A generalized exposition of first-species counterpoint theory

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Abstract: We generalize Mazzola’s first-species counterpoint model to arbitrary rings and obtain some counting results that enrich the theory of admitted successors, pointing to a structural approach, beyond computations. The original motivations of the model, as well as all technical passages, are carefully reviewed so as to provide a complete exposition of its essential features. Based on this exposition, we suggest several improvements and stress the value of a possible simplification.

Keywords: counterpoint; rings; modules; combinatorics

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1 Introduction

This article has several purposes.

First, it intends to provide a rigorous, but strongly motivated, generalized exposition of a model for first-species counterpoint, based on ring and module theory. This kind of counterpoint is the simplest one and the didactic basis of Renaissance counterpoint theory, as taught by Johann J. Fux in [7]. The model was introduced by Mazzola in [13], where the discussion confined to the case when the ground ring is $\mathbb{Z}_{12}$; a ring that can be used to model the algebraic behavior of the twelve intervals between tones in the chromatic scale of Western musical tradition. Then, the model was re-exposed with some additional computational results by Hichert in [12, Part VII]. Further generalizations to the case when the rings are of the form $\mathbb{Z}_n$ were considered in [1, 10], and then included in a collaborative compendium of mathematical counterpoint theory and its computational aspects [3]. The motivation for such a generalization to $\mathbb{Z}_n$ was the existence of microtonal scales with more than twelve tones, which have been used for making real music [2].

Second, at the same time, we generalize the model to noncommutative rings. However, though this generalization can sound difficult to the reader, the essential feature to bear in mind, while reading this paper, is the fact that two given elements of the ground ring need not commute. On the contrary, dropping the commutativity hypothesis offers the possibility of defining counterpoint notions on noncommutative rings of generalized intervals, which can be worth exploring in musical terms; see Example 3.10 and Section 10. From a more theoretic perspective, this generalization has the advantage that in some proofs, the essential arguments, as well as the necessity of the commutative hypothesis for the ground ring, are identified. For example, in Section 8.1 we obtain some counting formulas for the cardinality of deformed consonance sets, and a maximization criterion of these cardinals, thanks to the commutativity hypothesis. Anyway, the paper can be read by assuming that all rings are commutative, since there is no specialized result from noncommutative ring theory used here. A basic course on rings and modules is the only prerequisite.

Third, the previous literature on this subject has a strong computational flavour. In this paper, we emphasize the necessity of a structural and conceptual approach to the model. The main goal would be to provide a theoretic proof of the Counterpoint Theorem [3, Cf. Theorem 3.1]. This theorem gives the exact number of admitted successors of a counterpoint interval and was established by using Hichert’s algorithm. The aforementioned counting results could simplify this algorithm. Also, they allow us to compute the admitted successors, by hand, in the case of Renaissance counterpoint. Thus, we point to understand why the number of admissible successors of a counterpoint interval is that obtained by Hichert’s algorithm. We believe that a profound comprehension of Renaissance counterpoint and its generalizations pass through a structural understanding of their models, or rather, of counterpoint itself.

Regarding other mathematical models of counterpoint, Mazzola’s school has an important
counterpart in D. Tymoczko. Tymoczko’s model [15, Appendix] is based on orbifolds and is oriented towards voice leading. Moreover, it works by means of a geometric reading of the usual counterpoint rules, in contrast to the predictive character of Mazzola’s model, which follows the principle that these rules obey profound relations based on musical symmetries. There has been a polemic around Mazzola’s and Tymoczko’s models. Tymoczko’s initial critique can be found in [15] and a response by Mazzola and the second author can be found in [4].

We organize this paper as follows. We start with the basic definitions: symmetries (natural operations in music) in Section 2, hierarchy of dichotomies and polarities (consonance/dissonance partitions and their symmetries) in Section 3, and dual numbers (modelling two voices in counterpoint) in Section 4. In particular, in Section 3.4, we prove that several dichotomies are strong, in contrast to previous approaches, where the authors advised the reader to check the strong dichotomy property with the computer. Then, in Section 5, we discuss global and local symmetries of consonances/dissonances, which, according to Mazzola, are the responsible for tension/resolution forces in counterpoint. This allows us to provide a formal definition of first-species counterpoint in Section 5.3. In Section 6, we simplify this definition, by reducing it to the case when the cantus firmus is the note c and use a small group of symmetries. Based on this simplification, we prove that, in any notion of counterpoint, it is always possible to find successors of a given consonant interval. This result is the Little theorem of counterpoint. A careful review of a crucial lemma (Lemma 7.1) in the proof of this theorem leads us to the counting formulas and the maximization criterion in Section 8 for deformed consonance sets. This section is the main contribution of this article. With these tools, in Section 9, we obtain the admitted successors of consonant intervals in Renaissance counterpoint, without the computer. In Section 10, we explore a noncommutative notion of counterpoint. Finally, in Section 11, we compare the strengths and the weaknesses of Mazzola’s model, and subsequently suggest some improvements and, above all, a simplification of the model. The appendix includes the proof of a variation of the rearrangement inequality, which is crucial for the maximization criterion, and a brief discussion on the categorical groupoid of intervals of a ring, which is useful for deciding when a partition of a ring into two equipotent sets has an affine symmetry.

Throughout this paper, $R$ denotes an arbitrary ring with unity, not necessarily commutative. When needed, commutativity hypothesis on $R$ is explicitly stated. Similarly, modules considered in this paper are right $R$-modules, unless we indicate otherwise.

2 Ring symmetries

Affine homomorphisms are the natural correspondences that occur in music. They are the natural formalization that encompasses musical transformations like transposition and inversion in the ring $\mathbb{Z}_{12}$ of tones.\footnote{The ring $\mathbb{Z}_{12}$ models the twelve tones of the chromatic scale and, at the same time, the twelve intervals between these tones. The distinction is usually clear from the context. On the other hand, they come from affine geometry, understood as the study of the invariant properties under the action of the group of affine
automorphisms of the plane \( \mathbb{R}^2 \) regarded as a real vector space. However, in music, we are interested in the variations of musical material (scales, intervals) with symmetries. The basic variation in counterpoint is the prohibition of parallelisms, whereas the fundamental invariant of affine geometry is parallelism between lines.

Let \( R \) be a ring. Given two \( R \)-modules \( M \) and \( N \), an affine homomorphism from \( M \) to \( N \) is the composite \( e^a \circ f \), where \( f : M \to N \) is an \( R \)-homomorphism, \( a \in N \), and \( e^a : N \to N : x \mapsto x + a \) is the translation associated with \( a \). We write a typical affine homomorphism \( e^a \circ f \) as \( e^a f \), for short. Observe that the composition rule for affine homomorphisms is \((e^a f) \circ (e^b g) = e^{f(b) + a} fg\).

The monoid (with respect to the composition) \( \text{End}_R(M) \) of \( R \)-endomorphisms of an \( R \)-module \( M \) is a ring with the usual sum of homomorphisms. However, the monoid of affine endomorphisms of \( M \) is not a ring since the distributivity fails, except when \( M \) is the trivial module. We denote by \( \text{Aut}_R(M) \) the group of \( R \)-automorphisms of an \( R \)-module \( M \).

An affine homomorphism \( e^a f : M \to N \) is an isomorphism if and only if \( f \) is an \( R \)-isomorphism.\(^2\) Thus, affine automorphisms of \( R \) are of the form \( e^a f \), where \( f \in \text{Aut}_R(R) \) and \( a \in R \). On the other hand, recall that there is a ring isomorphism

\[
\text{End}_R(R) \cong R,
\]

which identifies an element \( r \in R \) with the \( R \)-endomorphism obtained by left multiplication with \( r \). This isomorphism restricts to a group isomorphism

\[
\text{Aut}_R(R) \cong R^*,
\]

which establishes a bijective correspondence between \( R \)-automorphisms of \( R \) and invertible elements of \( R \). From this discussion we conclude that an affine automorphism of \( R \) is the function associated with the linear polynomial \( bx + a \), where \( b \in R^* \) and \( a \in R \).

In what follows, we will call affine automorphisms of \( R \) symmetries of \( R \) and denote the group of symmetries of a ring \( R \) by \( \text{Sym}(R) \). We use the notation \( e^a b \) for the symmetry associated with \( bx + a \). Two symmetries \( e^a b \) and \( e^{a'} b' \) are equal if and only if \( a = a' \) and \( b = b' \). Observe that \( e^0 1 \) is the identity symmetry and that the composition of symmetries is given by the formula

\[
e^a b \circ e^{a'} b' = e^{ba' + a bb'}.
\]

The inverse of a symmetry \( e^a b \) is \( e^{-b^{-1} a b^{-1}} \).

### 3 Basic motivations and definitions

In Renaissance counterpoint we divide the ring \( \mathbb{Z}_{12} \) of intervals into two disjoint subsets, namely the set \( K \) of consonances and the set \( D \) of dissonances. The consonances are unison, minor third, major third, perfect fifth, minor sixth, and major sixth. The dissonances are unison, minor third, major third, perfect fifth, minor sixth, and major sixth. The dissonances are

\(^2\)First, suppose that \( e^b g : N \to M \) is the inverse of \( e^a f \). Then \((e^a f)(e^b g) = e^{f(b) + a} fg = id_N \) and \((e^a f)(e^b g) = e^{g(a) - b} gf = id_M \). Therefore, \( fg = id_N, gf = id_M \), and \( b = -g(a) \). Conversely, if \( fg = id_N \) and \( gf = id_M \), then \((e^a f)(e^{-g(a)} g) = id_N \) and \((e^{-g(a)} g)(e^a f) = id_M \).
minor second, major second, perfect fourth, tritone, minor seventh, and major seventh. The corresponding mathematical definitions are

\[ K = \{0, 3, 4, 7, 8, 9\} \text{ and } D = \{1, 2, 5, 6, 10, 11\} \]

A composition of first-species counterpoint consists of two voices, cantus firmus and discantus, whose notes have the same length and where each interval between the voices is a consonance. See [7, pp. 19-29] for details.

The germ of Mazzola’s model is the observation that there is a unique symmetry \( p \) of the ring \( \mathbb{Z}_{12} \), namely \( e^{2\pi i/5} \), that sends consonances to dissonances. In fact,

\[ e^{2\pi i/5}(0) = 2, \quad e^{2\pi i/5}(3) = 5, \quad e^{2\pi i/5}(4) = 10, \quad e^{2\pi i/5}(7) = 1, \quad e^{2\pi i/5}(8) = 6, \quad e^{2\pi i/5}(9) = 11. \]

We postpone the proof of the uniqueness to Section 3.4. Moreover, \( e^{2\pi i/5} \circ e^{2\pi i/5} = e^{0\pi i/5} = \text{id} \), so \( e^{2\pi i/5} \) is its own inverse. Two important structural implications of this uniqueness property, for mathematical counterpoint theory, can be appreciated in Theorem 5.4 and Lemma 6.4.

Up to now, we have the intervals, consonances, and dissonances of counterpoint, but we need to model the voices in a composition. We achieve this by means of the dual numbers ring \( \mathbb{Z}_{12}[\epsilon] \), which is the ring of linear polynomials \( c + d\epsilon \), with \( c, d \in \mathbb{Z}_{12} \), in an indeterminate \( \epsilon \) satisfying the relation \( \epsilon^2 = 0 \). The element \( c + d\epsilon \), called a contrapuntal interval, represents a pitch class \( c \), of the cantus firmus, together with the interval \( d \) between \( c \) and the pitch class \( c + d \) from the superior discantus. As already said, in a piece of Renaissance counterpoint, it is mandatory that \( d \) be a consonance. For instance, the dual number \( 2 + 7\epsilon \) in \( \mathbb{Z}_{12}[\epsilon] \) comes from the following musical example.

![Musical Example]

\( \text{contrapunctus} \quad \{2 + 7\epsilon\} \quad \text{2 + 7} \epsilon \)

\( \text{cantus firmus} \)

Thus, we have a partition of the ring \( \mathbb{Z}_{12}[\epsilon] \) into contrapuntal consonances and dissonances, namely \( \{K[\epsilon], D[\epsilon]\} \), where \( X[\epsilon] \) consists of all \( c + d\epsilon \) with \( d \in X \) for \( X = K, D \). We would want properties for this partition analogous to those of \( \{K, D\} \). Certainly, the symmetry \( e^{2\pi i/5} \) of \( \mathbb{Z}_{12}[\epsilon] \) is a quite natural \(^4\) extension of \( e^{2\pi i/5} \) sending \( K[\epsilon] \) to \( D[\epsilon] \) since \( e^{2\pi i/5} \) is simple and acts on the interval part \( d \) of a dual number \( c + d\epsilon \) just as \( e^{2\pi i/5} \). But in this case it is not the unique sending \( K[\epsilon] \) to \( D[\epsilon] \). For example, \( e^{2\pi i/5 + 1} \) also does.

We thus have two kinds of partitions \( \{K, D\} \) of a ring \( R \) that are important for counterpoint: those for which there is a symmetry \( p \) of \( R \) that sends \( K \) to \( D \) (dual numbers case) and those for which such a symmetry exists and is unique (intervals case).

We want to compose a piece of first-species counterpoint, which is a suitable finite sequence of dual numbers \( \xi_1, \ldots, \xi_n \) in \( K[\epsilon] \) with cantus firmus and discantus in a mode.\(^5\)

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\(^3\)We discuss the choice of dual numbers to model this situation in Section 4.

\(^4\)A precise statement of this naturality claim is Proposition 5.5.

\(^5\)A musical introduction to ecclesiastical modes is [9, pp. 59-82]. We consider modes from a more mathematical perspective in Definition 5.8.
However, not all sequences are valid since counterpoint has some rules. These rules establish when $\xi_{i+1}$ is an admitted successor of $\xi_i$ for $i = 0, \ldots, n - 1$. In the model to be exposed in the following sections, we aim to predict these rules and formalize the idea of an admitted successor so that, given a cantus firmus, we can compose a discantus with the aid of the rules and our own musicality. It is important to stress that the model is not an automatic or mechanic process for producing compositions, although the symmetries and the rules can provide a framework for such a mechanization [3, 10].

3.1 Dichotomies

We start to develop our theory by formalizing the common properties of the rings of intervals $\mathbb{Z}_{12}$ and of dual numbers $\mathbb{Z}_{12}[\epsilon]$ that are relevant for counterpoint.

Let $R$ be a ring.

- A partition $\{K, D\}$ of $R$ is a dichotomy of $R$ if $|K| = |D|$. Note that a finite ring $R$ has dichotomies if and only if its cardinality is even.

- A self-complementary dichotomy of $R$ is a triple $(K, D, p)$, where $\{K, D\}$ is a partition of $R$, $p$ is a symmetry of $R$, and $p(K) = D$. Note that necessarily $\{K, D\}$ is a dichotomy of $R$.

The triples $(K, D, e^{2\pi i})$ and $(K[\epsilon], D[\epsilon], e^{2\pi i})$ from Renaissance counterpoint are self-complementary dichotomies. The dichotomy $\{(0, 1, 3, 6, 8, 11), (2, 4, 5, 7, 9, 10)\}$ of $\mathbb{Z}_{12}$ is not part of a self-complementary dichotomy of $\mathbb{Z}_{12}$ as proved in Appendix.

