ASYMPTOTIC PROPERTIES OF STEADY PLANE SOLUTIONS OF THE NAVIER-STOKES EQUATIONS IN THE EXTERIOR OF A HALF-SPACE

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Abstract. Motivated by Gilbarg-Weinberger’s early work on asymptotic properties of steady plane solutions of the Navier-Stokes equations on a neighborhood of infinity [19], we investigate asymptotic properties of steady plane solutions of this system on a half-neighborhood of infinity with finite Dirichlet integral and Navier-slip boundary condition, and obtain that the velocity of the solution grows more slowly than \( \sqrt{\log r} \), while the pressure converges to 0 along each ray passing through the origin.

Keywords: Navier-Stokes equations, Liouville type theorem, asymptotic behavior, exterior of a half-space

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1. Introduction

The problem of a rigid body through a viscous liquid originates with the pioneering work of Stokes in 1851 [39] on the effect of internal friction on the movement of a pendulum in a liquid, which is modeled by the incompressible Navier-Stokes equations in a planar exterior domain (see also the introduction in [15]). When the influence of the boundary walls of the container was disregarded, in the fundamental work by Leray [30], where the solution of the two dimensional exterior problem for the Navier-Stokes equations was constructed with finite Dirichlet integral, it is remaining that whether the constructed solution satisfies the asymptotic behavior at \( \infty \). Gilbarg-Weinberger [19] described the asymptotic behavior of the velocity, the pressure and the vorticity, where they showed that 
\[
\lim_{r \to \infty} \int_0^{2\pi} |u(r, \theta) - \bar{u}|^2 d\theta = 0
\]
for some constant vector \( \bar{u} \). Later, Amick [1] proved that \( u \in L^\infty \) under zero boundary condition. Recently, Korobkov-Pileckas-Russo in [27] and [26] obtained that
\[
\lim_{|x| \to \infty} u(x) = \bar{u}.
\]

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More references on the existence and asymptotic behavior of solutions in an exterior domain, we refer to [2, 13, 14, 16, 17, 24, 28, 32–34] and the references therein. For the generalized q-energy of \(|\nabla u|_q < \infty\), asymptotic behavior of solutions or Liouville type theorems are considered in [29, 42, 43]. On the other hand, a body moving within an incompressible fluid parallel to a wall in an otherwise unbounded domain is modeled in a planar exterior domain in a half space with appropriate boundary conditions on the wall. Hillairet-Wittwer [22] obtained the existence and uniqueness of weak solution with the boundary condition at infinity by considering the Oseen model, which implies the faster decay at infinity, see also Boeckle-Wittwer [4], Guo-Witter-Zhou [21] for the recent progress.

In this paper, we investigate the asymptotic properties of Leray solution with finite Dirichlet integral without the infinite boundary condition. In details, consider an arbitrary solution \(\{w', p'\}\) of the following two-dimensional steady Navier-Stokes equations

\[
\begin{aligned}
\Delta w' - (w' \cdot \nabla) w' &= \nabla p', \\
\nabla \cdot w' &= 0
\end{aligned}
\]  

(1.1)

in the exterior of the upper half-space, \(\mathbb{R}^2_+ \setminus \overline{B_{r_0}}\) with \(r_0 > 0\), where \(w'(x, y) = (u'(x, y), v'(x, y))\) represents fluid velocity, \(p'(x, y)\) is the fluid pressure and \(B_{r_0}\) is a ball centered at 0.

There are several possibilities for the boundary condition on \(\Gamma = \{(x, y); y = 0, |x| > r_0\}\). The widely used is the following Dirichlet or no-slip boundary condition:

\[
w'(x, y) = 0 \quad \text{on } \partial B_1.
\]

At this case, Liouville type theorem was proved by Seregin in [35] for the ancient solutions of the time-dependent case. More Liouville type theorems are obtained by Guo-Wang in [20] by assuming different generalized weak solutions. We refer to [3, 23, 37, 41] and the references therein for some results on singularity or classification of solutions under this boundary condition. However, in the case where the obstacles have an approximate limit, the Dirichlet boundary conditions are no longer valid (see for example [36]). Due to the roughness of the boundary and the viscosity of the fluid, it is usually assumed that there is a stagnant fluid layer near the boundary, which allows the fluid to slip. This situation seems to match the reality. In 1827, Navier [12] considered the slip phenomena and proposed the Navier-slip boundary conditions:

\[
\begin{aligned}
\frac{\partial w'}{\partial n} + \alpha(x) \nabla w' &= 0, \\
2[D(w'(x, y)) \cdot n]_\tau + \alpha(x) w'_\tau &= 0,
\end{aligned}
\]

(1.2)

where \(D(w'(x, y))\) is the stress tensor of fluid, \(n\) and \(\tau\) are the unit outer normal vector and tangential vector of the boundary, \(\alpha(x)\) is a physical parameter, which is a \(L^\infty\) function on the boundary. For the flat boundary of \(\Gamma = \{(x, y); y = 0, |x| > r_0\}\), motivated by the above condition of (1.2), generally we try to consider the body
suspended in a linear shear flow as in [11] or [10], which showed the forces generated due to wall and shear effects in the absence of relative motion,

\[ v'(x,0) = b, \partial_2 u'(x,0) = a, \text{ on } \Gamma, \tag{1.3} \]

where the constants \( a, b \in \mathbb{R} \). Specially, let \( w = w' - (ay, b) = (u, v) \) and \( p = p' + abx \), then

\[
\begin{cases}
\Delta w - (w \cdot \nabla)w - (ay\partial_1 + b\partial_2)w - (av, 0) = \nabla p, \\
\nabla \cdot w = 0
\end{cases}
\tag{1.4}
\]

which satisfies the boundary condition:

\[ v(x,0) = 0, \partial_2 u(x,0) = 0, \text{ on } \Gamma. \tag{1.5} \]

Let \( \omega = u_y - v_x \) be the vorticity of a velocity vector \( w = (u, v) \) satisfying (1.4). Then the vorticity equation of (1.4) follows:

\[ \Delta \omega - w \cdot \nabla \omega - (ay\partial_1 + b\partial_2)\omega = 0, \text{ in } \Omega_0, \tag{1.6} \]

and \( \omega = 0 \) on \( \Gamma \) due to (1.5), where

\[ \Omega_0 = \{(r, \theta), r > r_0, 0 < \theta < \pi\} = \mathbb{R}_+^2 \setminus \overline{B_{r_0}}, \quad r_0 > 0, \]

by assuming \( r = \sqrt{x^2 + y^2} \) with \( x = r \cos \theta \) and \( y = r \sin \theta \).

Our main results are as follows. The first result is about the growth estimate of the velocity.

**Theorem 1.1.** Let \( w \in C^2(\Omega_0) \) be a solution of the Navier-Stokes equations (1.4) with the Navier-slip boundary condition (1.5) and

\[ \int_{\Omega_0} |\nabla w|^2 dx dy < \infty, \tag{1.7} \]

then

\[ \lim_{r \to \infty} \frac{|w(r, \theta)|}{\sqrt{\log r}} = 0. \tag{1.8} \]

**Remark 1.1.** The growth rate in this case is similar as that in [19]. However, Cauchy integral formula in [19] doesn’t seem to work in this case due to the boundary effect. We used point-wise behavior theorem in [13] by proving the \( L^q \) estimate of approximating vorticity functions with \( q > 2 \). It is still unknown whether the convergence of the velocity holds in the \( L^\infty \) sense as Korobkov-Pileckas-Russo in [27] and [26]. The condition of \( w \in C^2(\Omega_0) \) can be relaxed to \( w \in C^2_{\text{loc}}(\Gamma) \), since the solution with Dirichlet energy (1.7) is regular in the interior.

For the special half-plane, the Liouville type theorem also holds similar to the whole space.
**Theorem 1.2.** Let \( \{w, p\} \) be a solution of the Navier-Stokes equations (1.4) with the Navier-slip boundary condition (1.5) defined over the upper half space \( \mathbb{R}^2_+ \). Assume \( w \in C^2(\mathbb{R}^2_+) \) and
\[
\int_{\mathbb{R}^2_+} |\nabla w|^2 \, dx \, dy < \infty.
\]
(1.9)
Then \( w \) and \( p \) are constants in \( \mathbb{R}^2_+ \).

**Remark 1.2.** It’s proved by using the monotonicity property of \( G(r) = \frac{1}{2} \int_0^\pi |w(r, \theta)|^2 \, d\theta \) and the vanishing vorticity. When the Navier-slip boundary condition (1.5) is replaced by the no-slip condition, which seems to be more difficult than the high dimensional case in \( \mathbb{R}^3 \) or \( \mathbb{R}^3_+ \) with the no-slip boundary condition, which is an open problem as stated in [15] and [40]. We refer to Galdi [15], Chae [8] and Seregin [36], where some Liouville type theorems are proved by assuming that \( u \in L^{9/2}(\mathbb{R}^3) \), \( \Delta u \in L^{6/5}(\mathbb{R}^3) \) and \( u \in \text{BMO}^{-1}(\mathbb{R}^3) \), respectively. For more related discussion, we refer to [7, 9, 25, 41] and the references therein.

Another major feature of this paper lies in the estimates of the pressure, and we obtained the following results by two different approaches.

**Theorem 1.3.** Let \( \{w, p\} \) be a solution of the Navier-Stokes equations (1.4) with the Navier-slip boundary condition (1.5). Assume that \( w \in C^2(\Omega_0) \), (1.7) hold, \( a = 0 \) and \( bu(r, 0) = bu(r, \pi) \) for \( r > r_0 \). Then there hold

(i) \[
|p(r, \theta)| = o(\log r),
\]
uniformly for all \( \theta \in [0, \pi] \).

(ii) up to a constant
\[
\lim_{r \to \infty, \theta = \theta_0} p(r, \theta) = 0,
\]
where \( \theta_0 \in [0, \pi] \).

**Remark 1.3.** The pressure’s decay is not better than the exterior domain of \( \mathbb{R}^2 \). The main obstacle comes from the boundary effect. Case (i) of Theorem 1.3 is proved by the help of Brezis-Gallouet inequality, and the proof of Case (ii) in Theorem 1.3 is following the route of [19] but using Green representation formula of harmonic function on the half-annulus domain.

**Remark 1.4.** Combining the two results of Theorem 1.3, it seems to be unknown that whether \( p \) converges to zero at infinity along all the rays uniformly. For example, Theorem 1.3 doesn’t work when \( \theta = \frac{1}{r} \), \( r = K \) and \( p = p(K, \frac{1}{r}) = 1 \), where \( K \in \mathbb{N} \).

