Estimates of Certain Exit Probabilities for $p$-Adic Brownian Bridges

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Abstract
For each prime $p$, a diffusion constant together with a positive exponent specify a Vladimirov operator and an associated $p$-adic diffusion equation. The fundamental solution of this pseudo-differential equation gives rise to a measure on the Skorokhod space of $p$-adic valued paths that is concentrated on the paths originating at the origin. We calculate the first exit probabilities of paths from balls and estimate these probabilities for the Brownian bridges.

Keywords Exit probabilities · Brownian motion · $p$-Adic diffusion · Brownian bridges

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1 Introduction

The study of diffusion equations in the $p$-adic setting began with Vladimirov’s introduction in [21] of a pseudo-differential operator acting on certain complex valued
functions with domain in the $p$-adic numbers that is, in many respects, an analog of the classical Laplacian. He further investigated in [22] the spectral properties of this operator, now known as the Vladimirov operator. Both Taibleson in [19] and Saloff-Coste in [16] wrote even earlier about pseudo-differential operators in the context of local fields, and Saloff-Coste studied such operators in [17] in the more general setting of local groups. Kochubei gave in [14] the fundamental solution to the $p$-adic analog of the diffusion equation, with the operator introduced by Vladimirov replacing the Laplace operator. Albeverio and Karwowski further investigated diffusion in the $p$-adic setting in [2], constructing a continuous time random walk on $\mathbb{Q}_p$, computing its transition semigroup and infinitesimal generator, and showing among other things that the associated Dirichlet form is of jump type. Varadarajan further explored diffusion in the non-Archimedean setting in [20], discussing an analog to the diffusion equation where the functions have domains contained in $[0, \infty) \times S$, where $S$ is a finite dimensional vector space over a division ring which is finite dimensional over a local field of arbitrary characteristic. In the current work, we calculate the exit probabilities of $p$-adic Brownian paths from balls centered at the origin. Theorem 4.7 gives estimates for the exit probabilities of the $p$-adic Brownian bridges and is the principle novelty of the paper. It is significant at least in part for its potential application in estimating expected values of functionals on path space. The intuitive nature of these estimates given the exact calculation of the exit probabilities for $p$-adic Brownian motion belies the nontrivial technical challenge in obtaining these estimates.

The current work finds immediate application in an upcoming study of Adelic path measures and both diffusion and Schrödinger operators in the Adelic setting. Diffusion in the $p$-adic setting has a multitude of applications. For example, ultrametricity arises in the theory of complex systems and many references cited by [13, Chapter 4] study an area where estimates on exit probabilities should be significant, for example, the works [3–6] of Avetisov, Bikulov, Kozyrev, and Osipov dealing with $p$-adic models for complex systems. The estimates for the $p$-adic Brownian bridges are particularly significant to the study of Feynman–Kac integrals in the $p$-adic setting since these integrals involve integration of functionals on path spaces with respect to measures on the Brownian bridges. Digeres, Varadarajan, and Varadhan in [11] studied the spectral properties of the Hamiltonians associated with a large class of quantum system using path integral methods and proved the convergence of finite dimensional real quantum systems to their continuum limits. Albeverio, Gordon, and Khrennikov studied finite approximation of quantum systems in the setting of locally compact abelian groups in [1], obtaining some of the results of [11] in this very general setting. In [7], Bakken, Digeres, and Weisbart improved upon some of the results of [1], but in the restricted setting of configuration spaces that are local fields. The results of [7] relied on path integral methods, mirroring the approach in [11] to obtain uniformity in the estimates. Due to the importance of path integral methods in the study of quantum systems, we should expect application of the current work to the further study of quantum systems in the $p$-adic setting. In an upcoming article, we apply the current work to the study of Brownian motion in the adelic setting.

Fix $p$ to be a prime number. Section 2 reviews the basic properties of the $p$-adic numbers, denoted henceforth by $\mathbb{Q}_p$. It also gives an overview of diffusion in the $p$-adic setting. Take the stochastic process $X$ to be the $p$-adic Brownian motion associated
with a diffusion constant $\sigma$ and a Vladimirov operator with exponent $b$. The sample paths for $X$ are in the probability space $(D([0, \infty) : Q_p), P)$, where $D([0, \infty) : Q_p)$ is the Skorokhod space of paths valued in $Q_p$ and $P$ is a probability measure on $D([0, \infty) : Q_p)$ that is concentrated on the paths originating at 0. Section 3 presents the main equalities of the paper, exact calculations of the exit probabilities of a $p$-adic Brownian motion from balls of fixed radii containing 0. Finally, Sect. 4 provides estimates of the first exit probabilities from balls for the Brownian bridges conditioned to originate at a fixed point in the given ball and to be at a fixed point in the same ball at a later point in time.

2 Brownian Motion in the $p$-Adic Setting

Gouvêa’s book [12] provides an accessible introduction to $p$-adic analysis. For a further study of $p$-adic numbers and analysis in the $p$-adic setting, see the book [18] of Ramakrishnan and Valenza, and the earlier work [24] of Weil. A standard reference in the field of $p$-adic mathematical physics, the book [23] of Vladimirov, Volovich, and Zelenov offers an excellent introduction to $p$-adic analysis and both quantum theory and diffusion in the $p$-adic setting as does Kochubei’s book [15], which provides a greater focus on $p$-adic diffusion. Much of the current section is taken from [8] with only minor modification and is included here for the reader’s convenience and to improve some of the prior exposition. Billingsley’s book [9] is a standard reference in probability theory that supplements the relevant theory that this section summarizes.

2.1 Basic Facts about $Q_p$

Let $p$ be a fixed prime number. Denote by $| \cdot |$ the absolute value on $Q_p$ and denote, respectively, by $B_k(x)$ and $S_k(x)$ the ball and the circle of radii $p^k$, the compact open sets

$$B_k(x) = \{ y \in Q_p : |y - x| \leq p^k \} \quad \text{and} \quad S_k(x) = \{ y \in Q_p : |y - x| = p^k \}.$$ 

