General relativistic, nonstandard model for the
dark sector of the Universe.

P.C. Stichel\textsuperscript{1)} and W.J. Zakrzewski\textsuperscript{2)}

\textsuperscript{1)}Fakultät für Physik, Universität Bielefeld, D-33501 Bielefeld, Germany,
e-mail:peter@physik.uni-bielefeld.de

\textsuperscript{2)}Department of Mathematical Sciences, University of Durham,
Durham DH1 3LE, UK
e-mail: W.J.Zakrzewski@durham.ac.uk

Abstract

We present a general relativistic version of the self-gravitating fluid model for the dark sector of the Universe (darkon fluid) introduced in Phys. Rev. 80 (2009) 083513 and extended and reviewed in Entropy (2013) 559. This model contains no free parameters in its Lagrangian. The resulting energy-momentum tensor is dustlike with a nontrivial energy-flow. In an approximation valid at sub-Hubble scales we find that the cosmic acceleration is not attributed to any kind of negative pressure but it is due to a dynamically determined sign change of the energy density. This property turns out to be equivalent to a time-dependent, sign changing curvature. The cosmological equations obtained agree with those of the nonrelativistic model but they are given a new physical interpretation. Furthermore we have derived the self-consistent equation to be satisfied by the nonrelativistic gravitational potential produced by a galactic halo in our model from a weak field limit of a generalized Tolman-Oppenheimer-Volkoff equation.
1 Introduction

It is now pretty clear that the present Universe undergoes a phase of accelerated expansion (see the recent reviews [1], [2]). On the other hand there exists an overwhelming evidence for the existence of gravitational effects on all cosmological scales (termed “dark matter”) which cannot be explained by the gravitation of standard matter in the framework of General Relativity (GR) (see the review [3]). All of these data are in good agreement with a Λ-cold dark matter (CDM) cosmology (see [1], [2] and the literature cited therein). But this ΛCDM model suffers, at least, from the following insufficiencies:

- Interpreted as the energy density of the vacuum the experimental value of Λ turns out to be too small by a factor of $10^{54}$ (see [4]).
- None of the proposed DM-constituents has been observed (cp. [5])
- There is a CDM-controversy on small scales [6].

Other observations which are in disagreement with the ΛCDM model have been recently listed by Kroupa [7].

One can find in the literature a large number of papers explaining either the accelerated expansion and/or dark matter by changing either the geometrical part of Einstein’s field equations (termed modified gravity) or the matter part (addition of some scalar and/or tensor fields). We will not comment on either of these attempts (for details see e.g. the reviews [1], [2] and the literature cited therein). But we want to point out that all these proposals are of a phenomenological nature, they contain either some new parameters or even free functions. So what we need is some new (i.e. unconventional) physics which, however, should be based on known physical principles (e.g. symmetry). Such a model containing no new constants in its Lagrangian and based on Galilean symmetry (minimal gravitational coupling of massless Galilean particles in agreement with the equivalence principle) has been presented in [8], further developed in [9] and reviewed and extended.
in [10]. This nonrelativistic, unified model for the dark sector of the Universe, termed darkon fluid model, describes successfully observational data for the transition from a decelerating to an acceleration phase of the Universe as well as the flat behaviour of galactic rotation curves [8], [10].

One of the aims of the present paper is to present a general relativistic version of this model and to relate some approximate solutions of it to the corresponding solutions of its nonrelativistic counterpart.

The paper is organized as follows: To get a self-consistent paper and to have an appropriate starting point for its relativistic generalization we present in section 2 a short review of the nonrelativistic model [8]-[10]. In Section 3 we treat first the special relativistic generalization of the free model, discuss the different options to consider classical spin contributions and, after a Belinfante transformation, we introduce gravitation by the principle of minimal coupling. Also we discuss there the energy conditions. In Section 4 we consider the solutions of the coupled system of the Einstein field equations and the relativistic darkon fluid equations of motion at sub-Hubble scales which lead to the cosmological solutions treated in section 5. We discuss the Friedmann equations which turn out to be completely different from the FLRW model (changing sign of the energy density resp. of the time-dependent curvature). In section 6 we treat the same coupled system of equations in non-comoving coordinates and derive the self-consistent equation for the halo-gravitational potential derived in [10], as a weak-field limit of the Tolman-Oppenheimer-Volkoff equation. Some final remarks are presented in section 7.

2 Nonrelativistic, self-gravitating darkon fluid

In [8] we have introduced nonrelativistic massless ‘particles’ as a dynamical realization of the unextended Galilei group. These ‘particles’ move in an enlarged twelve-dimensional phase space [10] consisting of

- The ‘particle’ trajectory $\vec{x}(t)$
• the momentum $\vec{p}(t)$, canonically conjugate to $\vec{x}(t)$

• the velocity vector $\vec{y}(t)$

and

• the reduced boost vector $\vec{q}(t)$ (called ‘pseudo-momentum’), canonically conjugate to $\vec{y}(t)$.

In accordance with the Galilean algebra the corresponding ‘one-particle’ Hamiltonian $H$ is given by

$$H_0 = p_i y_i,$$

(1)

corresponding to, by a Legendre transformation, the Lagrangian

$$L_0 = p_i (\dot{x}_i - y_i) + q_i \dot{y}_i,$$

(2)

and so giving the equations of motion (EOMs):

$$\dot{x}_i = y_i, \quad \dot{p}_i = 0, \quad \dot{q}_i = -p_i, \quad \dot{y}_i = 0.$$

(3)

But such a ‘particle’ is not a classical particle in the usual sense as it is not detectable by any finite-sized macroscopic measurement device because

• momentum and velocity vector are independent of each other and we have no ability to measure the momentum,

• the boost vector $\vec{q}$ has, for fixed position $\vec{x}$ and velocity $\vec{y}$ an arbitrary $i.e.$ un-determined length.

For these reasons we have called these ‘particles’ darkons \[^9\]; as they exist only as elements of a self-gravitating fluid whose action is a substitute for what is usually called ‘dark energy’ and ‘dark matter’.

