Lagrangianity for log extendable overconvergent $F$-isocrystals

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Abstract

In the framework of Berthelot’s theory of arithmetic $\mathcal{D}$-modules, we prove that Berthelot’s characteristic variety associated with a holonomic $\mathcal{D}$-modules endowed with a Frobenius structure has pure dimension. As an application, we get the lagrangianity of the characteristic variety of a log extendable overconvergent $F$-isocrystal.

Introduction

Let $\mathcal{V}$ be a complete discrete valued ring of mixed characteristic $(0, p)$, $\pi$ be a uniformizer, $K$ its field of fractions, $k$ its residue field which is supposed to be perfect. $X$ be a smooth formal $\mathcal{V}$-scheme (the topology is the $p$-adic one), $K$ its field of fractions, $k$ its residue field which is supposed to be perfect. A $k$-variety is a separated reduced scheme of finite type over $k$. We will...

Convention, notation of the paper

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1 Convention and preliminaries on filtered modules

We use here the terminology of Laumon in [Lau85 A.1]. For instance, a filtered ring \((D, D_i)\) is a ring \(D\), unitary, non necessary commutative, with an increasing filtration indexed by \(\mathbb{Z}\). If there is no risk of confusion, we will simply say “complete” for “ind-pro-complete”. In this section, \((D, D_i)\) will be a complete filtered ring such that \(\text{gr}(D, D_i)\) is a left and right noetherian ring.

**Definition 1.1.**

1. A filtered \((D, D_i)\)-module \((M, M_i)\) is said to be good if \((M_i)_{i \in \mathbb{Z}}\) is a good filtration (with the definition of good filtration given in [Lau85 A.1.0]). From the proposition [Lau85 A.1.1], a good filtered \((D, D_i)\)-module is complete and is a \(D\)-module of finite type.

2. Following [Gro61 2.1.2], we say that a filtered \((D, D_i)\)-module \((M, M_i)\) is free if \((M, M_i)\) is a direct sum of some filtered \((D, D_i)\)-modules of the form \((D, D_i)(n)\) (where \(n\) is some integer and \((D, D_i)(n) := (D, D_i+n)\)).

**1.2 (Strictness).** We remark that Lemma [Lau83 3.3.2] is still valid in our context if we require that our filtered \((D, D_i)\)-modules are complete. For instance, let \(0 \rightarrow (M', M'_i) \rightarrow (M, M_i) \xrightarrow{f} (M'', M''_i) \rightarrow 0\) be a sequence of good filtered \((D, D_i)\)-modules such that \(g \circ f = 0\). Then \(0 \rightarrow \text{gr}M' \rightarrow \text{gr}M \rightarrow \text{gr}M'' \rightarrow 0\) is exact if and only if, for any \(i \in \mathbb{Z}\), the sequences \(0 \rightarrow M'_i \rightarrow M_i \rightarrow M''_i \rightarrow 0\) are exact. When this property is satisfied, we will say that \(0 \rightarrow (M', M'_i) \rightarrow (M, M_i) \rightarrow (M'', M''_i) \rightarrow 0\) is an “exact sequence of good filtered \((D, D_i)\)-modules”.

Let \(u: (M, M_i) \rightarrow (N, N_i)\) be a morphism of good filtered \((D, D_i)\)-modules. From [Lau85 A.1.1.2], we notice that the filtered \((D, D_i)\)-modules \(\ker u\) and \(\text{Coker} u\) are good. We introduce the following notion of strict morphism of good filtered \((D, D_i)\)-modules: a morphism \(u: (M, M_i) \rightarrow (N, N_i)\) of good filtered \((D, D_i)\)-modules is strict if the canonical morphism \(\text{Coim} u \rightarrow \text{Im} u\) is an isomorphism of filtered \((D, D_i)\)-modules. The morphism \(u\) is strict if and only if, for any \(i \in \mathbb{Z}\), we have \(u(M_i) = u(M) \cap N_i\). Moreover, we have the exact sequences of good filtered \((D, D_i)\)-modules:

\[
0 \rightarrow \ker u \rightarrow (M, M_i) \rightarrow \text{Coim} u \rightarrow 0,
0 \rightarrow \text{Im} u \rightarrow (N, N_i) \rightarrow \text{Coker} u \rightarrow 0.
\]  

(1.2.1)

By applying the functor \(\text{gr}\) to the exact sequences[1.2.1] we get that \(u\) is strict if and only if \(\text{ker} \text{gr}(u) = \text{gr} \ker(u)\) and \(\text{coker} \text{gr}(u) = \text{gr} \text{coker}(u)\). If \(u\) is strict then we have also \(\text{im} \text{gr}(u) = \text{gr} \text{im}(u)\). With this definition of strictness, we check that the category of good filtered \((D, D_i)\)-modules is exact (see the definition in [Lau85 1.0]).

**Remark 1.3.** Let \((M, M_i)\) be a filtered \((D, D_i)\)-module. We remark that \((M, M_i)\) is a good filtered \((D, D_i)\)-module if and only if there exists a strict epimorphism of the form \(u: (L, L_i) \rightarrow (M, M_i)\), where \((L, L_i)\) is a free filtered \((D, D_i)\)-module of finite type.

**Notation 1.4 (Localisation).** Let \(f\) be a homogeneous element of \(\text{gr}D\). We denote by \((D, D_i)_{(f)}\) the complete filtered ring of \((D, D_i)\) relatively to \(S_1(f) := \{ f^n, n \in \mathbb{N} \} \subset \text{gr}D\) (see the definition after [Lau85 Corollaire A.2.3.4]).