At last, we also obtain the decay estimate of the vorticity.
Theorem 1.4. Let \( \{w, p\} \) be a solution of the Navier-Stokes equations (1.4) with the Navier-slip boundary condition (1.5). Assume that \( w \in C^2(\Omega_0), \) (1.7) hold and \( a = 0. \) Then
\[
\lim_{r \to \infty} \frac{r^{\frac{3}{4}}}{(\log r)^{\frac{3}{8}}} |\omega(r, \theta)| = 0,
\]
where \( \theta \in [0, \pi] \).

The paper is organized as follows, in Section 2, we recall and introduce some technical lemmas. In Section 3, we are aimed to the proof of Theorem 1.1, which is the decay estimate of the velocity. Theorem 1.2, Theorem 1.3 and Theorem 1.4 are proved in Section 4—Section 6, respectively.

Throughout this article, \( C \) denotes a constant only depending on the known finite norms of solutions (e.g. \( \|\nabla w\|_{L^2} \)) and may be different from line to line.

2. Preliminary lemmas

The following lemma is similar to the exterior domain \( B_{r_0}^c \) of Lemma 2.1 in [19], one could make the same arguments as [19] in such an exterior angular region. In fact, one may also do an extension to the whole space by noting the following \( C^{0,1} \) is enough and we omitted it.

Lemma 2.1. Let \( f \in C^1(\Omega_0) \) and have finite Dirichlet integral
\[
\int_{\Omega_0} |\nabla f|^2 dxdy < \infty.
\]
Then
\[
\lim_{r \to \infty} \frac{1}{\log r} \int_0^\pi f(r, \theta)^2 d\theta = 0.
\]

It immediately follows that a special subsequence is pointwise convergent, which is also the same as Lemma 2.2 in [19] and is stated as follows.

Lemma 2.2. Let \( f = (f_1, f_2) \), where \( f_1 \) and \( f_2 \) satisfy the hypotheses of Lemma 2.1. Then there is a sequence \( r_n, r_n \in (2^n, 2^{n+1}) \), such that
\[
\lim_{n \to \infty} \frac{|f(r_n, \theta)|^2}{\log r_n} = 0,
\]
where \( \theta \in [0, \pi] \).

As in [19], we can also obtain the bounded-ness of the \( L^2 \) norm of \( \nabla \omega \).

Lemma 2.3. Let \( w = (u, v) \) satisfy the Navier-Stokes equations (1.4) and Navier-slip boundary condition (1.5) in \( \Omega_0 \). Assume \( w \in C^2(\Omega_0) \) and have finite Dirichlet integral of (1.7). Then
\[
\int_{\Omega_0} |\nabla \omega|^2 dxdy < \infty.
\]
Proof. For a large number $R_0 > 0$, choose $r_0 < r_1 < \frac{R_0}{2} < \rho < R \leq R_0$ and a non-negative $C^2$ cut-off function $\eta(r)$ such that

$$\eta(r) = \begin{cases} 1, & r_1 < r < \rho, \\ 0, & r \leq r_0, r \geq R, \end{cases} \quad (2.5)$$

$$|\nabla^k \eta| \leq \frac{C}{(r_1 - r_0)^k}, \text{ as } r_0 < r < r_1,$$

and

$$|\nabla^k \eta| \leq \frac{C}{(R - \rho)^k}, \text{ as } \rho < r < R,$$

for $k = 1, 2$. Let $h(\omega) = \omega^2$ and $B_+^r = B_+^r(0)$ stands for the upper ball centered at 0 and of radius $r$. Then by (1.4) and (1.6) we have

$$\operatorname{div}[\eta^6 \nabla h(\omega) - h(\omega)\nabla(\eta^6) - \eta^6 h(\omega)w] = 2\eta^6 |\nabla \omega|^2 - h(\omega)[\Delta(\eta^6) + w \cdot \nabla(\eta^6)] + 2\eta^6 \omega(ay\partial_1 + b\partial_2)\omega.$$

Integration by parts over $B_R^+ \setminus B_{r_0}^+$ yields that

$$\int_{B_R^+ \setminus B_{r_0}^+} \eta^6 |\nabla \omega|^2 \, dxdy = \frac{1}{2} \int_{B_R^+ \setminus B_{r_0}^+} \omega^2(\Delta \eta^6 + w \cdot \nabla \eta^6) \, dxdy + \frac{1}{2} \int_{B_R^+ \setminus B_{r_0}^+} \nabla \eta^6 \cdot (ay, b)\omega^2 \, dxdy. \quad (2.6)$$

Note that $\eta = 1$ for $r_1 < r < \rho$ and (2.6) implies that

$$\int_{B_R^+ \setminus B_{r_0}^+} |\nabla \omega|^2 \, dxdy \leq \int_{B_R^+ \setminus B_{r_0}^+} \eta^6 |\nabla \omega|^2 \, dxdy \quad (2.7)$$

$$\leq \frac{1}{2} \int_{B_R^+ \setminus B_{r_0}^+} \omega^2(\Delta \eta^6 + w \cdot \nabla \eta^6) \, dxdy + \frac{1}{2} \int_{B_R^+ \setminus B_{r_0}^+} \nabla \eta^6 \cdot (ay, b)\omega^2 \, dxdy$$

$$+ C(r_1, \|w\|_{L^\infty(B_{r_1}^+ \setminus B_{r_0}^+)}, \|\nabla \omega\|_{L^\infty(B_{r_1}^+ \setminus B_{r_0}^+)},) \quad (2.8)$$

where we used

$$\int_{B_{r_1}^+ \setminus B_{r_0}^+} \omega^2(\Delta \eta^6 + w \cdot \nabla \eta^6) \, dxdy + \int_{B_{r_1}^+ \setminus B_{r_0}^+} \nabla \eta^6 \cdot (ay, b)\omega^2 \, dxdy \leq C(r_1, \|w\|_{L^\infty(B_{r_1}^+ \setminus B_{r_0}^+)}, \|\nabla \omega\|_{L^\infty(B_{r_1}^+ \setminus B_{r_0}^+)}).$$

First, we consider the second part of the right integral in (2.7)

$$\frac{1}{2} \int_{B_R^+ \setminus B_{r_0}^+} \nabla \eta^6 \cdot (ay, b)\omega^2 \, dxdy \leq \frac{CR}{R - \rho} \int_{B_R^+ \setminus B_{r_0}^+} |\nabla \omega|^2 \, dxdy \leq \frac{CR}{R - \rho}. \quad (2.9)$$
The first part of the right integral over the upper half annulus $B_{\bar{R}}^+ \setminus B_{\rho}^+$ in (2.7) is controlled by

$$
\left| \int_{B_{\bar{R}}^+ \setminus B_{\rho}^+} \omega^2 \Delta \eta^6 \, dx \, dy \right| + \left| \int_{B_{\bar{R}}^+ \setminus B_{\rho}^+} \omega^2 (\mathbf{w} - \bar{\mathbf{w}}) \cdot \nabla \eta^6 \, dx \, dy \right|
+ \left| \int_{B_{\bar{R}}^+ \setminus B_{\rho}^+} \omega^2 \bar{\mathbf{w}} \cdot \nabla \eta^6 \, dx \, dy \right| \doteq T_1 + T_2 + T_3,
$$

where

$$
\bar{\mathbf{w}}(r) = \frac{1}{\pi} \int_0^\pi \mathbf{w}(r, \theta) \, d\theta.
$$

For $T_1$, we get

$$
T_1 \leq \frac{C}{(R - \rho)^2} \int_{B_{\bar{R}}^+ \setminus B_{\rho}^+} \omega^2 \, dx \, dy \leq \frac{C}{(R - \rho)^2} \int_{B_{\bar{R}}^+ \setminus B_{\rho}^+} |\nabla \mathbf{w}|^2 \, dx \, dy \leq \frac{C}{(R - \rho)^2}. \tag{2.10}
$$

For $T_2$, by Schwarz’s inequality there holds

$$
T_2 \leq 6 \int_{B_{\bar{R}}^+ \setminus B_{\rho}^+} |\mathbf{w} - \bar{\mathbf{w}}||\nabla \eta||\eta^5|\omega^2 \, dx \, dy
\leq \frac{C}{R - \rho} \left( \int_\rho^R \int_0^\pi |\mathbf{w} - \bar{\mathbf{w}}|^2 \, d\theta \, dr \right)^{\frac{1}{2}} \left( \int_{B_{\bar{R}}^+ \setminus B_{\rho}^+} \eta^{10} \omega^4 \, dx \, dy \right)^{\frac{1}{2}}.
$$

Using Wirtinger’s inequality

$$
\int_0^\pi |\mathbf{w} - \bar{\mathbf{w}}|^2 \, d\theta \leq \int_0^\pi |\mathbf{w}_\theta|^2 \, d\theta,
$$

one can obtain that

$$
\left( \int_\rho^R \int_0^\pi |\mathbf{w} - \bar{\mathbf{w}}|^2 \, d\theta \, dr \right)^{\frac{1}{2}} \leq \left( \int_\rho^R \int_0^\pi |\mathbf{w}_\theta|^2 \, d\theta \, dr \right)^{\frac{1}{2}} \leq R \left( \int_{B_{\bar{R}}^+ \setminus B_{\rho}^+} |\nabla \mathbf{w}|^2 \, dx \, dy \right)^{\frac{1}{2}} \leq CR.
$$

Moreover, since $\mathbf{w} \in C^2(\overline{\Omega}_0)$, we know that $\nabla \omega \in C(\overline{\Omega}_0)$. It follows that

$$
\int_{B_{\bar{R}}^+ \setminus B_{\rho}^+} |\nabla \omega|^2 \, dx \, dy \leq C(\|\nabla^2 \mathbf{w}\|_{L^\infty(B_{\bar{R}}^+ \setminus B_{\rho}^+)})
$$

Then by Gagliardo-Nirenberg inequality [31], we have

$$
\left( \int_{B_{\bar{R}}^+ \setminus B_{\rho}^+} \eta^{10} \omega^4 \, dx \, dy \right)^{\frac{1}{2}}
\leq C\|\eta^2 \omega\|_{L^2(B_{\bar{R}}^+)} \|\nabla (\eta^2 \omega)\|_{L^2(B_{\bar{R}}^+)}
\leq C\|\eta^2 \omega\|_{L^2(B_{\bar{R}}^+)} \|\nabla \eta^2\omega + \eta^2(\nabla \omega)\|_{L^2(B_{\bar{R}}^+)}
$$
\[
\leq C \| \omega \|_{L^2(B_R^+ \setminus B_0^+)} \left( C + \frac{C}{R - \rho} + \| \nabla \omega \|_{L^2(B_R^+ \setminus B_0^+)} + \| \nabla \omega \|_{L^2(B_{R/2}^+ \setminus B_0^+)} \right)
\]
\[
\leq C (\| \nabla^2 w \|_{L^\infty(B_{R/2}^+ \setminus B_0^+)} ) \left( 1 + \frac{1}{R - \rho} + \| \nabla \omega \|_{L^2(B_{R/2}^+ \setminus B_0^+)} \right).
\]