To introduce the coupling to gravitation represented by the field strength $g_i(\vec{x}, t)$ we have to require, in agreement with Einstein’s equivalence principle, the validity of Newton’s law

$$\ddot{x}_i(t) = g_i(\vec{x}(t), t),$$

(4)
which will be realized if we add to $L_0$ an interaction part (minimal coupling)

$$L_{int} = -q_i g_i. \quad (5)$$

An important property of our darkons is the appearance of a macroscopic spin: The conserved total angular momentum is given by the sum of the usual orbital angular momentum and a 2nd term which we call, for convenience, spin [10] (see also Mathisson [11])

$$J_i = \epsilon_{ikl} (x_k p_l + y_k q_l). \quad (6)$$

Note that the two terms in (6) act separately as generators of rotations in the $\{\vec{x}, \vec{p}\}$ resp. $\{\vec{y}, \vec{q}\}$ parts of the phase space.

To promote this ‘one-particle’ picture to a self-gravitating fluid we replace the ‘one-particle’ phase space coordinates $A_i = \{x_i, p_i, q_i, y_i\}$ by the continuum labeled by $\vec{\xi} \in R^3$ (comoving coordinates) $A_i(t) \to A_i(\vec{\xi}, t)$.

The Lagrangian for our darkon fluid then becomes

$$L = \int d^3\xi \left[ p_i (\dot{x}_i - y_i) + q_i (\dot{y}_i - g_i) \right] + L_{field} \quad (7)$$

where, as usual

$$L_{field} = -\frac{1}{8\pi G} \int d^3 x g^2_i(\vec{x}, t). \quad (8)$$

The Lagrangian (7) is invariant w.r.t. infinitesimal relabeling transformations $\vec{\xi} \to \vec{\xi} + \vec{\alpha}(\vec{\xi})$ with $\vec{\nabla}_\xi \cdot \vec{\alpha} = 0$ leading to the conservation law [9], [10]

$$\dot{\theta}_i = 0 \quad \text{where} \quad \theta_i \equiv -\frac{\partial x_k}{\partial \xi_i} q_k + \frac{\partial x_k}{\partial \xi_i} \dot{q}_k \quad (9)$$

which, after elimination of the momentum field $p_i$, allows us to reduce the EOM for $q_i$ to a 1-st order equation [9], [10]. Then by means of the usual transformations from comoving coordinates $\vec{\xi}$ to the fixed ones $\vec{x} = \vec{x}(\vec{\xi}, t)$ we obtain from the Lagrangian formulation (7) the Eulerian formulation given by the Lagrangian [10]

$$L = \int d^3 x \left[ \nu q_i (D_t u_i - g_i) - \theta (\dot{\nu} + \partial_k (\nu u_k)) + n \alpha D_t \beta \right] + L_{field}, \quad (10)$$
where $n(x, t)$ denotes the ‘particle’ density. We have introduced the auxiliary field

$$\theta_i(\vec{x}, t) \equiv \frac{\partial \xi_k}{\partial x_i} \theta_k(\vec{\xi})|_{\vec{\xi}=\vec{\xi}(\vec{x}, t)}$$

and its Clebsch-parameterization $\theta_i = \partial_i \theta + \alpha \partial_i \beta$.

Furthermore $\vec{u}$ denotes the velocity field $u_k(\vec{x}, t) \equiv \dot{x}_k(\vec{\xi}, t)|_{\vec{\xi}=\vec{\xi}(\vec{x}, t)}$ and $D_t$ the convective derivative $D_t \equiv \frac{\partial}{\partial t} + u_k \frac{\partial}{\partial x_k}$.

Note that the Hamiltonian corresponding to the Lagrangian (10) is not bounded from below. In [8] we have argued that this does not lead to any stability problems.

The equations of motion (EOMs) following from the Lagrangian (10) have been solved for

- the isotropic, homogeneous case (cosmology) in [8] resp. [10] (see also section 5 of this paper),
- the spherically symmetric, steady state case modeling halos [10] (see also section 6 of this paper).

### 3 General relativistic approach

#### 3.1 Nongravitating, special relativistic case

We start our discussion with the nongravitating i.e. special relativistic case for two reasons:

- to discuss the notion of (zero) rest-mass in our enlarged phase space,
- to discuss the role of the spin-term within the energy-momentum tensor (EMT).

The relativistic generalization of the action corresponding to the free matter part of the Lagrangian (10) is then given by (we use the Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$)

$$S = \int d^4x (nq_{\nu}Du^\nu - \theta \partial_\nu (nu^\nu) + n\alpha D\beta),$$

(11)
where we have defined the relativistic version of the convective derivative by $D \equiv u^\lambda \partial_\lambda$. We also require that the velocity field $u^\nu$ obeys the usual constraint $u_\nu u^\nu = -1$.

From the Lagrangian (11) we derive the Euler-Lagrange EOMs

$$\partial_\nu (nu^\nu) = 0, \quad Du^\nu = 0,$$

$$D\alpha = D\beta = D\theta = 0, \quad Dq_\lambda = q_\nu \partial_\lambda u^\nu + \theta_\lambda$$

with

$$\theta_\lambda \equiv \partial_\lambda \theta + \alpha \partial_\lambda \beta.$$  

It is easy to see that

- the four-momentum vector field, defined analogously to the EOM $p_i = -\dot{q}_i$ in (3) by $p_\mu \equiv -Dq_\mu$, is space-like (from the EOMs one deduces that $p^\lambda u_\lambda = 0$),

- the second EOM in (13) is invariant w.r.t. the gauge transformation $q_\lambda \to q_\lambda + \epsilon u_\lambda$, i.e. we can fix the gauge by choosing $q_\lambda u^\lambda = 0$ so that $q_\lambda$ becomes space-like.

- the EOMs are invariant w.r.t. the shift symmetry $q_\lambda \to q_\lambda + c_\lambda$, $\theta \to \theta - c_\lambda u^\lambda$, where $c_\lambda$ is a constant vector field. Note that this kind of shift symmetry is characteristic for Galileon theories (cp section 2.1 in [12]).