Denote by $\mathbb{Z}_p$ the ring of integers, the unit ball in $Q_p$. Note that for any integer $a$, the set $p^{-a}\mathbb{Z}_p$ is the ball $B_a(0)$. The field $Q_p$ is a locally compact abelian group under the usual addition and is a totally disconnected topological space. Let $\mu$ be the Haar measure on the additive group $Q_p$, normalized to be 1 on $\mathbb{Z}_p$. Define for each $x$ in $Q_p$ the unique function

$$a_x : \mathbb{Z} \rightarrow \{0, 1, \ldots, p - 1\}$$

that has the property that

$$x = \sum_{k \in \mathbb{Z}} a_x(k)p^k.$$
Note that there is a natural number $N$ with the property that $k$ is larger than $N$ implies that $a_x(-k)$ is 0. Define by $\{x\}$ the rational number

$$\{x\} = \sum_{k<0} a_x(k) p^k,$$

where the given sum is necessarily a finite sum. With $S^1$ denoting the multiplicative group of unit complex numbers, denote by $\chi$ the rank 0 character

$$\chi : \mathbb{Q}_p \to S^1 \text{ by } \chi(x) = e^{2\pi \sqrt{-1} \{x\}}.$$ 

The field $\mathbb{Q}_p$ is self-dual, meaning that for any character $\phi$ on $\mathbb{Q}_p$, there is an $\alpha$ in $\mathbb{Q}_p$ so that for all $x$ in $\mathbb{Q}_p$,

$$\phi(x) = \chi(\alpha x).$$

The Fourier transform $\mathcal{F}$ on $L^2(\mathbb{Q}_p)$ is the unitary extension to all of $L^2(\mathbb{Q}_p)$ of the operator initially defined for each $f$ in $L^1(\mathbb{Q}_p) \cap L^2(\mathbb{Q}_p)$ by

$$(\mathcal{F}\{)(x) = \int_{\mathbb{Q}_p} \chi(-xy)f(y) \, d\mu(y).$$

The inverse Fourier transform $\mathcal{F}^{-1}$ acts on any $f$ in $L^1(\mathbb{Q}_p) \cap L^2(\mathbb{Q}_p)$ by

$$(\mathcal{F}^{-1}f)(y) = \int_{\mathbb{Q}_p} \chi(xy)f(x) \, d\mu(x).$$

Denote by $SB(\mathbb{Q}_p)$ the Schwartz–Bruhat space of complex valued, compactly supported, locally constant functions on $\mathbb{Q}_p$. This set of functions is the $p$-adic analog of the set of complex valued, compactly supported, smooth functions on $\mathbb{R}$ with the important difference that, unlike in the real case, $SB(\mathbb{Q}_p)$ is invariant under the Fourier transform.

### 2.2 Path Spaces

Let $\mathcal{S}$ be a Polish space. Denote by $F([0, \infty) : \mathcal{S})$ the set of all functions mapping $[0, \infty)$ to $\mathcal{S}$. Refer to this space as the space of all paths on $\mathcal{S}$ with domain $[0, \infty)$. For each $t$ in $[0, \infty)$, suppose that $\mathcal{S}_t$ is equal to $\mathcal{S}$. For any finite sequence $(t_1, \ldots, t_k)$ of points in $[0, \infty)$, denote by $\pi_{t_1, \ldots, t_k}$ the canonical projection

$$\pi_{t_1, \ldots, t_k} : \prod_{t \in [0, \infty)} \mathcal{S}_t \to S^k.$$ 

These projections are the length $k$ projections. The set of paths with $k$ conditions is the set of all inverse images of Borel subsets of $S^k$ under length $k$ projections. The set
of cylinder sets of finite type is the union over all \( k \) in the natural numbers of the sets of paths with \( k \) conditions, and the set of cylinder sets is the \( \sigma \)-algebra generated by the cylinder sets of finite type. The simple cylinder sets do not form an algebra, but they do form a \( \pi \)-system that generates the cylinder sets.

If a premeasure on the set of cylinder sets of finite type with paths valued in a Polish space is consistent, then the Kolmogorov Extension Theorem guarantees the existence an extension of the premeasure to a measure on the cylinder sets of \( F([0, \infty) : S) \). Analytical investigations of path spaces often require specialization to more a restrictive space. Define for each \( t \) in \([0, \infty)\), the random variable \( X_t \) on the probability space \((F([0, \infty) : S), P)\) by

\[
X_t(\omega) = \omega(t) \quad \text{with} \quad \omega \in D([0, \infty) : S).
\]

The stochastic process \( X \) is a function defined by

\[
X : t \mapsto X_t. \tag{2}
\]

If \( X \) satisfies certain moment estimates due to Chentsov in [10], then it has a version in the set of Skorokhod paths \( D([0, \infty) : S) \), the set of cádlág functions from \([0, \infty)\) to \( S \) equipped with the Skorokhod metric. The diffusion process we will study will have sample paths in this space.

### 2.3 Diffusion in the \( Q_p \) Setting

Fix a positive real number \( b \). Define the multiplication operator \( M \) to act on \( SB(Q_p) \) by

\[
(Mf)(x) = |x|^b f(x).
\]

The pseudo-Laplace operator \( \tilde{\Delta} \) with exponent \( b \) acts on \( SB(Q_p) \) by

\[
(\tilde{\Delta} f)(x) = (\mathcal{F}^{-1} M \mathcal{F} \{ \}) (x). \tag{3}
\]

This operator is densely defined and the Fourier transform of a multiplication operator, hence an essentially self-adjoint operator from \( SB(Q_p) \) to \( L^2(Q_p) \). Denote once again by \( \tilde{\Delta} \) the unique self-adjoint extension of the operator given by (3) to its maximal domain in \( L^2(Q_p) \). Extend \( \tilde{\Delta} \) to act on any complex valued function

\[
f : R_+ \times Q_p \to C
\]

with the property that \( f(t, \cdot) \) is in \( D(\tilde{\Delta}) \) for each positive \( t \) by fixing \( t \) and viewing \( f(t, \cdot) \) as a function only of \( Q_p \). Denote this extension by \( \Delta \), the Vladimirov operator with exponent \( b \). Define similarly the Fourier and inverse Fourier transforms on functions on \( R_+ \times Q_p \) that for each positive \( t \) are square integrable over \( Q_p \) by computing
the given transform of the function for fixed positive $t$. Fix $\sigma$ to be a positive real number and refer to it as a diffusion constant. The pseudo-differential equation

$$
\frac{df(t, x)}{dt} = -\sigma \Delta f(t, x)
$$

has as its fundamental solution the function

$$
\rho(t, x) = \left( \mathcal{F}^{-1} e^{-\sigma |\cdot|^2} \right)(x).
$$

A minor modification of the more general arguments of [20] shows that the function $f(t, x)$ is a probability density function that gives rise to a probability measure $P$ on $D([0, \infty) : \mathbb{Q}_p)$ that is concentrated on the set of paths originating at 0. Define the probability of a cylinder set of finite type in the following way. For each non-negative $t$, let $X_t$ be the function on $D([0, \infty) : \mathbb{Q}_p)$ given by (2). For any strictly increasing finite sequence of time points $(t_1, \ldots, t_k)$ in $(0, \infty)$ and any finite sequence of Borel sets $(U_0, \ldots, U_k)$ of $\mathbb{Q}_p$, if $U_0$ contains 0, then

$$
P(X_0 \in U_0 \cap X_{t_1} \in U_1 \cap \cdots \cap X_{t_k} \in U_k)
= \int_{U_1} \cdots \int_{U_k} \rho(t_1, x_1) \rho(t_2 - t_1, x_2 - x_1) \cdots \rho(t_k - t_{k-1}, x_k - x_{k-1})
\, d\mu(x_k) \cdots d\mu(x_1).
$$