Hence we obtain the estimate
\[
T_2 \leq C (\| \nabla^2 w \|_{L^\infty(B_{R/2}^+ \setminus B_0^+)} ) \frac{R}{R - \rho} \left( 1 + \frac{1}{R - \rho} + \left( \int_{B_{R/2}^+ \setminus B_0^+} |\nabla \omega|^2 dxdy \right)^{\frac{1}{2}} \right)^{\frac{3}{2}}. \tag{2.11}
\]

For \(T_3\), it follows from Lemma 2.1 that
\[
\bar{w}(r) = o \left( \sqrt{\log r} \right),
\]
and hence
\[
T_3 \leq \int_{B_{R/2}^+ \setminus B_0^+} |\bar{w}| |\nabla \eta^6| \omega^2 dxdy \leq \frac{C (\log R)^{\frac{3}{2}}}{R - \rho}. \tag{2.12}
\]

Combining (2.7)-(2.12), by Young’s inequality we get
\[
\int_{B_{R/2}^+ \setminus B_0^+} |\nabla \omega|^2 dxdy \leq \frac{1}{2} \int_{B_{R/2}^+ \setminus B_0^+} |\nabla \omega|^2 dxdy + C (\| w \|_{C^2(B_{R/2}^+ \setminus B_0^+)} ) \left( \frac{R^2}{(R - \rho)^2} + \frac{(\log R)^{\frac{3}{2}}}{R - \rho} + 1 \right).
\]

Using Giaquinta iteration in [18], we conclude that
\[
\int_{B_{R/2}^+ \setminus B_0^+} |\nabla \omega|^2 dxdy \leq C (\| w \|_{C^2(B_{R/2}^+ \setminus B_0^+)} ) \left( \frac{R^2}{(R - \rho)^2} + 1 \right).
\]

Take \(\rho = \frac{R_0}{2}\) and \(R = R_0\), and letting \(R_0 \to \infty\), the proof is complete.

**Lemma 2.4.** Under the hypotheses of Lemma 2.3 we have
\[
\lim_{r \to \infty} r^2 |\omega(r, \theta)| = 0, \tag{2.13}
\]
where \(\theta \in [0, \pi]\).

**Proof.** Using the polar coordinate transformation and Cauchy inequality, we obtain
\[
\int_{2^n}^{2^{n+1}} \frac{dr}{r} \int_0^\pi (r^2 \omega^2 + 2r |\omega \omega_\theta|) d\theta \leq \int_{r > 2^n, 0 < \theta < \pi} (2\omega^2 + |\nabla \omega|^2) dxdy.
\]

Hence by the integral theorem of the mean, there exists an \(r_n \in (2^n, 2^{n+1})\) such that
\[
\int_0^\pi \left[ r_n^2 \omega(r_n, \theta)^2 + 2r_n |\omega(r_n, \theta)\omega_\theta(r_n, \theta)| \right] d\theta \leq \frac{1}{\log 2} \int_{r > 2^n, 0 < \theta < \pi} (2\omega^2 + |\nabla \omega|^2) dxdy. \tag{2.14}
\]
Note that
\[ \omega(r_n, \theta)^2 - \frac{1}{\pi} \int_0^\pi \omega(r_n, \theta)^2 d\theta \leq 2 \int_0^\pi |\omega(r_n, \theta)\omega_\theta(r_n, \theta)| d\theta. \]

It follows from (2.14) and Lemma 2.3 that
\[ 0 \leq r_n \omega(r_n, \theta)^2 \leq \int_0^\pi \left[ r_n^2 \omega(r_n, \theta)^2 + 2 r_n |\omega(r_n, \theta)\omega_\theta(r_n, \theta)| \right] d\theta \leq \frac{1}{\log 2} \int_{r>2^n, 0<\theta<\pi} (2\omega^2 + |\nabla \omega|^2) dxdy \to 0, \ n \to \infty, \]
which implies
\[ \lim_{n \to \infty} \left[ r_n \max_{\theta \in [0, \pi]} \omega(r_n, \theta)^2 \right] = 0. \tag{2.15} \]

By (1.5), we have \( \omega(x, 0) = 0 \) on \( \Gamma \). Let \( D \) denote the half-annulus region with \( r_n < r < r_{n+1} \). Since \( \omega \) a solution of the equation (1.6), it satisfies the maximum principle in \( D \). Noting that \( r_{n+1} \leq 4r_n \), we infer that for \( r \in (r_n, r_{n+1}) \)
\[ r \max_{\theta \in [0, \pi]} \omega(r, \theta)^2 \leq r_{n+1} \max_{\theta \in [0, \pi]} \omega(r, \theta)^2 \leq r_{n+1} \max_{\theta \in [0, \pi]} \omega(r_n, \theta)^2, \]
which implies the desired result (2.13) due to (2.15).

At last, we introduce the Brezis-Gallouet inequality (see Lemma 2 in [5], or Lemma 3.1 in [6]).

**Lemma 2.5.** Let \( f \in H^2(\Omega) \), where \( \Omega \) is a bounded domain or an exterior domain with compact smooth boundary. Then there exists a constant \( C_\Omega \) depending only on \( \Omega \), such that
\[
\| f \|_{L^\infty(\Omega)} \leq C_\Omega \| f \|_{H^1(\Omega)} \log \left( e + \frac{\| \Delta f \|_{L^2(\Omega)}}{\| f \|_{H^1(\Omega)}} \right),
\]
or
\[
\| f \|_{L^\infty(\Omega)} \leq C_\Omega (1 + \| f \|_{H^1(\Omega)}) \log \left( e + \| f \|_{L^2(\Omega)} \right).
\]

Note that the second inequality can be obtained immediately from the first one by arguments whether \( \| f \|_{H^1(\Omega)} < 1 \).

The following lemma is from Theorem II.9.1 in [15].

**Lemma 2.6.** Let \( \Omega \subset \mathbb{R}^2 \) be an exterior domain and let
\[ \nabla f \in L^2 \cap L^p(\Omega), \]
for some \( 2 < p < \infty \). Then
\[ \lim_{|x| \to \infty} \frac{|f(x)|}{\sqrt{\log(|x|)}} = 0, \]
uniformly.

3. Decay of the velocity

Proof of Theorem 1.1. For \( r = \sqrt{x^2 + y^2} \), we take two cut-off functions \( \varphi(x, y) \) and \( \eta(x, y) \) as follows:

\[
\varphi(r) = \begin{cases} 
1, & r < R \\
0, & r > 2R
\end{cases}, \quad \eta(r) = \begin{cases} 
1, & r > 3r_0 \\
0, & r < 2r_0
\end{cases}
\]

Firstly, we show that

\[
\omega \in L^p(\Omega_0), \quad \forall p \geq 2. \tag{3.1}
\]

Noting that \( |\nabla \eta| \leq C \) and \( |\nabla \varphi| \leq \frac{C}{R} \) for a constant \( C \) independent of \( R \). Then using (2.4), (1.7) and Gagliardo-Nirenberg inequality, for any \( p \geq 2 \), we have

\[
\|\omega \varphi \eta\|_{L^p(\mathbb{R}^2_+)} \leq C \|\omega \varphi \eta\|_{L^2(\mathbb{R}^2_+)}^{\frac{2}{p}} \|\nabla (\omega \varphi \eta)\|_{L^2(\mathbb{R}^2_+)}^{1-\frac{2}{p}}
\]

\[
\leq C \|\omega\|_{L^2(\mathbb{R}^2_+ \setminus B_{2r_0}^+)} \left( \|\nabla \omega\|_{L^2(\mathbb{R}^2_+ \setminus B_{2r_0}^+)} + \|\nabla \varphi\|_{L^2(\mathbb{R}^2_+ \setminus B_{2r_0}^+)} + \|\nabla \eta\|_{L^2(\mathbb{R}^2_+ \setminus B_{2r_0}^+)} \right)
\]

\[
\leq C \left( C + \frac{C}{R} \right)
\]

Since \( \varphi = 1 \) as \( R \to \infty \). Hence let \( R \to \infty \), we get

\[
\|\omega\|_{L^p(\mathbb{R}^2_+ \setminus B_{3r_0}^+)} \leq C.
\]

Moreover, due to \( w \in C^2(\overline{\Omega_0}) \), there holds \( \omega \in C^1(\overline{\Omega_0}) \). Then there holds

\[
\|\omega\|_{L^p(B_{3r_0}^+ \setminus B_{r_0}^+)} \leq C.
\]

Thence (3.1) holds.

Secondly, set \( \dot{w}(x, y) = (\dot{u}(x, y), \dot{v}(x, y)) \), where

\[
\dot{u}(x, y) = \begin{cases} 
u(x, y), & y \geq 0 \\
u(x, -y), & y < 0
\end{cases}, \quad \dot{v}(x, y) = \begin{cases} v(x, y), & y \geq 0 \\
v(x, -y), & y < 0
\end{cases}
\]

Then

\[
\dot{w} \in C^{0,1}_{\text{loc}}(\mathbb{R}^2 \setminus B_{r_0}),
\]

and

\[
\dot{w} = w, \quad \text{in } \Omega_0.
\]

Moreover

\[
\ddot{w}(x, y) = \begin{cases} \partial_y u(x, y) - \partial_x v(x, y), & y \geq 0 \\
-(\partial_y u)(x, -y) + (\partial_x v)(x, -y), & y < 0
\end{cases}
\]