The fact that $p_\mu$ is a space-like vector field could easily lead to the wrong conclusion, that our darkons are tachyons (cp. appendix B in [8]). But, as argued by Weyssenhoff and Raabe [13] in a similar context, we should define the rest-mass as the energy in the rest system of the ‘particle’ given by $m_0 = -u^\lambda p_\lambda$, which, however, vanishes in our case.

The Poincare invariance leads to the existence of two conserved currents (cp. Appendix A in [14]).
• From translational invariance we get the canonical, nonsymmetric energy-momentum tensor (EMT)

\[ \hat{T}^\mu{}^\nu = np^\mu u^\nu \] (15)

• from the Lorentz invariance we get

\[ J^{\nu,\alpha\beta} = x^\alpha \hat{T}^{\beta\nu} - x^\beta \hat{T}^{\alpha\nu} + S^{\nu,\alpha\beta}, \] (16)

where the spin tensor is given in our case by

\[ S^{\nu,\alpha\beta} = nu^\nu(u^\alpha q^\beta - u^\beta q^\alpha) \] (17)

The conservation law \( \partial_\nu \hat{T}^\mu{}^\nu = 0 \) follows immediately from the EOMs. Furthermore, the EOMs also give us

\[ \partial_\nu S^{\nu,\alpha\beta} = \hat{T}^{\alpha\beta} - \hat{T}^{\beta\alpha} \] (18)

and so give

\[ \partial_\nu J^{\nu,\alpha\beta} = 0. \]

Note that the relativistic fluid described by the action (11) is not a Weyssenhoff-fluid [13] as the spin tensor (17) does not obey the Frenkel condition [15] \( S^{\nu,\alpha\beta} u_\beta = 0 \). Instead we get from (17) \( S^{\nu,\alpha\beta} u_\beta = nu^\nu q^\alpha \).

### 3.2 General relativistic dynamics

According to Hehl [16] we have now two possibilities for coupling our relativistic fluid to gravity

• To gauge away the spin tensor by a Belinfante transformation [17]

\[ \hat{T}^{\mu\nu} \rightarrow T^{\mu\nu} = \hat{T}^{\mu\nu} + \frac{1}{2} \partial_\lambda(S^{\mu,\nu\lambda} + S^{\lambda,\nu\mu} + S^{\nu,\mu\lambda}) \] (19)

leading to a conserved, symmetric EMT

\[ T^{\mu\nu} = n(u^{\mu} p^\nu + u^{\nu} p^\mu) + \partial_\lambda(nu^\mu u^\nu q^\lambda) \] (20)

This EMT may then be used as a source term in Einstein’s field equations after we have performed the substitutions (21) (see eq. (29)).
• Consider spin as a dynamical variable by relating the spin tensor to the torsion tensor in the framework of a Riemann-Cartan space-time and use the canonical EMT [15] as the source term in Einstein’s field equations.

In this paper we prefer to use the first possibility as in this case we can reproduce, at sub-Hubble scales, the cosmological equations which are valid for the nonrelativistic darkon fluid (section 5). To realize this we have to apply the principle of minimal gravitational coupling (cp. [18]): So we perform the substitutions

$$\eta_{\mu\nu} \to g_{\mu\nu} \quad \text{and} \quad \partial_{\lambda} \to \nabla_{\lambda}$$

in the special relativistic action (10). Here $g_{\mu\nu}$ is the metric tensor and $\nabla_{\lambda}$ is the covariant derivative $\nabla_{\lambda} A^{\nu} = \partial_{\lambda} A^{\nu} + \Gamma_{\lambda\sigma}^{\nu} A^{\sigma}$ where the elements of the connection $\Gamma_{\lambda\sigma}^{\nu}$ are given by the Christoffel symbols.

To obtain also Einstein’s field equations from the principle of least action we have to consider the total action

$$S = \int d^4x \sqrt{-g}(nq_{\nu}u^{\lambda}\nabla_{\lambda}u^{\nu} - \theta \nabla_{\nu}(nu^{\nu}) + n\alpha u^{\lambda} \partial_{\lambda} \beta) + S_{EH},$$

(22)

with the Einstein-Hilbert action given by the well-known expression

$$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R,$$

(23)

where $g$ is the determinant of $g_{\mu\nu}$, $R$ is the Ricci scalar, $n$ is the particle density and $u^{\lambda}$ resp. $q^{\lambda}$ are the velocity field resp. the relativistic generalization of the pseudo-momentum field. The scalar fields $\theta$, $\alpha$ and $\beta$ are Lagrange-multiplier fields which originate from the relabelling symmetry (see the nonrelativistic Lagrangian formulation in section 2).

From the action (22) we derive the darkon fluid EOMs (which, alternatively, may be obtained by applying the substitution rule (21) to the special-relativistic EOMs (12) and (13))

$$\nabla_{\lambda}(nu^{\lambda}) = 0, \quad u^{\lambda}\nabla_{\lambda}u^{\nu} = 0$$

(24)
\[ u^\lambda \nabla_\nu q^\nu = q_\lambda \nabla_\nu u^\lambda + \theta_\nu \]  
(25)

and

\[ u^\lambda \nabla_\lambda \theta_\nu + \theta_\lambda \nabla_\nu u^\lambda = 0 \quad \text{with} \quad u^\nu \theta_\nu = 0 \]  
(26)

where by (14) \( \theta_\nu \equiv \partial_\nu \theta + \alpha \partial_\nu \beta \) and Einstein’s field equations are the standard ones

\[ G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G T^{\mu\nu}. \]  
(27)

Here \( R^{\mu\nu} \) is the Ricci tensor and the EMT \( T^{\mu\nu} \) is given by (29) given below.

We find again that the fields \( q_\lambda \) and \( p_\lambda \equiv -u^\nu \nabla_\nu q_\lambda \) obey the constraints

\[ u^\lambda q_\lambda = u^\lambda p_\lambda = 0 \]  
(28)

and so they are space-like (recall that \( u^\lambda \) is time-like, normalized by \( u^\lambda u_\lambda = -1 \)).