We would like to construct self-complementary dichotomies other than the basic one of Renaissance counterpoint, so as to create new notions of counterpoint. Given an arbitrary finite ring $R$ of even cardinality, we could start with a symmetry $p$ of $R$ and aim to construct a self-complementary dichotomy $(K, D, p)$. We pick some $k_1 \in R$ to be in $K$, and define $d_1 := p(k_1)$, with $d_1$ to be in $D$. Then, if $|R| > 2$, we take some $k_2 \in R \setminus \{k_1, d_1\}$ to be in $K$ and define $d_2 := p(k_2)$ to be in $D \setminus \{d_1\}$, and so on. Note that this process produces a dichotomy $\{K, D\}$ provided $p$ has no fixed points and is finite since $R$ is. We say that a symmetry $p$ without fixed points is a derangement.

The following proposition contains some concluding properties of self-complementary dichotomies.

Proposition 3.1. Each self-complementary dichotomy $(K, D, p)$ of $R$ has the following properties:

1. The identity $p(D) = K$ holds, so $(D, K, p)$ is a self-complementary dichotomy.

2. The symmetry $p$ is a derangement.

Proof. 1. Since $p$ is a bijection, the properties of the inverse image imply that $p(D) = p(R \setminus K) = R \setminus p(K) = R \setminus D = K$. 2. Let $x \in R$. If $x \in K$, then $x \neq p(x) \in D$. If $x \in D$, then, by 1, $x \neq p(x) \in K$.

Thus, self-complementary dichotomies are strongly related to derangements.

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6This definition is essentially the same of an autocomplementary dichotomy in [12, Definition 92].
3.2 Strong dichotomies and polarities

We formalize the additional uniqueness property of the Renaissance dichotomy \((K, D, e^2)\) with the following definitions. Let \(R\) be a ring. A self-complementary dichotomy \((K, D, p)\) of \(R\) is a strong dichotomy of \(R\) if \(p\) is the unique symmetry of \(R\) such that \(p(K) = D\). In such a case, we also say that \(p\) is a polarity.

As we will show in the next sections, once we have a strong dichotomy of a ring (made up of generalized intervals), we have an induced self-complementary dichotomy of the respective dual numbers ring (made up from contrapuntal intervals between two voices), and an associated theory of admitted successors. Thus, strong dichotomies lead to new counterpoint worlds for composing non-traditional counterpoint.

We need to determine whether a given self-complementary dichotomy \((K, D, p)\) is strong. We approach this problem by determining the number of symmetries sending \(K\) to \(D\). Given another symmetry \(q\) with \(q(K) = D\), note that

\[ q^{-1} \circ p(K) = q^{-1}(D) = K \]

and hence \(q^{-1} \circ p\) is in the stabilizer \(\theta(K)\), where

\[ \theta(K) := \{g \in \text{Sym}(R) \mid g(K) = K\}. \]

This suggests the right action of \(\theta(K)\), by composition, on the set \(\text{Sym}(K, D)\) of symmetries of \(R\) sending \(K\) to \(D\), as shown in the following diagram.

\[ \circ : \text{Sym}(K, D) \times \theta(K) \to \text{Sym}(K, D) \]

\[ (q, g) \mapsto q \circ g \]

As we have observed, every \(q \in \text{Sym}(K, D)\) is in the same orbit of \(p\). Also, the stabilizer of \(p\) under this action is the identity since \(p \circ g = p\) implies \(g = id_R\). Hence, the following proposition.

**Proposition 3.2.** Let \((K, D, p)\) be a self-complementary dichotomy of a ring \(R\). There is a bijective correspondence between \(\text{Sym}(K, D)\) and \(\theta(K)\). The bijection sends an element \(g \in \theta(K)\) to \(p \circ g\).

**Proof.** The function \(\theta(K) \to \text{Sym}(K, D) : g \mapsto p \circ g\) is surjective since the action is transitive and is injective since the stabilizer of \(p\) is trivial.

Thus, we have reduced the study of \(\text{Sym}(K, D)\) to that of the stabilizer group \(\theta(K)\) of \(K\). Besides, we have determined when a self-complementary dichotomy is strong, as established in the following corollary. We say that a partition \(\{K, D\}\) of \(R\) is rigid if \(\theta(K)\) is the trivial group.

**Corollary 3.3.** A self-complementary dichotomy \((K, D, p)\) is a strong dichotomy if and only if \(\{K, D\}\) is rigid.
Now, our objective is to prove a pair of lemmas that help us to decide whether a given dichotomy is rigid, by discarding non-identity symmetries that could be in $\theta(K)$. These lemmas could be a first approximation to a more conceptual argument.\footnote{This conceptual argument can be related to the groupoid of intervals of $K$, briefly introduced in Appendix.} Consider the left action $\cdot$, by multiplication, of the group $R^*$ on $R$.

**Lemma 3.4.** Let $\{K, D\}$ be a dichotomy of a ring $R$. Suppose that $a \in R$ and that $C$ is either $K$ or $D$. If there is $r \in R$ such that

1. the orbit $R^*r$ of $r$ under the action $\cdot$ is contained in $C$ and
2. $r + a \notin C$,

then $e^a b \notin \theta(K)$ for each $b \in R^*$.

**Proof.** If $r + a \notin C$ for $r$ as above, then for each $b \in R^*$, $b(b^{-1} \cdot r) + a \notin C$ with $b^{-1} \cdot r \in R^*r \subseteq C$, and hence $e^a b \notin \theta(C) = \theta(K)$. \hfill $\square$

In particular, if $r = 0$, we obtain the following result.

**Lemma 3.5.** Suppose that $C$ is either $K$ or $D$ and that $0 \in C$. Each symmetry of the form $e^a b$ with $a \in R \setminus C$ is not in $\theta(K)$. In particular, $e^a b \notin \theta(K)$.

Before using these lemmas to check that some dichotomies are strong, we first study a possible way to construct self-complementary dichotomies that are good candidates to strong ones.

### 3.3 Constructing strong dichotomies: quasipolarities

A symmetry $p$ of a ring $R$ is **involutive** if $p \circ p = id$. If $p = e^a b$, by Equation (2), this condition is equivalent to $e^{ba+a}b^2 = e^0 1$, that is, to $ba + a = 0$ and $b^2 = 1$. In the next proposition we prove that all polarities are involutive, so if we want a polarity to open up a counterpoint world, we could start with an involutive derangement, then define a suitable self-complementary dichotomy following the strategy in Section 3.1, and finally check whether the uniqueness property holds.

**Proposition 3.6.** If a self-complementary dichotomy $(K, D, p)$ is strong, then $p$ is involutive.

**Proof.** If $(K, D, p)$ is a self-complementary dichotomy and $p$ has the uniqueness property, then $p \circ p(K) = p(D) = K$ by 1 in Proposition 3.1, and hence $p \circ p = id$ by Corollary 3.3. \hfill $\square$

The converse of this proposition does not hold since, in the Renaissance dichotomy $(K[e], D[e], e^{2i} 5)$, $e^{2i} 5$ is involutive but the dichotomy is not strong, as already observed.

We call involutive derangements **quasipolarities**. A **quasipolarization** of $R$ is a self-complementary dichotomy $(K, D, p)$ of $R$ where $p$ is involutive. Therefore, in view of Proposition 3.1, $(K, D, p)$ is a quasipolarization if and only if $p$ is a quasipolarity.
Note that not all self-complementary dichotomies are quasipolarizations. For example, the self-complementary dichotomy \((\{0, 2, 4, 6, 8, 10\}, \{1, 3, 5, 7, 9, 11\}, e^1)\) of \(\mathbb{Z}_{12}\) is not a quasipolarization since \(e^1 \circ e^1 = e^2 \neq id\).

Thus, the construction of quasipolarizations is a good beginning if we want a strong dichotomy, though quasipolarizations need not be strong dichotomies.

### 3.4 Examples of strong dichotomies

In this section, we use Lemmas 3.4 and 3.5 to open up a series of counterpoint worlds. First, consider the case when \(R\) is the commutative ring \(\mathbb{Z}_{12}\) and different examples of quasipolarizations that we will check to be strong dichotomies. The set of orbits of the action of the group \(\{1, 5, 7, 11\}\) of invertible elements of \(\mathbb{Z}_{12}\) on \(\mathbb{Z}_{12}\) is

\[\{\{0\}, \{1, 5, 7, 11\}, \{2, 10\}, \{3, 9\}, \{4, 8\}, \{6\}\}\]

In the following example we prove the claim that the Renaissance quasipolarization is a strong dichotomy.

**Example 3.7 (Renaissance counterpoint).** Consider the dichotomy \((K, D, e^{25})\), where \(K = \{0, 3, 4, 7, 8, 9\}\) and \(D = \{1, 2, 5, 6, 10, 11\}\). Let us prove that the dichotomy is strong. According to Corollary 3.3, it is enough to show that our dichotomy is rigid, or equivalently, that \(\theta(K) = \{e^01\}\). In fact, by Lemma 3.5, \(e^a b \notin \theta(K)\) for each symmetry \(e^a b\) with \(a \in D\). The cases when \(a \in K\) remain. If \(a = 0\), then since \(5 \times 7 = 11 \notin K\), \(7 \times 7 = 1 \notin K\), and \(11 \times 7 = 5 \notin K\), we conclude that the symmetries of the form \(e^0b\) with \(b = 5, 7, 9\) are not in \(\theta(K)\). Finally, since

\[3 + 8 = 8 + 3 = 11 \notin K, 4 + 9 = 9 + 4 = 1 \notin K, \text{ and } 3 + 7 \notin K,\]

Lemma 3.4 implies that all symmetries \(e^a b\) with \(a \in \{3, 4, 7, 8, 9\}\) are not in \(\theta(K)\). We have exhausted all possibilities, except the identity, and hence \(\theta(K)\) is trivial.

**Example 3.8 (Scriabin’s mystic chord, see [5]).** Consider the dichotomy \((K, D, e^{511})\), where \(K = \{0, 2, 4, 6, 9, 10\}\) and \(D = \{1, 3, 5, 7, 8, 11\}\). Let us show that \(e^{511}\) is a polarity by using the same strategy of the preceding example.

As before, \(e^a b \notin \theta(K)\) for each symmetry \(e^a b\) with \(a \in D\), and we need to discard the cases when \(a \in K\). If \(a = 0\), then since \(5 \times 4 = 8 \notin K, 7 \times 9 = 3 \notin K\), and \(11 \times 4 = 8 \notin K\), the symmetries of the form \(e^0 b\) with \(b = 5, 7, 11\) are not in \(\theta(K)\). Also, since

\[6 + 2 = 2 + 6 = 8 \notin K, 5 + 4 = 9 \notin D, 2 + 9 = 11 \notin K, \text{ and } 10 + 10 = 8 \notin K,\]

Lemma 3.4 implies that all symmetries \(e^a b\) with \(a \in \{2, 4, 6, 9, 10\}\) are not in \(\theta(K)\). Hence, \(\theta(K)\) is trivial.

The next computation establishes the existence of at least a strong dichotomy in \(\mathbb{Z}_{2k}\) for each \(k \geq 3\). These dichotomies are the beginning of a lot of microtonal counterpoint worlds, which were first studied in [1]. Recall that all invertible elements of \(\mathbb{Z}_{2k}\) are odd since they are coprime with \(2k\).
**Example 3.9** (Cf. [3, Proposition 2.1] and [1, Proposition 2.6]). Let $k \geq 3$. In $\mathbb{Z}_{2k}$, consider the dichotomy $(K, D, e^{-1}(-1))$, where

$$K = \{0, 1, 3, ..., 2k - 5, 2k - 3\} \text{ and } D = \{-1, 2k - 2, 2k - 4, ..., 4, 2\}.$$ 

We already know (Lemma 3.5) that $e^a b \notin \theta(K)$ for each symmetry $e^a b$ with $a \in D$. If $a = 0$, then $b(-1) = -b \neq -1$ with $-b$ odd whenever $b \in \mathbb{Z}_{2k}^* \setminus \{1\}$ and hence $-b \notin D$, so the symmetries of the form $e^0 b$ with $b \in \mathbb{Z}_{2k}^* \setminus \{1\}$ are not in $\theta(D) = \theta(K)$. Also, since all orbits of even numbers, except the orbit $\{0\}$ of 0, under the action of $\mathbb{Z}_{2k}^*$ on $\mathbb{Z}_{2k}$ are contained in $D$, the equations

$$2 + 1 = 3 \notin D, ..., 2 + (2k - 5) = 2k - 3 \notin D \text{ and } 4 + (2k - 3) = 1 \notin D,$$

imply (Lemma 3.4) that all symmetries $e^a b$ with $a \in \{1, 3, ..., 2k - 5, 2k - 3\}$ are not in $\theta(K)$. Hence, $\theta(K)$ is trivial. \hfill \diamond

Now, a noncommutative example.

**Example 3.10 (A noncommutative strong dichotomy).** Let $R$ be the noncommutative ring of all upper triangular matrices $2 \times 2$ with entries in $\mathbb{Z}_2$. The symmetry $e^I$, where $I$ is the identity matrix, is a quasipolarity, so we can construct self-complementary dichotomies with the procedure of Section 3.1. A possible choice is the dichotomy with $K = \{0, A_1, A_2, A_3\}$, where

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{and } A_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

In this case, $R^* = \{I, A_3\}$ and $A_1$ and $A_2$ are invariant under the left action of $R^*$. Let us show that $K$ is rigid. As usual, $e^A B \notin \theta(K)$ for each symmetry $e^A B$ with $A \in D$. The cases when $A \in K$ remain. If $A = 0$, then since $A_3 A_3 = I \notin K$, the symmetry $e^0 A_3$ is not in $\theta(K)$. Also,

$$A_2 + A_1 = A_1 + A_2 \notin K \text{ and } A_1 + A_3 \notin K,$$

so, by Lemma 3.4, all symmetries $e^A B$ with $A \in \{A_1, A_2, A_3\}$ are not in $\theta(K)$. Thus, $\theta(K)$ is trivial, and $(K, R \setminus K, e^I)$ is a strong dichotomy. The reader can obtain other examples of strong dichotomies from quasipolarities of $R$, whenever the chosen $K$ is not invariant under the left action of $A_3$, by using a similar argument. \hfill \diamond

Finally, note that there is no strong dichotomy of $\mathbb{Z}_4$ to define a counterpoint world with a four-tone scale (which can be thought as a diminished seventh chord). In fact, the possible dichotomies $\{\{0, 1\}, \{2, 3\}\}$, $\{\{0, 2\}, \{1, 3\}\}$, and $\{\{0, 3\}, \{1, 2\}\}$ are invariant under $e^I(-1)$, $e^2$, and $e^{-1}(-1)$, respectively, so they are not rigid or strong (Corollary 3.3).

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\(^8\)Thanks to the hypothesis $k \geq 3$, the six distinct elements 0, 1, 3, $-1$, 2, and 4 are in $\mathbb{Z}_{2k}$. 

11
4 The dual numbers ring

Given an arbitrary ring $R$, we can construct the polynomial ring $R[x]$ and the two-sided ideal $\langle x^2 \rangle$ consisting of all (left or right) multiples of $x^2$. Thus, we have the quotient ring $R[x]/\langle x^2 \rangle$, which we call the ring of dual numbers associated with $R$. If we denote by $\epsilon$ the class of $x$, then each element of the ring of dual numbers can uniquely be written as $c + d\epsilon$, and hence it makes sense to denote the ring of dual numbers by $R[\epsilon]$. The ring $R[\epsilon]$ is noncommutative if $R$ is.