Let \( \varphi \) is standard mollifier operator, \( \varepsilon > 0 \), set \( \varphi_\varepsilon = \varepsilon^{-2} \varphi(\frac{x}{\varepsilon}) \). Thus

\[
\varphi_\varepsilon \ast \dot{w} = \dot{w}_\varepsilon \in C^\infty(\mathbb{R}^2 \setminus B_{2r_0}).
\]
For the cut-off function \( \phi = 1 \) outside \( B_{3r_0} \) and \( \phi = 0 \) in \( B_{2r_0} \), noting that
\[
\nabla \perp \cdot (\hat{\omega}_\varepsilon \phi) = \hat{\omega}_\varepsilon + \hat{w}_\varepsilon \cdot \nabla \perp \phi,
\]
where \( \hat{\omega}_\varepsilon = \nabla \perp \cdot \hat{w}_\varepsilon \).
Putting the operation \( \nabla \perp \) into the both side of the above equation, we have
\[
\Delta (\hat{\omega}_\varepsilon \phi) = \nabla \perp (\hat{\omega}_\varepsilon \phi + \hat{w}_\varepsilon \cdot \nabla \perp \phi) + \nabla \cdot (\hat{\omega}_\varepsilon \phi).
\]
Noting that \( \Delta \phi = -\nabla \cdot (\hat{\omega}_\varepsilon \phi) \), we get
\[
\hat{\omega}_\varepsilon = \nabla \perp \cdot \hat{w}_\varepsilon.
\]
Noting that \( \nabla \cdot \hat{w}_\varepsilon = 0 \) by triangle inequality, we have
\[
\Delta (\hat{\omega}_\varepsilon \phi) = \nabla \perp (\hat{\omega}_\varepsilon \phi + \hat{w}_\varepsilon \cdot \nabla \perp \phi) + \nabla \cdot (\hat{\omega}_\varepsilon \phi).
\]
Thus with the help of Lemma 2.6, we have
\[
\nabla (\hat{\omega}_\varepsilon \phi) = \nabla \hat{\omega}_\varepsilon \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty,
\]
by Calderon-Zygmund estimates, we have
\[
\| \nabla (\hat{\omega}_\varepsilon \phi) \|_{L^p(\mathbb{R}^2)} \leq C \| \hat{\omega}_\varepsilon \phi + \hat{w}_\varepsilon \cdot \nabla \perp \phi \|_{L^p(\mathbb{R}^2)} + C \| \nabla \cdot (\hat{\omega}_\varepsilon \phi) \|_{L^p(\mathbb{R}^2)}, \quad p > 1.
\]
Noting that \( \nabla \cdot \hat{w}_\varepsilon = 0 \), we have
\[
\| \nabla \hat{\omega}_\varepsilon \|_{L^p(\mathbb{R}^2) / \mathcal{B}_{2r_0}} \leq C \| \hat{\omega}_\varepsilon \phi \|_{L^p(\mathbb{R}^2)} + C \| \hat{\omega}_\varepsilon \phi \|_{L^p(\mathbb{R}^2)}.
\]
Since \( \hat{w}_\varepsilon \in C^{0,1}(\mathbb{R}^2 \setminus B_{2r_0}) \) uniformly, there holds \( \| \hat{w}_\varepsilon \|_{L^p(\mathbb{R}^2 \setminus B_{2r_0})} \leq C \). Using the definition of \( \phi \), we have
\[
\| \nabla \hat{\omega}_\varepsilon \|_{L^p(\mathbb{R}^2 \setminus B_{2r_0})} \leq C + C \| \hat{\omega}_\varepsilon \phi \|_{L^p(\mathbb{R}^2)} \leq C + \| \hat{\omega} \|_{L^p(\mathbb{R}^2 \setminus B_{2r_0})}.
\]
Let \( \varepsilon \rightarrow 0 \), by Lebesgue's dominated convergence theorem and (3.1), we have
\[
\| \nabla \hat{\omega} \|_{L^p(\mathbb{R}^2 \setminus B_{2r_0})} \leq C + C \| \hat{\omega} \|_{L^p(\mathbb{R}^2 \setminus B_{2r_0})} \leq C + C \| \omega \|_{L^p(\mathbb{R}^2 \setminus B_{2r_0})} \leq C, \quad \forall \ p \geq 2.
\]

4. Liouville type theorem in the half-plane

**Proof of Theorem 1.2.** In the case of \( \mathbb{R}^2_+ \), it follows from Lemma 2.4 that \( \omega \rightarrow 0 \) at infinity. By (1.5), \( \omega(x,0) = 0 \). Hence \( \omega \equiv 0 \) in \( \mathbb{R}^2_+ \) due to the maximum principle. Noting that \( \Delta u = \omega_y \) and \( \Delta v = -\omega_x \), there holds \( \Delta w = (\Delta u, \Delta v) = 0 \) in \( \mathbb{R}^2_+ \).

Recall \( B^+_r = \left\{ \sqrt{x^2 + y^2} < r : y > 0 \right\} \) and \( \partial B^+_r = \Gamma_1 + \Gamma_2 \), where
\[
\Gamma_1 = \left\{ (x, y) : -r \leq x \leq r, y = 0 \right\}, \quad \Gamma_2 = \left\{ (x, y) : \sqrt{x^2 + y^2} = r, y > 0 \right\}.
\]
Due to (1.3), we get
\[
\int_{\Gamma_1} n \cdot \nabla w \cdot wd\sigma = -\int_{-r}^r \frac{\partial}{\partial y} u(x, 0) \cdot u(x, 0) dx = 0.
\]
Then by the integration by parts formula, we have
\[
0 = \int_{B^+_r} -\Delta w \cdot w \, dxdy
- \int_{\Gamma_1 + \Gamma_2} \mathbf{n} \cdot \nabla w \cdot w \, ds
= \int_{B^+_r} |\nabla w|^2 \, dxdy - \frac{1}{2} \int_0^\pi \frac{\partial}{\partial r} [w(r, \theta)|^2] d\theta \cdot r,
\]
where \( \mathbf{n} \) denotes outward unit normal to \( \partial B^+_r \). Write
\[
G(r) = \frac{1}{2} \int_0^\pi |w(r, \theta)|^2 d\theta,
\]
and
\[
C_0 = \int_{B^+_{r_0}} |\nabla w|^2 \, dxdy,
\]
where \( r > r_0 \). Then
\[
rG'(r) \geq C_0
\]
for any \( r > r_0 \). Integration it over \( (r_0, r) \), we have
\[
G(r) \geq G(r_0) + C_0 \log \frac{r}{r_0}.
\]
Lemma 2.1 shows that \( G(r) = o(\log r) \), hence
\[
C_0 = \int_{B^+_{r_0}} |\nabla w|^2 \, dxdy = 0.
\]
Letting \( r_0 \to \infty \), it follows that \( \nabla w = 0 \) in \( \mathbb{R}^2_+ \) and \( w \) is constant. The proof is complete.

5. Asymptotic behavior of the pressure

Since \( \Delta u = \omega_y \) and \( \Delta v = -\omega_x \), the Navier-Stokes equations can be written in the form
\[
\begin{cases}
\omega_y + (u + ay)v_y - (v + b)u_y - av = p_x, \\
-\omega_x - (u + ay)v_x + (v + b)u_x = p_y,
\end{cases}
\]
\[ u_x + v_y = 0. \tag{5.1} \]
Specially, let \( a = 0 \), we get
\[
\begin{cases}
\omega_y + uv_y - (v + b)u_y = p_x, \\
-\omega_x - uv_x + (v + b)u_x = p_y,
\end{cases}
\]
\[ u_x + v_y = 0. \tag{5.2} \]
it follows that
\[
p_r = \frac{1}{r} [\omega_\theta + uv_\theta - (v + b)u_\theta]. \tag{5.3} \]
The first lemma is about the convergence of the square norm for a subsequence, which is similar as in [19].

**Lemma 5.1.** Let \( \{w, p\} \) be a solution of the Navier-Stokes equations (1.4) with the Navier-slip boundary condition (1.5). Assume that \( w \in C^2(\Omega_0) \), (1.7) hold and \( a = 0 \), there exists a sequence \( \{R_n\} \), \( R_n \in (2^{2n}, 2^{2n+1}) \), such that

\[
\lim_{n \to \infty} \int_0^\pi |p(R_n, \theta) - \bar{p}(R_n)|^2 d\theta = 0. \tag{5.4}
\]

**Proof.** We first show that for any \( r_1 > \max(r_0, 1) \)

\[
\int_{r>r_1, 0<\theta<\pi} \frac{|\nabla p|^2}{\log r} dxdy < \infty.
\]

Using (5.2) and Cauchy inequality

\[
|\nabla p|^2 = p_x^2 + p_y^2 = [\omega_y + uv_y - (v + b)u_y]^2 + [-\omega_x - uv_x + (v + b)u_x]^2 
\leq 4|\nabla \omega|^2 + 8u^2|\nabla v|^2 + 4(2v^2 + 2b^2)|\nabla u|^2 
\leq 4|\nabla \omega|^2 + 16|w|^2|\nabla w|^2 + 8b^2|\nabla w|^2.
\]

From (1.8) in Theorem 1.1, we know \( |w(r, \theta)|^2 = o(\log r) \), thus

\[
\int_{r>r_1, 0<\theta<\pi} \frac{|\nabla p|^2}{\log r} dxdy 
\leq 4 \int_{r>r_1, 0<\theta<\pi} \frac{|\nabla \omega|^2}{\log r} dxdy + 16 \int_{r>r_1, 0<\theta<\pi} \frac{|w|^2}{\log r} |\nabla w|^2 dxdy + 8b^2 \int_{r>r_1, 0<\theta<\pi} \frac{|\nabla w|^2}{\log r} dxdy 
\leq \frac{4}{\log r_1} \int_{r>r_1, 0<\theta<\pi} |\nabla \omega|^2 dxdy + C \int_{r>r_1, 0<\theta<\pi} |\nabla w|^2 dxdy + \frac{8b^2}{\log r_1} \int_{r>r_1, 0<\theta<\pi} |\nabla w|^2 dxdy 
\leq C \left( \int_{r>r_1, 0<\theta<\pi} |\nabla \omega|^2 dxdy + \int_{r>r_1, 0<\theta<\pi} |\nabla w|^2 dxdy \right).
\]