### 3.3 Energy-momentum tensor (EMT)

The EMT (20), after having performed the substitutions (21), becomes

\[ T^{\mu\nu} = -n \left[ u^\mu u^\lambda \nabla_\lambda q^\nu + (\mu \leftrightarrow \nu) \right] + \nabla_\lambda (nu^\mu u^\nu q^\lambda). \]  
(29)

By using the darkon fluid EOMs the expression (29) can be brought in to its canonical form (see [19])

\[ T^{\mu\nu} = \rho u^\mu u^\nu + k^\mu u^\nu + k^\nu u^\mu, \]  
(30)

where

\[ \rho = \nabla_\lambda (nq^\lambda) \quad \text{and} \quad k^\mu = n \left( q^\lambda (\partial_\lambda u^\mu - \partial^\mu u_\lambda) - \theta^\mu \right) \]  
(31)

are the energy density resp. the energy flow vector seen by an observer comoving with the darkon fluid.

Usually the vector \( k^\mu \) is called the ‘heat-flow vector’. But such a terminology assumes, at least implicitly, that we have a description of \( k^\mu \) and \( \rho \) in terms of a relativistic, irreversible thermodynamics (for the general framework see [20], for an application to cosmology see [21]). But \( \rho \) and \( k^\mu \) are
completely fixed in our case by the darkon fluid EOMs. So it is an open question whether they are accessible to a thermodynamic description or whether the arising energy flow is due to the generation of gravitational radiation. It is outside the scope of the present paper to consider this question.

Note that the expression (31) for the energy density $\rho$ is not positive definite! So at least the weak energy condition is violated. But, as will be shown in section 5, exactly this property of our model is crucial for the model’s explanation of the accelerated expansion of the Universe. Energy conditions are constraints on the EMT of a general relativistic fluid which, originally, has been thought as being necessary for the fluid ‘to be physically reasonable’ (see [22] and the literature cited within). But it is well known that e.g. the introduction of ‘dark energy’ within the FLRW model (negative pressure with $\rho + 3p < 0$) violates the strong energy condition. This is in agreement with a very recent and general discussion in the framework of extended theories of gravitation [23], which comes to the conclusion that the violation of energy conditions is a general property in the presence of dark energy.

4 Solutions at sub-Hubble scales

The aim of this section is to derive approximate solutions at sub-Hubble scales for the coupled system of the darkon fluid EOMs and the Einstein-field equations in spherically symmetric geometry. The cosmological solutions, to be considered in the next section, arise as a particular case. The non-accelerated fluid motion (24) allows the consideration of synchronous comoving ($u^\mu = \delta^\mu_0$), spherically symmetric coordinates defined by the metric

$$ds^2 = -dt^2 + B^2(t, r)dr^2 + Y^2(t, r)d\Omega^2.$$  \hspace{1cm} (32)

For this metric the space-like vectors $q_\mu$ and $\theta_\mu$ have only a non-vanishing radial component

$$q_\mu = qs_\mu, \quad \theta_\mu = \bar{\theta}s_\mu \quad \text{with} \quad s_\mu \equiv (0, B).$$  \hspace{1cm} (33)
Then the remaining darkon fluid equations (24), (25) and (26) have the following form resp. solutions

\[ n(t, r) = n_0(r) B, \quad \dot{B}(t, r) = \dot{\alpha}(r) B, \]

where \( n_0(r) \) resp. \( \dot{\alpha}(r) \) are arbitrary functions of \( r \). The energy density \( \rho \) defined by (31), then becomes, in terms of \( q \) and the metric

\[ \rho = \frac{1}{B Y^2} \left( \frac{n_0 q}{B} \right)', \]

and obeys, due to the 2nd eq. in (34), the local energy conservation equation

\[ \dot{\rho} + \rho \left( \frac{\dot{B}}{B} + 2 \frac{\dot{Y}}{Y} \right) - \frac{1}{B Y^2} \left( \frac{\alpha(r)n_0(r)}{B^2} \right)' = 0, \]

where and in what follows \( ' \) denotes the derivative w.r.t. \( r \).

Note that the velocity field \( u^\mu \) has vanishing vorticity in the spherically symmetric case. Therefore the energy flow vector \( k^\mu \) reduces to

\[ k^\mu = -n\theta^\mu. \]

The Einstein-field equations (27) now become (cp. eq. (7) with \( A = 1 \) in [24])

\[ 2 \frac{\dot{B}}{B} \frac{\dot{Y}}{Y} + \frac{1 + \dot{Y}^2}{Y^2} - \frac{Y''}{Y^2 B^2} - 2 \frac{Y'}{B} \left( \frac{Y'}{B} \right)' - \frac{\kappa}{B Y^2} \left( \frac{n_0 q}{B} \right)' = 0, \]

\[ 2 \frac{\ddot{Y}}{Y} + \frac{1 + \dot{Y}^2}{Y^2} - \frac{Y''}{Y^2 B^2} = 0 \]

\[ \frac{\ddot{B}}{B} + \frac{\ddot{Y}}{Y} + \frac{\dot{B}\dot{Y}}{BY} - \frac{1}{BY} \left( \frac{Y'}{B} \right)' = 0, \]

\[ -\frac{\kappa}{2} \frac{n_0(r)\alpha(r)}{B^2} = Y B^2 \left( \frac{Y'}{B} \right)', \]

with \( \kappa \equiv 8\pi G \) where (38), (39), (40) and (41) represent, respectively, the 00, \( rr \), tangential and 0\( r \)-components of (27).