If 0 is not in $U_0$, then

$$
P(X_0 \in U_0 \cap X_{t_1} \in U_1 \cap \cdots \cap X_{t_k} \in U_k) = 0.
$$

This premeasure on the simple cylinder sets extends to the measure on $D([0, \infty) : \mathbb{Q}_p)$. Denote once again and henceforth by $P$ this probability measure. Define the stochastic process $X$ as in (2) to be the function

$$
X : I \times D([0, \infty) : \mathbb{Q}_p) \to \mathbb{Q}_p \text{ by } (t, \omega) \mapsto X_t(\omega) = \omega(t).
$$

The probability measure $P$ on $D([0, \infty) : \mathbb{Q}_p)$ gives full measure to paths originating at 0. For each $x$ in $\mathbb{Q}_p$, define $P_x$ to be the probability measure given by the same density function that defines $P$ but conditioned to give full measure to the paths originating at $x$. If $A$ is a cylinder set, then

$$
P_x(A) = P(A - x).
$$

For each $y$ in $\mathbb{Q}_p$ and each positive $T$, the arguments of [20] guarantee the existence of the probability measures concentrated on the $p$-adic Brownian bridges, namely, the measures $P_{T, x, y}$ which are given by the measures $P_x$ conditioned so that paths almost surely take value $y$ at time $T$. These conditioned measures form a continuous family of probability measures depending on the starting and ending points, as [20] discusses.
in its more general setting but with the diffusion constant $\sigma$ restricted to be equal to 1. There is no obstruction to allowing for a more general diffusion constant, which \cite{8} discusses.

### 3 First Exit Probabilities for $p$-Adic Path Spaces

This section presents an exact calculation of the probability that a $p$-adic Brownian path remains within a ball until time $T$. Such an event is the complement of a first exit event occurring within the specified time interval. This calculation allows for estimates of the same quantity in the context of the Brownian bridges. Denote by $\alpha$ the quantity

$$\alpha = 1 - \frac{p^b - 1}{p^{b+1} - 1}. \tag{8}$$

Given a measurable subset $A$ of $\mathbb{Q}_p$ and a function $f$ that is integrable over $A$, compress notation by writing

$$\int_A f(x) \, dx = \int_A f(x) \, d\mu(x)$$

and retain this notation throughout this and the next section. Let $X$ be the stochastic process given by (6). Define by $||X||_T$ the value

$$||X||_T = \sup_{0 \leq t \leq T} |X_t|. \tag{9}$$

**Theorem 3.1**  For any non-negative real number $T$ and integer $a$,

$$P(||X||_T \leq p^a) = e^{-\sigma \alpha T p^{-ab}}.$$

**Proof** The non-Archimedean property of the absolute value on $\mathbb{Q}_p$ guarantees the equality of the sets $A_1$ and $A_2$ where

$$A_1 = \{(x_1, x_2, \ldots, x_n) : |x_1| \leq p^a, |x_2| \leq p^a, \ldots, |x_n| \leq p^a\}$$

and

$$A_2 = \{(x_1, x_2, \ldots, x_n) : |x_1| \leq p^a, |x_2 - x_1| \leq p^a, \ldots, |x_n - x_{n-1}| \leq p^a\}.$$

For any increasing finite sequence $(t_1, \ldots, t_n)$ in $[0, T]$, the equality of $A_1$ and $A_2$ implies that

$$P\left(\max_{t_j} (|X_{t_1}|, \ldots, |X_{t_n}|) \leq p^a\right) = P\left(\max_{t_j} (|X_{t_1}|, |X_{t_1} - X_{t_2}|, \ldots, |X_{t_n} - X_{t_{n-1}}|) \leq p^a\right). \tag{9}$$
For each natural number \( j \) in \([1, N]\), denote by \( t_j \) the number
\[
  t_j = \frac{jT}{N}.
\]
Define the random variable \( X_0 \) to be identically 0. For each \( j \), the increments \( X_{t_j} - X_{t_{j-1}} \) are independent and identically distributed, implying that
\[
P\left( \max_{t_j} \left( \left| X_{t_1} \right|, \ldots, \left| X_{t_n} \right| \right) \leq p^a \right) = \prod_{1 \leq j \leq N} P\left( \left| X_{t_1} \right| \leq p^a \cap \cdots \cap \left| X_{t_n} \right| \leq p^a \right)
= P\left( \left| X_{t_1} \right| \leq p^a \right)^N = P\left( \left| X_{\frac{T}{N}} \right| \leq p^a \right)^N.
\] (10)

Define the function \( B \) on the non-negative real numbers by
\[
B(t) = P\left( |X_t| \leq p^a \right).
\]
The twice continuous differentiability in \( t \) of \( B \) implies that
\[
B\left( \frac{T}{n} \right) = 1 + \frac{B'(0)T}{n} + O\left( \frac{1}{n^2} \right).
\]
and so
\[
\lim_{n \to \infty} B\left( \frac{T}{n} \right)^n = e^{TB'(0)}.
\]
The right continuity of the sample paths together with (9) and (10) guarantees that
\[
P\left( ||X||_T \leq p^a \right) = \lim_{n \to \infty} P\left( \max_j \left( \left| X_{\frac{T}{n}} \right|, \left| X_{\frac{2T}{n}} \right|, \ldots, \left| X_{\frac{jT}{n}} \right|, \ldots, \left| X_{\frac{T}{n}} \right| \right) \leq p^a \right)
= \lim_{n \to \infty} B\left( \frac{T}{n} \right)^n
= e^{TB'(0)}.
\] (11)

Denote, respectively, by \( B_r \) and \( S_r \) the sets \( B_r(0) \) and \( S_r(0) \) and by \( 1_r \) the characteristic function on \( B_r \). A straightforward modification of the arguments in [20] shows that, for positive \( t \), the density function \( \rho(t, x) \) for the random variable \( X_t \) satisfies the equality
\[
\rho(t, x) = \sum_{r \in \mathbb{Z}} e^{-\sigma tp^r b} \int_{S_r} (xy) \, dy
= \sum_{r \in \mathbb{Z}} \left( e^{-\sigma tp^b} - e^{-\sigma tp^{(r+1)b}} \right) \int_{B_r} (xy) \, dy
= \sum_{r \in \mathbb{Z}} \left( e^{-\sigma tp^b} - e^{-\sigma tp^{(r+1)b}} \right) p^r 1_{-r}(x).
\] (12)
Use equation (12) for $\rho$ to obtain the equality

$$B(t) = P(|X_t| \leq p^a) = \int_{B_a} \rho(t, x) \, dx$$
$$= \sum_{r \in \mathbb{Z}} \left( e^{-\sigma t p^b} - e^{-\sigma t p^{(r+1)b}} \right) p^r \int_{B_a} 1_{-r}(x) \, dx$$
$$= e^{-\sigma t p^{-ab}} + \sum_{r \leq -a-1} p^{a+r} \left( e^{-\sigma t p^b} - e^{-\sigma t p^{(r+1)b}} \right).$$ \hspace{1cm} (13)

