Improved Upper Bounds for the Hot Spots Constant of Lipschitz Domains

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Abstract
The Hot Spots constant for bounded smooth domains was recently introduced by Steinerberger (2021) as a means to control the global extrema of the first nontrivial eigenfunction of the Neumann Laplacian by its boundary extrema. We generalize the Hot Spots constant to bounded Lipschitz domains and show that it leads to a necessary and sufficient condition for the weak Hot Spots conjecture HS2 of Bañuelos and Burdzy (J. Funct. Anal. 164(1), 1–33, 1999). We also derive a new general formula for a dimension-dependent upper bound that can be tailored to any specific class of domains. This formula is then used to compute upper bounds for the Hot Spots constant of the class of all bounded Lipschitz domains in \(\mathbb{R}^d\) for both small \(d\) and for asymptotically large \(d\) that significantly improve upon the existing results.

Keywords
Hot Spots conjecture · Neumann problem · Shape functional · Exit time

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1 Introduction
The Hot Spots conjecture is a famous question regarding the location of the extrema of eigenfunctions corresponding to the first nonzero eigenvalue of the Neumann Laplacian on a bounded Euclidean domain \(D\) (nonempty connected open set) with sufficiently regular

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boundary. Loosely speaking, the conjecture asserts that the maximum and minimum of such eigenfunctions—the hot and cold spots—are attained on the boundary $\partial D$ and not inside $D$. It was first proposed by J. Rauch during a special program in partial differential equations and related topics held at Tulane University in the spring of 1974; see [34].

There are many cases where the Hot Spots conjecture is known to be true. Any bounded domain in $\mathbb{R}$ is a finite interval and it is a trivial matter to check that the Hot Spots conjecture holds. The same can be said for other domains whose eigenfunctions are known explicitly, such as parallelepipeds, balls, and annuli, or whose eigenfunctions can be factored, such as bounded cylinders in $\mathbb{R}^{d+1}$ with arbitrary Lipschitz cross sections; see [21]. Other cases have required highly nontrivial proofs to verify that the Hot Spots conjecture holds. Some notable examples are convex planar domains with two axes of symmetry [19], lip domains [1], sufficiently thin tubular neighborhoods of curves in arbitrary 2-dimensional Riemannian manifolds [24], and triangles [20]; see also [2, 4, 12, 31, 35]. However, several planar counterexamples to the Hot Spots conjecture have been constructed; see [6, 10, 11, 22]. All of these counterexamples feature multiply connected domains and it is widely believed that the Hot Spots conjecture holds for arbitrary convex domains in $\mathbb{R}^d$.

A novel approach to tackling the Hot Spots conjecture is through a Hot Spots constant which was recently introduced by Steinerberger in [36]; see also [22]. The basic idea for a fixed domain $D$ is to examine the quotient $\sup_{x \in D} \varphi_2(x) / \sup_{x \in \partial D} \varphi_2(x)$ for an eigenfunction $\varphi_2$ corresponding to the first nonzero eigenvalue $\mu_2$ of the Neumann Laplacian on $D$. Then the supremum of this quotient over all such $\varphi_2$ is called the Hot Spots constant of $D$. If one can bound this constant from above by 1, then the Hot Spots conjecture holds for $D$.

Finding accurate estimates of the Hot Spots constant that hold over a general class of domains is a compelling question, and in the words of Steinerberger [36], this is tantamount to asking:

How ‘wrong’ can the Hot Spots conjecture be?

Steinerberger goes on to prove that

$$\sup_{x \in D} \varphi_2(x) \leq 58.35 \cdot \sup_{x \in \partial D} \varphi_2(x)$$

holds for any bounded domain $D \subset \mathbb{R}^d$ with smooth boundary. This shows that the Hot Spots constant for this class of domains is at most 58.35. Moreover, he shows that this upper bound can be improved as the dimension $d$ increases and it asymptotically approaches $\sqrt{e} \approx 3.89$ as $d \to \infty$. On the other hand, a numerical example of Kleefeld [22] shows that the Hot Spots constant is at least 1.001.

The main result of this paper is to extend Steinerberger’s results to bounded domains $D \subset \mathbb{R}^d$ with Lipschitz boundary and to significantly improve the upper bound on the Hot Spots constant.

**Theorem** (Simplified Version) Suppose $D \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary and let $\varphi_2$ be an eigenfunction corresponding to the first nonzero eigenvalue of the Neumann Laplacian on $D$. Then

$$\sup_{x \in D} \varphi_2(x) \leq 5.1043 \cdot \sup_{x \in \partial D} \varphi_2(x).$$

Furthermore, for asymptotically large dimension, the constant in this upper bound can be improved to $\sqrt{e} \approx 1.6487$. More precisely, there exists a sequence of positive real numbers
\( \{\epsilon_n\}_{n\geq1} \) converging to zero such that, if \( D \subset \mathbb{R}^d \) is a bounded domain with Lipschitz boundary and \( \varphi_2 \) is as above, then

\[
\sup_{x \in D} \varphi_2(x) \leq (\sqrt{e} + \epsilon_d) \cdot \sup_{x \in \partial D} \varphi_2(x).
\]

This result follows from Theorems 1 and 2 below, which establish upper bounds for the \( d \)-dimensional Hot Spots constant for both small values of \( d \) and for asymptotically large \( d \). In addition, we obtain the following results.

- We derive a new general formula for a dimension-dependent upper bound for the Hot Spots constant that can be tailored to any specific class of bounded Lipschitz domains; see Theorem 3.
- We formalize the idea of a Hot Spots constant with a definition that is applicable to bounded Lipschitz domains (see Eq. 4) and that leads to a necessary and sufficient condition for a weak version of the Hot Spots conjecture, namely HS2 of [2]; see Proposition 1.

The rest of the paper is organized as follows. We discuss the conjecture in more detail in Section 2.1 and formally define the Hot Spots constant in Section 2.2. Our main results are stated in Section 3. We collect some useful facts about Neumann Laplacian eigenvalues and eigenfunctions in Section 4.1. In Section 4.2, we review how Neumann Laplacian eigenfunctions can be studied using reflecting Brownian motion and also prove a key preliminary result. Finally, we prove our main results in Section 5.

## 2 The Hot Spots Constant

### 2.1 The Hot Spots Conjecture and HS2

Before describing the conjecture in more detail, we recall some basic facts about Neumann Laplacian eigenvalues and eigenfunctions; see Section 4.1 for further details. Consider a bounded domain \( D \subset \mathbb{R}^d \) with Lipschitz boundary (Lipschitz domain) and let \( \Delta \) be the Laplacian acting in \( L^2(D) \) with Neumann boundary conditions. In this case it is well known that \( -\Delta \) has a discrete spectrum, that is, there exists a sequence of eigenvalues \( 0 = \mu_1 < \mu_2 \leq \mu_3 \leq \ldots \) along with an \( L^2(D) \) orthonormal basis \( \{1/\sqrt{|D|} \equiv \varphi_1, \varphi_2, \varphi_3, \ldots\} \) of eigenfunctions which satisfy

\[
\begin{cases}
-\Delta \varphi_j = \mu_j \varphi_j & \text{in } D \\
\frac{\partial \varphi_j}{\partial \nu} = 0 & \text{on } \partial D
\end{cases}
\]

for each \( j \in \mathbb{N} \). Here \( \nu \) denotes the outward unit normal and the boundary condition is understood to hold at almost every \( x \in \partial D \) with respect to \( (d - 1) \)-dimensional surface measure. We will occasionally have to deal with eigenvalues of the Dirichlet Laplacian and these will be denoted by \( \lambda_j \). In case of possible ambiguity, we specify the domain by writing \( \mu_j(D) \) or \( \lambda_j(D) \).