It follows from (2.4) in Lemma 2.3 and (1.7) that

\[
\int_{r>r_1, 0<\theta<\pi} \frac{|\nabla p|^2}{\log r} < \infty.
\]
for any $r_1 > \max(r_0, 1)$. By the integral theorem of the mean and Wirtinger’s inequality there is an $R_n \in (2^{2^n}, 2^{2^{n+1}})$ such that

$$\log 2 \int_0^\pi |p(R_n, \theta) - \bar{p}(R_n)|^2 d\theta = \int_{2^{2^n}}^{2^{2^{n+1}}} \frac{1}{r \log r} \int_0^\pi |p(r, \theta) - \bar{p}(r)|^2 d\theta$$

$$\leq \int_{2^{2^n}}^{2^{2^{n+1}}} \int_0^\pi \frac{p^2_{\theta}}{r \log r} d\theta dr$$

$$\leq \int_{2^{2^n} < r < 2^{2^{n+1}}, 0 < \theta < \pi} \frac{|\nabla p|^2}{\log r} dxdy \to 0, \ \ n \to \infty.$$
We have the representation
\[ p(r, \theta) - \bar{p}(r) = p(R_n \bar{r}, \theta) - \bar{p}(R_n \bar{r}) \]
\[ = - \int_{1 < \bar{r} < R_0, 0 < \theta < \pi} G(\bar{r}, \theta; \bar{p}, \varphi) R_n^2 H(R_n \bar{p}, \varphi) \bar{p} d\rho d\varphi \]
\[ + \int_{\bar{p} = 1, 0 < \theta < \pi} \frac{1}{\bar{\rho}} \frac{\partial G}{\partial \varphi} (\bar{r}, \theta; \bar{p}, \varphi) (p(R_n \bar{p}, \varphi) - \bar{p}(R_n \bar{p})) \bar{p} d\varphi \]
\[ - \int_{1 < \bar{r} < R_0, \theta = 0} \frac{1}{\bar{\rho}} \frac{\partial G}{\partial \varphi} (\bar{r}, \theta; \bar{p}, \varphi) (p(R_n \bar{p}, \varphi) - \bar{p}(R_n \bar{p})) \bar{p} d\varphi \]
\[ + \int_{1 < \bar{r} < R_0, \theta = \pi} \frac{1}{\bar{\rho}} \frac{\partial G}{\partial \varphi} (\bar{r}, \theta; \bar{p}, \varphi) (p(R_n \bar{p}, \varphi) - \bar{p}(R_n \bar{p})) \bar{p} d\varphi \]
\[ - \int_{\bar{p} = R_0, 0 < \theta < \pi} \frac{\partial G}{\partial \rho} (\bar{r}, \theta; \bar{p}, \varphi) (p(R_n \bar{p}, \varphi) - \bar{p}(R_n \bar{p})) \bar{p} d\varphi \]
\[ = J_1 + J_2 + J_3 + J_4 + J_5, \]

where \( G = G(\bar{r}, \theta; \bar{p}, \varphi) \) is the harmonic Green’s function for the upper half annulus \( 1 < \bar{r} < R_0, 0 < \theta < \pi \). \( G \) can be written in the form (see, for example, p.140, problem 4 with answer on p.418 in [15])

\[
G(\bar{r}, \theta; \bar{p}, \varphi) = \begin{cases} 
\sum_{k=1}^{\infty} \frac{(\rho^k - \bar{\rho}^k)}{\pi k (R_0^k - R_0^{-k})} \left[ \left( \frac{R_0}{\bar{\rho}} \right)^k - \left( \frac{\bar{\rho}}{R_0} \right)^k \right] \sin k \theta \sin k \varphi, & \bar{r} < \bar{\rho}, \\
\sum_{k=1}^{\infty} \frac{(\bar{\rho}^k - \bar{\rho}^{-k})}{\pi k (R_0^k - R_0^{-k})} \left[ \left( \frac{R_0}{\bar{\rho}} \right)^k - \left( \frac{\bar{\rho}}{R_0} \right)^k \right] \sin k \theta \sin k \varphi, & \bar{r} > \bar{\rho}.
\end{cases}
\]

Next, we use variable substitution for (5.8). Let \( \rho = R_n \bar{\rho} \) and noting that \( R_0 = \frac{R_n}{R_m}, \bar{r} = \frac{r}{R_n} \). For \( J_1 \), we have

\[
J_1 = - \int_{1 < \bar{r} < R_0, 0 < \theta < \pi} G(\bar{r}, \theta; \bar{p}, \varphi) R_n^2 H(R_n \bar{p}, \varphi) \bar{p} d\rho d\varphi
\]
\[ = - \int_{1 < \rho < R_0, 0 < \theta < \pi} \sum_{k=1}^{\infty} \frac{(\rho^k - \bar{\rho}^k)}{\pi k (R_0^k - R_0^{-k})} \left[ \left( \frac{R_0}{\rho} \right)^k - \left( \frac{\bar{\rho}}{R_0} \right)^k \right] \sin k \theta \sin k \varphi R_n^2 H(R_n \bar{p}, \varphi) \bar{p} d\rho d\varphi
\]
\[ = - \int_{1 < \rho < R_0, 0 < \theta < \pi} \sum_{k=1}^{\infty} \frac{(\rho^k - \bar{\rho}^k)}{\pi k (R_0^k - R_0^{-k})} \left[ \left( \frac{R_0}{\rho} \right)^k - \left( \frac{\bar{\rho}}{R_0} \right)^k \right] \sin k \theta \sin k \varphi R_n^2 H(R_n \bar{p}, \varphi) \bar{p} d\rho d\varphi
\]
\[ = - \int_{R_n < \rho < R_0, 0 < \theta < \pi} \sum_{k=1}^{\infty} \frac{(\rho^k - R_n^{2k} / \rho^k)}{\pi k (R_m^k - R_m^{-k})} (r^k - R_m^2 / r^k) \sin k \theta \sin k \varphi H(\rho, \varphi) \rho d\rho d\varphi
\]
\[ - \int_{r < R_n, 0 < \theta < \pi} \sum_{k=1}^{\infty} \frac{(r^k - R_n^{2k} / r^k)}{\pi k (R_m^k - R_m^{-k})} (\rho^k - R_m^2 / \rho^k) \sin k \theta \sin k \varphi H(\rho, \varphi) \rho d\rho d\varphi.
\]
For $J_2$, noting that $\bar{\rho} = 1 < \bar{r}$, we get

$$J_2 = \int_{\bar{\rho}=1,0<\theta<\pi} \frac{\partial G}{\partial \rho}(\bar{r}, \theta; \bar{\rho}, \varphi)(p(R_n\bar{\rho}, \varphi) - \bar{p}(R_n\bar{\rho}))\bar{\rho}d\varphi$$

$$= \int_{\bar{\rho}=1,0<\theta<\pi} \frac{\partial}{\partial \bar{\rho}} \left\{ \sum_{k=1}^{\infty} \frac{\rho^k - \bar{\rho}^k}{\pi k (R_0^2 - R_0^2)} \left[ (R_0^2/\bar{\rho})^k - (\bar{r}^2/\bar{\rho})^k \right] \sin k\theta \sin k\varphi \right\} [p(R_n\bar{\rho}, \varphi) - \bar{p}(R_n\bar{\rho})]d\varphi$$

$$= \int_{\bar{\rho}=R_0,0<\theta<\pi} \frac{\partial}{\partial \rho} \left[ - \sum_{k=1}^{\infty} \frac{(\rho^k - R_m^2/\rho^k)}{\pi k (R_m^2 - R_n^2)} (r^k - R_m^2/r^k) \sin k\theta \sin k\varphi \right] [p(\rho, \varphi) - \bar{p}(\rho)]\rho d\varphi.$$

For $J_5$, noting that $\bar{\rho} = R_0 > \bar{r}$, we obtain

$$J_5 = - \int_{\bar{\rho}=R_0,0<\theta<\pi} \frac{\partial G}{\partial \rho}(\bar{r}, \theta; \bar{\rho}, \varphi)(p(R_n\bar{\rho}, \varphi) - \bar{p}(R_n\bar{\rho}))\bar{\rho}d\varphi$$

$$= - \int_{\bar{\rho}=R_0,0<\theta<\pi} \frac{\partial}{\partial \bar{\rho}} \left\{ \sum_{k=1}^{\infty} \frac{\rho^k - \bar{\rho}^k}{\pi k (R_0^2 - R_0^2)} \left[ (R_0^2/\bar{\rho})^k - (\bar{r}^2/\bar{\rho})^k \right] \sin k\theta \sin k\varphi \right\} [p(R_n\bar{\rho}, \varphi) - \bar{p}(R_n\bar{\rho})]d\varphi$$

$$= - \int_{\bar{\rho}=R_0,0<\theta<\pi} \frac{\partial}{\partial \rho} \left[ - \sum_{k=1}^{\infty} \frac{(\rho^k - R_m^2/\rho^k)}{\pi k (R_m^2 - R_n^2)} (r^k - R_m^2/r^k) \sin k\theta \sin k\varphi \right] [p(\rho, \varphi) - \bar{p}(\rho)]\rho d\varphi.$$

Since $G(\bar{r}, \theta; \bar{\rho}, \varphi) = 0$ for $\theta = 0, \pi$. It follows that $J_3 = J_4 = 0$. Hence the representation of $G(r, \theta; \rho, \varphi)$ is follows:

$$G(r, \theta; \rho, \varphi) = \begin{cases} \sum_{k=1}^{\infty} \frac{(\rho^k - R_m^2/\rho^k)}{\pi k (R_m^2 - R_n^2)} (r^k - R_m^2/r^k) \sin k\theta \sin k\varphi, & r < \rho, \\ \sum_{k=1}^{\infty} \frac{(\rho^k - R_m^2/\rho^k)}{\pi k (R_m^2 - R_n^2)} (r^k - R_m^2/r^k) \sin k\theta \sin k\varphi, & r > \rho. \end{cases}$$

Moreover, (5.8) can be written in the following form

$$p(r, \theta) - \bar{p}(r) = - \int_{A_{nm}^+} G(r, \theta; \rho, \varphi)H(\rho, \varphi)\rho d\rho d\varphi$$

$$+ \int_{\rho=R_n,0<\theta<\pi} \frac{\partial G}{\partial \rho}(r, \theta; \rho, \varphi)(p(\rho, \varphi) - \bar{p}(\rho))\rho d\varphi$$

$$- \int_{\rho=R_m,0<\theta<\pi} \frac{\partial G}{\partial \rho}(r, \theta; \rho, \varphi)(p(\rho, \varphi) - \bar{p}(\rho))\rho d\varphi.$$

(5.9)
Define

$$G^{(2)}(r; \rho_1, \varphi_1; \rho_2, \varphi_2)$$

$$= \int_0^\pi G(r, \theta; \rho_1, \varphi_1)G(r, \theta; \rho_2, \varphi_2)\,d\theta$$

$$= \sum_{k=1}^\infty \frac{(r^k - R_n^{2k}/r^k)^2(\rho_1^k - R_n^{2k}/\rho_1^k)(\rho_2^k - R_n^{2k}/\rho_2^k)\sin k\varphi_1 \sin k\varphi_2}{2\pi k^2(R_n^{2k} - R_n^{2k})^2}$$

when \( r < \rho_1, r < \rho_2 \), with similar expressions for the other cases. Therefore

$$|G^{(2)}(r; \rho_1, \varphi_1; \rho_2, \varphi_2)| \leq \sum_{k=1}^\infty \left( \frac{\rho_1^k - R_n^{2k}/\rho_1^k}{2\pi k^2(R_n^{2k} - R_n^{2k})^2} \right)^2$$

$$:= \hat{G}^{(2)}.$$ 

Due to \( r > R_n \), we find \((r^k - R_n^{2k}/r^k)^2\) is monotonically increasing with respect to \( r \). When \( r = \rho_1 \) or \( r = \rho_2 \), \( \hat{G}^{(2)} \) attains at the maximum. If \( \rho_1 < \rho_2 \), then

$$\hat{G}^{(2)} \leq \sum_{k=1}^\infty \frac{\rho_1^k - R_n^{2k}/\rho_1^k}{2\pi k^2(R_n^{2k} - R_n^{2k})^2}.$$ 

