NORMAL CONFORMAL KILLING FORMS

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Abstract. We introduce in this paper normal twistor equations for differential forms and study their solutions, the so-called normal conformal Killing forms. The twistor equations arise naturally from the canonical normal Cartan connection of conformal geometry. Reductions of its holonomy are related to solutions of the normal twistor equations. The case of decomposable normal conformal holonomy representations is discussed. A typical example with an irreducible holonomy representation are the so-called Fefferman spaces. We also apply our results to describe the geometry of solutions with conformal Killing spinors on Lorentzian spin manifolds.

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1. INTRODUCTION

A classical object of interest in differential geometry is conformal symmetry. Typical examples for conformal symmetry arise from the flow of Killing and conformal Killing vector fields on a semi-Riemannian manifold. The notion of conformal vector fields has a natural generalisation to differential forms and spinor fields, namely the so-called conformal Killing forms and spinors (cf. [Kas68], [Tec69], [Pen71], [PR86], [Lic88], [BFGK91], [Sem01]).

We want to introduce in this paper a special class of conformal Killing forms, which we call the normal conformal Killing forms (shortly: nc-Killing forms). These objects are solutions of certain twistor equations, which are conformally covariant, and moreover, they are subject to a normalisation condition. Their existence reflects a special part of the conformal symmetry for a metric (or conformal structure) on a semi-Riemannian manifold.

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The normal twistor equations are induced by the canonical normal Cartan connection of conformal geometry. This canonical connection is a well-known object in conformal geometry (cf. [Kob72]). It lives on the principal fibre bundle, which has a parabolic subgroup of the Möbius group as structure group. Thereby, the parabolic subgroup consists of those conformal transformations of the conformally flat model (Möbius space), which fix the point at infinity. The normalisation condition for the normal conformal Cartan connection is expressed in form of the $\kappa$-tensor
\[
K_g = \frac{1}{n-2} \left( \frac{\text{scal}_g}{2(n-1)} - \text{Ric}_g \right),
\]
which is a basic curvature tensor in conformal geometry (here given in terms of a metric $g$).

The normal conformal Cartan connection can be naturally extended to a usual principal fibre bundle connection with the Möbius group as structure group. Using this extended normal conformal connection, the notion of normal holonomy of a conformal structure can be introduced. It is the structure group to which the normal conformal connection can be ‘maximally’ reduced. Since nc-Killing forms find their interpretation as parallel sections in certain tractor bundles with respect to the covariant derivative induced by the extended normal conformal connection, the existence of nc-Killing forms is apparently related to the holonomy group of the normal conformal connection.

Similar as for the holonomy theory of the Levi-Civita connection in (semi-)Riemannian geometry, the holonomy of the normal conformal connection can be used for the characterisation of the underlying conformal geometry. It turns out in the course of our discussion that a decomposable normal conformal holonomy representation is related to the conformal Einstein condition on (semi)-Riemannian spaces furnished with classical geometric structures such as Sasaki structures, nearly-Kähler structures etc. In particular, it is possible to relate decomposability of the normal conformal holonomy representation to the existence of a certain product metric in the conformal class of a space. The irreducible holonomy representations forbid the conformal Einstein condition. A well-known example where this happens are the Fefferman spaces in Lorentzian geometry, which arise by construction from CR-geometries. The approach via the holonomy discussion of the normal conformal connection will allow us to derive a certain geometric description for conformal spaces admitting solutions of the normal twistor equations. In particular, we will be able to give an improved geometric characterisation of conformal spin spaces admitting conformal Killing spinors (cf. [BL03]).

The road map for our investigations of nc-Killing forms and normal conformal holonomy is as follows. In the paragraphs 2 to 4 we develop the basic notions and facts for the construction of the canonical normal connection of conformal geometry and present finally the normal twistor equations for differential forms. In paragraph 5 we derive integrability conditions in terms of curvature conditions for the existence of solutions. In paragraph 6 we study solutions for nc-Killing forms on Einstein spaces (cf. Theorem 1). As we will see, this is a natural thing to do in view of the normality condition in form of the $\kappa$-tensor $K$. In paragraph 7 we discuss the simplest form of solutions, the decomposable twistors, and we will understand in paragraph 8 that the solutions on Einstein spaces are the basic building blocks, which appear for the decomposable case (Theorem 2). In paragraph 9 we discuss as a showcase the normal conformal holonomy representations and solutions of the twistor equations on 4-dimensional Riemannian and Lorentzian manifolds. Finally, we use our results to discuss the conformal Killing spinor equation on Lorentzian spin manifolds (cf. paragraph 10; Theorem 5).
2. The representations

Let $\mathbb{R}^{r,s}$ denote the (pseudo)-Euclidean space of signature $(r, s)$ with dimension $n = r + s$. The Lie algebra of conformal Killing vector fields on $\mathbb{R}^{r,s}$ is isomorphic to $\mathfrak{so}(r+1, s+1)$. We explain here the usual action of $\mathfrak{so}(r+1, s+1)$ on the spaces of $p$-forms $\Lambda^p_{r+1,s+1}$ over $\mathbb{R}^{r+1,s+1}$ in terms of 2-forms with respect to the irreducible parts of the subrepresentations belonging to the subalgebra $\mathfrak{so}(r,s)$. The latter one is the Lie algebra of the special orthogonal group $\text{SO}(r,s)$, which is isomorphic to the set of Killing vector fields on $\mathbb{R}^{r,s}$ having a zero at the origin, i.e., these are generators of orthogonal rotations. In the following, we denote by $b$ and $z$ the mappings between $\mathbb{R}^{r,s}$ and its dual $\mathbb{R}^{r,s*}$, which are naturally induced via the metric product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{r,s}$. Moreover, we denote by $e = (e_1, \ldots, e_n)$ the standard orthonormal basis in $\mathbb{R}^{r,s}$ such that $\varepsilon_i := \langle e_i, e_i \rangle = -1$ for $i < r+1$.

The space of 2-forms on $\mathbb{R}^{r,s}$ is naturally isomorphic to $\mathfrak{so}(r,s)$ via the mapping

$$\iota : \Lambda^2_{r,s} \to \mathfrak{so}(r,s) \subset \mathfrak{gl}(n).$$

$$\omega \mapsto (x \mapsto (x \cdot \omega)^2)$$

The natural action of $\Lambda^2_{r,s}$ on $\alpha \in \Lambda^p_{r,s}$ is then given by

$$e^b_i \wedge e^b_j \circ \alpha = -e^b_i \wedge (e_j \cdot \alpha) + e^b_j \wedge (e_i \cdot \alpha)$$

$$= e_j \cdot \alpha (e^b_i \wedge \alpha) - e_i \cdot \alpha (e^b_j \wedge \alpha).$$

The Lie algebra $\mathfrak{so}(r+1, s+1)$ of the group of conformal transformations on the conformal compactification space $S^{r,s}$ of $\mathbb{R}^{r,s}$ (Möbius space of signature $(r,s)$) is $|1|$-graded:

$$\mathfrak{so}(r+1, s+1) = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+,\]$$

where $\mathfrak{g}_- \cong \mathbb{R}^{r,s}$, $\mathfrak{g}_0 \cong \mathfrak{co}(r,s)$ and $\mathfrak{g}_+ \cong \mathbb{R}^{r,s*}$ (see their brackets below). To set up explicit identifications for these three subspaces, let $(e_t, e_s, e_1, \ldots, e_n)$ be an orthonormal frame of $\mathbb{R}^{r+1,s+1}$, where $e_t$ is timelike, $e_s$ spacelike and the $e_i$'s are the basis of $\mathbb{R}^{r,s}$. We denote $e_- = \frac{1}{\sqrt{2}}(e_s - e_t)$ and $e_+ = \frac{1}{\sqrt{2}}(e_s + e_t)$. Then we identify

$$\iota : \mathbb{R}^{r,s} \to \mathfrak{g}_-,$$

$$x \mapsto e^b_t \wedge e^b_s$$

$$\iota : \mathbb{R}^{r,s} \to \mathfrak{g}_+,$$

$$y^b \mapsto e^b_+ \wedge y^b$$

$$\iota : \mathbb{R} \oplus \mathfrak{so}(r,s) \to \mathfrak{g}_0,$$

$$l, \omega \mapsto l \cdot e^b_- \wedge e^b_+ + \omega.$$

Besides the usual bracket on $\mathfrak{g}_0 \cong \mathfrak{co}(r,s)$, the non-vanishing Lie brackets are

$$[\omega, x] = (x \cdot \omega)^2, \quad [\omega, y^b] = y^b \cdot \omega \quad \text{and} \quad [x, y^b] = (x, y) \cdot e^b_- \wedge e^b_+ + x^b \wedge y^b,$$

where $x \in \mathfrak{g}_-$, $y^b \in \mathfrak{g}_+$ and $\omega \in \mathfrak{g}_0$. The brackets $[\mathfrak{g}_-, \mathfrak{g}_-]$ and $[\mathfrak{g}_+, \mathfrak{g}_+]$ all vanish.

An arbitrary $(p+1)$-form $\alpha \in \Lambda^p_{r+1,s+1}$ on $\mathbb{R}^{r+1,s+1}$ decomposes into

$$\alpha = e^b_- \wedge \alpha_- + \alpha_0 + e^b_+ \wedge \alpha_+ + e^b_\tau \wedge \alpha_\tau$$

with uniquely determined forms $\alpha_-, \alpha_+ \in \Lambda^p_{r,s}$, $\alpha_0 \in \Lambda^p_{r+1,s+1}$ and $\alpha_\tau \in \Lambda^p_{r+1,s+1}$. This split sum is with respect to the decomposition of $\Lambda^p_{r+1,s+1}$ into the irreducible submodules

$$\Lambda^p_{r,s} \oplus \Lambda^p_{r+1,s} \oplus \Lambda^p_{r,s} \oplus \Lambda^p_{r+1,s+1}.$$
of the restricted action to $so(r,s)$. The action of $so(r + 1, s + 1)$ on $\Lambda_{r+1,s+1}^{p+1}$ with respect to this decomposition is given by
\[
\begin{align*}
\tilde{e}_-^p \wedge e_0^q & = 0 \\
\tilde{e}_-^p \wedge e_0^q \circ \alpha_- & = 0 \\
\tilde{e}_+^p \wedge e_0^q \circ \alpha_- & = -e_0^p \wedge (e_i \cdot 0 \alpha_0) \\
\tilde{e}_+^p \wedge e_0^q \circ \alpha_+ & = e_0^p \wedge e_0^q \circ \alpha_+ \\
\tilde{e}_-^p \wedge e_0^q \circ \alpha_+ & = e_0^p \wedge \alpha_+ + e_0^p \wedge (e_i \cdot 1 \alpha_+) \\
\end{align*}
\]
for $\tilde{e}_-^p \wedge e_0^q \in g_-$. For $\tilde{e}_-^p \wedge e_0^q \in g_+$ we have
\[
\begin{align*}
\tilde{e}_-^p \wedge e_0^q \circ \alpha_- & = e_0^p \wedge (e_i \cdot 1 \alpha_0) \\
\tilde{e}_+^p \wedge e_0^q \circ \alpha_- & = e_0^p \wedge e_0^q \circ \alpha_+ \\
\tilde{e}_+^p \wedge e_0^q \circ \alpha_- & = e_0^p \wedge \alpha_+ + e_0^p \wedge (e_i \cdot 1 \alpha_+) \\
\tilde{e}_+^p \wedge e_0^q \circ \alpha_+ & = 0 \\
\end{align*}
\]
and it is
\[
\tilde{e}_-^p \wedge e_0^q \circ \alpha = -e_0^p \wedge \alpha_- + e_0^p \wedge \alpha_+ .
\]
The action of $so(r,s)$ on the components of $\alpha$ is the usual one.

3. The normal conformal connection

Let $(M^{n,r}, g)$ be an oriented (pseudo)-Riemannian manifold, where $g$ is a metric of signature $(r, n-r)$. The metric $g$ induces a conformal structure $c := \lbrack g \rbrack$ on $M^{n,r}$, which is by definition the equivalence class of metrics, which differ from $g$ only by multiplication with a positive function in $C^\infty(M)$. Such a conformal structure on $M$ is equivalently defined by a reduction of the general linear frame bundle $GL(M)$ to a principal fibre bundle $G_0(M)$ with structure group $CO(r, s) = \mathbb{R}_+ \times SO(r, s)$. The canonical form with values in $\mathbb{R}^{r+s} \cong g_-$ reduced to $G_0(M)$ is denoted by $\theta_-$. Moreover, the metric $g$ gives rise to the Levi-Civita connection form $\omega_{LC}$ on $G_0(M)$.

The conformal structure $c = \lbrack g \rbrack$ on $M$ is equivalently defined by a $P$-reduction $P(M)$ of the second order frame bundle $GL(2)(M)$, where the structure group $P$ is the parabolic subgroup of the Möbius group $SO(r + 1, s + 1)$ with Lie algebra
\[
P := g_0 \oplus g_1 .
\]
The principal fibre bundle $P(M)$ inherits an invariant canonical form $\theta = \theta_- + \theta_0$ from $GL(2)(M)$. Thereby, it is
\[
d\theta_- = \lbrack \theta_- , \theta_0 \rbrack ,
\]
i.e., the canonical form has no torsion (cf. [Kob72], [CSS97]).

Now let $\omega$ be an arbitrary Cartan connection on $P(M)$ with values in $so(r + 1, s + 1) = g_- \oplus g_0 \oplus g_1$. Its curvature 2-form is defined by
\[
\Omega = d\omega + \frac{1}{2}[\omega, \omega]
\]
and the corresponding curvature function with values in $g_-^* \otimes g_-^* \otimes g$ is given in a point $u \in P(M)$ by
\[
\nu(u)(x, y) := \Omega(\omega_{NC}^{-1}(x), \omega_{NC}^{-1}(y))(u), \quad x, y \in g_- .
\]
Since the map $ad : g_0 \rightarrow gl(g_-)$ is injective, the $g_0$-part $\nu_0$ of the curvature function can be seen as function on $P(M)$ with values in $g_-^* \otimes g_-^* \otimes g_-^* \otimes g_+^*$.

It is a well-known fact in conformal geometry that there exists a unique Cartan connection
\[
\omega_{NC} = \omega_- \oplus \omega_0 \oplus \omega_1
\]
on $P(M)$ with the following properties (cf. [Kob72], [CSS97]):
(1) It is
\[ \omega_{-1} = \theta_{-1} \quad \text{and} \quad \omega_0 = \theta_0 , \]
i.e., the torsion of \( \omega_{NC} \) vanishes and

\[ \text{tr}(\nu_0)(x, y) := \sum_{i=1}^{n} \nu_0(e_i, x)(y)(e_i^\flat) = 0 , \quad x, y \in \mathfrak{g}_- , \]
i.e., the trace of the \( \mathfrak{g}_0 \)-part of the curvature function is trivial.

The so defined Cartan connection \( \omega_{NC} \) on the reduced bundle \( P(M) \) is called the canonical normal Cartan connection of conformal geometry and is the basic object for all considerations in this paper.