The partial sums of the infinite sum (13) as well as the partial sums of the derivatives of the terms of (13) converge uniformly to continuous functions of time, justifying a term by term differentiation of the infinite sum. Term by term differentiation of (13) implies that

$$B'(0) = -\sigma p^{-ab} + \sigma \sum_{r \leq -a-1} p^{a+r} \left( p^{(r+1)b} - p^b \right)$$
$$= -\sigma p^{-ab} + \sigma p^a (p^b - 1) \sum_{r \leq -a-1} p^{r(1+b)}$$
$$= -\sigma p^{-ab} + \sigma p^a (p^b - 1) p^{-(a+1)(1+b)} \cdot \frac{1}{1 - p^{-(1+b)}}$$
$$= -\sigma p^{-ab} \left(1 - \frac{p^b - 1}{p^{1+b} - 1}\right) = -\sigma \alpha p^{-ab}.$$

\hspace{1cm} (14)

Equalities (11) and (14) together imply that

$$P \left( ||X||_T \leq p^a \right) = e^{-\sigma \alpha T p^{-ab}}. \quad \square$$

A change of variables together with (7) implies the following further result that for every non-negative real number $T$ and $x$ in $Q_p$,

$$P_x \left( ||X - x||_T \leq p^a \right) = e^{-\sigma \alpha T p^{-ab}}.$$

\hspace{1cm} (15)

Every point of a ball in $Q_p$ is the center of the ball, which implies the following corollary to Theorem 3.1.

**Corollary** For every non-negative real number $T$, for any $x$ in $Q_p$ and for any $x'$ in $B_a(x)$,

$$P_x \left( ||X - x'||_T \leq p^a \right) = e^{-\sigma \alpha T p^{-ab}}.$$
4 First Exit Probabilities for the Brownian Bridges

For any positive \( t \), denote by \( D_{t,x,y}((0, \infty) : Q_p) \) the set of all paths in the \( Q_p \) valued Skorokhod space on \([0, \infty)\) that start at \( x \) at time 0 and are at \( y \) at time \( t \), the set of \( p \)-adic Brownian bridges with the given endpoints. The measure \( P_{t,x,y} \) is the measure \( P_x \) conditioned so that \( D_{t,x,y}(I : Q_p) \) has full measure. Let \( r \) henceforth denote an index variable that varies over the set of integers and let \( a \) be an integer. The probabilities associated with first exit times for \( p \)-adic Brownian paths imply a similar inequality for the Brownian bridges, in particular, that if \( y \) is in \( p^{-a} \mathbb{Z}_p \) and \( t \) is in \((0, T]\), then

\[
P_{t,x,y}(||X - x||_T \leq p^a) \geq P_x(||X - x||_T \leq p^a).
\]

The proof requires several calculations that the lemmata below present. To compress notation, recall that \( p^{-a} \mathbb{Z}_p \) is the set \( B_a \) and use the latter notation for a ball henceforth.

**Lemma 4.1** Suppose that \( \rho \) is the fundamental solution to (4). For all positive \( t \),

\[
\int_{B_a} \rho(t, x) \, dx = p^a \int_{B-a} e^{-\sigma t|x|^b} \, dx.
\]

**Proof** Integrate \( \rho(t, \cdot) \) over \( B_a \) using the simplified expression for \( \rho \) given by (12) to obtain

\[
\int_{B_a} \rho(t, x) \, dx = \sum_{r \in \mathbb{Z}} \left( e^{-\sigma tp^r b} - e^{-\sigma tp^{r+1} b} \right) p^r \int_{B_a} 1_{-r}(x) \, dx
\]

\[
= \sum_{r \leq -a} \left( e^{-\sigma tp^r b} - e^{-\sigma tp^{r+1} b} \right) p^r p^a + \sum_{r > -a} \left( e^{-\sigma tp^r b} - e^{-\sigma tp^{r+1} b} \right)
\]

\[
= e^{-\sigma tp^{-ab}} + \left( e^{-\sigma tp^{-(a+1)b}} - e^{-\sigma tp^{-ab}} \right) p^{-1}
\]

\[
+ \left( e^{-\sigma tp^{-(a+2)b}} - e^{-\sigma tp^{-(a+1)b}} \right) p^{-2} + \ldots
\]

\[
= e^{-\sigma tp^{-ab}} (1 - p^{-1}) + e^{-\sigma tp^{-(a+1)b}} p^{-1} (1 - p^{-1})
\]

\[
+ e^{-\sigma tp^{-(a+2)b}} p^{-2} (1 - p^{-1}) + \ldots
\]

\[
= p^a \left( \int_{S-a} e^{-\sigma t|x|^b \, dx} + \int_{S-a-1} e^{-\sigma t|x|^b \, dx} + \int_{S-a-2} e^{-\sigma t|x|^b \, dx} + \ldots \right)
\]

\[
= p^a \int_{B-a} e^{-\sigma t|x|^b \, dx}.
\]

\[\square\]

**Lemma 4.2** Suppose that \( t \) and \( t' \) are both positive real numbers. For all \( z \) in \( Q_p \),

\[
\int_{Q_p} \chi(zy) e^{-\sigma t'|y|^b} \left( \int_{B_a} e^{-\sigma t|y+w|^b} \, dw \right) \, dy
\]

\[
> \int_{Q_p} \chi(zy) e^{-\sigma (t+t')|y|^b} \, dy \int_{B_a} e^{-\sigma t|w|^b} \, dw.
\]
Proof  Decompose the integral over $Q_p$ into a sum of integrals over disjoint circles to obtain the equalities

\[
\int_{Q_p} \chi(z) e^{-\sigma |y|^b} \left( \int_{B_a} e^{-\sigma t |y+w|^b} dw \right) dy = p^a \sum_{r > a} \int_{S_r} \chi(z) e^{-\sigma |y|^b} \left( \int_{B_a} e^{-\sigma t |y+w|^b} dw \right) dy
\]

\[
= \sum_{r > a} e^{-\sigma t p r^b} \int_{S_r} \chi(z) \left( \int_{B_a} e^{-\sigma t |y+w|^b} dw \right) dy
\]

\[
+ \sum_{r \leq a} e^{-\sigma t p r^b} \int_{S_r} \chi(z) \left( \int_{B_a} e^{-\sigma t |y+w|^b} dw \right) dy
\]

\[
= p^a \sum_{r > a} e^{-\sigma (t+t') p r^b} \left( \int_{B_r} \chi(z) dy - \int_{B_{r-1}} \chi(z) dy \right)
\]

\[
+ \sum_{r \leq a} e^{-\sigma t' p r^b} \int_{B_r} e^{-\sigma t |y|^b} \left( \int_{B_r} \chi(z) dy - \int_{B_{r-1}} \chi(z) dy \right)
\]

\[
= p^a \sum_{r > a} \left( e^{-\sigma (t+t') p r^b} - e^{-\sigma (t+t') p(r+1)^b} \right) \int_{B_r} \chi(z) dy - p^a e^{-\sigma t p r^b} \int_{B_r} \chi(z) dy
\]

\[
+ \sum_{r \leq a} \left( e^{-\sigma t' p r^b} - e^{-\sigma t' p(r+1)^b} \right) \int_{B_r} \chi(z) dy \int_{B_a} e^{-\sigma t |w|^b} dw
\]

\[
+ e^{-\sigma t p a b} \int_{B_a} \chi(z) dy \int_{B_a} e^{-\sigma t |w|^b} dw = *.
\]

