Extremal solutions for a broad spectrum of nonlinear elliptic systems

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Abstract
In this paper we study extremal solutions for the Dirichlet problem

\[
\begin{cases}
-\mathcal{L}u = \Lambda F(x,u) \quad &\text{in } \Omega, \\
u = 0 &\text{on } \partial \Omega,
\end{cases}
\]

where $\Omega$ is a bounded $C^{1,1}$ domain in $\mathbb{R}^n$, $n \geq 2$, $u = (u_1, \ldots, u_m) : \overline{\Omega} \to \mathbb{R}^m$, $m \geq 1$, $\mathcal{L}u = (\mathcal{L}_1 u_1, \ldots, \mathcal{L}_m u_m)$, where each $\mathcal{L}_i$ denotes a uniformly elliptic linear operator of second order in nondivergence form in $\Omega$, $\Lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$, $F = (f_1, \ldots, f_m) : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ and $\Lambda F(x,u) = (\lambda_1 f_1(x,u), \ldots, \lambda_m f_m(x,u))$.

For a general class of maps $F$ we prove that there exists a hypersurface $\Lambda^*$ in $\mathbb{R}^m_+ := (0, +\infty)^m$ such that tuples $\Lambda \in \mathbb{R}^m_+$ below $\Lambda^*$ correspond to stable minimal positive strong solutions to the above system. Already for tuples above $\Lambda^*$, there is no nonnegative strong solution. The shape of the hypersurface $\Lambda^*$ depends on growth on $u$ of the vector nonlinearity $F$ and is also discussed.

When $\Lambda \in \Lambda^*$ and each operator $\mathcal{L}_i$ has slightly smooth coefficients, the problem admits a unique minimal nonnegative weak solution, which is called extremal solution. In addition, when $F$ is a potential field and each $\mathcal{L}_i$ is the Laplace operator, we discuss on the strong regularity of extremal solutions for any $m \geq 1$ in dimensions $n = 2$ and $n = 3$ for convex domains and $2 \leq n \leq 9$ for balls.

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1 Introduction and main statements

The present work concerns mainly with existence, regularity and stability of solution of the elliptic system to $m$-parameters

$$\begin{cases}
-\mathcal{L}_i u_i = \lambda_i f_i(x, u_1, \ldots, u_m) & \text{in } \Omega, \ i = 1, \ldots, m, \\
u_i = 0 & \text{on } \partial \Omega, \ i = 1, \ldots, m,
\end{cases}$$

where $\Omega$ is a bounded open subset of $\mathbb{R}^n$ with $C^{1,1}$ boundary, $n \geq 2$ and for each $i = 1, \ldots, m$, $f_i : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory function, $m \geq 1$, $\lambda_i$ is a positive number and $\mathcal{L}_i$ denotes a linear differential operator of second order in $\Omega$ of the form

$$\mathcal{L}_i = a^i_{kl}(x) \partial_{kl} + b^i_j(x) \partial_j + c^i(x).$$

We assume that each $\mathcal{L}_i$ is uniformly elliptic, that is, there exist positive constants $C_0$ and $c_0$ such that

$$c_0 |\xi|^2 \leq a^i_{kl}(x) \xi_k \xi_l \leq C_0 |\xi|^2, \ \forall \xi \in \mathbb{R}^n, \ x \in \Omega, \ i = 1, \ldots, m.$$ 

We also consider coefficients $a^i_{kl} \in C(\Omega), b^i_j, c^i \in L^\infty(\Omega)$ and denote by $b > 0$ a constant so that

$$|b^i_j(x)|, |c^i(x)| \leq b, \ \forall x \in \Omega, \ i = 1, \ldots, m.$$ 

Moreover, assume that each $\mathcal{L}_i$ satisfies the strong maximum principle in $\Omega$, or equivalently, its principal eigenvalue $\mu_1(-\mathcal{L}_i, \Omega)$ is positive for $i = 1, \ldots, m$.

Throughout work, the above problem will be represented shortly as

$$\begin{cases}
-\mathcal{L} u = \Lambda F(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases} \quad (1)$$

with the natural notations $u = (u_1, \ldots, u_m)$, $\mathcal{L} u = (\mathcal{L}_1 u_1, \ldots, \mathcal{L}_m u_m)$, $\Lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}_+^m := (0, +\infty)^m$, $F = (f_1, \ldots, f_m)$ and $\Lambda F(x, u) = (\lambda_1 f_1(x, u), \ldots, \lambda_m f_m(x, u))$.

For $m = 1$, the classical Dirichlet problem

$$\begin{cases}
-\mathcal{L} u = \lambda f(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases} \quad (2)$$

has been well studied since the 1960s for $C^1$ functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

(a) $f(0) > 0$ ($f$ is positive at the origin);

(b) $f(s) \leq f(t)$ for every $0 \leq s \leq t$ ($f$ is nondecreasing);

(c) $\lim_{t \to +\infty} \frac{f(t)}{t} = +\infty$ ($f$ is superlinear at infinity).

In fact, it all started with the seminal case treated by Gelfand in his celebrated work [14] where $\mathcal{L}$ is the Laplace operator $\Delta$, $f(u) = e^u$ and $\Omega$ is the unit Euclidean ball $B$. For more general operators and under the conditions (a)-(c), Keller and Cohen [8], Keller and Keener [17] and Crandall and Rabinowitz [11] established
the existence of a positive parameter $\lambda^*$ such that for any $0 < \lambda < \lambda^*$, the problem (2) possesses a stable minimal positive strong solution $u_\lambda$. Moreover, the family of solutions $(u_\lambda)$ is increasing and differentiable with respect to $\lambda$. Stability here means that the first eigenvalue $\mu_1(-\Delta - \lambda f'(u_\lambda))$, corresponding to the linearized problem, is positive. Already for $\lambda > \lambda^*$, they showed that there is no nonnegative strong solution. When $\mathcal{L} = \Delta$, by taking the pointwise limit of $(u_\lambda)$ as $\lambda \uparrow \lambda^*$, Brezis, Cazenave, Martel and Ramiandrisoa proved (see Lemma 5 of [2]) that a unique minimal nonnegative weak solution $u^* \in L^1(\Omega)$ exists for $\lambda = \lambda^*$, so called extremal solution of (2), and no nonnegative weak solution exist for $\lambda > \lambda^*$ under the extra condition that $f$ is a convex function.

By adapting some ideas of [2], we also deduce that extremal solution for $\lambda = \lambda^*$ regarding operators in nondivergence form with slightly smoother coefficients, namely $\mathcal{L} = a_{kl}(x)\partial_{kl} + b_j(x)\partial_j + c(x)$ satisfying $a_{kl} \in C^2(\overline{\Omega})$ and $b_j \in C^1(\overline{\Omega})$ for every $k, l, j = 1, \ldots, n$. In fact, denote by $\mathcal{L}^*$ the adjoint operator of $\mathcal{L}$. Note that $0 < \mu_1 := \mu_1(-\mathcal{L}, \Omega) = \mu_1(-\mathcal{L}^*, \Omega)$. Let $\phi_1^* \in W^{2,n}(\Omega) \cap W^{1,n}_{0}(\Omega)$ be a positive eigenfunction of $-\mathcal{L}^*$ associated to $\mu_1$. By (a) and (c) there exists a constant $C > 0$ such that $f(t) \geq \frac{2\mu_1}{\lambda^*}t - C$ for all $t \geq 0$. Using this inequality, we get

$$\lambda \int_{\Omega} f(u_\lambda)\phi_1^*dx = \int_{\Omega} u_\lambda(-\mathcal{L}^*\phi_1^*)dx = \mu_1 \int_{\Omega} u_\lambda \phi_1^*dx \leq \frac{1}{2} \lambda^* \int_{\Omega} f(u_\lambda)\phi_1^*dx + \frac{1}{2} \lambda^* C \int_{\Omega} \phi_1^*dx.$$ 

Letting $\lambda \uparrow \lambda^*$ and using (b), we derive that $\lim_{\lambda \to \lambda^*} \int_{\Omega} f(u_\lambda)\phi_1^*dx$ exists and is finite. We then consider the positive strong solution $\zeta^* \in W^{2,n}(\Omega) \cap W^{1,n}_{0}(\Omega)$ of the problem

$$\begin{cases} -\mathcal{L}^* u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega \end{cases}$$

and notice by Hopf’s Lemma that $\zeta^* \leq C\phi_1^*$ in $\Omega$ for some constant $C > 0$. Thus, we obtain

$$\int_{\Omega} u_\lambda dx = \lambda \int_{\Omega} f(u_\lambda)\zeta^*dx \leq \lambda^* C \int_{\Omega} f(u_\lambda)\phi_1^*dx,$$

so that $(u_\lambda)$ is uniformly bounded in $L^1(\Omega)$. Since $u_\lambda$ is increasing on $\lambda$, it follows that the limit $u^* = \lim_{\lambda \to \lambda^*} u_\lambda$ exists in $L^1(\Omega)$. Moreover, again by (b), we have $f(u^*) \in L^1(\Omega, \delta(x)dx)$, where $\delta(x)$ represents the distance function of $x$ to $\partial \Omega$. Then, one easily concludes that $u^*$ is a minimal nonnegative weak solution of (2). The part of uniqueness follows directly from Lemma 5.1.

A fundamental question raised by Brezis and Vázquez [3] is whether the extremal solution $u^*$ is bounded and so classical. There are important answers concerning the case $\mathcal{L} = \Delta$. More previously, in 1973, Joseph and Lundgren [16] showed that $u^*$ is not bounded when $f(u) = e^u$ and $\Omega$ is the unit ball in $\mathbb{R}^n$ of dimension $n \geq 10$, because $\lambda^* = 2(n - 2)$ and $u^*(x) = -2 \log |x|$ and, in 1980, Mignot and Puel [23] proved the boundedness of $u^*$ for any dimension $n \leq 9$. Henceforth, other quite relevant results were established during the last two decades: In 2000, Nedev [26] showed that the extremal solution $u^*$ is bounded in dimensions $n \leq 3$ whenever $f$ is convex; still for such nonlinearities, Cabré and Capella [5] proved in 2006 that $u^*$ is bounded when $\Omega$ is the unit ball in $\mathbb{R}^n$ and $n \leq 9$; regarding more general functions $f$, Cabré [4] established in 2010 the boundedness of $u^*$ for dimensions $n \leq 4$ with $\Omega$ convex in case $n \in \{3, 4\}$; in 2013, Villegas [31] removed the convexity of $\Omega$ in dimension $n = 4$, however, convexity of $f$ is required; lastly, more
recently, Cabré, Figalli, Ros-Oton and Serra [6] proved that \( u^* \) is bounded in dimensions \( n \leq 9 \) whenever \( f \) is convex and, moreover, the dimension \( n = 9 \) is optimal in this case.

Part of this work is dedicated to the study for any \( m \geq 1 \) of existence of a hypersurface \( \Lambda^* \) decomposing \( \mathbb{R}^m_+ \) into two connected components \( \mathcal{A} \) and \( \mathcal{B} \) such that the system (1) admits a stable minimal positive strong solution whenever \( \Lambda \in \mathcal{A} \) and has no nonnegative strong solution for \( \Lambda \in \mathcal{B} \). If further the coefficients of \( L_i \) are a little more smooth, a unique minimal nonnegative weak solution (naturally called extremal solution) exists for every \( \Lambda \in \Lambda^* \). In addition, we establish its boundedness in dimensions \( n = 2 \) and \( n = 3 \) for maps \( F \) of potential type.

In order to state the requirements satisfied by the map \( F \), we introduce some suitable notations. For \( a = (a_1, \ldots, a_m), \quad b = (b_1, \ldots, b_m) \in \mathbb{R}^m \) and \( \gamma = (\gamma_1, \ldots, \gamma_m) \in \mathbb{R}^m_+ \), consider the product notations

\[
ab = (a_1 b_1, \ldots, a_m b_m), \quad \Pi \gamma = \prod_{i=1}^m \gamma_i
\]

and the nonlinear shift

\[
S_\gamma(a) = \left(|a_2|^{\gamma_1-1} a_2, |a_3|^{\gamma_2-1} a_3, \ldots, |a_m|^{\gamma_{m-1}-1} a_m, |a_1|^{\gamma_m-1} a_1 \right).
\]

The order symbol \(<\) (\(\leq\)) to be used between vectors means: \( a < b \) (or \( a \leq b \)) iff \( a_i < b_i \) (or \( a_i \leq b_i \)) for every \( i = 1, \ldots, m \).

The map \( F : \Omega \times \mathbb{R}^m \to \mathbb{R}^m \) of interest in this paper satisfies some very general regularity conditions, namely, \( F(x, \cdot) : \mathbb{R}^m \to \mathbb{R}^m \) is continuous for \( x \in \Omega \) almost everywhere and \( F(\cdot, t) : \Omega \to \mathbb{R}^m \) belongs to \( L^n(\Omega; \mathbb{R}^m) := L^n(\Omega) \times \cdots \times L^n(\Omega) \) for every \( t \in \mathbb{R}^m_+ \). Furthermore, \( F \) satisfies three assumptions closely related to the corresponding scalar (a)-(c):

(A) \( F(x, 0) > 0 \) for \( x \in \Omega \) almost everywhere \((0 = (0, \ldots, 0) \in \mathbb{R}^m) \) \( (F \) is positive at the origin);

(B) \( F(x, s) \leq F(x, t) \) for \( x \in \Omega \) almost everywhere and \( s, t \in \mathbb{R}^m \) with \( 0 \leq s \leq t \) \( (F \) is nondecreasing);

(C) There exist a map \( \rho \in L^n(\Omega; \mathbb{R}^m) \) such that \( \rho(x) > 0 \) for \( x \in \Omega \) almost everywhere and a tuple \( \alpha \in \mathbb{R}^m_+ \) with \( \Pi \alpha = 1 \) such that for any \( \kappa > 0 \), we have

\[
F(x, t) \geq \kappa \rho(x) S_\alpha(t)
\]

for \( x \in \Omega \) almost everywhere and \( t \in \mathbb{R}^m_+ \) with \( |t| > M \), where \( M > 0 \) is a constant depending on \( \kappa \) \( (F \) is “superlinear” at infinity).

Notice that the hypothesis (C) provides simultaneously a coupling of the system (1) through the lower comparison of \( F(\cdot, t) \) by the shift \( S_\alpha(t) \) and a kind of superlinearity for maps \( F \) expressed by the condition \( \Pi \alpha = 1 \).

We highlight below some important prototype examples of maps \( F \) that fulfill the above assumptions.

