ABSTRACT

A geometric global formulation of the higher-order Lagrangian formalism for systems with finite number of degrees of freedom is provided. The formalism is applied to the study of systems with groups of Noetherian symmetries.
1. Introduction

This paper is a continuation of [1] in which a generalization of the usual Lagrangian formalism for first-order Lagrangian systems was provided. The ideas of [1] were suggested by the reformulation of the Lagrangian formalism for systems with finite number of degrees of freedom due to Poincaré, Cartan and Lichnerowicz (see for instance [2]) and Souriau [3]. The main idea is to use for the kinematical description the first-order jet construction and for the dynamical description a 2-form $\sigma$ having as characteristic system exactly the Euler-Lagrange equations. This formalism is particularly suited for the description of Lagrangian systems with groups of Noetherian symmetries.

In this paper we will provide a generalization of these ideas to higher-order Lagrangian systems. As in [1] we will consider the most general situation in which the basic kinematic manifold $S$ does not have a canonical fibered structure over the time axis: $S \rightarrow T$. The case when this fibration exists has been extensively studied in the literature. We distinguish two approaches which are however closely related. One line of argument tries to generalize the Poincaré-Cartan 1-form [4],[5] and another uses a 2-form which has as associated system the Euler-Lagrange equations [6]-[8]. Our approach is very similar with the second point of view. Some formal differences with respect to this approach should be noted. The first one has been alluded above, namely we do not need the fiber bundle structure $S \rightarrow T$. Another difference consists in the choice of the kinematics. In [6]-[8] one starts form the Euler-Lagrange equations; if the order of these equations is $s$ then one works on the $s$-order jet bundle extension of $S$: $J^s(S)$. On the other hand the notion of kinematics is intimately connected with the number of “initial conditions” which uniquely determine the evolution of the system. We will prove that if the “initial conditions” refer to the velocities of order up to $r$, then one can work on the $r$-order jet bundle extension of $S$: $J^r(S)$. In this way all reference to dynamics dissapears from the construction of kinematics, which is rather natural. Moreover, in this way we agree with the usual approach in the physics literature. For instance, for the case when the “initial conditions” refer to the initial position and velocity ($r = 1$) one usually works on the first-order jet bundle extension of $S$ (see e.g. [1]). Let us note that in general $s = r+1$ so our approach reduces the order of the jet bundle extension by 1. This simplification can be of some importance in practical computations. We are able to prove that the 2-form in our paper and the 2-form in [6]-[8] are very closely related.

In Section 2 we present the general formalism. We will stress on the possibility of defining globally the 2-form $\sigma$ in the same spirit as in [1] and will we give a natural definition for the notion of regularity closely related with the one in [6]-[8]. We will analyse afterwards Lagrangian systems with groups of Noetherian symmetries. In the end we will comment on the relation of our approach with the one in [6]-[8].

In Section 3 we illustrate the usefulness of the framework for the cases $r = 2$ and $r = 3$ with Galilei and Poincaré invariance as in [1].

In Section 4 we analyse again the case of Poincaré invariance but we use a homogeneous formalism in the same spirit as in [9], [10].
2. General Formalism for Higher-Order Lagrangian Systems

2.1 We emphasise again that we will consider only the case of finite number of degrees of freedom. Suppose $S$ is a $(N + 1)$-dimensional manifold interpreted as the “space-time” manifold of the system. $S$ will be called the kinematical manifold of the system. We remind briefly how one can construct the $s$-order jet bundle extension of $S$. One starts from the projective tangent bundle:

$$J^1_1(S) \equiv PT(S)$$

with the canonical projection $\pi_0 : J^1_1(S) \to S$. For $k > 0$, $J^k_1(S)$ will be a bundle over $J^k_1(S)$: $\pi_k : J^{k+1}_1(S) \to J^k_1(S)$. If $x_k \in J^k_1(S)$ then the fiber over $x_k$ consists of the straight lines of $T_{x_k}(J^k_1(S))$ which projects into elements of $J^k_1(S)$ i.e.

$$J^{k+1}_1(S)_{x_k} = \{ R \cdot v | v \in T_{x_k}(J^k_1(S)), (\pi_k)_*v \in J^k_1(S)_{\pi_{k-1}(x_k)} \}. \quad (2.1)$$

We need a convenient system of local charts on $J^r_1(S)$. Suppose $(t, q^A_0)$, $A = 1, ..., N$ is a local system of coordinates on the open set $U_0 \subset S$. Then one proves by recurrence that a system of charts on $J^k_1(S)$, $k = 1, ..., r$ exists and it is determined as follows:

1) one has a system of open sets $U_k \subset J^k_1(S), k = 1, ..., r$ such that

$$U_{k+1} \subseteq (\pi_k)^{-1}(U_k), \quad k = 0, ..., r - 1;$$

2) on $U_k$ one has local coordinates $(t, q^A_0, ..., q^A_k), A = 1, ..., N$ such that if the coordinates of $x_k \in J^k_1(S)$ are $(t, (q^A_0)_0, ..., (q^A_k)_0)$, then the straight line in $T_{x_k}(J^k_1(S))$ corresponding to $(t, (q^A_0)_0, ..., (q^A_k+1)_0)$ is generated by the vector:

$$\left( \frac{\delta}{\delta t} \right)_k \equiv \left( \frac{\partial}{\partial t} \right)_{x_k} + \sum_{i=0}^k (q^A_{i+1})_0 \left( \frac{\partial}{\partial q^A_i} \right)_{x_k}. \quad (2.2)$$

Here $k = 0, ..., r - 1$ and the summation convention over the dummy indices is used.

Indeed, one has the relationship:

$$\left( (\pi_k)_*\left( \frac{\delta}{\delta t} \right)_k \right) = \left( \frac{\delta}{\delta t} \right)_{k-1}, \quad (2.3)$$

and the assertion follows from the definition (2.1).

It is clear that $q^A_1$ will be interpreted as the coordinates of the velocity, $q^A_2$ as the coordinates of the accelerations (i.e. velocities of order 2) and so on.

Then, by definition, an evolution space over $S$ is any open subbundle $E$ of the bundle $J^r_1(S)$.

2.2 Let $E \subseteq J^r_1(S)$ be an evolution space over $S$. We define:

$$\land_{LS}(E) \equiv \{ \sigma \in \land^2(E) | i_{V_1} i_{V_2} \sigma = 0, \forall V_i \in Vect(E) \text{ s.t. } (\pi_r)_*V_i = 0 \text{ } i = 1, 2 \}. \quad (2.4)$$

It is clear that in the local coordinates above, $\forall \sigma \in \land_{LS}(E)$ has the expression:

$$\sigma = \sum_{i=0}^{r-1} \sigma^i_{AB} dq^A_i \land \delta q^B_i + \sigma^A dq^A_i \land dt + \frac{1}{2} \sum_{i,j=0}^{r-1} \tau^i_{AB} \delta q^A_i \land \delta q^B_j + \sum_{i=0}^{r-1} \tau^i_A \delta q^A_i \land dt. \quad (2.5)$$

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Here we have denoted:

\[ \delta q_i^A \equiv dq_i^A - q_{i+1}^A dt \quad (i = 0, ..., r - 1), \quad (2.6) \]

and we can suppose that:

\[ \tau_{ij}^{ij} = -\tau_{ji}^{ij}. \quad (2.7) \]

Now we have a key result:

**Proposition 1:** In the notations (2.5) the equations:

\[ \sigma_A = 0 \quad (2.8) \]
\[ \sigma_{AB}^i = 0 \quad (i = 1, ..., r - 1) \quad (2.9) \]
\[ \tau_A^i = 0 \quad (i = 1, ..., r - 1) \quad (2.10) \]

are globally defined.

