A Review on Realization Theory for Infinite-Dimensional Systems

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Abstract

We give an introduction to the realization theory for infinite-dimensional systems. That is, we show that for any function $G$, analytic and bounded in the right half of the complex plane, there exist (unbounded) operators $A, B, C$ such that $G(s_1) - G(s_2) = (s_2 - s_1)C(s_1I - A)^{-1}(s_2I - A)^{-1}B$. Here $A$ is the infinitesimal generator of a strongly continuous semigroup on a Hilbert space, and $B$ and $C$ are admissible input and output operators, respectively. Our results summarise and clarify the results as found in the literature, starting more than 40 years ago.

1 Introduction

Already since the beginning of infinite-dimensional systems theory, there has been interest in the state-space realization problem. The state-space realization problem is the problem of finding, for a given function, a system in state-space form whose transfer function equals this given function. If we start with a rational function, then it is well-known that we can always find a system with a finite-dimensional state-space whose transfer function is the given rational function. Moreover, it is always possible to find a controllable and observable realization and all controllable and observable realizations are equivalent, i.e., let $(A_1, B_1, C_1, D_1)$ and $(A_2, B_2, C_2, D_2)$ be two realizations of the transfer function $G(s)$ that are controllable and observable, then there exists an invertible matrix $S$ such that $A_1 = SA_2S^{-1}, B_1 = SB_2, C_1 = C_2S^{-1},$ and $D_1 = D_2$.

Since every finite-dimensional system has a rational transfer function, for a non-rational transfer function it is only possible to find a (state-space) realization with

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an infinite-dimensional state space. For functions that are analytic and bounded in 
some right-half plane, the realization problem was investigated by a number of 
people, e.g. Baras and Brockett [BB73], Fuhrmann [Fuh81], Helton [Hel76], 
Yamamoto [Yam81, Yam82], Salamon [Sal89] and Weiss [Wei89a, Wei97]. Here we present the 
realization theory in the language of well-posed linear systems.

We end this section with some notation and well-known results, see e.g. [CZ95, 
Appendix A.6], Duren [Dur00] or Rudin [Ru03].

\[
C^s_\delta := \{ z \in \mathbb{C} \mid \text{Re } z > \delta \}, \quad \delta \in \mathbb{R},
\]

\[
C_s := \{ z \in \mathbb{C} \mid \text{Re } z < s \}, \quad s \in \mathbb{R},
\]

\[
H_\infty(\Omega) := \{ f : \Omega \to \mathbb{C} \mid f \text{ is holomorphic and bounded}, \ \Omega \subset \mathbb{C},
\]

\[
H_2(C^s_\delta) := \{ f : C^s_\delta \to \mathbb{C} \mid f \text{ holomorphic and sup}_{r > \delta} \int_{-\infty}^{\infty} |f(r+i\omega)|^2 d\omega < \infty,\}
\]

\[
H_2(C^-_s) := \{ f : C^-_s \to \mathbb{C} \mid f \text{ holomorphic and sup}_{r < \delta} \int_{-\infty}^{\infty} |f(r+i\omega)|^2 d\omega < \infty,\}
\]

Furthermore, a holomorphic function \(G : C^s_\delta \to \mathbb{C}\) is called *inner* if \(|G(z)| \leq 1\) 
for \(z \in C^s_\delta\), and \(|G(it)| = 1\) for almost every \(t \in \mathbb{R}\).

Clearly, every inner function is an element of \(H_\infty(C^s_\delta)\). Hence we may define \(G(it)\) 
as the non-tangential limits of \(G(z)\). This limit exists almost everywhere, and so the condition, \(|G(it)| = 1\) a.e., makes sense.

On \(H_2(C^s_\delta)\) and \(H_2(C^-_s)\) we define the following inner product from \(L_2(i\mathbb{R})\):

\[
\langle f, g \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(i\omega)\overline{g(i\omega)} d\omega.
\]  

Then the following holds

1. \(H_2(C^s_\delta)^\perp = H_2(C^-_s)\);
2. \(H_2(C^s_\delta) \oplus H_2(C^-_s) = L_2(i\mathbb{R})\).
3. The Laplace transform is an isometric isometry between \(L_2(0, \infty)\) and \(H_2(C^s_\delta)\).
   Similarly, \(L_2(-\infty, 0)\) and \(H_2(C^-_s)\) are isometrically isomorph.
4. The Fourier transform is an isometric isomorphism between \(L_2(\mathbb{R})\) and \(L_2(i\mathbb{R})\).

The latter two results are known as Paley-Wiener theorem.

## 2 Well-posed linear systems and realization theory

A quite large and well-studied class of infinite-dimensional linear systems is the class 
of well-posed linear systems introduced by Salamon [Sal89] and Weiss [Wei89a, 
Wei89b]. By now there are excellent books on well-posed linear systems, see e.g. 
[Sta05] or [TW09].

