A locally integrable multi-dimensional billiard system

D. Treschev

1 Introduction

Let \( D \subset E \) be a domain with the smooth boundary \( S = \partial D \) in the Euclidean space \( E = \mathbb{R}^{n+1}, \ n \geq 1 \). Assuming that the closure \( \bar{D} \) is compact, we define the billiard system in \( D \) as follows. A particle moves with a unit constant velocity inside \( D \). The reflection from the boundary is elastic.

The word elastic means a well-known relation between the velocities of the particle before and after the impact ("the angle of the incidence equals the angle of the reflection"), but we will use an equivalent variational equation. Consider the generating function (the discrete Lagrangian)

\[
L : \mathcal{P} \to \mathbb{R}, \quad \mathcal{P} = S \times S, \quad L(a, b) = |a - b|.
\]

Then \( a, b, c \in S \) are 3 consecutive impact points iff

\[
\partial_b (L(a, b) + L(b, c)) = 0, \quad \partial_b = \frac{\partial}{\partial b}.
\]

(1.1)

Billiard systems were introduced by Birkhoff [3] and since that time occupied a noticeable part of the dynamics. Many results and references on the subject, as a rule in the case \( n = 1 \), can be found in [11, 12].

Now we define the billiard map \( \beta : \mathcal{P} \to \mathcal{P} \). A pair of consecutive impact points \( (a, b) \) is transformed to \( (b, c) = \beta(a, b) \) such that \( a, b, c \) satisfy (1.1). This means that \( \beta \) determines a discrete Lagrangian system (a general definition of a discrete Lagrangian system is contained in [5]). In particular, \( \beta \) is symplectic.

In this paper we consider the case when the domain \( D \) is symmetric in all coordinate hyperplanes. We assume that \( E \) splits into the direct product \( \mathbb{R}^n \times \mathbb{R} \), where the subspaces \( \mathbb{R}^n \times \{0\} \subset E \) and \( \{0\} \times \mathbb{R} \subset E \) are called horizontal and vertical respectively. The map \( \beta \) has the periodic orbit \( \gamma \) of period 2 lying on the vertical coordinate axis. We are interested in the local dynamics near \( \gamma \).

Consider the ball \( B = \{x \in \mathbb{R}^n : |x| < d\} \). We determine \( S \) locally by the graphs

\[
S_- = \{(x, f(x)) : x \in B\}, \quad S_+ = \{(x, -f(x)) : x \in B\},
\]

where

\[
f : B \to \mathbb{R} \text{ is a negative even function.}
\]

(1.2)

Our question is as follows.
Is it possible to choose $f$ so that the corresponding billiard map $\beta$ is locally (near $\gamma$) conjugated to a linear map?

Before a discussion of the motivations and results, we reformulate this question in a technically more convenient form.

Let $I : E \to E$, $I^2 = \text{id}$ be the symmetry in the horizontal hyperplane. Then $I$ can be naturally extended to an involution of the phase space $\mathcal{P}$: $(a, b) \mapsto (Ia, Ib)$. Slightly abusing the notation, we denote this involution as $I$. We define the quotient space $\hat{\mathcal{P}} = \mathcal{P}/I$ and the natural projection $\text{pr} : \mathcal{P} \to \hat{\mathcal{P}}$.

The maps $I, \beta : \mathcal{P} \to \mathcal{P}$ commute. Therefore there exists a unique map $\hat{\beta} : \hat{\mathcal{P}} \to \hat{\mathcal{P}}$ such that

$$\hat{\beta} \circ \text{pr} = \beta.$$

The quotient map $\hat{\beta}$ is more convenient for the local study because $\gamma$ projects to a fixed point $\hat{\gamma}$ of $\hat{\beta}$. Hence we can propose another (equivalent) version of the question $Q$.

$\hat{Q}$. Is it possible to choose $f$ so that the corresponding quotient billiard map $\hat{\beta}$ is locally (near $\hat{\gamma}$) conjugated to a linear map?

This linear map $\rho$ is symplectic and coincides with the linearization of $\hat{\beta}$ at $\hat{\gamma}$. Let $\lambda_1^{\pm 1}, \ldots, \lambda_n^{\pm 1}$ be the eigenvalues of $\rho$.

In this paper we consider only the most interesting situation when $\hat{\gamma}$ is linearly stable. Then the eigenvalues lie on the unit circle. Explicit conditions of linear stability of a billiard orbit of period two (in general, not necessarily symmetric case) are well-known in the case $n = 1$ (see for example, [11]), some geometric interpretations of these conditions are given in [8]. In the case $n = 2$ such stability conditions can be found in [9].

If the eigenvalues form a resonant set i.e.,

$$\lambda_1^{k_1} \cdots \lambda_n^{k_n} = 1 \quad \text{for some nonzero vector } k = (k_1, \ldots, k_n) \in \mathbb{Z}^n,$$

then there is no hope to have a positive answer to $\hat{Q}$ (see Section 4). In the nonresonant case we show (Theorem 1) that $f$ can be obtained as a formal Taylor series. The most intriguing question on the convergence of this series remains open. But numerical analysis makes reasonable the following

**Conjecture 1** If the set $\lambda_1, \ldots, \lambda_n$ is nonresonant and moreover, satisfies good Diophantine properties, the series, presenting $f$, is locally convergent. The same is true for the series, presenting the conjugacy map.

Consider the nonresonant case. If Conjecture 1 is true, we obtain a local real-analytic billiard system with linear quasi-periodic dynamics. In this case $f$ turns out to be convex. This locally defined convex function $f$ can be smoothly continued up to a function $F$ such that graphs of $\pm F$ determine a smooth closed hypersurface $S \subset E$, the boundary of a convex domain $D \subset E$. Hence we obtain a globally defined billiard system such that the corresponding phase space $\mathcal{P}$ contains an open domain $\mathcal{D} \subset \mathcal{P}$ filled by quasiperiodic motions with the same set of frequencies. In particular, the only periodic orbit in $\mathcal{D}$ is $\gamma$.

