23           | 5.24           | -3.10         |
| $V_{II}^{(3)}$ | 1.19          | 5.70         | -1.79           | -1.79        | 5.13           | 5.23           | 5.24           | -3.10         |
| $V_{III}^{(3)}$ | 1.16          | 5.40         | -1.73           | -1.73        | 5.13           | 5.23           | 5.24           | -3.10         |
| $V_4^{(3)}$ | 1.09           | 4.78         | -1.64           | -1.64        | 5.24           | 5.24           | 5.24           | -3.10         |
| $V_5^{(3)}$ | 3.93           | 56.18        | -5.90           | -5.90        | -3.10         | -3.10         | -3.10         | -3.10         |
| $V_I$     | 1.22           | 4.56         | -1.41           | -1.60        | 3.71           | 3.71           | 3.71           | 3.71           |
| $V_{III}$ | 0.61           | 2.00         | -0.83           | -0.87        | 5.06           | 5.06           | 5.06           | 5.06           |

C. Comparison with experiments and conclusions

In tables XIV, XV all the results are summarized and the retardation corrections included for both protons and neutrons. The calculation of the additional terms, within the quark models we are discussing, is rather straightforward because they are basically related to the electric dipole sums already calculated in the previous sections and to the charge proton and neutron radii. In particular (cfr. Eqs. (3)) and (5))

$$\Delta \alpha_{p,n}^\text{p,n} = \frac{e^2}{3M} \langle r_{p,n}^2 \rangle_{\text{ch}},$$

$$\Delta \beta_{p,n}^\text{p,n} = -\frac{3}{2M} m_0(D_2') = -\frac{e^2}{2M} \left( \langle r_{p,n}^2 \rangle_{\text{ch}} + 2 \langle r_{n,p}^2 \rangle_{\text{ch}} \right),$$

$$\beta_{\text{dia}}^{p} = -\frac{e^2}{6m} \left[ \langle r_{p}^2 \rangle_{\text{ch}} + \frac{2}{3} \langle r_{n}^2 \rangle_{\text{ch}} \right],$$

$$\beta_{\text{dia}}^{n} = -\frac{e^2}{6m} \left[ \langle r_{n}^2 \rangle_{\text{ch}} + \frac{2}{3} \langle r_{p}^2 \rangle_{\text{ch}} \right],$$

where also the $SU(6)$ breaking effects have been included.

Let us discuss first the electric polarizability.

The values closest to the experimental data are obtained for the hyperradial potential models, where the three-body contributions are included in the structure of the interaction. The peculiar potential $V_5^{(3)}$ introduced because it reproduces the electric form factor of the proton, shows the classical limitation of the quark models: a huge polarizability when one ask to the quark degrees of freedom to cover all the charge spatial distribution replacing also the effects of the meson cloud. This is a well known problem and it can be easily understood looking at the approximate relation (3): by replacing the charge root mean square radius with its experimental value one gets $\alpha_E \approx 42 \times 10^{-4}$ fm$^3$. On the contrary if one wants to reproduce the excitation energy of the dipole states $\omega_{h.o.} \approx 600$ MeV, and one obtains $\alpha_E \approx 3.3 \times 10^{-4}$ fm$^3$! In order to overcome this contradiction we want to investigate better the predictions of the quark models which are able to reproduce the spectrum of baryons and look at the quantitative amount of meson contribution one needs to fill the gap between quark model contribution and the experimental data.
In the case of hyperradial TBM potentials let us take the results coming from the linear plus confining "Coulomb" (i.e. the interaction $V_1^{(3)}$) as the typical example. Which contribution should one need from mesons to reproduce the experimental polarizabilities of neutron and proton? Including the meson contributions into the retardation corrections (7) is rather simple by taking the experimental value of the proton charge radius: one obtains the prediction for the dynamic polarizability $\bar{\alpha}_p^E = (5.60 + 3.78) \times 10^{-4} = 9.38 \times 10^{-4}$ fm$^3$ which is clamming for a contribution of the mesons in the range of $\approx 2.7 \times 10^{-4}$ fm$^3$. For the neutron the static value of the polarizability is known experimentally (cfr. section I) and the meson contribution is found to be much larger $\approx 7.3 \times 10^{-4}$ fm$^3$ in agreement with the calculation of ref. [13] where the difference between the neutron and proton static polarizability is found $5 \times 10^{-4}$ fm$^3$ and ascribed to mesonic degrees of freedom. The net result is that the contribution of the quarks to the proton polarizability is not small and comparable with the one coming from the meson cloud. This conclusion is basically valid for all the TBM $V_1^{(3)} - V_4^{(3)}$ fitted on the excitation spectrum and it favours the QCD "inspired" form $-\tau/\xi + k_1 \xi$, i.e. the $V_1^{(3)}$ potential model. On the contrary the potential models where the three-body forces are included phenomenologically to reproduce the position of the Roper resonance, predict quite low values of the static polarizability therefore asking for a much larger meson contribution [14].