While it follows immediately from the hypoellipticity of the Laplacian that solutions of (2) are in \( C^\infty(D) \), it turns out that they are also Hölder continuous; see [28, Proposition 3.6]. This implies that each eigenfunction can be extended uniquely to a function in \( C^0(\mathring{D}) \). In particular, each \( \varphi_j \) can be treated as a function defined pointwise on \( \mathring{D} \).
Returning now to the conjecture, we will actually be considering a slightly weaker version, namely HS2 of [2]. Like the original conjecture, HS2 asserts that the maximum and minimum of eigenfunctions corresponding to $\mu_2$ are attained on $\partial D$, but this version does not rule out the possibility that global extrema may also occur inside $D$. More precisely, we have the following definition.

**Definition 1** We say that HS2 holds for a bounded Lipschitz domain $D$ if for every Neumann Laplacian eigenfunction $\varphi_2$ corresponding to $\mu_2$ and all $y \in D$, we have

$$\inf_{x \in \partial D} \varphi_2(x) \leq \varphi_2(y) \leq \sup_{x \in \partial D} \varphi_2(x).$$

Similarly, we say that HS2 holds for a subclass $\mathcal{D}$ of bounded Lipschitz domains if HS2 holds for every domain $D \in \mathcal{D}$.

### 2.2 The Hot Spots Constant

Steinerberger worked exclusively with smooth domains but remarked that this assumption could be relaxed. Here we give a precise definition of the Hot Spots constant for Lipschitz domains and show that it is well-defined; see [5] for a definition of Lipschitz domain. Suppose $D$ is a bounded Lipschitz domain and let $\varphi_2$ be a Neumann Laplacian eigenfunction corresponding to $\mu_2$. Then we claim that $\varphi_2$ takes positive and negative values on both $D$ and $\partial D$. The claim for $D$ is an immediate consequence of the orthogonality of $\varphi_2$ and the constant eigenfunction $\varphi_1$, while the claim for $\partial D$ follows from an argument of Pleijel [32]; see Lemma 2 below for a precise statement and proof. Since $\varphi_2$ is continuous on $\overline{D}$, these claims imply that both

$$1 \leq \frac{\sup_{x \in D} \varphi_2(x)}{\sup_{x \in \partial D} \varphi_2(x)} < \infty \quad \text{and} \quad 1 \leq \frac{\inf_{x \in D} \varphi_2(x)}{\inf_{x \in \partial D} \varphi_2(x)} < \infty$$

must hold. With this in mind, we define the Hot Spots constant of $D$ by

$$\mathcal{C}(D) := \sup \left\{ \frac{\sup_{x \in D} \varphi_2(x)}{\sup_{x \in \partial D} \varphi_2(x)} \left| \begin{array}{l}
\varphi_2 \text{ is a Neumann Laplacian eigenfunction} \\
\text{corresponding to } \mu_2
\end{array} \right. \right\}. \quad (4)$$

**Remark 1** If $\varphi_2$ is an eigenfunction corresponding to $\mu_2$, then so is $-\varphi_2$. Hence changing both of the inner supremums in (4) to infimums over the same sets yields an equivalent definition of $\mathcal{C}(D)$.

The idea of a Hot Spots constant can also be applied to a class of domains. More precisely, suppose that $\mathcal{D}$ is a subclass of bounded Lipschitz domains. Then with a slight abuse of notation, we define the Hot Spots constant of this class by

$$\mathcal{C}(\mathcal{D}) := \sup_{D \in \mathcal{D}} \mathcal{C}(D). \quad (5)$$

In case $\mathcal{D}$ is the class of all bounded Lipschitz domains in $\mathbb{R}^d$, we simply write $\mathcal{C}_d$ for $\mathcal{C}(\mathcal{D})$ and refer to this as the $d$-dimensional Hot Spots constant. More specifically, we have

$$\mathcal{C}_d := \sup \left\{ \mathcal{C}(D) \left| D \subset \mathbb{R}^d \text{ is a bounded Lipschitz domain} \right. \right\}. \quad (6)$$
The following proposition, whose proof is given in Section 5.4, verifies that the Hot Spots constant can indeed be used as a necessary and sufficient condition for the weak Hot Spots conjecture HS2 that was stated in Definition 1.

**Proposition 1** Suppose that \( D \) is a bounded Lipschitz domain and that \( \mathcal{D} \) is a subclass of such domains. Then HS2 holds for \( D \) (respectively \( \mathcal{D} \)) if and only if \( \mathcal{C}(D) \leq 1 \) (respectively \( \mathcal{C}(\mathcal{D}) \leq 1 \)).

Interest in the Hot Spots constant goes beyond the conjecture. To see why this is so, let \( s > 0 \) be a scaling parameter. Notice that (2) implies that if \( \mu_2 \) and \( \varphi_2(x) \) correspond to a domain \( D \subset \mathbb{R}^d \), then \( \frac{1}{s^2} \mu_2 \) and \( \frac{1}{s^d} \varphi_2(\frac{x}{s}) \) are the analogous eigenvalue and eigenfunction which correspond to \( D \) scaled by \( s \). It follows that the Hot Spots constant is scale invariant, hence the mapping \( D \mapsto \mathcal{C}(D) \) is an example of a shape functional. This raises the question of identifying the maximal domains, if any, which maximize \( \mathcal{C} \) over the class of bounded Lipschitz domains in \( \mathbb{R}^d \). In other words, are there domains \( D \subset \mathbb{R}^d \) such that \( \mathcal{C}(D) = \mathcal{C}_d \)? If so, what do they look like? No less interesting is the characterization of the minimal domains where \( \mathcal{C}(D) = 1 \), as these are precisely the domains where HS2 holds. We refer to the monographs [16, 17] and references therein for more results on shape functionals and related extremal problems.

Not much is known about the \( d \)-dimensional Hot Spots constant or the existence of maximal domains beyond the trivial case \( d = 1 \). Here we know that \( \mathcal{C}_1 = 1 \), so every bounded domain is both maximal and minimal. We also know from any of the counterexamples mentioned above that \( \mathcal{C}_2 > 1 \). Indeed, Kleefeld [22] uses superconvergent numerical methods to show that \( \mathcal{C}_2 \geq 1 + 10^{-3} \), and he hints that this approach may also be applicable in \( \mathbb{R}^3 \). On the other hand, bounding \( \mathcal{C}_d \) from above is the central topic of the present paper.

### 3 Main Results

The first of our main results are bounds on the \( d \)-dimensional Hot Spots constant \( \mathcal{C}_d \). When compared with (1), we see that these bounds offer a 10-fold improvement over the existing result.