An element $c + d\epsilon \in R[\epsilon]$ is invertible if and only if $c$ is invertible in $R$. In fact, if $(c' + d')c + d\epsilon = 1$ and $(c + d)(c' + d')\epsilon = 1$, then $c' + (c'd + d'c)\epsilon = 1$ and $cc' + (cd' + dc')\epsilon = 1$, and hence $c' = c^{-1}$ ($c$ is invertible with two-sided inverse $c'$). Reciprocally, if $c$ is invertible, define $c' = c^{-1}$ and $d' = -c'dc'$. Thus, $c' + d'\epsilon$ is a two-sided inverse for $c + d\epsilon$.

Recall, from Section 3, that an element $c + d\epsilon$ of $\mathbb{Z}_{12}[\epsilon]$ represents a generalized pitch class $c$, from the cantus firmus, together with the interval $d$ formed between $c$ and a pitch class from a superior discantus. However, we must ask why we choose the dual numbers ring structure on the elements $c + d\epsilon$ and not another. Mazzola requires [13, p. 558] the characterizing relation $\epsilon^2 = 0$ because it entails two voices and it is consistent with two competing interpretations for three voices (cantus firmus, discantus, and countertenor), namely the English theory, which regards intervals only in relation to the cantus firmus, and the continental theory, which considers all intervalllic relationships between the voices.

We have the following proposition regarding dichotomies of the dual numbers ring $R[\epsilon]$ induced by dichotomies of a ring $R$. Given subsets $X, Y \subseteq R$, we define

$$X + Y\epsilon = \{c + d\epsilon \in R[\epsilon] \mid c \in X \text{ and } d \in Y\}.$$  

Also, we denote by $X[\epsilon]$ the set $R + X\epsilon$.

**Proposition 4.1.** Let $\{K, D\}$ be a dichotomy of a ring $R$.

1. The pair $\{K[\epsilon], D[\epsilon]\}$ is a dichotomy of $R[\epsilon]$.
2. If $(K, D, e^a b)$ is a self-complementary dichotomy of $R$, then $(K[\epsilon], D[\epsilon], e^{ac} b)$ is a self-complementary dichotomy of $R[\epsilon]$.
3. If $(K, D, e^a b)$ is a quasipolarization of $R$, then $(K[\epsilon], D[\epsilon], e^{ac} b)$ is a quasipolarization of $R[\epsilon]$.

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9See [11, p. 3] for details.

10In this representation we make use of the fact that $R$ can be regarded as a subring of $R[x]/\langle x^2 \rangle$.

11More specifically, in the continental theory we would consider the polynomial $a + eb + \epsilon^2 c$, where $a$ is the cantus firmus and $b$ and $c$ the intervals between the cantus firmus and the discantus and between the discantus and the countertenor, respectively. On the other hand, we would have $a + eb$ and $a + \epsilon(b + c)$ in the English theory. However, we could disagree with this argument because it is not clear why we take the algebraically dependent variables $\epsilon$ and $\epsilon^2$ to separate two, in principle, independent intervals.

Section 5 provides a more convincing argument for using dual numbers. They allow a certain analogy with tangent bundles and their local symmetries, the latter being responsible for nature forces, and, in particular, tension/resolution forces in counterpoint, according to Mazzola’s ideas.
As previously commented, strong dichotomies need not induce strong dichotomies on dual numbers.

Proof. 1. Exercise. 2. This follows from 1 and the equalities

\[ e^{atb}(K[\epsilon]) = \{e^{atb}(r + k\epsilon) \mid r \in R \text{ and } k \in K\} \]
\[ = \{br + (bk + a)\epsilon \mid r \in R \text{ and } k \in K\} \]
\[ = bR + (e^{atb}(K))\epsilon \]
\[ = R + D\epsilon \]
\[ = D[\epsilon]. \]

As to the fourth equality above, note that \(bR = R\) since \(b\) is invertible. 3. This follows from 2 and the equation

\[ e^{atb} \circ e^{atb} = e^{(ba+a)\epsilon}b^2 = e^01, \]

which is a consequence of the identity \(e^{atb} \circ e^{atb} = e^01\), equivalent to \(ba + a = 0\) and \(b^2 = 1\). □

5 Global and local symmetries

Once we have the dual numbers, which model the two voices of first-species counterpoint, the next step is the theory of admitted successors of a given contrapuntal consonance \(\xi\), with \(\xi = c + k\epsilon\) and \(k\) consonance of a given strong dichotomy \((K, D, e^{atb})\). The basic principle of the model is the alternation between consonances and dissonances as generator of the movement from \(\xi\) to its successor. In fact, we introduce a dissonance character within consonances and apply the alternation principle as follows. We deform the induced dichotomy \(\{K[\epsilon], D[\epsilon]\}\) of the dual numbers into a new one \(\{g(K[\epsilon]), g(D[\epsilon])\}\), by applying a symmetry \(g\) so that \(\xi\) is a deformed dissonance, that is, \(\xi \in g(D[\epsilon])\), which must move to a deformed consonance \(\eta \in g(K[\epsilon])\) by alternation. Moreover, since in a counterpoint piece all intervals are consonances, we require that \(\eta \in g(K[\epsilon]) \cap K[\epsilon]\). See Figure 1. This explains the first condition and part of the third one of contrapuntal symmetry on \(g\).

However, the model also establishes a hierarchy of tensions among the alternations between contrapuntal intervals and their successors, thanks to global and local symmetries. In
contemporary physics, local symmetries are the responsible for the fundamental nature forces (gravitation, weak nuclear, electromagnetic, strong nuclear), via breaking of global symmetries of physical situations. The precise language for describing local (or gauge) symmetries is differential geometry, in particular, fiber bundles [8, Section 5.3]. We try to translate, informally, this situation to counterpoint as follows. We regard the dual numbers projection onto their cantus firmus part as a sort of tangent bundle over $R$ (see Figure 2), the latter playing the role of a manifold. The number $\epsilon$ represents an infinitesimal tangent vector and $x + R\epsilon$ representing its associated tangent space at $x$, which we denote by $I_x$. In this way, Mazzola regards the discantus as a tangential approximation to the cantus firmus. This offers a more mathematical explanation of the use of dual numbers in counterpoint.

To explain global and local symmetries in our context, let us consider two different families of partitions of the fibers of $R[\epsilon]$. First, take the partition $\{x + K\epsilon, x + D\epsilon\}$ of each $I_x$ into real consonances and dissonances. Second, consider the partition $\{g(K[\epsilon]) \cap I_x, g(D[\epsilon]) \cap I_x\}$ of each $I_x$ into consonances and dissonances deformed by a symmetry $g$ of $R[\epsilon]$. We will show that there is a unique symmetry $p^*[\epsilon]$, naturally induced by the original polarity $e'^b$ (Section 5.1), that leaves invariant $I_z$ and interchanges $K[\epsilon]$ and $D[\epsilon]$; in particular, it interchanges consonances $z + K\epsilon$ and dissonances $z + D\epsilon$ within the fiber $I_z$. This symmetry acts by translation on fibers and sends consonances in a fiber to dissonances in the translated fiber. Moreover, if we require that $p^*[\epsilon]$ sends $g(K[\epsilon])$ to $g(D[\epsilon])$ (second condition of contrapuntal symmetry on $g$), then this symmetry also sends deformed consonances to translated deformed dissonances. According to Mazzola [12, pp. 648-649], $p^*[\epsilon]$ is a global symmetry with respect to the first family of partitions since consonances can be translated, that is, $y + (x + K\epsilon) = (y + x) + K\epsilon$, and hence $p^*[\epsilon]$ acts on consonances by translation plus consonance/dissonance interchange. In contrast, $p^*[\epsilon]$ is usually a local symmetry with

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12 There is a main reason for such an analogy of $R$ with a variety: the Abelian group $\mathbb{Z}_{12}$ can be regarded as a combinatorial torus via the decomposition $\mathbb{Z}_{12} \cong \mathbb{Z}_4 \times \mathbb{Z}_3$. See [12, 3] for details on the torus of thirds.

13 This is a subtle point: there can be non-identity symmetries $g$ of $R[\epsilon]$ such that $p^*[\epsilon]$ is global. Details in Section 5.2.
Figure 3: The action of the global symmetry $p^z[\epsilon]$, namely translation plus consonance/dissonance interchange (see the left-hand cylinder), can be translated along fibers, in contrast to that of the global symmetry $p^z[\epsilon]$ (right-hand cylinder), which does not correspond to translation plus deformed consonance/dissonance interchange, and strongly depends on the parameter $x$.

respect to the second family of partitions (deformed consonances/dissonances), since this translation property does not hold, and hence $p^z[\epsilon]$ does not act on deformed consonances by translation plus deformed consonance/dissonance interchange; see Figure 3. This means that local symmetries highly depend on the parameter $x$, whereas global symmetries are essentially constant with respect to $x$. The mathematical details of the global and the local character of $p^z[\epsilon]$ are in Section 5.2.

In Mazzola’s model, this breaking of regularity of local symmetries produces an extra tension between a contrapuntal interval and its successor beyond the mere alternation. However, the global or local character of contrapuntal symmetries is a further feature rather than a precondition in their definition. The second condition only paves the way for this feature, by means of a consistent interchange of deformed consonances and dissonances, between fibers. The crucial point is the third condition of contrapuntal symmetry on $g$, which establishes that the sets of successors $g(K[\epsilon]) \cap K[\epsilon]$ must have the maximum number of elements. This ensures a maximum of alternations, which could be interpreted as a maximum independence of voices—an important principle of counterpoint.

With these reflections at hand, we define first-species counterpoint in Section 5.3.

5.1 Fiber-invariant symmetries

The interchange of contrapuntal consonances and dissonances and the invariance of the fiber $I_z$ are the basic properties of $p^z[\epsilon]$. In the case when $z = 0$, $p^0[\epsilon]$ is the natural extension $e^{\alpha \beta}$ studied in Proposition 4.1. We next prove that $p^z[\epsilon]$ is uniquely determined, for strong dichotomies, and that $p^z[\epsilon]$ is the conjugate $e^z p^0[\epsilon] e^{-z}$.

**Definition 5.1.** Let $g$ be a symmetry of $R[\epsilon]$ and $x \in R$. We say that $g$ is $x$-invariant if it leaves invariant the tangent space $I_x$, that is, if $g(I_x) = I_x$.

We first compute some candidates to $p^x[\epsilon]$. 
Proposition 5.2. A symmetry $g$ of $R[e]$, where $g = e^{u+ve}(c+de)$, is $x$-invariant if and only if $u = (1-c)x$.

Proof. Note that

$$g(x + Re) = e^{u+ve}(c + de)(x + Re) = cx + u + (cR + dx + v)e = cx + u + Re.$$ 

Thus, $g$ is $x$-invariant if and only $cx + u = x$, the latter condition being equivalent to $u = (1-c)x$. 

We denote by $H_x$ the group$^{14}$ of all $x$-invariant symmetries of $R[e]$. In the case when $x = 0$, we simply write $H$ instead of $H_0$. The conjugation automorphism $e^x \circ (-) \circ e^{-x}$ of Sym$(R[e])$ restricts to an isomorphism $H \rightarrow H_x$ for each $x$, as established in the following proposition.

Proposition 5.3. For each $x \in R$, the conjugation homomorphism $e^x \circ (-) \circ e^{-x} : H \rightarrow H_x$ is an isomorphism of groups.

Proof. First, let us prove that the conjugation homomorphism has its images in $H_x$. If $h \in H$, then

$$e^x \circ h \circ e^{-x}(x + Re) = e^x \circ h(Re) = e^x(Re) = x + Re.$$ 

Moreover, the inverse of $e^x \circ (-) \circ e^{-x}$ is $e^{-x} \circ (-) \circ e^x$. 

In turn, this conjugation restricts to bijections $H(K[e], D[e]) \rightarrow H_x(K[e], D[e])$ between respective sets of symmetries sending $K[e]$ to $D[e]$ (check).

The uniqueness in the following theorem offers an explanation of the structural role of the strong dichotomy $(K, D, p)$.

Theorem 5.4. Let $(K, D, e^a b)$ be a strong dichotomy of $R$. For each cantus firmus $x \in R$, there is a unique $x$-invariant symmetry $p^x[e]$, defined by $p^x[e] = e^x \circ e^{ac} b \circ e^{-x} = e^{(1-b)x + ac} b$, such that $(K[e], D[e], p^x[e])$ is a self-complementary dichotomy of $R[e]$. In particular, by Lemma 5.6,

$$p^x[e](x + K) = x + D.$$ 

Proof. Since self-complementary dichotomies $(K[e], D[e], p')$ with $p'$ $x$-invariant are in correspondence (conjugation) with self-complementary dichotomies $(K[e], D[e], p)$ where $p$ is 0-invariant, the proof of the theorem reduces to the case when $x = 0$ (Proposition 5.5). In that case, $p^0[x]$ is the natural extension $e^{ac} b$, so $p^x[e] = e^x \circ e^{ac} b \circ e^{-x} = e^{(1-b)x + ac} b$. 

Proposition 5.5. Let $(K, D, e^a b)$ be a strong dichotomy of $R$. The symmetry $e^{ac} b$ is the unique 0-invariant symmetry such that $(K[e], D[e], e^{ac} b)$ is a self-complementary dichotomy of $R[e]$. 

$^{14}$In fact, this set is a group since it is an stabilizer one.
Proof. Let $h$ be a 0-invariant symmetry, where $h = e^{a+ve}(c + de)$. By Proposition 5.2, $u = 0$. Moreover, if $h(K[\epsilon]) = D[\epsilon]$, from Lemma 5.6, we obtain that $cK + v = D$, so the condition that $p$ is a polarity implies that $c = b$ and $v = a$. Further, we claim that $d = 0$. In fact, by the condition that $h(K[\epsilon]) = D[\epsilon]$,

$$e^{ve}(c + de)(1 + K\epsilon) = c + (cK + v + d)\epsilon = c + (D + d)\epsilon \subseteq D[\epsilon]$$

and

$$e^{ve}(c + de)(-1 + K\epsilon) = -c + (cK + v - d)\epsilon = -c + (D - d)\epsilon \subseteq D[\epsilon],$$

so $D + d \subseteq D$ and $D - d \subseteq D$ and hence $D + d \subseteq D$ and $D \subseteq D + d$. Thus, $D = D + d$ and, since both $K$ and $D$ are rigid (Corollary 3.3), $d = 0$.

Up to now, we know that each 0-invariant symmetry $h$ such that $h(K[\epsilon]) = D[\epsilon]$ is necessarily $e^{a+e}$. It remains to show that $(K[\epsilon], D[\epsilon], e^{a+e})$ is a self-complementary dichotomy. This was done in Proposition 4.1. \hfill \Box

Lemma 5.6. If $h$ is an $x$-invariant symmetry of $R[\epsilon]$ such that $h(K[\epsilon]) = D[\epsilon]$, then $h(x + K\epsilon) = x + D\epsilon$.

