Weak amenability of weighted measure algebras and their second duals

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Abstract. In this paper, we study the weak amenability of weighted measure algebras and prove that $M(G, \omega)$ is weakly amenable if and only if $G$ is discrete and every bounded quasi-additive function is inner. We also study the weak amenability of $L^1(G, \omega)^{\ast\ast}$ and $M(G, \omega)^{\ast\ast}$ and show that the weak amenability of these Banach algebras are equivalent to finiteness of $G$. This gives an answer to the question concerning weak amenability of $L^1(G, \omega)^{\ast\ast}$ and $M(G, \omega)^{\ast\ast}$.

1 Introduction

Let $G$ be a locally compact group with an identity element $e$. Let us recall that a continuous function $\omega: G \to [1, \infty)$ is called a weight function if for every $x, y \in G$

$$\omega(xy) \leq \omega(x) \omega(y) \quad \text{and} \quad \omega(e) = 1.$$ 

Let $C_0(G, 1/\omega)$ be the space of all functions $f$ on $G$ such that $f/\omega \in C_0(G)$, the space of all bounded continuous functions on $G$ that vanish at infinity. Let also $M(G, \omega)$ be the Banach space of all complex regular Borel measures $\mu$ on $G$ for which $\omega\mu \in M(G)$, the measure algebra of $G$. It is well-known that $M(G, \omega)$ is the dual space of $C_0(G, 1/\omega)$ [4, 21, 26], see [22, 24] for study of weighted semigroup measure algebras; see also [13, 14, 17]. Note that $M(G, \omega)$ is a Banach algebra with the norm $\|\mu\|_{\omega} := \|\omega\mu\|$ and the convolution product “$*$” defined by

$$\mu * \nu(f) = \int_G \int_G f(xy) \, d\mu(x) \, d\nu(y) \quad (\mu, \nu \in M(G, \omega), f \in C_0(G, 1/\omega)).$$

Let $L^1(G, \omega)$ be the Banach space of all Borel measurable functions $f$ on $G$ such that $\omega f \in L^1(G)$, the group algebra of $G$. Then $L^1(G, \omega)$ with the convolution product “$*$” and the norm $\|f\|_{1, \omega} = \|\omega f\|_1$ is a Banach algebra.

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A Borel measurable function $p$ from $G \times G$ into $\mathbb{C}$ is called \textit{quasi-additive} if for almost every where $x, y, z \in G$

$$p(xy, z) = p(x, yz) + p(y, zx).$$

If there exists $h \in L^\infty(G, 1/\omega)$ such that

$$p(x, y) = h(xy) - h(yx)$$

for almost every where $x, y \in G$, then $p$ is called \textit{inner}. Let $D(G, \omega)$ be the set of all quasi-additive functions $p$ on $G$ such that

$$\sup_{x,y} |p(x, y)| \omega^\otimes(x, y) < \infty.$$ 

We denote by $I(G, \omega)$ the set of inner quasi-additive functions. For $\mu \in M(G, \omega)$, let $L^\infty(|\mu|, \omega)$ be the Banach space of all $\omega-$bounded Borel measurable functions $p$ on $G$ such that $\|p\|_{\omega, \mu} = \|p/\omega\|_\mu < \infty$. An element

$$P = (p_\mu)_{\mu} \in \Pi\{L^\infty(|\mu|, \omega) : \mu \in M(G, \omega)\}$$

is called a $\omega-$\textit{generalized function} on $G$ if

$$\sup\{\|p_\mu\|_{\omega, \mu} : \mu \in M(G, \omega)\} < \infty$$

and for every $\mu, \nu \in M(G, \omega)$ with $|\mu| \ll |\nu|$ we have $p_\mu = p_\nu$, $|\mu| - a.e..$ The space of all $\omega-$\textit{generalized function} on $G$ is denoted by $GL(G, 1/\omega)$. It is well-known from [25] that $GL(G, 1/\omega)$ is the dual of $M(G, \omega)$ for the pairing

$$\langle (p_\mu)_\mu, \nu \rangle = \int_G p_\nu \, d\nu.$$

A function $F = (F_{\mu \otimes \nu})_{\mu, \nu \in M(G, \omega)} \in GL(G \times G, \omega^\otimes)$ is called a \textit{generalized quasi-additive function} if

$$F_{(\mu \ast \nu) \otimes \eta}(xy, z) = F_{\mu \otimes (\nu \ast \eta)}(x, yz) + F_{\nu \otimes (\eta \ast \mu)}(y, zx)$$

for all $\mu, \nu, \eta \in M(G, \omega)$ and $x, y, z \in G$. The set of all generalized quasi-additive functions is denoted by $GD(G, \omega)$. If there exists $p = (p_\mu) \in GL(G, 1/\omega)$ such that for every $\mu, \nu \in M(G, \omega)$ and for almost every where $x, y \in G$

