GROSSLY DETERMINED SOLUTIONS FOR A BOLTZMANN-LIKE EQUATION

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Abstract. In gas dynamics, the connection between the continuum physics model offered by the Navier-Stokes equations and the heat equation and the molecular model offered by the kinetic theory of gases has been understood for some time, especially through the work of Chapman and Enskog, but it has never been established rigorously. This paper established a precise bridge between these two models for a simple linear Boltzmann-like equation. Specifically a special class of solutions, the grossly determined solutions, of this kinetic model are shown to exist and satisfy closed form balance equations representing a class of continuum model solutions.

1. Introduction. The Maxwell–Boltzmann (or Boltzmann) equation models the dynamics of a dilute gas:

\[
\frac{\partial F}{\partial t} + \sum_{j=1}^{3} v_j \frac{\partial F}{\partial x_j} = C(F, F)
\]

where \( C(F, F) \) is the collisions operator. The unknown \( F(t, x, v) \) is the molecular density function of the gas. We require \( F(t, x, v) : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R} \) to be a non-negative integrable function with respect to \( v \). Define \( n(t, x) := \int_{\mathbb{R}^3} F(t, x, v) \, dv \) where \( V = \mathbb{R}^3 \) represents “velocity” space. Then, \( F(t, x, v)/n(t, x) \) is a probability distribution with respect to \( v \). Specifically, we interpret this distribution as the probability of seeing a molecule of velocity \( v \) (in \( \mathbb{R}^3 \)) at position \( x \) (the point \( x \) in \( \mathbb{R}^3 \)) at time \( t \).

The collisions operator \( C(F, F) \) is normally a bilinear integral operator which acts only on the velocity variables \( v \). Different models of intermolecular interaction (often called the encounter problem) yield different forms of \( C(F, F) \), but there is a commonality to all collisions operators in the full theory. Specifically, collisions operators are required to satisfy the properties of conservation of mass, momentum and energy.

In Fundamentals of Maxwell’s Kinetic Theory of a Simple Monotonic Gas [26], C. Truesdell and R. G. Muncaster wrote a text designed to put the Maxwell-Boltzmann
equation on both firm mathematical and historical ground. In the epilogue of the text, the authors discuss what they term the main open problems of kinetic theory. Specifically, they discuss the need for a more detailed existence and uniqueness theory, the impact of the Boltzmann $H$-theorem on the “trend to equilibrium” of a gas, and they discuss a concept of their own invention – grossly determined solutions. In the 35 years since their writing, a great deal has been accomplished in regards to existence theory for both the homogeneous [4, 23, 13, 24] and the inhomogeneous Boltzmann equation [22, 14, 3] under varying assumptions about the collisions operator. Implications of the $H$-theorem also continue to be a great source of scholarly interest. In [8], Cercignani directly addresses the problem of existence as stated in [26] and, in doing so, reframes Truesdell and Muncaster’s question about the $H$-theorem leading to great productivity (see [27, 12]). Until now, the main problem on grossly determined solutions has not received much attention.

In contrast to the Maxwell–Boltzmann equation, the Navier–Stokes equations model the dynamics of a gas via physical fields of the gas:

$$\rho (v_t + v \cdot \nabla v) = \nu \nabla^2 v + \mu \Delta v + \rho f$$

$$\rho_t + (\rho v)_i = 0$$

where $\nu$ (the bulk viscosity), $\mu$ (the shear viscosity), and $f$ (the external force) are given or defined via balance laws and constitutive equations. Here, the unknowns are the velocity and density fields, $v$ and $\rho$. In [26, Ch. XXIII], C. Truesdell and R. G. Muncaster remark that – no matter which model of gas flow you begin with – the ultimate goal is the same: determine the density, velocity and temperature fields of the gas. They then note that many of the known exact solutions of Boltzmann’s equation – such as those solutions derived from Hilbert’s iteration (see [10, pg. 316] or [26, Ch. XXII]), or the Chapman and Enskog procedure (see [19, pg. 86]) – shared the property that the solution class could be represented as being dependent on one (or more) of the gas’s physical properties. This led them to define the concept of a grossly determined solution: a solution which is determined at any given instant by the gross conditions (mass density, velocity, temperature) of the gas at that time. In their epilogue, the authors suggest that these concepts may lead to a new way forward:

1. In general, can we determine a set of conservation laws that define the gross field properties?
2. Can we use these conservation laws to determine the class of grossly determined solutions to the problem?
3. If one could find the class of general solutions, can we show that the general solutions evolve asymptotically in time to the class of grossly determined solutions?

In addition to finding a new, richer class of solutions to the Boltzmann equation (a microscopic/atomic level model of gas flow), the class of grossly determined solutions would now be in terms akin to the solutions of the Navier-Stokes equations (a macroscopic/gross fields model of gas dynamics). In spirit, this type of research is already being done. For example, relaxations and generalizations of the Chapman-Enskog procedure to the Navier-Stokes equations [25] or the Burnett equations [21, 17] are attempting to accomplish the same goal as grossly determined solutions. However, to date, no one has explicitly explored Truesdell and Muncaster’s conjecture.
The goal of this paper is to prove that grossly determined solutions exist for a linearized form of the Boltzmann equation, demonstrating steps (1) and (2) above. In a forthcoming proof-of-concept paper [5], step (3) will be established for the fully one-dimensional version of (2). The following theorem is the main result of this paper.

**Main Theorem.** Consider the model of fluid flow

\[
\frac{\partial f}{\partial t}(t, x, v) + v \cdot \nabla f(t, x, v) = -f(t, x, v) + \int_{\mathbb{R}^3} f(t, x, w) \phi(\omega) \, dw
\]

where \( f(t, x, v) \) is the molecular density function of the gas and \( \phi \) is the probability density function \( \phi(v) := e^{-\gamma v^2/\pi^{3/2}} \). Let \( \rho(t, x) \) represent the density function of the gas:

\[
\rho(t, x) := \int_{\mathbb{R}^3} f(t, x, v) \phi(v) \, dv.
\]

Then a solution to equation (2) is given by

\[
f(t, x, v) = \int_{\mathbb{R}} K_v(y) \rho(t, x - y) \, dy
\]

where \( \rho \) satisfies the associated continuity equation:

\[
\partial_t \rho + \nabla \cdot M = 0
\]

in which \( M = \langle M_1, M_2, M_3 \rangle \) is the mass-flux field

\[
M_i(t, x) = \int_{\mathbb{R}^3} v_i f(t, x, v) \phi(v) \, dv, \ i = 1, 2, 3.
\]

The Fourier transform form of \( \rho \) that solves the continuity equation (4) is

\[
\hat{\rho}(t, \xi) = \hat{\rho}_0(\xi)e^{-ik(\xi)t}
\]

where \( \hat{\rho}(t, \xi) \) has support within \( |\xi| \in (0, \sqrt{\pi}) \) and \( \hat{\rho}_0(\xi) \) denotes the Fourier transform of the density function at \( t = 0 \).

Then the Fourier transform form of the solution to (2) is given by

\[
\hat{f}(t, \xi, v) = \left( 1 + i[(v \cdot \xi) - k(\xi)] \right) \hat{\rho}_0(\xi)e^{-ik(\xi)t}
\]

where \( k(\xi) = (-1 + |\xi|C(|\xi|))i \) and \( c = C(|\xi|) \) is defined implicitly by

\[
|\xi| = \int_{\mathbb{R}} \frac{c\phi(v)}{\epsilon^2 + \nu^2} \, dv.
\]

Section 2 of this paper gives an extremely brief introduction to the Maxwell-Boltzmann equation and the role of balance laws in the kinetic theory. In Section 3, we will justify why the partial integro-differential equation (2) is an appropriate proxy for the full one-dimensional Boltzmann equation. Section 4 derives the class of grossly determined solutions stated in Main Theorem 1.

