ASYMPTOTICS IN A TWO-SPECIES CHEMOTAXIS SYSTEM WITH LOGISTIC SOURCE

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(Communicated by Zhian Wang)

Abstract. This paper deals with nonnegative solutions of a fully parabolic two-species chemotaxis system with competitive kinetics under homogeneous Neumann boundary conditions in a N-dimensional bounded smooth domain with reasonably smooth nonnegative initial data. In a previous paper of Bai & Winkler (2016), the equilibrium of the global bounded classical solution was shown in both coexistence and extinction cases. We extend this result to weak solutions and prove these solutions globally exist and finally converge to the same semi-trivial steady state in a certain sense.

1. Introduction. The parabolic-parabolic Keller-Segel system with logistic-type growth restrictions was introduced to describe chemotactic migration, where certain bacteria diffuse and are oriented to the higher density of chemical signal, interfered with the time scales of cell proliferation and death. This prototypical chemotaxis model with Neumann initial-boundary value conditions

\[
\begin{aligned}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + \kappa u - \mu u^2, & x \in \Omega, \ t > 0, \\
    v_t &= \Delta v - v + u, & x \in \Omega, \ t > 0,
\end{aligned}
\]

has been widely studied in the past decades. Here the function \(u\) stands for the density of cells, \(v\) denotes the density of a chemical substance with positive constants \(\chi, \kappa, \mu\). If \(n \leq 2\), then the system (1) processes global bounded classical solution for any \(\mu > 0\) [19]. If \(n \geq 3\) and the quotient \(\frac{\mu}{\chi}\) is sufficiently large, solutions to (1) are globally bounded [25] and approach the unique nontrivial spatially homogeneous equilibrium in the large time limit [27, 7]. Moreover, convergence rates can be found in [10]. Global weak solutions also exist for arbitrarily small \(\mu > 0\) and when \(n = 3\) and \(\kappa\) is not too large, the weak solutions become classical solutions after finite time, their large time behavior has also been considered when \(\kappa \leq 0\) [15].

Results on the corresponding quasilinear chemotaxis system with logistic source are given in [23, 29]. In the case \(\kappa = \mu = 0\), this is the classical Keller-Segel system proposed by Keller and Segel [14] (see e.g. the survey [4] for a broader overview). When \(n = 1\), all solutions are globally and uniformly bounded [18]. When \(n = 2\), solutions are globally bounded for suitably small \(\int_{\Omega} u_0 dx\) [17] and
there exist unbounded solutions for large $\int_{\Omega} u_0 dx$ [11, 12]. When $n \geq 3$, there exist bounded solutions [24] and blow-up solutions within finite time [26]. Relevant works on asymptotic behavior of solutions to the classical Keller-Segel system can be consulted in [24, 6, 9].

As for two-species with logistic-type growth restrictions, we consider a fully parabolic two-species chemotaxis system with competitive kinetics

\[
\begin{aligned}
&u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla w) + \mu_1 u(1 - u - a_1 v), \quad x \in \Omega, t > 0, \\
v_t = \Delta v - \chi_2 \nabla \cdot (v \nabla w) + \mu_2 v(1 - v - a_2 u), \quad x \in \Omega, t > 0, \\
w_t = \Delta w - \alpha u + \beta v, \quad x \in \Omega, t > 0,
\end{aligned}
\]

\[
\begin{aligned}
&\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0, \\
w(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x), \quad x \in \Omega,
\end{aligned}
\]

in a bounded smooth domain $\Omega \subset \mathbb{R}^n$ with positive parameters $\chi_1, \chi_2, \mu_1, \mu_2, \alpha, \beta$. The initial data $u_0, v_0$ and $w_0$ are given nonnegative functions fulfilling

\[
\begin{aligned}
&u_0 \in C^0(\bar{\Omega}), \quad u_0 \geq 0, u_0 \not\equiv 0, \\
v_0 \in C^0(\bar{\Omega}), \quad v_0 \geq 0, v_0 \not\equiv 0, \\
w_0 \in W^{1,\infty}(\Omega), \quad w_0 \geq 0, \quad \text{in } \Omega.
\end{aligned}
\]

Without chemotactic mechanism, the associated ODE model

\[
\begin{aligned}
&w' = \mu_1 u(1 - u - a_1 v), \quad t > 0, \\
v' = \mu_2 v(1 - v - a_2 u), \quad t > 0,
\end{aligned}
\]

describes the competition of two species corresponding to a Lotka-Volterra-type kinetics. It is known that if $a_1 \in (0, 1), a_2 \in (0, 1)$ and both species are initially positive, the unique positive steady state $(\frac{1-a_1}{1-a_1a_2}, \frac{1-a_2}{1-a_1a_2})$ to this system is globally asymptotically stable [1]. If $a_1 \geq 1, a_2 \in (0, 1)$, the solution of (4) satisfies $u(t) \to 0$ and $v(t) \to 1$ as $t \to \infty$ [30]. In light of this, we concentrate on the large time behavior including the influence of chemotaxis on the dynamics of biological species. In the parabolic-elliptic counterpart of (2), when both $a_1 \in (0, 1)$ and $a_2 \in (0, 1)$ along with $\mu_1, \mu_2$ sufficiently large, notice that the unique positive spatially homogeneous steady state is given by

\[
\begin{aligned}
u_* &:= \frac{1-a_1}{1-a_1a_2}, \\
v_* &:= \frac{1-a_2}{1-a_1a_2}, \\
w_* &:= \frac{\alpha(1-a_1) + \beta(1-a_2)}{1-a_1a_2},
\end{aligned}
\]

with $(u_*, v_*, w_*)$ solving the linear algebraic system

\[
\begin{aligned}
&1 - u_* - a_1 v_* = 0, \\
&1 - v_* - a_2 u_* = 0, \\
&w_* + \alpha u_* + \beta v_* = 0,
\end{aligned}
\]

[22] derives that this unique positive spatially homogeneous equilibrium $(u_*, v_*, w_*)$ is globally asymptotically stable. While $a_1 > 1, a_2 \in (0, 1)$ and $\mu_1, \mu_2$ are sufficiently large, using delicate comparison techniques [20] detects that this system possesses a uniquely determined global-in-time classical solution toward the semitrivial steady state $(0, 1, \beta)$. Moreover, the chemotactic interactions between species and two chemicals have been analyzed in [21, 13]. In the fully parabolic case, global boundedness of solutions to a variant of (2) is studied in [31]. [3] shows that if $n \leq 2$, the system (2) admits a unique global bounded classical solution. Whenever $n \geq 1$, if $a_1 \in (0, 1), a_2 \in (0, 1)$ and $\mu_1, \mu_2$ are suitably large, global bounded classical solutions with suitably regular initial data convergence to $(u_*, v_*, w_*)$ uniformly in $\Omega$ in the large time limit where $(u_*, v_*, w_*)$ being the constant from (5). If $a_1 \geq 1, a_2 \in (0, 1)$ and $\mu_2$ is large enough, each global bounded classical solution...
Theorem 1.1. Let \(a_1 \in (0, 1), a_2 \in (0, 1)\) and \(\chi_1, \chi_2, \mu_1, \mu_2, \alpha, \beta > 0\) be such that
\[
\frac{\mu_1 a_1}{\mu_2 a_2} \chi_2^2 < \frac{16 \mu_1 a_1 (1 - a_1 a_2)}{a_1 \alpha^2 + a_2 \beta^2 - 2 a_1 a_2 \alpha \beta}.
\]
Assume that \(\Omega \subset \mathbb{R}^n (n \geq 1)\) is a bounded smooth domain with initial data fulfilling (3). Then the weak solution \((u, v, w)\) to (2) obtained in Lemma 3.6 satisfies
\[
\int_t^{t+1} \|u(\cdot, s) - u_*\|_{L^2(\Omega)} ds + \int_t^{t+1} \|v(\cdot, s) - v_*\|_{L^2(\Omega)} ds
\]
\[
+ \int_t^{t+1} \|w(\cdot, s) - w_*\|_{L^2(\Omega)} ds \to 0
\]
as \(t \to \infty\), with \((u_*, v_*, w_*)\) as in (5).