$$F_{\mu \otimes \nu}(x, y) = p_{\mu \ast \nu}(xy) - p_{\nu \ast \mu}(yx),$$

then $F$ is said to be a \textit{generalized inner quasi-additive function}. The set of all generalized inner quasi-additive functions is denoted by $GI(G, \omega)$. 
Let $A$ be a Banach algebra and $D : A \to A^*$ be a bounded linear operator. Then $D$ is called cyclic if $\langle D(a), a \rangle = 0$ for all $a \in A$. Let us recall that a bounded linear operator $D : A \to A^*$ is called a derivation if

$$D(ab) = D(a) \cdot b + a \cdot D(b)$$

for all $a, b \in A$. The space of all bounded continuous derivations from $A$ into $A^*$ is denoted by $\mathcal{Z}(A, A^*)$. If every element of $\mathcal{Z}(A, A^*)$ is cyclic, then $A$ is called cyclically weakly amenable, however, $A$ is called weakly amenable if every derivation $D \in \mathcal{Z}(A, A^*)$ is inner; that is, there exists $z \in A^*$ such that for every $a \in A$

$$D(a) = \text{ad}_z(a) := z \cdot a - a \cdot z.$$  

The weak amenability of group algebras have been study by several authors. For example, Brown and Moran [2] studied the weak amenability of measure algebra of locally compact Abelian groups and showed that if zero is the only continuous point derivation of $M(G)$, then $G$ is discrete. Note that if $G$ is discrete, then $M(G)$ is weakly amenable, because in this case $M(G) = \ell^1(G)$ is always weakly amenable [8]. One can prove that if $d$ is a non-zero continuous point derivation of $M(G)$ at

$$\varphi \in \Delta(M(G)) \cup \{0\},$$

then the map $\mu \mapsto d(\mu) \varphi$ is a continuous non-inner derivation from $M(G)$ into $M(G)^*$. In other words, $M(G)$ is not weakly amenable. These facts give rise to the conjecture that for a locally compact group $G$, the Banach algebra $M(G)$ is weakly amenable if and only if $G$ is discrete; or equivalently, zero is the only continuous point derivation of $M(G)$ at a character. Dales, Ghahramani and Helemskii [3] proved this conjecture. Some authors investigated the weak amenability of the second dual of Banach algebras. For instance, Ghahramani, Loy and Willis [7] proved that if $G$ is a locally compact Abelian group and $L^1(G)^{**}$ is weakly amenable, then $G$ is discrete. Forrest [6] investigated the weak amenability of the dual of a topological introverted subspace $X$ of $VN(G)$. Under certain conditions, he showed that if $A(G)^{**}$ is weakly amenable, then every Abelian subgroup of $G$ is finite. As a consequence of this result, he improved the result of Ghahramani, Loy and Willis. In fact, for a locally compact Abelian group $G$, he proved that weak amenability of $L^1(G)^{**}$ is equivalent to the finiteness of $G$. Lau and Loy [9] considered a left introverted subspace of $L^\infty(G)$ containing $AP(G)$, say $X$, and studied weak amenability of $X^*$. One can obtain the result of weak amenability of $L^1(G)^{**}$ from Lau-Loy's theorem. Finally, Dales, Lau and Strauss [5] proved that $L^1(G)^{**}$ is weakly amenable if and only if there is no non-zero continuous point derivation of $L^1(G)^{**}$ at the discrete augmentation character; or equivalently, $G$ is finite.

This paper is organized as follow. In Section 2 we study the weak amenability of $M(G, \omega)$ and show that $M(G, \omega)$ is weakly amenable if and only if $G$ is discrete and every bounded quasi-additive function is inner. We also prove that cyclic weak
amenability and point amenability of $M(G, \omega)$ are equivalent to weak amenability of it. Section 3 is devoted to studies of the weak amenability of second dual of $L^1(G, \omega)$ and $M(G, \omega)$. We proved that $L^1(G, \omega)^{**}$ is weakly amenable if and only if $M(G, \omega)^{**}$ is weakly amenable; or equivalently, $G$ is finite. We verify that cyclic weak amenability and point amenability of $L^1(G, \omega)^{**}$ and $M(G, \omega)^{**}$ are equivalent to finiteness of $G$.

2 Weighted measure algebras

Let $\omega_i$ be a weight function on locally compact group $G_i$ for $i = 1, 2$. Define the weight function $\omega_1 \otimes \omega_2$ on $G_1 \times G_2$ by

$$\omega_1 \otimes \omega_2(x_1, x_2) = \omega_1(x_1)\omega_2(x_2)$$

for all $x_1 \in G_1$ and $x_2 \in G_2$. In the case where, $G_1 = G_2 = G$ and $\omega_1 = \omega_2 = \omega$, we set $\omega^\otimes = \omega_1 \otimes \omega_2$. The following result is needed to prove our results.