2. Background.

2.1. **The collisions operator and the summational invariants.** The collisions operator is normally a homogeneous operator of degree 2. (i.e. \( L[u] = \alpha^2 L[u] \)). For an intuition of the structure of \( C(F, F) \), consider two particles \( P \) and \( P^* \) and let \( v \) and \( v^* \) and \( v' \) and \( v' \) be the pre- and post- collision velocities of the particles \( P \) and \( P^* \), respectively. Let \( F(t, x, v) \) be the molecular density function for the gas. For notational convenience, let \( F(v') = F(t, x, v') \), \( F(v'_*) = F(t, x, v'*) \), etc.
We have introduced new unknowns $v'$ and $v'_*$ into our problem. These can be derived from the Encounter Problem [26, Ch. VI], the modeling of the interaction of two particles in otherwise empty space. In this framework, under appropriate assumptions, the encounter problem is akin to solving a two-body problem. Thus, we can interpret $v'$ and $v'_*$ as

$$v' = V'(v, v'_*, s_1, s_2)$$

and

$$v'_* = V'_*(v, v_*, s_1, s_2)$$

where $S = \mathbb{R}^2$ is a parameter space representing the spatial trajectories of the molecules $P$ and $P^*$.

The net increase in the density of molecules of velocity $v$ by collisions is modeled as being proportional to the difference $F(v')F(v'_*) - F(v)F(v_*)$. To ensure that this difference is itself a molecular density function, we modify by an appropriate weight function $w$. This results in the collisions operator

$$C(F, F)(v) = \int_{V_*} \int_{S} w(F(v')F(v'_*) - F(v)F(v_*)) dSd(v_*)$$

(8)

While the derivation of the collisions operator and its properties are rife with motivational and simplifying assumptions, we will take the viewpoint that the following conservation properties are axiomatic.

**Proposition 1. Properties of the Collisions Operator**

1. (conservation of mass condition)

$$\int_{V} C(F, F)d\mathbf{v} = 0$$

2. (conservation of momentum condition)

$$\int_{V} v_i C(F, F)d\mathbf{v} = 0$$

where $v_i$ is any component of the molecular velocity

3. (conservation of energy condition)

$$\int_{V} |\mathbf{v}|^2 C(F, F)d\mathbf{v} = 0$$

where $|\mathbf{v}|^2 = v_1^2 + v_2^2 + v_3^2$ is kinetic energy (modulo a constant).

The quantities $1$, $v_i$, and $|\mathbf{v}|^2$ are called the **summational invariants**. The summational invariant conditions are derived from using $C(F, F)$ and the assumption that the total mass, momentum and energy before a collision are equal to those same quantities after a collision.

Equipped with the above conservation properties, the collisions operator has another additional characteristic.

**Proposition 2.** $C(F, F) = 0$ if and only if $F$ is a Maxwellian (normal) distribution.

2.2. **Balance equations / conservation laws derived from the Boltzmann equation.** In the classical theory, the summational invariants of the collisions operator are used to derive the balance equations associated with continuum fluid dynamics. Here, the Boltzmann equation is converted into a system of PDEs that are dependent upon the gross field properties of the gas.

Recall that $F(t, x, \mathbf{v})$ is a non-normalized, probability distribution with respect to $\mathbf{v}$. From this, we establish the gross (physical) properties of density, momentum (velocity) and energy. Let $m$ be the molecular mass. Then

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1The interaction of two particles need not be dependent on the pre- and post- velocities alone. For example, in a finer model, molecules may be assumed to be non-spheres and the interaction between two molecules will now depend upon spatial orientation in addition to position. See [26, Ch. VI].
1. the density function (0th moment): $\rho(t, x) = \int_V mF(t, x, v)dv$

2. the $i$th component of the momentum (1st moment):
   $$\bar{v}_i(t, x)\rho(t, x) = \int_V mv_i F(t, x, v)dv$$

3. the energy function (contracted 2nd moment):
   $$e(t, x)\rho(t, x) = \int_V \frac{m|v|^2}{2} F(t, x, v)dv.$$

Now, beginning with the Boltzmann Equation
   $$\frac{\partial F}{\partial t} + 3\sum_{j=1}^{3} v_j \frac{\partial F}{\partial x_j} = C(F, F)$$
we use the moments to derive the field equations.

**Proposition 3. The Balance Equations**

1. (the continuity equation)
   $$\frac{\partial \rho}{\partial t} + \sum_{j=1}^{3} \frac{\partial}{\partial x_j} (\rho(t, x)\bar{v}_i(t, x)) = 0$$

2. $\frac{\partial}{\partial t} (\rho(t, x)\bar{v}_i(t, x)) + \sum_{j=1}^{3} \frac{\partial}{\partial x_j} (P_{ij}(t, x)) = 0$ where $P_{ij}(t, x) = \int_V mv_iv_j Fdv$

3. $\frac{\partial}{\partial t} (e(t, x)\rho(t, x)) + \sum_{j=1}^{3} \frac{\partial}{\partial x_j} (T_j) = 0$ where $T_j(x, t) = \int_V \frac{m|v|^2}{2} v_j Fdv.$

**Proof.** We include the proof of the continuity equation to motivate some of the computations in the following section. The others are unimportant to this paper and are omitted.

To derive the continuity equation, multiply the Boltzmann Equation by the constant $m$. Integrate over the velocity space $V$:

$$\int_V m\frac{\partial F}{\partial t} dv + \int_V \sum_{j=1}^{3} mv_j \frac{\partial F}{\partial x_j} dv = \int_V mC(F, F)dv$$

$$\frac{\partial}{\partial t} \left( \int_V mFdv \right) + \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left( \int_V mv_j Fdv \right) = m \int_V (1)C(F, F)dv.$$

By derivation of the density function above and properties of the collision condition, we obtain

$$\frac{\partial \rho}{\partial t} + \sum_{j=1}^{3} \frac{\partial}{\partial x_j} (\rho(t, x)\bar{v}_j(t, x)) = 0.$$

The balance equations have introduced new unknown functions. The term $P = [P_{ij}]$ in balance equation (2) is called the stress tensor. In traditional kinetic theory of gas texts (versus elasticity), this term is called the pressure tensor. (The pressure tensor is the negative of the stress tensor.) Similarly, one can interpret the function $T = (T_1, T_2, T_3)$ as an energy flux vector. In the classical theory, assumptions are now made about the gas with the goal of representing these tensors back in terms of density, momentum and energy (i.e. constitutive relations). In other words, the system of PDEs that comprise the balance laws are now a closed system in terms...
of the density, momentum and energy functions. The ultimate goal of this exercise is that we now hope that this new system of PDEs in the gross fields alone are solvable via classical PDE methods.

3. Derivation of an approximation of the Boltzmann equation.

3.1. Approximating the collisions operator. We begin with the Maxwell–Boltzmann equation
\[ \partial_t F + v \cdot \nabla F = C(F, F) \]
where \( F(t, x, v) : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R} \). We seek to replace \( C \) with a term \( \tilde{C} \) that simplifies the equation, but still retains some of the basic characteristics of the full collisions operator.

In Truesdell and Muncaster’s text [26, Ch. VII], alternative forms of the collisions operator are explored. We first note that the collisions operator can be written more generally as a symmetric bilinear operator:
\[ C(G, H)(v) = \frac{1}{2} \int_{V_*} \int_S w[G'(v')H(v'') + G(v'')H'(v') - G(v')H(v'') - G(v'H(v))]dSdv_* \]
Or, more simply denoted,
\[ C(G, H)(v) = \frac{1}{2} \int_{V_*} \int_S w[G'_*H'_* + G'_*H' - G_*H_* - G_*H]dSdv_* \quad (9) \]
where \( G \) and \( H \) are any functions such that the integral is finite. Note that if we let \( F = G = H \), then the above simplifies to equation (8), the original collisions operator.