**Theorem 1** The \( d \)-dimensional Hot Spots constant \( \mathcal{C}_d \) defined by (6) has the following upper bounds.

| \( d \) | upper bound for \( \mathcal{C}_d \) |
|-------|------------------|
| 2     | 5.1043           |
| 3     | 3.5288           |
| 4     | 3.0200           |
| 10    | 2.3314           |
| 100   | 1.8809           |

The table from Theorem 1 indicates that the upper bounds for the \( d \)-dimensional Hot Spots constants have a decreasing trend. Our next main result extrapolates this trend by establishing an asymptotic upper bound for \( \mathcal{C}_d \) as \( d \to \infty \). This both generalizes and lowers the asymptotic upper bound from [36].
Theorem 2 The $d$-dimensional Hot Spots constant $C_d$ defined by (6) has the asymptotic upper bound

$$\limsup_{d \to \infty} C_d \leq \sqrt{e} \approx 1.6487.$$ 

Theorems 1 and 2 are obtained from a more general theorem that gives a variational-type formula for an upper bound for the Hot Spots constant which can be tailored to any class of domains. Before stating this result, we need to introduce the concept of a $V$-function and its corresponding $V$-bound. This is a convenient formulation of a recent result of H. Vogt [38]. We also need to highlight some facts about the ratio $\frac{\mu_2}{\lambda_1}$. Both of these ideas will feature prominently in the statement and proof of our next result.

Towards this end, let $D \subset \mathbb{R}^d$ be a domain without any boundedness or boundary regularity assumptions and let $\mathbb{P}_x$ be the law under which $W = (W_t : t \geq 0)$ is $d$-dimensional Brownian motion starting at $x \in D$ and running at twice the usual speed. We define the first exit time of $W$ from $D$ by

$$\tau_D := \inf\{t \geq 0 : W_t / \notin D\},$$

with the usual convention of $\inf\{\emptyset\} = \infty$.

It is well known that the right tail of $\tau_D$ under $\mathbb{P}_x$ has an exponential rate of decay given by the principal eigenvalue $\lambda_D$ of the Dirichlet Laplacian on $D$. More precisely, it follows from [37, Theorem 3.1.2] that for any $x \in D$ we have

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}_x(\tau_D > t) = -\lambda_D$$

where

$$\lambda_D := \inf_{f \in C^\infty_c(D) \setminus \{0\}} \frac{\int_D |\nabla f|^2 \, dy}{\int_D f^2 \, dy}.$$ 

Here $C^\infty_c(D)$ is the space of smooth functions on $D$ with compact support. In general, $\lambda_D$ needn’t be a true eigenvalue but merely the bottom of the spectrum. However, if the Dirichlet Laplacian on $D$ has a discrete spectrum, then $\lambda_D = \lambda_1$.

Now we can give a precise definition of the $V$-function and $V$-bound. In what follows, $\dim D$ denotes the topological dimension of the Euclidean domain $D$.

**Definition 2** Let $\mathcal{D}$ be a subclass of Euclidean domains. We call a function $V : (0, 1] \times \mathbb{N} \to [1, \infty)$ a $V$-function for the class $\mathcal{D}$ if for all domains $D \in \mathcal{D}$, the corresponding $V$-bound

$$\sup_{x \in D} \mathbb{P}_x(\tau_D > t) \leq V(\epsilon, \dim D)e^{-(1-\epsilon)\lambda_D t}$$

holds for all $0 < \epsilon \leq 1$ and $t \geq 0$.

It follows from [38, Theorem 2.1] that

$$(\epsilon, d) \mapsto 2^{1/4} \left(\frac{1 + 1/\sqrt{\epsilon}}{2}\right)^{d/2}$$

is a $V$-function for the class of all Euclidean domains. Moreover, [3, Proposition 2.4] shows that

$$(\epsilon, d) \mapsto \epsilon^{d/4} \frac{\sqrt{2}}{(2d)^{d/4}} \sqrt{\frac{\Gamma(d)}{\Gamma(d/2)}} \left(\frac{1 + 1/\sqrt{\epsilon}}{2}\right)^{d/2}$$

is a $V$-function for the same class that yields a better $V$-bound.
Remark 2 It would be interesting to find a $V$-function tailored specifically to convex domains that improves upon (9).

We use the $V$-bounds corresponding to (8) and (9) to replace an estimate used in [36] that Steinerberger deems particularly wasteful. These $V$-bounds have wide applicability to other problems featuring eigenvalues and exit times and have played a role in several recent results such as improving spectral bounds for the torsion function of Brownian motion [3, 38] and symmetric stable processes [29], estimating the loss of torsional rigidity due to a Brownian fracture [8], and establishing bounds for extremal problems related to the conformal Skorokhod embedding [27].

As pointed out in [36, Section 3.2], when both the Dirichlet and Neumann Laplacians on a domain $D \subset \mathbb{R}^d$ have a discrete spectrum, then a combination of the Faber-Krahn and Szegő-Weinberger inequalities yields the upper bound

\[
\frac{\mu_2(D)}{\lambda_1(D)} \leq \frac{P_{d/2,1}^2}{J_{d/2-1,1}},
\]

where $J_{d/2-1,1}$ is the first positive root of the Bessel function $J_{d/2-1}$ and $P_{d/2,1}$ is the first positive root of the derivative of $x^{1-d/2}J_{d/2}(x)$. Equality in (10) holds only when $D$ is a ball in $\mathbb{R}^d$; see [17, Section 7.6.3] for a quantitative improvement that takes into account how much $D$ deviates from a ball.

Next we define a version of the upper bound (10) that is specific to a subclass of bounded Lipschitz domains.

**Definition 3** Let $\mathcal{D}$ be a subclass of bounded Lipschitz domains. We call a function $r : \mathbb{N} \to (0, \infty)$ a ratio upper bound for the class $\mathcal{D}$ if

\[
\frac{\mu_2(D)}{\lambda_1(D)} \leq r(\dim D)
\]

holds for all domains $D \in \mathcal{D}$.

Now we can state our next main result which is a dimension-dependent upper bound for the Hot Spots constant. This bound is rather general and can be tailored to any specific class of bounded Lipschitz domains by the appropriate choice of a $V$-function and ratio upper bound. In what follows, we denote by $\mathcal{U}^d$ the class of bounded Lipschitz domains of dimension $d$, and by $\mathcal{U}^*$ the class of bounded Lipschitz domains of dimension greater than 1.

**Theorem 3** Let $\mathcal{D} \subset \mathcal{U}^*$ be a given subclass of bounded Lipschitz domains and define $S := \{d \in \mathbb{N} \mid \mathcal{D} \cap \mathcal{U}^d \neq \emptyset\}$. Suppose that $V$ and $r$ are a $V$-function and ratio upper bound, respectively, for the class $\mathcal{D}$. If $r(d) < 1$ for all $d \in S$, then the Hot Spots constant of $\mathcal{D}$ defined by (5) has the upper bound

\[
C(\mathcal{D}) \leq \sup_{d \in S} \inf_{0 < \epsilon < 1-r(d)} e^{r(d)a} \left(1 + \frac{r(d)V(\epsilon, d)}{1 - \epsilon - r(d)} e^{-(1-\epsilon)a}\right).
\]

**Remark 3** Regardless of the particulars of the class $\mathcal{D}$, the $V$-functions (8) and (9) are always valid. Likewise, Lemma 1 below shows that we can always take

\[
r(d) = \frac{4d + 8}{d(d + 8)}.
\]
Remark 4 We exclude domains of dimension 1 for technical reasons. As noted above in Section 2.2, $c(D) = 1$ if $\dim D = 1$ so this is of no consequence.