The estimate

\[
e^{-\sigma t p a b} < p^a \int_{B_a} e^{-\sigma t |w|^b} dw < 1,
\]

and (16) together imply that

\[
* > \sum_{r > a} \left( e^{-\sigma (t+t') p r^b} - e^{-\sigma (t+t') p(r+1)^b} \right) \int_{B_r} \chi(z) dy \int_{B_a} e^{-\sigma t |w|^b} dw
\]

\[
+ \sum_{r \leq a} \left( e^{-\sigma t' p r^b} - e^{-\sigma t' p(r+1)^b} \right) \int_{B_r} \chi(z) dy \int_{B_a} e^{-\sigma t |w|^b} dw.
\]
For any \( r \) in \( \mathbb{Z} \),

\[
g(s) = \int_{pr^b}^{p(r+1)b} e^{-sx} \, dx = e^{-sp^b} - e^{-sp(r+1)b}
\]

is decreasing in the variable \( s \) since

\[
g'(s) = \int_{pr^b}^{p(r+1)b} -xe^{-sx} \, dx < 0.
\]

The fact that \( g \) is decreasing in \( s \) implies that

\[
* > \sum_{r \in \mathbb{Z}} \left( e^{-\sigma(t+t')p^b} - e^{-\sigma(t+t')p(r+1)b} \right) \int_{B_r} \chi(zy) \, dy \int_{B_a} e^{-\sigma|w|^b} \, dw
\]

\[
= \sum_{r \in \mathbb{Z}} e^{-\sigma(t+t')p^b} \int_{S_r} \chi(zy) \, dy \int_{B_a} e^{-\sigma|w|^b} \, dw
\]

\[
= \sum_{r \in \mathbb{Z}} \int_{S_r} \chi(zy) e^{-\sigma(t+t')|y|^b} \, dy \int_{B_a} e^{-\sigma|w|^b} \, dw
\]

\[
= \int_{Q_p} \chi(zy) e^{-\sigma(t+t')|y|^b} \, dy \int_{B_a} e^{-\sigma|w|^b} \, dw.
\]

\[\square\]

**Proposition 4.3** Suppose that \( \rho \) is the fundamental solution to (4). For all positive real numbers \( t \) and \( t' \) and all \( z \) in \( Q_p \),

\[
\int_{B_a} \rho(t, x) \rho(t', z - x) \, dx > \rho(t + t', z) \int_{B_a} \rho(t, x) \, dx.
\]

**Proof** Write both \( \rho(t, x) \) and \( \rho(t', z - x) \) as the Fourier transforms of exponential functions and rearrange terms to obtain the equalities

\[
\int_{B_a} \rho(t, x) \rho(t', z - x) \, dx
\]

\[
= \int_{B_a} \left( \int_{Q_p} \chi(xw)e^{-\sigma|w|^b} \, dw \right) \left( \int_{Q_p} \chi((z - x)y)e^{-\sigma|y|^b} \, dy \right) \, dx
\]

\[
= \int_{B_a} \left( \int_{Q_p} \int_{Q_p} \chi(xw) \chi((z - x)y)e^{-\sigma|w|^b} e^{-\sigma|y|^b} \, dw \, dy \right) \, dx
\]

\[
= \int_{Q_p} \int_{Q_p} \left( \int_{B_a} \chi(xw) \chi((z - x)y)e^{-\sigma|w|^b} e^{-\sigma|y|^b} \, dx \right) \, dw \, dy
\]

\[
= \int_{Q_p} \int_{Q_p} \left( \int_{B_a} \chi(xw) \chi(zy) e^{-\sigma|w|^b} e^{-\sigma|y|^b} \, dx \right) \, dw \, dy
\]

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\[
\int_Q p \int_{Q_p} ( \int_{B_a} \chi(zy) \chi(x(w - y)) e^{-\sigma t|w|} e^{-\sigma t'|y|} \, dx) \, dw \, dy \\
= \int_Q p \int_{Q_p} \chi(zy) \left( \int_{B_a} \chi(x(w - y)) \, dx \right) e^{-\sigma t|w|} e^{-\sigma t'|y|} \, dw \, dy.
\]

The equality
\[
\int_{B_a} \chi(xa) \, dx = p^a \mathbb{1}_{B-a}(a)
\]
together with (17) implies that
\[
\int_{B-a} \rho(t, x) \rho(t', z - x) \, dx \\
= \int_Q p \int_{Q_p} \chi(zy) p^a \mathbb{1}_{B-a}(w - y) e^{-\sigma t|w|} e^{-\sigma t'|y|} \, dw \, dy \\
= p^a \int_Q p \int_{B-a} \chi(zy) e^{-\sigma t|y+w|} e^{-\sigma t'|y|} \, dw \, dy \\
= p^a \int_Q p \int_{B-a} \chi(zy) e^{-\sigma t'|y|} \left( \int_{B-a} e^{-\sigma t|y+w|} \, dw \right) \, dy \\
> \int_Q p \int_{B-a} \chi(zy) e^{-\sigma(t+t')|y|} \, dy \cdot p^a \int_{B-a} e^{-\sigma t|w|} \, dw \\
= \rho(t + t', z) p^a \int_{B-a} e^{-\sigma t|w|} \, dw = \rho(t + t', z) \int_{B-a} \rho(t, x) \, dx
\]

where Lemma 4.2 implies the inequality and Lemma 4.1 implies the ultimate equality. \qed

**Corollary** Suppose that \( \rho \) is the fundamental solution to (4). For all positive real numbers \( t \) and \( t' \) and all \( z \) in \( B_a \),
\[
\int_{B_a} \rho(t, x) \rho(t', z - x) \, dx > \rho(t + t', z) \int_{B_a} \rho(t', x) \, dx.
\]

**Proof** Use a change of variables to obtain the equality
\[
\int_{B_a} \rho(t, x) \rho(t', z - x) \, dx = \int_{B_a} \rho(t, z - u) \rho(t', u) \, du.
\]
The result then follows immediately from Proposition 4.3. \qed