Theorem 1.2. Let \(a_1 \geq 1, a_2 \in (0, 1)\) and \(\chi_1, \chi_2, \mu_1, \mu_2, \alpha, \beta > 0\) satisfy the relation
\[
\chi_2^2 < \frac{16 \mu_2 a_2 (1 - a'_1 a_2)}{a'_1 \alpha^2 + a_2 \beta^2 - 2 a'_1 a_2 \alpha \beta}
\]
for some \(a'_1 \in (1, a_1]\) such that \(a'_1 a_2 < 1\). Suppose that \(\Omega \subset \mathbb{R}^n (n \geq 1)\) is a bounded domain with smooth boundary and (8) holds. Then the weak solution \((u, v, w)\) to (2) constructed in Lemma 3.6 complies with
\[
\int_t^{t+1} \|u(\cdot, s)\|_{L^2(\Omega)} ds + \int_t^{t+1} \|v(\cdot, s) - 1\|_{L^2(\Omega)} ds
\]
\[
+ \int_t^{t+1} \|w(\cdot, s) - \beta\|_{L^2(\Omega)} ds \to 0
\]
as \(t \to \infty\).

2. Existence of approximate solutions. We first introduce its regularized problem as follows:
\[
\begin{aligned}
&u_{\epsilon t} = \Delta u_{\epsilon} - \chi_1 \nabla \cdot (u_{\epsilon} \nabla w_{\epsilon}) + \mu_1 u_{\epsilon} (1 - u_{\epsilon} - a_1 v_{\epsilon}) - \frac{1}{\epsilon} u_{\epsilon}^2 \ln b u_{\epsilon}, \quad x \in \Omega, t > 0, \\
v_{\epsilon t} = \Delta v_{\epsilon} - \chi_2 \nabla \cdot (v_{\epsilon} \nabla w_{\epsilon}) + \mu_2 v_{\epsilon} (1 - v_{\epsilon} - a_2 u_{\epsilon}) - \frac{1}{\epsilon} v_{\epsilon}^2 \ln c v_{\epsilon}, \quad x \in \Omega, t > 0, \\
w_{\epsilon t} = \Delta w_{\epsilon} - v_{\epsilon} + \alpha u_{\epsilon} + \beta v_{\epsilon}, \quad x \in \Omega, t > 0, \\
\frac{\partial w_{\epsilon}}{\partial \nu} = \frac{\partial u_{\epsilon}}{\partial \nu} = \frac{\partial v_{\epsilon}}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0, \\
u_0(x) = u_0(x), v_0(x) = v_0(x), w_0(x) = w_0(x), \quad x \in \Omega,
\end{aligned}
\]
for \(\epsilon \in (0, 1)\) with \(b, c > 0\) to be determined in Section 4 since the precise value of \(b, c\) is of no importance for the global existence of weak solutions. To see the global solvability to (10), we have to count on the well known Amann’s theory [2], which shows that for nonnegative initial data \(u_0 \in C^0(\bar{\Omega}), v_0 \in C^0(\bar{\Omega})\) and \(w_0 \in W^{1,\infty}(\Omega)\), a unique triple of classical solutions \((u_{\epsilon}, v_{\epsilon}, w_{\epsilon})\) exists in \(\Omega \times (0, T_{\text{max}})\) for \(T_{\text{max}} \in (0, \infty]\), and
\[
\limsup_{t \nearrow T_{\text{max}}} \left\{ \|u_{\epsilon}(\cdot, t)\|_{L^\infty(\Omega)} + \|v_{\epsilon}(\cdot, t)\|_{L^\infty(\Omega)} \right\} = \infty
\]
provided $T_{\max} < \infty$. Thereupon what remained to prove is the boundedness of $u_\varepsilon$ and $v_\varepsilon$ in $\Omega \times (0,T_{\max})$. In preparation, the following boundedness properties can be easily checked.

**Lemma 2.1.** For each $\varepsilon \in (0,1)$, the solution of (10) satisfies

$$
\int_\Omega u_\varepsilon(\cdot,t) \leq m_1 := \max\{ \int_\Omega u_0, |\Omega| \}
$$

and

$$
\int_\Omega v_\varepsilon(\cdot,t) \leq m_2 := \max\{ \int_\Omega v_0, |\Omega| \}
$$

for all $t > 0$.

**Proof.** Integrating the first equation in (10) and using Hölder’s inequality yield

$$
\frac{d}{dt} \int_\Omega u_\varepsilon \leq \mu_1 \int_\Omega u_\varepsilon - \mu_1 \int_\Omega u_\varepsilon^2 \leq \mu_1 \int_\Omega u_\varepsilon - \frac{\mu_1}{|\Omega|} (\int_\Omega u_\varepsilon)^2.
$$

An ODE comparison argument implies (12). Moreover, applying the same argument as above to the second equation in (10) produces (13).

To pursue their boundedness property, we need to improve the regularity of $u_\varepsilon$ and $v_\varepsilon$ in a higher $L^p$ space, which is dependent on the maximal Sobolev regularity with time weighted function [5, 7]. However, the boundary condition on initial data does not meet the requirement of this regularity, we could use any positive time instead, which according to Amann’s theory [2] satisfies the boundary condition hereafter in the regularity. More specifically, for any given $t_0 \in (0, T_{\max})$ and $t_0 \leq 1$, we obtain $(u_\varepsilon(\cdot,t_0), v_\varepsilon(\cdot,t_0), w_\varepsilon(\cdot,t_0)) \in C^2(\overline{\Omega})$ with $\partial_\nu w_\varepsilon(\cdot,t_0) = 0$ on $\partial\Omega$, and can choose $M > 0$ such that

$$
\sup_{0 \leq \tau \leq t_0} \| w_\varepsilon(\cdot,t) \|_{L^\infty(\Omega)} \leq M, \quad \| \Delta w_\varepsilon(\cdot,t_0) \|_{L^\infty(\Omega)} \leq M.
$$

(14)

**Lemma 2.2.** Let $\chi_1, \chi_2, \mu_1, \mu_2, \alpha, \beta > 0$ and let $b, c > 0$. Suppose that $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ is a nonnegative classical solution of (10) on $t \in (0,T_{\max})$ with initial data contented with (3). For any $p > 1$ and each $\varepsilon \in (0,1)$, there is a constant $C_{b,c,\varepsilon} > 0$ such that

$$
\| u_\varepsilon(\cdot,t) \|_{L^p(\Omega)} + \| v_\varepsilon(\cdot,t) \|_{L^p(\Omega)} \leq C_{b,c,\varepsilon}
$$

(15)

for all $t \in (0,T_{\max})$.