**Proof:** Let \( U_0, \bar{U}_0 \in S \) with local coordinates \( (t, q_0^A) \) and \( (\bar{t}, \bar{q}_0^A) \) respectively. Let \( (t, q_0^A, ..., q_k^A) \) and \( (\bar{t}, \bar{q}_0^A, ..., \bar{q}_k^A) \) the local systems of coordinates on \( J_1^r(S) \) constructed as in Subsection 2.1. Let us suppose that \( U_0 \cap \bar{U}_0 \neq \emptyset \) and that the change of charts has the form:

\[ \bar{t} = f(t, \bar{q}_0); \quad \bar{q}_i^A = g_i^A(t, \bar{q}). \quad (2.11) \]

We must show that:

\[ \sigma_A = 0, \quad \sigma_{AB}^i = 0, \quad \tau_A^i = 0 \quad (i = 1, ..., r - 1) \]

iff

\[ \bar{\sigma}_A = 0, \quad \bar{\sigma}_{AB}^i = 0, \quad \bar{\tau}_A^i = 0 \quad (i = 1, ..., r - 1). \]

In fact it is sufficient to prove only one implication. First one needs the change of charts induced on \( J_1^r(S) \), i.e. the functions \( \bar{q}_i^A \quad (i = 1, ..., r) \). One can show by induction that one has:

\[ \bar{q}_i^A = g_i^A(t, q_1, ..., q_i) \quad (i = 1, ..., r). \quad (2.12) \]

The functions \( g_i^A \) are determined recursively from the formulae:

\[ g_{k+1}^A = \left( \frac{\delta f}{\delta t} \right)^{-1} \frac{\delta g_k^A}{\delta t} \quad (k = 0, ..., r - 1), \quad (2.13) \]

where:

\[ \frac{\delta}{\delta t} \equiv \left( \frac{\delta}{\delta t} \right)_{r-1} = \frac{\partial}{\partial t} + \sum_{k=0}^{r-1} q_{k+1}^A \frac{\partial}{\partial q_k^A}. \quad (2.14) \]

From (2.13) one determines that \( \delta \bar{q}_i^A \quad (i = 0, ..., r - 1) \) has the following structure:

\[ \delta \bar{q}_i^A = \sum_{j=0}^{i} g_{ij}^A \delta q_j^B \]
for some functions $g_{iB}^A$. Now the desired implication follows. Q.E.D.

2.3 By definition, a Lagrange-Souriau form on the evolution space $E \subseteq J^r_t(S)$ is any closed 2-form $\sigma \in \wedge_{LS}(E)$ verifying (2.8) - (2.10). If we take into account these relations, the expression (2.5) simplifies to:

$$\sigma = \sigma_{AB} dq^A_i \wedge \delta q^B_0 + \frac{1}{2} \sum_{i,j=0}^{r-1} \tau_{AB}^{ij} \delta q^A_i \wedge \delta q^B_j + \tau_A \delta q^A_0 \wedge dt. \quad (2.15)$$

In practical computations we will need the explicit content of closedness condition:

$$d\sigma = 0. \quad (2.16)$$

The computations are rather tedious but elementary so we provide only the final result. We have two distinct cases. For $r = 1$ we get, as expected, the results obtained in [1] (namely eqs. (2.10)-(2.15) from [1]).

The case $r > 1$ in which we are interested in this paper is a little more complicated; we get:

$$\frac{\partial \sigma_{AB}}{\partial q^C_r} = 0. \quad (2.17)$$

$$\frac{\delta \sigma_{AB}}{\delta t} = \frac{\partial \tau_A}{\partial q^B_r} + \tau_{AB}^{0,r-1} = 0. \quad (2.18)$$

$$\tau_{AB}^{1,r-1} = \sigma_{BA}. \quad (2.19)$$

$$\tau_{AB}^{r-1,i} = 0 \quad (i = 2, ..., r - 1). \quad (2.20)$$

$$\frac{\partial \sigma_{AB}}{\partial q^C_0} - \frac{\partial \sigma_{AC}}{\partial q^B_0} + \frac{\partial \tau_{00}}{\partial q^A_0} = 0. \quad (2.21)$$

$$\frac{\partial \sigma_{AB}}{\partial q^C_i} + \frac{\partial \tau_{0i}}{\partial q^A_r} = 0 \quad (i = 1, ..., r - 1). \quad (2.22)$$

$$\frac{\partial \tau_{AB}^{ij}}{\partial q^C_r} = 0 \quad (i, j = 1, ..., r - 1). \quad (2.23)$$

$$\frac{\delta \tau_{AB}^{00}}{\delta t} + \frac{\partial \tau_{AB}}{\partial q^A_0} - \frac{\partial \tau_A}{\partial q^B_0} = 0. \quad (2.24)$$

$$\frac{\delta \tau_{AB}^{0i}}{\delta t} + \tau_{AB}^{0,i-1} - \frac{\partial \tau_A}{\partial q^B_i} = 0 \quad (i = 1, ..., r - 1). \quad (2.25)$$

$$\frac{\delta \tau_{AB}^{ij}}{\delta t} + \tau_{AB}^{i-1,j} - \tau_{BA}^{j-1,i} = 0 \quad (i, j = 1, ..., r - 1). \quad (2.26)$$

$$\frac{\partial \tau_{AB}^{ij}}{\partial q^C_k} + \frac{\partial \tau_{BC}}{\partial q^A_i} + \frac{\partial \tau_{CA}}{\partial q^B_j} = 0 \quad (i, j, k = 0, ..., r - 1). \quad (2.27)$$
with the convention that equation (2.20) disappears for \( r = 2 \).

We will call (2.17)-(2.27) the structure equations.

2.4 A Lagrangian system on \( S \) is, by definition, any couple \( (E, \sigma) \) where \( E \subseteq J^r_1(S) \) is an evolution space over \( S \) and \( \sigma \) is a Lagrange-Souriau form on \( E \).

We will give now the main concepts involved in the study of Lagrangian systems. There is a close resemblance with [1].

Two Lagrangian systems \( (E_1, \sigma_1) \) and \( (E_2, \sigma_2) \) on \( S \) are called equivalent if there exists a diffeomorphism \( \phi \in \text{Diff}(S) \) such that \( \dot{\phi}(E_1) = E_2 \) and:

\[
(\dot{\phi})^*\sigma_2 = \sigma_1.
\] (2.28)

Here \( \dot{\phi} \in \text{Diff}(J^r_1(S)) \) is the natural lift of \( \phi \).

To introduce the notion of non-degeneracy we need:

**Proposition 2:** Let \( \sigma \) be a Lagrange-Souriau form written locally as in (2.15). Then the condition:

\[
\text{det}(\sigma_{AB}) \neq 0,
\] (2.29)

is globally defined.

**Proof:** Using the expressions of the change of charts \( (t, q^A_0, \ldots, q^A_r) \to (f, g^A_0, \ldots, g^A_r) \) form Proposition 1, it is rather easy to get the transformation law for the functions \( \sigma_{AB} \):

\[
\sigma_{AB} = \bar{\sigma}_{CD} J^C_A J^D_B \left( \frac{\delta f}{\delta t} \right)^{-r},
\]

where

\[
J^A_B = \frac{\partial g^A_0}{\partial q^B_0} - g^A_1 \frac{\partial f}{\partial q^B_0}
\]

So, it remains to show that the inversability of the change of charts implies:

\[
\text{det}(J) \neq 0
\]

which is not very complicated. Q.E.D.

We say that the Lagrangian system \( (E, \sigma) \) is non-degenerated if the condition (2.29) is true.

An evolution is any immersion \( \gamma : T \to S \), where \( T \) is some one-dimensional manifold (the “time” manifold). Usually \( T = \mathbb{R} \). If \( \gamma \) is an evolution and \( \dot{\gamma} : T \to J^r_1(S) \) is the natural lift of \( \gamma \), we say that \( \gamma \) verifies the Euler-Lagrange equations of motion if:

\[
(\dot{\gamma})^* i_Z \sigma = 0.
\] (2.30)

for any vector field \( Z \) on \( E \).

It can be proven that if \( \sigma \) is exhibited in the form (2.15) then the Euler-Lagrange equations are:

\[
\sigma_{BA} \circ \dot{\gamma} \frac{d^{r+1} x^B}{dt^{r+1}} - \tau_A \circ \dot{\gamma} = 0.
\] (2.31)
where $\gamma$ is chosen of the form $\gamma : t \mapsto (t, x^A(t))$. So, it is clear that these equations are, in general, of order $s = r + 1$. Moreover if $(E, \sigma)$ is a non-degenerated Lagrangian system, then it follows that $\gamma$ is determined by the “initial condition” $x^A_0(t_0), \ldots, x^A_r(t_0)$ (at least locally). So, the non-degeneracy condition expresses in an abstract form the notion of (Newtonian) determinism.

Usually, one has a causality relationship on $S$ (i.e. an order relation on $S$) and then it is natural to require that the points in $\text{Im}(\gamma)$ are in a causality relationship.

We also note that the phase space can be constructed following Souriau idea [3]. Namely if $\text{dim}(\text{Ker}(\sigma))$ is constant on $E$, the phase space is the characteristic foliation of $(E, \sigma)$.

We close this Subsection with the formulation of the notion of (Noetherian) symmetries. Let $(E, \sigma)$ be a Lagrangian system over $S$. A symmetry of $(E, \sigma)$ is any diffeomorphism $\phi \in \text{Diff}(S)$ such that: (a) $\dot{\phi}(E) = E$; (b) if $\gamma$ verifies the Euler-Lagrange equations, then $\phi \circ \gamma$ is also a solution of these equations.