Billiard system in the domain $\mathcal{D}$ is locally integrable. The problem of integrability of billiard systems is widely discussed. According to the Birkhoff conjecture any domain on
a plane bounded by a smooth closed curve generates an integrable billiard system iff this curve is an ellipse. Partial results confirming this conjecture are contained in [1, 2, 11], (see also [7], where an analog of the Birkhoff conjecture for outer billiard systems is proven). Usually the billiard integrability is discussed in the context of the existence of a first integral polynomial in the momenta, see a recent survey and a collection of new results in [10]. Local first integrals which should exist for the billiard system in $D$ are real-analytic, but certainly not polynomial in the momenta.

Because of a strong degeneracy of the billiard dynamics in $D$ we expect that the spectrum of the Laplace operator in $D$ (say, with the Dirichlet boundary conditions) can be very special. Thereby an interesting question appears on possible values of the quantity $\text{meas } D/\text{meas } P$. Numeric computations in the case $n = 1$ show that this ratio can exceed 50%.

Now we present some discussion of numeric results for $n = 1$ and $n = 2$.

The case $n = 1$. This situation is studied in [13] and [14]. We consider the normalization $f(0) = -1/2$. Taking $\lambda = \lambda_1 = e^{i\alpha}$, where $\alpha/\pi$ is an irrational number, we compute the sequence of coefficients $f_{2j}$, where

$$f(x) = \sum f_{2j}x^{2j}. \quad (1.3)$$

Now we present some conjectures motivated by results of numeric computations.

1. For Diophantine $\alpha/(2\pi) \in (0.3, 0.5)$ the ratio $b_j = f_{2j}/f_{2j-2}$ admits the asymptotic expansion

$$b_j = b_{\infty} \left(1 + \frac{\sigma}{j} + O\left(\frac{1}{j^2}\right)\right).$$

2. Graph of $b_{\infty}^{-1/2}$ as a function of $\alpha/(2\pi)$ is presented in Fig. 1.

![Figure 1](image)

**Figure 1**: The graph of $b_{\infty}^{-1/2}$ as a function of $\alpha/(2\pi)$. Two “gaps” correspond to the resonances $\frac{\alpha}{2\pi} = 3/10$ and $\frac{\alpha}{2\pi} = 1/3$.

This function is not defined for rational $\alpha/\pi$, but looks smooth. Probably, this function is Whitney smooth, [15].

3. Independently of $\alpha$ numerically $\sigma = -3/2$. If this really the case, we have:

$$f_{2j} = C \frac{b_{\infty}^j}{j^{3/2}} \left(1 + O\left(\frac{1}{j}\right)\right).$$
This would imply that series (1.3) converges on the boundary of the convergence disk $|x| \leq x_\ast = b^{-1/2}$ and has at the points $x = \pm x_\ast$ singularities of type $\sqrt{(x_\ast \mp x)}$, in particular, the tangent line to the graph at these points is vertical.

This means that the graph of $f$ can be probably continued through the points $(\pm x_\ast, f(\pm x_\ast))$ up to a longer real-analytic curve.

**The case $n = 2$.** Putting $f(0) = -1/2$, we compute the Taylor coefficients $a_{j_1j_2}$ ($j_1$, $j_2$ are even), where

$$f(x) = \frac{1}{2} \sum_{j_1,j_2 \in 2\mathbb{Z}_+} a_{j_1j_2} x_1^{j_1} x_2^{j_2}, \quad a_{00} = -1, \quad \mathbb{Z}_+ = \{0, 1, \ldots\}.$$  

The coefficients $a_{0k}$ and $a_{k0}$ can be computed from the case $n = 1$ because sections of the billiard domain by the vertical planes $x_1 = 0$ and $x_2 = 0$ give solutions of the problem with $n = 1$.

Take for example, $\lambda_1 = e^{i\alpha_1}$, $\lambda_2 = e^{i\alpha_2}$, where $\alpha_1/(2\pi)$ and $\alpha_2/(2\pi)$ are quadratic irrationals:

$$\alpha_1 = 2\pi \cdot (3, 3, 1, 1, 1, 1, \ldots), \quad \alpha_2 = 2\pi \cdot (2, 5, 2, 2, 2, \ldots)$$

(chain fractions). For any even $k$ we present the line

$$\frac{a_{0k}}{\sqrt{C_k}} \frac{a_{2k-2}}{\sqrt{C_k}} \ldots \frac{a_{k0}}{\sqrt{C_k}}$$

The binomial multipliers $1/\sqrt{C_k}$ are motivated by the Bombieri norm on the space of homogeneous polynomials [6]. Here are numeric data beginning from $k = 4$ (we save only 5 digits):

$$\begin{align*}
.50276, & 1.0749, 1.8853 \\
.38788, & 1.1811, 1.9557, 3.6123 \\
.36853, & 1.5808, 2.7866, 4.5700, 8.6479 \\
.39228, & 2.3233, 4.4113, 7.3709, 12.080, 23.183 \\
.44643, & 3.6066, 7.3683, 12.798, 20.965, 34.380, 66.587 \\
.53202, & 5.8039, 12.711, 23.049, 38.630, 62.628, 102.77, 200.34
\end{align*}$$

Conjecture [4] for $n = 1$ predicts an exponential growth of the numbers $a_{0k}$ (the left-hand side of the table) and $a_{k0}$ (the right-hand side of the table). We see that the numbers in each line grow monotonically. This suggests an extension of Conjecture [4] to the case of arbitrary $n \geq 1$.