**Proof.** Multiplying the first equation in (2) by $p u_\varepsilon^{p-1}$ and testing the second equation against $\frac{w_\varepsilon}{\mu_2} p u_\varepsilon^{p-1}$, we then integrate by parts to see

$$
\frac{d}{dt} \int_\Omega u_\varepsilon^p + \frac{\mu_1}{\mu_2} \int_\Omega v_\varepsilon^p
$$

$$
= p \int_\Omega u_\varepsilon^{p-1} \cdot (\Delta u_\varepsilon - \chi_1 \nabla \cdot (u_\varepsilon \nabla w_\varepsilon) + \mu_1 u_\varepsilon (1 - u_\varepsilon - a_1 v_\varepsilon) - \varepsilon u_\varepsilon^2 \ln b u_\varepsilon)
$$

$$
+ \frac{\mu_1}{\mu_2} p \int_\Omega v_\varepsilon^{p-1} \cdot (\Delta v_\varepsilon - \chi_2 \nabla \cdot (v_\varepsilon \nabla w_\varepsilon) + \mu_2 v_\varepsilon (1 - v_\varepsilon - a_2 u_\varepsilon) - \varepsilon v_\varepsilon^2 \ln c v_\varepsilon)
$$
\[
\leq -(p-1) \int_{\Omega} \varepsilon u_{\varepsilon}^p (\nabla u_{\varepsilon})^2 - \chi_1 (p-1) \int_{\Omega} \varepsilon u_{\varepsilon}^p \Delta u_{\varepsilon} + \mu_1 p \int_{\Omega} \varepsilon u_{\varepsilon}^p - \mu_1 p \int_{\Omega} \varepsilon u_{\varepsilon}^{p-1} \\
- \frac{\mu_1}{\mu_2} (p-1) \int_{\Omega} \varepsilon \nabla v_{\varepsilon}^p - \frac{\mu_1}{\mu_2} \chi_2 (p-1) \int_{\Omega} \varepsilon \nabla v_{\varepsilon}^p \Delta u_{\varepsilon} + \mu_1 p \int_{\Omega} \varepsilon v_{\varepsilon}^p - \mu_1 p \int_{\Omega} \varepsilon v_{\varepsilon}^{p-1} \\
- \varepsilon \int_{\Omega} \varepsilon v_{\varepsilon}^{p-1} \ln b u_{\varepsilon} - \varepsilon \int_{\Omega} \varepsilon v_{\varepsilon}^{p-1} \ln c v_{\varepsilon}
\]
for all \( t \in (0, T_{\text{max}}) \). For any \( \delta_1 > 0 \), Young’s inequality gives

\[
\chi_1 (p-1) \int_{\Omega} \varepsilon u_{\varepsilon}^p \Delta u_{\varepsilon} \leq \frac{\mu_1 p (p-1)}{p+1} \delta_1^{p+1} \int_{\Omega} \varepsilon u_{\varepsilon}^{p-1} + \chi_1 (p-1) \int_{\Omega} \varepsilon u_{\varepsilon}^{p-1} \Delta u_{\varepsilon}^{p+1}
\]
on \( (0, T_{\text{max}}) \). Take \( \delta_1 = \left( \frac{\mu_1 (p+1)}{2(p-1)} \right)^{\frac{p}{p+1}} \), the above inequality implies

\[
\chi_1 (p-1) \int_{\Omega} \varepsilon u_{\varepsilon}^p \Delta u_{\varepsilon} \leq \frac{\mu_1 p (p-1)}{p+1} \int_{\Omega} \varepsilon u_{\varepsilon}^{p-1} + \chi_1 (p-1) \int_{\Omega} \varepsilon u_{\varepsilon}^{p-1} \Delta u_{\varepsilon}^{p+1}
\]
on \( (0, T_{\text{max}}) \) with \( k_1 := \frac{p}{p+1} \left( \frac{2(p-1)}{p+1} \right)^p \). Analogously, for any \( \delta_2 > 0 \) we write

\[
\frac{\mu_1}{\mu_2} \chi_2 (p-1) \int_{\Omega} \varepsilon v_{\varepsilon}^p \Delta v_{\varepsilon} \leq \frac{\mu_1 p (p-1)}{p+1} \delta_2^{p+1} \int_{\Omega} \varepsilon v_{\varepsilon}^{p-1} + \frac{\mu_1 p (p-1)}{p+1} \int_{\Omega} \varepsilon v_{\varepsilon}^{p-1} \Delta v_{\varepsilon}^{p+1}
\]
on \( (0, T_{\text{max}}) \). Let \( \delta_2 = \left( \frac{p+1}{2(p+1)} \right)^{\frac{p}{p+1}} \), the above inequality shows

\[
\frac{\mu_1}{\mu_2} \chi_2 (p-1) \int_{\Omega} \varepsilon v_{\varepsilon}^p \Delta v_{\varepsilon} \leq \frac{\mu_1 p (p-1)}{p+1} \int_{\Omega} \varepsilon v_{\varepsilon}^{p-1} + \frac{\mu_1 p (p-1)}{p+1} \int_{\Omega} \varepsilon v_{\varepsilon}^{p-1} \Delta v_{\varepsilon}^{p+1}
\]
on \( (0, T_{\text{max}}) \) with \( k_2 := 2(p^{-1} \left( \frac{p}{p+1} \right)^{p+1} + 2 \). Moreover, we deduce from the Gagliardo-Nirenberg inequality and Young’s inequality that

\[
\int_{\Omega} \varepsilon u_{\varepsilon}^p = \| u_{\varepsilon}^p \|_{L^2 (\Omega)} \leq C_0 \left( \| \nabla u_{\varepsilon}^p \|_{L^2 (\Omega)}^2 + \| u_{\varepsilon}^p \|_{L^{2(1-a)} (\Omega)}^{2(1-a)} + \| u_{\varepsilon}^p \|_{L^{2(1-a)} (\Omega)}^2 \right) \leq \frac{4(p-1)}{p+1} \| \nabla u_{\varepsilon}^p \|_{L^2 (\Omega)}^2 + C'_1 \| u_{\varepsilon}^p \|_{L^2 (\Omega)}^2
\]
where constants \( C_0, C'_1 > 0 \) depend on \( n, p, \Omega \) with \( a = \frac{p-1}{p+1} \in (0, 1) \). Therefore,

\[
(p+1) \int_{\Omega} \varepsilon u_{\varepsilon}^p \leq \frac{4(p-1)}{p+1} \| \nabla u_{\varepsilon}^p \|_{L^2 (\Omega)}^2 + C_1 \| u_{\varepsilon}^p \|_{L^2 (\Omega)}^p = p(p-1) \int_{\Omega} \varepsilon u_{\varepsilon}^{p-2} | \nabla u_{\varepsilon} |^2 + C_1 \| u_{\varepsilon}^p \|_{L^2 (\Omega)}^p
\]
on \( (0, T_{\text{max}}) \) with a constant \( C_1 := C_1 (n, p, \Omega) > 0 \). Similarly, we find

