An improved bound for the star discrepancy of sequences in the unit interval

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Abstract

It is known that there is a constant $c > 0$ such that for every sequence $x_1, x_2, \ldots$ in $[0, 1)$ we have for the star discrepancy $D_N^*$ of the first $N$ elements of the sequence that $ND_N^* \geq c \cdot \log N$ holds for infinitely many $N$. Let $c^*$ be the supremum of all such $c$ with this property. We show $c^* > 0.065664679\ldots$, thereby slightly improving the estimates known until now.

1 Introduction and statement of the result

Let $x_1, x_2, \ldots$ be a point sequence in $[0, 1)$. By $D_N^*$ we denote the star discrepancy of the first $N$ elements of the sequence, i.e.,

$$D_N^* = \sup_{x \in [0, 1]} \left| \frac{A_N(x)}{N} - x \right|,$$

where $A_N(x) := \#\{1 \leq n \leq N | x_n < x \}$. The sequence $x_1, x_2, \ldots$ is uniformly distributed in $[0, 1)$ iff $\lim_{N \to \infty} D_N^* = 0$.

In 1972 W. M. Schmidt [7] showed that there is a positive constant $c$ such that for all sequences $x_1, x_2, \ldots$ in $[0, 1)$ we have

$$D_N^* > c \cdot \frac{\log N}{N}$$

for infinitely many $N$. The order $\frac{\log N}{N}$ is best possible. There are many known sequences for which $D_N^* \leq c' \cdot \frac{\log N}{N}$ holds for all $N$ with an absolute

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constant $c'$. For all necessary details on discrepancy and low-discrepancy sequences see the monographs [2] or [3].

So it makes sense to define the one-dimensional star discrepancy constant $c^*$ to be the supremum over all $c$ such that (1) holds for all sequences $x_1, x_2, \ldots$ in $[0, 1)$ for infinitely many $N$. Or, in other words,

$$c^* := \inf_{\omega} \limsup_{N \to \infty} \frac{N D_N^*(\omega)}{\log N},$$

where the infimum is taken over all sequences $\omega = x_1, x_2, \ldots$ in $[0, 1)$, and where $D_N^*(\omega)$ denotes the star discrepancy of the first $N$ elements of $\omega$.

The currently best known estimates for $c^*$ are

$$0.0646363 \ldots \leq c^* \leq 0.222 \ldots$$

The upper bound was given by Ostromoukhov [6] (thereby slightly improving earlier results of Faure (see, for example, [1])). The lower bound was given by Larcher [3].

It is the aim of this paper to improve the above lower bound for $c^*$. That is, we prove

**Theorem 1.** For the one-dimensional star discrepancy constant we have

$$c^* \geq 0.065664679 \ldots$$

The idea of the proof follows a method introduced by Liardet [4] which was also used by Tijdeman and Wagner in [8] and by Larcher in [3].

## 2 Main ideas and proof of Theorem 1

We will heavily make use of the idea, the notation, and most of the results used and obtained in [3]. In this paper we extend the analysis carried out in the aforementioned paper. In this section we therefore repeat the most important notation and facts from [3] and explain how we extend the method to prove Theorem 1.

We consider a finite point set $P = \{x_1, x_2, \ldots, x_N\}$ in $[0, 1)$ with $N = [a^t]$ for some real $a$, $3 \leq a \leq 3.7$, and some $t \in \mathbb{N}$. Further, we divide the index-set $A = \{1, 2, \ldots, N\}$ into index-subsets $A_0, A_1, A_2$, where $A_0 = \{1, 2, \ldots, [a^{t-1}]\}$, $A_2 = \{[a^t] - [a^{t-1}] + 1, \ldots, [a^t]\}$, and $A_1 = A \setminus (A_0 \cup A_2)$.

For simplicity, let us first of all assume that $a^t$ and $a^{t-1}$ are integers (of course this can only happen if $a = 3$). For $x \in [0, 1)$ we consider the
discrepancy function \( D_n(x) := \#\{i \leq n| x_i < x\} - nx = A_n(x) - nx \) and we define the function \( f(x) := \max_{n \in A_2} D_n(x) - \max_{n \in A_1} D_n(x) \).