The further plan of the paper is as follows. In Section [2] we obtain equations (the conjugacy equations) from which the function $f$ and the conjugacy map can be computed. We discuss basic symmetries of this equation and some properties of its (formal) solutions in Section [3]. A further analysis of the solutions is contained in Section [4] where we prove a theorem about the existence of a formal solution in the non-resonant case. Other symmetries of the formal solution are discussed in Section [5]. Finally we present another form of the conjugacy equation which contains the unknown functions polynomially. We used this equation in the numeric analysis of the Taylor series for $f$ and the conjugacy map.
2 Conjugacy equation

2.1 Conjugacy map

We consider billiard trajectories which hit \( S_- \) and \( S_+ \) alternatively. If \( a \) and \( b \) are two consecutive impact points of a trajectory then

\[
\begin{align*}
  \mathbf{a} &= \left( \begin{array}{c} a \\ f(a) \end{array} \right), \quad \mathbf{b} = \left( \begin{array}{c} b \\ -f(b) \end{array} \right) \quad \text{or} \quad \mathbf{a} = \left( \begin{array}{c} a \\ -f(a) \end{array} \right), \quad \mathbf{b} = \left( \begin{array}{c} b \\ f(b) \end{array} \right), \quad a, b \in B.
\end{align*}
\]

In both cases

\[
L(\mathbf{a}, \mathbf{b}) = |\mathbf{a} - \mathbf{b}| = \hat{L}(a, b) = \left( |a - b|^2 + (f(a) + f(b))^2 \right)^{1/2}.
\]

(2.1)

Here the passage from the points \( \mathbf{a}, \mathbf{b} \in S \) to their projections to \( B \) corresponds to the passage from \( \beta \) to the quotient map \( \hat{\beta} \). Note that in these coordinates

\[
\hat{\gamma} = (0, 0) \in B \times B.
\]

Hence \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) are 3 consecutive points on a trajectory\(^1\) of \( \beta \) iff \( a, b, c \) are 3 consecutive points on a trajectory of \( \hat{\beta} \) iff \( (1.1) \) holds or equivalently,

\[
\frac{\partial}{\partial b} (\hat{L}(a, b) + \hat{L}(b, c)) = 0.
\]

More explicitly,

\[
\frac{b - a + \nabla f(b)(f(a) + f(b))}{L(a, b)} + \frac{c - a + \nabla f(b)(f(c) + f(b))}{L(b, c)} = 0.
\]

(2.2)

Given a collection of complex numbers

\[
\lambda_1, \ldots, \lambda_n, \quad |\lambda_1| = \ldots = |\lambda_n| = 1
\]

(2.3)

consider the linear symplectic map

\[
\rho : \mathbb{C}^n \rightarrow \mathbb{C}^n, \quad z = (z_1, \ldots, z_n) \mapsto \rho(z) = (\lambda_1 z_1, \ldots, \lambda_1 z_n).
\]

The billiard map is locally conjugated with \( \rho \) iff there exists a diffeomorphism \( X : U \rightarrow B \), where \( U \subset \mathbb{C}^n \) and \( B \subset B \times B \) are neighborhoods of zero\(^2\) such that the following diagram

\[
\begin{array}{ccc}
  U & \xrightarrow{\rho} & U \\
  \downarrow X & & \downarrow X \\
  B & \xrightarrow{\hat{\beta}} & B
\end{array}
\]

\(^1\) which hits \( S_\pm \) alternatively

\(^2\) Below we assume that \( U \) is \( \rho \)-invariant.
commutes. The map \( X \) has no relation with the complex structure on \( \mathbb{C}^n \). Hence we have to use \((z, \bar{z}) = (z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)\) as coordinates on \( \mathbb{C}^n \). We put
\[
X(z, \bar{z}) = (a, b) = (\chi_-(z, \bar{z}), \chi(z, \bar{z})).
\]

Let \( \beta(a, b) = (b, c) \) and \( a = \chi_-(z, \bar{z}) \). Then
\[
b = \chi(z, \bar{z}) = \chi_\circ \rho(z, \bar{z}), \quad c = \chi \circ \rho(z, \bar{z}),
\]
and \( a, b, c \) are connected by (2.2). This implies
\[
\chi = \chi_\circ \rho \quad \tag{2.4}
\]
and
\[
\partial_2 L(\chi \circ \rho^{-1}, \chi) + \partial_1 L(\chi, \chi \circ \rho) = 0, \quad \tag{2.5}
\]
where \( \partial_1 \) and \( \partial_2 \) are defined by
\[
\partial_2 L(x, y) = \frac{\partial L}{\partial y}(x, y), \quad \partial_1 L(x, y) = \frac{\partial L}{\partial x}(x, y).
\]

In principle, the functions \( f, \chi \) can be computed from (2.5). But (2.5) turns out to be overdetermined w.r.t. \( f \). This produces some difficulties in the proof of the existence of a solution. To avoid these difficulties, below we replace (2.5) by an equivalent system, free of such problems.

### 2.2 Averaging

We define the action of the torus \( T^n = \mathbb{R}^n/(2\pi \mathbb{Z}^n) \) on \( \mathbb{C}^n \) by the equation
\[
T \ni \alpha \mapsto \rho_\alpha : \mathbb{C}^n \to \mathbb{C}^n, \quad \rho_\alpha z = e^{i\alpha} z.
\]

For any function \( g : U \to \mathbb{C} \) defined on a set \( U \subset \mathbb{C}^n \), invariant with respect to the action of \( T^n \), we put \( \rho_\alpha^* g = g \circ \rho_\alpha \) and define the average
\[
\langle g \rangle : U \to \mathbb{C}, \quad \langle g \rangle = \frac{1}{(2\pi)^n} \int_{T^n} \rho_\alpha^* g \, d\alpha, \quad \llbracket g \rrbracket = g - \langle g \rangle.
\]

For any \( g = \sum_{k', k'' \in \mathbb{Z}_+^n} g_{k'k''} z^{k'} \bar{z}^{k''} \) we have the identities
\[
\langle g \rangle = \sum_{k \in \mathbb{Z}_+^n} g_{kk} (zz)^k, \quad \langle g \rangle = \langle \rho^* g \rangle.
\]

Averaging can be applied to differential forms as well. In particular, we have the identity
\[
d\langle g \rangle = \langle dg \rangle \quad \text{for any function } g : U \to \mathbb{C}. \quad \tag{2.6}
\]
2.3 The forms $\nu, \hat{\nu}$

Consider the forms $\mu, \hat{\mu}$ on $B \times B$:

$$\mu = \partial_2 L(x_-, x) \, dx = \sum_{j=1}^{n} \partial x_j L(x_-, x) \, dx_j, \quad \hat{\mu} = \partial_1 L(x_-, x) \, dx_- = \sum_{j=1}^{n} \partial x_{j-} L(x_-, x) \, dx_{j-}.$$ 