We want to describe the normal conformal Cartan connection \( \omega_{NC} \) in terms of the metric \( g \) in the conformal class \( c \). First, we notice that if \( \pi : P(M) \rightarrow G_0(M) \) denotes the natural projection then \( \theta_- \) projects to the canonical form on \( G_0(M) \subset GL^{(1)}(M) \). Furthermore, the \( G_0 \)-equivariant lifts \( \sigma \) of \( G_0(M) \) to \( P(M) \) correspond bijectively to the Weyl connections \( \omega^\sigma \) on \( G_0(M) \) by

\[ \omega^\sigma = \sigma^* \theta_0 . \]

In particular, if \( \sigma^g \) is the equivariant lift induced by the Levi-Civita connection \( \omega_{LC}^g \) then the \( \mathfrak{g}_0 \)-part of \( \omega_{NC} \) is related to \( \omega_{LC}^g \) by \( \omega_{LC}^g = \sigma^* \theta_0 \). It remains to determine the \( \mathfrak{g}_1 \)-part of \( \omega_{NC} \) with respect to \( g \). This part must be calculated from the trace-free condition on the curvature function \( \nu_0 \) and the result is

\[ \omega_1 = -\Gamma \circ \theta_{-1} , \]
where the function \( \Gamma : P(M) \rightarrow \mathfrak{g}_{-1} \otimes \mathfrak{g}_1 \) is the pullback of the so-called \( \varrho \)-tensor on \( (M, g) \), which is given by the expression

\[ K_g = \frac{1}{n - 2} \left( \frac{\text{scal}_g}{2(n-1)} - \text{Ric}_g \right) . \]

Thereby, \( \text{Ric}_g \) denotes the Ricci-tensor and \( \text{scal}_g \) is the scalar curvature of \( g \).

In short, we see that \( \omega_{NC} \) is given with respect to \( g \in c \) by \( \theta_- \), \( \omega_{LC}^g \) and \( K_g \). This description is invariant in the sense that for all \( \hat{g} \in c \) the connection \( \omega_{NC} \) is determined by these data in the same way. This can explain the importance of the \( \varrho \)-tensor in conformal geometry. It transforms naturally in the conformal class.

However, there is a basic construction, which assigns to every Cartan connection on a principal fibre bundle, a usual principal fibre bundle connection through extension. In our case of conformal Cartan geometry, this can be done as follows. Let

\[ \mathfrak{M}(M) = P(M) \times_P \text{SO}(r + 1, s + 1) \]

be the extended bundle with structure group \( \text{SO}(r + 1, s + 1) \). We call this bundle the Möbius frame bundle. With respect to a metric \( g \) and the inclusion of \( \mathfrak{so}(r, s) \) in \( \mathfrak{so}(r + 1, s + 1) \) as described above, we can express this bundle also as

\[ \mathfrak{M}(M) = \text{SO}(M, g) \times_{\text{SO}(r, s)} \text{SO}(r + 1, s + 1) , \]

where \( \text{SO}(M, g) \) is the orthonormal frame bundle to \( g \) on \( M \). Then a local frame \( s = (s_1, \ldots, s_n) \) on \( M \), which is a local section in \( \text{SO}(M, g) \), has a natural extension to a section \( s_c = (s_-, s_+, s_1, \ldots, s_n) \) in \( \mathfrak{M}(M) \). Thereby, the \( (s_c)_i \)'s can be seen as tractors (sections) in \( T\mathfrak{M}(M) \cong \Lambda_0^2(M) \) (see below).

The Cartan connection \( \omega_{NC} \) can now be extended to a usual principal connection on \( \mathfrak{M}(M) \) by right translation on the fibres. We denote this normal conformal connection on \( \mathfrak{M}(M) \) also by \( \omega_{NC} = \omega_{-1} \oplus \omega_0 \oplus \omega_1 \). We have already calculated the components of the connection \( \omega_{NC} \) with respect to the metric \( g \). Let \( \mathfrak{s} \) be a local
frame on \((M, g)\). Then we have the following expression for the local connection form on \((M, g)\):

\[
\omega_{NC} \circ ds_e(X) = e_\varepsilon \wedge \theta_-(X)^b + \omega_{LC} \circ ds_e(X) - e_\varepsilon \wedge \theta_-(K_g(X))^b, \quad X \in TM,
\]

where \((e_-, e_+, e_1, \ldots, e_n)\) is the standard basis in \(\mathbb{R}^{r+1,s+1}\) and \(\theta_-\) is evaluated at \(s\). The reason for using the extended approach for \(\omega_{NC}\) is because, in the following, we would like to use the usual notion of holonomy for a connection on a principal fibre bundle. We denote the holonomy of \(\omega_{NC}\) on \(\mathfrak{M}(M)\) over the conformal space \((M, c)\) with \(\text{Hol}(\omega_{NC}, c)\) (or just \(\text{Hol}(\omega_{NC})\) if there is no ambiguity).

Moreover, with the approach of principal connection forms we can introduce covariant derivatives to \(\omega_{NC}\) on vector bundles with structure group \(\text{SO}(r+1, s+1)\) associated to \(\mathfrak{M}(M)\) in the usual manner. In particular, \(\omega_{NC}\) induces derivatives \(\nabla^{NC}\) on the Möbius \(p\)-form bundles (tractor bundles) defined as

\[
\Lambda^p_{\mathfrak{M}}(M) := \mathfrak{M}(M) \times \Lambda^p_{r+1,s+1}.
\]

With respect to the metric \(g\) these bundles split into sums of the usual \(p\)-form bundles on \(M\):

\[
\Lambda^{p+1}_{\mathfrak{M}}(M) = \Lambda^p(M) \oplus \Lambda^{p+1}(M) \oplus \Lambda^{p-1}(M) \oplus \Lambda^p(M).
\]

The covariant derivative \(\nabla^{NC}\) acts on sections in these bundles with respect to the above splitting by the matrix expression

\[
\nabla^{NC}_X \alpha = \begin{pmatrix}
\nabla^{LC}_X & -X \cdot & X^b \wedge & 0 \\
-K(X)^b \wedge & \nabla^{LC}_X & 0 & X^b \wedge \\
K(X) \cdot & 0 & \nabla^{LC}_X & X \cdot \\
0 & K(X) \cdot & K(X)^b \wedge & \nabla^{LC}_X
\end{pmatrix} \begin{pmatrix}
\alpha_- \\
\alpha_0 \\
\alpha_+ \\
\alpha_\mp
\end{pmatrix}.
\]

Thereby, \(\nabla^{LC}\) denotes the Levi-Civita connection. This expression is calculated straightforwardly from the local form of \(\omega_{NC}\) and the formulae for the action of \(\mathfrak{so}(r+1, s+1)\) on \(\Lambda^{p+1}_{r+1,s+1}\).

4. THE TWISTOR EQUATIONS

Let \((M^{n,r}, g)\) be an oriented (pseudo)-Riemannian manifold and let \(\Lambda^{p+1}_{\mathfrak{M}}(M)\) be the associated bundle of \((p + 1)\)-forms to the principal fibre bundle \(\mathfrak{M}(M)\) with normal conformal covariant derivative \(\nabla^{NC}\). We call a section \(\alpha \in \Omega^{p+1}_{\mathfrak{M}}(M)\) a (normal) twistor iff \(\nabla^{NC}\alpha = 0\). The twistor \(\alpha\) corresponds via the metric \(g\) to a set of differential forms on \(M^{n,r}\):

\[
\alpha \leftrightarrow (\alpha_-, \alpha_0, \alpha_\mp, \alpha_+),
\]

where \(\alpha_-, \alpha_+ \in \Omega^p(M)\), \(\alpha_0 \in \Omega^{p+1}(M)\) and \(\alpha_\mp \in \Omega^{p-1}(M)\). The condition \(\nabla^{NC}\alpha = 0\) is then equivalent to the set of conformally covariant equations given by

\[
\nabla^{LC}_X \alpha_- - X \cdot \alpha_0 + X^b \wedge \alpha_\mp = 0 \quad (1)
\]
\[
-K(X)^b \wedge \alpha_- + \nabla^{LC}_X \alpha_0 + X^b \wedge \alpha_+ = 0 \quad (2)
\]
\[
K(X) \cdot \alpha_- + \nabla^{LC}_X \alpha_\mp + X \cdot \alpha_+ = 0 \quad (3)
\]
\[
K(X) \cdot \alpha_0 + K(X)^b \wedge \alpha_\mp + \nabla^{LC}_X \alpha_+ = 0. \quad (4)
\]
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We calculate from \( \alpha_- \) of a given solution \( \alpha \) the remaining differential forms in order to get equations for \( \alpha_- \) only. It is

\[
d = \sum_{i=1}^{n} \varepsilon_i s_i^\flat \wedge \nabla^L \text{C} \quad \text{and} \quad d^* = -\sum_{i=1}^{n} \varepsilon_i s_i \cdot \nabla^L \text{C}
\]

the exterior differential resp. the codifferential with respect to a local orthonormal frame \( s \). The equations \( 1 \) - \( 3 \) imply for a twistor \( \alpha \) of degree \( p + 1 \) that

\[
d\alpha_- = (p + 1)\alpha_0, \quad d^*\alpha_- = (n - p + 1)\alpha_\mp
\]

\[
\frac{1}{p+1} d^*\alpha_- = (n - p)\alpha_+ - \sum_{i} \varepsilon_i s_i \cdot \mathcal{I}(K(s_i)^\flat \wedge \alpha_-)
\]

\[
\frac{1}{n(p+1)} dd^*\alpha_- = -p\alpha_+ - \sum_{i} \varepsilon_i s_i^\flat \wedge (K(s_i) \cdot \alpha_-)
\]

For \( n \neq 2p \) the sum of the latter two equations results to

\[
\alpha_+ = \frac{1}{n-2p} \cdot \left\{ -\frac{\text{scal}}{2(n-1)} \alpha_- + \frac{1}{p+1} d^*\alpha_- + \frac{1}{n-p+1} dd^*\alpha_- \right\}
\]

which is

\[
\alpha_+ = \frac{1}{n-2p} (\nabla^\star \nabla - \frac{\text{scal}}{2(n-1)})\alpha_-
\]

where \( \nabla^\star \nabla \) denotes the Bochner-Laplacian. In the middle dimensional case \( n = 2p \) we have

\[
\alpha_+ = \frac{1}{n} \cdot \left\{ \frac{1}{p+1} (d^*d - dd^*)\alpha_- + \sum_{i} \varepsilon_i \cdot (s_i \cdot \mathcal{I}(K(s_i)^\flat \wedge \alpha_-) - s_i^\flat \wedge (K(s_i) \cdot \alpha_-)) \right\}
\]

We observe that \( \alpha_- \equiv 0 \) if and only if the twistor \( \alpha \) is trivial.

With the so derived expressions for the components of a twistor \( \alpha \) we now formulate the normal twistor equations for a \( p \)-form \( \alpha_- \) on a (pseudo)-Riemannian manifold \( (M^{n-r}, g) \). They are

\[
0 = \nabla^L X \alpha_- - \frac{1}{p+1} X \cdot \mathcal{I} d\alpha_- + \frac{1}{n-p+1} X^\flat \wedge d^*\alpha_- \quad (5)
\]

\[
0 = -K(X)^\flat \wedge \alpha_- + \frac{1}{p+1} \nabla^L \alpha_- + X^\flat \wedge \nabla^\mathcal{K} \alpha_- \quad (6)
\]

\[
0 = K(X) \cdot \mathcal{I} \alpha_- + \frac{1}{n-p+1} \nabla^L d^*\alpha_- + X \cdot \mathcal{I} \nabla^\mathcal{K} \alpha_- \quad (7)
\]

\[
0 = \frac{1}{p+1} K(X) \cdot \mathcal{I} d\alpha_- + \frac{1}{n-p+1} K(X)^\flat \wedge d^*\alpha_- + \nabla^L \nabla^\mathcal{K} \alpha_- \quad (8)
\]

whereby we set

\[
\nabla^\mathcal{K} := \frac{1}{n-2p} \cdot \left\{ -\frac{\text{scal}}{2(n-1)} \mathcal{I} + \nabla^\star \nabla \right\} \quad \text{for } n \neq 2p
\]

and

\[
\nabla^{n/2} := \frac{1}{n} \cdot \left\{ \frac{1}{p+1} (d^*d - dd^*) + \sum_{i} \varepsilon_i \cdot (s_i \cdot \mathcal{I}(K(s_i)^\flat \wedge \cdot) - s_i^\flat \wedge (K(s_i) \cdot \cdot)) \right\}
\]

In the following, we say that a \( p \)-form \( \alpha_- \in \Omega^p(M) \), which satisfies the (normal) twistor equations \( 5 \) - \( 8 \), is a normal conformal Killing \( p \)-form (or shortly a nc-Killing \( p \)-form). The conformal covariance of the equations implies that if \( \alpha_- \) is a
nc-Killing $p$-form to $g$ on $M$ then the rescaled $p$-form
\[ \tilde{\alpha}_- := e^{-\phi} \alpha_- \]
is nc-Killing with respect to the conformally changed metric $\tilde{g} = e^{-2\phi} \cdot g$.

However, the equations [14] - [8] are not only conformally covariant, but a further natural symmetry appears. Let $\ast$ denote the Hodge-star operator on $\Lambda^*(M)$ defined by
\[ \alpha_- \wedge \ast \alpha_- = g(\alpha_-, \alpha_-) dM, \]
where $dM$ denotes the volume form of $(M^{n,r}, g)$. It is
\[ \ast|_{\Lambda^p} = (-1)^{p(n-p)+r} \quad \text{and} \quad d^\ast = (-1)^{n(p-1)+r+1} \ast d \ast. \]

There is also a 'Hodge' operator $\ast_{2\mathbb{R}}$ on $\Lambda^*_M(M)$ defined in the same manner:
\[ \alpha \wedge \ast_{2\mathbb{R}} \alpha = c_{2\mathbb{R}}(\alpha, \alpha) dM, \]
where $dM_{2\mathbb{R}} := -e^\beta \wedge e^\alpha \wedge dM$ and $c_{2\mathbb{R}}$ is the obvious $\text{SO}(r+1, s+1)$-invariant scalar product on $\Lambda^*_{2\mathbb{R}}(M)$. The operator $\ast_{2\mathbb{R}}$ is parallel:
\[ \nabla^{NC} \ast_{2\mathbb{R}} = \ast_{2\mathbb{R}} \nabla^{NC}. \]

Therefore, if $\alpha$ is a $(p+1)$-twistor then $\ast_{2\mathbb{R}} \alpha$ is a $(n-p+1)$-twistor. The twistor $\ast_{2\mathbb{R}} \alpha$ corresponds to the set
\[( ( -1 )^p \ast \alpha_- , \ast \alpha_\mp , - \ast \alpha_0 , ( -1 )^{p+1} \ast \alpha_+ ) \]
of differential forms. This shows that if $\alpha_-$ is a nc-Killing $p$-form then $\ast \alpha_-$ is a nc-Killing $(n-p)$-form. Indeed, with
\[ \ast ( X \ast \beta^p ) = ( -1 )^{p+1} X^b \wedge \ast \beta \quad \text{and} \quad \ast ( X^b \wedge \beta^p ) = ( -1 )^p X \ast \beta, \]
and since $\ast \square \ast n-p$ is anti-commuting, the normal twistor equations [5] - [8] are seen to be $\ast$-invariant as well.

Finally, we remark that for a 1-form $\alpha_-$, equation [5] just means that the dual to $\alpha_-$ is a conformal vector field. In general, solutions of [5] are known as conformal Killing $p$-forms (cf. [Kas68], [Tat69], [Sem01]). Equation [5] is Hodge $\ast$-invariant itself. The additional equations [6] - [8] then impose further conditions on a conformal Killing $p$-form to be 'normal'.

