A-INFINITY OBSTRUCTION THEORY

FERNANDO MURO

Abstract. We define a new obstruction theory for the extension of truncated $A$-infinity algebra structures. Obstructions lie in the new terms of a spectral sequence which extends Bousfield–Kan’s fringed spectral sequence of a certain tower of fibrations. The obstructions living in the first and second pages are classical. We compute the second page of our spectral sequence, including the differential, in terms of Hochschild cohomology and universal Massey products. We put into practice our theory by showing with an example that obstructions can be explicitly computed beyond the second page.

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1. INTRODUCTION

An $A$-infinity algebra $(X, d, m_2, m_3, \ldots, m_n, \ldots)$ over a field is a cochain complex of vector spaces $(X, d)$ equipped with a sequence of morphisms of graded vector spaces

\[ m_n : X \otimes \cdots \otimes X \to X, \]

of degree $2 - n$ satisfying certain equations. Differential graded algebras are $A$-infinity algebras with $m_3 = m_4 = \cdots = m_n = \cdots = 0$ ($m_2$ is the product). An $A$-infinity algebra is minimal if the differential of the underlying complex vanishes $d = 0$. Any differential graded algebra $A$ is quasi-isomorphic to a minimal $A$-infinity

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structure on its cohomology $(H^*X, 0, m_2, m_3, \ldots, m_n, \ldots)$ where $m_2$ is the induced associative algebra structure. Therefore, if we aim at understanding the differential graded algebras with fixed cohomology graded algebra $A = (H^*X, m_2)$ we should consider the minimal $A$-infinity structures on it.

In order to investigate such structures, there are well known obstruction theories which decide when a truncated minimal $A$-infinity structure $(X, 0, m_2, \ldots, m_k)$, also called $A_k$-structure, can be extended to an $A_{k+1}$-structure, see for instance [LH03 Ch. B]. In the way we have posed the problem, obstructions are Hochschild cochains. They live in a Hochschild complex, which is usually huge, and must vanish as cochains, not just in cohomology. Hence, they are of little interest. If we allow for the modification of the last piece of the $A_k$-structure, $m_k$, then the obstruction lies in the Hochschild cohomology of the graded algebra $A = (X, m_2)$, more precisely in

$$HH^{k+1,2-k}(A).$$

The first degree in Hochschild cohomology $HH^{p,q}(A)$ is the Hochschild degree and the second degree is the internal degree coming from the grading of $A$. The first possibly non-trivial obstruction occurs when $k = 4$. In that case $m_3$ represents a Hochschild cohomology class

$$\{m_3\} \in HH^{3,-1}(A),$$

that we call universal Massey product [Kad82] [BKS04], and the obstruction is the Gerstenhaber square

$$Sq(\{m_3\}) \in HH^{5,-2}(A),$$

which in characteristic $\neq 2$ is simply

$$Sq(\{m_3\}) = \frac{1}{2}[\{m_3\}, \{m_3\}].$$

If the Hochschild cohomology of $A$ vanishes in total degree 3 then all $A_k$-structures extend to $A$-infinity structures, since recipients of obstructions vanish. However this happens very seldom, and obstructions are rather complicated to compute when they live in a non-trivial vector space.

In this paper, given a minimal $A_k$-algebra $(X, 0, m_2, \ldots, m_k)$, we define the first $\left\lfloor \frac{k+1}{2} \right\rfloor$ pages of a spectral sequence $E^{st}_r$, that we therefore call truncated, which is concentrated in (most of) the right half plane, and an obstruction living in $E_r^{k-1,k-2}$ to the existence of an $A_{k+1}$-extension after possibly perturbing $m_{k-r+2}, \ldots, m_k$. The second page of the spectral sequence is

$$E^s_2 = HH^{s+2,-t}(A)$$

for $t \geq s > 0$ and for $s \geq 2$ and $t \in \mathbb{Z}$, and the obstruction for $r = 2$ is the classical one (for $r = 1$ we obtain the uninteresting Hochschild cochains). Hence, the more perturbations we allow, the bigger are the chances of getting a trivial obstruction. Even the recipient is more likely to be trivial! We also prove that, for $k \geq 5$, the differential of the second page in the previous range is up to sign the Gerstenhaber bracket with the universal Massey product

$$d_2 = \pm[\{m_3\}, -].$$

In particular, if we know the algebraic structure of the Hochschild cohomology (see Section 2 for a remainder) we can directly compute most of the third page $E^3_{st}$. Note that the differential $\pm[\{m_3\}, -]$ indeed squares to zero since $k \geq 5$ and $\{m_3\}$ has
odd total degree so $d_2^2 = \{\{m_3\}, \{\{m_3\}, -\}] = [\text{Sq}(\{m_3\}), -\}$ by standard identities in a Gerstenhaber algebra, therefore the obstruction $\text{Sq}(\{m_3\}) = 0$ vanishes.

With this enhanced obstruction theory, we can prove theorems like the following one.

**Theorem 1.1.** Let $(X, 0, m_2, m_3, m_4)$ be a minimal $A_4$-algebra such that the universal Massey product satisfies $\text{Sq}(\{m_3\}) = 0$ and that the cup product

$$\{m_3\} \sim - : HH^p(A) \rightarrow HH^{p+3,q-1}(A)$$

in the Hochschild cohomology of the graded algebra $A = (X, m_2)$ is bijective for $p \geq 2$ and $q \in \mathbb{Z}$. Then, up to $A$-infinity quasi-isomorphism with identity linear part, there is a unique $A$-infinity algebra $(X, 0, m_2, m_3', m_4', \ldots)$ with the same universal Massey product $\{m_3\} = \{m_3'\}$.

This theorem, proved in Section 9, rather than imposing strong vanishing conditions, is tailored to be applied to graded algebras with a rich structure on Hochschild cohomology. Therefore, it is in the antipodes of classical results and opens a completely new range of applicability.

The previous theorem can be for instance applied in the following situation. Assume $\Lambda$ is an ungraded algebra fitting in an exact sequence of $\Lambda$-bimodules

$$\Lambda_{\sigma} \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow \Lambda,$$

where $P_i$ is projective, $i = 0, 1, 2$, and $\Lambda_{\sigma}$ is $\Lambda$ with right $\Lambda$-module structure twisted by an algebra automorphism $\sigma : \Lambda \cong \Lambda$, e.g. a (deformed) preprojective algebra of generalized Dynkin type [BES07]. If we consider the graded algebra

$$A = \frac{\Lambda \langle x^{\pm 1} \rangle}{(x^{\lambda} - \sigma(x^{\lambda}))} = \bigoplus_{n \in \mathbb{Z}} \Lambda_{\sigma^n}, \quad |x| = -1,$$

and apply $A \otimes_{k(x^{\pm 1})} A^{\text{op}} \otimes_{A \otimes A^{\text{op}}} -$ to the previous exact sequence we obtain a class in $HH^3(A) \cong \text{Ext}_{A \otimes A^{\text{op}}}(A, A)$, represented by some $m_3$, such that $\{m_3\} \sim -$ satisfies the assumption in Theorem 1.1. Hence, an $A$-infinity structure on $A$ with universal Massey product $\{m_3\}$ exists (and is essentially unique) if and only if $\text{Sq}(\{m_3\}) = 0$. In Section 10 we explicitly compute examples of this kind. In work in progress, we apply Theorem 1.1 to the existence and uniqueness of models for finite 1-Calabi-Yau triangulated categories [Ami07], for which there are still non-standard examples in characteristic 2 which are not known to have models.

The truncated spectral sequence, far from being a mere artefact to contain obstructions, has the following homotopical meaning. The space of DG-algebra structures on a given complex $X = (X, d)$ is weakly equivalent to the space of $A_\infty$-structures, which is the mapping space

$$\text{Map}(A_\infty, \mathcal{E}(X))$$

in the model category $\text{dgOp}$ of differential graded (non-symmetric) operads, see [Mur11, Theorem 1.1] or [Lyu11, Proposition 1.8], from the $A$-infinity operad $A_\infty$ to the endomorphism operad of $X$. Vertices in this space are honest $A$-infinity structures on $X$ and two of them lie in the same component if and only if there is an $A_\infty$-map with identity linear part between them (in particular an $A$-infinity quasi-isomorphism). This space is the homotopy limit of a tower of fibrations

$$\cdots \rightarrow \text{Map}(A_{n+1}, \mathcal{E}(X)) \rightarrow \text{Map}(A_n, \mathcal{E}(X)) \rightarrow \cdots$$
Figure 1. Range of definition of the Bousfield–Kan fringed spectral sequence of a tower of fibrations.

whose layers are the spaces of $A_n$-algebra structures, $n \geq 2$.

A base point $\phi$ in $\text{Map}(A_\infty, E(X))$, i.e. an $A_\infty$-structure on $X$, induces base points the in tower by restriction. The Bousfield–Kan fringed spectral sequence [BK72, Ch. IX, §4] of such a tower of based fibrations is a spectral sequence concentrated in the upper half of the bisection of the first quadrant $t \geq s \geq 0$ (Fig. 1). It is somewhat anomalous since, despite it mostly consists of abelian groups $E^r_{st}$, $t - s \geq 2$, the terms are just groups for $t - s = 1$ and plain pointed sets for $t - s = 0$. Moreover, the terms of the angle bisector $E^r_{ss}$ are not given by the homology of differentials $d_{r-1}$.

For our particular tower of fibrations, we extend in Section 5 the Bousfield–Kan fringed spectral sequence to most of the right half plane (Fig. 2). More precisely, we extend it to the half plane $s \geq 2r - 2$. There are homogeneous definitions of the terms $E^r_{st}$ in the red and blue regions which coincide in the overlap. They are not defined in the white region. The blue region consists of vector spaces. Moreover, we show that the groups $E^r_{r,s+1}$ are abelian for all $s \geq 0$ and we endow the terms $E^r_{ss}$ with an abelian group structure for $s \geq r - 1$.

Differentials
\[ d_r : E^r_{st} \longrightarrow E^r_{s+r,t+r-1}, \]
which are group or vector space homomorphisms according to the nature of the source, are defined except for $t = s < r$. This includes the Bousfield–Kan differentials in the red region, all possible differentials in the blue region, as well as some differentials departing from the fringed line which jump over the white region (Fig. 2). The term $E^r_{s+1}$ is the homology of $d_r$ whenever the incoming and outgoing differentials are defined (the incoming differential is taken to be $0 \rightarrow E^r_{st}$ if $s < r$). This covers most of the terms, it just excludes the pointed sets $E^r_{r+1}$, $s < r$, and the vector spaces $E^r_{r+1}$, $2r \leq s < 3r - 2$, $t < s - 1$, below the fringed line, even below the line $t - s = -1$ where obstructions live (Fig. 3).

The limit terms $E^r_{\infty}$ are only defined for $t \geq s \geq 0$ since the vertical blue line moves to the right as $r$ increases, hence they are just Bousfield–Kan’s. This does not diminish the relevance of our extension since its extra structure facilitates computations. Convergence issues were satisfactorily treated by Bousfield and Kan [BK72, Ch. IX, §5], the term $E^r_{\infty}$ contributes to $\pi_{t-s}(\text{Map}(A_\infty, E(X)), \phi)$. Our
computations are suitable to prove a very strong notion of convergence in some specific cases, like in the following result.

**Theorem 1.2.** Under the hypotheses of Theorem 1.1, if \( \phi: A_\infty \to \mathcal{E}(X) \) corresponds to the \( A \)-infinity algebra \( (X, 0, m_2, m_3, m_4, \ldots) \), the terms \( E_3^{st} \) of the extended spectral sequence related to the homotopy groups of \( \text{Map}(A_\infty, \mathcal{E}(X)) \) based at \( \phi \) vanish for \( s \geq 2 \). Hence the spectral sequence collapses at the third page for \( t > s \), it satisfies Mittag-Leffler convergence, and we have short exact sequences,

\[
E_3^{1, t+1} \hookrightarrow \pi_t(\text{Map}(A_\infty, \mathcal{E}(X)), \phi) \to E_3^{0, t} , \quad t \geq 1.
\]

This theorem is also proved in Section 9. See Section 10 for an example satisfying these assumptions where we can even compute the terms \( E_3^{st} \), \( t > s = 0, 1 \).

We fully calculate the second page of the extended spectral sequence, terms and differentials, which is essentially given by Hochschild cohomology and by the Gerstenhaber bracket with the universal Massey product, as indicated above, see Section 6. In particular, the terms \( E_2^{st} \) and the differential \( d_2 \) only depend on the \( A_3 \)- and \( A_5 \)-structures underlying \( \phi \), respectively. We show more generally that the page \( E_r^{st} \) and the differential \( d_r \) only depend on the underlying \( A_{2r-1} \)- and \( A_{2r+1} \)-structures. This observation leads to the construction of the aforementioned truncated spectral sequence in Section 7. Then we proceed in Section 8 with the definition of the obstructions, which lie in the diagonal line \( t - s = -1 \) right below the fringed line, i.e. in the new part.

All these constructions are actually defined for an arbitrary target DG-operad \( U \), but the computation of the second page is specially neat for \( U = \mathcal{E}(X) \).

We construct the extended spectral sequence from the following simple observation. It is possible to construct the operad \( A_{r+s} \) directly from \( A_r \) for \( s \geq r \geq 0 \) via a cofiber sequence

\[
\Sigma_{A_m} B_{m,r,s} \to A_r \to A_{r+s}
\]

in the category \( A_m \downarrow \text{dgOp} \) of differential graded operads under \( A_m \), \( r \geq m \geq s \), where \( B_{m,r,s} \) is a relatively linear DG-operad, see Sections 8 and 9. All constructions
in this paper are based on these cofiber sequences, in other closely related cofiber sequences, and in their interrelationship.

Bousfield defined [Bou89] an extension of the fringed spectral sequence when the tower of fibrations comes from a cosimplicial space, and obstructions therein. We tried to define a cosimplicial space for our tower of fibrations in order to apply Bousfield’s result, but in the end it was more complicated and less suitable for computations. Our extension goes further beyond Bousfield’s. We believe this is just due to the particular features of the category of DG-operads. In fact, we conjecture that it is possible to obtain versions of our results in the category of operads of spaces, and that Bousfield’s range of extension is optimal in that case. We also think that Bousfield’s extension could be conceptually explained from the results recently obtained by Mathew and Stojanoska [MS15] on the homotopy fibers of the bonding maps of the tower of fibrations of a cosimplicial space.

We work all the time over an arbitrary commutative ground ring \( \mathbb{k} \). We have restricted to fields in the introduction to avoid projectivity hypotheses. We use mostly, but not exclusively, chain complexes, i.e. complexes \( \mathbb{A} \) with differentials of degree \(-1\), and we indicate the degree of a homogeneous component of a graded \( \mathbb{k} \)-module in the subscript \( A_n \). The degree of a homogeneous element \( x \) will be denoted by \( |x| \). We sometimes change to cohomological degrees by reversing signs \( A^n = A_{-n} \). This also changes the sign of the degree of differentials. Although the introduction has been completely written with cohomological degrees, it would be complicated to restrict to just chain or cochain complexes in the body of the paper. Hochschild cohomology has of course a cohomological grading, while the operad \( A_\infty \) is naturally homological since it arises as the cellular chain complex of associahedra. Such an unnecessary restriction would also cause confusion with some references. All operads will be non-symmetric. Graded means \( \mathbb{Z} \)-graded.

2. Algebraic background on operads and Hochschild cohomology

A graded operad or DG-operad \( \mathcal{O} \) is a sequence of graded modules or chain complexes \( \{ \mathcal{O}(n) \}_{n \geq 0} \), where \( \mathcal{O}(n) \) is called the arity \( n \) component, equipped with
operadic compositions,
\[ o_i : O(p) \otimes O(q) \to O(p + q - 1), \quad 1 \leq i \leq p, \ q \geq 0, \]

and an identity element \( id = id_O \in O(1) \) satisfying
\[ x o_i (y o_j z) = (x o_i y) o_{i+j-1} z; \]
\[ (x o_i y) o_j z = (-1)^{|y||z|} (x o_j z) o_{i+n-1} y, \quad j < i, \quad z \in O(n); \]
\[ id o_1 x = x = o_i id. \]

In the DG-case, operadic compositions being chain maps translates in the operadic Leibniz rule,
\[ d(x o_i y) = d(x) o_i y + (-1)^{|x|} x o_i d(y), \]

which implies \( d(id) = 0 \). The differential of \( O \) is sometimes denoted by \( d_O \) in order to avoid confusion.

Given a graded operad \( O \), an \( O \)-module \( M \) in the sense of [Mar96, Definition 1.4] is a sequence of graded modules \( \{M(n)\}_{n \geq 0} \) equipped with compositions,
\[ o_i : M(p) \otimes O(q) \to M(p + q - 1), \quad o_i : O(p) \otimes M(q) \to M(p + q - 1), \quad 1 \leq i \leq p, \ q \geq 0, \]
satisfying the same laws as graded operads when one of the variables is in \( M \) and the rest in \( O \). This is the same as a linear \( O \)-module [BHT97, Definition 2.13] or an infinitesimal \( O \)-bimodule [MV09, §3.1]. Any graded operad is a module over itself and restriction of scalars along graded operad maps is defined in the obvious way.

The category of \( O \)-modules is graded abelian.

Recall that the suspension \( \Sigma X \) of a chain complex \( X \) is defined by the existence of a natural degree +1 isomorphism
\[ \sigma : x = \Sigma X. \]

Suspension is an invertible functor. The suspension of an \( O \)-module \( M \) is defined by the aritywise suspension of complexes, imposing that \( \sigma : x = \Sigma M \) is a degree +1 map of \( O \)-modules, i.e., given \( x \in O \) and \( y \in M \),
\[ \sigma(x o_i y) = (-1)^{|y|} x o_i \sigma(y), \quad \sigma(y o_i x) = \sigma(y o_i x). \]

A sequence of graded sets can be regarded as a set \( S \) equipped with a map \( S \to \mathbb{N} \times \mathbb{Z} : x \mapsto \text{arity of } x, |x| \), where \(|x|\) is the degree of \( x \). We denote by \( F(S) \) the free graded operad on \( S \), which satisfies the obvious universal property.

In order to simplify formulas, we will later use the following algebraic structures which underly operads.

A graded or DG-brace algebra is a graded module or chain complex \( B \) equipped with maps called braces, \( n \geq 1 \),
\[ B \otimes (n + 1) \to B, \]
\[ x_0 \otimes x_1 \otimes \cdots \otimes x_n \mapsto x_0 \{x_1, \ldots, x_n\}, \]
satisfying the brace relation,
\[ x\{y_1, \ldots, y_p\}\{z_1, \ldots, z_q\} = \sum_{0 \leq i_0 \leq i_1 \leq \cdots \leq i_p \leq q} (-1)^{\sum_{j=0}^{p} i_j} x\{z_{i_0+1}, \ldots, z_{i_1}\}, \ldots, y_p\{z_{i_p+1}, \ldots, z_{j_p}\}, z_{j_p+1}, \ldots, z_q\}. \]
The sign is simply determined by the Koszul sign rule, \( \epsilon = \sum_{k=1}^{p} \sum_{i=1}^{k} |y_k| \cdot |z_l| \). In the DG-case, the fact that braces are chain maps is equivalent to the brace Leibniz rule,

\[
d(x_0\{x_1, \ldots, x_n\}) = d(x_0)\{x_1, \ldots, x_n\} + \sum_{i=1}^{n} (-1)^{\sum_{j=0}^{i-1} |x_j|} x_0\{x_1, \ldots, d(x_i), \ldots, x_n\}.
\]

Any graded or DG-brace algebra has an underlying graded or DG-Lie algebra structure with Lie bracket defined by

\[
[x, y] = x\{y\} - (-1)^{|x||y|} y\{x\}.
\]

For \(|x|\) odd, or for all \(x\) if \(2 = 0 \in k\), the following equation holds,

(2.1) \[
[x\{x\}, y] = [x, [x, y]].
\]

If \(\mathcal{O}\) is a graded or DG-operad, then \(\oplus_{n \geq 0} \mathcal{O}(n)\) is a graded or DG-brace algebra with the following structure, \(x_i \in \mathcal{O}(j_i)\), \(0 \leq i \leq n\),

(2.2) \[
x_0\{x_1, \ldots, x_n\} = \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq j_0} (\cdots ((x_0 \circ_{i_1} x_1) \circ_{i_2+j_i-1} x_2) \cdots) \circ_{i_n+j_i+\cdots+j_{n-1}-(n-1)} x_n.
\]

Note that

(2.3) \[
x_0\{x_1, \ldots, x_n\} = 0, \quad n > j_0,
\]

since the previous summation is empty in this case. In particular, the brace subalgebra structure on \(\oplus_{n \geq 1} \mathcal{O}(n)\) extends to \(\prod_{n \geq 1} \mathcal{O}(n)\), completing with respect to the arity filtration. Observe also that

\[
\mathcal{O}(j_0)_{d_0} \otimes \cdots \otimes \mathcal{O}(j_n)_{d_n} \rightarrow \mathcal{O}(j_0 + \cdots + j_n - n)_{d_0+\cdots+d_n},
\]

\[
x_0 \otimes x_1 \otimes \cdots \otimes x_n \mapsto x_0\{x_1, \ldots, x_n\},
\]

i.e. braces in \(\oplus_{n \geq 0} \mathcal{O}(n)\) are degree-homogeneous, as required in any brace algebra, but if we regard the arity as another degree, then it has degree \(-n\) with respect to it. Concerning signs in the brace relation, arity does not play any role. As a consequence, the induced Lie bracket satisfies

\[
\{-, -\} : \mathcal{O}(p)_d \otimes \mathcal{O}(q)_{e} \rightarrow \mathcal{O}(p+q-1)_{d+e},
\]

and \([x, y] = 0\) if \(x, y \in \mathcal{O}(0)\).

A degree \(-1\) multiplication on a graded operad \(\mathcal{O}\) is an element \(m_2 \in \mathcal{O}(2)_{-1}\) satisfying \(m_2\{m_2\} = 0\). By (2.1),

\[
[m_2, -] : \mathcal{O}(n)_d \rightarrow \mathcal{O}(n+1)_{d-1}
\]

is a differential in \(\oplus_{n \geq 0} \mathcal{O}(n)\). Moreover, by the Jacobi identity \(\oplus_{n \geq 0} \mathcal{O}(n)\) equipped with this differential is actually a DG-Lie algebra. It is not however a DG-brace algebra. Actually the failure in being a DG-brace algebra yields relations in homology between the Lie bracket and the additional operation we now define.

The cup product

\[
\sim : \mathcal{O}(p)_d \otimes \mathcal{O}(q)_{e} \rightarrow \mathcal{O}(p+q)_{d+e-1}
\]

in \(\oplus_{n \geq 0} \mathcal{O}(n)\), defined by

\[
x \sim y = (-1)^{|x|}m_2\{x, y\},
\]
is associative since, by the brace relation,

\[0 = m_2(m_2\{x, y, z\}) = m_2(m_2\{x, y\}, z) + (-1)^{|z|}m_2\{x, m_2\{y, z\}\}.\]

The rest of terms appearing in the brace relation vanish by (2.3), since \(m_2\) has arity 2. This product yields a differential graded algebra structure on \(\oplus_{n \geq 0}O(n)\) if we shift degrees by \(-1\). The Leibniz rule

\[(2.4) \quad [m_2, x \rightsquigarrow y] = [m_2, x] \rightsquigarrow y + (-1)^{|x|-1}x \rightsquigarrow [m_2, y]\]

follows straightforwardly from the brace relation and from the identity \(m_2\{m_2\}\).