Noting that \(|\rho_2^k - R_n^{2k}/\rho_2^k|\) is monotonically decreasing with respect to \( \rho_2 \), \( \hat{G}^{(2)} \) reaches its maximum at \( \rho_1 = \rho_2 \). Let

$$h(r) = (r^k - R_n^{2k}/r^k)(r^k - R_n^{2k}/r^k).$$

An easy computation which lets \( h'(r) = 0 \) shows that \( r = (R_mR_n)^{\frac{1}{2}} \). In conclusion, the maximum with respect to \( \rho_1 \) and \( \rho_2 \) of \( \hat{G}^{(2)} \) occurs when \( \rho_1 = \rho_2 = r = (R_mR_n)^{\frac{1}{2}} \). Thus

$$|G^{(2)}(r; \rho_1, \varphi_1; \rho_2, \varphi_2)| \leq \sum_{k=1}^\infty \frac{[(R_mR_n)^{\frac{k}{2}} - R_m^{2k}/(R_mR_n)^{\frac{k}{2}}]^2[(R_mR_n)^{\frac{k}{2}} - R_m^{2k}/(R_mR_n)^{\frac{k}{2}}]^2}{2\pi k^2(R_m^{2k} - R_n^{2k})^2}$$

$$= \sum_{k=1}^\infty \frac{[R_m^{-\frac{k}{2}}R_n^{-\frac{k}{2}}(R_m - R_n)^2]^2[R_m^{\frac{k}{2}}R_n^{-\frac{k}{2}}(R_m - R_n)^2]^2}{2\pi k^2(R_m^k + R_n^k)^2(R_m^k - R_n^k)^2}$$

$$= \sum_{k=1}^\infty \frac{(R_m^k - R_n^k)^2}{2\pi k^2(R_m^k + R_n^k)^2} \leq \sum_{k=1}^\infty \frac{1}{k^2} = C_1.$$
Also
\[
\frac{\partial G(r, \theta; \rho, \varphi)}{\partial \rho} \bigg|_{\rho = R_m} = \frac{\partial}{\partial \rho} \left[ - \sum_{k=1}^{\infty} \frac{(r^k - R_{m}^{2k} / r^k)}{\pi k (R_{m}^{2k} - R_{m}^{2k})} \left( \rho^k - \frac{R_{m}^{2k}}{\rho^k} \right) \sin k \theta \sin k \varphi \right] \bigg|_{\rho = R_m} \\
= - \sum_{k=1}^{\infty} \frac{(r^k - R_{n}^{2k} / r^k) (k \rho^{k-1} + k R_{m}^{2k} / \rho^{k+1})}{\pi k (R_{m}^{2k} - R_{n}^{2k})} \sin k \theta \sin k \varphi \bigg|_{\rho = R_m} \\
= - \sum_{k=1}^{\infty} \frac{2(r^k - R_{m}^{2k} / r^k) R_{m}^{k-1}}{\pi (R_{m}^{2k} - R_{n}^{2k})} \sin k \theta \sin k \varphi
\]
and thus, if \( R_n < r \leq R_{m-2} \), \( 0 < \theta < \pi \), we have
\[
\int_0^\pi |G_\rho(r, \theta; R_{m}, \varphi)|^2 R_m^2 d\varphi \\
= \sum_{k=1}^{\infty} \frac{4(r^k - R_{m}^{2k} / r^k)^2 R_{m}^{2k}}{\pi^2 (R_{m}^{2k} - R_{n}^{2k})^2} \sin^2 k \theta \int_0^\pi \sin^2 k \varphi d\varphi \\
\leq \sum_{k=1}^{\infty} \frac{2(r^k - R_{m}^{2k} / r^k)^2 R_{m}^{2k}}{\pi (R_{m}^{2k} - R_{n}^{2k})^2} \\
\leq \sum_{k=1}^{\infty} \frac{2(R_{m-2}/R_{m})^{2k}}{\pi [1 - (R_n/R_m)^2]^{2k}} \\
\leq \sum_{k=1}^{\infty} 2^{-2^{n+1}k} \leq C_2.
\]

Similarly, if \( R_{n+2} \leq r < R_{m} \), \( 0 < \theta < \pi \), we have
\[
\frac{\partial G(r, \theta; \rho, \varphi)}{\partial \rho} \bigg|_{\rho = R_n} = - \sum_{k=1}^{\infty} \frac{2(r^k - R_{m}^{2k} / r^k) R_{n}^{k-1}}{\pi (R_{m}^{2k} - R_{n}^{2k})} \sin k \theta \sin k \varphi
\]
and
\[
\int_0^\pi |G_\rho(r, \theta; R_{n}, \varphi)|^2 R_{n}^2 d\varphi \\
= \sum_{k=1}^{\infty} \frac{4(r^k - R_{m}^{2k} / r^k)^2 R_{n}^{2k}}{\pi^2 (R_{m}^{2k} - R_{n}^{2k})^2} \sin^2 k \theta \int_0^\pi \sin^2 k \varphi d\varphi \\
\leq \sum_{k=1}^{\infty} \frac{2(R_{n}/R_{n+2})^{2k}}{\pi [1 - (R_n/R_m)^2]^{2k}} \\
\leq \sum_{k=1}^{\infty} 2^{-2^{n}k} \leq C_3.
\]
By Hölder inequality, it follows from (5.9) that for \( r \in [R_{n+2}, R_{m-2}], 0 < \theta < \pi, m \geq n + 5 \)

\[
\int_0^\pi |p(r, \theta) - \bar{p}(r)|^2 d\theta 
\leq 3 \int_0^\pi \int_0^\pi \int_{R_n}^{R_m} G(r, \theta; \rho, \varphi) H(\rho, \varphi) |p(\rho, \varphi) - \bar{p}(\rho)| R_n d\varphi d\theta 
+ 3 \int_0^\pi \int_0^\pi \int_{R_n}^{R_m} \frac{\partial G}{\partial \rho}(r, \theta; R_n, \varphi) |p(R_n, \varphi) - \bar{p}(R_n)| R_n d\varphi d\theta 
+ 3 \int_0^\pi \int_0^\pi \int_{R_n}^{R_m} \frac{\partial G}{\partial \rho}(r, \theta; R_m, \varphi) |p(R_m, \varphi) - \bar{p}(R_m)| R_m d\varphi d\theta 
\]

\[
\leq 3 \int_0^\pi \int_{R_n}^{R_m} \int_{R_n}^{R_m} \int_0^\pi G(r, \theta; \varphi_1, \varphi_2) G(r, \theta; \varphi_1, \varphi_2) d\theta d\varphi_1 d\varphi_2 
\int_{R_n}^{R_m} \left| \frac{\partial G}{\partial \rho}(r, \theta; R_n, \varphi) \right|^2 R_n d\varphi \int_0^\pi |p(R_n, \varphi) - \bar{p}(R_n)|^2 d\varphi d\theta 
+ 3 \int_0^\pi \int_{R_n}^{R_m} \int_{R_n}^{R_m} \left| \frac{\partial G}{\partial \rho}(r, \theta; R_m, \varphi) \right|^2 R_m d\varphi \int_0^\pi |p(R_m, \varphi) - \bar{p}(R_m)|^2 d\varphi d\theta 
\]

\[
\leq 3\pi C_1 \left( \int_{R_n}^{R_m} |H| |dxdy| \right)^2 + 3\pi C_3 \int_0^\pi |p(R_n, \theta) - \bar{p}(R_n)|^2 d\theta 
+ 3\pi C_2 \int_0^\pi |p(R_m, \varphi) - \bar{p}(R_m)|^2 d\varphi.
\]

By letting \( m \to \infty \) and using (5.4) and (5.7), we obtain an upper bound on the left member for \( r > 2^{2n+2} \), and this bound approaches zero as \( n \to \infty \). From this we infer

\[
\lim_{r \to \infty} \int_0^\pi |p(r, \theta) - \bar{p}(r)|^2 d\theta = 0.
\]

The third lemma is about the convergence of the average pressure.

**Lemma 5.3.** Under the assumptions of Theorem 1.3, the average pressure

\[
\bar{p}(r) = \frac{1}{\pi} \int_0^\pi p(r, \theta) d\theta
\]

has a limit at infinity

\[
\lim_{r \to \infty} \bar{p}(r) = p_\infty < \infty.
\] (5.10)

**Proof.** First, Navier-slip boundary condition tells us that \( \omega(x, 0) = 0 \) and \( v(x, 0) = 0 \) due to (1.5). Hence

\[
\int_0^\pi \omega_0 d\theta = 0,
\]
and
\[ \int_0^{\pi} (uv_\theta + vu_\theta) d\theta = \int_0^{\pi} \bar{u}v_\theta d\theta = 0, \]

where
\[ \bar{u}(r) = \frac{1}{\pi} \int_0^{\pi} u(r, \theta) d\theta. \]

Besides, due to \( bu(r, 0) = bu(r, \pi) \), we average (5.3) to find that
\[
\bar{p}'(r) = \frac{1}{\pi r} \int_0^{\pi} \left[ \omega_\theta + uv_\theta - (v + b)u_\theta \right] d\theta \\
= \frac{1}{\pi r} \int_0^{\pi} [2uv_\theta - (vu_\theta + uv_\theta)] d\theta \\
= \frac{2}{\pi r} \int_0^{\pi} (u - \bar{u})v_\theta d\theta.
\] (5.11)

Integrating this inequality with respect to \( r \) over \((r_1, r_2) \) \((r_2 \geq r_1 \geq r_0)\), then by Cauchy and Wirtinger inequalities, we find
\[
|\bar{p}(r_2) - \bar{p}(r_1)| = \frac{2}{\pi} \int_{r_1}^{r_2} \int_0^{\pi} \frac{(u - \bar{u})v_\theta}{r} d\theta dr \\
\leq \frac{1}{\pi} \int_{r_1}^{r_2} \int_0^{\pi} \frac{|w - \bar{w}|^2 + |w_\theta|^2}{r} d\theta dr \\
\leq C \int_{r_1}^{r_2} \int_0^{\pi} \frac{|w_\theta|^2}{r} d\theta dr \\
\leq C \int_{r > r_1} |\nabla w|^2 dxdy.
\]

Since the right member of this inequality tends to zero as \( r_1 \to \infty \), it follows that \( \bar{p}(r) \) has a limit \( p_\infty \), as asserted.