Remark 5 If $D \subset \mathcal{U}^d$ for some fixed dimension $d$, then the upper bound can be simplified by omitting the supremum appearing in (11).

4 Preliminary Results

4.1 Neumann Laplacian Eigenvalues and Eigenfunctions

Unlike the case of Dirichlet boundary conditions, the boundedness of $D$ alone is not sufficient to imply a discrete spectrum for the Neumann Laplacian. Indeed, pathological planar domains consisting of a series of “rooms and passages” can be constructed to demonstrate this; see [15]. On the other hand, when $D$ is a bounded Lipschitz domain, it follows from the compactness of the embedding $H^1(D) \hookrightarrow L^2(D)$ that the Neumann Laplacian has a discrete spectrum; see [16, Theorem 1.2.8].

A control on the ratio $\frac{\mu_2(D)}{\lambda_1(D)}$ is an essential component of Theorem 3 and the following lemma gives a simple dimension-dependent upper bound that is valid for all bounded Lipschitz domains. This result can be seen as a refinement of [36, Lemma 3] when $d \geq 4$ and is an easy consequence of some known estimates of Bessel zeros.

Lemma 1 If $D \subset \mathbb{R}^d$ is a bounded Lipschitz domain, then we have

$$\frac{\mu_2(D)}{\lambda_1(D)} < \frac{4d + 8}{d(d + 8)}.$$

Proof of Lemma 1 From Equation (1’) of [26] we have

$$p_{d/2,1}^2 < d + 2$$

and from Equation (1) of [25] we have

$$j_{d/2-1,1}^2 > \frac{1}{4d(d + 8)}.$$

The lemma follows from substituting these bounds into (10).

As discussed in Section 2.2, both (3) and the proof of Proposition 1 assume that $\varphi_2$ must change sign on $\partial D$. The following lemma confirms that this is indeed true. The proof borrows an idea of Pleijel [32] from the planar case.

Lemma 2 Suppose $D \subset \mathbb{R}^d$ is a bounded Lipschitz domain and let $\varphi_2$ be a Neumann Laplacian eigenfunction corresponding to $\mu_2$. Then $\varphi_2$ takes positive and negative values on $\partial D$.

Proof of Lemma 2 The claim is trivial for $d = 1$ so assume $d \geq 2$. Since $\varphi_2$ must take positive and negative values on $D$, it follows from Courant’s nodal domain theorem [16, Theorem 1.3.2] that the open subsets of $\mathbb{R}^d$ defined by

$$D_+ := \{x \in D : \varphi_2(x) > 0\} \quad \text{and} \quad D_- := \{x \in D : \varphi_2(x) < 0\}$$

are both nonempty and connected. Hence $D_+$ and $D_-$ are both bounded domains in $\mathbb{R}^d$. 

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Next, consider the boundaries of $D_+$ and $D_-$, namely $\partial D_+$ and $\partial D_-$, which are both subsets of $\overline{D}$. Notice that the continuity of $\varphi_2$ on $\overline{D}$ implies that $\varphi_2 \geq 0$ on $\partial D_+$ and $\varphi_2 \leq 0$ on $\partial D_-$. We claim that there exist points $x_+ \in \partial D_+$ and $x_- \in \partial D_-$ such that $\varphi_2(x_+) > 0$ and $\varphi_2(x_-) < 0$. To see that this is true, suppose for a contradiction that $\varphi_2$ vanishes on $\partial D_+$. Then the restriction of $\varphi_2$ to $D_+$ is the (nonnormalized) first eigenfunction of the Dirichlet Laplacian on $D_+$ with eigenvalue $\lambda_1(D_+)$ equal to $\mu_2(D)$. Hence the domain monotonicity property of Dirichlet Laplacian eigenvalues [16, Section 1.3.2] implies that

$$\mu_2(D) = \lambda_1(D_+) \geq \lambda_1(D).$$

However, this contradicts the fact that $\mu_2(D) < \lambda_1(D)$; see Lemma 1 or [13]. The same argument can be used on $\partial D_-$ and this establishes the existence of the points $x_+$ and $x_-$. Lastly, we show that $x_+ \in \partial D$ and $x_- \in \partial D$. Suppose for a contradiction that $x_+ \notin \partial D$. Then we must have $x_+ \in D$. Since $\varphi_2(x_+) > 0$, this implies that $x_+ \in D_+$ by definition. However, this contradicts the fact that $x_+ \in \partial D_+$. Arguing similarly for $x_-$ proves the lemma. 

### 4.2 Reflecting Brownian Motion

The connection between reflecting Brownian motion (RBM) and boundary value problems for the Neumann Laplacian goes back at least to [18]; see also [9, 23]. In [7], Bass and Hsu construct RBM as a continuous strong Markov process on the closure of a bounded Lipschitz domain $D$; see also [33, Remark 2.4]. While Fukushima [14] had already constructed RBM for arbitrary bounded domains, in general this process actually lives on a certain compactification of the domain and not necessarily on $\overline{D}$ itself. Regardless of the boundary regularity, the transition density $p(t, x, y)$ of RBM, also called the Neumann heat kernel of the domain, is known to be smooth in the interior of $D$. However, if the boundary of $D$ is Lipschitz, then $p(t, x, y)$ can be extended to a continuous function on $(0, \infty) \times \overline{D} \times \overline{D}$; see [7, Remark 4.1 and Lemma 4.3].

For the rest of this section, let $D$ be a bounded Lipschitz domain and let $\mathbb{P}_x$ and $\mathbb{E}_x$ be the law and expectation under which $X = (X_t : t \geq 0)$ is $\overline{D}$-valued RBM starting at $x \in \overline{D}$ and running at twice the usual speed. Then it can be shown that for each $t > 0$, the transition density of $X$ is the kernel of a positive self-adjoint Hilbert–Schmidt integral operator mapping $L^2(D)$ into $L^2(D)$. In particular, the orthonormal basis for $L^2(D)$ satisfying (2) can be used to give the Hilbert–Schmidt expansion

$$p(t, x, y) = \sum_{j=1}^{\infty} e^{-\mu_j t} \varphi_j(x) \varphi_j(y), \quad t > 0, \quad x, y \in \overline{D},$$

(12)

where the convergence is absolute and uniform on $[t_0, \infty) \times \overline{D} \times \overline{D}$ for every $t_0 > 0$; see [30, Theorem 10] for a detailed proof. An important consequence of this expansion is the eigenfunction identity

$$\mathbb{E}_x[\varphi_j(X_t)] = \int_D p_t(x, y) \varphi_j(y) dy = e^{-\mu_j t} \varphi_j(x), \quad t \geq 0, \quad x \in \overline{D},$$

(13)

which holds for all $j \in \mathbb{N}$. The interchange of integration and summation necessary to deduce (13) from the orthogonality of the eigenfunctions when $t > 0$ can be justified by the uniform convergence of the right-hand side of (12).

The following lemma is the key ingredient in the proof of Theorem 3. It is inspired by Lemma 1 of [36] and the proof begins similarly by splitting the expectation appearing in
(13) into two parts depending on whether or not $X$ has hit the boundary $\partial D$ by time $t$. Hence we need to define the exit time of the RBM $X$ from $D$. This is analogous to the definition (7) for ordinary Brownian motion $W$. Note that the exit times of both processes are 0 by definition when starting from $\partial D$, and both have the same law when starting from inside $D$ (recall that both processes run at twice the usual speed). These processes only differ after hitting $\partial D$ so their exit times from $D$ are equivalent for the purposes of this lemma.