The homology of \(\oplus_{n \geq 0}O(n)\) is therefore a graded Lie algebra, and also a graded associative algebra with respect to the degree shifted by \(-1\). The associative algebra structure is actually commutative in the graded sense,

\[x \rightsquigarrow y = (-1)^{(|x|-1)(|y|-1)}y \rightsquigarrow x.\]

This holds since, at the level of chains, the brace relation yields

\[(2.5) \quad x \rightsquigarrow y - (-1)^{|x||y|}y \rightsquigarrow x = -(-1)^{|x|}([m_2, x\{y\}] - [m_2, x]\{y\} - (-1)^{|x|}[m_2, y]).\]

Both algebraic structures in the homology of \(\oplus_{n \geq 0}O(n)\) are related by the following derivation equation

\[\{x, y \rightsquigarrow z\} = [x, y] \rightsquigarrow z + (-1)^{|x||y|-1}y \rightsquigarrow [x, z].\]

This follows from the fact that, on chains, the brace relation gives rise to

\[\{x, y \rightsquigarrow z\} - [x, y] \rightsquigarrow z - (-1)^{|x||y|-1}y \rightsquigarrow [x, z]
= (-1)^{|x|}([m_2, x\{y, z\}] - [m_2, x]\{y, z\} - (-1)^{|x|}[m_2, y]).\]

For \(d\) odd, or for all \(d\) if \(2 = 0 \in \mathbb{k}\), the brace squaring quadratic operation

\[\text{Sq}: O(n)_d \rightarrow O(2n - 1)_{2d}, \quad \text{Sq}(x) = x\{x\},\]

passes to homology, where it is called Gerstenhaber square,

\[\text{Sq}: H_d(O(n)) \rightarrow H_{2d}(O(2n - 1)).\]

This can be checked by using (2.5) and (2.4). Even at the level of chains, it satisfies

\[\text{Sq}(x + y) = \text{Sq}(x) + \text{Sq}(y) + [x, y],\]

\[[\text{Sq}(x), y] = [x, [x, y]].\]

Here we use (2.4) for the second equation. Note that the Gerstenhaber square vanishes in arity 0 by (2.3). The relation between the Gerstenhaber square and the cup product is given by the following formula in homology, which holds if \(|x|\) and \(|y|\) are odd, or if \(2 = 0 \in \mathbb{k}\),

\[\text{Sq}(x \rightsquigarrow y) = \text{Sq}(x) \rightsquigarrow y^2 + x \rightsquigarrow [x, y] \rightsquigarrow y + x^2 \rightsquigarrow \text{Sq}(y).\]

This formula is a consequence of the following equation at the level of chains,

\[\text{Sq}(x \rightsquigarrow y) - \text{Sq}(x) \rightsquigarrow y^2 = x \rightsquigarrow [x, y] \rightsquigarrow y - x^2 \rightsquigarrow \text{Sq}(y)\]

\[= |m_2, x \rightsquigarrow (y\{x, y\})| + x \rightsquigarrow (y\{m_2, x\}, y) - x \rightsquigarrow (y\{x, m_2, y\})
+ |m_2, (x\{x, y\}) \rightsquigarrow y| + (x\{m_2, x\}, y) \rightsquigarrow y - (x\{x, m_2, y\}) \rightsquigarrow y
+ |m_2, (x\{x\}) \rightsquigarrow (y\{y\})| + (x\{m_2, x\}) \rightsquigarrow (y\{y\}) - (x\{x\}) \rightsquigarrow (y\{m_2, y\})
- |m_2, x \rightsquigarrow y\{x, y\}|.\]
The most prominent example of operad with degree $-1$ multiplication is the endomorphism operad of the suspension of a graded associative algebra. Recall that the *endomorphism operad* $\mathcal{E}(X)$ of a chain complex $X$ is given by

$$\mathcal{E}(X)(n) = \text{Hom}(X^\otimes n, X),$$

where Hom stands for the internal morphism object in the category of chain complexes. The composition product $\circ_i$ is given by the composition of multilinear maps at the $i^{\text{th}}$ slot and the operadic identity is the identity map.

If $A$ is a graded associative algebra, the *shifted multiplication*

$$m_2: \Sigma A \otimes \Sigma A \longrightarrow \Sigma A$$

is the map defined by

$$m_2(\sigma x \otimes \sigma y) = (-1)^{|x|} \sigma(x \cdot y).$$

This map $m_2 \in \mathcal{E}(\Sigma A)(2)_{-1}$ is a degree $-1$ multiplication for $\mathcal{E}(\Sigma A)$. In fact, the equation $m_2\{m_2\} = 0$ is equivalent to the associativity of the graded algebra structure.

The *Hochschild complex* $C^{p,q}(A)$ of $A$ is the $(\mathbb{N} \times \mathbb{Z})$-graded complex (Hochschild degree, internal degree) given by the endomorphism operad with the following alternative grading

$$C^{p,q}(A) = \mathcal{E}(\Sigma A)(p)_{-p-q+1}.$$

Note that the differential has bidegree $(0, 1)$, hence the total complex is of cohomological type. The *Hochschild cohomology* $HH^{p,q}(A)$ endowed with the cup product is a graded commutative algebra with respect to the total degree,

$$\cup: HH^{p,q}(A) \otimes HH^{s,t}(A) \longrightarrow HH^{p+s,q+t}(A),$$

and a graded Lie algebra for the total degree shifted by $-1$,

$$[-,-]: HH^{p,q}(A) \otimes HH^{s,t}(A) \longrightarrow HH^{p+s-1,q+t}(A).$$

The Gerstenhaber square is defined in even total degree, or everywhere if $2 = 0 \in k$,

$$\text{Sq}: HH^{p,q}(A) \longrightarrow HH^{2p-2q}(A).$$

Both the Lie bracket and the Gerstenhaber square vanish when restricted to Hochschild degree 0. Summing up, if we denote the total degree of $x \in HH^{p,q}(A)$ by $|x| = p+q$, 
then the following equations hold in Hochschild cohomology,

\[(x \triangleright y) \triangleright z = x \triangleright (y \triangleright z),\]

\[x \triangleright y = (-1)^{|x||y|} y \triangleright x,\]

\[[x, y] = -(-1)^{|x|-1} [y, |x|],\]

\[x, x] = 0, \quad |x| \text{ odd,}\]

\[[x, [y, z]] = [[x, y], z] + (-1)^{|x|-1} [y, [x, z]],\]

\[x, [x, x]] = 0, \quad |x| \text{ even,}\]

\[x, y \triangleright z = [x, y] \triangleright z + (-1)^{(|x|-1)|y|} y \triangleright [x, z],\]

\[\text{Sq}(x + y) = \text{Sq}(x) + \text{Sq}(y) + [x, y], \quad |x|, |y| \text{ even or } 2 = 0 \in k,\]

\[\text{Sq}(x \triangleright y) = \text{Sq}(x) \triangleright y^2 + x \triangleright [x, y] \triangleright y + x^2 \triangleright \text{Sq}(y), \quad \text{idem,}\]

\[\text{[Sq}(x, y) = [x, [x, y]], \quad |x| \text{ even or } 2 = 0 \in k.\]

Other sources use operads \(O\) with degree 0 multiplication and the endomorphism operad \(\mathcal{E}(A)\) of an associative algebra \(A\), rather than its suspension. The great disadvantage of that approach is that it is necessary to introduce complicated signs in order to consider the appropriate brace algebra structure on \(\bigoplus_{n \geq 0} O(n)\). Nevertheless, both approaches are equivalent via the operadic suspension \(\Lambda: \mathsf{dgOp} \to \mathsf{dgOp}\), which is an automorphism of the category of DG-operads satisfying \(\Lambda \mathcal{E}(X) \cong \mathcal{E}(\Sigma X)\) and preserving all the homotopical structure that we review in the following section, compare [Mur15, Definition 2.4 and Remark 2.5].

3. SOME HOMOTOPY THEORY OF DG-OPERADS

We endow the category \(\mathsf{dgOp}\) of DG-operads with the model structure transferred from the projective model structure on chain complexes. This model structure exists by [Lyu11, Mur11]. Weak equivalences are quasi-isomorphisms and fibrations are surjections, in particular all objects are fibrant. Cofibrations in \(\mathsf{dgOp}\) are the retracts of cellular maps, defined below. We also consider the category \(O \downarrow \mathsf{dgOp}\) of DG-operads under a fixed DG-operad \(O\) and the category \(O \downarrow \mathsf{dgOp} \downarrow O\) of DG-operads over and under \(O\), whose objects are DG-operads equipped with an inclusion \(O \to P\) and a retraction \(P \to O\), also called based objects. These categories inherit a model structure from \(\mathsf{dgOp}\). The trivial map in \(O \downarrow \mathsf{dgOp}\) from a based object is the composite \(P \to O \to Q\).

A map of DG-operads \(O \to P\) is cellular if, on underlying graded operads, it is the inclusion of the first factor of a coproduct \(P = O \amalg \mathcal{F}(S)\) where the second factor is free on a sequence of graded sets \(S\) endowed with a continuous increasing filtration \(\{S_\beta\}_{\beta \leq \alpha}\), \(\alpha\) an ordinal, such that \(d(S_{\beta+1}) \subset O \amalg \mathcal{F}(S_\beta)\) for all \(\beta < \alpha\). We also say that \(P\) is cellular relative to \(O\), or just relatively cellular if \(O\) is understood.

The sub-\(O\)-module \(\langle S \rangle_O \subset O \amalg \mathcal{F}(S)\) generated by \(S\) is free. The suspension of a free \(O\)-module is free,

\[\Sigma(S)_O = \langle \sigma S \rangle_O.\]

A linear DG-operad is a relatively cellular DG-operad as above whose differential (co)restricts to

\[S \xrightarrow{(d_0, d_1)} O \oplus \langle S \rangle_O \subset P.\]
If \( d_0 = 0 \) we say that it is \textit{strictly linear}. The differential of a linear DG-operad also (co)restricts to

\[
\langle S \rangle_O \xrightarrow{\left( \begin{array}{c} d_0 \\ d_1 + d_{O,S} \end{array} \right)} O \oplus \langle S \rangle_O \subset P.
\]

Here \( \bar{d}_0 \) and \( \bar{d}_1 \) are the unique \( O \)-module morphisms extending \( d_0 \) and \( d_1 \) to the free \( O \)-module \( \langle S \rangle_O \) and \( d_{O,S} : \langle S \rangle_O \to \langle S \rangle_O \) is the only \( k \)-linear map satisfying \( d(S) = 0 \) and the operadic Leibniz rule if one element in \( x \circ_i y \) is in \( O \) and the other one is in \( \langle S \rangle_O \), i.e. it is the (co)restriction of the differential of the DG-operad coproduct \( O \amalg F(S) \) where \( F(S) \) is endowed with the trivial differential.

As in any model category, (functorial) cylinders in \( O \Downarrow \text{dgOp} \) exist. Nevertheless, we are more interested in small cylinders \([\text{Mur}15]\) than in functorial ones. Any relatively cellular DG-operad \( P \) has a relative cylinder \( I_O P \) with underlying graded operad

\[(3.1) \quad I_O P = O \amalg F(i_0 S \amalg \sigma S \amalg i_1 S).\]

Here \( i_0 S \) and \( i_1 S \) are plain copies of \( S \). The differential on these generators is determined by the fact that the obvious relative inclusion maps \( i_0, i_1 : P \to I_O P \) are DG-maps. On suspended generators, the differential has a complicated recursive formula. However, if \( P \) is linear it is simply given by

\[d(\sigma x) = i_0 x - i_1 x - \sigma d_1(x), \quad x \in S.\]

The \textit{projection of the cylinder} \( I_O P \to P \), which identifies \( i_0 S = i_1 S = S \) and sends \( \sigma S \) to zero, is a DG-map in general. Given a map \( f : P \to Q \) of cellular DG-operads, a map \( I_O f : I_O P \to I_O Q \) is a \textit{cylinder} of \( f \) if it is compatible with \( f \) and the inclusions and projection of the cylinders.

The relative \textit{torus} \( T_O P \) is the coequalizer of the morphisms \( i_0, i_1 : P \to I_O P \). It has underlying graded operad

\[(3.2) \quad T_O P = P \amalg F(\sigma S).\]

If \( P \) is linear

\[d(\sigma x) = -\sigma d_1(x), \quad x \in S.\]

In general, the torus is an operad under \( P \) by means of the inclusion \( i_0 = i_1 : P \subset T_O P \), and it is based by the map \( T_O P \to P \) induced by the projection of the cylinder, which is the identity on the first factor of the previous coproduct and trivial on \( \sigma S \).

Assume now that \( P \) is based. For instance, any strictly linear DG-operad is canonically based by the map \( P \to O \) under \( O \) which vanishes on \( S \). The \textit{mapping cone} \( C_f \) of a map \( f : P \to Q \) relative to \( O \) is the colimit of

\[Q \xleftarrow{f} P \xrightarrow{i_0} I_O P \xleftarrow{i_1} P \to O.\]

As a graded operad \( C_f \) is \( Q \amalg F(\sigma S) \). If \( P \) is strictly linear, the differential is defined by the fact that the inclusion of the first factor \( Q \to C_f \) is a DG-map and

\[d(\sigma x) = f(x) - \sigma d(x), \quad x \in S.\]

In particular, \( C_f \) would be linear under \( Q \).
If $f: \mathcal{P} \to \mathcal{Q}$ is cellular, the mapping cone of $f$ is naturally weakly equivalent to the quotient $\mathcal{Q}/\mathcal{P}$, i.e. the colimit of

$$\mathcal{Q} \leftarrow \mathcal{P} \to \mathcal{O}.$$

If the graded operad underlying $\mathcal{Q}$ is $\mathcal{P} \amalg \mathcal{F}(S') = \mathcal{O} \amalg \mathcal{F}(S) \amalg \mathcal{F}(S')$, then the underlying graded operad of $\mathcal{Q}/\mathcal{P}$ is $\mathcal{O} \amalg \mathcal{F}(S')$ endowed with the unique differential which turns the projection onto the quotient

$$\text{proj.}: \mathcal{Q} \to \mathcal{Q}/\mathcal{P}$$

into a DG-map. This projection is obviously defined on $S$ as the retraction $\mathcal{P} \to \mathcal{O}$ (i.e. trivially if $\mathcal{P}$ is strictly linear), and it is defined as the identity on $\mathcal{O} \amalg \mathcal{F}(S')$.

The relative suspension $\Sigma_\mathcal{O}\mathcal{P}$ is the mapping cone of the retraction $\mathcal{P} \to \mathcal{O}$, i.e. it is obtained from the cylinder by quotienting out the boundary. As a graded operad it is

$$\Sigma_\mathcal{O}\mathcal{P} = \mathcal{O} \amalg \mathcal{F}(\sigma S).$$

If $\mathcal{P}$ is strictly linear the differential is given by

$$d(\sigma x) = -\sigma d(x), \quad x \in S,$$

so it is again strictly linear. Actually, since $\sigma$ is an isomorphism, any strictly linear DG-operad $\mathcal{P}$ can be canonically desuspended. The desuspension is

$$\Sigma^{-1}_\mathcal{O}\mathcal{P} = \mathcal{O} \amalg \mathcal{F}(\sigma^{-1} S)$$

as a graded operad, and the differential is given by

$$d(\sigma^{-1} x) = -\sigma^{-1} d(x), \quad x \in S.$$

Hence, strictly linear DG-operads are a curious example of infinite suspensions in the otherwise unstable pointed model category $\mathcal{O} \downarrow \text{dgOp} \downarrow \mathcal{O}$. Note also that the relative torus of a strictly linear DG-operad $\mathcal{P}$ is

$$(3.3) \quad T_\mathcal{O}\mathcal{P} = \Sigma_\mathcal{O}\mathcal{P} \cup_\mathcal{O} \mathcal{P}.$$  

The canonical up-to-homotopy cogroup structure of a strictly linear DG-operad $\mathcal{P}$, coming from the fact that it is a suspension, is given as follows, $x \in S$,

$$\text{comultiplication}: \mathcal{P} \to \mathcal{P} \cup_\mathcal{O} \mathcal{P}, \quad \text{coinversion}: \mathcal{P} \to \mathcal{P}, \quad x \mapsto j_1 x + j_2 x, \quad x \mapsto -x.$$

Here, $j_1$ and $j_2$ are the inclusions of the factors of the coproduct $\mathcal{P} \cup_\mathcal{O} \mathcal{P}$ under $\mathcal{O}$. The counit is the retraction $\mathcal{P} \to \mathcal{O}$. Note that this structure is actually strict, not just up to homotopy. Moreover, $\mathcal{P}$ is a co-$\mathbb{k}$-module object in $\mathcal{O} \downarrow \text{dgOp}$ via the following relative morphisms, $\alpha \in \mathbb{k}$, $x \in S$,

$$\mathcal{P} \to \mathcal{P}, \quad x \mapsto \alpha x.$$

Therefore, sets of (homotopy classes of) maps out of a strictly linear DG-operad are $\mathbb{k}$-modules in a natural way.

Let $\mathcal{Q}$ be another strictly linear DG-operad with underlying graded operad $\mathcal{Q} = \mathcal{O} \amalg \mathcal{F}(S')$. A DG-operad map $f: \mathcal{P} \to \mathcal{Q}$ is strictly linear if $f(S) \subseteq \langle S' \rangle_\mathcal{O}$. Strictly linear maps are based. The model theoretic suspension of $f$, obtained in general by first choosing a cylinder of $f$ and then quotienting out the boundary, is the relative map

$$\Sigma_\mathcal{O}f: \Sigma_\mathcal{O}\mathcal{P} \to \Sigma_\mathcal{O}\mathcal{Q}.$$
defined as

\[(\Sigma_O f)(\sigma x) = \sigma f(x), \quad x \in S.\]

This map is again strictly linear. Indeed, strictly linear maps can be desuspended as above, so they are infinite suspensions. Moreover, they are strictly compatible with the previous co-(k-module) structure. Hence, strictly linear maps induce k-modules homomorphisms between sets of (homotopy classes of) maps from strictly linear DG-operads.

The symbol \(\text{Map}_O\) will denote mapping spaces in \(O \downarrow \text{dgOp}\). In order to have an explicit model in mind, we think of \(\text{Map}_O(\mathcal{P}, \mathcal{U})\) as the simplicial set of maps from a cofibrant resolution of the source (or just from the source if it happens to be cofibrant) to a simplicial resolution of the target. We denote the set of components by

\[[\mathcal{P}, \mathcal{U}]_O = \pi_0 \text{Map}_O(\mathcal{P}, \mathcal{U}),\]

which is the set of (left) homotopy classes of maps \(\mathcal{P} \to \mathcal{U}\) if \(\mathcal{P}\) is cofibrant. When the source is strictly linear, the mapping space is based by the trivial map and we can use the desuspension above to enhance \(\text{Map}_O(\mathcal{P}, \mathcal{U})\) to an \(\Omega\)-spectrum \(\text{Map}^s_O(\mathcal{P}, \mathcal{U})\) defined by

\[\text{Map}^s_O(\mathcal{P}, \mathcal{U})_n = \text{Map}_O(\Sigma^{-n} \mathcal{P}, \mathcal{U}) = \text{Map}_O(\Sigma \Sigma^{-1} \mathcal{P}, \mathcal{U}) \simeq \Omega \text{Map}_O(\Sigma^{-1} \mathcal{P}, \mathcal{U}) = \Omega \text{Map}_O(\mathcal{P}, \mathcal{U})_{n+1}, \quad n \geq 0.\]

By the previous observation, this is actually a spectrum in the category of simplicial \(k\)-modules, i.e. an incarnation of an \(Hk\)-module spectrum, where \(Hk\) is the Eilenberg–MacLane ring spectrum of \(k\), compare [Shi07]. This construction is functorial in the second variable with respect to relative maps and on the first variable with respect to strictly linear maps.

If \(\mathcal{P}\) is just cellular, we can still use the relative torus to compute the loop space of \(\text{Map}_O(\mathcal{P}, \mathcal{U})\) at any \(f: \mathcal{P} \to \mathcal{U}\),

\[\Omega_f \text{Map}_O(\mathcal{P}, \mathcal{U}) \simeq \text{Map}_\mathcal{P}(\mathcal{T}_O \mathcal{P}, \mathcal{U}).\]

Note the different subscripts of Map. This weak equivalence is natural in \(\mathcal{U}\) in the obvious way. As for naturality in \(\mathcal{P}\), given a map between relatively cellular DG-operads \(g: \mathcal{Q} \to \mathcal{P}\), we first consider the map \(\mathcal{T}_O \mathcal{Q} \to \mathcal{T}_O \mathcal{P}\) obtained from a cylinder \(I_O f\) by taking coequalizers of \(i_0\) and \(i_1\) and then change the base of the source along \(g\), obtaining in this way a based map under \(\mathcal{P}\)

\[T_O g: T_O \mathcal{Q} \cup \mathcal{Q} \mathcal{P} \to T_O \mathcal{P}.\]

This morphism induces a map

\[(T_O g)^*: \text{Map}_{\mathcal{P}}(T_O \mathcal{P}, \mathcal{U}) \to \text{Map}_{\mathcal{Q}}(T_O \mathcal{Q}, \mathcal{U}) = T_O \text{Map}_{\mathcal{Q}}(\mathcal{Q}, \mathcal{U})\]

which identifies with \(g^*: \Omega_f \text{Map}_\mathcal{P}(\mathcal{P}, \mathcal{U}) \to \Omega_f \text{Map}_\mathcal{Q}(\mathcal{Q}, \mathcal{U})\). Beware, however, that \(T_O\) is not a strict functor (just up to higher coherent homotopies) since our relative cylinder is not (known to be) strictly functorial.

The following lemma shows how to decompose a linear DG-operad as a mapping cone.
Lemma 3.5. Let $\mathcal{O} \subset \mathcal{P} \subset \mathcal{Q}$ be DG-operads such that the inclusion $\mathcal{P} \subset \mathcal{Q}$ is cellular, $\mathcal{Q}$ has underlying graded operad $\mathcal{P} \amalg \mathcal{F}(S)$, and its differential $d_\mathcal{Q}$ (co)restricts to

$$S \xrightarrow{(d_0, d_1)} \mathcal{P} \oplus \langle S \rangle \mathcal{O} \subset \mathcal{P} \oplus \langle S \rangle \mathcal{P} \subset \mathcal{Q},$$

in particular $\mathcal{Q}$ is linear relative to $\mathcal{P}$. Then there is a unique strictly linear DG-operad $\mathcal{Q}_\mathcal{P}$ with underlying graded operad $\mathcal{O} \amalg \mathcal{F}(S)$ such that the differential of $\mathcal{Q}_\mathcal{P}$ on $S$ coincides with $d_1$. Moreover, there is a cofiber sequence in $\mathcal{O} \downarrow \text{dgOp}$

$$\Sigma^{-1} \mathcal{Q}_\mathcal{P} \xrightarrow{f_{\mathcal{P}, \mathcal{Q}}} \mathcal{P} \xrightarrow{\text{incl.}} \mathcal{Q}$$

where $f_{\mathcal{P}, \mathcal{Q}}$ is defined by $f_{\mathcal{P}, \mathcal{Q}}(\sigma^{-1}x) = d_0(x)$, $x \in S$.