The results on the magnetic susceptibilities

The predicted paramagnetic susceptibility is the same for all the TBM potentials $|\beta_{\text{para}}^p| = 8.7 \times 10^{-4}$ fm$^3$. In particular for the proton the inclusion of meson spatial distribution on the correction (78) and on the diamagnetic contribution (79) yields to $\beta_M^p = (8.7 - 9.0) \times 10^{-4}$ fm$^3$ and therefore to a mesonic contribution $2.4 \pm 1.3 \times 10^{-4}$ fm$^3$, a result consistent with the analysis of ref. [1] where the $M1$ pion photoproduction has been investigated.

### TABLE XV. Retardation corrections to the electric polarizability and magnetic susceptibility of the neutron calculated for the hyperradial potential models (71) - (75) and the models (62)-(63). The units are $10^{-4}$ fm$^3$.

| potentials  | $\Delta \alpha_{E_n}^p$ | $\bar{\alpha}_E^p$ | $\Delta \beta^n$ | $\beta_{\text{dia}}^n$ | $\beta_M^n$ |
|------------|-------------------------|---------------------|-----------------|-------------------------|-------------|
| $V_1^{(3)}$ | 0                       | 5.60                | -2.04           | -1.36                   | 5.29        |
| $V_2^{(3)}$ | 0                       | 4.50                | -1.78           | -1.19                   | 5.72        |
| $V_3^{(3)}$ | 0                       | 4.24                | -1.73           | -1.16                   | 5.81        |
| $V_4^{(3)}$ | 0                       | 3.69                | 1.64            | -1.09                   | 5.97        |
| $V_5^{(3)}$ | 0                       | 52.9                | -5.90           | -3.93                   | -1.16       |
| $V_I$      | -0.10                   | 3.24                | -1.41           | -0.99                   | 4.32        |
| $V_{III}$  | -0.02                   | 1.37                | -0.83           | -0.57                   | 5.37        |
The neutron susceptibility has been experimentally derived by the knowledge of the static electric polarizability, adding retardation corrections and using the dispersion relation (2), the precision is rather limited. The mesonic contribution can be extracted from \( \beta_n^M = (8.7 - 6.8) \times 10^{-4} \) fm\(^3\) and it results to be \( 1.9 \pm 3.6 \times 10^{-4} \) fm\(^3\).

The potential models \( V_1 - V_{III} \) give analogous results. The remarkable difference is that their complicate spin structure can be easily seen looking at the mean excitation energy \( m_1(\mu_z)/m_0(\mu_z) \approx 370 \text{ MeV} \) and 467 MeV respectively which is larger than the \( M_{\Delta} - M_N \) indicating that the \( SU(6) \)-breaking mechanism introduces a larger excitation spectrum: an effect which results also in the width of the predictions for \( \beta_M \) in table [XIII].

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$$m_3(D'_{\text{hyp}}) = \lim_{\epsilon \to 0} \left[ \frac{4}{9} \frac{e^2}{m^2} \frac{M_\Delta - M_N}{\epsilon^2 \pi \sqrt{\pi}} \int d\xi \xi^2 |\psi_0(\xi)|^2 \right] \mathcal{I}(\epsilon^2), \quad (82)$$

with

$$\mathcal{I}(\epsilon^2) = \int_0^1 dt \int d\xi \xi^2 \sqrt{1-t^2} \left( 2\xi^2 t^2 - \frac{3}{2} \xi^2 \right) \xi^5 e^{-\xi^2 t^2/\epsilon^2} |\psi_0(\xi)|^2, \quad (83)$$

and the convergence has to be checked numerically.
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