**Proposition 2.1** Let $\omega_i$ be a weight function on locally compact group $G_i$ for $i = 1, 2$. Then

$$M(G_1, \omega_1)^\hat{\otimes} M(G_2, \omega_2) = M(G_1 \times G_2, \omega_1 \otimes \omega_2).$$

**Proof.** Let $\eta_i \in M(G_i, \omega_i)$, for $i = 1, 2$. Then for every $f \in C_0(G_1 \times G_2)$, we have

$$\langle \eta_1 \otimes \eta_2, f \rangle = \int_{G_1} \int_{G_2} f(x, y) \, d\eta_1(x)d\eta_2(y).$$

It is easy to prove that

$$\eta_1 \otimes \eta_2 \in C_0(G_1 \times G_2, 1/\omega_1 \otimes \omega_2)^* = M(G_1 \times G_2, \omega_1 \otimes \omega_2).$$

Conversely, let $\eta \in M(G_1 \times G_2, \omega_1 \otimes \omega_2)$. In view of Theorem Lusin’s theorem, there exists sequences $(f_n)$ and $(g_n)$ in the unit ball $C_c(G_1, 1/\omega_1)$ and $C_c(G_2, 1/\omega_2)$ with compact support, respectively, such that for almost every where $x \in G_1$ and $y \in G_2$

$$f_n(x) \to 1 \quad \text{and} \quad g_n(y) \to 1$$

as $n \to \infty$. We define the functionals $\eta_1$ and $\eta_2$ by

$$\eta_1(f) = \lim_n \eta(f \otimes g_n) \quad \text{and} \quad \eta_2(g) = \lim_n \eta(f_n \otimes g)$$

for all $f \in C_0(G_1, 1/\omega_1)$ and $g \in C_0(G_2, 1/\omega_2)$. Then $\eta_1 \in M(G_1, \omega_1)$, $\eta_2 \in M(G, \omega_2)$. In fact,

$$|\eta_1(f)| \leq \|\eta\|\|f\|_{\infty, 1/\omega} \quad \text{and} \quad |\eta_2(g)| \leq \|\eta\|\|g\|_{\infty, 1/\omega}.$$
On the other hand,
\[ \eta_1 \otimes \eta_2(f \otimes g) = \lim_n \eta(f \otimes g_n)\eta(f_n \otimes g) \]
\[ = \lim_n \int_{G_1 \times G_2} f(x)f_n(x)g(y)g_n(y) \, d\eta(x, y). \]

Since \( \eta \) is bounded, it follows from Lebesgue dominated convergence theorem that
\( \|f_n \otimes g_n\|_{\infty, 1/\omega} \leq 1 \) and \( 1 \in L^1(\eta) \). Furthermore, \( f_n \otimes g_n(x, y) \to 1 \) for every \( x \in G_1, y \in G_2 \). For every \( f \in C_0(G_1, 1/\omega_1) \) and \( g \in C_0(G_2, 1/\omega_2) \)
\[ \eta_1 \otimes \eta_2(f \otimes g) = \int_{G_1 \times G_2} f \otimes g(x, y) \, d\eta(x, y) = \eta(f \otimes g). \]

It follows that for every \( h \in C_0(G_1, 1/\omega_1) \otimes C_0(G_2, 1/\omega_2) \)
\[ \eta_1 \otimes \eta_2(h) = \eta(h) \]
and so for every \( h \in C_0(G_1 \times G_2, 1/\omega_1 \otimes \omega_2) \)
\[ \eta_1 \otimes \eta_2(h) = \eta(h). \]

Therefore, \( \eta_1 \otimes \eta_2 = \eta. \)

For every \( f \in L^1(G, \omega) \), we define the seminorm \( T_f : M(G, \omega) \to [0, \infty) \) by
\[ T_f(\mu) = \|f \ast \mu\|_{1, \omega} + \|\mu \ast f\|_{1, \omega}. \]

The locally convex topology defined by the family of seminorms \( (T_f)_{f \in L^1(G, \omega)} \) is called the strict topology on \( M(G, \omega) \) with respect to \( L^1(G, \omega) \) (or briefly strict topology).

**Proposition 2.2** Let \( G \) be a locally compact group and \( \omega \) be a weight function on \( G \). If \( p \in D(G, \omega) \), then there exists a unique bounded derivation \( D \in Z(M(G, \omega), M(G, \omega)^*) \) such that \( p(x, y) = \langle D(\delta_x), \delta_y \rangle \) for all \( x, y \in G \), where \( \delta \) is the Dirac measure at \( \cdot \).