Then $d\mu = -d\hat{\mu}$ is the standard symplectic structure for the billiard system, see for example [5]. We define two 1-forms $\nu = X^* \mu$, $\hat{\nu} = X^* \hat{\mu}$ on $U \subset \mathbb{C}^n$. In more detail,

$$\nu = \partial_2 L(\chi \circ \rho^{-1}, \chi) \, d\chi, \quad \hat{\nu} = \partial_1 L(\chi \circ \rho^{-1}, \chi) \, d\chi \circ \rho^{-1}.$$ 

Equation (2.5) is equivalent to

$$\nu + \rho^* \hat{\nu} = 0. \quad (2.7)$$

**Lemma 2.1** Equation (2.7) implies

$$\langle L(\chi, \chi \circ \rho) \rangle = L(0, 0). \quad (2.8)$$

**Proof.** By (2.7)

$$0 = \langle \nu + \rho^* \hat{\nu} \rangle = \langle \nu + \hat{\nu} \rangle = \langle dL(\chi \circ \rho^{-1}, \chi) \rangle.$$ 

It remains to use identity (2.6). \hfill \blacksquare

2.4 Main equation

Instead of (2.5) or (2.7) we consider an equivalent system $[\nu + \rho^* \nu] = 0$, (2.8), or, in an explicit form

$$\left[\frac{\tau_- \chi + \tau_- f \circ \chi (\nabla f) \circ \chi}{L(\chi \circ \rho^{-1}, \chi)} + \frac{\tau_+ \chi + \tau_+ f \circ \chi (\nabla f) \circ \chi}{L(\chi, \chi \circ \rho)}\right] d\chi = 0, \quad (2.9)$$

$$\left\langle (\tau_- \chi)^2 + (\tau_- f \circ \chi)^2 \right\rangle^{1/2} = L(0, 0), \quad (2.10)$$

$$\tau_\pm \chi = \chi - \chi \circ \rho^\pm, \quad \tau_\pm f \circ \chi = f \circ \chi + f \circ \chi \circ \rho^\pm.$$ 

For any $j = 1, \ldots, n$ consider the maps

$$\kappa, \kappa_j : B \rightarrow B, \quad \kappa_j(x_1, \ldots, x_n) = (-1)^{\delta_j} x_1, \ldots, (-1)^{\delta_j} x_n, \quad \kappa = \kappa_1 \circ \ldots \circ \kappa_j.$$ 

Any map $\kappa_j$ is the symmetry in the $j$-th coordinate hyperplane and $\kappa$ is the central symmetry w.r.t. the origin.

Given the frequency vector (2.3) we search for a solution $(f, \chi)$ of (2.9), (2.10) satisfying the following properties.

---

3 At first glance it looks strange that for operators $\tau_\pm$ acting differently on $\chi$ and $f \circ \chi$ (the sign at the second term differs) the same notation is used. However below we will see that the function $\chi$ is odd in $z, \bar{z}$ while $f \circ \chi$ is even. Hence $\tau_\pm$ admit a universal formula $\tau_\pm = \text{id} + \iota^* \circ \rho^\pm \iota$, where $\iota$ is the central symmetry $(z, \bar{z}) \mapsto (-z, -\bar{z})$. 

7
(1) The function $f$ is real and (totally) even:

$$f \circ \kappa_j = f \quad \text{for any } j = 1, \ldots, n.$$ 

Its expansion in homogeneous forms $f^{(2k)}$ of degree $2k$ is

$$f = f^{(0)} + f^{(2)} + f^{(4)} + \ldots, \quad f^{(0)} = f_0 < 0, \quad f^{(2k)}(x) = \sum_{s \in \mathbb{Z}^n_+ : \|s\| = k} \mathcal{F}_{2s} x^{2s}, \quad \mathcal{F}_{2s} \in \mathbb{R}, \quad x^{2s} = x_1^{2s_1} \ldots x_n^{2s_n}, \quad \|s\| = s_1 + \ldots + s_n. \quad (2)$$

The conjugacy map $X$ is real:

$$\chi^{-}_{(z, \bar{z})} = \overline{\chi_{(z, \bar{z})}}, \quad \chi(z, \bar{z}) = \overline{\chi(z, \bar{z})}. \quad (2.11)$$

(3) The functions $\chi = (\chi_1, \ldots, \chi_n)$ are odd:

$$\chi_j \circ \kappa = -\chi_j, \quad j = 1, \ldots, n.$$ 

Their expansions in homogeneous forms $\chi_j^{(k)}$ of degree $k$ is

$$\chi_j = \chi_j^{(1)} + \chi_j^{(3)} + \ldots, \quad \chi^{(2k+1)}_j(z, \bar{z}) = \sum_{\|s'\| + \|s''\| = 2k+1} \chi_{s's''} z^{s'} \bar{z}^{s''}.$$ 

Note that two equations (2.11) are equivalent to each other and imply

$$\chi_{s's''} = \overline{\chi_{s''s'}} \quad \text{for any } s', s'' \in \mathbb{Z}^n_+.$$ 

(4) $\det \left( \frac{\partial (\chi_{-}\chi)}{\partial (z, \bar{z})} \right) \bigg|_{z = \bar{z} = 0} \neq 0.$

3 Symmetries

Definition 3.1 The rotation vector $\lambda$ is said to be nonresonant if the equation

$$\lambda_1^{k_1} \ldots \lambda_n^{k_n} = 1, \quad k_1, \ldots, k_n \in \mathbb{Z}$$

holds only in the trivial situation: $k_1 = \ldots = k_n = 0.$

In this section we study symmetries of equation (2.9), (2.10) with conditions (1)–(4).