**Proposition 4.4** Suppose that \( m \) is a natural number. Suppose that \( y \) is in \( B_a \) and \( n \) is a natural number large enough so that \( B_{-n}(y) \) is a subset of \( B_a \). If for each \( i \) in \( \{1, \ldots, m + 1\} \) the real number \( t_i \) is positive, then
\[
\int_{B_a} \cdots \int_{B_a} \int_{B_{-n}(y)} \rho(t_1, z_1) \rho(t_2, z_2 - z_1) \\
\cdots \rho(t_m, z_m - z_{m-1}) \rho(t_{m+1}, z - z_m) \, dz \, dz_m \cdots \, dz_1 \\
> \int_{B_{-n}(y)} \rho(t_1 + \cdots + t_m + t_{m+1}, z) \, dz \cdot \prod_{1 \leq i \leq m} \int_{B_a} \rho(t_i, z_i) \, dz_i.
\]

**Proof** If \( m \) is equal to 1, then the corollary to Proposition 4.3 implies inequality
\[
\int_{B_a} \rho(t_2, z_1) \rho(t_1, z - z_1) \, dx > \rho(t_1 + t_2, z) \int_{B_a} \rho(t_2, z_1) \, dz_1.
\] (18)

Integrate both sides of (18) to obtain,
\[
\int_{B_{-n}(y)} \int_{B_a} \rho(t_2, z_1) \rho(t_1, z - z_1) \, dz_1 \, dz \\
> \int_{B_{-n}(y)} \rho(t_1 + t_2, z) \int_{B_a} \rho(t_1, z_1) \, dz_1 \, dz \\
= \int_{B_{-n}(y)} \rho(t_1 + t_2, z) \, dz \int_{B_a} \rho(t_1, z_1) \, dz_1.
\] (19)

Switch the order of integration on the left hand side of (19) to verify the proposition when \( m \) is equal to 1.

Let \( z_0 \) be equal to 0 and let \( \ell \) be an arbitrary natural number. Suppose that for any increasing finite sequence \((t_1, \ldots, t_\ell)\) in \((0, \infty)\) of length \( \ell \),
\[
\int_{B_a} \cdots \int_{B_a} \int_{B_{-n}(y)} \rho(t_{\ell+1}, z - z_{\ell}) \prod_{1 \leq i \leq \ell} \rho(t_i, z_i - z_{i-1}) \, dz \, dz_{\ell} \cdots \, dz_1 \\
> \int_{B_{-n}(y)} \rho\left(t_{\ell+1} + \sum_{1 \leq k \leq \ell} t_k, z\right) \, dz \cdot \prod_{1 \leq i \leq \ell} \int_{B_a} \rho(t_i, z_i) \, dz_i.
\]

Suppose that \((t_1, \ldots, t_\ell, t_{\ell+1})\) is an increasing finite sequence in \((0, \infty)\) of length \( \ell + 1 \). Change variables by taking
\[
w_i = z_i - z_{i-1}
\]
to obtain the equality
\[
\int_{B_a} \cdots \int_{B_a} \int_{B_a} \int_{B_{-n}(y)} \rho(t_{\ell+2}, z - z_{\ell+1}) \\
\cdots \rho(t_{i}, z_i - z_{i-1}) \, dz \, dz_{\ell+1} \, dz_\ell \cdots \, dz_1 \\
= \int_{B_a} \cdots \int_{B_a} \int_{B_{-n}(y)} \rho(t_{\ell+2}, z - (w_1 + \cdots + w_\ell + w_{\ell+1}))
\]
\[
\prod_{i \leq 1 \leq \ell + 1} \rho(t_i, w_i) \, dz \, dw_{\ell + 1} \, dw_{\ell} \cdots dw_1 = *_1.
\]

Rearrange terms and then use the corollary to Proposition 4.3 to obtain the inequality
\[
*_1 = \int_{B_a} \cdots \int_{B_a} \left( \int_{B_{-n}(y)} \rho(t_{\ell + 2}, (z - w_1 - \cdots - w_\ell) - w_{\ell + 1}) \right) \prod_{i \leq 1 \leq \ell} \rho(t_i, w_i) \, dw_{\ell + 1} \bigg) \prod_{i \leq 1 \leq \ell} \rho(t_i, w_i) \, dw_\ell \cdots dw_1
\]
\[
> \int_{B_a} \cdots \int_{B_a} \left( \int_{B_{-n}(y)} \rho(t_{\ell + 2} + t_{\ell + 1}, (z - w_1 - \cdots - w_\ell)) \, dz \right) \prod_{i \leq 1 \leq \ell} \rho(t_i, w_i) \, dw_\ell \cdots dw_1 = *_2.
\]

Rearrange terms and then use the inductive hypothesis to obtain the inequality
\[
*_2 = \int_{B_a} \cdots \int_{B_a} \left( \int_{B_{-n}(y)} \rho(t_{\ell + 2} + t_{\ell + 1} + \sum_{1 \leq i \leq \ell} t_i, z) \, dz \right) \prod_{1 \leq i \leq \ell} \int_{B_a} \rho(t_i, z_i) \, dz_i \prod_{1 \leq i \leq \ell + 1} \int_{B_a} \rho(t_i, z_i) \, dz_i.
\]

Fubini’s Theorem therefore implies that
\[
\int_{B_a} \cdots \int_{B_a} \cdots \int_{B_{-n}(y)} \rho(t_{k + 2} + t_{k + 1} + \sum_{1 \leq i \leq \ell + 2} t_i, z) \, dz \cdots \prod_{1 \leq i \leq \ell + 1} \int_{B_a} \rho(t_i, z_i) \, dz_i,
\]
and so the axiom of induction implies the proposition. \Halmos

**Corollary** Suppose that \( m \) is a natural number. For all \( y \) in \( B_a \), if \( n \) is a natural number large enough so that \( B_{-n}(y) \) is a subset of \( B_a \), then
\[
P\left( X_t \in B_{-n}(y) \bigg| X_{\frac{m_{-1}}{m}} \in B_a \cap \cdots \cap X_{\frac{m_{-1}}{m}} \in B_a \cap X_t \in B_a \right)
\]
\[ P(X_t \in B_{-n}(y)). \]

**Proof** Ultrametricity of the valuation on \( \mathbb{Q}_p \) implies that
\[
P\left( X_{\frac{t}{m}} \in B_a \cap \cdots \cap X_t \in B_a \right)
= P\left( X_{\frac{t}{m}} - X_0 \in B_a \cap \cdots \cap X_{\frac{(m-1)t}{m}} \in B_a \right).
\]

Therefore,
\[
P\left( X_t \in B_{-n}(y) \mid X_{\frac{t}{m}} \in B_a \cap \cdots \cap X_{\frac{(m-1)t}{m}} \in B_a \right)
= \frac{P\left( X_t \in B_{-n}(y) \cap X_{\frac{t}{m}} \in B_a \cap \cdots \cap X_{\frac{(m-1)t}{m}} \in B_a \right)}{P\left( X_{\frac{t}{m}} - X_0 \in B_a \cap \cdots \cap X_{\frac{(m-1)t}{m}} \in B_a \right)}
= \frac{P\left( X_{\frac{t}{m}} - X_0 \in B_a \cap \cdots \cap X_t - X_{\frac{(m-1)t}{m}} \in B_a \right)}{P\left( X_{\frac{t}{m}} - X_0 \in B_a \right) \cdots P\left( X_t - X_{\frac{(m-1)t}{m}} \in B_a \right)}
\]

where independence of the increments of the process implies the ultimate equality.