Let \(T(t)\) be a \(C_0\)-semigroup on the separable Hilbert space \(H\), and let \(A\) its 
generator. We define the space \(H_{-1}\) to be the completion of \(H\) with respect to the norm

\[
\|x\|_{-1} := \|(\beta I - A)^{-1}x\|
\]

and the space \(H_1\) to be \(D(A)\) with the norm

\[
\|x\|_1 := \|(\beta I - A)x\|
\]

where \(\beta \in \rho(A)\), the resolvent set of \(A\). It is easy to verify that the topology of \(H_{-1}\) 
and \(H_1\) does not depend on \(\beta \in \rho(A)\). Moreover, \(\| \cdot \|_{1}\) is equivalent to the graph
norm on $D(A)$, so $H_1$ is complete. In Weiss [Wei89a, Remark 3.4] it is shown that $T(t)$ has a restriction to a $C_0$-semigroup on $H_1$ whose generator is the restriction of $A$ to $D(A)$, and $T(t)$ can be extended to a $C_0$-semigroup on $H_{-1}$ whose generator is an extension of $A$ with domain $H$. Therefore, we get

$$A \in \mathcal{L}(H_1, H) \quad \text{and} \quad A \in \mathcal{L}(H, H_{-1}).$$

$H_{-1}$ equals the dual of $D(A^*)$, where we have equipped $D(A^*)$ with the graph norm (see [Wei89a]). Following [Wei89a] and [Wei89b] we introduce admissible control operators and observation operators for $T(t)$.

**Definition 2.1**

1. Let $B \in \mathcal{L}(\mathbb{C}, H_{-1}) = H_{-1}$. For $t \geq 0$ we define the operator

$$B_t : L_2(0, \infty) \to H_{-1}$$

by

$$B_t u := \int_0^t T(t - \rho) Bu(\rho) d\rho.$$  

Then $B$ is called an admissible control operator for $T(t)$, if for some (and hence any) $t > 0$, $B_t \in \mathcal{L}(L_2(0, \infty), H)$.

2. Let $B$ be an admissible control operator for $T(t)$. $B$ is called an infinite-time admissible control operator for $T(t)$, if $T(\cdot) Bu(\cdot) : [0, \infty) \to H_{-1}$ is integrable for every $u \in L_2(0, \infty)$, and the operator $B_\infty : L_2(0, \infty) \to H_{-1}$, given by

$$B_\infty u := \int_0^\infty T(t) Bu(t) dt,$$

satisfies $B_\infty \in \mathcal{L}(L_2(0, \infty), H)$.

3. Let $C \in \mathcal{L}(H_1, \mathbb{C})$. Then $C$ is called an admissible observation operator for $T(t)$, if for some (and hence any) $t > 0$, there is some $K > 0$ such that

$$\|CT(\cdot)x\|_{L_2(0, t)} \leq K \|x\|, \quad x \in D(A).$$

4. Let $C$ be an admissible observation operator for $T(t)$. We call $C$ an infinite-time admissible observation operator if there is some $K > 0$ such that

$$\|CT(\cdot)x\|_{L_2(0, \infty)} \leq K \|x\|, \quad x \in D(A).$$

By definition, every infinite-time admissible control operator for $T(t)$ is an admissible control operator for $T(t)$. If $T(t)$ is exponentially stable, then the two notions of admissibility coincide. Similar statements hold for admissible observation operators for $T(t)$. Moreover, $B$ is an (infinite-time) admissible control operator for $T(t)$ if and only if $B^*$ is an (infinite-time) admissible observation operator for $T^*(t)$.

Let $C$ be an admissible observation operator for $T(t)$. Then for $t \geq 0$ the operator $\Psi_t \in \mathcal{L}(D(A), L_2(0, t))$, given by $\Psi_t x := CT(\cdot)x$, has a unique extension to $\mathcal{L}(H, L_2(0, t))$ (again denoted by $\Psi_t$). Similarly, if $C$ is an infinite-time admissible observation, then we can extend the operator $\Psi_\infty \in \mathcal{L}(D(A), L_2(0, \infty))$, given by $\Psi_\infty x := CT(\cdot)x$, to $\mathcal{L}(H, L_2(0, \infty))$.

**Definition 2.2** Let $B$ be an admissible control operator for $T(t)$, then

1. $(T, B)$ is exactly controllable in finite time if there exists a time $t_0$ such that $\text{Im} B_{t_0} = H$.

2. $(T, B)$ is approximately controllable if $\bigcup_{t \geq 0} \text{Im} B_t = H$. 

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3. $(T, B)$ is exactly controllable if $B$ be an infinite-time admissible control operator for $T(t)$ and $\text{Im} B_\infty = H$.

Let $C$ be an admissible observation operator for $T(t)$, then

1. $(T, C)$ is exactly observable in finite time if there exists a time $t_0$ such that $\|\Psi_{t_0} x\| \geq c\|x\|$ for some positive $c$ and every $x \in X$.

2. $(T, C)$ is approximately observable if $\cap_{t \geq 0} \ker \Psi_t = \{0\}$.

3. $(T, C)$ is exactly observable if $\Psi_\infty \in \mathcal{L}(H, L_2(0, \infty))$ and $\|\Psi_\infty x\| \geq c\|x\|$ for some positive $c$ and every $x \in X$.

If $B$ is infinite-time admissible, then it is easy to see that $(T, B)$ is approximately controllable if and only if $\text{Im} B_\infty = H$. A similar statement holds for the observation operator and approximate observability. It is easy to see, that controllability and observability are dual notions, i.e., $(T, B)$ is approximately (exactly) controllable if and only if $(T^*, B^*)$ is approximately (exactly) observable. We are now in the position to introduce well-posed linear systems.