(a) (Gauge symmetry). System (2.9), (2.10), (1)–(4) admits the following gauge symmetry. If $(f, \chi)$ is a solution, the pair $(f, \chi \circ s)$ is also a solution for any real $s : U \to U$ which commutes with $\rho$. Reality of the vector-function $s$ means that

$$s(z, \bar{z}) = (A_1(z, \bar{z}), \ldots, A_n(z, \bar{z}), \bar{A}_1(z, \bar{z}), \ldots, \bar{A}_n(z, \bar{z})). \quad (3.1)$$

Lemma 3.1 Suppose that $\lambda$ is nonresonant. Suppose that $s$, commuting with $\rho$, is real and can be expanded into a power series at zero. Then

$$s(z, \bar{z}) = (z_1 \psi_1, \ldots, z_n \psi_n, z_1 \bar{\psi}_1, \ldots, z_n \bar{\psi}_n), \quad \psi_j = \bar{\psi}_j(z_1 \bar{z}_1, \ldots, z_n \bar{z}_n). \quad (3.2)$$
Proof. We put
\[ s = \sum_{uv} s_{uv} z^u \bar{z}^v, \quad s_{uv} = (A_{uv1}, \ldots, A_{uvn}, B_{uv1}, \ldots, B_{uvn}), \quad u, v \in \mathbb{Z}_+^n. \]

Then the equation \( s \circ \rho = \rho \circ s \) for any \( j = 1, \ldots, n \) implies
\[ (\lambda^{u-v} - \lambda_j)A_{uvj} = 0, \quad (\lambda^{u-v} - \bar{\lambda}_j)B_{uvj} = 0, \quad \lambda^{u-v} = \lambda_1^{u_1-v_1} \ldots \lambda_n^{u_n-v_n}. \]

By nonresonant condition we obtain
\[ A_j(z, \bar{z}) = z_j \psi_j(z_1 \bar{z}_1, \ldots, z_n \bar{z}_n), \quad B_j(z, \bar{z}) = \bar{z}_j \vartheta_j(z_1 \bar{z}_1, \ldots, z_n \bar{z}_n). \]

It remains to use (3.1).

(b) Consider in (2.10) the homogeneous form of degree two in \( z, \bar{z} \):
\[ \left\langle \frac{1}{4f_0} \sum_j (\tau_- \chi_j^{(1)})^2 + \tau_- f^{(2)} \circ \chi^{(1)} \right\rangle = 0. \quad (3.3) \]

By using the notation
\[ \chi_j^{(1)} = \sum_{l=1}^n (c_{jl} z_l + \bar{c}_{jl} \bar{z}_l) \]

after simple transformations we obtain:
\[ \sum_{j,l} \left( 2 - \lambda_l - \lambda_l^{-1} + 8f_0 F_{e_j} \right) |c_{jl}|^2 z_l \bar{z}_l = 0, \]

where \( e_j \in \mathbb{Z}_+^n \) is the \( j \)-th unit vector: its \( j \)-th components equals 1 while all others vanish.

This means that for any \( l = 1, \ldots, n \)
\[ (2 - \lambda_l - \lambda_l^{-1} + 8f_0 F_{e_j}) |c_{jl}|^2 = 0. \]

Since \( \lambda_j \neq \lambda_k^{\pm 1} \) for \( j \neq k \), we see that for any \( l \) only one of coefficients \( c_{jl} \) may be nonzero, the one, corresponding to \( j \) such that \( 2 - \lambda_l - \lambda_l^{-1} + 8f_0 F_{e_j} = 0 \). By (4) for any \( j \) such \( j = j(l) \) exists. Without loss of generality we have: \( j(l) = l \). Hence,
\[ -8f_0 F_{e_j} = 2 - \lambda_j - \lambda_j^{-1}, \quad \chi_j^{(1)} = c_{jj} z_j + \bar{c}_{jj} \bar{z}_j, \quad c_{jj} \neq 0. \quad (3.4) \]

Direct computation shows that by equations (3.4) the homogeneous forms of degree 2 in (2.9) vanish:
\[ \frac{(\tau_- + \tau_+) \chi_j^{(1)} + 4f_0 \nabla_j f^{(2)} \circ \chi^{(1)}}{2|f_0|} d\chi_j^{(1)} = 0. \quad (3.5) \]

We can assume that the coefficients \( c_{jj} \) are real and positive. Indeed, by using the map \( s \) (3.2)
\[ s(z, \bar{z}) = (|c_{11}| c_{11}^{-1} z_1, \ldots, |c_{nn}| c_{nn}^{-1} z_n, |c_{11}| \bar{c}_{11}^{-1} \bar{z}_1, \ldots, |c_{nn}| \bar{c}_{nn}^{-1} \bar{z}_n) \]
for the gauge transformation \((f, \chi) \mapsto (f, \chi \circ s)\), we obtain:

\[(\chi_j \circ s)^{(1)} = |a_j|(z_j + \bar{z}_j), \quad a_j = |c_{jj}|.\]  (3.6)

(c) We define the maps (complex conjugacy of one coordinate)

\[\kappa_1, \ldots, \kappa_n : \mathbb{C}^n \to \mathbb{C}^n, \quad \kappa_j(z) = w, \quad w_j = \bar{z}_j, \quad w_k = z_k \text{ for any } k \neq j.\]

Let \(\rho_j : \mathbb{C}^n \to \mathbb{C}^n\) be the map

\[z \mapsto w = \rho_j(z), \quad w_l = (\kappa_j\lambda)_l z_l,\]

where \((\kappa_j\lambda)_l\) is the \(l\)-th coordinate of the vector \(\kappa_j\lambda\). We have the identity

\[\rho \circ \kappa_j = \kappa_j \circ \rho_j.\]

**Corollary 3.1** The pair \((f, \chi)\) is a solution of system (2.9), (2.10), corresponding to the frequency vector \(\lambda\) iff \((f, \chi \circ \kappa_j)\) is a solution of system (2.9), (2.10), corresponding to the frequency vector \(\kappa_j\lambda\).

**Corollary 3.2** Without loss of generality it is possible to assume that \(\arg \lambda_j \in (0, \pi)\) for any \(j = 1, \ldots, n\).

(d) We define the maps \(T_{jl} : \mathbb{C}^n \to \mathbb{C}^n\) which exchange the coordinates \(z_j\) and \(z_l\) in any vector \(z \in \mathbb{C}^n\). Let \(\rho_{jl} : \mathbb{C}^n \to \mathbb{C}^n\) be the map

\[z \mapsto w = \rho_{jl}(z), \quad w_k = (T\lambda)_k z_k, \quad k = 1, \ldots, n.\]

We have the identity

\[\rho \circ T_{jl} = T_{jl} \circ \rho_{jl}.\]

**Corollary 3.3** The pair \((f, \chi)\) is a solution of system (2.9), (2.10), corresponding to the frequency vector \(\lambda\) iff \((f, \chi \circ T_{jl})\) is a solution of system (2.9), (2.10), corresponding to the frequency vector \(T_{jl}\lambda\).