If $\phi \in \text{Diff}(S)$ verifies:

$$\dot{\phi}^* \sigma = \sigma$$

then, it is easy to prove that $\phi$ is a symmetry of the Lagrangian system. We call such type of symmetries, Noetherian symmetries. A Noetherian group of symmetries is an action of a group on $S$: $G \ni g \to \phi_g \in \text{Diff}(S)$ such that any $\phi_g$ is a Noetherian symmetry, i.e.:

$$\dot{\phi}_g^* \sigma = \sigma \quad (\forall g \in G).$$

Let us remark that in this case one must appropriately modify the notion of equivalence for Lagrangian systems (requiring that $\phi$ in (2.28) is a $G$-morphism) and the notion of causality (requiring that the order relation on $S$ is $G$-invariant).

As illustrated in [1], this formulation of the Lagrangian formalism allows an explicit classification of all Lagrangian systems $(E, \sigma)$ on $S$ having a certain group $G$ of Noetherian symmetries, for many groups $G$ of physical interest.

It is noteworthy that the entire structure induces the well-known symplectic formalism on the associated phase space [3].

2.5 Now we connect the abstract scheme developped above with the usual Lagrangian formalism. The way to do this is to first prove

**Proposition 3:** Let $(E, \sigma)$ be a Lagrangian system over $S$. Then the 2-form $\sigma$ can be written locally as follows:

$$\sigma = d\theta$$

where $\theta$ has the following expression:

$$\theta = Ldt + \sum_{i=0}^{r-1} f^{i+1}_A \delta q^A_i.$$

Here $L$ is a local function of $t$ and $q^A_i (i = 0, \ldots, r)$. We have denoted

$$f^{i+1}_A \equiv \sum_{k=0}^{r-1-i} (-1)^k \left( \frac{\delta}{\delta t} \right)^k \frac{\partial L}{\partial q^A_{i+1+k}}$$

(2.36)
for \( i = 0, \ldots, r - 1 \). Moreover, the following identities must be obeyed:

\[
\frac{\partial f^{i+1}_A}{\partial q^B_r} = 0 \quad (i = 1, \ldots, r - 1)
\]  

(2.37)

in the case \( r \geq 2 \).

**Proof:** From the closedness condition (2.16) we have locally (2.34) where \( \theta \) is generically of the form:

\[
\theta = L dt + \sum_{i=0}^{r-1} f^{i+1}_A \delta q^A_i + \theta_A dq^A_r.
\]

So, we have locally:

\[
\sigma = d\theta = \frac{\partial \theta_A}{\partial q^B_r} dq^B_r \wedge dq^A_r + \cdots
\]

where by \( \cdots \) we understand contributions from \( \Lambda_{LS}(E) \). Because \( \sigma \in \Lambda_{LS}(E) \) by definition, we must have: \( \frac{\partial \theta_A}{\partial q^B_r} = \frac{\partial \theta_B}{\partial q^A_r} \) so, \( \theta_A \) is of the form: \( \theta_A = \frac{\partial f}{\partial q^A_r} \).

If one redefines \( \theta \rightarrow \theta - df \), then (2.34) stays true, but the last contribution in the expression above of \( \theta \) dissappears. It means that \( \theta \) acquires the expression (2.35).

Next, one imposes the conditions (2.8)-(2.10) from the statement of Proposition 1 and gets besides (2.37) the following relations:

\[
\frac{\partial L}{\partial q^A_r} - f^r_A = 0
\]  

(2.38)

and

\[
\frac{\partial L}{\partial q^A_i} - \frac{\delta f^{i+1}_A}{\delta t} - f^i_A = 0 \quad (i = 1, \ldots, r - 1)
\]  

(2.39)

which can be used to prove recursively (2.36). Q.E.D.

We will call \( L \) a (local) *Lagrangian*. If \( \theta \) is of the form (2.35) we denote it by \( \theta_L \); we also denote \( \sigma_L \equiv d\theta_L \).

Now one recovers the usual Euler-Lagrange equations. Indeed, if \( \sigma = \sigma_L \) one easily shows that (2.31) becomes:

\[
\sum_{k=0}^{r} (-1)^k \frac{d^k}{dt^k} \left( \frac{\partial L}{\partial q^A_k} \circ \dot{\gamma} \right) = 0.
\]  

(2.40)

The identities (2.37) restrict the form of the Lagrangian \( L \) and can be used to “lower” the order of this function, i.e. to choose it independent of \( q^A_i \) for some \( i \geq c + 1 \) where \( c \leq r \). (This makes sense for \( r \geq 2 \).) Indeed, we have:

**Proposition 4:** Let \( (E, \sigma) \) be a Lagrangian system over \( S \), \( E \subseteq J^1_1(S) \) \((r \geq 2)\). Define:

\[
c \equiv \left\lceil \frac{r}{2} \right\rceil + 1.
\]  

(2.41)
Then there exists a local Lagrangian $L_{\text{min}}$ such that:

$$\frac{\partial L_{\text{min}}}{\partial q^A_i} = 0 \quad (i = c + 1, \ldots, r)$$ (2.42)

and instead of (2.35) and (2.36) we have:

$$\theta = L_{\text{min}} dt + \sum_{i=0}^{c-1} f^{i+1}_A \delta q^A_i$$ (2.43)

and

$$f^{i+1}_A \equiv \sum_{k=0}^{c-1-i} (-1)^k \left( \frac{\delta}{\delta t} \right)^k \frac{\partial L}{\partial q^{A+i+k}_i}.$$ (2.44)

Moreover in the case $r = 2c$, $L_{\text{min}}$ is linear in $q_c$:

$$\frac{\partial^2 L_{\text{min}}}{\partial q^A_c \partial q^B_c} = 0.$$ (2.45)

**Proof:** For $r = 2$ we have $c = 2 = r - 1$ so (2.35) and (2.36) coincide with (2.43) and (2.44) respectively. Moreover (2.37) and (2.38) give (2.45).

Suppose now that $r \geq 2$. From (2.38) it follows that $L$ has the form:

$$L = q^A_r f^r_A + L'$$

where $L'$ does not depend on $q_r$. We insert this expression into (2.39) for $i = r - 1$ and differentiate with respect to $q^B_c$; taking into account that $r \geq 2$ we have (2.37) and we get:

$$\frac{\partial f^r_A}{\partial q^B_r} = \frac{\partial f^r_A}{\partial q^B_r} \text{ so } f^r_A \text{ is of the following form: } f^r_A = \frac{\partial f^r_A}{\partial q^B_r} \text{ for some function } f^r \text{ which does not depend on } q_r.$$ If we redefine $\theta \rightarrow \theta - df^r$, then (2.34) stays true, but instead of (2.35) we have, up to some redefinitions:

$$\theta = L_{\text{min}} dt + \sum_{i=0}^{r-2} f^{i+1}_A \delta q^A_i$$

Now one iterates the procedure; the recurrence stops when $\theta$ acquires the expression (2.43) with $L_{\text{min}}$ verifying (2.42). Imposing again the conditions (2.8)-(2.10) we get (2.37) and (2.44). Now it is rather easy to show that if $r$ is odd then (2.37) are identically verified. If $r$ is even then (2.37) are identically verified for $i \geq 1$ and for $i = 1$ we obtain exactly (2.45). Q.E.D.

We will call $L_{\text{min}}$ a *minimal Lagrangian* and $c$ the *minimal order*.

One also has the following consequence of the result above:

**Proposition 5:** The non-degeneracy condition has the following local formulation:

- for $r$ odd:

$$\det \left( \frac{\partial^2 L_{\text{min}}}{\partial q^A_c \partial q^B_c} \right) \neq 0$$ (2.46)

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- for $r$ even:

$$
\text{det} \left( \frac{\partial^2 L_{\text{min}}}{\partial q_i^A \partial q_{c-1}^B} - A \leftrightarrow B \right) \neq 0.
$$

(2.47)

**Proof:** Consists in the computation of the functions $\sigma_{AB}$ in terms of $L_{\text{min}}$. Q.E.D.

One should compare Propositions 3-5 above with the similar results obtained in [7].

2.6 Next, one establishes that working with $\sigma$ instead of $L$ eliminates from the game the so-called total derivative Lagrangians. Indeed we have:

**Proposition 6:** The following three assertions are equivalent:

(a) $\sigma_L = 0$;

(b) the Euler-Lagrange equations for $(E, \sigma_L)$ are trivial;

(c) $L$ is of the form

$$
L = \frac{\delta \Lambda}{\delta t},
$$

(2.48)

with $\Lambda$ a function of $t$ and $q_i^A$ ($i = 0, ..., r - 1$).

**Proof:** The implication $(a) \iff (c)$ follows easily from the closedness condition (2.16).

The implication $(b) \iff (c)$ follows from (2.39) in a standard way by induction. Q.E.D.

Lagrangians of the type (2.48) are called *total derivative* Lagrangians.