Let \((M, M_i)\) be a good filtered \((D, D_i)\)-module. We put

\[
(M_{(f)}, M_{(f)}i) := (D_{(f)}, D_{(f)}i) \otimes_{(D, D_i)} (M, M_i),
\]  

(1.4.1)

the localized filtered module of \((M, M_i)\) with respect to \(S_1(f)\). We remind that \((M_{(f)}, M_{(f)}i)\) is also a good filtered \((D_{(f)}, D_{(f)}i)\)-module (see [Lau85 A.2.3.6]) and \(\text{gr}M_{(f)} \xrightarrow{\text{gr}f} \text{gr}(D_{(f)}) \otimes_{\text{gr}D} \text{gr}M\) (see [Lau85 A.1.1.3]).

The results and proofs of Malgrange in [Mal76 IV.4.2.3] (we can also find the proof in the book [HTT08 D.2.2]) can be extended without further problem in the context of complete filtered rings.
Lemma 1.5. Let $(M, M_i)$ be a good filtered $(D, D_i)$-module. Then there exists some free filtered $(D, D_i)$-modules of finite type $(L_n, L_{n,i})$ with $n \in \mathbb{N}$ and strict morphisms of good filtered $(D, D_i)$-modules $(L_{n+1}, L_{n+1,i}) \to (L_n, L_{n,i})$ and $(L_0, L_{0,i}) \to (M, M_i)$ such that $L_\ast \to M$ is a resolution of $M$ (in the category of $D$-modules).

We call such a resolution $(L_\ast, L_{\ast,i})$ a “good resolution” of $(M, M_i)$.

Proof. This is almost the same as [Mal76, IV.4.2.3.2]. For the reader, we remind the construction: with the remark [1.3] there exists a strict epimorphism of good filtered $(D, D_i)$-modules of the form $\phi_0: (L_0, L_{0,i}) \to (M, M_i)$, with $(L_0, L_{0,i})$ a free filtered $(D, D_i)$-module of finite type. Let $(M_1, M_{1,i})$ be the kernel of $\phi_0$ (in the category of good filtered $(D, D_i)$-modules): see [1.2]. Since $(M_1, M_{1,i})$ is good, there exists a strict epimorphism of the form $\phi_1: (L_1, L_{1,i}) \to (M_1, M_{1,i})$, with $(L_1, L_{1,i})$ a free filtered $(D, D_i)$-module of finite type. Hence, the morphism $(L_1, L_{1,i}) \to (L_0, L_{0,i})$ is strict. We go on similarly.

Remark 1.6. Let $(L_\ast, L_{\ast,i})$ be a good resolution of $(M, M_i)$. Then $gr(L_\ast, L_{\ast,i})$ is a resolution of $gr(M, M_i)$ by free $gr(D, D_i)$-modules of finite type (use the properties of strictness given in [1.2]).

Lemma 1.7. Let $K^\ast$ be a complex of abelian groups. Let $(F_iK^\ast)_{i \in \mathbb{Z}}$ be an increasing filtration of $K^\ast$. We put

$$F_iH'(K^\ast) := \text{Im}(H'(F_iK^\ast) \to H'(K^\ast)).$$  \hspace{1cm} (1.7.1)

Then $gr_i(H'(K^\ast))$ is a subquotient of $H'(gr_i(K^\ast))$.

Proof. For instance, we can look at the last seven lines of the proof of [HTT08, D.2.4] (or also at Malgrange’s description of the corresponding spectral sequence in [Mal76 IV.4.2.3.2]).

Proposition 1.8. Let $(M, M_i)$ be a good filtered $(D, D_i)$-module. Let $(N, N_i)$ be a filtered $(D, D_i)$-module. For any integer $r$, there exists a canonical filtration $F \text{Ext}^r_D(M, N)$ satisfying the following properties

1. $\text{Ext}^r_D(M, N) = \bigcup_{i \in \mathbb{Z}} F_i \text{Ext}^r_D(M, N)$.
2. $gr^r \text{Ext}^r_D(M, N)$ is a subquotient of $\text{Ext}^r_{grD}(grM, grN)$.
3. Suppose $(N, N_i) = (D, D_i)$. The canonical filtration $(F_i \text{Ext}^r_D(M, D))_{i \in \mathbb{Z}}$ of the right $D$-module $\text{Ext}^r_D(M, D)$ is a good filtration. In particular, $0 = \bigcap_{i \in \mathbb{Z}} F_i \text{Ext}^r_D(M, D)$. Moreover, we have the implication $\text{Ext}^r_{grD}(grM, grD) = 0 \Rightarrow \text{Ext}^r_D(M, D) = 0$.

Proof. From [1.5] and with its definition, there exists a good resolution $(L_\ast, L_{\ast,i})$ of $(M, M_i)$. We put $K^\ast := \text{Hom}_D(L_\ast, N)$. Since $L_\ast$ is a resolution of $M$ by projective $D$-modules, we get $H'(K^\ast) = \text{Ext}^1_D(M, N)$.