4 **Formal solution**

**Theorem 1** Suppose that the frequency vector \(\lambda\) (2.3) is nonresonant. Then for any \(f_0 < 0\) and \(a_1, \ldots, a_n > 0\) system (2.9), (2.10) (1)–(4) has a formal solution given by power series

\[f = \sum_{k=0}^{\infty} f^{(2k)}, \quad f^{(2k)}(x) = \sum_{\|\sigma\|=k} f_{2\sigma} x^{2\sigma}, \quad \chi = \sum_{k=0}^{\infty} \chi^{(2k+1)},\]  (4.1)

\[f^{(2)} = \sum_{j=1}^{n} F_{e_j} x_j^2, \quad \chi^{(1)}_j = a_j(z_j + \bar{z}_j), \quad F_{e_j} = \frac{2 - \lambda_j - \lambda_j^{-1}}{-8f_0}.\]  (4.2)
Proof of Theorem 7. According to (4.1)–(4.2) we are looking for \( f \) even in \( x \) and \( \chi \) odd in \( z, \bar{z} \). Hence even in \( z, \bar{z} \) homogeneous forms in (2.9), (2.10) will vanish. The forms of degree 2 in (2.9) and (2.10) have been analyzed in item (b), Section 3. In this way we obtain (4.2).

Suppose that we have computed \( f^{(2m)} \) and \( \chi^{(2m-1)} \) for all \( m < k \) by considering homogeneous forms in (2.9), (2.10) of degrees 2, 4, \ldots, \( 2k-2 \).

Taking in (2.9), (2.10) the homogeneous form of degree 2\( k \), we obtain:

\[
\left[ \frac{(\tau - \tau_+)\chi_j^{(2k-1)}}{2|f_0|} + \frac{2f_0(\nabla_j f^{(2k)} \circ \chi^{(1)} + \nabla_j f^{(2)} \circ \chi^{(2k-1)})}{2|f_0|} \right] d\chi_j^{(1)} = R_j^{(2k)} ,
\]

\[
\left\langle \frac{1}{4f_0} \sum_{j=1}^{n} 2\tau_+\chi_j^{(1)} \tau_-\chi_j^{(2k-1)} + \tau_- \sum_{j=1}^{n} 2\mathcal{F}_{e_j}\chi_j^{(1)} \chi_j^{(2k-1)} + \tau_- f^{(2k)} \circ \chi^{(1)} \right\rangle = Q_j^{(2k)},
\]

where the forms \( R_j^{(2k)} \) and the functions \( Q_j^{(2k)} \) are polynomials w.r.t. coefficients of \( f^{(2m)} \) and \( \chi^{(2m-1)} \) with \( m < k \).

\[
R_j^{(2k)} = [R_j^{(2k)}], \quad Q_j^{(2k)} = \langle Q_j^{(2k)} \rangle.
\]

By (3.5) the second fraction in the brackets \([ \ ]\) vanishes. Therefore

\[
\left[ \left( \lambda_j + \lambda_j^{-1} - \rho^* - \rho^{-1*} \right) \chi_j^{(2k-1)} + 2f_0 \nabla_j f^{(2k)} \circ \chi^{(1)} \right] d\chi_j^{(1)} = 2|f_0|R_j^{(2k)}, \quad (4.3)
\]

\[
\left\langle \sum_{j=1}^{n} \tau_-\chi_j^{(1)} \tau_-\chi_j^{(2k-1)} + 8f_0 \sum_{j=1}^{n} \mathcal{F}_{e_j}\chi_j^{(1)} \chi_j^{(2k-1)} \right. \\
\left. + 4f_0 \sum_{s \in \mathbb{Z}_+^k, \|s\|=k} \mathcal{F}_2\chi^{(1)}^{2s} \right\rangle = 2f_0Q_j^{(2k)}, \quad (4.4)
\]

Consider equation (4.4). The first term in the left-hand side equals

\[
\left\langle \sum_{j=1}^{n} a_j \left( (1 - \lambda_j^{-1})z_j + (1 - \lambda_j)\bar{z}_j \right) \sum_{\|l\|=k} \chi_j l^* l^* (1 - \lambda_j^{-1})z_j \bar{z}_j \right\rangle \\
= \sum_{j=1}^{n} a_j (1 - \lambda_j^{-1})(1 - \lambda_j) \sum_{\|l\|=k} \chi_j l^* e_j l + \chi_j l^* \bar{e_j} l z^l \bar{z}^l.
\]

The second term equals

\[
\left\langle 8f_0 \sum_{j=1}^{n} \mathcal{F}_{e_j}a_j (z_j + \bar{z}_j) \sum_{\|l\|=k, \|l^*\|=2k-1} \chi_j l^* l^* z^l \bar{z}^l \right\rangle \\
= 8f_0 \sum_{j=1}^{n} \frac{2 - \lambda_j - \lambda_j^{-1} - 8f_0}{-8f_0} a_j \sum_{\|l\|=k, \|l^*\|=2k-1} \chi_j l^* e_j l + \chi_j l^* \bar{e_j} l z^l \bar{z}^l.
\]
Hence the first two terms in the left-hand side of (4.1) cancel.

The third term equals

$$4f_0 \sum_{s \in \mathbb{Z}_0^q, \|s\|=k} \mathcal{F}_{2s} a^{2s} C_{2s_1}^{s_1} \cdots C_{2s_n}^{s_n} (z_1 \bar{z}_1)^{s_1} \cdots (z_n \bar{z}_n)^{s_n}.$$ 

Hence the coefficients $\mathcal{F}_{2s}$ are uniquely computed from (4.4).