Let us next show that, for isotropic coordinates,

\[ Y(t, r) = r B(t, r). \]
(39)–(41) enforce $\alpha(r) = 0$: By equating (39) and (40) in the isotropic case (42) we eliminate the time derivatives and obtain the well-known result (see [25])

$$
\left( \frac{1}{r} \left( \frac{1}{B} \right) \right) = 0
$$

(43)

with the solution

$$
B(t, r) = \frac{a(t)}{1 + \frac{r^2}{4} K(t)}
$$

(44)

where $a(t)$ resp. $K(t)$ are arbitrary functions of $t$. Now inserting (42), (44) into (39) leads by a straightforward calculation to $K(t) = K = \text{const}$ and, therefore, to the vanishing r.h.s. of (41). But then the EMT contains only a pure dust term which leads to a trivial cosmology (presence of only a decelerating phase).

Unfortunately we are unable to solve Einstein’s eq.s for $\alpha \neq 0$ exactly. So let us look for approximate solutions at sub-Hubble scales $\frac{r}{r_0} = \epsilon \ll 1$ ($r_0 = \text{Hubble radius}$) and take correspondingly for the derivatives (cp. [26])

$$
\partial_r = O(\epsilon^{-1}) \quad \text{and} \quad \partial_t = O(\epsilon^{-\frac{1}{2}}).
$$

(45)

From (39) we obtain the exact relation

$$
B = \frac{Y'}{(1 - b)^{\frac{3}{2}}}
$$

(46)

with

$$
b = -(2\ddot{Y}Y + \dot{Y}^2)
$$

(47)

Now we consider those metrics which have $Y(t, r) \propto r$ for small $r$. Then we have $b = O(\epsilon)$ and therefore $b$ may be treated as a perturbation.

Next we obtain, in leading order, from (41)

$$
\dot{b}(t, r) = 8\pi G \frac{n_0 \alpha(r)}{YY^2},
$$

(48)

which, when compared with (47), leads to the consistency relation

$$
- \left( \dot{Y}Y^2 \right) = 4\pi G \frac{n_0 \alpha}{Y^2}.
$$

(49)
It is easily seen that with (46-48) the 3rd Einstein eq. (40) is also fulfilled in leading order.

Instead of the 1st Einstein eq. (38), we use the Raychaudhuri-Ehlers equation (obtained by taking \((38)-(39)-2\times(40)\)) yielding, in leading order,

\[2\ddot{Y}Y' + \dot{Y}' = -\frac{4\pi G}{Y} \left(\frac{n_0q}{Y'}\right)'\]  

(50)

which, after multiplication by \(Y\), can be integrated to give

\[\ddot{Y}Y^2 = -4\pi G \frac{n_0q}{Y'} + f(t),\]  

(51)

where \(f(t)\) is an arbitrary function of \(t\). But if we differentiate (51) w.r.t. the time \(t\) and use the EOM (34) for \(q\), given in leading order by

\[\left(\frac{q}{Y}\right)' = \frac{\alpha}{Y'z}.\]  

(52)

we obtain, by comparison with the consistency relation (49), that \(f\) must be a constant. To get an analytic solution we put \(f\) equal to zero. So finally we have to solve the coupled system of equations (51) for \(f = 0\) and (52). To do this we consider a separation ansatz for \(Y\)

\[Y(t, r) = a(t)y(r)\]  

(53)

leading by (51) to a separable form for \(q\)

\[q(t, r) = q_0(t)q_1(r)\]  

(54)

where, due to (52), we may normalize \(q_1\) so that

\[q_1(r) = 4\pi G \frac{\alpha(r)}{y'(r)}\]  

(55)

Then we get for \(q_0\) the equation

\[\left(\frac{q_0}{a}\right)' = \frac{1}{4\pi Ga^2}.\]  

(56)

Finally from (51) we obtain

\[\ddot{a}a^3 = 4\pi GK_1q_0\]  

(57)
where \[ y^3 y^2 = -\frac{4\pi G}{K_1} \alpha n_0, \]
and \( K_1 \) is an arbitrary constant.

So the \( r \)-dependence of our solutions is completely specified by the choice of \( n_0(r) \) and \( \alpha(r) \) (for the case of cosmology we refer to the next section). We note that the separable forms of \( Y \) and \( q \) lead also to a separable form for the energy density \( \rho \) \( \text{(35)} \). Therefore the appearance of a Perpetuum Mobile of the third kind (continuous transfer of energy from one space region to another one) as advocated by Ivanov \( \text{[27]} \) is excluded.

\section{Cosmological solutions}

Note that \( \text{(56)} \) and \( \text{(57)} \) have exactly the form of the cosmological equations derived for the nonrelativistic darkon fluid in \( \text{[8]} \) resp. \( \text{[10]} \). But which choice has to be made for the two free functions \( n_0(r) \) and \( \alpha(r) \)?

For the cosmological solutions to be viable we have to require that the energy density \( \rho \) is a function of time only. Now taking \( \rho \) from \( \text{(35)} \) in leading order and using the ansätze \( \text{(53)}, \text{(54)} \) together with \( \text{(58)} \) we obtain

\[ \rho(t, r) = -\frac{K_1 q_0(t)}{a^4(t)} \left( y^3(r)y'(r) \right)' \]

So to get \( \rho = \rho(t) \) we have to choose \( y(r) = r \times \text{const.} \) as expected. By fixing the scale for \( r \) we can put this constant equal to one and we obtain

\[ \rho(t) = -\frac{3K_1 q_0}{a^4}. \]

Analogously, the requirement that \( n = n(t) \) leads, due to the 1st eq. in \( \text{(34)} \), to

\[ n_0(r) = r^2 n_{00} \]

and therefore, due to \( \text{(58)} \) to

\[ \alpha(r) = r\alpha_0 \]
with
\[-4\pi G n_{00} \alpha_0 = K_1, \tag{63}\]
where \(n_{00}\) and \(\alpha_0\) are arbitrary constants.

5.1 Metric and distances

From the foregoing results we conclude that, including 1st order corrections in \(B\), we arrive at the metric
\[
ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - r^2 K(t)} + r^2 d\Omega^2 \right), \tag{64}\]
where \(a(t)\) is the cosmological scale factor and \(K(t)\) is the time-dependent curvature determined from
\[
K(t) = -(2\ddot{a} a + \dot{a}^2). \tag{65}\]

Thus, in leading order, we have the FLRW metric and the standard distance formulae. As first order corrections have been obtained for \(B\) only (1st order corrections for \(Y\) require a solution of the Einstein equations in 2nd order) we are not yet able to calculate corrections to the distance formulae.