Akin to the traditional linearization technique (see [15]), we perturb a solution \( F \) about a Maxwellian density function. Let \( \phi(v) \) be a uniform Maxwellian (normal) distribution. Note our choice of \( \phi \) is independent of \( t \) and \( x \). Define the function
\[ F_\epsilon(t, x, v) := \phi(v)(1 + \epsilon f(t, x, v)), \epsilon > 0. \]
The function \( F_\epsilon \) can be interpreted as a slight deviation from the equilibrium solution \( \phi(v) \). Requiring \( F_\epsilon \) to be a solution to the Boltzmann equation, consider the action of \( C \) on \( F_\epsilon \):
\[ C(F_\epsilon, F_\epsilon) = C(\phi + \epsilon \phi f, \phi + \epsilon \phi f) = C(\phi, \phi) + \epsilon C(\phi, \phi f) + \epsilon C(\phi f, \phi) + \epsilon^2 C(\phi f, \phi f) \]
by the bilinearity of \( C \).

Since \( C(\phi, \phi) = 0 \) (because \( \phi \) is Maxwellian) and \( C(\phi f, \phi) = C(\phi, \phi f) \) (by symmetry of \( C \)),
\[ C(F_\epsilon, F_\epsilon) = 2\epsilon C(\phi, \phi f) + O(\epsilon^2). \]
Substituting \( F_\epsilon \) into the rest of the one dimensional Maxwell-Boltzmann equation leads one to consider the Boltzmann equation at first order
\[ \phi f_t + v \cdot \nabla (\phi f) = 2C(\phi, \phi f). \]
Using equation (9),
\[ 2C(\phi, \phi f) = \int_{V_*} \int_S w(\phi'(\phi f)'+\phi'(\phi f)' - \phi(\phi f)_* - \phi(\phi f))dSdv_* \]
\[ = -\phi(\phi f) \int_{V_*} \int_S w\phi_* dSdv_* - \phi \int_{V_*} \int_S w(\phi f)_* dSdv_* \]
and the Boltzmann equation at first order becomes

\[ \phi_f + \phi (v \cdot \nabla f) = - (\phi f) \int_{V_s} w(\phi f) dS d\mathbf{v}_s - \phi f \int_{S} w(\phi f) dS d\mathbf{v}_s \\
+ \int_{V_s} w(\phi f)_s + \phi_f (\phi f)' dS d\mathbf{v}_s. \]

(10)

Observe that if we drop the final term of the perturbed collisions operator in (10), we are left with a PIDE that will no longer satisfy all five conservation conditions of the full Boltzmann. However, it still satisfies the conservation of mass condition.

**Proposition 4.** Let \( \phi(v) \) be a Maxwellian (normal) distribution such that

\[ \int_{\mathbb{R}^3} \phi(v) dv = 1. \]

Consider a collisions operator of the form

\[ \tilde{C}(f) := -\phi(v)f(t, x, v) + \phi(v) \int_{V_s} \phi(v_s)f(t, x, v_s) d\mathbf{v}_s. \]

Then \( \tilde{C}(f)(v) \) satisfies the conservation of mass condition required of a Maxwell–Boltzmann collisions operator.

**Proof.**

\[
\int_{V} \tilde{C}(f) dv = -\int_{V} \phi(v)f(t, x, v) dv + \int_{V} \phi(v) dv \left[ \int_{V_s} \phi(v_s)f(t, x, v_s) d\mathbf{v}_s \right] \\
= -\int_{\mathbb{R}^3} \phi(v)f(t, x, v) dv + \int_{\mathbb{R}^3} \phi(v_s)f(t, x, v_s) d\mathbf{v}_s \quad \text{since } \phi \text{ is Maxwellian} \\
= 0.
\]

Again, we note that by disposing of the term \( \frac{1}{\phi} \int_{V_s} w(\phi f)' + \phi'_f (\phi f)' dS d\mathbf{v}_s \), we have removed the need to solve the associated two-body problem. In other words, while we will show that the operator \( \tilde{C}(f) \) has many of the important properties of the full collisions operator, we have essentially removed any “proper” collisions from this model.

Replacing the righthand side of (10) by \( \tilde{C}(f) \) results in the equation

\[ \phi(v)f_t(t, x, v) + \phi(v)(v \cdot \nabla f(t, x, v)) = -\phi(v)f(t, x, v) + \phi(v) \int_{V_s} \phi(v_s)f(t, x, v_s) d\mathbf{v}_s. \]

(12)

Since \( \phi(v) \neq 0 \) on all of \( \mathbb{R}^3 \), we can simplify further and state the final form of the model we will work with for the remainder of the paper.

**3.2. An approximation of the Boltzmann equation where the collisions operator conserves mass only.** Let \( x \in \mathbb{R}^3 \) represent the position of a molecule and let \( v \in \mathbb{R}^3 \) be the velocity of that molecule. Then the molecular density function \( f(t, x, v) \) is taken to satisfy the equation

\[ f_t(t, x, v) + v \cdot \nabla f(t, x, v) = -f(t, x, v) + \int_{\mathbb{R}^3} f(t, x, v) \phi(v) dv \]

(13)

where \( \phi(v) \) is the probability density function \( \phi(v) = e^{-v \cdot v/\pi^{3/2}}. \)
We should take a moment to point out the (13) is not entirely unique in the literature. Cercignani derived the one dimensional version \((x,v)\in\mathbb{R}, \phi(v) = e^{-v^2/\sqrt{\pi}}\) of (13) in splitting the one dimensional (in \(x\)) BGK equation. (See [9], [11], [7], or [18], for a more modern discussion.) In his work, Cercignani derived a representation of the general solution to the one dimensional version of (13) by extending Case's Method of Elementary Solutions [6].

While ideally we would be studying a problem with a more robust collisions operator, the extreme simplification employed above may make solutions of (13) applicable to a larger range of fields. For example, the model is close to the radiative transfer equation (see [1], [2]), in which the positive part of the collisions integral has been substituted by \(\int_{\mathbb{R}^3} f(t,x,v)\phi(v)\,dv\). The same kind of argument could be made for any entropy-dissipating kinetic model, such as Fokker-Planck-type equations, or simple models for granular media (see [16]).

3.3. Properties of \(C\). For the rest of this paper, we will be working with the simplified partial integro-differential equation (PIDE) (13). In keeping with the traditional approach, we need to understand the right-hand side of (13) as a collisions operator. Define \(C(f)\) as

\[
C(f) := -f(t,x,v) + \int_{\mathbb{R}^3} f(t,x,v)\phi(v)\,dv.
\]

In order to retain the conservation of mass condition, Proposition 4, our future work will require that we work with the weighted \(L_2\) inner product

\[
\langle f(v), g(v) \rangle_\phi := \int_{\mathbb{R}^3} f(v)g(v)\phi(v)\,dv.
\]

In this notation Proposition 4 takes the form

\[
\langle C(f), 1 \rangle_\phi = \int_{\mathbb{R}^3} \tilde{C}(f)\,dv = 0.
\]

Proposition 5. Properties of \(C\).

Let \(Cf := C(f)\) be the linear operator defined as in (14). Consider the variables \(t\) and \(x\) as fixed suppressed parameters and consider \(C(f)(v) := C(f)\) as an operator in the variable \(v\). Let \(\mathcal{F}\) be the class of functions such that

\[
\|f(v)\|_{2,\phi}^2 = \int_{\mathbb{R}^3} |f(v)|^2\phi(v)\,dv < \infty.
\]

1. If \(f \in \mathcal{F}\), then \(f(v)\phi(v)\) is \(L_1(\mathbb{R})\),
2. \(C(f) = 0\) if and only if \(f(v)\) is a constant.
3. \(C\) is a bounded self-adjoint operator; \(\langle Cf, g \rangle_\phi = \langle f, Cg \rangle_\phi\).
4. \(C\) is negative semi-definite; \(\langle f, Cf \rangle_\phi \leq 0\) for all real-valued \(f \in \mathcal{F}\). Additionally, \((f,Cf)_\phi = 0\) if and only if \(f\) is a constant.