Now turn to equation (4.3). The first term in the left-hand side equals

$$\sum_{l'=l''=2k-1, l'-l'' \neq \pm e_j} (\lambda_j + \lambda_j^{-1} - \lambda'^{-1} - \lambda''^{-1}) \chi_j l' l' l''.$$ 

The condition $l' - l'' \neq \pm e_j$ appears as a result of application of the operation $[\ ]$. It implies that the coefficients $\lambda_j + \lambda_j^{-1} - \lambda'^{-1} - \lambda''^{-1}$ do not vanish.

The second term in the left-hand side has been already computed from equation (4.4). Hence the coefficients $\chi_j l' l''$ and $\chi_j l'' l' \neq \pm e_j$ are computed uniquely from (4.3).

The coefficients $\chi_j l' e_j l$ and $\chi_j l'' e_j l$ can be chosen arbitrarily due to the gauge symmetry.

\section{5 Other symmetries}

When the existence theorem (Theorem 1) is proven, we can return to equation (2.5) which is equivalent to system (2.9), (2.10) but looks somewhat simpler. Now we can ignore overdeterminacy of (2.5) w.r.t. $f$.

In an explicit form (2.5) looks as follows:

$$\tau_- \chi + \tau_- f \circ \chi (\nabla f) \circ \chi \left. L(\chi \circ \rho^{-1}, \chi) \right| + \tau_+ \chi + \tau_+ f \circ \chi (\nabla f) \circ \chi \left. L(\chi, \chi \circ \rho) \right| = 0. \quad (5.1)$$

Direct computation shows that, analogously to (4.3), $f^{(2k)}$ and $\chi^{(2k-1)}$ satisfy the equations

$$(\lambda_j + \lambda_j^{-1} - \rho^* - \rho^{-1}) \chi_j^{(2k-1)} + 2f_0 \nabla_j f^{(2k)} \circ \chi^{(1)} = 2|f_0| P_j^{(2k)}, \quad (5.2)$$

where the functions $P_j^{(2k)}$ are polynomials w.r.t. coefficients of $f^{(2m)}$ and $\chi^{(2m-1)}$ with $m < k$.

**Proposition 5.1**

(1) If $\chi_j$ are chosen satisfying (4.2) then $\chi_j$ are odd in $z_j, \bar{z}_j$ and even in $z_l, \bar{z}_l$ for all $l \neq j$

$$\chi_j = -\chi_j \circ \kappa_j = \chi_j \circ \kappa_l \text{ for all } l \neq j. \quad (5.3)$$

(2) If the coefficients $\chi_{j l' e_j l}$ are odd in $z_j, \bar{z}_j$ (arbitrary due to the gauge symmetry) are chosen real, then all Taylor coefficients of $\chi$ are real.

**Proof.** (1) It is sufficient to note that if the homogeneous forms $\chi^{(2m-1)}$ with $m < k$ satisfy (5.3), the functions $P_j^{(2k)}$ also satisfy (5.3):

$$P_j^{(2k)} = -P_j^{(2k)} \circ \kappa_j = P_j^{(2k)} \circ \kappa_l \text{ for all } l \neq j.$$
(2) If the homogeneous forms $\chi^{(2m-1)}$ with $m < k$ satisfy (2.11) and coefficients of the forms $\chi^{(2m-1)}$ are real, the functions $P_j^{(2k)}$ are also real polynomials in $z$ and $\bar{z}$ with real coefficients.

The case $\lambda_1 = \ldots = \lambda_n = -1$ is resonant and one should not expect a positive answer to question $\hat{Q}$. Although in this case $\chi$ cannot be found, the function $f$ can be computed explicitly. Indeed, in this case

$$\tau_\pm \chi_j = 2\chi_j, \quad \tau_\pm f \circ \chi = 2f \circ \chi, \quad \hat{L}(\chi \circ \rho^{-1}, \chi) = \hat{L}(\chi, \chi \circ \rho) = 2\sqrt{h}, \quad h = f^2 \circ \chi + \chi^2.$$  

In this case in equations (5.1) we can regard $\chi_j$ as independent variables. These equations are equivalent to $\partial \chi_j h = 0$. Condition (1) implies that $h = f_0^2$, therefore $f(x) = -\sqrt{f_0^2 - x^2}$.

This formal computation shows that, in a certain sense, any sphere with the center at the origin is a limit solution of our problem when all $\lambda_j$ tend to $-1$. Numeric computations confirm this statement.

6 Another version of the main equation

In this section we present another version of the equations from which the functions $f$ and $\chi$ can be computed. For $n = 1$ these equations are the same as in [13, 14]. We used these equations for $n = 2$ in numeric computations since they contain the unknown functions polynomially.

6.1 Dynamical equations

Let $a, b, c \in E$ be three successive points of a billiard trajectory:

$$a = \left( \begin{array}{c} a \\ -f(a) \end{array} \right), \quad b = \left( \begin{array}{c} b \\ f(b) \end{array} \right), \quad c = \left( \begin{array}{c} c \\ -f(c) \end{array} \right).$$

(6.1)

The vector $n$ normal to $S_+$ at the point $b$ is

$$n = \left( \begin{array}{c} s \\ -1 \end{array} \right), \quad s_j = \partial x_j f(b), \quad j = 1, \ldots, n.$$  

The law of elastic reflection implies

$$a - b \parallel c - b - 2n \frac{\langle n, c - b \rangle}{n^2}$$

i.e., the vector $a - b$ is collinear to the vector symmetric to $c - b$ w.r.t. a plane orthogonal to $n$. In more detail,

$$a - b \parallel A(c - b), \quad A = n^2 I_{n+1} - 2nn^T = C^{-1}IC,$$
where
\[ I = \text{diag}(1, \ldots, 1, -1), \quad C = \begin{pmatrix} 0 & 0 & \ldots & 1 & s_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \ldots & 0 & s_2 \\ 1 & 0 & \ldots & 0 & s_1 \\ s_1 & s_2 & \ldots & s_n & -1 \end{pmatrix}. \]

Therefore we obtain:
\[ C(b - a) \parallel IC(b - c). \quad (6.2) \]

We refer to the map \((a, b) \mapsto \hat{\beta}(a, b) = (b, c)\) as the billiard map.