In [3] it was shown that the function \( f \) has the following properties:

(i) \( f(0) = f(1) = 0 \).

(ii) \( f \) is piecewise linear, piecewise monotonically decreasing, and \( |f| \) is bounded by \( a' \).

(iii) \( f \) is left-continuous and each discontinuity constitutes a positive jump.

(iv) The slope of \( f \) is always between \(-a'\) and \( s_0 := -a'^{-1}(a - 2)\).

(v) If \( f \) is continuous on \([x, y]\) then the slope of \( f(x) \) and \( f(y) \) can differ at most by \( a'^{-1} \).

(vi) \( f \) has discontinuities with a jump of height at least 1 in all points \( x_i \) with \( i \in A_1 \).

Further it was shown in [3, Lemma 2.11] that for given \( a \) and \( t \) there exists a function \( f_{\text{strong}}^*: [0, 1] \to \mathbb{R} \) satisfying (i)–(vi) for some \( x_1, \ldots, x_N \) (we say \( f_{\text{strong}}^* \) is strongly admissible) such that

\[
\int_0^1 |f_{\text{strong}}^*(x)| \, dx = \min_{g \text{ strongly admissible}} \int_0^1 |g(x)| \, dx,
\]

and (in [3, Lemma 2.14]) that for every \( \varepsilon > 0 \) and (now arbitrary) \( a \in [3, 4] \) and \( t \) with \( t \geq t(\varepsilon) \)

\[
\int_0^1 |f_{\text{strong}}^*(x)| \, dx \geq \frac{(a - 2)(8a + 3)}{8(1 - 2a)^2} - \varepsilon.
\]

Finally, we finished the proof of the Theorem in [3] in the following way:

It was shown that (see Section 3 in [3])

\[
\int_0^1 \left( \max_{n \in A} D_n(x) - \min_{n \in A} D_n(x) \right) \, dx \geq t \int_0^1 |f_{\text{strong}}^*(x)| \, dx \geq \left(\frac{(a - 2)(8a + 3)}{8(1 - 2a)^2} - \varepsilon\right) \geq \frac{\log N}{\log a} \cdot \left(\frac{(a - 2)(8a + 3)}{8(1 - 2a)^2} - \varepsilon\right) \geq 2 \log N \cdot 0.0646363 \ldots
\]
if we choose $a = 3.71866 \ldots$ and $N$ large enough. Hence there exist $x \in [0, 1]$ and $n \leq N$ with

$$D_n(x) \geq 0.0646363 \ldots \cdot \log N$$

and Theorem 1.1 from [3] follows.

To improve the above result from [3] in the present paper we proceed as follows: We show that $f$ has to satisfy an even more restrictive property (vi′) instead of property (vi) and we call a function $g$ satisfying (i)–(v) and (vi′) strictly admissible. Moreover, we show that there exists a strictly admissible function $f^\ast_{\text{strict}} : [0, 1] \to \mathbb{R}$ with

$$\int_0^1 |f^\ast_{\text{strict}}(x)| \, dx = \min_{g \text{ strictly admissible}} \int_0^1 |g(x)| \, dx$$

and

$$\int_0^1 |f^\ast_{\text{strict}}(x)| \, dx \geq \frac{(a - 2) \left(12a + 9 + (a - 2)(4a - 3) \log \left(1 + \frac{1}{a - 2}\right)\right)}{a \left(a - \frac{1}{2}\right)^2 \left(3 + (a - 2) \log \left(1 + \frac{1}{a - 2}\right)\right)} - \varepsilon$$

for all $a \in (3, 3.7]$ and $t \geq t(\varepsilon)$.

Note that, in the following, we will work with $a^t$ and $a^{t-1}$ as if they were integers and we will obtain the above result without “$-\varepsilon$” and for all $t \geq t_0$ in this case. For working with $[a^{t-1}]$ and $[a^t]$ instead of $a^{t-1}$ and $a^t$ we then easily obtain the stated result.