Proof. Under our assumptions,

$$h(x + K\epsilon) = h(K[\epsilon] \cap (x + Re)) = h(K[\epsilon]) \cap (x + Re) = D[\epsilon] \cap (x + Re) = x + D\epsilon.$$ \hfill \Box

5.2 The global and the local

Let us explain why $p^z[\epsilon]$ is a global symmetry of $R[\epsilon]$ with respect to the partition $\{x + K\epsilon, x + D\epsilon\}$ in each fiber. Given $x + r\epsilon$ in a fiber $I_x$,

$$p^z[\epsilon](x + r\epsilon) = x + (b-1)(x-z) + (br + a)\epsilon.$$  

Thus, $p^z[\epsilon]$ sends the fiber $I_x$ to $I_{x+(b-1)(x-z)}$, acts by translation by $(b-1)(x-z)$ units, and sends the set of consonances $x + K\epsilon$ to the set of dissonances $x + (b-1)(x-z) + D\epsilon$, corresponding to translation of $x + K\epsilon$ by $(b-1)(x-z)$ plus interchange of consonances and dissonances in $I_{x+(b-1)(x-z)}$. Note that this factorization (translation plus interchange) holds since translation acts on sets of consonances of the form $x + K\epsilon$, that is, $y + K\epsilon = (y - x) + x + K\epsilon$.

Nevertheless, $p^z[\epsilon]$ is usually local with respect to deformed consonances/dissonances $\{g(K[\epsilon]) \cap I_x, g(D[\epsilon]) \cap I_x\}$. In this case, we can send deformed consonances in $I_x$ to deformed dissonances in $I_{x+(b-1)(x-z)}$ by requiring that $p^z[\epsilon]$ sends $g(K[\epsilon])$ to $g(D[\epsilon])$. Clearly, under this condition,

$$p^z[\epsilon](g(K[\epsilon]) \cap I_x) = p^z[\epsilon](g(K[\epsilon])) \cap p^z[\epsilon](I_x) = g(D[\epsilon]) \cap I_{x+(b-1)(x-z)}.$$
But, unlike the case of real consonances, translation does not act on sets of deformed consonances. If \( g = e^{uc+ve}(c + dc) \), the consonances in \( g(K[\epsilon]) \cap I_x \) are those images \( e^{uc+ve}(c + dc)(r + ke) \in g(K[\epsilon]) \) in \( x + Re \), which satisfy

\[
e^{uc+ve}(c + dc)(r + ke) = cr + u + (ck + dr + v)\epsilon
\]

and \( cr + u = x \). Thus,

\[
g(K[\epsilon]) \cap I_x = x + (cK + dc^{-1}(x - u) + v)\epsilon.
\]

This means that, in general, \( g(K[\epsilon]) \cap I_y \) is not a translation of \( g(K[\epsilon]) \cap I_x \). We illustrate this fact with an example due to Mazzola.

**Example 5.7** (Cf. [12, p. 648] and [13, pp. 561-563]). Let \( R = \mathbb{Z}_{12} \), \( (K, D, e^25) \) the Renaissance strong dichotomy, and \( g = e^{8\epsilon}(5 + 4\epsilon) \). Here,

\[
p^0[\epsilon] \circ g = e^{2\epsilon}5 \circ e^{8\epsilon}(5 + 4\epsilon) = e^{6\epsilon}(1 + 4\epsilon) = e^{8\epsilon}(5 + 4\epsilon) \circ e^{2\epsilon}5 = g \circ p^0[\epsilon],
\]

and hence \( p^0[\epsilon](g(K[\epsilon])) = g(D[\epsilon]) \) (check). Moreover, in this case

\[
g(K[\epsilon]) \cap (x + \mathbb{Z}_{12}\epsilon) = x + (5K + 8x + 8)\epsilon,
\]

and hence \( 1 + (5K + 4)\epsilon \) (deformed consonances for \( y = 1 \)) is not a translation of \( 5 + 5K\epsilon \) (deformed consonances for \( x = 5 \)).

We also note a subtle fact from Equation (1): a symmetry \( g \) makes translation an action on the partitions of the form \( \{g(K[\epsilon]) \cap I_x, g(D[\epsilon]) \cap I_x\} \) if and only if \( d = 0 \). In fact, \( cK+dc^{-1}(x-u)+v = cK+dc^{-1}(y-u)+v \) for all \( x, y \in R \) is equivalent to \( K+c^{-1}dc^{-1}(x-u) = K+c^{-1}dc^{-1}(y-u) \) for all \( x, y \in R \) and to \( K+c^{-1}dc^{-1}(x-y) = K \) for all \( x, y \in R \). In particular, if \( x - y = 1 \) in the last equation, we obtain that \( c^{-1}dc^{-1} = 0 \) by the rigidity of \( K \) and hence \( d = 0 \). On the other hand, if \( d = 0 \), then \( g(K[\epsilon]) \cap I_x = x + (cK + v)\epsilon \) and we can translate the consonances through fibers. Thus, the condition \( d \neq 0 \) on a deformation symmetry is equivalent to its local character.

### 5.3 Defining first-species counterpoint

Now we can give the definition of admitted successor of a contrapunantal interval. We start with a contrapuntal consonance \( \xi \). We require it to be a dissonance deformed by a symmetry \( g \) (condition 1), which moves to a deformed consonance. Condition 3 ensures that this motion occurs within a maximum independence between cantus firmus and discantus. Condition 2 ensures that we have an appropriate environment for further determining the quality of the movement. In fact, local symmetries will detect tension forces.

**Definition 5.8.** Let \( R \) be a ring, \( (K, D, p) \) a strong dichotomy of \( R \), and \( (K[\epsilon], D[\epsilon], p^x[\epsilon]) \), for each \( x \in R \), the induced dichotomy of \( R[\epsilon] \) (Theorem 5.4).
• A contrapuntal symmetry for a consonance \( \xi \in K[\epsilon] \), where \( \xi = x + k\epsilon \), is a symmetry \( g \) of \( R[\epsilon] \) such that

1. \( \xi \in g(D[\epsilon]) \),
2. the triple \( (g(K[\epsilon]), g(D[\epsilon]), p^x[\epsilon]) \) is a self-complementary dichotomy, and
3. the cardinality of \( g(K[\epsilon]) \cap K[\epsilon] \) is maximum among all \( g \) satisfying 1 and 2.

Note that the contrapuntal symmetry for a given consonance is not required to be unique.

• An admitted successor of a consonance \( \xi \in K[\epsilon] \) is an element \( \eta \) of \( g(K[\epsilon]) \cap K[\epsilon] \) for some contrapuntal symmetry \( g \). See Figure 1.

• A piece of first-species counterpoint is a finite sequence \( \xi_1, \xi_2, \ldots, \xi_n \) of contrapuntal consonances, where \( \xi_{i+1} \) is an admitted successor of \( \xi_i \) for each \( i = 1, \ldots, n - 1 \). We require that cantus firmus and discantus belong to a certain subset \( X_t \) of \( R \) with a distinguished element \( t \in X \), called mode and tonic, respectively.

6 Simplified computation of admitted successors

In this section we characterize admitted successors of a contrapuntal consonance \( x + k\epsilon \) as the translations, by \( x \), of the admitted successors of \( k\epsilon \) associated with contrapuntal symmetries in \( H \). This offers and important simplification of the original computation.

First, we transfer symmetries satisfying conditions 1, 2, and 3, and admitted successors, between the consonances \( x + k\epsilon \) and \( k\epsilon \). The basic tool is the translation permutation \( e^x \circ (-) : \text{Sym}(R[\epsilon]) \rightarrow \text{Sym}(R[\epsilon]) \), whose inverse is \( e^{-x} \circ (-) \).

6.1 Transfer of the first condition

Let \( x + k\epsilon \) be a contrapuntal consonance. We define \( S_{x+k\epsilon}^1 \) as the set of all \( g \in \text{Sym}(R[\epsilon]) \) satisfying condition 1 in Definition 5.8, namely \( x + k\epsilon \in g(D[\epsilon]) \). Note that the translation \( e^x \circ (-) \) restricts to a bijection \( S_{k\epsilon}^1 \rightarrow S_{x+k\epsilon}^1 \). This means that \( k\epsilon \in g(D[\epsilon]) \) if and only if \( x + k\epsilon \in e^xg(D[\epsilon]) \).

6.2 Transfer of the second condition

We define \( S_{x}^2 \) as the set of all \( g \in \text{Sym}(R[\epsilon]) \) satisfying condition 2 in Definition 5.8, namely \( p^x[\epsilon](g(K[\epsilon])) = g(K[\epsilon]) \). The translation \( e^x \circ (-) \) restricts to a bijection \( S_{x}^2 \rightarrow S_{x}^2 \), as shown
by the following equivalences. Recall that $p^0[e] = e^{-x} \circ p^x[e] \circ e^x$ by Theorem 5.4.

$$
p^0[e](g(K[e])) = g(D[e]) \iff e^{-x}p^x[e]e^xg(K[e]) = g(D[e]) \iff p^x[e](e^xg(K[e])) = e^xg(D[e])
$$

### 6.3 Transfer of the third condition and admitted successors

Let $x + k\epsilon$ be a contrapuntal consonance and $S_{x+k\epsilon}$ the set of all $g \in \text{Sym}(R[\epsilon])$ satisfying 1 and 2 in Definition 5.8. Since the translation $e^x \circ (\cdot)$ restricts to bijections between sets of symmetries satisfying 1 and 2, respectively (Sections 6.1 and 6.2), then it restricts to a bijection $S_{k\epsilon} \rightarrow S_{x+k\epsilon}$.

On the other hand, note that

$$e^x(g(K[\epsilon]) \cap K[\epsilon]) = e^xg(K[\epsilon]) \cap e^x(K[\epsilon]) = e^xg(K[\epsilon]) \cap K[\epsilon].$$

In fact, the first equality holds since $e^x$, as a bijection, commutes with intersections. Also, the second equation holds because $X[\epsilon]$ (for any $X$) is invariant under transformations of the form $e^y$. In particular, this means that $|g(K[\epsilon]) \cap K[\epsilon]| = |e^xg(K[\epsilon]) \cap K[\epsilon]|$ and hence, in that sense, the bijection $S_{k\epsilon} \rightarrow S_{x+k\epsilon}$ preserves the cardinality of the sets of the form $g(K[\epsilon]) \cap K[\epsilon]$.

Thus, $|g(K[\epsilon]) \cap K[\epsilon]|$ is maximum, among all $g$ satisfying 1 and 2 in Definition 5.8, for the consonance $k\epsilon$, if and only if $e^xg(K[\epsilon]) \cap K[\epsilon]$ is for $x + k\epsilon$. This means that $S_{k\epsilon} \rightarrow S_{x+k\epsilon}$ restricts to a bijective correspondence between contrapuntal symmetries.

In particular, $g(K[\epsilon]) \cap K[\epsilon]$ is a set of admitted successors of $k\epsilon$ if and only if $e^xg(K[\epsilon]) \cap K[\epsilon]$ is for $x + k\epsilon$. This implies the following lemma.

**Lemma 6.1.** The sets of admitted successors of $x + k\epsilon$ can be computed as the sets of the form

$$e^x(g(K[\epsilon]) \cap K[\epsilon]),$$

where $g$ is a contrapuntal symmetry for $k\epsilon$.

### 6.4 Restricting symmetries

Given a symmetry $g$, we next prove that $g(K[\epsilon]) = h(K[\epsilon])$, where $h \in H$. Note that this immediately implies that $h$ is contrapuntal for $x + k\epsilon$ whenever $g$ is. This suggests to draw our attention to the contrapuntal symmetries in $H$.

Recall that $H$ consists of all 0-invariant symmetries of $R[\epsilon]$, that is, the symmetries of the form $e^{yx}(c + d\epsilon)$ with $c \in R^*$ (Proposition 5.2).
Proposition 6.2. Let $g$ be a symmetry of $R[\epsilon]$. There is $h_g \in H$ such that

$$g(X[\epsilon]) = h_g(X[\epsilon])$$

for any nonempty subset $X$ of $R$. Concretely, if $g = e^{u+ve}(c+de)$, then $h_g$ is $e^{(v-de^{-1}u)e}(c+de)$.

Proof. If $g = e^{u+ve}(c+de)$, then

$$g(X[\epsilon]) = \{cr + u + (cx + v + dr)e \mid r + x\epsilon \in X[\epsilon]\}.$$ 

Thus, since we can regard $cr + u$ as a parameter $r'$, ranging over $R$, of which $cx + v + dr$ depends because $r$ can be obtained as $c^{-1}r' - c^{-1}u$, we obtain that

$$g(X[\epsilon]) = \{r' + (cx + (v - dc^{-1}u) + dc^{-1}r')e \mid r' \in R \text{ and } x \in X\}.$$ 

The problem with this presentation is that it cannot be identified with the image of $X[\epsilon]$ under a symmetry in $H$, since $c$ in $cx$ should left multiply $r'$. For this reason, we must correct the parameter by defining a new one, say $s$, by means of the equation $cs = cr + u$. This choice yields

$$g(X[\epsilon]) = \{cs + (cx + v + d(s - c^{-1}u))e \mid s + x\epsilon \in X[\epsilon]\}$$

$$= \{cs + (cx + (v - dc^{-1}u) + ds)e \mid s + x\epsilon \in X[\epsilon]\}$$

$$= \{e^{(v-de^{-1}u)e}(c+de)(s + x\epsilon) \mid s + x\epsilon \in X[\epsilon]\}$$

$$= e^{(v-de^{-1}u)e}(c+de)(X[\epsilon]).$$

This means that the desired $h_g$ is $e^{(v-de^{-1}u)e}(c+de)$.

\[ \square \]

6.5 A first simplification

Our previous observations lead to a considerably simpler characterization of contrapuntal symmetries and admitted successors, only involving symmetries in $H$.

Proposition 6.3. The admitted successors of a consonance $x + ke \in K[\epsilon]$ can be computed as the elements of the sets of the form

$$e^x(h(K[\epsilon]) \cap K[\epsilon]),$$

where $h \in H$ and

(a) $ke \in h(D[\epsilon])$,

(b) the triple $(h(K[\epsilon]), h(D[\epsilon]), p^0[\epsilon])$ is a self-complementary dichotomy, and

(c) the cardinality of $h(K[\epsilon]) \cap K[\epsilon]$ is maximum among all $h \in H$ satisfying a and b.
Proof. By Lemma 6.1, it is enough to show that the collection of all sets of the form $g(K[e]) \cap K[e]$, with $g$ contrapuntal symmetry for $ke$, is equal to the set of all intersections $h(K'[e]) \cap K'[e]$ with $h$ satisfying (a), (b), and (c) above.

If $g$ is a contrapuntal symmetry for $ke$, then, since $g(K[e]) = h_g(K[e])$, $h_g$ is a contrapuntal symmetry and, in particular, satisfies (a) and (b). Also, $h_g$ satisfies (c) because $H \subseteq \text{Sym}(R[e])$. Moreover, $g(K[e]) \cap K[e] = h_g(K[e]) \cap K[e]$.

Conversely, if $h$ satisfies (a), (b), and (c), we claim that $h$ is a contrapuntal symmetry and hence $h(K[e]) \cap K[e]$ is an usual admitted successors set. To prove the claim, take $g$ satisfying 1 and 2 in Definition 5.8 for the consonance $ke$. The symmetry $g$ satisfies (a), (b), and $g(K[e]) = h_g(K[e])$, so

$$|g(K[e]) \cap K[e]| = |h_g(K'[e]) \cap K'[e]| \leq |h(K[e]) \cap K[e]|,$$

and $h$ is contrapuntal.

\[ \Box \]

6.6 Further simplification of the second condition

Now we prove that the condition (b) in Proposition 6.3 is just a commutativity one. Recall that the basic dichotomy $(K, D, e^b)$ is strong by hypothesis.