5.2 Friedmann equations

First of all we replace in our cosmological equations (56), (57) the unphysical quantity \(q_0\) through the energy density \(\rho\) by using (60). Then we obtain from

- (57) the 1st Friedmann eq.
\[
\frac{3\ddot{a}}{a} = -4\pi G \rho \tag{66}\]

- (56) the energy conservation equation in the form
\[
\dot{\rho} + 3 \frac{\dot{a}}{a} \rho + \frac{3K_1}{4\pi Ga^5} = 0 \tag{67}\]
Eliminating $\ddot{a}$ in the 2nd Friedmann eq. given by (65) by means of (66) we obtain the fundamental Friedmann eq. (cp. [28])

$$\dot{a}^2 + K(t) = \frac{8\pi G}{3} \rho a^2. \quad (68)$$

Next we express the last term in (67) through the curvature. To do this we differentiate (68) w.r.t. time $t$ and obtain, using (66) and (67)

$$\dot{K} = -\frac{2K_1}{a^3}. \quad (69)$$

Then (67) becomes

$$\dot{\rho} + 3\frac{\dot{a}}{a} \rho - \frac{3}{8\pi Ga^2} \dot{K} = 0, \quad (70)$$

i.e. the time-derivative of the curvature serves as a source term in the local energy conservation equation.

Obviously, as is well known, of the three equations (two Friedmann eq.s and the energy conservation eq.) only two are independent.

### 5.3 Solution of the Friedmann equations

Our original cosmological eq.s (56) and (57) had been solved in [8], [10] by means of two conservation laws $K_{2,3}$, which read, if we express them in terms of $\rho$

$$K_2 = \dot{a}K_1 - \frac{1}{2}g^2 \quad (71)$$

and

$$K_3 = \frac{g^3}{6} + K_2 g + \frac{K_1^2}{a}, \quad (72)$$

where we have defined

$$g(a) \equiv -\frac{4\pi G}{3} a^3 \rho(a). \quad (73)$$

Note that for $K_2 > 0$ (see below) we must have $K_1 > 0$ [10] in order to describe an expanding Universe.

Inserting (66) and (71) into (65) we obtain for the curvature

$$K(a) = -\left(2\frac{g(a)}{a} + \frac{1}{K_1^2}(K_2 + \frac{1}{2}g(a)^2)^2\right), \quad (74)$$
where we used the scale factor $a$ instead of the time $t$ as our independent variable.

However, it is more convenient to use the redshift $z$ ($(1 + z)^{-1} \equiv a$) as the independent variable. Then the cubic eq. for $g$ (72) becomes [10]

$$\frac{g^3}{6} + K_2 g = K_3 \left(1 - \frac{1 + z}{1 + z_t}\right),$$

(75)

where the transition redshift $z_t$ is defined by

$$1 + z_t = \frac{K_3}{K_1^2}.$$  

(76)

From (75), the first Friedmann eq. (66) and the choice $K_{2,3} > 0$ we conclude [8], [10]:

For $z > z_t$ we have $g < 0$ ($\rho > 0$) and so $\ddot{a} < 0$ (deceleration phase of the early Universe). On the other hand for $z < z_t$ we have $g > 0$ ($\rho < 0$) and so $\ddot{a} > 0$ (acceleration phase of the late Universe).

So we have learnt that the sign change of the energy density during evolution is for our model the dynamical reason for the transition from a decelerating to an accelerating phase of the Universe. This result differs completely from the standard FLRW-model where the cosmic acceleration is attributed to a negative pressure (e.g. a positive cosmological constant).

Now let us consider the behaviour of the curvature $K(z)$. From (75) we obtain for large redshifts ($z \gg 1$)

$$g(z) \sim -(6K_1^2 z)^{\frac{1}{3}}$$

(77)

leading, due to (74) to $K(z) \sim -\frac{\sigma(z)}{z} > 0$. On the other hand for $z < z_t$ we have $g > 0$ and therefore $K(z) < 0$.

Now, due to (69), $K(z)$ is an increasing function of $z$ and we have $K(z_t) < 0$ (note that $g(z_t) = 0$).

Thus we conclude:

The curvature $K(z)$ has a zero at some point $z_0$ with $z_0 > z_t$ and is positive for $z > z_0$ (spherical space) and becomes negative for $z < z_0$ (hyperbolic
space). Note that such a time dependence of the curvature with a possible sign change during evolution is known for the Stephani solution of Einsteins eq. \[29, 30\].

The sign changes of $\rho(z)$ resp. $K(z)$ are related to each other as can easily be read off from the fundamental Friedmann eq. (68): For positive curvature the energy density must be positive resp. for negative energy density the curvature must be negative.

What about the contribution of baryonic matter to the cosmological solutions? Suppose we describe baryonic matter, averaged over small scale inhomogeneities, by dust moving with the same four velocity then the darkon fluid. Then our cosmological equations contain only the total energy density $\rho$ given by the sum of the darkon fluid and the baryonic dust contribution. We are unable to discriminate between both contributions to $\rho$. This can be seen as follows: To take into account baryonic matter we have to add to the EMT (30) a dust contribution $T_{\mu\nu}^B$ ($\rho^B$ is the baryonic energy density)

$$T_{\mu\nu}^B = \rho^B u_\mu u_\nu,$$

(78)

which is separately covariantly conserved

$$\nabla_\rho T_{\mu\nu}^B g^{\rho\mu} = 0.$$

(79)

Then $\rho^B$ obeys the local energy conservation equation

$$\dot{\rho}^B + \rho^B \left( \frac{\dot{B}}{B} + 2 \frac{\dot{Y}}{Y} \right) = 0$$

(80)

and we have to add $\kappa \rho^B$ to the r.h.s. of the 1st Einstein-field eq. (38). So only the total energy density appears in (38). But the solution of the energy conservation eq. (70) for the darkon fluid is only fixed modulo a solution of the corresponding homogeneous eq. which is just given by (80).