Proof. Recall that \(\phi(v) = e^{-v^2/\sqrt{\pi}}\). Note that \(\phi^{1/2}(v) \in L_2(\mathbb{R})\) and that \(\|\phi^{1/2}(v)\|_2 = 1\). Then

\[
\|\phi^{1/2}(v)f(v)\|_1 = \|\phi^{1/2}(v)f^{1/2}(v)f(v)\|_1 \leq \|\phi^{1/2}(v)\|_2 \|f^{1/2}(v)f(v)\|_2 \quad \text{by Hölder’s inequality}
\]

\[
\leq \|f(v)\|_{2,\phi} \quad \text{by definition of } \mathcal{F}
\]

\(< \infty.\)
2. Let $C(f) = 0$. Then $f(\mathbf{v}) = \int_{\mathbb{R}^3} \phi(\mathbf{w}) f(\mathbf{w}) \, d\mathbf{w}$. Since $\phi(\mathbf{w}) f(\mathbf{w})$ is $L_1(\mathbb{R}^3)$, $f(\mathbf{v})$ must be a constant.

If $f(\mathbf{v})$ is constant, $C(f)(\mathbf{v}) = 0$ since $\int_{\mathbb{R}^3} \phi(\mathbf{v}) \, d\mathbf{v} = 1$.

3. First we will show that $C$ is a bounded operator on $\mathfrak{F}$.

$$|C f(\mathbf{v})| \leq |f(\mathbf{v})| + \int_{\mathbb{R}^3} |\phi(\mathbf{w}) f(\mathbf{w})| \, d\mathbf{w}$$

Then,

$$|C f(\mathbf{v})|^2 \leq |f(\mathbf{v})|^2 + 2|f(\mathbf{v})| \|f(\mathbf{v})\|_{2,\phi} + \|f(\mathbf{v})\|_{2,\phi}$$

and

$$\|C f(\mathbf{v})\|_{2,\phi}^2 = \int_{\mathbb{R}^3} |C f(\mathbf{v})|^2 \phi(\mathbf{v}) \, d\mathbf{v}$$

$$\leq \int_{\mathbb{R}^3} (|f(\mathbf{v})|^2 \phi(\mathbf{v}) + 2|f(\mathbf{v})| \|f(\mathbf{v})\|_{2,\phi} + \|f(\mathbf{v})\|_{2,\phi} \phi(\mathbf{v})) \, d\mathbf{v}$$

$$\leq \|f(\mathbf{v})\|_{2,\phi}^2 + 2\|f(\mathbf{v})\|_{2,\phi}^2 + \|f(\mathbf{v})\|_{2,\phi}^2$$

$$= 4\|f(\mathbf{v})\|_{2,\phi}^2.$$

Proving $C$ is self-adjoint is simply definition chasing:

$$\langle C f, g \rangle_\phi = \int_{\mathbb{R}^3} \left( -f(\mathbf{v}) + \int_{\mathbb{R}^3} f(\mathbf{w}) \phi(\mathbf{w}) \, d\mathbf{w} \right) g(\mathbf{v}) \phi(\mathbf{v}) \, d\mathbf{v}$$

$$= \int_{\mathbb{R}^3} \left( -f(\mathbf{v}) g(\mathbf{v}) \phi(\mathbf{v}) \right) \, d\mathbf{v} + \left( \int_{\mathbb{R}^3} f(\mathbf{w}) \phi(\mathbf{w}) \, d\mathbf{w} \right) \left( \int_{\mathbb{R}^3} g(\mathbf{v}) \phi(\mathbf{v}) \, d\mathbf{v} \right)$$

$$= \int_{\mathbb{R}^3} \left( -f(\mathbf{v}) g(\mathbf{v}) \phi(\mathbf{v}) \right) \, d\mathbf{v} + \left( \int_{\mathbb{R}^3} f(\mathbf{v}) \phi(\mathbf{v}) \, d\mathbf{v} \right)$$

$$= \int_{\mathbb{R}^3} f(\mathbf{v}) \left( -g(\mathbf{v}) + \int_{\mathbb{R}^3} g(\mathbf{w}) \phi(\mathbf{w}) \, d\mathbf{w} \right) \, d\mathbf{v}$$

$$= \langle f, C g \rangle_\phi.$$

4.

$$\langle C(f), f \rangle_\phi = \int_{\mathbb{R}^3} \left( -f(\mathbf{v}) + \int_{\mathbb{R}^3} \phi(\mathbf{v}) f(\mathbf{v}) \, d\mathbf{v} \right) f(\mathbf{v}) \phi(\mathbf{v}) \, d\mathbf{v}$$

$$= -\int_{\mathbb{R}^3} f^2(\mathbf{v}) \phi(\mathbf{v}) \, d\mathbf{v} + \left[ \int_{\mathbb{R}^3} f(\mathbf{v}) \phi(\mathbf{v}) \, d\mathbf{v} \right]^2.$$

To finish the proof, we interpret $\phi(\mathbf{v}) \, d\mathbf{v}$ as a probability measure over $\mathbb{R}^3$ and apply Jensen’s inequality via the convex function $x^2$. Hence

$$\langle C(f), f \rangle_\phi \leq -\int_{\mathbb{R}^3} f^2(\mathbf{v}) \phi(\mathbf{v}) \, d\mathbf{v} + \int_{\mathbb{R}^3} f^2(\mathbf{v}) \phi(\mathbf{v}) \, d\mathbf{v} = 0$$

While the majority of the properties of $C$ are not explicitly used in this paper, they are included to give credence to the model (2). The first three properties demonstrate that the new collisions operator retains much of the algebraic structure of the full Boltzmann collisions operator. On the other hand, Property 4 is about guaranteeing an irreversible process and a trend to equilibrium solutions in time. That is, Property 4 is equivalent to the Boltzmann $H$-theorem ([8], [16, Ch 2C]) for
this model. Without the negative semi-definiteness of $C$, we would not be able to make the claim of Step 3 in the conjecture of Truesdell and Muncaster – that the class of grossly determined solutions acts as the attracting set of solutions in time evolution – for this model.

4. The space of grossly determined solutions.

4.1. Introduction. In the full kinetic theory each solution of the Maxwell – Boltzmann equation leads immediately to a collection of fields that satisfy the five balance laws, Proposition 3. In classical gas dynamics one wishes to solve the five balance laws for the gross condition of the gas (density, momentum and energy) without any appeal to the kinetic theory. Solving the balance laws directly, however, is impossible as we have introduced additional unknown functions (the pressure tensor $P$ and the energy flux vector $T$). The goal of some classical iterative solution constructions (for example, the Chapman–Enskog procedure) has been to convert these new unknowns into functions of the gross condition of the gas and thereby “close” the balance laws and create PDEs that must be solved. Our goal here is similar, but at the level of the Maxwell–Boltzmann equation rather than at the level of the balance laws. Specifically one might hope to find a class of solutions for the molecular density $F$, the grossly determined solutions (GDS), that are completely determined by their own gross fields. For this class, then, $P$ and $T$ are functions of the gross fields and then the balance laws become a well defined system of PDEs that we can identify with classical gas dynamics.