We select below four examples of maps \( F \) satisfying the assumptions (A), (B) and (C) and associated to strongly coupled systems.
Example 1.1. Consider $F(x,t) = \rho(x)S_\beta \circ \mathcal{E}(t)$, where $\rho = (\rho_1, \ldots, \rho_m) \in L^\infty(\Omega; \mathbb{R}^m)$ is a positive map almost everywhere in $\Omega$, $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{R}_+^m$ is a $m$-tuple and $\mathcal{E}(t) := (e^{t_1}, \ldots, e^{t_m})$. In explicit form, we have
\[
F(x,t) = (\rho_1(x)e^{\beta_1t_2}, \ldots, \rho_{m-1}(x)e^{\beta_{m-1}t_m}, \rho_m(x)e^{\beta_mt_1}).
\]

Example 1.2. Consider $F(x,t) = \mathcal{P}_\alpha(\tau + \rho S_\beta(x,t))$, where $\rho = (\rho_1, \ldots, \rho_m)$, $\tau = (\tau_1, \ldots, \tau_m) \in L^\infty(\Omega; \mathbb{R}^m)$ are positive maps almost everywhere in $\Omega$, $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{R}_+^m$ are $m$-tuples such that $\Pi(\alpha \beta) > 1$ and $\mathcal{P}_\alpha(t) = (t_1^\alpha, \ldots, t_m^\alpha)$. In explicit form, we have
\[
F(x,t) = ((\rho_1(x)t_2^\alpha + \tau_1(x))^\alpha_1, \ldots, (\rho_{m-1}(x)t_m^\alpha + \tau_{m-1}(x))^\alpha_{m-1}, (\rho_m(x)t_1^\alpha + \tau_m(x))^\alpha_m).
\]

Example 1.3. Consider $F(x,t) = S_\beta(\tau(x) + A(x)t)$, where $A = [A_{ij}] \in L^\infty(\Omega; \mathbb{R}^{m^2})$ is a nonnegative matrix almost everywhere in $\Omega$ with positive diagonal entries, $\tau = (\tau_1, \ldots, \tau_m) \in L^\infty(\Omega; \mathbb{R}^m)$ is a positive map almost everywhere in $\Omega$ and $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{R}_+^m$ is a $m$-tuple such that $\Pi \beta > 1$. In explicit form, we have
\[
F(x,t) = \left(\sum_{j=1}^m A_{2j}(x)t_j + \tau_2(x)\right)^{\beta_1}, \ldots, \left(\sum_{j=1}^m A_{mj}(x)t_j + \tau_m(x)\right)^{\beta_{m-1}}, \left(\sum_{j=1}^m A_{1j}(x)t_j + \tau_1(x)\right)^{\beta_m}.
\]

Example 1.4. Consider $F(x,t) = \rho(x)\nabla f(t)$, where $\rho \in L^\infty(\Omega)$ is a positive function almost everywhere in $\Omega$ and $f(t) = \Pi_{i=1}^m f_i(t_i)$, $t = (t_1, \ldots, t_m)$, being $f_i : \mathbb{R} \to \mathbb{R}$ convex functions of $C^1$ class satisfying the scalar conditions (a), (b) and (c) and, in addition, $f_i'(0) > 0$ for every $i = 1, \ldots, m$.

Our first result states that

**Theorem 1.1.** Let $F : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ be a map such that $F(x, \cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is continuous for $x \in \Omega$ almost everywhere and $F(\cdot, t) : \Omega \to \mathbb{R}^m$ belongs to $L^\infty(\Omega; \mathbb{R}^m)$ for any $t \in \mathbb{R}^m$. Assume also that $F$ satisfies (A), (B) and (C). Then, there exists a set $\Lambda^*$ decomposing $\mathbb{R}_+^m$ into two sets $\Lambda$ and $B$ satisfying the following assertions:

(I) Problem (1) admits a minimal positive strong solution $u_\Lambda \in W^{2,n}(\Omega; \mathbb{R}^m) \cap W^{1,n}_0(\Omega; \mathbb{R}^m)$ for every $\Lambda \in A$. Moreover, the map $\Lambda \in A \mapsto u_\Lambda$ is nondecreasing and differentiable with respect to $\Lambda$;

(II) Problem (1) admits no nonnegative strong solution for any $\Lambda \in B$.

The next result provides some properties about regularity and shape of the separating set $\Lambda^*$.

**Theorem 1.2.** Let $F$ be as in Theorem 1.1. The set $\Lambda^*$ admits a parametrization $\Phi : \mathbb{R}_+^{m-1} \to \Lambda^* \subset \mathbb{R}_+^m$ of the form $\Phi(\sigma) = (\lambda^*(\sigma), \nu^*(\sigma)) := (\lambda^*(\sigma), \lambda^*(\sigma)\sigma)$, where $\lambda^*(\sigma)$ is a positive number for each $\sigma \in \mathbb{R}_+^{m-1}$. Moreover, we have:

(I) $\Phi$ is continuous;

(II) $\lambda^*(\sigma)$ is nonincreasing, that is, $\lambda^*(\sigma_1) \geq \lambda^*(\sigma_2)$ whenever $0 < \sigma_1 \leq \sigma_2$;
(III) For each $0 < \sigma_1 \leq \sigma_2$, there exists $i \in \{1, \ldots, m\}$ such that $\nu^*_i(\sigma_1) \leq \nu^*_i(\sigma_2)$. In particular, for $m = 2$, $\nu^*$ is nondecreasing;

(IV) $\lambda^*(\sigma) \to 0$ as $\sigma_i \to +\infty$ for each fixed $i = 1, \ldots, m - 1$.

As we shall see later, the sets $A$ and $B$ are related to the parametrization $\Phi(\sigma)$ as follows:

$$A = \{ (\lambda, \lambda \sigma) : 0 < \lambda < \lambda^*(\sigma), \sigma \in \mathbb{R}^{m-1}_+ \},$$

$$B = \{ (\lambda, \lambda \sigma) : \lambda > \lambda^*(\sigma), \sigma \in \mathbb{R}^{m-1}_+ \}.$$

In particular, Theorem 1.2 implies that the sets $A$ and $B$ are connected components of $\mathbb{R}^m_+$.

The first coordinate $\lambda^*(\sigma)$ of $\Phi(\sigma)$ may be bounded or unbounded depending on the growth on $t$ of the first coordinate $f_1(x, t)$ of the map $F(x, t)$. We discuss on this question in the following two remarks:

**Remark 1.1.** $\lambda^*(\sigma)$ is bounded if

$$f_1(x, t_1, 0, \ldots, 0) \geq \tau(x)t_1 \text{ for } x \in \Omega \text{ a.e. and } t_1 \geq 0,$$

where $\tau \in L^n(\Omega)$ and $\tau(x) > 0$ for $x \in \Omega$ almost everywhere. This is a consequence of the strong maximum principle satisfied, by assumption, by the operator $L_1$. Let $\varphi_1 \in W^{2,n}(\Omega) \cap W^{1,n}_0(\Omega)$ be a positive Dirichlet eigenfunction associated to the principal eigenvalue $\mu_1 = \mu_1(-L_1, \Omega, \tau)$ of the problem

$$\begin{cases}
-\mathcal{L}_1 \varphi &= \mu \tau(x) \varphi \text{ in } \Omega, \\
\varphi &= 0 \text{ on } \partial \Omega.
\end{cases}$$

The existence of the couple $(\mu_1, \varphi_1)$ follows as a byproduct of Theorem 6.4 of [18] and a simple approximation argument. We ensure that $\lambda \leq \mu_1$ for every $0 < \lambda < \lambda^*(\sigma)$, so that $\lambda^*(\sigma)$ is upper bounded by $\mu_1$ and the assertion follows. In fact, by contradiction, assume $\lambda > \mu_1$ for some $0 < \lambda < \lambda^*(\sigma)$ and set $\Delta = (\lambda, \lambda \sigma)$. As above claimed, one has $\Delta \in A$ and so by Theorem 1.1, the system (1) admits a minimal positive strong solution $u_\Delta = (u_1, \ldots, u_m)$. Consider now the set $S = \{ s > 0 : u_1 > s\varphi_1 \text{ in } \Omega \}$ which is clearly nonempty (by the Hopf’s lemma and strong maximum principle) and upper bounded. Let $s^* = \sup S > 0$. Then, $u_1 \geq s^*\varphi_1$ in $\Omega$ and

$$-\mathcal{L}_1(u_1 - s^*\varphi_1) = \lambda f_1(x, u_\Delta) - s^*\mu_1 \tau(x) \varphi_1$$

$$\geq \lambda \rho(x)u_1 - s^*\mu_1 \tau(x) \varphi_1$$

$$\geq s^*(\lambda - \mu_1) \tau(x) \varphi_1$$

$$> 0 \text{ in } \Omega$$

Again, by Hopf’s lemma and strong maximum principle, we get the contradictory inequality $u_1 \geq (s^* + \varepsilon)\varphi_1$ in $\Omega$ for $\varepsilon > 0$ small enough.
Remark 1.2. \( \lambda^*(\sigma) \) is unbounded if for any \( \lambda > 0 \) there exists \( t_0 = (t_1, \ldots, t_m) \in \mathbb{R}_+^m \) such that

\[
\lambda \|f_1(\cdot, t_0)\|_{L^n(\Omega)} \leq t_1.
\]

Let a fixed \( \lambda_0 > 0 \) and consider a constant \( C_0 > 0 \) such that, for any \( w \in W^{2,n}(\Omega) \cap W^{1,n}_0(\Omega) \),

\[
\|w\|_{L^\infty(\Omega)} \leq C_0 \|L_1w\|_{L^n(\Omega)}.
\]

By assumption, there exists \( t_0 = (t_1, \ldots, t_m) \in \mathbb{R}_+^m \) such that

\[
\lambda_0 C_0 \|f_1(\cdot, t_0)\|_{L^n(\Omega)} \leq t_1.
\]

For each \( i = 2, \ldots, m \), consider \( w_i \in W^{2,n}(\Omega) \cap W^{1,n}_0(\Omega) \) the strong solution of the problem

\[
\begin{cases}
-L_i w = f_i(x, t_0) & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega
\end{cases}
\]

Choose \( s > 0 \) such that \( \overline{w}_i := sw_i \leq t_i \) in \( \Omega \) for \( i = 2, \ldots, m \). Now let \( \overline{w}_1 \in W^{2,n}(\Omega) \cap W^{1,n}_0(\Omega) \) be the strong solution of

\[
\begin{cases}
-L_1 w = \lambda_0 f_1(x, t_1, \overline{w}_2, \ldots, \overline{w}_m) & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega
\end{cases}
\]

Thanks to the initial assumption and a priori estimate (3), we have \( \overline{w}_1 \leq t_1 \) in \( \Omega \), so that \( \overline{w}_0 := (\overline{w}_1, \ldots, \overline{w}_m) \leq t_0 \). But this inequality together with the definition of \( \overline{w}_i \) and the assumption (B) imply that \( \overline{w}_0 \) is a positive supersolution of

\[
\begin{cases}
-L u = \Lambda_0 F(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Lambda_0 = (\lambda_0, \lambda_0 \sigma) \in \mathbb{R}_+^m \) and \( \sigma = (s/\lambda_0, \ldots, s/\lambda_0) \in \mathbb{R}_+^{m-1} \). Then, using the fact that the zero map \( 0 = (0, \ldots, 0) \) is a subsolution of the above system, it follows from the super-subsolution method (see the proof of Theorem 1.1 in Section 3) that \( \Lambda_0 \in \mathcal{A} \), so that \( \lambda^*(\sigma) \geq \lambda_0 \). Since \( \lambda_0 > 0 \) is an arbitrary number, the desired conclusion follows.

For the existence on \( \Lambda^* \) we require that the coefficients \( a_{ki}^i \) and \( b_i^j \) of \( L_i \) are in \( C^2(\overline{\Omega}) \) and \( C^1(\overline{\Omega}) \), respectively, for every \( i = 1, \ldots, m \). Precisely, we have the following extended notion (see Definition 1 of [2]) of weak solution of (1):

**Definition 1.1.** Let \( \Lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m \). We say that \( u \in L^1(\Omega; \mathbb{R}^m) \) is a nonnegative weak solution of (1), if \( u \geq 0 \) almost everywhere in \( \Omega \),

\[
F(\cdot, u(\cdot))\delta(\cdot) \in L^1(\Omega; \mathbb{R}^m)
\]

and

\[
-\int_\Omega u L^* \zeta dx = \Lambda \int_\Omega F(x, u) \zeta dx
\]
for all $\zeta \in W^{2,n}(\Omega; \mathbb{R}^m) \cap W^{1,n}_0(\Omega; \mathbb{R}^m)$ such that $\mathcal{L}^* \zeta \in L^\infty(\Omega; \mathbb{R}^m)$.

Precisely, we have:

**Theorem 1.3.** Let $F : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ be a map such that $F(x, \cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is continuous for $x \in \Omega$ almost everywhere and $F(\cdot, t) : \Omega \to \mathbb{R}^m$ belongs to $L^a(\Omega; \mathbb{R}^m)$ for any $t \in \mathbb{R}^m$. Assume that $F$ satisfies (A), (B) and (C). Then, Problem (1) admits a unique minimal nonnegative weak solution $u^*_\Lambda$ for every $\Lambda \in \Lambda^*$.

We next introduce the concept of linearized stability for strong solutions of (1) which are known as steady states of the parabolic system

\[
\begin{cases}
\frac{\partial u}{\partial t} - \mathcal{L}u = AF(x,u) & \text{in } \Omega \times (0, +\infty), \\
u(x,0) = u_0(x) & \text{for } x \in \Omega, \\
u(x,t) = 0 & \text{for } (t,x) \in \partial \Omega \times (0, +\infty),
\end{cases}
\]

where $u(x,t) = (u_1(x,t), \ldots, u_m(x,t))$ and $F(x,t) = (f_1(x,t), \ldots, f_m(x,t))$.

Let $u$ be a steady state of the system (5). For a suitable notion of stability of $u$, one hopes naturally that for any globally defined strong solution of the form $v(x,t) = u(x) + e^{-\eta t}\varphi(x)$, one necessarily has $\eta > 0$. Assuming that $F(x, \cdot)$ is of $C^1$ class for $x \in \Omega$ almost everywhere, writing (5) for $v(x,t)$ and letting $t \to +\infty$, one gets an eigenvalue problem under the form of elliptic system, namely,

\[
\begin{cases}
-\mathcal{L}\varphi - \Lambda(A(x,u)\varphi) = \eta\varphi & \text{in } \Omega, \\
\varphi = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where $A(x,t)$ denotes the matrix $[Aij(x,t)]$ with entries

\[A_{ij}(x,t) := \frac{\partial f_i}{\partial x_j}(x,t).\]

**Definition 1.2.** A steady state $u$ of the problem (5) is said to be stable in the linearized sense, if the system (6) has a smallest positive eigenvalue $\eta_1$.

The linearized stability will be achieved for the minimal positive strong solutions $u_\Lambda$ with $\Lambda \in \mathcal{A}$ of Theorem 1.1 under the additional assumptions:

\[\sup_{|t| \leq a} |A(x,t)| \in L^n(\Omega) \text{ for every } a > 0\] (7)

and

(D) $A_{ij}(x,t) \geq 0$ for $x \in \Omega$ a.e. and $t > 0$ for any $i \neq j$ and $|\{x \in \Omega : A_{is_i}(x,t) > 0, \forall t > 0\}| > 0$ for every $i$, where $s$ is the shift permutation of $\{1, \ldots, m\}$, that is, $s_i = i + 1$ for $i = 1, \ldots, m - 1$ and $s_m = 1$.

It deserves to mention that the permutation $s$ has order $m$, that is, $m$ is the smallest integer $l$ such that $s^l = s \circ \ldots \circ s$ is equal the identity.

The assumptions (B) and (D) imply that the matrix $A(x,t)$ is nonnegative and, in particular, cooperative. Notice also that $A_{is_i}(\cdot, u_\Lambda) \neq 0$ for every $i$. In practice in examples, the nonlinear shift $S_\alpha$ and the shift permutation $s$ connect the assumptions (C) and (D).
Theorem 1.4. Let $F : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ be a map such that $F(x, \cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is of $C^1$ class for $x \in \Omega$ almost everywhere, $F(\cdot, t) : \Omega \to \mathbb{R}^m$ belongs to $L^n(\Omega; \mathbb{R}^m)$ for any $t \in \mathbb{R}^m$ and (7) is satisfied. Assume also that $F$ satisfies (A), (B), (C) and (D). Then, the minimal positive strong solution $u_\Lambda \in W^{2,n}(\Omega; \mathbb{R}^m) \cap W^{1,n}_0(\Omega; \mathbb{R}^m)$ is stable in the linearized sense for every $\Lambda \in \mathcal{A}$. Moreover, each eigenvalue $\mu_1$ of (6) is simple and admits a positive eigenfunction in $\Omega$.

Theorems 1.1, 1.2, 1.3 and 1.4 provide substantial improvements in the case $m = 2$, previously studied by Montenegro in [24] (see pages 406-408), by means of the weaker assumptions (A), (B), (C) and (D) and also extend the latter to $m \geq 3$ with new contributions.