We conclude that the fit to the experimental data for the Hubble function $H(z)$ and the corresponding determination of the values for the three constants $K_i (I = 1, 2, 3)$ presented in [10] contains besides the darkon fluid also the baryonic matter contribution (as regards other observational data we refer the reader to section 6.5 in [10]).
6 Non-comoving coordinates and modeling of halos

In this section we consider the darkon fluid moving in the radial direction relative to the cosmic rest system (CRS). The metric in the CRS is assumed to be given by Schwarzschild-like coordinates. We will

- derive the darkon fluid EOMs and the Einstein field equations by choosing the energy-frame (vanishing heat flux) for the CRS (see [31]),

- look for weak field solutions which arise to be equal to the nonrelativistic stationary solutions derived in [10] modeling halos.

6.1 Cosmic rest system (CRS)

Schwarzschild-like coordinates are defined by the spherically symmetric metric

\[
 ds^2 = -e^{2\phi(t,r)} dt^2 + e^{2\lambda(t,r)} dr^2 + r^2 d\Omega^2. \tag{81}
\]

This metric is assumed to be valid in the CRS defined by a time-like unit vector \( n^\mu \) and a space-like unit vector \( s^\mu \)

\[
 n^\mu \equiv (e^{-\phi}, 0), \quad s^\mu \equiv (0, e^{-\phi}) \tag{82}
\]

such that the CRS becomes the energy frame (vanishing heat flux) i.e. the EMT \((30)\) takes in the CRS frame the form

\[
 T^\mu{}^\nu = \rho^*(t,r)n^\mu n^\nu + p_r^*(t,r)s^\mu s^\nu, \tag{83}
\]

where \( \rho^* \), resp. \( p_r^* \) are the energy density, resp. the radial pressure in the CRS. Note that \((83)\) contains no transversal pressure \( p_t \) as \((30)\) is free of it (radial movement does not change \( p_t \)).

The darkon fluid is assumed to move with velocity \( v \) in the radial direction relative to the CRS. Then the four-velocity \( u^\mu \) resp. the vector \( \theta^\mu \) are given by

\[
 u^\mu = \gamma(n^\mu + vs^\mu), \quad \theta^\mu = \tilde{\theta}\gamma(vn^\mu + s^\mu), \tag{84}
\]
where \( \gamma(v) \equiv \left(1 - v^2\right)^{-\frac{1}{2}} \).

Comparing (30) with (83) and using (84) we obtain

\[
\rho^* = \frac{\rho}{1 + v^2} \quad \text{and} \quad p_r^* = -\rho^* v^2, \tag{85}
\]

where \( v(t, r) \) is determined by the requirement of the vanishing heat flux in the CRS

\[
\rho^* v - n\tilde{\theta} = 0. \tag{86}
\]

### 6.2 Einstein’s field equations

With the metric (81) and the EMT (83) we get for the Einstein-field equations (see [32] and the literature cited therein)

\[
\kappa \rho^* = \frac{1}{r^2} (r(1 - e^{-2\lambda}))', \tag{87}
\]

\[
\kappa p_r^* = \frac{1}{r^2} \left(-1 + e^{-2\lambda}(1 + 2r\phi')\right), \tag{88}
\]

\[
0 = \phi'' + \phi'^2 - \phi' \lambda' + \frac{\phi' - \lambda'}{r}, \tag{89}
\]

in which we had already used (90), and

\[
0 = \dot{\lambda}. \tag{90}
\]

Using (90) in (87) we get immediately

\[
\dot{\rho}^* = 0 \tag{91}
\]

### 6.3 Darkon fluid EOMs

From the darkon fluid EOMs (24–26) and (90) we get by using the metric (81) and eq. (84) for \( u^\mu \), resp. \( \theta^\mu \):

- the continuity eq. (1st eq. in (24)) becomes

\[
0 = e^{-\phi} \dot{n} + e^{-\lambda} \frac{(r^2 n v)'}{r^2} \tag{92}
\]
• the Euler eq. (2nd eq. in \([24]\)) becomes (cp. \([32]\), eq. (17))

\[
0 = e^{-\phi} \dot{\gamma}^2 + e^{-\lambda}(v v' \gamma^2 + \phi')
\]  

(93)

and

• by using \(q^\mu = q^{\gamma}(v n^\mu + s^\mu)\) we obtain from \((25)\)

\[
e^{-\phi} \gamma \dot{q} + \gamma e^{-\lambda}(v q' - q v') = \tilde{\theta},
\]  

(94)

where, due to \((26)\), \(\tilde{\theta}\) obeys the EOM

\[
e^{\lambda}(\tilde{\theta} \gamma) + (\tilde{\theta} \gamma ve^\phi)' = 0.
\]  

(95)

Finally we may express \(\rho^*\) defined by \((31)\) and \((85)\) in terms of the metric and the darkon fluid fields and we get

\[
\rho^* = \frac{\gamma}{r^2} e^{-\lambda} (r^2 q n)'.
\]  

(96)

Sometimes it is useful to use instead of the darkon fluid EOMs the EOMs for \(\rho^*\) resp. \(p^*_r\) which follow from the covariant conservation of the EMT

\[
\nabla_\mu T^{\mu\nu} = 0
\]  

(97)

We recall that \((97)\) can be derived either from the Bianchi identities for the Riemann tensor or directly from the darkon fluid EOMs. From the time-like part of \((97)\) we reproduce \((91)\) whereas the space-like part leads to the generalized Tolman-Oppenheimer-Volkoff (TOV) equation (see \([33]\)) which in our case takes the form

\[
(\rho^* + p^*_r)\phi' + 2 \frac{p^*_r}{r} + (p^*_r)' = 0.
\]  

(98)

Elimination of \(\tilde{\theta}\), a conservation law

By inserting \(\tilde{\theta}\) from \((86)\) into

• \((95)\) and using \((91)\), \((93)\) and \((98)\) we observe that \((95)\) is identically satisfied.
• (94) and using (96) for $\rho^*$ we obtain
\[ e^{-\phi} \dot{q} = e^{-\lambda} \frac{q}{nr^2} (vnr^2)' . \] (99)
Combining (92) with (99) leads to the conservation law
\[ (nq)' = 0 . \] (100)

6.4 Some exact relations

Here we derive some exact expressions which follow from the coupled system of Einstein field eq.s and darkon fluid EOMs.