Proof. There is a unique degree $-1$ self-map $d_{\mathcal{Q}_\mathcal{P}}$ of $\mathcal{O} \amalg \mathcal{F}(S)$ satisfying the operadic Leibniz rule defined by $d_\mathcal{Q}$ on $\mathcal{O}$ and by $d_1$ on $S$. Moreover, there is a unique graded operad map $f_{\mathcal{P}, \mathcal{Q}}: \mathcal{O} \amalg \mathcal{F}(\sigma^{-1}S) \to \mathcal{P}$ which restricts to the inclusion $\mathcal{O} \subset \mathcal{P}$ on the first factor and to $d_0\sigma$ on $\sigma^{-1}S$. We have to prove that $d_{\mathcal{Q}_\mathcal{P}}^2$ vanishes on $S$ and that the square

$$\begin{array}{ccc}
\Sigma^{-1} \mathcal{Q}_\mathcal{P} & \xrightarrow{f_{\mathcal{P}, \mathcal{Q}}} & \mathcal{P} \\
\downarrow d_{\mathcal{Q}_\mathcal{P}}^{-1} & \quad & \downarrow d_{\mathcal{P}} \\
\Sigma^{-1} \mathcal{O} \mathcal{Q}_\mathcal{P} & \xrightarrow{f_{\mathcal{P}, \mathcal{Q}}} & \mathcal{P}
\end{array}$$

commutes on $\sigma^{-1}S$. If we manage to do so, then it is clear that $\mathcal{Q}$ is the mapping cone of $f_{\mathcal{P}, \mathcal{Q}}$.

The map $d_{\mathcal{Q}_\mathcal{P}}$ (co)restricts on $\langle S \rangle \mathcal{O}$ to $\tilde{d}_1 + d_{\mathcal{O}, S}$. The (co)restriction of the differential $d_\mathcal{Q}$ to $\mathcal{P} \oplus \langle S \rangle \mathcal{O}$ is

$$\begin{pmatrix}
d_{\mathcal{P}} & d_0 \\
d_1 & d_{\mathcal{O}, S}
\end{pmatrix}.$$ 

Since $d_{\mathcal{Q}}^2 = 0$, this matrix squares to zero. We deduce that

$$(\tilde{d}_1 + d_{\mathcal{O}, S})^2 = 0,$$ 

$$d_{\mathcal{P}}d_0 = -\tilde{d}_0(\tilde{d}_1 + d_{\mathcal{O}, S}).$$

The first equation shows that $d_{\mathcal{Q}_\mathcal{P}}^2$ vanishes on $S$, actually on $\langle S \rangle \mathcal{O}$, and the second equation proves that the square

$$\begin{array}{ccc}
\langle \sigma^{-1}S \rangle \mathcal{O} & \xrightarrow{\tilde{d}_0\sigma} & \mathcal{P} \\
\downarrow -\sigma^{-1}(\tilde{d}_1 + d_{\mathcal{O}, S})\sigma & \quad & \downarrow d_{\mathcal{P}} \\
\langle \sigma^{-1}S \rangle \mathcal{O} & \xrightarrow{\tilde{d}_0\sigma} & \mathcal{P}
\end{array}$$

commutes. The latter is (co)restriction of the former, hence we are done. \qed

Remark 3.6. With the notation in the previous lemma, the definition of the relative torus of $\mathcal{Q}$ above clearly coincides with

$$T_{\mathcal{P}, \mathcal{Q}} = \Sigma_{\mathcal{O}} \mathcal{Q}_\mathcal{O} \cup_{\mathcal{O}} \mathcal{P}$$

as DG-operads under and over $\mathcal{P}$.

The following result is a kind of transitivity property for the mapping cone decomposition in the previous lemma.
Lemma 3.7. Let \( \mathcal{O} \subset \mathcal{P} \subset \mathcal{Q} \subset \mathcal{R} \) be DG-operads such that the inclusions \( \mathcal{P} \subset \mathcal{Q} \subset \mathcal{R} \) are cellular, \( \mathcal{Q} \) and \( \mathcal{R} \) have underlying graded operads \( \mathcal{P} \oplus F(S) \) and \( \mathcal{Q} \oplus F(S') \), respectively, and the differential of \( \mathcal{Q} \) factors as

\[
\begin{pmatrix}
  d_0 & d_0' \\
  d_1 & d_1'
\end{pmatrix}
\Rightarrow \mathcal{P} \oplus \langle S \rangle \oplus \langle S' \rangle \subset \mathcal{Q} \oplus \langle S' \rangle \subset \mathcal{R}.
\]

Then, with the notation in Lemma 3.5, \( \mathcal{Q}_P \subset \mathcal{R}_P \), \( \mathcal{R}_Q = \mathcal{R}_P / \mathcal{Q}_P \), and there is a staircase diagram in \( \mathcal{O} \downarrow \text{dgOp} \)

\[
\begin{array}{c}
\Sigma^{-1}_O \mathcal{R}_P \xrightarrow{f_{\mathcal{P}, \mathcal{R}}} \Sigma^{-1}_O \mathcal{R}_Q \\
\Sigma^{-1}_O \mathcal{Q}_P \xrightarrow{f_{\mathcal{P}, \mathcal{Q}}} \Sigma^{-1}_O \mathcal{Q} \\
\end{array}
\]

where the left ‘triangle’ commutes strictly and the square commutes up to the homotopy \( H : I_O \Sigma^{-1}_O \mathcal{R}_P \to \mathcal{Q} \) defined by \( H(x) = x \) if \( x \in S \) and \( H(y) = 0 \) if \( y \in S' \).

Proof. The assumptions imply that Lemma 3.5 applies to \( \mathcal{O} \subset \mathcal{P} \subset \mathcal{Q}, \mathcal{Q} \subset \mathcal{P} \subset \mathcal{R}, \) and \( \mathcal{O} \subset \mathcal{Q} \subset \mathcal{R} \), so the statement makes sense. They only slightly non-trivial part of the statement is the claim about the homotopy, that we now check.

There is a unique graded operad map

\[
H : I_O \Sigma^{-1}_O \mathcal{R}_P = \Sigma^{-1}_O \mathcal{R}_P \cup \Sigma^{-1}_O \mathcal{R}_P \mathcal{F}(S \amalg S') \to \mathcal{Q}
\]

such that \( Hi_0 = f_{\mathcal{P}, \mathcal{R}} \) composed with the inclusion \( \mathcal{P} \subset \mathcal{Q} \), \( Hi_1 \) is the desuspension of the projection onto the quotient \( \mathcal{R}_P \to \mathcal{R}_Q \) composed with \( f_{\mathcal{Q}, \mathcal{R}} \), and \( H \) is defined on \( S \) and \( S' \) as in the statement. We just have to check that

\[
\begin{array}{c}
I_O \Sigma^{-1}_O \mathcal{R}_P \xrightarrow{H} \mathcal{Q} \\
\Sigma^{-1}_O \mathcal{R}_P \xrightarrow{\partial H} \mathcal{Q}
\end{array}
\]

commutes on \( S \) and \( S' \).
The previous square (co)restricts to

\[
\begin{pmatrix}
\sigma^{-1} & 0 \\
0 & \sigma^{-1} \\
d_1 + d_O, S & d_1' \\
0 & d_2' + d_O, S' \\
-\sigma^{-1} & 0 \\
0 & -\sigma^{-1}
\end{pmatrix}
\]

It is straightforward to check that this square commutes, hence we are done. \(\square\)

If \(O\) is the initial DG-operad, relative notions are referred to as absolute, or simply omitting the word relative, and \(O\) is usually dropped from notation.

4. A DECOMPOSITION OF THE \(A\)-INFINITY OPERAD

Unlike in the introduction, rather than working with the \(A\)-infinity DG-operad and its truncations we work with their operadic suspensions [Mur15, Definition 2.4 and Remark 2.5], but we keep the same notation so as not to overload the paper with a ubiquitous symbol for this construction.

**Definition 4.1.** The cellular DG-operad \(A_\infty\) is, as a graded operad,

\[
A_\infty = F(\mu_2, \mu_3, \ldots, \mu_n, \ldots)
\]

with

- **arity of** \(\mu_n = n\),
  - \(|\mu_n| = -1\), \(n \geq 2\).

The differential is defined by the following equation in the graded brace algebra \(\prod_{n \geq 1} A_\infty(n) \ni \mu = (0, \mu_2, \mu_3, \ldots, \mu_n, \ldots)\),

\[
d(\mu) = \mu(\mu).
\]

Equivalently, if \(n \geq 2\),

\[
d(\mu_n) = \sum_{p+q=n+1}^{p,q \geq 2} \mu_p \{\mu_q\} = \sum_{p+q=n+1}^{p,q \geq 2} [\mu_p, \mu_q]
\]

\[
+ \mu_{n+1} \{\mu_{n+1}\} \text{ if } n \text{ is odd.}
\]

For \(m \geq 0\), \(A_m \subset A_\infty\) is the sub-DG-operad with underlying graded operad

\[
A_m = F(\mu_2, \ldots, \mu_m).
\]

It is well known that \(A_\infty\) is a DG-operad. Nevertheless, in order to warm up, we give a direct argument. It is clear that there is a unique degree \(-1\) self-map
\[ d = d_{\mathcal{A}_\infty} \] of \( \mathcal{A}_\infty \) satisfying the operadic Leibniz rule defined as above. It is a differential since, by the brace relation,
\[
\begin{align*}
    d^2 (\mu) &= d(\mu) \\
    &= d(\mu) - \mu(d(\mu)) \\
    &= \mu(\mu) - \mu(\mu) \\
    &= \mu(\mu) + \mu(\mu) + (-1)^{|\mu|^2} \mu(\mu) - \mu(\mu)
\end{align*}
\]
\[ = 0. \]

Clearly, \( \mathcal{A}_\infty \) is cellular taking \( S_n = \{ \mu_2, \ldots, \mu_n \} \), \( n < \omega \), and \( S = S_\omega = \bigcup_{n<\omega} S_n \).

**Definition 4.2.** The strictly linear DG-operad \( B_{\infty,1,\infty} \) relative to \( \mathcal{A}_\infty \) is, as a graded operad,
\[
    B_{\infty,1,\infty} = \mathcal{A}_\infty \sqcup \mathcal{F}(\bar{\mu}_2, \bar{\mu}_3, \ldots, \bar{\mu}_n, \ldots),
\]
where
\[
    \text{arity of } \bar{\mu}_n = n, \quad |\bar{\mu}_n| = -1, \quad n \geq 2.
\]
The differential is defined by the following equation in the graded brace algebra \( \prod_{n \geq 1} B_{\infty,1,\infty}(n) \ni \bar{\mu} = (0, \bar{\mu}_2, \bar{\mu}_3, \ldots, \bar{\mu}_n, \ldots) \),
\[
    d_{B_{\infty,1,\infty}}(\bar{\mu}) = [\mu, \bar{\mu}].
\]
Equivalently, if \( n \geq 2 \),
\[
    d_{B_{\infty,1,\infty}}(\bar{\mu}_n) = \sum_{p+q=n+1, p,q \geq 2} [\mu_p, \bar{\mu}_q].
\]

For \( r \geq 1 \), we define \( B_{\infty,1,1} \subset B_{\infty,1,\infty} \) as the strictly linear sub-DG-operad relative to \( \mathcal{A}_\infty \) with underlying graded operad
\[
    B_{\infty,1,1} = \mathcal{A}_\infty \sqcup \mathcal{F}(\bar{\mu}_2, \ldots, \bar{\mu}_r).
\]
The graded operad underlying the quotient \( B_{\infty,1,\infty} = B_{\infty,1,\infty}/B_{\infty,1,1} \) in \( \mathcal{A}_\infty \downarrow \operatorname{dgOp} \)
\[
    B_{\infty,1,\infty} = \mathcal{A}_\infty \sqcup \mathcal{F}(\bar{\mu}_{r+1}, \bar{\mu}_{r+2}, \ldots, \bar{\mu}_n, \ldots)
\]
and its differential is given by the following formula, \( n > r \),
\[
    d_{B_{\infty,1,\infty}}(\bar{\mu}_n) = \sum_{p+q=n+1, p \geq 2} [\mu_p, \bar{\mu}_q].
\]

For \( m \geq s \geq 0 \) we consider the linear sub-DG-operad \( B_{m,r,s} \subset B_{\infty,r,\infty} \) relative to \( \mathcal{A}_m \) with underlying graded operad
\[
    B_{m,r,s} = \mathcal{A}_m \sqcup \mathcal{F}(\bar{\mu}_{r+1}, \ldots, \bar{\mu}_{r+s}).
\]
The DG-operad \( B_{\infty,1,1} \) is well defined. Indeed, there is a unique degree \(-1\) self-map \( d = d_{B_{\infty,1,\infty}} \) of \( B_{\infty,1,\infty} \) satisfying the operadic Leibniz rule and defined as above. Moreover, it is a differential since,
\[
    d^2(\bar{\mu}) = d([\mu, \bar{\mu}])
\]
\[
    = [d(\mu), \bar{\mu}] + (-1)^{\mu}[\mu, d(\bar{\mu})]
\]
\[
    = [\mu(\mu), \bar{\mu}] - [\mu, [\mu, \bar{\mu}]]
\]
\[
    = 0.
\]
Here we use (2.1) and that $|\mu|$ is odd. For the cellularity of $B_{\infty,1,\infty}$ relative to $A_{\infty}$ we take $S_n = \{\bar{\mu}_2, \ldots, \bar{\mu}_n\}, n < \omega$, and $S = S_\omega = \bigcup_{n<\omega} S_n$. We should remark that $B_{m,r,s}$ is indeed a sub-DG-operad of $B_{\infty,r,\infty}$ since, for $r < n \leq r + s$, the index $p$ in the summation defining $d_{B_{\infty,r,\infty}}(\bar{\mu}_n)$ satisfies

\[(4.3)\quad p = n + 1 - q < r + s + 1 - r = s + 1 \leq m + 1,\]

i.e. $p \leq m$.

**Remark 4.4.** For $r \geq 1$ and $0 \leq s \leq m \leq n$, the following commutative square of cellular inclusions is a (homotopy) push-out in $\text{dgOp}$,

\[
\begin{array}{c}
A_m \xrightarrow{\text{push}} B_{m,r,s} \\
\downarrow \quad \uparrow \\
A_n \xrightarrow{} B_{n,r,s}
\end{array}
\]

**Proposition 4.5.** For $r \geq 1$ and $0 \leq s \leq m \leq n$, there is a homotopy cofiber sequence in $A_m \downarrow \text{dgOp}$

\[
\Sigma^{-1}_{A_m} B_{m,r,s} \xrightarrow{f_{m,r,s}} A_r \xrightarrow{\text{incl}} A_{r+s}
\]

where, for $r < n \leq r + s$,

\[
f_{m,r,s}(\sigma^{-1}\bar{\mu}_n) = \sum_{\substack{p+q=n+1 \\text{for} \quad 2 \leq p,q \leq r}} \mu_p \{\mu_q\} = \sum_{\substack{p+q=n+1 \\text{for} \quad 2 \leq p,q \leq r}} [\mu_p, \mu_q] + \mu_{n+1} \{\mu_{n+1}\} \text{ if } n \text{ is odd}.
\]

This follows by applying Lemma 3.5 to $A_m \subset A_r \subset A_{r+s}$. The inequalities $s \leq m \leq r$ guarantee that the hypotheses hold since for $r < n \leq r + s$, if $q$ in the summation defining $d(\mu_n)$ is $q > r$ then $p \leq m$ by (4.3), and the same if we exchange the roles of $p$ and $q$. Of course, in order to identify $A_{r+s}$ with the mapping cone of $f_{m,r,s}$ as explicitly defined in Section 3 we must identify $\bar{\mu}_n$ with $\mu_n$ in the mapping cone. However we cannot do this in $B_{m,r,s}$ since each symbol has a different meaning therein.

**Corollary 4.6.** For $r \geq 1$ and $0 \leq s \leq m \leq n$, the $k$-module $[B_{m,r,s},U]_{A_m}$ acts effectively and transitively on the set $[A_{r+s},U]_{A_r}$, provided the latter is non-empty.

For $m = r$ this follows from Proposition 4.5 and the fact that $A_r$ is initial in $A_r \downarrow \text{dgOp}$. For a general $m$ we use that $[B_{m,r,s},U]_{A_m} = [B_{r,r,s},U]_{A_r}$ by Remark 4.4.

We can also compute the relative torus of $A_{r+s}$, see Remarks 3.6 and 4.4.

**Corollary 4.7.** For $r \geq 1$ and $0 \leq s \leq r$, $T_{A_r} A_{r+s} = \Sigma_{A_{r+s}} B_{r+s,r,s}$. 

Proposition 4.8. Given $r \geq 1$, $s, t \geq 0$, and $s + t \leq m \leq r$, there is a staircase diagram

\[
\begin{array}{c}
A_r \xrightarrow{\text{incl.}} A_{r+s} \xrightarrow{\text{incl.}} A_{r+s+t} \\
f_{m,r,s+t} \downarrow H \downarrow f_{m,r,s+t} \\
\Sigma^{-1} \text{B}_{m,r,s+t} \xrightarrow{\text{incl.}} \Sigma^{-1} \text{B}_{m,r+s,t} \xrightarrow{\text{proj.}} \Sigma^{-1} \text{B}_{m,r+s,t} \xrightarrow{\text{incl.}} \Sigma^{-1} \text{B}_{m,r+s,t} \\
\end{array}
\]

where the ‘triangle’ on the left commutes and the square commutes up to the relative homotopy $H : I_{A_m} \Sigma^{-1} \text{B}_{m,r,s+t} \rightarrow A_{r+s}$ defined by the following properties: $H_{0}$ is $f_{m,r,s+t}$ composed with the inclusion, $H_{1}$ is the projection composed with $f_{m,r,s+t}$, and

\[
H(\bar{\mu}_n) = \begin{cases} 
\mu_n, & r < n \leq r + s, \\
0, & r + s < n \leq r + s + t.
\end{cases}
\]

This proposition is a consequence of Lemma 3.7.

Proposition 4.9. For $r \geq 1$, $s, t \geq 0$ and $m \geq s + t$, there is a homotopy cofiber sequence of strictly linear maps in $A_m \downarrow \text{dgOp} \downarrow A_m$

\[
\begin{array}{c}
\Sigma^{-1} \text{B}_{m,r,s+t} \xrightarrow{g_{m,r,s+t}} \text{B}_{m,r,s} \xrightarrow{\text{incl.}} \text{B}_{m,r,s} \xrightarrow{\text{incl.}} \text{B}_{m,r,s+t} \\
\end{array}
\]

where, for $r + s < n \leq r + s + t$,

\[
g_{m,r,s,t}(\sigma^{-1} \bar{\mu}_n) = \sum_{\substack{p+q=n+1 \\
p \geq 2 \\
r \geq q \leq r + s}} [\mu_p, \bar{\mu}_q].
\]

This result follows by applying Lemma 3.3 to $A_m \subset \text{B}_{m,r,s} \subset \text{B}_{m,r,s+t}$. The hypotheses of that lemma are clearly satisfied since $\text{B}_{m,r,s+t}$ is strictly linear relative to $A_m$.

We conclude this section with the computation of the tori (3.4) and the suspension of some maps.

Proposition 4.10. For $r \geq 1$, the torus of the map $f_{r,2r,1} : A_2 \xrightarrow{\Sigma^{-1}} \text{B}_{r,2r,1} \rightarrow A_2$ relative to $A_r$ is

\[
T_{A_r} f_{r,2r,1} = \Sigma A_2 \cdot g_{2r,2r,1} : \text{B}_{r,2r,1} \rightarrow \Sigma A_2 \cdot \text{B}_{2r,2r}.
\]

Proof. The tori of the source and the target have been computed in (3.3) and Corollary 3.7, respectively. We know in particular that the source and target of $T_{A_r} f_{r,2r,1}$ coincide with those of $\Sigma A_2 \cdot g_{2r,2r,1}$, see also (3.4).

We claim that there is a unique cylinder

\[
I_{A_r} f_{r,2r,1} : I_{A_r} \Sigma^{-1} \text{B}_{r,2r,1} \rightarrow I_{A_r} A_2
\]

satisfying

\[
(I_{A_r} f_{r,2r,1})(\bar{j}_{2r+1}) = - \sum_{\substack{p+q=2r+2 \\
2 \leq p \leq r \\
q > r}} [\mu_p, \sigma \mu_q] - i_0 \mu_{r+1} \{\sigma \mu_{r+1}\} + \sigma \mu_{r+1} \{i_1 \mu_{r+1}\}.
\]
It is clear that there is a unique map of graded operads compatible with \( f_{r,2r,1} \) and with the cylinder inclusions \( i_0 \) and \( i_1 \), and satisfying this formula. This map is also obviously compatible with the projection of the cylinder. We only need to check that it is a DG-map, i.e. that the previous formula is compatible with differentials.

The decomposition \( d(\mu_n) = d_0(\mu_n) + d_1(\mu_n) \), \( r < n \leq 2r \), in the linear DG-operad \( A_{2r} \) relative to \( A_r \) is given by

\[
d_0(\mu_n) = \sum_{p+q=n+1 \atop 2 \leq p, q \leq r} \mu_p \{\mu_q\}, \quad d_1(\mu_n) = \sum_{p+q=n+1 \atop 2 \leq p < r, \quad q > r} [\mu_p, \mu_q].
\]

Note that \( d(\mu_{r+1}) = d_0(\mu_{r+1}) \) and \( d_1(\mu_{r+1}) = 0 \), since if \( p, q \geq 2 \) and \( p + q = r + 2 \) then \( p, q \leq r \). In particular, \( d_{f_{A_r}A_{2r},i_0}(\mu_{r+1}) = d_{f_{A_r}A_{2r},i_1}(\mu_{r+1}) = d(\mu_{r+1}) \in A_r \).