Let $F_iK_n$ be the subset of the elements $\phi$ of $\text{Hom}_D(L_n, N)$ such that, for any integer $j \in \mathbb{Z}$, $\phi(L_{n,j}) \subset N_{n-j}$. Since $L_n$ is a $D$-module of finite type, we get $\bigcup_{i \in \mathbb{Z}} F_iK^n = K^n$. With the canonical induced filtration on $H'(K^\ast) = \text{Ext}^r_D(M, N)$ (see [1.7]), this yields the first property. Since $L_n$ is a free filtered $(D, D_i)$-modules of finite type, we check that the canonical morphism $grK^n \to \text{Hom}_{grD}(grL_n, grN)$ is an isomorphism. Since $grL_\ast$ is a resolution of $grM$ by projective $grD$-modules, we get $H'(grK^\ast) = \text{Ext}^r_{grD}(grM, grN)$. This implies the second point by using Lemma [1.7].

When $(N, N_i) = (D, D_i)$, the filtration $F_iK^n$ of the right $D$-module $K^n$ is a good filtration. We denote by $d_n: K^n \to K^{n+1}$ the canonical morphisms. From [Laux85, A.1.1.2], the induced filtrations on $\ker d_n$ and next on $\ker d_n/\text{Im}d_{n-1}$ (induced from the surjection $\ker d_n \to \ker d_n/\text{Im}d_{n-1}$) are good. We notice that this filtration on $\ker d_n/\text{Im}d_{n-1} = H^n(K^\ast)$ is the same as that defined at [1.7.1], which is the first assertion of the third point. With the second point, this yields the rest of the third point.
2 Purity of the characteristic variety of a holonomic $F\mathcal{D}$-modules

Lemma 2.1. Let $X$ be an affine smooth variety over $k$, $\mathcal{D} := \Gamma(X, \mathcal{D}_X^{(0)})$, $(\mathcal{D}_i)_{i \in \mathbb{N}}$ be its filtration by the order of the operators, $\mathcal{T}$ be an homogeneous element of $\text{gr}\mathcal{D}$. Let $(\mathcal{M}, M_i)$ and $(\mathcal{N}, N_i)$ be two good filtered $(\mathcal{D}(\mathcal{T}), \mathcal{D}(\mathcal{T}))$-modules and $r$ be an integer.

1. We have $\text{codimExt}^r_{\text{gr}\mathcal{T}}(\text{gr}\mathcal{M}, \text{gr}\mathcal{N}) \geq r$.

2. If $r < \text{codimgr}\mathcal{M}$ then $\text{Ext}^r_{\text{gr}\mathcal{N}}(\text{gr}\mathcal{M}, \text{gr}\mathcal{D}(\mathcal{T})) = 0$.

Proof. By construction (see [Lau85 A.2]), we get $\text{gr}\mathcal{D}(\mathcal{T}) \to (\text{gr}\mathcal{D} \mathcal{T})$. Then, this is well known (e.g. see [HTT08 D.4.4]).

2.2 (Localisation and $\pi$-adic completion). Let $X$ be an affine smooth $\mathcal{V}$-formal scheme, $X_n$ be the reduction of $X$ modulo $\pi^{n+1}$. We put $D := \Gamma(X, \mathcal{D}_X^{(0)})$ and $D_n := \Gamma(X, \mathcal{D}_X^{(0)}/\pi^n)$, these rings are canonically filtered by the order of the differential operators; we denote by $(D, D_n)$ and $(D_n, D_n)$ the (ind-pro) complete filtered rings. Let $f$ be an homogeneous element of $\text{gr} D$ and $f_n$ be its image in $\text{gr} D_n$. With the notation of [Abe14 2.3], we get the canonical isomorphism of (ind-pro) complete filtered ring

$$(D[f], D[f], i) \otimes_{\mathcal{V}} \mathcal{V}/\pi^{n+1} \sim (D[f_n], D_n).$$

(2.2.1)

We put $\widehat{D}_f[i]$ (be careful that this notation is slightly different from that of [1.4] as the $\pi$-adic completion of $D[f]$, i.e. $\widehat{D}_f[i] := \lim D[f]/\pi^{n+1} D[f] \to \lim D_n[i]$. Using Corollary [Lau85 A.1.1.1] and [Ber96 3.2.2.(iii)], we get from the isomorphism the noetherianity of $\widehat{D}_f[i]$.

Finally, when there is no confusion with the notation for any coherent $\widehat{D}_f$-module (resp. coherent $\widehat{D}_Q$-module) $M$ (resp. $N$), we set (by default in this new context) $M[f] := \widehat{D}_f[i] \otimes \hat{\mathcal{D}} M$ (resp. $N[f] := \widehat{D}_f[i] \otimes \hat{\mathcal{D}} N$).

Lemma 2.3. With the notation of 2.2 the homomorphism $\widehat{D} \to \widehat{D}_f$ is flat.

Proof. This is a consequence of [Ber96 3.2.3.(vii)], [Lau85 A.2.3.4.(ii)] and 2.2.1.

Remark 2.4. With the notation of 2.2 let $M$ be a coherent $\widehat{D}_f$-module. We put $M_n := M/\pi^{n+1} M$. From [Ber02 5.2.3.(iv)] there exists a good filtration $(M_n)_{n \in \mathbb{N}}$ of $M_n$ indexed by $\mathbb{N}$. We recall (see notation [1.4]) that we get a good filtered $(D_n[f_n], D_n[f_n], i)$-module by putting $(M_n[f_n], M_n[f_n], i) := (D_n[f_n], D_n[f_n], i) \otimes_{D_n, D_n} (M_n, M_n, i)$. Moreover, since $\widehat{D}_f[i]/\pi^{n+1} \to D_n[i]$ (use 2.2.1), then

$$M_n[f]/\pi^{n+1} M_n[f] \sim M_n[f_n].$$

(2.4.1)

From [Ber96 3.2.3.(v)], since $M[f]$ is a $\widehat{D}_f$-module of finite type, then $M[f]$ is complete for the $\pi$-adic topology. Hence, using 2.4.1 we get the canonical isomorphism of $\widehat{D}_f$-modules $M[f] \sim \lim M_n[f_n]$.

