The LOW Solution and Two-zero Textures of the Neutrino Mass Matrix

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Abstract

As a viable alternative to the LMA solution to the solar neutrino problem, the LOW solution is confronted with two-zero textures of the $3 \times 3$ neutrino mass matrix. We find out nine acceptable textures, from which instructive predictions can be obtained for the absolute values of neutrino masses, Majorana phases of CP violation, and effective masses of the tritium beta decay and neutrinoless double beta decay.

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In the standard electroweak model, three known neutrinos are assumed to be the exactly massless Weyl particles. This assumption has no conflict with all of today’s direct-mass-search experiments [1], but it is not guaranteed by any fundamental symmetry principle of particle physics. Indeed most extensions of the standard model, such as the SO(10) grand unified theories [2], do allow neutrinos to have tiny masses. If three neutrinos have non-degenerate masses, their mass eigenstates ($\nu_1, \nu_2, \nu_3$) may not coincide with their flavor eigenstates ($\nu_e, \nu_\mu, \nu_\tau$), leading to lepton flavor mixing.

The recent SNO results [3] provide compelling evidence that solar neutrinos undergo the flavor conversion ($\nu_e \rightarrow \nu_\mu, \nu_\tau$) during their travel to the earth. This anomaly, similar to the observed deficit of atmospheric $\nu_\mu$ neutrinos [4], is most likely due to neutrino oscillations – a quantum phenomenon which can naturally happen if neutrinos are massive and lepton flavors are mixed. In the framework of neutrino oscillations, a global analysis of current solar neutrino data indicates that the LMA solution is most favored [5]. As a viable alternative to the LMA solution, the LOW solution is accepted at about 3$\sigma$ level [6]. Before the LMA solution is firmly established as the correct solution to the solar neutrino problem, it is certainly meaningful and useful to study the LOW solution (as well as other alternatives) and its implication on neutrino masses and lepton flavor mixing parameters.

In this paper, we aim to confront the LOW solution with two-zero textures of the $3 \times 3$ neutrino mass matrix $M$ in the flavor basis where the charged lepton mass matrix is diagonal. The reason why we take into account texture zeros of $M$ is simple: $M$ totally involves nine physical parameters (three neutrino masses, three flavor mixing angles, and three CP-violating phases), but only six of them or their combinations (two neutrino mass-squared differences, three mixing angles and one CP-violating phase) can in principle be determined from neutrino oscillations. To recast the structure of $M$ from current experimental data, two or more extra constraints have to be assumed, either for some elements of $M$ (approach A) or for neutrino masses and CP-violating phases (approach B). Some recent works on texture zeros of $M$ done by a number of authors [7, 8, 9, 10] belong to approach A, while the systematic analysis of $M$ shown in Ref. [11] belongs to approach B. Such phenomenological attempts are important, because they may shed light on the underlying flavor symmetry and its breaking mechanism responsible for the structure of $M$, from which it is likely to obtain a deeper insight into the generation of neutrino masses and lepton flavor mixing.

The present work is different from the previous ones in several aspects. First of all, we concentrate on the LOW solution instead of the LMA solution. The former has not been confronted with two-zero textures of the neutrino mass matrix $M$ in the literature. Second, we carry out a careful numerical analysis of every two-zero pattern of $M$ to pin down its complete parameter space, because simple analytical approximations are sometimes unable to reveal the whole regions of relevant parameters allowed by current experimental data. Third, we quantitatively obtain the ranges of the absolute neutrino masses, the Majorana phases of CP violation, and the effective masses of the tritium beta decay and neutrinoless double beta decay. We find that nine of the fifteen two-zero textures of $M$ are compatible with the LOW solution, although two of them are only marginally allowed.