Compatibility with differentials is a consequence of the following equations since, in \( I_{A_r} \Sigma_{A_r}^{-1} B_{r,2r,1} \),

\[
d(\mu_{r+1}) = \sum_{p+q=2r+2 \atop 2 \leq p \leq r, \quad q > r} \big( [d(\mu_p), (\mu_q)] + [\mu_p, i_{0r} \mu_q + i_1 \mu_q + \sigma d_1(\mu_q)] \big)
\]

\[
-d(\mu_{r+1}) \{ \sigma \mu_{r+1} \} + i_0 \mu_{r+1} \{ i_0 \mu_{r+1} - i_1 \mu_{r+1} \}
\]

\[
+ (i_0 \mu_{r+1} - i_1 \mu_{r+1}) \{ i_1 \mu_{r+1} \} + \sigma \mu_{r+1} \{ d(\mu_{r+1}) \}
\]

\[
= i_0 f_{r,2r,1}(\sigma^{-1} \mu_{r+1}) - i_1 f_{r,2r,1}(\sigma^{-1} \mu_{r+1})
\]

\[
(\ast) = \sum_{p+q=2r+2 \atop p \geq 2 \leq q > r} \big( [d(\mu_p), \sigma \mu_q] + [\mu_p, \sigma d_1(\mu_q)] \big).
\]

Note that \( (\ast) = 0 \) since it is the same formula as \( d^2(\sigma \mu_{2r+1}) = 0 \) in \( \Sigma_{A_{r+1}} B_{r+1,r,r+1} \).

In the torus, the images along \( i_0 \) and \( i_1 \) are identified, so

\[
(T_A, f_{r,2r,1})(\mu_{2r+1}) = \sum_{p+q=2r+2 \atop 2 \leq p \leq r, \quad q > r} [\mu_p, \sigma \mu_q] - \mu_{r+1} \{ \sigma \mu_{r+1} \} + \sigma \mu_{r+1} \{ \mu_{r+1} \}
\]

\[
= \sum_{p+q=2r+2 \atop p \geq 2 \leq r} [\mu_p, \sigma \mu_q] = \Sigma A_{2r,2r,2r,2r,2r,2r}(\mu_{2r+1}).
\]

\[ \square \]

**Proposition 4.11.** The torus of the inclusion \( i : A_2 \subset A_3 \) is the inclusion

\[ Ti : \Sigma A_2 B_{3,1,1} \subset \Sigma A_2 B_{3,1,2}. \]

Moreover, the torus of \( f_{1,3,1} : \Sigma^{-1} B_{1,3,1} \rightarrow A_3 \) is the map under \( A_3 \)

\[ T f_{1,3,1} : B_{3,3,1} \longrightarrow \Sigma A_2 B_{3,1,2} \]

defined by

\[
(T f_{1,3,1})(\mu_4) = -[\mu_2, \sigma \mu_3] - [\mu_3, \sigma \mu_2] - \mu_2 \{ \sigma \mu_2, \sigma \mu_2 \}.
\]

Furthermore, the suspension of this based map is

\[ \Sigma A_2 T f_{1,3,1} = \Sigma^2 A_2 g_{3,1,2,1} : \Sigma A_2 B_{3,3,1} \longrightarrow \Sigma^2 A_2 B_{3,1,2}. \]
Proof. The torus of \( A_2 \) is \( TA_2 = \Sigma A_2 \mathcal{B}_{2,1,1} \) by Corollary 1.7, in particular \( TA_2 \cup A_2 \) \( A_3 = \Sigma A_3 \mathcal{B}_{3,1,1} \). The source of \( T_{f_{1,3,1}} \) is \( \mathcal{B}_{3,3,1} \) by (3.3), see also (3.4).

The torus of \( A_3 \) is not given by any previous result since it is not a linear DG-operad. However, we know by [Mur15] that the differential of the cylinder \( IA_3 \) in (3.1) on suspended elements is

\[
\begin{align*}
    d(\sigma \mu_2) &= i_0 \mu_2 - i_1 \mu_2,
    & d(\sigma \mu_3) &= i_0 \mu_3 - i_1 \mu_3 - \sigma \mu_2 \{i_1 \mu_2\} + i_0 \mu_2 \{\sigma \mu_2\},
\end{align*}
\]

Hence, the differential of the torus \( TA_3 \) in (3.2) on suspended elements is

\[
\begin{align*}
    d(\sigma \mu_2) &= 0,
    & d(\sigma \mu_3) &= -\sigma \mu_2 \{\mu_2\} + \mu_2 \{\sigma \mu_2\} = [\mu_2, \sigma \mu_2],
\end{align*}
\]
i.e. \( TA_3 = \Sigma A_3 \mathcal{B}_{3,1,2} \). It is now clear that the torus of the first inclusion in the statement is the second one.

The cylinder of \( \Sigma^{-1} \mathcal{B}_{1,3,1} \) is explicitly described in Section 3 since this is a strictly linear DG-operad. There is a unique graded operad map

\[
I_{f_{1,3,1}} : \Sigma^{-1} \mathcal{B}_{1,3,1} \rightarrow IA_3
\]
compatible with \( f_{1,3,1} \) and with the inclusions of the cylinder \( i_0 \) and \( i_1 \), and satisfying

\[
\begin{align*}
    (I_{f_{1,3,1}})(\bar{\mu}_4) &= -i_0 \mu_2 \{\sigma \mu_3\} - i_0 \mu_3 \{\sigma \mu_2\} - i_0 \mu_2 \{\mu_2, \sigma \mu_2\} \\
    &\quad + \mu_2 \{i_1 \mu_3\} + \sigma \mu_3 \{i_1 \mu_2\}.
\end{align*}
\]

This map is clearly compatible with the projection of the cylinder. Hence, in order to show that \( I_{f_{1,3,1}} \) is a cylinder it suffices to check the compatibility of the previous formula with differentials. This is a consequence of the following formulas since, in the cylinder of \( \Sigma^{-1} \mathcal{B}_{1,3,1} \), \( d(\bar{\mu}_4) = i_0 \sigma^{-1} \bar{\mu}_4 - i_1 \sigma^{-1} \bar{\mu}_4 \),

\[
\begin{align*}
    d(I_{f_{1,3,1}})(\bar{\mu}_4) &= i_0 \mu_2 \{i_0 \mu_3 - i_1 \mu_3 - \mu_2 \{i_1 \mu_2\} + i_0 \mu_2 \{\mu_2\}\} \\
    &\quad - i_0 \mu_2 \{i_0 \mu_2\} \{\sigma \mu_2\} + i_0 \mu_3 \{i_0 \mu_2 - i_1 \mu_2\} \\
    &\quad + i_0 \mu_2 \{i_0 \mu_2 - i_1 \mu_2, \sigma \mu_2\} + i_0 \mu_2 \{\sigma \mu_2, i_0 \mu_2 - i_1 \mu_2\} \\
    &\quad + (i_0 \mu_2 - i_1 \mu_2) \{i_1 \mu_3\} + \sigma \mu_2 \{i_1 \mu_2 \{i_1 \mu_2\}\} \\
    &\quad + (i_0 \mu_3 - i_1 \mu_3 - \sigma \mu_2 \{i_1 \mu_2\} + i_0 \mu_2 \{\sigma \mu_2\}) \{i_1 \mu_2\} \\
    &= i_0 f_{1,3,1}(\sigma^{-1} \bar{\mu}_4) - i_1 f_{1,3,1}(\sigma^{-1} \bar{\mu}_4).
\end{align*}
\]

Here we use the brace relation. Images along \( i_0 \) and \( i_1 \) are identified in the torus, so the formula for \( T f_{1,3,1} \) in the statement readily follows.

The map \( T f_{1,3,1} \) is not strictly linear, so we first need a cylinder in order to compute its suspension. We claim that there is a unique cylinder

\[
I_{A_3} T f_{1,3,1} : I_{A_3} \mathcal{B}_{3,3,1} \rightarrow I_{A_3} \Sigma A_3 \mathcal{B}_{3,3,1,2}
\]
satisfying

\[
(I_{A_3} T f_{1,3,1})(\sigma \bar{\mu}_4) = [\mu_2, \sigma^2 \bar{\mu}_3] + [\mu_3, \sigma^2 \bar{\mu}_2] + \mu_2 \{i_0 \sigma \bar{\mu}_2, \sigma^2 \bar{\mu}_2\} + \mu_2 \{\sigma^2 \bar{\mu}_2, i_1 \sigma \bar{\mu}_2\}.
\]

Clearly, there is a unique graded operad map compatible with \( T f_{1,3,1} \) and the inclusions of the cylinder and satisfying the previous formula. This map is compatible with the projection of the cylinder too. The following computation proves that the previous formula is compatible with differentials since, in \( I_{A_3} \mathcal{B}_{3,3,1} \),

\[
d(\sigma \bar{\mu}_4) =
\]
\[ i_0 \bar{\mu}_4 - i_1 \bar{\mu}_4, \]
\[
d(I_{A_3} T f_{1,3,1}) (\bar{\mu}_4) = - [\mu_2, i_0 \sigma \bar{\mu}_3 - i_1 \sigma \bar{\mu}_3 + [\mu_2, \sigma^2 \bar{\mu}_2])
+ [\mu_2 \{\mu_2\}, \sigma^2 \bar{\mu}_2] - [\mu_3, i_0 \sigma \bar{\mu}_2 - i_1 \sigma \bar{\mu}_2]
- \mu_2 \{i_0 \sigma \bar{\mu}_2, i_0 \sigma \bar{\mu}_2 - i_1 \sigma \bar{\mu}_2\} - \mu_2 \{i_0 \sigma \bar{\mu}_2 - i_1 \sigma \bar{\mu}_2, i_1 \sigma \bar{\mu}_2\}
= - [\mu_2, i_1 \sigma \bar{\mu}_3] - [\mu_3, i_0 \sigma \bar{\mu}_2 - \mu_2 \{i_0 \sigma \bar{\mu}_2, i_0 \sigma \bar{\mu}_2\}
+ [\mu_2, i_1 \sigma \bar{\mu}_2 + \mu_2 \{i_1 \sigma \bar{\mu}_2, i_1 \sigma \bar{\mu}_2\}
- [\mu_2, [\mu_2, \sigma^2 \bar{\mu}_2]] + [\mu_2 \{\mu_2\}, \sigma^2 \bar{\mu}_2]
+ \mu_2 \{i_0 \sigma \bar{\mu}_2, i_1 \sigma \bar{\mu}_2\} - \mu_2 \{i_0 \sigma \bar{\mu}_2 - i_1 \sigma \bar{\mu}_2\}
= i_0 (T f_{1,3,1}) (\bar{\mu}_4) - i_1 (T f_{1,3,1}) (\bar{\mu}_4).
\]

Here we use the formulas for the differentials of the cylinder and of the suspension of a strictly linear DG-operad, and also (2.1).

In the suspension, we kill the images of relative generators along \( i_0 \) and \( i_1 \), so
\[
(\Sigma_{A_3} T f_{1,3,1}) (\bar{\mu}_4) = [\mu_2, \sigma^2 \bar{\mu}_3] + [\mu_3, \sigma^2 \bar{\mu}_2] = (\Sigma^2_{A_3} g_{3,1,2,1}) (\bar{\mu}_4).
\]

\[ \square \]

5. The spectral sequence

In this section we define an extended spectral sequence for the computation of the homotopy groups of \( \text{Map}(A_\infty, \mathcal{U}) \) based at a fixed map \( \phi: A_\infty \to \mathcal{U} \). Our spectral sequence extends Bousfield–Kan’s fringed spectral sequence [BK72, Ch. IX, §4] of the tower of fibrations

\[
(\Sigma_{A_3} T f_{1,3,1}) (\bar{\mu}_4) = [\mu_2, \sigma^2 \bar{\mu}_3] + [\mu_3, \sigma^2 \bar{\mu}_2] = (\Sigma^2_{A_3} g_{3,1,2,1}) (\bar{\mu}_4).
\]

\[ \square \]

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\[
(\Sigma_{A_3} T f_{1,3,1}) (\bar{\mu}_4) = (\Sigma^2_{A_3} g_{3,1,2,1}) (\bar{\mu}_4).
\]

\[ \square \]

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\[
(\Sigma_{A_3} T f_{1,3,1}) (\bar{\mu}_4) = (\Sigma^2_{A_3} g_{3,1,2,1}) (\bar{\mu}_4).
\]

\[ \square \]
consecutive maps in the homotopy long exact sequence of the homotopy fiber sequence obtained by taking mapping space Map(−, U) on the cofiber sequence

\[
\Sigma^{-1} A_{1,s+1,1} \xrightarrow{f_{1,s+1,1}} A_{s+1} \xrightarrow{\text{incl.}} A_{s+2}
\]

from Proposition 4.5. The bottom quotient is simply a pointed set for \( t = s \), hence \( E_r^\infty \) is in principle just a pointed set. However, \( E_r^s \) is an abelian group for \( t > s \), even for \( t - s = 1 \).

For \( s \geq 2r - 2 \) and \( t \in \mathbb{Z} \), we choose \( m \geq r \) and define the \( k \)-module \( E_r^s \) as the homology of the following two composable solid arrows

\[
\pi_{t-s+1} \text{Map}^s_{A_m}(B_{m,s-r+2,r-1}, U) \quad \pi_{t-s} \text{Map}^s_{A_m}(B_{m,s+1,r}, U) \quad \pi_{t-s} \text{Map}^s_{A_m}(B_{m,s-r+2,2r-1}, U) \quad \pi_{t-s-1} \text{Map}^s_{A_m}(B_{m,s+2,r-1}, U)
\]

The full commutative diagram is defined for \( r \geq 1 \), \( s \geq r - 1 \), and \( m \geq 2r - 1 \). It is obtained from the following staircase diagram of strictly linear maps relative to \( A_m \), constructed by taking quotients in the three-step filtration of the top line

\[
\xymatrix{ B_{m,s-r+2,r-1} & B_{m,s-r+2,r} & B_{m,s-r+2,2r-1} \ar[d] \ar[d] \ar[d] \ar[d] \\
B_{m,s+1,1} & B_{m,s+1,r} & B_{m,s+2,r-1} }
\]

Here \( \twoheadrightarrow \) and \( \rightarrow \) denote inclusions and projections, respectively. Taking mapping \( Hk \)-module spectra \( \text{Map}^s_{A_m}(−, U) \) we obtain four linked fiber sequences. If we then take homotopy groups, we obtain a commutative diagram of \( k \)-modules with four intertwined long exact sequences. The full commutative diagram (5.4) is part of it. The composition of the two solid arrows is zero since in the rightmost path from top to bottom, the first two arrows are consecutive maps in one of the long exact sequences. If we exclude the rightmost vertex from (5.4), the diagram is defined for \( m \geq r \). This definition of \( E_r^t \) is independent of the choice of \( m \) by Remark 4.4.

The condition \( s \geq 2r - 2 \) is used below to show that both definitions of \( E_r^t \) coincide in the overlap.

The spectral sequence is concentrated in the right half plane. The first page \( E_1^r \) is defined in the whole right half plane, but \( E_r^s \), \( r \geq 2 \), is undefined in part of it. Fig. 2 shows in red and blue the regions of the right half plane where \( E_r^s \) is defined according to (5.3) and (5.4), respectively. The white part is where \( E_r^s \) is not defined. The red area is the range of definition of the Bousfield–Kan fringed spectral sequence. It is only this part that contributes to the homotopy groups of \( \text{Map}(A_\infty, U) \). The blue part helps to the computation of the critical red line \( s = t \), contributing to \( \pi_0 \), and is crucial for obstruction theory, as we will later see.
As Fig. 2 shows, there is a substantial overlap between the two formulas. The following result proves that there is no ambiguity.

**Lemma 5.6.** For \( r \geq 1 \) and \( t \geq s \geq 2r - 2 \) the two apparently different definitions of \( E^r_t \) coincide. Moreover, for \( t \geq s \geq r - 1 \geq 0 \) and \( m \geq r \), \( E^r_t \) is the homology of

\[
\text{Ker}[\pi_{t-s+1} \text{Map}(A_{s+1}, \mathcal{U}) \to \pi_{t-s+1} \text{Map}(A_{r+s+2}, \mathcal{U})]
\]

\[
\begin{array}{ccc}
\Sigma_{A_1}^{-1} B_{1,s+1,1} & \overset{f_{1,s+1,1}}{\longrightarrow} & A_{s+1} \\
\text{incl} & & \text{incl} \\
\Sigma_{A_m}^{-1} B_{m,s+1,1} & \overset{f_{m,s+1,1}}{\longrightarrow} & A_{s+1}
\end{array}
\]

so it is an abelian group even if \( t = s \).

**Proof.** Let \( r \geq 1 \) and \( t \geq s \geq 0 \). For \( 1 \leq m \leq s + 1 \), the commutative diagram of cofiber sequences from Proposition 4.3

\[
\begin{array}{ccc}
\Sigma_{A_1}^{-1} B_{1,s+1,1} & \overset{f_{1,s+1,1}}{\longrightarrow} & A_{s+1} \\
\text{incl} & & \text{incl} \\
\Sigma_{A_m}^{-1} B_{m,s+1,1} & \overset{f_{m,s+1,1}}{\longrightarrow} & A_{s+1}
\end{array}
\]

and Remark 4.4 yield the following commutative diagram of homotopy fiber sequences obtained by taking mapping spaces,

\[
\begin{array}{ccc}
\text{Map}_{A_m}(A_{s+2}, \mathcal{U}) & \longrightarrow & \text{Map}_{A_m}(A_{s+1}, \mathcal{U}) \\
\downarrow & & \downarrow \\
\text{Map}(A_{s+2}, \mathcal{U}) & \longrightarrow & \text{Map}(A_{s+1}, \mathcal{U})
\end{array}
\]

\[
\begin{array}{ccc}
\Sigma_{A_1}^{-1} B_{1,s+1,1} & \overset{f_{1,s+1,1}}{\longrightarrow} & A_{s+1} \\
\text{incl} & & \text{incl} \\
\Sigma_{A_m}^{-1} B_{m,s+1,1} & \overset{f_{m,s+1,1}}{\longrightarrow} & A_{s+1}
\end{array}
\]

Therefore the bottom arrow in (5.3) is the same as the following composite

\[
\begin{array}{ccc}
\pi_{t-s+1} \text{Map}_{A_m}(\Sigma_{A_m}^{-1} B_{m,s+1,1,1}, \mathcal{U}) & \longrightarrow & \pi_{t-s} \text{Map}_{A_m}(A_{s+2}, \mathcal{U}) \\
\downarrow & & \downarrow \\
\pi_{t-s} \text{Map}_{A_m}(A_{s+2}, \mathcal{U}) & \longrightarrow & \pi_{t-s} \text{Map}(A_{s+2}, \mathcal{U})
\end{array}
\]
Here the top arrow is a connecting homomorphism in the homotopy long exact sequence of the top homotopy fibration in \((5.8)\) and the middle one is induced by the left vertical arrow in \((5.8)\).

For \(r - 1 \leq m \leq s + 2\), the cofiber sequence from Proposition \(4.5\)
\[
\Sigma_{A_m}^{-1} B_{m, s+2, r-1} \xrightarrow{f_{m, s+2, r-1}} A_{s+2} \xrightarrow{\text{incl.}} A_{s+r+1}
\]
yields the following commutative diagram with three homotopy fiber sequences
\[
\begin{array}{c}
\text{Map}_{A_m}(A_{s+r+1}, U) \xrightarrow{\text{incl.}} \text{Map}_{A_m}(A_{s+2}, U) \xrightarrow{f_{m, s+2, r-1}} \text{Map}_{A_m}(\Sigma_{A_m}^{-1} B_{m, s+2, r-1}, U) \\
\text{Map}(A_{s+r+1}, U) \xrightarrow{\text{incl.}} \text{Map}(A_{s+2}, U) \\
\text{Map}(A_{m}, U) \xrightarrow{\text{incl.}} \text{Map}(A_{m}, U)
\end{array}
\]

Hence, taking homotopy groups and chasing the resulting diagram of long exact sequences we see that, for \(\max\{1, r - 1\} \leq m \leq s + 1\), the kernel of \((5.10)\) is the same as the kernel of \((5.9)\).

Here the top arrow is as in \((5.9)\).

For \(r \leq m \leq s + 1\), we consider the following staircase diagram from Proposition \(4.8\),
\[
\begin{array}{c}
\Sigma_{A_m}^{-1} B_{m, s+1, 1} \xrightarrow{f_{m, s+1, r}} A_{s+1} \xrightarrow{\text{incl.}} A_{s+2} \xrightarrow{\text{incl.}} A_{s+r+1} \\
\Sigma_{A_m}^{-1} B_{m, s+1, r} \xrightarrow{\text{incl.}} \Sigma_{A_m}^{-1} B_{m, s+2, r-1} \xrightarrow{\text{proj.}} \Sigma_{A_m}^{-1} B_{m, s+2, r-1} \\
\Sigma_{A_m}^{-1} B_{m, s+1, 1} \xrightarrow{\text{incl.}} \Sigma_{A_m}^{-1} B_{m, s+1, 1}
\end{array}
\]

Looking at the four intertwined long exact sequences obtained by taking \(\text{Map}_{A_m}(\cdot, U)\) and then homotopy groups, we observe that the composite \((5.10)\) coincides with the bottom solid arrow in \((5.4)\).

The strongest constraints on \(m\) are the last ones, \(r \leq m \leq s + 1\). Hence, an \(m\) satisfying all previous inequalities exists if and only if \(s \geq r - 1\). Moreover, the bottom arrow in \((5.4)\) is independent of the choice of \(m \geq r\). This proves the second claim in the statement.
The same argument as above shows that, for \( r - 1 \leq m \leq s - r + 2 \) we have a commutative diagram with three homotopy fiber sequences

\[
\begin{array}{c}
\text{Map}_{A_m}(A_{s+1}, U) \\
\downarrow \\
\text{Map}(A_{s+1}, U)
\end{array}
\quad \rightarrow 
\begin{array}{c}
\text{Map}_{A_m}(A_{s-r+2}, U) \\
\downarrow \\
\text{Map}(A_{s-r+2}, U)
\end{array}
\quad \rightarrow 
\begin{array}{c}
\text{Map}_{A_m}(\Sigma_{A_m}^{-1}B_{m,s-r+2,r-1}, U) \\
\downarrow \\
\text{Map}(A_{s-r+2}, U)
\end{array}
\quad \rightarrow 
\begin{array}{c}
\text{Map}(A_m, U)
\end{array}
\]

Taking homotopy groups and chasing the resulting diagram of long exact sequences we identify the image of the top arrow in (5.3) with the image of the composite

\[
\begin{array}{c}
\pi_{t-s+2} \text{Map}_{A_m}(\Sigma_{A_m}^{-1}B_{m,s-r+2,r-1}, U) \\
\downarrow \\
\pi_{t-s+1} \text{Map}_{A_m}(A_{s+1}, U)
\end{array}
\quad \rightarrow 
\begin{array}{c}
\pi_{t-s+1} f^*_{m,s+1} \text{Map}_{A_m}(A_{s+1}, U) \\
\downarrow \\
\pi_{t-s+1} \text{Map}_{A_m}(\Sigma_{A_m}^{-1}B_{m,s+1,1}, U)
\end{array}
\]

Here the top arrow is a connecting homomorphism in the homotopy long exact sequence of the top homotopy fiber sequence in the previous diagram. Moreover, for \( r \leq m \leq s - r + 2 \), proceeding as above with the following staircase diagram obtained from Proposition 4.8

we deduce that the composite (5.12) coincides with the top solid arrow in (5.4).