We endeavor to accomplish this goal for our model Boltzmann

\[
\frac{\partial f}{\partial t}(t, x, v) + v \cdot \nabla f(t, x, v) = -f(t, x, v) + \int_{\mathbb{R}^3} f(t, x, w) \phi(w) \, dw \tag{16}
\]

where $\phi(w)$ is the probability density function $\phi(v) := e^{-v \cdot v / \pi^{3/2}}$ (i.e. $\int_{\mathbb{R}^3} \phi(v) \, dv = 1$). That is, we will search for a set of grossly determined solutions for our simplified problem that represent a “classical” theory of gas dynamics embedded in our “kinetic” theory of gases.

4.2. Derivation of the continuity equation. By construction, we can define only one gross field. The mass-density is

\[
\rho(t, x) = m \int_{\mathbb{R}^3} f(t, x, v) \phi(v) \, dv.
\]

For simplicity we let $m = 1$ and define the density function $\rho(t, x)$:

\[
\rho(t, x) := \int_{\mathbb{R}^3} f(t, x, v) \phi(v) \, dv.
\]

As a result of the one gross field, we do not expect to be able to derive more than one balance law.

**Proposition 6.** The associated continuity equation is

\[
\partial_t \rho + \nabla \cdot M = 0 \tag{17}
\]

where $M = \langle M_1, M_2, M_3 \rangle$ is the vector-field

\[
M_i(t, x) = \int_{\mathbb{R}^3} v_i f(t, x, v) \phi(v) \, dv, \quad i = 1, 2, 3. \tag{18}
\]
Proof. By the definition of $\rho(t, x)$, we see that the model equation can be written

$$\partial_t f(t, x, v) + v \cdot \nabla f(t, x, v) = -f(t, x, v) + \rho(t, x).$$

To complete the proof, we integrate the model equation over the velocity space with respect to the $\phi$-weight and pull the derivatives out of the respective integrals. Note that, by construction of the collisions operator,

$$\int_{\mathbb{R}^3} [-f(t, x, v) + \rho(t, x)] \phi(v) dv = 0.$$

Similarly, since $f$ is $C^1$ in $t$, we have that

$$\int_{\mathbb{R}^3} \partial_t f(t, x, v) \phi(v) dv = \partial_t \left( \int_{\mathbb{R}^3} f(t, x, v) \phi(v) dv \right) = \partial_t \rho.$$

The mass-flux field $M$ is derived the same way.

$$\int_{\mathbb{R}^3} v \cdot \nabla f(t, x, v) \phi(v) dv = \sum \int_{\mathbb{R}^3} v_i f_{x_i}(t, x, v) \phi(v) dv$$

$$= \sum \partial_{x_i} \left( \int_{\mathbb{R}^3} v_i f(t, x, v) \phi(v) dv \right)$$

$$= \nabla \cdot M(t, x).$$

As we had in the traditional theory, a new unknown function $M$ has been added to the system. However, if we can describe $M$ as a function of $\rho$, then this will “close” the continuity equation in $\rho(t, x)$ and lead to the class of grossly determined solutions.

4.3. Derivation of the grossly determined solutions.

4.3.1. Observations and assumptions on the form of the GDS. For this problem, there is only one gross field property – mass density. In this setting, the question posited by Truesdell and Muncaster is “Could there be a special class of solutions of (16), each determined in some way by their own density field $\rho$?”

Assume that a solution $f$ is dependent on the density field $\rho(t, x)$. That is, $f(t, x, v) = G[\rho(t, x)](x, v)$. Then the mass-flux field is now implicitly a function of $\rho$,

$$M_i(t, x) = \int_{\mathbb{R}^3} v_i G[\rho(t, x)](x, v) \phi(v) dv.$$

Assuming we can ultimately solve for $G$, the continuity equation becomes a closed PDE in $\rho$ alone. Moreover, if we are able to determine $G$, we should be able to solve this PDE. Additionally, the gross field property can now be written

$$\rho(t, x) = \int_{\mathbb{R}} \phi(v) G[\rho(t, x)](x, v) dv$$

for all $\rho$. We now look for a way to find (or approximate) $G$.

By self-similarity conditions, since (16) is autonomous in $x$ (and $t$), one expects solutions $f$ to be invariant with respect to translations in $x$. Additionally, since the original problem is a linear PIDE, there is no harm in hoping to find solutions in which $G$ is linear in $\rho$. In Hörmander’s Linear Partial Differential Operators [20, pg 15], he proves an interesting representation theorem for linear maps of distributions:
Lemma 4.1. Let $U$ be a linear mapping of $C_0^\infty(\mathbb{R}^n)$ into $C^\infty(\mathbb{R}^n)$ which commutes with translations and is continuous in the sense that $U\psi_j \to 0$ in $C_0^\infty(\mathbb{R}^n)$ if the sequence $\psi_j \to 0$ in $C_0^\infty(\mathbb{R}^n)$. Then there exists one and only one distribution $u$ such that $U\psi = u * \psi$, $\psi \in C_0^\infty(\mathbb{R}^n)$.

Again, we have the freedom to create a solution (dependent on $\rho$) by any means necessary. As we are already embracing an ansatz, we will assume that “$G$ is continuous at zero”. In $G$’s current form, it is dependent on $x$ and $v$. If we can show that $G[\rho(t, \circ)](x, v)$ is invariant in $x$, then the lemma suggests we should look for grossly determined solutions $f$ that are convolutions with $\rho$.

Proposition 7. If a solution of the form $f(t, x, v) = G[\rho(t, \circ)](x, v)$ is invariant in the spacial dimension, then it can be written in the form $f(t, x, v) = G[\rho(t, x + \circ)](0, v)$. In other words, “the translation of a grossly determined solution yields another grossly determined solution” implies that the solution has the form $f(t, x, v) = G[\rho(t, x + \circ)](0, v)$.

Proof. Let $f(t, x, v) = G[\rho(t, \circ)](x, v)$. For fixed $y$, assume that $f(t, x + y, v)$ is another solution in this class. Then $f(t, x + y, v) = G[\rho_y(t, \circ)](x, v)$ for some different density field $\rho_y$. What is the connection between $\rho_y$ and $\rho$? We have

$$\rho(t, x) = \int_{\mathbb{R}^3} G[\rho(t, \circ)](x, v) \phi(v) \, dv = \int_{\mathbb{R}^3} f(t, x, v) \phi(v) \, dv.$$  

Then

$$\rho_y(t, x) = \int_{\mathbb{R}^3} G[\rho_y(t, \circ)](x, v) \phi(v) \, dv$$

$$= \int_{\mathbb{R}^3} \phi(v) f(t, x + y, v) \phi(v) \, dv$$

$$= \rho(t, x + y).$$

So, $f(t, x + y, v) = G[\rho(t, \circ + y)](x, v)$. Redefining the variables, we let $x = 0$ and $y = x$. Then

$$f(t, x, v) = G[\rho(t, x + \circ)](0, v).$$

Thus, by Hörmander’s lemma, $f$ is a convolution and can be represented in the form:

$$f(t, x, v) = \int_{\mathbb{R}^3} K_\nu(y) \rho(t, x - y) \, dy. \quad (19)$$

While in this context, $K_\nu(y)$ is being interpreted as the kernel in the spacial dimension, we use the notation $K_\nu$ to remember that this portion of the solution will also be dependent on velocity.