In the flavor basis where the charged lepton mass matrix is diagonal, the Majorana neutrino mass matrix can be expressed as

$$M = U \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} U^T,$$

(1)
where $\lambda_i$ (for $i = 1, 2, 3$) stand for the neutrino mass eigenvalues consisting of two nontrivial CP-violating phases ($\rho$ and $\sigma$), and $U$ is a CKM-like flavor mixing matrix containing three rotation angles ($\theta_x$, $\theta_y$ and $\theta_z$) and another CP-violating phase ($\delta$) [12]. Without loss of generality, we take the phase convention

$$\lambda_1 = m_1 e^{2i\rho}, \quad \lambda_2 = m_2 e^{2i\sigma}, \quad \lambda_3 = m_3$$

and $\lambda_3$ with $m_i$ being the physical neutrino masses, and parametrize $U$ as

$$U = \begin{pmatrix} c_x c_z & s_x c_z & s_z \\ -c_x s_y s_z - s_x c_y e^{-i\delta} & -s_x s_y s_z + c_x c_y e^{-i\delta} & s_y c_z \\ -c_x c_y s_z + s_x s_y e^{-i\delta} & -s_x c_y s_z - c_x s_y e^{-i\delta} & c_y c_z \end{pmatrix},$$

where $s_x \equiv \sin \theta_x$, $c_x \equiv \cos \theta_x$, and so on. In this parametrization, the neutrinoless double beta decay is associated with $\rho$ and $\sigma$, while CP violation in neutrino oscillations depends on $\delta$. Note that three mixing angles ($\theta_x, \theta_y, \theta_z$) can all be arranged to lie in the first quadrant. Arbitrary values between 0 and $2\pi$ (or between $-\pi$ and $+\pi$) are allowed for three CP-violating phases ($\delta, \rho, \sigma$).

The symmetric neutrino mass matrix $M$ totally has six independent complex entries. If two of them vanish, i.e., $M_{ab} = M_{pq} = 0$, we obtain two constraint equations:

$$M_{ab} = \sum_{i=1}^{3} (U_{ai} U_{bi} \lambda_i) = 0 ,$$

$$M_{pq} = \sum_{i=1}^{3} (U_{pi} U_{qi} \lambda_i) = 0 ,$$

where $a, b, p$ and $q$ run over $e, \mu$ and $\tau$, but $(p, q) \neq (a, b)$. Solving Eq. (4), we arrive at [8]

$$\lambda_1 = \frac{U_{a3} U_{b3} U_{p2} U_{q2} - U_{a2} U_{b2} U_{p3} U_{q3}}{U_{a2} U_{b2} U_{p1} U_{q1} - U_{a1} U_{b1} U_{p2} U_{q2}} ,$$

$$\lambda_2 = \frac{U_{a1} U_{b1} U_{p3} U_{q3} - U_{a3} U_{b3} U_{p1} U_{q1}}{U_{a2} U_{b2} U_{p1} U_{q1} - U_{a1} U_{b1} U_{p2} U_{q2}} ,$$

$$\lambda_3 = \sum_{i=1}^{3} (U_{pi} U_{qi} \lambda_i) = 0 .$$

This result implies that two neutrino mass ratios ($m_1/m_3, m_2/m_3$) and two Majorana-type CP-violating phases ($\rho, \sigma$) can fully be determined in terms of three mixing angles ($\theta_x, \theta_y, \theta_z$) and the Dirac-type CP-violating phase ($\delta$). Thus one may examine whether a two-zero texture of $M$ is empirically acceptable or not by comparing its prediction for the ratio of two neutrino mass-squared differences with the result extracted from current experimental data on solar and atmospheric neutrino oscillations:

$$R_\nu \equiv \frac{|m_2^2 - m_1^2|}{|m_3^2 - m_2^2|} \approx \frac{\Delta m_{\text{sun}}^2}{\Delta m_{\text{atm}}^2} .$$

The size of $R_\nu$ depends on which solution to the solar neutrino problem is taken.