The strongest constraints on \( m \) are the last ones, \( r - 1 \leq m \leq s - r + 2 \). Such an \( m \) exists if and only if \( s \geq 2r - 2 \).

\( \square \)

Remark 5.13. The green area in the following picture is where \( E_r^{st} \) admits the new description in the statement of Lemma 5.6.
The red dots are the pointed sets which do not have a canonical abelian group structure.

We can describe the red dots and the green corner as follows.

**Lemma 5.14.** For $0 \leq s = t \leq r - 1$, the homology of (5.3) is $$E^{rs}_r = \text{Im}([A_{s+r+1},U] \to [A_{s+2},U]) \cap \text{Ker}([A_{s+2},U] \to [A_{s+1},U]).$$

**Proof.** In this case $s-r+2 \leq 1$, so $A_{s-r+2}$ is the initial operad and $\text{Map}(A_{s-r+2},U)$ is contractible. Therefore, the exact sequence $$\pi_1 \text{Map}(A_{s+1},U) \to \pi_1 \text{Map}(\Sigma A_{s+1},U) \to \pi_0 \text{Map}(A_{s+2},U) \to \pi_0 \text{Map}(A_{s+1},U),$$ which is part of the homotopy long exact sequence of the homotopy fiber sequence obtained by taking $\text{Map}(-,U)$ on the cofiber sequence used to define (5.3), proves the claim. \hfill \Box

**Remark 5.15.** By [BK72, IX.3.1], the canonical map to the inverse limit $$[A_\infty,U] \to \lim_n [A_n,U]$$ induced by restrictions is surjective. Moreover, given $\phi: A_\infty \to U$, the fiber of the previous map at the image of $\phi$ is in bijection with the following first derived limit of fundamental groups $$\lim^1 \pi_1(\text{Map}(A_\infty,U),\phi)$$ as a pointed set.

We now define the differentials.

**Definition 5.16.** For $r \geq 1$, the abelian group homomorphism $$d_r: E^{st}_r \to E^{s+r,t+r-1}_r$$ is defined for the following values of $s$ and $t$ by choosing a representative in the subquotient $E^{st}_r$ and chasing the indicated diagram.
For \( t > s \geq 0 \), the diagram is

\[
\begin{array}{c}
\pi_{t-s+1} \text{Map}(\Sigma_{\mathcal{A}_1}^{-1}B_{1,s+1,1}, \mathcal{U}) \\
\downarrow \\
\pi_{t-s} \text{Map}(\mathcal{A}_{s+2}, \mathcal{U}) \\
\downarrow \\
\pi_{t-s} \text{Map}(\mathcal{A}_{s+r+1}, \mathcal{U}) \\
\downarrow \\
\pi_{t-s} \text{Map}(\Sigma_{\mathcal{A}_1}^{-1}B_{1,s+r+1,1}, \mathcal{U}).
\end{array}
\]

Here the first arrow is defined as the bottom arrow in (5.3) and the second arrow is defined by the inclusion \( \mathcal{A}_{s+2} \subset \mathcal{A}_{s+r+1} \). This diagram can be chased by the very definition of \( E_r^s \).

In the following five cases,

(a) \( r \geq 2, s \geq 2r - 2, \) and \( t \in \mathbb{Z} \);
(b) \( r = 1, s \geq 1, \) and \( t \in \mathbb{Z} \);
(c) \( t \geq s \geq r \geq 1; \)
(d) \( t > s = r - 1 \geq 0; \)
(e) \( r = 2, s = 0, \) and \( t > 1; \)

if we choose \( m \geq r + 1 \) the diagram is

\[
\begin{array}{c}
\pi_{t-s} \text{Map}_{\mathcal{A}_m}^s (\mathcal{B}_{m,s+1,1}, \mathcal{U}) \\
\downarrow \\
\pi_{t-s} \text{Map}_{\mathcal{A}_m}^s (\mathcal{B}_{m,s+1,r}, \mathcal{U}) \\
\downarrow \\
\pi_{t-s} \text{Map}_{\mathcal{A}_m}^s (\mathcal{B}_{m,s+r+1,1}, \mathcal{U}).
\end{array}
\]

Here the top arrow is induced by the inclusion \( \mathcal{B}_{m,s+1,1} \subset \mathcal{B}_{m,s+1,r} \) and the bottom arrow is a connecting homomorphism in the homotopy long exact sequence of the homotopy fiber sequence obtained by taking \( \text{Map}_{\mathcal{A}_m}^s (-, \mathcal{U}) \) on the homotopy cofiber sequence

\[
\mathcal{B}_{m,s+1,r} \xrightarrow{\text{incl}} \mathcal{B}_{m,s+1,r+1} \xrightarrow{\text{proj}} \mathcal{B}_{m,s+r+1,1}.
\]

The diagram can be chased in cases (a)–(d) since the kernel of the bottom solid arrow in (5.4) is the image of the incoming dashed arrow, which coincides with the top arrow in (5.18). Note that in cases (a) and (b), the differential \( d_r \) is a \( k \)-module homomorphism. The top arrow is actually defined for \( m \geq r \) and both arrows are independent of the choice of \( m \) by Remark 4.4.

We have defined \( d_r \) on all \( E_r^s \) except for \( 0 \leq s = t < r \), which are just \( r \) terms, depicted with red and green dots in the following figure,
The \( r - 1 \) red dots are plain pointed sets \( E_r^{s,s} \), \( 0 \leq s < r - 1 \), while the only green dot is the abelian group \( E_r^{0,r-1} \). In the next page \( E_r^{r-1,r-1} \) will be just a pointed set.

There are \( r - 2 \) differentials which cross over the white undefined area, linking an green term in the fringed line to a blue term, \( d_r^* : E_r^{s,s} \to E_r^{s+r+s+r-1} \), \( r \leq s < 2r - 2 \).

We have depicted the first one \( d_r^* : E_r^{r-1} \to E_r^{2r-2} \).

**Remark 5.19.** Alternatively, the bottom arrow in (5.18) is \( \pi_{t-s}g_{m,s+1,r,1} \) by Proposition 4.9. Expressing homotopy groups of mapping spectra as homotopy classes of maps from suspensions, we obtain that (5.18) coincides with

\[
\begin{align*}
[\Sigma^{t-s}B_{m,s+1,1,\mathcal{U}}]_{A_m} & \\
[\Sigma^{t-s}B_{m,s+1,r,\mathcal{U}}]_{A_m} & \Downarrow \quad (\Sigma^{t-s}g_{m,s+1,r,1})^* \\
[\Sigma^{t-s-1}B_{m,s+r+1,1,\mathcal{U}}]_{A_m} & 
\end{align*}
\]

We now show that the result of the diagram chases in Definition 5.16 represents an element in \( E_r^{s+r,t+r-1} \) which is independent of the choice of representatives in \( E_r^{st} \), and that both of them yield the same result in overlapping regions.

**Proposition 5.21.** The differential \( d_r^* \) is well defined in all cases and both definitions coincide in the overlap.

**Proof.** We start by proving that \( d_r^* \) is well defined. The first definition of \( d_r^* \) is Bousfield–Kan’s, so there is nothing to check. Cases (d) and (e) will follow from agreement, proven later, since the corresponding regions are fully contained in Bousfield–Kan’s. Let us consider the regions of the second definition which are not completely contained in the Bousfield–Kan region.
Choose $m \geq 2r$. In cases (a) and (b), the fact that $d_r : E_r^{s,t} = E_r^{s+r,t+r-1}$ is well defined follows from the properties of the following diagram, explained below,

$$
\begin{align*}
\pi_{t-s+1} \text{Map}_{A_m}^s (B_{m,s-r+2,r-1}, U) & \to \pi_{t-s-1} \text{Map}_{A_m}^s (B_{m,s+1,r+1}, U) & \pi_{t-s} \text{Map}_{A_m}^s (B_{m,s+2,r-1}, U) \\
\pi_{t-s} \text{Map}_{A_m}^s (B_{m,s+1,1}, U) & \leftarrow \pi_{t-s+1} \text{Map}_{A_m}^s (B_{m,s+1,r+1}, U) & \pi_{t-s-2} \text{Map}_{A_m}^s (B_{m,s+r+2,r-1}, U)
\end{align*}
$$

The homology of the left (resp. right) vertical solid arrows is $E_r^{s,t} = E_r^{s+r,t+r-1}$, and $d_r$ is defined by chasing the horizontal solid arrows. The numbered commutative triangles are defined by using the intertwined homotopy long exact sequences of the homotopy fibrations obtained by taking $\text{Map}_{A_m}^s (\cdot, U)$ on the staircase diagrams associated to the following three-step filtrations:

1. $B_{m,s-r+2,r-1} \subset B_{m,s-r+2,r} \subset B_{m,s-r+2,2r-1}$,
2. $B_{m,s-r+2,2r-1} \subset B_{m,s-r+2,2r-1} \subset B_{m,s-r+2,2r}$,
3. $B_{m,s+1,1} \subset B_{m,s+1,r} \subset B_{m,s+1,r+1}$,
4. $B_{m,s+1,r} \subset B_{m,s+1,r+1} \subset B_{m,s+1,2r}$.

In particular, the kernel of $a_{i,s,s}$ is the image of $b_{i,s,s}$, $i = 1, 2, 3$.

In case (c) we simply replace the top left corner with

$$
\begin{align*}
\pi_{t-s+1} \text{Map}(A_{s+1}, U) & \to \pi_{t-s+1} \text{Map}_{A_m}^s (B_{m,s+1,r+1}, U) \\
\pi_{t-s} \text{Map}_{A_m}^s (B_{m,s+1,1}, U) & \leftarrow \pi_{t-s+1} \text{Map}_{A_m}^s (B_{m,s+1,r+1}, U)
\end{align*}
$$

Here the two triangles commute by Proposition 4.8. We should, moreover, replace $\pi_{t-s+1} \text{Map}(A_{s+1}, U)$ on the top left corner of this diagram with the subgroup $\text{Ker}[\pi_{t-s+1} \text{Map}(A_{s+1}, U) \to \pi_{t-s+1} \text{Map}(A_{s-r+2}, U)]$, so that the homology of the resulting left vertical solid arrows in (5.22) is still $E_r^{s,t}$ by Lemma 5.6.
In cases (a), (b), and (c), agreement in the overlap follows from the following commutative diagram, where \( r + 1 \leq m \leq s + 1 \),

\[
\begin{align*}
\pi_{t-s} \text{Map}_{A_m}(B_{m,s+1,1}, \mathcal{U}) & \longrightarrow \pi_{t-s+1} \text{Map}(\Sigma_{A_1}^{-1}B_{1,s+1,1}, \mathcal{U}) \\
\pi_{t-s} \text{Map}_{A_m}(B_{m,s+1,r}, \mathcal{U}) & \longrightarrow \pi_{t-s} \text{Map}(A_{s+r+1}, \mathcal{U}) \\
\pi_{t-s-1} \text{Map}_{A_m}(B_{m,s+r+1,1}, \mathcal{U}) & \longrightarrow \pi_{t-s} \text{Map}(\Sigma_{A_1}^{-1}B_{1,s+r+1,1}, \mathcal{U}).
\end{align*}
\]

Here, the right (resp. left) vertical arrows yield the first (resp. second) definition of \( d_r: E_{r+1}^t \rightarrow E_{r+t+r-1}^t \). The top and bottom commutative pieces are respectively defined by the following two staircase diagrams from Proposition 4.8.

\[
\begin{align*}
A_{s+1} & \longrightarrow A_{s+2} \longrightarrow A_{s+r+1} \\
\Sigma_{A_m}^{-1}B_{m,s+1,r} & \longrightarrow \Sigma_{A_m}^{-1}B_{m,s+2,r-1} \\
\Sigma_{A_m}^{-1}B_{m,s+1,1} & \longrightarrow \Sigma_{A_m}^{-1}B_{m,s+r+1,1} \\
A_{s+1} & \longrightarrow A_{s+r+1} \longrightarrow A_{s+r+2} \\
\Sigma_{A_m}^{-1}B_{m,s+1,r+1} & \longrightarrow \Sigma_{A_m}^{-1}B_{m,s+r+1,1} \\
\Sigma_{A_m}^{-1}B_{m,s+1,r} & \longrightarrow \Sigma_{A_m}^{-1}B_{m,s+r+1,1} \\
\end{align*}
\]

In diagram (5.25) we just need \( r \leq m \leq s + 1 \), so it still makes sense for \( s = r - 1 \) and \( m = r \). Hence the top square in (5.24) also commutes in case (d). Moreover, we can replace the bottom square with the following one,

\[
\begin{align*}
\pi_{t-r} \text{Map}_{A_r}(B_{r,r,r}, \mathcal{U}) & \longrightarrow \pi_{t-r+1} \text{Map}(A_{2r}, \mathcal{U}) \\
\pi_{t-r+1} \text{Map}_{A_r}(A_{2r}, \mathcal{U}) & \longrightarrow \pi_{t-r+1} \text{Map}(\Sigma_{A_1}^{-1}B_{1,2r,1}, \mathcal{U}).
\end{align*}
\]
Here the two unlabeled arrows are induced by the following homotopy cofiber sequence in $\mathcal{A}_r \downarrow \text{dgOp}$ from Proposition 4.5:

$$
\Sigma_{A_r}^{-1} B_{r,r,r} \xrightarrow{f_{r,r,r}} A_r \xrightarrow{\text{incl.}} A_{2r}.
$$

Therefore the homomorphism $d_r : E_{r}^{p-1,t} \to E_{r}^{2p-1,t+r-1}$ can be obtained by chasing the left column in the following diagram:

$$
\begin{array}{c}
\pi_{t+r+1} \text{Map}_{A_r}(B_{r,r,1}, U) \xrightarrow{\cong} \pi_{t+r} \text{Map}_{A_{2r}}(\Sigma_{A_r} B_{2r, r, 1}, U) \\
\pi_{t+r+1} \text{Map}_{A_r}(B_{r,r,r}, U) \xrightarrow{\cong} \pi_{t+r} \text{Map}_{A_{2r}}(\Sigma_{A_r} B_{2r, r, r}, U) \\
\pi_{t+r+1} \text{Map}_{A_r}(A_{2r}, U) \xrightarrow{\cong} \pi_{t+r} \text{Map}_{A_{2r}}(\Sigma_{A_r} B_{2r, r, r}, U) \\
\pi_{t+r+1} \text{Map}_{A_r}(\Sigma_{A_r} B_{r, 2r, 1}, U) \xrightarrow{\cong} \pi_{t+r} \text{Map}_{A_{2r}}(B_{2r, 2r, 1}, U)
\end{array}
$$

Here, the horizontal arrows are the natural isomorphisms between the homotopy groups of a space and the shifted homotopy groups of its loop space. Loop spaces of mapping spaces are identified with mapping spaces from tori in the unbased case and from suspensions in the based case. We also use Remark 4.4. The top square, whose vertical maps are induced by DG-operad inclusions, is obviously commutative. The middle square commutes by (3.3) and Corollary 4.7, and the two other squares are commutative by Proposition 4.10. Hence agreement in case (d) follows, see Remark 5.19.

In the line (e) we have to chase (5.17), which in this case looks like the left column in the following diagram:

$$
\begin{array}{c}
\pi_{t+1} \text{Map}(\Sigma^{-1} B_{1,1,1}, U) \xrightarrow{\cong} \pi_{t-2} \text{Map}_{A_3}(\Sigma_{A_3}^3 B_{1,1,1}, U) \\
\pi_{t} \text{Map}(A_2, U) \xrightarrow{\cong} \pi_{t-2} \text{Map}_{A_3}(\Sigma_{A_3}^2 B_{1,1,1}, U) \\
\pi_{t} \text{Map}(A_3, U) \xrightarrow{\cong} \pi_{t-2} \text{Map}_{A_3}(\Sigma_{A_3}^2 B_{1,2,2}, U) \\
\pi_{t} \text{Map}(\Sigma^{-1} B_{1,3,1}, U) \xrightarrow{\cong} \pi_{t-2} \text{Map}_{A_3}(\Sigma_{A_3}^2 B_{3,3,1}, U)
\end{array}
$$

As above, the horizontal maps are the usual isomorphisms between the homotopy groups of a space and the shifted homotopy groups of its iterated loop space (triple on top and double in the rest). The top square commutes by (3.3) and Corollary 4.7, and the two other squares are commutative by Proposition 4.11. Therefore we obtain agreement in case (e).
We finally prove that the $E_{r+1}$ page is (mostly) given by the homology of the differential $d_r$.

**Proposition 5.26.** Whenever the two differentials

$$E_r^{s-r,t-r+1} \xrightarrow{d_r} E_r^{s,t} \xrightarrow{d_r} E_r^{s+r,t+r-1}$$

are defined, the composition vanishes and the homology is $E_{r+1}^{s,t}$. Furthermore, if only the outgoing differential $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$ is defined and $E_{r+1}^{s,t}$ is also defined then there is a canonical surjection

$$\text{Ker}[d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}] \to E_{r+1}^{s,t}$$

which is in fact bijective for $0 \leq s < r$ and $t > s$.

**Proof.** There is nothing to check for $r \geq 1$ and $t > s \geq 0$ (red region in Fig. 1) since this is Bousfield–Kan’s. Otherwise, we must analyze the image and the kernel of the incoming and outgoing differentials, respectively. We will use diagram (5.22) (also its version modified by (5.23)) and its properties from the proof of Proposition 5.21.

Suppose we are in case (a), (b) or (c). The kernel of $a_{3,s,t}$ in (5.22) is the image of $a_{1,s,t} = b_{3,s,t}$, hence the image of $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$ is $\text{Im} a_{2,s,t}/\text{Im} c_{s,t}$, if this equation holds, the homology of $E_{r+1}^{s,t}$ is defined then there is a canonical surjection $\text{Ker}[d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}] \to E_{r+1}^{s,t}$. Furthermore, using the homotopy cofiber sequence

$$E_{m,s+1} \xrightarrow{\text{incl}} E_{m,s+1,r+1} \xrightarrow{\text{proj}} E_{m,s+2,r}$$

we deduce that the image of $a_{1,s,t} b_{2,s,t}$ is the kernel of

$$e_{s,t} : \pi_{t-s} \text{Map}^*_{A_m}(B_{m,s+1,1}, U) \to \pi_{t-s} \text{Map}^*_{A_m}(B_{m,s+2,r}, U).$$

Changing variables, the previous paragraph also computes the image and the kernel of $d_r : E_r^{s-r,t-r+1} \to E_r^{s,t}$ when it fits in case (a), (b), or (c), i.e. if $r \geq 2$, $s \geq 3r - 2$, and $t \in \mathbb{Z}$; if $r = 1$, $s \geq 2$, and $t \in \mathbb{Z}$; or if $t + 1 \geq s \geq 2r > 2$. The image is $\text{Im} a_{2,s-r,t-r+1}/\text{Im} c_{s,t}$. Under these constraints, the outgoing differential in (5.27) is in case (a) or (b), so its kernel is $\text{Ker} e_{s,t}/\text{Im} c_{s,t}$. Therefore, $d_r^2 = 0$ is equivalent to $e_{s,t} a_{2,s-r,t-r+1} = 0$ and if this equation holds, the homology of (5.27) is $\text{Ker} e_{s,t}/\text{Im} a_{2,s-r,t-r+1}$. The maps $a_{2,s-r,t-r+1}$ and $e_{s,t}$ are precisely the composable homomorphisms defining $E_{r+1}^{s,t}$ according to the second formula in Definition 5.2 which is applicable, hence we are done with these cases.

Out of $t > r \geq 1$, if the incoming differential in (5.27) is undefined but $E_{r+1}^{s,t}$ is defined then $2r < s < 3r - 2$ and $t - s < -1$ (light blue region in Fig. 1). In particular, the outgoing differential is in case (a) or (b) and $E_{r+1}^{s,t} = \text{Ker} e_{s,t}/\text{Im} a_{2,s-r,t-r+1}$ is defined by the second formula in Definition 5.22. The kernel of $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$ is $\text{Ker} e_{s,t}/\text{Im} c_{s,t}$ and the identity in $\text{Ker} e_{s,t}$ induces a surjection $\text{Ker} e_{s,t}/\text{Im} c_{s,t} \to \text{Ker} e_{s,t}/\text{Im} a_{2,s-r,t-r+1}$ since $\text{Im} c_{s,t} \subset \text{Im} a_{2,s-r,t-r+1}$. Here we use the commutativity of (3) in (5.22) changing variables $s = s - r$ and $t = t - r + 1$.

There is just one remaining case out of $t > s \geq 0$ where both differentials in (5.27) are defined, the interval $r \leq s = t < 2r$ over the fringed line (green segment in Fig. 1). In this case, the outgoing differential is in case (a), (b), or (c), defined by the second formula in Definition 5.18 $E_{r+1}^{s,t}$ is defined as in the statement of Lemma 5.6 and the incoming differential is Bousfield–Kan’s, i.e. given by the first
Figure 4. Terms in page \( r + 1 \) related to differentials in page \( r \).
Terms in red, green, and dark blue are given by the homology of \( d_r \), while light blue terms are just quotients of the kernel of \( d_r \) \((r = 4)\).

The first and second pages

In this section we compute the first and second pages of the spectral sequence defined in the previous section (terms and differentials) in case \( \mathcal{U} = \mathcal{E}(X) \) is the endomorphism operad of a chain complex \( X \) with trivial differential. In many situations, e.g. if \( k \) is a field, even if the differential is nontrivial \( X \) may contain \( H_* X \) as a strong deformation retract. Such a strong deformation retraction gives rise to a quasi-isomorphism \( \mathcal{E}(H_* X) \to \mathcal{E}(X) \). Therefore the results in this section still apply, since the spectral sequence of the previous section is natural and homotopy invariant in the target operad \( \mathcal{U} \). Hence, for the sake of simplicity, we assume throughout this section that \( X = H_* X \).

A map \( \phi: \mathcal{A}_\infty \to \mathcal{E}(X) \) gives rise to a graded associative algebra structure on \( \Sigma^{-1} X \), denoted henceforth by \( A \), with shifted multiplication \( m_2 = \phi(\mu_2) \) in the sense of Section 2.

**Proposition 6.1.** The \( E_1 \)-terms of the spectral sequence in Section 3 for the computation of the homotopy groups of \( \text{Map}(\mathcal{A}_\infty, \mathcal{E}(X)) \) are the Hochschild chains,
\[
E_1^{st} = C^{s+2-t}(A), \quad s \geq 0, \quad t \in \mathbb{Z}.
\]
Moreover, the first page differential $d_1 : E_1^{st} \to E_1^{s+1,t}$ is, up to sign, the Hochschild differential,

$$d_1 = (-1)^{t-s}[m_2, -],$$

whenever it is defined, i.e. for $s \geq 1$ and $t \in \mathbb{Z}$, and for $t > s \geq 0$.