\[
\frac{\mu_1}{\mu_2} (p+1) \int_{\Omega} \varepsilon v_{\varepsilon}^p \leq \frac{4 \mu_1 (p-1)}{\mu_2 p} \| \nabla v_{\varepsilon}^p \|_{L^2 (\Omega)}^2 + C_2 \| v_{\varepsilon}^p \|_{L^2 (\Omega)}^p
\]
\[
= \frac{\mu_1 p (p-1)}{\mu_2 p} \int_{\Omega} \varepsilon v_{\varepsilon}^{p-2} | \nabla v_{\varepsilon} |^2 + C_2 \| v_{\varepsilon}^p \|_{L^2 (\Omega)}^p
\]
on \((0, T_{\text{max}})\) with some \(C_2 := C_2(\mu_1, \mu_2, n, p, \Omega) > 0\). Combining (16)-(19), it follows
\[
\frac{d}{dt} \left( \int_{\Omega} u^p \frac{\mu_1}{\mu_2} \int_{\Omega} v^p \right) + (p + 1) \left( \int_{\Omega} u^p \frac{\mu_1}{\mu_2} \int_{\Omega} v^p \right)
\leq \mu_1 \left( \frac{\chi_1}{\mu_1} + \frac{\chi_2}{\mu_2} \right) \int_{\Omega} |\Delta w_\varepsilon|^{p+1} - \frac{\mu_1 p}{2} \int_{\Omega} u^{p+1} - \frac{\mu_1 p}{2} \int_{\Omega} v^{p+1}
\leq \mu_1 \left( \frac{\chi_1}{\mu_1} + \frac{\chi_2}{\mu_2} \right) \int_{\Omega} |\Delta w_\varepsilon|^{p+1} + \mu_1 p \int_{\Omega} u^{p+1} - \varepsilon \int_{\Omega} u^{p+1} \ln bv_\varepsilon + \mu_1 p \int_{\Omega} v^{p+1} - \varepsilon \frac{\mu_1}{\mu_2} p \int_{\Omega} v^{p+1} \ln cv_\varepsilon
\leq C_1 \int_{\Omega} \bar{v}_\varepsilon^p + C_2 \int_{\Omega} v^p
\]
for all \(t \in (0, T_{\text{max}})\). Let \(t_0 \in (0, T_{\text{max}})\), making use of Gronwall’s inequality to the above inequality at
\[
\int_{\Omega} u^p(\cdot, t) + \frac{\mu_1}{\mu_2} \int_{\Omega} v^p(\cdot, t)
\leq e^{-(p+1)(t-t_0)} \left( \int_{\Omega} u^p(\cdot, t_0) + \frac{\mu_1}{\mu_2} \int_{\Omega} v^p(\cdot, t_0) \right)
\]
for all \(t \in (t_0, T_{\text{max}})\). Moreover, the maximal Sobolev regularity (Lemma 3.3 [5]) indicates
\[
\int_{t_0}^{t} e^{-(p+1)(t-s)} \int_{\Omega} |\Delta w_\varepsilon|^{p+1}
\leq \alpha k \int_{t_0}^{t} e^{-(p+1)(t-s)} \int_{\Omega} u^{p+1} + \beta k \int_{t_0}^{t} e^{-(p+1)(t-s)} \int_{\Omega} v^{p+1}
\leq \alpha k e^{-(p+1)(t-t_0)} (\|w_\varepsilon(\cdot, t_0)\|^{p+1}_{L^{p+1}(\Omega)} + \|\Delta w_\varepsilon(\cdot, t_0)\|^{p+1}_{L^{p+1}(\Omega)})
\]
with a constant \(k := k(\varepsilon) > 0\) for all \(t \in (t_0, T_{\text{max}})\). Thus insert the above inequality into (20) to find
\[
\int_{\Omega} u^p(\cdot, t) + \frac{\mu_1}{\mu_2} \int_{\Omega} v^p(\cdot, t)
\leq e^{-(p+1)(t-t_0)} \left( \int_{\Omega} u^p(\cdot, t_0) + \frac{\mu_1}{\mu_2} \int_{\Omega} v^p(\cdot, t_0) \right)
\]

\[
+ \mu_1 \left( \alpha k \left( \frac{\chi_1}{\mu_1} + \frac{\chi_2}{\mu_2} \right) - \frac{p}{2} \right) \int_{t_0}^{t} e^{-(p+1)(t-s)} \int_{\Omega} u^{p+1}
\]

Let $u$ be given by Lemma 2.2, there exists a constant $C$ depending on $L^\infty_{loc}(\Omega)$ such that

$$
\begin{align*}
\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} & \leq C_{b,c,\varepsilon} \\
\|w\|_{L^\infty(\Omega)} & \leq C_{b,c,\varepsilon}
\end{align*}
$$

with $C := |\Omega|$ for all $t \in (t_0, T_{\max})$. Combining Lemma 2.1 and (14) with (21), we conclude the assertion. \hfill \Box

Whereupon applying the argument from Lemma 4.4 in [16] we get the $L^\infty$ norm of $u_\varepsilon$ and $v_\varepsilon$.

**Lemma 2.3.** Let $\chi_1, \chi_2, \mu_1, \mu_2, \alpha, \beta > 0$ and let $b, c > 0$. Then for the classical solution of (10) on $t \in (0, T_{\max})$ given by Lemma 2.2, there exists a constant $C_{b,c,\varepsilon} > 0$ such that for each $\varepsilon \in (0, 1)$

$$
\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{b,c,\varepsilon}
$$

in $t \in (0, T_{\max})$.

3. **Weak solutions.** The first notion that we need is the definition of weak solutions.

**Definition 3.1.** A triple of nonnegative functions

$$
\begin{align*}
u \in L^2_{loc}([0, \infty) ; L^2(\Omega)) \\
v \in L^2_{loc}([0, \infty) ; L^2(\Omega)) \\
w \in L^2_{loc}([0, \infty) ; W^{1,2}(\Omega))
\end{align*}
$$
will be called a weak solution of (2) if the following identities

\[-\int_0^\infty \int_\Omega u_\varphi_t - \int_\Omega u_\varphi(\cdot, 0) = \int_0^\infty \int_\Omega u_\Delta + \chi_1 \int_0^\infty \int_\Omega u \nabla w \cdot \nabla \varphi + \mu_1 \int_0^\infty \int_\Omega u \varphi \]

\[-\mu_1 \int_0^\infty \int_\Omega u_\varphi^2 - \mu_1 a_1 \int_0^\infty \int_\Omega u \varphi \]

\[-\int_0^\infty \int_\Omega v_\varphi_t - \int_\Omega v_\varphi(\cdot, 0) = \int_0^\infty \int_\Omega v_\Delta + \chi_2 \int_0^\infty \int_\Omega v \nabla w \cdot \nabla \varphi + \mu_2 \int_0^\infty \int_\Omega v \varphi \]

\[-\mu_2 \int_0^\infty \int_\Omega v_\varphi^2 - \mu_2 a_2 \int_0^\infty \int_\Omega u \varphi \]

as well as

\[-\int_0^\infty \int_\Omega w_\varphi_t - \int_\Omega w_\varphi(\cdot, 0) = -\int_0^\infty \int_\Omega \nabla w \cdot \nabla \varphi - \int_0^\infty \int_\Omega w \varphi \]

\[\quad + \alpha \int_0^\infty \int_\Omega u \varphi + \beta \int_0^\infty \int_\Omega v \varphi \]

(25)

hold for all test functions \( \varphi \in C_0^\infty (\bar{\Omega} \times [0, \infty)) \).

Our next preparation will provide some basic estimates on three components of solutions to (10).

**Lemma 3.2.** Let \( b, c > 0 \) and \( m_1, m_2 \) be as defined in Lemma 2.1. Then for all \( \varepsilon \in (0, 1) \) the inequality

\[\int_0^T \int_\Omega u_\varepsilon^2 + \frac{\varepsilon}{\mu_1} \int_0^T \int_\Omega u_\varepsilon^2 \ln b u_\varepsilon \leq m_1 (T + \frac{1}{\mu_1}) \]

(26)

holds together with

\[\int_0^T \int_\Omega v_\varepsilon^2 + \frac{\varepsilon}{\mu_2} \int_0^T \int_\Omega v_\varepsilon^2 \ln c v_\varepsilon \leq m_2 (T + \frac{1}{\mu_2}) \]

(27)

for \( T > 0 \).

**Proof.** We observe from the first equation in (10) that

\[\frac{d}{dt} \int_\Omega u_\varepsilon + \mu_1 \int_\Omega u_\varepsilon^2 + \varepsilon \int_\Omega u_\varepsilon^2 \ln b u_\varepsilon \leq \mu_1 \int_\Omega u_\varepsilon \]

for \( t > 0 \). Thus (26) is a consequence of (28) after time-integration and (12). By the similar argument as above to the second equation in (10), (27) holds.