Lemma 2.5. We keep notation 2.2. Let $\mathcal{N}$ be a coherent $\widehat{D}_X^{(0)}$-module, $\text{Car}(0)(\mathcal{N})$ its characteristic variety of level 0 (see the definition in [Ber02 5.2.5]). We put $N := \Gamma(X, \mathcal{N})$. The following assertions are equivalent

1. $D(f_0) \cap \text{Car}(0)(\mathcal{N}) = \emptyset$.

2. $N[f] = 0$.

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Proof. From [Ber96, 3.4.5], there exists a coherent $\overline{D}$-module without $p$-torsion $M$ such that $M_{Q} \simto N$. Since the extension $\overline{D} \rightarrow \overline{D}[f]$ is flat (see (2.5)), we get that $M_{f}$ is also without $p$-torsion ($p$ is in the center of $\overline{D}$ and $\overline{D}[f]$). This yields that $N_{f} = 0$ if and only if $M_{f} = 0$. Let $\overline{M} := M / \pi M$. From [2.4.1] we have $M_{f} / \pi M M_{f} \simto \overline{M}_{f}$. Hence, $M_{f}$ is 0 if and only if $\overline{M}_{f} = 0$ (e.g. see [Ber96, 3.2.2.(ii)]). From [Ber02, 5.2.3.(iv)], there exists a good filtration $(\overline{M}_{f})_{f \in \mathbb{N}}$ of $\overline{M}$ indexed by $\mathbb{N}$. From the remark 2.4 this induces canonically the (ind-pro) complete $(\overline{D}_{f}, \overline{D}_{f}[i])$-module $(\overline{M}_{f}, \overline{D}_{f}[i])$. Since $(\overline{D}_{f}[i])$ is (ind-pro) complete, then the equalities $\overline{M}_{f} = 0$ and $gr(\overline{D}_{f}[i], \overline{D}_{f}[i]) = 0$ are equivalent. Also, $(gr\overline{M})_{f} = 0$ if and only if $D(f) \cap gr(\overline{M}) = 0$. Since $(gr\overline{M})_{f} \simto gr(\overline{M}_{f})$ (see [Lau85, A.1.1.3]) and since by definition $\text{Car}^{0}(N) = supp(gr(\overline{M}, \overline{M}))$, we conclude the proof. □

Remark 2.6. Let $A = \oplus_{i \in \mathbb{N}} A_{i}$ be a graded ring. Let $I$ be a graded ideal. Let $a_{1}, \ldots, a_{r}$ be some homogeneous generators of $I$. We notice that $|\text{Spec}A| \backslash V(I) = \bigcup_{i=1}^{r} D(a_{i})$.

The following proposition is the analogue of [Kas77] 2.11:

Proposition 2.7. Let $X$ be a smooth $V$-formal scheme. Let $N$ be a coherent $\widehat{D}_{X,Q}$-module and $V$ be an irreducible component of codimension $r$ of $\text{Car}^{0}(N)$, the characteristic variety of level 0 of $N$ (see [Ber02, 5.2.5]). Then, $\text{Car}^{0}(\mathcal{E}xt_{\overline{D}_{X,Q}}^{r}(N, \widehat{D}_{X,Q}))$ contains $V$.

Proof. We follow the proof of [Kas77, 2.11]: first, we can suppose $X$ affine with local coordinates. We set $D := \Gamma(X, \widehat{D}_{X,Q}^{0})$, $N := \Gamma(X, N)$, $\overline{D} := \Gamma(X, \widehat{D}_{X,Q}^{0})$. Let $M$ be a coherent $\overline{D}$-module without $p$-torsion such that $M_{Q} \simto N$. Let $\overline{M} := M / \pi M$. From [Ber02, 5.2.3.(iv)], there exists a good filtration $(\overline{M}_{f})_{f \in \mathbb{N}}$ of $\overline{M}$ indexed by $\mathbb{N}$. By definition, we have $\text{Car}^{0}(N) = supp(gr(\overline{M}, \overline{M}))$ (we recall that this is independent on the choice of the good filtration). Let $\eta$ be the generic point of $V$. From [Mat80, 7.D and 10.B.i)], the irreducible components of $supp(gr\overline{M})$ are of the form $V(J)$ with $J$ a homogeneous ideal. Let $Z$ be the union of the irreducible components of $supp(gr\overline{M})$ which do not contain $\eta$. Then, we get from the remark 2.6 that there exists a homogeneous element $f \in gr D$ such that $\eta \in D(f)$ and $D(f) \cap Z = 0$ (in other words, $D(f) \cap \text{Car}^{0}(N) = D(f) \cap V \neq 0$).

Now, suppose absurdly that $\eta \not\in \text{Car}^{0}(\mathcal{E}xt_{\overline{D}_{X,Q}}^{r}(N, \widehat{D}_{X,Q}))$. Using the same arguments as above, there exists a homogeneous element $g \in gr D$ such that $\eta \in D(g)$ and $D(g) \cap \text{Car}^{0}(\mathcal{E}xt_{\overline{D}_{X,Q}}^{r}(N, \widehat{D}_{X,Q})) = \emptyset$. We put $h = fg$.