**Proof.** Let \( p \in D(G, \omega) \). Then \( \Gamma(D_1) = p \) for some \( D_1 \in Z(L^1(G, \omega), L^\infty(G, 1/\omega)). \)

By Proposition 2.1.6 [23], there exists \( D_2 \in Z(M(G, \omega), L^\infty(G, 1/\omega)) \) such that \( D_2 \) is strict-weak* continuous and \( D_2|_{L^1(G, \omega)} = D_1 \). Hence for every \( f \in L^1(G, \omega) \),
\[ \langle D_2(\delta_x), f \rangle = \lim \langle D_1(e_\alpha \ast \delta_x), f \rangle \]
\[ = \lim \int_G \int_G p(x, y)(e_\alpha \ast \delta_x)(z)f(z) \, dz \, dy \]
\[ = \int_G \int_G p(x, y)e_\alpha(zx^{-1})f(y) \, dz \, dy. \]

On the other hand, there exists a linear functional \( T_1 : L^1(G \times G, \omega^\odot) \to \mathbb{C} \) such that
\[ \langle T_1, f \otimes g \rangle = \langle D_1(f), g \rangle \]
for all $f, g \in L^1(G, \omega)$. Since $L^1(G \times G, \omega^\otimes)$ is a closed ideal in $M(G \times G, \omega^\otimes)$, it follows that $T_1$ has a strict continuous extension, say $T_2 : M(G, \omega) \otimes M(G, \omega) \to \mathbb{C}$. Define $D : M(G, \omega) \to M(G, \omega)^*$ by

$$\langle D(\mu), \nu \rangle = \langle T_2, \mu \otimes \nu \rangle$$

for all $\mu, \nu \in M(G, \omega)$. If $(e_\alpha)$ is a bounded approximate identity of $L^1(G, \omega)$, then for every $x \in G$, $e_\alpha \ast \delta_x \to \delta_x$ in the strict topology. So

$$T_2(e_\alpha \ast \delta_x \otimes e_\alpha \ast \delta_y) \to T_2(\delta_x \otimes \delta_y).$$

Therefore,

$$\langle D(\delta_x), \delta_y \rangle = \lim \langle T_2(e_\alpha \ast \delta_x \otimes e_\alpha \ast \delta_y) \rangle = \lim \langle D_2(e_\alpha \ast \delta_x), e_\alpha \ast \delta_y \rangle = p(x, y),$$

as claimed. \hfill \Box

In the following, let $\mathcal{I}_{nn}(M(G, \omega), M(G, \omega)^*)$ be the set of all inner derivations from $M(G, \omega)$ into $M(G, \omega)^*$, and let $\mathcal{B}(M(G, \omega), M(G, \omega)^*)$ be the space of bounded linear operators from $M(G, \omega)$ into $M(G, \omega)^*$. Define the isometric isomorphism $\Gamma$ from Banach space $\mathcal{B}(M(G, \omega), M(G, \omega)^*)$ onto $(M(G, \omega) \otimes M(G, \omega))^*$ by

$$\langle \Gamma(T), \mu \otimes \nu \rangle = \langle T(\mu), \nu \rangle,$$

$M(G, \omega) \otimes M(G, \omega)$ is the projective tensor product of $M(G, \omega)$; see Proposition 13 VI in [1].

**Proposition 2.3** Let $G$ be a locally compact group and $\omega$ be a weight function on $G$. Then the following statements hold.

(i) The function $\Gamma : \mathcal{Z}(M(G, \omega), M(G, \omega)^*) \to GD(G, \omega)$ is an isometric isomorphism. Furthermore, $\Gamma(\mathcal{I}_{nn}(M(G, \omega), M(G, \omega)^*)) = GI(G, \omega)$.

(ii) If $D \in \mathcal{Z}(M(G, \omega), M(G, \omega)^*)$, then for every $\mu \in M(G, \omega)$ there exists $F = (F_{\mu \otimes \nu})_\nu \in GD(G, \omega)$ such that $D(\mu) = (p_{\mu, \nu})_\nu$ and $p_{\mu, \nu}(y) = \int_G F_{\mu \otimes \nu}(x, y) \, d\mu(x)$ for almost every where $y \in G$.

**Proof.** Let $D \in \mathcal{Z}(M(G, \omega), M(G, \omega)^*)$. Then $D \in \mathcal{B}(M(G, \omega), M(G, \omega)^*)$. Putting $A = B = M(G, \omega)$ in the definition of $\Gamma$, we have

$$F := \Gamma(D) \in (M(G, \omega) \otimes M(G, \omega))^* = GL(G \times G, 1/\omega^\otimes)$$

and

$$\langle D(\mu), \nu \rangle = \langle F, \mu \otimes \nu \rangle = \int_G F_{\mu \otimes \nu}(x, y) \, d(\mu \otimes \nu)(x, y) = \int_G \int_G F_{\mu \otimes \nu}(x, y) \, d\mu(x) \, d\nu(y).$$
On the other hand, if $P = (p_{\mu})_{\mu \in M(G,\omega)}$, then
\[
\langle \text{ad}_P(\mu), \nu \rangle = \langle P \cdot \mu - \mu \cdot P, \nu \rangle = \langle P, \mu \ast \nu \rangle - \langle P, \nu \ast \mu \rangle = \int_G p_{\mu \ast \nu}(x) \, d\mu(x) d\nu(y) = \int_G \int_G (p_{\mu \ast \nu}(xy) - p_{\nu \ast \mu}(yx)) \, d\mu(x) d\nu(y).
\]