Lemma 3 Suppose $D$ is a bounded Lipschitz domain and let $\tau_D$ denote the first exit time of $D$ by Brownian motion running at twice the usual speed. Fix any $j \in \mathbb{N}$ and let $\varphi_j$ be a Neumann Laplacian eigenfunction corresponding to $\mu_j$. Then there exists at least one $x_0 \in \overline{D}$ such that $\varphi_j(x_0) = \sup_{x \in D} \varphi_j(x)$. Moreover, for all such $x_0$ and all $t \geq 0$, we have

$$1 \leq e^{\mu_j t} \mathbb{P}_{x_0}(\tau_D > t) \left( 1 - \frac{\sup_{x \in D} \varphi_j(x)}{\sup_{x \in D} \varphi_j(x)} \right) + \left( 1 + \int_0^t \mu_j e^{\mu_j s} \mathbb{P}_{x_0}(\tau_D > s) ds \right) \frac{\sup_{x \in \partial D} \varphi_j(x)}{\sup_{x \in D} \varphi_j(x)}.$$  

Proof of Lemma 3 For any $t \geq 0$ and $x \in \overline{D}$, we can rewrite the eigenfunction identity (13) as

$$e^{-\mu_j t} \varphi_j(x) = \mathbb{E}_x \left[ \varphi_j(X_t) 1_{\tau_D > t} \right] + \mathbb{E}_x \left[ \varphi_j(X_t) 1_{\tau_D \leq t} \right].$$  

We estimate the first term on the right-hand side of (15) by

$$\mathbb{E}_x \left[ \varphi_j(X_t) 1_{\tau_D > t} \right] \leq \sup_{y \in D} \varphi_j(y) \mathbb{P}_x(\tau_D > t).$$  

For the second term, we use the strong Markov property, identity (13), and the fact that $X_{\tau_D} \in \partial D \mathbb{P}_x$-almost surely to obtain

$$\mathbb{E}_x \left[ \varphi_j(X_t) 1_{\tau_D \leq t} \right] = \mathbb{E}_x \left[ \mathbb{E}_{X_{\tau_D}} \left[ \varphi_j(X_{t-\tau_D}) 1_{\tau_D \leq t} \right] \right] = \mathbb{E}_x \left[ e^{-\mu_j (t-\tau_D)} \varphi_j(X_{\tau_D}) 1_{\tau_D \leq t} \right] \leq \sup_{y \in \partial D} \varphi_j(y) e^{-\mu_j t} \mathbb{E}_x \left[ e^{\mu_j \tau_D} 1_{\tau_D \leq t} \right].$$  

Substituting both (16) and (17) into (15) leads to the inequality

$$\varphi_j(x) \leq \sup_{y \in D} \varphi_j(y) e^{\mu_j t} \mathbb{P}_x(\tau_D > t) + \sup_{y \in \partial D} \varphi_j(y) \mathbb{E}_x \left[ e^{\mu_j \tau_D} 1_{\tau_D \leq t} \right],$$  

which holds for any $t \geq 0$ and $x \in \overline{D}$. Since $\varphi_j$ is continuous on the compact set $\overline{D}$, there exists some $x_0 \in \overline{D}$ such that $\varphi_j(x_0) = \sup_{y \in D} \varphi_j(y)$. Hence setting $x = x_0$ in (18) and then dividing both sides by $\sup_{y \in \partial D} \varphi_j(y) > 0$ gives us

$$1 \leq e^{\mu_j t} \mathbb{P}_{x_0}(\tau_D > t) \left( 1 - \frac{\sup_{y \in \partial D} \varphi_j(y)}{\sup_{y \in D} \varphi_j(y)} \right) + \mathbb{E}_{x_0} \left[ e^{\mu_j \tau_D} 1_{\tau_D \leq t} \right] \frac{\sup_{y \in \partial D} \varphi_j(y)}{\sup_{y \in D} \varphi_j(y)}.$$  

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Next we find an alternative expression for the expectation appearing on the right-hand side of (19) that will be more amenable to estimation later on in the proof of Theorem 3. For any $t \geq 0$ and $x \in D$, we have $\mathbb{P}_x$-almost surely

$$\int_0^t \mu_j e^{\mu_j s} \mathbb{1}_{\{\tau_D > s \}} ds = \int_0^{\tau_D \wedge t} \mu_j e^{\mu_j s} ds = e^{\mu_j (\tau_D \wedge t)} - 1 = e^{\mu_j \tau_D} \mathbb{1}_{\{\tau_D \leq t \}} + e^{\mu_j t} \mathbb{1}_{\{\tau_D > t \}} - 1.$$ 

Taking expectations and using Tonelli’s theorem on the left-hand side leads to

$$\int_0^t \mu_j e^{\mu_j s} \mathbb{P}_x(\tau_D > s) ds = \mathbb{E}_x \left[ e^{\mu_j \tau_D} \mathbb{1}_{\{\tau_D \leq t \}} + e^{\mu_j t} \mathbb{P}_x(\tau_D > t) - 1 \right].$$

All of these terms are finite so we can rearrange this equation to yield

$$\mathbb{E}_x \left[ e^{\mu_j \tau_D} \mathbb{1}_{\{\tau_D \leq t \}} \right] = 1 - e^{\mu_j t} \mathbb{P}_x(\tau_D > t) + \int_0^t \mu_j e^{\mu_j s} \mathbb{P}_x(\tau_D > s) ds. \quad (20)$$

Finally, we set $x = x_0$ in (20), substitute this into (19), and then collect the $\mathbb{P}_x(\tau_D > t)$ terms together to produce the desired inequality

$$1 \leq e^{\mu_j t} \mathbb{P}_{x_0}(\tau_D > t) \left( 1 - \sup_{y \in \partial D} \frac{\varphi_j(y)}{\sup_{y \in D} \varphi_j(y)} \right) + \left( 1 + \int_0^t \mu_j e^{\mu_j s} \mathbb{P}_{x_0}(\tau_D > s) ds \right) \sup_{y \in \partial D} \frac{\varphi_j(y)}{\sup_{y \in D} \varphi_j(y)}.$$

\[ \square \]

5 Proofs of the Main Results

In this section we give the proofs of Theorems 1, 2, 3 and Proposition 1. We begin with the proof of Theorem 3 and use this result to prove Theorems 1 and 2. Lastly, Proposition 1 will be derived as a consequence of Lemma 2.

5.1 Proof of Theorem 3

**Proof of Theorem 3** In order to prove inequality (11), we verify that

$$C(D) \leq \inf_{a \geq 0, 0 < \epsilon < 1 - r(d)} e^{r(d)a} \left( 1 + \frac{r(d) V(\epsilon, d)}{1 - \epsilon - r(d)} e^{-(1-\epsilon)a} \right) \quad (21)$$

holds for every $D \in \mathcal{D}$. Towards this end, let $D$ be an arbitrary domain in $\mathcal{D}$ and put $d = \dim D \geq 2$. In what follows, the eigenvalues $\lambda_1$ and $\mu_2$ always pertain to $D$.