As for the LOW solution, we have $3.5 \times 10^{-8} \text{ eV}^2 < \Delta m_{\text{sun}}^2 < 1.2 \times 10^{-7} \text{ eV}^2$ versus $1.2 \times 10^{-3} \text{ eV}^2 \leq \Delta m_{\text{atm}}^2 \leq 4.8 \times 10^{-3} \text{ eV}^2$ at the $3\sigma$ confidence level [8], which leads to $7.3 \times 10^{-6} < R_\nu < 1.0 \times 10^{-4}$. The mixing angles of solar and atmospheric neutrino oscillations read as $0.43 \leq \tan^2 \theta_{\text{sun}} \leq 0.86$ and $0.3 \leq \sin^2 \theta_{\text{atm}} \leq 0.7$, obtained from the same global analysis [8]. We then arrive at the ranges of $\theta_x (\approx \theta_{\text{sun}})$ and $\theta_y (\approx \theta_{\text{atm}})$:
33.2° < θ_x < 42.8° and 33.2° ≤ θ_y ≤ 56.8°. The third mixing angle θ_z is restricted by the CHOOZ experiment [13]: θ_z ≈ θ_{chz} < 13.3° extracted from the upper limit sin^2 2θ_{chz} < 0.2. There is no experimental constraint on the CP-violating phase δ. Hence we take δ from 0° to 360° in our numerical calculations.

There are totally fifteen distinct topologies for the structure of M with two independent vanishing entries, as shown in Tables 1 and 2. We work out the explicit expressions of λ_1/λ_3 and λ_2/λ_3 for each pattern of M by use of Eq. (5), and list the results in the same tables. With the input values of θ_x, θ_y, θ_z and δ mentioned above, we calculate the ratio R_ν and examine whether it is in the range allowed by current data. This criterion has been used before [8] to pick the phenomenologically favored patterns of M in the LMA case.

Nine of the fifteen two-zero textures of M listed in Table 1 are found to be in accord with the LOW solution as well as the atmospheric neutrino data. They can be classified into four categories: A (with A_1 and A_2), B (with B_1, B_2, B_3 and B_4), C, and D (with D_1 and D_2). The point of this classification is that the textures of M in each category result in similar physical consequences, which are almost indistinguishable in practice. The other six patterns of M (categories E and F) listed in Table 2 cannot coincide with current experimental data. In particular, the exact neutrino mass degeneracy (m_1 = m_2 = m_3) is predicted from three textures of M belonging to category F.

Now let us focus on the nine phenomenologically acceptable textures of M. As R_ν is required to be very small, the space of four input parameters (θ_x, θ_y, θ_z and δ) may strongly be constrained for a specific two-zero pattern of M. To be more concrete, we take patterns A_1, B_1, C and D_1 as four typical examples for numerical illustration. Our results for sin^2 2θ_{chz} versus δ and θ_y versus θ_x are shown Figs. 1 – 4. Some comments are in order.

(1) For pattern A_1, arbitrary values of δ are allowed if sin^2 2θ_{chz} is tiny (≤ 0.002). When the values of δ are taken to be around 180°, however, sin^2 2θ_{chz} can be large enough, up to its experimental upper limit. The mixing angles θ_x and θ_y may take any values in the ranges allowed by current data. Therefore we conclude that pattern A_1 is favored in phenomenology with little fine-tuning. A similar conclusion can be drawn for pattern A_2.

(2) For pattern B_1, δ is essentially unconstrained if sin^2 2θ_{chz} is extremely close to zero; and only δ ≈ 90° or δ ≈ 270° is acceptable if sin^2 2θ_{chz} deviates somehow from zero. Except θ_y ≠ 45°, there is no further constraint on the parameter space of (θ_x, θ_y). We conclude that pattern B_1 with maximal CP violation (i.e., sin δ ≈ ±1) is phenomenologically favored. So are patterns B_2, B_3 and B_4.

(3) For pattern C, δ = 90° or δ = 270° is forbidden. Furthermore, θ_y = 45° is forbidden. We see that the allowed parameter space of (δ, θ_{chz}) and that of (θ_x, θ_y) are rather large. Hence pattern C is also favored in phenomenology.