In the very same way as in [3] and as described above we then obtain $D_n(x) \geq 0.065664679 \ldots \cdot \log N$ for some $x \in [0, 1]$ and $n \geq N$ by choosing $a = 3.62079 \ldots$. Consequently, Theorem 1 follows.

So it remains to prove the two main auxiliary results, namely, that a stronger property (vi′) for $f$ as well as the lower bound for $\int_0^1 |f^\ast_{\text{strict}}(x)| \, dx$ as stated above hold. This is carried out in the next section. For the proofs of these two results we will have to use some facts already obtained in [3], again.

### 3 Proof of the auxiliary results

**Lemma 1.** Let $j \in A_2$, i.e., $j = a^t - a^{t-1} + k$ for some integer $k$, $1 \leq k < a^{t-1}$, and assume that $f(x) = \max_{n \in A_2} D_n(x) - \min_{n \in A_0} D_n(x)$ has a discontinuity in $x_j$. Let further $l_j, r_j \in A$ such that $\mathcal{P} \cap (x_{l_j}, x_{r_j}) = \{x_j\}$. If there exists an $\overline{x} \in (x_j, x_{r_j})$ such that, in $\overline{x}$ has slope $s(\overline{x}) > s_0 - k$ then $f(\overline{x}) = \min_{x \in (x_j, x_{r_j})} f(x)$ for all $x \in [x_{l_j}, x_{r_j}]$. Here, again, $s_0 = a^t(a - 2)$ as defined in property (iv) above.
Remark. The meaning of Lemma 1 is illustrated in Figure 1. Using the same notation $f(x)$ lies above the line with slope $s_0$ reaching back from the point $(\bar{x}, f(\bar{x}))$ (dashed) in case the slope of $f$ (solid) becomes flatter than $s_0 - k$.

![Figure 1](image-url)

Proof of Lemma 1. Let $x, \bar{x}$ be like above with $s(\bar{x}) > s_0 - k$. We set $\bar{n}_i = n_i(\bar{x})$ and $\bar{n}_i = n_i(x)$ such that $D_{\bar{n}_i}(\bar{x}) = \max_{n_i \in A_i} D_n(\bar{x})$ and $D_{n_i}(x) = \max_{n \in A_i} D_n(x)$. So $f(\bar{x}) = D_{\bar{n}_2}(\bar{x}) - D_{n_0}(\bar{x})$ and $f(x) = D_{n_2}(x) - D_{n_0}(x)$.

First we show that $\bar{n}_2 < j$. Indeed, we have

$$a^{t-1} - \bar{n}_2 \geq \bar{n}_0 - \bar{n}_2 = s(\bar{x}) > s_0 - k = -a^{t-1}(a - 2) - k.$$

Thus, $\bar{n}_2 < a^{t-1} + k = j$.

Since $A_n$ does not change its value in $x_j$ for $n < j$, $D_{n_2}$ does not have a jump in $x_j$. Consequently, $D_{n_2}(\bar{x}) = D_{n_2}(x) - D_{n_2}(\bar{x})$. This observation yields

$$D_{n_2}(\bar{x}) - D_{n_2}(\bar{x}) \geq D_{n_2}(x) - D_{n_2}(\bar{x}) = n_2(x - \bar{x}).$$

By the same argument we additionally obtain

$$D_{n_2}(x) - D_{n_0}(x) \leq D_{n_2}(x) - D_{n_0}(x) = n_0(x - \bar{x}).$$

Alltogether

$$f(x) - f(\bar{x}) = (D_{n_2}(x) - D_{n_2}(\bar{x})) - (D_{n_2}(x) - D_{n_2}(\bar{x})) \geq (\bar{n}_2 - n_0)(x - \bar{x}) \geq -s_0(x - \bar{x})$$

and the result follows. $\square$
In addition to the new property of $f$ obtained in Lemma 1 one can easily convince oneself that $f$ is continuous at $x_1$. This result is not very effectful yet nice for calculation purposes. We will use this fact in the following concept of strict admissibility.