Finally, we connect with the usual definition of the Noetherian symmetries. Indeed, suppose that the transformation $\phi$ in (2.32) has the following form in local coordinates:

$$
\phi(t, q_0) = (f(t, q_0), g_0(t, q_0)) \quad \text{and} \quad \sigma = \sigma_L.
$$

Then one easily establish using Propositions 1 and 5 that $L$ verifies the following identity:

$$
\frac{\delta f}{\delta t} L \circ \dot{\phi} - L = \frac{\delta \Lambda}{\delta t}.
$$

(2.49)

Let us suppose that $\theta$ is globally defined on $E$. Then one can introduce the *action functional*:

$$
A(\gamma) \equiv \int (\dot{\gamma})^* \theta.
$$

(2.50)

Then the solutions of the Euler-Lagrange equations are extremals of this functional and moreover (2.49) is equivalent to the usual definition of Noetherian symmetries:

$$
A(\phi \circ \gamma) = A(\gamma) + \text{a trivial action}.
$$

(2.51)

where by a *trivial action* we mean an action giving trivial Euler-Lagrange equations of motions. (Such actions are also called boundary terms).

We have recovered the whole structure of the Lagrangian formalism. However, we did not need a configuration space like in the usual formulations.

Let us stress once again that the concept of Lagrangian is only a local one. Even if we can take $U = E$, $L$ remains a chart dependent object and cannot be considered, in general, as a function on $E$.

2.7 We close this Section with some comments regarding the relation of our approach with the one in [6]-[8]. As stated in the Introduction, in this approach the kinematics takes place in $J^1_1(S) = J^{1+1}_1(S)$. 
Instead of (2.15) one takes $\sigma$ of the form:

\[
\sigma = E_A \delta q_A^0 \wedge dt + \frac{1}{2} \sum_{i,j=0}^{r} F_{AB}^{ij} \delta q_i^A \wedge \delta q_j^B
\]  

(2.52)

where we have implicitly assumed that (2.6) has been extended for $i = r$ also (which is possible in $J^{r+1}_1(S)$). Inspecting all the results of Subsections 2.2-2.6 we conclude that they stay true in this case also (with some appropriate modifications). For instance the global definition of $\sigma$ involves the following conditions:

(a) $i_V \sigma = 0$ for any $V \in \text{Vert}(J^{r+1}_1(S))$ such that $(\pi_{r+1})_* V = 0$.

(b) $\sigma$ does not contain the differential forms $\delta q_i^A \wedge dt$ $(i = 1, \ldots, r)$.

In rest, one must take care to replace everywhere $\frac{\delta}{\delta t} \equiv (\frac{\delta}{\delta t})_{r-1}$ with $\frac{\delta}{\delta t} \equiv (\frac{\delta}{\delta t})_r$.

A final observation is the following one. By some combinatorics one can eliminate completely the variables $q_{r+1}$ from the expression (2.52) and one obtains exactly the expression (2.15). So, (2.52) is the pull-back of (2.15).

3. Lagrangian Systems with Groups of Noetherian Symmetries

A. We consider first the case $r = 2$. The kinematic manifold will be taken as $S = R \times R^{n-1}$ with global coordinates $(t, q_A^1)$ $(A = 1, \ldots, n-1)$. Then $J_1^2 \cong R \times R^{n-1} \times R^{n-1}$ with global coordinates $(t, q_0^A, q_1^A, q_2^A)$ $(A = 1, \ldots, n-1)$.

3A.1 In this case the closedness conditions (2.17)-2.27) reduces to:

\[
\frac{\partial \sigma_{AB}}{\partial q_C^c} = 0.
\]  

(3.1)

\[
\frac{\delta \sigma_{AB}}{\delta t} - \frac{\partial \tau_{AB}}{\partial q_B} + \tau_{AB}^{01} = 0.
\]  

(3.2)

\[
\tau_{AB}^{11} = \sigma_{BA}.
\]  

(3.3)

\[
\frac{\partial \sigma_{AB}}{\partial q_C^c} - \frac{\partial \sigma_{AC}}{\partial q_B} + \frac{\partial \tau_{BC}^{00}}{\partial q_2^A} = 0.
\]  

(3.4)

\[
\frac{\partial \sigma_{AB}}{\partial q_1^c} + \frac{\partial \tau_{BC}^{01}}{\partial q_2^A} = 0.
\]  

(3.5)

\[
\frac{\partial \tau_{AB}^{11}}{\partial q_2^c} = 0.
\]  

(3.6)

\[
\frac{\delta \tau_{AB}^{00}}{\delta t} + \frac{\partial \tau_{AB}}{\partial q_0^A} - \frac{\partial \tau_{A}}{\partial q_0^B} = 0.
\]  

(3.7)

\[
\frac{\delta \tau_{AB}^{01}}{\delta t} + \tau_{AB}^{00} - \frac{\partial \tau_{A}}{\partial q_1^B} = 0.
\]  

(3.8)
\[ \frac{\delta t^{11}_{AB}}{\delta t} + \tau^{01}_{AB} - \tau^{01}_{BA} = 0. \] (3.9)

\[ \frac{\partial \tau^{ij}_{AB}}{\partial q^A_k} + \frac{\partial \tau^{jk}_{BC}}{\partial q^B_i} + \frac{\partial \tau^{ki}_{CA}}{\partial q^C_j} = 0 \quad (i, j, k = 0, 1). \] (3.10)

3A.2 We first study the functions \( \sigma_{AB} \). It is not hard to prove from the relations above that one has the antisymmetry property:

\[ \sigma_{AB} = -\sigma_{BA} \] (3.11)

and also:

\[ \frac{\partial^2 \sigma_{AB}}{\partial q^1_i \partial q^1_j} = 0. \] (3.12)

3A.3 We postulate now invariance with respect to spatio-temporal translations and rotations; the action of these transformations on \( S \) is:

\[ \phi_{R,\eta,a}(t, q_0) = (t, +\eta, Rq_0 + a). \] (3.13)

Here \( \eta \in R, a \in R^{n-1} \) and \( R \in SO(n-1) \). The lift of \( \phi_{R,\eta,a} \) to \( J^2_1(S) \) is:

\[ \dot{\phi}_{R,\eta,a}(t, q_0, q_1, q_2) = (t + \eta, Rq_0 + a, Rq_1, Rq_2). \] (3.14)

We require that:

\[ (\dot{\phi}_{R,\eta,a})^* \sigma = \sigma. \] (3.15)

It is easy to see that (3.15) is equivalent to the independence of the functions \( \sigma_{AB} \), \( \tau^{ij}_{AB} \) and \( \tau_A \) of the variables \( t \) and \( q_0 \) and with their rotation covariance.

In particular, if we take into account the relation (3.1) it follows that \( \sigma_{AB} \) depends only of the variable \( q_1 \). Then (3.12) shows that \( \sigma_{AB} \) is a polynomial of the form:

\[ \sigma_{AB} = \sigma_{ABC}q^C_1 + \sigma'_{AB} \]

with \( \sigma_{ABC} \) and \( \sigma'_{AB} \) some constants.

Because of the rotation covariancve, the antisymmetry property (3.11) and (3.10) (for \( i = j = k = 1 \)), \( \sigma_{AB} \) can be non-zero iff \( n = 3 \); in this case we have:

\[ \sigma_{AB} = \kappa \varepsilon_{AB}. \] (3.16)

From (3.17) it follows that:

\[ \tau_{AB} = -\kappa \varepsilon_{AB}. \] (3.17)

We still have to determine the functions \( \tau^{00}_{AB} \), \( \tau^{01}_{AB} \) and \( \tau_A \). This is not very difficult. One obtains from the closedness condition that \( \tau^{00}_{AB} \) is constant, \( \tau^{01}_{AB} \) depends only of \( q_1 \) and verifies \( \frac{\partial \tau^{01}_{AB}}{\partial q^1_i} = B \leftrightarrow C \), and \( \tau_A \) is given by:

\[ \tau_A = \tau^{01}_{AB}q^B_2 + \tau^{00}_{AB}q^B_1 + t_A \].
with $t_A$ constants. Then Lorentz covariance fixes:

$$
\tau_{AB}^{00} = \lambda \varepsilon_{AB} \quad (\lambda \in R). \tag{3.18}
$$

$$
\tau_{AB}^{01} = \delta_{AB} f(q_1^2) + 2q_1 A q_1 B f'(q_1^2). \tag{3.19}
$$

and

$$
\tau_{A} = \tau_{AB}^{01} q_B^2 + \tau_{AB}^{00} q_1^B. \tag{3.20}
$$

Here $f$ is an arbitrary smooth function.

The solution to our problem is given by (3.16)-(3.20).

It is easy to prove that $\sigma = \sigma_L$ with $L$ of the form

$$
L(q_1, q_2) = -\frac{1}{2} \kappa \varepsilon_{AB} q_A^B q_2^B + l(q_1^2) + \frac{1}{2} \lambda \varepsilon_{AB} q_0^A q_1^B \tag{3.21}
$$

with $l$ such that $f = -2l'$.

