On the Number of Hyperedges in the Hypergraph of Lines and Pseudo-discs

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Abstract
Consider a hypergraph whose vertex set is a family of \( n \) lines in general position in the plane, and whose hyperedges are induced by intersections with a family of pseudo-discs. We prove that the number of \( t \)-hyperedges is bounded by \( O(t^2 n^2) \) and that the total number of hyperedges is bounded by \( O(n^3) \). Both bounds are tight.

1 Introduction
A family \( \mathcal{F} \) of simple Jordan regions in \( \mathbb{R}^2 \) is called a family of pseudo-discs if for any \( c_1, c_2 \in \mathcal{F}, |\partial(c_1) \cap \partial(c_2)| \leq 2 \), where \( \partial(c) \) is the boundary of \( c \). Given a set \( P \) of points in \( \mathbb{R}^2 \) and a family \( \mathcal{F} \) of pseudo-discs, define the geometric hypergraph \( H(P, \mathcal{F}) \) whose vertices are the points of \( P \), and any pseudo-disc \( c \in \mathcal{F} \) defines a hyperedge of all points contained in \( c \).

The family of hypergraphs \( H(P, \mathcal{F}) \) – for a general \( \mathcal{F} \) and in the special case where all elements of \( \mathcal{F} \) are convex – have been studied extensively (see, e.g., [1, 3, 6, 9, 13]). In particular, it was proved in [7] that for any \( P, \mathcal{F} \), the Delaunay graph of \( H(P, \mathcal{F}) \) (namely, the restriction of \( H \) to hyperedges of size 2) is planar, and that for any fixed \( t \), the number of hyperedges of \( H(P, \mathcal{F}) \) of size \( t \) is bounded by \( O(t^2 |P|) \). This result was generalized in [11] (see also [4]) to the case where \( P \) is a family of pseudo-discs instead of points, and the hyperedges are defined by non-empty intersections of any element in \( \mathcal{F} \) with the elements of \( P \).

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In this note we consider hypergraphs \( H = H(\mathcal{L}, \mathcal{F}) \) whose vertex set \( V(H) = \mathcal{L} \) is a family of lines in the plane, and whose hyperedges are induced by intersections with a family \( \mathcal{F} \) of pseudo-discs. Namely, any \( c \in \mathcal{F} \) defines the hyperedge
\[
e_c = \{ \ell \in \mathcal{L} : \ell \cap c \neq \emptyset \} \in \mathcal{E}(H).
\]
We assume that the geometric objects are in general position, in the sense that no 3 lines pass through a common point, no line passes through an intersection point of two boundaries of pseudo-discs.

Unlike the hypergraphs of points w.r.t. pseudo-discs, \( H(P, \mathcal{F}) \), the number of hyperedges in a hypergraph \( H(\mathcal{L}, \mathcal{F}) \), of lines w.r.t. pseudo-discs, of any fixed size, may be quadratic in the number of vertices. Such a hypergraph was demonstrated in a beautiful paper of Aronov et al. [5]. They showed that for any family \( \mathcal{L} \) of lines, if \( \mathcal{F} \) consists of the inscribed circles of the triangles formed by any triple of lines, then for any \( t \geq 3 \), the number of \( t \)-hyperedges (i.e., hyperedges of size \( t \)) in \( H(\mathcal{L}, \mathcal{F}) \) is exactly \( \binom{n-t+2}{2} \).

For any fixed \( t \), there exist hypergraphs \( H(\mathcal{L}, \mathcal{F}) \) in which the number of \( t \)-hyperedges is larger than in the construction of Aronov et al. [5], even when \( \mathcal{F} \) is allowed to contain only discs (as some of those discs might not be inscribed in a triangle formed by the lines). We prove that the number of \( t \)-hyperedges cannot be significantly larger for any hypergraph \( H(\mathcal{L}, \mathcal{F}) \) of lines with respect to pseudo-discs. Specifically, we prove:

**Theorem 1.1.** Let \( \mathcal{L} \) be a family of \( n \) lines in the plane, let \( \mathcal{F} \) be a family of pseudo-discs, and assume both families are in general position. Then
\[
|\{ e \in \mathcal{E}(H(\mathcal{L}, \mathcal{F})) : |e| = t \}| = O_t(n^2).
\]

Our techniques combine probabilistic and planarity arguments, together with exploiting properties of arrangements of lines, in particular the zone theorem.