**Definition 1.** A function $g : [0, 1] \to \mathbb{R}$ is called strictly admissible if it satisfies conditions (i)–(v) and the following additional condition (vi’):

There exists a set $\Gamma = \{\xi_1, \xi_2, \ldots, \xi_{a-1}\} \subset [0, 1)$ such that:

a) If $g$ has a jump in $\xi$ then $\xi \in \Gamma$.

b) There exists a set $\Gamma_1 \subset \Gamma$, $|\Gamma_1| = a^{l-1}(a - 2)$, such that $f$ has a jump of height at least one in each $\xi \in \Gamma_1$.

c) There exist $a^{l-1} - 1$ further points $\{\xi_1, \xi_2, \ldots, \xi_{a^{l-1}-1}\} =: \Gamma_2$ with the following property:

For each $1 \leq n < a^{l-1}$ let $\xi_{i_n}, \xi_{r_n} \in \Gamma \cup \{0, 1\}$ such that $\Gamma \cap (\xi_{i_n}, \xi_{r_n}) = \{\xi_k\}$. Now, if there is an $x \in (\xi_{i_n}, \xi_{r_n})$ with

$$s(x) > s_0 - n$$

then

$$g(x) \geq g(\xi) - s_0(\xi - x)$$

for all $x \in [\xi_{i_n}, \xi_{r_n})$. Here, $s(x)$ denotes the slope of $g$ in $x$.

From [3] and from Lemma 1 it follows that $f$ is strictly admissible. The space of strictly admissible functions, again, is obviously closed with respect to pointwise convergence. Hence, there exists $f_{\text{strict}}$ strictly admissible with

$$\int_0^1 |f(x)| \, dx \geq \min_{g \text{ strictly admissible}} \int_0^1 |g(x)| \, dx = \int_0^1 |f_{\text{strict}}(x)| \, dx.$$

We intend to estimate $\int_0^1 |f_{\text{strict}}(x)| \, dx$ from below. To this end we have to derive some properties of $f_{\text{strict}}$.

**Lemma 2.** Let $f_{\text{strict}}$ have a discontinuity in $\gamma$. Then there exist two zeros $\alpha, \beta$ of $f_{\text{strict}}$ with $\alpha < \gamma < \beta$ such that $\gamma$ is the only discontinuity in the interval $(\alpha, \beta)$.

**Proof.** First of all, if $\gamma$ is the only point at which $f_{\text{strict}}$ has a jump, the claim is fulfilled with $\alpha = 0$ and $\beta = 1$. Hence it suffices to show the following statement: Let $f_{\text{strict}}$ have two successive discontinuities in, say, $a_1$ and $a_2$, $0 < a_1 < a_2 < 1$. Then $f_{\text{strict}}$ has a zero in the interval $(a_1, a_2)$. 


For contradiction we assume $f^{\ast}_{\text{strict}} > 0$ on $(a_1, a_2)$ (the case $f^{\ast}_{\text{strict}} < 0$ can be treated quite similarly). In what follows, we will construct a strictly admissible function $\tilde{f}$ such that

$$
\int_0^1 |\tilde{f}(x)| \, dx < \int_0^1 |f^{\ast}_{\text{strict}}(x)| \, dx,
$$

which clearly contradicts the definition of $f^{\ast}_{\text{strict}}$.

Naturally, we need to take special care in constructing $\tilde{f}$ if either $a_1 \in \Gamma_2$ or $a_2 \in \Gamma_2$ which was defined in Definition 1. Moreover, if we manage to preserve the height of any existing jump in any other case then (vi’b) is automatically fulfilled for $\tilde{f}$.