Lemma 6.4. The triple $(h(K[e]), h(D[e]), p^0[e])$ is a self-complementary dichotomy, for $h \in H$, if and only if $p^0[e] \circ h = h \circ p^0[e]$.

Proof. In fact, given $h \in H$, we have the following equivalences.

$$p^0[e](h(K[e])) = h(D[e]) \iff p^0[e] \circ h(K[e]) = h \circ p^0[e](K[e]) \iff p^0[e] \circ h = h \circ p^0[e].$$

The first equivalence holds since $p^0[e](K[e]) = D[e]$ (Theorem 5.4 or Proposition 4.1). The second equivalence holds by Lemma 6.5.

\[ \Box \]

Lemma 6.5. The stabilizer of $K[e]$ is $\{e^u \mid u \in R\}$. In particular, $K[e]$ is rigid with respect to $H$, and $f(K[e]) = f'(K[e])$ implies $f = f'$ whenever $f, f' \in H$.

Proof. On the one hand, note that $\{e^u \mid u \in R\}$ is contained in the stabilizer of $K[e]$.

On the other hand, suppose that $e^{u+ve}(c + d\epsilon)(K[e]) = K[e]$. Then, by intersecting both sides of the equation with the fiber $I_x$ we obtain

$$e^{u+ve}(c + d\epsilon)(K[e]) \cap (x + R\epsilon) = K[e] \cap (x + R\epsilon),$$

and hence

$$\{cr + u + (ck + dr + v)\epsilon \mid r \in R \text{ and } k \in K\} \cap (x + R\epsilon) = x + K\epsilon. \quad (2)$$
But \( \{ cr + u + (ck + dr + v) \epsilon \mid r \in R \text{ and } k \in K \} \) intersects \( x + Re \) just when \( cr + u = x \), that is, if and only if \( r = c^{-1}(x - u) \). Thus, Equation (2) implies

\[
cK + dc^{-1}(x - u) + v = K
\]

for each \( x \in R \). By taking \( x = u \), we obtain \( cK + v = K \) and hence \( v = 0 \) and \( c = 1 \) since \( K \) is rigid (Corollary 3.3). Then, by taking \( x = c + u \), we obtain \( K + d = K \), so \( d = 0 \). This means that the initial symmetry is just \( e^u \).

Moreover, \( K[\epsilon] \) is rigid with respect to \( H \) since the unique element of \( H \) of the form \( e^u \) is the identity \( e^0 \).

Finally, suppose that \( f(K[\epsilon]) = f'(K[\epsilon]) \) with \( f, f' \) in the group \( H \). This implies that \( f'^{-1} \circ f(K[\epsilon]) = K[\epsilon] \), so \( f'^{-1} \circ f = id \), and \( f = f' \).

\[\square\]

6.7 Main theorem

If we replace condition (b) in Proposition 6.3 by its equivalent obtained in Lemma 6.4, we immediately obtain the main theorem.

**Theorem 6.6.** Let \( R \) be a ring and \((K, D, p)\) a strong dichotomy of \( R \). The admitted successors of a consonance \( x + k \epsilon \in K[\epsilon] \) can be computed as the elements of the sets of the form

\[
e^x(h(K[\epsilon]) \cap K[\epsilon]),
\]

where \( h \in H \) and

1. \( k \epsilon \in h(D[\epsilon]) \),
2. \( p^0[\epsilon] \circ h = h \circ p^0[\epsilon] \), and
3. the cardinality of \( h(K[\epsilon]) \cap K[\epsilon] \) is maximum among all \( h \in H \) satisfying 1 and 2.

This theorem says that, to compute the admitted successors of a consonance \( x + k \epsilon \), it is enough to do so for \( k \epsilon \), and then apply the transposition \( e^x \) (Equation (3)). In this simplification, we only use symmetries in \( H \) and reduce the condition 2 to a suitable commutativity.

7 The little theorem of first-species counterpoint

Let \( R \) be a finite ring, \((K, D, p)\) a strong dichotomy of \( R \) with \( p = e^a b \), and \((K[\epsilon], D[\epsilon], p^0[\epsilon])\) the induced dichotomy of \( R[\epsilon] \), with \( p^0[\epsilon] = e^{ax} b \); see Proposition 4.1 or Theorem 5.4.

According to Theorem 6.6, it is desirable to count the number of elements of the sets of the form \( h(K[\epsilon]) \cap K[\epsilon] \), for \( h \in H \), so as to choose suitable maximum cardinals. However, this task is difficult in general. In the Little theorem of counterpoint we establish that each consonance has at least \(|K|^2\) admitted successors. For example, in Renaissance counterpoint, each consonance has at least 36 admitted successors. Therefore, we know that there is plenty of choices to compose a counterpoint piece, though the present computation does not provide
the admitted successors explicitly, which requires a greater effort. We obtain a more complete computation of cardinals in Section 8, and a maximization criterion in Section 8.2.

If $e^{ve}(c + d\epsilon)$, $c \in R^*$, belongs to $H$ and $r + k\epsilon \in K[\epsilon]$, then

$$e^{ve}(c + d\epsilon)(r + k\epsilon) = cr + (ck + v + dr)\epsilon.$$ 

Thus,

$$e^{ve}(c + d\epsilon)(K[\epsilon]) = \bigcup_{r \in R} cr + (cK + v + dr)\epsilon$$

and

$$e^{ve}(c + d\epsilon)(K[\epsilon]) \cap K[\epsilon] = \bigcup_{r \in R} cr + ((cK + v + dr) \cap K)\epsilon$$

(4)

because $e^0 c$ is a permutation of $R$, and hence

$$|e^{ve}(c + d\epsilon)(K[\epsilon]) \cap K[\epsilon]| = \sum_{r \in R} |(cK + v + dr) \cap K|.$$ (5)

The proof idea of the little theorem of counterpoint is the following. Equation (5) gives an expression for $|h(K[\epsilon]) \cap K[\epsilon]|$, whenever $h \in H$. As we show in Section 7.3, the exact value of the sum in Equation (5) is $|K|^2$ if $d = 1$. Thus, given a consonance $k\epsilon$, if we prove that there is at least an $h \in H$, with $d = 1$, satisfying the conditions 1 and 2 in Theorem 6.6, then we deduce that the number of admissible successors is at least $|K|^2$ by the maximum property of admitted successors sets associated with contrapuntal symmetries. So as to prove the existence of such an $h$, we first need to give concrete criteria for deciding when an $h \in H$ satisfies 1 or 2. We do this in Sections 7.1 and 7.2. Finally, we translate our result to any consonance $x + k\epsilon$, by using the bijective translation $e^x$.

7.1 The first condition criterion

Regarding the condition 1 in Theorem 6.6, given $h = e^{ve}(c + d\epsilon)$, we have the following equivalences, where $p$ is the polarity $e^a b$ of $(K, D, e^a b)$.

$$k\epsilon \in h(D[\epsilon]) \iff k \in cD + v$$

$$\iff k \in c \cdot p(K) + v$$

$$\iff v = k - c \cdot p(s) \text{ for some } s \in K$$

7.2 The second condition criterion

As to 2 in Theorem 6.6, given $h = e^{ve}(c + d\epsilon)$, we have the following equivalences.

$$p^0[\epsilon] \circ h = h \circ p^0[\epsilon] \iff e^{ae} b \circ e^{ve}(c + d\epsilon) = e^{ve}(c + d\epsilon) \circ e^{ae} b$$

$$\iff e^{(bv + a)\epsilon}(bc + bde) = e^{(ca + v)\epsilon}(cb + dbe)$$

$$\iff bv + a = ca + v, \ bc = cb, \text{ and } bd = db$$

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7.3 The little theorem

We can prove the existence of an $h \in H$ with $d = 1$ that satisfies 1 and 2 in Theorem 6.6, for the consonance $k\epsilon$. In fact, according to Sections 7.1 and 7.2, it is enough to show that the following system of equations, in the unknowns $c \in \mathbb{R}$, $v \in \mathbb{R}$, and $s \in K$, has at least a solution for $d = 1$.

\[
\begin{align*}
  v &= k - cbs - ca \\
  bv + a &= ca + v \\
  bc &= cb \\
  bd &= db
\end{align*}
\]

Certainly, it has the solution $s = k$, $c = b$ and $v = -ba$ (note that $b^2 = 1$ since $eab$ is involutive by Proposition 3.6). To conclude, $e^{-ba}(b + \epsilon)$ is the desired $h$.

We are almost done, but before, we need a lemma that allows to give the exact value of $|h(K[\epsilon]) \cap K[\epsilon]|$ for the $h$ obtained.

**Lemma 7.1.** Let $K$ and $K'$ be subsets of a finite Abelian group $R$. The equation

\[
\sum_{r \in R} |(K' + r) \cap K| = |K'| |K|
\]

holds.

**Proof.** Let $\chi_K : R \rightarrow \{0, 1\}$ be the characteristic function of $K$.

\[
\begin{align*}
  \sum_{r \in R} |(K' + r) \cap K| &= \sum_{r \in R} \sum_{s \in K'} \chi_K(s + r) \\
  &= \sum_{s \in K'} \sum_{r \in R} \chi_K(s + r) \\
  &= \sum_{s \in K'} \sum_{r' \in R} \chi_K(r') \\
  &= \sum_{s \in K'} |K| = |K'| |K|
\end{align*}
\]

\[
\square
\]

**Lemma 7.2.** Let $R$ be a finite ring and $(K, D, p)$ a strong dichotomy. Each consonance $k\epsilon$ has at least $|K|^2$ admitted successors.

**Proof.** If $h \in H$ satisfies $d = 1$, then, according to Lemma 7.1,

\[
|h(K[\epsilon]) \cap K[\epsilon]| = \sum_{r \in R} |(cK + v + r) \cap K| = |cK + v||K| = |K|^2.
\]

Hence, $h$, with $h = e^{-ba}(b + \epsilon)$, satisfies 1, 2, and the previous equation. Now, let $N$ be the number of admitted successors of $k\epsilon$. If $h'$ satisfies the conditions of Theorem 6.6, for $k\epsilon$, then

\[
N \geq |h'(K[\epsilon]) \cap K[\epsilon]| \geq |h(K[\epsilon]) \cap K[\epsilon]| = |K|^2.
\]

\[
\square
\]
An upper bound for a single contrapuntal symmetry can be obtained as well. First, suppose that \( d \neq 0 \). Note that

\[
\sum_{r \in \mathbb{R}} |(cK + v + dr) \cap K| = \rho \sum_{r \in \text{Im}(d \cdot \_)} |(cK + v + r) \cap K|,
\]

where \( d \cdot \_ : \mathbb{R} \rightarrow \mathbb{R} \) is the \( \mathbb{R} \)-endomorphism that sends an element \( r \in \mathbb{R} \) to \( dr \), and \( \rho = |\text{Ker}(d \cdot \_)| \).

In the case when \( e^v + rc \) is not the identity \( e^01 \), note that \( |(cK + v + r) \cap K| \) is at most \( |K| - 1 \) since \( K \) is rigid. On the other hand, there is at most a value of \( r \) in \( \text{Im}(d \cdot \_) \) that makes \( e^v + rc \) the identity, and hence there is at most an \( r \) such that \( |(cK + v + r) \cap K| = |K| \). Thus,

\[
\rho \sum_{r \in \text{Im}(d \cdot \_)} |(cK + v + r) \cap K| \leq \rho \left[ |\text{Im}(d \cdot \_)| - 1 \right] (|K| - 1) + |K| = 2|K|^2 - 2|K| + \rho \leq 2|K|^2 - 2|K| + |K| = 2|K|^2 - |K|.
\]

As to the last inequality note that \( \rho \), as a divisor of \( |\mathbb{R}| \) that is not \( 15 |\mathbb{R}| \), must be less than or equal to \( |K| \) (which coincides with \( |\mathbb{R}|/2 \), the greatest divisor of \( |\mathbb{R}| \) different from \( |\mathbb{R}| \).

Now suppose that \( d = 0 \). So as to find an upper bound for

\[
\sum_{r \in \mathbb{R}} |(cK + v) \cap K|
\]

we consider two cases. If \( c \neq 1 \), then \( e^v c \) is not the identity \( e^01 \). If \( c = 1 \), then, since we require \( h \) (for \( h = e^v c \)) to satisfy condition 1 in Theorem 6.6, by Section 7.1, \( v = k - p(s) \) for some \( s \in K \). This means that \( v \neq 0 \) since \( p(s) \in D \), and hence \( e^v c \) is not the identity. In both cases, \( |(cK + v) \cap K| \leq |K| - 1 \) since \( K \) is rigid. Thus,

\[
\sum_{r \in \mathbb{R}} |(cK + v) \cap K| \leq |\mathbb{R}|(|K| - 1) = 2|K|(|K| - 1).
\]

To sum up, collecting the results for \( d \neq 0 \) and \( d = 0 \) and using Theorem 6.6, we obtain the upper bound in the following theorem.

**Theorem 7.3 (Little theorem of counterpoint).** Let \( \mathbb{R} \) be a finite ring and \( (K, D, p) \) a strong dichotomy. Each consonance \( x + ke \) has at least \( |K|^2 \) admitted successors, and at most \( 2|K|^2 - |K| \) for a single contrapuntal symmetry.

---

15\( \)In fact \( \rho \), which is by definition \( |\text{Ker}(d \cdot \_)| \), is equal to \( |\mathbb{R}| \) if and only if \( d = 0 \)–but we are assuming \( d \neq 0 \).
Proof. By Theorem 6.6, \( e^x \) is a bijection between the sets of admitted successors of \( k\epsilon \) and \( x + k\epsilon \). Thus, the number of admitted successors of \( x + k\epsilon \) is equal to the number of admitted successors of \( k\epsilon \). The lower bound now follows from Lemma 7.2. The upper bound corresponds to the preceding discussion. \( \square \)

8 Counting formulas and maximization

Now we generalize Lemma 7.1 so as to have a better approximation to the exact value of the terms in Equation (5).

Lemma 8.1. Let \( K \) and \( K' \) be subsets of a finite ring \( R \) and \( d \in R \). The equation

\[
\sum_{r \in R} |(K' + dr) \cap K| = \rho \sum_{\gamma \in \text{Coker}(d \cdot \_)} |K'_\gamma||K_\gamma|
\]

(6)

holds, where \( \text{Coker}(d \cdot \_ ) = R/\text{Im}(d \cdot \_ ) \),

\[
X_\gamma = \{ x \in X \mid [x] = \gamma \}
\]

for each \( X \subseteq R \), and \( \rho = |\text{Ker}(d \cdot \_ )| \).

Proof.

\[
\sum_{r \in R} |(K' + dr) \cap K| = \rho \sum_{r \in \text{Im}(d \cdot \_ )} |(K' + r) \cap K|
\]

\[
= \rho \sum_{r \in \text{Im}(d \cdot \_ )} \sum_{s \in K'} \chi_K(s + r)
\]

\[
= \rho \sum_{s \in K'} \sum_{r \in \text{Im}(d \cdot \_ )} \chi_K(s + r)
\]

\[
= \rho \sum_{s \in K'} |K'_s||K_s|
\]

\[
= \rho \sum_{\gamma \in K'/\text{Im}(d \cdot \_ )} |K'_\gamma||K_\gamma|
\]

(7)

As to the last equality, note that if \( \gamma \) is not in the image \( K'/\text{Im}(d \cdot \_ ) \), of \( K' \) under the canonical projection onto the cokernel, then \( |K'_\gamma| = 0 \). \( \square \)

The following corollary illustrates the lemma in the important case when the ring is \( \mathbb{Z}_n \).