In addition, we show that for any choice of \( \mathcal{L} \) and \( \mathcal{F} \), the total number of hyperedges in \( H(\mathcal{L}, \mathcal{F}) \) does not exceed \( O(n^3) \). This upper bound is tight, since the total number of hyperedges in the hypergraph presented by Aronov et al. [5] is \( \binom{n}{3} \).

**Proposition 1.2.** Let \( \mathcal{L} \) be a family of \( n \) lines in the plane, let \( \mathcal{F} \) be a family of pseudo-discs, and assume both families are in general position. Then \( |\mathcal{E}(H(\mathcal{L}, \mathcal{F}))| = O(n^3) \).

# 2 Preliminaries

In this section we present previous results and simple lemmata that will be used in our proofs.

## 2.1 Pseudo-discs

The two following lemmata are standard useful tools when handling families of pseudo-discs:

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1For the difference between hypergraphs induced by pseudo-discs and hypergraphs induced by discs, see [10] and the references therein.
Lemma 2.1 (Lemma 1 in [15], based on [16]). Let \( F \) be a family of pseudo-discs, \( D \in F, x \in D \). Then \( D \) can be continuously shrunk to the point \( x \), such that at each moment during the shrinking process, the family obtained from \( F \) remains a family of pseudo-discs.

Lemma 2.2 (Lemma 2 in [15]). Let \( B \) be a family of pairwise disjoint closed connected sets in \( \mathbb{R}^2 \). Let \( F \) be a family of pseudo-discs. Define a graph \( G \) whose vertices correspond to the sets in \( B \) and connect two sets \( B, B' \in B \) if there is a set \( D \in F \) such that \( D \) intersects \( B \) and \( B' \) but not any other set from \( B \). Then \( G \) is planar, hence \(|E(G)| < 3|V(G)|\).

2.2 Arrangements and zones

A finite set \( L \) of lines in \( \mathbb{R}^2 \) determines an arrangement \( A \). The 0-dimensional faces of \( A \) (namely, the intersections of two distinct lines from \( L \)), are called the vertices of \( A \), the 1-dimensional faces are called the edges of \( A \), and the 2-dimensional faces are the cells of \( A \). Clearly, all cells are convex. The cell complexity of a cell \( f \) in \( A \), denoted by \( \text{comp}(f) \), is the number of lines incident with the cell. The zone of an additional line \( \ell \), is the set of faces of \( A \) intersected by \( \ell \). The complexity of a zone is the sum of the cell complexities of the faces in the zone of \( \ell \), i.e., total number of edges of these faces, counted with multiplicities.

Theorem 2.3 (Zone Theorem [8]). In an arrangement of \( n \) lines, the complexity of the zone of a line is \( O(n) \).

The best possible upper bound in the theorem is \( \lfloor 9.5(n-1) \rfloor - 3 \), obtained by Pinchasi [14].

We shall use a generalization of the theorem, for which an extra definition is needed. Given an arrangement \( A \) and a line \( \ell \), the 1-zone of \( \ell \) is defined as the zone of \( \ell \), and for \( t > 1 \) the \( t \)-zone of \( \ell \) is defined as the set of all faces adjacent to the \( (t-1) \)-zone, that do not belong to any \( i \)-zone for \( i < t \). The \((\leq t)\)-zone of \( \ell \) is the union of the \( i \)-zones of \( \ell \) for all \( 1 \leq i \leq t \).

The following generalization of the zone theorem was given as Exercise 6.4.2 in [12]. Its proof can be found in [17, Prop. 1].

Lemma 2.4 ([17]). Let \( A \) be an arrangement of \( n \) lines. Then for any \( t \), the \((\leq t)\)-zone of any additional line \( \ell \) contains at most \( O(tn) \) vertices.

By planarity, this implies:

Corollary 2.5. Let \( A \) be an arrangement of \( n \) lines. Then for any \( t \), the \((\leq t)\)-zone of any additional line \( \ell \) has complexity \( C_{\leq t}(\ell) = O(tn) \).

2.3 Leveraging from 2-hyperedges to \( t \)-hyperedges

The following lemma allows bounding the number of \( t \)-hyperedges in a hypergraph \( H = (V, E) \) in terms of the number of its 2-hyperedges (i.e., the size of its Delaunay sub-hypergraph) and its VC-dimension.