We stress once again that the only physically interesting case in the non-degenerated one which occurs only for $\kappa \neq 0$.

**Remark** If we impose in addition Galilei invariance it can be proved as in [1] that $l$ can be chosen of the form $l(q_1^2) = \frac{1}{2} l' m q_1^2$ with $m \in R$ and $\lambda = 0$.

If we impose in addition the invariance with respect to pure Lorentz transformations we must have $\kappa = 0$ which contradicts the non-degeneracy.

B. Next we consider the case $r = 3$. We take the same kinematic manifold $S$ as before and we identify $J^3(S) \cong R \times R^{n-1} \times R^{n-1} \times R^{n-1}$ with coordinates $(t, q_0, q_1, q_2, q_3)$. We will consider as before Galilei and Poincaré invariance.

3B.1 In this case the closedness conditions (2.17)-(2.27) become:

$$
\frac{\partial \sigma_{AB}}{\partial q_C^j} = 0. \tag{3.22}
$$

$$
\frac{\delta \sigma_{AB}}{\delta t} - \frac{\partial \tau_{A}}{\partial q_B^3} + \tau_{AB}^{02} = 0. \tag{3.23}
$$

$$
\tau_{AB}^{12} = \sigma_{BA}. \tag{3.24}
$$

$$
\tau_{AB}^{22} = 0. \tag{3.25}
$$

$$
\frac{\partial \sigma_{AB}}{\partial q_0^C} - \frac{\partial \sigma_{AC}}{\partial q_0^B} + \frac{\partial \tau_{BC}^{00}}{\partial q_3^B} = 0. \tag{3.26}
$$

$$
\frac{\partial \sigma_{AB}}{\partial q_1^C} + \frac{\partial \tau_{BC}^{01}}{\partial q_3^A} = 0. \tag{3.27}
$$

$$
\frac{\partial \sigma_{AB}}{\partial q_2^C} + \frac{\partial \tau_{BC}^{02}}{\partial q_3^A} = 0. \tag{3.28}
$$

$$
\frac{\partial \tau_{AB}^{ij}}{\partial q_r^C} = 0 \quad (i, j = 1, 2). \tag{3.29}
$$
\[ \frac{\delta \tau_{00}^{AB}}{\delta t} + \frac{\partial \tau_{B}}{\partial q_{0}^{A}} - \frac{\partial \tau_{A}}{\partial q_{0}^{B}} = 0. \]  
(3.30)

\[ \frac{\delta \tau_{01}^{AB}}{\delta t} + \tau_{AB}^{00} - \frac{\partial \tau_{A}}{\partial q_{1}^{B}} = 0. \]  
(3.31)

\[ \frac{\delta \tau_{02}^{AB}}{\delta t} + \tau_{AB}^{01} - \frac{\partial \tau_{A}}{\partial q_{2}^{B}} = 0. \]  
(3.32)

\[ \frac{\delta \tau_{ij}^{AB}}{\delta t} + \tau_{AB}^{i-1,j} - \tau_{BA}^{j-1,i} = 0 \quad (i, j = 1, 2). \]  
(3.33)

\[ \frac{\partial \tau_{ij}^{AB}}{\partial q_{k}^{C}} + \frac{\partial \tau_{jk}^{BC}}{\partial q_{i}^{A}} + \frac{\partial \tau_{ki}^{CA}}{\partial q_{j}^{B}} = 0 \quad (i, j, k = 0, 1, 2). \]  
(3.34)

3B.2 As in Subsection 3A.2 we concentrate first on the functions \( \sigma_{AB} \). It is not very hard to derive from the closedness conditions the following consequences:

(3.35) 
\[ \sigma_{AB} = \sigma_{BA} \]

and

(3.36) 
\[ \frac{\partial \sigma_{AB}}{\partial q_{2}^{C}} = \frac{\partial \sigma_{AC}}{\partial q_{2}^{B}} \]

3B.3 Let us impose the invariance with respect to the Galilei transformations:

(3.37) 
\[ \phi_{R,v,\eta,a}(t, q_{0}) = (t + \eta, Rq_{0} + tv + a). \]

(We use the same notation as in Subsection 3A.3; in addition we have \( v \in \mathbb{R}^{n-1} \)).

The lift of these transformation is:

(3.38) 
\[ \hat{\phi}_{R,v,\eta,a}(t, q_{0}, q_{1}, q_{2}, q_{3}) = (t + \eta, Rq_{0} + tv + a, Rq_{1} + v, Rq_{2}, Rq_{3}). \]

The invariance condition

(3.39) 
\[ (\hat{\phi}_{R,v,\eta,a})^{*}\sigma = \sigma \]

is equivalent to the fact that the functions \( \sigma_{AB} \), \( \tau_{AB}^{ij} \) and \( \tau_{A} \) do not depend on \( t, q_{0} \) and \( q_{1} \) and are also rotation covariant.

Because \( \sigma_{AB} \) depend only of \( q_{2} \) (see (3.22)) we can use (3.35) and (3.36) to prove that these function have the generic form:

(3.40) 
\[ \sigma_{AB} = \delta_{AB}F(q_{2}^{2}) + 2q_{2A}q_{2B}F'(q_{2}^{2}) \]

for some smooth function \( F \). The condition of non-degeneracy is

\[ F \neq 0 \]

which will be assumed in the following.
3B.4 Let us define the Lagrangian:

\[ L_0(q_2) \equiv l(q_2^2) \]  

(3.41)

and take \( l \) such that

\[ F = -2l' \]  

(3.42)

(see (3.40)). Then one can easily prove that:

\[ (\sigma_{L_0})_{AB} = \sigma_{AB}. \]  

(3.43)

Moreover \( L_0 \) verifies a relation of the type (2.49) for every transformation \( \phi_{R,v,\eta,a} \) with \( \Lambda = 0 \). So, we have:

\[ (\dot{\phi}_{R,v,\eta,a})^* \sigma_{L_0} = \sigma_{L_0}, \]  

(3.44)

The relations above suggest to define an auxiliary Lagrange-Souriau 2-form:

\[ \sigma' \equiv \sigma - \sigma_{L_0}. \]  

(3.45)

Then we have:

\[ (\dot{\phi}_{R,v,\eta,a})^* \sigma' = \sigma' \]  

(3.46)

and

\[ \sigma'_{AB} = 0. \]

3B.5 We are reduced to a simpler problem, namely the case when:

\[ \sigma_{AB} = 0. \]  

(3.47)

This brings substantial simplifications to the closedness conditions (3.22)-(3.34). Taking into account the Galilei invariance (3.46) also we obtain the following solution:

\[ \tau_{00}^{00} = 0 \]  

(3.48)

\[ \tau_{AB}^{01} = -m \delta_{AB} \quad (m \in R) \]  

(3.49)

\[ \tau_{AB}^{02} = \tau_{AB}^{11} = \kappa \varepsilon_{AB} \quad (n = 3). \]  

(3.50a)

\[ \tau_{AB}^{02} = \tau_{AB}^{11} = 0 \quad (n > 3) \]  

(3.50b)

and

\[ \tau_A = -m q_{2A} + \tau_{AB}^{02} q_3^B. \]  

(3.51)

It remains to note that we have \( \sigma = \sigma_{L_1} \) where:

\[ L_1(q_1, q_2) = \frac{1}{2} m q_1^2 + \frac{1}{2} \kappa \varepsilon_{AB} q_1^A q_2^B \]  

(3.52)

where the last contribution shows up only for \( n = 3 \).
3B.6 Form the last two Subsections it follows that \( \sigma = \sigma_L \) where:

\[
L = L_0 + L_1 = l(q_2^2) + \frac{1}{2}mq_1^2 + \frac{1}{2}r_{AB}^0 q_1^A q_2^B. \tag{3.53}
\]

3B.7 We turn now to the Poincaré invariance; this means that we keep the invariance with respect to \( \phi_{R,0,\eta,a} \) (see (3.37)) but instead of \( \phi_{1,v,0,0} \) we consider:

\[
\phi_{\beta,\chi}(t, q_0) = (cosh(\chi)t + sinh(\chi)\beta \cdot q_0, q_0 + \beta[(cosh(\chi) - 1)\beta \cdot q_0 + t sinh(\chi)]) \tag{3.54}
\]

for \( \beta \in S^2 \) and \( \chi \in R \) and we impose:

\[
(\dot{\phi}_{\beta,\chi})^* \sigma = \sigma. \tag{3.55}
\]