Hence, we have $\eta \in D(h)$ and $D(h) \cap \text{Car}^{0}(\mathcal{E}xt_{\overline{D}_{X,Q}}^{r}(N, \widehat{D}_{X,Q})) = D(h) \cap V = 0$. By the same arguments as above, there exists a coherent (right) $\overline{D}_{X,Q}$-module, from Theorem A and B of Berthelot (see [Ber96, 3.3]), we get the equality $\Gamma(X, \mathcal{E}xt_{\overline{D}_{X,Q}}^{r}(N, \widehat{D}_{X,Q})) = \mathcal{E}xt_{\overline{D}_{X,Q}}^{r}(N, \overline{D}_{X,Q})$. From 2.3 this implies $\mathcal{E}xt_{\overline{D}_{X,Q}}^{r}(N, \overline{D}_{X,Q}) \simto \mathcal{E}xt_{\overline{D}_{X,Q}}^{r}(N, \overline{D}_{X,Q}) = 0$. Since the extension $\overline{D}_{Q} \rightarrow \overline{D}[i]_{Q}$ is flat (see 2.3), we get $\mathcal{E}xt_{\overline{D}_{Q}}^{r}(N, \overline{D}[i]_{Q}) \simto \mathcal{E}xt_{\overline{D}_{Q}}^{r}(N, \overline{D}[i]_{Q}) = 0$.

2) a) Since $\text{Car}^{0}(N) = supp(gr\overline{M})$, then $D(h) \cap \text{Car}^{0}(N) = supp((gr\overline{M})_{h})$. Since we have also $D(h) \cap \text{Car}^{0}(N) = D(h) \cap V$, then in particular we get $\text{Codim}(gr\overline{M})_{h} = r$. Since $(gr\overline{M})_{h} = gr(M[h])$, then from 2.1 for any $i < r$ we obtain $\mathcal{E}xt_{\overline{D}_{h}}^{r}(gr(M[h]), gr(\overline{M}_{h})) = 0$. From 1.3, this yields that for $i < r$, $\mathcal{E}xt_{\overline{D}_{h}}^{r}(gr(M[h]), gr(\overline{M}_{h})) = 0$.

On the other hand, from 2.1 we get for any $i > r$ the inequality $\text{Codim}(\mathcal{E}xt_{\overline{D}_{h}}^{r}(gr(M[h]), gr(\overline{M}_{h}))) > r$. Hence, by reducing $D(h)$ if necessary (use again the remark 2.6), for any $i > r$ we get $\mathcal{E}xt_{\overline{D}_{h}}^{r}(gr(M[h]), gr(\overline{M}_{h})) = 0$ and then $\mathcal{E}xt_{\overline{D}_{h}}^{r}(M[h], \overline{D}_{h}[i]) = 0$. To sum up, we have found an homogeneous element $h \in gr D$ such that $\eta \in D(h)$ and for $i \neq r$, $\mathcal{E}xt_{\overline{D}_{h}}^{r}(gr(M[h]), gr(\overline{M}_{h})) = 0$.

2) b) Now, since $M[h]$ is without p-torsion, $\mathcal{R}Hom_{\overline{D}_{h}}(M[h], \overline{D}_{h}[i]) \otimes_{\overline{D}_{h}} \overline{D}_{h}[i] \simto \mathcal{R}Hom_{\overline{D}_{h}}(gr(M[h]), gr(\overline{M}_{h}))$. From the exact sequence of universal coefficients (e.g. see the beginning of the proof of [Vir00, 1.5.8]), we get the inclusion $\mathcal{E}xt_{\overline{D}_{h}}^{r}(M[h], \overline{D}_{h}[i]) \otimes_{\overline{D}_{h}} \overline{D}_{h}[i] \hookrightarrow \mathcal{E}xt_{\overline{D}_{h}}^{r}(gr(M[h]), gr(\overline{M}_{h}))$. Hence, for any $i \neq r$, from the step 2) a) of the proof, we obtain
the vanishing Ext\[^i\]_{\hat{D}_{[\mathfrak{h}]}}(M_{[\mathfrak{h}]}, \hat{D}_{[\mathfrak{h}]}) \otimes_{\hat{D}_{[\mathfrak{h}]}} \mathcal{T}_{[\mathfrak{h}]0} = 0. By using [Ber96 3.2.2.(ii)], since Ext\[^i\]_{\hat{D}_{[\mathfrak{h}]}}(M_{[\mathfrak{h}]}, \hat{D}_{[\mathfrak{h}]}) is a coherent \(\hat{D}_{[\mathfrak{h}]}\)-module, for \(i \neq r\) we have Ext\[^i\]_{\hat{D}_{[\mathfrak{h}]}}(M_{[\mathfrak{h}]}, \hat{D}_{[\mathfrak{h}]}) = 0 and then Ext\[^i\]_{\hat{D}_{[\mathfrak{h}],[Q]}(N_{[\mathfrak{h}],[Q]}(\hat{D}_{[\mathfrak{h}],[Q]) = 0 (because \(\hat{D}_{[\mathfrak{h}]}) \rightarrow \hat{D}_{[\mathfrak{h}],[Q}\) is flat).