Corollary 8.2. Let \( K \) and \( K' \) be subsets of \( \mathbb{Z}_n \) and \( d \in \mathbb{Z}_n \). The equation

\[
\sum_{r=0}^{n-1} |(K' + dr) \cap K| = \rho \sum_{i=0}^{\rho-1} |K'_i||K_i|
\]

(7)

holds, where

\[
X_i = \{ x \in X \mid x \equiv i \pmod{\rho} \}
\]

for each \( X \subseteq \mathbb{Z}_{12} \) and \( \rho = \gcd(d, n) \).
Proof. Note that (isomorphism and index theorems for groups)
\[ |\text{Ker}(d \cdot _)| = |R/\text{Im}(d \cdot _)| = |R/\text{Im}(d \cdot _)| = |\text{Coker}(d \cdot _)|. \]

Now, by [6, Theorem 6.14], \( R/\text{Im}(d \cdot _) = \mathbb{Z}_{12}/d\mathbb{Z}_{12} = \mathbb{Z}_{12}/\rho\mathbb{Z}_{12} \cong \mathbb{Z}_\rho \), where \( \rho = \gcd(d, n) \), so \( |\text{Ker}(d \cdot _)| = |\text{Coker}(d \cdot _)| = \rho \). Moreover, \([x] = \gamma\) for \( \gamma \in \text{Coker}(d \cdot _) \) if and only if \( x \equiv i \pmod{\rho} \) by identifying \( \gamma \) and \( i \) through the isomorphism \( \text{Coker}(d \cdot _) \cong \mathbb{Z}_\rho \).

Just two curiosities. In Lemma 8.1, the case \( d = 1 \) corresponds essentially to Lemma 7.1. The case \( d = 0 \) is just the formula
\[ \sum_{r \in R} |K' \cap K| = |R||K' \cap K|. \]

Example 8.3. Let \((K, D, e^25)\) be the Renaissance strong dichotomy of \( \mathbb{Z}_{12} \), with \( K = \{0, 3, 4, 7, 8, 9\} \) and \( D = \{1, 2, 5, 6, 10, 11\} \). The following table contains the particular values of the right-hand side of Equation (5), by using Equation (7) with \( K' = cK + v \), for \( d \) ranging over \( \mathbb{Z}_{12} \). After the table, we briefly justify each row.

| \(d\) | \(\rho\) | \(\sum_{r=0}^{11} |(K' + dr) \cap K|\) |
|---|---|---|
| 0 | 12 | 12\((K'_0| + K'_3| + K'_4| + K'_7| + K'_8| + K'_9|)\) |
| 1, 5, 7, 11 | 1 | 36 |
| 2, 10 | 2 | 36 |
| 3, 9 | 3 | 3\((K'_0|3 + K'_1|2 + K'_2|)\) |
| 4, 8 | 4 | 4\((K'_0|3 + K'_1| + K'_3|2)\) |
| 6 | 6 | 6\((K'_0| + K'_1| + K'_2| + K'_4|)\) |

Case \( d = 0 \). The number of elements in \( K \) congruent to \( i \) modulo 12 is just given by the characteristic function of \( K \) as a subset of \( \mathbb{Z}_{12} \).

Case \( d = 1, 5, 7, 11 \). Since all integers are congruent modulo 1, we obtain the expression \( |K'||K| \) for the sum, whose exact value is \( 6 \times 6 \).

The case \( d = 2, 10 \) is quite interesting. The integers congruent to \( 0 \) (respectively \( 1 \)) modulo 2 are the even (respectively odd) ones. Now, there are three even numbers \( (0, 4, \) and \( 8\)\) and three odd numbers \( (3, 7, \) and \( 9\)\) in \( K \). Thus, the counting formula becomes \( 2(|K'_0|3 + |K'_1|3) \). But multiplying \( K \) by \( c \) (which is always odd because it is invertible) does not alter the parity of its elements, and adding \( v \) to \( cK \) does not alter the number of odd or even elements. For this reason, \( |K'_0| = |K'_1| = 3 \) and the exact value of the counting formula is \( 2((3 \times 3) + (3 \times 3)) \).

Case \( d = 3, 9 \). In this case, \( |K'_0| = |\{0, 3, 9\}| = 3, |K'_1| = |\{4, 7\}| = 2, |K'_2| = |\{8\}| = 1, \) and the remaining terms of the form \( K_i \) are empty.

Case \( d = 4, 8 \). Here \( |K'_0| = |\{0, 4, 8\}| = 3, |K'_1| = |\{9\}| = 1, |K'_3| = |\{3, 7\}| = 2, \) and the remaining terms of the form \( K_i \) are empty.

Case \( d = 6 \). Here \( |K'_0| = |\{0\}| = 1, |K'_1| = |\{7\}| = 1, |K'_2| = |\{8\}| = 1, |K'_3| = |\{3, 9\}| = 2, |K'_4| = |\{4\}| = 1, \) and the remaining terms of the form \( K_i \) are empty. \(\Box\)
Another important consequence of Equation (6)
\[
\sum_{r \in R} |(K' + dr) \cap K| = \rho_d \sum_{r \in \mathrm{Im}(d \cdot \_)} |(K' + r) \cap K|
\]
is that if \(\mathrm{Im}(d \cdot \_) = \mathrm{Im}(d' \cdot \_}\), then
\[
\rho_d = |\ker(d \cdot \_)| = |R|/|\mathrm{Im}(d \cdot \_)| = |R|/|\mathrm{Im}(d' \cdot \_)| = \rho_{d'}
\]
and hence
\[
\sum_{r \in R} |(K' + dr) \cap K| = \sum_{r \in R} |(K' + d'r) \cap K|.
\]

Moreover, in the case when the ring is \(\mathbb{Z}_n\), we observe from Equation (7) that we can reduce the computation of the sums of the form \(\sum_{r \in R} |(K' + dr) \cap K|\) for all \(d\) with the same \(\rho\) (recall that \(\rho = \gcd(d, n)\)) to the computation of the sum \(\sum_{r \in R} |(K' + \rho r) \cap K|\), since these sums coincide.

### 8.1 Main counting formulas

Now, according to Equation (5) we will focus on the case when \(K' = cK + v\). Let us assume that \(R\) is commutative. For each \(d\), the sequence \(\{|K'_{\gamma}\}\) is a rearrangement\(^{16}\) of \(\{|K_{\gamma}\}\), where \(\gamma\) ranges over \(\ker(d \cdot \_\)\). In fact, note first that

\[
|K'_{[r]}| = \{|e^v c(k) | k \in K \text{ and } [e^v c(k)] = [r] \text{ in } \ker(d \cdot \_\)}|
\]

\[
= \{|e^v c(k) | k \in K, c k + v - r \in dR\}|
\]

\[
= \{|e^v c(k) | k \in K, k - (c^{-1} r - c^{-1} v) \in dR\}|
\]

\[
= \{|e^v c(k) | k \in K, k - (e^v c)^{-1}(r) \in dR\}|
\]

\[
= \{|e^v c(k) | k \in K, [k] = [(e^v c)^{-1}(r)] \text{ in } \ker(d \cdot \_\)}|
\]

\[
= \{|[k] \in K | [k] = [(e^v c)^{-1}(r)]\} = |K[(e^v c)^{-1}(r)]|.
\]

Second, by the commutativity of \(R\), the function (actually a ring symmetry)

\[
[e^v c] : \ker(d \cdot \_\) \rightarrow \ker(d \cdot \_\)
\]

is well defined (check) with inverse given by the function induced by \((e^v c)^{-1}\), so \([e^v c]\) and \([(e^v c)^{-1}]\) are permutations of the cokernel. Thus, regarding \(\{|K'_{\gamma}\}\) and \(\{|K_{\gamma}\}\) as functions from the cokernel to \(\mathbb{N}\), the former is the composite of the latter with \([(e^v c)^{-1}]\). Also, note that \([(e^v c)^{-1}] = [e^{-v} c^{-1}] = e^{[e^{-v}][c]} c^{-1}. This proves the main counting formula.

\(^{16}\)A rearrangement of a function (sequence) \(f : S \rightarrow X\) is a function of the form \(f \circ \sigma\), where \(\sigma\) is a permutation of \(S\).
Theorem 8.4 (Main counting formulas). Let $R$ be a commutative ring, $K \subseteq R$, and $e^v(c + de)$ a symmetry in $H$. The equation

$$|e^v(c + de)(K[\epsilon]) \cap K[\epsilon]| = \rho \sum_{\gamma \in \text{Coker}(d \cdot \_)} |K_{[e^v(c)^{-1}][\gamma]}||K_{\gamma}| = \rho \sum_{\alpha \in \text{Coker}(d \cdot \_)} |K_{\alpha}||K_{[e^v(c)(\alpha)]}| \quad (8)$$

holds, where $K_{\gamma} = \{ x \in K \mid [x] = \gamma \}$ and $\rho = |\text{Ker}(d \cdot \_)|$. In particular, if $R = \mathbb{Z}_n$, the right-hand term of Equation (8) coincides with

$$\rho \sum_{i=0}^{\rho-1} |K_i||K_{e^v c(i)}|, \quad (9)$$

where $K_i = \{ x \in K \mid x \equiv i \pmod{\rho} \}$, $\rho = \gcd(d, n)$, and $e^v c$ is reduced modulo $\rho$.

Proof. Combine the previous discussion with Equation (5), Lemma 8.1, and Corollary 8.2. The second equality in Equation (8) follows from the first one and the change of variable $\alpha = [(e^v c)^{-1}](\gamma)$.

8.2 Maximization criterion

The following maximization criterion helps to find symmetries $h$ such that $|h(K[\epsilon]) \cap K[\epsilon]|$ is maximum among all symmetries with $h = e^v(c + de)$ and $d$ fixed. It is important to emphasize that it need not find symmetries with maximum values among all symmetries satisfying conditions 1 and 2 of contrapuntal symmetry, but the criterion is very useful to discard a number of symmetries whose values are not maximum.

Theorem 8.5 (Maximization criterion). Assume the hypotheses of Theorem 8.4. The right-hand side of Equation (8)

$$\rho \sum_{\alpha \in \text{Coker}(d \cdot \_)} |K_{\alpha}||K_{[e^v c(\alpha)]}|$$

is maximum, for $e^v c$ ranging over all symmetries of $R$, if and only if $|K_{[e^v c(\alpha)]}| = |K_{\alpha}|$ for each $\alpha \in \text{Coker}(d \cdot \_)$. In particular, the sum is maximum if $e^v c \equiv e^0 1 \pmod{d}$, that is,\footnote{Definition: we say that $x \equiv y \pmod{d}$ if $[x] = [y]$ in $R/\text{Im}(d \cdot \_)$.} if $v \equiv 0 \pmod{d}$ and $c \equiv 1 \pmod{d}$. Further, the maximum value is

$$\rho \sum_{\gamma \in \text{Coker}(d \cdot \_)} |K_{\gamma}|^2.$$

Proof. Apply the rearrangement inequality (Theorem 11.1) to the sequences $(|K_{\gamma}|)_{\gamma}$ and $(|K_{[e^v c(\gamma)]}|)_{\gamma}$, the latter being a rearrangement of the former by Section 8.4.
8.3 Steps for the computation of contrapuntal symmetries

The steps for calculating the contrapuntal symmetries in $H$, of the form $e^{\epsilon c}(c + d\epsilon)$, for a consonance of the form $ke$ is the following. We start with a strong dichotomy $(K, D, e^a b)$.

1. Solve the following system of equations in the unknowns $c \in R^\times$, $v$, and $d$.

   \[
   \begin{cases}
   bv + a = ca + v \\
   bc = cb \\
   bd = db
   \end{cases}
   \]

   This corresponds to condition 2 of contrapuntal symmetry (Section 7.2).

2. For each consonance $ke$, among the symmetries of the form $e^{\epsilon c}(c + d\epsilon)$ obtained in 1, choose those with $v \in k - cD$. This corresponds to condition 1 of contrapuntal symmetry (Section 7.1). The reason for first computing the symmetries satisfying condition 2 is that those symmetries are usually less than those satisfying condition 1, so we perform less operations.

3. For each consonance $ke$, among the symmetries $e^{\epsilon c}(c + d\epsilon)$ obtained in 2, choose those such that the right-hand term of Equation (6)

   \[
   \rho \sum_{\gamma \in \text{Coker}(d \_)} |(cK + v)_\gamma||K_\gamma|
   \]

   with $K' = cK + v$ is maximum. This ensures that the condition 3 holds by Equation (5).

   In the case when $R$ is commutative, maximize the right-hand side of Equation (8).

   In the case when $R = \mathbb{Z}_n$, maximize Equation (9). So as to discard a number of symmetries, if $R$ is commutative, Theorem 8.5 can be used.

   In the next section we apply the previous steps to computing the contrapuntal symmetries in the case of first-species Renaissance counterpoint.

9 First-species Renaissance counterpoint

Let $(K, D, e^25)$ be the Renaissance strong dichotomy of $\mathbb{Z}_{12}$, with $K = \{0, 3, 4, 7, 8, 9\}$ and $D = \{1, 2, 5, 6, 10, 11\}$.

9.1 Condition 2

By the commutativity of $\mathbb{Z}_{12}$, the condition 2 for symmetries $e^{\epsilon c}(c + d\epsilon) \in H$, $c \in R^\times$, reduces to solve the equation

\[
5v + 2 = c2 + v,
\]

which is equivalent to

\[
4v = 2(c - 1).
\]
If \( c = 1, 7 \), then the equation becomes \( 4v = 0 \), so \( v = 0, 3, 6, 9 \). If \( c = 5, 11 \), then the equation becomes \( 4(v - 2) = 0 \) and hence \( v = 2, 5, 8, 11 \). The solutions are summarized in the following table.

\[
\begin{array}{c|c}
\text{c} & \text{v} \\
\hline
1, 7 & 0, 3, 6, 9 \\
5, 11 & 2, 5, 8, 11 \\
\end{array}
\]

### 9.2 Condition 1

Now, among these solutions, we choose, for each consonance \( k \in \mathbb{Z} \), those satisfying \( v \in k - cD \).