4.3.2. Solving for the kernel $K_\nu(y)$. Assume that $f(t, x, v) = \int_{\mathbb{R}^3} K_\nu(y) \rho(t, x - y) \, dy$ and substitute $f$ into (16). Both the convolution solution form of $f$ and the transport term $v \cdot \nabla f$ suggest that we should take the Fourier transform of the PIDE with respect to the spacial variable. Define

$$\hat{g}(t, \xi, v) := \int_{\mathbb{R}^3} e^{-i(\xi \cdot x)} g(t, x, v) \, dx.$$
When we apply the Fourier transform to the PIDE, with the convolution solution inserted, it yields
\[ \hat{K}_v(\xi) \frac{\partial \hat{\rho}}{\partial t}(t, \xi) + i(v \cdot \xi) \hat{K}_v(\xi) \hat{\rho}(t, \xi) = -\hat{K}_v(\xi) \hat{\rho}(t, \xi) + \hat{\rho}(t, \xi). \]  
(20)

Additionally, we can transform the gross field property. Using the convolution solution, the density becomes
\[ \rho(t, x) = \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} K_v(y) \rho(t, x - y) \, dy \right) \phi(v) \, dv. \]

Under the transform, we get
\[ \hat{\rho}(t, \xi) = \int_{\mathbb{R}^3} \hat{K}_v(\xi) \hat{\rho}(t, \xi) \phi(v) \, dv. \]
Or,
\[ \hat{\rho}(t, \xi) \left( 1 - \int_{\mathbb{R}^3} \hat{K}_v(\xi) \phi(v) \, dv \right) = 0. \]  
(21)

Upon the support of \( \hat{\rho}(t, \xi) \), equation (21) requires that
\[ \int_{\mathbb{R}^3} \hat{K}_v(\xi) \phi(v) \, dv = 1. \]  
(22)

Note that (22) shows the first constraint that the transformed kernel satisfy. Namely \( \hat{K}_v(\xi) \) must satisfy a normalization condition.

Lastly, we need the transform of the continuity equation (17) under the convolution solution assumption. We have
\[ \partial_t \rho + \nabla \cdot M = 0 \]
where the mass-flux field is now
\[ M_i(t, x) = \int_{\mathbb{R}^3} v_i \left( \int_{\mathbb{R}^3} K_v(y) \rho(t, x - y) \, dy \right) \phi(v) \, dv. \]

Under the transform, \( \nabla \cdot M \) becomes
\[ \hat{\nabla} \cdot M = \sum \int_{\mathbb{R}^3} v_i \left( i \xi_i \hat{K}_v(\xi) \hat{\rho}(t, \xi) \right) \phi(v) \, dv \]
\[ = i \hat{\rho}(t, \xi) \sum \int_{\mathbb{R}^3} v_i \xi_i \hat{K}_v(\xi) \phi(v) \, dv \]
\[ = i \hat{\rho}(t, \xi) \int_{\mathbb{R}^3} (\xi \cdot v) \hat{K}_v(\xi) \phi(v) \, dv. \]

Then the transformed continuity equation is
\[ \partial_t \hat{\rho}(t, \xi) + i \hat{\rho}(t, \xi) \int_{\mathbb{R}^3} (\xi \cdot v) \hat{K}_v(\xi) \phi(v) \, dv = 0. \]

To simplify this expression a bit, define
\[ k(\xi) := \int_{\mathbb{R}^3} (\xi \cdot v) \hat{K}_v(\xi) \phi(v) \, dv. \]  
(23)

Then the continuity equation can be written
\[ \frac{\partial \hat{\rho}}{\partial t}(t, \xi) + ik(\xi) \hat{\rho}(t, \xi) = 0 \]  
(24)
and we see immediately that, up to determining \( \hat{K}_v(\xi) \), we have a separable ODE in \( \hat{\rho} \). Specifically, given an initial density condition \( \rho(0, \xi) \), we see that the transformed representation of \( \hat{\rho}(t, x) \) is
\[
\hat{\rho}(t, \xi) = \hat{\rho}_0(\xi)e^{-ik(\xi)t}
\]
where \( \hat{\rho}_0(\xi) := \hat{\rho}(0, \xi) \).

We now use (24) to determine a second representation of \( \hat{K}_v(\xi) \), this one in terms of \( k(\xi) \). Solving for \( \partial_t \hat{\rho}(t, \xi) \), we can now write the transformed PIDE (20) independently of any time derivative.

\[
\hat{K}_v(\xi)(-ik(\xi)\hat{\rho}(t, \xi)) + i(v \cdot \xi)\hat{K}_v(\xi)\hat{\rho}(t, \xi) = -\hat{K}_v(\xi)\hat{\rho}(t, \xi) + \hat{\rho}(t, \xi).
\]

Again, on the support of \( \hat{\rho}(t, \xi) \), we can simplify to
\[
\hat{K}_v(\xi)(-ik(\xi)) + i(v \cdot \xi)\hat{K}_v(\xi) = -\hat{K}_v(\xi) + 1.
\]

This results in a representation of \( \hat{K}_v(\xi) \) in terms of \( k(\xi) \),
\[
\hat{K}_v(\xi) = \frac{1}{1 + i[(v \cdot \xi) - k(\xi)]}.
\]

Moreover, apart from knowing \( k \), we have an explicit form of \( \hat{K}_v \) in the variable \( v \) alone.

We seem to have found a second constraint condition (25) required of \( \hat{K}_v(\xi) \) in addition to the normalization condition (22). Additionally, the recursive nature of (25) looks problematic. Recall, by (23), \( k(\xi) \) is defined using \( \hat{K}_v(\xi) \). I claim that the two conditions are equivalent. In fact, this equivalence is at the heart of why we can ultimately solve for \( \hat{K}_v(\xi) \).

**Lemma 4.2.** Let \( \hat{K}_v(\xi) \) be defined recursively as in (23) and (25). Then
\[
\int_{\mathbb{R}^3} \hat{K}_v(\xi)\phi(v) \, dv = 1.
\]

**Proof.** Consider \( \int_{\mathbb{R}^3} \hat{K}_v(\xi)\phi(v) \, dv \) where \( \hat{K}_v(\xi) \) is defined as in (25).

\[
\int_{\mathbb{R}^3} \hat{K}_v(\xi)\phi(v) \, dv = \int_{\mathbb{R}^3} \frac{\phi(v) \, dv}{1 + i[(v \cdot \xi) - k(\xi)]}
\]
\[
= \int_{\mathbb{R}^3} \frac{\phi(v)(1 + i[(v \cdot \xi) - k(\xi)]) - \phi(v)(i[(v \cdot \xi) - k(\xi)]) \, dv}{1 + i[(v \cdot \xi) - k(\xi)]}
\]
\[
= 1 - i \int_{\mathbb{R}^3} \frac{\phi(v)[(v \cdot \xi) - k(\xi)] \, dv}{1 + i[(v \cdot \xi) - k(\xi)]}
\]
\[
= 1 - i \left( \int_{\mathbb{R}^3} \frac{(v \cdot \xi)\phi(v) \, dv}{1 + i[(v \cdot \xi) - k(\xi)]} - \int_{\mathbb{R}^3} \frac{k(\xi)\phi(v) \, dv}{1 + i[(v \cdot \xi) - k(\xi)]} \right)
\]
\[
= 1 - i \left( k(\xi) - k(\xi) \int_{\mathbb{R}^3} \hat{K}_v(\xi)\phi(v) \, dv \right),
\]

again showing that we require \( \int_{\mathbb{R}^3} \hat{K}_v(\xi)\phi(v) \, dv = 1 \).