By using (90), (91) and (100) we conclude from (96) that
\[ \dot{v} = 0 \] (101)
and therefore the 2nd eq. in (85) leads to
\[ \dot{p}_r^* = 0 , \] (102)
which, when used in the 2nd Einstein eq. (88) gives
\[ \dot{\phi}' = 0 . \] (103)

Eq. (103) can be easily integrated to give
\[ \phi(t, r) = \phi_0(r) + \phi_1(t) , \] (104)
where $\phi_0$ and $\phi_1$ are arbitrary functions of $r$ and $t$, respectively. Finally, by using (101) and (104), the Euler eq. (93) can be integrated to give
\[ \phi_0(r) = \frac{1}{2} \log(1 - v^2(r)) . \] (105)

6.5 Weak field limit for the Tolman-Oppenheimer-Volkoff (TOV) equation

As a weak field limit we understand a space-time described by a small perturbation of the Minkowski metric at sub-Hubble scales (cp. 34).
To be specific we follow the procedure of Green and Wald \cite{26} and put \((\epsilon \ll 1)\)

\[
\phi = O(\epsilon), \quad \lambda = O(\epsilon), \quad v = O(\epsilon^{\frac{1}{2}}), \quad \partial_r = O(\epsilon^{-1}), \quad \partial_t = O(\epsilon^{-\frac{1}{2}}).
\]

(106)

Then we obtain in leading order:

- From (98)
  \[
  \rho \phi'_0 - 2\frac{\rho v^2}{r} - (\rho v^2)' = 0.
  \]
  (107)

- From the 2nd, resp. 3rd Einstein eq. (88) resp. (89)
  \[
  \lambda(r) = r\phi'_0(r),
  \]
  (108)

which, when combined with the 1st Einstein eq. (87), leads to the Poisson eq.

\[
4\pi G\rho = \frac{1}{r^2} (r^2 \phi'_0)'.
\]

(109)

- From (105)
  \[
  \phi_0 = -\frac{1}{2} v^2.
  \]
  (110)

If we now insert (109) and (110) into (107) we obtain

\[
3(r^2 \phi'_0)' \phi'_0 + 2\phi_0 (r^2 \phi'_0)' = 0.
\]

(111)

Multiplying (111) by \((-2\phi_0)^{\frac{1}{2}}\) (integrating factor) we obtain

\[
\left((-2\phi_0)^{\frac{1}{2}} (r^2 \phi'_0)\right)' = 0,
\]

(112)

which, after integration, leads to the following nonlinear ordinary differential equation for the gravitational potential \((\beta = \text{const.})\)

\[
(r^2 \phi'_0)' = \frac{\beta}{2} (-2\phi_0)^{-\frac{3}{2}},
\]

(113)

which was derived in \cite{10} as the stationary solution of the spherically symmetric, nonrelativistic darkon fluid equations.
6.6 Modeling halos

In [10] we used the numerical solutions of (113) to determine the circular motion of a star in the potential $\phi_0$ given by the formula (see [35])

$$\frac{\dot{v}^2(r)}{r} = \phi'_0(r),$$

(114)

where $\dot{v}$ is the rotational velocity of the star. Thus, if all stars of a galaxy are in circular motion the graph of $\dot{v}$ gives the galactic rotation curve. We recall that the results reported in [10] are in qualitative agreement with observational data.

7 Final remarks

In this paper we have generalized our nonrelativistic darkon fluid model (NDFM), introduced in [8] and enlarged and reviewed in [10], to the framework of General Relativity. Our relativistic model contains, as is the case for the NDFM, no free parameters in its Lagrangian. This feature distinguishes our model, to the best of our knowledge, from all other models for dark energy resp. dark matter. The relativistic model reproduces, at sub-Hubble scales, the cosmological equations derived from the NDFM (section 5) and in the weak field limit the nonlinear differential equation satisfied by the gravitational potential for stationary solutions of the NDFM (section 6). We recall that the NDFM predicts qualitatively correct values of the late time cosmic acceleration as well as the flat behaviour of galactic rotation curves [8], [10]. Note that the derivation of already known results from the relativistic model has led to new insights resp. physical interpretations: our cosmological equations (background model) are different from the FLRW model. The cosmic acceleration is not attributed to a negative pressure (e.g. a positive cosmological constant) but it is due to a dynamically determined sign change of the energy density. This property turns out to be equivalent to a time-dependent, sign changing curvature. In this our relativistic model is very different from the model of dipolar dark matter and dark energy advocated.
by Blanchet and Tiec [36], [37]. These authors consider in [36] a relativistic action which, to some extent, is equivalent to ours but it differs mainly by the addition of an ad hoc internal force depending on the polarization field. This phenomenological internal force mimics a cosmological constant. Thus, their background model is the ΛCDM model which is completely different from our background model.

We have managed to derive the nonrelativistic gravitational potential produced by a galactic halo in our model from a weak field limit of the generalized Tolman-Oppenheimer-Volkoff (TOV) equation. But in contrast to the original application of TOV (hydrostatic equilibrium within a star) we have derived and applied the generalized TOV to a non-equilibrium situation given by non-comoving coordinates.

The main aim of the present paper was to present a general relativistic version of the NDFM and to look at its approximations which reproduce either the cosmological or the stationary solutions of the NDFM. This we have achieved but we are aware of the fact that further work on the consequences of the relativistic model is called for.

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