Now, by the argument used in the proof of Theorem 2.3 in [18], it can be shown that the statement (i) holds. For (ii), assume that $D \in Z(M(G,\omega), M(G,\omega)^*)$ and $\mu \in M(G,\omega)$. Then
\[
D(\mu) = (p_{\mu,\nu})_{\nu} \times GL(G, 1/\omega).
\]

Thus $D(\mu) = (p_{\mu,\nu})_{\nu}$ for some $(p_{\mu,\nu})_{\nu} \in GL(G, 1/\omega)$. Hence for every $\nu \in M(G,\omega)$, we have
\[
\langle D(\mu), \nu \rangle = \int_G p_{\mu,\nu} \, d\nu.
\]

This together with (1) shows that
\[
p_{\mu,\nu}(y) = \int_G F_{\mu \otimes \nu}(x, y) \, d\mu(x)
\]
for almost every where $y \in G$. \hfill \square

We are now in a position to prove the main result of this section.

**Theorem 2.4** Let $G$ be a locally compact group and $\omega$ be a weight function on $G$. Then the following assertions are equivalent.

(a) $M(G,\omega)$ is weakly amenable.

(b) For every $D \in Z(M(G,\omega), M(G,\omega)^*)$ there exists $P = (p_{\mu})_{\mu \in M(G,\omega)}$ such that $\langle D(\mu), \nu \rangle = \int_G \int_G (p_{\mu \ast \nu}(xy) - p_{\nu \ast \mu}(yx)) \, d\mu(x) d\nu(y)$ for all $\mu, \nu \in M(G,\omega)$.

(c) Every generalized quasi-additive function is inner.

(d) $M(G)$ is weakly amenable and $D(G,\omega) = I(G,\omega)$.

(e) $G$ is discrete and every non-inner quasi-additive function in $L^\infty(G, 1/\omega)$ is unbounded.

**Proof.** The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a) follow from Proposition 2.3. By Theorem 1.2 in [3] the implication (d) $\Leftrightarrow$ (e) holds. Also, the implication (e) $\Rightarrow$ (a) follows from Corollary 2.5 in [18]. For (a) $\Rightarrow$ (e), let $M(G,\omega)$ be weakly amenable and $\varphi$ be a character of $M(G)$. If $d$ is a continuous point derivation at $\varphi$ on $M(G)$, then $d|_{M(G,\omega)}$ is a continuous point derivation of $M(G,\omega)$ at $\varphi|_{M(G,\omega)}$. Hence $d$ is zero on $M(G,\omega)$. Since $M(G,\omega)$ is dense in $M(G)$, we have $d = 0$ on $M(G)$ which is implies $G$ is discrete.
Apply Theorem 2.4 in [18] to conclude that $D(G, \omega) = I(G, \omega)$. \hfill \Box

From Theorem 4.8 in [19] and Theorem 2.4 and its proof, we may prove the next result.

**Corollary 2.5** Let $G$ be a locally compact group. Then the following assertions are equivalent.

(a) $M(G, \omega)$ is weakly amenable.
(b) $M(G, \omega)$ is cyclically weakly amenable.
(c) $M(G, \omega)$ is point amenable.
(d) $G$ is discrete and every non-inner quasi-additive function in $L^\infty(G, 1/\omega)$ is unbounded.

An elementary computation shows that the functions $\omega'$ and $\omega^*$ defined by

$$\omega'(x) = \omega(x^{-1}) \quad \text{and} \quad \omega^*(x) = \omega \otimes \omega'(x, x)$$

are weight functions on $G$. Combining Theorem 2.4 and the result of [18] we have the following result.

**Corollary 2.6** Let $\omega$ and $\omega_0$ be weight functions on a locally compact group $G$. Then the following statements hold.

(i) If $\omega \leq m\omega_0$ for some $m > 0$, $M(G, \omega_0)$ is weakly amenable and $I(G, \omega_0) = D(G, \omega_0)$, then $M(G, \omega)$ is weakly amenable.

(ii) If $\omega$ and $\omega_0$ are equivalent, then weak amenability of $M(G, \omega)$ is equivalent to weak amenability of $M(G, \omega_0)$.

(iii) $M(G, \omega')$ is weakly amenable if and only if $M(G, \omega)$ is weakly amenable.

(iv) If $M(G, \omega^*)$ is weakly amenable and $I(G, \omega^*) = D(G, \omega^*)$, then $M(G, \omega)$ is weakly amenable.

(v) If $G$ is Abelian, then $M(G, \omega^*)$ is weakly amenable if and only if $M(D, \omega^{\otimes})$ is weakly amenable, where $D := \{(x, x^{-1}) : x \in G\}$.