First of all, we notice that $f^{\ast}_{\text{strict}}$ cannot have a bend at, say, $y \in (a_1, a_2)$ such that the slope before the bend is greater than afterwards. We say $f^{\ast}_{\text{strict}}$ has a bend in $y$ if $f^{\ast}_{\text{strict}}$ is continuous in $y$ and if it changes its slope in $y$. Indeed, let $\delta > 0$ such that the slope of $f^{\ast}_{\text{strict}}$ is constant on $(y-\delta, y)$ as well as on $(y, y+\delta]$. Then, as can be seen in Figure 2, we may interchange those pieces such that the resulting function $\tilde{f}$ (solid) remains continuous in $[y-\delta, y+\delta]$. Its absolute integral, however, is smaller than that of $f^{\ast}_{\text{strict}}$ (dashed). Thus, we need only consider bends where $f^{\ast}_{\text{strict}}$ becomes flatter.

Let now $a_2 \notin \Gamma_2$. We choose $\delta_1 > 0$ such that the slope of $f^{\ast}_{\text{strict}}$ is a constant $s_1$ on $(a_2, a_2 + \delta_1)$. Furthermore, we set

$$
s = \min \{s^*(x) : x \in (a_1, a_2 + \delta_1)\},
$$

where $s^*$ denotes the slope of $f^{\ast}_{\text{strict}}$ and where we define $s^*(a_2)$ as its left limit. Now, let $0 < \delta \leq \min \{-2f^{\ast}_{\text{strict}}(a_2)/(s_1 + s), \delta_1\}$. With this choice of $\delta$ we have

$$
f^{\ast}_{\text{strict}}(a_2) + s\delta > -f^{\ast}_{\text{strict}}(a_2 + \delta).
$$
In this case we may thus construct \( \tilde{f} \) by moving the discontinuity to \( \tilde{a}_2 = a_2 + \delta \). The missing part of \( \tilde{f} \) on the left of \( \tilde{a}_2 \) of length \( \delta \) is then chosen such that \( \tilde{f} \) is continuous in \( a_2 \) and such that it has constant slope \( s \). This construction is visualized in Figure 3 (again \( f_{\text{strict}}^* \) is represented by the dashed and \( \tilde{f} \) by the solid line). This choice for the slope guarantees that the height of the jump is preserved and, additionally, property (vi’.c) from Definition 1 too, cannot be violated by this construction if \( a_1 \in \Gamma_2 \).

Certainly, the same construction also works if \( a_2 = \xi_{k_n} \in \Gamma_2 \) for a suitable \( k_n \) with \( s^* \leq -a^{l-1}(a-2) - n \) between \( a_2 \) and the next discontinuity of \( f_{\text{strict}}^* \).

However, if there is some point \( x > a_2 \) before the next jump of \( f_{\text{strict}}^* \) with \( s^*(x) > -a^{l-1}(a-2) - n \) we have to proceed differently. In this case, we keep the discontinuity at \( a_2 \) and take the smallest such \( x \), call it \( \overline{x} \). We define

\[
\tilde{f}(x) := \begin{cases} 
  s_0(x - \overline{x}) + f_{\text{strict}}^*(\overline{x}) & \text{if } x \in [\overline{x} - \delta, \overline{x}), \\
  s^*(\overline{x})(x - \overline{x}) + f(\overline{x} - \delta) & \text{if } x \in [a_2, \overline{x} - \delta), \\
  f_{\text{strict}}^*(x) & \text{else,}
\end{cases}
\]

where \( \delta > 0 \) is such that we still have a positive jump in \( a_2 \). Recall that a discontinuity always constitutes a positive jump, hence this is possible. Figure 4 shows \( \tilde{f} \) (solid) as well as \( f_{\text{strict}}^* \) (dashed) in this case. Notice that, again,

\[
\int_0^1 |\tilde{f}(x)| \, dx < \int_0^1 |f_{\text{strict}}^*(x)| \, dx
\]

and that (vi’.c) from Definition 2 is not violated for \( a_2 \). Additionally, the condition on \( \delta \) guarantees that (vi’.c) is not violated for \( a_1 \) if \( a_1 \in \Gamma_2 \) either. Moreover, we need not take care of the height of the jump in \( a_2 \), since \( \Gamma_1 \) and \( \Gamma_2 \) are disjoint. The dotted line represents the line with slope \( s_0 \) reaching back from \( \{\overline{x}, f_{\text{strict}}^*(\overline{x})\} \) which occurs in Definition 2.