Our final results concern the boundedness of extremal solutions $u^*_\Lambda$ of (1) for any $m \geq 1$ in the special case that all $\mathcal{L}_i$ are Laplace operators and $F : \mathbb{R}^m \to \mathbb{R}^m$ is a potential field. More precisely, given a function $f : \mathbb{R}^m \to \mathbb{R}$ of $C^2$ class, we consider the following problem for the gradient field $F = \nabla f$:

$$\begin{align*}
-\Delta u &= \Lambda \nabla f(u) \quad \text{in} \quad \Omega, \\
u &= 0 \quad \text{on} \quad \partial \Omega.
\end{align*}$$

(8)

The system (8) is called symmetric in the sense that the matrices $A(x, t) = \text{Hess} f(t)$ associated to the linearized problem are symmetric.

For $m = 2$, Cowan and Fazly [10] showed that the extremal solutions $u^*_\Lambda$ with $\Lambda \in \Lambda^*$ are bounded for convex domains in dimensions $n = 2$ and $n = 3$ and for functions $f$ of separable variables, that is, $f(u_1, u_2) = f_1(u_1)f_2(u_2)$, where $f_1$ and $f_2$ satisfy some assumptions which lead to the conditions (A), (B), (C) and (D). When $\Omega$ is the unit ball in $\mathbb{R}^n$, the authors also proved the boundedness of the corresponding extremals for dimensions $2 \leq n \leq 6$. Previous results involving regularity of extremal solutions can be found in [9].

When $m \geq 3$ and $\Omega = B$, Fazly [12] considered symmetric systems and established that the stable minimal classical solutions $u_\Lambda = (u_1, \ldots, u_m)$, which are radial for any $\Lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{A}$, satisfy

(i) $\sum^{m}_{i=1} \frac{|u_i(r)|}{\sqrt{\Lambda_i}} \leq C_{n,m} \sum^{m}_{i=1} \frac{1}{\sqrt{\Lambda_i}} \|u_i\|_{H^1(B(B_{1/2})} \quad \text{for} \quad r \in (0, 1], \quad \text{if} \quad 2 \leq n \leq 9$;

(ii) $\sum^{m}_{i=1} \frac{|u_i(r)|}{\sqrt{\Lambda_i}} \leq C_{n,m}(1 + |\log r|) \sum^{m}_{i=1} \frac{1}{\sqrt{\Lambda_i}} \|u_i\|_{H^1(B(B_{1/2})} \quad \text{for} \quad r \in (0, 1], \quad \text{if} \quad n = 10$;

(iii) $\sum^{m}_{i=1} \frac{|u_i(r)|}{\sqrt{\Lambda_i}} \leq C_{n,m}r^{-\frac{n}{2} + \sqrt{n-1} + 2} \sum^{m}_{i=1} \frac{1}{\sqrt{\Lambda_i}} \|u_i\|_{H^1(B(B_{1/2})} \quad \text{for} \quad r \in (0, 1], \quad \text{if} \quad n \geq 11$

for some positive constant $C_{n,m}$ independent of $r$, where $B_\delta$ denotes the ball in $\mathbb{R}^n$ of radius $\delta$ centered at the origin. These three estimates were previously obtained for $m = 1$ by Villegas [30].

On the other hand, a straightforward argument produces a bound for the above terms $\|u_i\|_{H^1(B(B_{1/2})}$ for $\Lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{A}$ near the extremal set $\Lambda^*$. In fact, let $\varphi \in C^\infty(\mathbb{R}^n)$ be a fixed function such that $0 \leq \varphi \leq 1$, $\varphi = 0$ in $B_{1/4}$ and $\varphi = 1$ in $B \setminus \overline{B_{1/2}}$. Multiplying the $i$-th equation of (8), $-\Delta u_i = \lambda_i f_{u_i}(u_\Lambda)$, by $u_i \varphi^2$, integrating by parts and using Young’s inequality and that $u_i$ is radially decreasing, we derive

$$\begin{align*}
\int_{B \setminus \overline{B_{1/4}}} |\nabla u_i|^2 \varphi^2 dx &= \lambda_i \int_{B \setminus \overline{B_{1/4}}} f_{u_i}(u_\Lambda) u_i \varphi^2 dx + \int_{\partial B_{1/4}} \frac{\partial u_i}{\partial \nu} u_i \varphi^2 d\sigma - \int_{B \setminus \overline{B_{1/4}}} 2u_i \varphi \nabla u_i \cdot \nabla \varphi dx \\
&\leq \lambda_i \int_{B \setminus \overline{B_{1/4}}} f_{u_i}(u_\Lambda) u_i dx + 2 \int_{B \setminus \overline{B_{1/4}}} |\nabla \varphi|^2 u_i^2 dx + \frac{1}{2} \int_{B \setminus \overline{B_{1/4}}} |\nabla u_i|^2 \varphi^2 dx.
\end{align*}$$

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Thus,

$$\int_{B \setminus B_{1/2}} |\nabla u_i|^2 \, dx \leq 2\lambda_i \int_{B \setminus B_{1/4}} f_i(u_i) u_i \, dx + C \int_{B \setminus B_{1/4}} u_i^2 \, dx. \quad (9)$$

In addition, we also have

$$\sup_{B \setminus B_{1/4}} u_i(x) = u_i(1/4) \leq \frac{1}{|B_{1/4} \setminus B_{1/8}|} \int_{B_{1/4} \setminus B_{1/8}} u_i(x) \, dx \leq C_n \|u_i\|_{L^1(B)} \leq C_n \|u_\Lambda\|_{L^1(B;\mathbb{R}^m)}. \quad (10)$$

Therefore, for a fixed $\sigma \in \mathbb{R}^{m-1}$, Lemma 5.3 and the estimates (9) and (10) provide, for any $\lambda^*(\sigma) < \lambda < \lambda^*(\sigma)$,

$$\|u_\Lambda\|_{H^1((B \setminus B_{1/2});\mathbb{R}^m)} \leq C$$

for some positive constant $C$ independent of $\lambda$. In particular, if $(\lambda_k)$ is an increasing sequence converging to $\lambda^*(\sigma)$, then the above estimate holds for $\Lambda_k = (\lambda, \lambda_k \sigma)$ with $k$ large enough. So, plugging the radial solutions $u_{\Lambda_k}$ in the above inequalities (i), (ii) and (iii) obtained by Fazly and after letting $k \to +\infty$, we deduce the following result for the extremal solution $u_\Lambda^*$:

**Theorem 1.5.** Let $F : \mathbb{R}^m \to \mathbb{R}^m$ be a potential field of $C^1$ class satisfying the assumptions (A), (B), (C) and (D). If $\Omega = B$, then the extremal solution $u_\Lambda^*$ of (8) is radial for any $\Lambda \in \Lambda^*$ and, in addition, satisfies

(I) $u_\Lambda^* \in L^\infty(B)$, if $2 \leq n \leq 9$;

(II) $u_\Lambda^*(r) \leq C(1 + |\log r|)$ for $r \in (0, 1]$, if $n = 10$;

(III) $u_\Lambda^*(r) \leq C r^{-\frac{m+\sqrt{n-1}}{2}}$ for $r \in (0, 1]$, if $n \geq 11$,

where $C > 0$ is a constant depending only on $n$, $m$ and $\Lambda$.

Inspired on the insightful developing done in the scalar context by Cabré in [4], we also study the regularity of extremal solutions of (8) for more general $C^{1,1}$ domains $\Omega$.

Let $F = \nabla f$ be a field of $C^1$ class satisfying (A), (B), (C) and (D) and let $u_\Lambda$ be the stable minimal positive strong solution of (8) associated to $\Lambda = (\lambda_1, \ldots, \lambda_m) \in A$. Based on some important ideas of Fazly and Ghoussoub in [13], we provide for completeness the proof that linearized stability of $u_\Lambda$ implies a fundamental geometric Poincaré inequality (see Proposition 7.2). Namely, for any test map $\psi = (\psi_1, \ldots, \psi_m) \in H^1_0(\Omega; \mathbb{R}^m)$,

$$\sum_{i,j} \int_\Omega f_{i,j}(u_\Lambda) \psi_i \psi_j \, dx \leq \sum_i \frac{1}{\lambda_i} \int_\Omega |\nabla \psi_i|^2 \, dx. \quad (11)$$

Such an inequality has been proved in [12] and [13] for $\Lambda = (1, \ldots, 1)$ and the proof for arbitrary $\Lambda \in A$ is carried out in the same line. As we shall see, the validity of (11) plays a key role in the proof of boundedness of $u_\Lambda^*$ when $n = 2$ or $n = 3$.

In a precise way:

**Theorem 1.6.** Let $F : \mathbb{R}^m \to \mathbb{R}^m$ be a potential field of $C^1$ class satisfying (A), (B), (C) and (D). Assume that $\Omega$ is convex and $n = 2$ or $n = 3$. Then, the extremal solution $u_\Lambda^*$ of (8) is bounded for any $\Lambda \in \Lambda^*$. 

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Theorem 1.6 fairly improves the regularity result established by Cowan and Fazly for potential fields with $m = 2$ (see Theorem 2.1 of [10]) and is a new contribution for $m \geq 3$ in more general domains.

The paper is organized into six sections. In Section 2 we investigate principal eigenvalues for a class of nonlinear elliptic systems and introduce an essential ingredient, namely, the so-called related principal spectral hypersurface. Section 3 is dedicated to the proof of Theorem 1.1 which provides an extremal separating set with respect to existence of positive strong solutions. Section 4 is devoted to the proof of Theorem 1.2, which provides some qualitative properties of this set. In Section 5 we prove Theorem 1.3 which ensures the existence of extremal solution on the separating set. In Section 6 we prove the linearized stability of minimal positive strong solutions stated in Theorem 1.4. Finally, in Section 7 we prove Theorem 1.6 which guarantees the regularity of extremal solutions under convexity of domain in dimensions $n = 2$ and $n = 3$.

2 The principal spectral hypersurface

Consider the following eigenvalue problem

\[
\begin{cases}
-L\varphi &= \Lambda \rho(x) S_\alpha \varphi \quad \text{in } \Omega, \\
\varphi &= 0 \quad \text{on } \partial \Omega,
\end{cases}
\]

(12)

where $\rho \in L^m(\Omega; \mathbb{R}^m)$ is a map satisfying $\rho(x) > 0$ for $x \in \Omega$ almost everywhere, $m \geq 1$ and $\alpha \in \mathbb{R}^m_+$ satisfies $\Pi \alpha = 1$.

The tuple $\Lambda \in \mathbb{R}^m$ is said to be an eigenvalue of (12) if the problem admits a nontrivial strong solution $\varphi \in W^{2,m}(\Omega; \mathbb{R}^m) \cap W^{1,m}_0(\Omega; \mathbb{R}^m)$ which by Sobolev embedding is in $C^1(\overline{\Omega}; \mathbb{R}^m)$. Here $C^1(\overline{\Omega}; \mathbb{R}^m)$ stands for the Banach space $\{ u \in C^1(\overline{\Omega}; \mathbb{R}^m) : u = 0 \text{ on } \partial \Omega \}$ endowed with the norm

$$
\| u \|_{C^1(\overline{\Omega}; \mathbb{R}^m)} := \sum_{i=1}^m \| u_i \|_{C^1(\overline{\Omega})},
$$

where $C^1(\overline{\Omega}; \mathbb{R}^m)$ denotes the product space $C^1(\overline{\Omega}) \times \cdots \times C^1(\overline{\Omega})$. Furthermore, if all components of $\varphi$ are positive in $\Omega$, then $\Lambda$ is called a principal eigenvalue of (12). We denote by $\Gamma_\alpha(\Omega, \rho)$ the set of all principal eigenvalues of the above problem.

The central question discussed in this section concerns the characterization of the set $\Gamma_\alpha(\Omega, \rho)$. In particular, we will show that this set makes up a smooth hypersurface in $\mathbb{R}^m$ which will be referred as principal hypersurface of $\mathbb{R}^m$ basically for two reasons. Firstly, the above problem is an extension of the Dirichlet eigenvalue problem for uniformly elliptic operators which corresponds to $m = 1$ and $\rho = 1$ in $\Omega$. So, in this case, the set $\Gamma_\alpha(\Omega, \rho)$ is single and represents the principal eigenvalue of these operators. Secondly, when $m = 2$ it has been proved in [25] that the set $\Gamma_\alpha(\Omega, \rho)$ is the image of a smooth curve satisfying a number of qualitative properties.

Strong maximum principle and Hopf’s lemma for uniformly elliptic operators, here denoted respectively by (SMP) and (HL), will play key tools in our proofs.

In the sequel, we will use a nonlinear version of Krein-Rutman Theorem (see for example Chang [7] or Mahadevan [20]) to show that the set $\Gamma_\alpha(\Omega, \rho)$ can be seen as the inverse image of a regular value by a smooth function $H : \mathbb{R}^m_+ \to \mathbb{R}$. 
Proposition 2.1. Let \( \rho \in L^n(\Omega; \mathbb{R}^m) \) be a positive map almost everywhere in \( \Omega \) and \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}_+^m \) be a tuple satisfying \( \Pi \alpha = 1 \) with \( m \geq 1 \). Then, there exist a function \( H : \mathbb{R}_+^m \to \mathbb{R} \) and a positive constant \( \lambda_* \) such that

\[
\Gamma_{\alpha}(\Omega, \rho) = \{ \Lambda \in \mathbb{R}_+^m : H(\Lambda) = \lambda_* \}.
\]

Moreover, the function \( H \) is given by

\[
H(\Lambda) = H(\lambda_1, \ldots, \lambda_m) := \lambda_1\lambda_2^{\alpha_1} \cdots \lambda_m^{\alpha_{m-1}}.
\]

Proof. Consider the Banach spaces

\[
E_0 = C_0(\overline{\Omega}) := \{ u \in C(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega \},
\]

\[
E_1 = C_0^1(\overline{\Omega}) := \{ u \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega \}
\]

endowed with the usual norms \( \| \cdot \|_{E_0} := \| \cdot \|_{C(\overline{\Omega})} \) and \( \| \cdot \|_{E_1} := \| \cdot \|_{C^1(\overline{\Omega})} \), respectively.

For each \( i = 1, \ldots, m \), let \( T_i : E_0 \to E_1 \) be the operator defined by \( T_iu = v \), where \( v \in W^{2,n}(\Omega) \cap W^{1,n}_0(\Omega) \subset E_1 \) is the unique strong solution of the problem

\[
\begin{aligned}
-L_i v &= \rho_i(x)|u|^{\alpha_i-1}u \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial\Omega
\end{aligned}
\]

(13)

By the Calderón-Zygmund elliptic theory, each map \( T_i \) is well-defined and continuous.

Let \( J \) be the inclusion from \( E_1 \) into \( E_0 \) which is clearly compact. We now define the composition operator \( T : E_0 \to E_0 \) by \( T := T_1 \circ \cdots \circ T_m \circ J \). Notice that \( T \) is continuous and compact. In addition, the existence and uniqueness of solution of the problem (13) yield the \( \alpha_i \)-homogeneity property of \( T_i \), that is,

\[
T_i(\tau u) = \tau^{\alpha_i}T_i(u)
\]

for all \( \tau > 0 \) and \( u \in E_0 \). So, thanks to the assumption \( \Pi \alpha = 1 \), it follows that \( T \) is positively 1-homogeneous.