It is better to consider (3.55) in the infinitesimal form:

\[
L_{\beta,\sigma} \sigma = \sigma \tag{3.56}
\]

where \( X_{\beta} \in Vect(J_3^1(S)) \) is the vector field associated to the uni-parametric group action \( R \ni \chi \rightarrow \dot{\phi}_{\beta,\chi} \in Diff(E) \) which can be easily computed:

\[
X_{\beta} = \beta \cdot q_0 \frac{\partial}{\partial t} + t \beta \cdot \frac{\partial}{\partial q_0} + [\beta - (\beta \cdot q_1)q_1] \cdot \frac{\partial}{\partial q_1} - \\
[(\beta \cdot q_2)q_1 + 2(\beta \cdot q_1)q_2] \cdot \frac{\partial}{\partial q_2} - [3(\beta \cdot q_1)q_2 + 3(\beta \cdot q_2)q_2 + (\beta \cdot q_3)q_1] \cdot \frac{\partial}{\partial q_3}. \tag{3.57}
\]

3B.8 We want to obtain from (3.56) another relation on \( \sigma_{AB} \) which replaces the independence of \( q_1 \) from the Galilean case. To this purpose we use an well known relation \( L = di_X + i_X d \) and (2.16) to write (3.56) as follows:

\[
di_{X_{\beta}} \sigma = 0. \tag{3.58}
\]

From this relation we select the coefficient of the differential form \( dq_3^A \wedge \delta q_0^B \) and take into account that \( \beta \in S^2 \) is arbitrary; we get:

\[
3\sigma_{AB}q_{1C} + \sigma_{AD}q_1^D \delta_{BC} + \sigma_{BD}q_1^D \delta_{AC} - \frac{\partial \sigma_{AB}}{\partial q_1^D} (\delta_{CD} - q_{1C}q_1^D) + \frac{\partial \sigma_{AB}}{\partial q_2^D} (q_1^D q_2C + 2q_2^D q_1C) = 0. \tag{3.59}
\]

Using the rotation covariance and the symmetry property (3.35) it is easy to see that (3.59) is compatible with the following structure of \( \sigma_{AB} \):

\[
\sigma_{AB} = \delta_{AB}a + q_1a q_{1B} + q_2a q_{2B} + (q_1a q_{2B} + q_2a q_{1B})d. \tag{3.60}
\]

where \( a, b, c \) and \( d \) are smooth functions of the invariants:

\[
\xi_1 \equiv q_1^2, \quad \xi_2 \equiv q_2^2, \quad \xi_{12} \equiv q_1 \cdot q_2. \tag{3.61}
\]
We first impose (3.36) and obtain in particular:

\[ c = 2 \frac{\partial a}{\partial \xi_2} \]  
(3.62)

and

\[ d = \frac{\partial a}{\partial \xi_{12}}. \]  
(3.63)

Next, we impose (3.59); after some computations we obtain an equation of the following type:

\[ q_{1C}X_{AB} + q_{2C}Y_{AB} + Z_A\delta_{BC} + Z_B\delta_{AC} = 0 \]

which is easily seen to be equivalent to:

\[ X_{AB} = 0 \]  
(3.64)

\[ Y_{AB} = 0 \]  
(3.65)

and

\[ Z_A = 0. \]  
(3.66)

From (3.66) we get in particular:

\[ a - (1 - \xi_1)b + \xi_{12}d = 0. \]  
(3.67)

Taking into account (3.62) and (3.63) it follows that it is sufficient to determine the function \( a \). To this purpose we consider the coefficient of the tensor \( \delta_{AB} \) in (3.64) and (3.65) and get:

\[ 3a - 2(1 - \xi_1) \frac{\partial a}{\partial \xi_1} + 3\xi_{12} \frac{\partial a}{\partial \xi_{12}} + 4\xi_2 \frac{\partial a}{\partial \xi_2} = 0 \]  
(3.68)

and

\[ (1 - \xi_1) \frac{\partial a}{\partial \xi_{12}} + 2\xi_{12} \frac{\partial a}{\partial \xi_2} = 0. \]  
(3.69)

It is easy to prove that the solution of these equations in the domain \( \xi \neq 1 \) is of the form

\[ a(\xi_1, \xi_2, \xi_{12}) = |1 - \xi_1|^{-3/2} F \left( \frac{(1 - \xi_1)\xi_2 + \xi_{12}^2}{(1 - \xi_1)^3} \right). \]  
(3.70)

If one requires the non-degeneracy one must have \( F \neq 0 \) which we will suppose in the following. Now the smoothness condition for \( a \) compels us to choose as evolution space on of the following form:

\[ E^\eta \equiv \{(t, q_0, q_1, q_2, q_3)|\text{sign}(1 - q_1^2) = \eta\} \quad (\eta = \pm). \]  
(3.71)

From (3.62), (3.63) and (3.67) one determines \( b, c \) and \( d \) respectively and this elucidates completely the structure of the function \( \sigma_{AB} \).
3B.9 We use the same trick as in 3B.4. Define the Lagrangian $L_0$ by:

$$L_0(q_1, q_2) \equiv |1 - q_1^2|^{1/2} l \left( \frac{(1 - q_1^2) q_2^2 + (q_1 \cdot q_2)^2}{(1 - q_1^2)^3} \right)$$  \hspace{1cm} (3.72)$$

such that:

$$F = -2l'. \hspace{1cm} (3.73)$$

Then it is not very hard to prove that:

$$(\sigma_{L_0})_{AB} = \sigma_{AB}. \hspace{1cm} (3.74)$$

Moreover $L_0$ verifies a relation of the type (2.49) with $\Lambda = 0$ for every transformation of the type $\phi_{R,0,\eta,a}$ and $\phi_{\beta,\chi}$, so we have:

$$(\dot{\phi}_{R,0,\eta,a})^* \sigma_{L_0} = \sigma_{L_0} \quad (\dot{\phi}_{\beta,\chi})^* \sigma_{L_0} = \sigma_{L_0}. \hspace{1cm} (3.75)$$

If we define the auxiliary Lagrange-Souriau form:

$$\sigma' \equiv \sigma - \sigma_{L_0} \hspace{1cm} (3.76)$$

we have:

$$(\dot{\phi}_{R,0,\eta,a})^* \sigma' = \sigma' \quad (\dot{\phi}_{\beta,\chi})^* \sigma' = \sigma'. \hspace{1cm} (3.77)$$

and

$$\sigma'_{AB} = 0$$

3B.10 So again we have succeeded to reduce ourselves to a simpler problem, namely one with

$$\sigma_{AB} = 0. \hspace{1cm} (3.78)$$

The computations are more easy now and we give only the final result: $\sigma = \sigma_{L_1}$ where:

$$L_1(q_1, q_2) = -m|1 - q_1^2|^{1/2} + \frac{1}{2}\kappa(q_0 - tq_1) \hspace{1cm} (3.79)$$

where $m, \kappa \in R$ and $\kappa$ can be non-zero only for $n = 2$.

3B.11 Combining the results of the last two Subsection we obtain the final result: the most general Lagrange-Souriau form for $r = 3$ with Poincaré invariance is $\sigma = \sigma_L$ where:

$$L(q_1, q_2) \equiv |1 - q_1^2|^{1/2} l \left( \frac{(1 - q_1^2) q_2^2 + (q_1 \cdot q_2)^2}{(1 - q_1^2)^3} \right) + \frac{1}{2}\kappa(q_0 - tq_1). \hspace{1cm} (3.80)$$
4. The Homogeneous Formalism

4.1 We will give an alternative formulation of the whole problem using the so-called homogeneous formalism. This method was used in [9],[10] for the analysis of the extended objects (with Poincaré and Galilei invariance).

The idea is the following one. Suppose the kinematic manifold of the system is \(M: \dim(M) = n\). Instead of working on \(J^r_1(M)\), one takes in the general formalism from Section 1 \(S = R \times M\), works on \(J^r_1(S)\) and imposes in addition the condition that the reparametrization invariance is a Noetherian symmetry.