**Lemma 3.3.** Let \( m_1, m_2 \) be taken from Lemma 2.1. Then for each \( \varepsilon \in (0, 1) \) the third component \( w_\varepsilon \) of the solution of (10) satisfies

\[\frac{1}{2} \int_0^T \int_\Omega w_\varepsilon^2 + \int_0^T \int_\Omega |\nabla w_\varepsilon|^2 \leq \frac{1}{2} \int_\Omega w_0^2 + \left( \frac{\alpha_1^2 m_1}{\mu_1} + \frac{\beta_2^2 m_2}{\mu_2} \right) + (\alpha_1^2 m_1 + \beta_2^2 m_2)T \]

(29)

and

\[\int_0^T \int_\Omega |\Delta w_\varepsilon|^2 \leq \int_\Omega |\nabla w_0|^2 + 2 \left( \frac{\alpha_1^2 m_1}{\mu_1} + \frac{\beta_2^2 m_2}{\mu_2} \right) + 2(\alpha_1^2 m_1 + \beta_2^2 m_2)T \]

(30)

for all \( T > 0 \).
Proof. Testing the third equation in (10) against \( w_\varepsilon \) and using Young’s inequality yield
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega w_\varepsilon^2 + \frac{1}{2} \int_\Omega w_\varepsilon^2 + \int_\Omega |\nabla w_\varepsilon|^2 \leq \alpha^2 \int_\Omega u_\varepsilon^2 + \beta^2 \int_\Omega v_\varepsilon^2
\]  
(31)
for \( t > 0 \). Therefore (29) results from (31) after time-integration and Lemma 3.2. Moreover,
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla w_\varepsilon|^2 + \int_\Omega |\nabla w_\varepsilon|^2 + \frac{1}{2} \int_\Omega |\Delta w_\varepsilon|^2 \leq \alpha^2 \int_\Omega u_\varepsilon^2 + \beta^2 \int_\Omega u_\varepsilon^2
\]  
(32)
for \( t > 0 \). (32) after time-integration and Lemma 3.2 indicate (30).

Then collecting these bounds can easily gain the following estimates.

Lemma 3.4. Let \( b, c > 0 \), then for all \( T > 0 \) there are constants \( C_3, C_4 > 0 \) contented with
\[
\int_0^T \int_\Omega u_\varepsilon^2 \ln(u_\varepsilon + 1) + \varepsilon \int_0^T \int_\Omega u_\varepsilon^2 \ln b u_\varepsilon \ln(u_\varepsilon + 1) + \int_0^T \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + 1)^2} \leq C_3(T)
\]
and
\[
\int_0^T \int_\Omega v_\varepsilon^2 \ln(v_\varepsilon + 1) + \varepsilon \int_0^T \int_\Omega v_\varepsilon^2 \ln c v_\varepsilon \ln(v_\varepsilon + 1) + \int_0^T \int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon + 1)^2} \leq C_4(T)
\]
for all \( \varepsilon \in (0, 1) \). In Particular, the families \( \{u_\varepsilon^2 \ln b u_\varepsilon\}_{\varepsilon \in (0,1)} \), \( \{v_\varepsilon^2 \ln c v_\varepsilon\}_{\varepsilon \in (0,1)} \), \( \{u_\varepsilon^2\}_{\varepsilon \in (0,1)} \) and \( \{v_\varepsilon^2\}_{\varepsilon \in (0,1)} \) are equi-integrable over \( \Omega \times (0,T) \).

Proof. An application of the first equation in (10) we write
\[
\frac{d}{dt} \int_\Omega u_\varepsilon \ln(u_\varepsilon + 1)
\]
\[= -\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon + 1} + \chi_1 \int_\Omega \frac{u_\varepsilon \nabla u_\varepsilon \cdot \nabla w_\varepsilon}{u_\varepsilon + 1} + \mu_1 \int_\Omega u_\varepsilon \ln(u_\varepsilon + 1) - \mu_1 \int_\Omega u_\varepsilon^2 \ln(u_\varepsilon + 1)
\]
\[- a_1 \mu_1 \int_\Omega u_\varepsilon v_\varepsilon \ln(u_\varepsilon + 1) - \varepsilon \int_\Omega u_\varepsilon^2 \ln b u_\varepsilon \ln(u_\varepsilon + 1) - \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + 1)^2}
\]
\[+ \chi_1 \int_\Omega \frac{u_\varepsilon \nabla u_\varepsilon \cdot \nabla w_\varepsilon}{(u_\varepsilon + 1)^2} + \mu_1 \int_\Omega \frac{u_\varepsilon^2}{u_\varepsilon + 1} - \mu_1 \int_\Omega u_\varepsilon^2
\]
\[- a_1 \mu_1 \int_\Omega \frac{u_\varepsilon^2 v_\varepsilon}{u_\varepsilon + 1} - \varepsilon \int_\Omega \frac{u_\varepsilon^3}{u_\varepsilon + 1} \ln b u_\varepsilon \]
for \( t > 0 \). By straightforward integration over time,
\[
\mu_1 \int_0^T \int_\Omega u_\varepsilon^2 \ln(u_\varepsilon + 1) + \varepsilon \int_0^T \int_\Omega u_\varepsilon^2 \ln b u_\varepsilon \ln(u_\varepsilon + 1) + \int_0^T \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + 1)^2}
\]
\[\leq \int_\Omega u_0 \ln(u_0 + 1) + \chi_1 \int_\Omega \frac{u_\varepsilon \nabla u_\varepsilon \cdot \nabla w_\varepsilon}{u_\varepsilon + 1} + \chi_1 \int_\Omega \frac{u_\varepsilon \nabla u_\varepsilon \cdot \nabla w_\varepsilon}{(u_\varepsilon + 1)^2}
\]
\[+ \mu_1 \int_0^T \int_\Omega u_\varepsilon \ln(u_\varepsilon + 1) + \mu_1 \int_0^T \int_\Omega u_\varepsilon + \varepsilon \int_0^T \int_\Omega \frac{u_\varepsilon^3}{u_\varepsilon + 1} \ln b u_\varepsilon
\]
(35)
for all $T > 0$ and $\varepsilon \in (0, 1)$. Based on $0 \leq \ln(s + 1) \leq s$ for all $s \geq 0$ and Young’s inequality,
\[ \chi_1 \int_0^T \int_\Omega \frac{u_\varepsilon \nabla u_\varepsilon \cdot \nabla w_\varepsilon}{u_\varepsilon + 1} = \chi_1 \int_0^T \int_\Omega \nabla u_\varepsilon \cdot \nabla w_\varepsilon - \chi_1 \int_0^T \int_\Omega \nabla \ln(u_\varepsilon + 1) \cdot \nabla w_\varepsilon \]
\[ = -\chi_1 \int_0^T \int_\Omega u_\varepsilon \Delta w_\varepsilon + \chi_1 \int_0^T \int_\Omega \ln(u_\varepsilon + 1) \Delta w_\varepsilon \]
\[ \leq \chi_1 \int_0^T \int_\Omega u_\varepsilon^2 + \chi_1 \int_0^T \int_\Omega |\Delta w_\varepsilon|^2 \]

and
\[ \chi_1 \int_0^T \int_\Omega u_\varepsilon \nabla u_\varepsilon \cdot \nabla w_\varepsilon \frac{1}{u_\varepsilon + 1} \leq \frac{1}{2} \int_0^T \int_\Omega (u_\varepsilon + 1)^2 + \frac{\chi_1}{2} \int_0^T \int_\Omega |\nabla w_\varepsilon|^2 \]

for all $T > 0$ and $\varepsilon \in (0, 1)$. Hence,
\[ \mu_1 \int_0^T \int_\Omega u_\varepsilon^2 \ln(u_\varepsilon + 1) + \varepsilon \int_0^T \int_\Omega u_\varepsilon^2 \ln b u_\varepsilon \ln(u_\varepsilon + 1) + \frac{1}{2} \int_0^T \int_\Omega (u_\varepsilon + 1)^2 \]
\[ \leq \int_\Omega u_\varepsilon \ln(u_\varepsilon + 1) + \chi_1 \int_0^T \int_\Omega u_\varepsilon^2 + \chi_1 \int_0^T \int_\Omega |\Delta w_\varepsilon|^2 + \frac{\chi_1}{2} \int_0^T \int_\Omega |\nabla w_\varepsilon|^2 \]
\[ + \mu_1 \int_0^T \int_\Omega u_\varepsilon^2 + \mu_1 \int_0^T \int_\Omega u_\varepsilon + \frac{1}{2b^2 e^T} \]

for all $T > 0$ and $\varepsilon \in (0, 1)$ due to $s^2 \ln b s > -\frac{1}{2b^2 e}$ for all $s \geq 0$. Inserting (12), (26), (29) and (30) into (36) obtains (33). Moreover, using the second equation in (10) and following the similar discussion as above result in (34). \qed

In addition, we also need time-derivatives estimates for an Aubin-Lions type compactness argument.