Let us recall the classical definition of VC-dimension. A subset \( \mathcal{V}' \subseteq \mathcal{V} \) is shattered if all its subsets are realized by hyperedges, meaning \( \{\mathcal{V}' \cap e : e \in \mathcal{E}\} = 2^{\mathcal{V}'} \). The VC-dimension of \( H \), denoted by \( VC(H) \), is the cardinality of a largest shattered subset of \( \mathcal{V} \), or \(+\infty\) if arbitrarily large subsets are shattered.
Lemma 2.6 (Theorem 6 (ii),(iii) in [2]). Let $H = (\mathcal{V}, \mathcal{E})$ be an $n$-vertex hypergraph. Suppose that there exists an absolute constant $c$ such that for every $\mathcal{V}' \subset \mathcal{V}$, the Delaunay graph of the sub-hypergraph induced by $\mathcal{V}'$ has at most $c|\mathcal{V}'|$ edges. Then the VC-dimension $d$ of $H$ is at most $2c + 1$, and the number of hyperedges of size at most $t$ in $H$ is $O(t^{d-1}n)$.

The lemma generalizes similar results proved in [4, 7] for hypergraphs of pseudo-discs with respect to pseudo-discs. The assertion regarding the VC-dimension is a simple observation. (Indeed, if a set of $d$ vertices is shattered, then we have $\binom{d}{2} \leq cd$, and thus, $d - 1 \leq 2c$, or equivalently, $d \leq 2c + 1$.) The assertion regarding the number of hyperedges is more involved.

3 The number of $t$-hyperedges in $H(\mathcal{L}, \mathcal{F})$

In this section we prove Theorem 1.1. We prove the following stronger statement:

Proposition 3.1. Let $\mathcal{L}$ be a family of $n$ lines in the plane, let $\mathcal{F}$ be a family of pseudo-discs, and assume both families are in general position. Then for each $\ell \in \mathcal{L}$,

$$|\{e \in E(H(\mathcal{L}, \mathcal{F})) : |e| = t, \ell \in e\}| = O_t(n).$$

Consequently, $|\{e \in E(H(\mathcal{L}, \mathcal{F})) : |e| = t\}| = O_t(n^2)$.

Proof of Proposition 3.1. First we prove the statement for hyperedges of size 3, and then we leverage the result to general hyperedges.

3-hyperedges. Fix a line $\ell$. We observe that for a pseudo-disc $c$ that defines a 3-hyperedge $\{\ell, \ell', \ell''\}$ there exists a cell of $A(\mathcal{L} \setminus \{\ell\})$ which is in the $\leq 2$-zone of $\ell$ in $A(\mathcal{L} \setminus \{\ell\})$ such that $c$ intersects two edges of this cell where one of these edges is on $\ell'$ and the second is on $\ell''$. With every such pseudo-disk $c$ we associate one such cell $f_c$ and one such pair of edges of this cell, and denote this pair by $e_c$.

Define a graph $G = (V, E)$ whose vertices are all edges in the $(\leq 2)$-zone of $\ell$ in $A(\mathcal{L} \setminus \{\ell\})$, and whose edges are the pairs $e_c$ associated with the pseudo-disks that define a 3-hyperedge. Note that for any hyperedge $e = \{\ell, \ell', \ell''\}$ we choose exactly one pair of edges of $A(\mathcal{L} \setminus \{\ell\})$ - one is on $\ell'$ and one is on $\ell''$ - that form a corresponding edge of $G$. Thus by construction, $|E|$ is equal to the number of 3-hyperedges containing $\ell$, and so, we want to prove that $|E| = O(n)$.

Consider a single cell $f$ of $A(\mathcal{L} \setminus \{\ell\})$. For each pseudo-disk $c$ that defines a 3-hyperedge containing $\ell$ and has $f_c = f$, $c$ does not intersect any other edge of $f$ besides the two edges in $e_c$ (as otherwise, $c$ would intersect at least 4 lines of $\mathcal{L}$). Hence, the restriction of $G$ to the edges of the cell $f$ (after removing their endpoints), satisfies the assumptions of Lemma 2.2. Thus, by Lemma 2.2, the subgraph of $G$ induced by the edges of $f$ is planar, and hence, its number of edges is at most 3 times the complexity of $f$. Summing over all cells in the $(\leq 2)$-zone of $\ell$, we obtain $|E| \leq 3\sum f \text{comp}(f) = O(n)$ by Corollary 2.5 and therefore, $|E| = O(n)$, as asserted.
**t-hyperedges.** Fix a line \( \ell \), and consider the hypergraph \( H' \) whose vertex set is \( \mathcal{L} \setminus \{ \ell \} \) and whose edge set is \( \{ e \setminus \{ \ell \} : e \in \mathcal{E}(H), \ell \in e \} \). The 2-hyperedges of \( H' \) correspond to 3-hyperedges of \( H \) containing \( \ell \), and thus, by the first step, their number is \( O(n) \). Furthermore, for any \( \mathcal{L}' \subset \mathcal{L} \setminus \{ \ell \} \), the number of 2-hyperedges in the restriction of \( H' \) to \( \mathcal{L}' \) is \( O(|\mathcal{L}'|) \), by the same argument. Therefore, \( H' \) satisfies the assumptions of Lemma \ref{lem:vc}, which implies that the VC-dimension \( d \) of \( H' \) is constant, and that the number \( C_{t-1} \) of \( (t-1) \)-hyperedges of \( H' \) is \( O(d^{d-1}n) \).