If the local coordinates on \(S\) are \((\tau, X^\mu) \ (\mu = 1, ..., n)\) then the Lagrange-Souriau 2-form writes:

\[
\sigma = \sigma_{\mu\nu} dX^\mu_i \wedge \delta X^\nu_0 + \frac{1}{2} \sum_{i,j=0}^{r-1} \tau_{ij}^\mu \delta X^\mu_i \wedge \delta X^\nu_j + \tau_\mu \delta X^\mu_0 \wedge d\tau \tag{4.1}
\]

where:

\[
\delta X^\mu_i \equiv dX^\mu_i - X^\mu_{i+1} d\tau \ (i = 0, ..., r - 1). \tag{4.2}
\]

The action of a reparametrization on \(S\) is constructed as follows: let \(f \in Diff(R)\); then we have

\[
\phi_f(\tau, X) = (f(\tau), X). \tag{4.3}
\]

The lift of \(\phi_f\) to \(J^r_1(S)\) is:

\[
\dot{\phi}_f(\tau, X_0, ..., X_r) = (f(\tau), X_0, g_1(\tau, X), ..., g_r(\tau, X)) \tag{4.4}
\]

where the functions \(g_i \ (i = 1, ..., r)\) are recursively constructed from:

\[
g_{i+1}^\mu = \frac{1}{f'} \frac{\delta g_i^\mu}{\delta \tau} \ (i = 0, ..., r - 1). \tag{4.5}
\]

Here:

\[
\frac{\delta}{\delta \tau} \equiv \frac{\partial}{\partial \tau} + \sum_{i=0}^{r-1} X_{i+1}^\mu \frac{\partial}{\partial X_i^\mu} \tag{4.6}
\]

and we have denoted for uniformity:

\[
g_0(\tau, X) \equiv X_0 \tag{4.7}
\]

Then we postulate reparametrization invariance as follows:

\[
(\dot{\phi}_f)^* \sigma = \sigma. \tag{4.8}
\]

4.2 We will illustrate the method for a physically interesting case: \(r = 3\) and \(M\) is the Minkowski space.

First, we list the closedness conditions for \(\sigma\) (see (2.17)-(2.27)):

\[
\frac{\partial \sigma_{\mu\nu}}{\partial X_3^\mu} = 0. \tag{4.9}
\]
\[
\frac{\delta \sigma_{\mu \nu}}{\delta \tau} - \frac{\partial \tau_{\mu}}{\partial X_3^\nu} + \tau_{\mu \nu}^{02} = 0. \tag{4.10}
\]

\[
\tau_{\mu \nu}^{12} = \sigma_{\nu \mu}. \tag{4.11}
\]

\[
\tau_{\mu \nu}^{22} = 0. \tag{4.12}
\]

\[
\frac{\partial \sigma_{\mu \nu}}{\partial X_0^\mu} - \frac{\partial \sigma_{\mu \rho}}{\partial X_0^\rho} + \frac{\partial \tau_{00}^{\mu \rho}}{\partial X_3^\mu} = 0. \tag{4.13}
\]

\[
\frac{\partial \sigma_{\mu \nu}}{\partial X_1^\mu} + \frac{\partial \tau_{01}^{\mu \rho}}{\partial X_3^\mu} = 0. \tag{4.14}
\]

\[
\frac{\partial \sigma_{\mu \nu}}{\partial X_2^\mu} + \frac{\partial \tau_{02}^{\mu \rho}}{\partial X_3^\mu} = 0. \tag{4.15}
\]

\[
\frac{\partial \tau_{ij}^{ij}}{\partial X_3^\rho} = 0 \quad (i, j = 1, 2). \tag{4.16}
\]

\[
\frac{\delta \tau_{00}^{\mu \nu}}{\delta \tau} + \frac{\partial \tau_{\nu}}{\partial X_0^\mu} - \frac{\partial \tau_{\mu}}{\partial X_0^\nu} = 0. \tag{4.17}
\]

\[
\frac{\delta \tau_{01}^{\mu \nu}}{\delta \tau} + \tau_{\mu \nu}^{00} - \frac{\partial \tau_{\mu}}{\partial X_1^\nu} = 0. \tag{4.18}
\]

\[
\frac{\delta \tau_{02}^{\mu \nu}}{\delta \tau} + \tau_{\mu \nu}^{01} - \frac{\partial \tau_{\mu}}{\partial X_2^\nu} = 0. \tag{4.19}
\]

\[
\frac{\delta \tau_{ij}^{ij}}{\delta \tau} + \tau_{ij}^{i-1 j} - \tau_{ij}^{j-1 i} = 0 \quad (i, j = 1, 2). \tag{4.20}
\]

\[
\frac{\partial \tau_{ij}^{ik}}{\partial X_k^\rho} + \frac{\partial \tau_{ij}^{jk}}{\partial X_1^\rho} + \frac{\partial \tau_{ij}^{ki}}{\partial X_2^\rho} = 0 \quad (i, j, k = 0, 1, 2). \tag{4.21}
\]

Next, we compute from (4.5) the functions \( g_i \) appearing in (4.4); we get:

\[
g_i^\mu = \sum_{j=1}^{3} g_i^j X_j^\mu \quad (i = 1, 2, 3) \tag{4.22}
\]

where:

\[
g_i^j = \frac{1}{(f')^i} \quad (i = 1, 2, 3) \tag{4.23}
\]

and

\[
g_2^1 = -\frac{f'''}{(f')^3}, \quad g_3^1 = \frac{3f'' - f'''f'}{(f')^5}, \quad g_3^2 = -\frac{3f'''}{(f')^4}. \tag{4.24}
\]

4.3 Finally, we impose Poincaré invariance in the target space. The action of this group on \( S \) is:

\[
\phi_{L,a}(\tau, X_0) = (\tau, LX_0 + a) \tag{4.25}
\]
with the lift to $E \subseteq J'_1(S)$:

$$\dot{\phi}_{L,a}(\tau, X_0, X_1, X_2, X_3) = (\tau, L X_0 + a, L X_1, L X_2, L X_3).$$  \hfill (4.26)

The Poincaré invariance is formulated as:

$$(\dot{\phi}_{L,a})^* \sigma = \sigma$$  \hfill (4.27)

and it is easily seen to be equivalent to the independence of the functions $\sigma_{\mu\nu}$, $\tau_{ij}$, and $\tau_\mu$ of the variable $X_0$ and also their Lorentz covariance.

4.4 As in Section 3 we concentrate on the analysis of the functions $\sigma_{\mu\nu}$. We have from 4.2 and 4.3 that these functions do not depend on $X_3$ and $X_0$ respectively, so they are functions of $\tau$, $X_1$ and $X_2$.

We get more conditions from the reparametrization invariance (4.8). Indeed the coefficient of the differential form $dX_3^\mu \wedge \delta X_0^\nu$ in this relation provides:

$$g_{33}^3 \sigma_{\mu\nu} \circ \dot{\phi}_f = \sigma_{\mu\nu}$$

or explicitly (see (4.4) and (4.22)-(4.24)):

$$\frac{1}{(f')^3} \sigma_{\mu\nu} \left( f(\tau), \frac{1}{f'} X_1, X_2 - \frac{f''}{f'} X_1 \right) = \sigma_{\mu\nu}(\tau, X_1, X_2).$$  \hfill (4.28)

If we take in (4.28) $f(\tau) = \tau + a$ ($a \in R$) we obtain that $\sigma_{\mu\nu}$ does not depen on $\tau$ so it is really a function of $X_1$ and $X_2$. If we take in (4.28) $f(\tau) = \lambda \tau$ ($\lambda \in R_+$) we get an homogeneity property:

$$\lambda^{-3} \sigma_{\mu\nu}(\lambda^{-1} X_1, \lambda^{-2} X_2) = \sigma_{\mu\nu}(X_1, X_2) \quad (\forall \lambda \in R_+)$$  \hfill (4.29)

and (4.28) simplifies to

$$\sigma_{\mu\nu}(X_1, X_2 - \alpha X_1) = \sigma_{\mu\nu}(X_1, X_2) \quad (\forall \alpha \in R).$$  \hfill (4.30)

To (4.29) and (4.30) one must add relations of the type (3.35) and (3.36) which are obtained in the same way:

$$\sigma_{\mu\nu} = \sigma_{\nu\mu}$$  \hfill (4.31)

and

$$\frac{\partial \sigma_{\mu\nu}}{\partial X_2^\rho} = \frac{\partial \sigma_{\mu\nu}}{\partial X_2^\nu}. \hfill (4.32)$$

4.5 The analysis of (4.28)-(4.32) proceeds as in section 3. First, from the Lorentz covariance of $\sigma_{\mu\nu}$ and (4.30) one has the following generic expression:

$$\sigma_{\mu\nu} = \delta_{\mu\nu} A + X_1_\mu X_1_\nu B + X_2_\mu X_2_\nu C + (X_1_\mu X_2_\nu + X_2_\mu X_1_\nu) D$$  \hfill (4.33)
with \(A, B, C\) and \(D\) smooth functions of the invariants:

\[
\zeta_1 \equiv X_1^2, \quad \zeta_2 = \equiv X_2^2, \quad \zeta_{12} \equiv X_1 \cdot X_2.
\] (4.34)

First one uses (4.32) and obtain:

\[
C = 2 \frac{\partial A}{\partial \zeta_2}
\] (4.35)

\[
D = \frac{\partial A}{\partial \zeta_{12}}
\] (4.36)

and:

\[
2 \frac{\partial B}{\partial \zeta_2} = \frac{\partial^2 A}{\partial \zeta_{12}^2}.
\] (4.37)