(4) For pattern D_1, δ is restricted to be around 0° or 360°. In particular, the region 90° ≤ δ ≤ 270° is entirely excluded. sin^2 2θ_{chz} > 0.012 holds for the allowed range of δ. Different from patterns A_1, B_1 and C, pattern D_1 requires relatively strong correlation between θ_x and θ_y (e.g., small values of θ_y are associated with large values of θ_x in the allowed parameter space). In this sense, we argue that pattern D_1 is less natural in phenomenology, although it has not been ruled out by current experimental data. A similar argument can be made for pattern D_2.

At this point, it is worthwhile to compare between LOW and LMA solutions against two-zero patterns of the neutrino mass matrix M. The main difference between two solutions

1Note that categories A, B and C correspond to those given in Refs. [6, 8] for the LMA solution.
A two-zero texture of $M$ has a number of interesting predictions, in particular, for the absolute neutrino masses and the Majorana phases of CP violation \textsuperscript{3}. With the help of Eq. (5), one may calculate the mass ratios $m_1/m_3 = |\lambda_1/\lambda_3|$ and $m_2/m_3 = |\lambda_2/\lambda_3|$ as well as the Majorana phases $\rho = \arg(\lambda_1/\lambda_3)/2$ and $\sigma = \arg(\lambda_2/\lambda_3)/2$. The absolute neutrino mass $m_3$ can be determined from

$$m_3 = \frac{1}{\sqrt{1 - \left(\frac{m_2}{m_3}\right)^2}} \sqrt{\Delta m^2_{\text{atm}}}. \quad (7)$$

Therefore a full determination of the mass spectrum of three neutrinos is actually possible. Then we may obtain definite predictions for the effective mass of the tritium beta decay,

$$\langle m \rangle_e = m_1 c_x^2 c_z^2 + m_2 s_x^2 c_z^2 + m_3 s_z^2; \quad (8)$$

and that of the neutrinoless double beta decay,

$$\langle m \rangle_{ee} = |m_1 c_x^2 c_z^2 e^{2i\rho} + m_2 s_x^2 c_z^2 e^{2i\sigma} + m_3 s_z^2|.$$

It is clear that the Dirac phase $\delta$ has no contribution to $\langle m \rangle_{ee}$. Note that CP- and T-violating asymmetries in normal neutrino oscillations are controlled by $\delta$ or the rephasing-invariant parameter $J = s_x c_x s_y c_y s_z c_z^2 \sin \delta$ [14]. Because of the smallness of $\Delta m^2_{\text{sun}}$ in the LOW case, however, there is no hope to measure leptonic CP violation in the terrestrial long-baseline neutrino oscillation experiments [16]. Whether $\langle m \rangle_e$ and $\langle m \rangle_{ee}$ can be measured remains an open question. The present experimental upper bounds are $\langle m \rangle_e < 2.2$ eV [1] and $\langle m \rangle_{ee} < 0.35$ eV [17] at the 90% confidence level. The proposed KATRIN experiment [18] is possible to reach the sensitivity $\langle m \rangle_e \sim 0.3$ eV, and a number of next-generation experiments for the neutrinoless double beta decay [19] is possible to probe $\langle m \rangle_{ee}$ at the level of 10 meV to 50 meV.

We perform a numerical calculation of $m_2/m_3$ versus $m_1/m_3$, $\sigma$ versus $\rho$, $\langle m \rangle_{ee}$ versus $\langle m \rangle_e$, and $J$ versus $m_3$ for patterns A\textsubscript{1}, B\textsubscript{1}, C, and D\textsubscript{1}. The results are shown in Figs. 1 – 4. Some discussions are in order.