3) From steps 1) and 2), we have checked that \(\mathbb{R}Hom_{\hat{D}_{[\mathfrak{h}],[Q]}(N_{[\mathfrak{h}],[Q]}(\hat{D}_{[\mathfrak{h}],[Q]) = 0. By using the biduality isomorphism (see [Vir00 I.3.6] and notice that \(N_{[\mathfrak{h}]}) is a perfect complex because so is \(N\) and because the extension \(\hat{D}_{[\mathfrak{h}]) \rightarrow \hat{D}_{[\mathfrak{h}],[Q}\) is flat), we get \(N_{[\mathfrak{h}]}) = 0, which is absurd following Lemma 2.3 because \(\eta \in D(\mathfrak{h})).\)

\[\text{Theorem 2.8.}\] Let \(X\) be a smooth \(\mathcal{V}\)-formal scheme. Let \(r\) be an integer, \(N\) be a coherent \(\hat{D}_{X,[Q]}\) module such that \(\mathbb{E}xt\[^i\]_{\hat{D}_{X,[Q]}(N, \hat{D}_{X,[Q]) = 0}\) for any \(s \neq r\). Then, the characteristic variety \(\text{Car}^{(0)}(N)\) of \(N\) is purely of codimension \(r\).

\[\text{Proof.}\] If \(V\) is an irreducible component of \(\text{Car}^{(0)}(N)\) of codimension \(s\), then from 2.7 we get \(\mathbb{E}xt\[^s\]_{\hat{D}_{X,[Q]}(N, \hat{D}_{X,[Q]) \neq 0}\) since it contains \(V\). Hence \(s = r\).

\[\text{Corollary 2.9.}\] Let \(X\) be a smooth integral \(\mathcal{V}\)-formal scheme of dimension \(d\). Let \(N \neq 0\) be a holonomic \(F \cdot \hat{D}_{X,[Q}\) module. Then, the characteristic variety \(\text{Car}(N)\) of \(N\) is purely of codimension \(d\).

\[\text{Proof.}\] The is a consequence of Virrion’s holonomicity characterization (see Theorem [Vir00 III.4.2] and of Theorem 2.8)

3 Lagrangianity for log-extendable overconvergent isocrystal

\[\text{Notation 3.1.}\] Let \(f : X \rightarrow Y\) be a morphism of smooth varieties. We denote by \(T^*X\) (resp. \(T^*Y\)) the cotangent space of \(X\) (resp. \(Y\)), \(\rho_f : X \times_Y T^*Y \rightarrow T^*X\) the canonical morphism. We set \(T^*_X Y := \rho_f^{-1}(T^*_X Y)\). Similar to as in the very beginning of [HTT08 E.2], we define the canonical 1-form \(\alpha_X\) of \(T^*X\) that we will denote by \(\alpha_X \in \Omega^1_{T^*X}: f_{1, \ldots, t_d}\) are local coordinates of \(X\), we get local coordinates \(t_1, \ldots, t_d, \xi_1, \ldots, \xi_d\) of \(T^*X\), where \(\xi_i\) is the element associated with \(\partial_i\) the derivation with respect to \(t_i\). In this local coordinate system, we get \(\alpha_X = \sum_{i=1}^d \xi_i dt_i\).

\[\text{Definition 3.2.}\] Let \(X\) be a smooth formal scheme over \(\mathcal{V}\) and \(X\) be its special fiber.

1. Let \(E\) be a subvariety of \(T^*X\). The restriction \(\alpha_X\) on \(E\) is denoted by \(\alpha_E\). As before Proposition [Kas77 2.3], we say that \(E\) is isotropic if there exists an open dense subset \(U\) of \(E\) such that \(\alpha_X|U = 0\). When \(X\) is purely of dimension \(d\), we say that \(E\) is Lagrangian if \(E\) is isotropic and purely of codimension \(d\). In general, we say that \(E\) is Lagrangian is the restriction of \(E\) on the irreducible components of \(T^*X\) are Lagrangian in the previous sense. For instance, if \(Z\) is a smooth closed survariety of \(T^*X\), then \(\mathcal{Z}^d\) is Lagrangian.

2. Let \(E\) be a coherent \(F \cdot \hat{D}_{X,[Q}\) module. We say that \(E\) is Lagrangian (resp. isotropic) if \(\text{Car}(E)\) is Lagrangian (resp. isotropic). From 2.9 \(\mathcal{E}\) is Lagrangian if and only if \(\mathcal{E}\) is holonomic and isotropic.

3. Let \(E\) be a complex of \(F \cdot \hat{D}_{log}(\hat{D}_{X,[Q}\) ). We put \(\text{Car}(E) := \cup_n \text{Car}(\mathcal{H}^n(E))\). We say that \(E\) is Lagrangian if for any integer \(n\), the characteristic variety of \(\mathcal{H}^n(E)\) is Lagrangian, i.e. if \(\text{Car}(E)\) is Lagrangian.