5. CURVATURE CONDITIONS

We derive here integrability conditions for the existence of nc-Killing $p$-forms on a (pseudo)-Riemannian manifold $(M^{n,r}, g)$ in terms of curvature expressions. We denote by $s = (s_1, \ldots, s_n)$ a local frame. Let
\[ R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \]
be the Riemannian curvature tensor, where $X, Y, Z \in T M$ are tangent vectors. By contraction, we obtain
\[ \text{Ric}(X) = \sum_{i=1}^n \varepsilon_i \cdot R(X, s_i) s_i, \quad \text{scal} = \text{tr}(\text{Ric}), \]
the Ricci tensor and the scalar curvature. The $g$-tensor is $K = \frac{1}{n-r} (\text{scal} - tr(\text{Ric}))$. The trace-free part of the Riemannian curvature tensor is the Weyl tensor $W$, which can be expressed by
\[ W = R - g \star K, \]
where $\star$ denotes the Kulkarni-Nomizu product. Moreover, we have the Cotton-York tensor $C$, which is the anti-symmetrisation of the covariant derivative of the $g$-tensor:
\[ C(X, Y) := (\nabla_X K)(Y) - (\nabla_Y K)(X). \]
Furthermore, we find the Bach tensor

\[
B(X, Y) = \sum_{i=1}^{n} \varepsilon_i \cdot \nabla_{s_i} C(X, Y, s_i) - \sum_{i=1}^{n} \varepsilon_i \cdot W(K(s_i), X, Y, s_i),
\]

where \(C(X, Y, Z) := C_X(Y, Z) = g(C(Y, Z), X)\). The Weyl tensor considered as a symmetric map on the space of 2-forms is conformally invariant. The Bach tensor is symmetric and divergence-free. Moreover, we have the Bianchi identities

\[
R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0,
\]

\[
\nabla_X R(Y, Z) + \nabla_Y R(Z, X) + \nabla_Z R(X, Y) = 0
\]

for all \(X, Y, Z \in TM\), which also imply

\[
\sum_{i=1}^{n} \varepsilon_i \cdot \nabla_{s_i} W(X, Y, Z, s_i) = (3 - n) \cdot C(Z, X, Y) \quad \text{and} \quad \sum_{i=1}^{n} \varepsilon_i \cdot C(s_i, s_i, X) = 0.
\]

The Kulkarni-Nomizu product \(g \ast K\) acts on 2-forms by

\[
g \ast (s_i ^b \wedge s_j ^b) = s_i ^b \wedge K(s_j) ^b - s_j ^b \wedge K(s_i) ^b.
\]

We calculate now the curvature of the normal conformal covariant derivative \(\nabla_{NC}^n\) on \(\Lambda^{p+1}_{2n}(M)\). For this, let \(\alpha = (\alpha_-, \alpha_0, \alpha_+, \alpha_{\pm})\) be a smooth section in \(\Lambda^{p+1}_{2n}(M)\). It is

\[
(\nabla_{NC}^n \nabla_{NC}^n \alpha)_- = \nabla_{NC}^l \nabla_{NC}^l \alpha_- - Y \mathcal{J} \alpha_0 + Y^b \wedge \alpha_{\pm}
\]

\[
- X \mathcal{J} (-K(Y)^b \wedge \alpha_- + \nabla_{NC}^l \alpha_0 + Y^b \wedge \alpha_+)
\]

\[
+ X^b \wedge (K(Y) \mathcal{J} \alpha_- + \nabla_{NC}^b \alpha_+ + Y \mathcal{J} \alpha_+)
\]

\[
(\nabla_{NC}^n \nabla_{NC}^n \alpha)_0 = -K(X)^b \wedge (\nabla_{NC}^l \alpha_- - Y \mathcal{J} \alpha_0 + Y^b \wedge \alpha_+)
\]

\[
+ \nabla_{NC}^l (-K(Y)^b \wedge \alpha_- + \nabla_{NC}^l \alpha_0 + Y^b \wedge \alpha_+)
\]

\[
+ X^b \wedge (K(Y) \mathcal{J} \alpha_0 + K(Y)^b \wedge \alpha_{\pm} + \nabla_{NC}^b \alpha_+)
\]

\[
(\nabla_{NC}^n \nabla_{NC}^n \alpha)_+ = K(X) \mathcal{J} (-K(Y)^b \wedge \alpha_- + \nabla_{NC}^l \alpha_0 + Y^b \wedge \alpha_+)
\]

\[
+ \nabla_{NC}^l (K(Y) \mathcal{J} \alpha_- + \nabla_{NC}^b \alpha_+ + Y \mathcal{J} \alpha_+)
\]

\[
+ X^b \wedge (K(Y) \mathcal{J} \alpha_0 + K(Y)^b \wedge \alpha_{\pm} + \nabla_{NC}^b \alpha_+)
\]

\[
(\nabla_{NC}^n \nabla_{NC}^n \alpha)_+ = K(X) \mathcal{J} (-K(Y)^b \wedge \alpha_- + \nabla_{NC}^l \alpha_0 + Y^b \wedge \alpha_+)
\]

\[
+ K(X)^b \wedge (K(Y) \mathcal{J} \alpha_- + \nabla_{NC}^b \alpha_+ + Y \mathcal{J} \alpha_+)
\]

\[
+ \nabla_{NC}^l (K(Y) \mathcal{J} \alpha_0 + K(Y)^b \wedge \alpha_{\pm} + \nabla_{NC}^b \alpha_+)
\]
and we obtain
\[
(R^\nabla(X,Y) \circ \alpha) - (R^\nabla(X,Y) \circ \alpha)
+ \left( (X \cdot (K(Y)^\flat \wedge \alpha_-) - Y \cdot (K(X)^\flat \wedge \alpha_-))
+ (X^b \wedge (K(Y)^\flat \alpha_-) - Y^b \wedge (K(X)^\flat \alpha_-))ight)
\]
\[
= W(X,Y) \circ \alpha_-
\]
\[
(R^\nabla(X,Y) \circ \alpha)_0 = W(X,Y) \circ \alpha_0 - C(X,Y)^b \wedge \alpha_- 
\]
\[
(R^\nabla(X,Y) \circ \alpha)_{\mp} = W(X,Y) \circ \alpha_+ + C(X,Y) \cdot \alpha_-
\]
\[
(R^\nabla(X,Y) \circ \alpha)_{+} = W(X,Y) \circ \alpha_+ + C(X,Y)^b \wedge \alpha_+
\]
i.e., the curvature takes the matrix form
\[
R^\nabla = \begin{pmatrix}
W & 0 & 0 & 0 \\
-C(X,Y)^b \wedge & W & 0 & 0 \\
C(X,Y)^\flat \cdot & 0 & W & 0 \\
0 & C(X,Y)^\cdot & C(X,Y)^b \wedge & W \\
\end{pmatrix}
\]
As integrability condition for the existence of a twistor \(\alpha\) we obtain
\[
W(X,Y) \circ \alpha_- = 0 
\]
\[
W(X,Y) \circ \alpha_0 = C(X,Y)^b \wedge \alpha_- 
\]
\[
W(X,Y) \circ \alpha_\mp = -C(X,Y)^\cdot \alpha_- 
\]
\[
W(X,Y) \circ \alpha_+ = -C(X,Y)^\cdot \alpha_0 - C(X,Y)^b \wedge \alpha_+ 
\]
By taking the divergence on both sides of these equations we get
\[
(n - 4) \cdot C_T \circ \alpha_- = 0 
\]
\[
(n - 4) \cdot C_T \circ \alpha_0 = -B(T)^b \wedge \alpha_- 
\]
\[
(n - 4) \cdot C_T \circ \alpha_\mp = B(T)^\cdot \alpha_- 
\]
\[
(n - 4) \cdot C_T \circ \alpha_+ = B(T)^\cdot \alpha_0 + B(T)^b \wedge \alpha_+ 
\]
The curvature conditions for the nc-Killing \(p\)-form \(\alpha_-\) take the form:
\[
W(X,Y) \circ \alpha_- = 0 \quad (9)
\]
\[
W(X,Y) \circ \alpha_0 = (p + 1) \cdot C(X,Y)^b \wedge \alpha_- \quad (10)
\]
\[
W(X,Y) \circ d^* \alpha_- = -(n - p + 1) C(X,Y) \cdot \alpha_- \quad (11)
\]
\[
W(X,Y) \circ \Box_p \alpha_- = -\frac{1}{p + 1} C(X,Y)^\cdot \alpha_- - \frac{1}{n - p + 1} C(X,Y)^b \wedge d^* \alpha_- \quad (12)
\]
Of course, the sets of integrability conditions are conformally covariant and invariant under the Hodge *-operator.
6. Normal conformal Killing \( p \)-forms on Einstein manifolds

We consider in this paragraph solutions of the normal twistor equations (8) - (8) on Einstein manifolds. Before we start with this, we want to state a criterion when a space \((8)\) on Einstein manifolds. Before we start with this, we want to state a criterion when a metric \( \tilde{g} \) in the conformal class \( c = [g] \), which satisfies
\[
\text{Ric}_{\tilde{g}} = \frac{\text{scal}_0}{n} \cdot \tilde{g}.
\]

For this, let us assume that \( f_- = \alpha_- \) is a nc-Killing function (= 0-form) without zeros. We have mentioned before that the rescaled function \( \tilde{\alpha}_- = \frac{1}{f_-} \cdot \alpha_- = 1 \) is nc-Killing with respect to the metric \( \tilde{g} = \frac{1}{f_-} \cdot g \). From the twistor equations (8) - (8), it follows immediately
\[
K_{\tilde{g}} = - \frac{\text{scal}_0}{2n(n-1)} \cdot \tilde{g} ,
\]
which means that \( \tilde{g} \) is Einstein. On the other hand, every constant function on an Einstein space is nc-Killing. The criterion then says that a metric is conformally Einstein (i.e., there is an Einstein metric in the conformal class \( [g] \)) if and only if there exists at least one nc-Killing function without zeros.

Obviously, in case that \( f_- \) has a zero the rescaling in the way as above is not possible. Indeed, examples of nc-Killing functions on non-conformally Einstein spaces are well known (cf. [KR96]). However, since in general the set of zeros of nc-Killing forms is singular on the underlying conformal space, we can at least say that the existence of a nc-Killing function \( f_- \) implies that, up to singularities, an Einstein metric exists in the conformal class \( [g] \). This is exactly the case when the holonomy group of the conformal connection \( \omega_{NC} \) fixes at least one vector in \( \mathbb{R}^r+1 \). In case that this vector is lightlike the ‘Einstein scaling’ is Ricci-flat. The timelike case is for \( \text{scal} > 0 \), the spacelike when \( \text{scal} < 0 \). By the way, the normal twistor equations for a function \( f_- \) are in general equivalent to
\[
\text{Hess}(f_-) = f_- \cdot K_0 \quad \text{and} \quad K(X)(f_-) = \frac{1}{n} X(\frac{\text{scal}}{2(n-1)} f_-)
\]
for all \( X \in TM \), where \( K_0 \) denotes the trace-free part of the \( g \)-tensor.

Now we assume that \((M^{n,r}, g)\) is a (pseudo)-Riemannian Einstein manifold. The constant functions are nc-Killing on Einstein manifolds. In particular, the 1-form \( o := s^b - \frac{\text{scal}}{2(n-1)n} s^b_+ \), which obviously satisfies the equations (1) - (4), is the normal twistor in \( \Omega^1_{\text{g}}(M) \), which corresponds to the set of forms \((1,0,0,-\frac{\text{scal}}{2(n-1)n})\), i.e., \( o_- = 1 \) is the constant unit function. Furthermore, let
\[
\alpha = s^- \wedge \alpha_- + \alpha_0 + s^b_\wedge s^b_+ \wedge \alpha_+ + s^b_+ \wedge \alpha_+.
\]
be an arbitrary normal twistor of degree \((p+1)\) with \( do_- \neq 0 \) on \( M \). It follows immediately that \( o \wedge \alpha \) is a \((p+2)\)-twistor. This twistor corresponds to the set
\[
(\alpha_0, 0, \alpha_+ + \frac{\text{scal}}{2(n-1)n} \alpha_- - \frac{\text{scal}}{2(n-1)n} \alpha_0)
\]
of differential forms, which shows that \( do_- \) is a (closed) nc-Killing \((p+1)\)-form. The \((n-p-1)\)-form \(*do_-\) is then nc-Killing and coclosed.

In general, the set of twistor equations (5) - (5) reduces for a coclosed \( p \)-form \( \beta_- \) on an Einstein space to
\[
\nabla^LC_X \beta_- = \frac{1}{p+1} \cdot X \cdot \nabla d\beta_-,
\]
\[
\nabla^LC_X d\beta_- = -\left(\frac{p+1}{n \cdot (n-1)} \cdot X^b \wedge \beta_- \right),
\]
which implies $\Delta_p \beta_- = \frac{(p+1)(n-p)\text{scal}}{n(n-1)} \beta_-$ for the Laplacian $\Delta_p = dd^* + d^*d$. A differential form that satisfies these two equations above (not only in the Einstein case) is called a special Killing $p$-form to the Killing constant $-\frac{(p+1)\text{scal}}{n(n-1)}$ (cf. [Sem01]).

There is a nice way to describe the geometry of spaces with special Killing forms for non-zero Killing constant using the cone construction. We explain this approach briefly as next. In a further step we show that we can extend this approach to describe all nc-Killing forms on Einstein spaces with non-zero scalar curvature as parallel forms on an 'ambient' metric.

The cone metric with scaling $b \neq 0$ is defined on the space $\mathbb{R}_+ \times M$ as

$$\bar{g}_b := b dt^2 + t^2 g .$$

This metric has either signature $(r, s + 1)$ or $(r + 1, s)$. We have the following result for special Killing forms.

**Proposition 1.** (cf. [Sem01]) Let $(M^{n,r}, g)$ be a (pseudo)-Riemannian manifold and $\hat{M}_b$ its cone with metric $\bar{g}_b$ to the constant $b = \frac{(n-1)n}{\text{scal}} \neq 0$. Then the special Killing $p$-forms on $M^{n,r}$ with Killing constant $-\frac{(p+1)\text{scal}}{n(n-1)}$ correspond bijectively to the parallel $(p+1)$-forms on the cone $\hat{M}_b$. The correspondence is given by

$$\beta_- \in \Omega^p(M) \quad \mapsto \quad t^p dt \wedge \beta_- + \frac{\text{sign}(b) \cdot t^{p+1}}{p+1} d\beta_- \in \Omega^{p+1}(\hat{M}_b) .$$

However, the metric $\bar{g}_b$ on $\mathbb{R}_+^2 \times M$ of signature $(r + 1, s + 1)$ in dimension $n + 2$ defined by

$$\bar{g}_b := b (dt^2 - ds^2) + t^2 g$$

is appropriate to describe all normal twistors on Einstein spaces ($\text{Ric} \neq 0$) as parallel forms.