Consider the positive cone of \( E_j \) given by \( K_j = \{ u \in E_j : u \geq 0 \text{ in } \Omega \} \) for \( j = 1, 2 \), so \( E_j \) is an ordered Banach space with respect to \( K_j \). We assert that \( T \) is strongly monotone with respect to \( K_0 \). Indeed, the positivity of \( \rho_i \) and the \( (\text{SMP}) \) imply that each operator \( T_i \) is monotone, so that \( T \) is monotone too. Moreover, it follows that \( T_iu_2 - T_iu_1 \in K_0 \setminus \{0\} \) whenever \( u_2 - u_1 \in K_0 \setminus \{0\} \). Consequently, \( \tilde{u}_2 - \tilde{u}_1 \in K_0 \setminus \{0\} \) provided that \( u_2 - u_1 \in K_0 \setminus \{0\} \), where \( \tilde{u} = T_2 \circ \cdots \circ T_m(u) \). Thus, the \( (\text{HL}) \) implies that \( T_1\tilde{u}_2 - T_1\tilde{u}_1 \in K_1 \), where \( K_1 \) denotes the interior of \( K_1 \). In other words, \( T_1u_2 - T_1u_1 \in K_1 \subset K_0 \) and the claim follows. Therefore, the positively 1-homogeneous operator \( T \) fulfills all the assumptions required by the nonlinear Krein-Rutman theorem. More specifically, by the fairly complete version of this result provided in Theorem 1.4 of [7], there exists a number \( \lambda_* > 0 \) such that \( \lambda_*^{-1} \) is the unique principal eigenvalue of \( T \) and is the largest among all real eigenvalues of \( T \). Moreover, all eigenfunctions of \( T \) associated to the eigenvalue \( \lambda_*^{-1} \) are multiple of a fixed eigenfunction \( \varphi_* \in K_1 \).
We now are ready to conclude the proof. Let \( \Lambda = (\lambda_1, \ldots, \lambda_m) \in \Gamma_\alpha(\Omega, \rho) \subset \mathbb{R}_m^+ \) and \( \varphi = (\varphi_1, \ldots, \varphi_m) \in C_0^1(\overline{\Omega}; \mathbb{R}^m) \) be a positive eigenfunction associated to \( \Lambda \). Then, from each equation of the problem (12), we deduce that \( \varphi_i = \lambda_i T_i \varphi_{i+1} \) for \( i = 1, \ldots, m - 1 \) and \( \varphi_m = \lambda_m T_m \varphi_1 \). Arguing step by step with replacement, we derive

\[
\varphi_1 = \lambda_1 \lambda_2^{\alpha_1} \cdots \lambda_m^{\alpha_1 \cdots \alpha_m - 1} T_1 \circ \cdots \circ T_m(\varphi_1) = H(\Lambda) T \varphi_1 .
\]

Since \( \varphi_1 \in K_0 \setminus \{0\} \), the above equality implies that \( H(\Lambda)^{-1} \) is a principal eigenvalue of \( T \), so that \( H(\Lambda) = \lambda_* \), by uniqueness.

Conversely, let \( \Lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}_m^+ \) be such that \( H(\Lambda) = \lambda_* \). Choose an eigenfunction \( \varphi_* \in K_0 \) of \( T \) associated to the principal eigenvalue \( \lambda_*^{-1} \). Define by recurrence the functions \( \varphi_m = \lambda_m T_m \varphi_* \) and \( \varphi_i = \lambda_i T_i \varphi_{i+1} \) for \( i = 1, \ldots, m - 1 \). Then, \( \varphi = (\varphi_1, \ldots, \varphi_m) \in C_0^1(\overline{\Omega}; \mathbb{R}^m) \) and, by the (SMP), the map \( \varphi \) is positive in \( \Omega \). On the other hand, we also have

\[
\varphi_1 = \lambda_1 \lambda_2^{\alpha_1} \cdots \lambda_m^{\alpha_1 \cdots \alpha_m - 1} T_1 \circ \cdots \circ T_m(\varphi_1) = H(\Lambda) T \varphi_* .
\]

Finally, using that \( H(\Lambda) = \lambda_* \) and \( \varphi_* \) is an eigenfunction of \( T \) corresponding to \( \lambda_*^{-1} \), the above equality yields \( \varphi_1 = \varphi_* \) and this proves that \( \Lambda \) is a principal eigenvalue of the problem (13). In other words, we conclude that \( \Lambda \in \Gamma_\alpha(\Omega, \rho) \).

In a natural way, the principal \((m - 1)\)-hypersurface \( \{ \Lambda \in \mathbb{R}_m^+ : H(\Lambda) = \lambda_* \} \) will be represented by the notation \( \Lambda_1 \). Notice that \( \Lambda_1 \) decomposes \( \mathbb{R}_m^+ \) into two unbounded connected components.

An immediate consequence of Proposition 2.1 is

**Corollary 2.1.** For each vector \( \sigma = (\sigma_1, \ldots, \sigma_{m-1}) \in \mathbb{R}_m^{m-1} \), there exists a unique number \( \theta_* = \theta_*(\sigma) > 0 \) such that

\[
(\theta_*, \theta_* \sigma_1, \ldots, \theta_* \sigma_{m-1}) \in \Lambda_1 .
\]

Moreover, we have

\[
\theta_*(\sigma) = \left( \frac{1}{\sum_{i=1}^{m-1} \prod_{k=1}^{i-1} \alpha_k} \right) \prod_{i=1}^{m-1} \sigma_i^{\prod_{k=1}^{i-1} \alpha_k} .
\]

In the most part of this work, we will use the parametrization \( \sigma \in \mathbb{R}_m^{m-1} \mapsto (\theta_*(\sigma), \theta_*(\sigma) \sigma) \) of \( \Lambda_1 \) described in this corollary.
3 The construction of the extremal set $\Lambda^*$

Throughout this and the next section it will be assumed that $F(x, \cdot)$ is continuous for $x \in \Omega$ almost everywhere and $F(\cdot, t) \in L^n(\Omega; \mathbb{R}^m)$ for $t \in \mathbb{R}^m_+$ and $F$ satisfies the requirements (A), (B) and (C).

This section is devoted to the study about existence of a $(m - 1)$-hypersurface $\Lambda^*$ decomposing $\mathbb{R}^m_+$ into two sets corresponding to existence and nonexistence of positive strong solution of (1). The nonlinear eigenvalue problem (12) to $\Lambda$-parameters considered in the previous section will play a fundamental role in the construction of the set $\Lambda^*$.

We start by defining the set

$$\mathcal{A}_0 = \{ \Lambda \in \mathbb{R}^m_+ : (1) \text{ admits a positive strong solution } u \in W^{2, n}(\Omega; \mathbb{R}^m) \cap W^{1, n}_0(\Omega; \mathbb{R}^m) \}.$$ 

The following lemma implies that $\mathcal{A}_0$ is non-empty.

**Lemma 3.1.** The inclusion $(B_{\varepsilon_0}(0) \cap \mathbb{R}^m_+) \subset \mathcal{A}_0$ holds for $\varepsilon_0 > 0$ sufficiently small.

**Proof.** A basic fact to be used in the proof is that $F(\cdot, u) \in L^n(\Omega; \mathbb{R}^m)$ whenever $u \in L^\infty(\Omega; \mathbb{R}^m)$. Let $u_k$ be defined recursively by $u_1 = 0$ and $u_{k+1} = \Lambda(-\mathcal{L})^{-1}(F(x, u_k))$ for $k \geq 1$. Notice that the sequence $(u_k)$ is well defined in $W^{2, n}(\Omega; \mathbb{R}^m) \cap W^{1, n}_0(\Omega; \mathbb{R}^m)$ thanks to the assumptions on $F$ and the $L^n$ Calderón-Zygmund standard theory. Fix $t_0 \in \mathbb{R}^m_+$ and choose $\varepsilon_0 > 0$ such that $\varepsilon_0(-\mathcal{L})^{-1}(F(\cdot, t_0)) \leq t_0$ in $\Omega$. For $\Lambda \in B_{\varepsilon_0}(0) \cap \mathbb{R}^m_+$, the condition (B) gives

$$u_2(x) = \Lambda(-\mathcal{L})^{-1}(F(x, 0)) \leq \varepsilon_0((-\mathcal{L})^{-1}(F(x, t_0)) \leq t_0 \text{ for } x \in \Omega.$$ 

In addition, the hypothesis (A) and (SMP) imply that $u_2 > 0 = u_1$ in $\Omega$. Using again (B) and (SMP), we derive

$$u_3(x) = \Lambda(-\mathcal{L})^{-1}(F(x, u_2)) \leq \varepsilon_0((-\mathcal{L})^{-1}(F(x, t_0)) \leq t_0 \text{ for } x \in \Omega$$

and

$$u_3(x) = \Lambda(-\mathcal{L})^{-1}(F(x, u_2)) \geq \Lambda(-\mathcal{L})^{-1}(F(x, u_1)) = u_2(x) \text{ for } x \in \Omega.$$ 

In short, $u_1 \leq u_2 \leq u_3 \leq t_0$ in $\Omega$. Proceeding inductively with the aid of (B) and (SMP), we conclude that the sequence $(u_k)$ is pointwise nondecreasing and uniformly upper bounded by $t_0$. Thus, $(u_k)$ converges pointwise and in $L^q(\Omega; \mathbb{R}^m)$ to $u$ for any $q \geq 1$. Invoking again the $L^n$ Calderón-Zygmund theory, it follows that $u_k$ converges to $u$ in $W^{2, n}(\Omega; \mathbb{R}^m)$ and so $\Lambda \in \mathcal{A}_0$. $\square$

The idea behind the construction of $\Lambda^*$ lies in considering points on each half-line in $\mathbb{R}^m_+$ starting from the origin at the direction $(1, \sigma)$, namely, $\mathcal{T}_\sigma = \{ \lambda > 0 : (\lambda, \lambda \sigma) \in \mathcal{A}_0 \}$, where $\sigma \in \mathbb{R}^{m-1}_+$. By Lemma 3.1, the set $\mathcal{T}_\sigma$ is nonempty for every $\sigma \in \mathbb{R}^{m-1}_+$. We assert that the segment $\mathcal{T}_\sigma$ is bounded for each fixed $\sigma \in \mathbb{R}^{m-1}_+$. The following auxiliary result will be useful in the proof of this claim:
**Lemma 3.2.** Assume the assumptions (A), (B) and (C) hold. Then, there exist a positive map \( \rho_0 \in L^n(\Omega; \mathbb{R}^m) \), a tuple \( \alpha \in \mathbb{R}^m_+ \) with \( \Pi \alpha = 1 \) and a constant \( C_0 > 0 \) such that, for \( x \in \Omega \) almost everywhere and \( t \in \mathbb{R}^m_+ \),

\[
C_0 F(x, t) \geq \rho_0(x) S_\alpha(t).
\]

**Proof.** Applying (C) for \( \kappa = 1 \), we obtain a positive map \( \rho \in L^n(\Omega; \mathbb{R}^m) \), a tuple \( \alpha \in \mathbb{R}^m_+ \) with \( \Pi \alpha = 1 \) and a constant \( M > 0 \) such that

\[
F(x, t) \geq \rho(x) S_\alpha(t)
\]

for \( x \in \Omega \) almost everywhere and \( t \in \mathbb{R}^m_+ \) with \( |t| > M \). On the other hand, by continuity and compactness, there exists a constant \( D_0 > 0 \) so that

\[
S_\alpha(t) \leq D_0(1, \ldots, 1)
\]

for all \( t \in \mathbb{R}^m_+ \) with \( |t| \leq M \). Combining this conclusion with the assumptions (A) and (B), we derive

\[
D_0 F(x, t) \geq D_0 F(x, 0) \geq F(x, 0) S_\alpha(t)
\]

for \( x \in \Omega \) almost everywhere and \( t \in \mathbb{R}^m_+ \) with \( |t| \leq M \). Thus, the proof of Lemma 3.2 follows by choosing the positive constant \( C_0 = \max\{1, D_0\} \) and the positive map \( \rho_0(x) = \min\{\rho(x), F(x, 0)\} \).

\[\square\]

The main result of this section is as follows:

**Lemma 3.3.** The set \( \mathcal{T}_\sigma \) is bounded for every \( \sigma \in \mathbb{R}^{m-1}_+ \).

**Proof.** Let a fixed \( \sigma \in \mathbb{R}^{m-1}_+ \) and \( \Lambda = (\lambda, \lambda \sigma) \in \mathcal{A}_0 \). By the definition of \( \mathcal{A}_0 \), the problem (1) admits a positive strong solution \( u \). Invoking Lemma 3.2 in (1), we get

\[
-\mathcal{L} u \geq \frac{1}{C_0} \Lambda \rho_0(x) S_\alpha(u),
\]

where \( \rho_0 \in L^n(\Omega; \mathbb{R}^m) \) is a positive map and \( \alpha \in \mathbb{R}^m_+ \) is a tuple such that \( \Pi \alpha = 1 \). Applying Corollary 2.1 to the eigenvalue problem (13) with weight \( \rho_0 \), we get a principal eigenvalue \( \Lambda_0 = (\theta_\ast, \theta_\ast \sigma) \in \mathbb{R}^m_+ \) and an associated positive eigenfunction \( \varphi_0 \). We claim that \( \Lambda \leq C_0 \Lambda_0 \). Otherwise, we set \( \hat{\alpha}_i = \Pi_{k=i}^m \alpha_k \) and denote \( \hat{\alpha} = (\hat{\alpha}_1, \ldots, \hat{\alpha}_m) \) and \( s^{\hat{\alpha}} = (s^{\hat{\alpha}_1}, \ldots, s^{\hat{\alpha}_m}) \). We now consider the set \( \mathcal{S} = \{s : u > s^{\hat{\alpha}} \varphi_0 \text{ in } \Omega\} \), which is nonempty by (HL) and (SMP) and also upper bounded. Let \( s^\ast = \sup \mathcal{S} > 0 \). Assuming by contradiction that \( \Lambda > C_0 \Lambda_0 \), using the previous inequality and that \( u \geq (s^\ast)^{\hat{\alpha}} \varphi_0 \) in \( \Omega \), we derive

\[
-\mathcal{L}(u - (s^\ast)^{\hat{\alpha}} \varphi_0) \geq \frac{1}{C_0} \Lambda \rho_0(x) S_\alpha(u) - (s^\ast)^{\hat{\alpha}} \Lambda_0 \rho_0(x) S_\alpha(\varphi_0)
\]

\[
\geq \left( \frac{1}{C_0} \Lambda - \Lambda_0 \right) \rho_0(x)(s^\ast)^{\hat{\alpha}} S_\alpha(\varphi_0) > 0 \text{ in } \Omega
\]
Here we use that \( S_\alpha((s^*)^\alpha \varphi_0) = (s^*)^\alpha S_\alpha(\varphi_0) \). Finally, the (HL) and (SMP) applied to the above inequality yield the contradiction \( u > (s^* + \varepsilon^\alpha \varphi_0) \) in \( \Omega \) for \( \varepsilon > 0 \) small enough. Therefore, we deduce that \( \Lambda \leq C_0 \Lambda_0 \), so that \( \lambda \leq C_0 \theta_\ast \). \( \square \)

It follows from Lemmas 3.1 and 3.3 that the positive number \( \lambda^\ast(\sigma) = \sup T_\sigma \)

is well defined for each \( \sigma \in \mathbb{R}^{m-1}_+ \). Introduce the sets

\[
\begin{align*}
A &= \{ (\lambda, \lambda \sigma) : 0 < \lambda < \lambda^\ast(\sigma), \ \sigma \in \mathbb{R}^{m-1}_+ \}, \\
B &= \{ (\lambda, \lambda \sigma) : \lambda > \lambda^\ast(\sigma), \ \sigma \in \mathbb{R}^{m-1}_+ \}, \\
\Lambda^\ast &= \{ (\lambda^\ast(\sigma), \lambda^\ast(\sigma) \sigma) : \sigma \in \mathbb{R}^{m-1}_+ \}.
\end{align*}
\]

The set \( \Lambda^\ast \) is called extremal set associated to the problem (1).

For ending the proof of Theorem 1.1 with the above-introduced sets, it remains to show that \( A \subset A_0 \), which is done in the next result.

**Lemma 3.4.** The set \( A_0 \) contains \( A \).

**Proof.** Let a fixed \( \sigma \in \mathbb{R}^{m-1}_+ \). Given \( \Lambda_0 = (\lambda_0, \lambda_0 \sigma) \in A_0 \), it suffices to show the inclusion \( (0, \lambda_0) \subset T_\sigma \).

Take \( 0 < \lambda < \lambda_0 \) and set \( \Lambda = (\lambda, \lambda \sigma) \). Let \( u_{\Lambda_0} \) be a positive strong solution of (1) corresponding to \( \Lambda_0 \).