**Proof.** By Definition 5.2 for any $m \geq 1$,

$$E_1^{st} = \pi_{t-s} \text{Map}_{A_m}(B_{m,s+1,1}, \mathcal{E}(X)) = [\Sigma_{m} B_{m,s+1,1}, \mathcal{E}(X)]_{A_m}.$$

The operad $B_{m,s+1,1} = A_m \amalg \mathcal{F}(\bar{\mu}_{s+2})$ is the coproduct of $A_m$ with the free graded operad on a single generator of arity $s + 2$ and degree $-1$ endowed with the trivial differential. Hence $\Sigma_{A_m} B_{m,s+1,1} = A_m \amalg \mathcal{F}(\sigma^{t-s} \bar{\mu}_{s+2})$, where the second factor is free as a graded operad on a generator of the same arity and degree $t - s - 1$ with trivial differential. Since $X$ and hence $\mathcal{E}(X)$ have trivial differential, we have a $k$-module isomorphism

$$[\Sigma_{A_m} B_{1,s+1,1}, \mathcal{E}(X)]_{A_m} \cong \mathcal{E}(X)(s + 2)t_{s-1} = C^{s+2,-t}(A) : \varphi \mapsto \varphi(\sigma^{t-s} \bar{\mu}_{s+2}).$$

All differentials of the first page fit into the cases where the second formula in Definition 5.16 applies. The top map in (5.20) is the identity for $r = 1$. Hence $d_1$ is given by the bottom homomorphism, which is induced by the map relative to $A_m$ defined by

$$(\Sigma_{A_m} g_{m,s+1,1,1})(\sigma^{t-s} \bar{\mu}_{s+3}) = \sigma^{t-s} g_{m,s+1,1,1}(\sigma^{-1} \bar{\mu}_{s+3}) = \sigma^{t-s}[\mu_2, \bar{\mu}_{s+2}] = (-1)^{t-s}[\mu_2, \sigma^{t-s} \bar{\mu}_{s+2}].$$

This proves the claim. \hfill \square

Denote by $Z^{p,q}(A)$ the modules of cocycles in the Hochschild complex.

**Corollary 6.2.** For $s > 0$, the $E_2$-terms of the previous spectral sequence are

$$E_2^{st} = HH^{s+2,-t}(A)$$

whenever they are defined, i.e. $t \geq s \geq 1$ or $s \geq 2$ and $t \in \mathbb{Z}$. Moreover, for $t > s = 0$,

$$E_1^{0t} = Z^{2,-t}(A).$$

The remaining $E_2$-term is the following.

**Proposition 6.3.** The pointed set $E_2^{00}$ is the set of graded associative algebra structures on $\Sigma^{-1}X$ and the base point is $A$.

**Proof.** The term $E_2^{00}$, computed in Lemma 5.14, is just the image of the map

$$[A_3, \mathcal{E}(X)] \longrightarrow [A_2, \mathcal{E}(X)]$$

induced by the inclusion $A_2 \subset A_3$. Moreover, by Proposition 5.5 this image coincides with the kernel of

$$f^\prime_{1,2,1} : [A_2, \mathcal{E}(X)] \longrightarrow [\Sigma^{-1}B_{1,2,1}, \mathcal{E}(X)].$$

A simple computation as in the proof of Proposition 6.1 shows that this map is the following quadratic map between $k$-modules,

$$C^{2,0}(A) = \mathcal{E}(X)(2, -1) \longrightarrow C^{2,0}(A) = \mathcal{E}(X)(3, -2) : m \mapsto m\{m\}.$$
This proposition follows since a degree $-1$ map $m: X \otimes X \to X$ is the shifted multiplication of a graded associative algebra structure on $\Sigma^{-1}X$ if and only if $m\{m\} = 0$.

We now compute the differential of the second page of our spectral sequence.

The map $\phi: A_\infty \to E(X)$ takes $\mu_3$ to a Hochschild cocycle $m_3 = \phi(\mu_3) \in C^{s,-1}(A)$, since $\phi$ is a DG-operad map, $d(\mu_4) = [\mu_2, \mu_3]$, and $X$ and hence $E(X)$ have trivial differential, so $0 = [m_2, m_3]$. Its Hochschild cohomology class, called universal Massey product, will be denoted by

$$\{m_3\} \in HH^{3,-1}(A).$$

**Theorem 6.4.** The second page differential $d_2: E_2^{s,t} \to E_2^{s+2,t+1}$ of the spectral sequence in Section 5 for the computation of the homotopy groups of $\text{Map}(A_\infty, E(X))$ is, for $t > s \geq 1$ and for $s \geq 2$ and $t \in \mathbb{Z}$,

$$d_2 = (-1)^{t-s}\{m_3, -\}: HH^{s+2,-t}(A) \to HH^{s+4,-t-1}(A);$$

for $s = 0$ and $t > 1$,

$$d_2: Z^{2,-t}(A) \to HH^{2,-t-1}(A) \xrightarrow{(-1)^t\{m_3, -\}} HH^{4,-t-1}(A);$$

and for $s = 0$ and $t = 1$,

$$d_2: Z^{2,-1}(A) \to HH^{2,-1}(A) \xrightarrow{- \sigma_{s-1}} HH^{4,-2}(A).$$

**Proof.** The differential $d_2$ is given by the second chasing in Definition 5.10 except for $(s, t) = (0, 1)$. We will use the alternative description in Remark 5.19.

For $(s, t) \neq (0, 1)$, we take $x \in E_2^{s,t} = HH^{s+2,-t}(A)$ and choose a representing Hochschild cocycle $\tilde{x} \in Z^{s+2,-t}(A) \subset C^{s+2,-t}(A)$ if $s > 0$ or directly take the Hochschild cocycle $x = \tilde{x} \in E_2^{s,t} = Z^{2,-t}(A)$ if $s = 0$. We then consider the map $\tilde{x}: \Sigma_{A_m}^{s+2} B_{m+1,1} \to E(X)$ defined by $\tilde{x}(\sigma^{t-s}\bar{\mu}_{s+2}) = \tilde{x}$ and by the restriction of $\phi$ on $A_m$, compare the proof of Proposition 6.1. This map is an element in the topmost $k$-module in (5.20).

We can define an extension $\hat{x}: \Sigma_{A_m}^{s} g_{m,s+1,2} \to E(X)$ of $\tilde{x}$ by $\hat{x}(\sigma^{t-s}\bar{\mu}_{s+3}) = 0$, since $d(\sigma^{t-s}\bar{\mu}_{s+3}) = [\mu_2, \sigma^{t-s}\bar{\mu}_{s+2}]$ and $\hat{x}([\mu_2, \sigma^{t-s}\bar{\mu}_{s+2}]) = [\phi(\mu_2), \hat{x}(\sigma^{t-s}\bar{\mu}_{s+2})] = [m_2, \bar{x}] = 0$ by the cocycle condition. The map $\hat{x}$ is a preimage of $\tilde{x}$ along the top arrow in (5.20).

The element $d_2(x)$ is represented by

$$\hat{x}(\Sigma_{A_m}^{s} g_{m,s+1,2,1})(\sigma^{t-s-1}\bar{\mu}_{s+4}) = \hat{x}\sigma^{t-s}(g_{m,s+1,2,1}(\sigma^{t-s-1}\bar{\mu}_{s+4})$$

$$= \hat{x}\sigma^{t-s}([\mu_2, \bar{\mu}_{s+3}] + [\mu_3, \bar{\mu}_{s+2}])$$

$$= (-1)^{t-s} \hat{x}([\mu_2, \sigma^{t-s}\bar{\mu}_{s+3}] + [\mu_3, \sigma^{t-s}\bar{\mu}_{s+2}])$$

$$= (-1)^{t-s} ([\phi(\mu_2), \hat{x}(\sigma^{t-s}\bar{\mu}_{s+3})]$$

$$+ [\phi(\mu_3), \hat{x}(\sigma^{t-s}\bar{\mu}_{s+2}])$$

$$= (-1)^{t-s} ([m_2, 0] + [m_3, \bar{x}])$$

$$= (-1)^{t-s}[m_3, \bar{x}].$$

This concludes the proof for $(s, t) \neq (0, 1)$.

In the remaining case, $d_2: E_2^{s,1} \to E_2^{s+2,2}$, we must chase (5.17). Proceeding exactly as in the last paragraph of the proof of Proposition 5.21, we identify (5.17) for
(s, t) = (0, 1), which is the left column in the following diagram, with the right column,

\[
\begin{array}{c}
\pi_2 \text{Map}(\Sigma^{-1}B_{1,1,1}, \mathcal{U}) \cong \pi_0 \text{Map}_{\mathcal{A}_3}(\Sigma \mathcal{A}_3 B_{3,1,1}, \mathcal{U}) \\
\downarrow \cong \\
\pi_1 \text{Map}(\mathcal{A}_2, \mathcal{U}) \cong \pi_0 \text{Map}_{\mathcal{A}_3}(\Sigma \mathcal{A}_3 B_{3,1,1}, \mathcal{U}) \cong \lfloor \Sigma \mathcal{A}_3 B_{3,1,1}, \mathcal{U} \rfloor_{\mathcal{A}_3}
\end{array}
\]

\[
\begin{array}{c}
\pi_1 \text{Map}(\mathcal{A}_3, \mathcal{U}) \cong \pi_0 \text{Map}_{\mathcal{A}_3}(\Sigma \mathcal{A}_3 B_{3,1,2}, \mathcal{U}) \cong \lfloor \Sigma \mathcal{A}_3 B_{3,1,2}, \mathcal{U} \rfloor_{\mathcal{A}_3}
\end{array}
\]

\[
\begin{array}{c}
\pi_1 \text{Map}(\Sigma^{-1}B_{1,3,1}, \mathcal{U}) \cong \pi_0 \text{Map}_{\mathcal{A}_3}(\Sigma \mathcal{A}_3 B_{3,3,1}, \mathcal{U}) \cong \lfloor \Sigma \mathcal{A}_3 B_{3,3,1}, \mathcal{U} \rfloor_{\mathcal{A}_3}
\end{array}
\]

The map \(T_{f_1,3,1}\) has been computed in Proposition 4.11. Hence, given a cocycle \(x = \bar{x} \in E_2^{01} = Z^{2,-1}(A)\) as above for \((s, t) = (0, 1)\) we define the map \(\hat{x}\) and its extension \(\hat{x}\) as indicated and \(d_2(x)\) is represented by

\[
\hat{x}(T_{f_1,3,1})(\bar{\mu}_4) = \hat{x}(-[\mu_2, \sigma \bar{\mu}_3] - [\mu_3, \sigma \bar{\mu}_2] - \mu_2(\sigma \bar{\mu}_2, \sigma \bar{\mu}_3))
\]

\[
= - (\phi(\mu_2), \hat{x}(\sigma \bar{\mu}_3)) - (\phi(\mu_3), \hat{x}(\sigma \bar{\mu}_2)) - \phi(\mu_2)\{\hat{x}(\sigma \bar{\mu}_2), \hat{x}(\sigma \bar{\mu}_3)\}
\]

\[
= - [m_2, 0] - [m_3, \bar{x}] - m_2\{x, x\}
\]

\[
= - [m_3, x] - x \sim x.
\]

This proves the remaining case. \(\square\)

**Remark 6.5.** In all pages of our spectral sequence, most terms are \(k\)-modules, concretely those in the blue region. This region shrinks on each step, and the \(E_\infty\) terms over the fringed line are in general just abelian groups. All \(E_{r+1}\) terms over the fringed line are given by the homology of the differential \(d_r\). It is therefore reasonable to wonder why differentials force the loss of the \(k\)-module structure. An illustrative example is \(d_2: E_2^{01} \rightarrow E_2^{22}\), computed in Theorem 6.3. The source and the target are \(k\)-modules, but \(d_2\) is the composition of a \(k\)-module projection with the sum of the \(k\)-linear map \(x \mapsto -[m_3, x]\) and the quadratic map \(x \mapsto -x \sim x\). The quadratic part is an abelian group homomorphism with order 2 image since \(\sim\) is graded commutative and \(x\) is in odd Hochschild total degree, but it is not \(k\)-linear, it actually squares scalars. The term \(E_3^{01} \subset Z^{2,-1}(A)\), which is the kernel of \(d_2: E_2^{01} \rightarrow E_2^{22}\), is the abelian group formed by the Hochschild cocycles whose cohomology classes satisfy the following kind of Maurer–Cartan equation

\[
x \sim x + [m_3, x] = 0.
\]

However, if 2 is invertible in \(k\) then \(x \sim x = 0\) and the second page differential is defined in a homogeneous way.

### 7. The truncated spectral sequence

In this section we show that, if we just have a map \(\phi: \mathcal{A}_k \rightarrow \mathcal{U}, k \geq 3\), that we fix, we can still construct the terms of the first \(\lfloor \frac{k-1}{2} \rfloor\) pages and the differentials of the first \(\lceil \frac{k-1}{2} \rceil\) pages of the spectral sequence in Section 6. This truncated spectral
sequence turns out to related to the Bousfield–Kan fringed spectral sequence of the tower of fibrations

$$
\cdots = \text{Map}(A_k, \mathcal{U}) = \text{Map}(A_k, \mathcal{U}) \to \text{Map}(A_{k-1}, \mathcal{U}) \to \cdots \to \text{Map}(A_2, \mathcal{U}),
$$

which stabilizes after \( k - 2 \) steps. We also observe that the first and second pages of the truncated spectral sequence, whenever they are defined, are given as in Section 6. Proofs are essentially the same as in Sections 5 and 6. Here we just take care of the specific issues.

Let us start with the terms.

**Definition 7.2.** Let \( 1 \leq r \leq \frac{k+1}{2} \) (this last number need not be an integer, but \( r \) must be an integer). We define \( E_{r}^{st} \) in the following cases as indicated:

(a) If \( t \geq s \geq 0 \) and \( s \leq k - r - 1 \), then \( E_{r}^{st} \) is defined as the homology of \((5.3)\).

It is an abelian group for \( t > s \) and a pointed set for \( t = s \).

(b) If \( t \geq s \geq r - 1 \) then the abelian group \( E_{r}^{st} \) is the homology of \((5.4)\) for any \( r \leq m \leq k \).

(c) If \( s \geq 2r - 2 \) and \( t \in \mathbb{Z} \), the \( k \)-module \( E_{r}^{st} \) is the homology of \((5.4)\), \( r \leq m \leq k \).

See Fig. 5.

**Remark 7.3.** The bound \( r \leq \frac{k+1}{2} \) comes from the fact that, for the definition of \( E_{r}^{st} \) in case (c), we must ensure that the solid arrows in \((5.4)\) compose to 0. For this we need to consider the full diagram \((5.4)\), which is defined from mapping spectra in \( A_{m} \downarrow \text{dgOp} \) for \( m \geq 2r - 1 \). In order to place \( \mathcal{U} \) in this comma category, we must require \( m \leq k \), so \( 2r - 1 \leq k \).

In (a) we require \( s \leq k - r - 1 \) since the mapping space \( \text{Map}(A_{s+r+1}, \mathcal{U}) \) appears in \((5.3)\) based at the restriction of \( \phi \), so we need \( s + r + 1 \leq k \).

Let \( \tilde{E}_{r}^{st} \) denote the terms of the Bousfield–Kan spectral sequence of the tower of fibrations \((7.1)\).

**Lemma 7.4.** The terms \( E_{r}^{st} \) are well defined and apparently different definitions coincide in the possible overlaps. Moreover, there are natural morphisms

$$
E_{r}^{st} \rightarrow \tilde{E}_{r}^{st}, \quad t \geq s \geq 0, \quad 1 \leq r \leq \frac{k+1}{2},
$$

which are bijective for \( s \leq k - r - 1 \), injective for \( k - r \leq s < k - 1 \), and the only map to \( E_{r}^{st} = 0 \) for \( s \geq k - 1 \).

**Proof.** Given \( r \geq 1 \) and \( t \geq s \geq 0 \), the term \( \tilde{E}_{r}^{st} \) is trivial for \( s \geq k - 1 \), it is the homology of

$$
\text{Ker}[\pi_{t-s+1} \text{Map}(A_{s+1}, \mathcal{U}) \to \pi_{t-s+1} \text{Map}(A_{s-r+2}, \mathcal{U})]
$$

which are bijective for \( s \leq k - r - 1 \), injective for \( k - r \leq s < k - 1 \), and the only map to \( E_{r}^{st} = 0 \) for \( s \geq k - 1 \).

for \( k - r \leq s < k - 1 \), and it is \( E_{r}^{st} \) for \( s \leq k - r - 1 \). In particular \( E_{r}^{st} \) is well defined in case (a). The only difference between \((7.5)\) and \((5.3)\) is the subscript of
Figure 5. Regions (a), (b), and (c) in Definition 7.2 are depicted in red, green, and blue, respectively. The red dots denote plain pointed sets, the rest consists of abelian groups and moreover \( k \)-modules in the blue region. The green and blue regions always overlap. If the red region overlaps the blue region, then it also overlaps the green region. All other possible configurations can happen, as these pictures show for \( r = 4 \) and \( k = 14, 9, 7 \), respectively. The last case is kind of special, it only happens for \( k \) odd and \( r = \frac{k+1}{2} \), and the vertical white band has always width 1. In particular all terms in and over the fringed line are always defined.

\( \mathcal{A} \) in the source of the homomorphism in the denominator of the bottom object. It is \( k \) in (7.5) and \( s + r + 1 \) in (5.3).

Let us check that the definition in the green region (b) makes sense, i.e. that the two arrows in (5.7) compose to 0. Using (5.11), which makes sense for \( r \leq m \leq s+1 \)
and in particular in case (b), we see that the composition of \([5,7]\) factors as

\[
\begin{array}{ccc}
\text{Ker}\left[\pi_{t-s+1} \text{Map}(A_{s+1}, U) \rightarrow \pi_{t-s+1} \text{Map}(A_{s-r+2}, U)\right] \\
\downarrow \\
\pi_{t-s+1} \text{Map}(A_{s+1}, U) \\
\downarrow \\
\pi_{t-s+1} \text{Map}(\Sigma^{-1}B_{1,s+1,r}, U) \\
= \pi_{t-s} \text{Map}^s_{A_m}(B_{m,s+1,r}, U) \\
\downarrow \\
\pi_{t-s} \text{Map}^s_{A_m}(B_{m,s+1,1}, U) \\
\downarrow \\
\pi_{t-s-1} \text{Map}^s_{A_m}(B_{m,s+2,r-1}, U)
\end{array}
\]

Here the last two arrows are consecutive maps in the homotopy long exact sequence of the homotopy fiber sequence obtained by taking \(\text{Map}^s_{A_m}(\cdot, U)\) on the homotopy cofiber sequence

\[
B_{m,s+1,1} \rightarrow B_{m,s+1,r} \rightarrow B_{m,s+2,r-1}.
\]

Hence we are done.

Let us prove that \(E^s_t\) is a submodule of \(\bar{E}^s_t\) for \(k-r \leq s \leq k-1\) and \(1 \leq r \leq \frac{k+1}{2}\).

We can check as in the proof of Lemma \([5,6]\) that in this range \(\bar{E}^s_t\) coincides with the homology of the staircase diagram

\[
\begin{array}{ccccccc}
B_{m,s+1,1} & \rightarrow & B_{m,s+1,k-s-1} & \rightarrow & B_{m,s+1,r} \\
\downarrow & & \downarrow & & \downarrow \\
B_{m,s+2,k-s-2} & \rightarrow & B_{m,s+2,r-1} & \rightarrow & B_{m,k,r+s-k+1}
\end{array}
\]

where \(k - s - 1 \leq m \leq k\). Here the bottom arrow is induced by the homotopy cofiber sequence

\[
B_{m,s+1,1} \rightarrow B_{m,s+1,k-s} \rightarrow B_{m,s+2,k-s-2},
\]

and the top arrow is the same as in \([5,7]\), i.e. \(E^s_t\) and \(\bar{E}^s_t\) are defined as quotients with the same denominator, so we just have to check that the numerator of the former is contained in the numerator of the latter. Under the standing constraints in this paragraph, \(0 < k - s - 1 \leq k - (k-r) - 1 < r\) and the following staircase diagram

\[
\begin{array}{ccc}
B_{m,s+1,1} & \rightarrow & B_{m,s+1,k-s-1} & \rightarrow & B_{m,s+1,r} \\
\downarrow & & \downarrow & & \downarrow \\
B_{m,s+2,k-s-2} & \rightarrow & B_{m,s+2,r-1} & \rightarrow & B_{m,k,r+s-k+1}
\end{array}
\]
proves that the bottom map in (7.6) factors through the bottom map in (5.3), so
the kernel of the latter is contained in the kernel of the former.

Consistency of Definition 7.2 in case (c) has been checked in Remark 7.3. Agree-
ment in the possible overlaps follows as in the proof of Lemma 5.6. □

We now define the differentials of the truncated spectral sequence.

**Definition 7.7.** For \( 1 \leq r \leq \frac{k-1}{2} \), the abelian group homomorphism

\[
d_r : E^s_r \to E^{s+r,t+r-1}_r
\]

is defined for the following values of \( s \) and \( t \) by choosing a representative in the
subquotient \( E^s_r \) and chasing the indicated diagram.

1. For \( t > s \geq 0 \) and \( s \leq k - r - 1 \), the diagram is (5.17).
2. In the five cases (a)–(e) of Definition 5.16 if we choose \( r + 1 \leq m \leq k \) the
diagram is (5.18).

In cases (2) (a) and (b), \( d_r \) is a \( k \)-module homomorphism.

The upper bound for \( r \) is needed since, in order to check \( d^2_r = 0 \) along the lines
of Proposition 5.26, the terms \( E^{s+r,t+r-1}_r \) must be defined.

Denote by \( \bar{d}_r : \bar{E}^s_r \to \bar{E}^{s+r,t+r-1}_r \) the differentials of the Bousfield–Kan spectral
sequence of the tower of fibrations (7.1).

**Proposition 7.8.** The differential \( d_r \) is a well defined homomorphism. Moreover,
for \( 1 \leq r \leq \frac{k-1}{2} \) and \( t > s \geq 0 \) the following squares commutes,

\[
\begin{array}{c}
E^s_r \xrightarrow{d_r} \bar{E}^s_r \\
\downarrow \quad \downarrow \bar{d}_r \end{array}
\]

\[
\begin{array}{c}
E^{s+r,t+r-1}_r \xrightarrow{\bar{d}_r} \bar{E}^{s+r,t+r-1}_r \\
\end{array}
\]

Here the horizontal arrows are the maps in Lemma 7.4.

**Proof.** If \( t > s \geq 0 \) and \( s < k - r - 1 \), the horizontal maps in the statement are
injective by Lemma 7.4 and \( \bar{d}_r \) is defined by chasing (5.11). Hence, if we check that
\( \bar{d}_r(E^s_r) \subset E^{s+r,t+r-1}_r \) we can simultaneously conclude that \( d_r \) is well defined and
that the square commutes. For this, it suffices to prove that for \( r \leq m \leq k \) the
composite

\[
\pi_{t-s} \text{Map}(A_{s+r+1}, \mathcal{U}) \xrightarrow{\pi_{t-s-1} F_{s+r+1,1}} \pi_{t-s} \text{Map}(\Sigma^{-1} A_{s+r+1,1}, \mathcal{U}) \xrightarrow{\pi_{t-s-1} \text{Map}(B_{m,s+r+2}, \mathcal{U})} \pi_{t-s-2} \text{Map}(B_{m,s+r+2,1}, \mathcal{U})
\]

vanishes, since \( E^{s+r,t+r-1}_r \) is defined as the homology of (5.7) after changing vari-
ables \( s = s+r \), \( t = t+r-1 \). Using the following staircase diagram from Proposition
which is defined for \( r \leq m \leq s + r + 1 \), it is easy to see as in previous cases that the composite in the former diagram factors through two consecutive maps in a homotopy long exact sequence, so it vanishes.