**Proposition 2.** Let $(M^{n,r}, g)$ be an Einstein space and $(\hat{M}_b, \bar{g}_b)$ its ambient metric with $b = \frac{(n-1)n}{\text{scal}} \neq 0$. There is a natural and bijective correspondence between normal twistors $\alpha \in \Omega^{n+1} M$ and parallel forms $\vec{\alpha} \in \Omega^{n+1}(\hat{M}_b)$. Moreover, the holonomy groups of the normal conformal connection and the Levi-Civita connection on $\hat{M}_b$ coincide, i.e.,

$$\text{Hol}(\omega_{NC}, c) \cong \text{Hol}(\bar{g}_{\vec{\beta}, b}) .$$

**Proof.** We prove the statement that the holonomies coincide. For this, we embed $M$ in $\hat{M}_b$ by $i : M \to M \times \{(1, 1)\}$. The 1-tistor on $M$ which defines the Einstein structure is $\mathbf{s}_b = \frac{\text{scal}}{2(n-1)n} \mathbf{s}_+$. Next we define an isometric map between the tangent tractors on $M$ and the tangent vectors at $i(M) \subset \hat{M}_b$ by assigning

$$s_i \quad \to \quad s_i \in T\hat{M}_b|_{M \times \{(1, 1)\}}$$

$$\sqrt{|\beta|} \cdot (s_- + \frac{\text{scal}}{2(n-1)n} \mathbf{s}_+) \quad \to \quad \sqrt{|\beta|} \cdot \partial \ell \quad \in T\hat{M}_b|_{M \times \{(1, 1)\}}$$

$$\sqrt{|\beta|} \cdot (s_- - \frac{\text{scal}}{2(n-1)n} \mathbf{s}_+) \quad \to \quad \sqrt{|\beta|} \cdot \partial \ell \quad \in T\hat{M}_b|_{M \times \{(1, 1)\}}$$

It can be easily calculated that the resulting bundle isomorphism between $\Omega(M)$ and $SO(\hat{M}_b)|_{M \times \{(1, 1)\}}$ has the property $i^* \omega_{\text{LC}} = \omega_{NC}$, i.e., the connection forms are identified in this way. Moreover, all elements in the holonomy of the metric $\bar{g}$ are generated by the horizontal lifts of paths which move solely on the level set $M \times \{(1, 1)\}$. This shows that $\text{Hol}(\omega_{NC}, c) \cong \text{Hol}(\omega_{LC})$.

The above map between the tangent tractors on $M$ and the vectors on $\hat{M}_b$ also provides the correspondence between the normal twistors on $M$ and the parallel forms on the ambient space $\hat{M}_b$. $\square$
In particular, one can see from Proposition 2 that every nc-Killing form on an Einstein space with non-zero scalar curvature is the sum of a closed and a coclosed nc-Killing form. The holonomy groups of the Levi-Civita connections on the cone $\hat{M}_b$ and the ambient $M_b$ are obviously identical, which also means $\text{Hol}(\omega_{NC}) = \text{Hol}(\omega_{\hat{g}_b}^{LC})$.

In the Riemannian case a geometric characterisation of complete spaces $(M^n, g)$ with positive scalar curvature admitting special Killing forms was established by using the above correspondence with the cone and the holonomy classification for simply connected, irreducible and non-locally symmetric spaces (cf. [Bar93, Sem01]). Thereby, we remember to the fact that if the holonomy of a Riemannian cone $\hat{M}_b$, $b > 0$, over a complete Riemannian space $M$ is reducible then the cone is automatically flat. However, it is not difficult to extend the geometric characterisation to non-complete spaces with special Killing forms when the cone is reducible, but not flat. In this case the metric $g$ on $M$ turns out to be (locally) a certain warped-product. Combining Proposition 1 and 2 results to the following.

**Theorem 1.** a) Let $(M^n, g)$ be a simply connected and complete Riemannian Einstein space of positive scalar curvature admitting a nc-Killing $p$-form. Then $M^n$ is either

1. the round (conformally flat) sphere $S^n$,
2. an Einstein-Sasaki manifold of odd dimension $n \geq 5$ with a special Killing 1-form $\alpha_-$,
3. an Einstein-3-Sasaki space of dimension $n = 4m + 3 \geq 7$ with three independent special Killing 1-forms $\alpha^1_-, \alpha^2_- \text{ and } \alpha^3_-,$
4. a nearly Kähler manifold of dimension 6, where the Kähler form $\omega_-$ is a special Killing 2-form or
5. a nearly parallel $G_2$-manifold in dimension 7 with its fundamental form $\gamma_-$ as special Killing 3-form.

b) If the space $M^n$ is not complete and the cone reducible then the metric $g$ has up to a constant scaling factor (locally) the form

$$dt^2 + \sin^2(t) \cdot k + \cos^2(t) \cdot h,$$

where $k^p$ and $h^q$ are arbitrary Riemannian Einstein metrics of positive scalar curvature on spaces with dimension $p \text{ resp. } q$ ($0 \leq q \leq n - 1$). The scaled volume forms

$$\sin^{-p} \cdot \text{dvol}_k \text{ and } \cos^{-q} \cdot \text{dvol}_h$$

to $k$ and $h$ are special Killing of degree $p \text{ resp. } q$.

Similarly, for nc-Killing forms on Riemannian Einstein spaces $(M^n, g)$ of negative scalar curvature one has to consider the cone with Lorentzian metric (indeterminate of signature $(1, n)$). In this case the Lorentzian cone either has weakly irreducible or decomposable holonomy. In both cases one can show that $(M^n, g)$ admits (locally) certain warped-product structures (cf. [Bau89]). In general, there is no classification of possible holonomy groups of the Levi-Civita connection for pseudo-Riemannian spaces. This implicates the lack of a further geometric characterisation of pseudo-Riemannian Einstein spaces with nc-Killing forms.

A parallel form on a Ricci-flat metric $g$ is nc-Killing. The lightlike 1-twistor $o = s_- \hat{=} s_-$ gives rise to the constant nc-Killing function $f_- = 1$. The holonomy of the normal conformal connection to $[g]$ will be weakly irreducible, in general. A reproduction of the normal conformal connection with its holonomy by some ambient metric with its Levi-Civita connection and the corresponding holonomy is not done here in the general situation.
Finally, we want to take a closer look on the warped-product structure in Theorem 1 for the case when the Riemannian cone is reducible and has one parallel vector \( \hat{P} \). Through the correspondence with the cone, the vector \( \hat{P} \) can be seen to give rise to a function \( f_{\hat{P}} \) on \((M^n, g)\), which satisfies the second order differential equation
\[
\nabla df_{\hat{P}} = -\frac{f_{\hat{P}}}{c} \cdot g,
\]
i.e., the vector \( \text{grad}(f_{\hat{P}}) \) is a conformal gradient field. It is well-known that the existence of a conformal gradient field gives rise to a (local) warped-product structure on the Riemannian space \( M^n \) (cf. [KR97]). In Theorem 1 this is the special case when the metric \( h \) or \( k \) vanishes (\( p \) or \( q = 0 \)). Moreover, in case that \( f_{\hat{P}} \) or \( \text{grad}(f_{\hat{P}}) \) has a zero on an Einstein space its sectional curvature is constant.

Furthermore, conformal gradient fields are known to generate conformal transformations between Einstein spaces (cf. [Bri25], [Küh88]). This can be understood with our approach in the following way. The function \( f_{\hat{P}} \) on an Einstein space \((M^n, g)\) is a nc-Killing function (or special Killing function) and we mentioned already above that the scaling of the metric \( g \) by a nc-Killing function without zeros gives rise to a conformal transformation to another Einstein metric.

To summarise, an Einstein space \((M^n, g)\) has always nc-Killing functions (the constant functions). In case that there is in addition a non-constant nc-Killing function \( f_- \) without zeros, whose corresponding twistor is \( \alpha \in \Omega_{2^p+1}(M) \), we find a conformal transformation to a further Einstein metric \( \tilde{g} \in \mathbb{G} \). In particular, the normal 2-twistor \( \alpha \wedge \alpha \) corresponds to the conformal gradient field \( \text{grad}(f_-) \), which gives rise to the warped product structure. More generally, a set of \( j \) “independent” nc-Killing functions on an Einstein space \( M^n \) induces \( j - 1 \) different conformal transformations to further Einstein metrics in \( \mathbb{G} \) and \( j - 1 \) different ways of expressing warped-product structures. In case that \( M^n \) is the \( n \)-sphere \( S^n \), there is the constant nc-Killing function \( \alpha_- \) on \( S^n \) and there are \( n + 1 \) further “independent” nc-Killing functions, each of them with an isolated zero on \( S^n \), which give rise to \( n + 1 \) conformal transformations to Einstein metrics with constant sectional curvature up to a singularity (stereographic projections). This is the conformally flat case, where the number of “independent” nc-Killing functions on a space is the maximal one (i.e., \( n + 2 \)).

7. Solutions with decomposable twistors

In this paragraph we want to investigate conformal spaces \((M^n, [g])\), which admit decomposable normal twistors
\[
\alpha = \alpha_1 \wedge \cdots \wedge \alpha_{p+1} \in \Omega_{2^p+1}(M),
\]
i.e., the \( \alpha_i \)’s are 1-forms in \( \Omega_1(M) \). The existence of such a twistor implies that the holonomy representation of the normal conformal connection \( \omega_{NC} \) has an invariant (non-trivial) subspace in \( \mathbb{R}^{p+1,s+1} \). That means the representation is not irreducible. We remember that we studied in the previous paragraph, which was about Einstein metrics, the case of a 1-twistor. Of course, a 1-twistor is always decomposable.

First we want to observe here an easy generation principle for coclosed nc-Killing forms from a given one. So let \( \alpha_- \) be such a coclosed nc-Killing form on a space \( H \) of dimension \( p \) with Einstein metric \( h \) of scalar curvature \( \text{scal}_h \). Now we consider the product metric
\[
g = h \times l,
\]
where \( l \) is a metric on a space \( L \) of dimension \( q \), and we produce the pullback of \( \alpha_- \) to \( M = H \times L \). Obviously, the first of the normal twistor equations (1) for the
Lemma 1. Let \( \alpha_- \) be a conformal Killing p-form with \( \| \alpha_- \|^2 \neq 0 \) on a space \((M^n, g)\) satisfying the following three properties:

1. \( \alpha_- \) is decomposable, i.e., \( \alpha_- = \alpha_1 \wedge \ldots \wedge \alpha_p \) is a \( \wedge \)-product of p 1-forms,
2. there is \( A \in \Gamma(TM) \) such that \( d\alpha_- = A^\flat \wedge \alpha_- \) and
3. there is \( B \in \Gamma(TM) \) such that \( d^*\alpha_- = B \cdot \alpha_- \).

Then it exists a rescaled metric \( \tilde{g} \) in the conformal class \([g]\) such that the rescaled conformal Killing form \( \tilde{\alpha}_- \) is parallel. In particular, if \( 0 < p < n \) then \( \tilde{g} \) is (locally) a product metric \( h \times l \).

**Proof.** First we observe that the three assumptions are invariant under conformal rescaling, e.g. it is \( d(e^\phi\alpha_-) = A^\flat \wedge e^\phi\alpha_- \) with \( A = d\phi + A \). Moreover, since \( \| \alpha_- \|^2 \neq 0 \), we can scale the metric \( g \) such that \( \alpha_- \) has constant non-zero length. For simplicity, we assume that \( g \) is already in this scaling. Then it is

\[
0 = X(g(\alpha_-, \alpha_-)) = 2g(\nabla_X \alpha_- , \alpha_-) = 2g(\frac{1}{p+1}X \cdot d\alpha_- - \frac{1}{n-p+1}X^\flat \wedge d^*\alpha_- , \alpha_-) .
\]

But from the assumptions, we see that this is only possible if \( A, B = 0 \), i.e., \( \alpha_- \) is closed and coclosed which means that it is parallel, since it is a conformal Killing form. Moreover, \( \alpha_- \) is decomposable and this shows that \( g \) is (locally) a product metric \( h \times l \) for the case when \( \deg(\alpha_-) \neq 0, n \).

This lemma generalizes the well-known fact that a conformal vector field, which is hypersurface orthogonal, is parallel with respect to some metric in the conformal class. We also remark at this point that in general a conformal Killing p-form \( \alpha_- \) is conformally equivalent to a parallel form for some metric \( \tilde{g} = e^{-2\phi} \cdot g \) in the conformal class if and only if

\[
d\alpha_- = (p+1) \cdot d\phi \wedge \alpha_- \quad \text{and} \quad d^*\alpha_- = -(n-p+1) \cdot \text{grad}(\phi) \cdot \alpha_- .
\]

This shows that \( A, B \neq 0 \) in the above lemma are actually parallel vectors (if they exist). Moreover, for an (anti)-selfdual form the latter two equations are equivalent and this can be used to show that any (anti)-selfdual conformal Killing \((n/2)\)-form \( \alpha_- \) with length function \( e^\phi \) on a Riemannian space \( M^n \) is parallel with respect to \( \tilde{g} = e^{-2\phi} g \) (cf. [Sem01]).

Now we are prepared to consider nc-Killing forms \( \alpha_- \) with corresponding decomposable twistor. Let \( \alpha \) be such a twistor of degree \( p+1 \). The first statement
that we can make says that the four corresponding differential forms $\alpha_-, \alpha_0, \alpha_+$ and $\alpha_+$ all are decomposable as well. For example, it is

$$\alpha_- = s_+ \cdot \mathcal{J} (s_- \cdot \mathcal{J} (s_+ \wedge \alpha))$$

and henceforth, $\alpha_-$ is obviously decomposable. But we can say even more. The twistor $\alpha$ has two different normal forms with respect to a fixed frame $(s_-, s_+, s_1, \ldots, s_n)$. In the first case, it is $s_- \cdot \mathcal{J} (s_+ \cdot \mathcal{J} \alpha) = 0$ and the corresponding normal form is given by

$$\alpha = (a \cdot s_- + b \cdot s_+ + c \cdot t_{p+1}^b) \wedge t_1^b \wedge \ldots \wedge t_p^b,$$

where the $t_i$’s are orthogonal to each other and are contained in the span of the $s_i$’s, $i = 1, \ldots, n$, and $a, b$ and $c$ are some constants. In the second case, it is $s_- \cdot \mathcal{J} (s_+ \cdot \mathcal{J} \alpha) \neq 0$ and we have the normal form

$$\alpha = (a \cdot s_- + b \cdot t_p^b + c \cdot t_{p+1}^b) \wedge (d \cdot s_+ + t_1^p) \wedge \ldots \wedge t_p^b.$$

From these two normal forms we can see that for a twistor $\alpha$ there are vectors $A, B$ such that
d$\alpha_- = A \wedge \alpha_-$ and $d^* \alpha_- = B \cdot \mathcal{J} \alpha_-$. Indeed, we can apply now Lemma 1 and obtain the following result.

**Lemma 2.** Let $\alpha_-$ be a nc-Killing p-form on $(M, g)$ with $|\alpha_-|^2 \neq 0$ such that the corresponding normal twistor $\alpha$ is decomposable. Then there exists $\tilde{g}$ in the conformal class $[g]$ such that the rescaled form $\tilde{\alpha}_-$ is parallel.