Immediately it follows from Lemma 5.3 and Lemma 5.2 that the following conclusion holds.

**Corollary 5.1.** Under the assumptions of Theorem 1.3, we have
\[
\lim_{r \to \infty} \int_0^{\pi} |p(r, \theta) - p_\infty|^2 d\theta = 0. \] (5.12)

**Proof of Case (i) in Theorem 1.3.** For \( r > r_0 \), define
\[ \tilde{p}(\tilde{x}) = r^2 p(r \tilde{x}) = r^2 p(x), \]
where \( x \in B_{2r}^+ \setminus B_r^+ \) and \( r \) is large enough. Without loss of generality, we still consider \( B_r^+ \), since one can mollify the domain such that it’s regular. By Lemma 2.5, we have

\[
\|\tilde{p}\|_{L^\infty(B_{2r}^+ \setminus B_r^+)} \leq C(1 + \|\tilde{p}\|_{H^1(B_{2r}^+ \setminus B_r^+)} \sqrt{\log(e + \|\Delta \tilde{p}\|_{L^2(B_{2r}^+ \setminus B_r^+)})} \\
= C(1 + \|\tilde{p}\|_{L^2(B_{2r}^+ \setminus B_r^+) + \|\nabla \tilde{p}\|_{L^2(B_{2r}^+ \setminus B_r^+)} \sqrt{\log(e + \|\Delta \tilde{p}\|_{L^2(B_{2r}^+ \setminus B_r^+)})}.
\]

Due to the scaling, we have

\[
\|\tilde{p}\|_{L^\infty(B_{2r}^+ \setminus B_r^+)} = r^2 \|p\|_{L^\infty(B_{2r}^+ \setminus B_r^+)} , \\
\|\tilde{p}\|_{L^2(B_{2r}^+ \setminus B_r^+)} = r \|p\|_{L^2(B_{2r}^+ \setminus B_r^+)} , \\
\|\nabla \tilde{p}\|_{L^2(B_{2r}^+ \setminus B_r^+)} = r^2 \|\nabla p\|_{L^2(B_{2r}^+ \setminus B_r^+)} , \\
\|\nabla^2 \tilde{p}\|_{L^2(B_{2r}^+ \setminus B_r^+)} = r^3 \|\nabla^2 p\|_{L^2(B_{2r}^+ \setminus B_r^+)} .
\]

Hence

\[
r^2 \|p\|_{L^\infty(B_{2r}^+ \setminus B_r^+)} \leq C(1 + r \|p\|_{L^2(B_{2r}^+ \setminus B_r^+) + r^2 \|\nabla p\|_{L^2(B_{2r}^+ \setminus B_r^+)} \sqrt{\log(e + r^3 \|\Delta p\|_{L^2(B_{2r}^+ \setminus B_r^+)})}.
\]

From Lemma 5.3 we know \( |p_\infty| \leq C \) and by (5.12) in Corollary 5.1 we have

\[
\|p\|_{L^2(B_{2r}^+ \setminus B_r^+)} \leq 2 \int_0^{2r} \int_0^{\pi} |p(\rho, \theta) - p_\infty|^2 d\theta d\rho + 2 \int_r^{2r} \int_0^{\pi} |p_\infty|^2 \rho d\theta d\rho \leq o(r) \int_r^{2r} \rho d\rho + Cr^2.
\]

Then

\[
\|p\|_{L^2(B_{2r}^+ \setminus B_r^+)} \leq Cr^2.
\]

Since \( |\Delta w|^2 = |\nabla \omega|^2 \), then by Lemma 2.3, \( w = o(\sqrt{\log r}) \) and Navier-Stokes equation (1.1), there holds

\[
\|\nabla p\|_{L^2(B_{2r}^+ \setminus B_r^+)} \leq C \int_{B_{2r}^+ \setminus B_r^+} |\Delta w|^2 dx dy + C \int_{B_{2r}^+ \setminus B_r^+} |w \cdot \nabla w|^2 dx dy + C \int_{B_{2r}^+ \setminus B_r^+} |\nabla w|^2 dx dy \\
\leq C \int_{B_{2r}^+ \setminus B_r^+} |\nabla \omega|^2 dx dy + C[o(\log r) + 1] \int_{B_{2r}^+ \setminus B_r^+} |\nabla w|^2 dx dy \\
\leq C[1 + o(\log r)].
\]
Besides, similar to the second step of Theorem 1.1 using (3.2) and (3.3), we also get
\[ \| \nabla^2 (\hat{w}_\epsilon \phi) \|_{L^2(\mathbb{R}^2)} \leq C \| \nabla (\hat{w}_\epsilon \phi + \hat{w}_\epsilon \cdot \nabla \phi) \|_{L^2(\mathbb{R}^2)} + C \| \nabla \cdot (\hat{w}_\epsilon \phi) \|_{L^2(\mathbb{R}^2)} \]
\[ = C \| \phi \nabla \hat{w}_\epsilon + \hat{w}_\epsilon \phi + \nabla (\hat{w}_\epsilon \cdot \nabla \phi) \|_{L^2(\mathbb{R}^2)} + C \| \nabla (\nabla \phi \hat{w}_\epsilon) \|_{L^2(\mathbb{R}^2)} \]
\[ \leq C \| \nabla \hat{w}_\epsilon \|_{L^2(\mathbb{R}^2 \setminus B_{2r_0})} + C \| \hat{w}_\epsilon \|_{L^2(B_{3r_0} \setminus B_{2r_0})} + C \| \nabla \hat{w}_\epsilon \|_{L^2(B_{3r_0} \setminus B_{2r_0})}. \]

Since \( \hat{w}_\epsilon \in C^\infty(\mathbb{R}^2 \setminus B_{2r_0}) \), we know \( \| \hat{w}_\epsilon \|_{L^2(B_{3r_0} \setminus B_{2r_0})} \leq C \). Using (1.7), we have
\[ \| \nabla \hat{w}_\epsilon \|_{L^2(B_{3r_0} \setminus B_{2r_0})} \leq C. \]

Therefore, \( \| \nabla^2 \hat{w}_\epsilon \|_{L^2(\mathbb{R}^2 \setminus B_{2r_0})} \leq C + C \| \nabla \hat{w}_\epsilon \|_{L^2(\mathbb{R}^2 \setminus B_{2r_0})} \leq C + C \| \nabla \hat{w}_\epsilon \|_{L^2(\mathbb{R}^2 \setminus B_{2r_0})}. \]

Let \( \epsilon \to 0 \), by Lebesgue’s dominated convergence theorem and Lemma 2.3, we have
\[ \| \nabla^2 \hat{w}_\epsilon \|_{L^2(\mathbb{R}^2 \setminus B_{3r_0})} \leq C + C \| \nabla \hat{w}_\epsilon \|_{L^2(\mathbb{R}^2 \setminus B_{2r_0})} \leq C + C \| \nabla \hat{w}_\epsilon \|_{L^2(\mathbb{R}^2 \setminus B_{2r_0})} \leq C. \]

Then by (5.2)1,2 and Gagliardo-Nirenberg inequality, we have
\[ \int_{B_{3r_0}^+ \setminus B_{2r_0}^+} |\Delta p|^2 dxdy \]
\[ \leq C \int_{B_{3r_0}^+ \setminus B_{2r_0}^+} (u^2 v_x^2 + u^2 v_y^2) dxdy \]
\[ \leq C \int_{B_{3r_0}^+ \setminus B_{2r_0}^+} |\nabla w|^4 dxdy \]
\[ \leq C \int_{B_{3r_0}^+ \setminus B_{2r_0}^+} |\nabla w|^2 dxdy \int_{B_{3r_0}^+ \setminus B_{2r_0}^+} |\nabla^2 w|^2 dxdy \]
\[ + Cr^{-2} \int_{B_{3r_0}^+ \setminus B_{2r_0}^+} |\nabla w|^2 dxdy \]
\[ \leq C(1 + r^{-2}). \]

Therefore using (5.13), one can get
\[ r^2 \| p \|_{L^\infty(B_{3r}^+ \setminus B_{2r}^+)} \leq C(1 + Cr^2 + Cr^2 \sqrt{1 + o(\log r)}) \sqrt{\log(e + Cr^3 \sqrt{1 + r^{-2}})} \]
\[ \leq r^2 o(\log r), \]
which implies
\[ \| p \|_{L^\infty(B_{3r}^+ \setminus B_{2r}^+)} \leq o(\log r). \]

The proof is complete.

**Proof of Case (ii) in Theorem 1.3.** Let the point \( P(2R, \theta) \) be the origin of a new system of polar coordinates \((r', \theta')\) and suppose that \( R > r_0 \). In these new coordinates we still have
\[ p_{rr'} = \frac{1}{r'}[\omega_{rr'} + uw_{rr'} - (v + b)u_{rr'}]. \]
from the Navier-Stokes equations. Integrating with respect to \( r' \) over \((0, r')\)

\[
p(P) = p(r', \theta') + \int_0^{r'} \frac{1}{\rho}[(v+b)u_{\theta'} - uv_{\theta'} - \omega_{\theta'}]d\rho.
\]  

(5.14)

**Proof of Case I**: \( p(2R, \theta) \) with \( \theta = 0, \pi \). As shown in Figure 1:

![Figure 1](image)

**Figure 1.** The case of \( \theta = 0, \pi \).

Let’s consider the case of \( \theta = 0 \), and \( \theta = \pi \) is similar. Integrating (5.14) with respect to \( \theta' \) over \([0, \pi]\)

\[
\pi p(P) = \int_0^{\pi} p(r', \theta')d\theta' + \int_0^{r'} \int_0^{\pi} \frac{1}{\rho}[(v+b)u_{\theta'} - uv_{\theta'} - \omega_{\theta'}]d\rho d\theta'.
\]

Due to \( \omega(x, 0) = 0, v(x, 0) = 0 \) and \( bu(r, 0) = bu(r, \pi) \). Similar to the (5.11) calculation

\[
p(P) = \frac{1}{\pi} \int_0^{\pi} p(r', \theta')d\theta' + \frac{2}{\pi} \int_0^{r'} \int_0^{\pi} \frac{1}{\rho} [\ddot{u}(\rho) - u(\rho, \theta')] v_{\theta'}(\rho, \theta')d\rho d\theta',
\]

where

\[
\ddot{u}(r') = \frac{1}{\pi} \int_0^{\pi} u(r', \theta')d\theta'.
\]