Figure 3:

\[\text{Figure 3:}\]

\begin{center}
\includegraphics[width=0.5\textwidth]{figure3.png}
\end{center}
Thus, $f_{\text{strict}}^*$ consists of parts $Q$, each of which is defined on an interval $[\alpha, \beta]$ with $f_{\text{strict}}^*(\alpha) = f_{\text{strict}}^*(\beta) = 0$ and such that there is exactly one discontinuity in $(\alpha, \beta)$, see Figure 4.

In the following we determine the number of such $Q$’s for $f_{\text{strict}}^*$.

**Lemma 3.** The function $f_{\text{strict}}^*$ has exactly $a^t - 1$ discontinuities.

**Proof.** Assume that the total number of discontinuities of $f_{\text{strict}}^*$ is less than $a^t - 1$. Then, in the following, we will define a strictly admissible function $\tilde{f}$ from $f_{\text{strict}}^*$ whose absolute integral is smaller than that of $f_{\text{strict}}^*$. Let $\Gamma^*$ be the set $\Gamma$ from property (vi’). Let $\xi^* \in \Gamma^*$ such that $f_{\text{strict}}^*$ is continuous in $\xi^*$. The definition of $\Gamma^*_1$ (i.e., the set $\Gamma_1$ for $f_{\text{strict}}^*$) guarantees $\xi^* \not\in \Gamma^*_1$. Assume that $\xi^* \in \Gamma^*_2$ (the case $\xi^* \in \Gamma^*_0 := \Gamma^* \setminus (\Gamma^*_1 \cup \Gamma^*_2)$ can be treated quite analogously).

Now choose $\gamma \in \Gamma^*$ such that $f_{\text{strict}}^*$ has a jump in $\gamma$. We show that $\gamma \in \Gamma^*_1$ and that $f_{\text{strict}}^*$ has a jump of height 1 in $\gamma$ (case d) below). Indeed, a priori we are in one of the following four cases:

a) $\gamma \in \Gamma^*_0$,

b) $\gamma \in \Gamma^*_2$,

c) $\gamma \in \Gamma^*_1$ with a jump of height greater than 1, or

d) $\gamma \in \Gamma^*_1$ with a jump of height exactly equal to 1 in $\gamma$.

Assume that $\gamma \in \Gamma^*_2$ (case b). By Lemma 2 $\gamma$ is isolated by two successive zeros of $f_{\text{strict}}^*$. Hence (3) from property (vi’) cannot hold, and therefore (2) from the same property does not hold either. Consequently, (see Fig. 6) we can take a point $\xi$ on the left of $\gamma$ and insert a short piece of minimal slope on $[\xi, \gamma)$ without interfering with property (vi’). Again, the dashed line represents $f_{\text{strict}}^*$ and the solid one the resulting new function $\tilde{f}$. The new set $\Gamma$ is the set $\Gamma^*$ with $\xi^*$ replaced by $\xi$. 

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Figure 4:  

Figure 5:
This construction also works for case a) in the same way, and, with some special care, i.e., the jump of \( \tilde{f} \) in \( \gamma \) maintains a height of at least one, for case c) too.

Consequently, \( f_{\text{strict}}^* \) can only have the \( a^t(a - 2) \) jumps at the positions given by \( \Gamma_i^* \). All these jumps have height exactly equal to one and there are absolutely no further jumps. Obviously, \( f_{\text{strict}}^* \) cannot have slope \(-a^t\) everywhere, since then

\[
0 > a^t(a - 2) - a^t = f_{\text{strict}}^*(1),
\]

a contradiction to property (i). Thus, there exists an interval \([\delta_1, \delta_2]\) such that \( f_{\text{strict}}^* > 0 \) (or \( f_{\text{strict}}^* < 0 \)) on \([\delta_1, \delta_2]\) and its slope is greater than \(-a^t\). We choose \( \delta' \in (\delta_1, \delta_2) \) sufficiently close to \( \delta_1 \) (or to \( \delta_2 \)) and define