The parallel nc-Killing p-form that is guaranteed by Lemma 2 is decomposable. If its degree is different from 0 and $n$, it gives rise (locally) to a product metric $h \times l$ in the conformal class $[g]$. Moreover, the normal twistor equations 2 and 3 show that the factors $h, l$ are Einstein and we can conclude that $\tilde{g}$ is a product of Einstein metrics with

$$\text{scal}_l = -\frac{(n-p)(n-p-1)}{p \cdot (p-1)} \cdot \text{scal}_h.$$

We also want to discuss the case when $\alpha_-$ is a lightlike nc-Killing form whose corresponding twistor is decomposable. We use the following convention. If a decomposable p-form $\gamma$ on $\mathbb{R}^{n \times a}$ is the $\wedge$-product of lightlike 1-forms only, i.e., the corresponding subspace to $\gamma$ in $\mathbb{R}^{n \times a}$ is totally lightlike then we call the decomposable p-form totally lightlike (isotropic) as well. There is a version of Lemma 1 for totally lightlike p-forms.

**Lemma 3.** Let $\alpha_-$ be a totally lightlike conformal Killing p-form on a space $(M, g)$ with the following two properties:

1. There is $A \in \Gamma(TM)$ such that $d\alpha_- = A \wedge \alpha_-$ and
2. there is $B \in \Gamma(TM)$ such that $d^* \alpha_- = B \cdot \mathcal{J} \alpha_-.$

Then it exists (locally) a rescaled metric $\tilde{g}$ in the conformal class $[g]$ such that the rescaled nc-Killing form $\tilde{\alpha}_-$ is parallel. In particular, the holonomy of the Levi-Civita connection to $\tilde{g}$ is reducible with a fixed lightlike subspace.

**Proof.** First, we show that we can assume $\alpha_-$ to be a closed form. This is for the following reason. The differential form $d\alpha_-$ is decomposable and closed. Hence, by Frobenius’ there are (local) coordinates $(x_1, \ldots, x_n)$ such that $d\alpha_- = dx_1 \wedge \ldots \wedge dx_{p+1}.$ Moreover, since $\alpha_-$ is decomposable we can choose these coordinates such that $\alpha_- = f \cdot dx_1 \wedge \ldots \wedge dx_p,$ where $f$ is a function in the $x_1, \ldots, x_{p+1}.$ By rescaling the metric with the function $f$ we find that $\tilde{\alpha}_- = f^{-1} \alpha_-$ is a closed nc-Killing form.
Now let $\alpha_- = l_1 \wedge \ldots \wedge l_p$ be a totally isotropic and closed conformal Killing form with $d^* \alpha_- = t \cdot l_1 \wedge \ldots \wedge l_{p-1}$, where the $l_i$'s are mutually orthogonal lightlike 1-forms and $t$ is some function. Then we calculate in an arbitrary point $m \in M$:

$$
0 = X(g(\bar{l}_1 \ldots \bar{l}_{p-1} \ldots \bar{l}_1 \ldots \bar{l}_p)) = 2 \cdot g(\bar{l}_1 \ldots \bar{l}_{p-1} \ldots \bar{l}_1 \ldots \bar{l}_p) = 2 \cdot (-1)^p \cdot g(tX^1, l_p) \quad \text{for all} \ X \in T_p M ,
$$

where we have chosen lightlike 1-forms $\bar{l}_i$ with $\nabla \bar{l}_i(m) = 0$ and $g_m(l_i, \bar{l}_i) = 1$ and $g_m(l_i, \bar{l}_j) = 0$ for $i \neq j$.

But this is only possible for all $X \in TM$ if $t \equiv 0$, i.e., $d^* \alpha_- = 0$. Henceforth, $\alpha_-$ is parallel and totally isotropic. This proves the statements of the lemma. □

Using this lemma and the normal form description for decomposable twistors with respect to some $(s_-, s_+)$ leads us to the next result, which is for lightlike nc-Killing forms.

**Lemma 4.** Let $\alpha_-$ be a totally isotropic nc-Killing $p$-form on $(M, g)$ with decomposable twistor. Then there is (at least locally) a metric $\tilde{g}$ in the conformal class such that the rescaled form $\tilde{\alpha}_-$ is parallel.

We can say even more than stated in the lemma. With respect to the metric $\tilde{g}$, where $\tilde{\alpha}_-$ is totally isotropic and parallel the corresponding twistor takes the form

$$
\alpha = (s_-^a + a \cdot s_+^b) \wedge l_1 \wedge \ldots \wedge l_p
$$

for some constant $a$. However, if $a \neq 0$ then $\beta = l_1 \wedge \ldots \wedge l_p$ would be a twistor itself, since the totally lightlike subspace, which uniquely belongs to $\alpha_-$, is parallel with respect to $\nabla^{NC}$. This is not possible, because the fact that $\beta$ is a twistor means $\beta_-$ is zero and $d\beta_- \neq 0$. For this reason, the constant $a$ must be zero (so that $\alpha$ is totally isotropic), which implies that the scalar curvature of $\tilde{g}$ is zero. Furthermore, the twistor equations (2) and (3) show that the Ricci tensor of $\tilde{g}$ maps into the totally lightlike subspace of the tangent space that corresponds to the nc-Killing form $\tilde{\alpha}_-$. Then the metric $\tilde{g}$ has reducible holonomy with an invariant lightlike subspace (that is not dilated under the action). And the holonomy is possibly undecomposable (that is the generic case). The derived results so far sum up to the following proposition.

**Proposition 3.** Let $(M, c)$ be a simply connected conformal space and $\text{Hol}(\omega_{NC})$ the corresponding holonomy group of the normal conformal connection $\omega_{NC}$. The holonomy group $\text{Hol}(\omega_{NC})$ fixes a decomposable $(p + 1)$-form on $\mathbb{R}^{p+1}$, if and only if one of the following cases occurs.

1. There is a product $h^p \times l^q$ of Einstein metrics in $c$ with

$$
sca l h = -\frac{g(g-1)}{p(p-1)} \cdot sca l _h .
$$

If $sca l _h \neq 0$ then $\text{Hol}(\omega_{NC})$ fixes a non-degenerate subspace and if $sca l _h = 0$ then $\text{Hol}(\omega_{NC})$ fixes a degenerate subspace of dimension $p+1$ with exactly one lightlike direction.

2. There is $g \in c$ with totally isotropic Ricci tensor and parallel totally isotropic form. The group $\text{Hol}(\omega_{NC})$ fixes a totally isotropic subspace (without dilation) of dimension at least 2.
induces an isomorphism of the connection $\omega$ of the embedding, which is just the bundle $r$ group $\text{SO}(\cdot)$ extended connection $\omega$ to a local frame $(\cdot)$.

$\text{Let } M \subset \text{SO}(\cdot)$ be the bundles $\text{M}$ extended principal fibre bundles with $B(L)$ and the holonomy representation decomposes $\text{H}l$ and the reducibility of the stabiliser in case that the normal conformal holonomy is decomposable. For this description the reducibility of the stabiliser $\text{S}_\alpha := \text{Stab}(\alpha)$ of a twistor $\alpha$ to a given nc-Killing form $\alpha$ is used to decompose the underlying conformal geometry. The corresponding (weakly) irreducible’ parts are the building blocks of solutions. In Theorem 1 we discussed already examples for those ‘irreducible’ normal conformal geometries.

Lemma 5. Let $\text{Hol}(\omega_{\text{NC}}, c)$ be the normal conformal holonomy of a simply connected space $(M, c)$. We assume that there is a product of Einstein metrics $h^p \times l^q$ in $c$ with $\text{scal}_i = -\frac{q(q-1)}{p(p-1)} \cdot \text{scal}_h \neq 0$. Then it is

$$\text{Hol}(\omega_{\text{NC}}, c) = \text{Hol}(\omega_{\text{NC}}, [h]) \times \text{Hol}(\omega_{\text{NC}}, [l])$$

and the holonomy representation decomposes $\mathbb{R}^{r+1,s+1}$ in two non-degenerate subspaces $V_1 \oplus V_2$.

Proof. Over $M$ we have the $\text{SO}(r+1,s+1)$-bundle $\mathfrak{M}(M)$ with connection $\omega_{\text{NC}}$. Let $H$ and $L$ denote the spaces where $h$ and $l$ live. We can pull back the bundles $\mathfrak{M}(H)$ and $\mathfrak{M}(L)$ with their normal connections to $M$ to obtain a $\text{SO}(r_1 + 1, s_1 + 1) \times \text{SO}(r_2 + 1, s_2 + 1)$-bundle with connection $\omega$. The structure group of the latter bundle sits in $\text{SO}(r+2,s+2)$. We denote the corresponding extended principal fibre bundle with $B(H, L)$. Since $h$ and $l$ are Einstein, the extended connection $\omega$ can be reduced to a subbundle $T$ of $B(H, L)$ with structure group $\text{SO}(r+1,s+1)$.

We define now a bundle embedding of $\mathfrak{M}(M)$ in $B(H, L)$. This embedding induces an isomorphism of the connection $\omega_{\text{NC}}$ and the reduction of $\omega$ to the image of the embedding, which is just the bundle $T$. The map can be given with respect to a local frame $(s_-, s_+, s_1, \ldots, s_n)$, which fits to the scaling $g = h \times l$ such that $a = (s_1, \ldots, s_p)$ spans $TH$ and $b = (s_{p+1}, \ldots, s_n)$ spans $TL$, in the following way. Let

$$s_+^b + \frac{-\text{scal}_g}{2(n-2p)(n-1)} \cdot s_-^b \mapsto a_-^b + \frac{\text{scal}_h}{2p(p-1)} \cdot a_+^b$$

$$s_-^b + \frac{\text{scal}_g}{2(n-2p)(n-1)} \cdot s_+^b \mapsto b_-^b + \frac{\text{scal}_l}{2(n-p)(n-p-1)} \cdot b_+^b$$

$$s_i \mapsto a_i, \quad i = 1, \ldots, p$$

$$s_i \mapsto b_i, \quad i = p + 1, \ldots, n.$$
calculation to see that
\[
\omega_{NC} \circ ds_e(X) = s^\perp_+ \wedge X^\flat + \omega_{LC} \circ ds_e(X) - s^\perp_+ \wedge K_g(X)^\flat
\]
\[
= a^\perp_- \wedge X^\flat + \omega_{LC} \circ da_e(X) - a^\perp_- \wedge K_h(X)^\flat
+ b^\perp_- \wedge X^\flat + \omega_{LC} \circ db_e(X) - b^\perp_- \wedge K_i(X)^\flat
= \omega \circ d(a_e + b_e)(X) = \omega \circ da_e(X) + \omega \circ db_e(X)
\]
and this shows that $\text{Hol}(\omega_{NC}, c)$ is the product of the normal conformal holonomies on $[h]$ and $[l]$. The condition on the subrepresentations $V_1$ and $V_2$ to be non-degenerate is clear from Proposition 3.

The subspaces $V_1$ and $V_2$ in the lemma are naturally identified with the tangent spaces of the cones over $h$ and $l$ at every point via the mapping given in the proof. There is also a version of Lemma 5 when $g = h \times l$ in $c$ is a product of Ricci-flat metrics (i.e., $g$ itself is Ricci-flat). In this case it still holds $\text{Hol}(\omega_{NC}, c) = \text{Hol}(\omega_{NC}, [h]) \times \text{Hol}(\omega_{NC}, [l])$.

We extend now Proposition 3 to a generalised form to make a statement for the case when the holonomy representation decomposes into an arbitrary number of non-degenerate components:

\[
\mathbb{R}^{r+s+1} = \bigoplus_i V_i, \quad i = 1, \ldots, v.
\]

Indeed, then it is possible to decompose the conformal space into further parts and their normal conformal holonomies as well. However, by applying this method one must pay attention to the fact that all the scalings of the conformal structure when the metric on the base manifold becomes a product in the conformal class will be different ones, in general. Hence, the conformal structure is not just given by a simple product of several metrics.

**Proposition 4.** Let $(M, c)$ be a conformal space and let $\mathbb{R}^{r+s+1} = \bigoplus_i V_i$ be a decomposition of the representation of $\text{Hol}(\omega_{NC})$ into (weakly) irreducible submodules $V_i$ with $\dim V_i > 1$ for all $i$. Then there exist (locally) metrics $g_i$, which are Einstein with $\text{scal}_{g_i} \neq 0$ of signature $(r_i, s_i)$ and $\text{Hol}(\omega_{NC}, [g_i])$ acts (weakly) irreducible on a subspace of codimension $1$ in $\mathbb{R}^{r+s+1}$. Moreover, there are functions $\phi_i$ such that

\[
c = \left[ \sum_i \phi_i \cdot g_i \right].
\]

The holonomy group $\text{Hol}(\omega_{NC})$ is equal to the product

\[
\text{Hol}(\omega_{NC}, c) = \Pi_i \text{Hol}(\omega_{NC}, [g_i]).
\]

The statement of Proposition 3 also implies that the cone $\hat{g}_i$ of every factor $g_i$ in the decomposition has (weakly) irreducible holonomy with respect to the Levi-Civita connection (since $V_i$ can be identified with the tangent spaces of the cone $\hat{g}_i$ at every point).

Now we can consider the situation when a nc-Killing form $\alpha_-$ on a space $(M, c)$ exists such that the stabiliser $S_\alpha$ of the corresponding twistor $\alpha$ in $\text{SO}(r + s + 1)$ acts decomposable on $\mathbb{R}^{r+s+1}$. In general, we have the following decomposition.

**Lemma 6.** Let $\alpha$ be a $p$-form on $\mathbb{R}^{r+s+1}$ with stabiliser $S_\alpha \subset \text{SO}(r + 1, s + 1)$ and let

\[
\bigoplus_i V_i^\alpha = \mathbb{R}^{r+s+1}, \quad i = 1, \ldots, \nu
\]

be the decomposition of the $S_\alpha$-module $\mathbb{R}^{r+s+1}$ into (weakly) irreducible components (without dilation). Then there are unique $p$-forms $\alpha_i \in \Lambda^p(V_i^\alpha)$ for all $i$, which are stable under $S_\alpha$ such that $\alpha = \sum_i \alpha_i$. 
Let $\alpha_i \neq 0$ for some $i$ as in Lemma 6 with the property that $V_i^\alpha$ is (weakly) irreducible. Then we obtain a factor $h$ in an appropriate scaled metric $g = h \times l \in c$, and the conformal structure $[h]$ admits a nc-Killing form $\alpha_{i-}$, which comes from the twistor $\alpha_i$ on $V_i^\alpha$. Thereby, we identify $V_i^\alpha$ with the tangent space of the cone over $h$ at every point. The pullback $\alpha_{i-}$ to the total space $(M, g)$ is a nc-Killing form itself and it holds

$$Y^\flat \wedge \alpha_{i-} = -\nabla_Y \alpha_{i-} + Y \cdot \alpha_0 = 0 \quad \text{for all } Y \perp TH.$$ 

That implies $\alpha_{i-} = 0$ and hence, $\alpha_{i-}$ is coclosed for $g$. It is also coclosed on $(H, h)$. We obtain the geometric characterisation in case that the normal conformal holonomy is decomposable (but the conformal structure not Einstein).

**Theorem 2.** Let $\alpha_-$ be a nc-Killing form on $(M, c)$ and $\oplus_i V_i^\alpha$ (dim$V_i^\alpha > 1$) a decomposition of $\mathbb{R}^{r+1,s+1}$ with respect to the stabiliser $S_\alpha$ into (weakly) irreducible components and let $c = [\sum e^{\phi_i} \cdot g_i]$ be a corresponding representation of the conformal class (as in Proposition 4). It is $\alpha_- = \sum_i \alpha_{i-}$ and if the component $\alpha_i$ on $V_i^\alpha$ to the twistor $\alpha$ is non-trivial then $\alpha_{i-}$ is a special Killing form for $g_i$.