(1) For pattern A\textsubscript{1}, $\rho \approx \delta/2$ or $\rho \approx \delta/2 - 180^\circ$ and $\sigma \approx \rho \pm 90^\circ$ hold. Two neutrino mass ratios lie in the ranges $0.001 \leq m_1/m_3 \leq 0.3$ and $0.003 \leq m_2/m_3 \leq 0.3$, and the absolute value of $m_3$ is in the range $0.035$ eV $\leq m_3 \leq 0.071$ eV. As $\langle m \rangle_{ee} = 0$ is a direct consequence of texture A\textsubscript{1}, we calculate the sum of three neutrino masses $\sum m_i$ instead of $\langle m \rangle_{ee}$. The result is $0.035$ eV $\leq \sum m_i \leq 0.11$ eV, in contrast with $7.13 \times 10^{-5}$ eV $\leq \langle m \rangle_e \leq 0.022$ eV. The rephasing-invariant of CP violation $J$ is found to lie in the range $-0.049 \leq J \leq 0.048$. Similar predictions are expected for pattern A\textsubscript{2}. 

\textsuperscript{3} A specific two-zero texture of $M$ may be in accord with both LMA and LOW solutions, although the relevant parameter space for LOW is usually smaller than that for LMA (in particular, when the input parameters $\theta_{\text{sun}}$, $\theta_{\text{atm}}$ and $\theta_{\text{che}}$ take values at the same confidence level for two solutions). A careful numerical analysis of the LMA solution shows that patterns D\textsubscript{1} and D\textsubscript{2} can marginally be allowed [14], like the LOW case. It is difficult to observe this point from simple analytical approximations made in Refs. [7, 8]. In the spirit of naturalness, however, we expect that categories A, B and C of two-zero patterns of $M$ are more favorable than category D in either LMA or LOW case.
In summary, we have confronted the LOW solution with two-zero textures of the neutrino mass matrix. Nine patterns of \( M \) can be classified into four distinct categories, found to be acceptable in phenomenology. Compared with categories A, B and C, category D seems less favored by current experimental data. This situation is similar to the LMA case. We expect that new data to be accumulated from solar and atmospheric neutrino experiments will allow us to isolate fewer two-zero patterns of \( M \) which are phenomenologically favored.

Of course, to pin down the correct solution to the solar neutrino problem is urgent and important. We have shown that LOW and LMA solutions are essentially compatible with the same textures of \( M \), although the parameter space for the former is usually smaller or a bit contrived. One may carry out a similar analysis for the VO solution to the solar neutrino problem, which has \( R_\nu \sim 10^{-7} \), by use of the analytical results presented in Tables 1 and 2. In this case, some fine-tuning of the input parameters is unavoidable to make \( R_\nu \) strongly suppressed. Once we are aware of the true solution, some better understanding of two-zero textures of the neutrino mass matrix will be available.

Finally it is worth remarking that a specific texture of lepton mass matrices may not be preserved to all orders or at any energy scales in the unspecified interactions from which lepton masses are generated. Nevertheless, those phenomenologically favored textures at low energy scales, no matter whether they are of the two-zero form or other forms \([20, 21]\), are possible to provide enlightening hints at the underlying dynamics of lepton mass generation at high energy scales.
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Table 1: Nine patterns of the neutrino mass matrix $M$ with two independent vanishing entries, which are \textit{compatible} with the LOW solution and other empirical hypotheses. The analytical results for two ratios of three neutrino mass eigenvalues $\lambda_1/\lambda_3$ and $\lambda_2/\lambda_3$ are given in terms of four flavor mixing parameters $\theta_x$, $\theta_y$, $\theta_z$ and $\delta$.