Finally, the number of \( t \)-hyperedges of \( H \) that contain \( \ell \) is equal to \( C_{t-1} \). This completes the proof. \( \square \)

### 4 The total number of hyperedges in \( H(\mathcal{L}, \mathcal{F}) \)

In this section we prove Proposition \ref{prop:hyperedges}.

**Proof of Proposition** \ref{prop:hyperedges} By Lemma \ref{lem:shrinking}, we can shrink the pseudo-discs one by one, such that the shrinking of each pseudo-disc \( c \in \mathcal{F} \) is stopped when it becomes tangent to two lines. (Formally, first \( c \) is shrunk until the first time it is tangent to some line in \( \mathcal{L} \), and then it is shrunk towards the tangency point until the next time it is tangent to some line in \( \mathcal{L} \).) By the general position assumption, we can perform the shrinking process in such a way that the obtained geometric objects (i.e., lines and shrinked pseudo-discs) are also in general position. We replace each \( c \in \mathcal{F} \) by its shrunk copy. Let \( \mathcal{F}' \) be the obtained family. Then \( H(\mathcal{L}, \mathcal{F}) = H(\mathcal{L}, \mathcal{F}') \), and by a tiny perturbation we can assume that all tangencies are in a point.

For any two lines \( \ell_1, \ell_2 \in \mathcal{L} \), denote by \( \mathcal{F}'(\ell_1, \ell_2) \) the set of all pseudo-discs in \( \mathcal{F}' \) that are tangent to both \( \ell_1 \) and \( \ell_2 \). We claim that for any \( \ell_1, \ell_2 \in \mathcal{L} \), \( |\mathcal{E}(H(\mathcal{L}, \mathcal{F}'(\ell_1, \ell_2)))| = O(n) \), and this implies \( |\mathcal{E}(H)| = O(n^3) \), the assertion of Proposition \ref{prop:hyperedges}.

To show this, for any \( c \in \mathcal{F}'(\ell_1, \ell_2) \), we define \( x_{\ell_1, \ell_2}(c) = c \cap \ell_1 \subset \mathbb{R}^2 \) and \( y_{\ell_1, \ell_2}(c) = c \cap \ell_2 \subset \mathbb{R}^2 \) (see Figure \ref{fig:hyperedges}). In each of the four wedges that \( \ell_1, \ell_2 \) form, we define a linear order relation on the elements of \( \mathcal{F}'(\ell_1, \ell_2) \): \( c < c' \) if the segment \( [x_{\ell_1, \ell_2}(c), y_{\ell_1, \ell_2}(c)] \) is completely above the segment \( [x_{\ell_1, \ell_2}(c'), y_{\ell_1, \ell_2}(c')] \) (that is, if the points \( x_{\ell_1, \ell_2}(c), y_{\ell_1, \ell_2}(c) \) are closer to the intersection point within the wedge than the points \( x_{\ell_1, \ell_2}(c'), y_{\ell_1, \ell_2}(c') \), respectively).

First, we claim that this relation is well defined, since for \( c \neq c' \) two such segments never intersect. Indeed, assume to the contrary they intersect, so that \( y_{\ell_1, \ell_2}(c') \) is above \( y_{\ell_1, \ell_2}(c) \), while \( x_{\ell_1, \ell_2}(c') \) is below \( x_{\ell_1, \ell_2}(c) \). The pseudo-disc \( c \) divides the remainder of the wedge into two connected components – the part ‘above’ it and the part ‘below’ it. Now, consider the points \( x_{\ell_1, \ell_2}(c'), y_{\ell_1, \ell_2}(c') \). In the boundary of \( c' \), these points are connected by two curves. As these points are in different connected components w.r.t. \( c \), each of these curves intersects \( c \) at least twice, which means that \( c, c' \) intersect at least 4 times, a contradiction.