We remember to the fact that if dim$V_i^\alpha = 1$ for some $i$ then the conformal structure is Einstein and we find a characterisation of solutions via the ambient metric (cf. Proposition 2 Theorem 1). In the next, paragraph we will come across the case when $Hol(\omega_{NC})$ acts irreducible on $\mathbb{R}^{r+1,s+1}$. We also obtain the following conclusion concerning coclosed nc-Killing forms.

**Corollary 1.** Let $\alpha_-$ be a nc-Killing form without zeros on $(M, c)$ such that the stabiliser $S_\alpha$ decomposes $\mathbb{R}^{r+1,s+1}$ to $V_1 \oplus V_2$ with dim$V_1,2 > 1$. Then $\alpha_-$ is coclosed with respect to the corresponding scaling $h \times l \in c$.

| $\mathbb{R}^{r+1,s+1} = \oplus_i V_i$ | scaling of $c$ | nc-Killing form $\alpha_-$ |
|--------------------------------------|----------------|----------------------------|
| $V_i$ non-degenerate, $dimV_i > 1 \ \forall \ i$ | $\sum e^{\phi_i} g_i$, $g_i$ Einstein | $\alpha_- = \sum_i \alpha_{i-}$, $\alpha_- \sim$ coclosed |
| $V_i$ non-degenerate, $dimV_i = 1$ | $g \in c$ Einstein, $scal_g \neq 0$ | $\alpha_- = \beta_- + \gamma_-$, $d^* \beta_- = d\gamma_- = 0$ |
| undecomposable, fixed 1-form | $g \in c$ Ricci-flat | $\alpha_-$ or $d\alpha_-$ parallel |
| undecomposable, fixed p-form, $1 < p < n + 1$ | non-Einstein | $\alpha_-$ or $d\alpha_-$ parallel |
| irreducible | non-Einstein | $\alpha_-$ nc-Killing $d^* \alpha_- = ?$ |
| undecomposable, (fixed subspace with dilation) | non-Einstein | non |

**Table 1.** Normal conformal holonomy actions, natural scalings of the underlying conformal structure and properties of the occurring nc-Killing forms thereof.

We can also understand now that if a nc-Killing form $\alpha_-$ without zeros on a non-conformal Einstein space can not be scaled (locally) to a coclosed form then the stabiliser $Stab(\alpha)$ of the corresponding twistor acts irreducible or undecomposable.
on $\mathbb{R}^{n+1,1}$. In case that the space is conformally Einstein ($\text{scal} \neq 0$) we mentioned already that every nc-Killing form is the sum of a closed and a coclosed form in the Einstein scaling. In Table 4 we give an overview of the situation that we tried to explain here.

9. Solutions in dimension 4

After the general discussion so far, we want to study in this paragraph solutions of the normal twistor equations for differential forms on 4-dimensional Riemannian and Lorentzian manifolds and their corresponding possible normal conformal holonomy groups.

9.1. The Riemannian case. Let $(M^4, g)$ be an oriented Riemannian 4-space. We discuss nc-Killing forms according to their degree.

First, if there is a normal conformal function $f_- = \alpha_-$ without zero then $M^4$ is a conformal Einstein space. The 1-form

$$o = \tilde{s}_-^b - \text{scal} \frac{1}{2(n-1)n} s_+^{b} \in \Omega^1_{\text{nc}}(M)$$

is parallel. That implies that for $\text{scal} > 0$ the holonomy $\text{Hol}(\omega_{NC})$ of the normal conformal connection $\omega_{NC}$ is reduced at least to the subgroup $SO(5)$ of the Möbius group $SO(1, 5)$. For negative scalar curvature ($\text{scal} < 0$) the holonomy is reduced to a subgroup of $SO(1, 4)$. For Ricci-flat spaces the holonomy $\text{Hol}(\omega_{NC})$ is contained in the stabiliser of a single lightlike vector in $\mathbb{R}^{1,5}$.

Next we take a look at the nc-Killing 1-forms. So let $\alpha_-$ be a nc-Killing 1-form on $(M^4, g)$ with $V_- = \alpha^2_-$ its dual conformal vector field and $f = g(V_-, V_-)$ the square length. We assume for the moment that $f$ has no zero. Then the conformally changed metric $\tilde{g} = f^{-1} \cdot g$ has $V_-$ as a Killing vector field, i.e., $L_{V_-} \tilde{g} = 0$. Moreover, $V_-$ is divergence-free and the dual $\tilde{\alpha}_- = \frac{1}{\sqrt{g}} \alpha_-$ is coclosed with respect to $\tilde{g}$. Now there are two possibilities. Either $\tilde{\alpha}_-$ is parallel or it is not closed. In the first case one can see from the twistor equations that $\tilde{g}$ is Ricci-flat with a parallel vector, i.e., $(M^4, \tilde{g})$ is (conformally) flat.

In the latter case we can find locally an orthonormal frame $\tilde{s}$ such that

$$\tilde{\alpha}_- = \tilde{s}_1^2 \quad \text{and} \quad d\tilde{\alpha}_- = h \cdot \tilde{s}_2^2 \wedge \tilde{s}_3^2$$

for some local function $h$. The integrability conditions (9) and (10) then say that

$$\tilde{s}_1(X, Y) W_{\tilde{g}} = 0 \quad \text{and} \quad \tilde{s}_1(X, Y) (C(X, Y) \wedge \tilde{\alpha}_-) = 0$$

for all $X, Y \in TM$. Moreover, it is

$$C_{\tilde{g}}(X, Y) \cdot \tilde{\alpha}_- = 0$$

by (11), which together implies $C_{\tilde{g}} \equiv 0$. Therefore, we have $W_{\tilde{g}}(X, Y) \circ d\tilde{\alpha}_- = 0$ for all $X$ and $Y$, which means that $W_{\tilde{g}}(\tilde{s}_2^2 \wedge \tilde{s}_3^2) = k \cdot \tilde{s}_2^2 \wedge \tilde{s}_3^2$ for some function $k$ and $W_{\tilde{g}}(\tilde{s}_1^2 \wedge \tilde{s}_3^2) = 0$ in all other cases. But since $tr W_{\tilde{g}} = 0$, the function $k$ must be zero and therefore, it is $W_{\tilde{g}} \equiv 0$, i.e., $M^4$ is conformally flat. Altogether, we can conclude that any 4-space admitting a nc-Killing 1-form with or without zeros is conformally flat.

Finally, we have to consider the case when $\alpha_-$ is a nc-Killing 2-form. We assume without loss of generality that $\alpha_-$ is selfdual. Moreover, we have seen already that if $\alpha_-$ has no zeros, we can also assume that $\tilde{\alpha}_-$ is parallel with respect to some conformally changed metric $\tilde{g}$ (cf. [Sem01] and paragraph 7). Therefore, $\tilde{g}$ is a Kähler metric. There are two cases. First, the metric $\tilde{g}$ is Einstein, which is only possible if $\tilde{g}$ is Ricci-flat, i.e., $\tilde{g}$ has holonomy $SU(2)$ or is flat. In the former case
the holonomy of $\omega_{NC}$ is the stabiliser group $\text{Stab}(e^\flat_\omega \wedge \omega_o)$, where $\omega_o$ denotes the standard Kähler form on $\mathbb{R}^4$.

In case that $\tilde{g}$ is not Einstein it is locally up to a constant scaling factor a product of the sphere $S^2$ with the hyperbolic space form $H^2$. The volume forms of the factors of this product are the nc-Killing 2-forms. The stabiliser in $\text{SO}(1, 5)$ of the corresponding twistors is $\text{SO}(3) \times \text{SO}(1, 2)$. However, the product $S^2 \times H^2$ is already conformally flat, i.e., the holonomy $\text{Hol}(\omega_{NC})$ is trivial.

**Theorem 3.** Let $(M^4, [g])$ be a Riemannian 4-space with conformal structure $[g]$ and let $\alpha_-$ be a nc-Killing form without zero of arbitrary degree then at least one of the following cases occurs (up to a conformal scaling factor)

1. $\deg(\alpha_-) = 0$ and $M^4$ is Einstein,
2. $\deg(\alpha_-) = 2$ and $M^4$ is Ricci-flat and Kähler or
3. $M^4$ is flat.

In particular, the holonomy group of $\omega_{NC}$ for a simply connected space $M^4$ is contained in one of the following subgroups of the Möbius group $\text{SO}(1, 5)$:

- $\text{SO}(5)$
- $\text{SO}(1, 4)$
- $\text{Stab}(e^\flat_\omega)$
- $\text{Stab}(e^\flat_\omega \wedge \omega_o)$
- $\{e\}$.

Obviously, irreducible representations of subgroups of the Möbius group do not occur in the Riemannian case. It is a matter of further investigation, which proper subgroups of the stated groups in Theorem 3 can really occur as holonomy groups. For example, it is possible that there is a conformal geometry such that $\text{Hol}(\omega_{NC}) \subset \text{SO}(1, 4)$ acts weakly irreducible on $\mathbb{R}^{1, 4}$ with dilation. It is also possible that $\text{Hol}(\omega_{NC})$ acts weakly irreducible on $\mathbb{R}^{1, 5}$ with dilation, although in this case no nc-Killing form exists. At least, we can say if $\text{Hol}(\omega_{NC})$ has an invariant subspace without dilation then the geometry is Einstein or conformally flat. Table 2 gives an overview of possible holonomy groups.

| $\text{Hol}(\omega_{NC})$ sitting in | (local conformal) geometry $[g]$ | nc-Killing form |
|-------------------------------------|-------------------------------|----------------|
| $\text{SO}(5)$                     | Einstein, $\text{scal} > 0$  | one function  |
| $\text{SO}(1, 4)$                  | Einstein, $\text{scal} < 0$  | one function  |
| $\text{Stab}(e^\flat_\omega)$     | Ricci-flat                    | one function  |
| $\text{Stab}(e^\flat_\omega \wedge \omega_o)$ | Ricci-flat, Kähler            | Kähler form   |
| $\{e\}$                           | conformally flat              | maximal        |
| $\text{Stab}(\mathbb{R} \cdot e^\flat_\omega)$ | ?                             | non           |
| $\text{SO}(1, 5)$                  | generic case                  | non            |

Table 2. Partial holonomy list for the normal conformal connection of Riemannian spaces in dimension 4.

As we mentioned before it is well known that there exits a selfdual nc-Killing 2-form with zero on the conformal completion of the Eguchi-Hanson metric, which is not any longer conformally equivalent to an Einstein metric (cf. [KR 96]). Nevertheless, the holonomy $\text{Hol}(\omega_{NC})$ in this case is equal to $\text{Stab}(e^\flat_\omega \wedge \omega_o)$.  

9.2. The Lorentzian case. We turn now to the case of a Lorentzian 4-manifold \((M^{4,1}, g)\). We choose the signature \((- + ++)\). If there is a nc-Killing function without zero the space \(M^{4,1}\) is conformally Einstein. In case that there is a timelike or spacelike nc-Killing 1-form \(\alpha_-\) one can easily show (as in the Riemannian case) that \(M^{4,1}\) is already conformally flat.

Here in the Lorentzian case we must also take into consideration the case when the length \(f := g(\alpha_-, \alpha_-)\) of a nc-Killing 1-form \(\alpha_-\) vanishes identically, i.e., the dual conformal vector field \(V_-\) is everywhere null. We consider such a field locally and without zeros. Then we can assume that \(g\) is scaled such that \(V_-\) is a Killing vector field on \(M^{4,1}\). There are two possible cases. Either

\[\alpha_- \wedge d\alpha_- \neq 0\]

or this twist 3-form vanishes identically. If it vanishes then \(V_-\) is parallel in the scaling \(g\). In particular, in dimension 4 that means \(g\) is a pp-wave with vanishing scalar curvature. The corresponding twistor takes the form \(e_\perp^\flat \wedge \alpha_-\) determining a totally lightlike plane in \(\mathbb{R}^2,4\).

In the other case the twist \(\alpha_- \wedge d\alpha_-\) does not vanish. This case is also well known. The underlying metric \(g\) is a so-called Fefferman metric (cf. \([\text{Fef76}], [\text{Gra87}]\)). Equivalently, a Fefferman metric is determined by the existence of a lightlike nc-Killing vector field \(V_-\) with the property

\[\text{Ric}(V_-, V_-) = \text{const} > 0.\]

| \(\text{Hol}(\omega_{NC})\) sitting in | local conformal geometry \([g]\) | nc-Killing form |
|---------------------------------|---------------------------------|----------------|
| \(\text{SO}(1,4)\)             | Einstein, scal \(> 0\)          | one function   |
| \(\text{SO}(2,3)\)             | Einstein, scal \(< 0\)          | one function   |
| \(\text{Stab}(e_-^\perp)\)     | Ricci-flat                      | one function   |
| \(\text{Stab}(e_-^\perp \wedge l), l \text{ null, } l \perp e_-^\perp\) | pp-wave                        | 1-form without twist, \(\text{Ric}(V_-, V_-) = 0\) |
| \(\text{SU}(1,2)\)             | Fefferman spaces                | 1-form with twist, \(\text{Ric}(V_-, V_-) > 0\) |
| \(\{e\}\)                     | conformally flat                | maximal        |
| \(\text{Stab}(3\text{-form})\) | ?                               | 2-form         |
| undecomposable with dilation    | ?                               | non            |
| \(\text{SO}(2,4)\)             | generic case                    | non            |

Table 3. Possible holonomy groups for the normal conformal connection of a Lorentzian space in dimension 4.

Fefferman spaces are locally constructed as 1-dimensional fibre bundles over strict pseudoconvex CR-manifolds \((N^3, W, J, \gamma)\), where the direction of the fibre is that of \(V_-\), i.e., lightlike, the 2-form \(d\alpha_-\) is related to the complex structure \(J : W \to W\) on the CR-manifold \(N\) and the 1-form \(\alpha_+\) is related to \(\gamma\), which is dual to the Reeb vector field of the pseudoconvex CR-structure. The normal twistor \(\alpha\) in \(\Omega^2_M(M)\) belonging to \(\alpha_-\) takes a very natural form. It corresponds to the standard
(pseudo)-Kähler form $\omega_o$ on $\mathbb{R}^{2,4}$. This implies that the holonomy of $\omega_{NC}$ is included in $SU(1, 2)$ for every Fefferman space. The group $SU(1, 2)$ acts irreducible on $\mathbb{R}^{2,4}$. This characterisation by the holonomy group can be seen as a conformally invariant definition for Fefferman spaces. It directly implies that if a Fefferman space has the full holonomy group $SU(1, 2)$ it is not conformally Einstein.