Next, one uses (4.30) and obtains in particular:

\[
A \circ \phi_\alpha = A
\] (4.38)

\[
B \circ \phi_\alpha + \alpha^2 C \circ \phi_\alpha - 2\alpha D \circ \phi_\alpha = B
\] (4.39)

where:

\[
\phi_\alpha(\zeta_1, \zeta_2, \zeta_{12}) \equiv (\zeta_1, \zeta_2 - 2\alpha \zeta_{12} + \alpha^2 \zeta_1, \zeta_{12} - \alpha \zeta_1).
\] (4.40)

Finally, from the homogeneity property (4.29) one obtains for \(A\):

\[
\lambda^{-3} A(\lambda^{-2} \zeta_1, \lambda^{-4} \zeta_2, \lambda^{-3} \zeta_{12}) = A(\zeta_1, \zeta_2, \zeta_{12}) \quad (\forall \lambda \in R_+).
\] (4.41)

This shows that for \(\zeta_1 \neq 1\), \(A\) must be of the form:

\[
A(\zeta_1, \zeta_2, \zeta_{12}) = |\zeta_1|^{-3/2} A_1 \left( |\zeta_1|^{-3/2} \zeta_{12}, |\zeta_1|^{-2} \zeta_2 \right)
\]

for some smooth function \(A_1\). But then (4.38) gives rather easily that in fact we have:

\[
A(\zeta_1, \zeta_2, \zeta_{12}) = |\zeta_1|^{-3/2} a \left( \frac{\zeta_1 \zeta_2 - \zeta_{12}^2}{\zeta_{12}^3} \right)
\] (4.42)

for some smooth function \(a\). The functions \(C\) and \(D\) are obtained from (4.35) and (4.36) respectively:

\[
C(\zeta_1, \zeta_2, \zeta_{12}) = 2|\zeta_1|^{-7/2} a' \left( \frac{\zeta_1 \zeta_2 - \zeta_{12}^2}{\zeta_{12}^3} \right)
\] (4.43)

\[
D(\zeta_1, \zeta_2, \zeta_{12}) = -2|\zeta_1|^{-7/2} \zeta_2 \zeta_1^{-1} a' \left( \frac{\zeta_1 \zeta_2 - \zeta_{12}^2}{\zeta_{12}^3} \right)
\] (4.44)
It remains to determine $B$ from (4.37) and (4.39). The result is:

$$
B(\zeta_1, \zeta_2, \zeta_{12}) = 2|\zeta_1|^{-11/2}\zeta_{12}^{1/2} \left( \frac{\zeta_1 \zeta_2 - \zeta_{12}^2}{\zeta_1^3} \right) - 2|\zeta_1|^{-3/2}\zeta_1^{-1} a \left( \frac{\zeta_1 \zeta_2 - \zeta_{12}^2}{\zeta_1^3} \right) + b_0|\zeta_1|^{-5/2}
$$

for some $b_0 \in \mathbb{R}$.

We stress once again that (4.42)-(4.45) have been derived only for $\zeta_1 \neq 1$. The condition of non-degeneracy and smoothness compel us to take as evolution spaces one of the following two possibilities:

$$
E^\eta \equiv \{(\tau, X_0, X_1, X_2, X_3)| \text{sign}(X_1^2) = \eta\} \quad (\eta = \pm).
$$

(4.46)

4.6 We use now the same trick as in Section 3 namely, we consider the Lagrangean:

$$
L_0(X_1, X_2) \equiv |X_1^2|^{1/2}F \left( \frac{X_2^2 - (X_1 \cdot X_2)^2}{(X_1^2)^3} \right)
$$

(4.47)

which is well defined on $E^\eta$ and take $F$ such that:

$$
a = -2F'.
$$

(4.48)

Then one can obtain after some computations that:

$$
\sigma_{\mu\nu} - (\sigma_{L_0})_{\mu\nu} = b_0|\zeta_2|^{-5/2}X_{1\mu}X_{1\nu}.
$$

(4.49)

This relation suggests to define as before:

$$
\sigma' \equiv \sigma - \sigma_{L_0}.
$$

(4.50)

Then we will have from (4.50):

$$
\sigma'_{\mu\nu} = b_0|\zeta_2|^{-5/2}X_{1\mu}X_{1\nu}.
$$

(4.51)

It is also not very hard to prove that $L_0$ verifies a relation of the type (2.49) with $\Lambda = 0$ for every transformation $\phi_f$ and $\phi_{L,a}$.

So we have:

$$
(\dot{\phi}_f)^*\sigma_{L_0} = \sigma_{L_0}
$$

(4.52)

and

$$
(\dot{\phi}_{L,a})^*\sigma_{L_0} = \sigma_{L_0}
$$

(4.53)

From (4.8) and (4.52) it follows that:

$$
(\dot{\phi}_f)^*\sigma' = \sigma'
$$

(4.54)

and from (4.27) and (4.53) that:

$$
(\dot{\phi}_{L,a})^*\sigma' = \sigma'
$$

(4.55)
4.7 The analysis performed above shows that the starting problem can be reduced to a simpler one, namely when:

\[ \sigma_{\mu \nu} = b_0 |\zeta_1^2|^{-5/2} X_{1\mu} X_{1\nu} \]

But a careful analysis of the reparametrization invariance (4.8) shows that in fact we must have \( b_0 = 0 \). So, in the end it remains to analyse the case:

\[ \sigma_{\mu \nu} = 0. \] (4.56)

We outline briefly this analysis. First, one uses the closedness conditions to prove that \( \tau^{11}_{\mu \nu} = \tau^{02}_{\nu \mu} = \text{const.} \). Then (4.8) compels us to take:

\[ \tau^{11}_{\mu \nu} = \tau^{02}_{\nu \mu} = 0. \] (4.57)

Next, one shows that \( \tau^{00}_{\mu \nu} = \text{const.} \). Antisymmetry and Lorentz covariance gives:

\[ \tau^{00}_{\mu \nu} = \kappa \varepsilon_{\mu \nu} \quad (n = 2) \] (4.58a)

\[ \tau^{00}_{\mu \nu} = 0 \quad (n > 2). \] (4.58b)

We also obtain from the closedness condition that \( \tau^{01}_{\mu \nu} \) depends only on \( X_1 \), is symmetric and verifies \( \frac{\partial \tau^{01}_{\mu \nu}}{\partial X_1^\rho} = \nu \leftrightarrow \rho \). Combining with the Lorentz covariance we get:

\[ \tau^{01}_{\mu \nu} = \delta_{\mu \nu} A(X_1^2) + 2 X_{1\mu} X_{1\nu} A'(X_1^2). \]

But (4.8) provides a homogeneity property on \( A \) so we end up with \( E = E^n \) and:

\[ \tau^{01}_{\mu \nu} = \delta_{\mu \nu} \text{msign}(X_1^2) |X_1^2|^{1/2} - X_{1\mu} X_{1\nu} m |X_1^2|^{-3/2} \quad (m \in \mathbb{R}). \] (4.59)

Finally, Lorentz covariance and the closedness condition determine \( \tau_{\mu} \) to be:

\[ \tau_{\mu} = \tau^{01}_{\mu \nu} X_2^\nu + \tau^{00}_{\mu \nu} X_1^\nu. \] (4.60)

It remains to note that \( \sigma = \sigma_{L_1} \) where:

\[ L_1(X_1) = -m |X_1^2|^{1/2} + \frac{1}{2} \tau^{00}_{\mu \nu} X_0^\mu X_1^\nu. \] (4.61)

So, in the notations of Subsection 4.6, \( \sigma = \sigma_L \) where the Lagrangian \( L \) has the structure:

\[ L(X_1, X_2) \equiv |X_1^2|^{1/2} F \left( \frac{X_1^2 X_2^2 - (X_1 \cdot X_2)^2}{(X_1^2)^3} \right) + \frac{1}{2} \tau^{00}_{\mu \nu} X_0^\mu X_1^\nu \] (4.62)

where, of course, the last contribution shows out \( \text{iff } n = 2 \).
Remarks: 1) The first contribution in (4.62) has been extensively investigated in [11], where one proves a weaker result, namely that $L$ is the generic form of a strictly invariant Lagrangian i.e. a Lagrangian verifying relations of the type (2.49) with $\Lambda = 0$ for $\phi_f$ and $\phi_{L,a}$. Our analysis proves a much stronger result, namely that every Lagrangian which is invariant up to a total derivative under these transformations (i.e. the transformations are Noetherian symmetries) is equivalent to a Lagrangian of the type (4.62).

2) If we take in (4.62) $\tau = X^0_0$ we reobtain (3.80).

5. Conclusions
We have proved that the main results of [1] extends to higher order Lagrangian theories and we have illustrated the method on the study of Lagrangian systems with groups of Noetherian symmetries.

It is plausible that one can extend these results to classical field theories using ideas from [12],[13].

In this way one would be able to obtain in a systematic way (using the homogeneous formalism) Lagrangians for extended objects of the same type as (4.62). Higher-order Lagrangians for extended objects have been already used in the literature [14],[15].

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