Let $\phi : G \to G$ be a group epimorphism and $\omega$ be a weight function on $G$. Then the function $\overrightarrow{\omega} : G \to [1, \infty)$ defined by $\overrightarrow{\omega}(x) = \omega(\phi(x))$ is a weight function on $G$.

For every quasi-additive function $p$, let $\mathcal{S}(p)$ be the quasi-additive function defined by

$$\mathcal{S}(p)(x, y) = p(\phi(x), \phi(y)) \quad (x, y \in G).$$

Theorem 2.4 together with Proposition 4.1 and Theorem 4.6 in [18] proves the next result.
Corollary 2.7 Let $\omega$ be weight function on locally compact group $G$. Then the following statements hold.

(i) If $\phi: G \to G$ is a continuous group epimorphism, $M(G, \overline{\omega})$ is weakly amenable and $\mathfrak{S}(I(G, \omega)) = I(G, \overline{\omega})$, then $M(G, \omega)$ is weakly amenable.

(ii) If $G$ is Abelian and $M(G, \overline{\omega})$ is weakly amenable, then $M(H, \omega|_H)$ is weakly amenable, where $H$ is a subgroup of $G$.

Corollary 2.8 Let $\omega_i$ be a weight function on a locally compact group $G_i$, for $i = 1, 2$. Then the following assertions are equivalent.

(a) $M(G_1, \omega_1) \hat{\otimes} M(G_2, \omega_2)$ is weakly amenable.
(b) $M(G_1, \omega_1) \hat{\otimes} M(G_2, \omega_2)$ is cyclically weakly amenable.
(c) $M(G_1, \omega_1) \hat{\otimes} M(G_2, \omega_2)$ is point amenable.
(d) $M(G_i, \omega_i)$ is weakly amenable and $G_i$ is discrete, for $i = 1, 2$.

Proof. Let $M(G_1, \omega_1) \hat{\otimes} M(G_2, \omega_2)$ be point amenable. Since $M(G_i, \omega_i)$ is unital, for $i = 1, 2$, from Proposition 2.1 we infer that then $M(G_1 \times G_2, \omega_1 \otimes \omega_2)$ is point amenable. By Theorem 2.4, $G_1 \times G_2$ is discrete. It follows that $G_i$ is discrete, for $i = 1, 2$. Hence $M(G_i, \omega_i) = \ell^1(G_i, \omega_i)$ and so

$$\ell^1(G_1, \omega_1) \hat{\otimes} \ell^1(G_1, \omega_1) = M(G_1, \omega_1) \hat{\otimes} M(G_2, \omega_2)$$

is weakly amenable. In view of Corollary 4.8 in [18], $\ell^1(G_i, \omega_i)$ is weakly amenable. So (c) implies (d).

Let $M(G_i, \omega_i)$ is weakly amenable and $G_i$ is discrete, for $i = 1, 2$. By Corollary 2.5, $M(G_i, \omega_i)$ is point amenable, for $i = 1, 2$. It follows from Theorem 4.1 in [20] and Proposition 2.1 that

$$M(G_1, \omega_1) \hat{\otimes} M(G_2, \omega_2) = M(G_1 \times G_2, \omega_1 \otimes \omega_2)$$

is point amenable. Again, apply Corollary 2.5 to conclude that $M(G_1, \omega_1) \hat{\otimes} M(G_2, \omega_2)$ is weakly amenable. That is, (d) implies (a). □

As a consequence of Corollary 2.8, we give the next result.

Corollary 2.9 Let $\omega_i$ be a weight function on a locally compact discrete group $G_i$, for $i = 1, 2$. Then the following assertions are equivalent.

(a) $\ell^1(G_1, \omega_1) \hat{\otimes} \ell^1(G_2, \omega_2)$ is weakly amenable.
(b) $\ell^1(G_1, \omega_1) \hat{\otimes} \ell^1(G_2, \omega_2)$ is cyclically weakly amenable.
(c) $\ell^1(G_1, \omega_1) \hat{\otimes} \ell^1(G_2, \omega_2)$ is point amenable.
(d) $\ell^1(G_i, \omega_i)$ is weakly amenable and $G_i$ is discrete, for $i = 1, 2$. 


We say that $T \in M(G, \omega)^*$ vanishes at infinity if for every $\varepsilon > 0$, there exists a compact subset $K$ of $G$, for which $|\langle T, \mu \rangle| < \varepsilon$, where $\mu \in M(G, \omega)$ with $|\mu|(K) = 0$ and $\|\mu\|_\omega = 1$. We denote by $M_s(G, \omega)$ the subspace of $M(G, \omega)^*$ consisting of all functionals that vanish at infinity. In the case where, $\omega(x) = 1$ for all $x \in G$, we write $M_s(G, \omega) := M_s(G)$.