In the following table, we organize the results of the operations involved. Specifically, we compute \( 3\mathbb{Z}_{12} \cap (k - cD) \) for \( c = 1, 7 \) and \( (2 + 3\mathbb{Z}_{12}) \cap (k - cD) \) for \( c = 5, 11 \).

\[
\begin{array}{c|c|c|c|c|c|c}
\text{c} & \text{k} & 0 & 3 & 4 & 7 & 8 \\
\hline
1 & \{6\} & \{9\} & \{3, 6\} & \{6, 9\} & \{3, 6, 9\} & \{3\} \\
5 & \{2, 5, 11\} & \{2, 5, 8\} & \{2, 11\} & \{2, 5\} & \{2\} & \{2, 8, 11\} \\
7 & \{6\} & \{9\} & \{6, 9\} & \{0, 9\} & \{3, 6, 9\} & \{3\} \\
11 & \{2, 5, 11\} & \{2, 5, 8\} & \{2, 5\} & \{5, 8\} & \{2\} & \{2, 8, 11\} \\
\end{array}
\]

We can easily fill in the table as follows. There are six entries, namely those labelled by \((c, k)\) with \( c = 1, 5 \) and \( k = 0, 4, 8 \), which we can start from, the other being obtained by using certain symmetries. In fact, note that

\[
7(3\mathbb{Z}_{12} \cap (k - D)) = 3\mathbb{Z}_{12} \cap (7k - 7D),
\]

\[
7((2 + 3\mathbb{Z}_{12}) \cap (k - 5D)) = (2 + 3\mathbb{Z}_{12}) \cap (7k - 11D),
\]

\[
\pm 3 + (3\mathbb{Z}_{12} \cap (k - cD)) = 3\mathbb{Z}_{12} \cap (\pm 3 + k - cD),
\]

and

\[
\pm 3 + ((2 + 3\mathbb{Z}_{12}) \cap (k - cD)) = (2 + 3\mathbb{Z}_{12}) \cap (\pm 3 + k - cD).
\]

For example, the entry \((1, 3)\) is obtained by adding 3 to the entry \((1, 0)\), and the entry \((7, 0)\) is obtained by multiplying the entry \((1, 0)\) by 7.

### 9.3 Maximization

In the case of the Renaissance dichotomy, Theorem 8.5 allows us to obtain the following table of maximum values, without the restrictions of conditions 1 and 2 of contrapuntal symmetry, and their corresponding symmetries. We do not include the value \( d = 0 \) since in that case the maximum is not useful, because the identity does not satisfy the condition 1. After the table we justify the results.
If $\rho = 3$, since all entries of the vector $(|K_0|, |K_1|, |K_2|)$ are different, then the unique rearrangement that coincides with it is the composition with $e^01$, and hence the maximum is only taken for the identity modulo 3. The same is true for $\rho = 4$. If $\rho = 6$, then $(|K_0|, |K_1|, |K_2|, |K_3|, |K_4|, |K_5|) = (1, 1, 1, 2, 1, 0)$. Now, if a symmetry $e^v c$ modulo 6 induces a rearrangement that leaves the vector invariant, then the rearrangement leaves $K_3$ invariant, but $e^v c(3) = 3 + v = 3$ and hence $v = 0$. Moreover, $e^05(1) = 5$ and hence it does not induce a rearrangement that leaves $K_1$ invariant, so $e^v c = e^01$.

**Contrapuntal symmetries for $0\epsilon$.**

According to the table in Section 9.2, there are exactly two symmetries for $\rho = 6$, namely $e^6e(1 + 6\epsilon)$ and $e^6e(7 + 6\epsilon)$, with $e^v c$ congruent to $e^01$ modulo 6. This allows us to discard the cases $\rho = 3, 2, 1$. Up to now the maximum is 48, and it remains to examine the cases $\rho = 4, 12$.

In the case when $\rho = 4$, the residues modulo 4 of the candidates in the table produce the following sums, calculated with Equation (9).

\[
\begin{array}{c|c|c|c}
  c & v & 1 & 2 & 3 \\
  \hline
  1 & 36 & 16 & 36 \\
  3 & 24 & 20 & 48 \\
\end{array}
\]

Thus, we have find two new symmetries with sum 48 (the maximum up to now), namely $e^{11e}(11 + 8\epsilon)$ and $e^{11e}(11 + 4\epsilon)$. Here, $e^{1111} \equiv e^{33} \pmod{4}$.

It remains to examine the case $\rho = 12$. We have the following sums.

\[
\begin{array}{c|c|c|c|c|c}
  c & v & 6 & 2 & 5 & 11 \\
  \hline
  1 & 24 & & & & \\
  7 & 36 & & & & \\
  5 & 0 & 36 & 36 & & \\
  11 & 12 & 24 & 48 & & \\
\end{array}
\]

This means that the maximum sum is 48 and that there is yet another symmetry $e^{11e}11$ for our account.

**Contrapuntal symmetries for $3\epsilon$.**
According to the second table in Section 9.2, there are exactly two symmetries for $\rho = 4$, namely $e^8(5 + 4\epsilon)$ and $e^8(5 + 8\epsilon)$, with $e^v c$ congruent to $e^0 1$ modulo 4. We thus discard the cases $\rho = 6, 3, 2, 1$. Up to now the maximum is 56.

The remaining case is $\rho = 12$. We have the following sums.

| $c$ | $v$ | 9 | 2 | 5 | 8 |
|-----|-----|---|---|---|---|
| 1   |     | 36|   |   |   |
| 7   |     | 24|   |   |   |
| 5   |     | 0 | 36| 48|   |
| 11  |     | 12| 24| 36|   |

The maximum is 56 and there are no more contrapuntal symmetries.

**Contrapuntal symmetries for $4\epsilon$**

There are exactly two symmetries for $\rho = 6$, namely $e^6(1 + 6\epsilon)$ and $e^6(7 + 6\epsilon)$, with $e^v c$ congruent to $e^0 1$ modulo 6. This allows us to discard the cases $\rho = 3, 2, 1$. Up to now the maximum is 48. The sums are less than 48 for $\rho = 4$ as shown in the following table.

| $c$ | $v$ | 1 | 2 | 3 |
|-----|-----|---|---|---|
| 1   |     | 16| 36|   |
| 3   |     | 24| 20|   |

The case $\rho = 12$ yields the following sums.

| $c$ | $v$ | 3 | 6 | 2 | 11| 9 | 5 |
|-----|-----|---|---|---|---|---|---|
| 1   |     | 36| 24|   |   |   |   |
| 7   |     | 0 | 36|   |   |   |   |
| 11  |     | 12| 24|   |   |   |   |

Hence, the maximum is 48 and there are no more contrapuntal symmetries.

**Contrapuntal symmetries for $7\epsilon$**

The case $\rho = 12$ yields the following sums.

| $c$ | $v$ | 0 | 6 | 2 | 8 | 9 | 5 |
|-----|-----|---|---|---|---|---|---|
| 1   |     | 24|   |   | 36|   |   |
| 7   |     | 60|   |   | 24|   |   |
| 11  |     | 36|   |   | 24|   |   |
Thus, we discard all remaining cases for $\rho$ and the maximum is 60, corresponding to a unique contrapuntal symmetry $e^{0}7$.

**Contrapuntal symmetries for $8\epsilon$**

There are exactly two symmetries for $\rho = 6$, namely $e^{6}(1 + 6\epsilon)$ and $e^{6}(7 + 6\epsilon)$, with $e^{v}c$ congruent to $e^{0}1$ modulo 6. This allows us to discard the cases $\rho = 3, 2, 1$. Up to now the maximum is 48.

The sums for $\rho = 4$ are the same of the consonance $0\epsilon$. In the table from Section 9.2, we observe that there are two additional symmetries, namely $e^{3}(7 + 4\epsilon)$ and $e^{3}(7 + 8\epsilon)$, with $e^{v}c$ congruent to $e^{3}3$ modulo 4.

The case $\rho = 12$ yields the following sums.

| $c$ | $v$ | 2   | 3   | 6   | 9   |
|-----|-----|-----|-----|-----|-----|
| 1   |     | 36  | 24  | 36  |     |
| 5   |     | 0   |     |     |     |
| 7   |     | 48  | 36  | 24  |     |
| 11  |     | 12  |     |     |     |

Hence, the maximum is 48 and there is an additional contrapuntal symmetry $e^{3}7$.

**Contrapuntal symmetries for $9\epsilon$**

There are exactly two symmetries for $\rho = 4$, namely $e^{8}(5 + 4\epsilon)$ and $e^{8}(5 + 8\epsilon)$, with $e^{v}c$ congruent to $e^{0}1$ modulo 4. We thus discard the cases $\rho = 6, 3, 2, 1$. Up to now the maximum is 56.

For $\rho = 12$ we have the following sums.

| $c$ | $v$ | 3   | 2   | 11  | 8   |
|-----|-----|-----|-----|-----|-----|
| 1   |     | 36  |     |     |     |
| 7   |     | 48  |     |     |     |
| 5   |     | 0   | 36  | 48  |     |
| 11  |     | 12  | 48  | 36  |     |

The maximum is 56 and there are no more contrapuntal symmetries.

### 9.4 Admitted successors

We finally can obtain the list of admitted successors, by directly using Equation (4) for each contrapuntal symmetry $h$ obtained in Section 9.3; see Table 1. In this table we can appreciate the utility of the maximization criterion (Theorem 8.5). The contrapuntal symmetries $e^{6}(1 + 6\epsilon)$ (three times) and $e^{6}(7 + 6\epsilon)$ (three times) are equal to the identity modulo 6, and the
contrapuntal symmetries $e^{8\epsilon}(5 + 8\epsilon)$ (two times) and $e^{8\epsilon}(5 + 4\epsilon)$ (two times) are equal to the identity modulo 4. In these cases, the criterion was used to discard many symmetries $h$ such that the cardinalities of the associated deformed consonances sets $h(K[\epsilon]) \cap K[\epsilon]$ were not maximum.

On the other hand, the **prohibition of parallel fifths** is an important conclusion that we can draw from the table. In fact, the consonance $7\epsilon$ has as set of admitted successors $\mathbb{Z}_{12} + (K \setminus \{7\})\epsilon$, that is, any interval can follow a fifth, except a fifth. In the following section, we show that the predictions of the model are congruent with the examples from [7]. However, a rigorous comparison between the allowances/prohibitions of the model and Fux is to be eventually communicated, because the study mentioned in [12, Section 31.4.1] was not published.

### 9.5 Five musical examples from Fux’s Gradus ad Parnassum

Under the results of Table 1, in this section we analyse the five examples of first-species counterpoint provided in [7, Chapter one] in which cantus firmus is the lower voice.\(^\text{18}\) In these examples, all modes are essentially the diatonic scale set, $X = \{0, 2, 4, 5, 7, 9, 11\}$, with the tonics 2 (first example), 4 (second example), 5 (third example), 7 (fourth example), and 9 (fifth example). These tonics correspond to the notes d (Dorian mode), e (Phrygian), f (Lydian), g (Mixolydian), and a (Eolian), respectively. However, these modes can suffer alterations related to musical considerations. For instance, according to [7, p. 28], in the modes $X_2$ (Dorian), $X_7$ (Mixolydian), and $X_9$ (Eolian), the seventh degrees must be raised, especially in the last but one measure. The prohibition of the interval 6 (tritone) in the melodies is also accepted in this context. It is important to stress that these musical features are not prerequisites of the model, and are congruent with its results.

A direct verification shows that all successors in Examples 1 to 4, corresponding to Figures 5, 11, 13 and 15 in [7] are predicted by the model. In Example 4, the model also predicts the *battuta* octave correction. Only the fifth example, corresponding to Figure 23 in [7], has a successor that is not predicted by the model. It occurs in the bars 5-6. The interval $4\epsilon$ (major third) goes to $4 + 8\epsilon$ (minor sixth), but no contrapuntal symmetry for $4\epsilon$ predicts $4 + 8\epsilon$ as a successor.

### 10 A non-commutative counterpoint

Consider the same data of Example 3.10. There, we have the strong dichotomy $(K, D, e')$ of the ring $R$ of upper triangular matrices on $\mathbb{Z}_2$, with $K = \{0, A_1, A_2, A_3\}$ and $D = \{I, B_1, B_2, B_3\}$, where $B_i = A_i + I$ for $i = 1, 2, 3$. In this case, $b = I$ and $a = I$.

\(^{18}\)We will omit the remaining cases, which are related with the problem of hanging counterpoint [3, Section 2.2.1], in our exposition of the essential features of the model.
| $k$ | $|h(K[\epsilon]) \cap K[\epsilon]|$ | $h$ | Admitted successors of $k\epsilon$ |
|-----|---------------------------------|-----|----------------------------------|
| 0   | 48                              | $e^{6\epsilon}(1 + 6\epsilon)$ | $r + \{3, 9\}\epsilon$, $r$ even  
 $r + K\epsilon$, $r$ odd |
|     |                                 | $e^{6\epsilon}(7 + 6\epsilon)$ | $r + \{3, 7, 9\}\epsilon$, $r$ even  
 $r + (K \setminus \{7\})\epsilon$, $r$ odd |
|     |                                 | $e^{11\epsilon}(11 + 8\epsilon)$ | $\{0, 3, 6, 9\} + \{3, 4, 7, 8\}\epsilon$  
 $\{1, 4, 7, 10\} + \{0, 3, 7, 8\}\epsilon$  
 $\{2, 5, 8, 11\} + \{0, 3, 4, 7\}\epsilon$ |
|     |                                 | $e^{11\epsilon}(11 + 4\epsilon)$ | $\{0, 3, 6, 9\} + \{3, 4, 7, 8\}\epsilon$  
 $\{1, 4, 7, 10\} + \{0, 3, 4, 7\}\epsilon$  
 $\{2, 5, 8, 11\} + \{0, 3, 7, 8\}\epsilon$ |
|     |                                 | $e^{11\epsilon}11$             | $\mathbb{Z}_{12} + \{3, 4, 7, 8\}\epsilon$ |
| 3   | 56                              | $e^{8\epsilon}(5 + 8\epsilon)$ | $\{0, 3, 6, 9\} + \{0, 4, 7, 8\}\epsilon$  
 $\{1, 4, 7, 10\} + (K \setminus \{7\})\epsilon$  
 $\{2, 5, 8, 11\} + (K \setminus \{9\})\epsilon$ |
|     |                                 | $e^{8\epsilon}(5 + 4\epsilon)$ | $\{0, 3, 6, 9\} + \{0, 4, 7, 8\}\epsilon$  
 $\{1, 4, 7, 10\} + (K \setminus \{9\})\epsilon$  
 $\{2, 5, 8, 11\} + (K \setminus \{7\})\epsilon$ |
| 4   | 48                              | $e^{6\epsilon}(1 + 6\epsilon)$ | see $k = 0$ |
|     |                                 | $e^{6\epsilon}(7 + 6\epsilon)$ | |
| 7   | 60                              | $e^{9\epsilon}7$              | $\mathbb{Z}_{12} + (K \setminus \{7\})\epsilon$ |
|     |                                 | $e^{3\epsilon}7$              | $\mathbb{Z}_{12} + \{0, 3, 4, 7\}\epsilon$ |
| 8   | 48                              | $e^{6\epsilon}(1 + 6\epsilon)$ | see $k = 0$ |
|     |                                 | $e^{6\epsilon}(7 + 6\epsilon)$ | |
|     |                                 | $e^{3\epsilon}(7 + 4\epsilon)$ | $\{0, 3, 6, 9\} + \{0, 3, 4, 7\}\epsilon$  
 $\{1, 4, 7, 10\} + \{3, 4, 7, 8\}\epsilon$  
 $\{2, 5, 8, 11\} + \{0, 3, 7, 8\}\epsilon$ |
|     |                                 | $e^{3\epsilon}(7 + 8\epsilon)$ | $\{0, 3, 6, 9\} + \{0, 3, 4, 7\}\epsilon$  
 $\{1, 4, 7, 10\} + \{0, 3, 7, 8\}\epsilon$  
 $\{2, 5, 8, 11\} + \{3, 4, 7, 8\}\epsilon$ |
| 9   | 56                              | $e^{8\epsilon}(5 + 8\epsilon)$ | see $k = 3$ |
|     |                                 | $e^{8\epsilon}(5 + 4\epsilon)$ | |

Table 1: Contrapuntal symmetries and admitted successors for the cantus firmus 0. We obtain the admitted successors of $x + k\epsilon$ by adding $x$ to the cantus firmus of the results.
10.1 Condition 2
We solve the equation 

\[ v + I = c + v, \]

so \( c = I, \) and \( v \) and \( d \) range over \( R. \)

10.2 Condition 1
Now, among the previous solutions, we take, for each consonance \( k \epsilon, \) those satisfying

\[ v \in k - D = k + D. \]

The following table contains the results.

| \( k \) | \( k + D \) | \( A_1 \) | \( A_2 \) | \( A_3 \) |
|-------|-------|-------|-------|-------|
| 0     | \( D \) | \( \{B_1, I, A_3, A_2\} \) | \( \{B_2, A_3, I, A_1\} \) | \( \{B_3, A_2, A_1, I\} \) |

10.3 Maximization
Since \( c = 1, \) and 1 commutes with all elements of \( R, \) the maximization criterion (Theorem 8.5) remains valid in this case, and we use it.