Finally, we shortly mention some statements about the situation for nc-Killing 2-forms in the Lorentzian setting. The spaces $S^2 \times H^1,1$ and $H^2 \times S^1,1$ have parallel and decomposable nc-Killing 2-forms, which are the volume forms of the factors in the product. The stabilisers of these are $SO(1, 2) \times SO(1, 2)$ in the first and $SO(3) \times SO(2, 1)$ in the second case. However, these product spaces are again conformally flat, i.e., if the holonomy $Hol(\omega_{NC})$ is contained in those stabilisers then it is trivial. But there may appear other 3-twistors on a Lorentzian 4-space. To investigate this situation, it could be useful to know that there exists a complete orbit type classification of the space $\Lambda^3 \mathbb{R}^6$ of 3-forms under the action of $GL(6)$ (cf. [Rei07, Hit00]). From these orbit types one can try to calculate the orbit types with respect to the action of the structure group $SO(2, 4)$. And then one could investigate whether further stabiliser groups of those 3-form orbits appear that are the holonomy groups to some special normal conformal geometries. In Table 4 we list the cases for nc-Killing forms and holonomies of $\omega_{NC}$ for a Lorentzian 4-manifold that we mentioned here in our discussion.

10. Application to conformal Killing spinors

The discussion of the normal twistor equations for differential forms so far shows that there are methods to describe their solutions and many of their underlying geometric structures are well known objects and do occur in the mathematical literature as subject to substantial work. We want to use our acquired results to study a topic in conformal differential geometry, which itself was subject to various investigations during the last 30 years, namely the twistor equation for spinor fields. There is a systematic investigation of this twistor equation in Riemannian spin geometry (cf. [Lic88, BECK91]). However, the origin of the twistor equation for spinors was in the theory of General Relativity (cf. [Pen67, PR86]) and nowadays, there has been done considerable work for twistor spinors in the Lorentzian setting, too (cf. [Bau00, BL03]). The remarkable point of the twistor equation for spinors, and this can explain why this equation plays an important role, is the fact that all its solutions are automatically normal (in the sense that we used here already). We start with recalling the very basic facts about spinors and their twistor equation.

Let $(M^n, g)$ be a semi-Riemannian spin manifold of dimension $n \geq 3$. We denote by $S$ the spinor bundle and by $\mu : T^*M \otimes S \to S$ the Clifford multiplication. The 1-forms with values in the spinor bundle decompose into two subbundles

$$T^*M \otimes S = V \oplus Tw,$$

where $V$, being the orthogonal complement to the ‘twistor bundle’ $Tw := Ker\mu$, is isomorphic to $S$. We obtain two differential operators of first order by composing the spinor derivative $\nabla^S$ with the orthogonal projections onto each of these subbundles, the Dirac operator $D$

$$D : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) = \Gamma(S \oplus Tw) \xrightarrow{pr_S} \Gamma(S)$$

and the twistor operator $P$

$$P : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) = \Gamma(S \oplus Tw) \xrightarrow{pr_Tw} \Gamma(Tw).$$
Both operators are conformally covariant. More exactly, if \( \tilde{g} = e^{-2\phi} g \) is a conformal change of the metric, the Dirac and the twistor operator satisfy

\[
\begin{align*}
D_{\tilde{g}} &= e^{\frac{\phi}{2}} D_g e^{-\frac{\phi}{2}} \\
P_{\tilde{g}} &= e^{\phi} P_g e^{-\phi}. 
\end{align*}
\]

A spinor field is called conformal Killing spinor if it lies in the kernel of the twistor operator \( P \). Alternatively, a spinor field \( \varphi_- \in \Gamma(S) \) is a conformal Killing spinor if and only if

\[
\nabla_X^S \varphi_- + \frac{1}{n} X \cdot D \varphi_- = 0 \quad \text{for all vector fields } X.
\]

This is the twistor equation for spinors. A conformal Killing spinor \( \varphi_- \) with respect to \( g \) rescales by

\[
\tilde{\varphi}_- := e^{-1/2\phi} \cdot \varphi_-
\]
to a conformal Killing spinor with respect to the conformally changed metric \( \tilde{g} := e^{-2\phi} \cdot g \). Thereby, we use the canonical identification of the spinor bundles over \((M, g)\) and \((M, \tilde{g})\). We say that \( \varphi_- \) is conformally equivalent to \( \varphi_- \).

Obviously, each parallel spinor \( (\nabla^S \varphi_- = 0) \) is a twistor spinor. Another special class of twistor spinors are the Killing spinors \( \varphi_- \), which satisfy the equation

\[
\nabla_X^S \varphi_- = \lambda X \cdot \varphi_-
\]

for all \( X \in TM \) and some fixed \( \lambda \in \mathbb{C} \setminus \{0\} \) (this can be seen as corresponding notion to Killing differential forms). The dimension of the space of twistor spinors is a conformal invariant and bounded by

\[
\dim \ker P \leq 2 \text{rank} S = 2 \frac{n+1}{2} =: d_n.
\]

If \( \dim \ker P = d_n \), then \((M^n, g)\) is conformally flat. Conversely, if \((M^n, g)\) is simply connected and conformally flat, then \( \dim \ker P = d_n \).

Now, we focus our attention on the case of Lorentzian signature \((- + \ldots +)\). Let \((M^n, g)\) be an oriented and time-oriented Lorentzian spin manifold. On the spinor bundle \( S \) there exists an indefinite non-degenerate inner product \( \langle , \rangle \) such that

\[
\begin{align*}
\langle X \cdot \varphi_-, \psi_- \rangle &= \langle \varphi_, X \cdot \psi_- \rangle \quad \text{and} \\
X(\langle \varphi_-, \psi_- \rangle) &= \langle \nabla_X^S \varphi_-, \psi_- \rangle + \langle \varphi_-, \nabla_X^S \psi_- \rangle,
\end{align*}
\]

for all vector fields \( X \) and all spinor fields \( \varphi_-, \psi_- \) (cf. \([\text{Bau81}]\)). Each spinor field \( \varphi_- \in \Gamma(S) \) defines a vector field \( V_{\varphi_-} \) on \( M \), the so-called Dirac current, by the relation

\[
g(V_{\varphi_-}, X) := - \langle X \cdot \varphi_-, \varphi_- \rangle
\]

for all \( X \in TM \). The vector \( V_{\varphi_-} \) is causal and future-directed. The zero sets of \( \varphi_- \) and \( V_{\varphi_-} \) coincide, i.e., for a non-trivial spinor the associated field is non-trivial. (This is a very useful fact specific for Lorentzian geometry.) If \( \varphi_- \) is a twistor spinor, then \( V_{\varphi_-} \) is a conformal Killing field. The dual of \( V_{\varphi_-} \) is denoted by \( \alpha_{\varphi_-} \).

We have the following known geometric structure result for Lorentzian spaces with conformal Killing spinors. Thereby, we call a space with a parallel lightlike vector field a Brinkmann space. The notion of Fefferman spaces that appeared in the preceding section can be extended to every even-dimensional manifold \((n > 2)\) (cf. \([\text{Gra87}, \text{Bau88}]\)). We just say here in short, a Fefferman space \((M^{2m}, [g])\) is a Lorentzian space with a conformal structure such that the normal conformal holonomy group \( \text{Hol}(\omega_{NC}, [g]) \) sits in \( \text{SU}(1, m) \). A Lorentzian Einstein-Sasaki structure on an odd-dimensional manifold \((M^{2m+1}, g)\) lifts to a Kähler structure on its Ricci-flat cone, i.e., the holonomy \( \text{Hol}(\omega_{NC}) \) sits in \( \text{SU}(1, m) \), which is itself a subgroup of \( \text{SO}(2, 2m+1) \).
Proposition 5. (BLAM) Let \((M^n, g)\) be a Lorentzian spin manifold and let \(\varphi_- \in \Gamma(S)\) be a conformal Killing spinor without zeros and let \(V_{\varphi_-}\) be its associated vector field.

a) In case that \(V_{\varphi_-}\) is lightlike there are exactly two different cases:

1) The twist 3-form \(t_{\varphi_-} := \alpha_{\varphi_-} \wedge \text{d}\alpha_{\varphi_-}\) vanishes identically and \(\varphi_-\) is (locally) conformally equivalent to a parallel spinor, whose associated lightlike vector field defines a Brinkmann space.

2) The twist \(t_{\varphi_-}\) does not vanish and \((M^n, g)\) is a Fefferman space.

b) In case that \(g(V_{\varphi_-}, V_{\varphi_-}) = \text{const} < 0\) and \(\varphi_-\) is a Killing spinor (i.e., \(D\varphi_- = -n\lambda\varphi_-\)) the space \((M^n, g)\) is Einstein-Sasaki.

c) If the length function \(\langle \varphi_-, \varphi_- \rangle\) has no zero then the metric \(g\) can be rescaled to the Einstein metric \(\bar{g} := \frac{1}{\langle \varphi_-, \varphi_- \rangle} \cdot g\). In particular, there exists at least one Killing spinor on \((M, \bar{g})\).

The results in Proposition 5 do not give a complete answer to the case when the associated field \(V_{\varphi_-}\) is timelike. It is only mentioned the Einstein-Sasaki case. It is our interest to handle the geometric description for all timelike cases of \(V_{\varphi_-}\). Thereby, both cases that \((M^n, g)\) is conformally Einstein resp. is not conformally Einstein are possible and of interest. For this purpose we can apply now our results from the discussion of normal conformal Killing forms that we have developed in the previous paragraphs. That this approach is reasonable is justified by the fact that, as we mentioned already, every conformal Killing spinor is normal. We can see this in the following way (for arbitrary signature).

Assume that \((M^n, g)\) is a spin manifold. Then there is a spin Möbius bundle on \(M\), which we denote by \(\mathfrak{M}_{\text{Spin}}(M)\). This is a principal fibre bundle with structure group \(\text{Spin}(r + 1, s + 1)\) and is a double cover of \(\mathfrak{M}(M)\) respecting the right multiplication on the fibres and the natural homomorphism

\[
\lambda : \text{Spin}(r + 1, s + 1) \to \text{SO}(r + 1, s + 1).
\]

The normal conformal connection form \(\omega_{\text{NC}}\) on \(\mathfrak{M}(M)\) lifts to a unique connection form on the spin Möbius bundle \(\mathfrak{M}_{\text{Spin}}(M)\). Moreover, this connection form induces a covariant derivative \(\nabla^{\text{NC}}\) on the tractor spinor bundle \(S_{\mathfrak{M}}\) defined by

\[
S_{\mathfrak{M}} := \mathfrak{M}_{\text{Spin}}(M) \times \rho \Delta_{r+1,s+1},
\]

where \((\Delta_{r+1,s+1}, \rho)\) is the spinor representation in signature \((r + 1, s + 1)\).

With respect to the metric \(g\), this spinor bundle splits into the sum of two usual spinor bundles over \((M^n, g)\), \(S_{\mathfrak{M}} = S \oplus S\). The equation \(\nabla^{\text{NC}} \varphi = 0\) translates to

\[
\nabla^S_X \varphi_- + X \cdot \varphi_+ = 0
\]

\[
\nabla^S_X \varphi_+ - \frac{1}{n} K(X) \cdot \varphi_- = 0,
\]

where \(\varphi_-\) and \(\varphi_+\) are smooth spinor fields on \((M, g)\). From the two twistor equations, it follows that \(\varphi_+ = \frac{1}{n} \cdot D\varphi_-\) and the normal twistor equations for spinors \(\varphi_- \in \Gamma(S)\) take the following form:

\[
\nabla^S_X \varphi_- + \frac{1}{n} X \cdot D\varphi_- = 0
\]

\[
\nabla^S_X D\varphi_- - \frac{n}{2} K(X) \cdot \varphi_- = 0.
\]

We recognise that the first of the two equations is just the conformal Killing spinor equation as introduced before.

Until now everything works analogous to the normal twistor equations for differential forms. However, it is easy to see that the second equation for spinors is implied by the first equation alone, the conformal Killing spinor equation. This is
in contrast to the case of differential forms, where the additional normal twistor equations are not implicated by the conformal Killing equation. Here for spinors we calculate:

\[
\nabla^S \nabla^S_{s_j} \varphi_- + \frac{1}{n} s_j \cdot \nabla^S \nabla^S_{s_i} \varphi_- = 0
\]

\[
\nabla^S \nabla^S_{s_i} \varphi_- + \frac{1}{n} s_i \cdot \nabla^S \nabla^S_{s_j} \varphi_- = 0
\]

which results to

\[
R^S(s_j, s_i) \cdot \varphi_- = - \frac{1}{n} s_i \cdot \nabla^S \nabla^S_{s_j} \varphi_- + \frac{1}{n} s_j \cdot \nabla^S \nabla^S_{s_i} \varphi_- .
\]

Using that \(\sum_k s_k \cdot R^S(s_i, s_k) \varphi_- = - \frac{1}{2} \text{Ric}(s_i) \varphi_- \) and \(D^2 \varphi_- = \frac{n-\text{scal}}{4(n-1)} \varphi_- \), it follows the second twistor equation \([13]\) for spinors.

**Theorem 4.** Let \((M, c)\) be a conformal spin space of signature \((r, s)\). The sets of conformal Killing spinors \(\varphi_- \in \Gamma(S)\) (i.e., \(\nabla^S_X \varphi_- + \frac{1}{n} X \cdot \nabla^S \varphi_- = 0\)) and normal twistor spinors \(\varphi \in \Gamma(S_{\mathbb{R}})\) (i.e., \(\nabla^{\text{NCS}} \varphi = 0\)) are naturally identified by the mapping

\[
\varphi_- \mapsto \varphi = (\varphi_- , \frac{1}{n} \cdot D \varphi_- ) .
\]

In general, a spinor field gives rise to differential forms of degree \(p\). In Lorentzian geometry we have already introduced as special case the Dirac current. The general construction is as follows. Let \(\varphi_- \) be a spinor field on \((M^n, g)\). The corresponding \(p\)-forms \(\alpha^p_{\varphi_-}\) are defined by the relation

\[
g(\alpha^p_{\varphi_-} , X^p) := i^{p(p-1)+r+1} (X^p \cdot \varphi_- , \varphi_- ) \quad \text{for all } X^p \in \Lambda^p(M) .
\]

The so defined \(p\)-forms are not non-trivial in general. This depends on the given spinor. The same method applies to attach tractor \((p+1)\)-forms to some tractor spinor \(\varphi\). Moreover, the tractor forms associated to a twistor spinor are parallel with respect to \(\nabla^{\text{NCS}}\), i.e., they are normal twistor forms. The corresponding induced nc-Killing forms are just the associated differential forms of the induced conformal Killing spinor. We want to direct our attention to the Lorentzian case again. Our discussion gives rise to the following corollary to Theorem 4.

**Corollary 2.** Let \(\varphi_- \) be a conformal Killing spinor on a Lorentzian spin manifold \((M^n, g)\). Then the Dirac current \(V_{\varphi_-}\) is a non-trivial and causal nc-Killing vector field. Its dual \(\alpha_{\varphi_-}\) is a nc-Killing 1-form.

This corollary is now our starting point for the geometric description of Lorentzian spaces admitting conformal Killing spinors, in particular, those with timelike Dirac current.

We have learned in paragraph 8 that the twistor to an nc-Killing form either has an irreducible acting stabiliser on a subspace of codimension 0 or 1 in \(\mathbb{R}^{r+1,s+1}\) or else, for example in the non-degenerate case, there exists a product metric in the conformal class. We want to use this philosophy here and show that the rank \(r \kappa(\varphi_-)\) of the nc-Killing 1-form \(\alpha_{\varphi_-}\) to a conformal Killing spinor \(\varphi_-\) determines whether or not there is (in the non-degenerate case) a product in the conformal class with respect to which the nc-Killing 1-form \(\alpha_{\varphi_-}\) restricts to the factors. Thereby, the rank \(r \kappa(\alpha_-)\) of an arbitrary 1-form \(\alpha_-\) is defined to be the unique natural number such that

\[
\alpha_- \wedge (d \alpha_-)^{r \kappa(\alpha_-)} \neq 0 \quad \text{and} \quad \alpha_- \wedge (d \alpha_-)^{r \kappa(\alpha_-)+1} = 0 .
\]

We notice that for an nc-Killing 1-form \(\alpha_-\) and all natural numbers \(l\) it is

\[
(\alpha_{l+1}^+) = \alpha_- \wedge (d \alpha_-)^l ,
\]
i.e., the form $\alpha_\perp \wedge (d\alpha_\perp)^l$ is itself an nc-Killing form and it corresponds to the twistor $\alpha^{l+1}$.