Our starting point is inequality (14) of Lemma 3 applied to any Neumann Laplacian eigenfunction $\varphi_2$ corresponding to $\mu_2$. In this case (3) holds so we can substitute upper bounds for $\mathbb{P}_{x_0}(\tau_D > t)$ and $\mathbb{P}_{x_0}(\tau_D > s)$ into (14) while still preserving the inequality. For this we use the $V$-bound corresponding to $V$ which is valid for $D$ by hypothesis; see Definition 2. Since this is typically a poor bound for small $s$, we split the integral into two
parts and then use the trivial upper bound of 1 when $s$ is between 0 and $u$, with $u \in [0, t]$ being a parameter. So for any $\delta, \epsilon \in (0, 1 - r(d))$ and $t \geq u \geq 0$, we can write

$$1 \leq V(\delta, d)e^{-\left((1 - \delta)\lambda_1 - \mu_2\right)t} \left(1 - \frac{\sup_{x \in \partial D} \varphi_2(x)}{\sup_{x \in D} \varphi_2(x)}\right) + \left(e^{\mu_2 u} + \mu_2 V(\epsilon, d) \int_u^t e^{-\left((1 - \epsilon)\lambda_1 - \mu_2\right)s} ds\right) \frac{\sup_{x \in \partial D} \varphi_2(x)}{\sup_{x \in D} \varphi_2(x)}.$$ 

The boundedness assumption on $D$ implies that $\lambda_1 > 0$, so we can set $u = \frac{a}{\lambda_1}$ and $t = \frac{b}{\lambda_1}$ for any $b \geq a \geq 0$ to get

$$1 \leq V(\delta, d)e^{-\left(1 - \delta - \frac{\mu_2}{\lambda_1}\right)b} \left(1 - \frac{\sup_{x \in \partial D} \varphi_2(x)}{\sup_{x \in D} \varphi_2(x)}\right) + \left(e^{\frac{\mu_2}{\lambda_1}a} + \frac{\mu_2}{\lambda_1} V(\epsilon, d) \int_{\frac{a}{\lambda_1}}^{\frac{b}{\lambda_1}} e^{-\left(1 - \epsilon\right)\lambda_1} ds\right) \frac{\sup_{x \in \partial D} \varphi_2(x)}{\sup_{x \in D} \varphi_2(x)}.$$ 

Applying the change of variables $s \mapsto \frac{s}{\lambda_1}$ to the integral results in

$$1 \leq V(\delta, d)e^{-\left(1 - \delta - \frac{\mu_2}{\lambda_1}\right)b} \left(1 - \frac{\sup_{x \in \partial D} \varphi_2(x)}{\sup_{x \in D} \varphi_2(x)}\right) + \left(e^{\frac{\mu_2}{\lambda_1}a} + \frac{\mu_2}{\lambda_1} V(\epsilon, d) \int_{\frac{a}{\lambda_1}}^{\frac{b}{\lambda_1}} e^{-\left(1 - \epsilon\right)\lambda_1} ds\right) \frac{\sup_{x \in \partial D} \varphi_2(x)}{\sup_{x \in D} \varphi_2(x)}.$$ 

(22)

Now it is obvious that the ratio upper bound $r(d)$ can be substituted for the four instances of $\frac{\mu_2}{\lambda_1}$ that appear in (22) while still preserving the inequality; see Definition 3. To shorten what would otherwise be an exceedingly lengthy expression, we set

$$\rho(\delta, d) := 1 - \delta - r(d)$$

and define $\rho(\epsilon, d)$ analogously. We revert back to the original notation at the end of the proof. Noting that $\rho(\epsilon, d) \neq 0$, we can also evaluate the integral in (22) to obtain

$$1 \leq V(\delta, d)e^{-\rho(\delta, d)b} \left(1 - \frac{\sup_{x \in \partial D} \varphi_2(x)}{\sup_{x \in D} \varphi_2(x)}\right) + \left(e^{r(d)a} + \frac{r(d) V(\epsilon, d)}{\rho(\epsilon, d)} (e^{-\rho(\epsilon, d)a} - e^{-\rho(\epsilon, d)b})\right) \frac{\sup_{x \in \partial D} \varphi_2(x)}{\sup_{x \in D} \varphi_2(x)}.$$ 

Again noting that (3) holds, this inequality can be rearranged into

$$\left(1 - V(\delta, d)e^{-\rho(\delta, d)b}\right) \frac{\sup_{x \in \partial D} \varphi_2(x)}{\sup_{x \in D} \varphi_2(x)} \leq \left(e^{r(d)a} - V(\delta, d)e^{-\rho(\delta, d)b} + \frac{r(d) V(\epsilon, d)}{\rho(\epsilon, d)} (e^{-\rho(\epsilon, d)a} - e^{-\rho(\epsilon, d)b})\right).$$ 

(23)
Since $\rho(\delta, d) > 0$, for $b \geq 0$ large enough we have
\[ 1 - V(\delta, d)e^{-\rho(\delta, d)b} > 0, \] (24)
and this allows us to transform (23) into the inequality
\[
\sup_{x \in D} \varphi_2(x) \leq e^{r(d)a} - V(\delta, d)e^{-\rho(\delta, d)b} + \frac{r(d)V(\epsilon, d)}{\rho(\epsilon, d)} \left( e^{-\rho(\epsilon, d)a} - e^{-\rho(\epsilon, d)b} \right) \frac{1 - V(\delta, d)e^{-\rho(\delta, d)b}}{1 - V(\delta, d)e^{-\rho(\delta, d)b}}.
\] (25)

In principle, we could try to optimize inequality (25) over $\delta, \epsilon \in (0, 1-r(d))$ and $b \geq a \geq 0$ satisfying the constraint (24). In practice, however, the optimal value of $b$ is always so large as to suggest letting $b \to \infty$. This leads to a much simpler inequality that is nearly as good as the original. More specifically, since $\rho(\delta, d) = 1 - \delta - r(d)$ and $\rho(\epsilon, d) = 1 - \epsilon - r(d)$ are both positive, letting $b \to \infty$ in (25) results in
\[
\sup_{x \in D} \varphi_2(x) \leq e^{r(d)a} \left( 1 + \frac{r(d)V(\epsilon, d)}{1 - \epsilon - r(d)} e^{-(1-\epsilon)a} \right),
\]
which holds for any $0 < \epsilon < 1 - r(d)$, any $a \geq 0$, and any Neumann Laplacian eigenfunction $\varphi_2$ corresponding to $\mu_2$. Thus we have verified that (21) holds for $D$ as desired.