We start by computing \( \text{Im}(r \cdot (-)), \rho, K/\text{Im}(r \cdot (-)), \) and \( (|K_\gamma|_\gamma) \) in the following table. The ordering on \( \gamma \) is that written for the classes in \( K/\text{Im}(r \cdot (-)). \) As before, the maximization criterion is not useful for the case \( d = 0 \) since the identity does not satisfy the condition 1.

| \( r \) | \( rR \) | \( \rho \) | \( K/\text{Im}(r \cdot (-)) \) | \( (|K_\gamma|_\gamma) \) |
|-------|-------|-------|----------------|----------------|
| \( A_1, A_2 \) | \( \{0, A_1, A_2, B_3\} \) | 2 | \( \{A_1R, A_1R + A_3\} \) | (3, 1) |
| \( A_3, I \) | \( R \) | 1 | \( \{R\} \) | (4) |
| \( B_i \) for \( i = 1, 2, 3 \) | \( \{0, B_i\} \) | 4 | \( \{0, B_i\}, \{A_i, I\}, \{A_1, A_2, A_3\} \setminus \{A_i\} \) | (1, 1, 2, 0) |

We have the following maximum values with the restriction \( c = 1. \) In the case \( \rho = 2, \) the unique \( [e^v] \) inducing the identity rearrangement of \( (3, 1) \) is the identity modulo \( A_1. \) In the case \( d = B_1, \) if \( [e^v] \) induces an identity rearrangement of \( (1, 1, 3, 0), \) then necessarily \( [e^v] \) leaves invariant \( K_{(A_2, A_3)}, \) which is equal to \( \{A_2, A_3\}. \) The possibilities are either \( v = 0 \) or \( v = B_1, \) that is, \( e^v \) is the identity modulo \( B_1. \) The cases \( d = B_2 \) and \( d = B_3 \) are similar.

| \( d \) | \( \rho \) | maximum \( \sum_{r \in R} \left| (K + v + dr) \cap K \right| \) | \( v \) |
|-------|-------|----------------|-------|
| \( A_1, A_2 \) | 2 | 20 | \( 0, A_1, A_2, B_3 \) |
| \( A_3, I \) | 1 | 16 | any |
| \( B_1 \) | 4 | 24 | \( 0, B_1 \) |
| \( B_2 \) | 4 | 24 | \( 0, B_2 \) |
| \( B_3 \) | 4 | 24 | \( 0, B_3 \) |
Now we must compute the cardinals for \( d = 0 \), that is, those of the form \( 8|(K + v) \cap K| \) with \( v \neq 0 \) by the table in Section 10.2. We already have \( K + I + k = D + k \) for \( k \in K \), by the same table. Moreover, \( K + A_i = \{0, A_i\} \cup (\{B_1, B_2, B_3\} \setminus \{B_i\}) \). Thus, \( 8|(K + v) \cap K| = 0 \) if \( v = I \) and \( 8|(K + v) \cap K| = 16 \) if \( v \notin \{id, 0\} \), so the maximum cardinals (subject to conditions 1 and 2) occur for the symmetries with maximum value 24 in the preceding table. The following table shows the contrapuntal symmetries directly obtained.

| \( k \)   | \( h \)            |
|--------|------------------|
| 0      | \( e^{B_i \epsilon}(I + B_i \epsilon) \) for \( i = 1, 2, 3 \) |
| \( A_i \) (with \( i = 1, 2, 3 \)) | \( e^{B_i \epsilon}(I + B_i \epsilon) \) |

### 10.4 Admitted successors

The direct use of Equation (4) for each contrapuntal symmetry \( h \) obtained in Section 10.3 produces the following table. In this example, the maximization criterion is useful to determine all contrapuntal symmetries.

| \( k \)   | \( h \)            | admitted successors of \( k \epsilon \) |
|--------|------------------|---------------------------------|
| 0      | \( e^{B_i \epsilon}(I + B_i \epsilon) \), \( 1 \leq i \leq 3 \) | see \( k = A_i \) |
| \( A_i \) (with \( i = 1, 2, 3 \)) | \( e^{B_i \epsilon}(I + B_i \epsilon) \) | \( \{A_3, I, B_1, B_2\} + K \epsilon \) \( \{0, A_1, A_2, B_3\} + (\{A_1, A_2, A_3\} \setminus \{A_i\}) \epsilon \) |

Note that some parallelisms of the consonance \( A_i \epsilon \) are forbidden in this notion of counterpoint. The counterpoint notion that results for multiplication on the right is very similar.\(^{19}\)

### 11 Final reflections on Mazzola’s model

The validity of the Little theorem of counterpoint for noncommutative rings stresses several important features of Mazzola’s model. First, the little theorem shows that the model is nontrivial in the noncommutative case (at least \(|K| \) admitted successors always), so it is worth experimenting with new examples of counterpoint; for instance, we can perform counterpoint in noncommutative rings of square matrices. Second, the possibility of generalization also stresses a certain profundity and intrinsic generality in the little theorem. The present noncommutative generalization suggests a structural richness beyond algorithmic computations, which is hinted at by the counting formulas and the maximization criterion. Moreover, these two tools allow us to make a reasonable calculation of contrapuntal symmetries by hand—a task closer to craftsmanship of counterpoint. A lot of combinatorics is to be studied regarding the revelation of profound algebraic structures behind counterpoint.

Other strengths of Mazzola’s model are the following.

1. In contrast to psychological or descriptive approaches, it has a very simple but powerful idea as philosophy: counterpoint rules obey profound relations based on musical symmetries.

\(^{19}\)To be more exact, this notion is obtained by applying the previous theory to the opposite ring \( R^{op} \) instead of \( R \). In that case, all right \( R \)-modules considered become left \( R \)-modules.
2. Its astonishing agreement with Palestrina’s Missa Papae Marcellus, as studied in [14].
3. It predicts the prohibition of parallel fifths, among other rules (Section 9.5), as a consequence of the partition of the twelve intervals into consonances and dissonances.
4. It connects contrapuntal tension/resolution forces with fundamental nature forces, via a crucial idea in contemporary physics, namely local symmetries, which are formalized thanks to differential geometry, in particular, tangent bundles. Mazzola’s analogous algebraic language is based on dual numbers.
5. It paves the way to many other forms of counterpoint, which opens up a lot of fields of musical experimentation.

However, the previous mathematical discussions suggest the following weaknesses of the model.

1. The model is not natural. Though the mathematical justification is interesting, it is not entirely clear why we use the structure of the dual numbers ring to model intervals and not another structure, even if the product in the dual numbers ring has a musical interpretation [3, Section 2.2.1], though incidentally. Mazzola’s musical justification of dual numbers was discussed in Section 4. Moreover, a mathematical model of counterpoint should clarify and simplify the classical theory in music treatises. We would expect a teachable model, which stresses the essential ideas behind counterpoint so that a consistent theory can be established, in contrast to the usual pedagogical model based on unjustified rules.
2. The use of local symmetries is unclear. In Mazzola’s works, there is no formal definition of global and local symmetry. In this paper, we tried to provide such a definition based on Mazzola’s hints. However, we observed that local and global symmetries give further explanations on the nature of the tension between an interval and its successors but do not play an essential role in Definition 5.8. For example, the contrapuntal symmetry \( e^{07} \), for \( 7 \in \mathbb{Z}_{12} \), obtained in Section 9.4 is not local (definition in Section 5.2) but it is responsible for the parallel fifths prohibition. This suggests a simplification of the condition 2, the latter being related to local/global symmetries according to 5.2. For the same reason, we could question the use of dual numbers as a bundle analogue, since this analogy has a clear relation to local/global symmetries.
3. It only deals with first-species counterpoint and a convincing theory of the remaining species, and the cases of three or more voices, are not at hand.

Based on the previous remarks, we suggest a simplification of the first-species counterpoint model, sharing the strengths of Mazzola’s one and correcting their weaknesses.

\[20\text{Dual numbers are essentially pairs } (c, d) \text{ (cantus and interval with the discantus) of elements of } \mathbb{Z}_{12}. \text{ A more natural structure on these pairs is that induced by the usual addition and product of tones in } \mathbb{Z}_{12}, \text{ that is, the ring structure. Curiously, the induced structure is that of the quotient ring } \mathbb{Z}_{12}[x]/\langle x^2 - x \rangle, \text{ which is similar to the dual numbers one but has a slightly different product.}\]
Appendix

Theorem 11.1 (Rearrangement inequality, variation). Suppose given an ordered list of real numbers

\[ a_1 \geq a_2 \geq \cdots \geq a_n. \]

The inequality

\[ \sum_{i=1}^{n} a_i^2 \geq \sum_{i=1}^{n} a_i a_{\sigma(i)} \]

holds for each permutation \( \sigma \) of \( \{1, \ldots, n\} \). Moreover, the equality holds if and only if \((a_1, \ldots, a_n) = (a_{\sigma(1)}, \ldots, a_{\sigma(n)})\).

Proof. Induction on \( n \). The case \( n = 1 \) is just the equality \( a_2 = a_2 \). Now suppose the result for \( n - 1 \). We consider two cases.

First case: \( a_{\sigma(1)} = a_1 \). We can assume \( \sigma(1) = 1 \) (switch \( \sigma(1) \) and 1) since this do not alter the sum \( \sum_{i=1}^{n} a_i a_{\sigma(i)} \). In this case,

\[ \sum_{i=1}^{n} a_i a_{\sigma(i)} = a_1^2 + \sum_{i=2}^{n} a_i a_{\sigma(i)} = a_1^2 + \sum_{i=1}^{n-1} a_i' a_{\sigma'(i)}, \]

where \( a_i' = a_{i+1} \) and \( \sigma'(i) = \sigma(i+1) - 1 \) for each \( i = 1, \ldots, n - 1 \). Thus, by applying the induction hypothesis,

\[ \sum_{i=2}^{n} a_i a_{\sigma(i)} \leq \sum_{i=2}^{n} a_i^2 \]

and the equality holds if and only if \((a_2, \ldots, a_n) = (a_{\sigma(2)}, \ldots, a_{\sigma(n)})\). Hence, by adding \( a_1^2 \) on both sides,

\[ \sum_{i=1}^{n} a_i a_{\sigma(i)} \leq \sum_{i=1}^{n} a_i^2 \]

and the equality holds if and only if \((a_1, \ldots, a_n) = (a_{\sigma(1)}, \ldots, a_{\sigma(n)})\).

Second case: \( a_{\sigma(1)} \neq a_1 \). Note that there is \( k \) with \( a_{\sigma(k)} = a_1 \) such that \( a_k < a_1 \); otherwise \( \{k \mid a_{\sigma(k)} = a_1\} \subseteq \{k \mid a_k = a_1\} \) and, since these sets have the same finite cardinality, they are equal, which is a contradiction since 1 is in the latter but not in the former. Thus,

\[ \sum_{i=1}^{n} a_i a_{\sigma(i)} = a_1 a_{\sigma(1)} + a_k a_{\sigma(k)} + \sum_{i \in \{1, \ldots, n\} \setminus \{1, k\}} a_i a_{\sigma(i)} \]

and, if \( \sigma' \) is obtained from \( \sigma \) by switching \( \sigma(1) \) and \( \sigma(k) \), then

\[ \sum_{i=1}^{n} a_i a_{\sigma'(i)} - \sum_{i=1}^{n} a_i a_{\sigma(i)} = (a_1 - a_k)(a_{\sigma(k)} - a_{\sigma(1)}) = (a_1 - a_k)(a_1 - a_{\sigma(1)}) > 0. \]
This means that
\[ \sum_{i=1}^{n} a_i a_{\sigma(i)} < \sum_{i=1}^{n} a_i a_{\sigma'(i)} \leq \sum_{i=1}^{n} a_i^2 , \]
where the right-hand inequality corresponds to the first case. Finally, note that here the inequality is always strict and \((a_1, \ldots, a_n) \neq (a_{\sigma(1)}, \ldots, a_{\sigma(n)})\).

The following is a short reflection about the problem of deciding when a dichotomy \(\{K, D\}\) of a ring \(R\) is not self-complementary.

Let \(X \subseteq R\). We associate with \(X\) a categorical groupoid of intervals \(G(X)\) as follows. Its set of objects is \(X\). For each \(x, y \in X\), there is a unique morphism from \(x\) to \(y\), namely the triple \((x, y, y - x)\). We define the composition by

\[ (y, z, z - y) \circ (x, y, y - x) = (x, z, z - y + y - x) = (x, z, z - x). \]

The identities are of the form \((x, x, 0)\). The inverse of \((x, y, y - x)\) is \((y, x, x - y)\).

Given a self-complementary dichotomy \((K, D, e^a b)\) of \(R\), we have an induced groupoid isomorphism \(F : G(K) \to G(D)\) sending \((x, y, y - x)\) to \((e^a b(x), e^a b(y), b(y - x))\), which acts linearly. Hence, a self-complementary dichotomy induces a linear correspondence between the intervals of \(K\) and \(D\).

This fact can be used to prove that the dichotomy \(\{K, D\}\) with \(K = \{0, 1, 3, 6, 8, 11\}\) and \(D = \{2, 4, 5, 7, 9, 11\}\) is not self-complementary. In fact, the sequence \((11, 0, 1)\) of elements of \(K\) has the maximum length among sequences \((x_1, \ldots, x_n)\) such that \(x_{i+1} - x_i = 1\) for \(i = 0, \ldots n - 1\). If a symmetry \(e^a b\) sends \(K\) to \(D\), then the groupoid isomorphism \(F\) sends \((11, 0, 1)\) to a sequence \((d_1, d_2, d_3)\) in \(D\) with maximum length among sequences \((y_1, \ldots, y_n)\) such that \(y_{i+1} - y_i = b\). But the sequences of maximum length with \(y_{i+1} - y_i = b\) are \((4, 5)\) (for \(b = 1\)), \((5, 4)\) (for \(b = 11\)), \((11, 4, 9, 2, 7)\) (for \(b = 5\)), and \((7, 2, 9, 4, 11)\) (for \(b = 7\)), whose length is not 3; a contradiction.

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\(^{21}\)It is the unique possible definition if the correspondence on objects is \(e^a b\).
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