The next point that is important for our discussion is the fact that there exists a complete normal form classification for skew-adjoint endomorphisms on the pseudo-Euclidean space $\mathbb{R}^{2,n}$ with signature $(2, n)$ (cf. [Bou92]). These normal forms can be written down explicitly and their stabiliser groups in $SO(2, n)$ can be calculated. We are interested here in those skew-adjoint endomorphisms whose stabiliser is maximal (in the sense that the stabiliser is not properly contained in the stabiliser of any other skew-adjoint operator). The space of skew-adjoint endomorphisms is naturally identified with the space of 2-forms on $\mathbb{R}^{2,n}$. We present in the following list all normal forms for 2-forms on $\mathbb{R}^{2,n}$ with maximal stabiliser and with the additional condition that the corresponding skew-adjoint operators map all causal vectors to causal vectors. The latter requirement characterises those 2-forms, which are associated to a spinor in the module $\Delta_{2,n}$ for signature $(2, n)$ (cf. [Lei03]).

1. $l_1 \wedge l_2$ with $l_1$ and $l_2$ spanning a totally lightlike subspace in $\mathbb{R}^{2,n}\ast$,
2. $l_1 \wedge t_1$ with $l_1$ lightlike and $t_1$ an orthogonal timelike 1-form to $l_1$,
3. $\omega_o$ the standard symplectic form on $\mathbb{R}^{2,n}$,
4. $\omega_o|_V$ the standard symplectic form on a non-degenerate subspace $V$ of $\mathbb{R}^{2,n}$ with signature $(2, p - 1)$, where $p < n$.

Moreover, we can say that every 2-form $\alpha$ determines a unique normal form with maximal stabiliser, which contains $\text{Stab}(\alpha)$.

We use these normal forms in the following way. Let us assume that $(M^n, g)$ is a Lorentzian spin manifold with conformal Killing spinor $\varphi_\perp$. The corresponding twistor spinor $\varphi \in \Gamma(S^2g)$ is parallel with respect to $\nabla^{NCS}$ and induces a non-trivial twistor 2-form $\alpha_\varphi$. The holonomy of the connection $\omega_{NC}$ lies necessarily in the stabiliser belonging to the normal form of the 2-twistor $\alpha_\varphi$. This implies that there is a twistor 2-form $\alpha$ on $M$ whose corresponding normal form is one of those in the list, since they have maximal stabiliser. Let us discuss the different cases.

First, we assume that the normal form to $\alpha$ is $l_1 \wedge l_2$. In this case the corresponding nc-Killing 1-form is lightlike and hypersurface orthogonal ($\alpha^2 = 0$). We can conclude that the underlying conformal structure is represented by a Brinkmann metric. However, for the particular case $l_1 \wedge l_2$, one can even show that $\alpha_{\perp \varphi} = \alpha$ (cf. [Lei03]). This implies that the twistor spinor $\varphi_\perp$ is (locally) conformally equivalent to a parallel spinor on the Brinkmann space, which induces the lightlike parallel vector. In the second case, we can conclude, since $\alpha^2 = 0$, that the underlying conformal structure is that of a static spacetime with parallel spinor, i.e., it is

$$[g] = [-dt^2 + h],$$

where $h$ is a Riemannian metric with parallel spinor. The third case of a symplectic form $\omega_o$ is the Fefferman case when the rank of $\alpha_\perp$ is $(n/2) - 1$. It remains to investigate the fourth case when the normal form is the standard symplectic form on a proper subspace of $\mathbb{R}^{2,n}$. This is exactly the unknown case that we aim to describe here.

So let us assume that $(M^n, g)$ is a simply connected Lorentzian spin manifold with a conformal Killing spinor $\varphi_\perp$, whose corresponding twistor spinor $\varphi$ induces a twistor 2-form $\alpha_\varphi$ that has a stabiliser in

$$U(1, \frac{p-1}{2}) \times SO(n - p + 1).$$

Then the 1-form $\alpha_\perp$ is timelike and there exists a 2-twistor $\alpha$ with a restricted symplectic form as normal form, which induces a nc-Killing form $\alpha_\perp$ on $M$ of rank

$$rk(\alpha_\perp) = rk(\alpha_\varphi) = \frac{p-1}{2}.$$
In particular, the twistor $\alpha^{r_k(\alpha_-)}+1$ is decomposable of degree $p + 1$. The corresponding subspace of $\mathbb{R}^{2,n}$ is non-degenerate and we can conclude that there is a product metric $h \times l$ in $[g]$, where $h$ and $l$ are Einstein metrics on simply connected spin manifolds $H^p$ and $L^{n-p}$ with dimension $p$ resp. $n - p$. Moreover, $\alpha_-$ is a coclosed nc-Killing 1-form of maximal rank $\frac{2n - 1}{2}$ on $(H, h)$, and this implies that $h$ is an Einstein-Sasaki metric. In fact, $(H, h)$ is a simply connected Einstein-Sasaki space admitting a conformal Killing spinor $\psi_{H-}$, which induces the nc-Killing 1-form $\alpha_-$. However, the existence of the conformal Killing spinor $\varphi_-$ should also impose a condition on $(L, l)$, and therefore, it still remains to discuss the geometry of the Riemannian Einstein spin manifold $(L, l)$.

For this we observe the following. The representation $\mathbb{R}^{2,n}$ splits into the subspaces $V$ and $W$ under the projection of the stabiliser group $\text{Stab}(\varphi)$ to $\text{SO}(2, n)$. On $V$ lives the symplectic form $\omega_o$ with signature $(2, p - 1)$. Let $\Delta_V$ and $\Delta_W$ denote the spinor modules over $V$ resp. $W$. As representations spaces of $\text{Spin}(2, p - 1) \times \text{Spin}(n - p + 1)$, it is

$$\Delta_V \otimes \Delta_W = \Delta_{2,n} \quad \text{for } n \text{ odd},$$

$$\Delta_V \otimes \Delta_W = \Delta_{\pm 2,n} \quad \text{for } n \text{ even},$$

where $\Delta^\pm$ denotes the half spinor modules in even dimensions. The splitting with respect to $\text{Stab}(\varphi)$ gives rise to a decomposition of the tractor spinor bundle as

$$S_{3\mathbb{R}} = S_V \oplus S_W.$$

We observe now that if we choose the product metric $h \times l$ on $M = H \times L$ in the conformal class $[g]$ then we can naturally identify $S_V$ and $S_W$ with the spinor bundles over the cones $(\hat{H}, \hat{h})$ resp. $(\hat{L}, \hat{l})$ restricted to the bases:

$$S_V|_H \cong S_{\hat{H}}|_H \quad \text{and} \quad S_W|_L \cong S_{\hat{L}}|_L.$$

We also know that it holds

$$\text{Hol}(\omega_{NC}) = \text{Hol}(\hat{h}) \times \text{Hol}(\hat{l}).$$

Moreover, we have the following general fact. Let $\rho$ be a representation of a product group $G_1 \times G_2$ and $\rho_1, \rho_2$ be representations of $G_1$ resp. $G_2$ such that

$$\rho \cong \rho_1 \otimes \rho_2.$$

Then the representation $\rho$ has a fixed vector if and only if $\rho_1$ and $\rho_2$ both have fixed vectors. In our situation that means the stabiliser $\text{Stab}(\varphi) \subset \text{Spin}(2, n)$ fixes a spinor both in $\Delta_V$ and $\Delta_W$ and this proves that we can find parallel spinors on the cones $\hat{H}$ and $\hat{L}$. The parallel spinor spinor on $\hat{H}$ gives rise to the Killing spinor $\psi_{H-}$ inducing the Lorentzian Einstein Sasaki structure on $H$, as we discussed it already. And now we can also see that there exists a Killing spinor $\psi_{L-}$ on the Riemannian Einstein spin space $L$ of positive scalar curvature. The associated vector to the spinor $\psi_{L-}$ on the Riemannian space $L$ vanishes, since the symplectic form $\omega_o$ lives on $V$ only. All together, this leads us to the following characterisation result. Thereby, we say that a conformal Killing spinor $\varphi_-$ has no singularities on $M$ if it has no zero and the function $\|\alpha_{\varphi_-}\|^2 = g(\alpha_{\varphi_-}, \alpha_{\varphi_-})$ either has no zero or is identically zero.

**Theorem 5.** Let $\varphi_-$ be a conformal Killing spinor without singularities on a simply connected Lorentzian spin manifold $(M^n, g)$ of dimension $n$. Let $\alpha_{\varphi_-}$ be the dual of the Dirac current with rank $\text{rk}(\alpha_{\varphi_-})$ and length $\|\alpha_{\varphi_-}\|^2$. The following cases occur.

1. It is

$$\text{rk}(\alpha_{\varphi_-}) = 0 \quad \text{and} \quad \|\alpha_{\varphi_-}\|^2 = 0$$
and then $\varphi_-$ is locally conformally equivalent to a parallel spinor on a Brinkmann space with lightlike Dirac current.

(2) It is

$$rk(\alpha_\varphi^-) = 0 \quad \text{and} \quad \|\alpha_\varphi^-\|^2 < 0$$

and then $[g] = [-dt^2 + h]$, where $h$ is a Ricci-flat Riemannian metric admitting a parallel spinor.

(3) The dimension $n$ is odd and the rank

$$rk(\alpha_\varphi^-) = (n - 1)/2$$

is maximal. Then $\varphi_-$ is conformally equivalent to a Killing spinor on a Lorentzian Einstein-Sasaki manifold. In this scaling the Dirac current is timelike of constant length.

(4) The dimension $n$ is even and

$$rk(\alpha_\varphi^-) = (n - 2)/2$$

Then it is $(d\alpha_\varphi^-)^{n/2} \neq 0$ and $(M, g)$ is a Fefferman space with twistor spinor $\varphi_-$ and $\|\alpha_\varphi^-\|^2 = 0$.

(5) It is

$$0 < rk(\alpha_\varphi^-) < (n - 2)/2$$

and then there is a product metric $h \times l$ in $[g]$, whereby $h$ is an Einstein-Sasaki metric on a Lorentzian space $H$ of dimension $p := 2 \cdot rk(\alpha_\varphi^-) + 1$ admitting a Killing spinor $\psi_H^-$ and $l$ is an Einstein metric with Killing spinor $\psi_L^-$, whose associated 1-form is trivial, on the Riemannian space $L$ with positive scalar curvature $\text{scal}_l = -\frac{(n-p)(n-p-1)}{p(p-1)}\text{scal}_h$. The spinor $\psi_H^- \otimes \psi_L^- \in \Gamma(S^g)$ is a twistor spinor on $M$ with timelike Dirac current (of constant negative length in the scaling $h \times l$).

We want to make a remark to the last point of Theorem 5. Indeed, the formula (15) for the holonomy shows in general that if $g = h \times l$ is a product metric such that $h$ and $l$ admit Killing spinors $\psi_H^-$ and $\psi_L^-$ with Killing numbers $\lambda_H = \pm i\lambda_L$ then the tensor product $\psi_H^- \otimes \psi_L^- \in \Gamma(S^g)$ is a conformal Killing spinor on $h \times l$. One can also prove this fact by confirming the twistor equation directly using the spinor connection

$$\nabla^{S,g} = \nabla^{S,h} \otimes 1 + 1 \otimes \nabla^{S,l}.$$ 

However, for this one has to work out carefully the correct identification for an appropriate tensor product of the Clifford algebras $Cl(1, p-1)$ or $Cl(p-1, 1)$ with $Cl(0, n-p+1)$ or $Cl(n-p+1, 0)$ on the one side and the Clifford algebra $Cl(1, n-1)$ or $Cl(n-1, 1)$ on the other side.

11. Further questions and outlook

We are concerned in this paper with the study of solutions of the normal twistor equations in conformal geometry and their relation to the normal conformal holonomy representation. The discussion shows that for decomposable and weakly irreducible holonomy representations without dilation the conformal geometry of the underlying space of a solution can be described by (products of) special geometries on Einstein spaces or at least Ricci-isotropic spaces. Those geometric structures are well-known and were subject to substantial work in the literature in the past and, therefore, there is considerable knowledge that could be applied here for further and more particular interest in those solutions. Moreover, there is a well-known case in the literature, which has in our context an irreducible normal conformal holonomy representation. We mean the Fefferman spaces in pseudo-Riemannian geometry. However, at this point we already
come across a quite natural question that does not seem to have a complete answer.

What is the geometric structure of a conformal space that has irreducible normal conformal holonomy representation?

Or more general:

How does the complete list of possible normal conformal holonomy groups look like and what kind of conformal structures do they describe?

We want to mention two particular examples to illustrate the questions that we ask for. First, there is the exceptional group $G_2 \subset SO(3, 4)$ in the split case of dimension 7. The group $SO(3, 4)$ is the Möbius group for a conformal space in signature $(2, 3)$. The first question then asks for the underlying geometry (and their existence) that belongs to a conformal space, which has normal conformal holonomy group $G_2$.

A further example of an interesting holonomy representation according to the second (more general) question is the case of a weakly irreducible holonomy group with dilation in the Möbius group $SO(1, n + 1)$, which then belongs to conformal Riemannian geometry. (Those spaces do not have solutions for the normal twistor equations.) One can ask this question in the first instance for 3-dimensional Riemannian space. It seems that a normal conformal holonomy classification even in this case is not completely known.

In general, conformal Killing forms were already introduced and studied in the works [Kas68, Tac69]. Recently, there was a systematic investigation on this topic by U. Semmelmann (cf. [Sem01]). In particular, this work shows the construction of a certain Killing connection, with respect to whom all conformal Killing forms find an interpretation as parallel sections in a certain differential form bundle. (In fact, it is the same bundle as our Möbius form bundle.) It arises now the question what properties does this Killing connection have, in particular, which structure group is attached to it, and whether this connection is somehow related to a conformally invariant connection (e.g. the normal one) on the Möbius frame bundle with structure group $SO(r + 1, s + 1)$.

An extension of our investigations that leads in a similar direction, as proposed for the Killing connection above, is the idea of dropping the normalisation condition for our twistor equations and do investigations for more general conformal Killing forms. For such an attempt one could ask what conformally invariant connections on $\mathcal{M}(M)$, that induce the twistor equations, are reasonable to look at? For example, does there always exists a conformal connection with structure group in $SO(r + 1, s + 1)$, which induces appropriate twistor equations to a given conformal Killing form, or what other kind of structure groups should be considered. However, these ideas are bit loose here for the moment. Nevertheless, there is the question whether the existence of a conformal Killing form gives rise to some twistor with a certain stabiliser group and some other tensor(s), which describe uniquely the underlying conformal geometry of a space, and are useful in order to find a systematic construction principle for those solutions.

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