**Definition 2.3** $(T, B, C, G)$ is called a well-posed linear system if the following holds

1. $T(t)$ is a $C_0$-semigroup,

2. $B$ is an admissible control operator for $T(t)$,

3. $C$ is an admissible observation operator for $T(t)$,

4. There exists a constant $\rho > 0$ such that $G \in \mathcal{H}_\infty(\mathbb{C}_\rho^+)$ and

\[
\frac{G(s) - G(z)}{z - s} = C(sI - A)^{-1}(zI - A)^{-1}B, \quad s, z \in \mathbb{C}_\rho^+, s \neq z, \tag{2}
\]

where $A$ is the generator of $T(t)$.

Here $G$ is called the transfer function. A transfer function is determined by $(T, B, C)$ up to an additive constant operator. Note, that this definition of a well-posed linear system is not the standard one introduced by Weiss [Wei89a], but it is equivalent to Weiss’s definition, see Curtin and Weiss [CW89]. Moreover, this definition of a well-posed linear system is equivalent to Salamon’s definition of a time-invariant, linear control system, see Salamon [Sal89] and Weiss [Wei94a].

Following, Salamon [Sal89], the system trajectory of a well-posed linear system $(T, B, C, G)$ is given by

\[
x(t, x_0, u) := T(t)x_0 + \int_0^t T(t - \rho)Bu(\rho)\,d\rho, \quad t \geq 0,
\]

\[
y(t, x_0, u) := C(x(t, x_0, u) - (\mu I - A)^{-1}Bu(t)) + G(\mu)u(t), \quad t \geq 0.
\]

Here $u \in L_2(0, \infty)$ denotes the input of the system, $x_0 \in H$ denotes the initial state, $x(t, x_0, u)$ denotes the state of the system at time $t$, and $y(t, x_0, u)$ denotes the output at time $t$. Note, that the definition of $y(t, x_0, u)$ does not depend on $\mu \in \rho(A)$, where $\rho(A)$ denotes the resolvent set of $A$. Moreover, if additionally $(T, B, C, G)$ is a regular system, i.e., $D := \lim_{s \to +\infty, s \in \mathbb{R}} G(s)$ exists, then we are able to choose $D$ as the feedthrough term and $y(t, x_0, u)$ is given by

\[
y(t, x_0, u) = Cx(t, x_0, u) + Du(t), \quad t \geq 0.
\]

Note, that in general $D$ does not exists.
Definition 2.4 A well-posed linear system \((T, B, C, G)\) is called exponentially stable, if \(T\) is an exponentially stable \(C_0\)-semigroup. The system is called infinite-time admissible if \(B\) is an infinite-time admissible control operator and \(C\) is an infinite-time admissible observation operator for \(T(t)\).

Furthermore, it will be called approximately controllable, exactly controllable, or exactly controllable in finite-time if \((T, B)\) has that property. Similar properties are defined concerning observability.

Definition 2.5 Let \(G \in H_\infty(C_0^+)\) for some \(\delta > 0\). We say that \(G\) has a realization as a well-posed linear system if there exists a well-posed linear system \((T, B, C, G)\).

Salamon [Sal89] proved that every \(G \in H_\infty(C_0^+)\) has a realization as a well-posed linear system. His realization is the well-known shift realization. The shift realization has already been studied by many mathematicians, see for example Baras and Brockett [BB73], Helton [Hel76], Fuhrmann [Fuh81], Yamamoto [Yam81] [Yam82], and Weiss [Wei97]. Before we state the proof, we would like to present the motivation behind this proof.

Assume that \(G\) is the Laplace transform of a continuous function \(h \in L_1(0, \infty) \cap L_2(0, \infty)\). Then finding a state-space realization just means finding a triple \((T(t), B, C)\) such that

\[
h(t) = CT(t)B, \quad t \geq 0.
\]

To establish this equality, the idea is very simple. We choose the state space \(L_2(0, \infty)\), and define for \(z \in L_2(0, \infty)\) the left-shift semigroup

\[
[T(t)z](x) = z(t + x), \quad x \geq 0.
\]

Furthermore, for a continuous functions in \(z \in L_2(0, \infty)\) we define

\[
Cz = z(0).
\]

If we now choose \(B = h\), then by combining (4) and (5) we see that

\[
CT(t)B = [T(t)h](0) = h(t), \quad t \geq 0.
\]

This is precisely equality (3), and hence we have constructed a realization.

For transfer functions that do not have a smooth inverse Laplace transform, there are difficulties in defining \(CT(t)B\). However, as we shall show, the realization is still possible with these choices of \(T(t), B\) and \(C\). In order to prove the realization, it is easier to work with the Laplace transforms of the above object. Thus \(L_2(0, \infty)\) becomes \(H_2(C_0^+)\), and \(T(t), B\) and \(C\) become their equivalent counterpart on this space.

Theorem 2.6 Every function \(G \in H_\infty(C_0^+)\) has a realization.

Proof: As state space \(H\) we choose

\[
H := H_2(C_0^+).
\]

Our \(C_0\)-semigroup is given by

\[
T(t)x = P_{H_2(C_0^+)}[e^{t}x(\cdot)], \quad x \in H,
\]

where \(P_{H_2(C_0^+)}\) is the orthogonal projection from \(L_2(i\mathbb{R})\) to \(H_2(C_0^+)\). This semigroup has as infinitesimal generator

\[
(Ax)(s) = sx(s) - \dot{x}(0), \quad x \in D(A),
\]
with
\[ D(A) = \{ x \in H \mid s \mapsto sx(s) - \hat{x}(0) \in H_2(C_0^+) \}, \]
where \( \hat{x} \) denotes the inverse Laplace transform of \( x \).