Since \( \Lambda \leq \Lambda_0 \), we have

\[-L u_{\Lambda_0} = \Lambda_0 F(x, u_{\Lambda_0}) \geq \Lambda F(x, u_{\Lambda_0}) \text{ in } \Omega.\]

In other words, the map \( u_{\Lambda_0} \) is a supersolution of the problem (1). Let \( \{u_k\}_{k \geq 1} \) be the sequence of maps constructed in the proof of Lemma 3.1. Proceeding in a similar manner to that proof, one easily concludes that \( \{u_k\}_{k \geq 1} \) is a pointwise nondecreasing sequence that satisfies \( u_k \leq u_{\Lambda_0} \) in \( \Omega \) for all \( k \geq 1 \). Then, the same argument applies as in Lemma 3.1 and so we deduce that the problem (1) has a positive strong solution. Therefore, \( \Lambda \in A_0 \), so that \( \lambda \in T_\sigma \). This finishes the proof. \( \square \)

### 4 Qualitative properties of \( \Lambda^\ast \)

Consider the map \( \Phi : \mathbb{R}^{m-1}_+ \to \Lambda^\ast \subset \mathbb{R}^m_+ \) defined by

\[ \Phi(\sigma) = (\lambda^\ast(\sigma), \nu^\ast(\sigma)) := (\lambda^\ast(\sigma), \lambda^\ast(\sigma) \sigma) \]

which parameterizes the set \( \Lambda^\ast \), where \( \lambda^\ast(\sigma) \) was defined in the previous section.

This section focuses on the qualitative properties satisfied by \( \Lambda^\ast \) by means of the map \( \Phi \).

We first prove the continuity of \( \Phi \) which corresponds to the part (I) of Theorem 1.2.

**Proposition 4.1.** The hypersurface \( \Lambda^\ast \) is continuous.
Proof. For the proof of the continuity of $\Phi$, it suffices to show the continuity of $\lambda^*$. Assume by contradiction that $\lambda^*(\sigma)$ is discontinuous at some $\sigma \in \mathbb{R}^{m-1}_+$. Then, there exist a number $\varepsilon > 0$ and a sequence $(\sigma_k)$ converging to $\sigma$ such that, for any $k \geq 1$,

$$|\lambda^*(\sigma_k) - \lambda^*(\sigma)| \geq \varepsilon.$$ 

Module a subsequence, we can assume that

$$\lambda^*(\sigma_k) \leq \lambda^*(\sigma) - \varepsilon$$

or

$$\lambda^*(\sigma_k) \geq \lambda^*(\sigma) + \varepsilon$$

for $k$ sufficiently large. For our argument of contradiction, it is enough to consider only the first case, once the second one is carried out in an analogous way.

Take positive numbers $\underline{\lambda}$ and $\overline{\lambda}$ so that

$$\lambda^*(\sigma_k) < \underline{\lambda} < \lambda < \overline{\lambda} \leq \lambda^*(\sigma)$$

In particular,

$$\lambda^*(\sigma_k) < \underline{\lambda} < \overline{\lambda} \leq \lambda^*(\sigma)$$

for $k$ large enough.

Let $\underline{\Delta}_k = (\underline{\lambda}, \underline{\lambda} \sigma_k)$ and $\overline{\Delta} = (\overline{\lambda}, \overline{\lambda} \sigma)$. From the above inequality and Lemma 3.4, it follows that (1) has a positive strong solution $\underline{\pi}$ corresponding to $\overline{\Delta}$. Moreover, since $\overline{\lambda} \geq \underline{\lambda}_k$, we have

$$\begin{cases} -\mathcal{L}\underline{\pi} \geq \underline{\lambda}_k F(x, \underline{\pi}) & \text{in } \Omega \\ \underline{\pi} = 0 & \text{on } \partial \Omega, \end{cases}$$

which is equivalent to saying that $\underline{\pi}$ is a positive supersolution of (1) associated to $\underline{\Delta}_k$. Hence, (1) has a positive strong solution corresponding to $\underline{\Delta}_k$, so that $\overline{\lambda} \leq \lambda^*(\sigma_k)$ for $k$ large enough, contradicting the above reverse inequality.

For understanding the asymptotic behavior of the hypersurface $\Lambda^*$, we need to study the behavior of $\Phi$’s components. Proposition 4.2 below corresponds to the part (II) of Theorem 1.2.

**Proposition 4.2.** The function $\lambda^*(\sigma)$ is nonincreasing on $\sigma$.

**Proof.** Assume by contradiction that the claim is false, that is, $\lambda^*(\sigma_1) < \lambda^*(\sigma_2)$ for some $\sigma_1 < \sigma_2$ in $\mathbb{R}^{m-1}_+$. Choose positive numbers $\underline{\lambda}$ and $\overline{\lambda}$ such that

$$\lambda^*(\sigma_1) < \underline{\lambda} < \overline{\lambda} < \lambda^*(\sigma_2),$$

then
\[ \lambda^*(\sigma_1) \sigma_1 < \lambda \sigma_1 < \lambda^*(\sigma_2) \sigma_2. \]

Setting \( \underline{\Lambda} = (\underline{\lambda}, \lambda \sigma_1) \) and \( \bar{\Lambda} = (\bar{\lambda}, \bar{\lambda} \sigma_1) \), one has \( \underline{\Lambda} < \bar{\Lambda} \). Then, applying exactly the same argument of the previous proposition to these inequalities, we readily arrive at a contradiction. \( \square \)

We now prove the part (III) of Theorem 1.2.

**Proposition 4.3.** For any \( 0 < \sigma_1 \leq \sigma_2 \), there exists \( i \in \{1, \ldots, m\} \) such that \( \nu_i^*(\sigma_1) \leq \nu_i^*(\sigma_2) \).

**Proof.** Assume by contradiction that there exists \( 0 < \sigma_1 \leq \sigma_2 \) such that \( \nu_i^*(\sigma_2) < \nu_i^*(\sigma_1) \). But these inequalities imply that \( \lambda^*(\sigma_2) < \lambda^*(\sigma_1) \) and \( \lambda^*(\sigma_2) \sigma_2 < \lambda^*(\sigma_1) \sigma_1 \). Then, we can take positive numbers \( \overline{\lambda} \) and \( \underline{\lambda} \) such that

\[ \lambda^*(\sigma_2) < \overline{\lambda} \sigma_2 < \lambda^*(\sigma_1) \sigma_1 \]

and

\[ \lambda^*(\sigma_2) \sigma_2 < \underline{\lambda} \sigma_1 < \lambda^*(\sigma_1) \sigma_1. \]

Let \( \underline{\Lambda} = (\underline{\lambda}, \underline{\lambda} \sigma_2) \) and \( \bar{\Lambda} = (\bar{\lambda}, \bar{\lambda} \sigma_1) \). Thereby, one has \( \underline{\Lambda} < \bar{\Lambda} \). Proceeding as in the proof of the Proposition 4.1, we conclude that \( \underline{\lambda} \leq \lambda^*(\sigma_2) \), which is clearly a contradiction. \( \square \)

Finally, by using the Proposition 4.2, we establish the asymptotic behavior of the hypersurface \( \Lambda^* \) stated in the part (IV) of Theorem 1.2.

**Proposition 4.4.** The limit \( \lambda^*(\sigma) \rightarrow 0 \) as \( \sigma_i \rightarrow +\infty \) occurs for each fixed \( i = 1, \ldots, m-1 \).

**Proof.** Let fixed \( i = 1, \ldots, m-1 \). It was proved in Lemma 3.3 that \( \lambda^*(\sigma) \leq C_0 \mu_1(\sigma) \) for all \( \sigma \in \mathbb{R}^{n-1}_+ \), where \( C_0 > 0 \) is a constant independent of \( \sigma \). Using the characterization of \( \mu_1(\sigma) \) provided in Corollary 2.1, we derive \( \mu_1(\sigma) \rightarrow 0 \) as \( \sigma_i \rightarrow +\infty \) and so follows the conclusion. \( \square \)

## 5 Weak solutions on \( \Lambda^* \)

The proof of Theorem 1.3 requires \( L^1 \) a priori estimates for \( C^1(\overline{\Omega}) \) solutions of (1) corresponding to \( \Lambda \) in a neighborhood of the hypersurface \( \Lambda^* \).

Let \( \delta(x) = \text{dist}(x, \partial \Omega) \). We recall that the \( \delta \)-weighted \( L^1 \) space is given by

\[ L^1(\Omega, \delta(x)dx) = L^1(\Omega, \delta) := \left\{ h : \Omega \rightarrow \mathbb{R} : \int_{\Omega} |h(x)| \delta(x) dx < +\infty \right\} \]

and endowed with the norm \( \|h\|_{L^1(\Omega, \delta)} = \int_{\Omega} |h(x)| \delta(x)dx \).

The next two lemmas are useful tools in our proofs. The first one is a straightforward and direct adaptation of the proof of Lemma 1 of [2] since its essential ingredient is the maximum principle for the laplacian which, by assumption, is also satisfied by \( L \). The second one is a consequence of global estimates for Green functions associated to elliptic operators on \( C^{1,1} \) domains established in [1], [15] and [32].
Lemma 5.1. Let \( \mathcal{L} = a_{kl}(x)\partial_{kl} + b_j(x)\partial_j + c(x) \) be a uniformly elliptic operator such that \( a_{kl} \in C^2(\Omega) \), \( b_j \in C^1(\Omega) \) for \( k, l, j = 1, \ldots, n \), and \( c \in L^\infty(\Omega) \). Assume \( \mu_1(-\mathcal{L}, \Omega) > 0 \). Given \( h \in L^1(\Omega, \delta) \), the problem

\[
\begin{aligned}
-\mathcal{L}v &= h \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial\Omega
\end{aligned}
\]  

(14)

admits a unique weak solution \( v \in L^1(\Omega) \). Moreover,

\[ \|v\|_{L^1(\Omega)} \leq C_1\|h\|_{L^1(\Omega, \delta)} \]

for some positive constant \( C_1 \) independent of \( h \). In addition, if \( h \geq 0 \) almost everywhere in \( \Omega \), then \( v \geq 0 \) almost everywhere in \( \Omega \).

Lemma 5.2. Let \( \mathcal{L} = a_{kl}(x)\partial_{kl} + b_j(x)\partial_j + c(x) \) be a uniformly elliptic operator such that \( a_{kl} \in C^2(\Omega) \), \( b_j \in C^1(\Omega) \) for \( k, l, j = 1, \ldots, n \), and \( c \in L^\infty(\Omega) \). Assume \( \mu_1(-\mathcal{L}, \Omega) > 0 \). If \( h \in L^\infty(\Omega) \) and \( h \geq 0 \) almost everywhere in \( \Omega \), then the strong solution \( v \in W^{2,n}(\Omega) \cap W^{1,n}_0(\Omega) \) of (14) satisfies

\[ v(x) \geq C_2\delta(x)\|h\|_{L^1(\Omega, \delta)} \]

for \( x \in \Omega \) almost everywhere, where the positive constant \( C_2 \) is independent of \( h \).

Proof. Let \( G_\mathcal{L} \) and \( G_\Delta \) be Green’s functions corresponding to \( -\mathcal{L} \) and \( -\Delta \) with zero Dirichlet condition at the boundary. It follows independently from [1] and [15] that there exists a constant \( C > 0 \), depending on \( \Omega \) and coefficients of \( \mathcal{L} \), such that, for any \( (x, y) \in \Omega \times \Omega \),

\[ \frac{1}{C}G_\Delta(x, y) \leq G_\mathcal{L}(x, y) \leq CG_\Delta(x, y). \]

On the other hand, as proved in [32], there exists a constant \( \tilde{C} > 0 \), depending only on \( \Omega \), such that, for any \( (x, y) \in \Omega \times \Omega \),

\[ G_\Delta(x, y) \geq \tilde{C}\delta(x)\delta(y). \]

Therefore, from the last two estimates, there exists a constant \( C_2 > 0 \), depending on \( \Omega \) and coefficients of \( \mathcal{L} \), such that, for any \( (x, y) \in \Omega \times \Omega \),

\[ G_\mathcal{L}(x, y) \geq C_2\delta(x)\delta(y). \]

Therefore, the strong solution \( v \in W^{2,n}(\Omega) \cap W^{1,n}_0(\Omega) \) of (14) satisfies

\[ v(x) = \int_\Omega G_\mathcal{L}(x, y)h(y)dy \geq C_2\delta(x) \left( \int_\Omega h(y)\delta(y) \, dy \right). \]

This ends the proof.

Before presenting our next lemma, it will be useful to mention a consequence of the assumptions (A) and (C). Let \( \kappa, \rho, \alpha \) and \( M \) be as in (C). Thanks to the positivity of \( F \) and \( \rho \) and the compactness of \( \{t \in \mathbb{R}^n : |t| \leq M\} \), there exists a constant \( B > 0 \) depending on \( \kappa \), so that
\[ F(x,t) \geq \kappa \rho(x) S_\alpha(t) - B \rho(x) \]  
(15)

for \( x \in \Omega \) almost everywhere and \( t \in \mathbb{R}_+^m \).

In the following result we establish a \textit{a priori} estimate to solutions of (1) by using Lemmas 5.1 and 5.2.

**Lemma 5.3.** Let a fixed \( \sigma \in \mathbb{R}^{m-1}_+ \) and \( L_i = a_{kl}^i(x) \partial_{kl} + b_{kj}^i(x) \partial_j + c^i(x) \) be a uniformly elliptic operator such that \( a_{kl}^i \in C_2(\overline{\Omega}) \), \( b_{kj}^i \in C^1(\overline{\Omega}) \) and \( c^i \in L^\infty(\Omega) \) for \( i = 1, \ldots, m \) and \( k, l, j = 1, \ldots, n \). Let \( F: \Omega \times \mathbb{R}^m \to \mathbb{R}^m \) be a map such that \( F(x, \cdot) : \mathbb{R}^m \to \mathbb{R}^m \) is continuous for \( x \in \Omega \) almost everywhere and \( F(\cdot, t) : \Omega \to \mathbb{R}^m \) belongs to \( L^n(\Omega; \mathbb{R}^m) \) for any \( t \in \mathbb{R}^m \). Assume also that \( F \) satisfies (A), (B) and (C). Then, there exists a positive constant \( C_3 \) such that, for any \( \frac{1}{2} \lambda^*(\sigma) < \lambda < \lambda^*(\sigma) \),

\[ \|u\|_{L^1(\Omega; \mathbb{R}^m)} \leq C_3 \]

for every positive strong solution \( u \in W^{2,n}(\Omega; \mathbb{R}^m) \cap W^{1,n}_0(\Omega; \mathbb{R}^m) \) of (1) associated to \( \Lambda = (\lambda, \lambda\sigma) \).

**Proof.** Assume by contradiction that there is no \textit{a priori} bound of solutions. Let \((u_k)\) be a sequence in \(W^{2,n}(\Omega; \mathbb{R}^m) \cap W^{1,n}_0(\Omega; \mathbb{R}^m)\) of positive strong solutions of (1) associated to \( \Lambda_k = (\lambda_k, \lambda_k\sigma) \) with \( \frac{1}{2} \lambda^*(\sigma) < \lambda_k < \lambda^*(\sigma) \) satisfying \( \|u_k\|_{L^1(\Omega; \mathbb{R}^m)} \to +\infty \).

Let \( s_1 = 1 \) and \( s_{i+1} = \alpha_i s_i \) for \( i = 1, \ldots, m \). Note that \( s_{m+1} = s_1 \), since \( \Pi \alpha = 1 \). Observing that all \( s_i \) are positive, there exists \( l \in \{1, \ldots, m\} \) such that, modulo a subsequence,

\[ \|u_{l+1}^k\|_{L^1(\Omega)} \to +\infty \]

and

\[ \|u_{l+1}^k\|_{L^1(\Omega)}^{s_{l+1}} \geq \|u_l^k\|_{L^1(\Omega)}^{s_l} \]

for all \( i \neq l + 1 \).