\[
\tilde{f}(x) = \begin{cases} 
    f_{\text{strict}}^*(\delta_1) - a^t(x - \delta_1) & \text{if } x \in (\delta_1, \delta'], \\
    f_{\text{strict}}^*(x) & \text{else},
  \end{cases}
\]

or

\[
\tilde{f}(x) = \begin{cases} 
    f_{\text{strict}}^*(\delta_2) - a^t(x - \delta_2) & \text{if } x \in (\delta', \delta_2], \\
    f_{\text{strict}}^*(x) & \text{else},
  \end{cases}
\]

respectively. See Figures 7 and 8.

From the above results we obtain that \( f_{\text{strict}}^* \) has to be of the following form: It divides \([0, 1]\) into \( a^t - 1 \) parts \([\alpha, \beta]\) with \( f_{\text{strict}}^*(\alpha) = f_{\text{strict}}^*(\beta) = 0 \), and \( f_{\text{strict}}^* \) has exactly one discontinuity \( \gamma \in (\alpha, \beta) \). We say that \([\alpha, \beta]\) is of type \( Q_i \) if \( \gamma \in \Gamma_i^* \) for \( i = 0, 1, 2 \).

From [3], equation (2), we know that, for an interval of type \( Q_0 \) (this corresponds to the type \( Q'' \) in the abovementioned paper), we have

\[
\int_{\alpha}^{\beta} |f_{\text{strict}}^*(x)| \, dx \geq \chi^2 \frac{a^t(a - 2)}{4}, \quad \chi = \beta - \alpha,
\]
and from [3] Lemma 2.12] and the considerations following the proof of this lemma we know that for an interval of type \(Q_1\) (this corresponds to the type \(Q'\) in the abovementioned paper) we have

\[
\int_{\alpha}^{\beta} |f_{\text{strict}}^*(x)| \, dx \geq \frac{\chi (4 - a^{t-1} \chi)}{16}, \quad \chi = \beta - \alpha.
\]

Moreover, we know from [3] Lemma 2.10] that for \(f_{\text{strict}}^*\) all \(a^{t-1}\) intervals \(Q_0\) have the same length and all \(a^t - 2a^{t-1}\) intervals \(Q_1\) have the same length.

**Lemma 4.** For \(1 \leq n \leq a^{t-1} - 1\) let \(Q_2^{(n)}\) be given by the interval \([\alpha, \beta)\). Then we have

\[
\int_{Q_2^{(n)}} |f_{\text{strict}}^*(x)| \, dx \geq (\beta - \alpha)^2 \frac{|s_0|(n + |s_0|)}{2(n + 2|s_0|)}.
\]

**Proof.** This follows from the remark preceeding Lemma 4 and simple calculations.

To finish the proof of our theorem we finally show:

**Lemma 5.** For all \(3 \leq a \leq 3.7\) we have

\[
\int_{0}^{1} |f_{\text{strict}}^*(x)| \, dx \geq \frac{(a - 2) \left(12a + 9 + (a - 2)(4a - 3) \log \left(1 + \frac{1}{a-2}\right)\right)}{16 \left(a - \frac{1}{2}\right)^2 \left(3 + (a - 2) \log \left(1 + \frac{1}{a-2}\right)\right)}.
\]