The space $M_s(G, \omega)$ is a norm closed subspace of $M(G, \omega)^*$. It is proved that $M_s(G, \omega)^*$ with the first Arens product is a Banach algebra [16]. For each $f \in L^1(G, \omega)$, we may consider $f$ as a linear functional in $M_s(G, \omega)^*$. One can prove that $L^1(G, \omega)$ is a closed ideal in $M_s(G, \omega)^*$ and $M_s(G, \omega)^* = L^1(G, \omega)$ if and only if $G$ is discrete [16]; see [15] for the case $\omega = 1$.

**Corollary 2.10** Let $G$ be a locally compact group. Then the following assertions are equivalent.

(a) $M_s(G, \omega)^*$ is weakly amenable.

(b) $M_s(G, \omega)^*$ is cyclically weakly amenable.

(c) $M_s(G, \omega)^*$ is point amenable.

(d) $G$ is discrete and every non-inner quasi-additive function in $L^\infty(G, 1/\omega)$ is unbounded.

**Proof.** Let $M_s(G, \omega)^*$ be point amenable. Since $M(G, \omega)$ is a direct summand of $M_s(G, \omega)^*$, by Theorem 3.7 in [20], $M(G, \omega)$ is point amenable. Hence $G$ is discrete and every non-inner quasi-additive function in $L^\infty(G, 1/\omega)$ is unbounded. Thus (c) implies (d). It is easy to see that if $G$ discrete, then

$$M_s(G, \omega)^* = L^1(G, \omega)^* = M(G, \omega).$$

It follows that (d) implies (a). \qed

Let $L^\infty(G, 1/\omega)$ be the space of all Borel measurable functions $f$ on $G$ with $f/\omega \in L^\infty(G)$, the Lebesgue space of bounded Borel measurable functions on $G$. Let also $L^\infty_0(G, 1/\omega)$ denote the subspace of $L^\infty(G, 1/\omega)$ consisting of all functions $f \in L^\infty(G, 1/\omega)$ that vanish at infinity. It is proved that $L^\infty_0(G, 1/\omega)$ is left introverted in $L^\infty(G, 1/\omega)$.

So $L^\infty_0(G, 1/\omega)^*$ is a Banach algebra with the first Arens product [10]; see also [4, 11, 12, 21].

**Corollary 2.11** Let $G$ be a locally compact group. Then the following assertions are equivalent.

(a) $L^\infty_0(G, 1/\omega)^*$ is weakly amenable.

(b) $L^\infty_0(G, 1/\omega)^*$ is cyclically weakly amenable.

(c) $L^\infty_0(G, 1/\omega)^*$ is point amenable.

(d) $G$ is discrete and every non-inner quasi-additive function in $L^\infty(G, 1/\omega)$ is unbounded.
Corollary 2.12 Let $\omega_i$ be a weight function on a locally compact group $G_i$, for $i = 1, 2$. Then the following assertions are equivalent.

(a) $M_*(G_1, \omega_1)^*$ and $M_*(G_2, \omega_2)^*$ are weakly amenable.
(b) $L_0^\infty(G, 1/\omega)^*$ and $L_0^\infty(G, 1/\omega)^*$ are weakly amenable.
(c) $M_*(G_1, \omega_1)^* \hat{\otimes} M_*(G_2, \omega_2)^*$ is weakly amenable and $G_i$ is discrete, for $i = 1, 2$.
(d) $L_0^\infty(G, 1/\omega)^* \hat{\otimes} L_0^\infty(G, 1/\omega)^*$ is weakly amenable and $G_i$ is discrete, for $i = 1, 2$.

Proof. Assume that $M_*(G_1, \omega_1)^*$ and $M_*(G_2, \omega_2)^*$ are weakly amenable. By Corollary 2.10, $G_i$ is discrete and $M(G_i, \omega_i) = M_*(G_i, \omega_i)^*$ is weakly amenable, for $i = 1, 2$. It follows from Corollary 2.8 that

$$M_*(G_1, \omega_1)^* \hat{\otimes} M_*(G_2, \omega_2)^* = M(G_1, \omega_1) \hat{\otimes} M(G_2, \omega_2)$$

is weakly amenable. So (a) implies (c).

Let $M_*(G_1, \omega_1)^* \hat{\otimes} M_*(G_2, \omega_2)^*$ be weakly amenable and $G_i$ is discrete, for $i = 1, 2$. This implies that $M(G_1, \omega_1) \hat{\otimes} M(G_2, \omega_2)$ is weakly amenable. Thus $M_*(G_i, \omega_i)^* = M(G_i, \omega_i)$ is weakly amenable. Hence (c) implies (a). Similarly, (b) and (d) are equivalent. □

Let $LUC(G, 1/\omega)$ be the space of all continuous function $f$ on $G$ such that $f/\omega$ is a left uniformly continuous functions on $G$; for study of this space see [27]. Let $WAP(\mathfrak{a})$ be the space of all weakly almost periodic functionals on Banach algebra $\mathfrak{a}$, that is, $f \in \mathfrak{a}^*$ such that the map $a \mapsto af$ from $\mathfrak{a}$ into $\mathfrak{a}^*$ is weakly compact, where $\langle af, b \rangle = \langle f, ba \rangle$ for all $b \in \mathfrak{a}$.