Combining (23) and (25) we find a representation of \( k(\xi) \) that suppresses \( \hat{K}_v \):
\[
k(\xi) = \int_{\mathbb{R}^3} \frac{v \cdot \xi}{1 + i[(v \cdot \xi) - k(\xi)]} \phi(v) \, dv.
\]

(26)
At this point, we are going to make what appears to be a second ansatz. We are going to assume that $k(\xi)$ is purely imaginary. This conjecture is motivated from the work of Cercignani. In [11], a form of the general solution of the one dimensional version of (16) was found by taking the Laplace transform in the time-dimension and then looking for separable solutions. In that work, the separation parameters (or generalized eigenvalues) were shown to have two discrete spectral points and the continuous spectrum of $\mathbb{R}$. The goal of this current paper is not to determine the general solution, but to demonstrate that the subclass of grossly determine solutions exist. Here, under the Fourier transform, the continuous spectrum $\mathbb{R}$ is equivalent to the imaginary axis in $\mathbb{C}$. We proceed with the assumption that $k(\xi) = ai$, for some real-valued function $a(\xi)$. Then (26) becomes

$$ai = \int_{\mathbb{R}^3} \frac{v \cdot \xi}{(1 + a) + i(v \cdot \xi)} \phi(v) \, dv$$

$$= \int_{\mathbb{R}^3} \frac{(v \cdot \xi)[(1 + a) - i(v \cdot \xi)]}{(1 + a)^2 + (v \cdot \xi)^2} \phi(v) \, dv$$

$$(1 + a) \int_{\mathbb{R}^3} \frac{v \cdot \xi}{(1 + a)^2 + (v \cdot \xi)^2} \phi(v) \, dv - i \int_{\mathbb{R}^3} \frac{(v \cdot \xi)^2}{(1 + a)^2 + (v \cdot \xi)^2} \phi(v) \, dv.$$

For our ansatz on $k(\xi)$ to hold, we need to demonstrate that the integral associated with the real-part of the last expression is zero. Demonstrating this simply requires a well-chosen change of variables. We want a change of variables $v = Hv$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that

$$v \cdot \xi = g(\xi) w_1 \quad \text{and} \quad v \cdot v = w_1^2 + h(w_2, w_3, \xi).$$

As we will need to compute $v \cdot v$ under the change of variable and

$$v \cdot v = Hw \cdot Hw = w \cdot H^THw,$$

we are lead to consider the construction of $H$ as an orthogonal matrix $Q$. After experimentation, we see that the matrix

$$Q = \begin{bmatrix}
\xi_1 & \xi_2 & \xi_3 \\
\frac{\xi_1}{\sqrt{\xi_1^2 + \xi_2^2}} & -\frac{\xi_2}{\sqrt{\xi_1^2 + \xi_2^2}} & \frac{\xi_3}{\sqrt{\xi_1^2 + \xi_2^2}} \\
\frac{\xi_2}{\sqrt{\xi_1^2 + \xi_2^2}} & \frac{\xi_1}{\sqrt{\xi_1^2 + \xi_2^2}} & \frac{\xi_3}{\sqrt{\xi_1^2 + \xi_2^2}} \\
\frac{\xi_3}{\sqrt{\xi_1^2 + \xi_2^2}} & \frac{\xi_2}{\sqrt{\xi_1^2 + \xi_2^2}} & \frac{\xi_1}{\sqrt{\xi_1^2 + \xi_2^2}}
\end{bmatrix},$$

where $E = \sqrt{\xi_1^2 + \xi_2^2}$. This ensures that $Q^TQ = I$ (orthogonal matrix), and therefore

$$v \cdot v = Qw \cdot Qw = w \cdot Q^TQw = w \cdot w.$$

Moreover,

$$Q^T \xi = \begin{bmatrix}
\xi_1 & \xi_2 & \xi_3 \\
\frac{\xi_1}{\sqrt{\xi_1^2 + \xi_2^2}} & -\frac{\xi_2}{\sqrt{\xi_1^2 + \xi_2^2}} & \frac{\xi_3}{\sqrt{\xi_1^2 + \xi_2^2}} \\
\frac{\xi_2}{\sqrt{\xi_1^2 + \xi_2^2}} & \frac{\xi_1}{\sqrt{\xi_1^2 + \xi_2^2}} & \frac{\xi_3}{\sqrt{\xi_1^2 + \xi_2^2}} \\
\frac{\xi_3}{\sqrt{\xi_1^2 + \xi_2^2}} & \frac{\xi_2}{\sqrt{\xi_1^2 + \xi_2^2}} & \frac{\xi_1}{\sqrt{\xi_1^2 + \xi_2^2}}
\end{bmatrix}^T \begin{bmatrix}
\xi_1 \\
\xi_2 \\
\xi_3
\end{bmatrix} = \begin{bmatrix}
|\xi| \\
0 \\
0
\end{bmatrix}.$$
Returning to the integral of the real-part of \( k(\xi) \) under the \( ai \) assumption, we have

\[
\int_{\mathbb{R}^3} \frac{(v \cdot \xi)\phi(v)}{(1 + a)^2 + (v \cdot \xi)^2} dv = \int_{\mathbb{R}^3} \frac{|\xi|w_1}{(1 + a)^2 + (|\xi|w_1)^2} \cdot \frac{e^{-w_1^2 - w_2^2 - w_3^2}}{\pi^{3/2}} dw = \frac{1}{|\xi|^{3/2}} \int_{\mathbb{R}^3} \frac{w_1}{c^2 + w_1^2} e^{-w_1^2} dw_1 w_2 \quad \text{where } c = 1 + a.
\]

As the integrand in \( w \)-space is separable, we have

\[
\int_{\mathbb{R}^3} \frac{(v \cdot \xi)\phi(v)}{(1 + a)^2 + (v \cdot \xi)^2} dv = \frac{1}{|\xi|^{3/2}} \int_{\mathbb{R}} \frac{w_1}{c^2 + w_1^2} e^{-w_1^2} dw_1 = 0,
\]
since the resultant integrand in \( w_1 \) is an odd-function. In the end, we have reduced the \( k(\xi) = ai \) condition to the equation

\[
a = -\int_{\mathbb{R}^3} \frac{(v \cdot \xi)\phi(v)}{(1 + a)^2 + (v \cdot \xi)^2} dv.
\]

We will now show that this last equation results in a constraint on the freedom of \( \xi \) in our subclass of solutions.

Consider the impact of the change of variables \( Q \) on the last equation. By essentially the same arithmetic above, we see that

\[
a = -\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{w_1^2}{c^2 + w_1^2} e^{-w_1^2} dw_1 \quad \text{where } c = 1 + a.
\]

Thinking of this last equation as a function of \( c \), we get

\[
-1 + c|\xi| = -\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{w_1^2}{c^2 + w_1^2} e^{-w_1^2} dw_1
\]

\[
1 - c|\xi| = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{-c^2 + c^2 + w_1^2}{c^2 + w_1^2} e^{-w_1^2} dw_1 = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{-c^2}{c^2 + w_1^2} e^{-w_1^2} dw_1 \quad \text{and} \quad \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-w_1^2} dw_1.
\]

As the last integral is just the standard normal density function over \( \mathbb{R} \), we see that

\[
-c|\xi| = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{-c^2}{c^2 + w_1^2} e^{-w_1^2} dw_1.
\]

This results in a very specific constraint on \( |\xi| \),

\[
|\xi| = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{c}{c^2 + w_1^2} e^{-w_1^2} dw_1.
\]

To better understand this, let us define the function \( \Xi(c) \) as follows:

\[
\Xi(c) := \int_{\mathbb{R}} \frac{c\phi(v)}{c^2 + v^2} dv. \tag{27}
\]

Note that we now are able to represent \( |\xi| \) as a parametric function of \( c \). To examine the values of \( |\xi| \) defined over the range of \( c \), we begin with the following graphical observation, Figure 1.

It appears that for the solution class, our transform variable is bounded. In fact, we can show that \( |\xi| = \Xi(c) \in (0, \sqrt{\pi}) \).