In particular, for \( i = l \), we have

\[ \|u_l^k\|_{L^1(\Omega)} \to +\infty \]

and

\[ \|u_l^k\|_{L^1(\Omega)}^{\alpha_l} \geq \|u_l^k\|_{L^1(\Omega)} \]

(16)

Let \( \zeta_l^* \in W^{2,n}(\Omega) \cap W^{1,n}_0(\Omega) \) be the strong solution of

\[
\begin{aligned}
-\mathcal{L}_i^* \zeta_l^* &= 1, \quad \text{in} \quad \Omega, \\
\zeta_l^* &= 0 \quad \text{on} \quad \partial \Omega.
\end{aligned}
\]

Take \( \kappa \) in (15) so that

\[ \kappa (C_1 C_2)^{\alpha_l} \lambda_k \sigma_{l-1} \int_{\Omega} \rho_l(x) \delta(x)^{\alpha_l} \zeta_l^*(x) dx \geq 2 \]

for every \( k \geq 1 \), where \( C_1 \) and \( C_2 \) are the positive constants given in Lemmas 5.1 and 5.2, respectively.

Let \( e_k = \|u_{l+1}^k\|_{L^1(\Omega)} \). Thanks to (16) and (15), we get

\[ 1 \geq e^{-\alpha_l}_k \int_{\Omega} u_{l+1}^k dx = e^{-\alpha_l}_k \int_{\Omega} (-\mathcal{L}_i u_{l+1}^k) \zeta_l^* dx = e^{-\alpha_l}_k \lambda_k \sigma_{l-1} \int_{\Omega} f_l(x, u_k) \zeta_l^* dx \]

\[ \geq e^{-\alpha_l}_k \kappa \lambda_k \sigma_{l-1} \int_{\Omega} \rho_l(x) \left( u_{l+1}^k \right)^{\alpha_l} \zeta_l^* dx - B e^{-\alpha_l}_k \lambda_k \sigma_{l-1} \int_{\Omega} \rho_l(x) \zeta_l^* dx \]

20
for every $k \geq 1$. We then estimate the term $e_k^{-\alpha_l} \int_\Omega \rho_l(x) \left( u_{l+1}^k \right)^{\alpha_l} \zeta^*_l dx$. It follows from Lemmas 5.1 and 5.2 that

$$u_{l+1}^k(x) \geq C_2 \delta(x) \int_\Omega \lambda_k \sigma_l f_l(x, u_k) \delta(x) dx \geq C_1 C_2 \delta(x) \| u_{l+1}^k \|_{L^1(\Omega)},$$

so that

$$e_k^{-\alpha_l} \int_\Omega \rho_l(x) \left( u_{l+1}^k \right)^{\alpha_l} \zeta^*_l dx \geq (C_1 C_2)^{\alpha_l} \int_\Omega \rho_l(x) \delta(x)^{\alpha_l} \zeta^*_l dx.$$  

Finally, we clearly get a contradiction by passing the limit $k \to +\infty$ in the inequalities

$$1 \geq \kappa \lambda_k e_k^{-\alpha_l} \sigma_{l-1} \int_\Omega \rho_l(x) \left( u_{l+1}^k \right)^{\alpha_l} \zeta^*_l dx - B e_k^{-\alpha_l} \lambda_k \sigma_{l-1} \int_\Omega \rho_l(x) \zeta^*_l dx$$

$$\geq \kappa \left( C_1 C_2 \right)^{\alpha_l} \lambda_k \sigma_{l-1} \int_\Omega \rho_l(x) \delta(x)^{\alpha_l} \zeta^*_l (x) dx - B e_k^{-\alpha_l} \lambda_k \sigma_{l-1} \int_\Omega \rho_l(x) \zeta^*_l dx$$

$$\geq 2 - O \left( e_k^{-\alpha_l} \right).$$

**Proof of Theorem 1.3.** Let $\Lambda = (\lambda^*(\sigma), \lambda^*(\sigma)) \in \Lambda^*$ for some fixed $\sigma \in \mathbb{R}^{m-1}$. In order to show the existence of a minimal weak solution of (1) corresponding to $\Lambda$, we take a sequence $(\lambda_k)$ so that $0 < \lambda_k < \lambda^*(\sigma)$ and $\lambda_k \uparrow \lambda^*(\sigma)$. For each $k \geq 1$, let $u_k \in W^{2,n}(\Omega; \mathbb{R}^m) \cap W^{1,n}_0(\Omega; \mathbb{R}^m)$ be the minimal positive strong solution associated $\Lambda_k := (\lambda_k, \lambda_k \sigma)$. By the part (I) of Theorem 1.1, the sequence $(u_k)$ is nondecreasing on $k$. Then, by the assumption (B), the sequence $(F(x, u_k))$ is also nondecreasing. In addition, by taking the strong solution $\zeta^* \in W^{2,n}(\Omega; \mathbb{R}^m) \cap W^{1,n}_0(\Omega; \mathbb{R}^m)$ of the problem

$$\begin{cases} -\mathcal{L}^* \zeta^* = 1 & \text{in } \Omega, \\
\zeta^* = 0 & \text{on } \partial \Omega \end{cases}$$

as a test function, we derive the relation

$$\Lambda_k \int_\Omega F(x, u_k) \zeta^* dx = \int_\Omega u_k dx.$$  

Thus, invoking Lemma 5.3, we conclude that $(u_k)$ and $(F(\cdot, u_k) \delta(\cdot))$ are bounded in $L^1(\Omega; \mathbb{R}^m)$. Therefore, by the monotone convergence theorem, $(u_k)$ and $(F(\cdot, u_k) \delta(\cdot))$ converge to $u^*$ and $F(\cdot, u^*) \delta(\cdot)$ in $L^1(\Omega; \mathbb{R}^m)$, respectively. So, letting $k \to +\infty$ in the equality

$$\int_\Omega u_k (-\mathcal{L}^* \zeta) dx = \Lambda_k \int_\Omega F(x, u_k) \zeta dx,$$

where $\zeta \in W^{2,n}(\Omega; \mathbb{R}^m) \cap W^{1,n}_0(\Omega; \mathbb{R}^m)$ satisfies $\mathcal{L}^* \zeta \in L^\infty(\Omega; \mathbb{R}^m)$, it follows that $u^*$ is a weak solution of (1) associated to $\Lambda$.

Finally, we easily see that the weak solution $u^*$ is minimal. Indeed, let $v$ be another nonnegative weak solution of (1) associated to $\Lambda$. By the construction of $u_k$ in the proof of Lemma 3.1 and, by Lemma 5.2, one easily deduces that $u_k \leq v$ almost everywhere in $\Omega$, which lead easily to the minimality of $u^*$. \qed
6 Stability of minimal solutions

We now concentrate special attention to the proof of Theorem 1.4. We recall that the minimal positive strong solution \( u = u_\Lambda \) of (1) for \( \Lambda \in A \) is said to be a stable steady state of (5) in the linearized sense, if the eigenvalue problem (6) has a smallest positive eigenvalue \( \eta_1 \).

We start with a fundamental result on existence of positive supersolution for the linearized problem of (7) at \( u = u_\Lambda \).

**Lemma 6.1.** Let \( F: \Omega \times \mathbb{R}^m \to \mathbb{R}^m \) be a map such that \( F(x, \cdot): \mathbb{R}^m \to \mathbb{R}^m \) is of \( C^1 \) class for \( x \in \Omega \) almost everywhere, \( F(\cdot, t): \Omega \to \mathbb{R}^m \) belongs to \( L^n(\Omega; \mathbb{R}^m) \) for any \( t \in \mathbb{R}^m \) and (7) is satisfied. Assume also that \( F \) satisfies (A), (B), (C) and (D). Then, for any \( \Lambda \in A \), the eigenvalue problem (6) with \( u = u_\Lambda \) has a positive strong supersolution \( \psi \in W^{2,n}(\Omega; \mathbb{R}^m) \cap W^{1,n}_0(\Omega; \mathbb{R}^m) \) for \( \eta = 0 \).

**Proof.** Let a fixed \( m \)-tuple \( \Lambda = (\lambda_1, \ldots, \lambda_m) \in A \). Since \( u = u_\Lambda \) is differentiable with respect to \( \Lambda \), we set \( \psi = (\psi_1, \ldots, \psi_m) \), where

\[
\psi_i = \frac{\partial u_i}{\partial \lambda_1}
\]

for \( i = 1, \ldots, m \). It is clear that \( \psi = 0 \) on \( \partial \Omega \), once \( u = 0 \) on \( \partial \Omega \). Furthermore, deriving the \( i \)th equation of (1) in relation to \( \lambda_1 \), we get

\[
-L_i \psi_i = \frac{\partial}{\partial \lambda_1} [-L_i u_i] = \sum_{j=1}^m \lambda_i f_i(x, u) \frac{\partial u_j}{\partial \lambda_1} + \delta_{ij} f_i(x, u) \text{ in } \Omega.
\]

In other words, \( \psi \in W^{2,n}(\Omega; \mathbb{R}^m) \cap W^{1,n}_0(\Omega; \mathbb{R}^m) \) satisfies

\[
\begin{aligned}
-L \psi - \Lambda(A(x, u) \psi) &= G(x) \quad \text{in } \Omega,
\psi &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( A(x, t) \) is the \( m \times m \) matrix with entries

\[
A_{ij}(x, t) := \frac{\partial f_i}{\partial t_j}(x, t)
\]

and \( G(x) = (g_1(x), g_2(x), \ldots, g_m(x)) = (f_1(x, u(x)), 0, \ldots, 0) \) for \( x \in \Omega \). Moreover, \( \psi \geq 0 \) in \( \Omega \), since each component of \( u \) is nondecreasing with respect to \( \Lambda \).

Note that the assumptions (A) and (B) imply that \( F \) is positive. So, noting that \( g_1 \) is positive almost everywhere in \( \Omega \) and the matrix \( A(x, u_\Lambda) \) has nonnegative entries in \( \Omega \) (by (B) and (D)), the strong maximum principle applied to the above first equation produces \( \psi_1 > 0 \) in \( \Omega \). Using again the nonnegativity of \( A_{ij}(x, u_\Lambda) \) for any \( i, j \) and that \( A_{is}(x, u_\Lambda) \), where \( s \) is the shift permutation of \( \{1, \ldots, m\} \), is non identically zero for all \( i \), we apply the strong maximum principle to the remaining equations in order to guarantee that \( \psi \) is a positive strong supersolution of (6) for \( \eta = 0 \). This concludes the proof of lemma. \( \square \)

Since the entries of the matrix \( A(\cdot, u_\Lambda) \) belongs to \( L^n(\Omega) \) and existence of principal eigenvalues of elliptic systems of type (6) are only available in the \( L^\infty \) setting (see for example [19] and [28], among other references therein), we shall combine Lemma 6.1 with an approximation procedure in order to prove the desired stability.
Proof of Theorem 1.4. Let $u_\Lambda$ be the minimal positive strong solution of (1) corresponding to $\Lambda \in \mathcal{A}$. For $x \in \Omega$ and $i,j = 1, \ldots, m$, we set $A_{ij}(x):=A_{ij}(x,u_\Lambda(x))$. It is clear, by (7), (B) and (D), that $A_{ij} \in L^n(\Omega)$, is nonnegative and $A_{is_i} \neq 0$ for all $i,j$, where $s$ is the shift permutation of $\{1, \ldots, m\}$. So, for each $i,j$, there exists a nondecreasing sequence of nonnegative functions $(A_{ij}^k) \subset L^\infty(\Omega)$ converging to $A_{ij}$ in $L^n(\Omega)$ such that $A_{is_i} \neq 0$. Denote the matrix $[A_{ij}^k(x)]$ by $A_k^k(x)$ for $x \in \Omega$. Thanks to monotonicity of the entries of $A_k^k(x)$, the positive supersolution $\psi$ exhibited in Lemma 6.1 satisfies $-\mathcal{L}\psi - \Lambda(A_k^k(x)\psi) > 0$ in $\Omega$. Then, by Theorem 2.1 of [19], the problem

$$
\begin{cases}
-\mathcal{L}\varphi - \Lambda(A_k^k(x)\varphi) = \eta^k \varphi & \text{in } \Omega, \\
\varphi = 0 & \text{on } \partial\Omega
\end{cases}
$$

admits a principal eigenvalue $\eta^k > 0$ with corresponding positive eigenfunction $\varphi^k = (\varphi_1^k, \ldots, \varphi_m^k)$ which we normalize by

$$
\|\varphi^k\|_{C(\overline{\Omega};\mathbb{R}^m)} = 1.
$$

We ensure that the sequence $(\eta^k)$ is decreasing. Otherwise, if the inequality $\eta^k \leq \eta_1^{k+1}$ holds for some $k \in \mathbb{N}$, we introduce the nonempty set $S := \{s > 0 : \varphi^{k+1} > s\varphi^k \text{ in } \Omega\}$ and the finite number $s^* := \sup S > 0$. Using that $\varphi^k$ is positive in $\Omega$, $\varphi^{k+1} \geq s^*\varphi^k$ in $\Omega$ and $A_{ij}(x)$ is increasing on $k$ for $i \neq j$, we can set into the vector language

$$
-\mathcal{L}(\varphi^{k+1} - s^*\varphi^k) = \Lambda(A^{k+1}(x)\varphi^{k+1}) - s^*\Lambda(A_k^k(x)\varphi^k) + \eta^{k+1}_1\varphi^{k+1} - s^*\eta^{k}_1\varphi^k
\geq \Lambda(\varphi^{k+1} - s^*\varphi^k) + \eta^{k+1}_1\varphi^{k+1} - s^*\eta^{k}_1\varphi^k
\geq \Lambda(\varphi^{k+1} - s^*\varphi^k) + \eta^{k}_1\varphi^{k+1} - s^*\eta^{k}_1\varphi^k
\geq \Lambda(\varphi^{k+1} - s^*\varphi^k)
$$

in $\Omega$. Since $\varphi^{k+1} = s^*\varphi^k$ on $\partial\Omega$, the strong maximum principle for cooperative elliptic systems (e.g. Theorem 2.1 of [19]) and the Hopf’s lemma applied to each operator of $\mathcal{L}$ yield the contradiction $\varphi^{k+1} > (s^* + \varepsilon)\varphi^k$ in $\Omega$ for $\varepsilon > 0$ small enough. So, we conclude that $(\eta^1_k)$ is decreasing. Consequently, the limit $\eta_1 := \lim_{k \to +\infty} \eta^1_k \geq 0$ exists. Note also that the sequence of positive eigenfunctions $(\varphi^k)$ converges in $C(\overline{\Omega};\mathbb{R}^m)$ to a positive map $\varphi = (\varphi_1, \ldots, \varphi_m)$. This last claim follows readily from the $L^n$ Calderon-Zygmund elliptic theory, the normalization condition, the nonnegativity of $A(x)$ and $A_{is_i} \neq 0$. In particular, $\eta_1$ is an eigenvalue of the problem (6) with associated positive eigenfunction $\varphi$.

The next step is to prove that $\eta_1$ is positive. Assume by contradiction that $\eta_1 = 0$. Again we consider the auxiliary nonempty set $S := \{s > 0 : \psi > s\varphi \text{ in } \Omega\}$ and the positive number $s^* := \sup S < \infty$. Using that $\psi \geq s^*\varphi$ in $\Omega$, we get for $x \in \Omega$,
\[-L_1(\psi_1 - s^*\varphi_1) = \lambda_1 \sum_{j=1}^{k} A_{1j}(x)\psi_j - s^*\lambda_1 \sum_{j=1}^{k} A_{1j}(x)\varphi_j + g_1(x) \]

\[= \lambda_1 \sum_{j=1}^{k} A_{1j}(x)(\psi_j - s^*\varphi_j) + g_1(x) \]

\[\geq g_1(x) > 0.\]

Consequently, $\psi_1 > (s^* + \varepsilon)\varphi_1$ in $\Omega$ for $\varepsilon > 0$ small. Using this conclusion in the remaining equations together with the nonnegativity of $A(x)$ and the fact that $A_{i,i} \neq 0$, we obtain the contradiction $\psi > (s^* + \varepsilon)\varphi$ in $\Omega$ for $\varepsilon > 0$ small. Therefore, we have $\eta_1 > 0$.