3.3. Let \(X\) be an affine smooth variety over \(k\) admitting local coordinates \(t_1, \ldots, t_d\). We denote by \(X^{(m)}\) the base change of \(X\) by the \(m\)th power of Frobenius of \(S := \text{Spec} k\), by \(F^m : X \rightarrow X^{(m)}\) the relative Frobenius morphism. From the equalities [Ber96 1.1.3.1, 2.2.4.(iii)], we compute that for any \(j < m\) we have in \(\mathcal{D}_{X/S}\) the equality \((\partial_t)^{(p^m j)^{m}} p = 0). From [Ber96 1.1.3.(ii), 2.2.4.(iii)], for any \(k\) and \(l\), we compute that there exists \(u \in \mathbb{Z}_p\) such that \((\partial_t)^{(p^m k)^{m}} l = u\partial_t)^{(p^m k)^{m}}\). Let \(\tilde{\xi}^{(m)}_t\) be the class of \(\partial_t)^{(p^m k)^{m}}\) in \((\mathcal{G} \mathcal{D}_{X/S})_{red}. Hence, with the formula [Ber96 2.2.5.1], we check that \((\mathcal{G} \mathcal{D}_{X/S})_{red} = \{(\xi^{(m)}_t)^k \mathcal{O}_X\).
From [Ber02, 5.2.2], the canonical morphism \((\gr D^{(m)}_{X/S})_{\red} \to F^m \gr D^{(0)}_{X^m/S}\) induced by the morphism \(D^{(m)}_{X/S} \to F^m D^{(0)}_{X^m/S}\) of left \(D^*_X/S\)-modules is an isomorphism. We remark that this is consequence of above computations and of [Ber00, 2.2.4.3] which states that the image of \(\delta_{i}^{e-k/m}\) via \(D^{(m)}_{X/S} \to F^m D^{(0)}_{X^m/S}\) is \(1 \otimes \delta_{i}^{e-k/m}\) if \(p^m\) divides \(k\) and otherwise is \(0\) and then \(\xi_{p}(m)\) is sent to \(1 \otimes \xi_{p}\), where \(\xi_{p}\) is the class of \(\delta_{i}\) in \(\gr D^{(0)}_{X^m/S}\) isomorphic to \(\gr D^{(0)}_{X/S}\).

**Proposition 3.4.** Let \(X\) be an integral separated smooth formal \(\mathcal{V}\)-scheme, \(Z\) be a strict normal crossing divisor of \(X\). We put \(X_\# := (X/Z)\) the corresponding smooth log formal \(\mathcal{V}\)-scheme. Let \(\mathcal{G}\) be a coherent \(D^\times_{X, \mathcal{Q}}\)-module which is also a locally projective \(\mathcal{O}_{X, \mathcal{Q}}\)-module of finite type (i.e., following [Car09] 4.9 and 4.15, \(\mathcal{G}\) is a convergent isocrystal on \(X_\#\)). Let \(E := (\mathcal{O})(\mathcal{G})\) be the induced isocrystal on \(X / Z\) convergent along \(Z\). We suppose that \(\mathcal{G}\) is endowed with a Frobenius structure. Let \(Z_1, \ldots, Z_r\) be the irreducible components of \(Z\). For any subset \(I\) of \(\{1, \ldots, r\}\), we put \(Z_I := \cap_{i \in I} Z_i\). Then we have the inclusion

\[ |\text{Car}(E)| \subset \bigcup_{I \subset \{1, \ldots, r\}} T^\bullet Z_I X, \tag{3.4.1}\]

where \(T^\bullet Z \equiv X\) is the standard notation (see (2.7)). In particular, with the remark [3.2] and since we know that \(E\) is holonomic (see (CT12)), this implies that \(|\text{Car}(E)|\) is Lagrangian.

**Proof.** This is local so we can suppose \(X\) affine with local coordinates \(t_1, \ldots, t_d\) such that \(Z_i = V(t_i)\) for \(i = 1, \ldots, r\). We proceed by induction on \(r\). For \(r = 0\) (i.e., \(Z\) is empty), this is already known (see the example after [Ber02, 5.2.7]). Suppose now \(r \geq 1\). From [Car09, 4.12], there exists a coherent \(\hat{\mathcal{D}}^{(0)}_{X, \mathcal{Q}}\)-module \(\mathcal{G}^{(0)}\) which is also \(\mathcal{O}_X\)-coherent and such that \(\mathcal{G}^{(0)}(\mathcal{G}) \rightarrow \mathcal{G}\). We can suppose that \(\mathcal{G}^{(0)}\) has no \(p\)-torsion (indeed, from [Ber02, 3.3], the subsheaf of \(\mathcal{G}(\mathcal{G})\) of \(p\)-torsion elements is a coherent \(\hat{\mathcal{D}}^{(0)}_{X, \mathcal{Q}}\)-module and also a coherent \(\mathcal{O}_X\)-module). With the notation [Car09, 5.1], \(\mathcal{G}(\mathcal{V})\) is also a coherent \(D^\times_{X, \mathcal{Q}}\)-module and a locally projective \(\mathcal{O}_{X, \mathcal{Q}}\)-module of finite type. Hence, from [Car09, 4.14], we get \(\hat{\mathcal{D}}^{(m)}_{X, \mathcal{Q}} \otimes \hat{\mathcal{D}}^{(0)}_{X, \mathcal{Q}} \rightarrow \mathcal{G}(\mathcal{G})\). We denote by \(\mathcal{H}(m)\) the quotient of \(\hat{\mathcal{D}}^{(m)}_{X, \mathcal{Q}} \otimes \hat{\mathcal{D}}^{(0)}_{X, \mathcal{Q}} \mathcal{G}(\mathcal{G})\) by its \(p\)-torsion part. The latter isomorphism implies that \(\mathcal{H}(m) \rightarrow \mathcal{G}(\mathcal{G})\). By using [Ber02, 3.4.5] and [Car09, 4.12], it follows that \(\mathcal{H}(m)\) is isogeneous to a coherent \(\hat{\mathcal{D}}^{(m)}_{X, \mathcal{Q}}\)-module which is also \(\mathcal{O}_X\)-coherent. Since \(\Gamma(X, \mathcal{O}_X)\) is noetherian and \(\mathcal{H}(m)\) has no \(p\)-torsion, we get that \(\Gamma(X, \mathcal{H}(m))\) is a \(\Gamma(X, \mathcal{O}_X)\)-module of finite type. Since \(\mathcal{H}(m)\) is a coherent \(\hat{\mathcal{D}}^{(m)}_{X, \mathcal{Q}}\)-module, this yields that \(\mathcal{H}(m)\) is also \(\mathcal{O}_X\)-coherent (this is a log-variation of [Car06b, 2.2.13] and its check is identical).