The rest of cases fit into (2) (a), (b), or (c), and the proof that \( d_r \) is well defined is identical to Lemma 5.21. Moreover, for \( s \geq k - r - 1 \) the sink of the square in the statement is \( E_{s+r+1,t+r-1}^{s+1} = 0 \), so it obviously commutes. □

The differentials \( d_r \) are suitable to compute (most of) the following page \( E_{r+1} \).

**Proposition 7.9.** For \( 1 \leq r \leq \frac{k-1}{2} \), the statement of Proposition 5.26 holds for the truncated spectral sequence.

The proof of is essentially as in Proposition 5.26, replacing the tower of fibrations (5.1) with (7.1).

Since \( k \geq 3 \), taking \( \mathcal{U} = \mathcal{E}(X) \), the map \( \phi: A_k \to \mathcal{E}(X) \) gives rise to a graded associative algebra structure on \( \Sigma^{-1}X \), denoted henceforth by \( A \), with shifted multiplication \( m_2 = \phi(\mu_2) \) in the sense of Section 2.

**Theorem 7.10.** If \( k \geq 3 \), the computation in Theorem 6.1 of \( E_1^{st} \), for \( s \geq 0 \) and \( t \in \mathbb{Z} \), and of \( d_1: E_1^{st} \to E_2^{s+1,t} \), for \( s \geq 1 \) and \( t \in \mathbb{Z} \) and for \( t > s = 0 \), holds in the truncated spectral sequence for \( \mathcal{U} = \mathcal{E}(X) \).

The proof of this theorem is the same as the proof of Theorem 6.1.

**Corollary 7.11.** In the situation of Theorem 7.10, the computation in Corollary 6.2 of \( E_2^{st} \), for \( t \geq s \geq 0 \) and for \( s \geq 2 \) and \( t \in \mathbb{Z} \), holds for the truncated spectral sequence.

The following result admits the same proof as Proposition 6.3.

**Proposition 7.12.** In the conditions of Theorem 7.10, the computation in Proposition 6.5 of \( E_0^{00} \) is also valid for the truncated spectral sequence.

For \( k \geq 4 \), \( m_3 = \phi(\mu_3) \in C^3(A) \) is a Hochschild cocycle whose Hochschild cohomology class, called universal Massey product, will be denoted by \( \{m_3\} \in HH^3(A) \).

**Theorem 7.13.** If \( k \geq 5 \), the computation in Theorem 6.4 of \( d_2: E_2^{st} \to E_3^{s+2,t+1} \) for \( t > s \geq 0 \), for \( s \geq 2 \) and \( t \in \mathbb{Z} \), and for \( t > s = 0 \), is also true in the truncated spectral sequence for \( \mathcal{U} = \mathcal{E}(X) \).

The proof of this theorem is exactly the same as the proof of Theorem 6.4. The lower bound \( k \geq 5 \) is necessary for two reasons. On the one hand, the truncated spectral sequence differential \( d_2 \) is defined just for \( k \geq 5 \). On the other hand,
Theorem 7.13 identifies $d_2 = \pm [\{m_3\}, -]$ in most cases, so we could think that $k \geq 4$ should actually be enough to define $d_2$, since the universal Massey product $\{m_3\}$ is defined for those values of $k$. Nevertheless, we also need that $\text{Sq}(\{m_3\}) = 0$ so that the differential $[\{m_3\}, -]$ squares to zero, and this happens precisely for $k \geq 5$, as we will check in the following section.

Remark 7.14. Given $\phi: A_k \to \mathcal{U}$ and $l \leq k$, it is clear that the (truncated) spectral sequence of $\phi$ restricted to $A_l$ coincides with the spectral sequence of $\phi$ up to the $E_{1,1}^{k-l+1}$ terms and $d_1^{k-l}$ differentials, even if $k = \infty$.

8. Obstruction theory

In this section we define obstructions to the extension of morphisms $\phi: A_k \to \mathcal{U}$, $k \geq 3$, which live in the truncated spectral sequence constructed in the previous section. Obviously, the chances of being trivial increase with the page number.

Theorem 8.1. Given a map $\phi: A_k \to \mathcal{U}$ in $dgOp$ and $[\frac{k+1}{2}] \leq l \leq k$, its precomposition $\phi f_{m,k,1}: \Sigma_{A_m}^{-1} B_{m,k,1} \to \mathcal{U}$, $1 \leq m \leq k$, with the map $f_{m,k,1}$ in Proposition 7.5 represents an obstruction element in a term of the truncated spectral sequence of Section 7, see Definition 7.2,

$$\Theta_{kl}(\phi) \in E_{k-l+1}^{k-1,k-2},$$

which vanishes if and only if there is a map $\phi': A_{k+1} \to \mathcal{U}$ such that the restrictions of $\phi$ and $\phi'$ to $A_l$ coincide.

The term $E_{k-l+1}^{k-1,k-2}$ lies right below the fringed line $t-s = 0$, not in the range of definition of the Bousfield–Kan spectral sequence of Fig. 7.1. It is in the blue region of Fig. 5, more precisely it is given by Definition 7.2 (c) since $k-l+1 \geq 1$ and $k-1 \geq 2(k-l+1)-2$. These inequalities follow from the upper and lower bounds for $l$ in the statement, respectively.

Proof of Theorem 8.1. By the construction of $E_{k-l+1}^{k-1,k-2}$ in Definition 7.2 (c), in order to check that $\phi f_{m,k,1}$ represents an element in that term it suffices to show that the composite

$$\begin{align*}
\pi_0 \text{Map}_{A_m}(A_k, \mathcal{U}) \\
\downarrow \pi_0 f_{m,k,1} \\
\pi_0 \text{Map}_{A_m}(\Sigma_{A_m}^{-1} B_{m,k,1}, \mathcal{U}) \\
= \pi_{-1} \text{Map}^*_{A_m}(B_{m,k,1}, \mathcal{U}) \\
\downarrow \text{bottom arrow in 7.3} \\
\pi_{-2} \text{Map}^*_{A_m}(B_{m,k+1,k-l}, \mathcal{U})
\end{align*}$$

is trivial, i.e. it maps the whole set on top to the zero element of the bottom $k$-module. Using the following staircase diagram from Proposition 4.8, $k-l+1 \leq $
we see that the previous composite factors through two consecutive maps in the homotopy long exact sequence of the homotopy fiber sequence of spectra obtained by taking \( \text{Map}_{A_m}(-, \mathcal{U}) \) on

\[
B_{m,k,1} \Rightarrow B_{m,k,k-l+1} \Rightarrow B_{m,k+1,k-l},
\]

so it follows.

We now consider a new staircase diagram from Proposition 4.8

This diagram is defined for \( k-l+1 \leq m \leq l \). Such an \( m \) exists by the lower bound for \( l \) in the statement. Using the definition of \( E_{k-l+1}^{k-1,k-2} \) and the homotopy long exact sequence associated as above to the following homotopy cofiber sequence of strictly linear DG-operads

\[
B_{m,l,k-l} \Rightarrow B_{m,l,k-l+1} \Rightarrow B_{m,k,l-k},
\]

we see that \( \Theta_{kl}(\phi) \in E_{k-l+1}^{k-1,k-2} \) vanishes if and only if the precomposition of \( \phi f_{m,k,1} \) with the projection in the previous staircase diagram is null-omotopic. This implies that the restriction of \( \phi \) to \( A_l \) extends to the mapping cone of \( f_{m,l,k-l+1} \), which is \( A_{k+1} \).

Remark 8.2. It might seem that a big \( l \) is better since the map \( \phi' \) would be closer to \( \phi \). However this is not the case since the obstruction then lives in the initial pages of the spectral sequence, which have less chances to vanish. For instance, if \( l = k \) the obstruction lies in the first page, which is huge, and if \( l = k-1 \) the obstruction is in the second page, in the part given by Hochschild cohomology if \( \mathcal{U} = \mathcal{E}(X) \) and \( X \) has trivial differential. This part may be still big, but it is at least a Morita invariant of the graded associative algebra structure \( A \) on \( \Sigma^{-1}X \) with shifted multiplication \( m_2 = \phi(\mu_2) \).

Obstructions in the first and second pages are classical. They are defined for all \( k \geq 3 \). In order to get obstructions in farther pages, \( k \) must be bigger. For a fixed
Proves that, if \( \Theta_{k,l+1}(\phi) \in E_{k-l+1}^{k-1,k-2} \).

For a given \( k \), all obstructions are related. It is known that the classical obstruction in the first page is a Hochschild cocycle representing the obstruction in the second page. More generally, the fact that the representative is independent of \( l \) proves that, if \( \left\lfloor \frac{k+1}{2} \right\rfloor \leq l < k \), the obstruction

\[
\Theta_{k,l+1}(\phi) \in E_{k-l}^{k-1,k-2}
\]

is a cycle \( d_{k-l}(\Theta_{k,l+1}(\phi)) = 0 \) of the \((k-l)\)th page which represents \( \Theta_{kl}(\phi) \in E_{k-l}^{k-1,k-2} \) in the following page.

For \( k = 4 \) and \( \mathcal{U} = \mathcal{E}(X) \) the endomorphism operad of a complex \( X \) with trivial differential, \( \Sigma^{-1}X \) has a graded algebra structure \( A \) with shifted multiplication \( m_2 = \phi(\mu_2) \) and we have the following explicit computation of the first meaningful obstruction, see Remark 8.5 in terms of the universal Massey product \( \{m_3\} \in HH^{3,-1}(A) \) and the Gerstenhaber square in Hochschild cohomology. We use the identification of the \( E_2 \)-terms of the truncated spectral sequence with Hochschild cohomology groups in Corollary 7.11.

**Proposition 8.3.** Given \( \phi: A_4 \to \mathcal{E}(X) \),

\[
\Theta_{4,3}(\phi) = \text{Sq}(\{m_3\}) \in E_2^{3,2} = HH^{5,-2}(A).
\]

**Proof.** A representing cocycle for \( \Theta_{4,3}(\phi) \) is \( \phi f_1 A_4(\sigma^{-1})\bar{\mu}_5 = \phi([\mu_2,\mu_4] + \mu_3\{m_3\}) = [m_2,m_4] + m_3\{m_3\} \in C^{5,-2}(A) \), see Proposition 6.3 and compare the proof of Proposition 6.1. The Hochschild cohomology class of this element is \( \text{Sq}(\{m_3\}) \), represented by \( m_3\{m_3\} \), since \( [m_2,m_4] \) is a Hochschild coboundary. \( \square \)

**Remark 8.4.** If 2 is invertible in \( k \) then the relations in a Gerstenhaber algebra show that

\[
\text{Sq}(\{m_3\}) = \frac{1}{2}[\{m_3\},\{m_3\}].
\]

**Remark 8.5.** Let \( X \) be a complex with trivial differential equipped with a map \( \phi: A_3 \to \mathcal{E}(X) \), so that the second page of the truncated spectral sequence is defined.

The set of homotopy classes of maps \( A_2 \to \mathcal{E}(X) \) which extend to \( A_3 \) is \( E_2^{00} \) by Lemma 5.14 i.e. the set of graded associative algebra structures on \( \Sigma^{-1}X \) by Proposition 6.3. It is pointed at the structure \( A \) with shifted multiplication \( \phi(\mu_2) = m_2 \).

Moreover, the set of homotopy classes of maps \( \psi: A_3 \to \mathcal{E}(X) \) with \( \psi(\mu_2) = m_2 = \phi(\mu_2) \) which extend to \( A_4 \) is \( E_2^{11} = HH^{3,-1}(A) \), again by Lemma 5.14. The bijection sends \( \psi \) to its universal Toda bracket \( \{\psi(\mu_3)\} \). By Proposition 8.3 the subset of homotopy classes which extend to \( A_5 \) is the zero locus of the quadratic map

\[
\text{Sq}: HH^{3,-1}(A) \to HH^{5,-2}(A).
\]
9. Vanishing and collapse

In this section we give a criterion under which the third page of the (truncated) spectral sequence $E^3_{st}$ is concentrated in a certain horizontal interval $0 \leq s < u$. This implies a collapse at page number $u$, except in the critical points $0 \leq t = s < u$.

We deduce uniqueness results for $A$-infinity algebra structures. We also relate the final page of the spectral sequence to a filtration of the homotopy groups of the space of $A$-infinity algebra structures. These results are used in the final section in an explicit computation.

Definition 9.1. Given a graded algebra $A$ with shifted multiplication $m_2: \Sigma A \otimes \Sigma A \to \Sigma A$, the Euler derivation $\delta: \Sigma A \to \Sigma A \in C^{1,0}(A) = \mathcal{E}(\Sigma A)(1)_0$ is the $(1,0)$-cochain defined as

$$\delta(x) = (1 - |x|) \cdot x, \quad x \in \Sigma A.$$ 

Proposition 9.2. Given a Hochschild cochain $y \in C^{p,q}(A) = \mathcal{E}(\Sigma A)(p)_{p-q+1}$,

$$[\delta, y] = q \cdot y.$$ 

Proof. On the one hand,

$$[\delta, y] = \delta(y) - y\delta = \delta \circ_1 y - \sum_{i=1}^p y \circ_i \delta.$$ 

On the other hand, given $x_1, \ldots, x_n \in \Sigma A$,

$$(\delta \circ_1 y)(x_1, \ldots, x_n) = \delta(y(x_1, \ldots, x_n)) = \left( p + q - \sum_{i=1}^p |x_n| \right) \cdot y(x_1, \ldots, x_n),$$

$$(y \circ_i \delta)(x_1, \ldots, x_n) = y(x_1, \ldots, \delta(x_i), \ldots, x_n) = (1 - |x_i|) \cdot y(x_1, \ldots, x_n).$$

Hence the formula in the statement follows. 

Corollary 9.3. The Euler derivation $\delta$ is a Hochschild cocycle.

Proof. Simply note that $[m_2, \delta] = -[\delta, m_2] = 0 \cdot m_2 = 0$ since $m_2 \in C^{2,0}(A)$.

We denote the Hochschild cohomology class of the Euler derivation by

$$\{\delta\} \in HH^{1,0}(A).$$

We call it Euler class. This class is in general non-trivial, as the following observation shows.

Corollary 9.4. If the Euler class of a graded algebra $A$ is trivial $\{\delta\} = 0$ then $HH^{*,q}(A) = 0$ for any $q \in \mathbb{Z}$ which is invertible in $k$.

This follows from the formula in Proposition 9.2 in cohomology.

Proposition 9.5. Given $y \in HH^{n-1}(A)$ with $n$ odd, and $x \in HH^{p,q}(A)$, then

$$(y \sim x) = [y, \{\delta\} \sim x] + \{\delta\} \sim [y, x].$$

Proof. Using Proposition 9.2 and the laws of a Gerstenhaber algebra reviewed in Section 2

$$[y, \{\delta\}] = -[\{\delta\}, y] = y,$$

$$[y, \{\delta\} \sim x] = [y, \{\delta\}] \sim x - \{\delta\} \sim [y, x] = y \sim x - \{\delta\} \sim [y, x].$$
Remark 9.6. The formula in the previous proposition shows that \( \{ \delta \} \sim - \) is a chain null-homotopy for \( y \sim - \), if we think of \([y, -]\) as a differential. Indeed, \([y, -]\) is a differential if \( \text{Sq}(y) = 0 \) and \( y \sim - \) is a chain map since
\[
[y, y \sim x] = [y, y] \sim x + y \sim [y, x] = 2 \text{Sq}(y) + y \sim [y, x] = y \sim [y, x].
\]
This is applied below for \( y = \{ m_3 \} \) the universal Massey product of an \( A_k \)-algebra, \( k \geq 5 \).

Proposition 9.7. The cup square of the Euler class vanishes,
\[
\{ \delta \}^2 = \{ \delta \} \sim \{ \delta \} = 0 \in HH^{2,0}(A).
\]
Proof. Consider the cochain \( \beta \in C^{1,0}(A) = \mathcal{E}(X)(1)_0 \) defined by
\[
\beta(\sigma x) = \frac{|x|(|x| - 1)}{2} \cdot \sigma x, \quad \sigma x \in \Sigma A.
\]
We have
\[
\delta \sim \delta = m_2 \{ \delta, \delta \} = m_2 \delta, \delta,
\]
\[
[m_2, \beta] = m_2 \beta - \beta m_2 = m_2 \circ_1 \beta + m_2 \circ_2 \beta - \beta \circ_1 m_2.
\]
Given \( \sigma x, \sigma y \in \Sigma A, \)
\[
m_2(\delta, \delta)(\sigma x, \sigma y) = m_2(\delta(\sigma x), \delta(\sigma y))
\]
\[
= (-|x|)(-|y|)m_2(\sigma x, \sigma y)
\]
\[
= (-1)^{|x|} |x| |y| \sigma(x \cdot y),
\]
\[
(m_2 \circ_1 \beta)(\sigma x, \sigma y) = m_2(\beta(\sigma x), \sigma y)
\]
\[
= \frac{|x|(|x| - 1)}{2} m_2(\sigma x, \sigma y)
\]
\[
= (-1)^{|x|} \frac{|x|(|x| - 1)}{2} \sigma(x \cdot y),
\]
\[
(m_2 \circ_2 \beta)(\sigma x, \sigma y) = m_2(\sigma x, \beta(\sigma y))
\]
\[
= \frac{|y|(|y| - 1)}{2} m_2(\sigma x, \sigma y)
\]
\[
= (-1)^{|y|} \frac{|y|(1 - |y|)}{2} \sigma(x \cdot y),
\]
\[
(\beta \circ_1 m_2)(\sigma x, \sigma y) = \beta(m_2(\sigma x, \sigma y))
\]
\[
= (-1)^{|x|} \beta(\sigma(x \cdot y))
\]
\[
= (-1)^{|x|} \frac{|x| + |y|(|x| + |y| - 1)}{2} \sigma(x \cdot y).
\]
Now it is easy to see that
\[
\delta \sim \delta + [m_2, \beta] = 0.
\]
Hence \( \{ \delta \} \sim \{ \delta \} = \{ \delta \sim \delta \} = 0 \). \qed
Proof of Theorems 1.1 and 1.2. Let \( \phi_0 : A_4 \to \mathcal{E}(X) \) be the map corresponding to \((X, 0, m_2, m_3, m_4)\). By Theorem 8.1 and Proposition 8.3, we can find an extension \( \phi_1 : A_5 \to \mathcal{E}(X) \) of the restriction of \( \phi_0 \) to \( A_3 \). We now compute the truncated spectral sequence of this extension \( \phi_1 \), as defined in Section 7.

Excluding \( E_2^{00}, E_2^{01}, \) and \( E_2^{11} \), the second page of the truncated spectral sequence splits into two families of cochain complexes \( C^*_n \) and \( D^*_n \), \( n \in \mathbb{Z} \),

\[
C^*_n \quad D^*_n \quad C^*_2 \quad D^*_2
\]

with the following homogeneous descriptions for \( n \geq 2 \) and \( n \leq 1 \),

\[
C^m_n = \begin{cases} 
E_2^{2m,n+m}, & m \geq 1 \text{ or } m = 0 \text{ and } n \geq 2; \\
0, & \text{elsewhere};
\end{cases} \\
D^m_n = \begin{cases} 
E_2^{2m+1,n+m}, & m \geq 1 \text{ or } m = 0 \text{ and } n \geq 2; \\
0, & \text{elsewhere}.
\end{cases}
\]

The differential is of course \( d_2 \) in all cases. By the computation of most \( E_3 \) terms as the homology of \( d_2 \) in Proposition 7.3, for \( n \geq 2 \) and for \( m \geq 2 \) we have

\[
H^m C^*_n = E_3^{2m,n+m}, \quad H^m C^*_n = E_3^{2m+1,n+m}.
\]

In order to compute these groups we relate the previous two families of cochain complexes to two new families \( \bar{C}^*_n \) and \( \bar{D}^*_n \) which agree with them almost everywhere and can be depicted as follows:
Here, on any coordinate \((s, t), s \geq 0, t \in \mathbb{Z}\), we place the Hochschild cohomology group \(HH^{s+2,-t}(A)\). Therefore,
\[
\bar{C}^m_n = HH^{2m+2,-n-m}(A), \quad \bar{D}^m_n = HH^{2m+3,-n-m}(A).
\]
The differential is \(\{m_3\}, \cdot\) in all cases.

The cup product with the universal Massey product \(\{m_3\} \dashv \cdot\) induces cochain maps
\[
f : \bar{C}^*_n \to \bar{D}^{*+1}_n, \quad g : \bar{D}^*_n \to \bar{C}^{*+2}_{n-1},
\]
depicted below in red and blue, respectively.

By our assumptions on \(\{m_3\} \dashv \cdot\), \(f\) and \(g\) are injective, the cokernel of \(f\) is concentrated in degree \(* = -1\), and the cokernel of \(g\) is concentrated in degrees \(* = -2, -1\). Therefore, the associated long exact sequences induce the following isomorphisms in cohomology for \(m \geq 1\),
\[
f_* : H^m \bar{C}^*_n \cong H^{m+1} \bar{D}^*_n, \quad g_* : H^m \bar{D}^*_n \cong H^{m+2} \bar{C}^*_n.
\]
By Proposition 9.5 and Remark 9.6, $f$ and $g$ are null-homotopic, so for $m \geq 1$
\[ H^m C_n^* = 0 = H^m D_n^*. \]

By the computation of the second page of the truncated spectral sequence in Corollary 7.11 and Theorem 7.13, there are cochain maps
\[ h: C_n^* \rightarrow \bar{C}_n^*, \quad k: D_n^* \rightarrow \bar{D}_n^*, \]
such that, for $n \geq 2$, $k$ is an isomorphism and $h$ is surjective with kernel concentrated in degree 0. Moreover, for $n \leq 1$ both $h$ and $k$ are injective with cokernel concentrated in degree 0. Using the associated long exact sequences in cohomology, we deduce that, for $n \geq 2$ and $m \geq 1$, and for $n \leq 1$ and $m \geq 2$,
\[ H^m C_n^* = 0 = H^m D_n^*. \]

This proves that $E^{st}_3 = 0$

unless $s \leq 1$ or $(s, t) = (2, 2)$.