**Lemma 3.5.** Let $b, c > 0$ with $m \in \mathbb{N}$ such that $m > \frac{n}{2}$, then for all $\varepsilon \in (0, 1)$ and $T > 0$ the solution of (10) satisfies
\[ \|\partial_t \ln(u_\varepsilon(\cdot, t) + 1)\|_{L^1((0, T); (W^{m-2}_{0, \varepsilon}(\Omega))^{*})} \leq C_5(T + 1) \]
(37)
\[ \|\partial_t \ln(v_\varepsilon(\cdot, t) + 1)\|_{L^1((0, T); (W^{m-2}_{0, \varepsilon}(\Omega))^{*})} \leq C_6(T + 1) \]
(38)
as well as
\[ \|w_{\varepsilon t}(\cdot, t)\|_{L^1((0, T); (W^{m-2}_{0, \varepsilon}(\Omega))^{*})} \leq C_7(T + 1) \]
(39)
with constants $C_5, C_6, C_7 > 0$.

**Proof.** For fixed $t > 0$ and arbitrary $\psi \in W^{m, 2}_{0}(\Omega)$, employing the first equation in (10) we compute
\[ \left| \int_\Omega \partial_t \ln(u_\varepsilon(\cdot, t) + 1) \cdot \psi \right| \]
Lemma 3.6. Let $\chi_1, \chi_2, \mu_1, \mu_2, \alpha, \beta > 0$ with $b, c > 0$. Suppose that $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ is a solution to (10) with initial data complying with (3). There is a sequence $u_\varepsilon(\cdot, t)$, $v_\varepsilon(\cdot, t)$, and $w_\varepsilon(\cdot, t)$ for all $t > 10$ such that $u_\varepsilon(\cdot, t)$ and $v_\varepsilon(\cdot, t)$ are bounded in $L^\infty_c(\Omega)$ and $\int_\Omega \varepsilon w_\varepsilon(\cdot, t) = 0$. Then, for all $t > 10$, we have

$$
\int_\Omega u_\varepsilon(\cdot, t) \leq C \left( \int_\Omega |\nabla u_\varepsilon|^2 + \int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega |\nabla w_\varepsilon|^2 + \int_\Omega u_\varepsilon + \int_\Omega v_\varepsilon + \int_\Omega w_\varepsilon + 1 \right)
$$

for some constant $C > 0$. Therefore, there exists a global weak solution to (2) for all $t > 0$. 

**Proof.** By the uniform boundedness principle, we have $u_\varepsilon(\cdot, t)$, $v_\varepsilon(\cdot, t)$, and $w_\varepsilon(\cdot, t)$ are bounded in $L^\infty_c(\Omega)$ and $\int_\Omega \varepsilon w_\varepsilon(\cdot, t) = 0$. Then, for all $t > 10$, we have

$$
\int_\Omega u_\varepsilon(\cdot, t) \leq C \left( \int_\Omega |\nabla u_\varepsilon|^2 + \int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega |\nabla w_\varepsilon|^2 + \int_\Omega u_\varepsilon + \int_\Omega v_\varepsilon + \int_\Omega w_\varepsilon + 1 \right)
$$

for some constant $C > 0$. Therefore, there exists a global weak solution to (2). 

**Remark.** By an appropriate approximation procedure of its corresponding regularized system (10), we are able to construct a global weak solution to (2). 

**Example.** Let $\chi_1, \chi_2, \mu_1, \mu_2, \alpha, \beta > 0$ with $b, c > 0$. Suppose that $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ is a solution to (10) with initial data complying with (3). There is a sequence $u_\varepsilon(\cdot, t)$, $v_\varepsilon(\cdot, t)$, and $w_\varepsilon(\cdot, t)$ for all $t > 10$ such that $u_\varepsilon(\cdot, t)$ and $v_\varepsilon(\cdot, t)$ are bounded in $L^\infty_c(\Omega)$ and $\int_\Omega \varepsilon w_\varepsilon(\cdot, t) = 0$. Then, for all $t > 10$, we have

$$
\int_\Omega u_\varepsilon(\cdot, t) \leq C \left( \int_\Omega |\nabla u_\varepsilon|^2 + \int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega |\nabla w_\varepsilon|^2 + \int_\Omega u_\varepsilon + \int_\Omega v_\varepsilon + \int_\Omega w_\varepsilon + 1 \right)
$$

for some constant $C > 0$. Therefore, there exists a global weak solution to (2). 

**Conclusion.** By an appropriate approximation procedure of its corresponding regularized system (10), we are able to construct a global weak solution to (2).
Finally, multiplying both sides of the equations in (10) by and (46). Combining (43) and (46) with (49) respectively, (52) and (53) conclude.

ward extraction process in (29) proves (49). Moreover, (51) is directly from (43) 3.1 and using convergence properties asserted in Lemma 3.6 obtain (23)-(25).

\[
{\varepsilon_j}_{j \in \mathbb{N}} \subset (0, 1), \ v_j \searrow 0 \text{ as } j \to \infty \text{ such that}
\]
\[
u_{\varepsilon_j} \to u \quad \text{a.e. in } \Omega \times (0, \infty) \tag{42}
\]
\[
u_{\varepsilon_j} \to u \quad \text{in } L^2_{\text{loc}}(\Omega \times [0, \infty)) \tag{43}
\]
\[
\varepsilon_j u_{\varepsilon_j}^2 \ln \nu_{\varepsilon_j} \to 0 \quad \text{in } L^1_{\text{loc}}(\Omega \times [0, \infty)) \tag{44}
\]
\[
u_{\varepsilon_j} \to \nu \quad \text{a.e. in } \Omega \times (0, \infty) \tag{45}
\]
\[
u_{\varepsilon_j} \to \nu \quad \text{in } L^2(\Omega \times [0, \infty)) \tag{46}
\]
\[
\varepsilon_j u_{\varepsilon_j}^2 \ln \nu_{\varepsilon_j} \to 0 \quad \text{in } L^1_{\text{loc}}(\Omega \times [0, \infty)) \tag{47}
\]
\[
\nu_{\varepsilon_j} \to v \quad \text{a.e. in } \Omega \times (0, \infty) \tag{48}
\]
\[
\nu_{\varepsilon_j} \to v \quad \text{in } L^2((0, \infty); W^{1,2}(\Omega)) \tag{49}
\]
\[
\nu_{\varepsilon_j} \to w \quad \text{a.e. in } \Omega \times (0, \infty) \tag{50}
\]
\[
\nu_{\varepsilon_j} \to w \quad \text{in } L^1(\Omega \times [0, \infty)) \tag{51}
\]
\[
\nu_{\varepsilon_j} \nu_{\varepsilon_j} \to u \nu \quad \text{in } L^1_{\text{loc}}(\Omega \times [0, \infty)) \tag{52}
\]
\[
\nu_{\varepsilon_j} \nu_{\varepsilon_j} \to v \nu \quad \text{in } L^1_{\text{loc}}(\Omega \times [0, \infty)) \tag{53}
\]

for some triple of nonnegative functions \((u, v, w)\) which form a global weak solution to (2) in the sense of Definition 3.1.