We denote by \(M(m) := \hat{\mathcal{D}}^{(m)}_{X, \mathcal{Q}} \otimes \hat{\mathcal{D}}^{(0)}_{X, \mathcal{Q}} \mathcal{H}(m)\) and by \(N(m)\) the quotient of \(M(m)\) by its \(p\)-torsion part. We put \(\overline{M}(m) := M(m) / \pi M(m) \), \(\overline{N}(m) := N(m) / \pi N(m)\). Since we have the epimorphism \(M(m) \rightarrow \overline{M}(m)\), from [Ber02, 5.2.4.4(i)] and its notation, we get \(\text{Car}(M(m)) \subset \text{Car}(M(m))\). From [Car09, 4.14], we get the second isomorphism \(D^\dagger_{X, \mathcal{Q}} \otimes \hat{\mathcal{D}}^{(m)}_{X, \mathcal{Q}} \mathcal{N}(m) \rightarrow D^\dagger_{X, \mathcal{Q}} \otimes \hat{\mathcal{D}}^{(m)}_{X, \mathcal{Q}} \mathcal{G}(\mathcal{G}) \rightarrow D^\dagger_{X, \mathcal{Q}} \otimes \hat{\mathcal{D}}^{(m)}_{X, \mathcal{Q}} \mathcal{G}(\mathcal{G})\). Since \(\mathcal{G}\) has a Frobenius structure, it follows from [Car09, 5.24(ii)] and [CT12, 2.2.9] that we have the isomorphism \(D^\dagger_{X, \mathcal{Q}} \otimes \hat{\mathcal{D}}^{(m)}_{X, \mathcal{Q}} \mathcal{G}(\mathcal{G}) \rightarrow \mathcal{E}\). Hence, \(\overline{M}(m) := \hat{\mathcal{D}}^{(m)}_{X, \mathcal{Q}} \otimes \hat{\mathcal{D}}^{(0)}_{X, \mathcal{Q}} \mathcal{H}(m)\) is \(\mathcal{O}_X\)-coherent and we check that \(\overline{M}(m)\) is a good filtration of \(\overline{M}(m)\) (see [Ber02, 5.2.3]). By definition, this implies \(\text{Car}(M(m)) = \text{Supp}(\oplus_{i \in \mathbb{N}} (\overline{M}(m) / \overline{M}(m))\)). For \(d \geq i > r\), we remark that \(\delta_i^{e-k/m} \in D^{(m)}_{X, \mathcal{Q}}\). Hence, we get in \(T^{l}(m) X = \text{Spec}(\gr D^{(m)}_{X, \mathcal{Q}})_{\red}\) the inclusion \(\text{Car}(m) (\overline{M}(m)) \subset \cap_{l=i+1}^{d} V(\xi_{l}(m))\) (see the notation of (3.3). Since, modulo the homeomorphism between \(T^{l}(m) X\) and \(T^{l} X\) of [Ber02] 5.2.2.1).
Let \( Z_{\text{min}} := \bigcap_{i=1}^r Z_i \). Then, we get \( \tau^{-1}(\text{Car}(E)) \subset \tau^{-1}(\bigcap_{i=1}^r V(\xi_i)) = T_{\text{min}}^* X \), where \( \tau : Z_{\text{min}} \times_X T^* X \to T^* X \) is the canonical immersion.

For any \( i = 1, \ldots, r \), we put \( X_i := \bar{X} \setminus Z_i \). From the induction hypothesis, we get the first inclusion \( \text{Car}(E|X_i) \subset \bigcup_{i \in I} T_{\xi_i}^* X_i \), where the union runs through subsets \( I \) of \( \{1, \ldots, r\} \) which do not contain \( i \). Since \( (X_i) \) is a open covering of \( \bar{X} \setminus Z_{\text{min}} \) and since \( \tau^{-1}(\text{Car}(E)) \subset T_{\text{min}}^* X \), we conclude.

\[ \square \]

**Remark 3.5.** Let \( E' \to E \to E'' \to E'[1] \) be an exact triangle of \( F \cdot \mathcal{D}^-_{\text{coh}}(\mathcal{D}^+_X \mathcal{O}_X) \). Using [Ber02, 5.2.4.(i) and 5.2.7], we get the equality \( \text{Car}(E) = \text{Car}(E') \cup \text{Car}(E'') \). This yields that \( E' \) and \( E'' \) are Lagrangian if and only if \( E \) is Lagrangian. Hence, to check the Lagrangianity of overholonomic \( F \)-complexes, we reduce by devissage to the case of overcoherent \( F \)-isocrystals (see [Car06a]). One can hope to get the Lagrangianity of any overcoherent \( F \)-isocrystals by using Kedlaya’s semistable reduction theorem ([Ked11]) and the latter result [5.4]. But this is not so clear and it remains an open question.

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