We now show that $\eta_1$ is the smallest eigenvalue of (6). In fact, assume that (6) admits an eigenvalue $\eta \in (-\infty, \eta_1)$ with corresponding eigenfunction $\phi$. Without loss of generality, assume that at least one component of $\phi$ is positive somewhere in $\Omega$. As usual, consider the set $S := \{s > 0 : \phi > s\phi \text{ in } \Omega\}$ and the well-defined positive number $s^* := \sup S$. The assumption $\eta < \eta_1$ and similar arguments to those used above based on maximum principles give the contradiction $\phi > (s^* + \varepsilon)\varphi$ in $\Omega$ for $\varepsilon > 0$ small. Indeed, it suffices to note that

\[-L(\varphi - s^*\phi) = \Lambda(A(x)\varphi) - s^*\Lambda(A(x)\phi) + \eta\varphi - s^*\eta\phi \]

\[= \Lambda(A(x)(\varphi - s^*\phi)) + \eta\varphi - s^*\eta\phi \]

\[\geq \eta\varphi - s^*\eta\phi \]

\[> \eta(\varphi - s^*\phi) \text{ in } \Omega,\]

which implies that $L(\varphi - s^*\phi) < 0$ in $\Omega$ if $\eta \geq 0$ and $(L + \eta)(\varphi - s^*\phi) < 0$ in $\Omega$ if $\eta < 0$, and the desired contradiction is achieved in both cases, since the operators of $L + \eta$ satisfy strong maximum principles when $\eta < 0$.

Finally, we assert that $\eta_1$ is simple. Let $\tilde{\varphi}$ be another eigenfunction associated to $\eta_1$. As above, we can assume that some component of $\tilde{\varphi}$ is positive somewhere in $\Omega$. Thus, we introduce the nonempty set $S := \{s > 0 : \varphi > s\tilde{\varphi} \text{ in } \Omega\}$ and the positive number $s^* := \sup S$. It is clear that $\varphi \geq s^*\tilde{\varphi}$ in $\Omega$. We claim that $\varphi = s^*\tilde{\varphi}$ in $\Omega$. Otherwise, $\varphi_l \neq s^*\tilde{\varphi}_l$ for some $l \in \{1, \ldots, m\}$. On the other hand, as we already know, $L_l(\varphi_l - s^*\tilde{\varphi}_l) \leq 0$ in $\Omega$, so that $\varphi_l > (s^* + \varepsilon)\tilde{\varphi}_l$ in $\Omega$ for $\varepsilon > 0$ small. Hence, by using the assumptions (7), (B) and (D) as done above, one easily deduces the contradiction $\varphi > (s^* + \varepsilon)\tilde{\varphi}$ in $\Omega$ for $\varepsilon > 0$ small. This ends the proof. \qed

### 7 Regularity of the extremal solutions for $n = 2$ and $n = 3$

In this section we consider the problem

\[
\begin{cases}
-\Delta u = \Lambda \nabla f(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]

(17)
where $\Delta u = (\Delta u_1, \ldots, \Delta u_m)$, $\Lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}_+^m$, $f : \mathbb{R}_+^m \to \mathbb{R}$ is a positive $C^2$ function and $\Lambda \nabla f(u) = (\lambda_1 f_{u_1}(u), \ldots, \lambda_m f_{u_m}(u))$ with $m \geq 1$.

Assume the potential field $F = \nabla f$ satisfies the assumptions (A), (B) and (C). Moreover, assume that $\text{Hess } f(t) > 0$ for all $t \in \mathbb{R}_+^m$. This condition implies in particular that the Jacobian matrix of $F$ satisfies (D).

Under these conditions, we consider the parameter $\Lambda = (\lambda^*(\sigma), \lambda^*(\sigma)\sigma) \in \Lambda^*$ for a fixed $\sigma \in \mathbb{R}^{n^{-1}_m}$. Let $(\lambda_k)$ be a sequence converging to $\lambda^*(\sigma)$ so that $0 < \lambda_k < \lambda^*(\sigma)$ for all $k \geq 1$. Then, each minimal positive strong solution $u_{\Lambda_k}$ for $\Lambda_k = (\lambda_k, \lambda_k \sigma)$ is stable and the $L^1$-limit

$$\lim_{k \to +\infty} u_{\Lambda_k} = u^*$$

is the extremal solution associated to $\Lambda$. In this section we will prove that $u^*$ is bounded when $\Omega$ is convex and $n = 2$ or $n = 3$. The key point here is an estimate for the solutions $u_{\Lambda_k}$ in the space $W^{1,4}$ in a neighborhood of $\partial \Omega$.

**Proposition 7.1.** Let $f \in C^2(\mathbb{R}^m)$ be a positive function satisfying $\text{Hess } f(t) > 0$ for all $t \in \mathbb{R}_+^m$ with $m \geq 1$. Assume that the potential field $F = \nabla f$ satisfies the assumptions (A), (B) and (C). Let also $\Lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{A}$ and $u_{\Lambda} = (u_1, \ldots, u_m)$ be the stable minimal positive strong solution of (17) associated to $\Lambda$. Assume that $n = 2$ or $n = 3$. Then, for any $t > 0$,

$$\|u_i\|_{L^\infty(\Omega)} \leq t + \frac{C(n)}{t} |\Omega|^{\frac{4-m}{2m}} \sqrt{\lambda_i} \left( \sum_{k} \frac{1}{\lambda_k} \int_{\{u_k < t\}} |\nabla u_k|^4 dx \right)^{1/2}$$

$$+ C(n) |\Omega|^{\frac{4-m}{2m}} \sqrt{\lambda_i} \left( \sum_{k < l} \int_{\Omega \{u_k \geq t, u_l \geq t\}} f_{u_k u_l}(u) \nabla u_k \nabla u_l dx \right)^{1/2}, \quad (18)$$

where $\{u_k < t\} = \{x \in \Omega \mid u_k(x) < t\}$.

The proof of Proposition 7.1 requires two essential tools. The first one is the vector Poincaré inequality (11) satisfied by the stable minimal positive strong solution $u_{\Lambda}$ of (17) for any $\Lambda \in \mathcal{A}$. The second one is an extension to potential elliptic systems of the Sternberg-Zumbrun inequality obtained in the scalar context in [27]. Both results are stated and proved below.

**Proposition 7.2.** Let $f \in C^2(\mathbb{R}^m)$ be a positive function satisfying $\text{Hess } f(t) > 0$ for all $t \in \mathbb{R}_+^m$ with $m \geq 1$. Assume that the potential field $F = \nabla f$ satisfies the assumptions (A), (B) and (C). Let also $\Lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{A}$ and $u_{\Lambda}$ be the stable minimal positive strong solution of (17) associated to $\Lambda$. Then, the inequality (11) holds for every test map $\psi = (\psi_1, \ldots, \psi_m) \in H^{1,2}_0(\Omega; \mathbb{R}^m)$.

**Proof.** Let $u_{\Lambda}$ be the minimal positive strong solution of (8) for $\Lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{A}$. By Theorem 1.4, the problem

$$\begin{cases}
-\Delta \varphi - \Lambda(\text{Hess } f(u_{\Lambda})\varphi) = \mu \varphi & \text{in } \Omega, \\
\varphi = 0 & \text{on } \partial \Omega
\end{cases}$$
Consider a cutoff map $\psi = (\psi_1, \ldots, \psi_m) \in C_0^\infty(\Omega; \mathbb{R}^m)$ be such an eigenfunction. Then, each component of $\varphi$ satisfies

$$-\Delta \varphi_i - \lambda_i \sum_j f_{u_i u_j}(u_\Lambda) \varphi_j = \mu_1 \varphi_i \text{ in } \Omega.$$ 

On the one hand, integrating by parts and applying Cauchy-Schwarz and Young’s inequalities, we obtain

$$\int_\Omega \Delta \varphi_i \psi_i^2 dx - \lambda_i \sum_j \int_\Omega f_{u_i u_j}(u_\Lambda) \varphi_j \psi_i^2 dx = \mu_1 \int_\Omega \psi_i^2 dx \geq 0.$$ 

On the other one, we have

$$\sum_{i,j} \int_\Omega f_{u_i u_j}(u_\Lambda) \frac{\varphi_j}{\varphi_i} \psi_i^2 dx = \sum_{i,j} \int_\Omega f_{u_i u_j}(u_\Lambda) \frac{\varphi_j}{\varphi_i} \psi_i^2 dx + \sum_{i,j} \int_\Omega f_{u_i u_j}(u_\Lambda) \frac{\varphi_i}{\varphi_j} \psi_j^2 dx + \sum_i \int_\Omega f_{u_i u_i}(u_\Lambda) \psi_i^2 dx \geq - \sum_{i,j} \int_\Omega f_{u_i u_j}(u_\Lambda) \frac{\varphi_i}{\varphi_j} \psi_j^2 dx + \sum_i \int_\Omega f_{u_i u_i}(u_\Lambda) \psi_i^2 dx = \sum_{i,j} \int_\Omega f_{u_i u_j}(u_\Lambda) \psi_i \psi_j dx.$$ 

Thus, for any $\psi = (\psi_1, \ldots, \psi_m) \in C_0^\infty(\Omega; \mathbb{R}^m)$, we conclude that

$$\sum_{i,j} \int_\Omega f_{u_i u_j}(u_\Lambda) \psi_i \psi_j dx \leq \sum_i \frac{1}{\lambda_i} \int_\Omega |\nabla \psi_i|^2 dx.$$ 

Finally, the above inequality easily extends by density to test maps $\psi \in H_0^1(\Omega; \mathbb{R}^m)$. \square
Proposition 7.3. Let $f \in C^2(\mathbb{R}^m)$ be a positive function satisfying \(\text{Hess} \ f(t) > 0\) for all $t \in \mathbb{R}^m$ with $m \geq 1$. Assume that the potential field $F = \nabla f$ satisfies the assumptions (A), (B) and (C). Let $\Lambda = (\lambda_1, \ldots, \lambda_m) \in A$ and $u_\Lambda = (u_1, \ldots, u_m)$ be the stable minimal positive strong solution of (17). Then, for any Lipschitz function $\eta_\delta : \Omega \to \mathbb{R}$ satisfying $\eta_\delta \big|_{\partial \Omega} = 0,$

$$\sum_i \frac{1}{\lambda_i} \int_{\Omega \cap \{|\nabla u_i| \neq 0\}} (|\nabla u_i|^2 + |A_i|^2|\nabla u_i|^2) \eta_i^2 \, dx \leq \sum_i \frac{1}{\lambda_i} \int_{\Omega} |\nabla u_i|^2|\nabla \eta_i|^2 \, dx + \sum_{i \neq j} \int_{\Omega} f_{u_i u_j}(u)|\nabla u_i||\nabla u_j|(\eta_i - \eta_j)^2 \, dx,$$

where $\nabla_T$ denotes the tangential gradient along a level set of $u_i$ and

$$|A_i|^2 = \sum_{l=1}^{n-1} \kappa_{i,l}^2$$

with $\kappa_{i,l}$ being principal curvatures of the level sets of $u_i$ passing through $x \in \Omega \cap \{|\nabla u_i| \neq 0\}$.

Proof. Taking $\psi_i = c_i \eta_i$, $i = 1, \ldots, m$, as a test functions in the Poincaré type inequality in (11), where

$$c_i = \sqrt{|\nabla u_i|^2 + \varepsilon^2}$$

and $\eta_\delta : \Omega \to \mathbb{R}$ is a Lipschitz function such that $\eta_\delta \big|_{\partial \Omega} = 0,$ we obtain

$$\sum_{i,j} \int_{\Omega} f_{u_i u_j}(u)c_i c_j \eta_i \eta_j \, dx \leq \sum_i \frac{1}{\lambda_i} \int_{\Omega} (c_i^2 |\nabla \eta_i|^2 + \nabla c_i \cdot \nabla (c_i \eta_i^2)) \, dx.$$ 

Integrating by parts, we get

$$\sum_{i,j} \int_{\Omega} f_{u_i u_j}(u)c_i c_j \eta_i \eta_j \, dx + \sum_i \frac{1}{\lambda_i} \int_{\Omega} c_i \eta_i^2 \Delta c_i \, dx \leq \sum_i \frac{1}{\lambda_i} \int_{\Omega} c_i^2 |\nabla \eta_i|^2 \, dx.$$ 

A simple calculation shows that

$$c_i \Delta c_i = \sum_k \Delta(\partial_k u_i) \partial_k u_i + \sum_{k,l} (\partial_{kl} u_i)^2 - \sum_k \left( \sum_{l} \partial_{kl} u_i \frac{\partial u_i}{\sqrt{|\nabla u_i|^2 + \varepsilon^2}} \right)^2.$$ 

Since $\Delta u_i = -\lambda_i f_{u_i}(u)$ em $\Omega$, we have $\Delta(\partial_k u_i) = -\lambda_i \sum_{r=1}^{m} f_{u_i u_r}(u) \partial_k u_r$. So,

$$\sum_k \Delta(\partial_k u_i) \partial_k u_i = -\lambda_i \sum_{r=1}^{m} f_{u_i u_r}(u) \nabla u_r \cdot \nabla u_i.$$ 

Therefore,

$$c_i^2 f_{u_i u_i}(u) + \frac{1}{\lambda_i} c_i \Delta c_i = \varepsilon^2 f_{u_i u_i}(u) + |\nabla u_i|^2 f_{u_i u_i} - \sum_{r=1}^{m} f_{u_i u_r}(u) \nabla u_r \cdot \nabla u_i$$

$$+ \frac{1}{\lambda_i} \left\{ \sum_{k,l} (\partial_{kl} u_i)^2 - \sum_k \left( \sum_{l} \partial_{kl} u_i \frac{\partial u_i}{\sqrt{|\nabla u_i|^2 + \varepsilon^2}} \right)^2 \right\}$$

\[27\]
and this leads to
\[
\sum \frac{1}{\lambda_i} \int_{\Omega} c_i^2 |\nabla \eta_i|^2 dx \geq \sum_{i \neq j} \int_{\Omega} f_{ui,uj}(u)c_i c_j \eta_i \eta_j dx + \sum_i \int_{\Omega} \varepsilon^2 \eta_i^2 f_{ui,ui}(u)dx \\
+ \sum_i \int_{\Omega} \left( |\nabla u_i|^2 f_{ui,ui}(u) - \sum_{r=1}^m f_{ui,ur} \nabla u_r \cdot \nabla u_i \right) \eta_i^2 dx \\
+ \sum_i \frac{1}{\lambda_i} \int_{\Omega} \left\{ \sum_{k,l} (\partial_{kl} u_i)^2 - \sum_k \left( \sum_l \partial_{kl} u_i \frac{\partial u_i}{|\nabla u_i|} \right)^2 \right\} \eta_i^2 dx.
\]

Estimating the last integral, we derive
\[
\sum \frac{1}{\lambda_i} \int_{\Omega} c_i^2 |\nabla \eta_i|^2 dx \geq \sum_{i \neq j} \int_{\Omega} f_{ui,uj}(u)c_i c_j \eta_i \eta_j dx + \sum_i \int_{\Omega} \varepsilon^2 \eta_i^2 f_{ui,ui}(u)dx \\
+ \sum_i \int_{\Omega} \left( |\nabla u_i|^2 f_{ui,ui}(u) - \sum_{r=1}^m f_{ui,ur} \nabla u_r \cdot \nabla u_i \right) \eta_i^2 dx \\
+ \sum_i \frac{1}{\lambda_i} \int_{\Omega \cap (\nabla u_i \neq 0)} \left\{ \sum_{k,l} (\partial_{kl} u_i)^2 - \sum_k \left( \sum_l \partial_{kl} u_i \frac{\partial u_i}{|\nabla u_i|} \right)^2 \right\} \eta_i^2 dx. \tag{19}
\]

We now use two differential identities. The first one can be found in [27] asserts that
\[
\sum_{k,l} (\partial_{kl} u_i)^2 - \sum_k \left( \sum_l \partial_{kl} u_i \frac{\partial u_i}{|\nabla u_i|} \right)^2 = |\nabla_T |\nabla u_i||^2 + |A_i|^2 |\nabla u_i|^2, \tag{20}
\]
where $\nabla_T$ and $|A_i|$ are as in the statement of proposition. The second one states for $\nabla u_i \neq 0$ that
\[
\sum_i |\nabla u_i|^2 f_{ui,ui}(u) \eta_i^2 - \sum_{i,r} f_{ui,ur}(u) \nabla u_r \cdot \nabla u_i \eta_i^2 = - \sum_{i<j} f_{ui,uj}(u) \nabla u_i \cdot \nabla u_j (\eta_i^2 + \eta_j^2). \tag{21}
\]

Replacing (20) and (21) in (19), we obtain
\[
\sum \frac{1}{\lambda_i} \int_{\Omega} c_i^2 |\nabla \eta_i|^2 dx \geq \sum_i \int_{\Omega} \varepsilon^2 \eta_i^2 f_{ui,ui}(u)dx \\
+ \sum_i \int_{\Omega} f_{ui,ui}(u)(2c_i c_j \eta_i \eta_j - \nabla u_i \cdot \nabla u_j (\eta_i^2 + \eta_j^2))dx \\
+ \sum_i \frac{1}{\lambda_i} \int_{\Omega \cap (\nabla u_i \neq 0)} (|\nabla_T |\nabla u_i||^2 + |A_i|^2 |\nabla u_i|^2) \eta_i^2 dx.
\]

Letting $\varepsilon \downarrow 0$ and using Cauchy-Schwarz’s inequality, we derive
\[
\sum \frac{1}{\lambda_i} \int_{\Omega \cap (\nabla u_i \neq 0)} (|\nabla_T |\nabla u_i||^2 + |A_i|^2 |\nabla u_i|^2) \eta_i^2 dx \leq \sum_i \frac{1}{\lambda_i} \int_{\Omega} |\nabla u_i|^2 |\nabla \eta_i|^2 dx \\
+ \sum_{i<j} \int_{\Omega} f_{ui,uj}(u) |\nabla u_i| |\nabla u_i| (\eta_i - \eta_j)^2 dx.
\]
Now, we are ready to prove the estimate (18).