Proof. Remembering Lemma 3.4 and Lemma 3.5 there is a constant \(C_8(T) > 0\) such that

\[
\|\ln(u \varepsilon(\cdot, t) + 1)\|_{L^2((0,T); W^{1,2}(\Omega))} + \|\partial_t \ln(u \varepsilon(\cdot, t) + 1)\|_{L^1((0,T); (W^{m,2}(\Omega))^*)} \leq C_8
\]

with \(m \in \mathbb{N}\) such that \(m > \frac{9}{2}\). A variant of Aubin-Lions Lemma [8] shows strong precompactness of \(\{\ln(u \varepsilon + 1)\}_{\varepsilon \in (0,1)}\) in the space \(L^2(\Omega \times (0, T))\), which further ensures (42) upon a subsequence. Adopting Lemma 3.4, \(\{u_{\varepsilon_j}^2\}_{\varepsilon \in (0,1)}\) is weakly convergent in \(L^1(\Omega \times (0, T))\). (42) and Lemma A.3 [28] enable us to find a subsequence such that

\[
u_{\varepsilon_j}^2 \to u^2 \quad \text{in } L^1(\Omega \times (0, T)) \tag{54}
\]

Thereupon the combination of (54) and passing a further subsequence along (26), (43) holds. Lemma 3.4 suggests that \(\{u_{\varepsilon_j}^2 \ln \nu_{\varepsilon_j}\}_{\varepsilon \in (0,1)}\) is equi-integrable, and particularly weakly sequentially precompact in \(L^1(\Omega \times (0, T))\). Thus (42) and Egorov’s theorem indicate (44). Furthermore, using Lemma 3.2, Lemma 3.4, Lemma 3.5 and repeating the above argument deduce (45)-(47).

We see from Lemma 3.3 and Lemma 3.5 that

\[
\|w \varepsilon(\cdot, t)\|_{L^2((0,T); W^{1,2}(\Omega))} + \|\partial_t w \varepsilon(\cdot, t)\|_{L^1((0,T); (W^{m,2}(\Omega))^*)} \leq C_9
\]

with some constant \(C_9(T) > 0\) and \(m \in \mathbb{N}\) such that \(m > \frac{9}{2}\). Taking into account the Aubin-Lions Lemma once again, \(\{w \varepsilon\}_{\varepsilon \in (0,1)}\) is strongly precompact in \(L^2(\Omega \times (0, T))\) as well as (48) is valid along a further subsequence. A straightforward extraction process in (29) proves (49). Moreover, (51) is directly from (43) and (46). Combining (43) and (46) with (49) respectively, (52) and (53) conclude. Finally, multiplying both sides of the equations in (10) by \(\varphi\) as selected in Definition 3.1 and using convergence properties asserted in Lemma 3.6 obtain (23)-(25).
4. Asymptotic behavior. Inspired by [3], we could use the energy functional employed in [3] to obtain the large time stabilization of the solution to (2) constructed in Lemma 3.6. Different from previous sections which for arbitrary $b, c > 0$ there exist weak solutions in $\Omega \times (0, \infty)$, we have to select specific $b, c > 0$ in (10) below for stabilization results.

4.1. Coexistence. In the case of weak competition, which means both $a_1 \in (0, 1)$ and $a_2 \in (0, 1)$, applying the energy functional given in Lemma 3.2 [3] and the limit procedure, we establish the convergence result of weak solutions.

Lemma 4.1. Let $a_1 \in (0, 1), a_2 \in (0, 1)$ with $(u_*, v_*, w_*)$ being as defined in (5) and let (6) be true. Then for any global bounded classical solution $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ of (10) emanating from $(u_0, v_0, w_0)$ contended with (3), denoting functions

$$E_1(t) = \int_\Omega (u_\varepsilon - u_* - u_* \ln \frac{u_\varepsilon}{u_*}) + \frac{\mu_1 a_1}{\mu_2 a_2} \int_\Omega (v_\varepsilon - v_* - v_* \ln \frac{v_\varepsilon}{v_*})$$

$$+ (u_* \chi_1^2 + \frac{\mu_1 a_1}{\mu_2 a_2} v_* \chi_2^2) \frac{\delta}{2} \int_\Omega (w_\varepsilon - w_*)^2$$

$$F_1(t) = \int_\Omega (u_\varepsilon - u_*)^2 + \int_\Omega (v_\varepsilon - v_*)^2 + \int_\Omega (w_\varepsilon - w_*)^2$$

for all $t > 0$ and $\varepsilon \in (0, 1)$, then we choose $b = \frac{1}{u_*} > 0$ and $c = \frac{1}{v_*} > 0$ such that

$$E_1(t) \geq 0 \quad (55)$$

and

$$\frac{d}{dt} E_1(t) \leq -\lambda F_1(t) \quad (56)$$

for all $t > 0$ with constants $\delta > 0$ and $\lambda > 0$.

Proof. (55) can be confirmed by the reasoning in Lemma 3.2 [3].

On account of (6), we can pick $\delta > 0$ satisfying

$$\frac{1}{4} < \delta < \frac{4\mu_1 a_1 (1 - a_1 a_2)}{(a_1 \alpha^2 + a_2 \beta^2 - 2a_1 a_2 \alpha \beta)(u_* \chi_1^2 + \frac{\mu_1 a_1}{\mu_2 a_2} v_* \chi_2^2)} \quad (57)$$

Denote

$$E_{11}(t) = \int_\Omega (u_\varepsilon - u_* - u_* \ln \frac{u_\varepsilon}{u_*}),$$

$$E_{12}(t) = \int_\Omega (v_\varepsilon - v_* - v_* \ln \frac{v_\varepsilon}{v_*}),$$

$$E_{13}(t) = \frac{1}{2} \int_\Omega (w_\varepsilon - w_*)^2,$$

for all $t > 0$. Let $b = \frac{1}{u_*} > 0$ and $c = \frac{1}{v_*} > 0$, by virtue of (10) and Young’s inequality, we derive

$$\frac{d}{dt} E_{11}(t) = \int_\Omega (1 - \frac{u_*}{u_\varepsilon}) u_{\varepsilon t}$$

$$= -u_* \int_\Omega \frac{\nabla u_\varepsilon}{u_\varepsilon}^2 + u_* \chi_1 \int_\Omega \frac{\nabla u_\varepsilon \cdot \nabla w_\varepsilon}{u_\varepsilon} + \mu_1 \int_\Omega (u_\varepsilon - u_*) (1 - u_\varepsilon - a_1 v_\varepsilon)$$

$$- \varepsilon \int_\Omega u_\varepsilon (u_\varepsilon - u_*) \ln bu_\varepsilon$$
Since $\delta > \frac{4}{3}$, owing to (57). In accordance with Sylvester’s criterion, $Q$ is positive definite. Thus there exists $\lambda > 0$ such that

$$X(x,t) \cdot (Q \cdot X(x,t)) \geq \lambda |X(x,t)|^2$$

for all $x \in \Omega$ and $t > 0$. Inserting this inequality into (58) finishes the proof.
4.2. Dominance of the second species. In the case when \( a_1 \geq 1 > a_2 \), applying the energy functional as that in Lemma 3.4 [3] and following a similar discussion as that in Section 4.1, we will verify the weak solution in Lemma 3.6 stabilizing towards \((0, 1, \beta)\) as time approaching to infinity.