Furthermore, we define
\[ B = G, \]
and
\[ Cx = \hat{x}(0) \quad \text{for } x \in D(A). \]

Having made these choices, we still have to prove that they form a well-posed linear system with transfer function \( G \). We begin by showing that \( T(t) \) is a \( C_0 \)-semigroup on \( H \).

1. Let \( S(t) \) be the right-shift semigroup on \( H \), i.e.
\[ S(t)x := e^{-t}x(\cdot), \quad x \in H. \]

It is easy to see that this is a \( C_0 \)-semigroup, and that the adjoint of this \( C_0 \)-semigroup equals \( T(t) \). As \( H \) is a Hilbert space, the adjoint of a \( C_0 \)-semigroup is again a \( C_0 \)-semigroup. Hence, \( T(t) \) is a \( C_0 \)-semigroup.

2. Now we shall derive a simple formula for the resolvent operator of \( A \). Since the right-shift semigroup has growth bound zero, so has its adjoint, \( T(t) \), and thus we have that the open right-half is contained in the resolvent set of \( A \). Take \( \beta \in C_0^+ \), and \( x, y \in H \), then
\[
\langle y, (\beta I - A)^{-1}x \rangle = \int_0^\infty \langle y, e^{-\beta t}T(t)x \rangle dt \\
= \int_0^\infty e^{-\beta t} \langle T(t)^*y, x \rangle dt \\
= \int_0^\infty e^{-\beta t} \frac{1}{2\pi} \int_{-\infty}^\infty e^{-i\omega t} y(i\omega) \overline{x(i\omega)} d\omega dt \\
= \frac{1}{2\pi} \int_{-\infty}^\infty \left[ \frac{1}{\beta + i\omega} y(i\omega) \overline{x(i\omega)} \right] d\omega \\
= \frac{1}{2\pi} \int_{-\infty}^\infty y(i\omega) \overline{x(i\omega)} \left[ \frac{x(i\omega)}{\beta - i\omega} - \frac{x(\beta)}{\beta - i\omega} \right] d\omega \\
= \langle y, \frac{x(\cdot) - x(\beta)}{\beta - \cdot} \rangle.
\]

Here we have used that \( x(\beta)/x(\cdot) \) is in \( H_2(C_0^-) = [H_2(C_0^+)]^\perp \). Furthermore, it is easy to see that \( \frac{x(\cdot) - x(\beta)}{\beta - \cdot} \) is an element of \( H_2(C_0^+) \). So we conclude that
\[
[(\beta I - A)^{-1}x](s) = \frac{x(s) - x(\beta)}{\beta - s} \quad \text{for } s \neq \beta. \tag{6}
\]

3. The inverse Laplace transform of the function given in (6) is given by
\[
\hat{z}(t) = \int_t^\infty \hat{x}(\tau)e^{-\beta \tau} d\tau.
\]
Therefore, \( \hat{\xi} \) is a continuous function on \([0, \infty)\), and the value in zero equals \( x(\beta) \).

From (6) we can identify \( H \) with \( D(A) \) and an easy calculation shows that

\[
(Ax)(s) = sx(s) - \hat{x}(0), \quad x \in D(A),
\]

with

\[
D(A) = \{ x \in H \mid s \mapsto sx(s) - \hat{x}(0) \in H_2(\mathbb{C}_0^+) \}.
\]

4. Using (6) we can identify \( H_{-1} = D(A^*)' \) with the space

\[
\left\{ f : \mathbb{C}_0^+ \to \mathbb{C} \mid s \mapsto \frac{f(s) - f(\beta)}{\beta - s} \in H \text{ for some } \beta \in \mathbb{C}_0^+ \right\}.
\]

Moreover, the sesquilinear form \( \langle \cdot, \cdot \rangle_{D(A^*) 	imes D(A^*)}' \) is given by

\[
\langle f, g \rangle_{D(A^*) 	imes D(A^*)}' = \int_{-\infty}^{\infty} f(i\omega)g'(i\omega) \, d\omega, \quad f \in D(A^*), g \in D(A^*)'.
\]

5. From (9) it follows directly that \( H_{\infty}(\mathbb{C}_0^+) \) can be seen as a subspace of \( H_{-1} \), and so \( B = G \) is an element of \( H_{-1} \).

Now we show that \( B \) is admissible. Take \( y \in D(A^*) \)

\[
\langle y, \int_0^\infty T(t)Bu(t)dt \rangle_{D(A^*) 	imes D(A^*)}'
\]

\[
= \int_0^\infty \langle y, T(t)Bu(t) \rangle_{D(A^*) 	imes D(A^*)}' \, dt
\]

\[
= \int_0^\infty \langle T(t)^*y, Bu(t) \rangle_{D(A^*) 	imes D(A^*)}' \, dt
\]

\[
= \int_0^\infty \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t}y(i\omega)\overline{G(i\omega)u(t)} \, d\omega \, dt
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} y(i\omega)\overline{G(i\omega)u(-j\omega)} \, d\omega
\]

\[
= \langle y, P_{H_2} (G(\cdot)\hat{u}(\cdot)) \rangle.
\]

Since for every \( u \in L_2(0, \infty) \), we have that \( \hat{u}(\cdot) \in L_2(i\mathbb{R}) \), and since \( G \) is bounded on the imaginary axis, we have that \( \hat{G}(\cdot)\hat{u}(\cdot) \in L_2(i\mathbb{R}) \). Thus we have that

\[
\int_0^\infty T(t)Bu(t)dt = P_{H_2(\mathbb{C}_0^+)} (G(\cdot)\hat{u}(\cdot))
\]

is well-defined for every \( u \in L_2(0, \infty) \) with values in \( H \). Thus \( B \) is an admissible control operator for \( T \).