| Pattern of $M$ | Results of $\lambda_1/\lambda_3$ and $\lambda_2/\lambda_3$ |
|---------------|----------------------------------------------------------|
| $A_1$ : $(0 \ 0 \ \times)$ \ $(0 \ \times \ \times)$ \ $(0 \times \ \times)$ | $\begin{align*}
\lambda_1 &= \frac{s_x}{c_x} \frac{s_x c_y}{c_x s_y} e^{i\delta} - s_z \\
\lambda_2 &= -\frac{s_x}{c_x} \frac{c_x s_y}{s_x c_y} e^{i\delta} + s_z \\
\lambda_3 &= +\frac{s_x}{c_x} \frac{c_x c_y}{s_x s_y} e^{i\delta} - s_z
\end{align*}$ |
| $A_2$ : $(\times \ \times \ \ 0)$ \ $(0 \ \times \ \times)$ \ $(0 \times \ \times)$ | $\begin{align*}
\lambda_1 &= \frac{s_x c_y}{s_x s_y} \left(\frac{2 c_x c_y - s_x^2 - c_x^2}{s_x^2 - c_x^2} + \frac{c_y s_x}{c_y s_x} \left(\frac{s_x^2 s_y^2}{c_x^2 s_x^2} + c_y e^{i\delta} + s_z \right)\right) e^{2i\delta} \\
\lambda_2 &= -\frac{s_x c_y}{s_x s_y} \left(\frac{2 c_x c_y - s_x^2 - c_x^2}{s_x^2 - c_x^2} + \frac{c_y s_x}{c_y s_x} \left(\frac{s_x^2 s_y^2}{c_x^2 s_x^2} + c_y e^{i\delta} + s_z \right)\right) e^{2i\delta} \\
\lambda_3 &= +\frac{s_x c_y}{s_x s_y} \left(\frac{2 c_x c_y - s_x^2 - c_x^2}{s_x^2 - c_x^2} + \frac{c_y s_x}{c_y s_x} \left(\frac{s_x^2 s_y^2}{c_x^2 s_x^2} + c_y e^{i\delta} + s_z \right)\right) e^{2i\delta}
\end{align*}$ |
| $B_1$ : $(\times \ \times \ \ 0)$ \ $(0 \ \times \ \times)$ \ $(0 \times \ \times)$ | $\begin{align*}
\lambda_1 &= \frac{s_x c_x c_y}{s_x s_y} \left(\frac{2 c_x c_y - s_x^2 - c_x^2}{s_x^2 - c_x^2} + \frac{c_y s_x}{c_y s_x} \left(\frac{s_x^2 s_y^2}{c_x^2 s_x^2} + c_y e^{i\delta} + s_z \right)\right) e^{2i\delta} \\
\lambda_2 &= -\frac{s_x c_x c_y}{s_x s_y} \left(\frac{2 c_x c_y - s_x^2 - c_x^2}{s_x^2 - c_x^2} + \frac{c_y s_x}{c_y s_x} \left(\frac{s_x^2 s_y^2}{c_x^2 s_x^2} + c_y e^{i\delta} + s_z \right)\right) e^{2i\delta} \\
\lambda_3 &= +\frac{s_x c_x c_y}{s_x s_y} \left(\frac{2 c_x c_y - s_x^2 - c_x^2}{s_x^2 - c_x^2} + \frac{c_y s_x}{c_y s_x} \left(\frac{s_x^2 s_y^2}{c_x^2 s_x^2} + c_y e^{i\delta} + s_z \right)\right) e^{2i\delta}
\end{align*}$ |
| $B_2$ : $(\times \ \times \ \ 0)$ \ $(0 \ \times \ \times)$ \ $(0 \times \ \times)$ | $\begin{align*}
\lambda_1 &= \frac{s_x c_x c_y}{s_x s_y} \left(\frac{2 c_x c_y - s_x^2 - c_x^2}{s_x^2 - c_x^2} + \frac{c_y s_x}{c_y s_x} \left(\frac{s_x^2 s_y^2}{c_x^2 s_x^2} + c_y e^{i\delta} + s_z \right)\right) e^{2i\delta} \\