Let us check that $E^{22}_3 = 0$ too, so the $E_3$ terms are concentrated in the band $t \geq s = 0, 1$,

Consider the following commutative diagram of solid arrows, which are at least abelian group homomorphisms,

(9.8) $\begin{array}{ccc}
Z^{2,-1}(A) & \xrightarrow{\alpha} & HH^{4,-2}(A) \\
\downarrow & & \downarrow \setminus \{m_3\} \setminus \{m_3\} \\
HH^{2,-1}(A) & \xrightarrow{\beta} & HH^{5,-2}(A)
\end{array} \xrightarrow{[m_3],-} HH^{6,-3}(A)$

\[ \xrightarrow{[m_3],-} HH^{7,-3}(A) \xrightarrow{[m_3],-} HH^{9,-4}(A) \]

Here
\[ \alpha(x) = x \setminus x + \{m_3\} \setminus x = x^2 + \{m_3\} \setminus x, \]
\[ \beta(x) = \{m_3\} \setminus x + \{\delta\} \setminus x \setminus x = \{m_3\} \setminus x + \{\delta\} \setminus x^2. \]

The only slightly non-trivial part is the commutativity of the trapezoidal subdiagram, which follows from Proposition 9.2 and the laws of a Gerstenhaber algebra,
since we know that \[ \{ m_3 \}, \{ m_3 \} \sim x = \{ m_3 \} \sim \{ m_3, x \}, \]
\[ \{ m_3 \}, \{ \delta \} \sim x \sim x = \{ m_3 \}, \{ \delta \} \sim x \sim x \\
+ (-1)^{(3-1-1)-(1+0)} \{ \delta \} \sim \{ m_3, x \} \sim x \\
+ (-1)^{(3-1-1)-(1+0)+(3-1-1)-(2-1)} \{ \delta \} \sim x \sim \{ m_3, x \} \\
= \{ m_3 \} \sim x \sim x \\
- \{ \delta \} \sim \{ m_3, x \} \sim x \\
+ (-1)^{(2-1)-(3-1)+(2-1)-1} \{ \delta \} \sim \{ m_3, x \} \sim x \\
= \{ m_3 \} \sim x \sim x. \]

The cohomology of the top (resp. bottom) sequence of \((9.8)\) at the middle term is \(E^2_3\) (resp. \(H^2D_1^* = 0\)), by Proposition 9.4 Corollary 9.11 and Theorem 9.13. If we manage to prove that \(\beta\) is an isomorphism, then \((9.8)\) induces an isomorphism in middle homology \(E^2_3 \cong H^2D_1^* = 0\), and hence we are done. For this, we are going to define a commutative diagram of abelian group homomorphisms,

\[
\begin{array}{cccc}
\{ m_3 \}^3 \sim & \sim & \sim & \sim \\
HH^{2,-1}(A) \xrightarrow{\beta} HH^{5,-2}(A) \xrightarrow{\gamma} HH^{11,-4}(A) \xrightarrow{\lambda} HH^{23,-8}(A)
\end{array}
\]

We will deduce that \(\gamma \beta\) and \(\lambda \gamma\) are bijective, hence \(\beta\) is an isomorphism. The homomorphisms \(\gamma\) and \(\lambda\) are defined as

\[
\gamma(x) = \{ m_3 \}^2 \sim x - \{ \delta \} \sim x^2, \\
\lambda(x) = \{ m_3 \}^4 \sim x + \{ \delta \} \sim x^2.
\]

Commutativity follows from

\[
\gamma \beta(x) = \gamma(\{ m_3 \} \sim x + \{ \delta \} \sim x^2) \\
= \{ m_3 \}^3 \sim x + \{ m_3 \}^2 \sim \{ \delta \} \sim x^2 - \{ \delta \} \sim \{ m_3 \} \sim x + \{ \delta \} \sim x^2) \]
\[
= \{ m_3 \}^3 \sim x + \{ m_3 \}^2 \sim \{ \delta \} \sim x^2 - \{ \delta \} \sim \{ m_3 \}^2 \sim x^2 \\
= \{ m_3 \}^3 \sim x, \]
\[
\lambda \gamma(x) = \lambda(\{ m_3 \}^2 \sim x - \{ \delta \} \sim x^2) \\
= \{ m_3 \}^6 \sim x - \{ m_3 \}^4 \sim \{ \delta \} \sim x^2 + \{ \delta \} \sim \{ m_3 \}^2 \sim x - \{ \delta \} \sim x^2) \]
\[
= \{ m_3 \}^6 \sim x - \{ m_3 \}^4 \sim \{ \delta \} \sim x^2 + \{ \delta \} \sim \{ m_3 \}^4 \sim x^2 \\
= \{ m_3 \}^6 \sim x.
\]

Here we use Proposition 9.7 and that the cup product is graded commutative, and in particular \(2x^2 = 0\) in both cases.

By Theorem 8.1 starting with \(\phi_1\), defined in the first paragraph of this proof, we can inductively construct maps \(\phi_n: A_{n+4} \to E(X)\) extending \(\phi_{n-1}\) restricted to \(A_{n+1}\), \(n \geq 2\), since \(\Theta_{n+3,n+1}(\phi_{n-1}) \in E_3^{n+2,n+1} = 0\). Here we use Remark 7.14. Therefore, there is a unique a map \(\phi: A_{\infty} \to E(X)\) whose restriction to \(A_{n+2}\) coincides with the restriction of \(\phi_n\), \(n \geq 1\).
Let us check uniqueness. If $\psi: A_\infty \to \mathcal{E}(X)$ is another such map, we prove by induction that the restrictions of $\phi$ and $\psi$ to $A_{n+2}$ are homotopic for all $n \geq 1$. For $n = 1$ it follows since they have the same universal Massey products, see Remark 8.5. If it is true for $n - 1$, $n \geq 2$, then by Lemma 5.13 the restrictions of $\phi$ and $\psi$ to $A_{n+2}$ are elements of $E_{n+1}^{n+1} = 0$, so they are homotopic. This proves that the images of $\phi$ and $\psi$ along the canonical surjection

$$[A_\infty, \mathcal{E}(X)] \to \lim_n [A_n, \mathcal{E}(X)]$$

in Remark 5.15 coincide. The fiber of the image of $\phi$ is a singleton by Remark 5.15 since the previous computation shows that $\lim_n E_n^{s+1} = 0$ for all $s \geq 0$, therefore $\lim_n E_n = 0$ (Map$(A_n, \mathcal{E}(X)), \phi)) = 0$ [BK72] IX.5.4]. This shows that $\phi$ is homotopic to $\psi$.

The short exact sequences in the statement of Theorem 1.2 follow from Mittag-Leffler (and hence complete) convergence [BK72] IX.5.4 and IX.5.5].

**Remark 9.9.** If $s = 0, 1$ and $t - s \geq 1$, the terms $E_3^{s,t}$ are, $t \geq 2$,

$$E_3^{1t} = \text{Ker} \left( HH^{3,-t}(A) \xrightarrow{[\text{tms},-]} HH^{5,-t-1}(A) \right),$$

$$E_3^{0t} = \text{Ker} \left( Z^{2,-t}(A) \xrightarrow{[\text{tms},-]} HH^{2,-t}(A) \xrightarrow{[\text{tms},-]} HH^{3,-t-1}(A) \right),$$

$$E_3^{01} = \text{Ker} \left( Z^{2,-1}(A) \xrightarrow{Z^{2,-1}(A)} HH^{2,-1}(A) \xrightarrow{[\text{tms},x]} HH^{4,-2}(A) \right),$$

by Corollary 7.11 and Theorem 7.13.

10. AN EXPLICIT EXAMPLE

Let $k$ be a field, $A_0 = k[e]/(e^2)$ the ungraded algebra of dual numbers, and

$$A = \frac{A_0(x^\pm 1)}{(x^\pm 1)} = \frac{k[e, x^\pm 1]}{(e^2, x^\pm 1)},$$

where $|x| = 1$ (and of course $|e| = 0$). This graded algebra is commutative only if $\text{char} k = 2$.

The graded algebra $HH^{0,*}(A)$ is the center of $A$ with cohomological grading, i.e. $HH^{0,*}(A) = A$ if $\text{char} k = 2$ and

$$HH^{0,*}(A) = \frac{k[e,x,x^\pm 2]}{(ex)^2}$$

otherwise. The cohomological degrees of $ex$ and $x^2$ are $-1$ and $-2$, respectively.

In order to compute the whole Hochschild cohomology of $A$, we use the well known isomorphism

$$HH^{*,*}(A) \cong \text{Ext}^{*,*}_{A \otimes A^{op}}(A, A)$$

which identifies the cup product with the Yoneda product. The Hochschild degree $*$ corresponds to the length of the Yoneda extensions and $*$ comes from the fact that $A$ is graded and we are considering graded $A$-bimodules, i.e. graded left $A \otimes A^{op}$-modules. In this section, for the sake of simplicity, the cup product is denoted by plain juxtaposition, $a \cup b = ab$.

Consider the following extension

$$A \xrightarrow{i} A \otimes_{k(x^\pm 1)} A \twoheadrightarrow A$$

(10.1)
where \( i(1) = x \otimes \epsilon + \epsilon \otimes x \) and the projection is the product in \( A \).

**Lemma 10.2.** The \( A \)-bimodule \( A \otimes_{k(x \pm 1)} A \) has projective dimension 1 and the map \( i \) in (10.1) induces the trivial map on \( \text{Ext}^1_{A \otimes A^{op}}(-, A) \),

\[ i^* = 0: \text{Ext}^1_{A \otimes A^{op}}(A \otimes_{k(x \pm 1)} A, A) \rightarrow \text{Ext}^1_{A \otimes A^{op}}(A, A). \]

**Proof.** The bottom sequence in the following diagram is a specific length 1 projective resolution of \( A \otimes_{k(x \pm 1)} A \),

\[
\cdots \rightarrow A \otimes A \oplus A \otimes A (x \otimes x + x \otimes 1 \otimes x) \rightarrow A \otimes A \rightarrow A \rightarrow \cdots
\]

\[
(0 \otimes x + x \otimes x) \rightarrow x \otimes 1 \otimes x
\]

The top projective resolution of \( A \) can be constructed by combining (10.1) with the projective resolution of \( A \otimes_{k(x \pm 1)} A \). The two projections are given by the product in \( A \). The maps in the commutative square of the middle are given by left multiplication in \( A \otimes A^{op} \) by the indicated element (or matrix of elements). Recall that the product in \( A^{op} \) is defined by \((x, y) \mapsto (-1)^{|x||y|}yx\), so the product in \( A \otimes A^{op} \) is \((x_1 \otimes x_2)(y_1 \otimes y_2) = (-1)^{|x_1||y_2|}x_1y_1 \otimes y_2x_2 \). Applying \( \text{Hom}_{A \otimes A^{op}}(-, A) \) to the left vertical arrow yields 0 since for any \( a \in A \), \( xa \epsilon + (-1)^{|a|} \epsilon ax = 0 \).  

The \( k \)-module \( HH^{1,*}(A) \) is the Lie algebra of derivations of \( A \) modulo inner derivations. The extension (10.1) represents an element

\[ \{e\} \in HH^{1,1}(A), \]

also represented by the derivation \( e: A \rightarrow A \) defined by \( e(x) = 0 \) and \( e(\epsilon) = x^{-1} \). Lemma 8 proves that

\[ \{e\} \sim -: HH^{p,q}(A) \rightarrow HH^{p+1,q+1}(A) \]

is an isomorphism for all \( p \geq 1 \). Since \( x^2 \in HH^{0,-2}(A) \) is a unit, it is only left to compute \( HH^{1,0}(A) \) and \( HH^{1,-1}(A) \). A straightforward explicit computation with derivations proves that \( HH^{1,*}(A) \), as an \( HH^{0,*}(A) \)-module, is freely generated by the two elements

\[ \{\delta\} \in HH^{1,0}(A), \quad \{e\} \in HH^{1,1}(A), \]

in any characteristic. The Lie left \( HH^{1,*}(A) \)-module structure on \( HH^{0,*}(A) \) is defined by derivation evaluation. It is immediate to check, actually at the level of cochains, that

\[ \text{Sq}(\{e\}) = 0, \]

in particular \([\{e\}, \{e\}] = 2\text{Sq}(\{e\}) = 0\), and moreover, if \( \text{char} k = 2 \),

\[ \text{Sq}(\{\delta\}) = \{\delta\}. \]

The previous computations and Propositions 9.2 and 9.7 fully determine \( HH^{*,*}(A) \) as a Gerstenhaber algebra. In particular, as a commutative algebra with respect to the total degree, it has the following nice presentation,

\[ HH^{*,*}(A) = k[\epsilon x, x^\pm 2, \{\delta\}, \{e\}], \quad \text{char} k \neq 2; \]

\[ HH^{*,*}(A) = k[\epsilon x, x^\pm 1, \{\delta\}, \{e\}] / (\epsilon^2, \{\delta\}^2), \quad \text{char} k = 2. \]
In the first case, the generators have total degrees $-1$, $-2$, 1, and 2, respectively, and in the second case 0, $\mp 1$, 1, and 2. If char $k \neq 2$, the dimension of $HH^{p,q}(A)$ is 1 for $p = 0$ and 2 if $p > 0$, and if char $k = 2$ dimensions double.

Let us consider the quadratic map

\[ Sq: HH^{3,-1}(A) \to HH^{5,-2}(A). \]

If char $k \neq 2$, bases of the source and the target are given by

\[ x^4(e)^3, ex^3(\delta)(e)^2 \in HH^{3,-1}(A), \quad x^6(\delta)(e)^4, ex^7(e)^5 \in HH^{5,-2}(A), \]

and, using the previous description of the Gerstenhaber algebra $HH^{\ast,\ast}(A)$ and the general laws of these algebras, it is easy to see that the Gerstenhaber square is given by

\[ Sq(\alpha x^4(e)^3 + \beta ex^3(\delta)(e)^2) = 3\alpha\beta x^6(\delta)(e)^4 - \alpha\beta ex^7(e)^5, \quad \alpha, \beta \in k. \]

Therefore, by Remark 8.5, the set of homotopy classes of maps $A_4 \to \mathcal{E}(\Sigma A)$ with underlying graded algebra $A$ which extend to $A_4$ is the 2-dimensional vector space $HH^{3,-1}(A)$ and the subset of maps extending to $A_5$ is the zero locus of $\alpha\beta = 0$ with respect to the previous basis, i.e. the coordinate axes,

Moreover, each map $A_3 \to \mathcal{E}(\Sigma A)$ corresponding to a point over the horizontal axis different from the origin, $\alpha x^4(e)^3$, $\alpha \in k^\times$, extends uniquely to $A_\infty \to \mathcal{E}(\Sigma A)$ by Theorem 1.1.

If char $k = 2$, the bases are

\[ x^4(e)^3, ex^4(e)^3, x^3(\delta)(e)^2, ex^3(\delta)(e)^2 \in HH^{3,-1}(A), \]

\[ x^6(\delta)(e)^4, ex^6(\delta)(e)^4, x^7(e)^5, ex^7(e)^5 \in HH^{5,-2}(A), \]

and the quadratic map is given by

\[
\begin{align*}
Sq(\alpha_1 x^4(e)^3 + & \alpha_2 ex^4(e)^3 + \alpha_3 x^3(\delta)(e)^2 + \alpha_4 ex^3(\delta)(e)^2) \\
= & \alpha_1\alpha_2 x^6(\delta)(e)^4 + \alpha_2\alpha_3 ex^6(\delta)(e)^4 \\
& + (\alpha_1\alpha_2 + \alpha_1\alpha_3)x^7(e)^5 + (\alpha_2^2 + \alpha_2\alpha_3 + \alpha_1\alpha_4)ex^7(e)^5.
\end{align*}
\]

In this case, the homotopy classes of maps $A_3 \to \mathcal{E}(\Sigma A)$ with underlying graded algebra $A$ which extend to $A_4$ is the 4-dimensional vector space $HH^{3,-1}(A)$, and the subset of maps extending to $A_5$ is the union of the following two planes intersecting at the origin,

\[
\begin{align*}
\{ & \alpha_1 = 0, \\
\alpha_2 = 0, \\
\alpha_3 = 0, \\
\alpha_4 = 0. \}
\end{align*}
\]

By Theorem 1.1 any point in the second plane which is not in $\alpha_1 = 0$, i.e. $\alpha_1 x^4(e)^3 + \alpha_2 ex^4(e)^3 + \alpha_3 x^3(\delta)(e)^2$, $\alpha_1 \in k^\times$, $\alpha_2, \alpha_3 \in k$, corresponds to a map which extends uniquely to $A_\infty \to \mathcal{E}(\Sigma A)$. Here we use that $ex^4(e)^3$ and $x^3(\delta)(e)^2$ are nilpotent, since $e^2 = 0 = (\delta)^2$. 

Let \( \phi : A_\infty \to \mathcal{E}(\Sigma A) \) be the only homotopy class with universal Massey product \( x^4\{e\}^3 \). In order to compute the \( E_1 \) terms of the spectral sequence converging to the homotopy groups of \( \text{Map}(A_\infty, \mathcal{E}(\Sigma A)) \) based at \( \phi \), see Theorem 1.2 and Remark 0.9 we calculate the homomorphisms

\[
[x^4\{e\}^3, -] : HH^{p,q}(A) \rightarrow HH^{p+2,q-1}(A).
\]

Assume that \( \text{char } k \neq 2 \). If \( p-q \) is even, bases of source and target are given by

\[
x^{p-q}\{e\}^p, e x^{p-q-1}\{\delta\}\{e\}^{p-1} \in HH^{p,q}(A),
\]

\[
x^{p-q+2}\{\delta\}\{e\}^{p+1}, e x^{p-q+3}\{e\}^{p+2} \in HH^{p+2,q-1}(A),
\]

and

\[
[x^4\{e\}^3, x^{p-q}\{e\}^p] = 0,
\]

\[
[x^4\{e\}^3, e x^{p-q-1}\{\delta\}\{e\}^{p-1}] = 3x^{p-q+2}\{\delta\}\{e\}^{p+1} + (-1)^{p-q-1}e x^{p-q+3}\{e\}^{p+2}.
\]

If \( p-q \) is odd, they are given by

\[
x^{p-q-1}\{\delta\}\{e\}^{p-1}, e x^{p-q}\{e\}^p \in HH^{p,q}(A),
\]

\[
x^{p-q+3}\{e\}^{p+2}, e x^{p-q+2}\{\delta\}\{e\}^{p+1} \in HH^{p+2,q-1}(A),
\]

and

\[
[x^4\{e\}^3, x^{p-q-1}\{\delta\}\{e\}^{p-1}] = (-1)^{p-q-1}x^{p-q+3}\{e\}^{p+2},
\]

\[
[x^4\{e\}^3, e x^{p-q}\{e\}^p] = 3x^{p-q+3}\{e\}^{p+2}.
\]

With these formulas, the vanishing of \( E_1^{s,t} \), \( s \geq 2 \), checked in the proof of Theorems 1.1 and 1.2 follows from the very computation of the \( E_2 \) page in Corollary 6.2 and Theorem 6.4. Moreover, for \( t \geq 2 \), \( E_2^{1,t} \) is 1-dimensional, generated by \( x^{3+t}\{e\}^3 \) if \( t \) is odd, and by \( 3x^{2+t}\{\delta\}\{e\}^2 - e x^{3+t}\{e\}^3 \) if \( t \) is even. Furthermore, for \( t \geq 1 \), \( E_3^{0,t} \) if the inverse image along the natural projection \( Z^{2,-t}(A) \to HH^{2,-t}(A) \) of the 1-dimensional subspace generated by \( x^{2t}\{e\}^2 \) if \( t \) is even and by \( 3x^{t+1}\{\delta\}\{e\} - e x^{t+2}\{e\}^2 \) if \( t \) is odd. This is a vector space of dimension \( 2^{8t} \), so the homotopy groups of \( \text{Map}(A_\infty, \mathcal{E}(\Sigma A)) \) based at \( \phi \), \( t \geq 1 \),

\[
E_3^{1,t+1} \hookrightarrow \pi_1(\text{Map}(A_\infty, \mathcal{E}(X)), \phi_\infty) \to E_3^{0,t},
\]

are extensions of an infinite dimensional vector space by a vector space of dimension 1. We should remark that, nevertheless, these homotopy groups are in principle just abelian groups for \( t \geq 2 \) and a group for \( t = 1 \).

If \( \text{char } k = 2 \), everything is very similar, except for the computation of \( E_3^{0,1} \), which is the kernel of the abelian group homomorphism

\[
HH^{2,-1}(A) \to HH^{4,-2}(A) : x \mapsto [x^4\{e\}^3, x] + x \sim x.
\]

These \( k \)-vector spaces have bases

\[
x^2\{\delta\}\{e\}, e x^2\{\delta\}\{e\}, x^3\{e\}^2, e x^3\{e\}^2 \in HH^{2,-1}(A),
\]

\[
x^6\{e\}^4, e x^6\{e\}^4, x^5\{\delta\}\{e\}^3, e x^5\{\delta\}\{e\}^3 \in HH^{4,-2}(A),
\]

The \( \mathbb{F} \)-vector spaces have bases
the $k$-vector space homomorphism $[x^4(e)^3, -]$ is determined by the following formulas,
\[
[x^4(e)^3, x^2(\delta)(e)] = x^6(e)^4,
\]
\[
[x^4(e)^3, \varepsilon x^2(\delta)(e)] = \varepsilon x^6(e)^4 + x^8(\delta)(e)^4,
\]
\[
[x^4(e)^3, x^3(e)^2] = 0,
\]
\[
[x^4(e)^3, \varepsilon x^3(e)^2] = x^6(e)^4,
\]

i.e.
\[
[x^4(e)^3, \alpha_1 x^2(\delta)(e) + \alpha_2 \varepsilon x^2(\delta)(e) + \alpha_3 x^3(e)^2 + \alpha_4 \varepsilon x^3(e)^2]
= (\alpha_1 + \alpha_4)x^6(e)^4 + \alpha_2 \varepsilon x^6(e)^4 + \alpha_2 x^8(\delta)(e)^4,
\]

and the cup square abelian group homomorphism is given by
\[
(\alpha_1 x^2(\delta)(e) + \alpha_2 \varepsilon x^2(\delta)(e) + \alpha_3 x^3(e)^2 + \alpha_4 \varepsilon x^3(e)^2)^2 = \alpha_3^2 x^6(e)^4.
\]

Hence $E_3^{01}$ is the inverse image along the natural projection $Z^{2,-1}(A) \twoheadrightarrow HH^{2,-1}(A)$
of the abelian subgroup given by the elements
\[
\alpha^2(\delta)(e) + \beta x^3(e)^2 + (\alpha + \beta^2) \varepsilon x^3(e)^2 \in HH^{2,-1}(A), \quad \alpha, \beta \in k.
\]

This subgroup is actually the additive group of the $k$-vector space $k^2$, but its embedding in $HH^{2,-1}(A)$ does not preserve the scalar product.

This example can be computed over any commutative ring $k$, and it is meaningful since $A$ is free as a $k$-module, so any $A$-infinity algebra with homology graded algebra $A$ has a minimal model. The computation is more tedious since direct summands of the form $k/2$ and $\text{Tor}_3^2(\mathbb{A},\mathbb{Z}/2)$ show up.

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Universidad de Sevilla, Facultad de Matemáticas, Departamento de Álgebra, Avda. Reina Mercedes s/n, 41012 Sevilla, Spain

E-mail address: fmuro@us.es

URL: http://personal.us.es/fmuro