Corollary 2.13 Let $WAP(L^1(G, \omega))^*$ or $LUC(G, \omega)^*$ be 0-point amenable. Then $G$ is discrete.

Let $\mathfrak{a}$ be one of the Banach algebras $M(G, \omega), M_*(G, \omega)^*, L_0^\infty(G, 1/\omega)^*, WAP(L^1(G, \omega))^*$ or $LUC(G, \omega)^*$.

Proposition 2.14 Let $G$ be a locally compact group. If $\mathfrak{a}$ is cyclically amenable, then every element of $CD(G, \omega)$ is inner.

Proof. Let $M(G, \omega)$ be cyclically amenable. Since $L^1(G, \omega)$ is a direct summand of $M(G, \omega)$, by Theorem 3.7 in [20], the Banach algebra $L^1(G, \omega)$ is cyclically amenable. It follows from Theorem 5.6 in [18] that every element of $CD(G, \omega)$ is inner. For the other cases, we only need to recall that

$$\mathfrak{a} = M(G, \omega) \oplus \mathfrak{B}$$

for some closed subspace $\mathfrak{B}$ of $\mathfrak{a}$. □
3 The second dual of Banach algebras

The main result of this section is the following which solves an open problem posed in [9].

**Theorem 3.1** Let $G$ be a locally compact group. Then the following assertion are equivalent.

(a) $L^1(G, \omega)^{**}$ is weakly amenable.
(b) $L^1(G, \omega)^{**}$ is cyclically weakly amenable.
(c) $L^1(G, \omega)^{**}$ is point amenable.
(d) $G$ is finite.

**Proof.** Let $\iota: L^1(G, \omega) \rightarrow L^1(G)$ be the inclusion map. Since $L^1(G, \omega)$ is dense in $L^1(G)$, $\iota$ is a continuous homomorphism with dense range. So $\iota^{**}: L^1(G, \omega)^{**} \rightarrow L^1(G)^{**}$ is epimorphism. Hence if $L^1(G, \omega)^{**}$ is point amenable, then by Theorem 2.1 in [20] the Banach algebra $L^1(G)^{**}$ is point amenable. It follows that every continuous point derivation of $L^1(G)^{**}$ at the discrete augmentation character is zero. From Theorem 11.17 in [5] infer that $G$ is finite. So (c)$\Rightarrow$(d). The implications (a)$\Rightarrow$(b)$\Rightarrow$(c) follows from Theorem 4.1 in [19].

**Corollary 3.2** Let $G$ be a locally compact group. Then the following assertion are equivalent.

(a) $M(G, \omega)^{**}$ is weakly amenable.
(b) $M(G, \omega)^{**}$ is cyclic amenable.
(c) $M(G, \omega)^{**}$ is point amenable.
(d) $G$ is finite.

**Proof.** Let $M(G, \omega)^{**}$ is point amenable. By Proposition 5.2 in [20], the Banach algebra $M(G, \omega)$ is point amenable. In view of Corollary 2.5, $G$ is discrete. Hence $L^1(G, \omega)^{**}$ is weakly amenable. Now, apply Theorem 3.1.

Let us recall that if there exists a compact invariant neighborhood of $e$ in $G$, then $G$ is called an $[IN]$-group. The following result is an improvement of Theorem 3.4 in [9].

**Theorem 3.3** Let $G$ be a connected locally compact group. If either $G_d$ is amenable or $G$ is an $[IN]$-group, then the following assertions are equivalent.

(a) $L^1(G, \omega)^{**}$ is weakly amenable.
(b) $M(G, \omega)$ is weakly amenable.
(c) $G = \{e\}$. 
Proof. Let $L^1(G,\omega)^{**}$ be weakly amenable. Since 
\[ L^1(G,\omega)^{**} = M(G,\omega) \oplus C_0(G,\omega)^\perp \]
and $C_0(G,\omega)^\perp$ is an ideal in $L^1(G,\omega)^{**}$, we have $M(G,\omega)$ is weakly amenable. So (a) $\Rightarrow$ (b). Let’s show that (b) $\Rightarrow$ (c). To this end, let $M(G,\omega)$ be weakly amenable. It follows from Theorem 2.4 that $G$ discrete and $M(G)$ is weakly amenable. If $G_d$ is amenable, then from Theorem 3.3 in [9] we infer that $G = \{e\}$. If $G$ is an $[IN]$—group, then by Theorem 3.4 in [9], $G$ is compact. Since $G$ is also discrete, it follows that $G$ is finite. Hence $G_d$ is amenable. Thus $G = \{e\}$. So (b) $\Rightarrow$ (c). The implication (c) $\Rightarrow$ (a) is clear. □

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