\lambda_2 &= -\frac{s_x c_x c_y}{s_x s_y} \left(\frac{2 c_x c_y - s_x^2 - c_x^2}{s_x^2 - c_x^2} + \frac{c_y s_x}{c_y s_x} \left(\frac{s_x^2 s_y^2}{c_x^2 s_x^2} + c_y e^{i\delta} + s_z \right)\right) e^{2i\delta} \\
\lambda_3 &= +\frac{s_x c_x c_y}{s_x s_y} \left(\frac{2 c_x c_y - s_x^2 - c_x^2}{s_x^2 - c_x^2} + \frac{c_y s_x}{c_y s_x} \left(\frac{s_x^2 s_y^2}{c_x^2 s_x^2} + c_y e^{i\delta} + s_z \right)\right) e^{2i\delta}
\end{align*}$ |
| $C$ : $(\times \ \times \ \ 0)$ \ $(0 \ \times \ \times)$ \ $(0 \times \ \times)$ | $\begin{align*}
\lambda_1 &= -\frac{c_x}{s_x} \left(\frac{s_x^2 s_y^2 - c_x^2}{s_x^2 - c_x^2} + \frac{c_y s_x}{c_y s_x} \left(\frac{s_x^2 s_y^2}{c_x^2 s_x^2} + c_y e^{i\delta} + s_z \right)\right) e^{2i\delta} \\
\lambda_2 &= +\frac{s_x c_x s_y}{s_x s_y} \left(\frac{s_x^2 s_y^2 - c_x^2}{s_x^2 - c_x^2} + \frac{c_y s_x}{c_y s_x} \left(\frac{s_x^2 s_y^2}{c_x^2 s_x^2} + c_y e^{i\delta} + s_z \right)\right) e^{2i\delta} \\
\lambda_3 &= -\frac{c_x}{s_x} \left(\frac{s_x^2 s_y^2 - c_x^2}{s_x^2 - c_x^2} + \frac{c_y s_x}{c_y s_x} \left(\frac{s_x^2 s_y^2}{c_x^2 s_x^2} + c_y e^{i\delta} + s_z \right)\right) e^{2i\delta}
\end{align*}$ |
| $D_1$ : $(\times \ \times \ \ 0)$ \ $(0 \ \times \ \times)$ \ $(0 \times \ \times)$ | $\begin{align*}
\lambda_1 &= -\frac{s_x}{s_z} \left(\frac{c_x s_y}{s_x c_y} + c_x s_y s_z e^{i\delta}\right) e^{i\delta} \\
\lambda_2 &= +\frac{s_x}{s_z} \left(\frac{c_x s_y}{c_x s_y} + c_x s_y s_z e^{i\delta}\right) e^{i\delta} \\
\lambda_3 &= -\frac{s_x}{s_z} \left(\frac{c_x s_y}{s_x c_y} + c_x s_y s_z e^{i\delta}\right) e^{i\delta}
\end{align*}$ |
| $D_2$ : $(\times \ \times \ \ 0)$ \ $(0 \ \times \ \times)$ \ $(0 \times \ \times)$ | $\begin{align*}
\lambda_1 &= +\frac{s_x}{s_z} \left(\frac{c_x s_y}{s_x c_y} + c_x s_y s_z e^{i\delta}\right) e^{i\delta} \\
\lambda_2 &= -\frac{s_x}{s_z} \left(\frac{c_x s_y}{c_x s_y} + c_x s_y s_z e^{i\delta}\right) e^{i\delta} \\
\lambda_3 &= +\frac{s_x}{s_z} \left(\frac{c_x s_y}{s_x c_y} + c_x s_y s_z e^{i\delta}\right) e^{i\delta}
\end{align*}$ |
Table 2: Six patterns of the neutrino mass matrix $M$ with two independent vanishing entries, which are *incompatible* with the LOW solution and other empirical hypotheses. The analytical results for two ratios of three neutrino mass eigenvalues $\lambda_1/\lambda_3$ and $\lambda_2/\lambda_3$ are given in terms of four flavor mixing parameters $\theta_x$, $\theta_y$, $\theta_z$ and $\delta$.