6. Part 2 implies that for \( x \in D(A) \), \( \hat{x}(0) \) is well-defined. This immediately proves that \( C \) is a well-defined operator on \( D(A) \). From part 3 and equation (6), we see that

\[
C(\beta I - A)^{-1}x = x(\beta).
\]

Since \( C(\beta I - A)^{-1}x \) is the Laplace transform of \( CT(\cdot)x \), we get from the equation above that

\[
CT(t)x = \hat{x}(t),
\]
for every \( x \in H = H_2(\mathbb{C}_+^+) \). From Paley-Wiener theorem we get that the \( L_2(0, \infty) \)-norm of \( \hat{x} \) equals the \( H \)-norm of \( x \). In other words, for every \( x \in H \) we have that \( \Psi_\infty x := CT(t)x \in L_2(0, \infty) \). Hence \( C \) is an admissible observation operator for \( T \).

Note that we even have
\[
\| \Psi_\infty x \| = \| x \|. \tag{13}
\]
Thus \( (T, C) \) is exactly observable.

7. We now show that \( G \) is a transfer function of \( (T, B, C) \). We have to show that \( (2) \) holds. Combining equation \( (11) \) with \( (6) \) gives that
\[
C(sI - A)^{-1}(\beta I - A)^{-1}B = [(\beta I - A)^{-1}B](s) = [(\beta I - A)^{-1}G](s) = \frac{G(s) - G(\beta)}{\beta - s}.
\]
Thus we have constructed a realization of the transfer function \( G \).

For a \( G \in H_\infty(\mathbb{C}_0^+) \) the Hankel operator with symbol \( G \) is defined as the operator \( H_G : L_2(0, \infty) \rightarrow L_2(0, \infty) \) given by
\[
\widehat{H_G}u := P_{H_2(\mathbb{C}_0^+)}(G(\cdot)\hat{u}(\cdot)), \quad u \in L_2(0, \infty),
\tag{14}
\]
where \( \hat{\cdot} \) denotes the Laplace transform. If \( (T, B, C, G) \) is a realization of \( G \), \( B \) is an infinite-time admissible control operator, and \( C \) is an infinite-time admissible observation operator for \( T(t) \) we get
\[
H_G = \Psi_\infty B_\infty.
\]
From the Hankel operator we can derive special results.

**Lemma 2.7** If the Hankel operator with symbol \( G \) has closed range, then all infinite-time admissible, approximately controllable and approximately observable realizations are equivalent, i.e., if \( (T_1, B_1, C_1, G) \) with state space \( H_1 \), and \( (T_2, B_2, C_2, G) \) with state space \( H_2 \) are both infinite-time admissible, approximately controllable and approximately observable realizations, then there exists a bounded, invertible operator \( S \in \mathcal{L}(H_1, H_2) \) such that
\[
T_2 = ST_1S^{-1}, \quad B_2 = SB_1, \quad \Psi_2 = \Psi_1S^{-1}.
\]
Furthermore, if a realization of \( G \) is exactly controllable and exactly observable, then \( H_G \) has closed range.

**Proof** The first part is shown in Proposition 6.2 of Ober and Wu [OW96].

Let us now assume that \( G \) has an exactly controllable and exactly observable realization. This implies, that \( G \) can be written in the form
\[
H_G = \Psi_\infty B_\infty,
\]
where \( B_\infty \in \mathcal{L}(L_2(0, \infty), H) \), \( \Psi_\infty \in \mathcal{L}(H, L_2(0, \infty)) \), \( B_\infty \) is surjective and \( \| \Psi_\infty x \| \geq c\| x \| \), for some positive \( c \). The surjectivity of \( B_\infty \) implies that the range of \( H_G \) equals the range of \( \Psi_\infty \), and the open mapping theorem shows that the range of \( \Psi_\infty \) is closed. Thus \( H_G \) has closed range.

Let us remark that the above result is not true if \( B_\infty \) and/or \( \Psi_\infty \) are not bounded operators on \( L_2(0, \infty) \) and \( H \), respectively. An example can be found at the end of this section.
Corollary 2.8 Suppose $H_G$ has closed range and there exists an exactly controllable and exactly observable realization. Then every infinite-time admissible, approximately controllable and approximately observable realization is exactly controllable and exactly observable.

If the transfer function is inner, then there exists a realization which is exactly controllable and exactly observable.

Theorem 2.9 Let $G \in H_\infty(C_0^+$) be an inner function. Then there exists an exactly controllable and exactly observable well-posed linear system $(T, B, C, G)$.

Furthermore, $G$ has an exactly controllable and exactly observable realization with the $C_0$-semigroup having the additional property that

1. it is exponentially stable if and only if $\inf_{\Re z \in (0, \alpha)} |G(z)| > 0$ for some $\alpha > 0$.
2. it is a group if and only if $\inf_{\Re z > \rho} |G(z)| > 0$ for some $\rho > 0$.

Proof Theorem 2.6 shows there exists a realization of $G$. We use the notation as introduced in the proof of Theorem 2.6. Let $V$ denote the closed subspace of $H_2(C_0^+)$ defined as

$$V = [G H_2(C_0^+)]^\perp,$$

where the orthogonal complement is taken in $H_2(C_0^+)$. We shall show that the realization $(T|_V, G, C|_V)$ has the desired properties.