Multiply this relation by \( r' \) and integrate from 0 to \( R \), and we find

\[
p(P) = \frac{2}{\pi R^2} \int_0^{R} \int_0^{\pi} p(r', \theta')r'd\theta'd\theta' + \frac{4}{\pi R^2} \int_0^{r'} \int_0^{\pi} \frac{\ddot{u}(\rho) - v_{\theta'}}{\rho} r'd\theta'd\rho d\theta' = I_1 + I_2.
\]  

(5.15)
Note that the upper half disc $r' < R, 0 < \theta' < \pi$ is contained in the upper half annulus $R < r < 3R, 0 < \theta < \pi$. For $I_1$, using Schwarz inequality and (5.12)

$$|I_1|^2 \leq \frac{2}{\pi R^2} \int_0^R \int_{\theta < \pi} p^2(r' \rho \theta')^2 \left( \int_0^R \int_{\theta < \pi} r' \rho \theta' \right)$$

$$\leq \frac{2}{\pi R^2} \int_R^{3R} r dr \left\{ \max_{R < \rho < 3R} \int_0^\pi p(r, \theta)^2 d\theta \right\}$$

$$= \frac{8}{\pi} \max_{R < \rho < 3R} \int_0^\pi p(r, \theta)^2 d\theta \to 0, \text{ as } R \to \infty.$$ 

For $I_2$, using Cauchy and Wirtinger inequalities

$$| \int_0^\pi (\bar{u} - u)v \theta' d\theta' | \leq \int_0^\pi \frac{|u - \bar{u}|^2 + |v \theta'|^2}{2} d\theta'$$

$$\leq \int_0^\pi \frac{|w - \bar{w}|^2 + |w \theta'|^2}{2} d\theta'$$

$$\leq \int_0^\pi |w \theta'|^2 d\theta' \leq \int_0^\pi \rho^2 |\nabla w|^2 d\theta'.$$

Consequently, we have

$$I_2 \leq \frac{4}{\pi R^2} \int_0^R \int_0^{\rho \theta'} \int_0^{\rho \theta'} |\nabla w(\rho, \theta')|^2 \rho d\rho d\theta' r' dr'$$

$$\leq \frac{4}{\pi R^2} \int_{\rho \theta' < 3R, 0 < \theta < \pi} |\nabla w|^2 dxdy \int_0^R \rho d\rho$$

$$= \frac{2}{\pi} \int_{\rho \theta' < 3R, 0 < \theta < \pi} |\nabla w|^2 dxdy \to 0, \text{ as } R \to \infty.$$ 

It follows from (5.15) that

$$|p(2R, 0)| \to 0,$$

as $R \to \infty$. Thus the proof of Case I is complete.

**Proof of Case II**: $P(2R, \theta)$ with $\theta \in (0, \pi)$. As shown in Figure 2:
Integrating (5.14) with respect to $\theta'$ over $[0, 2\pi]$

$$p(P) = \frac{1}{2\pi} \int_0^{2\pi} p(r', \theta') d\theta' + \frac{1}{\pi} \int_0^{R\sin \theta} \int_0^{2\pi} \frac{1}{\rho} [\hat{u}(\rho) - u(\rho, \theta')] v_{\theta'}(\rho, \theta') d\rho d\theta',$$

where

$$\hat{u}(r') = \frac{1}{2\pi} \int_0^{2\pi} u(r', \theta') d\theta'.$$

Multiply this relation by $r'$ and integrate from 0 to $R\sin \theta$

$$p(P) = \frac{1}{\pi R^2 \sin^2 \theta} \int_0^{R\sin \theta} \int_0^{2\pi} p(r', \theta') r' d\theta' dr' + \frac{2}{\pi R^2 \sin^2 \theta} \int_0^{R\sin \theta} \int_0^{r'} \int_0^{2\pi} \frac{(\hat{u} - u)v_{\theta'}}{\rho} r' d\theta' d\rho dr' \equiv I_1' + I_2'.$$

Noting that the disc $r' < R\sin \theta, 0 < \theta' < 2\pi$ is contained in the upper half annulus $2R - R\sin \theta < r < 2R + R\sin \theta, 0 < \theta < \pi$. Then similar to the Case I, for $I_1'$, by Schwarz’s inequality and (5.12), we get

$$|I_1'|^2 \leq \left( \frac{1}{\pi R^2 \sin^2 \theta} \right)^2 \left( \int_0^{R\sin \theta} \int_0^{2\pi} p^2 r' d\theta' dr' \right) \left( \int_0^{R\sin \theta} \int_0^{2\pi} r' d\theta' dr' \right)$$

$$\leq \frac{1}{\pi R^2 \sin^2 \theta} \int_{2R-R\sin \theta < r < 2R+R\sin \theta, 0 < \theta < \pi} p^2 dx dy$$

$$\leq \frac{1}{\pi R^2 \sin^2 \theta} \int_{2R-R\sin \theta}^{2R+R\sin \theta} r dr \left\{ \max_{2R-R\sin \theta < r < 2R+R\sin \theta} \int_0^{2\pi} p(r, \theta)^2 d\theta \right\}$$

$$= \frac{4}{\pi \sin \theta} \max_{2R-R\sin \theta < r < 2R+R\sin \theta} \int_0^{2\pi} p(r, \theta)^2 d\theta \to 0,$$ as $R \to \infty$. 

Figure 2. The case of $\theta \in (0, \pi)$. 

- $Q_1 = (2R - R\sin \theta, 0)$
- $Q_2 = (2R + R\sin \theta, 0)$
For $I'_2$, using Cauchy and Wirtinger’s inequalities, we have

\[
I'_2 \leq \frac{2}{\pi R^2 \sin^2 \theta} \int_0^{R \sin \theta} \int_0^{r} \int_0^{2\pi} |\nabla w(\rho, \theta^\prime)|^2 \rho d\rho d\theta^\prime r^\prime dr^\prime
\]

\[
\leq \frac{2}{\pi R^2 \sin^2 \theta} \int_{2R-R \sin \theta < r < 2R+R \sin \theta, 0 < \theta < \pi} |\nabla w|^2 dxdy \int_0^{R \sin \theta} \rho d\rho
\]

\[
= \frac{1}{\pi} \int_{2R-R \sin \theta < r < 2R+R \sin \theta, 0 < \theta < \pi} |\nabla w|^2 dxdy \to 0, \text{ as } R \to \infty.
\]

Hence from (5.16), we obtain $\forall \theta \in (0, \pi)$, there holds

\[
|P(2R, \theta)| \to 0
\]

as $R \to \infty$. Hence we prove the case of II. Combining the results of Case I and Case II, the proof is complete.

6. Decay of the vorticity.

**Lemma 6.1.** Under the assumptions of Theorem 1.4, we have

\[
\int_{r > r_1, 0 < \theta < \pi} \frac{r}{(\log r)^{1/2}} |\nabla \omega|^2 dxdy < \infty (r_1 > \max(r_0, 2 - r_0)). \quad (6.1)
\]

**Proof.** Choose $R > r_1 > \max(r_0, 2 - r_0)$ and two non-negative $C^2$ cut-off functions $\xi_1$ and $\xi_2$ such that

\[
\xi_1(r) = \begin{cases} 
0, & r \leq \frac{1}{2}(r_0 + r_1), \\
1, & r \geq r_1 
\end{cases}, \quad \xi_2(r) = \begin{cases} 
1, & r \leq 1, \\
0, & r \geq 2.
\end{cases} \quad (6.2)
\]

Let

\[
\eta(r) = \xi_1(r)\xi_2\left(\frac{r}{R}\right)\frac{r}{(\log r)^{1/2}},
\]

\[
h(\omega) = \omega^2.
\]

Clearly $\eta(r)$ vanishes near $r = r_0$ and near $r = \infty$. Noting that $a = 0$, similar to the discussion for (2.6), we have

\[
2 \int_{r > r_0, 0 < \theta < \pi} \eta |\nabla \omega|^2 dxdy = \int_{r > r_0, 0 < \theta < \pi} \omega^2 (\Delta \eta + \omega \cdot \nabla \eta) dxdy
\]

\[
+ \int_{r > r_0, 0 < \theta < \pi} \nabla \eta \cdot (0, b) \omega^2 dxdy. \quad (6.3)
\]

One verifies easily that there is a constant $C$ independent of $R$ such that

\[
|\Delta \eta| \leq C, \quad |\nabla \eta| \leq \frac{C}{(\log r)^{1/2}}.
\]
Noting that $\eta = \frac{r}{\log r}$ for $r_1 < r < R$, it follows from (6.3) and (1.8)

\[ \int_{r_1 < r < R, 0 < \theta < \pi} \frac{r}{\log r} |\nabla \omega|^2 dxdy \]

\[ \leq \frac{1}{2} \int_{r > r_0, 0 < \theta < \pi} \omega^2 (\Delta \eta(r) + \omega \cdot \nabla \eta(r)) dxdy + \frac{1}{2} \int_{r > r_0, 0 < \theta < \pi} \nabla \eta(r) \cdot (0, b) \omega^2 dxdy \]

\[ \leq C \int_{r > r_0, 0 < \theta < \pi} \omega^2 [1 + \frac{|\omega|}{(\log r)^{\frac{1}{2}}}] dxdy + C \int_{r > r_0, 0 < \theta < \pi} \frac{1}{(\log r)^{\frac{1}{2}}} \omega^2 dxdy \]

\[ \leq C(1 + \frac{1}{(\log r_0)^{\frac{1}{2}}}) \int_{r > r_0, 0 < \theta < \pi} \omega^2 dxdy < \infty. \]

Letting $R \to \infty$, we obtain (6.1).

Using Lemma 6.1 one can improve the result of Lemma 2.4.

**Proof of Theorem 1.4.**

Note that for $2^n > r_0$

\[ \int_{2^n}^{2^{n+1}} \frac{dr}{r} \int_0^\pi \left( r^2 \omega^2 + 2 \frac{r^{\frac{3}{2}}}{(\log r)^{\frac{1}{2}}} |\omega_\theta| \right) d\theta \]

\[ \leq \int_{2^n < r < 2^{n+1}, 0 < \theta < \pi} (\omega^2 + 2 \frac{r^{\frac{3}{2}}}{(\log r)^{\frac{1}{2}}} |\omega||\nabla \omega|) dxdy \]

\[ \leq \int_{r > 2^n, 0 < \theta < \pi} (2\omega^2 + 2 \frac{r}{(\log r)^{\frac{1}{2}}} |\nabla \omega|^2) dxdy. \]

Using (6.1) and proceeding exactly as in the proof of Lemma 2.4 we obtain (1.12).

The proof is complete.

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