**Proof.** Due to Lemma 4 and the remarks preceeding it we have to minimize
the right hand-side of

\[ \int_0^1 |f^\ast_{\text{strict}}(x)| \, dx \geq a^{t-1} \cdot \chi_0 \frac{a^{t-1}(a-2)}{4} + a^{t-1}(a-2) \cdot \frac{\chi_1(4-a^{t-1}\chi_1)}{16} + \sum_{n=1}^{a^{t-1}-1} \left( \chi_2^{(n)} \right)^2 \frac{|s_0|(n+|s_0|)}{2(n+2|s_0|)} \]

\[ = a^{t-1} \cdot \chi_0^2 \tilde{A}_0 + a^{t-1}(a-2) \cdot \frac{\chi_1(4-a^{t-1}\chi_1)}{16} + \sum_{n=1}^{a^{t-1}-1} \left( \chi_2^{(n)} \right)^2 \tilde{A}_n \]

with respect to \( \chi_0, \chi_1, \chi_2^{(n)} \geq 0 \) (these quantities denote the lengths of the intervals \( Q_0, Q_1, Q_2^{(n)} \)) under the constraint

\[ a^{t-1}\chi_0 + a^{t-1}(a-2)\chi_1 + \sum_{n=1}^{a^{t-1}-1} \chi_2^{(n)} = 1. \]

The Lagrangian approach immediately implies \( \tilde{A}_0\chi_0 = \tilde{A}_n\chi_2^{(n)} \) for all \( 1 \leq n < a^{t-1} \). The constraint therefore yields

\[ \chi_0 = \frac{1 - a^{t-1}(a-2)\chi_1}{a^{t-1} + \sum_{n=1}^{a^{t-1}-1} \tilde{A}_n}. \]

Moreover, the denominator in the above equation simplifies to

\[ a^{t-1} + \sum_{n=1}^{a^{t-1}-1} \tilde{A}_n = a^{t-1} + \sum_{n=1}^{a^{t-1}-1} \left( 1 - \frac{n}{2(|s_0| + n)} \right) = \]

\[ = 2a^{t-1} - 1 - \frac{1}{2} \sum_{n=|s_0|+1}^{a^{t-1}-1+|s_0|} \left( 1 - \frac{|s_0|}{n} \right) = \frac{1}{2} \left( 3a^{t-1} - 1 + |s_0| \sum_{n=|s_0|+1}^{a^{t-1}-1+|s_0|} \frac{1}{n} \right). \]

The latter sum can be bounded by \( \log(1 + 1/(a-2)) \) from above. We summarize our intermediate findings and obtain

\[ \int_0^1 |f^\ast_{\text{strict}}(x)| \, dx \geq \]

\[ \frac{(a-2)(1 - a^{t-1}(a-2)\chi_1)^2}{2 \left( 3 + (a-2) \log \left( 1 + \frac{1}{a-2} \right) \right)} + a^{t-1}(a-2) \frac{\chi_1(4-a^{t-1}\chi_1)}{16} =: p(\chi_1). \]
Now, our goal is to minimize the function $p$. We immediately see that $p$ is a polynomial of degree two and its leading coefficient is positive for all $3 < a \leq 3.7$. Thus, it attains its minimum at its only critical point

$$
\chi_{\text{crit}} = a^{-1-t} \frac{2 \left( 4a - 11 - (a-2) \log \left( 1 + \frac{1}{a-2} \right) \right)}{29 + 8a(a-4) - (a-2) \log \left( 1 + \frac{1}{a-2} \right)}. \tag{11}
$$

On the other hand, from the proof of Lemma 2.13 in [3] we know that we have the following bounds for $\chi_{1}$

$$
\chi_{\text{min}} := \frac{a^{1-t}}{a - \frac{1}{2}} \leq \chi_{1} \leq \frac{a^{1-t}}{a - \frac{3}{2}}.
$$

We will show that $\chi_{\text{crit}} \leq \chi_{\text{min}}$. Indeed, it can easily be verified that the denominator of $\chi_{\text{crit}}$ is positive. Thus, $\chi_{\text{crit}} > \chi_{\text{min}}$ iff

$$
0 > 3a - 9 - (a-1)(a-2) \log \left( 1 + \frac{1}{a-2} \right) =: q(a).
$$

We observe that $q(3.7) < 0$ and, additionally, that $q'(a) > 0$ for all $a \in (3, 3.7]$. Hence $\chi_{1} = \frac{a^{1-t}}{a - \frac{1}{2}}$ and by inserting this value into the function $p$ the result follows.

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