**Lemma 4.2.** Let \( a_1 \geq 1, a_2 \in (0, 1) \) and (8) be valid. Suppose that \((u_\varepsilon, v_\varepsilon, w_\varepsilon)\) is any global bounded classical solution of (10) and \((u_0, v_0, w_0)\) satisfies (3). Define functions

\[
F_2(t) := \int_\Omega u_\varepsilon + \int_\Omega (v_\varepsilon - 1 - \ln v_\varepsilon) + \frac{\delta}{2} \int_\Omega (w_\varepsilon - \beta)^2
\]

\[
F_2(t) := \int_\Omega u_\varepsilon^2 + \int_\Omega (v_\varepsilon - 1)^2 + \int_\Omega (w_\varepsilon - \beta)^2
\]

for all \( t > 0 \) and \( \varepsilon \in (0, 1) \) with some \( a'_1 \in (1, a_1] \) such that \( a'_1 a_2 < 1 \). Then for fixed \( b = \frac{1}{\mu(a'_1 - 1)} > 0 \) and \( c = \frac{1}{a'_1} > 0 \), there exist constants \( \delta > 0 \) and \( \lambda > 0 \) fulfilling

\[
E_2(t) \geq 0
\]

and

\[
\frac{d}{dt} F_2(t) \leq -\lambda F_2(t) - \frac{\mu_2 a_2 (a'_1 - 1)}{a'_1} \int_\Omega u_\varepsilon - \frac{\mu_2 a_2}{\mu_1 a'_1} \cdot \varepsilon \int_\Omega u_\varepsilon^2 \ln b u_\varepsilon
\]

for all \( t > 0 \).

**Proof.** Notice that (60) is obvious according to Lemma 3.2 [3].

Recalling (8), we can choose \( \delta > 0 \) contented with

\[
\frac{1}{4} < \delta < \frac{4\mu_2 a_2 (1 - a'_1 a_2)}{\chi^2 (a'_1 a^2 + a_2 \beta^2 - 2a'_1 a_2 \alpha \beta)}.
\]
Denote
\[ E_{21}(t) := \int_\Omega u_\varepsilon, \]
\[ E_{22}(t) := \int_\Omega (v_\varepsilon - 1 - \ln v_\varepsilon), \]
\[ E_{23}(t) := \frac{1}{2} \int_\Omega (w_\varepsilon - \beta)^2, \]
for all \( t > 0 \). Fix \( b = \frac{1}{\mu_1(a_1^* - 1)} > 0 \) and \( c = 1 > 0 \), we invoke (10) and Young’s inequality to compute
\[
\frac{d}{dt} E_{21}(t) = \mu_1 \int_\Omega u_\varepsilon(1 - u_\varepsilon - a_1 v_\varepsilon) - \varepsilon \int_\Omega u_\varepsilon^2 \ln b u_\varepsilon \\
\leq \mu_1 \int_\Omega u_\varepsilon(1 - u_\varepsilon - a_1' v_\varepsilon) - \varepsilon \int_\Omega u_\varepsilon^2 \ln b u_\varepsilon \\
= -\mu_1(a_1^* - 1) \int_\Omega u_\varepsilon - \mu_1 \int_\Omega u_\varepsilon^2 - \mu_1 a_1' \int_\Omega u_\varepsilon(v_\varepsilon - 1) - \varepsilon \int_\Omega u_\varepsilon^2 \ln b u_\varepsilon,
\]
\[
\frac{d}{dt} E_{22}(t) = -\int_\Omega \left| \nabla v_\varepsilon \right|^2 + \lambda_2 \int_\Omega \frac{\nabla v_\varepsilon \cdot \nabla w_\varepsilon}{v_\varepsilon} + \mu_2 \int_\Omega (v_\varepsilon - 1)(1 - v_\varepsilon - a_2 u_\varepsilon) \\
- \varepsilon \int_\Omega v_\varepsilon(v_\varepsilon - 1) \ln c v_\varepsilon \\
\leq \frac{\lambda_2}{4} \int_\Omega \left| \nabla w_\varepsilon \right|^2 - \mu_2 \int_\Omega (v_\varepsilon - 1)^2 - \mu_2 a_2 \int_\Omega u_\varepsilon(v_\varepsilon - 1),
\]
\[
\frac{d}{dt} E_{23}(t) = -\int_\Omega \left| \nabla w_\varepsilon \right|^2 - \int_\Omega (w_\varepsilon - \beta)^2 + \alpha \int_\Omega (w_\varepsilon - \beta) u_\varepsilon + \beta \int_\Omega (w_\varepsilon - \beta)(v_\varepsilon - 1),
\]
for all \( t > 0 \) due to the fact \( s(s - 1) \ln s \geq 0 \) for any \( s > 0 \). Using \( \delta > \frac{1}{4} \) and collecting the above terms show
\[
\frac{d}{dt} E_{21}(t) \leq -\int_\Omega X \cdot (Q' \cdot X) - \frac{\mu_2 a_2(a_1^* - 1)}{a_1'} \int_\Omega u_\varepsilon - \frac{\mu_2 a_2}{\mu_1 a_1'} \cdot \varepsilon \int_\Omega u_\varepsilon^2 \ln b u_\varepsilon \tag{63}
\]
for all \( t > 0 \), where the vector function
\[ X(x, t) := (u_\varepsilon(x, t), v_\varepsilon(x, t) - 1, w_\varepsilon(x, t) - \beta) \]
for all \( x \in \Omega \) and \( t > 0 \), and the constant matrix \( Q' \)
\[
Q' := \begin{pmatrix}
\frac{\mu_2 a_2}{a_1} & \mu_2 a_2 & -\frac{\lambda_2 a_1}{2} \\
\mu_2 a_2 & \mu_2 & -\frac{\lambda_2 a_1}{2} \\
-\frac{\lambda_2 a_1}{2} & -\frac{\lambda_2 a_1}{2} & -\frac{\lambda_2 a_1}{2}
\end{pmatrix}.
\]
By Sylvester’s criterion and (62), \( Q' \) is positive definite. Hence
\[ X(x, t) \cdot (Q' \cdot X(x, t)) \geq \lambda |X(x, t)|^2 \]
with some \( \lambda > 0 \) for all \( x \in \Omega \) and \( t > 0 \). Combining this inequality with (63), (61) concludes. \( \square \)
Proof of Theorem 1.2. Since
\[ \varepsilon \int_{\Omega} u_{\varepsilon}^2 \ln b u_{\varepsilon} = \varepsilon \int_{\{u_{\varepsilon} \geq \frac{1}{b}\}} u_{\varepsilon}^2 \ln b u_{\varepsilon} + \varepsilon \int_{\{u_{\varepsilon} < \frac{1}{b}\}} u_{\varepsilon}^2 \ln b u_{\varepsilon} \]
\[ \geq -\frac{1}{b} \int_{\Omega} u_{\varepsilon} \]
\[ \geq -\frac{1}{b} \int_{\Omega} u_{\varepsilon} \]
for all \( t > 0 \), take \( b = \frac{1}{\mu_{1}(a_1 - 1)} > 0 \) we deduce from Lemma 4.2 that
\[ \frac{d}{dt} F_2(t) \leq -\lambda F_2(t) - \frac{\mu_2 a_2 (a_1' - 1)}{a_1} \int_{\Omega} u_{\varepsilon} + \frac{\mu_2 a_2}{\mu_1 a_1'} \frac{1}{b} \int_{\Omega} u_{\varepsilon} \]
\[ = -\lambda F_2(t) - \frac{\mu_2 a_2}{a_1'} ((a_1' - 1) - \frac{1}{b\mu_1}) \int_{\Omega} u_{\varepsilon} \]
\[ \leq -\lambda F_2(t) \]
for all \( t > 0 \). Together with (60), this indicates that
\[ \int_{0}^{\infty} F_2(t) dt = \int_{0}^{\infty} \int_{\Omega} (u_{\varepsilon} - 0)^2 + \int_{0}^{\infty} \int_{\Omega} (v_{\varepsilon} - 1)^2 + \int_{0}^{\infty} \int_{\Omega} (w_{\varepsilon} - \beta)^2 \leq C \]
with some \( C > 0 \) independent of \( \varepsilon \in (0, 1) \). By similar reasoning as in the proof of Theorem 1.1, (9) holds.

Acknowledgments. This work was supported by the National Natural Science Foundation of China (No. 11771354) and the Fundamental Research Funds for the Central Universities, Shaanxi NSF (No. S2017-ZRJJ-MS-0104). The authors are grateful to Professor Michael Winkler for his valuable suggestions, which largely improve this paper.

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Received May 2020; revised August 2020.

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