We begin by showing that $V$ is $T(t)$-invariant.

1. Take an arbitrary $x \in H$ and $v \in V$, then

$$\langle Gx, T(t)v \rangle = \langle (T(t))^* Gx, v \rangle = \langle e^{-t} G(\cdot)x(\cdot), v(\cdot) \rangle = \langle G(\cdot)e^{-t}x(\cdot), v(\cdot) \rangle = 0,$$

since $e^{-t}x \in H$, and so $G(\cdot)e^{-t}x(\cdot) \in V^\perp$.

The set of all $w$ that can be written as $Gx$ is dense in $V^\perp = \overline{GH_2(C_0^+)}$, thus we have that $T(t)V \subset V$.

From this we see that $T_V(t)$ defined as the restriction to $V$ of $T(t)$ is a $C_0$-semigroup on $V$.

2. Now we show that $B := G$ is an admissible control operator for $T_V(t)$.

From 10 we see that

$$B_\infty u = \int_0^\infty T(t)Bu(t)dt = P_{H_2(C_0^+)}(G(\cdot)\hat{u}(-\cdot)).$$

If we can show that this expression maps into $V$, then we are done. Take an $x \in H$, and consider

$$\langle Gx, B_\infty u \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(i\omega)x(i\omega)\overline{G(i\omega)\hat{u}(-i\omega)}d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(i\omega)\overline{\hat{u}(-i\omega)}d\omega = 0,$$

where we have used $x \in H = H_2(C_0^+)$, and $\hat{u}(-\cdot) \in H_2(C_0^+)^\perp$.

So we have shown that $B_\infty u$ is orthogonal to any $Gx$ with $x \in H_2(C_0^+)$. This proves that $B_\infty$ maps into $V$. 

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3. Since $C$ is admissible for $T(t)$ it is directly clear that $C|_V$ defined as
\[ C_V x := \hat{x}(0) = Cx, \quad x \in V \]
is admissible for $T_V(t)$.

4. Combining the above results we see that we found a second realization of $G$. We shall now prove that it is exactly controllable. Note that from equations (13) and (15) it follows that $(T_V, C_V)$ is exactly observable.

5. To show that the range of $B_\infty$ is closed we prove that it is a partial isometry. That is for every $u \in L_2(0, \infty)$ with $u \perp \ker B_\infty$ there holds
\[ \|B_\infty u\| = \|u\|. \] (16)

Let $\hat{v} \in H_2(C_0^\infty)$, and define for $s \in C_0^\infty$, $G^t(s) = \overline{G(-s)}$. It is easy to see that $G^t \in H_\infty(C_0^\infty)$ and also that $G^t \hat{v} \in H_2(C_0^\infty)$. For $s \in C_0^\infty$ define $\hat{q}(s) = G^t(-s)\hat{v}(-s)$. Then $\hat{q} \in H_2(C_0^\infty)$. Finally, we denote by $q$ the inverse Laplace transform of $\hat{q}$. We claim that $q \in \ker B_\infty$.

From (10)
\[ B_\infty q = P_{H_2(C_0^\infty)}(G(\cdot)\hat{q}(\cdot)) = P_{H_2(C_0^\infty)}(G(\cdot)G^t(\cdot)\hat{v}(\cdot)) \]
\[ = P_{H_2(C_0^\infty)}(G(\cdot)\overline{G(\cdot)}\hat{v}(\cdot)) = P_{H_2(C_0^\infty)}(\hat{v}(\cdot)) = 0, \]
where we have used that $G$ is inner, and that $\hat{v} \in H_2(C_0^\infty)$.

For $u \in L_2(0, \infty)$ with $u \perp \ker B_\infty$ we show next that
\[ \langle G(\cdot)\hat{u}(\cdot), \overline{P_{H_2(C_0^\infty)}(G(\cdot)\hat{u}(\cdot))} \rangle = 0. \] (17)

We denote by $\hat{v} = P_{H_2(C_0^\infty)}(G(\cdot)\hat{u}(\cdot))$, and thus the inner product (17) becomes
\[ \langle G(\cdot)\hat{u}(\cdot), \hat{v}(\cdot) \rangle = \langle \hat{u}(\cdot), \overline{G(\cdot)}\hat{v}(\cdot) \rangle \]
\[ = \langle \hat{u}(\cdot), G^t(\cdot)\hat{v}(\cdot) \rangle \]
\[ = \langle \hat{u}(\cdot), G^t(\cdot)\hat{v}(\cdot) \rangle = \langle \hat{u}(\cdot), \hat{q}(\cdot) \rangle. \]

The later is zero by the fact the inner product in Laplace domain equals the inner product in time domain, and that $u \perp \ker B_\infty$, $q \in \ker B_\infty$.