### 6.2 Conjugacy equation

Question \(\hat{Q}\) is equivalent to the following one. Does the following conjugation equation
\[ \hat{\beta} \circ X = X \circ \rho, \quad \hat{\beta} = \hat{\beta}(f) \quad (6.3) \]
have a solution \((f, X)\), where \(f\) satisfies (1.2) and \(X = (\chi_-, \chi) : U \to B^2\) is a diffeomorphism of a neighborhood \(U \subset \mathbb{R}^{2n}\) to its image.

Equations (6.3), (6.4), and (6.5) imply
\[ C\tau_-(\begin{pmatrix} \chi \\ f \circ \chi \end{pmatrix}) \parallel IC\tau_+(\begin{pmatrix} \chi \\ f \circ \chi \end{pmatrix}). \quad (6.6) \]

For \(n = 1\) equation (6.6) takes the form
\[ \begin{pmatrix} \tau_- f \circ \chi + s \tau_- f \circ \chi \\ s \tau_- - \tau_- f \circ \chi \end{pmatrix} \parallel \begin{pmatrix} \tau_+ f \circ \chi + s \tau_+ f \circ \chi \\ -s \tau_+ + \tau_+ f \circ \chi \end{pmatrix}, \]
which implies
\[ 2f' \circ \chi \left( \tau_- f \circ \chi \tau_+ f \circ \chi - \tau_- \chi \tau_+ \chi \right) + (1 - f'^2 \circ \chi) \left( \tau_- \chi \tau_+ f \circ \chi + \tau_+ \chi \tau_- f \circ \chi \right) = 0. \quad (6.5) \]

This equation is obtained in [13]. Its analysis is contained in [13, 14].

If \(n = 2\), (6.6) takes the form
\[ \begin{pmatrix} \tau_- f \circ \chi + s_2 \tau_- f \circ \chi \\ \tau_- \chi_1 + s_1 f \circ \chi \\ \tau_- \chi_1 + s_2 \tau_- \chi_2 - \tau_- f \circ \chi \end{pmatrix} \parallel \begin{pmatrix} \tau_+ f \circ \chi + s_2 \tau_+ f \circ \chi \\ \tau_+ \chi_1 + s_1 \tau_+ f \circ \chi \\ -s_1 \tau_+ + \tau_+ \chi_2 + \tau_+ f \circ \chi \end{pmatrix}, \]
which implies 2 equations:
\[ 2\partial_1 f \circ \chi \left( \tau_- f \circ \chi \tau_+ f \circ \chi - \tau_- \chi_1 \tau_+ \chi_1 \right) + (1 - (\partial_1 f \circ \chi)^2) \left( \tau_- \chi_1 \tau_+ f \circ \chi + \tau_+ \chi_1 \tau_- f \circ \chi \right) - \partial_2 f \circ \chi \left( \tau_+ \chi_2 \tau_- f \circ \chi + \tau_- \chi_2 \tau_+ f \circ \chi \right) = 0, \quad (6.6) \]
\[ 2\partial_2 f \circ \chi \left( \tau_- f \circ \chi \tau_+ f \circ \chi - \tau_- \chi_2 \tau_+ \chi_2 \right) + (1 - (\partial_2 f \circ \chi)^2) \left( \tau_- \chi_2 \tau_+ f \circ \chi + \tau_+ \chi_2 \tau_- f \circ \chi \right) - \partial_1 f \circ \chi \left( \tau_+ \chi_1 \tau_- f \circ \chi + \tau_- \chi_1 \tau_+ f \circ \chi \right) = 0. \quad (6.7) \]
Our numeric results in the case $n = 2$ presented in Section 1 are based on the analysis of system (6.6), (6.7).

References

[1] A. Avila, J. De Simoi, and V. Kaloshin. An integrable deformation of an ellipse of small eccentricity is an ellipse. Annals of Mathematics, Vol. 184 (2016), Issue 2, P. 527-558.

[2] M. Bialy and A.E. Mironov Angular Billiard and Algebraic Birkhoff conjecture. arXiv:1601.03196

[3] Birkhoff G. D., Dynamical systems. Amer. Math. Soc. Colloquium Publications V. 9, New York, 1927.

[4] S.V. Bolotin, Integrable Birkhoff billiards, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 2 (1990), 33-36, 105 (in Russian); translated in Mosc. Univ. Mech. Bull. 45:2 (1990), 10-13.

[5] S. V. Bolotin, D. V. Treschev, The anti-integrable limit, Russian Math. Surveys, 70:6 (2015), 975-1030.

[6] B. Beauzamy, E. Bombieri, P. Enflo, H.L. Montgomery. Products of polynomials in many variables. Journal of Number Theory 1990, Vol. 36 (2), 219-245.

[7] A. Glutsyuk and E. Shustin On polynomially integrable planar outer billiards and curves with symmetry property. arXiv:1607.07593

[8] V. V. Kozlov, Two-link billiard trajectories: extremal properties and stability, J. Appl. Math. Mech., 64:6 (2000), 903-907.

[9] V. V. Kozlov, Problem of stability of two-link trajectories in a multidimensional Birkhoff billiard, Proc. Steklov Inst. Math., 273 (2011), 196-213.

[10] V.V. Kozlov, Polynomial conservation laws for the Lorentz gas and the Boltzmann-Gibbs gas, Russian Math. Surveys, 71:2 (2016), 253-290.

[11] V. V. Kozlov, D. V. Treschev, Billiards. A genetic introduction to the dynamics of systems with impacts, Translations of Mathematical Monographs, 89, Amer. Math. Soc., Providence, RI, 1991.

[12] S. Tabachnikov. Geometry and Billiards, Student Mathematical Library, Volume 30, Providence, RI - Amer. Math. Soc, 2005.

[13] D. Treschev, Billiard map and rigid rotation, Phys. D, 255 (2013), 31-34.

[14] D. V. Treschev, On a Conjugacy Problem in Billiard Dynamics, Proc. Steklov Inst. Math., 289 (2015), 291-299.
[15] H. Whitney, Analytic extensions of functions defined in closed sets, Transactions of the American Mathematical Society, American Mathematical Society (1934), 36 (1): 63-89.