Proof of Proposition 7.1. Given \( \Lambda \in \mathcal{A} \), let \( u_\Lambda = (u_1, \ldots, u_m) \) be the stable minimal positive strong solution of (17) associated to \( \Lambda \). For each \( i = 1, \ldots, m \), set \( T_{u_i} := \|u_i\|_{L^\infty(\Omega)} \) and \( \Gamma^u_s := u_i^{-1}(s) \) for \( s \in [0, T_{u_i}] \). By Sard’s theorem, almost every \( s \in (0, T_{u_i}) \) is a regular value of \( u_i \). Choose in Proposition 7.3, \( \eta_i(x) = \varphi(u_i(x)) \), where \( \varphi \) is a Lipschitz function in \([0, +\infty)\) such that \( \varphi(0) = 0 \). By the coarea formula, we have

\[
\frac{1}{\lambda_i} \int_{\Omega} |\nabla u_i|^2 |\nabla \eta_i|^2 \, dx = \frac{1}{\lambda_i} \int_{\Omega} |\nabla u_i|^4 \varphi'(u_i)^2 \, dx = \frac{1}{\lambda_i} \int_{0}^{T_{u_i}} \left( \int_{\Gamma^u_s} |\nabla u_i|^3 dV_s \right) \varphi'(s)^2 \, ds.
\]

This equality and Proposition 7.3 together imply

\[
\sum_i \frac{1}{\lambda_i} \int_{0}^{T_{u_i}} \left( \int_{\Gamma^u_s} |\nabla u_i|^3 dV_s \right) \varphi'(s)^2 \, ds + \sum_{i<j} \int_{\Omega} f_{u_i u_j}(u) |\nabla u_i| |\nabla u_j| (\varphi(u_i) - \varphi(u_j))^2 \, dx
\]

\[
\geq \sum_i \frac{1}{\lambda_i} \int_{\Omega \setminus \{|\nabla u_i| \neq 0\}} (|\nabla u_i| |\nabla u_i|^2 + |A_i|^2 |\nabla u_i|^2) \varphi(u_i)^2 \, dx
\]

\[
\geq \sum_i \frac{1}{\lambda_i} \int_{0}^{T_{u_i}} \left( \int_{\Gamma^u_s} |A_i|^2 |\nabla u_i| dV_s \right) \varphi(s)^2 \, ds.
\]

More specifically, taking

\[
\varphi(s) = \begin{cases} 
  s/t & \text{if } 0 \leq s < t, \\
  1 & \text{if } t \leq s
\end{cases}
\]

in the above inequality, we get

\[
\frac{1}{\lambda_i} \int_{t}^{T_{u_i}} \int_{\Gamma^u_s} |A_i|^2 |\nabla u_i| dV_s \, ds 
\leq \sum_k \frac{1}{\lambda_k} \int_{0}^{T_{u_k}} \left( \int_{\Gamma^u_s} |A_k|^2 |\nabla u_k| dV_s \right) \varphi(s)^2 \, ds
\]

\[
\leq \sum_k \frac{1}{\lambda_k t^2} \int_{0}^{t} \left( \int_{\Gamma^u_s} |A_k|^2 |\nabla u_k| dV_s \right) \, ds
\]

\[
+ \sum_{k<l} \int_{\Omega} f_{u_k u_l}(u) |\nabla u_k| |\nabla u_l| (\varphi(u_k) - \varphi(u_l))^2 \, dx.
\]

In conclusion, the expression of \( \varphi \) and the coarea formula provide

\[
\frac{1}{\lambda_i} \int_{t}^{T_{u_i}} \int_{\Gamma^u_s} |A_i|^2 |\nabla u_i| dV_s \, ds 
\leq \sum_k \frac{1}{\lambda_k t^2} \int_{\{u_k < t\}} |\nabla u_k|^4 \, dx
\]

\[
+ \sum_{k<l} \int_{\Omega \setminus \{u_k < t, u_l \geq t\}} f_{u_k u_l}(u) |\nabla u_k| |\nabla u_l| \, dx.
\]
Denote by $|\Gamma^u_s|$ the volume of $\Gamma^u_s$ and $H_i$ the mean curvature function of $\Gamma^u_s$. For any $n \geq 2$, the geometric inequality

$$|\Gamma^u_s|^{\frac{n-2}{n}} \leq C(n) \int_{\Gamma^u_s} |H_i|dV_s$$

holds almost every $s$. In dimension $n = 2$, the set $\Gamma^u_s$ is a regular curve for almost every $s$ and (23) follows from the theory of plane curves. For $n \geq 3$, the inequality (23) is a consequence of Theorem 2.1 of \[22\] by Michael and Simon (see also Mantegazza \[21\], Proposition 5.2).

On the other hand, the isoperimetric inequality ensures that

$$V_i(s) := |\{u_i > s\}| \leq C(n)|\Gamma^u_s|^{\frac{n}{n-1}}.$$  

(24)

Joining (23) and (24) and applying Hölder’s inequality, we derive

$$V_i(s)^{\frac{n-2}{n}} \leq C(n) \int_{\Gamma^u_s} |H_i|dV_s$$

\[
\leq C(n) \left\{ \int_{\Gamma^u_s} |A_i|^2|\nabla u_i|dV_s \right\}^{1/2} \left\{ \int_{\Gamma^u_s} \frac{dV_s}{|\nabla u_i|} \right\}^{1/2}.
\]

Here it was used that $|H_i| \leq |A_i|$. Thus, we have

$$\frac{1}{\sqrt{\lambda_i}}(T_{u_i} - t) = \frac{1}{\sqrt{\lambda_i}} \int_t^{T_{u_i}} ds$$

\[
\leq \frac{1}{\sqrt{\lambda_i}} \int_t^{T_{u_i}} C(n) \left\{ \int_{\Gamma^u_s} |A_i|^2|\nabla u_i|dV_s \right\}^{1/2} \left\{ V_i(s)^{\frac{2(2-n)}{n}} \int_{\Gamma^u_s} \frac{dV_s}{|\nabla u_i|} \right\}^{1/2} ds
\]

\[
\leq C(n) \left\{ \frac{1}{\lambda_i} \int_t^{T_{u_i}} \int_{\Gamma^u_s} |A_i|^2|\nabla u_i|dV_s ds \right\}^{1/2} \left\{ \int_t^{T_{u_i}} V_i(s)^{\frac{2(2-n)}{n}} \int_{\Gamma^u_s} \frac{dV_s}{|\nabla u_i|} ds \right\}^{1/2}.
\]

Thanks to the estimate (22), we arrive at

$$\frac{1}{\sqrt{\lambda_i}}(T_{u_i} - t) \leq C(n) \left\{ \sum_k \frac{1}{\lambda_i k^2} \int_{\{u_k < t\}} |\nabla u_k|^4 dx + \sum_{k < l} \int_{\{u_k \geq t, u_l \geq t\}} f_{u_k u_l}(u) |\nabla u_k||\nabla u_l|dx \right\}^{1/2}$$

\[
\times \left\{ \int_t^{T_{u_i}} V_i(s)^{\frac{2(2-n)}{n}} \int_{\Gamma^u_s} \frac{dV_s}{|\nabla u_i|} ds \right\}^{1/2}.
\]

Since the function $V_i(t)^{\frac{2-n}{n}}$ is nonincreasing for $n \leq 3$, again applying the coarea formula, we get

$$-V_i'(s) = \int_{\Gamma^u_s} \frac{dV_s}{|\nabla u_i|}$$

for $s \in (0, T_{u_i})$ almost everywhere. We also have

$$\left| \Omega \right|^{\frac{2-n}{n}} \geq V_i(t)^{\frac{2-n}{n}} = V_i(s)^{\frac{2-n}{n}} \bigg|_{s=T_{u_i}}^t \geq \frac{4-n}{n} \int_t^{T_{u_i}} V_i(s)^{\frac{2(2-n)}{n}} (-V_i'(s)) ds.$$
Hence, we establish that
\[
\frac{4 - n}{n} \int_{T^{u_i}} V_i(s)^{\frac{2(2-n)}{n}} \int_{T^{u_i}} \frac{dV_s}{|\nabla u_i|} ds \leq |\Omega|^{4-n}.
\]
Finally, using this inequality, we deduce that
\[
\|u_i\|_{L^\infty(\Omega)} \leq t + C(n)|\Omega|^{\frac{4-n}{2n}} \sqrt{t} \left( \sum_k \frac{1}{\lambda_k t^2} \int_{\{|u_k| < t\}} |\nabla u_k|^4 dx \right)^{1/2} + \sum_{k<l} \int_{\Omega \setminus \{|u_k \geq t, u_l \geq t\}} f_{u_k u_l}(u)|\nabla u_k| |\nabla u_l| dx.
\]

The next proposition consists of two estimates in a neighborhood of \(\partial \Omega\) for positive strong solutions of (17).

**Proposition 7.4.** Let \(u\) be a positive strong solution of (17). Denote \(\Omega_{\varepsilon} = \{ x \in \Omega : \delta(x) < \varepsilon \}\). Assume that \(\Omega\) is convex and \(n \geq 2\). Then, there exists constants \(\varepsilon, D_1, D_2 > 0\) depending only on the domain \(\Omega\) such that, for any \(i = 1, \ldots, m\),

(i) \(\|u_i\|_{L^\infty(\Omega_{\varepsilon})} \leq D_2 \|u_i\|_{L^1(\Omega)}\);

(ii) \(u_i(x) \geq D_1 \delta(x)\) for every \(x \in \Omega\).

The tool used in the proof of the first assertion is the well-known moving planes method. We refer for example to Troy \[29\] where the global estimate (i) is proved on convex domains for any \(m \geq 1\) and \(n \geq 2\). The claim (ii) is a direct consequence from Lemmas 5.1 and 5.2.

Propositions 7.1 and 7.4 are essential ingredients in the proof of the following result:

**Proposition 7.5.** Let \(f \in C^2(\mathbb{R}^m)\) be a positive function satisfying \(\text{Hess} f(t) > 0\) for all \(t \in \mathbb{R}^m\) with \(m \geq 1\). Assume that the potential field \(F = \nabla f\) satisfies the assumptions (A), (B) and (C). Assume that \(\Omega\) is convex and \(n = 2\) or \(n = 3\). Moreover, let \(u_\Lambda = (u_1, \ldots, u_m)\) be the stable minimal positive strong solution of (17) associated to \(\Lambda \in \mathcal{A}\). Then, there exists a constant \(C_0 > 0\), depending on \(\Omega, \varepsilon, D_1, D_2, \Lambda, \|\nabla f\|_{L^\infty(\mathbb{R}^m; \mathbb{R}^m)}\) and \(\|\text{Hess} f\|_{L^\infty(\mathbb{R}^m; \mathbb{R}^{m^2})}\), such that
\[
\|u_i\|_{L^\infty(\Omega)} \leq C_0, \quad (25)
\]
where \(\varepsilon, D_1\) and \(D_2\) are given in Proposition 7.4 and \(r = \rho \|u_\Lambda\|_{L^1(\Omega; \mathbb{R}^m)}\).

**Proof.** In Proposition 7.1, take
\[
t = D_1 \frac{\varepsilon}{2}.
\]
By the part (ii) of Proposition 7.4, for \(x \in \{u_i < t\}\), we have
$$D_1 \delta(x) \leq u_i(x) < t = D_1 \frac{\varepsilon}{2},$$

so that

$$\delta(x) < \frac{\varepsilon}{2}.$$ 

Thus,

$$\{ u_i < t \} \subset \Omega_{\varepsilon/2} \text{ and } \Omega \setminus \{ u_k \geq t, u_l \geq t \} \subset \Omega_{\varepsilon/2}$$

for all $i, k, l = 1, \ldots, m$. By Sobolev embedding, it suffices to establish the conclusion for $\| u_i \|_{W^{1,4}(\Omega_{\varepsilon/2})}$. Notice that $u_\Lambda$ satisfies

$$\begin{cases}
-\Delta u_\Lambda &= \Lambda \nabla f(u_\Lambda) \text{ in } \Omega_{\varepsilon}, \\
u_\Lambda &= 0 \text{ on } \partial \Omega.
\end{cases}$$

Moreover, $\partial \Omega \cup \Omega_{\varepsilon/2}$ is a precompact subset of $\partial \Omega \cup \Omega_{\varepsilon}$ and both are smooth. On the other hand, by the part (i) of Proposition 7.4, we have

$$\| f_{u_i}(u_1, \ldots, u_m) \|_{L^\infty(\Omega_{\varepsilon})} \leq \| \nabla f \|_{L^\infty(\overline{B_r};\mathbb{R}^m)}.$$ 

So, global $L^p$ Calderón-Zygmund estimate applied to each equation of (27) yields

$$\| u_i \|_{W^{1,4}(\Omega_{\varepsilon/2})} \leq C_1$$

for some constant $C_1 > 0$ depending on $\Omega, \varepsilon, D_1, D_2, \Lambda$ and $\| \nabla f \|_{L^\infty(\overline{B_r};\mathbb{R}^m)}$. Therefore, Proposition 7.1, (26) and the above estimate give, for any $i = 1, \ldots, m,$

$$\| u_i \|_{L^\infty(\Omega)} \leq C_0.$$ 

Finally, the boundedness of the extremal solution $u^*$ follows from proposition 7.5.

**Proof of Theorem 1.6.** If $u^*$ is the extremal solution of (17) associated to $\Lambda = (\lambda^*(\sigma), \lambda^*(\sigma)) \in \Lambda^*$. Take a sequence $(\lambda_k)$ converging to $\lambda^*(\sigma)$ so that $0 < \lambda_k < \lambda^*(\sigma)$. Let $u_{\Lambda_k}$ the minimal stable positive strong solution associated to $\Lambda_k = (\lambda_k, \lambda_k \sigma)$. Since $(u_{\Lambda_k})$ converges pointwise almost everywhere in $\Omega$ and in $L^1(\Omega)$ to $u^*$, letting $k \to +\infty$ in (25), we conclude that $u^* \in L^\infty(\Omega)$.

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