Now we prove the equality (16).
\[ \|B_\infty u\|^2 = \langle P_{H_2(C_0^\infty)}(G(\cdot)\hat{u}(\cdot)), P_{H_2(C_0^\infty)}(G(\cdot)\hat{u}(\cdot)) \rangle \]
\[ = \langle G(\cdot)\hat{u}(\cdot), P_{H_2(C_0^\infty)}(G(\cdot)\hat{u}(\cdot)) \rangle \]
\[ = \langle G(\cdot)\hat{u}(\cdot), G(\cdot)\hat{u}(\cdot) \rangle - \langle G(\cdot)\hat{u}(\cdot), P_{H_2(C_0^\infty)}(G(\cdot)\hat{u}(\cdot)) \rangle \]
\[ = \langle G(\cdot)\hat{u}(\cdot), G(\cdot)\hat{u}(\cdot) \rangle - 0 = \langle \hat{u}(\cdot), \hat{u}(\cdot) \rangle = \|u\|^2, \]
where we have used (17) and isometry of Laplace and Fourier transforms, i.e. Paley-Wiener theorem. From (16) it follows directly that the range of $B_\infty$ is closed. If we can show that the range is dense in $V$, then we have proved the assertion. Suppose that the range is not dense in $V$, then there exists a $v \in V$ such that $v \in \overline{\text{Im} B_\infty}^*$. So we have
\[ 0 = \langle v, B_\infty u \rangle \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} v(i\omega)\overline{G(i\omega)\hat{u}(-i\omega)}d\omega \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(i\omega)v(i\omega)\overline{\hat{u}(-i\omega)}d\omega. \]
Since this holds for every \( \hat{u}(-\cdot) \in H_2(C_0^-) \), we have that
\[
x := \overline{G} v \in H_2(C_0^+)
\]
Since \( G \) is inner, we get from the above equation that
\[
v = Gx,
\]
for the \( x \in H_2(C_0^+) \). This means that \( v \in V^\perp \), and since it is an element of \( V \) it must be zero. This proves that the range of \( B_\infty \) equals \( V \).

6. We see that it remains to prove that the semigroup has the additional properties.

Gearhart [Gea78, Theorem 2.1 and Theorem 2.2] or Moeller [Moe62, Theorem 3.1 and Theorem 3.2] shows \( \sigma(T(1)) \subseteq \{ z \in \mathbb{C} \mid |z| < e^{-\alpha} \} \) if and only if \( \inf_{Re \, z \in (0, \alpha)} |G(z)| > 0 \). Thus this proves part 1.

Moreover, from Gearhart [Gea78, Theorem 3.4] we get that also part 2 of the theorem holds.

Next we give an example, which shows that Lemma 2.7 does not hold without the assumption of infinite-time admissibility.

**Example 2.10** Let \( A_0 \) be an infinitesimal generator on the infinite-dimensional Hilbert space \( H \) that satisfies \( A_0 = -A_0^\ast \), and let \( b \in H \). Define the operator \( B_0 \in L(H) \) as \( B_0 u = b \cdot u \). We assume that the system \( (A_0, B_0^\ast) \) is approximately observable. It is well-known that \( G(s) := 1 - B_0^\ast(sI - A_0 + \frac{1}{2} B_0 B_0^\ast)^{-1} B_0 \) is an inner function and Lemma 2.7 together with Theorem 2.9 show that the Hankel operator \( H_G \) has closed range. Clearly, \( (T_0, B_0, B_0^\ast, G) \) is a well-posed linear system, where \( T_0 \) is the semigroup generated by \( A_0 - \frac{1}{2} B_0 B_0^\ast \). By the special structure of the system, we have that \( (T_0, B_0, B_0^\ast, G) \) is approximately controllable and approximately observable. However, since \( B_0 \) is compact, we have that \( (A_0 - \frac{1}{2} B_0 B_0^\ast, B_0^\ast) \) is not exactly observable in finite time.

We are now going to construct another realization. We begin by considering the realization \( (T, B, C, G) \) as constructed in Theorem 2.6 and Theorem 2.9. Note, that \( (T, B, C, G) \) is exactly controllable, exactly observable and the state space for the realization in Theorem 2.9 is given by
\[
V = [GH_2(C_0^+)]^\perp,
\]
where the orthogonal complement in taken in \( H_2(C_0^+) \). We define \( V_1 \) as the closure of \( V \) in the topology of \( H_2(C_0^+) \). Note that this space is isometric isomorph (via Laplace transform) with the weighted \( L_2 \)-space
\[
\{ f \in L_2^{loc}(0, \infty) \mid \int_0^\infty e^{-t} |f(t)|^2 \, dt < \infty \}.
\]

The semigroup on \( V \) is given by
\[
T(t)v = P_{H_2(C_0^+)}(e^t v),
\]
It is easy to see that the inverse Laplace transform of \( T(t)v \) is given by
\[
(T(t)v)(\tau) = \hat{v}(\tau + t), \quad \tau \geq 0.
\]
Now we shall calculate the norm of $T(t)v$ in the new state space $V_1$.

\[
\|T(t)v\|_{V_1}^2 = \|(T(t)v)\|_{V_1}^2 = \|\hat{v}(\cdot + t)\|_{V_1}^2 = \int_0^\infty |e^{-\tau}\hat{v}(\tau + t)|^2 d\tau
\]

\[
= e^{2t} \int_0^\infty |e^{-\tau}\hat{v}(\tau)|^2 d\tau = e^{2t} \|v\|_{V_1}^2.
\]

This implies that for every $t \geq 0$, we can extend the operator $T(t)$ to $V_1$. Since $T(t)$ is a $C_0$-semigroup on $V$, and since $V$ is dense in $V_1$, we have that the extension is again a $C_0$-semigroup. We denote this new semigroup by $T_1(t)$.