### 5.2 Proof of Theorem 1

**Proof of Theorem 1** We use Theorem 3 with the $V$-function (9) to compute the upper bounds for the $d$-dimensional Hot Spots constant $C_d$ for the specific values of $d$ given in Theorem 1. The table below shows how the ratio of $p_{d/2,1}^2$ to $j_{d/2-1,1}^2$ is used for the ratio upper bound $r(d)$ as indicated by (10). These roots were calculated to high precision with Mathematica using `FindRoot` and the built-in function `BesselJ`. Moreover, the numbers were rounded up or down as appropriate to ensure that $r(d)$ is an upper bound. The near-optimal pairs $(\epsilon, a)$ were also found with Mathematica using `NMinimize`.

| $d$ | $p_{d/2,1}^2$ | $j_{d/2-1,1}^2$ | $r(d)$ | $\epsilon$ | $a$ | upper bound for $C_d$ |
|-----|---------------|-----------------|--------|-------------|-----|---------------------|
| 2   | 3.3900        | 5.7831          | 0.5862 | 0.0929      | 1.0081 | 5.1043             |
| 3   | 4.3330        | 9.8696          | 0.4391 | 0.1485      | 1.2205 | 3.5288             |
| 4   | 5.2896        | 14.681          | 0.3604 | 0.1903      | 1.4325 | 3.0200             |
| 10  | 11.160        | 57.582          | 0.1939 | 0.3359      | 2.5846 | 2.3314             |
| 100 | 101.02        | 3144.1          | 0.0322 | 0.6894      | 16.219 | 1.8809             |

### 5.3 Proof of Theorem 2

**Proof of Theorem 2** We start by using Theorem 3 with a suitable $V$-function $V$ and ratio upper bound $r$ to write
\[
C_d \leq e^{r(d)a} \left( 1 + \frac{r(d)V(\epsilon, d)}{1 - \epsilon - r(d)} e^{-(1-\epsilon)a} \right).
\] (26)
which holds for any $\epsilon \in (0, 1 - r(d))$ and $a \geq 0$ when $d \geq 2$. We take as a $V$-function

$$V(\epsilon, d) := 2^{1/4} \left( \frac{1 + 1/\sqrt{\epsilon}}{2} \right)^{d/2}$$

from (8), and use Lemma 1 to justify the choice of

$$r(d) := \frac{4d + 8}{d(d + 8)}$$

when $d$ is large enough so that $r(d) < 1$.

Next we make specific choices for $\epsilon$ and $a$ as functions of the dimension $d$. Our choices are of a simple form, yet incorporate some flexibility in order to get the sharpest result. More specifically, we take

$$\epsilon = \epsilon_d := (1 + cd^\alpha)^{-2}$$

and

$$a = a_d := kd^\beta.$$ 

Clearly we must have $c, k > 0$, while any $\beta \in \mathbb{R}$ is valid. Moreover, the table from the proof of Theorem 1 suggests that we choose $\alpha < 0$ and $\beta > 0$. However, in order to determine the precise ranges of $c$ and $\alpha$ that ensure $\epsilon_d \in (0, 1 - r(d))$, we need to examine the asymptotic behavior of $\epsilon_d$ as $d \to \infty$. This can be deduced from the Taylor expansion of $(1 + x)^{-2}$ at $x = 0$, namely

$$\epsilon_d = 1 - 2cd^\alpha + O(d^{2\alpha}) \text{ as } d \to \infty. \quad (27)$$

Hence if $\alpha \in (-1, 0)$, or if $\alpha = -1$ and $c > 2$, then $\epsilon_d \in (0, 1 - \frac{4}{d})$ for $d$ large enough.

We will also need the asymptotic behavior of $V(\epsilon_d, d)$ as $d \to \infty$. For this we can use the Taylor expansion of $\log(1 + x)$ at $x = 0$ to write

$$V(\epsilon_d, d) = 2^{1/4} \left( 1 + \frac{c}{2} d^\alpha \right)^{d/2}$$

$$= 2^{1/4} \exp \left( \frac{d}{2} \log \left( 1 + \frac{c}{2} d^\alpha \right) \right)$$

$$= 2^{1/4} \exp \left( \frac{d}{2} \left( \frac{c}{2} d^\alpha + O(d^{2\alpha}) \right) \right) \text{ as } d \to \infty$$

$$= 2^{1/4} \exp \left( \frac{c}{4} d^{\alpha + 1} + O(d^{2\alpha + 1}) \right) \text{ as } d \to \infty. \quad (28)$$

Now we substitute the asymptotics (27) and (28) into (26) under the restrictions $\alpha \in (-1, 0)$ and $c, k, \beta > 0$ to yield

$$C_d \leq e^{4kd^{\beta - 1}} \left( 1 + \frac{4}{d} 2^{1/4} e^{\frac{c}{2} d^{\alpha + 1} + O(d^{2\alpha + 1})} e^{-2ckd^{\alpha + \beta} + O(d^{2\alpha + \beta})} \right) \text{ as } d \to \infty. $$

Since the expression appearing within the large parentheses is bounded below by 1 for $d$ large enough, if $\beta > 1$ then the first exponential factor will grow unboundedly with $d$ and render the estimate meaningless. On the other hand, if $\beta < 1$, then the $\frac{1}{4} d^{\alpha + 1}$ term will dominate the $-2ckd^{\alpha + \beta}$ term inside the exponential, also leading to an estimate that grows unboundedly with $d$. Hence we must have $\beta = 1$. With this parameter choice we can now write

$$C_d \leq e^{4k} \left( 1 + \frac{2^{9/4} e^{\frac{c}{2} (1 - 2k)d^{\alpha + 1} + O(d^{2\alpha + 1})}}{2cd^{\alpha + 1} - 4 + O(d^{2\alpha + 1})} \right) \text{ as } d \to \infty.$$
Noting again that the expression appearing within the large parentheses is bounded below by 1 for \( d \) large enough, we see that choosing \( k = \frac{1}{8} \) along with any \( c > 0 \) and \( \alpha \in (-1, -\frac{1}{2}] \) will lead to the best possible asymptotic bound that can be deduced from (26) and our choices of \( V \)-function and ratio upper bound. In this case we have

\[
\limsup_{d \to \infty} C_d \leq \lim_{d \to \infty} \sqrt{e} \left( 1 + \frac{2^{9/4} e^{O(d^{2\alpha+1})}}{2cd^{\alpha+1} - 4 + O(d^{2\alpha+1})} \right) = \sqrt{e}.
\]

### 5.4 Proof of Proposition 1

**Proof of Proposition 1** First suppose that \( C(D) \leq 1 \). In light of (3), this is actually equivalent to \( C(D) = 1 \). Hence by (4) and Remark 1, if \( \varphi_2 \) is a Neumann Laplacian eigenfunction corresponding to \( \mu_2 \), then both

\[
\sup_{x \in D} \varphi_2(x) \geq \frac{1}{\inf_{x \in \partial D} \varphi_2(x)} \leq 1
\]

must hold. By Lemma 2, we know that \( \varphi_2 \) takes positive and negative values on \( \partial D \), so \( \sup_{x \in \partial D} \varphi_2(x) > 0 \) and \( \inf_{x \in \partial D} \varphi_2(x) < 0 \). Thus (29) implies that

\[
\sup_{x \in D} \varphi_2(x) \leq \sup_{x \in \partial D} \varphi_2(x) \quad \text{and} \quad \inf_{x \in D} \varphi_2(x) \geq \inf_{x \in \partial D} \varphi_2(x).
\]

Since (30) holds for any \( \varphi_2 \), it follows from Definition 1 that HS2 holds for \( D \).

Conversely, suppose that HS2 holds for \( D \). Then by Definition 1 we have

\[
\sup_{x \in D} \varphi_2(x) \leq \sup_{x \in \partial D} \varphi_2(x),
\]

which by Lemma 2 implies

\[
\sup_{x \in D} \varphi_2(x) \leq \sup_{x \in \partial D} \varphi_2(x) \leq 1.
\]

Since (31) holds for any \( \varphi_2 \), it follows from (4) that \( C(D) \leq 1 \).