Second, we claim that in each wedge, every line in \( \mathcal{L} \) intersects a subset of consecutive elements of \( \mathcal{F}'(\ell_1, \ell_2) \) under the order \( < \). Indeed, assume that some line \( \ell \) intersects two pseudo-discs \( c_1, c_2 \), as depicted in Figure \ref{fig:hyperedges}. We want to show it must intersect \( c_2 \) as well. Like above, \( c_2 \) divides the wedge (without it) into two connected components. By the same argument as above, \( c_1 \) cannot intersect the component below \( c_2 \) (as otherwise, it would cross \( c_2 \) four times). Similarly, \( c_3 \) cannot intersect the component above \( c_2 \). Thus, either \( \ell \) intersects...
Figure 1: Illustration for the proof of Proposition 1.2 - \( c_1, c_2, c_3 \) are tangent to the lines \( \ell_1, \ell_2 \), and \( c_1 \preceq c_2 \preceq c_3 \).

at least one of \( c_1, c_3 \) inside \( c_2 \), or \( \ell \) contains a point above \( c_2 \) and a point below \( c_2 \). In both cases, \( \ell \) must intersect \( c_2 \).

Finally, by passing over all elements of \( \mathcal{F}'(\ell_1, \ell_2) \) in each wedge, from the smallest to the largest, according to the order \( \preceq \), the number of times that the hyperedge defined by the current pseudo-disc is changed is linear in \( |\mathcal{L}| \). Indeed, any such change is caused by appearance or disappearance of some line, and each line in \( \mathcal{L} \) appears at most once and disappears at most once, along the process. Therefore, in each wedge, \( |\mathcal{E}(H(\mathcal{L}, \mathcal{F}'(\ell_1, \ell_2))))| = O(n) \), and summing over all pairs \( \{\ell_1, \ell_2\} \in \mathcal{L}, \) we get \( |\mathcal{E}(H)| = O(n^3) \).

\[ \square \]

5 Open Problems

We conclude this note with a few open problems.

Hypergraph of lines and inscribed pseudo-discs. A natural question is whether the arguments of Aronov et al. [5] can be extended from discs to pseudo-discs. We have found that all their arguments would go through if we knew that every triangle has an inscribed pseudo-disc. More precisely, we would need that for any triangle formed by three sides \( a, b, c \), there is a pseudo-disc \( d \in \mathcal{F} \), contained in the closed triangle, that intersects every side in exactly one point, or if there is no such \( d \in \mathcal{F} \), then we can add such a new pseudo-disc \( d \) to \( \mathcal{F} \) such that \( \mathcal{F} \cup \{d\} \) still forms a pseudo-disc family. Unfortunately, it seems that such a theory has not been developed yet, not even for \( \mathcal{F} \) all whose elements are convex.

We note that for the related problem regarding circumscribed pseudo-discs, even a stronger
result is known. Specifically, it was shown in [16, Thm. 5.1] that for any three points \( a, b, c \), there is a pseudo-disc \( d \in \mathcal{F} \) such that \( a, b, c \in \partial d \), or if there is no such \( d \in \mathcal{F} \), then we can add such a new pseudo-disc \( d \) to \( \mathcal{F} \) such that \( \mathcal{F} \cup \{d\} \) still forms a pseudo-disc family.

**Dependence on \( t \) in Theorem 1.1.** While we showed the quadratic dependence on \( n \) in Theorem 1.1 to be tight, the dependence on \( t \) is not clear. It seems plausible that

\[
|\{e \in \mathcal{E}(H(\mathcal{L}, \mathcal{F})) : |e| = t\}| = O(tn^2),
\]

but we have not been able to prove this. On the other hand, even the stronger upper bound \( O(n^2) \) for any fixed \( t \), that would immediately imply Proposition 1.2 might hold.

**Analogue of Lemma 2.6 for 3-sized hyperedges.** It seems plausible that one can prove the following analogue of Lemma 2.6 for 3-sized hyperedges: If in some hypergraph on \( n \) vertices, for any induced hypergraph, the number of 3-sized hyperedges is quadratic in the number of vertices, then for any fixed \( t \), the number of \( t \)-sized hyperedges is \( O_t(n^2) \). Such a strong leveraging lemma would allow an easier proof of Theorem 1.1.

### Acknowledgements

The authors are grateful to Rom Pinchasi for inspiring and helpful suggestions, to Stefan Felsner for suggesting to use Lemma 2.4 and for other valuable suggestions, and to Manfred Scheucher for useful discussions on arrangements of pseudo-discs.

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