Next we construct a well-posed realization of $G$ with state space equal to $V_1$. As semigroup we take $T_1$, and as $B_1$ we take $B_1 = G$. From part 2. in Theorem 2.9 we see that the corresponding $B_{1,\infty}$ satisfies

\[
B_{1,\infty} = iB_{\infty},
\]

where $i$ denotes the inclusion of $V$ into $V_1$. Thus $B_1$ is an infinite-time admissible control operator for $T_1(t)$. Furthermore, since the range of $B_{\infty}$ equals $V$, and since $V$ is dense in $V_1$, we have that $(T_1, B_1)$ is approximately controllable.

As observation operator we take

\[
C_1 x := \hat{x}(0), \quad x \in V_1.
\]

Hence the extension of $C$ to $V_1$. We have to show that this is an admissible observation operator for $T_1(t)$. For $v \in V$, we have that $C_1 T_1(t)v = CT(t)v = \hat{v}(t)$, and thus

\[
\int_0^1 |\hat{v}(t)|^2 dt \leq e^2 \int_0^1 |e^{-t}\hat{v}(t)|^2 dt \leq e^2 \int_0^\infty |e^{-\tau}\hat{v}(\tau)|^2 d\tau = e^2 \|v\|_{V_1}^2.
\]

Since $V$ is dense in $V_1$, we conclude that $C_1$ is an admissible observation operator for $T_1(t)$. From the definition of $C_1$ it is clear that $(T_1, C_1)$ is approximately observable.

By (18), we see that

\[
i(zI - A)^{-1}B = (zI - A)^{-1}B_1,
\]

where $A_1$ is the infinitesimal generator of $T_1(t)$. Hence

\[
C_1(sI - A_1)^{-1}(zI - A_1)^{-1}B_1 = C_1(sI - A)^{-1}(zI - A)^{-1}B
\]

\[
= C_1(sI - A)^{-1}(zI - A)^{-1}B = C(sI - A)^{-1}(zI - A)^{-1}B = G(s) - G(z),
\]

Thus we have proved that $(T_1, B_1, C_1, G)$ is realization of $G$ as well. Furthermore, this realization is approximately controllable and observable. Thus the realizations $(T_0, B_0, B_0^*, G)$, $(T, B, C, G)$ and $(T_1, B_1, C_1, G)$ are all approximately controllable and approximately observable.

We now assume that all approximately controllable and approximately observable realizations are equivalent. The equivalence of $(T, B, C, G)$ and $(T_1, B_1, C_1, G)$ implies that the topologies of $V$ and $V_1$ would be equivalent. Since $V_1$ is the closure of $V$ in the topology of $V_1$, this implies that $V = V_1$. In particular, there holds

\[
\int_0^\infty e^{-2t}|f(t)|^2 dt \geq K \int_0^\infty |f(t)|^2 dt,
\]

for all $f$ whose Laplace transform lies in $V_1$, where $K$ is independent of $f$. Using this we see that

\[
K \int_0^\infty |f(t)|^2 dt \leq \int_0^\infty e^{-2t}|f(t)|^2 dt.
\]
\[
= \int_0^{t_0} e^{-2t} |f(t)|^2 dt + \int_{t_0}^{\infty} e^{-2t} |f(t)|^2 dt \\
\leq \int_0^{t_0} |f(t)|^2 dt + e^{-2t_0} \int_{t_0}^{\infty} |f(t)|^2 dt \\
\leq \int_0^{t_0} |f(t)|^2 dt + e^{-2t_0} \int_{0}^{\infty} |f(t)|^2 dt.
\]

Hence for all \( t_0 \)
\[
[K-e^{-2t_0}] \int_0^{\infty} |f(t)|^2 dt \leq \int_0^{t_0} |f(t)|^2 dt
\]

Thus for \( t_0 \) sufficiently large
\[
\int_0^{t_0} |f(t)|^2 dt \geq K_1 \int_0^{\infty} |f(t)|^2 dt \tag{19}
\]

for all \( f \)'s whose Laplace transform lies in \( V_1 \). For the output \( y \) we have that
\[
y(t) = C_1 T_1(t)v = \tilde{v}(t)
\]

Hence with (19), we obtain that
\[
\int_0^{t_0} |y(t)|^2 dt \geq K_1 \int_0^{\infty} |y(t)|^2 dt = K_1 \int_0^{\infty} |\tilde{v}(t)|^2 dt = K_1 \|v\|^2_{V} \geq K_1 \|v\|^2_{\tilde{V}_1}.
\]

Thus \((C_1,T_1)\) is exactly observable in finite-time. The equivalence of the systems \((T_0,B_0,B_0^*,G)\) and \((T_1,B_1,C_1,G)\) implies that the realization \((T_0,B_0,B_0^*,G)\) is also exactly observable in finite time. However, this is not possible, since \(B_0^*\) is compact. Concluding we see that the realization \((T_1,B_1,C_1,G)\) cannot be equivalent with \((T,B,C,G)\) whereas \((T,B,C,G)\) is exactly controllable and exactly observable, and \((T_1,B_1,C_1,G)\) is approximately controllable and approximately observable. Note that the realization \((T_1,B_1,C_1,G)\) does not have an infinite-time admissible observation operator.

### 3 Closing remarks

The origin of this paper dates back to the research for the article [JZ02]. For that we needed that realization theory written down in the language of well-posed systems. Since that was not done before, we decided to do it ourselves. Hence we do not claim originality, but hope that this manuscript clarifies the ideas behind realization theory. For more reading we refer to Chapter 9 of [Sta05]. Since there is a close link between properties of realizations and Hankel operators, the book of Peller [Pel03] is also recommended.

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