Next suppose that we have a class of domains \( \mathcal{D} \) with \( C(D) \leq 1 \). It follows from (5) that \( C(D) \leq 1 \) for each \( D \in \mathcal{D} \). We have already shown that this implies HS2 holds for each \( D \in \mathcal{D} \). Hence by Definition 1, HS2 holds for \( \mathcal{D} \). Conversely, suppose that HS2 holds for \( \mathcal{D} \). Hence HS2 holds for each \( D \in \mathcal{D} \). We have already shown that this implies \( C(D) \leq 1 \) for each \( D \in \mathcal{D} \). Now it follows from (5) that \( C(D) \leq 1 \).

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References

1. Atar, R., Burdzy, K.: On Neumann eigenfunctions in lip domains. J. Amer. Math. Soc. 17(2), 243–265 (2004). MR 2051611
2. Bañuelos, R., Burdzy, K.: On the “hot spots” conjecture of J. Rauch, J. Funct. Anal. 164(1), 1–33 (1999). MR 1694534
3. Bañuelos, R., Mariano, P., Wang, J.: Bounds for exit times of Brownian motion and the first Dirichlet eigenvalue for the Laplacian, arXiv:2003.06867 (2020)
4. Bañuelos, R., Pang, M., Pascu, M.: Brownian motion with killing and reflection and the “hot-spots” problem. Probab. Theory Related Fields 130(1), 56–68 (2004). MR 2092873
5. Bass, R.F.: Probabilistic Techniques in Analysis, Probability and Its Applications (New York). Springer-Verlag, New York (1995). MR 1329542
6. Bass, R.F., Burdzy, K.: Fiber Brownian motion and the “hot spots” problem. Probab. Theory Related Fields 105(1), 25–58 (2000). MR 1788041
7. Bass, R.F., Hsu, P.: Some potential theory for reflecting Brownian motion in Hölder and Lipschitz domains. Ann. Probab. 19(2), 486–508 (1991). MR 1106272
8. van den Berg, M., den Hollander, F.: Torsional rigidity for cylinders with a Brownian fracture. Bull. Lond. Math. Soc. 50(2), 321–339 (2018). MR 3830123
9. Brosamler, G.A.: A probabilistic solution of the Neumann problem. Math. Scand. 38(1), 137–147 (1976). MR 408009
10. Burdzy, K.: The hot spots problem in planar domains with one hole. Duke Math. J. 129(3), 481–502 (2005). MR 2169871
11. Burdzy, K., Werner, W.: A counterexample to the “hot spots” conjecture. Ann. Math. (2) 149(1), 309–317 (1999). MR 1680567
12. Doumerc, Y., Moriarty, J.: Exit problems associated with affine reflection groups. Probab. Theory Related Fields 145(3-4), 351–383 (2009). MR 2529433
13. Filonov, N.: On an inequality for the eigenvalues of the Dirichlet and Neumann problems for the Laplace operator. Algebra i Analiz 16(2), 172–176 (2004). MR 2068346
14. Fukushima, M.: A construction of reflecting barrier Brownian motions for bounded domains. Osaka Math. J. 4, 183–215 (1967). MR 231444
15. Hempel, R., Seco, L.A., Simon, B.: The essential spectrum of Neumann Laplacians on some bounded singular domains. J. Funct. Anal. 102(2), 448–483 (1991). MR 1140635
16. Henrot, A.: Extremum Problems for Eigenvalues of Elliptic Operators, Frontiers in Mathematics. Basel, Birkhäuser Verlag (2006). MR 2251558
17. Henrot, A. (ed.): Shape Optimization and Spectral Theory. De Gruyter Open, Warsaw (2017). MR 3681143
18. Ikeda, N.: On the construction of two-dimensional diffusion processes satisfying Wentzell’s boundary conditions and its application to boundary value problems. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. 33, 367–427 (1960/61). MR 126883
19. Jerison, D., Nadirashvili, N.: The “hot spots” conjecture for domains with two axes of symmetry. J. Amer. Math. Soc. 13(4), 741–772 (2000). MR 1775736
20. Judge, C., Mondal, S.: Euclidean triangles have no hot spots. Ann. of Math. (2) 191(1), 167–211 (2020). MR 4045963
21. Kowohl, B.: Rearrangements and Convexity of Level Sets in PDE, Lecture Notes in Mathematics, p. 1150. Springer, Berlin (1985). MR 810619
22. Kleeved, A.: The hot spots conjecture can be false: some numerical examples. Adv. Comput. Math. 47(6), 31 (2021). Paper No. 85, MR 4349243
23. Korostelev, A.P.: The probability distribution of the solution of the problem with an oblique derivative. Teor. Verojatnost. i Primenen. 18, 172–176 (1973). MR 0312571
24. Krejčířik, D., Tušek, M.: Location of hot spots in thin curved strips. J. Differential Equations 266(6), 2953–2977 (2019). MR 3912674
25. Lorch, L.: Some inequalities for the first positive zeros of Bessel functions. SIAM J. Math. Anal. 24(3), 814–823 (1993). MR 1215440
26. Lorch, L., Szego, P.: Bounds and monotonicities for the zeros of derivatives of ultraspherical Bessel functions. SIAM J. Math. Anal. 25(2), 549–554 (1994). MR 1266576
27. Mariano, P., Panzo, H.: Conformal Skorokhod embeddings and related extremal problems. Electron. Commun. Probab. 25(Paper No. 42), 11 (2020). MR 4112773
28. Nittka, R.: Regularity of solutions of linear second order elliptic and parabolic boundary value problems on Lipschitz domains. J. Differential Equations 251(4-5), 860–880 (2011). MR 2812574
29. Panzo, H.: Spectral upper bound for the torsion function of symmetric stable processes. Proc. Amer. Math. Soc. 150(3), 1241–1255 (2022)
30. Pascu, M.N.: Probabilistic Approaches to Eigenvalue Problems, Ph.D. thesis, University of Connecticut (2001)
31. Pascu, M.N.: Scaling coupling of reflecting Brownian motions and the hot spots problem. Trans. Amer. Math. Soc. 354(11), 4681–4702 (2002). MR 1926894
32. Pleijel, Å.: Remarks on Courant’s nodal line theorem. Comm. Pure Appl. Math. 9, 543–550 (1956). MR 80861
33. Ramasubramanian, S.: Reflecting Brownian motion in a Lipschitz domain and a conditional gauge theorem. Sankhyä Ser. A 63(2), 178–193 (2001). MR 1897449
34. Rauch, J.: Five problems: an introduction to the qualitative theory of partial differential equations, Partial differential equations and related topics (Program, Tulane Univ., New Orleans, La., 1974), Springer-Verlag, Berlin, 1975, pp. 355–369. Lecture Notes in Math., Vol. 446, MR 0509045
35. Siudeja, B.: Hot spots conjecture for a class of acute triangles. Math. Z. 280(3-4), 783–806 (2015). MR 3369351
36. Steinerberger, S.: An upper bound on the Hot Spots constant. To appear in Rev. Mat. Iberoam. arXiv:2106.03677 (2021)
37. Sznitman, A.-S.: Brownian motion, Obstacles and Random Media, Springer Monographs in Mathematics. Springer, Berlin (1998). MR 1717054
38. Vogt, H.: $L_\infty$-estimates for the torsion function and $L_\infty$-growth of semigroups satisfying Gaussian bounds. Potential Anal. 51(1), 37–47 (2019), MR 3981441

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