Write the above probabilities as integrals to obtain
\[
P\left( X_t \in B_{-n}(y) \mid X_{\frac{t}{m}} \in B_a \cap \cdots \cap X_{\frac{(m-1)t}{m}} \in B_a \right)
= \int_{B_a} \cdots \int_{B_a} \int_{B_{-n}(y)} \rho\left( \frac{t}{m}, z \right) \rho\left( \frac{t}{m}, z_2 - z_1 \right) \cdots \rho\left( \frac{t}{m}, z_{m-1} - z_{m-2} \right)
\cdot \rho\left( \frac{t}{m}, z - z_{m-1} \right) dz \, dz_{m-1} \cdots dz_1 \cdot \frac{1}{\prod_{1 \leq i \leq m} \int_{B_a} \rho\left( \frac{t}{m}, z_i \right) dz_i}
\]
\[
> \int_{B_{-n}(y)} \rho\left( \frac{t}{m} + \cdots + \frac{t}{m}, z \right) dz \cdot \prod_{1 \leq i \leq m} \int_{B_a} \rho\left( \frac{t}{m}, z_i \right) dz_i \cdot \frac{1}{\prod_{1 \leq i \leq m} \int_{B_a} \rho\left( \frac{t}{m}, z_i \right) dz_i}
\]
\[
= \int_{B_{-n}(y)} \rho(t, z) dz = P(X_t \in B_{-n}(y)),
\]

where Proposition 4.4 implies the inequality above. \( \square \)
Proposition 4.5 For all \( y \) in \( B_a \), if \( n \) is a positive integer that is large enough so that \( B_{-n}(y) \) is a subset of \( B_a \), then

\[
P(X_t \in B_{-n}(y) \mid ||X||_t \leq p^a) > P(X_t \in B_{-n}(y)).
\]

Proof Denote by \( ||X||_t^{(m)} \) the random variable given by

\[
||X||_t^{(m)} = \max \left\{ |X_i| : i \in \left\{ \frac{t}{m}, \frac{2t}{m}, \ldots, \frac{(m-1)t}{m}, t \right\} \right\}.
\]

Since

\[
||X||_t^{(m)} \leq p^a
\]

if and only if for all \( i \) in \( \{1, \ldots, m\} \),

\[
X_{\frac{t}{m}} \in B_a,
\]

the corollary to Proposition 4.4 implies that

\[
P(X_T \in B_{-n}(y) \mid ||X||_t^{(m)} \leq p^a)
\]

\[
> P(X_t \in B_{-n}(y)) \frac{P\left( (X_t \in B_{-n}(y)) \cap (||X||_t^{(m)} \leq 1) \right)}{P\left( ||X||_t^{(m)} \leq p^a \right)}
\]

\[
= \frac{P\left( X_{\frac{t}{m}} \in B_a \cap \cdots \cap X_{\frac{(m-1)t}{m}} \in B_a \cap X_t \in B_{-n}(y) \right)}{P\left( ||X||_t^{(m)} \leq p^a \right)}
\]

\[
= \frac{P\left( X_{\frac{t}{m}} \in B_a \cap \cdots \cap X_{\frac{(m-1)t}{m}} \in B_a \cap X_t \in B_{-n}(y) \right)}{P\left( ||X||_t^{(m)} \leq p^a \right)}
\]

\[
\cdot P\left( X_{\frac{t}{m}} \in B_a \cap \cdots \cap X_{\frac{(m-1)t}{m}} \in B_a \right)
\]

\[
> P(X_t \in B_{-n}(y)) \frac{P\left( X_{\frac{t}{m}} \in B_a \cap \cdots \cap X_{\frac{(m-1)t}{m}} \in B_a \right)}{P\left( ||X||_t^{(m)} \leq p^a \right)},
\]

hence

\[
P(X_T \in B_{-n}(y) \mid ||X||_t^{(m)} \leq p^a)
\]

\[
> P(X_t \in B_{-n}(y)) \frac{P\left( X_{\frac{t}{m}} \in B_a \cap \cdots \cap X_{\frac{(m-1)t}{m}} \in B_a \right)}{P\left( ||X||_t^{(m)} \leq p^a \right)}.
\]
The right continuity of the Skorokhod paths implies that
\[
\lim_{m \to \infty} P\left( ||X||_t^{(m)} \leq p^a \right) = P\left( ||X||_t \leq p^a \right),
\] (21)
that
\[
\lim_{m \to \infty} P\left( X_t \in B_{-n}(y) \cap ||X||_t^{(m)} \leq p^a \right) = P\left( X_t \in B_{-n}(y) \cap ||X||_t \leq p^a \right),
\] (22)
and that
\[
\lim_{m \to \infty} P\left( \frac{X_t}{m} \in B_a \cap \cdots \cap \frac{X_{(m-1)t}}{m} \in B_a \right) = P\left( ||X||_t^{(m)} \leq p^a \right).
\] (23)

Use the fact that
\[
P\left( X_t \in B_{-n}(y) \cap ||X||_t^{(m)} \leq p^a \right) \neq 0
\]
and
\[
P\left( X_t \in B_{-n}(y) \cap ||X||_t \leq p^a \right) \neq 0
\]
together with the equalities (21), (22), and (23) to obtain the equalities
\[
\lim_{m \to \infty} P\left( X_t \in B_{-n}(y) \right) \frac{P\left( \frac{X_t}{m} \in B_a \cap \cdots \cap \frac{X_{(m-1)t}}{m} \in B_a \right)}{P\left( ||X||_t^{(m)} \leq p^a \right)} = P\left( X_t \in B_{-n}(y) \right)
\]
and
\[
\lim_{m \to \infty} P\left( X_T \in B_{-n}(y) \right) \frac{P\left( ||X||_t^{(m)} \leq p^a \right)}{P\left( ||X||_t \leq p^a \right)} = P\left( X_T \in B_{-n}(y) \right) \frac{P\left( ||X||_t \leq p^a \right)}{P\left( X_t \in B_{-n}(y) \right)}.
\]

These two equalities together with (20) imply the proposition.