**Proposition 8.** Let \( \Xi(c) \) be defined as in (27). Then \( \lim_{c \to 0^+} \Xi(c) = \sqrt{\pi} \).
Proof. Note that for this limit, \( c > 0 \). Then

\[
\lim_{c \to 0^+} \Xi(c) = \lim_{c \to 0^+} \int_\mathbb{R} \frac{c}{\sqrt{\pi}} \frac{e^{-v^2}}{c^2 + v^2} \, dv
\]

when \( v = cu \)

\[
= \frac{1}{\sqrt{\pi}} \int_\mathbb{R} \frac{1}{1 + u^2} \, du
\]

\[
= \arctan(u) \bigg|_0^\infty
\]

\[
= \sqrt{\pi}.
\]

It is also clear that \( \lim_{c \to \infty} \Xi(c) = 0 \). We conclude that \( |\xi| \in (0, \sqrt{\pi}) \).

We have reached a point in the calculations where, if we can represent \( c \) as a function of \( |\xi| \), we would be able to unwind the above calculations and find a representation of the transformed solution. We now seek the inverse of \( \Xi(c) \).

Graphically, the function \( \Xi(c) \) appears to be a strictly decreasing function. We will show that \( \Xi(c) \) is strictly decreasing, thus proving that \( \Xi(c) \) is a one-to-one function. Hence \( \Xi(c) \) is invertible.

**Proposition 9.** \( \Xi(c) \) is a strictly decreasing function.

**Proof.** Without loss of generality, let \( c_1 > c_2 \) and \( c_i \in (0, \infty) \). Then

\[
\Xi(c_1) - \Xi(c_2) = \int_\mathbb{R} \frac{c_1}{c_1^2 + v^2} \phi(v) \, dv - \int_\mathbb{R} \frac{c_2}{c_2^2 + v^2} \phi(v) \, dv
\]

\[
= (c_1 - c_2) \int_\mathbb{R} \frac{(-c_1c_2 + v^2)\phi(v)}{(c_1^2 + v^2)(c_2^2 + v^2)} \, dv.
\]

Note that \( \frac{(-c_1c_2 + v^2)\phi(v)}{(c_1^2 + v^2)(c_2^2 + v^2)} \) is an even function in \( v \). It will be sufficient to understand the resultant integral on \((0, \infty)\). Note that the integrand is negative on \((0, \sqrt{c_1c_2})\) and positive on \((\sqrt{c_1c_2}, \infty)\). Splitting the integral, we have

\[
\int_0^{\sqrt{c_1c_2}} \frac{(-c_1c_2 + v^2)\phi(v)}{(c_1^2 + v^2)(c_2^2 + v^2)} \, dv + \int_{\sqrt{c_1c_2}}^{\infty} \frac{(-c_1c_2 + v^2)\phi(v)}{(c_1^2 + v^2)(c_2^2 + v^2)} \, dv.
\]
Bounding the negative integral below and the positive integral above results in
\[
\int_0^\infty \frac{(-c_1 c_2 + v^2)\phi(v)}{(c_1^2 + v^2)(c_2^2 + v^2)} \, dv 
\leq \phi(\sqrt{c_1 c_2}) \left[ \int_0^{\sqrt{c_1 c_2}} \frac{(-c_1 c_2 + v^2)}{(c_1^2 + v^2)(c_2^2 + v^2)} \, dv + \int_{\sqrt{c_1 c_2}}^\infty \frac{(-c_1 c_2 + v^2)}{(c_1^2 + v^2)(c_2^2 + v^2)} \, dv \right] 
\leq 2\phi(\sqrt{c_1 c_2}) \left[ 2(\arctan \sqrt{c_2/c_1} - \arctan \sqrt{c_1/c_2}) - \pi \right].
\]

Then
\[
\Xi(c_1) - \Xi(c_2) = (c_1 - c_2) \int_\mathbb{R} \frac{(-c_1 c_2 + v^2)\phi(v)}{(c_1^2 + v^2)(c_2^2 + v^2)} \, dv 
\leq 2\phi(\sqrt{c_1 c_2}) \left[ 2(\arctan \sqrt{c_2/c_1} - \arctan \sqrt{c_1/c_2}) - \pi \right] < 0
\]
since \(\arctan \sqrt{c_2/c_1} - \arctan \sqrt{c_1/c_2} < \pi/2\). (Recall \(c_i \neq 0\).) Hence, we have shown that \(\Xi(c)\) is a strictly decreasing function. \(\square\)

4.3.3. The solution class of grossly determined solutions. We are now ready to prove Main Theorem 1.

Proof of Main Theorem 1. By Claim 9, \(\Xi(c)\) is invertible. Define \(C(|\xi|) := \Xi^{-1}(|\xi|)\). Then \(|\xi| = \Xi(c)\) defines \(c\) implicitly as \(c = \Xi^{-1}(|\xi|) = C(|\xi|)\) for values \(|\xi| \in (0, \sqrt{\pi})\). Unwinding the preceding computations, we can now show that a class of grossly determined solutions exists:

1. The parameter \(c = C(|\xi|)\) exists as an invertible function of \(|\xi|, \xi \in B'(0, \sqrt{\pi})\), the punctured open sphere of radius \(\sqrt{\pi}\) centered at the origin.
2. The function \(k(\xi)\) can be represented as \(k(\xi) = (-1 + |\xi|c)i\), since \(c = (1 + a)/|\xi|\) and \(k = ai\).
3. On \(B'(0, \sqrt{\pi})\), \(k(\xi)\) exists and:
   (a) \(K_\xi\) exists by (25),
   (b) \(\hat{\rho}\) exists via solving the PDE (24) and the solution is
   \(\hat{\rho}(t, \xi) = \hat{\rho}_0(\xi)e^{-ik(\xi)t}\).
4. Off of \(B'(0, \sqrt{\pi})\), equation (21) requires \(\hat{\rho}(t, \xi)\) to be zero. Hence, \(\hat{\rho}(t, \xi)\) has support exclusively in \(B'(0, \sqrt{\pi})\).
5. We now have the representation of
   \[K_\xi \hat{\rho} = \left( \frac{1}{1 + i(|\xi| - k(\xi))} \right) \hat{\rho}_0(\xi)e^{-ik(\xi)t}.\]

Thus a class of grossly determined solutions, each solution dependent upon its own density field, is given by
\[f(t, x, v) = \int_{\mathbb{R}^3} K_\xi(y)\hat{\rho}(t, x - y) \, dy.\]
4.4. Behavior of the grossly determined solutions. It is clear from the form of $\hat{f}$, that the class of grossly determined solutions will be $C^\infty$. This is due to the form of the kernel $\hat{K}_v$ and the fact that $\hat{\rho}$ and $\hat{K}_v$ are of compact support in $\xi$. This regularity result is in keeping with known grossly determined solutions: such as Maxwellian distribution (where density, velocity and temperature are treated as fixed parameters), and the solutions due to the iterative method of Hilbert or the Chapman-Enskog procedure. Unfortunately, there is nothing immediate from the form of $\hat{f}$ that suggests time evolution towards a Maxwellian. This is also in keeping with known grossly-determined solutions. In [26, pg. 285], Muncaster constructs a grossly determined solution that does not approach a Maxwellian density in the course of time. In other words, there is no approach to equilibrium for the grossly determined solution he constructed. This fact does not violate the conjecture of Truesdell and Muncaster in any way. Their conjecture is that it is the entire subclass of grossly determined solutions that act as the limit set for the class of general solutions. In [5], we demonstrate that the conjecture holds for the one dimensional version of (2) originally explored by Cercignani. Showing the same for (2) remains an open-problem.

5. Conclusions. In the terms of Truesdell and Muncaster’s conjectures on grossly determined solutions, we have established the existence of a class of grossly determined solutions for a Boltzmann-like equation. Specifically, given the density of a gas at an initial time, we are able to state the convolution solution (for all time) for an inhomogeneous transport equation with modified linearized collisions operator. In a companion paper, we will demonstrate that the class of general solutions to the one dimensional version of (2) does have the property that, in time, each member decays to a solution from the subclass of grossly determined solutions.

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