| Pattern of $M$ | Results of $\lambda_1/\lambda_3$ and $\lambda_2/\lambda_3$ |
|---------------|---------------------|
| $E_1 : \begin{pmatrix} 0 & \times & \times \\
\times & 0 & \times \\
\times & \times & \times \end{pmatrix}$ | $\frac{\lambda_1}{\lambda_3} = -\frac{1}{c_y c_z^2} \cdot \frac{s_x^2 s_y^2 (c_z^2 - s_z^2) - c_x c_y s_x^2}{(s_z^2 - c_z^2) c_y + 2 s_x c_z s_y s_x e^{i \delta}} e^{2i \delta}$ |
| $E_2 : \begin{pmatrix} 0 & \times & \times \\
\times & \times & \times \\
\times & \times & 0 \end{pmatrix}$ | $\frac{\lambda_1}{\lambda_3} = -\frac{1}{s_y c_z^2} \cdot \frac{s_x^2 c_y (c_z^2 - s_z^2) - c_x s_y s_x^2}{(s_z^2 - c_z^2) s_y - 2 s_x c_z s_y s_x e^{i \delta}} e^{2i \delta}$ |
| $E_3 : \begin{pmatrix} 0 & \times & \times \\
\times & \times & 0 \\
\times & 0 & \times \end{pmatrix}$ | $\frac{\lambda_1}{\lambda_3} = -\frac{1}{c_z^2} \cdot \frac{s_x s_y c_y (c_z^2 - s_z^2) + c_x s_z^2}{(c_z^2 - s_z^2) s_y c_y + s_x c_z (c_y^2 - s_y^2) s_x e^{i \delta}} e^{2i \delta}$ |
| $F_1 : \begin{pmatrix} \times & 0 & 0 \\
0 & \times & \times \\
0 & \times & \times \end{pmatrix}$ | $\frac{\lambda_1}{\lambda_3} = 1$ |
| $F_2 : \begin{pmatrix} \times & 0 & \times \\
0 & \times & 0 \\
0 & \times & \times \end{pmatrix}$ | $\frac{\lambda_1}{\lambda_3} = \frac{c_x c_y + s_x s_y s_z e^{-i \delta}}{s_x c_y + c_x s_y s_z e^{i \delta}} e^{2i \delta}$ |
| $F_3 : \begin{pmatrix} \times & \times & 0 \\
\times & \times & 0 \\
0 & 0 & \times \end{pmatrix}$ | $\frac{\lambda_1}{\lambda_3} = \frac{c_x s_y - c_y s_x s_z e^{-i \delta}}{s_x s_y - c_x c_y s_z e^{i \delta}} e^{2i \delta}$ |
Figure 1: Pattern $A_1$ of the neutrino mass matrix $M$: allowed regions of $\sin^2 2\theta_{\text{chz}}$ versus $\delta$, $\theta_y$ versus $\theta_x$, $\sigma$ versus $\rho$, $m_2/m_3$ versus $m_1/m_3$, $\sum m_i$ versus $\langle m \rangle_e$, and $J$ versus $m_3$. 
Figure 2: Pattern B$_1$ of the neutrino mass matrix $M$: allowed regions of $\sin^2 2\theta_{\text{chz}}$ versus $\delta$, $\theta_y$ versus $\theta_x$, $\sigma$ versus $\rho$, $m_2/m_3$ versus $m_1/m_3$, $\langle m \rangle_{ee}$ versus $\langle m \rangle_e$, and $J$ versus $m_3$. 

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Figure 3: Pattern C of the neutrino mass matrix $M$: allowed regions of $\sin^2 2\theta_{\text{chz}}$ versus $\delta$, $\theta_y$ versus $\theta_x$, $\sigma$ versus $\rho$, $m_2/m_3$ versus $m_1/m_3$, $\langle m \rangle_{ee}$ versus $\langle m \rangle_e$, and $J$ versus $m_3$. 
Figure 4: Pattern D₁ of the neutrino mass matrix $M$: allowed regions of $\sin^2 2\theta_{\text{cha}}$ versus $\delta$, $\theta_y$ versus $\theta_x$, $\sigma$ versus $\rho$, $m_2/m_3$ versus $m_1/m_3$, $\langle m \rangle_{ee}$ versus $\langle m \rangle_e$, and $J$ versus $m_3$. 