**Proposition 4.6** For all y in B_a, if n is a positive integer that is large enough so that B_{-n}(y) is a subset of B_a, then
\[
P\left( ||X||_t \leq p^a \mid X_t \in B_{-n}(y) \right) > P\left( ||X||_t \leq p^a \right).
\]

**Proof** Proposition 4.5 implies that
\[
P\left( ||X||_t \leq p^a \mid X_t \in B_{-n}(y) \right)
= \frac{P\left( ||X||_t \leq p^a \cap X_t \in B_{-n}(y) \right)}{P\left( X_t \in B_{-n}(y) \right)}
= \frac{P\left( X_t \in B_{-n}(y) \cap ||X||_t \leq p^a \right) P\left( ||X||_t \leq p^a \right)}{P\left( ||X||_t \leq p^a \right) P\left( X_t \in B_{-n}(y) \right)}
\]
\[ P(X_t \in B_{-n}(y) \mid ||X||_t \leq p^a) \frac{P(||X||_t \leq p^a)}{P(X_t \in B_{-n}(y))} > P(X_t \in B_{-n}(y)) \frac{P(||X||_t \leq p^a)}{P(X_t \in B_{-n}(y))} = P(||X||_t \leq p^a). \]

\[ \square \]

**Theorem 4.7**  Suppose that \( x \) is in \( Q_p \) and \( y \) is in \( B_{a} + x \). For all \( t \) in \( (0, T] \),

\[ P_{t,x,y}(||X - x||_T \leq p^a) \geq P_x(||X - x||_T \leq p^a). \]

**Proof** Denote by \( z \) the \( p \)-adic number \( y - x \). Write the measure on the Brownian bridges as a limit of conditional measures with respect to events of nonzero probability to obtain

\[ P(||X||_t \leq p^a \mid X_t = z) = \lim_{n \to \infty} P(||X||_t \leq p^a \mid X_t \in B_{-n}(y)) \geq \lim_{n \to \infty} P(||X||_t \leq p^a) = P(||X||_t \leq p^a), \tag{24} \]

where Proposition 4.6 implies the inequality (24). Independence of the increments of the process implies that

\[ P(||X||_T \leq p^a \mid X_t = z) = P(||X||_t \leq p^a \mid X_t = z) P_z(||X||_{T-t} \leq p^a) \geq P(||X||_t \leq p^a) P_z(||X||_{T-t} \leq p^a) = P(||X||_t \leq p^a) P(||X||_{T-t} \leq p^a) \]

\[ = P(||X||_T \leq p^a), \tag{25} \]

where the inequality (25) follows from (24). The inequality

\[ P(||X||_T \leq p^a \mid X_t = z) \geq P(||X||_T \leq p^a) \tag{26} \]

together with the equalities

\[ P(||X||_T \leq p^a \mid X_t = z) = P_{t,0,y-x}(||X||_T \leq p^a) = P_{t,x,y}(||X||_T \leq p^a) \]

and

\[ P(||X||_T \leq p^a) = P_x(||X - x||_T \leq p^a), \]

imply that

\[ P_{t,x,y}(||X - x||_T \leq p^a) \geq P_x(||X - x||_T \leq p^a). \]

\[ \square \]
References

1. Albeverio, S., Gordon, E.I., Khrennikov, AYu.: Finite-dimensional approximations of operators in the Hilbert spaces of functions on locally compact abelian groups. Acta Appl. Math. 64(1), 33–73 (2000)
2. Albeverio, S., Karwowski, W.: A random walk on $p$-adics - the generator and its spectrum. Stochastic Process. Appl. 5(3), 1–22 (1994)
3. Avetisov, V.A., Bikulov, Akh.: On the ultrametricity of the fluctuation dynamic mobility of protein molecules. Proc. Steklov Inst. Math. 265(1), 75–81 (2009)
4. Avetisov, V.A., Bikulov, Akh., Kozyrev, S.V.: Application of $p$-adic analysis to models of breaking of replica symmetry. J. Phys. A 32(50), 8785–8791 (1999)
5. Avetisov, V.A., Bikulov, Akh., Kozyrev, S.V.: Description of logarithmic relaxation by a model of a hierarchical random walk. Dokl. Akad. Nauk 368(2), 164–167 (1999)
6. Avetisov, V.A., Bikulov, Akh., Osipov, V.A.: $p$-Adic description of characteristic relaxation in complex systems. J. Phys. A 36(15), 4239–4246 (2003)
7. Bakken, E., Digernes, T., Weisbart, D.: Brownian motion and finite approximations of quantum systems over local fields. Rev. Math. Phys. 29(5), 1750016 (2017)
8. Bakken, E., Weisbart, D.: $p$-Adic Brownian motion as a limit of discrete time random walks. Commun. Math. Phys. 369(2), 371–402 (2019)
9. Billingsley, P.: Probability and Measure, 3rd edn. John Wiley and Sons, New York (1995)
10. Chentsov, N.N.: Weak convergence of stochastic processes whose trajectories have no discontinuities of the second kind and the “heuristic” approach to the Kolmogorov-Smirnov tests. Theory Prob. Appl. 1(1), 140–144 (1956)
11. Digernes, T., Varadarajan, V.S., Varadhan, S.R.S.: Finite approximations to quantum systems. Rev. Math. Phys. 6(4), 621–648 (1994)
12. Gouveia, F.Q.: $p$-Adic Numbers. Universitext. Springer, Berlin, Heidelberg (1997)
13. Khrennikov, A., Kozyrev, S., Zúñiga-Galindo, W.A.: Ultrametric equations and its applications. Encyclopedia of mathematics and its applications, vol. 168. Cambridge University Press, Cambridge (2018)
14. Kochubei, A.N.: Parabolic equations over the field of $p$-adic numbers. Math. USSR Izvestiya 39, 1263–1280 (1992)
15. Kochubei, A. N.: Pseudo-Differential Equations and Stochastics over non-Archimedean Fields. Monographs and Textbooks in Pure and Applied Mathematics. vol. 244, Marcel Dekker Inc., New York (2001)
16. Saloff-Coste, L.: Opérateurs pseudo-différentiels sur un corps local. C. R. Acad. Sci. Paris Sér. I(297), 171–174 (1983)
17. Saloff-Coste, L.: Opérateurs pseudo-différentiels sur certains groupes totalement discontinus. Studia Math. 83, 205–228 (1986)
18. Ramakrishnan, D., Valenza, R.J.: Fourier Analysis on Number Fields. Springer, New York (1999)
19. Taibleson, M.H.: Fourier Analysis on Local Fields. N.J.: University of Tokyo Press, Tokyo, Princeton University Press, Princeton (1975)
20. Varadarajan, V.S.: Path integrals for a class of $p$-adic Schrödinger equations. Lett. Math. Phys. 39(2), 97–106 (1997)
21. Vladimirov, V.S.: Generalized functions over the field of $p$-adic numbers. Russian Math. Surveys 43(5), 19–64 (1988)
22. Vladimirov, V.S.: On the spectrum of some pseudo-differential operators over $p$-adic number field. Algebra Anal. 2, 107–124 (1990)
23. Vladimirov, V.S., Volovich, I.V., Zelenov, E.I.: $p$-Adic Analysis and Mathematical Physics. World Scientific, Singapore (1994)
24. Weil, A.: Basic Number Theory. Springer, Berlin, Heidelberg, New York (1967)

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